COEULERIAN GRAPHS

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Abstract. We suggest a measure of “Eulerianness” of a finite directed graph and define a class of “coEulerian” graphs. These are the graphs whose Laplacian lattice is as large as possible. As an application, we address a question in chip-firing posed by Björner, Lovász, and Shor in 1991, who asked for “a characterization of those digraphs and initial chip configurations that guarantee finite termination.” Björner and Lovász gave an exponential time algorithm in 1992. We show that this can be improved to linear time if the graph is coEulerian, and that the problem is NP-complete for general directed multigraphs. We define a notion of “multiEulerian tour” of a directed graph, characterize when such tours exist, and count them by generalizing the BEST theorem.

1. Introduction

In this paper \( G = (V, E) \) will always denote a finite directed graph, with loops and multiple edges permitted. We assume throughout that \( G \) is strongly connected: for each \( v, w \in V \) there are directed paths from \( v \) to \( w \) and from \( w \) to \( v \). Trung Van Pham [24] introduced the quantity

\[
M_G = \gcd\{\kappa(v) | v \in V\}
\]

where \( \kappa(v) \) is the number of spanning trees of \( G \) oriented toward \( v \). We will see that \( M_G \), which we will call the Pham index of the graph \( G \), can be interpreted as a measure of “Eulerianness”.

A finite directed multigraph \( G \) is called Eulerian if it has an Eulerian tour (a closed path that traverses each directed edge exactly once). We are going to take the view that Eulerianness is an algebraic property of the graph Laplacian \( \Delta \) acting on integer-valued functions \( f \in \mathbb{Z}^V \) by

\[
\Delta f(v) = d_v f(v) - \sum_{\text{head}(e)=v} f(\text{tail}(e)).
\]

Here \( d_v \) is the outdegree of vertex \( v \). The context is the following well-known equivalence, where \( 1 \) denotes the constant function \( 1(v) = 1 \) for all \( v \in V \).
Proposition 1.1. The following are equivalent for a strongly connected directed multigraph \( G = (V, E) \).

1. \( \ker(\Delta : \mathbb{Z}^V \to \mathbb{Z}^V) = \mathbb{Z}1 \).
2. \( M_G = \kappa(v) \) for all \( v \in V \).
3. \( G \) is Eulerian.

Our main result is in some sense dual to Proposition 1.1: it gives several equivalent characterizations of the graphs with Pham index 1. These coEulerian graphs are the farthest from being Eulerian.

Our motivation for considering coEulerian graphs and the Pham index comes from chip-firing, which we now describe. A chip configuration on \( G \), or simply configuration for short, is a function \( \sigma : V \to \mathbb{Z} \). If \( \sigma(v) > 0 \) we think of a pile of \( \sigma(v) \) chips at vertex \( v \), and if \( \sigma(v) < 0 \) we think of a hole waiting to be filled by chips. Denoting by \( d_v \) the outdegree of vertex \( v \), we say that \( v \) is stable for \( \sigma \) if \( \sigma(v) < d_v \), and active for \( \sigma \) otherwise. A vertex \( v \) can fire by sending one chip along each outgoing edge, resulting in the new configuration

\[
\sigma' = \sigma - \Delta \delta_v
\]

where \( \Delta \) is the graph Laplacian (1) and \( \delta_v(w) \) is 1 if \( v = w \) and 0 otherwise. Concretely, we may think of \( \sigma, \sigma' \) as column vectors and \( \Delta \delta_v \) as a column of the matrix

\[
\Delta_{vw} = \begin{cases} 
-d_{vw}, & v \neq w \\
-1 & v = w
\end{cases}
\]

where \( d_{vw} \) denotes the number of directed edges of \( G \) from \( w \) to \( v \). More generally, we can specify a firing vector \( x \in \mathbb{N}^V \) and fire each vertex \( v \) a total of \( x(v) \) times, resulting in the configuration \( \sigma' = \sigma - \Delta x \). Here and throughout, \( \mathbb{N} = \{0, 1, 2, \ldots\} \).

A legal firing sequence is a finite sequence of configurations \( \sigma_0, \ldots, \sigma_k \) such that each \( \sigma_i \) for \( i = 1, \ldots, k \) is obtained from \( \sigma_{i-1} \) by firing a vertex that is active for \( \sigma_{i-1} \). A configuration \( \sigma \) is called stable if \( \sigma(v) < d_v \) for all \( v \in V \). We say that \( \sigma \) stabilizes if there is legal firing sequence \( \sigma = \sigma_0, \ldots, \sigma_k \) such that \( \sigma_k \) is stable.

Björner, Lovász, and Shor posed the following problem in 1991 [6].

The halting problem for chip-firing:

Given the adjacency matrix of a finite, strongly connected multigraph \( G \) and a chip configuration \( \sigma \) on \( G \) with \( \sigma \geq 0 \),

Decide whether \( \sigma \) stabilizes.

Write \( |\sigma| = \sum_{v \in V} \sigma(v) \) for the total number of chips. This quantity is conserved by firing (since \( |\Delta \delta_v| = 0 \) for all \( v \in V \)). The maximal stable configuration

\[
\sigma_{\max}(v) = d_v - 1
\]

has \( |\sigma_{\max}| = \#E - \#V \). By the pigeonhole principle, any configuration \( \sigma \) with \( |\sigma| > \#E - \#V \) has at least one unstable vertex, so such \( \sigma \) does not stabilize. A natural question arises: Which directed graphs have the property that every chip configuration of \( \#E - \#V \) chips stabilizes? These graphs are the subject of our main result.
We write $\mathbb{Z}_0^V$ for the set of $\sigma \in \mathbb{Z}^V$ such that $|\sigma| = 0$.

**Theorem 1.2.** The following are equivalent for a strongly connected directed multigraph $G = (V, E)$.

1. $\text{Im}(\Delta : \mathbb{Z}^V \to \mathbb{Z}^V) = \mathbb{Z}_0^V$.
2. $M_G = 1$.
3. A chip configuration $\sigma$ on $G$ stabilizes if and only if $|\sigma| \leq \#E - \#V$.
4. For some $s \in V$, the sandpile group $K(G, s)$ is cyclic with generator $\overline{\beta}_s$.
5. For all $s \in V$, the sandpile group $K(G, s)$ is cyclic with generator $\overline{\beta}_s$.

Items (1) and (2) are in some sense dual to their counterparts in Proposition 1.1, so we propose the term **coEulerian** for a graph satisfying the equivalent conditions of Theorem 1.2. The sandpile group $K(G, s)$ and $\overline{\beta}_s$ are defined below in Section 2.

**Figure 1.** Example of a coEulerian graph: a path of length $n$ with edge multiplicities 2 to the right and 3 to the left. It has $\kappa(v) = \pi(v) = 2^v 3^{n-v}$ spanning trees oriented toward $v$, so its Pham index is $M_G = \gcd(2^n, 2^{n-1}3, \ldots, 3^n) = 1$.

**1.1. History.** One of the earliest results in chip-firing is the following observation of Tardos.

**Lemma 1.3.** [28, Lemma 4] Let $\sigma$ be a configuration on an undirected graph $G$. If there is a legal firing sequence for $\sigma$ in which every vertex of $G$ fires at least once, then $\sigma$ does not stabilize.

Tardos used Lemma 1.3 to prove that for any configuration $\sigma$ on an undirected $n$-vertex graph, if $\sigma$ stabilizes then it does so in $O(n^4)$ firings. Eriksson showed, however, that on a directed graph a configuration may require an exponential number of firings to stabilize [14]. Björner and Lovász [5] generalized the “at least once” condition of Lemma 1.3 to directed graphs as follows.

**Lemma 1.4.** [5] For every strongly connected multigraph $G$ there is a unique primitive $\pi \in \mathbb{N}^V$ such that $\Delta \pi = 0$. If there is a legal firing sequence for $\sigma$ in which every vertex $v$ fires at least $\pi(v)$ times, then $\sigma$ does not stabilize.

This gives a procedure for deciding the **HALTING PROBLEM FOR CHIP-FIRING**: perform legal firings in any order until either you reach a stable configuration or the criterion of Lemma 1.4 certifies that $\sigma$ will not stabilize. There is only one problem: the values $\pi(v)$ may be exponentially large. Figure 1 shows a coEulerian graph on vertex set $\{0, 1, \ldots, n\}$ with $\pi(v) = 2^v 3^{n-v}$. The algorithm just described would run for exponential time on this graph, but Theorem 1.2 gives a much faster algorithm to
decide the halting problem for chip-firing on any coEulerian graph: count the total number of chips and compare to \( \#E - \#V \). As far as we are aware, this is the first progress on the halting problem for chip-firing on directed graphs since the work of Björner and Lovász [5].

1.2. Related work. Pham [24] introduced the index \( M_G \) to answer a question posed in [18]: Which directed graphs \( G \) have the property that all unicycles of \( G \) lie in the same orbit of the rotor-router operation? He showed that \( G \) has this property if and only if \( M_G = 1 \), and that in general the number of orbits is \( M_G \).

The Halting Problem for Chip-Firing is a special case of the halting problem for a class of automata networks called abelian networks. A polynomial time algorithm to decide if a given abelian network halts on all inputs appears in [8], where it is remarked that the problem of deciding whether a given abelian network halts on a given input is a subtler problem. The Halting Problem for Chip-Firing is of this latter type (the “input” to the abelian network is the chip configuration \( \sigma \)).

The next section is devoted to the proofs of Proposition 1.1 and Theorem 1.2. In Section 3 we define a notion of multiEulerian tour for directed graphs, and count such tours. In Section 4 we show that despite its being easy for Eulerian and coEulerian graphs, the Halting Problem for Chip-Firing on finite directed multigraphs is NP-complete in general. One ingredient in the proof is Theorem 4.1, which expresses an arbitrary \((n - 1)\)-dimensional lattice in \( \mathbb{Z}_0^n \) as the Laplacian lattice of a strongly connected multigraph.

2. Sandpiles and the Halting Problem

To prove Theorem 1.2 we will compare chip-firing with and without a sink vertex. This kind of comparison appears also in the study of the abelian sandpile threshold state [21], and in the extension of the Biggs-Merino polynomial to Eulerian graphs [23] and to all strongly connected graphs [10]. Sections 2.1 and 2.2 review the relevant background on chip-firing and the sandpile group. In Section 2.3 we relate the sandpile groups with and without sink, and in Section 2.4 we prove the results stated in the introduction.

2.1. Background. The following result frees us from considering only legal firing sequences in looking for an answer to the halting problem for chip-firing.

**Lemma 2.1.** (Least Action Principle, [7, Lemma 4.3]) Let \( \sigma \) be a chip configuration on a finite directed graph. Then \( \sigma \) stabilizes if and only if there exists an \( x \in \mathbb{N}^V \) such that \( \sigma - \Delta x \) is stable.

A sizable portion of the ground soon to be covered is motivated by the following principle: in looking for a stabilizing firing sequence, instead of firing willy-nilly we can establish some structure by designating a special vertex \( s \) as the sink, which fires only if no other vertex is active. We fire active, nonsink vertices until all nonsink vertices are stable. At this point if the sink is stable we are done; otherwise, we fire the sink (once) and repeat.

The reduced Laplacian \( \Delta_s \) is the matrix obtained by deleting the row and column of \( \Delta \) corresponding to the sink \( s \). To emphasize the distinction between \( \Delta \) and \( \Delta_s \), we
will sometimes refer to \( \Delta \) as the **total Laplacian**. In what follows we will sometimes identify the vertex set \( V \) with \( \{1, \ldots, n\} \) and set \( s = n \).

**Definition 2.2.** Let \( G = (V, E) \) be a finite strongly connected multigraph and fix \( s \in V \). The **sandpile group** of \( G \) with sink \( s \) is the group quotient

\[
K(G, s) = \mathbb{Z}^{n-1}/\Delta_s \mathbb{Z}^{n-1}
\]

where \( \Delta_s \mathbb{Z}^{n-1} \) is the integer column-span of \( \Delta_s \).

A **sandpile** is a chip configuration \( \eta \in \mathbb{Z}^{n-1} \) indexed by the nonsink vertices. When we wish to emphasize that a chip configuration is defined also at the sink, we call it a **total configuration**. One can imagine that a sandpile is composed of sand grains which behave just like chips except that they are small enough to disappear down the sink. The definitions “stable” and “firing vector” have obvious analogues for sandpiles: a sandpile \( \eta \) is stable if \( \eta(i) < d_i \) for all \( v_i \neq s \); and firing vectors for sandpiles live in \( \mathbb{Z}^{n-1} \). The sandpile group treats two sandpiles as equivalent if one can be obtained from the other by firing nonsink vertices. We write \( \overline{\eta} \) for the equivalence class of \( \eta \) in \( K(G, s) \).

On a strongly connected graph, every sandpile stabilizes, and its stabilization does not depend on the order of firings [18, Lemmas 2.2 and 2.4]; we denote the stabilization of \( \eta \) by \( \eta^\circ \). Next we recall the connection between sandpiles and spanning trees.

**Definition 2.3.** An **oriented spanning tree** of a directed graph \( G = (V, E) \) rooted at \( s \in V \) is a spanning subgraph \( T = (V, A) \) such that

1. Every vertex \( v \neq s \) has outdegree 1 in \( T \).
2. \( s \) has outdegree 0 in \( T \).
3. \( T \) has no oriented cycles.

Hence an oriented spanning tree has as its limbs edges that point toward the root. Let \( \kappa(s) \) denote the number of oriented spanning trees in \( G \) rooted at \( s \).

**Theorem 2.4.** (Matrix tree theorem [27, Theorem 5.6.8] and [18, Lemma 2.8]) For a finite strongly connected multigraph \( G \) and a vertex \( s \),

\[
\kappa(s) = \det \Delta_s = \#K(G, s).
\]

Note that if \( G \) is strongly connected then it has at least one spanning tree rooted at \( s \), so \( \Delta_s \) is invertible; since the rows of \( \Delta \) sum to 0, this implies that \( \Delta \) has rank \( n - 1 \).

There is a natural representative for each equivalence class of \( K(G, s) \). To describe this representative, we say that a sandpile \( \eta \) is **accessible** if from any other sandpile it is possible to obtain \( \eta \) by adding a nonnegative number of sand grains at each vertex and then selectively firing active vertices. A sandpile that is both stable and accessible is called **recurrent**.

**Theorem 2.5.** [18, Cor. 2.16] The set \( \text{Rec}(G, s) \) of all recurrent sandpiles is an abelian group under the operation

\[
\eta \oplus \xi := (\eta + \xi)^\circ
\]
and it is isomorphic via the inclusion map to the sandpile group $K(G, s)$.

The **recurrent identity element** $e_s \in \text{Rec}(G, s)$ is the unique recurrent sandpile in $\Delta_s \mathbb{Z}^{n-1}$. The recurrent representative $\eta_{\text{rec}}$ of a sandpile $\eta$ can be found by adding the identity and stabilizing:

$$\eta_{\text{rec}} = (\eta + e_s)^\circ.$$  

Dhar’s burning test [11] determines whether a given sandpile on an Eulerian graph is recurrent. Speer [26] generalized the burning test to directed graphs. Dhar’s and Speer’s tests are closely related to Lemmas 1.3 and 1.4 respectively.

### 2.2. Cyclic subgroups of the sandpile group.

For $s, v \in V$ let $\beta_s(v) = d_{sv}$, the number of directed edges from $s$ to $v$. In accordance with our principle of controlled sink firing, given a recurrent sandpile $\eta$ we are interested in

$$C_\eta = \{ (\eta + k\beta_s)^\circ : k \in \mathbb{N} \},$$

the set of sandpiles obtainable from $\eta$ by firing the sink $s$ some nonnegative number of times and then stabilizing. Note that starting with a recurrent sandpile, adding sand grains to the nonsink vertices and then stabilizing results in another recurrent sandpile; so all sandpiles in $C_\eta$ are recurrent. Note that

$$(\eta + \beta_s)^\circ = (\eta + e_s + \beta_s)^\circ = \eta \oplus \gamma_s$$

where $\gamma_s = (e_s + \beta_s)^\circ$ is the recurrent representative of $\beta_s$. It follows that

$$C_\eta = \eta \oplus \langle \gamma_s \rangle$$

where $\langle \gamma_s \rangle$ denotes the cyclic subgroup of $\text{Rec}(G, s)$ generated by $\gamma_s$.

To investigate these cosets of $\langle \gamma_s \rangle$, we recall the period vector introduced by Björner and Lovász.

**Definition 2.6.** [5] Given a graph $G$ with total Laplacian $\Delta$, a vector $p \in \mathbb{N}^n$ is called a **period vector** for $G$ if $\Delta p = 0$. A period vector is **primitive** if the greatest common divisor of its entries is 1.

In other words, a period vector $p$ has the property that firing each vertex $v \in V$ a total of $p(v)$ times results in no net movement of chips. The following lemma sums up some useful properties of period vectors.

**Lemma 2.7.** [5, Prop. 4.1] A strongly connected multigraph $G$ has a unique primitive period vector $\pi_G$. All entries of $\pi_G$ are strictly positive, and all period vectors of $G$ are of the form $k\pi_G$ for $k = 1, 2, \ldots$. Moreover, if $G$ is Eulerian, then $\pi_G = 1$.

A consequence of the strict positivity of $\pi_G$ that we will use several times is that $\Delta \mathbb{Z}^n = \Delta \mathbb{N}^n$.

We now introduce a very special period vector. Recall that $\kappa(v)$ denotes the number of spanning trees of $G$ oriented toward $v$.

**Lemma 2.8** ([1, 9]). $\Delta \kappa = 0$. 
Recall the **Pham index** \( M = M_G \), defined as the greatest common divisor of the spanning tree counts \( \{ \kappa(v) | v \in V \} \). By Lemmas 2.7 and 2.8, the vector \( \pi = \frac{1}{M} \kappa \) is the unique primitive period vector of \( G \).

Next we argue that \( \pi(s) = \text{ord}(\gamma_s) \), the order of \( \gamma_s \) in the group \( \text{Rec}(G, s) \). Noting that \( \beta_s \) is the restriction of \(-\Delta \delta_s \) to the nonsink vertices, we have that \( m \beta_s \in \Delta_s \mathbb{Z}^{n-1} \) if and only if there is a vector \( x \in \mathbb{Z}^{n-1} \) such that

\[
\Delta(x + m \delta_s) = 0.
\]

(The equality in the sink coordinate follows from the equality in the nonsink coordinates because the sum of all the coordinates is 0.) Thus \( m \beta_s \in \Delta_s \mathbb{Z}^{n-1} \) if and only if there is a period vector \( p \) with \( p(s) = m \), which by Lemma 2.8 happens if and only if \( \pi(s) \) divides \( m \). Thus \( \pi(s) \) is the order of \( \beta_s \) in \( K(G, s) \), which by Theorem 2.5 is the order of \( \gamma_s \) in \( \text{Rec}(G, s) \). Recalling that \( \pi(s) = \kappa(s)/M \), we conclude the following.

**Lemma 2.9.** [24, Lemma 6] For any choice of sink \( s \), we have that

\[
\text{ord}(\gamma_s) = \kappa(s)/M = \# \text{Rec}(G, s)/M
\]

Thus, \( M = \# \text{Rec}(G, s)/\langle \gamma_s \rangle \) is the number of distinct cosets of \( \langle \gamma_s \rangle \) in \( \text{Rec}(G, s) \).

### 2.3. Comparison of sandpile groups with and without sink

We now investigate the structure of the quotient group \( \text{Rec}(G, s)/\langle \gamma_s \rangle \). Recall that \( \beta_s \) is the sandpile \( \beta_s(v) = d_{sv} \), where \( s \) is the designated sink vertex and that \( \beta_s \) is the equivalence class of \( \beta_s \) in \( K(G, s) \). As before we write \( \mathbb{Z}^n_0 \) for the group of vectors in \( \mathbb{Z}^n \) with coordinates summing to 0.

**Theorem 2.10.** For any strongly connected multigraph \( G \) and any vertex \( s \),

\[
\text{Rec}(G, s)/\langle \gamma_s \rangle \cong K(G, s)/\langle \beta_s \rangle \cong \mathbb{Z}^n_0/\Delta \mathbb{Z}^n.
\]

The meat of the proof for this theorem is packaged in the following workhorse lemma. To translate between sandpiles and total configurations, we introduce some notation: Given a vector \( x \in \mathbb{Z}^n \), we denote by \( \bar{x} \) the restriction of \( x \) to the nonsink vertices; and given \( \eta \in \mathbb{Z}^{n-1} \), we write \( \eta_k \) for the extension of \( \eta \) to \( \mathbb{Z}^n \) such that \( |\eta_k| = k \).

**Lemma 2.11.** Let \( \sigma, \tau \in \mathbb{Z}^n \) with \( |\sigma| = |\tau| \). Then the following are equivalent.

1. \( \sigma \equiv \tau \text{ mod } \Delta \mathbb{Z}^n \)
2. \( \bar{\sigma} \equiv \bar{\tau} \text{ mod } \Delta_s \mathbb{Z}^{n-1} + \mathbb{Z} \beta_s \)
3. \( (\bar{\sigma} + e_s)^\circ \in (\bar{\tau} + e_s)^\circ \oplus \langle \gamma_s \rangle \)

**Proof.** (1 \( \iff \) 2) Assume (1), and let \( m = |\sigma| = |\tau| \). Recall that \( \sigma_k \) denotes the extension of \( \sigma \) to \( \mathbb{Z}^n \) such that \( |\sigma_k| = k \). We observe that (1) holds if and only if there is an \( x \in \mathbb{Z}^n \) such that \( \sigma = \tau - \Delta x \). If \( \sigma = \tau - \Delta x \), then

\[
\sigma = \tau - \Delta x = \tau - \Delta \begin{bmatrix} \bar{x} \\ 0 \end{bmatrix} - \Delta \begin{bmatrix} 0 \\ x(s) \end{bmatrix} = \tau - \Delta \begin{bmatrix} \Delta_s \bar{x} \\ a \end{bmatrix} - x(s)c_s
\]

where \( c_s \) denotes the column of \( \Delta \) corresponding with the sink and \( a \) is the dot product of the \( n \)th row of \( \Delta \) with \( (\bar{x}, 0) \). Since \( \beta_s(i) = -c_s(i) \) for each \( i \neq s \), it follows that
\[ \hat{\sigma} = \hat{\tau} - \Delta_s \hat{x} + x(s)\beta_s. \]

Going the other way, we assume that \( \hat{\sigma} = \hat{\tau} - k\beta_s - \Delta_s \hat{x} \) for some \( k \in \mathbb{N} \) and \( \hat{x} \in \mathbb{N}^{n-1} \). Let \( \sigma' \) be the total configuration

\[ \sigma' = \tau - \Delta \left[ \begin{array}{c} \hat{x} \\ k \end{array} \right]. \]

Then \( \sigma'(i) = \hat{\sigma}(i) \) for all \( i \neq s \) and \( |\sigma'| = |\tau| \). Since \( \sigma(s) \) is determined by \( |\sigma| \) and \( |\sigma| = |\tau| \), we have that \( \sigma' = \sigma \).

(2 \iff 3) Note that (3) is equivalent to the existence of an \( x \) such that \((\hat{\sigma} + e_s)^{\circ} \equiv (\hat{\tau} + e_s)^{\circ} + x(s)(\beta_s + e_s)^{\circ} - \Delta_s \hat{x} \) which in turn is equivalent to the congruence \( \hat{\sigma} \equiv \hat{\tau} \mod \Delta_s \mathbb{Z}^{n-1} + \mathbb{Z}\beta_s \).

**Proof of Theorem 2.10.** Define a map \( \phi : K(G, s)/\langle \beta_s \rangle \to \mathbb{Z}_0^n/\Delta \mathbb{Z}^n \) sending

\[ \overline{\eta} \mod \langle \beta_s \rangle \mapsto \eta_0 \mod \Delta \mathbb{Z}^n. \]

Let \( \eta, \xi \in \mathbb{Z}^{n-1} \). If \( \overline{\eta} \equiv \overline{\xi} \mod \langle \beta_s \rangle \), then by Lemma 2.11 we have that \( \eta_0 \equiv \xi_0 \mod \Delta \mathbb{Z}^n \), so that \( \phi \) is well-defined. The equation \( \eta_0 + \xi_0 = (\eta + \xi)_0 \) is immediate from the definition, so that \( \phi \) is a homomorphism. The map \( \phi \) is also surjective, since for each \( \sigma \in \mathbb{Z}_0^n \) there is a corresponding \( \hat{\sigma} \in \mathbb{Z}^{n-1} \), and \( \phi(\hat{\sigma} \mod \langle \beta_s \rangle) = \sigma \mod \Delta \mathbb{Z}^n \). We now show that \( \phi \) is injective to complete the proof that \( \phi \) is an isomorphism. Suppose that \( \sigma \equiv \tau \mod \Delta \mathbb{Z}^n \). Then by Lemma 2.11 we have that \( \overline{\sigma} \equiv \overline{\tau} \mod \langle \beta_s \rangle \) and the theorem is proved.

In the Eulerian case, we recover the following well-known result.

**Corollary 2.12.** [18, Lemma 4.12] Let \( G \) be a finite Eulerian graph. Then for any vertex \( s \) we have that \( K(G, s) \cong \mathbb{Z}_0^n/\Delta \mathbb{Z}^n \). In particular, the sandpile group of \( G \) is independent of choice of sink up to isomorphism.

**Proof.** Since the outdegree and indegree of \( v \) are equal we see that firing \( v \) and all of its neighbors once leaves \( v \) with the same number of chips as before. Hence \( 1 \) is the primitive period vector for \( G \). Then \( \langle \beta_s \rangle \) is trivial and the isomorphism \( K(G, s) \cong \mathbb{Z}_0^n/\Delta \mathbb{Z}^n \) follows by Theorem 2.10.

### 2.4. Eulerian and CoEulerian Graphs.

We conclude this section by proving the two results stated in the introduction.

**Proof of Proposition 1.1.** (1 \implies 3) The equation \( \Delta 1 = 0 \) implies that the outdegree and indegree of each vertex are equal, so \( G \) has an Eulerian tour by [27, Theorem 5.6.1].

(3 \implies 2) As previously noted, the group \( \langle \gamma_v \rangle \) is trivial for any vertex \( v \) in an Eulerian graph. The implication then follows from Lemma 2.9.

(2 \implies 1) This follows from Lemma 2.8.

**Proof of Theorem 1.2.** (3 \implies 4) We prove the contrapositive. Assume there is a sink \( s \) such that \( K(G, s) \neq \langle \beta_s \rangle \), and fix the number of chips on \( G \) to be \( m = \#E - \#V \). Our assumption implies that there are two distinct cosets \( C_1 \) and \( C_2 \) of \( \langle \beta_s \rangle \) such that \( \sigma_{\max} \in C_1 \). Choosing an \( \eta \in C_2 \), we remark that \( \eta_m \) stabilizes if and only if \( \eta_m \equiv \sigma_{\max} \mod \Delta \mathbb{Z}^n \). (since \( \sigma_{\max} \) is the only stable total configuration with
m chips). This occurs if and only if $\eta \in \bar{\sigma}_{\text{max}} \oplus \langle \beta_g \rangle = C_1$ by Lemma 2.11, so we see that $\eta_m$ does not stabilize.

(4 $\implies$ 3) Suppose $\langle \beta_g \rangle = K(G,s)$, and let $\sigma$ be a total configuration with $|\sigma| \leq \#E - \#V$. Then $\sigma = \tau - \delta$ for some $\tau, \delta \in \mathbb{Z}^n$ where $|\tau| = \#E - \#V$ and $\delta \geq 0$. Observe that $\pi \equiv \bar{\sigma}_{\text{max}} \text{ mod } \Delta \mathbb{Z}^{n-1} + \mathbb{Z} \beta_g$ so that $\pi \equiv \sigma_{\text{max}} \text{ mod } \Delta \mathbb{Z}^n$ by Lemma 2.11. It follows that $\sigma \equiv \sigma_{\text{max}} - \delta \text{ mod } \Delta \mathbb{Z}^n$. Using that $\Delta \mathbb{Z}^n = \Delta \mathbb{N}^n$, we conclude from Lemma 2.1 that $\sigma$ stabilizes.

(2 $\iff$ 4 and 2 $\iff$ 5) These equivalences follow from Lemma 2.9.

(1 $\iff$ 4) This equivalence follows from Theorem 2.10. $\Box$

3. Multi-Eulerian tours

**Definition 3.1.** Fix $\pi \in \mathbb{N}^V$. A $\pi$-Eulerian tour of $G$ is a closed path that uses each directed edge $e$ of $G$ exactly $\pi_{\text{tail}(e)}$ times.

We will see shortly that every strongly connected $G$ has a $\pi$-Eulerian tour for suitable $\pi$, and count the number of such tours. To do this, we recall the BEST theorem (named for its discoverers: de Bruijn, Ehrenfest, Smith and Tutte) counting 1-Eulerian tours of an Eulerian directed multigraph $G$. Write $\epsilon_1(G,e)$ for the number of $\pi$-Eulerian tours of $G$ starting with a fixed edge $e$.

**Theorem 3.2.** (BEST [13, 29]) If $G$ is Eulerian, then

$$\epsilon_1(G,e) = \kappa_v \prod_{v \in V} (d_v - 1)!$$

where $d_v$ is the outdegree of $v$; vertex $v$ is the tail of edge $e$, and $\kappa_v$ is the number of spanning trees of $G$ oriented toward $w$.

The purpose of this section is to record the following generalization of the BEST theorem to all strongly connected graphs. The proof is a straightforward application of the BEST theorem.

**Theorem 3.3.** Let $G = (V,E)$ be a strongly connected directed multigraph with Laplacian $\Delta$, and let $\pi \in \mathbb{N}^V$. Then $G$ has a $\pi$-Eulerian tour if and only if $\Delta \pi = 0$. If $\Delta \pi = 0$, then the number of $\pi$-Eulerian tours starting with edge $e$ is given by

$$\epsilon_\pi(G,e) = \kappa_w \prod_{v \in V} \frac{(d_v \pi_v - 1)!}{(\pi_v - 1)!}$$

where $d_v$ is the outdegree of $v$; vertex $v$ is the tail of edge $e$, and $\kappa_w$ is the number of spanning trees of $G$ oriented toward $w$.

Note that the ratio on the right side is a multinomial coefficient and hence an integer.

**Proof.** We define a multigraph $\tilde{G}$ by replacing each edge $e$ of $G$ from $u$ to $v$ by $\pi_u$ edges $e^1_v, \ldots, e^{\pi_u}_v$ from $u$ to $v$. Each vertex $v$ of $\tilde{G}$ has outdegree $d_v \pi_v$ and indegree $\sum_{u \in V} \pi_u d_{uv}$, so $\tilde{G}$ is Eulerian if and only if $\Delta \pi = 0$.

If $(e^1_1, \ldots, e^m_m)$ is a 1-Eulerian tour of $\tilde{G}$, then $(e_1, \ldots, e_m)$ is a $\pi$-Eulerian tour of $G$. Conversely, for each $\pi$-Eulerian tour of $G$ beginning with a fixed vertex $w$, the
occurrences of each edge $e$ in the tour can be labeled with an arbitrary permutation of $\{1, \ldots, \pi_{\text{tail}(e)}\}$ to obtain a 1-Eulerian tour of $\tilde{G}$. Hence

$$\epsilon_1(\tilde{G}, w) = \epsilon_\pi(G, w) \prod_{v \in V} (\pi_v!)^{d_v}.$$ 

In particular, $G$ has a $\pi$-Eulerian tour if and only if $\tilde{G}$ is Eulerian.

To complete the counting in the case when $\tilde{G}$ is Eulerian, the BEST theorem gives the number of Eulerian tours of $\tilde{G}$ starting with a fixed edge $e^1$ with tail $w$, namely

$$\epsilon_1(\tilde{G}, e^1) = \tilde{\kappa}_w \prod_{v \in V} (d_v \pi_v - 1)!,$$

where

$$\tilde{\kappa}_w = \kappa_w \prod_{v \neq w} \pi_v$$

is the number of spanning trees of $\tilde{G}$ oriented toward $w$, since each spanning tree of $G$ oriented toward $w$ gives rise to $\prod_{v \neq w} \pi_v$ spanning trees of $\tilde{G}$.

By cyclically shifting the tour, the number of $\pi$-Eulerian tours of $G$ starting with a given edge $e = (w, v)$ does not depend on $v$. Hence

$$\frac{\epsilon_\pi(G, e)}{\epsilon_\pi(G, w)} = \frac{1}{d_w}, \quad \frac{\epsilon_1(\tilde{G}, e^1)}{\epsilon_1(\tilde{G}, w)} = \frac{1}{d_w \pi_w}.$$ 

We conclude that

$$\epsilon_\pi(G, e) = \frac{1}{d_w} (d_w \pi_w) \tilde{\kappa}_w \prod_{v \in V} \frac{(d_v \pi_v - 1)!}{(\pi_v!)^{d_v}}$$

which together with (2) completes the proof. \qed

A special class of multi-Eulerian tours are the simple rotor walks [25, 18, 19, 24]. In a simple rotor walk, we specify an ordering of the set of outgoing edges from each vertex $v$, and the successive exits from $v$ repeatedly cycle through this ordering. If $G$ is Eulerian then a simple rotor walk on $G$ eventually settles into an Eulerian tour which it traces repeatedly. More generally, if $G$ is strongly connected then a simple rotor walk on $G$ eventually settles into a $\pi$-Eulerian tour where $\pi$ is the primitive period vector of $G$.

### 4. Computational complexity

Björner and Lovász [5, Corollary 4.9] showed that the halting problem for chip-firing can be decided in polynomial time for Eulerian multigraphs. By Theorem 1.2, it can be decided in linear time for coEulerian multigraphs. The purpose of this section is to show that despite these two easy cases, the problem is NP-complete for general directed multigraphs.

To see that it is in NP, let $\sigma$ be a nonnegative halting chip configuration on a strongly connected directed multigraph $G = (V, E)$, and let $x(v)$ be the number of times vertex $v$ fires. By Lemma 2.1 the vector $x$ is a certificate that $\sigma$ halts. Why does this certificate have polynomial size? By Lemma 1.4 we have $x(v) < \pi(v)$ for
some vertex $v$. Moreover for any directed edge $(u_1, u_2)$ the vertex $u_2$ receives at least $x(u_1)$ chips from $u_1$ and so $u_2$ fires at least $x(u_1)/d_2$ times. For any vertex $u$, by inducting along a path from $u$ to $v$ we find that $x(u) \leq D x(v)$ where $D = \prod_{w \in V} d_w$ is the product of all outdegrees. By Lemmas 2.7 and 2.8 relating the primitive period vector $\pi$ to the spanning tree count vector $\kappa$, we have $\pi(v) \leq \kappa(v) \leq D$, so all entries of $x$ are at most $D^2$. Noting that $\log D \leq \sum_{u,v \in V} \log d_{uv}$, which is the size of description of the adjacency matrix, we conclude that the halting problem for chip-firing is in $NP$.

To show that it is also $NP$-hard, our starting point is the following decision problem considered by Amini and Manjunath [2].

**Nonnegative rank:**

**Given** a basis of an $(n-1)$-dimensional lattice $L \subset \mathbb{Z}_0^n$ and a vector $\sigma \in \mathbb{Z}^n$

**Decide** whether there is a vector $\tau \in \mathbb{N}^n$ such that $\sigma - \tau \in L$.

If there exists such a $\tau$, then $\sigma$ is said to have nonnegative rank relative to $L$. In [2, Theorem 7.2] NONNEGATIVE RANK is shown to be $NP$-hard by reducing from the problem of deciding whether a given simplex with rational vertices contains an integer point. (To give a little context, the term “rank” is inspired by the Riemann-Roch theorem of Baker and Norine [4]. Asadi and Backman [3] extend parts of the Baker-Norine theory to directed graphs. Kiss and Tóthméresz [20] show that computing the Baker-Norine rank—a harder problem than deciding whether it is nonnegative—is already $NP$-hard when $L$ is the Laplacian lattice of a simple undirected graph.)

The link between chip-firing and NONNEGATIVE RANK is provided by the following variant of a theorem of Perkinson, Perlman and Wilmes [22].

**Theorem 4.1.** Given an $(n-1)$-dimensional lattice $L \subset \mathbb{Z}_0^n$, there exists a strongly connected multigraph with Laplacian $\Delta$ such that

$$L = \Delta \mathbb{Z}^n.$$ 

Moreover, $\Delta$ can be computed from a basis of $L$ in polynomial time, and all entries of $\Delta$ are bounded in absolute value by $nd$ where $d = \det L$.

The inspiration for Theorem 4.1 is [22, Theorem 4.11], which expresses an arbitrary $(n-1)$-dimensional lattice in $\mathbb{Z}^{n-1}$ as a reduced Laplacian lattice $\Delta_s \mathbb{Z}^{n-1}$. Modifying its proof to express $L \subset \mathbb{Z}_0^n$ as a total Laplacian lattice is straightforward; we give the details below.

In our application it will be essential to compute the Laplacian matrix $\Delta$ from a basis of $L$ in polynomial time (in the length of description of the basis). It is not evident whether [22, Algorithm 4.13] runs in polynomial time, due to possible blow up of the matrix entries in repeated applications of the Euclidean algorithm [15, 16]. As detailed below, this numerical blow up can be avoided by the usual trick of computing modulo the determinant $d$.

To see how we will apply Theorem 4.1, note that strong connectivity implies

$$\Delta \mathbb{Z}^n = \Delta \mathbb{N}^n.$$
since the period vector of Lemma 2.7 is strictly positive. Thus, a vector $\sigma \in \mathbb{Z}^n$ has nonnegative rank relative to $L = \Delta \mathbb{Z}^n$ if and only if there exists $x \in \mathbb{N}^n$ such that

$$\sigma + \Delta x \geq 0.$$ 

Now by Lemma 2.1, such an $x$ exists if and only if the chip configuration $\sigma_{\text{max}} - \sigma$ stabilizes. To summarize, a polynomial time computation of $\Delta$ given a basis for $L$ yields a polynomial time Karp reduction from nonnegative rank to the halting problem for chip-firing on a finite directed multigraph, showing that the latter is NP-hard.

**Corollary 4.2.** THE HALTING PROBLEM FOR CHIP-FIRING is NP-complete.

It remains to prove Theorem 4.1. Recall that an $m \times m$ integer matrix $U$ is called unimodular if $\det U = \pm 1$. Any nonsingular square integer matrix $A$ has a Hermite normal form

$$H = AU$$

where $U$ is a unimodular integer matrix, and $H = (h_{ij})$ is lower-triangular with integer entries satisfying

$$0 < h_{ii}, \quad 1 \leq i \leq m$$

$$0 \leq h_{ij} < h_{ii}, \quad 1 \leq j < i \leq m.$$ 

The existence and uniqueness of $H$ was proved by Hermite [17]. The Hermite normal form is useful to us because $H$ can be computed from $A$ in polynomial time [12] and $HZ^m = A(UZ^m) = AZ^m$ by the unimodularity of $U$. Let

$$d = |\det A| = \det H = \prod_{i=1}^{m} h_{ii}.$$ 

We will use the following observations about the column span $AZ^m$.

**Lemma 4.3.** [12, Cor. 2.3] Let $B$ be a lower triangular $m \times m$ matrix whose columns are in $AZ^m$ and whose diagonal entries satisfy $b_{ii} = h_{ii}$ for all $i$. Then $BZ^m = AZ^m$.

**Lemma 4.4.** [12, Prop. 2.5] $dZ^m \subset AZ^m$.

We will apply these lemmas with $m = n - 1$. Note that an $n \times n$ integer matrix is the total Laplacian of a directed multigraph if and only if (i) the entries of each column sum to zero, (ii) the diagonal entries are nonnegative, and (iii) the off-diagonal entries are nonpositive.

Given an $n \times (n - 1)$ integer matrix whose columns are a $\mathbb{Z}$-basis of the $(n - 1)$-dimensional lattice $L \subset \mathbb{Z}^n_0$, let $A$ be the result of removing the last row of $M$. Since each column of $M$ sums to zero, $A$ is nonsingular. Let $H = AU$ be the Hermite normal form of the $(n - 1) \times (n - 1)$ nonsingular matrix $A$.

The hypotheses of Lemma 4.3 are trivially satisfied when $B = H$; and if $B$ satisfies the hypotheses of Lemma 4.4 then by Lemma 4.4 it will continue to do so if we subtract $d$ from an entry below the diagonal. Using this operation we can make the
entries immediately below the diagonal sufficiently negative so that the sum of the entries in each column is nonpositive. Namely, let $B = (b_{ij})$ where

$$b_{ij} = \begin{cases} h_{ij} - k_j d, & i = j + 1 \\ h_{ij}, & \text{else} \end{cases}$$

and $k_j$ for each $j = 1, \ldots, n - 2$ is a nonnegative integer such that

$$(k_j - 1)d < \sum_{i=1}^m h_{ij} \leq k_j d.$$  \hfill (3)

Now let

$$\Delta = \begin{bmatrix} +d & -h_{11} & 0 & 0 & 0 & \cdots & 0 \\ 0 & + & -h_{22} & 0 & 0 & \cdots & 0 \\ 0 & - & + & -h_{33} & 0 & \cdots & 0 \\ 0 & - & - & + & -h_{44} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & - & - & - & - & \cdots & -h_{mm} \\ -d & - & - & - & - & \cdots & +h_{mm} \end{bmatrix}$$

be the $n \times n$ matrix with upper right corner $-B$, the column vector $de_1 - de_n$ appended on the left, and a row appended on the bottom such that the entries of each column sum to zero. By the choice of $k_j$ in (3), the bottom row of $\Delta$ is nonpositive, except for its rightmost entry $h_{mm}$. Therefore $\Delta$ satisfies the conditions (i)-(iii) above. Since the entries immediately above the diagonal of $\Delta$ are negative, as is $\Delta_{n1}$, the matrix $\Delta$ is the Laplacian of a strongly connected multigraph (it has the Hamiltonian cycle $1 \rightarrow n \rightarrow n - 1 \rightarrow \cdots \rightarrow 1$). By Lemma 4.4 the first column of $\Delta$ belongs to $L$. Moreover, since both $L$ and $\Delta \mathbb{Z}^n$ are contained in $\mathbb{Z}_0^n$ and $B \mathbb{Z}^{n-1} = A \mathbb{Z}^{n-1}$, the integer span of the remaining columns of $\Delta$ is $L$. Thus $L = \Delta \mathbb{Z}^n$. Since each entry of $H$ is at most $d = \prod h_{ii}$, each entry of $\Delta$ has magnitude at most $nd$, completing the proof of Theorem 4.1.

4.1. Simple directed graphs. Let us point out a sense in which the NP-hardness of Corollary 4.2 is rather weak: The directed graphs for which the HALTING PROBLEM FOR CHIP-FIRING is hard may have large edge multiplicities. This is because the Laplacian $\Delta$ of Theorem 4.1 may have large entries, which in turn is because the lattice $L$ in a hard instance of NONNEGATIVE RANK has large determinant. An interesting question is whether the HALTING PROBLEM FOR CHIP-FIRING remains NP-hard when restricted to simple directed graphs, those with edge multiplicities in $\{0, 1\}$.

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