A receding horizon scheme for discrete-time polytopic linear parameter varying systems in networked architectures

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Abstract. This paper proposes a Model Predictive Control (MPC) strategy to address regulation problems for constrained polytopic Linear Parameter Varying (LPV) systems subject to input and state constraints in which both plant measurements and command signals in the loop are sent through communication channels subject to time-varying delays (Networked Control System (NCS)). The results here proposed represent a significant extension to the LPV framework of a recent Receding Horizon Control (RHC) scheme developed for the so-called robust case. By exploiting the parameter availability, the pre-computed sequences of one-step controllable sets inner approximations are less conservative than the robust counterpart. The resulting framework guarantees asymptotic stability and constraints fulfillment regardless of plant uncertainties and time-delay occurrences. Finally, experimental results on a laboratory two-tank test-bed show the effectiveness of the proposed approach.

1. Introduction
The term Networked Control System [2, 16] describes a complex process: the plant to be regulated, actuators, sensors and the controller, all exchanging data via a communication channel. Relevant contributions on this subject ranges from Lyapunov strategies for nonlinear MIMO systems [25] to state/observer based methodologies for linear continuous time NCSs [20].

Here we point out our attention to the class of constrained RHC strategies which are an extremely appealing methodology for dealing with the NCS stabilization problem thanking to its intrinsic capability to generate, at each time instant, a sequence of virtual inputs which can be transmitted within a single data-packet [9, 23, 21]. In [9], the authors consider a receding horizon strategy for nonlinear systems under wireless and asynchronous measurements. In order to regulate the state towards an equilibrium point while minimizing a given performance index, a Lyapunov-based model predictive controller is designed by explicitly taking into account data losses both in the optimization problem formulation and in the controller implementation. This proposed scheme allows an explicit characterization of the stability region and guarantees that such a set is invariant for the closed-loop system under data-losses. Gupta and Quevedo [23] extend a control scheme for nonlinear plants, popular in real-time systems to tolerate the presence of time-varying processing resources (such as variable delays, packet losses/drops etc.) and known as anytime algorithm. In [21], by following similar ideas as in [9] a nonlinear RHC scheme exploiting a Network Delay Compensation strategy is proposed to efficiently manage the simultaneous presence of constraints, model uncertainties, time-varying transmission delays and data-packet losses. The main aim consists in overcoming the inherent difficulties related to a MPC design based on linear process models and the joint presence of constraints, nonlinearities and NCS features. Moving from
these considerations in [10] a discrete time RHC strategy for NCSs which are described by means of uncertain polytopic linear plants, subject to time-varying delays and data-losses, is developed. There, the framework prescribes that measurements and control commands need to be sent over communications links which bring to varying transmission delays and packet dropouts between plant-controller and vice-versa. The key idea is the following: pre-computed inner approximations sequences of the one-step ahead state prediction sets are on-line exploited as target sets for the actual state prediction vector to compute the commands to be applied to the plant in a receding horizon fashion and the time-varying delays are taken into account by resorting to both Independent-of-Delay (IOD) and Delay Dependent (DD) stability concepts [13]. Following the same lines of [10], a more general framework based on polynomial approach has been presented in [11, 12].

The class of LPV systems has attracted a significant attention within the time-delay systems research field: delay-independent analysis and synthesis [26], delay-dependent stability conditions [24], $H_{\infty}$ controller design problems for NCSs with packet dropouts [15], continuous-time networked control systems with random sensor-delays [18] and references therein. However, from these contributions it is noted that very few efforts have been paid on the study of efficient strategies for dealing with constrained NCS-LPV systems subject to time-varying delays on the communication medium, which mainly motivates this present study.

In this paper, the main aim is to extend the networked control scheme of [10] to the LPV framework. As it is well-known, the LPV system paradigm provides an interesting modelling framework which is capable to describe special classes of non-linear and/or time-varying dynamics. Because of its relevance in practical applications, several Model Predictive Control (MPC) schemes have been recently developed for the LPV framework. For a comprehensive literature on the matter see e.g. [5, 22, 8]. One difficulty dealing with LPV plants governed by linear scheduled control laws is that the one-step ahead state prediction set is non-convex. Here, a tight outer convex approximation proposed in [7] will be exploited for control design purposes. Then, the networked control strategy is adapted to the class of constrained polytopic LPV systems by exploiting the outer convex approximations of the one-step state prediction set. Specifically, one-step controllable set sequences are formally determined used for managing time-delay occurrences on the signals sent through transmission channels.

Effectiveness and applicability of the proposed control framework are shown via simulation studies on a cascaded two-tank process.

![Networked control system structure](image)

**Figure 1.** Networked control system structure

### 2. Problem formulation

In the sequel, we shall consider the networked control systems depicted in Fig. 1 where both the communication channels are subject to time-varying induced delays.

A discrete-time polytopic LPV system is used to describe the process dynamics

$$x_p(t + 1) = \Phi(\alpha(t))x_p(t) + G(\alpha(t))u(t)$$

(1)

where $t \in \mathbb{Z}_+ := \{0, 1, \ldots\}$, $x_p(t) \in \mathbb{R}^n$ denotes the state plant and $u(t) \in \mathbb{R}^m$ the control input. The
measurable time-varying parameter vector $\alpha(t) \in \mathbb{R}^l$ belongs to the unit simplex
\[
P_i := \left\{ \alpha \in \mathbb{R}^l : \sum_{i=1}^l \alpha_i = 1, \alpha_i \geq 0 \right\}
\] (2)

while the system matrices $\Phi(\alpha)$ and $G(\alpha)$ to
\[
\Sigma(P_i) := \left\{ (\Phi(\alpha), G(\alpha)) = \sum_{i=1}^l \alpha_i (\Phi_i, G_i), \alpha \in P_i \right\}
\] (3)

with the pairs $(\Phi_i, G_i)$ denoting the polytope vertices of $\Sigma(P_i)$, viz. $(\Phi_i, G_i) \in \text{vert} \{\Sigma(P_i)\}, i = 1, \ldots, I$.

Moreover, the following constraints are prescribed
\[
u(t) \in U, \forall t \geq 0, \quad U := \{u \in \mathbb{R}^m | u^T u \leq \pi\},
\] (4)
\[
x_p(t) \in X, \forall t \geq 0, \quad X := \{x_p \in \mathbb{R}^n | x_p^T x_p \leq \chi\},
\] (5)

with $\pi > 0, \chi > 0$, and $U, X$ compact subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively.

The actuator buffer is used to store the last applied command, hereafter named $u^R_{-1}$, whereas at each time instant $t$ the logic is instructed to apply the current command $u(t)$ if more recent than $u^R_{-1}$.

**Assumption 1** Let us assume that:
- $\tau_{sa}(t) : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ such that $\tau_{sa}(t) \leq \bar{\tau}_s, \forall t \geq 0, \bar{\tau}_s \in \mathbb{Z}_+$;
- $\tau_{sc}(t) : \mathbb{Z}_+ \rightarrow \mathbb{Z}_+$ such that $\tau_{sc}(t) \leq \bar{\tau}_c, \forall t \geq 0, \bar{\tau}_c \in \mathbb{Z}_+$.

By using the following definitions:

**Definition 1** At each time instant $t$, the age of the state measurement used by the controller to compute the command input is defined as
\[
\tau_m(t) \leq \tau_{ca}(t)
\]

**Definition 2** At each time instant $t$, the age of the command used by the actuator is defined as
\[
\tau_c(t) \leq \tau_{sc}(t)
\]

we have that the following age cumulative induced delay should be considered:
\[
\tau_{NL}(t) = \tau_m(t) + \tau_c(t - \tau_m(t))
\]

in order to compute the control action $u(t)$ on the plant-controller link.

Then to properly comply with time-delay occurrences, it is mandatory to equip the NCS with a timestamp mechanism and a clock synchronization procedure capable to provide at each time instant $t$ sensor and actuator latencies and the actuator latency $\tau_m(t)$ and $\tau_c(t)$. As it is well-known, the round-trip delay $\tau_c(t - \tau_m(t))$ cannot be available at the controller side, therefore in the sequel we shall consider the upper bound $\bar{\tau}_c$ on the controller-actuator link for design purposes and the following network induced delay is exploited:
\[
\tau(t) = \tau_m(t) + \bar{\tau}_c, \forall t
\] (6)

This paper focuses on the problem of designing a RHC strategy that is capable via the control action
\[
u(\cdot) = g(x_p(\cdot), \alpha(\cdot))
\] (7)
to asymptotically stabilizes the closed-loop system under the following time-delay occurrences:
- **Sensor-to-Controller link** - each time-delay occurrence is bounded: $\tau(t) < \bar{\tau}$, with the upper bound $\bar{\tau}$ the maximum admissible time interval (MATI) [25];
- **Controller-to-Actuator link** - at each time instant $t$ the actuator receives a control action $u(t)$. 

From now on we refer to the above statement as the Network Constrained Stabilization (NCS) problem. It will be addressed by exploiting the basic dual-mode predictive scheme based on the set-theoretic approach, see [1], [7] and references therein. Essentially, the control scheme consists of the following phases:

- **Off-line** - Stabilizing state-feedback control laws for (1) subject to (4), (5) and (6) are first derived by resorting to DD and IOD stability concepts. Then, the corresponding working regions are enlarged by deriving sets of states that can be steered into DD and IOD terminal ellipsoids in a finite number of steps.

- **On-line** - At each time $t$, the “smallest” ellipsoidal set (DD or IOD regions) compatible with the current cumulative delay instance $\tau(t)$ is first determined. Then, the command action is computed by minimizing a running cost under the condition that the one-step state prediction belongs to the successor set [3].

### 3. Outer approximation of the one-step ahead state prediction set under scheduled control laws

In this section, the availability at each time instant of the system parameter $\alpha(t)$ is used to characterize the one-step ahead state sets, see [7] for details.

To this end, we shall consider the unconstrained evolutions of the polytopic system (1) and shall define the one-step ahead state prediction set as

$$X^+ := \{ x^+_p \mid x^+_p = \Phi(\alpha)x_p + G(\alpha)u, \, \forall \alpha \in P_i \} \quad (8)$$

where $x_p \in \mathbb{R}^n$ is a given initial state and $u \in \mathbb{R}^m$ the command input. By using the following class of scheduled control laws

$$u(t) = \sum_{j=1}^{l} \alpha_j(t)u_j(t) \quad (9)$$

with $u_j(t) \in \mathbb{R}^m$, each element $x^+ \in X^+$ can be rewritten as follows

$$x^+_p = \sum_{i=1}^{l} \sum_{j=1}^{l} \alpha_i\alpha_j \Phi_i x_p + \sum_{i=1}^{l} \sum_{j=1}^{l} \alpha_i \alpha_j G_i u_j, \, \forall \alpha \in P_i \quad (10)$$

Then, it is straightforwardly to show that expression (10) can be rewritten as

$$x^+_p = \left( \begin{array}{c} (I_l^T \otimes I_{n \times n}) \left( \begin{array}{c} \Phi_1 \\ \vdots \\ \Phi_l \end{array} \right) \end{array} \right) x^+ \left( \begin{array}{c} G_1 \\ \vdots \\ G_l \end{array} \right) \left( \begin{array}{c} \alpha \alpha^T \otimes I_{m \times m} \end{array} \right) \tilde{u}, \forall \alpha \in P_i \quad (11)$$

where $\otimes$ denotes the Kronecker product and $I_l \triangleq [1 \ldots 1]^T \in \mathbb{R}^l$ and $\tilde{u}^T = [u_1^T \ldots u_l^T]^T \in \mathbb{R}^{lm}$. Since the above expression shows a quadratic dependence of $x^+$ from the parameter vector $\alpha \in P_i$, the dyadic rank-one matrix $\alpha \alpha^T$, the region $X^+$ is not a convex set. With this in mind, a standard way to proceed is to search for an appropriate convexification of $X^+$ by solving the following problem:

**Outer Approximation Problem (OAP)** - Determine a collection of matrices $\Pi_i \in \mathbb{R}^{l \times l}, i = 1, \ldots, l_c$ such that

$$\{ \alpha \alpha^T \mid \alpha \in P_i \} \subseteq \Pi \quad (12)$$

where $\Pi := \text{conv}\{\Pi_1, \ldots, \Pi_{l_c}\}$ is the set of all convex combinations of $\{\Pi_1, \ldots, \Pi_{l_c}\}$.

Note that the integer $l_c$ represents the minimum number of vertices of the outer convexification complying with (12). Then, the one-step ahead state prediction set (8) can be embedded into

$$X^+_l := \{ x^+_p \in \mathbb{R}^n \mid x^+_p = \tilde{A}(\tilde{\alpha})x_p + \tilde{B}(\tilde{\alpha})\tilde{u}, \, \forall \tilde{\alpha} \in P_i \} \quad (13)$$

that is generated by means of the following multi-model linear system

$$x_p(t+1) = \tilde{\Phi}(\tilde{\alpha}(t))x_p(t) + \tilde{G}(\tilde{\alpha}(t))\tilde{u}(t), \, \tilde{\alpha}(t) \in P_i \quad (14)$$
whose a more refined region than (18), can be derived by resorting to the following result:

\[
\Phi(\alpha) = \sum_{i=1}^{l} \alpha_i \Phi_i, \quad \Phi_i = \begin{pmatrix} 1_l \otimes I_{n \times n} \end{pmatrix} (\Pi_i \otimes I_{n \times n}) \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_l \end{bmatrix}
\]

with \( \Phi_i \) being the polytopic set \( \Pi_i \) and \( \alpha, \Phi \). Then it is straightforward to show that the polytopic set

\[
\Omega_{(l+1)/2} = \mathcal{P}_{(l+1)/2}
\]

encloses vectors of the form \( \alpha^T = \alpha_1^T, \ldots, \alpha_l^T, 2\alpha_{l+1}^T, \ldots, 2\alpha_n^T \), \( \forall \alpha \in \mathcal{P}_l \) and the following family of vertices for \( \Pi \) is obtained

\[
\text{vert} \{ \Pi \} = \left\{ \left( e_i e_j^T \right)_{i=1}^{l}, \left( \frac{1}{2} \left( e_i e_j^T + e_j e_i^T \right) \right)_{i=1}^{l} \right\}
\]

In [7] it has been proved that a polytope, which embeds the vector \( \alpha^T, \forall \alpha \in \mathcal{P}_l \) into a more refined region than (18), can be derived by resorting to the following result:

\[
2\alpha_i \leq \frac{1}{2}, \quad i, j = 1, \ldots, l, \forall \alpha \in \mathcal{P}_l,
\]

i.e. the intersection of the unit simplex \( \mathcal{P}_{(l+1)/2} \) with the family of half-spaces (20):

\[
\Omega_{(l+1)/2} := \left( \mathcal{P}_{(l+1)/2} \right) \cap \bigcap_{i=1}^{l(l+1)/2} \left\{ \alpha \in \mathbb{R}^{l(l+1)/2} \left| \alpha_i \leq 1/2 \right. \right\}
\]

whose \( \mathcal{P} \cdot l \cdot c \cdot \) vertices are given by

\[
\text{vert} \{ \Omega \} = \left\{ \left( e_i \right)_{i=1}^{l}, \left( \frac{1}{2} \left( e_i e_j^T + e_j e_i^T \right) \right)_{i=1}^{l} \right\}
\]

Finally, based on (22), the following more tight polytopic outer approximation of \( \alpha^T, \forall \alpha \in \mathcal{P}_l \) results

\[
\text{vert} \{ \Pi \} = \left\{ \left( e_i e_j^T \right)_{i=1}^{l}, \left( \frac{1}{4} \left( e_k e_s^T + e_s e_k^T \right) + \frac{1}{2} e_i e_j^T \right)_{i=1}^{l} , \left( \frac{1}{4} \left( e_k e_s^T + e_s e_k^T \right) + \frac{1}{4} \left( e_i e_j^T + e_j e_i^T \right) \right)_{i=1}^{l} \right\}
\]

\[
\left\{ \left( \frac{1}{4} \left( e_k e_s^T + e_s e_k^T \right) + \frac{1}{4} \left( e_i e_j^T + e_j e_i^T \right) \right)_{i=1}^{l} \right\}
\]

4. Networked LPV MPC strategy

In this section, the availability of \( \alpha(t) \) will be taken into account by referring to the multi-model state space description (14)-(16) with the uncertainty characterized by the polytope vertices (23). This means that the NCS problem is solved by resorting to an uncertain (robust) paradigm where the hypothesised parameter knowledge is used to develop a more insight state space description.
4.1. Off-line Phase

In the first part of this section, for the sake of notational simplicity and without loss of generality the DD and IOD acronyms will be dropped out.

Recall that given the plant (1) under a time-delay free scenario the sets of states $i$-step controllable to $T$ are determined via the following recursion:

\[
T_0 := T \subset X
\]

\[
T_i := \{ x_p \in X : \exists \bar{u} \text{ such that } u_j \in \mathcal{U}, j = 1, \ldots, i; \forall \alpha \in \mathcal{P}_i, \Phi(\bar{\alpha}) x_p + \bar{G}(\bar{\alpha}) \bar{u} \in T_{i-1} \} \tag{24}
\]

where $T_i$ is the set of states that can be steered into $T_{i-1}$ using a single control move [3]. To extend such a concept to the proposed framework, it is important to notice that the one-step state prediction needs to be evaluated along the model

\[
x(t+1) = \Phi(\bar{\alpha}(t)) x(t) + \bar{G}(\bar{\alpha}(t)) \bar{u}(t) \tag{25}
\]

with $\bar{u}(t)$ chosen according to a delayed state plant information, i.e. $x(t) = x_p(t - \tau(t))$. Therefore, at each time instant $t$ there exists a difference between the process measurement $x_p(t)$ and the model state $x(t)$ that must be adequately addressed. By following the developments of [10], we will resort to the following transition map

\[
x(t+1) = \Phi(\bar{\alpha}(t)) x(t - \tau(t)) + \bar{G}(\bar{\alpha}(t)) \bar{u}(t), \tau(t) \in [0, \bar{\tau}] \tag{26}
\]

in order to compute the sequence of sets $T_i$.

By considering the auxiliary state $y(t)$ the following descriptor form results

\[
\begin{bmatrix}
x(t+1) \\
0
\end{bmatrix} = \begin{bmatrix}
y(t) + x(t) \\
-\gamma(t) + \Phi(\bar{\alpha}(t)) x(t - \tau(t)) + \bar{G}(\bar{\alpha}(t)) \bar{u}(t) - x(t)
\end{bmatrix} \tag{27}
\]

Hence, by noticing that $x(t - \tau(t)) = x(t) - \sum_{j=t-\tau(t)}^{t-1} y(j)$ and by imposing the worst-case occurrence on the time-varying delay $\tau(t)$, i.e. $\tau(t) = \bar{\tau}$, $\forall t$, we have that

\[
\hat{E} \dot{x}(t+1) = \Phi(\bar{\alpha}(t)) \dot{x}(t) + \bar{G}(\bar{\alpha}(t)) \bar{u}(t) - \bar{G}(\bar{\alpha}(t)) \sum_{j=t-\tau}^{t-1} y(j) \tag{28}
\]

where $\dot{x}(t) = [x(t)^T y(t)^T]^T$, $\hat{E} = \text{diag}(I, 0)$,

\[
\Phi(\bar{\alpha}(t)) = \begin{bmatrix} I & I \\ \Phi(\bar{\alpha}(t)) - I & -I \end{bmatrix}, \quad \bar{G}(\bar{\alpha}(t)) = \begin{bmatrix} 0 \\ \bar{G}(\bar{\alpha}(t)) \end{bmatrix}, \quad \bar{G}_y(\bar{\alpha}(t)) = \begin{bmatrix} 0 \\ \Phi(\bar{\alpha}(t)) \end{bmatrix}
\]

The following proposition provides inner approximations of the exact one-step controllable sets related to the description (26).

**Proposition 1** Let $T_0 \neq \emptyset \subset X$ be a given robustly invariant ellipsoidal region complying with the input constraints and $\hat{x}_\text{aug} = [x^T y^T z_1^T z_2^T]^T$ the augmented state describing the dynamics (28) with $z_1, z_2 \in \mathbb{R}^n$ accounting for all the cumulative sum vectors $y(j)$, $j = t - \bar{\tau}, \ldots, t - 1$. Then, the ellipsoidal sets sequence

\[
E_0 = T_0
\]

\[
E_i = \text{Proj}_x \{ \text{In}[\hat{x}_\text{aug} \in \mathbb{R}^{4n} \text{ with } x \in X, z_1, z_2 \in E_{i-1} : \exists \bar{u} \text{ such that } u_j \in \mathcal{U}, j = 1, \ldots, i; \forall \alpha \in \mathcal{P}_i, \text{Proj}_x \{ \Phi(\bar{\alpha}) \hat{x}_\text{aug} + \bar{G}(\bar{\alpha}) \bar{u} \in E_{i-1} \} ] \}
\]

\[
\hat{\Phi}(\bar{\alpha})_\text{aug} = \begin{bmatrix} I & I & 0 & 0 \\ \Phi(\bar{\alpha}) - I & -I & -\tau\Phi(\bar{\alpha}) & \tau\Phi(\bar{\alpha}) \\ 0 & 0 & 0 & I \end{bmatrix}, \quad \hat{G}(\bar{\alpha})_\text{aug} = \begin{bmatrix} 0 & \bar{G}(\bar{\alpha}) & 0 & 0 \end{bmatrix}^T
\]

if non-empty, satisfies $E_i \subset T_i$. 

Proof - The proof follows mutatis mutandis the same arguments of [10].

From now on we will assume that for (14) there exist

(i) IOD and DD stabilizing feedback control laws

\[ \bar{u}(t) = K_{IOD}x_p(t - \tau(t)), \forall \tau(t) \leq \tau, \]
\[ \bar{u}(t) = K_{DD}x_p(t - \tau(t)), \forall \tau(t) \leq \tau_{max} \leq \tau \]

(ii) IOD and DD non-empty robust positively invariant regions for the closed-loop state evolutions

\[ \mathcal{E}_{IOD} \triangleq \{ x_p \in \mathbb{R}^n | x_p^T Q_{IOD} x_p \leq 1 \} \subset \mathbb{R}^n, \]
\[ \mathcal{E}_{DD} \triangleq \{ x_p \in \mathbb{R}^n | x_p^T Q_{DD} x_p \leq 1 \} \subset \mathbb{R}^n \]

complying with input and state constraints (4)-(5), see [13, 10] for technical details.

Finally, the same reasoning on the time-delay occurrence management outlined in [10] can be here applied: two nested one-step controllable ellipsoidal sequences with \( N + 1 \) elements (\( N > 0 \)) \( \{ T_{IOD}^i \}_{i=0}^N \) and \( \{ T_{DD}^i \}_{i=0}^N \), have to be computed by using recursions (29) under the additional condition

\[ T_{DD}^0 \leq T_{IOD}^i \]

(30)

4.2. On-line phase

We will resort to a time-stamp mechanism in order to characterize the data-packet dispatch along the sensor-to-controller and controller-to-actuator channels. Specifically, the event \((x_p, t)\) denotes the state measurement \(x_p\) sent at the time instant \(t\), whereas \((u, t)\) has the following meaning: \(u\) is the command sent by the controller and marked by the stamp \(t\). Moreover, the stored command \(u_{-1}^k\) is time stamped with \(t_{-1}\).

On the controller side, at each instant \(t\) the computation of the control action \(u(t)\) is performed by using the following arguments:

- Consider the most recent data-packet \((x_p, t)\);
- Determine the set containing \(x_p\): if \(x_p \in T_{IOD}^i\) then the set \(T_{IOD}^i\) is selected, otherwise find the smallest index \(i\) such that \(x_p \in T_{DD}^i\);
- Solve the following optimization:
  - if \(x_p(t - \tau(t)) \in T_{IOD}^i\):
    \[ \ddot{u}(t) = \arg \min_{\bar{u}} J_{(t)}(x_p(t - \tau(t)), \bar{u}) \]
    subject to
    \[ \bar{\Phi}(\bar{\alpha})x_p(t - \tau(t)) + \bar{G}(\bar{\alpha})\bar{u} \in T_{IOD}^i, \]
    \[ \bar{\alpha} \in \mathcal{P}, u \in \mathcal{U}, j = 1, \ldots, l, \]
  - else \(x_p(t - \tau(t)) \in T_{DD}^i\):
    \[ \ddot{u}(t) = \arg \min_{\bar{u}} J_{(t)}(x_p(t - \tau(t)), \bar{u}) \]
    subject to
    \[ \bar{\Phi}(\bar{\alpha})x_p(t - \tau(t)) + \bar{G}(\bar{\alpha})\bar{u} \in T_{DD}^i, \]
    \[ \bar{\alpha} \in \mathcal{P}, u \in \mathcal{U}, j = 1, \ldots, l, \]

where

\[ J_{(t)}(x(t), \bar{u}) = \max_{j} \| \Phi_j x_p(t - \tau(t)) + \bar{G}_j \bar{u} \|_{R_{(t)}^j}^2 + \| \bar{u} \|_{R_u}^2 \]

(35)

with \( R_u = R_{(t)}^j > 0 \), and \( P_{(t)}^{-1} > 0 \) the shaping matrix of the one-step controllable set \( T_{(t)}^{-1} \). The apex notation \( (\cdot) \) refers to both IOD and DD acronyms.
Notice that when the delayed state belongs to some \( T_i^{DD} \), the command \( u(t) \) can be computed using the optimization (31)-(32) with \( T_i^{IOD} \) in place of \( T_i^{DD} \) and the feasibility is retained thanking to (30).

All the above developments allows one to write down a computable MPC scheme:

**NCS-LPV-MPC-Algorithm**

**Initialization**

0.1 Given the scalars \( \tau_c, \tau_{\text{max}} \) and \( \bar{\tau} \), compute the non-empty robust invariant ellipsoidal regions \( T_0^{DD} \subset \mathbb{R}^n, T_0^{IOD} \subset \mathbb{R}^n \) and the stabilizing state feedback gains \( K_{DD}, K_{IOD} \) satisfying (4)-(5) \( \forall \tau(t) \leq \bar{\tau} \).

0.2 Generate and store the sequences of \( N \) one-step controllable sets \( T_i^{DD} \) and \( T_i^{IOD} \) complying with (30).

**On-line phase**

**SENSOR SIDE** - for all \( t \in \mathbb{Z}_+ \)

1.1 send the packet \((x_p,t_{x_p})\) with \( t_{x_p} = t \) the time-stamp;

**CONTROLLER SIDE** - for all \( t \in \mathbb{Z}_+ \) and for all arrived packets \((x_p,t_{x_p})\)

1.1 compute \( \tau(t) = \tau_m(t) + \tau_c \) with \( \tau_m(t) = t - t_{x_p} \);

1.1.1 If there exists \( i(t) := \min\{i : x_p \in T_i^{IOD}\} \) then

   a- If \( i(t) = 0 \) then \( \bar{u}(t) = K_{IOD}x_p \)

   b- else solve (31)-(32);

1.1.2 else find \( i(t) := \min\{i : x_p \in T_i^{DD}\} \)

   a- If \( i(t) = 0 \) then \( \bar{u}(t) = K_{DD}x_p \)

   b- else solve (33)-(34);

1.2 Send the packet \((\bar{u},t_{\bar{u}})\).

**ACTUATOR SIDE** - for all \( t \in \mathbb{Z}_+ \) and for all arrived packets \((\bar{u},t_{\bar{u}})\)

1.1 If \( t_{\bar{u}} > t_{x_{\bar{u}}} \) then \( u^{\bar{u}}_{\bar{u},1} := \bar{u}, t_{x_{\bar{u},1}} := t_{\bar{u}} \) and apply \( \bar{u} \);

1.2 else apply \( u^{\bar{u}}_{\bar{u},-1} \).

The next proposition proves feasibility retention and closed-loop stability of the proposed NCS-LPV-MPC Algorithm.

**Proposition 2** Let the set sequences \( T_i^{DD} \) and \( T_i^{IOD} \) be non-empty and \( x_p(0) \in T_i^{DD} \cup T_i^{IOD} \). Then, the NCS-LPV-MPC algorithm always satisfies the constraints and ensures asymptotic stability for all bounded time-delay occurrences.

**Proof** - On the controller side, existence of solutions at time \( t \) implies existence of solutions at time \( t + 1 \), because the optimization problems in steps 1.1.1 and 1.1.2 are always feasible and under the boundedness of \( \tau_c(t) \) the timely arrival feature of the computed command \( \bar{u}(t) \) is guaranteed. Let us consider without loss of generality that at the generic instant \( t \) the state \( x_p(t - \tau(t)) \in T_i^{DD} \). First note that by construction there exists an input vector \( \bar{u} \) satisfying (4)-(5) such that the one-step state evolution \( \Phi(\bar{u})x_p(t - \tau(t)) + \bar{G}(\bar{u})\bar{u} \) belongs to \( T_i^{DD} \) for all \( \tau(t) \) with \( \bar{\tau} \). Then, thanking to the requirement (30) and to recursions (29), at the next time instant \( t + 1 \) \( u(t) \) drives \( \Phi(\bar{u})x_p(t - \tau(t)) + \bar{G}(\bar{u})\bar{u} \) into \( T_i^{DD} \). Therefore, the existence of a solution \( \bar{u}(t + 1) \) for steps 1.1.1 and 1.1.2 is ensured.

On the actuator side the feasibility is guaranteed by using the properties of the ellipsoidal sequences \( T_i^{DD} \) and \( T_i^{IOD} \). In fact, at each time instant the actuator logic is able to apply an admissible control input because the last stored command \( u^{\bar{u}}_{\bar{u},1} \) is updated at most after \( \bar{\tau} \) time instants. The asymptotic stability follows from the same arguments. \( \square \)
Table 1. Tanks and constraints values

| Parameter | Value        | Parameter | Value        |
|-----------|--------------|-----------|--------------|
| $S_1$     | 2500 cm$^2$  | $S_2$     | 1600 cm$^2$  |
| $A_1$     | 9 cm$^2$     | $A_2$     | 4 cm$^2$     |
| $\bar{h}_1$ | 35 cm       | $\bar{h}_2$ | 200 cm      |
| $\bar{h}_1$ | 1 cm         | $\bar{h}_2$ | 10 cm       |

Table 2. Parameter values

| Parameter | Value        |
|-----------|--------------|
| $g$       | 980 cm/(sec$^2$) |
| $\rho$    | 0.001 Kg/(cm$^3$) |
| $V_{\text{max}}$ | 4           |
| $T_c$     | 0.5 sec      |

5. A cascaded two-tank system

A two-tank process (see Figure 2) is used to evaluate the performance of the proposed NCS-LPV-MPC algorithm. A schematic diagram of the process is shown in Figure 2.b and the goal is to regulate the water level $h_2(t)$ (plant output) at a given set-point by acting on the incoming water flow via the supply pump voltage $V(t)$ (plant input).

The cascaded two-tank model is described by the following non-linear differential equations

\[
\begin{align*}
\rho S_1 \dot{h}_1 &= -\rho A_1 \sqrt{2gh_1} + u \\
\rho S_2 \dot{h}_2 &= \rho A_1 \sqrt{2gh_1} - \rho A_2 \sqrt{2gh_2}
\end{align*}
\]  

(36)

where $u$ is the water flow supplied by the pump whose command is the voltage $V$, $S_i$, $i = 1, 2$, are the tank cross sections, $h_i$ the water levels in the tanks, $A_i$ the pipe sections connecting the tanks, $g$ the gravity constant, and $\rho$ the water density. A simple static equation is used to model the relationship between the input voltage $V(t)$ and the incoming water flow $u(t)$

\[
u(t) = \begin{cases} 
V(t) & \text{if } V(t) \geq 0 \\
0 & \text{if } V(t) < 0
\end{cases}
\]  

(37)

Moreover, the following constraints are enforced (see Tables I-II)

\[
h_1 \leq h_1 \leq \bar{h}_1, \ h_2 \leq h_2 \leq \bar{h}_2, \ 0 \leq V \leq V_{\text{max}}
\]  

(38)

An LPV multi-model description of (36) has been exploited by following the lines indicated in [6]. Hence, the plant has been discretized with a sampling time $T_c = 0.5$ sec. and it is required that $h_2$ tracks a set-point $r$ whose value changes from 105 to 107 at $t = 100$ sec. (see Fig. 8 - dashed line). It is supposed that the initial water levels $h_1$ and $h_2$ are equal to $[11 \ 112]^T$.

In order to implement the NCS-LPV-MPC strategy, a test rig has been built up: it includes two computers and the two tank process (see Fig. 3). The notebook marked as A in Fig. 3 is a Toshiba
Satellite A665-S6058 Intel® Core™ i5-450M Processor (3M cache, 2.40 GHz 4gb RAM) and acts as the software interface of both the actuator and sensor units within the control scheme of Fig. 1. The second PC (an Alienware M15x with processor Intel® Core™ i7 940XM (8M Cache, 2.13 GHz 4GB RAM)) has the role of the controller and runs Matlab 2012b®. The signals are physically transmitted between two Intranet IP addresses 160.97.27.112 and 160.97.27.54, which are both located on the department WiFi Network DEIS42c3 physically implemented on Ubiquiti® PicoStation™ M2 HP (see Fig. 5) and the TCP protocol is used. Moreover to simulate the network communication latency, delays have been artificially added into the Intranet environment (see Fig. 4) so that the network MATI is \( \bar{\tau} = 1.8 \text{ sec} \) (i.e. 4 time units in the proposed framework), while \( \tau_{\text{max}} = 1 \text{ sec} \) (2 time units) has been estimated by applying the heuristics described in [10].

Finally, in order to characterize the set of admissible disturbances/measurement errors acting on the tanks, the following region

\[
D = \{ d \in \mathbb{R}^2 | d^T d \leq 0.5 \}
\]

has been considered and used in the proposed RHC setting as suggested in [7]. For this simulation, we have used the scenario of Fig. 4 which depicts the cumulative time-delay occurring at each time instant on the network. A family of 50 regions \( T_i \) centered at \([20.74 105]^T\) has been off-line computed by using recursions (29), see Fig. 5.

All the relevant results achieved under the action of the proposed NCS-LPV-MPC control strategy are summarized in Figs. 6-8. First, it is worth to remark that both input and state constraints are always fulfilled. Then, Fig. 8 shows a good capability of the proposed scheme to reach the prescribed set-points regardless of any time-delay occurrence on the packet transmission. Finally, it is important to underline that the on-line burdens, evaluated by computing the average CPU time (seconds per step), confirms a modest computational load (0.0283 sec.) when compared to the sampling time \( T_c \) (about one order less). This is relevant because it can be viewed as a tangible clue that the proposed approach can be affordable in practical applications.

6. Conclusions

In this paper, a receding horizon strategy for LPV polytopic networked systems subject to input state and communication constraints has been proposed. First, by exploiting ellipsoidal calculus and viability theory, a novel fast ellipsoidal MPC scheme capable to properly manage time-delay occurrences on the network channels is proposed. Then, it is worth to underline that the requested computational effort and resources requirements appear modest so rendering more simple the applicability of such an approach.
The effectiveness of the proposed scheme has been shown by means of an illustrative example that has put in evidence the **NCS-LPV-MPC** capability to on-line take care of time-varying delays on both sensor-to-controller and controller-to-actuator links.

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