THE MODULI SPACES OF SHEAVES ON K3 SURFACES ARE IRREDUCIBLE SYMPLECTIC VARIETIES

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Abstract. We show that the moduli spaces of sheaves on a projective K3 surface are irreducible symplectic varieties, and that the same holds for the fibers of the Albanese map of moduli spaces of sheaves on an Abelian surface.

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1. INTRODUCTION AND MAIN RESULTS

A holomorphic symplectic form on a complex manifold \(X\) is an everywhere nondegenerate, closed, holomorphic 2-form on \(X\). A complex manifold admitting a holomorphic symplectic form is called holomorphic symplectic manifold. We let \(h^{p,0}(X)\) be the dimension of the vector space \(H^0(X, \Omega^p_X)\).

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A connected compact Kähler manifold $X$ is an irreducible symplectic manifold if it is holomorphic symplectic, simply connected and $h^{2,0}(X) = 1$. In particular, an irreducible symplectic manifold has even complex dimension and trivial canonical bundle. For this kind of manifolds several results are known, namely:

1. the $\mathbb{Z}$—module $H^2(X, \mathbb{Z})$ is free, has a pure weight-two Hodge structure and a nondegenerate integral quadratic form $q$ of signature $(3, b_2(X) - 3)$, called the Beauville form of $X$ (see [2]);

2. there is a positive rational number $F_X$, called Fujiki constant of $X$, such that for every $\alpha \in H^2(X, \mathbb{Z})$ we have
   $$\int_X \alpha^{2n} = F_X q(\alpha)^n,$$
   where $2n$ is the complex dimension of $X$ (see [12]). As a consequence, the Beauville form and the Fujiki constant are deformation invariant;

3. a Local Torelli Theorem holds: if we let $Def(X)$ be the base of a Kuranishi family of $X$ and
   $$\Omega_X := \{ \alpha \in \mathbb{P}(H^2(X, \mathbb{C})) | q(\alpha) = 0, \ q(\alpha + \overline{\alpha}) > 0 \},$$
   there is a holomorphic map $p : Def(X) \longrightarrow \Omega_X$, called the period map, which is a local biholomorphism (see again [2]);

4. a version of a Global Torelli Theorem was shown by Verbitsky and Markman: two irreducible symplectic manifolds $X$ and $Y$ are bimeromorphic if and only if there is a parallel transport operator between $H^2(X, \mathbb{Z})$ and $H^2(Y, \mathbb{Z})$ which is a Hodge isometry (see [53], [31], [20]).

Irreducible symplectic manifolds are one of the three types of manifolds which are building blocks for compact Kähler manifolds with numerically trivial canonical bundle. The Bogomolov Decomposition Theorem asserts that if $X$ is a connected compact Kähler manifold with numerically trivial canonical bundle, then there is a finite étale cover $\tilde{X}$ of $X$ such that
   $$\tilde{X} = T \times \prod_{i=1}^{n} X_i \times \prod_{j=1}^{m} Y_j,$$
   where $T$ is a complex torus, the $X_i$’s are irreducible Calabi-Yau manifolds (i. e. compact, connected, simply connected manifolds with trivial canonical bundle and such that $H^0(X_i, \Omega_{X_i}^p) = 0$ for all $0 < p < \text{dim}(X_i)$), and the $Y_j$’s are irreducible symplectic manifolds.

There are very few known deformation classes of irreducible symplectic manifolds, namely:
(1) a compact, connected smooth complex surface is an irreducible symplectic manifold if and only if it is a K3 surface;

(2) if $S$ is a K3 surface and $n \in \mathbb{N}$, $n \geq 2$, the Hilbert scheme $\text{Hilb}^n(S)$ of $n$ points on $S$ is an irreducible symplectic manifold of dimension $2n$ and second Betti number 23 (see [2], Théorème 3 and Proposition 6);

(3) if $T$ is a 2-dimensional complex torus and $n \in \mathbb{N}$, $n \geq 2$, then the fibers $\text{Kum}^n(T)$ of the sum morphism $\text{Hilb}^{n+1}(T) \to T$ are irreducible symplectic manifolds of dimension $2n$ and second Betti number 7 (see [2], Théorème 4 and Proposition 8);

(4) there are two more deformation classes: $OG_6$, in dimension 6 and with second Betti number 8, and $OG_{10}$, in dimension 10 and with second Betti number 24 (see [42], [13] and [48]).

A possible way to obtain new examples of varieties behaving like irreducible symplectic manifolds is to enlarge the family of varieties we are considering by including singular varieties. This is a very natural step, in particular in view of the Minimal Model Program.

Indeed, if $X$ is a connected complex projective manifold with $\kappa(X) = 0$, if the MMP works for $X$ then it produces a birational map $X \dashrightarrow Y$, where $Y$ has terminal singularities and nef canonical divisor. Assuming the abundance conjecture, we get that a multiple of $K_Y$ is trivial. In conclusion, in the classification of projective varieties whose Kodaira dimension is 0 it is then central to extend the Bogomolov decomposition to normal projective varieties having terminal singularities and torsion (i.e. numerically trivial by Theorem 8.2 of [21]) canonical divisor.

1.1. Bogomolov decomposition in the singular setting. In order to state the Bogomolov decomposition in the singular projective setting, we need to introduce the singular analogues of irreducible Calabi-Yau and symplectic manifolds. Before doing this, we introduce the following notation: if $X$ is a normal variety and $X_{\text{reg}}$ is the smooth locus of $X$ whose open embedding in $X$ is $j : X_{\text{reg}} \to X$, for every $p \in \mathbb{N}$ such that $0 \leq p \leq \dim(X)$ we let

$$\Omega^p_X := j_* \Omega^p_{X_{\text{reg}}} = \left( \wedge^p \Omega_X \right)^{**},$$

whose global sections are called reflexive $p$–forms on $X$. A reflexive $p$–form on $X$ is then a holomorphic $p$–form on $X_{\text{reg}}$.

If $f : Y \to X$ is a finite, dominant morphism between two irreducible normal varieties, then there is a morphism of reflexive sheaves $f^* \Omega^p_X \to \Omega^p_Y$, induced by the usual pull-back morphism of forms on the smooth loci, giving a morphism $f^! : H^0(X, \Omega^p_X) \to H^0(Y, \Omega^p_Y)$, called reflexive pull-back morphism.
We first recall the definitions of symplectic form and symplectic variety (see [3]).

**Definition 1.1.** Let $X$ be a normal variety.

1. A **symplectic form** on $X$ is a closed reflexive 2-form $\sigma$ on $X$ which is non-degenerate at each point of $X_{\text{reg}}$.
2. If $\sigma$ is a symplectic form on $X$, the pair $(X, \sigma)$ is a **symplectic variety** if for every resolution $f : \tilde{X} \to X$ of the singularities of $X$, the holomorphic symplectic form $\sigma_{\text{reg}} := \sigma|_{X_{\text{reg}}}$ extends to a holomorphic 2-form on $\tilde{X}$.
3. If $(X, \sigma)$ is a symplectic variety and $f : \tilde{X} \to X$ is a resolution of the singularities over which $\sigma_{\text{reg}}$ extends to a holomorphic symplectic form, we say that $f$ is a **symplectic resolution**.

A symplectic variety has canonical, and hence rational, singularities, and trivial canonical bundle. Conversely, by Theorem 6 of [38] a normal variety having rational Gorenstein singularities and whose regular locus carries a holomorphic symplectic form is a symplectic variety.

Moreover, a normal variety having a symplectic form and whose singular locus has codimension at least 4 is a symplectic variety (see [10]), and a symplectic variety has terminal singularities if and only if its singular locus has codimension at least 4 (Corollary 1 of [36]).

We now define irreducible Calabi-Yau and irreducible symplectic varieties following [17]. If $X$ and $Y$ are two irreducible normal projective varieties, a **finite quasi-étale morphism** $f : Y \to X$ is a finite morphism which is étale in codimension one.

**Definition 1.2.** Let $X$ be an irreducible normal projective variety with trivial canonical divisor and canonical singularities, of dimension $d \geq 2$.

1. The variety $X$ is **irreducible Calabi-Yau** if for every $0 < p < d$ and for every finite quasi-étale morphism $Y \to X$, we have $H^0(Y, \Omega^p_Y) = 0$.
2. The variety $X$ is **irreducible symplectic** if it has a symplectic form $\sigma \in H^0(X, \Omega^2_X)$, and for every finite quasi-étale morphism $f : Y \to X$ the exterior algebra of reflexive forms on $Y$ is spanned by $f^{[*]} \sigma$.

**Remark 1.3.** An irreducible normal projective variety $X$ of dimension $n$ has rational singularities and trivial canonical divisor if and only if it has rational Gorenstein singularities and admits a holomorphic $2n$-form which does not vanish at every $p \in X_{\text{reg}}$ (see Corollary 5.24 of [28]).
Remark 1.4. If $X$ is a normal projective variety, by definition of $\Omega^p_X$ we have $H^0(X, \Omega^p_X) = H^0(X_{\text{reg}}, \Omega^p_{X_{\text{reg}}})$. Greb, Kebekus, Kovács and Peternell prove in Theorem 1.4 of [16] that if $X$ is a quasi-projective variety with klt singularities and $\pi: \widehat{X} \to X$ is a log-resolution, then for every $p \in \mathbb{N}$ such that $0 \leq p \leq \dim(X)$ the sheaf $\pi_* \Omega^p_{\widehat{X}}$ is reflexive. This implies in particular (see Observation 1.3 therein) that $H^0(X, \Omega^p_X) \cong H^0(\widehat{X}, \Omega^p_{\widehat{X}})$.

The definition of irreducible symplectic variety is motivated by the description of the algebra of holomorphic forms of an irreducible symplectic manifold, which is spanned by a holomorphic symplectic form.

By Proposition A.1 of [22], a smooth irreducible symplectic variety is an irreducible symplectic manifold. Moreover, by Corollary 13.3 of [15] an irreducible symplectic variety $X$ is simply connected. In particular, the $\mathbb{Z}$–module $H^2(X, \mathbb{Z})$ is free.

Remark 1.5. A stronger definition of irreducible symplectic and Calabi-Yau varieties avoiding finite quasi-étale covers could be obtained by replacing the condition about finite quasi-étale morphisms with the condition that the fundamental group of the regular locus is trivial (see also section 1.4 of [15]).

Remark 1.6. If an irreducible symplectic variety $X$ admits a symplectic resolution $Y$, then $Y$ is an irreducible symplectic manifold. Indeed as the singularities of $X$ are canonical (and hence klt), by [52] we have $\pi_1(X) \cong \pi_1(Y)$. By Corollary 13.3 of [15] we know that $X$ is simply connected, hence $Y$ is simply connected. Finally, by Theorem 1.4 of [16] we have $h^0(Y, \Omega^2_Y) = h^0(X, \Omega^2_X)$, which is 1 as $X$ is an irreducible symplectic variety.

Definition 1.2 proves to be the good one in view of the Bogomolov decomposition theorem in the singular projective setting. Höring and Peternell (see Theorem 1.5 of [18]) show that if $X$ is an irreducible normal projective variety with terminal singularities (or more generally with klt singularities and smooth in codimension 2) and has numerically trivial canonical bundle, then it admits a finite quasi-étale cover $\widehat{X}$ such that

$$\widehat{X} = T \times \prod_{i=1}^n X_i \times \prod_{j=1}^m Y_j,$$

where $T$ is a complex torus, the $X_i$’s are irreducible Calabi-Yau varieties and the $Y_j$’s are irreducible symplectic varieties, and the $X_i$’s and the $Y_j$’s have terminal singularities (see Proposition 5.20 of [28]).
The proof of the Bogomolov decomposition theorem in the singular projective setting stated in [18] consists of three major parts: one is the holonomy decomposition obtained by Greb, Guenancia and Kebekus in [15]; a second one is an algebraic integrability theorem of Druel, proved in [6]; the final ingredient is Theorem 1.1 of [18]. Less general versions of the Bogomolov decomposition theorem in the projective singular setting were previously obtained in [17], [6] and [7].

Remark 1.7. Motivated by Remark 1.5, it is natural to ask whether \( \pi_1(X_{reg}) \) of an irreducible symplectic or Calabi-Yau variety \( X \) is finite. If this holds true, it would then be natural to define irreducible symplectic and Calabi-Yau varieties as proposed in Remark 1.5. Theorem 1.5 of [18] would still hold under these stronger definitions.

Moreover, it would give a strong constraint on \( \pi_1(X_{reg}) \) for a connected projective variety \( X \) having terminal singularities and trivial canonical bundle (in analogy to that given by the Bogomolov decomposition theorem on \( \pi_1(X) \) for a projective manifold \( X \) with trivial canonical bundle).

All the examples of irreducible symplectic varieties we present in this paper have smooth locus with finite fundamental group.

1.2. Beauville form for irreducible symplectic varieties. Another class of varieties appearing in the literature is the following, motivated by the work of Namikawa, in particular by [38] and [39] (see [51], [25]).

Definition 1.8. A connected projective symplectic variety \( (X, \sigma) \) is a Namikawa symplectic variety if \( h^1(X, \mathcal{O}_X) = 0 \) and \( h^0(X, \Omega^2_X) = 1 \).

A first example of Namikawa symplectic varieties is that of projective symplectic varieties having a symplectic resolution which is an irreducible symplectic manifold, as the following shows:

Proposition 1.9. If \( X \) is a connected projective symplectic variety admitting a symplectic resolution of the singularities which is an irreducible symplectic manifold, then \( X \) is a Namikawa symplectic variety.

Proof. Let \( \tilde{X} \rightarrow X \) be a symplectic resolution where \( \tilde{X} \) is an irreducible symplectic manifold. As \( X \) has canonical (and hence klt) singularities, by Theorem 1.4 of [16] we get

\[
h^0(X, \Omega^2_X) = h^0(\tilde{X}, \Omega^2_{\tilde{X}}) = 1,
\]

where the last equality follows from \( \tilde{X} \) being irreducible symplectic. Moreover, as \( X \) has rational singularities we have

\[
h^1(X, \mathcal{O}_X) = h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0,
\]
where the last equality follows again from the fact that \( \tilde{X} \) is irreducible symplectic (and hence smooth, projective and simply connected). \( \square \)

A second example of Namikawa symplectic varieties is that of irreducible symplectic varieties, as the following shows:

**Proposition 1.10.** An irreducible symplectic variety is a Namikawa symplectic variety.

**Proof.** If \( X \) is an irreducible symplectic variety, we need to prove that \( X \) is a symplectic variety with \( h^1(X, \mathcal{O}_X) = 0 \) and \( h^0(X, \Omega^2_X) = 1 \). The fact that \( h^0(X, \Omega^2_X) = 1 \) is immediate from the definition of irreducible symplectic variety.

Moreover, as \( X \) has canonical singularities and trivial canonical bundle, it has rational Gorenstein singularities. As it carries a holomorphic symplectic form on its regular locus, by Theorem 6 of [38] it follows that \( X \) is a symplectic variety.

Finally, let \( f : \tilde{X} \to X \) be a smooth projective resolution of the singularities. Then we have the following chain of equalities:

\[
\begin{align*}
    h^1(X, \mathcal{O}_X) &= h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \quad \text{as } X \text{ has rational singularities} \\
    &= h^0(\tilde{X}, \Omega^1_{\tilde{X}}) \quad \text{as } \tilde{X} \text{ is smooth projective} \\
    &= h^0(X, \Omega^1_X) \quad \text{by Theorem 1.4 of [16] since } X \text{ is klt} \\
    &= 0 \quad \text{as } X \text{ is irreducible symplectic}
\end{align*}
\]

thus concluding the proof. \( \square \)

**Remark 1.11.** In particular, an irreducible symplectic variety is a symplectic variety, and in general it does not admit any symplectic resolution (this is the case, as instance, of many of the examples we present in this paper). In particular, a Namikawa symplectic variety does not in general have a symplectic resolution.

A Namikawa symplectic variety is not necessarily an irreducible symplectic variety, as the following examples show.

**Example 1.12.** The \( m \)-th symmetric product \( Y = \text{Sym}^m(X) \) of a projective K3 surface \( X \) is a Namikawa symplectic variety. Indeed, it is a symplectic variety having \( Z = \text{Hilb}^m(X) \) as a resolution of the singularities, hence \( Y \) is Namikawa symplectic by Proposition 1.9.

Anyway, \( Y \) has a finite quasi-étale cover \( f : X^m \to Y \), and the exterior algebra of reflexive forms on \( X^m \) is not generated by any reflexive 2-form if \( m > 1 \). It follows that \( Y \) is not irreducible symplectic.
Example 1.13. Let $A$ be an Abelian surface, $m \geq 2$ an integer and $Y = \text{Sym}^m(A)$. We let $f : A^m \to A$ and $\tilde{f} : Y \to A$ be the sum morphisms, and set $A^m_0 := f^{-1}(0)$ and $Y_0 := \tilde{f}^{-1}(0)$. We know that $Y$ and $Y_0$ are symplectic varieties, and $Z := \text{Hilb}^m(A)$ and $Z_0 := \text{Kum}^m(A)$ are their respective symplectic resolutions. By Proposition 1.9 it follows that $Y_0$ is a Namikawa symplectic variety.

Anyway, $Y_0$ has a finite quasi-étale cover $\pi_0 : A^m_0 \to Y_0$, and we have $A^m_0 \cong A^{m-1}$. Hence the algebra of reflexive forms on $A^m_0$ is not generated by any reflexive 2-form, so $Y_0$ is not irreducible symplectic.

Namikawa symplectic varieties share many features with irreducible symplectic manifolds. A first case showing this is given by a symplectic variety $X$ admitting a symplectic resolution $f : Y \to X$ which is an irreducible symplectic manifold. Then $f^* : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ is an inclusion of mixed Hodge structures, since $X$ has rational singularities: the $\mathbb{Z}$-module $H^2(X, \mathbb{Z})$ is then free, has a pure weight-two Hodge structure and a nondegenerate integral quadratic form of signature $(3, b_2(X) - 3)$. Bakker and Lehn proved in [1] a Local and a Global Torelli Theorem in this situation.

As already seen before, as a consequence of the MMP we are more interested in symplectic varieties having terminal singularities (hence with no symplectic resolutions). By Corollary 1 of [36], a Namikawa symplectic variety $X$ which is $\mathbb{Q}$-factorial has terminal singularities if and only if its singular locus has codimension at least 4. For a Namikawa symplectic variety $X$ of this type the following results hold:

1. the (free part of the) $\mathbb{Z}$-module $H^2(X, \mathbb{Z})$ has a pure weight-two Hodge structure and a non-degenerate quadratic form $q$ of signature $(3, b_2(X) - 3)$, called the \textit{Beauville form} of $X$ (see [38] and [39]).
2. the deformations of $X$ are unobstructed and locally trivial (see [36] and [37]);
3. there is a positive rational number $F_X$, called \textit{Fujiki constant} of $X$, verifying the same equality as the Fujiki constant of an irreducible symplectic manifold (see [51]). As a consequence, the Beauville form and the Fujiki constant are deformation invariant;
4. a Local Torelli Theorem holds (see Theorem 8 of [38]).

The case of Namikawa symplectic varieties having terminal singularities but which are not $\mathbb{Q}$-factorial has been studied in [25].

Remark 1.14. By the Theorem of [36], the singular locus of a connected symplectic variety cannot have any irreducible component of
codimension $3$, hence either the singular locus has codimension at least 4, or it has a component of codimension 2.

1.3. **Notation and main results of the paper.** The aim of the present paper is to provide families of irreducible symplectic varieties using moduli spaces of sheaves on K3 or Abelian surfaces.

The result we get, which will be stated precisely in this section, shows that for generic polarizations a moduli space of semistable sheaves on a projective K3 surface is an irreducible symplectic variety, with the only exception of symmetric products of K3 surfaces.

Similarly, and always for generic polarizations, the fibers of the Albanese morphism of a moduli space of semistable sheaves on an Abelian surface are irreducible symplectic varieties, unless the moduli space is a symmetric product of Abelian surfaces.

In particular, we get a finite number of deformation classes of irreducible symplectic varieties in each dimension.

In the following, $S$ will denote a projective K3 surface or an Abelian surface, and we let $\epsilon(S) := 1$ if $S$ is K3, and 0 if $S$ is Abelian. We will denote $\rho(S)$ the rank of the Néron-Severi group $NS(S)$ of $S$.

An element $v \in \tilde{H}(S, \mathbb{Z}) := H^{2*}(S, \mathbb{Z})$ will be written $v = (v_0, v_1, v_2)$, where $v_i \in H^{2i}(S, \mathbb{Z})$, and $v_0, v_2 \in \mathbb{Z}$. It will be called *Mukai vector* if $v_0 \geq 0$, $v_1 \in NS(S)$ and if $v_0 = 0$, then either $v_1$ is the first Chern class of an effective divisor, or $v_1 = 0$ and $v_2 > 0$.

Recall that $\tilde{H}(S, \mathbb{Z})$ has a pure weight-two Hodge structure and a lattice structure with respect to the Mukai pairing $(.,.)$ (see [21], Definitions 6.1.5 and 6.1.11). We let $v^2 := (v, v)$ for every Mukai vector $v$, and we call $\tilde{H}(S, \mathbb{Z})$ the *Mukai lattice* of $S$.

If $F$ is a coherent sheaf on $S$, we define its *Mukai vector* as

$$v(F) := \text{ch}(F) \sqrt{\text{td}(S)} = (rk(F), c_1(F), ch_2(F) + \epsilon(S)rk(F)).$$

Let now $v$ be a Mukai vector on $S$ and suppose that $H$ is a $v$–generic polarization (see section 2.1 for the definition). We write $M_v(S, H)$ (resp. $M^s_v(S, H)$) for the moduli space of Gieseker $H$–semistable (resp. $H$–stable) sheaves on $S$ with Mukai vector $v$. If $S$ is Abelian and $v^2 > 0$, we have a dominant isotrivial fibration $a_v : M_v(S, H) \rightarrow S \times \hat{S}$ (see section 4.1 of [56]), where $\hat{S}$ is the dual of $S$. We let $K_v(S, H) := a_v^{-1}(0_S, \mathcal{O}_S)$, and $K^*_v(S, H) := K_v(S, H) \cap M^s_v(S, H)$. If no confusion on $S$ and $H$ is possible, we drop them from the notation.

We write $v = mw$, where $m \in \mathbb{N}$ and $w$ is a primitive Mukai vector on $S$. If $M^s_v \neq \emptyset$, then it is a holomorphic symplectic quasi-projective manifold of dimension $v^2 + 2$ (see [33]).
If $m = 1$ and $S$ is $K3$, then $M_v^s \neq \emptyset$ if and only if $v^2 \geq -2$ (see Theorem 0.1 of [55]). If $S$ is Abelian, then $M_v^s \neq \emptyset$ if and only if $v^2 \geq 0$ (see Theorem 0.1 of [56], and compare with section 2.4 of [23]). If $w^2 > 0$, then $M_v$ and $K_v$ are normal, irreducible projective varieties (see Theorem 4.4 of [23] and Remark A.1 of [46]).

We introduce the following definition (we let $\mathbb{N}^* := \mathbb{N}\backslash\{0\}$):

**Definition 1.15.** Let $S$ be a projective $K3$ or Abelian surface, $v$ a Mukai vector, $H$ an ample divisor on $S$ and $m, k \in \mathbb{N}^*$. We say that $(S, v, H)$ is an $(m, k)$—triple if the following conditions are verified:

1. the polarization $H$ is primitive and $v$—generic;
2. we have $v = mw$, where $w$ is primitive and $w^2 = 2k$;
3. if $w = (0, w_1, w_2)$ and $\rho(S) > 1$, then $w_2 \neq 0$.

If $(S, v, H)$ is an $(m, k)$—triple, then $M_v$ is a nonempty, irreducible, normal projective variety of dimension $2m^2k + 2$ (see Theorem 4.4 of [23]), which is symplectic and whose regular locus is $M_v^s$. If $S$ is Abelian and $(m, k) \neq (1, 1)$, then $K_v$ is a nonempty, irreducible, normal projective variety of dimension $2m^2k - 2$, which is symplectic and whose regular locus is $K_v^s$. If $(m, k) = (1, 1)$, then $M_v$ is isomorphic to the Abelian 4—fold $S \times \hat{S}$ and $K_v$ is just a point.

**Remark 1.16.** In the definition of $(m, k)$—triple we excluded $k \leq 0$, but we have a precise description of $M_v$ in this case.

1. If $k < 0$ and $S$ is $K3$, then $M_v$ is either empty (if $k < -1$) or a point (if $k = -1$, see [34]).
2. If $k < 0$ and $S$ is Abelian, then $M_v = \emptyset$ (see [56]).
3. If $k = 0$ and $S$ is $K3$, then $M_w$ is a $K3$ surface (see [31]) and $M_v \simeq \text{Sym}^m(M_w)$ (see section 1 of [23]). If $m \geq 2$, then $M_v$ is a Namikawa symplectic variety which is not irreducible symplectic (see Example 1.12).
4. If $k = 0$ and $S$ is Abelian, then $M_w$ is an Abelian surface (see [31]) and $M_v \simeq \text{Sym}^m(M_w)$ (see section 1 of [23]). Then $M_v$ is not Namikawa symplectic since $h^{0,1}(M_v) \neq 0$, and the fiber of the sum morphism $\text{Sym}^m(M_w) \longrightarrow M_w$ is a point (if $m = 1$) or a Namikawa symplectic variety which is not irreducible symplectic (if $m \geq 2$, see Example 1.13).

The first result we will prove is the following:

**Theorem 1.17.** Let $m, k \in \mathbb{N}^*$, and let $(S_1, v_1, H_1)$ and $(S_2, v_2, H_2)$ be two $(m, k)$—triples.
(1) If $S_1$ and $S_2$ are both K3 surfaces or both Abelian surfaces, then $M_{v_1}$ and $M_{v_2}$ are deformation equivalent, and the deformation is locally trivial.

(2) If $S_1$ and $S_2$ are two Abelian surfaces, then $K_{v_1}$ and $K_{v_2}$ are deformation equivalent, and the deformation is locally trivial.

**Remark 1.18.** The deformation relating $M_{v_1}$ and $M_{v_2}$ in Theorem 1.17 is obtained using only deformations of the moduli spaces along curves (induced by deformations of the corresponding $(m, k)$–triples: see section 2.3 for the definition), and isomorphisms between moduli spaces induced by Fourier-Mukai transforms.

As a consequence of Theorem 1.17 we get a single deformation class for every pair $(m, k)$. The proof of Theorem 1.17 is the content of section 2 of the present paper.

For $m = 1$, Theorem 1.17 is due to several authors (see [33], [2], [41], [55], [56]). For $(2, 1)$–triples, the proof of Theorem 1.17 is contained in [45].

The deformation equivalence in point (1) of Theorem 1.17 is basically due to Yoshioka: for Mukai vectors with positive rank, this is Proposition 3.6 of [58]; the rank 0 case is not explicitly stated, but can be obtained as in Corollary 3.5 of [39]. Since it is an important result which plays a central role in our paper, we decided to include a complete proof. The local triviality of the deformation follows from the Main Theorem of [37].

Yoshioka’s original proof of the deformation equivalence involves two main technical tools: deformations of K3 or Abelian surfaces inducing deformations of the moduli spaces, and Fourier-Mukai transforms. As a by-product of his proof one gets non-emptiness and normality of the moduli spaces. Based on his proof of the deformation equivalence of the moduli spaces, and using a different argument to deal with particular cases, Yoshioka proves also their irreducibility (see Theorem 3.18 of [58]).

The proof of point (1) of Theorem 1.17 we propose is a re-elaborated version of Yoshioka’s proof, that we tried to keep as elementary as possible. It uses the same technical tools together with Theorem 4.4 of [23], which proves the irreducibility and the normality of the moduli spaces independently of [58] and [59] (and which implies the irreducibility and the normality of the $K_v$’s, as shown in Remark A.1 of [46]).

We remark that the proof of Theorem 4.4 of [23] uses the fact that if $v$ is primitive and $v^2 \geq 0$, then $M_v \neq \emptyset$, that was proved by Yoshioka in [59] and [56], independently of [58] and [59] (compare with section 2.4 of [23]).
The use of Theorem 4.4 of [23] allows us to minimize the technical tools involved in the proof of the deformation equivalence of moduli spaces: the only Fourier-Mukai transforms we will use are those corresponding to tensorization with line bundles, the one whose kernel is the ideal of the diagonal (for K3 surfaces), and the one whose kernel is the Poincaré bundle (for Abelian surfaces), and we only need to check the preservation of the semistability under these functors in the most natural direction (see section 2.4).

The aim of section 3 is to show the following, which is the main result of this paper:

**Theorem 1.19.** Let \( m, k \in \mathbb{N}^* \) and \((S, v, H)\) be an \((m, k)-\)triple.

1. If \( S \) is K3, then \( M_v \) is an irreducible symplectic variety.
2. If \( S \) is Abelian and \((m, k) \neq (1, 1)\), then \( K_v \) is an irreducible symplectic variety.

**Remark 1.20.** If \( S \) is K3 and \( k \leq 0 \), then the only case in which \( M_v \) is an irreducible symplectic variety is when \( m = 1 \) and \( k = 0 \), in which case \( M_v \) is a K3 surface. In all other cases \( M_v \) is either empty, or a point, or a Namikawa symplectic variety which is not irreducible symplectic (see Remark 1.16). If \( S \) is Abelian and \( k \leq 0 \), then the analogue of \( K_v \) in this case is not irreducible symplectic (see again Remark 1.16). If \( m = k = 1 \), then \( K_v \) is just a point.

**Remark 1.21.** Theorem 1.19 provides an answer to Question 14.10 of [15]. Together with Remark 1.16 it says that all moduli spaces of sheaves on a projective K3 surface (and all the fibers of the Albanese morphism of moduli spaces of sheaves on Abelian surfaces) are irreducible symplectic varieties, with the only exception of symmetric products. Moreover, we have the following cases:

1. If \( S \) is K3, then \( M_v \) is smooth if and only if \( m = 1 \) (and in this case it is deformation equivalent to \( \text{Hilb}^{k+1}(S) \)). If \( S \) is Abelian, then \( K_v \) is smooth if and only if \( m = 1 \) (moreover, if \( k = 1 \) it is just a point, if \( k = 2 \) it is a K3 surface, and if \( k \geq 3 \) it is deformation equivalent to \( \text{Kum}^{k-1}(S) \));
2. If \( S \) is K3, then \( M_v \) has a symplectic resolution if and only if \((m, k) = (2, 1)\) (which is deformation equivalent to \( \text{OG}_{10} \)). If \( S \) is Abelian, then \( K_v \) has a symplectic resolution if and only if \((m, k) = (2, 1)\) (which is deformation equivalent \( \text{OG}_6 \));
3. in all other cases \( M_v \) and \( K_v \) have terminal singularities.

The proof of Theorem 1.19 uses Theorem 1.17 to reduce to a surface \( S \) such that \( NS(S) = \mathbb{Z} \cdot h \), where \( h \) is the first Chern class of an ample divisor \( H \) with \( H^2 = 2k \). Taking \( v = m(0, h, 0) \), if \( S \) is a K3 surface
we show that $M_v$ and $M'_v$ are simply connected; if $S$ is Abelian, we show that $K_v$ and $K'_v$ are simply connected (if $(m, k) \neq (2, 1)$). We then calculate the numbers $h^0(M_v, \Omega^1_{M_v})$ and $h^0(K_v, \Omega^1_{K_v})$ by comparing them with $h^0(M'_{v'}, \Omega^1_{M'_{v'}})$ and $h^0(K'_{v'}, \Omega^1_{K'_{v'}})$ where $v'$ is the primitive Mukai vector $v' = (0, mh, 1 - m^2 k)$.

Remark 1.22. The irreducible symplectic varieties we get with Theorem 1.19 have all simply connected smooth locus, up to one exception, namely when $S$ is an Abelian surface and $(m, k) = (2, 1)$: in this case the fundamental group of $K'_v$ is $\mathbb{Z}/2\mathbb{Z}$ (see section 4 of [32], or Theorem 3.7). In any case, all the irreducible symplectic varieties we obtain have smooth locus with finite fundamental group.

As a corollary of Theorem 1.19 by [38] and [39] one concludes the following (see the discussion after Remark 1.6):

**Theorem 1.23.** Let $m, k \in \mathbb{N}^*$ and $(S, v, H)$ an $(m, k)$–triple.

1. If $S$ is K3, on $H^2(M_v, \mathbb{Z})$ there is a nondegenerate integral quadratic form of signature $(3, b_2(M_v) - 3)$ and a compatible pure weight-two Hodge structure.

2. If $S$ is Abelian and $(m, k) \neq (1, 1)$, on $H^2(K_v, \mathbb{Z})$ there is a nondegenerate integral quadratic form of signature $(3, b_2(K_v) - 3)$ and a compatible pure weight-two Hodge structure.

For $(1, k)$–triples this follows from the fact that $M_v$ and $K_v$ are irreducible symplectic manifolds. For $(2, 1)$–triples this is a consequence of the fact that $M_v$ and $K_v$ have a symplectic resolution (see Remark 1.6 and [45]).

In both cases there is a Hodge isometry between $v^\perp$ and $H^2(M_v, \mathbb{Z})$ if $S$ is K3, and a Hodge isometry between $v^\perp$ and $H^2(K_v, \mathbb{Z})$ if $S$ is Abelian (see [41, 55, 56, 45]).

**Remark 1.24.** A natural question is if the same holds for every $(m, k)$–triple. We will come back to this problem in a forthcoming paper.

2. **Deformations of moduli spaces**

In this section we study how moduli spaces vary under deformations and isomorphisms. In section 2.1 we recall the notions of $v$–generic polarizations, $v$–walls and $v$–chambers. section 2.2 is devoted to the morphism $a_v$ in the case of Abelian surfaces. In section 2.3 we introduce deformations of moduli spaces induced by deformations of $(m, k)$–triples along smooth, connected curves. In section 2.4 we study isomorphisms between moduli spaces coming from Fourier-Mukai transforms.
Section 2.5 is devoted to the proof of Theorem 1.17, which is the main result of this chapter. Our proof of Theorem 1.17 is heavily based on Theorem 4.4 of [23] which asserts that if $H$ is $v$–generic, then the moduli space $M_v(S, H)$ is a normal, irreducible projective variety of expected dimension. Theorem 4.4 of [23] is based on the non-emptiness of moduli spaces of sheaves for generic polarizations and primitive Mukai vectors of positive square (Theorems 0.1 and 8.1 of [56], compare with section 2.4 of [23]).

These assumptions could be avoided (following Yoshioka) using Theorem 1.17 of [33] and stronger versions of Lemmas 2.22, 2.24, 2.25 and 2.26 (see Remarks 2.16 and 2.27).

2.1. Generic polarizations. In what follows, $S$ will always denote a projective K3 or Abelian surface, and $v = (v_0, v_1, v_2)$ a Mukai vector on $S$. We will furthermore suppose that when $\rho(S) > 1$, if $v_0 = 0$ then $v_2 \neq 0$ (the case of $v = (0, v_1, 0)$ will be briefly discussed in Remark 2.6 and Example 2.7).

We associate to each Mukai vector $v$ of this form a set $W_v$ of divisors on $S$, whose definition depends on $v_0$: the case $v_0 > 0$ will be different than the case $v_0 = 0$.

If $v_0 > 0$, we let

$$|v| = \frac{v_0^2}{4} (v, v) + \frac{v_0^{2v(S)+2}}{2}.$$ 

The rational number $|v|$ only depends on $v_0$ and $v^2$. If $|v| > 0$, we define

$$W_v := \{ D \in NS(S) \mid - |v| \leq D^2 < 0 \},$$

and we let $W_v := \emptyset$ if $|v| = 0$. We notice that if $v = mw$ for $m \in \mathbb{N}^*$ and $w$ is a primitive Mukai vector such that $w^2 = 2k > 0$, then $|v| > 0$.

If $v_0 = 0$, for every pure sheaf $\mathcal{E}$ of Mukai vector $v$, and $\mathcal{F} \subseteq \mathcal{E}$ of Mukai vector $u = (0, u_1, u_2)$, the divisor associated to the pair $(\mathcal{E}, \mathcal{F})$ is defined as $D := u_2 v_1 - v_2 u_1$. The set $W_v$ will then be the set of the non-numerically trivial divisors associated to all the possible pairs of this type.

A primitive ample divisor $H$ on $S$ will be called polarization\footnote{By a slight abuse of notation, the line bundle $\mathcal{O}_S(H)$ will be usually denoted $H$, and will still be called polarization.}. The set $W_v$ is used to define the notion of $v$–generic polarizations as follows:

**Definition 2.1.** A polarization $H$ is $v$–generic if $H \cdot D \neq 0$ for every $D \in W_v$. 

If \( \rho(S) = 1 \), then the ample generator of the Picard group of \( S \) is \( v \)-generic for every \( v \). If \( \rho(S) \geq 2 \), there can be polarizations which are not \( v \)-generic, and to characterize them we introduce \( v \)-walls and \( v \)-chambers. We let \( \text{Amp}(S) \) be the ample cone of \( S \).

**Definition 2.2.** If \( D \in W_v \), the \( v \)-wall associated to \( D \) is
\[
D^\perp := \{ \alpha \in \text{Amp}(S) \mid D \cdot \alpha = 0 \}.
\]

The \( v \)-wall associated to \( D \in W_v \) is a hyperplane in \( \text{Amp}(S) \). If \( v_0 > 0 \), by Theorem 4.C.2 of [21] the set of \( v \)-walls is locally finite in \( \text{Amp}(S) \). If \( v_0 = 0 \), it is even finite (see section 1.4 of [56]).

**Definition 2.3.** Suppose that \( \rho(S) \geq 2 \). A connected component of \( \text{Amp}(S) \setminus \bigcup_{D \in W_v} D^\perp \) is called \( v \)-chamber.

By definition, a polarization is \( v \)-generic if and only if it lies in a \( v \)-chamber.

**Remark 2.4.** If \( H \) is \( v \)-generic and \( \mathcal{F} \) is an \( H \)-semistable sheaf with Mukai vector of the form \( pv \) for some \( p \in \mathbb{Q} \). In particular, if \( v \) is primitive and \( H \) is \( v \)-generic, any \( H \)-semistable sheaf of Mukai vector \( v \) is \( H \)-stable (compare section 2.4 of [23]).

An important property of generic polarizations is that changing polarization inside a \( v \)-chamber does not affect the moduli space. More precisely, we have the following (see [47], or section 4.C in [21]):

**Proposition 2.5.** Suppose that \( \rho(S) \geq 2 \) and that \( v \) is a Mukai vector on \( S \). Let \( \mathcal{C} \) be a \( v \)-chamber, and suppose that \( H, H' \in \mathcal{C} \). A sheaf \( \mathcal{E} \) of Mukai vector \( v \) is \( H \)-(semi)stable if and only if it is \( H' \)-(semi)stable. As a consequence, we have natural identifications \( M_v(S, H) = M_v(S, H') \), \( M_v^*(S, H) = M_v^*(S, H') \), \( K_v(S, H) = K_v(S, H') \) and \( K_v^*(S, H) = K_v^*(S, H') \).

**Remark 2.6.** The definition of \( v \)-generic polarization makes perfect sense for Mukai vectors of type \( v = (0, v_1, 0) \), but it is not well-adapted to our goals. Indeed, if \( v = (0, v_1, 0) \), by defining \( W_v \) as before we see that \( D \in W_v \) is of the form \( D = bv_1 \) for some \( b \neq 0 \). As \( v_1 \) is the first Chern class of an effective divisor, we get \( H \cdot D \neq 0 \) for all \( D \in W_v \), and hence every polarization would be \( v \)-generic.

Now, the definition of \( v \)-genericity is motivated by the fact that if \( v \) is a primitive Mukai vector and \( H \) is \( v \)-generic, then a \( H \)-semistable sheaf with Mukai vector \( v \) is \( H \)-stable: this holds if \( v = (0, v_1, v_2) \) where \( v_2 \neq 0 \), or if \( v = (0, v_1, 0) \) and \( \rho(S) = 1 \) (as a consequence of
Remark 2.4], but it is no longer true if \( v = (0, v_1, 0) \) and \( \rho(S) \geq 2 \) as the following example (due to Yoshioka) shows.

**Example 2.7.** Let \( S \) be an elliptic K3 surface with \( NS(S) = \mathbb{Z} \sigma \oplus \mathbb{Z} f \), where \( \sigma \) is the class of a section \( \Sigma \) and \( f \) is the class of a fiber \( F \). We consider the primitive Mukai vector \( v = (0, \sigma + f, 0) \) on \( S \), and we let \( j_\Sigma \) and \( j_F \) be the inclusions of \( \Sigma \) and \( F \) in \( S \) respectively. Moreover, let \( N \in Pic^{-1}(\Sigma) \) and \( M \in Pic^0(F) \): the sheaves \( j_\Sigma^*N \) and \( j_F^*M \) are \( H \)-stable with respect to any polarization \( H \), and we have \( v(j_\Sigma^*N) = (0, \sigma, 0) \) and \( v(j_F^*M) = (0, f, 0) \). The sheaf \( j_\Sigma^*N \oplus j_F^*M \) is then \( H \)-semistable of Mukai vector \( v = (0, \sigma + f, 0) \), but it is not \( H \)-stable, since \( j_\Sigma^*N \) and \( j_F^*M \) are both \( H \)-destabilizing.

We conclude this section with the behaviour of genericity with respect to tensorization with a line bundle. We need the following notation: if \( v \) is a Mukai vector on \( S \) and \( L \in Pic(S) \), we let \( v_L := v \cdot ch(L) \). If \( L = O_S(D) \) for some divisor \( D \), then we let \( v_D := v_L \). Notice that if \( v(\mathcal{F}) = v \), then \( v(\mathcal{F} \otimes L) = v_L \).

**Lemma 2.8.** Let \( v \) be a Mukai vector and \( H \) a polarization on \( S \).

1. If \( v = (r, \xi, a) \) with \( r > 0 \) and \( L \in Pic(S) \), we have that \( H \) is \( v \)-generic if and only if it is \( v_L \)-generic.
2. If \( v = (0, \xi, a) \) and \( d \in \mathbb{Z}^* \), we have that \( H \) is \( v \)-generic if and only if it is \( v_{dH} \)-generic.

**Proof.** If \( v = (r, \xi, a) \) with \( r > 0 \), notice that

\[
    v_L = (r, \xi + rc_1(L), a + \xi \cdot L + rL^2/2).
\]

Hence \( v \) and \( v_L \) have the same rank, and it is easy to see that \( v^2 = v_L^2 \). It follows that \( |v| = |v_L| \): this implies that \( W_v = W_{v_L} \), so \( H \) is \( v \)-generic if and only if it is \( v_L \)-generic.

If \( v = (0, \xi, a) \), notice that \( v_{dH} = (0, \xi, a + d\xi \cdot H) \). There is a bijection between \( W_v \cup \{0\} \) and \( W_{v_{dH}} \cup \{0\} \) mapping the divisor associated to a pair \( (\mathcal{F}, \mathcal{E}) \) to the one associated to the pair \( (\mathcal{F} \otimes O_S(dH), \mathcal{E} \otimes O_S(dH)) \).

If \( D \in W_v \) is associated to \( (\mathcal{F}, \mathcal{E}) \), and if \( v(\mathcal{E}) = (0, \zeta, b) \), we then have \( D = b\zeta - a\xi \). The divisor associated to \( (\mathcal{F} \otimes O_S(dH), \mathcal{E} \otimes O_S(dH)) \) is then

\[
    D' = D + d(\xi \cdot H)\zeta - d(\zeta \cdot H)\xi,
\]

and the bijection maps \( D \) to \( D' \) (and conversely). We then see that \( D \cdot H = D' \cdot H \), hence \( H \) is \( v \)-generic if and only if it is \( v_{dH} \)-generic. \( \square \)

If \( v = (0, \xi, a) \) and \( L \) is not a multiple of \( H \), then it is not in general true that \( H \) is \( v \)-generic if and only if it is \( v_L \)-generic (as the following example shows): we still have a bijection between \( W_v \cup \{0\} \) and \( W_{v_L} \cup \{0\} \) mapping a divisor \( D \) to a divisor \( D' \), but in general \( D \cdot H \neq D' \cdot H \).
Example 2.9. We consider the elliptic K3 surface $S$ of Example 2.7 and the same notation for $\Sigma, F, j_\Sigma$ and $j_F$, but we let $v := (0, \sigma + f, 1)$. We consider the polarization $H$ such that $c_1(H) = \sigma + af$, and the line bundle $L$ such that $c_1(L) = -af$. If $a > 0$, then $H$ is $v$–generic. We show that it is not $v_L$–generic.

Indeed, let $N \in Pic^1(\Sigma)$ and $M \in Pic^{-1}(F)$: then the coherent sheaf $\mathcal{F} := j_\Sigma^* N \oplus j_F^* M$ has Mukai vector $v$, so the Mukai vector of $\mathcal{F} \otimes L$ is $v_L = (0, \sigma + f, 1 - a)$. The subsheaf $\mathcal{E} = j_F^* M$ of $\mathcal{F}$ has Mukai vector $(0, f, -1)$, so the Mukai vector of $\mathcal{E} \otimes L$ is $(0, f, -1)$. The associated divisor is then $D' = -\sigma + (a - 2)f \in W_{v_L}$, and as easily seen we have $D' \cdot H = 0$: the polarization $H$ is then not $v_L$–generic.

2.2. Yoshioka’s fibration. Here we recall the definition and the main properties of the morphism $a_v : M_v(S, H) \to S \times \hat{S}$ introduced by Yoshioka in [60], and relate it to another morphism used in [43].

We let $S$ be an Abelian surface, $\hat{S}$ its dual and $\mathcal{P}$ the Poincaré line bundle on $S \times \hat{S}$. Fix a Mukai vector $v$ and a $v$–generic polarization $H$ on $S$, and let $M_v$ be the moduli space of $H$–semistable sheaves on $S$ with Mukai vector $v$. We furthermore fix a coherent sheaf $\mathcal{F}_0$ with Mukai vector $v$.

We recall that for a smooth projective variety $X$, the Grothendieck group $K(X)$ has a ring structure: if $F$ and $G$ are two locally free sheaves, we let $[F] + [G] = [F \oplus G]$ and $[F] \cdot [G] = [F \otimes G]$; if $F$ and $G$ are coherent but not locally free, replace them by a finite locally free resolution of both.

If $f : X \to Y$ is a morphism of smooth projective varieties, then we have the ring morphisms of pull-back $f^* : K(Y) \to K(X)$ and the group morphism of push-forward $f_! : K(X) \to K(Y)$. Moreover, the determinant map $\det : K(X) \to Pic(X)$ is well-defined.

2.2.1. Yoshioka’s fibration. We now define the morphism $a_v$ following Yoshioka (see [60] and [56]). For every coherent sheaf $\mathcal{F}$ on $S$ of Mukai vector $v$, we set

$$a_v(\mathcal{F}) := \det(p_{\hat{S}}(p^*_S([\mathcal{F}] - [\mathcal{F}_0]) \cdot ([\mathcal{P}] - [\mathcal{O}_{S \times \hat{S}}])) \in Pic^0(\hat{S}),$$

where $p_S$ and $p_{\hat{S}}$ are the two projections of $S \times \hat{S}$ onto $S$ and $\hat{S}$ respectively, and $Pic^0(\hat{S})$ is the group of topologically trivial line bundles on $\hat{S}$. Letting

$$F : D^b(S) \to D^b(\hat{S}), \quad F(E^*) := Rp_{\hat{S}}(p^*_S E^* \otimes \mathcal{P})$$

be the Fourier-Mukai functor with kernel $\mathcal{P}$, then we have

$$a_v(\mathcal{F}) = \det(F(\mathcal{F})) \otimes \det(F(\mathcal{F}_0))^\vee \in Pic^0(\hat{S}).$$
Notice that we have an isomorphism $\text{Pic}^0(\hat{S}) \cong S$, hence we have a morphism $\alpha_v : M_v \longrightarrow S$.

We then let

$$a_v : M_v \longrightarrow S \times \hat{S}, \quad a_v(F) := (\alpha_v(F), \det(F) \otimes \det(F_0)^\gamma).$$

Let now $K_v = K_v(S, H) := a_v^{-1}(0_S, \mathcal{O}_S)$, where $0_S$ is the zero of the Abelian group $S$: if $v^2 > 0$, the morphism

$$\tau_v : K_v \times S \times \hat{S} \longrightarrow M_v, \quad \tau(E, p, L) := \tau_p^*(E) \otimes L,$$

is a finite étale cover (for a proof of this, see section 4.2 of [56]).

2.2.2. O’Grady’s fibration. Another morphism $b_v : M_v \longrightarrow S \times \hat{S}$ was used by O’Grady in [43]. For $\gamma \in CH_0(S)$, we let $\Sigma(\gamma) \in S$ be the sum of the points of the support of a representative of $\gamma$, counted with multiplicities (i.e. the Albanese image of $\gamma$). For a coherent sheaf $F$ on $S$ we let $c_2(F) \in CH_0(S)$ be the second Chern class of $F$, and we set

$$\beta(F) := \Sigma(c_2(F)).$$

The morphism $b_v$ is defined as follows: we have

$$b_v : M_v \longrightarrow S \times \hat{S}, \quad b_v(F) := (\beta(F), \det(F) \otimes \det(F_0)^\gamma).$$

The relation between $a_v$ and $b_v$ is explained in the following

**Lemma 2.10.** There is an automorphism $g : S \longrightarrow S$ such that

$$b_v = (g \times \text{id}_{\hat{S}}) \circ a_v.$$

**Proof.** To prove this, we just need to show that for every $F_1, F_2 \in M_v$, we have $a_v(F_1) = a_v(F_2)$ if and only if $b_v(F_1) = b_v(F_2)$. Equivalently, we just need to show that for every $F_1, F_2 \in M_v$ such that $\det(F_1) \simeq \det(F_2)$, we have $a_v(F_1) = a_v(F_2)$ if and only if $\beta(F_1) = \beta(F_2)$.

Suppose first that $\det(F_1) \simeq \det(F_2)$ and $\alpha_v(F_1) = \alpha_v(F_2)$, and let $\Gamma := [F_1] - [F_2] \in K(S)$. As $v(F_1) = v(F_2) = v$ and $\det(F_1) \simeq \det(F_2)$, the only nontrivial Chern class of $\Gamma$ (in the Chow ring of $S$) is $c_2(\Gamma)$.

Moreover, there is a representative of $c_2(\Gamma)$ of the form

$$\tilde{\Gamma} := \sum_{i=1}^n p_i - \sum_{i=1}^n q_i,$$

where $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$ are points of $S$. We then notice that $\Gamma \in K(S)$ has the same rank and Chern classes of the class

$$\Gamma' := \left[ \bigoplus_{i=1}^n \mathbb{C}_{p_i} - \bigoplus_{i=1}^n \mathbb{C}_{q_i} \right] \in K(S).$$
Notice that if we let $\tilde{F} : K(S) \rightarrow K(\tilde{S})$ be the morphism induced by $F$ on the level of the Grothendieck groups, we have
\[ \det[\tilde{F}(\Gamma')] = \bigotimes_{i=1}^{n} P_{p_i} \otimes \mathcal{P}_{q_i}. \]

As $\det(\tilde{F}(\Gamma))$ depends only on the rank and the Chern classes of $\Gamma$ in the Chow ring of $\tilde{S}$, we get
\[ \det(\tilde{F}([\mathcal{F}_1])) \otimes \det(\tilde{F}([\mathcal{F}_2])) = \det(\tilde{F}(\Gamma)) = \det(\tilde{F}(\Gamma')) = \bigotimes_{i=1}^{n} P_{p_i} \otimes \mathcal{P}_{q_i} = \mathcal{P}_{\sum_{i=1}^{n} (p_i - q_i)} = \mathcal{P}_T, \]
where the equality $\bigotimes_{i=1}^{n} P_{p_i} \otimes \mathcal{P}_{q_i} = \mathcal{P}_{\sum_{i=1}^{n} (p_i - q_i)}$ follows from the fact that the map
\[ S \rightarrow \tilde{S}, \quad p \mapsto \mathcal{P}_p \]
is a group isomorphism.

Now, notice that as $\det(\mathcal{F}_1) = \det(\mathcal{F}_2)$, we have that $\alpha_v(\mathcal{F}_1) = \alpha_v(\mathcal{F}_2)$ if and only if $\det(\tilde{F}([\mathcal{F}_1])) = \det(\tilde{F}([\mathcal{F}_2]))$. The previous equalities give that this holds if and only if $\mathcal{P}_T = \mathcal{O}_S$. But is equivalent to $\Sigma([\mathcal{T}]) = 0$, where $[\mathcal{T}]$ is the class of $\mathcal{T}$ in $CH_0(\mathcal{S})$. As this class is $c_2(\mathcal{T})$, we finally get that $\alpha_v(\mathcal{F}_1) = \alpha_v(\mathcal{F}_2)$ if and only if $\Sigma(c_2(\mathcal{T})) = 0$. This last condition is equivalent to $\Sigma(c_2(\mathcal{F}_1)) = \Sigma(c_2(\mathcal{F}_2))$, i. e. to $\beta(\mathcal{F}_1) = \beta(\mathcal{F}_2)$, concluding the proof.

As a consequence, we see that $b_v$ is an isotrivial fibration.

2.3. Deformations of $(m, k)$–triples. We introduce the main construction we use in the following. Let $(S, v, H)$ be an $(m, k)$–triple, $T$ a smooth, connected curve, and use the following notation: if $f : Y \rightarrow T$ is a morphism and $\mathcal{L} \in Pic(Y)$, for every $t \in T$ we let $Y_t := f^{-1}(t)$ and $\mathcal{L}_t := \mathcal{L}|_{Y_t}$.

Definition 2.11. Let $(S, v, H)$ be an $(m, k)$–triple, and write the Mukai vector $v = m(r, \xi, a)$, where $\xi = c_1(L)$. A deformation of $(S, v, H)$ along $T$ is a triple $(\mathcal{X}, \mathcal{H}, \mathcal{L})$, where:
\begin{enumerate}
\item $\mathcal{X}$ is a projective, smooth deformation of $S$ along $T$, i. e. there is a smooth, projective, surjective map $f : \mathcal{X} \rightarrow T$ such that $\mathcal{X}_t$ is a projective surface for every $t \in T$, and there is $0 \in T$ such that $\mathcal{X}_0 \simeq S$;
\item $\mathcal{H}$ is a line bundle on $\mathcal{X}$ such that $\mathcal{H}_t$ is ample for every $t \in T$ and such that $\mathcal{H}_0 \simeq H$;
\item $\mathcal{L}$ is a line bundle on $\mathcal{X}$ such that $\mathcal{L}_0 \simeq L$.
\end{enumerate}

Remark 2.12. If $(S, v, H)$ is an $(m, k)$–triple and $(\mathcal{X}, \mathcal{H}, \mathcal{L})$ is a deformation of $(S, v, H)$ along a smooth, connected curve $T$, then $(\mathcal{X}_t, v_t, \mathcal{H}_t)$ is an $(m, k)$–triple if and only if $\mathcal{H}_t$ is $v_t$–generic (see
Remark 2.13 of [45]. Moreover, we recall that the subset of \( T \) given by those \( t \in T \) such that \( \mathcal{H}_t \) is not \( v_t \)-generic is locally closed by Lemmas 2.6 and 2.11 of [45].

**Remark 2.13.** Let \((S, v, H)\) be an \((m, k)\)-triple where \( v = m(r, \xi, a) \), with \( r > 0 \) and \( \xi = c_1(L) \). Let \( T \) be a smooth, connected curve, and \( f : \mathcal{X} \to T \) a smooth, projective deformation of \( S \) such that \( \mathcal{X}_0 \simeq S \) for some \( 0 \in T \). On \( \mathcal{X} \) consider two line bundles \( \mathcal{H} \) and \( \mathcal{L} \) such that \( \mathcal{H}_0 \simeq H \) and \( \mathcal{L}_0 \simeq L \). Then \((\mathcal{X}, \mathcal{H}, \mathcal{L})\) is a deformation of \((S, v, H)\) along \( T \) if and only if \( \mathcal{H}_t \) is ample for every \( t \in T \). As the set of \( t \in T \) such that \( \mathcal{H}_t \) is ample is Zariski open in \( T \), by removing a finite number of points from \( T \) we can always assume that \((\mathcal{X}, \mathcal{H}, \mathcal{L})\) is a deformation of \((S, v, H)\) along \( T \).

We let \( \phi : \mathcal{M} \to T \) be the relative moduli space of semistable sheaves and \( \phi^s : \mathcal{M}^s \to T \) the relative moduli space of stable sheaves associated to a deformation \((\mathcal{X}, \mathcal{H}, \mathcal{L})\) of an \((m, k)\)-triple \((S, v, H)\) along a smooth, connected curve \( T \). This means that for every \( t \in T \) we have \( \mathcal{M}_t = M_{v_t}(\mathcal{X}_t, \mathcal{H}_t) \) and \( \mathcal{M}_t^s = M_{v_t}^s(\mathcal{X}_t, \mathcal{H}_t) \).

If \( S \) is Abelian, let \( \mathcal{X} \to T \) be the dual family, i.e. the smooth projective family whose fiber over \( t \in T \) is the dual of \( \mathcal{H}_t \). Consider the following condition:

\((\ast)\) the morphism \( \phi : \mathcal{M} \to T \) has a section, and \( \mathcal{X} \to T \) is a \( T \)-group scheme.

If the condition \((\ast)\) holds, we have a \( T \)-morphism \( a_v : \mathcal{M} \to \mathcal{X} \times_T \mathcal{X} \) such that for every \( t \in T \) the restriction morphism \( a_{v_t, \mathcal{M}_t} \) is the Yoshioka fibration defined in section 2.2. If

\[ Z := \{(0_{\mathcal{X}_t}, \mathcal{O}_{\mathcal{X}_t}) \in \mathcal{X}_t \times_T \mathcal{X}_t \mid t \in T \} \subseteq \mathcal{X} \times_T \mathcal{X}, \]

the restriction of the family \( \mathcal{M} \) to \( Z \) is denoted \( \mathcal{X} \): restricting the morphism \( \phi \) to \( \mathcal{X} \) we get a morphism \( \phi_0 : \mathcal{X} \to T \), whose fiber over \( t \in T \) is \( K_v(\mathcal{X}_t, \mathcal{H}_t) \). A similar definition, but using \( \mathcal{M}^s \) instead of \( \mathcal{M} \), gives the family \( \phi_0^s : \mathcal{X}^s \to T \).

**Remark 2.14.** The condition \((\ast)\) is always verified up to a finite étale cover of \( T \).

The first result we need is that the families \( \mathcal{M} \) and \( \mathcal{X} \) are \( T \)-flat over a Zariski open neighborhood of any \( t \in T \) such that \((\mathcal{X}_t, v_t, \mathcal{H}_t)\) is an \((m, k)\)-triple. This is the content of the following Lemma:

**Lemma 2.15.** Let \((S, v, H)\) be an \((m, k)\)-triple, \( T \) a smooth, connected curve, \((\mathcal{X}, \mathcal{H}, \mathcal{L})\) a deformation of \((S, v, H)\) along \( T \), and assume that
if $S$ is Abelian then condition $(\ast)$ holds. Suppose that $t \in T$ is such that $(\mathcal{X}_t, v_t, \mathcal{H}_t)$ is an $(m, k)$–triple.

(1) The morphisms $\phi : \mathcal{M} \to T$ and $\phi_0 : \mathcal{H} \to T$ are flat at $t$.

(2) The morphisms $\phi^s : \mathcal{M}^s \to T$ and $\phi_0^s : \mathcal{H}^s \to T$ are smooth at $t$.

**Proof.** This is a consequence of the fact that $\mathcal{M}$ (resp. $\mathcal{H}$) is connected (since $T$ and the fibers are connected), and that the fibers are irreducible (by Theorem 4.4 of [23] for $\mathcal{M}$, and Remark A.1 of [46] for $\mathcal{H}$), reduced and of the same dimension. □

**Remark 2.16.** Instead of using Theorem 4.4 of [23], Lemma 2.15 could be proved using Theorem 1.17 of [33], which asserts that a simple sheaf on a K3 or Abelian surface $S$ can be extended along a smooth deformation of $S$ on a polydisk $\Delta$, as soon as its first Chern class remains of type $p_1, q_1$ along the deformation, and that in this case the relative moduli space of simple sheaves along $\Delta$ is smooth.

If $(S, v, H)$ is an $(m, k)$–triple, then choosing a nontrivial deformation of it along a smooth, connected curve $T$ we get a flat, projective deformation $\phi : \mathcal{M} \to T$ of $M_v$, and a smooth quasi-projective deformation $\phi^s : \mathcal{M}^s \to T$ of $M_v^s$. Moreover, if $S$ is Abelian we get a flat, projective deformation $\phi_0 : \mathcal{H} \to T$ of $K_v$, and a smooth quasi-projective deformation $\phi_0^s : \mathcal{H}^s \to T$ of $K_v^s$. We now prove that this deformation is locally trivial:

**Lemma 2.17.** Let $(S, v, H)$ be an $(m, k)$–triple, $T$ a smooth connected curve, $(\mathcal{X}, \mathcal{L}, \mathcal{H})$ a deformation of $(S, v, H)$ along $T$, and assume that if $S$ is Abelian then condition $(\ast)$ holds.

(1) If $p \in \mathcal{M}$ and $t := \phi(p)$ is such that $(\mathcal{X}_t, v_t, \mathcal{H}_t)$ is an $(m, k)$–triple, then $(\mathcal{M}, p) \simeq (\mathcal{M}_t, p) \times (T, t)$ as germs of analytic spaces.

(2) If $p \in \mathcal{H}$ and $t := \phi_0(p)$ is such that $(\mathcal{X}_t, v_t, \mathcal{H}_t)$ is an $(m, k)$–triple, then $(\mathcal{H}, p) \simeq (\mathcal{H}_t, p) \times (T, t)$ as germs of analytic spaces.

**Proof.** If $m = 1$, then $\phi$ is a smooth, projective morphism, and there is nothing to prove. If $m = 2$ and $k = 1$, this is Proposition 2.16 of [45]. For the remaining cases, by [23] $M_v$ and $K_v$ are symplectic varieties which are locally factorial, and by Corollary 1 of [36] they have terminal singularities. The Main Theorem of [37] tells us that for every $p \in M_v$ (resp. $p \in K_v$) and for every $n \in \mathbb{N}$, the infinitesimal $n$–th order deformation of $M_v$ (resp. of $K_v$) induced by $\phi$ (resp. by $\phi_0$), which is flat by Lemma 2.15, is locally trivial at $p$: the statement follows hence by Corollary 0.2 of [11]. □
As a corollary of this, using the Thom First Isotopy Lemma (see Theorem 3.5 of [5]) we have the following:

**Lemma 2.18.** Let \((S, v, H)\) be an \((m, k)\)-triple, \(T\) a smooth connected curve, \((\mathcal{X}, \mathcal{L}, \mathcal{H})\) a deformation of \((S, v, H)\) along \(T\) and assume that if \(S\) is Abelian then condition (*) holds.

1. If \(p \in \mathcal{M}\) and \(t := \phi(p)\) is such that \((\mathcal{X}_t, v_t, \mathcal{H}_t)\) is an \((m, k)\)-triple, there is an analytic open neighborhood \(U \subseteq T\) of \(t\) such that \(\phi^{-1}(U)\) is homeomorphic to \(\mathcal{M}_t \times U\).
2. If \(p \in \mathcal{H}\) and \(t := \phi_0(p)\) is such that \((\mathcal{X}_t, v_t, \mathcal{H}_t)\) is an \((m, k)\)-triple, there is an analytic open neighborhood \(U \subseteq T\) of \(t\) such that \(\phi_0^{-1}(U)\) is homeomorphic to \(\mathcal{H}_t \times U\).

2.4. **Isomorphisms between moduli spaces.** Here we describe several isomorphisms that will be frequently used in the proof of Theorem 1.17. All of them are induced by Fourier-Mukai transforms, either the tensorization with a line bundle or the one whose kernel is the ideal sheaf of the diagonal (for K3 surfaces) or the Poincaré bundle (for Abelian surfaces).

2.4.1. **Isomorphisms from tensorization with line bundles.** Let \(S\) be a projective K3 or Abelian surface, and let \(v = m(r, \xi, a)\) be a Mukai vector. Recall that if \(L \in \text{Pic}(S)\), we defined \(v_L := v \cdot \text{ch}(L)\), and that if \(D\) is a divisor on \(S\), we let \(v_D := v_{\mathcal{O}_S(D)}\) (see section 2.1, Lemma 2.8).

**Definition 2.19.** Let \(v, v' \in \tilde{H}(S, \mathbb{Z})\) be two Mukai vectors, and let \(v = (r, \xi, a), v' = (r', \xi', a')\).

1. If \(H\) is a polarization on \(S\), we say that \(v\) and \(v'\) are \(H\)-equivalent if there is \(s \in \mathbb{Z}\) such that \(v' = v_{sH}\).
2. If \(r, r' > 0\), we say that \(v\) and \(v'\) are equivalent if there is \(L \in \text{Pic}(S)\) such that \(v' = v_L\).

The following is the main result about isomorphisms induced by tensorization with a line bundle\(^2\), which shows that moduli spaces of sheaves corresponding to equivalent (or \(H\)-equivalent) Mukai vectors are isomorphic (and the isomorphism is induced by tensorization with a suitable line bundle).

**Lemma 2.20.** Let \(S\) be a K3 or Abelian surface, \(v = m(r, \xi, a)\) a Mukai vector and \(H\) an ample line bundle on \(S\).

\(^2\)In the following Lemma, by a slight abuse of notation we let \(K_v\) denote not only the fiber of \(a_v : \mathcal{M}_v \rightarrow S \times \hat{S}\) over \((0_S, \mathcal{O}_S)\), but also any other fiber. This is justified since \(a_v\) is an isotrivial fibration, so all its fibers are isomorphic.
(1) For every \(d \in \mathbb{Z}\) the morphism
\[
M_v(S, H) \longrightarrow M_{v+dH}(S, H), \quad \mathcal{F} \mapsto \mathcal{F} \otimes \mathcal{O}_S(dH)
\]
is an isomorphism, which induces isomorphisms \(M^s_v \cong M^s_{v+dH}\),
\(K_v \cong K_{v+dH}\), and \(K_v^s \cong K_{v+dH}^s\).

(2) If \(r > 0\), \(L \in \text{Pic}(S)\) and \(H\) is \(v\)-generic, the morphism
\[
M_v(S, H) \longrightarrow M_{vL}(S, H), \quad \mathcal{F} \mapsto \mathcal{F} \otimes L
\]
is an isomorphism, which induces isomorphisms \(M^s_v \cong M^s_{vL}\),
\(K_v \cong K_{vL}\) and \(K_v^s \cong K_{vL}^s\).

Proof. First, notice that \(v(\mathcal{F} \otimes L) = v(\mathcal{F}) \cdot \text{ch}_L\). To prove the first point of the statement, it is enough to remark that a sheaf \(\mathcal{F}\) of Mukai vector \(v\) is \(H\)-\((semi)\)stable if and only if \(\mathcal{F} \otimes \mathcal{O}_S(dH)\) is \(H\)-\((semi)\)stable.

For the second point, we need to show that if \(\mathcal{F}\) is \(H\)-\((semi)\)stable, then \(\mathcal{F} \otimes L\) is \(H\)-\((semi)\)stable. This is proved for stable sheaves by Yoshioka (see Lemma 1.1 of [56]), and the proof goes through for semistable sheaves.

If \(S\) is Abelian, by definition of the morphism \(a_v\) (see section 2.2) we have that if \(\mathcal{F}_1\) and \(\mathcal{F}_2\) are in the same fiber of \(a_v\), then \(\mathcal{F}_1 \otimes L\) and \(\mathcal{F}_2 \otimes L\) are in the same fiber of \(a_{vL}\). As \(a_v\) and \(a_{vL}\) are both isotrivial fibrations, the isomorphism between \(M_v\) and \(M_{vL}\) obtained by tensorization with \(L\) induces an isomorphism between \(K_v\) and \(K_{vL}\). \(\Box\)

For Mukai vectors of rank 0 it is in general not true that tensoring with any line bundle induces an isomorphism between moduli spaces, as the following example shows.

Example 2.21. We let \(S\) be an elliptic K3 surface as in Example 2.1 and we use the same notation for \(\Sigma, F, j_\Sigma\) and \(j_F\). For \(a \gg 0\), consider the polarization \(H\) with \(c_1(H) = \sigma + af\) and the Mukai vector \(v = (0, \sigma + f, a - 1)\). As \(\sigma \gg 0\) we have that \(H\) is \(v\)-generic.

We let \(N \in \text{Pic}^a(S)\) and \(M \in \text{Pic}^1(F)\), so that \(j_\Sigma^*N\) and \(j_F^*M\) are both \(H\)-stable, and \(v(j_\Sigma^*N) = (0, \sigma, a - 2)\) and \(v(j_F^*M) = (0, f, 1)\). The coherent sheaf \(\mathcal{F} := j_\Sigma^*N \oplus j_F^*M\) is \(H\)-semistable and its Mukai vector is \(v\). We let \(\mathcal{G} := j_F^*M\).

If \(L := \mathcal{O}_S(\Sigma)\), then \(\mathcal{F} \otimes L\) is not \(H\)-\(\text{semistable}\): indeed \(\mathcal{G} \otimes L\) is a subsheaf of \(\mathcal{F} \otimes L\), \(v(\mathcal{F} \otimes L) = (0, \sigma + f, a - 2)\) and \(v(\mathcal{G} \otimes L) = (0, f, 2)\), so \(p_H(\mathcal{G} \otimes L) > p_H(\mathcal{F} \otimes L)\) since
\[
p_H(\mathcal{F} \otimes L, n) = n + \frac{a - 2}{a - 1}, \quad p_H(\mathcal{G} \otimes L, n) = n + 2.
\]
2.4.2. Isomorphisms from Fourier-Mukai transforms: K3 surfaces. We now recall two basic results, originally due to Yoshioka, about isomorphisms between moduli spaces of sheaves over K3 surfaces coming from Fourier-Mukai transforms. Yoshioka’s theorems are stated in a more general setting. Here we present simplified adapted proofs for the convenience of the reader. In this section we will only consider the Fourier-Mukai transform whose kernel is the ideal of the diagonal.

We need the following notation: if $S$ a projective K3, we let $\Delta$ be the diagonal and $I$ the ideal of $\Delta$. We have an exact sequence of coherent sheaves on $\hat{S}$:

$$0 \to \mathcal{I} \to \mathcal{O}_{\hat{S}} \to \mathcal{O}_{\Delta} \to 0$$

(1)

We let $\pi_1, \pi_2 : \hat{S} \to S$ be the two projections and consider

$$F : D^b(S) \to D^b(S), \quad F(E^*) := R\pi_2_*(\pi_1^*E^* \otimes \mathcal{I}),$$

$$\hat{F} : D^b(S) \to D^b(S), \quad \hat{F}(E^*) := R\hat{\mathcal{H}}\text{om}_{\pi_1}(\mathcal{I}, \pi_2^*E^*).$$

By [4] we know that $F$ is an equivalence of triangulated categories; moreover, the functor $F_{\hat{\pi}_2}$ is the right and left adjoint to $F$, so that $F \circ \hat{F} = [-2]$ (see Proposition 1.26 of [19]). We say that a coherent sheaf $G$ verifies WIT(0) (resp. WIT(2)) with respect to $F$ if $F(G) = F^0(G)$ (resp. $F(G) = F^2(G)[-2]$).

If $G$ is a coherent sheaf on $S$, then the functor $R\pi_{2*}(\pi_1^*G \otimes \mathcal{L} \cdot)$ applied to the exact sequence (1) gives the long exact sequence of coherent sheaves on $S$

$$0 \to F^0(G) \to \mathcal{O}_S \otimes H^0(G) \xrightarrow{ev} G \to$$

(2)

$$\to F^1(G) \to \mathcal{O}_S \otimes H^1(G) \to 0 \to$$

$$\to F^2(G) \to \mathcal{O}_S \otimes H^2(G) \to 0$$

and similarly the functor $R\hat{\mathcal{H}}\text{om}_{\pi_1}(\cdot, \pi_2^*G)$ applied to the exact sequence (1) gives the long exact sequence of coherent sheaves on $S$

$$0 \to \mathcal{O}_S \otimes H^0(G) \to \hat{F}^0(G) \to$$

(3)

$$\to 0 \to \mathcal{O}_S \otimes H^1(G) \to \hat{F}^1(G) \to$$

$$\to \mathcal{G} \to \mathcal{O}_S \otimes H^2(G) \to \hat{F}^2(G) \to 0$$

We use the following notation: if $v = (r, \xi, a)$, we let $\tilde{v} := (a, -\xi, r)$. The first result we need is the following (see Theorem 3.18 of [57]).

**Lemma 2.22.** Let $S$ be a K3 surface such that $\text{Pic}(S) = \mathbb{Z} \cdot H$, where $H$ is an ample line bundle such that $H^2 = 2l$. Fix furthermore $r, k \in \mathbb{N}^*$, and let $v = (r, nh, a)$ be a Mukai vector on $S$ such that $v^2 = 2k$, where $h := c_1(H)$. 

(1) There is $n_0 \in \mathbb{N}$ such that if $n > n_0$, every $H-$semistable (resp. $H-$stable) sheaf $\mathcal{E}$ with Mukai vector $v$ on $S$ verifies WIT(0) with respect to $F$, and the sheaf $F^0(\mathcal{E})$ is $H-$semistable (resp. $H-$stable) with Mukai vector $\bar{v}$.

(2) The functor $F$ induces isomorphisms $M_v \simeq M_{\bar{v}}$ and $M^s_v \simeq M^s_{\bar{v}}$.

Proof. The first point of the statement implies that the functor $F$ induces an injective morphism $f : M^{(s)}_v \rightarrow M^{(s)}_{\bar{v}}$. By Theorem 4.4 of [23] these moduli spaces are irreducible of the same dimension: it follows that $f$ is an isomorphism.

It then only remains to prove the first point of the statement. We divide the proof in several steps, and we will present it only for semistable sheaves (for stable sheaves it is similar).

Step 1: the sheaf $\mathcal{E}$ verifies WIT(0) with respect to $F$. We first remark that if $s \in \mathbb{N}$, then $v_sH = (r, n, h, a)$ where $n_s = n + rs$ and $a_s = a + 2lns + rls^2$. As $M_v \simeq M_{v_{sH}}$ by point (1) of Lemma 2.20 and as $n \equiv n_s$ mod $r$, the number of equivalence classes of Mukai vectors on $S$ of rank $r$ and square $2k$ is at most $r$.

The family of semistable sheaves with fixed Mukai vector $v$ is bounded: it follows that there is $T \in \mathbb{N}$ such that if $s > T$, for an $H-$semistable sheaf $\mathcal{E}$ with Mukai vector $v$ we have $H^1(\mathcal{E} \otimes \mathcal{O}_S(sH)) = H^2(\mathcal{E} \otimes \mathcal{O}_S(sH)) = 0$ and the evaluation morphism

$$H^0(\mathcal{E} \otimes \mathcal{O}_S(sH)) \otimes \mathcal{O}_S \rightarrow \mathcal{E} \otimes \mathcal{O}_S(sH)$$

is surjective.

Now, as seen before the set of equivalence classes of Mukai vector on $S$ of rank $r$ and square $2k$ is finite. Moreover, the tensorization with $H$ preserves the semistability and induces an isomorphism between moduli spaces of sheaves (by Lemma 2.20). From this, it follows that there is uniform bound $T \in \mathbb{N}$, depending only on $r$, $k$ and $l$, such that if $n > T$ then each $H-$semistable sheaf $\mathcal{E}$ with Mukai vector $v = (r, nh, a)$ is such that $H^1(\mathcal{E}) = H^2(\mathcal{E}) = 0$ and the evaluation morphism $H^0(\mathcal{E}) \otimes \mathcal{O}_S \rightarrow \mathcal{E}$ is surjective. By the exact sequence (2) for $\mathcal{E}$ we get $F^1(\mathcal{E}) = F^2(\mathcal{E}) = 0$, so that $F(\mathcal{E}) = F^0(\mathcal{E})$.

Step 2: the sheaf $F^0(\mathcal{E})$ is locally free, and its Mukai vector is $\bar{v}$. By Step 1, the exact sequence (2) applied to $\mathcal{E}$ gives the exact sequence

$$0 \rightarrow F^0(\mathcal{E}) \rightarrow H^0(\mathcal{E}) \otimes \mathcal{O}_S \rightarrow \mathcal{E} \rightarrow 0.$$  

We then see that $v(F^0(\mathcal{E})) = \bar{v}$.

As $\mathcal{E}$ is a torsion-free and $S$ is a surface, the projective dimension of $\mathcal{E}$ is at most 1. The exact sequence (4) is a resolution of $\mathcal{E}$ where $H^0(\mathcal{E}) \otimes \mathcal{O}_S$ is locally free and $F^0(\mathcal{E})$ is torsion-free. If $F^0(\mathcal{E})$ was not locally-free, then we would get a locally free resolution of length 2 of $\mathcal{E}$.
(obtained by replacing $F^0(\mathcal{E})$ by a locally free resolution), and hence the projective dimension of $\mathcal{E}$ would be 2. It follows that $F^0(\mathcal{E})$ has to be locally free.

**Step 3: the sheaf $F^0(\mathcal{E})$ is $H$–semistable.** We proceed by contradiction, supposing that $F^0(\mathcal{E})$ is not $H$–semistable. Then there is a subsheaf $\mathcal{G}_1 \subseteq F^0(\mathcal{E})$ such that $p_H(\mathcal{G}_1) > p_H(F^0(\mathcal{E}))$, and we can choose it to be $H$–stable and having maximal reduced Hilbert polynomial. Such a $\mathcal{G}_1$ is the first term of a Jordan-Hölder filtration of the first term of a Harder-Narasimhan filtration of $F^0(\mathcal{E})$.

The sheaf $\mathcal{G}_1$ is locally free. Indeed, if it was not locally free, $\mathcal{G}_1^{**}$ would be a locally free subsheaf of $F^0(\mathcal{E})$ with $p_H(\mathcal{G}_1^{**}) > p_H(\mathcal{G}_1)$, contradicting the maximality of $p_H(\mathcal{G}_1)$.

We write $v(\mathcal{G}_1) = (a_1, -n_1 h, r_1)$. As $\mathcal{G}_1$ is locally free and contained in $F^0(\mathcal{E})$ (which is locally free of rank $a$), we get $0 < a_1 < a$. As

$$p_H(F^0(\mathcal{E}), m) = m^2 - \frac{2n}{a} m + \frac{a + r}{a}, \quad p_H(\mathcal{G}_1, m) = m^2 - \frac{2n_1}{a_1} m + \frac{a_1 + r_1}{a_1},$$

and as $p_H(\mathcal{G}_1) > p_H(F^0(\mathcal{E}))$, we have either $n_1/a_1 < n/a$ or $n_1/a_1 = n/a$ and $r_1/a_1 > r/a$.

We claim that $n_1 > 0$. Indeed, we have $\mathcal{G}_1 \subseteq F^0(\mathcal{E}) \subseteq H^0(\mathcal{E}) \otimes \mathcal{O}_S$, and as $H^0(\mathcal{E}) \otimes \mathcal{O}_S$ is $H$–semistable of slope 0, we have $\mu_H(\mathcal{G}) \leq 0$: but since $\mu_H(\mathcal{G}) = -2l n_1/a_1$, we get $n_1 \geq 0$.

Moreover, if $n_1 = 0$, then $\mathcal{G}_1$ would contain a $\mu_H$–stable locally free subsheaf $\mathcal{G}_1'$ such that $\mu_H(\mathcal{G}_1') = 0$. Then $\mathcal{G}_1'$ is a subsheaf $H^0(\mathcal{E}) \otimes \mathcal{O}_S$, so there is a nontrivial morphism $g : \mathcal{G}_1' \to \mathcal{O}_S$.

This morphism is injective (since if the kernel was not trivial, then it would be a subsheaf of $\mathcal{G}_1'$ with slope 0, contradicting the $\mu_H$–stability of $\mathcal{G}_1'$), and as $\mathcal{G}_1'$ is locally free we get that $g$ is an isomorphism. It follows that $\mathcal{O}_S \subseteq F^0(\mathcal{E})$: this means that $F^0(\mathcal{E})$ has a section, which is impossible (since $F^0(\mathcal{E})$ is the kernel of an evaluation map). This proves that $n_1 > 0$.

To resume, if $F^0(\mathcal{E})$ is not $H$–semistable, then it has an $H$–stable subsheaf $\mathcal{G}_1$ of Mukai vector $v(\mathcal{G}_1) = (a_1, -n_1 h, r_1)$ where $a_1, n_1 > 0$ and either $n_1/a_1 < n/a$, or $n_1/a_1 = n/a$ and $r_1/a_1 > r/a$. Notice that in particular $n_1 < n$.

Now, let $\mathcal{G}_2$ be the quotient of $F^0(\mathcal{E})$ by its subsheaf $\mathcal{G}_1$, so that we have an exact sequence

$$0 \to \mathcal{G}_1 \to F^0(\mathcal{E}) \to \mathcal{G}_2 \to 0.$$
Applying the functor $\hat{F}$ to it, and by using the fact that $\hat{F} \circ F = [-2]$, then we get the two exact sequences

\[(5) \quad 0 \to \hat{F}^0(\mathcal{G}_2) \to \hat{F}^1(\mathcal{G}_1) \to 0\]

and

\[(6) \quad 0 \to \hat{F}^1(\mathcal{G}_2) \to \hat{F}^2(\mathcal{G}_1) \to \mathcal{E} \to \hat{F}^2(\mathcal{G}_2) \to 0\]

The sheaf $\mathcal{G} := \hat{F}^2(\mathcal{G}_1)/\hat{F}^1(\mathcal{G}_2)$ is then a subsheaf of $\mathcal{E}$. We show that $p_H(\mathcal{G}) > p_H(\mathcal{E})$: it follows that $\mathcal{E}$ is not $H$–semistable, completing the contradiction argument.

To do so, let us first show that $\mathcal{G}_1$ verifies WIT(2) with respect to $\hat{F}$.

First, notice that

\[c_1(\mathcal{G}_2) = c_1(F^0(\mathcal{E})) - c_1(\mathcal{G}_1) = (n_1 - n)h,\]

so that $\mu_H(\mathcal{G}_2) < 0$ (since $n_1 < n$). More generally, all the subsheaves of $\mathcal{G}_2$ have strictly negative first Chern class. Indeed, the starting term of the Harder-Narasimhan filtration of $F^0(\mathcal{E})$ has the same reduced Hilbert polynomial of $\mathcal{G}_1$, and $\mu_H(\mathcal{G}_1) < 0$.

Hence all the direct summands of $gr(F^0(\mathcal{E}))$ and of $gr(\mathcal{G}_2)$ (i. e. the quotients of the Harder-Narasimhan filtration of $F^0(\mathcal{E})$ and of $\mathcal{G}_2$) have all strictly negative first Chern class, so this holds for all subsheaves of $F^0(\mathcal{E})$ and of $\mathcal{G}_2$. In particular, we see that $H^0(\mathcal{G}_2) = 0$.

The exact sequence (5) applied to $\mathcal{G}_2$ then gives $\hat{F}^0(\mathcal{G}_2) = 0$: by the exact sequence (5) we then get $\hat{F}^1(\mathcal{G}_1) = 0$, so that $\mathcal{G}_1$ verifies WIT(2) with respect to $\hat{F}$.

We now claim that $r_1 > 0$. Indeed, as $\mathcal{G}_1$ verifies WIT(2) with respect to $\hat{F}$, by the exact sequence (5) applied to $\mathcal{G}_1$ we see that $v(\hat{F}^2(\mathcal{G}_1)) = (r_1, n_1 h, a_1)$, hence $r_1 \geq 0$. If $r_1 = 0$, the morphism $\hat{F}^2(\mathcal{G}_1) \to \mathcal{E}$ in the exact sequence (5) would be trivial (since $\mathcal{E}$ is torsion-free and $\hat{F}^2(\mathcal{G}_1)$ is torsion). As $\hat{F}$ is fully faithful, this would imply that the inclusion morphism $\mathcal{G}_1 \to F^0(\mathcal{E})$ is trivial, getting a contradiction: it follows that $r_1 > 0$.

Moreover, the exact sequence (5) applied to $\mathcal{G}_2$ shows that $\hat{F}^1(\mathcal{G}_2)$ is an extension of a subsheaf of $\mathcal{G}_2$ by $\mathcal{O}_S \otimes H^1(\mathcal{G}_2)$: it follows that the first Chern class of $\hat{F}^1(\mathcal{G}_2)$ is negative.

Now, notice that as $\mathcal{G} = \hat{F}^2(\mathcal{G}_1)/\hat{F}^1(\mathcal{G}_2)$, we have

\[c_1(\mathcal{G}) = c_1(\hat{F}^2(\mathcal{G}_1)) - c_1(\hat{F}^1(\mathcal{G}_2)).\]

As $c_1(\hat{F}^1(\mathcal{G}_2))$ is negative, it follows that $p_H(\mathcal{G}) > p_H(\hat{F}^2(\mathcal{G}_1))$. If $p_H(\hat{F}^2(\mathcal{G}_1)) > p_H(\mathcal{E})$, we are done.
To show this, recall that \( v(\mathcal{F}^2(\mathcal{G}_1)) = (r_1, n_1 h, a_1) \), hence we get
\[
 p_H(\mathcal{E}) = m^2 + 2n_m + \frac{a + r}{r}, \quad p_H(\mathcal{F}^2(\mathcal{G}_1)) = m^2 + \frac{2n_1}{r_1}m + \frac{a_1 + r_1}{r_1}.
\]
Moreover, recall that:

1. we have \( r, n, a > 0 \) and \( \ln^2 - ra = v^2/2 = k > 0 \);
2. we have \( r_1, n_1, a_1 > 0 \);
3. we have \( a_1 < a \), and either \( n_1/a_1 < n/a \), or \( n_1/a_1 = n/a \) and \( r_1/a_1 > r/a \);
4. we have \( \ln^2 - r_1a_1 = v(\mathcal{G}_1)^2/2 \geq -1 \), since \( \mathcal{G}_1 \) is \( H \)-stable.

By Lemma 2.23 below, it follows that there is \( n_0 \in \mathbb{N} \) such that for every \( n > n_0 \), we have \( n_1/r_1 \geq n/r \), and if \( n_1/r_1 = n/r \), then \( a_1/r_1 > a/r \).

But this means that \( p_H(\mathcal{F}^2(\mathcal{G}_1)) > p_H(\mathcal{E}) \), completing the proof. \( \square \)

We now prove the following, which is used to conclude the proof of the previous Lemma:

**Lemma 2.23.** Fix \( k, l, r \in \mathbb{N}^* \), and let \( n, a, r_1, a_1, n_1 \in \mathbb{N}^* \) be such that the following conditions are fulfilled:

1. \( \ln^2 - ra = k \);
2. \( \ln^2 - r_1a_1 \geq -1 \);
3. \( a_1 < a \);
4. \( n_1/a_1 < n/a \) or \( n_1/a_1 = n/a \) and \( r_1/a_1 > r/a \).

If \( n > 32r^3k \), then either \( n_1/r_1 > n/r \), or \( n_1/r_1 = n/r \) and \( a_1/r_1 > a/r \).

**Proof.** We let \( k_1 := \ln^2 - r_1a_1 \), so that \( k_1 \geq -1 \). As \( n_1/a_1 \leq n/a \), it follows that \( n_1/n \leq a_1/a \). Moreover, as \( a_1 = \frac{\ln^2 - k_1}{r_1} \) and \( a = \frac{\ln^2 - k}{r} \), we get the inequality
\[
\frac{n_1}{n} \leq \frac{r}{r_1} \cdot \frac{n_1}{n} \cdot \frac{n_1 - k_1}{k_1}.
\]
This implies that
\[
1 \leq \frac{r}{r_1} \cdot \frac{n_1 - k_1}{k_1} \cdot \frac{k_1}{n - k_1}.
\]

We claim that as \( n > 32r^3k \) we have \( r > r_1 \). Indeed, as \( n > 32r^3k \geq 3k \) and \( k_1 \geq -1 \), we get
\[
\frac{n_1 - k_1}{n - k_1} \leq \frac{n_1 + 1}{n_1} \leq \frac{n_1 + 1}{n_1} \leq \frac{n_1 + 1}{n_1} \leq \frac{n_1}{n_1} + \frac{2}{3}
\]
where the last inequality follows from the fact that \( n > n_1 \) (so that \( n - n_1 \geq 1 \)).
As $n > n_1$ and $n > 3k$, the last term of the inequality (8) is strictly smaller than 1: this is trivial if $n_1 \geq 2$; if $n_1 = 1$, then $n > n_1 + 1$ (as $n \geq 3$), and hence the last inequality in equation (8) is strict. In any case we get

$$\frac{n_1 - \frac{k}{ln_1}}{n - \frac{k}{n}} < 1,$$

hence the inequality (7) gives $r > r_1$.

We now write the inequality (7) in a different form. More precisely, we have

$$1 \leq \frac{n_1}{r_1} \cdot \frac{1 - \frac{k}{ln_1}}{1 - \frac{k}{ln^2}} \leq \frac{n_1}{r} \cdot \frac{1 + \frac{1}{ln_1}}{1 - \frac{k}{ln^2}},$$

where the last equality follows from $k_1 \geq -1$. As $n > 32r^3k$ we see that $1 - \frac{k}{ln^2} > 0$, hence the previous inequality becomes

(9) $$\frac{n_1}{r_1} \cdot \frac{1 - \frac{k}{ln_1}}{1 + \frac{1}{ln_1}} \leq 1 - \frac{k}{ln^2} + \frac{1}{ln_1},$$

We first want to show that $n_1/r_1 \geq n/r$. As the first term of the inequality (9) is an integral multiple of $\frac{1}{r_1 n}$, by the inequality (9) it is enough to show that

$$\frac{k}{ln^2} + \frac{1}{ln_1} \leq \frac{1}{r_1 n}.$$

To do so, notice first that $1 + \frac{1}{ln_1} > 1$, and that as $1/ln_1^2 \leq 1$ we have

$$\frac{1}{1 + \frac{1}{ln_1^2}} \leq \frac{1}{2}.$$

Moreover, as $n > 32r^3k$, we get $\frac{k}{ln^2} < 1/4$. We then find that

$$1 - \frac{k}{ln^2} + \frac{1}{ln_1} = 1 - \frac{k}{ln^2} + \frac{1}{ln_1} < \frac{1}{4},$$

so that we finally get

$$\frac{n_1}{r_1} \geq \frac{n}{4r}.$$

This implies that $n_1 \geq \frac{n}{4r}$, hence we get

$$\frac{1}{ln_1^2} \leq \frac{1}{ln^2} \leq \frac{1}{ln^2} \frac{32r^3k}{16r^2} = \frac{1}{2l rnk} \leq \frac{1}{2r n}.$$
Lemma 2.24. Let $M$ be a Mukai vector on $S$ which allows us to pass from Mukai vectors of rank 0 to Mukai vectors of strictly positive rank (see Proposition 3.14 of [57] for a proof for stable sheaves). We use the following notation: if $(v, \xi, a)$ is such an object, write $h := c_1(H)$, and suppose that $H$ is $v$–generic and $\tilde{v}$–generic.

1. There is $a_0 \in \mathbb{N}$ such that if $a > a_0$, every $H$–semistable (resp. $H$–stable) sheaf $E$ with Mukai vector $v$ on $S$ verifies $\text{WIT}(0)$ with respect to $F$, and the sheaf $F(E)$ is $H$–semistable (resp. $H$–stable) with Mukai vector $\tilde{v}$.

2. The functor $F$ induces isomorphisms $M_v \cong M_{\tilde{v}}$ and $M_v^s \cong M_{\tilde{v}}^s$.

Proof. As in the proof of Lemma 2.22, the second point follows from the first, since the moduli spaces are irreducible and of the same dimension (by Theorem 4.4 of [23]). We are then left to show the first point of the statement, and again we present only the proof for semistable sheaves, the proof for stable ones being similar.

Notice that if $v = (0, \xi, a)$ and $s \in \mathbb{N}$, then $v_s = (0, \xi, a_s)$ where $a_s = a + s\xi \cdot H$. Hence $a_s \equiv a \mod \xi \cdot H$: as by point (1) of Lemma 2.20 the tensorization with $H$ preserves the $H$–semistability and induces an isomorphisms between moduli spaces, we have $M_v \cong M_{(0, \xi, a)}$.

It follows that the number of $H$–equivalence classes of Mukai vectors of rank 0 and first Chern class $\xi$ is finite (its cardinality is at most $\xi \cdot H$).

As in the proof of Lemma 2.22 this implies that there is $T \in \mathbb{N}$ such that if $a > T$ then all the sheaves $E$ of Mukai vector $v = (0, \xi, a)$ are such that $H^1(E) = H^2(E) = 0$ and the evaluation map $H^0(E) \otimes \mathcal{O}_S \longrightarrow E$ is surjective. This implies that $F(E) = F(E)$, which is a locally free sheaf as the projective dimension of $E$ is 1, and that $v(F(E)) = \tilde{v}$ (see Step 1 and Step 2 of the proof of Lemma 2.22).
We are then only left to prove that $F^0(\mathcal{E})$ is $H$—semistable, and again the proof is by contradiction, so we start by supposing that $F^0(\mathcal{E})$ is not $H$—semistable. Hence it has an $H$—stable locally free subsheaf $\mathcal{G}'$ such that $p_H(\mathcal{G}') > p_H(F^0(\mathcal{E}))$. We write $v(\mathcal{G}') = (a', -\xi, r')$, where $0 < a' < a$.

We notice that $\xi'$ is effective and $\xi' \neq 0$. Indeed, we have that $\text{det}(\mathcal{G}') \subseteq \mathcal{O}_S$, so $\mu_H(\mathcal{G}') \leq 0$. If $\xi'$ was not effective, as seen in Step 3 of Lemma 2.22 a $\mu_H$—stable subsheaf of $\mathcal{G}'$ would be isomorphic to $\mathcal{O}_S$, so that $\mathcal{G}'$ would have a section. As $\mathcal{G}' \subseteq F^0(\mathcal{E})$, this would imply that $F^0(\mathcal{E})$ has a section, which is not possible as $F^0(\mathcal{E})$ is the kernel of an evaluation map.

We let $d := \xi \cdot H$ and $d' := \xi' \cdot H$ (hence $d, d' > 0$, since $\xi$ and $\xi'$ are effective). Writing explicitly the reduced Hilbert polynomials of $\mathcal{G}'$ and of $\mathcal{E}$, as $p_H(\mathcal{G}') > p_H(F^0(\mathcal{E}))$ we get that either $d'/a' < d/a$, or $d'/a' = d/a$ and $r'/a' > 0$. Notice that as $d'/a' \leq d/a$, we get $d'/a' \leq a'/a < 1$.

We claim that if $a > 0$, then $r' \leq 0$: this will imply that $d'/a' < d/a$ (as otherwise we have $d'/a' = d/a$ and $r'/a' > 0$, so $r > 0$). Indeed, as $\mathcal{G}'$ is $H$—stable we have $v(\mathcal{G}')^2 \geq -2$. Hence, if $r' \geq 1$ we get

$$-2 \leq v(\mathcal{G}')^2 = (\xi')^2 - 2a'r' \leq (\xi')^2 - 2a'.$$

As $d'/d \leq a'/a$, it follows that $a' \geq a\frac{d'}{d}$, so that

$$-2 \leq (\xi')^2 - 2a\frac{d'}{d}.$$

Now, for every $d \in \mathbb{N}$, the set of numerical equivalence classes of effective curves $C$ on $S$ such that $C \cdot H \leq d$ is finite. It follows that the set

$$\left\{ \frac{d \cdot C^2 + 2}{2C \cdot H} \mid C \subseteq S \text{ is an effective curve, } C \cdot H < d \right\}$$

has a maximum, denoted $M_d$. Now, suppose that $a > M_d$. As $\xi'$ is an effective divisor such that $\xi' \cdot H < d$, we have

$$(\xi')^2 - 2a\frac{d'}{d} < (\xi')^2 - 2d\frac{(\xi')^2 + 2}{2d'} \cdot \frac{d'}{d} = -2,$$

getting a contradiction.

Now, the same proof of Lemma 2.22 shows that $\mathcal{G}'$ verifies WIT(2) with respect to $\hat{F}$, and $\hat{F}^2(\mathcal{G}')$ is a sheaf whose Mukai vector is $(r', \xi', a')$. It follows that $r' \geq 0$, and since $r' \leq 0$ we get $r' = 0$. As $\mathcal{G}'$ $H$—destabilizes $F^0(\mathcal{E})$, this implies that $d'/a' < d/a$. 

If we now let $\mathcal{G}''$ be the quotient of $F^0(\mathcal{E})$ by $\mathcal{G}'$, by applying the functor $\hat{F}$ to the exact sequence

$$0 \to \mathcal{G}' \to F^0(\mathcal{E}) \to \mathcal{G}'' \to 0$$

we get the exact sequence

$$0 \to \hat{F}^1(\mathcal{G}'') \to \hat{F}^2(\mathcal{G}') \to \mathcal{E} \to \hat{F}^2(\mathcal{G}'') \to 0.$$

Now, the exact sequence (3) applied to $\mathcal{G}''$ shows that $\hat{F}^1(\mathcal{G}'')$ is an extension of a subsheaf of $\mathcal{G}''$ by a locally free sheaf. As $\mathcal{G}''$ is torsion-free, it follows that $\hat{F}^1(\mathcal{G}'')$ is torsion-free. As it is a subsheaf of $\hat{F}^2(\mathcal{G}')$, whose rank is $r'=0$, it has to be trivial. The previous exact sequence then becomes

$$0 \to \hat{F}^2(\mathcal{G}') \to \mathcal{E} \to \hat{F}^2(\mathcal{G}'') \to 0.$$

As $\hat{F}^2(\mathcal{G}')$ is a subsheaf of $\mathcal{E}$, it follows that $\xi' \neq 0$ (otherwise $\mathcal{E}$ would have 0-dimensional torsion).

But now notice that $p_H(\mathcal{E}, m) = m + a/d$, $p_H(\hat{F}^2(\mathcal{G}'), m) = m + a'/d'$, and recall that $d'/a' < d/a$, so $a'/d' > a/d$. Hence $p_H(\hat{F}^2(\mathcal{G}')) > p_H(\mathcal{E})$, so $\mathcal{E}$ is not $H$-semistable, which is not possible. \hfill $\square$

2.4.3. Isomorphisms from Fourier-Mukai functors: Abelian surfaces.

In the previous section we have seen that if $S$ is a projective K3 surface, then the Fourier-Mukai transform whose kernel is the ideal of the diagonal of $S \times S$ induces isomorphisms between moduli spaces of semistable sheaves (under conditions on the Mukai vectors).

In this section we prove that similar results hold true when the base surface is Abelian, but in this case the Fourier-Mukai transform one has to use the Poincaré bundle as kernel: again these results are originally due to Yoshioka, and we present here the proofs for the convenience of the reader.

Let $S$ be an Abelian surface, $\hat{S}$ its dual and $\mathcal{P}$ the Poincaré line bundle in $S \times \hat{S}$. We then consider the two functors

$$F : D^b(S) \to D^b(\hat{S}), \quad F(\mathcal{E}^\bullet) := R\pi_{\hat{S}*}(\pi_S^*E^\bullet \otimes \mathcal{P}),$$

$$\hat{F} : D^b(\hat{S}) \to D^b(S), \quad \hat{F}(\mathcal{E}^\bullet) := \tau^* R\pi_S(\pi_S^*E^\bullet \otimes \mathcal{P}),$$

where $\pi_S$ and $\pi_{\hat{S}}$ are the two projections of $S \times \hat{S}$ onto $S$ and $\hat{S}$ respectively, and $\tau : \hat{S} \to S$ is the involution acting as $-1$. By [4] we know that $F$ is an equivalence; moreover we have $F \circ \hat{F} = [-2]$ (see Theorem 2.2 of [35]).
If \( L \in \text{Pic}(S) \), we let \( \hat{L} := \det(-[F(L)]) \in \text{Pic}(\hat{S}) \); moreover, if \( \xi = c_1(L) \), we let \( \hat{\xi} := c_1(\hat{L}) \). If \( H \) is an ample line bundle, then \( \hat{H} \) is ample (see Proposition 3.11 of [35]).

We use the following notation: if \( v = (r, \xi, a) \), we let \( \hat{v} := (a, -\hat{\xi}, r) \) (which is then a Mukai vector on \( \hat{S} \)). If \( H \) is a \( v \)-generic polarization on \( S \), then \( \hat{H} \) is a \( \hat{v} \)-generic polarization on \( \hat{S} \) (see Remark 1.1 of [57]).

The following Lemma is an analogue for Abelian surfaces of Lemma 2.22 (see Theorem 3.18 of [57]):

**Lemma 2.25.** Let \( S \) be an Abelian surface such that \( \text{NS}(S) = \mathbb{Z} \cdot h \), where \( h \) is the first Chen class of an ample divisor \( H \) with \( H^2 = 2l \). Fix furthermore \( r, k \in \mathbb{N}^* \), and let \( v = (r, nh, a) \) be a Mukai vector on \( S \) such that \( v^2 = 2k \).

1. There is \( n_0 \in \mathbb{N} \) such that if \( n > n_0 \), every \( H \)-semistable (resp. \( H \)-stable) sheaf \( E \) with Mukai vector \( v \) on \( S \) verifies WIT(0) with respect to \( F \), and the sheaf \( F^0(E) \) is \( \hat{H} \)-semistable (resp. \( \hat{H} \)-stable) with Mukai vector \( \hat{v} \).

2. The functor \( F \) induces isomorphisms \( M_v(S, H) \cong M_{\hat{v}}(\hat{S}, \hat{H}) \), and \( M_v^s \cong M_{\hat{v}}^s \), \( K_v \cong K_{\hat{v}} \) and \( K_v^s \cong K_{\hat{v}}^s \).

**Proof.** The structure of the proof is exactly the same as the one of Lemma 2.22, but with some modifications that we explain here for the convenience of the reader. Again, we give a proof only for semistable sheaves, as for stable sheaf the proof is similar.

First, as in the proof of Lemma 2.22 we notice that it is enough to show the first point of the statement. Indeed, this implies that there is an injective morphism \( M_v \rightarrow M_{\hat{v}} \) sending a sheaf \( E \) to \( F^0(E) \). But now by Theorem 4.4 of [23] we know that \( M_v \) and \( M_{\hat{v}} \) are both irreducible of the same dimension, hence this inclusion is an isomorphism.

Notice that if \( E, E_0 \in M_v(S, H) \), as \( \imath^* \circ \hat{F} : D^b(\hat{S}) \rightarrow D^b(S) \) is the Fourier-Mukai transform with kernel \( \mathcal{P} \), by definition of \( a_{\hat{v}} \) we have

\[
a_{\hat{v}}(F(E)) = (\det(\imath^* \hat{F}(F(E)))) \otimes (\det(\imath^* \hat{F}(F(E_0))))^\vee, \det(F(E)) \otimes \det(F(E_0))^\vee = (\imath^*(\det(F) \otimes (\det(E_0))^\vee), \det(F(E)) \otimes \det(F(E_0))^\vee) = \varepsilon(a_v(E)),
\]

where \( \varepsilon : S \times \hat{S} \rightarrow \hat{S} \times S, \varepsilon(p, q) := (\hat{r}(q), p) \), and \( \hat{r} : \hat{S} \rightarrow \hat{S} \) is the involution acting as \(-1\).

It follows that \( E_1 \) and \( E_2 \) lie in the same fiber of \( a_v \) if and only if \( F(E_1) \) and \( F(E_2) \) lie in the same fiber of \( a_{\hat{v}} \). As \( a_v \) and \( a_{\hat{v}} \) are isotrivial fibrations, it follows that the functor \( F \) induces an injection \( K_v \rightarrow K_{\hat{v}} \).
Since by Remark A.1 of [46] we know that $K_v$ and $K_r$ are irreducible and of the same dimension, this inclusion is an isomorphism.

In conclusion, it only remains to prove the first point of the statement: the proof will have the same structure of that of Lemma 2.22.

The proof of the fact that $E$ verifies WIT(0) with respect to $F$, and that $F^0(E)$ has Mukai vector $\tilde{v}$ is exactly as in the proof of Step 1 of Lemma 2.22. That proof shows moreover that up to tensoring $E$ by a line bundle of degree 0, we get that $H^1(E) = H^2(E) = 0$. By cohomology and base change, it follows that $F^0(E)$ is locally free (see Step 2 of the proof of Lemma 2.22).

As in Step 3 of the proof of Lemma 2.22, we now prove that $F^0(E)$ is $H$-semistable using a contradiction argument. More precisely, we suppose that $F^0(E)$ is not $H$-semistable, and we deduce that $E$ is not $H$-semistable (contradicting the assumption on $E$). Let $v = (r, nh, a)$, so that $\tilde{v} = (a, -nh, r)$.

To do so, let $G_1$ be a subsheaf of $F^0(E)$ with $p_{\tilde{H}}(G_1) > p_{\tilde{H}}(F^0(E))$, and as in Lemma 2.22 we can choose it to be $\tilde{H}$-stable, locally free and with maximal reduced Hilbert polynomial. We let $v(G_1) = (a_1, -n_1 \hat{h}, r_1)$, and again we have $0 < a_1 < a$.

The first claim is that $n_1 > 0$. To show this, let $Z \subseteq S$ be a reduced 0-dimensional subscheme whose degree is $d \gg 0$. One can choose $Z$ so that for every $L \in \text{Pic}^0(S)$ no section of $E \otimes L$ vanishes along $Z$. We let $E_Z$ be the restriction of $E$ to $Z$, and consider the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow E \longrightarrow E_Z \longrightarrow 0.$$  

By construction, for every $L \in \text{Pic}^0(S)$ we have $H^0(\mathcal{K} \otimes L) = 0$.

Applying $F$ to the previous exact sequence we then get an inclusion $F^0(E) \subseteq F^0(E_Z)$. Now, notice that $F^0(E_Z)$ is a direct sum of line bundles of degree 0 on $S$: as $G_1 \subseteq F^0(E)$, we then get an inclusion of $G_1$ in a direct sum of line bundles of degree 0 on $S$: it then follows that $\mu_H(G_1) \leq 0$, i.e. $n_1 \geq 0$ (since a direct sum of line bundles of degree 0 is $H$-semistable of $H$-slope 0).

If $n_1 = 0$ we then get $\mu_H(G_1) = 0$. As in the proof of Lemma 2.22, this would lead to find a subsheaf $\tilde{G}_1$ of $G_1$ which is isomorphic to a line bundle $L$ of degree 0 (which is one of the direct summands of $F(E_Z)$).

This implies that the sheaf $\mathcal{H} := p^*_S(\mathcal{E}) \otimes \mathcal{P} \otimes p^*_S(\tilde{L})$ has a global section: but this is impossible since $p_S^* \mathcal{H} = 0$. This concludes the proof of the claim.

The remaining part of the proof is as in Step 3 of Lemma 2.22. We notice that here we can apply again Lemma 2.22 since $G_1$ being $H$-stable we get that $v(G_1) \geq 0$ (and hence even bigger than $-2$).
The second Lemma we need is the following, allowing us to pass from Mukai vectors of rank 0 to Mukai vectors of strictly positive rank (this is the analogue for Abelian surfaces of Lemma 2.24 and see again Proposition 3.14 of \[57\] for a proof in the case of stable sheaves).

**Lemma 2.26.** Let $S$ be an Abelian surface, $H$ a polarization on $S$ and fix $k \in \mathbb{N}$. Let $\xi$ be the first Chern class of an effective divisor on $S$ such that $\xi^2 = 2k$. Let $v = (0, \xi, a)$ be a Mukai vector on $S$, write $h := c_1(H)$, and suppose that $H$ is $v$–generic.

1. There is $a_0 \in \mathbb{N}$ such that if $a > a_0$, every $H$–semistable (resp. $H$–stable) sheaf $E$ with Mukai vector $v$ on $S$ verifies WIT(0) with respect to $F$, and the sheaf $F(E)$ is $\hat{H}$–semistable (resp. $\hat{H}$–stable) with Mukai vector $\hat{v}$.

2. The functor $F$ induces an isomorphism $M_v(S, H) \simeq M_{\hat{v}}(\hat{S}, \hat{H})$, and $M^s_v \simeq M^s_{\hat{v}}$, $K_v \simeq K_{\hat{v}}$, $K^s_v \simeq K^s_{\hat{v}}$.

**Proof.** As in the proof of Lemma 2.24, the second point of the statement is a consequence of the first one. Again we give an argument only for semistable sheaves, the case of stable sheaves being analogue.

The proof is almost identical to that of Lemma 2.24 using the functors $F$ and $\hat{F}$ given before. Again one shows that if $a$ is sufficiently big, then an $H$–semistable sheaf $E$ of Mukai vector $v = (0, \xi, a)$ verifies WIT(0) with respect to $F$, and $F^0(E)$ is torsion-free and has Mukai vector $\hat{v} = (a, -\xi, 0)$.

The proof is again by contradiction: we suppose that $F^0(E)$ is not $\hat{H}$–semistable, and we deduce that $E$ is not $\hat{H}$–semistable. Hence, let $G' \subseteq F^0(E)$ be $\hat{H}$–stable, $\hat{H}$–destabilizing, with maximal reduced Hilbert polynomial, and let $v(G') = (a', -\xi', r')$.

As in the proof of Lemma 2.24 we still have $0 < a' < a$. Moreover, we have that $\xi'$ numerically equivalent to an nontrivial effective divisor. Indeed, there is a (possibly nonreduced) 0–dimensional subscheme $Z$ of $S$ of very high degree which is contained in the support of $\xi$ such that $G' \subseteq F^0(E) \subseteq F^0(E_Z)$, where $E_Z$ is the restriction of $E$ to $Z$.

The sheaf $F^0(E_Z)$ is semistable and the factors of its Jordan–Hölder filtration are line bundles of degree 0 on $\hat{S}$. Arguing as in the proof of Lemma 2.24 we see that $\det(G')$ is contained in some $L \in \text{Pic}^0(\hat{S})$, so that $\det(G')$ is numerically equivalent to the opposite of a nontrivial effective divisor, or $\det(G') \in \text{Pic}^0(\hat{S})$. This last possibility can be excluded as in the proof of Lemma 2.24 so as $c_1(\det(G')) = -\hat{\xi}$, it follows that $\hat{\xi}$ is numerically equivalent to an effective nontrivial divisor.
The remaining part of the proof is as for Lemma 2.22 (since here as $G'$ is $\tilde{H}$–stable we have $v(G') \geq 0$, and hence bigger than $-2$).

\[ \Box \]

**Remark 2.27.** Lemmas 2.22, 2.24, 2.25, 2.26 prove that the functor $F$ sends a semistable sheaf of Mukai vector $v$ to a semistable sheaf of Mukai vector $\tilde{v}$. Under the same hypothesis, it can be proved that the functor $\tilde{F}$ sends a semistable sheaf of Mukai vector $\tilde{v}$ to a semistable sheaf of Mukai vector $v$. However, the proof of this is somehow less elementary and seems to be more naturally proved in the context of derived categories, as one needs to analyse how $\tilde{F}$ acts on complexes that a priori are not sheaves (see Theorem 3.1 and Proposition 3.2 of [59]). This stronger statement guarantees that $M_v \nparallel H$ if and only if $M_\tilde{v} \nparallel H$, and implies the second point of the previous Lemmas without assuming Theorem 4.4 of [23].

### 2.5. The proof of Theorem 1.17

This section is devoted to the proof of Theorem 1.17: the goal is to show that if $(S, v, H)$ is an $(m, k)$–triple, then $M_v$ (resp. $K_v$) is deformation equivalent to the moduli space associated to an $(m, k)$–triple which is independent of $(S, v, H)$. Before giving the proof, we provide several lemmas we will need.

#### 2.5.1. Changing polarization and first Chern class

We first show the following Lemma, which allows us, if the rank of the Mukai vector is strictly positive, to suppose that the first Chern class of the Mukai vector is a multiple of the polarization. As a consequence, this will allow us to suppose the Néron-Severi group of $S$ to have rank 1.

**Lemma 2.28.** Let $(S, v, H)$ be an $(m, k)$–triple where $v = m(r, \xi, a)$ is such that $r > 0$, and let $g := \gcd(r, \xi)$. Suppose that $\rho(S) \geq 2$, and let $\mathcal{C}$ be the $v$–chamber such that $H \in \mathcal{C}$. Then there exists a Mukai vector $v' = m(r, \xi', a')$ and a polarization $H'$ in $\mathcal{C}$ such that:

1. $v'$ is equivalent to $v$;
2. $(H')^2 \geq 0$;
3. $\xi' = gc_1(H')$.

In particular $M_v(S, H) \simeq M_{v'}(S, H')$ and $M_v^s(S, H) \simeq M_{v'}^s(S', H')$. If $S$ is Abelian, we have $K_v(S, H) \simeq K_{v'}(S, H)$ and $K_v^s(S, H) \simeq K_{v'}^s(S, H)$.

**Proof.** This is a generalization of Lemma II.6 of [41]. First, notice that as $H \in \mathcal{C}$, and as changing polarization inside $\mathcal{C}$ does not change the moduli space (by Proposition 2.5), up to changing polarization in $\mathcal{C}$ we can suppose that $\xi \notin \mathbb{R} \cdot c_1(H)$. 

Now, let \( d \in \mathbb{N} \) and \( v' := v \cdot ch(O_S(dH)) \). Then \( v' \) is equivalent to \( v \), and
\[
v' = m(r, \xi + r d c_1(H), a + d \xi \cdot H + r d^2 H^2 / 2).
\]
We first notice that if \( d \gg 0 \), then \( \xi + r d c_1(H) \in \mathcal{C} \). Moreover, writing \( r = gs \) and \( \xi = g \zeta \), where \( gcd(s, \zeta) = 1 \) (since \( g = gcd(r, \xi) \)), we have
\[
\xi' := \xi + r d c_1(H) = g \zeta + g s d c_1(H) = g(\zeta + s d c_1(H)).
\]
We now let \( H' \) be an ample divisor such that \( c_1(H') = \zeta + s d c_1(H) \). As \( d \gg 0 \), we have
\[
(H')^2 = \zeta^2 + 2 s d \zeta \cdot H + s^2 d^2 H^2 / 2 \gg 0.
\]
Since \( H' \in \mathcal{C} \), we just need to prove that \( H' \) is primitive for some choice of \( d \gg 0 \).

To show this, write \( \zeta = p \zeta' \) for a primitive class \( \zeta' \). As \( gcd(s, \zeta) = 1 \), we have \( gcd(s, p) = 1 \): it follows that if \( d \gg 0 \) is such that \( gcd(p, d) = 1 \), then \( gcd(p, sd) = 1 \). The class \( p \zeta' + s d c_1(H) \) is then a primitive element in the rank 2 sublattice \( \Lambda \) of \( NS(S) \) spanned by \( \zeta' \) and \( c_1(H) \).

Letting \( S(\Lambda) \) be the saturation of \( \Lambda \) in \( NS(S) \), as \( c_1(H) \) is primitive we get that \( S(\Lambda)/\Lambda \) is cyclic and spanned by the class of an element \( \beta \in S(\Lambda) \). It follows that every \( \alpha \in \Lambda \) which is not primitive in \( NS(S) \) has to be a multiple of \( \beta \).

The classes \( p \zeta' + s d c_1(H) \) and \( p \zeta' + s d' c_1(H) \) are not multiple to each other unless \( d = d' \): it then follows that there is at most one \( d \in \mathbb{Z} \) which is such that \( d \gg 0 \), \( gcd(p, d) = 1 \) and \( p \zeta' + s d c_1(H) \) is not primitive in \( NS(S) \). As a consequence, there is \( d \gg 0 \) such that \( c_1(H') = \zeta + s d c_1(H) \) is primitive.

To conclude the proof, we just need to notice that if \( S \) is Abelian, then by point (1) of Lemma 2.20, the tensorization with \( O_S(dH) \) induces an isomorphism between the fibers of the corresponding Yoshioka fibrations. \( \square \)

2.5.2. Deformation to elliptic surfaces. Elliptic surfaces having a section and whose Picard number is 2 prove to be particularly useful, as in this case we have a privileged class of polarizations, called \( v \)-suitable. Let \( Y \) be an elliptic K3 or Abelian surface such that \( NS(Y) = \mathbb{Z} \cdot f \oplus \mathbb{Z} \cdot \sigma \), where \( f \) is the class of a fiber and \( \sigma \) is the class of a section. Let \( v \) be a Mukai vector on \( Y \), and recall the following definition (see [41]):

**Definition 2.29.** A polarization \( H \) on \( Y \) is called \( v \)-suitable if \( H \) is in the unique \( v \)-chamber whose closure contains \( f \).
We have an easy numerical criterion to guarantee that a polarization on $Y$ is $v$–suitable (see Lemma I.0.3 of [41] for K3 surfaces, and point (2) of Lemma 2.24 of [45] for Abelian surfaces):

**Lemma 2.30.** Let $Y$ be an elliptic K3 surface with $NS(Y) = \mathbb{Z} \cdot \sigma \oplus \mathbb{Z} \cdot f$, where $\sigma$ is a section and $f$ is a fibre, and let $v = (r, \xi, a)$ be a Mukai vector on $Y$ such that $r > 0$. Let $H$ be a polarization, and suppose that $c_1(H) = \sigma + tf$ for some $t \in \mathbb{Z}$.

1. If $Y$ is K3, then $H$ is $v$–suitable if $t \geq |v| + 1$.
2. If $Y$ is Abelian, then $H$ is $v$–suitable if $t \geq |v|$.

In the next Lemma, by deforming an $(m, k)$–triple $(S, v, H)$ to an $(m, k)$–triple $(Y, v', H')$ where $Y$ elliptic surface and $H'$ is $v'$–suitable, we show that the deformation class of $M_v$ (resp. $K_v$) depends only on the rank $r$ of $v$ (when $r > 0$ and prime to the first Chern class of $v$).

**Lemma 2.31.** For $i = 1, 2$ let $(S_i, v_i, H_i)$ be an $(m, k)$–triple where either $S_1$ and $S_2$ are K3 surfaces, or $S_1$ and $S_2$ are Abelian surfaces. Write $v_i = m(r_i, \xi_i, a_i)$ for $i = 1, 2$, and suppose furthermore that the following conditions are verified:

1. $r_1 = r_2 =: r > 0$;
2. $gcd(r, \xi_1) = gcd(r, \xi_2) = 1$;

Then $M_{v_1}$ and $M_{v_2}$ (resp. $K_{v_1}$ and $K_{v_2}$) are deformation equivalent, and similarly $M^s_{v_1}$ and $M^s_{v_2}$ (resp. $K^s_{v_1}$ and $K^s_{v_2}$) are deformation equivalent.

**Proof.** The argument we present here was first used by O’Grady in [41] and by Yoshioka in [55] for primitive Mukai vectors, and by the authors in [45] in the case of $m = 2$ and $k = 1$.

First, we can always assume $\rho(S_i) > 1$. Indeed, consider a non-trivial smooth, projective deformation $\mathcal{X}_i$ of $S_i$ along an open 1–dimensional disc $\Delta$, and let $0 \in \Delta$ be such that $\mathcal{X}_i|_0 \simeq S_i$. By the Main Theorem of [44], the locus of $t \in \Delta$ such that $\rho(\mathcal{X}_{i,t}) > 1$ is dense in the classical topology of $\Delta$.

If $\mathcal{H}_i \in Pic(\mathcal{X}_i)$ is a deformation of $H_i$ and $\mathcal{L}_i \in Pic(\mathcal{X}_i)$ is a deformation of a line bundle $L_i \in Pic(S_i)$ such that $c_1(L_i) = \xi_i$, then $(\mathcal{X}_{i,t}, v_{i,t}, \mathcal{H}_{i,t})$ is an $(m, k)$–triple for all but a finite number of $t \in \Delta$ (see Remark 2.13): hence there is $t \in \Delta$ such that $\rho(\mathcal{X}_{i,t}) > 1$ and $(\mathcal{X}_{i,t}, v_{i,t}, \mathcal{H}_{i,t})$ is an $(m, k)$–triple.

By Lemma 2.28 we suppose $v_i = m(r, c_1(H_i), a_i)$, where $H_i$ is ample and $H_i^2 = 2d_i$ with $d_i > 0$. Let $Y$ be a K3 (resp. Abelian) surface admitting an elliptic fibration and such that $NS(Y) = \mathbb{Z} \cdot \sigma \oplus \mathbb{Z} \cdot f$, where $f$ is the class of a fiber, and $\sigma$ is the class of a section.
For $i = 1, 2$, because of the connectedness of the moduli spaces of polarized K3 or Abelian surfaces, there is a smooth, connected curve $T_i$ and a deformation $(\mathcal{X}_i, L_i, \mathcal{H}_i)$ over $T_i$ of the $(m, k)$–triple $(S_i, v_i, H_i)$ such that there is $t \in T_i$ with the property $(\mathcal{X}_i, t, v_{i,t}, H_{i,t}) = (Y, v_i', H'_i)$, where

1. $c_1(H'_i) = \sigma + p_i f$, where $p_i = d_i + 1 \gg 0$.
2. $v'_i = m(p_i, c_1(H'_i), a_i)$.

Let $\xi'_i := c_1(H'_i)$. Notice that $(v'_1)^2 = (v'_2)^2$ and they have the same rank: hence $|v'_1| = |v'_2|$, so by Lemma 2.30 a polarization is $v'_i$–suitable if and only if it is $v'_2$–suitable. As $p_i \gg 0$, by Lemma 2.30 we have that $H'_i$ is $v'_i$–suitable for $i = 1, 2$, hence $H'_1$ and $H'_2$ are in the same chamber $C$. By Proposition 2.5 we then change to a common generic polarization $H \in C$, which is $v'_1$–generic for $i = 1, 2$.

As $(v'_1)^2 = (v'_2)^2$, we have $(\xi'_1)^2 - 2ra_1 = (\xi'_2)^2 - 2ra_2$, and as

$$(\xi'_i)^2 = (\sigma + p_i f)^2 = 2(p_i - 1),$$

we then get the equation

$$p_1 = p_2 + r(a_1 - a_2).$$

Letting $l := a_1 - a_2$ and $F$ a fiber of the elliptic fibration, we get

$$v'_2 \cdot ch(O_Y(lF)) = m(r, \sigma + p_2 f, a_2) \cdot (1, lF, 0) = m(r, \sigma + p_1 f, a_1) = v'_1,$$

where the second equality follows from equation (10). By point (2) of Lemma 2.20 we see that $M_{v'_1}(Y, H) \simeq M_{v'_2}(Y, H)$, concluding the proof for the moduli spaces.

To conclude the proof, suppose that $S_1$ and $S_2$ are both Abelian surfaces. Up to considering an étale cover of $T_1$ and $T_2$, the deformations of $(m, k)$–triples along $\Delta$, $T_1$ and $T_2$ induce flat deformations of $K_{v_1}$ and $K_{v_2}$, so that $K_{v_1}$ is deformation equivalent to $K_{v'_1}(Y, H)$.

As by point (2) of Lemma 2.20 the tensorization with $O_Y(lF)$ induces an isomorphism between the fibers of the corresponding Yoshioka fibrations, we are done.

**Remark 2.32.** In order to relate $M_{v_1}$ and $M_{v_2}$ (resp. $M_{v_1}^{s}$ and $M_{v_2}^{s}$), or similarly for the fibers of the Yoshioka fibration for Abelian surfaces) in the previous proof, we only used deformations of $(m, k)$–triples along a smooth, connected curve, and isomorphisms between moduli spaces given by tensorization with a line bundle.

**Remark 2.33.** Lemma 2.31 remains true if we replace conditions (1) and (2) with the three following conditions:

1. $r_1 = r_2 =: r$
(2) \(\gcd(r, \xi_1) = \gcd(r, \xi_2) =: g\)
(3) \(a_1 \equiv a_2 \mod g\).

The proof is exactly as before: we first deform \((S_i, v_i, H_i)\) to an \((m, k)\)-triple \((Y, v'_i, H)\), where \(Y\) and \(H\) are as in the proof of Lemma 2.31 and where \(v'_i = (r, gc_1(H'_i), a_i)\) (here again the \(H'_i\)'s are as before).

The fact that \(pv_1q \equiv pv_2q \mod g\) gives the equation

\[gp_1 + gp_2 + r'(a_1 - a_2),\]

where \(r' = r/g\) (generalizing equation (10)). Writing \(a_1 - a_2 = lg\), a simple calculation gives \(v'_2 \cdot ch(O_Y(1F)) = v'_1\).

2.5.3. **An intermediate result on Mukai vectors.** The following numerical Lemma will allow us to compare moduli spaces of semistable sheaves with different ranks on a K3 or Abelian surface whose Néron-Severi group has rank 1. Together with Lemmas 2.22, 2.24, 2.25 and 2.26 this will allow us to show that moduli spaces of semistable sheaves with different ranks are deformation equivalent.

**Lemma 2.34.** Let \((S, v, H)\) be an \((m, k)\)-triple, where \(S\) is a projective K3 or an Abelian surface such that \(NS(S) = \mathbb{Z} \cdot h\), with \(h = c_1(H)\). Write \(v = m(r, nh, a)\), and suppose \(r > 0\). For every \(s \in \mathbb{Z}\) let

\[v_s := v \cdot ch(O_S(sH)) = m(r, ns h, a_s).\]

(1) For every \(N \in \mathbb{N}\) there is \(s > N\) such that \(n_s > 0\) and \(\gcd(n_s, a_s) = 1\).

(2) If \(n = 1\) and \(a = 0\), then for every \(N \in \mathbb{N}\) there is \(s > N\) such that \(n_s > 0\), \(\gcd(n_s, a_s) = 1\) and \(a_s \in 2k\mathbb{Z}\).

**Proof.** Write \(H^2 = 2l\). It is easy to see that we have

\[n_s = n + rs = n_{s-1} + r\]

and

\[a_s = a + 2lns + rs^2 = a + lns + lsn_s.\]

We let \(\tilde{a}_s := a + lns\), so that \(a_s = \tilde{a}_s + lsn_s\). From this equality, it follows that \(\gcd(n_s, a_s) = 1\) if and only if \(\gcd(n_s, \tilde{a}_s) = 1\).

Now, we let

\[A := \begin{bmatrix} n & r \\ a & ln \end{bmatrix},\]

which is a primitive element of \(M_2(\mathbb{Z})\) since \(\gcd(r, n, a) = 1\). Moreover, we have

\[A \cdot \begin{bmatrix} 1 \\ s \end{bmatrix} = \begin{bmatrix} n_s \\ \tilde{a}_s \end{bmatrix}.\]
If \( p \) is a prime number dividing both \( n_s \) and \( \bar{a}_s \), viewing \( A \) as the matrix representing a linear map \( f : (\mathbb{Z}/p\mathbb{Z})^2 \rightarrow (\mathbb{Z}/p\mathbb{Z})^2 \), by equation (13) we see that \( f \) is not invertible, hence \( \det(A) \equiv 0 \mod p \). We then conclude that if \( p \) is a prime number dividing both \( n_s \) and \( a_s \), then \( p \) divides \( \det(A) \). As \( \det(A) = k \), we then see that any prime number dividing both \( n_s \) and \( a_s \) has to divide \( k \).

We let \( \{p_1, \ldots, p_d\} \) be the prime factors of \( k \). Suppose that \( s \in \mathbb{N} \) is such that \( n_s \) and \( a_s \) are not relatively prime, and let \( q \) be a prime number dividing both \( n_s \) and \( a_s \), i.e. dividing both \( n_s \) and \( \bar{a}_s \). Then there is \( j \in \{1, \ldots, d\} \) such that \( q = p_j \).

Notice that as \( A \) is a primitive element in \( M_2(\mathbb{Z}) \), there is a linear combination of the lines of \( A \) which is a primitive element of \( \mathbb{Z}^2 \), i.e. there are \( b, c \in \mathbb{Z} \) such that the vector

\[
\begin{bmatrix} e & f \end{bmatrix} : = b \begin{bmatrix} n & r \end{bmatrix} + c \begin{bmatrix} a & ln \end{bmatrix}
\]

is primitive in \( \mathbb{Z}^2 \).

As we are supposing \( n_s \) and \( a_s \) to be divisible by \( p_j \) for some \( j \in \{1, \ldots, d\} \), it follows that there is \( j \in \{1, \ldots, d\} \) such that

\[
e + sf = bn + ca + s(br + cln) = bn_s + c\bar{a}_s \equiv 0 \mod p_j
\]

so \( s \) has to satisfy one of the \( d \) congruences \( e + sf \equiv 0 \mod p_j \).

By the Chinese Remainder Theorem there is \( T \in \mathbb{N} \) which verifies all these congruences. Moreover, the integers verifying at least one of them are of the form \( T + m_1p_1 + \cdots + m_dp_d \) for some \( m_1, \ldots, m_d \in \mathbb{Z} \).

Now, consider \( N \in \mathbb{N} \), and suppose that \( s > N \) is such that \( n_s \) and \( a_s \) are not relatively prime. It then follows that then

\[
s \in B : = \{T + m_1p_1 + \cdots + m_dp_d > N \mid m_1, \ldots, m_d \in \mathbb{Z}\}.
\]

Now, notice that \( B \) is a proper subset of the set of integers bigger than \( N \): it follows that there \( s > N \) such that \( n_s \) and \( a_s \) are relatively prime, concluding the proof of the first point of the statement.

For the second point, notice that if \( n = 1 \) and \( a = 0 \) (so that \( l = k \)), then equation (11) gives \( n_s = 1 + rs \), and equation (12) gives \( a_s = 2ls + rst^2 = 2ks + rks^2 \). If \( s \) is a very big even integer, we then see that \( n_s \gg 0 \) and \( a_s \in 2k\mathbb{Z} \). It only remains to prove that we can choose \( s \) to be very big, even, and such that \( n_s \) and \( a_s \) are relatively prime.

If \( p \) is a prime number dividing \( a_s \) and \( n_s \), as we saw before \( p \) has to divide \( k \). Letting \( s = 2ks' \) for some \( s' \in \mathbb{Z} \), then any prime dividing \( k \) cannot divide \( n_s = 1 + 2ks't \), hence there can be no prime number dividing both \( n_s \) and \( a_s \), and we are done. \( \square \)
2.5.4. The proof of Theorem 1.17. We now proceed with the proof of:

**Theorem 1.17.** Let \( m, k \in \mathbb{N}^* \), and let \((S_1, v_1, H_1)\) and \((S_2, v_2, H_2)\) be two \((m, k)\)-triples.

1. If \( S_1 \) and \( S_2 \) are both K3 surfaces or both Abelian surfaces, then \( M_{v_1} \) and \( M_{v_2} \) are deformation equivalent, and the deformation is locally trivial.
2. If \( S_1 \) and \( S_2 \) are two Abelian surfaces, then \( K_{v_1} \) and \( K_{v_2} \) are deformation equivalent, and the deformation is locally trivial.

**Proof.** The proof for K3 and for Abelian surfaces is formally the same. Let \((S, v, H)\) be an \((m, k)\)-triple, and write \( v = m(r, \xi, a) \). For every \( l \in \mathbb{N}^* \) we let \( X_l \) be a projective K3 or Abelian surface such that \( NS(X_l) = \mathbb{Z} \cdot h_l \), where \( h_l = c_1(H_l) \) and \( H_l \) is an ample divisor with \( H_l^2 = 2l \). Let \( u_l := m(0, h_l, 0) \), where \( (0, h_l, 0) \) is a primitive Mukai vector on \( S \) of square \( 2l \), so that \((X_l, u_l, H_l)\) is an \((m, l)\)-triple.

We show that \( M_u(S, H) \) is deformation equivalent to \( M_{u_k}(X_k, H_k) \). The equivalence is obtained using deformations of the moduli spaces induced by deformations of the corresponding \((m, k)\)-triple along smooth, connected curves, and isomorphism between moduli spaces induced by tensor products with line bundles, and by the Fourier-Mukai transforms whose kernel is the ideal of the diagonal (for K3 surfaces) or the Poincaré bundle (for Abelian surfaces).

As the deformations we use are locally trivial (by Lemma 2.17), we conclude that the deformation equivalence is locally trivial, so this will conclude the proof of point (1) of the statement.

Moreover, since the Yoshioka fibration is preserved under tensorization with line bundles (by Lemma 2.20) and by the Fourier-Mukai transforms whose kernel is the Poincaré bundle (by Lemmas 2.25 and 2.26), and since up to an étale cover the deformation of an \((m, k)\)-triple along a curve induces a locally trivial deformation of the fibers of the Yoshioka fibration (by Lemmas 2.15 and 2.17), point (2) of the statement is implied by point (1).

We divide the proof in several steps: in the first, we reduce to the case \( r > 0 \); in the second, we reduce to the case where \( r \) and \( \xi \) are relatively prime; in the third we reduce to the case where \( r \) is a multiple of \( 2k \); the fourth step concludes the proof.

**Step 1: reduction to \( r > 0 \).** Suppose first that \( v = m(0, \xi, a) \), where \( \xi \) is effective. For \( d \in \mathbb{Z} \), we let \( v_d := v_{dH} = m(0, \xi, a_d) \), where \( a_d = a + d\xi \cdot H \). As \( \xi \) is effective and \( H \) is ample, we have \( \xi \cdot H > 0 \): it follows that if \( d \gg 0 \), then \( a_d \gg 0 \). Moreover, we can suppose that \( H \) is both \( v_d \)-generic and \( \tilde{v}_d \)-generic.
Now, by point (1) of Lemma 2.20 we have $M_v \simeq M_{v_d}$. As $a_d \gg 0$, by Lemma 2.24 we get $M_{v_d} \simeq M_{\tilde{v}_d}$. But $\tilde{v}_d = m(a_d, \xi, 0)$ and $a_d > 0$: hence it is enough to show the theorem for Mukai vector of strictly positive rank.

**Step 2: reduction to $r \gg 0$ and prime with $\xi$.** By Step 1, we suppose $r > 0$. By Lemma 2.28 and Proposition 2.5 we can then suppose $v = m(r, \xi, a)$ with $r > 0$, $\xi = gc_1(H)$ (where $g := \gcd(r, \xi)$), and $H^2 \gg 0$. We let $H^2 = 2l$, and consider the $(m, k)$–triple $(X_l, v', H_l)$ where $v' = m(r, gh_l, a)$.

As the moduli spaces of polarized K3 or Abelian surfaces are connected, the moduli spaces $M_v(S, H)$ and $M_{v'}(X_l, H_l)$ are deformation equivalent. Let now $s \in \mathbb{Z}$, and write $v'_s := v'_s H_l = m(r, g_h, a_s)$: by point (1) of Lemma 2.34 there is $s \in \mathbb{Z}$ such that $g_h, a_s \gg 0$ and $\gcd(g_h, a_s) = 1$. By point (1) of Lemma 2.20 we have $M_{v'} \simeq M_{v'_s}$.

As $g_h \gg 0$, by Lemma 2.22 we get $M_{v'_s} \simeq M_{\tilde{v}'_s}$. But since we have $\tilde{v}'_s = m(a_s, g_h, r)$, $\gcd(a_s, g_h) = 1$ and $a_s \gg 0$, we conclude that it is sufficient to prove the theorem for Mukai vectors having rank $r \gg 0$ prime with the first Chern class.

**Step 3: reduction to $r \in 2k\mathbb{Z}$, $r \gg 0$ and prime with $\xi$.** By Step 2, by deforming to a surface $S'$ such that $\rho(S') \geq 2$, and by using Lemma 2.28 we can suppose $v = m(r, c_1(H), a)$, and $r \gg 0$. We consider the $(m, k)$–triple $(X_k, v'', H_k)$ where $v'' = m(r, h_k, 0)$. By Lemma 2.31 we know that $M_v(S, H)$ and $M_{v''}(X_k, H_k)$ are deformation equivalent.

Let now $s \in \mathbb{Z}$, and $v''_s := v'' s H_k = m(r, n_s h_k, a_s)$, where $n_s = 1 + rs$ and $a_s = 2ks + rk^2$. By point (2) of Lemma 2.34 we can choose $s$ such that $n_s \gg 0$, $a_s \in 2k\mathbb{Z}$ and $\gcd(n_s, a_s) = 1$. By point (1) of Lemma 2.20 we know that $M_{v''} \simeq M_{v''_s}$.

Moreover, as $n_s \gg 0$, by Lemma 2.22 we have $M_{v''_s} \simeq M_{\tilde{v''}_s}$. But $\tilde{v''}_s = m(a_s, n_s h_k, r)$, and $a_s \in 2k\mathbb{Z}$. In conclusion, we just need to prove the theorem for Mukai vectors having rank $r \gg 0$ which is prime with the first Chern class, and which is a multiple of $2k$.

**Step 4: conclusion.** By Step 3, we just need to consider an $(m, k)$–triple $(S, v, H)$ where $v = m(r, \xi, a)$, with $r$ prime with $\xi$ and such that $r = 2kp$ for $p \gg 0$. We show that $M_v(S, H)$ is deformation equivalent to $M_{u_k}(X_k, H_k)$.

By Lemma 2.31 we know that $M_v(S, H)$ is deformation equivalent to $M_{v''}(X_k, H_k)$, where $v'' = m(2kp, h_k, 0)$. As $p \gg 0$, by Lemma 2.24 we have $M_{v''} \simeq M_{\tilde{v''}}$, where $\tilde{v''} = m(0, h_k, 2kp)$. But now notice that $\tilde{v''} \cdot ch(O_{X_k}(-pH_k)) = u_k$, so by point (1) of Lemma 2.20 we have $M_{\tilde{v''}} \simeq M_{u_k}$, concluding the proof. □
Remark 2.35. The proof of Theorem 1.17 is based on Theorem 4.4 of [23], asserting that if \( v = mw \), then \( M_v \) is nonempty and irreducible as long as \( M_w \) is nonempty. This result is indeed used in the proof of Lemmas 2.15, 2.22, 2.24, 2.25 and 2.26.

In the proof of all these Lemmas the use of Theorem 4.4 of [23] can be avoided (see Remarks 2.16 and 2.27): by doing so, the proof of Theorem 1.17 then implies that if \((S, v, H)\) is an \((m, k)\)-triple, then \(M_v \neq \emptyset\), since it shows that it is deformation equivalent to \(M_w(X_k, H_k)\), which is nonempty (since the linear system \(|mH_k|\) contains irreducible curves).

3. THE MODULI SPACES ARE IRREDUCIBLE SYMPLECTIC VARIETIES

This section is devoted to the proof of Theorem 1.19: if \((S, v, H)\) is an \((m, k)\)-triple, then \(M_v\) and \(K_v\) are irreducible symplectic varieties.

To do so, we first show in section 3.1 that if \(S\) is a projective K3 surface, then \(M_v\) and \(M_v^a\) are simply connected. Similarly, if \(S\) is an Abelian surface, then \(K_v\) and \(K_v^a\) are simply connected (with the exception of \((m, k) = (2, 1)\), where \(K_v\) is still simply connected, but the fundamental group of \(K_v^a\) is \(\mathbb{Z}/2\mathbb{Z}\)).

This will allow us to show that the exterior algebra of reflexive forms on any finite quasi-étale cover \(f: Y \to M_v\) (resp. \(f: Y \to K_v\)) is generated by the reflexive pull-back of a symplectic form on \(M_v\) (resp. on \(K_v\)): this will be done in section 3.2, by showing that for a particular \((m, k)\)-triple \((S, v, H)\) there is a rational dominant map from from a moduli space \(M_u\) (resp. \(K_u\)) with primitive Mukai vector, to the moduli space \(M_v\) (resp. \(K_v\)).

3.1. Simple connectedness. We first show in this section that the moduli spaces \(M_v\) and \(M_v^a\) (resp. \(K_v\) and \(K_v^a\)) are simply connected. We will divide the proof if this in two main parts: the first one is devoted to the case of K3 surfaces; in the second we will consider Abelian surfaces. In both cases, the proof has the same structure: we first show the simple connectedness of the moduli space for a particular \((m, k)\)-triple, and then use Theorem 1.17 to conclude.

3.1.1. The case of K3 surfaces. Let \(X\) be a projective K3 surface with \(\text{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_X(H)\), where \(H\) is an ample divisor such that \(H^2 = 2k\). We let \(h := c_1(H)\), and we choose \(m \in \mathbb{N}^a\).

We let \(V\) be the open subset of \(|mH|\) of smooth curves, and \(U\) the open subset of \(|mH|\) of integral curves. For \(u = m(0, h, 0)\) we will consider the morphism \(p_u: M_u \to |mH|\) sending a sheaf to its Fitting subscheme (see Corollary 20.5 of [8], and [29]).
First we show that the subset of reducible curves in $|mH|$ is a divisor if and only if $m = 2$ and $k = 1$:

**Lemma 3.1.** Let $X$ be a K3 surface such that $\text{Pic}(X) = \mathbb{Z}.H$, where $H$ is an ample line bundle such that $H^2 = 2k$. Let $m \in \mathbb{N}^*$ and consider the subset $R \subseteq |mH|$ parameterizing reducible curves. If $(m, k) \neq (2, 1)$, then $	ext{codim}_{mH}(R) \geq 2$.

**Proof.** As $R$ parameterizes the reducible curves in the linear system $|mH|$, we have

$$R = \bigcup_{1 \leq m_1, m_2 \leq m, \ m_1 + m_2 = m} |m_1H| \times |m_2H|.$$ 

As $	ext{dim}(|pH|) = 1 + kp^2$ for every $p \in \mathbb{N}^*$, we get

$$\text{dim}(|m_1H| \times |m_2H|) = 2 + k(m_1^2 + m_2^2), \quad \text{dim}(|mH|) = 1 + km^2,$$

so $|m_1H| \times |m_2H|$ has codimension $2m_1m_2k - 1$ in $|mH|$.

In order for $R$ to have codimension 1 in $|mH|$, there must be $1 \leq m_1, m_2 \leq m$ such that $m_1 + m_2 = m$, and such that $2m_1m_2k - 1 = 1$. Hence $m_1, m_2, k = 1$, so that $m = 2$ and $k = 1$. Thus, if $(m, k) \neq (2, 1)$ we get $\text{codim}_{mH}(R) \geq 2$. \hfill $\square$

Now, let $\mathcal{J}_V := p_u^{-1}(V)$ and $\mathcal{J}_U := p_u^{-1}(U)$, which are two open subsets of $\mathcal{M}_u$. We notice that if $C \in V$, then $\mathcal{F} \in p_u^{-1}(C)$ if and only if there is $L \in \text{Pic}(C)$ of degree $m^2k$ such that $\mathcal{F} = j_*L$, where $j : C \hookrightarrow X$ is the inclusion. In particular, we have an isomorphism

$$p_u^{-1}(C) \cong \text{Pic}^{m^2k}(C)$$

obtained by mapping $\mathcal{F} = j_*L$ to $L$.

Moreover, if $C \in U$, then $\mathcal{F} \in p_u^{-1}(C)$ if and only if $\mathcal{F} = j_*L$, where $j : C \hookrightarrow X$ is the inclusion, and $L$ is a rank one torsion-free sheaf on $C$ of degree $m^2k$, i. e. such that $\chi(L) = 0$. We notice that all these sheaves are $H$–stable of Mukai vector $u$, hence we have

$$\mathcal{J}_V \subseteq \mathcal{J}_U \subseteq \mathcal{M}_u^s \subseteq \mathcal{M}_u.$$

We start by showing the following (the proof is a generalization of the argument proposed in section 4 of [42]).

**Proposition 3.2.** The moduli spaces $\mathcal{M}_u$ and $\mathcal{M}_u^s$ are simply connected.

**Proof.** If $m = 1$, then $\mathcal{M}_u = \mathcal{M}_u^s$: this is an irreducible symplectic manifold, and we are done. For $(m, k) = (2, 1)$, see section 4 of [42].

---

3Here and in what follows, if $C$ is a smooth projective curve and $d \in \mathbb{Z}$, we let $\text{Pic}^d(C)$ be the set of line bundles of degree $d$ on $C$. 

---

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For $m \geq 2$, notice that $(X, u, H)$ is an $(m, k)$-triple, hence $M_u$ is a normal, irreducible complex variety (by Theorem 4.4 of [23]). Since for a normal projective variety the inclusion of an open subvariety induces a surjection on the fundamental groups (see Proposition 2.10 of [26]) we have a surjective map $\pi_1(M_u^s) \to \pi_1(M_u)$.

The chain of inclusions
\[ J_V \hookrightarrow J_U \hookrightarrow M_u^s \hookrightarrow M_u \]
of smooth open subvarieties of $M_u$ given before induces then a chain of surjections
\[ \pi_1(J_V) \to \pi_1(J_U) \to \pi_1(M_u) \to \pi_1(M_u). \]

We then just need to show that $\pi_1(j)$ is the trivial map.

To show this, notice that the homotopy exact sequence of the fibration $p_u : J_V \to V$ gives the exact sequence
\[ \pi_1(p_u^{-1}(C)) \to \pi_1(J_V) \to \pi_1(V) \to \{1\}, \]
where $C \in V$. As remarked above we have $p_u^{-1}(C) \simeq \text{Pic}^{m^2k}(C)$, hence the exact sequence is
\[ (14) \quad \pi_1(\text{Pic}^{m^2k}(C)) \to \pi_1(J_V) \to \pi_1(V) \to \{1\}. \]

We start by proving the following:

**Lemma 3.3.** The morphism $\pi_1(j) \circ j_C : \pi_1(\text{Pic}^{m^2k}(C)) \to \pi_1(J_U)$ is trivial.

**Proof.** Let $\ell \subseteq |mH|$ be a generic line, and suppose it is generated by two smooth curves intersecting transversally. By Lemma 3.1, we can suppose that all the curves in $\ell$ are reduced and irreducible.

If $\pi : \tilde{X} \to X$ is the blow-up of $X$ along the base locus $Bs(\ell)$ of $\ell$, then $\tilde{X}$ is the total space of $\ell$: this means that for every $s \in \tilde{X}$ there is a unique curve $C_s$ of $\ell$ such that $s \in C_s$, where $\tilde{C}_s$ is the proper transform of $C_s$. We have a natural fibration $p_\ell : \tilde{X} \to \ell$ mapping $s \in \tilde{X}$ to $C_s$.

We now define an embedding $g : \tilde{X} \to J_U$ of fibrations over $\ell$. First, fix $p \in Bs(\ell)$, and let $d := 1 + m^2k$. Let $s \in \tilde{X}$: then $\pi(s) \in C_s$, and consider the rank one torsion-free sheaf $L_s := \mathcal{T}_{\pi(s)} \otimes \mathcal{O}_{C_s}(dp)$. Notice that the degree of $L_s$ is $m^2k$ on $C_s$; if $j_s : C_s \to X$ is the embedding of $C_s$ in $X$, we have $j_{ss}(L_s) \in J_U$, and we let $g(s) := j_sL_s$. The inclusion
$g$ then fits in a commutative diagram (where $i$ is the inclusion)

$$
\begin{array}{ccl}
\tilde{X} & \xrightarrow{g} & \mathcal{J}_U \\
p_v & & p_u \\
\ell & \xrightarrow{i} & U
\end{array}
$$

Notice that if $t \in \ell$ is a generic point and $C$ is the corresponding curve in $\ell$, then $p^{-1}_v(t) = \tilde{C}$, the proper transform of $C$ under the blow-up map, while $p^{-1}_u(t) \simeq Pic^m(C)$. The restriction $g_t : \tilde{C} \longrightarrow Pic^m(C)$ of $g$ to $p^{-1}_v(t)$ can be identified with the Abel-Jacobi map from $C$ to its Jacobian. It then induces a surjective morphism $\pi_1(g_t) : \pi_1(\tilde{C}) \longrightarrow \pi_1(Pic^m(C))$.

Now, let $C \in \ell$ be a smooth curve. We have a commutative diagram

$$
\begin{array}{ccl}
\tilde{C} & \xrightarrow{g_t} & Pic^m(C) \\
i & & \downarrow \pi_1 \\
\tilde{X} & \xrightarrow{g} & \mathcal{J}_U \uparrow 
\end{array}
$$

indicating a commutative diagram

$$
\begin{array}{ccl}
\pi_1(\tilde{C}) & \xrightarrow{\pi_1(g_t)} & \pi_1(Pic^m(C)) \\
\pi_1(i) & & \downarrow \pi_1(j) \circ j_C \\
\pi_1(\tilde{X}) & \xrightarrow{\pi_1(g)} & \pi_1(\mathcal{J}_U)
\end{array}
$$

As $\pi_1(\tilde{X}) = \{1\}$ and the morphism $\pi_1(g_t)$ is surjective, it follows that $\pi_1(j) \circ j_C$ is trivial, thus concluding the proof.

An immediate consequence of Lemma 3.3 is that the surjective morphism $\pi_1(j)$ factors through a surjective morphism

$$
\pi_1(j) : \pi_1(\mathcal{J}_V)/\text{im}(j_C) \longrightarrow \pi_1(\mathcal{J}_U).
$$

The exact sequence (14) gives an isomorphism between $\pi_1(V)$ and $\pi_1(\mathcal{J}_V)/\text{im}(j_C)$, hence we get a surjective map $\iota : \pi_1(V) \longrightarrow \pi_1(\mathcal{J}_U)$, which is then trivial if and only if $\pi_1(j)$ is trivial: we then just need to show that $\iota$ is trivial.

To do so, consider the generic line $\ell \subseteq |mH|$ of the proof of Lemma 3.3: all the curves parametrized by $\ell$ are reduced and irreducible, and we can suppose that it is transversal to $W := U \setminus V$, where $\ell \cap W := \{x_1, \ldots, x_p\}$ is given by smooth points of $W$. 
As $\ell$ is generic, by the Zariski Main Theorem (see Theorem 3.22 of [54]), the inclusion of $\ell \setminus W$ in $V$ gives a surjection $\pi_1(\ell \setminus W) \rightarrow \pi_1(V)$, hence we finally get a surjective morphism $\iota_\ell : \pi_1(\ell \setminus W) \rightarrow \pi_1(J_U)$, and we just need to show that $\iota_\ell$ is trivial. More precisely, if $\gamma_1, \cdots, \gamma_p$ are the generators of $\pi_1(\ell \setminus W)$, we need to show that $\iota_\ell(\gamma_i)$ is trivial.

Now, notice that the fibration $p_\ell : \tilde{X} \rightarrow \ell$ has a section $\sigma_\ell$ (fixing $p \in Bs(\ell)$, we let $\sigma_\ell(t)$ be the unique intersection between $\pi^{-1}(p)$ and $p_\ell^{-1}(t)$). Hence every $\gamma_i$ has a lifting $\tilde{\gamma}_i$ in $\pi_1(\tilde{X})$, and by construction its image in $\pi_1(J_U)$ under $\pi_1(g)$ is $\iota_\ell(\gamma_i)$. But as $\pi_1(\tilde{X}) = \pi_1(X)$ (since $\pi : \tilde{X} \rightarrow X$ is a blow-up) and as $\pi_1(X)$ is trivial (since $X$ is K3), it follows that $\iota_\ell(\gamma_i) = 0$. □

The main consequence of Proposition 3.2 is that the moduli spaces of (semi)stable sheaves associated to $(m, k)$—triples are simply connected:

**Theorem 3.4.** Let $(S, v, H)$ be an $(m, k)$—triple where $S$ is a projective K3 surface. Then $M_v$ and $M_v^s$ are simply connected.

**Proof.** For $m = 1$, we have $M_v = M_v^s$: this is an irreducible symplectic manifold, and we are done.

Fix now $m \geq 2$ and $k \geq 1$. By point (1) of Theorem 1.17 the moduli spaces arising from $(m, k)$—triples on K3 surfaces are all deformation equivalent. As this deformation equivalence is obtained using only isomorphism of moduli spaces (coming from Fourier-Mukai transforms) and deformations of the moduli spaces induced by deformations of $(m, k)$—triples, by point (1) of Lemma 2.18 these deformation equivalent moduli spaces are also homeomorphic.

It is then enough to prove that $M_v(S, H)$ is simply connected for one $(m, k)$—triple $(S, v, H)$. By Proposition 3.2 this holds for the $(m, k)$—triple $(X, u, H)$, and we are done. □

### 3.1.2. The case of Abelian surfaces

Let $A$ be an Abelian surface with $NS(A) = \mathbb{Z} \cdot h$, where $h = c_1(H)$ and $H$ is an ample divisor such that $H^2 = 2k$. We let $h := c_1(H)$, $m \in \mathbb{N}$ and $u := m(0, h, 0)$.

Let $Y_{mH}$ be the Hilbert scheme of curves on $A$ which are deformation of curves in $|mH|$, and let $p_u : M_u \rightarrow Y_{mH}$ be the morphism mapping a sheaf to its Fitting subscheme. We moreover let $p^K_u : K_u \rightarrow |mH|$ be the restriction of $p_u$ to $K_u$.

First we show that the subset of reducible curves in $|mH|$ is a divisor if and only if $m = 2$ and $k = 1$:

**Lemma 3.5.** Let $A$ be an Abelian surface such that $NS(A) = \mathbb{Z} \cdot h$, where $h = c_1(H)$ and $H$ is an ample line bundle such that $H^2 = 2k$. 
Let $m \in \mathbb{N}^*$ and consider the subset $R \subseteq |mH|$ parameterizing reducible curves. If $(m, k) \neq (2, 1)$, then $\text{codim}_{|mH|}(R) \geq 2$.

**Proof.** If $C \in |mH|$ is reducible, there are $L \in \mathcal{S}, m_1, m_2 \in \mathbb{N}^*$ such that $m = m_1 + m_2$, and two curves $C_1 \in |m_1H + L|$ and $C_2 \in |m_2H - L|$ such that $C = C_1 + C_2$. It follows that

$$R = \bigcup_{1 \leq m_1, m_2 \leq m, L \in \mathcal{S}} (m_1H + L) \times |m_2H - L|.$$ 

Notice that $\dim(|mH|) = km^2 - 1$ and

$$\dim((m_1H + L) \times |m_2H - L|) = k(m_1^2 + m_2^2) - 2,$$

so

$$\dim\left(\bigcap_{L \in \mathcal{S}} (m_1H + L) \times |m_2H - L|\right) = k(m_1^2 + m_2^2).$$

The codimension of $\bigcap_{L \in \mathcal{S}} (m_1H + L) \times |m_2H - L|$ is then $2km_1m_2 - 1$ in $|mH|$. 

In order for $R$ to have codimension 1 in $|mH|$, there must be $1 \leq m_1, m_2 \leq m$ such that $m_1 + m_2 = m$, and such that $2m_1m_2k - 1 = 1$. Hence $m_1, m_2, k = 1$, so that $m = 2$ and $k = 1$. Thus, if $(m, k) \neq (2, 1)$ we get $\text{codim}_{|mH|}(R) \geq 2$. \hfill $\Box$

We now prove the following, giving the simple connectedness of $K_v$ and $K_u^s$ for particular $(m, k)$-triples.

**Proposition 3.6.** If $(m, k) \neq (2, 1)$, then $K_u$ and $K_u^s$ are simply connected.

**Proof.** If $m = 1$, then $K_u = K_u^s$, and this is either a point (if $k = 1$) or an irreducible symplectic manifold (if $k > 1$), and we are done.

For $m \geq 2$, notice that $(A, u, H)$ is an $(m, k)$-triple, hence $K_u$ is a normal, irreducible complex variety (see Remark A.1 of [46]). As a consequence we have a surjective map $\pi_1(K_u^s) \longrightarrow \pi_1(K_u)$ (see Proposition 2.10 of [26]): it will be sufficient to prove that $K_u^s$ is simply connected.

To show this, let $p_{u,K_u^s} : K_u^s \longrightarrow |mH|$ be the restriction of $p_u^K$ to $K_u^s$. By the Theorem in section 1.1, Part II of [14], the fundamental group of a smooth connected variety admitting a dominant mapping to $\mathbb{P}^N$ (for some $N$) is generated by the fundamental group of the inverse image of a generic line in $\mathbb{P}^N$. As a consequence, if $\ell \subseteq |mH|$ is a generic line and $K^0 := p_{u,K_u^s}^{-1}(\ell) \subseteq K_u^s$, we have a surjective morphism $\pi_1(K^0) \longrightarrow \pi_1(K_u^s)$. It is then enough to show that $K^0$ is simply connected.
As \( \ell \) is generic in \(|mH|\), by Bertini’s Theorem we know that \( K^0 \) is smooth. Moreover, by Lemma 3.5 all the curves parameterized by \( \ell \) are reduced and irreducible. It then follows that \( K^0 = (p_u^K)^{-1}(\ell) \).

To show that \( K^0 \) is simply connected, we show that \( K^0 \) is a fiber of an isotrivial fibration, and then use the homotopy exact sequence of this fibration to conclude. The domain of this isotrivial fibration will be \( M^0 := p_u^{-1}(\ell) \) (which is a subset of \( M_u \)), that will be identified with the relative compactified Jacobian of \( \ell \). By construction, there is an inclusion \( f : K^0 \longrightarrow M^0 \) fitting in a commutative diagram

\[
\begin{array}{ccc}
K^0 & \xrightarrow{f} & M^0 \\
p^0_K & \downarrow & p^0 \\
\ell & \xrightarrow{id} & \ell
\end{array}
\]

where \( p^0_K \) is the restriction of \( p^K_u \) to \( K^0 \), and \( p^0 \) is the restriction of \( p_u \) to \( M^0 \).

We now let \( \sigma : M^0 \longrightarrow A \) be the restriction to \( M^0 \) of the map \( \beta : M_u \longrightarrow A \) defined in section 2.2, mapping a sheaf \( F \) to the Albanese image of \( c_2(F) \). As the determinant of \( F \in M^0 \) is represented by the Fitting subscheme of \( F \), which is a divisor in \(|mH|\), by Lemma 2.10 we have

\[
K^0 = M^0 \cap K_u = M^0 \cap b^{-1}_u(0_A, \mathcal{O}_A) = \sigma^{-1}(0_A),
\]

where \( b_u : M_u \longrightarrow A \times \hat{A} \) is the O’Grady fibration of \( M_u \) defined in section 2.2.

Next, we claim that \( \sigma : M^0 \longrightarrow A \) is an isotrivial fibration. Indeed, if \( L \in \text{Pic}^0(A) \) is represented by a divisor \( D \), and \( \delta \) is a 0–cycle of degree 0 on \( A \) representing \( mH \cdot D \) in the Chow ring of \( A \), then the tensorization with \( L \) induces an automorphism of \( M_u \) mapping \( K^0 \) to \( \sigma^{-1}(\delta) \). It follows that the connected algebraic group \( \text{Pic}^0(A) \) acts transitively on the fibers of the projective morphism \( \sigma \): this implies that \( \sigma \) is an isotrivial fibration.

Finally, notice that \( K^0 \) is connected. Indeed, it is the inverse image, under the dominant map \( p^K_u : K_u \longrightarrow |mH| \), of a linear space of the projective space \(|mH|\): by Theorem 1.1 of [13], it follows that \( K^0 \) is connected.

To resume, we have an isotrivial fibration \( \sigma : M^0 \longrightarrow A \), and \( K^0 \) is one of the fibers. The homotopy exact sequence associated to this fibration gives then

\[
\pi_2(A) \longrightarrow \pi_1(K^0) \xrightarrow{\pi_1(f)} \pi_1(M^0) \xrightarrow{\pi_1(\sigma)} \pi_1(A) \longrightarrow \{1\},
\]
where the last term comes from the fact that $K^0$ is connected. As $A$ is an Abelian surface, we have $\pi_2(A) = \{1\}$, hence in order to show that $K^0$ is simply connected, we just need to prove that the morphism $\pi_1(\sigma): \pi_1(M^0) \longrightarrow \pi_1(A)$ is injective.

To do so, suppose that $\ell$ is generated by two smooth curves intersecting transversally at a finite number of points. Let $Bs(\ell)$ be the base locus of $\ell$, and $\pi: \tilde{A} \longrightarrow A$ the blow-up of $A$ along $Bs(\ell)$.

The surface $\tilde{A}$ is the total space of $\ell$: for every $a \in \tilde{A}$ there is a unique curve $C_a \in \ell$ such that $a \in \tilde{C}_a$, where $\tilde{C}_a$ is the proper transform of $C_a$ under $\pi$. We then have a fibration $p_\ell: \tilde{A} \longrightarrow \ell$, mapping $a \in \tilde{A}$ to the point of $\ell$ corresponding to $C_a$.

There is a natural morphism $g: \tilde{A} \longrightarrow M^0$ of fibrations over $\ell$ obtained as follows: first, choose $p \in Bs(\ell)$, and let $d := m^2k + 1$. For every $a \in \tilde{A}$, the rank 1 torsion-free sheaf $\mathcal{I}_{g(a)} \otimes \mathcal{O}_{C_a}(dp)$ has degree $m^2k$. We then let $g(a) := \mathcal{I}_{g(a)} \otimes \mathcal{O}_{C_a}(dp)$, so to have a commutative diagram

$$
\begin{array}{ccc}
\tilde{A} & \longrightarrow & M^0 \\
p_\ell \downarrow & & \downarrow p^0 \\
\ell & \longrightarrow & \ell
\end{array}
$$

If $t \in \ell$ is a generic point, the curve $C$ corresponding to $t$ is smooth, $p_\ell^{-1}(t) = \tilde{C}$ and $(p^0)^{-1}(t) \simeq Pic^{m^2k}(C)$. Let $p_1, \ldots, p_n \in \ell$ be the points corresponding to singular curves. The fundamental group of $M^0$ is generated by $\pi_1(Pic^{m^2k}(C))$ and by liftings $\gamma_1, \ldots, \gamma_n$ of the generators $\gamma_1, \ldots, \gamma_n$ of $\pi_1(\ell \backslash \{p_1, \ldots, p_n\})$.

Moreover, the morphism $g_\ell: \tilde{C} \longrightarrow Pic^{m^2k}(C)$ given by the restriction of $g$ to $p_\ell^{-1}(t)$ can be identified to the Abel-Jacobi map from $C$ to its Jacobian: it then induces a surjective map $\pi_1(\tilde{C}) \longrightarrow \pi_1(Pic^{m^2k}(C))$.

As $\tilde{C} \subseteq \tilde{A}$, it follows that $\pi_1(M^0)$ is generated by $\pi_1(\tilde{A})$ and by the $\gamma_1, \ldots, \gamma_n$. Now, notice that the fibration $p_\ell: \tilde{A} \longrightarrow \ell$ has a section: fixing $p \in Bs(\ell)$, this section is obtained by mapping $t \in \ell$ to the unique intersection point of $\pi^{-1}(p)$ and $p_\ell^{-1}(t)$.

We can then choose the liftings $\gamma_1, \ldots, \gamma_n$ to be in the image of $\pi_1(\tilde{A})$ in $\pi_1(M^0)$. As $\sigma \circ g: \tilde{A} \longrightarrow A$ induces an isomorphism between $\pi_1(\tilde{A})$ and $\pi_1(A)$, this concludes the proof. \(\square\)

Theorem \[1.17\] allows us to generalize Proposition \[3.6\] to all $(m, k)$-triples over Abelian surfaces (with the exception of $(2, 1)$-triples).
Theorem 3.7. Let \((S, v, H)\) be an \((m, k)\)-triple where \(S\) is an Abelian surface.

1. If \((m, k) \neq (2, 1)\), then \(K_v\) and \(K_s^v\) are simply connected.
2. If \((m, k) = (2, 1)\), then \(K_v\) is simply connected, and \(\pi_1(K_v^s) = \mathbb{Z}/2\mathbb{Z}\).

Proof. For \(m = 1\), we have \(K_v = K_s^v\), which is either a point (if \(k = 1\)) or an irreducible symplectic manifold (if \(k \geq 2\)): the statement is then clear in this case.

Fix now \(m \geq 2\) and \(k \geq 1\), and suppose that \((m, k) \neq (2, 1)\). By point (2) of Theorem 1.17, the fibers of the Yoshioka fibration of the moduli spaces arising from \((m, k)\)-triples on Abelian surfaces are all deformation equivalent. As this deformation equivalence is obtained using only isomorphism of moduli spaces (coming from Fourier-Mukai transforms) and deformations of the moduli spaces induced by deformations of \((m, k)\)-triples, by point (2) of Lemma 2.18 the homeomorphism type of \(K_{nw}\) only depends on \(m\) and \(k = w^2/2\).

It is then enough to find one particular \((m, k)\)-triple \((S, v, H)\) for which \(K_v\) and \(K_v^s\) are simply connected. By Proposition 3.6, this holds for the \((m, k)\)-triple \((A, u, H)\), so we are done in this case.

If \((m, k) = (2, 1)\), then \(K_v\) admits a symplectic resolution \(\tilde{K}_v\), which is an irreducible symplectic manifold by point (2) of Theorem 1.6 in [45]. As \(K_v\) has canonical singularities, by [52] we have \(\pi_1(K_v^s) = \pi_1(\tilde{K}_v)\): it follows that \(K_v\) is simply connected.

By Theorem 4.2 and Proposition 5.3 of [32], we know that \(K_v^s\) has a 2 : 1 étale cover from an open subset \(U\) of an irreducible symplectic manifold \(Y\) which is deformation equivalent to a Hilbert scheme of 3 points on a K3 surface. This open subset \(U\) is obtained by removing from \(Y\) 256 copies of \(\mathbb{P}^3\) and one copy of a desingularization of the singular locus of \(K_v\). It follows that the complement of \(U\) has codimension at least 2 in \(Y\), so that \(\pi_1(U) = \pi_1(Y) = \{1\}\). It then follows that the fundamental group of \(K_v^s\) is \(\mathbb{Z}/2\mathbb{Z}\).

Recall that if \(X\) is a normal projective variety having at most rational singularities, it is possible to define the Albanese variety \(Alb(X)\) as the Albanese variety of any desingularization \(\tilde{X}\) of \(X\), and the construct the Albanese morphism \(alb : X \rightarrow Alb(X)\) by descending the usual Albanese morphism of \(\tilde{X}\) (see Proposition 2.3 of [50], and Lemma 8.1 of [24]).

As a consequence of Theorem 3.7, we show in the next result that the Yoshioka fibration is the Albanese morphism of the moduli space \(M_v\).
Corollary 3.8. Let \((S, v, H)\) be an \((m, k)\)-triple where \(S\) is an Abelian surface. The morphism \(a_v : M_v \to S \times \hat{S}\) is the Albanese morphism of \(M_v\).

Proof. For \(m = 1\), the map \(a_v\) is the Albanese map by point (1) of Theorem 0.1 in [56]. We then suppose \(m \geq 2\).

We begin by considering \((m, k) = (2, 1)\), in which case we know by Théorème 1.1 of [30] that \(M_v\) admits a symplectic resolution of the singularities \(\pi : \tilde{M} \to M_v\), which is obtained by blowing up the singular locus \(\Sigma\) with reduced structure.

Now, for every \((p, L) \in S \times \hat{S}\) the fiber \(K_{p, L} := a_v^{-1}(p, L)\) is a singular symplectic variety whose singular locus is \(\Sigma_{p, L} := \Sigma \cap K_{p, L}\), and \(\tilde{K}_{p, L} := \pi^{-1}(K_{p, L})\) is the symplectic resolution of \(K_{p, L}\), which is an irreducible symplectic manifold by point (2) of Theorem 1.6 of [45]. It follows that \(a_v \circ \pi : \tilde{M} \to S \times \hat{S}\) is the Albanese morphism of \(\tilde{M}\), so that \(a_v : M_v \to S \times \hat{S}\) is the Albanese morphism of \(M_v\).

Let us now finally consider the case \((m, k) \neq (2, 1)\), and \(m \geq 2\). Let \(\pi : \tilde{M} \to M_v\) be a desingularization of \(M_v\), where \(\tilde{M}\) is a smooth projective variety. The inclusion \(j : M_v^s \to \tilde{M}\) induces a surjective morphism \(\pi_1(j) : \pi_1(M_v^s) \to \pi_1(\tilde{M})\). If we now let \(a_v^s : M_v^s \to S \times \hat{S}\) be the restriction of \(a_v\) to \(M_v^s\), then \(a_v^s\) is an isotrivial fibration whose fibers are all isomorphic to \(K_v^s\).

As \(K_v^s\) is simply connected (since \((m, k) \neq (2, 1)\), by the previous part of the proof), it follows that (all the fibers of \(a_v^s\) are simply connected, hence it induces an isomorphism \(\pi_1(a_v^s) : \pi_1(M_v^s) \to \pi_1(S \times \hat{S})\).

Now, notice that \(a_v \circ \pi \circ j = a_v^s\), hence \(\pi_1(j)\) is injective, and hence an isomorphism. But this implies that \(\pi_1(a_v \circ \pi)\) is an isomorphism, so that \(a_v \circ \pi : \tilde{M} \to S \times \hat{S}\) is the Albanese morphism for \(\tilde{M}\). It then follows that \(a_v : M_v \to S \times \hat{S}\) is the Albanese morphism of \(M_v\), concluding the proof. \(\square\)

3.2. The proof of Theorem 1.19. We are finally in the position to show Theorem 1.19, i.e. that \(M_v\) and \(K_v\) are irreducible symplectic varieties. Before doing this, we calculate the dimension of the space of reflexive \(p\)-forms for particular \((m, k)\)-triples.

Lemma 3.9. Let \((S, v, H)\) be an \((m, k)\)-triple, and suppose that \(NS(S) = \mathbb{Z} \cdot h\) and \(v = m(0, h, 0)\), where \(h := c_1(H)\).
(1) If $S$ is K3 and $p \in \mathbb{N}$ is such that $0 \leq p \leq 2m^2k + 2$, then
\[
h^0(M_v, \Omega^{[p]}_{M_v}) = \begin{cases} 
1 & \text{if } p \text{ is even} \\
0 & \text{if } p \text{ is odd}
\end{cases}
\]

(2) If $S$ is Abelian and $p \in \mathbb{N}$ is such that $0 \leq p \leq 2m^2k - 2$, then
\[
h^0(K_v, \Omega^{[p]}_{K_v}) = \begin{cases} 
1 & \text{if } p \text{ is even} \\
0 & \text{if } p \text{ is odd}
\end{cases}
\]

Proof. Suppose first that $S$ is K3. We let $u := (0, mh, 1 - m^2k)$, which is a primitive Mukai vector on $S$.
If $C \in |mH|$ is an integral curve and $j : C \to S$ is the inclusion, for every $L \in \text{Pic}^1(C)$ the sheaf $j_\ast L$ is $H$-stable of Mukai vector $u$. The sheaves of this type form an open subset $U$ of $M_u$.
Moreover, if $L \in \text{Pic}^1(C)$ then $L^\otimes m^2k \in \text{Pic}^{m^2k}(C)$, hence $j_\ast (L^\otimes m^2k)$ is an $H$-stable sheaf of Mukai vector $v$. We then have a rational map
\[
g : M_u \dashrightarrow M_v, \quad g(j_\ast L) := j_\ast L^\otimes m^2k.
\]

We first show that $g$ is dominant. To do so, consider the two fibrations $p_u : M_u \to |mH|$ and $p_v : M_v \to |mH|$ sending a sheaf to its Fitting subscheme. If $C \in |mH|$ is smooth, we have $p_u^{-1}(C) \simeq \text{Pic}^1(C)$ and $p_v^{-1}(C) \simeq \text{Pic}^{m^2k}(C)$, hence $p_u^{-1}(C), p_v^{-1}(C) \simeq \text{Pic}^2(C)$.

The restriction of $g$ to $p_u^{-1}(C)$ is the multiplication by $m^2k$ on $\text{Pic}^1(C)$, hence it is surjective. This shows that if $V \subseteq |mH|$ is the open subset of smooth curves, then $g$ sends $p_u^{-1}(V)$ surjectively to $p_v^{-1}(V)$. As $M_u$ and $M_v$ are two projective varieties which are both irreducible and of the same dimension, it follows that $g$ is dominant.

Now, notice that as $M_v$ has canonical singularities, by Theorem 1.4 of [1934] for every $p$ we have
\[
h^0(M_v, \Omega_{M_v}^{[p]}) = h^0(\hat{M}_v, \Omega_{\hat{M}_v}^{[p]}),
\]
where $\hat{M}_v$ is a resolution of the singularities of $M_v$. As $g$ is dominant, we get $h^0(\hat{M}_v, \Omega_{\hat{M}_v}^{[p]}) \leq h^0(M_u, \Omega_{M_u}^{[p]})$ for every $p$.

Now, since $u$ is primitive we know that $M_u$ is an irreducible symplectic manifold, hence $h^0(M_v, \Omega_{M_v}^{[p]}) = 1$ if $p$ is even and 0 otherwise, so that $h^0(M_v, \Omega_{M_v}^{[p]}) = 0$ if $p$ is odd, and $h^0(M_v, \Omega_{M_v}^{[p]}) \leq 1$ if $p$ is even. Since $h^0(M_v, \Omega_{M_v}^{[p]}) \geq 1$ if $p$ is even as $M_v$ is a symplectic variety, we are done.

If now $S$ is Abelian, the proof is exactly the same, but replacing $M_v$ with $K_v$ and $M_u$ with $K_u$. \qed

We now prove the following:
Theorem 1.19. Let \( m, k \in \mathbb{N}^* \) and \((S,v,H)\) an \((m,k)\)-triple.

1. If \( S \) is K3, then \( M_v \) is an irreducible symplectic variety.
2. If \( S \) is Abelian and \((m,k) \neq (1,1)\), then \( K_v \) is an irreducible symplectic variety.

Proof. We begin by considering \( S \) to be a K3 surface. If \( m = 1 \), then \( M_v \) is an irreducible symplectic manifold by Theorem 0.1 of [55].

If \( m \geq 2 \), then \( M_v \) is a symplectic variety, and let \( \sigma \) be a symplectic form on it. We have to show that if \( f : Y \to M_v \) is a finite quasi-étale morphism, then the exterior algebra of reflexive forms on \( Y \) is spanned by \( f^\ast \sigma \).

Let then \( f : Y \to M_v \) be a finite quasi-étale cover, which then induces a finite quasi-étale cover of \( M_v^s \). As a finite quasi-étale morphism of a smooth variety is étale, and as \( M_v^s \) is simply connected by Theorem 3.4, it follows that \( f \) is an isomorphism.

We then just need to show that the exterior algebra of reflexive forms on \( M_v \) if spanned by \( \sigma \). This follows if we show that \( h^0(M_v, \Omega^{[p]}_{M_v}) = 1 \) if \( p \) is even, and \( h^0(M_v, \Omega^{[p]}_{M_v}) = 0 \) if \( p \) is odd.

For this, let \( S' \) be a projective K3 surface with \( Pic(S') = \mathbb{Z} \cdot H' \), where \( H' \) is an ample line bundle with \((H')^2 = 2k\), and let \( \nu' := m(0, h', 0) \), where \( h' = c_1(H') \). Then \((S', \nu', H')\) is an \((m,k)\)-triple, and by Lemma 3.9 we have \( h^0(M_{\nu'}, \Omega^{[p]}_{M_{\nu'}}) = 1 \) if \( p \) is even, and \( h^0(M_{\nu'}, \Omega^{[p]}_{M_{\nu'}}) = 0 \) if \( p \) is odd.

By Theorem 1.17 \( M_v \) and \( M_{\nu'} \) are deformation equivalent, and the deformation is locally trivial. It follows that they have resolutions \( \tilde{M}_v \) and \( \tilde{M}_{\nu'} \) of the singularities which are smooth and deformation equivalent as smooth varieties. In particular, their Hodge numbers are equal. By Theorem 1.4 of [16] we then have

\[
h^0(M_v, \Omega^{[2]}_{M_v}) = h^0(\tilde{M}_v, \Omega^{[2]}_{\tilde{M}_v}) = h^0(\tilde{M}_{\nu'}, \Omega^{[2]}_{\tilde{M}_{\nu'}}) = h^0(M_{\nu'}, \Omega^{[2]}_{M_{\nu'}}),
\]

and we are done.

If \( S \) is an Abelian surface, the proof is identical if \((m,k) \neq (2,1)\), replacing \( M_v \) with \( K_v \), \( M_{\nu'} \) by \( K_{\nu'} \), and using point (1) of Theorem 3.7 instead of Theorem 3.4. The case \((m,k) = (2,1)\) has to be treated differently.

If \((m,k) = (2,1)\), by point (2) of Theorem 3.7 we have \( \pi_1(K_v^s) = \mathbb{Z}/2\mathbb{Z} \), hence \( K_v \) has only two possible finite quasi-étale covers (up to isomorphism): one is \( K_v \) itself, and the other will be denoted \( Y_v \). We need to show that the exterior algebra of reflexive forms on \( K_v \) and \( Y_v \) are spanned by the reflexive pull-back of a symplectic form on \( K_v \). To
do so, it will be enough to show that both $K_v$ and $Y_v$ are birational to an irreducible symplectic manifold.

For $K_v$, by point (2) of Theorem 1.6 in [45] we know that $K_v$ has a symplectic resolution which is an irreducible symplectic manifold (deformation equivalent to $OG_6$). For $Y_v$, by Proposition 5.3 of [32] we know that it is birational to an irreducible symplectic manifold (deformation equivalent to $Hilb^3(K3)$). This concludes the proof.

We conclude with the proof of the following:

**Theorem 1.23.** Let $m, k \in \mathbb{N}^*$ and $(S, v, H)$ an $(m, k)$-triple.

(1) If $S$ is $K3$, on $H^2(M_v, \mathbb{Z})$ there is a nondegenerate integral quadratic form of signature $(3, b_2(M_v) - 3)$ and a compatible pure weight-two Hodge structure.

(2) If $S$ is Abelian and $(m, k) \neq (1, 1)$, on $H^2(K_v, \mathbb{Z})$ there is a nondegenerate integral quadratic form of signature $(3, b_2(K_v) - 3)$ and a compatible pure weight-two Hodge structure.

**Proof.** We give a proof only for $K3$ surfaces: the proof for Abelian surfaces is exactly the same, replacing $M_v$ with $K_v$. If $m = 1$, this holds since $M_v$ is an irreducible symplectic manifold. For $(m, k) = (2, 1)$, this is Theorem 1.7 of [45].

For all other cases, $M_v$ is an irreducible symplectic variety by point (1) of Theorem 1.19 hence Namikawa symplectic by Proposition 1.10. Its singular locus has codimension at least 4, and by [23] it is locally factorial.

By point (2) of Theorem 8 in [38] on $H^2(M_v, \mathbb{R})$ there is then a quadratic form $q$, which is nondegenerate and has the prescribed signature by Corollary 8 of [39]. The integrality and the compatibility are proved as in the smooth case (see [2]).

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