HOOK LENGTH PROPERTY OF \( d \)-COMPLETE POSETS VIA \( q \)-INTEGRALS

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Abstract. The hook length formula for \( d \)-complete posets states that the \( P \)-partition generating function for them is given by a product in terms of hook lengths. We give a new proof of the hook length formula using \( q \)-integrals. The proof is done by a case-by-case analysis consisting of two steps. First, we express the \( P \)-partition generating function for each case as a \( q \)-integral and then we evaluate the \( q \)-integrals. Several \( q \)-integrals are evaluated using partial fraction expansion identities and others are verified by computer.

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1. Introduction

The classical hook length formula due to Frame, Robinson and Thrall [1] states that for a partition $\lambda$ of $n$, the number $f^\lambda$ of standard Young tableaux of shape $\lambda$ is given by

\begin{equation}
    f^\lambda = \frac{n!}{\prod_{x \in \lambda} h(x)},
\end{equation}

where $h(x)$ is the hook length of the cell $x$ in $\lambda$. One can naturally consider the shape $\lambda$ as a poset $P$ on the cells in $\lambda$. Then the $P$-partition generating function for the poset also has the following hook length formula:

\begin{equation}
    \sum_{\sigma : P \to \mathbb{N}} q^{\lvert \sigma \rvert} = \prod_{x \in P} \frac{1}{1 - q^{h(x)}},
\end{equation}

where the sum is over all $P$-partitions $\sigma$. See Section 2 for the definition of $P$-partitions. Using the theory of $P$-partitions, one can see that (2) implies (1). There are also hook length formulas for the $P$-partition generating functions for the posets coming from shifted shapes and forests.

Proctor [9] introduced $d$-complete posets, which include the posets of shapes, shifted shapes and forests. Proctor [8] proved that every $d$-complete poset can be decomposed into irreducible $d$-complete posets. He then classified all irreducible $d$-complete posets into 15 classes.

To every element $x$ in a $d$-complete poset a positive integer $h(x)$ called the hook length is assigned. The $d$-complete posets have the following hook length property.

**Theorem 1.1 (Hook Length Formula for $d$-complete posets).** For any $d$-complete poset $P$, we have

\begin{equation}
    \sum_{\sigma : P \to \mathbb{N}} q^{\lvert \sigma \rvert} = \prod_{x \in P} \frac{1}{1 - q^{h(x)}},
\end{equation}

where the sum is over all $P$-partitions $\sigma$.

Theorem 1.1 was proved by Proctor [10], Nakada [6, 7], and generalized by Ishikawa and Tagawa [2, 3] to “leaf posets”. However, their proofs are only sketched in conference proceedings, and full proofs of the hook length formula (Theorem 1.1) are not yet available in the literature.

In this paper we give a new and complete proof of Theorem 1.1 using $q$-integrals. Our proof is based on case-by-case analysis. Several cases are proved by using partial fraction expansion identities and the remaining cases are verified by computer. We outline our proof of Theorem 1.1 as follows.

We introduce semi-irreducible $d$-complete posets, which contain all irreducible $d$-complete posets. First, we show that in order to prove Theorem 1.1, it is sufficient to consider the semi-irreducible $d$-complete posets. Then we show that the $P$-partition generating function for each semi-irreducible $d$-complete poset can be written as a $q$-integral. Therefore, in order to compute the generating functions for 15 classes of semi-irreducible $d$-complete posets, we need to evaluate the corresponding 15 $q$-integrals. The first two of the 15 classes are shapes and shifted shapes which are well known to have the hook length formula. Hence we focus on considering the remaining 13 classes, but we also provide a proof for the class of shifted shapes using the $q$-integral technique. Among the $q$-integrals corresponding to the 13 classes, 2 of them have an arbitrary number of integration variables and other 11 $q$-integrals have a fixed number of integration variables. In this paper we verify these 11 $q$-integrals by computer and compute the remaining 2 $q$-integrals by hand. In the process of proving the $q$-integrals by hand, we utilize several partial fraction expansion identities (cf. [5], [16]).

The rest of this paper is organized as follows. In Section 2, we give necessary definitions. In Section 3, we prove some properties of $P$-partition generating functions which are used later. In Section 4, we introduce semi-irreducible $d$-complete posets and prove that it suffices to consider them for showing the hook length formula for the $d$-complete posets. In Section 5, we consider a certain class of posets that includes all semi-irreducible $d$-complete posets. We express the $P$-partition generating function for an arbitrary poset in this class as a $q$-integral. In Section 6, using the result in the previous section we express the $P$-partition generating function for each
semi-irreducible $d$-complete poset as a $q$-integral. In Section 7 we evaluate the $q$-integrals obtained in the previous section.

2. Preliminaries

In this section, we introduce basic definitions and notation that are used throughout this paper. We also recall some properties of $d$-complete posets. For the details, we refer the readers to [8, 11].

2.1. Basic definitions and notation. We will use the following notation for $q$-series:

\[(a;q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \quad (a_1, a_2, \ldots, a_k; q)_n = (a_1; q)_n \cdots (a_k; q)_n.\]

A partition is a weakly decreasing sequence $\mu = (\mu_1, \ldots, \mu_k)$ of nonnegative integers. Each $\mu_i$ is called a part of $\mu$. The size $|\mu|$ of $\mu$ is the sum of its parts. If the size of $\mu$ is $n$, we say that $\mu$ is a partition of $n$ and write $\mu \vdash n$. The length of $\mu$ is the number of nonzero parts in $\mu$. We denote the set of partitions of length at most $n$ by $\text{Par}_n$.

The staircase partition $(n-1, n-2, \ldots, 1, 0)$ is denoted by $\delta_n$. For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, the alternant $a_\lambda(x_1, \ldots, x_n)$ is defined by

\[a_\lambda(x_1, \ldots, x_n) = \det(x_\lambda^j i^i)_{i,j=1}^n.\]

Two partitions are considered to be the same if they have the same nonzero parts. For example, $(4, 3, 1) = (4, 3, 1, 0) = (4, 3, 1, 0, 0)$. Thus, for a partition $\mu = (\mu_1, \ldots, \mu_k)$, we use the convention $\mu_i = 0$ for $i > k$ if it is necessary. For two partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$, we define $\lambda + \mu$ to be the partition $(\lambda_1, \ldots, \lambda_k + \mu)$, where $k = \max(r, s)$. For example, $(3, 1, 1) + (4, 3, 2, 1, 0) = (7, 4, 3, 1, 0)$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition. The Young diagram of $\lambda$ is the left-justified array of squares in which there are $\lambda_i$ squares in row $i$. The Young poset of $\lambda$ is the poset whose elements are the squares in the Young diagram of $\lambda$ with relation $x \leq y$ if $x$ is weakly below and weakly to the right of $y$. See Figure 1. The transpose of $\lambda$ is the partition $\lambda'$ whose $i$th part $\lambda'_i$ is the number of parts of $\lambda$ at least $i$.

![Figure 1](image1.png)

**Figure 1.** The Young diagram of $\lambda = (4, 3, 1)$ on the left and its Young poset on the right.

Throughout this paper, Hasse diagrams are rotated $45^\circ$ counterclockwise unless otherwise stated. Therefore, smaller elements are located weakly to the right and weakly below larger elements.

If $\lambda$ has no equal nonzero parts, $\lambda$ is called strict. For a strict partition $\lambda$, the shifted Young diagram of $\lambda$ is the diagram obtained from the Young diagram of $\lambda$ by shifting the $i$th row to the right by $i - 1$ units. The shifted Young poset of $\lambda$ is defined similarly. See Figure 2.

![Figure 2](image2.png)

**Figure 2.** The shifted Young diagram of $\lambda = (6, 4, 1)$ on the left and its shifted Young poset on the right.
By abuse of notation, we identify a partition \( \lambda \) with its Young diagram and also with its Young poset if there is no possible confusion. For a strict partition \( \lambda \), the shifted Young diagram of \( \lambda \) is denoted by \( \lambda^* \). Similarly, the shifted Young poset of \( \lambda \) will also be written as \( \lambda^* \).

For a Young diagram or a shifted Young diagram \( \lambda \), a semistandard Young tableau of shape \( \lambda \) is a filling of \( \lambda \) with nonnegative integers such that the integers are weakly increasing in each row and strictly increasing in each column. A reverse plane partition of shape \( \lambda \) is a filling of \( \lambda \) with nonnegative integers such that the integers are weakly increasing in each row and each column. See Figures 3 and 4.

Let \( \text{SSYT}(\lambda) \) be a strict partition. For \( \lambda \in \text{SSYT}(\lambda^*) \) or \( \lambda \in \text{RPP}(\lambda^*) \), the leftmost entry in each row is called a diagonal entry. We define the reverse diagonal sequence \( \text{rdiag}(T) \) to be the sequence of diagonal entries in the non-increasing order. For example, if \( T \) is the semistandard Young tableau obtained from \( \lambda \) by adding \( i-1 \) to each entry in the \( i \)th row. For example, the reverse plane partitions and the semistandard Young tableaux in Figures 3 and 4 are mapped to each other by this bijection.

Let \( \lambda \) be a strict partition. For \( T \in \text{SSYT}(\lambda^*) \) or \( T \in \text{RPP}(\lambda^*) \), the leftmost entry in each row is called a diagonal entry. We define the reverse diagonal sequence \( \text{rdiag}(T) \) to be the sequence of diagonal entries in the non-increasing order. For example, if \( T_1 \) and \( T_2 \) are the semistandard Young tableau and the reverse plane partition in Figure 4 respectively, then \( \text{rdiag}(T_1) = (3, 1, 0) \) and \( \text{rdiag}(T_2) = (1, 0, 0) \).

Now we recall basic properties of \( P \)-partitions. See [14, Chapter 3] for more details.

Let \( P \) be a poset with \( n \) elements. A \( P \)-partition is a map \( \sigma : P \to \mathbb{N} \) such that \( x \leq_P y \) implies \( \sigma(x) \geq \sigma(y) \). In other words, a \( P \)-partition is just an order-reversing map from \( P \) to \( \mathbb{N} \).

For an integer \( m \geq 0 \), we denote by \( \mathcal{P}_{\geq m}(P) \) the set of all \( P \)-partitions \( \sigma \) with \( \min(\sigma) \geq m \). We also define \( \mathcal{P}(P) = \mathcal{P}_{\geq 0}(P) \). For a \( P \)-partition \( \sigma \), the size \( |\sigma| \) of \( \sigma \) is defined by

\[
|\sigma| = \sum_{x \in P} \sigma(x).
\]

For a poset \( P \), we define \( \text{GF}_q(P) \) to be the \( P \)-partition generating function:

\[
\text{GF}_q(P) = \sum_{\sigma \in \mathcal{P}(P)} q^{|\sigma|}.
\]

The following definitions allow us to build \( d \)-complete posets starting from a chain.

**Definition 2.1.** Let \( P \) be a poset containing a chain \( C = \{x_1 < x_2 < \cdots < x_n\} \). For \( \lambda \in \text{Par}_n \), we denote by \( D(P, C, \lambda) \) the poset obtained by taking the disjoint union of \( P \) and \( (\lambda + \delta_{n+1})^* \) and identifying \( x_n, x_{n-1}, \ldots, x_1 \) with the diagonal elements of \( (\lambda + \delta_{n+1})^* \).

\[
\begin{array}{c|ccc}
0 & 1 & 1 & 2 \\
1 & 2 & 4 \\
4 & & & \\
\hline
0 & 1 & 1 & 2 \\
1 & 2 & 4 \\
& & 2 \\
\end{array}
\]

**Figure 3.** A semistandard Young tableau of shape \((4, 3, 1)\) on the left and a reverse plane partition of shape \((4, 3, 1)\) on the right.

\[
\begin{array}{c|ccc}
0 & 0 & 1 & 2 \quad 2 \\
1 & 2 & 2 & 4 \\
3 & & & \\
\hline
0 & 0 & 1 & 2 \quad 2 \\
0 & 1 & 1 & 3 \\
& & 1 \\
\end{array}
\]

**Figure 4.** A semistandard Young tableau of shifted shape \((6, 4, 1)\) on the left and a reverse plane partition of shifted shape \((6, 4, 1)\) on the right.
Definition 2.2. Let \( n \) and \( k \) be positive integers. Let 
\[
X = \{(\lambda^{(i)}, n_i, s_i) : 1 \leq i \leq k\},
\]
where \( n_i \) and \( s_i \) are positive integers with \( s_i + n_i - 1 \leq n \), \( \lambda^{(i)} \in \text{Par}_{n_i} \). We define \( P_n(X) \) to be the poset constructed as follows. Let \( P_0 \) be a chain \( x_1 < x_2 < \cdots < x_n \) with \( n \) elements, called diagonal entries. For \( 1 \leq i \leq k \), we define \( P_i = D(P_{i-1}, C_i, \lambda^{(i)}) \) where \( C_i = \{x_{s_i}, x_{s_i+1}, \ldots, x_{n_i}\} \). Then we define \( P_n(X) = P_k \). We also define \( P_m^m(X) \) to be the poset obtained from \( P_n(X) \) by attaching a chain with \( m \) elements above \( x_n \). See Figure 5 for an example. We say that an element \( y \in P_n(X) \) is of level \( i \) if \( y \leq x_i \) and \( y \not\leq x_{i-1} \), see Figure 6.

\[ \text{Figure 5. The poset } P_3^2(X) \text{ for } X = \{((2), 1, 1), ((2), 1, 2), ((6, 5, 3), 3, 1), (\emptyset, 2, 2)\}, \text{ where } \lambda = (4, 2) \text{ and } \mu = (5, 4, 2). \]

\[ \text{Figure 6. The poset } P_3(X) \text{ for } X = \{((1), 1, 1), ((4, 2), 2, 1), ((5, 4, 2), 3, 1), (\emptyset, 2, 2)\}. \]

This poset has 6 elements of level 1, 11 elements of level 2 and 9 elements of level 3.

In this paper the diagonal entries of \( P_n^m(X) \) will be represented by black squares in the Hasse diagram as shown in Figures 5 and 6.

2.2. \( d \)-complete posets. In this subsection we define \( d \)-complete posets following [8].

Let \( P \) be a poset. A diamond in \( P \) consists of four elements \( w, x, y, z \) where \( z \) covers \( x \) and \( y \), and both \( x \) and \( y \) cover \( w \). In this case, \( z \) is called the top, \( w \) is called the bottom, and \( x \) and \( y \) are the sides.

For \( k \geq 3 \), a double-tailed diamond poset \( d_k(1) \) can be constructed from a diamond, by attaching a chain of length \( k - 3 \) above the top \( z \), and below the bottom \( w \). Thus \( d_k(1) \) has \( 2k - 2 \) elements. The \( k - 2 \) elements above the two incomparable elements \( x \) and \( y \) are called neck elements. For \( k \geq 3 \), an interval \([w, z]\) is called a \( d_k \)-interval if it is isomorphic to \( d_k(1) \). Moreover, for \( k \geq 4 \), an interval is a \( d_k^{-1} \)-interval if it is isomorphic to \( d_k(1) \setminus \{t\} \), where \( t \) is the maximal element of \( d_k(1) \).

Now we define \( d \)-complete posets and the hook lengths of their elements.

Definition 2.3. A poset \( P \) is called \( d \)-complete if it satisfies the following conditions for any \( k \geq 3 \):

1. If \( I \) is a \( d_k^{-1} \)-interval, then there exists an element \( z \) such that \( z \) covers the maximal element(s) of \( I \) and \( I \cup \{z\} \) is a \( d_k \)-interval,
2. If \([w, z]\) is a \( d_k \)-interval then \( z \) does not cover any elements of \( P \) outside \([w, z]\), and
There are no $d_k$-intervals which differ only in the minimal elements.

**Definition 2.4.** Let $P$ be a $d$-complete poset. For any element $z \in P$, we define its hook length, denoted by $h(z)$, recursively as follows:

1. If $z$ is not in the neck of any $d_k$-interval, then $h(z)$ is the number of elements in $P$ which are less than or equal to $z$, i.e., the number of elements of the principal order ideal generated by $z$.
2. If $z$ is in the neck of some $d_k$-interval, then we can take the unique element $w \in P$ such that $[w, z]$ is a $d_\ell$-interval, for some $\ell \leq k$. If we let $x$ and $y$ be the two incomparable elements in this $d_\ell$-interval, then $h(z) = h(x) + h(y) - h(w)$.

We say that a $d$-complete poset $P$ has the hook length property if $P$ satisfies the hook length formula in Theorem 1.1.

It is not hard to see from the definition that, if $P$ is $d$-complete and connected, then $P$ has a unique maximal element $\hat{1}$ and for each $w \in P$, every saturated chain from $w$ to $\hat{1}$ has the same length (see [8, Proposition in §3]). A top tree element $x \in P$ is an element such that every element $y \geq x$ is covered by at most one other element, and the top tree of $P$ consists of all top tree elements.

Given $P$ a connected $d$-complete poset with top tree $T$, an element $y \in P$ is called acyclic if $y \in T$ and it is not in the neck of any $d_k$-interval for any $k \geq 3$.

**Definition 2.5.** For $d$-complete posets $P_1$ and $P_2$, suppose that $P_1$ has an acyclic element $x$ and $P_2$ has the unique maximal element $y$. The slant sum $P_1^x \setminus y P_2$ of $P_1$ and $P_2$ at $x$ and $y$ is the poset obtained by taking the disjoint union of $P_1$ and $P_2$ with additional covering relation $x < y$.

A $d$-complete poset $P$ is called irreducible if it is connected and it cannot be written as a slant sum of two non-empty $d$-complete posets. In [8], Proctor showed that any connected $d$-complete poset $P$ can be uniquely decomposed into a slant sum of irreducible $d$-complete posets. Furthermore, he classified all irreducible $d$-complete posets which are connected and contain at least two elements into 15 disjoint classes. For the list of 15 classes of irreducible $d$-complete posets, see [8, Table 1].

### 3. Some properties of $P$-partitions

In this section we prove two lemmas that will be used later in this paper.

**Definition 3.1.** For a poset $P$, let $P^+$ be the poset obtained from $P$ by adding a new element which is greater than all elements in $P$. If $P$ has a unique maximal element, we define $P^-$ to be the poset obtained from $P$ by removing the maximal element.
Note that \((P^+)^- = P\) for any poset \(P\). If \(P\) has a unique maximal element, \((P^-)^+ = P\). There is a simple relation between \(\text{GF}_q(P^+\) and \(\text{GF}_q(P)\).

**Lemma 3.2.** For a poset \(P\) with \(p\) elements, we have

\[
\text{GF}_q(P^+) = \frac{1}{1 - q^{p+1}} \text{GF}_q(P).
\]

**Proof.** Let \(z\) be the unique maximal element of \(P^+\). For \(\sigma \in \mathcal{P}(P^+)\), if \(\sigma(z) = k\) and \(\sigma'\) is the restriction of \(\sigma\) to \(P\), we have \(\sigma' \in \mathcal{P}_{\geq k}(P)\). Thus

\[
\text{GF}_q(P^+) = \sum_{\sigma \in \mathcal{P}(P^+)} q^{\vert \sigma \vert} = \sum_{k=0}^{\infty} \sum_{\sigma' \in \mathcal{P}_{\geq k}(P)} q^{\vert \sigma' \vert + k} = \sum_{k=0}^{\infty} \sum_{\tau \in \mathcal{P}(P)} q^{\vert \tau \vert + kp + k} = \frac{1}{1 - q^{p+1}} \text{GF}_q(P). \quad \Box
\]

**Definition 3.3.** Let \(P\) be a poset in which there is a unique maximal element \(y_1\) and a specified element \(y_2\) covered by \(y_1\). For integers \(m, k \geq 1\), we define \(D_{m,k}(P)\) to be the poset obtained from \(P\) by adding a disjoint chain \(z_m > \cdots > z_1 > z_0 > z_{-1} > \cdots > z_{-k}\) and a new element \(y_0\) with additional covering relations \(z_1 > y_0, z_0 > y_1, z_{-1} > y_2\) and \(y_0 > y_1\). See Figure 8. We also define \(D_{k}(P)\) to be the poset obtained from \(D_{m,k}(P)\) by removing the elements \(z_m, \ldots, z_1, z_0\).

**Figure 8.** The posets \(D_{m,k}(P)\) on the left and \(D_{k}(P)\) on the right.

The following lemma will be used later to decompose the \(P\)-partition generating function of \(d\)-complete posets.

**Lemma 3.4.** Let \(P = \{y_1, y_2, \ldots, y_p\}\) be a poset in which \(y_1\) is the unique maximal element and \(y_2\) is covered by \(y_1\). Then

\[
\text{GF}_q(D_{m,k}(P)) = \frac{1}{(q^{p+k+1}; q)_m + 2} \left( \frac{q^{p+1}}{(q; q)_{m+1}} \text{GF}_q(P^+) + (1 - q^{2p+2k+2}) \text{GF}_q(D_{k}(P)) \right).
\]

**Proof.** By applying Lemma 3.2 repeatedly, we have

\[
\text{GF}_q(D_{m,k}(P)) = \frac{1}{(q^{p+k+1}; q)_m} \text{GF}_q(D_{m,k}(P) - \{z_1, \ldots, z_m\}).
\]

Note that \(D_{m,k}(P) - \{z_1, \ldots, z_m\}\) is the poset obtained from \(D_{k}(P)\) by adding a new element \(y_0\) which covers \(y_1\). Let \(A_1\) (resp. \(A_2\)) be the set of \(P\)-partitions \(\sigma\) of \(D_{m,k}(P) - \{z_1, \ldots, z_m\}\) such that \(\sigma(y_0) < \sigma(z_0)\) (resp. \(\sigma(y_0) \geq \sigma(z_0)\)). Then

\[
\text{GF}_q(D_{m,k}(P) - \{z_1, \ldots, z_m\}) = \sum_{\sigma \in A_1} q^{\vert \sigma \vert} + \sum_{\sigma \in A_2} q^{\vert \sigma \vert}.
\]

By considering the value of \(\sigma(y_0)\) separately, one can see that

\[
\sum_{\sigma \in A_1} q^{\vert \sigma \vert} = \sum_{\sigma \in A_1} q^{t} \sum_{\sigma \in \mathcal{P}_{\geq t+1}(D_{k}(P))} q^{\vert \sigma \vert} = \sum_{t \geq 0} q^{t} \sum_{\sigma \in \mathcal{P}(D_{k}(P))} q^{\vert \sigma \vert} = \frac{q^{p+k+1}}{1 - q^{p+k+2}} \text{GF}_q(D_{k}(P)).
\]

Let \(D'_{k}(P)\) be the poset obtained from \(D_{k}(P)\) by adding a new element \(y_0\) and the covering relations \(y_1 < y_0 < z_0\). By Lemma 3.2

\[
\sum_{\sigma \in A_2} q^{\vert \sigma \vert} = \sum_{\sigma \in \mathcal{P}(D'_{k}(P))} q^{\vert \sigma \vert} = \frac{1}{1 - q^{p+k+2}} \sum_{\sigma \in \mathcal{P}(D'_{k}(P)^-)} q^{\vert \sigma \vert} = \frac{1}{1 - q^{p+k+2}} \left( \sum_{\sigma \in D_{k}(P)} q^{\vert \sigma \vert} + \sum_{\sigma \in B_2} q^{\vert \sigma \vert} \right),
\]

where \(B_2\) is the set of \(P\)-partitions \(\sigma\) of \(D_{k}(P)^-\) such that \(\sigma(y_0) < \sigma(z_0)\).
where $B_1$ (resp. $B_2$) is the set of $\sigma \in \mathcal{P}(\mathcal{D}_k^i(P))$ such that $\sigma(y_0) \geq \sigma(z_{-1})$ (resp. $\sigma(y_0) < \sigma(z_{-1})$). First, observe that
\[ \sum_{\sigma \in B_1} q^{\lvert \sigma \rvert} = GF_q(D_k(P)). \]

For the sum over $\sigma \in B_2$, by considering the value of $\sigma(z_{-1})$ separately, we have
\[
\sum_{\sigma \in B_2} q^{\lvert \sigma \rvert} = \sum_{t \geq 0} q^t \sum_{\sigma \in \mathcal{P}_{\geq t+1}(P^+)} q^{\lvert \sigma \rvert} \sum_{\sigma' \in \mathcal{P}_{\geq t}(C_{k-1})} q^{\lvert \sigma' \rvert}
= \sum_{t \geq 0} q^{t+(t+1)(p+1)+(k-1)} \sum_{\sigma \in \mathcal{P}(P^+)} q^{\lvert \sigma \rvert} \sum_{\sigma' \in \mathcal{P}(C_{k-1})} q^{\lvert \sigma' \rvert}
= \frac{q^{k+1}}{1 - q^{p+k+1} - \frac{1}{(q; q)_{k-1}}} GF_q(P^+),
\]
where $C_{k-1}$ is a chain with $k-1$ elements. By combining the above identities, we obtain the stated formula for $GF_q(D_{m,k}(P))$. \hfill \Box

4. SEMI-IRREDUCIBLE D-COMPLETE POSETS

In this section we introduce semi-irreducible d-complete posets. We show that in order to prove the hook length property of the d-complete posets it suffices to consider the semi-irreducible d-complete posets.

**Definition 4.1.** A d-complete poset $P$ is semi-irreducible if it is obtained from an irreducible d-complete poset by attaching a chain with arbitrary number of elements (possibly 0) below each acyclic element.

We give the complete description of 15 semi-irreducible d-complete poset classes in Section 6 upon the computation of the $q$-integrals. These include all irreducible d-complete posets. See Appendix A for the figures of the semi-irreducible d-complete posets.

We will prove the following theorem by a case-by-case analysis in Sections 6 and 7.

**Theorem 4.2.** Every semi-irreducible d-complete poset has the hook length property.

Now we show that the above theorem implies the hook length properties of the d-complete posets. First we need the following lemma.

**Lemma 4.3.** Let $P_0$ be an irreducible d-complete poset with $k$ acyclic elements $y_1, \ldots, y_k$. Suppose that $P_1, \ldots, P_k$ are (possibly empty) connected d-complete posets having the hook length property. Let $P$ be the poset obtained from $P_0$ by attaching $P_i$ below $y_i$ for each $1 \leq i \leq k$, i.e.,
\[ P = (\cdots (P_0 y_{v_1} \backslash y_{v_1} P_1 y_{v_2} \backslash y_{v_2} P_2) \cdots y_{v_k} \backslash y_{v_k} P_k), \]
where $v_i$ is the unique maximal element of $P_i$. Then $P$ also has the hook length property.

**Proof.** First, observe that
\[
GF_q(P) = \sum_{\sigma \in \mathcal{P}(P)} q^{\lvert \sigma \rvert} = \sum_{t_1, \ldots, t_k \geq 0} \sum_{\sigma_0 \in \mathcal{P}(P_0)} \prod_{i=1}^k q^{\lvert \sigma_{i} \rvert} q^{\lvert \sigma_{i} \rvert} q^{\lvert \sigma_{i} \rvert} q^{\lvert \sigma_{i} \rvert} q^{\lvert \sigma_{i} \rvert} q^{\lvert \sigma_{i} \rvert}
\]
Let $p_i = \lvert P_i \rvert$ for $1 \leq i \leq k$. Since each $P_i$ has the hook length property, we have
\[
\sum_{\sigma_i \in \mathcal{P}_{\geq t_i}(P_i)} q^{\lvert \sigma_i \rvert} = q^{t_i p_i} \sum_{\sigma_i \in \mathcal{P}(P_i)} q^{\lvert \sigma_i \rvert} = q^{t_i p_i} \prod_{u \in P_i} \frac{1}{1 - q^{h(u)}}.
\]
Then by (3) and (4) we have
\[
GF_q(P) = \sum_{t_1, \ldots, t_k \geq 0} \sum_{\sigma_0 \in \mathcal{P}(P_0)} \prod_{i=1}^k q^{t_i p_i} \prod_{u \in P_i} \frac{1}{1 - q^{h(u)}}.
\]
For an integer \( n \geq 0 \), denote by \( C_n \) a chain with \( n \) elements. Using the relation
\[
\sum_{\sigma_i \in \mathcal{P}_{\geq t_i}(C_{p_i})} q^{\left|\sigma_i\right|} = \frac{q^{i_p}}{(q; q)_{p_i}},
\]
we can rewrite (5) as
\[
\text{(6)} \quad \text{GF}_q(P) = \sum_{t_1, \ldots, t_k \geq 0} \sum_{\sigma_i \in \mathcal{P}(P_0)} q^{\left|\sigma\right|} \prod_{i=1}^{k} \left( (q; q)_{p_i} \sum_{\sigma_i \in \mathcal{P}_{\geq t_i}(C_{p_i})} q^{\left|\sigma_i\right|} \prod_{u \in C_{p_i}} \frac{1}{1 - q^{h(u)}} \right).
\]

Let \( P' \) be the semi-irreducible \( d \)-complete poset obtained from \( P_0 \) by attaching \( C_{p_i} \) below \( y_i \) for each \( i \). Then
\[
\text{(7)} \quad \text{GF}_q(P') = \sum_{t_1, \ldots, t_k \geq 0} \sum_{\sigma_i \in \mathcal{P}(P_0)} q^{\left|\sigma\right|} \prod_{i=1}^{k} \sum_{\sigma_i \in \mathcal{P}_{\geq t_i}(C_{p_i})} q^{\left|\sigma_i\right|} \prod_{u \in C_{p_i}} \frac{1}{1 - q^{h(u)}}.
\]

By (6) and (7), we obtain
\[
\text{(8)} \quad \text{GF}_q(P) = \text{GF}_q(P') \prod_{i=1}^{k} \left( (q; q)_{p_i} \prod_{u \in C_{p_i}} \frac{1}{1 - q^{h(u)}} \right).
\]

On the other hand, by Theorem 4.2
\[
\text{(9)} \quad \text{GF}_q(P') = \prod_{u \in P_0} \frac{1}{1 - q^{h(u)}} = \prod_{u \in P_0} \frac{1}{1 - q^{h(u')}} \prod_{i=1}^{k} \prod_{u \in C_{p_i}} \frac{1}{1 - q^{h(u)}} = \prod_{u \in P_0} \prod_{i=1}^{k} \frac{1}{1 - q^{h(u')}} \prod_{i=1}^{k} \frac{1}{1 - q^{h(u)}}.
\]

By (8) and (9), we obtain
\[
\text{(10)} \quad \text{GF}_q(P) = \prod_{u \in P_0} \frac{1}{1 - q^{h(u')}} \prod_{i=1}^{k} \prod_{u \in C_{p_i}} \frac{1}{1 - q^{h(u)}}.
\]

Since \( y_i \)'s are acyclic elements, we have \( h_{P'}(u) = h_{P}(u) \) for all \( u \in P_0 \). Thus the right hand side of (10) is equal to \( \prod_{u \in P_1} 1/(1 - q^{h(u)}) \) and \( P \) has the hook length property.

Assuming Theorem 4.2 we can prove the hook length formula of the \( d \)-complete posets.

**Theorem 4.4.** Every \( d \)-complete poset has the hook length property.

**Proof.** Let \( P \) be a \( d \)-complete poset. We prove the theorem by induction on the number of irreducible components of \( P \). If \( P \) is disconnected, by induction hypothesis each connected component has the hook length property. Now we assume that \( P \) is connected. If \( P \) is irreducible, the theorem is true by Theorem 4.2. Suppose that \( P \) is not irreducible. Let \( P_0 \) be the irreducible component of \( P \) containing the unique maximum element of \( P \). Then \( P \) is obtained from \( P_0 \) by attaching several (possibly empty) \( d \)-complete posets below each acyclic element of \( P_0 \). By induction hypothesis, the attached \( d \)-complete posets have the hook length property. By Lemma 4.3 \( P \) also has the hook length property, which completes the proof.

The rest of this paper is devoted to prove Theorem 4.2.

5. **\( q \)-integrals**

In this section we prove that the \( P \)-partition generating function for \( P_n(X) \) can be expressed as a \( q \)-integral, where \( P_n(X) \) is the poset in Definition 2.2.

The **\( q \)-integral** of a function \( f(x) \) over \([a, b] \) is defined by
\[
\int_a^b f(x) d_q x = (1 - q) \sum_{i=0}^{\infty} \left( f(bq^i) - f(aq^i) \right),
\]
where it is assumed that \( 0 < q < 1 \) and the sum absolutely converges.
For a multivariable function \( f(x_1, \ldots, x_n) \) and a partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \), we denote
\[
f(q^\lambda) = f(q^{\lambda_1}, \ldots, q^{\lambda_n}).
\]

We define the multivariate \( q \)-integral over the simplex \( \{(x_1, \ldots, x_n) : 0 \leq x_1 \leq \cdots \leq x_n \leq 1\} \) by
\[
\int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} f(x_1, \ldots, x_n) dx_1 \cdots dx_n = \int_0^1 \int_0^{x_1} \cdots \int_0^{x_n} f(x_1, \ldots, x_n) dx_1 \cdots dx_n.
\]

By Lemma 5.1, the left hand side is equal to \( \lambda \) Lemma 5.4.

Let \( \mu \leq \lambda \). We have
\[
\int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} f(x_1, \ldots, x_n) dx_1 \cdots dx_n = (1 - q)^n \sum_{\mu \in \text{Par}_n} q^{||\mu||} f(q^{\mu}).
\]

**Remark 5.2.** The definition of a multiple \( q \)-integral naturally extends to a multiple \( q \)-integral over a convex polytope (see [4]). The \( q \)-integral in Lemma 5.1 is over the order polytope of a chain. Kim and Stanton [4, Theorem 4.1] showed that for a poset \( P \), the \( q \)-integral over the order polytope of \( P \) is the generating function for the \( (P, \omega) \)-partitions, where \( \omega \) is the labeling of \( P \) determined by the order of integration.

The following lemma will be used in Section 7.

**Lemma 5.3.** Let \( f(x_1, \ldots, x_{n-1}) \) be a homogeneous function of degree \( d \) in variables \( x_1, \ldots, x_{n-1} \), i.e., \( f(tx_1, \ldots, tx_{n-1}) = t^d f(x_1, \ldots, x_{n-1}) \). Then
\[
\int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} f(x_1, \ldots, x_{n-1}) x_1^k dx_1 \cdots dx_n x_n = \frac{1 - q}{1 - q^{n+k+d}} \int_{0 \leq x_1 \leq \cdots \leq x_{n-1} \leq 1} f(x_1, \ldots, x_{n-1}) dx_1 \cdots dx_{n-1}.
\]

**Proof.** By Lemma 5.1 the left hand side is equal to
\[
(1 - q)^n \sum_{\mu \geq \cdots \geq \mu_n \geq 0} q^{||\mu||+k\mu_n} f(q^{\mu_1}, \ldots, q^{\mu_{n-1}})
\]

Considering \( \lambda = (\lambda_1, \ldots, \lambda_{n-1}) \) given by \( \lambda_i = \mu_i - \mu_n \), the above sum can be written as
\[
(1 - q)^n \sum_{\mu_0 = 0}^{\infty} q^{\mu_0(n+k+d)} \sum_{\lambda \in \text{Par}_{n-1}} q^{||\lambda||} f(q^{\lambda}).
\]

By Lemma 5.1 again, the above sum is equal to the right hand side of the formula in this lemma. \( \square \)

Recall that that every semi-irreducible \( d \)-complete poset can be written as \( P_n^m(X) \). By Lemma 3.2 we have
\[
\text{GF}_q(P_n^m(X)) = \frac{1}{(q^{P_n(X)} + 1, q)_m} \text{GF}_q(P_n(X)).
\]

We will show that \( \text{GF}_q(P_n(X)) \) can be written as a \( q \)-integral. First, we need some lemmas.

**Lemma 5.4.** Let \( n \) and \( k \) be positive integers and
\[
X = \{(\lambda^{(i)}, n_i, s_i) : 1 \leq i \leq k\},
\]
where \( n_i \) and \( s_i \) are positive integers with \( s_i + n_i - 1 \leq n \) and \( \lambda^{(i)} \in \text{Par}_{n_i} \). For \( \mu = (\mu_1, \ldots, \mu_n) \in \text{Par}_n \), let
\[
\mu^{[i]} = (\mu_{s_i}, \mu_{s_i+1}, \ldots, \mu_{s_i+n_i-1}).
\]

Then we have
\[
\text{GF}_q(P_n(X)) = q^{-\sum_{i=1}^n (n-i)\ell_i} \sum_{\mu \in \text{Par}_n} q^{||\mu||} \prod_{i=1}^k \sum_{T \in \text{SSYT}((\delta_{n_i+1, \lambda^{(i)}}),\text{rdiag}(T) = \mu^{[i]})} q^{T[-||\mu^{[i]}||]},
\]
where \( \ell_i \) is the number of elements of level \( i \) in \( P_n(X) \).

**Proof.** Let \( P_0 \) be the set of \( P_n(X) \)-partitions. Let \( P_1 \) be the set of \( P_n(X) \)-partitions \( \tau \) with the condition that if an element \( x \) of level \( i \) covers an element \( y \) of level \( i - 1 \), then \( \tau(x) < \tau(y) \). Let \( \sigma \in P_0 \) be the map \( \tau : P_n(X) \rightarrow \mathbb{N} \) given by \( \tau(x) = \sigma(x) + n - \ell(x) \), where \( \ell(x) \) is the level of \( x \). It is not hard to verify that the map \( \tau \) is a bijection from \( P_0 \) to \( P_1 \) with \( |f(\sigma)| = |\sigma| + \sum_{i=1}^n (n - i)\ell_i \). Thus, we have

\[
\text{GF}_q(P_n(X)) = q^{-\sum_{i=1}^n (n-i)\ell_i} \sum_{\tau \in P_1} q^{|\tau|}. 
\]

For every \( \tau \in P_1 \), we have \( \tau(x_1) > \tau(x_2) > \cdots > \tau(x_n) \). By the construction of \( P_n(X) \), one can easily see that

\[
\sum_{\tau \in P_1} q^{|\tau|} = \sum_{\mu \in \text{Par}_n} q^{|\mu|} \prod_{i=1}^k \sum_{T \in \text{SSYT}((\delta_n + \lambda)^*)} q^{|T|-|\mu[i]|}, 
\]

which completes the proof.

**Lemma 5.5.** For \( \lambda, \mu \in \text{Par}_n \) we have

\[
\sum_{T \in \text{SSYT}((\delta_n + \lambda)^*)} q^{|T|-|\mu|} = \frac{(-1)^{\left(\frac{n}{2}\right)} a_{\lambda+\delta_n}(q^n)}{\prod_{j=1}^{n} (q; q)_{\lambda_j+n-j}}. 
\]

**Proof.** If \( \mu \) has two equal parts, then both sides are zero. Thus it suffices to consider the case \( \mu = \nu + \delta_n \) for some \( \nu \in \text{Par}_n \). Subtracting \( i - 1 \) from every entry in the \( i \)-th row gives a bijection from \( \text{SSYT}((\delta_n + \lambda)^*) \) to \( \text{RPP}((\delta_n + \lambda)^*) \). Thus

\[
\sum_{T \in \text{SSYT}((\delta_n + \lambda)^*)} q^{|T|} = \sum_{T \in \text{RPP}((\delta_n + \lambda)^*)} q^{|T|+\sum_{i=1}^n (i-1)(\lambda_i+n+1-i)}. 
\]

In [4, Theorem 8.7], it is shown that

\[
\sum_{T \in \text{RPP}((\delta_n + \lambda)^*)} q^{|T|} = \frac{q^{\nu+\delta_n}-\sum_{i=1}^n (i-1)(\lambda_i+n+1-i)}{\prod_{j=1}^{n} (q; q)_{\lambda_j+n-j}}. 
\]

By the above two identities, we obtain the lemma.

The following result implies that \( \text{GF}_q(P_n(X)) \) can be expressed as a \( q \)-integral.

**Theorem 5.6.** Let \( n \) and \( k \) be positive integers and

\[
X = \{ (\lambda^{(i)}, n_i, s_i) : 1 \leq i \leq k \},
\]

where \( n_i \) and \( s_i \) are positive integers with \( s_i + n_i - 1 \leq n \), \( \lambda^{(i)} \) is a partition with \( n_i \) parts. Suppose that for every \( 1 \leq j \leq n - 1 \), there is \( 1 \leq i \leq k - 1 \) such that \( s_i \leq j < j + 1 \leq s_i + n_i - 1 \). Then

\[
\text{GF}_q(P_n(X))
\]

\[
= \frac{q^{-\sum_{i=1}^n (n-i)\ell_i}}{(1-q)^{n}} \int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} \prod_{i=1}^k \frac{(-1)^{\left(\frac{n}{2}\right)} a_{\lambda^{(i)}+\delta_n}(x_{s_i}, x_{s_i+1}, \ldots, x_{s_i+n_i-1})}{\prod_{j=1}^{n} (q; q)_{\lambda_j^{(i)}+n_i-j}} dx_1 \cdots dx_n,
\]

where \( \ell_i \) is the number of elements of level \( i \) in \( P_n(X) \).

**Proof.** Let

\[
f(x_1, \ldots, x_n) = \prod_{i=1}^k \frac{(-1)^{\left(\frac{n}{2}\right)} a_{\lambda^{(i)}+\delta_n}(x_{s_i}, x_{s_i+1}, \ldots, x_{s_i+n_i-1})}{\prod_{j=1}^{n} (q; q)_{\lambda_j^{(i)}+n_i-j}}.
\]
By Lemmas 5.4 and 5.5 the left hand side is
\[ \text{GF}_q(P(X)) = q^{-\sum_{i=1}^{n}(n-i)\ell_i} \sum_{\mu \in \text{Par}_n, \mu \text{ strict}} q^{\|\mu\|} f(q^{\mu}). \]

On the other hand, the right hand side is
\[ \frac{q^{-\sum_{i=1}^{n}(n-i)\ell_i}}{(1-q)^n} \int_{0 \leq x_1 \leq \ldots \leq x_n \leq 1} f(x_1, \ldots, x_n) d_q x_1 \cdots d_q x_n = q^{-\sum_{i=1}^{n}(n-i)\ell_i} \sum_{\mu \in \text{Par}_n} q^{\|\mu\|} f(q^{\mu}). \]

If \( \mu_i = \mu_j \) for some \( i < j \), then we have \( \mu_i = \mu_{i+1} \). In this case, by the given condition, we have \( f(q^{\mu}) = 0 \). Therefore, we only need to consider the partitions \( \mu \in \text{Par}_n \) which are strict. Hence, we obtain the desired identity. \( \square \)

6. \( P \)-partition generating functions to \( q \)-integrals

In this section we express the \( P \)-partition generating function for each semi-irreducible \( d \)-complete poset as a \( q \)-integral using Theorem 5.6. See Appendix A for the list of all semi-irreducible \( d \)-complete posets and their figures.

Note that for a partition \( \lambda \in \text{Par}_n \), we have
\[ \prod_{u \in \lambda} \frac{1}{1-q^{h(u)}} = \frac{\prod_{1 \leq i < j \leq n} (1-q^{\lambda_i+\lambda_j+1})}{\prod_{i=1}^{n} (q;q)_{\lambda_i+n-i}}, \]

(11) \[ \prod_{u \in (\lambda+\delta_n+1)^\ast} \frac{1}{1-q^{h(u)}} = \frac{\prod_{1 \leq i < j \leq n} (1-q^{\lambda_i+\lambda_j+1})}{\prod_{i=1}^{n} (q;q)_{\lambda_i+n-i} \prod_{1 \leq i \leq j \leq n} (1-q^{2n+1-i-j+\lambda_i+\lambda_j})}. \]

In this section, when a \( d \)-complete poset is given, \( \ell_i \) denotes the number of elements of level \( i \) in the poset.

6.1. Class 1: Shapes. A semi-irreducible \( d \)-complete poset of class 1 is \( P_n(X_1) \), where \( n \geq 3 \) and
\[ X_1 = \{(\lambda, n, 1), (\mu, n, 1)\}, \]
with \( \lambda, \mu \in \text{Par}_n \), see Figure 10. For \( 1 \leq i \leq n \), we have \( \ell_i = \lambda_{n+1-i} + \mu_{n+1-i} + 2i - 1 \). By Theorem 5.6
\[ \text{GF}_q(P_n(X_1)) = \frac{q^{-\binom{2}{2}}-\sum_{i=1}^{n-1} (\lambda_{i+1}+\mu_{i+1})}{(1-q)^n} \int_{0 \leq x_1 \leq \ldots \leq x_n \leq 1} a_{\lambda+\delta_n}(x_1, \ldots, x_n) a_{\mu+\delta_n}(x_1, \ldots, x_n) d_q x_1 \cdots d_q x_n. \]

Since
\[ \prod_{u \in P_n(X_1)} \frac{1}{1-q^{h(u)}} = \frac{\prod_{1 \leq i < j \leq n} (1-q^{\lambda_i+\lambda_j+1})(1-q^{\mu_i+\mu_j+1})}{\prod_{i=1}^{n} (q;q)_{\lambda_i+n-i} (q;q)_{\mu_i+n-i} \prod_{i,j=1}^{n} (1-q^{\lambda_i+\lambda_j+2n-i-j+1})}, \]
the hook length property for class 1 is equivalent to
\[ \int_{0 \leq x_1 \leq \ldots \leq x_n \leq 1} a_{\lambda+\delta_n}(x_1, \ldots, x_n) a_{\mu+\delta_n}(x_1, \ldots, x_n) d_q x_1 \cdots d_q x_n \]
\[ = q^{\binom{2}{2}+\sum_{i=1}^{n-1} (\lambda_{i+1}+\mu_{i+1})} \frac{\prod_{1 \leq i < j \leq n} (1-q^{\lambda_i+\lambda_j+1})(1-q^{\mu_i+\mu_j+1})}{\prod_{i,j=1}^{n} (1-q^{\lambda_i+\lambda_j+2n-i-j+1})}. \]

This is the special case \( k = 1 \) of Warnaar’s integral [15, Theorem 1.4].
6.2. Class 2: Shifted shapes. A semi-irreducible $d$-complete poset of class 2 is $P_n(X_2)$, where $n \geq 4$ and 

$$X_2 = \{(\mu, n, 1)\},$$

with $\mu \in \text{Par}_n$, see Figure 11. For $1 \leq i \leq n$, we have $\ell_i = \mu_{n+1-i} + 1$. By Theorem 5.6

$$\text{GF}_q(P_n(X_2)) = \frac{q^{-\sum_{i=1}^{n-1} \mu_{n+1-i}}}{(1-q)^n} \prod_{1 \leq i < j \leq n} (1-q^{\mu_i - \mu_j + 1}),$$

which is proved in [4, Theorem 8.16] using the connection between reverse plane partitions and $q$-integrals. In Section 4 we give a new proof of [12] by explicitly computing the $q$-integral.

6.3. Class 3: Birds. A semi-irreducible $d$-complete poset of class 3 is $P_n^m(X_3)$, where 

$$X_3 = \{((\lambda, 1), (\mu, b, 1), (m), 1, 1)\},$$

with $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$, see Figure 12.

We have $\ell_1 = \lambda_2 + \mu_2 + m + 1$, $\ell_2 = \lambda_1 + \mu_1 + 3$. Thus

$$\text{GF}_q(P(X_3)) = \frac{q^{-\sum_{i=1}^{n-1} \mu_{n+1-i} + 1}}{(1-q)^n} \prod_{1 \leq i < j \leq n} (1-q^{\mu_i - \mu_j + 1}),$$

The hook length property for Class 3 is equivalent to

$$\text{GF}_q(P^m(X_3)) = \frac{q^{-(\lambda_2 + \mu_2 + m + 1)}}{(1-q)^{\lambda_1 + \mu_1 + 1}} \prod_{1 \leq i < j \leq n} (1-q^{\mu_i - \mu_j + 1}),$$

or

$$\int_{0 \leq x_1 \leq \cdots \leq x_n} a_{\lambda + \delta_2}(x_1, \ldots, x_n) \prod_{1 \leq i < j \leq n} (1-q^{\mu_i - \mu_j + 1}) \prod_{1 \leq i < j \leq n} (1-q^{\mu_i - \mu_j + 1}).$$

This formula has been verified by computer.

6.4. Class 4: Insects. A semi-irreducible $d$-complete poset of class 4 is $P_{n+1}^m(X_4)$, where $n \geq 2$, $k \geq 0$ and

$$X_4 = \{((\lambda, n-1, 1), (\mu, n+1, 1), (k), 1, n)\},$$

with $\lambda \in \text{Par}_{n-1}$ and $\mu \in \text{Par}_{n+1}$, see Figure 13. In this poset, $\ell_j = \lambda_{n-j} + \mu_{n-j+2} + 2j - 1$ for $1 \leq j \leq n - 1$, $\ell_n = \mu_2 + n + k$ and $\ell_{n+1} = \mu_1 + n + 1$. By applying Lemma 3.2 and Theorem 5.6 we obtain

$$\text{GF}_q(P^m_{n+1}(X_4)) = \frac{1}{(q^{1+|\lambda|+n^2+k+3}; q)_{\lambda_1+\mu_2}} \text{GF}_q(P^m_{n+1}(X_4)),$$

where

$$\text{GF}_q(P_{n+1}(X_4)) = \frac{1}{(q^{1+|\lambda|+n^2+k+3}; q)_{\lambda_1+\mu_2}} \prod_{0 \leq x_1 \leq \cdots \leq x_n+1} (1-q)^n \prod_{1 \leq i < j \leq n} \frac{a_{\lambda_1}(x_1, \ldots, x_n)}{(q;q)_k}.$$
An explicit computation of the hook lengths in the poset $P_{n+1}^{λ_1+n-2}(X_4)$ gives

$$\prod_{u \in P_{n+1}^{λ_1+n-2}(X_4)} \frac{1}{1 - q^{h(u)}} = \frac{1}{(q; q)_{λ_1+n-1}} \prod_{i=1}^{n+1} \frac{1}{1 - q^{λ_1+λ_i+n(n-1)+k+1}} \prod_{1 \leq i < j \leq n+1} (1 - q^{λ_1-λ_i+j})^n \prod_{1 \leq i < j \leq n+1} (1 - q^{λ_i+λ_j+2n-i+j+1})$$

Thus the hook length property for class 4 is equivalent to

$$(13) \int_{0 \leq x_1 \leq \cdots \leq x_{n+1} \leq 1} x_1^k a_{λ+δ_{n-1}}(x_1, \ldots, x_{n-1}) a_{μ+δ_{n-1}}(x_1, \ldots, x_{n-1}) d_q x_1 \cdots d_q x_{n+1} = (-1)^{\sum_{i=1}^{n} (i+1)α_i + n(n-1)/2} \prod_{i=1}^{n+1} (1 - q^{λ_1+λ_i+n(n-1)+k+1}) \prod_{1 \leq i < j \leq n+1} (1 - q^{λ_1-λ_i+j})^n \prod_{1 \leq i < j \leq n+1} (1 - q^{λ_i+λ_j+2n-i+j+1}) \int_{0 \leq x_1 \leq \cdots \leq x_{n+1} \leq 1} x_1^k a_{μ+δ_{n-1}}(x_1, x_2) a_{λ+δ_{n-1}}(x_1, x_2) d_q x_1 d_q x_2 d_q x_3.$$

We prove this identity in Section 7.5.

6.5. Class 5: Tailed inets. A semi-irreducible $d$-complete poset of class 5 is $P_{λ_1}^{λ_1+1}(X_5)$ with $λ \in \text{Par}_2, \ μ \in \text{Par}_3$ and

$$X_5 = \{λ, 2, 1), (μ, 3, 1), (θ, 2, 2), ((1, 1, 1)\}.$$ See Figure 4. In this poset, $λ_1 = λ_2 + μ_3 + 2, \ λ_2 = λ_1 + μ_2 + 3$ and $λ_3 = μ_1 + 4$. In this case,

$$\text{GF}_q(P_{λ_1}^{λ_1+1}(X_5)) = \frac{1}{(q^{λ_1+μ_1+10}; q)_{λ_1+1}} \text{GF}_q(P_3(X_5)),$$

where

$$\text{GF}_q(P_3(X_5)) = \frac{q^{-\sum_{i=1}^{n} (i+1)(λ_i+μ_{i+1})+7}}{(1 - q)^3} \int_{0 \leq x_1 \leq \cdots \leq x_{n+1} \leq 1} \frac{-a_{λ+δ_2}(x_1, x_2)}{(q; q)_{λ_1+1}(q; q)_{λ_2}} \prod_{i=1}^{n} \frac{-a_{μ+δ_2}(x_1, x_2, x_3)}{(q; q)_{μ_1+3-j}} \prod_{i=1}^{n} \frac{-a_{δ_2}(x_1, x_2, x_3)}{(q; q)_{δ_1+δ_1+1}} d_q x_1 d_q x_2 d_q x_3.$$

Then the hook length property for class 5 is equivalent to the following identity

$$\int_{0 \leq x_1 \leq \cdots \leq x_{n+1} \leq 1} x_1 a_{δ_2}(x_2, x_3) a_{λ+δ_2}(x_1, x_2) a_{μ+δ_2}(x_1, x_2, x_3) d_q x_1 d_q x_2 d_q x_3 = (-1)^{\sum_{i=1}^{n} (i+1)(λ_i+μ_{i+1})+7} (1 - q^4) \prod_{i=1}^{n} (1 - q^{λ_1-λ_i+1})(1 - q^{λ_1+μ_1+10})(1 - q^{λ_1+μ_3+9}) \prod_{i=1}^{n} (1 - q^{λ_1-λ_i+1})(1 - q^{λ_1+μ_1+7-i-j}) \prod_{i=1}^{n} (1 - q^{λ_1+μ_3+7-i-j}) \prod_{i=1}^{n} (1 - q^{λ_1+μ_3+4-i-j}).$$

This formula has been verified by computer.
6.6. **Class 6: Banners.** A semi-irreducible \(d\)-complete poset of class 6 is \(P^m_4(X_6)\) with \(m \geq 0\) and
\[
X_6 = \{(\mu, 4, 1), (m, 1, 2)\}
\]
for \(\mu \in \text{Par}_4\). See Figure 15. In this poset, we have \(\ell_1 = \mu + 1, \ell_2 = \mu_3 + m + 2, \ell_3 = \mu_2 + 3\) and \(\ell_4 = \mu + 4\). Then
\[
\text{GF}_q(P^m_4(X_6)) = \frac{1}{(q^{\mu_1+m+11}; q)_m} \text{GF}_q(P_4(X_6)),
\]
where
\[
\text{GF}_q(P_4(X_6)) = \frac{q^{-(\sum_{i=1}^3 \mu_i+1+2m+10)}}{(1-q)^4} \int_{0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 1} a_{\mu+\delta_4}(x_1, x_2, x_3, x_4) \prod_{j=1}^4 (q; q)_{\mu_j+4-j} \times \frac{a_{\delta_3}(x_2)}{(q; q)_m} d_q x_1 d_q x_2 d_q x_3 d_q x_4.
\]
Hence the hook length property for class 6 is equivalent to the following identity
\[
\int_{0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 1} x_2^2 a_{\mu+\delta_4}(x_1, x_2, x_3, x_4) d_q x_1 d_q x_2 d_q x_3 d_q x_4 = \frac{q^{-(\sum_{i=1}^3 \mu_i+1+2m+10)}}{(1-q)^4} \prod_{i=1}^4 (1-q^{\mu_i+5-i}) \prod_{1 \leq i < j \leq 4} 1 - q^{\mu_i-\mu_j+j-i}.
\]
This formula has been verified by computer.

6.7. **Class 7: Nooks.** In this case, a semi-irreducible \(d\)-complete poset of class 7 is \(P^{\lambda_1+2}_4(X_7)\) with
\[
X_7 = \{(\lambda, 3, 1), (\emptyset, 2, 1), (\mu, 3, 2), (\emptyset, 2, 3)\},
\]
where \(\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \text{Par}_3\) and \(\mu = (\mu_1, \mu_2, 0) \in \text{Par}_2\). See Figure 16. Also, we have \(\ell_1 = \lambda_3 + 1, \ell_2 = \lambda_2 + 3, \ell_3 = \lambda_1 + \mu_2 + 4\) and \(\ell_4 = \mu_1 + 4\). Thus,
\[
\text{GF}_q(P^{\lambda_1+2}_4(X_7)) = \frac{1}{(q^{\lambda_1+\mu+13}; q)_{\lambda_1+2}} \text{GF}_q(P_4(X_7)),
\]
where
\[
\text{GF}_q(P_4(X_7)) = \frac{q^{-(\sum_{i=1}^3 \lambda_i+\mu+13)}}{(1-q)^4} \int_{0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 1} a_{\lambda+\delta_3}(x_1, x_2, x_3) \prod_{i=1}^4 (q; q)_{\lambda_i+3-i} \times \frac{a_{\delta_2}(x_1, x_2)}{(q; q)_{\mu_1+2}} \frac{a_{\delta_3}(x_2, x_3, x_4)}{(q; q)_{\mu_2+1}} \frac{a_{\delta_4}(x_3, x_4)}{(1-q)} d_q x_1 d_q x_2 d_q x_3 d_q x_4.
\]
The hook length property for class 7 means the following identity
\[
\int_{0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 1} (x_1 - x_2)(x_3 - x_4) a_{\lambda+\delta_3}(x_1, x_2, x_3) a_{\mu+\delta_3}(x_2, x_3, x_4) d_q x_1 d_q x_2 d_q x_3 d_q x_4
\]
\[
= \frac{q^{3 \sum_{i=1}^3 \lambda_i+\mu+13}}{(q^{\lambda_1+\mu+10}; q)_3} \prod_{i=1}^3 (1-q^{\lambda_i+\mu+15-i}) \prod_{1 \leq i < j \leq 3} (1-q^{\lambda_i-\lambda_j+j-i}) \times \prod_{i=1}^3 \prod_{j=1}^3 (1-q^{\lambda_i+\mu+13+3i+j}).
\]
This formula has been verified by computer.
6.8. **Class 8: Swivels.** For a semi-irreducible \( d \)-complete poset of class 8, there are 4 subclasses:

- **Class 8-(1):** \( P_4^{\lambda_1+2}(X_8^{(1)}) \) for \( \lambda \in \text{Par}_3 \) and
  \[ X_8^{(1)} = \{(\lambda, 3, 1), ((2), 1, 1), (\emptyset, 2, 1), (\emptyset, 3, 2), (\emptyset, 2, 3)\} \],

- **Class 8-(2):** \( P_5^{\lambda_1+3}(X_8^{(2)}) \) for \( \lambda \in \text{Par}_4 \) and
  \[ X_8^{(2)} = \{(\lambda, 4, 1), ((1, 0), 2, 1), (\emptyset, 2, 2), (\emptyset, 3, 3), (\emptyset, 2, 4)\} \],

- **Class 8-(3):** \( P_5^{\lambda_1+3}(X_8^{(3)}) \) for \( \lambda \in \text{Par}_4 \) and
  \[ X_8^{(3)} = \{(\lambda, 4, 1), ((1, 1), 2, 1), (\emptyset, 2, 2), (\emptyset, 3, 3), (\emptyset, 2, 4)\} \],

- **Class 8-(4):** \( P_6^{\lambda_1+4}(X_8^{(4)}) \) for \( \lambda \in \text{Par}_4 \) and
  \[ X_8^{(4)} = \{(\lambda, 5, 1), ((\emptyset, 3, 1), (\emptyset, 2, 3), (\emptyset, 3, 4), (\emptyset, 2, 5)\} \].

See Figures 17, 18, and 19. We consider the 4 subclasses separately.

**Class 8-(1):** In this case, \( \ell_1 = \lambda_3 + 3 \), \( \ell_2 = \lambda_2 + 3 \), \( \ell_3 = \lambda_1 + 4 \) and \( \ell_4 = 4 \). Then,

\[
\text{GF}_q(P_4^{\lambda_1+2}(X_8^{(1)})) = \frac{1}{(q^{\lambda_1+15}; q)_{\lambda_1+2}} \text{GF}_q(P_4(X_8^{(1)})),
\]

where

\[
\text{GF}_q(P_4(X_8^{(1)})) = \frac{q^{-\binom{\lambda_1+19}{1}}}{(1-q)^3} \int_{0 \leq x_1 \leq \cdots \leq x_4 \leq 1} -a_{\lambda, \delta_5}(x_1, x_2, x_3) \cdot \frac{a_{(2), \delta_5}(1)}{(1-q)(1-q^2)} \times \frac{-a_{\delta_5}(x_1, x_2)}{1-q} \cdot \frac{-a_{\delta_5}(x_2, x_3, x_4)}{(1-q)^2(1-q^2)} \cdot \frac{-a_{\delta_5}(x_3, x_4)}{1-q} \cdot dq x_1 \cdots dq x_4.
\]

Hence, the hook length property for class 8-(1) is equivalent to

\[
(14) \quad \int_{0 \leq x_1 \leq \cdots \leq x_4 \leq 1} x_1^2(x_1 - x_2)(x_2 - x_3 - x_4)dq x_1 \cdots dq x_4
\]

\[
= \frac{q^{\sum_{i=1}^{3} \lambda_i + 19}(1-q)^9(1+q)}{(q^{\lambda_1+10}; q)^3(1-q^{\lambda_1+14})} \cdot \prod_{i=1}^{3} \frac{1 - q^{\lambda_1+17-i}}{(q^{\lambda_1+6-i}; q)_2(q^{\lambda_1+5+i}; q)_2} \prod_{1 \leq i < j \leq 3} (1 - q^{\lambda_1-\lambda_j+j-i}).
\]

This formula has been verified by computer.

**Class 8-(2):** In this case, \( \ell_1 = \lambda_4 + 1 \), \( \ell_2 = \lambda_3 + 4 \), \( \ell_3 = \lambda_2 + 4 \), \( \ell_4 = \lambda_1 + 5 \) and \( \ell_5 = 4 \). Then,

\[
\text{GF}_q(P_5^{\lambda_1+3}(X_8^{(2)})) = \frac{1}{(q^{\lambda_1+19}; q)_{\lambda_1+3}} \text{GF}_q(P_5(X_8^{(2)})),
\]

where

\[
\text{GF}_q(P_5(X_8^{(2)})) = \frac{q^{-\binom{\lambda_1+29}{1}}}{(1-q)^5} \int_{0 \leq x_1 \leq \cdots \leq x_5 \leq 1} a_{\lambda_1+\delta}(x_1, x_2, x_3, x_4, x_5) \cdot \frac{-a_{(1,0)+\delta_5}(x_1, x_2)}{(1-q)(1-q^2)} \times \frac{-a_{\delta_5}(x_1, x_2, x_3)}{1-q} \cdot \frac{-a_{\delta_5}(x_3, x_4, x_5)}{(1-q)^2(1-q^2)} \cdot \frac{-a_{\delta_5}(x_3, x_4)}{1-q} \cdot dq x_1 \cdots dq x_5.
\]

Hence, the hook length property for class 8-(2) is equivalent to

\[
(15) \quad \int_{0 \leq x_1 \leq \cdots \leq x_5 \leq 1} (x_1^2 - x_2^2)(x_2 - x_3)(x_4 - x_5)dq x_1 \cdots dq x_5
\]

\[
= \frac{q^{\sum_{i=1}^{5} \lambda_i + 29}(1-q)^{11}(1+q)^2}{(q^{\lambda_1+15}; q)^4(1-q^{\lambda_1+18})} \cdot \prod_{i=1}^{4} \frac{1}{(q^{\lambda_1+5-i}; q^2)(1-q^{\lambda_1+9+i})} \prod_{1 \leq i < j \leq 4} (1 - q^{\lambda_1-\lambda_j+j-i}).
\]

This formula has been verified by computer.

**Class 8-(3):** In this case, \( \ell_1 = \lambda_4 + 2 \), \( \ell_2 = \lambda_3 + 4 \), \( \ell_3 = \lambda_2 + 4 \), \( \ell_4 = \lambda_1 + 5 \) and \( \ell_5 = 4 \). Then,

\[
\text{GF}_q(P_5^{\lambda_1+3}(X_8^{(3)})) = \frac{1}{(q^{\lambda_1+20}; q)_{\lambda_1+3}} \text{GF}_q(P_5(X_8^{(3)})),
\]
where
\[ GF_q(P_6(X_8^{(3)})) = \frac{q^{-(\sum_{i=1}^{4} \lambda_i + 33)}}{(1-q)^5} \int_{0 \leq x_1 \leq \ldots \leq x_6 \leq 1} \frac{a_{\lambda+\delta_1}(x_1, x_2, x_3, x_4)}{\prod_{i=1}^{4} (q; q)_{\lambda_i+4-i}} \frac{-a_{(1,1)+\delta_1}(x_1, x_2)}{(1-q)(1-q^2)} \times \frac{-a_{\delta_1}(x_2, x_3)}{1-q} \cdot \frac{-a_{\delta_1}(x_3, x_4, x_5)}{1-q} \cdot \frac{-a_{\delta_1}(x_4, x_5)}{1-q^2} = \frac{1}{(q; q)_{6}^{2}} \cdot \frac{1}{(1-q^2)} \cdot \frac{1}{1-q} \cdot \frac{1}{1-q^2} \cdot d_q x_1 \cdots d_q x_5. \]

Hence, the hook length property for class 8-(3) is equivalent to
\begin{equation}
\int_{0 \leq x_1 \leq \ldots \leq x_5 \leq 1} \frac{a_{\lambda+\delta_1}(x_1, x_2, x_3, x_4, x_5)}{\prod_{i=1}^{4} (q; q)_{\lambda_i+4-i}} \cdot \frac{-a_{(1,1)+\delta_1}(x_1, x_2, x_3, x_4, x_5)}{(1-q)(1-q^2)} \times \frac{-a_{\delta_1}(x_2, x_3, x_4, x_5)}{1-q} \cdot \frac{-a_{\delta_1}(x_4, x_5)}{1-q^2} = \frac{1}{(q; q)_{6}^{2}} \cdot \frac{1}{(1-q^2)} \cdot \frac{1}{1-q} \cdot \frac{1}{1-q^2} \cdot d_q x_1 \cdots d_q x_5.
\end{equation}

This formula has been verified by computer.

Class 8-(4): In this case, \( \ell_1 = \lambda_5 + 1, \ell_2 = \lambda_4 + 3, \ell_3 = \lambda_3 + 5, \ell_4 = \lambda_2 + 5, \ell_5 = \lambda_1 + 6 \) and \( \ell_6 = 4 \). Then,
\[ GF_q(P_6^{\lambda_1+4}(X_8^{(4)})) = \frac{1}{(q; q)_{5}^{2}} \cdot \frac{1}{(1-q^2)} \cdot \frac{1}{1-q} \cdot \frac{1}{1-q^2} \cdot d_q x_1 \cdots d_q x_6. \]

Hence, the hook length property for class 8-(4) is equivalent to
\begin{equation}
\int_{0 \leq x_1 \leq \ldots \leq x_6 \leq 1} \frac{a_{\lambda+\delta_1}(x_1, \ldots, x_6)}{\prod_{i=1}^{5} (q; q)_{\lambda_i+4-i}} \cdot \frac{-a_{(1,1)+\delta_1}(x_1, x_2, x_3)}{(1-q)(1-q^2)} \times \frac{-a_{\delta_1}(x_3, x_4, x_5)}{1-q} \cdot \frac{-a_{\delta_1}(x_4, x_5, x_6)}{1-q^2} = \frac{1}{(q; q)_{6}^{2}} \cdot \frac{1}{(1-q^2)} \cdot \frac{1}{1-q} \cdot \frac{1}{1-q^2} \cdot d_q x_1 \cdots d_q x_6.
\end{equation}

We provide a proof of the hook length property for class 8-(4) in Theorem 7.2 by decomposing the \( P \)-partition generating function for \( P_6^{\lambda_1+4}(X_8^{(4)}) \) using Lemma 3.4.

6.9. Class 9: Tailed swivels. For a semi-irreducible \( d \)-complete poset of class 9, there are 2 subclasses:

- Class 9-(1): \( P_5^{\lambda_1+3}(X_9^{(1)}) \), with \( \lambda \in \text{Par}_5 \) and
  \[ X_9^{(1)} = \{ (\lambda, 4, 1), ((1), 1, 1), ((1, 0), 2, 1), (\emptyset, 2, 2), (\emptyset, 3, 3), (\emptyset, 2, 4) \}. \]

- Class 9-(2): \( P_5^{\lambda_1+3}(X_9^{(2)}) \), with \( \lambda \in \text{Par}_5 \) and
  \[ X_9^{(2)} = \{ (\lambda, 4, 1), ((1), 1, 1), ((1, 1), 2, 1), (\emptyset, 2, 2), (\emptyset, 3, 3), (\emptyset, 2, 4) \}. \]

See Figures 21 and 22.

Class 9-(1): We have \( \ell_1 = \lambda_4 + 2, \ell_2 = \lambda_3 + 4, \ell_3 = \lambda_2 + 4, \ell_4 = \lambda_1 + 5 \) and \( \ell_5 = 4 \). Then,
\[ GF_q(P_5^{\lambda_1+3}(X_9^{(1)})) = \frac{1}{(q; q)_{5}^{2}} \cdot \frac{1}{(1-q^2)} \cdot \frac{1}{1-q} \cdot \frac{1}{1-q^2} \cdot d_q x_1 \cdots d_q x_6. \]

where
\[ GF_q(P_5(X_9^{(1)})) = \frac{q^{-(\sum_{i=1}^{4} \lambda_i + 34)}}{(1-q)^5} \int_{0 \leq x_1 \leq \ldots \leq x_5 \leq 1} \frac{a_{\lambda+\delta_1}(x_1, x_2, x_3, x_4)}{\prod_{i=1}^{4} (q; q)_{\lambda_i+4-i}} \cdot \frac{a_{(1,1)+\delta_1}(x_1)}{1-q}. \]
This formula has been verified by computer.

Hence the hook length property for class 9-(1) is equivalent to the identity

\[
\int_{0 \leq x_1 \leq \cdots \leq x_5 \leq 1} x_1 (x_2^2 - x_3^2) (x_2 - x_3)(x_4 - x_5) a_{\delta_1}(x_3, x_4, x_5) a_{\lambda + \delta_1}(x_1, x_2, x_3, x_4, x_5) \prod_{1 \leq i < j \leq 4} \frac{1 - q^{\lambda_i - \lambda_j - j - i}}{1 - q^{\lambda_i + \lambda_j + 13 - i - j}} d_q x_1 \cdots d_q x_5.
\]

This formula has been verified by computer.

Class 9-(2): We have \( \ell_1 = \lambda_4 + 3, \ell_2 = \lambda_3 + 4, \ell_3 = \lambda_2 + 4, \ell_4 = \lambda_1 + 5 \) and \( \ell_5 = 4 \). Then,

\[
GF_q(P_5^{\lambda_1+3}(X_9^{(2)})) = \frac{1}{(q^{\lambda_1+21}; q)_{\lambda_1+3}} \prod_{1 \leq i < j \leq 4} \frac{1 - q^{\lambda_i - \lambda_j - j - i}}{1 - q^{\lambda_i + \lambda_j + 13 - i - j}}
\]

Thus the hook length property for class 9-(2) is equivalent to the identity

\[
\int_{0 \leq x_1 \leq \cdots \leq x_5 \leq 1} x_1^2 (x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_4 - x_5) a_{\delta_1}(x_3, x_4, x_5) a_{\lambda + \delta_1}(x_1, x_2, x_3, x_4, x_5) \prod_{1 \leq i < j \leq 4} \frac{1 - q^{\lambda_i - \lambda_j - j - i}}{1 - q^{\lambda_i + \lambda_j + 13 - i - j}} d_q x_1 \cdots d_q x_5.
\]

This formula has been verified by computer.

6.10. Class 10: Tagged swivels. A semi-irreducible \( d \)-complete poset of class 10 is \( P_6^{\lambda_1+4}(X_{10}) \) with

\[
X_{10} = \{(\lambda, 5, 1), (\emptyset, 2, 1), ((1), 2, 2), (\emptyset, 2, 3), (\emptyset, 3, 4), (\emptyset, 2, 5)\},
\]

where \( \lambda \in \text{Par}_5 \) and \( \ell_1 = \lambda_5 + 1, \ell_2 = \lambda_4 + 3, \ell_3 = \lambda_3 + 5, \ell_4 = \lambda_2 + 5, \ell_5 = \lambda_1 + 6, \ell_6 = 4 \). See Figure 23. In this case,

\[
GF_q(P_6^{\lambda_1+4}(X_{10})) = \frac{1}{(q^{\lambda_1+25}; q)_{\lambda_1+4}} \prod_{1 \leq i < j \leq 4} \frac{1 - q^{\lambda_i - \lambda_j - j - i}}{1 - q^{\lambda_i + \lambda_j + 14 - i - j}}
\]

Thus the hook length property for class 10 is equivalent to the identity

\[
\int_{0 \leq x_1 \leq \cdots \leq x_6 \leq 1} (x_1 - x_2)(x_2^2 - x_3^2)(x_3 - x_4)(x_4 - x_5)(x_5 - x_6) a_{\delta_1}(x_3, x_4, x_5, x_6) a_{\lambda + \delta_1}(x_1, \ldots, x_5) d_q x_1 \cdots d_q x_6.
\]

Hence the hook length property for class 10 is equivalent to the identity

\[
(18) \quad \int_{0 \leq x_1 \leq \cdots \leq x_6 \leq 1} (x_1 - x_2)(x_2^2 - x_3^2)(x_3 - x_4)(x_4 - x_5)(x_5 - x_6) a_{\delta_1}(x_3, x_4, x_5, x_6) a_{\lambda + \delta_1}(x_1, \ldots, x_5) d_q x_1 \cdots d_q x_6
\]
We provide a proof of the hook length property for class 10 in Theorem 7.3 by decomposing the $P$-partition generating function for $P_{n+1}^{\lambda_3+4}(X_{10})$ using Lemma 3.4.

### 6.11. Class 11: Swivel shifteds.

A semi-irreducible $d$-complete poset of class 11 is $P_{n+1}^{\lambda_1+n-1}(X_{11})$ with $n \geq 3$, $k \geq 1$, $\epsilon \in \{0, 1\}$ and

$$X_{11} = \{(\lambda, n, 1), ((k-1), 3, n-1), (\emptyset, 2, n), ((\epsilon), 1, 1)\} \cup \bigcup_{i=1}^{n-2} \{((\emptyset, 2, i))\}.$$  

In this case we have $\ell_1 = \lambda_n + 1$, $\ell_j = \lambda_{n+1-j} + j + 1$, for $2 \leq j \leq n$, $\ell_{n+1} = k + 3$. See Figure 24. Then

$$GF_q(P_{n+1}^{\lambda_1+n-1}(X_{11})) = \frac{1}{(q^{\lambda_1+n+\frac{1}{2}n(n+1)}; q)_{\lambda_1+n-1}} GF_q(P_{n+1}(X_{11})),$$

where

$$GF_q(P_{n+1}(X_{11})) = \frac{q^{-\sum_{i=1}^{n} i\lambda_i + \frac{1}{2}n(n^2+3n-1)}}{(1-q)^n} \prod_{i=1}^{n-2} \frac{(-1)^{i}(\sum_{i=1}^{n} i\lambda_i + \frac{1}{2}n(n+1))}{(q^{\lambda_1+n+\frac{1}{2}n(n+1)}; q)_{\lambda_1+n-i}} \times \prod_{i=1}^{n-2} \frac{-a_{\lambda_i}(x_1, \ldots, x_n)}{(1-q)(q; q)_{k+1}} \prod_{i=1}^{n-2} \frac{-a_{\lambda_i}(x_1, x_{n+1})}{1-q} d_qx_1 \cdots d_qx_{n+1}.$$  

Hence, the hook length property for the poset of class 11 is equivalent to the following identity

$$\int_{0 \leq x_1 \leq \cdots \leq x_{n+1} \leq 1} a_{\lambda_i}(x_1, \ldots, x_n) \cdot a_{(k-1)+3}(x_{n-1}, x_n) \cdot (x_n - x_{n+1})$$

$$\times \prod_{i=1}^{n-2} (x_i - x_{i+1}) d_qx_1 \cdots d_qx_{n+1}$$

$$= (-1)^{n+1} q^{\sum_{i=1}^{n} i\lambda_i + \frac{1}{2}n(n^2+3n-1)} (1-q)^{n+2} q^{2\sum_{i=1}^{n} i\lambda_i + \frac{1}{2}n(n+1)} \prod_{1 \leq i < j \leq n} \frac{1-q^{\lambda_i-\lambda_j+j-1}}{1-q^{n+4-i-j+\lambda_i+\lambda_j}}.$$  

We provide a proof of the hook length property for class 11 in Theorem 7.7 by decomposing the $P$-partition generating function for $P_{n+1}^{\lambda_1+n-1}(X_{11})$ using Lemma 3.4.

### 6.12. Class 12: Pumps.

A semi-irreducible $d$-complete poset of class 12 is $P_{n+1}^{\lambda_1+3}(X_{12})$, where

$$X_{12} = \{(\emptyset, 3, 1), (\emptyset, 2, 1), (\lambda, 4, 2), (\emptyset, 2, 3), (\emptyset, 3, 4), (\emptyset, 2, 5)\},$$

with $\lambda \in \Par_3$. See Figure 25. In this case, $\ell_1 = 1$, $\ell_2 = 3$, $\ell_3 = \lambda_3 + 4$, $\ell_4 = \lambda_2 + 4$, $\ell_5 = \lambda_1 + 5$, $\ell_6 = 4$. Then

$$GF_q(P_{n+1}^{\lambda_1+3}(X_{12})) = \frac{1}{(q^{\lambda_1+22}; q)_{\lambda_1+3}} GF_q(P_{6}(X_{12})).$$

where

$$GF_q(P_{6}(X_{12})) = \frac{q^{-\sum_{i=1}^{3} i\lambda_i + 42}}{(1-q)^6} \int_{0 \leq x_1 \leq \cdots \leq x_6 \leq 1} a_{\lambda_i}(x_1, x_2, x_3) a_{\lambda_i+5}(x_1, x_2) a_{\lambda_i+21}(x_1, x_2) a_{\lambda_i+35}(x_1, x_2) a_{\lambda_i+49}(x_1, x_2)$$

$$\times \prod_{i=1}^{3} (q; q)_{\lambda_i+4-i} \frac{1-q}{(1-q)^2(1-q^2)} d_qx_1 \cdots d_qx_6.$$  

The hook length property for class 12 is equivalent to the following identity

$$\int_{0 \leq x_1 \leq \cdots \leq x_6 \leq 1} (x_1 - x_2)(x_3 - x_4)(x_5 - x_6) a_{\delta_3}(x_1, x_2, x_3) a_{\delta_4}(x_4, x_5, x_6)$$

$$\times a_{\lambda_i+4}(x_2, x_3, x_4, x_5) d_qx_1 \cdots d_qx_6.$$
Thus, (19) is equivalent to

\[ \lambda \text{ that the resulting integral can be verified by our testing code.} \]

By [4, Lemma 5.7] we have

\[ \int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}} f(x, y) \, dx \, dy = \prod_{i=1}^{3} \frac{1 - q^{\lambda_{i} + \lambda_{i} + 25 - i}}{(q^{\lambda_{i} + \lambda_{i} + 15 - i}; q)_{3}} \prod_{1 \leq i < j \leq 3} \frac{1 - q^{\lambda_{i} - \lambda_{j} + 1}}{(q^{\lambda_{i} + \lambda_{j} + 15 - i - j}; q)_{2}}. \]

It takes too much time to verify (19) using our testing code. In what follows we modify (19) so that the resulting integral can be verified by our testing code.

Using [4, Lemma 5.7] we can change the order of integration to obtain that the left hand side of (20) is equivalent to

\[ \int_{D} f(x_1, x_2, x_3, x_4, x_5) \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5 = \prod_{i=1}^{3} \frac{1 - q^{\lambda_{i} + \lambda_{i} + 25 - i}}{(q^{\lambda_{i} + \lambda_{i} + 15 - i}; q)_{3}} \prod_{1 \leq i < j \leq 3} \frac{1 - q^{\lambda_{i} - \lambda_{j} + 1}}{(q^{\lambda_{i} + \lambda_{j} + 15 - i - j}; q)_{2}}, \]

where \( D \) is the list of inequalities \( 0 \leq x_1 \leq x_2 \leq y_1 \leq x_3 \leq y_2 \leq y_3 \leq x_4 \leq x_5 \leq x_6 \leq 1 \) and

\[ f(x_1, x_2, x_3, x_4, x_5) = (x_1 - x_2)(x_3 - x_4)(x_5 - x_6)a_{\delta_{\lambda_{1}}}(x_1, x_2, x_3)a_{\delta_{\lambda_{2}}}(x_4, x_5, x_6). \]

Using [4, Lemma 5.7], we can change the order of integration to obtain that the left hand side of (20) is equal to

\[ \int_{D} f(x_1, x_2, x_3, x_4, x_5) a_{\lambda_{1} + \delta_{\lambda_{2}}}(y_1, y_2, y_3) \, dy_1 \, dy_2 \, dy_3 \, dx_1 \, dx_2 \, dx_3 \, dx_4 \, dx_5 = q^{-|\lambda| - 6} \int_{D} f(x_1, x_2, x_3, x_4, x_5) a_{\lambda_{1} + \delta_{\lambda_{2}}}(y_1, y_2, y_3) \, dy_1 \, dy_2 \, dy_3. \]

Hence, (20) is equivalent to

\[ \mathcal{X} = \{1, 2, \ldots, 13\} \]

using our testing code. Interestingly, it has many factors:

\[ \mathcal{X} \]

By using (22), we have verified (21) by our testing code.

6.13. Class 13: Tailed pumps. A semi-irreducible \( d \)-complete poset of class 13 is \( P_{6}^{\lambda_{1} + 3}(X_{13}) \), where

\[ X_{13} = \{(1, 1, 1), (\emptyset, 2, 1), (\emptyset, 3, 1), (\emptyset, 4, 2), (\emptyset, 2, 3), (\emptyset, 3, 4), (\emptyset, 2, 5)\}, \]

with \( \lambda \in \mathbb{N} \). See Figure 26. We have \( \ell_1 = 2, \ell_2 = 3, \ell_3 = 4, \ell_4 = \lambda_2 + 4, \ell_5 = \lambda_1 + 5, \ell_6 = 4 \). In this case,

\[ \text{GF}_{q}(P_{6}^{\lambda_{1} + 3}(X_{13})) = \frac{1}{(q^{\lambda_{1} + 23}, q)_{\lambda_{1} + 3}} \text{GF}_{q}(P_{6}(X_{13})), \]

where

\[ P_{6}(X_{13}) = \{1, 2, \ldots, 13\} \]

using our testing code.
where
\[
\begin{align*}
GF_q(P_6(X_{13})) &= \frac{q^{-(\sum_{i=1}^4 \lambda_i + 47)}}{(1 - q)^6} \int_0 \cdots \int_0 a_{(1) + \delta_1}(x_1) \cdot \frac{1}{1 - q} - a_{\delta_2}(x_1, x_2) \cdot \frac{1}{1 - q} - a_{\delta_3}(x_1, x_2, x_3) \cdot \frac{1}{(1 - q)^2(1 - q^2)} \\
&\times \frac{a_{(\lambda_1, \lambda_2, 0, 0) + \delta_4}(x_2, x_3, x_4, x_5)}{(1 - q)^4(1 - q^2)} \cdot \frac{1}{1 - q} - a_{\delta_2}(x_4, x_5, x_6) \cdot \frac{1}{1 - q} - a_{\delta_3}(x_5, x_6) \cdot \frac{1}{1 - q} - a_{\delta_2}(x_5, x_6) \cdot \frac{1}{1 - q} \\
&\times (1 - q) \prod_{i=1}^6 (q; q)_{\lambda_i + 4 - i} \cdot \frac{1}{1 - q} = \frac{1}{1 - q} d_q x_1 \cdots d_q x_6.
\end{align*}
\]

The hook length property for class 13 is equivalent to the identity
\[
\begin{align*}
\int_{0 \leq x_1 \leq \cdots \leq x_6 \leq 1} x_1(x_1 - x_2)(x_3 - x_4)(x_5 - x_6) a_{\delta_1}(x_1, x_2, x_3) &\delta_2(x_4, x_5, x_6) \\
&\times (1 - q)^{\sum_{i=1}^4 \lambda_i + 47} (1 + q)^2 (1 - q^\lambda_1 - q^{\lambda_2 + 1}) (1 - q^{\lambda_1 + 23}) \\
&\times \left(1 - q^\lambda_{1 + 3 - i} x_2 \cdots x_6 \prod_{i=1}^2 (q; q)^{\lambda_i + 4 - i} (q; q)^{\lambda_i + 9 - i} \right) \\
&\times \frac{1}{1 - q} d_q x_1 \cdots d_q x_6.
\end{align*}
\]

This formula has been verified by computer.

6.14. **Class 14:** Near bats. A semi-irreducible \(d\)-complete poset of class 14 is \(P_{6}^{m+3}(X_{14})\), with \(X_{14} = \{(2, 1, 1), (0, 2, 1), (\emptyset, 3, 1), ((m), 4, 2), (0, 2, 3), (\emptyset, 3, 4), (\emptyset, 2, 5)\}\), and \(m \geq 0\). See Figure 27. We have \(\ell_1 = 3, \ell_2 = 3, \ell_3 = 4, \ell_4 = 4, \ell_5 = m + 5, \ell_6 = 4\). Then,
\[
GF_q(P_{6}^{m+3}(X_{14})) = \frac{1}{(q^{m+24}; q)_{m+3}} GF_q(P_6(X_{14})),
\]

where
\[
\begin{align*}
GF_q(P_6(X_{14})) &= \frac{q^{-(m+52)}}{(1 - q)^6} \int_0 \cdots \int_0 a_{(2) + \delta_1}(x_1) \cdot \frac{1}{1 - q} - a_{\delta_2}(x_1, x_2) \cdot \frac{1}{1 - q} - a_{\delta_3}(x_1, x_2, x_3) \cdot \frac{1}{1 - q} - a_{\delta_4}(x_1, x_2, x_3) \cdot \frac{1}{1 - q} \\
&\times \frac{a_{(m, 0, 0, 0) + \delta_4}(x_2, x_3, x_4, x_5)}{(q; q)_{m+3} (q; q)^{2}(1 - q^2)} \cdot \frac{1}{1 - q} - a_{\delta_2}(x_3, x_4) \cdot \frac{1}{1 - q} - a_{\delta_3}(x_4, x_5, x_6) \cdot \frac{1}{1 - q} - a_{\delta_2}(x_5, x_6) \cdot \frac{1}{1 - q} \\
&\times (1 - q)^{\sum_{i=1}^4 \lambda_i + 47} (1 + q^2)(1 - q^\lambda_1 - q^{\lambda_2 + 1}) (1 - q^{\lambda_1 + 23}) \\
&\times \prod_{i=1}^2 (q; q)^{\lambda_i + 4 - i} (q; q)^{\lambda_i + 9 - i} \frac{1}{1 - q} d_q x_1 \cdots d_q x_6.
\end{align*}
\]

Thus the hook length property for the posets of class 14 is equivalent to the identity
\[
\begin{align*}
\int_{0 \leq x_1 \leq \cdots \leq x_6 \leq 1} x_1^2(x_1 - x_2)(x_3 - x_4)(x_5 - x_6) a_{\delta_1}(x_1, x_2, x_3) &\delta_2(x_4, x_5, x_6) \\
&\times (1 - q)^{m+52} (1 - q)^{20} (1 + q)^2(q^{m+1}; q)_{3} (q^{m+24}; q)_{2} (1 + q^{m+13}) \\
&\times \frac{1}{(q; q)^{11} (q^4; q)_{5} (q^{m+9}; q)_{9} d_q x_1 \cdots d_q x_6}.
\end{align*}
\]

This formula has been verified by computer.

6.15. **Class 15:** Bat. A semi-irreducible \(d\)-complete poset of class 15 is \(P_{6}^{3}(X_{15})\), with \(X_{15} = \{(3, 1, 1), (0, 2, 1), (\emptyset, 3, 1), (\emptyset, 4, 2), (0, 2, 3), (\emptyset, 3, 4), (\emptyset, 2, 5)\}\), and \(\ell_1 = 4, \ell_2 = 3, \ell_3 = 4, \ell_4 = 4, \ell_5 = 5, \ell_6 = 4\). See Figure 28. Then,
\[
GF_q(P_{6}^{3}(X_{15})) = \frac{1}{(q^{25}; q)_{3}} GF_q(P_6(X_{15})),
\]

where
\[
\begin{align*}
GF_q(P_6(X_{15})) &= \frac{q^{57}}{(1 - q)^6} \int_0 \cdots \int_0 a_{(3) + \delta_1}(x_1) \cdot \frac{1}{1 - q} - a_{\delta_2}(x_1, x_2) \cdot \frac{1}{1 - q} - a_{\delta_3}(x_1, x_2, x_3) \cdot \frac{1}{1 - q} \\
&\times \frac{a_{\delta_4}(x_2, x_3, x_4, x_5)}{(1 - q)(q; q)_{2}(q; q)_{3}} \cdot \frac{1}{1 - q} - a_{\delta_2}(x_3, x_4) \cdot \frac{1}{1 - q} - a_{\delta_3}(x_4, x_5, x_6) \cdot \frac{1}{1 - q} - a_{\delta_2}(x_5, x_6) \cdot \frac{1}{1 - q} \\
&\times (1 - q)^{\sum_{i=1}^4 \lambda_i + 47} (1 + q)^2(1 - q^\lambda_1 - q^{\lambda_2 + 1}) (1 - q^{\lambda_1 + 23}) \\
&\times \prod_{i=1}^2 (q; q)^{\lambda_i + 4 - i} (q; q)^{\lambda_i + 9 - i} \frac{1}{1 - q} d_q x_1 \cdots d_q x_6.
\end{align*}
\]

Hence the hook length property for the posets of class 15 is equivalent to the identity
\int_{0 \leq x_1 \leq \cdots \leq x_6 \leq 1} x_1^3(x_1 - x_2)(x_3 - x_4)(x_5 - x_6) a_{\delta_3}(x_1, x_2, x_3) a_{\delta_3}(x_4, x_5, x_6) \times a_{\delta_4}(x_2, x_3, x_4, x_5) d_q x_1 \cdots d_q x_6
\quad = \frac{(-1)q^{37}(1 - q)^{18}(1 + q)^3(q; q)_{2}^{2}(q^{25}; q)_{4}}{(q; q)_{17}(q^{3}; q)_{9}(1 - q^{9})}.

This formula has been verified by computer.

7. Evaluation of the q-integrals

In this section we complete the proof of the hook length property of d-complete posets by evaluating the corresponding q-integrals obtained in the previous section.

Observe that since the q-integral for class \( i \in \{3, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15\} \) is an integral of a polynomial with a fixed number of variables, it can be verified by computer. We say that these q-integrals are of finite type. The q-integrals of finite type except classes 8-(4), 10 and 12 can be verified by a brute-force computation within a reasonable amount of time. Recall that we modified the q-integral for class 12 in Section 6.12 so that the resulting q-integral can be verified by a brute-force computation. In this section we do similar things for classes 8-(4) and 10. After necessary modification, all q-integrals of finite type can be verified by a brute-force computation.

We provide testing code which computes the following integral when \( n \) is given as a specific integer and \( f \) is a polynomial

\[ \int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} f(x_1, \ldots, x_n) d_q x_1 \cdots d_q x_n. \]

The testing code evaluates the above q-integrals simply expanding \( f(x_1, \ldots, x_n) \) and using the facts

\[ \int_{a}^{b} x^k d_q x = \frac{(1 - q)(b^{k+1} - a^{k+1})}{1 - q^{k+1}}, \]

\[ \int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} x_1^{a_1} \cdots x_n^{a_n} d_q x_1 \cdots d_q x_n = \prod_{i=1}^{n} \frac{1 - q}{1 - q^{a_1 + \cdots + a_i + 1}}. \]

We have verified all q-integrals (modified if necessary) of finite type using our testing code and a computer with 3.5 GHz Intel Core i7 and 32 GB memory, which took about 11 hours in total as shown below.

sage: load("check_d_complete.sage")
sage: test_all()
Testing Class3()
True
It took 0 hours 0 minutes 0.804966926575 seconds.
Testing Class5()
True
It took 0 hours 0 minutes 0.267915010452 seconds.
Testing Class6()
True
It took 0 hours 0 minutes 0.391582965851 seconds.
Testing Class7()
True
It took 0 hours 0 minutes 17.5302219391 seconds.
Testing Class8_1()
True
It took 0 hours 0 minutes 7.67528510094 seconds.
Testing Class8_2()
True
It took 2 hours 6 minutes 50.673628979 seconds.
Testing Class8_3()
True
It took 1 hours 40 minutes 1.43686294556 seconds.
Testing Class8_4()
True
It took 0 hours 0 minutes 10.554230878 seconds.
Testing Class9_1()
True
It took 1 hours 57 minutes 22.2945978642 seconds.
Testing Class9_2()
True
It took 1 hours 56 minutes 15.6300830841 seconds.
Testing Class10()
True
It took 0 hours 0 minutes 3.3731648922 seconds.
Testing Class12()
True
It took 0 hours 24 minutes 39.6105351448 seconds.
Testing Class13()
True
It took 2 hours 41 minutes 19.9908559322 seconds.
Testing Class14()
True
It took 0 hours 15 minutes 59.6924049854 seconds.
Testing Class15()
True
It took 0 hours 0 minutes 7.89000296593 seconds.
All checked!
It took 11 hours 3 minutes 17.8191730976 seconds.

Since the $q$-integrals for classes 1, 2, 4 and 11 are not of finite type, they have to be proved by hand. Note that the hook length properties for class 1 (shapes) and class 2 (shifted shapes) are well known. Hence it remains to consider classes 4 and 11. We also give another proof of the hook length formula for class 2. To prove the hook length property for classes 2, 4 and 11, we utilize some partial fraction expansion identities. Throughout this section we use the following notation.

For a partition $\mu \in \text{Par}_n$ and an integer $1 \leq \ell \leq n$, we define

$$\hat{\mu}^{(\ell)}_i = \begin{cases} 
\mu_i + 1 & \text{if } i < \ell, \\
\mu_{i+1} & \text{if } i \geq \ell.
\end{cases}$$

The following lemma is useful in this section.

**Lemma 7.1.** Let $f(x_1, \ldots, x_{n-1})$ be a homogeneous function of degree $d$ in variables $x_1, \ldots, x_{n-1}$, i.e., $f(tx_1, \ldots, tx_{n-1}) = t^d f(x_1, \ldots, x_{n-1})$. Then for a partition $\mu \in \text{Par}_n$, we have

$$\int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} x_1^k f(x_1, \ldots, x_{n-1}) a_{\mu+\delta_n}(x_1, \ldots, x_n) dx_1 \cdots dx_n = \frac{1 - q}{1 - q^{d+\binom{n+1}{2}}} \sum_{\ell=1}^{n} (-1)^{n-\ell} \int_{0 \leq x_1 \leq \cdots \leq x_{n-1} \leq 1} f(x_1, \ldots, x_{n-1}) dx_1 \cdots dx_{n-1} \times a_{\hat{\mu}^{(\ell)} + \delta_{n-1}}(x_1, \ldots, x_{n-1}) dx_1 \cdots dx_{n-1}. $$

**Proof.** Since

$$a_{\mu+\delta_n}(x_1, \ldots, x_n) = \sum_{\ell=1}^{n} (-1)^{n-\ell} x_{n+\ell}^{-d} a_{\hat{\mu}^{(\ell)} + \delta_{n-1}}(x_1, \ldots, x_{n-1}).$$
the left hand side is equal to
\[ \sum_{\ell=1}^{n} (-1)^{n-\ell} \int_{0 \leq x_1 \leq \cdots \leq x_{n-1} \leq 1} x_{n}^{\mu_{\ell} + n-\ell} f(x_1, \ldots, x_{n-1}) a_{\hat{\mu}(\ell)}^{\hat{\delta}_{n-1}} (x_1, \ldots, x_{n-1}) d_q x_1 \ldots d_q x_{n-1}. \]

Since \( f(x_1, \ldots, x_{n-1}) a_{\hat{\mu}(\ell)}^{\hat{\delta}_{n-1}} (x_1, \ldots, x_{n-1}) \) is a homogeneous polynomial in \( x_1, \ldots, x_{n-1} \) whose degree is
\[ |\hat{\mu}(\ell)| + \left( \frac{n-1}{2} \right) + d = |\mu| - \mu + \ell - 1 + \left( \frac{n-1}{2} \right) + d, \]
by Lemma 5.3

\[ \int_{0 \leq x_1 \leq \cdots \leq x_{n-1} \leq 1} x_{n}^{\mu_{\ell} + n-\ell} f(x_1, \ldots, x_{n-1}) a_{\hat{\mu}(\ell)}^{\hat{\delta}_{n-1}} (x_1, \ldots, x_{n-1}) d_q x_1 \ldots d_q x_{n-1} \]
\[ = \frac{1 - q}{1 - q^{n+\delta_{n-1}+\ell}} \int_{0 \leq x_1 \leq \cdots \leq x_{n-1} \leq 1} f(x_1, \ldots, x_{n-1}) a_{\hat{\mu}(\ell)}^{\hat{\delta}_{n-1}} (x_1, \ldots, x_{n-1}) d_q x_1 \ldots d_q x_{n-1}, \]
which completes the proof. \( \square \)

7.1. Class 8 : Swivels.

**Theorem 7.2.** The hook length property holds for class 8.

*Proof.* In Section 6.8, we have classified the semi-irreducible \( d \)-complete posets of class 8 in 4 subclasses, and explicitly expressed the hook length property in terms of the \( q \)-integral in (14), (15), (16) and (17). The \( q \)-integrals corresponding to the posets \( P_5^{\lambda_1+2}(X_8^{(1)}) \), \( P_5^{\lambda_1+3}(X_8^{(2)}) \) and \( P_5^{\lambda_1+4}(X_8^{(3)}) \), for \( \lambda \in \text{Par}_5 \), have been verified by computer. Here we deal with the poset \( P_6^{\lambda_1+4}(X_8^{(4)}) \). We have expressed the hook length property in (17), however we compute the \( q \)-integral of the poset \( P_6^{\lambda_1+4}(X_8^{(4)}) \) in a different way by utilizing Lemma 3.4.

In terms of the notation defined in Definition 3.3, the poset \( P_6^{\lambda_1+4}(X_8^{(4)}) \) can be expressed as
\[ D_{\mu_1+4,1}(Q), \text{ where } Q = P_5(X)^{-} \text{ for } \]
\[ X = \{ (\mu, 5, 1), (\emptyset, 2, 1) \} \]
and \( \mu = \lambda + (1^5) \), i.e., \( \mu_i = \lambda_i + 1 \). See Figure 9 for the posets \( Q^+ \) and \( D_1(Q) \). Note that the discrepancy between \( \lambda \) in Figure 20 and \( \mu \) in Figure 9 comes from moving the square dots (diagonal entries) up by one.

![Figure 9](image_url)

**Figure 9.** The posets \( Q^+ \) on the left and \( D_1(Q) \) on the right.
Then by Lemma 3.3
\[
\text{GF}_q(D_{\mu_1+4,1}(Q)) = \frac{1}{(q^{|\mu|+17}; q)_{\mu_1+6}} (q^{|\mu|+16} \text{GF}_q(Q^+) + (1 - q^{2|\mu|+34}) \text{GF}_q(D_1(Q))).
\]

Since \( Q^+ = P_3(X) \) and \( D_1(Q) = P_3(X') \) for \( X \) in (24) and \( X' = \{(\mu, 5, 1), (\emptyset, 2, 1), (\emptyset, 2, 4)\} \), by Theorem 5.6
\[
\text{GF}_q(Q^+) = \frac{(-1)^{5-\ell}(1-q)}{1-q^{|\mu|+16}} \int \sum_{x_1 \leq \ldots \leq x_5 \leq 1} (x_1 - x_2)a_{\mu+\delta_5}(x_1, \ldots, x_5)d_qx_1 \cdots d_qx_5.
\]

By Lemma 7.1
\[
(25) \int \sum_{x_1 \leq \ldots \leq x_5 \leq 1} (x_1 - x_2)a_{\mu+\delta_5}(x_1, \ldots, x_5)d_qx_1 \cdots d_qx_5
\]
\[
= \frac{(-1)^{5-\ell}(1-q)}{1-q^{|\mu|+16}} \int \sum_{0 \leq x_1 \leq \ldots \leq x_4 \leq 1} (x_1 - x_2)a_{\hat{\mu}+\delta_4}(x_1, x_2, x_3, x_4)d_qx_1 \cdots d_qx_4,
\]

\[
(26) \int \sum_{x_1 \leq \ldots \leq x_5 \leq 1} x_4(x_1 - x_2)a_{\mu+\delta_5}(x_1, \ldots, x_5)d_qx_1 \cdots d_qx_5
\]
\[
= \frac{(-1)^{5-\ell}(1-q)}{1-q^{|\mu|+17}} \int \sum_{0 \leq x_1 \leq \ldots \leq x_4 \leq 1} x_4(x_1 - x_2)a_{\hat{\mu}+\delta_4}(x_1, x_2, x_3, x_4)d_qx_1 \cdots d_qx_4,
\]

\[
(27) \int \sum_{x_1 \leq \ldots \leq x_5 \leq 1} x_5(x_1 - x_2)a_{\mu+\delta_5}(x_1, \ldots, x_5)d_qx_1 \cdots d_qx_5
\]
\[
= \frac{(-1)^{5-\ell}(1-q)}{1-q^{|\mu|+17}} \int \sum_{0 \leq x_1 \leq \ldots \leq x_4 \leq 1} (x_1 - x_2)a_{\hat{\mu}+\delta_4}(x_1, x_2, x_3, x_4)d_qx_1 \cdots d_qx_4.
\]

The \( q \)-integrals with 4 variables in (25), (26) and (27) can be explicitly computed by computer as follows. For any partition \( \nu \in \text{Par}_4 \) and \( \epsilon \in \{0, 1\} \), we have
\[
(28) f(\nu, \epsilon) := \int \sum_{0 \leq x_1 \leq \ldots \leq x_4 \leq 1} x'_4(x_1 - x_2)a_{\nu+\delta_4}(x_1, x_2, x_3, x_4)d_qx_1 \cdots d_qx_4
\]
\[
= \frac{(-1)q^{12+\sum_{i=1}^{\epsilon}(i-1)}q^{5(1-q^{\nu+j+i})\prod_{1 \leq i < j \leq 4}(1-q^{\nu_i-\nu_j+j-i})}}{\prod_{i=1}^{5}(1 - q^{|\nu_i|+\nu_{i+1}+\nu_{i+2}+\nu_{i+3}+\nu_{i+4}})(1 - q^{|\mu|+\delta_4})}. \]

We note that \( f(\nu, \epsilon) \) in (28) does not seem to have a nice factorization formula when \( \epsilon \geq 2 \).

On the other hand, by explicitly computing the hook lengths of the elements in \( P_0^{\mu_1+4}(X_8^{(4)}) = D_{\mu_1+4,1} \), we get
\[
\prod_{u \in P_0^{\mu_1+4}(X_8^{(4)})} \frac{1}{1-q^{h(u)}} = \frac{\prod_{i=1}^{5}(1 - q^{|\nu|+\mu_{i+4}+23})\prod_{1 \leq i < j \leq 5}(1 - q^{\mu_i-\mu_j+j-i})}{(q^{|\mu|+16}, q)_{\mu_1+7}\prod_{i=1}^{\mu_1+7}(q, q)_{\mu_1+7-i}\prod_{1 \leq i < j \leq 5}(1 - q^{|\mu|+\mu_i-\mu_j+i+j+4})}.
\]

Using the above observations we obtain that the hook length property for class 8-(4) is equivalent to
function satisfies the relation Figure 23) can be also expressed as
\[ D \]
computation, we express the poset in a different way using Lemma 3.4.

We have verified \((29)\) by computer. \(\square\)

7.2. Class 10 : Tagged Swivels.

**Theorem 7.3.** The hook length property holds for class 10.

**Proof.** Note that we derived the q-integer identity which implies the hook length property for the semi-irreducible d-complete posets of class 10 in \([18]\). As we did in the proof of Theorem 7.2 rather than proving \([18]\) by computing the q-integral directly, for the sake of the simplicity of the computation, we express the poset in a different way using Lemma 3.4.

Let \( Q = P_6^0(X)^- \) and \( D_1(Q) = \ell \), where
\[ X = \{(\mu, 5, 1), ((1), 1, 2)\} \quad X' = \{(\mu, 5, 1), ((1), 1, 2), (\emptyset, 2, 4)\}. \]

By Theorem 5.6
\[ GF_q(D_{5+4,1}(Q)) = \frac{1}{(q^{16} + q^{17}) \sum_{x=1}^\infty} \int_{\nu \in Par_5} x_2 a_{\mu+6} \left(x_1, ..., x_5\right) q x_1 \cdots q x_5. \]

By Lemma 7.1
\[ \int_{0 \leq x_1 \leq \cdots \leq x_5 \leq 1} x_2 a_{\mu+6} \left(x_1, ..., x_5\right) q x_1 \cdots q x_5 \]
[= \sum_{\ell=1}^5 \left(-1\right)^{5-\ell} (1 - q) q^{(\mu)_{\ell+6}} \int_{0 \leq x_1 \leq \cdots \leq x_4 \leq 1} x_2 a_{\ell+6} \left(x_1, x_2, x_3, x_4\right) q x_1 \cdots q x_4]. \]

The q-integrals with 4 variables in the above 3 equations can be explicitly computed by computer: For \( \nu \in Par_5 \) and an integer \( m \geq 0 \), we have
\[ g(\nu, m) := \int_{0 \leq x_1 \leq \cdots \leq x_4 \leq 1} x_2 a_{\mu+6} \left(x_1, x_2, x_3, x_4\right) q x_1 \cdots q x_4. \]
On the other hand, by explicitly computing the hook lengths of the elements in $P_{6}^{4+4}(X_{10}) = D_{\mu+4,1}(Q)$, we get

$$
\prod_{u \in P_{6}^{4+4}(X_{10})} \frac{1}{1 - q^{h(u)}} = \frac{1}{(1 - q)(q^{\bar{\mu}|+17}; q)_{\mu+6}} \prod_{i=1}^{5} \frac{1 - q^{\mu|+\mu-i+23}}{(1 - q^{\mu|+\mu-i+10+i})(q; q)_{\mu+6-i}} \prod_{1 \leq i < j \leq 5} \frac{1 - q^{\mu_i-\mu_j+j-i}}{1 - q^{\mu_i+\mu_j-i-j+13}}.
$$

Summarizing the above observations, we obtain that the hook length property for class 10 is equivalent to

$$
\sum_{\ell=1}^{5} (-1)^{5-\ell} q^{-\sum_{i=1}^{5}(i-1)\mu_i} \frac{1 - q^{\mu|+(\bar{\mu}^{(1)}_{\ell})}}{1 - q^{\mu|+(\bar{\mu}^{(1)}_{\ell})+k}} \prod_{1 \leq i < j \leq 5} (1 - q^{\mu_i-\mu_j+j-i}) \prod_{1 \leq i < j \leq n} (1 - q^{2n+1-i-j+\mu_i+\mu_j+1}).
$$

We have verified (30) by computer.

\[\square\]

7.3. Class 2: Shifted shapes. In Section [7], we have shown that the hook length property for class 2 is equivalent to [12]. We give a proof of the following lemma which is slightly more general than [12].

**Lemma 7.4.** For nonnegative integers $n, k$ and $\mu \in \text{Par}_n$, we have

$$
\int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} (-1)^{\bar{\mu}^{(1)}} d_{\bar{\mu}^{(1)}}(x_1, \ldots, x_n) x_1^{k_1} \cdots x_n^{k_n} = (1 - q)^n q^{\bar{\mu}^{(1)}} \frac{1 - q^{\bar{\mu}^{(1)}+k}}{1 - q^{\bar{\mu}^{(1)}+k+d}} \int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} (1 - q)^n q^{\bar{\mu}^{(1)}+\sum_{i=1}^{n}(i-1)\mu_i} \prod_{1 \leq i < j \leq n} (1 - q^{\mu_i-\mu_j+j-i}) \prod_{1 \leq i < j \leq n} (1 - q^{2n+1-i-j+\mu_i+\mu_j+1}) \prod_{1 \leq i < j \leq n} (1 - q^{\bar{\mu}^{(1)}-\bar{\mu}^{(1)}+j-i}) \prod_{1 \leq i < j \leq n} (1 - q^{\bar{\mu}^{(1)}+\bar{\mu}^{(1)}+j-i}).
$$

**Proof.** By Lemma 7.1, the left hand side of (31) can be written as

$$
\int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} (-1)^{\bar{\mu}^{(1)}} d_{\bar{\mu}^{(1)}}(x_1, \ldots, x_n) x_1^{k_1} \cdots x_n^{k_n}.
$$

Then, by induction, proving (31) is equivalent to showing the following identity

$$
(1 - q)^n q^{\bar{\mu}^{(1)}+\sum_{i=1}^{n}(i-1)\mu_i} \frac{1 - q^{\bar{\mu}^{(1)}+k}}{1 - q^{\bar{\mu}^{(1)}+k+d}} \prod_{1 \leq i < j \leq n} (1 - q^{\mu_i-\mu_j+j-i}) \prod_{1 \leq i < j \leq n} (1 - q^{2n+1-i-j+\mu_i+\mu_j+1}) = \frac{1}{1 - q^{\bar{\mu}^{(1)}+\sum_{i=1}^{n}(i-1)\mu_i}} \sum_{\ell=1}^{n} (-1)^{1-\ell} (1 - q)^n q^{\bar{\mu}^{(1)}+\sum_{i=1}^{n}(i-1)\bar{\mu}_{\ell}^{(1)}} \prod_{1 \leq i < j \leq n} (1 - q^{\bar{\mu}^{(1)}-\bar{\mu}_{\ell}^{(1)}+j-i}) \prod_{1 \leq i < j \leq n} (1 - q^{\bar{\mu}^{(1)}+\bar{\mu}_{\ell}^{(1)}+j-i}),
$$

or,

$$
q^{\bar{\mu}^{(1)}} (1 - q^{\bar{\mu}^{(1)}+\sum_{i=1}^{n}(i-1)\mu_i}) = \sum_{\ell=1}^{n} (-1)^{1-\ell} q^{\sum_{i=1}^{n}(i-1)\bar{\mu}_{\ell}^{(1)}} \sum_{i=1}^{n}(i-1)\bar{\mu}_{\ell}^{(1)}.
$$
By considering the parts of $\hat{\mu}^{(\ell)}$ specified in (23), it is not difficult to check
\begin{equation}
- \sum_{i=1}^{n} (i-1)\mu_i + \sum_{j=\ell+1}^{n-1} (i-1)\hat{\mu}_i^{(j)} = \binom{\ell-1}{2} - (\ell-1)\mu_{\ell} - \sum_{j=\ell+1}^{n} \mu_j,
\end{equation}
\begin{equation}
\prod_{1 \leq i < j \leq n-1} (1 - q^{n+i-j+1}) \prod_{1 \leq i < j \leq n} (1 - q^{n+i-j+1}) = \prod_{i=1, i \neq \ell}^{n} (1 - q^{2n+1-i+j+\mu_i+\mu_{i+1}})
\end{equation}
and
\begin{equation}
\prod_{1 \leq i < j \leq n-1} (1 - q^{2n+1-i-j+\mu_i+\mu_{i+1}}) \prod_{1 \leq i < j \leq n} (1 - q^{2n-1-i-j+\hat{\mu}_i^{(j)}+\hat{\mu}_{j+1}^{(j)}}) = (1 - q^{2n+1-\ell+\mu_{\ell}}) \prod_{i=1, i \neq \ell}^{n} (1 - q^{2n+2-\ell-i+\mu_i+\mu_{i+1}}).
\end{equation}
Eventually, (31) is equivalent to
\begin{equation}
q^{\binom{n+1}{2}+|\mu|} (1 - q^{\binom{n+1}{2}+|\mu|}) = \sum_{\ell=1}^{n} q^{n+1-\ell+\mu_{\ell}} (1 - q^{n+1-\ell+\mu_{\ell}}) \prod_{i=1, i \neq \ell}^{n} \frac{1 - a_ia_{\ell}}{1 - a_{\ell}/a_i}.
\end{equation}
If we let $a_j = q^{n+1-j+\mu_j}$, then this identity can also be written as
\begin{equation}
a_1 \cdots a_n (1 - a_1 \cdots a_n) = \sum_{\ell=1}^{n} a_{\ell} (1 - a_{\ell}) \prod_{i=1, i \neq \ell}^{n} \frac{1 - a_ia_{\ell}}{1 - a_{\ell}/a_i}.
\end{equation}
To prove (36), we note the following partial fraction expansion [12 p.81-83] (cf. [5] (7.13) in Lemma 7.9)
\begin{equation}
\prod_{i=1}^{n} \frac{1 - tx_iy_i}{1 - tx_i} = y_1 \cdots y_n + \sum_{\ell=1}^{n} \frac{1 - y_{\ell}}{1 - tx_{\ell}} \prod_{i=1, i \neq \ell}^{n} \frac{1 - x_iy_i/x_{\ell}}{1 - x_i/x_{\ell}}.
\end{equation}
Take $x_i \mapsto a_i$ and $y_i \mapsto 1/a_i^2$ in (37) and get
\begin{equation}
1 - \prod_{i=1}^{n} a_i(a_i - t) = \sum_{\ell=1}^{n} \frac{1 - a_{\ell}^2}{1 - ta_{\ell}} \prod_{i=1, i \neq \ell}^{n} \frac{1 - a_ia_{\ell}}{1 - a_{\ell}/a_i}.
\end{equation}
Then (36) is the result obtained by subtracting $t = -1$ case of (38) from the $t = 0$ case of (38). \qed

We remark that Lemma 7.4 proves the shifted version of Warnaar’s $q$-integral (cf. [4] Theorem 8.16)) via explicit computation of the $q$-integral.

7.4. Class 4 : Insets.

**Theorem 7.5.** The hook length property holds for class 4.

**Proof.** In Section 6.4, we figured that the hook length property for class 4 is equivalent to (13).

By Lemma 7.1, the left hand side of (13) is
\begin{equation}
\int_{0 \leq x_1 \leq \cdots \leq x_{n+1} \leq 1} x_n^k a_{\lambda+\delta_{n-1}}(x_1, \ldots, x_{n-1}) a_{\mu+\delta_{n+1}}(x_1, \ldots, x_{n+1}) d_q x_1 \cdots d_q x_{n+1}
\end{equation}
\begin{equation}
= \frac{1 - q}{1 - q^{|\lambda|+|\mu|+k+n^2+2}} \sum_{\ell=1}^{n+1} (-1)^{n+1-\ell} Y_\ell,
\end{equation}
where
\begin{equation}
Y_\ell = \int_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} x_n^k a_{\lambda+\delta_{n-1}}(x_1, \ldots, x_{n-1}) a_{\hat{\mu}^{(\ell)}+\delta_{n}}(x_1, \ldots, x_{n}) d_q x_1 \cdots d_q x_{n}.
\end{equation}
Now we compute $Y_\ell$ for a fixed integer $\ell$. Let $Q = P_n(X)$, where

$$X = \{ (\lambda, n - 1, 1), (\hat{\mu}^{(\ell)}, n, 1), ((k), 1, n) \}.$$  

Then the number $\ell_i$ of elements of level $i$ in $Q$ is $\ell_i = \lambda_i + \hat{\mu}^{(\ell)}_i + 2i - 1$ for $1 \leq i \leq n - 1$ and $\ell_n = \hat{\mu}^{(\ell)}_1 + n + k$. By Theorem 5.6

$$Y_\ell = (-1)^{n-1} q^{-\sum_{i=1}^{n-1} (\lambda_i + \hat{\mu}^{(\ell)}_i) - \frac{1}{2} n(n-1)(2n-1)} \prod_{i=1}^{n-1} \frac{1}{(q;q)_{\lambda_i + n-i}} \prod_{i=1}^{n} \frac{1}{(q;q)_{\hat{\mu}^{(\ell)}_i + n-i}}.$$  

On the other hand, $Q^-$ is the disjoint union of $\nu$ and a chain with $k$ elements, where $\nu$ is the Young poset obtained from the rectangular shape $(n-1)^n$ by attaching $\mu$ to the right and the transpose of $\lambda$ at the bottom. Thus, by Lemma 3.2 and the hook length formula for a shape,

$$\text{GF}_q(Q) = \frac{1}{1 - q^{\lambda} [\mu] + [\mu] + n(n-1) + k+1} \text{GF}_q(Q^-)$$

$$= \frac{1}{1 - q^{\lambda} [\mu] - \mu + \ell + n^2 - n + k} \prod_{1 \leq i \leq n} (q;q)_{\lambda_i + n-i} \prod_{1 \leq i \leq n} \frac{1}{(q;q)_{\hat{\mu}^{(\ell)}_i + n-i}} \prod_{1 \leq i \leq n} (1 - q^{\lambda}_i - \mu_i + j-i).$$

By (40) and (41),

$$Y_\ell = (-1)^{n-1} q^{-\sum_{i=1}^{n-1} (\lambda_i + \hat{\mu}^{(\ell)}_i) - \frac{1}{2} n(n-1)(2n-1)} \prod_{1 \leq i \leq n} (1 - q^{\lambda}_i - \mu_i + j-i) \prod_{1 \leq i \leq n} (1 - q^{\lambda}_i - \mu_i + j-i).$$

By (39) and (12) and the identities (32), (33) and

$$\prod_{1 \leq i \leq n} (1 - q^{\lambda}_i + \mu_i + 2n - j + 1) \prod_{1 \leq i \leq n} (1 - q^{\lambda}_i + \mu_i + 2n - i - j) = \prod_{1 \leq i \leq n} (1 - q^{\lambda}_i + \mu_i + 2n - j + 1),$$

we can rewrite (13) as

$$\prod_{j=1}^{n+1} (1 - q^{\lambda}_i + [\mu] + \mu_i + n^2 - n + k + 1)$$

$$= \sum_{\ell=1}^{n+1} q^{-[\lambda] - [\mu] + \mu_i + n^2 - n + k} \prod_{1 \leq j \leq n, j \neq \ell} (1 - q^{\lambda}_i - [\mu] + n^2 - n + k + 1) \prod_{j=1}^{n+1} (1 - q^{\lambda}_i + [\mu] + n^2 - n + k + 1).$$

To prove (43), we remark a partial fraction expansion [10, p.451]

$$\prod_{j=1}^{n+1} (1 - b_j/t) = \prod_{\ell=1}^{n+1} (1 - a_\ell/t) \prod_{j=1}^{n+1} (1 - a_\ell/b_j)$$

for $b_1 \cdots b_{n+1} = a_1 \cdots a_n t$.

Let $n \rightarrow n + 1$ and

$$a_1 = q^{-|\mu| - n - 2 - k}, \quad 1 \leq i \leq n + 1,$$

$$b_1 = q^{\lambda} - i + n - 1 - k, \quad 1 \leq i \leq n - 1,$$

$$b_n = t c, \quad b_{n+1} = t c, \quad b_{n+2} = 1/c^2$$

for some $c$, and

$$t = q^{-[\lambda] - [\mu] - n^2 - 2 - 2k}.$$
With this substitution, we can check that the condition \( b_1 \cdots b_{n+2} = a_1 \cdots a_{n+1} t \) is satisfied, and the partial fraction expansion becomes
\[
\frac{\prod_{j=1}^{n-1}(1 - b_j/t)(1 - c^2(1 - tc)^2)}{\prod_{j=1}^{n+1}(1 - a_j/t)} = \sum_{\ell=1}^{n+1} \frac{(-a_\ell) \prod_{j=1}^{n-1}(1 - a_\ell/b_j)}{(1 - a_\ell/t) \prod_{j=1, j\neq \ell}^{n+1}(1 - a_\ell/a_j)}.
\]
If we divide both sides by \( c^2 \) and take the limit \( c \to \infty \), then we obtain
\[
\frac{\prod_{j=1}^{n-1}(1 - b_j/t)}{\prod_{j=1}^{n+1}(1 - a_j/t)} = \sum_{\ell=1}^{n+1} \frac{(-a_\ell) \prod_{j=1}^{n-1}(1 - a_\ell/b_j)}{(1 - a_\ell/t) \prod_{j=1, j\neq \ell}^{n+1}(1 - a_\ell/a_j)},
\]
or, by substituting \( a_i \)'s, \( b_j \)'s and \( t \),
\[
\frac{\prod_{j=1}^{n-1}(1 - q^{\lambda+\mu+n^2-n-j+k+1})/b_j}{\prod_{j=1}^{n+1}(1 - q^{\lambda+\mu+n^2-n-j+k+1})/a_j} = \sum_{\ell=1}^{n+1} \frac{-q^{-n-\mu_\ell+\ell-k-2}}{1 - q^{\lambda+\mu_\ell+n^2-n+k+\ell}} \cdot \frac{\prod_{j=1, j\neq \ell}^{n+1}(1 - q^{\mu_\ell+\ell-k-j})}{\prod_{j=1}^{n+1}(1 - q^{\mu_\ell+\ell-k-j})},
\]
which is equivalent to (43).

**Remark 7.6.** The partial fraction expansion (43) that we use is actually written in an elliptic form in [13], and (41) can be derived by setting \( p = 0 \) in equation (4.3) of Rosengren's paper [13]. We remark that the elliptic partial fraction identity plays an essential role in the theory of elliptic hypergeometric series associated to the root system \( A_n \), see [13].

### 7.5 Class 11: Swivel shifted shapes

**Theorem 7.7.** The hook length property holds for class 11.

**Proof.** First of all, let us assume that the element \( A \) in Figure 24 is not there (i.e., \( \epsilon = 0 \)). Let \( \mu = \lambda + (1^n) \), i.e., \( \mu = \lambda_i + 1 \), for \( 1 \leq i \leq n \). Observe that \( P_{n+1+\mu+n-1}^\lambda(11) = D_{\mu+n-2,k}(Q) \), where \( Q = ((\mu + \delta_{n+1})^*)^* \). Since \( Q^* = (\mu + \delta_{n+1})^* \), we have
\[
GF_q(Q^*) = \prod_{u \in (\mu + \delta_{n+1})^*} \frac{1}{1 - q^{h(u)}}.
\]
By applying Lemma 3.4 the P-partition generating function for the poset \( P_{n+1+\mu+n-1}^\lambda(11) \) can be written as
\[
GF_q(P_{n+1+\mu+n-1}^\lambda(11)) = \frac{1}{(q)_{\mu+n} (q;q)_{\mu+n}} \left( \frac{1}{1 - q^{h(u)}} \prod_{u \in (\mu + \delta_{n+1})^*} \frac{1}{1 - q^{h(u)}} \right) + (1 - q^{2\mu+2\delta_{n+1}+2k}) GF_q(D_k(Q))
\]
Note that \( D_k(Q) = P_n(X) \) for \( X = (\mu, \mu, (k-1, 0), 2, n-1) \). Thus, by Theorem 5.6
\[
GF_q(D_k(Q)) = \frac{1}{(1 - q^n(q;q)_k \prod_{i=1}^n (q; q)_{\mu+i-1})} \cdot \frac{1}{(1 - q^{\mu+\mu+n-1})} \cdot \prod_{0 \leq x_1 \leq \cdots \leq x_n \leq 1} (-1)^n a_{\mu+\delta_n}(x_1, \ldots, x_n)(x_{n-k} - x_{n-k-1})_q x_1 \cdots x_n.
\]
By Lemmas 1.7, 1.7 and (32), we have
\[
GF_q(D_k(Q)) = Y - Z,
\]
where
\[
Y = \frac{1}{(q;q)_k \prod_{i=1}^n (q; q)_{\mu+n-i}} \cdot \frac{1 - q^{\mu+(\mu+n)/2}}{1 - q^{\mu+(\mu+n)/2+k}} \cdot \prod_{1 \leq i < j \leq n} (1 - q^{-\mu_j+j-\mu_i}) \cdot \prod_{1 \leq j \leq n} (1 - q^{\mu - 2\mu_j + j + \mu_i + 1}).
\]
\[ Z = \frac{q^{-n(\frac{n}{2})}}{1 - q^{n|\mu|+(\frac{n}{2})+k}} \cdot \frac{1}{(q; q)_k} \prod_{i=1}^{n} (q; q)_{\mu_i+n-i} \times \sum_{\ell=1}^{n} (-1)^{\ell-1} q^{\ell(\frac{n}{2})-\ell(\ell-1)\mu_{\ell}-\sum_{j=\ell+1}^{n} \mu_j} \frac{1 - q^{\ell(\frac{n}{2})+k}}{1 - q^{\ell(\frac{n}{2})+k}} \prod_{1 \leq i < j \leq n - 1} (1 - q^{\ell(\frac{n}{2})+k}) \prod_{1 \leq i \leq j \leq n - 1} (1 - q^{2n-1-i-j}) \cdot \prod_{u \in (\mu + \delta_n)} \frac{1}{1 - q^{h(u)}}. \]

where \( \tilde{\mu}(\ell) \) is as defined in [23]. By [11], we have

\[ Y = \frac{1}{(q; q)_k} \cdot \frac{1 - q^{n|\mu|+(\frac{n}{2})}}{1 - q^{n|\mu|+(\frac{n}{2})+k}} \prod_{u \in (\mu + \delta_n)} \frac{1}{1 - q^{h(u)}}. \]

By [11], (33) and (34), we have

\[ Z = \frac{q^{-|\mu|-(\frac{n}{2})}}{1 - q^{n|\mu|+(\frac{n}{2})+k}} \cdot \frac{1}{(q; q)_k} \prod_{u \in (\mu + \delta_n)} \frac{1}{1 - q^{h(u)}} \times \sum_{\ell=1}^{n} q^{\ell(\frac{n}{2})+k} \frac{1 - q^{\ell(\frac{n}{2})+k}}{1 - q^{\ell(\frac{n}{2})+k}} \prod_{i=1,i \neq \ell}^{n} \frac{1 - q^{2n-1-i-\mu_i+\mu_{\ell}}}{1 - q^{\mu_{\ell}-\mu_i+j-\ell}}. \]

On the other hand, specific computation of the hook lengths of the elements in \( P_{n+1}^{\lambda_1+n-1}(X_{11}) \) gives

\[ \prod_{u \in P_{n+1}^{\lambda_1+n-1}(X_{11})} \frac{1}{1 - q^{h(u)}} = \prod_{j=1}^{n} \prod_{\mu_i+\delta_n} \frac{1}{1 - q^{h(u)}} = \frac{(q; q)_{k-1} (q^{\mu|\mu|+(\frac{n}{2})+k}; q)_{\mu+n} \prod_{j=1}^{n} (1 - q^{\mu|\mu|+(\frac{n}{2})+k+j+1})}{(q; q)_{k-1} (q^{\mu|\mu|+(\frac{n}{2})+k}; q)_{\mu+n} \prod_{j=1}^{n} (1 - q^{\mu|\mu|+(\frac{n}{2})+k+j+1})} \prod_{u \in (\mu + \delta_n)} \frac{1}{1 - q^{h(u)}}. \]

Then by (45), (46), (47), (48) and (49), to prove the hook length property of the semi-irreducible \( d \)-complete posets of class 11, we need to prove the identity

\[ \prod_{j=1}^{n} \frac{1 - q^{(\frac{n}{2})+|\mu|+\mu_j+k+\frac{n(n+3)}{2}+\mu_{\ell}+\delta_n+1}}{1 - q^{(\frac{n}{2})+|\mu|+\mu_j+k+\mu_{\ell}+\delta_n+1}} = \frac{1 - q^{2|\mu|+(\frac{n}{2})+k}}{1 - q^{k}} \times \prod_{j=1,j \neq \ell}^{n} \frac{1 - q^{2n-2-j-\ell+\mu_j+\mu_\ell}}{1 - q^{\mu_j-\mu_\ell+j-\ell}}. \]

By replacing the fraction

\[ \frac{1 - q^{(\frac{n}{2})+|\mu|+\mu_{\ell}+\delta_n+1}}{1 - q^{(\frac{n}{2})+|\mu|+\mu_{\ell}+\delta_n+1}} = 1 - q^{(\frac{n}{2})+|\mu|+\mu_{\ell}+\delta_n+1} \frac{1 - q^{k}}{1 - q^{(\frac{n}{2})+|\mu|+\mu_{j}+\delta_n+1}}; \]

we can rewrite (50) as

\[ \prod_{j=1}^{n} \frac{1 - q^{(\frac{n}{2})+|\mu|+\mu_j+k+\frac{n(n+3)}{2}+\mu_{\ell}+\delta_n+1}}{1 - q^{(\frac{n}{2})+|\mu|+\mu_j+k+\mu_{\ell}+\delta_n+1}} = \frac{1 - q^{2|\mu|+(\frac{n}{2})+k}}{1 - q^{k}} \times \prod_{j=1,j \neq \ell}^{n} \frac{1 - q^{2n-2-j-\ell+\mu_j+\mu_\ell}}{1 - q^{\mu_j-\mu_\ell+j-\ell}}. \]
By (35), we know that
\[ q^{(\ell)} \mu(1 - q^{n(2) \over 2}) = \sum_{\ell=1}^{n} q^{\mu - \ell + 1} (1 - q^{n+1-\ell+\mu}) \prod_{j=1, j \neq \ell}^{n} \frac{1 - q^{2n+2-j-\ell+\mu}_j + \mu}{1 - q^{\mu-\ell+\mu}_j - j - \ell}, \]
thus [31] can be again changed to
\[ (52) \prod_{j=1}^{n} \frac{1 - q^{\mu+\mu_j+k+n(n+3)}_j - j + 1}{1 - q^{\mu-\mu_j+k+n(n+3)}_j - j + 1} \]
\[ = q^{\mu+(n+1) \over 2} + (1 + q^{\mu+(n+1) \over 2} + k) \prod_{\ell=1}^{n} \frac{1 - q^{n+1-\ell+\mu}_j}{1 - q^{\mu}+\mu_j+\mu - k - 1} \prod_{j=1, j \neq \ell}^{n} \frac{1 - q^{2n+2-j-\ell+\mu_j+\mu}}{1 - q^{\mu-\ell+\mu}_j - j - \ell}. \]
To rewrite the left hand side of (52), we use the identity (37) with
\[ x_i \mapsto q^{-\mu_i-1-n}, \quad y_i \mapsto q^{2\mu_i-2i+2+2n}, \quad \text{and} \quad t = q^{\mu+k+(n+1) \over 2}. \]
Then
\[ (53) \prod_{j=1}^{n} \frac{1 - q^{\mu+\mu_j+k+n(n+3)}_j - j + 1}{1 - q^{\mu-\mu_j+k+n(n+3)}_j - j + 1} \]
\[ = q^{\mu+2(n+1) \over 2} + \sum_{\ell=1}^{n} \frac{1 - q^{2n+2-2\ell+2\mu}_j}{1 - q^{\mu+\mu_j+\mu - k - 1}} \prod_{j=1, j \neq \ell}^{n} \frac{1 - q^{2n+2-j-\ell+\mu_j+\mu}}{1 - q^{\mu-\ell+\mu}_j - j - \ell}. \]
Using (53) for the left hand side of (52) and simplifying the terms gives
\[ q^{\mu+(n+1) \over 2} - q^{2\mu+2(n+1) \over 2} \]
\[ = \sum_{\ell=1}^{n} \frac{1 - q^{2n+2-2\ell+2\mu}_j}{1 - q^{\mu+\mu_j+\mu - k - 1}} \prod_{j=1, j \neq \ell}^{n} \frac{1 - q^{2n+2-j-\ell+\mu_j+\mu}}{1 - q^{\mu-\ell+\mu}_j - j - \ell} \]
\[ - (1 + q^{\mu+(n+1) \over 2} + k) \sum_{\ell=1}^{n} \frac{1 - q^{n+1-\ell+\mu}_j}{1 - q^{\mu+\mu_j+\mu - k - 1}} \prod_{j=1, j \neq \ell}^{n} \frac{1 - q^{2n+2-j-\ell+\mu_j+\mu}}{1 - q^{\mu-\ell+\mu}_j - j - \ell}, \]
or
\[ q^{\mu+(n+1) \over 2} (1 - q^{\mu+(n+1) \over 2}) = \sum_{\ell=1}^{n} q^{n+1-\ell+\mu} (1 - q^{n+1-\ell+\mu}) \prod_{j=1, j \neq \ell}^{n} \frac{1 - q^{2n+2-j-\ell+\mu_j+\mu}}{1 - q^{\mu-\ell+\mu}_j - j - \ell}, \]
which is exactly (35).
If the element $A$ exists in the poset $P_{n+1}^{\lambda_1+n-1}(X_{11})$ (i.e., when $\epsilon = 1$), then we have $n + 1$ many integration variables and $\mu$ is equal to $\lambda$. In this case, the above computation still holds, by replacing $n$ by $n + 1$ and keeping $\mu = \lambda$.

**APPENDIX A. FIGURES OF SEMI-IRREDUCIBLE $d$-COMPLETE POSETS**

Here, we provide the pictures of all 15 classes of semi-irreducible $d$-complete posets. In each picture, when there is $\lambda$, the Young poset of the transpose of $\lambda$ is drawn. There are no restrictions on $\lambda$ and $\mu$ except their lengths.

Our description is slightly different from the original description in [8, Table 1] as follows:

- **Class 8:** In [8, Table 1], the circled element in Figures 17, 18, 19, and 20 may or may not be in the poset. In our setting, if the circled element is missing, the poset belongs to Class 10 in Figure 23.
- **Class 9:** In [8, Table 1], the circled element in Figures 21 and 22 may or may not be in the poset. In our setting, if the circled element is missing, the poset belongs to Class 10 in Figure 23.
- **Class 10:** In Class 10, the number of diagonal entries is at least 4. In [8, Table 1], this number is at least 6. Our classification for the number of diagonal entries being 4 or 5 correspond to Class 8 and class 9 in Figures 17, 18, 19, 20, 21, and 22 with the circled element removed.
Hook length property of $d$-complete posets via $q$-integrals

$\lambda = (\lambda_1, \ldots, \lambda_n)$

$\mu = (\mu_1, \ldots, \mu_n)$

Figure 10. A semi-irreducible $d$-complete poset of class 1, $P_n(X_1)$ for $n \geq 2$, $\lambda, \mu \in \text{Par}_n$ and $X_1 = \{(\lambda, n, 1), (\mu, n, 1)\}$. This is irreducible if and only if $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$.

$\mu = (\mu_1, \ldots, \mu_n)$

Figure 11. A semi-irreducible $d$-complete poset of class 2, $P_n(X_2)$ for $n \geq 4$, $\mu \in \text{Par}_n$ and $X_2 = \{(\mu, n, 1)\}$. This is irreducible if and only if $\mu_1 = \mu_2$.

$\mu = (\mu_1, \mu_2)$

$\lambda = (\lambda_1, \lambda_2)$

Figure 12. A semi-irreducible $d$-complete poset of class 3, $P_m^2(X_3)$ for $m \geq 0$, $\lambda, \mu \in \text{Par}_2$ and $X_3 = \{(\lambda, 2, 1), (\mu, 2, 1), ((m), 1, 1)\}$. This is irreducible if and only if $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$. 

\[
\begin{align*}
\mu &= (\mu_1, \ldots, \mu_{n+1}) \\
\lambda &= (\lambda_1, \ldots, \lambda_{n-1})
\end{align*}
\]

**Figure 13.** A semi-irreducible \(d\)-complete poset of class 4, \(P_{n+1}^{\lambda_1 + n - 2}(X_4)\) for \(n \geq 2, k \geq 0, \lambda \in \text{Par}_{n-1}, \mu \in \text{Par}_{n+1}\) and \(X_4 = \{(\lambda, n-1, 1), (\mu, n+1, 1), ((k), 1, n)\}\). This is irreducible if and only if \(k = 0\) and \(\mu_1 = \mu_2\).

\[
\begin{align*}
\mu &= (\mu_1, \mu_2, \mu_3) \\
\lambda &= (\lambda_1, \lambda_2)
\end{align*}
\]

**Figure 14.** A semi-irreducible \(d\)-complete poset of class 5, \(P_{n}^{\lambda_1 + 1}(X_5)\) for \(\lambda \in \text{Par}_2, \mu \in \text{Par}_3\) and \(X_5 = \{((\lambda, 2, 1), (\mu, 3, 1), (0, 2, 2), ((1), 1, 1)\}\}. This is irreducible if and only if \(\mu_1 = \mu_2\).

\[
\begin{align*}
\mu &= (\mu_1, \mu_2, \mu_3, \mu_4) \\
\lambda &= \emptyset, (1, 2), (1, 1)
\end{align*}
\]

**Figure 15.** A semi-irreducible \(d\)-complete poset of class 6, \(P_{n}^{m}(X_6)\) for \(m \geq 0\) and \(\mu \in \text{Par}_4\) with \(\mu_4 \geq 1\) and \(X_6 = \{(\mu, 4, 1), ((m), 1, 2)\}\). This is irreducible if and only if \(\mu_1 = \mu_2\).
Figure 16. A semi-irreducible $d$-complete poset of class 7, $P_4^{\lambda_1+2}(X_7)$ for $\lambda \in \text{Par}_3, \mu \in \text{Par}_2$ and $X_7 = \{(\lambda, 3, 1), (\emptyset, 2, 1), (\mu, 3, 2), (\emptyset, 2, 3)\}$. This is irreducible if and only if $\mu_1 = \mu_2$.

Figure 17. A semi-irreducible $d$-complete poset of class 8-(1), $P_5^{\lambda_1+2}(X_8^{(1)})$ for $\lambda \in \text{Par}_3$ and $X_8^{(1)} = \{(\lambda, 3, 1), ((2), 1, 1), (\emptyset, 2, 1), (\emptyset, 3, 2), (\emptyset, 2, 3)\}$. This poset is always irreducible.

Figure 18. A semi-irreducible $d$-complete poset of class 8-(2), $P_5^{\lambda_1+3}(X_8^{(2)})$ for $\lambda \in \text{Par}_4$ and $X_8^{(2)} = \{(\lambda, 4, 1), ((1, 0), 2, 1), (\emptyset, 2, 2), (\emptyset, 3, 3), (\emptyset, 2, 4)\}$. This poset is always irreducible.
\( \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \)

Figure 19. A semi-irreducible \( d \)-complete poset of class 8-(2), \( P_{\lambda}^{\lambda_1 + 3}(X_8^{(3)}) \) for \( \lambda \in \text{Par}_4 \) and \( X_8^{(3)} = \{(\lambda, 4, 1), ((1, 1), 2, 1), (\emptyset, 2, 2), (\emptyset, 3, 3), (\emptyset, 2, 4)\} \). This poset is always irreducible.

\( \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \)

Figure 20. A semi-irreducible \( d \)-complete poset of class 8-(3), \( P_{\lambda}^{\lambda_1 + 4}(X_8^{(4)}) \) for \( \lambda \in \text{Par}_5 \) and \( X_8^{(4)} = \{(\lambda, 5, 1), (\emptyset, 3, 1), (\emptyset, 2, 3), (\emptyset, 3, 4), (\emptyset, 2, 5)\} \). This poset is always irreducible.
Figure 21. A semi-irreducible $d$-complete poset of class 9-(1), $P_{5}^{\lambda_1+3}(X_{9}^{(1)})$ for $\mu \in \text{Par}_4$ and $X_{9}^{(1)} = \{(\lambda, 4, 1), ((1), 1, 1), ((1, 0), 2, 1), (\emptyset, 2, 2), (\emptyset, 3, 3), (\emptyset, 2, 4)\}$. This poset is always irreducible.

Figure 22. A semi-irreducible $d$-complete poset of class 9-(2), $P_{5}^{\lambda_1+3}(X_{9}^{(2)})$ for $\lambda \in \text{Par}_4$ and $X_{9}^{(2)} = \{(\lambda, 4, 1), ((1), 1, 1), ((1, 0), 2, 1), (\emptyset, 2, 2), (\emptyset, 3, 3), (\emptyset, 2, 4)\}$. This poset is always irreducible.
Figure 23. A semi-irreducible $d$-complete poset of class 10, $P_6^{\lambda_1+4}(X_{10})$ for
\( \lambda \in \text{Par}_5 \) and $X_{10} = \{(\lambda, 5, 1), (\emptyset, 2, 1), ((1), 2, 2), (\emptyset, 2, 3), (\emptyset, 3, 4), (\emptyset, 2, 5)\}$. This poset is always irreducible.

Figure 24. A semi-irreducible $d$-complete poset of class 11, $P_{n+1}^{\lambda_1+n-1}(X_{11})$ for
\( n \geq 3, k \geq 1 \) and $\lambda \in \text{Par}_n$, $\epsilon \in \{0, 1\}$ and $X_{11} = \{((\lambda, n, 1), ((k-1), 3, n-1), (\emptyset, 2, n), ((\epsilon), 1, 1)) \cup \bigcup_{i=1}^{n-2}((\emptyset, 2, i))\}$. The element with label $A$ may or may not be in the poset depending on $\epsilon$. This poset is irreducible if and only if $k = 1$. 
Figure 25. A semi-irreducible $d$-complete poset of class 12, $P_{6}^{\lambda_{1}+3}(X_{12})$ for $\lambda \in \text{Par}_3$ and $X_{12} = \{(\emptyset, 3, 1), (\emptyset, 2, 1), (\lambda, 4, 2), (\emptyset, 2, 3), (\emptyset, 3, 4), (\emptyset, 2, 5)\}$. This poset is always irreducible.

Figure 26. A semi-irreducible $d$-complete poset of class 13, $P_{6}^{\lambda_{1}+3}(X_{13})$ for $\lambda \in \text{Par}_2$ and $X_{13} = \{((1), 1, 1), (\emptyset, 2, 1), (\emptyset, 3, 1), (\lambda, 4, 2), (\emptyset, 2, 3), (\emptyset, 3, 4), (\emptyset, 2, 5)\}$. This poset is always irreducible.

Figure 27. A semi-irreducible $d$-complete poset of class 14, $P_{6}^{m+3}(X_{14})$ for $m \geq 0$ and $X_{14} = \{((2), 1, 1), (\emptyset, 2, 1), (\emptyset, 3, 1), ((m), 4, 2), (\emptyset, 2, 3), (\emptyset, 3, 4), (\emptyset, 2, 5)\}$. This poset is always irreducible.
The unique semi-irreducible $d$-complete poset of class $15$, $P^3_{15}(X_{15})$ for $X_{15} = \{((3), 1, 1), (\emptyset, 2, 1), (\emptyset, 3, 1), (\emptyset, 4, 2), (\emptyset, 2, 3), (\emptyset, 3, 4), (\emptyset, 2, 5)\}$. This poset is irreducible.

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