Fast Bayesian Inference in Nonparametric Double Additive Location-Scale Models With Right- and Interval-Censored Data

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Abstract

Penalized B-splines are routinely used in additive models to describe smooth changes in a response with quantitative covariates. It is typically done through the conditional mean in the exponential family using generalized additive models with an indirect impact on other conditional moments. Another common strategy consists in focusing on several low-order conditional moments, leaving the complete conditional distribution unspecified. Alternatively, a multi-parameter distribution could be assumed for the response with several of its parameters jointly regressed on covariates using additive expressions.

Our work can be connected to the latter proposal for a right- or interval-censored continuous response with a highly flexible and smooth nonparametric density. We focus on location-scale models with additive terms in the conditional mean and standard deviation. Starting from recent results in the Bayesian framework, we propose a quickly converging algorithm to select penalty parameters from their marginal posteriors. It relies on Laplace approximations to the conditional posterior of the spline parameters. Simulations suggest that the so-obtained estimators own excellent frequentist properties and increase efficiency as compared to approaches with a working Gaussian hypothesis. We illustrate the methodology with the analysis of imprecisely measured income data.

Keywords: Location-scale model ; Dispersion model; Imprecise data ; Interval-censoring ; P-splines ; Laplace approximation ; Smooth density estimation.

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1 Introduction

Additive models are flexible alternatives to the classical linear regression model to describe in a flexible way the effect of quantitative covariates on various aspects of a response distribution. Early proposals focused on the conditional mean with limited assumptions on the conditional distribution of the response \cite{BreimanFriedman1985}. That idea was used to extend generalized linear models (GLM, \cite{NelderWedderburn1972}) and the analysis of nonnormal data (such as counts or proportions) in the framework of the exponential family of distributions: additive terms enter the GLM linear predictor (connecting covariates to a pre-specified function of the conditional mean) for a fixed value of the dispersion parameter, yielding generalized additive models (GAM) \cite{HastieTibshirani1986,HastieTibshirani1990,Wood2017}. Further extensions are possible by enabling covariates to also affect other aspects of the response distribution such as dispersion, skewness and kurtosis, see \cite{LambertLindsey1999} for early work on this with the four parameters of the stable distribution simultaneously modelled and \cite{RigbyStasinopoulos2005} for an extension to a large choice of parametric distributions. \cite{Lee06} and \cite{GijbelsProsdocimi2012} considered joint additive models for location and dispersion within, respectively, the exponential and the double-exponential families of distributions, while \cite{Croux2012} relied on a (robustified) extended quasi-likelihood method.

Our paper will focus on double additive models for the conditional mean and standard deviation in location-scale models with a nonparametric error distribution. The response will be assumed continuous and possibly subject to right or interval censoring. Nonparametric inference from censored data in location-scale models has been investigated by many authors, see e.g. \cite{FanGijbels1994} for early work using local polynomials and \cite{HeuchenneVanKeilegom2010} with the references therein for some more recent work. These methods typically focus on the estimation of the conditional location and can only handle the estimation of the smooth effects of a very limited number of covariates. Additive models based on P-splines \cite{EilersMarx1996,LangBrezger2004} are preferred here for their excellent properties \cite{EilersMarx2010} and the possibility to handle a large number of additive terms. They are used to specify the joint effect of covariates on location and dispersion in the framework of the location-scale model, see Section 2. A nonparametric error distribution with an underlying smooth hazard function and fixed moments will be assumed for the standardized error term, see Section 2.5. In the absence of right censoring, a
location-scale model with a small number of additive terms and a quartile-constrained error density (instead of the hazard here) was considered in Lambert (2013) to analyse interval-censored data, with inference relying on a numerically demanding MCMC algorithm. We show how Laplace approximations to the conditional posterior (of blocks) of spline parameters can be combined to bring fast and reliable estimation of the additive terms in the location and dispersion models, and provide a smooth estimate of the underlying error hazard function under moment constraints. These approximations are the cornerstones in the derivation of the marginal posteriors for the penalty parameters and smoothness selection, see Sections 2.4 and 2.5.5. The resulting estimation procedures are motivated using Bayesian arguments and shown to own excellent frequentist properties, see Section 3 and Supplementary Material C. They are extremely fast and can handle a large number of additive terms within a few seconds even with pure R code. The methodology is illustrated in Section 4 with the analysis of right- and interval-censored income data in a survey. We conclude the paper with a discussion in Section 5.

2 Additive location-scale model

Consider a vector \((Y, z, x)\) where \(Y\) is a univariate continuous response, \(z\) a \(p\)-vector of categorical covariates, and \(x\) a \(J\)-vector of quantitative covariates. The response could be subject to right censoring, in which case one only observes \((T, \Delta)\), where \(T = \min\{Y, C\}\), \(\Delta = I(Y \leq C)\) and \(C\) denotes the right censoring value that we shall assume independent of \(Y\) given the covariates. The response could also be interval-censored, meaning that it is only known to lie within an interval \((Y^L, Y^U)\).

Such settings are not only common in survival analysis when studying the time elapsed between a clearly defined time origin and an event of interest, but also in surveys when the respondent reports a quantitative response by pointing one interval or semi-interval in the partition of the variable support.

We consider here a location-scale model,

\[
Y = \mu(z, x) + \sigma(z, x)\varepsilon
\]

(2.1)

to describe the distribution of the response conditionally on the covariates, where \(\mu(z, x)\) denotes the conditional location, \(\sigma(z, x)\) the conditional dispersion, and \(\varepsilon\) an error term independent of \(z\) and \(x\) assumed to have fixed 1st and 2nd order moments. One could for example assume that \(\mathbb{E}(\varepsilon) = 0\) and \(\mathbb{V}(\varepsilon) = 1\). The latter conditions lead to interpretation of \(\mu(z, x)\) and
\[ \sigma(z, x) \] as the conditional mean and standard deviation, respectively. Other 
constraints are possible such as in Lambert (2013) where \( \varepsilon \) was assumed to 
have a zero median and a unit interquantile range, implying that \( \mu(z, x) \) and 
\( \sigma(z, x) \) had to be interpreted as the conditional median and interquantile
range.

Assume that independent copies \((y_i, z_i, x_i) \) \((i = 1, \ldots, n)\) are observed 
on \( n \) units with the possibility of right or interval censoring on \( y_i \), as 
described above. We consider additive models for the conditional location and 
dispersion of the response:

\[
\begin{align*}
(\mu(z_i, x_i))_{i=1}^n &= \left( \beta_0 + \sum_{k=1}^p \beta_k z_{ik} + \sum_{j=1}^J f^\mu_j(x_{ij}) \right)_{i=1}^n = Z \beta + \sum_{j=1}^J \mathbf{f}^\mu_j \\
&= (2.2) \\
(\log \sigma(z_i, x_i))_{i=1}^n &= \left( \delta_0 + \sum_{k=1}^p \delta_k z_{ik} + \sum_{j=1}^J f^\sigma_j(x_{ij}) \right)_{i=1}^n = Z \delta + \sum_{j=1}^J \mathbf{f}^\sigma_j \\
&= (2.3)
\end{align*}
\]

where \( f^\mu_j(\cdot) \) and \( f^\sigma_j(\cdot) \) denote smooth additive terms quantifying the effect 
of the \( j \)-th quantitative covariate on the conditional mean and dispersion, 
\( f^\mu_j = (f^\mu_j(x_{ij}))_{i=1}^n \) and \( f^\sigma_j = (f^\sigma_j(x_{ij}))_{i=1}^n \) their values over units stacked in 
vectors, \( Z \) the \( n \times (1 + p) \) design matrix with a column of 1’s for the 
intercept and one column per additional categorical covariate. For simplicity and 
without restriction, we assume that the quantitative covariates take values in 
\((0, 1)\). This can be achieved for \( x_j \) by relocating and rescaling it using e.g. the 
following linear transform, \((x_j - \min_i\{x_{ij}\})/(\max\{x_{ij}\} - \min_i\{x_{ij}\})\). Now 
consider a basis of \((L + 1)\) cubic B-splines \( \{s^\ell_j(\cdot)\}_{\ell=1}^{L+1} \) associated to equally 
spaced knots on \((0, 1)\). They are recentered for identification purposes in the 
additive model using \( s^\ell_j(\cdot) = s^\ell_0(\cdot) - \int_0^1 s^\ell_0(u)du \) \((\ell = 1, \ldots, L)\). Then, 
the additive terms in the conditional location and dispersion models can be 
approximated using linear combinations of these (recentered) B-splines, 
\( f^\mu_j = \left( \sum_{\ell=1}^L s^{\ell}(x_{ij}) \theta^\mu_{j,\ell} \right)_{i=1}^n = \mathbf{S}_j \theta^\mu_j, \ f^\sigma_j = \left( \sum_{\ell=1}^L s^{\ell}(x_{ij}) \theta^\sigma_{j,\ell} \right)_{i=1}^n = \mathbf{S}_j \theta^\sigma_j, \)
where \( [\mathbf{S}_j]_{i,\ell} = s^{\ell}(x_{ij}), \ (\theta^\mu_{j,\ell})_\ell = \theta^\mu_{j,\ell} \) and \( (\theta^\sigma_{j,\ell})_\ell = \theta^\sigma_{j,\ell} \). Hence, using 
vectorial notations, the expressions for the conditional location and dispersion 
in \((2.2)\) and \((2.3)\) can be rewritten as \((\mu_i = \mu(z_i, x_i))_{i=1}^n = \mathbf{X} \psi^\mu, \)
\((\sigma_i = \sigma(z_i, x_i))_{i=1}^n = \exp(\mathbf{X} \psi^\sigma)\) with design matrix 
\( \mathbf{X} = [\mathbf{Z}, \mathbf{S}_1, \ldots, \mathbf{S}_J] = [\mathbf{Z}, \mathbf{S}] \in \mathbb{R}^{n \times q}; \) matrices of spline parameters (with one column per 
additive term) \( \Theta^\mu = [\theta^\mu_1, \ldots, \theta^\mu_J], \Theta^\sigma = [\theta^\sigma_1, \ldots, \theta^\sigma_J] \) in \( \mathbb{R}^{L \times J}; \) vectors of 
(stacked) regression parameters \( \psi^\mu = (\beta, \text{vec}(\Theta^\mu)), \ \psi^\sigma = (\delta, \text{vec}(\Theta^\sigma)) \)
in \( \mathbb{R}^q \), where \( q = (1 + p + JL) \). With \( p_1 \) (resp. \( p_2 \)) covariates and a B-spline basis of size \( L_1 \) (resp. \( L_2 \)) shared by each of the \( J_1 \) (resp. \( J_2 \)) additive terms in the location (resp. dispersion) model, we would end up with design matrices \( X^\mu = [Z^\mu, S^\mu] \in \mathbb{R}^{n \times q_1} \) (resp. \( X^\sigma = [Z^\sigma, S^\sigma] \in \mathbb{R}^{n \times q_2} \)) with \( q_1 = (1 + p_1 + J_1 L_1) \) (resp. \( q_2 = (1 + p_2 + J_2 L_2) \)) such that \( (\mu_i)_{i=1}^n = X^\mu \psi^\mu \), \( (\sigma_i)_{i=1}^n = \exp(X^\sigma \psi^\sigma) \).

### 2.1 Penalized log-likelihood for the joint regression model

Estimation of the regression parameters and of the additive terms (for given penalty parameters) can be made using penalized likelihood. Denote by \( f_\epsilon(\cdot; \phi) \) (resp. \( S_\epsilon(\cdot; \phi) \)) the conditional density (resp. survival function) of the standardized error term \( \epsilon \) in (2.1) with a possible dependence on a set of parameters \( \phi \). The contribution \( \ell_i = \ell_i(\psi^\mu, \psi^\sigma, \phi; D) \) of unit \( i \) to the log-likelihood will depend on the censoring status of the observed response \( y_i \):

- Uncensored \( y_i = t_i \): then, the corresponding standardized error term \( e_i \) is equal to \( r_i = (y_i - \mu_i)/\sigma_i \) with log-likelihood contribution \( \ell_i = -\log \sigma_i + \log f_\epsilon(r_i) \).

- Right-censored at \( y_i > t_i \): then, the corresponding standardized error term is \( e_i > r_i = (t_i - \mu_i)/\sigma_i \) with log-likelihood contribution \( \ell_i = \log S_\epsilon(r_i) \).

- Interval-censored with \( y_i \in (y^L_i, y^R_i) \): then, the log-likelihood contribution is \( \ell_i = \log (S(r^L_i) - S(r^R_i)) \) as \( e_i \in (r^L_i, r^R_i) \) where \( r^L_i = (y^L_i - \mu_i)/\sigma_i \) and \( r^R_i = (y^R_i - \mu_i)/\sigma_i \).

Smoothness of the additive terms can be tuned by penalizing changes in differences of neighbour spline parameters (Eilers and Marx 1996, 2010). In a frequentist framework, this can be done by adding one penalty (to the log-likelihood) per additive term. When penalizing second-order differences in the location model, the penalty for the \( j \)th additive term (\( j = 1, \ldots, J_1 \)) becomes \( \lambda^\mu_j \sum_{\ell=1}^{L_1-2} ((\theta^\mu_{\ell+2,j} - \theta^\mu_{\ell+1,j}) - (\theta^\mu_{\ell+1,j} - \theta^\mu_{\ell,j}))^2 = \lambda^\mu_j \sum_{\ell} (D^\mu \theta^\mu)^2_{\ell} = \theta^\mu_j (\lambda^\mu_j D^\mu \theta^\mu)_{\ell} \), where \( D^\mu \) denotes the corresponding difference matrix and \( \lambda_j^\mu \) the associated penalty matrix. At the limit, as \( \lambda_j^\mu \to +\infty \), the estimated second-order differences will tend to zero, forcing the estimate of the function \( f^\mu_j(x_j) \) to be linear. Similar penalties with penalty parameters \( \lambda_j^\sigma \) can be defined for each additive term in the dispersion model.
2.2 Bayesian specification

In a Bayesian framework, similar penalties arise through the specification of conditional priors for the spline parameters (Lang and Brezger, 2004), yielding for the \( j \)th additive terms in the location and dispersion models, 
\[
p(\theta_j^\mu | \lambda_j^\mu) \propto \exp\left(-\frac{1}{2} \theta_j^\mu \top \left(\lambda_j^\mu P^\mu\right) \theta_j^\mu\right),
\]
\[
p(\theta_j^\sigma | \lambda_j^\sigma) \propto \exp\left(-\frac{1}{2} \theta_j^\sigma \top \left(\lambda_j^\sigma P^\sigma\right) \theta_j^\sigma\right).
\]

Assuming joint Normal priors for the intercepts and the regression parameters sequentially and conditionally on the error density, 
\[
\text{Assuming joint Normal priors for the intercepts and the regression parameters associated to the other covariates } z, \quad \beta \sim \mathcal{N} \left( \tilde{b}, (Q^\mu)^{-1} \right), \quad \delta \sim \mathcal{N} \left( \tilde{d}, (Q^\sigma)^{-1} \right), \quad \text{the joint priors for the regression and spline parameters in } \psi^\mu \text{ and } \psi^\sigma \text{ induce Gaussian Markov random fields (GMRF) (Rue and Held, 2005) as they can be written as}
\]
\[
p(\psi^\mu | \lambda^\mu) \propto \exp\left(-\frac{1}{2} (\psi^\mu - b) \top K^\mu_{\lambda}(\psi^\mu - b)\right); \quad p(\psi^\sigma | \lambda^\sigma) \propto \exp\left(-\frac{1}{2} (\psi^\sigma - d) \top K^\sigma_{\lambda}(\psi^\sigma - d)\right),
\]
where \( b = (\tilde{b}, 0_{L_1 L_2}), K^\mu_{\lambda} = \text{diag}(Q^\mu, P^\mu), P^\mu = \Lambda^\mu \otimes P^\mu, [\Lambda^\mu]_{jj'} = \delta_{jj'} \lambda_j^\mu, \)
\( d = (\tilde{d}, 0_{L_2 L_2}), K^\sigma_{\lambda} = \text{diag}(Q^\sigma, P^\sigma), P^\sigma = \Lambda^\sigma \otimes P^\sigma \) and \([\Lambda^\sigma]_{jj'} = \delta_{jj'} \lambda_j^\sigma\). Then the joint posterior for the parameters is
\[
p(\psi^\mu, \psi^\sigma, \lambda^\mu, \lambda^\sigma, \phi | D) \propto L(\psi^\mu, \psi^\sigma, \phi; D) \ p(\psi^\mu | \lambda^\mu) \ p(\psi^\sigma | \lambda^\sigma) \ p(\lambda^\mu) \ p(\lambda^\sigma) \ p(\phi).
\]

2.3 Estimation of \( \psi^\mu \) and \( \psi^\sigma \)

The estimation of the regression parameter \( \psi^\mu \) and \( \psi^\sigma \) will be made sequentially and conditionally on the error density \( f_e(\cdot; \phi) \) and the penalty parameters \( \lambda^\mu \) and \( \lambda^\sigma \). It is based on the following decomposition of their joint conditional posterior:
\[
p(\psi^\mu, \psi^\sigma | \lambda^\mu, \lambda^\sigma, \phi, D) = p(\psi^\mu | \psi^\sigma, \lambda^\mu, \phi, D) \ p(\psi^\sigma | \lambda^\mu, \lambda^\sigma, \phi, D). \quad (2.4)
\]

The conditional posterior for the location parameters is given by
\[
p(\psi^\mu | \psi^\sigma, \lambda^\mu, \phi, D) \propto L(\psi^\mu, \psi^\sigma, \phi; D) \ p(\psi^\mu | \lambda^\mu), \quad (2.5)
\]
while the final expression for
\[
p(\psi^\sigma | \lambda^\mu, \lambda^\sigma, \phi, D) = \frac{p(\psi^\mu, \psi^\sigma | \lambda^\mu, \lambda^\sigma, \phi, D)}{p(\psi^\mu | \lambda^\mu, \lambda^\sigma, \phi, D)} \propto L(\psi^\mu, \psi^\sigma, \phi; D) \ p(\psi^\sigma | \lambda^\sigma) \ |\Sigma_{\lambda}^{1/2}.
\]

(2.6)
is obtained by using the Laplace approximation \( N \left( \tilde{\psi}_\lambda, \tilde{\Sigma}_\lambda \right) \) in the denominator and evaluating it at the posterior mode \( \tilde{\psi}_\lambda \). Indeed, given the Normality assumption for the prior \( (\psi^\sigma | \lambda^\mu ) \), the conditional posterior in the denominator will be approximately Normal, see Rue and Martino (2009) for arguments in the general context of Gaussian random fields.

Estimates for the regression parameters will be obtained by alternating the maximization of (2.5) and (2.6) till convergence. For \( \tilde{\psi}_\mu \), this is done for given values of the other parameters using a Newton-Raphson (N-R) algorithm built upon the gradient and (minus) Hessian of the log of (2.5),

\[
U^\lambda_{\psi^\mu}(\psi^\mu) = \frac{\partial \log p(\psi^\mu | \lambda^\mu, \psi^\sigma, \phi, D)}{\partial \psi^\mu} = \mathbf{X}^\top \omega^\mu - K^\mu_\lambda \psi^\mu, \\
-H^\lambda_{\psi^\mu}(\psi^\mu) = - \frac{\partial^2 \log p(\psi^\mu | \lambda^\mu, \psi^\sigma, \phi, D)}{\partial \psi^\mu \partial \psi^\mu} = \mathbf{X}^\top W^\mu \mathbf{X}^\mu + K^\mu_\lambda,
\]

with \( \omega^\mu \in \mathbb{R}^n \) and \( W^\mu = \text{diag}(w^\mu) \in \mathbb{R}^{n \times n} \) given in Appendix A. At convergence, it yields the conditional posterior mode \( \tilde{\psi}_\mu \) and variance-covariance matrix \( \tilde{\Sigma}_\lambda = \left( -H^\lambda_{\psi^\mu}(\tilde{\psi}_\mu) \right)^{-1} \) in the above mentioned Laplace approximation. The estimates for \( \tilde{\psi}_\sigma \) are also obtained using a N-R algorithm based on the gradient and (minus) Hessian of the log of (2.6),

\[
U^\lambda_{\psi^\sigma}(\psi^\sigma) = \frac{\partial \log p(\psi^\sigma | \lambda^\sigma, \psi^\mu, \phi, D)}{\partial \psi^\sigma} = \mathbf{X}^\sigma^\top \omega^\sigma - K^\sigma_\lambda \psi^\sigma + \frac{\partial E^\mu_\lambda}{\partial \psi^\sigma}, \\
-H^\lambda_{\psi^\sigma}(\psi^\sigma) = - \frac{\partial^2 \log p(\psi^\sigma | \lambda^\sigma, \psi^\mu, \phi, D)}{\partial \psi^\sigma \partial \psi^\sigma} = \mathbf{X}^\sigma^\top W^\sigma \mathbf{X}^\sigma + K^\sigma_\lambda - \frac{\partial^2 E^\mu_\lambda}{\partial \psi^\sigma \partial \psi^\sigma},
\]

with \( E^\mu_\lambda = \frac{1}{2} \log |\tilde{\Sigma}_\lambda^\mu| \) (see Appendix B for its partial derivatives) and \( \omega^\sigma \in \mathbb{R}^n \), \( W^\sigma = \text{diag}(w^\sigma) \in \mathbb{R}^{n \times n} \) defined in Appendix A. It leads to Algorithm 1 for the estimation of the regression parameters \( \psi^\mu \) and \( \psi^\sigma \).

**Algorithm 1. Estimation of \( \psi^\mu \) and \( \psi^\sigma \)**

At each iteration of their respective Newton-Raphson algorithm (and conditionally on the values of the other model parameters):

1. Compute the standardized residuals \( r_i = (y_i - \mu_i(\tilde{\psi}_\lambda^\mu)) / \sigma_i(\tilde{\psi}_\lambda^\sigma) \) accompanied by their observation indicators \( d_i \), some of these residuals being right-censored \( (d_i = 0) \) or interval-censored with \( r_i \in (r_i^L, r_i^R) \).

2. Recompute vectors \( \omega^\mu, \omega^\sigma, w^\mu, w^\sigma \) and, hence, the diagonal matrices \( W^\mu = \text{diag}(w^\mu) \) and \( W^\sigma = \text{diag}(w^\sigma) \) using (A.1) and (A.2).
3. Update the location and dispersion parameters $\psi^\mu_\lambda$ and $\psi^\sigma_\lambda$ using \((2.7)\) and \((2.8)\), $\psi^\mu_\lambda \leftarrow \psi^\mu_\lambda - (H^\lambda_{\psi^\mu})^{-1} U^\lambda_{\psi^\mu}$; $\psi^\sigma_\lambda \leftarrow \psi^\sigma_\lambda - (H^\lambda_{\psi^\sigma})^{-1} U^\lambda_{\psi^\sigma}$, with step-halving when found necessary through the monitoring of $p(\psi^\mu|\psi^\sigma,\lambda^\mu,\phi,D)$ and $p(\psi^\sigma|\lambda^\mu,\lambda^\sigma,\phi,D)$, respectively.

At convergence, after a few iterations, one obtains the conditional posterior modes $\hat{\psi}^\mu_\lambda$ and $\hat{\psi}^\sigma_\lambda$ with negative inverse Hessians $\Sigma^\mu_\lambda = (H^\lambda_{\psi^\mu}(\hat{\psi}^\mu_\lambda))^{-1}$ and $\Sigma^\sigma_\lambda = (H^\lambda_{\psi^\sigma}(\hat{\psi}^\sigma_\lambda))^{-1}$.

### 2.4 Selection of the penalty parameters $\lambda^\mu$ and $\lambda^\sigma$

Let $\psi = (\psi^\mu, \psi^\sigma)$ and $\lambda = (\lambda^\mu, \lambda^\sigma)$. Starting from the joint posterior for the model parameters, we have (with an implicit dependence on the standardized error distribution and its parameter(s) $\phi$) the following identity for the marginal posterior of $\lambda$:

$$p(\lambda|D) = \frac{p(\psi, \lambda|D)}{p(\psi|\lambda, D)}.$$  \(2.9\)

Given the conditional GMRF prior for $\psi$, we conclude that the conditional posterior in the denominator is approximately Gaussian \cite{Rue2009}. Using a Laplace approximation, we obtain $(\psi|\lambda, D) \sim \mathcal{N}(\hat{\psi}_\lambda, \Sigma_\lambda)$, where $\hat{\psi}_\lambda$ denotes the conditional posterior mode of $\psi$ (obtained using Algorithm 1) and

$$\Sigma_\lambda^{-1} = -\sum_{i=1}^{n} \left[ \frac{\partial^2 \ell_i}{\partial \psi^\mu \partial \psi^\mu} \right] + \left[ K^\mu_\lambda . . K^\sigma_\lambda \right],$$

see also \cite{Tierney1986} for general arguments for such an approximation to the marginal posterior of $\lambda$. Evaluating the RHS of \((2.9)\) at $\hat{\psi}_\lambda$ with the preceding Laplace approximation, we approximate $p(\lambda|D)$ by $\tilde{p}(\lambda|D) \propto p(\hat{\psi}_\lambda, \lambda|D) \left| \Sigma^{-1}_\lambda \right|^{-1/2}$ \cite{Wood2017}. \cite{Gressani2018} obtained a similar starting expression to build their proposal for the selection of penalty parameters in an additive regression model with a parametric error distribution. \cite{Gressani2018} also followed that strategy in the context of a cure survival model with splines used to specify the baseline hazard function for susceptible subjects. Ignoring the cross-derivatives in $\Sigma^{-1}_\lambda$ yields

$$\tilde{p}(\lambda|D) \propto p(\hat{\psi}_\lambda, \lambda|D) \left| X^{\mu^\top} W^\mu X^{\mu} + K^{\mu^\mu}_\lambda \right|^{-1/2} \left| X^{\sigma^\top} W^\sigma X^{\sigma} + K^{\sigma^\sigma}_\lambda \right|^{-1/2},$$  \(2.10\)
with $W^\mu$ and $W^\sigma$ given in Appendix A. Dropping the $\mu$ or $\sigma$ superscript and letting

$$M = S^T W S - S^T W Z(Z^T W Z + Q)^{-1} Z^T W S, \quad (2.11)$$

each determinant in (2.10) can be rewritten as

$$|X^T W X + K_\lambda| = |Z^T W Z + Q| |M + P_\lambda|,$$

where only the last factor directly depends on the penalty parameters $\lambda$. Combined with (2.10) and taking $\lambda_j^\mu \sim G(1, b^\mu = 10^{-4})$, we conclude that

$$\log \tilde{p}(\lambda^\mu|\lambda^\sigma, D) \doteq \log p(\hat{\psi}_\lambda, \lambda|D) - \frac{1}{2} \log |M^\mu + P_\lambda^\mu| \quad (2.12)$$

$$= \ell(\hat{\psi}_\lambda; D) + \sum_{j=1}^{J_1} \left\{ \frac{L_1 - r}{2} \log \lambda_j^\mu - \left( b^\mu + \frac{1}{2}(\hat{\theta}_j^\mu)^\top P^\mu \hat{\theta}_j^\mu \right) \lambda_j^\mu \right\}$$

$$- \frac{1}{2} \log |M^\mu + P_\lambda^\mu|.$$  

The indirect dependence of the log-likelihood and of $M^\mu$ on $\lambda^\mu$ (through $\hat{\psi}_\lambda$ and $W^\mu$) will be ignored during the computation of the gradient $U_{\lambda^\mu}$ and Hessian $H_{\lambda^\mu}$ as (non reported) numerical simulations suggest that this dependence is moderate. Practically, in an iterative maximization of (2.12) using the N-R algorithm, we fix $\ell(\hat{\psi}_\lambda; D)$ and $M^\mu$ at their values $\hat{\ell}$ and $\hat{M}^\mu$ at the beginning of the iteration, and compute the gradient and Hessian of

$$\log \tilde{p}(\lambda^\mu|\lambda^\sigma, D) = \hat{\ell} + \sum_{j=1}^{J_1} \left\{ \frac{L_1 - r}{2} \log \lambda_j^\mu - \left( b^\mu + \frac{1}{2}(\hat{\theta}_j^\mu)^\top P^\mu \hat{\theta}_j^\mu \right) \lambda_j^\mu \right\}$$

$$- \frac{1}{2} \log |\hat{M}^\mu + P_\lambda^\mu|.$$  

Let $\hat{R}_j^\mu = \hat{R}_j^\mu(\lambda^\mu) = (\hat{M}^\mu + P_\lambda^\mu)^{-1} \left((1_j 1_j^\top) \otimes P^\mu\right)$ for $j = 1, \ldots, J_1$ where $1_j$ denotes the $j$th unit vector. Then, using results on the derivative of determinants and after some algebra, on can show that

$$\left( U_{\lambda^\mu}(\lambda^\mu) \right)_j = \frac{\partial \log \tilde{p}(\lambda^\mu|\lambda^\sigma, D)}{\partial \lambda_j^\mu} = \frac{L_1 - r}{2\lambda_j^\mu} - \left( b^\mu + \frac{1}{2}(\hat{\theta}_j^\mu)^\top P^\mu \hat{\theta}_j^\mu \right) - \frac{1}{2} \text{tr} \left( \hat{R}_j^\mu \right),$$

$$-[H_{\lambda^\mu}(\lambda^\mu)]_{jk} = - \frac{\partial^2 \log \tilde{p}(\lambda^\mu|\lambda^\sigma, D)}{\partial \lambda_j^\mu \partial \lambda_k^\mu} = \frac{L_1 - r}{2(\lambda_j^\mu)^2} \delta_{jk} - \frac{1}{2} \text{tr} \left( \hat{R}_j^\mu \hat{R}_k^\mu \right). \quad (2.13)$$

Similar expressions can be obtained for $(\lambda^\sigma|\lambda^\mu, D)$ by switching the role of $\mu$ and $\sigma$ as superscripts. The penalty parameters are selected to maximize (2.12) and its counterpart for $\lambda^\sigma$ using Algorithm 2, yielding $\hat{\lambda}^\sigma$ and $\hat{\lambda}^\sigma$.  

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Algorithm 2. Selection of $\lambda^\mu$ and $\lambda^\sigma$

Let $g(\nu) = \log \tilde{p}(\lambda^\mu|\lambda^\sigma, D)$ where $\lambda^\mu = \lambda_{\text{min}} + \exp(\nu)$ with $\lambda_{\text{min}}$ denoting the smallest desirable value for the penalty parameter of an additive term. Using the chain rule, one can show that $(\tilde{U}_{\nu})_j = \exp(\nu_j)(\tilde{U}_{\lambda^\nu})_j$ and $(\tilde{H}_{\nu})_{jk} = \exp(\nu_j + \nu_k)(\tilde{H}_{\lambda^\nu})_{jk} + \delta_{jk}\exp(\nu_j)(\tilde{U}_{\lambda^\nu})_j$ for $1 \leq j, k \leq J_1$. We propose to select $\lambda^\mu$ by maximizing $g(\nu)$ using a Newton-Raphson algorithm with at each iteration:

1. (a) Given current values for $\lambda^\mu$ and $\hat{\theta}^\mu_{\lambda}$, compute the gradient $\tilde{U}_\nu$ and Hessian matrix $\tilde{H}_\nu$ using (2.13);
   (b) Update: $\nu \leftarrow \nu - \frac{1}{\tilde{H}_\nu^{-1}} \tilde{U}_\nu$; $\lambda^\mu \leftarrow \lambda_{\text{min}} + \exp(\nu)$;

2. Update $\hat{\psi}^\mu_{\lambda}$ using Algorithm 1, $\tilde{M}^\mu$ using (2.11), yielding $\hat{\theta}^\mu_{j,\lambda}$ and $\hat{R}^\mu_j$, giving at convergence $\hat{\lambda}^\mu = \exp(\hat{\nu})$.

The same procedure with the superscripts $\sigma$ and $\mu$ interchanged yields $\hat{\lambda}^\sigma$.

2.5 Nonparametric pivotal density

2.5.1 Density specification

Besides classical parametric choices for the distribution of the standardized error term $\epsilon$, nonparametric forms could be preferred. Here, we propose to specify that distribution through the associated hazard $h_\epsilon(\cdot)$ function using a linear combination of $K$ B-splines, $\log h_\epsilon(r) = \sum_{k=1}^K b_k(r)\phi_k$, where $\{b_k(\cdot) : k = 1, \ldots, K\}$ denotes a large B-spline basis associated to an equidistant grid of knots on the support of the distribution. Given the constraints $E(\epsilon) = 0$ and $V(\epsilon) = 1$, one can practically assume (using Chebyshev’s theorem) that (most of) the probability mass is on $(r_{\text{min}}, r_{\text{max}}) = (-6, 6)$, say. Our approach is to some extent connected to the proposal made by Cai et al. (2002) with a (truncated) linear spline basis in a mixed model framework. We go further here by considering interval-censored data and moment constraints for the underlying density function. Note that starting from the hazard function to estimate the underlying distribution does not imply that the underlying variable must be positive. The only requirement is the designation of a (conservative) lower bound for the support of the standardized error term. A spline approximation to the log-density could also be considered (Eilers and Marx, 1996; Kooperberg and Stone, 1991; Lambert and Eilers, 2009; Lambert, 2011), but a construct based on the hazard function turns out to be analytically more convenient to handle censored data, see below.
2.5.2 Density estimation from i.i.d. right-censored data

We now detail how we propose to estimate the spline coefficients \( \phi \) in the framework of Bayesian P-splines from potentially right- or even interval-censored data.

Denote by \( \{ J_j = [a_{j-1}, a_j) \}_{j=1}^{J} \) a partition of \((r_{\min}, r_{\max})\) into a very large number \( J \) of bins of equal width \( \Delta \) with midpoints \( \{ u_j \}_{j=1}^{J} \). Given a random sample of \( n \) i.i.d. observations \( r_i (i = 1, \ldots, n) \) for a potentially right-censored (coded by \( d_i = 0 \) and 1 otherwise) variable \( \epsilon \), let \( k_j = \sum_{i=1}^{n} k_{ij} \) and \( n_j = \sum_{i=1}^{n} n_{ij} \) with \( k_{ij} = 1(r_i \in J_j) \) and \( n_{ij} = 1(r_i \geq a_{j-1}) = 1(r_i \in \cup_{s \geq j} J_s) \). The log-likelihood for the estimation of the spline parameters \( \phi = (\phi_1, \ldots, \phi_K) \) from right-censored data can be written as

\[
\ell(\phi|D) = \sum_{i=1}^{n} \left\{ d_i \log h_\epsilon(r_i) - H_\epsilon(r_i) \right\} \approx \sum_{j=1}^{J} (k_j \log h_j - n_j h_j \Delta) \tag{2.14}
\]

with \( h_j = h_\epsilon(u_j) = \exp\left\{ \sum_{k=1}^{K} b_k(u_j) \phi_k \right\} \) where the approximation in (2.14) comes from data binning and quadrature to approximate the cumulated hazard function. Following Eilers and Marx (1996), we penalize third order \((r = 3)\) differences of successive spline parameters, yielding the penalized log-likelihood, \( \ell_p(\phi|\tau, D) = \ell(\phi|D) - \tau^2 \phi^\top P \phi \), with penalty matrix \( P \) of rank \((K-r)\). Given the expressions for the gradient and Hessian,

\[
U_\tau(\phi) = \frac{\partial \ell_p}{\partial \phi} = B^\top (k - nh \Delta) - \tau P \phi ; \tag{2.15}
\]

\[
-H_\tau(\phi) = -\frac{\partial^2 \ell_p}{\partial \phi \partial \phi^\top} = B^\top \text{diag}(nh \Delta) B + \tau P , \tag{2.16}
\]

where \( [B]_{jk} = b_k(u_j), k = (k_j)_{j=1}^{J}, n = (n_j)_{j=1}^{J}, h = (h_j)_{j=1}^{J} \), one can use the (fast converging) Newton-Raphson procedure to obtain spline parameter estimates for a given value of the penalty parameter \( \tau \), with at each iteration, \( \phi \leftarrow \phi - (H_\tau(\phi))^{-1} U_\tau(\phi) \), yielding at convergence \( \hat{\phi}_\tau \).

2.5.3 Inclusion of interval-censored data

The contribution of interval-censored units to \( k_j \) and \( n_j \) can also be included and reevaluated at every iteration of the preceding Newton-Raphson procedure. Denote the hazard and density estimates from the previous iteration by \( \tilde{h}_\epsilon(\cdot) \) and \( \tilde{f}_\epsilon(\cdot) = \tilde{h}_\epsilon(\cdot) \exp(-\tilde{H}_\epsilon(\cdot)) \), and let \( \tilde{n}_j = \int_{J_j} \tilde{f}_\epsilon(r)dr \approx \tilde{f}_\epsilon(u_j) \Delta \). Consider an interval-censored observation \( r_i \in (r_i^L, r_i^R) \) and let
\[ \mathcal{G}_i = \{ j : J_j \cap (r_i^L, r_i^R) \neq \emptyset \} \]. Then, the contribution of unit \( i \) to the previously defined \( k_j \) and \( n_j \) are given by \( k_{ij} = \pi_j / \sum_{s \in \mathcal{G}_i} \pi_s \mathbb{1}(j \in \mathcal{G}_i) \) and \( n_{ij} = \mathbb{1}(j < \min \mathcal{G}_i) + \sum_{s=j}^{\max \mathcal{G}_i} \pi_s / \sum_{s \in \mathcal{G}_i} \pi_s \mathbb{1}(j \in \mathcal{G}_i) \), respectively. At convergence, the procedure in Section 2.5.2 with, now, interval-censored data entering the computation of \( k_j \) and \( n_j \) will provide an estimate \( \hat{\phi} \) of the spline parameters \( \phi \) for given \( \tau \) and, hence, of the density estimate underlying the potentially right- or interval-censored observations.

2.5.4 Density estimation with moment constraints

Constraints on the mean and variance of the underlying distribution can also be forced. More generally, consider a set of (potentially) nonlinear constraints \( F_s(\phi) = f_s \) \( (s = 1, \ldots, S) \) shortly denoted vectorially by \( \mathbf{F}(\phi) = \mathbf{f} \).

At every iteration of the preceding Newton-Raphson procedure, we suggest to linearize each constraint using a first-order Taylor expansion about the current estimate \( \tilde{\phi} \) of the spline parameters, \( \tilde{F}_s(\phi) = F_s(\phi) + \mathbf{v}_s^\top (\phi - \tilde{\phi}) \) with \( \mathbf{v}_s = \frac{\partial F_s(\phi)}{\partial \phi} \). Hence, letting \( \mathbf{V} = [\tilde{\mathbf{v}}_1, \ldots, \tilde{\mathbf{v}}_S]^\top \in \mathbb{R}^{S \times K} \), a linearized version of the constraints is \( \mathbf{V} \phi = \mathbf{c} \) with \( \mathbf{c} = \mathbf{V} \tilde{\phi} + (\mathbf{f} - \mathbf{F}(\tilde{\phi})) \).

The estimation of the spline parameters under these linearized constraints can be made using the Lagrangian

\[
G(\phi, \omega) = \ell_p(\phi | \tau, \mathbf{D}) - \omega^\top (\mathbf{V} \phi - \mathbf{c}),
\]

(2.17)

with Lagrange multipliers \( \omega \). Practically, at every iteration of a Newton-Raphson procedure, the preceding values \( (\tilde{\phi}, \tilde{\omega}) \) of the spline parameters and Lagrange multipliers are updated using

\[
\begin{pmatrix}
\tilde{\phi} \\
\tilde{\omega}
\end{pmatrix}
\leftarrow
\begin{pmatrix}
\frac{\partial^2 \ell_p(\phi | \tau, \mathbf{D})}{\partial \phi \partial \phi^\top} & -\mathbf{V}^\top \\
-\mathbf{V} & 0
\end{pmatrix}^{-1}
\begin{pmatrix}
\frac{\partial \ell_p(\phi | \tau, \mathbf{D})}{\partial \phi} \\
-\mathbf{V} \tilde{\phi} + \mathbf{c}
\end{pmatrix},
\]

(2.18)

with partial derivatives of the penalized log-likelihood given in (2.15) and (2.16). Now consider specific constraints on the spline parameters based on the first two moments (\( S = 2 \)) of the density, remembering that \( f(u_j) = \)
Let \( h_j \exp(-H_j) \) (and letting \( \Delta \to 0^+ \)):

\[
\mathbb{E}(\epsilon) = \mu_\epsilon = 0 \Leftrightarrow F_1(\phi) = \sum_{j=1}^{J} u_j h_j \exp(-H_j) \Delta = 0 = f_1;
\]

\[
\mathbb{V}(\epsilon) = \sigma_\epsilon^2 = 1 \Leftrightarrow F_2(\phi) = \sum_{j=1}^{J} u_j^2 h_j \exp(-H_j) \Delta = f_2.
\]

Let \( \tilde{h}_j = \tilde{h}_\epsilon(u_j), \tilde{H}_j = \sum_{\ell \leq j} \tilde{h}_j \Delta, \tilde{f}_j = \tilde{h}_j \exp(-\tilde{H}_j) \) and \( b_{jk} = b_k(u_j) \).

Then, one can show that

\[
\tilde{V}_{1k} = \frac{\partial F_1(\phi)}{\partial \phi_k} = \sum_{j=1}^{J} u_j \tilde{f}_j \Delta \left( b_{jk} - \sum_{\ell \leq j} b_{k\ell} \tilde{h}_\ell \Delta \right)
\]

\[
\tilde{V}_{2k} = \frac{\partial F_2(\phi)}{\partial \phi_k} = \sum_{j=1}^{J} u_j^2 \tilde{f}_j \Delta \left( b_{jk} - \sum_{\ell \leq j} b_{k\ell} \tilde{h}_\ell \Delta \right) - 2 F_1(\phi) \tilde{V}_{1k}.
\]

Combining these last results with the elements from Sections 2.5.2 and 2.5.3, one can estimate the spline parameters underlying the hazard and, hence, the density, for given (potentially) right- or interval-censored data and penalty parameter \( \tau \). The following section is devoted to the selection of \( \tau \).

### 2.5.5 Selection of the penalty parameter \( \tau \)

Given the following priors,

\[
\tau \sim \mathcal{G}(1, b) \quad ; \quad p(\phi | \tau) \propto \tau^{-\frac{K-r}{2}} \exp \left( -\frac{\tau}{2} \phi^\top P \phi \right), \quad (2.19)
\]

the joint posterior for the spline and the penalty parameters \((\phi, \tau)\) are

\[
p(\phi, \tau | D) \propto \exp\{\ell(\phi | D)\} p(\phi | \tau) p(\tau) = \exp\{\ell_p(\phi | \tau, D)\} \tau^{-\frac{K-r}{2}} p(\tau). \quad (2.20)
\]

Using the same arguments as in Section 2.4 for \((\psi | \lambda, D)\), the conditional posterior for the spline parameters, \(p(\phi | \tau, D) \propto \exp\{\ell_p(\phi | \tau, D)\}\), can be shown to be approximately

\[
(\phi | \tau, D) \sim \mathcal{N} \left( \hat{\phi}_\tau, \hat{\Sigma}_\tau \right), \quad (2.21)
\]

where \( \hat{\phi}_\tau \) denotes the conditional posterior mode (equal to the penalized MLE of \( \phi \) given \( \tau \), see Sections 2.5.2 and 2.5.3), \( \hat{\Sigma}_\tau = H_{\tau}(\hat{\phi}_\tau) = B^\top W_{\tau} B + \tau P \), cf. Eq. (2.16), with \( W_{\tau} = \text{diag}(w_{\tau}) \), \( w_{\tau} = nh_{\tau} \Delta \) and \( h_{\tau} \) giving the estimated hazard at the bin midpoints when \( \phi = \hat{\phi}_\tau \). Given that the number of observations \((k)_j\) in bin \( J_j \) has expected value \((w)_j = (nh_{\tau} \Delta)_j\), one might reasonably approximate the last variance-covariance matrix by
\[ \hat{\Sigma}_\tau^{-1} \approx B^\top W B + \tau P \] with \( W = \text{diag}(k) \), thereby restricting its explicit dependence on \( \tau \) to the \( \tau P \) term. The marginal posterior for \( \tau \) is given by

\[ p(\tau|D) = \frac{p(\phi, \tau|D)}{p(\phi_\tau, \tau|D)} \propto p(\hat{\phi}_\tau, \tau|D) |B^\top W B + \tau P|^{-1/2} \] (2.22)

with the approximation coming from (2.21) and the substitution of \( W_\tau \) by \( W \). Now consider a singular value decomposition of penalty matrix, \( P = U \Upsilon \Upsilon \Upsilon \Upsilon U^\top \), where \( U = [U_1 \ U_0] \), \( U^\top U = I_K \), \( \Upsilon = \text{blockdiag}(\Upsilon_1, 0_r) \), with the last \( r \) diagonal elements of \( \Upsilon = \text{diag}(\upsilon) \) being zero. Then, using properties of determinants and defining \( \tilde{B} = W^{1/2} U \tilde{B}_1 = W^{1/2} U_1 \), \( \tilde{B}_0 = W^{1/2} U_0 \), \( M = \tilde{B}_1^\top \tilde{B}_1 - \tilde{B}_1^\top \tilde{B}_0 (\tilde{B}_0^\top \tilde{B}_0)^{-1} \tilde{B}_0^\top \tilde{B}_1 \), one has

\[ |B^\top W B + \tau P| = |\tilde{B}_0^\top \tilde{B}_0| |\Upsilon_1|^\tau K_{r-} \prod_{j=1}^{K-r} \left( 1 + \frac{n \tilde{m}_j}{\tau} \right) \] (2.23)

where \( \tilde{M} = \frac{1}{n} \Upsilon_1^{-1/2} M \Upsilon_1^{-1/2} \) has eigenvalues \( \{\tilde{m}_j\}_{j=1}^{K-r} \) independent of \( \tau \).

Combining (2.19), (2.20), (2.22) and (2.23), one has

\[ \log p(\tau|D) = \ell(\hat{\phi}_\tau|\tau, D) + \log p(\tau) - \frac{1}{2} \sum_{j=1}^{K-r} \log \left( 1 + \frac{n \tilde{m}_j}{\tau} \right) \]

\[ = \ell(\hat{\phi}_\tau|D) - \tau \left( b + \frac{1}{2} \hat{\phi}_\tau^\top P \hat{\phi}_\tau \right) - \frac{1}{2} \sum_{j=1}^{K-r} \log \left( 1 + \frac{n \tilde{m}_j}{\tau} \right) , \] (2.24)

suggesting Algorithm 3 to select \( \tau \).

**Algorithm 3.** Density estimation (selection of \( \tau \) and computation of \( \hat{\phi}_\tau \))

**Principle:** the algorithm alternates the following two steps till convergence:

1. For a given value of the penalty parameter \( \tau \), select the spline parameters \( \phi \) to maximize \( p(\phi|\tau, D) \) under the moments constraints \( \mathbb{E}(\epsilon) = 0 \) and \( \mathbb{V}(\epsilon) = 1 \);

2. Update \( \tau \) to maximize the approximation (2.24) to \( \log p(\tau|D) \).

**Practically:** repeat till convergence:

1. Given the current estimate for \( \tau \), maximize the Lagrangian in (2.17) by repeating the Newton-Raphson step in (2.18) till convergence to \( \hat{\phi}_\tau \).
2. Update $\tau$ by using the fixed-point method on the partial derivative of (2.24) w.r.t. $\tau$ set to zero. Practically, repeat till convergence

$$\tau \leftarrow \sum_{j=1}^{K-r} \frac{n\tilde{m}_j}{\tau + n\tilde{m}_j} / \left(2b + \hat{\phi}_\tau P\hat{\phi}_\tau\right).$$

At convergence, it yields $(\hat{\tau}, \hat{\phi})$ and the estimated hazard $\hat{h}_\epsilon(\cdot) = \exp\left(\sum_{k=1}^{K} b_k(\cdot)\hat{\phi}_k\right)$.

For example, with a dataset of size $n = 1000$ including 40% uncensored, 40% interval-censored and 20% right-censored data, the selection of $\tau$ and the estimation of $K = 50$ B-spline parameters (an unnecessary very large $K$ used to challenge Algorithm 3) took 6 iterations and one tenth of a second using pure R code on a small desktop computer.

### 2.6 Algorithm for fitting the NP additive location-scale model

We now have all the necessary ingredients for fitting the nonparametric double additive location-scale model (NP-DALSM) from possibly right- or even interval-censored data. The algorithm is iterative and alternates the estimation of the error density (Step 1), of the regression and spline parameters in the location (Step 2) and dispersion (Step 3) submodels, selection of the penalty parameters for the additive terms in location and dispersion (Step 4), see Algorithm 4.

**Algorithm 4. Global Algorithm: Fitting the NP additive location-scale model**

Iterate the following steps till convergence:

1. **Estimation of the error hazard and density:**
   
   (a) Given the current estimates for the regression and splines parameters, compute the standardized residuals $r_i = \frac{y_i - \hat{\mu}(\Psi^K)}{\hat{\sigma}(\Psi^K)}$ accompanied by their observation indicators $d_i$, some of these residuals being right-censored $(d_i = 0)$ or interval-censored with $r_i \in (r_i^L, r_i^R)$.
   
   (b) Use Algorithm 3 on these data to update the estimates of the error hazard function $\hat{h}_\epsilon$ and density $f_\epsilon$. It is based on the estimation and selection of the underlying spline parameters $\phi$ and penalty parameter $\tau$.

2. **Estimation of $\psi^\mu$:** given the current values of the other parameters and in particular of the penalty parameter vector $\lambda^\mu$ for the additive
terms in the location submodel, \( \psi^\mu = (\beta, \text{vec}(\Theta^\mu)) \) is reevaluated to maximize \( p(\psi^\mu|\psi^\sigma, \lambda^\mu, \phi, D) \propto L(\psi^\mu, \psi^\sigma, \phi; D) \ p(\psi^\mu|\lambda^\mu) \) using the Newton-Raphson (N-R) procedure described in Algorithm 1 with the current estimate for \( \psi^\mu \) as starting value.

3. **Estimation of \( \psi^\sigma \):** given the current values of the other parameters and in particular of the penalty parameter vector \( \lambda^\sigma \) for the additive terms in the dispersion submodel, \( \psi^\sigma = (\delta, \text{vec}(\Theta^\sigma)) \) is reevaluated to maximize \( p(\psi^\sigma|\lambda^\mu, \lambda^\sigma, \phi, D) \propto L(\psi^\mu, \psi^\sigma, \phi; D) \ p(\psi^\sigma|\lambda^\sigma) \left| \tilde{\Sigma}_\lambda^{1/2} \right|^{1/2} \) using the Newton-Raphson (N-R) procedure described in Algorithm 1 with the current estimate for \( \psi^\sigma \) as starting value.

4. **Selections of \( \lambda^\mu \) and \( \lambda^\sigma \):** the penalty parameters in the additive terms are chosen to maximize \( \log \tilde{p}(\lambda^\mu|\lambda^\sigma, D) \) and \( \log \tilde{p}(\lambda^\sigma|\lambda^\mu, D) \) using Algorithm 2.

Possible starting values are obtained by:
- Assuming a Gaussian error distribution ;
- Discarding right-censored data and setting interval-censored ones to their midpoint value, yielding a reduced response vector \( \tilde{y} \) with an associated design matrix \( \tilde{X}^\mu \) for the additive location submodel ;
- Setting the elements in penalty vectors \( \lambda^\mu \) and \( \lambda^\sigma \) to a moderately large value (100, say) ;
- Estimating \( \psi^\mu \) using penalized LS: \( \psi^\mu \leftarrow \left( \tilde{X}^{\mu\top} \tilde{X}^\mu + K^\mu_{\lambda} \right)^{-1} \tilde{X}^{\mu\top} \tilde{y} ; \)
- Fixing \( \psi^\sigma \) to zero, except its first component \( \delta_0 \) set to the log of the mean squared error.

Convergence is very fast with the suggested initial conditions. One major advantage of our proposal is that it does not require backfitting as regression and spline parameters are updated simultaneously within the location and dispersion submodels. An additional remarkable feature is the joint update of the (log of the) penalty parameters using a Newton-Raphson procedure based on approximate analytical expressions for the gradient and Hessian of their marginal posterior. And last but not least, the error distribution is also estimated through the underlying (log-)hazard expressed as a linear combination of (penalized) P-splines with a penalty parameter selected to maximize its posterior density. The whole procedure is able to handle right- or interval-censored response data.
3 Simulation Study

An extended simulation study was made to evaluate the performances of the proposed algorithm to fit the nonparametric additive location-scale model. The data were simulated with conditional location and dispersion given by, respectively,

\[
\mu(z^\mu, x^{\mu}) = (\beta_0 + \beta_1 z_1^\mu + \beta_2 z_2^\mu) + f_1^\mu(x_1^{\mu}) + f_2^\mu(x_2^{\mu}), \quad (3.1)
\]

\[
\log \sigma(z^\sigma, x^{\sigma}) = (\delta_0 + \delta_1 z_1^\sigma + \delta_2 z_2^\sigma) + f_1^\sigma(x_1^{\sigma}) + f_2^\sigma(x_2^{\sigma}). \quad (3.2)
\]

Different combinations of sample sizes \(n (= 1500, 500, 250)\), right censoring (RC = 0%, 25%, 50%) rates and interval censoring (IC = 0%, 25%, 50%) rates were considered. The standardized error term (with mean 0 and variance 1) in (2.1) was taken to have a Normal mixture distribution, \(\epsilon \sim .8 \mathcal{N}(-0.414, 0.538^2) + .2 \mathcal{N}(1.655, 0.646^2)\), see Fig.3 in the Supplementary Material. For each of the \(n\) units, the pair of covariates \((p_1 = p_2 = 2)\) with linear effects in (3.1) and (3.2) were independently generated from Bernoulli and Normal distributions, \(z_1^\mu, z_1^\sigma \sim \text{Bern}(0.6)\); \(z_2^\mu, z_2^\sigma \sim \mathcal{N}(0, 1)\), with regression parameters \(\beta = (1.6, .3, .75)\), \(\delta = (-.5, -.03, .01)\). Two \((= J_1 = J_2)\) additive terms per regression submodel were added, \(f_1^\mu(x) = .113 - 4\sqrt{2} \sin(1.2 \pi x), f_2^\mu(x) = .586 - .3(x^2 + 3)^{-1}\), \(f_1^\sigma(x) = -0.158 + 0.15x + 0.25x^2, f_2^\sigma(x) = 12(x - 0.5)^3\), with \(x_1^\mu, x_2^\mu, x_1^\sigma, x_2^\sigma\) generated independently from a uniform distribution on \((0, 1)\), see the solid curves on Fig.6 in the Supplementary Material for a graphical representation. For each of the \(n\) units, covariates were first sampled to define the underlying first and second order (conditional) moments in (3.1) and (3.2), yielding \(\mu_i\) and \(\sigma_i\) for the \(i\)th unit. The associated uncensored response was then obtained using \(y_i = \mu_i + \sigma_i e_i\) with \(e_i\) sampled from the Normal mixture. Right censoring was created randomly and independently of the underlying response and covariates using an exponential distribution \(C_i \sim \text{Exp}(\lambda)\) with \(\lambda\) selected to reach the desired percentage \(RC\) of right censored responses. The observed response was then defined as \(t_i = \min\{y_i, c_i\}\) with observation indicator \(\delta_i = I(c_i > y_i)\). The non right-censored data (for which \(\delta_i = 1\)) were subsequently interval-censored with probability \(IC/(1-RC)\) with, then, \(y_i\) only reported to lie in \((y_i^L, y_i^R)\) where \(y_i^L = y_i - 1.5u_i \sigma(Y)\) and \(y_i^R = y_i + 1.5(1-u_i) \sigma(Y)\) with \(u_i \sim U(0,1)\), yielding an interval of width equal to 1.5 the marginal standard deviation of the response.

The double additive location-scale model (DALSM) was fitted by assuming a nonparametric (NP) or a Normal (\(\mathcal{N}\)) density for the error term. Under the working Normality hypothesis, the sandwich estimator (White
was preferred over the model-based one for the variance-covariance of the regression and spline parameter estimates. A report on the detailed simulation results can be found in Supplementary Material C. In summary, our simulation study suggests that the proposed NP estimation strategy enables to quantify the effects of covariates on location and dispersion with negligible biases and important efficiency gains as compared to an approach assuming normality. Uncertainty in the estimation is properly quantified, except when the sample size is small (as compared to the number of parameters in the model). Then, the effective coverage of credible intervals can be smaller than the nominal value. In these cases, MCMC with proposals built using approximated posteriors resulting from the algorithm in Section 2.6 would generate more reliable quantification of uncertainty, but at a higher computational cost. The error density is properly estimated in the absence of right censoring even with a rather small sample size and a large interval censoring rate. But the combination of a small $n$ and a large right censoring rate somehow decrease the quality of the expected reconstruction as the available information on the error distribution becomes sparse and incomplete. Then, the smallest component in the Normal mixture tends to be flattened around its mode.

### 4 Application

The proposed application involves interval- and right-censored responses. The data of interest come from the European Social Survey [European Social Survey Round 8 Data 2016](https://www.europeansocialsurvey.org). We focus on the money available per person in Belgian households for respondents aged 25-55 when the main source of income comes from wages or salaries ($n = 756$). Each person reports the total net monthly income of the household in one of 10 decile-based intervals: 1: $<1120$ ($n_1 = 8$), 2: $[1120, 1399]$ ($n_2 = 13$), 3: $[1400, 1719]$ ($n_3 = 47$), 4: $[1720, 2099]$ ($n_4 = 53$), 5: $[2100, 2519]$ ($n_5 = 82$), 6: $[2520, 3059]$ ($n_6 = 121$), 7: $[3060, 3739]$ ($n_7 = 167$), 8: $[3740, 4529]$ ($n_8 = 126$), 9: $[4530, 5579]$ ($n_9 = 74$), 10: $\geq 5580$ euros ($n_{10} = 65$).

We model the relation of the available income per person (91.4% are interval-censored, 8.6% right-censored) to the availability of (at least) 2 salaries (64.2%) in the household, the age (Age: $41.0 \pm 8.83$ years) and the number of years of full-time education completed (Educ: $14.9 \pm 3.34$ years) by the respondent. That individualized income is obtained by dividing the household one by the OECD-modified equivalence scale [Hagenaars et al. 1994](https://www.unece.org/fileadmin/DAM統計/ece/ceg/2010/2010_00125.pdf), as recommended by the Statistical Office of the European Union.
The first adult in the household contributes to 1.0 to that scale, each person aged at least 14 adds .5 to it, while each younger member brings an extra .3 to the household weight. For example, a respondent aged 31 declaring a household net monthly income in the interval (3060, 3740) euros with a partner aged 34 and 4 children aged 15, 10, 9 and 3 would be associated to an OECD-modified scale of 2.9 and an interval-censored response of (1055.2, 1289.7) euros (available per person).

The nonparametric double additive location-scale model (NP-DALSM) described in Section 2 with the flexible error density from Section 2.5 was fitted using Algorithm 4: 10 (=L) and 20 (=K) B-splines were taken to model the additive terms and the log hazard of the error distribution, respectively. The response was rescaled in thousand euros, while quantitative covariates were relocated and rescaled to take values in (0, 1) before running the algorithm. It converged after 10 iterations in about 2 seconds using pure R code. Parameter estimates quantifying the effect of the TwoIncomes binary indicator on the conditional mean and the log of the standard deviation can be found in Table 1, suggesting an average increase of 266 euros per person in the household when two members of the household work (conditionally on Age and Educ), while the effect on dispersion is not statistically significant.

The effects of Age and Educ on the conditional mean and dispersion can be visualized on the first and second rows of Fig. 1 respectively, with the corresponding estimated additive terms. The money available per household member tends to decrease with age (see $f^\mu_1(Age)$) between 25 and 40 (most likely due the arrival of children in the family) and to increase afterwards (probably thanks to wage increase with seniority and the departure

| Fixed effects | Location | Dispersion |
|---------------|----------|------------|
| Intercept     | 1.589    | -0.430     |
| TwoIncomes    | 0.266    | -0.020     |

| Additive Location | Dispersion |
|-------------------|------------|
| terms             | e.d.f.     | e.d.f.     |
| Age               | 3.69       | 2.40       |
| Educ              | 3.55       | 3.86       |

Table 1 – Belgian income data (ESS 2016): fixed effect estimates and effective degrees of freedom (e.d.f.) (with 95% credible intervals) for the additive terms in the NP double additive location-scale model.
Figure 1 – Belgian income data (ESS 2016): estimated additive terms in the NP additive location-scale model with pointwise 95% credible intervals; Row 1 (effects on location): \( f_1^\mu (\text{Age}) \) and \( f_2^\mu (\text{Educ}) \) in euros; Row 2 (relative effects on dispersion): \( \exp ( f_1^\sigma (\text{Age})) \) and \( \exp ( f_2^\sigma (\text{Educ})) \); Estimated error density (solid line) compared to the standard Normal (dashed line).
Figure 2 – Belgian income data (ESS 2016): fitted conditional deciles for the income per person in two-income households.
of children). The dispersion, reported as the exponential of the additive term, \( \exp(f_1'(\text{Age})) \), significantly increases with \( \text{Age} \) with an acceleration over 45. However, the dominating effect comes from the education level of the respondent with approximately a difference of 1000 euros (in expected available income per person) between a less educated (6 years) and a highly educated (20 years) one, see \( f_2''(\text{Educ}) \). The effect on dispersion is also large, see \( \exp(f_2''(\text{Educ})) \), with essentially an important contrast between less and highly educated respondents, the latter group showing the largest heterogeneity. Indeed, while most low skilled persons have difficulties to find a job or are confined to low-pay professions, a university degree offers a large variety of opportunities from a moderately paid civil servant job to a manager position in a multinational corporation in the chemical, pharmaceutical or financial sectors. The estimated density for the error term can also be seen at the bottom of Fig.1 with a right-skewed shape clearly distinguishable from the Gaussian one typically assumed in parametric location-scale regression models. The resulting estimates for the deciles of the income available per person for varying education levels and ages are pictured on Fig.2. Interval- and right-censored data are represented as intervals and dashed semi-intervals, respectively (with horizontal noise added to untie respondents sharing the same age). The precedingly discussed combined non-linear impacts of age and education level on the distribution of the available income per person are now clearly visible.

5 Discussion

The proposed nonparametric double-additive location-scale model (NP-DALSM) is a fast and efficient alternative to parametric location-scale models. Unlike moment-based estimation approaches such as the generalized method of moments (see e.g. [Wang et al., 2014]), it provides a full estimation of the conditional distribution of the response, that can be used to understand and visualize how it is qualitatively and quantitatively affected by covariates. The density of the error distribution is estimated from possibly right- or interval-censored responses under moment constraints. The penalty parameters controlling the smoothness of the additive terms in the location and dispersion submodels are automatically selected using approximations to their marginal posteriors. These are obtained by substituting Laplace approximations to the conditional posteriors of the spline parameters, see Section 2.3.

Simulations suggest that the effects of covariates are properly estimated
with no significant biases in the estimation of regression parameters and additive terms. The determinant in (2.6) plays an important role in the process as its neglect would lead to non negligible biases in the estimation of the dispersion part. Its role is comparable to the correction brought by restricted maximum likelihood (REML) in more elementary settings or in (adjusted) estimating functions, see e.g. Jørgensen and Knudsen (2004). Biases in the estimation of the intercepts can appear under large right censoring rates, while the additive terms tend to be over-smoothed (as it should) when information becomes sparse. It can for example result from the combination of large right censoring rates and small sample sizes (as compared to the large number of parameters to be estimated).

The nonparametric specification with P-splines of (the log-hazard function underlying) the error density markedly increases the efficiency of regression parameter and additive term estimates over results under a working Normality hypothesis, and reduces the risk of misleading conclusions following from a misspecified nonnormal parametric density. While our proposal extends to nonparametric errors and interval-censored settings some aspects of the remarkable work by Wood and Fasiolo (2017) or Wood (2017), several issues still need to be studied in that specific framework. Model validation is one topic, with the presence of interval-censored data complicating the capacity to diagnose misspecification from partially observed residuals. Model selection should also be investigated. Obvious starting solutions would consist in computing information criteria such as AIC and BIC with the number of parameters replaced by effective dimensions (Komárek et al., 2005). The uncertainty in the selection of the penalty parameters can also be accounted for, see Wood et al. (2016) or Wood (2017, Section 6.11) for additional perspectives. More elaborate procedures for testing the necessity to include an additive term (in location or dispersion) or to opt for a simpler linear form could be developed in our framework. From a Bayesian perspective, they should be built using a combination of the conditional posterior for the spline parameters of the additive term of interest and the marginal posterior for the associated penalty parameter. Nonlinear and smooth interactions between covariates could also be added to the location and dispersion parts in the same way as Lee and Durbán (2011) and Rodríguez-Álvarez et al. (2018) with the conditional mean in mixed models.
A Expressions for $\omega^\mu, \omega^\sigma, W^\mu, W^\sigma$

Rewriting the error density as $f_\epsilon(\cdot) = h_\epsilon(\cdot) \exp[-H_\epsilon(\cdot)]$ where $H_\epsilon(\cdot) = -\log S_\epsilon(\cdot)$ and $h_\epsilon(\cdot) = f_\epsilon(\cdot)/S_\epsilon(\cdot)$, we obtain the following expressions (depending on the censoring status of the response) for the elements of $\omega^\mu$, $\omega^\sigma$ in $\mathbb{R}^n$ and for the diagonal elements $w^\mu$, $w^\sigma$ in the $n \times n$ matrices $W^\mu = \text{diag}(w^\mu)$, $W^\sigma = \text{diag}(w^\sigma)$:

**Uncensored or right-censored** $t_i$: if $d_i$ is the censoring indicator, then

$$\omega_i^\mu = -\frac{1}{\sigma_i} \left( d_i \frac{h_i'}{h_i} - h_i \right) ; \quad \omega_i^\sigma = -d_i r_i \frac{h_i'}{h_i} - d_i + r_i h_i ,$$

$$w_i^\mu = \frac{1}{\sigma_i^2} \left\{ d_i \left( \frac{h_i'}{h_i} \right)^2 - d_i \frac{h_i''}{h_i} + h_i' \right\} , \quad \text{(A.1)}$$

$$w_i^\sigma = d_i \left\{ \left( \frac{h_i'}{h_i} \right)^2 r_i^2 + \frac{h_i'}{h_i} r_i - \frac{h_i''}{h_i} r_i^2 \right\} + h_i' r_i^2 + h_i r_i ,$$

where $h_i = h_\epsilon(r_i)$, $h_i' = \frac{dh_i(r_i)}{dr}$, $h_i'' = \frac{d^2h_i(r_i)}{dr^2}$ ;

**Interval-censored with** $y_i \in (y_i^L, y_i^R)$:

$$\omega_i^\mu = \frac{1}{\sigma_i} \frac{f_\epsilon(r_i^L) - f_\epsilon(r_i^R)}{S_\epsilon(r_i^L) - S_\epsilon(r_i^R)} ; \quad \omega_i^\sigma = \frac{r_i^L f_\epsilon(r_i^L) - r_i^R f_\epsilon(r_i^R)}{S_\epsilon(r_i^L) - S_\epsilon(r_i^R)} ,$$

$$w_i^\mu = \frac{1}{\sigma_i^2} \left[ \frac{f_\epsilon(r_i^L) g(r_i^L) - f_\epsilon(r_i^R) g(r_i^R)}{S_\epsilon(r_i^L) - S_\epsilon(r_i^R)} \right] + \left\{ \frac{f_\epsilon(r_i^L) - f_\epsilon(r_i^R)}{S_\epsilon(r_i^L) - S_\epsilon(r_i^R)} \right\}^2 ,$$

$$w_i^\sigma = \left\{ \frac{r_i^L f_\epsilon(r_i^L)m(r_i^L) - r_i^R f_\epsilon(r_i^R)m(r_i^R)}{S_\epsilon(r_i^L) - S_\epsilon(r_i^R)} \right\} + \left\{ \frac{r_i^L f_\epsilon(r_i^L) - r_i^R f_\epsilon(r_i^R)}{S_\epsilon(r_i^L) - S_\epsilon(r_i^R)} \right\}^2 , \quad \text{(A.2)}$$

where $g(r) = h_\epsilon'(r)/h_\epsilon(r) - h_\epsilon(r)$ and $m(r) = 1 + r g(r)$. 

B Gradient and Hessian of $E_\lambda^\mu$

Denote the $i$th row of $\mathbf{X}_i^\mu$ (resp. $\mathbf{X}_i^\sigma$) by the column vector $\mathbf{x}_i^\mu$ (resp. $\mathbf{x}_i^\sigma$). Let us drop the “$\sim$” sign to simplify notation and set $\Sigma_\lambda^\mu = (\mathbf{X}_i^\mu^T W^\mu \mathbf{X}_i^\mu + K_\lambda^\mu)^{-1}$. One has

$$E_\lambda^\mu = \frac{1}{2} \log |\Sigma_\lambda^\mu| = -\frac{1}{2} \log \left| \sum_{i=1}^n w_i^\mu \mathbf{x}_i^\mu \mathbf{x}_i^\mu^T + K_\lambda^\mu \right| .$$
Let $A_k = \mathbf{X}^\mu \text{diag} \left( \frac{\partial w_i^\mu}{\partial \psi_k^\sigma} \right) \mathbf{X}^\mu$ and $A_{k\ell} = \mathbf{X}^\mu \text{diag} \left( \frac{\partial^2 w_i^\mu}{\partial \psi_k^\sigma \partial \psi_\ell^\sigma} \right) \mathbf{X}^\mu$ for $1 \leq k, \ell \leq q$. Reminding that for an arbitrary positive definite matrix $\mathbf{M}_t$, $\frac{\partial}{\partial t} \log \left| \mathbf{M}_t \right| = \text{tr} \left( \mathbf{M}_t^{-1} \frac{\partial \mathbf{M}_t}{\partial t} \right)$, $\frac{\partial}{\partial t} \mathbf{M}_t^{-1} = -\mathbf{M}_t^{-1} \frac{\partial \mathbf{M}_t}{\partial t} \mathbf{M}_t^{-1}$, and using $\frac{\partial w_i^\mu}{\partial \psi_k^\sigma} \approx -2w_i^\mu x_{ik}$, one can show that

$$
\frac{\partial E_k^\mu}{\partial \psi_k^\sigma} = -\frac{1}{2} \sum_{i=1}^{n} x_i^\mu \Sigma_{\lambda} x_i^\mu \frac{\partial w_i^\mu}{\partial \psi_k^\sigma} = -\frac{1}{2} \text{tr} \left( \Sigma_{\lambda} A_k \right) \approx \sum_{i=1}^{n} w_i^\mu \left( x_i^\mu \Sigma_{\lambda} x_i^\mu \right) x_{ik}^\sigma
$$

$$
\frac{\partial^2 E_k^\mu}{\partial \psi_k^\sigma \partial \psi_\ell^\sigma} = \frac{1}{2} \sum_{i=1}^{n} \left( x_i^\mu \Sigma_{\lambda} x_i^\mu \right) \frac{\partial^2 w_i^\mu}{\partial \psi_k^\sigma \partial \psi_\ell^\sigma} - \frac{1}{2} \text{tr} \left( \Sigma_{\lambda}^\mu A_k \Sigma_{\lambda}^\mu A_\ell \right) = \frac{1}{2} \text{tr} \left( \Sigma_{\lambda}^\mu A_{k\ell} \right) - \frac{1}{2} \text{tr} \left( \Sigma_{\lambda}^\mu A_k \Sigma_{\lambda}^\mu A_\ell \right)
$$

C Detailed simulation results

The double additive location-scale model (DALSM) was fitted by assuming a nonparametric (NP) or a Normal ($\mathcal{N}$) density for the error term with 10 ($=L$) B-splines (associated to equidistant knots on $(0, 1)$) to reconstruct each of the additive terms and 20 ($=K$) B-splines (associated to equidistant knots on $(-6, 6)$) to estimate the (log of the hazard function underlying the) nonnormal error density. Figures 3, 4 and 5 report on the estimation of the regression parameters $\beta$ and $\delta$ for each of the three sample sizes for the nine possible combinations of right and interval censoring rates. The boxplots inform us on the (sampling) distribution of the parameter estimates (in grey for NP and white for $\mathcal{N}$) over the $S = 500$ replicates, R.E. indicates the Relative Efficiency (defined as the ratio of the mean squared errors) under a working normality hypothesis (a value smaller than 1.0 suggesting than the NP assumption is preferable), while E.C. reports the Effective Coverage of 95% credible intervals (computed as $\hat{\theta} \pm 1.96 \text{s.e.}(\hat{\theta})$). Whatever the considered sample size, the bias in the estimation of the regression parameters is practically zero under the proposed NP approach, except for the intercept $\beta_0$ in the location part when all data are censored (with IC=RC=50%) and for the intercept $\delta_0$ in the dispersion part where negative biases increasing with the RC rate tend to appear. Larger biases appear for the intercepts under the same circumstances when assuming Normality for the error term. In addition, mean squared errors are always (resp. nearly always) markedly larger under the Normality hypothesis when $n = 1500$ (resp. $n = 500$ or 250) (as revealed by the reported R.E. values below 1.00 under $\mathcal{N}$). For settings with negligible biases and when $n = 1500$, the effective coverages of credible intervals are close to their nominal value 95% whatever the considered
assumption on the error distribution, suggesting that the standard errors were properly quantified and the posterior distribution of the parameters close to normality. When \( n = 500 \) and biases are negligible, the coverages of credible intervals are satisfactory for the location parameters, but tend to be slightly smaller than the nominal value for the dispersion parameters under the NP hypothesis. When the sample size is small (\( n = 250 \), as compared to the model complexity and the amount of censoring), while efficiency gains are still observable for the NP approach, the effective coverages of credible intervals are nearly always below the results achieved under the normality working hypothesis. Our results (not shown here) indicates an under-estimation under NP of the posterior standard deviation of the regression parameters when information is sparse.

Report on the estimation of the additive terms can be found in Tables 2, 3, and 4. Whatever the sample size and censoring rates, the absolute biases averaged over the covariate support \((0, 1)\) are very small, at the exception of \( f_2^\sigma(x) \) for values of \( x \) close to zero when the sample size is small (\( n = 250 \)) and the right and interval censoring rates are large. Then, given the sparse information available, additive term estimates naturally tend to be oversmoothed. It probably explains part of the bias reported during the estimation of the intercept \( \beta_0 \) or \( \delta_0 \). This is illustrated in Fig. 6 and 7 when the interval censoring rate is 0% or 50%, respectively, for increasing right censoring rates. The wider dark grey envelope (connecting successive intervals containing 95% of the additive term estimates \( f_µ^j(x) \) or \( f_σ^j(x) \) over the \( S \) replicates) also indicate that the working Normality hypothesis for the error term yields less efficient estimates than under the NP assumption (with light-grey envelopes). This is confirmed numerically by the relative efficiency values reported in the preceding tables. The effective coverages of 95% credible intervals for \( f_µ^j(x) \) or \( f_σ^j(x) \) averaged over the support \((0, 1)\) of the covariate and the \( S \) replicates are close to their nominal values, except when information is sparse as it naturally results in over-smoothing.

The estimates of the NP error density (averaged over the \( S \) replicates) are given in Fig. 8 for different combinations of right- and interval censoring rates. When the sample is large and in the absence of right censoring, the density is very well estimated with an excellent performance of the selection procedure for the underlying smoothness parameter (cf. Section 2.5.5). Large right censoring rates have an important negative effect on the quality of the reconstruction as it reduces the ability to detect or position the second mode of the target density. Combined with a large interval censoring rate and a small sample size, it can even result in a right-skewed unimodal average density estimate (see the dotted curve at the bottom right of the
figure) with the smallest component in the Normal mixture tending to be flattened around its mode.

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Figure 3 – Simulation study \((n = 1500)\): estimation of the regression parameters in the double additive location-scale model over \(S = 500\) replicates: boxplot of the point estimates under a nonparametric (grey) or Normal (white) error term, Relative Efficiency (R.E.) under the working Normality hypothesis, Effective Coverage (E.C.) of 95% credible intervals.
Figure 4 – Simulation study ($n = 500$): estimation of the regression parameters in the double additive location-scale model over $S = 500$ replicates: boxplot of the point estimates under a nonparametric (grey) or Normal (white) error term, Relative Efficiency (R.E.) under the working Normality hypothesis, Effective Coverage (E.C.) of 95% credible intervals.
Figure 5 – Simulation study \((n = 250)\): estimation of the regression parameters in the double additive location-scale model over \(S = 500\) replicates: boxplot of the point estimates under a nonparametric (grey) or Normal (white) error term, Relative Efficiency (R.E.) under the working Normality hypothesis, Effective Coverage (E.C.) of 95% credible intervals.
Table 2 - Simulation study \((n = 1500)\): estimation of the additive terms in the double additive location-scale model over \(S = 500\) replicates for varying right censoring (RC) and interval censoring (IC) rates: Mean absolute bias, Root mean integrated squared error (RMISE), Relative Efficiency with an assumed Normal (\(N\)) or nonparametric (NP) error term, Mean effective coverage of 95% credible intervals.

| IC        | MA-Bias  | RMISE | Rel.Eff. | Coverage | 95% CI |
|-----------|----------|-------|----------|----------|--------|
| \(0\%)    |          |       |          |          |        |
| RC:       |          |       |          |          |        |
| 0%        |          |       |          |          |        |
|           | NP       | \(N\) | \(N\)    | \(N\)    | \(N\)  |
|           | 0.003    | 0.003 | 0.004    | 0.002    | 0.007  |
|           | 0.003    | 0.003 | 0.006    | 0.002    | 0.011  |
|           | 0.015    | 0.017 | 0.016    | 0.025    | 0.036  |
|           | 0.022    | 0.024 | 0.022    | 0.035    | 0.048  |
|           | 1.000    | 1.000 | 1.000    | 1.000    | 1.000  |
|           | 0.471    | 0.507 | 0.521    | 0.493    | 0.566  |
|           | 0.959    | 0.958 | 0.959    | 0.950    | 0.952  |
|           | 0.003    | 0.003 | 0.004    | 0.002    | 0.009  |
|           | 0.003    | 0.003 | 0.006    | 0.002    | 0.012  |
|           | 0.017    | 0.020 | 0.021    | 0.026    | 0.038  |
|           | 0.023    | 0.026 | 0.023    | 0.037    | 0.050  |
|           | 1.000    | 1.000 | 1.000    | 1.000    | 1.000  |
|           | 0.517    | 0.572 | 0.552    | 0.508    | 0.587  |
|           | 0.958    | 0.954 | 0.959    | 0.951    | 0.952  |
|           | 0.003    | 0.003 | 0.004    | 0.002    | 0.010  |
|           | 0.003    | 0.003 | 0.006    | 0.002    | 0.013  |
|           | 0.019    | 0.023 | 0.025    | 0.029    | 0.042  |
|           | 0.025    | 0.029 | 0.025    | 0.038    | 0.053  |
|           | 1.000    | 1.000 | 1.000    | 1.000    | 1.000  |
|           | 0.587    | 0.649 | 0.503    | 0.551    | 0.632  |
|           | 0.956    | 0.960 | 0.929    | 0.955    | 0.945  |
|           | 0.003    | 0.003 | 0.004    | 0.002    | 0.010  |
|           | 0.003    | 0.003 | 0.006    | 0.002    | 0.013  |
|           | 0.019    | 0.023 | 0.025    | 0.029    | 0.042  |
|           | 0.025    | 0.029 | 0.025    | 0.038    | 0.053  |
|           | 1.000    | 1.000 | 1.000    | 1.000    | 1.000  |
|           | 0.587    | 0.649 | 0.503    | 0.551    | 0.632  |
|           | 0.956    | 0.960 | 0.929    | 0.955    | 0.945  |
|           | 0.003    | 0.003 | 0.004    | 0.002    | 0.010  |
|           | 0.003    | 0.003 | 0.006    | 0.002    | 0.013  |
|           | 0.019    | 0.023 | 0.025    | 0.029    | 0.042  |
|           | 0.025    | 0.029 | 0.025    | 0.038    | 0.053  |
|           | 1.000    | 1.000 | 1.000    | 1.000    | 1.000  |
|           | 0.587    | 0.649 | 0.503    | 0.551    | 0.632  |
|           | 0.956    | 0.960 | 0.929    | 0.955    | 0.945  |
Table 3 – Simulation study ($n = 500$): estimation of the additive terms in the double additive location-scale model over $S = 500$ replicates for varying right censoring (RC) and interval censoring (IC) rates: Mean absolute bias, Root mean integrated squared error (RMISE), Relative Efficiency with an assumed Normal ($\mathcal{N}$) or nonparametric (NP) error term, Mean effective coverage of 95% credible intervals.

| IC   | $f_1^I(x)$ | $f_2^I(x)$ | $f_1^T(x)$ | $f_2^T(x)$ |
|------|-------------|-------------|-------------|-------------|
|      | 0%          | 25%         | 50%         | 0%          | 25%         | 50%         | 0%          | 25%         | 50%         |
| 0%   | MA-Bias     | NP          | 0.004       | 0.004       | 0.004       | 0.005       | 0.005       | 0.007       | 0.002       | 0.003       | 0.004       | 0.018       | 0.021       | 0.025       |
|      | RMISE       | $\mathcal{N}$ | 0.006       | 0.006       | 0.006       | 0.007       | 0.008       | 0.008       | 0.005       | 0.005       | 0.008       | 0.022       | 0.028       | 0.038       |
| 0%   | Rel.Eff.    | $\mathcal{N}$ | 0.027       | 0.032       | 0.039       | 0.026       | 0.030       | 0.036       | 0.046       | 0.055       | 0.069       | 0.064       | 0.075       | 0.089       |
| 0%   | Coverage    | $\mathcal{N}$ | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       |
| 0%   | 95% CI      | $\mathcal{N}$ | 0.944       | 0.935       | 0.919       | 0.922       | 0.911       | 0.906       | 0.918       | 0.910       | 0.900       | 0.899       | 0.883       | 0.882       |
| 0%   | MA-Bias     | NP          | 0.004       | 0.004       | 0.004       | 0.005       | 0.006       | 0.009       | 0.003       | 0.002       | 0.006       | 0.020       | 0.024       | 0.036       |
| 0%   | RMISE       | $\mathcal{N}$ | 0.006       | 0.006       | 0.006       | 0.008       | 0.008       | 0.010       | 0.005       | 0.005       | 0.008       | 0.024       | 0.032       | 0.048       |
| 25%  | Rel.Eff.    | $\mathcal{N}$ | 0.030       | 0.036       | 0.046       | 0.029       | 0.034       | 0.042       | 0.051       | 0.060       | 0.076       | 0.069       | 0.082       | 0.105       |
| 25%  | Coverage    | $\mathcal{N}$ | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       |
| 25%  | 95% CI      | $\mathcal{N}$ | 0.940       | 0.932       | 0.930       | 0.917       | 0.912       | 0.926       | 0.905       | 0.921       | 0.929       | 0.890       | 0.885       | 0.903       |
| 25%  | MA-Bias     | NP          | 0.004       | 0.005       | 0.006       | 0.006       | 0.008       | 0.010       | 0.003       | 0.003       | 0.007       | 0.021       | 0.028       | 0.060       |
| 25%  | RMISE       | $\mathcal{N}$ | 0.007       | 0.006       | 0.007       | 0.008       | 0.009       | 0.010       | 0.005       | 0.006       | 0.010       | 0.026       | 0.036       | 0.067       |
| 50%  | Rel.Eff.    | $\mathcal{N}$ | 0.033       | 0.041       | 0.056       | 0.032       | 0.038       | 0.052       | 0.055       | 0.063       | 0.083       | 0.073       | 0.093       | 0.138       |
| 50%  | Coverage    | $\mathcal{N}$ | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       | 1.000       |
| 50%  | 95% CI      | $\mathcal{N}$ | 0.949       | 0.946       | 0.947       | 0.942       | 0.940       | 0.934       | 0.939       | 0.942       | 0.927       | 0.937       | 0.918       | 0.889       |

Mean effective coverage of 95% credible intervals.
Table 4 – Simulation study ($n = 250$): estimation of the additive terms in the double additive location-scale model over $S = 500$ replicates for varying right censoring (RC) and interval censoring (IC) rates: Mean absolute bias, Root mean integrated squared error (RMISE), Relative Efficiency with an assumed Normal ($\mathcal{N}$) or nonparametric (NP) error term, Mean effective coverage of 95% credible intervals.

| IC | $f_1^1(x)$ | $f_1^2(x)$ | $f_2^1(x)$ | $f_2^2(x)$ |
|----|-------|-------|-------|-------|
| MA-Bias | NP | 0.006 | 0.007 | 0.007 | 0.007 |
|       | $\mathcal{N}$ | 0.008 | 0.008 | 0.008 | 0.008 |
|       | RMISE | NP | 0.041 | 0.048 | 0.059 | 0.053 |
|       |       | $\mathcal{N}$ | 0.053 | 0.059 | 0.069 | 0.053 |
|       | Rel.Eff. | NP | 1.000 | 1.000 | 1.000 | 1.000 |
|       |       | $\mathcal{N}$ | 0.602 | 0.647 | 0.740 | 0.602 |
|       | Coverage | NP | 0.879 | 0.875 | 0.869 | 0.924 |
|       |       | $\mathcal{N}$ | 0.924 | 0.925 | 0.914 | 0.924 |
|       | 95% CI | NP | 0.922 | 0.919 | 0.920 | 0.922 |
|       |       | $\mathcal{N}$ | 0.922 | 0.919 | 0.920 | 0.922 |

| IC | $f_1^1(x)$ | $f_1^2(x)$ | $f_2^1(x)$ | $f_2^2(x)$ |
|----|-------|-------|-------|-------|
| MA-Bias | NP | 0.005 | 0.007 | 0.007 | 0.007 |
|       | $\mathcal{N}$ | 0.008 | 0.008 | 0.008 | 0.008 |
|       | RMISE | NP | 0.045 | 0.053 | 0.069 | 0.057 |
|       |       | $\mathcal{N}$ | 0.057 | 0.065 | 0.077 | 0.057 |
|       | Rel.Eff. | NP | 1.000 | 1.000 | 1.000 | 1.000 |
|       |       | $\mathcal{N}$ | 0.645 | 0.675 | 0.811 | 0.645 |
|       | Coverage | NP | 0.885 | 0.884 | 0.904 | 0.922 |
|       |       | $\mathcal{N}$ | 0.922 | 0.919 | 0.920 | 0.922 |
|       | 95% CI | NP | 0.922 | 0.919 | 0.920 | 0.922 |
|       |       | $\mathcal{N}$ | 0.922 | 0.919 | 0.920 | 0.922 |
Figure 6 – Simulation study \((n = 250)\): averaged estimated additive terms (over \(S = 500\) replicates) in the absence of interval censoring, but for increasing right censoring rates and by assuming a NP (dashed line) or a Normal (dotted line) error term. Envelopes (light grey: NP; dark grey: Normal) result from consecutive intervals containing 95% of the \(S\) estimates for \(f_j^\mu(x)\) or \(f_j^\sigma(x)\) with \(x\) in \((0, 1)\).
Figure 7 – Simulation study ($n = 250$): averaged estimated additive terms (over $S = 500$ replicates) under a 50% interval censoring rate combined with increasing right censoring rates and by assuming a NP (dashed line) or a Normal (dotted line) error term. Envelopes (light grey: NP ; dark grey: Normal) result from consecutive intervals containing 95% of the $S$ estimates for $f_{\mu_j}(x)$ or $f_{\sigma_j}(x)$ with $x$ in $(0,1)$. 
Figure 8 – Simulation study: estimated error densities in the double additive location-scale model (averaged over the $S = 500$ replicates) using a NP error term for different combinations of sample sizes, right- (RC) and interval censoring (IC) rates.