FREE CUBIC IMPLICATION ALGEBRAS

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To the memory of our mentor and friend
Gian-Carlo Rota

ABSTRACT. We construct free cubic implication algebras with finitely many generators, and determine the size of these algebras.

1. Introduction

In [3] Metropolis and Rota introduced a new way of looking at the face lattice of an $n$-cube based on its symmetries. Subsequent work has lead to a purely equational representation of these lattices – the varieties of MR and cubic implication algebras. [2] describes the variety of Metropolis-Rota implication algebras (MR algebras). [1] gives an equational description of cubic implication algebras and implicitly proves that the face lattices of $n$-cubes generate the variety. Therefore free cubic implication algebras must exist. In this paper we give an explicit construction for the free cubic implication algebra on $m$ generators and determine its size.

The argument comes in several parts. First we produce a candidate for the free algebra on $k+1$ generators by looking at embeddings into interval algebras and choosing a minimal one. Thus the free algebra is embedded into a known cubic implication algebra. We compute the size of this cubic implication algebra. From [1] we know that every finite cubic implication algebra is a finite union of interval algebras and we know that the overlaps are also interval algebras. The size of an interval algebra is easy to determine so we can use an inclusion-exclusion argument to determine the size of the cubic algebra.

The next part is to show that our candidate for the free algebra is generated by the images of the generators and so the embedding is onto. Again we use the facts that our cubic implication algebra is a finite union of interval algebras and each interval algebra is the set of $\Delta$-images of a Boolean algebra to reduce the problem to showing that certain atoms in a well-chosen Boolean algebra are generated.

We start by recalling some basic definitions and facts about cubic algebras – the reader is referred to [1] for more details.

Definition 1.1. A cubic implication algebra is a join semi-lattice with one and a binary operation $\Delta$ satisfying the following axioms:

a. if $x \leq y$ then $\Delta(y, x) \lor x = y$;

b. if $x \leq y \leq z$ then $\Delta(z, \Delta(y, x)) = \Delta(\Delta(z, y), \Delta(z, x))$;

c. if $x \leq y$ then $\Delta(y, \Delta(y, x)) = x$;

d. if $x \leq y \leq z$ then $\Delta(z, x) \leq \Delta(z, y)$;

Let $xy = \Delta(1, \Delta(x \lor y, y)) \lor y$ for any $x, y$ in $L$. Then:

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Definition 1.2. An MR implication algebra is a cubic implication algebra satisfying the MR-axiom:

\[ \Delta(x, a) \vee b < x \text{ iff } a \wedge b \text{ does not exist.} \]

Example 1.1. Let \( B \) be a Boolean algebra, then the interval algebra of \( B \) is

\[ \mathcal{I}(B) = \{ [a, b] \mid a \leq b \text{ in } B \} \]

ordered by inclusion. The operations are defined by

\[ 1 = [0, 1] \]
\[ [a, b] \vee [c, d] = [a \wedge c, b \vee d] \]
\[ \Delta([a, b], [c, d]) = [a \vee (b \wedge \overline{c}), b \wedge (a \vee \overline{d})] \].

It is straightforward to show \( \mathcal{I}(B) \) is an MR-algebra. Additional details may be found in \[1\].

Example 1.2. Let \( X \) be any set. The signed set algebra of \( X \) is the set

\[ \mathcal{S}(X) = \{ \langle A, B \rangle \mid A, B \subseteq X \text{ and } A \cap B = \emptyset \} \]

The operations are

\[ 1 = \langle \emptyset, \emptyset \rangle \]
\[ \langle A, B \rangle \vee \langle C, D \rangle = \langle A \cap C, B \cap D \rangle \]
\[ \Delta(\langle A, B \rangle, \langle C, D \rangle) = \langle A \cup C \setminus B, B \cup D \setminus A \rangle \].

\( \mathcal{S}(X) \) is isomorphic to \( \mathcal{S}(\wp(X)) \) by \( \langle A, B \rangle \mapsto [A, X \setminus B] \). All finite MR-algebras are isomorphic to some signed set algebra (and hence to some interval algebra).

As part of the representation theory in \[1\] we had the following definitions and lemma:

Definition 1.3. Let \( L \) be a cubic implication algebra and \( a \in L \). Then the localization of \( L \) at \( a \) is the set

\[ L_a = \{ \Delta(y, x) \mid a \leq x \leq y \} \]

Associated with localization is the binary relation \( \preceq \) we can define by

\[ a \preceq b \text{ iff } b \in L_a \]

or by the equivalent internal definitions:

\[ a \preceq b \text{ iff } a \leq \Delta(a \vee b, b) \]
\[ a \preceq b \text{ iff } b = (a \vee b) \wedge (\Delta(1, a) \vee b). \]

In \[1\] we establish the equivalence of these three definitions and make great use of this relation in getting a representation theorem for cubic algebras. The next lemma is the part of that representation theory that we need in order to understand the free algebra construction that follows.

Lemma 1.4. Let \( L \) be any cubic implication algebra. Then \( L_a \) is an atomic MR-algebra, and hence isomorphic to an interval algebra.
Definition 2.1. Let $X$ be a set, the set of generators.
(i) Let $\mathcal{F} r(X)$ denote the free cubic implication algebra with generators $X$.
(ii) Let $X' = \{ s_i \mid x \in X \} \cup \{ t_i \mid x \in X \}$ and let $B$ be the Boolean algebra generated by $X'$ with the relations $s_x \leq t_x$ for all $x \in X$. Let $x' = (s_x, t_x) \in \mathcal{I}(B) = \mathcal{L}$, and let $\mathcal{L}_i(X) = \bigcup_{x \in X} \mathcal{L}_x$.
(iii) Let $\mathcal{F}_k$ be the free Boolean algebra with the $2k + 2$ generators $\{ s_0, \ldots, s_k \} \cup \{ t_0, \ldots, t_k \}$.

It is not the case that $\mathcal{L}_i(X)$ is the free cubic implication algebra, it is too large. But it serves as a prototype for discussing the construction of a free cubic implication algebra. It also allows us to compute an upper bound to the size of the free algebra.

Lemma 2.2. $\mathcal{F} r(\{x_0, \ldots, x_k\})$ is finite with size at most $3^{2k+2}$. 

Proof. This is because $\mathcal{F} r(\{x_0, \ldots, x_k\})$ embeds into $\mathcal{I}(\mathcal{F}_k)$ – letting $x_i = (s_i, t_i)$.

The idea of this proof is crucial – we embed the free algebra into an interval algebra and determine properties of the free algebra from the embedding.

Suppose that $X = \{ a_0, a_1, \ldots, a_k \}$ is finite. Let $\mathcal{F} r(X)$ embed into $\mathcal{I}(C) = \mathcal{L}$ as an upper segment for some Boolean algebra $C$. Let $e(a_i) = (s_i, t_i) = a'_i$ for $0 \leq i \leq k$. Define inductively

$$
\delta_0 = a'_0 \\
\delta_{i+1} = \delta_i \lor a'_{i+1} \\
= \delta_i \land \Delta(\delta_i \lor a'_{i+1}, a'_{i+1})
$$

Then it must be the case that $\mathcal{F} r(X)$ embeds into $\mathcal{L}_\delta$, since we have that $\delta_i \leq a'_i$ for all $i$ and the image of $\mathcal{F} r(X)$ in $\mathcal{I}(C)$ is the set $\bigcup_{i=0}^k \mathcal{L}_\delta$. Thus we may as well assume that $\delta_k$ is a vertex, and furthermore that it is $[0, 0]$ as all vertices are interchangeable by a cubic isomorphism. Furthermore, we see that if $B^*$ is the subalgebra generated by the $s_i$’s and the $t_i$’s, then in fact $\mathcal{F} r(X)$ embeds into $\mathcal{I}(B^*)_{\delta_k}$.

So now we construct a new candidate for $\mathcal{F} r(X)$, where $X$ is the finite set $\{ a_0, a_1, \ldots, a_k \}$. Let $B_X$ be the Boolean algebra generated by $\{ s_0, \ldots, s_k \} \cup \{ t_0, \ldots, t_k \}$ with the relations $s_i \leq t_i$ for all $0 \leq i \leq k$ and $\delta_k = [0, 0]$.

By the above argument, we see that

$$|\mathcal{F} r(X)| \leq \left| \bigcup_{i=0}^k \mathcal{L}_{[s_i, t_i]} \right|.$$

We will compute the cardinality of the right-hand-side and show that the intervals $[s_i, t_i]$ cubically generate $\bigcup_{i=0}^k \mathcal{L}_{[s_i, t_i]}$ and so $\mathcal{F} r(X) \simeq \bigcup_{i=0}^k \mathcal{L}_{[s_i, t_i]}$.

3. Getting better relations

The relation $\delta_k = [0, 0]$ is not easy to use, so we will recast it as a series of statements about the $s_i$’s and the $t_i$’s.

First a fact about interval algebras that we will often make use of in the following argument. It is easily verified from the definitions above.
If \(a = [a, b]\) and \(w = [u, v]\) are any two intervals then

\[
a \land \Delta(a \lor w, w) = [a \lor (b \land \overline{v}), b \land (a \lor \overline{w})].
\]

Now define inductively a sequence from \(\mathcal{B}_X\) as follows:

\[
\begin{align*}
\sigma_0 &= s_0; & \tau_0 &= t_0; \\
\sigma_{i+1} &= \sigma_i \lor (\tau_i \cap \overline{t}_{i+1}); & \tau_{i+1} &= \sigma_i \lor (\tau_i \cap \overline{s}_{i+1}).
\end{align*}
\]

It is not hard to see that \(\sigma_i \leq \tau_i\) for all \(i\) and that \(\delta_i = [\sigma_i, \tau_i]\). So our extra condition can now be rewritten as \(\sigma_k = \tau_k = 0\). This is still rather unsatisfactory. Instead of using these relations we will produce another set that give useful information more directly. We do this by defining a larger class of relations that are used to show that the desired relations capture the ones above and no more.

**Definition 3.1.** Let \(0 \leq l \leq k\) in \(\mathbb{N}\), and \(t, \alpha\) in \(\mathcal{B}_X\). Then

\[
\begin{align*}
R_{l,k}(t, \alpha) : & \quad t \leq \bigvee_{j=l}^{i} s_j \lor t_{i+1} \lor \alpha \\
Q_{l,k}(t, \alpha) : & \quad t \leq \bigvee_{j=l}^{k} s_j \lor \alpha.
\end{align*}
\]

We now demonstrate that these are the desired relations.

**Lemma 3.2.**

If \(\tau_k \leq \alpha\) then \(t_0 \leq \bigvee_{j=1}^{k} s_j \lor \alpha\).

**Proof.** For \(k = 0\) this just says that \(s_0 \leq \alpha\).

If general we have

\[
\begin{align*}
\tau_{k+1} &= \sigma_k \lor (\tau_k \land \overline{s}_{k+1}) \leq \alpha & \iff & \sigma_k \leq \alpha \text{ and } \tau_k \land \overline{s}_{k+1} \leq \alpha \\
& \quad \Rightarrow \tau_k \leq s_{k+1} \lor \alpha \\
& \quad \Rightarrow t_0 \leq \bigvee_{j=1}^{k} s_j \lor s_{k+1} \lor \alpha \\
& \quad \Rightarrow t_0 \leq \bigvee_{j=1}^{k} s_j \lor \alpha.
\end{align*}
\]

\[\square\]

**Lemma 3.3.**

If \(\sigma_k \leq \alpha\) then \(t_0 \leq \bigvee_{j=1}^{i} s_j \lor t_{i+1} \lor \alpha\) for all \(i < k\) and \(s_0 \leq \alpha\).

**Proof.** For \(k = 0\) this just says that \(s_0 \leq \alpha\).
In general we have
\[ \sigma_{k+1} = \sigma_k \lor (\tau_k \land \tau_{k+1}) \leq \alpha \iff \sigma_k \leq \alpha \text{ and } \tau_k \leq t_{k+1} \lor \alpha \]

\[ \Rightarrow t_0 \leq \bigvee_{j=1}^{i} s_j \lor t_{i+1} \lor \alpha \text{ for all } i < k; \]
\[ s_0 \leq \alpha; \text{ and }\]
\[ f_0 \leq \bigvee_{j=1}^{k} s_j \lor t_{k+1} \lor \alpha \]
\[ \Rightarrow t_0 \leq \bigvee_{j=1}^{i} s_j \lor t_{i+1} \lor \alpha \text{ for all } i \leq k; \text{ and }\]
\[ s_0 \leq \alpha. \]

This shows the necessity of these relations, now we show they are also sufficient.

**Lemma 3.4.** Suppose that \( i < k \) and

- \( \sigma_i \leq \alpha \);
- \( R_{i+1,k,j}(\tau_i, \alpha) \) for all \( i \leq j < k \); and
- \( Q_{i+1,k}(\tau_i, \alpha) \).

Then

- \( \sigma_{i+1} \leq \alpha \);
- \( R_{i+2,k,j}(\tau_{i+1}, \alpha) \) for all \( i + 1 \leq j < k \); and
- \( Q_{i+2,k}(\tau_{i+1}, \alpha) \).

**Proof.** We have \( \tau_i \leq \bigvee_{p=i+2}^{i} s_p \lor t_{j+1} \lor \alpha \) for all \( i \leq j < k \).

Taking \( j = i \) we have that \( \tau_i \land t_{i+1} \leq \alpha \). As \( \sigma_i \leq \alpha \) we therefore get \( \sigma_{i+1} = \sigma_i \lor (\tau_i \land \tau_{i+1}) \leq \alpha \).

For \( j > i \) we get \( \tau_i \land \tau_{i+1} \leq \bigvee_{p=i+2}^{j} s_p \lor t_{j+1} \lor \alpha \), and so \( \tau_{i+1} = \sigma_i \lor (\tau_i \land \tau_{i+1}) \leq \alpha \lor \bigvee_{p=i+2}^{j} s_p \lor t_{j+1} \lor \alpha = \bigvee_{p=i+2}^{j} s_p \lor t_{j+1} \lor \alpha \).

From \( Q_{i+1,k}(\tau_i, \alpha) \) we have \( \tau_i \leq \bigvee_{p=i+1}^{k} s_p \lor \alpha \) and so \( \tau_i \land \tau_{i+1} \leq \tau_i \leq \bigvee_{p=i+2}^{k} s_p \lor \alpha \) which gives \( \tau_{i+1} \leq \tau_i \leq \bigvee_{p=i+2}^{k} s_p \lor \alpha \). \( \square \)

**Corollary 3.5.** Suppose that

- \( s_0 \leq \alpha \);
- \( R_{1,k,j}(t_0, \alpha) \) for all \( 1 \leq j < k \); and
- \( Q_{1,k}(t_0, \alpha) \).

Then

- \( \sigma_k \leq \alpha \); and
- \( \tau_k \leq \alpha \).

**Proof.** It follows immediately from the lemma, and noting that \( Q_{k+1,k}(\tau_k, \alpha) \) is the same as \( \tau_k \leq \alpha \). \( \square \)
Proposition 3.6. \( \sigma_k = \tau_k = 0 \iff s_0 \leq 0 \) if

\[ R_{1,k,F}(0,0) \text{ for all } 1 \leq j < k; \text{ and} \]

\[ Q_{1,k,F}(0,0). \]

Proof. Immediate from the last corollary. \( \Box \)

Definition 3.7. Let \( B_k \) be the Boolean algebra generated by \( \{s_0, \ldots, s_k\} \cup \{t_0, \ldots, t_k\} \) with the relations

\[ Z: \quad s_0 \leq 0; \]

\[ S_i: \quad s_i \leq t_i, \text{ for all } i \leq k; \]

\[ R_i: \quad R_{1,k,F}(0,0) \text{ for all } 1 \leq j < k; \text{ and} \]

\[ Q_i: \quad Q_{1,k,F}(0,0). \]

Let \( \mathcal{L}^{(k)} = \mathcal{F}(B_k) \) and \( \mathcal{L}(X) = \bigcup_{i=0}^{k} \mathcal{L}^{(i)} \).

We aim to compute the size of \( \mathcal{L}(X) \) as this provides an upper bound to the size of \( \mathcal{F}r(X) \). This starts with a lot of atom counting in \( B_k \).

4. Looking At Atoms

In this section we aim to see how atoms are produced in \( B_k \) as a preliminary to counting them. This is based upon our knowledge of atoms in \( F_k \) and the ideal we quotient out by to get \( B_k \). This ideal is generated by the set

\[ S_k = \{s_0\} \cup \{s_i \land \bar{t}_i \mid 0 \leq i \leq k\} \cup \{t_0 \land \bigwedge_{j=1}^{i} \bar{s}_j \land \bar{t}_{i+1} \mid 0 \leq i < k\} \cup \{t_0 \land \bigcap_{j=0}^{k} \bar{s}_j\}. \]

The elements of \( S_k \) come in four different kinds. For ease of reference we name them as

\[ s_0 \]

\[ u_i = s_i \land \bar{t}_i \quad \text{for } 0 \leq i \leq k \]

\[ r_i = t_0 \land \bigwedge_{j=1}^{i} \bar{s}_j \land \bar{t}_{i+1} \quad \text{for } 0 \leq i \leq k-1 \]

\[ q_k = t_0 \land \bigcap_{j=0}^{k} \bar{s}_j. \]

There is one new atom that comes from the failure of \( Q_{k-1} \).

Lemma 4.1. Let \( a_k = t_0 \land \bigwedge_{i=0}^{k-1} \bar{s}_i \). Then \( a_k \) is an atom in \( B_k \).

Proof. We recall from the usual construction of the free Boolean algebra on \( \{s_0, \ldots, s_k\} \cup \{t_0, \ldots, t_k\} \) that every atom has the form \( \bigwedge_{j=0}^{k} e_j s_j \land \bigwedge_{j=0}^{k} d_j \bar{t}_j \) where \( e_j \) and \( d_j \) are \( \pm 1 \) and \( 1a = a, -1a = \bar{a} \).

In \( B_k \) we have \( s_0 = 0 \) so that \( e_0 = -1 \). For atoms below \( a_k \) we have \( \delta_0 = 1 \) and \( e_j = -1 \)

\[ \text{for all } j < k. \]

By \( Q_k \) we have \( a_k \land \bar{s}_k = 0 \) so that \( a_k \leq s_k \). Also for \( 0 \leq i < k \) we have \( a_k \land \bar{t}_{i+1} \leq t_0 \land \bigwedge_{j=1}^{i} \bar{s}_j \land \bar{t}_{i+1} = 0 \) by \( R_{k,F} \) so that \( a_k \leq t_i \) for all \( i \). This implies \( a_k = a_k \land s_k \land \bigwedge_{j=1}^{k} t_j \) is an atom or zero.

If \( a_k = 0 \) then so does \( a'_k = a_k \land s_k \land \bigwedge_{j=0}^{k} t_j \). Therefore (in \( F_k \)) we have \( a'_k = \bigwedge_{j=0}^{k-1} \bar{s}_j \land s_k \land \bigwedge_{j=0}^{k} t_j \) is the ideal generated by \( S_k \). As \( a'_k \) is an atom in \( F_k \) this means that \( a'_k \) must be below one of the elements of \( S_k \).

- \( a'_k \leq \bar{s}_0 \) so it is not below \( s_0 \).
- $\alpha'_i \leq \overline{\text{Q}}_i$ for $i = 0, \ldots, k - 1$ so that $\alpha'_i$ is not below any $u_i$.
- $\alpha'_k \leq s_k$ so that $\alpha'_k$ is not below $q_k$.
- $\alpha'_k \leq t_i$ for $i = 0, \ldots, k$ so that $\alpha'_k$ is not below $r_i$.

Hence $\alpha'_k$ cannot be in this ideal. \hfill $\Box$

The remainder of the analysis is an investigation of the change from $B_{k-1}$ to $B_k$. The last lemma is the most important change as in $B_{k-1}$ we have $a_k = 0$ by $Q_{k-1}$.

We need also note that $R_0$ is independent of $k$ so another change is that $R_{k-1}$ comes into effect. As $Q_k$ implies $R_{k-1}$ since $s_k \leq t_k$ this is not noticeable until $B_{k+1}$, so we really only have $R_{k-2}$ and $Q_k$ to worry about.

Further we need to note that the relations $R_i$ and $Q_k$ only affect the interval $[0, t_0]$ and do nothing in $[0, \overline{t}_0]$ – this is because they are all of the form $t_0 \wedge \text{something} = 0$.

**Lemma 4.2.** Let $a \in B_{k-1}$ be an atom. Then $a$ is split into three atoms in $B_k$.

**Proof.** We will work in $F_k$ as $B_k$ is a quotient of this algebra. We will also assume that $a$ is an atom in $F_{k-1}$ so that

$$a = \bigwedge_{i=0}^{k-1} \varepsilon_i s_i \wedge \bigwedge_{i=0}^{k-1} \delta_i t_i.$$  

Of course we have $\varepsilon_0 = -1$ as $s_0 = 0$ in $B_{k-1}$.

In $F_k$ $a$ splits into four parts $-a_{00} = a \wedge (s_k \wedge t_k), a_{01} = a \wedge (s_k \wedge \overline{t}_k), a_{10} = a \wedge (\overline{s}_k \wedge t_k)$ and $a_{11} = a \wedge (\overline{s}_k \wedge \overline{t}_k)$. In $B_k$ we have $a_{01} = 0$ as $s_k \leq t_k$. We need to show that none of the others are made zero in $B_k$.

If one of them – call it $a_{pq} –$ is zero in $B_k$ then, in $F_k$ it must be in the ideal generated by $S_k$.

As $a_{pq}$ is an atom in $F_k$ it must be the case that $a_{pq}$ is smaller than something in $S_k$. This means that $\delta_0 = 1$ as everything in $S_k$ is below $t_0$.

Suppose that $a_{pq} \leq t_0 \wedge \bigwedge_{j=1}^{k-1} \overline{s}_j \wedge \overline{t}_{i+1}$ for some $0 \leq i < k - 1$. As $a_{pq} > 0$ this means that $\varepsilon_j = -1$ for $0 \leq j < i$ and $\delta_{i+1} = -1$. But now we have $a \leq t_0 \wedge \bigwedge_{j=1}^{i} \overline{s}_j \wedge \overline{t}_{i+1}$ and so $a = 0$ in $B_{k-1}$ – contradiction.

Hence $a_{pq} \leq t_0 \wedge \bigwedge_{j=1}^{k-1} \overline{s}_j \wedge \overline{t}_k$ or $a_{pq} \leq t_0 \wedge \bigwedge_{j=1}^{k} \overline{s}_j$. Either of these implies $\varepsilon_j = -1$ for $0 \leq j < k - 1$ and so $a = 0$ in $B_{k-1}$ – contradiction. \hfill $\Box$

Lastly we need to observe that the new atom $a_k = t_0 \wedge \bigwedge_{i=1}^{k-1} \overline{s}_i$ is not one of the atoms produced as in the last lemma – since if $a = \bigwedge_{i=0}^{k-1} \varepsilon_i s_i \wedge \bigwedge_{i=0}^{k-1} \delta_i t_i$ is an atom of $B_{k-1}$ and $a_k \leq a$ in $B_k$ then we have $a_k \wedge a > 0$ in $F_k$ and so $a = a_k$ in $F_k$. But this implies $\delta_0 = 1$ and $\varepsilon_j = -1$ for $0 \leq j < k - 1$ and so $a = 0$ in $B_{k-1}$ – contradiction.

The atoms as produced in the way described above fall into natural groupings. It helps to understand the counting arguments we give in the next section if we know how these groupings come about.

**Definition 4.3.** Let

- $T_{00} = \langle \{1\}, \cdot, = \rangle$
- $A_{00} = \{1\}$
- $T_{11} = \langle \{t_0\}, \cdot, = \rangle$
- $A_{11} = \{t_0\}$
- $T_{01} = \{\overline{t}_0, s_1, \overline{t}_1, s_1 \wedge t_1, s_1 \wedge \overline{t}_0, \overline{t}_1, \overline{t}_0, s_1 \wedge t_1, \overline{s}_1 \wedge t_0, (s_1, \overline{t}_0)\} = \}$
- $A_{01} = \{\overline{t}_0, s_1, \overline{t}_1, s_1 \wedge t_1\}$
A few tree diagrams will help us see what this definition is really about.

\[ T_{ij} = \begin{cases} 
\{ (t_0 \land \bigwedge_{l=1}^{i-1} \overline{s_l}, s_1, t_0 \land \bigwedge_{l=1}^{i-1} \overline{s_l} \land s_i \land s_{i+1}, t_0 \land \bigwedge_{l=1}^{i-1} \overline{s_l} \land s_j \land \overline{t_{i+1}}, \\
(t_0 \land \bigwedge_{l=1}^{i-1} \overline{s_l} \land s_i \land (\overline{s}_{i+1} \land t_{i+1}) \} \cup \{ (t_0 \land \bigwedge_{l=1}^{i-1} \overline{s_l} \land s_i \land s_{i+1}, t_0 \land \bigwedge_{l=1}^{i-1} \overline{s_l} \land s_j \} \cup \\
(t_0 \land \bigwedge_{l=1}^{i-1} \overline{s_l} \land s_i \land \overline{t_{i+1}}, t_0 \land \bigwedge_{l=1}^{i-1} \overline{s_l} \land s_j \} \cup \\
(t_0 \land \bigwedge_{l=1}^{i-1} \overline{s_l} \land s_i \land (\overline{s}_{i+1} \land t_{i+1}), t_0 \land \bigwedge_{l=1}^{i-1} \overline{s_l} \land s_j \} \cup \} = \\
T_{i,j-1} \cup \bigcup_{\alpha \in A_{i,j-1}} \{ \alpha \land s_j, \alpha \land \overline{t_j}, \alpha \land (\overline{s_j} \land t_j) \}, \quad j \neq i + 1 \\
\leq_{i,j-1} \cup \bigcup_{\alpha \in A_{i,j-1}} \{ \alpha \land s_j, \beta \}, \alpha \land (\overline{s_j} \land t_j), \delta \} \cup = \quad j > i + 1 \\
\bigcup_{\alpha \in A_{i,j-1}} \{ \alpha \land s_j, \alpha \land \overline{t_j}, \alpha \land (\overline{s_j} \land t_j) \} \cup = \quad j > i + 1 \\
r_{i,j-1} \cup \bigcup_{\alpha \in A_{i,j-1}} \{ \alpha \land s_j, \alpha \land \overline{t_j}, \alpha \land (\overline{s_j} \land t_j) \} \cup = \quad j > i + 1 \\
\end{cases} \]

T_{00} : • 1

T_{11} : • t_0

T_{01} :

\[ \overline{t_0} \]

\[ \overline{t_0} \land s_1 \quad \overline{t_0} \land \overline{t_1} \quad \overline{t_0} \land (\overline{s_1} \land t_1) \]

\[ = \]

T_{22} : • t_0 \land \overline{s_1}

\[ = \]

T_{12} : • t_0 \land s_1

\[ t_0 \land s_1 \land s_2 \quad t_0 \land s_1 \land \overline{t_2} \quad t_0 \land s_1 \land (\overline{s_2} \land t_2) \]

\[ = \]
The reader is invited to produce the next layer of trees.

We note that \( \bigcup_{j=0}^{k} A_{jk} \) is the set of all atoms of \( B_k \).

5. Counting Atoms and Other Things

Now we want to count just how many atoms there are and in what locations they may be found. This leads to a calculation of the size of \( \mathcal{L}(X) \).

**Definition 5.1.** Let

\[ \alpha_k(t) = \text{the number of atoms below } t \text{ in } B_k. \]

**Lemma 5.2.**

\[ \alpha_k(1) = \frac{1}{2}(3^{k+1} - 1). \]

**Proof.** \( B_0 = 2 \) has one atom.

From the lemmas 4.1 and 4.2, we have \( \alpha_k(1) = 3\alpha_{k-1}(1) + 1 \) from which we have the desired formula. \( \square \)

**Lemma 5.3.**

\[ \alpha_k((\overline{s}_{j_1} \land t_{j_1}) \land \ldots \land (\overline{s}_{j_n} \land t_{j_n})) = \frac{1}{2}(3^{k+1-n} - 1) \]

if all the \( \langle s_{j_i}, t_{j_i} \rangle \) are distinct.
Proof. The proof is the same as above – the first case where the formula makes sense is \( B_{n-1} \) and in this case \( (\overline{x}_j \land t_{j_i}) \land \ldots \land (\overline{x}_k \land t_{k_i}) \leq t_0 \land \wedge_{i=0}^{n-1} \overline{x}_i = 0 \) and so there are no atoms below it, and \( 0 = \frac{1}{2}(3^{(n-1)+1-n} - 1) \).

Let

\[ t = (\overline{x}_j \land t_{j_i}) \land \ldots \land (\overline{x}_k \land t_{k_i}). \]

There are two cases in general –

\textbf{No} \( j_i = k \): Going from \( B_{k-1} \) to \( B_k \) the new atom is below \( t \) and all other atoms split in three so we have \( \alpha_k(t) = 3\alpha_{k-1}(t) + 1 \) which gives the desired formula.

\textbf{Some} \( j_i = k \): Without loss of generality \( i = n \). This case is different as the new atom \( a_k \) in \( B_k \) is not below \( \overline{x}_k \land t_k \). But then every atom in \( B_{k-1} \) splits into three, one of which is below \( \overline{x}_k \land t_k \). Therefore \( \alpha_k((\overline{x}_j \land t_{j_i}) \land \ldots \land (\overline{x}_k \land t_{k_i}) \))\ is equal to \( \alpha_{k-1}((\overline{x}_j \land t_{j_i}) \land \ldots \land (\overline{x}_{k-1} \land t_{k-1})) \) and so equals \( \frac{1}{2}(3(k-1)+1-n-1) = \frac{1}{2}(3^{k+1-(n+1)} - 1) \). \( \square \)

As we are really interested in intervals we need the following observation

\textbf{Lemma 5.4.} Let \( B \) be a finite Boolean algebra, and \( x \in \mathcal{I}(B) \). Then the number of atoms in \([x, 1]\) is equal to the number of atoms in \( B \) less the number of atoms below \( \overline{x}_0 \land x_1 \).

\textbf{Proof.} Let \( x = [x_0, x_1] \). We note that if \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are two finite Boolean algebras then the number of atoms in \( \mathcal{A}_1 \times \mathcal{A}_2 \) equals the number of atoms in \( \mathcal{A}_1 \) plus the number of atoms in \( \mathcal{A}_2 \).

Since we have

\[ [x, 1] \cong [0, x_0] \times [x_1, 1] \]
\[ \cong [0, x_0] \times [0, \overline{x}_1] \]

and

\[ B \cong [0, x_0] \times [0, \overline{x}_1] \times [0, \overline{x}_0 \land x_1] \]

the result is immediate. \( \square \)

Now we want to compute the size of \( \mathcal{L}(X) \). As this is a union of interval algebras we will use an inclusion-exclusion calculation to find its size. We recall that in general for cubic algebras that

\[ \mathcal{L}_a \cap \mathcal{L}_b = \mathcal{L}_{a \Delta (a \lor b)} \]

is another interval algebra – \( \square \) theorem 4.6. Using the last lemma it is relatively easy to compute the size of these intervals.

\textbf{Definition 5.5.} Let \( x \) be an interval in \( B_k \). Let

\[ \alpha'_k(x) = \text{the number of atoms below } \overline{x}_0 \land x_1; \]
\[ \alpha'_1(x) = \text{the number of atoms in } [x, 1]. \]

\textbf{Lemma 5.6.} Let \( x = [x_0, x_1] \) be an interval in a finite Boolean algebra \( B \). Let \( n \) be the number of atoms in \([x, 1]\). Then

\[ |\mathcal{I}(B)_x| = 3^n. \]

\textbf{Proof.} This is just a special case of the fact that if \( |B| = 2^n \) then \( |\mathcal{I}(B)| = 3^n \) as \( \mathcal{I}(B)_x \cong \mathcal{I}([x, 1]). \) \( \square \)

Next the natural points of intersection.
**Definition 5.7.** Let $I \subseteq \{0, \ldots, k\}$. Let $i_0 < \cdots < i_l$ be an increasing enumeration of $I$. Then

$$
\eta_{i_0}^I = [s_{i_0}, t_{i_0}]
$$

$$
\eta_{i_{j+1}}^I = \eta_{i_j}^I \lor \Delta(\eta_{i_j}^I \lor [s_{i_{j+1}}, t_{i_{j+1}}], [s_{i_{j+1}}, t_{i_{j+1}}]) \quad \text{if } j < l
$$

$$
\eta_l^I = \eta_{i_l}^I.
$$

**Lemma 5.8.** Let $J \subseteq \{0, \ldots, k\}$. For all $i$ such that $0 \leq i < |J|

$$
\eta_i^J = [s_{j_0} \land \bigwedge_{p=1}^i (s_{j_p} \lor \overline{t_{j_p}}), t_{j_0} \lor \bigvee_{p=1}^i (\overline{s}_{j_p} \land t_{j_p})].
$$

**Proof.** The proof is by induction on $i$ – it is clearly true for $i = 0$. The superscript $J$ will be suppressed. Let

$$
a_i = s_{j_0} \land \bigwedge_{p=1}^i (s_{j_p} \lor \overline{t_{j_p}})
$$

$$
b_i = t_{j_0} \lor \bigvee_{p=1}^i (\overline{s}_{j_p} \land t_{j_p})
$$

$$
\eta_{i+1} = \eta_i \lor \Delta(\eta_i \lor [s_{j_{i+1}}, t_{j_{i+1}}], [s_{j_{i+1}}, t_{j_{i+1}}])
$$

$$
= [a_i, b_i] \lor \Delta([a_i \land s_{j_{i+1}}, b_i \lor t_{j_{i+1}}], [s_{j_{i+1}}, t_{j_{i+1}}])
$$

$$
= [a_i, b_i] \lor [(a_i \land s_{j_{i+1}}) \lor (b_i \land t_{j_{i+1}}), (b_i \lor t_{j_{i+1}}) \land (a_i \lor s_{j_{i+1}})]
$$

$$
= [a_i \land (s_{j_{i+1}} \lor \overline{t_{j_{i+1}}}), b_i \lor (\overline{s}_{j_{i+1}} \land t_{j_{i+1}})]
$$

$$
= [a_{i+1}, b_{i+1}].
$$

□

The next step is to compute the number of atoms in $\eta_i^J$, i.e. below $\overline{a}_i \land b_i$.

**Lemma 5.9.**

$$
\alpha_i^J(\eta_i^J) = 3^i \left(\frac{2}{3}\right)^i.
$$

**Proof.** This is an inclusion-exclusion argument – let $S_{k,i,J} = [\overline{s}_{j_p} \land t_{j_p} \mid 0 \leq p \leq i]$ for all $J \subseteq \{0, \ldots, k\}$ and $i < |J|$. Again the superscript $J$ is omitted.
Then inclusion-exclusion gives us

\[ \alpha^*(\eta_i) = \sum_{j=1}^{i+1} (-1)^{j+1} \left( \sum_{A \subseteq \{k+1, \ldots, k+i\}} \alpha_k(\bigcap_j A) \right) \]

\[ = \sum_{j=1}^{i+1} \binom{i+1}{j} (-1)^{j+1} \frac{1}{2} (3^{k+1-j} - 1) \]

\[ = \frac{1}{2} (3^{k+1} - 1) - \frac{1}{2} \sum_{j=0}^{i+1} \binom{i+1}{j} (-1)^j (3^{k+1-j} - 1) \]

\[ = 3^{k+1} \sum_{j=0}^{i+1} \binom{i+1}{j} \left( \frac{1}{3} \right)^j - \sum_{j=0}^{i+1} \binom{i+1}{j} (-1)^j \]

\[ = 3^{k+1} \left( 1 - \frac{1}{3} \right)^{i+1} - (1 - 1)^{i+1} \]

\[ = 3^{k+1} \left( \frac{2}{3} \right)^{i+1}. \]

Thus

\[ \alpha'_k(\eta_i) = \alpha_k(1) - \alpha^*(\eta_i) \]

\[ = \frac{1}{2} (3^{k+1} - 1) - \left[ \frac{1}{2} (3^{k+1} - 1) - 3^k \left( \frac{2}{3} \right)^i \right] \]

\[ = 3^k \left( \frac{2}{3} \right)^i. \]

\[ \square \]

**Definition 5.10.** Let

\[ \Phi(k, l) = 3^k \left( \frac{2}{3} \right)^l. \]

Now at last we are able to compute the size of \( \mathcal{L}(X) = \bigcup_{i=0}^k \mathcal{L}^{(k)}_{[r, k]} \).

For ease of reading let \( \mathcal{M}_i = \mathcal{L}^{(k)}_{[r, k]} \). Then inclusion-exclusion gives us that

\[ |\mathcal{L}(X)| = \sum_{j=1}^{k+1} (-1)^{j+1} \left( \sum_{A \subseteq \{0, \ldots, r\} \atop |A|=j} \left| \bigcap_{j \in A} \mathcal{M}_j \right| \right) \]

\[ = \sum_{j=1}^{k+1} \binom{k+1}{j} (-1)^{j+1} 3^{\Phi(k, j-1)}. \]
6. The other direction

Now we turn to showing that the algebra we have constructed is the free cubic implication algebra. Since we know that the free algebra embeds into \( \mathcal{L}(X) \), it suffices to show that \( \mathcal{L}(X) \) is generated by the intervals \( I_i = [s_i, t_i] \) for \( i = 0, \ldots, k \) using only cubic operations. In fact it suffices only to show that the elements covering the \( I_i \) are all cubically generated as these are atoms of \([I_i, 1]\) and so generate \([I_i, 1]\) with only joins. All other elements in \( \mathcal{L}_{I_i} \) are then \( \Delta \)-images of two elements of \([I_i, 1]\) and so all of \( \mathcal{L}(X) \) is obtained.

Let \( \mathcal{F}_k \) now denote the subalgebra of \( \mathcal{L}(X) \) generated by \( \{I_i | 0 \leq i \leq k\} \). We are trying to show that \( \mathcal{F}_k = \mathcal{L}(X) \).

We start with the easiest case.

**Lemma 6.1.** Every possible atom above \( I_0 \) is in \( \mathcal{F}_k \).

**Proof.** The atoms above \( I_0 \) are of the form \([0, t_0 \vee a]\) where \( a \leq \overline{t}_0 \) is an atom of \( \mathcal{B}_2 \).

To see we get these we note that

\[
[0, t_0] \vee [s_i, t_i] = [0, t_0 \vee t_i]
\]

and

\[
([0, t_0] \vee [\overline{t}_i, \overline{t}_0]) \rightarrow [0, t_0] = [0, t_0 \vee \overline{t}_i] \rightarrow [0, t_0]
\]

If \( a \leq \overline{t}_0 \) is any atom then \( a = \overline{t}_0 \wedge \bigwedge_i e_i s_i \wedge \bigvee_i \delta_i t_i \) and therefore \( t_0 \vee a = \bigwedge_i (e_i s_i \vee t_0) \wedge \bigvee_i (\delta_i t_i \vee t_0) \) and as we have each \([0, t_0 \vee t_i]\) and \([0, t_0 \vee s_i]\) we get the desired meets and complements and hence \([0, t_0 \vee a]\). \( \square \)

The rest of the proof consists of carefully showing that all the atoms in \([I_i, 1]\) are obtained.

**Definition 6.2.** Let \( a \in \mathcal{B}_k \) be an atom and \([x, y] \in \mathcal{I}(\mathcal{B}_k) \). Then

(a) \( a \) is left-associated with \([x, y]\) iff \( a \leq x \).

(b) \( a \) is right-associated with \([x, y]\) iff \( a \leq \overline{y} \).

(c) \( a \) is associated with \([x, y]\) iff \( a \leq x \) or \( a \leq \overline{y} \).

The idea of association is that an atom of \([x, y], 1\) is either \([x \land \overline{y}, y]\) for some \( a \) an atom below \( x \), or \([x, y \lor a]\) for some atom \( a \leq \overline{y} \). Thus \( a \) is associated with \([x, y]\) iff \( a \) produces an atom in \([x, y], 1\).

**Lemma 6.3.** Let \( a \) be associated with \([x, y] \geq [u, v] \). Then \( a \) is associated with \([u, v]\).

**Proof.** \([u, v] \leq [x, y] \iff x \leq u \) and \( y \leq v \iff x \leq u \) and \( \overline{v} \leq \overline{y} \). The result is now clear. \( \square \)

**Lemma 6.4.** Let \( a \) be an atom of \( \mathcal{B}_k \) and \([x, y], [u, v] \) be two intervals. Then

(a) \( a \) is left-associated with \([x, y] \lor [u, v]\) iff \( a \) is left-associated with both \([x, y]\) and \([u, v]\).

(b) \( a \) is right-associated with \([x, y] \lor [u, v]\) iff \( a \) is right-associated with both \([x, y]\) and \([u, v]\).

**Proof.** The result is immediate as \( a \leq x \lor u\) iff \( a \leq x \) and \( a \leq u \); and \( a \leq \overline{y} \lor \overline{v}\) iff \( a \leq \overline{y} \) and \( a \leq \overline{v} \). \( \square \)

**Lemma 6.5.** Let \( a \) be an atom of \( \mathcal{B}_k \) and \([x, y] \) be an interval. Then \( a \) is associated with \([x, y]\) iff \( a \) is associated with \( \Delta(1, [x, y]) \).

**Proof.** \( \Delta(1, [x, y]) = [\overline{y}, \overline{x}] \) so that \( a \leq x \) iff \( a \leq \overline{x} \) and \( a \leq \overline{y} \) iff \( a \leq \overline{y} \). \( \square \)

Our intent is to show that all the desired atoms in \([I_i, 1]\) are cubically generated from the \( I_j \). Because we need to come back to this point so often we have the following definition.
Definition 6.6. Let $a$ be an atom of $\mathcal{B}_k$ associated with an interval $[x, y] \in \mathcal{L}(X)$. Then $a$ is cubically assigned to $[x, y]$ iff

- $a \leq x$ and $[x \land \overline{a}, y]$ is cubically generated from the $I_j$; or
- $a \leq \overline{y}$ and $[x, y \lor a]$ is cubically generated from the $I_j$.

Our task is made a little easier by the fact that we only need to show a cubic assignment once in order to get enough cubic assignments.

Lemma 6.7. Let $a$ be an atom of $\mathcal{B}_k$ associated with $[x, y]$ and $[u, v]$ both in $\mathcal{F}_k$. Suppose that $a$ is cubically assigned to $[x, y]$. Then $a$ is cubically assigned to $[u, v]$.

Proof. First it is easy to see that $a$ is cubically assigned to $[x, y]$ iff $a$ is cubically assigned to $\Delta(1, [x, y])$.

This means that we need only deal with the case that $a$ is left-associated with both $[x, y]$ and $[u, v]$.

Since $a$ is left-associated with both intervals it is also left-associated with $[x \land u, y \lor v]$ and so $[x \land u \land \overline{a}, y \lor v] > [x \land u, y \lor v]$. Then we can form $[x \land u, y \lor v] \rightarrow [u, v] = [u \land \overline{a}, v]$.

The next step is to show that each atom does get cubically assigned to some interval in $\mathcal{F}_k$. Let $k \in \mathbb{N}$ and $\mathcal{S}(k)$ be the signed set algebra on $\{0, \ldots, k\}$. We define a function $R: \{\text{atoms of } \mathcal{B}_k\} \rightarrow \mathcal{S}(k)$ by induction on $k$:

$k = 0$: The only atom is 1 which is assigned $(\emptyset, \{0\})$;

$k = i + 1$: If $a \in \mathcal{B}_i$ is an atom and $R(a) = (R_0, R_1)$ then

$$R(a \land s_k) = (R_0 \cup [k], R_1)$$

$$R(a \land \overline{t}_k) = (R_0, R_1 \cup \{k\})$$

$$R(a \land (\overline{s}_k \land t_k)) = R(a).$$

For the new atom $R(t_0 \land \bigwedge_{j=1}^k \overline{s}_j) = (\{k\}, \emptyset)$.

Note that the function $R$ is not onto, but we do not need it to be.

Lemma 6.8. If $R(a) = (R_0, R_1)$ then

(a) $R_0 = \left\{ j \mid a \leq s_j \right\}$

(b) If $j = \min(R_0 \cup R_1)$ then

$$a = (t_0 \land \bigwedge_{i=0}^{j-1} \overline{s}_i) \land \bigwedge_{p \in R_0 \cup \{j+1\}} s_p \land \bigwedge_{q \in R_1} t_q \land \bigwedge_{r \in R_0 \cup R_1 \setminus \{j\}} (\overline{s}_r \land t_r).$$

Proof. This is true for all atoms in $\mathcal{B}_0$ as 1 = $t_0$ is the only atom and $R(1) = (\emptyset, \{0\})$.

Suppose that both (a) and (b) are true for all atoms in $\mathcal{B}_k$ and let $a$ be an atom of $\mathcal{B}_{k+1}$.

If $a = t_0 \land \bigwedge_{j=0}^k \overline{s}_j$ is the new atom then we have (by the R-rules) that $a \leq t_i$ for all $0 \leq i \leq k + 1$, and $a \leq s_{k+1} \land t$ by the Q-rule. Clearly also we have $a \leq \overline{s}_i$ for all $0 \leq i \leq k$ so that

$$\left\{ j \mid a \leq s_j \right\} = \{k + 1\}$$

$$\left\{ j \mid a \leq t_j \right\} = \emptyset$$
so that $R(a)$ is as asserted.

It is clear that $a$ has the desired form.

If $a = a' \wedge s_{k+1}$ and $a'$ is a $\mathcal{B}_k$-atom which we assume is expressed as in (b). We know inductively that $R(a')$ is as asserted and $a \leq s_{k+1}$ and $a \leq a'$ so that

$$R_0 \subseteq \{ j \mid a \leq s_j \}$$

$$R_1 \subseteq \{ j \mid a \leq \overline{t_j} \}.$$

We also know that if $j \notin R_0$ and $j \leq k$ then $a' \wedge s_j = 0$ in $\mathcal{B}_k$ and this is preserved in $\mathcal{B}_{k+1}$.

Likewise with $j \notin R_1$. Thus we get the desired equalities.

A similar argument works if $a = a' \wedge \overline{t}_{k+1}$.

If $a = a' \wedge (\overline{t}_{k+1} \wedge \overline{t}_{k+1})$ then $a \notin s_{k+1}$ and $a \notin \overline{t}_{k+1}$ so the sets do not change. □

**Corollary 6.9.** Let $a, b$ be atoms in $\mathcal{B}_k$ such that $R(a) = R(b)$. Then $a = b$.

**Proof.** This is clear from part (b) of the lemma. □

Now we turn to looking at getting atoms assigned to intervals in $\mathcal{L}(X)$.

**Definition 6.10.** Let $(A_0, A_1) \in \mathcal{J}(k)$. We define the interval

$$J(A_0, A_1) = \bigvee_{j \in A_0} I_j \lor \bigvee_{t \in A_1} \Delta(1, I_t).$$

We note that $J(A_0, A_1)$ is in $\mathcal{F}_k$.

**Lemma 6.11.** Let $a$ be an atom of $\mathcal{B}_k$. Then $a$ is left-associated with $J(R(a))$.

**Proof.** This is immediate from lemma 6.8(a). □

**Lemma 6.12.** An atom $a$ is associated with an interval of the form $J(A_0, A_1)$ iff $R(a) \leq (A_0, A_1)$ or $R(a) \leq (A_1, A_0)$.

**Proof.** Note that $(A_1, A_0) = \Delta(1, (A_0, A_1))$.

The left to right direction is clear as $a$ is associated with $J(R(a))$ and $R(a) \leq (A_0, A_1)$ implies $J((A_0, A_1)) \leq J(R(a))$ so we can apply lemma 6.5. The other half follows from lemma 6.5.

Suppose that $a$ is associated with $J(A_0, A_1)$. We may assume that $a$ is left-associated – otherwise use $\Delta(1, J(A_0, A_1)) = J(A_1, A_0)$.

Then we have that $a \leq s_j$ for all $j \in A_0$ and $a \leq \overline{t_j}$ for all $j \in A_1$ – by the definition of $J(A_0, A_1)$. Hence we have

$$A_0 \subseteq \{ j \mid a \leq s_j \} = R_0$$

and

$$A_1 \subseteq \{ j \mid a \leq \overline{t_j} \} = R_1.$$

Thus we have $R(a) \leq (A_0, A_1)$. □

**Lemma 6.13.** Let $(A_0, A_1) \in \mathcal{J}(k)$ with $(A_0, A_1) < (0, 0)$. Then there is some atom $a \in \mathcal{B}_k$ such that either $R(a) = (A_0, A_1)$ or $R(a) = (A_1, A_0)$.

**Proof.** Let $A_0 \cup A_1$ be enumerated as $j_1 < j_2 < \cdots < j_n$.

$j_1 = 0$: Then we will assume that $j_1 = 0 \in A_1$ – else switch the order. Then we define

$$a = \bigwedge_{j \in A_0} s_j \wedge \bigwedge_{j \in A_1} \overline{t_j} \wedge \bigwedge_{j \in A_0 \cup A_1} (\overline{s_j} \wedge \overline{t_j}).$$
This is a non-zero atom as it is below \( \overline{t}_0 \) and nothing gets killed here except by the rules \( s_i \leq t_i \). Clearly also we have

\[
\begin{align*}
\{ j \mid a \leq s_j \} &= A_0 \\
\text{and} \quad \{ j \mid a \leq \overline{t}_j \} &= A_1
\end{align*}
\]

so that \( R(a) = \langle A_0, A_1 \rangle \).

\[ j_1 > 0: \] Then we will assume that \( j_1 \in A_0 \) - else switch the order. Then we define

\[
a = (t_0 \land \bigwedge_{i=0}^{j_1-1} \overline{s}_i) \land \bigwedge_{p \in (R_0 \setminus \{j_1+1\})} s_p \land \bigwedge_{q \in R_1} \overline{t}_q \land \bigwedge_{r \in (R_0 \cup R_1)} (\overline{t}_r \land t_r) .
\]

This is nonzero as nothing below \( t_0 \land \bigwedge_{i=0}^{j_1-1} \overline{s}_i \) gets killed in any \( \mathcal{B}_j \) for \( j > j_1 \). Clearly also we have

\[
\begin{align*}
\{ j \mid a \leq s_j \} &= A_0 \\
\text{and} \quad \{ j \mid a \leq \overline{t}_j \} &= A_1
\end{align*}
\]

so that \( R(a) = \langle A_0, A_1 \rangle \).

\[ \square \]

Now we can prove that the desired assignments exist.

**Lemma 6.14.** Let \( a \) be any atom. Then \( a \) is cubically assigned to \( J(R(a)) \).

**Proof.** The proof is by induction on the rank of \( R(a) = \langle R_0, R_1 \rangle \) in \( \mathcal{J}(k) \).

- **Rank 0:** Then there is only one atom associated with \( J(R(a)) \) and in particular the interval \( [J(R(a)), 1] \) has only two elements, and it’s unique atom is \([0, 1]\). Since this atom has to come from \( a \) we see that \( a \) is cubically assigned to \( J(R(a)) \).

- **Rank > 0:** Then for every \( \langle B_0, B_1 \rangle < J(R(a)) \) there is an atom \( b \) such that \( R(b) \) is equal to either \( \langle B_0, B_1 \rangle \) or \( \langle B_1, B_0 \rangle \) and is therefore associated with and cubically assigned to \( J(B_0, B_1) \). By lemma 6.12 \( a \) is the only other atom associated with \( J(R(a)) \).

For each atom \( b \) associated with \( J(R(a)) \) let \( x_b \in [J(R(a)), 1] \) be the corresponding atom. Then by induction we cubically have \( x_b \) for all \( b \neq a \) and hence

\[
x_a = \bigvee_{b \neq a} x_b \to J(R(a))
\]

is cubically generated. Thus \( a \) is cubically assigned to \( J(R(a)) \).

\[ \square \]

**Theorem 6.15.**

\[ \mathcal{L}(X) \cong \mathcal{F} r(X) \]

**Proof.** Since we have \( \mathcal{F} r(X) \) embeds into \( \mathcal{L}(X) \) and every atom \( a \) in \( \mathcal{B}_k \) is cubically assigned to \( J(R(a)) \in \mathcal{L}(X) \) we see that every element of \( \mathcal{L}(X) \) is generated by the intervals \( I_i \). Hence the embedding must be onto.

\[ \square \]

This completes our description of free cubic implication algebras. One nice consequence of this result is that if \( M \) is an MR-algebra and \( X \subseteq M \) is any set, then the cubic subalgebra generated by \( X \) is upwards closed in the MR-subalgebra generated by \( X \). This is true because our result shows that it is true for finite free algebras.
References

[1] C. G. Bailey and J. S. Oliveira, An Axiomatization for Cubic Algebras, Mathematical Essays in Honor of Gian-Carlo Rota (B. E. Sagan and R. P. Stanley, eds.), Birkhäuser, 1998., pp. 305–334.

[2] A Universal Axiomatization of Metropolis-Rota Implication Algebras, in preparation, available at arXiv:0902.0157v1[math.CO]

[3] N. Metropolis and G.-C. Rota, Combinatorial Structure of the faces of the n-Cube, SIAM J. Appl.Math. 35 (1978), 689–694.

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