Perfect fluid flows on $\mathbb{R}^d$ with growth/decay conditions at infinity

R. McOwen $^1$ · P. Topalov $^1$

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Abstract
We study the well-posedness and the spatial behavior at infinity of perfect fluid flows on $\mathbb{R}^d$ with initial velocity in a scale of weighted Sobolev spaces that allow spatial growth/decay at infinity as $|x|^{\beta}$ with $\beta < 1/2$. Moreover, for initial velocity with sufficient spatial decay, we show that the solution of the Euler equation generically develops an asymptotic expansion at infinity with non-vanishing asymptotic terms that depend analytically on time and the initial data. For initial data in the Schwartz space, we identify the evolution space of the fluid velocity with a certain space of symbols.

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1 Introduction
The motion of an incompressible perfect fluid in $\mathbb{R}^d$, $d \geq 2$, is described by the Euler equation

$$\begin{cases}
    u_t + (u \cdot \nabla) u = -\nabla p, \\
    \text{div } u = 0, \\
    u|_{t=0} = u_0,
\end{cases}$$

(1)

where $u(t, x)$ is the velocity field and $p(t, x)$ is the scalar pressure. In this paper, we consider the possibility of a spatial growth and decay of solutions of (1) in the context of a class of weighted Sobolev spaces that have been used by many authors because they have nice mapping properties for the Laplace operator. To define these, assume $1 < p < \infty$, $\delta \in \mathbb{R}$, and $m$ is a nonnegative integer. Let $W^{m, p}_\delta(\mathbb{R}^d)$ denote the Banach
space obtained as the closure of $C^\infty_c(\mathbb{R}^d)$, i.e. the smooth functions with compact support, in the norm
\[
\| f \|_{W^m_\delta} = \sum_{|\alpha| \leq m} \| \langle x \rangle^{\delta + |\alpha|} \partial^\alpha f \|_{L^p}.
\] (2)

Here $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index with $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $\partial^\alpha$ denotes the partial derivative $\partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_d}_{x_d}$, and $\langle x \rangle = \sqrt{1 + |x|^2}$. We shall write $W^{0,p}_\delta(\mathbb{R}^d)$ as $L^p_\delta(\mathbb{R}^d)$, i.e. a weighted $L^p$-space. It was shown in [2,23] that $f \in W^{m,p}_\delta(\mathbb{R}^d)$ for $m > d/p$ implies $f \in C^k(\mathbb{R}^d)$ for $0 \leq k < m - d/p$ with
\[
\sup_{x \in \mathbb{R}^d} \langle x \rangle^{\delta + d/p + |\alpha|} \| \partial^\alpha f(x) \| \leq C \| f \|_{W^m_\delta}.
\] (3a)

and in fact
\[
\| \langle x \rangle^{\delta + d/p + |\alpha|} \| \partial^\alpha f(x) \| \| \to 0 \quad \text{as} \quad |x| \to \infty \quad \text{for} \quad |\alpha| < m - d/p.
\] (3b)

(Note also that $1/\langle x \rangle^\beta \in W^{m,p}_\delta(\mathbb{R})$ for $m \geq 0$ and $\delta + d/p < \beta$.) Of course, we are interested in vector fields, so we shall denote by $W^{m,p}_\delta = W^{m,p}_\delta(\mathbb{R}^d, \mathbb{R}^d)$ vector fields $u = (u_1, \ldots, u_d)$ with components $u_k \in W^{m,p}_\delta(\mathbb{R}^d)$. In fact, henceforth we shall use the notation $W^{m,p}_\delta$ regardless of whether we are considering scalar, vector, or even matrix-valued functions. For $m \geq 1$ we are particularly interested in divergence free vector fields, so we denote these by
\[
\hat{W}^{m,p}_\delta := \{ u \in W^{m,p}_\delta \mid \text{div} \ u = 0 \}.
\] (4)

Note that $\hat{W}^{m,p}_\delta$ is a closed subspace of $W^{m,p}_\delta$. Our first result concerns weights $\delta \in \mathbb{R}$ in the range
\[
-1/2 < \delta + d/p < d - 1.
\] (5)

Using (3b) we see that (5) allows vector fields $u$ with mild growth, i.e. $|u(x)| = O(|x|^\beta)$ as $|x| \to \infty$ for $0 < \beta < 1/2$, or decay, i.e. $|u(x)| = O(|x|^\beta)$ for $-(d-1) < \beta < 0$.

**Theorem 1.1** Assume $m > 1 + d/p$, $1 < p < \infty$, $d \geq 2$, and the weight $\delta$ satisfies (5). Then, for any given $\rho > 0$ there exists $\tau > 0$ such that for any $u_0 \in \hat{W}^{m,p}_\delta$ with $\| u_0 \|_{W^{m,p}_\delta} < \rho$ there exists a unique\(^1\) solution
\[
u \in C([0, \tau], \hat{W}^{m,p}_\delta) \cap C^1([0, \tau], \hat{W}^{m-1,p}_\delta)
\]

of the Euler equation (1) such that $|\nabla p(t, x)| = o(1)$ as $|x| \to \infty$ for $t \in [0, \tau]$. The solution depends continuously on the initial data $u_0 \in \hat{W}^{m,p}_\delta$. Moreover, for any fixed $t \in [0, \tau]$ the pressure $p(t)$ is uniquely determined up to an additive constant.

\(^1\) The uniqueness statements in this paper are considered within the described class of solutions.
Theorem 1.1 (as well as Theorem 1.2(a) formulated below) generalizes the result of Cantor [8] which only applies for \( d \geq 3, \ p > d/(d - 2) \), and \( 1 + d/p < \delta + d/p < d - 1 \). Theorem 1.1 naturally poses the following question: what happens for \( \delta + d/p \geq d - 1 \)? The complete answer is provided by Theorem 1.2 below and relies on the asymptotic spaces introduced in [23,24]. For technical reasons, we require greater regularity, namely \( m > 3 + d/p \). Take \( \chi \in C^\infty(\mathbb{R}) \) such that \( \chi(\rho) = 0 \) for \( \rho \leq 1 \), \( \chi(\rho) = 1 \) for \( \rho \geq 2 \), and \( 0 \leq \chi(\rho) \leq 1 \) for \( 1 \leq \rho \leq 2 \). We also set \( r := |x| \) and \( \theta := x/|x| \) for \( x \neq 0 \). For \( \rho > 0 \) denote by \( B_{\hat{W}_\delta}^{m,p}(\rho) \) the open ball of radius \( \rho \) centered at the origin in \( \tilde{W}_\delta^{m,p} \).

**Theorem 1.2** Assume that \( m > 3 + d/p, \ 1 < p < \infty, \) and \( d \geq 2 \). Then, we have:

(a) For any weight \( \delta \in \mathbb{R} \) with \( 0 < \delta + d/p < d + 1 \) and for any given \( \rho > 0 \) there exists \( \tau > 0 \) such that for any divergence free vector field \( u_0 \in B_{\hat{W}_\delta}^{m,p}(\rho) \) there exists a unique solution of the Euler equation

\[
\begin{align*}
    u & \in C([0, \tau], \tilde{W}_\delta^{m,p}) \cap C^1([0, \tau], \tilde{W}_\delta^{m-1,p}).
\end{align*}
\]

(b) For any weight \( \delta \in \mathbb{R} \) with \( \delta + d/p \geq d + 1 \) and for any given \( \rho > 0 \) there exists \( \tau > 0 \) such that for any divergence free vector field \( u_0 \in B_{\hat{W}_\delta}^{m,p}(\rho) \) there exists a unique solution of the Euler equation of the form\(^2\)

\[
\begin{align*}
    u(x, t) &= \chi(r) \sum_{d+1 \leq k \leq \delta + d/p} \frac{a_k(\theta, t)}{r^k} + f(x, t), \quad x \in \mathbb{R}^d, \ t \in [0, \tau], \quad (6)
\end{align*}
\]

where

\[
\begin{align*}
    f & \in C([0, \tau], \tilde{W}_\delta^{m,p}) \cap C^1([0, \tau], \tilde{W}_\delta^{m-1,p}), \quad (7)
\end{align*}
\]

\[
\begin{align*}
    a_k & \in C^1 \left([0, \tau], C(S^{d-1}, \mathbb{R}^d)\right), \quad d + 1 \leq k \leq \delta + d/p, \quad (8)
\end{align*}
\]

and for any given \( t \in [0, \tau] \), the components of \( a_k(\theta, t) \) are eigenfunctions of the Laplace operator \( -\Delta_S \) on the unit sphere \( S^{d-1} \) with eigenvalue \( \lambda_{k-d+2} = k(k - d + 2) \).

The solution in (a) and (b) depends continuously on the initial data \( u_0 \in \tilde{W}_\delta^{m,p} \). It is global in time for \( d = 2 \). Moreover, the asymptotic coefficients \( a_k \), \( d + 1 \leq k \leq \delta + d/p \), in (6) are analytic as functions of time and the initial data, i.e. as maps \( a_k : [0, \tau] \times B_{\hat{W}_\delta}^{m,p}(\rho) \to C(S^{d-1}, \mathbb{R}^d) \).

**Remark 1.1** Theorem 1.1 and Theorem 1.2(a) imply that the Euler equation is well-posed in \( \tilde{W}_\delta^{m,p} \) for \( -1/2 < \delta + d/p < d + 1 \) and \( m > 3 + d/p \). Note that the bound \( d + 1 \) is sharp. In fact, Theorem 1.2(b) and Proposition 1.1 stated below show that the

\[^2\text{In particular, by (3b), the solution } u \text{ has an asymptotic expansion at infinity of order equal to the integer part of } \delta + d/p \text{ with remainder in } \tilde{W}_\delta^{m,p}.\]
range cannot be extended to $\delta + d/p \geq d + 1$. The bound $-1/2$ appears since we require $\nabla p(t, x) = o(1)$ as $x \to \infty$.

**Remark 1.2** The continuous dependence on the initial data in Theorem 1.2(b) means that the data-to-solution map $u_0 \mapsto (a_{d+1}, \ldots, a_N; f)$,

$$B_{\tilde{W}^m,p}(\rho) \to C([0, \tau], C(S^{d-1}, \mathbb{R}^d)^{N-d} \times \tilde{W}^{m,p}_\delta) \cap C^1([0, \tau], C(S^{d-1}, \mathbb{R}^d)^{N-d} \times \tilde{W}^{m-1,p}_\delta),$$

(9)

is continuous, where $N$ is the integer part of $\delta + d/p$. In this way, for $\delta + d/p \geq d + 1$ the Euler equation is well-posed in the sense described in (9). Altogether, with this extended notion of well-posedness, the Euler equation is well-posed in the weighted Sobolev space $\tilde{W}^{m,p}_\delta$ for any $\delta + d/p > -1/2$.

**Remark 1.3** The analyticity in time established e.g. in [27] (cf. also [29]) concerns the solution of the Euler equation in Lagrangian coordinates. In contrast, the analyticity of the coefficients $a_k$ for $d + 1 \leq k < \delta + d/p$ in Theorem 1.2 concerns the solution of the Euler equation (1) written in Eulerian coordinates. Note that the remainder $f$ in (6) is not necessarily analytic in time.

Let us now discuss the asymptotic terms appearing in the solution (6) given by Theorem 1.2(b). The following proposition implies that for generic initial data $u_0$ in $\tilde{W}^{m,p}_\delta$ with $\delta + d/p \geq d + 1$, all asymptotic terms in (6) do appear and do not vanish generically. More specifically, we have

**Proposition 1.1** Assume that $m > 3 + d/p$, $1 < p < \infty$, and $d \geq 2$. Then, under the assumptions of Theorem 1.2(b), there exists an open dense set $\mathcal{N}$ in $\tilde{W}^{m,p}_\delta$ such that for any initial data $u_0 \in \mathcal{N}$ and for any $d + 1 \leq k < \delta + d/p$ and $1 \leq j \leq d$ the $j$-th component $a^j_k(t)$ of the asymptotic coefficient $a_k(t) \in C(S^{d-1}, \mathbb{R}^d)$ of the solution (6) does not vanish in $C(S^{d-1}, \mathbb{R})$ for all but finitely many $t \in [0, \tau]$.

Our next result shows that the interval of existence $[0, \tau]$, $\tau > 0$, in Theorem 1.2 can be chosen independent of the regularity exponent $m > 3 + d/p$ and the weight $\delta + d/p > 0$. More specifically, we have the following "no gain no loss" result.

**Proposition 1.2** Take a regularity exponent $m_0 > 3 + d/p$, $1 < p < \infty$, a weight $\delta_0 + d/p > 0$, a radius $\rho > 0$, and let $[0, \tau]$, $\tau > 0$, be the interval of existence of the solution (6) of the Euler equation (1) with initial data $u_0 \in B_{\tilde{W}^{m_0,p}_\delta}(\rho)$, given by Theorem 1.2. Then, for any $m \geq m_0$, $\delta \geq \delta_0$, and for any initial data $u_0 \in B_{\tilde{W}^{m_0,p}_\delta}(\rho) \cap \tilde{W}^{m,p}_\delta$ there exists a unique solution (6) of the Euler equation (1) that is defined on $[0, \tau]$, satisfies (7), (8), and depends continuously on the initial data (see (9)).

---

3 Since the asymptotic sum in (6) vanishes for $\delta + d/p < d + 1$, formula (6) continues to hold also in the case of Theorems 1.1 and 1.2(a).
Solutions in spaces of symbols Proposition 1.2 implies that the Euler equation (1), with initial data in the Schwartz space $S = S(R^d, R^d)$, is locally well-posed in the space $T^\infty$ of $C^\infty$ vector fields on $R^d$ that allow an infinite asymptotic expansion

$$u(x) \sim \sum_{k \geq d+1} \frac{a_k(\theta)}{r^k} \quad \text{as } |x| \to \infty \quad (10)$$

with coefficients $a_k(\theta), k \geq d+1$, as in Theorem 1.2(b). The asymptotic formula (10) means that for any $N \geq d$ and for any multi-index $\alpha \in Z^d_{\geq 0}$ there exists a constant $C_{N, \alpha} > 0$ such that for any $x \in R^d$,

$$\left| \partial^\alpha \left( u(x) - \chi(r) \sum_{d+1 \leq k \leq N} \frac{a_k(\theta)}{r^k} \right) \right| \leq \frac{C_{N, \alpha}}{\langle x \rangle^{N+1+|\alpha|}}. \quad (11)$$

Hence, we can think of $T^\infty$ as a space of symbols (cf. [4,15]). The best constants $C_{N, \alpha}$ for $N \geq d$ and $|\alpha| \leq N$ in (11) equip the space of symbols $T^\infty$ with a countable set of semi-norms that induce a Fréchet topology on $T^\infty$. Denote by $\hat{S}$ the space of divergence free vector fields in $S$. We have the following

**Theorem 1.3** For any $u_0 \in \hat{S}$ there exist $\tau > 0$ and a unique solution of the Euler equation $u \in C^1([0, \tau], T^\infty)$ that depends continuously on the initial data. The asymptotic coefficients $a_k(t) \in C(S^{d-1}, R^d)$ for $k \geq d+1$ depend analytically on $t \in [0, \tau]$ and the initial data.

The theorem follows directly from Theorem 1.2 and Proposition 1.2. In Sect. 5 we show that, for generic initial data in $\hat{S}$, any asymptotic term in the infinite asymptotic expansion (10) of the solution $u \in C^1([0, \tau], T^\infty)$ given by Theorem 1.3 does not vanish in $C(S^{d-1}, R)$ for all but finitely many $t \in [0, \tau]$.

**Corollary 1.1** There exists a dense set $N$ in $\hat{S}$ such that for any initial data $u_0 \in N$ and for any $k \geq d+1$ and $1 \leq j \leq d$ the $j$-th component $a_j^k(t)$ of the asymptotic coefficient $a_k(t) \in C(S^{d-1}, R^d)$ in the asymptotic expansion (10) of the solution $u \in C^1([0, \tau], T^\infty)$ given by Theorem 1.3 does not vanish in $C(S^{d-1}, R)$ for all but finitely many $t \in [0, \tau]$.

In this sense, the space of symbols $T^\infty$ is the minimal evolution space for the Euler equation with initial data in the Schwartz space.

**Discussion and motivation** Here we put the results discussed above in the context of asymptotic spaces. The motivation for introducing the asymptotic spaces in [23] was to answer the following questions: What would happen if we allow the initial data $u_0$ of the Euler equation to be of the form

$$u_0(x) = \chi(r) \sum_{0 \leq k < b+d/p} \frac{a_k(\theta)}{r^k} + f(x), \quad f \in W^m,p_{\delta}, \quad a_k \in C(S^{d-1}, R^d). \quad (12)$$
as suggested by Proposition B.1 in Appendix B? More specifically, is it true that (12) is preserved by the Euler flow? The answer of this question is negative and is given in [24]: Due to the specifics of the non-linear term of the equation, if we do not impose restrictions on the spherical Fourier modes of the asymptotic coefficients $a_k \in C(S^{d-1}, \mathbb{R}^d)$ in (12), then, generically, the solution of the Euler equation develops logarithmic asymptotic terms as in formula (90a) in Appendix C. This is the reason for introducing in [23, 24] the asymptotic spaces $\mathcal{A}_{N, \ell}^{m,p}$ with log terms that are invariant with respect to the Euler equation. As shown in [30], logarithmic asymptotic terms do not appear (at least in the two dimensional case) if one imposes additional restrictions on the spherical Fourier modes of the asymptotic coefficients of $u_0$.

In contrast to these works, in the present paper we do not assume that the initial data has an asymptotic expansion at infinity. Instead, we assume only that $u_0 \in \tilde{W}_{{\delta}}^{m,p}$, and hence $u_0(x) = O(1/|x|^{\delta + d/p})$ as $|x| \to \infty$. Our goal is to study the non-local properties of the Euler equation by characterizing the spatial asymptotic behavior of its solutions as a function of the spatial decay of the initial data $u_0$, in the framework given by the scale of weighted Sobolev spaces $W_{{\delta}}^{m,p}$ with weight $\delta \in \mathbb{R}$. The choice of the function spaces $W_{{\delta}}^{m,p}$ is natural since the regularity $m > d/p$ and the weight $\delta \in \mathbb{R}$ control the spatial decay of the functions and their derivatives, and since the scale of $W_{{\delta}}^{m,p}$ spaces has nice mapping properties with respect to the Laplace operator on $\mathbb{R}^d$ (cf. Proposition B.1 in Appendix B). This characterization is given by Theorem 1.1, Theorem 1.2, and Proposition 1.1, and is summarized by formula (6), which holds for all weights $\delta + d/p > -1/2$. The formula shows that there is a countable set of threshold values of $\delta + d/p$ at $d+1, d+2, \ldots$ such that for any $d + 1 \leq k \leq \delta + d/p$ the solution of the Euler equation, generically, develops an asymptotic term of the form $a_k(\theta, t)/r^k$. In particular, we see that the space $\tilde{W}_{{\delta}}^{m,p}$ is preserved by the Euler flow for $-1/2 < \delta + d/p < d + 1$ but it is not preserved for $\delta + d/p \geq d + 1$. Informally speaking, this means that if the initial data $u_0$ decays faster than $O(1/|x|^{d+1})$, then the solution of the Euler equations develops non-vanishing asymptotic terms as described in (6). In particular, the solution is of order $O(1/|x|^{d+1})$ — a fact first noticed for $d = 3$ and small $t > 0$ in [11]. As mentioned in Remark 1.1, the lower limit $-1/2$, appearing in Theorem 1.1, is sharp if one assumes that $\nabla p = o(1)$ as $|x| \to \infty$. However, we do not exclude that, under additional assumptions on the class of solutions, the solution map of the Euler equation can be extended below that limit. Finally, note that asymptotic-type spaces with a single asymptotic term of order $O(1/r)$ were introduced and studied in dimension two: we refer to the affine space $E_m$ in [9, Definition 1.3.3], as well as to the radial-energy decomposition in [21, Definition 3.1].

Let us now discuss briefly the advantage given by the asymptotic expansion (6) for the solutions of the Euler equation. Choose $\delta \in \mathbb{R}$ such that $\delta + d/p \geq d + 2$, $m > l + d/p$ for some integer $l \geq 3$, and let $N$ be the integer part of $\delta + d/p$. Take $u_0 \in \tilde{W}_{{\delta}}^{m,p}$. Then, by Theorem 1.2, there exists $\tau > 0$ and a unique solution $u$ of the Euler equation on the interval $[0, \tau]$ such that (6), (7), and (8) hold. It then follows from (3a) that there exists a constant $C > 0$ such that for any $t \in [0, \tau]$ and for any $|x| \geq 2$ we have

4 Recall that for $m > d/p$ the elements of $W_{{\delta}}^{m,p}$ are of order $O(1/|x|^{\delta + d/p})$ as $|x| \to \infty$. © Springer
\[
\left| \partial^a \left( u(x, t) - \sum_{d+1 \leq k \leq N-1} \frac{a_k(\theta, t)}{r^k} \right) \right| \leq \frac{C}{|x|^{N+|\alpha|}}, \quad 0 \leq |\alpha| \leq l. \tag{13}
\]

Since the solution depends continuously on the initial data, the estimate (13) holds locally uniformly on the initial data \( u_0 \in \dot{W}^{m,p}_{\delta} \). Therefore, up to terms of order \( O \left( 1/|x|^N \right) \), the solution \( u(x, t) \) of the Euler equation (together with its spatial derivatives of order \( \leq l \)) is represented by its asymptotic part \( \sum_{d+1 \leq k \leq N-1} \frac{a_k(\theta, t)}{r^k} \), uniformly in \( t \in [0, \tau] \) and locally uniformly on the initial data in \( \dot{W}^{m,p}_{\delta} \). Moreover, Theorem 1.2 imposes strong restrictions on the asymptotic coefficients, as each asymptotic coefficient \( a_k(t) \) is an eigenfunction of the Laplace operator \( -\Delta_S \) on the unit sphere \( S^{d-1} \) with eigenvalue \( \lambda_k = k(k - d + 2) \), and it depends analytically on \( \tau \in [0, \tau] \). In particular, each \( a_k \) considered as a map \( a_k : S^{d-1} \times [0, \tau] \to \mathbb{R}^d \) is real analytic.

**Structure & proofs.** We prove Theorem 1.1 in Lagrangian coordinates (cf. [1,12,33]). The first step is to define for \( 1 < p < \infty, m > d/p + 1 \), and for any weight \( \gamma \in \mathbb{R} \) with \( \gamma + d/p > -1 \), a set \( D^{m,p}_\gamma \) of maps \( \mathbb{R}^d \to \mathbb{R}^d \) of the form \( \varphi(x) = x + f(x) \) where \( f \in W^{m,p}_\gamma \). (Note that then, by (3a), \( f(x) = O(|x|^{1-\varepsilon}) \) for \( \varepsilon > 0 \).) This is done in Sect. 2 and Appendix A, where we show that \( D^{m,p}_\gamma \) is a group of diffeomorphisms of \( \mathbb{R}^d \) that satisfies additional regularity properties summarized in Theorem 2.1 and Lemma 2.1. The second step is the proof of the smoothness of the map (24) for weights \( 0 < \delta + d/p < d - 1 \) and \( \delta + d/p \neq 1 \) (cf. Proposition 3.1 in Sect. 3). This is done by a detailed analysis of the mapping properties of the Laplace operator in the scale of \( W \)-spaces. One of the main issues that appears here (and throughout the whole paper) is that the Laplace operator \( \Delta : W^{m+2,p}_{\delta} \to W^{m,p}_{\delta+2} \) is not Fredholm for certain integer values of \( \delta + d/p \). Theorem 1.1 is then proved in Sect. 4. The main idea is based on the observation that the non-linear term of the Euler equation has a specific “weight-gaining” property expressed by the inequalities (32) and (33). This allows us to prove the smoothness of the Euler vector field (38) in the case when \( -1/2 < \delta + d/p < d - 1 \) (cf. Theorem 4.1). Note that the proof of Theorem 1.1 is not based on the theory of asymptotic spaces developed in [23,24,30] and requires only regularity \( m > 1 + d/p \). In contrast, the asymptotic spaces are used as an important tool for proving the asymptotic expansion (6) in Theorem 1.2. The proof is done in several steps: The first step is to generalize the main results in [24,30] to asymptotic spaces with log terms and remainder in \( W^{m,p}_{\gamma N} \) with weight \( N \leq \gamma_N + d/p < N + 1 \) and \( N \geq 0 \). These generalizations are done in Appendices C and D. Then we use the conservation of vorticity, the properties of the solutions of the Euler equation in asymptotic spaces, and Proposition B.1 in Appendix B to show that for non-integer weights \( \delta + d/p > 0 \) the solution has the asymptotics described in Theorem 1.2. The proof of the important case of integer weights (the threshold values of \( \delta + d/p \)) needs a separate argument since Proposition B.1 does not hold for such values of \( \delta + d/p \). The strategy is to argue by approximation. We first approximate \( u_0 \in \dot{W}^{m,p}_{\delta} \) by a sequence of initial data \( u_{0j} \in \dot{W}^{m,p}_{\delta^j} \) where \( \delta' + d/p \notin \mathbb{Z} \) and \( 0 < \delta' - \delta < 1 \), and then show that in the limit, when \( u_{0j} \to u_0 \) as \( j \to \infty \) in \( W^{m,p}_{\delta} \), the solutions of the Euler equation with initial data \( u_{0j} \) converge to a solution of the Euler equation with...
asymptotics described in Theorem 1.2. The proof of the convergence is based on the decomposition formula (55) that leads to the crucial integral representation (57) for the Fourier modes of the asymptotic coefficient $a_k$ with $k = \delta + d/p$ (see Remark 5.1). The proof of the analyticity of the asymptotic coefficients for non-integer weights is based on the integral relation (54). The case of integer weights is again obtained by approximation. In Sect. 5 we also prove Proposition 1.1 and Corollary 1.1, which imply that the asymptotic terms appearing in Theorem 1.2 are generic. The proof is based on an important non-vanishing result given by Lemma 5.1. Note also that the requirement $m > 3 + d/p$ in Theorem 1.2 is technical and can be replaced by $m > 1 + d/p$ but that would require improvement of the regularity assumptions in [23] (see Remark 2.4 in [24]).

Related work First we list works that consider (1) but only for $d = 2$. Solutions of the Euler equation for (not necessarily bounded) domains in $\mathbb{R}^2$ were constructed by Wolibner in [33]. Kato and Ponce [17,18], showed the well-posedness of (1) as the zero-viscosity limit of the Navier–Stokes equation in the standard $L^p$-Sobolev spaces $H^{m,p}(\mathbb{R}^2)$ for $m > 1 + d/p$; this requires the vector fields to vanish at infinity. To allow the velocity field to be merely bounded at infinity, several authors also considered conditions on the vorticity $\omega = \text{curl } u$. For example, Serfaty [28] showed that for $u_0, \omega_0 \in L^\infty$, there is a unique solution $u \in L^\infty$ of (1) with the pressure $p$ determined up to an additive constant and satisfying $p = o(|x|)$ as $|x| \to \infty$; under the same conditions, Kelliher [19] gave a characterization of the behavior of the solution $u$ as $|x| \to \infty$. We also refer to the recent paper by Misoiolek and Yoneda [25]. Under a decay condition on the vorticity, Benedetto, Marchioro, and Pulvirenti [3] allowed a sublinear growth of the solution, namely if $|u_0(x)| = O(|x|^\beta)$ as $|x| \to \infty$ where $\beta < 1$ and $\omega_0 \in L^p \cap L^\infty(\mathbb{R}^2)$ for $p < 2/\beta$, then the unique solution $u$ retains these properties as $|x| \to \infty$. Very recently, Elgindi and Jeong [13] allowed up to linear growth for $u$, namely $|u(x)| = O(|x|)$ as $|x| \to \infty$, but instead of requiring $\omega$ to decay at infinity, they imposed symmetry conditions. Without any decay or symmetry conditions on $\omega \in L^\infty(\mathbb{R}^2)$, Cozzi and Kelliher [10] allowed growth $|u(x)| = O(|x|^\beta)$ as $|x| \to \infty$ for $\beta < 1/2$, which is similar to the growth allowed in our Theorem 1.1 (although our initial conditions and solutions require $|\omega(x)| = O(|x|^{\beta-1})$ as $|x| \to \infty$). Most of the results considered above do not discuss the well-posedness of the obtained solutions.

Now we describe works that consider (1) for $d \geq 2$ and that are not related to the asymptotic spaces already discussed above. Kato [16] showed the well-posedness of (1) as the zero-viscosity limit of the Navier–Stokes equation in the standard $L^2$-Sobolev spaces $H^m(\mathbb{R}^3)$ for $m \geq 3$. Following the approach of Ebin and Marsden [12], Cantor [8] showed the well-posedness of (1) in the weighted Sobolev spaces $W^m_{\delta,p}$, but only for $d \geq 3, p > d/(d-2)$, and $1 + d/p < \delta + d/p < d - 1$. The fact that the solution of (1) decays at the rate $O(1/|x|^{d+1})$ when the initial data has compact support or is rapidly decaying at infinity was first noticed in [11] for $d = 3$ and small $t > 0$; they do not obtain a full asymptotic expansion and do not study dependence on initial data (cf. also [5,6,20] and the references therein for related results on the Navier–Stokes equation). The well-posedness results of [8,16] all involve velocity fields that vanish at infinity. However, to our best knowledge, Theorem 1.1 is the only result that allows classical solutions of (1) to grow as $|x| \to \infty$ in dimension greater than two.
Finally, let us discuss previous work on the existence of solutions of partial differential equations in spaces of symbols (cf. Theorem 1.3). In the one dimensional case, spaces of symbols were considered by Bondareva and Shubin in [4] where they proved that the Korteweg–de Vries equation (KdV) allows a unique solution in such spaces. A similar result was proved in [15] for the modified KdV equation. Note that the space of symbols $\mathcal{T}^\infty$ appears naturally in the case of the Euler equation due to the non-local nature of its right hand side.

2 Spaces of maps on $\mathbb{R}^d$

We will use groups of diffeomorphisms on $\mathbb{R}^d$ whose behavior at infinity is modeled on the weighted Sobolev spaces $W_{\delta}^{m,p}$. This was done in [7] for $\delta \geq 0$, and in [23] for $\delta + d/p > 0$ in the context of asymptotic spaces. However, to include non-local nature of its right hand side.

\[
\mathcal{D}_{\gamma}^{m,p} := \{ \varphi : \mathbb{R}^d \to \mathbb{R}^d \mid \varphi = \text{id} + w, \ w \in W_{\gamma}^{m,p} \text{ and } \det(d\varphi) > 0 \}. \tag{14}
\]

(Throughout this paper we denote the identity map on $\mathbb{R}^d$ by id and the identity matrix by 1.) For $\varphi = \text{id} + w \in \mathcal{D}_{\gamma}^{m,p}$, the above restrictions on $m$ and $\gamma$ imply by (3a) that $|w(x)| = O(|x|^{1-\mu})$ as $|x| \to \infty$ for some $\mu > 0$. Moreover, we have $d\varphi = I + dw$ with $|dw(x)| = O(|x|^{-\mu})$ as $|x| \to \infty$. This and the fact that $m > 1 + d/p$ imply that there exists $\varepsilon > 0$ such that $\det(d\varphi) > \varepsilon > 0$. By the Hadamard–Levy’s theorem, one then sees that $\mathcal{D}_{\gamma}^{m,p}$ consists of orientation preserving $C^1$-diffeomorphisms of $\mathbb{R}^d$ (cf. Corollary A.1 in Appendix A). Note that $\mathcal{D}_{\gamma}^{m,p}$ can be identified with an open set in $W_{\gamma}^{m,p}$. In this way, $\mathcal{D}_{\gamma}^{m,p}$ is a Banach manifold modeled on $W_{\gamma}^{m,p}$. Recall from [23, Proposition 2.2, Lemma 2.2] that, for $m > d/p$ and any $\delta_1, \delta_2 \in \mathbb{R}$, pointwise multiplication of functions $(f, g) \mapsto fg$ defines a continuous map

\[
W_{\delta_1}^{m,p} \times W_{\delta_2}^{m,p} \to W_{\delta_1+\delta_2+d/p}^{m,p} \tag{15}
\]

and for any $m \geq 0$, $\delta \in \mathbb{R}$, $1 \leq j \leq d$,

\[
\partial_{x_j} : W_{\delta}^{m+1,p} \to W_{\delta}^{m,p} \tag{16}
\]

is bounded. Now we want to show that $\mathcal{D}_{\gamma}^{m,p}$ is a topological group under composition of maps. Since for $\varphi, \psi \in \mathcal{D}_{\gamma}^{m,p}$, $\varphi = \text{id} + w$ we have that $\varphi \circ \psi = \psi + w \circ \psi$, we see that the continuity of $(\varphi, \psi) \mapsto \varphi \circ \psi$ is equivalent to the continuity of the map $(w, \psi) \mapsto w \circ \psi, W_{\gamma}^{m,p} \times \mathcal{D}_{\gamma}^{m,p} \to W_{\gamma}^{m,p}$.

However, it will be important to also consider $w \circ \varphi$ for $w \in W_{\delta}^{m,p}$ with weights $\delta$ different from $\gamma$. We have the following

**Theorem 2.1** Assume that $m > 1 + d/p$, $\gamma + d/p > -1$ and $\delta \in \mathbb{R}$.

(a) The composition $(w, \varphi) \mapsto w \circ \varphi$ is continuous as a map $W_{\delta}^{m,p} \times \mathcal{D}_{\gamma}^{m,p} \to W_{\delta}^{m,p}$ and $C^1$ as a map $W_{\delta}^{m+1,p} \times \mathcal{D}_{\gamma}^{m,p} \to W_{\delta}^{m,p}$.
(b) The inverse \( \varphi \mapsto \varphi^{-1} \) is continuous as a map \( \mathcal{D}_\gamma^{m,p} \rightarrow \mathcal{D}_\gamma^{m,p} \) and \( C^1 \) as a map \( \mathcal{D}^{m+1,p}_\gamma \rightarrow \mathcal{D}^{m,p}_\gamma \).

Theorem 2.1 and its proof are analogous to Propositions 5.1 and 5.2 in [23] (cf. also the proof of [30, Theorem 2.1]); for the details we refer to Appendix A.

For fixed \( \varphi \in \mathcal{D}^{m,p}_\gamma \), we can operate on functions and vector fields by right translation:

\[ R_\varphi(v) := v \circ \varphi. \] (17)

In fact, under the hypotheses of Theorem 2.1, we see that \( R_\varphi : W^{m,p}_\delta \rightarrow W^{m,p}_\delta \) is linear, bounded, and has a bounded inverse, \( R_{\varphi^{-1}} \). Hence, we have

**Corollary 2.1** Assume \( m \geq m_0 > 1 + d/p \) and \( \gamma + d/p > -1 \). For any \( \varphi \in \mathcal{D}^{m,p}_\gamma \) and \( \delta \in \mathbb{R} \), the map \( R_\varphi : W^{m_0,p}_\delta \rightarrow W^{m_0,p}_\delta \) is a linear isomorphism.

We can also conjugate differential operators with \( R_\varphi \) and its inverse. For example, suppose \( \varphi \in \mathcal{D}^{m,p}_\gamma \), \( f \in W^{m_0,p}_\delta \) is scalar-valued, and \( v \in W^{m_0,p}_\delta \) is a vector field. Consider the maps

\[
\begin{align*}
(\varphi, f) & \mapsto R_\varphi \circ \nabla \circ R_{\varphi^{-1}}(f), \quad (18a) \\
(\varphi, v) & \mapsto R_\varphi \circ \operatorname{div} \circ R_{\varphi^{-1}}(v), \quad (18b) \\
(\varphi, f) & \mapsto \Delta_\varphi(f) := R_\varphi \circ \Delta \circ R_{\varphi^{-1}}(f). \quad (18c)
\end{align*}
\]

These maps are not just continuous in \( \varphi \) and \( f \) (or \( v \)) as asserted in Theorem 2.1, but their special structure makes them smooth\(^5\).

**Lemma 2.1** Assume \( \gamma + d/p > -1 \) and \( \delta \in \mathbb{R} \).

(a) If \( m \geq m_0 > 1 + d/p \) then (18a) is smooth as a map \( \mathcal{D}^{m,p}_\gamma \times W^{m_0,p}_\delta(\mathbb{R}^d) \rightarrow W^{m_0-1,p}_\delta \) and (18b) is smooth as a map \( \mathcal{D}^{m,p}_\gamma \times W^{m_0,p}_\delta \rightarrow W^{m_0-1,p}_{\delta+1} \).

(b) If \( m \geq m_0 > 2 + d/p \) then (18c) is smooth as a map \( \mathcal{D}^{m,p}_\gamma \times W^{m_0,p}_\delta \rightarrow W^{m_0-2,p}_{\delta+2} \).

**Proof of Lemma 2.1** In (a) we have \( \varphi, f \in C^1 \), so we can use the chain rule \( d(f \circ \varphi) = (df \circ \varphi) \cdot \, d\varphi \) and the relationship \( d(\varphi^{-1}) = [(d\varphi) \circ \varphi^{-1}]^{-1} \), i.e. the inverse of the Jacobian matrix, to conclude

\[ R_{\varphi} \circ d \circ R_{\varphi^{-1}}(f) = df \cdot (d\varphi)^{-1}. \]

If we write \( \varphi = \text{id} + w \), with \( w \in \mathcal{W}^{m,p}_\gamma \), then \( d\varphi = I + dw \) where the matrix \(dw\) has components in \( \mathcal{W}^{m-1,p}_{\gamma+1} \). But, using the adjoint formula for the inverse of a matrix, we see that \( (d\varphi)^{-1} \) is a matrix with components which are obtained by taking sums and products of elements of \( d\varphi \), and a product by \( 1/\det(d\varphi) \). Then, one sees from (15) (see e.g. the proof of [30, Lemma 2.3]) that \( (d\varphi)^{-1} = I + T(w) \) where

---

\(^5\) By “smooth” we mean \( C^\infty \)-smooth. However, these maps will actually be shown to be real analytic.
Let \( w \mapsto T(w) \), \( W^m_{\gamma} \to W^{m-1}_{\gamma+1} \), be real-analytic. Moreover, since \( \gamma + 1 + d/p > 0 \), the multiplication by \( I + T(w) \) defines a real analytic map \( W^{m_0-1}_{\delta+1} \to W^{m_0-1}_{\delta+1} \).

In summary, we see from (15) and (16) that the mapping \((\phi, f) \mapsto d f \cdot (d\phi)^{-1}\)

\[ D^m_{\delta} \times W^{m_0}_{\delta} \to W^{m_0-1}_{\delta+1} \]

is real-analytic. This confirms that (18a) is real analytic and similar calculations show (18b) is real analytic. By rewriting (18c) as

\[ (\phi, f) \mapsto (R\phi \circ \text{div} \circ \gamma^{-1})(R\phi \circ \nabla \circ \gamma^{-1})(f), \]

where \((\phi, f) \in D^m_{\delta} \times W^{m_0}_{\delta}, \) we see that (b) follows from (a).

The reason to introduce the group of diffeomorphisms is to reduce the partial differential equation in \( W^m_{\delta} \) to an ordinary differential equation in \( D^m_{\delta} \times W^m_{\delta} \).

The following will be used in Sect. 4 as part of that analysis.

**Proposition 2.1** For \( m > 1 + d/p \) and \( \gamma + d/p > -1 \), assume \( u \in C([0, \tau], W^m_{\gamma}) \).

Then there is a unique solution \( \phi \in C^1([0, \tau], D^m_{\gamma}) \) of the equation

\[
\left\{ \begin{array}{l}
\dot{\phi} = u \circ \phi, \\
\phi|_{t=0} = \text{id}.
\end{array} \right.
\]

This proposition can be proved exactly like Proposition 2.1 in [24] (cf. [12] for the case of compact manifolds). We simply observe here that the condition \( m > 1 + d/p \) is used with Theorem 2.1(a) to view \( u \circ \phi \) as a \( C^1 \) vector field on \( D^m_{\gamma} \) and obtain a solution \( \phi \in C^1([0, \tau], D^m_{\gamma}) \); a bootstrap argument is then used to show \( \phi \in C^1([0, \tau], D^m_{\gamma}) \).

### 3 Mapping properties of the Laplacian and its inverse

The mapping on scalar functions

\[ \Delta : W^m_{\delta} \to W^{m-2}_{\delta+2} \]

for \( m \geq 2 \) was studied in [22] and found to have the following properties:

- **(A)** it is an isomorphism for \( 0 < \delta + d/p < d - 2 \) when \( d \geq 3 \),
- **(B)** it is surjective for \(-1 < \delta + d/p < 0\) with the space of constant functions \( \mathcal{N}_0 \) as a nullspace,
- **(C)** it is injective with cokernel \( \mathcal{N}_0 \) for \( d - 2 < \delta + d/p < d - 1 \). Hence, if \( g \in W^{m-1}_{\delta+2} \) satisfies \( \int_{\mathbb{R}^d} g \, dx = 0 \), then \( \Delta^{-1} g \in W^m_{\delta} \).

In fact, it was also shown in [22] that for \(-k - 1 < \delta + d/p < -k \) where \( k = 0, 1, 2, \ldots \), the nullspace of (20) consists of harmonic polynomials of degree \( k \) and these appear as the cokernel of (20) for \( d - 2 + k < \delta + d/p < d - 1 + k \). In [23] it was shown that the presence of cokernel for \( \delta + d/p > d - 2 \) leads to asymptotics when inverting \( \Delta \) on \( W^{m-2}_{\delta+2} \) (cf. Proposition B.1 in Appendix B for an extended version of

\[ \text{Springer} \]
this result); this was the genesis of the asymptotic spaces $A_{n,N}^{m,p}$ studied there. These asymptotic spaces will be used in a subsequent section of this paper; but for now, let us explore further consequences of (A) and (B).

In view of (B), for $m \geq 1$ the map $\Delta : W_{\delta - 1}^{m+1,p} \to W_{\delta + 1}^{m-1,p}$ has nontrivial nullspace $\mathcal{N}_0$ for $0 < \delta + d/p < 1$, so we can form the quotient space, $W_{\delta - 1}^{m+1,p} / \mathcal{N}_0$. This is a Banach space under the usual quotient norm, and we can reformulate (B) as the statement:

$$\Delta : W_{\delta - 1}^{m+1,p} / \mathcal{N}_0 \to W_{\delta + 1}^{m-1,p} \text{ is an isomorphism for } 0 < \delta + \frac{d}{p} < 1. \quad (21)$$

Note that $\nabla : W_{\delta - 1}^{m+1,p} / \mathcal{N}_0 \to W_{\delta}^{m,p}$ is well-defined and bounded, so $\nabla \circ \Delta^{-1} : W_{\delta + 1}^{m-1,p} \to W_{\delta}^{m,p}$ is bounded. Combining this with the fact (see case (A)) that $\Delta : W_{\delta - 1}^{m+1,p} \to W_{\delta + 1}^{m-1,p}$ is an isomorphism for $1 < \delta + \frac{d}{p} < d - 1 \quad (22)$

we conclude that $\nabla \circ \Delta^{-1} : W_{\delta + 1}^{m-1,p} \to W_{\delta}^{m,p}$ is bounded for $0 < \delta + \frac{d}{p} < d - 1$ but $\delta + \frac{d}{p} \neq 1$. Hence, we have

**Lemma 3.1** For $m \geq 1$, $0 < \delta + \frac{d}{p} < d - 1$, and $\delta + \frac{d}{p} \neq 1$, the operator

$$\nabla \circ \Delta^{-1} : W_{\delta + 1}^{m-1,p} \to W_{\delta}^{m,p} \quad (23)$$

is bounded.

Assume that $d \geq 2$, $m > 1 + d/p$, and $\gamma + d/p > -1$. For $0 < \delta + d/p < d - 1$ and $\delta + d/p \neq 1$ consider the map

$$(\varphi, f) \mapsto R_{\varphi} \circ \nabla \circ \Delta^{-1} \circ R_{\varphi^{-1}}(f), \quad \mathcal{D}_{\gamma}^{m,p} \times W_{\delta + 1}^{m-1,p} \to W_{\delta}^{m,p}, \quad (24)$$

which is well defined by Corollary 2.1 and Lemma 3.1. We have

**Proposition 3.1** For $0 < \delta + d/p < d - 1$ and $\delta + d/p \neq 1$ the map (24) is well defined and smooth.

**Proof of Proposition 3.1** In view of Corollary 2.1, (21), and (22), we can write

$$R_{\varphi} \circ \nabla \circ \Delta^{-1} \circ R_{\varphi^{-1}}(f) = (R_{\varphi} \circ \nabla \circ R_{\varphi^{-1}}) \circ (R_{\varphi} \circ \Delta^{-1} \circ R_{\varphi^{-1}})(f) \quad (25)$$

where

$$(\varphi, f) \mapsto R_{\varphi} \circ \Delta^{-1} \circ R_{\varphi^{-1}}(f),$$

$$\mathcal{D}_{\gamma}^{m,p} \times W_{\delta + 1}^{m-1,p} \to \begin{cases} W_{\delta - 1}^{m+1,p} / \mathcal{N}_0 & \text{for } 0 < \delta + \frac{d}{p} < 1, \\ W_{\delta - 1}^{m+1,p} & \text{for } 1 < \delta + \frac{d}{p} < d - 1, \end{cases} \quad (26)$$
and
\[(\varphi, g) \mapsto R_\varphi \circ \nabla \circ R_{\varphi^{-1}}(g), \quad D^{m,p}_\gamma \times W^{m+1,p}_\delta \rightarrow W^{m,p}_\delta.\] (27)

It follows from Lemma 2.1(a) that the map (27) is smooth. We will prove the smoothness of the map (26) by following the proof of Proposition 5.1 in [24]. Consider the spaces \(F := W^{m+1,p}_{\delta+1}\) and
\[E := \begin{cases} W^{m+1,p}_{\delta-1}/\mathcal{N}_0 & \text{for } 0 < \delta + d/p < 1, \\ W^{m+1,p}_{\delta-1} & \text{for } 1 < \delta + d/p < d-1. \end{cases}\]

Let \(GL(E, F)\) be the group of linear isomorphisms \(G : E \rightarrow F\) considered as a subspace of \(\mathcal{L}(E, F)\), the Banach space of all bounded linear maps \(E \rightarrow F\). By using Neumann series, one sees that \(GL(E, F)\) is an open set in \(\mathcal{L}(E, F)\) and the map
\[G \mapsto G^{-1}, \quad GL(E, F) \rightarrow GL(F, E),\] (28)
is real analytic. It follows from (21), (22), and Corollary 2.1 that for any \(\varphi \in D^{m,p}_\gamma\),
\[\Delta_\varphi \equiv R_\varphi \circ \Delta \circ R_{\varphi^{-1}} \in GL(E, F) \quad \text{and} \quad \Delta^{-1}_\varphi = R_\varphi \circ \Delta^{-1} \circ R_{\varphi^{-1}} \in GL(F, E).\]

By Lemma 2.1(b), the map (18c) is real analytic. Since this map is linear in its second argument we conclude that the map
\[\varphi \mapsto \Delta_\varphi \equiv R_\varphi \circ \Delta \circ R_{\varphi^{-1}}, \quad D^{m,p}_\gamma \rightarrow GL(E, F),\]
is real analytic. Composing this with the real analytic map (28) we find that
\[\varphi \mapsto \Delta^{-1}_\varphi, \quad D^{m,p}_\gamma \rightarrow GL(F, E),\]
is real analytic. Hence the map (26) is smooth. Finally, the smoothness of the map (24) follows from the factorization (25) and the fact that, by Lemma 2.1(a), the map
\[(\varphi, g) \mapsto R_\varphi \circ \nabla \circ R_{\varphi^{-1}}(g), \quad D^{m,p}_\gamma \times E \rightarrow F,\]
is well defined and smooth. \(\square\)

4 The case \(-1/2 < \delta + d/p < d - 1\)

Let \(u \in C([0, \tau], W^{m,p}_{\delta}) \cap C^1([0, \tau], W^{m-1,p}_{\delta}), m > 1 + d/p, \delta \in \mathbb{R}\), be a solution of the Euler equation (1). Applying \(\text{div}\) to both sides of \(u_t + (u \cdot \nabla u) u = -\nabla p\) and using \(\text{div} u = 0\), we obtain
\[\Delta p = -Q(u),\] (29)
where

\[ Q(u) := \text{tr} \left( (du)^2 \right) = \text{div} (u \cdot \nabla u); \tag{30} \]

here \((du)^2\) denotes the square of the Jacobian matrix and \(\text{tr}\) denotes the trace of a matrix. Note that \(Q\) maps vector functions to scalar functions; in fact, we have the following mapping property.

**Lemma 4.1** If \(m > 1 + d/p\), then \(u \mapsto Q(u)\) defines a smooth map

\[ Q : W_{\delta}^{m,p} \to W_{2\delta+2+d/p}^{m-1,p} \quad \text{for any } \delta \in \mathbb{R}. \tag{31} \]

**Proof of Lemma 4.1** Note that differentiation is continuous \(\partial_k : W_{\delta}^{m,p} \to W_{\delta+1}^{m-1,p}\). Recall from (15) that for \(m > d/p\) and any \(\delta_1, \delta_2 \in \mathbb{R}\), pointwise multiplication of functions \((f, g) \mapsto fg\) defines a continuous map

\[ W_{\delta_1}^{m,p} \times W_{\delta_2}^{m,p} \to W_{\delta_1+\delta_2+d/p}^{m,p}. \]

Since \(Q\) involves only differentiation, pointwise products, and sums of elements of \(du\), we then conclude that (31) is real analytic, and hence smooth. \(\square\)

For future use, let us see what happens when \(Q\) is conjugated with \(\varphi \in \mathcal{D}_\gamma^{m,p}\):

**Lemma 4.2** If \(m > 1 + d/p\) and \(\gamma + d/p > -1\), then the map \((\varphi, v) \mapsto R_\varphi \circ Q \circ R_{\varphi^{-1}}(v)\) is smooth \(\mathcal{D}_\gamma^{m,p} \times W_{\delta}^{m,p} \to W_{2\delta+2+d/p}^{m-1,p} \quad \text{for any } \delta \in \mathbb{R}. \)

**Proof of Lemma 4.2** The proof of Lemma 4.2 follows from the formula

\[ R_\varphi \circ Q \circ R_{\varphi^{-1}}(v) = \text{tr} \left( [R_\varphi \circ d \circ R_{\varphi^{-1}}(v)]^2 \right) \]

and using Lemma 2.1(a) to show \((\varphi, v) \mapsto R_\varphi \circ d \circ R_{\varphi^{-1}}(v)\) is smooth. \(\square\)

Note that we will be applying Lemma 4.2 below with \(\gamma = \delta\).

Now, to use (29) to determine \(\nabla\) as found in Sect. 3 to \(Q(u)\). In fact, since we only need \(\nabla p\) in (1), we will now determine the mapping properties of \(u \mapsto \nabla \circ \Delta^{-1} \circ Q(u)\) on \(W_{\delta}^{m,p}\), which maps vector fields to vector fields. It follows from Lemma 3.1 that

\[ \nabla \circ \Delta^{-1} : W_{\kappa+1}^{m-1,p} \to W_{\kappa}^{m,p} \tag{32a} \]

is bounded provided \(m \geq 1\) and

\[ 0 < \kappa + d/p < d - 1, \quad \kappa + d/p \neq 1. \tag{32b} \]
The strategy is now to choose $\kappa$ so that $W_{\frac{\delta+2}{p}}^{m-1,p} \subseteq W_{\kappa+1}^{m-1,p}$ and $W_{\kappa}^{m,p} \subseteq W_{\delta}^{m,p}$, as well as (32b). This can be done by choosing

$$\max(0, \delta + d/p) < \kappa + d/p < \min\left(d - 1, 2(\delta + d/p) + 1\right) \quad \text{and} \quad \kappa + d/p \neq 1.$$  \hspace{1cm} (33)

**Proposition 4.1** For $m > 1 + d/p$ and $-1/2 < \delta + d/p < d - 1$, the mapping

$$\nabla \circ \Delta^{-1} \circ Q : W_{\delta}^{m,p} \rightarrow W_{\delta}^{m,p} \hspace{1cm} (34)$$

is smooth.

**Proof of Proposition 4.1** One can easily confirm that the condition $-1/2 < \delta + d/p < d - 1$ implies $\max(0, \delta + d/p) < \min\left(d - 1, 2(\delta + d/p) + 1\right)$, so we can choose $\kappa$ to satisfy all conditions in (33), and therefore $\nabla \circ \Delta^{-1} \circ Q : W_{\delta}^{m,p} \rightarrow W_{\kappa}^{m,p} \subseteq W_{\delta}^{m,p}$. \hfill $\square$

The above analysis enables us to eliminate $p$ from (1) and to rewrite it as

$$\begin{aligned}
\left\{ u_t + (u \cdot \nabla) u = \nabla \circ \Delta^{-1} \circ Q(u), \quad u\big|_{t=0} = u_0. \right.
\end{aligned} \hspace{1cm} (35)$$

Notice that the condition $\operatorname{div} u = 0$ for $t > 0$ has also been eliminated. We have the following lemma.

**Lemma 4.3** Assume that $m > 1 + d/p$ and $-1/2 < \delta + d/p < d - 1$. If the curve $u \in C([0, \tau], W_{\delta}^{m,p}) \cap C^1([0, \tau], W_{\delta}^{m-1,p})$ satisfies the Euler equation (1) with pressure satisfying $|\nabla p(x, t)| = o(1)$ as $|x| \to \infty$ for $t \in [0, \tau]$ then $u \in C([0, \tau], W_{\delta}^{m,p}) \cap C^1([0, T], W_{\delta}^{m-1,p})$ satisfies (35). Conversely, if $u \in C([0, \tau], W_{\delta}^{m,p}) \cap C^1([0, T], W_{\delta}^{m-1,p})$ satisfies (35) with $u_0 \in W_{\delta}^{m,p}$ then it satisfies (1) so that $|\nabla p(x, t)| = o(1)$ as $|x| \to \infty$ and for any $t \in [0, \tau]$ we have that $p(t) = \Delta^{-1} \circ Q(u(t))$ up to an additive constant. \hfill $^6$

The proof of this lemma follows the lines of the proof of Lemma 4.1 in [24], so we will not give the details.

Now we want to replace (35) with an ordinary differential equation on the tangent bundle of $D_{\delta}^{m,p}$. The differential structure of $D_{\delta}^{m,p}$ is inherited from $W_{\delta}^{m,p}$ in a natural way, so $D_{\delta}^{m,p}$ may be viewed as a Banach manifold modeled on the Banach space $W_{\delta}^{m,p}$. This allows us to identify the tangent bundle $TD_{\delta}^{m,p}$ with the product space:

$$TD_{\delta}^{m,p} = D_{\delta}^{m,p} \times W_{\delta}^{m,p}. \hspace{1cm} (36)$$

\hfill $^6$ Here $\Delta^{-1} : W_{\kappa+1}^{m-1,p} \rightarrow W_{\kappa-1}^{m+1,p}/N_0$ with $0 < \kappa + d/p < 1$ (cf. (21)).
Next we define the Euler vector field $\mathcal{E}$ on the tangent bundle $D_{\delta}^{m,p} \times W_{\delta}^{m,p}$. For $\varphi \in D_{\delta}^{m,p}$ and $v \in W_{\delta}^{m,p}$, we introduce

$$\mathcal{E}_2(\varphi, v) = R_{\varphi} \circ \nabla \circ \Delta^{-1} \circ Q \circ R_{\varphi^{-1}}(v).$$ (37)

For $-1/2 < \delta + d/p < d - 1$ we can use Proposition 4.1 to conclude $\mathcal{E}_2(\varphi, v) \in W_{\delta}^{m,p}$.

So if we define $\mathcal{E}(\varphi, v) = (\varphi, \mathcal{E}_2(\varphi, v))$, we obtain a map

$$\mathcal{E} : D_{\delta}^{m,p} \times W_{\delta}^{m,p} \to D_{\delta}^{m,p} \times W_{\delta}^{m,p}.$$ (38)

To have a unique integral curve, we need $\mathcal{E}$ to be at least Lipschitz continuous on $T D_{\delta}^{m,p}$. In fact, we will show $\mathcal{E}$ is smooth on $T D_{\delta}^{m,p}$ for certain values of $\delta$.

**Theorem 4.1** The vector field (38) is smooth for $d \geq 2$, $m > 1 + d/p$, and $-1/2 < \delta + d/p < d - 1$.

**Proof of Theorem 4.1** We factor $\mathcal{E}_2$ as follows:

$$\mathcal{E}_2(\varphi, v) = \left(R_{\varphi} \circ \nabla \circ \Delta^{-1} \circ R_{\varphi^{-1}}\right) \circ \left(R_{\varphi} \circ Q \circ R_{\varphi^{-1}}\right)(v).$$

Recall from the proof of Proposition 4.1 the decomposition

$$W_{\delta}^{m,p} \overset{Q}{\longrightarrow} W_{2\delta+2+d/p}^{m-1,p} \overset{\iota}{\longrightarrow} W_{\kappa+1}^{m-1,p} \overset{\nabla \circ \Delta^{-1}}{\longrightarrow} W_{\kappa}^{m,p} \overset{\iota}{\longrightarrow} W_{\delta}^{m,p},$$

where $\iota$ denotes inclusion and $\kappa$ has been chosen to satisfy (33). Conjugation by $R_{\varphi}$ of $Q : W_{\delta}^{m,p} \to W_{2\delta+2+d/p}^{m-1,p}$ is smooth by Lemma 4.2 (for any $\delta \in \mathbb{R}$). The conjugation by $R_{\varphi}$ of $\nabla \circ \Delta^{-1} : W_{\kappa+1}^{m-1,p} \to W_{\kappa}^{m,p}$ is smooth by Proposition 3.1. □

The system of ordinary differential equations associated with the Euler vector field is

$$\left\{ \begin{array}{l}
(\dot{\varphi}, \dot{v}) = \mathcal{E}(\varphi, v), \\
(\varphi, v)|_{t=0} = (\text{id}, u_0).
\end{array} \right.$$ (39)

The relationship of $(\varphi, v)$ to the solution of (35) is provided by the following.

**Proposition 4.2** The map

$$(\varphi, v) \mapsto u := R_{\varphi^{-1}}(v)$$ (40)

provides a continuous, bijective correspondence between solutions $(\varphi, v) \in C^1([0, \tau], T D_{\delta}^{m,p})$ of (39) and solutions $u \in C([0, \tau], W_{\delta}^{m,p}) \cap C^1([0, \tau], W_{\delta}^{m-1,p})$ of (35).

This proposition follows from Proposition 2.1 and can be proved just like Lemma 7.1 in [24], so we will not repeat the details here.

We can now prove Theorem 1.1.
Perfect fluid flows on $\mathbb{R}^d$ with growth/decay conditions at infinity

Proof of Theorem 1.1 Under the assumption that $|\nabla p(t, x)| = o(1)$ as $|x| \to \infty$ we obtain from Lemma 4.3 and Proposition 4.2 that the Euler equation (1) is equivalent to the dynamical system (39) on $TD_{\delta}^{m,p}$. We know by Theorem 4.1 that $\mathcal{E}$ is smooth on $TD_{\delta}^{m,p}$. Hence, by the standard theory of ordinary differential equations in Banach spaces, for any $u_0 \in W_{\delta}^{m,p}$ there exists $\tau = \tau(\|u_0\|_{W_{\delta}^{m,p}}) > 0$ and a unique solution $(\varphi, v) \in C^1([0, \tau], TD_{\delta}^{m,p})$ of (39) that depends smoothly on the initial data. The uniqueness and the continuous dependence of $u$ on initial data then follows from the uniqueness and the continuous dependence of the solutions of (39) on the initial data together with Lemma 4.3 and Proposition 4.2. The statement on the pressure also follows from Lemma 4.3.

5 Proofs of Theorem 1.2, Propositions 1.1, and 1.2

We will deduce Theorem 1.2 using results in [24] concerning the well-posedness of the Euler equations in asymptotic spaces. We will give here a brief description of the asymptotic spaces $A_{n,N;0}^{m,p}$ that we need; for more details, see Appendix C of this paper or the papers [23,24].

For integers $m > d/p$ and $N \geq n \geq 0$, $A_{n,N;0}^{m,p}$ is a Banach whose elements are vector fields on $\mathbb{R}^d$ of the form

$$u(x) = \chi(r) \left( \frac{a_0^0(\theta) + \cdots + a_0^n(\theta)(\log r)^n}{r^n} + \cdots + \frac{a_N^0(\theta) + \cdots + a_N^n(\theta)(\log r)^n}{r^N} \right) + f(x).$$

(41)

Here $a_j^k \in H^{m+1+N-k,p}(S^{d-1}, \mathbb{R}^d) \subseteq C(S^{d-1}, \mathbb{R}^d)$ for $n \leq k \leq N$ and $0 \leq j \leq k$; $H^{m,p}$ denotes the standard $L^p$ Sobolev space of order $m$. The remainder function $f(x)$ satisfies $f \in W_{\gamma N}^{m,p}$ for a weight

$$\gamma_N := N + \gamma_0 \quad \text{where} \quad 0 \leq \gamma_0 + d/p < 1,$$

(42)

which by (3b) implies $f(x) = o(r^{-N})$ as $r \to \infty$. If $n = 0$, we use the abbreviation $A_{N;0}^{m,p}$ and we let $\hat{A}_{N;0}^{m,p}$ denote the closed subspace of divergence free vector fields in $A_{N;0}^{m,p}$. By $A_{n,N}^{m,p}$ (resp. $A_{n,N;0}^{m,p}$) we denote the closed subspace of $A_{n,N}^{m,p}$ (resp. $A_{n,N;0}^{m,p}$) that consists of vector fields of the form (41) without log terms. We refer to Appendix C for more details.

Actually, in [23,24], strict inequality $0 < \gamma_0 + d/p < 1$ was assumed in (42). In this case, it was shown in Theorem 1.1 in [24] that for any $m > 3 + d/p$ and $\rho > 0$ there exists $\tau > 0$ such that for any $u_0 \in \hat{A}_{N;0}^{m,p}$ with $\|u_0\|_{A_{N;0}^{m,p}} < \rho$ there exists a unique solution of the Euler equation

$$u \in C([0, \tau], \hat{A}_{N;0}^{m,p}) \cap C^1([0, \tau], \hat{A}_{N;0}^{m-1,p}),$$

(43)

that depends continuously on the initial data $u_0$. We will use this to prove our Theorem 1.2.
**Proof of Theorem 1.2** Let $N$ denote the integer part of $\delta + d/p > 0$ and $\gamma_0 := \delta - N$. Then $N \geq 0$ and $0 \leq \gamma_0 + d/p < 1$. Since $\delta = \gamma N$, $W_{\delta}^{m,p}$ is the remainder space for $A_{N;0}^{m,p}$. In particular, $W_{\delta}^{m,p}$ is a closed subspace in $A_{N;0}^{m,p}$. Consequently, for any $\rho > 0$ there exists $\tau > 0$ such that for any $u_0 \in W_{\delta}^{m,p}$ with $\|u_0\|_{W_{\delta}^{m,p}} < \rho$ the Euler equation (1) has a unique solution (43) that depends continuously on the initial data in $W_{\delta}^{m,p}$. If $0 < \gamma_0 + d/p < 1$ we refer as above to [24, Theorem 1.1], while for $\gamma_0 + d/p = 0$ we refer to Proposition C.1 in Appendix C.

We first assume $\delta + d/p \notin \mathbb{Z}$, so $0 < \gamma_0 + d/p < 1$. Since the case when $d = 2$ follows from Corollary 1.1 and Corollary 1.2 in [30], we will concentrate our attention on the case when $d \geq 3$. To see that the asymptotic terms appearing in the solution (43) are of the form described in Theorem 1.2, we argue as follows. Since $m > 3 + d/p$ it follows from (43) and the Sobolev embedding theorem that $u \in C([0, \tau], C^2(\mathbb{R}^d, \mathbb{R}^d))$. By applying the curl operator to (1) we then see that $\omega_t + L_u \omega = 0$ where $L_u \omega$ denotes the Lie derivative of the vorticity form $\omega := d(u^\flat)$ with respect to the time dependent vector field $u$. (Here $d$ denotes the exterior differentiation of the one form $u^\flat$ obtained by lowering the indices of the vector field $u$ with the help of the Euclidean metric on $\mathbb{R}^d$.) This implies that the pull-back $\varphi(t)^* (\omega(t))$ of the vorticity form $\omega(t)$ with respect to the flow map $\varphi(t) : \mathbb{R}^d \to \mathbb{R}^d$ generated by $u$ is independent of $t \in [0, \tau]$. Recall from Corollary 2.2 in [24] that

$$\varphi \in C^1([0, \tau], \hat{AD}_{N;0}^{m,p}),$$

(44)

where $\hat{AD}_{N;0}^{m,p}$ denotes the group of volume preserving asymptotic diffeomorphisms of $\mathbb{R}^d$ (see Section 2 in [24]). Hence, for any $t \in [0, \tau]$ we have that

$$\omega(t) = (d\psi(t))^T (\omega(0) \circ \psi(t)) (d\psi(t)), \quad \psi(t) := \varphi(t)^{-1},$$

(45)

where $d\psi$ denotes the Jacobian matrix of $\psi(t) : \mathbb{R}^d \to \mathbb{R}^d$,

$$\omega = (\omega_{\alpha j})_{1 \leq \alpha, j \leq d}, \quad \omega_{\alpha j} = \frac{\partial u_j}{\partial x_\alpha} - \frac{\partial u_\alpha}{\partial x_j},$$

(46)

is the matrix of the components of the vorticity form $\omega(t)$, and $(\cdot)^T$ denotes the transpose of a matrix. Recall from Theorem 9.1 in [24] that $\hat{AD}_{N;0}^{m,p}$ is a real analytic submanifold in the group $AD_{N;0}^{m,p}$ of asymptotic diffeomorphisms of $\mathbb{R}^d$. The elements of $AD_{N;0}^{m,p}$ are $C^{1}$-diffeomorphisms of $\mathbb{R}^d$ of the form $id + w$ where $w \in A_{N;0}^{m,p}$. Since $\hat{AD}_{N;0}^{m,p}$ (and $AD_{N;0}^{m,p}$) is a topological group (cf. [24, Corollary 2.1]) we obtain from (44) that $\psi \in C([0, \tau], \hat{AD}_{N;0}^{m,p})$. By combining this with (46), the fact that $\omega(0) = d(u_0^\flat) \in W_{\delta+1}^{m-1,p}$, Corollary 6.1 (b) in [23], 8 and the conservation law (45),

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7 This is a re-expression of the conservation of vorticity theorem.

8 Note that $AD_{N;0}^{m,p} \subseteq AD_0^{m,p}$ and the inclusion is bounded.
we conclude from the pointwise multiplication properties of $W$-spaces (see (15) and (16)) that

\[ \omega \in C([0, \tau], W^{m-1,p}_{\delta+1}). \quad (47) \]

Since the solution (43) is divergence free, we obtain from (46) that for any $1 \leq j \leq d$,

\[
(\text{div } \omega)_j := \sum_{\alpha=1}^{d} \partial_\alpha \omega_{\alpha,j} = \Delta u_j - \partial_j (\text{div } u) = \Delta u_j
\]

where $\partial_\alpha$ denotes the distributional partial derivative in the direction $x_\alpha$. In view of the assumption that $\delta + d/p \notin \mathbb{Z}$, we then conclude that for any $t \in [0, \tau]$,

\[
\begin{align*}
    u(t) &= \Delta^{-1} \text{div } \omega(t), \\
    &= \sum_{d-2 \leq k < \delta + d/p} a_k(\theta, t) r^k + f(t), \quad t \in [0, \tau],
\end{align*}
\]

where $\Delta^{-1} : W^{m-2,p}_{\delta+2} \rightarrow \mathcal{A}^{m,p}_N$ is the inverse of the Laplace operator given by Proposition B.1 in Appendix B (cf. Lemma A.3 (b) in [24]) and $\mathcal{A}^{m,p}_N \subseteq \mathcal{A}^{m,p}_{N,0}$ is the asymptotic space without log terms. Note that by Proposition 1.1 in [24] the leading term $u_0$ in the asymptotic expansion (41) with $n = 0$ of the solution (43) vanishes, since $u_0 \in W^{m,p}_\delta$. This, together with Proposition B.1 in Appendix B, then implies that

\[
\begin{align*}
    u(t) &= \chi(r) \sum_{d-2 \leq k < \delta + d/p} a_k(\theta, t) r^k + f(t), \quad t \in [0, \tau],
\end{align*}
\]

where $f(t) \in W^{m,p}_\delta$ and $a_k(\theta, t)$ with $d-2 \leq k \leq N$ is an eigenfunction of the Laplace operator $-\Delta_S$ on the unit sphere $S^{d-1}$ with eigenvalue $\lambda_{k-d+2} = k(k-d+2)$. In fact, it follows from Remark 1.3 in [24] (or, alternatively, it can be deduced from (80) and the integral identity (66) below) that the three leading terms in (49) do not appear, so $u(t)$ is of the form (6). This completes the proof of items (a) and (b) of Theorem 1.2.

The continuous dependence of (6) on the initial data $u_0 \in \hat{W}^{m,p}_\delta$ (as described in Remark 1.2) follows from the continuous dependence of the solution (43) and the fact that $W^{m,p}_\delta$ is a closed subspace in $\mathcal{A}^{m,p}_{N,0}$.

Let us now prove the analyticity of the asymptotic coefficients $a_k$, $d+1 \leq k \leq \delta + d/p$, in the case when $\delta + d/p \geq d + 1$ and $\delta + d/p \notin \mathbb{Z}$. Take an integer $k \in \mathbb{Z}$ such that

\[
d + 1 \leq k \leq \delta + d/p
\]

and $1 \leq l \leq \nu(k') \equiv \dim \mathcal{H}_{k'}$ and set $k' = k - d + 2$. It then follows from (48) and Proposition B.1 in Appendix B that, for any given $t \in [0, \tau]$, the Fourier coefficient

9 Since $\delta + d/p \notin \mathbb{Z}$ we have that $k < \delta + d/p$. 

\[
\chi(r) \sum_{d-2 \leq k < \delta + d/p} a_k(\theta, t) r^k + f(t), \quad t \in [0, \tau],
\]

\[
\text{Since } \delta + d/p \notin \mathbb{Z} \text{ we have that } k < \delta + d/p.
\]
\( \hat{a}_{k';l}(t) \) in the Fourier expansion of \( a_k(t) \in C(S^{d-1}, \mathbb{R}^d) \) can be obtained from the integral formula
\[
\hat{a}_{k';l}(t) = -C_k' \int_{\mathbb{R}^d} H_{k';l} \text{div} \, \omega(t) \, dx,
\]
where \( C_k' := 1/(2k' + d - 2) \), \( (\text{div} \, \omega)_j \equiv \sum_{\alpha=1}^d \partial_\alpha \omega_{\alpha j} \), and \( H_{k';l} \) is a harmonic polynomial of degree \( k' \geq 0 \) (see Proposition B.1 for the precise definition of \( H_{k';l} \)). It follows from (47) that for any \( t \in [0, \tau] \),
\[
\omega(t) \in W^{m-1}_{\delta+1} \, p \quad \text{and} \quad \text{div} \, \omega(t) \in W^{m-2}_{\delta+2} \, p.
\]
Note that by Hölder’s inequality
\[
W^m_\delta \subseteq L^1
\]
for \( m \geq 0 \) and \( \delta + d/p > d \). By combining (52) with the fact that \( d + 1 \leq k < \delta + d/p \) we see from (53) that we have enough decay to apply Stokes’ theorem to the integral in (51) and obtain from (45) that
\[
\hat{a}_{k';l}(t) = C_k' \int_{\mathbb{R}^d} (\nabla H_{k';l})^T \omega(t) \, dx
\]
where we changed the variables and used that \( (d\psi) \circ \varphi = (d\varphi)^{-1} \) and that \( \varphi(t) \) is volume preserving diffeomorphism by (44). By taking \( \tau > 0 \) smaller if necessary, we obtain from Proposition 9.2 in [24] that the map
\[
(t, u_0) \mapsto \varphi(t; u_0), \quad [0, \tau] \times B_{\mathbb{R}^d}^m(p, \rho) \rightarrow \hat{A}D^m_{N;0},
\]
is analytic. By combining this with (54), the properties (15), (16), and the fact that \( H_{k';l} \) is a polynomial, we conclude that \( \hat{a}_{k';l} : [0, \tau] \times B_{\mathbb{R}^d}^m(p, \rho) \rightarrow \mathbb{R} \) is analytic.

The global existence of the solution (63) in the case when \( d = 2 \) and \( \delta + 2/p > 0 \) is not integer follows from Corollary 1.1 and 1.2 in [30]. This completes the proof of Theorem 1.2 in the case when \( \delta + d/p > 0 \) is not integer. Note that the argument above cannot be readily applied in the case when \( \delta + d/p > 0 \) is integer since Proposition B.1 in Appendix B does not hold for integer values of \( \delta + d/p \).

Before turning to the case when \( \delta + d/p \in \mathbb{Z} \) note that for any \( t \in [0, \tau] \) we have \( d\varphi(t) = id + dw(t) \) where \( dw(t) \in A^{m-1}_{1, N+1; -1} \) (cf. Appendix C). This implies that for any integer \( k \) such that \( d + 1 \leq k \leq \delta + d/p \) the integrand \( I_k'(\varphi(t), \omega(0)) \) in (54)
can be written as
\[ I_k'(\varphi(t), \omega(0)) = (\nabla H_{k';l})^T \omega(0) + R_k'(\varphi(t), \omega(0)) \]  
(55)

where \( \omega(0) \equiv du_0^b \) and \( R_k'(\varphi(t), \omega(0)) \in W^{m-1,p}_{(\delta+1)-(k'-1)+1} \subset L^1 \) by (53). Moreover, for any \( \delta \in \mathbb{R} \) with \( \delta + d/p \geq d + 1 \) and for any integer \( d + 1 \leq k \leq \delta + d/p \) the map
\[ (\varphi, u_0) \mapsto R_k'(\varphi, du_0^b), \quad AD_{N:0}^{m,p} \times B_{W^m,p}(\rho) \to L^1, \]  
(56)
is analytic. By combining this with (54) we conclude that for any non-integer \( \delta + d/p \geq d + 1 \) for any \( d + 1 \leq k \leq \delta + d/p \) and for any \( 1 \leq l \leq v(k') \) we have that
\[ \tilde{a}_{k';l}(t; u_0) = C_k' \int_{\mathbb{R}^d} R_k'(\varphi(t; u_0), du_0^b) \, dx \]  
(57)
where we used that for \( d + 1 \leq k < \delta + d/p \) we have from (53) that \( (\nabla H_{k';l})^T (du_0^b) \in W^{m-1,p}_{\delta+1-k'+1} \subset L^1 \) and, by the Stokes’ theorem, (46), and the fact that \( H_{k';l} \) is a harmonic polynomial,
\[ \int_{\mathbb{R}^d} (\nabla H_{k';l})^T (du_0^b) \, dx = \left( \int_{\mathbb{R}^d} \sum_{\alpha=1}^d (\partial_\alpha H_{k';l})(\partial_\alpha u_0 j - \partial j u_0\alpha) \, dx \right)_{1 \leq j \leq d} \]
\[ = -\int_{\mathbb{R}^d} (\Delta H_{k';l}) u_0 \, dx + \int_{\mathbb{R}^d} H_{k';l} \nabla (\text{div } u_0) \, dx = 0 \]
for \( u_0 \in W_\delta^m \).

\[ \square \]

Remark 5.1 Note that for \( \varphi(t; u_0) \in AD_{N:0}^{m,p} \) (with \( N \) the integer part of \( \delta + d/p \)) the expression on the right side of (57) is well defined for all values of \( \delta + d/p \geq d + 1 \) and for any \( d + 1 \leq k \leq \delta + d/p \). In particular, it is well defined for \( \delta + d/p \in \mathbb{Z} \) and \( k = \delta + d/p \), and depends analytically on \( u_0 \in W_\delta^m \). (In contrast, \( du_0^b \in W^{m-1,p}_{\delta+1} \) implies that the term \( (\nabla H_{k';l})^T (du_0^b) \) in (55) belongs to \( W^{m-1,p}_{(\delta+1)-(k'+1)} \) that is not a subset in \( L^1 \) for \( \delta + d/p \in \mathbb{Z} \) and \( k = \delta + d/p \). In particular, (54) is not well defined for \( \delta + d/p \in \mathbb{Z} \) and \( k = \delta + d/p \).)

With this preparation, we now turn to the case when \( \delta + d/p \in \mathbb{Z} \) so \( \delta + d/p \geq 1 \) and \( \gamma_0 + d/p = 0 \); we need to include \( d = 2 \) since integral values of \( \delta + d/p \) were not studied in [30]. Since \( \gamma_N = \delta \), we see as above that \( W_\delta^m \) is the remainder space for the asymptotic space \( A_{1,N:0}^{m,p} \). In particular, \( W_\delta^m \) is a closed subspace in \( A_{1,N:0}^{m,p} \). By Proposition C.1, for any \( \rho > 0 \) there exists \( \tau > 0 \) such that for any \( u_0 \in W_\delta^m \subset A_{1,N:0}^{m,p} \) with \( \| u_0 \|_{W_\delta^m} < \rho \) there exists a unique solution of the Euler equation
\[ u \in C([0, \tau], A_{1,N:0}^{m,p}) \cap C^1([0, \tau], A_{1,N:0}^{m-1,p}) \]
and \( \varphi \in C^1([0, \tau], \hat{A}D^{m,p}_{1,N:0}) \) (cf. Corollary 2.2 in [24]) that depend continuously on the initial data \( u_0 \in \dot{W}^{m,p}_{\delta} \) with \( \| u_0 \|_{\dot{W}^{m,p}_{\delta}} < \rho \). Now, take \( \delta' > \delta \) so that \( \delta' - \delta < 1 \). Then, \( \delta' + d/p \not\in \mathbb{Z} \) and the space \( \dot{W}^{m,p}_{\delta'} \) is dense in \( \dot{W}^{m,p}_{\delta} \). Let \( (u_{0j})_{j \geq 1} \) be a sequence of initial data in \( \dot{W}^{m,p}_{\delta'} \) such that \( \| u_{0j} \|_{\dot{W}^{m,p}_{\delta'}} < \rho \) and

\[
 u_{0j} \xrightarrow{\dot{W}^{m,p}_{\delta'}} u_0 \quad \text{as} \quad j \to \infty.
\]  

Let

\[
 u_j \in C([0, \tau], \mathcal{A}^{m,p}_{1,N:0}) \cap C^1([0, \tau], \mathcal{A}^{m-1,p}_{1,N:0})
\]

be the corresponding solutions of the Euler equation in \( \mathcal{A}^{m,p}_{1,N:0} \) and let

\[
 (\varphi_j, v_j) \in C^1([0, \tau], \hat{A}D^{m,p}_{1,N:0} \times \mathcal{A}^{m,p}_{1,N:0})
\]

be the integral curve of the smooth Euler vector field (98) (cf. Appendix C) with initial data \( (\text{id}, u_0) \in \hat{A}D^{m,p}_{1,N:0} \times \mathcal{A}^{m,p}_{1,N:0} \). We then conclude from (45), the fact that \( \mathcal{A}D^{m,p}_{1,N:0} \) is a topological group, \( \mathcal{A}D^{m,p}_{1,N:0} \subseteq \mathcal{A}D^{m,p}_{0} \), Corollary 6.1 (b) in [23], the properties (15), (16), and \( \omega_{0j} \equiv d(u_{0j}^b) \in W^{m-1,p}_{\delta'+1} \), that for any \( j \geq 1 \),

\[
 \omega_{0j} \equiv d(u_{0j}^b) \in C([0, \tau], W^{m-1,p}_{\delta'+1}).
\]

Since \( \delta' + d/p > 0 \) and \( \delta' + d/p \not\in \mathbb{Z} \), we then obtain from \( u_j(t) = \Delta^{-1} \text{div} \omega_j(t) \), where \( \omega_j(t) \equiv d(u_{0j}^b(t)) \) and \( t \in [0, \tau] \), and Proposition B.1 in Appendix B (see formula (86), and Proposition 3.3 all in [30] for the case when \( d = 2 \), that for any given \( t \in [0, \tau] \) the solution \( u_j(t) \) has an asymptotic expansion of the form

\[
 u_j(t) = \chi(t) \sum_{d+1 \leq k \leq \delta + d/p} \frac{a_k(\theta, t)}{k!} + f_j(t), \quad f_j(t) \in W^{m,p}_{\delta},
\]

with asymptotic coefficients as described in Theorem 1.2. In view of the continuous dependence of (60) on the initial data we then conclude from (58) that for any given \( t \in [0, \tau] \),

\[
 u_j(t) \xrightarrow{\mathcal{A}^{m,p}_{1,N:0}} u(t) \quad \text{and} \quad \varphi_j(t) \xrightarrow{\mathcal{A}D^{m,p}_{1,N:0}} \varphi(t) \quad \text{as} \quad j \to \infty.
\]
This together with (61) and the definition of the norm in $A_{m,p}^{m,p}$ (cf. [24, Section 2]) then implies that $f_j \xrightarrow{W_{m,p}^{m,p}_\delta} f$ and $a_{kj} \xrightarrow{L^\infty} a_k$ as $j \to \infty$ and hence,

$$u(t) = \chi(r) \sum_{d+1 \leq k \leq \delta + d/p} \frac{a_k(\theta, t)}{r^k} + f(t), \quad f(t) \in W_{m,p}^{m,p}, (63)$$

Since $a_{kj} \xrightarrow{L^\infty} a_k$ as $j \to \infty$ and since $a_{kj} \in C(S^{d-1}, \mathbb{R}^d)$ is an eigenfunction of the Laplace operator $-\Delta_S$ on $S^{d-1}$ with eigenvalue $\lambda_{k'} = k'(k' + d - 2)$, we conclude that the limit function $a_k$ on the sphere is again an eigenfunction of $-\Delta_S$ with the same eigenvalue. The well-posedness of the solution $u$ in the sense of (9) follows from the continuous dependence of $u$ on the initial data in the asymptotic space $A_{m,p}^{m,p}$.

Moreover, it follows from (62) and Remark 5.1 that for any $d + 1 \leq k \leq \delta + d/p$, $1 \leq l \leq v(k')$, and for any $t \in [0, \tau]$ the Fourier coefficient $\hat{a}_{k';j}(t)$ of the asymptotic coefficient $a_k(t)$ in (63) is given by (57). The analyticity of the asymptotic coefficients $a_k$, $d + 1 \leq k \leq \delta + d/p$, then follows from (57) and the analyticity of the map (56). Finally, the global existence in the case when $d = 2$ and $\delta + 2/p > 0$ is an integer follows from Proposition D.1 in Appendix D and the approximation argument applied above.

The proof of Proposition 1.1 is based on the following non-vanishing lemma.

**Lemma 5.1** Assume that $m > 3 + d/p$, $1 < p < \infty$. Then for any $k' \geq 0$, $1 \leq j \leq d$, and for any $\epsilon > 0$ there exists a homogeneous harmonic polynomial $H_{k'}^j(x)$ of degree $k'$ in $x \in \mathbb{R}^d$ and a divergence free vector field $u_0 \in C_c^\infty(\mathbb{R}^d)$ with support in the annulus $\epsilon < |x| < 2\epsilon$ such that

$$M_{k'}^j(u_0) := \int_{\mathbb{R}^d} H_{k'}^j \partial_j(Q(u_0)) \, dx \neq 0$$

where $Q(u_0) \equiv \text{tr} (du_0)^2$.

**Proof of Lemma 5.1** Assume that $m > 3 + d/p$, $1 < p < \infty$. Let $P(x)$ be a homogeneous harmonic polynomial in $x \in \mathbb{R}^d$ of degree $k' \geq 0$. Then, for any initial data with compact support $u_0 \in C_c^\infty$ and for any given index $1 \leq j \leq d$ we have

$$M_{k'}^j(u_0) = \int_{\mathbb{R}^d} P \partial_j(Q(u_0)) \, dx = -\int_{\mathbb{R}^d} (\partial_j P) \, Q(u_0) \, dx \quad (64)$$

where, by (30),

$$Q(u_0) = \text{tr} (du_0)^2 = \text{div} (u_0 \cdot \nabla u_0). \quad (65)$$

10 No summation over $j$ in the formula is assumed.
Further, by (64), (65), and the Stokes’ theorem,\\
\[ M_{j}^{k}(u_{0}) = - \int_{\mathbb{R}^{d}} (\partial_{j} P) \text{div}(u_{0} \cdot \nabla u_{0}) \, dx = \int_{\mathbb{R}^{d}} \sum_{1 \leq \alpha, \beta \leq d} (\partial_{\alpha} \partial_{j} P)(\partial_{\beta} u_{0\alpha})u_{0\beta} \, dx \]
\[ = \int_{\mathbb{R}^{d}} \sum_{1 \leq \alpha, \beta \leq d} (\partial_{\alpha} \partial_{j} P) \partial_{\beta}(u_{0\alpha}u_{0\beta}) \, dx = - \int_{\mathbb{R}^{d}} \sum_{1 \leq \alpha, \beta \leq d} (\partial_{\beta} \partial_{\alpha} \partial_{j} P)(u_{0\alpha}u_{0\beta}) \, dx \]
\[ = - \int_{\mathbb{R}^{d}} \sum_{1 \leq \alpha, \beta \leq d} (\partial_{\beta} \partial_{\alpha} \partial_{j} P)(u_{0\alpha}u_{0\beta}) \, dx \] (66)

where we used that \( \sum_{\beta=1}^{d} \partial_{\beta} u_{0\beta} \equiv \text{div} u_{0} = 0 \) and the identity \( \text{div}(f X) = (df)(X) + f \text{div} X \) that holds for any \( C^{\infty} \)-smooth vector field \( X \) and a scalar function \( f \) on \( \mathbb{R}^{d} \). In particular, we confirm from (66) that\[ M_{j}^{k}(u_{0}) = 0 \quad \text{for } 0 \leq k' \leq 2. \]

Let us now assume that \( k' \geq 3 \). Then, we fix the indexes \( 1 \leq \alpha, \beta \leq d, \alpha \neq \beta \), and set\[ u_{0} := - (\partial_{\beta} H) \partial_{\alpha} + (\partial_{\alpha} H) \partial_{\beta} \] (67)
where the (scalar) Hamiltonian \( H \in C^{\infty}_{c} \) will be specified later. Note that the vector field \( u_{0} \) in (67) is automatically divergence free. It then follows from (66) and (67) that for any choice of the homogeneous harmonic polynomial \( P \) of degree \( k' \geq 3 \) and for any choice of the Hamiltonian \( H \in C^{\infty}_{c} \) we have that\[ M_{k'}^{P}(u_{0}) = \int_{\mathbb{R}^{d}} \left\{ 2(\partial_{\alpha}^{2} \partial_{\beta} P) (\partial_{\alpha} H)(\partial_{\beta} H) - (\partial_{\alpha} \partial_{\beta}^{2} P) (\partial_{\alpha} H)^{2} - (\partial_{\alpha}^{3} P) (\partial_{\beta} H)^{2} \right\} \, dx \] (68)

where \( u_{0} \) is given by (67). It is useful to introduce the complex variables \( z := x_{\alpha} + ix_{\beta} \) and \( \bar{z} := x_{\alpha} - ix_{\beta} \) and the Cauchy operators \( \partial_{z} := \frac{1}{2}(\partial_{\alpha} - i \partial_{\beta}) \) and \( \partial_{\bar{z}} := \frac{1}{2}(\partial_{\alpha} + i \partial_{\beta}) \). Then, \( \partial_{\alpha} = \partial_{z} + \partial_{\bar{z}}, \partial_{\beta} = i(\partial_{z} - \partial_{\bar{z}}) \) and hence\[ \partial_{\alpha} H = H_{z} + H_{\bar{z}} \quad \text{and} \quad \partial_{\beta} H = i(H_{z} - H_{\bar{z}}) \] (69)
where \( H_{z} \equiv \partial_{z} H \) and \( H_{\bar{z}} \equiv \partial_{\bar{z}} H \). We now set\[ P(x) := z^{k'} + \bar{z}^{k'}, \quad k' \geq 3, \] (70)
and note that \( P \) is a harmonic polynomial (since we can write \( \Delta = 4\partial_{z} \partial_{\bar{z}} + \sum_{\mu \neq \alpha, \beta} \partial_{\mu}^{2} \)). The integrand in (68) can then be written as\[ 2(\partial_{\alpha}^{2} \partial_{\beta} P) (\partial_{\alpha} H)(\partial_{\beta} H) - (\partial_{\alpha} \partial_{\beta}^{2} P) (\partial_{\alpha} H)^{2} - (\partial_{\alpha}^{3} P) (\partial_{\beta} H)^{2} = \]
\[ = 2i(\partial_{\alpha}^{2} \partial_{\beta} P)(H_{z}^{2} - H_{\bar{z}}^{2}) - (\partial_{\alpha} \partial_{\beta}^{2} P)(H_{z} + H_{\bar{z}})^{2} + (H_{z} - H_{\bar{z}})^{2} \]
Perfect fluid flows on \( \mathbb{R}^d \) with growth/decay conditions at infinity 1475

\[
4i \partial_\alpha \partial_\beta (\partial_\zeta P) H_z^2 - 4i \partial_\alpha \partial_\beta (\partial_{\bar{\zeta}} P) H_{\bar{z}}^2 = 4 \left( (\partial_\zeta^3 P) H_z^2 + (\partial_{\bar{\zeta}}^3 P) H_{\bar{z}}^2 \right)
\]

(71)

where we used (69), (70), and the fact that \( \partial_\zeta^2 P = -\partial_{\bar{\zeta}}^2 P \). We now choose

\[
H(x) := R(x) a(\varrho) + b(\varrho_1, \varrho_2), \quad \varrho := |x|^2, \quad \varrho_1 := z\bar{z}, \quad \varrho_2 := |x|^2 - |z|^2,
\]

(72)

where \( R(x) \) is a homogeneous polynomial in \( x \in \mathbb{R}^d \) and \( a(\varrho) \) and \( b(\varrho_1, \varrho_2) \) are \( C^\infty \) functions of their arguments \( a : \mathbb{R} \to \mathbb{R}, b : \mathbb{R}^2 \to \mathbb{R} \), such that, when considered as functions of \( x \in \mathbb{R}^d \), they have non-empty support in the annulus \( \varepsilon \leq |x| \leq 2\varepsilon \) for a given \( \varepsilon > 0 \). Then,

\[
H_z = a R_z + a' \varrho \bar{R} + b'_1 \varrho_1
\]

(73)

where \( a'_\varrho \) denotes the derivative of \( a \in C^\infty_c(\mathbb{R}) \) and \( b'_1 \) denotes the partial derivative of \( b(\varrho_1, \varrho_2) \) with respect to \( \varrho_1 \). Further, we set in (72),

\[
R(x) := z^\ell + \bar{z}^\ell, \quad \ell \geq 2,
\]

(74)

and then obtain from (73) that

\[
H_z^2 = (2\ell a b'_1 |z|^2 + 2a' b'_1 |z|4)z^{\ell-2} + \ldots
\]

(75)

where \( \ldots \) stand for a sum of terms of the form

\[
c(\varrho_1, \varrho_2) |z|^{\mu} z^{2\ell-2} \quad \text{or} \quad c(\varrho_1, \varrho_2) |z|^{\mu} \bar{z}^{v}, \quad \nu \geq 2, \quad \mu \in \mathbb{Z}_{\geq 0},
\]

(76)

where \( c(\varrho_1, \varrho_2) \) (different for each term) has support in the annulus \( \varepsilon \leq |x| \leq 2\varepsilon \). Let us now set \( \ell := k' - 1, k' \geq 3 \). Then, it follows from (68), (70), (71), (75), and (76), that

\[
M' K^\varrho(u_0) = 4 \int_{\mathbb{R}^{d-2}} \left( \int_{\mathbb{R}^2} \left\{ (\partial_\zeta^3 P) H_z^2 + (\partial_{\bar{\zeta}}^3 P) H_{\bar{z}}^2 \right\} dx_\alpha dx_\beta \right) dx' = C \int_{\mathbb{R}^{d-2}} \left( \int_0^\infty \int_0^{2\pi} b'_1(\varrho_1, \varrho_2) (\ell a(\varrho) + r^2 a'_\varrho(\varrho)) r^{2\ell-1} d\varrho d\varrho_1 \right) dx' = C \pi \int_{\mathbb{R}^{d-2}} \left( \int_0^\infty b'_1(\varrho_1, \varrho_2) (\ell a(\varrho) + \varrho_1 a'_\varrho(\varrho)) \varrho_1^{\ell-1} d\varrho_1 \right) dx' = C \pi \int_{\mathbb{R}^{d-2}} \left( \int_0^\infty \partial_{\varrho_1}(b(\varrho_1, \varrho_2)) \partial_{\varrho_1}(a(\varrho)\varrho'^1) d\varrho_1 \right) dx'
\]

(77)
where \( \varrho = |x|^2 = \varrho_1 + \varrho_2, \varrho_1 = r^2, \varrho_2 = |x'|^2 \), and \( C \) is a non-zero constant depending on \( k' \geq 3 \). Finally, by choosing

\[
b(\varrho_1, \varrho_2) := a(\varrho)\varrho_1^\ell	ag{78}\]

we obtain from (77) that for any \( k' \geq 3 \) and for any \( 1 \leq \alpha \leq d \) and

\[
\beta := \begin{cases} 
d, & \alpha \neq d \\
1, & \alpha = d
\end{cases}
\]

we have that \( M_k^\alpha(u_0) \neq 0 \) where \( u_0 \) is given by (67) with Hamiltonian

\[
H(x) = \left( (z^{k' - 1} + \bar{z}^{k' - 1}) + |z|^{2k' - 2}\right) \tilde{a}(\varrho), \quad k' \geq 3,
\]

by (72), (74), and (78). Finally, we take \( \alpha = j \) and set \( H^i_j(x) := P(x) \) with \( P(x) = z^k + \bar{z}^k \). This completes the proof of the lemma.

**Proof of Proposition 1.1** Assume that \( m > 3 + d/p, 1 < p < \infty \). Let us first consider the case when \( \delta + d/p \geq d + 1 \) and \( \delta + d/p \not\in \mathbb{Z} \). Since the subcase when \( d = 2 \) follows from Proposition 1.2 (ii) in [30], we will concentrate our attention on the case when \( d \geq 3 \). Take \( d + 1 \leq k < \delta + d/p \) and \( 1 \leq j \leq d \). By Lemma 5.1 there exists a homogeneous harmonic polynomial \( H^i_j \) and a divergence free vector field \( u_0 \in C_c^\infty \) such that (see (66))

\[
M_k^j(u_0) = \int_{\mathbb{R}^d} H^i_j(x) \partial_j (Q(u_0)) \, dx = -\int_{\mathbb{R}^d} \sum_{1 \leq \alpha, \beta \leq d} \left( \partial_\beta \partial_\alpha \partial_j H^i_j \right) (u_{0\alpha}u_{0\beta}) \, dx
\]

(79)

does not vanish. Since \( d + 1 \leq k < \delta + d/p \), we conclude from (53) that \( (\partial_\beta \partial_\alpha \partial_j H^i_j) (u_{0\alpha}u_{0\beta}) \in L^1 \) for any \( u_0 \in W^{m,p}_\delta \) and for any \( 1 \leq \alpha, \beta \leq d \). Hence, the right side in (79) defines a non-vanishing bounded quadratic form on \( W^{m,p}_\delta \rightarrow \mathbb{R} \).

In particular, there exists an open dense set \( \mathcal{N} \) in \( W^{m,p}_\delta \) such that \( M_k^j(u_0) \neq 0 \) for any \( u_0 \in \mathcal{N} \). Let us now take \( u_0 \in \mathcal{N} \) (not necessarily the one from Lemma 5.1) and let \( u \) be the solution (6) of the Euler equation given by Theorem 1.2(b) with initial data \( u_0 \). Let \( N \) be the integer part of \( \delta + d/p \). Then, by Lemma 4.1 in [24], the solution \( u \in C([0, \tau], A^{m,p}_N) \cap C^1([0, \tau], A^{m-1,p}_N) \) satisfies the equation

\[
|t| + (u \cdot \nabla)u = \Delta^{-1}(\nabla \circ Q(u)), \quad u|_{t=0} = u_0.	ag{80}
\]

By taking \( t = 0 \) in (80) and then comparing the asymptotic terms in (80) we obtain from Proposition B.1 in Appendix B (cf. (87)) and \( (u_0 \cdot \nabla)u_0 \in W^{m-1,p}_\delta \) that

\[
\left( \partial_t a_k^i \right|_{t=0}, h^i_j \right)_{L^2(S^{d-1}, \mathbb{R})} = C_{d,k} \int_{\mathbb{R}^d} H^i_j(x) \partial_j (Q(u_0)) \, dx \equiv C_{d,k} M_k^j(u_0)	ag{81}
\]
where \( \partial_t a_{k}^{j} \big|_{t=0} \) is the \( t \)-derivative of the \( j \)-th component \( a_{k}^{j} \) of the \( k \)-th asymptotic coefficient of the expansion (6) of \( u \), \( C_{d,k} > 0 \) is a constant, and \( h_{k,j}'(\theta) := H_{k,j}^{j}(x)/r^{k'} \), \( k' \equiv k - d + 2 \geq 3 \). It now follows from (81) that the Fourier coefficient of \( \partial_t a_{k}^{j} \big|_{t=0} \) corresponding to the spherical harmonic \( h_{k,j}^{j} \) does not vanish. (We choose such an orthonormal basis in the eigenspace \( \mathcal{H}_{k'} \) of the Laplace operator \( -\Delta_{S} \) on the unit sphere \( S^{d-1} \) with eigenvalue \( \lambda_{k}(k'+d-2) \) (cf. Proposition B.1) that includes the normalized eigenfunction \( h_{k,j}^{j} \).) By combining this with the fact that the \( j \)-th component \( a_{k}^{j}(t) \in C \left(S^{d-1}, \mathbb{R}\right), t \in [0, \tau], \) of the asymptotic coefficient \( a_{k} \) of the solution (6) is analytic in time (cf. Theorem 1.2) we complete the proof of the proposition in the case when \( \delta + d/p \neq 3 \).

Finally, consider the case when \( \delta + d/p \geq 3 + 1 \) is an integer. Arguing by approximation, we conclude from the continuous dependence of \( u \) on the initial data in \( W_{d}^{m,p} \) that the integral formula (81) continues to hold. Since the quadratic form (79) is bounded in \( W_{d}^{m,p} \), the arguments above show that the proposition also holds in the case when \( \delta + d/p \in \mathbb{Z} \). The case when \( d = 2 \) follows in the same way from Lemma 4.3 in [30] and Proposition C.1 in Appendix C.  

Let us now prove Proposition 1.2.

Proof of Proposition 1.2 For a fixed weight \( \delta_{0} + d/p > 0 \) the independence of the interval of existence \([0, \tau], \tau > 0\), on the choice of the regularity exponent \( m \geq m_{0} \) follows as in Proposition 4.1 in [32] (see [12] for the original argument) and the analyticity of the map (71) in [24]. The independence of the interval of existence on the choice of the weight \( \delta \geq \delta_{0} \), for a given regularity exponent \( m \geq m_{0} \), can then be deduced from the preservation of vorticity (45), as in the proof of Theorem 1.2. We will omit the details of this proof.

Finally, we prove Corollary 1.1 stated in the Introduction.

Proof of Corollary 1.1 For any \( k \geq d + 1 \) and \( 1 \leq j \leq d \) consider the homogeneous harmonic polynomial \( H_{k,j}^{j} \) given by Lemma 5.1. It follows from (81) that for any initial data \( u_{0} \in \mathcal{S} \) the Fourier coefficient of \( \partial_t a_{k}^{j} \big|_{t=0} \) corresponding to the spherical harmonic \( h_{k,j}^{j}(\theta) \equiv H_{k,j}^{j}(x)/r^{k'} \) equals \( C_{d,k} M_{k,j}^{j}(u_{0}) \) where \( k' \equiv k - d + 2 \) and \( C_{d,k} > 0 \) is a constant. In view of (79), \( M_{k,j}^{j} : W_{d}^{m,p} \to \mathbb{R} \) is a bounded quadratic form on \( W_{d}^{m,p} \) for any \( m > 3 + d/p \), \( 1 < p < \infty \), and \( \delta + d/p > 3 + 1 \). This, together with Lemma 5.1, then implies that for any \( k \geq d + 1 \) and \( 1 \leq j \leq d \) we have that \( M_{k,j}^{j} : \mathcal{S} \to \mathbb{C} \) is a non-vanishing analytic map. Hence, for any \( k \geq d + 1 \) and \( 1 \leq j \leq d \), the zero set \( Z_{k,j}^{j} := \{ u \in \mathcal{S} \mid M_{k,j}^{j}(u) = 0 \} \) is nowhere dense in \( \mathcal{S} \). Since, \( \mathcal{S} \) is a complete metric space we then obtain from the Baire category theorem that the set

\[
\mathcal{N} := \bigcap_{k \geq d+1, 1 \leq j \leq d} \left( \mathcal{S} \setminus Z_{k,j}^{j} \right)
\]

is dense in \( \mathcal{S} \). This implies that for any \( k \geq d + 1 \) and \( 1 \leq j \leq d \), the asymptotic coefficient \( a_{k}^{j}(t) \) does not vanish in \( C \left(S^{d-1}, \mathbb{R}\right) \) for \( t > 0 \) taken sufficiently small.
Since, by Theorem 1.3, \( a^j_k \) depends analytically on \( t \in [0, \tau] \) we conclude that it vanishes only at finitely many \( t \in [0, \tau] \). This completes the proof of Corollary 1.1. \( \square \)

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Declarations

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A Auxiliary results

Most of the results in this Appendix are only generalizations to \( \gamma + d/p > -1 \) of results in [23] that assumed \( \gamma + d/p > 0 \). Rather than repeat the detailed proofs given in [23], we will simply describe how to generalize them to the case \( \gamma + d/p > -1 \).

In one instance, we generalize a statement from [14]. The following lemma follows directly from (3a).

Lemma A.1 If \( w \in W^{m,p}_\gamma \) with \( m > d/p \) and \( \gamma + d/p > -1 \), then

\[
C_1(x) \leq |x + w(x)| \leq C_2(x) \quad \text{for } x \in \mathbb{R}^d,
\]

where \( C_1, C_2 > 0 \) may be chosen locally uniformly for \( w \in W^{m,p}_\gamma \).

Local uniformity means that for any \( w_0 \in W^{m,p}_\gamma \) there exists an open neighborhood \( U \) of \( w_0 \) in \( W^{m,p}_\gamma \) such that the inequality (82) holds for any \( w \in U \).

Our next result is analogous to Lemma 6.3 in [23]. Let \( |d\varphi(x)| \) denote the sum of the absolute values of the elements of the matrix \( d\varphi(x) \) for \( x \in \mathbb{R}^d \).

Lemma A.2 If \( \varphi = \text{id} + w \in D^{m,p}_\gamma \) with \( m > 1 + d/p \) and \( \gamma + d/p > -1 \), then

\[
|d\varphi(x)| \leq C \quad \text{and} \quad 0 < \varepsilon \leq \det(d\varphi(x)) \quad \text{for } x \in \mathbb{R}^d,
\]

where \( C \) may be chosen uniformly for \( \|w\|_{W^{m,p}_\gamma} \leq M \) and \( \varepsilon \) may be chosen locally uniformly for \( w \in W^{m,p}_\gamma \).

Proof of Lemma A.2 It follows from (3a) that for \( m > 1 + d/p \) and \( \delta + d/p \geq -1 \),

\[
W^{m-1,p}_{\delta+1} \subseteq L^\infty
\]
is bounded. Take $\varphi = \text{id} + w \in D^{m,p}_\gamma$ with $m > 1 + d/p$ and $\gamma + d/p > -1$. Then, the first inequality in (83) follows from the boundedness of the inclusions

$$W^{m,p}_\gamma \hookrightarrow W^{m-q,p}_{\gamma+1} \subseteq L^\infty$$

and the fact that $d\varphi = I + dw$. The second inequality in (83) follows in a similar way from (84), $d\varphi = I + dw$, and (3a), since the latter implies that $|dw| = C \|w\|_{W^{m-1,p}_\delta} / (\delta + p/d)^{1+1}$ with $(\delta + p/d) + 1 > 0$. The estimates above a locally uniform for $\varphi \in D^{m,p}_\gamma$. \Box

The following corollary follows from Lemma A.2 and Hadamard–Levy’s theorem.

**Corollary A.1** If $\varphi_0 = \text{id} + w_0 \in D^{m,p}_\gamma$ where $m > 1 + d/p$, $\gamma + d/p > -1$, and $\tilde{w} \in W^{m,p}_\gamma$ with $\|\tilde{w}\|_{W^{m,p}_\gamma}$ sufficiently small, then $\varphi_0 + \tilde{w} \in D^{m,p}_\gamma$. In particular, the set of maps $D^{m,p}_\gamma$ can be identified with an open set in $W^{m,p}_\gamma$. Hence, $D^{m,p}_\gamma$ is a Banach manifold modeled on $W^{m,p}_\gamma$.

For $\varphi \in D^{m,p}_\gamma$, we know that $\varphi^{-1}$ exists but we need estimates at infinity in order to conclude that $\varphi^{-1} \in D^{m,p}_\gamma$. The following is a first step and is analogous to Lemma 6.4 in [23].

**Lemma A.3** If $\varphi = \text{id} + w \in D^{m,p}_\gamma$ where $m > 1 + d/p$ and $\gamma + d/p > -1$, then

$$|d(\varphi^{-1})(x)| \leq C \quad 0 < \varepsilon \leq \det (d(\varphi^{-1})(x)) \quad \text{for } x \in \mathbb{R}^d,$$

where $C$ and $\varepsilon$ may be chosen locally uniformly for $w \in W^{m,p}_\gamma$.

**Proof of Lemma A.3** The lemma follows from Lemma A.2 and the formula

$$d(\varphi^{-1}) = [(d\varphi) \circ \varphi^{-1}]^{-1}$$

for the Jacobian matrix of $\varphi^{-1}$. \Box

Now let us consider compositions of maps as in Theorem 2.1. We begin with an estimate.

**Lemma A.4** Suppose $m > 1 + d/p$, $\gamma + d/p > -1$ and $\varphi \in D^{m,p}_\gamma$. Then for every $0 \leq k \leq m$ and $\delta \in \mathbb{R}$, we have

$$\|w \circ \varphi\|_{W^{k,p}_\delta} \leq C \|w\|_{W^{k,p}_\delta} \quad \text{for all } f \in W^{k,p}_\delta,$$

where $C$ may be taken locally uniformly for $\varphi \in D^{m,p}_\gamma$.

**Proof of Lemma A.4** This is proved by induction using Lemma A.3 for a change of integration variable when $k = 0$ and Proposition 2.2 in [23] to handle products in the induction step. For details see the proof of Lemma 6.5 in [23]. \Box
Lemma A.5 Assume $m > 1 + d/p$, $γ + d/p > −1$, $δ ∈ \mathbb{R}$, and $f ∈ C^∞_c(\mathbb{R}^d)$. If $φ, φ_k ∈ D^m_Y$ with $φ_k → φ$ in $D^{m,p}_Y$ as $k → ∞$, then $f ∘ φ_k → f ∘ φ$ in $W^m_{δ,p}(\mathbb{R}^d)$.

Proof of Lemma A.5 The only difference from the proof of Lemma 6.6 in [23] is that $w_k(x)$ and $w(x)$ in $φ_k(x) = x + w_k(x)$ and $φ(x) = x + w(x)$ are of order $O(|x|^{1−ε})$ for some $ε > 0$ instead of just being bounded. But the rest of the proof in [23] can be used here without change. □

Lemmas A.4 and A.5 can be combined to obtain the continuity of the composition.

Corollary A.2 Suppose $m > 1 + d/p$, $γ + d/p > −1$ and $δ ∈ \mathbb{R}$. Then the composition $(f, φ) → f ∘ φ$, $W^{m,p}_δ$ is continuous.

Proof of Corollary A.2 The details are the same as in the proof of Corollary 6.1 in [23]. □

Next we investigate when the composition is $C^1$.

Lemma A.6 Assume $m > 1 + d/p$, $γ + d/p > −1$, $δ ∈ \mathbb{R}$, and take $φ_0 ∈ D^m_Y$. For $f ∈ W^{m+1,p}_δ$ and $φ ∈ D^m_Y$ sufficiently close to $φ_0$, we have

$$\|f ∘ φ − f ∘ φ_0\|_{W^{m,p}_δ} ≤ C \|f\|_{W^{m+1,p}_δ} \|φ − φ_0\|_{W^{m,p}_δ},$$

where $C > 0$ can be taken uniformly for all $φ$ in an open neighborhood of $φ_0$.

Proof of Lemma A.6 The details are the same as in the proof of Lemma 6.7 in [23]. □

Corollary A.3 Suppose $m > 1 + d/p$, $γ + d/p > −1$ and $δ ∈ \mathbb{R}$. Then the composition $(f, φ) → f ∘ φ$, $W^{m+1,p}_δ$ is $C^1$.

Proof of Corollary A.3 Using the above lemmas, Corollary A.3 can be proved following the proof of Proposition 5.1 in [23] (cf. also Appendix B in [32]). □

This completes the proof of Theorem 2.1(a). Let us now prove Theorem 2.1(b).

Lemma A.7 If $φ ∈ D^m_Y$ where $m > 1 + d/p$ and $γ + d/p > −1$, then $φ^{-1} ∈ D^m_Y$.

Proof of Lemma A.7 Let $φ = id + w$ and $φ^{-1} = id + u$. To show $φ^{-1} ∈ D^m_Y$, we need to show $∂^α u ∈ L^p_{γ+|α|}$ for all $|α| ≤ m$. For $α = 0$ we use the change of variables $x = φ(y)$ to compute

$$\int_{\mathbb{R}^d} (x)^{γ,p} |u(x)|^p dx = \int_{\mathbb{R}^d} (φ(y))^{γ,p} |φ^{-1}(x) − x|^p dx = \int_{\mathbb{R}^d} (φ(y))^{γ,p} |w(y)|^p det(∂φ(y)) dy ≤ C \int (y)^{γ,p} |w(y)|^p dy < ∞,$$

where we have used (82), (83), and $w = u ∘ φ ∈ W^{m,p}_Y$ (cf. Lemma A.4). For $1 ≤ |α| ≤ m$, we can proceed as in (28) in [14] to show

$$∂^α (φ^{-1} − id) = F^{(α)} ∘ φ^{-1},$$

(85)
where \( F(\alpha) : \mathbb{R}^d \to \mathbb{R}^d \) is in \( W^{m-|\alpha|,p}_\gamma \). Then, by (85), for any \( \alpha \) with \( 0 \leq |\alpha| \leq m \),

\[
\int (x)^{(|\gamma|+|\alpha|)} p |\partial^\alpha (\varphi^{-1}(x) - x)|^p \, dx = \int (\varphi(y))^{(|\gamma|+|\alpha|)} p |F(\alpha)(y)|^p \, dy \leq C \int (y)^{(|\gamma|+|\alpha|)} p |F(\alpha)(y)|^p \, dy < \infty.
\]

Hence \( \varphi^{-1} - \text{id} \in W^{m,p}_\gamma \). This implies that \( \varphi \in D^{m,p}_\gamma \).

The continuity of the map \( \varphi \mapsto \varphi^{-1}, D^{m,p}_\gamma \to D^{m,p}_\gamma \), now follows from Corollary A.2 and Theorem 2 in [26]. The last statement in Theorem 2.1 (b) can be proved by following the proof of Proposition 2.13 in [14].

**Remark A.1** Note that the regularity assumption \( m > 1 + d/p \) in Lemma A.7 above is weaker than the regularity assumption \( m > 3 + d/p \) in the analogous [23, Lemma 7.2] and [30, Lemma 2.5]. This happens since in [23,30] one deals with the asymptotic expansion of \( \varphi^{-1} \), that complicates the situation. Note however, that the results in [23,30] can be extended to the case when \( m > 1 + d/p \) by expanding their proofs.

**B Inverting the Laplace operator**

In this Appendix we present, in an extended form, a basic result about the inversion of the Laplace operator in weighted Sobolev spaces (see Lemma A.3 in [24]). Denote by \( S' \) the space of tempered distributions in \( \mathbb{R}^d \).

**Proposition B.1** Assume that \( d \geq 3 \) and \( m \geq 0 \) with \( 1 < p < \infty \). Then, for any \( g \in W^{m,p}_\delta(\mathbb{R}^d, \mathbb{R}) \) with weight \( \delta \in \mathbb{R} \) such that \( \delta + d/p > 0, \delta + d/p \not\in \mathbb{Z} \), there exists a unique (up to adding a constant term) solution \( u \) in \( S' \cap L^\infty \) of the Poisson equation

\[
\Delta u = g
\]

such that

\[
u_k(\theta) = \frac{a_k(\theta)}{r^k} f, \quad f \in W^{m+2,p}_\delta,
\]

where \( a_k(\theta) \) is an eigenfunction of the Laplace operator \( -\Delta_S \) on the unit sphere \( S^{d-1} \) in \( \mathbb{R}^d \) with eigenvalue \( \lambda_{k-d+2} = k(k - d + 2) \). If we fix for any \( k' := k - d + 2 \geq 0 \) with \( d - 2 \leq k < \delta + d/p \) an orthonormal basis

\[
\{ h_{k',l}(\theta) \mid 1 \leq l \leq \nu(k') \}, \quad \nu(k') := \dim \mathcal{H}_{k'},
\]

As noted in [26], the completeness condition in [26, Theorem 2] can be replaced by local completeness.

For the case when \( d = 2 \) we refer to Proposition 3.3 in [30].
of the eigenspace $\mathcal{H}_{k'}$ of $-\Delta_S$ with eigenvalue $\lambda_{k'} = k'(k' + d - 2)$ and expand

$$a_k(\theta) = \sum_{l=1}^{v(k')} \hat{a}_{k';l} h_{k';l}(\theta)$$

in the Fourier modes, then the Fourier coefficient $\hat{a}_{k';l}$ equals

$$\hat{a}_{k';l} = -\frac{1}{2k' + d - 2} \int_{\mathbb{R}^d} g(x) H_{k';l}(x) \, dx, \quad 1 \leq l \leq v(k'),$$

where $H_{k';l}(x) := h_{k';l}(\theta)r^{k'}$ is the homogeneous harmonic polynomial (of degree $k' \geq 0$) that corresponds to the Fourier mode $h_{k';l}(\theta)$.\(^{13}\) If we coordinatize the linear space $I^m_{\delta,p}$ of functions of the form (86) by the Fourier coefficients of $a_k(\theta)$, $d - 2 \leq k < \delta + d / p$, and the reminder $f \in W^m_{\delta,p}$ then the map

$$\Delta^{-1} : W^{m+2,p}_{\delta+2} \to I^{m,p}_{\delta}$$

is an isomorphism of Banach spaces.

**Remark B.1** Note that the eigenvalues of the Laplace operator on the unit sphere $S^{d-1}$ in $\mathbb{R}^d$ are highly degenerate: we have that $\dim \mathcal{H}_0 = 1$ (these are all constants), $\dim \mathcal{H}_1 = d$ (all linear polynomials restricted to $S^{d-1}$), and

$$\dim \mathcal{H}_{k'} = \binom{d - 1 + k'}{d - 1} - \binom{d - 3 + k'}{d - 1}, \quad k' \geq 2,$$

where the second binomial coefficient above vanishes when $d = 2$.

**Proof of Proposition B.1** The fact that there exists a unique solution $u \in \mathcal{H}' \cap L^\infty$ of the form (86) follows from [24, Lemma A.3(b)] (cf. also [22]) and the fact that there is a bijective correspondence between the eigenfunctions of the Laplace operator $-\Delta_S$ on the unit sphere $S^{d-1}$ in $\mathbb{R}^d$ with eigenvalue $\lambda_{k'} = k'(k' + d - 2)$ and the restriction to $S^{d-1}$ of harmonic polynomials of degree $k' \geq 0$ ([31, §22.2]). Let us now prove the integral relation (87). To this end, take $d - 2 \leq k < \delta + d / p$, $d - 2 \leq n < \delta + d / p$, $1 \leq l_1 \leq v(k')$, $1 \leq l_2 \leq d(n')$ with $n' := n - d + 2$, and consider the $k'$-th asymptotic term $A_{k';l} := \chi h_{k';l} / r^{k'}$ in (86). Then, it follows from the second Green’s identity and the fact that the eigenspaces $\mathcal{H}_{k'}$ and $\mathcal{H}_{n'}$ are $L^2$-orthogonal on $S^{d-2}$ for $k' \neq n'$, that

$$\int_{\mathbb{R}^d} \Delta(A_{k';l_1}) H_{n';l_2} \, dx = \lim_{R \to \infty} \int_{S^{d-1}} \left( \frac{\partial A_{k';l_1}}{\partial r} H_{n';l_2} - A_{k';l_1} \frac{\partial H_{n';l_2}}{\partial r} \right) \, d\sigma_R$$

$$= C(k, n) \lim_{R \to \infty} R^{n' - k'} \int_{S^{d-1}} h_{k';l_1} h_{n';l_2} \, d\sigma_1 = C(k, n) \delta_{kn}$$

(88)

\(^{13}\) Note that $g h_{k';l} \in L^1(\mathbb{R}^d, \mathbb{R})$. 
where $C(k, n) := -(k' + n' + d - 2) \neq 0$, $S_{R}^{d-1}$ is the sphere of radius $R$, and $\delta_{k}n$ is the Kronecker delta. Now, consider the remainder $f \in W_{b}^{m+2, p}$ of the solution $u$ in (86). For any $d - 2 \leq n < \delta + d / p$ and $1 \leq l \leq d(n')$ we have that $\Delta(f)H_{n'; l} \in W_{b}^{m+2, p}$. This, together with the estimate (3a) implies that $\Delta(f)H_{n'; l} = O(1/r^{\delta+(d/p)+2-n'})$, and hence $\Delta(f)H_{n'; l} \in L^1(\mathbb{R}^d)$ by the estimate $\delta + (d/p) + 2 - n' > d$. By arguing in the same way as above, we also have

\[
\int_{\mathbb{R}^d} \Delta(f)H_{n'; l} \, dx = \lim_{R \to \infty} \int_{S_R^d} \left( \frac{\partial f}{\partial r} H_{n'; l} - f \frac{\partial H_{n'; l}}{\partial r} \right) \, d\sigma_R
\]

\[
= \lim_{R \to \infty} \int_{S_R^d} O(1/R^{\delta+(d/p)-n'+1}) \, d\sigma_R = \lim_{R \to \infty} O(1/R^{\delta+(d/p)-n}) = 0, \quad (89)
\]

where we used that $n < \delta + d/p$ and the estimate (3a) on the decay of $f$ at infinity. Finally, the integral formula (87) follows from (88), (89), and (86).

\section{C Asymptotic spaces $A_{n, N; \ell}^{m, p}$}

In this section we will discuss the asymptotic spaces with log terms that are used in the proof of Theorem 1.2. For integers $m > d/p$, $0 \leq n \leq N$, and $\ell \geq -n$, let $A_{n, N; \ell}^{m, p}$ denote functions $u$ on $\mathbb{R}^d$ of the form

\[
\chi(r) \left( \frac{a_0^0(\theta)}{r^n} + \cdots + \frac{a_n^{n+\ell}(\theta)(\log r)^{n+\ell}}{r^n} + \cdots + \frac{a_N^0(\theta)}{r^N} + \cdots + \frac{a_N^{n+\ell}(\theta)(\log r)^{N+\ell}}{r^N} \right) + f(x),
\]

(90a)

where $a_k^j \in H^{m+1+N-k, p}(S_{S_R^d}^{d-1}, \mathbb{R}^d)$ for $0 \leq j \leq k + \ell$, $0 \leq n \leq k \leq N$, and $f \in W_{\gamma N}^{m, p}$ with $N \leq \gamma N + d/p < N + 1$ so, by (3b), $f(x) = o(r^{-N})$ as $r \to \infty$. This is a Banach space with norm

\[
\|u\|_{A_{n, N; \ell}^{m, p}} = \sum_{0 \leq j \leq k + \ell, n \leq k \leq N} \|a_k^j\|_{H^{m+1+N-k, p}} + \|f\|_{W_{\gamma N}^{m, p}}.
\]

(90b)

Note that $W_{\gamma N}^{m, p}$ is a closed subspace of $A_{n, N; \ell}^{m, p}$. It is easy to confirm that the following inclusions are bounded for $N \leq \gamma N + d/p < N = 1$:

\[
A_{n_1, N_1; \ell_1}^{m, p} \subseteq A_{n, N; \ell}^{m, p} \quad \text{if } n_1 \geq n, \ N_1 \geq N, \ \ell \geq \ell_1 \geq -n, \quad (91a)
\]

\[
\partial x_j : A_{n, N; \ell}^{m, p} \to A_{n+1, N+1; \ell-1}^{m-1, p} \quad \text{if } m > 1 + d/p, \ 1 \leq j \leq d. \quad (91b)
\]

Moreover, if $n = n_1 + n_2$, $\ell = \ell_1 + \ell_2$, and $N < \min(N_1 + n_2, N_2 + n_1)$, then pointwise multiplication $(u, v) \mapsto uv$,

\[
A_{n_1, N_1; \ell_1}^{m, p} \times A_{n_2, N_2; \ell_2}^{m, p} \to A_{n, N; \ell}^{m, p}, \quad (91c)
\]
is a bounded bilinear map. When \( \ell_i = -n_i \) there are no log terms in the leading asymptotic and we have the sharper version with \( N = \min(N_1 + n_2, N_2 + n_1) \), \( n = n_1 + n_1 \):

\[
\mathcal{A}_{n_1, N_1; -n_1}^{m, p} \times \mathcal{A}_{n_2, N_2; -n_2}^{m, p} \to \mathcal{A}_{n, N; \ell}^{m, p}
\]

(91d)

These may be combined to conclude that

\[
\mathcal{A}_{n, N; \ell}^{m, p} \text{ is a Banach algebra}
\]

(91e)
in the case when \( n \geq 1 \), or when \( \ell = -n \). For more details, see Appendix B in [23] (for the case when \( N < \gamma_N < N + 1 \)).

Analogous to (14), for \( m > 1 + d/p \) we introduce asymptotic diffeomorphisms

\[
\mathcal{AD}_{n, N; \ell}^{m, p} := \{ \varphi : \mathbb{R}^d \to \mathbb{R}^d \mid \varphi = \text{id} + w, \ w \in \mathcal{A}_{n, N; \ell}^{m, p} \text{ and } \det(d\varphi) > 0 \}.
\]

(92)

Analogous to Theorem 4.1 above, Theorem 6.1 in [24] shows for \( m > 3 + d/p \) and \( 0 \leq n \leq \min(d + 1, N) \) that the Euler vector field

\[
\mathcal{E} : \mathcal{AD}_{n, N; 0}^{m, p} \times \mathcal{A}_{n, N; 0}^{m, p} \to \mathcal{AD}_{n, N; 0}^{m, p} \times \mathcal{A}_{n, N; 0}^{m, p}
\]

(93)
is smooth. Theorem 1.1 in [24] uses this vector field as we did in Sect. 4 to show the existence of a unique solution as in (43). However, all results in [24] are under the assumption \( N < \gamma_N + d/p < N + 1 \).

We now show that this solvability of the Euler equations in \( \mathcal{A}_{n, N; 0}^{m, p} \) also holds for \( N \leq \gamma_N + d/p < N + 1 \), at least when \( n = 1 \).

**Proposition C.1** Assume \( m > 3 + d/p, N \geq 1 \), and \( N \leq \gamma_N < N + 1 \). Then, for any given \( \rho > 0 \) there exists \( \tau > 0 \) such that for any \( u_0 \in \hat{\mathcal{A}}_{1, N; 0}^{m, p} \) with \( \| u_0 \| \mathcal{A}_{1, N; 0}^{m, p} < \rho \) there exists a unique solution of the Euler equation

\[
u \in C([0, \tau], \hat{\mathcal{A}}_{1, N; 0}^{m, p}) \cap C^1([0, \tau], \hat{\mathcal{A}}_{1, N; 0}^{m, -1, p}),
\]

(94)

that depends continuously on the initial data \( u_0 \).

**Proof of Proposition C.1** As previously stated, this was proved in [24] when \( N < \gamma_N + d/p < N + 1 \), so we need only consider the case when \( \gamma_N + d/p = N \). To do this, let us change notation within this proof and denote by \( \mathcal{A}_{n, N; 0}^{m, p} \) the asymptotic space \( \mathcal{A}_{n, N; \ell}^{m, p} \) with \( \gamma_N + d/p = N \); we reserve the notation \( \hat{\mathcal{A}}_{n, N; 0}^{m, p} \) for an asymptotic space with a remainder \( f \in W_{\gamma_N}^{m, p} \) with \( N < \gamma_N < N + 1 \). In particular, referring to (91a), we have the following bounded inclusions for any \( \ell \geq -n \):

\[
\mathcal{A}_{n, N; \ell}^{m, p} \subseteq \mathcal{Q}_{n, N; \ell}^{m, p} \subseteq \mathcal{A}_{n, N; -1; \ell}^{m, p}.
\]

(95)

We will need to take the square of elements in \( \mathcal{Q}_{n, N; \ell}^{m, p} \). For \( n + \ell > 0 \), we know from (91c) and (91d) that the log terms in the leading asymptotic prevent us from keeping...
the same order of decay; but decreasing this order of decay by one enables us to also embed in a space with $0 < \gamma_0 + d/p < 1$:

$$u \mapsto u^2 \text{ is smooth } \mathcal{A}^{m,p}_{n,N;\ell} \rightarrow \mathcal{A}^{m,p}_{2n,N+n+1;2\ell}. \quad (96)$$

Using (91b), for $u \in \mathcal{A}^{m,p}_{1,N;0}$ we have $du \in \mathcal{A}^{m-1,p}_{2,N+1;1}$ so (96) implies that $Q(u) = \text{tr} (du)^2$ defines a bounded quadratic polynomial map $Q : \mathcal{A}^{m,p}_{1,N;0} \rightarrow \mathcal{A}^{m-1,p}_{4,N+2;2} \subseteq \mathcal{A}^{m-1,p}_{2,N+2;2}$. Consequently, the maps

$$Q : \mathcal{A}^{m,p}_{1,N;0} \rightarrow \mathcal{A}^{m-1,p}_{2,N+2;2} \quad \text{and} \quad \nabla \circ Q : \mathcal{A}^{m,p}_{1,N;0} \rightarrow \mathcal{A}^{m-2,p}_{3,N+3;3} \quad (97)$$

are smooth. We can now apply Proposition 3.1 and (17b) in [24] to conclude that the linear map $\Delta^{-1} : \mathcal{A}^{m-2,p}_{3,N+3;3} \rightarrow \mathcal{A}^{m,p}_{1,N+1;0}$ is bounded and injective. Combined with (97) and the embeddings $\mathcal{A}^{m,p}_{1,N+1;0} \subseteq \mathcal{A}^{m,p}_{1,N;0} \subseteq \mathcal{A}^{m,p}_{1,N;0}$ we see that

$$\Delta^{-1} \circ \nabla \circ Q : \mathcal{A}^{m,p}_{1,N;0} \rightarrow \mathcal{A}^{m,p}_{1,N;0}$$

is smooth. The arguments in [24, Section 4, 5, and 6] (cf. Lemma 5.2, Proposition 5.1, Lemma 6.1, and Theorem 6.1 in [24]) then show that the associated conjugate map

$$(\varphi, f) \mapsto (R_\varphi \circ \Delta^{-1} \circ R_{\varphi^{-1}}) \circ (R_\varphi \circ \nabla \circ Q \circ R_{\varphi^{-1}})(f),$$

$$\mathcal{A}^{m,p}_{1,N;0} \times \mathcal{A}^{m,p}_{1,N;0} \rightarrow \mathcal{A}^{m,p}_{1,N;0} \times \mathcal{A}^{m,p}_{1,N;0},$$

is smooth. This implies that the Euler vector field $\mathcal{E}$ is smooth as a map

$$\mathcal{E} : \mathcal{A}^{m,p}_{1,N;0} \times \mathcal{A}^{m,p}_{1,N;0} \rightarrow \mathcal{A}^{m,p}_{1,N;0} \times \mathcal{A}^{m,p}_{1,N;0}. \quad (98)$$

The arguments in the proof of Theorem 1.1 above then complete the proof of the proposition (cf. also Section 7 in [24]). \qed

### D Global existence in the case when $d = 2$

In this section we generalize Theorem 1.1 in [30] and prove that for $d = 2$ the solution of the Euler equation (1) has a unique global in time solution in the asymptotic space $Z^{m,p}_N$ with weight $\gamma_N$ such that $\gamma_N + d/p > 0$ is integer.\footnote{In particular, we obtain an alternative proof of Proposition C.1 in the case $d = 2$.} For a given $a \in \mathbb{R}$ denote by $[a]$ the integer part of $a$. We will follow the notation introduced in [30, Section 2].

For a given $1 < p < \infty$, $m > 2/p$, and $\delta + 2/p > 0$ we set $N := [\delta + 2/p]$, $\gamma_N := \delta$, and consider the space of complex valued functions of $z \in \mathbb{C}$,

$$Z^{m,p}_{n,N} := \left\{ \chi \sum_{n \leq k+l \leq N} \frac{a_{kl}}{z^k \bar{z}^l} + f \bigg| f \in W^{m,p}_{\gamma_N}, a_{kl} \in \mathbb{C} \right\}, \quad (99)$$
where \(0 \leq n \leq N + 1\) and where we omit the summation term if \(n = N + 1\) and set \(\mathcal{Z}_{n,N}^{m,p} = W_{\gamma_n}^{m,p}\). We also set \(\mathcal{Z}_N^{m,p} = \mathcal{Z}_{0,N}^{m,p}\). The space (99) is a closed subspace in the asymptotic space \(\mathcal{A}_N^{m,p}\) of vector fields on \(\mathbb{R}^2\) that satisfies Proposition 2.1 and 2.2 in [30]. Note however, that Proposition 3.3 and Theorem 3.2 in [30] does not hold for integer \(\delta + 2/p\). As a consequence, the proof of the global well-posedness of the Euler equation for \(d = 2\) in [30, Section 5] does not apply for integer values of \(\delta + 2/p\). Following [30, Section 2] we denote the group of diffeomorphisms of \(\mathbb{R}^2\) modeled on \(\mathcal{Z}_N^{m,p}\) by \(\mathcal{Z}D_N^{m,p}\). First, we prove the following lemma.

**Lemma D.1** Take \(m > 3 + 2/p\), a non-integer \(\delta + 2/p > 0\), and let \(\hat{\delta}\) be the lowest integer \(\hat{\delta} > \delta\) such that \(\hat{\delta} + 2/p \in \mathbb{Z}\). Then, for a given volume preserving \(\varphi \in \mathcal{Z}D_M^{m,p}\) (with \(\gamma_M := \delta\), \(M := \lfloor \delta + 2/p \rfloor\)) and \(u_0 \in \mathcal{Z}_N^{m,p}\) (with \(\gamma_N := \hat{\delta}\), \(N := \hat{\delta} + 2/p\)) we have that

\[
\left( R_\varphi \circ \partial_{\hat{\delta}}^{-1} \circ R_{\varphi^{-1}} \right)(\partial_{\hat{\delta}} u_0) = u_0 + R(\varphi, u_0), \quad R(\varphi, u_0) \in \mathcal{Z}_{1,M+1}^{m,p},
\]

where the map \(R : \mathcal{Z}D_M^{m,p} \times \mathcal{Z}_N^{m,p} \to \mathcal{Z}_{M+1}^{m,p}\) is analytic and \(\partial_{\hat{\delta}}\) denotes the Cauchy operator \(\partial_{\hat{\delta}} : \mathcal{Z}_{1,M}^{m,p} \to \mathcal{Z}_{M+1}^{m,p}\).

**Proof of Lemma D.1** Since \(u_0 \in \mathcal{Z}_N^{m,p}\) (with \(\gamma_N = \delta\)) we have that \(\partial_{\hat{\delta}} u_0 \in \mathcal{Z}_{1,M+1}^{m,p}\) and, by Proposition 3.4 in [30], \(\left( R_\varphi \circ \partial_{\hat{\delta}}^{-1} \circ R_{\varphi^{-1}} \right)(\partial_{\hat{\delta}} u_0)\) is well defined and belongs to \(\mathcal{Z}_{1,M}^{m,p}\). By setting \(w := \left( R_\varphi \circ \partial_{\hat{\delta}}^{-1} \circ R_{\varphi^{-1}} \right)(\partial_{\hat{\delta}} u_0)\) we then obtain from Lemma 2.4 in [30] that

\[
\left( R_\varphi \circ \partial_{\hat{\delta}} \circ R_{\varphi^{-1}} \right)(w) = \partial_{\hat{\delta}} u_0.
\]

This, together with formula (54) in [30] and the fact that \(\varphi = \text{id}_C + u \in \mathcal{Z}D_M^{m,p}\) is volume preserving, then implies that \(\partial_{\hat{\delta}} w + (\partial_{\hat{\delta}} w)(\partial_{\hat{\delta}} u_0) - (\partial_{\hat{\delta}} w)(\partial_{\hat{\delta}} u_0) = \partial_{\hat{\delta}} u_0\), or equivalently,

\[
w = u_0 + \partial_{\hat{\delta}}^{-1}\left[(\partial_{\hat{\delta}} w)(\partial_{\hat{\delta}} u_0) - (\partial_{\hat{\delta}} w)(\partial_{\hat{\delta}} u_0)\right].
\]

Since, by Lemma 3.5 in [30], \(w \equiv \left( R_\varphi \circ \partial_{\hat{\delta}}^{-1} \circ R_{\varphi^{-1}} \right)(\partial_{\hat{\delta}} u_0) \in \mathcal{Z}_{1,M}^{m,p}\) depends analytically on \((\varphi, u_0) \in \mathcal{Z}D_M^{m,p} \times \mathcal{Z}_N^{m,p}\), we obtain from Proposition 2.2 in [30] that

\[
(\partial_{\hat{\delta}} w)(\partial_{\hat{\delta}} u_0) - (\partial_{\hat{\delta}} w)(\partial_{\hat{\delta}} u_0) \in \mathcal{Z}_{M+2}^{m,p+1}
\]

and depends analytically on \((\varphi, u_0) \in \mathcal{Z}D_M^{m,p} \times \mathcal{Z}_N^{m,p}\). By combining this with Theorem 3.2 in [30] we then see that

\[
R(\varphi, u_0) := \partial_{\hat{\delta}}^{-1}\left[(\partial_{\hat{\delta}} w)(\partial_{\hat{\delta}} u_0) - (\partial_{\hat{\delta}} w)(\partial_{\hat{\delta}} u_0)\right] \in \mathcal{Z}_{1,M+1}^{m,p+1}
\]

\[\text{15 Theorem 3.2 in [30] states that this map is an isomorphism; see (16) in [30] for the definition of the space } \mathcal{Z}_{M+1}^{m,p+1}.\]
Perfect fluid flows on $\mathbb{R}^d$ with growth/decay conditions at infinity

and depends analytically on $(\varphi, u_0) \in ZD_M^{m,p} \times Z_N^{m,p}$. This completes the proof of the lemma.

Now, we are ready to prove

**Proposition D.1** Assume that $m > 3 + 2/p$, $\delta + 2/p > 0$ is an integer, and $d = 2$. Then, for any $u_0 \in \mathbb{Z}_N^{m,p}$ (with $\gamma_N := \delta$ and $N := \delta + 2/p$) the Euler equation (1) has a unique global in time solution $u \in C([0, \infty), \mathbb{Z}_N^{m,p}) \cap C^1([0, \infty), \mathbb{Z}_N^{m-1,p})$ that depends continuously on the initial data (cf. [30, Theorem 1.1] for the case when $\gamma_N + 2/p$ is not integer).

**Proof of Proposition D.1** Assume that $\delta + 2/p > 0$ is an integer and choose $\delta^- \in \mathbb{R}$ such that $0 < \delta - \delta^- < 1$ and $\delta^- + 2/p > 0$. Take $u_0 \in \mathbb{Z}_N^{m,p}$ (with $\gamma_N = \delta$). Since $\mathbb{Z}_N^{m,p}$ is a subspace in $Z_M^{m,p}$ (with $\gamma_M := \delta^-$ and $M := [\delta^- + 2/p] = N - 1$) and since $\delta^- + 2/p$ is not integer, we conclude from [30, Theorem 1.1] that there exists a unique solution of the Euler equation

$$u \in C([0, \infty), Z_M^{m,p}) \cap C^1([0, \infty), Z_M^{m-1,p})$$

that depends continuously on the initial data $u_0 \in Z_N^{m,p}$. By [30, Proposition 4.2], $\varphi \in C^1([0, \infty), Z_D^{m,p})$ where $\varphi = u \circ \varphi$, $\varphi|_{t=0} = u_0$, and it depends continuously on the initial data $u_0 \in Z_N^{m,p}$. The preservation of vorticity (cf. formula (76) in [30]) and Lemma D.1 then imply that

$$u = \partial_z^{-1}((\partial_z u_0) \circ \varphi^{-1}) = R_{\varphi^{-1}} \circ (R_{\varphi} \circ \partial_z^{-1} \circ R_{\varphi^{-1}})(\partial_z u_0)$$

$$= u_0 \circ \varphi^{-1} + \mathcal{R}(\varphi, u_0) \circ \varphi^{-1}$$

(102)

where $\mathcal{R}(\varphi, u_0) \in Z_1^{m,p}$ and it depends analytically on $(\varphi, u_0) \in Z_D^{m,p} \times Z_N^{m,p}$. Since $\gamma_M + 1 > \delta$ we have that $Z_M^{m,p} \subseteq Z_N^{m,p}$. By Proposition 2.3 and Proposition 2.4 in [30] we then obtain that

$$u \in C([0, \infty), Z_N^{m,p}) \cap C^1([0, \infty), Z_N^{m-1,p})$$

and it depends continuously on the initial data $u_0 \in Z_N^{m,p}$. This completes the proof of the proposition. \qed

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