TOPOLOGICAL CONJUGACY OF LINEAR SYSTEMS ON LIE GROUPS

ADRIANO DA SILVA
Departamento de Matemática
Universidade de Campinas, Campinas, Brazil

ALEXANDRE J. SANTANA
Departamento de Matemática
Universidade Estadual de Maringá, Maringá, Brazil

SIMÃO N. STELMASTCHUK
Universidade Federal do Paraná
Jandáia do Sul, Brazil

Abstract. In this paper we study a classification of linear systems on Lie groups with respect to the conjugacy of the corresponding flows. We also describe stability according to Lyapunov exponents.

1. Introduction. Conjugacy and stability are some of the central notions in the classical theory of dynamical systems and are also present in many others contexts, for example, control theory, topological dynamics and Lie groups theory. The study of topological classification has a long history, for example, from Poincaré [7] it is clear that Poincaré knew that orthogonal $2 \times 2$ matrices are topologically conjugate if and only if they are linearly conjugate, in Kuiper and Robbin [6] was studied the problem of classifying linear operators using topological conjugacy and in Robbin [8] was studied topological conjugacy and structural stability for discrete dynamical systems. The main idea of this subject are to establish conditions to find a homeomorphism between the state spaces that preserves trajectories and to find conditions for stability in term of eigenvalues and Lyapunov exponents.

In this direction, we comment briefly on some works that deal conjugacy and stability in our context. In the case of dynamical systems, i.e., if we consider the state space as Euclidean space, a well-known result says that in case of hyperbolicity, two linear autonomous differential equations are topologically conjugate if and only if the dimensions of the stable subspaces coincide (see, e.g., Colonius and Kliemann [2] and Robinson [9]). Now in control theory, Ayala, Colonius, and Kliemann [1] and Colonius and Santana [3] studied these concepts in case of control systems. Using concepts and techniques from topological dynamics they classified the control flows of these control systems. In Lie groups, Kawan, Rocío and Santana (see [5])

2010 Mathematics Subject Classification. 22E20, 54H20, 34D20, 37B99.
Key words and phrases. Topological conjugacy, linear vector fields, flows, Lie groups, Lyapunov exponents.

The first author was supported by CAPES grant n° 4792/2016-PRO and partially supported by FAPESP grant n° 2013/19756 – 8. The second author was partially supported by Fundação Araucária grant n° 20134003. This work was partially supported by CNPq/Universal grant n° 476024/2012-9.
studied the flows of nonzero left invariant vector fields on Lie groups with respect
to topological conjugacy. Using the fundamental domain method, it was showed
that on a simply connected nilpotent Lie group any such flows are topologically
conjugate. Combining this result with the Iwasawa decomposition, they proved
that on a noncompact semisimple Lie group the flows of two nilpotent or abelian
fields are topologically conjugate. Finally, taking the state space as affine groups
they showed that the conjugacy class of a left invariant vector field does not depend
on its Euclidean component.

In our paper we consider flows given by an infinitesimal automorphism, that is,
we consider the linear system on a connected Lie group
\[ \dot{g}(t) = \mathcal{X}(g(t)), \]
where the drift \( \mathcal{X} \) is an infinitesimal automorphism. Then following a similar ap-
proach of the classical result we classify the linear flows according to the decompo-
sition of the state space in stable, unstable and central Lie subgroups. Moreover,
given a fixed point \( g \in G \) of the linear flow given by \( \mathcal{X} \), we define when
\( g \) is sta-
bly, asymptotically stable and exponentially stable. We characterize these stability
properties using Lyapunov exponents.

The paper is organized as follows, in the second section we establish some results,
prove that the stable and unstable subgroups are simply connected and characterize
the stable and unstable elements as attractors and repellers. In the third section we
prove an important result of the paper, that is, given two linear vector fields we give
conditions on it and on their stable spaces in order to ensure that their respective
flows are conjugated. Finally, in the last section we study the Lyapunov stability.

2. Stable and unstable Lie subgroups. In this section we present notations
and basic tools to prove that the stable, unstable components of the state space are
simply connected. Moreover in the main result of this section we characterize the
stable and unstable elements as attractors and repellers respectively.

**Definition 2.1.** Let \( \{ \varphi_t \}_{t \in \mathbb{R}} \) be a flow of automorphisms on a connected Lie group
\( G \). We say that \( \varphi_t \) is **contracting** if there are constants \( c, \mu > 0 \) such that
\[ \| (d\varphi_t)_e X' \| \leq ce^{-\mu t}\| X' \| \quad \text{for any } X \in \mathfrak{g}; \]
We say that \( \varphi_t \) is **expanding** if \( (\varphi_t)^{-1} = \varphi_{-t} \) is contracting.

The next results relate dynamical properties of the flows of automorphisms with
topological properties.

**Proposition 1.** Let \( G \) be a connected Lie group and \( (\varphi_t) \) a flow of automorphism
on \( G \). If \( \varphi_t \) is contracting or expanding, the Lie group \( G \) is simply connected.

**Proof.** Assume that \( \varphi_t \) is contracting, since the expanding case is analogous. Let
\( \tilde{G} \) be the connected simply connected covering of \( G \) and consider \( D \) as the central
discrete subgroup of \( \tilde{G} \) such that \( G = \tilde{G}/D \).

Since the canonical projection \( \pi : \tilde{G} \to G \) is a covering map and \( \tilde{G} \) is simply
connected, we can lift \( (\varphi_t)_{t \in \mathbb{R}} \) to a flow \( (\tilde{\varphi}_t)_{t \in \mathbb{R}} \) on \( \tilde{G} \) such that
\[ \pi \circ \tilde{\varphi}_t = \varphi_t \circ \pi, \quad \text{for all } t \in \mathbb{R}. \]
In particular \( \tilde{\varphi}_t(D) = D \) for all \( t \in \mathbb{R} \). As \( D \) is discrete and \( \tilde{\varphi}_t \) is continuous, we have
that \( \tilde{\varphi}_t(x) = x \) for all \( x \in D \) and \( t \in \mathbb{R} \). However, as \( \varphi_t \) is contracting, \( \tilde{\varphi}_t \) is also
contracting and so, \( \tilde{\varphi}_t(x) \to \hat{e} \) when \( t \to \infty \), where \( \hat{e} \in \tilde{G} \) is the identity element.
Note that $D$ is discrete and $\varphi_t$-invariant, then for $x \in D$ and $t > 0$ large enough we have that $\varphi_t(x) = e$, hence $x = e$. Therefore, $D = \{e\}$ and so $\hat{G} = G$. \hfill \square

Next we define the concept of linear vector fields.

**Definition 2.2.** Let $G$ be a connected Lie group. A vector field $\mathcal{X}$ is said to be **linear** if its associated flow $(\varphi_t)_{t \in \mathbb{R}}$ is a one-parameter subgroup of $\text{Aut}(G)$.

**Remark 1.** By considering $\mathbb{R}^n$ as an abelian Lie group, any invertible $n \times n$ matrix $A$ induces a linear vector field as $\mathcal{X}_A(g) := Ag$. Its associated linear flow is given by matrix exponential $(e^{\lambda A})_{t \in \mathbb{R}}$.

Associated with the linear vector field $\mathcal{X}$ we have the connected $\varphi$-invariant Lie subgroups $G^+$, $G^0$ and $G^-$ with Lie algebras given, respectively, by

$$g^+ = \bigoplus_{\alpha; \text{Re}(\alpha) > 0} g_\alpha, \quad g^0 = \bigoplus_{\alpha; \text{Re}(\alpha) = 0} g_\alpha \quad \text{and} \quad g^- = \bigoplus_{\alpha; \text{Re}(\alpha) < 0} g_\alpha,$$

where $\alpha$ is an eigenvalue of the derivation $\mathcal{D}$ associated with $\mathcal{X}$ and $g_\alpha$ its generalized eigenspace.

We consider also $G^{+0}$ and $G^{-0}$ as the connected $\varphi$-invariant Lie subgroups with Lie algebras $g^{+0} = g^+ \oplus g^0$ and $g^{-0} = g^- \oplus g^0$ respectively. The next proposition contains important properties of the above subgroups. Its proof can be found in [1], Proposition 2.9.

**Proposition 2.** It holds:

1. $G^{+0} = G^+G^0 = G^0G^+$ and $G^{-0} = G^-G^0 = G^0G^-$;
2. $G^+ \cap G^- = G^{+0} \cap G^- = G^{-0} \cap G^+ = \{e\}$;
3. All the above subgroups are closed in $G$;
4. If $G$ is solvable then

$$G = G^{+0}G^- = G^{-0}G^+.$$  \hfill (2)

Moreover, the singularities of $\mathcal{X}$ are in $G^0$.

The subgroups $G^+$, $G^0$ and $G^-$ are called, respectively, the **unstable**, **central** and **stable** subgroups of the linear flow $\varphi_t$. We denote the restriction of $\varphi_t$ to $G^+$ and $G^-$, respectively, by $\varphi^+_t$ and $\varphi^-_t$. Such restrictions are in fact automorphisms of $G^+$ and $G^-$ since such subgroups are closed. The next proposition gives us another topological property of such subgroups.

**Proposition 3.** The Lie subgroups $G^+$ and $G^-$ are simply connected.

**Proof.** Since $(d\varphi_t)_e = e^\mathcal{D}$, restricted to $g^+$ and to $g^-$, has only eigenvalues with positive and negative real parts, respectively, we have that $(\varphi^+_t)_{t \in \mathbb{R}}$ is an expanding flow of automorphisms and $(\varphi^-_t)_{t \in \mathbb{R}}$ a contracting flow of automorphisms. Then Proposition [1] implies that $G^+$ and $G^-$ are simply connected. \hfill \square

**Remark 2.** We should remark that when $G$ is a compact Lie group and $\varphi_t$ is a flow of automorphisms associated to the linear vector field $\mathcal{X}$ then $G = G^0$ and therefore $\varphi_t$ cannot be contracting or expanding.

Next we prove a technical lemma that is necessary in the next sections.

**Lemma 2.3.** Suppose that $G$ is a connected Lie group. Let $H_1, H_2$ be closed subgroups of $G$ such that $G = H_1H_2$ and $H_1 \cap H_2 = \{e\}$. Given a sequence $(x_n)$ in $G$, consider the unique sequences $(h_1, x_{n_1})$ in $H_1$ and $(h_2, x_{n_2})$ in $H_2$ such that $x_n = h_1h_2$. Then, $x_n \to x$ if and only if $h_{1,n} \to h_1$ for $i = 1, 2$, where $x = h_1h_2$. 


Proof. Certainly if \( h_{i,n} \to h_i \) in \( G \), \( i = 1, 2 \), we have that \( x_n = h_{1,n} h_{2,n} \to h_1 h_2 = x \) in \( G \). Suppose that \( x_n \to x \). By considering \( h_{1,n} = h_1^{-1} h_{1,1} \) and \( h_{2,n} = h_2^{-1} h_{2,1} \) we can assume that \( x_n \to e \) and we need to show that \( h_{i,n} \to e, \ i = 1, 2 \). Let \( V_{i} \) be a neighborhood of \( e \in H_i, \ i = 1, 2 \) and consider the neighborhood \( W = V_1 V_2 \) of \( e \in G \). As \( x_n \to e \), there is \( N \in \mathbb{N} \) such that for \( n \geq N \) we have \( x_n \in W \). Then \( h_{i,2} \in V_i, \ i = 1, 2 \) showing that \( h_{i,n} \to e \) for \( i = 1, 2 \).

Next we characterize the stable and unstable elements as attractors and repellers respectively. First we introduce the concept of hyperbolic linear vector field.

**Definition 2.4.** Let \( X \) be a linear vector field. We say that \( X \) is hyperbolic if its associated derivation \( D \) is hyperbolic, that is, \( D \) has no eigenvalues with zero real part.

**Remark 3.** We should notice that a necessary condition for the existence of a hyperbolic linear vector field on a connected Lie group \( G \) is that \( G \) is a nilpotent Lie group. This follows from the fact that if \( g_\alpha \) is the generalized eigenspace associated with the eigenvalue \( \alpha \) of \( D \), then

\[
[g_\alpha, g_\beta] \subset g_{\alpha + \beta}
\]

if \( \alpha + \beta \) is an eigenvalue of \( D \) (\( g_{\alpha + \beta} = 0 \) if \( \alpha + \beta \) is not eigenvalue of \( D \)).

To define the attractors and repellers we consider \( g \) the left invariant Riemannian distance on \( G \). As \( \varphi_t \circ L_g = L_{\varphi_t(g)} \circ \varphi_t \) we have

\[
g(\varphi_t(g), \varphi_t(h)) \leq ||(d\varphi_t)_e|| g(g, h), \quad \text{for any } g, h \in G.
\]

In particular, since \((d\varphi_t^-)_e = e^{tD}a^-\) has only eigenvalues with negative real part, there are constants \( c, \mu > 0 \) such that

\[
g(\varphi_t^-(g), \varphi_t^-(h)) \leq c e^{-\mu t} g(g, h), \quad \text{for any } g, h \in G^-, t \geq 0.
\]

(3)

Analogously, we have that

\[
g(\varphi_t^+(g), \varphi_t^+(h)) \geq c e^{\mu t} g(g, h), \quad \text{for any } g, h \in G^+, t \geq 0.
\]

(4)

Now we can characterize the stable and unstable elements as attractors and repellers.

**Theorem 2.5.** If \( X \) is a hyperbolic linear vector field on \( G \) then

1. It holds that \( g \in G^- \) if and only if \( \lim_{t \to -\infty} \varphi_t(g) = e \).
2. It holds that \( g \in G^+ \) if and only if \( \lim_{t \to -\infty} \varphi_t(g) = e \).

**Proof.** We prove only the first assertion, since the proof of second one is analogous. Let \( g \in G^- \). It is clear that \( \varphi_t(g) = \varphi_t^-(g) \). From equation (3) it follows that

\[
g(\varphi_t^-(g), e) = g(\varphi_t^-(g), \varphi_t^-(e)) \leq e^{-\mu t} g(g, e).
\]

Taking \( t \to -\infty \) we have \( g(\varphi_t^-(g), e) \to 0 \), that is, \( \lim_{t \to -\infty} \varphi_t(g) = e \).

Conversely, let \( g \in G^- \) and assume that \( \lim_{t \to -\infty} \varphi_t(g) = e \). By Proposition 2 \( g \) can be written uniquely as \( g = g^- g^+ \) with \( g^\pm \in G^\pm \). By the left invariance of the metric we have that

\[
g(\varphi_t(g^+), e) = g(\varphi_t(g^-) \varphi_t(g^+), \varphi_t(g^-)) = g(\varphi_t(g^-) \varphi_t(g^+), \varphi_t(g^-))
\]

\[
\leq g(\varphi_t(g), e) + g(e, \varphi_t(g^-)).
\]

Since \( g^- \in G^- \) we have

\[
\lim_{t \to -\infty} \varphi_t(g^-) = \lim_{t \to -\infty} \varphi_t(g) = e
\]
implying that \( \lim_{t \to \infty} \varphi_t(g^+) = e \). Hence, using the inequality \( 4 \) it follows that 
\( g^+ = e \), then 
\( g = g^- \in G^- \).

3. Conjugation between linear flows. Let \( A, B \) be are real \( n \times n \)-matrices. It is well known that \( e^{tA} \) and \( e^{tB} \) are topologically conjugated if and only if the dimensions of their unstable (and hence the dimension their stable) subspaces coincide (see for instance \([2], \) Theorem 2.2.5), that is, there exists a homeomorphism \( \xi : \mathbb{R}^n \to \mathbb{R}^n \) such that

\[
\xi(e^{tA}v) = e^{tB}\xi(v), \quad \text{for any } v \in \mathbb{R}^n.
\]

Our aim in this section is to extend such result for general linear flows. From now on we consider \( X \) and \( Y \) as linear vector fields on connected Lie groups \( G \) and \( H \) respectively, denote their linear flows by \((\varphi_t)_{t \in \mathbb{R}}\) and \((\psi_t)_{t \in \mathbb{R}}\) and their associated derivation by \( D \) and \( F \), respectively. We say that \( X \) and \( Y \) are topologically conjugated if there exists a homeomorphism \( \pi : G \to H \) such that

\[
\pi(\varphi_t(g)) = \psi_t(\pi(g)), \quad \text{for all } g \in G.
\]

With the following lemma we can prove a theorem that establishes conditions to have conjugation between the restricted flows.

**Lemma 3.1.** It holds that \( \dim g^+ = \dim h^+ \) if and only if there exists a homeomorphism \( \xi : g^+ \to h^+ \) such that

\[
\xi(e^{tD^+}X) = e^{tF^+}\xi(X), \quad \text{for any } X \in g^+,
\]

where \( D^+ \) and \( F^+ \) are the restrictions of \( D \) and \( F \) to \( g^+ \) and \( h^+ \), respectively.

**Proof.** Consider \( \dim g^+ = n \) and \( \dim h^+ = m \). Then there are two isomorphisms \( S : g^+ \to \mathbb{R}^n \) and \( T : h^+ \to \mathbb{R}^m \). Thus we define the linear maps \( \tilde{D} : \mathbb{R}^n \to \mathbb{R}^n \) by \( \tilde{D} = SD^+S^{-1} \) and \( \tilde{F} : \mathbb{R}^m \to \mathbb{R}^m \) by \( \tilde{F} = TF^+T^{-1} \). It is straightforward to see that \( \tilde{D} \) has the same eigenvalues of \( D^+ \) and the same is true for the eigenvalues of \( \tilde{F} \) and \( F \).

Suppose now that \( n = m \), then there exists a homeomorphism \( \zeta : \mathbb{R}^n \to \mathbb{R}^n \) such that

\[
\zeta(e^{t\tilde{D}}X) = e^{t\tilde{F}}\zeta(X), \quad \text{for any } X \in \mathbb{R}^n.
\]

Defining \( \xi : g^+ \to h^+ \) by \( \xi = T^{-1}\zeta S \) we see that

\[
\xi^{-1}e^{tF^+}\xi(X) = S^{-1}\zeta^{-1}e^{t(TF^+T^{-1})}\zeta S(X) = S^{-1}\zeta^{-1}e^{t\tilde{F}}\zeta S(X) = e^{tD^+}(X).
\]

Conversely, suppose that there exists a homeomorphism \( \xi : g^+ \to h^+ \) such that

\[
\xi(e^{tD^+}X) = e^{tF^+}\xi(X), \quad \text{for all } X \in g^+.
\]

The map \( \zeta : \mathbb{R}^n \to \mathbb{R}^m \) given by \( \zeta(v) = T\xi S^{-1}(v) \) is a homeomorphism, then \( \dim g^+ = n = m = \dim h^+ \).

In a similar way, we can prove the above result in the stable case.

**Theorem 3.2.** The unstable subgroups of \( \varphi_t \) and \( \psi_t \) have the same dimension if and only if \( \varphi_t^+ \) and \( \psi_t^+ \) are conjugated. Analogously, the dimensions of their stable subgroups agree if and only if \( \varphi_t^- \) and \( \psi_t^- \) are conjugated.
Proof. We prove the unstable case, since the stable is analogous. Let us assume that the unstable subgroups of $\varphi_t$ and $\psi_t$ have the same dimension. Therefore, $\dim g^+ = \dim h^+$ and by the above lemma, there exists $\xi : g^+ \to h^+$ such that

$$\xi(e^{x}X) = e^{\xi(x)} \xi(X), \quad \text{for all } X \in g^+.$$  

By Proposition\textsuperscript{8} the subgroups $G^+$ and $H^+$ are simply connected. Since they are also nilpotent Lie groups, their exponential maps $\exp_{G^+}$ and $\exp_{H^+}$ are diffeomorphisms which implies that the map

$$\pi : G^+ \to H^+, \quad g \in G^+ \mapsto \pi(g) := \exp_{H^+} (\xi(\exp_{G^+}^{-1}(g)))$$  

is well defined. By the very definition, we have that $\pi$ is a homeomorphism. Now we show that $\pi$ conjugates $\varphi^+_t$ and $\psi_t^+$. Since $\varphi^+_t \circ \exp_{G^+} = \exp_{G^+} \circ e^{\varphi^+_t}$ and $\psi^+_t \circ \exp_{H^+} = \exp_{H^+} \circ e^{\psi^+_t}$ we have, for any $g \in G^+$, that

$$\pi(\varphi^+_t(g)) = \exp_{H^+} (\xi(\exp_{G^+}^{-1}(\varphi^+_t(g)))) = \exp_{H^+} (\xi(e^{\varphi^+_t} (\exp_{G^+}^{-1}(g))))$$

$$= \exp_{H^+} (\xi(e^{\varphi^+_t} (\exp_{G^+}^{-1}(g)))) = \psi^+_t (\pi(g)).$$

Conversely, suppose that there exists a homeomorphism $\pi : G^+ \to H^+$ such that

$$\pi(\varphi^+_t(g)) = \psi_t (\pi(g)), \quad g \in G^+.$$  

Since $G^+$ and $H^+$ are connected, nilpotent and simply connected, it follows that the map $\xi : g^+ \to h^+$ given by

$$\xi(X) = \exp_{H^+}^{-1} (\pi(\exp_{G^+}(X))), \quad \text{for any } X \in g^+$$

is well defined and is in fact a homeomorphism between $g^+$ and $h^+$. Moreover, $\xi$ is a conjugacy between $e^{\varphi^+_t}$ and $e^{\psi^+_t}$. In fact,

$$\xi(e^{\varphi^+_t} X) = \exp_{H^+}^{-1} (\pi(\exp_{G^+}(\varphi^+_t X))) = \exp_{H^+}^{-1} (\pi(\varphi^+_t (\exp_{G^+}(X))))$$

$$= \exp_{H^+}^{-1} (\psi^+_t (\pi(\exp_{G^+}(X)))) = e^{\psi^+_t} (\exp_{H^+}^{-1} (\pi(\exp_{G^+}(X)))) = e^{\psi^+_t} (\xi(X)).$$

By Lemma\textsuperscript{3,1} we have $\dim g^+ = \dim h^+$. Then the unstable subgroups of $\varphi_t$ and $\psi_t$ have the same dimension.

Now we have the main result of the paper.

**Theorem 3.3.** Suppose that $X$ and $Y$ are hyperbolic. If the stable and unstable subgroups of $\varphi_t$ and $\psi_t$ have the same dimension, then $\varphi_t$ and $\psi_t$ are conjugated.

**Proof.** By the above theorem, there are homeomorphisms $\pi_u : G^+ \to H^+$, that conjugates $\varphi^+_t$ and $\psi_t$, and $\pi_s : G^- \to H^-$ that conjugates $\varphi^-_t$ and $\psi^-_t$. Since $G$ is nilpotent, Proposition\textsuperscript{2} item 4. implies that $G = G^+ G^-$ and item 2. that $G^+ \cap G^- = e_G$. Consequently, any $g \in G$ has a unique decomposition $g = g^+ g^-$ with $g^+ \in G^+$ and $g^- \in G^-$. The same is true for $H$. Therefore, the map $\pi : G \to H$ given by

$$g = g^+ g^- \in G^+ G^- \mapsto \pi(g) := \pi_u(g^+) \pi_s(g^-) \in H^+ H^- = H$$

is well defined and has inverse $\pi^{-1}(h^+ h^-) = \pi^{-1}_u(h^+) \pi^{-1}_s(h^-)$.

Now we prove that $\pi$ is continuous. Take $(x_n)$ a sequence in $G$ and assume that $x_n \to x$. By Proposition\textsuperscript{2} item 4. there are sequences $(g^+_n)$ in $G^+$ and $(g^-_n)$ in $G^-$ such that $x_n = g^+_n g^-_n$. If $x = g^+ g^-$ we have by Lemma\textsuperscript{2,3} that $x_n \to x$ if and only if $g^+_n \to g^+$ in $G^+$. Since $\pi_u$ and $\pi_s$ are homeomorphism we have that $\pi_u(g^+_n) \to \pi_u(g^+)$ and $\pi_s(g^-_n) \to \pi_u(g^-)$. Using Lemma\textsuperscript{2,3} for $H$ we have

$$\pi(x_n) = \pi_u(g^+_n) \pi_s(g^-_n) \to \pi_u(g^+) \pi_u(g^-) = \pi(x).$$
showing that $\pi$ is continuous. Analogously $\pi^{-1}$ is continuous.

To finish note that $\pi$ conjugates $\varphi_t$ and $\psi_t$. In fact
$$\pi(\varphi_t(g)) = \pi_u(\varphi_t^+(g^+))\pi_s(\varphi_t^-(g^-)) = \psi_t(\pi_u(g^+))\psi_t(\pi_s(g^-)) = \psi_t(\pi(g)).$$

\[ \square \]

**Corollary 1.** Suppose that $X$ and $Y$ are hyperbolic and $G = H$. If the stable or the unstable subgroups of $\varphi_t$ and $\psi_t$ have the same dimension, then $\varphi_t$ and $\psi_t$ are conjugated.

**Proof.** In fact, if $G_1^+$ and $G_2^+$ are, respectively, the unstable subgroups of $\varphi_t$ and $\psi_t$ and, $G_1^-$ and $G_2^-$, respectively, their stable subgroups, then
$$\dim G_1^+ + \dim G_1^- = \dim G = \dim G_2^+ + \dim G_2^-$$

implying that $\dim G_1^+ = \dim G_2^+$ if and only if $\dim G_1^- = \dim G_2^-$. \[ \square \]

In order to prove the converse of Theorem 3.3 we need the following technical lemma.

**Lemma 3.4.** Let $X$ and $Y$ be hyperbolic linear vector fields on $G$ and $H$, respectively. If the homeomorphism $\pi : G \to H$ conjugates $\varphi_t$ and $\psi_t$, then $\pi(e_G) = e_H$.

**Proof.** By Remark 3 we have that $G$ and $H$ are nilpotent Lie group and therefore, by Proposition 2 item 4., the unique fixed points of the flows $\varphi_t$ and $\psi_t$ are $e_G \in G$ and $e_H \in H$, respectively. Then, the following equality
$$\psi_t(\pi(e_G)) = \pi(\varphi_t(e_G)) = \pi(e_G).$$

shows the lemma. \[ \square \]

**Theorem 3.5.** Take the hyperbolic linear vector fields $X$ and $Y$. If $\varphi_t$ and $\psi_t$ are conjugated, then their stable and unstable subgroups have the same dimension.

**Proof.** Let $\pi : G \to H$ be a homeomorphism such that
$$\pi(\varphi_t(g)) = \psi_t(\pi(g)).$$

From Theorem 3.2 it is sufficient to show that $\varphi_t^\pm$ and $\psi_t^\pm$ are conjugated. We only show that $\varphi_t^-$ and $\psi_t^-$ are conjugated. We begin by showing that $\pi(G^-) = H^-$. Take $g \in G^-$. From Lemma 3.4 and Theorem 2.5 it follows that
$$e_H = \pi(e_G) = \pi \left( \lim_{t \to \infty} \varphi_t(g) \right) = \lim_{t \to \infty} \pi(\varphi_t(g)) = \lim_{t \to \infty} \psi_t(\pi(g)).$$

Again by Theorem 2.5 we have that $\pi(g) \in H^-$ showing that $\pi(G^-) \subset H^-$. Analogously we show that $\pi^{-1}(H^-) \subset G^-$. If we consider the restriction $\pi_s := \pi|_{G^-}$ we have that $\pi_s$ is a homeomorphism between $G^-$ and $H^-$ and it conjugates $\varphi_t^-$ and $\psi_t^-$. From Theorem 3.2 we have that the stable subgroups of $\varphi_t$ and $\psi_t$ have the same dimension. \[ \square \]

4. **Some stability properties.** In this section we study some stability properties of a linear flow on a Lie group $G$.

First we define the Lyapunov exponent at $g \in G$ in direction to $v \in T_gG$ by
$$\lambda(g, v) = \limsup_{t \to \infty} \frac{1}{t} \log(||(d\varphi_t)_g(v)||),$$

where the norm $|| \cdot ||$ is given by the left invariant metric.
Our next step is to show the invariance of the Lyapunov exponent. In fact, since that \( \varphi_t \circ L_g = L_{\varphi_t(g)} \circ \varphi_t \), it follows that
\[
(d\varphi_t)_g(v) = (d\varphi_t)_g((dL_g)_e \circ (dL_{g^{-1}})_g(v)) = (dL_{\varphi_t(g)})_e \circ (d\varphi_t)_g((dL_{g^{-1}})_g(v)).
\]
As \( \| \cdot \| \) is a left invariant norm we have that
\[
\lambda(g, v) = \limsup_{t \to \infty} \frac{1}{t} \log(\| (dL_{\varphi_t(g)})_e \circ (d\varphi_t)_g((dL_{g^{-1}})_g(v)) \|)
\]
\[
= \limsup_{t \to \infty} \frac{1}{t} \log(\| (d\varphi_t)_e((dL_{g^{-1}})_g(v)) \|)
\]
\[
= \lambda(e, (dL_{g^{-1}})_g(v)).
\]
It is clear that for \( v \in g \) we obtain \( \lambda(g, v(g)) = \lambda(e, v(e)) \). In other words, taking \( v \in g \) we obtain
\[
\lambda(e, v) = \limsup_{t \to \infty} \frac{1}{t} \log(\| e^{t \mathcal{D}}(v) \|).
\]

One important property to the characterization of Lyapunov exponents is: consider \( u, v \in g \), then \( \lambda(e, u + v) \leq \max\{\lambda(e, u), \lambda(e, v)\} \) and the equality is true if \( \lambda(e, u) \neq \lambda(e, v) \).

Now denote by \( \lambda_1, \ldots, \lambda_k \) the \( k \) distinct values in of the real parts of the derivation \( \mathcal{D} \). We have then that
\[
g = \bigoplus_{i=1}^k g_{\lambda_i}, \quad \text{where} \quad g_{\lambda_i} := \bigoplus_{\alpha; \Re(\alpha) = \lambda_i} g_{\alpha}
\]

**Theorem 4.1.** It holds that
\[
\lambda(e, v) = \lambda \Leftrightarrow v \in g_\lambda := \bigoplus_{\alpha; \Re(\alpha) = \lambda} g_{\alpha}
\]

**Proof.** We first suppose that \( v \in g_\lambda \), then
\[
\lambda(e, v) = \limsup_{t \to \infty} \frac{1}{t} \log(\| e^{t \mathcal{D}}(v) \|)
\]
\[
= \limsup_{t \to \infty} \frac{1}{t} \log(\| e^{t \alpha} e^{(\mathcal{D} - \alpha I)}(v) \|)
\]
\[
= \Re(\alpha) + \limsup_{t \to \infty} \frac{1}{t} \log(\| e^{(\mathcal{D} - \alpha I)}(v) \|).
\]
Since \( e^{(\mathcal{D} - \alpha I)}(v) = \sum_{i=0}^d \frac{t^i}{i!} (\mathcal{D} - \alpha I)^i(v) \) is a polynomial, it follows that
\[
\limsup_{t \to \infty} \frac{1}{t} \log(\| e^{(\mathcal{D} - \alpha I)}(v) \|) = 0
\]
which gives us \( \lambda(e, v) = \Re(\alpha) = \lambda \) as stated.

Conversely, suppose that \( \lambda(e, v) = \lambda \) and that \( v \notin g_\lambda \). Assume that \( \lambda = \lambda_1 \) and write \( v = v_2 + v_3 + \ldots + v_k \) with \( v_i \in g_{\lambda_i} \) for \( i = 2, \ldots, k \). Since \( \lambda(e, v_i) = \lambda_i \neq \lambda_j = \lambda(e, v_j) \), for \( i, j = 1, \ldots, n \) with \( i \neq j \), the above lemma assures that
\[
\lambda_1 = \lambda(e, v) = \lambda(e, v_2 + v_3 + \ldots + v_k)
\]
\[
= \max\{\lambda(e, v_2), \lambda(e, v_3), \ldots, \lambda(e, v_k)\}
\]
\[
= \max\{\lambda_2, \lambda_3, \ldots, \lambda_k\}.
\]
Since \( \lambda_1 \neq \lambda_i \) for \( i = 2, \ldots, k \) we have a contradiction. \( \square \)

Now we recall the following concepts in the context of Lie groups.
Definition 4.2. Let $g \in G$ be a fixed point of $\mathcal{X}$. We say that $g$ is

1) **stable** if for all $g$-neighborhood $U$ there is a $g$-neighborhood $V$ such that $\varphi_t(V) \subset U$ for all $t \geq 0$;

2) **asymptotically stable** if it is stable and there exists a $g$-neighborhood $W$ such that $\lim_{t \to \infty} \varphi_t(x) = g$ whenever $x \in W$;

3) **exponentially stable** if there exist $c, \mu$ and a $g$-neighborhood $W$ such that for all $x \in W$ it holds that
   
   \[ g(\varphi_t(x), g) \leq ce^{-\mu t} g(x, g), \quad \text{for all} \quad t \geq 0; \]

4) **unstable** if it is not stable.

Note that, since property 3) is local, it does not depend on the metric that we choose on $G$.

The next result is important for the main results of this section.

Lemma 4.3. Let $\mathcal{X}$ and $\mathcal{Y}$ be linear vector fields on the Lie groups $G$ and $H$, respectively, and $\pi : G \to H$ be a continuous map that commutes the linear flows of $\mathcal{X}$ and $\mathcal{Y}$. If the fixed point $g$ of $\mathcal{X}$ is stable (asymptotically stable) and there is a $g$-neighborhood $U$ such that $V = \pi(U)$ is open in $H$ and the restriction $\pi|_U$ is a homeomorphism, then the fixed point $\pi(g)$ of $\mathcal{Y}$ is stable (asymptotically stable). Moreover, if $\pi$ is a covering map the converse also holds.

**Proof.** Assume that $g$ is stable for $\mathcal{X}$ and let $U'$ be a $\pi(g)$-neighborhood. There exists a $g$-neighborhood $U$ such that $\pi$ restricted to $U$ is a homeomorphism and $\pi(U) \subset U'$. By the stability, there exists a $g$-neighborhood $V$ such that $\varphi_t(V) \subset U$ for all $t \geq 0$. Consequently $V' = \pi(V)$ is a $\pi(g)$-neighborhood and it holds that

\[ \varphi_t(V') = \varphi_t(\pi(V)) = \pi(\varphi_t(V)) \subset \pi(U) \subset U', \quad \text{for all} \quad t \geq 0 \]

showing that $\pi(g)$ is stable for $\mathcal{Y}$.

If $g$ is asymptotically stable, there is a $g$-neighborhood $W$ such that $\lim_{t \to \infty} \varphi_t(x) = g$ for any $x \in W$. We can assume that $W$ is small enough such that $\pi$ restricted to $W$ is a homeomorphism. Then $W' = \pi(W)$ is a $\pi(g)$-neighborhood and

\[ \lim_{t \to \infty} \varphi_t(x) = \lim_{t \to \infty} \pi(\varphi_t(x)) = \pi(\lim_{t \to \infty} \varphi_t(x)) = \pi(g) \]

showing that $\pi(g)$ is asymptotically stable for $\mathcal{Y}$.

Now suppose that $\pi$ is a covering map and that $\pi(g)$ is stable for $\mathcal{Y}$. Since $\pi$ is a covering map, there is a distinguished $\pi(g)$-neighborhood $U'$, that is, $\pi^{-1}(U') = \bigcup_{\alpha} U_\alpha$ is a disjoint union in $G$ such that $\pi$ restricted to each $U_\alpha$ is a homeomorphism onto $U'$. Let $U$ be a given $g$-neighborhood and assume that $U$ is the component of $\pi^{-1}(U')$ that contains $g$. By stability, there exists a $\pi(g)$-neighborhood $V'$ such that $\varphi_t(V') \subset U'$ for all $t \geq 0$. Let $V \subset U$ be a $g$-neighborhood such that $\pi(V) \subset V'$. For $x \in V$ it holds that

\[ \pi(\varphi_t(x)) = \varphi_t(\pi(x)) \in \varphi_t(V') \subset U', \quad \text{for all} \quad t \geq 0 \]

and consequently $\varphi_t(x) \in \pi^{-1}(U')$ for all $t \geq 0$. Since $\pi^{-1}(U')$ is a disjoint union and $x \in V \subset U$ we have that $\varphi_t(x) \in U$ for all $t \geq 0$. As $x \in V$ is arbitrary, it follows that $\varphi_t(V) \subset U$ for all $t \geq 0$, showing that $g$ is stable for the linear vector field $\mathcal{X}$.

The asymptotically stability follows, as above, from the fact that $\pi$ has a continuous local inverse. \qed
The following theorem characterizes asymptotic and exponential stability at the identity $e \in G$ for a linear vector field in terms of the eigenvalues of $\mathcal{D}$.

**Theorem 4.4.** For a linear vector field $\mathcal{X}$ the following statements are equivalents:

(i) The identity $e \in G$ is asymptotically stable;
(ii) The identity $e \in G$ is exponentially stable;
(iii) All Lyapunov exponents of $\varphi_t$ are negative;
(iv) The stable subgroup $G^-$ satisfies $G = G^-$.

**Proof.** Since $G = G^-$ if and only if $\mathfrak{g} = \mathfrak{g}^-$ we have that (iii) and (iv) are equivalent, by Theorem 4.1. Moreover, by equation (3) we have that (iii) and (iv) implies (ii) and (ii) certainly implies (i). We just need to show that (i) implies (iv).

First we prove that if $e \in G$ is asymptotically stable, then $G$ is nilpotent. In fact, let $U$ be a neighborhood of $0 \in \mathfrak{g}$ such that $\exp$ restricted to $U$ is a diffeomorphism and such that $\exp(U) \subset W$. For any $X \in \ker \mathcal{D}$ let $\delta > 0$ small enough such that $g = \exp(\delta X) \in W$. Since $\varphi_t(g) = g$ for any $t \in \mathbb{R}$ the asymptotic assumption implies that $g = e$, consequently $X = 0$ and then $\ker \mathcal{D} = \{0\}$. The derivation $\mathcal{D}$ is then invertible which implies that $\mathfrak{g}$ is a nilpotent Lie algebra and so $G$ is a nilpotent Lie group.

Now we show that if $e \in G$ is asymptotically stable, then $G = G^-$. To do this, note that the derivation $\mathcal{D}$ on the Lie algebra $\mathfrak{g}$ can be identified with the linear vector field on $\mathfrak{g}$ given by $X \mapsto \mathcal{D}(X)$. Its associated linear flow is given by $e^{t\mathcal{D}}$. As $G$ is a nilpotent Lie group we have that $\exp : \mathfrak{g} \to G$ is a covering map. Moreover, since $\varphi_t \circ \exp = \exp \circ e^{t\mathcal{D}}$ it follows that $e \in G$ is asymptotically stable if and only if $0 \in \mathfrak{g}$ is asymptotically stable for the linear vector field induced by $\mathcal{D}$.

To finish the proof, note that $0 \in \mathfrak{g}$ is asymptotically stable if and only if $\mathcal{D}$ has only eigenvalues with negative real part (2, Theorem 1.4.8), that is, $\mathfrak{g} = \mathfrak{g}^-$ implying that $G = G^-$ and concluding the proof.

**Remark 4.** Note that the above result shows that local stability is equal to global stability. Moreover, we can see in the proof that if $e \in G$ is asymptotically stable for a linear vector field $\mathcal{X}$ then $G$ is a simply connected nilpotent Lie group.

The next result concerns the stability of a linear vector field.

**Theorem 4.5.** The identity $e \in G$ is stable for the linear vector field $\mathcal{X}$ if $G = G^{-0}$ and $\mathcal{D}$ restricted to $\mathfrak{g}^0$ is semisimple.

**Proof.** First note that $G = G^{-0}$ if and only if $\mathfrak{g} = \mathfrak{g}^{-0}$. By Theorem 4.1.10 in [2], the conditions that $\mathfrak{g} = \mathfrak{g}^{-0}$ and that $\mathcal{D}|\mathfrak{g}^0$ is semisimple is equivalent to $0 \in \mathfrak{g}$ be stable for the linear vector field induced by $\mathcal{D}$. Since $\exp$ is a local diffeomorphism around $0 \in \mathfrak{g}$ we have, by Lemma 4.3, that the stability of $0 \in \mathfrak{g}$ for $\mathcal{D}$ implies that $e \in G$ is stable for $\mathcal{X}$.

Now we give a partial converse of the above theorem.

**Theorem 4.6.** If $e \in G$ is stable for the linear vector field $\mathcal{X}$ then $G = G^{-0}$. Moreover, if $\exp_{G^0} : \mathfrak{g}^0 \to G^0$ is a covering map then $e \in G$ stable for $\mathcal{X}$ implies also that $\mathcal{D}|\mathfrak{g}^0$ is semisimple.

**Proof.** By equation (3), the only element in $G^+$ that has bounded positive $\mathcal{X}$-orbit is the identity. Therefore, if $e \in G$ is stable then $G^+ = \{e\}$ and consequently $G = G^{-0}$.
Since $G^0$ is $\varphi$-invariant, the linear flow $X$ induces a linear vector field $X_{G^0}$ on $G^0$ such that the associated linear flow is the restriction $(\varphi_t)|_{G^0}$. Moreover, as $G^0$ is a closed subgroup, it follows that the stability of $e \in G$ for $X$ implies that $e \in G^0$ is stable for the restriction $X_{G^0}$.

If we assume that $\exp_{G^0}$ is a covering map, we have by Lemma 4.3 that $e \in G^0$ is stable for $X_{G^0}$ if and only if $0 \in g^0$ is stable for $D|_{g^0}$. Then by Theorem 1.4.10 of [2] we have that $D|_{g^0}$ is semisimple.

When $G$ is a nilpotent Lie group the subgroup $G^0$ is also nilpotent and so the map $\exp_{G^0} : g^0 \to G^0$ is a covering map. We have then the following.

Corollary 2. If $G$ is a nilpotent Lie group then $e \in G$ is stable if and only if $G = G^{-0}$ and $D|_{g^0}$ is semisimple.

Remark 5. Another example where we have that the stability of the linear vector field $X$ on the neutral element implies that $D|_{g^0}$ is semisimple is when $G$ is a solvable Lie group and $\exp : g \to \tilde{G}$ is a diffeomorphism, where $\tilde{G}$ is the simply connected covering of $G$.

Remark 6. We conclude by observing that our work can be an initial step for the topological conjugacy study of control systems of type

$$\dot{g}(t) = X(g(t)) + \sum_{i=1}^{n} u_i(t)X_i(g(t)),$$

where $X$ is an infinitesimal automorphism of the connected Lie group $G$, $X_i$ are right invariant vector fields on $G$ and $u(t) = (u_1(t), \ldots, u_n(t)) \in \mathbb{R}^n$ belongs to the class of unrestricted admissible control functions.

REFERENCES

[1] V. Ayala, F. Colonius and W. Kliemann, On topological equivalence of linear flows with applications to bilinear control systems, J. Dyn. Control Syst., 13 (2007), 337–362.
[2] F. Colonius and W. Kliemann, Dynamical Systems and Linear Algebra, American Mathematical Society, 2014.
[3] F. Colonius and A. J. Santana, Topological conjugacy for affine-linear flows and control systems, Commun. Pure Appl. Anal., 10 (2011), 847–857.
[4] A. Da Silva, Controllability of linear systems on solvable Lie groups SIAM J. Control Optim., 54 (2016), 372–390.
[5] C. Kawan, O. G. Rocio and A. J. Santana, On topological conjugacy of left invariant flows on semisimple and affine Lie groups, Proyecciones, 30 (2011), 175–188.
[6] N. H. Kuiper and J. W. Robbin, Topological classification of linear endomorphisms Invent. Math., 19 (1973), 83–106.
[7] H. Poincaré, Sur Les Courbes Définies Par Les Equations Différentielles, In Oeuvres de H. Poincaré I, Gauthier-Villars, Paris, 1928.
[8] J. W. Robbin, Topological conjugacy and structural stability for discrete dynamical systems Bull. Amer. Math. Soc., 78 (1972), 923–952.
[9] C. Robinson, Dynamical Systems. Stability, Symbolic Dynamics, and Chaos, 2nd Edition, CRC Press, London, 1999.

Received November 2016; revised January 2017.

E-mail address: ajsilva@ime.unicamp.br
E-mail address: ajsantana@uem.br
E-mail address: simnaos@gmail.com