Constraining alternative gravity theories using the solar neutrino problem

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Abstract
The neutrino flavor oscillation is studied in some classes of alternative gravity theories in a plane specified by $\theta = \pi/2$, exploiting the spherical symmetry and general equations for oscillation phases are given. We first calculate the phase in a general static spherically symmetric model and then we discuss some spherically symmetric solutions in alternative gravity theories. Among them we discuss the effect of a cosmological term in the Schwarzschild-(anti)de Sitter solution, which is the vacuum solution in $F(R)$ theory and the effect of charge and the Gauss–Bonnet coupling parameter on the oscillation phase is presented. Finally, we discuss a charged solution with a spherical symmetry in $F(R)$ theory and also its implication to the oscillation phase. We calculate the oscillation length and transition probability in these spherically symmetric spacetimes and have presented a graphical representation for the transition probability with various choices for parameters in our theory. From this, we have constrained parameters appearing in these alternative theories using standard solar neutrino results.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Neutrino oscillation is a very rich and interesting problem in its own right. This problem is not only connected to modern particle physics but also to cosmology, astrophysics and other diverse branches, having interesting phenomenological consequences. Mass neutrino mixing and oscillations were first studied and proposed by Pontecorvo [1], then Mikheyev, Smirnov and Wolfenstein (MSW for short) had discussed the effect of transformation of one neutrino...
flavor into another in a varying density medium [2, 3]. Recently, the mass neutrino oscillation has been a hot topic and there have been many theoretical [4–10] as well as experimental [11–15] studies. The neutrino oscillation first formulated in flat spacetime has been extended to curved spacetime [16–21] and it has been used to test the equivalence principle recently [22]. Oscillation phase along the geodesic line will produce a factor of 2 in the high energy limit when compared to the value along a null geodesic. This factor of 2 exists in both flat and Schwarzschild spacetime as shown by several authors [19, 23–25]. The issue regarding this factor of 2 is due to the difference between the time-like and null geodesics. Also, there exist some alternative mechanisms to take into account the effect of gravitational field on neutrino flavor oscillation [26, 27]. Neutrino oscillation in non-inertial frame has also drawn some attention recently [28, 29]. There has been extensive study of Neutrino oscillation in spacetime with both curvature and torsion [30, 31].

In recent years, there has been a boost in the research on application of Neutrino oscillation in various astrophysical contexts. The Pulsar kicks mechanism, based on spin flavor conversion of neutrino, which is propagating in a gravitational field has been discussed extensively in [32]. Observations suggest that pulsars have a very high proper motion with reference to surrounding stars. These suggest that pulsars undergo some kind of impulse or kick. In spite of many proposals in this direction this still remains an open issue. The above pulsar kick mechanism can also be explained by introducing neutrinospheres and resonant oscillation $\nu_e \rightarrow \nu_{\mu, \tau}$ between these neutrinospheres. This kind of Neutrino oscillation in the presence of strong magnetic field leads to such high proper motion of pulsars [33].

Other studies on Neutrino oscillation mainly focusses on the fact that though the mass squared differences and mixing angles are well observed, the absolute value of neutrino masses are not properly known, leading to neutrino masses being hierarchical or quasi-degenerate in nature. Along with there are extensive works on the mixing angle $\theta_{13}$, CP violation in neutrino oscillation and the effects of non-vanishing 1–2 mixing. There exists a number of theoretical models, for example, considering neutrino masses to be degenerate at some seesaw scale, large mixing angle for solar and atmospheric neutrinos using renormalization group equations in order to address these issues [34–39].

There have been some recent developments regarding different astrophysical aspects of alternative gravity theories [40, 41]. In this paper, we consider neutrino oscillation in some classes of alternative gravity theories. For simplicity, we discuss only spherically symmetric solutions in alternative theories of gravitation, but interestingly they all turn out to have important implications. We have derived quiet generally for all spherically symmetric solutions of a particular form (see equation (9)) that $\Phi_{\text{geod}}^k / \Phi_{\text{null}}^k = 2$ and we have only taken the high energy limit and not weak field approximation.

We have discussed three spherically symmetric solutions in this paper. First one corresponds to the vacuum solution to $F(R)$ gravity, which is the Schwarzschild–(anti-)de Sitter solution and has great importance today regarding the cosmological constant. We have put a bound on the cosmological parameter in this solution from the present day solar neutrino data. Secondly, we consider Einstein–Maxwell–Gauss–Bonnet (EMGB) gravity in five dimensions and a spherically symmetric solution has been discussed [42]. There we put bounds on the GB parameter $\alpha$ using solar neutrino oscillation data. Finally, we discuss charged solution in $F(R)$ gravity and different parameters have been estimated.

Finally, we calculate the proper oscillation length in all these spherically symmetric spacetime. The oscillation length is found proportional to $E_{\text{loc}} = E / \sqrt{g_{00}}$, the local energy measurement. A decrease in local energy leads to a decrease in the oscillation length as the neutrino travels out the gravitational field. Thus, blueshift of the oscillation length occurs in contrast to redshift for light signal, which is an interesting result.
The paper is organized as follows. In section 2, we give a brief review of neutrino oscillation in flat spacetime; next, in section 3, we discuss the neutrino oscillation in general static spherically symmetric spacetime. Then we consider neutrino oscillation in different classes of alternative gravity theories and proper oscillation length in these theories. The paper ends with a discussion on our results. Throughout the paper we have used the units $G = c = \hbar = 1$ and $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$.

2. Neutrino oscillation in flat spacetime

In this section, we shall briefly review some properties of two-flavor neutrino oscillation in flat spacetime which will be helpful for later developments. In standard treatment, the flavor basis eigenstate, denoted by $|\nu_\alpha\rangle$, is actually a superposition of the mass basis eigenstates $|\nu_k\rangle$ such that they are connected by a unitary transformation \[18\],

$$|\nu_\alpha\rangle = \sum_k U_{\alpha k} \exp[-i\Phi_k] |\nu_k\rangle \quad (1)$$

where

$$\Phi_k = E_k t - \vec{p}_k \cdot \vec{x}, \quad (k = 1, 2) \quad (2)$$

and the unitary matrix $U_{\alpha k}$ comprises the transformation between flavor and mass basis. Here $E_k$ and $\vec{p}_k$ correspond to the energy and momentum of the mass eigenstates $|\nu_k\rangle$. For a neutrino which is produced at some spacetime point, $A(t_A, \vec{x}_A)$, and detected at another spacetime point $B(t_B, \vec{x}_B)$, the phase as presented in equation (2) can be generalized to a coordinate-independent form and can become suitable for application in a curved spacetime. This could be given by [18, 43]

$$\Phi_k = \int_A^B p_{\mu}^{(k)} \frac{dx^\mu}{ds} \quad (3)$$

where the 4-momentum is given by

$$p_{\mu}^{(k)} = m_k g_{\mu\nu} \frac{dx^\nu}{ds} \quad (4)$$

and $m_k$ is the rest mass corresponding to the mass eigenstate $|\nu_k\rangle$; $g_{\mu\nu}$ and $s$ correspond to the metric tensor and an affine parameter, respectively. In the literature, the mass eigenstates are usually taken to be the energy eigenstates with a common energy, up to $O(m/E)$. We use the approximation $E \gg M$ and assume massless trajectory implying that the neutrino travels along the null trajectory. For two flavor mixing $\nu_e - \nu_\mu$, we can write

$$\nu_e = \cos \theta \nu_1 + \sin \theta \nu_2, \quad \nu_\mu = -\sin \theta \nu_1 + \cos \theta \nu_2 \quad (5)$$

where $\theta$ is the vacuum mixing angle. The oscillation probability that the neutrino which is produced as $|\nu_e\rangle$ but detected as $|\nu_\mu\rangle$ is given by [44]

$$P(\nu_e \rightarrow \nu_\mu) = |\langle \nu_e | \nu_\mu(x, t) \rangle|^2 = \sin^2(2\theta) \sin^2 \left( \frac{\Phi_{kj}}{2} \right) \quad (6)$$

where $\Phi_{kj} = \Phi_k - \Phi_j$ is the phase shift for neutrino flavor oscillation. The Phase can also be expressed in terms of energy and position of creation and detection of the neutrino such that [18]

$$\Phi_k \simeq m_k^2 |\vec{x}_B - \vec{x}_A| (2E_0)^{-1} \quad (7)$$

with $E_0$ being the energy for a massless neutrino. So, the phase shift which is responsible for oscillation is given by

$$\Phi_{kj} \simeq \Delta m_{kj}^2 |\vec{x}_B - \vec{x}_A| (2E_0)^{-1} \quad (8)$$

where $\Delta m_{kj}^2 = m_k^2 - m_j^2.$
3. Neutrino oscillation in a general static spherically symmetric spacetime

In this section, we shall discuss the neutrino oscillation along both null and time-like geodesics in a general static spherically symmetric spacetime with metric ansatz [45]

$$\text{d}x^2 = f(r) \text{d}t^2 - f(r)^{-1} \text{d}r^2 - r^2 \text{d}\Omega_2^2.$$  \hfill (9)

We have restricted this discussion to four dimensions only; however, it can be generalized to higher dimension in a straightforward manner. We shall restrict our motion in the $\theta = \pi/2$ plane; due to spherical symmetry this would not hinder the general nature of the metric.

The components of the canonical momenta of $k$th massive neutrino in equation (4) are

\begin{align*}
p^{(k)}_i &= p^{(k)}_0 = m_0 f(r)t = m_0 E_k \\
p^{(k)}_r &= -m_0 f(r)^{-1}\dot{r} \\
p^{(k)}_\phi &= -m_0 r^2 \dot{\phi} = -m_0 \dot{\delta}_k
\end{align*}  \hfill (10)

where we have introduced $\dot{t} = \text{d}t/\text{d}s$, $\dot{r} = \text{d}r/\text{d}s$ and $\dot{\phi} = \text{d}\phi/\text{d}s$. The metric components do not depend on $t$, thus, we have a conserved energy per particle mass given by $E_k$ and the metric components also do not depend on $\phi$ leading to conserved angular momenta per particle mass $I_k$. We could have $t = E_k f(r)^{-1}$ and $\phi = I_k r^{-2}$. The phase along null geodesic from point A to point B is given by [18, 43]

$$\Phi_k^{\text{null}} = \int_A^B p^{(k)}_r \text{d}x^\mu = \int_A^B \left( p^{(k)}_0 \text{d}r + p^{(k)}_\phi \text{d}\phi + p^{(k)}_r \text{d}r \right)$$

$$= \int_A^B \left( p^{(k)}_0 \text{d}t/\text{d}r + p^{(k)}_\phi \text{d}\phi/\text{d}r + p^{(k)}_r \text{d}r \right) \text{d}r.$$  \hfill (11)

In the literature, the neutrino is usually taken to travel along the null line. Thus, we shall calculate the phase along light-ray trajectory from A to B. The Lagrangian appropriate for the motion in $\theta = \pi/2$ plane is

$$L = \frac{1}{2} (f(r)\dot{t}^2 - f(r)^{-1}\dot{r}^2 - r^2 \dot{\phi}^2).$$  \hfill (12)

The Hamiltonian could be given by

$$H = E_k \dot{t} - I_k \dot{\phi} + m_0 \dot{\delta}_k \dot{r} = \delta_1 = \text{constant}. $$  \hfill (13)

From independence of Hamiltonian on time $t$ we can easily derive the following result:

$$2H = E_k \dot{t} - I_k \dot{\phi} + m_0 \dot{\delta}_k \dot{r} = \delta_1 = \text{constant}. $$  \hfill (14)

We can take $\delta_1 = 1$ for time-like geodesics and $\delta_1 = 0$ for null geodesics without any loss of generality. Substituting for $\dot{r}$, $\phi$ and $p^{(k)}_r$ from equation (10) in equation (14) for null geodesics leads to the radial equation of motion

$$\dot{r} = E_k \sqrt{1 - \frac{f(r)l_k^2}{r^2 E_k^2}}$$  \hfill (15)

Now we define a new function such that

$$V(r) = 1 - \frac{f(r)l_k^2}{r^2 E_k^2}.$$  \hfill (16)

From this we have calculated the equations governing $t$ and $\phi$ as

$$\frac{dt}{dr} = \frac{1}{f(r)\sqrt{V}}, \quad \frac{d\phi}{dr} = \frac{l_k}{E_k r^2 \sqrt{V}}.$$  \hfill (17)

The on-mass shell condition corresponds to

$$m_k^2 = g_{\mu\nu} p^{(k)}_\mu p^{(k)}_\nu = p^{(k)}_0 p^{(k)}_r + p^{(k)} \delta^{(k)}_\phi + p^{(k)}_r p^{(k)}_r.$$  \hfill (18)
Using equation (10) into the on-mass shell condition we readily obtain
\[ p^{(k)r} = m_k \sqrt{E_k^2 V - f(r)}. \]  
(19)

Then using equations (19), (17), (15) and (10) in equation (11) for phase we readily obtain
\[ \Phi^\text{null}_k \simeq \int_A^B \frac{m_k dr}{2E_k \sqrt{V}}. \]  
(20)

The phase as presented in equation (20) is a general result. For different \( f(r) \) the function \( V(r) \) changes and hence the phase. If \( f(r) = 1 - 2M/r \), then in the high energy limit we should have, \( V \sim 1 \), hence the phase has the following expression:
\[ \Phi^\text{null}_k = \int_A^B \frac{m_k dr}{2E_k} = \frac{m_k^2}{2p_0^2} (r_B - r_A) \]  
(21)

which is the phase in Schwarzschild spacetime [18]. An ultra-relativistic neutrino travels with speed very close to that of light and hence is considered to travel along the null line. However, there exists significant difference between massive neutrino and photon, which becomes important while determining the key features of neutrino oscillation. Thus, for a more general situation we should calculate the phase along time-like geodesics. An extra factor of 2 as mentioned earlier is obtained as we compare the time-like geodesic with null geodesic in high energy limit. This factor originates due to the fact that we have treated neutrino to be massive, while calculated the phase along the null and the time-like trajectory. Thus, this factor of 2 is a consequence of neutrino mass. For time-like geodesic, setting \( \delta_1 = 1 \), we can derive from equation (14)
\[ E_k \dot{t} - l_k \dot{\phi} + m_k^{-1} p^{(k)r} = 1. \]  
(22)

Note that the equations for \( \dot{t} \) and \( \dot{\phi} \) are same for time-like geodesics [46]. However the radial equation becomes
\[ \dot{r} = \sqrt{E_k^2 V - f(r)}. \]  
(23)

Then we have obtained expressions for \( dt/dr \) and \( d\phi/dr \) for time-like geodesics given by
\[ \frac{dr}{dr} = \frac{E_k}{f \sqrt{E_k^2 V - f(r)}}, \quad \frac{d\phi}{dr} = \frac{l_k}{r^2 \sqrt{E_k^2 V - f(r)}}. \]  
(24)

Using the on-mass shell condition we readily obtain
\[ p^{(k)r} = m_k \sqrt{E_k^2 V - f(r)}. \]  
(25)

Thus, the phase along the time-like geodesic has the following expression:
\[ \Phi^\text{geod}_k = \int_A^B \frac{m_k dr}{\sqrt{E_k^2 V - f(r)}}. \]  
(26)

In the high energy limit the above expression reduces to
\[ \Phi^\text{geod}_k \simeq \int_A^B \frac{m_k dr}{E_k \sqrt{V}} = 2\Phi^\text{null}_k. \]  
(27)

The factor of 2 exists in neutrino phase calculation for flat [24], Schwarzschild [19, 23] and Kerr–Newmann [46] spacetime. Here we again found that factor of 2 for a general static spherically symmetric spacetime. This factor appears since there exists an intrinsic difference between time-like and null geodesics. In deriving the null phase we have used 4-momentum which is along the time-like geodesic and \( r \) which is along the null geodesic. However, for the time-like phase we have derived both the quantities keeping them along time-like geodesics; this leads to that factor of 2, which is a general feature of any curved spacetime.
4. Neutrino oscillation in some classes of alternative gravity theories

Current theoretical models of cosmology have two fundamental problems, namely inflation and the late time acceleration of the universe. The usual scenario used to explain both of these accelerating epochs is to develop acceptable dark energy models, which includes: scalar, spinor, cosmological constant and higher dimensions. Even if such a model seems to be partially successful, it is mainly hindered by the coupling with the usual matter and hence its compatibility with standard elementary particle theories.

However, another natural choice is the classical generalization of general relativity (GR), which is called modified gravity or alternative gravity theory [47–50]. Thus, a gravitational alternative is needed to explain both inflation and dark energy seems reasonable on the ground of the expectation that GR is an approximation valid at a small curvature. The sector of modified gravity theory which contains the gravitational terms, relevant at high energy has produced the inflationary epoch. During evolution the curvature decreases and hence GR describes to a good approximation the intermediate universe. With a further decrease of the curvature as the sub-dominant terms gradually grow we observe a transition from deceleration to cosmic acceleration. There exist many models including traditional $F(R)$, string-inspired models, scalar tensor theories, Gauss–Bonnet theory and many others. In the next subsections we shall discuss neutrino oscillations in three spherically symmetric solutions for different alternative gravity theories.

4.1. Neutrino oscillation in $F(R)$ gravity

GR is widely accepted as one of the fundamental theory relating matter energy density to geometric properties of the spacetime. The standard cosmological model can explain the evolution of the universe except inflation and late time cosmic acceleration, as already mentioned. Although many scalar field models have been proposed earlier in the framework of string theory and super-gravity to explain inflation, Cosmic Microwave Background radiation does not show any evidence in favor of some model. The same kind of approach has also been taken to explain cosmic acceleration by introducing different dark energy models where concrete observation is still missing.

Thus, one of the simplest choice is the modification of GR action by introducing a term $F(R)$ in the Lagrangian, where $F$ is some arbitrary function of the scalar curvature $R$. There exist two methods for deriving field equations; first, we can vary the action with respect to the metric tensor $g_{\mu\nu}$ and the other method which is called the Palatini method is not discussed here. In $F(R)$ gravity [51–54], the Einstein–Hilbert action

$$S_{\text{EH}} = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi} + L_{\text{matter}}\right),$$

gets replaced by an action appropriate for the introduction of the function of scalar curvature

$$S_{F(R)} = \int d^4x\sqrt{-g}\left(\frac{F(R)}{16\pi} + L_{\text{matter}}\right).$$

Varying this action we readily obtain the corresponding field equation in this gravity theory to be given by

$$\frac{1}{2}g_{\mu\nu}F(R) - R_{\mu\nu}F'(R) - g_{\mu\nu}\Box F'(R) + \nabla_\mu \nabla_\nu F'(R) = -4\pi T_{\text{matter}\mu\nu}.$$  

Now we shall discuss two class of solutions for the above set of Einstein equations involving vacuum solution and charged black hole solution in $F(R)$ gravity.
4.1.1. Vacuum solution in \( F(R) \) gravity. Several solutions (sometimes exact) to this field equation have been obtained; however, due to complicated nature, the number of such exact solutions is much less than that in classical GR. There exists a (A)dS–Schwarzschild solution that corresponds to a vacuum solution \( (T = 0) \) for which the Ricci scalar is covariantly constant. This also corresponds to \( R_{\mu\nu} \propto g_{\mu\nu} \). Since \( \Box F'(R) = 0 \) for this case equation (30) reduces to the following algebraic equation:

\[
0 = 2F(R) - RF'(R).
\]

It is evident that the model \( F(R) \propto R^2 \) satisfies the above equation \([50, 53, 54]\). Hence the (A)dS–Schwarzschild is an exact vacuum solution to this situation with the respective line element given by

\[
d\xi^2 = \left(1 - \frac{2M}{r} \mp \frac{r^2}{\Lambda} \right) dt^2 - \left(1 - \frac{2M}{r} \mp \frac{r^2}{\Lambda} \right)^{-1} dr^2 - r^2 d\Omega^2.
\]

Here the minus(plus) sign corresponds to (anti-)de Sitter space, \( M \) is the mass of the black hole and \( \Lambda \) is the length parameter of (anti-)de Sitter space, which is related to the scalar curvature \( R = \pm \frac{12}{\Lambda^2} \) (the plus sign corresponds to de Sitter space and minus sign corresponds to anti-de Sitter space).

The phase along null line is given by

\[
\Phi_{k}^{\text{null}} = \int_{A}^{B} \frac{m_{d}dr}{2E_{k} \left[1 - \left(\frac{1 - \frac{2M}{r} \mp \frac{r^2}{\Lambda}}{rE_{k}}\right)^{\xi}\right]}.
\]

and the phase along the geodesic line has the following expression:

\[
\Phi_{k}^{\text{geod}} = \int_{A}^{B} \frac{m_{d}dr}{\sqrt{E_{k}^2 \left[1 - \left(\frac{1 - \frac{2M}{r} \mp \frac{r^2}{\Lambda}}{rE_{k}}\right) - \left(1 - \frac{2M}{r} \mp \frac{r^2}{\Lambda} \right)^{-1}\right]}},
\]

4.1.2. Charged solution in \( F(R) \) gravity. In this section, we consider charged solutions for \( F(R) \) gravity having the form given by \( F(R) = R - \lambda \exp(-\xi R) \)[55]. All such viable modifications in gravity must pass through all the tests from the large scale structure of the universe to solar system. When the correction factor to the Einstein gravity action is of an exponential form, it is possible to show that it does not contradict solar system tests \([56]\). Also in addition the solution from this model is mostly identical to that from Einstein gravity except a change in Newton’s constant \([57]\). Using the function given by \( F(R) \) we readily obtain a topological charged solution in which the function \( f(r) \) as given by equation (9) leads to \([55]\)

\[
f(r) = 1 - \frac{\Lambda}{3} r^2 - \frac{M}{r} + \frac{Q^2}{r^2}.
\]

In this solution, we should set some parameters and following the approach as presented in \([55]\) we easily read off the parameters such that the following relations are satisfied:

\[
1 + \frac{\lambda \xi}{e^{\xi R}} = 0
\]

\[
\frac{\lambda}{e^{\xi R}} + \frac{R}{2} \left(\frac{\lambda \xi}{e^{\xi R}} - 1\right) = 0
\]

(36)

with the solutions \( \lambda = Re^{-1} \) and \( \xi = -1/R \). We can also reverse the argument i.e. setting \( \lambda = Re^{-1} \) and \( \xi = -1/R \) and deriving that equation (35) satisfies field equations. To interpret
the charge term we need scalar-tensor representation of $F(R)$ gravity theory. Then the neutrino phase along the null line has the following expression:

$$\Phi^\text{null}_k = \int_A^B \frac{m_l dr}{2E_k \sqrt{\left(1 - \frac{f(r)}{r^2 E_k^2}\right)}}$$

(37)

while that along the geodesic goes by the following expression:

$$\Phi^\text{geod}_k = \int_A^B \frac{m_l dr}{\sqrt{E_k^2 \left(1 - \frac{f(r)}{r^2 E_k^2}\right)} - f(r)}$$

(38)

where $E_k$ and $l_k$ are the energy and angular momentum, respectively, of the neutrino and $f(r)$ is given by equation (35).

### 4.2. Neutrino oscillation in Einstein–Maxwell–Gauss–Bonnet gravity

Theories with extra spatial dimension have been an area of considerable interest since the original work of Kaluza and Klein. The advent of string theory boosts this issue which predicts the presence of an extra spatial dimension. Among the large number of alternatives the Brane world scenario is considered as a strong candidate which has theoretical basis in some underlying string theory. Usually, the effect of string theory on classical gravitational theories [58, 59] is investigated using a low energy effective action, which in addition to the Einstein–Hilbert action contains squares and higher powers of the curvature term. However, the field equations become of fourth order and bring in ghosts [60]. In this context, Lovelock [61] showed that if higher curvature terms appear in a particular combination in the action, the field equation becomes of second order.

In EMGB gravity, the action in the five-dimensional spacetime $(M, g_{\mu\nu})$ can be written as

$$S = \frac{1}{2} \int_M d^5x \sqrt{-g} \left[R + \alpha L_{\text{GB}} + L_{\text{matter}}\right].$$

(39)

where $L_{\text{GB}} = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2$ is the GB Lagrangian and $L_{\text{matter}} = F_{\mu\nu}F^{\mu\nu}$ is the Lagrangian for the matter part i.e. electromagnetic field. Here $\alpha$ is the coupling constant for the GB term having the dimension $(\text{length})^2.$ As $\alpha$ is regarded as the inverse string tension, so we must have $\alpha \geq 0.$

The gravitational and electromagnetic field equations are obtained by varying the above action with respect to $g_{\mu\nu}$ and $A_\mu$ (see [42]),

$$G_{\mu\nu} - \alpha H_{\mu\nu} = T_{\mu\nu}$$

$$\nabla_\mu F^{\mu}_\alpha = 0$$

(40)

where $T_{\mu\nu} = 2F_{\alpha\beta}F^{\alpha\beta}_{\mu\nu} - \frac{1}{2}F_{\mu\alpha\beta}F^{\alpha\beta}_{\nu} - \frac{1}{2}F_{\nu\alpha\beta}F^{\alpha\beta}_{\mu}$ is the electromagnetic field tensor.

A spherically symmetric solution to the above field equations has been obtained by [62] and the line element has the following expression:

$$ds^2 = -g(r) dr^2 + \frac{dr^2}{g(r)} + r^2 d\Omega^2,$$

(41)

where the metric co-efficient is

$$g(r) = K + \frac{r^2}{4\alpha} \left[1 \pm \sqrt{1 + \frac{8\alpha(m + 2\alpha |K|)}{r^4} - \frac{8\alpha q^2}{3r^6}}\right].$$

(42)
Here $K$ is the curvature, $m + 2\alpha | K |$ is the geometrical mass of the spacetime and $d\Omega^2$ is the metric of a 3D hyper-surface such that

$$d\Omega^2 = d\theta_1^2 + \sin^2 \theta_1 (d\theta_2^2 + \sin^2 \theta_2 d\theta_3^2).$$

(43)

The range is given by $\theta_1, \theta_2 : [0, \pi]$. We have assumed that there is a constant charge $q$ at $r = 0$ and the vector potential be $A_\mu = \Phi(r) \delta_\mu^0$ such that $\Phi(r) = -\frac{q}{2r}$.

In this metric the metric function $g(r)$ will be real for $r \geq r_0$ where $r_0^2$ is the largest real root of this cubic equation:

$$3z^3 + 24\alpha (m + 2\alpha | K |)z - 8\alpha q^2 = 0.$$  

(44)

By a transformation of the radial co-ordinates one can show that $r = r_0$ is an essential singularity of the spacetime [62].

The phase along the null line has the explicit form

$$\Phi_{null}^k = \int_{A}^{B} \frac{m_k \, dr}{2E_k \sqrt{\left(1 - \frac{g(r)}{rE_k^2}\right)}}$$

(45)

and the phase along the geodesic line could be given by

$$\Phi_{geod}^k = \int_{A}^{B} \frac{m_k \, dr}{\sqrt{E_k^2 \left(1 - \frac{g(r)}{rE_k^2}\right)} - g(r)}$$

(46)

where $g(r)$ is given by equation (42).

5. Proper oscillation length

The propagation of a neutrino is well understood in terms of its proper length. However, the quantity $dr$ that appears in equation (20) is only a coordinate. The proper distance has the following expression [63]:

$$dl = \sqrt{\frac{g_{0\nu}g_{0\nu}}{g_{00}}} - g_{\mu\nu} \, dx^\mu \, dx^\nu.$$  

(47)

For the metric ansatz as presented in equation (9) and then using equation (16) we readily obtain the proper distance to be given by

$$dl = dr[f(r)V]^{-1/2}.$$  

(48)

For convenience, we shall adopt the differential form of (20)

$$d\Phi_{null}^k = \frac{m_k \, dr}{2E_k \sqrt{V}}.$$  

(49)

Substituting (48) we readily obtain

$$d\Phi_{null}^k = \frac{m_k^2}{2p_0^2} \sqrt{f} \, dl$$

(50)

where it is generally assumed that the mass and energy eigenstates are identical with a common energy. The equal energy assumptions are taken to be correct by some authors [19, 25] and has been studied carefully in papers [64]. Also, it is adopted quite widely in a great deal of literature that $p_0$ will represent the common energy of mass eigenstates. The condition of equal momentum has also been adopted in order to study the neutrino oscillation. In flat spacetime, both conditions represent the same neutrino oscillation results. Due to the time and space translation invariance the free particle energy and momentum are conserved. In
curved, stationary spacetime, the energy is conserved along the geodesic due to the existence of a time-like Killing vector field. However, $\partial/\partial r$ is not a Killing vector field and hence the momentum $p_r$ is not conserved. Thus, it is difficult to study neutrino oscillation under the equal momentum assumption in a curved spacetime. In this section, we shall consider phase along the null line. Hence the phase shift determining the oscillation could be given by

$$d\Phi_{\text{null}}^{kj} = d\Phi_{\text{null}}^{k} - d\Phi_{\text{null}}^{j} = \frac{\Delta m_{kj}^2}{2p_0} \sqrt{f} \frac{dl}{d\phi_{\text{null}}}$$  \hspace{1cm} (51)$$

where $\Delta m_{kj}^2 = m_k^2 - m_j^2$. Equation (51) can be rewritten as

$$\frac{dl}{d\phi_{\text{null}}} = \frac{4\pi p_0}{\Delta m_{kj}^2} \sqrt{f(r)} = \frac{4\pi p_0^{\text{loc}}}{\Delta m_{kj}^2}. \hspace{1cm} (52)$$

The term $\frac{4\pi p_0}{\Delta m_{kj}^2} \sqrt{f(r)}$ in (52) has been interpreted as the oscillation length $L_{\text{OSC}}$ (which is actually defined by the proper distance as the phase shift $\Phi_{\text{null}}^{kj}$ changes by $2\pi$) which is measured by the observer at rest at a position $r$, and $p_0^{\text{loc}} = p_0/\sqrt{f(r)}$ is the local energy. As $r$ approaches infinity, $p_0^{\text{loc}}$ approaches the energy $p_0$ measured by an observer at infinity. Thus, the neutrino oscillation length in a black hole spacetime is given by following the metric ansatz (9) as

$$L_{\text{OSC}}^{\text{grav}} = \frac{4\pi p_0}{\Delta m_{kj}^2} \sqrt{f(r)} \hspace{1cm} (53)$$

while for a flat spacetime it reduces to

$$L_{\text{OSC}}^{\text{flat}} = \frac{4\pi p_0}{\Delta m_{kj}^2}. \hspace{1cm} (54)$$

Hence we can define a quantity which measures the fractional change in the oscillation length due to the presence of gravity

$$\delta l_1 = \frac{L_{\text{OSC}}^{\text{grav}} - L_{\text{OSC}}^{\text{flat}}}{L_{\text{OSC}}^{\text{flat}}} = 1 - \frac{1}{\sqrt{f(r)}} - 1. \hspace{1cm} (55)$$

Another quantity of interest is the shift in the oscillation length due to these alternative theories compared with the vacuum Schwarzschild solution in Einstein GR and can be computed as

$$\delta l_2 = \frac{L_{\text{OSC}}^{\text{alter}} - L_{\text{OSC}}^{\text{sch}}}{L_{\text{OSC}}^{\text{sch}}} = 1 - \frac{1}{\sqrt{f_{\text{alt}}(r)}} - \frac{1}{\sqrt{f_{\text{sch}}(r)}} \hspace{1cm} (56)$$

where $f_{\text{alt}}(r)$ is the metric element for the alternative gravity theory and $f_{\text{sch}}(r)$ is the metric element for Schwarzschild theory i.e. $1 - 2M/r$. Next, we shall calculate these quantities for the spherically symmetric solution used previously in this paper and hence put bounds on the parameters.

5.1. Vacuum solution in $F(R)$ gravity

We now consider a proper oscillation length for neutrino oscillation in the vacuum solution for $F(R)$ gravity. As pointed out in section 4.1.1 the vacuum solution in $F(R)$ gravity actually comes form $F(R) \propto R^2$ and hence is completely different from the usual vacuum solution in Einstein theory for which $F(R) \propto R$. Thus, the vacuum solution in $F(R)$ gravity has the same structure as the (A)dS–Schwarzschild solution but is obtained from an $R^2$ Lagrangian compared to $R$ in Einstein theory and differs from the standard (A)dS–Schwarzschild solution.
Figure 1. The variation of the two quantities \( \delta l_1 \) and \( \delta l_2 \) defined in equations (57) and (58) for the vacuum solution in \( F(R) \) gravity with a radial coordinate for different choice of physical parameters i.e. \( M \) and \( L \).

in GR [50, 53, 54]. The quantities defined in equations (55) and (56) lead to the following expressions in this gravity theory:

\[
\delta l_1 = \frac{1}{\sqrt{1 - \frac{2M}{r} \mp \frac{r^2}{L^2}}} - 1
\] (57)

and

\[
\delta l_2 = \frac{1}{\sqrt{1 - \frac{2M}{r} \mp \frac{r^2}{L^2}}} - \frac{1}{\sqrt{1 - \frac{2M}{r}}}
\] (58)

These two quantities are being plotted in figure 1, for the vacuum solution presented in this section. From the figures we observe that \( \delta l_1 \) have the same asymptotic nature for all choice of parameters, which is also valid for \( \delta l_2 \). In this context, we would like to constrain our parameters in the theory. For this purpose we consider the oscillation probability of the neutrino to convert from one flavor to another. For this purpose we use the data of solar neutrino oscillation, which is a two-flavor neutrino oscillation discussed in this paper. We present how the oscillation probability vary with the energy of the neutrino for different choice of parameters. This variation of the oscillation probability is presented in figure 2.

Now we present the data for solar neutrino in a tabular form and using the oscillation probability expression we get bounds on the cosmological parameter \( L \).

5.2. Proper oscillation length in Einstein–Maxwell–Gauss–Bonnet gravity

We consider proper oscillation length for neutrino oscillation in the EMGB gravity. The quantities defined in equations (55) and (56) lead to the following expressions in this gravity theory given by

\[
\delta l_1 = \frac{1}{\sqrt{K + \frac{\ell^2}{4\alpha} \left[ 1 \pm \sqrt{1 + \frac{8\alpha (m + 2\alpha |K|)}{\ell^2}} - \frac{8\alpha |K|}{3\ell^2} \right]}} - 1
\] (59)
Figure 2. The above figure illustrates the variation of $e \to e$ probability with neutrino energy measured in MeV for an energy window. Different graphs are for different choice of $L$ in the vacuum solution for $F(R)$ gravity. The length is taken to be 180 km.

and

$$
\delta l_2 = \frac{1}{\sqrt{K + \frac{r^2}{4\alpha} \left[ 1 \pm \sqrt{1 + \frac{8\alpha (m + 2\alpha K)}{r^4} - \frac{8\alpha q^2}{3r^4}} \right]}} - \frac{1}{\sqrt{1 - \frac{2M}{r}}}. \quad (60)
$$

These two quantities defined above are plotted in figure 3, for the charged solution presented in this section. From the figures we observe that $\delta l_1$ and $\delta l_2$ have the same asymptotic behavior.

We would like to constrain the string tension parameter $\alpha$ in the theory for some specific value of the charge (the charge in the astrophysical situation being very small, so we have also taken charges in that order). For this purpose we consider the oscillation probability of the neutrino to convert from one flavor to another as measured on earth. The solution not being asymptotically flat has a non-negligible contribution even on the earth. For this purpose we use the data of solar neutrino oscillation, which is a two-flavor neutrino oscillation in the gravitational field discussed in this paper. This variation of oscillation probability is presented in figure 4.

Now the data for solar neutrino is presented in a tabular form and using the oscillation probability expression we get bounds on the string tension $\alpha^{-1}$.

5.3. Proper oscillation length in charged solution in $F(R)$ gravity

In this section, proper oscillation length for neutrino oscillation in the charged $F(R)$ gravity has been discussed. The quantities defined in equations (55) and (56) lead to the following expressions in this gravity theory such that

$$
\delta l_1 = \frac{1}{\sqrt{1 - \frac{4}{3} r^2 - \frac{M}{r} + \frac{q^2}{r^2}} - 1} \quad (61)
$$
Figure 3. Figures show the variation of the two quantities $\delta l_1$ and $\delta l_2$ given by equations (59) and (60) for solution in EMGB gravity with a radial coordinate. Different graphs indicate different choices of parameters in the theory, namely, $M$, $q$ and $\alpha$.

Figure 4. The variation of $e \rightarrow e$ probability with energy in MeV for an energy window and length of 180 km. Different graphs are for different choices of parameter in EMGB gravity namely, $\alpha$.

and

$$
\delta l_2 = \frac{1}{\sqrt{1 - \frac{\Delta}{\lambda} r^2 - \frac{M}{r} + \frac{Q^2}{r^2}}} - \frac{1}{\sqrt{1 - \frac{2M}{r}}}.
$$

(62)

These two quantities defined above are plotted in figure 5, for the charged solution presented in this section. From the figures we observe that $\delta l_1$ and $\delta l_2$ have the same asymptotic behavior.
Figure 5. The variation of the two quantities $\delta l_1$ and $\delta l_2$ as presented in equations (61) and (62) for charged solution in $F(R)$ gravity with a radial coordinate. Different graphs are for different choices of parameters namely, $M$, $q$ and $\Lambda$.

Figure 6. The variation of $e \rightarrow e$ probability with energy in MeV for a length of 180 km. Different graphs are for different choices of parameters in the charged $F(R)$ solution namely, $\Lambda$.

We would like to constrain the parameter $\Lambda$ in the theory for some specific value of the charge (charge in the astrophysical situation being very small, so we have taken charges in that order). For this purpose we consider the oscillation probability of the neutrino converting from one flavor to another as measured on earth. For this purpose we use the data of solar neutrino oscillation, which is a two-flavor neutrino oscillation in gravitational field as discussed in this paper. This variation of oscillation probability is presented in figure 6.
The best bound on the parameter $\alpha$ appearing in the vacuum solution for $F(R)$ gravity is given from table 1 as $L^{-1} < 1.895 \times 10^{-12}$. This is obtained using the SNO phase III results on solar neutrino oscillation. Also the string tension parameter $\alpha^{-1}$ in EMGB gravity has the best bounded value as $\alpha^{-1} < 5.759 \times 10^{-24}$ obtained from SNO phase II data on the neutrino flux (see table 2). In a similar manner we have the best bound on $|\Lambda|$ as $|\Lambda| < 0.798 \times 10^{-23}$ for SNO phase II neutrino oscillation data (see table 3).

6. Concluding remarks

In this paper, we have discussed and given analytical expression for the phase of mass neutrino propagating along both null and time-like geodesic in a general static spherically symmetric spacetime. Then we apply our phase expression in three spherically symmetric solutions in a propagating along both null and time-like geodesic in a general static spherically symmetric spacetime. In this paper, we have discussed and given analytical expression for the phase of mass neutrino propagating along both null and time-like geodesic in a general static spherically symmetric spacetime. Then we apply our phase expression in three spherically symmetric solutions in a propagating along both null and time-like geodesic in a general static spherically symmetric spacetime. These phase expressions are being evaluated in the equatorial plane $\theta = \pi/2$ using spherical symmetry. Using the phase expression we have calculated neutrino oscillation probability and hence put bounds on these alternative theories. Thus, this work not only shows an alternative way to constrain parameters in alternative gravity theories but also shows neutrino phase and deviation from standard results through the quantities $\delta l_1$ and $\delta l_2$, respectively. By setting different parameters to zero we have matched our result to flat and Schwarzschild spacetime. We have also shown quiet generally that the phase along null geodesic and time-like geodesic have a factor of 2 in any general spherically symmetric spacetime.

In the last section, we have presented the variation of the oscillation length for introducing extra parameters in our theory in comparison with flat and Schwarzschild spacetime. Also from the difference in the oscillation probability due to the presence of alternative gravity can be used to constrain the parameters. For this purpose we have used data on neutrino flux as measured by different experiments and then put bounds on these parameters. The bounds on these parameters appear quiet small and hence have not been observed experimentally yet.

### Table 1. $^8$B solar neutrino results from real time experiments. The predictions of BPS08(GS) and SHP11(GS) standard solar models are also shown. The errors are the statistical errors. Bounds on the cosmological parameter are estimated.

| Experiment   | Reaction | $^8$B $\nu$ flux | Bound on cosmological parameter $L^{-1}$ |
|--------------|----------|------------------|-----------------------------------------|
| Kamiokande [65] | $v\bar{e}$ | $2.80 \pm 0.19$ | $<2.235 \times 10^{-12}$ |
| Super-K I [66] | $v\bar{e}$ | $2.38 \pm 0.02$ | $<2.325 \times 10^{-12}$ |
| Super-K II [67] | $v\bar{e}$ | $2.41 \pm 0.05$ | $<2.308 \times 10^{-12}$ |
| Super-K III [68] | $v\bar{e}$ | $2.32 \pm 0.04$ | $<2.364 \times 10^{-12}$ |
| SNO Phase I [69] | CC | $1.76^{+0.06}_{-0.05}$ | $<2.412 \times 10^{-12}$ |
| (pure D$_2$O) | $v\bar{e}$ | $2.39^{+0.24}_{-0.25}$ | $<2.323 \times 10^{-12}$ |
| NC | | $5.99^{+0.44}_{-0.43}$ | $<2.029 \times 10^{-12}$ |
| SNO Phase II [70] | CC | $1.68 \pm 0.06$ | $<2.423 \times 10^{-12}$ |
| (NaCl in D$_2$O) | $v\bar{e}$ | $2.35 \pm 0.22$ | $<2.328 \times 10^{-12}$ |
| NC | | $4.94 \pm 0.21$ | $<2.040 \times 10^{-12}$ |
| SNO Phase III [15] | CC | $1.67^{+0.05}_{-0.04}$ | $<2.425 \times 10^{-12}$ |
| ($^3$He counters) | $v\bar{e}$ | $1.77^{+0.24}_{-0.23}$ | $<2.411 \times 10^{-12}$ |
| NC | | $5.54^{+0.31}_{-0.30}$ | $<1.895 \times 10^{-12}$ |
| Borexino [71] | $v\bar{e}$ | $2.4 \pm 0.4$ | $<2.312 \times 10^{-12}$ |
| SSM [BPS08(GS)] [72] | – | $5.94 (1 \pm 0.11)$ | – |
| SSM [SHP11(GS)] [73] | – | $5.58 (1 \pm 0.14)$ | – |
However, these results are derived for two generation neutrino oscillation; also in sun only one type of neutrino is produced (namely, the electron type) and being of low energy we need not consider all the three generations. In recent times, there have been many works concerning ultra-high energy neutrino from AGN and other energetic astrophysical sources in the universe. So it would be quiet natural to discuss three generation neutrino oscillation in gravitational field and apply the results of neutrino oscillation for these high energy neutrinos.

Table 2. $^8$B solar neutrino results from real time experiments. The predictions of BPS08(GS) and SHP11(GS) standard solar models are also shown. The errors are the statistical errors. Bounds on string tension is estimated.

| Experiment     | Reaction | $^8$B $\nu$ flux | Bound on string tension $\alpha^{-1}$ |
|----------------|----------|------------------|--------------------------------------|
| Kamiokande [65]| $\nu e$  | $2.80 \pm 0.19$  | $<5.935 \times 10^{-24}$              |
| Super-K I [66] | $\nu e$  | $2.38 \pm 0.02$  | $<6.213 \times 10^{-24}$              |
| Super-K II [67]| $\nu e$  | $2.41 \pm 0.05$  | $<6.125 \times 10^{-24}$              |
| Super-K III [68]| $\nu e$ | $2.32 \pm 0.04$  | $<6.311 \times 10^{-24}$              |
| SNO Phase I [69]| CC     | $1.76_{-0.05}^{+0.06}$ | $<6.438 \times 10^{-24}$            |
| (pure D$_2$O)  | $\nu e$  | $2.39_{-0.24}^{+0.25}$ | $<6.313 \times 10^{-24}$            |
| NC            |          | $5.09_{-0.43}^{+0.5}$ | $<5.892 \times 10^{-24}$            |
| SNO Phase II [70]| CC   | $1.68 \pm 0.06$  | $<6.523 \times 10^{-24}$              |
| (NaCl in D$_2$O)| $\nu e$ | $2.35 \pm 0.22$  | $<6.281 \times 10^{-24}$              |
| NC            |          | $4.94 \pm 0.21$  | $<5.759 \times 10^{-24}$              |
| SNO Phase III [15]| CC   | $1.67_{-0.04}^{+0.05}$ | $<6.425 \times 10^{-24}$            |
| ($^3$He counters)| $\nu e$ | $1.77_{-0.24}^{+0.25}$ | $<6.312 \times 10^{-24}$            |
| NC            |          | $5.54_{-0.31}^{+0.33}$ | $<5.891 \times 10^{-24}$            |
| Borexino [71] | $\nu e$  | $2.4 \pm 0.4$    | $<6.391 \times 10^{-24}$              |
| SSM [BPS08(GS)] [72] | –   | $5.94(1 \pm 0.11)$ | –                                     |
| SSM [SHP11(GS)] [73] | –   | $5.58(1 \pm 0.14)$ | –                                     |

Table 3. $^8$B solar neutrino results from real time experiments. The predictions of BPS08(GS) and SHP11(GS) standard solar models are also shown. The errors are the statistical errors. Bounds on the parameter $\Lambda$ is estimated.

| Experiment     | Reaction | $^8$B $\nu$ flux | Bound on parameter $|\Lambda|$ in charged theory |
|----------------|----------|------------------|-----------------------------|
| Kamiokande [65]| $\nu e$  | $2.80 \pm 0.19$  | $<0.812 \times 10^{-23}$    |
| Super-K I [66] | $\nu e$  | $2.38 \pm 0.02$  | $<1.209 \times 10^{-23}$    |
| Super-K II [67]| $\nu e$  | $2.41 \pm 0.05$  | $<1.189 \times 10^{-23}$    |
| Super-K III [68]| $\nu e$ | $2.32 \pm 0.04$  | $<1.321 \times 10^{-23}$    |
| SNO Phase I [69]| CC     | $1.76_{-0.05}^{+0.06}$ | $<1.536 \times 10^{-23}$    |
| (pure D$_2$O)  | $\nu e$  | $2.39_{-0.24}^{+0.25}$ | $<1.413 \times 10^{-23}$    |
| NC            |          | $5.09_{-0.43}^{+0.5}$ | $<0.912 \times 10^{-23}$    |
| SNO Phase II [70]| CC   | $1.68 \pm 0.06$  | $<1.623 \times 10^{-23}$    |
| (NaCl in D$_2$O)| $\nu e$ | $2.35 \pm 0.22$  | $<1.193 \times 10^{-23}$    |
| NC            |          | $4.94 \pm 0.21$  | $<0.798 \times 10^{-23}$    |
| SNO Phase III [15]| CC   | $1.67_{-0.05}^{+0.05}$ | $<1.415 \times 10^{-23}$    |
| ($^3$He counters)| $\nu e$ | $1.77_{-0.24}^{+0.25}$ | $<1.472 \times 10^{-23}$    |
| NC            |          | $5.54_{-0.31}^{+0.33}$ | $<0.921 \times 10^{-23}$    |
| Borexino [71] | $\nu e$  | $2.4 \pm 0.4$    | $<1.371 \times 10^{-23}$    |
| SSM [BPS08(GS)] [72] | –   | $5.94(1 \pm 0.11)$ | –                                     |
| SSM [SHP11(GS)] [73] | –   | $5.58(1 \pm 0.14)$ | –                                     |
Since these neutrinos are generated in strong gravitational field, they may provide the behavior of gravity at such high field limit. We left these issues to be discussed in some future work.

Also, another interesting feature of this problem is the blueshift of neutrino phase. The oscillation length for any spherically symmetric spacetime is proportional to local energy, interpreted as neutrino climbing out of the gravitational potential well. However, in this work we have used some solutions that are AdS-like on infinity thus decreasing the oscillation length. Thus, our result is valid for any spherically symmetric solution not only in four dimension but in any spacetime dimensions. Also, the constraints on the parameters are very interesting regarding their cosmological interpretation, which again makes neutrino oscillation in curved spacetime a very interesting and profound problem in physics.

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