Abstract. In this paper we continue our study of a multidimensional random walk with zero mean and finite variance killed on leaving a cone. We suggest a new approach that allows one to construct a positive harmonic function in Lipschitz cones under minimal moment conditions. This approach allows also to obtain more accurate information about the behaviour of the harmonic function not far from the boundary of the cone. We also prove limit theorems under new moment conditions.

1. Introduction, main results and discussion

1.1. Notation and assumptions. Consider a random walk \( \{S(n), n \geq 1\} \) on \( \mathbb{R}^d \), \( d \geq 1 \), where
\[ S(n) = X(1) + \cdots + X(n) \]
and \( \{X(n), n \geq 1\} \) is a sequence of independent copies of a random vector \( X = (X_1, X_2, \ldots, X_d) \). Denote by \( S^{d-1} \) the unit sphere in \( \mathbb{R}^d \) centred at the origin and by \( \Sigma \) an open and connected subset of \( S^{d-1} \). Let \( K \) be the cone generated by the rays emanating from the origin and passing through \( \Sigma \), i.e. \( \Sigma = K \cap S^{d-1} \).

Let \( \tau_x \) be the exit time from \( K \) of the random walk with starting point \( x \in K \), that is,
\[ \tau_x = \inf\{n \geq 1 : x + S(n) \notin K\}. \]

In [9, 12] we studied asymptotics for
\[ P(\tau_x > n), \quad n \to \infty, \]
constructed a positive harmonic function for \( S(n) \) killed at leaving \( K \) and prove conditional limit theorems for this random walk. An important role in our approach was played by the harmonic function of the Brownian motion killed at the boundary of \( K \) which can be described as the unique (up to a constant factor), strictly positive on \( K \) solution of the following boundary problem:
\[ \Delta u(x) = 0, \quad x \in K \quad \text{with boundary condition } u|_{\partial K} = 0. \]

This function is homogeneous of a certain order \( p > 0 \), that is \( u(x) = |x|^p u(|x|) \) for \( x \in K \). The function \( u(x) \) and the constant \( p \) can be found as follows. When \( d = 1 \) then there are only two non-trivial cones: \( K = (0, \infty) \) and \( K = (-\infty, 0) \). For these cones the harmonic function is given by \( u(x) = |x| \) and, clearly, \( p = 1 \).

2020 Mathematics Subject Classification. Primary 60G50; Secondary 60G40, 60F17.

Key words and phrases. Random walk, exit time, harmonic function, conditioned process.

This research was partially supported by the Ministry of Science and Higher Education of the Russian Federation, agreement 075-15-2019-1620 date 08/11/2019. D. Denisov was supported by a Leverhulme Trust Research Project Grant RPG-2021-105. V. Wachtel was partially supported by DFG.
Assume now that $d \geq 2$. Let $L_{S^{d-1}}$ be the Laplace-Beltrami operator on $S^{d-1}$ and assume that $\Sigma$ is regular with respect to $L_{S^{d-1}}$. Under this assumption, there exists a complete set of orthonormal eigenfunctions $m_j$ and corresponding eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ satisfying

\[
L_{S^{d-1}} m_j(\sigma) = -\lambda_j m_j(\sigma), \quad \sigma \in \Sigma \\
m_j(\sigma) = 0, \quad \theta \in \partial \Sigma.
\]  

Then

\[
p = \sqrt{\lambda_1 + (d/2 - 1)^2 - (d/2 - 1)} > 0
\]

and the positive harmonic function $u(x)$ of the Brownian motion is given by

\[
u(x) = |x|^p m_1 \left( \frac{x}{|x|} \right), \quad x \in K.
\]  

Note that (2) implies that

\[
u(x) \leq C |x|^p.
\]  

We refer to [2, 3] for some further details about the function $u$ and the properties of the exit times of Brownian motion from the cone $K$.

If $S(n)$ is a one-dimensional random walk with zero mean and finite variance and if $K = (0, \infty)$ then the function $V(x) := E[-S(\tau_x)]$ is harmonic for $\{S(n)\}_{n \geq 0}$ killed at leaving $K$. Typically one proves the finiteness of $E[-S(\tau_x)]$ via the Wiener-Hopf factorisation. In the Appendix of the present paper we give an alternative proof of this fact by constructing an appropriate supermartingale. This is a simplified version of the approach used in this paper for construction of a positive harmonic function. It is worth mentioning that if $S(n)$ is a one-dimensional oscillating random walk then, without further restrictions on its increments, the renewal function of weak descending ladder heights is harmonic.

In [9] (for a particular case of Weyl chambers see [8]) we were considering cones for which the function $u$ can be extended to a harmonic function in a bigger cone. Under this rather restrictive assumption on $K$ we showed that if $E[|X|^p \wedge (2 + \varepsilon)]$ is finite for some $\varepsilon > 0$ then the limit

\[
V(x) = \lim_{n \to \infty} E[u(x + S(n)); \tau_x > n]
\]

is finite for all $x \in K$ and this function is harmonic for the random walk killed at leaving $K$, that is

\[
V(x) = E[V(x + S(1)); \tau_x > 1], \quad x \in K.
\]

We also showed that

\[
\lim_{r \to \infty} \frac{V(r\sigma)}{u(r\sigma)} = 1
\]

for every $\sigma \in \Sigma$. This relation does not give us any information on the behaviour of $V$ close to the boundary of the cone $K$.

Next in [12] we suggested two alternative constructions of $V$. These new constructions allow one to weaken the geometric restrictions on $K$: it suffices to assume that $K$ is either convex or $C^2$ and star-like. A further advantage of these constructions is the estimate

\[
|V(x) - u(x)| \leq C \left( 1 + \frac{|x|^p}{(d(x))^\gamma} \right),
\]
where
\[ d(x) = \text{dist}(x, \partial K) \]
and \( \gamma \) is a sufficiently small positive number. This implies that \( V(x) \sim u(x) \) if \( x \to \infty \) in a such way that \( d(x) \geq |x|^{1-\delta} \) for a sufficiently small \( \delta > 0 \).

The purpose of the present paper is twofold. First, we will relax the moment assumption \( E[|X|^p] < \infty \). It is worth mentioning that if \( p > 2 \) then, as it has been noticed in [9], the assumption \( E[|X|^p] < \infty \) is optimal. Therefore, the question is whether one can replace \( E[X^{2+\varepsilon}] < \infty \) by a weaker condition in the case when \( p \leq 2 \). Second, we will analyze the behaviour of the harmonic function \( V(x) \) for all \( x \) such that \( d(x) \to \infty \).

In this paper we use martingale techniques to construct harmonic functions. This approach is similar to the approach in [22, 23, 24], who used supermartingales to obtain heat kernel estimates for random walks in cones and Lipschitz domains satisfying rather strong moment assumptions.

1.2. **Main result.** It turns out that if one wishes to impose weaker moment assumptions then more restrictive smoothness conditions on \( K \) are required. Throughout we will impose

**Assumption (G).** Cone \( K \) is Lipschitz and star-like, that is there exists \( x_0 \in \Sigma \) such that \( \text{dist}(x_0, x_0 + K) > 0 \). Furthermore, we assume that there exist a constant \( C > 1 \) such that
\[
\begin{align*}
  u(x) &\leq C|x|^{p-1}d(x), \quad x \in K \quad (4) \\
  u(x) &\geq C^{-1}|x|^{p-1}d(x), \quad x \in K. \quad (5)
\end{align*}
\]

**Remark 1.** Assumption (G) holds when \( \Sigma \) is \( C^{1,\alpha} \), see [13] for the definition of \( C^{1,\alpha} \). It holds even in a slightly more general case when \( \Sigma \) is a Lyapunov-Dini surface, as can be seen from [25]. To prove (4) we can consider first the bounded region \( D = \{ x \in K : 0.5 < |x| < 2 \} \). Then the bound \( u(x) \leq Cd(x) \) follows from [25, Theorem 2.4]. For that we need to notice that the bound in this result is proved locally. Since \( u = 0 \) is continuously differentiable at the boundary \( \{ x \in \partial K \cap : 0.75 < |x| < 1.25|x| \} \). Using the homogeneity of \( u \) we can extend the bound to the whole cone. Similarly the lower bound follows from [25, Theorem 2.5] by noticing that this theorem holds locally as well and implies immediately the lower bound for the Green function in the region \( D \). Then, the Boundary Harnack Principle implies the bound for harmonic functions. Finally we use the homogeneity of \( u \) to extend the bound to the whole cone.

We will also need the following moment assumptions.

**Assumption (M).**
\[
\begin{align*}
  (M1) \quad &E[X_i] = 0, i = 1, \ldots d; \\
  (M2) \quad &E[X_i^2] = 1, i = 1, \ldots d \text{ and } E[X_i X_j] = 0, 1 \leq i < j \leq d; \\
  (M3) \quad &E[|X|^2 \text{log}(1 + |X|)] < \infty. \text{ In the case } p > 2 \text{ we additionally assume that } E[|X|^p] \text{ is finite.}
\end{align*}
\]

**Theorem 2.** Let the assumptions (G) and (M) hold. Then the function
\[
V(x) := \lim_{n \to \infty} E[u(x + S(n)); \tau_x > n]
\]
is finite and harmonic for \( \{S(n)\} \) killed at leaving \( K \), i.e.,

\[
V(x) = E[V(x + S(n)); \tau_x > n], \quad x \in K, \; n \geq 1.
\]

Furthermore, if \( p \geq 1 \) then

\[
V(x) \sim u(x) \quad \text{for } x \in K \text{ with } d(x) \to \infty,
\]

and

\[
\sup_{x \in K, |x| = o(n^{p/(2(p-1))})} \left| \frac{E[u(x + S(n)); \tau_x > n] - V(x)}{1 + u(x)} \right| \to 0, \quad n \to \infty.
\]

If \( p < 1 \) then

\[
V(x) \sim u(x) \quad \text{for } x \in K \text{ when } d(x)|x|^{p-1} \to \infty.
\]

In our earlier papers \([9, 12]\) it has been shown that the function \( V(x) \) from (6) is well defined under a slightly stronger moment condition:

\[
E[|X|^{2+\delta}] < \infty \text{ for some } \delta > 0 \text{ instead of } E[|X|^2 \log(1 + |X|)] < \infty.
\]

We will illustrate in examples below that the latter condition is optimal in some sense. On the other hand, in Theorem 2 we impose different geometric restrictions on the cone \( K \). Namely, we replace \( C^2 \) assumption on the cone by a milder Assumption (G). However the alternative convexity assumption seems to require additional moments and is not covered by Theorem 2. An advantage of the current approach is that it allows one to extend the arguments to the case of Markov chains in a more straightforward way, see \([6]\). Construction and behaviour of harmonic functions of Markov chains were considered various situations by many authors, see \([5, 10, 14, 15, 19]\) and references therein for some of publications. Another improvement of Theorem 2 over \([9, 12]\) is a more accurate information about the asymptotics behaviour of \( V(x) \) near the boundary of the cone.

**Theorem 3.** Let the assumptions (G) and (M) hold true. Assume also that \( p \geq 1 \). Then

(a) there exist positive constants \( C \) and \( R \) such that

\[
P(\tau_x > n) \leq C \frac{u(x + Rx_0)}{n^{p/2}} \quad \text{for all } x \in K,
\]

where \( x_0 \) is defined in (G);

(b) uniformly in \( x \in K \) such that \( |x| \leq \frac{\sqrt{n}}{\log n} \),

\[
P(\tau_x > n) \sim \frac{V(x)}{n^{p/2}}, \quad n \to \infty,
\]

and

\[
P \left( \frac{x + S(n)}{\sqrt{n}} \in D \middle| \tau_x > n \right) \to c \int_D u(z) e^{-|z|^2/2} dz, \quad n \to \infty,
\]

for every compact \( D \subset K \).

1.3. **Discussion of the \( |X|^2 \log(1 + |X|) \)-condition.** In this section we shall consider some specific examples of random walks, which will show that the condition \( E[|X|^2 \log(1 + |X|)] < \infty \) is rather close to the minimal one. We restrict our attention to the cone

\[
K = \{ x \in \mathbb{R}^2 : |x_2| < x_1 \},
\]
which is a 2-dimensional Weyl chamber of type $D$. The harmonic function of the Brownian motion killed at leaving $K$ is easy to compute:

$$u(x) = x_1^2 - x_2^2.$$ 

Therefore, $p = 2$ for this cone.

Let $W = (W_1, W_2)$ be a random vector with the following distribution:

$$P(W_1 = 0, W_2 = 1) = P(W_1 = 0, W_2 = -1) = \frac{1}{4},$$

$$P(W_1 = k, W_2 = 0) = p_k, \quad k = -1, 0, 1, 2, \ldots,$$  

where the numbers $\{p_k\}$ are such that

$$\sum_{k=-1}^{\infty} p_k = \frac{1}{2}, \quad \sum_{k=-1}^{\infty} kp_k = 0 \quad \text{and} \quad \sum_{k=-1}^{\infty} k^2p_k = \frac{1}{2}.$$ 

Then one has the equalities

$$E[W_1] = E[W_2] = E[W_1W_2] = 0 \quad \text{and} \quad E[W_1^2] = E[W_2^2] = \frac{1}{2}. \tag{12}$$

**Example 4.** We assume that the increments of $S(n)$ are independent copies of the vector $\sqrt{2}W$. It follows from (12) that the vector $\sqrt{2}W$ has zero mean and that its covariance matrix is equal to the identity matrix. This implies that the sequence $u(x + S(n))$ is a martingale. Furthermore, it is immediate from the definition of $W$ that

$$u(x + S(\tau_x)) = 0 \quad \text{for every} \quad x \in (\sqrt{2}\mathbb{Z}^2) \cap K.$$ 

Using this observation, we obtain

$$E[u(x + S(n))1\{\tau_x > n\}|F_{n-1}]$$

$$= E[u(x + S(n))1\{\tau_x > n - 1\} - u(x + S(n))1\{\tau_x = n\}|F_{n-1}]$$

$$= 1\{\tau_x > n - 1\}E[u(x + S(n))|F_{n-1}] - E[u(x + S(\tau_x))1\{\tau_x = n\}|F_{n-1}]$$

$$= u(x + S(n-1))1\{\tau_x > n - 1\}.$$ 

In particular,

$$u(x) = E[u(x + S(1))1\{\tau_x > 1\}], \quad x \in (\sqrt{2}\mathbb{Z}^2) \cap K.$$ 

Thus, $u(x)$ is harmonic for $S(n)$ killed at leaving $K$. So, a harmonic function may exist without further moment restrictions. We now show that the relation

$$P(\tau_x > n) \sim \frac{u(x)}{n} \tag{13}$$

may fail in this case. To this end we assume that the numbers $\{p_k\}$ are such that

$$\log mE[W_1^2; W_1 \geq m] \to \infty \quad \text{as} \quad m \to \infty. \tag{14}$$

Consider now the stopping time

$$\sigma_n := \inf\{k \geq 1 : X_1(k) \geq 2n^2\}.$$ 

It is easy to see that if $\tau_x > \sigma_n = j$ then

$$S_1(k) \geq X_1(j) - \sqrt{2}(k - j) \quad \text{and} \quad |S_2(k)| \leq \sqrt{2}k \quad \text{for all} \quad k \geq j.$$ 

In particular,

$$\{\tau_x > n, \sigma_n = j\} = \{\tau_x > j - 1, \sigma_n = j\}$$
and, for all \( n \) large enough,
\[
\lim_{n \to \infty} P(\tau_x > j - 1, \sigma_n > j) = 0.
\]
Therefore, \( \{\tau_x > j - 1, \sigma_n = j\} \). As a result we have
\[
\E[u(x + S(n)); \tau_x > n] \geq \E[u(x + S(n)); \tau_x > n, \sigma_n \leq n]
\]
\[
\geq \sum_{j=1}^{n} P(\tau_x > j - 1, \sigma_n > j - 1) \frac{1}{2} \E[X_1^2(j); X_1(j) \geq 2n^2]
\]
\[
= \E[W_1^2; W_1 \geq \sqrt{2n^2}] \sum_{j=1}^{n} P(\tau_x > j - 1, \sigma_n > j - 1).
\]
Furthermore, for every \( j \leq n \) one has
\[
P(\tau_x > j - 1, \sigma_n > j - 1) \geq P(\tau_x > j - 1) - P(\sigma_n \leq j - 1)
\]
\[
\geq P(\tau_x > j - 1) - (j - 1)P(W_1 \geq \sqrt{2n^2}).
\]
Therefore,
\[
\E[u(x + S(n)); \tau_x > n] \geq \E[W_1^2; W_1 \geq \sqrt{2n^2}] \sum_{j=1}^{n} P(\tau_x > j - 1) - n^2P(W_1 \geq \sqrt{2n^2})
\]
\[
= \E[W_1^2; W_1 \geq \sqrt{2n^2}] \sum_{j=1}^{n} P(\tau_x > j - 1) + o(1). \tag{15}
\]
If (13) holds then
\[
\sum_{j=1}^{n} P(\tau_x > j - 1) \sim \kappa u(x) \log n.
\]
Plugging this into (15) and taking into account (14), we conclude that
\[
\E[u(x + S(n)); \tau_x > n] \to \infty.
\]
This contradicts to the harmonicity of \( u(x) \). Consequently, (13) can not hold for walks satisfying (14).

We next show that the limit of the sequence \( \E[u(x + S(n)); \tau_x > n] \) may be infinite.

**Example 5.** Now we assume that the increments of \( S(n) \) are independent copies of the vector \( \sqrt{2}(W_2, W_1) \). In this case we have
\[
u(x + S(\tau_x)) \leq 0 \quad \text{for all} \quad x \in (\sqrt{2}\Z^2) \cap K. \tag{16}
\]
By the optional stopping theorem for the martingale \( u(x + S(n)) \),
\[
u(x) = \E[u(x + S(\tau_x \wedge n))]
\]
\[
= \E[u(x + S(n)); \tau_x > n] + \E[u(x + S(\tau_x)); \tau_x \leq n].
\]
Consequently,
\[
\E[u(x + S(n)); \tau_x > n] = u(x) - \E[u(x + S(\tau_x)); \tau_x \leq n].
\]
(16) allows one to apply the monotone convergence theorem. As a result we have
\[
\lim_{n \to \infty} \E[u(x + S(n)); \tau_x > n] = u(x) - \E[u(x + S(\tau_x))]. \tag{17}
\]
For every \( y \in K \) we have

\[
\begin{align*}
\mathbf{E}[-u(y + X); y + X \notin K] &= \mathbf{E}[(y_2 + X_2)^2 - (y_1 + X_1)^2; y + X \notin K] \\
&= \mathbf{E}[(y_2 + X_2)^2 - (y_1 + X_1)^2; X_2 > y_1 - y_2] \\
&= \mathbf{E}[(y_2 + X_2)^2 - y_1^2; X_2 > y_1 - y_2] \\
&\geq \frac{1}{2} \mathbf{E}[(y_2 + X_2)^2; X_2 > \sqrt{2}y_1 - y_2] \\
&\geq \frac{1}{8} \mathbf{E}[X_2^2; X_2 > 2(|y_1| + |y_2|)].
\end{align*}
\]

Therefore,

\[
\begin{align*}
\mathbf{E}[-u(x + S(\tau_x))] &= \sum_{n=0}^{\infty} \int_{K} \mathbf{P}(x + S(n) \in dy; \tau_x > n) \mathbf{E}[-u(y + X); y + X \notin K] \\
&\geq \frac{1}{8} \sum_{n=0}^{\infty} \int_{K} \mathbf{P}(x + S(n) \in dy; \tau_x > n) \mathbf{E}[X_2^2; X_2 > 2(|y_1| + |y_2|)] \\
&= \frac{1}{4} \sum_{n=0}^{\infty} \int_{K} \mathbf{P}(x + S(n) \in dy; \tau_x > n) \mathbf{E}[W_1^2; W_1 > \sqrt{2}(|y_1| + |y_2|)] \\
&\geq \frac{1}{4} \sum_{n=0}^{\infty} \mathbf{P}(|x_1 + S_1(n)| + |x_2 + S_2(n)| \leq n^2; \tau_x > n) \mathbf{E}[W_1^2; W_1 > \sqrt{2}n^2].
\end{align*}
\]

(18)

It is immediate from the definition of the vector \( W \) that

\[
|x_2 + S_2(n)| < x_1 + S_1(n) \leq x_1 + \sqrt{2}n
\]
on the event \( \{\tau_x > n\} \). This yields the existence of \( n_0 = n_0(x) \) such that

\[
\mathbf{P}(|x_1 + S_1(n)| + |x_2 + S_2(n)| \leq n^2; \tau_x > n) = \mathbf{P}(\tau_x > n), \quad n \geq n_0.
\]

(19)

If (14) is valid and

\[
\mathbf{P}(\tau_x > n) \geq \frac{c(x)}{n}
\]

then

\[
\lim_{n \to \infty} \mathbf{E}[u(x + S(n)); \tau_x > n] = \infty.
\]

Combining this with (17), we conclude that

\[
\lim_{n \to \infty} \mathbf{E}[u(x + S(n)); \tau_x > n] = \infty.
\]

\[\diamond\]

It turns out that if the increments of the walk are independent copies of of the vector \( \sqrt{2}(W_2, W_1) \) then the asymptotic behaviour of \( \mathbf{P}(\tau_x > n) \) can be studied without constructing a positive harmonic function.

**Proposition 6.** For the random walk from Example 5 we have:

(a) the sequence \( E_n := \mathbf{E}[u(x + S(n)); \tau_x > n] \) is monotone increasing and slowly varying;

(b) as \( n \to \infty \),

\[
\mathbf{P}(\tau_x > n) \sim \frac{E_n}{n};
\]

(c) the limit of \( E_n \) is finite if and only if \( \mathbf{E}[W_1^2 \log(1 + W_1)] \) is finite.
2. Preliminary estimates

In this section we will present some preliminary bounds.

2.1. Estimates for the harmonic and Green functions. Here we collect some information that will be used in the sequel.

Recall that \( d(x) = \text{dist}(x, \partial K) \) and note that following simple bound holds

\[ d(x) \leq |x|, \quad x \in K. \]

We will use the standard multi-index notation for partial derivatives, that is for \( \alpha = (\alpha_1, \ldots, \alpha_d) \) we put

\[ |\alpha| = \alpha_1 + \cdots + \alpha_d \]

\[ \partial^\alpha f(x) = \frac{\partial |\alpha|}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} f(x). \]

We shall always assume that \( u(x) = 0 \) for all \( x \not\in K \).

We will make use of the following result, see [12, Lemma 2.1].

Lemma 7. There exists a constant \( C = C(d) \) such that for \( x \in K \) and \( \alpha \) with \( |\alpha| \leq 3 \),

\[ \left| \frac{\partial^\alpha u(x)}{\partial x^\alpha} \right| \leq C \frac{u(x)}{d(x)^{|\alpha|}}. \]  

We will also make use of the following result proved in [12, Lemma 2.3].

Lemma 8. Assume that equation (4) holds. Let \( x \in K \). Then,

\[ |u(x + y) - u(x)| \leq C |y| \left( |x|^{p-1} + |y|^{p-1} \right) \]  

and, for \(|y| \leq |x|/2\),

\[ |u(x + y) - u(x)| \leq C |y||x|^{p-1}. \]  

For \( p < 1 \) and \( x \in K \),

\[ |u(x + y) - u(x)| \leq C |y|^p. \]  

This lemma has been formulated under slightly different assumptions. However, it can be seen from the proof that assumption (4) is sufficient.

Let \( G(x, y) \) be the Green function of Brownian motion killed on leaving the cone \( K \). For all \( y \in K \) this function is harmonic in \( x \neq y \) and vanishes at the boundary of the cone.

Lemma 9. Let the assumption (G) hold. Then, for any \( A > 0 \) there exists a constant \( C_A \) such that

\[ G(x, y) \leq C_A \widehat{G}(x, y), \]  

where

\[ \widehat{G}(x, y) := \begin{cases} \frac{u(x)u(y)}{|y|^{p-2} + |x-y|^{p-2}}, & |x| \leq |y|, |x-y| \geq A|y|, \\
\frac{u(x)u(y)}{|y|^{p-2} + |x-y|^{p-2}}, & |y| \leq |x|, |x-y| \geq A|y|, \\
\frac{u(x)u(y)}{|y|^{p-1}|x-y|^{p-1} + 1}, & d(y)/2 \leq |x - y| \leq A|y|, \\
\frac{1}{|x-y|^{p-1}} I(d > 2) + \ln \left( \frac{d(y)}{|x-y|} \right) I(d = 2), & |x - y| < d(y)/2. \end{cases} \]
Proof. When $\Sigma$ is of class $C^2$ the required bound was proved in [1, Lemma 1 and Lemma 4]. The result was derived from [17] (see also [25, Theorem 2.3]) by transferring the estimate
$$G(x, y) \leq \frac{d(x)d(y)}{|x - y|^d}$$
for the Green function proved for bounded Lyapunov domains to an estimate for cones. It is clear that the same proof will work provided condition (5) holds.

Alternatively, for $d \geq 3$ the bound follows from the following result proved in [16, Remark 3.1],
$$G(x, y) \leq C \frac{u(x)u(y)}{\max(|x|, |y|)^2|x - y|^d}.$$ 
Indeed when $\Sigma$ is Lipschitz the cone $K$ is uniform. Then the result follows from the assumptions (4) and (5). The case $d = 2$ follows from [1, Lemma 1 and Lemma 4]. For $d = 2$ the last line in the bound is an estimate of the Green function for cone $K$ can be derived via the Green function for the ball $B(y, d(y)/2)$. □

2.2. Estimates for the error term. Let $u(x) = 0$ when $x \notin K$ and let
$$f(x) = E[u(x + X)] - u(x). \quad (26)$$
The following lemma will be convenient in the construction to follow.

Lemma 10. Assume that $E[|X|^2] < \infty$ and, in addition,
$$\begin{cases} E[|X|^p] < \infty, & p > 2; \\ E[|X|^2 \log(1 + |X|)] < \infty, & p = 2. \end{cases}$$
Then, there exists a slowly varying, monotone decreasing differentiable function $\gamma(t)$ such that
$$E[|X|^p; |X| > t] = o(\gamma(t)t^{p-2}) \quad (27)$$
and
$$\int_1^\infty x^{-1}\gamma(x)dx < \infty. \quad (28)$$
Proof. We consider first the case $p \geq 1$. It is clear that
$$t \mapsto \frac{E[|X|^p; |X| > t]}{t^{p-1}}$$
is monotone decreasing.

Furthermore, by the Fubini theorem,
$$\int_{1}^{\infty} \frac{E[|X|^p; |X| > t]}{t^{p-1}} dt = E \left[ |X|^p \int_{1}^{\infty} t^{1-p} dt; |X| > 1 \right]$$
$$\leq \begin{cases} \frac{E[|X|^2]}{(2-p)}, & p < 2; \\ E[|X|^2 \log(1 + |X|)], & p = 2; \\ \frac{E[|X|^p]}{p-2}, & p > 2. \end{cases}$$
Thus we may apply the result of [4]: there exists a slowly varying function $\ell(t)$ such that $\ell(t)/t$ is integrable and
$$\frac{E[|X|^p; |X| > t]}{t^{p-1}} \leq \ell(t)/t.$$ 
Equivalently,
$$E[|X|^p; |X| > t] \leq \ell(t)t^{p-2}. $$
It is possible to choose a decreasing, slowly varying function $\gamma(t)$ such that $\gamma(t)/t$ is integrable and $\ell(t) = o(\gamma(t))$. This completes the proof in the case $p \geq 1$.

When $p < 1$ then, using the Markov inequality, we get

$$\frac{\mathbb{E}[|X|^p; |X| > t]}{t^{p-1}} = t^{1-p} \mathbb{E} \left[ \frac{|X|}{|X|^{1-p}}; |X| > t \right] \leq \mathbb{E}[X; |X| > t].$$

Now it remains to apply the already proven estimate for $p = 1$.

Finally note that it is not difficult to achieve differentiability of $\gamma$ by adjusting it.

**Lemma 11.** Assume that equation (4) holds and let $f$ be defined by (26). Then,

$$|f(x)| = o(\beta(x)),$$

as $d(x) \to \infty$,

where

$$\beta(x) := |x|^{p-1} \frac{\gamma(d(x))}{d(x)}$$

and function $\gamma(x)$ is constructed in Lemma 10.

**Proof.** We will start with a Taylor theorem for a thrice differentiable function $U$. Let $y$ be such that $|y| \leq d(x)/2$. Then,

$$\left| U(x + y) - U(x) - \nabla U(x) \cdot y - \frac{1}{2} \sum_{i,j} \frac{\partial^2 U(x)}{\partial x_i \partial x_j} y_i y_j \right| \leq R_{2,0}(x)|y|^{2+\theta},$$

where

$$R_{2,0}(x) = \sup_{i,j} [U_{x_i x_j}]_{\theta, B(x,d(x)/2)} < \infty.$$ 

Here, we let for $\theta \in (0, 1]$ and open $D$,

$$[f]_{\theta, D} = \sup_{y, z \in D, y \neq z} \frac{|f(y) - f(z)|}{|y - z|^{\theta}}.$$

We can further proceed as follows

$$\left| \mathbb{E} \left[ U(x + X) - U(x) - \frac{1}{2} \Delta U(x) \right] \right|$$

$$= \left| \mathbb{E} \left[ U(x + X) - U(x); |X| \leq d(x)/2 \right] - \frac{1}{2} \Delta U(x) \right|$$

$$+ \left| \mathbb{E} \left[ U(x + X) - U(x); |X| > d(x)/2 \right] \right|$$

$$\leq \left| \mathbb{E} \left[ \left( \nabla U(x) \cdot X + \frac{1}{2} \sum_{i,j} \frac{\partial^2 U(x)}{\partial x_i \partial x_j} X_i X_j \right) 1(|X| \leq d(x)/2) \right] - \frac{1}{2} \Delta U(x) \right|$$

$$+ R_{2,0}(x) \mathbb{E} \left[ |X|^{2+\theta}; |X| \leq d(x)/2 \right] + \left| \mathbb{E} \left[ U(x + X) - U(x); |X| > d(x)/2 \right] \right|. $$
Now rearrange the terms and make use of the assumptions $E[X_i] = 0, \text{cov}(X_i, X_j) = \delta_{i,j}$ to obtain

$$
\left| E \left[ U(x + X) - u(x) - \frac{1}{2} \Delta U(x) \right] \right|
\leq \left| E \left[ \left( \nabla U(x) \cdot X + \frac{1}{2} \sum_{i,j} \frac{\partial^2 U(x)}{\partial x_i \partial x_j} X_i X_j \right) 1(|X| > d(x)/2) \right] \right|
+ R_{2,0}(x)E \left[ |X|^{2+\theta}; |X| \leq d(x)/2 \right] + \left| E \left[ U(x + X) - U(x); |X| > d(x)/2 \right] \right|.
$$

We will now make use of (31) with $U = u$ and $\theta = 1$. Applying the inequalities (20), (3) and (4) one can estimate the remainder as follows,

$$
|R_{2,1}(x)| \leq |x|^{p-1}d(x)^{-2}.
$$

Also, using (21) and (22), we can estimate

$$
\left| E \left[ u(x + X) - u(x); |X| > d(x)/2 \right] \right|
\leq C E[|X|^p; |X| > |x|/2] + C|x|^{p-1}E[|X|; |X| > d(x)/2].
$$

Thus, it follows from (31) that

$$
|f(x)| \leq \left| E \left[ \left( \nabla u(x) \cdot X + \frac{1}{2} \sum_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} X_i X_j \right) 1(|X| > d(x)/2) \right] \right|
+ C|x|^{p-1}d(x)^{-2}E \left[ |X|^3; |X| \leq d(x)/2 \right]
+ C E[|X|^p; |X| > |x|/2] + C|x|^{p-1}E[|X|; |X| > d(x)/2].
$$

The partial derivatives of the function $u$ in the first term can be estimated via Lemma 7 and (4), which results in the following estimate

$$
|E[\nabla u(x) \cdot X; |X| > d(x)/2]| \leq C|x|^{p-1}E[|X|; |X| > d(x)/2].
$$

We estimate the terms with the second derivative using (20) and (4),

$$
\frac{1}{2} \left| E \left[ \sum_{i,j} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} X_i X_j; |X| > d(x)/2 \right] \right| \leq C|x|^{p-1}d(x)^{-1}E[|X|^2; |X| > d(x)/2].
$$

Then,

$$
|f(x)| \leq C \left[ |x|^{p-1}E[|X|; |X| > d(x)/2] + |x|^{p-1}d(x)^{-1}E[|X|^2; |X| > d(x)/2] \right.
+ \left. |x|^{p-1}(d(x))^{-2}E[|X|^3; |X| \leq d(x)/2] \right]
+ E[|X|^p; |X| > |x|/2] + |x|^{p-1}E[|X|; |X| > d(x)/2].
$$
The term with the moment of the order 3 can be estimated using (27) with $p = 2$,
\[
E \left[ |X|^3; |X| \leq \frac{d(x)}{2} \right] \leq 3 \int_0^{d(x)/2} y^2 P(|X| > y) dy
\]
\[
\leq A^3 + 3 \int_A^{d(x)/2} y^2 E\left[ |X|^2; |X| > y \right] dy
\]
\[
\leq A^3 + \varepsilon_A \int_A^{d(x)/2} \gamma(y) dy \leq A^3 + \varepsilon_A C d(x) \gamma(d(x)),
\]
where $\varepsilon_A \to 0$ as $A \to \infty$ and we have used the slow variation of $\gamma$ in the final step. Hence,
\[
E \left[ |X|^3; |X| \leq \frac{d(x)}{2} \right] = o(1) d(x) \gamma(d(x)), \quad d(x) \to \infty.
\]
Then, for $p \geq 2$, using the Markov inequality we can further simplify the right-hand-side as follows,
\[
|f(x)| \leq C E[ |X|^p; |X| > |x| ] + o(1) |x|^{p-1} \gamma(d(x)) \frac{d(x)}{d(x)}.
\]
Applying Lemma 10 we obtain
\[
|f(x)| \leq o(1) \left( \gamma(x) |x|^{p-2} + |x|^{p-1} \frac{\gamma(d(x))}{d(x)} \right)
\]
\[
\leq o(1) |x|^{p-1} \frac{\gamma(d(x))}{d(x)},
\]
since $d(x) \leq |x|$. For $p < 2$ we have the following estimate
\[
|f(x)| \leq o(1) |x|^{p-1} \gamma(d(x)) \frac{d(x)}{d(x)}.
\]

3. CONSTRUCTION OF A NON-NEGATIVE SUPERMARTINGALE

For $x \in K$ let $\beta(x)$ be the function defined in (29). Let
\[
U_\beta(y) = \int_K G(x,y) \beta(x) dx.
\]
By the definition of the Green function
\[
\Delta_\beta U(y) = -\beta(y).
\]
Using (24) one can show that $U_\beta$ is well defined. For that we will estimate the integral in (33) in a sequence of Lemmas. Pick $A < 1$ and $C_A$ such that Lemma 9 holds.

**Lemma 12.** Let the assumption (G) hold. Then, there exists a function $\varepsilon_1(R) \to 0$ such that for $y \in K : |y| > R$,
\[
I_1(y) := \int_{K \cap \{|x| \leq |y|, |x-y| \geq A|y|\}} \hat{G}(x,y) \beta(x) dx \leq \varepsilon_1(R) u(y).
\]

**Proof.** Put $d_{\Sigma}(\sigma) = \text{dist}(\sigma, \partial \Sigma)$. Then there exists a constant $c_0 > 0$ such that
\[
d(r \sigma) < rd_{\Sigma}(\sigma) < c_0 d(r \sigma).
\]
We have, using (4),
\[
\frac{I_1(y)}{u(y)} \leq C|y|^{2-d-2p} \int_{K \cap \{|x| \leq |y|\}} u(x)|x|^{p-1} \frac{\gamma(d(x))}{d(x)} \, dx
\]
\[
\leq C|y|^{2-d-2p} \int_{K \cap \{|x| \leq |y|\}} |x|^{2p-2} \gamma(d(x)) \, dx
\]
\[
\leq C|y|^{2-d-2p} \int_{\Sigma} d\sigma \int_{0}^{|y|} r^{2p+d-3} \gamma(d(r\sigma)) \, dr.
\]
Then we can split the \( \Sigma = \Sigma_0 \cup \Sigma_1 \) in such a way that for \( \Sigma_0 \) contains all \( \sigma \in \Sigma \) with the distance \( d_\Sigma(\sigma) \leq \varepsilon_0 \), where \( \varepsilon_0 \) is sufficiently small to ensure that
\[
C|y|^{2-d-2p} \int_{\Sigma_0} d\sigma \int_{0}^{|y|} r^{2p+d-3} \gamma(d(r\sigma)) \, dr
\]
\[
\leq \sup_{t \geq 0} \gamma(t) C|y|^{2-d-2p} \int_{\Sigma_0} d\sigma \int_{0}^{|y|} r^{2p+d-3} \, dr < \text{Area}(\Sigma_0) \leq \varepsilon/2.
\]
Next using (36) and regular variation of \( \gamma \),
\[
C|y|^{2-d-2p} \int_{\Sigma_1} d\sigma \int_{0}^{|y|} r^{2p+d-3} \gamma(d(r\sigma)) \, dr
\]
\[
\leq C|y|^{2-d-2p} \int_{0}^{|y|} r^{2p+d-3} \gamma(c_0^{-1} r \varepsilon_0) \, dr
\]
\[
\leq C\gamma(|y|) \leq \varepsilon/2,
\]
for \( |y| > R \) and sufficiently large \( R \) due to the fact that \( \gamma(t) \to 0 \).
\( \square \)

**Lemma 13.** Let the assumption (G) hold. Then, there exists a function \( \varepsilon_2(R) \to 0 \) such that for \( y \in K : |y| > R \),
\[
I_2(y) := \int_{K \cap \{|x| \geq |y|, |x-y| \geq 4|y|\}} \hat{G}(x,y) \beta(x) \, dx \leq \varepsilon_2(R) u(y). \quad (37)
\]

**Proof.** To prove the statement it is sufficient to show that for any \( \varepsilon > 0 \) there exists \( R > 0 \) such that \( I_2(y) \leq \varepsilon u(y) \) for all \( y : |y| > R \). We have, using (4),
\[
\frac{I_2(y)}{u(y)} \leq C \int_{K \cap \{|x| \geq |y|\}} u(x)|x|^{2-d-2p}|x|^{p-1} \frac{\gamma(d(x))}{d(x)} \, dx
\]
\[
\leq C \int_{K \cap \{|x| \geq |y|\}} |x|^{-d} \gamma(d(x)) \, dx
\]
\[
\leq C \int_{\Sigma} d\sigma \int_{|y|}^{\infty} r^{-1} \gamma(c_0^{-1} r d_\Sigma(\sigma)) \, dr
\]
\[
\leq C \int_{\Sigma} d\sigma \int_{|y|c_0^{-1} d_\Sigma(\sigma)}^{\infty} r^{-1} \gamma(r) \, dr.
\]
Let $c_1 = \text{diam}(\Sigma)$. Then,

$$C \int \sigma \int_0^\infty \frac{r^{-1}\gamma(r)}{r_0^{-1}(|y|d\Sigma(r))} dr = C \int_0^\infty r^{-1}\gamma(r) \int_{\Sigma \cap \{c_0^{-1}d\Sigma(r) < r\}} d\sigma dr$$

$$\leq C \int_0^\infty r^{-1}\gamma(r) \min \left(\frac{r_0}{|y|}c_1\right) dr \leq \varepsilon,$$

since

$$\int_0^\infty r^{-1}\gamma(r) \min \left(\frac{r_0}{|y|}c_1\right) dr = \frac{c_0}{|y|} \int_0^{c_0|y|} \gamma(r) dr + c_1 \int_{c_0|y|}^\infty \gamma(r) r^{-1} dr$$

can be made small for $|y| > R$ and sufficiently large $R$ due to the convergence of the integral and the fact that $\gamma$ converges to 0.

\[\square\]

**Lemma 14.** Let the assumption \((G)\) hold. Then, there exists a bounded monotone function $\varepsilon_3(R) \to 0$ such that for $y \in K : d(y) > R$,

$$I_3(y) := \int_{K \cap \{d(y)/2 < |x-y| < A|y|\}} G(x,y)\beta(x) dx \leq \varepsilon_3(R)u(y). \quad (38)$$

**Proof.** First note that $|x-y| < A|y|$ with $A < 1$ implies that

$$(1 - A)|y| < |x| < (1 + A)|y|.$$

We have,

$$\frac{I_3(y)}{u(y)} \leq C|y|^{1-p} \int_{K \cap \{d(y)/2 < |x-y| < A|y|\}} \frac{u(x)|x|^{1-p}}{|x-y|^d} \beta(x) dx$$

$$\leq C|y|^{1-p} \int_{K \cap \{d(y)/2 < |x-y| < A|y|\}} \frac{d(x)}{|x-y|^d} |x|^{p-1} \frac{\gamma(d(x))}{d(x)} dx$$

$$\leq C \int_{K \cap \{d(y)/2 < |x-y| < A|y|\}} \frac{\gamma(d(x))}{|x-y|^d} dx.$$

Splitting the integral into regions we obtain

$$\frac{I_3(y)}{u(y)} \leq C \sum_{n=0}^{[\log_2 A|y|/d(y)]} \int_{K \cap \{2^n d(y)/2 < |x-y| < 2^n d(y)\}} \frac{\gamma(d(x))}{|x-y|^d} dx$$

$$\leq \sum_{n=0}^{[\log_2 A|y|/d(y)]} \frac{C}{2^n d(y)^d} \int_{K \cap \{2^n d(y)/2 < |x-y| < 2^n d(y)\}} \gamma(d(x)) dx$$

$$\leq \sum_{n=0}^{[\log_2 A|y|/d(y)]} \frac{C}{2^{4n} d(y)^d} (2^n d(y))^{d-1} \int_0^{2^n d(y)} \gamma(r) dr$$

$$\leq C \sum_{n=0}^{[\log_2 A|y|/d(y)]} \gamma(2^n d(y)),$$

using the slow variation of $\gamma$ in the last inequality and the definition of $A$. Then,

$$\frac{I_3(y)}{u(y)} \leq C \int_0^\infty \gamma(2^{-1}d(y)) dt \leq C \int_1^\infty \frac{\gamma(zd(y))}{z} dz = C \int_0^\infty \frac{\gamma(zd(y))}{z} dz.$$
We obtain immediately that for $y : d(y) > R$,
\[
\frac{I_3(y)}{u(y)} \leq C \int_{R/2}^{\infty} \frac{\gamma(zd(y))}{z} dz,
\]
which is finite and converges to 0 as $R \to \infty$. \hfill \Box

**Lemma 15.** Let the assumption (G) hold. Then, there exists a bounded monotone decreasing function $\varepsilon_4(R) \to 0$, as $R \to \infty$, such that for $y \in K : d(y) > R$,
\[
I_4(y) := \int_{K \cap \{|x-y|<d(y)/2\}} \hat{G}(x,y)\beta(x) dx \leq \varepsilon_4(R) u(y). \quad (39)
\]

**Proof.** We consider the case $d \geq 3$, as the case $d = 2$ is similar. For $|x-y| \leq d(y)/2$ we use the bound
\[
I_4(y) \leq C \int_{K \cap \{|x-y|\leq d(y)/2\}} \frac{1}{|x-y|^{d-2}} \beta(x) dx
\]
\[
\leq C \int_{K \cap \{|x-y|\leq d(y)/2\}} \frac{|x|^{p-1}}{|x-y|^{d-2}} \gamma(d(x)) dx
\]
Note that $|x-y| \leq d(y)/2$ implies that $\frac{|y|}{2} \leq |x| \leq \frac{3|y|}{2}$ and $\frac{d(y)}{2} \leq d(x) \leq \frac{3d(y)}{2}$. Using these estimates and (5) we obtain
\[
\frac{I_4(y)}{u(y)} \leq \frac{C \gamma(d(y))}{d(y)^2} \int_{\{|x-y|\leq d(y)/2\}} \frac{dx}{|x-y|^{d-2}} \leq C \gamma(d(y)),
\]
implying the statement of the theorem since $\gamma$ is monotone. \hfill \Box

**Lemma 16.** Let the assumption (G) hold. Then, $U_\beta(y)$ is finite. Moreover, there exists a bounded and monotone function $\varepsilon(R) \to 0$, as $R \to \infty$, such that for $y \in K : d(y) > R$, the following estimate is valid
\[
U_\beta(y) \leq \varepsilon(R) u(y). \quad (40)
\]

**Proof.** The statement follows from the bound $G(x,y) \leq C \hat{G}(x,y)$ and Lemma 12–Lemma 15. \hfill \Box

We will also need estimates for the derivatives $U_\beta$.

**Lemma 17.** Let the assumption (G) hold. Then, there exists a function $\varepsilon(R) \to 0$ such that for $y \in K : d(y) > R$ we have
\[
|U_\beta(y)| \leq \varepsilon(R) u(y), \quad (41)
\]
\[
\left| \frac{\partial U_\beta(y)}{\partial y_i} \right| \leq \varepsilon(R) \frac{u(y)}{d(y)} + Cd(y) \beta(y), \quad (42)
\]
\[
\left| \frac{\partial^2 U_\beta(y)}{\partial y_i \partial y_j} \right| \leq \varepsilon(R) \frac{u(y)}{d(y)^2} + C \beta(y), \quad (43)
\]
\[
[(U_\beta)_{y_i y_j}]_{\theta, B(y, \frac{1}{2} d(y))} \leq \varepsilon(R) \frac{u(y)}{d(y)^2 \theta} + C \beta(y) \frac{d(y)^\theta}{d(y)^\theta}, \quad (44)
\]
where $\theta \in (0,1)$.
Lemma 18. Assume that $\gamma$ where we used the mean value theorem and the assumption that e.g. [13, Section 14.6],

Proof. The first bound (41) is simply (40).

For other bounds we make use of Theorem 4.6 in [13]. We apply this theorem to the concentric balls $B_r(y)$ and $B_{2r}(y)$, where $r = \frac{1}{3}d(y)$ to obtain that

$$r(U_\beta(y))_{y_1} + r^2(U_\beta)_{y_1 y_1}(y) \leq C(d, \theta) \left( \sup_{x : x \in B(y, \frac{4}{3}d(y))} U_\beta(x) + r^2 + \theta |\beta|_{\theta, B(y, 2r)} \right)$$

and

$$r^2 + \theta |(U_\beta)_{y_1 y_1}|_{\theta, B(y, r)} \leq C(d, \theta) \left( \sup_{x : x \in B(y, \frac{4}{3}d(y))} U_\beta(x) + r^2 + \theta |\beta|_{\theta, B(y, 2r)} \right).$$

It is known that the distance $d(x)$ to the boundary of $K$ is uniformly Lipschitz, see e.g. [13, Section 14.6],

$$|d(x) - d(z)| \leq |x - z|, \quad x, z \in K.$$

We have, for $x, z \in B \left(y, \frac{1}{3}d(y)\right)$,

$$\frac{|\beta(x) - \beta(z)|}{|x - z|^\theta} \leq \frac{|x|^p - |z|^p}{|x - z|^\theta} \frac{\gamma(d(z))}{d(z)} + |z|^p - \frac{|\gamma(d(x))/d(x) - \gamma(d(z))/d(z)|}{|x - z|^\theta}$$

$$\leq C |y|^p - \frac{\gamma(d(y))}{d(y)^{1-\theta}} d(y)^{1-\theta} + C |y|^p - \frac{\gamma(d(y))}{d(y)^{2-\theta}} d(y)^{2-\theta}$$

$$\leq C \frac{\beta(y)}{d(y)},$$

where we used the mean value theorem and the assumption that $\gamma$ is differentiable.

Lemma 18. Assume that (4) holds. For any $\varepsilon > 0$ there exists $R > 0$ such that for $x \in K$ with $d(x) > R$ we have

$$|U_\beta(x + y) - U_\beta(x)| \leq \varepsilon |y||x|^{p-1},$$

(45)

for $|y| \leq |x|/2$.

Proof. The proof follows the same arguments as the proof of [12, Lemma 2.3] The only change is that we use (42) to estimate the first derivative. \hfill \Box

Put

$$f_\beta(x) := E[U_\beta(x + X)] - U_\beta(x).$$

Lemma 19. For any $\varepsilon > 0$ there exists $R > 0$ such that for $x \in K$ with $d(x) > R$ the following bound holds

$$\left| f_\beta(x) + \frac{1}{2} \beta(x) \right| \leq \varepsilon \beta(x).$$

Proof. The proof goes exactly as in Lemma 11. We will apply (31) with $U = U_\beta$. The remainder $R_{2, \theta}(x)$ can be estimated using the the inequality (44),

$$R_{2, \theta}(x) = C_\theta \left( \varepsilon \frac{u(x)}{d(x)^{2+\theta}} + \frac{\beta(x)}{d(x)^\theta} \right).$$

Also, using Lemma 16 and (45), we can estimate

$$|E(U_\beta(x + X) - U_\beta(x)); |X| > d(x)|$$

$$\leq C \varepsilon(R) E[|X|^p; |X| > |x|/2] + C \varepsilon(R) |x|^{p-1} E[|X|; |X| > d(x)/2],$$
where \( \varepsilon(R) \to 0 \). Then, applying (31) with \( U = U_\beta \) and using the fact that \( \Delta U_\beta(x) = -\beta(x) \) we obtain,

\[
\left| f_\beta(x) + \frac{1}{2} \beta(x) \right| \leq \left| E \left[ \left( \nabla U_\beta(x) \cdot X + \frac{1}{2} \sum_{i,j} \frac{\partial^2 U_\beta}{\partial x_i \partial x_j} X_i X_j \right) 1(|X| > d(x)/2) \right] \right| \\
+ CR_{2,\rho}(x)E \left[ |X|^{2+\theta}; |X| \leq d(x)/2 \right] \\
+ C\varepsilon(R)E[|X|^p; |X| > |x|/2] + C\varepsilon(R)|x|^{p-1}E[|X|; |X| > d(x)/2].
\]

The partial derivatives of the function \( U_\beta \) in the first term can be estimated via Lemma 17, which results in the following estimate

\[
|E[\nabla U_\beta(x) \cdot X; |X| > d(x)/2]| \leq C \left( \varepsilon(R) \frac{u(x)}{d(x)} + C\beta(x)d(x) \right) E[|X|^2; |X| > d(x)/2].
\]

Using (27) with \( p = 1 \) we obtain

\[
|E[\nabla U_\beta(x) \cdot X; |X| > d(x)/2]| \leq C\varepsilon(R)\beta(x).
\]

We can estimate the terms with the second derivative as follows.

\[
\frac{1}{2} \left| E \left[ \sum_{i,j} \frac{\partial^2 u(x)}{\partial x_i \partial x_j} X_i X_j; |X| > d(x)/2 \right] \right| \\
\leq C \left( \varepsilon(R) \frac{u(x)}{d(x)^2} + C\beta(x) \right) E[|X|^2; |X| > d(x)/2].
\]

Then,

\[
\left| f_\beta(x) + \frac{1}{2} \beta(x) \right| \leq C \left( \varepsilon(R)\beta(x) + \left( \varepsilon(R) \frac{u(x)}{d(x)^2} + C\beta(x) \right) \frac{\gamma(d(x))}{d(x)} \right) \\
+ R_{2,\rho}(x)E \left[ |X|^{2+\theta}; |X| \leq d(x)/2 \right] \\
+ \varepsilon(R)E[|X|^p; |X| > |x|/2] + \varepsilon(R)|x|^{p-1}E[|X|; |X| > d(x)/2].
\]

Using the fact that \( \varepsilon(R) \to 0 \) as \( R \to \infty \) and Lemma 10 we arrive at the conclusion.

**Lemma 20.** For every \( c \leq 1/2 \) there exists \( R > 0 \) such that the sequence

\[
Y^{(c)}_n := (u(x + Rx_0 + S(n))) + U_\beta(x + Rx_0 + S(n))1\{\tau_x > n\} \\
+ c \sum_{k=0}^{n-1} \beta(x + Rx_0 + S(k))1\{\tau_x > k\}
\]

is a supermartingale.

**Proof.** Set, for brevity,

\[
V_\beta(y) := u(Rx_0 + y) + U_\beta(Rx_0 + y).
\]

Then

\[
E[Y^{(c)}_n - Y^{(c)}_{n-1} | \mathcal{F}_{n-1}] \\
= E[V_\beta(x + S(n))1\{\tau_x > n\} - V_\beta(x + S(n-1))1\{\tau_x > n-1\} | \mathcal{F}_{n-1}] \\
+ c\beta(x + Rx_0 + S(n-1))1\{\tau_x > n-1\}.
\]
Since the functions $u$ and $U_\beta$ are nonnegative, we have the inequality

$$ E[Y_n^{(c)} - Y_{n-1}^{(c)} | \mathcal{F}_{n-1}] $$

$$ \leq 1 \{ \tau_x > n - 1 \} E[V_\beta(x + S(n)) - V_\beta(x + S(n-1)) | \mathcal{F}_{n-1}] $$

$$ + c\beta(x + Rx_0 + S(n-1))1 \{ \tau_x > n - 1 \} $$

$$ = (f(x + Rx_0 + S(n-1)) + f_\beta(x + Rx_0 + S(n-1)))1 \{ \tau_x > n - 1 \} $$

$$ + c\beta(x + Rx_0 + S(n-1))1 \{ \tau_x > n - 1 \}. $$

According to Lemma 11 and Lemma 19 there exists $R > 0$ such that

$$ |f(y + Rx_0)| \leq \left( \frac{1}{8} - \frac{c}{4} \right) \beta(y + Rx_0) $$

and

$$ |f_\beta(y + Rx_0) + \frac{1}{2} \beta(y + Rx_0)| \leq \left( \frac{1}{8} - \frac{c}{4} \right) \beta(y + Rx_0) $$

for all $y \in K$. Therefore,

$$ E[Y_n^{(c)} - Y_{n-1}^{(c)} | \mathcal{F}_{n-1}] \leq - \left( \frac{1}{4} - \frac{c}{2} \right) \beta(x + Rx_0 + S(n-1))1 \{ \tau_x > n - 1 \}. $$

This completes the proof. \qed

**Lemma 21.** If $R$ is large enough then

$$ E \left[ \tau_{x}^{-1} \sum_{k=0}^{\tau_x-1} \beta(x + Rx_0 + S(k)) \right] \leq 3V_\beta(x). \quad (47) $$

Furthermore, there exists $\varepsilon(R) \downarrow 0$ such that

$$ E \left[ \sum_{k=0}^{\tau_x-1} |f(x + Rx_0 + S(k))| \right] \leq \varepsilon(R)u(x + Rx_0). \quad (48) $$

**Proof.** Let $R$ be so large that $(Y_n^{(1/3)})_{n \geq 0}$ defined in (46) is a supermartingale. Then,

$$ E[Y_n^{(1/3)}] \leq Y_0^{(1/3)} = V_\beta(x). $$

Since $u$ and $U_\beta$ are non-negative,

$$ \frac{1}{3} E \left[ \sum_{k=0}^{n-1} \beta(x + Rx_0 + S(k))1 \{ \tau_x > k \} \right] \leq V_\beta(x). $$

Letting here $n \to \infty$ and noting that

$$ \sum_{k=0}^{\infty} \beta(x + Rx_0 + S(k))1 \{ \tau_x > k \} = \sum_{k=0}^{\tau_x-1} \beta(x + Rx_0 + S(k)) $$

we obtain (47).

Estimate (48) follows from (47), (41) and from Lemma 11. \qed
4. Construction of the harmonic function

Consider the sequence

\[ Z_n := u(x + Rx_0 + S(n \wedge \tau_x)) - \sum_{k=0}^{n \wedge \tau_x - 1} f(x + Rx_0 + S(k)), \quad n \geq 0. \]

Then

\[
\begin{align*}
E[Z_n - Z_{n-1} | F_{n-1}] &= E[(Z_n - Z_{n-1})1\{\tau_x > n - 1\} | F_{n-1}] \\
&= 1\{\tau_x > n - 1\}E[(u(x + Rx_0 + S(n)) - u(x + Rx_0 + S(n - 1))) | F_{n-1}] \\
&\quad - 1\{\tau_x > n - 1\}f(x + Rx_0 + S(n - 1)).
\end{align*}
\]

Recalling that \( f(y) = E[u(y + X)] - u(y) \), we conclude that \( Z_n \) is a martingale. By the martingale property,

\[
E[u(x + Rx_0) = E[Z_0] = E[Z_n]
\]

\[
= E \left[ u(x + Rx_0 + S(n)) - \sum_{k=0}^{n-1} f(x + Rx_0 + S(k)); \tau_x > n \right]
\]

\[
+ E \left[ u(x + Rx_0 + S(\tau_x)) - \sum_{k=0}^{\tau_x - 1} f(x + Rx_0 + S(k)); \tau_x \leq n \right].
\]

Consequently,

\[
E[u(x + Rx_0 + S(n)); \tau_x > n]
\]

\[
= u(x + Rx_0) - E[u(x + Rx_0 + S(\tau_x)); \tau_x \leq n]
\]

\[
+ E \left[ \sum_{k=0}^{\tau_x - 1} f(x + Rx_0 + S(k)); \tau_x \leq n \right]
\]

\[
+ E \left[ \sum_{k=0}^{n-1} f(x + Rx_0 + S(k)); \tau_x > n \right]. \tag{49}
\]

By the monotone convergence theorem,

\[
\lim_{n \to \infty} E[u(x + Rx_0 + S(\tau_x)); \tau_x \leq n] = E[u(x + Rx_0 + S(\tau_x))]. \tag{50}
\]

The estimate (48) allows one to apply the dominated convergence theorem. As a result we have

\[
\lim_{n \to \infty} E \left[ \sum_{k=0}^{\tau_x - 1} f(x + Rx_0 + S(k)); \tau_x \leq n \right] = E \left[ \sum_{k=0}^{\tau_x - 1} f(x + Rx_0 + S(k)) \right] \tag{51}
\]

and

\[
\lim\sup_{n \to \infty} \left| E \left[ \sum_{k=0}^{\tau_x - 1} f(x + Rx_0 + S(k)); \tau_x > n \right] \right|
\]

\[
\leq \lim\sup_{n \to \infty} E \left[ \sum_{k=0}^{\tau_x - 1} |f(x + Rx_0 + S(k))|; \tau_x > n \right] = 0. \tag{52}
\]
Combining (49)—(52), we conclude that
\[
\lim_{n \to \infty} E[u(x + Rx_0 + S(n)); \tau_x > n]
= u(x + Rx_0) - E[u(x + Rx_0 + S(\tau_x))] + E \left[ \sum_{k=0}^{\tau_x-1} f(x + Rx_0 + S(k)) \right].
\] (53)

Recalling that \(u(x)\) is nonnegative and taking into account (48), we see that this limit is finite.

We now show that the sequence \(E[u(x + S(n)); \tau_x > n]\) has the same limit. According to (21),
\[
|u(x + Rx_0 + S(n)) - u(x + S(n))| \leq C (R|x + Rx_0 + S(n)|^{p-1} + R^p).
\]
Therefore,
\[
\left| E[u(x + Rx_0 + S(n)); \tau_x > n] - E[u(x + S(n)); \tau_x > n] \right| \leq CR E[|x + Rx_0 + S(n)|^{p-1}; \tau_x > n] + CR^p P(\tau_x > n).
\] (54)

We next notice that (47) implies that
\[
\varepsilon_n := E[\beta(x + Rx_0 + S(n)); \tau_x > n] \to 0.
\]
Then there exists a sequence \(a_n \uparrow \infty\) such that
\[
E[|x + Rx_0 + S(n)|^{p-1}; d(x + Rx_0 + S(n)) \leq a_n, \tau_x > n]
\leq \frac{a_n}{\gamma(a_n)} E[\beta(x + Rx_0 + S(n)); \tau_x > n]
= \frac{a_n}{\gamma(a_n)} \varepsilon_n \to 0.
\] (55)

Moreover, using (5) and (53), we obtain
\[
E[|x + Rx_0 + S(n)|^{p-1}; d(x + Rx_0 + S(n)) > a_n, \tau_x > n]
\leq \frac{C}{a_n} E[u(x + Rx_0 + S(n)); \tau_x > n] \to 0.
\] (56)

Combining (54)—(56) and using the finiteness of \(\tau_x\), we conclude that
\[
\left| E[u(x + Rx_0 + S(n)); \tau_x > n] - E[u(x + S(n)); \tau_x > n] \right| \to 0.
\]

From this and from (53) we get (6) with
\[
V(x) = u(x + Rx_0) - E[u(x + Rx_0 + S(\tau_x))] + E \left[ \sum_{k=0}^{\tau_x-1} f(x + Rx_0 + S(k)) \right].
\] (57)

Due to (48),
\[
V(x) \leq Cu(x + Rx_0).
\]

This upper bound allows one to repeat the arguments from Lemma 13 in [9] and to conclude that \(V\) is harmonic for \(S(n)\) killed at leaving the cone \(K\). This completes the proof of the first statement of Theorem 2.
5. Upper bounds for $P(\tau_x > n)$ in the case $p \geq 1$

We start with the following estimate for the tails of the distribution function of the exit time $\tau_{xbm}^x$ from $K$ of Brownian motion.

**Lemma 22.** Assume that $p \geq 1$ and that the cone $K$ is Lipschitz and starlike. Then there exists a constant $C$ such that

$$P(\tau_{xbm}^x > t) \leq Cu(x)^{t/p/2}, \quad x \in K. \quad (58)$$

**Proof.** For convex cones the proof can be found in (0.4.1) of [22] We will show how the proof of part (ii) in Theorem 1 of [22] at page 344 can be modified to show the result under our assumptions.

First note that for $x \in x_0 + K$ we have $d(x) \geq c_0$ for some $c_0$. Then, for $x \in x_0 + K \subset K$,

$$P(\tau_{xbm}^x > 1) \leq \frac{1}{c_0} d(x) \leq Cd(x)|x|^{p-1} \leq Cu(x),$$

where we also used (5) and the assumption $p \geq 1$. Now note that that the rest of the proof in [22] does not use convexity and uses only the assumption that $K$ is Lipschitz. Thus, repeating the rest of the proof of part (ii) in Theorem 1 of [22] at page 344 in exactly the same way we obtain that

$$P(\tau_{xbm}^x > 1) \leq Cu(x).$$

Then, the statement follows by the scaling property of Brownian motion. \hfill \Box

**Lemma 23.** Assume that the conditions of Theorem 3 are valid. Then,

$$P(\tau_x > n) \leq C\frac{V_\beta(x)}{n^{p/2}} \leq C\frac{u(x + Rx_0)}{n^{p/2}}.$$  

**Proof.** Choose $b_n = o(\sqrt{n})$ so that

$$P\left(\sup_{u \leq n} |S([u]) - B(u)| \geq b_n\right) = o\left(\left(\frac{b_n}{\sqrt{n}}\right)^p\right). \quad (59)$$

Using this approximation, we get

$$P(\tau_y > k) \leq P(\tau_{xbm}^{y + b_n x_0} > k) + o\left(\left(\frac{b_n}{\sqrt{n}}\right)^p\right), \quad k \leq n.$$  

Then, using (58),

$$P(\tau_y > k) \leq C \frac{u(y + b_n x_0)}{k^{p/2}} + o\left(\left(\frac{b_n}{\sqrt{n}}\right)^p\right), \quad k \leq n.$$  

If $d(y) \geq b_n$ then, using (21) and (5), we obtain

$$u(y + b_n x_0) \leq u(y) + C b_n (|y|^{p-1} + b_n^{p-1}) \leq Cu(y).$$

As a result, for all $y \in K$ with $d(y) \geq b_n$,

$$P(\tau_y > n/4) \leq C\frac{u(y)}{n^{p/2}}.$$  

Define

$$\nu_n := \inf\{k \geq 1 : d(x + S(k)) \geq b_n\}.$$
Then, by the Markov property,
\[
\mathbb{P}(\tau_x > n, \nu_n \leq 3n/4) \leq C \mathbb{E}[u(x + S(\nu_n)); \tau_x > \nu_n, \nu_n \leq 3n/4]
\]
\[
\leq C \mathbb{E}[u(x + R\varepsilon_0 + S(\nu_n)); \tau_x > \nu_n, \nu_n \leq 3n/4]
\]
\[
\leq C \mathbb{E}[V_\beta(x + S(\nu_n)); \tau_x > \nu_n, \nu_n \leq 3n/4]
\]
\[
\leq C \frac{\mathbb{E}[U_{\nu_n}^{(0)}; \nu_n \leq 3n/4]}{n^{p/2}}, \tag{60}
\]
where \( Y_n^{(0)} \) is defined as in Lemma 20. Since this sequence is a supermartingale,
\[
\mathbb{E}[U_{\nu_n}^{(0)}; \nu_n \leq 3n/4] \leq \mathbb{E}[U_{\nu_n}^{(0)}] \leq Y_{\nu_n}^{(0)} = V_\beta(x).
\]
Therefore,
\[
\mathbb{P}(\tau_x > n, \nu_n \leq 3n/4) \leq C \frac{V_\beta(x)}{n^{p/2}}. \tag{61}
\]
Set
\[
K_n := \{ y \in K : d(y) \geq b_n \}
\]
and
\[
c_n := [b_n^2].
\]
Then, by the Markov property,
\[
\mathbb{P}(\tau_x > n, \nu_n > 3n/4)
\]
\[
\leq \mathbb{P}(\tau_x > n/2, x + S(n/2 + jc_n) \in K \setminus K_n, \text{ for all } j \leq n/4c_n)
\]
\[
\leq \mathbb{P}(\tau_x > n/2) \left( \sup_{y \in K \setminus K_n} \mathbb{P}(y + S(c_n) \in K \setminus K_n) \right)^{n/4c_n}
\]
Repeating now the arguments from the proof of Lemma 14 in [9], we obtain
\[
\mathbb{P}(\tau_x > n, \nu_n > 3n/4) \leq \mathbb{P}(\tau_x > n/2)e^{-cn/b_n^2}.
\]
Combining this with (61), we have
\[
\mathbb{P}(\tau_x > n) \leq C \frac{V_\beta(x)}{n^{p/2}} + \mathbb{P}(\tau_x > n/2)e^{-cn/b_n^2}.
\]
Set \( \gamma_j := c2^j/b_n^2 \). Then, letting \( n = 2^N \) in the previous bound, we have
\[
\mathbb{P}(\tau_x > 2^N) \leq C \frac{V_\beta(x)}{2^{Np/2}} + e^{-\gamma N} \mathbb{P}(\tau_x > 2^{N-1}).
\]
Iterating this estimate \( m \) times, we obtain
\[
\mathbb{P}(\tau_x > 2^N) \leq C \frac{V_\beta(x)}{2^{Np/2}} \left( 1 + \sum_{k=0}^{m-2} 2^{(k+1)p/2} \prod_{j=0}^{k} e^{-\gamma N-j} \right)
\]
\[
+ \prod_{j=0}^{m-1} e^{-\gamma N-j} \mathbb{P}(\tau_x > 2^{N-m}).
\]
Choosing here \( m = \lceil N/2 \rceil \) and noting that \( \min_{j \leq N/2} \gamma_{N-j} \) converges to infinity as \( N \to \infty \), we infer that
\[
\prod_{j=0}^{m-1} e^{-\gamma_{N-j}} = o(2^{-Np/2})
\]
and
\[
\sup_{N \geq 1} \sum_{k=0}^{m-2} 2^{(k+1)p/2} \prod_{j=0}^{k} e^{-\gamma_{N-j}} < \infty.
\]
This finishes the proof.

We can now give a proof of a strengthened version of one of results in McConnell [18].

**Corollary 24.** For every \( 0 < q < p \),
\[
\mathbb{E} \left[ \max_{k \leq \tau_x} |S(k)|^q \right] \leq C(V_\beta(x))^{q/p}. \tag{62}
\]

**Proof.** For every \( q < p \), by the Burkholder-Davis-Gundy inequality,
\[
\mathbb{E} \left[ \max_{k \leq \tau_x} |S(k)|^q \right] \leq C \mathbb{E} \left[ (\tau_x)^{q/2} \right].
\]

Using Lemma 23, we have
\[
\mathbb{E} \left[ (\tau_x)^{q/2} \right] = \frac{q}{2} \int_0^\infty u^{q/2-1} \mathbb{P}(\tau_x > u) du
\]
\[
\leq K^{q/2} + \frac{q}{2} \int_K^\infty u^{q/2-1} \mathbb{P}(\tau_x > u) du
\]
\[
\leq K^{q/2} + CV_\beta(x) \frac{q}{2} \int_K^\infty u^{q/2-p/2-1} du
\]
\[
= K^{q/2} + CV_\beta(x) \frac{q}{p-q} K^{(q-p)/2}.
\]

Taking here \( K = (V_\beta(x))^{2/p} \), we have
\[
\mathbb{E} \left[ (\tau_x)^{q/2} \right] \leq C(V_\beta(x))^{q/p}.
\]

**Lemma 25.** If \( u(x + S(n))1\{\tau_x > n\} \) is a submartingale then
\[
\mathbb{P}(\tau_x > n) \leq C \frac{\mathbb{E}[u(x + S(n)); \tau_x > n]}{n^{p/2}}.
\]

**Proof.** By the first inequality in (60),
\[
\mathbb{P}(\tau_x > n, \nu_n \leq 3n/4) \leq c \frac{\mathbb{E}[u(x + S(\nu_n)); \tau_x > \nu_n, \nu_n \leq 3n/4]}{n^{p/2}}
\]
\[
\leq c \frac{\mathbb{E}[u(x + S(\nu_n \wedge (3n/4))); \tau_x > \nu_n \wedge 3n/4]}{n^{p/2}}.
\]

Due to the submartingale property of \( u(x + S(n))1\{\tau_x > n\} \),
\[
\mathbb{E}[u(x + S(\nu_n \wedge (3n/4))); \tau_x > \nu_n \wedge 3n/4] \leq \mathbb{E}[u(x + S(n)); \tau_x > n].
\]
Consequently,
\[ P(\tau_x > n, \nu_n \leq 3n/4) \leq C \frac{E[u(x + S(n)); \tau_x > n]}{n^{p/2}}. \]
Repeating now the second part of the proof of the previous lemma and using the
monotonicity of \( E[u(x + S(n)); \tau_x > n] \), we get the desired result. □

6. Asymptotic properties of the harmonic function.

In order to show the equivalence \( V(x) \sim u(x) \) we first notice that the assumption
\( d(x) \to \infty \) implies that \( u(x) \sim u(x + Rx_0) \) for every fixed \( R \) when \( u(x) \to +\infty. \)
Next, by (48),
\[
\left| \frac{V(x)}{u(x + Rx_0)} - 1 \right| \leq \frac{E[u(x + Rx_0 + S(\tau_x))]}{u(x + Rx_0)} + \frac{E \left[ \sum_{k=0}^{\tau_x - 1} f(x + Rx_0 + S(k)) \right]}{u(x + Rx_0)} + \varepsilon(R). \tag{63}
\]
When \( p \geq 1 \) and \( x + S(\tau_x) \notin K \), we obtain from (21) that
\[ u(x + Rx_0 + S(\tau_x)) \leq CR|x + Rx_0 + S(\tau_x)|^{p-1} \leq CR^p + C|x|^{p-1} + C|S(\tau_x)|^{p-1}. \tag{64} \]
It follows from (62) that
\[ E[|S(\tau_x)|^q] \leq C(1 + |x|^q), \quad q < p. \tag{65} \]
Consequently,
\[ E[u(x + Rx_0 + S(\tau_x))] \leq CR^p + C|x|^{p-1}. \]
Applying now (5), we get
\[ E[u(x + Rx_0 + S(\tau_x))] \leq CR^p + \frac{C}{d(x)} u(x) \leq CR^p + \frac{C}{d(x)} u(x + Rx_0) \]
and, consequently,
\[ \frac{E[u(x + Rx_0 + S(\tau_x))]}{u(x + Rx_0)} \to 0. \tag{66} \]
When \( p < 1 \) the random variable \( u(x + Rx_0 + S(\tau_x)) \) is bounded. Hence (66) holds
when \( u(x) \to +\infty \), or, equivalently, when \( d(x)|x|^{p-1} \to +\infty \).
Combining convergence in (66) with (63) and using the relation \( u(x) \sim u(x + Rx_0) \), we conclude that
\[ \limsup \left| \frac{V(x)}{u(x)} - 1 \right| \leq \varepsilon(R). \]
Letting here \( R \to \infty \) we complete the proof.

We now turn to the proof of the uniform convergence in the case \( p \geq 1 \). We first
prove the following auxiliary result:
\[
E[|S(n)|^{p-1}; \tau_x > n] + E[|S(\tau_x)|^{p-1}; \tau_x > n] \leq 2E[\max_{k \leq \tau_x} |S(k)|^{p-1}; \tau_x > n] = o(V_{\beta}(x)) \tag{67}
\]
uniformly in \( x \in K \). Fix some \( p' > p \). Then \( \frac{1}{q'} := 1 - \frac{1}{p'} > 1 - \frac{1}{p} \) and, consequently, \( q' < p/(p-1) \). Using now the Hölder inequality, (62) with \( q = q'(p-1) \) and Lemma 23, we get

\[
\mathbb{E}[\max_{k \leq \tau_x} |S(k)|^{p-1}; \tau_x > n] \leq \left( \mathbb{E}[\max_{k \leq \tau_x} |S(k)|^{q'(p-1)}] \right)^{1/q'} \mathbb{P}^{1/p'}(\tau_x > n)
\]

\[
\leq CV^{1-1/p}(x) \frac{V^{1/p'}_{\beta}(x)}{n^{p/2p'}}.
\]

Since \( V^{1/p'-1/p}(x) \) is bounded, we get (67).

Combining (54), (67) and Lemma 23, we have

\[
\left| \mathbb{E}[u(x + R\tau_x + S(n)); \tau_x > n] - \mathbb{E}[u(x + S(n)); \tau_x > n] \right| \leq C(R^n + R|x|^{p-1}) \mathbb{P}(\tau_x > n) + \mathbb{E}[|S(n)|^{p-1}; \tau_x > n]
\]

\[
\leq C \frac{R^n + R|x|^{p-1}}{n^{p/2}} V_{\beta}(x) + o(V_{\beta}(x)).
\]

Therefore,

\[
\left| \mathbb{E}[u(x + R\tau_x + S(n)); \tau_x > n] - \mathbb{E}[u(x + S(n)); \tau_x > n] \right| = o(V_{\beta}(x))
\]

uniformly in \( x \in K \) such that \( |x| = o(n^{p/2(p-1)}) \). This implies that we are left to show that

\[
\mathbb{E}[u(x + R\tau_x + S(n)); \tau_x > n] - V(x) = o(V_{\beta}(x))
\]

uniformly in \( x \in K \) such that \( |x| = o(n^{p/2(p-1)}) \). Combining (57) and (49), we have

\[
\mathbb{E}[u(x + R\tau_x + S(n)); \tau_x > n] - V(x)
\]

\[
= \mathbb{E}[u(x + R\tau_x + S(\tau_x)); \tau_x > n] - \mathbb{E}\left[ \sum_{k=n}^{\tau_x-1} f(x + R\tau_x + S(k)); \tau_x > n \right].
\]

Using (64) and (67) we obtain

\[
\mathbb{E}[u(x + R\tau_x + S(\tau_x)); \tau_x > n]
\]

\[
\leq C(R^n + R|x|^{p-1}) \mathbb{P}(\tau_x > n) + \mathbb{E}[|S(\tau_x)|^{p-1}; \tau_x > n] = o(V_{\beta}(x)),
\]

for \( x \in K \) such that \( |x| = o(n^{p/2(p-1)}) \). For \( y \in K \) now put

\[
Q(y) := \sum_{k=0}^{\tau_x-1} f(y + R\tau_x + S(k)).
\]

and note that, by the Markov property,

\[
\mathbb{E}\left[ \sum_{k=n}^{\tau_x-1} f(x + R\tau_x + S(k)); \tau_x > n \right] = \mathbb{E}[Q(x + S(n)); \tau_x > n].
\]

By Theorem 2, as \( d(x) \to \infty \), \( V(x) \sim u(x) \) and, therefore, \( Q(x) = o(u(x)) \) as \( d(x) \to \infty \). Consider an increasing sequence \( \gamma_n \uparrow \infty \) and split

\[
\mathbb{E}[Q(x + S(n)); \tau_x > n]
\]

\[
= \mathbb{E}[Q(x + S(n)); \tau_x > n, d(x + S(n)) \geq \gamma_n]
\]

\[
+ \mathbb{E}[Q(x + S(n)); \tau_x > n, d(x + S(n)) < \gamma_n].
\]
Then, for some \( \varepsilon_n \downarrow 0 \),

\[
E[Q(x+S(n)); \tau_x > n, d(x+S(n)) \geq \gamma_n] \leq \varepsilon_n E[u(x+S(n)); \tau_x > n] \\
\leq C \varepsilon_n E[V_\beta(x+S(n)); \tau_x > n] \leq C \varepsilon_n V_\beta(x) = o(V_\beta(x))
\]

uniformly in \( x \in K \), where we have also used the supermartingale property of \( V_\beta(x) \).

By (47),

\[
E[Q(x+S(n)); \tau_x > n, d(x+S(n)) < \gamma_n] \\
\leq 3E[V_\beta(x+S(n)); \tau_x > n, d(x+S(n)) < \gamma_n] \\
\leq CE[u(x+Rx_0+S(n)); \tau_x > n, d(x+S(n)) < \gamma_n] \\
\leq C \gamma_n E[|x+Rx_0+S(n)|^{p-1}; \tau_x > n].
\]

Then, for \( \gamma_n \uparrow \infty \) sufficiently slowly, by Lemma 23 and (67),

\[
E[Q(x+S(n)); \tau_x > n, d(x+S(n)) < \gamma_n] \\
\leq C \gamma_n (|x+Rx_0|^{p-1}P(\tau_x > n) + E[|S(n)|^{p-1}; \tau_x > n]) \\
= o(V_\beta(x)),
\]

uniformly in \( x \in K \) such that \( |x| = o(n^{p/(2(p-1))}) \). Thus, the proof of Theorem 2 is complete.

7. Proof of Theorem 3 for cones with \( 1 \leq p < 2 \).

Part (a) is immediate from Lemma 23.

Choose \( m = m(n) \) so that \( m(n) = o(n) \) and \( b_n^2 = o(m(n)) \), where \( b_n \) is defined by (59). Fix some \( \varepsilon < 1, A > 1 \) and define

\[
K_1 := \{ y \in K : d(y) \leq \varepsilon \sqrt{m}, |y| \leq A \sqrt{m} \}, \\
K_2 := \{ y \in K : d(y) > \varepsilon \sqrt{m}, |y| \leq A \sqrt{m} \}, \\
K_3 := \{ y \in K : |y| > A \sqrt{m} \}.
\]

Then, by the Markov property at time \( m \),

\[
P \left( \frac{x+S(n)}{\sqrt{n}} \in D, \tau_x > n \right) \\
= \int_{K_1} P(x+S(m) \in dy, \tau_x > m) P \left( \frac{y+S(n-m)}{\sqrt{n}} \in D, \tau_y > n-m \right) \\
+ \int_{K_2} P(x+S(m) \in dy, \tau_x > m) P \left( \frac{y+S(n-m)}{\sqrt{n}} \in D, \tau_y > n-m \right) \\
+ \int_{K_3} P(x+S(m) \in dy, \tau_x > m) P \left( \frac{y+S(n-m)}{\sqrt{n}} \in D, \tau_y > n-m \right).
\] (69)
As a result we have an upper bound in Lemma 23 for the tail of $|S(n)|^p/n$ uniformly in $|x| < \sqrt{m}$. Using the upper bound from (a), we obtain
\[
\int_{K_1} P(x + S(m) \in dy, \tau_x > m) P \left( \frac{y + S(n - m)}{\sqrt{n}} \in D, \tau_y > n - m \right)
\leq \int_{K_1} P(x + S(m) \in dy, \tau_x > m) P(\tau_y > n - m)
\leq \frac{C}{(n - m)^{p/2}} \int_{K_1} P(x + S(m) \in dy, \tau_x > m) u(y + Rx_0)
\leq \frac{C}{n^{p/2}} \int_{K_1} P(x + S(m) \in dy, \tau_x > m) u(y + Rx_0).
\]

It follows from (4) that
\[
u(y) \leq u(y + Rx_0) \leq C |y + Rx_0|^{p-1} d(y + Rx_0) \leq C \varepsilon A^{p-1} m^{p/2}, \quad y \in K_1.
\]

Therefore,
\[
\int_{K_1} P(x + S(m) \in dy, \tau_x > m) P \left( \frac{y + S(n - m)}{\sqrt{n}} \in D, \tau_y > n - m \right)
\leq C \varepsilon A^{p-1} \frac{m^{p/2}}{n^{p/2}} P(\tau_x > m) \leq C \varepsilon A^{p-1} \frac{u(x + Rx_0)}{n^{p/2}}
\]

and
\[
\int_{K_1} P(x + S(m) \in dy, \tau_x > m) u(y) \leq C \varepsilon A^{p-1} u(x + Rx_0).
\]

Using the upper bound from (a) once again, we have
\[
\int_{K_3} P(x + S(m) \in dy, \tau_x > m) P \left( \frac{y + S(n - m)}{\sqrt{n}} \in D, \tau_y > n - m \right)
\leq \frac{C}{n^{p/2}} \int_{K_3} P(x + S(m) \in dy, \tau_x > m) u(y + Rx_0)
\leq \frac{C}{n^{p/2}} \mathbb{E} \left[ |S(n)|^p ; \tau_x > m, |S(m)| > A \sqrt{m}/2 \right],
\]

uniformly in $|x| < \sqrt{m}$. Applying now the Markov inequality, we obtain
\[
\int_{K_3} P(x + S(m) \in dy, \tau_x > m) P \left( \frac{y + S(n - m)}{\sqrt{n}} \in D, \tau_y > n - m \right)
\leq \frac{C}{n^{p/2}} (A \sqrt{m})^{p-2} \mathbb{E} \left[ |S(\tau_x \wedge m)|^2 \right].
\]

Applying the Optional Stopping Theorem to the martingale $|S(n)|^2 - dn$ and the upper bound in Lemma 23 for the tail of $\tau_x$, we conclude that
\[
\mathbb{E} \left[ |S(\tau_x \wedge m)|^2 \right] \leq C \mathbb{E}[\tau_x \wedge m] \leq C u(x + Rx_0)m^{1-p/2}.
\]

As a result we have
\[
\int_{K_3} P(x + S(m) \in dy, \tau_x > m) P \left( \frac{y + S(n - m)}{\sqrt{n}} \in D, \tau_y > n - m \right)
\leq \frac{C}{n^{p/2}} A^{p-2} u(x + Rx_0).
\]
By the same arguments,
\[
\int_{K_2} P(x + S(m) \in dy, \tau_x > m) u(y) \leq \frac{C}{n^{p/2}} A^{p-2} u(x + Rx_0). \tag{73}
\]

Finally, (59) implies that
\[
P \left( \frac{y + S(n - m)}{\sqrt{n}} \in D, \tau_y > n - m \right) \sim P \left( \frac{y + B(n - m)}{\sqrt{n}} \in D, \tau^{bm} > n - m \right)
\]
\[
\sim \kappa \frac{u(y)}{n^{p/2}} \int_D u(z) e^{-|z|^2/2} dz
\]
uniformly in \(y \in K_2\). Consequently,
\[
\int_{K_2} P(x + S(m) \in dy, \tau_x > m) P \left( \frac{y + S(n - m)}{\sqrt{n}} \in D, \tau_y > n - m \right)
\]
\[
\sim \kappa \frac{u(y)}{n^{p/2}} \int_D u(z) e^{-|z|^2/2} dz \int_{K_2} P(x + S(m) \in dy, \tau_x > m) u(y).
\]

Combining this with (71) and (73), we get
\[
\left| \int_{K_2} P(x + S(m) \in dy, \tau_x > m) P \left( \frac{y + S(n - m)}{\sqrt{n}} \in D, \tau_y > n - m \right)
\]
\[
- \kappa \frac{u(y)}{n^{p/2}} \int_D u(z) e^{-|z|^2/2} dz E[u(x + S(m)); \tau_x > m]
\]
\[
\leq Cu(x + R x_0) \frac{\varepsilon A^{p-1} + A^{p-2}}{n^{p/2}} + o \left( \frac{1}{n^{p/2}} \right). \tag{74}
\]

Plugging (70), (72) and (74) into (69), we obtain
\[
\left| P \left( \frac{x + S(n)}{\sqrt{n}} \in D, \tau_x > n \right) - \kappa \frac{u(x + S(m))}{n^{p/2}} \int_D u(z) e^{-|z|^2/2} dz E[u(x + S(m)); \tau_x > m] \right|
\]
\[
\leq Cu(x + R x_0) \frac{\varepsilon A^{p-1} + A^{p-2}}{n^{p/2}} + o \left( \frac{1}{n^{p/2}} \right),
\]
uniformly in \(|x| \leq \sqrt{m}\). Letting here first \(\varepsilon \to 0\) and then \(A \to \infty\) and recalling that \(E[u(x + S(m)); \tau_x > m] \to V(x)\) uniformly in \(|x| \leq \sqrt{m}\), we finally arrive at the relation
\[
P \left( \frac{x + S(n)}{\sqrt{n}} \in D, \tau_x > n \right) \sim \frac{\kappa V(x)}{n^{p/2}} \int_D u(z) e^{-|z|^2/2} dz,
\]
which holds uniformly in \(|x| = o(\sqrt{n})\). Even simpler arguments give
\[
P (\tau_x > n) \sim \frac{\kappa V(x)}{n^{p/2}}.
\]
Thus, (b) is proven for \(1 \leq p < 2\).
8. Proof of Theorem 3 for $p \geq 2$.

According to Corollary 3.2 in Sakhanenko [21], one can construct $S(n)$ and a
Brownian motion $B(t)$ on a joint probability space so that

$$
P \left( \sup_{u \leq n} |S([u]) - B(u)| > x \right) \leq Cn \left( \mathbf{E} \min \left\{ \frac{|X|^2}{x^2}, \frac{|X|^3}{x^3} \right\} + \mathbf{E} \min \left\{ \frac{|B(1)|^2}{x^2}, \frac{|B(1)|^3}{x^3} \right\} \right).
$$

We next notice that the assumption $\mathbf{E}[|X|^2 \log |X|] < \infty$ implies that

$$
\mathbf{E} \left[ |X|^2; |X| > x \right] \leq \frac{1}{\log x} \mathbf{E} \left[ |X|^2 \log |X|; |X| > x \right] = o \left( \frac{1}{\log x} \right)
$$

and

$$
\mathbf{E} \left[ |X|^3; |X| \leq x \right] \leq x^{3/4} + \mathbf{E} \left[ |X|^3; x^{1/4} < |X| \leq x \right]
$$

$$
\leq x^{3/4} + \frac{x}{\log x} \mathbf{E} \left[ |X|^2 \log |X|; x^{1/4} < |X| \leq x \right] = o \left( \frac{x}{\log x} \right).
$$

Consequently,

$$
\mathbf{E} \min \left\{ \frac{|X|^2}{x^2}, \frac{|X|^3}{x^3} \right\}
$$

$$
= \frac{1}{x^3} \mathbf{E} \left[ |X|^3; |X| \leq x \right] + \frac{1}{x^2} \mathbf{E} \left[ |X|^3; |X| > x \right] = o \left( \frac{1}{x^2 \log x} \right).
$$

Obviously, the same bound holds for $X$ replaced by $B(1)$. As a result,

$$
P \left( \sup_{u \leq n} |S([u]) - B(u)| > x \right) = o \left( \frac{n}{x^2 \log x} \right).
$$

Then we can choose $x = b_n = o \left( \frac{\sqrt{n}}{\log^{1/2} n} \right)$ such that

$$
P \left( \sup_{u \leq n} |S([u]) - B(u)| > x \right) = o \left( \frac{1}{\log^{1/2} n} \right) = o \left( \frac{b_n^2}{n} \right). \quad (75)
$$

Lemma 26. Set $\delta_n = \log^{-p/8} n$. Uniformly in $y \in K$ such that $d(y) \geq \delta_n \sqrt{n}$ and

$|y| \leq \varepsilon_n \sqrt{n}$ for some $\varepsilon_n \downarrow 0$,

$$
P \left( \frac{y + S(n)}{\sqrt{n}} \in D, \tau_y > n \right) = (\kappa + o(1)) \left( \int_D u(z) e^{-|z|^2/2} dz \right) \frac{u(y)}{n^{p/2}},
$$

where $D$ is either a compact subset of $K$ or $D = K$.

Proof. Since the proof is very similar to the beginning of the proof of Lemma 23,
we omit it. In the case $p > 2$, a stronger result has been proven in Lemma 20 of [9]. □

Define

$$
K_1 := \{ y \in K : d(y) \leq \delta_n \sqrt{n}, |y| \leq \varepsilon_n \sqrt{n} \},
$$

$$
K_2 := \{ y \in K : d(y) > \delta_n \sqrt{n}, |y| \leq \varepsilon_n \sqrt{n} \},
$$

$$
K_3 := \{ y \in K : |y| > \varepsilon_n \sqrt{n} \}.
$$
Set also 
\[ m = m(n) = \left\lfloor \frac{n}{\log^{1/2} n} \right\rfloor. \]

Then, by the Markov property at time \( m \),

\[
P \left( \frac{x + S(n)}{\sqrt{n}} \in D, \tau_x > n \right) 
= \int_{K_2} P(x + S(m) \in dy, \tau_x > m) P \left( \frac{y + S(n - m)}{\sqrt{n}} \in D, \tau_y > n - m \right)
+ \int_{K_1 \cup K_3} P(x + S(m) \in dy, \tau_x > m) P \left( \frac{y + S(n - m)}{\sqrt{n}} \in D, \tau_y > n - m \right).
\]

(76)

It is immediate from Lemma 26 that

\[
\int_{K_2} P(x + S(m) \in dy, \tau_x > m) P \left( \frac{y + S(n - m)}{\sqrt{n}} \in D, \tau_y > n - m \right)
= \frac{\kappa + o(1)}{n^{p/2}} \int_D u(z) e^{-|z|^2/2} dz \int_{K_2} P(x + S(m) \in dy, \tau_x > m) u(y)
= \frac{\kappa + o(1)}{n^{p/2}} \int_D u(z) e^{-|z|^2/2} dz (E[u(x + S(m)); \tau_x > m] 
- E[u(x + S(m)); \tau_x > m, x + S(m) \in K_1 \cup K_3]).
\]

Using the upper bound from (a), we obtain

\[
\int_{K_1 \cup K_3} P(x + S(m) \in dy, \tau_x > m) P \left( \frac{y + S(n - m)}{\sqrt{n}} \in D, \tau_y > n - m \right)
\leq \int_{K_1 \cup K_3} P(x + S(m) \in dy, \tau_x > m) P (\tau_y > n - m)
\leq \frac{C}{(n - m)^{p/2}} \int_{K_1 \cup K_3} P(x + S(m) \in dy, \tau_x > m) u(y + Rx_0)
\leq \frac{C}{n^{p/2}} E[u(x + Rx_0 + S(m)); \tau_x > m, x + S(m) \in K_1 \cup K_3].
\]

If we show that
\[
E[u(x + Rx_0 + S(m)); \tau_x > m, x + S(m) \in K_1 \cup K_3] = o(n^{p/2} u(x + Rx_0)) \quad (77)
\]
uniformly in \( x \in K \) with \( |x| = o \left( \frac{\sqrt{n}}{\log n} \right) \) then

\[
P \left( \frac{x + S(n)}{\sqrt{n}} \in D, \tau_x > n \right) = \frac{\kappa + o(1)}{n^{p/2}} \int_D u(z) e^{-|z|^2/2} dz E[u(x + S(m)); \tau_x > m].
\]

Combining this with Theorem 2, we get the desired relation. Thus, we are left to prove (77).

It follows from (4) that

\[
u(y + Rx_0) \leq C|y + Rx_0|^{p-1} d(y + Rx_0) \leq C \varepsilon_n^{p-1} \delta_n n^{p/2} \quad y \in K_1.
\]

Therefore,

\[
E[u(x + Rx_0 + S(m)); \tau_x > m, x + S(m) \in K_1] \leq C \varepsilon_n^{p-1} \delta_n n^{p/2} P(\tau_x > m).
\]
Applying the upper bound from part (a) and recalling the definition of $m$, we conclude that
\[
E[u(x + Rx_0 + S(m)); \tau_x > m, x + S(m) \in K_1] \\
\leq C\varepsilon_n^{p-1}\delta_n(n/m)^{p/2}u(x + Rx_0) = C\varepsilon_n^{p-1}u(x + Rx_0).
\] (78)

Now let
\[
l = \left[ \frac{\sqrt{n}}{2n\log n} \right].
\]

We have,
\[
E[u(x + Rx_0 + S(m)); \tau_x > m, x + S(m) \in K_3] \\
\leq E[V_\beta(x + Rx_0 + S(m)); \tau_x > m, x + S(m) \in K_3] \\
=: E_1 + E_2,
\]
where
\[
E_1 = E[V_\beta(x + Rx_0 + S(m)); \tau_x > m, |x + S(l)| > \varepsilon_n\sqrt{n}/2]
\]
and
\[
E_2 = E[V_\beta(x + Rx_0 + S(m)); \tau_x > m, |x + S(l)| \leq \varepsilon_n\sqrt{n}/2, |S(m) - S(l)| > \varepsilon_n\sqrt{n}/2].
\]

Since $V_\beta(y + S(k))1\{\tau_y > k\}$ is a supermartingale,
\[
E_1 \leq E[V_\beta(x + Rx_0 + S(l)); \tau_x > l, |x + S(l)| > \varepsilon_n\sqrt{n}/2] \\
\leq E[V_\beta(x + Rx_0 + S(l)); \tau_x > l, |x + S(l)| > \varepsilon_n\sqrt{n}/2, \max_{k \leq l} |X(k)| \leq t_n] \\
+ \sum_{k=1}^{l} E[V_\beta(x + Rx_0 + S(l)); \tau_x > l, |x + S(l)| > \varepsilon_n\sqrt{n}/2, |X(k)| > t_n] \\
\leq E[V_\beta(x + Rx_0 + S(l)); |x + S(l)| > \varepsilon_n\sqrt{n}/2, \max_{k \leq l} |X(k)| \leq t_n] \\
+ \sum_{k=1}^{l} E[V_\beta(x + Rx_0 + S(l)); \tau_x > l, |X(k)| > t_n],
\]
where
\[
t_n = \varepsilon_n^{\sqrt{n}}/\log n.
\]

Since $V_\beta(y + Rx_0) \leq Cu(y + Rx_0) \leq C|y + Rx_0|^p$,
\[
E[V_\beta(x + Rx_0 + S(l)); |x + S(l)| > \varepsilon_n\sqrt{n}/2, \max_{k \leq l} |X(k)| \leq t_n] \\
\leq C|x + Rx_0|^p u(x + Rx_0) + E[|S(l)|^p; |S(l)| > \varepsilon_n\sqrt{n}/4, \max_{k \leq l} |X(k)| \leq t_n] \\
\leq C\left( \frac{|x + Rx_0|^p}{|S(l)|^{p/2}} u(x + Rx_0) + E[|S(l)|^p; |S(l)| > \varepsilon_n\sqrt{n}/4, \max_{k \leq l} |X(k)| \leq t_n] \right).
\]

By the Fuk-Nagaev inequality, see Lemma 22 in [9], for $z \geq \frac{\varepsilon_n^{p/2}\sqrt{n}}{n}$,
\[
P(|S(l)| > z, \max_{k \leq l} |X(k)| \leq t_n) \leq 2d \left( \frac{e\sqrt{d}d}{z^2 t_n} \right)^{z/\sqrt{d}t_n} \leq 2d \left( \frac{e\sqrt{d}\varepsilon_n^{p/2}\sqrt{n}}{z} \right)^{\log n/\sqrt{d}n}.
\]
This implies that
\[
\mathbb{E}[|S(l)|^p; |S(l)| > \epsilon_n\sqrt{n}/4, \max_{k \leq l} |X(k)| \leq t_n] \\
= \left(\frac{\epsilon_n\sqrt{n}}{4}\right)^p \mathbb{P}\left(|S(l)| > \frac{\epsilon_n\sqrt{n}}{4}, \max_{k \leq l} |X(k)| \leq t_n\right) \\
+ p \int_{\epsilon_n\sqrt{n}/4}^{\infty} 2^{p-1} \mathbb{P}(|S(l)| > z, \max_{k \leq l} |X(k)| \leq t_n) dz \\
\leq C(\epsilon_n) \left(\frac{\epsilon_n\sqrt{n}}{4}\right)^p (4\epsilon_n \sqrt{d\epsilon_n} \log \frac{n}{\sqrt{\epsilon_n}}) = o(1).
\]

As a result, uniformly in \(x \in K\) such that \(|x| = o(\sqrt{l})\),
\[
\mathbb{E}[V_\beta(x + Rx_0 + S(l)); |x + S(l)| > \epsilon_n\sqrt{n}/2, \max_{k \leq l} |X(k)| \leq t_n] = o(u(x + Rx_0)). \tag{79}
\]

Using the supermartingale property of \(V_\beta\), we get
\[
\mathbb{E}[V_\beta(x + Rx_0 + S(l)); \tau_x > l, |X(k)| > t_n] \\
\leq \mathbb{E}[V_\beta(x + Rx_0 + S(k)); \tau_x > k, |X(k)| > t_n] \\
\leq C\mathbb{E}[u(x + Rx_0 + S(k)); \tau_x > k, |X(k)| > t_n].
\]

Using (21), we get
\[
\mathbb{E}[u(x + Rx_0 + S(k)); \tau_x > k, |X(k)| > t_n] \\
\leq \mathbb{E}[V_\beta(x + Rx_0 + S(k-1)); \tau_x > k-1 |X| > t_n] \\
+ C\mathbb{E}[|x + Rx_0 + S(k-1)|^{p-1}; \tau_x > k-1 |X| > t_n] \\
+ \mathbb{P}(\tau_x > k-1)\mathbb{E}[|X|^p; |X| > t_n].
\]

By the supermartingale property of \(V_\beta\),
\[
\mathbb{E}[V_\beta(x + Rx_0 + S(k-1)); \tau_x > k-1 |X| > t_n] \leq Cu(x + Rx_0)\mathbb{P}(|X| > t_n).
\]

Consequently,
\[
\sum_{k=1}^{l} \mathbb{E}[V_\beta(x + Rx_0 + S(k-1)); \tau_x > k-1 |X| > t_n] \\
\leq Cu(x + Rx_0)l\mathbb{P}(|X| > t_n) = o(u(x + Rx_0)), \tag{80}
\]

where the last estimate follows from the bound
\[
l\mathbb{P}(|X| > t_n) \leq \frac{l}{t_n^2 \log t_n} \mathbb{E}[|X|^2 \log |X|] = O(\epsilon_n).
\]

Next,
\[
\mathbb{E}[|x + Rx_0 + S(k-1)|^{p-1}; \tau_x > k-1] \\
\leq |x + Rx_0|^{p-1}\mathbb{P}(\tau_x > k-1) + \mathbb{E}[|S(k-1)|^{p-1}; \tau_x > k-1] \\
\leq C|x + Rx_0|^{p-1}u(x + Rx_0) \frac{1}{kp/2} + \mathbb{E}[|S(k-1)|^{p-1}; \tau_x > k-1].
\]

We shall consider the cases \(p = 2\) and \(p > 2\) separately.
If \( p = 2 \) then
\[
\sum_{k=1}^l |x + Rx_0| \frac{u(x + Rx_0)}{k} E[|X|; |X| > t_n] \\
\leq C|x + Rx_0|u(x + Rx_0) \log t_n E[|X|^2 \log |X|; |X| > t_n] \\
\leq C u(x + Rx_0) |x + Rx_0| \log n \frac{E[|X|^2 \log |X|; |X| > t_n]}{\varepsilon_n}.
\]
If \( \varepsilon_n \) converges to zero sufficiently slow then
\[
\frac{E[|X|^2 \log |X|; |X| > t_n]}{\varepsilon_n} \to 0
\]
and, consequently,
\[
\sum_{k=1}^l |x + Rx_0| \frac{u(x + Rx_0)}{k} E[|X|; |X| > t_n] = o(u(x + Rx_0))
\]
uniformly in \( x \in K \) such that \( |x| \leq \frac{\sqrt{\pi}}{\log n} \).

If \( p > 2 \) then
\[
\sum_{k=1}^l |x + Rx_0|^{p-1} \frac{u(x + Rx_0)}{k^{p/2}} E[|X|; |X| > t_n] \\
\leq C|x + Rx_0|^{p-1} u(x + Rx_0) \frac{E[|X|^p; |X| > t_n]}{t_n^{p-1}} \\
\leq C u(x + Rx_0) \left( \frac{|x + Rx_0| \log n}{\sqrt{n}} \right)^{p-1} \frac{E[|X|^p; |X| > t_n]}{\varepsilon_n^{p-1}}.
\]
This bound implies that, for any \( p \geq 2 \), uniformly in \( x \in K \) with \( |x| \leq \frac{\sqrt{\pi}}{\log n} \),
\[
\sum_{k=1}^l |x + Rx_0|^{p-1} \frac{u(x + Rx_0)}{k^{p/2}} E[|X|; |X| > t_n] = o(u(x + Rx_0)) \tag{81}
\]
provided that \( \varepsilon_n \to 0 \) sufficiently slow.

In the case \( p = 2 \) we also have
\[
E[|S(k)|; \tau_x > k] \leq \sqrt{k \log k} P(\tau_x > k) + \frac{1}{\sqrt{k \log k}} E[|S(k)|^2, \tau_x > k] \\
\leq C \frac{u(x + Rx_0) \log^{1/2} k}{\sqrt{k}} + \frac{1}{\sqrt{k \log k}} E[|S(\tau_x \wedge k)|^2].
\]
Noting that \( |S(k)|^2 - dn \) is a martingale and using part (a) of the theorem, we get
\[
E[|S(\tau_x \wedge k)|^2] = \frac{d}{E[\tau_x \wedge k]} \leq cu(x + Rx_0) \log k.
\]
Consequently,
\[
E[|S(k)|; \tau_x > k] \leq C \frac{u(x + Rx_0) \log^{1/2} k}{\sqrt{k}}.
\]
Therefore,
\[
\sum_{k=1}^{l} \mathbb{E}[|S(k - 1)|; \tau_x > k - 1] \leq Cu(x + Rx_0) \frac{\sqrt{\log l} \mathbb{E}[|X|^2 \log |X|; |X| > t_n]}{t_n \log t_n} \leq Cu(x + Rx_0) \varepsilon_n^{1/2} = o(u(x + Rx_0)).
\]

If \( p > 2 \) then using the upper bound for \( P(\tau_x > k) \) we obtain
\[
\mathbb{E}[|S(k)|^{p-1}; \tau_x > k] \leq k^{p/2-1/2} P(\tau_x > k) + k^{-r/2} \mathbb{E}[|S(k)|^{p-1+r}; \tau_x > k] \leq Cu(x + Rx_0) k^{-1/2} + k^{-r/2} \mathbb{E}[\max_j |S(j)|^{p-1+r}]
\]
for every \( r \in (0, 1) \). Applying (62), we obtain
\[
\mathbb{E}[|S(k)|^{p-1}; \tau_x > k] \leq Cu(x + Rx_0) k^{-r/2}.
\]

Then, summing up over \( k \), we get
\[
\sum_{k=1}^{l} \mathbb{E}[|S(k - 1)|^{p-1}; \tau_x > k - 1] \mathbb{E}[|X|; |X| > t_n] \leq Cu(x + Rx_0) l^{1-r/2} t_n^{1-p} \mathbb{E}[|X|^p; |X| > t_n] = o(u(x + Rx_0))
\]
if we choose \( r \) so that \( p + r - 3 > 0 \). This shows that
\[
\sum_{k=1}^{l} \mathbb{E}[|S(k - 1)|^{p-1}; \tau_x > k - 1] \mathbb{E}[|X|; |X| > t_n] = o(u(x + Rx_0)) \tag{82}
\]
for all \( p \geq 2 \).

Considering again the case \( p = 2 \), we have by Lemma 23,
\[
\sum_{k=1}^{l} P(\tau_x > k - 1) \mathbb{E}[|X|^2; |X| > t_n] \leq Cu(x + Rx_0) \log l \mathbb{E}[|X|^2; |X| > t_n] \leq Cu(x + Rx_0) \log l \mathbb{E}[|X|^2 \log |X|; |X| > t_n] \leq Cu(x + Rx_0) \mathbb{E}[|X|^2 \log |X|; |X| > t_n] = o(u(x + Rx_0)).
\]

And if \( p > 2 \) then
\[
\sum_{k=1}^{l} P(\tau_x > k - 1) \mathbb{E}[|X|^p; |X| > t_n] \leq Cu(x + Rx_0) \mathbb{E}[|X|^p; |X| > t_n] = o(u(x + Rx_0)).
\]

Therefore,
\[
\sum_{k=1}^{l} P(\tau_x > k - 1) \mathbb{E}[|X|^p; |X| > t_n] = o(u(x + Rx_0)) \tag{83}
\]
for all \( p \geq 2 \). Combining (79)—(83), we conclude that
\[
E_1 = o(u(x + Rx_0)) \tag{84}
\]
uniformly in \( x \in K \) such that \( |x| \leq \sqrt{n}/\log n \).
We now turn to $E_2$. It follows from (21) that $u(x+y) \leq u(x)+C|y|^p$ for $|x| \leq |y|$. Using this bound, we get

$$E_2 \leq CE[u(x+Rx_0+S(m)); \tau_x > m, |x+S(l)| \leq \varepsilon_n \sqrt{n}/2, |S(m)-S(l)| > \varepsilon_n \sqrt{n}/2]$$

$$\leq CE[V_\beta(x+Rx_0+S(l)); \tau_x > l|P(|S(m)-S(l)| > \varepsilon_n \sqrt{n}/2)$$

$$+ CP(\tau_x > l)E[|S(m-l)|^p; |S(m-l)| > \varepsilon_n \sqrt{n}/2].$$

Combining the supermartingale property of $V_\beta$ with the Chebyshev inequality, we conclude that

$$E[V_\beta(x+Rx_0+S(l)); \tau_x > l|P(|S(m)-S(l)| > \varepsilon_n \sqrt{n}/2)$$

$$\leq V_\beta(x)P(|S(m)-S(l)| > \varepsilon_n \sqrt{n}/2) = o(u(x+Rx_0)).$$

Therefore,

$$E_2 \leq o(u(x+Rx_0)) + CP(\tau_x > l)E[|S(m-l)|^p; |S(m-l)| > \varepsilon_n \sqrt{n}/2]$$

$$\leq o(u(x+Rx_0)) + C\frac{u(x+Rx_0)}{l^{p/2}}E[|S(m-l)|^p; |S(m-l)| > \varepsilon_n \sqrt{n}/2].$$

So, it remains to show that

$$\frac{1}{l^{p/2}}E[|S(m-l)|^p; |S(m-l)| > \varepsilon_n \sqrt{n}/2] \to 0. \quad (85)$$

Fix some $r > 2p\sqrt{d}$. Then, by Corollary 23 in [9] with $y = x/r$,

$$P(|S(k)| > x) \leq 2d \left(\frac{er\sqrt{d}k}{x^2}\right)^{r/\sqrt{d}} + kP(|X| > x/r).$$

This implies that

$$E[|S(k)|^p; |S(k)| > \varepsilon_n \sqrt{n}/2]$$

$$= (\varepsilon_n \sqrt{n}/2)^pP(|S(k)| > \varepsilon_n \sqrt{n}/2) + p \int_{\varepsilon_n \sqrt{n}/2}^\infty z^{p-1}P(|S(k)| > z)dz$$

$$\leq rP\beta E[|X|^p; |X| > \varepsilon_n \sqrt{n}/2] + C(p, d, r)(\varepsilon_n^2 n)^{p/2} \left(\frac{k}{\varepsilon_n^2 n}\right)^{r/\sqrt{d}}.$$

Consequently,

$$\frac{1}{l^{p/2}}E[|S(m-l)|^p; |S(m-l)| > \varepsilon_n \sqrt{n}/2]$$

$$\leq rP\beta m \left(\frac{|X|^p}{l}\right) \left(\frac{m}{\varepsilon_n^2 n}\right)^{r/\sqrt{d}}.$$

Recalling the definitions of $m$ and $l$, we get

$$\left(\frac{\varepsilon_n^2 n}{l}\right)^{p/2} \left(\frac{m}{\varepsilon_n^2 n}\right)^{r/\sqrt{d}} \leq C\varepsilon_n^{-p/2} \log^{p/2} n(\varepsilon_n^2 \log^{1/4} n)^{-r/\sqrt{d}} = o(1)$$

when $\varepsilon_n$ decreases to zero sufficiently slow.
If \( p = 2 \) then, by the Chebyshev inequality,
\[
\frac{m}{\lVert p \rVert^2} \mathbb{E}[|X|^p; |X| > \varepsilon_n \sqrt{n}/2] \leq C \log^{1/3} n \varepsilon_n^{-3} \mathbb{E}[|X|^2; |X| > \varepsilon_n \sqrt{n}/2]
\]
\[
\leq C \log^{-1/4} n \varepsilon_n^{-3} \mathbb{E}[|X|^2 \log |X|; |X| > \varepsilon_n \sqrt{n}/2] = o(1).
\]
In the case \( p > 2 \) it suffices to notice that \(
\frac{m}{\lVert p \rVert^2} \to 0.
\)
Therefore, \( E_2 = o(u(x + Rx_0)) \).
Combining this with (84) we obtain (77). Thus, the proof is finished.

9. Proof of Proposition 6

By the optional stopping theorem for the martingale \( u(x + S(n)) \),
\[
u(x) = \mathbb{E}[u(x + S(\tau_x \land k)]
\]
\[
= \mathbb{E}[u(x + S(k)); \tau_x > k] + \mathbb{E}[u(x + S(\tau_x)); \tau_x \leq k], \quad k \geq 1.
\]
Thus, for all \( n > m \geq 1 \),
\[
E_n - E_m = \mathbb{E}[u(x + S(n)); \tau_x > n] - \mathbb{E}[u(x + S(m)); \tau_x > m]
\]
\[
= \mathbb{E}[-u(x + S(\tau_x)); \tau_x \in (m, n]]
\]
\[
:= \Sigma_1 + \Sigma_2,
\]
where
\[
\Sigma_1 = \sum_{k=m}^{n-1} \sum_{y \in K \cap \{y: |y| \leq A \sqrt{n}\}} \mathbb{P}(x + S(k) = y, \tau_x > k) \mathbb{E}[-u(y + X); y + X \notin K]
\]
and
\[
\Sigma_2 = \sum_{k=m}^{n-1} \sum_{y \in K \cap \{y: |y| > A \sqrt{n}\}} \mathbb{P}(x + S(k) = y, \tau_x > k) \mathbb{E}[-u(y + X); y + X \notin K].
\]
Since \( u(x + S(\tau_x)) \leq 0 \) the sequence \( E_n \) is increasing.

We start with an upper bound for \( \Sigma_2 \). We first notice that for every \( y \in K \) one has
\[
\mathbb{E}[-u(y + X); y + X \notin K] = \mathbb{E}[(y_2 + X_2)^2 - y_2^2; X_2 > y_1 - y_2]
\]
\[
\leq \mathbb{E}[2y_2^2 + 2X_2^2 - y_2^2; X_2 > y_1 - y_2]
\]
\[
\leq |y|^2 \mathbb{P}(X_2 > y_1 - y_2) + 2 \mathbb{E}[X_2^2]
\]
\[
\leq |y|^2 \mathbb{P}(\tau_y = 1) + 1.
\]
Therefore,
\[
\Sigma_2 \leq \sum_{k=m}^{n-1} \sum_{y \in K \cap \{y: |y| > A \sqrt{n}\}} \mathbb{P}(x + S(k) = y, \tau_x > k) \left(|y|^2 \mathbb{P}(\tau_y = 1) + 1 \right)
\]
\[
\leq \mathbb{E} \left([x + S(\tau_x - 1)]^2; \tau_x \in (m, n], |x + S(\tau_x - 1)| \geq A \sqrt{n} \right)
\]
\[
+ \sum_{k=m}^{n-1} \mathbb{P} \left(|x + S(k)| \geq A \sqrt{n}, \tau_x > k \right).
\]
(87)

For every \( k < \tau_x \) one has the inequality
\[
|x_2 + S_2(k)| < x_1 + S_1(k).
\]
Moreover, in view of (90), for all
\[ \max_{k<\tau_x} |x + S(k)|^q \leq 2^{q/2} \max_{k<\tau_x} |x_1 + S_1(k)|^q \]
for every \( q > 2 \). Then, by the Burkholder-Davis-Gundy inequality,
\[
E \left[ \max_{k<\tau_x} |x + S(k)|^q; \tau_x \leq n \right] \leq 2^{q/2} E \left[ \max_{k<\tau_x} |x_1 + S_1(k)|^q; \tau_x \leq n \right] \\
\leq 2^{3q/2} |x|^q + 2^{3q/2} E \left[ \max_{k<\tau_x} |S_1(k)|^q; \tau_x \leq n \right] \\
\leq C(|x|^q + E(\tau_x \wedge n)^{q/2}).
\]
According to Lemma 25,
\[ P(\tau_x > k) \leq C(x) \frac{E_k}{k} \leq C(x) \frac{E_n}{k}, \quad k \leq n. \tag{88} \]
This implies that
\[ E((\tau_x \wedge n)^{q/2}) \leq C(x) E_n n^{q/2-1}. \tag{89} \]
As a result we have
\[
E \left[ \max_{k<\tau_x} |x + S(k)|^q; \tau_x \leq n \right] \leq C(x) E_n n^{q/2-1}. \tag{90} \]
By the same arguments,
\[ E [ |x + S(n)|^q; \tau_x > n ] \leq C(x) E_n n^{q/2-1}. \tag{91} \]
Using the Markov inequality and taking into account (89), we obtain
\[
E \left[ |x + S(\tau_x - 1)|^2; \tau_x \in (m, n], |x + S(\tau_x - 1)| \geq A\sqrt{n} \right] \\
\leq (A\sqrt{n})^{2-q} E \left[ \max_{k<\tau_x} |x + S(k)|^q; \tau_x \leq n \right] \\
\leq C(x) A^{-2} E_n. \tag{92} \]
Moreover, in view of (90), for all \( k \leq n, \)
\[ P(|x + S(k)| \geq A\sqrt{n}, \tau_x > k) \leq (A\sqrt{n})^{-q} E [ |x + S(k)|^q; \tau_x > k ] \leq \frac{C(x) E_n}{A^q n}. \]
Consequently,
\[ \sum_{k=m}^{n-1} P(|x + S(k)| \geq A\sqrt{n}, \tau_x > k) \leq \frac{C(x) E_n}{A^q}. \tag{93} \]
Plugging (91) and (92) into (87), we have
\[ \Sigma_2 \leq \frac{C(x) E_n}{A^q}. \tag{94} \]
To estimate \( \Sigma_1 \) we derive an upper bound for the local probability \( P(x + S(n) = y, \tau_x > n) \). Set \( m = \lceil 3n/4 \rceil \). Then
\[
P(x + S(n) = y; \tau_x > n) \\
= \sum_{z \in K} P(x + S(m) = z; \tau_x > m) P(z + S(n - m) = y; \tau_x > n - m) \\
= \sum_{z \in K} P(x + S(m) = z; \tau_x > m) P(y - S(n - m) = z; \tau_y' > n - m) \\
\leq P(\tau_y' > n - m) \max_{z \in K} P(x + S(m) = z; \tau_x > m),
\]
where
\[
\tau_y' = \inf \{ k \geq 1 : y - S(k) \notin K \}.
\]
Replacing \( K \) by a half-plane and using \([9, \text{Lemma 3}]\), one has
\[
P(\tau'_y > n - m) \leq \frac{Cd(y)}{\sqrt{n-m}} \leq \frac{C(y_1 - |y_2|)}{\sqrt{n}}.
\]  
(94)

Furthermore, by Lemma 27 in \([9]\),
\[
\max_{z \in K} P(x + S(m) = z; \tau_x > m) \leq \frac{C}{m} P(\tau_x > m/2).
\]

Taking now into account (88), we have
\[
\max_{z \in K} P(x + S(m) = z; \tau_x > m) \leq C \frac{E_n}{n^{3/2}}.
\]

Combining this with (94), we arrive at the bound
\[
P(x + S(m) = y; \tau_x > n) \leq C(y_1 - |y_2|) \frac{E_n}{n^{3/2}}, \quad y \in K.
\]  
(95)

For every \( y \in K \) with \( |y| \leq A \sqrt{n} \) we have
\[
E[-u(y + X); y + X \notin K] = E[(y_2 + X_2)^2 - y_1^2; X_2 > y_1 - y_2]
\]
\[
\leq 2y_2 E[X_2; X_2 > y_1 - y_2] + E[X_2^2; X_2 > y_1 - y_2]
\]
\[
\leq 2A \sqrt{n} E[X_2; X_2 > y_1 - y_2] + E[X_2^2; X_2 > y_1 - y_2].
\]

Combining this with (95), we get
\[
\Sigma_1 \leq \sum_{k=m}^{n-1} \sum_{y \in \mathcal{K} \cap \{ y ; |y| \leq A \sqrt{n} \}} (y_1 - |y_2|) (2A \sqrt{n} E[X_2; X_2 > y_1 - y_2] + E[X_2^2; X_2 > y_1 - y_2])
\]
\[
\leq C \frac{E_n}{m^{3/2}} \sum_{y \in \mathcal{K} \cap \{ y ; |y| \leq A \sqrt{n} \}} (y_1 - |y_2|) (2A \sqrt{n} E[X_2; X_2 > y_1 - y_2] + E[X_2^2; X_2 > y_1 - y_2])
\]
\[
\leq C \frac{E_n}{m^{3/2} A \sqrt{n}} \sum_{y \in \mathcal{K} \cap \{ y ; |y| \leq A \sqrt{n} \}} E[X_2^2; X_2 > y_1 - y_2].
\]

The existence of the second moment implies that the sum is \( o(n) \). Therefore,
\[
\Sigma_1 = o \left( \frac{E_n m^{3/2}}{A^{3/2}} \right).
\]  
(96)

Applying this estimate and (93) to (86), we conclude that, if \( m \sim cn \) for some \( c \in (0,1] \) then
\[
\limsup_{n \to \infty} \left| \frac{E_m}{E_n} - 1 \right| \leq \frac{C(x)}{A^{3/2}}.
\]

Letting now \( A \to \infty \) we finish the proof of (a).

To prove the part (b) we choose \( m = m(n) \) such that \( \frac{m}{n} \to 0 \) and \( E_m \sim E_n \).

Then, by the Markov property,
\[
P(\tau_x > n) = \sum_{y \in K} P(x + S(m) = y, \tau_x > m | \tau_y > n - m).
\]  
(97)
Fix $\varepsilon < 1, A > 1$ and set

\[
K_1 := \{y \in K : |y| \leq A\sqrt{m}, d(y) \leq \varepsilon \sqrt{m}\},
\]
\[
K_2 := \{y \in K : |y| \leq A\sqrt{m}, d(y) > \varepsilon \sqrt{m}\},
\]
\[
K_3 := \{y \in K : |y| > A\sqrt{m}\}.
\]

By the Brownian approximation, if $\frac{m}{n} \to 0$ sufficiently slow,

\[
\mathbf{P}(\tau_y > n - m) \sim \frac{u(y)}{n}
\]

uniformly in $y \in K_2$. Therefore,

\[
\sum_{y \in K_2} \mathbf{P}(x + S(m) = y, \tau_x > m) \mathbf{P}(\tau_y > n - m)
= \frac{C}{n} \mathbf{E}[u(x + S(m)); y + S(m) \in K_2, \tau_x > m] + o\left(\frac{E_n}{n}\right). \quad (98)
\]

Applying (88) and using the Optional Stopping Theorem, we obtain

\[
\sum_{y \in K_3} \mathbf{P}(x + S(m) = y, \tau_x > m) \mathbf{P}(\tau_y > n - m)
\leq \frac{C}{n} \sum_{y \in K_3} \mathbf{P}(x + S(m) = y, \tau_x > m) \mathbf{E}[u(y + S(n - m)); \tau_y > n - m]
= \frac{C}{n} \sum_{y \in K_3} \mathbf{P}(x + S(m) = y, \tau_x > m) \left(u(y) - \mathbf{E}[u(y + S(\tau_y)); \tau_y \leq n - m]\right)
\leq \frac{C}{n} \mathbf{E}[u(x + S(m)); |x + S(m)| > A\sqrt{m}, \tau_x > m]
- \frac{C}{n} \mathbf{E}[u(x + S(\tau_x)); \tau_x \in (m, n)].
\]

It follows from the part (a) that the last expectation is $o(E_n)$. Consequently,

\[
\sum_{y \in K_3} \mathbf{P}(x + S(m) = y, \tau_x > m) \mathbf{P}(\tau_y > n - m)
\leq \frac{C}{n} \mathbf{E}[u(x + S(m)); |x + S(m)| > A\sqrt{m}, \tau_x > m] + o\left(\frac{E_n}{n}\right). \quad (99)
\]

By the same arguments,

\[
\sum_{y \in K_1} \mathbf{P}(x + S(m) = y, \tau_x > m) \mathbf{P}(\tau_y > n - m)
\leq \frac{C}{n} \mathbf{E}[u(x + S(m)); x + S(m) \in K_1, \tau_x > m] + o\left(\frac{E_n}{n}\right). \quad (100)
\]

Putting together (97)—(100), we get

\[
\left|\mathbf{P}(\tau_x > n) - \frac{E_n}{n}\right|
\leq \frac{C}{n} \mathbf{E}[u(x + S(m)); x + S(m) \in K_1 \cup K_3, \tau_x > m] + o\left(\frac{E_n}{n}\right). \quad (101)
\]

If $y \in K_1$ then $u(y) \leq C\varepsilon Am$. From this observation and from (88) we get

\[
\mathbf{E}[u(x + S(m)); x + S(m) \in K_1, \tau_x > m] \leq C\varepsilon Am \mathbf{P}(\tau_x > m) \leq C\varepsilon AE_n.
\]
Furthermore, it follows from (90) that
\[ E[u(x + S(m)); x + S(m) \in K_3, \tau_x > m] \]
\[ \leq C(A\sqrt{m})^{2-q}E[|x + S(m)|^q; \tau_x > m] \leq \frac{C}{A^{q-2}}E_n. \]

As a result,
\[ E[u(x + S(m)); x + S(m) \in K_1 \cup K_3, \tau_x > m] \leq C (A\varepsilon + A^{2-q}) E_n. \]

Plugging this into (101), we have
\[ \left| P(\tau_x > n) - \frac{E_n}{n} \right| \leq C \left( A\varepsilon + A^{2-q} \right) \frac{E_n}{n} + o \left( \frac{E_n}{n} \right). \]

Letting first \( \varepsilon \to 0 \) and then \( A \to \infty \), we arrive at the desired relation.

If \( E[W_1^2 \log(1 + |W_1|)] < \infty \) then the finiteness of \( \lim_{n \to \infty} E_n \) follows from Theorem 2. Thus, it remains to show that \( E_n \to \infty \) for walks with \( E[W_1^2 \log(1 + |W_1|)] = \infty \). It follows from (86) that
\[ E_{2n} - E_n = E[-u(x + S(\tau_x)); \tau_x \in (n, 2n)] \]
\[ = \sum_{k=n}^{2n-1} \int_K P(x + S(k) \in dy, \tau_x > k)E[-u(y + X); y + X \notin K]. \]

Repeating the arguments from the proof of (18) and taking into account (19), we get
\[ E_{2n} - E_n \geq \frac{1}{4} E[W_1^2; W_1 > \sqrt{2n^2}] \sum_{k=n}^{2n-1} P(\tau_x > k). \]

Using parts (a) and (b), we conclude that there exists \( n_0 \) such that
\[ E_{2n} - E_n \geq \frac{\sqrt{2}}{8} E[W_1^2; W_1 > \sqrt{2n^2}] E_n, \quad n \geq n_0. \]

In other words,
\[ E_{2n} \geq \left( 1 + \frac{\sqrt{2}}{8} E[W_1^2; W_1 > \sqrt{2n^2}] \right) E_n, \quad n \geq n_0. \]

Therefore,
\[ \lim_{n \to \infty} E_n \geq \prod_{m=0}^{\infty} \left( 1 + \frac{\sqrt{2}}{8} E[W_1^2; W_1 > \sqrt{2n^2}] \right). \]

But the product on the right hand side is infinite if \( E[W_1^2 \log(1 + |W_1|)] = \infty \). This completes the proof of the proposition.

**APPENDIX A. CONSTRUCTION OF A POSITIVE SUPERMARTINGALE FOR A RANDOM WALK ON THE POSITIVE HALF LINE**

Here we will construct explicitly a positive supermartingale for a one-dimensional random walk on a positive half-line. This construction has already appeared in [20].

Let \( X, X_1, X_2, \ldots \) be i.i.d. random variables with the distribution function \( F \), zero mean \( E[X_1] = 0 \) and finite variance \( \sigma^2 := E[X_1^2] < \infty \). Consider the random walk \( S_0 = 0 \) and
\[ S_n = X_1 + \cdots + X_n, \quad n \geq 1. \]

Let \( \tau_x := \inf\{n \geq 1 : x + S_n < 0\}. \)
Fix some positive constants $A, R$ and define functions for $x \geq 0$,
\[
\beta(x) = P(X \leq -x) = F(-x),
\]
\[
\beta^I(x) = E[(x + X_1)_{-}] = \int_{-\infty}^{\infty} \beta(y) dy,
\]
\[
\beta^{II}(x) = \int_{-\infty}^{\infty} \beta(y) dy, \quad m(x) = A \int_{0}^{x} \beta^{II}(x) dy
\]
and
\[
V(x) = \begin{cases} 
  x + R + m(x), & x \geq 0 \\
  0, & \text{otherwise. }
\end{cases} \tag{102}
\]

**Lemma 27.** Let $E[X_1] = 0$ and $E[X_1^2] < \infty$. Then, for the function $V$ defined in (102),
\[
E[V(x + X_1); \tau_x > 1] \leq V(x) - \beta(x), \quad x \geq 0. \tag{103}
\]

**Proof.** Put
\[
\Delta(x) := E[V(x + X_1); \tau_x > 1] - V(x).
\]

Then,
\[
\Delta(x) = E[x + X_1, X_1 > -x] - x - RP(X_1 \leq -x) + E[m(x + X_1), X_1 > -x] - m(x)
\]
\[
= -E[x + X_1, X_1 \leq -x] - R\beta(x) - m(x)\beta(x) + \int_{-x}^{\infty} F(dy)(m(x + y) - m(x))
\]
\[
= \beta^I(x) - R\beta(x) - m(x)\beta(x)
\]
\[
+ \int_{-x}^{0} F(dy)(m(x + y) - m(x)) + \int_{0}^{\infty} F(dy)(m(x + y) - m(x)).
\]

Integrating twice the integrals by parts we obtain,
\[
\Delta(x) = \beta^I(x) - R\beta(x) - \int_{-x}^{0} dyF(y)m^I(x + y) + \int_{0}^{\infty} dyF(y)m^I(x + y)
\]
\[
= \beta^I(x) - R\beta(x) - m'(x)\beta^I(0) + A\beta^I(x)\sigma^2 / 2
\]
\[
+ \int_{-x}^{0} dy\beta^I(-y)m''(x + y) + m'(x)F^I(0) + \int_{0}^{\infty} dyF^I(y)m''(x + y),
\]
where $\sigma^2 = E[(X_1^-)^2]$. Since $E[X_1] = 0$,
\[
\beta^I(0) - F^I(0) = 0.
\]

Therefore,
\[
\Delta(x) = \beta^I(x) - R\beta(x) + A\beta^I(x)\sigma^2 / 2
\]
\[
+ \int_{0}^{\infty} dyF^I(y)m''(x + y) + \int_{-x}^{0} dy\beta^I(-y)m''(x + y).
\]
We can estimate the second integral as follows,
\[
\int_{-x}^{0} dy \beta^I(-y) m''(x + y) = -A \int_{0}^{x} dy \beta^I(y)\beta^I(x - y) = -2A \int_{0}^{x/2} dy \beta^I(y)\beta^I(x - y) \leq -2A\beta^I(x) (\beta^{II}(0) - \beta^{II}(x/2)) = -A\sigma^2\beta^I(x) + 2A\beta^I(x)\beta^{II}(x/2).
\]

Noting now that \(m''\) is negative and, consequently, \(\int_{0}^{\infty} dy \beta^I(y)\beta^I(x + y) = -2\beta^I(x)\beta^{II}(x/2)\), we infer that
\[
\Delta(x) \leq \beta^I(x) - R\beta(x) - \frac{A}{2}\sigma^2\beta^I(x) + 2A\beta^I(x)\beta^{II}(x).
\]
Choosing now \(A = \frac{4}{\sigma^2}\), we arrive at the inequality
\[
\Delta(x) \leq 2A\beta^I(x)\beta^{II}(x) - \beta^I(x) - R\beta(x) \quad (104)
\]

The finiteness of \(E(X_1^-)^2\) implies that \(\beta^{II}(x) \to 0\) as \(x \to \infty\). In particular, there exists \(x_0\) such that \(2A\beta^I(x)\beta^{II}(x) - \beta^I(x) \leq 0\) for all \(x \geq x_0\). Increasing \(R\), we can make the right hand side in (104) negative for \(x < x_0\). Increasing \(R\) by 1 we can achieve that \(\Delta(x) \leq -\beta(x)\) for all \(x \geq 0\). \(\square\)

Lemma 27 immediately implies that the sequence \(V(x + S_n) \mathbb{1}\{\tau_x > n\}\) is a positive supermartingale. As a consequence,
\[
E[V(x + S_n); \tau_x > n] \leq V(x), \quad n \geq 1.
\]

Since \(V(y) > y\) for all \(y > 0\), we also have the inequality
\[
E[x + S_n; \tau_x > n] \leq V(x), \quad n \geq 1. \quad (105)
\]

By the martingale property of \(S_n\),
\[
x = E(x + S_{\tau_x \wedge n}) = E[x + S_{\tau_x}; \tau_x \leq n] + E[x + S_n; \tau_x > n].
\]
From this equality and (105) we obtain
\[
-\frac{1}{n} E[x + S_{\tau_x}; \tau_x \leq n] \leq V(x) - x = m(x) + R, \quad n \geq 1.
\]
Leaving here \(n \to \infty\) we finally get the integrability of the overshoot:
\[
-\frac{1}{n} E[x + S_{\tau_x}] \leq m(x) + R.
\]

Acknowledgment. The authors gratefully acknowledge hospitality of the Saint-Petersburg Department of Steklov Institute, where a significant part of this work has been done. Special thanks are due to Elena Skripka for the smooth organisation of the visit.
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