Lyapunov functions and asymptotic analysis of a complex analogue of the second Painlevé equation

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Abstract. We consider the ordinary differential equation on the real axis that is a complex analogue of the second Painlevé equation. The solutions with a growing amplitude at positive infinity and the solutions that tend to zero at negative infinity are investigated. By applying Lyapunov function method we analyze the stability of such solutions and construct the long-term asymptotics for general solutions.

1. Introduction
Consider the ordinary differential equation:

\[ \frac{d^2 w}{dz^2} + (2|w|^2 - z)w = 0, \quad z \in \mathbb{R}, \quad w \in \mathbb{C}, \]

(1)

that is a complex analogue of the second Painlevé equation. Such an equation appears in the description of the transition layer in the problems of self-steepening [1, 2] and self-focusing [3]. In this paper, we study the solutions with unbounded growing amplitude as \( z \to +\infty \) and the solutions that tend to zero as \( z \to -\infty \). It follows from [4] that the formula

\[ f = (\log w)' \]

connects such functions with complex singular solutions of the second Painlevé equation

\[ f'' = 2f^3 - 2zf + \kappa, \]

where \( \kappa = 2(zw^\ast - w^\ast z) + 1 \). Our goal is the construction of asymptotics for general solutions to equation (1) as \( z \to \pm \infty \).

Note that the substitution \( w(z) = R(z) \exp(i\Phi(z)) \) in equation (1) leads to the following system:

\[ R'' - R(\Phi_z)^2 + (2R^2 - z)R = 0, \quad R\Phi'' + 2R'\Phi' = 0. \]

It follows from the second equation that

\[ \Phi(z) = c_2 + c_1 \int_{z_0}^{z} (R(\zeta))^{-2} d\zeta, \quad z_0 = \text{const}, \]

(2)

with arbitrary parameters \( c_1, c_2 \). The substitution of the expression for \( \Phi(z) \) in the first equation leads to the ordinary differential equation for \( R(z) \):

\[ R'' + (2R^2 - z)R - c_1^2 R^{-3} = 0. \]

(3)

Asymptotic solutions of (3) can be constructed in the form:

\[ R_+ = \sqrt{\frac{z}{2}} + \sum_{k=2}^{\infty} a_k z^{-3k/2+1/2}, \quad z \to +\infty; \quad R_- = \sum_{k=0}^{\infty} b_k (-z)^{-3k/2-1/4}, \quad z \to -\infty. \]

(4)
Substituting these series in (3) and grouping the expressions of the same powers of $z$ and $-z$ give the recurrence relations for determining the constant coefficients $a_k$ and $b_k$. In particular, 
\[ a_2 = (1+16c_1^2)/(8\sqrt{2}), \quad b_0 = |c_1|^{1/2}, \quad b_1 = -|c_1|^{3/2}/2. \] 
The existence of solutions $R_+(z)$ as $z > z_*$, $z_*$ = const $> 0$ and $R_-(z)$ as $z < -z_*$ with such asymptotics follows from [5].

In the first step, we investigate the stability of the particular solutions $R_+(z)$ and $R_-(z)$ as $z \to +\infty$ and $z \to -\infty$, respectively. The stability will ensure the existence of a family of solutions to equation (3) with a similar long-term behaviour. In the last step, asymptotics for such solutions will be constructed by averaging method [6, 7] with some modifications.

2. Stability of solutions $R_-(z)$ and $R_+(z)$

**Theorem 1** The solution $R_+(z)$ to equation (3) is asymptotically stable as $z \to +\infty$. Moreover, for all $\epsilon \in (0, 1)$ there exist $\delta_0 > 0$ and $z_0 > 0$ such that for all $(R_0, R_1): (R_+(z_0) - R_0)^2 + (R'_+(z_0) - R_1)^2 < \delta_0^2$ the solution to equation (3) with initial data $R(z_0) = R_0$, $R'(z_0) = R_1$ has the asymptotics:

\[ R(z) = \sqrt{\frac{z}{2}} + O\left(z^{\frac{3(1-\epsilon)}{4(1+\epsilon)}}\right), \quad R'(z) = \frac{1}{2}\sqrt{\frac{1}{2z}} + O\left(z^{\frac{3(1-\epsilon)}{4(1+\epsilon)}}\right), \quad z \to +\infty. \] (5)

**Proof.** In equation (3) we make the change of variables:

\[ R(z) = R_+(z) + \frac{z^{1/2}r(t)}{\sqrt{2}}, \quad t = \frac{2z^{3/2}}{3}, \] (6)

and for new functions $r(t)$, $v(t) = r'(t)/\sqrt{2}$ we study the stability of the fixed point $(0, 0)$ in the following system:

\[ \frac{dr}{dt} = \partial_r H(r, v, t), \quad \frac{dv}{dt} = -\partial_v H(r, v, t) - t^{-1}v, \] (7)

where

\[ H = \frac{v^2}{\sqrt{2}} + \frac{r^2(3p_+^2 - 1)}{2\sqrt{2}} + \frac{r^3p_+}{\sqrt{2}} + \frac{r^4}{4\sqrt{2}} + t^{-2}r^2 \left(\frac{4\sqrt{2}v^2(2r + 3p_+)}{9p_+^2(r + p_+)^2} - \frac{1}{18\sqrt{2}}\right), \]

$p_+(t) = \sqrt{2}R_+(z)z^{-1/2} = 1 + O(t^{-2})$ as $t \to \infty$. It is easy to calculate that $H(r(v, v, t) = (r^2 + v^2 + r^3 + r^4)/\sqrt{2} + O(d^2)O(t^{-2})$, uniformly as $t \to \infty$ and $d = \sqrt{r^2 + v^2} \to 0$. We see that the fixed point $(0, 0)$ is a center in the asymptotic limit ($t \to \infty$), that, however, does not guaranty the stability in the complete system [8]. A Lyapunov function candidate is constructed of the form [9, 10]:

\[ V(r, v, t) = \sqrt{2}\left[H(r, v, t) + t^{-1}rv\right]. \] (8)

The function $V$ is positive definite in the vicinity of the fixed point: $V = d^2 + O(d^3) + O(d^2)O(t^{-2})$ as $t \to \infty$ and $d \to 0$. The total derivative of this function with respect to $t$ along the trajectories of system (7) has the following asymptotics:

\[ \frac{dV}{dt} \bigg|_{(7)} \overset{def}{=} \partial_r V + \partial_v V \partial_r H + \partial_v V [-\partial_v H - t^{-1}v] = -t^{-1}[d^2 + O(d^3)][1 + O(t^{-1})]. \]

Therefore for all $\epsilon > 0$ there exist $d_0 > 0$ and $t_0 > 0$ such that the function $V$ satisfies the inequalities: $(1 - \epsilon)d^2 \leq V(r, v, t) \leq (1 + \epsilon)d^2$, $dV/dt\bigg|_{(7)} \leq -t^{-1}\sigma V$, $\sigma = (1 - \epsilon)/(1 + \epsilon)$,
as $d \leq d_0$ and $t \geq t_0$. Moreover, for all $\varepsilon \in (0,d_0)$ there exists $d_\varepsilon = \varepsilon \sqrt{\sigma/2}$ such that
\[ \sup_{d \leq d_\varepsilon} V(r,v,t) \leq (1 + \varepsilon)d_\varepsilon^2 \leq (1 - \varepsilon)d_\varepsilon^2 \leq \sup_{d \geq d_\varepsilon} V(r,v,t) \text{ as } t > t_0. \]
The last estimates and the negativity of the total derivative of the function $V$ ensure that any solution of system (7) with initial data from the ball $d \leq d_\varepsilon$ cannot leave $\varepsilon$-neighbourhood of the equilibrium as $t > t_0$. Consider the solution $r(t), v(t)$ to system (7) with initial data $r(0) + v(t_0) \leq d_0^2$, then the function $z(t) = V(r(t), v(t), t)$ satisfies the inequality $dz/dt \leq -\sigma t^{-\sigma} z$ as $t > t_0$. Integrating the last expression with respect to $t$, we obtain $0 \leq z(t) \leq \ell_0 \tau^{-\sigma}$ with the parameter $\ell_0$, depending on $d_0$ and $t_0$. This implies the asymptotic estimate: $r^2(t) + v^2(t) = O(t^{-\sigma})$ as $t \to \infty$. Returning to the original variables we derive asymptotic estimates (5) as $z \to +\infty$.

**Theorem 2** The solution $R_-(z)$ to equation (3) is stable as $z \to -\infty$. Moreover, there exist $\delta_0 > 0$ and $z_0 > 0$ such that for all $(R_0, R_1): (R_-(z_0) - R_0)^2 + (R_-(z_0) - R_1)^2 < \delta_0^2$ the solution to equation (3) with initial data $R_-(z_0) = R_0, R'_-(z_0) = R_1$ has the asymptotics:
\[ R(z) = O\left((-z)^{-1/4}\right), \quad R'(z) = O\left((-z)^{1/4}\right), \quad z \to -\infty. \]

**Proof.** The exchange of variables $z = -y, R(z) = R_-(z) + y^{-1/4}r(t)/2, t = 2y^{3/2}/3$ leads to the non-autonomous Hamiltonian system:
\[ \frac{dr}{dt} = \partial_r H(r,v,t), \quad \frac{dv}{dt} = -\partial_v H(r,v,t), \]
where
\[ H(r,v,t) = v^2 + \frac{r^2}{4} + \frac{c^2r^2(2q_\ast + r)}{q_\ast^2(2q_\ast + r)^2} + t^{-1}\left(q_\ast^2q_\ast^2 + \frac{r^4}{3} + \frac{r^4}{24}\right) + t^{-2}\frac{5r^2}{144}. \]
$q_\ast(t) = R_\ast y^{1/4} = b_0 + 2b_1t^{-1/3} + O(t^{-2})$ as $t \to \infty$. For the new variables $r(t)$ and $v(t) = r'(t)/2$ we study the stability of the fixed point $(0,0)$ in the Hamiltonian system (10). Note that the function $H(r,v,t)$ has the following asymptotics as $t \to \infty$ and $d \to 0$:
\[ H(r,v,t) = r^2 + v^2 + O(d^3) + t^{-1}\left[2|c_1|r^2 + O(d^3)\right] + O(t^{-2})O(d^2). \]
As above, the fixed point $(0,0)$ is a center in the asymptotic limit. A Lyapunov function candidate for system (10) is constructed in the form: $V(r,v,t) = H(r,v,t) - t^{-2}|c_1|r/2$. The function $V(r,v,t)$ is positive in a punctured neighborhood of the fixed point: $V = d^2[1 + O(d) + O(t^{-1})]$ as $t \to \infty$ and $d \to 0$. The total derivative of $V(r,v,t)$ with respect to $t$ along the trajectories of system (10) has the following asymptotics:
\[ dV/dt|_{t=0} = -t^{-2}|c_1|r^2 + v^2 + O(d^3) + O(t^{-2})O(t^{-3}). \]
Moreover, for all $\delta > 0$ there exist $d_0 > 0$ and $t_0$ such that $(1 - \delta)d^2 \leq V(r,v,t) \leq (1 + \delta)d^2$. $dV/dt|_{t=0} \leq -|c_1|\sigma t^{-2}V$ as $d \leq d_0$ and $t \geq t_0$. Consider the solution to system (7) with initial data from the ball $d \leq d_0$, then the function $\ell(t) = V(r(t), v(t), t)$ satisfies the inequality: $dz/dt \leq -t^{-2}\sigma|c_1|\ell$. Hence $0 \leq \ell(t) \leq \ell_0$ as $t > t_0$ and $r^2(t) + v^2(t) = O(1)$ as $t \to \infty$. Returning to the original variables we derive asymptotic estimates (9) as $z \to -\infty$.

3. Asymptotics for general solutions
The stability of the particular solutions $R_+(z), R_-(z)$ ensures the existence of a family of solutions to (3) with a similar behaviour as $z \to +\infty$ and $z \to -\infty$. At this section we construct the asymptotics for such solutions by averaging method with a use of the Lyapunov functions found in the previous section.

**Theorem 3** For all $c_1 \in \mathbb{R}$ equation (3) has two-parameter family of solutions $R_+(z; c_3, c_4)$ with the asymptotics:
\[ R(z) = \sqrt{\frac{z}{2}} + z^{-1/4} \sum_{k=0}^{\infty} \rho_k(S(z); c_3)z^{-3k/4}, \quad R'(z) = z^{1/4} \sum_{k=0}^{\infty} \rho_k(S(z); c_3)z^{-3k/4} \]
as \( z \to +\infty \), where the functions \( \rho_k(S;c_3), \vartheta_k(S;c_3) \) are \( 2\pi \)-periodic in \( S \), \( S(z) = c_4 + 2\sqrt{2}z^{3/2}/3 + 3c_3^2(\log z)/(4\sqrt{2}) \), \( \rho_0 = c_3 \cos S \), \( \vartheta_0 = -c_3\sqrt{2}\sin S \).

**Proof.** In the first step, we make change of variables (6) in system (3) and for new functions \( r(t), v(t) = r'(t)/\sqrt{G} \) we consider the non-autonomous near-Hamiltonian system (7) in a neighbourhood of the stable fixed point \((0,0)\). Let us consider a Hamiltonian system: \( r' = \partial_s H_0, \ v' = -\partial_r H_0 \), where \( H_0(r,v) = \lim_{t \to \infty} H(r,v,t) \) and \( H_0 = (v^2 + r^2 + r^3 + r^4)/\sqrt{2} \). It is easy to see, that the level lines \( \{(r,v): H_0(r,v) = h\} \) define a family of closed curves on the phase space \((r,v)\) parameterized by the parameter \( h \in (0,h_0) \), \( h_0 = \text{const} > 0 \). To each closed curve there corresponds a \( T(h) \)-periodic solution \( r_0(t,h), v_0(t,h) \) of the Hamiltonian system such that \( T(h) = 2\pi/\omega(h) \), \( \omega(h) = \sqrt{2} + 3h/8 + O(h^2) \) as \( h \to 0 \). Consider the transformation of variables \( r = \tilde{r}_0(s,h), \ v = \tilde{v}_0(s,h) \), where \( \tilde{r}_0(s,h) = r_0(s/\omega(h), h), \ \tilde{v}_0(s,h) = v_0(s/\omega(h), h) \).

Then in new variables \((s,h)\) system (7) takes the form: \( dh/dt = -\omega(h)[\partial_s \tilde{H} + t^{-1}\partial_t \tilde{r}_0 \tilde{v}_0], \ ds/dt = \omega(h)[\partial_t \tilde{H} + t^{-1}\partial_t \tilde{r}_0 \tilde{v}_0], \) where \( \tilde{H}(s,h,t) \equiv H(\tilde{r}_0(s,h), \tilde{v}_0(s,h), t) \) is \( 2\pi \)-periodic functions with respect to \( s \). Moreover, the last system is asymptotically Hamiltonian with decaying non-Hamiltonian terms as \( t \to \infty \). Solutions of this systems can be investigated by the averaging method. To simplify asymptotic constructions, we make the additional transformation of variables. Introduce a new variable \( \ell(t) \) associated with the Lyapunov function (8) such that \( \ell(t) = \bar{V}(s(t),h(t),t) \), where \( \bar{V}(s,h,t) \equiv V(\tilde{r}_0(s,h), \tilde{v}_0(s,h), t) \) is \( 2\pi \)-periodic function in \( s \). It follows the properties of the Lyapunov function that \( \bar{V} = \sqrt{2}h[1 + O(t^{-1})], \ d\bar{V}/dt|_?(?) = -t^{-1}[\sqrt{2}h + O(h^{3/2})] + O(h^0O(t^{-2}) \) as \( t > t_0, h < h_0, \) uniformly with respect to \( s \in \mathbb{R} \). Hence the transformation \((s,h) \mapsto (s,\ell)\) is invertible: \( \partial_h \bar{V} \neq 0 \). System (7) in new variables takes the form:

\[
\frac{d\ell}{dt} = \bar{F}(s,\ell,t), \quad \frac{ds}{dt} = \bar{G}(s,\ell,t), \tag{12}
\]

where \( \bar{F}(s,\bar{V},t) = \partial_s \bar{V} \partial_s \bar{H} + \partial_s \bar{V} (\partial_s \bar{H} - t^{-1} \partial_t \bar{V}_0) + \partial_s \bar{V}, \ \bar{G}(s,\bar{V},t) = \omega(h)[\partial_s \bar{H} + t^{-1}\partial_t \bar{r}_0 \bar{v}_0]. \) It is not difficult to deduce the asymptotics of the functions \( \bar{F} \) and \( \bar{G} \) as \( t \to \infty \): \( \bar{F} = \sum_{k=1}^{\infty} f_k(s,\ell)t^{-k}, \ \bar{G} = g_0(\ell) + \sum_{k=1}^{\infty} g_k(s,\ell)t^{-k} \), where \( f_k(s,\ell) \) and \( g_k(s,\ell) \) are \( 2\pi \)-periodic functions with respect to \( s \), \( f_1(s,\ell) = t^{-1}(1+O(\ell^{1/2}), g_0(\ell) = \omega(\ell/\sqrt{2}) = \sqrt{2}+3\ell^2/(8\sqrt{2})+O(\ell^{3/2}), g_1(s,\ell) = -\sin \ell \cos s+O(\ell) \) as \( \ell \to 0, s \in [0,2\pi) \). To construct asymptotic solutions to system (12), it is convenient to single out a Hamiltonian part:

\[
\frac{d\ell}{dt} = -\partial_s \mathcal{H} + \bar{F}, \quad \frac{ds}{dt} = \partial_t \mathcal{H}, \tag{13}
\]

where \( \mathcal{H}(s,\ell,t) = \int_0^\ell G(s,l,t)dl, \ \bar{F}(s,\ell,t) = F(s,\ell,t) + \partial_s \mathcal{H}(s,\ell,t) \) are \( 2\pi \)-periodic with respect to \( s \). The asymptotic solution to the first equation in (13) is sought in the form: \( \ell(t) = \lambda(t) + \Lambda(s,\lambda(t),t) \), where \( \lambda(t) \) is the solution of the averaged equation:

\[
\frac{d\lambda}{dt} = \langle \bar{F}(s,\lambda + \Lambda(s,\lambda,t),t) \rangle_s, \tag{14}
\]

and \( \Lambda(s,\lambda,t) \) is \( 2\pi \)-periodic function in \( s \) with a zero average \( \langle \Lambda(s,\lambda,t) \rangle_s = 0 \), and satisfies the equation \( \partial_s \mathcal{H}(s,\lambda + \Lambda) + \partial_t \mathcal{H}(s,\lambda + \Lambda,t)\partial_s \Lambda + \partial_t \Lambda = \mathcal{F}(s,\lambda + \Lambda) - [1 + \partial_s \Lambda]\langle \bar{F}(s,\lambda + \Lambda,t) \rangle_s \). We integrate the last equation with respect to \( s \) and
choose the constant of integration in such a way that the result has a zero average:

\[
\mathcal{H}(s, \lambda + \Lambda, t) - \left\langle \mathcal{H}(s, \lambda + \Lambda, t) \right\rangle_s + \int_s^{s*} \partial_l \Lambda(\sigma, \lambda, t) \, d\sigma - \left\langle \int_s^{s*} \partial_l \Lambda(\sigma, \lambda, t) \, d\sigma \right\rangle_s = \\
\int_s^{s*} \mathcal{F}(\sigma, \lambda + \Lambda, t) - \left\langle \mathcal{F}(\sigma, \lambda + \Lambda, t) \right\rangle_s \, d\sigma - \left\langle \int_s^{s*} \mathcal{F}(\sigma, \lambda + \Lambda, t) \, d\sigma \right\rangle_s - \left\langle \mathcal{F}(s, \lambda + \Lambda, t) \right\rangle_s \left( \int_s^{s*} \partial_l \Lambda(\sigma, \lambda, t) \, d\sigma - \left\langle \int_s^{s*} \partial_l \Lambda(\sigma, \lambda, t) \, d\sigma \right\rangle_s \right),
\]

(15)

\(s_s = \text{const.}\) The asymptotic solution of (15) is sought in the form: \(\Lambda(s, \lambda, t) = \sum_{k=1}^{\infty} \lambda_k(s, \lambda) t^{-k}\).

Substituting this series into equation (15), and equating the terms of the same power of \(t\), we obtain the chain of equations: \(g_0(\lambda) \lambda_k = L_k(s, \lambda) - \langle L_k(s, \lambda) \rangle_s\), \(k \geq 1\), where each function \(L_{k+1}\) is expressed through \(\lambda_1, \ldots, \lambda_k\). For example, \(L_1 = \int_s^{s*} [f_1 - (f_1)_s] \, d\sigma\), \(L_2 = \int_s^{s*} \left\{ \sum_{k=1}^{s} (f_2 + \lambda_1(1 + \partial_t f_1 + \partial_s g_1)) - \sum_{k=1}^{s} \partial_s \lambda_k \right\} \, d\sigma - \lambda_1 g_1 - \lambda_1^2 \partial_t g_0 / 2\). It can easily be checked that all coefficients \(\partial_k\) are uniquely determined in the class of \(2\pi\)-periodic functions with zero average (\(\lambda_k(s, \lambda) \rangle_s = 0\).

The asymptotic solution of the equation for the angle \(s\) is sought in the form \(s(t) = \psi(t) + \Psi(\psi(t), \lambda(t), t)\), where

\[
\frac{d\psi}{dt} = \left\langle \partial_l \mathcal{H}(\phi + \Psi(\phi, \lambda, t), \lambda + \Lambda(\phi + \Psi(\phi, \lambda, t), \lambda, t), t) \right\rangle_{\phi},
\]

(16)

and the function \(\Psi\) satisfies the equation: \(\left\langle \partial_l \Psi + 1 \right\rangle \left\langle \partial_l \mathcal{H}(\phi + \Psi(\phi, \lambda + \Lambda, t)) \right\rangle_{\phi} + \partial_l \Psi = \partial_l \mathcal{H}(\psi + \Psi(\lambda, \lambda + \Lambda, t)) - \partial_s \Psi \mathcal{F}(s, \lambda + \Lambda, t) \rangle_s\), with the additional condition: \(\langle \Psi \rangle_{\psi} = 0\).

Asymptotics for \(\Psi\) are constructed in the form: \(\Psi(\psi, \lambda, t) = \sum_{k=1}^{\infty} \psi_k(\psi, \lambda) t^{-k}\). Substituting the series into the last equation and grouping the expressions of the same power of \(t\) give the following chain of differential equations: \(g_0(\lambda) \partial_k \psi_k = P_k(\psi, \lambda) - \langle P_k(\psi, \lambda) \rangle_{\psi} \rangle, \quad k \geq 1\). Note that each function \(P_{k+1}\) is expressed through \(\psi_1, \ldots, \psi_k\). For example, \(P_1 = g_1 + \partial_s \lambda_1, \quad P_2 = g_2 + \partial_s \lambda_1, \lambda_1 + \partial_s \lambda_0 \lambda_2 + \partial_s^3 \lambda_0 \lambda_1^2 / 2 + (1 + \partial_s g_1) \psi_1 - \partial_s \psi_1 (f_1 + g_1 + \partial_s \lambda_0 \lambda_1 + \partial_s \int_0^\ell g_1 \, dl) \rangle_{\psi}\).

It follows that all coefficients \(\psi_k\) are uniquely determined in the class of \(2\pi\)-periodic functions with \(\langle \psi_k(\psi, \lambda) \rangle_{\psi} = 0\).

In the last step, we integrate the averaged equations (14) and (16). Note that the stability of the trivial solution to system (7) ensures that \(t(t) \to 0\) as \(t \to t_0\). Therefore, we can use the asymptotic behavior of the functions \(F(s, \ell, t)\) and \(G(s, \ell, t)\) as \(\ell \to 0\) and \(t \to \infty\) in the integration of the averaged equations. Thus we have

\[
\frac{d\lambda}{dt} = -t^{-1}[\lambda + O(\lambda^{3/2})][1 + O(t^{-1})].
\]

It follows from the last equation that \(\lambda(t) = \ell_0^2 t^{-1} + \sum_{k=2}^{\infty} \ell_k t^{-k}\) as \(t \to \infty\), where \(\ell_0\) is an arbitrary parameter and the coefficients \(\ell_k = \text{const}\) are uniquely determined. The averaged equation (16) for the angle is integrated trivially: \(\psi(t) = \psi_0 + \int_t^{t_0} \langle \partial_\tau \mathcal{H}(\psi + \Psi(\psi, \lambda(\tau), \tau), \lambda(\tau) + \Lambda(\psi + \Psi(\psi, \lambda(\tau), \lambda(\tau), \tau)), \tau) \rangle_{\psi} \, d\tau\), where \(\psi_0 = \text{const}\). The asymptotics of \(\psi(t)\) is defined by
the expansions for $\lambda$, $\Lambda$, and $\Psi$. Thus we have $\psi(t) = \psi_0 + \sqrt{2}t + \omega_{-1} \log t + \sum_{k=1}^{\infty} s_k t^{-k}$ as $t \to \infty$, where $\psi_0$ is an arbitrary parameter, $\omega_{-1} = 3b_0^2/(8\sqrt{2})$ and the coefficients $s_k$ are uniquely determined. Note that the justification of the constructed asymptotics follows from [11]. Thus, a two-parameter family of solutions $\ell(t; \ell_0, \psi_0)$, $s(t; \ell_0, \psi_0)$ to system (12) has the following asymptotics: $\ell(t) = \ell_0 t^{-1} + \sum_{k=2}^{\infty} \hat{s}_k(\tilde{S}(t)) t^{-k}$, $s(t) = \tilde{S}(t) + \sum_{k=1}^{\infty} \hat{s}_k(\tilde{S}(t)) t^{-k}$ as $t \to \infty$, where $\hat{s}_k$ and $\tilde{s}_k$ are $2\pi$-periodic functions in $\tilde{S}$, and the function $\tilde{S}(t)$ has the following form: $\tilde{S}(t) = \psi_0 + \sqrt{2}t + \omega_{-1} \log t$. The transformation formulas lead to the asymptotic estimates for solutions to system (7): $r(t) = \ell_0 t^{-1/2} \left[ \cos \tilde{S}(t) + O(t^{-1}) \right]$, $v(t) = -\ell_0 t^{-1/2} \left[ \sin \tilde{S}(t) + O(t^{-1}) \right]$. Returning to the original variables we obtain estimates (11).

**Theorem 4** For all $c_1 \neq 0$ equation (3) has two-parameter family of solutions $R_-(z; c_3, c_4)$ with the asymptotics:

$$R(z) = (-z)^{-1/4} \sum_{k=0}^{\infty} \rho_k(S(z); c_3)(-z)^{-3k/2}, \quad R'(z) = (-z)^{-1/4} \sum_{k=0}^{\infty} \vartheta_k(S(z); c_3)(-z)^{-3k/2},$$

as $z \to -\infty$, where the functions $\rho_k(S; c_3)$, $\vartheta_k(S; c_3)$ are $2\pi$-periodic in $S$, $S(z) = c_4 + \omega_0(c_1, c_3)(-z)^{3/2} + \omega_{-1}(c_1, c_3) \log(-z)$, $\rho_0 = |c_1|^{1/2} + (c_3/2) \cos S$, $\vartheta_0 = -c_3 \sin S$, $\omega_k(c_1, c_3) = \text{const}$.

**Proof.** The arguments are similar to that of Theorem 3.

The asymptotic estimates for $\Phi(z)$ are derived from the expansions for $R(z)$ as $z \to \pm\infty$. In particular, the solutions $R_+(z)$, $R_-(z)$ with asymptotics (4) lead to

$$\Phi_+ = 2c_1 \log z + c_2 + c_1 \sum_{k=2}^{\infty} \alpha_k z^{-3k/2}, \quad z \to +\infty,$$

$$\Phi_- = -\frac{2\text{sgn}(c_1)}{3}(-z)^{3/2} - c_1 \log(-z) + c_2 + c_1 \sum_{k=1}^{\infty} \beta_k(-z)^{-3k/2}, \quad z \to -\infty.$$

Here the coefficients $\alpha_k$, $\beta_k$ are uniquely defined, for example, $\alpha_2 = 4a_2 \sqrt{2}/3$, $\beta_1 = -2(2b_2 b_0 - 3b_0^2)/(3b_0^2)$.

Thus, Theorems 3 and 4 with (2) give the asymptotics for general solutions to equation (1) as $z \to \pm\infty$. The connection formulae for asymptotics of equation (1) have not been considered here. This will be discussed elsewhere.

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