Construction and application of algebraic dual polynomial representations for finite element methods

V. Jain*, Y. Zhang†, A. Palha‡, M. Gerritsma§

*Delft University of Technology, Faculty of Aerospace Engineering, P.O. Box 5058, 2600 GB Delft, The Netherlands

Abstract

Given a polynomial basis $\Psi_i$ which spans the polynomial vector space $P$, this paper addresses the construction and use of the algebraic dual space $P'$ and its canonical basis for finite element methods. These dual spaces supplemented with boundary conditions obey the De Rham cohomology if the primal spaces also form a De Rham sequence. It is shown that duality pairing between primal and dual representations reduces to the vector product of the degrees of freedom. This is demonstrated with the application of dual basis to a constrained minimization problem by a multi-element, mixed formulation of the Poisson equation in 3D. Well-posedness for this mixed formulation can be done algebraically in terms of the degrees of freedom only. The method is also applied to a pair of Dirichlet-Neumann problems and it is shown that the finite dimensional approximations satisfy the duality properties for these problems on any arbitrary grid. These two test problems will also show that one does not explicitly need to construct the dual basis, but merely exploit its properties.

Keywords: Finite element method, Spectral elements, Algebraic dual polynomials, Riesz Representation Theorem

1. Introduction

With every linear vector space $\mathcal{V}$ we have the algebraic dual $\mathcal{V}' = \mathcal{L}(\mathcal{V}, \mathbb{R})$, see [2, §2.10] or [3, §2.10]. If $d_V$ is the dimension of the space $\mathcal{V}$, and $e_i, i = 1, \ldots, d_V$, forms a basis for $\mathcal{V}$, then one can construct a canonical basis $e^*_j, i = 1, \ldots, d_V$, which satisfies $\langle e^*_i, e_j \rangle := e^*_i(e_j) = \delta_{ij}$. The finite dimensional polynomial spaces we use in finite element methods also form a linear vector space and therefore the existence of an algebraic dual polynomial space directly follows from functional analysis, or more precisely in the finite dimensional case, from linear algebra, [4, §3.F].

Earlier use of dual bases has been in isogeometric methods for projection of B-splines [5, 6]. In standard finite elements they have been used for mortar methods [7, 8] as test functions for coupling of discontinuous finite elements - resulting in representation of the discrete mortar map as a diagonal matrix only. In recent works these ideas have been combined for isogeometric mortar methods in [9]. For other implementations in isogeometric methods see [10, 11]. Different methods for construction of dual basis have also been discussed in [12], and for construction of dual splines in [13].

In this paper we will present the use of dual basis for mimetic spectral element methods. We set up a De Rham sequence for primary spaces and for each space in this complex we construct a dual space. The sequence of these dual spaces with boundary conditions also forms a De Rham complex. The construction of a dual basis used in this work is similar to the inverse Gram constructions described in [8, 12]. Let $P$ be a finite dimensional function space with basis $\Psi_i, i = 1, \ldots, d_P$, then we will give a construction of the dual space $P'$ and its canonical dual basis $\Psi'_f$.

As applications we will demonstrate the use of dual basis on a constrained minimization problem of the Poisson equation in 3D. We also show the discrete well posedness of this problem which is easily expressed in terms of degrees of freedom only. It will be shown that the use of an algebraic dual basis results in a very sparse matrix where

*Corresponding author. Tel. +31 15 2789670.
Email address: V.Jain@tudelft.nl (V. Jain)

Preprint submitted to Elsevier  February 28, 2020
two of the sub-matrices consist of 1, −1 and 0 only and do not change with the shape and size of the element. This observation is relevant for incompressible flow equation where we encounter a similar div-grad pair. These techniques may also be valuable in electromagnetism to represent the involution constraint $\text{div} \, B = 0$ in a way that is very sparse and independent of the shape and size of the mesh.

We will also solve for the pair of Dirichlet-Neumann problems discussed in [1]. The duality of this pair is used to prove well posedness of the class of DPG methods [14–16]. We will prove this duality in a discrete setting. This is in general not trivial. We use a representation in primal degrees of freedom for the Neumann problem and a representation in terms of dual degrees of freedom for the Dirichlet problem. It is shown that the duality relation $\phi^h = \text{div} \, q^h$ continues to hold point-wise in these finite-dimensional approximations, on arbitrary grids, through the use of algebraic dual polynomials. In addition, we prove that $\|h^h\|_{H^1} = \|\phi^h\|_{H^1} = \|q^h\|_{H^1(\text{div})}$ holds, just as in the continuous setting.

The construction of dual polynomial spaces in the one dimensional case is presented in Section 2. In this section it is also shown how nodal sampling and edge sampling from polynomial spaces extend to Sobolev spaces. The derivative of a dual representation will also be given in this section. In Section 3 this construction of dual spaces is extended to two dimensions. In Section 4 we define the primal and dual spaces for three dimensions. In Section 5 a short overview of nodal Lagrange polynomials is given, such that the linear map, $p \in P \mapsto (\gamma_{\text{ndo}}(p), \ldots, \gamma_{N_{\text{ndo}}}(p)) \in \mathbb{R}^{\text{ndo}}(p)$, is bijective.

The linear functionals $\{\gamma_i\}$ are called the local degrees of freedom. The following Proposition taken from [18] defines the basis functions:

**Proposition 1.** There exists a basis $\{\varphi_1, \ldots, \varphi_{N_{\text{ndo}}}\}$ in $P$ such that

$$N_i(\varphi_j) = \delta_{ij}, \quad 1 \leq i, j \leq \text{ndo}.$$  

**Example 1.** Consider the interval $K = [-1, 1] \subset \mathbb{R}$. Let $\xi_i \in K$, $i = 0, \ldots, N$, be the roots of the polynomial $(1 - \xi^2)L_N^0(\xi)$, where $L_N(\xi)$ is the Legendre polynomial of degree $N$ and $L_N^0(\xi)$ its derivative. These nodes are referred to as the Gauss-Lobatto-Legendre (GLL) points, [20]. Let $P$ be the space of polynomials of degree $N$ defined on the interval $K$. For any $p \in P$ define the degrees of freedom by

$$\gamma_i(p) := p(\xi_i), \quad i = 0, \ldots, N.$$  

(1)

Because polynomials are continuous, (1) is well-defined. The superscript ‘0’ in $N_0^0$ indicates that we sample the polynomial $p$ in points. The basis which satisfies the Kronecker-delta property from Proposition 1 is given by the set of Lagrange polynomials through the GLL-points

$$h_i(\xi) = \frac{(\xi^2 - 1)L_N^0(\xi)}{N(N + 1)L_N(\xi_i)(\xi_i - \xi)}, \quad i = 0, 1, \ldots, N.$$
This example also corresponds to [18, Prop.1.34] for \(d = 1\).

**Remark 1.** Note that the degrees of freedom are linear functionals on \(\mathcal{P}\). The nodal sampling of functions in \(\mathcal{P}\) is essentially the Dirac delta distribution which is well defined when the vector \(\mathcal{P}\) consists of continuous functions, see [3, Example 2.10.2]. Extension of this functional to Sobolev spaces in this way is in general not possible. The extension to Sobolev spaces will be given in Definition 4.

**Example 2.** Let \(K\) and \(\xi\) be defined as in Example 1. Let \(Q\) be the space of polynomial degree \((N - 1)\). The degrees of freedom will be defined in this case by

\[
N_i^1(p) = \int_{\xi_{i-1}}^{\xi_i} p(\xi) \, d\xi, \quad i = 1, \ldots, N. \tag{2}
\]

For polynomials the integral in (2) is well-defined. The superscript ‘1’ in \(N_i^1\) expresses the fact that the degrees of freedom are associated to line segments \([\xi_{i-1}, \xi_i]\). The basis functions, \(e_j(\xi)\), which satisfy the Kronecker-delta property from Proposition 1 need to satisfy

\[
N_i^1(e_j) = \int_{\xi_{i-1}}^{\xi_i} e_j(\xi) \, d\xi = \delta_{ij}.
\]

**Lemma 1.** The basis functions \(e_j(\xi)\) on the GLL-grid defined in Example 2 are given by

\[
e_j(\xi) = -\sum_{k=0}^{j-1} \frac{dh_k}{d\xi}(\xi), \quad j = 1, \ldots, N, \tag{3}
\]

where \(h_k(\xi)\) are the Lagrange polynomials defined in Example 1.

**Proof.**

\[
\int_{\xi_{i-1}}^{\xi_i} e_j(\xi) \, d\xi = -\sum_{k=0}^{j-1} \int_{\xi_{i-1}}^{\xi_i} dh_k(\xi) = -\sum_{k=0}^{j-1} [h_k(\xi_i) - h_k(\xi_{i-1})] = \delta_{ij},
\]

where we repeatedly use the Kronecker-delta property of the Lagrange polynomials. If the Lagrange polynomials \(h_k(\xi)\) are polynomials of degree \(N\), then \(dh_k(\xi)/d\xi\) is a polynomial of degree \((N - 1)\). It is easy to show that \(e_j\) forms a basis for \(Q\). \(\square\)

**Corollary 1.** From (3) it follows that

\[
\frac{dh_j}{d\xi} = e_j(\xi) - e_{j+1}(\xi) \, .
\]

So if \(p \in \mathcal{P}\) is expanded in terms of Lagrange polynomials as

\[
p(\xi) = \sum_{i=0}^{N} N_i^0(p) h_i(\xi),
\]

then its derivative is given by

\[
\frac{dp}{d\xi}(\xi) = \sum_{i=0}^{N} N_i^0(p) \frac{dh_i}{d\xi} = \sum_{i=0}^{N} N_i^0(p) [e_i(\xi) - e_{i+1}(\xi)] = \sum_{i=1}^{N} \left( N_i^0(p) - N_{i-1}^0(p) \right) e_i(\xi), \tag{4}
\]

where we used that \(e_0(\xi) = e_{N+1}(\xi) = 0\).
Let $E^{1,0}$ be the $N \times (N + 1)$ matrix
\[
E^{1,0} = \begin{pmatrix}
-1 & 1 & & & \\
-1 & 1 & & & \\
& & \ddots & & \\
& & & -1 & 1 \\
& & & & -1 & 1
\end{pmatrix},
\]
then we can write (4) as
\[
\frac{dp}{d\xi}(\xi) = \sum_{i=0}^{N} \sum_{j=0}^{N} E^{1,0}_{i,j} N^{0}(p) e_{i}(\xi).
\]
Taking the derivative of a nodal expansion changes the nodal degrees of freedom discussed in Example 1 to integral degrees of freedom discussed in Example 2. The matrix $E^{1,0}$ is called the incidence matrix, which converts nodal degrees of freedom to integral degrees of freedom. Differentiation is a map $d/d\xi : P \mapsto Q$.

2.1. Construction of dual basis
Consider the finite element constructed in Example 1. Any element $p \in P$ can be represented as
\[
p(\xi) = \sum_{i=0}^{N} N^{0}_i(p) h_i(\xi),
\]
where $N^{0}_i(p)$ are the nodal degrees of freedom and $h_i(\xi)$ are the associated basis functions. To simplify the notation, we will write this as
\[
p(\xi) = \Psi^{0}(\xi) N^{0}(p),
\]
where
\[
\Psi^{0}(\xi) = (h_0(\xi) \ h_1(\xi) \ldots h_{N-1}(\xi) \ h_N(\xi)) \quad \text{and} \quad N^{0}(p) = \begin{pmatrix}
N^{0}_0(p) \\
N^{0}_1(p) \\
\vdots \\
N^{0}_{N-1}(p) \\
N^{0}_N(p)
\end{pmatrix}.
\]
Let $p, q \in P$ be both represented as in (5), then the $L^2$-inner product is given by
\[
(p, q)_{L^2(K)} := \int_{K} pq \ dK = N^{0}(p)^T M^{0}(0) N^{0}(q).
\]
Here $M^{0}(0)$ denotes the mass matrix (or the Gram matrix) associated with the nodal basis functions
\[
M^{0}(0) = \int_{K} \Psi^{0}(\xi)^T \Psi^{0}(\xi) \ dK.
\]
**Definition 2.** Let $N^{0}(p)$ be the degrees of freedom for $p \in P$, then the dual degrees of freedom, $\tilde{N}^{1}(p)$ for $p \in P$ are defined by
\[
N^{0}(q)^T \tilde{N}^{1}(p) := N^{0}(q)^T M^{0}(0) N^{0}(p), \quad \forall q \in P.
\]
Therefore
\[
\tilde{N}^{1}(p) = M^{0}(0) N^{0}(p).
\]
The dual degrees of freedom $\tilde{N}^{1}(p)$ are linear functionals acting on the primal degrees of freedom $N^{0}(p)$. Linearity of the functional follows from the linearity of the $L^2$ inner-product.
Remark 2. In Definition 2 the superscript ‘1’ on $\tilde{N}$ corresponds to the geometric dual of ‘0’ of the one-dimensional domain $K = [-1, 1]$. In general the dual of $N^0$ in $\mathbb{R}^d$ is denoted by $\tilde{N}^d$.

Corollary 2. The dual basis functions are given by
\[
\tilde{\Psi}^1(\xi) = \Psi^0(\xi) M^{(0)}^{-1}.
\] (6)

Proof.
\[
p(\xi) = \Psi^0(\xi) N^0(p) = \Psi^0(\xi) M^{(0)}^{-1} M^{(0)} N^0(p) = \tilde{\Psi}^1(\xi) \tilde{N}^1(p).
\] (7)

Remark 3. Note that in (7) an element $p \in \mathcal{P}$ can be represented in $\mathcal{P}$ and in $\mathcal{P}'$. This is due to the fact that $L^2(K)$ is the pivot space in this duality relation, see also [3, Ex.6.7.2].

Corollary 3. The mass matrix $\tilde{M}^{(1)}$ is the inverse of the mass matrix $M^{(0)}$.

Proof.
\[
\tilde{M}^{(1)} := \int_K \tilde{\Psi}^1(\xi)^T \tilde{\Psi}^1(\xi) \, dK \overset{(6)}{=} M^{(0)^{-1}} \int_K \Psi^0(\xi)^T \Psi^0(\xi) \, dK \cdot M^{(0)}^{-1} \overset{(2.1)}{=} M^{(0)^{-1}}.
\]

(a) Lagrange polynomials

(b) Dual Lagrange polynomials

Figure 1: The nodal Lagrange polynomial basis functions and the associated dual polynomials for $N = 4$.

In Figure 1 the Lagrange polynomials through the GLL-points and the associated dual polynomials are presented for $N = 4$.

Analogous to the construction of the dual nodal polynomials, we can also construct the dual polynomials to the edge functions. Let an element $p \in \mathcal{Q}$ be represented as
\[
p(\xi) = \sum_{i=1}^{N} N^1_i(p) e_i(\xi).
\]

In the simplified notation this can be written as
\[
p(\xi) = \Psi^1(\xi) N^1(p),
\]
with
\[
\Psi^1(\xi) = (e_1(\xi) e_2(\xi) \ldots e_{N-1}(\xi) e_N(\xi)) \quad \text{and} \quad N^1(p) = \begin{pmatrix} N^1_1(p) \\ N^1_2(p) \\ \vdots \\ N^1_{N-1}(p) \\ N^1_N(p) \end{pmatrix}.
\]

Similarly, we can write the \(L^2\)-inner product for two functions \(p, q \in Q\) expanded in this way as
\[
(p, q)_{L^2(K)} = N^1(p)^T M^{(1)} N^1(q),
\]
with \(M^{(1)}\) the mass matrix (or the Gram matrix) associated with the edge polynomials
\[
M^{(1)} = \int_K \Psi^1(\xi)^T \Psi^1(\xi) dK.
\]

**Definition 3.** Let \(N^1(p)\) be the degrees of freedom for \(p \in Q\), then the associated dual degrees of freedom \(\tilde{N}^0(p)\) are defined as
\[
N^1(q)^T \tilde{N}^0(p) := N^1(q)^T M^{(1)} N^1(p) \quad \forall q \in Q.
\]

Therefore
\[
\tilde{N}^0(p) = M^{(1)} N^1(p).
\]

Here again we follow Remark 2 to denote the superscript ‘0’ on \(\tilde{N}^0\) corresponding to geometric dual of ‘1’ for \(K = [-1, 1]\). In the \(d\)-dimensional case the dual degrees of freedom of \(N^1\) will be denoted by \(\tilde{N}^{d-1}\), see also Section 3.

Following Corollary 2, the dual edge functions are then given by
\[
\tilde{\Psi}^0(\xi) := \Psi^1(\xi) M^{(1)^{-1}}.
\]

In Figure 2 the edge polynomials \(e_i(\xi)\) and their dual polynomials \(\tilde{e}_i(\xi)\) are shown for \(N = 3\).

![Figure 2: The edge polynomial basis functions and the associated dual polynomials for \(N = 3\).](image)

From the Definitions 2 and 3 we see that the dual degrees of freedom act as linear functionals on the primal degrees of freedom. These two definitions essentially are a particular form of the Riesz Representation Theorem, [19, §2.4] or [2, §3.8]. We have, in this case that
\[
N^0(p)^T M^{(0)} N^0(p) = \tilde{N}^1(p)^T \tilde{M}^{(1)} \tilde{N}^1(p) \quad \text{and} \quad N^1(p)^T M^{(1)} N^1(p) = \tilde{N}^0(p)^T \tilde{M}^{(0)} \tilde{N}^0(p).
\]
The mass matrices $M^{(0)}$ and $M^{(1)}$ which map the primal degrees of freedom to the dual degrees of freedom are called the Riesz maps, [3, §6.4]. A direct consequence is that
\[ \|N^k(p)\|_{\mathbb{H}^k}^2 = N^k(p)^T M^{(k)} N^k(p) = \tilde{N}^{-k}(p)^T \tilde{M}^{(1-k)} \tilde{N}^{-k}(p) = \|\tilde{N}^{-k}(p)\|_{\mathbb{H}^{1-k}}^2, \quad k = 0, 1, \]
which just states that the Riesz map preserves the norm. One can compare this construction with covariant and contravariant representation of vectors. Let $v = \psi e_i \in \mathcal{V}$ be the contravariant representation and $\alpha = \alpha_i e^i \in \mathcal{V}^*$ a covariant representation, then for every $\alpha \in \mathcal{V}$ there exists a $v_\alpha \in \mathcal{V}$ such that $\langle \alpha, w \rangle = (v_\alpha, w)$, for all $w \in \mathcal{V}$. This is the Riesz representation theorem. Compare this with Definitions 2 and 3. In components the connection between $\alpha$ and $v_\alpha$ is written as $\alpha_i = g_{ij} v'_a$, where $g_{ij} = (e_i, e_j)$ is the metric tensor. If we compare this with $\tilde{N}^{-k}(p) = M^{k} N^k(p)$ for all $p \in \mathcal{P}, Q$ and $k = 0, 1$, we see that the mass matrix plays the role of the metric tensor $g_{ij}$. Note also that in this case we have that $\langle e'_i, e_j \rangle = \delta_{ij}$, which states that $e'_i$ is a canonical dual basis of $e_j$. A similar relation holds for the primal and dual polynomials.

**Lemma 2.** Let $\Psi^k(\xi)$ and $\tilde{\Psi}^1-k(\xi)$ be the primal and the dual basis as defined above, then these bases are bi-orthogonal with respect to each other
\[ \int_{\mathcal{K}} \tilde{\Psi}^1-k(\xi)^T \Psi^k(\xi) d\mathbf{K} = \mathbb{I}, \quad k = 0, 1, \]
where $\mathbb{I}$ is the $(N + 1) \times (N + 1)$ identity matrix for $k = 0$ and the $N \times N$ identity matrix for $k = 1$.

**Proof.** Using the definition of the dual basis, $\tilde{\Psi}^1-k(\xi) = M^{(k)-1} \Psi^k(\xi)$, gives,
\[ \int_{\mathcal{K}} \tilde{\Psi}^1-k(\xi)^T \Psi^k(\xi) d\mathbf{K} = M^{(k)-1} \int_{\mathcal{K}} \Psi^k(\xi)^T \Psi^k(\xi) d\mathbf{K} = M^{(k)-1} M^{(k)} = \mathbb{I}. \]

In Remark 1 it was stated that nodal sampling of a function is only possible in the space $\mathcal{P}$ of continuous functions. In a Sobolev space the elements consist of equivalence classes of functions that satisfy an integral equation and in this case nodal sampling may not be defined.

**Example 3.** Let $\xi_i$ be the GLL points which were defined in Example 1. Consider the functions
\[ f(\xi) = 1 \quad \text{and} \quad g(\xi) = \begin{cases} 1 & \text{if } \xi \neq \xi_i \\ 0 & \text{if } \xi = \xi_i \end{cases} \quad \text{for } i = 0, \ldots, N, \quad \xi \in [-1, 1]. \]

As elements of $L^2([-1, 1])$ the functions $f$ and $g$ are the same, but $N^{0}(f) \neq N^{0}(g)$ if nodal sampling would have been used. For a well-posed degree of freedom, we require that the operation should be independent of the representation we take from an equivalence class.

**Lemma 3.** Let $p \in \mathcal{P}$, then the nodal degrees of freedom are given by
\[ N^0(p) = \int_{\mathcal{K}} \tilde{\Psi}^1(\xi)^T p(\xi) d\mathbf{K}. \]

**Proof.** Every $p \in \mathcal{P}$ can be written as $p(\xi) = \Psi^0(\xi) N^0(p)$, therefore
\[ \int_{\mathcal{K}} \tilde{\Psi}^1(\xi)^T p(\xi) d\mathbf{K} = \int_{\mathcal{K}} \tilde{\Psi}^1(\xi)^T \Psi^0(\xi) N^0(p) d\mathbf{K} = N^0(p), \]
where in the last equality we used Lemma 2.

Example 3 demonstrated that nodal sampling of a $f \in L^2([-1, 1])$ is not well-defined. But Lemma 3 allows us to extend nodal sampling to square integrable functions.
Definition 4. For \( f \in L^2([-1, 1]) \) we define the nodal degrees of freedom by
\[
\mathcal{N}^\delta(f) := \int_K \Psi^i(\xi)^T f(\xi) \, dK,
\]
where we assume the exact evaluation of the Lebesgue integral.

Corollary 4. Using now the fact that \( \Psi^i(\xi) = \Psi^0(\xi) M^{(0)} \) this ‘nodal sampling’ can be written as
\[
\mathcal{N}^\delta(f) = M^{(0)-1} \int_K \Psi^0(\xi)^T f(\xi) \, dK,
\]
which is just the \( L^2 \)-projection of \( f \) onto the basis functions. Analogously we have
\[
\mathcal{N}^1(f) := \int_K \Psi^0(\xi)^T f(\xi) \, dK = M^{(1)-1} \int_K \Psi^1(\xi)^T f(\xi) \, dK,
\]
\[
\tilde{\mathcal{N}}^0(f) := \int_K \Psi^i(\xi)^T f(\xi) \, dK \quad \text{and} \quad \tilde{\mathcal{N}}^1(f) := \int_K \Psi^0(\xi)^T f(\xi) \, dK.
\]

2.2. Differentiation of dual variables

Let \( q \) be expanded in Lagrange polynomials and \( \phi \) in dual edge polynomials. The objective of this section is to define \( d\phi/d\xi \). We will do this with the help of integration by parts
\[
\int_K q \frac{d\phi}{d\xi} \, dK = -\int_K \phi \frac{dq}{d\xi} \, dK + \int_{\partial K} q \phi \, dK. \tag{9}
\]
Let \( q \) and \( \phi \) be expanded as
\[
q(\xi) = \sum_{i=0}^N N_i^\delta(q) h_i(\xi) \quad \text{and} \quad \phi(\xi) = \sum_{j=0}^N \tilde{N}_j^0(\phi) \tilde{w}_j(\xi).
\]
Then, using (4), we have
\[
\int_K \frac{dq}{d\xi} \, dK = \mathcal{N}^\delta(q)^T E^{1,0} \tilde{\mathcal{N}}^\delta(\phi) = \mathcal{N}^\delta(q)^T E^{1,0} \tilde{N}_0^0(\phi). \tag{10}
\]
where we used Lemma 2 in the last step.

Using (10) in (9) gives
\[
\int_K q \frac{d\phi}{d\xi} \, dK = -\mathcal{N}^\delta(q)^T E^{1,0} \tilde{N}_0^0(\phi) + \mathcal{N}_0^\delta(q) \tilde{N}_N^0(\phi) - \mathcal{N}_0^\delta(q) \tilde{N}_0^0(\phi), \quad \forall q \in \mathcal{P}, \tag{11}
\]
where \( \tilde{N}_N^0(\phi) \) and \( \tilde{N}_{N+1}^0(\phi) \) are the nodal values of \( \phi(\xi) \) at the end points \( \xi = -1 \) and \( \xi = 1 \), respectively. Since (11) needs to hold for all \( q \in \mathcal{P} \), we define the degrees of freedom of derivative of the dual representation of \( \phi \) as
\[
\tilde{\mathcal{N}}^1 \left( \frac{d\phi}{d\xi} \right) := -E^{1,0} \tilde{N}_0^0(\phi) + \tilde{N}_N^0(\phi) - \tilde{N}_0^0(\phi). \tag{12}
\]

With these degrees of freedom we can expand the derivative of \( \phi \) as
\[
\frac{d\phi}{d\xi} = \tilde{\Psi}^1(\xi) \tilde{\mathcal{N}}^1 \left( \frac{d\phi}{d\xi} \right) = \tilde{\Psi}^1(\xi) \left[ -E^{1,0} \tilde{N}_0^0(\phi) + \tilde{N}_N^0(\phi) - \tilde{N}_0^0(\phi) \right]. \tag{13}
\]
Remark 4. Note that while \( \phi(\xi) \) is a polynomial of degree \( (N - 1) \), its derivative, as defined by (13), is a polynomial of degree \( N \).
3. Two-dimensional dual spaces

In order to address more challenging problems, it is important to consider in more detail the case $K = [-1, 1]^2 \subset \mathbb{R}^2$. For $d := \dim K = 2$, we have the two sets of function spaces that obey the De Rham cohomology [21, 22]

$$H(\text{curl}; K) \xrightarrow{\nabla \cdot} H(\text{div}; K) \xrightarrow{\nabla} L^2(K) \quad \text{and} \quad H^1(K) \xrightarrow{\nabla} H(\text{curl}; K) \xrightarrow{\nabla \times} L^2(K).$$

We will introduce three different finite element spaces in the sense of Definition 1 such that the associated discrete functional spaces, $C_h(K) \subset H(\text{curl}; K)$, $D_h(K) \subset H(\text{div}; K)$, $S_h(K) \subset L^2(K)$, and the corresponding dual spaces $	ilde{S}_h^0(K)$, $	ilde{D}_h^0(K)$, $	ilde{C}_h^0(K)$ obey the discrete De Rham complex.

3.1. The function space $C_h(K) \subset H(\text{curl}; K)$

3.1.1. Primal finite element

Let $\xi, \eta \in [-1, 1], i = 0, \ldots, N$, be Gauss-Lobatto-Legendre (GLL) points, and $P$ denote the space of polynomials of degree $N$ defined on the interval $[-1, 1]$, see Example 1. Consider now the polynomial tensor product space $C_h(K) := P \otimes P$. Given the set $x$ of 2D nodes $x_k$ defined as $x_k := \{x_{i(N+1)+j} = (\xi, \eta) \mid i, j = 0, \ldots, N\}$, we can introduce for any $p \in C_h(K)$ the degrees of freedom as

$$N^0_k(p) := p(x_k), \quad k = 0, \ldots, (N + 1)^2 - 1.$$

The basis which satisfies the Kronecker-delta property from Proposition 1 is given by the Lagrange (or nodal) polynomials, $\epsilon^{(0)}_k, k = 0, \ldots, (N+1)^2 - 1$, through the two-dimensional GLL-nodes $x_{i(N+1)+j} = (\xi, \eta), i, j = 0, \ldots, N$, such that

$$\epsilon^{(0)}_{i(N+1)+j}(\xi, \eta) := h_i(\xi)h_j(\eta), \quad i, j = 0, \ldots, N;$$

where $h_i$ are the 1D nodal interpolants introduced in Example 1. A visual representation of these basis functions for $N = 2$ is presented in Figure 3a.

3.1.2. The dual finite element

The construction of the dual basis functions follows the ideas presented in Section 2.1. Here we outline the direct application to the 2D case of constructing the dual basis of the space $\tilde{C}_h$. The degrees of freedom of the dual element are given by

$$\tilde{N}^2(p) := M^{(0)}N^0(p),$$
where
\[ \Psi^0(\xi, \eta) = \left( e^{(0)}_0(\xi, \eta), \ldots, e^{(0)}_{(N+1)^2-1}(\xi, \eta) \right) \quad \text{and} \quad \mathcal{M}^{(0)} = \int_{K} \Psi^0(\xi, \eta) \Psi^0(\xi, \eta) \, dK. \]

Since the dual basis functions \( \tilde{e}_j^{(2)} \) need to satisfy the Kronecker-delta property
\[ \mathcal{N}^2(\tilde{e}_j^{(2)}) = \delta_{ij}, \]
by Corollary 2 we have that the dual basis functions can be expressed in terms of the primal basis functions as
\[ \tilde{\Psi}^2(\xi, \eta) := \left( \tilde{e}_0^{(2)}, \ldots, \tilde{e}_{(N+1)^2-1}^{(2)} \right) = \left( e^{(0)}_0, \ldots, e^{(0)}_{(N+1)^2-1} \right) \mathcal{M}^{(0)-1} = \Psi^0(\xi, \eta) \mathcal{M}^{(0)-1}. \]

A visual representation of these basis functions for \( N = 2 \) is presented in Figure 3b.

### 3.2. The function space \( D^h(K) \subset H(\text{div}; K) \)

#### 3.2.1. Primal finite element

Let \( \xi_i, \eta_j \in [-1, 1], i = 0, \ldots, N, \) be the GLL points, and \( P \) and \( Q \) denote the space of polynomials of degree \( N \) and \( N - 1 \), respectively, defined on the interval \([-1, 1]\), as in Example 1 and 2. Consider now the polynomial tensor product spaces \( Q^1_\xi := P \otimes Q \) and \( Q^1_\eta := Q \otimes P \). We introduce for any polynomial vector field \( \mathbf{p} \in Q^1_\xi \times Q^1_\eta \) the degrees of freedom as
\[
\begin{align*}
N_{i\eta_j}(\mathbf{p}) & := \int_{[\xi_i, \xi_{i+1}]} \mathbf{p} \cdot e_\xi \, d\eta, \quad i = 0, \ldots, N \quad \text{and} \quad j = 1, \ldots, N, \\
N_{i-1}(\eta_j)N_{j+1}(N+1)(\mathbf{p}) & := \int_{[\eta_j, \eta_{j+1}]} \mathbf{p} \cdot e_\eta \, d\xi, \quad i = 1, \ldots, N \quad \text{and} \quad j = 0, \ldots, N,
\end{align*}
\]
where \( e_\xi \) and \( e_\eta \) are the unit vectors in the \( \xi \)- and \( \eta \)-directions, respectively. In a polynomial vector space these integrals are well-defined.

It is possible to show, see [23–26], that the basis functions which satisfy the Kronecker-delta property from Proposition 1 are the 2D edge polynomials, \( e^{(1)}_k \), \( k = 1, \ldots, 2N(N+1) \), defined as
\[
\Psi^1(\xi, \eta) = \begin{cases} 
\epsilon^{(1)}_{N\xi j}(\xi, \eta) := h_i(\xi) e_j(\eta) e_\xi, & \text{if } i = 0, \ldots, N \text{ and } j = 1, \ldots, N, \\
\epsilon^{(1)}_{i\eta(N+1)+j+1}(\xi, \eta) := e_i(\xi) h_j(\eta) e_\eta, & \text{if } i = 1, \ldots, N \text{ and } j = 0, \ldots, N,
\end{cases}
\]
where \( h_i \) are the 1D nodal interpolants introduced in Example 1, and \( e_j \) are the 1D edge interpolants introduced in Example 2. A visual representation of these basis functions for \( N = 2 \) is presented in Figure 4a.

Let \( \omega^h \in C^0(\Omega) \) be represented as
\[
\omega^h = \sum_{i=0}^{N} \sum_{j=0}^{N} \omega_{ij} h_i(\xi) h_j(\eta),
\]
then
\[
\text{curl } \omega^h = \left( \frac{\partial \omega_j}{\partial \eta} - \frac{\partial \omega_i}{\partial \xi} \right),
\]
using Corollary 1, we have
\[
\text{curl } \omega = \left( \sum_{i=0}^{N} \sum_{j=0}^{N} (\omega_{ij} - \omega_{i,j-1}) h_i(\xi) e_j(\eta) - \sum_{i=0}^{N} \sum_{j=0}^{N} (\omega_{i,j} - \omega_{i-1,j}) e_i(\xi) h_j(\eta) \right) = \Psi^1(\xi, \eta) \mathcal{M}^{1,0} \mathcal{N}^0(\omega^h).
\]
This implies that \( \mathcal{R}(\text{curl}; C^0(\Omega)) \subset D^h(\Omega) \), where \( \mathcal{R}(\text{curl}; C^0(\Omega)) \) denotes the range of the curl operator applied to elements from \( C^0(\Omega) \). This is a necessary requirement for \( C^0(\Omega) \) and \( D^h(\Omega) \) to form a finite dimensional De Rham sequence, (24).
(a) The primal basis edge polynomials
(b) The dual basis edge polynomials

Figure 4: Visualization of the primal basis functions, \( \epsilon^{(1)}_{N+1,j}(\xi, \eta) \) (top left), \( \epsilon^{(1)}_{(i-1)(N+1)+j+1}(\xi, \eta) \) (bottom left), and dual basis functions, \( \tilde{\epsilon}^{(1)}_{N+1,j}(\xi, \eta) \) (top right), and \( \tilde{\epsilon}^{(1)}_{(i-1)(N+1)+j+1}(\xi, \eta) \) (bottom right), for the spaces \( \mathcal{D}(K) \) and \( \tilde{\mathcal{D}}(K) \) with \( N = 2 \).
3.2.2. Dual finite element

The construction of the dual basis functions of the space $\tilde{D}^h(K)$ is done in the same manner as for the dual basis functions of the space $C^h(K)$. In this case, the dual basis functions can be expressed in terms of the primal basis functions as

$$
\Psi^1(\xi, \eta) := \left[ \begin{array}{cccccc}
\tilde{e}_1^{(1)} & \cdots & \tilde{e}_{N+1}^{(1)} \\
0 & \cdots & 0 \\
\tilde{e}_{N+1}^{(1)} & \cdots & \tilde{e}_{2N+1}^{(1)}
\end{array} \right],
$$

with

$$
\mathbb{M}^{(1)} := \int_K \tilde{e}_i^{(1)}(\xi, \eta) \cdot \tilde{e}_j^{(1)}(\xi, \eta) dK, \quad i, j = 1, \ldots, 2N(N+1).
$$

A visual representation of these basis functions for $N = 2$ is presented in Figure 4b.

The degrees of freedom of the dual element are given by

$$\tilde{N}^1(p) := \mathbb{M}^{(1)} N^1(p).$$

In the case of orthogonal mesh we have that $\mathbb{M}^{(1)}$ is block diagonal

$$
\mathbb{M}^{(1)} = \left( \begin{array}{cc}
\mathbb{M}^{(1,2)} & 0 \\
0 & \mathbb{M}^{(1,0)}
\end{array} \right),
$$

with

$$
\mathbb{M}^{(1,i)} := \int_K \tilde{e}_i^{(1)}(\xi, \eta) \cdot \tilde{e}_j^{(1)}(\xi, \eta) dK, \quad i, j = 1, \ldots, N(N+1),
$$

$$
\mathbb{M}^{(1,n)} := \int_K \tilde{e}_i^{(1)}(\xi, \eta) \cdot \tilde{e}_j^{(1)}(\xi, \eta) dK, \quad i, j = N(N+1) + 1, \ldots, 2N(N+1).
$$

3.3. The function space $S^h(K) \subset L^2(K)$

3.3.1. Primal finite element

Once again, let $\xi, \eta \in [-1, 1], i = 0, \ldots, N$, be the GLL points, and $Q$ represent the space of polynomials of degree $N – 1$ on the interval $[-1, 1]$. Consider now the polynomial tensor product space $Q^2 := Q \otimes Q$. The degrees of freedom for this finite element can be introduced for any polynomial $p \in Q^2$ as

$$
N^2(p) := \int_{-1}^{\eta_i} \int_{-1}^{\eta_j} p d\xi d\eta, \quad i, j = 1, \ldots, N.
$$

These integrals are well-defined in a polynomial space. It is possible to demonstrate, see [23–26], that the basis functions which satisfy the Kronecker-delta property from Proposition 1 are the surface polynomials, $\epsilon_k^{(2)}, k = 1, \ldots, N^2$, defined as

$$
\epsilon_{(i-1)N+j}^{(2)}(\xi, \eta) := \epsilon_i(\xi) \epsilon_j(\eta), \quad i, j = 1, \ldots, N,
$$

where, as before, $\epsilon_i$ are the 1D edge interpolants introduced in Example 2. A visual representation of these basis functions for $N = 2$ is presented in Figure 5a.

An element from $q^h \in D^h(K)$ can be represented in the basis functions of $D^h(K)$ as (14)

$$
q^h(\xi, \eta) = \left( \begin{array}{c}
\sum_{i=0}^{N} \sum_{j=1}^{N} u_{i,j} h_i(\xi) \epsilon_j(\eta) \\
\sum_{i=1}^{N} \sum_{j=1}^{N} v_{i,j} h_i(\xi) \epsilon_j(\eta)
\end{array} \right).
$$

If we take the divergence of this vector field and use (4) repeatedly, we have

$$
\text{div} \ q^h(\xi, \eta) = \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1} \right] \epsilon_i(\xi) \epsilon_j(\eta).
$$
So, we see that the divergence modifies the degrees of freedom (the expansion coefficients) and changes the basis functions from basis functions in $D^h(K)$ to basis functions for $S^h(K)$. We can write this as

$$\text{div} \; q^h(\xi, \eta) = \Psi_0^2(\xi, \eta) E_2^{1,0} N^1(q^h),$$

where the incidence matrix $E_2^{1,0}$ is a sparse matrix which only contains the non-zero entries, $-1$ and $1$, as can be seen from (18).

Application of (4) shows that $\mathcal{R}(\text{div}; D^h(K)) = S^h(K)$, which is required for the spaces $D^h(K)$ and $S^h(K)$ to be part of the finite dimensional De Rham sequence (24). Since we have that $\text{curl} \; \omega^h \equiv 0$ for all $\omega^h \in C^h(K)$, we have, by combining (15) and (19) that

$$\text{curl} \; \omega^h = \Psi_0^2(\xi, \eta) E_2^{1,0} N^0(\omega^h) \equiv 0.$$ 

Since $\Psi_0^2(\xi, \eta)$ forms basis for $S^h(K)$ it implies that $E_2^{1,0} N^0(\omega^h) \equiv 0$. If, in addition, this needs to hold for all $\omega^h \in C^h(K)$, we need to have $E_2^{1,0} N^0 \equiv 0$, which is a well-known property of incidence matrices in mimetic methods.

### 3.3.2. Dual finite element

The dual basis functions of the space $\tilde{S}^h(K)$ follow the same steps as performed for the spaces $\tilde{C}^h(K)$ and $\tilde{D}^h(K)$. The degrees of freedom for the dual element are given by

$$\tilde{N}^0(p) := M^{(2)} N^2(p),$$

with

$$M_{ij}^{(2)} := \int_K \epsilon_i^{(2)}(\xi, \eta) \epsilon_j^{(2)}(\xi, \eta) \, dK, \quad i, j = 1, \ldots, N^2.$$ 

The associated dual basis functions are expressed in terms of the primal basis functions as

$$\tilde{\Psi}_0^0(\xi, \eta) := (\epsilon_0^{(2)} \left| \epsilon_{N^2}^{(2)}\right) = (\epsilon_1^{(2)} \ldots \epsilon_{N^2}^{(2)}) M^{(2)^{-1}} := \Psi_0^2(\xi, \eta) M^{(2)^{-1}}.$$ 

A visual representation of these basis functions for $N = 2$ is presented in Figure 5b.
3.4. Dual De Rham sequence

In the multi-dimensional case we can extend the derivative of dual representations, (12), in the following: For \( q^h \in D^h(K) \) and \( s^h \in \tilde{S}^h(K) \), we have, using (19)

\[
\int_K s^h \, \text{div} q^h \, dK = \widetilde{N}^0(s^h) \mathbb{E}^{2,1} N^1(q^h) .
\]

Since

\[
\int_K s^h \, \text{div} q^h \, dK = \int_{\partial K} (q^h \cdot n) \, s^h \, d\Gamma - \int_K \text{grad} s^h \cdot q^h \, dK .
\]

This gives

\[
N^1(q^h)^T \tilde{N}^1(\text{grad } s^h) = \int_K \text{grad } \tilde{s}^h \cdot q^h \, dK = -\int_K \tilde{s}^h \, \text{div} q^h \, dK + \int_{\partial K} (q^h \cdot n) \, s^h \, d\Gamma = -N^1(q^h)^T \mathbb{E}^{2,1} \tilde{N}^0(s^h) + \sum_{i=1}^N \left[ s_{i,N}(q_{i})_{h,N} - s_{i,0}(q_{i})_{h,0} \right] + \sum_{j=1}^N \left[ s_{N,i}(q_{j})_{N,i} - s_{0,i}(q_{j})_{N,i} \right] \]

(20)

where \( \mathbb{N} \) is a very sparse matrix with non-zero entries \(-1\) and \(+1\) only. Since this equality needs to hold for all \( q^h \in D^h(K) \) we have

\[
\tilde{N}^1(\text{grad } s^h) = -\mathbb{E}^{2,1} \tilde{N}^0(s^h) + \mathbb{N} \tilde{N}^0(s^h) .
\]

Note that the degrees of freedom for \( q^h \cdot n \) along the boundary are already contained in the space of degrees of freedom for \( D^h(K) \), but the degrees of freedom for \( s^h \) along the boundary are not in \( \tilde{S}^h(K) \), therefore these have to be explicitly added. If we set these trace variables along the boundary to zero, we refer to this space as \( \tilde{S}^h_0(K) \) and (20) reduces to

\[
\tilde{N}^1(\text{grad } s^h) = -\mathbb{E}^{2,1} \tilde{N}^0(s^h) + \mathbb{N} \tilde{N}^0(s^h) .
\]

We therefore have that the gradient is a mapping \( \tilde{S}^h_0(K) \rightarrow \overline{D}^h \approx \overline{D}^0_0 \).

Likewise, we have that for all \( \omega^h \in C^h(K) \) and \( q^h \in \overline{D}^h(K) \) according to (15)

\[
\int_K q^h \cdot \text{curl } \omega^h \, dK = \tilde{N}^1(q^h)^T \mathbb{E}^{1,0} N^0(\omega^h) .
\]

Then the identity

\[
\int_K q^h \cdot \text{curl } \omega^h \, dK = -\int_{\partial K} (q^h \cdot n) \omega^h \, d\Gamma + \int_K \omega^h \, \text{curl } q^h \, dK ,
\]

gives

\[
N^0(\omega^h)^T \tilde{N}^2(\text{curl } q^h) = \int_K \omega^h \, \text{curl } q^h \, dK = \int_K q^h \cdot \text{curl } \omega^h \, dK + \int_{\partial K} (q^h \cdot n) \omega^h \, d\Gamma = N^0(\omega^h)^T \mathbb{E}^{1,0} \tilde{N}^1(q^h)^T + \sum_{i=1}^{N-1} \left[ \omega_{i,0}(q_{i})_{h,0} - \omega_{i,N}(q_{i})_{h,N} \right] + \sum_{j=1}^{N-1} \left[ \omega_{N,i}(q_{j})_{N,i} - \omega_{0,i}(q_{j})_{N,i} \right] + \omega_{N,N}(q_{N})_{N,N} - \omega_{0,N}(q_{N})_{h,N} + \omega_{N,0}(q_{0})_{h,0} - \omega_{0,0}(q_{0})_{h,0} \]

(21)

\[
\tilde{N}^1(q^h) = N^0(\omega^h)^T \mathbb{E}^{1,0} \tilde{N}^1(q^h)^T + N^0(\omega^h)^T \mathbb{N} \tilde{N}^1(q^h) .
\]

(22)
Note once again that in the boundary integral the degrees of freedom for \( \omega^h \) are contained in the space of the degrees of freedom for \( C^0(K) \), but the need to explicitly add the trace variables \( q^h \cdot n \). In this particular case, the situation is complicated near the corner of the domain where the normal changes direction. For a treatment of these degrees of freedom near corners, see [27]. The matrix \( N \) is the sparse trace matrix with entries \(-1\) and \(+1\) only.

If we set the trace variable \( q^h \cdot n \) to zero along the boundary \( \partial K \), we refer to this space as \( \tilde{D}_0^h(K) \) and (22) reduces to
\[
N^0(\omega^h) \tilde{N}^2(\omega^h) = N^0(\omega^h) \tilde{N}^1(\omega^h) .
\]
If this relation needs to hold for all \( \omega^h \in C^0(K) \), then we have
\[
\tilde{N}^2(\omega^h) = \tilde{N}^1(\omega^h) .
\]

The basis functions in which these degrees of freedom are expanded are the dual polynomials in \( \tilde{C}^0(K) \). Therefore the curl applied to the \( \tilde{D}_0^h(K) \) is a map \( \tilde{D}_0^h(K) \rightarrow \tilde{C}^0(K) \). Therefore, the dual space supplemented with homogeneous boundary traces form a dual De Rham sequence
\[
\tilde{S}_0^h(K) \xrightarrow{\tilde{\nabla}^1 0^T} \tilde{D}_0^h(K) \xrightarrow{\tilde{\nabla}^0 0^T} \tilde{C}_0^h(K) \quad (23)
\]
In general
\[
C^h(K) \xrightarrow{\nabla^i} D^h(K) \xrightarrow{\nabla} S^h(K)
\]
\[
\xrightarrow{\delta^i} \xrightarrow{\delta^0} \tilde{S}_0^h(K) \quad (24)
\]

4. Three-dimensional dual spaces

Similar to Section 3 we can define primal and dual spaces for three dimensional problems. The De Rham sequence in this case is given by
\[
H^1(K) \xrightarrow{\text{grad}} H(\text{curl}; K) \xrightarrow{\text{curl}} H(\text{div}; K) \xrightarrow{\text{div}} L^2(K)
\]
We will define the four finite element spaces \( G^0(K) \subset H^1(K) \), \( C^0(K) \subset H(\text{curl}; K) \), \( D^0(K) \subset H(\text{div}; K) \), \( S^0(K) \subset L^2(K) \) and the corresponding dual spaces \( \tilde{G}_0^h(K) \), \( \tilde{C}_0^h(K) \), \( \tilde{D}_0^h(K) \), \( \tilde{S}_0^h(K) \) such that they obey the discrete De Rham cohomology.

Let \( \xi_i, \eta_j, \zeta_k, i, j, k = 0, \ldots, N \), be the GLL-points. Let \( P \) be the space of polynomials of degree \( N \) as introduced in Example 1 and \( Q \) be the space of polynomials of degree \( N - 1 \) as introduced in Example 2.

4.1. The function space \( G^0(K) \subset H^1(K) \)

Let \( G^0(K) := P \otimes P \otimes P \) be the tensor product space. Any element \( p \in G^0(K) \) can be represented by
\[
p(\xi, \eta, \zeta) = \Psi^0(\xi, \eta, \zeta) N^0(p) ,
\]
where \( N^0(p) \) are the degrees of freedom defined at set of 3D nodes \( x_{i(N+1)^2+j(N+1)+k} = (\tilde{\xi}_i, \tilde{\eta}_j, \tilde{\zeta}_k) \), \( i, j, k = 0, \ldots, N \) given by,
\[
N^0(p) := p(x_k) , \quad k = 0, \ldots, (N + 1)^3 - 1 ,
\]
and \( \Psi^0 \) are the Lagrange (or nodal) basis through \( x_k \) given by
\[
\Psi^0(\xi, \eta, \zeta) = \varepsilon^{(0)}_{i(N+1)^2+j(N+1)+k}(\xi, \eta, \zeta) := h_i(\xi) h_j(\eta) h_k(\zeta) \quad i = 0, \ldots, N; \ j = 0, \ldots, N; \ k = 0, \ldots, N .
\]

Following Corollary 2 the dual degrees of freedom for space \( \tilde{G}_0^h(K) \) are given by
\[
\tilde{N}^0(p) := M^{(0)} N^0(p) \quad \text{where} \quad M^{(0)} = \int_K \Psi^0(\xi, \eta, \zeta)^T \Psi^0(\xi, \eta, \zeta) \, dK , \quad (25)
\]
and the dual basis are given by,
\[
\tilde{\Psi}^0(\xi, \eta, \zeta) = \Psi^0(\xi, \eta, \zeta) M^{(0)^{-1}} .
\]
4.2. The function space $C^0(K) \subset H(\text{curl}; K)$

Consider the polynomial tensor spaces given by, $Q_{\xi} := Q \otimes P \otimes P$, $Q_{\eta} := P \otimes Q \otimes P$ and $Q_{\zeta} := P \otimes P \otimes Q$. We define the finite element space of edges in 3D as $C^0(K) := Q_{\xi} \times Q_{\eta} \times Q_{\zeta}$. Any vector field $\mathbf{e}^h \in C^0(K)$ can be represented as

$$e(\xi, \eta, \zeta) = \Psi^1(\xi, \eta, \zeta) N_1^1(c),$$

where the the degrees of freedom defined at edges are given by

$$N_1^1(c) = \left\{ \begin{array}{ll}
N_{j+N_1} + j(N_1 + 1) + (i-1)(N+1) + (j-1)(N+1) + (k-1)(N+1) & : \xi, \eta, \zeta \\
\mathcal{N}(i, j, k) & \text{for } i = 1, \ldots, N; j, k = 0, \ldots, N
\end{array} \right.,$$

and the basis that satisfies the Kronecker delta property, in the sense of integrals over the edges, are given by

$$\Psi^1(\xi, \eta, \zeta) = \left\{ \begin{array}{ll}
\mathcal{E}_{\xi, \eta, \zeta}^1 & : \xi, \eta, \zeta \\
\mathcal{E}_{\xi, \eta, \zeta}^1 & \text{for } i = 1, \ldots, N; j, k = 0, \ldots, N
\end{array} \right..$$

Let $p^h \in C^0(K)$ then following Corollary 1 the grad $p^h$ is given by

$$\text{grad } p^h(\xi, \eta, \zeta) = \Psi^1(\xi, \eta, \zeta) N_1^1(p^h),$$

expressed in terms of basis of $C^0(K)$. This implies that $\mathcal{R}(\text{grad}; C^0(K)) \subset C^0(K)$.

Following Corollary 2 the corresponding dual degrees of freedom in function space $C^0(K)$ are given by

$$\mathcal{N}_1^2(c) := M_1^1 N_1^1(c)$$

where

$$M_1^1 = \int_K \Psi^1(\xi, \eta, \zeta)^T \Psi^1(\xi, \eta, \zeta) dK,$$

and the dual basis are given by

$$\Psi_1^1(\xi, \eta, \zeta) = \Psi^1(\xi, \eta, \zeta) M_1^1.\$$

4.3. The function space $D^0(K) \subset H(\text{div}; K)$

We define the finite element space of degrees of freedom defined over surfaces as $D^0(K) := R_{\xi} \times R_{\eta} \times R_{\zeta}$, where, $R_{\xi} := P \otimes Q \otimes Q$, $R_{\eta} := Q \otimes P \otimes Q$ and $R_{\zeta} := Q \otimes Q \otimes P$. Any vector field $\mathbf{q} \in D^0(K)$ can be expressed as

$$q(\xi, \eta, \zeta) = \Psi^2(\xi, \eta, \zeta) N_2^2(q),$$

where, the degrees of freedom defined at the surfaces are given by

$$N_2^2 = \left\{ \begin{array}{ll}
N_{j(N_1 + 1) + k - 1}^2 & : \xi, \eta, \zeta \\
\mathcal{N}_{j(N_1 + 1) + k - 1}^2 & \text{for } i = 1, \ldots, N; j, k = 1, \ldots, N
\end{array} \right.,$$

and the basis function that satisfy the Kronecker delta property, as an integral over the surface, are given by

$$\Psi^2(\xi, \eta, \zeta) = \left\{ \begin{array}{ll}
\mathcal{E}_{\xi, \eta, \zeta}^2 & : \xi, \eta, \zeta \\
\mathcal{E}_{\xi, \eta, \zeta}^2 & \text{for } i = 1, \ldots, N; j, k = 1, \ldots, N
\end{array} \right..$$
Let \( c^h \in C^h(K) \), then the operation of discrete curl operator can be defined following Corollary 1 as
\[
\text{curl } q^h(\xi, \eta, \zeta) = \Psi^2(\xi, \eta) E^{2,1} N^1(q^h),
\]
expressed in terms of basis of \( D^h(K) \). This implies that \( \mathcal{R}(\text{curl}; C^h(K)) \subseteq D^h(K) \).

The corresponding dual degrees of freedom and dual basis functions, using Corollary 2, are given by
\[
\begin{align*}
\tilde{N}^1(p) &:= M^{(2)} N^2(p) \quad \text{where} \quad M^{(2)} = \int_K \Psi^2(\xi, \eta, \zeta)^T \Psi^2(\xi, \eta, \zeta) \, dK, \\
\tilde{\Psi}^2(\xi, \eta, \zeta) &= \Psi^2(\xi, \eta, \zeta) M^{(2)}^{-1}.
\end{align*}
\]

4.4. The function space \( S^h(K) \)

Let us define the finite element space of volumes in 3D as \( S^h(K) := Q \times Q \times Q \). We can express any polynomial \( f \in S^h(K) \) as
\[
f(\xi, \eta, \zeta) = \Psi^3(\xi, \eta, \zeta) N^3(f),
\]
where the degrees of freedom defined over a volume are given by
\[
N^3(f) = N^3_{i-1,j-1,k-1} := \int_{\xi_{i-1}}^{\xi_i} \int_{\eta_{j-1}}^{\eta_j} \int_{\zeta_{k-1}}^{\zeta_k} f \, d\xi \, d\eta \, d\zeta \quad \text{for} \quad i, j, k = 1, \ldots, N, \tag{27}
\]
and the basis functions that satisfy the Kronecker-delta property, in the sense of an integral over a volume, are given by
\[
\Psi^3(\xi, \eta, \zeta) = \epsilon_{i-1,j-1,k-1}(\xi, \eta, \zeta) \quad \text{for} \quad i, j, k = 1, \ldots, N.
\]
Let \( q^h \in D^h(K) \) be a divergence free vector field, then the discrete div operation can be defined using Corollary 1 as
\[
\text{div } q^h(\xi, \eta, \zeta) = \Psi^3(\xi, \eta, \zeta) E^{3,2} N^2(q^h),
\]
expressed in basis of \( S^h(K) \). This implies that \( \mathcal{R}(\text{div}; D^h(K)) \subseteq S^h(K) \).

The corresponding dual degrees of freedom and the dual basis are given by, using Corollary 2,
\[
\begin{align*}
\tilde{N}^0(f) &:= M^{(3)} N^3(f) \quad \text{where} \quad M^{(3)} = \int_K \Psi^3(\xi, \eta, \zeta)^T \Psi^3(\xi, \eta, \zeta) \, dK, \\
\tilde{\Psi}^3(\xi, \eta, \zeta) &= \Psi^3(\xi, \eta, \zeta) M^{(3)}^{-1}.
\end{align*}
\]

4.5. Discrete De Rham complex

Using the above defined spaces the discrete De Rham complex for primal spaces and the corresponding dual spaces can then be written as
\[
\begin{array}{cccccc}
G^h(K) & \xrightarrow{\text{grad}} & C^h(K) & \xrightarrow{\text{curl}} & D^h(K) & \xrightarrow{\text{div}} & S^h(K) \\
\downarrow \substack{\Psi^0 \; \text{grad} \; G^h(K) \; \text{grad} \; \downarrow \Psi^0 \; \text{grad} \; S^h(K)} & \downarrow \substack{\Psi^1 \; \text{curl} \; C^h(K) \; \text{curl} \; \downarrow \Psi^1 \; \text{curl} \; S^h(K)} & \downarrow \substack{\Psi^2 \; \text{div} \; D^h(K) \; \text{div} \; \downarrow \Psi^2 \; \text{div} \; S^h(K)} & \downarrow \substack{\Psi^3 \; \text{grad} \; S^h(K) \; \text{grad} \; \downarrow \Psi^3 \; \text{grad} \; G^h(K) \; \text{grad} \;} \\
G^h_0(K) & \leftarrow \xrightarrow{\text{div}} & C^h_0(K) & \leftarrow \xrightarrow{\text{curl}} & D^h_0(K) & \leftarrow \xrightarrow{\text{grad}} & S^h_0(K) \\
\end{array}
\]

(31)
5. Mixed formulation of the Poisson equation

So far we have introduced the construction of primal spaces and the corresponding dual spaces that obey the de Rham sequence. In this section we present an application of these spaces to a constrained minimization problem of the Poisson equation. We will compare the results from two formulations: 1) with primal spaces only, and 2) with primal and dual spaces. In this application it will be shown that the use of dual spaces can give much sparser systems with a lower condition number. Let \( K \subset \mathbb{R}^d \) for \( d = 3 \), then for \( \phi \in L^2(K) \) and \( q \in H(\text{div}; K) \) we define the functional

\[
\mathcal{L}(\phi, q; f, \hat{\phi}) := \int_K \frac{1}{2} |q|^2 \, dK + \int_K \phi (\text{div } q - f) \, dK - \int_{\partial K} \hat{\phi} \, (q \cdot n) \, d\Gamma,
\]

for prescribed functions \( f \in L^2(K) \) and \( \hat{\phi} \in H^{1/2}(\partial K) \). The optimality conditions for this functional are given by

\[
\begin{cases}
(p, q)_K + (\text{div } p, \phi)_K = \int_{\partial K} (p, n) \hat{\phi} \, d\Gamma & \forall \ p \in H(\text{div}; K) \\
(\phi, \text{div } q)_K = (\phi, f)_K & \forall \ \phi \in L^2(K).
\end{cases}
\]

(32)

This corresponds with a Poisson equation for \( \phi \) with Dirichlet boundary condition \( \phi = \hat{\phi} \) along the boundary. We will consider two different discretizations for this problem. For the first approximation we choose \((q^h, \phi^h) \in D^h(K) \times \tilde{S}^h(K)\), we will call this primal-primal formulation, while in the second case we approximate the solution as \((q^h, \phi^h) \in D^h(K) \times \tilde{S}^h(K)\), we will call this primal-dual formulation.

5.1. Primal-primal formulation

Let \( q^h \) be represented as in (26)

\[
q^h(\xi, \eta, \zeta) = \Psi^h(\xi, \eta, \zeta) N^2(q^h).
\]

Then, using (28), the divergence is given by

\[
\text{div } q^h(\xi, \eta, \zeta) = \Psi^h(\xi, \eta, \zeta) \tilde{N}^3(q^h).
\]

If we use this in the variational formulation (32), we get

\[
\begin{pmatrix}
M^{(2)} & \tilde{N}^2(q^h) \\
\tilde{N}^3(q^h) & M^{(3)}
\end{pmatrix}
\begin{pmatrix}
\tilde{N}^0(\phi^h) \\
\tilde{N}^0(f^h)
\end{pmatrix}
= \begin{pmatrix}
\Psi^h \\
\Psi^h
\end{pmatrix} \begin{pmatrix}
\tilde{N}^0_{\text{bot}}(\phi^h) \\
\tilde{N}^0_{\text{top}}(\phi^h) \\
\tilde{N}^0_{\text{left}}(\phi^h) \\
\tilde{N}^0_{\text{front}}(\phi^h) \\
\tilde{N}^0_{\text{back}}(\phi^h)
\end{pmatrix},
\]

(33)

where the degrees of freedom of \( f \) are obtained using (27)

\[
\tilde{N}^3(f) = N^3_{(i-1)N^2+(j-1)N^2+k-1}(f) := \int_{\xi_{i-1}}^{\xi_i} \int_{\eta_{j-1}}^{\eta_j} \int_{\zeta_{k-1}}^{\zeta_k} f \, d\xi \, d\eta \, d\zeta \quad i, j, k = 1, \ldots, N,
\]

(34)

and the degrees of freedom of the prescribed boundary condition \( \hat{\phi} \) are given by

\[
\tilde{N}^0_a(\phi^h) = N \tilde{N}^0(\hat{\phi}) = N
\begin{pmatrix}
\tilde{N}^0_{\text{bot}}(\hat{\phi}) \\
\tilde{N}^0_{\text{top}}(\hat{\phi}) \\
\tilde{N}^0_{\text{left}}(\hat{\phi}) \\
\tilde{N}^0_{\text{front}}(\hat{\phi}) \\
\tilde{N}^0_{\text{back}}(\hat{\phi})
\end{pmatrix}
\]

(35)

where \( N \) is the discrete unit normal vector that restricts the degrees of freedom of \( q^h \) to the boundary of the domain. For reference to orientation of faces see the Figure on the left of (35).
As an example, for single element in 2D case and \( N = 1 \), \( \mathbb{N} \) is given by

\[
\mathbb{N} = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]  \( (36) \)

where we have \((+)\) sign for the right and the top boundary degrees of freedom and the \((-)\) sign for the left and bottom boundary degrees of freedom based on the direction of the outward unit normal \( \mathbf{n} \), see [27, 28]. \( \Psi^2 \) are the 2D surface basis introduced in Section 3.3. The 2D integrals are boundary degrees of freedom obtained following Corollary 4.

The incidence matrix \( \mathbb{E}^{3,2} \) is a very sparse topological matrix which only contains entries \(-1\), \(1\) and \(0\), that do not depend on the shape, mesh size, or polynomial degree of the approximation, see [24–26]. All metric properties are contained in the mass matrices \( \mathbb{M}^{(2)} \) and \( \mathbb{M}^{(3)} \). For high order methods, these matrices are full matrices which destroy the sparsity of the incidence matrix with which they are multiplied in \((33)\). We refer to this formulation as the \textit{primal-primal} formulation, because both \( \mathbf{q} \) and \( \phi^h \) are expanded in primal basis functions. If the mesh is deformed, all sub-matrices in \((33)\) will change because the mass matrices \( \mathbb{M}^{(3)} \) will change and need to be recomputed.

### 5.2. Primal-dual formulation

Alternatively, we may approximate \( \phi^h \in \overline{S}^h(K) \). In this case the discrete system will be

\[
\begin{pmatrix}
\mathbb{M}^{(2)} & \mathbb{E}^{3,2T} \\
\mathbb{E}^{3,2} & 0
\end{pmatrix}
\begin{pmatrix}
\mathcal{N}^2(\mathbf{q}^h) \\
\tilde{\mathcal{N}}^0(\phi^h)
\end{pmatrix} = \begin{pmatrix}
\tilde{\mathcal{N}}^0_\Omega(\phi^h) \\
\mathcal{N}^1(f^h)
\end{pmatrix}  \quad (37)
\]

We see that if we expand \( \phi^h \) in terms of dual polynomials, the discrete divergence and gradient blocks in \((37)\) are very sparse and no longer depend on the metric of the mesh geometry. These matrices will be the same for all elements, even if these elements have different size or shape. That means they need to be set up for one element only. It is only the mass matrix \( \mathbb{M}^{(2)} \) that needs to be constructed for each element separately, unless all elements have the same size and shape.

**Remark 5.** We can immediately convert \((33)\) to \((37)\). The mass matrix \( \mathbb{M}^{(3)} \) in the second row of \((33)\) can be cancelled on both sides of the equation, while the mass matrix \( \mathbb{M}^{(3)} \) in the first row can be contracted with the degrees of freedom \( \mathcal{N}^1(\phi^h) \) to give \( \tilde{\mathcal{M}}^{(2)} \mathcal{N}^2(\phi^h) \), but these new unknowns are just the dual degrees of freedom \( \tilde{\mathcal{N}}^0(\phi^h) \) according to \((30)\).

### 5.3. The discrete inf-sup condition

At the continuous level one establishes well-posedness by showing that the divergence operator, \( \nabla \cdot \), from \( H(\text{div}; \Omega) \) into \( L^2(\Omega) \) is surjective in which case the Closed Range Theorem states that the operator \( \nabla \cdot \top \) is bounding and therefore injective, [29]. At the continuous level surjectivity of the divergence operator is proven through the auxiliary problem: For an arbitrary \( p \in L^2(\Omega) \) find \( \psi \in H^1_0(\Omega) \) such that

\[
\int_\Omega \nabla \psi \cdot \nabla \phi \ d\Omega = -\int_\Omega p \phi \ d\Omega , \quad \forall \phi \in H^1_0(\Omega) . \quad (38)
\]

The Lax-Milgram lemma ensures uniqueness of \( \psi \). If we set \( \mathbf{u}_p = \nabla \psi \) then we have \( \text{div} \mathbf{u}_p = p \) and since \( p \in L^2(\Omega) \) was arbitrary, this shows surjectivity of the divergence operator.
At the finite dimensional level the discrete inf-sup condition is derived in exactly the same way. Let \( \phi^h \in S^0_h(K) \), as in (24). Now prove that there is a \( q^h \in D^h(K) \) which is mapped by the divergence operator onto \( \phi^h \). Just as in the continuous setting, we use an auxiliary problem: For all \( \phi^h \in S^0_h(K) \) find \( \psi^h \in S^0_h(K) \) such that

\[
\int_K \text{grad} \psi^h \cdot \text{grad} \phi^h \, dK = - \int_K \phi^h \phi^h \, dK, \quad \forall \phi^h \in S^0_h(K).
\]

Using the dual basis functions this translates into

\[
\tilde{N}^0(\phi^h)^T E_{\alpha,n-1} [M^{(n-1)}]^{-1} E_{\alpha,n-1}^T \tilde{N}^0(\psi^h) = -\tilde{N}^0(\phi^h)^T M^{(n-1)} \tilde{N}^0(\psi^h),
\]

which has to hold for all vectors \( \tilde{N}^0(\psi^h) \) and therefore we have

\[
E_{\alpha,n-1} [M^{(n-1)}]^{-1} E_{\alpha,n-1}^T \tilde{N}^0(\psi^h) = -M^{(n-1)} \tilde{N}^0(\psi^h). \tag{40}
\]

Using (23) and (24) we define \( N^{n-1}(u^h) = -M^{(n-1)} E_{\alpha,n-1}^T \tilde{N}^0(\psi^h) \in D^h(K) \) in which case (40) can be written as

\[
E_{\alpha,n-1} N^{n-1}(u^h) = M^{(n-1)} \tilde{N}^0(\psi^h),
\]

or

\[
M^{(n)} E_{\alpha,n} N^{n-1}(u^h) = \tilde{N}^0(\psi^h), \tag{41}
\]

which states that for all vectors \( \tilde{N}^0(\psi^h) \) there exists a vector \( N^{n-1}(u^h) \) which is mapped by the discrete divergence, \( E_{\alpha,n-1} \) to degrees of freedom in \( S^h(K) \) and then mapped by the mass matrix \( M^{(n)} \) onto \( S^0_h(K) \).

**Remark 6.** Note that all basis functions in fact form a basis for the various spaces. For instance in the 2D case, if we have a rectangular mesh with \( n \) cells in the \( x \)-direction and \( m \) cells in the \( y \)-direction, then this mesh has \( 2nm + n + m \) edges and therefore \( 2nm + n + m \) linear independent vector fields can be represented on this mesh. The stream function is defined in the nodes of the mesh and we have \( (n+1)(m+1) \) nodes and therefore \( (n+1)(m+1) \) stream function. These generate only \( nM + n + m \) linearly independent divergence-free vector fields, because the stream function is determined up to a constant. So from the \( 2nm + n + m \) linearly independent vector fields that can be represented on this mesh, \( nM + n + m \) are divergence-free and the remaining \( NM \) linear independent vector fields span the range of the divergence operator. \( NM \) is the number of cells in this mesh where the potential is defined, so the dimension of the range of the divergence operator coincides with the dimension of the potential space, which allows for a one-to-one mapping.

### 5.4. Test case

In the following test case we compare the primal-primal formulation with the primal-dual formulation. Using the primal-dual formulation we obtain sparse algebraic formulation and lower condition number without compromising on quantitative results. Consider the domain (taken from [30]) shown in Figure 6. The deformed mesh coordinates \( (x, y, z) \in \Omega \) are obtained by transforming the orthogonal coordinates \( (\xi, \eta, \zeta) \in K \) with the mapping

\[
\begin{align*}
x &= \hat{x} + 0.03 \cos(3\pi\hat{x}) \cos(3\pi\hat{y}) \cos(3\pi\hat{z}) \quad &\hat{x} = 0.5(1 + \xi) \\
y &= \hat{y} + 0.03 \cos(3\pi\hat{x}) \cos(3\pi\hat{y}) \cos(3\pi\hat{z}) \quad &\hat{y} = 0.5(1 + \eta) \\
z &= \hat{z} + 0.03 \cos(3\pi\hat{x}) \cos(3\pi\hat{y}) \cos(3\pi\hat{z}) \quad &\hat{z} = 0.5(1 + \zeta)
\end{align*}
\]

We compare both the formulations with a manufactured solution \( \phi_{ex} = \sin(2\pi x) \sin(2\pi y) \sin(2\pi z) \) which gives

\[
f_{ex} = -\text{div}(\text{grad} \phi_{ex}),
\]

on a domain \( K \in [-1, 1]^3 \), and using Dirichlet boundary conditions over entire domain given by \( \phi = \phi_{ex|\partial K} \).

In Figure 7 we compare the sparsity pattern of primal-primal and primal-dual formulation for the algebraic formulations (33) and (37) for the given test case. In the top-left and the top-right figure we show the sparsity plot for a
Table 1: Condition number for the primal-primal formulation and the primal-dual formulation for polynomial degree N = 1, 3, 7.

| N   | Primal-Primal | Primal-Dual |
|-----|---------------|-------------|
| 1   | 361.6320      | 33.7474     |
| 3   | 8.4684e+3     | 218.9917    |
| 7   | 2.5062e+5     | 6.0411e+3   |

single element with N = 2. The non-zero entries in primal-dual formulation - 8586 are much less than in the primal-primal formulation - 14094. In the bottom-left and bottom-right we show similar comparison but now for the multiple element case with number of elements T = 2 × 2 × 2 and N = 2. The non-zero entries in primal-dual formulation - 70632 are much less than those in the primal-primal formulation.

In Table 1 we list the condition number of the algebraic system of the two formulations for T = 2 × 2 × 2 with polynomial degree N = 1, 3, 7. We observe that the condition number for the primal-primal formulation is significantly higher than that of the primal-dual formulation and grows faster. In this sense the use of dual polynomials can also be interpreted as a form of inverse type mixed preconditioning [31].

In the top-left plot of Figure 8 we show the $L^2$-error in the constraint (\(\text{div } q^h - f\)) and the interpolation error in the RHS term (\(f^h - f\)), for primal-dual formulation. On the x-axis we have the element length (in terms of unmapped \([-1, 1]^3\) domain) \(h = 2/\sqrt{T}\) and on the y-axis we have the $L^2$-error. We see that the error in constraint is equal to the interpolation error of the RHS term. The error converges optimally with \(O(N + 1)\).

In the top-right of Figure 8 we show the $L^2$-error in discrete constraint (\(\text{div } q^h - f^h\)) for both the formulations which is satisfied up to machine precision.

In the bottom-left plot of Figure 8 we see the error convergence of the fluxes in $H(\text{div})$-norm. The results from the primal-primal formulation and the primal-dual formulation coincide with each other and the error converges optimally with \(O(N + 1)\).

In the plot at the bottom right of Figure 8 we see the convergence in $L^2$-error of $\phi^h$. The results from the primal-primal formulation and the primal-dual formulation overlap. We see optimal convergence of errors of \(O(N + 1)\).

In terms of accuracy Figure 8 shows that the results from primal-primal formulation and primal-dual formulation are equal up to machine precision.

For the next test problem we choose a pair of div-grad problems that are dual to each other in a continuous setting.
We will show that this duality continues to hold true at discrete level by using the primal-dual representations for these problems.

6. The Dirichlet-Neumann problems

The mathematical theory of finite elements often makes use of the equivalence of dual problems. In general this equivalence no longer holds at the discrete level. In this section we want to show that this equivalence continues to hold at the finite dimensional level when dual representations are employed. In the proof of Lemma 2.2 in [1], for instance, use is made of an equivalence between a Dirichlet and a Neumann problem. We start with the two problems given by: Given \( \hat{\phi} \in H^{1/2}(\partial K) \)

1. The Dirichlet problem: Find \( \phi \in H^1(K) \) such that

\[
\begin{cases}
\phi = \hat{\phi} & \text{on } \partial K \\
-\text{div} (\text{grad } \phi) + \phi = 0 & \text{in } K
\end{cases}
\]  

(42)
2. The Neumann problem: Find \( q \in H(\text{div}; K) \) such that
\[
\begin{aligned}
\text{div} \, q &= \hat{\phi} \quad \text{on } \partial K \\
-\text{grad} \, (\text{div} \, q) + q &= 0 \quad \text{in } K.
\end{aligned}
\]
If \( \phi \) solves the Dirichlet problem (42), then \( q \) solves the Neumann problem (43), if and only if \( \phi = \text{div} \, q \). Furthermore, it follows that [1]
\[
||\hat{\phi}||_{H^1(\partial K)} = ||\phi||_{H^1(K)} = ||q||_{L(\text{div}; K)}.
\]

The finite dimensional problem is to find suitable finite dimensional function spaces \( \tilde{S}^h \subset H^1(K) \) and \( D^h \subset H(\text{div}; K) \) for

1. The Dirichlet problem: Find \( \phi^h \in \tilde{S}^h \) such that
\[
\begin{aligned}
\phi^h &= \Pi \hat{\phi} \quad \text{on } \partial K \\
-\text{div} \, (\text{grad} \phi^h) + \phi^h &= 0 \quad \text{in } K.
\end{aligned}
\]

2. The Neumann problem: Find \( q^h \in D^h \) such that
\[
\begin{aligned}
\text{div} \, q^h &= \Pi \hat{\phi} \quad \text{on } \partial K \\
-\text{grad} \, (\text{div} \, q^h) + q^h &= 0 \quad \text{in } K.
\end{aligned}
\]
where $\Pi_T \hat{\phi}$ is the projection of $\phi$ at the boundary, such that the solutions $\phi^h$ and $q^h$ will satisfy $\phi^h = \text{div} \, q^h$ identically in the element $K$. Furthermore, we wish to prove that in this case
\[
\|\phi^h\|_{H^2(\partial K)} = \|\phi^h\|_{H^1(K)} = \|q^h\|_{H(\text{div}; K)}.
\]

### 6.1. The Neumann problem

Consider $K \subset \mathbb{R}^d$, with $d = 2$. Then the variational formulation of the Neumann problem, (45), is given by: For $\phi^h \in H^{1/2}(\partial K)$ find $q^h \in D^h$ such that
\[
(\text{div} \, p^h, \text{div} \, q^h)_K + (p^h, q^h)_K = \int_{\partial K} (p^h \cdot n) \phi^h \, d\Gamma, \quad \forall p^h \in D^h.
\]

(46)

We represent $q^h$ as in (17)
\[
q^h(\xi, \eta) = \Psi^1(\xi, \eta) N^1(q^h).
\]

Then using (19) for divergence in the variational formulation (46) we obtain
\[
(\text{div} \, p^h, \text{div} \, q^h)_K + (p^h, q^h)_K = N^1(p^h)^T \mathbb{E}^{2,1} M^{(2)} \mathbb{E}^{2,1} N^1(q^h) + N^1(p^h)^T M^{(1)} N^1(q^h), \quad \forall p^h \in D^h.
\]

The boundary terms on the right hand side of (46) are evaluated in the same way as in (35) but now for 2D. Collecting all boundary terms, and using the fact that equality should hold for all $N^1(p^h)$ gives
\[
\mathbb{E}^{2,1} M^{(2)} \mathbb{E}^{2,1} N^1(q^h) + M^{(1)} N^1(q^h) = \tilde{N}_n^0(\hat{\phi}^h).
\]

### 6.2. The Dirichlet problem

Consider now the Dirichlet problem given by (44) on the domain $K \subset \mathbb{R}^d$, with $d = 2$. The variational formulation for this problem is given by: For $\phi^h \in H^{1/2}(\partial K)$ find $\phi^h \in S^h$, such that
\[
(\text{grad} \, \phi^h, \text{grad} \, \phi^h)_K + (\phi^h, \phi^h)_K = \int_{\partial K} \phi^h \frac{\partial \phi^h}{\partial n} \, d\Gamma.
\]

(48)

We discretize $\phi^h$ in terms of the dual degrees of freedom $\tilde{N}^0(\phi^h)$. Then the degrees of freedom of the gradient are given analogous to (12) by
\[
\tilde{N}^1(\text{grad} \, \phi^h) = -\mathbb{E}^{2,1} \tilde{N}^0(\phi^h) + \tilde{N}_n^0(\hat{\phi}^h),
\]

where $\tilde{N}_n^0(\hat{\phi}^h)$ are the degrees of freedom of the prescribed boundary condition. Then we know that the gradient of $\phi^h$ is given by
\[
\text{grad} \, \phi^h(\xi, \eta) = -\tilde{\Psi}^1(\xi, \eta) \left(-\mathbb{E}^{2,1} \tilde{N}^0(\phi^h) + \tilde{N}_n^0(\hat{\phi}^h)\right).
\]

(49)

The gradient of the test functions $\varphi^h$ is discretized similarly, but then the variations on the boundary are set to zero, therefore
\[
\text{grad} \, \varphi^h(\xi, \eta) = -\tilde{\Psi}^1(\xi, \eta) \mathbb{E}^{2,1} \tilde{N}^0(\varphi^h).
\]

If we use this in the variational formulation (48) we have
\[
(\text{grad} \, \varphi^h, \text{grad} \, \phi^h)_K + (\phi^h, \varphi^h)_K = \tilde{N}^0(\phi^h)^T \mathbb{E}^{2,1} M^{(1)} \mathbb{E}^{2,1} \tilde{N}^0(\phi^h) - \tilde{N}_n^0(\hat{\phi}^h) + \tilde{N}^0(\phi^h)^T M^{(0)} \tilde{N}^0(\phi^h) = 0.
\]

Note that the boundary conditions are strongly imposed in terms of the dual variables. Once again using the fact that equality should hold for all $\tilde{N}^0(\varphi^h)$ the discrete formulation is given by
\[
\mathbb{E}^{2,1} M^{(1)} \mathbb{E}^{2,1} \tilde{N}^0(\phi^h) + M^{(0)} \tilde{N}^0(\phi^h) = \mathbb{E}^{2,1} M^{(1)} \tilde{N}_n^0(\hat{\phi}^h).
\]

(50)
6.3. Relation between Dirichlet and Neumann problem

What we need to check now is that the solutions of (47) and (50) are related by \( \phi^h = \text{div} \, q^h \). This discrete relation translates into

\[
\tilde{N}^0(\phi^h) = M^{(2)}|E^{2,1}| N^1(q^h). \tag{51}
\]

In order to establish this relation, we fill in (51) in (50) to obtain

\[
E^{2,1} \tilde{M}^{(1)}|E^{2,1}| M^{(2)} E^{2,1} N^1(q^h) + \tilde{M}^{(0)}|E^{2,1}| E^{2,1} N^1(q^h) = E^{2,1} \tilde{M}^{(1)} N^1_0(\phi^h). \tag{52}
\]

We substitute (47) in (52) to get

\[
-E^{2,1} \tilde{M}^{(1)} N^1(q^h) + E^{2,1} \tilde{M}^{(1)} N^0_0(\phi^h) + \tilde{M}^{(0)}|E^{2,1}| E^{2,1} N^1(q^h) = E^{2,1} \tilde{M}^{(1)} N^0_0(\phi^h). \tag{53}
\]

Then we use the fact that \( \tilde{M}^{(1)} M^{(1)} = I \) and \( \tilde{M}^{(0)} M^{(2)} = I \) to get

\[
-E^{2,1} N^1(q^h) + E^{2,1} \tilde{M}^{(1)} N^0_0(\phi^h) + E^{2,1} N^1(q^h) = E^{2,1} \tilde{M}^{(1)} N^0_0(\phi^h),
\]

which proves the relation between the Dirichlet and the Neumann problem.

It remains to show that \( \|\phi^h\|_{H^1(K)} = \|q^h\|_{H(\text{div};K)} \). Using (49) we have that

\[
\|\phi^h\|^2_{H^1(K)} = \tilde{N}^0(\phi^h)T \tilde{M}^{(0)} N^0_0(\phi^h) + \tilde{N}^0(\phi^h)T E^{2,1} \tilde{M}^{(1)} \left[ \tilde{M}^{(1)} \tilde{N}^0_0(\phi^h) - E^{2,1} \tilde{M}^{(2)} E^{2,1} N^1(q^h) \right]. \tag{53}
\]

Since we have just established that, \( \tilde{N}^0(\phi^h) = M^{(2)}|E^{2,1}| N^1(q^h) \), we can insert this in (53)

\[
\|\phi^h\|^2_{H^1(K)} = N^1(q^h)T E^{2,1} \tilde{M}^{(2)} \tilde{M}^{(0)}|E^{2,1}| E^{2,1} N^1(q^h) + \tilde{N}^0(\phi^h)T E^{2,1} \tilde{M}^{(1)} \left[ \tilde{M}^{(1)} \tilde{N}^0_0(\phi^h) - E^{2,1} \tilde{M}^{(2)} E^{2,1} N^1(q^h) \right]
\] \[= N^1(q^h)T M^{(2)} E^{2,1} N^1(q^h) + N^1(q^h)T M^{(1)} M^{(1)} M^{(1)} N^1(q^h)
\] \[= N^1(q^h)T M^{(2)} E^{2,1} N^1(q^h) + N^1(q^h)T E^{2,1} M^{(2)} E^{2,1} N^1(q^h)
\]

\[
= \|q^h\|^2_{H(\text{div};K)},
\]

where we used again that \( \tilde{M}^{(0)} M^{(2)} = I \) and \( \tilde{M}^{(1)} M^{(1)} = I \) and the fact that the degrees of freedom \( q^h \) satisfy (47).

6.4. Test case

In this section we solve the Dirichlet (48) and the Neumann (46) problems on \( K \in [0, 1]^2 \) with one spectral element for a non-trivial boundary condition \( \hat{\phi} \) given by

\[
\hat{\phi} = \begin{cases} 
0 & \text{for } x = 0 \text{ and } y = 0 \\
-\sin(\pi y) & \text{for } x = 1 \\
-\ln(1 - 3x(1 - x)) & \text{for } y = 1
\end{cases}.
\]

For this test case we use a ‘standard’ orthogonal spectral element shown on the left, and a deformed spectral element shown on the right, of Figure 9.

The deformed mesh coordinates \((x, y)\) are obtained by transforming the orthogonal coordinates \((\xi, \eta)\) with the mapping

\[
\begin{align*}
    x &= \frac{1}{2} \left( 1 + \frac{1}{2} (\xi + c \sin(\pi \xi) \sin(\pi \eta)) \right) \\
y &= \frac{1}{2} \left( \eta + c \sin(\pi \xi) \sin(\pi \eta) \right)
\end{align*}
\]

where \( c \) is the deformation coefficient.
In Figure 10a the numerical solution $\text{div} \, q^h$ is shown on the orthogonal mesh, $c = 0$ for $N = 7$. The solution $\phi^h$ on the same mesh is visually indistinguishable from $\text{div} q^h$, Figure 10a, therefore in Figure 10b the difference between $\text{div} \, q^h$ and $\phi^h$ is shown. The difference between both solutions is of $O(10^{-14})$.

In Figure 11a $\text{div} q^h$ is plotted for the deformed grid with $c = 0.3$, for $N = 7$. On the deformed mesh we expect the solution to be less accurate than on the orthogonal mesh, but $\phi^h$ computed on the same mesh is graphically still identical to Figure 11a. The difference between $\text{div} \, q^h$ and $\phi^h$ is shown in Figure 11b. This confirms that for this test case the discrete equivalence (51) holds.

In order to corroborate that the norms $||\phi^h||_{H^1(K)}$ and $||q^h||_{H(\text{div},K)}$ are identical according to (54) for this specific problem, Table 2 lists these norms on three different meshes, the orthogonal mesh, $c = 0.0$, the slightly deformed mesh, $c = 0.15$ and the highly deformed mesh, $c = 0.3$. This table shows that on all mesh configuration and for all polynomial degrees we have $||\phi^h||_{H^1(K)} = ||q^h||_{H(\text{div},K)}$. All the three mesh configurations show convergence to a limiting value $||\phi||_{H^1(K)} = 2.35561$.

This work has been further extended to multi-element cases in [28, 32], three dimensional case in [33], another pair
of dual Dirichlet-Neumann problems in $H(\text{curl})$ spaces in [27], iso-geometric methods in [34] and a linear elasticity problem in [35].

7. Conclusions

In this paper a dual polynomial basis is constructed. The duality pairing between variables from a primal and a dual representation reduces to the vector product between the primal and dual degrees of freedom. The first example where the use of a dual representation is beneficial concerns the mixed formulation of the Poisson problem. We derive the $\text{inf-sup}$ stability condition simply in terms of degrees of freedom. We also show optimal convergence for multi-element test case on curved 3D domain. When a primal-dual formulation is used, two sub-matrices in the mixed formulation become very sparse, even though very high order methods are used and these two sub-matrices do not change when the mesh is deformed. The second example shows the equivalence of a Dirichlet-Neumann pair of equations (taken from [1]) at the discrete level. This equivalence is proven and illustrated by a test case.

We have also seen that the use of dual representations allows us to work directly with the degrees of freedom, without explicitly referring to the basis functions. It suffices to make use of its properties. This allows for a direct connection/comparison with staggered finite volume methods. In (37), $\mathbb{E}^{2,1}_1$ acts directly on the degrees of freedom $q^h$ and $\mathbb{E}^{2,1T}_1$ acts on the dual degrees of freedom for $\phi^h$. Only in post processing step we need to compute the dual polynomials but this can be done element-by-element.
In this paper the construction of dual polynomial basis is metric dependent. For example see (6) and (8) where the mass matrix terms $M^{(0)}$ and $M^{(1)}$ depend on the shape and size of the element. In future work we will present a construction of metric free dual polynomials that are independent of the shape and the size of the element by means of the wedge product instead of the inner product.

Acknowledgements

Varun Jain is supported by the Shell-NWO PhD75 Grant P/fom-d-69/3. Yi Zhang is supported by Chinese Scholarship Council No. 201607720010.

References

References

[1] C. Carstensen, L. Demkowicz, J. Gopalakrishnan, Breaking spaces and forms for the DPG method and applications including Maxwell equations, CAMWA 72 (2016) 494–522.
[2] E. Kreyszig, Introductory Functional Analysis with Applications, John Wiley & Sons, 1978.
[3] J. Oden, L. Demkowicz, Applied Functional Analysis, CRC Press, 2010.
[4] S. Axler, Linear Algebra Done Right, Springer, 2015.
[5] C. de Boor, G. J. Fix, Spline approximation by quas interpolants, Journal of Approximation Theory (1973) 19 – 45.
[6] C. de Boor, On local linear functionals which vanish at all B-splines but one, Theory of Approximation with Applications (1976) 120 – 145.
[7] B. Wohlmuth, A mortar finite element method using dual spaces for the Lagrange multiplier, SIAM Journal of Numerical Analysis (2000) 989–1012.
[8] P. Oswald, B. Wohlmuth, On polynomial reproduction of dual FE bases, Thirteenth International Conference on Domain Decomposition Methods (2001) 85–96.
[9] W. Dornisch, J. Stockler, J. Muller, Dual and approximate dual basis functions for B-splines and NURBS - Comparison and application for an efficient coupling of patches with isogeometric mortar methods, Computer Methods in Applied Mechanics and Engineering (2017) 449 – 496.
[10] L. Veiga, A. Buffa, G. Sangalli, R. Vázquez, Dual compatible splines on nontensor product meshes, Springer (2014) 15 – 26.
[11] M. Pinto, Moment preserving local spline projection operators, Constructive Approximation, Springer (2019) 1 – 21.
[12] P. Woźniak, Construction of dual bases, Journal of Computational and Applied Mathematics (2013) 75–85.
[13] P. Woźniak, Construction of dual B-spline functions, Journal of Computational and Applied Mathematics (2014) 301–311.
[14] L. Demkowicz, J. Gopalakrishnan, A class of discontinuous Petrov-Galerkin methods. Part I: The transport equation, Computer Methods in Applied Mechanics and Engineering (2010) 1558 – 1572.
[15] L. Demkowicz, J. Gopalakrishnan, A class of discontinuous Petrov-Galerkin methods. II: Optimal test functions, Numerical Methods Partial Differential (2011) 70 – 105.
[16] L. Demkowicz, J. Gopalakrishnan, Analysis of the DPG method for the Poisson equation, SIAM Journal of Numerical Analysis (2011) 1788 – 1809.
[17] P. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, 1978.
[18] A. Ern, J.-L. Guermond, Theory and Practice of Finite Elements, Springer, 2004.
[19] S. Brenner, L. Ridgway Scott, The Mathematical Theory of Finite Element Methods, Springer, 2008.
[20] C. Canuto, M. Hussaini, A. Quarteroni, T. Zang, Spectral Methods in Fluid Dynamics, Springer, 1988.
[21] D. N. Arnold, R. S. Falk, R. Winther, Finite element exterior calculus, homological techniques, and applications, Acta Numerica 15 (2006) 1–155.
[22] R. Hiptmair, Finite elements in computational electromagnetism, Acta Numerica (2002) 237 – 339.
[23] M. Gerritsma, Edge functions for spectral element methods, spectral and high order methods for partial differential equations, LNCS Springer (2011) 199–208.
[24] J. Kreeft, M. Gerritsma, Mixed mimetic spectral element method for Stokes flow: A pointwise divergence-free solution, Journal of Computational Physics 240 (2013) 284–309.
[25] A. Palha, P. P. Rebelo, R. Hiemstra, J. Kreeft, M. Gerritsma, Physics-compatible discretization techniques on single and dual grids, with application to the Poisson equation of volume forms, Journal of Computational Physics 257 (2014) 1394–1422.
[26] P. Bochev, M. Gerritsma, A spectral mimetic least-squares method, CAMWA 68 (2014) 1480–1502.
[27] M. Gerritsma, V. Jain, Y. Zhang, A. Palha, Algebraic dual polynomials for the equivalence of curl-curl problems, to appear in LNCS Springer.
[28] Y. Zhang, V. Jain, A. Palha, M. Gerritsma, The discrete Steklov-Poincaré operator using algebraic dual polynomials, Computational Methods in Applied Mathematics (2019) 645–661.
[29] D. Boiţă, F. Brezzi, M. Fortin, Mixed finite elements methods and applications, Springer Series in Computational Mechanics.
[30] M. F. Wheeler, G. Xue, I. Yotov, A multiscale mortar multipoint flux mixed finite element method, ESAIM: Mathematical Modelling and Numerical Analysis (2012) 759–796.
[31] K. Chen, M. J. Ablowitz, S. H. Davis, E. J. Hinch, A. Iserles, J. Ockendon, P. J. Olver, Matrix preconditioning techniques and applications, Cambridge University Press.
[32] V. Jain, J. Fisser, A. Palha, M. Gerritsma, A conservative hybrid method for Darcy flow, to appear in LNCS Springer.
[33] Y. Zhang, V. Jain, A. Palha, M. Gerritsma, Discrete equivalence of adjoint Neumann-Dirichlet div-grad and grad-div equations in curvilinear 3D domains, to appear in LNCS Springer.

[34] Y. Zhang, V. Jain, A. Palha, M. Gerritsma, The use of dual B-spline representations for the double De Rham complex of discrete differential forms, to appear in LNCS Springer.

[35] J. Fisser, Y. Zhang, V. Jain, M. Gerritsma, A mixed, mimetic, hybrid spectral element method for linear elasticity with pointwise linear- and angular momentum conservation, in preparation.