ON RADIAL AND CONICAL FOURIER MULTIPLIERS

YARYONG HEO  FÊDOR NAZAROV  ANDREAS SEEGER

Abstract. We investigate connections between radial Fourier multipliers on $\mathbb{R}^d$ and certain conical Fourier multipliers on $\mathbb{R}^{d+1}$. As an application we obtain a new weak type endpoint bound for the Bochner-Riesz multipliers associated to the light cone in $\mathbb{R}^{d+1}$, where $d \geq 4$, and results on characterizations of $L^p \to L^{p,\nu}$ inequalities for convolutions with radial kernels.

Introduction

This paper is a sequel to [9] in which the authors obtained a characterization of radial multipliers of $\mathcal{F}L^p(\mathbb{R}^d)$ provided that $1 < p < 2$ and the dimension $d$ is large enough. The main estimate in [9] was concerned with a convolution inequality for surface measure on spheres which in this paper we state as Hypothesis Sph($p_1, d$) for some $p_1 > 1$. Under this hypothesis we shall prove several equivalent statements on cone multipliers and radial Fourier multipliers.

In what follows we fix a radial $C^\infty(\mathbb{R}^d)$ function $\psi_0$ supported in a small ball of radius centered at the origin (say, of radius $\leq (100d)^{-1}$) whose Fourier transform vanishes at the origin to high order (say $100d$). We assume that $\hat{\psi}_0(\xi) \neq 0$ for $1/8 \leq |\xi| \leq 8$. Set

$$\psi = \psi_0 * \psi_0 \quad (0.1)$$

and, for $y \in \mathbb{R}^d$ and for $r \geq 1$, let $\sigma_r$ be surface measure on the sphere of radius $r$ centered at the origin, i.e.

$$\langle \sigma_r, f \rangle = r^{d-1} \int_{S^{d-1}} f(ry')d\sigma_1(y'). \quad (0.2)$$

1991 Mathematics Subject Classification. 42B15.

Y.H. supported by Korea Research Foundation Grant KRF-2008-357-C00002 and National Research Foundation of Korea Grant NRF-2009-0094068. F.N. supported in part by NSF grant 0800243. A.S. supported in part by NSF grant 0652890.
Hypothesis \( \text{Sph}(p, d) \). There is a constant \( C \) so that for every \( h \in L^p(\mathbb{R}^d \times \mathbb{R}^+; dy \, r^{d-1} dr) \) the inequality

\[
\left\| \int_{\mathbb{R}^d} \int_1^\infty h(y, r) \sigma_r * \psi(\cdot - y) \, dr \, dy \right\|_{L^p(\mathbb{R}^d)} \leq C \left( \int_{\mathbb{R}^d} |h(y, r)|^p \, dy \, r^{d-1} \, dr \right)^{1/p}
\]

holds.

Theorem 0.1. \((9)\) Hypothesis \( \text{Sph}(p, d) \) holds for \( d \geq 4, 1 \leq p < \frac{2(d-1)}{d+1} \).

1. Statement of results

In what follows \( L^{p, \nu} \) denotes the standard Lorentz space, and we shall usually assume that \( p \leq \nu \leq \infty \). We denote by \( \mathcal{F}_d f \) the \( \mathbb{R}^d \) Fourier transform of \( f \), defined by \( \mathcal{F}_df(\xi) = \int f(y)e^{-i\langle y, \xi \rangle} \, dy \). We shall also write \( \mathcal{F}f \) or \( \hat{f} \) if the dimension is clear from the context.

For each \( k \in \mathbb{Z} \) let \( \gamma_k \) be supported in \( (-1/4, 1/4) \). Define

\[
m(\xi, \tau) \equiv m_\gamma(\xi, \tau) = \sum_{k \in \mathbb{Z}} \gamma_k \left( \frac{|\xi| - \tau}{2^k} \right) \mathbb{1}_{[2^k, 2^{k+1})}(\tau)
\]

where \( \mathbb{1}_E \) denotes the characteristic function of \( E \). Let \( T \) denote the operator acting on Schwartz functions in \( \mathbb{R}^{d+1} \) by

\[
\mathcal{F}_{d+1}[Tf](\xi, \tau) = m_\gamma(\xi, \tau) \mathcal{F}_{d+1}f(\xi, \tau).
\]

Moreover, for each fixed \( \tau \in (0, \infty) \), define an operator \( T_\tau \) on functions in \( \mathbb{R}^d \) by

\[
\mathcal{F}_d[T_\tau f](\xi) = \gamma_k \left( \frac{|\xi| - \tau}{2^k} \right) \mathcal{F}_df(\xi), \quad \text{if} \ \tau \in [2^k, 2^{k+1}).
\]

Theorem 1.1. Let \( T, T_\tau \) be as in \((1.2), (1.3)\).

Suppose that \( 1 < p_1 < \frac{2d}{d+1} \) and suppose that Hypothesis \( \text{Sph}(p_1, d) \) holds. Let \( 1 < p < p_1, p \leq \nu \leq \infty \). Then the following statements are equivalent.

(i) \( T \) maps \( L^p(\mathbb{R}^{d+1}) \) boundedly to \( L^{p, \nu}(\mathbb{R}^{d+1}) \).

(ii) There is a constant \( C_p \) so that for all sequences \( \{\tau_k\}_{k=-\infty}^\infty \) satisfying \( \tau_k \in [2^k, 2^{k+1}) \), and for all \( f \in L^p(\mathbb{R}^d) \)

\[
\left\| \sum_{k \in \mathbb{Z}} \alpha_k T^{\tau_k}f \right\|_{L^{p, \nu}(\mathbb{R}^d)} \leq C_p \sup_{k \in \mathbb{Z}} \|\alpha_k\| \|f\|_{L^p(\mathbb{R}^d)}
\]
(iii) For every $k \in \mathbb{Z}$ there is a $\tau_k \in [2^k, 2^{k+1})$ such that $T_{\tau_k}$ maps $L^p(\mathbb{R}^d)$ boundedly to $L^{p,\nu}(\mathbb{R}^d)$, and such that $\sup_k \|T_{\tau_k}\|_{L^p \to L^{p,\nu}} < \infty$.

(iv) The functions $s \mapsto \hat{\gamma}_k(s) (1 + |s|)^{-\frac{d-1}{2}}$ belong to the weighted Lorentz space $L^{p,\nu}(\mathbb{R}, (1 + | \cdot |)^{d-1})$, with the uniform bound

\begin{equation}
\sup_{k \in \mathbb{Z}} \left\| \hat{\gamma}_k \right\|_{L^{p,\nu}(\mathbb{R}, (1 + | | \cdot | |)^{d-1})} < \infty.
\end{equation}

(v) The functions $F^{-1}_d[\gamma_k(|\cdot|)]$ belong to $L^{p,\nu}(\mathbb{R}^d)$, with the uniform bound

\begin{equation}
\sup_{k \in \mathbb{Z}} \|F^{-1}_d[\gamma_k(|\cdot|)]\|_{L^{p,\nu}(\mathbb{R}^d)} < \infty.
\end{equation}

From Theorem 0.1 we get

**Corollary 1.2.** Statements (i)-(v) in Theorem 1.1 are equivalent if $d \geq 4$, $1 < p < \frac{2(d-1)}{d+1}$, $p \leq \nu \leq \infty$.

The equivalence of (iv) $\iff$ (v) and the implication (iii) $\implies$ (iv) are in [5]. The implication (ii) $\implies$ (iii) is trivial. The implication (i) $\implies$ (iii) follows from a version of de Leeuw’s theorem, see Lemma 2.3. It is not presently clear how to deduce the global statement (ii) directly from (i), without going through (iv) or (v). The proofs of the main implications (iv) $\implies$ (i) and (iv) $\implies$ (ii) are given in [4], [5] they rely on Hypothesis Sph($p_1, d$).

As a consequence of the implication (iv) $\implies$ (i) we shall derive a new endpoint result for the so-called Bochner-Riesz multipliers for the cone, defined by

\begin{equation}
\rho_\lambda(\xi, \tau) = \left(1 - \frac{|\xi|^2}{\tau^2}\right)^\lambda_+.
\end{equation}

It is conjectured that $\rho_\lambda$ is a multiplier of $FL^p(\mathbb{R}^{d+1})$ if $\lambda > d(1/p - 1/2) - 1/2$ and $1 < p < \frac{2d}{d+1}$; this condition is necessary. This conjecture is open in the full $p$-range. The first sharp $L^p$ results for some range of $p$ were proved by T. Wolff [19] in two dimensions, with extensions and improvements in [13], [4], [6], [7], [9]. For the endpoint $\lambda = d(1/p - 1/2) - 1/2$ one conjectures a weak type $(p, p)$ inequality for $p < \frac{2d}{d+1}$. This endpoint inequality cannot be replaced by a stronger statement such as $L^p \to L^{p,\nu}$ boundedness for $\nu < \infty$.

In [6] we prove

**Corollary 1.3.** Let $d \geq 2$ and $p_1 > 1$, and suppose that Hypothesis Sph($p_1, d$) holds. Let $\rho_\lambda$ be as in (1.5). If $\lambda = d(1/p - 1/2) - 1/2$ and $1 < p < p_1$ then

\begin{equation}
\|F^{-1}[\rho_\lambda \hat{f}]\|_{L^{p,\infty}(\mathbb{R}^{d+1})} \leq C_p \|f\|_{L^p(\mathbb{R}^{d+1})}
\end{equation}
for all \( f \in L^p(\mathbb{R}^{d+1}) \). In particular, (1.6) holds for \( d \geq 4 \) and \( 1 < p < \frac{2(d-1)}{d+1} \).

**Remark.** Sharp bounds in \( H^p \), \( p < 1 \) and sharp bounds for the operator acting on functions of the form \( f(x,t) = f_0(|x|, t) \) can be found in Hong’s articles [10], [11]. More recently, Heo, Hong and Yang [8] proved a weak type \((1,1)\) inequality for a localized cone multiplier \( \chi(\tau)\rho(d-1)/2(\xi,\tau) \), in dimension \( d \geq 4 \). As a corresponding result for the global cone multiplier one can prove that for \( \text{Re}(\lambda) = (d-1)/2 \) the operator \( f \rightarrow F^{-1}_{d+1}[\rho_\lambda f] \) is bounded from the Hardy space \( H^1 \) to \( L^{1,\infty} \), under the assumption that \( \text{Sph}(p_1,d) \) holds for some \( p_1 > 1 \). This can be obtained by an analytic interpolation argument using the analytic family of multipliers \( \lambda \rightarrow \rho_\lambda \), the \( H^p \rightarrow L^{p,\infty} \) bounds in [10] for \( p < 1 \) and \( \text{Re}(\lambda) = d(1/p - 1/2) - 1/2 \), and the \( L^p \) result of Corollary 1.3. For the justification of the analytic interpolation one uses an adaptation of arguments in [15]. We shall not give details here.

The equivalence of condition (ii) in the theorem with conditions (iv) or (v) immediately yields a generalization of the main result in [9] to \( L^p \rightarrow L^{p,\nu} \) inequalities.

**Corollary 1.4.** Let \( p_1 > 1, 1 < p < p_1 < \frac{2d}{d+1} \) and assume that Hypothesis \( \text{Sph}(p_1, d) \) holds. Let \( m = m_0(|\cdot|) \) be a bounded radial function on \( \mathbb{R}^d \) and define \( T_m \) by

\[
F_d[T_m f] = m F_d f.
\]

Then, for any Schwartz function \( \eta \neq 0 \)

\[
(1.7) \quad \| T_m \|_{L^p \rightarrow L^{p,\nu}} \approx \sup_{t > 0} t^{d/p} \| T_m[\eta(t\cdot)] \|_{L^{p,\nu}}.
\]

Moreover, if \( \phi \in C_\infty^c \) is compactly supported in \((0, \infty)\) (and not identically zero) and \( \kappa_t \) denotes the Fourier transform on \( \mathbb{R} \) of the function \( \phi m_0(t\cdot) \) then

\[
(1.8) \quad \| T_m \|_{L^p \rightarrow L^{p,\nu}} \approx \sup_{t > 0} \left\| \frac{\kappa_t}{(1 + |r|)^{d-1/2}} \right\|_{L^{p,\nu}(\mathbb{R};(1+|r|)^{d-1}dr)} < \infty.
\]

The equivalence of the two conditions on the right hand sides of (1.7) and (1.8) with \( L^p_{\text{rad}} \rightarrow L^{p,\nu} \) boundedness (i.e. on radial functions, for \( p < \frac{2d}{d+1} \)) was proved in [5]. The \( L^p \) case \((p = \nu)\) for \( 1 < p < \frac{2(d-1)}{d+1} \) was proved in [9], moreover that article has already \( L^p \rightarrow L^{p,\nu} \) inequalities for radial multipliers which are compactly supported away from the origin.

2. Preliminaries

The following dyadic interpolation lemma is convenient in dealing with the short range estimation in [14], it is proved in §2 of [9].
Lemma 2.1. Let $0 < p_0 < p_1 < \infty$. Let $\{F_j\}_{j \in \mathbb{Z}}$ be a sequence of measurable functions on a measure space $\{\Omega, \mu\}$, and let $\{s_j\}$ be a sequence of nonnegative numbers. Assume that, for all $j$, the inequality

$$\|F_j\|_{L^p} \leq 2^{j \rho} M^{p\nu} s_j$$

holds for $\nu = 0$ and $\nu = 1$. Then for every $p \in (p_0, p_1)$, there is a constant $C = C(p_0, p_1, p)$ such that

$$\left\| \sum_j F_j \right\|_{L^p} \leq C^p M^p \sum_j 2^{j \rho} s_j.$$

We need a simple fact about Lorentz spaces.

Lemma 2.2. Let $(X_1, \mu_1)$, $(X_2, \mu_2)$ be $\sigma$-finite measure spaces, and let $\mu = \mu_1 \times \mu_2$ be the product measure on $X_1 \times X_2$. Then, for $1 \leq p < \infty$, $p \leq \nu \leq \infty$, and any $\mu$-measurable function $G$,

$$\|G\|_{L^{p,\nu}(X_1 \times X_2, \mu)} \leq C_{p,\nu} \left( \int \|G(x_1, \cdot)\|_{L^{p,\nu}(X_2, \mu_2)}^p \mu_1(\cdot) \right)^{1/p}.$$

The proof is a Fubini-type argument (in conjunction with Minkowski’s inequality in $\ell^{\nu/p}$), we refer to §9 of [9].

Finally we need a version of a restriction theorem for multipliers due to de Leeuw.

Lemma 2.3. Let $1 < p < \infty$ and $1 \leq p \leq \infty$ and let $m$ be a bounded continuous function in $\mathbb{R}^{d+1}$. Suppose that the operator $f \mapsto F_{d+1}[mF_{d+1}f]$ is bounded from $L^p,\nu_1$ to $L^p,\nu_2$ with operator norm $B$. Let, for $\xi \in \mathbb{R}^d$, $m_0(\xi) = m(\xi, 0)$. Then there is a constant $C$ independent of $m$ and $f$ such that

$$\|F_{d+1}^{-1}[m_0F_d f]\|_{L^p,\nu_2} \leq CB \|f\|_{L^p,\nu_1}.$$

Proof. This is just a modification of the proof given in [12]. By the hypothesis

$$\left| \int \int m(\xi, \tau) \hat{F}(\xi, \tau) \hat{G}(\xi, \tau) d\xi d\tau \right| \leq B \|F\|_{L^p,\nu_1} \|G\|_{L^{p',\nu'_2}}.$$

Now let $\chi$ be a Schwartz function on $\mathbb{R}$ whose Fourier transform is supported in $(-1, 1)$, so that $\hat{\chi}(0) \neq 0$. Given a small parameter $\delta$ we let $\chi_\delta(t) = \chi(\delta t)$, and, for $f \in L^{p_{1,\nu_1}}(\mathbb{R}^d)$, $g \in L^{p',\nu_2'}(\mathbb{R}^d)$ we define $F_\delta(x, t) = \delta \chi_\delta(t) f(x)$ and $G_\delta(x, t) = \chi_\delta(t) g(x)$. Observe that the inequality

$$\|h \otimes \chi_\delta\|_{L^{q,r}((\mathbb{R}^d)^2)} \leq C(\chi) \delta^{-1/q} \|h\|_{L^{q,r}(\mathbb{R}^{d+1})}$$

is immediate for $q = r$, by Fubini, and then holds for arbitrary $r$ by real interpolation. Thus $\|F_\delta\|_{L^p,\nu_1} \leq \delta^{1-1/p} \|f\|_{L^p,\nu_1}$ and $\|G_\delta\|_{L^{p',\nu'_2}} \leq \delta^{-1/p'} \|g\|_{L^{p',\nu'_2}}.$
Apply (2.2) with $F_\delta, G_\delta$ and let $\delta \to 0$. This yields
\[
[\hat{\chi}(0)]^2 \int m(\xi, 0) \hat{f}(\xi) \hat{g}(\xi) d\xi = \lim_{\delta \to 0} \int m(\xi, \tau) \hat{f}(\xi) \hat{g}(\xi) \delta^{-1} [\hat{\chi}(\delta^{-1} \tau)]^2 d\xi d\tau
\]
\[
\leq CB \|f\|_{L^p, \nu_1} \|g\|_{L^{p'}, \nu_2'}
\]
which implies the assertion. \hfill \Box

\section{3. Inequalities for spherical measures}

We shall now derive a consequence of Hypothesis Sph($p_1, d$) which will be used in conjunctions with atomic decompositions. Similar inequalities have been used in [9] but they were derived using the proof of (0.3) rather than (0.3) itself.

In what follows let $\ell$ be a nonnegative integer and, for $z = (z_1, \ldots, z_d) \in \mathbb{Z}^d$, let
\[
R_{\ell} = \{ x \in \mathbb{R}^d : 2^{\ell} z_i \leq x_i < 2^{\ell+1} z_i, i = 1, \ldots, d \}
\]
so that the $R_{\ell}$ form a tiling of $\mathbb{R}^d$ with dyadic cubes of sidelength $2^{\ell}$. We denote by $\chi_{R_{\ell}}$ the characteristic function of $R_{\ell}$. We denote variables in $\mathbb{Z}^d$ by $z = (z, z_{d+1})$ with $z \in \mathbb{Z}^d$. Let $I_{z_{d+1}, \ell} = [2^{\ell} z_{d+1}, 2^{\ell+1} z_{d+1})$ and let
\[
\chi_{z, \ell}(x, t) := \chi_{R_{\ell}}(x) \chi_{I_{z_{d+1}, \ell}}(t).
\]

For each $r > 0, z \in \mathbb{Z}^{d+1}$ we are given an $L^2(\mathbb{R}^{d+1})$ function $g_{z, r}$ depending continuously on $r$ such that
\[
\|g_{z, r}\|_{L^2(\mathbb{R}^{d+1})} \leq 1, \quad \text{for all } z, r.
\]
Moreover, for each positive integer $n$ we are given an $L^1$ function $\omega_n$ supported on $[1/2, 2]$ so that
\[
\sup_n \int_{1/2}^2 |\omega_n(\rho)| d\rho \leq 1.
\]

Let $\ell > 0$. We define an operator $S_{\ell}$ acting on functions $F$ on $\mathbb{Z}^{d+1} \times \mathbb{R}^+$ by
\[
S_{\ell} F(x, t) = \sum_{z} \sum_{n=\ell}^\infty \int_{2^n}^{2^{n+1}} F(z, r) \int_{1/2}^2 \omega_n(\rho) \int_{\mathbb{R}^d} \psi^* \sigma_{\rho r}(x-y) \chi_{z, \ell} g_{z, r}(y, t-r) dy d\rho dr.
\]

On the set $\mathbb{Z}^{d+1} \times \mathbb{R}^+$ we define the measure $\mu_d$ by
\[
\mu_d(E) = \int_0^\infty \sum_z r^{d-1} dr.
\]
for a measurable set $E \subset \mathbb{Z}^{d+1} \times \mathbb{R}^+$. 

**Proposition 3.1.** Suppose $d \geq 2$ and Hypothesis Sph($p_1, d$) holds for some $p_1 \in (1, 2)$. Let $g_{z,r}, \omega_n$ be as in (3.3), (3.4), $\ell > 0$, and define $S_\ell$ by (3.5). Then the inequality

$$
\|S_\ell F\|_{L^{p, \nu}(\mathbb{R}^{d+1})} \leq C_{p, \nu} 2^{\ell(d+1)(\frac{1}{p} - \frac{1}{2}) - \alpha} \|F\|_{L^{p, \nu}(\mathbb{Z}^{d+1} \times \mathbb{R}^+, \mu_d)}
$$

holds for (i) for $p = 1 = \nu$, with $\alpha = \frac{d-1}{2}$, (ii) for $p = p_1 = \nu$, with $\alpha = 0$ and, (iii), for

$$1 < p < p_1, \quad 0 < \nu \leq \infty \quad \text{with} \quad \alpha \leq \frac{d-1}{2} \frac{1 - \frac{1}{p_1}}{1 - \frac{1}{p}}.$$

**Proof.** Statement (iii) follows by real interpolation from the cases $p = \nu = p_1$ and $p = \nu = 1$.

We consider the case $p = p_1 = \nu$. In order to apply Hypothesis Sph($p_1, d$) we interchange the $\rho$- and the $r$-integrals and change variables $s = \rho \rho$. This yields

$$
S_\ell F(x, t) = \sum_z \sum_{n=\ell}^\infty \int_2^{\ell-1} \int_{s=2^n \rho}^{2^{n+1} \rho} F(z, \frac{s}{\rho}) \omega_n(\rho) \times
$$

$$
\int_{\mathbb{R}^d} \psi * \sigma_{s}(x-y) \chi_{[x, \ell]} g_{z, \frac{s}{\rho}}(y, t - \frac{s}{\rho}) dy ds d\rho,
$$

and thus

$$
S_\ell F(x, t) = \int_{2^{\ell-1}}^{\infty} \int_{\mathbb{R}^d} \psi * \sigma_s(x-y) V_\ell F(y, s, t) dy ds
$$

where

$$
V_\ell F(y, s, t) := \int_{\rho=1/2}^2 \sum_{n=\ell}^\infty \omega_n(\rho) \chi_{[2^n \rho, 2^{n+1} \rho)}(s) \sum_z F(z, \frac{s}{\rho}) \chi_{[x, \ell]} g_{z, \frac{s}{\rho}}(y, t - \frac{s}{\rho}) \rho^{-1} d\rho.
$$

For fixed $t$ we apply Hypothesis Sph($p_1, d$) and then integrate in $t$. This yields

$$
\|S_\ell F\|_{L^{p_1}(\mathbb{R}^{d+1})} = \left( \int_t \|S_\ell F(\cdot, t)\|_{L^{p_1}(\mathbb{R}^{d+1})} dt \right)^{1/p_1} \lesssim \left( \int \int_{2^{\ell-1}}^{\infty} \int |V_\ell F(y, s, t)|^{p_1} dy s^{d-1} ds dt \right)^{1/p_1}.
$$
We observe that if $2^\nu < s < 2^{\nu+1}$ then only the terms with $\nu - 1 \leq n \leq \nu + 1$ contribute to the $n$-sum in (3.8). Thus, for fixed $(y, t)$,

\[(\int_{2^\nu}^{2^{\nu+1}} |V_\ell F(y, s, t)|^{p_1} s^{d-1} ds)^{1/p_1} \leq \sum_{i = -1, 0, 1} (\int_{2^\nu}^{2^{\nu+1}} \left| \int_{\rho = 1/2}^{2} \omega_{\nu+i}(\rho) \sum_z F(z, \ell g_z, z) \right|^{p_1} s^{d-1} ds)^{1/p_1}.
\]

Now we have for fixed $\nu$

\[\left(\int_{2^\nu}^{2^{\nu+1}} \left| \int_{\rho = 1/2}^{2} \omega_{\nu+i}(\rho) \sum_z F(z, \ell g_z, z) \right|^{p_1} s^{d-1} ds \right)^{1/p_1} \leq \int_{1/2}^{2} \left| \omega_{\nu+i}(\rho) \right| \left(\int_{2^\nu}^{2^{\nu+1}} \left| \sum_z F(z, \ell g_z, z) \right|^{p_1} s^{d-1} ds \right)^{1/p_1} \frac{d\rho}{\rho}
\]

\[\leq \int_{1/2}^{2} \left| \omega_{\nu+i}(\rho) \right| \frac{d\rho}{\rho} \left(\int_{2^\nu}^{2^{\nu+2}} \left| \sum_z F(z, r) \left| \chi_{z, \ell g_z, r} \right| \right|^{p_1} r^{d-1} dr \right)^{1/p_1}
\]

\[\lesssim \left(\int_{2^\nu}^{2^{\nu+1}} \left| \sum_z F(z, r) \left| \chi_{z, \ell g_z, r} \right| \right|^{p_1} r^{d-1} dr \right)^{1/p_1}.
\]

We insert this back into (3.9) and obtain

\[(\int_{2^\nu}^{2^{\nu+1}} |V_\ell F(y, s, t)|^{p_1} s^{d-1} ds)^{1/p_1} \lesssim \left(\int_{2^\nu}^{2^{\nu+1}} \left| \sum_{z} F(z, r) \left| \chi_{z, \ell g_z, r} \right| \right|^{p_1} r^{d-1} dr \right)^{1/p_1}
\]

\[\leq \left(\int_{2^\nu}^{2^{\nu+2}} \left| \sum_{z} F(z, r) \left| \chi_{z, \ell g_z, r} \right| \right|^{p_1} r^{d-1} dr \right)^{1/p_1}.
\]

We take $L^{p_1}$ norms in $(y, t)$ and perform a shear transformation for fixed $r$ to get

\[\left(\int \int \int_{2^{\nu-1}}^{2^{\nu}} |V_\ell F(y, s, t)|^{p_1} s^{d-1} ds dt dy \right)^{1/p_1} \lesssim \left(\int_{2^{\nu-1}}^{2^{\nu}} \left| F(z, r) \right|^{p_1} \int \int \left| \chi_{z, \ell g_z, r} \right|^{p_1} dt dy r^{d-1} dr \right)^{1/p_1}.
\]

By Hölder’s inequality and our normalizing assumption (3.3) this is estimated by

\[\left(\int_{2^{\nu-1}}^{2^{\nu}} \left| \sum_{z} F(z, r) \right|^{p_1} 2^{\ell(d+1)(1-p_1/2)} \left| \chi_{z, \ell g_z, r} \right|^{p_1} r^{d-1} dr \right)^{1/p_1} \lesssim 2^{\ell(d+1)(1/p_1-1/2)} \left(\int \sum_{z} \left| F(z, r) \right|^{p_1} r^{d-1} dr \right)^{1/p_1}.
\]
This yields the assertion for \( p = p_1 = \nu \), with \( \alpha = 0 \).

For \( p = 1 \) we estimate

\[
\| S_\ell F \|_{L^1(\mathbb{R}^{d+1})} \lesssim \sum_{n=\ell}^{\infty} \sum_{\nu} \int_{2n}^{2n+1} |F(z, r)| \int_{t}^{t+2} |\omega_n(\rho)| \times \\
|\psi * \sigma_{\rho} \ast [\chi_{R_n, t} g_z, r](\cdot, t - r)| \| L^1(\mathbb{R}^{d+1}) dp \chi_{\mathbb{R}^{d+1}, t}(t - r) dt dr.
\]

The function \( \psi * \sigma_{\rho} \ast [\chi_{R_n, t} g_z, r](\cdot, t - r) \) is supported on a set of measure \( \leq \frac{C r^{d-1}}{\ell} \), namely an annulus of width \( \leq \frac{2\ell}{\ell - 1} \) built on a sphere of radius \( r\rho \). Moreover we have \( \|F_d[\psi * \sigma_{\rho}]\| \leq C r^{d-1} \) where \( C \) is independent of \( \rho \in [1/2, 2] \). Thus the last displayed expression can be estimated by

\[
\sum_{n=\ell}^{\infty} \sum_{\nu} \int_{2n}^{2n+1} |F(z, r)| \int_{t}^{t+2} |\omega_n(\rho)| \times \\
|\psi * \sigma_{\rho} \ast [\chi_{R_n, t} g_z, r](\cdot, t - r)| \| L^2(\mathbb{R}^{d}) dp \chi_{\mathbb{R}^{d+1}, t}(t - r) dt dr \\
\lesssim \sum_{n=\ell}^{\infty} \sum_{\nu} \int_{t}^{t+2} |\omega_n(\rho)| dp |F(z, r)| \times \\
2^{\ell/2} r^{d-1} \| [\chi_{R_n, t} g_z, r](\cdot, t - r) \|_{L^2(\mathbb{R}^{d})} \chi_{\mathbb{R}^{d+1}, t}(t - r) dt dr.
\]

Now we use (3.4), apply the Cauchy-Schwarz inequality in \( t \) and then use the normalizing assumption (3.3) to bound the last expression by

\[
2^{\ell/2} \sum_{z} \int |F(z, r)| r^{d-1} \times \\
2^{\ell/2} \left( \int \| [\chi_{R_n, t} g_z, r](\cdot, t - r) \|_{L^2(\mathbb{R}^{d})}^2 \chi_{\mathbb{R}^{d+1}, t}(t - r) dt \right)^{1/2} dr \\
\lesssim 2^{\ell} \sum_{z} \int |F(z, r)| r^{d-1} dr.
\]

This gives the assertion for \( p = \nu = 1 \), when \( \alpha = \frac{d-1}{2} \).

\]

4. The main estimate

We formulate our main estimate which will yield both the implications (iv) \( \implies \) (i) and (iv) \( \implies \) (ii) of Theorem 1.1. In this section \( \chi_1 \) will be a \( C^\infty \) function supported in \((5/8, 17/8)\) and \( \chi \) will be a \( C^\infty \) function supported in \((-4, 4)\). We now consider the convolution operator \( T \) on \( \mathbb{R}^{d+1} \) with multiplier

\[
m(\xi, \tau) = \sum_{k \in \mathbb{Z}} \chi_1(2^{-k}|\xi|) \chi(2^{-k}\tau) \Gamma_k \left( \frac{|\xi| - b_k}{2^k} \right).
\]
Theorem 4.1. Suppose that $1 < p_1 < \frac{d}{d+1}$ and that Hypothesis $\text{Sph}(p_1, d)$ holds. Let $m$ be as in (4.1), with $b_k \in \mathbb{R}$ and $|b_k| \leq 2$. Let $1 < p < p_1$ and $p \leq \nu \leq \infty$ and assume that

$$
\mathcal{C}_{p, \nu} := \sup_k \left\| \frac{\hat{\Gamma}_k}{(1 + |\cdot|)^{\frac{d-1}{2}}} \right\|_{L^{p, \nu}(\mathbb{R}, (1 + |\cdot|)^{d-1})} < \infty.
$$

Then

$$
\| \mathcal{F}^{-1} [m \hat{f}] \|_{L^{p, \nu}(\mathbb{R}^{d+1})} \lesssim \mathcal{C}_{p, \nu} \| f \|_{L^p(\mathbb{R}^{d+1})}.
$$

We apply the Fourier inversion formula on the real line to $\hat{\Gamma}_k$ and get

$$
m(\xi, \tau) = \sum_k \chi(2^{-k} \tau) \chi(2^{-k}|\xi|) \frac{1}{2\pi} \int_{-2}^2 \hat{\Gamma}_k(s) e^{i2^{-k}(|\xi| - b_k \tau)s} ds.
$$

By standard singular integral theory the convolution operator with Fourier multiplier

$$
\sum_k \chi(2^{-k} \tau) \chi(2^{-k}|\xi|) \frac{1}{2\pi} \int_{-2}^2 \hat{\Gamma}_k(s) e^{i2^{-k}(|\xi| - b_k \tau)s} ds
$$

is bounded on $L^p(\mathbb{R}^{d+1})$ for all $p \in (1, \infty)$. Therefore it suffices to consider the Fourier multiplier

$$
\sum_k \chi(2^{-k} \tau) \chi(2^{-k}|\xi|) \int_{2}^\infty \hat{\Gamma}_k(s) \exp(is2^{-k}(|\xi| - \tau)) ds
$$

and a similar multiplier involving an integration over $(-\infty, -2)$.

We note that these multipliers define bounded functions. Their $L^\infty(\mathbb{R}^{d+1})$ norms are bounded by

$$
\sup_k \int |\hat{\Gamma}_k(s)| \, ds \lesssim \sup_k \left\| \frac{\hat{\Gamma}_k}{(1 + |\cdot|)^{\frac{d-1}{2}}} \right\|_{L^{p, \infty}(\nu_d)}, \quad p < \frac{2d}{d+1};
$$

here $\nu_d$ denotes the measure

$$
d\nu_d(s) = (1 + |s|)^{d-1}ds.
$$

To see (4.5) note that the function $s \mapsto (1 + |s|)^{-\frac{d-1}{2}}$ belongs to $L^q(\nu_d)$ if $q > \frac{2d}{d+1}$. Thus, for $p < \frac{2d}{d+1}$, we have

$$
\int |w(s)|ds = \int \frac{|w(s)|\frac{d}{2}}{(1 + |s|)^{\frac{d-1}{2}} (1 + |s|)^{\frac{d-1}{2}}} \, d\nu_d(s) \lesssim \left\| \frac{w}{(1 + |\cdot|)^{\frac{d-1}{2}}} \right\|_{L^{p, \infty}(\nu_d)},
$$

which implies (4.5).

Now let $\vartheta$ be a $C^\infty$-function on the real line supported in $(1/8, 8)$ so that $\vartheta(s) = 1$ on $(1/5, 5)$ and observe that multiplication with $\vartheta(\beta|\xi|)$ does not
affect $\chi_1(|\xi|)$ as long as $1/2 \leq \beta \leq 2$. Thus we have to prove that the convolution operator with multiplier

\[
\sum_k \chi(\frac{\tau}{2^n}) \chi_1 \left( \frac{\xi}{2^n} \right) \sum_{n=1}^{\infty} \int_{2^n}^{2^{n+1}} \vartheta(2^{-n} s \frac{\xi}{2^n}) \hat{\Gamma}_k(s) \exp(is \frac{|\xi| - b_k \tau}{2^n}) \, ds
\]

is bounded on $L^p(\mathbb{R}^{d+1})$.

One can express $\mathcal{F}_d^{-1}[\exp(\pm i \cdot |\vartheta(2^{-n} \cdot |))]$ as an integral over spherical means plus an error term:

**Lemma 4.2.** For $n \geq 1$,

\[
\mathcal{F}_d^{-1}[\exp(\pm i \cdot |\vartheta(2^{-n} \cdot |))] = 2^n(d-1)/2 \int_{1/2}^{2} \omega_n^+(\rho) \sigma_\rho d\rho + \tilde{E}_n^\pm
\]

where $\omega_n^\pm$ is smooth on $(1/2, 2)$

\[
\sup_n \int |\omega_n^+(\rho)| d\rho < \infty,
\]

and, for any $N$,

\[
|\tilde{E}_n^\pm(x)| + 2^{-n} |\nabla \tilde{E}_n^\pm(x)| \leq C_N 2^{-nN} (1 + |x|)^{-N}.
\]

This can be proven by an application of the stationary phase method; a more direct argument is given in Lemma 10.2 in [9].

From the lemma we see that the convolution operator with multiplier (4.6) can be split as

\[
\sum_k K_k * f + \sum_{n=1}^{\infty} \sum_{k \in \mathbb{Z}} 2^{k(d+1)} E_{n,k}(2^k) * f
\]

where the main term is obtained by substituting the first term in (4.7) for $\mathcal{F}_d^{-1}[e^{\pm i \cdot |\vartheta(2^{-n} \cdot |)}]$ (cf. (4.11) below) and thus the rescaled term $E_{n,k}$ is given by

\[
E_{n,k}(x,t) = \int_{2^n}^{2^{n+1}} \hat{\Gamma}_k(s) \int \tilde{\zeta}(x-y, t-b_k s) s^{-d} \tilde{E}_n(s^{-1}y) \, dy \, ds
\]

where $\tilde{\zeta}(\xi, \tau) = \chi(\tau) \chi_1(|\xi|)$.

From (4.9) one gets

\[
|E_{n,k}(x,t)| + |\nabla E_{n,k}(x,t)| \leq C_N, 2^{-nN} \times \int_{s=2^n}^{2^{n+1}} |\hat{\Gamma}_k(s)| \int (1 + |x-y| + |t-b_k s|)^{-d-2} (1 + 2^{-n} |y|)^{-N/2} \, dy \, ds.
\]
We use \((1 + |x - y| + |t - b_k s|)^{-d - 2} \lesssim \left(\frac{1 + |y| + |b_k s|}{1 + |x| + |t|}\right)^{d + 2}\) and since \(|b_k| \leq 2\) this implies (assuming \(N\) is chosen sufficiently large, say \(N > 10^d\))
\[
|E_{n,k}(x,t)| + |\nabla E_{n,k}(x,t)| \lesssim \|\tilde{G}_k\|_{L^1(\mathbb{R})} 2^{-n}(1 + |x| + |t|)^{-d - 2}.
\]

From this estimate it follows easily that the operator \(E_{n}\) defined by
\[
E_{n} f = \sum_{k \in \mathbb{Z}} 2^{kd(d+1)} E_{n,k}(2^k \cdot) * f
\]
is a Calderón-Zygmund operator which is bounded on \(L^p(\mathbb{R}^{d+1})\) and the sum of the operator norms \(\sum_{n=1}^{\infty} \|E_{n}\|_{L^p \rightarrow L^p}\) is bounded by a constant only depending on \(p\).

We now consider the main term. This is the operator of convolution on \(\mathbb{R}^{d+1}\) with the kernel \(\sum_k K_k\) where
\[
(4.11) \quad F_{d+1}[K_k](\xi, \tau) = \chi_1(2^{-k}|\xi|)\chi(2^{-k}\tau) \times
\sum_{n=1}^{\infty} 2^{n \frac{d+1}{2}} \int_2^{2^{n+1}} \tilde{G}_k(s) e^{-ib_k s 2^{-k}\tau} \int_{1/2}^2 \omega_n(\rho) \mathcal{F}_d[\sigma_{\rho}](2^{-k}s\xi) \, d\rho \, ds.
\]

We now let \(\psi_0, \psi\) be \(C_0^\infty\)-functions as defined in the introduction and define \(\eta_0 \in S(\mathbb{R}^{d+1})\) by
\[
\eta_0(\xi, \tau) = \chi_1(|\xi|)\chi(\tau) \quad \text{by} \quad \frac{\chi_1(|\xi|)\chi(\tau)}{(\psi_0(\xi))^4} = \frac{\chi_1(|\xi|)\chi(\tau)}{(\psi(\xi))^2}.
\]

Define the dyadic Littlewood-Paley operator \(L_k\) by
\[
F_{d+1}[L_k f](\xi, \tau) = \eta_0(2^{-k}\xi, 2^{-k}\tau) F_{d+1}[f](\xi, \tau).
\]

Then
\[
K_k * f(x,t) = \int_2^\infty \int 2^{kd} H_{k,s}(2^k (x-y)) P_k L_k f(y, t - 2^{-k}b_k s) \, dy \, ds,
\]
where
\[
F_d[P_k g](\xi) = \psi(2^{-k}\xi) \mathcal{F}_d g(\xi)
\]
and
\[
H_{k,s}(x) = \sum_{n=1}^{\infty} 2^{n \frac{d+1}{2}} \tilde{G}_k(s) \chi[2^n, 2^{n+1}](s) \int_{\rho=1/2}^2 \omega_n(\rho)[\psi * s^{-d} \sigma_{\rho}(s^{-1} \cdot)] \, d\rho.
\]

(4.13) \quad \text{in (4.13) the \(*\) is used for convolution in } \mathbb{R}^d.
Atomic decompositions. As in [9] we use atomic decompositions constructed from a nontangential Peetre type maximal square function (cf. [14], [18] and [16]),
\[ S_f(x,t) = \left( \sum_k \sup_{|(y,s)| \leq 100(d+1)2^{-k}} |L_k f(x+y,t+s)|^2 \right)^{1/2}. \]
Then \( \|S f\|_p \leq C_p \|f\|_p \) for \( 1 < p < \infty \).

For fixed \( k \), we tile \( \mathbb{R}^{d+1} \) by the dyadic cubes of sidelength \( 2^{-k} \). We write \( L(Q) = -k \) if we want to indicate that the sidelength of a dyadic cube is \( 2^{-k} \). For each integer \( j \), we introduce the set
\[ \Omega_j = \{(x,t) : S f(x,t) > 2^j \}. \]
Let \( Q_j^k \) be the set of all dyadic cubes of sidelength \( 2^{-k} \) which have the property that \( |Q \cap \Omega_j| \geq |Q|/2 \) but \( |Q \cap \Omega_{j+1}| < |Q|/2 \). We also set
\[ \Omega_j^* = \{(x,t) : M \chi_{\Omega_j}(x,t) > 100^{-d-1} \} \]
where \( M \) is the Hardy-Littlewood maximal operator. \( \Omega_j^* \) is an open set containing \( \Omega_j \) and \( |\Omega_j^*| \lesssim |\Omega_j| \).

Let \( W_j \) is the set of all dyadic cubes \( W \) for which the 20-fold dilate of \( W \) is contained in \( \Omega_j^* \) and \( W \) is maximal with respect to this property. Clearly the interiors of these cubes are disjoint and we shall refer to them as Whitney cubes for \( \Omega_j^* \). For such a Whitney cube \( W \in W_j \) we denote by \( W^* \) the tenfold dilate of \( W \), and observe that the family of dilates \( \{W^* : W \in W_j\} \) have bounded overlap.

Note that each \( Q \in Q_j^k \) is contained in a unique \( W \in W_j \). For each \( W \in W_j \), set
\[ A_{k,W,j} = \sum_{Q \in Q_j^k : Q \subset \mathcal{W}} L_k f \chi_Q; \]
note that only terms with \( L(W) + k \geq 0 \) occur. Since any any dyadic cube \( W \) can be a Whitney cube for several \( \Omega_j^* \) we also define “cumulative atoms”,
\[ A_{k,W} = \sum_{j : W \in W_j} A_{k,W,j}. \]

Standard facts about these atoms are summarized in

**Lemma 4.3.** For each \( j \in \mathbb{Z} \) the following inequalities hold.

(i) \[ \sum_{W \in W_j} \sum_k \|A_{k,W,j}\|_2^2 \lesssim 2^{2j} \text{meas}(\Omega_j). \]
(ii) There is a constant $C_d$ such that for every assignment $W \mapsto k(W) \in \mathbb{Z}$, defined for $W \in \mathcal{W}_j$, and for $0 \leq p \leq 2$,
\[ \sum_{W \in \mathcal{W}_j} \text{meas}(W) \| A_{k(W),W,j} \|^p_\infty \leq C_d 2^{pj} \text{meas}(\Omega_j). \]

For the proof see Lemma 7.1 in [9] (or related statements in [1], [16]).

With this notation it is now our task to show the inequality
\[ (4.14) \quad \left\| \sum_{k} \sum_{j} \sum_{\ell \geq 0} \sum_{W \in \mathcal{W}_j} K_k \ast A_{k,W,j} \right\|_{L^p,\nu} \lesssim \mathcal{C}_{p,\nu} \| \mathcal{S} f \|_p. \]

Let
\[ (4.15) \quad \mathcal{W}_{k,s,W,j}(x,t) := \int 2^{kd} H_{k,s}(2^k \cdot - y) A_{k,W,j}(y,t - 2^{-k} b_k s) \, dy, \]
with $H_{k,s}$ in (4.13) and note that
\[ K_k \ast A_{k,W,j} = P_k \left[ \int_2^\infty \mathcal{W}_{k,s,W,j} \, ds \right], \]
with $P_k$ in (4.12).

The estimate (4.14) follows then from a short range and a long range inequality. The short range inequality is
\[ (4.16) \quad \left\| \sum_{k} \sum_{j} \sum_{\ell \geq 0} \sum_{W \in \mathcal{W}_j} P_k \left[ \int_2^{2^\ell} \mathcal{W}_{k,s,W,j} \, ds \right] \right\|_{L^p} \lesssim \sup_k \| \mathcal{\hat{T}}_k \|_{L^1(\mathbb{R})} \| \mathcal{S} f \|_p, \quad 1 < p < 2, \]
and implies the analogous $L^p \to L^{p,\nu}$ estimate since by assumption $\nu \geq p$. Recall that $\sup_k \| \mathcal{\hat{T}}_k \|_{L^1(\mathbb{R})} \lesssim \mathcal{C}_{p,\infty} \lesssim \mathcal{C}_{p,\nu}$ for $p < \frac{2d}{d+1}$, cf. (4.5).

The long range inequality is
\[ (4.17) \quad \left\| \sum_{k} \sum_{j} \sum_{\ell \geq 0} \sum_{W \in \mathcal{W}_j} P_k \left[ \int_2^{\infty} \mathcal{W}_{k,s,W,j} \, ds \right] \right\|_{L^{p,\nu}} \lesssim \mathcal{C}_{p,\nu} \| \mathcal{S} f \|_p. \]
The short range estimate. Since $\sum_j 2^{jp} \text{meas}(\Omega_j) \lesssim \|\mathcal{S}f\|_p^p$ it suffices by Lemma 2.1 to show that for fixed $j$ and $1 < p < 2$

\begin{equation}
(4.18) \quad \left\| \sum_k \sum_{\ell \geq 0} \sum_{W \in W_j} P_k \left[ \int_2^{2^\ell} \mathfrak{U}_{k,s,W,j} ds \right] \right\|_{L^p}^p \lesssim \sup_k \|\hat{\Gamma}_k\|_{L^1(\mathbb{R})}^p 2^{jp} \text{meas}(\Omega_j).
\end{equation}

Here we estimate an expression which is supported in $\Omega^*_j$. Thus the left hand side of (4.18) is dominated by

\begin{equation}
(4.19) \quad \text{meas}(\Omega^*_j)^{-1/p/2} \left\| \sum_k \sum_{\ell \geq 0} \sum_{W \in W_j} P_k \left[ \int_2^{2^\ell} \mathfrak{U}_{k,s,W,j} ds \right] \right\|_{L^2}^p
\end{equation}

which by the almost orthogonality of the operators $P_k$ is dominated by a constant times

\begin{equation}
(4.20) \quad \text{meas}(\Omega^*_j)^{-1/p/2} \left( \sum_k \sum_{\ell \geq 0} \sum_{W \in W_j} \left\| \int_2^{2^\ell} \mathfrak{U}_{k,s,W,j} ds \right\|_{L^2}^2 \right)^{p/2}.
\end{equation}

Now, for fixed $W$ with $L(W) = -k + \ell$, and for every $s \leq 2^\ell$, the expression $\mathfrak{U}_{k,s,W,j}$ is supported in the expanded cube $W^*$. The cubes $W^*$ with $W \in \Omega_j$ have bounded overlap, and therefore the expression (4.20) is dominated by a constant times

\begin{equation}
(4.21) \quad \text{meas}(\Omega^*_j)^{-1/p/2} \left( \sum_k \sum_{\ell \geq 0} \sum_{W \in W_j} \left\| \int_2^{2^\ell} \mathfrak{U}_{k,s,W,j} ds \right\|_{L^2}^2 \right)^{p/2}.
\end{equation}

Now we have for fixed $W$

$$
\left\| \int_2^{2^\ell} \mathfrak{U}_{k,s,W,j} ds \right\|_{L^2(\mathbb{R}^{d+1})} \lesssim \int_2^{2^\ell} \left( \int \left\| 2^{kd} H_{k,s}(\cdot, t - 2^{-k} b_k s) \right\|_{L^2(\mathbb{R}^d)}^2 dt \right)^{1/2} ds
\lesssim \int_2^{2^\ell} \left( \int \left\| \mathcal{F}_d[H_{k,s}] \right\|_{L^\infty(\mathbb{R}^d)}^2 \left\| A_{k,W,j}(\cdot, t - 2^{-k} b_k s) \right\|_{L^2(\mathbb{R}^d)}^2 dt \right)^{1/2} ds
= \int_2^{2^\ell} \left\| \mathcal{F}_d[H_{k,s}] \right\|_{L^\infty(\mathbb{R}^d)} ds \left\| A_{k,W,j} \right\|_{L^2(\mathbb{R}^{d+1})}
$$
and
\[
\int_{2^\ell}^{2^\ell+1} \|F_d[H_{k,s}]\|_\infty ds \lesssim \sum_{n=1}^{\ell-1} \int_{2^n}^{2^n+1} |\hat{\Gamma}_k(s)|^{2n} s^{-1} \int_{|\rho|=1/2} \omega_n(\rho) \|F_d[\psi \ast \hat{\sigma}_{\rho s}]\|_\infty d\rho ds.
\]

Since \(F_d[\psi \ast \hat{\sigma}_{\rho s}](\xi) = O(2^n(d-1)/2)\) uniformly in \(\rho \in (1/2, 2)\) and \(s \in [2^n, 2^{n+1}]\). Since \(\sup_n \|\omega_n\|_1 \leq 1\) we get
\[
\int_{2^\ell}^{2^\ell+1} \|F_d[H_{k,s}]\|_\infty ds \lesssim \int |\hat{\Gamma}_k(s)| ds
\]
and thus
\[
(4.22) \quad \|\int_{2^\ell}^{2^\ell+1} \mathfrak{V}_{k,s,W,j} ds\|_{L^2} \lesssim \int |\hat{\Gamma}_k(s)| ds \|A_{k,W,j}\|_2.
\]

We use this estimate in (4.21). By Lemma (4.3) we have
\[
\sum_k \sum_{\ell \geq 0} \sum_{W \in \mathcal{W}_j} \|A_{k,W,j}\|_2^2 \lesssim \sum_k \sum_{W \in \mathcal{W}_j} \|A_{k,W,j}\|_2^2 \lesssim 2^{2j} \text{meas}(\Omega_j).
\]

We combine this with (4.22). Since \(\text{meas}(\Omega_j^*) \lesssim \text{meas}(\Omega_j)\) it follows that the right hand side of (4.21) is dominated by a constant times
\[
\left[ \sup_k \int |\hat{\Gamma}_k(s)| ds \right]^p \text{meas}(\Omega_j) 2^{jp}
\]
which then yields (4.18) and finishes the proof of the short range estimate.

**The long range estimate.** It is now advantageous to use the cumulative atoms \(A_{k,W}\). If we let
\[
(4.23) \quad \mathfrak{V}_{k,s,W}(x,t) := \int 2^{kd} H_{k,s}(2^k(\cdot - y)) A_{k,W}(y,t - 2^{-k}b_k s) dy
\]
then \(\mathfrak{V}_{k,s,W} = \sum_j \mathfrak{V}_{k,s,W,j}\) and we have to show
\[
(4.24) \quad \left\| \sum_k \sum_{\ell \geq 0} \sum_{W, L(W)=-k+\ell} P_k \left[ \int_{2^\ell}^{\infty} \mathfrak{V}_{k,s,W} ds \right] \right\|_{L^{p,\nu}}^p \lesssim \mathfrak{C}_{p,\nu} \sum_j \text{meas}(\Omega_j) 2^{jp}.
\]
By Minkowski’s inequality this follows from estimates for fixed \( \ell > 0 \), with exponential decay:

\[
(4.25) \quad \left\| \sum_k \sum_{W: L(W) = -k + \ell} P_k \left[ \int_{2^\ell}^{\infty} \mathfrak{M}_{k,s,W} ds \right] \right\|_{L^p,\nu} \lesssim \mathcal{C}_{p,\nu} 2^{-\ell \alpha(p)} \left( \sum_j \text{meas}(\Omega_j) 2^j \right)^{1/p}.
\]

Here \( \alpha(p) > 0 \) for \( p < p_1 \) (in fact \( \alpha \) will be as in Proposition 3.1).

We interpolate an \( L^1(\ell^1) \to L^1 \) inequality and an \( L^2(\ell^2) \to L^2 \) inequality for the operators \( P_k \). Let \( \mathfrak{m} \) denote the measure on \( \mathbb{R}^{d+1} \times \mathbb{Z} \) defined as the product measure of Lebesgue measure on \( \mathbb{R}^{d+1} \) and counting measure on \( \mathbb{Z} \). Define for (suitable) functions \( h \) on \( \mathbb{R}^{d+1} \times \mathbb{Z} \) an operator \( P \) by

\[
Ph(x,t) = \sum_k \mathcal{F}_d^{-1} \left( \hat{\psi}(2^{-k} \cdot) \mathcal{F}h(\cdot,k) \right)(x,t).
\]

Then \( P \) maps \( L^1(\mathbb{R}^{d+1} \times \mathbb{Z}, \mathfrak{m}) \) to \( L^1(\mathbb{R}^{d+1}) \) and by orthogonality \( L^2(\mathbb{R}^{d+1} \times \mathbb{Z}, \mathfrak{m}) \) to \( L^2(\mathbb{R}^{d+1}) \); thus for \( 1 < p < 2 \)

\[
\|Ph\|_{L^p,\nu(\mathbb{R}^{d+1})} \lesssim \|h\|_{L^p,\nu(\mathbb{R}^{d+1} \times \mathbb{Z}, \mathfrak{m})}.
\]

Now by Lemma 2.2 this also implies, under the additional restriction \( \nu \geq p \),

\[
\left\| \sum_k P_k f_k \right\|_{L^p,\nu(\mathbb{R}^{d+1})} \lesssim \left( \sum_k \|f_k\|_{L^p,\nu(\mathbb{R}^{d+1})}^p \right)^{1/p}.
\]

Using this inequality we see that (4.25) follows from

\[
(4.26) \quad \sum_k \left\| \sum_{W: L(W) = -k + \ell} \int_{2^\ell}^{\infty} \mathfrak{M}_{k,s,W} ds \right\|_{L^p,\nu}^p \lesssim \mathcal{C}_{p,\nu} 2^{-\ell \alpha(p)} \sum_j \text{meas}(\Omega_j) 2^j.
\]

We need to rewrite \( \int_{2^\ell}^{\infty} \mathfrak{M}_{k,s,W} ds \) and also scale it in order to apply Hypothesis \( \text{Sph}(p_1, d) \) (or rather its consequence stated as Proposition 3.1). Note that

\[
\mathfrak{M}_{k,s,W}(x,t) = \int_{2^\ell}^{\infty} \int H_{k,s}(2^k x - y) A_{k,W}(2^{-k} y, 2^{-k}(2^k t - b_k s)) \, dy \, ds.
\]

If we set

\[
a_{k,W}(y,u) = A_{k,W}(2^{-k} y, 2^{-k} u)
\]

and

\[
v_{k,s,W}(x,t) = \int H_{k,s}(x - y) a_{k,W}(y, t - b_k s) \, dy \, ds
\]

then \( \mathfrak{M}_{k,s,W}(x,t) = v_{k,s,W}(2^k x, 2^k t) \) and of course we have

\[
(4.27) \quad \left\| \sum_{W: L(W) = -k + \ell} \int_{2^\ell}^{\infty} \mathfrak{M}_{k,s,W} \, ds \right\|_{L^p,\nu} = 2^{-k(d+1)/p} \left\| \sum_{W: L(W) = -k + \ell} \int_{2^\ell}^{\infty} v_{k,s,W} \, ds \right\|_{L^p,\nu}^p.
\]
Next (with * denoting convolution in $\mathbb{R}^d$)

\[(4.28) \quad \int_{2^\ell}^\infty \varphi_{n,r,W}(x,t) dr = \int_{1/2}^2 \sum_{n=\ell}^{2^{n+1}} \varphi_n(\rho) \int_{2^n}^{2^{n+1}} \Gamma_k(r) r^{1-d/2} d\rho \times A_k,W(x,t) dr d\rho. \]

We are now in the position to apply Proposition 3.1, with the choice of

\[F(z, r) = F_{k,\ell}(z, r) = \sum_{n=\ell}^{\infty} \Gamma_k(r) r^{1-d/2} \chi_{[2^n, 2^{n+1}]}(r) \sum_{W: L(W) = \ell-k} A_k,W(2^{-k} \cdot) \chi_{z,\ell} \|_{L^p}. \]

The sum in $W$ collapses as for given $z = (z_1, z_{d+1})$ there is a unique dyadic cube $W$ of sidelength $2^{\ell-k}$ so that the dyadic cube $2^kW = \{2^k(x, t) : (x, t) \in W\}$ is equal to $R_{\ell-k} \times I_{z_{d+1}, \ell}$. Also observe the pointwise estimate

\[\sum_{n=\ell}^{\infty} |\Gamma_k(r)| r^{1-d/2} \chi_{[2^n, 2^{n+1}]}(r) \lesssim \frac{|\Gamma_k(r)|}{(1 + |r|)^{d-1/2}}. \]

We now proceed to finish the proof of (4.26). By Proposition 3.1 and the Fubini-type Lemma 2.2 we get from (4.28)

\[\left\| \sum_{W: L(W) = \ell-k+\ell} \int_{2^\ell}^\infty \varphi_{k,s,W} ds \right\|_{L^p} \lesssim 2^{\ell((d+1)(\frac{1}{p} - \frac{1}{2}) - \alpha)} \|F_{k,\ell}\|_{L^p(\mathbb{R}^{d+1} \times \mathbb{R}^+, \mu_d)} \]

\[(4.29) \quad \lesssim 2^{\ell((d+1)(\frac{1}{p} - \frac{1}{2}) - \alpha)} \left\| \frac{|\Gamma_k|}{(1 + |r|)^{d-1/2}} \right\|_{L^p(\mathbb{R}, (1+|r|)^d dr)} \times \left( \sum_{W: L(W) = \ell-k+\ell} \|A_k,W(2^{-k} \cdot)\|_{L^p} \right) \lesssim \mathcal{C}_{p,\nu} 2^{-(\ell-k)(d+1)(\frac{1}{p} - \frac{1}{2})} \left( \sum_{W: L(W) = \ell-k+\ell} \|A_k,W\|_{L^p} \right)^{1/p}. \]

where $\alpha$ is as in Proposition 3.1. Combining (4.27) and (4.29) we obtain after a change of variables

\[(4.30) \quad \left( \sum_{W: L(W) = \ell-k+\ell} \int_{2^\ell}^\infty \sum_{W: L(W) = \ell-k+\ell} \int_{2^\ell}^\infty \|A_k,W\|_{L^p} \right)^{1/p} \lesssim \mathcal{C}_{p,\nu} 2^{-(\ell-k)(d+1)(\frac{1}{p} - \frac{1}{2})} \left( \sum_{W: L(W) = \ell-k+\ell} \|A_k,W\|_{L^p} \right)^{1/p}. \]
Note that for fixed $k$ and $W$, the functions $A_{k,W,j}$ live on disjoint sets (since the dyadic cubes of sidelength $2^{-k}$ are disjoint and each such cube is in exactly one family $Q_k^j$). Thus $\|A_{k,W}\|_2^p \lesssim \sum_j \|A_{k,W,j}\|_2^p$. We now sum in $k$ and obtain from (4.30)
\[
\left( \sum_k \left( \sum_{W: L(W) = -k + \ell} \mathcal{G}_{k,s,W} ds \right)^p \right)^{1/p} \lesssim \mathcal{C}_{p,\nu} 2^{-\alpha}\left( \sum_j \sum_{W \in W_j: L(W) = -k + \ell} \text{meas}(W)^{1-p/2} \|A_{k,W,j}\|_2^p \right)^{1/p}.
\]
Finally, using part (ii) of Lemma 4.3 we get
\[
\left( \sum_k \sum_j \sum_{W \in W_j: L(W) = -k + \ell} \text{meas}(W)^{1-p/2} \|A_{k,W,j}\|_2^p \right)^{1/p} \lesssim \left( \sum_j \sum_{W \in W_j} \text{meas}(W) \|A_{-L(W),W,j}\|_\infty^p \right)^{1/p} \lesssim \left( \sum_j \text{meas}(\Omega_j) 2^{jp} \right)^{1/p} \lesssim \|\mathcal{G}f\|_p.
\]
This finishes the proof of (4.26).

\[\square\]

5. Proof of Theorem 1.1

By the remarks in the introduction (following the statement of Corollary 1.2) it only remains to be shown that (iv) implies (i) and (ii). These implications quickly follow from Theorem 4.1.

For the implication (iv) \implies (i) we show, for the choices $b = 1$ and $b = \sqrt{2}$ that the multiplier
\[
m(\xi, \tau) = \sum_k \mathbb{1}_{[2^k, 2^{k+1})}(\tau) \gamma_k (2^{-k}(|\xi| - b\tau))
\]
defines an operator which is bounded from $L^p(\mathbb{R}^{d+1})$ to $L^{p,\nu}(\mathbb{R}^{d+1})$. The choice $b = \sqrt{2}$ and scaling in $\tau$ then also covers the multiplier
\[
m(\xi, \tau) = \sum_k \mathbb{1}_{[2^k, \sqrt{2}2^{k+1})}(\tau) \gamma_k (2^{-k}(|\xi| - \tau))
\]
and the assertion follows.
For the proof of (5.1) pick a smooth function $\chi_2$ which is equal to one on $[1, \sqrt{2}]$ and supported in $(9/10, 3/2)$. Recall that $\gamma_k$ is supported in $(-1/4, 1/4)$ and pick a smooth function $\chi_1$ which is equal to one on $(\frac{9}{10}b - \frac{1}{4}, \frac{3}{2}b + \frac{1}{4})$ and supported on $(\frac{9}{10}, 2b)$. Observe that, with these definitions

$$m(\xi, \tau) = \chi_E(\tau) \sum_k \chi_2(2^{-k}\tau)\chi_1(2^{-k}|\xi|)\gamma_k(2^{-k}(|\xi| - b\tau))$$

where $E = \bigcup_{k \in \mathbb{Z}}[2^k, 2^{k+\frac{1}{2}}]$. By the Marcinkiewicz multiplier theorem the convolution with multiplier $\chi_E(\tau)$ is bounded on $L^{p, \nu}(\mathbb{R}^{d+1})$ for all $1 < p < \infty, 0 < \nu \leq \infty$. Therefore it suffices to prove that under condition (1.4) the multiplier

$$m(\xi, \tau) = \sum_k \gamma_k(2^{-k}(|\xi| - \tau))\chi_2(2^{-k}\tau)\chi_1(2^{-k}|\xi|)$$

defines a convolution which maps $L^p(\mathbb{R}^d)$ to $L^{p, \nu}(\mathbb{R}^{d+1})$. But this follows immediately from Theorem 4.1 with the choice of $\Gamma_k = \gamma_k$, and $b_k = b$ ($= 1$ or $\sqrt{2}$) for all $k \in \mathbb{Z}$.

Next, for the implication $(iv) \implies (ii)$ we first note that since $\tau_k \in [2^k, 2^{k+\frac{1}{2}}]$ the term $\gamma(2^{-k}(|\xi| - \tau_k))$ vanishes for $|\xi| \notin \left(\frac{3}{2}2^k, \frac{5}{2}2^k\right)$. Now choose $\chi_1$ so that $\chi_1$ is supported in $(1/2, 5/2)$ and equal to one on $(3/4, 9/4)$. Then

$$\mathcal{F}_d[\sum_k \alpha_k T^{\tau_k}f](\xi) = \sum_k \alpha_k \gamma_k(2^{-k}|\xi| - 2^{-k}\tau_k)\chi_1(2^{-k}|\xi|)\mathcal{F}_d[f](\xi).$$

Now let $\chi$ be smooth and compactly supported in $(-4, 4)$. We claim that the multiplier transformation with Fourier multiplier

$$M(\xi, \tau) = \sum_k \gamma_k(2^{-k}(|\xi| - \tau_k))\chi_1(2^{-k}|\xi|)\chi(2^{-k}\tau)$$

maps $L^p(\mathbb{R}^{d+1})$ to $L^{p, \nu}(\mathbb{R}^{d+1})$. To see this we apply Theorem 4.1 with $\Gamma_k(s) = \alpha_k \gamma_k(s - 2^{-k}\tau_k)$ and $b_k = 0$ for all $k \in \mathbb{Z}$. The condition (1.4) for $\gamma_k$ is obviously equivalent with the condition (4.2) for $\Gamma_k$.

Now in (5.4) $\chi(\tau)$ may be chosen so that $\chi(0) = 1$. With this choice it follows by de Leeuw’s theorem (Lemma 2.3) that $\sum_k \alpha_k T^{\tau_k}$ maps $L^p(\mathbb{R}^d)$ to $L^{p, \nu}(\mathbb{R}^d)$. $\square$

6. THE CONE MULTIPLIER

Proof of Corollary 1.3. It suffices to consider the multiplier $\rho_\lambda(\xi, \tau)\chi(0, \infty)(\tau)$. We split for $\tau > 0$

$$\rho_\lambda(\xi, \tau) = \sum_{k \in \mathbb{Z}} 1_{[2^k, 2^{k+1}]}(\tau) \frac{2^{k\lambda}(\tau + |\xi|)^{\lambda}}{\tau^{2\lambda}} \left(\frac{\tau - |\xi|}{2^k}\right)_+.$$
Now let \( b \in C_c^\infty(\mathbb{R}) \) be supported in \((-1/4, 4)\) and satisfy \( b(s) = 1 \) for \(|s| \leq 1/8\). We can then write

\[
\rho_\lambda(\xi, \tau) \chi_{(0, \infty)}(\tau) = a_\lambda(\xi, \tau) \sum_{k \in \mathbb{Z}} \mathbb{1}_{[2^k, 2^{k+1})}(\tau) \gamma\left(\frac{\sqrt{2^k \tau + |\xi|}}{\sqrt{2^k}}\right) + \tilde{a}_\lambda(\xi, \tau)
\]

where

\[
a_\lambda(\xi, \tau) = \sum_{k \in \mathbb{Z}} \mathbb{1}_{[2^k, 2^{k+1})}(\tau) \left(\frac{2^k (\tau + |\xi|)}{2^k}\right)^\lambda b\left(\frac{|\xi| - \tau}{2^{k+1}}\right)
\]

\[
\tilde{a}_\lambda(\xi, \tau) = \sum_{k \in \mathbb{Z}} \mathbb{1}_{[2^k, 2^{k+1})}(\tau) \left(1 - \frac{|\xi|^2}{2^k \tau\tau}\right)^\lambda + b\left(\frac{|\xi| - \tau}{2^{k+1}}\right)
\]

and

\[
\gamma(u) = \begin{cases} (-u)^\lambda b(u) & \text{for } u < 0 \\ 0 & \text{for } u > 0 \end{cases}
\]

The multipliers \( a_\lambda \) and \( \tilde{a}_\lambda \) are treated by the Marcinkiewicz multiplier theorem. The associated convolution operators are thus bounded on \( L^{p, \nu} \) for all \( 1 < p < \infty, 0 < \nu \leq \infty \). Therefore the corollary follows if we can show that the convolution operator with multiplier

\[
\sum_{k \in \mathbb{Z}} \mathbb{1}_{[2^k, 2^{k+1})}(\tau) \gamma\left(2^{-k} (|\xi| - \tau)\right)
\]

maps \( L^p(\mathbb{R}^{d+1}) \) boundedly to \( L^{p, \infty}(\mathbb{R}^{d+1}) \). By Theorem 1.1 this is the case if

\[
\left\| \frac{\gamma}{(1 + |\cdot|)^{d-1}} \right\|_{L^{p, \infty}(\mathbb{R}, (1 + |r|)^{d-1}dr)} < \infty.
\]

But

\[
|\gamma(s)| \leq C(1 + |s|)^{-\lambda-1}
\]

and it is easy to check that \((1 + |\cdot|)^{-\lambda-1}\frac{d-1}{d\lambda} \) belongs to \( L^{p, \infty}(\mathbb{R}, (1 + |r|)^{d-1}dr) \) if and only if \( \lambda \geq d/p - (d + 1)/2 \).

References

[1] S.Y.A. Chang, R. Fefferman, A continuous version of duality of \( H^1 \) and \( \text{BMO} \) on the bidisc, Annals of Math. 112 (1980), 179–201.

[2] M. Christ, Weak type \( (1, 1) \) bounds for rough operators, Ann. of Math. (2) 128 (1988), no. 1, 19–42.

[3] C. Fefferman, E.M. Stein, Some maximal inequalities, Amer. J. Math. 93 (1971), 107–115.

[4] G. Garrigós, A. Seeger, Plate decompositions for cone multipliers, Proc. Edin. Math. Soc. 52 (2009), 1-21.

[5] Characterizations of Hankel multipliers, Math. Ann. 342, no. 1 (2008), 31-68.

[6] G. Garrigós, A. Seeger, W. Schlag, Improvements in Wolff inequality for decompositions of cone multipliers.
[7] Y. Heo, *Improved bounds for high dimensional cone multipliers*, Indiana Univ. Math. J., 58 (2009), no. 3, 1187–1202.
[8] Y. Heo, S. Hong, C. Yang, *An endpoint estimate for the cone multiplier*, Proc. Amer. Math. Soc. 138 (2010), 1333-1347.
[9] Y. Heo, F. Nazarov, A. Seeger, *Radial Fourier multipliers in high dimensions*, Acta Math., to appear.
[10] S. Hong, *Weak type estimates for cone multipliers on $H^p$ spaces, $p < 1$*, Proc. Amer. Math. Soc. 128 (2000), no. 12, 3529–3539.
[11] ______, *Some weak type estimates for cone multipliers*, Illinois J. Math. 44 (2000), no. 3, 496–515.
[12] M. Jodeit, *A note on Fourier multipliers*, Proc. Amer. Math. Soc. 27 (1971), 423–424.
[13] I. Laba, T. Wolff, *A local smoothing estimate in higher dimensions*, J. Anal. Math. 88 (2002), 149–171.
[14] J. Peetre, *On spaces of Triebel-Lizorkin type*, Ark. Mat. 13 (1975), 123–130.
[15] Y. Sagher, *On analytic families of operators*. Israel J. Math. 7 (1969), 350–356.
[16] A. Seeger, *Remarks on singular convolution operators*, Studia Math. 97 (1990), 91–114.
[17] T. Tao, *The weak-type endpoint Bochner-Riesz conjecture and related topics*, Indiana Univ. Math. J. 47 (1998), 1097–1124.
[18] H. Triebel, Theory of function spaces. Monographs in Mathematics, 78. Birkhäuser Verlag, Basel, 1983.
[19] T. Wolff, *Local smoothing type estimates on $L^p$ for large $p$*, Geom. Funct. Anal. 10 (2000), no. 5, 1237–1288.

Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA

E-mail address: heo@math.wisc.edu
E-mail address: nazarov@math.wisc.edu
E-mail address: seeger@math.wisc.edu