Asymptotic stability of \(N\)-soliton states of NLS

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Abstract

We prove the asymptotic stability and asymptotic completeness of an arbitrary number of weakly interacting solitons for NLS.

1 Introduction

The nonlinear Schrödinger equation

\[
\text{(NLS)} \quad i \frac{\partial \psi}{\partial t} = -\Delta \psi - \beta(|\psi|^2)\psi, \quad x \in \mathbb{R}^n
\]

has in general (exponentially) localized solutions in space, provided the nonlinearity has a negative (attractive) part. This is due to a remarkable cancellation of the dispersive effect of the linear part with the focusing caused by the attractive nonlinearity. To find such solutions, we look for time periodic solutions \(\psi \equiv e^{i\omega t} \phi_{\omega}(x)\). It follows that \(\phi_{\omega}\), if it exists, is a nonzero solution of the problem

\[
\text{(ENLS)} \quad -\omega \phi_{\omega} = -\Delta \phi_{\omega} - \beta(|\phi_{\omega}|^2)\phi_{\omega}.
\]

We shall refer to such solutions as nonlinear eigenfunctions. In general, for \(\phi_{\omega}\) to be localized (at least as \(L^2\) function) we need \(\omega > 0\).

The general existence theory for this elliptic problem has been studied in great detail, see the work of Coffman [Cof], Strauss [Str], and Berestycki, Lions [BL].

It is easy to see that if \(\psi_{\omega}(t, x) = e^{i\omega t} \phi_{\omega}(x)\) is a solution of NLS, then for any vector \(\vec{a} \in \mathbb{R}^n\) the function \(\phi_{\omega}(t, x - \vec{a})\) is also a solution. More generally, NLS is invariant under Galilean transformations

\[
\text{(1.1)} \quad \mathfrak{g}_{\vec{v}, D}(t) := e^{-i\vec{v} \cdot x - \frac{1}{2} |\vec{v}|^2 t} e^{i(t\vec{v} + D)p},
\]

and therefore we can construct solutions from \(\phi_{\omega}\) which are moving with arbitrary velocity \(\vec{v}\). As a result we obtain a family of exponentially localized solutions

\[
\psi_{\vec{v}, \gamma, D, \omega} = e^{i\vec{v} \cdot x - i\frac{1}{\omega}(|\vec{v}|^2 - \omega)t + i\gamma} \phi_{\omega}(x - \vec{v}t - D)
\]

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parametrized by a constant \((2n + 2)\)-dimensional vector \((\vec{v}, \gamma, D, \omega)\), which are known as solitons.

Solving the initial value problem for NLS requires understanding of two fundamental questions. The first is the existence of global in time solutions. Due to the focusing character of the nonlinear term the global existence theory ought to be based on the \(L^2\) conservation law. By the results of Kato [Kato] and Tsutsumi [Tsu] we can construct unique global solutions for any \(L^2\) initial data under the assumption that the nonlinearity \(\beta\) satisfies the condition \(|\beta(s)| \lesssim (1 + |s|)^q\) with \(q < \frac{2}{n}\).

The second problem is that of the asymptotic behavior of solutions as \(t \to +\infty\). Guided by the completely integrable models in one dimension we expect to have solutions with the asymptotic profile of \(N\) independently moving solitons:

\[
(1.2) \quad \psi(t, x) \approx \sum_{k=1}^{N} \psi_{\vec{v}_k, \gamma_k, D_k, \omega_k}(t, x).
\]

Moreover, given such a solution \(\psi(t, x)\) we expect that initial data \(\psi(0, x) + R\) for a suitably small perturbation \(R\) leads to a solution \(\tilde{\psi}\) with \(\|\psi(t, \cdot) - \tilde{\psi}(t, \cdot)\| \to 0\) as \(t \to \infty\) in an appropriate norm. The latter property is known as asymptotic stability. In the context of NLS one needs to modify (1.2) since one needs to make \((\vec{v}_k, \gamma_k, D_k, \omega_k)\) time-dependent. This fact was already observed in the context of orbital stability, see Weinstein [We1].

In this paper we give an affirmative answer to the question of existence and asymptotic stability of solutions with \(N\)-soliton profiles under the assumption of weak mutual interaction between the solitons. A superposition of \(N\) spatially separated moving solitons gives only an approximate solution of NLS. Our goal is to show that the initial data

\[
(1.3) \quad \psi_0(x) = \sum_{k=1}^{N} e^{i\vec{v}_k(0) \cdot x + i\gamma_k(0) \phi_{\omega_k}(0)(x - D_k(0))} + R_0,
\]

give rise to an asymptotically stable solution with the profile of \(N\) independent solitons, with perhaps different parameters \((\vec{v}_k, \gamma_k, D_k, \omega_k) \neq (\vec{v}_\ell(0), \gamma_\ell(0), D_\ell(0), \omega_\ell(0))\).

The function \(R_0\) is a perturbation satisfying a smallness assumption on its \(L^1 \cap L^2\) norm together with its derivatives. An important aspect of our main result is the assumption that the solitons are weakly interacting. This condition can be enforced in two ways. Firstly, one can assume that the initial shifts \(D_k(0)\) and the initial velocities \(v_k\) are chosen to model the case of non-colliding solitons:

\[
(1.4) \quad |D_k + \vec{v}_k t - D_{k'} - \vec{v}_{k'} t| \geq L + ct, \quad k \neq k'
\]

for some sufficiently large constant \(L\).

Alternatively, one can assume that the relative initial velocities of the solitons are large, i.e.,

\[
(1.5) \quad \min_{j \neq k} |\vec{v}_j - \vec{v}_k| > L.
\]

For the most part, we give details only for the case of (1.4) and leave the simple modifications required by (1.5) to the reader. Note that (1.5) does not rule out that the solitons collide. However, in view of (1.5) the time of interaction is of size \(L^{-1}\), and therefore the overall interaction remains weak.
We shall also require certain spectral assumptions on the nonlinear eigenstates $\phi_\omega$, which will be explained below.

We believe that the methods we use may be applied to other classes of equations with solitary type solutions and other symmetry groups (e.g. Lorentz instead of Galilean), if and whenever certain linear $L^p$ decay estimates can be verified for the linearized operators around one such soliton. A detailed analysis of such $L^p$ estimates for NLS was recently given in [RSS].

To explain our results we recall the precise notions of stability. Suppose we take the initial data of NLS to be an exact nonlinear eigenstate $\phi_\omega$, plus a small perturbation $R_0$. What is then the expected behavior of the solution? If the solution $\psi(t)$ stays near the soliton $\psi_\omega(t) = e^{i\omega t} \phi_\omega(x)$ up to a phase and translation for all times (in $H^1$ norm) we say that the soliton $\psi_\omega$ is \textit{orbitally stable}. If, as time goes to infinity, the solution in fact converges in $L^2$ to a nearby soliton plus radiation\footnote{a function with asymptotic behavior $e^{it\Delta} f$. Alternatively, we can and will replace the $L^2$ by the $L^\infty$ convergence. In the latter topology the contribution of the radiation can be ignored.} we say that the solution is \textit{asymptotically stable}.

Orbital stability of one soliton solutions has been subject of extensive work in the last 20 years. The first results date back to the work of Cazenave and Cazenave-Lions on logarithmic and monomial nonlinearities. The general case has been treated in the defining works of Shatah-Strauss [ShSt], Weinstein [We1] and [We2], and Grillakis-Shatah-Strauss [GSS1]. The general phenomena that has emerged from their results is that the orbital stability is essentially controlled by the sign of the quantity $\partial_\omega \|\phi_\omega\|_{L^2}$ (stable, if positive, and unstable, if negative). However, all these results addressed orbital stability of a special class of solitons generated by ground states: positive, radial solutions of the equations ENLS.

In [BL] Berestycki-Lions proved the existence of a ground state in three or more dimensions for any $\omega \neq 0$ under the conditions that the nonlinearity $\beta$ verifies $\lim_{s \to +\infty} \beta(s^2) s^{-\frac{4}{n-2}} \geq 0$ and such that there exists $0 < s_0 < \infty$, with $G(s_0) > 0$, for $G(s) \equiv 2 \int_0^s (\beta(\tau^2) \tau - \omega \tau) d\tau$. In fact, in their work ground states are found as minimizers of the constrained variational problem:

$$(1.6) \quad \inf J[u] = \inf \left\{ \int_{\mathbb{R}^n} |\nabla u|^2 : \int_{\mathbb{R}^n} G(u) = 1 \right\}.$$  

The question of uniqueness of a ground state has been studied in [McS], [Kw], [McL].

The asymptotic stability of one ground state soliton solutions of NLS and other equations was first shown for NLS with an extra attractive potential term in [SW1], [SW2], and [PW], for one NLS soliton in dimension one in [BP1] and in dimensions $n \geq 3$ in [Cu]; for NLS-Hartree see [FTY].

While the arguments for orbital stability were essentially based on Lyapunov type analysis and relied only on some limited information about the spectrum of the associated linear problem, the proofs of asymptotic stability required much more detailed properties of the related linearized systems. In particular, it led to the need to impose additional spectral assumptions on the linear operators associated with a soliton.

To describe the problem of linear stability (and spectral theory) we linearize NLS around a soliton
\( w = e^{i\theta} \phi_\omega \), using the ansatz

\[
(1.7) \quad \psi = e^{i\theta} \left( \phi_\omega(x - vt) + R(t, x - vt) \right)
\]

The resulting "linear" operator acting on \( R \), also has a term containing \( R \). After complexifying the space to \( (R, \bar{R}) \) we are left with a matrix non self-adjoint operator of the type

\[
(1.8) \quad \mathcal{H} = \begin{pmatrix} L_+ & W(x) \\ -W(x) & -L_+ \end{pmatrix}
\]

acting on \( L^2 \times L^2 \). Here

\[
L_+ = -\Delta + \omega - \beta(\phi_\omega^2) - \beta'(\phi_\omega^2)\phi_\omega^2, \quad W(x) = -\beta'(\phi_\omega^2)\phi_\omega^2
\]

The operator \( L_+ \) is a self-adjoint perturbation of \( -\Delta \) by an exponentially localized function, and \( W \) is an exponentially localized potential. Moreover, centering the perturbation \( R \) around the soliton as in (1.7) ensures that \( \mathcal{H} \) is time-independent. Note that by non self-adjointness of \( \mathcal{H} \) one can no longer guarantee that \( \|e^{it\mathcal{H}}\|_{2\to 2} < \infty \) where \( U(t) = e^{it\mathcal{H}} \) is given by \( i\partial_t U + \mathcal{H} U = 0, U(0) = \text{Id} \). In fact, the operator \( \mathcal{H} \) has a zero root space \( \mathcal{N} := \bigcup_{\ell \geq 1} \ker \mathcal{H}^\ell \) of dimension at least \( 2n + 2 \) containing the eigenfunction \( \phi_\omega \) as well as the elements generated from \( \phi_\omega \) by infinitesimal symmetries of the problem. We decompose

\[
L^2 \times L^2 = \mathcal{N} + \mathcal{N}^\perp
\]

and let \( P \) denote the projector on the second term in this decomposition (here \( \mathcal{N}^\ast = \bigcup_{\ell \geq 1} \ker (\mathcal{H}^\ast)^\ell \)). It is easy to see that \( \|U(t)f\|_2 \) grows polynomially for some \( f \in \mathcal{N} \). On the other hand, it is known from work of Weinstein [We1] that

\[
(\text{LS}) \quad \sup_t \|U(t)P\psi\|_{L^2} < \infty
\]

under certain conditions on the nonlinearity and provided \( \phi_\omega \) is the (positive) ground state of ENLS. Generally speaking, we refer to the property \( (\text{LS}) \) as linear stability.

In the case when \( \mathcal{H} \) has one negative eigenvalue and \( \phi_\omega \) is the unique ground state, one can show that the condition \( \sigma(\mathcal{H}) \subset \mathbb{R} \) is equivalent to the orbital stability condition \( \partial_\omega \|\phi_\omega\|_{L^2} > 0 \) (see [Gr], [BP1]). Although due to the lack of self-adjointness this is not sufficient for linear stability, additional arguments show that \( (\text{LS}) \) in fact holds just under the above conditions (see [We1], [GSS1]).

Linear stability of ground states has been considered for a large class of NLS in the work of Weinstein [We1], [We2], and Shatah, Strauss, Grillakis, [ShSt], [GSS1], [GSS2], [Gr], see also [SuSu], [Stu]. However, unconditional results were established only in the case of monomial nonlinearities \( \beta(s) = s^p \). Moreover, linear stability has been shown to be essentially equivalent to the orbital stability (except in the case of an \( L^2 \) critical nonlinearity \( \beta(s) = s^{2/n} \)).

Linear stability plays an essential role in the results on asymptotic stability. Moreover, the proofs of asymptotic stability of one-soliton solutions required even more stringent assumptions on the structure.
of the spectrum of $\mathcal{H}$, see Buslaev, Perelman [BP1], and Cuccagna [Cu]. This can be linked to the fact that on the linearized level asymptotic stability requires dispersive estimates of the type

$$\|U(t)P\psi_0\|_{L^\infty} \lesssim t^{-\frac{n}{2}} \|\psi_0\|_{L^1}$$

[Cu] or similar $L^2$-weighted decay estimates. To prove such estimates one needs to impose additional spectral conditions such as: absence of the discrete spectrum for $\mathcal{H}$ on the subspace $N^\perp$, absence of embedded eigenvalues, and absence of resonances at the edges of the continuous spectrum. The dispersive estimates for such matrix Hamiltonians in dimensions $n \geq 3$ were proved by Cuccagna by an extension of the method introduced by Yajima in the scalar case [Ya1]. In that approach, the decay estimates follows as a consequence of the proof of the $L^p \rightarrow L^p$, $\forall p \in [1, \infty]$ boundedness of the wave operators. In our recent work [RSS] we suggested a perhaps more straightforward approach for proving such estimates which instead relies on construction of the analytic extension of the resolvent of $\mathcal{H}$ and goes back to the work of Rauch [Rau] in the scalar case. This method, however, requires that all potential terms in the Hamiltonian $\mathcal{H}$ are exponentially localized functions, which perfectly fits the problem at hand. We refer to Hamiltonians verifying the required spectral assumptions (as well as the linear stability condition) as admissible. The proof of the dispersive estimates crucially relies on the time independence of the Hamiltonian $\mathcal{H}$, which was ensured by the choice of the ansatz.

In our work we choose the initial data of the form (1.3), with $\phi_{\omega_k}$ verifying the elliptic problem ENLS, satisfying the separation condition (1.4) and study the time asymptotic behavior of the corresponding solution of the time-dependent NLS.

As in the study of asymptotic stability of one-soliton solutions the first objective is the analysis of the linearized problem. It is natural to use the following ansatz for the solution:

$$\psi(t,x) = w + R(t,x) = \sum_{k=1}^{N} w_k + R(t,x), \quad w_k(t,x) := e^{i\theta_k} \phi_{\omega_k}(x - \vec{v}_k t - D_k)$$

where $\theta_k$ are the phases associated with $k$-th soliton and $\vec{v}_k, D_k, \omega_k$ are its parameters. We substitute this ansatz into the equation (NLS) and retain only those terms which are linear in $R, \bar{R}$. The resulting linear problem for the unknown $(R, \bar{R})$ is

$$i\partial_t U + \mathcal{H}(t)U = 0.$$ 

It contains a time-dependent complex Hamiltonian

$$\mathcal{H}(t) = \begin{pmatrix} L_+ & W(t,x) \\ -\overline{W(t,x)} & -L_+ \end{pmatrix}$$

where

$$L_+ = -\Delta - \beta(|w|^2) - \beta'(|w|^2)|w|^2, \quad W(t,x) = -\beta(|w|^2)w^2.$$ 

Because of the smallness assumption on the initial perturbation $R_0$ we can assume the the parameters of the final asymptotic profile will lie in a small neighborhood of the initial parameters

\[\text{which in the true ansatz are to be made time-dependent}\]
$(\vec{v}_k(0), \gamma_k(0), D_k(0), \omega_k(0))$. The separation condition on the initial parameters and the exponential localization of functions $\phi_{\omega_k}$ guarantees that $w$ is decomposed into a sum of functions of essentially disjoint support. We thus can replace the Hamiltonian $\mathcal{H}(t)$ with

$$\mathcal{H}(t) = \begin{pmatrix} -\Delta & 0 \\ 0 & \Delta \end{pmatrix} + \sum_{k=1}^{N} \begin{pmatrix} U_k(x - \vec{v}_k t - D_k) & W_k(x - \vec{v}_k t - D_k) \\ -\overline{W_k}(x - \vec{v}_k t - D_k) & -\overline{U_k}(x - \vec{v}_k t - D_k) \end{pmatrix}$$

where

$$U_k = -\beta(\phi_{\omega_k}^2(x)) - \beta'(\phi_{\omega_k}(x))\phi_{\omega_k}^2(x),$$
$$W_k = -e^{2i\theta_k} \beta(\phi_{\omega_k}^2(x))\phi_{\omega_k}^2(x)$$

This Hamiltonian belongs to the class of the so called matrix charge transfer Hamiltonians. Each of the Hamiltonians

$$\mathcal{H}_k(t) = \begin{pmatrix} -\Delta & 0 \\ 0 & \Delta \end{pmatrix} + \begin{pmatrix} U_k(x - \vec{v}_k t - D_k) & W_k(x - \vec{v}_k t - D_k) \\ -\overline{W_k}(x - \vec{v}_k t - D_k) & -\overline{U_k}(x - \vec{v}_k t - D_k) \end{pmatrix} = H_0 + V_k(x - \vec{v}_k t)$$

represents a linearization around the $k$th soliton. Moreover, a specially chosen Galilei transform of the type (1.1) maps a solution of the linear problem $i\partial_t U + \mathcal{H}_k(t)U = 0$ into a solution of the problem $i\partial_t U + \mathcal{H}_kU = 0$ with a time-independent Hamiltonian $\mathcal{H}_k$ of the form (1.8). Here $H_0 = \text{diag}(-\Delta, \Delta)$ and $V_k$ is an exponentially localized complex matrix potential.

Once again, at the linear level, the heart of the problem of asymptotic stability of $N$-solitons are the dispersive estimates for the solutions of the equation (1.11) with a matrix charge transfer Hamiltonian $\mathcal{H}(t)$. Observe that in contrast to the one soliton case, the linearized Hamiltonian is time-dependent.

Due to the separation condition the problem (1.11) admits "traveling" bound states generated by the discrete spectrum of each of the Hamiltonians $\mathcal{H}_k$. These bound states are formed by eigenfunctions and elements of the root space of $\mathcal{H}_k$ boosted by the Galilei transform corresponding to the parameters $\vec{v}_k, D_k$. The paper [RSS] establishes dispersive estimates

$$\|U(t)\psi_0\|_{L^2 + L^\infty} \lesssim (1 + t)^{-\frac{d}{2}} \|\psi_0\|_{L^1 \cap L^2}$$

for the solutions of the linear time-dependent Schrödinger equation (1.11) with a matrix charge transfer Hamiltonian $\mathcal{H}(t)$ of the type (1.12) in dimensions $d \geq 3$. These estimates hold under the assumption that each of the time-independent Hamiltonians $\mathcal{H}_k$ is admissible (requiring spectral assumptions and the linear stability condition) and that the solution $U(t)\psi_0$ is asymptotically orthogonal to all traveling bound states of $\mathcal{H}_k(t)$. The latter means that for all $k = 1, \ldots, N$

$$\|P_k(\mathcal{H}_k, t)U(t)\psi_0\|_{L^2} \rightarrow 0, \quad \text{as} \quad t \rightarrow +\infty$$

with $P_k(\mathcal{H}_k, t)$ denoting the time-dependent projection on the $k$-th subspace of traveling bound states, i.e., it is the conjugation of the spectral projection of the stationary operator $\mathcal{H}_k$ onto its bound states by suitable Galilei transforms. The estimate (1.13) is our main linear estimate. However, to ensure that the perturbation $R(t, x)$ in the decomposition (1.10) is asymptotically orthogonal to the subspace
of traveling bound states and thus decays in the linear approximation with the rate of $t^{-\frac{n}{2}}$, we need to allow the soliton parameters $\sigma_k = (\bar{v}_k, D_k, \gamma_k, \omega_k)$ to become time-dependent. This in turn makes it necessary to change the form of $w_k(t, x)$, for example, $x - \bar{v}_k t - D_k$ becomes

$$x - \int_0^t \bar{v}_k(s) \, ds - D_k(t).$$

The resulting nonlinear problem for $R$, after the complexification $Z = (R, \bar{R})$, takes the form

$$(1.14) \quad i\partial_t Z + H(t, \sigma(t))Z = \dot{\sigma} \cdot \partial_\sigma w + N(Z, w),$$

where $H(t, \sigma(t))$ is a time-dependent Hamiltonian. It is of the matrix charge transfer type provided $\sigma(t) = \text{const.}$ The term $\partial_\sigma w$ denotes the derivative of the solitary approximation $w(t, x; \sigma(t)) = \sum_{k=1}^N w_k(t, x; \sigma(t))$ with respect to its parameters $\sigma_k$, the term $\dot{\sigma}$ denotes the time derivative of the soliton parameters $\sigma_k$, and $N(Z, w)$ is a nonlinear term in $Z$. We introduce the notion of an admissible path $\sigma(t)$ in the space of parameters and a reference Hamiltonian $H(t, \sigma)$ at infinity corresponding to the matrix charge transfer Hamiltonian with fixed constants $\sigma_k = \sigma_k(t = \infty) = (\bar{v}_k, D_k, \gamma_k, \omega_k)$. At each time $t$ the solution $Z(t)$ is required to be orthogonal to the traveling bound states of the charge transfer Hamiltonians $H(t, \sigma = \sigma(t))$ obtained by fixing the parameters $\sigma_k = \sigma_k(t)$ at a given time $t$. This leads to the so called modulation equations for $\sigma_k$, which couple the PDE (1.14) for $Z$ with an ODE for the modulation parameters $\sigma_k(t)$. To impose the orthogonality condition we first need to verify that it is satisfied initially. Using standard arguments, see e.g. [BP1], one can ensure this property by modifying the soliton parameters $\sigma(0)$ slightly in the decomposition of the initial data $\psi_0$,

$$\psi_0(x) = \sum_{j=1}^N w_j(0, x; \sigma(0)) + R(0, x).$$

We later justify the orthogonality condition at any positive time by showing that it is propagated. To handle the nonlinear equation (1.14) we introduce the Banach spaces $\mathcal{X}_s$ and $\mathcal{Y}_s$ of functions of $(t,x)$

$$\|f\|_{\mathcal{X}_s} = \sup_{t \geq 0} \left( \|\psi(t, \cdot)\|_{H^s} + (1 + t)^{\frac{n}{2}} \sum_{k=0}^s \|\nabla^k f(t, \cdot)\|_{L^2 + L^\infty} \right)$$

$$\|F\|_{\mathcal{Y}_s} = \sup_{t \geq 0} \sum_{k=0}^s \left( \int_0^t \|\nabla^k F(\tau, \cdot)\|_{L^2} \, d\tau + (1 + t)^{\frac{n+1}{2}} \|\nabla^k F(t, \cdot)\|_{L^2} \right).$$

(1.15)

The space $\mathcal{X}_s$ is designed to control the solution $Z(t)$ itself, while the space $\mathcal{Y}_s$ takes care of the nonlinear terms appearing as inhomogeneous terms in the equation for $Z$. We rewrite the equation (1.14) replacing the Hamiltonian $H(t, \sigma(t))$ with the reference charge transfer Hamiltonian $H(t, \sigma)$. The solution operator of the corresponding linear problem

$$(1.16) \quad i\partial_t U + H(t, \sigma)U = F$$

maps $\mathcal{Y}_s \to \mathcal{X}_s$ for any integer $s$ uniformly in $\sigma$, provided that the solution is asymptotically orthogonal to all traveling bound state of $H(t, \sigma)$. We then show that the nonlinearity $F$ arising from the
inhomogeneous terms in the equation $Z$, written relative to the reference Hamiltonian $H(t, \sigma)$, maps $X_s \rightarrow Y_s$ for any $s > \left[ \frac{n}{2} \right] + 1$. Given the smallness assumption on the initial data for $Z = (R, \bar{R})$ this allows one to conclude the desired properties of $R$. The modulation equations for $\sigma(t)$ are then used in turn to control the path $\sigma(t)$. In particular, we show that there exist final values of the parameters $\sigma = \sigma(\infty)$, thus justifying the introduction of the reference Hamiltonian at infinity. The estimates for the inhomogeneous problem (1.16) phrased in terms of the mapping between $Y_s$ and $X_s$ are essentially the linear dispersive estimates for the time-dependent Schrödinger equation with a matrix charge transfer Hamiltonian $\mathcal{H}(t) = H(t, \sigma)$ proved in [RSS]. As it was mentioned before, these estimates only require the admissibility of the corresponding individual time-independent matrix Hamiltonians $\mathcal{H}_k$ (representing the linearization around each individual soliton). The admissibility conditions can be somewhat loosely divided into two categories:

1) Conditions related to linear stability: $\sup_t \| e^{it\mathcal{H}_k} f \|_{L^2_{-2}} < \infty$

2) Spectral assumptions on $\mathcal{H}_k$ (absence of the embedded eigenvalues, resonances, etc.).

Remark 1.1. We also require the absence of "spurious" eigenvalues. This means that we assume that all of the discrete spectrum of $\mathcal{H}_k$ is generated purely by the nonlinear eigenfunction $\phi_{\omega_k}$, i.e., it is described by the generalized 0 eigenspace space of $\mathcal{H}_k$ which has the precise dimension $(2n + 2)$ (coinciding with the dimension of the parameter space of $\sigma_k$). This is motivated by the requirement that the solution $Z$ has to be orthogonal to all traveling bound states of $\mathcal{H}_k$ for the dispersive estimates to hold, which can only be achieved by the choice of the parameters $\sigma_k$. In general, it is believed that the states corresponding to the spurious eigenvalues decay in time but the mechanism of this decay is purely nonlinear. We do not pursue this issue here.

The key ingredient in establishing the linear stability is known to be the monotonicity or convexity condition

\begin{equation}
(1.17) \quad \partial_{\omega} \| \phi_{\omega} \|_{L^2} > 0 \quad \text{or} \quad \langle L_+^{-1} \phi_{\omega}, \phi_{\omega} \rangle > 0.
\end{equation}

The known examples of when the monotonicity condition can be verified are limited to the case of the ground states $\phi$ corresponding to the monomial subcritical nonlinearities

\begin{equation}
(1.18) \quad \beta(s^2) = s^{p-1}, \quad 1 < p < 1 + \frac{4}{n}, \quad \forall \omega \neq 0
\end{equation}

and the nonlinearity of the mixed type (see [Sh])

\[ \beta(s^2) = s^2 - s^4 \]

for the values of $\omega$ close to $3/16$. In this paper we find a new class of nonlinearities satisfying condition (1.17). These nonlinearities lie "near" the subcritical monomials of (1.18) but vanish much faster near $s = 0$. More precisely we consider functions

\begin{equation}
(1.19) \quad \beta_\theta(s^2) = s^{p-1} \frac{f(s^2)}{\theta + f(s^2)}
\end{equation}

with a constant $\theta > 0$ and the function $f$ satisfying the conditions

\begin{equation}
(1.20) \quad C_1 s^{r+1} \leq |f(s^2)| \leq C_2 s^{r+1}, \quad s^2 |f'(s^2)| \leq C_2 |f(s^2)|, \quad p \in (1, 1 + \frac{4}{n}), \ r > -1
\end{equation}

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and prove that given a sufficiently small neighborhood \( U \) in the space of parameters \( \omega \) there exists a sufficiently small value \( \theta_0 \) such that for all \( \theta < \theta_0 \) and all \( \omega \in U \) the ground state of \( \beta_\theta \) corresponding to \( \alpha \) satisfies the monotonicity condition.

We note that the higher rate of vanishing of \( \beta(s^2) \) at \( s = 0 \) is important for asymptotic stability. In particular, it should be mentioned that if the power \( p \) in the monomial example is too low (\( p < 1 + \frac{2}{n} \)) even the scattering theory (asymptotic stability of a trivial 0 solution) fails.

We now describe the structure of the paper.

**Section 2.** contains the statement of the main result together with the definitions of some of the fundamental objects used in the proof. The latter include the definition of an ansatz \( \psi = w_\sigma + R \), separation condition on the initial data ensuring that the solitons are only weakly interacting, the notion of an admissible parameter path \( \sigma(t) \), and the spectral assumptions. It also contains the first discussion of the linearized Hamiltonians appearing in the later sections. We note here that traditionally the results on the asymptotic stability require the smallness assumption on the initial data in weighted Sobolev space. Our result uses instead the Sobolev space based on the intersection of \( L^1 \cap L^2(\mathbb{R}^n) \), which has a distinct advantage of being translation invariant.

**Section 3.** gives a detailed description of the linearization of the equation (1.11) around an \( N \)-soliton profile \( w_\sigma \) and introduces the notion of the reference charge transfer Hamiltonian at infinity.

**Section 4.** describes the structure of the nullspaces of the Hamiltonians \( H_j(\sigma) \), associated with the linearization on each individual nonlinear eigenfunction \( \phi_j \).

**Section 5.** recalls the dispersive estimates for solutions of the time-dependent Schrödinger equation with a charge transfer Hamiltonian.

**Section 6.** derives a system of ODE’s for the modulation parameters \( \sigma(t) = (\sigma_1, \ldots, \sigma_N) \) with \( \sigma_k = (\vec{v}_k, D_k, \gamma_k, \alpha_k) \) by requiring the complexified perturbation \( Z = (R, \bar{R}) \) to be orthogonal to the the unstable manifold comprised of the elements of the nullspaces of \( H_j(\sigma(t)) \).

**Section 7.** gives the bootstrap assumption on the size of the perturbation \( Z \) and the admissible path \( \sigma(t) \) and provides estimates on the difference between the linearized Hamiltonian \( H(\sigma(t)) \) and the reference Hamiltonian \( H(\sigma, t) \) at infinity.

**Section 8.** provides the solution of the modulation equations for the path \( \sigma(t) \).

**Section 9.** solves the nonlinear equation for the complexified perturbation \( Z \) in the space \( \mathcal{X}_s \) for \( s > \lceil \frac{n}{2} \rceil + 1 \). This includes algebra estimates designated to show that the \( \mathcal{Y}_s \)-norm of the nonlinear terms in the equation for \( Z \) can be controlled by the \( \mathcal{X}_s \)-norm of \( Z \) itself.

**Section 10.** proves the scattering result showing that as \( t \to \infty \) the solution \( \psi(t) \) decomposes into the sum of solitons and a solution of the free linear Schrödinger equation.

**Section 11.** discusses the existence for the coupled PDE-ODE system for \( Z \) and \( \sigma(t) \).

**Section 12.** returns to the detailed discussion of the associated linear problems and proves some of the assertions made in the first part of the paper. We give the precise definition of an admissible Hamiltonian and start the investigation of their spectral properties. In particular, in Section 11.2 we describe the spectrum of an admissible Hamiltonian and prove exponential decay of the elements of its generalized eigenspaces. In Section 11.3 we specialize to the admissible Hamiltonians arising from linearization around a nonlinear eigenfunction \( \phi \). We introduce and discuss the associated self-adjoint
operators $L_+$ and $L_-$ and show that the admissibility conditions on our Hamiltonian (excluding the assumptions on absence of the embedded spectrum and resonances) can be reduced to the monotonicity condition (1.17), and the statement that the null space of $L_-$ is spanned by $\phi$ while $L_+$ has a unique negative eigenvalue with the corresponding null space spanned by $\partial_{x_j}\phi$. Some of the arguments in this section follow those of [We1], [BP1].

Section 13. shows that for a very large class of nonlinearities the operator $L_+$, obtained by linearizing on a ground state, has a unique negative eigenvalue.

Section 14. establishes the desired properties of the operators $L_+$ and $L_-$ for a particular class of nonlinearities (namely those used in our main result) in the case when the nonlinear eigenfunction $\phi$ is a ground state.

For the most part of the paper (until Section 12) we do not specify the nature of nonlinear eigenstates $\phi_{\omega_k}$ of the elliptic problem (ENLS). In particular, we do not require them to be ground states. Instead, we choose to formulate a more general conditional result dependent upon verification of certain precise properties of the linearized operators $H_k$ associated with each $\phi_{\omega_k}$. It is only in Section 12 that we verify some of these assumptions in the case when $\phi_{\omega_k}$ are ground states. The reason for choosing this approach is to emphasize the method which allows us to handle weak interactions of (non-colliding) solitons generated by $\phi_{\omega_k}$, provided that certain properties of each individual eigenstate $\phi_{\omega_k}$ hold. We believe that our method will have an even wider range of applications than described here.

2 Statement of results

Consider the NLS

\[
i\partial_t\psi + \frac{1}{2}\Delta\psi + \beta(|\psi|^2)\psi = 0
\]

in $\mathbb{R}^n$, $n \geq 3$, with initial data

\[
\psi_0(x) = \sum_{j=1}^N w_j(0, x) + R_0(x).
\]

Here $w_j(0, x)$ are nonlinear eigenfunctions generating the solitons

\[
w_j(t, x) = w(t, x; \sigma_j(0)) = e^{i\theta_j(t, x)}\phi(x - x_j(t), \alpha_j(0))
\]

\[
\theta_j(t, x) = v_j(0) \cdot x - \frac{1}{2}(|v_j(0)|^2 - \alpha_j^2(0))t + \gamma_j(0)
\]

\[
x_j(t) = v_j(0)t + D_j(0).
\]

and $\phi = \phi(\cdot, \alpha)$ is a solution of

\[
\frac{1}{2}\Delta\phi - \frac{\alpha^2}{2}\phi + \beta(|\phi|^2)\phi = 0.
\]
The solitons $w_j$ as in (2.3) satisfy (2.1) with arbitrary constant parameters $\sigma_j(0) = (v_j(0), D_j(0), \gamma_j(0), \alpha_j(0))$. We assume that the nonlinearity $\beta$ satisfies for all integers $\ell \geq 0$

\begin{align}
|\beta(\ell)(s)| & \lesssim s^{\frac{p-1}{2}-\ell} + \text{ for } 0 \leq s \leq 1 \\
|\beta(\ell)(s)| & \lesssim s^{\frac{q-1}{2}-\ell} + \text{ for } s \geq 1
\end{align}

where $p \geq 2 + \frac{2}{n}$, $q < 1 + \frac{4}{n}$. The main results of our paper are the following theorems.

**Theorem 2.1.** Let $\beta$ be as in (1.19), (1.20) with a sufficiently small value of parameter $\theta > 0$. In addition, assume that $\beta(s^2)$ is a smooth function of $s$ and that $p + r \geq 1 + \frac{2}{n}$. Let

$$
\psi_0(x) = \sum_{j=1}^{N} w_j(0,x) + R_0(x),
$$

see (2.2) be initial data for NLS that satisfy the separation condition (2.17). Moreover, assume that $w_j$ are defined in terms of ground states (positive radial variational solutions) of ENLS. Finally, suppose that for all $\sigma \in \mathbb{R}^{N(2n+2)}$ with $|\sigma - \sigma(0)| < c$, the linearized operators $H_j(\sigma)$ from (2.24) have zero as their only eigenvalue, no resonances, and no embedded eigenvalues in their continuous spectrum. Then there exists a positive $\epsilon$ such that for $R_0$ satisfying the smallness assumption

$$
\sum_{k=0}^{s} \|\nabla^k R_0\|_{L^1 \cap L^2} < \epsilon
$$

for some integer $s > \frac{n}{2}$, there exists an admissible path $\sigma(t)$ with the limiting value $\sigma^\infty$ such that

$$
\left\| \psi(t,x) - \sum_{j=1}^{N} w_j(t,x;\sigma_j(t)) \right\|_{L^\infty_x} \lesssim (1 + t)^{-\frac{n}{2}}
$$

as $t \to \infty$. Moreover, there exists $u_0 \in L^2$ so that

$$
\left\| \psi(t,\cdot) - \sum_{j=1}^{N} w_j(t,x;\sigma^\infty) - e^{it\Delta} u_0 \right\|_{L^2} \to 0
$$

as $t \to \infty$.

**Remark 2.2.** We assume that $\alpha \neq 0$. The smallness of the parameter $\theta$ in the nonlinearity $\beta$ depends on the values of $\sigma(0)$.

Our methods also allow treating initial conditions that are defined in terms of nonlinear eigenfunctions which are not ground states as well as more general nonlinearities $\beta$. However, in contrast to the case of a ground state and the nonlinearities (1.19), in this case we cannot verify the convexity conditions as well as various conditions related to the spectrum of the linearized operators $H_j(\sigma)$. Therefore, we need to include them into the hypotheses.
Theorem 2.3. Let $\beta$ be as in (2.7) and (2.8). Impose the separation and convexity condition see (2.17) and (2.16), the spectral assumption from Definition 2.4. Suppose $\psi$ is the solution of (2.1) with initial condition (2.2) where the $w_j$ are generated by nonlinear eigenfunctions of ENLS (without assuming that they are ground states). Then there exists a positive $\epsilon$ such that for $R_0$ satisfying the smallness assumption (2.9) the conclusion of the previous theorem holds.

In the remainder of this section we shall discuss the theorem and its assumptions in more detail.

The soliton profiles $w_j(t,x;\sigma(t))$: In addition to the one-soliton solutions $w_j(t,x;\sigma_j)$ with constant $\sigma_j$ introduced above, we need functions

$$w_j(t,x;\sigma_j(t)) = e^{i\theta_j(t,x;\sigma_j(t))}\phi(x-x_j(t;\sigma(t)),\alpha_j(t)).$$

The phase $\theta_j(t,x;\sigma(t))$ and the path $x_j(t;\sigma(t))$ are defined in terms of the time-dependent parameters $\sigma_j(t) = (v_j(t),D_j(t),\gamma_j(t),\alpha_j(t))$ as follows:

$$\theta_j(t,x;\sigma(t)) = v_j(t) \cdot x - \int_0^t \frac{1}{2}(|v_j|^2 - \alpha_j^2)(s) \, ds + \gamma_j(t)$$

$$x_j(t;\sigma(t)) = \int_0^t v_j(s) \, ds + D_j(t).$$

Henceforth, the functions $w_j(t,x;\sigma(t))$ or simply $w_j(\sigma(t))$, correspond to the soliton moving along the time-dependent curve $\sigma(t)$ in the parameter space according to (2.11) and (2.12), while $w_j(\sigma)$ is the true soliton moving along the straight line determined by an arbitrary constant $\sigma$ as in (2.4) and (2.5).

Admissible paths $\sigma(t)$: We collect the individual parameter curves $\sigma_j(t)$ from above into a single curve $\sigma(t) := (\sigma_1(t),\ldots,\sigma_N(t)) \subset \mathbb{R}^{(2n+2)N}$. Given the initial value $\sigma(0)$ we introduce the set of admissible curves $\sigma(t)$ as those $C^1$ curves that remain in a small neighborhood of $\sigma(0)$ for all times and converge to their final value $\sigma(\infty) = \lim_{t \to +\infty} \sigma(t)$. We shall also impose the condition that for an admissible curve $\sigma(t)$

$$\int_0^\infty \int_s^\infty |\dot{\nu}_j(\tau) \cdot v_j(\tau) - \dot{\alpha}_j(\tau)\alpha_j(\tau)| \, d\tau \, ds < \infty, \quad \int_0^\infty \int_s^\infty |\dot{\nu}_j(\tau)| \, d\tau \, ds < \infty$$

for all $1 \leq j \leq N$. Given an admissible curve $\sigma(t)$ we define the constant vector $\sigma^\infty$ in the following fashion:

$$v_j^\infty = v_j(\infty), \quad D_j^\infty = D_j(\infty) - \int_0^\infty \int_s^\infty \dot{v}_j(\tau) \, d\tau \, ds,$$

$$\gamma_j^\infty = \gamma_j(\infty) + \int_0^\infty \int_s^\infty (\dot{\nu}_j(\tau) \cdot v_j(\tau) - \dot{\alpha}_j(\tau)\alpha_j(\tau)) \, d\tau \, ds, \quad \alpha_j^\infty = \alpha_j(\infty).$$

Convexity condition: We impose the convexity condition

$$(2.16) \quad \langle \partial_\alpha \phi(;\alpha),\phi(;\alpha) \rangle > 0, \quad \forall \alpha : \min_{j=1,\ldots,N} |\alpha - \alpha_j(0)| < c$$

for some positive constant $c$. As noted in the Introduction, the convexity condition is closely connected with the issue of orbital stability of the individual solitons $w_j(t,x;\sigma_j(0))$. 

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Separation conditions: Our theorem handles the case of so-called weakly interacting solitons. This means that the initial positions $D_j(0)$ and initial velocities $v_j(0)$ are such that for all $t \geq 0$ one has the (physical) separation condition

$$
|D_j(0) + v_j(0)t - D_\ell(0) - v_\ell(0)t| \geq L + ct, \quad \forall j \neq \ell = 1, \ldots, N
$$

with some sufficiently large constant $L$ and a positive constant $c$. Another assumption under which our theorems hold equally well is the condition of large relative velocities of the solitons. This means that

$$
\min_{j \neq \ell} |v_j - v_\ell| > L
$$

for some large $L$.

Let $\alpha_{\text{min}} = \min_{1 \leq j \leq k} \alpha_j(0) - c$. It will be understood henceforth that

$$
\alpha_{\text{min}}L \geq |\log \epsilon|.
$$

The small constant $\epsilon$ appears in our theorem as a measure of smallness of the initial perturbation $R_0$.

Spectral assumptions: We will write $w_j(\sigma(t)) = w_j(t, x; \sigma(t)) = e^{i\theta_j(\sigma(t))} \phi_j(\sigma(t))$ where

$$
\phi_j(\sigma(t)) = \phi(t, x; \sigma_j(t)) = \phi(x - x_j(t); \alpha_j(t)).
$$

Linearizing the equation (2.1) around the state $w = \sum_{j=1}^N w_j$, $\psi = w + R$ one obtains the following system of equations for $Z = (R/R)$:

$$
i\partial_t Z + H(t, \sigma(t))Z = F.
$$

Here $H(t, \sigma(t))$ is the time-dependent matrix Hamiltonian

$$
H(t, \sigma(t)) = H_0 +
$$

$$
\sum_{j=1}^N \left( \begin{array}{cc}
\beta(|w_j(\sigma(t))|^2) + \beta'(|w_j(\sigma(t))|^2)|w_j(\sigma(t))|^2 & \beta'(|w_j(\sigma(t))|^2)w_j^2(\sigma(t)) \\
-\beta'(|w_j(\sigma(t))|^2)w_j^2(\sigma(t)) & -\beta(|w_j(\sigma(t))|^2) - \beta'(|w_j(\sigma(t))|^2)|w_j(\sigma(t))|^2
\end{array} \right)
$$

$$
= H_0 + \sum_{j=1}^N V_j(t,x;\sigma(t)),
$$

$$
H_0 = \begin{pmatrix}
\frac{1}{2}\Delta & 0 \\
0 & -\frac{1}{2}\Delta
\end{pmatrix}
$$

with complex matrix time-dependent potentials $V_j(t, x; \sigma(t))$ dependent on $w_j$ and $\sigma(t)$. The right-hand side $F$ in (2.20) depends on $\sigma$, $w$, and nonlinearly on $Z$. For a given constant parameter vector $\sigma$ we shall introduce the Hamiltonian $H(t, \sigma)$

$$
H(t, \sigma) = H_0 +
$$

$$
\sum_{j=1}^N \left( \begin{array}{cc}
\beta(|w_j(\sigma)|^2) + \beta'(|w_j(\sigma)|^2)|w_j(\sigma)|^2 & \beta'(|w_j(\sigma)|^2)w_j^2(\sigma) \\
-\beta'(|w_j(\sigma)|^2)w_j^2(\sigma) & -\beta(|w_j(\sigma)|^2) - \beta'(|w_j(\sigma)|^2)|w_j(\sigma)|^2
\end{array} \right).
$$

We refer to Hamiltonians of the form (2.23) as matrix charge transfer Hamiltonians. They are discussed in more detail in Section 12, as well as in [RSS]. Recall that $w_j(\sigma)$ denotes the soliton moving along the
straight line determined by the constant parameters $\sigma_j$. The proof of our theorem relies on dispersive estimates for matrix charge transfer Hamiltonians that were obtained in [RSS], see also Section 12 below. For these estimates to hold, one needs to impose certain spectral conditions on the stationary Hamiltonians (2.24)

$$H_j(\sigma) := \left( \frac{1}{2} \Delta - \frac{\alpha_j^2}{\sigma_j^2} + \beta(\phi_j(\sigma)^2) + \beta'(\phi_j(\sigma)^2)\phi_j(\sigma)^2 ight.$$ \left. - \beta'(\phi_j(\sigma)^2)\phi_j(\sigma) \right) $$

where $\phi_j(\sigma) = \phi(x, \alpha_j)$, see (2.6). These Hamiltonians arise from the matrix charge transfer problem by applying a Galilei transform to the $j^{th}$ matrix potential in (2.23) so that this potential becomes stationary (strictly speaking, this also requires a modulation which leads to the spectral shift $\frac{\alpha_j^2}{\sigma_j^2}$ in (2.24)). We impose the spectral assumption as described by the following definition.

**Definition 2.4.** We say that the spectral assumption holds, provided for all $\sigma \in \mathbb{R}^{N(2n+2)}$ with $|\sigma - \sigma(0)| < c$ one has

- 0 is the only point of the discrete spectrum of $H_j(\sigma)$ and the dimension of the corresponding root space is $2n + 2$,
- each of the $H_j(\sigma)$ is admissible in the sense of Definition 12.1 below and the stability condition

$$\sup_t \| e^{itH_j(\sigma)} P_\sigma \|_{2 \to 2} < \infty,$$

see (12.3), holds (here $P_\sigma$ is the projection onto the scattering states associated with $H_j$, see (12.2)).

While the second condition is known to hold generically in an appropriate sense, see Section 12, the first condition is more restrictive and not believed to hold generically.

### 3 Reduction to the matrix charge transfer model

For the sake of simplicity we consider the case of two solitons, i.e., $N = 2$. Setting

$$w_1(\sigma(t)) + w_2(\sigma(t)) = w, \quad \psi = w + R,$$

where $w_j$ are as in (2.10), (2.11), and (2.12), we derive from (2.1) that

$$i\partial_t R + \frac{1}{2} \Delta R + (\beta(|w|^2) + \beta'(|w|^2)|w|^2)R + \beta'(|w|^2)w^2 R$$

$$= -(i\partial_t w + \frac{1}{2} \Delta w + \beta(|w|^2)w) + O(|w|^{p-2}|R|^2) + O(|R|^p),$$

using the assumptions (2.7), (2.8) on the nonlinearity $\beta$. Observe that

$$i\partial_t w + \frac{1}{2} \Delta w + \beta(|w|^2)w = -\sum_{j=1}^{2} \left[ (i\dot{v}_j(t) \cdot x + \dot{\gamma}_j(t)) w_j(\sigma(t)) + ie^{i\theta_j(\sigma(t))} \nabla \phi_j(\sigma(t)) \cdot \dot{D}_j(t) \right.$$

$$\left. - ie^{i\theta_j(\sigma(t))} \partial_\sigma \phi_j(\sigma(t)) \dot{\alpha}_j(t) \right] + O(w_1 w_2).$$

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In view of (3.1) one has
\[ i\partial_t R + \frac{1}{2}\Delta R + (\beta(|w|^2) + \beta'(|w|^2)|w|^2)R + \beta'(|w|^2)w^2 R \]
\[ = i\partial_t R + \frac{1}{2}\Delta R + \sum_{j=1}^{2} \left[ \beta(|w_j(\sigma(t))|^2) + \beta'(|w_j(\sigma(t))|^2)|w_j(\sigma(t))|^2 \right] R + \sum_{j=1}^{2} \beta'(|w_j(\sigma(t))|^2)w_j(\sigma(t))^2 R \]
\[ + O(w_1(\sigma(t))w_2(\sigma(t)))R. \]

Rewriting the equation (3.2) as a system for \( Z = (R, \bar{R}) \) therefore leads to
\[ (3.4) \quad i\partial_t Z + H(\sigma(t))Z = \dot{\Sigma}W(\sigma(t)) + O(w_1w_2)Z + O(w_1w_2) + O(|w|^p-2|Z|^2) + O(|Z|^p). \]

Here \( H(\sigma(t)) \) is the time-dependent matrix Hamiltonian from (2.21) and
\[ (3.5) \quad \dot{\Sigma}W(\sigma(t)) = \begin{pmatrix} f \\ -\bar{f} \end{pmatrix} \]

where
\[ (3.6) \quad f = \sum_{j=1}^{2} \left[ (\dot{v}_j(t) \cdot x + \dot{\gamma}_j(t))w_j(\sigma(t)) + ie^{i\theta_j(\sigma(t))}\nabla \phi_j(\sigma(t)) \cdot D_j(t) - ie^{i\theta_j(\sigma(t))}\partial_\alpha \phi_j(\sigma(t))\dot{\alpha}_j(t) \right] \]

We shall assume that \( \sigma(t) \) is an admissible path with initial values \( \sigma(0) \) in the sense of (2.13). Given an admissible curve \( \sigma(t) \) we introduce the reference Hamiltonian \( H(t, \sigma^\infty) \) “at infinity”
\[ (3.7) \quad H(t, \sigma^\infty) = H_0 + \sum_{j=1}^{2} \begin{pmatrix} \beta(|w_j(\sigma^\infty)|^2) + \beta'(|w_j(\sigma^\infty)|^2)|w_j(\sigma^\infty)|^2 & \beta'(|w_j(\sigma^\infty)|^2)w_j^2(\sigma^\infty) \\ -\beta'(|w_j(\sigma^\infty)|^2)\bar{w}_j^2(\sigma^\infty) & -\beta(|w_j(\sigma^\infty)|^2) - \beta'(|w_j(\sigma^\infty)|^2)|w_j(\sigma^\infty)|^2 \end{pmatrix} \]

where \( \sigma^\infty = (\sigma_1^\infty, \ldots, \sigma_N^\infty), \quad \sigma_j^\infty = (v_j^\infty, D_j^\infty, \gamma_j^\infty, \alpha_j^\infty) \) is the constant vector determined by the curve \( \sigma(t) \) as in (2.14) and (2.15):
\[ v_j^\infty = v_j(\infty), \quad D_j^\infty = D_j(\infty) - \int_0^\infty \int_s^\infty \dot{v}_j(\tau) \, d\tau \, ds, \]
\[ \gamma_j^\infty = \gamma_j(\infty) + \int_0^\infty \int_s^\infty (\dot{v}_j(\tau) \cdot v_j(\tau) - \dot{\alpha}_j(\tau)\alpha_j(\tau)) \, d\tau \, ds, \quad \alpha_j^\infty = \alpha_j(\infty). \]

Recall that \( w_j(\sigma^\infty) \) is the soliton moving along the straight line determined by the constant parameters \( \sigma_j^\infty \). For \( j = 1, \ldots, N \) we introduce the Hamiltonians
\[ (3.8) \quad H_j(t, \sigma^\infty) = H_0 + \begin{pmatrix} \beta(|w_j(\sigma^\infty)|^2) + \beta'(|w_j(\sigma^\infty)|^2)|w_j(\sigma^\infty)|^2 & \beta'(|w_j(\sigma^\infty)|^2)w_j^2(\sigma^\infty) \\ -\beta'(|w_j(\sigma^\infty)|^2)\bar{w}_j^2(\sigma^\infty) & -\beta(|w_j(\sigma^\infty)|^2) - \beta'(|w_j(\sigma^\infty)|^2)|w_j(\sigma^\infty)|^2 \end{pmatrix} \]

together with their stationary counterparts \( H_j(\sigma^\infty) \) as in (2.24) with \( \sigma = \sigma^\infty \).

The following lemma relates the evolutions corresponding to the Hamiltonians \( H_j(t, \sigma) \) and \( H_j(\sigma) \) for an arbitrary \( \sigma \) by means of a modified Galilean transformation.
Lemma 3.1. Let $U_j(t, \sigma)$ be the solution operator of the equation

$$i\partial_t U_j(t, \sigma) + H_j(t, \sigma)U_j(t, \sigma) = 0,$$

and $e^{iH_j(\sigma)}$ be the corresponding propagator for the time-independent matrix Hamiltonian $H_j(\sigma)$. Then

$$U_j(t, \sigma) = G_{v_j, D_j}^*(t)M_j^*(t, \sigma)e^{iH_j(\sigma)}M_j(0)G_{v_j, D_j}(0),$$

where $G_{v_j, D_j}(t)$ is the diagonal matrix Galilean transformation.

Proof. By definition,

$$i\dot{U}_j = i\dot{G}_{v_j, D_j}^*(t)M_j^*(t)e^{iH_j(\sigma)}M_j(0)G_{v_j, D_j}(0) + G_{v_j, D_j}^*(t)i\dot{M}_j^*(t)e^{iH_j(\sigma)}M_j(0)G_{v_j, D_j}(0).$$

Clearly,

$$i\dot{M}_j^*(t) = \left(\begin{array}{cc}
-\alpha^2/2 e^{i\alpha^2/2 t - i(v_j \cdot D_j + \gamma_j)} & 0 \\
0 & \alpha^2/2 e^{-i\alpha^2/2 t + i(v_j \cdot D_j + \gamma_j)}
\end{array}\right),$$

whereas one checks that

$$i\dot{G}_{v_j, D_j}^*(t) \left(\begin{array}{c} f_1 \\ f_2 \end{array}\right) = i \left(\begin{array}{c} \dot{G}_{v_j, D_j}^* f_1 \\ \dot{G}_{v_j, D_j}^* f_2 \end{array}\right) = \left(\begin{array}{c} -(v_j^2/2 - v_j \cdot p) g_{v_j, D_j}^* f_1 \\ (v_j^2/2 + v_j \cdot p) g_{v_j, D_j}^* f_2 \end{array}\right).$$

Finally, we need to move $H_j(\sigma)$ to the left in (3.13). We consider the differential operator separately from the matrix potential, i.e.,

$$H_j(t, \sigma) = H_0 + \left(\begin{array}{cc}
U_j(x - x_j(t)) & e^{2i\theta_j(t, x)}W_j(x - x_j(t)) \\
e^{-2i\theta_j(t, x)}W_j(x - x_j(t)) & -U_j(x - x_j(t))
\end{array}\right),$$

$$H_j(\sigma) = H_0 + \left(\begin{array}{cc}
\alpha^2/2 & 0 \\
0 & \alpha^2/2
\end{array}\right) + \left(\begin{array}{cc}
U_j & W_j \\
-W_j & -U_j
\end{array}\right).$$

where $x_j(t), \theta_j(t)$ are as in (2.5), (2.4), and $U_j = \beta(\phi_j(\sigma)^2) + \beta'(\phi_j(\sigma)^2)\phi_j(\sigma)^2$, $W_j = \beta'(\phi_j(\sigma)^2)\phi_j^2(\sigma)$. Note that, on the one hand, $M_j$ commutes with all matrices in (3.14) that do not involve $U_j, W_j$. On
the other hand, one has

\[ H_0 G_{v_j}^* (t) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} - G_{v_j}^* (t) \begin{pmatrix} H_0 + \left( -\frac{\alpha^2}{2} & 0 \\ 0 & -\frac{\alpha^2}{2} \right) \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \alpha^2 \sigma v_j, D (t) f_1 \\ -\frac{1}{2} \alpha^2 \sigma v_j, D (t) f_2 \end{pmatrix}. \]

Finally, we need to deal with the matrix potentials. Write \( \mathcal{M}(t) := \mathcal{M}(t, \sigma) = \begin{pmatrix} e^{-i\omega(t)/2} & 0 \\ 0 & e^{i\omega(t)/2} \end{pmatrix} \) and set \( \rho = t |\vec{v}_j|^2 + 2x \cdot \vec{v}_j \). Then (omitting the index \( j \) for simplicity)

\[ \mathcal{M}(t) G_{\vec{v}, D}(t) \begin{pmatrix} U(\cdot, -\vec{v}t - D) & e^{2i\theta} W(\cdot, -\vec{v}t - D) \\ -e^{-2i\theta} W(\cdot, -\vec{v}t - D) & -U(\cdot, -\vec{v}t - D) \end{pmatrix} = \begin{pmatrix} e^{-i\omega(t)/2} & 0 \\ 0 & e^{i\omega(t)/2} \end{pmatrix} \left( \begin{array}{c} \frac{\sigma v_j, D(t) U(\cdot, -\vec{v}t - D) f_1}{\sigma \sigma v_j, D(t) e^{2i\theta} W(\cdot, -\vec{v}t - D) f_2} \\ -\frac{\sigma v_j, D(t) e^{-2i\theta} W(\cdot, -\vec{v}t - D) f_1}{\sigma \sigma v_j, D(t) U(\cdot, -\vec{v}t - D) f_2} \end{array} \right) \]

Now \( 2\theta(t, -\vec{v}t - D) - \rho - \omega = 2\vec{v} \cdot x + \left| \vec{v}|^2 + \alpha^2 \right| t + 2\gamma + 2\vec{v} \cdot D - t \vec{v}|^2 - 2x \cdot \vec{v} - \omega = 0 \) by definition of \( \omega \), i.e., \( \omega = \alpha^2 t + 2\gamma + 2\vec{v} \cdot D \). Adding these expressions shows that

\[ i\bar{U}_j (t, \sigma) + H_j(t, \sigma) U_j(t, \sigma) = 0, \]

as claimed. \( \square \)

### 4 The root spaces of \( H_j(\sigma) \) and \( H_j^*(\sigma) \)

In view of Section 12 below (see in particular Definition 12.1 as well as (12.2)) we will need to understand the generalized eigenspaces of the stationary operators \( H_j(\sigma) \) from (14.3). By our spectral assumption, see Definition 2.4 above, only generalized eigenspaces at 0 are allowed. We denote these spaces by \( \mathcal{N}_j(\sigma) \) and refer to them as root spaces. Thus, \( \mathcal{N}_j(\sigma) = \ker \left( H_j(\sigma)^2 \right) \) and by (12.2) one has the direct (but not orthogonal) decomposition

\[ L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) = \mathcal{N}_j(\sigma) \oplus \mathcal{N}_j(\sigma)^*, \]

where \( \mathcal{N}_j(\sigma) = \ker \left( H_j^*(\sigma)^2 \right) \). The (nonorthogonal) projection onto \( \mathcal{N}_j(\sigma)^* \) associated with this decomposition is denoted by \( P_j(\sigma) \). While the evolution \( e^{itH_j(\sigma)} \) is unbounded on \( L^2 \) as \( t \to \infty \), it is
known in many cases that it remains bounded on Ran($P_j(\sigma)$). In Section 12 this is referred to as the linear stability assumption.

**Proposition 4.1.** Impose the hypotheses of Theorem 2.1 and let $H_j(\sigma)$ be as in (2.24). Then

- The nullspace $\mathcal{N}^*_j(\sigma)$ of $H_j^*(\sigma)$ is given by the following vector valued $2n+2$ functions $\xi^m_j(x;\sigma)$, $m = 1, \ldots, 2n+2$:

  \[
  \xi^m_j(x;\sigma) = \begin{pmatrix} u^m_j(x;\sigma) \\ \bar{u}^m_j(x;\sigma) \end{pmatrix},
  \]

  \[
  u^1_j(x;\sigma) = \phi_j(x;\sigma), \quad H_j^*(\sigma)\xi^1_j(\cdot;\sigma) = 0,
  \]

  \[
  u^2_j(x;\sigma) = i\frac{2}{\alpha_j}\partial_\sigma \phi_j(x;\sigma), \quad H_j^*(\sigma)\xi^2_j(\cdot;\sigma) = -i\xi^1_j(\cdot;\sigma),
  \]

  \[
  u^m_j(x;\sigma) = i\partial_{x_m-2}\phi_j(x;\sigma), \quad H_j^*(\sigma)\xi^m_j(\cdot;\sigma) = 0, \quad m = 3, \ldots, n+2,
  \]

  \[
  u^n_j(x;\sigma) = x_{m-n-2}\phi_j(x;\sigma), \quad H_j^*(\sigma)\xi^n_j(\cdot;\sigma) = -2i\xi^{m-n}_j(\cdot;\sigma), \quad m = n+3, \ldots, 2n+2
  \]

- Let

  \[
  J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
  \]

  Then $J$ is an isomorphism between the nullspaces of $H_j^*(\sigma)$ and $H_j(\sigma)$. In particular, the nullspace of $H_j(\sigma)$ has a basis \{ $J\xi^m_j(\cdot;\sigma)$ \mid $1 \leq m \leq 2n+2$ \}. Moreover, $\mathcal{N}^*_j(\sigma)$ is spanned by

  \[
  J \partial_\sigma W_j(t,x;\sigma) \quad \text{for} \quad 1 \leq r \leq 2n+2,
  \]

  where $W_j(t,x;\sigma) = (\begin{pmatrix} w_j(t,x;\sigma) \\ \bar{w}_j(t,x;\sigma) \end{pmatrix})$.

- One has the linear stability property

  \[
  \sup_t \left\| e^{itH_j(\sigma)}P_j(\sigma) \right\|_{2-2} < \infty
  \]

  where $P_j(\sigma)$ is the projection onto $\mathcal{N}^*_j(\sigma)^\perp$ as introduced above.

**Proof.**

For the case of monomial, subcritical nonlinearities these results go back to Weinstein’s work on modulational stability [We1].

5 Estimates for the linearized problem

In (3.4) we obtained the system

\[
\begin{align*}
&i\partial_t Z + H(t,\sigma^\infty)Z = (H(\sigma(t)) - H(t,\sigma^\infty))Z + \Sigma W(\sigma(t)) + O(w_1 w_2)Z + O(|w_1 w_2|Z) + O(|Z|^p) + O(Z^p),
\end{align*}
\]

The point of rewriting (3.4) in this form is to be able to use the dispersive estimates that were obtained in [RSS] for (perturbed) matrix charge transfer Hamiltonians, see also Sections 12 and Section 8 in [RSS].
Theorem 5.1. Let $Z(t,x)$ solve the equation

$$i\partial_t Z + H(t,\sigma)Z = F,$$
$$Z(0,\cdot) = Z_0(\cdot)$$

where the matrix charge transfer Hamiltonian $H(t,\sigma)$ satisfies the conditions of Definition 12.2. Assume that $Z$ satisfies

$$\|(\text{Id} - P_j(\sigma))M_j(\sigma,t)G_{v_j,D_j}(t)Z(t,\cdot)\|_{L^2} \leq B(1+t)^{-\frac{n}{2}}, \quad \forall j = 1,\ldots,k,$$

with some positive constant $B$, where $M_j(\sigma,t)$ and $G_{v_j,D_j}(t)$ are as in Lemma 3.1. Then $Z$ verifies the following decay estimate

$$\|Z(t)\|_{L^2+L^\infty} \lesssim (1+t)^{-\frac{n}{2}} \left(\|Z_0\|_{L^1\cap L^2} + \|F\| + B\right)$$

for $t > 0$ with

$$\|F\| := \sup_{t \geq 0} \left[ \int_0^t \|F(s)\|_{L^1} \, ds + (1+t)^{\frac{n}{2}+1}\|F(t)\|_{L^2}\right].$$

In addition, we also have the $L^2$ estimate

$$\|Z(t)\|_{L^2} \lesssim \|Z_0\|_{L^1\cap L^2} + \|F\| + B$$

For the proof see [RSS] and Section 12 below. In particular, note that (5.3) is related to the characterization of scattering states in Definition 12.4.

In the applications the inhomogeneous term $F$ is a nonlinear expression which depends on $Z$. Therefore, in addition to the estimates (5.4) and (5.5) we shall need corresponding estimates for the derivatives of $Z$.

For an integer $s \geq 0$ we define Banach spaces $X_s$ and $Y_s$ of functions of $(t,x)$

$$\|\psi\|_{X_s} = \sup_{t \geq 0} \left( \|\psi(t,\cdot)\|_{H^s} + (1+t)^{\frac{n}{2}} \sum_{k=0}^s \|\nabla^k \psi(t,\cdot)\|_{L^2+L^\infty}\right)$$

$$\|F\|_{Y_s} = \sup_{t \geq 0} \sum_{k=0}^s \left( \int_0^t \|\nabla^k F(\tau,\cdot)\|_{L^1} \, d\tau + (1+t)^{\frac{n}{2}+1}\|\nabla^k F(t,\cdot)\|_{L^2}\right)$$

The generalization of the estimates of Theorem 5.1 is given by the following theorem (see Section 9, in particular Proposition 9.3 in [RSS] for the proof).

Theorem 5.2. Under assumptions of Theorem 5.1 we have that for any integer $s \geq 0$

$$\|Z\|_{X_s} \lesssim \sum_{k=0}^s \|\nabla^k Z(0,\cdot)\|_{L^1\cap L^2} + \|F\|_{Y_s} + B$$
We apply Theorem 5.2 to the equation (5.1). This will, in particular, lead to our main result, i.e., that \( \|Z(t)\|_\infty \lesssim t^{-\frac{2}{5}} \) as \( t \to \infty \). We need to ensure that \( Z \) is a scattering solution relative to each of the channels of the charge transfer Hamiltonian \( H(t, \sigma) \), in the sense of the estimate (5.3). Analogous to Buslaev, Perelman [BP1] this will be accomplished by an appropriate choice of the path \( \sigma(t) \), to be made in the following section.

In order to prove existence of solutions \( Z \) and \( \sigma \) we require another version of Theorem 5.2, which follows easily from that result.

**Remark 5.3.** It is easy to see that a time-localized version of the previous theorem also holds. Indeed, let

\[
\|\psi\|_{L^2(T)} = \sup_{0 \leq t \leq T} \left( \|\psi(t, \cdot)\|_{H^s} + (1 + t)^{\frac{2}{5}} \sum_{k=0}^{s} \|\nabla^k \psi(t, \cdot)\|_{L^{2+L^\infty}} \right)
\]

(5.9) \[
\|F\|_{Y^s(T)} = \sup_{0 \leq t \leq T} \sum_{k=0}^{s} \left( \int_{0}^{t} \|\nabla^k F(\tau, \cdot)\|_{L^1} d\tau + (1 + t)^{\frac{2}{5} + 1} \|\nabla^k F(t, \cdot)\|_{L^2} \right).
\]

(5.10)

Then assuming (5.3) for \( 0 \leq t \leq T \) with a constant \( B_T \), one has

\[
\|Z\|_{X^s(T)} \lesssim \sum_{k=0}^{s} \|\nabla^k Z(0, \cdot)\|_{L^{1+L^2}} + \|F\|_{Y^s(T)} + B_T
\]

(5.11)

**Corollary 5.4.** Let \( \xi_j^m(t, x) \), \( 1 \leq m \leq 2n+2, 1 \leq j \leq N \), be a collection of smooth functions such that

\[
\sup_{t \geq 0} \|\mathcal{M}_j(\sigma, t)\mathcal{G}_{v_j, D_j}(t)\xi_j^m(t, x) - \xi_j^m(\cdot; \sigma)\|_{L^{1+L^2}} \leq \delta
\]

(5.12) for some small \( \delta > 0 \) and some given \( \sigma \). Let \( Z \) be a solution of

\[
i\partial_t Z + H(t, \sigma)Z = V(t, x)Z + F,
\]

(5.13)

\[
\langle Z(t), \xi_j^m(t, \cdot) \rangle = 0
\]

(5.14)

for all \( t \geq 0 \), where \( V(t, x) \) is a smooth function that satisfies \( \sup_{|\gamma| \leq s} \|\partial_x^\gamma V(t, \cdot)\|_{L^{1+L^\infty}} < \delta(1 + t)^{-1} \), with a nonnegative integer \( s \) for all \( t > 0 \). Then

\[
\|Z\|_{X^s} \lesssim \sum_{k=0}^{s} \|\nabla^k Z(0, \cdot)\|_{L^{1+L^2}} + \|F\|_{Y^s}.
\]

(5.15)

**Proof.** By (5.14),

\[
\left| \langle \mathcal{M}_j(\sigma, t)\mathcal{G}_{v_j, D_j}(t)Z(t), \xi_j^m(\cdot; \sigma) \rangle \right| \leq \left| \langle \mathcal{M}_j(\sigma, t)\mathcal{G}_{v_j, D_j}(t)Z(t), -\mathcal{M}_j(\sigma, t)\mathcal{G}_{v_j, D_j}(t)\xi_j^m(\cdot; \sigma) \rangle \right| \\
\leq \delta \|Z(t)\|_{2+\infty}.
\]

Therefore, for all \( 0 \leq t \leq T \),

\[
\left| \langle \mathcal{M}_j(\sigma, t)\mathcal{G}_{v_j, D_j}(t)Z(t), \xi_j^m(t, \cdot; \sigma) \rangle \right| \leq \delta(1 + t)^{-\frac{2}{5}} \sup_{0 \leq \tau \leq T} (1 + \tau)^{\frac{2}{5}} \|Z(\tau)\|_{2+\infty}
\]

\[
\leq \delta(1 + t)^{-\frac{2}{5}} \|Z\|_{X^s(T)} =: (1 + t)^{-\frac{2}{5}} B_T.
\]

20
Hence, using (5.11) one sees that
\[
\|Z\|_{X_s(T)} \lesssim \sum_{k=0}^{s} \|\nabla^k Z(0, \cdot)\|_{L^1 \cap L^2} + \|F\|_{Y_s(T)} + \|VZ\|_{Y_s(T)} + BT
\]
\[
\lesssim \sum_{k=0}^{s} \|\nabla^k Z(0, \cdot)\|_{L^1 \cap L^2} + \|F\|_{Y_s(T)} + \delta\|Z\|_{X_s(T)},
\]
and the desired conclusion follows.

6 Modulation equations

In their analysis of the stability relative to one soliton, Buslaev and Perelman [BP1], [BP2], and Cuccagna [Cu] derive the equations for \( \dot{\sigma} \) by imposing an orthogonality condition on the perturbation \( Z \) for all times. More precisely, they make the ansatz
\[
\psi = e^{i\theta(t,\sigma(t))}(w(\sigma(t)) + R)
\]
where \( e^{i\theta(t,\sigma(t))}w(\sigma(t)) \) is a single soliton evolving along a nonlinear set of parameters. The removal of the phase from the perturbation \( R \) leads to an equation which is simply the translation of the equation involving the stationary Hamiltonian (2.24) to the point \( vt + D \). This in turn makes it very easy to formulate the orthogonality conditions: At time \( t \), the function \( R(-vt + D) \) in (6.1) needs to be perpendicular to all elements of the generalized eigenspaces of all \( H_j(\sigma)^\ast \) as in (2.24), where \( \sigma \) is equal to the parameters \( \sigma(t) \) at time \( t \).

In the multi-soliton case the removal of the phases by means of this ansatz is not available, since distinct solitons carry distinct phases. As already indicated above, we work with the representation
\[
\psi(t) = \sum_{j=1}^{N} w_j(t, \sigma(t)) + R,
\]
which forces us to formulate the orthogonality condition in terms of a set of functions that is moving along with the \( w_j(t, \sigma(t)) \). We now define these functions.

**Definition 6.1.** Let \( \sigma(t) \) be an admissible path and define \( \theta_j(t, x; \sigma(t)) \) and \( x_j(t; \sigma(t)) \) as in (2.11) and (2.12). Also, set \( \phi_j(t, x; \sigma(t)) = \phi(x - x_j(t; \sigma(t)); \alpha_j(t)) \). Then we let
\[
\xi_j^m(t, x; \sigma(t)) = \begin{pmatrix} u_j^m(t, x; \sigma(t)) \\ \bar{u}_j^m(t, x; \sigma(t)) \end{pmatrix}
\]
with
\[
u_j^1(t, x; \sigma(t)) = w_j(t, x; \sigma(t)) = e^{i\theta_j(t, x; \sigma(t))} \phi_j(t, x; \sigma(t))
\]
\[
u_j^2(t, x; \sigma(t)) = \frac{2\alpha_j}{\alpha_j} e^{i\theta_j(t, x; \sigma(t))} \partial_{\alpha_j} \phi_j(t, x; \sigma(t))
\]
\[
u_j^m(t, x; \sigma(t)) = ie^{i\theta_j(t, x; \sigma(t))} \partial_{x_{m-2}} \phi_j(t, x; \sigma(t)) \quad \text{for} \quad 3 \leq m \leq n + 2
\]
\[
u_j^m(t, x; \sigma(t)) = e^{i\theta_j(t, x; \sigma(t))} \left(x_j^{m-2} - x^{m-2}_j(t; \sigma(t))\right) \phi_j(t, x; \sigma(t)) \quad \text{for} \quad n + 3 \leq m \leq 2n + 2.
\]
The following proposition should be thought of as a time-dependent version of Proposition 4.1. More precisely, if \( \sigma \) is a fixed set of parameters, then one can define an alternate set of vectors, \( \tilde{\xi}_j^m \), say, by applying appropriate Galilean transforms to the stationary vectors in Proposition 4.1. For example, take some \( \xi_j^m \) so that \( H_j^\sigma(\sigma)\xi_j^m = 0 \). Then the corresponding \( \tilde{\xi}_j^m \) satisfies

\[
i \partial_t \tilde{\xi}_j^m + H_j(t, \sigma)\tilde{\xi}_j^m = 0,
\]

with \( H_j(t, \sigma) \) as in (3.8). Naturally, one would therefore expect that

\[
i \partial_t \xi_j^m + H(\sigma(t))\xi_j^m = O(\dot{\sigma}_j) + O(e^{-ct}),
\]

where \( H(\sigma(t)) \) is as in (2.21) (the exponentially decaying term appears because of interactions between solitons). The following proposition shows that this indeed holds, but as in [Cu] we will work with a modified set of parameters \( \tilde{\sigma}_j(t) = (\sigma_j(t), \alpha_j(t), \tilde{\gamma}_j(t)) \) where

\[
(6.3) \quad \tilde{\gamma}_j(t) = \gamma_j(t) + \frac{1}{2} \sum_{m=1}^n \tilde{v}_j^m(t)x_j^m(t, \sigma(t)).
\]

The point of this modification is that the \( \tilde{\Sigma}W(\sigma(t)) \) term in (5.1) and (3.4) can be rewritten as

\[
\tilde{\Sigma}W(\sigma(t)) = \sum_{j=1}^k \left[ \tilde{\gamma}_j(t)J\xi_j^1(t, x; \sigma(t)) - \frac{\alpha_j}{2} \tilde{\gamma}_j(t)J\xi_j^2(t, x; \sigma(t)) \right] + \sum_{j=1}^k \sum_{m=1}^n \left[ \tilde{D}_j^m(t)J\xi_j^{m+2}(t, x; \sigma(t)) + \frac{1}{2} \tilde{v}_j^m(t)J\xi_j^{m+n+2}(t, x; \sigma(t)) \right],
\]

where \( \xi_j^m \) are as in Definition 6.1. This is of course due to the fact that passing to \( \tilde{\gamma}_j \) allows us to change from \( x \) to \( x - x_j(t; \sigma(t)) \) in (3.6).

**Proposition 6.2.** Let \( \sigma(t) \) be an admissible path and define \( \xi_j^m(t, x; \sigma(t)) \) as in Definition 6.1. Then

\[
(6.5) \quad i \partial_t \xi_j^1 + H_j^\sigma(\sigma(t))\xi_j^1 = O(\dot{\tilde{\sigma}}(|\phi_j| + |D\phi_j|))
\]

\[
(6.6) \quad i \partial_t \xi_j^2 + H_j^\sigma(\sigma(t))\xi_j^2 = i\xi_j^2 + O(\dot{\tilde{\sigma}}(|\phi_j| + |D\phi_j| + |D^2\phi_j|))
\]

\[
(6.7) \quad i \partial_t \xi_j^m + H_j^\sigma(\sigma(t))\xi_j^m = O(\dot{\tilde{\sigma}}(|\phi_j| + |D\phi_j| + |D^2\phi_j|)) \quad \text{for } 3 \leq m \leq n + 2
\]

\[
(6.8) \quad i \partial_t \xi_j^m + H_j^\sigma(\sigma(t))\xi_j^m = -2i\xi_j^{m-n} + O(\dot{\tilde{\sigma}}(|\phi_j| + |D\phi_j| + |D^2\phi_j|)) \quad \text{for } n + 3 \leq m \leq 2n + 2.
\]

Here \( D \) refers to either spatial derivatives \( \partial_{x^j} \) or derivatives \( \partial_{\alpha} \). Moreover, as in Definition 6.1, the function \( \phi_j \) needs to be evaluated at \( x - x_j(t; \sigma(t)) \), \( \alpha_j(t) \).

**Proof.** This is verified by direct differentiation of the functions in Definition 6.1. \( \square \)

The following proposition collects the modulation equations for the path \( \sigma(t) \) that are obtained by taking scalar products of (2.21) with the functions \( \xi_j^m \) from Definition 6.1. This will of course use (6.4). The modulation equations are derived from the orthogonality assumptions, see (6.9) below. Observe that these assumptions need not be satisfied at \( t = 0 \). Nevertheless, as in Buslaev and Perelman [BP1],

\[22\]
one shows by means of the implicit function theorem that one can replace the initial decomposition (2.2) by a nearby one which does satisfy the orthogonality condition. This uses the smallness of the initial perturbation $R_0$, as well as the separation conditions (2.17) or (2.18). The details can be found in Section 11, see Lemma 11.1. In that section it is also shown that, conversely, given the modulation of Proposition 6.3 the orthogonality condition will propagate if satisfied initially.

**Proposition 6.3.** Let $Z$ satisfy the system (3.4). Suppose that for all $t \geq 0$,

$$
(6.9) \quad \langle Z(t), \xi_j^m(t, \cdot; \sigma(t)) \rangle = 0 \quad \text{for all } j, m
$$

where $\xi_j^m$ is as in Definition 6.1. Then the path $\tilde{\sigma}(t) := (v_j(t), D_j(t), \tilde{\gamma}_j(t), \alpha_j(t))$, $j = 1, \ldots, n$ satisfies the following system of equations with matrix potentials $V_r(t, x; \sigma(t))$ as in (2.22):

$$
-2i\partial_j(t)\langle \phi_j(\sigma(t)), \partial_\alpha \phi_j(\sigma(t)) \rangle + O(\hat{\sigma}(t) || Z(t) ||_{L^2+L^\infty}) = \sum_{r \neq j} \langle V_r(t, \cdot; \sigma(t)) Z, \xi_j^1(t, \cdot; \sigma(t)) \rangle +
$$

$$
(6.10) \quad \left\langle \left( O(w_1 w_2) Z + O(w_1 w_2) + O(|w|^{p-2} |Z|^2) + O(|Z|^p) \right), \xi_j^1(t, \cdot; \sigma(t)) \right\rangle,
$$

$$
2i\partial_j(t)\langle \phi_j(\sigma(t)), \partial_\alpha \phi_j(\sigma(t)) \rangle + O(\hat{\sigma}(t) || Z(t) ||_{L^2+L^\infty}) = \sum_{r \neq j} \langle V_r(t, \cdot; \sigma(t)) Z, \xi_j^2(t, \cdot; \sigma(t)) \rangle +
$$

$$
\left\langle \left( O(w_1 w_2) Z + O(w_1 w_2) + O(|w|^{p-2} |Z|^2) + O(|Z|^p) \right), \xi_j^2(t, \cdot; \sigma(t)) \right\rangle,
$$

$$
\hat{\sigma}^m(t) \| \phi_j(\sigma(t)) \|^2 + O(\hat{\sigma}(t) || Z(t) ||_{L^2+L^\infty}) = \sum_{r \neq j} \langle V_r(t, \cdot; \sigma(t)) Z, \xi_j^{m+2}(t, \cdot; \sigma(t)) \rangle +
$$

$$
\left\langle \left( O(w_1 w_2) Z + O(w_1 w_2) + O(|w|^{p-2} |Z|^2) + O(|Z|^p) \right), \xi_j^{m+2}(t, \cdot; \sigma(t)) \right\rangle.
$$

**Proof.** Differentiating (6.9) yields

$$
\langle i \partial_t Z, \xi_j^m(t, \cdot; \sigma(t)) \rangle = \langle Z, i \partial_t \xi_j^m(t, \cdot; \sigma(t)) \rangle.
$$

Taking scalar products of (3.4) thus leads to

$$
\langle Z, i \partial_t \xi_j^m \rangle + \langle Z, H^*(\sigma(t)) \xi_j^m \rangle = \langle \hat{\Sigma} W(\sigma(t)), \xi_j^m \rangle + \langle O(w_1 w_2) Z + O(w_1 w_2) + O(|w|^{p-2} |Z|^2) + O(|Z|^p), \xi_j^m \rangle.
$$

In view of the explicit expressions (6.2) one has

$$
\langle J \xi_j^m(t, \cdot; \sigma(t)), \xi_j^1(t, \cdot; \sigma(t)) \rangle = -2i\langle \phi_j(\sigma(t)), \partial_\alpha \phi_j(\sigma(t)) \rangle
$$

$$
\langle J \xi_j^m(t, \cdot; \sigma(t)), \xi_j^1(t, \cdot; \sigma(t)) \rangle = 0 \quad \text{for } m \neq 2
$$

$$
\langle J \xi_j^m(t, \cdot; \sigma(t)), \xi_j^2(t, \cdot; \sigma(t)) \rangle = 0 \quad \text{for } m \neq 1
$$

$$
\langle J \xi_j^{m+2}(t, \cdot; \sigma(t)), \xi_j^{m+n+2}(t, \cdot; \sigma(t)) \rangle = -2i\| \phi_j(\sigma(t)) \|^2 \quad \text{for } 3 \leq m \leq n + 2.
$$

Therefore, the proposition follows by taking inner products in (6.4). Note that the terms containing $\hat{\sigma}(t) || Z(t) ||_{L^2+L^\infty}$ appear from Proposition 6.2. \[\square\]
7 Bootstrap assumptions

The proof of our main theorem relies on the bootstrap assumptions on the admissible path \( \sigma(t) \) and the size of the perturbation \( Z(t, x) = (R(t, x) - \overline{R(t, x)}) \) in the norms of the spaces \( X_s \) defined in (5.6).

**Bootstrap assumptions**

There exists a small constant \( \delta = \delta(\epsilon) \) dependent on the size of the initial data \( R_0 \) and the initial separation of the solitons \( w_j(0, x; \sigma(0)) \), see (2.17), and a sufficiently large constant \( C_0 \) such that for some integer \( s > \frac{n}{2} \)

\[
|\dot{\sigma}(t)| \leq \delta^2 (1 + t)^{-n}, \quad \forall t \geq 0, \tag{7.1}
\]

\[
\|Z\|_{X_s} \leq \delta C_0^{-1} \tag{7.2}
\]

**Remark 7.1.** The bootstrap assumption (7.1) together with the definition (6.3) implies that

\[
|\dot{\gamma}(t)| \leq \delta^2 (1 + t)^{-n+1} \tag{7.3}
\]

**Remark 7.2.** The bootstrap assumption (7.2) together with Lemma 9.4 implies that

\[
\|Z(t)\|_{L^\infty} \lesssim \delta C_0^{-1}(1 + t)^{-\frac{n}{2}}, \tag{7.4}
\]

\[
\|Z(t)\|_{H^s} \lesssim \delta C_0^{-1} \tag{7.5}
\]

The bootstrap assumption (7.1) strengthens the notion of the admissible path. In particular, it allows us to estimate the deviation between the path \( x_j(t; \sigma(t)) \) corresponding to the path \( \sigma(t) \) and the straight line \( x_j(t, \sigma^\infty) \) determined by the constant parameter \( \sigma^\infty \) which was defined from \( \sigma(t) \) in (2.14) and (2.15). This estimate will play an important role in our analysis.

**Lemma 7.3.** Let \( \sigma(t) \) be an admissible path satisfying the bootstrap assumption (7.1) and let \( \sigma^\infty \) be a constant parameter vector as in (2.14) and (2.15). Then

\[
|x_j(t; \sigma(t)) - tv^\infty_j - D^\infty_j| \lesssim \delta^2 (1 + t)^{-n+2} \tag{7.6}
\]

**Proof.** By our choice of \( v^\infty_j \) and \( D^\infty_j \) one has that

\[
|x_j(t; \sigma(t)) - tv^\infty_j - D^\infty_j| \lesssim \int_t^\infty \int_s^\infty |\dot{v}_j(\tau)| d\tau + \int_t^\infty |\dot{D}_j(s)| ds
\]

and the lemma follows from (7.1). \( \square \)

We then have the following corollary. To formulate it, we need the localizing functions

\[
\chi_0(x) = \exp\left(-\frac{1}{2}\alpha_{\min}(1 + |x|^2)^{\frac{1}{2}}\right) \tag{7.7}
\]

\[
\chi(t, x; \sigma^\infty) = \sum_{j=1}^k \chi_0(x - x_j(t; \sigma^\infty)).
\]

Here \( \alpha_{\min} > 0 \) satisfies \( \inf_{t \geq 0, 1 \leq j \leq k} \alpha_j(t) > \alpha_{\min} \) for any admissible path \( \sigma(t) \) starting at \( \sigma_0 \). The exponent \( \alpha_{\min} \) arises because of the decay rate of the ground state of (2.6).
Corollary 7.4. Let \( \sigma(t) \) be an admissible path satisfying the bootstrap assumption (7.1). With the parameters \( \sigma^\infty \) as in (2.14) and (2.15) one has

\[
(7.8) \quad |H(t, \sigma^\infty) - H(\sigma(t))| \lesssim \delta^2 (1 + t)^{2-n} \chi(t, x; \sigma^\infty),
\]

where \( H(t, \sigma^\infty) \) and \( H(\sigma(t)) \) are the Hamiltonians from (3.7) and (2.21).

Proof. The difference

\[
H(t, \sigma^\infty) - H(\sigma(t))
\]

is a sum of matrix valued potentials that are exponentially localized around the solitons \( w_j(\sigma(t)) \) or \( w_j(t, \sigma^\infty) \), respectively. By the previous lemma, we can assume that all the potentials are localized near the straight path \( x(t, \sigma^\infty) = (x_1(t; \sigma^\infty), \ldots, x_N(t; \sigma^\infty)) \). Since \( v_j^\infty = v_j(\infty), \alpha_j^\infty = \alpha_j(\infty) \),

\[
(7.9) \quad |[H(t, \sigma^\infty) - H(\sigma(t))]| \lesssim \sum_{j=1}^k |tv_j^\infty + D_j^\infty - \int_0^t v_j(s) \, ds - D_j(t)| \chi_0(x - x_j(t; \sigma^\infty))
\]

\[
(7.10) \quad + \sum_{j=1}^k \left| \frac{1}{2} \int_t^\infty \dot{v}_j(s) \cdot x \, ds - \frac{1}{2} \int_0^t \int_s^\infty (\dot{v}_j(s) \cdot v_j(s) - \dot{\alpha}_j(s) \alpha_j(s)) \, ds + \gamma_j - \gamma_j(t) \right| \chi_0(x - x_j(t; \sigma^\infty)).
\]

The term (7.9) arises as the difference of two paths, whereas (7.10) is the difference of the phases, i.e.,

\[
|e^{i\theta_j(t, x; \sigma^\infty)} - e^{i\theta_j(t, x; \sigma(t))}|.
\]

In view of the definitions of \( D_j, \gamma_j \) from (2.14) and (2.15) one has

\[
(7.11) \quad |H(t, \sigma^\infty) - H(\sigma(t))| \lesssim \sum_{j=1}^k \left( \int_t^\infty \int_s^\infty |\dot{v}_j(\tau)| \, d\tau + \int_t^\infty |\dot{D}_j(s)| \, ds \right) \chi_0(x - x_j(t, \sigma))
\]

\[
+ \sum_{j=1}^k \left( \int_t^\infty \int_s^\infty |\dot{v}_j(\tau) \cdot v_j(\tau) - \dot{\alpha}_j(s) \alpha_j(s)| \, d\tau \, ds + \int_t^\infty |\dot{\gamma}_j(s)| \, ds + \int_t^\infty |\dot{v}_j(s)| \, ds \, dx \right) \chi_0(x - x_j(t, \sigma^\infty))
\]

\[
\lesssim \delta^2 (1 + t)^{2-n} \chi(t, x; \sigma^\infty).
\]

For the final inequality one uses (7.3) and the fact that

\[
|x| \chi_0(x - x_j(t; \sigma^\infty)) \lesssim t.
\]

The corollary follows. \( \square \)
8 Solving the modulation equations

Our goal is to show that the system in Proposition 6.3 has a solution \( \dot{\sigma}(t) \) that satisfies the bootstrap assumptions (7.1). This requires some care, as the right-hand side in Proposition 6.3 involves the perturbation \( Z \). We will therefore first verify that the system of modulation equations is consistent with the bootstrap assumptions (7.1) and (7.2). In what follows, we will use both paths \( \tilde{\sigma}(t) \) and \( \sigma(t) \). By definition, see (6.3),

\[
\tilde{\gamma}_j(t) = - \int_t^\infty \left[ \tilde{\gamma}_j(s) + \frac{1}{2} \sum_{m=1}^n \tilde{e}_j^m(s)x_j^m(s;\sigma(s)) \right] ds.
\]

The integration is well-defined provided \( \tilde{\sigma} \) satisfies the bootstrap assumption. Indeed, in that case \( |v_j(t)| \lesssim (1 + t)^{-n} \) and since \( |x_j(t;\sigma(t))| \lesssim 1 + t \), the integral is absolutely convergent. Finally, recall the property (7.3) of the derivatives.

**Lemma 8.1.** Suppose the separation and convexity conditions hold, see (2.17) and (2.16). Let \( \tilde{\sigma}, Z \) be any choice of functions that satisfy the bootstrap assumptions for sufficiently small \( \delta > 0 \). If the inhomogeneous terms of the system (6.10) are defined by means of these functions, then this system has a solution \( \dot{\sigma} \) that satisfies (7.1) with \( \delta/2 \) for all times.

**Proof.** By the nonlinear stability condition (2.16), the left-hand side of (6.10) is of the form \( B_j(t)\tilde{\dot{\sigma}}_j(t) \) with an invertible matrix \( B_j(t) \). The \( O \)-term is a harmless perturbation of the matrix given by the main terms on the left-hand side, provided \( \delta \) is chosen sufficiently small. This easily follows from the smallness of \( Z \) given by (7.2). We need to verify that the right-hand side of (6.10) decays like \( \delta^2(1 + t)^{-n} \). We consider only the first equation in (6.10), the others being the same. The terms \( \langle V_r(t,\sigma)Z, \xi_j^r(t,.;\sigma(t)) \rangle \) for \( r \not= j \) and \( w_1w_2 \) are governed by the interaction of two different solitons. In view of the separation condition (2.17) and the exponential localization of the solitons, we have

\[
|\dot{\alpha}_j(t)| \lesssim e^{-\alpha_{\text{min}}(L+ct)}(1 + \|Z(t)\|_{L^2+L^\infty}^2 + \|Z(t)\|_{L^2+L^\infty}^p + \|Z(t)\|_{L^2+L^\infty}^p) \]

\[
\lesssim \delta C_0^{-1}(1 + t)^{-n} \left( \epsilon + \delta C_0^{-1} + \delta^{p-1}C_0^{-(p-1)} \right) \leq \left( \frac{\delta}{2} \right)^2 (1 + t)^{-n}
\]

where we have used the estimate (7.4), the condition (2.19), \( \Lambda_{\text{min}} \geq |\log \epsilon| \), and that \( p \geq 2 \).

More generally, the estimates leading up to (8.1) also yield the following result. The proof is implicit in the preceding one and is therefore omitted.

**Lemma 8.2.** Let \( \sigma(t) \) be an admissible path satisfying the bootstrap assumption (7.1) and \( Z(t) \) be an arbitrary function in \( X_s \). Define the function \( \tilde{\Sigma} \) as a solution of the equation

\[
\langle \tilde{\Sigma} W(\sigma(t)), \xi_j^m(t,.;\sigma(t)) \rangle = \langle G(Z(t),\sigma(t)), \xi_j^m(t,.;\sigma(t)) \rangle + \langle \Omega_j^m(t,.;\sigma(t)), Z(t,.) \rangle,
\]

where

\[
G(Z(t),\sigma(t)) = O(w_1(\sigma(t))w_2(\sigma(t))Z) + O(w_1(\sigma(t))w_2(\sigma(t)) + O(|w(\sigma(t))|^{p-2}|Z|^2) + O(|Z|^p),
\]

\[
\Omega_j^m(t,x;\sigma(t)) = O(\dot{\sigma}(|\phi_j| + |D\phi_j| + |D^2\phi_j|)) + \sum_{i \not= j} V_r(t,x;\sigma(t))\xi_j^m(t,x;\sigma(t)).
\]
Lemma 9.1. The following lemma shows that the orthogonality conditions (6.9) and the bootstrap assumptions (7.1), (7.2) imply that

\[ (9.1) \quad \langle Z(t), \xi^m_j(t, \cdot; \sigma(t)) \rangle = 0 \quad \text{for all } j, m \]

with \( \xi^m_j(t, x; \sigma(t)) \) is as in Definition 6.1. We next rewrite the equation (3.4) for \( Z \) in the form

\[ (9.2) \quad i \partial_t Z + H(t, \sigma(t)) Z = F, \]

\[ (9.3) \quad F = (H(t, \sigma^\infty) - H(\sigma(t))) Z + \hat{\Sigma} W(\sigma(t)) + O(w_1 w_2) Z + O(|w|^{p-2}) + O(|Z|^{p}) \]

with the reference Hamiltonian \( H(t, \sigma^\infty) \) as defined in (3.7). To verify the bootstrap assumption (7.2) we need to apply the the dispersive estimate for the inhomogeneous charge transfer problem stated in Theorem 5.2. The following lemma shows that the orthogonality conditions (6.9) and the bootstrap assumptions (7.1), (7.2) imply that \( Z \) is asymptotically orthogonal to the bound states of \( H^*_j(\sigma^\infty) \), as required in Theorem 5.2.

Lemma 9.1. Let \( Z \) be an arbitrary function in \( \mathcal{X}_s \) satisfying the orthogonality conditions (9.1) with respect to an admissible path \( \sigma(t) \) obeying the bootstrap assumption (7.1), and so that \( Z \) verifies the bootstrap assumption (7.2). Then \( Z \) is asymptotically orthogonal to the null spaces of the Hamiltonians \( H^*_j(\sigma^\infty) \) in the sense of (5.3). In fact,

\[ (9.4) \quad \| P_{N_j}(\sigma^\infty) G_{\psi^\infty, D^\infty_j} Z(t, \cdot) \|_{L^2} \lesssim \delta^3 (1 + t)^{-\frac{9}{2} - 1}, \quad \forall j = 1, ..., k \]

Proof. By the assumption \( Z(t) \) is orthogonal to the vectors \( \xi^m_j(t, x; \sigma(t)) \) introduced in Definition 6.1, while (9.4) is equivalent to the estimates

\[ |\langle G_{\psi^\infty, D^\infty_j} Z(t), \xi^m_j(\cdot; \sigma(\cdot), Z(t)) \rangle| \lesssim \delta^3 (1 + t)^{-\frac{9}{2} - 1}, \quad \forall j, m \]
Here $\xi_j^m(x;\sigma^\infty)$, defined in Proposition 4.1, refer to the elements of the null spaces $\mathcal{N}_j(\sigma^\infty)$ of the stationary Hamiltonians $H_j^s(\sigma^\infty)$. The desired estimate would then follow from the bootstrap assumption (7.2), in particular (7.4), and the inequality
\begin{equation}
|\mathcal{G}_j^+ - \mathcal{G}_j^-| \lesssim \delta^2(1 + t)^{-1}
\end{equation}
The vectors $\xi_j^m$ are composed of the functions derived from the bound state $\phi$. In particular, $\xi_j^1 = (\phi)$. Therefore,
\begin{equation}
|\mathcal{G}_j^+ - \mathcal{G}_j^-| \lesssim \delta^2(1 + t)^{-1}
\end{equation}
According to Lemma 7.3, $|x_j(t,\sigma(t)) - \mathcal{G}_j^-| \lesssim \delta^2(1 + t)^{-n+2}$. Similarly, (7.10) of Corollary 7.4 gives the estimate for the difference of the phases appearing in (9.6)
\begin{equation}
|e^{i\theta_j(t,x;\sigma^\infty)} - e^{i\theta_j(t,x;\sigma(t))}| \lesssim \delta^2(1 + t)^{-n+2}
\end{equation}
The estimate (9.5) follows immediately since $n \geq 3$. 

We now in the position to apply Theorem 5.2 to establish the improved $\mathcal{X}_s$ estimates for $Z(t)$.

**Lemma 9.2.** Let $Z$ be a solution of the equation (9.2) satisfying the bootstrap assumption (7.2) with some sufficiently small constants $\delta$ and $C_0^{-1}$. We also assume (due to Lemma 8.1) that the admissible path $\sigma(t)$ obeys the estimate (7.1). Then we have the following estimate
\begin{equation}
\|Z(t)\|_{\mathcal{X}_s} \lesssim \frac{\delta}{2}C_0^{-1}
\end{equation}
**Proof.** Perturbation $Z$ is a solution of the inhomogeneous charge transfer problem (9.2)
\begin{equation}
i\partial_t Z + H(t,\sigma^\infty)Z = F,
\end{equation}
with
\begin{equation}
F := (H(t,\sigma^\infty) - H(\sigma(t)))Z + \Sigma W(\sigma(t)) + O(w_1 w_2)Z + O(w_1 w_2) + O(|w|^p-2)\|Z\|^2| + O(|Z|^p)
\end{equation}
Lemma 9.1 shows that $Z$ is asymptotically orthogonal (with the constant $\delta^3$) to the null spaces of the Hamiltonians $H_j^s(\sigma^\infty)$. Therefore, Theorem 5.2 gives the estimate
\begin{equation}
\|Z(\cdot)\|_{\mathcal{X}_s} \lesssim \sum_{k=0}^{\infty} \|\nabla^k Z_0\|_{L^1 \cap L^2} + \|F\|_{\mathcal{Y}_s} + \delta^3
\end{equation}
with
\begin{equation}
\|F\|_{\mathcal{Y}_s} = \sup_{t \geq 0} \left( \sum_{k=0}^{\infty} \int_0^t \|\nabla^k F(\tau,\cdot)\|_{L^1} d\tau + (1 + t)^{\frac{n+1}{2}}\|F(t,\cdot)\|_{H^s} \right)
\end{equation}
By the assumptions on the initial data $\sum_{k=0}^{\infty} \|\nabla^k Z_0\|_{L^1 \cap L^2} \leq \epsilon << \delta$. Therefore, to obtain the conclusion of Lemma 9.2 it would suffice to verify that
\begin{equation}
\|F\|_{\mathcal{Y}_s} \lesssim \delta^2
\end{equation}
with $F$ defined as in (9.8). The estimate (9.11) relies on the following lemma and the bootstrap assumptions (7.2) on $Z(t)$. 

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Lemma 9.3. Let $\sigma(t)$ be an arbitrary admissible path satisfying the bootstrap assumption (7.1) and $Z(t)$ be an arbitrary function in $X_s$. Then the nonlinear expression $F$ defined in terms of the path $\sigma(t)$ and $Z(t)$ as in (9.8) obeys the estimate
\begin{equation}
\|F\|_{Y_s} \lesssim \delta^2 + \|Z\|_{X_s}^2 + \|Z\|_{X_s}^p
\end{equation}

9.1 Algebra estimates

In this section we establish several simple lemmas designed to ease the task of estimating the $Y_s$ norm of $F = F(Z, w, \sigma)$ in connection with the $X_s$ norms of $Z$.

We start by formulating a version of the Sobolev estimate tailored to the use of the space $L^2 + L^\infty$.

Lemma 9.4. Let $s$ be a positive integer. Then for any nonnegative integer $k \leq s$ and any $q \in [2, q_k]$, where
\begin{equation}
\frac{1}{q_k} = \frac{1}{2} - \frac{s-k}{n}, \quad \text{if } k > s - \frac{n}{2}
\end{equation}
\begin{equation}
q_k = \infty, \quad \text{if } k < s - \frac{n}{2}
\end{equation}
and $q \in [2, \infty)$ if $k = s - \frac{n}{2}$ the following estimates hold true
\begin{equation}
\|\nabla^k f\|_{L^2 + L^\infty} \lesssim \sum_{l=0}^{s} \|\nabla^l f\|_{L^2 + L^\infty} \leq (1 + t)^{-\frac{n}{2}} \|f\|_{X_s}
\end{equation}

In particular, if $s > \frac{n}{2}$
\begin{equation}
\|f\|_{L^\infty} \lesssim (1 + t)^{-\frac{n}{2}} \|f\|_{X_s}
\end{equation}

Proof. By duality and density it suffices to show that
\begin{equation}
\|f\|_{L^1 \cap L^2} \leq \sum_{l=0}^{s-k} \|\nabla^l f\|_{L^1 \cap L^2}
\end{equation}
The $L^1$ estimate is trivial while the the estimate for the $L^2$ norm follows from the standard Sobolev embedding $W^{k-l, q'} \subset L^2$, which holds for the range of parameters $(k, l, q)$ described in the Lemma.

Next are the estimates of the nonlinear quantities arising in (9.8) in terms of the $X_s$ norm.

Lemma 9.5. Let $\gamma(\tau)$ be a smooth function which obeys the estimates
\begin{equation}
|\gamma^{(\ell)}(\tau)| \lesssim \tau^{(\frac{1}{2} - \ell)_+}
\end{equation}
for some \( p \geq 2 + \frac{2}{n} \) and all non-negative integers \( \ell \). Here \( r_+ = r \) if \( r \geq 0 \) and \( r_+ = 0 \) if \( r < 0 \). Then for any \( s > \frac{n}{2} \) and any non-negative integer \( k \leq s \)

\[
\|\nabla^k (\gamma(\|f\|^2)f)\|_{L^1} \lesssim (1 + t)^{\frac{1}{2}} (\|f\|_{X_s}^p + \|f\|_{X_s}^{2k+1+(p-1-2k)}) ,
\]

(9.17)

\[
\|\nabla^k (\gamma(\|f\|^2)f)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{2}} (\|f\|_{X_s}^p + \|f\|_{X_s}^{2k+1+(p-1-2k)})
\]

(9.18)

In addition, if \( \gamma \) is a smooth function obeying (9.16) for some \( p \geq 2 \) and \( \zeta(x) \) is an exponentially localized smooth function, then for any \( q \in [1, 2] \)

\[
\|\nabla^k (\zeta(\|f\|^2)f)\|_{L^q} \lesssim (1 + t)^{-n} (\|f\|_{X_s}^p + \|f\|_{X_s}^{2k+1+(p-1-2k)}) .
\]

(9.19)

**Remark 9.6.** It will become clear from the proof below that if the function \( \gamma \) satisfies (9.16) for some \( p > 2 + \frac{2}{n} \), then the estimate (9.17) holds with a better rate of decay in \( t \). In particular,

\[
\int_0^\infty \|\nabla^k (\gamma(\|f\|^2)f)\|_{L^1} dt \lesssim \|f\|_{X_s}^p + \|f\|_{X_s}^{2k+1} .
\]

(9.20)

**Proof.** By Leibnitz’s rule

\[
\nabla^k (\gamma(\|f\|^2)f) = \sum_{\ell=0}^k \sum_{m_1+\ldots+m_{2\ell+1}=k} C_{\ell m} \gamma^{(\ell)}(\|f\|^2) \nabla^{m_1} f \nabla^{m_2} f \ldots \nabla^{m_{2\ell+1}} f
\]

with some positive integer constants \( C_{\ell m} \) and non-negative vectors \( \vec{m} = (m_1, \ldots, m_{2\ell+1}) \). We may assume that \( m_{2\ell+1} \geq m_2 \geq \ldots \geq m_1 \). Define

\[
q_{m_r} = \left( \frac{1}{2} - \frac{s - m_r}{n} \right)^{-1}
\]

(9.21)

for \( m_r \geq s - \frac{n}{2} \) and \( q_{m_r} = \infty \) otherwise. With the above definition the Sobolev embeddings

\[
H^s \subset W^{m_r, q_{m_r}} , \quad W^{s, 2} + W^{s, \infty} \subset W^{m_r, q_{m_r}} + W^{m_r, \infty}
\]

(9.22)

(recall that \( m_r \leq k \leq s \))

hold true by Lemma 9.4. Then

\[
\|\nabla^k (\gamma(\|f\|^2)f)\|_{L^1} \lesssim \|\gamma(\|f\|^2)\|_{L^1} \lesssim \|\gamma(\|f\|^2)\|_{L^{1+L^\infty}} \lesssim \|\gamma(\|f\|^2)\|_{L^{1+L^\infty}} + \|\nabla^k f\|_{L^{2+L^\infty}}
\]

\[
\sum_{\ell=1}^k \sum_{m_1+\ldots+m_{2\ell+1}=k} \|\gamma^{(\ell)}(\|f\|^2)\|_{L^{1+L^\infty}} \lesssim \|\nabla^k f\|_{L^{1+L^\infty}} + \|\nabla^k f\|_{L^{1+L^\infty}} + \|\nabla^k f\|_{L^{1+L^\infty}}
\]

(9.23)

where the final term inside the parentheses in (9.23) arises via the second embedding in (9.22). We claim for \( \ell \) that there exist two sets of parameters \( q_{m_r}^1 \) and \( q_{m_r}^2 \) for \( r = 1, \ldots, 2\ell \) such that

\[
2 \leq q_{m_r}^1, q_{m_r}^2 \leq q_{m_r} , \quad \forall r = 1, \ldots, 2\ell,
\]

\[
\sum_{r=1}^{2\ell} \frac{1}{q_{m_r}^1} = 1, \quad \sum_{r=1}^{2\ell} \frac{1}{q_{m_r}^2} = \frac{1}{q_{m_{2\ell+1}}^r} .
\]

\[3\text{In the case of } m_r = s - \frac{n}{2} \text{ the value of } q_{m_r} \text{ can be set arbitrarily large, but the embedding fails for } q_{m_r} = \infty. \text{ However, since the following argument has some “slack”, we can allow ourselves to still set } q_{m_r} = \infty \text{ for simplicity.}\]
Similarly, we estimate property (9.24) together with the estimate (9.16) we obtain provided that \( \ell > \frac{m}{2} \).

To prove the claim we let \( \tau \) be the number of \( m_r \) for \( r = 1, \ldots, 2\ell \) such that \( m_r \geq \frac{s - \frac{3}{2}}{2} \). Observe that

\[
\sum_{r=1: m_r \geq \frac{s - \frac{3}{2}}{2}} 2\ell m_r \leq k - m_{2\ell+1}
\]

Therefore,

\[
\sum_{r=1}^{2\ell} \frac{1}{q_{m_r}} \leq \frac{\tau}{2} - \frac{\tau s - k + m_{2\ell+1}}{n} \leq -\tau \left( \frac{s}{n} - \frac{1}{2} \right) + \frac{k - m_{2\ell+1}}{n}
\]

\[
\leq -\tau \left( \frac{s}{n} - \frac{1}{2} \right) - \frac{s - k}{n} + \frac{1}{2} \leq \frac{1}{q_{m_{2\ell+1}}}
\]

The inequality in the second line above follows since \( m_{2\ell+1} \geq \frac{s - \frac{3}{2}}{2} \), which holds if \( \tau > 0 \). On the other hand,

\[
\sum_{r=1}^{2\ell} \frac{1}{2} = \ell \geq 1
\]

and the claim immediately follows, provided that \( \ell > 0 \). Thus using the sequence \( q_{m_r}^1 \) to handle the \( L^1 \) norm in (9.23) and \( q_{m_r}^0 \) for the \( L^{d_{m_{2\ell+1}}} \) norm, we obtain

\[
\| \nabla^k \left( (|f|^2) f \right) \|_{L^1} \lesssim \| \gamma(|f|^2)|\nabla^k f|^1 - \frac{2}{n} \|_{L^1 \cap L^\frac{2m}{n+2}} \| \nabla^k f \|_{L^2 + L^\infty} + \sum_{\ell=1}^k \sum_{m_1 + \ldots + m_{2\ell+1} = k} \| \gamma^{(\ell)}(|f|^2) \|_{L^\infty} \| \nabla^{m_1} f \|_{L^{q_{m_1} + \ell}} \cdots \| \nabla^{m_{2\ell}} f \|_{L^{q_{m_{2\ell}} + \ell}} \| \nabla^k f \|_{L^2 + L^\infty}.
\]

By Hölder’s inequality

\[
\| \gamma(|f|^2)|\nabla^k f|^1 - \frac{2}{n} \|_{L^1 \cap L^\frac{2m}{n+2}} \lesssim \| \nabla^k f \|_{L^2}^{1 - \frac{2}{n}} \| f \|_{L^{p-1}} \| f \|_{L^{\frac{2m}{n+2}} \cap L^{2(p-1)}} \lesssim \| f \|_{H^s}^{p - \frac{2}{n}}
\]

provided that \( p \geq 2 + \frac{2}{n} \), which is dictated by the condition that \( (p - 1) \frac{2m}{n+2} \geq 2 \). Finally, using the property (9.24) together with the estimate (9.16) we obtain

\[
\| \nabla^k \left( (|f|^2) f \right) \|_{L^1} \lesssim \| f \|_{H^{s - \frac{2}{n}}} \| \nabla^k f \|_{L^2 + L^\infty} + \| f \|_{H^s}^{p - 1} + \| f \|_{H^s}^{2k + (p - 1 - 2k)} \| \nabla^k f \|_{L^2 + L^\infty}
\]

\[
\lesssim \ell^{-1} \left( \| f \|_{H^s}^{p - 1} + \| f \|_{H^{s - \frac{2}{n}}} \right) + \| f \|_{H^s}^{2k + (p - 1 - 2k)} \| \nabla^k f \|_{L^2 + L^\infty}.
\]

Similarly, we estimate

\[
(9.25) \quad \| \nabla^k \left( (|f|^2) f \right) \|_{L^2} \lesssim \| \gamma(|f|^2) \|_{L^{2} \cap L^{\infty}} \| \nabla^k f \|_{L^2 + L^\infty} + \sum_{\ell=1}^k \sum_{m_1 + \ldots + m_{2\ell+1} = k} \| \gamma^{(\ell)}(|f|^2) \|_{L^2 \cap L^{\frac{2m_{2\ell+1}}{n+2}}} \| \nabla^{m_1} f \|_{L^2 \cap L^{\frac{2m_{2\ell+1}}{n+2}}} \cdots \| \nabla^{m_{2\ell}} f \|_{L^2 \cap L^{\frac{2m_{2\ell}}{n+2}}} \| \nabla^k f \|_{L^2 + L^\infty} + \| \nabla^k f \|_{L^2 + L^\infty}.
\]
Furthermore, using definition (9.21) we have that
\[ q \] 
Then
\[ \| \gamma(|f|^2) \|_{L^2 \cap L^\infty} \leq \| f \|_{L^\infty \cap L^2}^{p-1} \leq \| f \|_{L^\infty \cap L^2}^{2-2} \| f \|_{L^2 \cap L^\infty} \lesssim (1 + t)^{-\frac{n}{2}(p-2)} \| f \|_{L^p}^{p-1}, \]
where the second inequality follows from interpolating \( L^2(p-1) \) between \( L^2 \) and \( L^\infty \) and the last inequality is a consequence of Lemma 9.4 and the definition of the space \( X_\gamma \). Thus
\[ (9.26) \]
\[ \| \gamma(|f|^2) \|_{L^2 \cap L^\infty} \| \nabla^k f \|_{L^2 + L^\infty} \lesssim (1 + t)^{-\frac{n}{2}(p-1)} \| f \|_{L^p}^p. \]
Furthermore, using definition (9.21) we have that
\[ \frac{2q_{m_{2\ell+1}}}{q_{m_{2\ell+1}} - 2} = \frac{n}{s - m_{2\ell+1}}. \]
Then
\[ (9.27) \]
\[ \| \gamma^{(\ell)}(|f|^2) \nabla^{m_1} f ... \nabla^{m_{2\ell}} f \|_{L^2 \cap L^{\frac{s-m_{2\ell+1}}{s}} \cap L^{\frac{s-m_{2\ell+1}+1}{s}}} \lesssim \| \nabla^{m_1} f \|_{L^{\frac{n}{s}} \cap L^{\frac{n}{s}}} \times \| \gamma^{(\ell)}(|f|^2) \nabla^{m_1} f ... \nabla^{m_{2\ell}} f \|_{L^2 \cap L^{\frac{s-m_{2\ell+1}}{s}} \cap L^{\frac{s-m_{2\ell+1}+1}{s}}} \]
Using the definition of \( q_{m_1} \) from (9.21) and the assumption that \( m_1 \leq m_{2\ell+1} \) we infer that the last norm reduces to the one of the space
\[ L^2 \cap L^{\frac{n}{s-m_{2\ell+1}}}, \quad \text{for } m_1 \leq s - \frac{n}{2}, \]
\[ L^2 \cap L^{\frac{n}{s-m_{2\ell+1}+(s-m_{2\ell+1})^{-2}}}, \quad \text{for } m_1 > s - \frac{n}{2}. \]
We now let \( \tau \) be the number of \( m_r \) for \( r = 2, ..., 2\ell \) such that \( m_r \geq s - \frac{n}{2} \). Observe that since \( s > \frac{n}{2} \) and \( s \geq k \)
\[ \sum_{r=2}^{2\ell} \frac{1}{q_{m_r}} \leq \frac{\tau}{2} - \frac{m_{2\ell+1} + m_1}{n} \]
\[ = -\tau \left( \frac{s}{n} - \frac{1}{2} \right) + \frac{k - m_{2\ell+1} - m_1}{n} \]
\[ \leq \min \left\{ \frac{s - m_{2\ell+1}}{n}, \frac{(s - m_{2\ell+1} - m_1) + s - \frac{n}{2}}{n} \right\} \]
On the other hand
\[ \sum_{r=2}^{2\ell} \frac{1}{2} = \ell - \frac{1}{2} \geq \frac{1}{2}. \]
It therefore follows that there exist 2 sets of parameters \( q_{m_r}^1 \) and \( q_{m_r}^2 \) for \( r = 1, ..., 2\ell \) such that
\[ (9.28) \]
\[ 2 \leq q_{m_r}^1, q_{m_r}^2 \leq q_{m_r}, \quad \forall r = 2, ..., 2\ell \]
\[ \sum_{r=2}^{2\ell} \frac{1}{q_{m_r}^1} = 2, \]
\[ \sum_{r=2}^{2\ell} \frac{1}{q_{m_r}^2} = \left( \frac{s - m_{2\ell+1} - m_1}{n} + \frac{s - \frac{n}{2}}{n} \right) \text{ or } \frac{s - m_{2\ell+1}}{n}. \]
In either case, with the help of Lemma 9.4, we can estimate
\[ \|\gamma(\ell)(|f|^2)\nabla^{m_2} f \ldots \nabla^{m_2} f\|_{L^2} \leq \|f\|_{H^s}^{(p-1-2\ell)_+ + 2\ell - 1} \]

It therefore follows that the second term in (9.25) is
\[ (9.29) \]
\[ \lesssim (1 + t)^{-n} \sum_{\ell=1}^k \|f\|_{X_s}^{(p-1-2\ell)_+ + 2\ell - 1}. \]

Now combining this with (9.26), and using the condition that \( p \geq 2 + \frac{2}{n} \), we infer that
\[ \|\nabla^k (\gamma(|f|^2)f\|_{L^2} \lesssim (1 + t)^{-n} f_{X_s} + \|f\|_{X_s}^{2k+1+(p-1-2k)_+}. \]

The proof of (9.19) proceeds along the lines of the argument for the \( L^2 \) estimate (9.18). We first observe that since \( \zeta(x) \) is an exponentially localized function, the \( L^q \) estimate for \( 1 \leq q \leq 2 \) can be reduced to the \( L^2 \) estimate. We then note that the condition that \( p \geq 2 + \frac{2}{n} \) was only used in the estimate (9.26) which now takes the form
\[ \|\nabla^k (\gamma(|f|^2)f\|_{L^2} \lesssim (1 + t)^{-n} f_{X_s} \]
\[ \lesssim (1 + t)^{-n} f_{X_s} + \|f\|_{X_s}^{2k+1+(p-1-2k)_+}. \]

The remaining estimates already have the desired form (9.29). \( \square \)

**9.2 \( L^1 \) estimates**

In this section and the following we prove Lemma 9.3. We start with the verification of
\[ \sum_{k=0}^s \int_0^\infty \|\nabla^k F(t, \cdot)\|_{L^1} dt \lesssim \delta^2 + \|Z\|_{X_s}^2 + \|Z\|_{X_s}^p \]

with \( F \) as in (9.8). By Corollary 7.4 we have
\[ (9.30) \]
\[ \left| H(t, \sigma^\infty) - H(\sigma(t)) \right| \lesssim \delta^2 (1 + t)^{2-n} \chi(t, x; \sigma^\infty), \]

where \( \chi(t, x; \sigma^\infty) \) is a smooth cut-off function localized around the union of the paths \( x_j(t; \sigma^\infty) = v_j^\infty t + D_j^\infty \). Moreover, the spatial derivatives of the above difference also satisfy the same estimates. Using the bootstrap assumptions (7.2) we obtain
\[ \sum_{k=0}^s \int_0^\infty \left\|\nabla^k \left( [H(\tau, \sigma^\infty) - H(\sigma(\tau))] Z(\tau) \right) \right\|_{L^1} dt \lesssim \delta^2 \sum_{k=0}^s \int_0^\infty \|\nabla^k Z(\tau)\|_{L^2+L^\infty} (1 + \tau)^{2-n} d\tau \]
\[ \lesssim \delta^2 \|Z\|_{X_s} \int_0^\infty (1 + \tau)^{2-n-\frac{2}{2}} d\tau \lesssim \delta^2 \|Z\|_{X_s}. \]

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The term $\dot{\Sigma}W(\sigma(t))$ obeys the point wise bound

$$|\dot{\Sigma}W(\sigma(t))| \lesssim \max_j |\dot{\sigma}_j(t)| \chi(t, x; \sigma^\infty)$$

This can be easily seen from the equation (6.4) and Lemma 7.3. The same estimate also holds for the spatial derivatives of the quantity above. Thus, with the help of the already verified estimate (7.1) we infer that

$$\sum_{k=0}^{s} \int_{0}^{\infty} \left\| \nabla^k \left( \dot{\Sigma}W(\sigma(t)) \right) \right\|_{L^1} \, d\tau \lesssim \delta^2 \int_{0}^{\infty} (1 + \tau)^{-n} \, d\tau \lesssim \delta^2$$

The estimates for the $O(w_1 w_2)Z$ and $O(w_1 w_2)$ terms in (9.8) are straightforward due to the separation and the exponential localization of the solitons $w_1$ and $w_2$, e.g.,

$$\sum_{k=0}^{s} \int_{0}^{\infty} \left\| O \left( \nabla^k (w_1 w_2) \right) \right\|_{L^1} \lesssim \int_{0}^{\infty} e^{-\alpha \min(L+\epsilon t)} \, d\tau \leq \frac{e^{-\alpha \min L}}{\alpha \min} \lesssim \frac{e}{\alpha \min} \lesssim \delta^2$$

Here we have used the separation assumption (2.17) and the condition (2.19), $\alpha \min L \geq |\log \epsilon|$.

The exponential localization of the multi-soliton state $w$, the bootstrap assumptions (7.2) and the estimate (9.19) of Lemma 9.5 yield the estimate

$$\sum_{k=0}^{s} \int_{0}^{\infty} \left\| O \left( \nabla^k |w|^{p-2} Z^2 \right) \right\|_{L^1} \, d\tau \lesssim \| Z \|^2_{\mathcal{X}_s} \int_{0}^{\infty} (1 + \tau)^{-n} \, d\tau \lesssim \| Z \|^2_{\mathcal{X}_s}.$$ 

Finally, with the help of (9.17) (more specifically using the improvement (9.20) of Remark 9.6), we obtain

$$\sum_{k=0}^{s} \int_{0}^{\infty} \left\| \nabla^k (Z^p(\tau)) \right\|_{L^1} \, d\tau \lesssim \| Z \|^p_{\mathcal{X}_s}$$

### 9.3 $L^2$ estimates

In this subsection we establish the estimate

$$\| F(t, \cdot) \|_{H^s} \lesssim (1 + t)^{-\frac{n}{2} - 1} \left( \delta^2 + \| Z \|^2_{\mathcal{X}_s} + \| Z \|^p_{\mathcal{X}_s} \right).$$

The arguments follows closely those of the previous section. Using the estimates (9.30), (9.19) and the bootstrap assumptions (7.2) we obtain

$$\left\| \left( H(\tau, \sigma^\infty) - H(\sigma(\tau)) \right) Z(t) \right\|_{H^s} \lesssim \delta^2 (1 + t)^{2-n} \sum_{k=0}^{s} \| \nabla^k Z(t, \cdot) \|_{L^2 + L^\infty} \lesssim \delta^2 \| Z \|_{\mathcal{X}_s} (1 + t)^{2-n-\frac{n}{2}} \lesssim \delta^2 \| Z \|_{\mathcal{X}_s} (1 + t)^{-\frac{n}{2} - 1}$$

where the last inequality follows since $n \geq 3$. Similar to (9.32)

$$\| \dot{\Sigma}W(\sigma(t)) \|_{H^s} \lesssim \delta^2 (1 + t)^{-n} \lesssim \delta^2 (1 + t)^{-\frac{n}{2} - 1}$$
The estimates for the $O(w_1w_2)Z$ and $O(w_1w_2)$ terms again follow from the separation and the exponential localization of the solitons $w_1$ and $w_2$.

\[(9.38)\]
\[\|O(w_1w_2)\|_{H^s} \lesssim e^{-\alpha_{\text{min}}(L+ct)} \lesssim \delta^2 (1 + t)^{-\frac{n}{2} - 1}.\]

The exponential localization of the multi-soliton state $w$ together with the estimate (9.19) of Lemma 9.5 and the bootstrap assumption (7.2), also give the estimate

\[(9.39)\]
\[\|O(w_1w_2)\|_{H^s} \lesssim \|Z\|_{X^s}^2 (1 + t)^{-n} \lesssim \|Z\|_{X^s}^2 (1 + t)^{-\frac{n}{2} - 1}.\]

Finally, using the estimate (9.18) of Lemma 9.5, we obtain

\[(9.40)\]
\[\|Z^p(t)\|_{H^s} \lesssim (1 + t)^{-\frac{n}{2} - 1}\|Z\|_{X^s}^p.\]

This completes the proof of Lemma 9.3.

**10 Scattering**

In this section we intend to prove the last part of Theorem 2.1. More precisely, we shall show that there exists $u_0 \in L^2$ so that

\[\psi(t, \cdot) = \sum_{j=1}^{N} w_j(t, x; \sigma^\infty) + R(t, x) + Q(t, x),\]

where the function $Q$ is spatially localized and decays in time with the rate $t^{-n+2}$ and thus $\|Q(t)\|_{L^2} \to 0$ as $t \to \infty$. The function $R$ verifies an inhomogeneous Schrödinger equation

\[i \partial_t R + \frac{1}{2} \Delta R = V(t, x) R + F, \quad R(0, x) = R_0(x)\]

with spatially exponentially localized potential $V$. According to Lemma 9.2 the inhomogeneous term $F$ satisfies the estimate

\[(10.1)\]
\[\|F(t, \cdot)\|_{L^2} \lesssim (1 + t)^{-\frac{n}{2} - 1} \delta^2,\]

while the solution $R$ has been shown to obey the dispersive estimate

\[(10.2)\]
\[\|R(t, \cdot)\|_{L^2 + L^\infty} \lesssim (1 + t)^{-\frac{n}{2}} \delta.\]

The standard scattering theory argument show that the desired function $u_0$ can be constructed as the limit

\[u_0 = R_0 - i \lim_{t \to \infty} \int_0^t e^{-i\frac{\delta}{2} \Delta} (V(s, \cdot) R(s, \cdot) + F(s, \cdot)) \, ds.\]

The estimates (10.1) and (10.2) together with the localization of $V$ guarantee the existence of the limit.
11 Existence

In Lemmas 8.1 and 9.2 we established the estimates

\begin{equation}
|\dot{\sigma}| \leq \frac{1}{4} \delta^2 (1 + t)^{-n}
\end{equation}

for the admissible path \( \sigma(t) \) and

\begin{equation}
\|Z\|_{L^2} \leq \delta^2
\end{equation}

for the solution \( Z(t, x) \) of the nonlinear inhomogeneous matrix charge transfer problem (9.2), under the bootstrap assumptions (7.1), (7.2)

\begin{equation}
|\dot{\sigma}| \leq \delta^2 (1 + t)^{-n},
\end{equation}

\begin{equation}
\|Z\|_{L^2} \leq \delta C_0^{-1}
\end{equation}

and the condition that \( Z \) is asymptotically orthogonal to the null spaces of the Hamiltonians \( H_j^{\pm}(\sigma) \) with the constant \( \delta^3 \). In this section we shall show that these are sufficient to establish the existence of the desired admissible path and the perturbation \( R \). We prove existence by iteration. We shall define a sequence of admissible paths \( \sigma^{(n)}(t) \) and approximate solutions \( Z^{(n)}(t) \) for \( n = 1, \ldots \) according to the following rules. First, we write the \( Z \) equation (9.2) and (9.3) in the form

\begin{equation}
i \partial_t Z + H(\sigma(t))Z = \dot{\Sigma}W(\sigma(t)) + G(Z, \sigma(t))
\end{equation}

where \( G(Z, \sigma(t)) = O(w_1 w_2)Z + O(w_1 w_2) + O(|w| |Z|^2) + O(|Z|^p) \). Set

\[ \sigma^{(1)}(t) = \sigma(0), \quad Z^{(0)} \equiv 0 \]

where \( \sigma^{(1)} \) is to be understood as the constant path coinciding with the initial data \( \sigma(0) \) common to all admissible paths. We now define functions \( Z^{(1)}(t, x) \) and \( \sigma^{(2)}(t) \) to be a solution of the following linear system

\begin{equation}
i \partial_t Z^{(1)} + H(\sigma^{(1)}(t))Z^{(1)} = \dot{\Sigma}^{(2)}W(\sigma^{(1)}(t)) + G(Z^{(0)}, \sigma^{(1)}(t)),
\end{equation}

\[ Z^{(1)}(0, x) = Z_0(x), \]

\begin{equation}
\left\langle \dot{\Sigma}^{(2)}W(\sigma^{(1)}(t)), \xi_j^m(t, \cdot; \sigma^{(1)}(t)) \right\rangle = \left\langle G(Z^{(0)}, \sigma^{(1)}(t)), \xi_j^m(t, \cdot; \sigma^{(1)}(t)) \right\rangle + \left\langle \Omega_j^m(t, \cdot; \sigma^{(1)}(t)), Z^{(1)}(t) \right\rangle.
\end{equation}

Here \( \Omega_j^m(t, x; \sigma^{(1)}(t)) \) is defined via the equation

\[ i \partial_t \xi_j^m(t, \cdot; \sigma^{(1)}(t)) + H^*(\sigma^{(1)}(t))\xi_j^m(t, \cdot; \sigma^{(1)}(t)) = S_k^m \xi_j^k(t, \cdot; \sigma^{(1)}(t)) + \Omega_j^m(t, \cdot; \sigma^{(1)}(t)), \]

where the matrix \( S \) collects the terms \( \xi_j^m(t, \cdot; \sigma^{(1)}(t)) \) on the right-hand sides of (6.5)-(6.8). Thus, using Propositions 6.2 and 6.3, we have

\[ \Omega_j^m(t, x; \sigma^{(1)}(t)) = O(\dot{\sigma}^{(1)}(0) + |D\phi_j| + |D^2\phi_j|) + \sum_{r \neq j} V_r(t, x; \sigma^{(1)}(t))\xi_j^m(t, x; \sigma^{(1)}(t)). \]
Observe that (11.6) arises from the nonlinear equation (11.5) by replacing \( \sigma(t) \) with the already defined path \( \sigma^{(1)}(t) \) as well as \( Z \) on the right-hand side with \( Z^{(0)} = 0 \). The equation (11.7) determining \( \sigma^{(2)}(t) \) ensures that \( \langle Z^{(1)}(t), \xi_j^m(t, \cdot; \sigma^{(1)}(t)) \rangle = 0 \). Indeed, taking scalar products of (11.6) with \( \xi_j^m(t, \cdot; \sigma^{(1)}(t)) \) and using Propositions 6.2 and 6.3 yields

\[
\frac{d}{dt} \Xi(t; \sigma^{(1)}(t)) + S \Xi(t; \sigma^{(1)}(t)) = 0,
\]

where \( \Xi(t; \sigma^{(1)}(t)) \) is the vector of \( \langle \xi_j^m(t, \cdot; \sigma^{(1)}(t)), Z^{(1)}(t, \cdot) \rangle \) and where \( S \) is the constant matrix defined above. As long as

\[
(11.8) \quad \Xi(0; \sigma^{(1)}(0)) = \Xi(0; \sigma(0)) = 0
\]
one therefore has \( \Xi(t; \sigma^{(1)}(t)) = 0 \) for all \( t \geq 0 \). Generally speaking, (11.8) need not be satisfied. However, using the fact that the initial perturbation \( R_0 \) is small, we proceed as in [BP1] Proposition 1.3.1 to show that one can modify the initial splitting in such a way that it does hold. More precisely, one has the following lemma.

**Lemma 11.1.** Let \( \psi(0) = \sum_{j=1}^N w_j(0, x; \sigma(0)) + R_0 \) with \( R_0 \) small as in (2.9) and assume either one of the separation conditions (2.17) or (2.18) as well as the convexity condition (2.16) in a small neighborhood \( U \) of \( \sigma(0) \). Then there exist \( \tilde{\sigma}(0) \in U \) such that in the decomposition

\[
\psi(0) = \sum_{j=1}^N w_j(0, x; \tilde{\sigma}(0)) + \tilde{R}_0
\]

the new perturbation \( \left( \frac{\tilde{R}_0}{R_0} \right) \) is orthogonal to the root spaces \( N_j^* \) of \( H_j^* \) and satisfies the smallness condition (2.9).

**Proof.** We need to solve the equation (with the solution being \( \tilde{\sigma}(0) \))

\[
(11.9) \quad \left( \Psi(0) - \sum_{j=1}^N W_j(0, \cdot; \sigma(0)), J \partial_\sigma W_\ell(0, \cdot; \sigma(0)) \right) = 0 \quad \text{for all} \quad 1 \leq r \leq 2n + 2,
\]

where \( J \) is the matrix from Proposition 4.1 and \( \Psi, W_j \) are the complexified versions of \( \psi, w_j \), respectively. One solves (11.9) by means of the implicit function theorem. Indeed, the derivative of the left-hand side of (11.9) is given by

\[
\left\langle - \sum_{j=1}^N \partial_\sigma w_j(0, \cdot; \sigma(0)), J \partial_\sigma W_\ell(0, \cdot; \sigma(0)) \right\rangle + \left\langle \Psi(0) - \sum_{j=1}^N W_j(0, \cdot; \sigma(0)), J \partial_\sigma^2 \sigma_k W_\ell(0, \cdot; \sigma(0)) \right\rangle.
\]

The second term is \( O(\varepsilon) \) where \( \varepsilon \) controls the size of the initial perturbation \( R_0 \) in \( L^1 \), say. On the other hand, the first term is separated from zero by virtue of the convexity condition, and either the
bounded below by the product of the matrices with $j$ the latter ensures that we are taking scalar products of quantities that are almost orthogonal by virtue are small because of either the (physical) separation condition (2.17) or the velocity condition (2.18). Indeed, $R$ are invertible, and since the original perturbation by the left-hand side of (11.9) contains zero, as claimed.

The remaining entries of the derivative matrix, which involve inner products with $W_j, W_k$ for $j \neq k$, are small because of either the (physical) separation condition (2.17) or the assumption of large relative velocities of the solitons (2.18). Indeed, as in Proposition 1.3.1 from [BP1] one sees that uniformly in the small neighborhood of $\alpha_\ell(0)$ that we are allowing, see the convexity condition (2.16).

The latter ensures that we are taking scalar products of quantities that are almost orthogonal by virtue of the large distances of their Fourier transforms. Hence the determinant of the derivative is essentially bounded below by the product of the matrices with $j = k$, see (11.10). This proves that the derivatives are invertible, and since the original perturbation $R_0$ is small, the image of the diffeomorphism given by the left-hand side of (11.9) contains zero, as claimed.

In general, we define

\begin{equation}
(11.11)
\end{equation}

\[
i \partial_t Z^{(n)} + H(\sigma^{(n)}(t))Z^{(n)} = \dot{\Sigma}^{(n+1)}W(\sigma^{(n)}(t)) + G(Z^{(n-1)}, \sigma^{(n)}(t))
\]

\[
Z^{(n)}(0,x) = Z_0(x)
\]

\begin{equation}
(11.12)
\end{equation}

\[
\langle \dot{\Sigma}^{(n+1)}W(\sigma^{(n)}(t)), \xi_j^m(t, :, \sigma^{(n)}(t)) \rangle = \left\langle G(Z^{(n-1)}, \sigma^{(n)}(t)), \xi_j^m(t, :, \sigma^{(n)}(t)) \right\rangle + \left\langle \Omega_j^m(t, :, \sigma^{(n)}(t)), Z^{(n)}(t) \right\rangle.
\]

Here

\[
G(Z^{(n-1)}, \sigma^{(n)}(t)) = O(w_1^{(n)}w_2^{(n)}) + O(w_1^{(n)}w_2^{(n)})Z^{(n-1)} + O(|w^{(n)}|^{p-2}|Z^{(n-1)}|^2) + O(|Z^{(n-1)}|^p)
\]

and $\Omega_j^m(t, x; \sigma^{(n)}(t))$ once again is defined via the equation

\[
i \partial_t \xi_j^m(t, :, \sigma^{(n)}(t)) + H^*(\sigma^{(n)}(t))\xi_j^m(t, :, \sigma^{(n)}(t)) = \mathcal{S}_k^m \xi_j^k(t, :, \sigma^{(n)}(t)) + \Omega_j^m(t, :, \sigma^{(n)}(t)),
\]

and has the form

\[
\Omega_j^m(t, x; \sigma^{(n)}(t)) = O(\dot{\sigma}^{(n)}(|\phi_j| + |D\phi_j| + |D^2\phi_j|)) + \sum_{r \neq j} V_r(t, x; \sigma^{(n)}(t)) \xi_j^m(t, x; \sigma^{(n)}(t)).
\]

Observe that by the same argument as in the case of $Z^{(1)}$, the perturbation $Z^{(n)}$ is orthogonal to the functions $\xi_j^m(t, x; \sigma^{(n)}(t))$. We shall assume that solutions $Z^{(n)}, \sigma^{(n)}$ of (11.11), (11.12) have already been constructed and we now proceed to estimate them. We will return to the issue of constructing those solutions at the end of this section. We shall now assume that $\sigma^{(n)}$ and $Z^{(n-1)}$ satisfy (11.3) and (11.4) and prove the estimates (11.1) and (11.2) for $\sigma^{(n+1)}$ and $Z^{(n)}$. First we estimate $\dot{\sigma}^{(n+1)}$ in terms of $Z^{(n)}$. Observe that $\dot{\sigma}^{(n+1)}$ verifies the system of ODE’s described in Lemma 8.2 with the function $G(Z^{(n)}(t), \sigma^{(n)}(t))$ and $\Omega_j^m(t, x; \sigma^{(n)}(t))$. Therefore,

\begin{equation}
(11.13)
\end{equation}

\[
|\dot{\sigma}^{(n+1)}(t)| \leq (1 + t)^{-n} \left(\frac{1}{4}\delta^2 + C\|Z^{(n)}\|_{L^2} + C\|Z^{(n)}\|_{L^4}^2\right)
\]
We now consider the $Z^{(n)}$ equation. First, by construction $Z^{(n)}$ is orthogonal to $\xi^m_j(t,x;\sigma^{(n)}(t))$. Observe also that $Z^{(n)}$ satisfies the equation
\[ i\partial_t Z^{(n)} + H(t,\sigma^{(n)}) Z^{(n)} = \left( H(t,\sigma^{(n)}) - H(\sigma^{(n)}(t)) \right) Z^{(n)} + \dot{\Sigma}^{(n+1)} W(\sigma^{(n)}(t)) + G(Z^{(n-1)},\sigma^{(n)}(t)) \]
Corollary 7.4 implies that
\[ H(t,\sigma^{(n)}) - H(\sigma^{(n)}(t)) = V(t,x), \]
where a smooth localized potential $V$ has the property that $\sup_{0 \leq |\gamma| \leq s} \|\partial^\gamma V(t,\cdot)\|_{L^1 \cap L^\infty} \lesssim \delta^2 (1+t)^{-n+2}$.
Moreover, the calculation leading to (9.6) of Lemma 9.1 shows that
\[ \|M_j(\sigma,t)G_{v_j}(t)\xi^m_j(t,\cdot;\sigma^{(n)}(t)) - \xi^m_j(\cdot;\sigma^{(n)}(\cdot))\|_{L^1 \cap L^2} \lesssim \delta^2 (1+t)^{-n+2} \]
since $\sigma^{(n)}$ is an admissible path satisfying the bootstrap assumptions. Therefore, by the results of Corollary 5.4, taking into account smallness of the initial data $Z^{(n)}(0) = Z_0$, we obtain
\begin{align}
\|Z^{(n)}\|_{X_s} \lesssim & \sum_{k=0}^s \|\nabla^k Z_0\|_{L^1 \cap L^2} + \|G(Z^{(n-1)},\sigma^{(n)})\|_{Y_s} + \|\dot{\Sigma}^{(n+1)} W(\sigma^{(n)}(t))\|_{Y_s} \\
\lesssim & \delta^2 + \|Z^{(n-1)}\|_{X_s}^2 + \|Z^{(n-1)}\|_{X_s}^p + \|Z^{(n)}\|_{X_s}^2 + \|Z^{(n)}\|_{X_s}^p,
\end{align}
(11.14)
\begin{align}
\lesssim & \delta^2 + \|Z^{(n)}\|_{X_s}^2 + \|Z^{(n)}\|_{X_s}^p,
\end{align}
(11.15)
where the inequality leading to (11.14) follows from the estimates on the nonlinear term $G(Z^{(n-1)}(t),\sigma^{(n)}(t))$ and $\dot{\Sigma}^{(n+1)} W(\sigma^{(n)}(t))$ obtained in Lemma 9.3. The bound on $\dot{\Sigma}^{(n+1)} W(\sigma^{(n)}(t))$ also uses the inequality (11.13). To pass to (11.15) we used the assumption that $Z^{(n-1)}$ satisfies (11.2). In the same way one obtains a local in time version of equation (11.15).
\[ \|Z^{(n)}\|_{X_s(T)} \lesssim \delta^2 + \|Z^{(n)}\|_{X_s(T)}^2 + \|Z^{(n)}\|_{X_s(T)}^p, \]
(11.16)
which by continuity in $T$ implies the desired estimate $\|Z^{(n)}\|_{X_s} \lesssim \delta^2$.

Thus, the sequence $Z^{(n)}$ is uniformly bounded and small in the space $X_s$ while $(1+t)^n \dot{\Sigma}^{(n)}$ is uniformly small point wise in time. Therefore, we can choose a convergent subsequence of the paths $\sigma^{(k)}(t) \rightarrow \sigma(t)$ and a weekly convergent in $H^s(\mathbb{R}^n)$ subsequence $Z^{(k)} \rightarrow Z$. We multiply the equation (11.11) by a smooth compactly supported function $\zeta(x)$, integrate over the entire space and pass to the limit using that on any compact set $Z^{(n)} \rightarrow Z$ strongly in $H^{s'}$ for any $s' < s$. In particular, since $s > \frac{n}{2}$, $Z^{(n)} \rightarrow Z$ point wise. It will follow that $Z$ is a solution of the equation
\begin{align}
& i\partial_t Z + H(\sigma(t)) Z = \dot{\Sigma} W(\sigma(t)) + G(Z(t),\sigma(t)), \\
& Z(0,x) = Z_0(x)
\end{align}
We also pass to the limit in the equation (11.12) to obtain
\[ \left\langle \dot{\Sigma} W(\sigma(t)),\xi^m_j(t,\cdot;\sigma(t)) \right\rangle = \left\langle G(Z(t),\sigma(t)),\xi^m_j(t,\cdot;\sigma(t)) \right\rangle + \left\langle \Omega^m_j(t,\cdot;\sigma(t)),Z \right\rangle. \]
Comparing equations (11.17) and (11.18) we conclude that
\[ \left\langle Z(t),\xi^m_j(t,\cdot;\sigma(t)) \right\rangle = 0 \]
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for all \( j, m \). Therefore, the function \( \psi = R + w_1 + w_2 \) solves the original NLS and by uniqueness, say in \( L^2 \), \( \psi \) is our original solution.

To show existence of the solution \( Z^{(n)}, \sigma^{(n+1)} \) of the linear system (11.11), (11.12) we first construct the solution on a small time interval. We note that the "system" (11.12) for \( \dot{\sigma}^{(n+1)} \) can be resolved algebraically due to the spatial separation of the paths \( \sigma^{(n)}(t) \). Therefore, for simplicity we can replace the system (11.11), (11.12) by the following caricature:

\[
  i \partial_t \sigma + \frac{1}{2} \Delta \sigma = V(t, x) \sigma + \omega(t) a(t, x) + g(t, x), \\
  \omega(t) = \langle z, b(t, \cdot) \rangle + f(t)
\]

Here \( V, a, b, f \) are sufficiently smooth given functions and \( g(t, \cdot) \in H^s \) uniformly in \( t \). We eliminate \( \omega(t) \) and infer that

\[
  i \partial_t \sigma + \frac{1}{2} \Delta \sigma = V(t, x) \sigma + \langle z, c(t, \cdot) \rangle a(t, x) + F(t, x)
\]

with some new smooth functions \( c, a \) and an \( H^s \) function \( F(t, \cdot) \). Using the standard energy estimates we obtain that

\[
  \| z(t) \|_{H^s} \leq \| z_0 \|_{H^s} + C_1 t \sup_{\tau \leq t} \| z(\tau) \|_{H^s} + C_2 t \sup_{\tau \leq t} \| F(\tau, \cdot) \|_{H^s},
\]

where the constants \( C_1 \) and \( C_2 \) depend on \( V, a, b, f \). Therefore, we can establish the existence of the solution on the time interval of size \( \frac{1}{2} C_1^{-1} \) by means of the standard contraction argument. Then we can repeat this argument indefinitely thus constructing a global *classical* solution.

## 12 The linearized problem

### 12.1 Estimates for matrix charge transfer models

In this section we recall some of the estimates from Sections 7 and 8 from our companion paper [RSS].

First, consider the case of a system with a single matrix potential:

\[
  (12.1) \quad i \partial_t \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} + \begin{pmatrix} H + U & -W \\ W & -H - U \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0
\]

with \( U, W \) real-valued and \( H = \frac{1}{2} \Delta - \mu, \mu > 0 \). We say that \( A := \begin{pmatrix} H + U & -W \\ W & -H - U \end{pmatrix} \) is *admissible* iff the conditions of the following Definition 12.1 hold.

**Definition 12.1.** Let \( A \) be as above with \( U, W \) real-valued and exponentially decaying. The operator \( A \) on \( \text{Dom}(A) = H^2(\mathbb{R}^n) \times H^2(\mathbb{R}^n) \subset \mathcal{H} := L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \) is admissible provided

- \( \text{spec}(A) \subset \mathbb{R} \) and \( \text{spec}(A) \cap (-\mu, \mu) = \{ \omega_\ell \mid 0 \leq \ell \leq M \} \), for some \( M < \infty \) where \( \omega_0 = 0 \) and all \( \omega_j \) are distinct eigenvalues. There are no eigenvalues in \( \text{spec}_{\text{ess}}(A) = (-\infty, -\mu] \cup [\mu, \infty) \).

- For \( 1 \leq \ell \leq M \), \( L_\ell := \ker(A - \omega_\ell)^2 = \ker(A - \omega_\ell) \), and \( \ker(A) \subsetneq \ker(A^2) = \ker(A^3) =: L_0 \). Moreover, these spaces are finite dimensional.
• The ranges \( \text{Ran}(A - \omega_{\ell}) \) for \( 1 \leq \ell \leq M \) and \( \text{Ran}(A^2) \) are closed.

• The spaces \( L_\ell \) are spanned by exponentially decreasing functions in \( \mathcal{H} \) (say with bound \( e^{-\varepsilon_0|x|} \)).

• The points \( \pm \mu \) are not resonances of \( A \).

• All these assumptions hold as well for the adjoint \( A^* \). We denote the corresponding (generalized) eigenspaces by \( L_\ell^* \).

We will discuss these conditions in detail in the following Subsection 12.2. It is possible to establish some of these properties by means of “abstract” methods (for example, the exponential decay of elements of generalized eigenspaces via a variant of Agmon’s argument, or the closedness of \( \text{Ran}(A - \omega_{\ell}) \) from Fredholm’s theory), whereas others can be reduced to statements concerning certain semi-linear elliptic operators \( L_+, L_- \), see (12.31) (for example, that the spectrum is real or that only 0 can have a generalized eigenspace). In a later section we will prove for a particular model that \( L_+, L_- \) have the required properties. One condition that we will not deal with in this paper is the absence of embedded eigenvalues in the essential spectrum. This property will remain an assumption.

It is shown in [RSS], Lemma 7.2 that under these conditions there is a direct sum decomposition

\[
\mathcal{H} = \sum_{j=0}^{M} L_j + \left( \sum_{j=0}^{M} L_j^* \right)^\perp
\]

and we denote by \( P_s \) the induced projection onto \( \left( \sum_{j=0}^{M} L_j^* \right)^\perp \). In general, \( P_s \) is non-orthogonal. The letter “s” here stands for “scattering” (subspace). It is known that \( \text{Ran}(P_s) \) plays the role of the scattering states for the evolution \( e^{itA} \). Indeed, the main result from Section 7 in [RSS] is that if \( A \) is admissible and the linear stability property

\[
\sup_t \| e^{itA} P_s \psi \|_{2 \rightarrow 2} < \infty
\]

holds, then one has the dispersive bound

\[
\| e^{itA} P_s \psi_0 \|_{L^2 + L^\infty} \lesssim |t|^{-\frac{1}{2}} \| \psi_0 \|_{L^1 \cap L^2}
\]

(if in addition \( \| \hat{V} \|_1 < \infty \), then the \( L^2 \) norm can be removed on the left-hand side). Next, we recall the notion of matrix charge transfer models from Section 8 in [RSS].

**Definition 12.2.** By a matrix charge transfer model we mean a system

\[
i \partial_t \tilde{\psi} + \begin{pmatrix} \frac{1}{2} \triangle & 0 \\ 0 & -\frac{1}{2} \triangle \end{pmatrix} \tilde{\psi} + \sum_{j=1}^{\nu} V_j (-\vec{v}_j t) \tilde{\psi} = 0
\]

\[
\tilde{\psi}|_{t=0} = \tilde{\psi}_0,
\]

where \( \vec{v}_j \) are distinct vectors in \( \mathbb{R}^3 \), and \( V_j \) are matrix potentials of the form

\[
V_j(t, x) = \begin{pmatrix} U_j(x) & e^{i\theta_j(t, x)} W_j(x) \\ e^{-i\theta_j(t, x)} W_j(x) & -U_j(x) \end{pmatrix},
\]
where \( \theta_j(t, x) = (|\vec{v}_j|^2 + \alpha_j^2)t + 2x \cdot \vec{v}_j + \gamma_j, \alpha_j, \gamma_j \in \mathbb{R}, \alpha_j \neq 0. \) Furthermore, we require that each
\[
H_j = \left( \begin{array}{cc}
\frac{1}{2} \Delta - \frac{1}{2} \alpha_j^2 + U_j & -W_j \\
-W_j & -\frac{1}{2} \Delta + \frac{1}{2} \alpha_j^2 - U_j
\end{array} \right)
\]
be admissible in the sense of Definition 12.1 and that it satisfy the linear stability condition (12.3).

It is clear that the Hamiltonian in (2.23) is of this form. As in Lemma 3.1 above one now verifies the following. The Galilean transforms \( G_{\vec{v}} := G_{\vec{v}, 0} \) are defined as in (3.11), i.e.,
\[
G_{\vec{v}}(t) \left( \begin{array}{c} f_1 \\ f_2 \end{array} \right) = \left( \frac{g_{\vec{v}, 0}(t) f_1}{g_{\vec{v}, 0}(t) f_2} \right)
\]
where \( g_{\vec{v}, 0}(t) = e^{-\frac{|\vec{v}|^2}{2}t}e^{-ix \cdot \vec{v}}e^{it\vec{v} \cdot \vec{p}}. \)

**Lemma 12.3.** Let \( \alpha \in \mathbb{R} \) and set
\[
A := \left( \begin{array}{cc}
\frac{1}{2} \Delta - \frac{1}{2} \alpha^2 + U & -W \\
-W & -\frac{1}{2} \Delta + \frac{1}{2} \alpha^2 - U
\end{array} \right)
\]
with real-valued \( U, W \). Moreover, let \( \vec{v} \in \mathbb{R}^3, \theta(t, x) = (|\vec{v}|^2 + \alpha^2)t + 2x \cdot \vec{v} + \gamma, \gamma \in \mathbb{R}, \) and define
\[
H(t) := \left( \begin{array}{cc}
\frac{1}{2} \Delta + U(\cdot - \vec{v}t) & -e^{i\theta(t, -\vec{v}t)W(\cdot - \vec{v}t)} \\
e^{-i\theta(t, -\vec{v}t)W(\cdot - \vec{v}t)} & -\frac{1}{2} \Delta - U(\cdot - \vec{v}t)
\end{array} \right).
\]

Let \( S(t), S(0) = Id, \) denote the propagator of the system
\[
i\partial_t S(t) + H(t)S(t) = 0.
\]

Finally, let
\[
(12.6) \quad M(t) = M_{\alpha, \gamma}(t) = \left( \begin{array}{cc}
e^{-i\omega(t)/2} & 0 \\
0 & e^{i\omega(t)/2}
\end{array} \right)
\]
where \( \omega(t) = \alpha^2t + \gamma. \) Then
\[
(12.7) \quad S(t) = G_{\vec{v}, 0}(t)^{-1}M(t)^{-1}e^{itA}M(0)G_{\vec{v}, 0}(0)
\]

**Proof.** One has
\[
(12.8) \quad i\partial_t M(t) G_{\vec{v}}(t) S(t) = \left( \begin{array}{cc}
\frac{i}{2} \omega & 0 \\
0 & -\frac{i}{2} \omega
\end{array} \right) M(t) G_{\vec{v}}(t) S(t) + M(t) i G_{\vec{v}}(t) S(t) - M(t) G_{\vec{v}}(t) H(t) S(t).
\]

Let \( \rho(t, x) = t|\vec{v}|^2 + 2x \cdot \vec{v}. \) One now checks the following properties by differentiation:
\[
(12.9) \quad M(t) i G_{\vec{v}}(t) = - \left( \begin{array}{cc}
\frac{1}{2} |\vec{v}|^2 + \vec{v} \cdot \vec{p} & 0 \\
0 & -\frac{1}{2} |\vec{v}|^2 + \vec{v} \cdot \vec{p}
\end{array} \right) M(t) G_{\vec{v}}(t)
\]
\[
M(t) G_{\vec{v}}(t) H(t) = \left( \begin{array}{cc}
\frac{1}{2} \Delta + U & -e^{i(\theta - \rho - \omega)W} \\
e^{-i(\theta - \rho - \omega)W} & -\frac{1}{2} \Delta - U
\end{array} \right) M(t) G_{\vec{v}}(t)
\]
\[
- \left( \begin{array}{cc}
\frac{1}{2} |\vec{v}|^2 + \vec{v} \cdot \vec{p} & 0 \\
0 & -\frac{1}{2} |\vec{v}|^2 + \vec{v} \cdot \vec{p}
\end{array} \right) M(t) G_{\vec{v}}(t).
\]

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The right-hand side of (12.9) arises as follows. First, the Galilei transform introduces a factor of $e^{-ix\cdot \vec v}$, which needs to be commuted with $\frac{1}{2}\triangle$. Since

$$
\frac{1}{2}\triangle (e^{-ix\cdot \vec v}f) = -\frac{1}{2}|\vec v|^2 e^{-ix\cdot \vec v}f - e^{-ix\cdot \vec v}i\vec v \cdot \vec \nabla f + \frac{1}{2}e^{-ix\cdot \vec v}\triangle f
$$

$$
= \frac{1}{2}|\vec v|^2 e^{-ix\cdot \vec v}f - i\vec v \cdot \vec \nabla (fe^{-ix\cdot \vec v}) + \frac{1}{2}e^{-ix\cdot \vec v}\triangle f
$$

$$
= \left( \frac{1}{2}|\vec v|^2 + \vec v \cdot \vec p \right) (fe^{-ix\cdot \vec v}) + \frac{1}{2}e^{-ix\cdot \vec v}\triangle f,
$$

one obtains the final term on the right-hand side of (12.9). It remains to check the terms involving the potentials (for simplicity $\theta(t, \cdot - \vec v) = \theta(t(t, \cdot - \vec v))$):

$$
\mathcal{M}(t)\mathcal{G}_\vec{v}(t)
\begin{pmatrix}
U(\cdot - \vec v\omega(t)/2) & -e^{i\theta(t, \cdot - \vec v)}W(\cdot - \vec v\omega(t)/2) \\
0 & -U(\cdot - \vec v\omega(t)/2)
\end{pmatrix}
\begin{pmatrix}
f_1 \\
f_2
\end{pmatrix}
$$

$$
= \begin{pmatrix}
e^{-i\omega(t)/2} & 0 \\
e^{i\omega(t)/2} & 0
\end{pmatrix}
\begin{pmatrix}
\mathcal{G}_\vec{v}(t)U(\cdot - \vec v\omega(t)/2)f_1 - \mathcal{G}_\vec{v}(t)e^{i\theta(t, \cdot - \vec v)}W(\cdot - \vec v\omega(t)/2)f_2 \\
\mathcal{G}_\vec{v}(t)e^{i\theta(t, \cdot - \vec v)}W(\cdot - \vec v\omega(t)/2)f_1 - \mathcal{G}_\vec{v}(t)U(\cdot - \vec v\omega(t)/2)f_2
\end{pmatrix}
$$

$$
= \begin{pmatrix}
U \mathcal{G}_\vec{v}(t)(e^{-i\omega(t)/2}f_1) & -W e^{-i\omega(t)/2}e^{i(\theta - \omega)}(\mathcal{G}_\vec{v}(t)e^{i\omega(t)/2}f_2) \\
W e^{i(\theta - \omega)} \mathcal{G}_\vec{v}(t)(e^{-i\omega(t)/2}f_1) - U \mathcal{G}_\vec{v}(t)e^{i\omega(t)/2}f_2
\end{pmatrix}
$$

\begin{pmatrix}
U & -e^{i(\theta - \omega)}W \\
e^{-i(\theta - \omega)}W & 0
\end{pmatrix}
\begin{pmatrix}
e^{-i\omega(t)/2} & 0 \\
e^{i\omega(t)/2} & 0
\end{pmatrix}
\begin{pmatrix}
\mathcal{G}_\vec{v}(t)f_1 \\
\mathcal{G}_\vec{v}(t)f_2
\end{pmatrix},
$$

as claimed. In view of our definitions, $\theta - \rho - \omega = 0$. Since $\omega = \alpha^2$, the lemma follows by inserting (12.9) into (12.8).

In order to prove our main dispersive estimates for such matrix charge transfer problems we need to formulate a condition which ensures that the initial condition belongs to the stable subspace. To do so, let $P_\alpha(H_j)$ and $P_\beta(H_j)$ be the projectors induced by the decomposition (12.2) for the operator $H_j$. Abusing terminology somewhat, we refer to $\text{ Ran}(P_\alpha(H_j))$ as the bound states of $H_j$.

**Definition 12.4.** Let $U(t)\vec{\psi}_0 = \vec{\psi}(t, \cdot)$ be the solution of (12.5). We say that $\vec{\psi}_0$ is a scattering state relative to $H_j$ if

$$
\|P_\alpha(H_j, t)U(t)\vec{\psi}_0\|_{L_2} \rightarrow 0 \text{ as } t \rightarrow +\infty.
$$

Here

$$
P_\beta(H_j, t) := \mathcal{G}_\vec{v}_j(t)^{-1}\mathcal{M}_j(t)^{-1}P_\beta(H_j)\mathcal{M}_j(t)\mathcal{G}_\vec{v}_j(t)
$$

with $\mathcal{M}_j(t) = \mathcal{M}_{\alpha_j, \gamma_j}(t)$ as in (12.6).

The formula (12.10) is of course motivated by (12.7). Clearly, $P_\beta(H_j, t)$ is the projection onto the bound states of $H_j$ that have been translated to the position of the matrix potential $V_j(\cdot - t\vec{v}_j)$. Equivalently, one can think of it as translating the solution of (12.5) from that position to the origin, projecting onto the bound states of $H_j$, and then translating back.

We now formulate our decay estimate for matrix charge transfer models, see Theorem 8.6 in [RSS].
**Theorem 12.5.** Consider the matrix charge transfer model as in Definition 12.2. Let $U(t)$ denote the propagator of the equation (12.5). Then for any initial data $\vec{\psi}_0 \in L^1 \cap L^2$, which is a scattering state relative to each $H_j$ in the sense of Definition 12.4, one has the decay estimates

\begin{equation}
\|U(t)\vec{\psi}_0\|_{L^\infty} \lesssim \langle t \rangle^{-\frac{3}{2}} \|\vec{\psi}_0\|_{L^1 \cap L^2}.
\end{equation}

For technical reasons, we need the estimate (12.11) for perturbed matrix charge transfer equations, as described in the following corollary. This is discussed in Remark 8.6 in [RSS].

**Corollary 12.6.** Let $\vec{\psi}$ be a solution of the equation

\begin{equation}
\begin{aligned}
&i\partial_t \vec{\psi} + \left( \begin{array}{cc}
\frac{1}{2}\Delta & 0 \\
0 & -\frac{1}{2}\Delta
\end{array} \right) \vec{\psi} + \sum_{j=1}^\nu V_j(\cdot - v_j^t)\vec{\psi} + V_0(t,x)\vec{\psi} = 0 \\
&\vec{\psi}|_{t=0} = \vec{\psi}_0,
\end{aligned}
\end{equation}

where everything is the same as in Definition 12.2 up to the perturbation $V_0(t,x)$ which satisfies

\[ \sup_t \|V_0(t,\cdot)\|_{L^1 \cap L^\infty} < \varepsilon. \]

Let $\tilde{U}(t)$ denote the propagator of the equation (12.12). Then for any initial data $\vec{\psi}_0 \in L^1 \cap L^2$, which is a scattering state relative to each $H_j$ in the sense of Definition 12.4 (with $U(t)$ replaced by $\tilde{U}(t)$), one has the decay estimates

\begin{equation}
\|\tilde{U}(t)\vec{\psi}_0\|_{L^\infty} \lesssim \langle t \rangle^{-\frac{3}{2}} \|\vec{\psi}_0\|_{L^1 \cap L^2}
\end{equation}

provided $\varepsilon$ is sufficiently small.

The corresponding inhomogeneous bound is stated in Section 12.

### 12.2 The spectral properties I: general arguments

In order for the linear estimates to apply, we need to impose the conditions in Definition 12.1 as well as the linear stability condition (12.3) on the operators from (2.24). The admissibility conditions of Definition 12.1 were motivated to a large extent by Buslaev and Perelman [BP1], who built on earlier work of Weinstein [We1]. We now analyze these conditions in detail. As before,

\begin{equation}
A := \left( \begin{array}{cc}
H + U & -W \\
W & -H - U
\end{array} \right) = B + V
\end{equation}

where $U, W$ are real-valued, $H = \frac{1}{2}\Delta - \mu$ with $\mu > 0$, and $V$ is the matrix potential consisting of $U, W$. In this subsection we deal with those properties that can be dealt with by means of general arguments, that make no use of any special structure of the operator.

**Lemma 12.7.** Let the matrix potential $V$ be bounded and go to zero at infinity. Then $(A - z)^{-1}$ is a meromorphic function in $\Omega := \mathbb{C} \setminus (-\infty, -\mu] \cup [\mu, \infty)$. The poles are eigenvalues of $A$ of finite multiplicity and $\text{Ran}(A - z)$ is closed for all $z \in \Omega$. Finally, the complement of $\Omega$ agrees with the essential spectrum of $A$, i.e., $\text{spec}_\text{ess}(A) = (-\infty, -\mu] \cup [\mu, \infty)$. 

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Proof. Suppose that \( z \in \Omega \). Then \( B - z \) is invertible, and \( A - z = \left( 1 + V(B - z)^{-1} \right)(B - z) \). Since \( V(B - z)^{-1} \) is analytic and compact in that region of \( z \)'s, the analytic Fredholm theorem implies that \( 1 + V(B - z)^{-1} \) is invertible for all but a discrete set of \( z \)'s in \( \Omega \). Furthermore, the poles are precisely eigenvalues of \( A \) of finite multiplicity. It is also a general property that the ranges \( \text{Ran}(1 + V(B - z)^{-1}) \) are closed. Indeed, if \( K \) is any compact operator on a Banach space, then it is well-known and also easy to see that \( \text{Ran}(I - K) \) is closed. Since \( B - z \) has a bounded inverse for all \( z \in \Omega \), this implies that \( \text{Ran}(A - z) \) is closed, as claimed. Conjugating by the matrix \( P = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \) leads to the Hamiltonians

\[
\tilde{A} := P^{-1}AP = i \begin{pmatrix} 0 & H + V_1 \\ -H - V_2 & 0 \end{pmatrix} = \tilde{B} + V, \quad \tilde{B} = i \begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix}, \quad V = i \begin{pmatrix} 0 & V_1 \\ -V_2 & 0 \end{pmatrix}
\]

where \( V_1 = U + W \) and \( V_2 = U - W \). The system (12.15) corresponds to writing a vector in terms of real and imaginary parts, whereas (12.28) corresponds to working with the solution itself and its conjugate. By means of the matrix \( J = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \) one can also write

\[
\tilde{B} = \begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} H + V_1 & 0 \\ 0 & H + V_2 \end{pmatrix} J.
\]

Since \( \tilde{B}^* = \tilde{B} \) it follows that \( \text{spec}(\tilde{B}) \subseteq \mathbb{R} \). One checks that for \( \Re z \neq 0 \)

\[
(\tilde{B} - z)^{-1} = (\tilde{B} + z) \begin{pmatrix} (H^2 - z^2)^{-1} & 0 \\ 0 & (H^2 - z^2)^{-1} \end{pmatrix} \begin{pmatrix} (H^2 - z^2)^{-1} & 0 \\ 0 & (H^2 - z^2)^{-1} \end{pmatrix} (\tilde{B} + z)
\]

(12.16)

\[
(\tilde{A} - z)^{-1} = (\tilde{B} - z)^{-1} - (\tilde{B} - z)^{-1} W_1 \left[ 1 + W_2 J(\tilde{B} - z)^{-1} W_1 \right]^{-1} W_2 J(\tilde{B} - z)^{-1}
\]

where \( W_1 \) and \( W_2 \) are the following matrix potentials that go to zero at infinity:

\[
W_1 = \begin{pmatrix} |V_1|^{\frac{1}{2}} & 0 \\ 0 & |V_2|^{\frac{1}{2}} \end{pmatrix}, \quad W_2 = \begin{pmatrix} |V_1|^{\frac{1}{2}} \text{sign}(V_1) & 0 \\ 0 & |V_2|^{\frac{1}{2}} \text{sign}(V_2) \end{pmatrix}.
\]

The inverse of the operator in brackets exists if \( z = it \) with \( t \) large, for example. Moreover, by the assumed decay of the potential the entire operator that is being subtracted from the right-hand side is compact in that case. One is therefore in a position to apply Weyl’s criterion, see Theorem XIII.14 in [RS4], whence

\[
\text{spec}_{\text{ess}}(A) = \text{spec}_{\text{ess}}(\tilde{A}) = (-\infty, -\mu] \cup [\mu, \infty).
\]

The identity (12.17) goes back to Grillakis [Gr]. \( \square \)

Next, we need to locate possible eigenvalues of \( A \) or equivalently, \( \tilde{A} \). This will not be done on the same general level, but require analysis of \( L_+, L_- \) from (12.31). But we first discuss another general property of the matrix operator \( A \).
Lemma 12.8. Let $A$ be as in (12.14) with $U,W$ continuous and $W$ exponentially decaying, whereas $U$ is only required to tend to zero. If $f \in \ker(A - E)^k$ for some $-\mu < E < \mu$ and some positive integer $k$, then $f$ decays exponentially.

Proof. We want to emphasize that the following result is “abstract” and does not rely on any special structure of the matrix potential or on any properties of $L_+$ or $L_-$. We will use a variant of Agmon’s argument [Ag]. More precisely, suppose that for some $-\mu < E < \mu$, there are $\psi_1, \psi_2 \in H^2(\mathbb{R}^n)$ so that

$$
(\Delta - \mu + U)\psi_1 - W\psi_2 = E\psi_1 \quad \text{and} \quad W\psi_1 + (-\Delta + \mu - U)\psi_2 = E\psi_2.
$$

As usual, $U,W$ are real-valued and exponentially decaying, $\mu > 0$. Suppose $|W(x)| \lesssim e^{-b|x|}$. Then define the Agmon metrics

$$
\rho_E^\pm(x) = \inf_{\gamma:0 \rightarrow x} L_{Ag}^\pm(\gamma)
$$

$$
L_{Ag}^\pm(\gamma) = \int_0^1 \min(\sqrt{(\mu \pm E - U(\gamma(t)))_+}, b/2) \|\dot{\gamma}(t)\| \, dt
$$

(12.20)

where $\gamma(t)$ is a $C^1$-curve with $t \in [0,1]$, and the infimum is to be taken over such curves that connect 0, $x$. These functions satisfy

$$
|\nabla \rho_E^\pm(x)| \leq \sqrt{(\mu \pm E - U(x))_+}.
$$

Moreover, one has $\rho_E^\pm(x) \leq b|x|/2$ by construction. Now fix some small $\varepsilon > 0$ and set $\omega^\pm(x) := e^{2(1-\varepsilon)\rho_E^\pm(x)}$. Our goal is to show that

$$
\int \left[ \omega^+(x)|\psi_1(x)|^2 + \omega^-(x)|\psi_2(x)|^2 \right] \, dx < \infty.
$$

(12.22)

Not only does this exponential decay in the mean suffice for our applications (cf. Section 7 in [RSS]), but it can also be improved to point wise decay using regularity estimates for $\psi_1, \psi_2$. We do not elaborate on this, see for example [Ag] and Hislop, Sigal [HiSig].

Fix $R$ arbitrary and large. For technical reasons, we set

$$
\rho_{E,R}^\pm(x) := \min \left( 2(1-\varepsilon)\rho_E^\pm(x), R \right), \quad \omega_R^\pm(x) := e^{\rho_{E,R}^\pm(x)}.
$$

Notice that (12.21) remains valid in this case, and also that $\rho_{E,R}^\pm(x) \leq \min(b|x|/2, R)$. Furthermore, by choice of $E$ there is a smooth functions $\phi$ that is equal to one for large $x$ so that

$$
\text{supp}(\phi) \subset \{\mu + E - U > 0\} \cap \{\mu - E - U > 0\}.
$$

It will therefore suffice to prove the following modified form of (12.22):

$$
\sup_R \int \left[ \omega_R^+(x)|\psi_1(x)|^2 + \omega_R^-(x)|\psi_2(x)|^2 \right] \phi^2(x) \, dx < \infty.
$$

(12.23)
All constants in the following argument will be independent of $R$. By construction, there is $\delta > 0$ such that

\begin{equation}
\label{eq:12.24}
\delta \int \omega_R^+(x) |\psi_1(x)|^2 \phi^2(x) \, dx \leq \int \omega_R^+(x)(\mu + E - U(x)) |\psi_1(x)|^2 \phi^2(x) \, dx
\end{equation}

\begin{equation*}
= \int \omega_R^+(x)(\Delta \psi_1 - W \psi_2)(x) \bar{\psi}_1(x) \phi^2(x) \, dx
\end{equation*}

\begin{equation}
\label{eq:12.25}
= - \int \nabla(\omega_R^+(x)\phi^2(x)) \nabla \psi_1(x) \bar{\psi}_1(x) \, dx - \int \omega_R^+(x)\phi^2(x) |\nabla \psi_1(x)|^2 \, dx
\end{equation}

\begin{equation}
\label{eq:12.26}
- \int \omega_R^+(x)W(x)\psi_1(x)\bar{\psi}_2(x)\phi^2(x) \, dx.
\end{equation}

As far as the final term (12.26) is concerned, notice that $\sup_{x,R} |\omega_R^+(x)\phi^2(x)W(x)| \leq 1$ by construction, whence $|\int (12.26)| \lesssim \|\psi_1\|_2 \|\psi_2\|_2$. Furthermore, by (12.21) and Cauchy-Schwarz, the first integral in (12.25) satisfies

\begin{equation}
\label{eq:12.27}
\left| \int \nabla(\omega_R^+(x)\phi^2(x)) \nabla \psi_1(x) \bar{\psi}_1(x) \, dx \right|
\end{equation}

\begin{equation*}
\leq 2(1 - \varepsilon) \left( \int \omega_R^+(x)(\mu + E - U(x))\phi^2(x) |\psi_1(x)|^2 \, dx \right)^{\frac{1}{2}} \left( \int \omega_R^+(x)\phi^2(x) |\nabla \psi_1(x)|^2 \, dx \right)^{\frac{1}{2}}
\end{equation*}

\begin{equation*}
+ 2 \left( \int \omega_R^+(x)\phi^2(x) |\nabla \psi_1(x)|^2 \, dx \right)^{\frac{1}{2}} \left( \int \omega_R^+(x)\phi^2(x) |\psi_1(x)|^2 \, dx \right)^{\frac{1}{2}}
\end{equation*}

Since the first integral in (12.27) is the same as that in (12.24), inserting (12.27) into (12.25) yields after some simple manipulations

\begin{equation*}
\varepsilon \int \omega_R^+(x)(\mu + E - U(x)) |\psi_1(x)|^2 \phi^2(x) \, dx \leq \varepsilon^{-1} \int \omega_R^+(x)\phi^2(x) |\psi_1(x)|^2 \, dx
\end{equation*}

\begin{equation*}
- \int \omega_R^+(x)\phi^2(x)W(x)\psi_2(x) \bar{\psi}_1(x) \, dx.
\end{equation*}

Since $\nabla \phi$ has compact support, and by our previous considerations involving $\omega_R^+ W$, the entire right-hand side is bounded independently of $R$, and thus also (12.24). A symmetric argument applies to the integral with $\psi_2$, and (12.23), (12.22) hold. This method also shows that functions belonging to generalized eigenspaces decay exponentially. Indeed, suppose $(A - E)\vec{g} = 0$ and $(A - E)\vec{f} = \vec{g}$. Then

\begin{equation*}
(\Delta - \mu + U)f_1 - Wf_2 = Ef_1 + g_1
\end{equation*}

\begin{equation*}
Wf_1 + (-\Delta + \mu - U)f_2 = Ef_2 + g_2
\end{equation*}

with $g_1, g_2$ exponentially decaying. Decreasing the value of $b$ in (12.20) if necessary allows one to use the same argument as before to prove (12.22) for $\vec{f}$. By induction, one then deals with all values of $k$ as in the statement of the lemma.

\section{12.3 The spectral properties II: reduction to $L_+, L_-$}

We now need to specialize $A$ from (12.14) to the form (2.24), i.e.,

\begin{equation}
\label{eq:12.28}
A = \begin{pmatrix}
\frac{1}{2}\Delta - \frac{g^2}{2} + \beta(\phi^2) + \beta'(\phi^2)\phi^2 \\
-\beta'(\phi^2)\phi^2 & -\frac{1}{2}\Delta + \frac{g^2}{2} - \beta(\phi^2) - \beta'(\phi^2)\phi^2
\end{pmatrix}.
\end{equation}

As shown in Section 2, these are the stationary Hamiltonians derived from the linearization of NLS, see (2.24) and Lemma 12.3. Let \( \alpha > 0 \) and \( \phi \) be a nonzero solution of
\[
\frac{1}{2} \triangle \phi - \frac{\alpha^2}{2} \phi + \beta(\phi^2) \phi = 0,
\]
i.e., \( \phi = \phi(\cdot; \alpha) \) for all \( \alpha \in (\alpha_0 - c_0, \alpha_0 + c_0) \). Moreover, \( \phi \) is smooth in both variables and
\[
\| \partial^a_\alpha \phi \|_{H^1(\mathbb{R}^n)} + \| \partial^{2}_\alpha \phi \|_{H^1(\mathbb{R}^n)} < \infty.
\]
Finally, we require that \( \phi \) is sufficiently rapidly decaying. A particular case would be a \( \phi \) which is positive and radially symmetric. Such a solution is known to exist and to be unique if \( \beta(u) = |u|^\sigma \) provided \( 0 < \sigma < \frac{2}{d-2} \) and is referred to as the “ground state”. It decays exponentially. However, in order to keep this section as general as possible, we do not require \( \phi \) to be the ground state. Let
\[
L_- := -\frac{1}{2} \triangle + \frac{\alpha^2}{2} - \beta(\phi^2), \quad L_+ := -\frac{1}{2} \triangle + \frac{\alpha^2}{2} - \beta(\phi^2) - 2\beta'(\phi^2) \phi^2
\]
with domains \( \text{Dom}(L_+) = \text{Dom}(L_-) = H^2(\mathbb{R}^n) \) so that
\[
\tilde{A} = \begin{pmatrix}
0 & -iL_- \\
iL_+ & 0
\end{pmatrix}
\]
with \( \text{Dom}(\tilde{A}) = H^2(\mathbb{R}^n) \times H^2(\mathbb{R}^n) \). Here \( \tilde{A} \) is obtained by conjugating \( A \) with the matrix \( P \), see (12.15). For simplicity, however, we no longer distinguish between \( A \) and \( \tilde{A} \), i.e., we set \( A = \tilde{A} \). The spectrum of \( L_\pm \) on \( [\mu, \infty) \) is purely absolutely continuous, and below \( \mu = \frac{\alpha^2}{2} > 0 \) there are at most a finite number of eigenvalues of finite multiplicity (by Birman-Schwinger, the assumed decay of \( \phi \) as well as \( \beta(0) = 0 \)). Clearly,
\[
L_- \phi = 0, \quad L_+ (\partial_j \phi) = 0, \quad 1 \leq j \leq n, \quad L_+ (\partial_\alpha \phi) = -\alpha \phi,
\]
where the final property is formal. We now collect some crucial properties discovered by M. Weinstein.

**Definition 12.9.** **Needed properties of the scalar elliptic operators \( L_+ \) and \( L_- \):**

Let \( \phi(\cdot; \alpha) \) be above, in particular assume that (12.30) holds. The kernels have the following explicit form:
\[
\ker(L_-) = \text{span}\{\phi\} \quad \text{and} \quad \ker(L_+) = \text{span}\{\partial_j \phi \mid 1 \leq j \leq n\}.
\]

The operator \( L_+ \) has a single negative eigenvalue \( E_1 \) with a unique ground state \( \psi > 0 \), whereas \( L_- \) is nonnegative, with 0 as an isolated eigenvalue. Furthermore, the convexity condition \( \langle \partial_\alpha \phi(\cdot; \alpha), \phi(\cdot; \alpha) \rangle > 0 \) holds, see (1.17).

In the following section, these conditions will be verified for a particular choice of nonlinearity. Note that according to this definition, \( \phi \) is the ground state of the linear operator \( L_- \) and as such is positive. In case of \( \phi(\cdot; \alpha) \) being the ground state of the nonlinear problem (12.29), these properties
have been shown to hold by Weinstein [We1] and [We2] in case of power nonlinearities, i.e., \( \beta(u) = |u|^{\sigma} \), \( 0 < \sigma < \frac{2}{d-2} \).

The purpose of this subsection is to reduce some of the admissibility conditions from Definition 12.1 to the properties of \( L_+, L_- \) from Definition 12.9. Recall from the previous subsection that several other properties hold in greater generality. We now collect those properties that follow from Definition 12.9 into a single proposition.

**Proposition 12.10.** Impose the spectral assumption on \( L_+ \) and \( L_- \) from Definition 12.9. Then

- \( \text{spec}(A) \subset \mathbb{R} \), the only eigenvalue that admits a generalized eigenspace is 0, and \( \text{Ran}(A^2) \) is closed.
- The linear stability condition holds.
- Equality holds in the relation concerning the root spaces (12.39) and (12.40). In particular, one has \( \text{ker}(A^2) = \text{ker}(A^3) \) and \( \text{ker}((A^*)^2) = \text{ker}((A^*)^3) \) and the rootspace \( \mathcal{N}(A^*) \) has dimension \( 2n + 2 \).

The proof of this proposition is split into several lemmas below.

**Lemma 12.11.** Impose the spectral assumption on \( L_+ \) and \( L_- \) from Definition 12.9. Then \( \text{spec}(A) \subset \mathbb{R} \), the only eigenvalue that admits a generalized eigenspace is 0, and \( \text{Ran}(A^2) \) is closed.

**Proof.** Consider

\[
A^2 = \begin{pmatrix} T^* & 0 \\ 0 & T \end{pmatrix}, \quad T = L_+L_-
\]

with domain \( H^4(\mathbb{R}^n) = W^{4,2}(\mathbb{R}^n) \). Following [BP1], we first show that any eigenvalue of \( T \), and therefore also of \( \text{spec}(A^2) \) is real, and then under the assumption (1.17), that it is nonnegative. Because of (12.18), the latter then implies that \( \text{spec}(A) \) is real, as required in Definition 12.1. Clearly, \( T\phi = 0 \).

Let \( \psi \notin \text{span}\{\phi\}, T\psi = E\psi \). Let \( \psi = \psi_1 + c\phi, \psi_1 \perp \phi \). Then

\[
L_+^{1 \over 2}L_+^{1 \over 2}L_-^{1 \over 2}\psi_1 = EL_+^{1 \over 2}\psi_1,
\]

so that \( L_-^{1 \over 2}\psi_1 \neq 0 \) is an eigenfunction of the symmetric operator \( L_+^{1 \over 2}L_+^{1 \over 2}L_-^{1 \over 2} \) (with domain \( H^4(\mathbb{R}^n) \)), and thus \( E \) is real. Hence any eigenvalue of \( A \) can only be real or purely imaginary. Since \( \phi \perp \text{ker}(L_+) \) by our assumption concerning \( L_+ \), the function

\[
g(E) := \langle (L_+ - E)^{-1}\phi, \phi \rangle
\]

is well-defined on an interval of the form \( (E_1, E_2) \) for some \( E_2 > 0 \). Moreover,

\[
g'(E) = \| (L_+ - E)^{-1}\phi \|^2 > 0
\]

so that \( g(E) \) is strictly increasing on the interval. Finally,

\[
g(0) = -\frac{1}{\alpha} \langle \partial_\alpha \phi, \phi \rangle < 0
\]

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in view of (12.33) and (1.17). Now suppose that \( A^2 \) has a negative eigenvalue. Then by the preceding, so does \( T \), and therefore also \( L_+^\frac{1}{2} L_+ L_-^\frac{1}{2} \). More precisely, the argument from before implies that there is \( \chi \in \ker(L_-)^\perp \), \( \chi \neq 0 \), so that

\[
(L_-^\frac{1}{2} L_+ L_-^\frac{1}{2} \chi, \chi) = (L_+ \psi, \psi) < 0
\]

with \( \psi = L_-^\frac{1}{2} \chi \). Let \( P_-^\perp \) denote the projection onto the orthogonal complement of \( \ker(L_-) = \text{span}(\phi) \).

By the Rayleigh principle this implies that the self-adjoint operator \( P_-^\perp L_+ P_-^\perp \) has a negative eigenvalue, say \( E_3 < 0 \). Thus \( L_+ \psi = E_3 \psi + c \phi \) for some \( \psi \perp \phi \). If \( c = 0 \), then \( E_3 = E_1 \) so that \( \psi > 0 \) as the ground state of \( L_+ \). But then \( \langle \phi, \psi \rangle > 0 \), which is impossible. So \( c \neq 0 \), and one therefore obtains

\[
(L_+ - E_3)^{-1} \phi = \frac{1}{c} \psi \implies g(E_3) = 0.
\]

But this contradicts (12.35) by strict monotonicity of \( g \). Thus \( A^2 \) does not have any negative eigenvalues, which implies that \( A \) does not have imaginary eigenvalues. Hence all eigenvalues of \( A \) are real, as desired.

We now turn to generalized eigenspaces. Suppose \( A \psi = E \psi + \chi \), where \( E \neq 0 \), \( (A - E) \chi = 0 \) and \( \chi \neq 0 \). This is equivalent to saying that \( A \) has a generalized eigenspace at \( E \). Then \( \psi, \chi \in \text{Dom}(A^2) \), and moreover

\[
(A^2 - E^2) \chi = 0, \quad (A^2 - E^2) \psi = (A - E) \chi + 2E \chi = 2E \chi,
\]

so that \( A^2 \) would have a generalized eigenspace at \( E \), and therefore also \( T \). Hence, suppose \( T \psi = E \psi \), with \( E \neq 0 \), \( \psi \neq 0 \). If \( (T - E) \chi = c \psi \) with \( c \neq 0 \), then

\[
(L_-^\frac{1}{2} L_+ L_-^\frac{1}{2} - E) L_-^\frac{1}{2} \chi_1 = c L_-^\frac{1}{2} \psi_1 \neq 0, \quad (L_-^\frac{1}{2} L_+ L_-^\frac{1}{2} - E)^2 L_-^\frac{1}{2} \chi_1 = c L_-^\frac{1}{2} (L_+ L_- - E) \psi = 0
\]

where \( \psi_1, \chi_1 \) denote the projections of \( \psi, \chi \) onto the orthogonal complement of \( \phi \). But \( L_-^\frac{1}{2} \psi_1 \neq 0 \) since \( E \neq 0 \) and thus \( E \) would have to be a generalized eigenvalue of \( L_-^\frac{1}{2} L_+ L_-^\frac{1}{2} \), which is impossible. So only \( E = 0 \) can have a generalized eigenspace. Here we used the property that \( L_-^\frac{1}{2} L_+ L_-^\frac{1}{2} \) is self-adjoint on its domain \( H^1(\mathbb{R}^n) \). While symmetry is obvious, self-adjointness on \( H^1(\mathbb{R}^n) \) requires a bit more care.

Suppose \( \langle L_-^\frac{1}{2} L_+ L_-^\frac{1}{2} f, g \rangle = \langle f, h \rangle \) for all \( f \in H^4(\mathbb{R}^n) \), and some fixed \( g, h \in L^2(\mathbb{R}^n) \). Taking \( f \in \ker(L_-) \) shows that \( P_- \hbar = h \), i.e., that \( h \in (\ker(L_-^\frac{1}{2}))^\perp \). By the Fredholm alternative applied to the self-adjoint operator \( L_\perp^\frac{1}{2} \), one can write \( h = L_\perp^{\frac{1}{2}} h_1 \) with some \( h_1 \in \text{Dom}(L_\perp^{\frac{1}{2}}) = H^1(\mathbb{R}^n) \). Note that \( h_1 \) is defined only up to an element in \( \ker(L_\perp^{\frac{1}{2}}) \), i.e., \( h_1 + c \phi \) has the same property for any constant \( c \). Thus

\[
\langle L_\perp^{\frac{1}{2}} L_+ L_\perp^{\frac{1}{2}} f, g \rangle = \langle f, L_\perp^{\frac{1}{2}} (h_1 + c \phi) \rangle = \langle L_\perp^{\frac{1}{2}} f, h_1 + c \phi \rangle
\]

for all \( f \in H^4(\mathbb{R}^n) \). Equivalently, setting \( f_1 = L_\perp^{\frac{1}{2}} f \), one has

\[
\langle L_\perp^{\frac{1}{2}} f_1, g \rangle = \langle f_1, h_1 + c \phi \rangle.
\]
Note that the class of \( f_1 \) are all functions in \( H^3(\mathbb{R}^n) \) with \( f_1 \perp \phi \). We now want to remove the latter restriction, which can be achieved by a suitable choice of \( c \). Indeed, in order to achieve

\[
\langle L_\perp^2 L_+(f_1 + \lambda \phi), g \rangle = \langle f_1 + \lambda \phi, h_1 + c\phi \rangle
\]

for all \( f_1 \in H^3(\mathbb{R}^n) \), \( f_1 \perp \phi \), \( \lambda \in \mathbb{C} \) one chooses \( c \) such that

\[
\langle L_\perp^2 L_+ \phi, g \rangle = \langle \phi, h_1 + c\phi \rangle,
\]

which can be done since \( \langle \phi, \phi \rangle > 0 \). Renaming \( h_1 + c\phi \) into \( h_1 \), one thus arrives at

\[
\langle L_\perp^2 L_+ f_1, g \rangle = \langle f_1, h_1 \rangle \quad \text{for all} \quad f_1 \in H^3(\mathbb{R}^n).
\]

Recall that \( h = L_\perp^2 h_1 \). One can now continue this procedure. Indeed, since (12.36) implies that \( h_1 \perp \ker(L_+) \), one can write \( h_1 = L_+ h_2 = L_+(h_2 + \sum_{j=1}^k c_j \psi_j) \), where \( \ker(L_+) = \text{span}\{\psi_j\}_{j=1}^k \) and \( h_2 \in H^3(\mathbb{R}^n) \) (in fact, \( \psi_j = \partial_j \phi \) by our assumption). As before, the constants \( \{c_j\} \) are chosen in such a way that

\[
\langle L_\perp^2 L_+ f_1 + \sum_{j=1}^k \lambda_j \psi_j, g \rangle = \langle L_+ f_1 + \sum_{j=1}^k \lambda_j \psi_j, h_2 + \sum_{j=1}^k c_j \psi_j \rangle
\]

for all \( \lambda_j \). This can be done because of the invertibility of the Gram matrix of \( \{\psi_j\}_{j=1}^k \). Hence

\[
\langle L_\perp^2 f_2, g \rangle = \langle f_2, h_2 \rangle \quad \text{for all} \quad f_2 \in H^1(\mathbb{R}^n).
\]

Moreover, \( h = L_\perp^2 L_+ h_2 \) with \( h_2 \in H^3(\mathbb{R}^n) \). By the self-adjointness of \( L_\perp^2 \) this implies that \( h_2 = L_\perp^2 g \). It follows that \( g \in H^4(\mathbb{R}^n) \) and \( h = L_\perp^2 L_+ L_\perp^2 g \) as desired.

Finally, we show that \( \text{Ran}(A^2) \) is closed. By (12.34) it suffices to show that the ranges of both \( T = L_+ L_- \) and \( T^* = L_- L_+ \) are closed with domain \( H^4(\mathbb{R}^n) \). We will first verify that these operators are closed on this domain. Indeed, they each can be written in the form \( \Delta^2 + F_1 \Delta + \Delta F_2 + F_3 \). Since for \( M \) large

\[
\|(\Delta^2 + M + F_1 \Delta + \Delta F_2 + F_3) f\|_2 \geq \|(\Delta^2 + M) f\|_2 - C\|f\|_{W^{2,2}} \geq \frac{1}{2}\|(\Delta^2 + M) f\|_2 \gtrsim \|f\|_{W^{4,2}};
\]

one concludes that \( T + M, T^* + M \) are closed, and therefore also \( T, T^* \). Next, let \( P_- \) and \( P_+ \) be the projections onto \( \ker(L_-) \) and \( \ker(L_+) \), respectively. Then \( L_- L_+ = L_- P_+^\perp L_+ P_+^\perp \). Now

\[
P_\perp^\perp L_+ f = L_+ f - \|\phi\|^{-2} \langle L_+ f, \phi \rangle \phi = L_+ \left( f - \|\phi\|^{-2} \langle f, L_+ \phi \rangle \phi \right),
\]

where we have written \( \phi = L_+ \tilde{\phi}, \tilde{\phi} \in \ker(L_+) = \text{Ran}(P_+^\perp) \) by virtue of the fact that

\[
\phi \in \text{Ran}(L_+) = \overline{\text{Ran}(L_+)} = \ker(L_+) = \text{span}\{\partial_j \phi\}_1^k.
\]

The last equality here is Weinstein’s characterization, more precisely, our assumption on \( L_+ \). Define

\[
\tilde{\phi} := P_+^\perp \phi_1, \quad Qf := f - \|\phi\|^{-2} \langle f, L_+ \phi \rangle \tilde{\phi}, \quad \tilde{Q}f := f - \|\phi\|^{-2} \langle f, P_+^\perp L_+ \phi \rangle \tilde{\phi}_1
\]

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so that (12.37) gives

\[ P_\perp^+L_+P_\perp^+ = L_+QP_\perp^+ \quad \implies \quad L_-L_+ = L_-L_+P_\perp^+\hat{Q}. \]

In particular,

\[ \|T^*f\|_2 = \|L_-P_\perp^+L_+P_\perp^+f\|_2 \geq c_1 \|P_\perp^+L_+P_\perp^+f\|_2 \]

\[ = c_1 \|L_+QP_\perp^+f\|_2 = c_1 \|P_\perp^+\hat{Q}f\|_2 \geq c_1c_2 \|P_\perp^+\hat{Q}f\|_2, \]

where the existence of \( c_1, c_2 > 0 \) follows from the self-adjointness of \( L_-^*, L_+^* \). Hence, if \( T^*f_n = T^*P_\perp^+\hat{Q}f_n \to h \) in \( L^2 \), then by (12.38) and linearity \( P_\perp^+\hat{Q}f_n \to g \) in \( L^2 \). Since \( T^* \) was shown to be closed, it follows that \( h = T^*g \), and \( \text{Ran}(T^*) \) is closed. A similar argument shows that \( \text{Ran}(T) \) is closed.

Next, we derive the linear stability assumption as well as the structure of the generalized eigenspaces of \( A \) and \( A^* \) from our spectral assumptions on \( L_-, L_+ \). From the spectral assumptions in Definition 12.9 as well as

\[ L_-\phi = 0, \quad L_-(x_j\phi) = -\partial_j\phi, \quad L_+(\partial_j\phi) = 0, \quad L_+(\partial_\alpha\phi) = -\alpha\phi \]

it follows that

\[ \ker(A) = \text{span}\left\{ \begin{pmatrix} 0 \\ \phi \\ \partial_j\phi \end{pmatrix} : 1 \leq j \leq n \right\} \]

\[ \ker(A^*) = \text{span}\left\{ \begin{pmatrix} \phi \\ 0 \\ \partial_\alpha\phi \end{pmatrix} : 1 \leq j \leq n \right\} \]

\[ \mathcal{N}(A) := \bigcup_{k=1}^{\infty} \ker(A^k) \supset \text{span}\left\{ \begin{pmatrix} 0 \\ \phi \\ \partial_\alpha\phi \\ \partial_j\phi \end{pmatrix} : 1 \leq j \leq n \right\} =: \mathcal{M} \]

\[ \mathcal{N}(A^*) := \bigcup_{k=1}^{\infty} \ker((A^*)^k) \supset \text{span}\left\{ \begin{pmatrix} \phi \\ 0 \\ x_j\phi \\ \partial_\alpha\phi \end{pmatrix} : 1 \leq j \leq n \right\} =: \mathcal{M}_*. \]

One of our goals is to show that equality holds in the last two relations. This is the same as the structure statement made in Proposition 4.1, but one needs to apply the matrix \( P = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \) to pass between these two representations.

Now suppose that \( i\partial_t\vec{\psi} + A\vec{\psi} = 0 \). This can be written as \( \partial_t\vec{\psi} + JM\vec{\psi} = 0 \) where \( J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \)

and \( M = \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix} \). Therefore,

\[ \frac{d}{dt}\langle \vec{\psi}, M\vec{\psi} \rangle = 2\Re\langle \partial_t\vec{\psi}, M\vec{\psi} \rangle = -2\Re\langle JM\vec{\psi}, M\vec{\psi} \rangle = 0 \]

by anti-selfadjointness of \( J \). In other words,

\[ Q(\vec{\psi}) := \langle L_+\psi_1, \psi_1 \rangle + \langle L_-\psi_2, \psi_2 \rangle \]
is constant in time if $\tilde{\psi}(t) = e^{itA}\tilde{\psi}(0)$ (here $\tilde{\psi} = (\psi_n)$). Although the previous calculation basically required classical solutions, it is clear that its natural setting are $H^1(\mathbb{R}^n)$-solutions. In that case one needs to interpret the form $Q(\tilde{\psi})$ via

$$
\langle L_+ \psi_1, \psi_1 \rangle = \frac{1}{2} \|\nabla \psi_1\|_2^2 + \frac{\alpha_2}{2} \|\psi_1\|_2^2 - \langle \beta(\phi^2)\psi_1, \psi_1 \rangle
$$

(12.41)

$$
\langle L_- \psi_2, \psi_2 \rangle = \frac{1}{2} \|\nabla \psi_2\|_2^2 + \frac{\alpha_2}{2} \|\psi_2\|_2^2 - \langle (\beta(\phi^2) + 2\beta'(\phi^2)\phi^2)\psi_2, \psi_2 \rangle.
$$

In what follows, we will tacitly make this interpretation whenever it is needed. The following lemmas are due to Weinstein [We1].

**Lemma 12.12.** Impose the spectral assumptions on $L_+, L_-$ from Definition 12.9. Then $\langle L_+ f, f \rangle \geq 0$ for all $f \in H^1(\mathbb{R}^n)$, $f \perp \phi$.

This is a special case of Lemma E.1 in [We1], and we refer the reader to that paper for the proof.

**Lemma 12.13.** Impose the spectral assumptions on $L_+, L_-$ from Definition 12.9. Then there exist constants $c = c(\alpha, \beta) > 0$ such that for all $\tilde{\psi} \in H^1(\mathbb{R}^n)$,

1. $\langle L_- \psi_2, \psi_2 \rangle \geq c\|\psi_2\|_2^2$ if $\psi_2 \perp \partial_\alpha \phi$, $\psi_2 \perp \partial_j \phi$

2. $\langle L_+ \psi_1, \psi_1 \rangle \geq c\|\psi_1\|_2^2$ if $\psi_1 \perp \phi$, $\psi_1 \perp x_j \phi$.

The constant $c(\alpha, \beta)$ can be taken to be uniform in $\alpha$ in the following sense: If $\alpha_0$ satisfies Definition 12.9, then there exists $\delta > 0$ so that 1. and 2. above hold for all $|\alpha - \alpha_0| < \delta$ with $c(\alpha, \beta) > \frac{1}{2}c(\alpha_0, \beta)$.

**Proof.** Consider the minimization problems

$$
in_{f \in H^1} \langle L_+ f, f \rangle \quad \text{subject to constraints} \quad \|f\|_2 = 1, \ f \perp \partial_\alpha \phi, \ f \perp \partial_j \phi
$$

(12.42)

$$
in_{g \in H^1} \langle L_+ g, g \rangle \quad \text{subject to constraints} \quad \|g\|_2 = 1, \ g \perp \phi, \ g \perp x_j \phi
$$

(12.43)

As usual, one would like to establish the existence of minimizers by means of passing to weak limits in minimizing sequences. While such sequences are bounded in $H^1(\mathbb{R}^n)$, this is not enough to guarantee strong convergence in $L^2(\mathbb{R}^n)$ because some (or all) of the $L^2$-mass might escape to infinity. Using the fact that the quadratic forms in question are perturbations of $\frac{1}{2}\|\nabla f\|_2^2 + \frac{\alpha_2}{2}\|f\|_2^2$ by a potential that decays at infinity, one can easily exclude that all the $L^2$-mass escapes to infinity. One then proceeds to show that the remaining piece of the limit, normalized to have $L^2$-norm one, is a minimizer. This, however, is a simple consequence of the non-negativity of $L_-$ and $L_+$, the latter under the constraint $f \perp \phi$, see Lemma 12.12 above. This argument is presented in all details in [We1], page 478 for the case of power nonlinearities. But the same argument also applies to the general nonlinearities considered here, and we do not write it out.

Assume therefore that $f_0$ is a minimizer of (12.42) with $\|f_0\|_2 = 1$, $f_0 \perp \partial_\alpha \phi$, $f_0 \perp \partial_j \phi$. Then

$$
L_- f_0 = \lambda_0 f_0 + c_0 \partial_\alpha \phi + \sum_{j=1}^n c_j \partial_j \phi
$$

(12.44)
for some Lagrange multipliers $\lambda_0, c_0, \ldots, c_n$. Clearly, $\lambda_0$ agrees with the minimum sought, and therefore it suffices to show that $\lambda_0 > 0$.

If $\lambda_0 = 0$, then taking the scalar product of (12.44) with $\phi$ implies that $c_0 = 0$ (using that $\langle \partial_k \phi, \phi \rangle > 0$). Taking scalar products with $x_k \phi$ shows that also $c_k = 0$ for $1 \leq k \leq n$. Thus $L^- f_0 = 0$, which would imply that $f_0 = \gamma \phi$ for some $\gamma \neq 0$. However, this is impossible because of $f_0 \perp \partial_\alpha \phi$.

Proceeding in the same manner for $L_+$, one arrives at the Euler-Lagrange equation

$$L_+ g_0 = \lambda_0 g_0 + c_0 \phi + \sum_{j=1}^n c_j x_j \phi.$$  

As before, $\lambda_0$ is the minimum on the left-hand side of (12.43) and thus $\lambda_0 \geq 0$ by Lemma 12.12. If $\lambda_0 = 0$, then taking scalar products with $\partial_k \phi$ leads to $c_k = 0$ for all $1 \leq k \leq n$. Hence $L_+ g_0 = c_0 \phi$ which implies that

$$g_0 = -\frac{c_0}{\alpha} \partial_\alpha \phi + \sum_{\ell=1}^n b_\ell \partial_\ell \phi.$$  

Taking scalar products of this line with $\phi$ and $x_j \phi$ shows that $c_0 = 0$ and $b_\ell = 0$ for all $1 \leq \ell \leq n$, respectively. But then $g_0 = 0$ which is impossible.

Since the constants $c(\alpha, \beta) > 0$ were obtained by contradiction, one has no control on their dependence on $\alpha$. However, let $|\alpha - \alpha_0| < \delta$ be as in Definition 12.9. Suppose $\|f\|_2 = 1$ satisfies $f \perp \phi(\cdot, \alpha)$, $f \perp x_j \phi(\cdot, \alpha)$. Then there is

$$h \in \text{span}\{\phi(\cdot, \alpha), \phi(\cdot, \alpha_0), x_j \phi(\cdot, \alpha), x_j \phi(\cdot, \alpha_0) : 1 \leq j \leq n\}$$

so that $f + h \perp \phi(\cdot, \alpha_0)$ and $f + h \perp x_j \phi(\cdot, \alpha_0)$. Moreover, since $\|\partial_\alpha \phi\|_{H^1(\mathbb{R}^n)} + \|\partial_\alpha^2 \phi\|_{H^1(\mathbb{R}^n)} < \infty$ one can take $\|h\|_{H^1(\mathbb{R}^n)}$ as small as desired provided $\delta$ is chosen small enough. One can therefore use inequality 2. from this lemma at $\alpha_0$ for $f + h$ to obtain a similar bound for $f$ at $\alpha$.  

The following corollary proves the crucial linear stability assumption contingent upon the spectral assumptions on $L_+, L_-$ from above (and thus, in particular, contingent upon the convexity condition). Strictly speaking, the following corollary gives a stronger statement than (12.3), since the range of $P_s$ is potentially smaller than needed for the stability to hold.

**Corollary 12.14.** Impose the spectral assumptions on $L_+, L_-$ from Definition 12.9. Then there exist constants $C = C(\alpha, \beta) < \infty$ so that for all $\tilde{\psi}_0 \in H^1(\mathbb{R}^n)$

\begin{equation}
(12.45) \quad \|e^{itA}\tilde{\psi}_0\|_{H^1(\mathbb{R}^n)} \leq C\|\tilde{\psi}_0\|_{H^1(\mathbb{R}^n)} \quad \text{provided} \quad \tilde{\psi} \in \mathcal{M}_s^+.
\end{equation}

Here $\mathcal{M}_s$ is the $A^*$-invariant subspace from (12.40). Moreover, the same bound holds for $H^s(\mathbb{R}^n)$-norms for any real $s$ with $s$-dependent constants (and thus in particular for $L^2(\mathbb{R}^n)$). Analogous statements hold for $e^{itA^*}$. Finally, the constants $C(\alpha, \beta)$ can be taken to be uniform in $\alpha$ in the following sense: If $\alpha_0$ satisfies Definition 12.9, then there exists $\delta > 0$ so that (12.45) holds for all $|\alpha - \alpha_0| < \delta$ with $C(\alpha, \beta) < 2C(\alpha_0, \beta)$.  

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Proof. Let \( \psi(t) = e^{itA}v_0 \). By Lemma 12.13 one has
\[
Q(\psi_0) = Q(\psi(t)) \geq c \|\psi(t)\|_{L^2(\mathbb{R}^n)}^2
\]
provided that \( \psi_0 \in \mathcal{M}_\pm \). Since clearly \( Q(\psi_0) \leq C\|\psi_0\|^2_{H^1(\mathbb{R}^n)} \) one concludes that (12.45) holds with \( L^2(\mathbb{R}^n) \) on the left-hand side. In order to pass to \( H^1(\mathbb{R}^n) \) write
\[
\langle L_- f, f \rangle = (1 - \varepsilon)\langle L_- f, f \rangle + \frac{\varepsilon}{2}\|\nabla f\|_2^2 + \frac{\varepsilon}{2}\|f\|_2^2 - \varepsilon \int_{\mathbb{R}^n} \beta(\phi^2(x))|f(x)|^2 \, dx
\]
(12.46)
\[
\geq \frac{\varepsilon}{2}\|\nabla f\|_2^2 + c(1 - \varepsilon)\|f\|_2^2 + \varepsilon(\alpha^2/2 - \|\beta\|_\infty)\|f\|_2^2,
\]
where the constant \( c \) in (12.46) is the one from Lemma 12.13. Taking \( \varepsilon \) small enough, one sees that the third term can be absorbed into the second. Thus the entire right-hand side of (12.46) admits the lower bound \( \frac{\varepsilon}{2}\|\nabla f\|_2^2 \). The same argument applies to \( L_+ \), and (12.45) follows. The uniformity statement concerning the constants \( C(\alpha, \beta) \) is an immediate consequence of the analogous statement in Lemma 12.13. To obtain (12.45) for all \( H^s \) spaces note first that
\[
\tag{12.47}
C^{-1}_\ell \|\tilde{\psi}\|_{H^{2s}(\mathbb{R}^n)} \leq \|(A + iM)^s\tilde{\psi}\|_2 \leq C\ell \|\tilde{\psi}\|_{H^{2s}(\mathbb{R}^n)}
\]
for all integers \( \ell \) and sufficiently large \( M = M(\ell) \). Indeed, to check the lower bound for \( \ell = 1 \) one can use \( (A + iM)^{-1} = (B + iM)^{-1}\left[1 + V(B + iM)^{-1}\right]^{-1} \). The inverse of the operator in brackets exists provided \( M \) is large and it is a bounded operator on \( L^2(\mathbb{R}^n) \). Taking powers of this relation allows one to deal with all \( \ell \geq 1 \). In our case \( V \) is \( C^\infty \) which is needed here). Since \( \mathcal{M}_s \) is \( A^\ast \)-invariant and therefore \( \mathcal{M}_\pm \) is \( A \)-invariant, inserting (12.47) into (12.45) allows one to pass to all odd integers \( s \). The case of general \( s \) then follows by interpolation. Finally, since all arguments in this section apply equally well to \( A^\ast \) as \( A \), the corollary follows.

\[ \square \]

This corollary has an important implication concerning the structure of the root spaces as required in Proposition 4.1.

**Corollary 12.15.** Impose the spectral assumptions on \( L_+, L_- \) from Definition 12.9. Then equality holds in the relation concerning the root spaces (12.39) and (12.40). In particular, one has \( \ker(A^2) = \ker(A^3) \) and \( \ker((A^\ast)^2) = \ker((A^\ast)^3) \).

**Proof.** Suppose \( \dim(\mathcal{N}(A)) > 2n + 2 \). Then there exists \( \tilde{v}_0 \in \mathcal{N}(A) \) such that \( \tilde{v}_0 \in \mathcal{M}_\pm \). This is because a system of \( 2n + 2 \) equations in \( 2n + 3 \) variables always has a nonzero solution. Since \( \partial_\alpha \phi \not\perp \phi \) and \( \partial_j \phi \not\perp x_j \phi \), one checks that \( \tilde{v}_0 \not\in \ker(A) \). Therefore, \( \tilde{v}_0 \in \ker(A^k) \setminus \ker(A^{k-1}) \) for some \( k \geq 2 \). Expanding \( e^{itA} \) into a series implies that \( \|e^{itA}\tilde{v}_0\|_2 > c t^{k-1} \) for some constant \( c > 0 \), which contradicts Corollary 12.14. Therefore, \( \dim(\mathcal{N}(A)) \leq 2n + 2 \). Since moreover \( \phi > 0 \) and \( \langle \partial_\alpha \phi, \phi \rangle > 0 \) imply that the \( 2n + 2 \) vectors on the right-hand side of (12.39) are linearly independent, equality must hold as claimed. Analogously for (12.40). \[ \square \]
13 Uniqueness of the negative eigenvalue of $L_+$

In this section we show that the assumption that the linear operator $L_+$ has a single simple negative eigenvalue of Definition 12.9 is automatically satisfied for a large class of nonlinearities.

We consider the ground state of the problem

\begin{equation}
- \frac{1}{2} \Delta \phi - \beta(|\phi|^2) \phi + \frac{\alpha^2}{2} \phi = 0
\end{equation}

constructed from the constrained minimization problem for the following functional:

\begin{equation}
J[u] = \left\{ \int_{\mathbb{R}^n} |\nabla u|^2 : W[u] = \int_{\mathbb{R}^n} G(u) = 1 \right\},
\end{equation}

where $G(s) = \int_0^s (\beta(s^2)s - \frac{\alpha^2}{2}s)$. If $w$ is a minimum it solves the equation

\begin{equation}
- \frac{1}{2} \Delta w - \lambda(\beta(w^2)w - \frac{\alpha^2}{2}w = 0
\end{equation}

where the Lagrange multiplier $\lambda$ is determined from the condition that $W[w] = 1$. We can then find a ground state via rescaling

\begin{equation}
\phi(x) = w(\lambda^{-\frac{1}{2}}x)
\end{equation}

for an appropriate choice of $\lambda$. In this section we prove the following:

**Theorem 13.1.** Let $\phi$ be the ground state of (13.1) constructed in the variational problem (13.2) with the nonlinearity $\beta$ verifying the condition that $|\beta(s^2) \leq C|s|^{p-1}$ with $p \in (1, 1 + \frac{4}{n-2})$ and that there exists $s_0$ such that $C(s_0) > 0$. Moreover, assume that

$$
\beta(s^2)s^2 \geq 2 \int_0^s \beta(\tau^2) \tau d\tau
$$

for all $s \geq 0$. Then the operator $L_+ = - \frac{1}{2} \Delta - \beta(\phi^2) - \beta'(\phi^2)\phi^2 + \frac{\alpha^2}{2}$ has a unique simple negative eigenvalue.

**Proof.** Let us consider a smooth family of functions $w_z$ satisfying the constraint $W[w_z] = 1$ for all $z$ and passing through a minimizer $w$, i.e., $w_0 = w$. Observe that since $w$ is a minimizer of the constrained variational problem we have that

$$
\frac{d}{dz} J[w_z]|_{z=0} = 0, \quad \frac{d^2}{dz^2} J[w_z]|_{z=0} \geq 0
$$

Denote

$$
\dot{w} = \frac{d}{dz} w_z|_{z=0}, \quad \ddot{w} = \frac{d^2}{dz^2} w_z|_{z=0}
$$

Then

$$
0 = \frac{d}{dz} J[w_z]|_{z=0} = 2 \int_{\mathbb{R}^n} \nabla w \nabla \dot{w}
$$
and

\begin{equation}
0 \leq \frac{d^2}{dz^2} J[w_z]|_{z=0} = 2 \int_{\mathbb{R}^n} |\nabla \dot{w}|^2 + \nabla w \nabla \ddot{w} = -2 \int_{\mathbb{R}^n} \Delta \dot{w} \ddot{w} + \Delta w \dot{w}
\end{equation}

Moreover, since the family \( w_z \) verifies the constraint \( W[w_z] = 1 \) we have

\[ \frac{d}{dz} W[w_z] = \frac{d^2}{dz^2} W[w_z] = 0 \]

Computing the derivatives we obtain

\begin{equation}
\int_{\mathbb{R}^n} (\beta(w^2)w - \frac{\alpha^2}{2}w) \dot{w} = 0,
\end{equation}

\begin{equation}
\int_{\mathbb{R}^n} (\beta(w^2) + 2\beta'(w^2)w^2 - \frac{\alpha^2}{2})|\dot{w}|^2 + (\beta(w^2)w - \frac{\alpha^2}{2}w) \ddot{w} = 0
\end{equation}

Now observe that the Euler-Lagrange equation (13.3) implies that

\[ -\int_{\mathbb{R}^n} \Delta w \ddot{w} = 2\lambda \int_{\mathbb{R}^n} (\beta(w^2)w - \frac{\alpha^2}{2}w) \dot{w} \]

On the other hand from (13.7) we have that

\[ \int_{\mathbb{R}^n} (\beta(w^2)w - \frac{\alpha^2}{2}w) \ddot{w} = -\int_{\mathbb{R}^n} (\beta(w^2) + 2\beta'(w^2)w^2 - \frac{\alpha^2}{2})|\dot{w}|^2 \]

Substituting this into (13.5) we obtain that

\begin{equation}
-2 \int_{\mathbb{R}^n} (\Delta \dot{w} + 2\lambda(\beta(w^2) + 2\beta'(w^2)w^2 - \frac{\alpha^2}{2})\dot{w}) \geq 0
\end{equation}

Recall the definition of the operator \( L_+ \) associated with a ground state \( \phi \):

\[ L_+ = -\frac{1}{2} \Delta - \beta(\phi^2) - 2\beta'(\phi^2)\phi^2 + \frac{\alpha^2}{2} \]

We know from (13.4) that \( \phi(x) = w(\lambda^{-\frac{1}{2}}x) \). Thus rescaling (13.8) leads to the inequality

\begin{equation}
\lambda^{\frac{n}{2}+1} \langle L_+ \dot{w}(\lambda^{-\frac{1}{2}}\cdot) , \dot{w}(\lambda^{-\frac{1}{2}}\cdot) \rangle \geq 0
\end{equation}

Furthermore, (13.6) implies that the function \( \dot{w}(\lambda^{-\frac{1}{2}}\cdot) \) can be chosen arbitrarily from the subspace orthogonal to the function

\[ \Psi = (\beta(\phi^2)\phi - \frac{\alpha^2}{2}\phi) \]

Let \( \Pi \) denote the orthogonal projection on the above subspace. Then (13.9) implies that the operator \( \Pi L_+ \Pi \geq 0 \). Now let \( \mu \) and \( u \) be correspondingly a negative eigenvalue of \( L_+ \) and the associated eigenfunction. Decompose

\[ u = v + \Psi \]

\[ ^{4}\text{It is easy to see that the condition that ensures that } \lambda > 0 \text{ is that} \]

\[ \beta(s^2)s^2 \geq 2 \int_0^s \beta(\tau^2)\tau d\tau \]

\[ ^{5}\text{One can always achieve this decomposition by normalizing}\ u \]
where \( \Pi v = v \). Then
\[
L_+ u = L_+ v + L_+ \Psi = \mu (v + \Psi)
\]
Projecting we obtain that
\[
(\Pi L_+ \Pi - \mu)v = -\Pi L_+ \Psi
\]
Since the operator \( \Pi L_+ \Pi \) is non-negative and \( \mu < 0 \) we have that \( (\Pi L_+ \Pi - \mu) \) is invertible and
\[
v = -(\Pi L_+ \Pi - \mu)^{-1}\Pi L_+ \Psi
\]
Therefore the eigenvalue \( \mu \) is simple. We now substitute \( v \) back into the eigenvalue equation for \( u \) and take a scalar product with \( \Psi \). We obtain that
\[
-\langle L_+ (\Pi L_+ \Pi - \mu)^{-1}\Pi L_+ \Psi, \Psi \rangle + \langle L_+ \Psi, \Psi \rangle = \mu \langle \Psi, \Psi \rangle
\]
We now assume that there is another negative eigenvalue \( \lambda \). Then the corresponding equation also holds with \( \lambda \) replacing \( \mu \). Subtracting we obtain
\[
-\langle L_+ (\Pi L_+ \Pi - \mu)^{-1}\Pi L_+ \Psi, \Psi \rangle + \langle L_+ (\Pi L_+ \Pi - \lambda)^{-1}\Pi L_+ \Psi, \Psi \rangle = (\mu - \lambda) \langle \Psi, \Psi \rangle
\]
Using the resolvent identity
\[
(\Pi L_+ \Pi - \mu)^{-1} - (\Pi L_+ \Pi - \lambda)^{-1} = (\mu - \lambda)(\Pi L_+ \Pi - \mu)^{-1}(\Pi L_+ \Pi - \lambda)^{-1}
\]
Thus,
\[
\langle (\Pi L_+ \Pi - \mu)^{-1}(\Pi L_+ \Pi - \lambda)^{-1}\Pi L_+ \Psi, \Pi L_+ \Psi \rangle = -\langle \Psi, \Psi \rangle
\]
Observe that \( (\Pi L_+ \Pi - \mu)^{-1}(\Pi L_+ \Pi - \lambda)^{-1} \) is a positive operator since \( \Pi L_+ \Pi \) is non-negative and \( \lambda, \mu < 0 \). Contradiction. \( \square \)

## 14 Verification of the needed properties of \( L_+ \) and \( L_- \) for a particular class of nonlinearities

The purpose of this section is to verify that the conditions imposed on \( L_+ \) and \( L_- \) in Definition 12.9 hold for a particular class of nonlinearities. This class is chosen so as to allow for the nonlinear analysis of the previous sections to go through.

\begin{equation}
\beta_\theta(s^2) = s^{p-1} \frac{f(s^2)}{\theta + f(s^2)}
\end{equation}

with a constant \( \theta > 0 \) and the function \( f \) satisfying the conditions
\begin{equation}
C_1 s^{r+1} \leq |f(s^2)| \leq C_2 s^{r+1}, \quad s^2 |f'(s^2)| \leq C_2 |f(s^2)|, \quad p \in (1, 1 + \frac{4}{n}), \quad r > -1
\end{equation}

The existence of a ground state for the problem
\begin{equation}
-\frac{1}{2} \Delta \phi - \beta(|\phi|^2) \phi + \frac{\alpha^2}{2} \phi = 0
\end{equation}

for \( \alpha \neq 0 \) had been established by Berestycki and Lions under the following conditions on the function \( \beta \):

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1. \(0 \geq \lim_{s \to +\infty} \beta(s)s^{-\frac{2}{n-2}} \geq +\infty\)

2. There exists \(s_0 > 0\) such that \(G(s_0) = \int_0^{s_0} \beta(s^2)s\,ds - \frac{\alpha^2}{4}s_0^2 > 0\)

Moreover, in the case when the function \(\beta(s)\) satisfies a stronger condition that

\[
\lim_{s \to +\infty} \beta(s)s^{-\frac{2}{n-2}} = 0
\]

a ground state can be constructed from a solution of the constrained minimization problem for the following functional:

\[
J[u] = \left\{ \int_{\mathbb{R}^n} |\nabla u|^2 : W[u] = \int_{\mathbb{R}^n} G(u) = 1 \right\}
\]

If \(w\) is a minimum it solves the equation

\[-\frac{1}{2}\Delta w - \lambda(\beta(w^2)w - \frac{\alpha^2}{2}w) = 0\]

where the Lagrange multiplier \(\lambda\) is determined from the condition that \(W[w] = 1\). We can then find a ground state via rescaling

\[
\phi(x) = w(\lambda^{-\frac{1}{2}}x)
\]

Observe that it is possible to choose \(w\) a positive spherically symmetric function. We now consider the case of the monomial subcritical nonlinearity \(\beta(s) = s^{\frac{p-1}{2}}\) with \(p < \frac{n+2}{n-2}\). By the results of Coffman, McLeod-Serrin, and Kwong there exists a unique positive radial solution of the equation (14.3) for \(\alpha \neq 0\). Let \(w\) denote the corresponding minimizer of the functional \(J\).

**Definition 14.1.** Given \(\gamma > 0\) and \(w\) the minimizer of \(J\) corresponding to the unique ground state \(\phi\) define

\[
\sigma(\gamma) = \inf \{ \theta : \text{for any positive non-increasing radial function } u \text{ with the property that } \|u - w\|_{H^1} \geq \theta \text{ we have that } J[u] \geq J[w] + \gamma \}
\]

We now make the following claim

**Lemma 14.2.** Let \(\beta(s^2) = s^{p-1}\) be a monomial nonlinearity with \(p \in (1, \frac{4}{n-2})\). Then the function \(\sigma(\gamma) \to 0\) as \(\gamma \to 0\).

*Proof.* We argue by contradiction. Assume that there exists a sequence \(\gamma_k \to 0\), a positive constant \(\sigma\), and positive radial functions \(u_{\gamma_k}\) such that \(\|u_{\gamma_k} - w\|_{H^1} \geq \theta\) but \(J[u_{\gamma_k}] < J[w] + \gamma_k\). Then the sequence \(u_{\gamma_k, \delta}\) is minimizing for the functional \(J\). This implies that

\[
\|\nabla u_{\gamma_k}\|_{L^2} \to \|\nabla w\|_{L^2}
\]

Using the constraint \(W[u_{\gamma_k}] = 1\) it is not difficult to show that the sequence \(u_{\gamma_k}\) is uniformly bounded in \(H^1\), see [BL]. Thus without loss of generality we assume that \(u_{\gamma_k} \to u\) weakly in \(H^1\) for some
radial non-increasing function \( u \). Therefore, \( u \) is another minimizer of the functional \( J \) and its rescaled version is a non-increasing radial solution of the equation (14.3). By the strong maximum principle it is positive\(^6\) and therefore a ground state. Since the ground state is unique, after rescaling back we conclude that \( u = w \). Therefore, we have constructed a sequence \( u_{\gamma_k} \) with the properties that

\[
\begin{align*}
(14.8) & \quad u_{\gamma_k} \to w \text{ weakly in } H^1, \\
(14.9) & \quad \nabla u_{\gamma_k} \to \nabla w \text{ in } L^2, \\
(14.10) & \quad \int_{\mathbb{R}^n} |u_{\gamma_k}|^{p+1} = 1 + \frac{\alpha^2}{4} \int_{\mathbb{R}^n} |u_{\gamma_k}|^2, \\
(14.11) & \quad \|u_{\gamma_k} - w\|_{H^1} \geq \theta
\end{align*}
\]

Since \( 2 < p + 1 \leq \frac{2n}{n-2} \), conditions (14.8) and (14.9) imply that

\[
\int_{\mathbb{R}^n} |u_{\gamma_k}|^{p+1} \to \int_{\mathbb{R}^n} |w|^{p+1}
\]

Thus from (14.10)

\[
\int_{\mathbb{R}^n} |u_{\gamma_k}|^2 \to \int_{\mathbb{R}^n} |w|^2
\]

and with the help of (14.8) and (14.9) we conclude that \( u_{\gamma_k} \to w \) in \( H^1 \). This contradicts (14.11).

---

### 14.1 Variational continuity

We now consider the ground state problem

\[
(\text{gr}_\theta) \quad - \frac{1}{2} \Delta \phi_{\theta} - \beta_{\theta}(|\phi_{\theta}|^2)\phi_{\theta} + \frac{\alpha^2}{2} \phi_{\theta} = 0
\]

for the nonlinearities

\[
\beta_{\theta}(s^2) = -|s|^{p-1} \frac{f(s^2)}{\theta + f(s^2)}
\]

where \( 1 < p < 1 + \frac{4}{n} \), the function \( f \) satisfies the following estimate

\[
C_1 s^q \leq |f(s^2)| \leq C_2 s^{q+1}
\]

for some positive constants \( C_1, C_2 \) and \( q > -1 \), and \( \theta > 0 \) is a positive number. Define

\[
(14.13) \quad G_{\theta}(\tau) = \int_0^\tau \beta_{\theta}(s^2)s \, ds - \frac{\alpha^2}{4} \tau^2, \\
(14.14) \quad W_{\theta}[u] = \int_{\mathbb{R}^n} G_{\theta}(u(x)) \, dx
\]

\(^6\)The minimizer \( u \) cannot be identically zero since one can show that the minimum is attained on the function satisfying the constraint \( W[u] = 1 \), see [BL].

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Lemma 14.3. We have the following estimate

\begin{equation}
|W_\theta[u] - W_0[u]| \lesssim \theta^{\frac{n-1}{p+q}} \int_{\mathbb{R}^n} (|u|^2 + |u|^{p+1})
\end{equation}

Proof. Estimate (14.15) immediately follows from the inequality

\begin{equation}
|G_\theta(\tau) - G_0(\tau)| \leq \tau^{p+1} \frac{\theta}{\theta + \tau^{q+1}} = \theta \tau^{p-q} \frac{\tau^{q+1}}{\epsilon + \tau^{q+1}}
\end{equation}

since the above expression can be bounded by

\begin{equation}
\min \{ \theta \tau^{p-q}, \tau^{p+1} \}
\end{equation}

Thus using the first term for the values of $\tau \geq \theta^{\frac{1}{p+q}}$, and the second term when $\tau \leq \theta^{\frac{1}{p+q}}$ we obtain (14.16).

We now consider the variational problem

\begin{equation}
J_\theta[u] = \{ \int_{\mathbb{R}^n} |\nabla u|^2 : W_\theta[u] = 1 \}
\end{equation}

Proposition 14.4. Let $\phi$ be the ground state of the problem (gr$_0$). Then for any sufficiently small $\theta > 0$ there exists a positive constant $\delta' = \delta'(\theta) \to 0$ as $\theta \to 0$, and a ground state $\phi_\theta$ of (gr$_\theta$) such that $\|\phi_\theta - \phi\|_{H^1} < \delta'$.

Proof. We start by choosing a sufficiently large constant $M$ such that for all sufficiently small $\theta$ any minimizer of $J_\theta$ is contained in a ball $B_{M/2}$ of radius $M/2$ in the space $H^1$. In particular, using (14.15) we will assume that for $u \in B_M$

\begin{equation}
|W_\theta[u] - W_0[u]| \lesssim \epsilon^{\frac{1}{p+q}}
\end{equation}

We now observe the following trivial property of the constraint functionals $W_\theta[u]$: for any $\theta \geq 0$ and an arbitrary $\mu \neq 0$

\begin{equation}
W_\theta[u(x)] = \mu^n W_\theta[u(\frac{x}{\mu})]
\end{equation}

We now fix a sufficiently small $\theta > 0$. Let $w$ be the minimizer of the variational problem $J = J_0$ corresponding to the unique ground state $\phi$. The function $w$ satisfies the constraint $W_0[w] = 1$. Therefore, using the rescaling property (14.19) and (14.18) we can show that there exists $\mu = \mu(w)$ with the property that

\begin{equation}
W_\theta[w(\frac{x}{\mu})] = 1,
\end{equation}

\begin{equation}
|\mu - 1| \leq \theta^{\frac{n-1}{p+q}}
\end{equation}

Moreover,

\begin{equation}
J_\theta[w(\frac{x}{\mu})] = \mu^{n-2} J_0[w(x)] = J_0[w] + O(\theta^{\frac{n-1}{p+q}})
\end{equation}
We now claim that there exists a small positive $\delta = \delta(\theta) \to 0$ as $\theta \to 0$, such that for any positive non-increasing radial function $u$ satisfying the constraint $W[\theta] = 1$ and the property that

$$\|u - w(\frac{x}{\mu})\|_{H^1} \geq \delta$$

we have

$$J[\theta] \geq J[\theta] + \theta^{\frac{p-1}{p+q}}$$

Assume for the moment that the claim holds. Then (14.22) and (14.23) imply that $J[\theta]$ has a minimizer in the $\delta$ neighborhood of the function $w(\frac{x}{\mu})$. We denote this minimizer by $w_\theta$. Then (14.20) implies that

$$\|w_\theta - w\|_{H^1} \leq \|w_\theta - w(\frac{x}{\mu})\|_{H^1} + \|w - w(\frac{x}{\mu})\|_{H^1} \leq \delta + \|w - w(\frac{x}{\mu})\|_{H^1}$$

Observe that $\|w - w(\frac{x}{\mu})\|_{H^1} \to 0$ as $\mu \to 1$, which follows by the density argument and the fact that it is easily satisfied on functions of compact support. Define the function $a_w(\epsilon)$:

$$a_w(\epsilon) := \sup_{|\mu| \leq \theta^{p+q}} \|w - w(\frac{x}{\mu})\|_{H^1}, \quad a_w(\theta) \to 0 \text{ as } \theta \to 0$$

Therefore,

$$\|w_\theta - w\|_{H^1} \leq \delta + a_w(\theta)$$

The functions $w_\theta, w$ are the solutions of the Euler-Lagrange equations

$$-\frac{1}{2} \Delta w_\theta - \lambda_\theta (\beta_\theta \beta_0 w_\theta)^2 w_\theta - \frac{\alpha^2}{2} w_\theta = 0,$$

$$-\frac{1}{2} \Delta w - \lambda (\beta_0 w)^2 w - \frac{\alpha^2}{2} w = 0,$$

where the Lagrange multipliers $\lambda_\theta, \lambda$ are determined from the conditions that $W[\theta] = W_0[w] = 1$. We multiply the equations (14.26) and (14.27) by $w_\theta$ and $w$ correspondingly, integrate by parts, and subtract one from another. Using the estimate

$$\int_{\mathbb{R}^n} |\beta_\theta (w_\theta)^2 w_\theta^2 - \beta_0 (w_\theta)^2 w_\theta| \lesssim \theta^{\frac{p-1}{p+q}},$$

which is essentially the same as the estimate (14.18), and the estimate (14.25) we obtain that

$$\lambda - \lambda_\theta \int_{\mathbb{R}^n} (\beta_\theta w_\theta^2 - \frac{\alpha^2}{2} w_\theta^2) = O(\delta^2) + O(\theta^{\frac{2(p-1)}{p+q}}) + a_w^2(\theta)$$

Recall that $\beta_0(w^2) = w^{p-1}$. The condition that $W[w] = 1$ implies that

$$\int_{\mathbb{R}^n} \left(\frac{1}{p+1} |w|^{p+1} - \frac{\alpha^2}{4} |w|^2 \right) = 1$$

In fact, the minimizer $w$ is smooth and localized in space and thus one could even give the precise dependence on $\mu$. 

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Thus,

\[ \int_{\mathbb{R}^n} (\beta_0(w)^2)w^2 - \frac{\alpha^2}{2} w^2 = 2 + \frac{p-1}{p+1} \int_{\mathbb{R}^n} |w|^{p+1} \geq 2 \]

This allows us to conclude that

\[ |\lambda - \lambda_\theta| \leq \delta^2 + \theta \frac{2(p-1)}{p+q} + a_w(\theta) \]

(14.29)

Finally, recall that the ground states \( \phi_\theta \) and \( \phi \) are obtained by the rescaling of the minimizers \( w_\theta \) and \( w \).

\[ \phi_\theta(x) = w_\theta(\lambda_\theta^{-\frac{1}{2}} x), \quad \phi(x) = w(\lambda^{-\frac{1}{2}} x) \]

Thus

\[ \|\phi_\theta - \phi\|_{H^1} \lesssim \|w_\theta(\lambda_\theta^{-\frac{1}{2}} x) - w(\lambda^{-\frac{1}{2}} x)\|_{H^1} \]

\[ = \left( \frac{n}{\theta} + \frac{n}{\lambda_\theta^{-1}} \right) \|w_\theta(x) - w\left(\frac{\lambda_\theta}{\lambda} x\right)\|_{H^1} \]

\[ \leq \left( \frac{n}{\theta} + \frac{n}{\lambda^{-1}} \right) \|w_\theta(x) - w(x)\|_{H^1} + \|w(x) - w\left(\frac{\lambda_\theta}{\lambda} x\right)\|_{H^1} \]

By (14.29) the constants \( \lambda_\theta \) are uniformly bounded in terms of the absolute constant \( \lambda \), which depends only on \( w \). Moreover, \( \lambda_\theta \to \lambda \) as \( \theta \to 0 \). We appeal again to the \( H^1 \) modulus of continuity of the minimizer \( w \) and define the function \(^8\)

\[ b_w(\theta, \delta) := \sup_{|\mu-1| \leq C(\delta^2 + \theta \frac{2(p-1)}{p+q} + a_w(\theta))} \|w(x) - w(\mu^{-\frac{1}{2}} x)\|_{H^1} \]

for some positive constant \( C \) dependent only on the minimizer \( w \). The function \( b_w(\theta, \delta) \to 0 \) as \( \theta, \delta \to 0 \). Therefore, since we have already proved in (14.25) that \( w_\theta \) is close to \( w \) in \( H^1 \), we obtain

\[ \|\phi_\theta - \phi\|_{H^1} \lesssim \delta + \theta \frac{p-1}{p+q} + a_w(\theta) + b_w(\theta, \delta) \]

(14.31)

Since by the claim \( \delta = \delta(\theta) \to 0 \) as \( \theta \to 0 \) and the functions \( a_w(\theta), b_w(\theta, \delta) \) also have this property we obtain the desired conclusion.

It remains to prove the claim (14.22), (14.23). Let \( u \) be as in the claim, i.e., \( u \in B_M \) and \( W_\theta[u] = 1 \), and

\[ \|u - w(\frac{x}{\mu})\|_{H^1} \geq \delta. \]

(14.32)

\(^8\)One can show that \( \alpha \neq 0 \) the Lagrange multiplier \( \lambda \neq 0 \). This follows from the following argument. By interpolation for \( p \leq \frac{n+2}{n-2} \)

\[ \int w^{p+1} \leq \|\nabla w\|^\frac{n+1}{2} \|w\|^\frac{p+1-n}{2} \]

Thus for \( n > 2 \) the power \( p + 1 - \frac{n+1}{2} \) is less than 2 and using Cauchy-Schwarz, constraint \( W[w] = 1 \) and the assumption that \( \alpha \neq 0 \), we can show that \( \|\nabla w\|_{L^2} \geq c \) for some positive constant \( c \). Repeating argument determining the Lagrange multiplier we verify that \( \lambda \neq 0 \)
for some $\delta$ to be chosen below. Similar to (14.20) we can find a constant $\nu = \nu(u)$ such that

\begin{align}
W_0[u(x)] &= 1, \\
J_0[u(x)] &= J_\theta[u] + O(\theta^{\frac{p}{p+q}}), \\
|\nu - 1| &\leq \theta^{\frac{p-1}{p+q}}
\end{align}

Using (14.20), (14.32), (14.34), and definition (14.24) we infer that

\begin{align}
\|u(x) - w\|_{H^1} &\geq \|u(x) - w(x)\|_{H^1} - \|w(x) - w(x)\|_{H^1} \\
&\geq \frac{\nu}{2} \delta - a_w(\theta) \\
&\geq \delta - \theta^{\frac{p-1}{p+q}} \delta - a_w(\theta)
\end{align}

We now use Lemma 14.2 for the variational problem $J = J_0$. This gives a function $\sigma(\gamma)$, with the property that $\sigma(\gamma) \to 0$ as $\gamma \to 0$, such that for any radial non-increasing positive $v$ with the property that $\|v - w\|_{H^1} \geq \sigma(\gamma)$ and $W_0[v] = 1$ we have $J_0[v] \geq J_0[w] + \gamma$. We set

$$
\gamma = \frac{5\theta^{p-1}}{p+q}, \quad \delta = \sigma(\gamma) + \theta^{\frac{p-1}{p+q}} + a_w(\theta)
$$

It follows from Definition 14.1 of $\sigma(\gamma)$ and (14.35) that with these choices, function $u(x)$ verifies the inequality

$$
J_0[u(x)] \geq J_0[w] + 5\theta^{\frac{p-1}{p+q}}
$$

Finally, using (14.21) and (14.33) we obtain

$$
J_\epsilon[u] \geq J_\theta[w(x)] + 3\theta^{\frac{p-1}{p+q}}
$$

It remains to note that the constant $\delta$ in has been chosen

$$
\delta = \sigma(\theta^{\frac{p-1}{p+q}}) + \theta^{\frac{p-1}{p+q}} + a_w(\theta)
$$

and by Lemma 14.2 and (14.24) goes to zero as $\theta \to 0$, as claimed. \qed

Recall definition of the operator $L^\theta_+$ associated with the ground state $\phi_\theta$.

\begin{align}
L^\theta_+ &= -\frac{1}{2} \Delta - \beta_\theta(\phi^2_\theta) - 2\beta'_\theta(\phi^2_\theta)\phi^2_\theta + \frac{\alpha^2}{2} \\
V_\theta &= \beta_\theta(\phi^2_\theta) + 2\beta'_\theta(\phi^2_\theta)\phi^2_\theta
\end{align}

Using the definition of $\beta_\theta$ we compute $V_\theta$ explicitly

\begin{align}
V_\theta &= p\phi^{p-1}_\theta \frac{f(\phi^2_\theta)}{\theta + f(\phi^2_\theta)} + 2\phi^{p+1}_\theta \frac{\theta f'(\phi^2_\theta)}{\theta + f(\phi^2_\theta)}
\end{align}

Uniform bounds on ground states $\phi_\theta$ guaranteed by the Proposition 14.4 imply the following result.
Lemma 14.5. Let function $f$ obey the assumptions that
\[ C_1 s^{q+1} \leq |f(s^2)| \leq C_2 s^{q+1}, \quad s^2 |f'(s^2)| \leq C_1 f(s^2) \]
for some positive constants $C_1, C_2$ and $q > -1$. Then for any $p \in (1, 1 + \frac{4}{n}]$ there exists a $r = r(p)$ in the interval $r \in \left[ \frac{p}{2}, \infty \right)$ such that
\[ (14.39) \quad \|V_\theta - V_0\|_{L^r} \to 0, \quad \theta \to 0 \]

Proof. We have a pointwise bound
\[ |V_\theta - p\phi_\theta^{p-1}| \lesssim \min \{ \phi_\theta^{p-1}, \theta \phi_\theta^{p-q-2} \} \leq \theta^0 + \phi_\theta^{p-1} \]
In addition, since $\phi_\theta \to \phi$ in $H^1$ we have that $\phi_\theta^{p-1} \to \phi^{p-1}$ in the space $L^{\frac{2}{p-1}} \cap L^{\frac{2n}{(n-2)(p-1)}}$. Since
\[ |V_\theta - V_0| \leq |V_\theta - p\phi_\theta^{p-1}| + p|\phi_\theta^{p-1} - \phi^{p-1}| \lesssim \theta^0 + \phi_\theta^{p-1} - \phi^{p-1} \]
we obtain the desired conclusion for any $r$ in the interval $r \in \left( \frac{2}{p-1}, \frac{2n}{(n-2)(p-1)} \right]$. The existence of the Lebesgue exponent $r$ in the desired interval now follows from the restriction $p \in (1, 1 + \frac{4}{n}]$ on the range of the exponent $p$.

Corollary 14.6. The operators
\[ (L_+^\theta - L_+)(-\triangle + 1)^{-1}: L^2 \to L^2, \quad (-\triangle + 1)^{-1}(L_+^\theta - L_+): L^2 \to L^2 \]
with the norm converging to 0 as $\theta \to 0$.

Proof. The difference $L_+^\theta - L_+ = V_\theta - V_0$. The result now follows from Lemma 14.5, Sobolev embeddings, and Hölder inequality.

14.2 Stability of the convexity condition

The goal of this section is to prove that the ground states $\phi_\theta$ of the problem
\[ \frac{1}{2} \Delta \phi + \beta_\theta(\phi^2) = \alpha^2 \phi \]
with the nonlinearity
\[ (14.40) \quad \beta_\theta(\phi^2) = |\phi|^{p-1} \phi \frac{f(\phi^2)}{\theta + f(\phi^2)} \]
\[ C_1 s^{q+1} \leq |f(s^2)| \leq C_2 s^{q+1}, \quad s^2 |f'(s^2)| \leq C_2 |f(s^2)|, \quad p \in (1, \frac{4}{n}), \quad q > -1 \]
verify the monotonicity condition (1.17), $\langle L_+^{\theta^{-1}} \phi_\theta, \phi_\theta \rangle > 0$. 65
Theorem 14.7. Let $\phi_0$ be ground states constructed in Proposition 14.4 with the nonlinearities $\beta_\theta$ satisfying (14.40). Then then for a given nonlinear eigenvalue $\alpha \neq 0$ and all sufficiently small $\theta$ the ground states $\phi_0$ verify the monotonicity condition
\begin{equation}
(14.41) \quad \langle L_+^{\theta -1} \phi_0, \phi_0 \rangle > 0
\end{equation}

Proof. Condition (14.41) is meaningful provided that $\phi_0$ is orthogonal to the kernel of $L_+^{\theta}$. We start by examining the spectrum of the operator $L_+^{\theta}$. The operator $L_+^{\theta}$ has a unique negative eigenvalue, the zero eigenvalue has multiplicity $n$ and the corresponding eigenspace is spanned by the function $\frac{\partial}{\partial x_i} \phi$, [We2]. The rest of the spectrum is contained in the set $[\alpha^2, \infty)$. Therefore, in the case $\alpha \neq 0$ the spectrum $\Sigma(L_+)$ of $L_+$ has an isolated discrete component (in fact two components). We can construct an eigenspace projector $P_0$ of an isolated component of the discrete spectrum
\begin{equation}
(14.42) \quad P_0 = \frac{1}{2\pi i} \int_\gamma (L_+ - z)^{-1} dz
\end{equation}
with an arbitrary curve $\gamma$ encircling the desired spectral set and such that $\gamma \cap \Sigma(L_+) = 0$. Consider now the resolvent of $L_+^{\theta}$ at $z$ such that $\text{dist}(z, \Sigma(L_+)) \geq C$ for some sufficiently small constant $C$, which only depends on $L_+$. We have
\begin{equation}
(14.43) \quad (L_+^{\theta} - z)^{-1} = (L_+ - z)^{-1} - (L_+^{\theta} - z)^{-1}(L_+^{\theta} - L_+)(L_+ - z)^{-1}
\end{equation}
It is not difficult to show that for such $z$
\[ \| (L_+ - z)^{-1} f \|_{H^2} \lesssim \| f \|_{L^2} \]
Therefore, using Corollary 14.6 we can conclude from (14.43) that for all sufficiently small $\theta h \geq 0$
\[ \| (L_+^{\theta} - z)^{-1} \| \leq 2\| (L_+ - z)^{-1} \| \]
and thus $z \notin \Sigma(L_+^{\theta})$. Moreover,
\begin{equation}
(14.44) \quad \| (L_+^{\theta} - z)^{-1} - (L_+ - z)^{-1} \| \leq c(\theta)
\end{equation}
for any $z : \text{dist}(z, \Sigma(L_+)) \geq C$. By Corollary 14.6 the constant $c(\theta) \to 0$ as $\theta \to 0$. Therefore, for the same path $\gamma$ as in (14.42) we can define
\begin{equation}
(14.45) \quad P_\theta = \frac{1}{2\pi i} \int_\gamma (L_+^{\theta} - z)^{-1} dz
\end{equation}
Moreover, for all sufficiently small $\theta \geq 0$ the rank of $P_\theta$ remains constant. Thus, for any sufficiently small $\theta$ the operator $L_+^{\theta}$ has a unique simple negative eigenvalue and a zero eigenspace of dimension $n$. Since we know that the functions $\frac{\partial}{\partial x_i} \phi_\theta$ are contained in that subspace, they, in fact, span it. Therefore, $\phi_\theta$ is orthogonal to the kernel of $L_+^{\theta}$ and the expression (14.41) is well defined.

For any sufficiently small $\theta \geq 0$ we set $Q_\theta$ to be a projection on the orthogonal complement of the null eigenspace of $L_+^{\theta}$. Let $\lambda \notin \cup_{\theta} \Sigma(L_+^{\theta})$. Define the operators
\begin{equation}
(14.46) \quad K_\theta(\lambda) := Q_0(L_+^{\theta} - \lambda)^{-1}Q_\theta - (L_+ - \lambda)^{-1}Q_0
\end{equation}
It follows from (14.44) and the properties of the spectrum of $L_+$ that for all small $\theta \geq 0$ and all $\lambda$ such that $|\lambda| \leq C$

\begin{equation}
\|(L_+^\theta - \lambda)^{-1}Q_\theta\| \leq \frac{1}{\text{dist} (\lambda, \Sigma(L_+^\theta) \setminus \{0\})} \leq C'
\end{equation}

for some universal constant $C'$, determined by the operator $L_+$. Also note that

\begin{equation}
\|Q_\theta - Q_0\| \leq c(\theta)
\end{equation}

This is a consequence of (14.44) and the definition

$$Q_\theta = I - \frac{1}{2\pi i} \int_{\gamma}(L_+^\theta - z')^{-1} dz'$$

with a short path $\gamma$ around the origin. Using the resolvent identity

$$(L_+ - z)^{-1} = (L_+ - z)^{-1} + (L_+ - z)^{-1}(L_+^\theta - L_+)(L_+^\theta - z)^{-1}$$

we obtain that for any $\lambda \notin \cup_\theta \Sigma(L_+^\theta)$

$$K_\theta(\lambda) = Q_0(L_+ - \lambda)^{-1}Q_\theta - (L_+ - \lambda)^{-1}Q_0 + Q_0(L_+ - \lambda)^{-1}(L_+^\theta - L_+)(L_+^\theta - \lambda)^{-1}Q_\theta$$

$$= (L_+ - \lambda)^{-1}Q_0(Q_\theta - Q_0) + (L_+ - \lambda)^{-1}Q_0(L_+^\theta - L_+)(L_+^\theta - \lambda)^{-1}Q_\theta$$

Using Corollary 14.6, (14.47), and (14.48) we infer that for any $\lambda \leq c$ and $\lambda \notin \cup_\theta \Sigma(L_+^\theta)$

\begin{equation}
\|K_\theta(\lambda)\| \leq c(\theta)(1 + \|L_+ - \lambda)^{-1}Q_0(-\Delta + 1)||) \leq c'(\theta)
\end{equation}

uniformly in $\lambda$. The last inequality follows since the operator norm of $(L_+ - \lambda)^{-1}Q_0(-\Delta + 1)$ is bounded by a universal constant dependent on $L_+$ only. This can be seen as follows. Since $V_0$ is a smooth potential and $(L_+ - \lambda)^{-1}Q_0$ is bounded on $L^2$ we can replace the operator $(-\Delta + 1)$ by $(L_+ - \lambda)$ and the result follows immediately.

We now test the operator $K_\theta(\lambda)$ on the ground state $\phi_\theta$. Using that $Q_\theta \phi_\theta = \phi_\theta$ we obtain

$$K_\theta(\lambda)\phi_\theta = Q_0(L_+^\theta - \lambda)^{-1}\phi_\theta - (L_+ - \lambda)^{-1}\phi + (L_+ - \lambda)^{-1}Q_0(\phi - \phi_\theta)$$

Coupling the above identity with $\phi$.

$$\langle \phi_\theta, (L_+^\theta - \lambda)^{-1}\phi_\theta \rangle - \langle \phi, (L_+ - \lambda)^{-1}\phi \rangle = \langle \phi, K_\theta(\lambda)\phi_\theta \rangle + \langle (\phi_\theta - \phi_0), (L_+^\theta - \lambda)^{-1}\phi_\theta \rangle$$

$$+ \langle \phi, (L_+ - \lambda)^{-1}Q_0(\phi - \phi_\theta) \rangle = O(c(\theta))$$

where we have used that $Q_\theta \phi_\theta = \phi_\theta$, the bound (14.47), and the estimate $\|\phi_\theta - \phi\|_{H^1} \leq c(\theta)$, which follows from Proposition 14.4. The above holds uniformly for all $|\lambda| \leq c$ and $\lambda \notin \cup_\theta \Sigma(L_+^\theta)$. Passing to the limit $\lambda \to 0$, say from the upper half-plane, we obtain that for all sufficiently small $\theta \geq 0$

$$\langle \phi_\theta, (L_+^\theta)^{-1}\phi_\theta \rangle = \langle \phi, L_+^{-1}\phi \rangle + O(c(\theta)) < 0$$

The last inequality follows since by the assumption $\phi$ is a stable ground state, i.e., $\langle \phi, L_+^{-1}\phi \rangle < 0$. \[\square\]
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