QUANTIZATION OF THE PROBABILITY DISTRIBUTION ON THE SIERPİŃSKI CARPET

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ABSTRACT. Quantization of a probability distribution is the process of estimating a given probability by a discrete probability that assumes only a finite number of levels in its support. Let $P$ be a Borel probability measure on $\mathbb{R}^2$ which has support the Sierpiński carpet generated by the four contractive similarity mappings on $\mathbb{R}^2$ with equal similarity ratios. For this probability measure, in this paper, the optimal sets of $n$-means and the $n$th quantization errors are determined for all $n \geq 2$. In addition, it is shown that the quantization dimension of the measure $P$ exists, and coincides with the Hausdorff dimension of the Sierpiński carpet, the Hausdorff dimension and the packing dimensions of the probability measure $P$, but the quantization coefficient for $P$ does not exist.

1. INTRODUCTION

The history of the theory and practice of quantization dates back to 1948. Since then quantization has become an important field in electrical engineering in connection with signal processing and data compression. Broadly speaking, quantization consists in replacing an actual large data set of size $N$ by a smaller set of prototypes of size $n \leq N$. The best choice is when loss of information about the initial data set is minimum. A good survey about the historical development of the theory has been provided by Gray and Neuhoff in [GN]. For more applied aspects of quantization the reader is referred to the book of Gersho and Gray (see [GG]). For mathematical treatment of quantization one may consult Graf-Luschgy’s book (see [GL]). Interested readers can also see [AW, DR, GKL, GL]. Let $\mathbb{R}^d$ denote the $d$-dimensional Euclidean space, $\| \cdot \|$ denote the Euclidean norm on $\mathbb{R}^d$ for any $d \geq 1$, and $n \in \mathbb{N}$. Then the $n$th quantization error for a Borel probability measure $P$ on $\mathbb{R}^d$ is defined by

$$V_n := V_n(P) = \inf \left\{ \int_{\mathbb{R}^d} \min_{a \in \alpha} \| x - a \|^2 dP(x) : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n \right\},$$

where the infimum is taken over all subsets $\alpha$ of $\mathbb{R}^d$ with $\text{card}(\alpha) \leq n$. If $\int \| x \|^2 dP(x) < \infty$ then there is some set $\alpha$ for which the infimum is achieved (see [GL]). Such a set $\alpha$ for which the infimum occurs and contains no more than $n$ points is called an optimal set of $n$-means. If $\alpha$ is a finite set, in general, the error $\int \min_{a \in \alpha} \| x - a \|^2 dP(x)$ is often referred to as the cost or distortion error for $\alpha$, and is denoted by $V(P; \alpha)$. Thus, $V_n := V_n(P) = \inf \{ V(P; \alpha) : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n \}$. It is known that for a continuous probability measure $P$ an optimal set of $n$-means always has exactly $n$ elements (see [GL]). The numbers

$$D(P) := \liminf_{n \to \infty} \frac{2 \log n}{-\log V_n(P)}, \quad \overline{D}(P) := \limsup_{n \to \infty} \frac{2 \log n}{-\log V_n(P)},$$

are respectively called the lower and upper quantization dimensions of the probability measure $P$. If $D(P) = \overline{D}(P)$, the common value is called the quantization dimension of $P$ and is denoted by $D(P)$. Quantization dimension measures the speed at which the specified measure of the error tends to zero as $n$ approaches to infinity. For any $s \in (0, +\infty)$, the numbers

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lim inf \( n \frac{2}{d} V_n(P) \) and \( \limsup n \frac{2}{d} V_n(P) \) are respectively called the s-dimensional lower and upper quantization coefficients for \( P \). If the s-dimensional lower and upper quantization coefficients for \( P \) are finite and positive, then \( s \) coincides with the quantization dimension of \( P \). The quantization coefficients provide us with more accurate information about the asymptotics of the quantization error than the quantization dimension. Given a finite subset \( \alpha \subset \mathbb{R}^d \), the Voronoi region generated by \( a \in \alpha \) is defined by

\[
M(a|\alpha) = \{ x \in \mathbb{R}^d : \| x - a \| = \min_{b \in \alpha} \| x - b \| \}
\]
i.e., the Voronoi region generated by \( a \in \alpha \) is the set of all points in \( \mathbb{R}^d \) which are closest to \( a \in \alpha \), and the set \( \{ M(a|\alpha) : a \in \alpha \} \) is called the Voronoi diagram or Voronoi tessellation of \( \mathbb{R}^d \) with respect to \( \alpha \). The generator \( a \in \alpha \) is called the centroid of its own Voronoi region with respect to the probability distribution \( P \), if

\[
a = \frac{1}{P(M(a|\alpha))} \int_{M(a|\alpha)} x \, dP = \frac{\int_{M(a|\alpha)} x \, dP}{\int_{M(a|\alpha)} dP}.
\]
The following proposition provides further information on the Voronoi regions generated by an optimal set of \( n \)-means (see [GG,GL1]).

**Proposition 1.1.** Let \( \alpha \) be an optimal set of \( n \)-means, \( a \in \alpha \), and \( M(a|\alpha) \) be the Voronoi region generated by \( a \in \alpha \), i.e., \( M(a|\alpha) = \{ x \in \mathbb{R}^d : \| x - a \| = \min_{b \in \alpha} \| x - b \| \} \). Then, for every \( a \in \alpha \), (i) \( P(M(a|\alpha)) > 0 \), (ii) \( P(\partial M(a|\alpha)) = 0 \), (iii) \( a = E(X : X \in M(a|\alpha)) \), and (iv) \( P\text{-almost surely the set } \{ M(a|\alpha) : a \in \alpha \} \text{ forms a Voronoi partition of } \mathbb{R}^d \).

**Remark 1.2.** For a Borel probability measure \( P \) on \( \mathbb{R}^d \), an optimal set of \( n \)-means forms a centroidal Voronoi tessellation of \( \mathbb{R}^d \); however, the converse is not true in general (see [DFG,GL,R]).

It is known that the classical Cantor set \( C \) is generated by the two contractive similarity mappings \( U_1 \) and \( U_2 \) given by \( U_1(x) = \frac{1}{3} x \) and \( U_2(x) = \frac{1}{3} x + \frac{2}{3} \) for all \( x \in \mathbb{R} \). Let \( P_e \) be a Borel probability measure on \( \mathbb{R} \) such that \( P_e = \frac{1}{2} P_e \circ U_1^{-1} + \frac{1}{2} P_e \circ U_2^{-1} \), where \( P_e \circ U_i^{-1} \) denotes the image measure of \( P_e \) with respect to \( U_i \) for \( i = 1,2 \) (see [H], Theorem 4.4(1) for a generalization of self-similar measure). Then, \( P_e \) has support the Cantor set \( C \). For this probability measure Graf and Luschgy determined the optimal sets of \( n \)-means and the \( n \)th quantization error for all \( n \geq 1 \) (see [GL2]). In this paper, we have considered a Sierpiński carpet, denoted by \( S \), which is generated by the four contractive similarity mappings \( S_1, S_2, S_3 \) and \( S_4 \) on \( \mathbb{R}^2 \) such that \( S_1(x_1, x_2) = \frac{1}{3}(x_1, x_2) \), \( S_2(x_1, x_2) = \frac{1}{3}(x_1, x_2) + (\frac{2}{3}, 0) \), \( S_3(x_1, x_2) = \frac{1}{3}(x_1, x_2) + (0, \frac{2}{3}) \), and \( S_4(x_1, x_2) = \frac{1}{3}(x_1, x_2) + (\frac{2}{3}, \frac{2}{3}) \) for all \( (x_1, x_2) \in \mathbb{R}^2 \). Let \( P \) be a Borel probability measure on \( \mathbb{R}^2 \) such that \( P = \frac{1}{4} P \circ S_1^{-1} + \frac{1}{4} P \circ S_2^{-1} + \frac{1}{4} P \circ S_3^{-1} + \frac{1}{4} P \circ S_4^{-1} \). Then, \( P \) has support the Sierpiński carpet \( S \). For this probability measure \( P \), in this paper, we have determined the optimal sets of \( n \)-means and the \( n \)th quantization errors for all \( n \geq 2 \). In addition, we have shown that the quantization dimension of the probability measure \( P \) exists, and equals the Hausdorff dimension of the Sierpiński carpet, which again equals the Hausdorff dimension and the packing dimensions of the probability measure \( P \), but the quantization coefficient for \( P \) does not exist. For the Cantor distribution, for any \( n \)-points, one can easily determine whether the \( n \)-points form a CVT (see [GL2]), but for the probability distribution supported by the Sierpiński carpet, considered in this paper, for \( n \)-points sometimes it is quite difficult whether the points form a CVT. The technique we utilized can be extended to determine the optimal sets and the corresponding quantization error for many other singular continuous probability measures on \( \mathbb{R}^2 \), such as probability measures on more general Sierpiński carpets, probability measures on Sierpiński gaskets, etc.
2. Preliminaries

In this section, we give the basic definitions, lemmas and proposition that will be instrumental in our analysis. By a string or a word $\sigma$ over an alphabet $I := \{1, 2, 3, 4\}$, we mean a finite sequence $\sigma := \sigma_1 \sigma_2 \cdots \sigma_k$ of symbols from the alphabet, where $k \geq 1$, and $k$ is called the length of the word $\sigma$. A word of length zero is called the empty word, and is denoted by $\emptyset$. By $I^*$ we denote the set of all words over the alphabet $I$ of some finite length $k$ including the empty word $\emptyset$. For any two words $\sigma := \sigma_1 \sigma_2 \cdots \sigma_k$ and $\tau := \tau_1 \tau_2 \cdots \tau_\ell$ in $I^*$, by $\sigma \tau := \sigma_1 \cdots \sigma_k \tau_1 \cdots \tau_\ell$, we mean the word obtained from the concatenation of the two words $\sigma$ and $\tau$. For any two words $\sigma, \tau \in I^*$, we say that $\sigma$ is an extension of $\tau$ if $\sigma = \tau \varepsilon$ for some word $\varepsilon \in I^*$.

The maps $S_i : \mathbb{R}^2 \to \mathbb{R}^2$, defined in the previous section, for $1 \leq i \leq 4$, will be the generating maps of the Sierpiński carpet. For $\sigma := \sigma_1 \sigma_2 \cdots \sigma_k \in I^k$, set $S_\sigma := S_{\sigma_1} \circ \cdots \circ S_{\sigma_k}$ and $J_\sigma := S_\sigma([0, 1] \times [0, 1])$. For the empty word $\emptyset$, by $S_\emptyset$, we mean the identity mapping on $\mathbb{R}^2$, and write $J := J_\emptyset = S_\emptyset([0, 1] \times [0, 1]) = [0, 1] \times [0, 1]$. The elements of the set $\{J_\sigma : \sigma \in \{1, 2, 3, 4\}^k\}$ are just the $4^k$ squares in the $k$th level in the construction of the Sierpiński carpet. The squares $J_{\sigma_1}, J_{\sigma_2}, J_{\sigma_3}$ and $J_{\sigma_4}$ into which $J_\sigma$ is split up at the $(k + 1)$th level are called the children of $J_\sigma$. By the center of $J_\sigma$, we mean the point of intersection of the two diagonals of $J_\sigma$. The set $S := \cap_{k \in \mathbb{N}} \cup_{\sigma \in \{1, 2, 3, 4\}^k} J_\sigma$ is the Sierpiński carpet and equals the support of the probability measure $P$ given by $P = \frac{1}{4} P \circ S_{1}^{-1} + \frac{1}{4} P \circ S_{2}^{-1} + \frac{1}{4} P \circ S_{3}^{-1} + \frac{1}{4} P \circ S_{4}^{-1}$.

Let us now give the following lemma.

**Lemma 2.1.** Let $f : \mathbb{R} \to \mathbb{R}^+$ be Borel measurable and $k \in \mathbb{N}$. Then,

$$\int f \, dP = \frac{1}{4^k} \sum_{\sigma \in I^k} \int f \circ S_\sigma \, dP.$$

**Proof.** We know $P = \frac{1}{4} P \circ S_{1}^{-1} + \frac{1}{4} P \circ S_{2}^{-1} + \frac{1}{4} P \circ S_{3}^{-1} + \frac{1}{4} P \circ S_{4}^{-1}$, and so by induction $P = \sum_{\sigma \in I^k} \frac{1}{4^k} P \circ S_\sigma^{-1}$, and thus the lemma is yielded. \( \square \)

We now prove the following lemma.

**Proposition 2.2.** Let $P_1, P_2$ be the marginal distributions of $P$, i.e., $P_1(A) = P(A \times \mathbb{R})$ for all $A \in \mathcal{B}$, and $P_2(B) = P(\mathbb{R} \times B)$ for all $B \in \mathcal{B}$, where $\mathcal{B}$ is the Borel $\sigma$-algebra on $\mathbb{R}$. Then, $P_1 = P_2 = P_e$, where $P_e$ is the Cantor distribution generated by $U_1$ and $U_2$ as defined in the previous section.

**Proof.** Let $S_{(1)}, S_{(2)}$ be the horizontal and vertical components of the transformations $S_i$ for $i = 1, 2, 3, 4$. Then, for any $(x_1, x_2) \in \mathbb{R}^2$ we have $S_{(1)}(x_1) = \frac{1}{3} x_1, S_{(2)}(x_2) = \frac{1}{3} x_2, S_{(21)}(x_1) = \frac{1}{3} x_1 + \frac{2}{3}, S_{(22)}(x_2) = \frac{1}{3} x_2, S_{(31)}(x_1) = \frac{1}{3} x_1, S_{(32)}(x_2) = \frac{1}{3} x_2 + \frac{2}{3}$, and $S_{(41)}(x_1) = \frac{1}{3} x_1 + \frac{2}{3}, S_{(42)}(x_2) = \frac{1}{3} x_2 + \frac{2}{3}$. But, $U_1(x) = \frac{1}{3} x$ and $U_2(x) = \frac{1}{3} x + \frac{2}{3}$ for all $x \in \mathbb{R}$, and so

$$S_{11} = S_{12} = S_{22} = S_{31} = U_1 \text{ and } S_{21} = S_{32} = S_{41} = S_{42} = U_2.$$

Again, for any $A \in \mathcal{B}$, we have

$$\left(\frac{1}{4} P_1 \circ S_{(1)}^{-1} + \frac{1}{4} P_1 \circ S_{(2)}^{-1} + \frac{1}{4} P_1 \circ S_{(3)}^{-1} + \frac{1}{4} P_1 \circ S_{(4)}^{-1}\right)(A) \text{ for all } A \in \mathcal{B}.$$
implying \( P_1 = \frac{1}{4} P_1 \circ S_{(11)}^{-1} + \frac{1}{4} P_1 \circ S_{(21)}^{-1} + \frac{1}{4} P_1 \circ S_{(31)}^{-1} + \frac{1}{4} P_1 \circ S_{(41)}^{-1}, \) which by the relation (1) yields 
\( P_1 = \frac{1}{2} P_1 \circ U_1^{-1} + \frac{1}{2} P_1 \circ U_2^{-1}. \) But, \( P_c \) is unique satisfying \( P_c = \frac{1}{2} P_c \circ U_1^{-1} + \frac{1}{2} P_c \circ U_2^{-1}, \) and hence \( P_1 = P_c. \) Similarly, we have \( P_2 = P_c. \) Thus, the proof of the proposition is complete.

For words \( \beta, \gamma, \ldots, \delta \) in \( I^* \), by \( a(\beta, \gamma, \ldots, \delta) \) we mean the conditional expectation of the random variable \( X \) given \( J_\beta \cup J_\gamma \cup \cdots \cup J_\delta, \) i.e.,

\[
a(\beta, \gamma, \ldots, \delta) = E(X \mid X \in J_\beta \cup J_\gamma \cup \cdots \cup J_\delta) = \frac{1}{P(J_\beta \cup \cdots \cup J_\delta)} \int_{J_\beta \cup \cdots \cup J_\delta} x dP.
\]

By \( \int x dP \) it is meant \( \int (x_1, x_2) dP. \) Let us now give the following lemma.

**Lemma 2.3.** Let \( X := (X_1, X_2) \) be a bivariate random variable with distribution \( P. \) Let \( E(X) \) and \( V(X) \) denote the expected vector and the expected squared distance of the random variable \( X. \) Then, \( E(X) = (E(X_1), E(X_2)) = (\frac{1}{2}, \frac{1}{2}) \) and \( V := V(X) = E||X - (\frac{1}{2}, \frac{1}{2})||^2 = \frac{1}{4}. \)

**Proof.** Since \( P_1 \) and \( P_2 \) are the marginal distributions of \( X := (X_1, X_2), \) the random variables \( X_1 \) and \( X_2 \) have distributions \( P_1 \) and \( P_2, \) respectively. Again, By Proposition 2.2, we have \( P_1 = P_2 = P, \) and hence both \( X_1 \) and \( X_2 \) are \( P \)-distributed random variables. Thus, by [GL2, Lemma 3.4], we obtain \( E(X_1) = E(X_2) = \frac{1}{2}, \) and \( V(X_1) = V(X_2) = \frac{1}{8}, \) and then

\[
E||X - (\frac{1}{2}, \frac{1}{2})||^2 = E(X_1 - \frac{2}{3})^2 + E(X_2 - \frac{2}{3})^2 = V(X_1) + V(X_2) = \frac{1}{4}.
\]

Hence, the lemma is yielded.

Now, the following two notes are in order.

**Note 2.4.** For any \( (a, b) \in \mathbb{R}^2, \)

\[
E||X - (a, b)||^2 = \int_{\mathbb{R}^2} [(x_1 - a)^2 + (x_2 - b)^2] dP(x_1, x_2)
\]

\[
= \int_{x_1 = -\infty}^{\infty} (x_1 - a)^2 \int_{x_2 = -\infty}^{\infty} dP(x_1, x_2) + \int_{x_2 = -\infty}^{\infty} (x_2 - b)^2 \int_{x_1 = -\infty}^{\infty} dP(x_1, x_2)
\]

\[
= \int_0^1 (x_1 - a)^2 dP_1(x_1) + \int_0^1 (x_2 - b)^2 dP_2(x_2) = E(X_1 - a)^2 + E(X_2 - b)^2
\]

\[
= V(X_1) + V(X_2) + (a - \frac{1}{2})^2 + (b - \frac{1}{2})^2 = V + ||(a, b) - (\frac{1}{2}, \frac{1}{2})||^2.
\]

In fact, for any \( \sigma \in I^k, k \geq 1, \) we have

\[
\int_{J_\sigma} ||x - (a, b)||^2 dP = \frac{1}{4^k} \int ||x - (a, b)||^2 dP \circ S_\sigma^{-1},
\]

which implies

\[
\int_{J_\sigma} ||x - (a, b)||^2 dP = \frac{1}{4^k} \left( \frac{1}{9} V + ||S_\sigma(\frac{1}{2}, \frac{1}{2}) - (a, b)||^2 \right).
\]

**Note 2.5.** From Lemma 2.3 it follows that the optimal set of one-mean is the expected vector and the corresponding quantization error is the expected squared distance \( V \) of the random variable \( X. \) For \( \sigma \in I^k, k \geq 1, \) since \( a(\sigma) = E(X : X \in J_\sigma), \) using Lemma 2.1 we have

\[
a(\sigma) = \frac{1}{P(J_\sigma)} \int_{J_\sigma} x dP(x) = \int_{J_\sigma} x dP \circ S_\sigma^{-1}(x) = \int S_\sigma(x) dP(x) = E(S_\sigma(X)).
\]

Since \( S_i \) are similarity mappings, it is easy to see that \( E(S_j(X)) = S_j(E(X)) \) for \( j = 1, 2, 3, 4 \) and so by induction, \( a(\sigma) = E(S_\sigma(X)) = S_\sigma(E(X)) = S_\sigma(\frac{1}{2}, \frac{1}{2}) \) for \( \sigma \in I^k, k \geq 1. \)
3. Optimal sets of $n$-means and the quantization errors for all $n \geq 2$

In this section, we first determine the optimal sets of $n$-means for $n = 2$ and $n = 3$. To determine the distortion error in the sequel we will frequently use the relation (3).

Lemma 3.1. The points in an optimal set of two-means can not lie on an oblique line of the Sierpiński carpet.

Proof. Let us consider the two-point set $\beta \subset \mathbb{R}^2$ given by $\beta := \{(\frac{1}{6}, \frac{1}{2}), (\frac{5}{6}, \frac{1}{2})\}$. Notice that the boundary of the Voronoi regions of the two points in $\beta$ is the line $x_1 = \frac{1}{2}$, i.e., $J_1 \cup J_3 \subset M((\frac{1}{6}, \frac{1}{2})|\beta)$ and $J_2 \cup J_4 \subset M((\frac{5}{6}, \frac{1}{2})|\beta)$. Let $V_{2,1}$ be the distortion error due to the set $\beta$. Then,

$$V_{2,1} = \int_{J_1 \cup J_3} \|x - (\frac{1}{6}, \frac{1}{2})\|^2 dP + \int_{J_2 \cup J_4} \|x - (\frac{5}{6}, \frac{1}{2})\|^2 dP = \frac{5}{36} = 0.138889.$$

In the Sierpiński carpet among all the oblique lines that pass through the point $(\frac{1}{2}, \frac{1}{2})$, the two diagonals have the maximum symmetry, i.e., with respect to the two diagonals the Sierpiński carpet is geometrically symmetric as well as symmetric with respect to the probability distribution $P$. By the symmetric with respect to the probability distribution $P$, it is meant that if the two basic squares of similar geometrical shape lie in the opposite sides of any of the diagonals, and are equidistant from the diagonal, then they have the same probability. Due to this, among all the pairs of two points which have the boundaries of the Voronoi regions oblique lines passing through the point $(\frac{1}{2}, \frac{1}{2})$, the two points which have the boundary of the Voronoi regions any of the two diagonals will give the smallest distortion error. Again, we know that the two points which give the smallest distortion error are the centroids of their own Voronoi regions. Let $(a_1, b_1)$ and $(a_2, b_2)$ be the centroids of the left half and the right half of the Sierpiński carpet with respect to the diagonal passing through the origin. We now look at the following two sums:

$$\frac{5}{24} + \frac{11}{144} + \frac{29}{864} + \frac{83}{5184} + \frac{245}{31104} + \frac{731}{186624} + \frac{2189}{1119744} + \cdots$$

$$= \frac{1}{4}\left(\frac{5}{6} + \frac{11}{6^2} + \frac{29}{6^3} + \frac{83}{6^4} + \frac{245}{6^5} + \frac{731}{6^6} + \frac{2189}{6^7} + \cdots\right)$$

$$= \frac{1}{4} \sum_{i=1}^{\infty} \frac{3^i + 2}{6^i} = \frac{1}{4} \sum_{i=2}^{\infty} \frac{1}{6^i} = \frac{1}{4} \cdot \frac{\frac{1}{2}}{\frac{1}{2} - \frac{1}{6}} = 1 \cdot \frac{\frac{1}{2}}{\frac{1}{2} - \frac{1}{6}} = \frac{7}{20}$$

and

$$\frac{1}{24} + \frac{7}{144} + \frac{25}{864} + \frac{79}{5184} + \frac{241}{31104} + \frac{727}{186624} + \frac{2185}{1119744} + \cdots$$

$$= \frac{1}{4}\left(\frac{1}{6} + \frac{7}{6^2} + \frac{25}{6^3} + \frac{79}{6^4} + \frac{241}{6^5} + \frac{727}{6^6} + \frac{2185}{6^7} + \cdots\right)$$

$$= \frac{1}{4} \sum_{i=1}^{\infty} \frac{3^i - 2}{6^i} = \frac{1}{4} \sum_{i=2}^{\infty} \frac{1}{6^i} = \frac{1}{4} \cdot \frac{\frac{1}{2}}{\frac{1}{2} - \frac{1}{6}} = 1 \cdot \frac{\frac{1}{2}}{\frac{1}{2} - \frac{1}{6}} = \frac{3}{20}$$

Thus, using (2), we have

$$(a_1, a_2) = E(X : X \in J_2 \cup (J_1 \cup J_2) \cup (J_1 \cup J_3) \cup (J_2 \cup J_3) \cup (J_4 \cup J_2) \cup (J_3 \cup J_2) \cup (J_4 \cup J_3) \cup \cdots)$$

$$= 2 \left(\frac{1}{4} S_2(\frac{1}{2}, \frac{1}{2}) + \frac{1}{48} \left(S_{12}(\frac{1}{2}, \frac{1}{2}) + S_{24}(\frac{1}{2}, \frac{1}{2})\right) + \frac{1}{48} \left(S_{112}(\frac{1}{2}, \frac{1}{2}) + S_{142}(\frac{1}{2}, \frac{1}{2})\right) + \frac{1}{4} \left(S_{412}(\frac{1}{2}, \frac{1}{2}) + S_{424}(\frac{1}{2}, \frac{1}{2})\right) + \cdots\right)$$

$$= 2 \left(\frac{5}{24} + \frac{11}{144} + \frac{29}{864} + \frac{83}{5184} + \frac{1}{24} + \frac{7}{144} + \frac{25}{864} + \frac{79}{5184} + \cdots\right) = 2 \left(\frac{7}{20} \cdot \frac{3}{20}\right).
i.e.,

\[ (a_1, a_2) = \left( \frac{7}{10}, \frac{3}{10} \right). \]

Similarly, one can show that

\[ (a_2, b_2) = E \left( X : X \in J_3 \cup (J_{13} \cup J_{43}) \cup (J_{113} \cup J_{143} \cup J_{413} \cup J_{443}) \cup \cdots \right) = \left( \frac{3}{10}, \frac{7}{10} \right). \]

Hence, if \( V_{2,2} \) is the distortion error due to the points \( \left( \frac{7}{10}, \frac{3}{10} \right) \) and \( \left( \frac{3}{10}, \frac{7}{10} \right) \), then we have

\[
V_{2,2} = \sum_{j=1}^{4} \int_{J_{j2} \cup J_{j4}} \| (x_1, x_2) - \left( \frac{7}{10}, \frac{3}{10} \right) \|^2 dP
+ \sum_{j=1}^{4} \int_{J_{j12} \cup J_{j42} \cup J_{j142} \cup J_{j412} \cup J_{j442} \cup J_{j444}} \| (x_1, x_2) - \left( \frac{7}{10}, \frac{3}{10} \right) \|^2 dP
\]

\[
= 2 \left( 0.0747492 \right) = 0.1494984 > V_{2,1},
\]

Since \( V_{2,2} > V_{2,1} \), the points in an optimal set of two-means cannot lie on an oblique line of the Sierpiński carpet. Thus, the lemma follows.

The following proposition gives all the optimal sets of two-means.

**Proposition 3.2.** The set \{\( \left( \frac{1}{6}, \frac{1}{2} \right), \left( \frac{5}{6}, \frac{1}{2} \right) \)\} and \{\( \left( \frac{1}{2}, \frac{1}{6} \right), \left( \frac{1}{2}, \frac{5}{6} \right) \)\} form two different optimal sets of two-means with quantization error \( V_2 = \frac{5}{30} = 0.138889 \).

**Proof.** By Lemma 3.1, the points in an optimal set of two-means cannot lie on an oblique line of the Sierpiński carpet. Thus, the two optimal points lie either on a horizontal line or on a vertical line. Let us first assume that they lie on a horizontal line. Let \( \alpha := \{(a, p), (b, p)\} \) be an optimal set of two-means. Since the optimal points are the centroids of their own Voronoi regions, by the properties of centroids, we have

\[
(a, p) P(M((a, p)|\alpha)) + (b, p) P(M((b, p)|\alpha)) = \left( \frac{1}{2}, \frac{1}{2} \right),
\]

which implies \( a P(M((a, p)|\alpha)) + b P(M((b, p)|\alpha)) = p P(M((a, p)|\alpha)) + p P(M((b, p)|\alpha)) = \frac{1}{2} \).

Thus, we see that \( p = \frac{1}{2} \), and the two optimal points \( (a, \frac{1}{2}) \) and \( (b, \frac{1}{2}) \) lie on the line \( x_2 = \frac{1}{2} \) and are in opposite sides of the point \( (\frac{1}{2}, \frac{1}{2}) \). Again, the optimal points are the centroids of their own Voronoi regions which implies \( 0 \leq a < \frac{1}{2} < b \leq 1 \). Thus,

\[
V_2 = \int_{c \in \alpha} \min_{c} \| x - c \|^2 dP = \int_{c \in \{a, b\}} \min_{c \in \{a, b\}} \| x - (c, \frac{1}{2}) \|^2 dP
\]

\[
= \int_{0, \frac{a+b}{2}} \| x - (a, \frac{1}{2}) \|^2 dP + \int_{\frac{a+b}{2}, 1} \| x - (b, \frac{1}{2}) \|^2 dP
\]

\[
= \int_{0, \frac{a+b}{2}} \left( (x_1 - a)^2 + (x_2 - \frac{1}{2})^2 \right) d(P_c \times P_c) + \int_{\frac{a+b}{2}, 1} \left( (x_1 - b)^2 + (x_2 - \frac{1}{2})^2 \right) d(P_c \times P_c)
\]

\[
= \int_{0, \frac{a+b}{2}} (x_1 - a)^2 dP_c + \int_{0, \frac{a+b}{2}} (x_1 - b)^2 dP_c + \int_{\frac{a+b}{2}, 1} (x_1 - a)^2 dP_c + \int_{\frac{a+b}{2}, 1} (x_1 - b)^2 dP_c
\]

\[
= \int_{c \in \{a, b\}} (x_1 - c)^2 dP_c + \int_{x_2 = \frac{1}{2}} (x_2 - \frac{1}{2})^2 dP_c.
\]
Notice that \( \int_{c \in \{a,b\}} (x-c)^2 dP_c \) represents the \( n \)th quantization error for the probability distribution \( P_c \) when \( n = 2 \), and so, by [GL2] Proposition 4.6, we have \( \int_{c \in \{a,b\}} (x-c)^2 dP_c = \frac{1}{72} \) and it occurs when \( a = \frac{1}{6} \) and \( b = \frac{5}{6} \). Moreover, \( \int (x_2 - \frac{1}{2})^2 dP_c = \frac{1}{8} \). Thus, we deduce that
\[
V_2 = 1 - \frac{1}{72} + \frac{1}{8} = \frac{5}{36} = 0.138889,
\]
and \( \{(\frac{1}{6}, \frac{1}{2}), (\frac{5}{6}, \frac{1}{2})\} \) is an optimal set of two-means. Due to symmetry, the set \( \{(\frac{1}{2}, \frac{1}{6}), (\frac{1}{2}, \frac{5}{6})\} \) also forms an optimal set of two-means. Hence, the proof of the proposition is complete. \( \square \)

We now prove the following two lemmas.

**Lemma 3.3.** The set \( \alpha_3 = \{(\frac{1}{6}, \frac{1}{2}), (\frac{5}{6}, \frac{1}{2}), (\frac{1}{2}, \frac{5}{6})\} \) forms a CVT with three-means and the corresponding distortion error is \( \frac{1}{12} \).

**Proof.** Recall that the boundaries of the Voronoi regions lie along the perpendicular bisectors of the line segments joining their centers. The perpendicular bisectors of the line segments joining each pair of points from the list \( \{(\frac{1}{6}, \frac{1}{2}), (\frac{5}{6}, \frac{1}{2}), (\frac{1}{2}, \frac{5}{6})\} \) are SW, TW and UW with equations respectively \( x_1 = \frac{1}{2} \), \( x_2 = \frac{1}{2}x_1 + \frac{1}{6} \) and \( x_2 = -\frac{1}{2}x_1 + \frac{5}{6} \), and they concur at the point \( W(\frac{1}{2}, \frac{5}{12}) \) as shown in Figure 1 (a). Thus, the three regions WUOS, WSAT and WTBCU form a Voronoi tessellation of the Sierpiński carpet. Let us denote the three regions respectively by \( M_1 \), \( M_2 \) and \( M_3 \). If \( (p_1, p_2), (q_1, q_2) \) and \( (r_1, r_2) \) are the centroids of these three regions respectively associated with the probability measure \( P \), we have
\[
(p_1, p_2) = \frac{1}{P(M_1)} \int_{M_1} xdP = \frac{1}{P(J_1)} \int_{J_1} xdP = \int xd(P \circ S_1^{-1}) = S_1(\frac{1}{2}, \frac{1}{2}) = (\frac{1}{6}, \frac{1}{6}),
\]
\[
(q_1, q_2) = \frac{1}{P(M_2)} \int_{M_2} xdP = \frac{1}{P(J_2)} \int_{J_2} xdP = \int xd(P \circ S_2^{-1}) = S_2(\frac{1}{2}, \frac{1}{2}) = (\frac{5}{6}, \frac{1}{6}),
\]
\[
(r_1, r_2) = \frac{1}{P(M_3)} \int_{M_3} xdP = \frac{1}{P(J_3 \cup J_4)} \int_{J_3 \cup J_4} xdP = \frac{1}{P(J_3 \cup J_4)} \left( \int_{J_3} xdP + \int_{J_4} xdP \right)
\]
\[
= \frac{1}{P(J_3 \cup J_4)} (P(J_3) \int xd(P \circ S_3^{-1}) + P(J_4) \int xd(P \circ S_4^{-1}))
\]
\[
= 2 \left( \frac{1}{4}S_3(\frac{1}{2}, \frac{1}{2}) + \frac{1}{4}S_4(\frac{1}{2}, \frac{1}{2}) \right) = (\frac{1}{2}, \frac{5}{6}).
\]
Thus, we see that the given set \( \alpha_3 \) forms a CVT with three-means. Then use (3), and calculate the corresponding distortion error as

\[
\int_{a \in \alpha_3} \min_{a} \|x - a\|^2 dP = \int_{J_1} \|x - (\frac{1}{6}, \frac{1}{6})\|^2 dP + \int_{J_2} \|x - (\frac{5}{6}, \frac{1}{6})\|^2 dP + \int_{J_3 \cup J_4} \|x - (\frac{1}{2}, \frac{5}{6})\|^2 dP
\]

\[
= \frac{1}{36} V + \frac{1}{36} (2V + \|S_3(\frac{1}{2}, \frac{1}{2}) - (\frac{1}{2}, \frac{5}{6})\|^2 + \|S_4(\frac{1}{2}, \frac{1}{2}) - (\frac{1}{2}, \frac{5}{6})\|^2) = \frac{1}{12}.
\]

Thus, the proof of the lemma is complete. \( \square \)

**Remark 3.4.** The elements in the set \( \alpha_3 = \{ (\frac{1}{6}, \frac{1}{6}), (\frac{5}{6}, \frac{1}{6}), (\frac{1}{2}, \frac{5}{6}) \} \) given by Lemma 3.3 form an isosceles triangle. Due to rotational symmetry there are four such sets giving the same distortion error \( \frac{1}{12} \).

**Lemma 3.5.** The set \( \beta_3 = \{ (\frac{5}{6}, \frac{1}{6}), (\frac{13}{30}, \frac{19}{30}), (\frac{19}{30}, \frac{13}{30}) \} \) forms a CVT with three-means and the corresponding distortion error is larger than \( \frac{1}{12} \).

**Proof.** The perpendicular bisectors of the line segments joining each pair of points from the list \( \{ (\frac{5}{6}, \frac{1}{6}), (\frac{13}{30}, \frac{19}{30}), (\frac{19}{30}, \frac{13}{30}) \} \) are SW, OW and TW with equations respectively \( x_2 = \frac{979}{460} - \frac{31x_1}{9} \), \( x_2 = x_1 \) and \( x_2 = \frac{979}{1800} - \frac{9x_1}{31} \), and they concur at the point \( W(\frac{979}{1800}, \frac{979}{1800}) \) as shown in Figure 1 (b). Let \( (p_1, p_2) \), \( (q_1, q_2) \) and \( (r_1, r_2) \) be the centroids of the three Voronoi regions with centers respectively \( P_1(\frac{13}{30}, \frac{19}{30}) \), \( Q_1(\frac{19}{30}, \frac{13}{30}) \) and \( R_1(\frac{5}{6}, \frac{1}{6}) \). Since the similarity mappings preserve the ratio of the distances of a point from any other two points, by (4) and (5), with respect to the probability measure \( P \) the centroids of the triangles \( OBC \) and \( OAB \) are obtained as \( S_1(\frac{3}{10}, \frac{7}{10}) = (\frac{3}{30}, \frac{7}{30}) \) and \( S_1(\frac{7}{10}, \frac{3}{10}) = (\frac{7}{30}, \frac{3}{30}) \), respectively. Therefore, using the definition of centroids, we have

\[
(p_1, p_2) = \frac{1}{P(\Delta OBC)} \left( P(J_3) \int_{J_3} xdP + P(\Delta OBC) \int_{\Delta OBC} xdP \right)
\]

\[
= \frac{1}{4} \left( \frac{1}{4} \left( \frac{1}{6}, \frac{5}{6} \right) + \frac{1}{8} \left( \frac{3}{30}, \frac{7}{30} \right) \right) = \left( \frac{13}{90}, \frac{19}{30} \right),
\]

\[
(q_1, q_2) = \frac{1}{P(\Delta OAB)} \left( P(J_2) \int_{J_2} xdP + P(\Delta OAB) \int_{\Delta OAB} xdP \right)
\]

\[
= \frac{1}{4} \left( \frac{1}{4} \left( \frac{1}{6}, \frac{5}{6} \right) + \frac{1}{8} \left( \frac{7}{30}, \frac{3}{30} \right) \right) = \left( \frac{19}{30}, \frac{13}{90} \right),
\]

\[
(r_1, r_2) = S_4(\frac{1}{2}, \frac{1}{2}) = \left( \frac{5}{6}, \frac{5}{6} \right).
\]
Thus, we see that the given set \( \beta_3 \) forms a CVT with three-means. Then use (3), and calculate the corresponding distortion error as

\[
\int \min_{a \in \beta_3} \| x - a \|^2 dP = \left( \text{distortion error due to the point } \left( \frac{5}{6}, \frac{5}{6} \right) \right) + 2 \left( \text{distortion error due to the point } \left( \frac{13}{90}, \frac{19}{30} \right) \right)
\]

\[
> \frac{1}{36} V + 2 \left( \int_{J_3} \| x - \left( \frac{13}{90}, \frac{19}{30} \right) \|^2 dP + \int_{J_{13}} \| x - \left( \frac{13}{90}, \frac{19}{30} \right) \|^2 dP 
\right)
\]

\[
+ \int_{J_{1113} \cup J_{1443}} \| x - \left( \frac{13}{90}, \frac{19}{30} \right) \|^2 dP + \int_{J_{1113} \cup J_{1433} \cup J_{1413} \cup J_{14113} \cup J_{141113} \cup J_{1411113} \cup J_{14111113} \cup J_{141111113}} \| x - \left( \frac{13}{90}, \frac{19}{30} \right) \|^2 dP
\]

\[
= \frac{1247143}{14929920} = 0.0835331 > 0.0833333 = \frac{1}{12}.
\]

Thus, the proof of the lemma is complete.

\[ \square \]

**Remark 3.6.** In the CVT \( \beta_3 \) given by Lemma 3.3, one point is the centroid of the child \( J_4 \) and the other two points are equidistant from the diagonal passing through the centroid. Due to rotational symmetry of the Sierpinski carpet, there are four such CVTs with three-means in which one point is the centroid of one of the children \( J_1, J_2, J_3 \) or \( J_4 \) and the other two points are equidistant from the diagonal passing through the centroid, and all have the same distortion error larger than \( \frac{1}{12} \).

The following proposition identifies the optimal sets of three-means and associated quantization error.

**Proposition 3.7.** Let \( \alpha_3 \) be the set given by Lemma 3.3. Then, \( \alpha_3 \) forms an optimal set of three-means with quantization error \( \frac{1}{12} \). The number of optimal sets of three-means is four.

**Proof.** The children at each level of the Sierpinski carpet construction are symmetrically distributed over the square, and they each have equal weight with respect to the probability measure \( P \), and so we can say that one point in an optimal set of three-means lies on a line of symmetry of the square and the other two points are equidistant from the line of symmetry. The square has two different kinds of symmetry: one is a diagonal of the square and one is a perpendicular bisector of the two opposite sides of the square. Comparing Lemma 3.3 and Remark 3.6, we can say that the set \( \alpha_3 \) given by Lemma 3.3 forms an optimal set of three-means with quantization error \( \frac{1}{12} \). By Remark 3.4, we see that the number of optimal sets of three-means is four.

**Remark 3.8.** Lemma 3.3 and Lemma 3.5 together show that under squared error distortion measure, the centroid condition is not sufficient for optimal quantization for singular continuous probability measures on \( \mathbb{R}^2 \), which is already known for absolutely continuous probability measures on \( \mathbb{R}^2 \) (see [DFG]), and for singular continuous probability measure on \( \mathbb{R} \) (see [R]).

The following proposition plays an important role in the paper.

**Lemma 3.9.** Let \( n \geq 4 \). Let \( \alpha_n \) be an optimal set of \( n \)-means such that \( \alpha_n \cap J_i \neq \emptyset \) for \( 1 \leq i \leq 4 \), and \( \alpha_n \) does not contain any point from \( J \setminus \bigcup_{i=1}^{4} J_i \). Set \( \beta_i := \alpha_n \cap J_i \) and \( n_i := \text{card}(\beta_i) \). Then, \( S_i^{-1}(\beta_i) \) is an optimal set of \( n_i \)-means. Moreover,

\[
V_n = \frac{1}{36} \left( V_{n_1} + V_{n_2} + V_{n_3} + V_{n_4} \right).
\]
Figure 2. Optimal configuration of $n$ points for $1 \leq n \leq 15$. 
Proof. By the hypothesis the sets $\beta_i$ are nonempty for all $1 \leq i \leq 4$. Since $\alpha_n$ does not contain any point from $J \setminus \cup_{i=1}^4 J_i$, we have $\alpha_n = \cup_{i=1}^4 \beta_i$. $\alpha_n$ is an optimal set of $n$-means, and so

$$V_n = \sum_{i=1}^4 \int_{J_i} \min_{a \in \alpha} \|x - a\|^2 dP = \sum_{i=1}^4 \int_{J_i} \min_{a \in \beta_i} \|x - a\|^2 dP.$$  

Now, using Lemma 2.1 and the definitions of the mappings $S_i$, we have

$$V_n = \frac{1}{36} \sum_{i=1}^4 \int_{J_i} \min_{a \in \beta_i} \|x - S_i^{-1}(a)\|^2 dP = \frac{1}{36} \sum_{i=1}^4 \int_{J_i} \min_{a \in S_i^{-1}(\beta_i)} \|x - a\|^2 dP.$$  

If $S_i^{-1}(\beta_i)$ is not an optimal set of $n_1$-means, then we can find a set $\gamma_1 \subset \mathbb{R}^2$ with $\text{card}(\gamma_1) = n_1$ such that

$$\int_{\gamma_1} \min_{a \in \gamma_1} \|x - a\|^2 dP < \int_{S_i^{-1}(\beta_i)} \min_{a \in S_i^{-1}(\beta_i)} \|x - a\|^2 dP.$$  

But, then $S_1(\gamma_1) \cup \beta_2 \cup \beta_3 \cup \beta_4$ will be a set of cardinality $n$, and

$$\int \min\{\|x - a\|^2 : a \in S_1(\gamma_1) \cup \beta_2 \cup \beta_3 \cup \beta_4\} dP$$

$$= \int J_1 \min_{a \in S_1(\gamma_1)} \|x - a\|^2 dP + \frac{1}{36} \sum_{i=2}^4 \int_{J_i} \min_{a \in S_i^{-1}(\beta_i)} \|x - a\|^2 dP$$

$$= \frac{1}{36} \int_{J_1} \min_{a \in S_1(\gamma_1)} \|x - S_1^{-1}(a)\|^2 dP + \frac{1}{36} \sum_{i=2}^4 \int_{J_i} \min_{a \in S_i^{-1}(\beta_i)} \|x - a\|^2 dP$$

$$= \frac{1}{36} \int_{J_1} \min_{a \in \gamma_1} \|x - a\|^2 dP + \frac{1}{36} \sum_{i=2}^4 \int_{J_i} \min_{a \in S_i^{-1}(\beta_i)} \|x - a\|^2 dP$$

$$< \frac{1}{36} \int_{J_1} \min_{a \in S_1^{-1}(\beta_1)} \|x - a\|^2 dP + \frac{1}{36} \sum_{i=2}^4 \int_{J_i} \min_{a \in S_i^{-1}(\beta_i)} \|x - a\|^2 dP.$$  

Thus by (6), we have $\int \min\{\|x - a\|^2 : a \in S_1(\gamma_1) \cup \beta_2 \cup \beta_3 \cup \beta_4\} dP < V_n$, which contradicts the fact that $\alpha_n$ is an optimal set of $n$-means, and so $S_i^{-1}(\beta_i)$ is an optimal set of $n_1$-means. Similarly, one can show that $S_i^{-1}(\beta_i)$ are optimal sets of $n_i$-means for all $2 \leq i \leq 4$. Thus, (6) implies $V_n = \frac{1}{36} (V_{n_1} + V_{n_2} + V_{n_3} + V_{n_4})$. This completes the proof of the lemma.

Let us now give the following proposition.

**Proposition 3.10. Let $n \geq 4$. Let $\alpha_n$ be an optimal set of $n$-means such that $\alpha_n \cap J_\sigma \neq \emptyset$ for $\sigma \in I(\ell(n))$ for some $\ell(n) \in \mathbb{N}$, and $\alpha_n$ does not contain any point from $J \setminus \cup_{\sigma \in I(\ell(n))} J_\sigma$. Set $\beta_\sigma := \alpha_n \cap J_\sigma$ and $n_\sigma := \text{card}(\beta_\sigma)$. Then, $S_\sigma^{-1}(\beta_\sigma)$ is an optimal set of $n_\sigma$-means. Moreover,

$$V_n = \frac{1}{36(\ell(n))} \sum_{\sigma \in I(\ell(n))} V_{n_\sigma}.$$  

**Figure 3.** Optimal configuration of $n$ points for $16 \leq n \leq 18$.  

Quantization of the probability distribution on the Sierpiński carpet 11
Proof. Since the similarity mappings preserve the ratio of the distances of a point from any other two points, the proof of the proposition follows by Lemma 3.9.

**Lemma 3.11.** Let $\alpha_n$ be an optimal set of $n$-means for $n \geq 4$. Then, $\alpha_n \cap J_i \neq \emptyset$ for $1 \leq i \leq 4$.

Proof. Recall that the Sierpiński carpet has four lines of symmetry: the two diagonals, the horizontal line $x_2 = \frac{1}{2}$, and the vertical line $x_1 = \frac{1}{2}$. Let $\alpha_n$ be an optimal set of $n$-means for $n \geq 4$. Consider the set $\beta := \{S_i(\frac{1}{2}, \frac{1}{2}) : 1 \leq i \leq 4\}$. Then, the distortion error due to the set $\beta$ is given by

$$\int \min_{a \in \beta} \|x-a\|^2 dP = \sum_{i=1}^{4} \int_{J_i} \|x-S_i(\frac{1}{2}, \frac{1}{2})\|^2 dP = \frac{1}{9} V = \frac{1}{36}.$$

Since $V_n$ is the quantization error for $n$-means for $n \geq 4$, we have $V_n \leq V_4 \leq \frac{1}{36} = 0.0277778$. Suppose that all the elements of $\alpha_n$ lie on the horizontal line $x_2 = \frac{1}{2}$. Then, for any $x \in \bigcup_{i=1}^{4} J_i$, $\min_{a \in \alpha_n} \|x-a\|^2 \geq (\frac{1}{2} - \frac{1}{9})^2 = \frac{49}{324}$, and so the distortion error is

$$\int \min_{a \in \alpha_n} \|x-a\|^2 dP > \sum_{i=1}^{4} \int_{J_i} \min_{a \in \alpha_n} \|x-a\|^2 dP \geq \sum_{i=1}^{4} \frac{49}{324} P(J_i) = \frac{49}{1296} = 0.0378086 > V_n,$$

which leads to a contradiction. Therefore, we can assume that all the elements of $\alpha_n$ can not lie on the horizontal line $x_2 = \frac{1}{2}$. Similarly, we can show that all the elements of $\alpha_n$ can not lie on the vertical line $x_1 = \frac{1}{2}$. Suppose that all the elements of $\alpha_n$ lie on the diagonal passing through the origin $(0, 0)$. Then, for any $x \in J_{22} \cup J_{33}$, $\min_{a \in \alpha_n} \|x-a\|^2 \geq \|\left(\frac{1}{8}, \frac{1}{8}\right) - (\frac{1}{2}, \frac{1}{2})\|^2 = \frac{49}{162}$, and so

$$\int \min_{a \in \alpha_n} \|x-a\|^2 dP > \int_{J_{22} \cup J_{33}} \min_{a \in \alpha_n} \|x-a\|^2 dP \geq 2 \cdot \frac{49}{162} \cdot \frac{1}{16} = \frac{49}{1296} = 0.0378086 > V_n,$$

which is a contradiction. So, we can assume that all the elements of $\alpha_n$ can not lie on any of the diagonals. Now, by the definition of centroid, we have

$$\sum_{i=1}^{4} \left(\sum_{j=1}^{4} a_i P(M((a_i, b_i)|\alpha_n)) = \left(\frac{1}{2}, \frac{1}{2}\right),
\right.$$

yielding $\sum_{i=1}^{4} a_i P(M((a_i, b_i)|\alpha_n)) = \frac{1}{2}$ and $\sum_{i=1}^{4} b_i P(M((a_i, b_i)|\alpha_n)) = \frac{1}{2}$ implying the fact that all the elements of $\alpha_n$ can not lie on one side of the horizontal line $x_2 = \frac{1}{2}$ or on one side of the vertical line $x_1 = \frac{1}{2}$. Now, we prove the following claim.

**Claim.** At least two of the points of $\alpha_n$ lie on one side of the horizontal line $x_2 = \frac{1}{2}$ and at least two of the points of $\alpha_n$ lie on the other side the horizontal line $x_2 = \frac{1}{2}$.

Suppose that there is only one point of $\alpha_n$ that lies above the line $x_2 = \frac{1}{2}$. Due to symmetry we can assume that the point lies on the vertical line $x_1 = \frac{1}{2}$. Then, for any $x \in J_{33} \cup J_{44}$, $\min_{a \in \alpha_n} \|x-a\|^2 \geq (\frac{1}{2} - \frac{1}{9})^2 = \frac{49}{324}$, and for any $x \in J_{31} \cup J_{32} \cup J_{41} \cup J_{42} \cup J_{43}$, $\min_{a \in \alpha_n} \|x-a\|^2 \geq (\frac{1}{2} - \frac{1}{3})^2 = \frac{1}{36}$, and so writing $A = J_{31} \cup J_{32} \cup J_{34} \cup J_{41} \cup J_{42} \cup J_{43}$, we have

$$\int \min_{a \in \alpha_n} \|x-a\|^2 dP > \int_{J_{31} \cup J_{44}} \min_{a \in \alpha_n} \|x-a\|^2 dP + \int_{A} \min_{a \in \alpha_n} \|x-a\|^2 dP \geq 2 \cdot \frac{49}{324} \cdot \frac{1}{16} + 6 \cdot \frac{1}{36} \cdot \frac{1}{16} = \frac{19}{648} = 0.029321 > V_n,$$

which leads to a contradiction. Thus, the claim is true. Similarly, we can prove that at least two of the points of $\alpha_n$ lie on one side of the vertical line $x_1 = \frac{1}{2}$ and at least two of the points of $\alpha_n$ lie on the other side the vertical line $x_1 = \frac{1}{2}$. Therefore, $\alpha_n$ contains points from each of the four quadrants $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$, $[\frac{1}{2}, 1] \times [0, \frac{1}{2}]$, $[0, \frac{1}{2}] \times [\frac{1}{2}, 1]$, and $[\frac{1}{2}, 1] \times [\frac{1}{2}, 1]$. Since the support of $P$ lies in $J_1 \cup J_2 \cup J_3 \cup J_4$ and $P$ is symmetrically distributed over $J$, we can assume that $\alpha_n$ contains points from each $J_i$ for $1 \leq i \leq 4$. In other words, $\alpha_n \cap J_i \neq \emptyset$ for $1 \leq i \leq 4$. □
Lemma 3.12. Let \( n \geq 4 \) and \( \alpha_n \) be an optimal set of \( n \)-means. Then, \( \alpha_n \cap (J \setminus \bigcup_{i=1}^{4} J_i) \) is an empty set.

Proof. Recall that the Sierpiński carpet has maximum symmetry with respect to any of its diagonals, and the lines \( x_1 = \frac{1}{2} \) and \( x_2 = \frac{5}{6} \). Assume that \( n = m4^{\ell(n)} \), where \( m, \ell(n) \in \mathbb{N} \). Then, due to symmetry and Lemma 3.11, the optimal set \( \alpha_n \) contains \( m4^{\ell(n)-1} \) elements from each of \( J_i \) implying \( \alpha_n \cap (J \setminus \bigcup_{i=1}^{4} J_i) \) is an empty set. Let us now consider the following cases:

Case 1. \( n = m4^{\ell(n)} + 1 \) for some \( m, \ell(n) \in \mathbb{N} \).

Suppose that \( \alpha_n \cap (J \setminus \bigcup_{i=1}^{4} J_i) \) is not empty. Then, due to symmetry we can assume that \( \alpha_n \) contains \( m4^{\ell(n)-1} \) elements from each of \( J_i \), and the remaining one element is the center \( (\frac{1}{2}, \frac{1}{2}) \) of the Sierpiński carpet. If the Voronoi region \( M(\frac{1}{2}, \frac{1}{2}) | \alpha_n \) of the point \( (\frac{1}{2}, \frac{1}{2}) \in \alpha_n \) does not contain any point from \( \bigcup_{i=1}^{4} J_i \), then \( P(M(\frac{1}{2}, \frac{1}{2}) | \alpha_n) = 0 \), which contradicts Proposition 1.1. Suppose that the Voronoi region \( M((\frac{1}{2}, \frac{1}{2}) | \alpha_n \) of the point \( (\frac{1}{2}, \frac{1}{2}) \in \alpha_n \) contains points from \( J_i \) implying the fact that there is a positive integer \( k \) such that \( J_{14k} \cup J_{23k} \cup J_{32k} \cup J_{14k} \subset M((\frac{1}{2}, \frac{1}{2}) | \alpha_n \), where for any \( 1 \leq i, j \leq 4 \) by \( ij^k \) it is meant the concatenation of \( i \) with \( j^k \), where \( j^k \) denotes the \( k \)-times concatenation of \( j \) with itself. Then, it can be seen that the distortion error due to the set \( \alpha_n \) is larger than the distortion error due to the set \( \alpha_n \) when the extra one point is moved to any of the children \( J_i \) for \( 1 \leq i \leq 4 \). This contradicts the fact that \( \alpha_n \) is an optimal set of \( n \)-means. Thus, \( \alpha_n \cap (J \setminus \bigcup_{i=1}^{4} J_i) \) is an empty set.

Case 2. \( n = m4^{\ell(n)} + 2 \) for some \( m, \ell(n) \in \mathbb{N} \).

Suppose that \( \alpha_n \cap (J \setminus \bigcup_{i=1}^{4} J_i) \) is not empty. Then, due to symmetry, we can assume that \( \alpha_n \) contains \( m4^{\ell(n)-1} \) elements from each of \( J_i \), and the remaining two elements are the points \( (\frac{1}{2}, \frac{1}{6}) \) and \( (\frac{1}{2}, \frac{5}{6}) \). If the Voronoi regions \( M((\frac{1}{2}, \frac{1}{6}) | \alpha_n \) and \( M((\frac{1}{2}, \frac{5}{6}) | \alpha_n \) do not contain any point from \( \bigcup_{i=1}^{4} J_i \), then it will contradict Proposition 1.1. Suppose that the Voronoi regions \( M((\frac{1}{2}, \frac{1}{6}) | \alpha_n \) and \( M((\frac{1}{2}, \frac{5}{6}) | \alpha_n \) contain points from \( \bigcup_{i=1}^{4} J_i \). Then, \( M((\frac{1}{2}, \frac{1}{6}) | \alpha_n \) will contain points from both \( J_1 \) and \( J_2 \). On the other hand, \( M((\frac{1}{2}, \frac{5}{6}) | \alpha_n \) will contain points from both \( J_3 \) and \( J_4 \). But, then it can be seen that the distortion error due to the set \( \alpha_n \) is larger than the distortion error due to the set \( \alpha_n \) when the point \( (\frac{2}{5}, \frac{2}{5}) \) is moved to either \( J_1 \) or \( J_2 \), and the point \( (\frac{2}{5}, \frac{2}{5}) \) is moved to either \( J_3 \) or \( J_4 \). This contradicts the fact that \( \alpha_n \) is an optimal set of \( n \)-means. Thus, \( \alpha_n \cap (J \setminus \bigcup_{i=1}^{4} J_i) \) is an empty set.

Case 3. \( n = m4^{\ell(n)} + 3 \) for some \( m, \ell(n) \in \mathbb{N} \).

Suppose that \( \alpha_n \cap (J \setminus \bigcup_{i=1}^{4} J_i) \) is not empty. Then, due to symmetry, without any loss of generality, we can assume that \( \alpha_n \) contains \( m4^{\ell(n)-1} + 1 \) elements from each of \( J_1 \) and \( J_2 \), and \( m4^{\ell(n)-1} \) elements from each of \( J_3 \) and \( J_4 \), and the remaining one element is the point \( (\frac{2}{5}, \frac{2}{5}) \). If the Voronoi region \( M((\frac{2}{5}, \frac{2}{5}) | \alpha_n \) does not contain any point from \( \bigcup_{i=1}^{4} J_i \), then it will contradict Proposition 1.1. Suppose that the Voronoi region \( M((\frac{2}{5}, \frac{2}{5}) | \alpha_n \) contains points from \( \bigcup_{i=1}^{4} J_i \). Then, \( M((\frac{2}{5}, \frac{2}{5}) | \alpha_n \) will contain points from both \( J_3 \) and \( J_4 \). But, then it can be seen that the distortion error due to the set \( \alpha_n \) is larger than the distortion error due to the set \( \alpha_n \) when the point \( (\frac{2}{5}, \frac{2}{5}) \) is moved to either \( J_3 \) or \( J_4 \). This contradicts the fact that \( \alpha_n \) is an optimal set of \( n \)-means. Thus, \( \alpha_n \cap (J \setminus \bigcup_{i=1}^{4} J_i) \) is an empty set in this case as well.

By Case 1, Case 2, and Case 3 the proof of the lemma is complete. \( \square \)
Let us now prove the following proposition.

**Proposition 3.13.** Let \( \alpha_n \) be an optimal set of \( n \)-means for \( n \geq 4^{\ell(n)} \) for some \( \ell(n) \in I^{(n)} \). Then, \( \alpha_n \cap J_\sigma \neq \emptyset \) for \( \sigma \in I^{(n)} \). Moreover, \( \alpha_n \cap (J_\sigma - \bigcup_{i=1}^4 J_{\sigma^{-i}}) \) for each \( \sigma \in I^{(n)} \) is an empty set, where \( \sigma^- \) is the word obtained from \( \sigma \) by deleting the last letter of \( \sigma \), i.e., if \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_n \), then \( \sigma^- = \sigma_1 \sigma_2 \cdots \sigma_{n-1} \).

**Proof.** First, assume that \( n = 4^{\ell(n)} \) for some \( \ell(n) \in \mathbb{N} \). If \( \ell(n) = 1 \), then the proposition reduces to Lemma 3.11 and Lemma 3.12. So, we assume that \( \ell(n) \geq 2 \). Then, by Lemma 3.11 and Lemma 3.12, and the symmetry of \( P \), the set \( \alpha_n \) contains \( 4^{\ell(n)-1} \) elements from each of \( J_{i_1} \), where \( 1 \leq i_1 \leq 4 \). Recall that \( P \circ S_{i_1}^{-1} \) is the image measure of \( P \) on \( J_{i_1} \), and the probability measure \( P \circ S_{i_1}^{-1} \) on \( J_{i_1} \) is also symmetric as the probability measure \( P \) on \( J \). Applying Lemma 3.11 and Lemma 3.12 on \( J_{i_1} \), we see that \( \alpha_n \) contains \( 4^{\ell(n)-2} \) elements from each of \( J_{i_1} \), where \( 1 \leq i_2 \leq 4 \). Proceeding in this way inductively, we see that \( \alpha_n \) contains one element from each of \( J_{i_1} \), \( i_2 \), and \( \ldots \), \( i_{\ell(n)-1} \) again, and proceeding in this way inductively, we can see that there exists a word \( \sigma \in I^{(n)} \), where \( \sigma \) is an extension of \( i_1 \), such that \( \alpha_n \) contains two elements from \( J_{i_1} \) and \( \alpha_n \) contains only one element from each \( J_{i_\tau} \) for \( \tau \in I^{(n)} \) with \( \tau \neq \sigma \). Thus, the proposition is true for \( n = 4^{\ell(n)} + 1 \). Next, assume that \( n = 4^{\ell(n)} + 2 \) for some \( \ell(n) \in \mathbb{N} \). Then, due to Lemma 3.11 and Lemma 3.12 and symmetry of \( P \), there exists two elements \( i_1, i_2 \in I \), such that \( \alpha_n \) contains \( 4^{\ell(n)-1} + 1 \) elements from each of \( J_{i_1} \) and \( J_{i_2} \), and \( 4^{\ell(n)-1} \) elements from each of \( J_{i_j} \) for \( j \in I \backslash \{i_1, i_2\} \). Applying Lemma 3.11 and Lemma 3.12 on both \( i_1 \) and \( i_2 \) again, and proceeding in this way inductively, we can see that there exist words \( \sigma, \tau \in I^{(n)} \), where \( \sigma \) is an extension of \( i_1 \), and \( \tau \) is an extension of \( i_2 \), such that \( \alpha_n \) contains two elements from each of \( J_{\sigma} \) and \( J_{\tau} \), and \( \alpha_n \) contains only one element from each \( J_{i_j} \) for \( \delta \in I^{(n)} \backslash \{\sigma, \tau\} \). Thus, the proposition is true for \( n = 4^{\ell(n)} + 2 \). Similarly, using Lemma 3.11 and Lemma 3.12 and symmetry of \( P \), we can prove that the proposition is true for any \( n \geq 4^{\ell(n)} \) for some \( \ell(n) \in \mathbb{N} \). \( \square \)

By Lemma 2.3, the set \( \alpha_1 = \{\left(\frac{1}{4}, \frac{1}{2}\right)\} \) is the only optimal set of one-mean, with quantization error \( V = \frac{1}{4} \). By Proposition 3.2 and Proposition 3.7, the sets \( \alpha_2 = \{\left(\frac{1}{4}, \frac{1}{2}\right), \left(\frac{3}{4}, \frac{1}{2}\right)\} \) and \( \alpha_3 = \{\left(\frac{1}{6}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{6}\right), \left(\frac{1}{2}, \frac{3}{4}\right)\} \) are optimal sets of two- and three-means with quantization error \( \frac{1}{36} \) and \( \frac{1}{12} \), respectively. Also, notice that the sets \( \alpha_2 \) and \( \alpha_3 \) are not the only optimal sets of two- and three-means; indeed, the total number of optimal sets of two-means is two and the total number of optimal sets of three-means is four. With this, the optimal sets of \( n \)-means for all \( n \geq 4 \), their numbers and the quantization error are given by the following theorem.

**Theorem 3.14.** Let \( P \) be a Borel probability measure on \( \mathbb{R}^2 \) supported by the Sierpiński carpet. Let \( n \in \mathbb{N} \) with \( n \geq 4 \). Let \( 1 \leq m \leq 3 \). Then,

(i) if \( n = m4^{\ell(n)} \) for some positive integer \( \ell(n) \), then \( \alpha_n = \{S_{\sigma}(\alpha_m) : \sigma \in I^{(n)}\} \) is an optimal set of \( n \)-means. The number of such sets is \( (2^{m-1})4^{\ell(n)} \) and the corresponding quantization error is given by

\[
V_n = \sum_{\sigma \in I^{(n)}} \int_{J_{\sigma m}} \min_{a \in S_{\sigma}(\alpha_m)} \|x - a\|^2 dP.
\]

(ii) if \( n = m4^{\ell(n)} + k \), where \( k \) is a positive integer such that \( 1 \leq k < 4^{\ell(n)} \) for some positive integer \( \ell(n) \), and \( t \subset I^{(n)} \) with \( \text{card}(t) = k \), then,

\[
\alpha_n(t) = \{S_{\sigma}(\alpha_m) : \sigma \in I^{(n)} \setminus t\} \cup \{S_{\sigma}(\alpha_{m+1}) : \sigma \in t\}
\]
is an optimal set of n-means. The number of such sets is \((2^{m-1})^{4^n \cdot k} \cdot 4^n C_k \cdot 2^m k\) if \(m = 1, 2\), and \((2^{m-1})^{4^n \cdot k} \cdot 4^n C_k\) if \(m = 3\); and the corresponding quantization error is given by

\[
V_n = \sum_{\sigma \in I(n) \setminus t} \int_{J_{\sigma}} \min_{a \in S_{\sigma}(a_m)} \|x - a\|^2 dP + \sum_{\sigma \in t} \int_{J_{\sigma}} \min_{a \in S_{\sigma}(a_{m+1})} \|x - a\|^2 dP,
\]

where \(uC_v = \binom{u}{v}\), the binomial coefficients.

**Proof.** Let \(m = 1, 2, 3\). Let \(n = m 4^n \) for some \(\ell(n) \in \mathbb{N}\). Then, by Proposition 3.13, it follows that \(\alpha_n\) contains \(m\) elements from each \(J_\sigma\) for \(\sigma \in I(n)\), which by Proposition 3.10 implies that \(S^{-1}_\sigma(\alpha_n \cap J_\sigma)\) is an optimal set of \(m\)-means, i.e., \(\alpha_n \cap J_\sigma = S_\sigma(\alpha_n)\), and so

\[
\alpha_n = \bigcup_{\sigma \in I(n)} S_\sigma(\alpha_m) = \{S_\sigma(\alpha_m) : \sigma \in I(n)\}.
\]

Since \(\alpha_m\) can be chosen in \(2^{m-1}\) different ways, the number of such sets is \((2^{m-1})^{4^n}\), and the corresponding quantization error is given by

\[
V_n = \int_{a \in \alpha_n} \min \|x - a\|^2 dP = \sum_{\sigma \in I(n) \setminus t} \int_{J_\sigma} \min_{a \in \alpha_n} \|x - a\|^2 dP + \sum_{\sigma \in t} \int_{J_\sigma} \min_{a \in \alpha_n} \|x - a\|^2 dP.
\]

Thus, \((i)\) is proved. To prove \((ii)\) we proceed as follows: Let \(n = m 4^n + k\) for some \(k, \ell(n) \in \mathbb{N}\) with \(1 \leq k < 4^n \) and \(m = 1, 2, 3\).

Let \(t \subset I(n)\) with \(\text{card}(t) = k\). Then, by Proposition 3.13, we can conclude that \(\alpha_n\) contains \(m\) elements from \(J_\sigma\) for each \(\sigma \in I(n) \setminus t\), and \((m + 1)\) elements from \(J_\sigma\) for \(\sigma \in t\). In other words, \(\alpha_n \cap J_\sigma = S_\sigma(\alpha_m)\) for \(\sigma \in I(n) \setminus t\), and \(\alpha_n \cap J_\sigma = S_\sigma(\alpha_{m+1})\) for \(\sigma \in t\). Thus,

\[
\alpha_n(t) = \{S_\sigma(\alpha_m) : \sigma \in I(n) \setminus t\} \cup \{S_\sigma(\alpha_{m+1}) : \sigma \in t\}.
\]

The corresponding quantization error is given by

\[
V_n = \int_{a \in \alpha_n} \min \|x - a\|^2 dP = \sum_{\sigma \in I(n) \setminus t} \int_{J_\sigma} \min_{a \in \alpha_n} \|x - a\|^2 dP + \sum_{\sigma \in t} \int_{J_\sigma} \min_{a \in \alpha_n} \|x - a\|^2 dP
\]

\[
= \sum_{\sigma \in I(n) \setminus t} \int_{J_\sigma} \min_{a \in S_{\sigma}(a_m)} \|x - a\|^2 dP + \sum_{\sigma \in t} \int_{J_\sigma} \min_{a \in S_{\sigma}(a_{m+1})} \|x - a\|^2 dP.
\]

Recall that \(\alpha_2\) can be chosen in two different ways, \(\alpha_3\) can be chosen in three different ways, and \(\alpha_4\) can be chosen in only one way. Thus, if \(m = 1\), the number of \(\alpha_n\) is \(4^n C_k \cdot 2^k\); if \(m = 2\), the number of \(\alpha_n\) is \(2^{4^n \cdot k} \cdot 4^n C_k \cdot 4^k\); and if \(m = 4\), the number of \(\alpha_n\) is \(4^{4^n \cdot k} \cdot 4^n C_k\). Hence, the proof of the theorem is complete.

\[\square\]

4. **Quantization Dimension and Quantization Coefficient**

In this section, we study the quantization dimension and the quantization coefficient for the probability measure \(P\) supported by the Sierpiński carpet. Let \(\beta\) be the Hausdorff dimension of the Sierpiński carpet, then \(4^n(\frac{1}{3})^\beta = 1\) which yields \(\beta = \frac{\log 4}{\log 3}\). In Theorem 4.1, we show that the quantization dimension of the probability measure \(P\) exists, and equals \(\beta\). In Theorem 4.3, we show that \(\beta\) dimensional quantization coefficient for \(P\) does not exist.

**Theorem 4.1.** Let \(P\) be a Borel probability measure on \(\mathbb{R}^2\) supported by the Sierpiński carpet. Then, \(\lim_{n \to \infty} \frac{2 \log n}{\log V_n} = \frac{\log 4}{\log 3}\), i.e., the quantization dimension of \(P\) exists and equals the Hausdorff dimension of the Sierpiński carpet.

**Proof.** By Theorem 3.14 and the equation (3), if \(n = 4^{\ell(n)}\) for some positive integer \(\ell(n)\), then

\[
V_n = \sum_{\sigma \in I(n)} \int_{J_\sigma} \|x - S_\sigma(\frac{1}{2}, \frac{1}{2})\|^2 dP = \frac{1}{9^{\ell(n)}} V = \frac{1}{9^{\ell(n)}} \frac{1}{4}.
\]
Let \( n \in \mathbb{N} \) and \( n \geq 4 \). Then, \( 4^\ell(n) \leq n < 4^\ell(n)+1 \) for some \( \ell(n) \in \mathbb{N} \). Hence, by (8), we have
\[
\begin{align*}
n^2V_n &= 4^{2\ell(n)}V_{4\ell(n)+1} = 4^{2\ell(n)} \cdot \frac{1}{9\ell(n)+1} \cdot \frac{1}{4} = \left(\frac{4}{3}\right)^{2\ell(n)} \frac{1}{36}, \\
n^2V_n &\leq 4^{2(\ell(n)+1)}V_{4\ell(n)+1} = 4^{2(\ell(n)+1)} \cdot \frac{1}{9\ell(n)+1} \cdot \frac{1}{4} = \left(\frac{4}{3}\right)^{2\ell(n)} \frac{1}{4},
\end{align*}
\]
implying
\[2\ell(n) \log \frac{4}{3} - \log 36 - 2 \log n \leq \log V_n \leq 2\ell(n) \log \frac{4}{3} + \log 4 - 2 \log n.\]
Thus,
\[
\lim_{n \to \infty} \frac{-\log V_n}{2 \log n} = -\log \frac{4}{3} \cdot \lim_{n \to \infty} \frac{\ell(n)}{\log n} + 1 = \log \frac{3}{4} \cdot \lim_{n \to \infty} \frac{\ell(n)}{\log n} + 1.
\]
Again, \( 4^\ell(n) \leq n < 4^\ell(n)+1 \) implies \( \lim_{n \to \infty} \frac{\ell(n)}{\log n} = \frac{1}{\log 4} \). Hence,
\[
\lim_{n \to \infty} \frac{-\log V_n}{2 \log n} = \log \frac{3}{4} \cdot \frac{1}{\log 4} + 1 = \log \frac{3}{4} \cdot \frac{1}{\log 3} \text{ implying } \lim_{n \to \infty} \frac{2 \log n}{-\log V_n} = \log \frac{4}{\log 3},
\]
which completes the proof of the theorem. \( \square \)

We need the following lemma to prove Theorem 4.3.

**Lemma 4.2.** Define the function \( f : [1, 2] \to \mathbb{R} \) by \( f(x) = \frac{1}{36} x^{\frac{2}{3}}(13-x) \). Then, \( f([1, 2]) = [\frac{1}{3}, \frac{11}{12}] \).

**Proof.** We have \( f'(x) = -\frac{x^{\frac{2}{3}}}{36}((\beta+2)x-36) \), and so, \( f'(x) > 0 \) if \( x < \frac{26}{2+\beta} \). Since 2 < \( \frac{26}{2+\beta} \), the function \( f \) is strictly increasing on the interval \([1, 2]\). Again, \( f(1) = \frac{1}{3} \) and \( f(2) = \frac{11}{12} \). Hence, \( f([1, 2]) = [\frac{1}{3}, \frac{11}{12}] \), which completes the proof of the lemma. \( \square \)

**Theorem 4.3.** \( \beta \)-dimensional quantization coefficient for \( P \) does not exist.

**Proof.** We need to show that \( \lim_{n \to \infty} n^\beta V_n \) does not exist. Let \((n_k)_{k \in \mathbb{N}}\) be a subsequence of the set of natural numbers such that \( 4^\ell(n_k) \leq n_k < 2 \cdot 4^\ell(n_k) \). To prove the theorem it is enough to show that the set of accumulation points of the subsequence \((n_k V_n)_{k \geq 1}\) equals \([\frac{1}{3}, \frac{11}{12}]\). Let \( y \in [\frac{1}{3}, \frac{11}{12}] \). We now show that \( y \) is a subsequential limit of the sequence \((n_k V_n)_{k \geq 1}\). Since \( y \in [\frac{1}{3}, \frac{11}{12}] \), \( y = f(x) \) for some \( x \in [1, 2] \). Set \( n_k = \lceil x 4^\ell \rceil \), where \( \lceil x 4^\ell \rceil \) denotes the greatest integer less than or equal to \( x 4^\ell \). Then, \( n_k < n_{k+1} \) and \( \ell(n_k) = \ell \), and there exists \( x_{k\ell} \in [1, 2] \) such that \( n_k = x_{k\ell} 4^\ell \). Recall that by \( \ell(n_k) = \ell \) it is meant that \( 4^\ell \leq n_k < 4^{\ell+1} \). Notice that if \( 4^\ell(n) \leq n \leq 4^\ell(n)+1 \), then by Theorem 3.14 we have
\[
V_n = (2 \cdot 4^\ell(n) - n) \cdot \frac{1}{36^\ell(n)+1} \cdot \frac{1}{4} + (n - 4^\ell(n)) \cdot \frac{5}{36} = \frac{1}{36^\ell(n)+1}(13 \cdot 4^\ell(n) - 4n).
\]
Thus, putting the values of \( n_k \) and \( V_{n_k} \), we obtain
\[
n_k^{\frac{2}{3}} V_{n_k} = n_k^{\frac{2}{3}} \cdot 4^{\frac{1}{36^\ell+1}}(13 \cdot 4^\ell - 4n_k) = x_{k\ell}^{\frac{2}{3}} 4^{\frac{1}{36^\ell+1}}(13 \cdot 4^\ell - 4x_{k\ell} 4^\ell) = x_{k\ell}^{\frac{2}{3}} 9^{\frac{1}{36^\ell+1}}(13 \cdot 4^\ell - 4x_{k\ell} 4^\ell),
\]
which yields
(8)
\[
n_k^{\frac{2}{3}} V_{n_k} = \frac{1}{36} x_{k\ell}^{\frac{2}{3}} (13 - 4x_{k\ell}) = f(x_{k\ell}).
\]
Again, \( x_{k\ell} 4^\ell \leq x 4^\ell < x_{k\ell} 4^\ell + 1 \), which implies \( x - \frac{1}{4^\ell} < x_{k\ell} \leq x \), and so, \( \lim x_{k\ell} = x \). Since, \( f \) is continuous, we have
\[
\lim_{\ell \to \infty} n_k^{\frac{2}{3}} V_{n_k} = f(x) = y,
\]
which yields the fact that \( y \) is an accumulation point of the subsequence \((n_k^{\frac{2}{3}} V_n)_{k \geq 1}\) whenever \( y \in [\frac{1}{3}, \frac{11}{12}] \). To prove the converse, let \( y \) be an accumulation point of the subsequence \((n_k^{\frac{2}{3}} V_n)_{k \geq 1}\).
Then, there exists a subsequence \((n_k^i V_{n_k})_{i \geq 1}\) of \((n_k^i V_{n_k})_{k \geq 1}\) such that \(\lim_{i \to \infty} n_k^i V_{n_k} = y\). Set \(\ell_k = \ell(n_k)\) and \(x_k = \frac{n_k}{4^k}\). Then, \(x_k \in [1, 2]\), and as shown in [S], we have

\[
\ell_k \rightarrow \ell(n_k) = f(x_k).
\]

Let \((x_{k_j})_{j \geq 1}\) be a convergent subsequence of \((x_k)_{k \geq 1}\), and then we obtain

\[
y = \lim_{i \to \infty} n_k^i V_{n_k} = \lim_{j \to \infty} n_k^i V_{n_k_{ij}} = \lim_{j \to \infty} f(x_{k_j}) \in \left[\frac{1}{3}, \frac{11}{12}\right].
\]

Thus, we have proved that the set of accumulation points of the subsequence \((n_k^i V_{n_k})_{k \geq 1}\) equals \(\left[\frac{1}{3}, \frac{11}{12}\right]\), and hence, the proof of the theorem is complete. □

**Remark 4.4.** Using the formula given by [MR, Theorem A], we see that the Hausdorff dimension and the packing dimension of the probability measure \(P\) are obtained as \(\frac{\log 4}{\log 3}\). Thus, Theorem 4.1 implies that the quantization dimension of the probability measure \(P\) coincides with the Hausdorff dimension of Sierpiński carpet, the Hausdorff dimension and the packing dimension of the probability measure \(P\) supported by the Sierpiński carpet.

**Remark 4.5.** Previously, Graf and Luschgy determined the optimal sets of \(n\)-means and the \(n\)th quantization error for a singular continuous probability measure supported by the classical Cantor set \(C\). In this paper, we determined the optimal sets of \(n\)-means and the \(n\)th quantization error for a singular continuous probability measure supported by a Sierpiński carpet. To the best of our knowledge, the work in this paper is the first advance to investigate the optimal quantizers for a singular continuous probability measure on \(\mathbb{R}^2\). The technique in this paper can be extended to determine the optimal sets of \(n\)-means and the \(n\)th quantization error for many other singular continuous probability measures generated by affine transformations in \(\mathbb{R}^2\).

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