DYNAMIC TRANSITIONS FOR THE S-K-T COMPETITION SYSTEM

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Abstract. This paper is concerned with dynamical transition for biological competition system modeled by the S-K-T equations. We study the dynamical behaviour of the S-K-T equations with two different boundary conditions. For the system under non-homogeneous Dirichlet boundary condition, we show that the system undergoes a mixed dynamic transition from the homogeneous state to steady state solutions when the bifurcation parameter cross the critical surface. For the system with Neumann boundary condition, we prove that the system undergoes a mixed dynamic transition, a jump transition and a continuous transition when the bifurcation parameter cross the critical number. Finally, two examples are provided to validate the effectiveness of the theoretical results.

1. Introduction. This paper considers the following biological competition model with self-diffusion and cross-diffusion,

\[
\begin{align*}
    u_t &= \Delta[(d_1 + \rho_{11}u + \rho_{12}v)u] + u(a_1 - b_1u - c_1v), \quad x \in \Omega, \ t > 0, \\
    v_t &= \Delta[(d_2 + \rho_{21}u + \rho_{22}v)v] + v(a_2 - b_2u - c_2v), \quad x \in \Omega, \ t > 0,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^n \) is a bounded smooth region which represents the species’ habitat, \( u \) and \( v \) are population densities of two competing species. \( a_i, b_i, c_i, d_i \ (i = 1, 2) \) and \( \rho_{ij} \ (i, j = 1, 2) \) are all positive constants. \( d_1 \) and \( d_2 \) represent random diffusion rates, \( \rho_{11} \) and \( \rho_{22} \) are self-diffusion rates which represent the intra-specific population pressures, \( \rho_{12} \) and \( \rho_{21} \) are the so-called cross-diffusion rates which represent inter-specific population pressures.

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The model (1) was first proposed by Shigesada, Kawasaki and Teramoto (S-K-T) [30] in 1979 for investigating the spatial segregation of two competing species under inter- and intra-species population pressures. Afterwards, it is receiving an increasing attention from many researchers in the past decades. Based on the abstract theory established in [1], it is known that the model (1) is related with analytic semigroup and can still be considered as a parabolic system. Moreover, the existence and non-existence of non-constant steady states of the model (1) with Neumann boundary condition were widely investigated in [15, 16, 17, 34]. Especially, the existence and instability of spiky steady states of the model (1) were obtained in [33, 37], and the existence and stability of non-constant positive solutions to the model (1) with Neumann boundary condition was derived in [34].

More recently, Du and Temam [5] proved the global existence of weak solutions of S-K-T systems in any space dimension \( n \geq 1 \) with a rather general condition on the coefficients, and also obtained a weak global attractor of the system (1) with the corresponding initial-boundary condition.

If \( \rho_{11} = \rho_{22} = 0 \), then the model (1) is reduced to the following system without self-diffusion

\[
\begin{align*}
  u_t &= \Delta[(d_1 + \rho_{12}v)u] + u(a_1 - b_1u - c_1v), \quad x \in \Omega, \ t > 0, \\
  v_t &= \Delta[(d_2 + \rho_{21}u)v] + v(a_2 - b_2u - c_2v), \quad x \in \Omega, \ t > 0.
\end{align*}
\]

(2)

For the existence of global solution and attractor of the model (2), we refer to [2, 17, 39, 40], and the existence and stability of steady states or traveling waves of the more general model, see other studies [28, 36, 37, 38] and the references therein. It is worth pointing out that Zhang and Liu [41] have derived two different types of dynamical transition for the S-K-T model (2) with Dirichlet boundary condition by using dynamical transition theory. Recently, Tan [32] investigated a free boundary problem describing S-K-T competition model with two competing species and obtained the global existence and uniqueness of solutions for the corresponding diffraction problem by approximation method, Galerkin method and Schauder fixed point theorem.

If \( \rho_{ij} = 0 \ (i, j = 1, 2) \), the system (1) can be reduced to the classical Lotka-Volterra competition model

\[
\begin{align*}
  u_t &= d_1\Delta u + (a_1 - b_1u - c_1v), \quad x \in \Omega, \ t > 0, \\
  v_t &= d_2\Delta v + v(a_2 - b_2u - c_2v), \quad x \in \Omega, \ t > 0.
\end{align*}
\]

(3)

which has been extensively and deeply investigated in the past several decades. Kan-on [8] showed that the system (3) with Neumann boundary condition has no non-constant positive steady state and that every nonnegative solution of the corresponding system with diffusion tends to some constant steady state except for the ‘strong competition’ case \( \frac{b_1}{d_2} < \frac{a_1}{d_1} < \frac{c_2}{c_1} \) as \( t \to \infty \). While the region \( \Omega \) is convex, Kishimoto and Weinberger [9] proved that the system (3) with Neumann boundary condition has no stable non-constant positive steady state for the strong competition case. Other interesting results related to the Lotka-Volterra competition model, one can refer to [3, 8, 9, 10, 17, 26, 29, 31, 35, 42] and references therein.

This paper studies the system (1) supplemented with the following two initial-boundary value conditions with non-homogeneous Dirichlet boundary condition and
Neumann boundary condition
\[
\begin{cases}
  u(x, 0) = u^0, \quad v(x, 0) = v^0, \\
  u|_{\partial \Omega} = u_0, \quad v|_{\partial \Omega} = v_0,
\end{cases}
\]
and
\[
\begin{cases}
  u(x, 0) = u^0, \quad v(x, 0) = v^0, \\
  \frac{\partial u}{\partial n}|_{\partial \Omega} = \frac{\partial v}{\partial n}|_{\partial \Omega} = 0,
\end{cases}
\]
where \( \partial / \partial n \) in (5) denotes the outward normal derivative on \( \partial \Omega \), \( u_0 = \frac{\alpha_1 \gamma_2 - \alpha_2 \gamma_1}{\delta_1 \gamma_2 - \delta_2 \gamma_1} \), \( v_0 = \frac{\alpha_2 \delta_1 - \alpha_1 \delta_2}{\delta_1 \gamma_2 - \delta_2 \gamma_1} \), and \( \alpha_i, \gamma_i, \delta_i (i = 1, 2) \) are the positive constant defined by (1). The Neumann boundary condition in (5) indicates that the species \( u \) and \( v \) have no material exchange with the outside world. And the Dirichlet boundary condition in (4) may be interpreted as the condition that the species \( u \) and \( v \) may stay in near the boundary with a state \( u = u_0, \ v = v_0 \). Note that Leung and Clark [12] investigated the bifurcation of equilibrium solutions for the competing-species reaction-diffusion equations with the non-homogeneous Dirichlet boundary condition by using the technique of upper and lower solutions. Other results related to the bifurcations and large-time asymptotic behavior of the competing-species reaction-diffusion equations with Dirichlet boundary data can be founded in [6, 11, 12, 13] and the reference therein.

The main objective of this article is to study dynamical transition for the S-K-T biological competing system (1) with initial-boundary condition (4) and (5). The main technique is dynamical transition theory, which was established by Ma and Wang [19, 26]. The theory studies dynamical transitions of dissipative systems in Nature. The key philosophy for dynamical transition theory is to search for all transition (bifurcation) states. The stability and basin of attraction for transition states provide naturally the mechanism of pattern formation associated with biological competing systems. It is worth mentioning that steady state bifurcation theory and dynamical transition theory have been used to solve many interesting mathematical and physical problems ([4, 7, 14, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 41]).

The paper is organized as follows. Section 2 introduces some preliminaries for dynamical transition (bifurcation) theory. Section 3.1 gives non-dimensional form for the system (1), and gets basic states of the corresponding stationary equations. Moreover, an abstract form for the system (1)-(4) is also gave and the principle of exchange of stabilities theorem are investigated in Subsection 3.2. Subsection 3.3 devotes to proving the main dynamical bifurcation theorems for the S-K-T model. In addition, section 4 mainly consider the dynamical transition of the system (1)-(5) similar to the Section 3. Finally, two simple examples are given to illustrate our main results.

2. Preliminaries. This section introduces dynamical transition theory for nonlinear dissipative systems developed by Ma and Wang [19, 26], which provides the basic theory for the following research of this paper.

Let \( H \) and \( H_1 \) be two Hilbert spaces, \( H_1 \subset H \) be a dense and compact inclusion. In science, nonlinear dissipative systems are generally governed by differential equations, which can be expressed in the following abstract form
\[
\begin{cases}
  \frac{d\omega}{dt} = L_\lambda \omega + G(\omega, \lambda), \\
  \omega(0) = \varphi,
\end{cases}
\]
where \( \omega : [0, \infty) \to H \) is an unknown function, and \( \lambda \) is the system parameter.

Assume that \( L_\lambda : H_1 \to H \) is a parameterized linear completely continuous field depending continuously on \( \lambda \) satisfying
\[
\begin{align*}
L_\lambda &= -A + B_\lambda \quad \text{is a sectorial operator,} \\
A &: H_1 \to H \quad \text{is a linear homeomorphism,} \\
B_\lambda &: H_1 \to H \quad \text{is a compact operator.}
\end{align*}
\]

Also, assume that \( G(\cdot, \lambda) : H_\sigma \to H \) is a \( C^r (r \geq 1) \) bounded mapping for some \( 0 \leq \sigma < 1 \), where \( H_\sigma \) is the fractional order space, and
\[
G(\omega, \lambda) = o(\|\omega\|_{H_\sigma}), \quad \lambda \in \mathbb{R}^1.
\]

Hereafter, we always assume that conditions (7) and (8) hold, which imply that the system (6) has a dissipative structure.

**Definition 2.1.** The system (6) is said to have a transition from \((\omega, \lambda) = (0, \lambda_0)\) at \( \lambda = \lambda_0 \) if the following two conditions are satisfied:

1. if \( \lambda < \lambda_0 \), \( \omega = 0 \) is locally asymptotically stable for (6);
2. if \( \lambda > \lambda_0 \), there exists a neighborhood \( U \subset H \) of \( \omega = 0 \) independent of \( \lambda \), such that for any \( \varphi \in U \setminus \Gamma_{\lambda} \), the solution \( \omega_{\lambda}(t, \varphi) \) of (6) satisfies that
\[
\begin{aligned}
&\limsup_{t \to \infty} \|\omega_{\lambda}(t, \varphi)\|_H \geq \delta(\lambda) > 0, \\
&\lim_{\lambda \to \lambda_0} \delta(\lambda) = \delta \geq 0,
\end{aligned}
\]
where \( \Gamma_{\lambda} \) is the stable manifold of \( \omega = 0 \), with \( \text{codim} \Gamma_{\lambda} \geq 1 \) in \( H \) for \( \lambda > \lambda_0 \).

Let \( \{\beta_j(\lambda) \in \mathbb{C} \mid j \in \mathbb{N}^*\} \) be the eigenvalues (counting multiplicity) of \( L_\lambda \), and assume that
\[
\begin{align*}
\text{Re}\beta_j(\lambda) \begin{cases}
< 0, & \text{if } \lambda < \lambda_0, \\
= 0, & \text{if } \lambda = \lambda_0, \quad \text{for any } 1 \leq j \leq m, \\
> 0, & \text{if } \lambda > \lambda_0,
\end{cases} \\
\text{Re}\beta_j(\lambda_0) < 0, & \text{for any } j \geq m + 1.
\end{align*}
\]

Then, under the assumption (9), we introduce the following lemma [26], which provides sufficient conditions and a basic classification for transitions of nonlinear dissipative systems.

**Lemma 2.2.** [26] Let the condition (9) hold. Then the system (6) must have a transition from \((\omega, \lambda) = (0, \lambda_0)\), and there is a neighborhood \( U \subset H \) of \( \omega = 0 \) such that the transition is one of the following three types:

1. **Continuous transition:** there exists an open and dense set \( \tilde{U}_{\lambda} \subset U \) such that for any \( \varphi \in \tilde{U}_{\lambda} \), the solution \( \omega_{\lambda}(t, \varphi) \) of (6) satisfies
\[
\lim_{\lambda \to \lambda_0} \limsup_{t \to \infty} \|\omega_{\lambda}(t, \varphi)\|_H = 0.
\]

2. **Jump transition:** for any \( \lambda_0 < \lambda < \lambda_0 + \varepsilon \) with some \( \varepsilon > 0 \), there is an open and dense set \( U_{\lambda} \subset U \) such that for any \( \varphi \in U_{\lambda} \),
\[
\limsup_{t \to \infty} \|\omega_{\lambda}(t, \varphi)\|_H \geq \delta > 0,
\]
where \( \delta > 0 \) is independent of \( \lambda \).
(3) Mixed transition: for any $\lambda_0 < \lambda < \lambda_0 + \varepsilon$ with some $\varepsilon > 0$, $U$ can be decomposed into two open (not necessarily connected) sets $U_1^\lambda$ and $U_2^\lambda$:

$$U = U_1^\lambda \cup U_2^\lambda, \quad U_1^\lambda \cap U_2^\lambda = \emptyset,$$

such that

$$\lim_{\lambda \to \lambda_0} \limsup_{t \to \infty} \|\omega(t, \varphi)\|_H = 0, \text{ for any } \varphi \in U_1^\lambda,$$

$$\limsup_{t \to \infty} \|\omega(t, \varphi)\|_H \geq \delta > 0, \text{ for any } \varphi \in U_2^\lambda,$$

where $U_1^\lambda$ and $U_2^\lambda$ are called metastable domain.

3. The transition under non-homogeneous Dirichlet boundary condition.

3.1. Mathematical setting. For simplicity, the domain $\Omega$ is taken as a three dimensional cuboid:

$$\Omega = (0, L_1) \times (0, L_2) \times (0, L_3), \text{ for } L_1 \neq L_2 \neq L_3.$$

3.1.1. Non-dimensional form. In order to make the equations (1) non-dimensional, let

$$u = \frac{d_2}{\rho_{21}} u', \quad v = \frac{d_2}{\rho_{22}} v', \quad x = lx', \quad t = \frac{l^2}{d_2} t'.$$

Meanwhile, define the following non-dimensional variables

$$d = \frac{d_1}{d_2}, \quad \lambda_1 = \frac{\rho_{11}}{\rho_{21}}, \quad \lambda_2 = \frac{\rho_{12}}{\rho_{22}},$$

$$\alpha_1 = \frac{a_1 l^2}{d_2}, \quad \delta_1 = \frac{b_1 l^2}{\rho_{21}}, \quad \gamma_1 = \frac{c_1 l^2}{\rho_{22}},$$

$$\alpha_2 = \frac{a_2 l^2}{d_2}, \quad \delta_2 = \frac{b_2 l^2}{\rho_{21}}, \quad \gamma_2 = \frac{c_2 l^2}{\rho_{22}},$$

where $l > 0$ is the length.

Omitting the primes, we derive the following non-dimensional form of equations (1),

$$\begin{cases}
u_t = d \Delta u + \lambda_1 \Delta u^2 + \lambda_2 \Delta uv + u(\alpha_1 - \delta_1 u - \gamma_1 v), & x \in \Omega, \ t > 0, \\
v_t = \Delta v + \Delta uv + \Delta v^2 + v(\alpha_2 - \delta_2 u - \gamma_2 v), & x \in \Omega, \ t > 0,
\end{cases} \quad (10)$$

here $u \geq 0, \ v \geq 0$ are unknown functions, and the following constants

$$d, \ \lambda_i, \ \alpha_i, \ \delta_i, \ \gamma_i \ (i = 1, 2),$$

are positive parameters.

The non-dimensional of the region $\Omega$ is written as

$$\Omega = (0, l_1) \times (0, l_2) \times (0, l_3), \text{ with } l_1 \neq l_2 \neq l_3.$$
3.1.2. Nonnegative basic states. Now, we study steady-state solutions for the model (10). It is easy to check that the equations (10) admit four physically realistic nonnegative trivial steady-state solutions:

\[
\omega_0 = (0,0)^T, \quad \omega_1 = \left( \frac{\alpha_1}{\delta_1}, 0 \right)^T, \quad \omega_2 = \left( 0, \frac{\alpha_2}{\gamma_2} \right)^T, \quad \omega_3 = (u_0, v_0)^T, \tag{11}
\]

where

\[
u_0 = \frac{\alpha_1 \gamma_2 - \alpha_2 \gamma_1}{\delta_1 \gamma_2 - \delta_2 \gamma_1}, \quad v_0 = \frac{\alpha_2 \delta_1 - \alpha_1 \delta_2}{\delta_1 \gamma_2 - \delta_2 \gamma_1}.
\]

Note that for the 0-Dirichlet boundary condition of the system (3.1), only the \(\omega_0 = (0,0)^T\) is the constant steady-state solutions. However, only positive solutions \((u_0 > 0, v_0 > 0)\) are of interest in the competition of biological population system (10) with (4). Hence, we need to make the following natural assumption to ensure a physical solution \((u_0 > 0, v_0 > 0)\),

\[ (H1) \frac{\gamma_1}{\gamma_2} < \frac{\alpha_1}{\alpha_2} < \frac{\delta_1}{\delta_2} \text{ or } \frac{\delta_1}{\gamma_2} < \frac{\alpha_1}{\alpha_2} < \frac{\gamma_1}{\gamma_2}. \]

Physically, only positive \(\omega_3\) is interest. The following mainly concentrates on bifurcation and transition problem of (10) at the steady-state solution \(\omega_3\) in (11).

For this purpose, take the translation

\[
u = \tilde{u} + u_0, \quad \nu = \tilde{v} + v_0. \tag{12}
\]

Omitting the widetilde, the problem (10) with (4) can be rewritten as

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \mu_1 \Delta u + \lambda_2 u_0 \Delta v - \delta_1 u_0 u - \gamma_1 u_0 v + \mu_2 \Delta u^2 + \lambda_2 \Delta u v - \delta_1 u^2 - \gamma_1 uv, \\
\frac{\partial v}{\partial t} &= v_0 \Delta u + \mu_2 \Delta v - \delta_2 v_0 u - \gamma_2 v_0 v + \Delta uv + \Delta v^2 - \delta_2 uv - \gamma_2 v^2,
\end{align*} \tag{13}
\]

with initial-boundary value conditions

\[
\begin{align*}
u(x,0) &= u^0 - u_0, \quad v(x,0) = v^0 - v_0, \\
u|_{\partial \Omega} &= v|_{\partial \Omega} = 0,
\end{align*} \tag{14}
\]

where

\[
\mu_1 = d + 2u_0 \lambda_1 + v_0 \lambda_2, \quad \mu_2 = 1 + u_0 + 2v_0. \tag{15}
\]

Then it suffices to study the bifurcation solution of (13) at the steady-state solution \(\omega = (0,0)^T\). Therefore, it only needs to focus on the pattern transition of (13)–(14) in the later discussion.

3.1.3. Abstract operator form. Now, define the following function spaces

\[
H = L^2(\Omega, \mathbb{R}^2), \quad H_1 = H^2(\Omega, \mathbb{R}^2) \cap H_0^1(\Omega, \mathbb{R}^2).
\]

Let

\[
\omega(t) = \begin{pmatrix} u(\cdot, t) \\ v(\cdot, t) \end{pmatrix}, \tag{16}
\]

and define operator \(-A_\lambda : H_1 \to H\) by

\[
-A_\lambda \omega = \begin{pmatrix} \mu_1 \Delta & \lambda_2 u_0 \Delta \\ v_0 \Delta & \mu_2 \Delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \tag{17}
\]
here the three parameters \( d, \lambda_1 \) and \( \lambda_2 \) should be represented by a single symbol \( \lambda \), i.e., \( \lambda = (d, \lambda_1, \lambda_2) \). Later, we regard the parameter \( \lambda \) as the control parameter.

Let \( B \) be the linear operator represented by

\[
B \omega = \begin{pmatrix} -\delta_1 u_0 & -\gamma_1 u_0 \\ -\delta_2 v_0 & -\gamma_2 v_0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.
\]

(18)

Clearly, the linear parts of equations (13) are equivalent to the following form

\[
L_\lambda \omega = -A_\lambda \omega + B \omega,
\]

(19)

and \( L_\lambda : H_1 \rightarrow H \) is a parameterized linear completely continuous field depending continuously on \( \lambda \).

Define \( G : H_1 \rightarrow H \) by

\[
G(\omega) = \begin{pmatrix} \lambda_1 \Delta u^2 + \lambda_2 \Delta uv - \delta_1 u_0 \Delta v - \gamma_1 u_0 \Delta v \\ \Delta uv + \Delta v^2 - \delta_2 uv - \gamma_2 v_0 \Delta v \end{pmatrix}.
\]

(20)

Then, it an be verified that (20) represents the nonlinear terms of equations (13). Therefore, the equations (13) with (14) can be rewritten as the following abstract equations

\[
\begin{cases}
\frac{d\omega}{dt} = L_\lambda \omega + G(\omega), \\
\omega(0) = \varphi,
\end{cases}
\]

(21)

where \( \varphi = (\varphi_1, \varphi_2)^T = (u^0 - u_0, v^0 - v_0)^T \).

3.2. Eigenvalues and principle of exchange of stabilities (PES). In this subsection, we shall calculate the eigenvalues of linearized eigenvalue equations of (13) and analysis the sign of the eigenvalues.

3.2.1. Eigenvalues and eigenvectors of \( L_\lambda \) and \( L_\lambda^* \). The linearized eigenvalue equations of (13) are given by

\[
\begin{align*}
\mu_1 \Delta u + \lambda_2 v_0 \Delta v - \delta_1 u_0 u - \gamma_1 u_0 v &= \beta u, \\
v_0 \Delta u + \mu_2 \Delta v - \delta_2 v_0 u - \gamma_2 v_0 v &= \beta v,
\end{align*}
\]

(22)

Let \( \rho_k \) and \( \psi_k \) satisfy the following eigenvalue equations

\[
\begin{cases}
-\Delta \psi_k = \rho_k \psi_k, \\
\psi_k|_{\partial \Omega} = 0, \\
\int_\Omega \psi_k^2 dx = 1.
\end{cases}
\]

(23)

Then,

\[
\rho_k = \pi^2 \left( \frac{k_1^2}{l_1^2} + \frac{k_2^2}{l_2^2} + \frac{k_3^2}{l_3^2} \right),
\]

\[
\psi_k = C_k \sin \frac{k_1 \pi x_1}{l_1} \sin \frac{k_2 \pi x_2}{l_2} \sin \frac{k_3 \pi x_3}{l_3},
\]

Here, \( k_1, k_2 \) and \( k_3 \) range over all integers such that \( k = |k_1| + |k_2| + |k_3| = 1, 2, \ldots \).

\( C_k > 0 \) is a constant to ensure \( \int_\Omega \psi_k^2 dx = 1 \). Clearly, \( 0 < \rho_1 < \rho_2 < \cdots < \rho_k < \cdots \).

Denote

\[
M_k(\lambda) = \begin{pmatrix} -\mu_1 \rho_k - \delta_1 u_0 & -\lambda_2 u_0 \rho_k - \gamma_1 u_0 & \\
-\nu_0 \rho_k - \delta_2 v_0 & -\mu_2 \rho_k - \gamma_2 v_0 \\
\end{pmatrix}.
\]

(24)
It is clear that $M_k(\lambda)$ is the coefficient matrix of (22). Then all eigenvalues $\beta_k = \beta_{ki} (i = 1, 2)$ of (24) satisfy

$$M_k(\lambda)\eta_{ki} = \beta_{ki}\eta_{ki}, \quad k = 1, 2, 3 \cdots,$$

(25)

where $\eta_{ki} \in \mathbb{R}^2$ is the eigenvector of $M_k(\lambda)$ corresponding to $\beta_{ki}$. Hence, the eigenvector $e_{ki}$ of (22) corresponding to $\beta_{ki}$ can be

$$e_{ki} = \eta_{ki}\psi_k, \quad k = 1, 2, 3 \cdots$$

(26)

By simple calculation, it can be verified that the eigenvalues $\beta_k = \beta_{ki}$ satisfy the following equation

$$\beta_k^2 + B_k\beta_k + C_k = 0,$$

(27)

where

$$B_k = \mu_1\rho_k + \mu_2\rho_k + \delta_1u_0 + \gamma_2v_0 > 0,$$

$$C_k = (\mu_1\rho_k + \delta_1u_0)(\mu_2\rho_k + \gamma_2v_0) - u_0v_0(\lambda_2\rho_k + \gamma_1)(\rho_k + \delta_2).$$

(28)

From (27), it follows that the expression of $\beta_{ki}$ can easily be written as

$$\beta_{k1} = \frac{-B_k + \sqrt{B_k^2 - 4C_k}}{2},$$

$$\beta_{k2} = \frac{-B_k - \sqrt{B_k^2 - 4C_k}}{2}.$$  

(29)

The direct calculation shows that the discriminant $\Delta_k = B_k^2 - 4C_k$ of equation (27) is always positive. Therefore, all eigenvalues $\beta_{ki} (i = 1, 2)$ are real.

The eigenvector corresponding to the eigenvalue $\beta_{ki}$ of $L_\lambda$ is taken as

$$e_{ki}(\lambda) = \left(\begin{array}{c}
\frac{\psi_k}{\lambda_2u_0\rho_k + \gamma_1u_0} \\
\frac{-\beta_{ki} + \mu_1\rho_k + \delta_1u_0 + \gamma_1v_0}{\lambda_2u_0\rho_k + \gamma_1u_0}
\end{array}\right),$$

(30)

which implies that $e_{k1}$ and $e_{k2}$ are linearly independent for $\beta_{k1} \neq \beta_{k2}$. Then the set of eigenvectors $\{e_{ki}\}_k$ forms a basis for $H$.

Let $L_\lambda^*$ and $M_k^*(\lambda)$ be the adjoint of $L_\lambda$ and $M_k(\lambda)$ respectively. It is known that the eigenvalues $\beta_{ki}^*$ of $M_k^*(\lambda)$ are complex conjugates of the eigenvalues $\beta_{ki}$ of $M_k(\lambda)$. Note that $\beta_{ki}$ are real, then $\beta_{ki}^* = \beta_{ki}$ ($i = 1, 2$). The eigenvector corresponding to the eigenvalue $\beta_{ki}$ of $L_\lambda^*$ is

$$e_{ki}^*(\lambda) = \left(\begin{array}{c}
-\frac{\psi_k}{v_0\rho_k + \delta_2v_0} \\
\frac{-\beta_{ki} + \mu_1\rho_k + \delta_1u_0 + \gamma_1v_0}{v_0\rho_k + \delta_2v_0}
\end{array}\right).$$

(31)

To facilitate calculation, let

$$h_{ki} = \frac{\beta_{ki} + \mu_1\rho_k + \delta_1u_0}{\lambda_2u_0\rho_k + \gamma_1u_0},$$

$$\overline{h_{ki}} = \frac{\beta_{ki} + \mu_1\rho_k + \delta_1u_0}{v_0\rho_k + \delta_2v_0}.$$  

(32)

Then the eigenvector $e_{ki}$ and its conjugate vector $e_{ki}^*$ can be expressed as

$$e_{ki}(\lambda) = \left(\begin{array}{c}
\psi_k \\
-h_{ki}\psi_k
\end{array}\right),$$

$$e_{ki}^*(\lambda) = \left(\begin{array}{c}
\psi_k \\
-\overline{h_{ki}}\psi_k
\end{array}\right).$$
Thus, the eigenvectors satisfy the following relations

\[ \langle e_{k_1}, e_{i_j}^* \rangle = \begin{cases} 0, & k \neq l \text{ or } i \neq j, \\ P_{k_1}, & k = l, i = j = 1, \\ P_{k_2}, & k = l, i = j = 2, \end{cases} \]

where

\[ P_{k_1} = 1 + h_{k_1} \bar{h}_{k_1} = \frac{\sqrt{B_k^2 - 4C_k}}{\beta_{k_1} + \mu_2 \rho_k + \gamma_2 v_0}, \]
\[ P_{k_2} = 1 + h_{k_2} \bar{h}_{k_2} = -\frac{\sqrt{B_k^2 - 4C_k}}{\beta_{k_2} + \mu_2 \rho_k + \gamma_2 v_0}. \] (33)

3.2.2. Principle of exchange of stabilities (PES). The linear stability and instability are precisely determined by the critical-crossing of the eigenvalues of (22), which is often called PES. For this purpose, we need to study the sign of the eigenvalues \( \beta_{k_1} \).

Form (29), it follows that \( \beta_{k_2} < 0 \) for any \( k \in \mathbb{N}^* \) and the sign of \( \beta_{k_1} \) depends on the sign of \( C_k \). Hereafter, we need to search for the critical number through \( C_k \).

Note that \( C_k = 0 \) can be rewritten as

\[ (\mu_2 \rho_k^2 + \gamma_2 v_0 \rho_k) d + 2u_0(\mu_2 \rho_k^2 + \gamma_2 v_0 \rho_k) \lambda_1 + [v_0(1 + 2v_0) \rho_k^2 + v_0(\gamma_2 v_0 - \delta_2 u_0) \rho_k] \lambda_2 = u_0(\gamma_1 v_0 - \mu_2 \delta_1) \rho_k + u_0 v_0(\gamma_1 \delta_2 - \gamma_2 \delta_1). \] (34)

For convenience, let

\[ E_k = \mu_2 \rho_k^2 + \gamma_2 v_0 \rho_k, \]
\[ F_k = 2u_0(\mu_2 \rho_k^2 + \gamma_2 v_0 \rho_k), \]
\[ J_k = v_0(1 + 2v_0) \rho_k^2 + v_0(\gamma_2 v_0 - \delta_2 u_0) \rho_k, \]
\[ H_k = u_0(\gamma_1 v_0 - \mu_2 \delta_1) \rho_k + u_0 v_0(\gamma_1 \delta_2 - \gamma_2 \delta_1). \] (35)

Clearly, the equality (34) can be expressed as

\[ E_k d + F_k \lambda_1 + J_k \lambda_2 = H_k. \]

As we mentioned in the assumption (H1), it is naturally assuming that the following case holds in this paper

\[ \left\{ \frac{\delta_1}{\delta_2} < \frac{\delta_1 \gamma_2 + \delta_2 \gamma_1}{2 \delta_2 \gamma_2} < \frac{\alpha_1}{\alpha_2} < \frac{\gamma_1}{\gamma_2}, \right. \] and
\[ \gamma_1 \text{ is large enough such that } \gamma_1 v_0 - \mu_2 \delta_1 > 0. \] (36)

This, together with (35), implies that \( \gamma_2 v_0 - \delta_2 u_0 > 0 \) in \( J_k, \gamma_1 v_0 - \mu_2 \delta_1 > 0 \) and \( \gamma_1 \delta_2 - \gamma_2 \delta_1 > 0 \) in \( H_k \).

It can be verified that the equation \( C_k = 0 \) defines a triangular cross-section \( \Lambda_k \) in the first quadrant of the \((d, \lambda_1, \lambda_2)\) coordinate system, i.e.,

\[ \Lambda_k = \{ \lambda = (d, \lambda_1, \lambda_2) \mid C_k = 0, \ \lambda_1 > 0, \ \lambda_2 > 0 \} , \ k \in \mathbb{N}^*, \] (37)

and each cross-section \( \Lambda_k \) separates the region that \( C_k > 0 \) (above the cross-section) from the region that \( C_k < 0 \) (below the cross-section). Define

\[ \Lambda_k^+ = \{ \lambda = (d, \lambda_1, \lambda_2) \mid C_k > 0, \ \lambda_1 > 0, \ \lambda_2 > 0 \} , \ k \in \mathbb{N}^*, \]
\[ \Lambda_k^- = \{ \lambda = (d, \lambda_1, \lambda_2) \mid C_k < 0, \ \lambda_1 > 0, \ \lambda_2 > 0 \} , \ k \in \mathbb{N}^*. \]
Then, for each fixed \( k \), near \( \lambda \in \Lambda_k \),
\[
\beta_{k1}(\lambda) \begin{cases} 
< 0, & \text{if } \lambda \in \Lambda_k^+, \\
0, & \text{if } \lambda \in \Lambda_k, \\
> 0, & \text{if } \lambda \in \Lambda_k^-.
\end{cases}
\] (38)

Moreover, let \( (d)_k^0, (\lambda)_k^0 \) and \( (\lambda)_k^0 \) be the intersection points of the triangular cross-section \( \Lambda_k \) and \( d \)-axis, \( \lambda_1 \)-axis, \( \lambda_2 \)-axis, respectively. Clearly,
\[
(d)_k^0 = \frac{H_k}{E_k} > 0, \quad (\lambda)_k^0 = \frac{H_k}{F_k} > 0, \quad (\lambda)_k^0 = \frac{H_k}{J_k} > 0,
\]
for all \( k \in \mathbb{N}^* \). From the assumption (36), it can be verified that \( (d)_k^0, (\lambda)_k^0 \) and \( (\lambda)_k^0 \) are decreasing about \( \rho_k \) as \( k \) increases. It is shown that each cross-section \( \Lambda_{k+1} \) lies below \( \Lambda_k \) for any \( k \geq 1 \) (see Fig. 1). This, together with (38), derives the following crucial lemma which characterizes the PES for (22).

**Figure 1.** cross-section graph of \( \Lambda_k \).

**Lemma 3.1.** If (36) holds, then the eigenvalues \( \beta_{ki} \) \( (i = 1, 2) \) of \( L_\lambda \) satisfy the following
\[
\beta_{11}(\lambda) \begin{cases} 
< 0, & \text{if } \lambda \in \Lambda_1^+, \\
0, & \text{if } \lambda \in \Lambda_1, \\
> 0, & \text{if } \lambda \in \Lambda_1^-,
\end{cases}
\]
\[
\beta_{k1}(\lambda) < 0, \quad \text{for any } k \neq 1, \lambda \in \Lambda_1,
\]
\[
\beta_{k2}(\lambda) < 0, \quad \text{for any } k \in \mathbb{N}^*, \lambda \in \Lambda_1.
\] (39)

**Proof.** From the assumption (36), it can be verified that \( \Lambda_k \) in (37) is a triangular cross-section in the first quadrant of the \((d, \lambda_1, \lambda_2)\) coordinate system. Also, let \( (d)_k^0 = \frac{H_k}{E_k}, \quad (\lambda)_k^0 = \frac{H_k}{F_k} \) and \( (\lambda)_k^0 = \frac{H_k}{J_k} \) be the intersection points of the triangular cross-section \( \Lambda_k \) and \( d \)-axis, \( \lambda_1 \)-axis, \( \lambda_2 \)-axis, respectively. It is clear that \( (d)_k^0, (\lambda)_k^0 \) and \( (\lambda)_k^0 \) are both greater than zero. Furthermore, it is not difficult to check that \( (d)_k^0, (\lambda)_k^0 \) and \( (\lambda)_k^0 \) are decreasing about \( \rho_k \) as \( k \) increases, implying that each \( \Lambda_k (k \geq 2) \) is contained in \( \Lambda_1^- \) and \( \Lambda_1 \) is included in \( \Lambda_1^+ (k \geq 2) \). Combining with (38) deduces the PES condition (39). \( \square \)
Remark 1. It is worth pointing out that the assumption (36) is replaced by the following condition
\[
\begin{align*}
\delta_1 &< \frac{\alpha_1}{\alpha_2} < \frac{\delta_1 \gamma_2 + \delta_2 \gamma_1}{2 \delta_2 \gamma_2} < \frac{\gamma_1}{\gamma_2}, \quad \text{and} \\
\gamma_1 &\text{ is large enough such that } \gamma_1 v_0 - \mu_2 \delta_1 > 0, \\
\gamma_2 &\text{ is large enough such that } \gamma_2 v_0 > (1 + 2 v_0) \rho_1 + \delta_2 u_0,
\end{align*}
\]
one can still derive the PES condition (39).

3.3. Main results and proof. The following theorems shall show the types of transition that the system (13)–(14) undergoes as the bifurcation parameter \( \lambda = (d, \lambda_1, \lambda_2) \) crosses the critical cross-section \( \Lambda_1 \) basing on Lemma 3.1.

By simple calculation, we have
\[
\int_\Omega \psi_1^3 dx \neq 0.
\]

Denote
\[
\alpha(\lambda) = \frac{P}{P_{11}},
\]
where
\[
P = \left[ (-\lambda_1 + \lambda_2 h_{11}) \rho_1 - \delta_1 + \gamma_1 h_{11}
\right. \\
+ h_{11} f_{11} (\gamma_2 h_{11} - \delta_2 - \rho_1 + h_{11} \rho_1)]
\int_\Omega \psi_1^3 dx,
\]
\[
P_{11} = \langle e_{11}, e_{11}^* \rangle,
\]
the expressions of \( h_{11}, f_{11}, \) and \( P_{11} \) are given by (32) and (33), \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( H \), and \( e_{11}^* \) is the conjugate vector of \( e_{11} \).

Now, we give the dynamical transition result for the system (13)–(14) under the condition (40).

**Theorem 3.2.** Suppose that (36) and (40) hold. If \( \alpha(\lambda) \neq 0 \), then the following assertions hold:

1. The system (13)–(14) has a mixed transition from the equilibrium state \( \omega = 0 \). More precisely, there exists a neighborhood \( U \subset H_1 \) of \( \omega = 0 \), such that \( U \) is separated into two disjoint open sets \( U_{1\lambda} \) and \( U_{2\lambda} \) by the stable manifold \( \Gamma_{\lambda} \) satisfying
   \[
   \begin{align*}
   (a) \quad & U = U_{1\lambda} + U_{2\lambda} + \Gamma_{\lambda}, \\
   (b) \quad & \text{the transition in } U_{1\lambda} \text{ is jump, and in } U_{2\lambda} \text{ is continuous;}
   \end{align*}
   \]
2. As \( \lambda \in \Lambda_1^+ \), the equilibrium state \( \omega = 0 \) is asymptotically stable and the system (13)–(14) bifurcates to a unique saddle point \( \omega(\lambda) \):
3. The system (13)–(14) bifurcates in \( U_{1\lambda} \) to a unique singular point \( \omega(\lambda) \) as \( \lambda \) crosses the critical cross-section \( \Lambda_1 \) from \( \Lambda_1^+ \) into \( \Lambda_1^- \), which is an attractor such that for any initial value \( \varphi \in U_{1\lambda} \),
\[
\lim_{t \to \infty} \| \omega(t, \varphi) - \omega(\lambda) \|_{H_1} = 0;
\]
4. The bifurcated singular point \( \omega(\lambda) \) can be expressed as
\[
\omega(\lambda) = -\frac{\beta_{11}}{\alpha(\lambda)} e_{11} + o(|\beta_{11}|).
Proof. The proof is divided into the following three steps.

**Step 1.** Space decomposition.

Based on the spectral theory of linear completely continuous field [26], the spaces $H$ and $H_1$ can be decomposed into the following form

$$H = E_1 \oplus E_2, \quad H_1 = E_1 \oplus E_2,$$

where $E_1 = \text{span}\{e_{11}\}$, $E_2 = E_{1}^{\perp}$.

Then, near $\lambda \in \Lambda_1$, the solution of equation (13) can be expressed in the form

$$\omega = x_{11} e_{11} + z, \quad z = \sum_{k \neq 1}^{\infty} x_k e_{k1} + \sum_{k=1}^{\infty} y_k e_{k2},$$

(42)

where $x_{11} e_{11} \in E_1$, $z \in E_2$, $e_{ki}$ $(i = 1, 2)$ is the eigenvector corresponding to the eigenvalue $\beta_{ki}$.

Thus, in the space $E_1$, the equations (13) can be reduced to

$$\langle e_{11}, e_{11}^* \rangle \frac{dx_1}{dt} = \langle L_\lambda(\omega), e_{11}^* \rangle + \langle G(\omega), e_{11}^* \rangle = \beta_{11} \langle e_{11}, e_{11}^* \rangle x_1 + \langle G(\omega), e_{11}^* \rangle.$$  

(43)

**Step 2.** Reduction equation.

From the condition (40), it does not need to consider the influence of the centre manifold function. That is to say, let $\Phi(x) = 0$ and $\omega = x_{11} e_{11}$ in (43). $\Phi(x) : E_1 \to E_2$ is the centre manifold function.

Then, we derive the following reduced bifurcation equation from (43)

$$\frac{dx_1}{dt} = \beta_{11} x_1 + \frac{\langle G(x_1 e_{11}), e_{11}^* \rangle}{\langle e_{11}, e_{11}^* \rangle}.$$  

(44)

By simple calculation, it follows that

$$\frac{\langle G(x_1 e_{11}), e_{11}^* \rangle}{\langle e_{11}, e_{11}^* \rangle} = \alpha(\lambda) x_1^2 + o(2),$$  

(45)

where

$$\alpha(\lambda) = \frac{P}{P_{11}},$$

$P$ and $P_{11}$ are defined as (41).

Therefore, the equation (44) can be rewritten as

$$\frac{dx_1}{dt} = \beta_{11} x_1 + \alpha(\lambda) x_1^2 + o(2).$$  

(46)

**Step 3.** Bifurcation analysis.

Noting that the transition of equations (13) and its local topological structure are completely determined by (46). If $\alpha(\lambda) \neq 0$, then the equation (46) has exactly a bifurcated solution as follows

$$x_1 = -\frac{\beta_{11}}{\alpha(\lambda)} + o(|\beta_{11}|).$$

Clearly, $x_1$ is a locally asymptotically stable singular point as $\lambda \in \Lambda_1^-$, and is also an unstable saddle point as $\lambda \in \Lambda_1^+$. Therefore,

$$\mathfrak{x}(\lambda) = -\frac{\beta_{11}}{\alpha(\lambda)} e_{11} + o(|\beta_{11}|),$$

$$\mathfrak{x}(\lambda) = -\frac{\beta_{11}}{\alpha(\lambda)} e_{11} + o(|\beta_{11}|),$$
is the bifurcated singular point of (13). Furthermore, it is a locally asymptotically stable point as $\lambda \in \Lambda_1^{-1}$ which implies that the system (13)–(14) has a continuous transition in $U_1^\lambda$. Meanwhile, the original equilibrium state loses its stability and the system (13)–(14) has a jump transition in $U_1^\lambda$, and $\varpi(\lambda)$ is an unstable saddle point as $\lambda \in \Lambda_1^+$. 

**Remark 2.** If $\int_\Omega \psi_1^2 dx \neq 0$, then the dynamical transitions of (13) are showed schematically in Fig. 2.

![Figure 2. The dynamical transitions of (13) when $\int_\Omega \psi_1^2 dx \neq 0$.](image)

4. The transition under Neumann boundary condition.

4.1. **Mathematical setting.** Firstly, similar to the section 3, we get that the non-dimensional form of (1) should be (10). For the system (10) with (5), the four nonnegative trivial steady-state solutions can be seen in (11). Physically speaking, only positive solutions ($u_0 > 0, v_0 > 0$) are of interest in the competition of biological population. Hence, the (H1) must hold in this section to ensure a physical solution. In fact, the system (10) with (5) will undergo transition at all four steady-state solutions in (11). The following mainly concentrates on bifurcation and transition problem of (10) at more general positive steady-state solution $\omega_3$ in (11). Similarly, by the translation (12), we can obtain the equations (13) with the following initial-boundary value conditions

$$\begin{cases}
  u(x, 0) = u_0 - u_0, \quad v(x, 0) = v_0 - v_0, \\
  \frac{\partial u}{\partial n}|_{\partial \Omega} = \frac{\partial v}{\partial n}|_{\partial \Omega} = 0.
\end{cases} \tag{47}$$

In the following, we only need to study the dynamical behaviour of the system (13)–(47).

4.1.1. **Abstract operator form.** Now, we define the following function spaces:

$$H = L^2(\Omega, \mathbb{R}^2),$$

$$H_1 = H_N^2(\Omega, \mathbb{R}^2)$$

$$= \{ \omega \in H^2(\Omega, \mathbb{R}^2) \mid \omega \text{ satisfies boundary condition in (5)} \}.$$ 

where

$$\omega(t) = \begin{pmatrix} u(\cdot, t) \\ v(\cdot, t) \end{pmatrix}, \tag{48}$$
Define the linear operators $A_{\lambda}$, $B_{\lambda}$, $L_{\lambda}$ and the nonlinear operator $G$, see section (3.1.3). Finally, the system (13)–(47) can be rewritten as the abstract equations (21).

4.2. Eigenvalues and principle of exchange of stabilities (PES). In this section, we shall calculate the eigenvalues of the linearized eigenvalue equations of (13) with the Neumann boundary condition and also analysis the sign of the eigenvalues.

The linearized eigenvalue equations of (13) are given by

$$
\begin{align*}
\mu_1 \Delta u + \lambda_2 u_0 \Delta v - \delta_1 u_0^2 - \gamma_1 u_0 v &= \beta u, \\
v_0 \Delta u + \mu_2 \Delta v - \delta_2 v_0^2 - \gamma_2 v_0 v &= \beta v, \\
\frac{\partial u}{\partial n} \big|_{\partial \Omega} &= \frac{\partial v}{\partial n} \big|_{\partial \Omega} = 0.
\end{align*}
$$

(49)

Let $\rho_k$ and $\psi_k$ satisfy the following eigenvalue equations

$$
\begin{align*}
- \Delta \psi_k &= \rho_k \psi_k, \\
\frac{\partial \psi_k}{\partial n} \big|_{\partial \Omega} &= 0, \\
\int_{\Omega} \psi_k^2 dx &= 1.
\end{align*}
$$

(50)

Then,

$$
\rho_k = \pi^2 \left( \frac{k_1^2}{l_1^2} + \frac{k_2^2}{l_2^2} + \frac{k_3^2}{l_3^2} \right),
$$

$$
\psi_k = C_k \cos \frac{k_1 \pi x_1}{l_1} \cos \frac{k_2 \pi x_2}{l_2} \cos \frac{k_3 \pi x_3}{l_3}.
$$

(51)

Here, $k_1$, $k_2$ and $k_3$ range over all integers such that $k = |k_1| + |k_2| + |k_3| = 0, 1, 2, \ldots$. $C_k > 0$ is a constant to ensure $\int_{\Omega} \psi_k^2 dx = 1$. Clearly, $0 = \rho_0 < \rho_1 < \rho_2 < \cdots < \rho_k < \cdots$.

Similar to subsection 3.2.1, we easily obtain the expression of $\beta_{ki}$ as follows

$$
\beta_{k1} = \frac{-B_k + \sqrt{B_k^2 - 4C_k}}{2},
$$

$$
\beta_{k2} = \frac{-B_k - \sqrt{B_k^2 - 4C_k}}{2},
$$

(52)

where $B_k$ and $C_k$ can be seen in (28). Moreover, we can show that all eigenvalues $\beta_{ki}$ ($i = 1, 2$) are real.

By calculation, the eigenvector corresponding to the eigenvalue $\beta_{ki}$ of $L_{\lambda}$ is taken as

$$
e_{ki}(\lambda) = \left( -\beta_{ki} + \mu_2 \rho_k + \delta_1 u_0 \psi_k \right),
$$

(53)

where $\psi_k$ ($k = 0, 1, 2, \cdots$) are defined in (51). Obviously, $e_{ki}$ and $e_{k2}$ are linearly independent when $\beta_{k1} \neq \beta_{k2}$. Then the set of eigenvectors $\{e_{ki} \}_{k,i}$ forms a basis for $H$. Note that the eigenvector corresponding to the eigenvalue $\beta_{ki}$ of $L_{\lambda}$ is

$$
e_{ki}^*(\lambda) = \left( -\frac{\beta_{ki} + \mu_2 \rho_k + \delta_1 u_0 \psi_k}{\rho_k} \right).
$$

(54)
Then the eigenvector \( e_{ki} \) and its conjugate vector \( e_{k}^\ast \) can be expressed as

\[
e_{ki}(\lambda) = \begin{pmatrix} \psi_k \\ -h_{ki} \psi_k \end{pmatrix},
\]

\[
e_{ki}^\ast(\lambda) = \begin{pmatrix} \psi_k \\ -\overline{h_{ki}} \psi_k \end{pmatrix},
\]

where \( h_{ki} \) and \( \overline{h_{ki}} \) are given by (32). Thus, the eigenvectors satisfy the following relations

\[
\langle e_{ki}, e_{kj}^\ast \rangle = \begin{cases} 
0, & k \neq l \text{ or } i \neq j, \\
P_{k1}, & k = l, i = j = 1, \\
P_{k2}, & k = l, i = j = 2,
\end{cases}
\]

where \( P_{k1} \) and \( P_{k2} \) are shown in (33).

4.2.1. Principle of exchange of stabilities (PES). The linear stability and instability are precisely determined by the critical-crossing of the eigenvalues of (49), which is often called PES. For this purpose, we need to study the sign of the eigenvalues \( \beta_{ki} \).

In view of (52), it is clear that \( \beta_{k2} < 0 \) for \( \forall \ k \in \mathbb{N} \), and the sign of \( \beta_{k1} \) depends on the sign of \( C_k \). Hereafter, we need to look for the critical number through \( C_k \).

It should be noted that, in this subsection, we choose \( \lambda_2 \) as the control parameter.

Noting that \( C_k \) can be regarded as a function of \( \rho \in \mathbb{R} (\rho > 0) \) and \( \rho_k \) are the discrete points on the \( \rho \) axis. i.e.

\[
C(\rho) = (\mu_1 \rho + \delta_1 u_0)(\mu_2 \rho + \gamma_2 v_0) - u_0 v_0 (\lambda_2 \rho + \gamma_1)(\rho + \delta_2) \\
= D \rho^2 + E \rho + F,
\]

where

\[
D = \mu_1 \mu_2 - \lambda_2 u_0 v_0, \\
E = A \lambda_2 + J, \\
A = v_0 (\gamma_2 v_0 - \delta_2 u_0), \\
J = (d + 2u_0 \lambda_1) \gamma_2 v_0 + \delta_1 u_0 + 2\delta_1 u_0 v_0 + u_0 (\delta_1 u_0 - \gamma_1 v_0), \\
F = u_0 v_0 (\delta_1 \gamma_2 - \delta_2 \gamma_1).
\]

It follows from (15) that \( D = (d + 2u_0 \lambda_1) \mu_2 + \lambda_2 v_0 (1 + 2v_0) > 0 \). For convenience, let \( D_1 = \mu_2 d + 2u_0 \lambda_1 \) and \( D_2 = v_0 (1 + 2v_0) \), then \( D \) can be rewritten as \( D = D_1 + \lambda_2 D_2 \).

As we mentioned in the assumption (H1), we only need to discuss the following two cases:

Case 1. \( \frac{\gamma_1}{\gamma_2} < \frac{\delta_1 \gamma_2 + \delta_2 \gamma_1}{2 \delta_1 \gamma_2} < \frac{\alpha_1}{\alpha_2} < \frac{\delta_1}{\delta_2} \).

Obviously, the equation (55) can be rewritten as the following form

\[
C(\rho) = D \left[ \rho + \frac{E}{2D} \right]^2 + \frac{4DF - E^2}{4D}.
\]

Then, the symmetry axis of function \( C(\rho) \) is \( \rho_c(\lambda) = -\frac{E}{2D} \) and the minimum \( C(\rho)_{\min} = F - \frac{E^2}{4D} \).

Assume that \( C(\rho)_{\min} = 0 \). Then we immediately get the following equation about \( \lambda_2 \):

\[
A^2 \lambda_2^2 + (2AJ - D_2 F) \lambda_2 + J^2 - 4D_1 F = 0.
\]
Owing to \( \frac{\gamma_1}{\gamma_2} < \frac{\delta_1 \gamma_2 + \delta_2 \gamma_1}{2 \delta_2 \gamma_2} < \frac{\gamma_1}{\gamma_2} < \frac{\delta_1}{\delta_2} \), \( A < 0 \), \( F > 0 \), \( J > 0 \), we easily verify that (58) has at least one positive root. Moreover, we take the largest root of the equation (58) as \( \lambda_2^0 \). Also, we can obtain that \( \rho_c(\lambda_2^0) > 0 \) when \( k = 0 \), \( C_0 > 0 \).

In addition, as \( \lambda_2 > \lambda_2^0 \), we can prove that \( C(\rho)_{\min} \) decreases and \( \rho_c(\lambda_2) \) increases with the increase of \( \lambda \) from \( \lambda_2^0 \). Hence, there must exists a \( k_0 \) such that
\[
\rho_{k_0} < \rho_c(\lambda_2) < \rho_{k_0+1}, \quad \text{and} \quad C(\rho_{k_0}) > 0, \ C(\rho_{k_0+1}) > 0.
\] (59)

On the one hand, if \( C(\rho_{k_0}) = 0 \), then we have
\[
\lambda_2^{k_0} = -\frac{J \rho_{k_0} + F + \frac{1}{3} \lambda_{k_0}^2}{D_2 \rho_{k_0}^2 + A \rho_{k_0}}.
\] (60)

On the other hand, it follows from \( C(\rho_{k_0} + 1) = 0 \) that
\[
\lambda_2^{k_0+1} = -\frac{J \rho_{k_0+1} + F + \frac{1}{3} \lambda_{k_0+1}^2}{D_2 \rho_{k_0+1}^2 + A \rho_{k_0+1}}.
\] (61)

Here \( \lambda_2^{k_0}, \lambda_2^{k_0+1} \) can be meaningful as long as \( \alpha_2 \) is small enough.

Now, we can take the critical number
\[
\lambda_2^* = \min\{\lambda_2^{k_0}, \lambda_2^{k_0+1}\}.
\] (62)

Then the following lemma characterizes the PES for (49).

**Lemma 4.1.** Assume that \( \frac{\gamma_1}{\gamma_2} < \frac{\delta_1 \gamma_2 + \delta_2 \gamma_1}{2 \delta_2 \gamma_2} < \frac{\gamma_1}{\gamma_2} < \frac{\delta_1}{\delta_2} \) and \( d_2 \) is small enough, then the eigenvalues \( \beta_k(i = 1, 2) \) of \( L_\lambda \) satisfy the following properties:

1. if \( \lambda_2^{k_0} \neq \lambda_2^{k_0+1} \), and \( \lambda_2^{k_0} < \lambda_2^{k_0+1} \), then
   \[
   \beta_{k_0} \begin{cases} < 0, & \lambda_2 < \lambda_2^*, \\ = 0, & \lambda_2 = \lambda_2^*, \\ > 0, & \lambda_2 > \lambda_2^*, \end{cases}
   \] (63)

   \[
   \beta_{k_1}(\lambda_2^*) < 0, \quad \forall \, k \neq k_0, \\
   \beta_{k_2}(\lambda_2^*) < 0, \quad \forall \, k \in \mathbb{N}.
   \]

2. if \( \lambda_2^{k_0} = \lambda_2^{k_0+1} \), then
   \[
   \beta_{k_0,1,2} \begin{cases} < 0, & \lambda_2 < \lambda_2^*, \\ = 0, & \lambda_2 = \lambda_2^*, \\ > 0, & \lambda_2 > \lambda_2^*, \end{cases}
   \] (64)

   \[
   \beta_{k_1}(\lambda_2^*) < 0, \quad \forall \, k \neq k_0, \ k_0 + 1, \\
   \beta_{k_2}(\lambda_2^*) < 0, \quad \forall \, k \in \mathbb{N}.
   \]

**Proof.** According to the assumption, \( C(\rho)_{\min} = 0 \) implies that the image of \( C(\rho) \) is tangent to the \( \rho \) axis and the tangent point is \( \rho_c(\lambda_2^0) \). It is clear that the symmetry axis \( \rho_c(\lambda_2^0) = -\frac{E}{2F} > 0 \). Hence, there must exists a \( k_0 \) satisfying (59).

Note that, \( C(\rho)_{\min} \) is a decreasing function about \( \lambda_2 \) as \( \lambda_2 > \lambda_2^0 \). With the increase of \( \lambda_2 \) from \( \lambda_2^0 \), there must be \( C(\rho_{k_0}) = 0 \) or \( C(\rho_{k_0} + 1) = 0 \). Furthermore, we take the critical number \( \lambda_2^* = \min\{\lambda_2^{k_0}, \lambda_2^{k_0+1}\} \).

If \( \lambda_2^{k_0} \neq \lambda_2^{k_0+1} \), without loss of generality, let \( \lambda_2^{k_0} = \lambda_2^* \). Near \( \lambda_2^* \), as \( d_2 \) is small enough, we can get \( C(\rho_k) < 0, \forall \, k \neq k_0, \ C(\rho_{k_0}) > 0 \), as \( \lambda_2 < \lambda_2^* \) and \( C(\rho_{k_0}) < 0 \), as \( \lambda_2 > \lambda_2^* \). Then we can derive (63).

Analogously, (64) can be derived. We omit the details.
Case 2. $\frac{\gamma_1}{\gamma_2} < \frac{2\delta_1 \gamma_1}{\delta_1 \gamma_2 + 2\delta_2 \gamma_1} < \frac{\alpha_1}{\alpha_2} < \frac{\delta_1 \gamma_2 + \delta_2 \gamma_1}{2\delta_2 \gamma_2} < \frac{\delta_1}{\delta_2}$.

In this case, $E > 0$. It is easy to see that changing $\lambda_2$ will not affect the sign of $C(\rho)$, which is always greater than zero. Namely $\beta_{ki} < 0$ for any $k \in \mathbb{N}$, $i = 1, 2$.

4.3. Main results and proofs. The following theorems will show the types of transition that the system (13)-(47) undergoes as the control parameter $\lambda_2$ crosses the critical number $\lambda_2^*$ basing on Lemma 4.1. Let $\rho_{k_0}$ in (60) and $\rho_{k_0+1}$ in (61) be the eigenvalues of (50), the $\psi_{k_0}$ be the eigenvector of (50) corresponding to $\rho_{k_0}$. Hereafter, we will give different transition theorems basing on the two cases about $\psi_{k_0}$. First, we consider the case that

$$\int_\Omega \psi_{k_0}^3 dx \neq 0. \quad (65)$$

We define

$$\overline{\alpha}(\lambda) = \frac{P}{P_{k_0}},$$

where

$$P = \left[-(\lambda_1 \rho_{k_0} + \delta_1) + (\lambda_2 \rho_{k_0} + \gamma_1) h_{k_0} \right.$$

$$- (\rho_{k_0} + \delta_2) h_{k_0 + 1} - (\rho_{k_0} + \gamma_2) h_{k_0}^2 \left. \int_\Omega \psi_{k_0}^3 dx, \right]$$

and the expressions of $h_{k_0}$, $h_{k_0 + 1}$, $P_{k_0}$ are defined as (32) and (33).

Then, under the condition (65), we have the following dynamic transition theorem.

**Theorem 4.2.** Assume that $\frac{\gamma_1}{\gamma_2} < \frac{2\delta_1 \gamma_1}{\delta_1 \gamma_2 + 2\delta_2 \gamma_1} < \frac{\alpha_1}{\alpha_2} < \frac{\delta_1 \gamma_2 + \delta_2 \gamma_1}{2\delta_2 \gamma_2} < \frac{\delta_1}{\delta_2}$, and $d_2$ is small enough. If $\overline{\alpha}(\lambda_2) \neq 0$, then we have the following assertions:

1. The system (13)-(47) has a mixed transition from $(0, \lambda_2^*)$. More precisely, there exists a neighborhood $U \subset H_1$ of $\omega = 0$, such that $U$ is separated into two disjoint open sets $U_1^{\lambda_2}$ and $U_2^{\lambda_2}$ by the stable manifold $\Gamma_{\lambda_2}$ satisfying

   (a) $U = U_1^{\lambda_2} + U_2^{\lambda_2} + \Gamma_{\lambda_2}$,
   
   (b) the transition in $U_1^{\lambda_2}$ is jump, and in $U_2^{\lambda_2}$ is continuous.

2. The system (13)-(47) bifurcates in $U_2^{\lambda_2}$ to a unique singular point $\overline{\omega}(\lambda_2)$ on $\lambda_2 > \lambda_2^*$, which is an attractor such that for any initial value $\varphi \in U_2^{\lambda_2}$,

   $$\lim_{t \to \infty} \|\omega(t, \varphi) - \overline{\omega}(\lambda_2)\|_{H_1} = 0, \quad \forall \varphi \in U_2^{\lambda_2}.$$

3. The system (13)-(47) bifurcates to a unique saddle point $\overline{\omega}(\lambda_2)$ on $\lambda_2 < \lambda_2^*$;

4. The bifurcated singular point $\overline{\omega}(\lambda_2)$ can be expressed as

   $$\overline{\omega}(\lambda_2) = -\frac{\beta_{k_0+1}}{\overline{\alpha}(\lambda_2)} c_{k_0+1} + o(|\beta_{k_0+1}|).$$

**Proof.** We shall prove the theorem in the following three steps.

**Step 1.** Space decomposition.

Based on the spectral theory of linear completely continuous field[26], the spaces $H$ and $H_1$ can be decomposed into the following form

$$H = E_1 \oplus E_2, \quad H_1 = E_1 \oplus E_2,$$

where $E_1 = \text{span}\{e_{k_0}\}$, $E_2 = E_1^\perp$. 

\[ \text{Hereafter, we will give different transition theorems basing on the two cases about } \psi_{k_0}. \text{ First, we consider the case that}\]
Then near $\lambda_*^2$, the solution of equation (13) can be expressed in the form
\[
\omega = x_1 e_{k_01} + z, \quad z = \sum_{k \neq k_0}^{\infty} x_k e_{k1} + \sum_{k=0}^{\infty} y_k e_{k2},
\]
where $x_1 e_{k_01} \in E_1$, $z \in E_2$, $e_{k_i}$ ($i = 1, 2$) is the eigenvector corresponding to the eigenvalue $\beta_{k_i}$.

Then in the space $E_1$, the equations (13) can be reduced to
\[
\langle e_{k_01}, e_{k_01}^* \rangle \frac{dx_1}{dt} = \langle L_\lambda(\omega), e_{k_01}^* \rangle + \langle G(\omega), e_{k_01}^* \rangle = \beta_{k_01} \langle e_{k_01}, e_{k_01}^* \rangle x_1 + \langle G(\omega), e_{k_01}^* \rangle.
\]

**Step 2. Reduction equation.**

According to the condition (65), we do not need to consider the influence of the centre manifold function. That is to say, we let $\Phi(x) = 0$ and $\omega = x_1 e_{k_01}$ in (68). $\Phi(x) : E_1 \rightarrow E_2$ is the centre manifold function.

Hence, we derive the following reduced bifurcation equation from (68)
\[
\frac{dx_1}{dt} = \beta_{k_01} x_1 + \frac{\langle G(x_1 e_{k_01}), e_{k_01}^* \rangle}{\langle e_{k_01}, e_{k_01}^* \rangle}.
\]

By simple calculation, we have
\[
\frac{\langle G(x_1 e_{k_01}), e_{k_01}^* \rangle}{\langle e_{k_01}, e_{k_01}^* \rangle} = \overline{\pi}(\lambda_2) x_1^2 + o(2),
\]

where
\[
\overline{\pi}(\lambda_2) = \frac{\mathcal{P}}{P_{k_01}},
\]
$P_{k_01}$ is defined in (33) and $\mathcal{P}$ is given by (66).

Thus, the equation (69) can be rewritten as
\[
\frac{dx_1}{dt} = \beta_{k_01} x_1 + \overline{\pi}(\lambda_2) x_1^2 + o(2).
\]

**Step 3. Bifurcation analysis.**

It is known that the transition of equations (13) and its local topological structure are completely determined by (71). If $\overline{\pi}(\lambda_2) \neq 0$, it is clear that (71) has exactly a bifurcated solution as follows
\[
\overline{x}_1 = -\frac{\beta_{k_01}}{\overline{\pi}(\lambda_2)} + o(|\beta_{k_01}|).
\]

Obviously, $\overline{x}_1$ is a locally asymptotically stable singular point on $\lambda_2 > \lambda_*^2$ and an unstable saddle point on $\lambda_2 < \lambda_*^2$. Therefore
\[
\overline{\omega}(\lambda_2) = -\frac{\beta_{k_01}}{\overline{\pi}(\lambda_2)} e_{k_01} + o(|\beta_{k_01}|),
\]

is the bifurcated singular point of (13). Furthermore, it is a locally asymptotically stable point on $\lambda_2 > \lambda_*^2$ which implies that the system (13)–(47) has a continuous transition in $U_{\lambda_*^2}$. Meanwhile, the original equilibrium state loses its stability and the system (13)–(47) has a jump transition in $U_{\lambda_*^2}$. And $\overline{\omega}(\lambda_2)$ is an unstable saddle point on $\lambda_2 < \lambda_*^2$. $\square$
Next, we consider the case that
\[
\int_{\Omega} \psi_{k_0}^3 \, dx = 0. 
\]  
(72)

Then, under the condition (72), we have the following theorem.

**Theorem 4.3.** Assume that \( \frac{a_1}{\gamma_2} < \frac{\delta_1 + \delta_2 \gamma_1}{2\delta_2 \gamma_2} < \frac{a_1}{\delta_2} \) and \( d_2 \) is small enough.

The following assertions hold:

1. If \( \bar{\alpha}(\lambda_2) > 0 \), then the system (13)–(47) has a jump transition from \((0, \lambda_2^*)\), and bifurcates to exactly two saddle points \( \omega^\pm(\lambda_2) \) on \( \lambda_2 < \lambda_2^* \);

2. If \( \bar{\alpha}(\lambda_2) < 0 \), then the system (13)–(47) has a continuous transition from \((0, \lambda_2^*)\), and bifurcates to two singular points \( \omega^\pm(\lambda_2) \) on \( \lambda_2 > \lambda_2^* \), which are asymptotically stable;

3. The bifurcated points \( \omega^\pm(\lambda_2) \) can be expressed as
\[
\omega^\pm(\lambda_2) = \pm \sqrt{-\frac{\beta_{k_0}}{\bar{\alpha}(\lambda_2)}} e_{k_0} + \phi \left( \pm \sqrt{-\frac{\beta_{k_0}}{\bar{\alpha}(\lambda_2)}} \right)^2 + o(|\beta_{k_0}|),
\]
where \( \phi = (\phi_1, \phi_2)^T \) in (76).

**Proof.** We shall prove this theorem in the following steps.

**Step 1.** Space decomposition.

This part is the same as the Step 1 in the proof of Theorem 2, so we omit it.

Analogously, in the space \( E_1 \), the equations (13) can be reduced to
\[
\langle e_{k_0}, e_{k_0}^* \rangle \frac{dx_1}{dt} = \langle L_\lambda(\omega), e_{k_0} \rangle + \langle G(\omega), e_{k_0} \rangle \\
= \beta_{k_0} \langle e_{k_0}, e_{k_0}^* \rangle x_1 + \langle G(\omega), e_{k_0} \rangle. 
\]  
(73)

**Step 2.** Centre-manifold reduction.

To evaluate the last term in (73), we need to know the centre manifold function \( \Phi(x) : E_1 \to E_2 \). Let \( \omega = x_1 e_{k_0} + \Phi(x_1) \). Then we use the asymptotic expression for the centre-manifold function near \( \lambda_2^* \) (see [26] Theorem B.4.1).

By direct circulation, we derive the centre-manifold function can be expressed as
\[
\Phi(x_1) = \sum_{k \neq k_0}^{\infty} -\frac{Q_k}{\beta_{k_1}} e_{k_1} + \sum_{k=0}^{\infty} -\frac{H_k}{\beta_{k_2}} e_{k_2},
\]
\[
= \left( \sum_{k \neq k_0}^{\infty} -\frac{Q_k}{\beta_{k_1}} \psi_k + \sum_{k=0}^{\infty} -\frac{H_k}{\beta_{k_2}} \psi_k \right) x_1^2, 
\]  
(74)

where
\[
Q_k = \left[ -(\lambda_1 \rho_k + \delta_1) + (\lambda_2 \rho_k + \gamma_1) \tilde{h}_{k_0_1} - (\rho_k + \delta_2) \tilde{h}_{k_0_1} \tilde{h}_{k_1} \right. \\
+ (\rho_k + \gamma_2) \tilde{h}_{k_0_1}^2 \tilde{h}_{k_1}, \\
H_k = \left[ -(\lambda_1 \rho_k + \delta_1) + (\lambda_2 \rho_k + \gamma_1) \tilde{h}_{k_0_1} - (\rho_k + \delta_2) \tilde{h}_{k_0_1} \tilde{h}_{k_2} \right. \\
+ (\rho_k + \gamma_2) \tilde{h}_{k_0_1}^2 \tilde{h}_{k_2} \right. \\
\]  
(75)

and \( h_{11}, \tilde{h}_{k_1}, \tilde{h}_{k_2} \) are given by (32).
Let

\[
\phi_1 = \sum_{k \neq k_0}^{\infty} \frac{Q_k}{\beta_{k_1}} \psi_k + \sum_{k=0}^{\infty} \frac{-H_k}{\beta_{k_2}} \psi_k,
\]

\[
\phi_2 = \sum_{k \neq k_0}^{\infty} \frac{Q_k h_{k_1}}{\beta_{k_1}} \psi_k + \sum_{k=0}^{\infty} \frac{H_k h_{k_2}}{\beta_{k_2}} \psi_k.
\]

Then

\[
\Phi(x_1) = (\phi_1, \phi_2)^T x_1^2 + o(2).
\]

Direct calculations yield to the following result

\[
\langle G(x_1 e_{k_01} + \Phi(x_1)), e_{k_01}^* \rangle = \tilde{\alpha}(\lambda_2) x_1^3 + o(3),
\]

where

\[
\tilde{\alpha}(\lambda_2) = \frac{\bar{P}}{P_{k_01}},
\]

\[
\bar{P} = (-\lambda_1 \rho_{k_0} + \bar{\tau}_{k_01} \rho_{k_0} - \gamma_1 + \delta_2 h_{k_01})(\phi_2 - h_{k_01} \phi_1) - 2\delta_1 \phi_1 - 2\lambda_1 \phi_1 \rho_{k_0} - 2\phi_2 h_{k_01} \bar{\tau}_{k_01} (\rho_{k_0} + \gamma_2),
\]

and the expression of \(P_{k_01}\) is defined as (33).

Combining (73) and (77), we deduce that the following reduced bifurcation equation

\[
\frac{dx_1}{dt} = \beta_{k_01} x_1 + \frac{\langle G(x_1 e_{k_01} + \Phi(x_1)), e_{k_01}^* \rangle}{\langle e_{k_01}, e_{k_01}^* \rangle},
\]

\[
= \beta_{k_01} x_1 + \tilde{\alpha}(\lambda_2) x_1^3 + o(3).
\]

**Step 3. Bifurcation analysis.**

Clearly, when \(\tilde{\alpha}(\lambda_2) > 0\), the equation (79) bifurcates two saddle points on \(\lambda_2 < \lambda_2^*\) and when \(\tilde{\alpha}(\lambda_2) < 0\), the equation (79) bifurcates two stable singular points on \(\lambda_2 > \lambda_2^*\). The bifurcated solutions can be expressed as

\[
x_1^\pm = \pm \sqrt{-\beta_{k_01} \alpha(\lambda)} + o(\sqrt{|\beta_{k_01}|}).
\]

It is known that the transition of equations (13) and its local topological structure are determined completely by (79). Therefore

\[
\omega^\pm(\lambda_2) = \pm \sqrt{-\beta_{k_01} \alpha(\lambda)} e_{k_01} + o\left(\sqrt{-\beta_{k_01} \alpha(\lambda)}\right),
\]

are the bifurcated singular points of (13). The stability of \(\omega^\pm(\lambda_2)\) is the same as that of \(x_1^\pm\).

**Remark 3.** If the PES condition (64) in Lemma 4.1 holds true, the the transition from two real eigenvalues occurs on \(\lambda_2 > \lambda_2^*\).

5. **Examples.** In this section, we illustrate the above analytical results with the following two examples on the unit interval \(\Omega = (0, 1)\).
5.1. An example with the Dirichlet boundary. Consider the following example:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta \left( \left( \frac{1}{500} + \frac{393}{25000} u + \frac{3}{1000} v \right) u + u \left( \frac{7}{10} - \frac{1}{2} u - \frac{33}{50} v \right) \right), \quad x \in (0,1), \\
\frac{\partial v}{\partial t} &= \Delta \left( \left( \frac{1}{10} + \frac{1}{2} u + \frac{3}{50} v \right) v + v \left( \frac{1}{10} - \frac{1}{2} u - \frac{3}{50} v \right) \right), \quad x \in (0,1),
\end{align*}
\]

(80)

with the initial-boundary value conditions (4).

It is worth noticing that the eigenvalues and eigenvectors of the negative Laplace operator to Dirichlet boundary conditions (see equation (23)) are

\[
\rho_k = k^2 \pi^2, \quad \psi_k = \sqrt{2} \sin(k \pi x), \quad k \in \mathbb{N}^*, \quad x \in (0,1).
\]

Based on the above analysis process in subsection 3.1 and subsection 3.2, we can find a critical triangular cross-section

\[
\Lambda_1 = \left\{(d, \lambda_1, \lambda_2) \mid 259.185d + 207.384\lambda_1 + 129.764\lambda_2 = 18.191, \quad d, \lambda_1, \lambda_2 > 0 \right\},
\]

and a critical control parameter

\[
\lambda = (d, \lambda_1, \lambda_2) = (0.02, 0.03144, 0.05) \in \Lambda_1.
\]

Moreover, the following PES condition is also obtained

\[
\beta_{11}(\lambda) \begin{cases} 
< 0, & \text{if } \lambda \in \Lambda_1^+, \\
= 0, & \text{if } \lambda \in \Lambda_1, \\
> 0, & \text{if } \lambda \in \Lambda_1^-,
\end{cases}
\]

\[
\beta_{k1}(\lambda) < 0, \quad \text{for any } k \neq 1, \quad \lambda \in \Lambda_1,
\]

\[
\beta_{k2}(\lambda) < 0, \quad \text{for any } k \in \mathbb{N}^*, \quad \lambda \in \Lambda_1.
\]

It is easy to check that \[\int_0^1 \psi_3^1 dx = \frac{2\sqrt{2}}{\pi} \neq 0.\] From Theorem 3.2, it immediately follows that the system (80) with (4) will bifurcate to a stable singular point as \(\lambda\) crosses the critical cross-section \(\Lambda_1\) from \(\Lambda_1^+\) into \(\Lambda_1^-\).

5.2. An example with the Neumann boundary. Consider the following example:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta \left( \left( \frac{1}{1000} + \frac{13}{1000} u + \frac{39}{25} v \right) u + u \left( \frac{1}{5} - \frac{13}{1000} u - \frac{13}{50} v \right) \right), \\
\frac{\partial v}{\partial t} &= \Delta \left( \left( \frac{1}{1000} + \frac{13}{1000} u + \frac{13}{1000} v \right) v + v \left( \frac{1}{50} - \frac{13}{1000} u - \frac{13}{50} v \right) \right),
\end{align*}
\]

(81)

with the initial-boundary value conditions (5), and \(x \in (0,1)\).

The eigenvalues and eigenvectors of the negative Laplace operator to Neumann boundary condition (see equations (50)) are

\[
\rho_k = k^2 \pi^2, \quad \psi_k = \sqrt{2} \cos(k \pi x), \quad k \in \mathbb{N}, \quad x \in (0,1).
\]

Based on the above analysis process in subsection 4.1 and subsection 4.2, the largest root of the equation (58) should be

\[
\lambda_0^2 \approx 359.04.
\]

Moreover, we can get the symmetry axis of function (57) can be

\[
\rho_c(\lambda_0^2) \approx 1.58.
\]

Obviously, there exist \(k_0 = 0\) such that \(0 < \rho_c(\lambda_0^2) < \rho_1\).
Therefore, we get the critical value from (61)

\[ \lambda^*_2 \approx 1236.87. \]

and the PES condition

\[
\begin{align*}
\beta_{11} &< 0, \quad \text{as} \quad \lambda_2 < \lambda^*_2, \\
\beta_{11} &= 0, \quad \text{as} \quad \lambda_2 = \lambda^*_2, \\
\beta_{11} &> 0, \quad \text{as} \quad \lambda_2 > \lambda^*_2, \\
\beta_k(\lambda^*_2) &< 0, \quad \text{for any} \quad k \neq 1, \\
\beta_k(\lambda^*_2) &< 0, \quad \text{for any} \quad k \in \mathbb{N}.
\end{align*}
\]

It is easy to check that \( \int_{\Omega} \psi_1^3 \, dx = 0. \) Based on the Theorem 3, we immediately get the system (81) with (5) will bifurcate to two stable points as \( \lambda_2 > 1236.87. \)

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