Schrödinger’s equation with gauge coupling derived from a continuity equation

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A quantization procedure without Hamiltonian is reported which starts from a statistical ensemble of particles of mass $m$ and an associated continuity equation. The basic variables of this theory are a probability density $\rho$, and a scalar field $S$ which defines a probability current $\vec{j} \equiv \rho \nabla S/m$. A first equation for $\rho$ and $S$ is given by the continuity equation. We further assume that this system may be described by a linear differential equation for a complex-valued state variable $\chi$. Using these assumptions and the simplest possible Ansatz $\chi(\rho, S)$, for the relation between $\chi$ and $\rho, S$, Schrödinger’s equation for a particle of mass $m$ in a mechanical potential $V(q, t)$ is deduced. For simplicity the calculations are performed for a single spatial dimension (variable $q$). Using a second Ansatz $\chi(\rho, S, q, t)$, which allows for an explicit $q, t$-dependence of $\chi$, one obtains a generalized Schrödinger equation with an unusual external influence described by a time-dependent Planck constant. All other modifications of Schrödinger’ equation obtained within this Ansatz may be eliminated by means of a gauge transformation. Thus, this second Ansatz may be considered as a generalized gauging procedure. Finally, making a third Ansatz, which allows for a non-unique external $q, t$-dependence of $\chi$, one obtains Schrödinger’s equation with electrodynamic potentials $\vec{A}, \phi$ in the familiar gauge coupling form. This derivation shows a deep connection between non-uniqueness, quantum mechanics and the form of the gauge coupling. A possible source of the non-uniqueness is pointed out.

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1 Introduction

Usually a physical system, which one wants to describe quantum mechanically, is first identified in the context of classical physics and then somehow transferred to the quantum mechanical domain; this process is referred to as “quantization”. Often the first step in the quantization process is the tacit assumption that a Hilbert space is associated with the examined system. Then, the remaining task is to find the proper algebra of operators. A more direct method which avoids this assumption, is to “derive” Schrödinger’s equation, i.e. to find premises which imply Schrödinger’s equation. This may be done in several ways. The method which was historically at the beginning of quantum mechanics [21] (“wave mechanics”) starts from the Hamilton-Jacobi equation of a classical system and tries to deduce from it - with the help of suitable modifications [3, 4, 7] - the corresponding quantum-mechanical equation for the time development of the system. Other premises leading to Schrödinger’s equation include special assumptions about the structure of momentum fluctuations [8] and the principle of minimum Fisher information [18].

In this paper, a new quantization procedure is reported which shares with the last two examples the property that it does not start from a single-particle picture but from a statistical ensemble. The simplest nontrivial system, a spinless particle of mass $m$ in nonrelativistic approximation is investigated. In order to define the subject of this work more precisely we start from the classical Hamilton-Jacobi equation for the action function $S(\vec{q}, t)$, which depends on the particle coordinates $q_k$ and the time $t$. It is given by

$$ \frac{\partial S(\vec{q}, t)}{\partial t} + \frac{1}{2m} \left( \frac{\partial S(\vec{q}, t)}{\partial \vec{q}} \right)^2 + V(\vec{q}, t) = 0, \quad (1) $$
if the movement takes place under the influence of a potential \( V(\vec{q}, t) \). The momentum field, that appears in Eq. (1) is given by

\[ p_k(\vec{q}, t) = \frac{\partial S(\vec{q}, t)}{\partial q_k}. \tag{2} \]

The fact that the Hamilton-Jacobi equation is the ideal starting point for the transition from classical physics to quantum mechanics is formally based on the following well-known reformulation of the time-dependent Schrödinger equation

\[ \frac{\hbar}{i} \frac{\partial}{\partial t} \psi(\vec{q}, t) + V(\vec{q}, t)\psi(\vec{q}, t) = \frac{\hbar^2}{2m} \nabla \psi(\vec{q}, t). \tag{3} \]

If the complex-valued variable \( \psi(\vec{q}, t) \) is, without any restrictions of generality, written in the form

\[ \psi(\vec{q}, t) = \sqrt{\rho(\vec{q}, t)} e^{i \frac{\hbar}{\sqrt{2}} S(\vec{q}, t)}, \tag{4} \]

then one obtains from Eq. (3), by calculating the real parts of both sides, the relation

\[ \frac{\partial \rho(\vec{q}, t)}{\partial t} + \frac{\partial \rho}{\partial \vec{q}} \frac{\partial S(\vec{q}, t)}{\partial \vec{q}} = 0, \tag{5} \]

which is a classical (no \( \hbar \) occurs) continuity equation for the probability density \( \rho \) and the probability current \( \rho \vec{p}/m \). Equating the imaginary parts of both sides of Eq. (3) one obtains the relation

\[ \frac{\partial S(\vec{q}, t)}{\partial t} + \frac{\hbar^2}{2m} \frac{\nabla \rho(\vec{q}, t)}{\sqrt{\rho(\vec{q}, t)}} = 0, \tag{6} \]

which differs from the Hamilton-Jacobi equation (1) only by the single term on the right hand side. Eq. (6) is sometimes referred to as quantum Hamilton-Jacobi equation.

Due to this similarity there have been attempts \cite{4, 7} to use Eq. (1) as a starting point and to derive Eq. (3), which presents the basis of quantum mechanics, by justifying introduction of the crucial quantum term appearing in Eq. (6). In the present work we go a different route, starting from assumptions which are simpler in certain respects. We postulate the existence of a statistical ensemble but do not start from the Hamilton-Jacobi equation itself. Instead, our basic postulate is the validity of a continuity equation (of the above type), interpreted as a local conservation law of probability. Since we have two unknown functions (\( \rho \) and \( S \)) and only one single equation, we clearly need further assumptions - and a second differential equation - in order to arrive at a mathematically well-defined problem. Our second assumption is very simple and of a purely formal nature. We require that both equations - the one already known and the second one still to be found - may be expressed mathematically as a single equation for a single complex state variable \( \chi \). This second assumption expresses something like the postulate of maximal mathematical simplicity. As we know, this postulate may be quite successful in physics, in particular if combined with other ideas.

Thus, the continuity equation has to be “extended” to the complex domain. This task may be described briefly as follows: Consider a complex-valued variable \( \chi \) which depends in an unspecified way on the real variables \( S \) and \( \rho \). Which differential equations for \( \chi \) exist, whose real (or imaginary) part agrees with the continuity equation and which functional dependencies \( \chi(\rho, S) \) are compatible with this requirement? Note that both the functional dependence of \( \chi \) and the shape of the differential equation are unknown “variables” of this problem. A detailed formulation of this problem is reported in the next section \cite{2}. The calculation, reported in section \cite{3} and appendix \cite{A} has been performed for simplicity for a single spatial dimension and for a reduced class of differential equations obeying several additional constraints. The result is Schrödinger’s equation for a particle in an external mechanical potential. Further, we formulate and justify in section \cite{4} the conjecture that all additional constraints except linearity may be omitted.
In section 3 the original Ansatz is extended by allowing for an additional, explicit space-time dependence of the state variable $\chi$. This is our first attempt to derive the minimal coupling rule, which is obeyed by essentially all fundamental interactions, in the present context. It turns out (details of the calculation are reported in appendix [3]) that a second, very unusual external influence, besides the potential $V$, appears in the Schrödinger equation. It takes the form of a time-dependent Planck constant. All other modifications due to the extended Ansatz are spurious, because they may be eliminated from the Schrödinger equation by a gauge transformation. The gauge field itself cannot be derived by means of this Ansatz. In section 5 our second attempt is undertaken to derive a gauge field. The Ansatz of the last section is once more extended by allowing for a non-unique space-time dependence of $\chi$. The result is Schrödinger’s equation for a charged particle in an external electromagnetic field.

Section 6 contains a detailed discussion of all assumptions and results and may be consulted in a first reading to obtain an overview of this work. It also contains remarks of a speculative nature concerning the relation between the classical theory of charged particles and fields on the one hand and the form of quantum mechanics and gauge coupling on the other hand. In the last section 7 one finds concluding remarks.

2 Formulation

We consider a statistical ensemble which is described by a probability density $\rho(\vec{q},t)$ and a probability current $\vec{j} = \rho \vec{p}/m$. We assume that the momentum field $\vec{p}$ may be written as the gradient of a scalar function $S(\vec{q},t)$, as in (2); this simplest possible form of $\vec{p}$ holds in Hamilton-Jacobi theory [20]. Our ensemble can thus be described by two variables, the real fields $\rho$ and $S$. From the continuity equation we obtain immediately Eq. (5) as a first differential equation for $\rho$ and $S$. The physical meaning of the function $S(\vec{q},t)$ is still unclear. Of course, in the classical limit of the theory to be constructed, $S(\vec{q},t)$ should agree with the action function of classical mechanics.

We are confronted with the problem of finding a second differential equation for $\rho$ and $S$. Two solutions of this problem may be found in nature, classical statistical mechanics and quantum theory. Let us compare these two solutions trying to learn something about the quantization process. The second equation is (1) in classical physics and (3) in quantum theory. One sees immediately that in quantum theory the additional term in (6) implies a dramatic change of the basic concepts. This term is not an externally controlled input parameter but describes a coupling between the identity of the examined object and the statistics. But let us put aside these physical aspects. Instead let us ask the more formal question in which of the two theories the task of formulating the basic equations has been solved in a simpler way. This is certainly quantum theory, because the two differential equations (5) and (6) can be combined to a single equation, the Schrödinger equation (3) which has on top of that a simple linear structure. A similar unification is not possible in classical physics (one finds [20] a nonlinear equation containing both $\psi$ and $\psi^\ast$). This situation suggests that this principle of simplicity may be used, turning the logical direction around, to derive quantum mechanics from classical mechanics or to use it at least as an essential part of the quantization process.

This principle of simplicity implies that the two differential equations for $\rho$ and $S,$ namely Eq. (5) and the second one which is still to be found, may be combined into a single equation for a two-component variable $\chi,$ which we assume to be a complex quantity. The variable $\chi$ replaces $\rho$ and $S$ and should of course be a function of $\rho$ and $S$. If we define $\chi$ in terms of its real and imaginary parts (which has the advantage of linearity in comparison with the polar representation) $\chi$ takes in the simplest case (more general relations will be considered in the next sections) the form

$$\chi(\rho, S) = \chi_1(\rho, S) + i\chi_2(\rho, S).$$

We assume furthermore that the differential equation for $\chi,$ which will be referred to as “state equation”, is a partial differential equation (with complex coefficients). The independent variables of this equation must be the usual space-time variables $q_\mu, t,$ which also occur in the continuity equation.

Each differential equation for $\chi$ whose real (or imaginary) part agrees with the continuity equation represents a possible extension of the latter. Its imaginary (or real) part provides us automatically with
a second differential equation for our two variables. Thus, each one of these equations defines, in a purely formal sense, a physical theory motivated by the principle of simplicity. We denote the class of differential equations defined in this way by \( C \). We know that \( C \) is not empty. It contains certainly Schrödinger’s equation (3); it is a calculation of few lines to derive the continuity equation (5) starting from (3). But the inverse problem, to start from the continuity equation (5) and to derive the set of all extensions, i.e. the set \( C \), may be less trivial. In particular one cannot assume that \( C \) contains only a single equation.

Being interested in the transition to quantum mechanics, we want to know under which conditions Schrödinger’s equation can be derived. For each quantization procedure the important question to ask is how the final result is obtained. For that reason we make no use of symmetry considerations of any kind in this paper. Thus, which additional conditions are necessary in order to single out, from the set \( C \), Schrödinger’s equation as only remaining equation? We introduce for brevity a symbol, say \( A \), to denote the set of all additional conditions defined in this way. \( C \) and \( A \) define together a quantization procedure. Obviously, this quantization procedure will only be convincing for a small number of (physically appealing) additional conditions, i.e. if \( A \) is “small”. What we are looking for is the smallest set of such assumptions.

If \( A \) should turn out to be empty, this would result in a very impressive quantization procedure; it would mean that quantum mechanics could be derived from only two assumptions, the continuity equation and the existence of a complex state variable [as well as the tacit assumption that the laws of nature may be formulated as differential equations of the conventional type, compare section 6]. In the next section it will be shown that this is not the case. However, the results indicate that only a single additional condition, the linearity of the differential equation, is required.

How to find concretely the set \( A \)? This set \( A \) is the smallest set of assumptions which eliminates all equations except the Schrödinger equation from the set \( C \). An obvious strategy is to start from a set of strong assumptions, say \( A_1 \), which defines a small class of differential equations (all being loosely speaking “similar” to Schrödinger’s equation). If the conditions implied by \( A_1 \) and \( C \) single out Schrödinger’s equation, one may eliminate some of the conditions contained in \( A_1 \) and test if the corresponding, smaller set \( A_2 \) of conditions may be used instead to lead to the same result. The final solution is the smallest possible set \( A \) obtained in this way.

In this paper a complete solution to this problem is not given. All explicit calculations are performed using a set \( A_1 \), defined by the following constraints for the state equation:

- linearity,
- a single space dimension (variable \( q \)),
- only derivatives of first order in \( t \),
- only derivatives up to the second order in \( q \),
- no mixed derivatives.

However, the structure of the solutions leads to the conjecture on \( A \) mentioned above, which is discussed in more detail in the next section.

3 No interaction

In the framework of the strategy set up in the last section the possibility of interaction, i.e. of an external influence on our single particle system, was not taken into account. In principle there are two possibilities to do that; we may either modify the set \( A \) or the definition of \( \chi \). In this work the second, more general, possibility will be chosen. In this sense, the simplest Ansatz \( \chi = \chi(\rho, S) \), dealt with in the present section, describes a situation without interaction. The meaning of the term interaction defined in this way may differ (and does in fact differ) from the conventional one; we continue to use it for simplicity.

We now have the following concrete mathematical problem: Find all differential equations obeying the conditions \( A_1 \) whose real part (or imaginary part - this can be fixed arbitrarily and does not represent
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a real constraint) agrees - after splitting off an arbitrary multiplicative factor - with the one-dimensional continuity equation. Our unknown variables are the coefficients of the differential equation, the state function $\chi$, and the factor $F$.

The basic complex-valued dynamic variable $\chi(\rho, S)$ is a function of the real variables $\rho(q, t)$ and $S(q, t)$ and is written in the above form (7). The multiplicative factor may be an arbitrary complex function $F(\rho, S, q, t)$. The coefficients of the linear differential equation are denoted by $a, b, d, e$ and are arbitrary complex functions of $q, t$. Then, the fundamental requirement implied by $C$ and $A_1$ takes the form

$$\Re \left[ F \left(a \frac{\partial}{\partial t} \chi + b \frac{\partial}{\partial q} \chi + d \frac{\partial^2}{\partial q^2} \chi + e \chi \right) \right] = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial q} \frac{1}{m} \frac{\partial S}{\partial q} + \frac{1}{m} \frac{\partial^2 S}{\partial q^2}. \tag{8}$$

The right hand side of (8) is given by the continuity equation.

We have only a single equation (8) and many unknowns, namely the complex functions $\chi(\rho, S), a(q, t), b(q, t), d(q, t), e(q, t)$. However, Eq. (8) is a very strong requirement because the variables $\rho, S$ are arbitrary. Consequently, the coefficients of $\rho, S$ - and of all derivatives of $\rho, S$ - must agree on both sides of (8). Using this fact and introducing real and imaginary parts of $a$, . . . , $e$ and $F$ according to $a = a_1 + ia_2, . . . , e = e_1 + ie_2$ and $F = F_1 + iF_2$, we obtain from Eq. (8) the following 10 conditions

$$\left( a_1 F_1 - a_2 F_2 \frac{\partial \chi_1}{\partial S} - (a_2 F_1 + a_1 F_2) \frac{\partial \chi_2}{\partial S} \right) = 0, \tag{9}$$
$$\left( d_1 F_1 - d_2 F_2 \frac{\partial^2 \chi_1}{\partial \rho^2} - (d_2 F_1 + d_1 F_2) \frac{\partial^2 \chi_2}{\partial \rho^2} \right) = 0, \tag{10}$$
$$\left( d_1 F_1 - d_2 F_2 \frac{\partial \chi_1}{\partial S} - (d_2 F_1 + d_1 F_2) \frac{\partial \chi_2}{\partial S} \right) = 0, \tag{11}$$
$$\left( b_1 F_1 - b_2 F_2 \frac{\partial \chi_1}{\partial \rho} - (b_2 F_1 + b_1 F_2) \frac{\partial \chi_2}{\partial \rho} \right) = 0, \tag{12}$$
$$\left( b_1 F_1 - b_2 F_2 \frac{\partial \chi_1}{\partial S} - (b_2 F_1 + b_1 F_2) \frac{\partial \chi_2}{\partial S} \right) = 0, \tag{13}$$
$$\left( d_1 F_1 - d_2 F_2 \frac{\partial \chi_1}{\partial \rho} - (d_2 F_1 + d_1 F_2) \frac{\partial \chi_2}{\partial \rho} \right) = 0, \tag{14}$$
$$\left( a_1 F_1 - a_2 F_2 \frac{\partial \chi_1}{\partial \rho} - (a_2 F_1 + a_1 F_2) \frac{\partial \chi_2}{\partial \rho} \right) = 1, \tag{15}$$
$$\left( d_1 F_1 - d_2 F_2 \frac{\partial^2 \chi_1}{\partial \rho \partial S} - (d_2 F_1 + d_1 F_2) \frac{\partial^2 \chi_2}{\partial \rho \partial S} \right) = \frac{1}{2m}, \tag{16}$$
$$\left( d_1 F_1 - d_2 F_2 \frac{\partial \chi_1}{\partial S} - (d_2 F_1 + d_1 F_2) \frac{\partial \chi_2}{\partial S} \right) - \frac{\rho}{m}, \tag{17}$$
$$\left( e_1 F_1 - e_2 F_2 \chi_1 - (e_2 F_1 + e_1 F_2) \chi_2 \right) = 0. \tag{18}$$

We have 12 unknown quantities and 10 equations, but the unknown variables $a_1, a_2, . . . , e_1, e_2$ and $F_1, F_2$ occur in (9)-(18) only in the combinations

$$\bar{a}_1 = a_1 F_1 - a_2 F_2, \quad \bar{a}_2 = a_2 F_1 + a_1 F_2, \tag{19}$$
$$\bar{b}_1 = b_1 F_1 - b_2 F_2, \quad \bar{b}_2 = b_2 F_1 + b_1 F_2, \tag{20}$$
$$\bar{d}_1 = d_1 F_1 - d_2 F_2, \quad \bar{d}_2 = d_2 F_1 + d_1 F_2, \tag{21}$$
$$\bar{e}_1 = e_1 F_1 - e_2 F_2, \quad \bar{e}_2 = e_2 F_1 + e_1 F_2. \tag{22}$$

Thus, we have 10 equations for 10 unknown quantities $\bar{a}_1, \bar{a}_2, . . . , \bar{e}_1, \bar{e}_2, \chi_1, \chi_2$. Most of the equations (9)-(18) look like differential equations. However, the coefficients appearing in these equations do also belong to our set of unknown variables. Thus, relations (9)-(18) may either be used as differential equations for the unknown functions $\chi_1(\rho, S), \chi_2(\rho, S)$ . . . or - if these functions are already known - as constraints for the unknown coefficients. In the latter case, one has relations which must hold for arbitrary $\rho, S$, i.e. for
each one of the considered conditions [out of the set (10)-(18)] the coefficients of all linear independent functions of $\rho, S$ must vanish separately. While this type of problem may be somewhat unusual, its solution is straightforward and does not require any unusual mathematical methods (its solution could possibly be simplified by using more sophisticated methods).

The first part of the calculation, the determination of $\chi_1, F_1$ is described in detail in appendix A. The result is

$$
\chi_1(\rho, S) = -\frac{2m|d|^2}{c_2} \sqrt{\rho} \cos \left( \frac{c_2}{2m|d|^2} S + C_5 \right) e^{-\frac{c_1}{2m|d|^2} S - C_6} + C_3,
$$

$$
\chi_2(\rho, S) = -\frac{2m|d|^2}{c_2} \sqrt{\rho} \sin \left( \frac{c_2}{2m|d|^2} S + C_5 \right) e^{-\frac{c_1}{2m|d|^2} S - C_6} + C_4,
$$

and

$$
F_1(\rho, S, q, t) = -\frac{\sqrt{\rho}}{m|d|^2} \left[ d_1 \sin \left( \frac{c_2}{2m|d|^2} S + C_5 \right) + d_2 \cos \left( \frac{c_2}{2m|d|^2} S + C_5 \right) \right] e^{-\frac{c_1}{2m|d|^2} S + C_6},
$$

$$
F_2(\rho, S, q, t) = -\frac{\sqrt{\rho}}{m|d|^2} \left[ d_1 \cos \left( \frac{c_2}{2m|d|^2} S + C_5 \right) - d_2 \sin \left( \frac{c_2}{2m|d|^2} S + C_5 \right) \right] e^{-\frac{c_1}{2m|d|^2} S + C_6},
$$

where $c_1, c_2$ are linear combinations of coefficients, defined in Eq. (14), and $|d|$ is the modulus of the complex number $d$.

Only a part of all conditions, namely (9), (10), (12), (13), (16), and (17) have been used in the course of the calculations leading to Eqs (23)-(26) (see appendix A). In these conditions only the coefficients $a_i$ and $d_i$ appear. Conditions (12), (13), containing the $b_i$, and condition (18), containing the $e_i$, have not been used. The conditions not yet used play the role of constraints for the constants of integration $C_3, C_4, C_5, C_6$ and the real- and imaginary parts of the coefficients $a, b, d, e$. We will, for brevity, refer to the totality of all these quantities as “parameters”.

Our 12 parameters are constants with respect to the variables $\rho, S$, but may be arbitrary functions of $q, t$, as far as the calculation reported in appendix A is concerned. However, they have to obey the basic requirement of this section, that $\chi$ depends only on $\rho, S$, and not on $q, t$. This implies that the following parameters or combinations of parameters

$$
C_3, C_4, C_5, C_6, \frac{c_1}{2m|d|^2}, \frac{c_2}{2m|d|^2}
$$

are constants (we use this term now in the usual sense of being independent of $q, t$).

In order to obtain the explicit form of the constraints for the parameters, Eqs. (23)-(26) have to be inserted in the conditions not yet used (11), (12), (13) and (18). These conditions hold for arbitrary values of $\rho, S$. This leads to $C_3 = C_4 = 0$ while no constraints exist for $C_5, C_6$. For the coefficients one obtains the conditions

$$
c_1 = a_1d_1 + a_2d_2 = 0, \quad (28)
$$

$$
d_1b_1 + d_2b_2 = 0, \quad (29)
$$

$$
d_1b_2 - d_2b_1 = 0, \quad (30)
$$

$$
a_2d_1 - a_1d_2 = r_1 (d_1^2 + d_2^2), \quad (31)
$$

$$
e_2d_1 - e_1d_2 = 0, \quad (32)
$$
where \( r_1 \) in (31) is an arbitrary real constant. As a consequence of (28)-(32) all parameters (with the exception of the arbitrary real constants \( C_5 \) und \( C_6 \)) may be expressed in terms of three arbitrary real functions \( d_1(q,t), d_2(q,t), f(q,t) \) and the real number \( r_1 \). In terms of the complex coefficients one obtains the following result: \( d \) is an arbitrary complex-valued function of \( q \) und \( t \), \( b = 0 \), and \( a \) and \( e \) are determined by \( d \) according to the linear relation

\[ a = ir_1 d, \quad e = fd. \]  

Now, all conditions have been taken into account. The complex quantities \( \chi \) and \( F \) may be written as

\[ \chi = -\frac{2m}{r_1} e^{-c_6 \sqrt{\rho} e^{i\left(\frac{r_1^2}{2m} S + C_5\right)}}, \]

\[ F = \sqrt{\rho} e^{c_6 e^{-i\left(\frac{r_1^2}{2m} S + C_5\right)}}, \]

and the differential equation for \( \chi \) takes the form

\[ ir_1 \frac{\partial \chi}{\partial t} + e \frac{\partial^2 \chi}{\partial q^2} + f d \chi = 0. \]  

If \( S \) obeys the continuity equation (33), then it has the dimension of an action. Consequently, the constant \( 2m/r_1 \) has the dimension of an action; we identify this constant with Planck’s constant \( \hbar \). The constant \( d \) may be canceled and the constant \( e \) has the dimension \( cm^{-2} \). We may now come back to a more conventional notation by introducing quantities \( \psi(q,t) \) and \( V(q,t) \), which replace \( \chi(q,t) \) and \( f(q,t) \) and are defined by

\[ \psi(q,t) = \sqrt{\rho(q,t)} e^{i\frac{S(q,t)}{2m}}, \quad V(q,t) = -\frac{\hbar^2}{2m} f(q,t). \]  

Then, Eq. (36) takes the form

\[ -\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial q^2} + V(q,t) \psi, \]  

which agrees with the one-dimensional Schrödinger equation for a particle of mass \( m \) in an external, mechanical potential \( V(q,t) \) (the terms containing \( C_5, C_6 \), which correspond to the usual freedom in amplitude and phase, have not been written down).

Thus, using the Ansatz \( \chi = \chi(\rho,S) \) and the set of conditions \( A_1 \) we found that only the one-dimensional Schrödinger equation yields the continuity equation as its real part. More precisely, we found an infinite number of such equations differing from each other by the choice of an arbitrary function \( V(q,t) \). We obtained a coupling to an external potential but no coupling to a gauge field; the meaning of our term “no interaction” should be adjusted accordingly. If the interaction is considered as a secondary aspect, the above assumptions define a quantization procedure, i.e. a way to perform the transition from classical physics to quantum mechanics.

Can we improve this quantization procedure by reducing the number of assumptions contained in \( A_1 \) ? Let us keep first the linear structure as well as the one-dimensionality of the equation and allow for higher derivatives with respect to \( t \) und \( q \). Thus, new terms appear in the fundamental condition (33). Will these new terms survive or will the corresponding coefficients vanish? All derivatives of \( \rho \) and \( S \), which occur in the continuity equation, on the right hand side of (33), appeared already in the differential equation defined by \( A_1 \). Therefore, all the new terms, due to the higher derivatives, will have to vanish (the corresponding coefficients are all solutions of homoneneous equations) just as, in the above calculation, the term proportional to the first derivative with respect to \( q \) had to vanish.

As a next step, let us allow for three spatial dimensions and keep only the condition of linearity. To get an idea what happens in this case, preliminary calculations using, instead of (33), the fundamental
condition

\[ \Re \left[ F \left( a \frac{\partial}{\partial t} \chi + b_k \frac{\partial}{\partial q_k} \chi + c_k \frac{\partial}{\partial t} \chi + d_{ik} \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_k} \chi + f \frac{\partial^2}{\partial t^2} \chi + e \chi \right) \right] = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial q_k} \frac{1}{m} \frac{\partial S}{\partial q_k} + \rho \frac{1}{m} \frac{\partial S}{\partial q_k} \partial q_k, \] (39)

have been performed (summation from 1 to 3 over double indices). The differential equation displayed in (39) contains all derivatives of \( q_k, t \) up to second order taking into account also the possibility of anisotropic coefficients. These (not completed) calculations indicate (i) that the anisotropy of the coefficients is, as one would expect intuitively, suppressed by the isotropy of the continuity equation, (ii) that the coefficients of the higher derivatives vanish, for the reason just mentioned, and (iii) that an exact calculation starting from (39), which is completely analogous to the one above, leads to the three-dimensional Schrödinger equation (3).

The remaining condition in \( A_1 \) is the linearity of the differential equation. It can not be eliminated; one realizes immediately that e.g. a cubic nonlinearity of the form \( r |\psi|^2 \psi \), where \( r \) is real, is compatible with the continuity equation. Thus we conjecture that our final quantization procedure is defined by a smallest set \( A \), which contains only one single condition, namely the linearity of the differential equation.

What is the physical meaning of the remaining condition of linearity? A plausible explanation is that linearity is a consequence of the probabilistic interpretation [9] of the wave function, which excludes (in general) predictions about single events. There is a large literature on the relation between linearity and indeterminism; we only mention Mackey’s axioms for statistical theories [14] and Caticha’s derivation of quantum mechanics from the rules of manipulating probability amplitudes [1]. On the other hand it is well-known that macroscopic quantum phenomena exist in nature, such as superconductivity and superfluidity, which may be successfully described by nonlinear terms in the corresponding complex state variables. This is not in conflict with the above explanation. In contrast to the true wave function, the state variables of these many-body theories do not possess a probabilistic interpretation. They lost their “immaterial” (probabilistic) meaning in the thermodynamic limit and may be directly measured in single experiments. The condition of linearity is required to exclude such theories from the consideration.

4 First attempt to derive interaction

The Ansatz \( \chi = \chi(\rho, S) \) used in the last section led, somewhat unexpectedly, to a term \( V \psi \) in Schrödinger’s equation (38), describing an interaction by means of an external mechanical Potential \( V \). A term describing coupling to a gauge field did, not unexpectedly, not appear. In this section we start to study the following question: Is it possible to obtain this type of coupling, which is obeyed by all fundamental interactions, in the present framework? If possible it requires, at any rate, a different, properly generalized Ansatz.

There is an obvious possibility to generalize our Ansatz with regard to an external influence: we may allow for an additional explicit \( q, t \)-dependence of our state variable \( \chi \), i.e. write

\[ \chi = \chi(\rho, S, q, t), \] (40)

instead of \( \chi = \chi(\rho, S) \). Such an extension seems quite natural if one wants to describe an external, otherwise unspecified influence on the system described by \( \chi \). We will study the consequences of (40) keeping all other assumptions unchanged.

Using (40) does not change the appearance of the basic condition (8) in a fundamental way but additional derivatives with respect to the explicit \( q, t \)-dependence have to be taken into account. To indicate this difference, we replace the symbols for the partial derivatives by symbols for total derivatives. Thus, our fundamental condition, generalized according to the new Ansatz (40) takes the form

\[ \Re \left[ F \left( a \frac{\mathrm{d}}{\mathrm{d}t} \chi + b \frac{\mathrm{d}}{\mathrm{d}q} \chi + c \frac{\mathrm{d}^2}{\mathrm{d}q^2} \chi + e \chi \right) \right] = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial q_k} \frac{1}{m} \frac{\partial S}{\partial q_k} + \rho \frac{1}{m} \frac{\partial^2 S}{\partial q^2}. \] (41)
The form of the factor $F(\rho, S, q, t)$ and the coefficients $a, b, d, e$ remain unchanged; the latter may be arbitrary functions of $q, t$. The problem defined by (41) contains now, in comparison to (3), an even larger number of unknown functions. Fortunately, it will turn out that the equations to determine the $\rho, S$-dependence and those for the $q, t$-dependence are “decoupled” in the sense that they may be solved one after the other.

Comparing the coefficients of the derivatives on both sides of Eq. (41) we now obtain the following 10 conditions

\begin{align}
\bar{a}_1 \frac{\partial \chi_1}{\partial S} - \bar{a}_2 \frac{\partial \chi_2}{\partial S} &= 0, \\
\bar{d}_1 \frac{\partial^2 \chi_1}{\partial \rho^2} - \bar{d}_2 \frac{\partial^2 \chi_2}{\partial \rho^2} &= 0, \\
\bar{d}_1 \frac{\partial^2 \chi_1}{\partial S^2} - \bar{d}_2 \frac{\partial^2 \chi_2}{\partial S^2} &= 0, \\
\bar{b}_1 \frac{\partial \chi_1}{\partial \rho} - \bar{b}_2 \frac{\partial \chi_2}{\partial \rho} + 2 \left( \bar{d}_1 \frac{\partial^2 \chi_1}{\partial \rho \partial q} - \bar{d}_2 \frac{\partial^2 \chi_2}{\partial \rho \partial q} \right) &= 0, \\
\bar{b}_1 \frac{\partial \chi_1}{\partial S} - \bar{b}_2 \frac{\partial \chi_2}{\partial S} + 2 \left( \bar{d}_1 \frac{\partial^2 \chi_1}{\partial S \partial q} - \bar{d}_2 \frac{\partial^2 \chi_2}{\partial S \partial q} \right) &= 0, \\
\bar{d}_1 \frac{\partial \chi_1}{\partial \rho} - \bar{d}_2 \frac{\partial \chi_2}{\partial \rho} &= 0, \\
\bar{a}_1 \frac{\partial \chi_1}{\partial \rho} - \bar{a}_2 \frac{\partial \chi_2}{\partial \rho} &= 1, \\
\bar{d}_1 \frac{\partial^2 \chi_1}{\partial \rho \partial S} - \bar{d}_2 \frac{\partial^2 \chi_2}{\partial \rho \partial S} &= \frac{1}{2m}, \\
\bar{d}_1 \frac{\partial \chi_1}{\partial S} - \bar{d}_2 \frac{\partial \chi_2}{\partial S} &= \frac{\rho}{m}, \\
\bar{e}_1 \chi_1 - \bar{e}_2 \chi_2 + \bar{a}_1 \frac{\partial \chi_1}{\partial t} - \bar{a}_2 \frac{\partial \chi_2}{\partial t} + \bar{b}_1 \frac{\partial \chi_1}{\partial q} - \bar{b}_2 \frac{\partial \chi_2}{\partial q} + \bar{d}_1 \frac{\partial^2 \chi_1}{\partial q^2} - \bar{d}_2 \frac{\partial^2 \chi_2}{\partial q^2} &= 0.
\end{align}

Again, each one of the conditions (42)-(51) may split into several sub-conditions, if on the left hand side several linear independent functions of $\rho, S$ occur. Comparing with the previous “interaction-free” conditions (9)-(18), we see that seven of the Eqs. (42)-(51) agree with corresponding equations in the set (9)-(18). Only (45), (46), and (51) differ from the corresponding previous conditions (12), (13), and (18) by new terms. These new terms appear as a consequence of the explicit $q, t$-dependence of $\chi_i$ and lead obviously to a coupling of previously independent coefficients.

A great simplification of the problem defined by (42)-(51) takes place as a consequence of the fact that conditions (12), (13) and (18) have not been used during the calculation of the $\rho, S$-dependence of $\chi$ in section 3) (only conditions containing $a$ and $d$ were required). Therefore all conditions in the set (42)-(51) which are necessary to obtain this relationship remain unchanged. Since the additional $q, t$-dependence does not affect this part of the calculation, the formal results for $\chi$ and $F$ from section 3) may be taken over without modification for the present calculation. Thus, for the real and imaginary parts of $\chi$ we obtain

\begin{align}
\chi_1(\rho, S, q, t) &= -\frac{2m|d|^2}{c_2} \sqrt{\rho} \cos \left( \frac{c_2}{2m|d|^2} S + C_5 \right) e^{-\frac{\rho}{2m^2|d|^2} \rho - C_6} + C_3, \\
\chi_2(\rho, S, q, t) &= -\frac{2m|d|^2}{c_2} \sqrt{\rho} \sin \left( \frac{c_2}{2m|d|^2} S + C_5 \right) e^{-\frac{\rho}{2m^2|d|^2} \rho - C_6} + C_4.
\end{align}

The real and imaginary parts of $F(\rho, S, q, t)$ remain completely unchanged and are given by Eqs. (25) and (26). Of course, there is an important difference between the results for $\chi$ and $F$ of the last and the present section. In Eqs. (52) and (53) not only the coefficients but also the integration constants
C_3, C_4, C_5, C_6 may be arbitrary functions of \( q, t \) (similar remarks apply to \( F \)). Those conditions, from the set [12]–[51], which have not yet been used present constraints for the spatial and temporal variation of all these parameters.

In order to find the explicit form of these constraints the results [52], [53], [54], and [55] have to be inserted in the conditions not yet used, namely [14], [15], [16], and [51]. The calculation reported in detail in appendix [3] leads to the following results for the parameters. The quantities \( C_5, C_6 \) are arbitrary functions of \( q, t \), while \( C_5 = C_4 = 0 \). The complex coefficient \( d \) is an arbitrary function of \( q, t \). The remaining coefficients are given by

\[
a = i 2 m \tilde{u} d, \\
b = \left( 2 \frac{\partial C_6}{\partial q} - i 2 \frac{\partial C_5}{\partial q} \right) d, \\
c = (H_1 + i H_2) d,
\]

where \( H_1 \) is an arbitrary real function of \( q, t \), and \( H_2 \) is given by

\[
H_2 = 2 m \tilde{u} \left( \frac{1}{\tilde{u}} \frac{\partial \tilde{u}}{\partial t} + \frac{\partial C_5}{\partial q} \right) - 2 \frac{\partial C_5}{\partial q} \frac{\partial C_6}{\partial q} - \frac{\partial^2 C_5}{\partial q^2}.
\]

The quantity \( \tilde{u} \), defined in Eq. [147], is an arbitrary function of \( t \). The above relations show that all coefficients are proportional to \( d \). Therefore, since \( C_3 = C_4 = 0 \), the coefficient \( d \) drops out of the differential equation. Thus, we are left with three arbitrary real functions, \( C_5, C_6, H_1 \), of \( q, t \) and and a single arbitrary function \( \tilde{u} \) of \( t \). All other parameters may be expressed in terms of these four functions.

The complex quantities \( \chi \) and \( F \) are given by

\[
\chi = - \frac{\sqrt{p}}{\tilde{u}} e^{-C_6 e^{(\tilde{u}S + C_5)}}, \\
F = \frac{\sqrt{p}}{m \tilde{u}} C e^{-i (\tilde{u}S + C_5)}.
\]

Inserting the results for \( a, b, c \) and dropping \( d \) the differential equation for \( \chi \) takes the form

\[
i 2 m \tilde{u} \frac{\partial \chi}{\partial t} + \left( \frac{1}{\tilde{u}} \frac{\partial \tilde{u}}{\partial t} + \frac{\partial C_6}{\partial q} \right) \frac{\partial \chi}{\partial q} + \frac{\partial^2 \chi}{\partial q^2} + \left( H_1 + i \left( 2 m \tilde{u} \left( \frac{1}{\tilde{u}} \frac{\partial \tilde{u}}{\partial t} + \frac{\partial C_6}{\partial q} \right) - 2 \frac{\partial C_5}{\partial q} \frac{\partial C_6}{\partial q} \right) \right) \chi = 0.
\]

From now on we use for simplicity the symbol for the partial derivative for all kinds of derivatives. At this point one may use the results [58], [59], to show that both sides of the fundamental condition [111] agree.

In order to compare with the previous, “interaction-free” equation [58] it is convenient to replace the fields \( \tilde{u}, H_1 \) by

\[
p(t) = \frac{1}{\tilde{u}}, \quad \tilde{V} = - \frac{H_1}{2m \tilde{u}^2}.
\]

Using these fields and rearranging terms, Eq. [60] takes the form

\[
- \frac{p}{\tilde{u}} \frac{\partial \chi}{\partial t} + 2 \frac{p^2}{m} \left( \frac{\partial C_6}{\partial q} - i \frac{\partial C_5}{\partial q} \right) \frac{\partial \chi}{\partial q} + \frac{p^2}{2m} \frac{\partial^2 \chi}{\partial q^2} - \tilde{V} \chi - \frac{p}{\tilde{u}} \left( - \frac{1}{\tilde{u}} \frac{\partial \tilde{u}}{\partial t} + \frac{\partial C_6}{\partial q} \right) \chi - \frac{p^2}{2m} \left( \frac{2}{\tilde{u}} \frac{\partial C_5}{\partial q} \frac{\partial C_6}{\partial q} + \frac{\partial^2 C_5}{\partial q^2} \right) \chi = 0.
\]

Comparing [62] and [58], one sees that now, instead of a single arbitrary function \( V(q, t) \) in [58], four arbitrary functions \( p(t), V(q, t), C_6(q, t), C_5(q, t) \) appear in [62]. The field \( p(t) \) has the dimension of an
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action and replaces \( h \), the field \( V(q,t) \) replaces the previous mechanical potential \( V(q,t) \). Each one of these four functions may play, in principle, the role of an “agent” for an external interaction. However it is possible (and we will see in a moment that this possibility applies) that some of these functions may be eliminated by means of a redefinition of the variable \( \chi \). Let us try first to eliminate the functions \( C_5, C_6 \).

Using the transformation

\[
\chi \Rightarrow \bar{\chi} = \chi e^{C_6-\imath C_5},
\]

Eq. (62) takes the form

\[
-\frac{p}{i} \frac{\partial \bar{\chi}}{\partial t} + \frac{p^2}{2m} \frac{\partial^2 \bar{\chi}}{\partial q^2} - V\bar{\chi} + \frac{1}{\imath} \frac{\partial p}{\partial t} \bar{\chi} = 0,
\]

if \( \bar{V} \) is replaced by \( V \) according to the relation

\[
\bar{V} = V - \frac{p^2}{2m} \left[ \left( \frac{\partial C_6}{\partial q} \right)^2 - \left( \frac{\partial C_5}{\partial q} \right)^2 + \frac{\partial^2 C_6}{\partial q^2} \right] - p \frac{\partial C_5}{\partial q}.
\]

Clearly, Eq. (63) is permitted because both \( \bar{V} \) and \( V \) are arbitrary functions. Eq. (64) is a generalized Schrödinger equation, which agrees with the standard form for time-independent \( p = h \). Obviously, the functions \( C_5, C_6 \) are - in the framework of the present Ansatz - physically meaningless, because they may be eliminated from the dynamic equation by means of a simple redefinition of the state variable.

If we use the arbitrary function \( V \) instead of \( \bar{V} \), the untransformed equation (62) takes the more familiar form

\[
-\frac{p}{i} \left( \frac{\partial}{\partial t} - \frac{1}{p} \frac{\partial p}{\partial t} + \frac{\partial C_6}{\partial q} - \frac{\partial C_5}{\partial q} \right) \chi + \frac{p^2}{2m} \left( \frac{\partial^2 C_6}{\partial q^2} - \frac{\partial^2 C_5}{\partial q^2} \right) \chi - V\chi = 0.
\]

This equation is very similar to the one obtained from the standard Schrödinger equation with the help of a gauge transformation; the difference is (besides the time-dependent field \( p \)) that a complex function \( C_5 + \imath C_6 \) appears instead of a real function \( C_5 \). In the present context, the gauge invariance of Schrödinger’s equation is a consequence of the insensitivity of the continuity equation against external disturbances. That part of the additional external \( q, t \)-dependence in \( \chi \), that may be expressed by variable \( C_5, C_6 \) has been eliminated by the requirement that the continuity equation remains valid. As a consequence of this requirement new “compensating” terms appeared in the coefficients. The present derivation opens a slightly different view on gauge theory. This point of view seems to be new, although the central role of the continuity equation and its symmetries for gauge theory is of course a well-established fact.

Can we continue this way and eliminate, as we did with \( C_5 \) and \( C_6 \), the arbitrary function \( p(t) \) too from Eq. (61), i.e. replace it by a constant \( h \)? This is apparently not the case. We may eliminate immediately the imaginary potential term (the one proportional to the time-derivative of \( p \)) from Eq. (61) by means of the transformation

\[
\bar{\chi} \Rightarrow \chi_0 = \bar{\chi} e^{-\imath p} = \frac{\bar{\chi}}{p}
\]

But the result obtained this way namely

\[
-\frac{p}{i} \frac{\partial \chi_0}{\partial t} + \frac{p^2}{2m} \frac{\partial^2 \chi_0}{\partial q^2} - V\chi_0 = 0
\]

does not agree with Schrödinger’s equation because still each \( h \) in (38) is replaced by an arbitrary function \( p(t) \) in Eq. (65). Of course, both equations agree if one sets approximately \( p(t) \approx \text{constant} = h \). Experimentally, this approximation seems to be valid; at least in the mostly investigated range of not too large time-intervals.

A time-dependence of Planck’s constant \( h \) has been the subject both of theoretical speculations and observations over large time periods [22]. It is not clear at present whether or not such a time-dependence exists. If it should turn out to be real, this would rise the fascinating question of its physical origin; obviously it does not fit into the existing scheme to describe interactions.
Let us recapitulate what we have done, paying particular attention to the question of a time-dependent $\hbar$. In the last section we chose the simplest possible Ansatz to construct Schrödinger’s equation. We found an infinite number of equations characterized by three arbitrary constants, $p$, $C_5$, $C_6$, and an arbitrary function $V(q,t)$. Let us recall the slightly rewritten result for $\chi$,

$$\chi = -p \sqrt{\rho e^{-\rho C_6 - iC_5 e^{\rho/2}}}.$$

(69)

In the present section we tried to make room for interactions without changing the basic framework. Essentially the only possibility to do that (see, however, the next section) was to allow for an arbitrary space-time dependence of the constants $p$, $C_5$, $C_6$. This is analogous to the usual procedure of “gauging”, which means postulating a space-time dependence of the parameters of the gauge group. The standard gauging procedure leads then, in a next step, with the help of the concept of “compensating fields”, to the introduction of the gauge coupling terms. Clearly, the present approach may be considered as a generalized gauging procedure; the proceeding is basically the same for all three constants. One could argue - in favour of the reality of the time-dependent $\hbar$ - that, in principle, this possibility to create interaction may be realized by nature for all parameters if it is realized for one of the constants (we know that it is realized for $C_6$, see the next section). On the other hand, the constants $C_5$, $C_6$, and $p$ in $\chi$ play a very different role [see Eq. (69)]. Gauging of $C_5$, $C_6$ is compensated by additional terms in the coefficients, gauging of $p$ cannot be compensated completely. Only the first $p$ in Eq. (69), which plays, similar to $C_6$, the role of a scaling factor for the amplitude, may be eliminated. Combining the two above gauge transformations creates a state function, which is given by

$$\chi_0 = \chi e^{C_5 - iC_5 - np} = -\sqrt{\rho e^{\rho/2}}.$$

(70)

The variable $\chi_0$ still contains the field $p(t)$. The latter presents the only real “interaction” created by the extended Ansatz (10). It plays, as expected, the role of an arbitrary, time-dependent scaling factor for the action $S$.

In the rest of this work we will neglect the time-dependence of $p$ and set $p = \hbar$. Then, in the framework of this approximation, the extended Ansatz (10) of the present section did not lead to any real interactions and we conclude that at any rate a different, more general Ansatz is required to derive the form of the gauge coupling. This will be done in the next section, where all the results derived in the present section, will turn out to be useful. There, we will continue with the threedimensional generalization of (60), which [for $p(t) = \hbar$] is given by

$$-\frac{\hbar}{i} \left( \frac{\partial}{\partial t} + \frac{\partial C_6}{\partial t} - i \frac{\partial C_5}{\partial t} \right) \chi + \frac{\hbar^2}{2m} \sum_{i=1}^3 \left( i \frac{\partial}{\partial q_i} + \frac{\partial C_6}{\partial q_i} - i \frac{\partial C_5}{\partial q_i} \right)^2 \chi - V \chi = 0.$$

(71)

The state function $\chi$ depends on $q_i$, $t$ and is given by

$$\chi = -\hbar \sqrt{\rho e^{-C_6 - i C_5 e^{\rho/2}}}.$$

(72)

We have (for $p = \hbar$) two possibilities to derive these equations. First, we expect that (71) and (72) may be derived in a straightforward manner from a three-dimensional generalization of the fundamental condition (44). As a second way to derive Eq. (71) one may of course perform simply a gauge transformation of the free Schrödinger equation. We will in the rest of this work always refer to (71) and (72) as derived according to the first possibility, i.e. from outside quantum mechanics. The difference is important. Choosing the first possibility, the role of the constants $C_5$, $C_6$ as “agents” of an external influence on the considered system is clear. Choosing the second possibility, these constants are primarily abstract group parameters without any direct physical significance (their role may, however, be suspected; e.g. one may show that $C_5$ does not depend on the wave function (10)).

5 Second attempt to derive interaction

The plausible looking Ansatz of the last section was not successful. Trying to find a better one, it may be helpful to review first briefly the standard methods of introducing the (abelian) gauge field in quantum
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mechanics. Formally, the gauge field is introduced with the help of the minimal coupling rule

$$\frac{\partial}{\partial \vec{q}} \Rightarrow \frac{\partial}{\partial \vec{q}} - i \frac{e}{\hbar} \vec{A},$$ (73)
$$\frac{\partial}{\partial t} \Rightarrow \frac{\partial}{\partial t} + \frac{i e}{\hbar} \phi,$$ (74)

where \(\vec{A}\) denotes the vector potential and \(\phi\) the scalar potential. Applying (73) and (74) to Eq. (3) one obtains Schrödinger’s equation for a particle in an electromagnetic field,

$$-\frac{\hbar}{i} \frac{\partial}{\partial t} \psi - e\phi \psi = -\frac{\hbar^2}{2m} \left( \frac{\partial}{\partial \vec{q}} - i \frac{e}{\hbar} \vec{A} \right)^2 \psi + V\psi.$$ (75)

If \(\psi\) is again written in the form (4) we obtain from (75) the two equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \vec{q}} \rho m \left( \frac{\partial S}{\partial \vec{q}} - \frac{e}{c} \vec{A} \right) = 0,$$ (76)
$$\frac{\partial S}{\partial t} + e\phi + V = -\frac{1}{2m} \left( \frac{\partial S}{\partial \vec{q}} - \frac{e}{c} \vec{A} \right)^2 + \frac{\hbar^2}{2m} \Delta \sqrt{\rho} \sqrt{\rho}.$$ (77)

Using the variables \(\rho\) and \(S\), there is also a minimal coupling rule to create the interaction terms in (76) and (77) from the interaction-free equations (5) and (6). It is obviously given by

$$\frac{\partial S}{\partial \vec{q}} \Rightarrow \frac{\partial S}{\partial \vec{q}} - \frac{e}{c} \vec{A},$$ (78)
$$\frac{\partial S}{\partial t} \Rightarrow \frac{\partial S}{\partial t} + e\phi.$$ (79)

How to justify the minimal coupling rule? The standard explanation is the “principle of local gauge invariance”. It requires that the parameters of the global symmetry group (the constants \(C_5, C_6\) in our notation) be arbitrary functions of \(q_k, t\) without destroying the form invariance of Schrödinger’s equation. This requires, thus the line of argument, introduction of a compensating field, the gauge field. However, as Eq. (66) shows, Schrödinger’s equation is already invariant under the local group; no matter whether or not the new fields have any physical effect. This has been pointed out [16] a few years after publication of the fundamental papers [26], [24] on gauge theory. The principle of gauge invariance alone, without additional assumptions, cannot explain the minimal coupling rule.

A second method to introduce a gauge field exists. There one postulates that the phase of the wave function (or rather a part of it) is non-integrable. This method goes back to Weyl [25] and London [12] and is discussed in many works. We mention only two, a compact presentation to be found in Dirac’s monopole paper [6] and a more detailed and clearly written discussion in a book by Kaempfer [10]. The basic idea of the non-integrable phase may be explained very quickly by starting from Eq. (75) and eliminating the potentials by means of a singular gauge transformation [10]. The result is an equation

$$-\frac{\hbar}{i} \frac{\partial \psi_C}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_C}{\partial q^2} + V(q, t)\psi_C,$$ (80)

which looks like the free Schrödinger equation but cannot be used as a differential equation because the wave function \(\psi_C\) is a multi-valued mathematical object with non-integrable phase; the notation \(\psi_C\) indicates an unspecified dependence on a path \(C\). The gauge field may be introduced exactly the other way round [6]. One starts from the free Schrödinger equation, postulates the existence of a non-integrable phase and eliminates this singular part by means of a gauge transformation, thereby creating the potentials in Eq. (75). How to justify the introduction of the non-integrable phase? Apparently, this question has not been discussed in the literature. This is surprising since the fact that Eq. (80) takes the form of the free Schrödinger equation is very remarkable and calls for an explanation.
This second method to introduce a gauge field, by means of a non-integrable phase, may be adapted for the present problem. A possible explanation - which is however still speculative in character - for the non-uniqueness of the phase will also be given (in the next section). Adapting this idea means that we allow in the Ansatz (40) for a non-unique explicit space-time dependence of the state function \( \chi \).

Introducing a non-unique mathematical object in a physical theory implies more or less automatically a further requirement, namely that all mathematical quantities of direct physical significance are well-behaved (unique) functions. Such a requirement (which is for well-behaved functions obvious and not worth mentioning), should be formulated explicitly in the present situation. We require, in particular, that the state function \( \chi \) itself is a unique (up to regular gauge transformations) function of \( q, t \) and that all derivatives are unique functions of \( q, t \); these requirements imply that we obtain a well-defined differential equation with unique coefficients. Thus, our once more extended Ansatz takes the form

\[
\chi = \chi(\rho, S; q, t), \quad (81)
\]

where the semicolon in (81) has been introduced to indicate the allowed non-uniqueness of the explicit \( q, t \)-dependence. It should be pointed out that the non-uniqueness is allowed but not required. It will be realized only if the consistency requirements (82) can be fulfilled (how this actually works will become clear in a moment).

We have to construct, using (81), (82), a state equation for \( \chi \). The basic construction scheme is the same as in the last section; in particular we start, from a fundamental condition which agrees formally with (41). Thus we have to solve

\[
\Re \left[ F \left( a \frac{d}{dt} + b \frac{d}{dq} + d \frac{d^2}{dq^2} + e \right) \chi \right] = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial q} \frac{1}{m} \frac{\partial S}{\partial q} + \rho \frac{1}{m} \frac{\partial^2 S}{\partial q^2}, \quad (83)
\]

together with the conditions (81) and (82). The problem defined by (81) - (83) is in principle completely independent from the problems dealt with in the last two sections. But all the formal mathematical results, obtained in the last two sections, may be taken over to the present problem; the new aspect - the non-uniqueness - does not change these results. The non-unique \( q, t \)-dependence of \( \chi \) may be expressed by three non-unique functions, namely \( p(t), C_5(q, t), C_6(q, t) \), and the derivatives of these functions have to be, according to condition (82) unique functions of \( q, t \). Thus we may immediately conclude that Eq. (83) implies relation (66). The latter will become a well-defined differential equation, if we are able to transform the non-unique quantity appearing in this equation [also called \( \chi \) in (66)] by proper manipulations into a unique function of \( q, t \). In other words, the only remaining problem is to fulfill condition (82). Before we proceed we neglect, as discussed in the last section, in (66) the time-dependence of \( p(t) \) and set \( p(t) = \hbar \). Further, we shall use in the following the three-dimensional Schrödinger equation (71) instead of (66). This is because the structures to be studied can only be represented adequately in a fourdimensional space-time continuum.

Our starting point is Eq. (72); obviously, condition (82) is not yet fulfilled. It is also clear that the functions \( C_5 \) and \( C_6 \) may be treated separately; given that we are unable to implement uniqueness, we may as well eliminate one or both of these functions by means of a singular gauge transformation (or simply set it equal to zero). Now all relations between our functions are fixed already; there is only a single possibility left to implement an additional condition: The functions \( \rho \) and \( S \) themselves may be defined to be non-unique functions of \( q, t \).

First we assume that \( C_6 \) cannot be implemented as a non-unique function and set \( C_6 = 0 \) in (71) and (72). The form of \( \chi \) shows, that the remaining non-uniqueness of \( C_5 \) may be compensated by postulating a non-unique variable \( S \) according to

\[
S = -\hbar C_5 + \hbar \varphi, \quad (84)
\]

where \( \varphi \) is a unique (up to regular gauge transformations) function of \( q, t \). This simple linear combination eliminates the non-uniqueness of \( C_5 \) by means of a non-unique \( S \) in favor of a unique \( \varphi \) and produces a
unique state function $\chi$. If, instead of $\varphi$, a quantity with the dimension of an action is required, we write $\varphi = \bar{S}/\hbar$. Eq. (5) shows that the replacement of $S$ by $\bar{S}$ according to

$$S \rightarrow \bar{S} = S + \hbar C_5$$

(85)

is compatible with the structure of the continuity equation and leads only to a redefinition of the probability current. Otherwise, this compensation procedure would not make sense.

Let us first follow the consequences of a non-unique $C_5$ before we come back to $C_5$. In order to obtain an explicit representation for $C_5$ we introduce for its derivatives, which are unique functions according to [80], the following symbols,

$$\frac{\partial C_5}{\partial \bar{q}_k} = \tilde{A}_k, \quad \frac{\partial C_5}{\partial t} = \bar{\phi}.$$  

(86)

Then, $C_5$ may be written as an integral over a path $C$,

$$C_5(\bar{q}, t; C) = \int_{\bar{q}_0, t_0; C}^{\bar{q}, t} [d\bar{q}_k \tilde{A}_k(\bar{q}, t') + dt' \bar{\phi}(\bar{q}, t')] .$$

(87)

The last formula (summation over double indices) represents one of the fundamental postulates of gauge theory [25], [6], [10]. The multi-valuedness of $C_5$ is expressed by the fact that $C_5$ does not only depend on the considered space-time point $\bar{q}, t$ but also on the path $C$ which leads from a (fixed) reference point $\bar{q}_0, t_0$ to $\bar{q}, t$. All conceivable values of $C_5$ may be obtained by specifying four real fields $\bar{\phi}(\bar{q}, t), \tilde{A}_k(\bar{q}, t), k = 1, 2, 3$.

These four fields are not independent from each other. In the following we reproduce some well-known results [6], [10] which are essential for the physical interpretation of these fields. We define a four-vector $x_\mu$, $\mu = 0, 1, 2, 3$ by $x_0 = v_0 t$, (where $v_0$ is an unknown constant with dimension of a velocity) and $x_k = q_k, k = 1, 2, 3$. The four-vector field $\tilde{A}_\mu$ is defined by $\tilde{A}_0(x_\mu) = \bar{\phi}(\bar{q}, t)/v_0$ and $\tilde{A}_k(x_\mu) = \bar{A}_k(q_k, t), k = 1, 2, 3$. Then, $C_5$ may be written in the form

$$C_5(\bar{q}, t; C) = \bar{C}_5(x; C) = \int_{x_\mu; C}^{x_\mu; \bar{C}} d\bar{x}_\mu \tilde{A}_\mu(x').$$

(88)

Integrating along an arbitrary closed path $C_0$ from $x_\mu$ to $x_\mu$ Stokes integral theorem implies

$$\int_{C_0} d\bar{x}_\mu \tilde{A}_\mu(x') = \int_{\bar{C}(x_\mu)} \int d\bar{x}_\mu \left( \frac{\partial \tilde{A}_\mu}{\partial x_\mu} - \frac{\partial \tilde{A}_\mu}{\partial x_\mu} \right),$$

(89)

where the antisymmetric tensor $df_{\mu\nu}$ characterizes the infinitesimal surface element of a surface $\bar{A}(C_0)$, which is bounded by our curve $C_0$ and otherwise arbitrary. Given that $C_5$ is multi-valued, the path integral on the left side of Eq. (89) must not vanish for arbitrary paths $C_0$. This implies that the integrand of the surface integral, which is denoted by $\tilde{F}_{\mu\nu}$, must not vanish for arbitrary space-time points. In other words, the set of points defined by

$$\tilde{F}_{\mu\nu} = \frac{\partial \tilde{A}_\mu}{\partial x_\mu} - \frac{\partial \tilde{A}_\mu}{\partial x_\mu} \neq 0,$$

(90)

must not be empty. At the points where Eq. (90) holds, the non-uniqueness of $C_5$ implies the non-commutativity of the derivatives with regard to $x_\mu$. If $\tilde{F}_{\mu\nu} = 0$ for all $x_\mu$, then the influence of $\tilde{A}_\mu$ (and $C_5$) may be eliminated by means of a regular gauge transformation. If, on the other hand, points exist where Eq. (90) holds true, then the values $\tilde{F}_{\mu\nu}$ takes at these points are invariant under regular gauge transformations. This means that $\tilde{F}_{\mu\nu}$ has a gauge invariant meaning and implies the possibility that $\tilde{F}_{\mu\nu}$ plays a role as (is proportional to) a classical field or force.

As a consequence of the definition (90) the field $\tilde{F}_{\mu\nu}$ obeys the differential equation

$$T_{\lambda\mu\nu} = \frac{\partial \tilde{F}_{\mu\nu}}{\partial x_\lambda} + \frac{\partial \tilde{F}_{\nu\lambda}}{\partial x_\mu} + \frac{\partial \tilde{F}_{\lambda\mu}}{\partial x_\nu} = 0.$$  

(91)
The left hand side $T_{\lambda\mu\nu}$ of Eq. (91) is antisymmetric in all three indices. Only four relations in Eq. (91) are independent. These may be written in the form

$$\varepsilon_{\kappa\lambda\mu\nu}\frac{\partial F_{\mu\nu}}{\partial x^\kappa} = 0. \tag{92}$$

The definition of $\varepsilon_{\kappa\lambda\mu\nu}$ may be found e.g. in the book by Landau [11]. If the essential components of $\bar{F}_{\mu\nu}$ are renamed according to

$$\bar{F}_{01} = -\bar{E}_1, \quad \bar{F}_{02} = -\bar{E}_2, \quad \bar{F}_{03} = -\bar{E}_3, \quad \bar{F}_{12} = +\bar{B}_3, \quad \bar{F}_{13} = -\bar{B}_2, \quad \bar{F}_{23} = +\bar{B}_1, \tag{93}$$

then the four relations (92) may be written as differential equations for the vectorfields $\bar{E}$ and $\bar{B}$ (with components $\bar{E}_i$ and $\bar{B}_i$),

$$\frac{\partial \bar{E}}{\partial t} = 0, \quad \frac{\partial}{\partial r} \times \bar{E} + \frac{1}{v_0} \frac{\partial \bar{B}}{\partial t} = 0. \tag{94}$$

Thus, the fields $\bar{E}_i$, $\bar{B}_i$ must be solutions of the homogeneous Maxwell equations, if the constant $v_0$ is identified with the velocity of light $c$. Of course, Eqs. (94) are not sufficient to determine $\bar{E}_i$, $\bar{B}_i$; some more equations are required. An obvious possibility is to identify $v_0$ with $c$ and to postulate that the rest of the equations is given by the two inhomogeneous Maxwell equations, i.e. to assume that $\bar{E}$ and $\bar{B}$ are proportional to the electric and magnetic field vectors $\vec{E}$ und $\vec{B}$ of Maxwell’s theory. This postulate, which is an essential part of Hermann Weyls first gauge theory, is closely related to the structure of Eq. (87).

We still have to find constants of proportionality with suitable dimensions. The “fields” $\bar{E}_i$ and $\bar{B}_i$ (both with dimension $cm^{-2}$) have to be connected, by means of proper constants of proportionality, to the two basic terms of the classical particle-field concept, force and field. If a field $\bar{E}_i$, which has the dimension of a force, is defined by means of $\alpha \bar{E}_i = \bar{E}_i$, then the constant $\alpha$ has the dimension $g^1 cm^3 sec^{-2}$. Using the available constants $m$, $h$, $c$ only a single combination with suitable dimension may be formed, namely $hc$. So we set $\alpha = hc$. In order to have an “objective” field $E_i$, whose existence does not depend on the presence or absence of a test particle, we have to introduce one more constant, the charge $e$. It is, like the mass $m$, part of the description of the individual particle. Thus we write $\bar{E}_i = eE_i$, where $e$ has the dimension $g^2 cm^2 sec^{-1}$. Combining both constants, and making similar considerations for the magnetic field, we obtain

$$\bar{E}_i = \frac{e}{hc} E_i, \quad \bar{B}_i = \frac{e}{hc} B_i, \quad \bar{F}_{\mu\nu} = \frac{e}{hc} F_{\mu\nu}. \tag{95}$$

Similar constants of proportionality occur if the potentials $\bar{A}_\mu$ are replaced by the standard potentials of electrodynamics $A_k, \phi$,

$$\bar{A}_0(x_\mu) = -\frac{e}{hc}\phi(q_k, t), \quad \bar{A}_k(x_\mu) = \frac{e}{hc}A_k(q_k, t). \tag{96}$$

Replacing in an analogous way $C_5$ by $C$ with the help of the relation

$$C_5(q, t; \mathcal{C}) = \frac{e}{hc}C(q, t; \mathcal{C}), \tag{97}$$

the multi-valued function $C$ may be written as a path integral with the usual potentials appearing in the integrand

$$C(q, t; \mathcal{C}) = \int_{q_0(t_0), t_0; \mathcal{C}}^{q(t), t} [dq_kA_k(q', t') - cdq'\phi(q', t')]. \tag{98}$$

Using these properly scaled variables, we obtain the following well-known relation between the components of $F_{\mu\nu}$ and the potentials $A_k, \phi$,

$$\bar{E} = -\frac{1}{c} \frac{\partial \bar{A}}{\partial t} - \frac{\partial \phi}{\partial r}, \quad \bar{B} = \frac{\partial}{\partial r} \times \bar{A}. \tag{99}$$
Schrödinger’s equation with gauge coupling . . .

Finally, with the help of the relations

$$\frac{\partial C_5}{\partial q_k} = \frac{e}{\hbar c} A_k, \quad \frac{\partial C_5}{\partial t} = -\frac{e}{\hbar} \phi,$$

(100)

(and with $C_6 = 0$) Schrödinger’s equation (102) for the variable

$$\psi = -\frac{\chi}{\hbar} = \sqrt{\rho e^{i\bar{S}/\hbar}}, \quad \bar{S} = S + \frac{e}{c} C,$$

(101)

takes the form

$$-\frac{i}{\hbar} \left( \frac{\partial}{\partial t} + \frac{e}{\hbar} \phi \right) \psi + \frac{\hbar^2}{2m} \sum_{i=1}^{3} \left( \frac{\partial}{\partial q_i} - \frac{e}{\hbar c} A_k \right)^2 \psi - V\psi = 0.$$

(102)

Thus, our second attempt to derive Schrödinger’s equation with gauge coupling turned out to be successful.

Now we come back to the question if a meaningful theory may be constructed with a non-zero multi-valued $C_6$. There is a standard argument on this point in the textbook literature. It says (using the present notation) that a non-zero $C_6$ is forbidden, because it leads (in contrast to a phase change) to a modification of the amplitude of the wave function and to a corresponding change in probability density. If this argument were true, a modification of the phase would be forbidden as well. The latter leads also to a modification of the amplitude - as a consequence of the coupling between phase and amplitude. However, while this argument is strictly speaking wrong it leads to the correct conclusion. We may proceed as we did with $C_5$, allowing for a multi-valued $\rho$ and defining a new probability density $\bar{\rho}$ by means of the relation

$$\bar{\rho} = \rho e^{-2C_6}.$$

(103)

The functions $\rho$ and $C_6$ are multi-valued while $\bar{\rho}$ is single-valued. We may represent $C_6$ again (as we did with $C_5$) as a path integral and the associated potentials would again fulfill the homogeneous Maxwell equations. But this is not sufficient. In order for the substitution $\rho \to \bar{\rho}$ (together with $S \to \bar{S}$) to make sense, it is necessary that the structure of the continuity equation remains intact. This means that performing the substitution $\rho \to \bar{\rho}$ in

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial q} \rho \left( \frac{\partial \bar{S}}{\partial q} - \frac{e}{c} \bar{A} \right) = 0,$$

(104)

produces a mathematically well-defined differential equation with unique coefficients, which still has the structure of a continuity equation (with possibly redefined density and current). This is not the case as insertion of (102) in (104) shows. Only the interaction mediated by a non-unique $C_5$ can be realized in nature.

6 Discussion

In this section we will give a summary of the paper and discuss its most important results. The calculations performed in section 3) led us to conjecture that Schrödinger’s equation may be derived from the following three assumptions

1. The continuity equations holds for a probability density $\rho$ and a probability current $\vec{j}$, which depends linearly on the gradient of a function $S$.

2. The system may be described by a complex state function $\chi$.

3. The state function $\chi$ obeys a linear differential equation.

We have not verified in detail this conjecture but given arguments supporting its validity in section 3); in the following discussion it will be considered as true.
The above three assumptions define a quantization procedure. The system that has been quantized was, however, not a particle but rather a statistical ensemble of particles. This quantization method does not require a Hamiltonian; instead the conservation law of probability (first assumption) represents the fundamental input of this theory, which may be roughly characterized as a statistical quantization method. The linearity (third assumption) of the state equation may also be understood in terms of the statistical nature of this theory. Finally, the second assumption of a complex-valued state function is the essential “non-classical” part of this theory. This is a formal requirement and its physical meaning and origin is a priori unclear. It is, however, a very simple assumption, much simpler than e.g. the canonical commutation relations, which are the essential nonclassical part of the conventional quantization method and are also of a formal nature.

We may ask why the present quantization method is conceptually simpler than the conventional one. The present method starts from a statistical ensemble while the conventional method starts from a single particle. Let us consider the classical limit of Schrödinger’s equation. This limit may be performed conveniently by setting \( h = 0 \) in Eqs. (5) and (6). The result is the Hamilton-Jacobi equation \( \frac{\partial \chi}{\partial t} = \mathcal{H} \) and the continuity equation \( \frac{\partial \rho}{\partial t} + \frac{\partial (\rho \chi)}{\partial q} = 0 \). The latter is obviously not eliminated by performing the limit \( h \to 0 \) because it does not contain the constant \( \hbar \). The classical limit (11), (15) of Schrödinger’s equation is a field theory, which describes an infinite number of particle trajectories. Each one of these trajectories may be calculated for given initial conditions in the framework of classical point mechanics [there is no coupling to \( \rho \) in Eq. (11)], but these trajectories are only realized with a certain probability which must be calculated with the help of (15). The classical limit of Schrödinger’s equation is not a particle theory but a statistical theory; this lends support to the ensemble interpretation of quantum theory (3). Therefore, for the inverse problem, the transition from classical physics to quantum theory, a statistical law may be a simpler (more natural) starting point than a particle law.

Three different functional forms for the dependence of \( \chi \) on \( \rho \) and \( S \) have been used in sections 3, 4, and 5. The first, simplest Ansatz \( \chi = \chi(\rho, S) \), in section 3, has been called “interaction free”. The result was Schrödinger’s equation for a particle (ensemble) in an external mechanical potential \( V(q, t) \).

In the second Ansatz \( \chi = \chi(\rho, S, q, t) \), in section 4, the possibility of an explicit dependence of \( \chi \) on \( q, t \) was taken into account. Each constant derived in section 3 becomes a possible “channel” for this additional \( q, t \)-dependence. One of these channels leads to a new and rather exotic “influence” on our system, which is formally described by a time-dependent Planck constant \( \hbar \). All other channels are eliminated by additional compensating terms in the coefficients. By means of a simple redefinition of the state variable the resulting equation may (for negligible variation of \( \hbar \)) be transformed to the free Schrödinger equation of section 3. This transformation agrees exactly with the usual quantum mechanical gauge transformation. Therefore, the Ansatz \( \chi = \chi(\rho, S, q, t) \) contains the usual gauging procedure as a special case. The gauge coupling terms itself cannot be derived within this Ansatz; it turns out, however, that it presents nevertheless an essential step towards its introduction.

The third Ansatz in section 5 has been written in the form \( \chi = \chi(\rho, S; q, t) \), which means that a multi-valued external space-time dependence of \( \chi \) is permitted. The idea for this Ansatz stems from the well-known quantum mechanical concept of a non-integrable phase. The functional form of all dependencies is already fixed by the calculations of section 3. It turns out that the only channel leading to an actual physical effect is given by the parameter \( C_5 \). Its multi-valuedness must be compensated for by the multi-valuedness of the quantity \( S \) (the generalization of the classical action), in order for the final phase of \( \chi \) to be well-behaved. Performing a variable substitution one obtains finally the correct gauge coupling terms in Schrödinger’s equation. In the standard formulation of gauge theory, the compensation effect is meant to restore the (locally destroyed) gauge symmetry, in the present formulation it restores the uniqueness of the state function. The last step to the introduction of the gauge field, performed in section 5, is formally nearly identical to the standard theory of non-integrable phases. There is, however, an important conceptual difference. The field \( C_5(q, t) \) of the present theory represents by definition an externally controlled influence on the considered system. The multi-valuedness of the action \( S(q, t) \), which describes the system, is a consequence of the multi-valuedness of \( C_5(q, t) \). The latter may be identified as reason while the former is the effect of the latter. This clear cause-reason relation is absent in the
standard theory of the non-integrable phase.

Let us come back to the question raised in section 5) how to justify, or “understand”, the form of the gauge coupling. We know that the non-uniqueness of \( C_5(q,t) \), which describes some external influence on our system, is a necessary condition for the occurrence of a gauge field. So, what is the reason for the non-uniqueness of \( C_5(q,t) \); where does it come from? The following is not a final answer to this question but rather a coherent collection of remarks intended to stimulate further research.

The non-uniqueness of a physical quantity means that it is impossible to express its effect in a “local” way, by stating its value on a particular space-time point. The term “local” needs further explanation. It applies practically to all of the conventional scheme of formulating physical laws by means of differential equations; in this scheme all the information required to predict (or retrodict) the behavior of a system is determined by stating its values (e.g. coordinates of particles or field values at all space points) at a particular instant of time. There is no universal law dictating that this “local” conventional scheme must be always true. It may well be that instead the values of the variables in a time interval, of finite extent, are required to predict the behavior of a system. Mathematically such theories are described e.g. by delay differential equations.

Let us consider the classical theory of charged particles and fields, having in mind a possible breakdown of the conventional scheme. Nearly all applications of this theory study one of two idealized situations. The first is calculating the fields produced by given sources, the second is calculating the trajectories of particles in given fields. Both types of problems are, strictly speaking, unrealistic, even if this does not cause any problems in macroscopic situations. Realistic problems have to take into account the mutual influence of trajectories and fields. Whenever this is done seriously [17], [13], [2] one encounters systems of delay-differential equations which are difficult to solve and whose mathematical structure is still to be analyzed in detail. But if one tries to simulate a realistic “nonlocal” problem in the framework of the conventional scheme, one encounters necessarily problems, in the form of non-unique or multi-valued predictions. This is what happens already for the simplest conceivable realistic problem of this type, a charged particle in an external electromagnetic field under the influence of its own radiation field [19], [23].

Now, quantum mechanics (as well as the present theory) uses, of course, the conventional “local” scheme of formulating physical laws. It is possible that the above explained “nonlocality” is - in a way still to be clarified in detail - responsible for the non-uniqueness of \( C_5(q,t) \). The latter plays, according to the derivation reported in section 5), a central role for the form of the gauge coupling. May we extend this idea about the origin of the form of the gauge coupling further, to quantum mechanics itself? The above calculation was in fact a simultaneous derivation of the fundamental equation of quantum theory and the form of the gauge coupling. Considering it that way, we made three (or two, if we neglect the possibility of a time-dependent Planck constant) different derivations of quantum theory, and may ask which one is most fundamental. The third derivation led to Schrödinger’s equation with the minimal coupling rule, which is obeyed by all interactions found in nature (including non-abelian generalizations and possibly excluding gravity). It should be considered as most fundamental, if quantum mechanics is to be understood as a step towards a realistic description of nature. If one adopts this point of view, Schrödinger’s equation depends also crucially on the assumption of a non-unique external influence. We conclude that a detailed study of the breakdown of the “local” formulation of realistic classical physics may be useful in order to achieve a deeper understanding of quantum mechanics.

This attitude towards quantum theory is not new, although a minority view. We mention, in particular, the work of Raju [17]. He revealed clearly crucial properties of classical particle-field systems which have been overlooked by the scientific community for nearly a century. Of course, such findings are also relevant for a careful characterization of quantum mechanical nonlocality. Nelson, using a quite different starting point, conjectured [15] that “..quantum fluctuations may be of electromagnetic origin.” This is to be understood in the sense that the stochastic mechanism, which underlies quantum mechanics, could be due to the “non-deterministic” behavior of charged particles, as a consequence of interaction with their own field [15]. This conjecture is - using a different language - very similar to the above. Finally we note, that there is a certain overlap of the present ideas with those underlying Stochastic Electrodynamics [5].
7 Concluding remarks

In this work Schrödinger’s equation with gauge coupling has been derived. The construction was based on the validity of a continuity equation and a linear differential equation for a complex-valued state variable. As an heuristic recipe this scheme may probably be extended to more general situations. Obvious possibilities to do that include a relativistic formulation or a multi-component version, which should be the analogon of non-abelian gauge theories. Other types of variables (quaternions) or even an application beyond the realm of special relativity seem conceivable. From a physical point of view, the most important generalization or modification of the present theory concerns the assumption of a complex state variable. This is a purely mathematical assumption which - despite of the outstanding structural properties of the field of complex numbers - should be replaced by a different, equivalent assumption which can be directly interpreted in physical terms. This is the most challenging question to be asked in the context of the present theory.

A Calculation of $\chi$ and $F$

Combining the derivative of (14) with respect to $\rho$ with (10), one obtains

$$\frac{\partial \bar{d}_1}{\partial \rho} \frac{\partial \chi_1}{\partial \rho} - \frac{\partial \bar{d}_2}{\partial \rho} \frac{\partial \chi_2}{\partial \rho} = 0. \quad (105)$$

Using (105) and (14) one obtains an elementary differential equation, which shows that the ratio of $\bar{d}_1$ and $\bar{d}_2$ does not depend on $\rho$; it is convenient to write this in the form

$$\frac{\bar{d}_1}{\bar{d}_2} = e^{f(S)}, \quad (106)$$

where $f(S)$ is an unknown function. If (16) is multiplied by $2\rho$ and afterwards combined with (17), one obtains a differential equation

$$2\rho \frac{\partial U}{\partial \rho} = U \quad (107)$$

for the $\rho$-dependence of a quantity $U(\rho, S)$, defined by

$$U(\rho, S) = \frac{\bar{d}_1}{\bar{d}_2} \frac{\partial \chi_1}{\partial S} - \frac{\partial \chi_2}{\partial S}. \quad (108)$$

Inserting the solution of (107) in (17) leads to the important intermediate result

$$\bar{d}_2(\rho, S) = \sqrt{\frac{\rho}{m}} e^{-h(S)}, \quad (109)$$

$$\bar{d}_1(\rho, S) = \sqrt{\frac{\rho}{m}} e^{-h(S)+f(S)}. \quad (110)$$

Thus, the dependence of $\bar{d}_1$ and $\bar{d}_2$ on $\rho$ is known while the two functions $h$ and $f$, which both depend only on $S$, are still to be found.

The solution of (14) and (15) with regard to $\frac{\partial \chi_1}{\partial \rho}$ and $\frac{\partial \chi_2}{\partial \rho}$ is given by

$$\frac{\partial \chi_1}{\partial \rho} = \frac{d_2}{d_2 \bar{a}_1 - d_1 \bar{a}_2}, \quad \frac{\partial \chi_2}{\partial \rho} = \frac{\bar{d}_1}{d_2 \bar{a}_1 - d_1 \bar{a}_2}. \quad (111)$$

The $\bar{a}_i$ may be expressed, using (19) and (21), by the $\bar{d}_i$; this determines the $\rho$-dependence of the right hand sides of the differential equations (111). Integrating (111) yields

$$\chi_1(\rho, S) = -\frac{2m|d|^2}{e_2} \sqrt{\rho} e^{\frac{h(S)-f(S)}{e_2}} + G(S), \quad (112)$$

$$\chi_2(\rho, S) = -\frac{2m|d|^2}{e_2} \sqrt{\rho} e^{\frac{f(S)}{e_2}} + H(S), \quad (113)$$
Schrödinger’s equation with gauge coupling . . .

where $|d|^2 = d_1^2 + d_2^2$ and $G(S), H(S)$ are unknown functions of $S$. In (112) and (113) the abbreviations
\begin{equation}
    c_1 = a_1d_1 + a_2d_2, \quad c_2 = a_2d_1 - a_1d_2,
\end{equation}
have been used. The remaining task is the calculation of the four functions $f, h, G, H$.

In order to do that we insert the solution of (107) for $U$ in (108) and combine it with (106). This leads to the relation
\begin{equation}
    e^{f(S)} \frac{\partial \chi_1}{\partial S} - \frac{\partial \chi_2}{\partial S} = e^{h(S)} \sqrt{\rho}.
\end{equation}
Using (115) and (9) we obtain the following derivatives of $\chi_1, \chi_2$ with respect to $S$.
\begin{align}
    \frac{\partial \chi_1}{\partial S} &= \sqrt{\rho} \frac{c_2e^{2f(S)} + c_1}{c_2e^{2f(S)} + c_2} e^{h(S)}, \\
    \frac{\partial \chi_2}{\partial S} &= \sqrt{\rho} \frac{c_1e^{2f(S)} - c_2}{c_2e^{2f(S)} + c_2} e^{h(S)}.
\end{align}
Next, we calculate the derivatives of $\chi_1$ and $\chi_2$ [see (112) and (113)] with respect to $S$ and equate the results with the derivatives as given by (116) and (117). From the resulting relation we may draw two conclusions. The first implies that $G(S)$ and $H(S)$ must be constants, say
\begin{equation}
    G(S) = C_3, \quad H(S) = C_4.
\end{equation}

The second implies two differential equations for $f(S)$ and $h(S)$, namely
\begin{align}
    - \frac{2m|d|^2}{c_2} \frac{\partial}{\partial S} \frac{1}{L_1(S)} &= \frac{c_2e^{2f(S)} + c_1}{c_2L_1(S)}, \\
    - \frac{2m|d|^2}{c_2} \frac{\partial}{\partial S} \frac{1}{L_2(S)} &= \frac{c_1e^{2f(S)} - c_2}{c_2L_1(S)}.
\end{align}
Here, the abbreviations
\begin{align*}
    L_1(S) &= e^{-h(S)} \left( 1 + e^{2f(S)} \right), \\
    L_2(S) &= e^{-f(S)} L_1(S)
\end{align*}
have been used. After a short rearrangement, Eqs. (119), (120) take the form
\begin{align}
    - \frac{\partial h}{\partial S} + \frac{\partial f}{\partial S} \frac{2e^{2f}}{1 + e^{2f}} &= \frac{1}{2m|d|^2} \left( c_2e^f + c_1 \right), \\
    \frac{\partial h}{\partial S} - \frac{\partial f}{\partial S} \frac{e^{f} - e^{-f}}{e^{f} + e^{-f}} &= \frac{1}{2m|d|^2} \left( c_2e^{-f} - c_1 \right),
\end{align}
which shows, that the two equations may be decoupled. Addition of Eqs. (121) and (122) yields a differential equation for $f(S)$ alone, namely
\begin{equation}
    \frac{\partial f}{\partial S} = \frac{c_2}{2m|d|^2} (e^f + e^{-f}),
\end{equation}
which may be solved easily. From (123) and (121) we obtain for the functions $f(S)$ and $h(S)$ the final results
\begin{align}
    f(S) &= \ln \tan \left( \frac{c_2}{2m|d|^2} S + C_5 \right), \\
    h(S) &= - \ln \cos \left( \frac{c_2}{2m|d|^2} S + C_5 \right) - \frac{c_1}{2m|d|^2} S - C_6,
\end{align}
containing the constants of integration $C_5, C_6$ and the coefficients $c_i, d_i$.

This determines the functional form of the variables $\chi$ and $F$, we were looking for. Using Eqs. (112), (113), (124), and (125) one obtains (23) and (24) for the real- and imaginary parts of $\chi$. For the real- and imaginary parts of $F$ one obtains with the help of (21), (109), and (110) the expressions (25) and (26).
B Calculation of parameters

Conditions (44), (45), (46) and (51) have not yet been used and play the role of constraints for our parameters. In this appendix we insert χ and F [see Eqs. (52), (53), (25), and (26)] in (44), (45) (46) and (51) and evaluate the resulting parameters.

We introduce the following abbreviations:

\[ U = \frac{c_2}{2m|d|^2} S + C_5, \]  
\[ V = \frac{c_1}{2m|d|^2} S + C_6, \]  
\[ W = \frac{2m|d|^2}{c_2} \sqrt{\rho}, \]  
\[ T = \frac{\sqrt{\rho}}{m|d|^2}. \]  

Using these abbreviations, the real and imaginary parts of χ and F are written as

\[ \chi_1 = -W \cos U e^{-V} + C_3, \]  
\[ \chi_2 = -W \sin U e^{-V} + C_4, \]  
\[ F_1 = T e^V (d_1 \sin U + d_2 \cos U), \]  
\[ F_2 = T e^V (d_1 \cos U - d_2 \sin U), \]

and the functions \( \bar{a}_i, \bar{b}_i, \bar{d}_i, \bar{e}_i, i = 1, 2 \) [see (19)-(22)] take the form

\[ \bar{a}_1 = T e^V (c_1 \sin U - c_2 \cos U), \]  
\[ \bar{a}_2 = T e^V (c_1 \cos U + c_2 \sin U), \]  
\[ \bar{b}_1 = T e^V (g_1 \sin U - g_2 \cos U), \]  
\[ \bar{b}_2 = T e^V (g_1 \cos U + g_2 \sin U), \]  
\[ \bar{d}_1 = |d|^2 T e^V \sin U, \]  
\[ \bar{d}_2 = |d|^2 T e^V \cos U, \]  
\[ \bar{e}_1 = T e^V (h_1 \sin U - h_2 \cos U), \]  
\[ \bar{e}_2 = T e^V (h_1 \cos U + h_2 \sin U), \]

where \( g_i, h_i \) are defined by

\[ g_1 = b_1 d_1 + b_2 d_2, \quad g_2 = b_2 d_1 - b_1 d_2, \]  
\[ h_1 = e_1 d_1 + e_2 d_2, \quad h_2 = e_2 d_1 - e_1 d_2. \]

Inserting \( \chi_i, F_i \) in (144) we obtain

\[ c_1 = 0. \]

The latter relation occurred already in section 3), it implies that the variable \( S \) appears only in the phase and not in the modulus of \( \chi \). From the conditions (45) and (46) we obtain the relations

\[ g_2 + 2|d|^2 \left( S \frac{\partial \hat{u}}{\partial q} + \frac{\partial C_5}{\partial q} \right) = 0, \]  
\[ g_1 - 2|d|^2 \frac{\partial C_6}{\partial q} = 0, \]

where the abbreviation

\[ \hat{u} = \frac{c_2}{2m|d|^2}. \]
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has been used. Eq. (144), i.e. \( V = C_6 \), has already been used in (146), and will also be used in the following to simplify the formulas. Since Eq. (145) must hold for arbitrary \( S \), condition (145) implies the two constraints

\[
\frac{\partial \tilde{u}}{\partial q} = 0, \\
\tag{148}
\]

(\( \tilde{u} \) depends only on \( t \)) and

\[
g_2 + 2|d|^2 \frac{\partial C_5}{\partial q} = 0. \\
\tag{149}
\]

The remaining condition (51) leads to a lengthy expression. We rewrite condition (51) in the following form

\[
T_c + T_a + T_b + T_d = 0, \\
\tag{150}
\]

\[
T_c = e_1 \chi_1 - e_2 \chi_2, \\
\tag{151}
\]

\[
T_a = a_y \frac{\partial \chi_1}{\partial t} - \tilde{a}_2 \frac{\partial \chi_2}{\partial t}, \\
\tag{152}
\]

\[
T_b = b_y \frac{\partial \chi_1}{\partial q} - \tilde{b}_2 \frac{\partial \chi_2}{\partial q}, \\
\tag{153}
\]

\[
T_d = d_y \frac{\partial^2 \chi_1}{\partial q^2} - \tilde{d}_2 \frac{\partial^2 \chi_2}{\partial q^2}. \\
\tag{154}
\]

associating a term with each one of the coefficients \( e, a, b, d \). Inserting \( \chi_i, F_i \) we obtain (using \( c_1 = 0 \))

\[
T_c = TW h_2 + T e^{C_6} \left[ (C_3 h_1 - C_4 h_2) \sin U - (C_4 h_1 + C_3 h_2) \cos U \right],
\tag{155}
\]

\[
T_a = -T \frac{\sqrt{p}}{\tilde{u}} c_y \left( \frac{1}{\tilde{u} \frac{\partial \tilde{u}}{\partial t}} + \frac{C_5}{\partial t} \right) + T e^{C_6} \left[ -c_1 \frac{\partial C_4}{\partial t} \sin U - c_2 \frac{\partial C_5}{\partial t} \cos U \right],
\tag{156}
\]

\[
T_b = T \frac{\sqrt{p}}{\tilde{u}} \left[ g_1 \left( S \frac{\partial \tilde{u}}{\partial q} + \frac{C_5}{\partial q} \right) - g_2 \left( \frac{1}{\tilde{u} \frac{\partial \tilde{u}}{\partial q}} + \frac{C_5}{\partial q} \right) \right] + T e^{C_6} \left[ \left( g_1 \frac{\partial C_4}{\partial q} - g_2 \frac{\partial C_5}{\partial q} \right) \sin U - \left( g_2 \frac{\partial C_3}{\partial q} + g_1 \frac{\partial C_4}{\partial q} \right) \cos U \right],
\tag{157}
\]

\[
T_d = -T \frac{\sqrt{p}}{\tilde{u}} |d|^2 \left[ 2 \left( \frac{1}{\tilde{u} \frac{\partial \tilde{u}}{\partial q}} + \frac{C_5}{\partial q} \right) \left( S \frac{\partial \tilde{u}}{\partial q} + \frac{C_5}{\partial q} \right) \right] - \left( S \frac{\partial^2 \tilde{u}}{\partial q^2} + \frac{\partial^2 C_5}{\partial q^2} \right) + T e^{C_6} |d|^2 \left[ \frac{\partial^2 C_3}{\partial q^2} \sin U - \frac{\partial^2 C_4}{\partial q^2} \cos U \right].
\tag{158}
\]

The left hand side of Eq. (150) is a sum of four terms, each one being proportional to a linear independent function of \( \rho, S \), namely \( \rho, \rho S, \sin S, \cos S \). As a consequence Eq. (150) leads to maximal four subconditions. The vanishing of the coefficient of \( \rho \) implies

\[
2 \frac{h_2}{c_2} - 2 \left( \frac{1}{\tilde{u} \frac{\partial \tilde{u}}{\partial t}} + \frac{C_5}{\partial t} \right) + 2 \frac{g_1}{c_2} \left( g_1 \frac{\partial C_4}{\partial q} - g_2 \left( \frac{1}{\tilde{u} \frac{\partial \tilde{u}}{\partial q}} + \frac{C_5}{\partial q} \right) \right) = 0,
\tag{159}
\]

The condition that the coefficient of \( \rho S \) vanishes does not lead to a new constraint; it is automatically fulfilled if Eq. (148) holds. Finally, the condition of a vanishing coefficient of \( \sin S, \cos S \) leads to the two
relations

\[ C_3 h_1 - C_4 h_2 - c_2 \frac{\partial C_4}{\partial t} + \left( g_1 \frac{\partial C_3}{\partial q} - g_2 \frac{\partial C_4}{\partial q} \right) + |d|^2 \frac{\partial^2 C_3}{\partial q^2} = 0, \]  
\[ (165) \]

\[ -C_4 h_1 - C_3 h_2 - c_2 \frac{\partial C_3}{\partial t} - \left( g_2 \frac{\partial C_3}{\partial q} + g_1 \frac{\partial C_4}{\partial q} \right) - |d|^2 \frac{\partial^2 C_4}{\partial q^2} = 0, \]  
\[ (166) \]

Now all constraints for the 12 real functions \( a_i, b_i, d_i, e_i, C_3, C_4, C_5, C_6 \) are known explicitly. These are Eqs. (164)-(166) and the above relations (144), (148), (146), (149). Using (148), (146) and (149), Eq. (164) takes the form

\[ 2 h_2 - 2 \left( \frac{\partial \bar{u}}{\partial t} + \frac{\partial C_6}{\partial t} \right) - \frac{g_1 g_2}{c_2} |d|^2 + 2 |d|^2 \frac{\partial^2 C_5}{c_2} \frac{\partial q}{\partial q^2} = 0. \]
\[ (167) \]

With the help of (146), (149) one obtains

\[ h_2 = 2m |d|^2 \bar{u} \left( \frac{\partial \bar{u}}{\partial t} + \frac{\partial C_6}{\partial t} \right) - 2 |d|^2 \frac{\partial C_5}{\partial q} \frac{\partial C_6}{\partial q} - |d|^2 \frac{\partial^2 C_5}{\partial q^2} - \frac{\partial^2 C_6}{\partial q^2}. \]
\[ (168) \]

The latter formula shows that \( h_2 \) is proportional to \( |d|^2 \), where the constant of proportionality is determined by the functions \( \bar{u}, C_5 \) and \( C_6 \). Exactly the same holds for \( c_1, c_2, g_1, g_2 \). The definitions of \( c_i, g_i, h_i \) show that it should be possible, for given \( c_i, g_i, h_i \), to express the \( a_i, b_i, e_i \) in terms of the \( d_i \). The resulting relations (for the complex quantities \( a, b, e \)) are given by Eqs. (54)-(56). The coefficients on the left hand side of Eq. (165) and (166) are arbitrary functions. This implies the (expected) trivial solutions

\[ C_3 = 0, \quad C_4 = 0 \]
\[ (169) \]

for these equations. As a consequence, the constant \( d \) drops out of the equations, the remaining arbitrary parameters being \( C_5, C_6, H_1, \bar{u} \).

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