Review

Basics of quantum plasmonics

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Abstract

The present work is a topical review of the theoretical research on the quantum theory of plasmons and plasmon–photon interaction. The plasmons are defined as the quanta of the quantized plasmonic field. The corresponding classical plasmonic field was constructed on the basis of the study of collective oscillations of the electron gas in the solid. The electron–electron Coulomb interaction is taken into account. The explicit forms of the plasmon–photon interaction Lagrangian in canonical quantum mechanics and the plasmon–photon interaction action functional in the functional integral approach are derived. They all show that the interaction processes are nonlocal ones. The physical origin of the nonlocality is the complex structure of plasmons as composite quasiparticles: they cannot be considered as point particles, as was assumed in all phenomenological theories.

Keywords: plasmon, plasmonics, quantization, functional integral, electron gas

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1. Introduction

The resonance phenomenon in collective oscillations of electron gas was demonstrated by early works on plasma oscillations in solids [1–4]. It was caused by the elementary collective excitations behaving like quasiparticles of a special type called plasmons. In recent years there has been significant progress in the study of the interaction processes in which plasmons play the essential role, and a new very promising area of quantum physics called plasmonics has emerged and was rapidly developed. The variety of plasmonic processes and phenomena is quite broad: the formation of hybrid systems consisting of semiconductor quantum dots and metallic nanoparticles [5, 6], interaction between a metallic nanoparticle and a dipole emitter [7], exciton–plasmon coupling (plexciton) [8–10], exciton–plasmon resonance energy transfer [11, 12], plasmon-enhanced light absorption [13, 14] and fluorescence [15–20], plasmonic–molecular resonance [21–28]. The results of the research on plasmonic processes have led to the creation of plasmonic nanomaterials with various efficient applications [29].

In our previous works [30–32] attempts were made to elaborate the quantum theory of plasmon enabling one to exactly derive the effective action functional, or the effective Hamiltonians in the special cases, of the interactions of plasmons with other quasiparticles in the matter as well as with the electromagnetic field, starting from first principles of electrodynamics and quantum physics. The present article is a topical review of the above-mentioned theoretical works with the extension to include also the derivation of the effective action functional of the interacting photon–plasmon system.

In section 2, applying the canonical quantization procedure of quantum mechanics, we present the theory of canonical quantization of the plasmonic field [31]. The construction of the quantum plasmonic field by means of the functional integral technique is performed in section 3 [30, 32]. In section 4 the effective action functional of the interacting photon–plasmon system is derived by means of the canonical quantization method in quantum mechanics. The same subject as that of section 4 is studied in section 5 in the framework of the functional integral approach. The conclusion and discussions are presented in section 6.

2. Quantum plasmonic field in canonical quantum mechanics

Consider the system of itinerant electrons moving in the electrostatic field generated by the positive charge of the ions...
in some crystal. Denote \( n(x, t) \) the electron density (number of electrons per unit volume) and \( n_0 \) its mean value averaged over both space and time. The average charge density \( -en_0 \) of electrons, \( -e \) being the negative electron charge, compensates the average positive charge of ions in the crystal, and the fluctuating charge density in the crystal is determined by the expression

\[
\rho(x, t) = -e [n(x, t) - n_0].
\]  

(1)

According to the Coulomb law, the charge distribution with the density (1) in a special region \( V \) generates a time-dependent electrical field with the potential

\[
\varphi(x, t) = \int_V \frac{\rho(x', t)}{|x - x'|} \mathrm{d}x'.
\]

(2)

From formula (2) it follows the Poisson equation

\[
\nabla^2 \varphi(x, t) = -4\pi \rho(x, t).
\]

(3)

The mutual interaction between electrons of the electron gas in the region \( V \) gives rise to the potential energy of the electron gas

\[
U(t) = \frac{1}{2} \int_V \int_V \mathrm{d}x \mathrm{d}x' \rho(x, t) \frac{1}{|x - x'|} \rho(x', t).
\]

(4)

which can be also represented in the form

\[
U(t) = \frac{1}{2} \int_V \mathrm{d}x \rho(x, t) \varphi(x, t),
\]

(5)

On the other side, as the consequence of the oscillating displacements of electrons, the fluctuation of the electron density \( n(x, t) \) generates the total kinetic energy of the electron gas. Denote \( \delta r(x, t) \) the displacement vector of the electron having the coordinate \( x \) at the time moment \( t \), and \( m \) the electron mass. Since the electron has the velocity

\[
\delta r(x, t) = \frac{\delta r(x, t)}{\partial t},
\]

(6)

the whole electron gas has following total kinetic energy

\[
T(t) = \frac{m}{2} \int_V \mathrm{d}x n(x, t) \delta r(x, t)^2.
\]

(7)

Consider now the conservation of the total electron number. Denote \( dN_S \) the number of electrons going through the boundary \( S \) of a region \( V \) from the inside of \( V \) into its outside during the time interval from \( t \) to \( t + \delta t \)

\[
dN_S = \int_S \mathrm{d}S \delta n(x, t),
\]

(8)

Using the Ostrogradski–Gauss formula, we rewrite \( dN_S \) as follows

\[
dN_S = \int_V \mathrm{d}x \nabla \cdot \int_V \mathrm{d}x \delta r(x, t) \delta r(x', t) \mathrm{d}t.
\]

(9)

The decrease of the electron number inside \( V \) during that time interval equals

\[
dN_V = \int_V \mathrm{d}x [n_0 - n(x, t)] \mathrm{d}t.
\]

(10)

Because of the conservation of the total electron number, there must be the equality

\[
dN_S = dN_V.
\]

(11)

From equations (1) and (9)–(11) it follows that

\[
\rho(x, t) = e V [n(x, t) \delta r(x, t)].
\]

(12)

For the convenience in performing the canonical quantization procedure we decompose the functions \( \rho(x, t) \), \( \varphi(x, t) \) and \( \delta r(x, t) \) into the Fourier series, using a basis consisting of the plane waves satisfying the periodic boundary conditions at the opposite surfaces of a cube with the volume \( V \) and normalized in this cube. We have the following formulae

\[
\rho(x, t) = \frac{1}{\sqrt{V}} \sum_k e^{ikx} \rho_k(t),
\]

(13)

\[
\varphi(x, t) = \frac{1}{\sqrt{V}} \sum_k e^{ikx} \varphi_k(t),
\]

(14)

\[
\delta r(x, t) = \frac{1}{\sqrt{V}} \sum_k e^{ikx} \sum_i \epsilon^{(i)}_k q^{(i)}_k(t),
\]

(15)

where \( \epsilon^{(i)}_k \) for each \( k \) are three unit vectors satisfying the conditions

\[
ke^{(i)}_k = \begin{cases} 0, & i = 1, 2 \\ k, & i = 3 \end{cases}
\]

(16)

\( k \) = \|k\|. Two terms with \( i = 1, 2 \) are the transverse displacements, while that with \( i = 3 \) is the longitudinal displacement along the direction of the wave vector \( k \). From equation (3) and formulae (13) and (14) we obtain

\[
\varphi_k(t) = \frac{4\pi}{k^2} \rho_k(t).
\]

(17)

Consider the approximation in which the fluctuating electron density \( n(x, t) \) in the expression of \( \rho(x, t) \) is replaced by its average value \( n_0 \)

\[
\rho(x, t) \approx en_0 \nabla \delta \varphi(x, t).
\]

(18)

In this case from the expressions (13) and (15) and equation (18) we obtain

\[
\rho_k(t) \approx ien_0 k \epsilon^{(3)}_k \rho^{(3)}_k(t).
\]

(19)

Therefore the potential energy (4) and the kinetic energy (7) are expressed in terms of the generalized coordinates of the system as follows

\[
U = 2\pi e^2 n_0^2 \sum_k \rho^{(3)}_k(t) \varphi^{(3)}_k(t),
\]

(20)

\[
T = \frac{1}{2} m n_0 \sum_{k = 1}^3 \epsilon^{(i)}_k \rho^{(i)}_k \varphi^{(i)}_k(t).
\]

(21)
\[ L = T - U = \frac{1}{2} \sum_k \left\{ n_0 m \sum_{i=1}^3 \dot{q}^{(i)}_k(t) \dot{q}^{(i)}_k(t) - 4\pi e^2 n_0^2 \dot{q}^{(3)}_k(t) \dot{q}^{(3)}_k(t) \right\}. \]  

(22)

The equations of motion are

\[ \frac{d^2 \dot{q}^{(i)}_k(t)}{dt^2} + 4\pi e^2 n_0^2 \delta_{ij} \dot{q}^{(j)}_k(t) = 0. \]  

(23)

It follows that the transverse coordinates \( \dot{q}^{(1)}_k(t) \) and \( \dot{q}^{(2)}_k(t) \) do not oscillate, while the longitudinal one \( \dot{q}^{(3)}_k(t) \) periodically oscillates with the angular frequency

\[ \omega_0 = \frac{4\pi e^2 n_0}{m}, \]  

(24)

which is the known plasma frequency of a free electron gas. Since the transverse coordinates do not oscillate, we discard them and replace the notation \( \dot{q}^{(3)}_k(t) \) by the simpler one \( \dot{q}_k(t) \). Lagrangian of the system becomes

\[ L = \frac{1}{2} \sum_k \left\{ n_0 m \dot{q}_k(t) \dot{q}_k(t) - 4\pi e^2 n_0^2 \dot{q}^2_k(t) \right\}. \]  

(25)

Now we demonstrate that the quantum mechanical system with the Lagrangian (25) can be considered as a classical field \( \sigma(x, t) \). Consider the Fourier transformation of this classical field and its time derivative

\[ \sigma(x, t) = \frac{1}{\sqrt{V}} \sum_k \hat{q}_k(t), \]  

(26)

\[ \dot{\sigma}(x, t) = \frac{1}{\sqrt{V}} \sum_k \hat{\dot{q}}_k(t). \]  

(27)

We have

\[ \int_V d^3 x \sigma(x, t)^2 = \sum_k \dot{q}_k(t) \dot{q}_k(t), \]  

(28)

\[ \int_V d^3 x \dot{\sigma}(x, t)^2 = \sum_k \dot{\dot{q}}_k(t) \dot{\dot{q}}_k(t). \]  

(29)

Now we apply the canonical quantization procedure to the real scalar field with the Lagrangian (30). Denote \( \delta L(t) \) and \( \delta I[\sigma] \) the variations of \( L(t) \) and \( I[\sigma] \) when the scalar field \( \sigma(x, t) \) and its time derivative \( \dot{\sigma}(x, t) \) are subjected to the infinitely small variations \( \delta \sigma(x, t) \) and \( \delta \dot{\sigma}(x, t) \),

\[ \delta \sigma(x, t) = \frac{\partial}{\partial t} \delta \sigma(x, t). \]  

(32)

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\[ \delta \sigma(x, t) = \frac{\partial}{\partial t} \delta \sigma(x, t). \]  

(33)

and therefore

\[ \delta I[\sigma] = \int_{-\infty}^{\infty} dt \delta \delta L(t) \]  

\[ = \int_V \int_{-\infty}^{\infty} dt \frac{\delta L(t)}{\delta \sigma(x, t)} \delta \sigma(x, t) \]  

\[ + \int_{-\infty}^{\infty} dt \int_V dx \frac{\delta L(t)}{\delta \dot{\sigma}(x, t)} \delta \dot{\sigma}(x, t). \]  

(34)

Because of the boundary condition

\[ \delta \sigma(x, t) \to 0 \quad \text{at} \quad t \to \pm \infty, \]  

(35)

the first term in the right-hand side of equation (34) vanishes, and we obtain

\[ \delta I[\sigma] = \int_{-\infty}^{\infty} dt \int_V dx \left[ \frac{\delta L(t)}{\delta \sigma(x, t)} - \frac{\partial}{\partial t} \right] \delta \sigma(x, t). \]  

(36)

On the other hand, by definition

\[ \delta I[\sigma] = \int_{-\infty}^{\infty} dt \int_V dx \frac{\delta L(t)}{\delta \sigma(x, t)} \delta \sigma(x, t). \]  

(37)

The comparison of formulae (36) and (37) gives

\[ \frac{\delta I[\sigma]}{\delta \sigma(x, t)} = \frac{\delta L(t)}{\delta \sigma(x, t)} - \frac{\partial}{\partial t} \left( \frac{\delta L(t)}{\delta \dot{\sigma}(x, t)} \right). \]  

(38)

From the extreme action principle
we obtain Lagrange functional equation
\[
\frac{\delta L(t)}{\delta \sigma(x, t)} - \frac{\partial}{\partial t} \left( \frac{\delta L(t)}{\delta \dot{\sigma}(x, t)} \right) = 0.
\] (40)

Considering \( \sigma(x, t) \) as the canonical coordinate of the field, we have following canonical momentum
\[
\pi(x, t) = \frac{\delta L(t)}{\delta \dot{\sigma}(x, t)} = \dot{\sigma}(x, t)
\] (41)

and Hamiltonian functional
\[
H = \int_V dx \sigma(x, t) \pi(x, t) - L
= \frac{1}{2} \int_V dx \left\{ \left( \frac{\partial \sigma(x, t)}{\partial t} \right)^2 + \omega_0^2 \sigma(x, t)^2 \right\}. \tag{42}
\]

After the canonical quantization procedure, the canonical coordinate \( \sigma(x, t) \) and momentum \( \pi(x, t) \) become the operators \( \hat{\sigma}(x, t) \) and \( \hat{\pi}(x, t) \), and we have following Hamiltonian operator of the quantized field
\[
\hat{H} = \frac{1}{2} \int_V dx \left\{ \left( \frac{\partial \hat{\sigma}(x, t)}{\partial t} \right)^2 + \omega_0^2 \hat{\sigma}(x, t)^2 \right\}. \tag{43}
\]

Now we expand the canonical field operator \( \hat{\sigma}(x, t) \) into the Fourier series of the orthogonal and normalized plane waves
\[
\hat{\sigma}(x, t) = \frac{1}{\sqrt{V}} \frac{1}{\sqrt{2\omega_0}} \sum_k \left[ e^{i(kx - \omega_0 t)} \hat{a}_k + e^{-i(kx - \omega_0 t)} \hat{a}_k^+ \right]. \tag{44}
\]

Substituting expression (44) of \( \hat{\sigma}(x, t) \) into the right hand side of formula (43), after lengthy standard calculations we obtain
\[
\hat{H} = \frac{1}{2} \sum_k \omega_0 \left( \hat{a}_k^+ \hat{a}_k + \hat{a}_k \hat{a}_k^+ \right). \tag{45}
\]

According to the equation (41), the canonical momentum \( \hat{\pi}(x, t) \) equals the time derivative of the canonical field \( \hat{\sigma}(x, t) \). Therefore \( \hat{\pi}(x, t) \) has following expansion
\[
\hat{\pi}(x, t) = -\frac{i}{\sqrt{V}} \sqrt{\frac{\omega_0}{2}} \sum_k \left[ e^{i(kx - \omega_0 t)} \hat{a}_k - e^{-i(kx - \omega_0 t)} \hat{a}_k^+ \right]. \tag{46}
\]

Inverting the expansions (44) and (46), we obtain the expressions of \( \hat{a}_k \) and \( \hat{a}_k^+ \) in terms of the canonical coordinate \( \hat{\sigma}(x, t) \) and momentum \( \hat{\pi}(x, t) \):
\[
\hat{a}_k = \frac{1}{\sqrt{V}} \int_V dx e^{-ikx} \left\{ \frac{\omega_0}{2} \hat{\sigma}(x, 0) \right\}
\]
\[
+ \frac{i}{\sqrt{2\omega_0}} \hat{\pi}(x, 0) \tag{47}
\]
and
\[
\hat{a}_k^+ = \frac{1}{\sqrt{V}} \int_V dx e^{-ikx} \left\{ \frac{\omega_0}{2} \hat{\sigma}(x, 0) \right\}
\]
\[
- \frac{i}{\sqrt{2\omega_0}} \hat{\pi}(x, 0) \tag{48}
\]

According to the canonical quantization rules, between the operators \( \hat{\sigma}(x, t) \) and \( \hat{\pi}(x, t) \) there exist the following equal-time canonical commutation relations
\[
[\hat{\sigma}(x, t), \hat{\sigma}(x', t)] = [\hat{\pi}(x, t), \hat{\pi}(x', t)] = 0,
[\hat{\sigma}(x, t), \hat{\pi}(x', t)] = -i\hbar (x-x'). \tag{49}
\]

Using two expressions (47) and (48) and commutation relations (49), we derive the following canonical commutation relations
\[
[\hat{a}_k, \hat{a}_l] = [\hat{a}_k^+, \hat{a}_l^+] = 0,
[\hat{a}_k, \hat{a}_l^+] = \delta_{kl}. \tag{50}
\]

These relations together with formula (45) for Hamiltonian \( \hat{H} \) show that \( \hat{a}_k \) and \( \hat{a}_k^+ \) are the destruction and creation operators of the quasiparticles with the energy \( \omega_0 \)—the plasmons. Thus, we have constructed the quantum field \( \hat{\sigma}(x, t) \) whose quanta are the plasmons—the quantum plasmonic field. In the harmonic approximation the plasmons are dispersionless. In order to establish the dispersion law of the plasmons by means of the canonical quantization procedure it is necessary to go beyond the harmonic approximation.

3. Quantum plasmonics field in functional integral formalism

3.1. Basic notions in functional integral method

Consider a physical structure in three-dimensional space, for any vector \( x \) in this space we introduce the vector
\[
x = (x, x_0) = (x, t)
\]
in the corresponding four-dimensional space–time and denote
\[
\int dx = \int dx \int dt.
\]
The integration domain in the three-dimensional space is the whole volume \( \Omega \) of the structure, while the integral over \( t \) is calculated from \( -\infty \) to \( +\infty \).

Let \( q(x) = q(x, t) \) be a real scalar function of \( x \). Considering each value \( q(x) \) as a component of a vector with the continuous vector index \( x \), we extend the notion of multiple integral and introduce the notion of functional integral as
follows

$$\int [D\psi] \cdots = \int \cdots \Pi d\psi, \quad (51)$$

Similarly, let $\psi_1 = \psi(x) = \psi(x, t)$ and $\psi_2 = \psi(x) = \psi(x, t)$
be two other hermitian conjugate spinor functions. Considering $\psi_1$ and $\psi_2$ as the spinor components of two vectors
with the continuous vector index $x$, we introduce the notion of functional integral over $\psi(x)$ and $\overline{\psi}(x)$ as the extension of
the definition of the multiple integral to the case of two uncountable sets of integration variables $\psi_1$ and $\psi_2$ with the continuous index $x$

$$\int [D\psi][D\overline{\psi}] \cdots = \int \cdots \Pi d\psi_1 d\psi_2, \quad (52)$$

The functional integral method was a powerful method for the
study of relativistic quantum field theory [33–36].

The subject of our study is the electron gas inside the
volume $\Omega$ of a physical structure. Because the electrons are
confined inside this volume, we impose on the functional integration variables the following boundary condition: they
must vanish outside volume $\Omega$ as well as on its surface.

The Bosonic functional integration variables $\varphi(x)$ are
commuting

$$\varphi(x)\varphi(y) = \varphi(y)\varphi(x), \quad (53)$$

while the fermionic ones $\psi(x)$ and $\overline{\psi}(x)$ anticommute each other

$$\psi(x)\overline{\psi}(y) + \overline{\psi}(y)\psi(x) = 0,$$
$$\psi(x)\psi(y) + \psi(y)\psi(x) = 0,$$
$$\overline{\psi}(x)\psi(y) + \psi(y)\overline{\psi}(x) = 0. \quad (54)$$

Functional integration variables $\psi(x)$ and $\overline{\psi}(x)$ are called the
Grassmann variables. Beside of these dynamical variables one
often uses also Grassmann parameter $\eta(x)$ and its hermitian conjugate $\overline{\eta}(x)$. They anticommute each other

$$\eta(x)\eta(y) + \eta(y)\eta(x) = 0,$$
$$\overline{\eta}(x)\overline{\eta}(y) + \overline{\eta}(y)\overline{\eta}(x) = 0,$$
$$\eta(x)\overline{\eta}(y) + \overline{\eta}(y)\eta(x) = 0. \quad (55)$$

and also anticommute with the dynamical Grassmann variables

$$\eta(x)\psi(y) + \psi(y)\eta(x) = 0,$$
$$\eta(x)\overline{\psi}(y) + \overline{\psi}(y)\eta(x) = 0,$$
$$\overline{\eta}(x)\psi(y) + \psi(y)\overline{\eta}(x) = 0,$$
$$\overline{\eta}(x)\overline{\psi}(y) + \overline{\psi}(y)\overline{\eta}(x) = 0. \quad (56)$$

Bosonic functional integration variable commutes with all
Grassmann dynamical variables and Grassmann parameters

$$\varphi(x)\psi(y) + \psi(y)\varphi(x) = 0,$$
$$\varphi(x)\overline{\psi}(y) + \overline{\psi}(y)\varphi(x) = 0,$$
$$\varphi(x)\eta(y) = \eta(y)\varphi(x),$$
$$\varphi(x)\overline{\eta}(y) = \overline{\eta}(y)\varphi(x). \quad (57)$$

In the calculations we often use the functional derivation operators $\frac{\delta}{\delta \eta(x)}$ and $\frac{\delta}{\delta \overline{\eta}(x)}$. They anticommute each other

$$\frac{\delta}{\delta \eta(x)} \psi(y) + \psi(y) \frac{\delta}{\delta \eta(x)} = 0,$$
$$\frac{\delta}{\delta \overline{\eta}(x)} \psi(y) + \psi(y) \frac{\delta}{\delta \overline{\eta}(x)} = 0,$$
$$\frac{\delta}{\delta \eta(x)} \overline{\psi}(y) + \overline{\psi}(y) \frac{\delta}{\delta \eta(x)} = 0,$$
$$\frac{\delta}{\delta \overline{\eta}(x)} \overline{\psi}(y) + \overline{\psi}(y) \frac{\delta}{\delta \overline{\eta}(x)} = 0. \quad (58)$$

but commute with the Bosonic integration variable

$$\frac{\delta}{\delta \eta(x)} \varphi(y) = \varphi(y) \frac{\delta}{\delta \eta(x)},$$
$$\frac{\delta}{\delta \overline{\eta}(x)} \varphi(y) = \varphi(y) \frac{\delta}{\delta \overline{\eta}(x)}. \quad (60)$$

From the above-mentioned anticommutativity property of fermionic integration variables $\psi(x)$ and $\overline{\psi}(x)$, Grassmann parameters $\eta(x)$ and $\overline{\eta}(x)$ and functional derivation operators $\frac{\delta}{\delta \eta(x)}$ and $\frac{\delta}{\delta \overline{\eta}(x)}$, it is straightforward to derive the following
formulae which are often used:

$$\frac{\delta}{\delta \eta(x)} \exp \left\{ \int dx \eta(x) \psi(x) \right\} = i \psi(y) \exp \left\{ \int dx \overline{\eta}(x) \psi(x) \right\},$$
$$\frac{\delta}{\delta \overline{\eta}(x)} \exp \left\{ \int dx \overline{\psi}(x) \eta(x) \right\} = -i \overline{\psi}(y) \exp \left\{ \int dx \overline{\psi}(x) \eta(x) \right\},$$
$$\frac{\delta^2}{\delta \eta(z) \delta \eta(y)} \exp \left\{ \int dx \left[ \overline{\psi}(x) \eta(x) + \overline{\eta}(x) \psi(x) \right] \right\} = \overline{\psi}(z) \psi(y) \exp \left\{ \int dx \left[ \overline{\psi}(x) \eta(x) + \overline{\eta}(x) \psi(x) \right] \right\},$$
$$\frac{\delta^2}{\delta \overline{\eta}(z) \delta \overline{\eta}(y)} \exp \left\{ \int dx \left[ \overline{\psi}(x) \eta(x) + \overline{\eta}(x) \psi(x) \right] \right\} = -\overline{\psi}(z) \psi(y) \exp \left\{ \int dx \left[ \overline{\psi}(x) \eta(x) + \overline{\eta}(x) \psi(x) \right] \right\},$$
$$\frac{\delta^2}{\delta \eta(z) \delta \overline{\eta}(y)} \exp \left\{ \int dx \left[ \overline{\psi}(x) \eta(x) + \overline{\eta}(x) \psi(x) \right] \right\} = -\overline{\psi}(z) \overline{\psi}(y) \exp \left\{ \int dx \left[ \overline{\psi}(x) \eta(x) + \overline{\eta}(x) \psi(x) \right] \right\}. \quad (61)$$
Denote \( H \left( -\frac{i}{\hbar} \frac{\partial}{\partial x}, x \right) \) the Hamiltonian of the electron with mass \( m \) moving in a confining potential \( U(x) \)

\[
H \left( -\frac{i}{\hbar} \frac{\partial}{\partial x}, x \right) = \frac{1}{2m} \left( -\frac{i}{\hbar} \frac{\partial}{\partial x} \right)^2 + U(x), \tag{62}
\]

and

\[
u(x - y) = u(x - y) \delta(x_0 - y_0), \tag{63}
\]

where \( u(x - y) \) is the potential energy of the Coulomb repulsion between two electrons

\[
u(x - y) = \frac{e^2}{|x - y|}, \tag{64}
\]

\( e \) is the electron charge. The key mathematical tools of the functional integral method in quantum plasmonics are the functional integrals

\[
Z^\nu[\eta, \bar{\eta}] = \int \left[ D\psi \right] \left[ D\bar{\psi} \right] \exp \left\{ i \int dx \left[ \bar{\psi}(x) \eta(x) + \bar{\eta}(x) \psi(x) \right] \right\} \times \exp \left\{ i \int dx \psi(x) \left[ \frac{i}{\hbar} \frac{\partial}{\partial x} - H \left( -\frac{i}{\hbar} \frac{\partial}{\partial x}, x \right) \right] \bar{\psi}(x) \right\}
\]

\[
\times \exp \left\{ -\frac{i}{2} \int dx \int dy \psi(x) \bar{\psi}(y) u(x - y) \psi(y) \bar{\psi}(y) \right\}, \tag{65}
\]

and

\[
Z^\nu_0 = Z^\nu[0,0] = \int \left[ D\psi \right] \left[ D\bar{\psi} \right] \exp \left\{ i \int dx \left[ \bar{\psi}(x) \eta(x) \right] \right\}
\]

\[
+ \hat{\eta}(x) \psi(x) \right\} \exp \left\{ i \int dx \psi(x) \left[ \frac{i}{\hbar} \frac{\partial}{\partial x} - H \left( -\frac{i}{\hbar} \frac{\partial}{\partial x}, x \right) \right] \bar{\psi}(x) \right\}
\]

\[
- \left( -\frac{i}{\hbar} \frac{\partial}{\partial x}, x \right) \right] \psi(x) \right) \left\}
\]

and

\[
Z^\nu_0 = Z^\nu[0,0] = \int \left[ D\psi \right] \left[ D\bar{\psi} \right] \exp \left\{ i \int dx \left[ \bar{\psi}(x) \eta(x) \right] \right\}
\]

\[
+ \hat{\eta}(x) \psi(x) \right\} \exp \left\{ i \int dx \psi(x) \left[ \frac{i}{\hbar} \frac{\partial}{\partial x} - H \left( -\frac{i}{\hbar} \frac{\partial}{\partial x}, x \right) \right] \bar{\psi}(x) \right\}
\]

\[
- \left( -\frac{i}{\hbar} \frac{\partial}{\partial x}, x \right) \right] \psi(x) \right) \left\}
\]

Expressions in formulae (65) and (67) are called the generating functionals.

The Grassmann dynamical variables \( \psi(x) \) and \( \bar{\psi}(x) \) can be considered as a fermionic field and its hermitian conjugate. In the case of the electron gas with the electron–electron Coulomb repulsion, the action functional of the field \( \psi(x) \) is

\[
I \left[ \psi, \bar{\psi} \right] = \int dx \bar{\psi}(x) \left[ \frac{i}{\hbar} \frac{\partial}{\partial x} - H \left( -\frac{i}{\hbar} \frac{\partial}{\partial x}, x \right) \right] \psi(x)
\]

\[
- \frac{1}{2} \int dx \int dy \bar{\psi}(x) \psi(x) (x - y) \bar{\psi}(y) \psi(y). \tag{69}
\]

If the electron–electron Coulomb repulsion is neglected, the action functional of the system becomes

\[
i_0 \left[ \psi, \bar{\psi} \right] = \int dx \bar{\psi}(x) \left[ \frac{i}{\hbar} \frac{\partial}{\partial x} - H \left( -\frac{i}{\hbar} \frac{\partial}{\partial x}, x \right) \right] \psi(x). \tag{70}
\]

Thus, the functional integrals \( Z^\nu \) and \( Z^\nu_0 \) of the electron gas are expressed in terms of its action functional \( I \left[ \psi, \bar{\psi} \right] \) and \( i_0 \left[ \psi, \bar{\psi} \right] \) as follows

\[
Z^\nu = \int \left[ D\psi \right] \left[ D\bar{\psi} \right] \exp \left\{ iI \left[ \psi, \bar{\psi} \right] \right\},
\]

\[
Z^\nu_0 = \int \left[ D\psi \right] \left[ D\bar{\psi} \right] \exp \left\{ i_{i_0} \left[ \psi, \bar{\psi} \right] \right\}. \tag{71}
\]

In general, for any physical system described by a set of the fields \( q_{\nu}(x) \), \( \nu = 1, 2, \ldots \), between the action functional \( I \left[ q_{\nu} \right] \) of this system and its functional integral \( Z^\nu \) \( \nu \), the relation of the form

\[
Z^\nu \nu = \int \Pi \left[ Dq_{\nu} \right] \exp \left\{ iI \left[ q_{\nu} \right] \right\} \tag{72}
\]

always holds.

### 3.2. Green functions

Now we define the Green functions and derive the relations between them. Since these relations represent the dynamics of the corresponding quantum field, the functional integral method can be considered as a new method of the field quantization.

Considering Grassmann dynamical variables \( \psi(x) \) and \( \bar{\psi}(x) \) as a fermionic field and its hermitian conjugate, we define the average (also called expectation value) of any product \( \psi(x_1) \psi(x_2) \psi(x_3) \psi(x_4) \cdots \psi(x_n) \psi(y_1) \psi(y_2) \psi(y_3) \psi(y_4) \cdots \psi(y_m) \) over all configurations of this field as follows:

\[
\langle \psi(x_1) \cdots \psi(x_n) \psi(y_1) \cdots \psi(y_m) \rangle \]

\[
= \frac{1}{Z^\nu} \int \left[ D\psi \right] \left[ D\bar{\psi} \right] \psi(x_1) \cdots \psi(x_n) \psi(y_1) \cdots \psi(y_m)
\]

\[
\times \exp \left\{ i \int dx \psi(x) \left[ \frac{i}{\hbar} \frac{\partial}{\partial x} - H \left( -\frac{i}{\hbar} \frac{\partial}{\partial x}, x \right) \right] \bar{\psi}(x) \right\}
\]

\[
\times \exp \left\{ -\frac{i}{2} \int dx \int dy \psi(x) \bar{\psi}(y) u(x - y) \psi(y) \bar{\psi}(y) \right\}, \tag{73}
\]

when the electron–electron Coulomb interaction is taken into
account, and
\[
\langle \psi(x_1) \ldots \psi(x_n) \bar{\psi}(y_1) \ldots \bar{\psi}(y_n) \rangle_0
= \frac{1}{Z_0} \int [D\psi][D\bar{\psi}] \psi(x_1) \ldots \psi(x_n) \bar{\psi}(y_1) \ldots \bar{\psi}(y_n) \]
\[
\times \exp \left\{ i \int dx \bar{\psi}(x) \left[ i \frac{\partial}{\partial x_0} - H \left( -i \frac{\partial}{\partial \mathbf{x}}, \mathbf{x} \right) \right] \psi(x) \right\}
\]
(74)
when the electron–electron Coulomb interaction is neglected and electron field is called the free field.

Let us consider in detail the case of free electron field. The average of the product \( \psi(y) \bar{\psi}(z) \) over all field configurations is called two–point (one–particle) Green function
\[
G(y, z) = \langle \bar{\psi}(y) \psi(z) \rangle_0.
\]
(75)
From the expression (67) of the generating functional \( Z_0^\eta \left[ \eta, \bar{\eta} \right] \) it follows that
\[
G(y, z) = - \frac{1}{Z_0^\eta} \frac{\delta^2 Z_0^\eta \left[ \eta, \bar{\eta} \right]}{\delta \eta \left( z \right) \delta \bar{\eta} \left( y \right)} \Big|_{\eta = \bar{\eta} = 0}.
\]
(76)
Similarly, the average of the product \( \bar{\psi}(y_1) \psi(y_2) \bar{\psi}(z_1) \psi(z_2) \) over all field configurations is called four-point (two-particle) Green function
\[
G(y_1, y_2, z_1, z_2) = \langle \bar{\psi}(y_1) \psi(y_2) \bar{\psi}(z_1) \psi(z_2) \rangle_0.
\]
(77)
It is expressed in terms of \( Z_0^\eta \left[ \eta, \bar{\eta} \right] \) as follows
\[
G(y_1, y_2, z_1, z_2) = \frac{1}{Z_0^\eta} \frac{\delta^4 Z_0^\eta \left[ \eta, \bar{\eta} \right]}{\delta \eta \left( z_1 \right) \delta \bar{\eta} \left( y_1 \right) \delta \eta \left( y_2 \right) \delta \bar{\eta} \left( z_2 \right)}.
\]
(78)
In order to establish the explicit form of the generating functional (67) we consider the Schrödinger equation
\[
H \left( -i \frac{\partial}{\partial \mathbf{x}}, \mathbf{x} \right) u_\alpha (\mathbf{x}) = E_\alpha u_\alpha (\mathbf{x})
\]
(79)
and introduce \( S(x, y) = S(\mathbf{x}, \mathbf{y}; x_0 = y_0) \) the solution of the inhomogeneous differential equation
\[
\left[ i \frac{\partial}{\partial x_0} - H \left( -i \frac{\partial}{\partial \mathbf{x}}, \mathbf{x} \right) \right] S(x, y) = \delta(x - y)
\]
\[
= \delta(x - y) \delta(x_0 - y_0).
\]
(80)
\( S(x, y) \) has the general form
\[
S(x, y; x_0 = y_0) = \frac{1}{2\pi} \int \text{d} \omega e^{i \omega (x_0 - y_0)} \sum_\alpha \frac{u_\alpha^* (x) u_\alpha (y)^*}{\omega - E_\alpha + i0}
\]
\[
+ i \sum_\alpha C_\alpha e^{-iE_\alpha (x_0 - y_0)} u_\alpha (x)^* u_\alpha (y)^*
\]
\[
= \frac{1}{2\pi} \int \text{d} \omega e^{i \omega (x_0 - y_0)} \sum_\alpha \frac{u_\alpha (x) u_\alpha (y)}{\omega - E_\alpha + i0}
\]
\[
\times \left[ 1 - \frac{C_\alpha}{\omega - E_\alpha + i0} + \frac{C_\alpha}{\omega - E_\alpha - i0} \right],
\]
(81)
where the constants \( C_\alpha \) are related with the parameters of the electron gas. Their physical meanings will be clarified latter.

Now let us perform the shift of the functional integration variables
\[
\psi(x) \rightarrow \psi(x) + \int \text{d} y S(x, y) \eta(y),
\]
\[
\bar{\psi}(x) \rightarrow \bar{\psi}(x) + \int \text{d} \bar{\eta} \bar{S}(x, y) \bar{\eta}(y),
\]
(82)
in the rhs of formula (68) for the constant \( Z_0^\eta \). The functional integral in this formula must be invariant under the shift (82) of the functional integration variables. After lengthy but standard transformations and calculations we obtain the following formula
\[
Z_0^\eta = \int [D\psi][D\bar{\psi}] \exp \left\{ i \int \text{d} x \left[ \bar{\psi}(x) \eta(x) \right.ight.
\]
\[
\times \left. \left. \left( -i \frac{\partial}{\partial x_0} - H \right) \right] \psi(x) \right\}
\]
\[
\times \exp \left\{ i \int \text{d} x \bar{\psi}(x) \left[ i \frac{\partial}{\partial x_0} - H \right.ight.
\]
\[
\times \left. \left. \left( -i \frac{\partial}{\partial \mathbf{x}}, \mathbf{x} \right) \right] \psi(x) \right\}
\]
\[
\times \exp \left\{ i \int \text{d} x \int \text{d} \bar{\eta} \left[ \bar{S}(x, y) \eta(y) \right. \right.
\]
\[
\left. \left. \left( -i \frac{\partial}{\partial \mathbf{x}}, \mathbf{x} \right) \right] \right\}.
\]
(83)
Comparing the rhs of relation (83) with the definition (67) of \( Z_0^\eta \left[ \eta, \bar{\eta} \right] \), we derive the explicit expression of this generating functional
\[
Z_0^\eta \left[ \eta, \bar{\eta} \right] = Z_0^\eta \exp \left\{ -i \int \text{d} x \int \text{d} \bar{\eta} \left[ S(x, y) \eta(y) \right. \right.
\]
\[
\left. \left. \left( -i \frac{\partial}{\partial \mathbf{x}}, \mathbf{x} \right) \right] \right\}.
\]
(84)
From expression (84) and formula (76) we obtain
\[
G(y, z) = iS(y, z).
\]
(85)
Similarly, from expression (84) and formula (78) we derive a relation between four-point and two-point Green functions
\[
G(y_1, y_2, z_1, z_2) = G(y_2, z_1) G(y_1, z_2) - G(y_1, z_1) G(y_2, z_2),
\]
(86)
which is the well-known Wick theorem for the four-point Green function of the free electron field. By means of the same reasonnings as those above presented, we can verify the validity of the Wick theorem for any \( 2n \) point Green function of the free electron field.

Now we consider again formula (81) for the function \( S(x - y; x_0 = y_0) \). Note that the limit \( \epsilon \rightarrow +0 \) of the average of the product \( \bar{\psi}(x, t + \epsilon) \psi(x, t) \) is the particle density \( n(x) \):
\[
n(x) = \lim_{\epsilon \rightarrow +0} \left\{ \bar{\psi}(x, t + \epsilon) \psi(x, t) \right\}
\]
\[
= -i \lim_{\epsilon \rightarrow +0} S(x, x; -\epsilon).
\]
(87)
From formula (81) for $S(x, x; -\epsilon)$ it follows that

$$n(x) = \sum_a C_a |u_a(x)|^2. \quad (88)$$

Thus, $C_a$ is the electron occupation number $n_a$ at the quantum state with the wave function $u_a(x)$: $C_a = n_a$.

### 3.3. Scalar field of collective oscillations in electron gas

Now we study the system of interacting electrons with the following functional integral

$$Z = \int [D\psi] [D\bar{\psi}] \exp \left\{ i \int dx \int dx' \psi(x) \frac{i}{\hbar} \frac{\partial}{\partial x} \right\} \left[ -H \left( -i \frac{\partial}{\partial x}, x \right) \bar{\psi}(x) \right\} \exp \left\{ -\frac{i}{2} \int dx \int dx' \bar{\psi}(x) \psi(x) u(x - x') \phi(x') \right\}. \quad (89)$$

The Coulomb term in the functional integral (89) is bilinear with respect to the electron density $\bar{\psi}(x) \psi(x)$. We linearize this bilinear interaction Hamiltonian by introducing a scalar field $\phi(x)$ playing the role of the order parameter of collective oscillations in the electron gas and using the following functional integral

$$Z^\phi = \int [D\phi] \exp \left\{ \int dx \int dx' \psi(x) \frac{i}{\hbar} \frac{\partial}{\partial x} \right\} \left[ -H \left( -i \frac{\partial}{\partial x}, x \right) \bar{\psi}(x) \right\] \exp \left\{ -\frac{i}{2} \int dx \int dx' \bar{\psi}(x) \psi(x) u(x - x') \phi(x') \right\}. \quad (90)$$

Performing a shift of the functional integration variable $\phi(x) \to \phi(x') - \bar{\psi}(x) \psi(x)$, we rewrite $Z^\phi$ in a new form

$$Z^\phi = \int [D\phi] \exp \left\{ \int dx \int dx' \psi(x) \frac{i}{\hbar} \frac{\partial}{\partial x} \right\} \left[ -H \left( -i \frac{\partial}{\partial x}, x \right) \bar{\psi}(x) \right\] \exp \left\{ -\frac{i}{2} \int dx \int dx' \bar{\psi}(x) \psi(x) u(x - x') \phi(x') \right\}. \quad (91)$$

From this relation we derive the celebrated Hubbard–Stratonovich transformation

$$\exp \left\{ -\frac{i}{2} \int dx \int dx' \bar{\psi}(x) \psi(x) u(x - x') \phi(x') \psi(x') \right\} = \frac{1}{Z^\phi} \int [D\phi] \exp \left\{ \int dx \int dx' \psi(x) \frac{i}{\hbar} \frac{\partial}{\partial x} \right\} \left[ -H \left( -i \frac{\partial}{\partial x}, x \right) \bar{\psi}(x) \right\] \exp \left\{ -\frac{i}{2} \int dx \int dx' \bar{\psi}(x) \psi(x) u(x - x') \phi(x') \right\}. \quad (92)$$

Substituting the expression in the rhs. of relation (93) instead of its lhs. which is a factor in the rhs. of formula (89), we transform the functional integral $Z$ into the form

$$Z = \frac{1}{Z^\phi} \int [D\phi] \exp \left\{ \int dx \int dx' \psi(x) \frac{i}{\hbar} \frac{\partial}{\partial x} \right\} \left[ -H \left( -i \frac{\partial}{\partial x}, x \right) \bar{\psi}(x) \right\] \exp \left\{ -\frac{i}{2} \int dx \int dx' \bar{\psi}(x) \psi(x) u(x - x') \phi(x') \right\} \times \left\{ \exp \left\{ -\frac{i}{2} \int dx \int dx' \bar{\psi}(x) \psi(x) u(x - x') \phi(x') \right\} \right\}. \quad (94)$$

Expanding the exponential function

$$\exp \left\{ -\frac{i}{2} \int dx \int dx' \bar{\psi}(x) \psi(x) u(x - x') \phi(x') \right\}$$

into a functional power series of the scalar field, we obtain

$$\exp \left\{ -\frac{i}{2} \int dx \int dx' \bar{\psi}(x) \psi(x) u(x - x') \phi(x') \right\} = F[\phi] = \sum_{n=0}^\infty F^{(n)}[\phi], \quad (96)$$

$$\exp \left\{ -\frac{i}{2} \int dx \int dx' \bar{\psi}(x) \psi(x) u(x - x') \phi(x') \right\} = F^{(0)}[\phi] = 1,$$

$$F^{(1)}[\phi] = \frac{i}{2} (iW^{(1)}[\phi])^2 + iW^{(2)}[\phi], \quad (99)$$

$$W^{(1)}[\phi] = -\int dx \int dx' n(x) u(x - x') \phi(x'), \quad (100)$$

$$\int dx \int dx' \int dy \int dy' S(x, y, x) S(y, y') u(x - x') u(y - y') \phi(x') \phi(y'), \quad (102)$$

$$F^{(3)}[\phi] = F^{(2)}[\phi] + iW^{(1)}[\phi], \quad (103)$$
It has the solution
\[ \varphi_0(y) = \int dy A^{-1}(y, x) \nu(x), \] (114)
where \( A^{-1}(y, x) \) is the kernel of the integral operator inverse to that with the kernel \( A(y, x) \):
\[ \int dz A(x, z) A^{-1}(z, y) = \int dz A^{-1}(x, z) A(z, y) = \delta(x - y). \] (115)
The extreme value of the action functional equals
\[ I_0[\varphi_0] = -\frac{1}{2} \int dx \int dy \varphi_0(x) A(x, y) \varphi_0(y). \] (116)
The fluctuations of the scalar field \( \varphi(x) \) around the extreme field \( \varphi_0(x) \) are described by the difference
\[ \zeta(x) = \varphi(x) - \varphi_0(x). \] (117)
In terms of this new field \( \zeta(x) \) the effective action \( I_0[\varphi] \) has the quadratic form
\[ I_0[\varphi + \zeta] = I_0[\varphi_0] + \frac{1}{2} \int dx \int dy \zeta(x) A(x, y) \zeta(y). \] (118)
The dynamical equation for the field \( \zeta(x) \) is
\[ \int dy A(x, y) \zeta(y) = 0. \] (119)
For testing the validity of above-presented functional integral formalism now we apply it to the simple case of a homogeneous electron gas in the three-dimensional space. In this case function \( S(x, y) \) depends only on the difference \( x - y \) of the coordinates and has the Fourier transformation
\[ S(x - y) = S(x - y; x_0 - y_0) = \frac{1}{(2\pi)^3} \int dk \int d\omega e^{i[k(x - y) - \omega(x_0 - y_0)]} \tilde{S}(k, \omega), \] (120)
where \( \tilde{S}(k, \omega) = 1 - n(k) \omega - E(k) + i0^+ + \frac{n(k)}{\omega - E(k) - i0^+}. \) (121)

From the principle of extreme action
\[ \frac{\delta I_0[\varphi]}{\delta \varphi(x)} = 0, \] (112)
We derive the dynamical equation for the scalar field \( \varphi_0(x) \) corresponding to the extreme action
\[ \int dy A(x, y) \varphi_0(y) = \nu(x). \] (113)

It has the solution
\[ \varphi_0(y) = \int dy A^{-1}(y, x) \nu(x), \] (114)
where

\[ \tilde{H}(\mathbf{k}, \omega) = \frac{i}{(2\pi)^3} \int d\mathbf{p} \int d\epsilon \; \tilde{S}(\mathbf{p} + \mathbf{k}, \epsilon + \omega) S(\mathbf{p}, \epsilon) \] (124)

and \( \tilde{u}(\mathbf{k}) \) is the Fourier transform of \( u(x) \)

\[ \tilde{u}(\mathbf{k}) = \frac{4\pi e^2}{k^2}. \] (125)

The dynamical equation (119) becomes

\[ 1 + \tilde{u}(\mathbf{k}) \tilde{H}(\mathbf{k}, \omega) = 0. \] (126)

This equation determines the dependence of \( \omega \) on \( k \)- the plasmon dispersion \( \omega(k) \). In order to establish the dispersion we must calculate \( \tilde{\Pi}(\omega) \).

By changing the integration variable \( \mathbf{p} \to -\mathbf{p} \) we rewrite expression (129) as follows

\[ \tilde{H}(\mathbf{k}, \omega) = \frac{1}{(2\pi)^3} \int d\mathbf{p} \frac{[1 - n(\mathbf{p} + \mathbf{k}/2)]n(\mathbf{p} - \mathbf{k}/2)}{E(\mathbf{p} + \mathbf{k}/2) - E(\mathbf{p} - \mathbf{k}/2) - \omega}. \] (130)

Because both \( E(\mathbf{p}) \) and \( n(\mathbf{p}) \) are the function of \( \mathbf{p}^2 \), we have

\[ n(-\mathbf{p}) = n(\mathbf{p}), \quad n(-\mathbf{p} + \mathbf{k}) = n(\mathbf{p} - \mathbf{k}), \]
\[ E(-\mathbf{p}) = E(\mathbf{p}), \quad E(-\mathbf{p} + \mathbf{k}) = E(\mathbf{p} - \mathbf{k}). \]

Therefore

\[ \tilde{H}(\mathbf{k}, \omega) = \frac{1}{(2\pi)^3} \int d\mathbf{p} \frac{[1 - n(\mathbf{p})]n(\mathbf{p} + \mathbf{k})}{E(\mathbf{p} + \mathbf{k}) - E(\mathbf{p} - \omega)}. \] (131)

By means of the substitution \( \mathbf{p} \to -\mathbf{p} \) we obtain

\[ \tilde{H}(\mathbf{k}, \omega) = \frac{1}{(2\pi)^3} \int d\mathbf{p} \frac{[1 - n(\mathbf{p})]n(\mathbf{p} - \mathbf{k})}{E(\mathbf{p} - \mathbf{k}) - E(\mathbf{p} - \omega)}. \] (132)

Combining (130) and (133), finally we derive the expression of \( \tilde{H}(\mathbf{k}, \omega) \)

\[ \tilde{H}(\mathbf{k}, \omega) = \frac{1}{(2\pi)^3} \int d\mathbf{p} \frac{[1 - n(\mathbf{p} + \mathbf{k}/2)]n(\mathbf{p} - \mathbf{k}/2)}{E(\mathbf{p} - \mathbf{k}/2) - E(\mathbf{p} + \mathbf{k}/2) - \omega}. \] (134)

Consider the case of the electron gas at vanishing absolute temperature \( T=0 \) and denote \( p_F \) the magnitude of the momentum of electrons at the Fermi surface. Because \( n(\mathbf{p}) \) is equal to unity inside and on the Fermi surface, i.e. at \( \mathbf{p} \leq p_F \), and vanishes outside this surface, i.e. at \( \mathbf{p} > p_F \), the domain of integration in the rhs of formula (1134) must be the common region of the volume inside the spherical surface \( \mathbf{p} = p_F^2 \), and the region outside the spherical surface \( \mathbf{p} = p_F^2 \). At small values of \( k \leq 2p_F \) two spheres \( \mathbf{p} - k/2 \leq p_F \) and \( \mathbf{p} + k/2 \leq p_F \) are overlapping (figure 1) and the domain of integration is the region bounded by two above-mentioned spherical surfaces.

Let us choose the direction of vector \( \mathbf{k} \) to be that of the vertical axis \( OZ \) in the rectangular coordinate system. Then for small values of \( k \), the domain of integration is the region bounded from above by the spherical surface \( \mathbf{p} = k/2 \) and bounded from below by the spherical surface

Figure 1. The integration domain \( \Omega \) is confined inside the region with the yellow color [32].

\[ p_F^2 = p_F^2 + k^2 - kp_x \cos \theta = p_F^2 + k^2 + kp_x \cos \theta, \]
\[ p_2 = \frac{k}{2} \cos \theta + \sqrt{p_F^2 - \frac{k^2}{4} \sin^2 \theta}, \]
\[ p_1 = -\frac{k}{2} \cos \theta + \sqrt{p_F^2 - \frac{k^2}{4} \sin^2 \theta}. \]
\[ \frac{p + k/2}{p_F} \] In this case formula (134) becomes

\[ \tilde{H}(k, \omega) = -\frac{k^2}{\omega m^2} 4\pi \int_0^{2\pi} \cos^2 \theta \times \left\{ \int p_{1}(\theta) 1 - (kp/m)^2 \cos^2 \theta p^3 dp \right\} d\theta, \quad (135) \]

where

\[ p_{1}(\theta) = \left[ p_{F}^2 - \frac{k^2}{4} \sin^2 \theta \right]^{1/2} + \frac{k}{2} \cos \theta, \]

\[ p_{2}(\theta) = \left[ p_{F}^2 - \frac{k^2}{4} \sin^2 \theta \right]^{1/2} - \frac{k}{2} \cos \theta. \quad (136) \]

Expanding the rhs of formula (135) into the power series of the small variable \( k^2 \) and limiting at the second order approximation, we obtain

\[ \tilde{H}(k, \omega) = -n_{0}k^2 \left( 1 + \frac{3}{2} \frac{p_{F}^2}{m^2} \frac{k^2}{\omega^2} \right), \quad (137) \]

where

\[ n_{0} = \frac{1}{(2\pi)^3} \frac{4\pi}{3} p_{F}^3 \quad (138) \]

is the electron density. Substituting expression (137) into the rhs of formula (123), using formula (125) and setting

\[ \omega_0^2 = \frac{4\pi e^3 n_{0}}{m}, \quad (139) \]

we obtain

\[ \tilde{A}(k, \omega) = \frac{4\pi^2}{k^2 \omega^2} \left( \omega^2 - \omega_0^2 - \frac{3}{5} \frac{p_{F}^2}{m^2} \frac{\omega_0^2}{\omega^2} k^2 \right). \quad (140) \]

Finally, we consider the dynamical equation (119). In this case kernel \( A(x - y) = A(x - y; x_0 - y_0) \) has the form (122) with the Fourier transform (123) determined by formula (140). Introducing the Fourier transformation of the scalar field \( \tilde{\zeta}(x, t) \)

\[ \tilde{\zeta}(x, t) = \frac{1}{(2\pi)^3} \int d\mathbf{k} \int d\omega \ e^{-i(kx - \omega t)} \tilde{\zeta}(k, \omega), \quad (141) \]

we can rewrite the dynamical equation (119) as follows

\[ \left( \omega^2 - \omega_0^2 - \frac{3}{5} \frac{p_{F}^2}{m^2} \omega_0^2 k^2 \right) \tilde{\zeta}(k, \omega) = 0. \quad (142) \]

Therefore between the energy \( \omega \) and the momentum \( k \) of the free plasmon we have following dispersion equation

\[ \omega^2 = \omega_0^2 + \frac{3}{5} \frac{p_{F}^2}{m^2} \omega_0^2 k^2. \quad (143) \]

In the first-order approximation with respect to the small ratio \( k^2/\omega_0^2 \) this dispersion equation has the following solution

\[ \omega^2 = \omega_0^2 + \frac{3}{5} \frac{p_{F}^2}{m^2} k^2. \quad (144) \]

In this approximation the Fourier transform \( \tilde{A}(k, \omega) \) of the kernel of the dynamical equation for free plasmon has the expression

\[ \tilde{A}(k, \omega) = \frac{4\pi^2}{k^2 \omega^2} \left( \omega^2 - \omega_0^2 - \gamma^2 k^2 \right) \quad (145) \]

with

\[ \gamma^2 = \frac{3}{5} \frac{p_{F}^2}{m^2}. \quad (146) \]

Note that the dispersion formula (144) was previously by many authors in conventional theory of plasma oscillations [37, 38]. The agreement of the above-presented result with that of the conventional theory of plasma oscillation demonstrates the validity of the functional integral method.

### 3.4. Quantization of plasmon field

Let us apply the presented results to the study of the effective action (118) of the homogeneous free electron gas in the three-dimensional space. Consider the quadratic part of the this effective action

\[ I^{(2)}[\zeta] = \frac{1}{2} \int dx \int dy \zeta(x)A(x - y)\zeta(y). \quad (147) \]

Using the Fourier transformation (122) of \( A(x - y) \) and Fourier transformation (141) of the wave function \( \zeta(x) \) we rewrite the functional (147) in the momentum representation

\[ I^{(2)}[\tilde{\zeta}] = \frac{1}{(2\pi)^3} \int d\mathbf{k} \int d\omega \frac{1}{2} \tilde{\zeta}(-\mathbf{k}, -\omega) \times \tilde{A}(\mathbf{k}, \omega) \tilde{\zeta}(\mathbf{k}, \omega). \quad (148) \]

Then using expression (145) of \( \tilde{A}(k, \omega) \) and setting

\[ \tilde{\sigma}(k, \omega) = \sqrt{\frac{4\pi\omega^2}{k^2 \omega^2}} \tilde{\zeta}(k, \omega), \quad (149) \]

we obtain the expression of the functional (148) in terms of the new function \( \tilde{\sigma}(k, \omega) \)

\[ I_{\tilde{\sigma}}[\sigma] = \frac{1}{(2\pi)^3} \int d\mathbf{k} \int d\omega \frac{1}{2} \tilde{\sigma}(-\mathbf{k}, -\omega) \times \left[ \frac{3}{5} \frac{p_{F}^2}{m^2} \omega_0^2 - \gamma^2 k^2 \right] \tilde{\sigma}(\mathbf{k}, \omega). \quad (150) \]

Considering \( \tilde{\sigma}(k, \omega) \) as the Fourier transforms of a new scalar field \( \sigma(x, t) \)

\[ \sigma(x, t) = \frac{1}{(2\pi)^3} \int d\mathbf{k} \int d\omega \ e^{i(kx - \omega t)} \tilde{\sigma}(k, \omega), \quad (151) \]

we obtain the following formula for the effective action of
this new scalar field

\[ I_{\text{eff}}[\sigma] = \int dt L_{\sigma}(t) \]  

(152)

with the effective Lagrangian

\[ L_{\sigma}(t) = \frac{1}{2} \int dx \left\{ \left[ \frac{\partial \sigma(x, t)}{\partial t} \right]^2 - \gamma^2 \right\} \times \left\{ \nabla \sigma(x, t)^2 - a_{\text{0}}^2 \sigma(x, t)^2 \right\}. \]  

(153)

It is straightforward to derive the Lagrange equation of this scalar field

\[ \frac{\partial^2 \sigma(x, t)}{\partial t^2} - \gamma^2 \nabla^2 \sigma(x, t) + a_{\text{0}}^2 \sigma(x, t) = 0. \]  

(154)

It follows that between the frequency \( \omega \) and the wave vector \( k \) of the plane waves of the scalar field \( \sigma(x, t) \) there must exist the relation

\[ \omega^2 = a_{\text{0}}^2 + \gamma^2 k^2. \]  

(155)

Finally we perform the quantization of the scalar field in the framework of the canonical quantization procedure. The canonical momentum \( \hat{\pi}(x, t) \) corresponding to the canonical coordinate \( \hat{\sigma}(x, t) \), by definition, is the operator

\[ \hat{\pi}(x, t) = -\frac{\delta L_{\sigma}}{\delta \dot{\sigma}(x, t)} = \hat{\pi}(x, t) \]  

(156)

with the notation

\[ \hat{\sigma}(x, t) = \frac{\partial \hat{\sigma}(x, t)}{\partial t}. \]  

(157)

The Hamiltonian of the system has the expression

\[ H_{\sigma} = \int dx \hat{\pi}(x, t) \hat{\sigma}(x, t) - L_{\sigma}(t) \]

\[ = \frac{1}{2} \int dx \left\{ \left[ \frac{\partial \sigma(x, t)}{\partial t} \right]^2 + \gamma^2 \left[ \nabla \sigma(x, t) \right]^2 \right\} \]

\[ + a_{\text{0}}^2 \sigma(x, t)^2 + \frac{1}{2} \int dx \left[ \nabla \sigma(x, t) \right]^2. \]  

(158)

According to the canonical quantization rule, between operators \( \hat{\sigma}(x, t) \) and \( \hat{\pi}(x, t) \) there exist the following equal-time commutation relations

\[ \left[ \hat{\sigma}(x, t), \hat{\pi}(x', t) \right] = \left[ \hat{\pi}(x, t), \hat{\sigma}(x', t) \right] = 0, \]

\[ \left[ \hat{\pi}(x, t), \hat{\pi}(x', t) \right] = -i \delta(x - x'). \]  

(159)

The expressions (153) and (158) for the Lagrangian and the Hamiltonian of the new scalar field \( \sigma(x, t) \) look like those of the Klein–Gordon real (hermitian) scalar field in relativistic quantum field theory except for scaling factor \( \gamma \) of the spatial coordinates. For the interpretation of the physical meaning of this scalar field we can apply the reasoning of the relativistic quantum field theory. In order to simplify the presentation we quantize the scalar field \( \sigma(x, t) \) in a cubic box with a very large volume \( V \), using the periodic boundary conditions, and expand the scalar field \( \hat{\sigma}(x, t) \) as follows

\[ \hat{\sigma}(x, t) = \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2\omega}} \left\{ \hat{a}_k e^{i(kx - \omega t)} + \hat{a}^+_k e^{-i(kx - \omega t)} \right\}. \]  

(160)

The corresponding expansion of the canonical momentum is

\[ \hat{\pi}(x, t) = -\frac{i}{\sqrt{V}} \sum_k \sqrt{\omega} \left\{ \hat{a}_k e^{i(kx - \omega t)} - \hat{a}^+_k e^{-i(kx - \omega t)} \right\}. \]  

(161)

Inverting the expansions (160) and (161) to express \( \hat{a}_k \) and \( \hat{a}^+_k \) in terms of \( \hat{\sigma}(x, 0) \) and \( \hat{\pi}(x, 0) \), we obtain

\[ \hat{a}_k = \frac{1}{\sqrt{V}} \int dxe^{-ikx} \left\{ \frac{\omega}{\sqrt{2}} \hat{\sigma}(x, 0) + \frac{i}{\sqrt{2\omega}} \hat{\pi}(x, 0) \right\}. \]  

(162)

\[ \hat{a}^+_k = \frac{1}{\sqrt{V}} \int dxe^{ikx} \left\{ \frac{\omega}{\sqrt{2}} \hat{\sigma}(x, 0) - \frac{i}{\sqrt{2\omega}} \hat{\pi}(x, 0) \right\}. \]  

(163)

From the canonical quantization rules (159) it follows the canonical commutation relations between operators \( \hat{a}_k \) and \( \hat{a}^+_k \). We obtain

\[ [\hat{a}_k, \hat{a}_l] = [\hat{a}^+_k, \hat{a}^+_l] = 0, \]

\[ [\hat{a}_k, \hat{a}^+_l] = \delta_{kl}. \]  

(164)

The Hamiltonian (158) is expressed in terms of these operators as follows

\[ H_{\sigma} = \frac{1}{2} \sum_k \left( \hat{a}^+_k \hat{a}_k + \frac{1}{2} \right) \omega. \]  

(165)

Remember that between frequency \( \omega \) and wave vector \( k \) there exists the relation (155).

Operators \( \hat{a}_k \) and \( \hat{a}^+_k \) are interpreted as the destruction and creation operators of the quanta of the quantized scalar field \( \hat{\sigma}(x, t) \). These quanta are called the plasmons, and formula (155) becomes the relation between energy and momentum of plasmon.

4. Plasmon–photon interaction in canonical quantum mechanics

Consider now the interaction of the electron gas with some external electromagnetic field. We choose to work in the transverse gauge of the electromagnetic field and denote \( A(x, t) \) its vector potential

\[ \nabla A(x, t) = 0. \]  

(166)

It was known [39] that the interaction Lagrangian has the following expression

\[ L_{\text{int}}(t) = \frac{e}{c} \int dx n(x, t) \delta \Phi(x, t) A(x, t), \]  

(167)

where \( c \) is the light velocity in the vacuum. Using relation (1) between the electron density \( n(x, t) \) and the charge density \( \rho(x, t) \) of the electron gas, we rewrite the expression in the
rh of equation (167) as follows

\[ L_{int}(t) = \frac{e\hbar}{c} \int dx \delta \hat{r}(x, t) A(x, t) \]

\[ - \frac{1}{c} \int dx \rho(x, t) \delta \hat{r}(x, t) A(x, t). \]  
(168)

In section 2 we have established the relationship between \( \delta \hat{r}(x, t) \) and the plasmonic field \( \sigma(x, t) \). Therefore in order to study the plasmon–photon interaction it is necessary to express \( \rho(x, t) \) in term of the components of the vector \( \delta \hat{r}(x, t) \). For this purpose we use formula (12), a consequence of the conservation of the total electron number in the electron gas. Using relation (1), we derive the following differential equation for the function \( \rho(x, t) \)

\[ \rho(x, t) = \frac{e\hbar}{c} \sum_i \int d\mathbf{q} \delta \hat{r}_i(x, t) - \rho(x, t) \sum_i \int d\mathbf{q} \delta \hat{r}_i \]

\[ \times (x, t) - \sum_i \delta \hat{r}_i(x, t) \int d\mathbf{q} \rho(x, t). \]  
(169)

Since this differential equation cannot be exactly solved, we look for \( \rho(x, t) \) in the form of a functional power series of the components \( \delta \hat{r}_i(x, t) \) and write the solution in the form

\[ \rho(x, t) = \sum_{n=0}^{\infty} \rho^{(n)}(x, t), \]  
(170)

where \( \rho(x, t) \) is the term of the \( \nu \)-th order with respect to the components \( \delta \hat{r}(x, t) \). Let us limit to the third order. By means of standard calculations it can be shown that

\[ \rho^{(1)}(x, t) = \frac{e\hbar}{c} \sum_i \int d\mathbf{q} \delta \hat{r}_i(x, t), \]  
(171)

\[ \rho^{(2)}(x, t) = \frac{e\hbar}{c} \sum_i \sum_j \int d\mathbf{q} \delta \hat{r}_i(x, t) \int d\mathbf{q} \delta \hat{r}_j(x, t), \]  
(172)

\[ \rho^{(3)}(x, t) = \frac{e\hbar}{c} \sum_i \sum_j \sum_k \int d\mathbf{q} \delta \hat{r}_i(x, t) \int d\mathbf{q} \delta \hat{r}_j(x, t) \int d\mathbf{q} \delta \hat{r}_k(x, t), \]  
(173)

Substituting expression (170)–(173) into the rhs of relation (168), we obtain the explicit expression of the interaction Lagrangian \( L_{int}(t) \) in term of the components \( \delta \hat{r}(x, t) \)

\[ L_{int}(t) = \sum_{n=1}^{\infty} L_{int}^{(n)}(t), \]  
(174)

where up to the fourth order we have

\[ L_{int}^{(1)}(t) = \frac{e\hbar}{c} \int dx \sum_i A_i(x, t) \delta \hat{r}_i(x, t), \]  
(175)

\[ L_{int}^{(2)}(t) = -\frac{e\hbar}{c} \int dx \sum_i A_i(x, t) \delta \hat{r}_i(x, t) \]

\[ \times \sum_j \int d\mathbf{q} V_j \delta \hat{r}_j(x, t), \]  
(176)

\[ L_{int}^{(3)}(t) = -\frac{e\hbar}{c} \int dx \sum_i A_i(x, t) \delta \hat{r}_i(x, t) \]

\[ \times \sum_j \sum_k \int d\mathbf{q} V_j \delta \hat{r}_j(x, t) \int d\mathbf{q} V_k \delta \hat{r}_k(x, t), \]  
(177)

\[ L_{int}^{(4)}(t) = -\frac{e\hbar}{c} \int dx \sum_i A_i(x, t) \delta \hat{r}_i(x, t) \]

\[ \times \sum_j \sum_k \sum_l \int d\mathbf{q} V_j \delta \hat{r}_j(x, t) \int d\mathbf{q} V_k \delta \hat{r}_k(x, t) \int d\mathbf{q} V_l \delta \hat{r}_l(x, t). \]  
(178)

In order to derive the relation between components \( \delta \hat{r}_i(x, t) \) and the plasmonic field \( \sigma(x, t) \) we use formula (15) with the longitudinal \( q_k^{(1)}(t) = q_{k}(t) \) but without the transverse ones

\[ \delta \hat{r}(x, t) = \frac{1}{\sqrt{V}} \sum_k e^{i\mathbf{k} \cdot \mathbf{x}} \hat{q}_{k}(t) \]  
(179)

as well as relation (29) between the generalized coordinate \( \hat{q}_{k}(t) \) and the Fourier component \( \delta \hat{r}(x, t) \) of the plasmonic field \( \sigma(x, t) \). Thus we have

\[ \delta \hat{r}(x, t) = \frac{1}{\sqrt{\rho_0 m}} \sum_k e^{i\mathbf{k} \cdot \mathbf{x}} \hat{q}_{k}(t). \]  
(180)

Expressing the Fourier component \( \delta \hat{r}(x, t) \) of the plasmonic field \( \sigma(x, t) \) in terms of this field

\[ \hat{q}_{k}(t) = \frac{1}{\sqrt{V}} \int dx e^{-i\mathbf{k} \cdot \mathbf{x}} \sigma(x, t), \]  
(181)

from equation (180) we obtain

\[ \delta \hat{r}(x, t) = \int dy F_i(x - y) \sigma(y, t), \]  
(182)

where

\[ F_i(x - y) = \frac{1}{\sqrt{\rho_0 m}} \frac{1}{\sqrt{V}} \sum_k e^{i\mathbf{k} \cdot \mathbf{x}} \sigma(y, t). \]  
(183)

Vector function \( F(x - y) \) with components (183) can be called the characteristic function or characteristic factor of the plasmon. Substituting expression (182) of the components \( \delta \hat{r}(x, t) \) into the rhs of relations (175)–(178), finally we obtain

\[ L_{int}^{(1)}(t) = \frac{e\hbar}{c} \int dx \int dy \sum_i A_i(x, t) \]

\[ \times \int dy_1 F_i(x - y_1) \sigma(y_1, t), \]  
(184)

\[ L_{int}^{(2)}(t) = -\frac{e\hbar}{c} \int dx \int dy_1 \int dy_2 \sum_i A_i(x, t) \]

\[ \times \int dy_1 F_i(x - y_1) \sum_j V_j F_j(x - y_2) \]

\[ \times \int dy_2 \sigma(y_2, t), \]  
(185)

\[ L_{int}^{(3)}(t) = \frac{e\hbar}{c} \int dx \int dy_1 \int dy_2 \int dy_3 \sum_i A_i(x, t) \]

\[ \times F_i(x - y_1) \sum_k \sum_j V_j F_j(x - y_2) \]

\[ \times \sigma(y_3, t), \]  
(186)
These formulae show that the interaction processes are non-local. The physical origin of this nonlocality is the complex structure of plasmon. Each plasmon is not a point particle, but it has the spatial extension around its center.

The plasmon–photon interaction Lagrangian (174) with the terms of the form (175)–(178) was derived when we use the electron–photon interaction Lagrangian (167) which is linear with respect to the vector potential A(x, t). If the nonlinear optical processes are taken into account, the electron–photon interaction Lagrangian must include also the terms nonlinear with respect to A(x, t). In this case the plasmon-photon interaction Lagrangian contains nonlinear terms describing the interaction processes with the participation of many photons.

5. Plasmon–photon interaction in functional integral technique

The most powerful tool for the theoretical study of plasmonic processes and phenomena is the functional integral technique. In order to apply this technique to the study of plasmon–photon interaction let us extend the reasoning presented in section 3 and substitute

\[
-i \frac{\partial}{\partial x} \rightarrow -i \frac{\partial}{\partial x} + \frac{e}{c} A(x, t),
\]

where \( e \) is the absolute value of the electron charge, in the expression (62) of the quantum mechanical Hamiltonian of the electron. Then instead of the functional integral Z of the expression (62) of the quantum mechanical Hamiltonian of the electron we must start from following functional integral of the electron gas interacting with some external transverse electromagnetic field \( A(x) \):

\[
Z = \frac{e^{N_e}}{Z_0^d} \int [D\phi] \left\{ \frac{1}{2} \int dx \int dx' \phi(x)u
\times (x - x')\phi(x') \right\} [D\phi] \exp \left\{ -i \int dx \int dx' \phi(x)u \left( x - x' \right) \right\}
\times \exp \left\{ i \int dx \phi^* (x) \left( \frac{\partial}{\partial x} - H \left( -i \frac{\partial}{\partial x} \right) \right) \phi (x) \right\},
\]

(188)

Then instead of formulae (95), (96), (105) and (106) we have following expression

\[
Z = \frac{Z_0^w}{Z_0^d} \int [D\phi] \exp \left\{ \frac{i}{2} \int dx \int dx' \phi(x)u
\times (x - x')\phi(x') \right\} \left\{ \frac{e}{2m} \int dx \right. \\
\times \int dx \int dx' \phi^* (x) \left( \frac{\partial}{\partial x} - H \left( -i \frac{\partial}{\partial x} \right) \right) \phi (x) \right\},
\]

(189)

where

\[
W^{(m, 0)} [\phi, A] = 0,
\]

(190)

\[
W^{(m, 1)} [\phi, A] = W^{(m)} [\phi], \quad n \geq 1,
\]

(191)

\[
W^{(0, 0)} [\phi, A] = 0,
\]

(192)

\[
W^{(0, 2)} [\phi, A] = -\frac{e^2}{2m} \int dx n(x) A(x)^2,
\]

(193)

\[
W^{(1, 1)} [\phi, A] = 0.
\]

(194)

The scalar field \( \phi(x) \) is expressed in terms of the plasmonic field \( \sigma(x, t) \) by means of formulae (117), (141), (149) and (151).

Thus the effective action

\[
I[\phi, A] = \sum_{n,m=1}^{\infty} W^{(n,m)} [\phi, A]
\]

(197)

of the interaction between the scalar plasmonic field \( \sigma(x, t) \) and the transverse electromagnetic field \( A(x, t) \) has been established. It shows that this interaction is nonlocal. Moreover, it is not instantaneous. From the above-presented formulae it is straightforward to derive the effective action corresponding to any plasmon–photon coupling vertex such as:

- photon absorption or emission by a plasmon \( (n=2, m=1) \),
- decay of a plasmon into two photons, stokes or anti-stokes Raman scattering with the creation or the destruction of a plasmon \( (n=1, m=2) \),
- photon–plasmon scattering, annihilation of two plasmons into two photons \( (n=2, m=2) \) and so on. This will be done in a subsequent work.
6. Conclusion and discussions

In the present work we have reviewed the main results of the rigorous theoretical study of the plasmonic processes in the interacting electron gas. Only general basic postulates (axioms), called also the first principles, of the quantum theory are accepted as the assumptions. Two different methods were applied: the quantum-mechanical canonical quantization method and the functional integral technique. The reasonsing of both methods were presented in this review.

The motivation to the research on the subject of the present review was explained in the introduction. The content of section 2 is the presentation on the quantum-mechanical quantization method and the functional integral technique. The rigorous theoretical study of the plasmonic processes in the interacting electron gas. Only general basic postulates (axioms), called also the first principles, of the quantum theory are accepted as the assumptions. Two different methods were applied: the quantum-mechanical canonical quantization method and the functional integral technique. The reasonsing of both methods were presented in this review. From the results of the study by both methods we can firmly conclude that all plasmonic interaction processes in the matter are nonlocal ones, in contrast to the local interaction Hamiltonian accepted in the phenomenological theories as the assumptions. Thus it is worth revising the interaction Hamiltonians accepted in references [5–28], and a lot of works should be done in the future.

However, the contents of the reviewed theoretical works are still not enough for the comparison of the theoretical predictions with the experimental data, because the electron–phonon interaction certainly plays the important role in the physical processes with the participation of plasmons, but until now there was no theoretical work on related subjects. We do hope that the present review will motivate the theoretical research on the contribution of electron–phonon interaction to the plasmonic processes.

We have seen that there exist two related but different methods for the study of the plasmon–photon interacting system: the canonical quantization method of quantum mechanics and the functional integral method of quantum field theory. In the simple case of the homogeneous and isotropic electron gas in bulk conducting solid the results of both methods are consistent. However, even for this simple case, in the framework of the functional integral method we can derive the formula determining the momentum dependence of the plasmon energy (the plasmon dispersion) by means of simple calculations, while in the canonical quantization method of quantum mechanics it is necessary to carry out complicated calculations of the perturbation theory and the renormalization procedure. Moreover, it is very difficult to apply the canonical quantization method of quantum mechanics to the study of many photonic processes and phenomena, while the functional integral technique is a very efficient theoretical tool for the study of all plasmonic processes and phenomena as well as the plasmonic devices and systems.

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