JET SCHEMES, ARC SPACES AND THE NASH PROBLEM

SHIHOKO ISHII

1. Introduction

The concepts jet scheme and arc space over an algebraic variety or an analytic space is introduced by Nash in his preprint in 1968 which is later published as [36]. The study of these spaces was further developed by Kontsevich, Denef and Loeser as the theory of motivic integration, see [28], [7], [8], [9], [10],[11]. These spaces are considered as something to represent the nature of the singularities of the base space. In fact, papers [12], [13], [34], [35] by Mustaţă, Ein and Yasuda show that geometric properties of the jet schemes determine certain properties of the singularities of the base space.

In this paper, we provide the beginners with the basic knowledge of these spaces and the Nash problem. One of powerful tools to work on these space is the motivic integration. But this paper does not step into this theory, as there are already very good introductory papers on the motivic integration by A. Craw [5], W. Veys [48] and F. Loeser [32]. We devote into the basic study of geometric structure of arc spaces and jet schemes. We also give an introduction to the Nash problem which was posed in [36].

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Throughout this paper the base field $k$ is algebraically closed field of arbitrary characteristic and a variety is an irreducible reduced scheme of finite type over $k$. A scheme of finite type over $k$ is always separated over $k$.

We omit the proofs of statements whose references are thought to be easily accessible. We assume the reader to have knowledge in the Hartshorne’s textbook [19].

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2. Construction of jet schemes and arc spaces

Definition 2.1. Let $X$ be a scheme of finite type over $k$ and $K \supset k$ a field extension. For $m \in \mathbb{N}$, a $k$-morphism $\text{Spec } K[t]/(t^{m+1}) \to X$ is called an $m$-jet of $X$ and a $k$-morphism $\text{Spec } K[[t]] \to X$ is called an arc of $X$. We denote the unique point of $\text{Spec } K[t]/(t^{m+1})$ by 0, while the closed point of $\text{Spec } K[[t]]$ by 0 and the generic point by $\eta$.

Theorem 2.2. Let $X$ be a scheme of finite type over $k$. Let $\text{Sch}/k$ be the category of $k$-schemes and $\text{Set}$ the category of sets. Define a contravariant functor $F^X_m : \text{Sch}/k \to \text{Set}$ by

$$F^X_m(Z) = \text{Hom}_k(Z \times_{\text{Spec } k} \text{Spec } k[t]/(t^{m+1}), X).$$

Then, $F^X_m$ is representable by a scheme $X_m$ of finite type over $k$, that is

$$\text{Hom}_k(Z, X_m) \cong \text{Hom}_k(Z \times_{\text{Spec } k} \text{Spec } k[t]/(t^{m+1}), X).$$

This $X_m$ is called the space of $m$-jets of $X$ or the $m$-jet scheme of $X$.

This proposition is proved in [4, p. 276]. In this paper, we prove this by a concrete construction for affine $X$ first and then patching them together for a general $X$. For our proof, we need some preparatory discussions.

2.3. Let $X$ be a $k$-scheme. Assume that $F^X_m$ is representable by $X_m$ for every $m \in \mathbb{N}$. Then, for $m < m'$, the canonical surjection $k[t]/(t^{m'+1}) \to k[t]/(t^{m+1})$ induces a morphism

$$\psi_{m', m} : X_{m'} \to X_m.$$

Indeed, the canonical surjection $k[t]/(t^{m'+1}) \to k[t]/(t^{m+1})$ induces a morphism

$$Z \times_{\text{Spec } k} \text{Spec } k[t]/(t^{m'+1}) \leftarrow Z \times_{\text{Spec } k} \text{Spec } k[t]/(t^{m+1}),$$
for an arbitrary $k$-scheme $Z$. Therefore we have a map $\text{Hom}_k(Z \times_{\text{Spec} k} \text{Spec} k[t]/(t^{m'+1}), X) \longrightarrow \text{Hom}_k(Z \times_{\text{Spec} k} \text{Spec} k[t]/(t^{m+1}), X)$ which gives the map

$$\text{Hom}_k(Z, X_{m'}) \longrightarrow \text{Hom}_k(Z, X_m).$$

Take, in particular, $X_{m'}$ as $Z$,

$$\text{Hom}_k(X_{m'}, X_{m'}) \longrightarrow \text{Hom}_k(X_{m'}, X_m)$$

then the image of $id_{X_{m'}} \in \text{Hom}(X_{m'}, X_{m'})$ by this map gives the required morphism.

This morphism $\psi_{m', m}$ is called a truncation map. In particular for $m = 0$, $\psi_{m', 0} : X_{m'} \longrightarrow X$ is denoted by $\pi_{m}$. When we need to specify the scheme $X$, we denote it by $\pi_{X_m}$.

Actually $\psi_{m', m}$ “truncates” a power series in the following sense: A point $\alpha$ of $X_{m'}$ gives an $m'$-jet $\alpha : \text{Spec} K[t]/(t^{m'+1}) \longrightarrow X$, which corresponds to a ring homomorphism $\alpha^* : A \longrightarrow K[t]/(t^{m'+1})$, where $A$ is the affine coordinate ring of an affine neighborhood of the image of $\alpha$. For every $f \in A$, let

$$\alpha^*(f) = a_0 + a_1 t + a_2 t^2 + \cdots + a_m t^m + \cdots + a_{m'} t^{m'},$$

then

$$\psi_{m', m}^*(\alpha)^*(f) = a_0 + a_1 t + a_2 t^2 + \cdots + a_m t^m.$$  

This fact can be seen by letting $Z = \{\alpha\}$ in the above discussion.

As we did already in the above argument, we denote the point of $X_m$ corresponding to $\alpha : \text{Spec} K[t]/(t^{m+1}) \longrightarrow X$ by the same symbol $\alpha$. Then, we should note that $\pi_{m}(\alpha) = \alpha(0)$.

**Proposition 2.4.** Let $f : X \longrightarrow Y$ be a morphism of $k$-schemes of finite type. Assume that the functor $F^X_m$ and $F^Y_m$ are representable by $X_m$ and $Y_m$, respectively. Then the canonical morphism $f_m : X_m \longrightarrow Y_m$ is induced for every $m \in \mathbb{N}$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
X_m & \longrightarrow & Y_m \\
\pi_{X_m} \downarrow & & \downarrow \pi_{Y_m} \\
X & \underset{f}{\longrightarrow} & Y
\end{array}
$$

**Proof.** Let $X_m \times \text{Spec} k[t]/(t^{m+1}) \longrightarrow X$ be the “universal family” of $m$-jets of $X$, i.e., it corresponds to the identity map in $\text{Hom}_k(X_m, X_m)$. By compositing this map and $f : X \longrightarrow Y$, we obtain a morphism

$$X_m \times \text{Spec} k[t]/(t^{m+1}) \longrightarrow Y,$$

which gives a morphism $X_m \longrightarrow Y_m$. Pointwise, this morphism maps an $m$-jet $\alpha \in X_m$ of $X$ to the composite $f \circ \alpha$ which is an $m$-jet
of \( Y \). To see this, just take a point \( \alpha \in X_m \) and see the image of \( \{\alpha\} \times \text{Spec} \ k[t]/(t^{m+1}) \to Y \). The commutativity of the diagram follows from this description.

**Proposition 2.5.** For \( k \)-schemes \( X \) and \( Y \), assume that the functor \( F^X_m \) and \( F^Y_m \) are representable by \( X_m \) and \( Y_m \), respectively. If \( f : X \to Y \) is an étale morphism, then \( X_m \simeq Y_m \times_Y X \), for every \( m \in \mathbb{N} \).

**Proof.** By the above proposition we have a commutative diagram:

\[
\begin{array}{ccc}
X_m & \xrightarrow{f_m} & Y_m \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

It is sufficient to prove that for every commutative diagram:

\[
\begin{array}{ccc}
Z & \to & Y_m \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

there is a unique morphism \( Z \to X_m \) which is compatible with the projections to \( X \) and \( Y_m \). Now we are given the following commutative diagram:

\[
\begin{array}{ccc}
Z & \to & Z \times_{\text{Spec} \ k} \text{Spec} \ k[t]/(t^{m+1}) \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

As \( f \) is étale, there is a unique morphism \( Z \times_{\text{Spec} \ k} \text{Spec} \ k[t]/(t^{m+1}) \to X \) which makes the two triangles commutative. This gives the required morphism:

\[
\square
\]

As a corollary of this proposition, we obtain the following lemma:

**Lemma 2.6.** Let \( U \subset X \) be an open subset of a \( k \)-scheme \( X \). Assume the functors \( F^X_m \) and \( F^U_m \) are representable by \( X_m \) and \( U_m \), respectively. Then, \( U_m = \pi^{-1}_{X_m}(U) \).

**Proof of Theorem 2.2** Since a \( k \)-scheme \( X \) is separated, the intersection of two affine open subsets is again affine. Therefore, by Lemma 2.6, it is sufficient to prove the representability of \( F^X_m \) for affine \( X \). Let \( X \) be \( \text{Spec} \ R \), where we denote \( R = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r) \). It is sufficient to prove the representability for an affine variety \( Z = \text{Spec} \ A \). Then, we obtain that

\[
(2.2.1) \quad \text{Hom}(Z \times \text{Spec} \ k[t]/(t^{m+1}), X) \simeq \text{Hom}(R, A[t]/(t^{m+1}))
\]
If we write \( \varphi(x_j) = a_j^{(0)} + a_j^{(1)} t + a_j^{(2)} t^2 + \ldots + a_j^{(m)} t^m \) for polynomials \( F_i^{(s)} \) in \( a_j^{(l)} \)'s. Then the above set (2.2.1) is represented as follows:

\[
\{ \varphi \in \text{Hom}(k[x_j, x_j^{(1)}, \ldots, x_j^{(m)}] | j = 1, \ldots, n], A) | \varphi(x_j^{(l)}) = a_j^{(l)}, F_i^{(s)}(a_j^{(l)}) = 0 \}
\]

If we write \( X_m = \text{Spec} k[x_j, x_j^{(1)}, \ldots, x_j^{(m)}]/(F_i^{(s)}(x_j^{(l)})) \), the last set is bijective to

\[
\text{Hom}(Z, X_m).
\]

Remark 2.7. The functor \( F_X \) is also representable even for \( k \)-scheme of non-finite type over \( k \). The existence of jet schemes for wider class of schemes is presented in [49].

Example 2.8. For \( X = \mathbb{A}^n_k \), it follows \( X_m = \mathbb{A}^{m+1}_k \). Indeed, this is the case that all \( f_i = 0 \), therefore all \( F_i^{(s)} = 0 \) in the proof of Proposition 2.2.

Example 2.9. Let \( X \) be a hypersurface in \( \mathbb{A}^3_k \) defined by \( f = xy + z^2 = 0 \). Then, \( X_2 \) is defined in \( \mathbb{A}^9_k \) by \( xy + z^2 = x^{(1)} y + x^{(1)} y + 2zz^{(1)} = x^{(2)} y + x^{(1)} y + xy^{(2)} + z^{(1)} z^{(1)} + 2zz^{(2)} = 0 \). One can see that \( X_2 \) is irreducible and not normal. Indeed, as \( X \setminus \{0\} \) is non-singular, \( \pi_2^{-1}(X \setminus \{0\}) \) is a 6-dimensional irreducible variety. On the other hand \( \pi_2^{-1}(0) \) is a hypersurface in \( \mathbb{A}^6 \), and therefore it is of dimension 5. Since \( X_2 \) is defined by 3 equations, every irreducible component of \( X_2 \) has dimension \( \geq 9 - 3 = 6 \). By this, \( \pi_2^{-1}(0) \) does not produce an irreducible component of \( X_m \), which yields the irreducibility of \( X_m \). Looking at the Jacobian matrix, one can see that the singular locus of \( X_2 \) is \( \pi_2^{-1}(0) \) which is of codimension one in \( X_2 \). Therefore, \( X_2 \) is not normal.

Let \( X_1 \) be the 1-jet scheme of \( X \). Then for every closed point \( x \in X \), the set of closed points of \( \pi_1^{-1}(x) \) is the set of morphisms \( \text{Spec} k[t]/(t^2) \to X \) with the image \( x \). This set is the Zariski tangent space of \( X \) at \( x \). Therefore, we can regard \( X_1 \) as the “tangent bundle” of \( X \).
Example 2.10. Let $X$ be a curve defined by $x^2 - y^2 - x^3 = 0$ in $\mathbb{A}^2_k$. Then $\pi_1^{-1}(X \setminus \{0\}) \to X \setminus \{0\}$ is $\mathbb{A}_k^1$-bundle, therefore $\pi_1^{-1}(X_{\text{reg}})$ is 2-dimensional. On the other hand $\pi_1^{-1}(0) \simeq \mathbb{A}_k^2$. Hence, $X_1$ has two irreducible components, $\pi_1^{-1}(X_{\text{reg}})$ and $\pi_1^{-1}(0)$.

Definition 2.11. The system $\{\psi_{m',m} : X_m \to X_m\}_{m<m'}$ is a projective system. Let $X_\infty = \lim_m X_m$ and call it the space of arcs of $X$ or arc space of $X$. Note that $X_\infty$ is not of finite type over $k$ if $\dim X > 0$.

Remark 2.12. One may be afraid that the projective limit scheme $\varprojlim_m X_m$ may not exist. But in our case we need not to worry, since for an affine scheme $X = \text{Spec} A$, the $m$-jet scheme $X_m = \text{Spec} A_m$ is affine for every $m$. Here, the morphisms $\psi_{m',m}^* : A_m \to A_{m'}$ corresponding to $\psi_{m',m}$ are direct system. It is well known that there is a direct limit $A_\infty = \varinjlim_m A_m$ in the category of $k$-algebras. The affine scheme $\text{Spec} A_\infty$ is our projective limit of $X_m$. For a general $k$-scheme $X$, we have only to patch affine pieces $\text{Spec} A_\infty$.

Using the representability of $F^X_m$ we obtain the following universal property of $X_\infty$:

Proposition 2.13. Let $X$ be a scheme of finite type over $k$. Then

$$\text{Hom}_k(Z, X_\infty) \simeq \text{Hom}_k(Z \times_{\text{Spec} k} \text{Spec} k[[t]], X)$$

for an arbitrary $k$-scheme $Z$, where $Z \times_{\text{Spec} k} \text{Spec} k[[t]]$ means the formal completion of $Z \times_{\text{Spec} k} \text{Spec} k[[t]]$ along the subscheme $Z \times_{\text{Spec} k} \{0\}$.

Proof. By the representability of $F^X_m$ we obtain an isomorphism of projective systems:

$$
\begin{align*}
\downarrow & \quad \downarrow \\
\text{Hom}_k(Z, X_{m+1}) & \simeq \text{Hom}_k(Z \times_{\text{Spec} k} \text{Spec} k[t]/(t^{m+2}), X) \\
\downarrow & \quad \downarrow \\
\text{Hom}_k(Z, X_m) & \simeq \text{Hom}_k(Z \times_{\text{Spec} k} \text{Spec} k[t]/(t^{m+1}), X)
\end{align*}
$$

Then, we obtain an isomorphism of the projective limits:

$$\text{Hom}_k(Z, \lim_m X_m) \simeq \text{Hom}_k(\lim_m (Z \times_{\text{Spec} k} \text{Spec} k[t]/(t^{m+1})), X),$$

which gives the required isomorphism. □

Remark 2.14. Consider the isomorphism of Proposition 2.13 in particular the case $Z = \text{Spec} A$ for a $k$-algebra $A$, we obtain

$$\text{Hom}_k(\text{Spec} A, X_\infty) \simeq \text{Hom}_k(\text{Spec} A[[t]], X).$$

Here, we note that in general

$$A \otimes_k k[[t]] \not\simeq A[[t]] \simeq A \hat{\otimes}_k k[[t]],$$
where $A \hat{\otimes}_k k[[t]]$ is the completion of $A \otimes_k k[[t]]$ by the ideal $(t)$. Indeed, for example, for $A = k[x]$, the ring $A[[t]]$ contains $\sum_{i=0}^{\infty} f_i(x)t^i$ such that $\deg f_i$ are unbounded, while $A \otimes_k k[[t]]$ does not contain such an element.

Now, consider the case $A = K$ for an extension field $K \supset k$, the bijection

$$\text{Hom}_k(\text{Spec} K, X_\infty) \simeq \text{Hom}_k(\text{Spec} K[[t]], X)$$

shows that a $K$-valued point of $X_\infty$ is an arc $\text{Spec} K[[t]] \to X$.

**Definition 2.15.** Denote the canonical projection $K \to X$ induced from the surjection $k[[t]] \to k[t]/(t^{m+1})$ by $\psi_m$ and the composite $\pi_m \circ \psi_m$ by $\pi$. When we need to specify the base space $X$, we write it by $\pi_X$.

A point $x \in X_\infty$ gives an arc $\alpha_x : \text{Spec} K[[t]] \to X$ and $\pi(x) = \alpha_x(0)$, where $K$ is the residue field at $x$. As the case of $m$-jets, we denote both $x \in X_\infty$ and $\alpha$ corresponding to $x$ by the same symbol $\alpha$.

For every $m \in \mathbb{N}$, $\psi_{\mu}(X_\infty)$ is a constructible set, since $\psi_m(X_\infty) = \psi_m(x, (X_
))$ for sufficiently big $m'$ ([18]).

**Definition 2.16.** Denote the canonical morphism $X \to X_m$ induced from the inclusion $k \hookrightarrow k[t]/(t^{m+1})$ ($m \in \mathbb{N} \cup \{\infty\}$) by $\sigma_m$. Here, we define $k[t]/(t^{m+1}) = k[[t]]$ for $m = \infty$. As $k \hookrightarrow k[t]/(t^{m+1})$ is a section of the projection $k[t]/(t^{m+1}) \to k$, our morphism $\sigma_m : X \to X_m$ is a section of $\pi_m : X_m \to X$.

For a point $x \in X$, let $K$ be the residue field at $x$, then $\sigma_m(x) : \text{Spec} K[t]/(t^{m+1}) \to X$ is an $m$-jet which factors through $\text{Spec} K \to X$ whose image is $x$. Therefore, $\sigma_m(x)$ is the constant $m$-jet at $x$, this is denoted my $x_m$.

**Example 2.17.** If $X = \mathbb{A}^n_k$, then $X_\infty = \text{Spec} k[x_j, x_j^{(1)}, x_j^{(2)} \ldots | j = 1, \ldots, n]$ which is isomorphic to $\mathbb{A}^\infty_k = \text{Spec} k[x_1, x_2, \ldots, x_i, \ldots]$. Here, we note that the set of closed points of $\mathbb{A}^\infty_k$ does not necessarily coincide with the set

$$k^\infty := \{(a_1, a_2, \ldots) | a_i \in k\}$$

(see the following theorem).

**Theorem 2.18** ([23], Proposition 2.10, 2.11). Every closed point of $\mathbb{A}^\infty_k$ is a $k$-valued point if and only if $k$ is not a countable field.

The concept “thin” in the following is first introduced in [14].

**Definition 2.19.** Let $X$ be a variety over $k$. We say that an arc $\alpha : \text{Spec} K[[t]] \to X$ is thin if $\alpha$ factors through a proper closed subvariety of $X$. An arc which is not thin is called a fat arc.
An irreducible subset \( C \) in \( X_\infty \) is called a \textit{thin set} if \( C \) is contained in \( Z_\infty \) for a proper closed subvariety \( Z \subset X \). An irreducible subset in \( X_\infty \) which is not thin is called a \textit{fat set}.

In case an irreducible subset \( C \) has the generic point \( \gamma \in C \) (i.e., the closure \( \overline{C} \) contains \( C \)), \( C \) is a fat set if and only if \( \gamma \) is a fat arc.

The following is proved in [24, Proposition 2.5]:

**Proposition 2.20** ([24] Proposition 2.5). Let \( X \) be a variety over \( k \) and \( \alpha : \text{Spec} \, K[[t]] \to X \) an arc. Then, the following hold:

(i) \( \alpha \) is a fat arc if and only if the ring homomorphism \( \alpha^* : \mathcal{O}_{X, \alpha(0)} \to K[[t]] \) induced from \( \alpha \) is injective;

(ii) Assume that \( \alpha \) is fat. For an arbitrary proper birational morphism \( \varphi : Y \to X \), \( \alpha \) is lifted to \( Y \).

**Remark 2.21.** A fat set in \( X_\infty \) for a variety \( X \) introduces a discrete valuation on the rational function field \( K(X) \) of \( X \). We do not give the construction of the valuation here. The reader may refer [24]. A Nash component (see the next section) is a fat set and the Nash map (see the next section) is just the correspondence to associate a fat set to the valuation induced from the fat set ([24]).

**Example 2.22.** One of typical examples of fat sets is an irreducible cylinder (i.e., the pull back \( \psi_m^{-1}(S) \) of a constructible set \( S \subset X_m \)) for a non-singular \( X \). Actually, take an \( m \)-jet \( \alpha_m : \text{Spec} \, k[t]/(t^{m+1}) \to X \) in \( C \), then, at a neighborhood of \( x = \alpha_m(0) = \pi_m(\alpha_m) \), \( X \) is étale over \( \mathbb{A}^n_k \). Therefore, we may assume that \( X = \mathbb{A}^n_k \) and \( x = 0 \). Assume that \( \psi_m^{-1}(\alpha_m) \) is thin, then it is contained in \( Z_\infty \) for some proper closed subset \( Z \subset X \). Let the \( m \)-jet \( \alpha_m \) corresponds to a ring homomorphism

\[
\alpha_m^* : k[x_1, \ldots, x_n] \to k[t]/(t^{m+1}), \quad \alpha_m^*(x_i) = \sum_{j=1}^{m} a_i^{(j)} t^j.
\]

Let \( x_i^{(j)} \) be an indeterminate for every \( i = 1, \ldots, n \) and \( j \geq m+1 \). Let \( \alpha^* : k[x_1, \ldots, x_n] \to k(x_i^{(j)} \mid i = 1, \ldots, n, j \geq m+1)[[t]] \) be an arc defined by

\[
\alpha^*(x_i) = \sum_{j=1}^{m} a_i^{(j)} t^j + \sum_{j=m+1}^{\infty} x_i^{(j)} t^j.
\]

Let \( \alpha^*(f) = F_0(a_i^{(j)}, x_i^{(j)}) + F_1(a_i^{(j)}, x_i^{(j)})t + \cdots + F_\ell(a_i^{(j)}, x_i^{(j)})t^\ell + \cdots \) for \( f \in I_Z \). Then, as \( x_i^{(j)} \)'s are indeterminates there is \( \ell \) such that \( F_\ell \neq 0 \). Hence, we obtain \( \alpha \in \psi_m^{-1}(C) \) such that \( \alpha \not\in Z_\infty \).
Example 2.23 ([6]). For a singular variety $X$, an irreducible cylinder is not necessarily fat. Indeed, let $X$ be the Whitney Umbrella that is a hypersurface defined by $xy^2 - z^2 = 0$ in $\mathbb{A}^3_k$. For $m \geq 1$, let
\[
\alpha_m^*: k[x, y, z]/(xy^2 - z^2) \longrightarrow k[t]/(t^{m+1})
\]
be the $m$-jet defined by $\alpha_m(x) = t, \alpha_m(y) = 0, \alpha_m(z) = 0$. Then, the cylinder $\psi_m^{-1}(\alpha_m)$ is contained in $\text{Sing}(X)_\infty$, where $\text{Sing}(X) = (y = z = 0)$. This is proved as follows: Let an arbitrary $\alpha \in \psi_m^{-1}(\alpha_m)$ be induced from
\[
\alpha^*: k[x, y, z] \longrightarrow k[[t]]
\]
with
\[
\alpha^*(x) = \sum_{j=1}^{\infty} a_j t^j, \alpha^*(y) = \sum_{j=1}^{\infty} b_j t^j, \alpha^*(z) = \sum_{j=1}^{\infty} c_j t^j,
\]
where we note that $a_1 = 1$. Then, the condition $\alpha^*(xy^2 - z^2) = 0$ implies that the initial term of $\alpha^*(xy^2)$ and that of $\alpha^*(z^2)$ cancel each other. If $\alpha^*(y) \neq 0$, then the order of $\alpha^*(xy^2)$ is odd, while if $\alpha^*(z) \neq 0$, the order of $\alpha^*(z^2)$ is even. Hence if $\alpha^*(y) \neq 0$ or $\alpha^*(z) \neq 0$, then the initial term of $\alpha^*(xy^2)$ and that of $\alpha^*(z^2)$ do not cancel each other. Therefore, $\alpha^*(y) = \alpha^*(z) = 0$, which shows that $\psi_m^{-1}(\alpha_m) \subset \text{Sing}(X)_\infty$.

3. Properties of jet schemes and arc spaces

3.1. Consider $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\} = \text{Spec} k[s, s^{-1}]$ as a multiplicative group scheme. For $m \in \mathbb{N} \cup \{\infty\}$, the morphism $k[t]/(t^{m+1}) \longrightarrow k[s, s^{-1}, t]/(t^{m+1})$ defined by $t \mapsto s \cdot t$ gives an action
\[
\mu_m: \mathbb{G}_m \times_{\text{Spec} k} \text{Spec} k[t]/(t^{m+1}) \longrightarrow \text{Spec} k[t]/(t^{m+1})
\]
of $\mathbb{G}_m$ on $\text{Spec} k[t]/(t^{m+1})$. Therefore, it gives an action
\[
\mu_{Xm}: \mathbb{G}_m \times_{\text{Spec} k} X_m \longrightarrow X_m
\]
of $\mathbb{G}_m$ on $X_m$. As $\mu_m$ is extended to a morphism: $\pi_m: \mathbb{A}^1 \times_{\text{Spec} k} \text{Spec} k[t]/(t^{m+1}) \longrightarrow \text{Spec} k[t]/(t^{m+1})$, we obtain the extension
\[
\pi_{Xm}: \mathbb{A}^1 \times_{\text{Spec} k} X_m \longrightarrow X_m
\]
of $\mu_{Xm}$.

Note that $\pi_{Xm}(\{0\} \times \alpha) = x_m$, where $x_m$ is the trivial $m$-jet on $x = \alpha(0) \in X$. Therefore, every orbit $\mu_{Xm}(\mathbb{G}_m \times \{\alpha\})$ contains the trivial $m$-jet on $\alpha(0)$ in its closure.

Proposition 3.2. For $m \in \mathbb{N} \cup \{\infty\}$, let $Z \subset X_m$ be an $\mathbb{G}_m$-invariant closed subset. Then the image $\pi_m(Z)$ is closed in $X$. In particular the image $\pi_m(Z)$ of an irreducible component of $Z \subset X_m$ is closed in $X$. 
Proof. Let $Z \subset X_m$ be an $\mathbb{G}_m$-invariant closed subset. Then, we obtain:

$$\overline{\pi}_{Xm}(\mathbb{A}^1 \times Z) = Z.$$ 

On the other hand, $\overline{\pi}_{Xm}(\{0\} \times Z) = \sigma_m \circ \pi_m(Z)$ by 3.1. Therefore, as $Z$ is closed, it follows that

$$Z \supset \sigma_m \circ \pi_m(Z) \supset \sigma_m(\pi_m(Z)),$$

which yields $\pi_m(Z) \supset \pi_m(Z)$. □

**Proposition 3.3.** Let $f : X \longrightarrow Y$ be a morphism of $k$-schemes of finite type. Then the canonical morphism $f_\infty : X_\infty \longrightarrow Y_\infty$ is induced such that the following diagram is commutative:

$$
\begin{array}{ccc}
X_\infty & \longrightarrow & Y_\infty \\
\pi_{Xm} \downarrow & & \downarrow \pi_{Ym} \\
X & \longrightarrow & Y
\end{array}
$$

Proof. The morphism $f_\infty$ is induced as the projective limit of $f_m$ ($m \in \mathbb{N}$). □

**Proposition 3.4.** Let $f : X \longrightarrow Y$ be a proper birational morphism of $k$-schemes of finite type such that $f|_{X \setminus W} : X \setminus W \simeq Y \setminus V$, where $W \subset X$ and $V \subset Y$ are closed. Then $f_\infty$ gives a bijection

$$X_\infty \setminus W_\infty \longrightarrow Y_\infty \setminus V_\infty.$$

Proof. Let $\alpha \in Y_\infty \setminus V_\infty$, then $\alpha(\eta) \in X \setminus V$. As $X \setminus W \simeq Y \setminus V$. We obtain the following commutative diagram:

$$
\begin{array}{ccc}
\text{Spec } K((t)) & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\text{Spec } K[[t]] & \overset{\alpha}{\longrightarrow} & X
\end{array}
$$

Then, as $f$ is a proper morphism, by the valuative criteria of properness, there is a unique morphism $\tilde{\alpha} : \text{Spec } K[[t]] \longrightarrow Y$ such that $f \circ \tilde{\alpha} = \alpha$. This shows the bijectivity as required. □

The following is the version for $m = \infty$ of Proposition 2.5:

**Proposition 3.5.** If $f : X \longrightarrow Y$ is an étale morphism, then $X_\infty \simeq Y_\infty \times_Y X$.

Proof. As $\lim_m (Y_m \times_Y X) = (\lim_m Y_m) \times_Y X$, the case $m = \infty$ is reduced to the case $m < \infty$ which is proved in Proposition 2.5. □

**Proposition 3.6.** There is a canonical isomorphism:

$$(X \times Y)_m \simeq X_m \times Y_m,$$

for every $m \in \mathbb{N} \cup \{\infty\}.$$


Proof. For an arbitrary \( k \)-scheme \( Z \),
\[
\text{Hom}_k(Z, X_m \times Y_m) \simeq \text{Hom}_k(Z, X_m) \times \text{Hom}_k(Z, Y_m),
\]
and the right hand side is isomorphic to
\[
\text{Hom}_k(Z \times_{\text{Spec}_k} \text{Spec} k[t]/(t^{m+1}), X) \times \text{Hom}_k(Z \times_{\text{Spec}_k} \text{Spec} k[t]/(t^{m+1}), Y)
\]
\[
\simeq \text{Hom}_k(Z \times_{\text{Spec}_k} \text{Spec} k[t]/(t^{m+1}), X \times Y).
\]
\[
\simeq \text{Hom}_k(Z, (X \times Y)_m).
\]
The case \( m = \infty \) follows from this. \( \square \)

**Proposition 3.7.** Let \( f : X \to Y \) be an open immersion (resp. closed immersion) of \( k \)-schemes of finite type. Then the induced morphism \( f_m : X_m \to Y_m \) is also an open immersion (resp. closed immersion) for every \( m \in \mathbb{N} \cup \{ \infty \} \).

**Proof.** The open case follows from Lemma 2.6 and Proposition 3.5. For the closed case, we may assume that \( Y \) is affine. If \( Y \) is defined by \( f_i \) \((i = 1, \ldots, r)\) in an affine space, then \( X \) is defined by \( f_i \) \((i = 1, \ldots, r, \ldots, u)\) with \( r \leq u \) in the same affine space. Then, \( Y_m \) is defined by \( F_i^{(s)} \) \((i = 1, \ldots, r, \ s \leq m)\) and \( X_m \) is defined by \( F_i^{(s)} \) \((i = 1, \ldots, r, \ldots, u, \ s \leq m)\) in the corresponding affine space. This shows that \( X_m \) is a closed subscheme of \( Y_m \). \( \square \)

**Remark 3.8.** In the above proposition we see that the property open or closed immersion of the base spaces is inherited by the morphism of the space of jets and arcs. But some properties are not inherited. For example, surjectivity and closedness are not inherited.

**Example 3.9.** There is an example that \( f : X \to Y \) is surjective and closed but \( f_\infty : X_\infty \to Y_\infty \) is neither surjective nor closed. Let \( X = \mathbb{A}^2_k \) and \( G = \langle \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{n-1} \end{pmatrix} \rangle \) be a finite cyclic subgroup in \( \text{GL}(2, \mathbb{C}) \) acting on \( X \), where \( n \geq 2 \) and \( \epsilon \) is a primitive \( n \)-th root of unity. Let \( Y = X/G \) be the quotient of \( X \) by the action of \( G \). Then, it is well known that the singularity appeared in \( Y \) is \( A_{n-1} \)-singularity. Then the canonical projection \( f : X \to Y \) is closed and surjective. We will see that these two properties are not inherited by \( f_\infty : X_\infty \to Y_\infty \). Let \( p \) be the image \( f(0) \in Y \). Then, by the commutativity
\[
\begin{array}{ccc}
X_\infty & \xrightarrow{f_\infty} & Y_\infty \\
\downarrow \pi_X & & \downarrow \pi_Y \\
X & \xrightarrow{f} & Y,
\end{array}
\]
we obtain $\pi_X^{-1}(0) = f^{-1}_\infty \circ \pi_Y^{-1}(p)$. Here, $\pi_X^{-1}(0)$ is irreducible, since $X$ is non-singular. On the other hand $\pi_Y^{-1}(p)$ has $(n-1)$-irreducible components by [36], [21]. Therefore the morphism $f_\infty$ is not surjective for $n \geq 3$. As $X \setminus \{0\} \to Y \setminus \{p\}$ is étale, the morphism

$$(X \setminus \{0\})_\infty \to (Y \setminus \{p\})_\infty$$

is also étale by Proposition 3.5. Since $Y_\infty$ is irreducible, $f_\infty$ is dominant. Therefore, $f_\infty$ is not closed.

Next we think of the irreducibility of the arc space or jet schemes. The following is proved in [27]. In [22] we gave another proof by using [21, Lemma 2.12] and a resolution of the singularities. Here we show a proof without a resolution.

**Theorem 3.10 ([27], [22]).** If characteristic of $k$ is zero, then the space of arcs of a variety $X$ is irreducible.

**Proof.** By [21, Lemma 2.12] we obtain the following:

1. Given any arc $\phi : \text{Spec} k'[[s]] \to X$, we construct an arc $\Phi$ such that $\phi \in \{\Phi\}$ and $\Phi(0) = \Phi(\eta) = \phi(\eta)$.
2. We construct an arc $\Psi$ such that $\Psi \in \{\Psi\}$ and $\Psi(\eta) \in X \setminus \text{Sing} X$.

Now for this $\Psi$ we apply the procedure (1) again, then we obtain a new arc $\Psi'$ such that $\Psi' \in \{\Psi'\}$ and $\Psi'(0) = \Psi'(\eta) = \Psi(\eta) \in X \setminus \text{Sing} X$. If we denote $\pi(\Psi') = \Psi'(0) = \lambda$, then $\Psi' \in \pi^{-1}(\lambda)$. As $\lambda \in X \setminus \text{Sing} X$, it follows that

$$\Psi' \in \pi^{-1}(\lambda) \subset \pi^{-1}(\rho),$$

where $\rho$ is the generic point of $X$. This yields $\phi \in \pi^{-1}(\rho)$ which is an irreducible closed subset. \qed

**Example 3.11 ([21], Example 2.13).** If the characteristic of $k$ is $p > 0$, $X_\infty$ is not necessarily irreducible. For example, the hypersurface $X$ defined by $x^p - y^p z = 0$ has an irreducible component in $(\text{Sing} X)_\infty$ which is not in the closure of $X_\infty \setminus (\text{Sing} X)_\infty$.

**Example 3.12 ([23]).** Let $X$ be a toric variety over an algebraically closed field of arbitrary characteristic. Then, $X_\infty$ is irreducible.

Next let us think of $m$-jet scheme. A space of $m$-jets is not necessarily irreducible even if the characteristic of $k$ is zero (see Example 2.10).

**Theorem 3.13 ([34]).** If $X$ is a variety of locally complete intersection over an algebraically closed field of characteristic zero, then $X_m$ is irreducible for all $m \geq 1$ if and only if $X$ has rational singularities.
Another story in which a geometric property of space of jets determines the singularities on the base space is as follows:

**Theorem 3.14 ([12]).** Let $X$ be a reduced divisor on a nonsingular variety over $\mathbb{C}$. $X$ has terminal singularities if and only if $X^m$ is normal for every $m \in \mathbb{N}$.

### 4. Introduction to the Nash problem

In this section, we assume the existence of resolutions of singularities. It is sufficient to assume that the characteristic of $k$ is zero. One of the most mysterious and fascinating problem in arc spaces is the Nash problem which was posed by Nash in his preprint in 1968. It is a question about the Nash components and the essential divisors. First we introduce the concept of essential divisors.

**Definition 4.1.** Let $X$ be a variety, $g : X_1 \to X$ a proper birational morphism from a normal variety $X_1$ and $E \subset X_1$ an irreducible exceptional divisor of $g$. Let $f : X_2 \to X$ be another proper birational morphism from a normal variety $X_2$. The birational map $f^{-1} \circ g : X_1 \to X_2$ is defined on a (nonempty) open subset $E^0$ of $E$. Because, by Zariski’s main theorem, the “undefined locus” of a birational map between normal varieties is of codimension $\geq 2$. The closure of $(f^{-1} \circ g)(E^0)$ is called the center of $E$ on $X_2$.

We say that $E$ appears in $f$ (or in $X_2$), if the center of $E$ on $X_2$ is also a divisor. In this case the birational map $f^{-1} \circ g : X_1 \to X_2$ is a local isomorphism at the generic point of $E$ and we denote the birational transform of $E$ on $X_2$ again by $E$. For our purposes $E \subset X_1$ is identified with $E \subset X_2$. Such an equivalence class is called an exceptional divisor over $X$.

**Definition 4.2.** Let $X$ be a variety over $k$ and let $\text{Sing} X$ be the singular locus of $X$. In this paper, by a resolution of the singularities of $X$ we mean a proper, birational morphism $f : Y \to X$ with $Y$ non-singular such that the restriction $Y \setminus f^{-1}(\text{Sing} X) \to X \setminus \text{Sing} X$ of $f$ is an isomorphism.

**Definition 4.3.** An exceptional divisor $E$ over $X$ is called an essential divisor over $X$ if for every resolution $f : Y \to X$ the center of $E$ on $Y$ is an irreducible component of $f^{-1}(\text{Sing} X)$.

For a given resolution $f : Y \to X$, the center of an essential divisor is called an essential component on $Y$.
Proposition 4.4. Let $f : Y \longrightarrow X$ be a resolution of the singularities of a variety $X$. The set

$$
\mathcal{E} = \mathcal{E}_{Y/X} = \left\{ \text{irreducible components of } f^{-1}(\text{Sing } X) \text{ which are centers of essential divisors over } X \right\}
$$
corresponds bijectively to the set of all essential divisors over $X$.

In particular, the set of essential divisors over $X$ is a finite set.

Proof. The map

$$
\{\text{essential divisors over } X\} \longrightarrow \mathcal{E}_{Y/X}, \quad E \mapsto \text{center of } E \text{ on } Y
$$
is surjective by the definition of essential components. To prove the injectivity, take an essential component $C$ and the blow up $Y' \longrightarrow Y$ with the center $C$. Then, there is a unique divisor $E \subset Y'$ dominating $C$. Let $Y'' \longrightarrow Y'$ be a resolution of the singularities of $Y'$. Then, $E$ is the unique exceptional divisor on $Y''$ that dominates $C$. Therefore, every exceptional divisor over $X$ with the center $C \subset Y$ has the center contained in $E$ on a resolution $Y''$ of the singularities of $X$. Therefore, by the definition of essential divisor, this $E$ is the unique essential divisor whose center on $Y$ is $C$. □

C. Bourvier and G. Gonzalez-Sprinberg also introduce “essential divisors” and “essential components” in [2] and [3], but we should note that the definitions are different from ours. In order to distinguish them we give different names to their “essential divisors” and “essential components”.

Definition 4.5 ([2], [3]). An exceptional divisor $E$ over $X$ is called a BGS-essential divisor over $X$ if $E$ appears in every resolution. An exceptional divisor $E$ over $X$ is called a BGS-essential component over $X$ if the center of $E$ on every resolution $f$ of the singularity of $X$ is an irreducible component of $f^{-1}(E')$, where $E'$ is the center of $E$ on $X$.

We will see how different they are from our essential divisors and essential components. First we see that they coincide for 2-dimensional case. To show this we need to introduce the concept minimal resolution.

Definition 4.6. A resolution $f : Y \longrightarrow X$ of the singularities of $X$ is called the minimal resolution if for any resolution $g : Y' \longrightarrow X$, there is a unique morphism $Y' \longrightarrow Y$ over $X$.

It is known that for a surface $X$ the minimal resolution $f : Y \longrightarrow X$ exists. It is characterized that $Y$ has no exceptional curve of the first kind over $X$.

For higher dimensional variety $X$, the minimal resolution does not necessarily exist. For example, $X = \{xy - zw = 0\} \subset \mathbb{A}^4$ has two
resolutions neither of which dominates the other. These two resolutions are obtained as follows: First take a blow-up $f : \tilde{Y} \to X$ at the origin of $X$ which has the unique singular point at the origin. Then, $f$ is a resolution of the singularity of $X$ and the exceptional divisor $E$ of $f$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Here we have two contractions $g_1 : Y_1 \to X$, $g_2 : Y_2 \to X$ whose restrictions are the first projection $p_1 : E = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ and the second projection $p_2 : E = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, respectively. The both $Y_i$'s are non-singular, therefore $f_i$'s are resolutions of the singularity of $X$. It is clear that there is no morphism between $Y_1$ and $Y_2$ over $X$.

**Proposition 4.7.** If $X$ is a surface, then each set of “essential divisors”, “BGS-essential divisors” and “BGS-essential components” are bijective to the set of the components of the fiber $f^{-1}(\text{Sing } X)$, where $f : Y \to X$ is the minimal resolution. These are also essential components on the minimal resolution.

**Remark 4.8.** Four concepts “essential divisor”, “essential component”, “BGS-essential divisor” and “BGS-essential component” are mutually different in general.

First, our essential component is different from the others, because it is a closed subset on a specific resolution and the others are all equivalence class of divisors.

Next, a BGS-essential divisor is different from a BGS-essential component or a essential divisor. Indeed, for $X = (xy - zw = 0) \subset \mathbb{A}^4_k$, the exceptional divisor obtained by a blow-up at the origin is the unique essential divisor and also the unique BGS-essential component, while there is no BGS-essential divisor, since $X$ has a resolution whose exceptional set is $\mathbb{P}^1_k$.

Finally a BGS-essential component and an essential divisor are different. Indeed, consider a cone generated by $(0,0,1), (2,0,1), (1,1,1), (0,1,1)$ in $\mathbb{R}^3$. It is well known that a cone generated by integer points in a real Euclidean space defines an affine toric variety (see [15], [38] for basic notion of toric variety). Let $X$ be the affine toric variety defined by this cone. Then the canonical subdivision adding a one dimensional cone $\mathbb{R}_{\geq 0}(1,0,1)$ is a resolution of $X$. As the singular locus of $X$ is of dimension one, there is no small resolution. Therefore, the divisor $D_{(1,0,1)}$ is the unique essential divisor, while $D_{(1,1,2)}$ and $D_{(2,1,2)}$ are BGS-essential components by the criterion [2, Theorem 2.3].

**Definition 4.9.** Let $X$ be a variety and $\pi : X_\infty \to X$ the canonical projection. An irreducible component $C$ of $\pi^{-1}(\text{Sing } X)$ is called a **Nash component** if it contains an arc $\alpha$ such that $\alpha(\eta) \not\in \text{Sing } X$. This is equivalent to that $C \not\subset (\text{Sing } X)_\infty$. 

Lemma 4.10 ([21]). If the characteristic of the base field $k$ is zero, then every irreducible component of $\pi^{-1}(\text{Sing } X)$ is a Nash component.

We note that for the positive characteristic case this lemma does not hold. Indeed, Example 3.11 is an example that $\pi^{-1}(\text{Sing } X)$ has an irreducible component which is not a Nash component.

Let $f : Y \to X$ be a resolution of the singularities of $X$ and $E_l$ ($l = 1, \ldots, r$) the irreducible components of $f^{-1}(\text{Sing } X)$. Now we are going to introduce a map $\mathcal{N}$ which is called the Nash map

\[
\left\{ \begin{array}{c}
\text{Nash components} \\
\text{of the space of arcs}
\end{array} \right\} \xrightarrow{\mathcal{N}} \left\{ \begin{array}{c}
\text{essential} \\
\text{components} \\
\text{on } Y
\end{array} \right\} \simeq \left\{ \begin{array}{c}
\text{essential} \\
\text{divisors} \\
\text{over } X
\end{array} \right\}.
\]

4.11 (construction of the Nash map). The resolution $f : Y \to X$ induces a morphism $f_{\infty} : Y_{\infty} \to X_{\infty}$ of schemes. Let $\pi_Y : Y_{\infty} \to Y$ be the canonical projection. As $Y$ is non-singular, $(\pi_Y)^{-1}(E_l)$ is irreducible for every $l$. Denote by $(\pi_Y)^{-1}(E_l)^{\circ}$ the open subset of $(\pi_Y)^{-1}(E_l)$ consisting of the points corresponding to arcs $\beta : \text{Spec } K[[t]] \to Y$ such that $\beta(\eta) \notin \bigcup_i E_i$. Let $C_i$ ($i \in I$) be the Nash components of $X$. Denote by $C_i^\circ$ the open subset of $C_i$ consisting of the points corresponding to arcs $\alpha : \text{Spec } K[[t]] \to X$ such that $\alpha(\eta) \notin \text{Sing } X$. As $C_i$ is a Nash component, we have $C_i^\circ \neq \emptyset$. The restriction of $f_{\infty}$ gives

\[
f_{\infty}' : \bigcup_{l=1}^r (\pi_Y)^{-1}(E_l)^{\circ} \to \bigcup_{i \in I} C_i^\circ.
\]

By Proposition 3.4, $f_{\infty}'$ is surjective. Hence, for each $i \in I$ there is a unique $l_i$ such that $1 \leq l_i \leq r$ and the generic point $\beta_{l_i}$ of $(\pi_Y)^{-1}(E_{l_i})^{\circ}$ is mapped to the generic point $\alpha_i$ of $C_i^\circ$. By this correspondence $C_i \to E_{l_i}$ we obtain a map

\[
\mathcal{N} : \left\{ \begin{array}{c}
\text{Nash components} \\
\text{of the space of arcs}
\end{array} \right\} \text{ through } \text{Sing } X \longrightarrow \left\{ \begin{array}{c}
\text{irreducible} \\
\text{components} \\
\text{of } f^{-1}(\text{Sing } X)
\end{array} \right\}.
\]

Lemma 4.12. The map $\mathcal{N}$ is an injective map to the subset $\{ \text{essential components on } Y \}$.

Proof. Let $\mathcal{N}(C_i) = E_{l_i}$. Denote the generic point of $C_i$ by $\alpha_i$ and the generic point of $(\pi_Y)^{-1}(E_i)$ by $\beta_i$. If $E_{l_i} = E_{l_j}$ for $i \neq j$, then $\alpha_i = f_{\infty}'(\beta_{l_i}) = f_{\infty}'(\beta_{l_j}) = \alpha_j$, a contradiction.

To prove that the $\{ E_{l_i} : i \in I \}$ are essential components on $Y$, let $Y' \to X$ be another resolution and $\tilde{Y} \to X$ a divisorial resolution...
which factors through both $Y$ and $Y'$. Let $E'_i \subset Y'$ and $\tilde{E}_i \subset \tilde{Y}$ be the irreducible components of the exceptional sets corresponding to $C_i$. Then, we can see that $E_i$ and $E'_i$ are the image of $\tilde{E}_i$. This shows that $\tilde{E}_i$ is an essential divisor over $X$ and therefore $E_i$ is an essential component on $Y$.

Problem 4.13. Is the Nash map

\[
\begin{align*}
\{ \text{Nash components} & \} \quad \overset{\mathcal{N}}{\longrightarrow} \quad \{ \text{essential components} & \} \quad \simeq \quad \{ \text{essential divisors} & \}
\end{align*}
\]

bijective?

After Nash’s preprint which posed this problem was circulated in 1968, Bouvier, Gonzalez-Sprinberg, Hickel, Lejeune-Jalabert, Nobile, Reguera-Lopez and others (see, [2], [17], [20], [29], [30], [31], [37], [42]) worked on the arc space of a singular variety related to this problem.

Recently for a toric variety of arbitrary dimension the Nash problem is affirmatively answered but is negatively answered in general by Ishii and Kollár in [21].

Here, we show known results for this problem.

Theorem 4.14 ([36]). The Nash problem is affirmatively answered for an $A_n$-singularity $(n \in \mathbb{N})$, where an $A_n$-singularity is the hypersurface singularity defined by $xy - z^{n+1} = 0$ in $\mathbb{A}^3_k$.

Theorem 4.15 ([42]). The Nash problem is affirmatively answered for a minimal surface singularity. Here, a minimal surface singularity means a rational surface singularity with the reduced fundamental cycle. The fundamental cycle is induced by M. Artin (see [1] for the definition).

Theorem 4.16 ([31], [43]). The Nash problem is affirmatively answered for a sandwiched surface singularity. Here, a sandwiched surface singularity means the formal neighborhood of a singular point on a surface obtained by blowing up a complete ideal in the local ring of a closed point on a non-singular algebraic surface. A complete ideal is defined by O. Zariski and Samuel (see [50], Vol II, Appendix 4), but the idea of a sandwiched singularity is that it is a singularity which is birationally sandwiched by non-singular surfaces.

These are results on rational surface singularities, the following gives affirmative answer for some non-rational surface singularities:

Theorem 4.17 ([40]). The Nash problem is affirmatively answered for a normal surface singularities with the reduced fiber $E$ of the singular
point on the minimal resolution such that $E_i \cdot E_i < 0$ for every irreducible component $E_i$ of $E$.

This result is generalized to a wider class of surface singularities in [33]. We omit the statement, since it is not simple.

The following results are for arbitrary dimension.

**Theorem 4.18** ([21]). The Nash problem is affirmatively answered for a toric singularity of arbitrary dimension.

**Theorem 4.19** ([24]). The Nash problem is affirmatively answered for a non-normal toric variety of arbitrary dimension.

We have a notion of the local Nash problem which is a slight modification of the Nash problem ([25]).

**Theorem 4.20** ([25]). The local Nash problem hold true for a quasi-ordinary singularities. Here, a quasi-ordinary singularity is a hypersurface singularity which is a finite cover over a non-singular variety with the normal crossing branch locus. We note that a quasi-ordinary singularity is not necessarily normal.

The paper [41] gives the affirmative answer to the Nash problem for a certain class of higher dimensional non-toric singularities.

So far we have seen the affirmative answers. But there are negative examples given in [21].

**Example 4.21.** Let $X$ be a hypersurface defined by $x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^6 = 0$ in $\mathbb{A}^5$. Then the number of the Nash components is one, while the number of the essential divisors is two. Therefore the Nash map is not bijective.

By the above example we can construct counter examples to the Nash problem for any dimension greater than 3. At this moment the Nash problem is still open for two and three dimensional variety. Now we can formulate a new version of the Nash problem:

**Problem 4.22.** What is the image of the Nash map? For two and three dimensional case, the image of the Nash map coincides with the set of essential divisors?

Related to this problem, we have one characterization of the image of the Nash map given by Reguera [44]. To formulate her result, we introduce the concept “wedge”.

**Definition 4.23.** Let $K \supset k$ be a field extension. A $K$-wedge of $X$ is a $k$-morphism $\gamma : \text{Spec } K[[\lambda, t]] \to X$. A $K$-wedge $\gamma$ can be identified to a $K[[\lambda]]$-point on $X_\infty$. We call the special arc of $\gamma$ the image in $X_\infty$.
of the closed point 0 of Spec $K[[\lambda]]$. We call the generic arc of $\gamma$ the image in $X_\infty$ of the generic point $\eta$ of Spec $K[[\lambda]]$.

**Theorem 4.24** ([44]). Let $E$ be an essential divisor over $X$ and $f : Y \to X$ a resolution of the singularities of $X$ on which $E$ appears. Let $\alpha \in X_\infty$ be the generic point of $f_\infty(\pi_Y^{-1}(E))$ and $k_E$ the residue field of $\alpha$. Then the following conditions are equivalent:

(i) $E$ belongs to the image of the Nash map;

(ii) For any resolution of the singularities $g : Y' \to X$ and for any field extension $K$ of $k_E$, any $K$-wedge $\gamma$ on $X$ whose special arc is $\alpha$ and whose generic arc belongs to $\pi_X^{-1}(\text{Sing } X)$, lifts to $Y'$;

(iii) There exists a resolution of the singularities $g : Y' \to X$ satisfying condition (ii).

As a corollary of this theorem, we also obtain Theorem 4.16.

There are some notions “the Nash problem for a pair $(X, Z)$” consisting of a variety $X$ and a closed subset $Z$ (see [39], [16]). It seems that these are on the way of developing.

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Department of Mathematics, Tokyo Institute of Technology, Oh-Okayama, Meguro, Tokyo, Japan
E-mail: ishii.s.ac@m.titech.ac.jp