Structural Robustness to Noise in Consensus Networks: Impact of Degrees and Distances, Fundamental Limits, and Extremal Graphs

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Abstract—We investigate how the graph topology influences the robustness to noise in undirected linear consensus networks. We measure the structural robustness by using the smallest possible value of steady state population variance of states under the noisy consensus dynamics with edge weights from the unit interval. We derive tight upper and lower bounds on the structural robustness of networks based on the average distance between nodes and the average node degree. Using the proposed bounds, we characterize the networks with different types of robustness scaling under increasing size. Furthermore, we show that there is a fundamental trade-off between the structural robustness and the average degree of networks. We then show that, random $k$-regular graphs (the degree of each node is $k$) with $n$ nodes typically have near optimal structural robustness among all the graphs with size $n$ and average degree $k$. We also show that when $k$ increases properly with $n$, random $k$-regular graphs maintain a structural robustness within a constant factor of the best possible value (corresponds to the complete graph) while also having the minimum average degree required for such robustness.

I. INTRODUCTION

Consensus networks, where the state of each node approaches a weighted average of the states of adjacent nodes, are used to model the diffusive couplings in a variety of natural and engineered systems such as biological systems, financial networks, social networks, communication systems, transportation systems, power grids, sensor networks, and robotic swarms. These systems typically operate in the face of various disturbances such as measurement/process noise, communication delays, component failures, misbehaving nodes, or malicious attacks (e.g., [2], [3], [4], [5], [6], [7]). Accordingly, a central question regarding such networks is how well they behave in the face of disturbances. This question has motivated many studies on the robustness of consensus networks. Graph measures such as connectivity (e.g., [8], [9]), expansion ratios (e.g., [10], [11]), Kirchoff index (e.g., [2], [3], [12]), and centrality (e.g., [13], [14]) have been used in the literature to quantify the robustness to different types of disturbances.

This paper is focused on the robustness of undirected consensus networks to noisy interactions. In such networks, each edge is endowed with some positive weight denoting the coupling strength between the corresponding nodes. We consider a setting with additive process noise, where the state of each node is attracted towards the weighted average of the states of its neighbors plus some independent and identically distributed (i.i.d.) white Gaussian noise with zero mean and unit covariance. We use the expected steady state population variance of states, which is a variant of the $H_2$-norm of the system with the output defined as the deviation of nodes from global average, as the measure of vulnerability to noise. Similar dynamics were considered in [2], [3] and it was shown that for any network with a given allocation of edge weights, the expected steady state variance can be expressed in terms of the weighted Laplacian eigenvalues. Some tight bounds on this robustness measure were presented in [15], [16]. In [17] and [18], the authors investigated the use of leader-follower control for improving the robustness of noisy consensus networks and presented algorithms for optimal leader selection.

In this paper, we first introduce the notion of structural robustness to noise, which assesses each network based on the smallest value of expected steady state variance that can be attained under the noisy consensus dynamics with edge weights from the unit interval. As such, the proposed notion aims to capture the limitation on robustness due to the network structure, which persists even under the best allocation of edge weights. We show that two simple graph measures, namely the average distance between nodes and the average node degree, define tight bounds on the proposed measure of structural robustness. We then use these novel bounds to obtain some fundamental graph topological limitations on structural robustness and characterize graphs with extremal robustness scaling. We also show that random $k$-regular graphs, which are graphs that are selected uniformly at random from the set of all graphs with $n$ nodes such that the number of edges incident to each node (degree) is equal to $k$, typically have near-optimal structural robustness among the graphs of same size and average degree. We support the theoretical results with numerical simulations. We also provide some connections between the proposed measure of structural robustness to noise and the connectivity-based measures of robustness to the targeted failures of nodes/edges in the Appendix. Specifically, the main contributions of this paper are as follows:

- We show that the average distance between nodes and the average node degree define tight upper and lower bounds on the proposed measure of structural robustness to noise. Using these bounds, we also provide a characterization of networks with extremal scaling of structural robustness, i.e., graph families such that the structural robustness gets arbitrarily worse (e.g., path graph) or arbitrarily better (e.g., complete graph) as the network size increases.
- We show that there is a fundamental trade-off between the structural robustness and the edge-sparsity of networks.
We express this trade-off in terms of tight bounds on the ratio of structural robustness of any given graph to the structural robustness of the complete graph (best among all connected graphs) and the star graph (best among the connected graphs with minimum average degree).

- We show that random \( k \)-regular graphs on \( n \) nodes have near-optimal structural robustness (with high probability), which approaches the optimal value with increasing \( k \), among all the graphs of size \( n \) and average degree \( k \) for any \( k \geq 3 \). Moreover, when \( k \) increases properly with size, random \( k \)-regular graphs maintain a desired level of near-optimal structural robustness while also having the minimum average degree required for such robustness.

The organization of this paper is as follows: Section II provides some graph theory preliminaries. Section III presents our results. Section IV provides the numerical simulations. Finally, Section V concludes the paper with some remarks and future directions.

II. PRELIMINARIES

A. Notation

We use \( \mathbb{R} \) and \( \mathbb{R}_+ \) to denote the set of real numbers and positive real numbers, respectively. For any finite set \( A \) with cardinality \( |A| \), we use \( \mathbb{R}^{|A|} \) (or \( \mathbb{R}_+^{|A|} \)) to denote the space of real-valued (or positive-real-valued) \( |A| \) - dimensional vectors. For any pair of vectors \( x, y \in \mathbb{R}^{|A|} \), we use \( x \leq y \) (or \( x < y \)) to denote the element-wise inequalities, i.e., \( x_i \leq y_i \) (or \( x_i < y_i \)) for all \( i = 1, 2, \ldots, |A| \). The all-ones and all-zeros vectors, their sizes being clear from the context, will be denoted by \( 1 \) and \( 0 \).

B. Graph Theory Basics

A graph \( G = (V, E) \) consists of a node set \( V = \{1, 2, \ldots, n\} \) and an edge set \( E \subseteq V \times V \). For an undirected graph, each edge is represented as an unordered pair of nodes. For each \( i \in V \), let \( N_i \) denote the neighborhood of \( i \), i.e.,

\[ N_i = \{ j \in V \mid (i, j) \in E \} \]

A path between a pair of nodes \( i, j \in V \) is a sequence of nodes \( \{i, j\} \) such that each pair of consecutive nodes are linked by an edge. For any node \( i \), the number \( d_i \) of its neighborhood, \( |N_i| \), is called its degree. Accordingly, the average node degree of a graph is

\[ \bar{d}(G) = \frac{1}{n} \sum_{i=1}^{n} d_i. \]

The distance between any two nodes \( i \) and \( j \), which is denoted by \( \delta_{ij} \), is equal to the number of edges on the shortest path between those nodes. The maximum distance between any two nodes, \( \max_{i,j \in V} \delta_{ij} \), is known as the diameter of the graph, and the average distance between the nodes is given as

\[ \bar{\delta}(G) = \frac{2}{n^2 - n} \sum_{1 \leq i < j \leq n} \delta_{ij}. \]

A graph is connected if there exists a path between every pair of nodes. For weighted graphs, we use \( w \in \mathbb{R}_+^{|E|} \) to denote the vector of edge weights and \( w_{ij} \in \mathbb{R}_+ \) to denote the weight of the edge \((i, j) \in E\). The adjacency matrix, \( A \), of a weighted graph is defined as

\[ [A_w]_{ij} = \begin{cases} w_{ij} & \text{if } (i, j) \in E \\ 0 & \text{otherwise}, \end{cases} \]

and the corresponding (weighted) graph Laplacian is

\[ [L_w]_{ij} = \begin{cases} \sum_{k \in N_i} A_{ik} & \text{if } i = j \\ -A_{ij} & \text{otherwise}, \end{cases} \]

In the remainder of the paper, we will use \( L \) to denote the unweighted Laplacian, i.e., the special case when \( w = 1 \).

C. Consensus Networks

Consensus networks can be represented as a graph, where the nodes correspond to the agents, and the weighted edges exist between the agents that are coupled through local interactions. For such a network \( G = (V, E) \), let the dynamics of each agent \( i \in V \) be

\[ \dot{x}_i(t) = \sum_{j \in N_i} w_{ij}(x_j(t) - x_i(t)) + \xi_i(t), \]

where \( x_i(t) \in \mathbb{R} \) denotes the state of \( i \), each \( w_{ij} \in \mathbb{R}_+ \) is a constant weight representing the strength of the coupling between \( i \) and \( j \), and \( \xi_i(t) \in \mathbb{R}_+^n \) is i.i.d. white Gaussian noise with zero mean and unit covariance, which is one of the standard noise models for agents that are independently affected by disturbances of same intensity due to various effects such as communication errors, noisy measurements, or quantization errors (e.g., see \[2, 3, 15, 17, 18\]). Accordingly, the overall dynamics of the agents can be expressed as

\[ \dot{x}(t) = -L_w x(t) + \xi(t), \]

where \( L_w \) denotes the weighted Laplacian. In a noise-free setting \( \xi(t) = 0 \) for all \( t \geq 0 \), the dynamics in (1) are known to result in a global consensus, \( \lim_{t \to \infty} x(t) = \text{span}\{1\} \), for any \( x(0) \in \mathbb{R}^n \) if and only if the graph is connected \[19, 20\]. In the noisy case, a perfect consensus can not be achieved. Instead, some finite steady state variance of \( x(t) \) is observed on connected graphs \[2, 3\]. Accordingly, the robustness of the network can be quantified through the expected population variance in steady state, i.e.,

\[ \mathcal{H}(G, w) := \lim_{t \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[(x_i(t) - \bar{x}(t))^2], \]

where \( \bar{x}(t) \in \mathbb{R} \) denotes the average of \( x_1(t), x_2(t), \ldots, x_n(t) \).

It can be shown that (e.g., see \[2, 3\]) \( \mathcal{H}(G, w) \) is equal to \( 1/n \) times the square of the \( H_2 \)-norm of the system in (1) from the input \( \xi(t) \) to the output \( y(t) \in \mathbb{R}^n \) defined as \( y_i(t) = x_i(t) - \bar{x}(t) \), and it satisfies

\[ \mathcal{H}(G, w) = \frac{1}{2n} \sum_{i=2}^{n} \frac{1}{\lambda_i(L_w)}, \]

where and \( 0 < \lambda_2(L_w) \leq \ldots \leq \lambda_n(L_w) \) denote the eigenvalues of the weighted Laplacian \( L_w \).
In this paper, we investigate how much the structure of the underlying graph (the edge set $E$) causes vulnerability to noise in consensus networks. We measure the structural vulnerability of any given network to noise based on the smallest possible value of $H(G, w)$, given that the edge weights should belong to the feasible set $W = \{w | 0 < w \leq 1\}$. Since multiplying all the weights by some $\alpha \in \mathbb{R}^+$ results in $L_{aw} = \alpha L_w$ and $H(G, aw) = H(G, w)/\alpha$ due to (2), it is possible to make $H(G, w)$ arbitrarily small for any network by just scaling up all the weights. By considering only weights in $(0, 1]$, we remove this possibility and focus on the impact of network structure.

**Definition (Structural Vulnerability and Robustness)** The structural vulnerability of an undirected consensus network $G = (V, E)$ to noise is the smallest possible value of $H(G, w)$ that is achievable under weights from the unit interval, i.e.,

$$H^*(G) := \min_{0 < w \leq 1} H(G, w).$$

(3)

The structural robustness to noise is quantified using the reciprocal of structural vulnerability, $1/H^*(G)$.

**Remark 1** In the remainder of this paper, for brevity we will say “structural robustness (or vulnerability)” without explicitly saying “to noise”. The term “structural robustness” is also used in the literature to refer to the robustness of a network’s connectivity to node or edge failures (e.g., [21], [12]). While some of which are also provided in the Appendix for interested readers, the distinction should be clear from the context.

**III. MAIN RESULTS**

In this section we provide the main results of this paper. We first express structural vulnerability in terms of the Laplacian eigenvalues. Then, we derive tight bounds on structural vulnerability based on the average node degrees and average distances. We also use these bounds to characterize graphs with extremal robustness properties. We then provide some results regarding the fundamental trade-off between sparsity and structural robustness and we show that random regular graphs have near-optimal structural robustness among the graphs of same size and sparsity.

**A. Connection to the Laplacian eigenvalues**

We start our derivations by providing $H^*(G)$ as a function of the (unweighted) Laplacian eigenvalues.

**Lemma 3.1.** For any connected undirected graph $G$,

$$H^*(G) = \frac{1}{2n} \sum_{i=2}^{n} \frac{1}{\lambda_i(L)}.$$  

(4)

where $L$ denotes the unweighted Laplacian of $G$.

**Proof.** For any connected undirected graph $G$, any weighted Laplacian is a positive semidefinite matrix. Increasing any of its weights or adding new edges leads to a new Laplacian that is equal to the initial Laplacian plus another matrix that is also a weighted Laplacian (a graph with just the added/strengthened edges). All the Laplacian eigenvalues monotonically (not necessarily strictly) increases under such an addition of a positive semidefinite matrix due to the Weyl’s inequality (e.g., see [22]). Hence, $H(G, w)$ is minimized for $w = 1$ within the feasible set of $\mathbb{R}$ for any connected undirected graph. The Kirchoff index satisfies

$$K_f(G) = n \sum_{i=2}^{n} \frac{1}{\lambda_i(L)},$$

where $L$ is the Laplacian of $G$. Accordingly, due to (4),

$$H^*(G) = \frac{K_f(G)}{2n^2}.$$  

(5)

The connection in (5) is particularly useful as it links the structural robustness to the rich literature in graph theory on Kirchhoff index. For instance, closed form expressions in terms of size are known for some graph families (e.g., see [24], [25], [26]). Using those results on Kirchhoff index we immediately obtain that the path ($P_n$), cycle ($C_n$), star ($S_n$), and complete ($K_n$) graphs of size $n$ have

$$H^*(P_n) = \frac{n^2 - 1}{12n},$$

$$H^*(C_n) = \frac{n^2 - 1}{24n},$$

$$H^*(S_n) = \frac{(n - 1)^2}{2n^2},$$

$$H^*(K_n) = \frac{(n - 1)^2}{2n^2}.$$  

(6) (7) (8) (9)

Furthermore, among all the connected undirected graphs with $n$ nodes, the Kirchoff index is minimized in the complete graph $K_n$ and maximized in the path graph $P_n$ (e.g., see [26]). As such, in light of (5), $K_n$ and $P_n$ are also the minimizer and maximizer of $H^*(G)$, respectively. There are also some results in the literature regarding the graphs with minimum Kirchoff index when there are additional constraints on topology. For example, among the graphs with a given size and diameter, the minimum Kirchoff index occurs in clique chains [26]. On the other hand, among the graphs with a given size and edge cut (graphs that can be disconnected by removing a given number of edges), clique-stars have the minimum Kirchoff index [27].

**B. Impact of Average Degree and Average Distance**

The structural vulnerability of any given network can be computed by using the Laplacian eigenvalues as in (4). Furthermore, the connection with the Kirchhoff index in (5) enables the identification of extremal graphs (path and complete) and provides closed form expressions in terms of network size for several graph families. However, it is not easy to use (4) or (5) for certain analysis and design applications in a systematic
and efficient way. For instance, finding an optimal way to add a
given number of edges to an arbitrary network to reduce the
\(\mathcal{H}^*(G)\) would require searching among all possibilities (e.g.,
see \cite{26}). Furthermore, while it is possible to see how \(\mathcal{H}^*(G)\) scales
with size for the special graph families with closed form
expressions, it is hard to do this for generic structures. One
way to overcome these type of difficulties is focusing on some
upper/lower bounds on \(\mathcal{H}^*(G)\) rather than its exact value.

Many upper and lower bounds on the Kirchhoff index
have been proposed in the literature by using graph measures
such as chromatic number, independence number, edge/node
connectivity, diameter, or degree distribution (e.g., see \cite{15},
\cite{28}, \cite{29}). These bounds typically require significant amount
of global information and/or computation, which limits their
applicability in large networks (e.g., see \cite{32}, \cite{31}). Motivated
by such limitations, we present a fundamental relationship
between the \(\mathcal{H}^*(G)\) and two aggregate measures, namely the
average node degree and the average distance between nodes.
Both of these measures can be computed/estimated efficiently
(e.g. in time sublinear in network size \cite{32}), possibly in
a distributed manner with partial information (e.g., \cite{33}). Furthermore, they have been widely used for the analysis and
design of various networks (not only consensus networks)
due to their significant impact on the overall structure and
performance (e.g., \cite{34}, \cite{35}, \cite{36}, \cite{37}). Our next result
shows that these two graph measures also have a fundamental
connection to the structural robustness of consensus networks.
Specifically, we show that the average node degree and the
average distance between nodes define upper and lower bounds
on \(\mathcal{H}^*(G)\). Furthermore, these bounds are tight and hold
with equality for complete graphs and trees (graphs with \(n\) nodes
and \(n - 1\) edges).

**Theorem 3.2.** For any connected undirected graph
\(G = (V, E)\) with \(n \geq 2\) nodes,
\[
\frac{(n - 1)^2}{2d(G)n^2} \leq \mathcal{H}^*(G) \leq \frac{\tilde{d}(G)(n - 1)}{4n},
\]
where \(\tilde{d}(G)\) is the average node degree, \(\tilde{d}(G)\) is the average
distance between the nodes. Moreover, the lower bound holds
with equality if and only if \(G\) is a complete graph, and the
upper bound holds with equality if and only if \(G\) is a tree.

**Proof.** (Lower bound:) Since the harmonic mean is always
less than or equal to the arithmetic mean, we have
\[
\frac{1}{n - 1}\sum_{i=2}^{n} \frac{1}{\lambda_i(L)} \leq \frac{1}{n - 1}\sum_{i=2}^{n} \frac{1}{\lambda_i(L)},
\]
where the left side is the harmonic mean and the right side
is the arithmetic mean of \(1/\lambda_2(L), 1/\lambda_3(L), \ldots, 1/\lambda_n(L)\).
Furthermore since \(L\) is a symmetric matrix, the sum of its
eigenvalues equals its trace, which is equal to the sum of node
degrees \(nd(G)\). Hence, (11) implies
\[
\frac{(n - 1)^2}{nd(G)} \leq \sum_{i=2}^{n} \frac{1}{\lambda_i(L)}.
\]

Due to \cite{3} and \cite{12},
\[
\mathcal{H}^*(G) = \frac{1}{2n}\sum_{i=2}^{n} \frac{1}{\lambda_i(L)} \geq \frac{(n - 1)^2}{2d(G)n^2}.
\]
Note that the harmonic mean equals the arithmetic mean if
and only if all the numbers are equal. Hence, (11) holds
with equality if and only if \(\lambda_1(L) = \lambda_2(L) = \ldots = \lambda_n(L)\).
Furthermore, all the positive Laplacian eigenvalues of a connected
graph are equal if and only if the graph is a complete graph
(e.g., see \cite{38}). Hence, (13) holds with equality if and only
if \(G\) is a complete graph. Alternatively, the lower bound can
also be proved by using (4) and the inequality shown in \cite{39}.

(Upper bound:) The Kirchhoff index satisfies
\[
K_f(G) \leq \sum_{1 \leq i < j \leq n} \delta_{ij},
\]
where the \(\delta_{ij}\) denotes the distance between nodes \(i\) and \(j\), and
(14) holds with equality if and only if \(G\) is a tree (e.g., see
\cite{26}). Since the sum of distances between the nodes satisfy
\[
\sum_{1 \leq i < j \leq n} \delta_{ij} = \frac{n(n - 1)\tilde{d}(G)}{2},
\]
(5) and (14) together imply
\[
\mathcal{H}^*(G) \leq \frac{\tilde{d}(G)(n - 1)}{4n}.
\]
Furthermore, since (14) holds with equality if and only if \(G\)
is a tree, the same is true for the inequality in (15).

**C. Graphs with Extremal Robustness Scaling**

One of the important considerations when designing large
scale networks is how the robustness of the system would
scale with its size. As indicated by \cite{3}-\cite{9}, different network
topologies may exhibit different robustness scaling properties.
For instance, while the structural vulnerability of complete
graph, \(\mathcal{H}^*(K_n)\), tends to zero as the network size increases
(see \cite{9}), the structural vulnerability of path graph,
\(\mathcal{H}^*(P_n)\), tends to infinity as the network size increases (see \cite{9}).
Apart from these two extremal cases of robustness scaling, there are
also networks (e.g., star graph) such that \(\mathcal{H}^*(G_n)\) converges
to some non-zero value as the network size increases. One
question of interest is then which topological properties deter-
mine how the structural robustness behaves as the size goes to
infinity. In this regard, the following result provides a graph
topological characterization of networks with different types
of robustness scaling.

**Theorem 3.3.** Let \(\{G_n\}_{n \in \mathbb{N}}\) denote an infinite sequence
of connected undirected graphs with \(n\) nodes. The structural
vulnerability of \(G_n\) tends to zero as \(n\) goes to infinity only
if the average node degree grows unbounded, i.e.,
\[
\lim_{n \to \infty} \mathcal{H}^*(G_n) = 0 \Rightarrow \lim_{n \to \infty} \tilde{d}(G_n) = \infty.
\]
Furthermore, the structural vulnerability grows unbounded
only if the average distance also grows unbounded, i.e.,
Furthermore, this upper bound is tight.

Proof. The upper bound follows from (8) and the lower bound in (10). The tightness follows from the fact that (17) is satisfied with equality for the complete graph, $G_n = K_n$, as per (8) and (9) since $\tilde{d}(K_n) = n - 1$.

Since $S_n$ has the best structural robustness achievable with the minimum number of edges a connected graph can have, (17) highlights the price of structural robustness in terms of sparsity. Any graph with significantly better structural robustness than the star graph of same size should have a proportionally high average degree.

E. Sparsity of Graphs with Bounded Robustness-Suboptimality

In light of Theorem 3.4 any graph has a fundamental limit on its structural robustness imposed by the average degree. A related question is then how sparse a graph can be while having a certain level of structural robustness relative to the best possible value (complete graph). Our next result addresses this question by giving a lower bound on the ratio $\mathcal{H}^*(G_n)/\mathcal{H}^*(K_n)$ using the average node degrees $\tilde{d}(G_n)$ and $\tilde{d}(K_n) = n - 1$.

Theorem 3.5. For any connected undirected graph with $n$ nodes, $G_n$, and the complete graph with $n$ nodes, $K_n$,

$$\frac{\mathcal{H}^*(G_n)}{\mathcal{H}^*(K_n)} \geq \frac{n - 1}{\tilde{d}(G_n)}.$$  \hspace{1cm} (18)

Furthermore, this lower bound is tight.

Proof. The lower bound follows from (8) and the lower bound in (10). The tightness can be shown by considering the case $G_n = K_n$, which results in $\tilde{d}(G_n) = n - 1$.

The bound in Theorem 3.5 can be used for the design of sparse yet robust networks. For example, consider a network design problem, where the goal is to build a network with the minimum number of edges that has a bounded robustness-suboptimality, i.e., $\mathcal{H}^*(G_n)/\mathcal{H}^*(K_n) \leq \alpha$ for some desired $\alpha \in [1, \infty)$. For instance, in a wireless sensor network, this design problem can be motivated by the goal of achieving robust distributed estimation with minimum communication due to energy and bandwidth considerations. In light of Theorem 3.5 such a network must have an average degree of at least $(n - 1)/\alpha$ since (18) implies

$$\frac{\mathcal{H}^*(G_n)}{\mathcal{H}^*(K_n)} \leq \alpha \Rightarrow \tilde{d}(G_n) \geq \frac{n - 1}{\alpha}.$$  \hspace{1cm} (19)

Accordingly, the design space can be narrowed down to the set of graphs with sufficiently many edges as per (19). However, finding networks with optimal structural robustness within this reduced search space is a combinatorial optimization problem, which becomes intractable as the network size increases. In the next subsection, we show that a specific family of graphs, namely the random regular graphs, are approximate optimizers of structural robustness under sparsity constraints.
F. Structural Robustness of Random Regular Graphs

In this subsection, we investigate the structural robustness of random regular graphs and show that they typically have almost optimal structural robustness among the graphs of same sparsity. A graph is called a \( k \)-regular graph if the number of edges incident to each node (degree) is equal to \( k \). For connected regular graphs with \( n \geq 3 \) nodes, the feasible values of \( k \) are \( \{2, 3, \ldots, n-1\} \) with the constraint that \( n \) and \( k \) can not be both odd numbers since the number of edges is equal to \( nk/2 \). The complete graph, which has the best structural robustness possible as given in (2), is the \( k \)-regular graph with \( k = n - 1 \). We will show that most \( k \)-regular graphs have desirable structural robustness properties, except for the special case of \( k = 2 \), which is the cycle graph \( C_n \). In light of (7), \( \mathcal{H}^*(C_n) \) clearly grows unbounded as the network size increases. Comparing (7) to (9), it can be seen that the structural vulnerability of a cycle graph is equal to the half of the path graph’s, i.e., \( \mathcal{H}^*(C_n) = \mathcal{H}^*(P_n)/2 \). Hence, the structural robustness of a cycle is always within a constant factor of the worst possible among the graphs of equal size.

On the other hand, the structural robustness of \( k \)-regular graphs for \( k \geq 3 \) is significantly different from the cycle graph’s structural robustness. As \( n \) goes to infinity, for \( k \geq 3 \) almost every \( k \)-regular graph has \( \lambda_2(L) \geq k - 2\sqrt{k-1} - \epsilon \) for any \( \epsilon > 0 \) (e.g., see (13) and the references therein). In light of (4), this property implies an upper bound on the structural vulnerability of those graphs since for any graph

\[
1/2n \sum_{i=2}^{n} \frac{1}{\lambda_i(L)} \leq \frac{n-1}{2n\lambda_2(L)}.
\]

Accordingly, for any integer \( k \geq 3 \) and \( \epsilon \in (0, k - 2\sqrt{k-1}) \)

\[
\lim_{n \to \infty} \Pr \left\{ \mathcal{H}^*(G_{n,k}) \leq \frac{n-1}{2n(k - 2\sqrt{k-1} - \epsilon)} \right\} = 1, \quad (20)
\]

where \( G_{n,k} \) is a random \( k \)-regular graph, i.e., a graph that is selected uniformly at random from the set of all \( k \)-regular graphs with \( n \) nodes. Since \( n \) and \( k \) cannot both be odd, for odd values of \( k \) the limit in (20) is determined along the sequence of even integers \( n \in \{k+1, k+3, \ldots\} \).

By combining (20) with the lower bound in (10) for \( \bar{d}(G) = k \), we can show that for large values of \( n \), with high probability, the structural vulnerability of random \( k \)-regular graphs \((k \geq 3)\) is within a constant factor of the smallest possible value among the graphs with the same size and average degree. Furthermore, this factor gets arbitrarily close to one as \( k \) increases. In other words, for large values of \( k \), random \( k \)-regular graphs have structural robustness arbitrarily close to the best possible (with that many edges) with arbitrarily high probability as the network size increases.

**Theorem 3.6.** For any integer \( k \geq 3 \) and \( \epsilon \in (0, k - 2\sqrt{k-1}) \)

\[
\lim_{n \to \infty} \Pr \left\{ \frac{\mathcal{H}^*(G_{n,k})}{\min_{G_i: \bar{d}(G_i) = k} \mathcal{H}^*(G_i)} \leq \frac{k}{k - 2\sqrt{k-1} - \epsilon} + \epsilon \right\} = 1, \quad (21)
\]

where \( G_{n,k} \) is a random \( k \)-regular graph.

**Proof.** Using the lower bound in (10) and (9), for any undirected graph \( G_n \) with \( n \) nodes and average degree \( \bar{d}(G_n) = k \),

\[
\min_{G_i: \bar{d}(G_i) = k} \mathcal{H}^*(G_i) \geq \frac{(n-1)^2}{2kn^2} \quad (22)
\]

Using (22) with (20), for any random \( k \)-regular graph with \( k \geq 3 \) and \( \epsilon \in (0, k - 2\sqrt{k-1}) \),

\[
\lim_{n \to \infty} \Pr \left\{ \frac{\mathcal{H}^*(G_{n,k})}{\min_{G_i: \bar{d}(G_i) = k} \mathcal{H}^*(G_i)} \leq \frac{2kn^2}{(2n^2 - 2n)(k - 2\sqrt{k-1} - \epsilon)} \right\} = 1. \quad (23)
\]

Note that the upper bound in (23) satisfies

\[
\lim_{n \to \infty} \frac{2kn^2}{(2n^2 - 2n)(k - 2\sqrt{k-1} - \epsilon)} = \frac{k}{k - 2\sqrt{k-1} - \epsilon}.
\]

Due to the definition of limit, there exists some \( n \) beyond which the upper bound in (23) is smaller than its limit plus \( \epsilon \). Consequently, we can replace the upper bound with that value and obtain (21). \( \Box \)

In light of Theorem 3.6, random \( k \)-regular graphs with sufficiently large \( k \) and size \( n \) have near optimal structural robustness among the graphs of same size and average degree. Fig. 1 illustrates how the approximation bound in (21) changes as a function of the degree \( k \). As shown in this figure, the approximation bound starts around 17.5 for \( k = 3 \), rapidly drops to 5 by \( k = 5 \) and to 2 by \( k = 15 \), and then keeps approaching one as \( k \) increases. Accordingly, random \( k \)-regular graphs are very good approximate solutions to the problem of optimization structural robustness subject to a sparsity constraint (upper bound on the number of edges).

We complement Theorem 3.6 by showing that, as the network size increases, the structural vulnerability of random \( k \)-regular graphs stays within a bounded proximity of the complete graph’s structural vulnerability for sufficiently large values of \( k \).
Theorem 3.7. For any constant \( \alpha \in [1, \infty) \) and any \( \epsilon > 0 \),
\[
\lim_{n \to \infty} \Pr \left\{ \frac{\mathcal{H}^*(\mathcal{G}_{n,k})}{\mathcal{H}^*(\mathcal{K}_n)} \leq \alpha + \epsilon \right\} = 1, \forall k \geq \frac{n - 1}{\alpha}, \tag{24}
\]
where \( \mathcal{K}_n \) is the complete graph and \( \mathcal{G}_{n,k} \) is a random \( k \)-regular graph.

Proof. For any random \( k \)-regular graphs with \( k \geq \frac{n - 1}{\alpha} \), (20) implies that
\[
\lim_{n \to \infty} \Pr \left\{ \mathcal{H}^*(\mathcal{G}_{n,k}) \leq \frac{n - 1}{2n} \left( \frac{n - 1}{\alpha} - 2\sqrt{\frac{n - 1 - \alpha}{\alpha} - \epsilon} \right) \right\} = 1, \tag{25}
\]
for any \( \epsilon \in (0, k - 2\sqrt{k - 1}) \). Using (25) together with (9), and without loss of generality setting \( \epsilon = 0.1 \), which is in \((0, k - 2\sqrt{k - 1})\) for all \( k \geq 3 \), we have
\[
\lim_{n \to \infty} \Pr \left\{ \frac{\mathcal{H}^*(\mathcal{G}_{n,k})}{\mathcal{H}^*(\mathcal{K}_n)} \leq \frac{n}{n - 1 - 2\sqrt{n - 1 - \alpha - 0.1}} \right\} = 1. \tag{26}
\]

Note that the upper bound in (26) approaches \( \alpha \) as \( n \to \infty \). Hence, for any \( \epsilon > 0 \) there is a sufficiently large value of \( n \) such that, the upper bound is smaller than \( \alpha + \epsilon \). Accordingly, we obtain (24). \( \square \)

Theorem 3.7 implies that the random regular graphs can approach the fundamental limit in (19) on \( \mathcal{H}^*(\mathcal{G}_n)/\mathcal{H}^*(\mathcal{K}_n) \) imposed by the sparsity of \( \mathcal{G}_n \). For example, for any constant \( \alpha \in [1, \infty) \) and even number of nodes \( n \) such that \( n \geq 3\alpha + 1 \), let \( \mathcal{G}_{n,k^*} \) be a random \( k^* \)-regular graph, where
\[
k^* = \left\lfloor \frac{n - 1}{\alpha} \right\rfloor. \tag{27}
\]

For such random regular graphs, as \( n \) increases, \( \mathcal{H}^*(\mathcal{G}_{n,k^*})/\mathcal{H}^*(\mathcal{K}_n) \) is upper bounded by \( \alpha \) with a very high probability due to (24). Furthermore, \( \mathcal{G}_{n,k^*} \) has an average degree of \( k^* \) that is equal or very close to the minimum required value of \( \frac{n - 1}{\alpha} \) as given in (19).

In summary, for \( k \geq 3 \), random \( k \)-regular graphs have near optimal structural robustness among the graphs of same size and sparsity. Furthermore, when \( k \) grows linearly with the network size, random \( k \)-regular graphs maintain a structural robustness within a bounded proximity of the best possible value (the complete graph’s) while also having the minimum average degree required to have such robustness.

It is also worth mentioning that random \( k \)-regular graphs (\( k \geq 3 \)) are known to have high node and edge expansion ratios, i.e., disconnecting a large component requires the failure of many nodes/edges. As such, in addition to their desirable structural robustness to noise, their connectivity is also very robust to the targeted failures of nodes and edges. We provide some discussion about this aspect in the Appendix of this paper. For further details on this subject and a distributed algorithm for building random regular graphs via local graph transformations, we refer the interested readers to [11] and the references therein.

IV. Simulation Results

We simulate the noisy consensus dynamics in (1) for uniform edge weights \( w = 1 \) on different network topologies and sizes to demonstrate their structural robustness. In each simulation, the network is initialized at \( x(0) = 0 \) and the variance of \( x(t) \) is observed under the noisy consensus dynamics as per (1), where \( \xi(t) \in \mathbb{R}^n \) is white Gaussian noise with zero mean and unit covariance.

A. Simulation Study 1

In the first set of simulations, we consider the path, star, random 3-regular, and complete graphs. We aim to numerically illustrate how the structural robustness of these graphs compare to each other and change with increasing network size. For each type we generate three networks of different sizes: \( n = 20, n = 40 \), and \( n = 60 \). The resulting state variances over time on each of these networks are shown in Fig. 2 for \( n = 20 \), Fig. 3 for \( n = 40 \), and Fig. 4 for \( n = 60 \). In Table I for each of these networks we provide the average of state variances over the simulation horizon and the theoretical value of structural vulnerability, which is computed using the Laplacian eigenvalues as per (4).

As shown in Table I the average state variances over the simulation horizon are very close to the theoretical values of structural vulnerability (shown in bold). For the path, star, and complete graphs, the empirical values can also be verified using (6), (8), and (9). For the random 3-regular graphs, the average distances are computed as \( 2.62 \) (\( n = 20 \)), \( 3.62 \) (\( n = 40 \)), and \( 4.09 \) (\( n = 60 \)). Using the average distances together with the average degrees, the lower and upper bounds in (10) are computed as \( 0.15 \) and \( 0.62 \) (\( G_{20,3} \)), \( 0.158 \) and \( 0.882 \) (\( G_{40,3} \)), \( 0.161 \) and \( 1.005 \) (\( G_{60,3} \)). For each random 3-regular graph, the observed average state variance is inside the corresponding interval and closer to the lower bound.

|          | \( n = 20 \) | \( n = 40 \) | \( n = 60 \) |
|----------|-------------|-------------|-------------|
| Path     | 1.662       | 3.32        | 5.21        |
|          | 1.663       | 3.33        | 4.99        |
| Star     | 0.45        | 0.478       | 0.485       |
|          | 0.45        | 0.475       | 0.483       |
| Random 3-regular | 0.239    | 0.286      | 0.305       |
|          | 0.237       | 0.287       | 0.305       |
| Complete | 0.024       | 0.0124      | 0.0085      |
|          | 0.024       | 0.0122      | 0.0082      |

Table I
AVERAGE OF STATE VARIANCES OVER TIME AND THE VALUE OF STRUCTURAL VULNERABILITY AS PER (4) (BOLD) FOR THE PATH, STAR, RANDOM 3-REGULAR, AND COMPLETE GRAPHS OF SIZES 20, 40, AND 60.
The results show that increasing network size amplifies the average state variance for the path graph, whereas it leads to a reduction the average state variance for the complete graph. Looking at the values in Table I, the average state variance is proportional to the network size $n$ for the path graph and it is proportional to $1/n$ for the complete graph. These observations are aligned with the analytical expressions in [6] and [9].

On the other hand, for the star graph the average state variance increases toward 0.5 as the network size increases, which is expected due to the limit of [8] as $n$ goes to infinity. For the random 3-regular graph, the average state variance shows some increase as the network grows. While there is no analytical expression for the structural vulnerability of random regular graphs as a function of their size, (20) implies that a random $k$-regular graph has $\mathcal{H}^*(G_{n,k})$ upper bounded by approximately $1/2(k - 2\sqrt{k - 1})$ with an arbitrarily high probability as $n$ increases. In all three sizes, the random 3-regular graph has better structural robustness than the star and the path graphs. Furthermore, the path and complete graphs exhibit the worst and best structural robustness as expected.

### B. Simulation Study 2

In the second set of simulations, we aim to illustrate how the structural robustness of random $k$-regular graphs with $k$ as in (27) change with increasing network size for a given $\alpha \in [1, \infty)$. As such, we investigate the performance of such graphs as an approximate solution to the combinatorial problem of designing a network with minimum sparsity that has $\mathcal{H}^*(G_n) \leq \alpha \mathcal{H}^*(K_n)$. For this simulation we pick $\alpha = 25$ and set $k$ as per (27) for four different sizes: $n = 100, n = 150, n = 200$, and $n = 250$. Accordingly, we simulate the noisy consensus dynamics on the random regular graphs $G_{100,4}, G_{150,5}, G_{200,8}$, and $G_{250,10}$. For each of these graphs, the resulting variance of states over time is shown in Fig. 3.

In these simulations, the average of state variances over the whole horizon were observed as 0.1818 ($G_{100,4}$), 0.1015 ($G_{150,5}$), 0.0717 ($G_{200,8}$), and 0.0556 ($G_{250,10}$). Note that different from the previous set of simulations, where the random regular $3$-regular graphs of increasing size exhibited increasing steady state population variance of states, we observe that the steady state dispersion shrinks when $k$ is increased in proportion to the network size. Such a positive dependence of structural robustness on increasing network size is similar to the robustness scaling of complete graphs.

In Table II, we provide the theoretical values of $\mathcal{H}^*(G_{n,k})$ and $\mathcal{H}^*(K_n)$, which are computed using the Laplacian eigenvalues of graphs as per [4]. We also provide their ratios, $\mathcal{H}^*(G_{n,k})/\mathcal{H}^*(K_n)$, in the last row of this table. The ratio
starts at 36.3 for $n = 100$ and monotonically drops to 27.8 by $n = 250$. These results indicate that $\mathcal{H}^*(G_{n,k})/\mathcal{H}^*(K_n)$ is approaching $\alpha$ in accordance with Theorem 3.7. Hence, such random $k$ regular graphs with $k$ as per (27) approximately maintain the required level of robustness with the minimum average degree possible as shown in (19).

V. CONCLUSION AND DISCUSSION

We investigated the structural robustness of undirected linear consensus networks to noisy interactions. We measured the structural robustness of a graph based on the smallest possible value of the expected steady state population variance of states under the noisy consensus dynamics with admissible edge weights in $[0, 1]$. We showed that the average distance and the average node degree in the underlying graph define tight bounds on the structural robustness. Using these novel bounds, we also presented some fundamental graph topological limitations on structural robustness and we investigated the graphs with extremal robustness properties.

The results of this paper provide some very useful insights into the analysis and design of robust consensus networks. For example, as per Theorem 3.3, maintaining a finite average distance (or diameter) is sufficient for a network growth process to ensure that the structural robustness does not get arbitrarily worse with increasing size. This relationship also highlights the importance of establishing long range connections in improving the robustness of networks. Similarly, any graph that has a structural robustness improving arbitrarily with increasing size (e.g., complete graph) must have an unbounded average degree, which may not be feasible in many applications due to physical limitations that impose edge-
sparsity. While edge-sparsity and structural robustness are both desirable in many applications, there is a fundamental trade-off between the two properties as we shown in Theorems 3.4 and 3.5. We have also shown that random regular graphs can achieve near-optimal trade-off between sparsity and robustness. In light of Theorem 3.6 random regular graphs typically have near-optimal structural robustness. Furthermore, as per Theorem 3.7, by setting the degree as in (27) it is possible to design random regular graphs that maintain (as the size grows) a desired level of near-optimal structural robustness while also having the minimum average degree required for such robustness.

As a future direction, we intend to extend our robustness analysis to the generalized case of directed graphs, where the interactions between nodes are not necessarily symmetric. We also plan to investigate the fundamental connections between the proposed notion of structural robustness and other graph metrics. Designing local graph transformation rules for distributed optimization of structural robustness, possibly subjected to some constraints on the network topology, is another direction we plan to explore. We also plan to investigate the fundamental trade-offs between the proposed measure of structural robustness and other system properties. For example, recently it was shown that the distances between the nodes have a major impact on the controllability of consensus networks and there are trade-offs between the controllability and robustness of such systems (e.g., [41], [42]). We believe that the results in this paper can be used for further investigation of such relationships between important system properties.

APPENDIX

This paper was focused on the robustness of consensus networks to noisy interactions. Another notion of robustness that is widely studied in the literature is the robustness of network connectivity to targeted (worst-case) node/edge failures. This notion is motivated by the fact that most networked systems (not only the consensus networks) rely on some flow (e.g., information, interactions, or substance) throughout the system, which can be realized properly only if the network is connected. Accordingly, there has been significant interest in using the connectivity-based graph measures for analyzing the robustness of networks to failures, proactively designing robust networks, improving robustness by adding/rewiring edges, or reactively recovering connectivity (e.g., [8], [9], [11], [12], [43]). Here, we provide some connections between the proposed measure of structural robustness to noise and the robustness to target node/edge failures. Through these connections, we also emphasize that random \( k \)-regular graphs are near optimal in terms of both notions of robustness among the graphs with same size and average degree for \( k \geq 3 \).

A graph is said to be \( k \)-node (or -edge) connected if at least \( k \) nodes (or edges) should be removed to render the graph disconnected. This basic measure treats connectivity as a binary property and does not take into account the size of disconnection. An arguably richer measure of connectivity is the edge (or node) expansion ratio (e.g., [44], [45]), which quantifies the ease of disconnecting a large part from the network by removing edges (or nodes). For any \( G = (V, E) \), the edge expansion ratio \( \phi_e(G) \), also known as the Cheeger constant or isoperimetric number, is defined as

\[
\phi_e(G) = \min_{S \subset V, |S| \leq \frac{|V|}{2}} \frac{|\{(i, j) \in E \mid i \in S, j \notin S\}|}{|S|}.
\] (28)

Similarly, the node expansion ratio \( \phi_n(G) \) is defined by considering the relative size of the node boundary rather than the edge boundary of \( S \subset V \), i.e.,

\[
\phi_n(G) = \min_{S \subset V, |S| \leq \frac{|V|}{2}} \frac{|\{j \in V \setminus S \mid i \in S, (i, j) \in E\}|}{|S|}.
\] (29)

It can be shown that for any connected graph \( G \), these two expansion ratios are related as

\[
d_{\text{max}}(G) \phi_n(G) \geq \phi_e(G) \geq \phi_n(G),
\] (30)

where \( d_{\text{max}}(G) \) is the maximum node degree of \( G \). If the expansion ratio of a graph is small, then it is possible to disconnect a large set of nodes by removing only a small number of edges (or nodes). Sparse graphs with expansion ratios staying bounded away from zero with increasing size are known as expander graphs. For further details on the expander graphs, we refer the readers to [46] and the references therein.

The edge expansion ratio is also closely tied to the smallest positive eigenvalue of the unweighted Laplacian (algebraic connectivity) via the Cheeger inequalities (e.g., see [47]), i.e,

\[
2\phi_e(G) \geq \lambda_2(L) \geq \frac{\phi_n^2(G)}{2d_{\text{max}}(G)}.
\] (31)

In light of (30) and (31), the expansion ratios and the algebraic connectivity are all similar in value for bounded-degree graphs, i.e., graphs with a constant upper bound on the node degrees. Accordingly, a bounded-degree graph is an expander if and only if the algebraic connectivity is bounded away from zero as size increases. For example, random \( k \)-regular graphs \( (k \geq 3) \) are expanders since almost all of them have \( \lambda_2(L) \geq k - 2\sqrt{k} - 1 - \epsilon \) for any \( \epsilon > 0 \) [40]. Hence, they are very robust not only to noise as we have shown in Theorem 3.6 but also to targeted node or edge removals. This desirable property separates them from many other graphs that are robust to some but not all of these three typical types of disturbances. For example, the star graph is fairly robust to noise and to targeted edge removals, but it is very fragile in the face of targeted node removals since removing the center node disconnects all the other nodes. Using (28) and (29), it can be shown that while the edge expansion ratio of the star graph is equal to one irrespective of size, its node expansion ratio is approximately \( 2/n \) (approaches zero with increasing size \( n \)). We show a star graph and a random 3-regular graph of same size, \( n = 20 \), in Fig. 6 to illustrate the contrast in their connectivity. For these two graphs we have computed the robustness measures as 1) star: \( H^* = 0.45, \phi_e = 1, \phi_n = 0.1 \), and 2) random 3-regular: \( H^* = 0.225, \phi_e = 0.6, \phi_n = 0.5 \). Accordingly, while the star has better robustness to targeted edge removals, the random 3-regular graph is better in the face of noisy interactions (smaller \( H^* \)). While the two graphs are somewhat similar in terms of these two measures, there is
a significant difference in their node expansion ratios, which would become even larger with increasing network size.

In order to obtain an explicit relationship between the expansion ratios and the structural robustness to noise, we first use (4) to link the algebraic connectivity to $\mathcal{H}^*(G)$ as

$$\mathcal{H}^*(G) = \frac{1}{2n} \sum_{i=2}^{n} \frac{1}{\lambda_i(L)} \leq \frac{n-1}{2n\lambda_2(L)}. \quad (32)$$

Using (31) and (32), we obtain

$$\mathcal{H}^*(G) \leq \frac{d_{\text{max}}(G)(n-1)}{\phi_n^2(G)n}. \quad (33)$$

Accordingly, there is a close relationship between the expansion ratios and the structural robustness to noise. Specifically, if a bounded-degree graph $G_n$ is an expander, then $\mathcal{H}^*(G_n)$ must remain finite as size increases due to (33). Note that $\mathcal{H}^*(G_n)$ is bounded away from zero for bounded-degree graphs as per Theorem 3.2. Hence, expanders necessarily have near optimal $\mathcal{H}^*(G_n)$ among the bounded-degree graphs of same size and sparsity. We have provided an explicit suboptimality bound in Theorem 3.6 for random $k$-regular graphs, and similar results can also be obtained for other expander families. It is also worth mentioning that, in addition to their desirable robustness properties, bounded-degree expanders have fast convergence (no matter their size) under the noise-free linear consensus dynamics since their algebraic connectivity, which determines the exponential convergence rate to consensus [43], is bounded away from zero.

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