WELL-POSEDNESS AND REGULARITY
FOR A FRACTIONAL TUMOR GROWTH MODEL

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Abstract. In this paper, we study a system of three evolutionary operator equations
involving fractional powers of selfadjoint, monotone, unbounded, linear operators having
compact resolvents. This system constitutes a generalization of a phase field system of
Cahn–Hilliard type modelling tumor growth that has been proposed in Hawkins-Daarud
et al. (Int. J. Numer. Math. Biomed. Eng. 28 (2012), 3–24) and investigated in recent

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papers co-authored by the present authors and E. Rocca. The model consists of a Cahn–Hilliard equation for the tumor cell fraction \( \phi \), coupled to a reaction-diffusion equation for a function \( S \) representing the nutrient-rich extracellular water volume fraction. Effects due to fluid motion are neglected. The generalization investigated in this paper is motivated by the possibility that the diffusional regimes governing the evolution of the different constituents of the model may be of different (e.g., fractional) type. Under rather general assumptions, well-posedness and regularity results are shown. In particular, by writing the equation governing the evolution of the chemical potential in the form of a general variational inequality, also singular or nonsmooth contributions of logarithmic or of double obstacle type to the energy density can be admitted.

1 Introduction

Let \( \Omega \subset \mathbb{R}^3 \) denote an open, bounded, and connected set with smooth boundary \( \Gamma \) and unit outward normal \( \mathbf{n} \), let \( T > 0 \) be given, and set \( Q_t := \Omega \times (0,t) \) for \( t \in (0,T) \) and \( Q := \Omega \times (0,T) \), as well as \( \Sigma := \Gamma \times (0,t) \). We investigate in this paper the evolutionary system

\[
\begin{align*}
\alpha \partial_t \mu + \partial_t \varphi + A^{2\sigma} \mu &= P(\varphi)(S - \mu) \quad \text{in } Q, \\
\mu &= \beta \partial_t \varphi + B^{2\sigma} \varphi + f(\varphi) \quad \text{in } Q, \\
\partial_t S + C^{2\tau} S &= -P(\varphi)(S - \mu) \quad \text{in } Q, \\
\mu(0) &= \mu_0, \quad \varphi(0) = \varphi_0, \quad S(0) = S_0, \quad \text{in } \Omega.
\end{align*}
\]

In the above system, \( \alpha > 0 \) and \( \beta > 0 \), and \( A^{2\sigma}, B^{2\sigma}, C^{2\tau} \), with \( r, \sigma, \tau > 0 \), denote fractional powers of the selfadjoint, monotone, and unbounded linear operators \( A, B, \) and \( C \), respectively, which are supposed to be densely defined in \( H := L^2(\Omega) \) and to have compact resolvents. Moreover, \( f \) denotes the derivative of a double-well potential \( F \). Typical and physically significant examples of \( F \) are the so-called classical regular potential, the logarithmic potential, and the double obstacle potential, which are given, in this order, by

\[
\begin{align*}
F_{\text{reg}}(r) &:= \frac{1}{4} (r^2 - 1)^2, \quad r \in \mathbb{R}, \\
F_{\text{log}}(r) &:= \begin{cases} 
(1 + r) \ln(1 + r) + (1 - r) \ln(1 - r) - c_1 r^2, & r \in (-1,1) \\
2 \log(2) - c_1, & r \notin [-1,1] 
\end{cases}, \\
F_{\text{2obs}}(r) &:= c_2 (1 - r^2) \quad \text{if } |r| \leq 1 \quad \text{and} \quad F_{\text{2obs}}(r) := +\infty \quad \text{if } |r| > 1.
\end{align*}
\]

Here, the constants \( c_i \) in (1.6) and (1.7) satisfy \( c_1 > 1 \) and \( c_2 > 0 \), so that the corresponding functions are nonconvex. In cases like (1.7), one has to split \( F \) into a nondifferentiable convex part \( F_1 \) (the indicator function of \([-1,1]\), in the present example) and a smooth perturbation \( F_2 \). Accordingly, in the term \( f(\varphi) \) appearing in (1.2), one has to replace the derivative \( F'_1 \) of the convex part \( F_1 \) by the subdifferential \( f_1 := \partial F_1 \) and interpret (1.2) as a differential inclusion or as a variation inequality involving \( F_1 \) rather than \( f_1 \). Furthermore, the function \( P \) occurring in (1.1) and (1.3) is nonnegative and smooth. Finally, the terms on the right-hand sides in (1.4) are prescribed initial data.