On the Lower Bound for the Higgs Boson Mass

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Abstract

We provide an alternative derivation of a lower bound on the mass of the Higgs boson which is somewhat simpler and more direct than the derivation based on the effective potential. For one TeV cutoff, the result is the same. For high scale cutoff, the lower bound is increased by slightly more than the expected uncertainty in the calculation.

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For the history of the subject we refer to reviews [1] and quote only two recent papers containing the latest refinements of the conventional approach [2]. Once the experimental lower bound on the top-quark mass exceeded 80 GeV, attention shifted from the original Linde-Weinberg bound, based on the properties of the one-loop effective potential for small \( \phi \), close to the minimum, to the large \( \phi \) behavior of the effective potential, as determined by renormalization group (RG) considerations.

\[
V_{\text{eff}}(\phi) = \frac{1}{4}\bar{\lambda}(t)(\xi(t)\phi)^4
\]

Here, \( \xi(t) \) is the anomalous dimension factor, and \( \lambda(t) \) is the \( \overline{MS} \) running coupling constant.

\[
t = \ln \frac{\phi}{M_0}, \quad \frac{d\bar{\lambda}(t)}{dt} = \beta_\lambda(\bar{\lambda}(t), \bar{g}(t))
\]

It is then argued that vacuum stability requires \( \bar{\lambda}(t) > 0 \) up to some high scale, \( \phi \sim M_{\text{GUT}}, \text{or } M_{\text{Pl}}, (M_0 \sim M_Z, \text{or } m_t, \text{ or } 246 \text{GeV}) \). To implement this condition, one has to know (or approximate) the \( \beta \)-function, integrate the RG differential equations starting from some initial values, \( \bar{\lambda}(0), \bar{g}^2(0) \), and relate the smallest acceptable \( \bar{\lambda}(0) \) to a physical Higgs mass.

We do appreciate this calculation, but questions may be raised about the perturbative nature, the scale ambiguity, and the conceptual basis. The exact \( \beta \)-function is not known, so one integrates the one-loop \( \beta \)-functions to get the "RG improved one-loop effective potential ". Since the coupling is not asymptotically free, the large \( t \) behavior of \( \bar{\lambda}(t) \) is not known. The best that can be (and is) achieved is perturbative self-consistency. For the indicated \( \bar{\lambda}(0) \), integrating the one-loop \( \beta \)-function gives \( \bar{\lambda}(t) \) which remains perturbative up to the high scales considered. The minimum \( \bar{\lambda}(0) \) depends on the minimum scale \( M_0 \) above. The physical Higgs mass does not. So there is some scale ambiguity. On the third point, we largely repeat the remarks of [3]. If one thinks of a nonperturbative formulation of the vacuum stability problem, in particular, a lattice formulation; the large \( \phi \) behavior of the effective potential is not the point. On the lattice, the exact effective potential is well defined and convex. The condition for vacuum stability is simply that the bare quartic coupling constant must be positive (\( \lambda_{\text{bare}} > 0 \)). Then the lower bound on
the Higgs mass is just the smallest output Higgs mass from the Monte Carlo simulation as one runs through the space of bare parameters in the broken symmetry phase.

We present a new derivation of the Higgs mass lower bound. It is also of the perturbative RG variety, and so also subject to the first concern above; but we have organized the calculation in a way which minimizes the scale ambiguity and makes no explicit reference to the effective potential. The essential input is that one is perturbing about the correct vacuum. A necessary condition for this is that the vev of the (shifted) field be zero, order by order in perturbation theory, and the renormalized mass squared of the shifted field be positive.

We start by computing the relation between the perturbative pole mass and the $\overline{\text{MS}}$ mass, for both the Higgs boson and the $t$-quark. The relation follows from the perturbative definition of the pole mass,

$$0 = \mathcal{D}^{-1}(M^*) = M^* - M^2 - Re \Sigma(M^2)$$

In this equation, $\mathcal{D}(q^2)$, and $\Sigma(q^2)$ are the two-point Green function and self-energy function, renormalized according to the $\overline{\text{MS}}$ prescription. $M^*$ is the perturbative pole mass. The result is

$$M^2 = \overline{M}^2 \{1 + \lambda (3I_{00}(M^2) + 9\mathcal{T}_{MM}(M^2)) + N_c \bar{y}^2 \left( \frac{M^2}{\overline{M}^2} - 4 \frac{m^2}{\overline{M}^2} \right) \mathcal{T}_{mm}(M^2) + 2(\zeta_v - 1) \}$$

$m$ is the $t$-quark mass, and $y$ is the $t$-quark Yukawa coupling ($m = \frac{y v}{\sqrt{2}}$). The contributions from the electroweak gauge sector, proportional to $g_2, g_1$, have also been calculated, but are not written out here. They will be included below. The term $\zeta_v - 1$ comes from a finite shift of the vev required in the $\overline{\text{MS}}$ scheme to enforce $\langle \hat{H} \rangle = 0$ through one-loop order. It will cancel out of the ratio computed below, so we do not have to give its value here. $\mathcal{I}_{ab}$ is the dimensionally regularized $\overline{\text{MS}}$ scalar one-loop two-point integral.

$$\mathcal{I}_{ab}(q^2) = \left[ d^{4-d} \int \frac{d^d l}{(2\pi)^d (l^2 - a^2)(l^2 - b^2)} \right]_{\overline{\text{MS}}}$$

Then (2) is

$$M^2 = \overline{M}^2 \{1 + \frac{\lambda}{16\pi^2} \left[ 12 \ln \frac{M^2}{\mu^2} - 24 + 3\sqrt{3} \pi \right] + N_c \bar{y}^2 \left( \frac{4}{\nu^2} \right) \left[ \ln \frac{m^2}{\mu^2} + f(r) \right] + 2(\zeta_v - 1) \}$$
where
\[
r = \frac{M}{m}, \quad f(r) = -2 + 2\sqrt{\frac{4 - r^2}{r^2}} \arctan\sqrt{\frac{r^2}{4 - r^2}}
\]
The corresponding calculation for the t-quark gives [4]
\[
m^* \equiv m = M^2 m^2 \left\{ 1 + \frac{\beta_0^2}{16\pi^2} \left[ 12 \ln \frac{m^2}{\mu^2} - 24 + 3\sqrt{3}\pi \right] + \frac{\beta_2^2}{16\pi^2} N_c (1 - \frac{4\pi}{r^2} + \ln \frac{m^2}{\mu^2} + f(r)) - \frac{3}{2} \ln \frac{m^2}{\mu^2} - \Delta(r) \right\} + \frac{\beta_s^2}{16\pi^2} C_F (6 \ln \frac{m^2}{\mu^2} - 8) + \frac{\beta_1^2}{2g^2} \text{ terms } + 2 - \text{loop}
\]
where
\[
\Delta(r) = -4 + \frac{r^2}{2} + \left( \frac{3}{2} r^2 - \frac{1}{4} r^4 \right) \ln r^2 + \frac{r}{2} (4 - r^2) \frac{\pi}{2} \arctan\sqrt{\frac{4 - r^2}{r^2}}
\]
We take the ratio of (2) to (5) and expand to one-loop order.
\[
\frac{M^*}{m^*} = \frac{M^2}{m^2} \left\{ 1 + \frac{\beta_0^2}{16\pi^2} \left[ 12 \ln \frac{M^2}{\mu^2} - 24 + 3\sqrt{3}\pi \right] + \frac{\beta_2^2}{16\pi^2} N_c (1 - \frac{4\pi}{r^2} + \ln \frac{m^2}{\mu^2} + f(r)) - \frac{3}{2} \ln \frac{m^2}{\mu^2} - \Delta(r) \right\}
\]

The \( \zeta_v - 1 \) terms, which also contain explicit dependence on \( \ln \mu^2 \), have cancelled out. A necessary condition for the \( \overline{MS} \) perturbation calculations to be defined in the broken symmetry phase is that \( \overline{M}^2, \overline{m}^2 \) be positive. Since the ratio of pole masses is positive, (5) satisfies the requirement perturbatively, for \( \mu \) around the weak scale. For large \( \mu^2 \), one has to provide a RG treatment of the large logarithms, just as in the conventional calculation involving the effective potential . In the broken symmetry phase, one can define the renormalized coupling constants such that the relation
\[
\frac{M^2}{m^2} = 4 \frac{\lambda}{y^2}
\]
is exact when all the quantities are either "star" (on-shell renormalization scheme) or "bar" (\( \overline{MS} \) renormalization scheme)[5]. Thus, not all quantities in (5) can be varied independently as functions of \( \mu \). We focus particularly on the \( \frac{4\pi}{r^2} \) multiplying the \( \ln \frac{m^2}{\mu^2} \).

Tracing its origin to a ratio of \( \overline{MS} \) masses in (5), we use (5) to replace the ratio of \( \overline{MS} \) masses by a ratio of \( \overline{MS} \) coupling constants. To leading (one-loop) order, the scale dependence of the ratio of \( \overline{MS} \) masses is determined by the coefficients of the explicit \( \ln \mu^2 \) terms in (5). For the other masses in (5), the difference between "star" and "bar" is higher order (combined with explicitly two-loop effects), as is the implicit \( \mu \) dependence of
the "bar" coupling constants. After these observations, and reinstating the $g_2^2, g_1^2$ terms, differentiating (8), we obtain

$$
\mu \frac{d}{d\mu}(\frac{\bar{y}}{\bar{y}}) = \frac{\bar{y}}{16\pi^2}[24(\frac{\bar{y}}{\bar{y}})^2 + (2N_c - 3 + 12C_F \frac{\bar{T}_2}{\bar{y}}) \frac{\bar{y}}{\bar{y}} - 2N_c
$$

$$
-(\frac{2g_2^2}{\bar{y}} + \frac{1}{6} \frac{g_1^2}{\bar{y}}) \frac{\bar{y}}{\bar{y}} + \frac{3}{8} (\frac{\bar{T}_2 + g_2^2}{\bar{y}})^2] + 2-loop
$$

We now give a sequence of estimates, of increasing refinement, of the ratio $\frac{M_h^2}{\bar{m}_t^2}$. Let $\frac{\bar{y}}{\bar{y}} = \rho$. (By (7), $\rho = \frac{\bar{r}^2}{\bar{r}}$). Let the right hand side of (8) be denoted $\beta_\rho$. Because of the $-2N_c$ term in (8), there is a critical value of $\rho$ below which $\beta_\rho$ becomes negative. And if the starting value of $\rho$ is below this value, as $\rho$ decreases the derivative becomes more negative, driving $\rho$ negative, unless some higher order effect intervenes. We will return to this possibility, but as our zeroth order estimate we take the critical value of $\rho$ for which one-loop $\beta_\rho$, evaluated with weak scale coupling constants, is zero. We take $g_s^2 = 1.366$ ($\alpha_s(174) = .109$) and $g^2 = 0.9990$ ($m_t = 174, v = 246.2$) and neglect the $g_2^2, g_1^2$ contributions. Then $\rho_c = 0.2019$, which gives $(\frac{\bar{M}_c^2}{\bar{m}_c^2})^2 = 0.8076$, or $\bar{M}_c = 156$, for $\bar{m} = 174$. If we include the contribution from the electroweak gauge couplings, $g_2^2, g_1^2$, the corresponding results are $\rho_c = 0.2068$, and $\bar{M}_c = 158$, a one percent shift.

The first refinement is to convert back from $(\frac{\bar{M}_c^2}{\bar{m}_c^2})_c$ to the ratio of squared perturbative pole masses by (6). Note that precisely for $\rho = \rho_c$, all of the $\ln \mu^2$ terms in (6) cancel, so there is no explicit dependence on $\mu$ in this correction. The result is

$$
(\frac{M_h^2}{m_t^2})_c = (\frac{\bar{M}_c^2}{\bar{m}_c^2})_c (1 - 0.101)
$$

which gives $M_h^2 \geq 148$.

We now turn to the effect of the running of the $\bar{MS}$ coupling constants, which appear as coefficients in (8), and the dependence of the lower bound on the cutoff (maximum value of $\frac{\mu}{\mu_0}$). One has to integrate coupled RG equations for five independent "coupling constants", $\bar{y}_s$, $\bar{g}_2^2, \bar{g}_1^2, \bar{y}^2$. Let $t = \ln \frac{\mu}{\mu_0}$. 

5
\[ \frac{d}{dt} \bar{g}_s^2 = -\frac{1}{16\pi^2} (22 - \frac{4}{3} N_g) \bar{g}_s^4 \]
\[ \frac{d}{dt} \bar{g}_2^2 = -\frac{1}{16\pi^2} [\frac{44}{9} N_g - \frac{1}{3} N_d] \bar{g}_2^4 \]
\[ \frac{d}{dt} \bar{g}_1^2 = \frac{1}{16\pi^2} [\frac{40}{9} N_g + \frac{1}{3} N_d] \bar{g}_1^4 \]
\[ \frac{d}{dt} \bar{y}^2 = \frac{1}{16\pi^2} [(3 + 2N_c) \bar{y}^4 - 12C_F \bar{g}_s^2 \bar{y}^2 - \frac{9}{2} \bar{g}_2^2 \bar{y}^2 - \frac{17}{6} \bar{g}_1^2 \bar{y}^2] \]
\[ \frac{d}{dt} (\rho) = \frac{\bar{y}^4}{16\pi^2} [24\rho^2 + (2N_c - 3 + 12C_F \bar{g}_2^4) \rho - 2N_c - \frac{9}{2} \bar{g}_2^2 \rho + \frac{1}{6} \bar{g}_1^2 \rho + \frac{3}{8} (\bar{g}_2^2 + \bar{g}_1^2)^2] \]

The first three equations are integrated trivially. If we neglect the \( \bar{g}_2, \bar{g}_1 \) contributions to the \( \bar{y} \) running, that equation can also be integrated analytically. But if one runs up to high scales, the electroweak gauge couplings become of same order as the QCD coupling constant; so we use NDSolve from Mathematica to provide an interpolating function solution for \( \bar{y}^2 \) which is substituted into the \( \bar{g}_2^2 \) equation, which is again integrated numerically by NDSolve.

If we run up to the Planck scale \( (m_{pl} \approx 10^{19} \text{ GeV}, t \approx 41) \) the smallest starting \( \rho(0) \) which does not lead to \( \rho(t) \) falling through zero before \( t = 41 \) is 0.2022, which is only slightly different from the critical value required to make the derivative zero at the weak scale. At the other extreme, if we only require the equations of the standard model to be consistent up to order of one TeV, the value of \( t \) which corresponds to one TeV depends on the choice of \( \mu_0 \). We choose \( \mu_0 = m_t \) as the most natural choice for relating \( m \) to \( m^* \). For \( m = 174 \), this corresponds to \( t_{\text{max}} = 1.75 \). Then the smallest starting \( \rho(0) \) which does not lead to \( \rho(t) \) falling through zero before \( t = 1.75 \) is 0.0520. Making the connection back to the ratio of pole masses by (6), we obtain the final result of this approach

\[ m_t^* = 174, \quad \frac{\pi_s(m_t)}{m_t} = 0.109 \]
\[ \mu_{\text{max}} \approx m_{\text{pl}} \quad M_h \geq 148 (141, 135) \]
\[ \mu_{\text{max}} \approx 1 TeV \quad M_h \geq 72 (72) \]

The numbers in parentheses are the corresponding results in the conventional approach. The lower bound obtained in the present approach is slightly higher in the high scale cutoff case, but not by much more than the difference between the results of two different
calculations in the conventional approach. The present derivation has the advantage that the zeroth order approximation, the value of $\frac{N}{m}$ obtained for the vanishing of the one-loop $\beta_\rho$ at the weak scale, differs by less than ten percent from the final value for the large cutoff limit. If one makes the corresponding zeroth order determination of the minimum $M$ in the conventional approach by setting $\beta_\lambda$ to zero at the weak scale, the result differs from the final large cutoff result by more than thirty percent. The significant difference is that $\beta_\rho$ contains the large QCD correction to $m$, while $\beta_\lambda$ does not. It is added in later as a correction when one integrates the coupled RG equations.

It is clearly desirable to have a large scale lattice simulation study of the combined Higgs-heavy quark-QCD sector. (Contributions from light quarks and electroweak gauge bosons are small, particularly if one doesn’t run up to some very high scale). We note that a quenched approximation simulation is not adequate for this problem. The term in (8) which triggers the possible instability is the $-2N_c$, clearly a contribution from an internal fermion closed loop.

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