Two particle realisation of the Poincare group with interaction

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Abstract

A relative position 4-vector is constructed for two spin-zero particles. Some advantages of this relative position over Bakamjian-Thomas are pointed out. The centre-of-mass (CM) and relative positions and momenta are an explicit realisation of the so-called non-canonical covariant representation. The Hamiltonian including potential terms is factorised into CM and relative components, the latter is a Lorentz scalar readily evaluated in the CM rest frame when the relative position, momentum are canonical conjugates.

Introduction

In the non-relativistic mechanics of two particle systems, the conversion of the individual particle generators into centre-of-mass (CM) and relative components has been a fruitful concept. Instead of the $(x_i, p_i)$ and $(x_j, p_j)$ conjugate pairs, one constructs the new conjugate pairs $(X, P)$ and $(\bar{x}, p)$, then the Hamiltonian can be written as the sum of CM and relative parts. Potentials which are scalar functions of $\bar{x} \equiv (x_i - x_j)$ can be inserted into the Hamiltonian such that the Galilei group algebra is maintained, and the two body problem is effectively reduced to one body with a potential.

In relativistic mechanics this procedure is more difficult. A relative position 4-vector $q_{BT}$ was found by Bakamjian and Thomas [1,2] which will be the starting point of our discussion. We will use covariant notation throughout, and all the CM and relative generators will be defined in terms of the individual particle generators $j^\lambda_\mu i, j^\lambda_\mu j, p^\lambda i, p^\lambda j$. Our central task is to introduce interaction terms into the two-particle Poincare generators

$$J^\lambda_\mu \equiv j^\lambda_\mu i + j^\lambda_\mu j, \quad P^\lambda \equiv p^\lambda i + p^\lambda j \quad (1.1)$$

(where $j^\lambda_\mu i, p^\lambda i$ are the free particle generators for particle $i$ etc) such the group algebra is maintained. More specifically we introduce interaction terms into the system energy-momentum 4-vector, ie $P^\lambda \rightarrow P^\lambda_{int}$, so that the usual relations characteristic of the Poincare group still hold:

$$\{J^\lambda_\mu, J^\rho_\nu\} = \eta^\lambda_\rho J^\mu_\nu + \eta^\mu_\nu J^\lambda_\rho - \eta^\lambda_\nu J^\mu_\rho - \eta^\mu_\rho J^\lambda_\nu \quad (1.2)$$

$$\{J^\lambda_\mu, P^\rho_{int}\} = \eta^\mu_\nu P^\lambda_{int} - \eta^\lambda_\nu P^\rho_{int}$$

$$\{P^\lambda_{int}, P^\rho_{int}\} = 0 \quad (1.3)$$

We use the classical Poisson brackets (Pb’s) instead of commutators at this stage, to avoid the technicalities of operator ordering. On quantisation the Pb’s are turned to commutators and a factor of $i$ included.

We use the following notation. For two 4-vectors $a, b : (a \wedge b)^\lambda_\mu \equiv a^\lambda b_\mu - a_\mu b^\lambda; \quad a \cdot b \equiv a^\lambda b_\lambda, \quad \eta^{00} = 1 = -\eta^{\alpha_\alpha}, \eta^{\lambda_\mu} = 0$ for $\lambda \neq \mu$.

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1 a 4-vector $a^\lambda$ must satisfy $\{J^\lambda_\mu, a^\rho\} = \eta^\mu_\nu a^\lambda - \eta^\lambda_\nu a^\mu$. 

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2 Relative position 4-vectors

The various 4-vector relative positions \( q^\lambda \) to be considered in this paper all satisfy the Poisson bracket relation

\[
\{ q^\lambda, \hat{P}^\mu \} = 0 \tag{2.1}
\]

where \( \hat{P}^\mu \equiv P^\mu / |P| \). It then follows that we are able to insert a potential \( V \) being a function of \( q \) into the system energy-momentum generators \( P^\lambda \) as follows

\[
P^\lambda \to P^\lambda_{\text{int}} \equiv \hat{P}^\lambda \left( |P| + V(q \cdot q) \right) \tag{2.2}
\]

and due to (2.1) the new \( P^\lambda_{\text{int}} \) interaction “Hamiltonians” will have zero Pb with each other:

\[
\{ P^\lambda_{\text{int}}, P^\mu_{\text{int}} \} = 0, \tag{2.3}
\]

and

\[
\{ J^\mu, P^\nu_{\text{int}} \} = \eta^{\mu \nu} P^\lambda_{\text{int}} - \eta^{\lambda \nu} P^\mu_{\text{int}}
\]

also follows if the potential \( V \) is a scalar function of \( q \). The Lorentz generators \( J^\lambda \) do not contain interaction terms, remaining the same as in (1.1). Furthermore we will show that the \( |P| \) in (2.2) can be expressed as a scalar function of the relative momentum \( v^\lambda \), then we will have factorised the Hamiltonian \( P^\lambda_{\text{int}} \) into CM and relative variables.

In using 4-vector notation we appear to have a surplus of components. But the \( q^\lambda \) and its conjugate \( v^\lambda \) (to be introduced in the next section) have the property that they are orthogonal to the system 4-momentum \( P \equiv p_i + p_j \), ie

\[
(q \cdot P) = (v \cdot P) = 0, \tag{2.4}
\]

so that \( q^0, v^0 \), can be regarded as dependant variables, and are zero in the system rest frame.

**The Bakamjian-Thomas relative position**

The Bakamjian-Thomas (BT) relative position generator is (see (5.6) of \cite{1})

\[
q_{\text{BT}}^\lambda \equiv \frac{j^\lambda_i P_\mu}{p_i \cdot P} - \frac{j^\lambda_j P_\mu}{p_j \cdot P} \tag{2.5}
\]

which is by construction orthogonal to the system momentum \( P^\lambda \), a time-like 4-vector. This means that \( q_{\text{BT}}^\lambda \) is space-like or null, then \( -(q_{\text{BT}} \cdot q_{\text{BT}}) \) is positive so can be regarded as a distance squared, which is also Lorentz invariant. It was shown in the original paper \cite{1} that in the non-relativistic limit \( -(q_{\text{BT}} \cdot q_{\text{BT}}) \simeq |x_i - x_j|^2 \).

Let us assume that a potential function of this distance, \( V(-q_{\text{BT}} \cdot q_{\text{BT}}) \), is included in the Hamiltonian, then the force on particle \( i \) is worked out from \( \{ V(-q_{\text{BT}} \cdot q_{\text{BT}}), p_i \} \). We can calculate from (2.5) and the individual particle Pb’s corresponding to (1.2,3) that

\[
\{ q_{\text{BT}}^\lambda, p_i^\mu \} = -\eta^{\lambda \mu} + \frac{q_{\text{BT}}^\lambda P^\mu}{p_i \cdot P} \tag{2.6}
\]

\[
\frac{1}{2} \{ -q_{\text{BT}} \cdot q_{\text{BT}}, p_i^\mu \} = q_{\text{BT}}^\mu - \frac{(p_i \cdot q_{\text{BT}}) P^\mu}{p_i \cdot P} \tag{2.7}
\]

\[
\equiv \frac{(q_{\text{BT}} \land P)^{\mu \rho}}{p_i \cdot P} p_{i \rho} \equiv F^{\mu \rho} p_{i \rho}, \tag{2.8}
\]
In the system rest frame when $P = 0$ the force tensor $F^{\mu \nu}$ only has “electric” components $(F^{10}, F^{20}, F^{30})$ and the force acts along the space component of $q_{BT}$. [To include “magnetic” forces due to the motion of particle $j$ the numerator of $F$ should be of the form $(q \wedge p_j)^{\mu \nu}$ instead of $(q_{BT} \wedge P)^{\mu \nu}$, which property we will show for the new relative position introduced below.]

We now introduce the 4-vector relative momentum $v^\lambda$ which is the usual relative momentum in the non-relativistic limit:

$$v^\lambda \equiv \frac{p_j \cdot P}{P^2} p_i^\lambda - \frac{p_j \cdot P}{P^2} p_i^\lambda \equiv (0, \frac{m_2 P_1 - m_2 P_j}{m_1 + m_2})$$

in the NR limit,

Alternative $v$ can be written as

$$v^\lambda = p^\lambda_{-} \equiv p^\lambda_{-} - \hat{P}^\lambda (p_{-} \cdot \hat{P})$$

with

$$p_{-} \equiv 1/2(p_i - p_j).$$

It can be shown that (see the appendix)

$$\{q^\lambda_{BT}, v^\mu\} = -\eta^\lambda\mu + \hat{P}^\lambda \hat{P}^\mu$$

and any such variables being orthogonal to $P$ and which satisfy the relation (2.11) we will call “covariant conjugates”, as in the system rest frame the components $q^0_{BT}, v^0$ are zero, and $\{q^a_{BT}, v^b\} = \delta^{ab}$ in the usual manner $(a, b = 1, 2, 3)$.

An alternative relative position

The construction of the $q$ proceeds as follows. First we introduce the 4-vector

$$q^\lambda = \frac{j^\lambda_{i} p_{j\mu} - j^\lambda_{j} p_{i\mu}}{p_i \cdot p_j}.$$ (2.12)

However this $q$ is not necessarily space-like, so next we project $q$ onto the $P$ hyperplane, defining

$$q_{-}^\lambda \equiv q^\lambda - \hat{P}^\lambda (q \cdot \hat{P}).$$ (2.13)

As the $q_{-}$ is by construction orthogonal to $P$, it is space-like or null and $\sqrt{-q_{-}^2} q_{-\lambda}$ defines a Lorentz invariant distance (which can also be shown to be the usual distance $|x_i - x_j|$ in the non-relativistic limit). We can now calculate the following Pb identities (as shown in the appendix)

$$\{q^\lambda_{-}, p^\mu_{i}\} = -\eta^{\lambda\mu} + \hat{P}^\lambda \hat{P}^\mu + p^\mu_{i} \left(\frac{v^\lambda_{-}}{p_i \cdot p_j}\right)$$ (2.14)

$$\{q^\lambda_{-}, p^\mu_{j}\} = \eta^{\lambda\mu} - \hat{P}^\lambda \hat{P}^\mu + p^\mu_{i} \left(\frac{v^\lambda_{-}}{p_i \cdot p_j}\right).$$ (2.15)

As above we calculate $\{-q_{-} \cdot q_{-}, p^\mu_{i}\}$ for the force on particle $i$ due to a potential $V(q_{-}^2)$:

$$\frac{1}{2} \{-q_{-} \cdot q_{-}, p^\mu_{i}\} = q^\mu_{-} - \frac{(v \cdot q_{-}) p^\mu_{j}}{p_i \cdot p_j}$$

$$= q^\mu_{-} - \left(\frac{p_i \cdot q_{-}}{p_i \cdot p_j}\right) p^\mu_{j}$$

$$= \frac{(q_{-} \wedge p_j)^{\mu \rho}}{p_i \cdot p_j} p_{i\rho} \equiv F^{\mu \rho} p_{i\rho}$$ (2.16)

and we see that the force tensor $F^{\mu \rho}$ now has “magnetic” components $(F^{23}, F^{31}, F^{12})$ due to the motion of particle $j$, and in fact $F'$ is of remarkably similar form to the electromagnetic field produced by particle $j$ if we put $V \propto (-q_{-}^2)^{-1/2}$ (see for example (14.15) of Jackson [3]).
Adding (2.9), (2.10) yields the Pb relation between $q_\perp$, $P$ :
\[\{q_\perp^\lambda, P^\mu\} = P^\mu \frac{v^\lambda}{p_i \cdot p_j} \Rightarrow \{q_\perp^\lambda, |P|\} = |P| \frac{v^\lambda}{p_i \cdot p_j} \quad (2.17a,b)\]
which means that
\[\{q_\perp^\lambda, \hat{P}^\mu\} \equiv \{q_\perp^\lambda, P^\mu/|P|\} = 0 \quad (2.18)\]
which last relation allows us to introduce potentials into the Hamiltonian as discussed in Sec. 1. To find the Pb relation between $q_\perp$ and $v$, we first subtract (2.15) from (2.14) yielding
\[\{q_\perp^\lambda, (p_- - \hat{P})\} = -v^\lambda \left( \frac{p_-}{p_i \cdot p_j} \right) \Rightarrow \{q_\perp^\lambda, v^\mu\} \equiv \{q_\perp^\lambda, p^\lambda_- - \hat{P}(p_- - \hat{P})\} = -v^\lambda \left( \frac{p_-}{p_i \cdot p_j} \right) + \hat{P}^\mu v^\lambda \left( \frac{p_-}{p_i \cdot p_j} \right) \]
\[= -v^\lambda \left( \frac{p_-}{p_i \cdot p_j} \right) - \frac{v^\lambda v^\mu}{p_i \cdot p_j}. \quad (2.19)\]
We see that $q_\perp, v$, do not qualify as covariant conjugates satisfying (2.11), because of the extra term on the RHS of (2.19).

There are two ways at arriving at a covariant conjugate pair. The first is to define
\[q'^\lambda \equiv q_\perp^\lambda - \left( \frac{q_\perp \cdot v}{v^2 + p_i \cdot p_j} \right) v^\lambda \quad (2.20)\]
then using (2.19) it can be readily shown that
\[\{q'^\lambda, v^\mu\} = -v^\lambda + \hat{P}^\mu \quad (2.21)\]
so that $q', v$ are covariant conjugates satisfying (2.11). But it turns out that $q'$ is none other that the Bakamjian-Thomas relative position that we have already encountered: the fact that
\[q' \equiv q_{BT}\]
which is rather a tedious calculation is shown in the appendix.

The second way to arrive at a covariant conjugate pair - which we will follow for the rest of this paper - is to rescale $q_\perp, v$ and define
\[\rho \equiv \frac{q_\perp}{|P|}, \quad \pi \equiv |P| v \quad (2.22)\]
then inspection of (2.12b) tells us that $\rho, \pi$, have the required Pb relation
\[\{\rho^\lambda, \pi^\mu\} = -v^\lambda + \hat{P}^\mu. \quad (2.23)\]
One reason for preferring the pair $(\rho, \pi)$ for the relative or internal variables is that potential functions of $\rho$ have the attractive feature of producing electromagnetic type forces (recalling (2.16)), instead of the pure electric type forces resulting from the BT relative position. In the next Section we will explore additional reasons for adopting $(\rho, \pi)$ instead of $(q_{BT}, v)$ as the relative position and relative momentum 4-vectors, including

(1) it is easier to factorise the Hamiltonian into CM and relative components when the relative component is expressed in terms of $\pi$ rather than $v$. And

(2) the CM and relative variables when these latter are expressed in terms of $(\rho, \pi)$ are maximally
independent and are an explicit realisation of the so-called ‘non-canonical covariant realisation’ (NCR) of [4], as discussed in Sec 4.

3. Factorisation of the Hamiltonian \( P_{\text{int}}^0 \)

As discussed at the beginning of Sec 2, the relation (2.1) allows us to introduce an interaction potential \( V \) being a scalar function of \(-\rho^2\) into the Hamiltonian
\[
P_{\text{int}}^0 = \hat{P}^0 \left(|P| + V(-\rho^2)\right)
\]
while maintaining the Pb relations of the Poincare group algebra. In this section we will first show that the \(|P|\) component in the Hamiltonian can be written in terms of \(\pi^2\), thus achieving a factorisation of the \(P_{\text{int}}^0\) into CM and relative generators. From (2.22), (2.10)
\[
\pi^2 = P^2 v^2 = P^2 [p^2 - (\hat{P} \cdot p_\perp)^2]
\]
\[
= \frac{1}{4} \left[ (m_i^2 + m_j^2 + 2p_i \cdot p_j)(m_i^2 + m_j^2 - 2p_i \cdot p_j) - (m_i^2 - m_j^2)^2 \right]
\]
\[
= m_i^2 m_j^2 - (p_i \cdot p_j)^2
\]
or
\[
p_i \cdot p_j = [m_i^2 m_j^2 - \pi^2]^{1/2}
\]
recalling that \(-\pi^2\) is positive due to \(\pi \cdot P = 0\). Then
\[
|P| = [m_i^2 + m_j^2 + 2p_i \cdot p_j]^{1/2}
\]
\[
= [m_i^2 + m_j^2 + 2|m_i^2 m_j^2 - \pi^2|^{1/2}]^{1/2}.
\]
In the non-relativistic regime when \(-\pi^2 \ll m_i^2 m_j^2\), then
\[
|P| \simeq \left[ m_i^2 + m_j^2 + 2m_i m_j - \frac{\pi^2}{m_i m_j} \right]^{1/2}
\]
\[
\simeq (m_i + m_j) - \frac{1}{2} \frac{\pi^2}{2m_i m_j (m_i + m_j)}
\]
\[
= (m_i + m_j) - \frac{1}{2} \frac{\pi}{\mu (m_i + m_j)} \cdot \frac{\pi}{(m_i + m_j)}
\]
\[(3.4)\]
where \(\mu\) is the usual reduced mass \(\mu = \frac{m_i m_j}{m_i + m_j}\). This is the standard non-relativistic expression for the energy taking into account that from (2.22) \(\pi\) has been rescaled by a factor of \(|P| \simeq m_i + m_j\), and that in the CM rest frame \(-\pi^2 \rho \rightarrow +\pi^a \pi^a = \pi^2 (a = 1, 2, 3)\).

Recalling the interaction Hamiltonian (3.1), the relative (internal) factor
\[
\left(|P| + V(\rho)\right)
\]
is a Lorentz scalar having the same value in any frame. This means that we can choose to evaluate it in the CM frame, when the \(\rho, \pi\) are conjugate variables, and we can follow the usual procedures in quantising the internal component of the Hamiltonian as discussed in Sec 5.

The CM factor \(\hat{P}^0\) in (3.1) is just the usual relativistic dilation factor \(\gamma\), which means that the two interacting particles are indeed behaving as one system. Bakamjian [5] noted the advantages of including the interaction terms in the 4-momentum vector (rather than in the boost generator as in the original BT paper) on physical grounds, in that the energy of interaction effectively increases the system rest mass, which in turn must contribute to the system momentum.
4 Relations between the CM and relative variables

The CM generators are \( R, \hat{P} \) where

\[
R^\lambda \equiv J^{\lambda \rho} \hat{P}_\rho, \quad \hat{P}^\lambda \equiv P^\lambda / |P|
\]

recalling from (1.1) that \( J \equiv j_i + j_j, \ P \equiv p_i + p_j \). The \( R^\lambda \) is essentially the Shirokov 4-vector (4) (but note that the Shirokov 4-vector position is \( \hat{P} \) rather than \( P \)). To obtain \( R \), so that \( R \) is the covariant conjugate to \( \hat{P} \) rather than \( \hat{P} \).

The relative generators are \( \rho, \pi \) which were defined in (2.22) and (2.13). The CM and relative generators are not in general independent as in the non-relativistic case, except in the system rest frame. The Pb relations below follow from the definitions of the generators in terms of \( j_i, j_j, p_i, p_j \), as an example we calculate \( \{ R^\lambda, \rho^\mu \} \) in the appendix.

\[
\{ R^\lambda, \hat{P}^\mu \} = -\eta_{\lambda \mu} + \hat{P}^\lambda \hat{P}^\mu \quad \{ \rho^\lambda, \pi^\mu \} = -\eta_{\lambda \mu} + \hat{P}^\lambda \hat{P}^\mu \quad (4.2)
\]

\[
\{ R^\lambda, \pi^\mu \} = J^{\lambda \mu} \quad \{ \rho^\lambda, \rho^\mu \} = 0 \quad (4.3)
\]

\[
\{ \hat{P}^\lambda, \hat{P}^\mu \} = 0 \quad \{ \pi^\lambda, \pi^\mu \} = 0 \quad (4.4)
\]

also the cross terms

\[
\{ R^\lambda, \pi^\mu \} = \hat{P}^\mu \pi^\lambda \quad \{ \hat{P}^\lambda, \rho^\mu \} = 0 \quad (4.5)
\]

\[
\{ R^\lambda, \rho^\mu \} = \hat{P}^\mu \rho^\lambda \quad \{ \pi^\lambda, \pi^\mu \} = 0 \quad (4.6)
\]

In the system rest frame when \( P = 0 \) then \( R^0 = \rho^0 = \pi^0 = 0 \), and the only Pb’s above which are non-zero are

\[
\{ R^a, \hat{P}^b \} = \delta^{ab}, \quad \{ \rho^a, \pi^b \} = \delta^{ab}, \quad \{ R^a, \hat{R}^b \} = J^{ab} \quad \text{when } P = 0.
\]

so that in this case \( \rho, \pi \) are conjugates to each other in the usual sense.

We can split the Lorentz generators \( J^{\lambda \mu} \) into external and relative parts \( L^{\lambda \mu}, M^{\lambda \mu} \) such that

\[
L^{\lambda \mu} + M^{\lambda \mu} = J^{\lambda \mu} \quad (4.7)
\]

where \( L^{\lambda \mu}, M^{\lambda \mu} \) are

\[
L^{\lambda \mu} = (R \wedge \hat{P})^{\lambda \mu} \quad M^{\lambda \mu} = (\rho \wedge \pi)^{\lambda \mu}. \quad (4.8)
\]

Note that (4.7) only holds for zero-spin particles, additional terms are required to make up the total angular momentum \( J \) if one or more of the particles has spin. For the Pb relations involving \( L, M \), we introduce the shorthand

\[
\eta_{\lambda \mu}^{\rho \sigma} \equiv \eta^{\rho \sigma} - \hat{P}^\rho \hat{P}^\sigma / \hat{P}^2 \quad (4.9)
\]

Then from the above relations (4.2-6) it follows that

\[
\{ L^{\lambda \mu}, R^\rho \} = \eta^{\rho \mu} R^\lambda - \eta^{\rho \sigma} R^\sigma - M^{\mu \rho} \hat{P}^\lambda + M^{\lambda \rho} \hat{P}^\mu \quad (4.10)
\]

\[
\{ M^{\lambda \mu}, R^\rho \} = M^{\mu \rho} \hat{P}^\lambda - M^{\lambda \rho} \hat{P}^\mu \quad (4.11)
\]

\[
\{ L^{\lambda \mu}, \hat{P}^\rho \} = \eta^{\rho \mu} \hat{P}^\lambda - \eta^{\lambda \rho} \hat{P}^\mu \quad (4.10)
\]

\[
\{ M^{\lambda \mu}, \hat{P}^\rho \} = 0 \quad (4.11)
\]

\[
\{ L^{\lambda \mu}, \rho^\rho \} = \hat{P}^\rho (\rho \wedge \hat{P})^{\lambda \mu} \quad (4.12)
\]

\[
\{ M^{\lambda \mu}, \rho^\rho \} = \eta^{\rho \mu} \rho^\lambda - \eta_{\lambda \rho} \rho^\mu \quad (4.12)
\]
\[
\{L^{\lambda\mu}, \pi^\rho\} = \vec{P}^\rho (\pi \wedge \vec{P})^{\lambda\mu}
\]
\[
\{M^{\lambda\mu}, \pi^\rho\} = \gamma_\perp^\rho \pi^\rho - \gamma_\perp^\lambda \pi^\mu
\]  
(4.13)

Adding the \(L^{\lambda\mu}, M^{\lambda\mu}\) pairs above, all generators have the correct 4-vector Pb relations with \(J^{\lambda\mu}\).

Finally from the above we can readily determine the Pb relations \(\{L^{\lambda\mu}, L^{\nu\rho}\}, \{L^{\lambda\mu}, M^{\nu\rho}\}\) and \(\{M^{\lambda\mu}, M^{\nu\rho}\}\). In particular
\[
\{M^{\lambda\mu}, M^{\nu\rho}\} = \eta_\perp^{\lambda\rho} M^{\mu\nu} + \eta_\perp^{\mu\nu} M^{\lambda\rho} - \eta_\perp^{\lambda\nu} M^{\mu\rho} - \eta_\perp^{\mu\rho} M^{\lambda\nu}.
\]  
(4.14)

Note that \(M \equiv (\rho \wedge \pi)\) is orthogonal to \(\vec{P}\), i.e. \(M^{\lambda\mu} P_\mu = 0\), which is necessary for the system to have space inversion invariance [4] and is a consequence of both \(\rho, \pi\) being orthogonal to \(\vec{P}\). In the CM rest frame only the components
\[
\{M^{31}, M^{31}, M^{12}\} \to \{S^1, S^2, S^3\}
\]
exist, and from (4.14) the \(S\) obey the same Pb’s as the usual spin 3-vector. Also from (4.11) we see that \(M^{\lambda\mu}\) is translation invariant as required for it to represent an internal angular momentum.

The Pb relations relations above mean that \(R, \vec{P}, \rho, \pi\) are an explicit realisation of the algebra which Rohrlich [4] labelled the non-canonical covariant realisation (NCR). We have constructed \(R, \vec{P}, \rho, \pi\) satisfying the NCR in terms of the individual particle generators \(j_i, j_j, p_i, p_j\). This realisation of the NCR is unique.

5 Quantisation and outlook

Much effort was spent by previous workers, for example [7], in finding CM and relative generators which are canonical (meaning that \(Q, P, q, p\) satisfy the canonical relations \(\{Q^a, P^b\} = \{q^a, p^b\} = \delta^{ab}\) ) in any inertial frame. Their approach was to find the relative position and momentum generators in the CM rest frame, then Lorentz boost these 3-vector generators to a general frame. This results in complicated expressions.

Our approach depends on being able to factorise the Hamiltonian into CM and relative components as in (3.1), i.e.
\[
P^0_{\text{int}} = \hat{P}^0 (|P| + V(\rho)) = \hat{P}^0 \mathcal{H}(\rho, \pi)
\]  
(5.1)

where the relative component \(\mathcal{H}(\rho, \pi)\) is a Lorentz scalar of \(\rho, \pi\). We then quantise \(\mathcal{H}\) in the CM rest frame, when the \(\rho, \pi\) are conjugate variables satisfying \(\{\rho^a, \pi^b\} = \delta^{ab}\), and because \(\mathcal{H}\) is a Lorentz scalar it will have the same eigenvalues in any other frame. The CM component \(\hat{P}^0 \equiv P^0/|P|\) is just the usual relativistic dilation factor \(\gamma\).

Below we outline the quantisation of a two particle system with a Coulomb potential
\[
V = -(-q_\perp \cdot q_\perp)^{-1/2} = -(-\rho \cdot \rho)^{-1/2} |P|^{-1}
\]
recalling from (2.22) that \(q_\perp = |P|\rho\). Then
\[
\mathcal{H} \equiv |P| + V = |P| - (-\rho \cdot \rho)^{-1/2} |P|^{-1}
\]  
= \[
\left[ m_i^2 + m_j^2 + 2|m_i^2 m_j^2 - \pi^2|^{1/2} \right]^{1/2} - (-\rho \cdot \rho)^{-1/2} \left[ m_i^2 + m_j^2 + 2|m_i^2 m_j^2 - \pi^2|^{1/2} \right]^{-1/2}
\]
and in the CM frame \(-\rho \cdot \rho \rightarrow \rho^2, -\pi \cdot \pi \rightarrow \pi^2\), then
\[
\mathcal{H} = \left[ m_i^2 + m_j^2 + 2|m_i^2 m_j^2 + \pi^2|^{1/2} \right]^{1/2} - |\rho|^{-1} \left[ m_i^2 + m_j^2 + 2|m_i^2 m_j^2 + \pi^2|^{1/2} \right]^{-1/2}
\]  
(5.2)
where \( \rho, \pi \) are conjugate variables satisfying \( \{ \rho^a, \pi^b \} = \delta^{ab} \). Putting \( \mathcal{H} = E \), this eigenvalue equation can be solved numerically if not in closed form. Further aspects of quantisation will be addressed elsewhere.

**Appendix**

**Proof of (2.11)**

First we calculate \( \{ q^\lambda_{BT}, p^\mu_i \} \):

\[
\{ q^\lambda_{BT}, p^\mu_i \} \equiv \left\{ \frac{\lambda p}{p_i \cdot P} - \frac{\lambda p}{p_j \cdot P} , p^\mu_i \right\}
\]

\[
= \left( \frac{p^\mu_i}{p_i \cdot P} \right) (\eta^\mu \rho^\lambda_i - \eta^\lambda \rho^\mu_i)
\]

\[
= -\eta^\lambda \mu + \frac{p^\lambda_i p^\mu_i}{p_i \cdot P}
\]

which is (2.6)

Then

\[
\{ q^\lambda_{BT}, \frac{1}{2}(p^\mu_i - p^\mu_j) \} = -\eta^\lambda \mu + \frac{1}{2} \left( \frac{p^\lambda_i}{p_i \cdot P} - \frac{p^\lambda_j}{p_j \cdot P} \right) P^\mu
\]

\[
\{ q^\lambda_{BT}, v^\mu \} \equiv \{ q^\lambda_{BT}, \frac{1}{2}(p^\mu_i - p^\mu_j) - \frac{1}{2} \hat{P}^\mu [(p_i - p_j) \cdot \hat{P}] \}
\]

\[
= -\eta^\lambda \mu + \hat{P}^\lambda \hat{P}^\mu
\]

which is (2.11)

**Proof of (2.14, 15)**

First we calculate \( \{ q^\lambda, p^\mu_i \} \) recalling \( q \) from (2.12):

\[
\{ q^\lambda, p^\mu_i \} \equiv \{ \lambda p_{j\mu} - \lambda p_{i\mu} , p^\mu_i \}
\]

\[
= \left( \frac{p_{j\mu}}{p_i \cdot P} \right) (\eta^{\mu} p^\lambda_j - \eta^\lambda p^{\mu}_j)
\]

\[
= -\eta^\lambda \mu + \frac{p^\lambda_j p^\mu_j}{p_i \cdot P}
\]

and

\[
\{ q \cdot \hat{P}, p^\mu_i \} = -\hat{P}^\mu + \frac{(p_i \cdot P) p^\mu_j}{p_i \cdot P_j}
\]

\[
\{ q^\lambda_{\perp}, p^\mu_i \} \equiv \{ q^\lambda - (q \cdot \hat{P}) \hat{P}^\lambda , p^\mu_i \}
\]

\[
= -\eta^\lambda \mu + \frac{p^\lambda_j p^\mu_j}{p_i \cdot P_j} - \hat{P}^\lambda \hat{P}^\mu + \frac{(p_i \cdot P) p^\mu_j}{p_i \cdot P_j}
\]

\[
= -\eta^\lambda \mu + \hat{P}^\lambda \hat{P}^\mu + \frac{p^\mu_j}{p_i \cdot P_j} (p^\lambda_j - (p_i \cdot P) \hat{P}^\lambda)
\]

\[
= -\eta^\lambda \mu + \hat{P}^\lambda \hat{P}^\mu + \frac{p^\mu_j}{p_i \cdot P_j} v^\lambda
\]

which is (2.14)
The equivalence of $q'$ and $q_{BT}$

We must show that, recalling (2.20),

$$q'_{\perp} - \left( \frac{q_{\perp} \cdot v}{v^2 + p_i \cdot p_j} \right) v^\lambda = q_{BT}^\lambda$$  \hspace{1cm} (2.20)

or equivalently

$$q_{BT}^\lambda + q_{BT} \cdot \frac{v}{p_i \cdot p_j} v = q_{\perp}.$$  \hspace{1cm} (A1)

The identity (A1) is most easily shown if we employ the auxiliary variable (the Shirokov position for particle $i$)

$$q_i^\lambda \equiv j_i^\lambda p_i / m_i^2 \Rightarrow q_i \cdot p_i = 0.$$  

then

$$q_{BT} = q_i - q_j - p_i \left( \frac{q_i \cdot P}{p_i \cdot P} \right) + p_j \left( \frac{q_j \cdot P}{p_j \cdot P} \right)$$  

and using identities such as

$$p_i = \hat{P}(p_i \cdot \hat{P}) + v, \quad p_j = \hat{P}(p_j \cdot \hat{P}) - v, \quad v^2 + p_i \cdot p_j = (p_i \cdot \hat{P})(p_j \cdot \hat{P})$$  

we finally arrive at

$$q_{BT} + \left( \frac{q_{BT} \cdot v}{p_i \cdot p_j} \right) v = (q_i - q_j) - \hat{P}[(q_i - q_j) \cdot \hat{P}] - v \left( \frac{q_i + q_j}{p_i \cdot p_j} \right) \cdot \hat{P}$$  \hspace{1cm} (A2)

and the RHS of (A2) can be shown to be equal to $q_{\perp}$ by expanding out similarly as above.

To calculate $\{R^\lambda, \rho^\mu\}$

Recall $R^\lambda \equiv J^{\lambda \rho} \hat{P}_\rho$, then

$$\{R^\lambda, \rho^\mu\} \equiv \{J^{\lambda \rho} \hat{P}_\rho, \rho^\mu\}$$

$$= (\eta^\rho^\mu \rho^\lambda - \eta^{\lambda \mu} \rho^\rho) \hat{P}_\rho$$

$$= \hat{P}^\mu \rho^\lambda$$  \hspace{1cm} as in (4.6)

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