The Cauchy problem for metric-affine $f(R)$-gravity in the presence of perfect-fluid matter

S Capozziello$^1$ and S Vignolo$^2$

$^1$Dipartimento di Scienze Fisiche, Università ‘Federico II’ di Napoli and INFN Sez. di Napoli, Compl. Univ. Monte S. Angelo Ed. N, via Cintia, I-80126 Napoli, Italy

$^2$DIPTEM Sez. Metodi e Modelli Matematici, Università di Genova, Piazzale Kennedy, Pad. D-16129 Genova, Italy

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Abstract

The Cauchy problem for metric-affine $f(R)$-gravity in the manner of Palatini and with torsion, in the presence of perfect fluid matter acting as a source, is discussed following the well-known Bruhat prescriptions for general relativity. The problem results in being well formulated and well posed when the perfect-fluid form of the stress–energy tensor is preserved under conformal transformations and the set of viable $f(R)$-models is not empty. The key role of conservation laws in the Jordan and in the Einstein frame is also discussed.

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1. Introduction

Several extensions of general relativity have recently acquired great interest in cosmology and in quantum field theory in order to cure the shortcomings of Einstein’s theory at ultra-violet and infra-red scales. In particular, such relativistic theories could result in being useful to address problems such as renormalization and regularization in quantum field theory and to address cosmological puzzles such as the observed huge amounts of dark energy and dark matter, up to now not probed at fundamental scales. Among these attempts, $f(R)$-gravity can be considered as a paradigm by which it is possible to preserve several good and well-established results of general relativity, without imposing ‘a priori’ the form of the gravitational action, chosen to be $f(R) = R$ in the Hilbert–Einstein case. For comprehensive reviews on the argument, see, e.g., [1–4]. From an epistemological viewpoint, considering $f(R)$-gravity is a statement of ‘ignorance’, we do not know the ‘true’ action of gravity but we know that it has to be constructed by curvature invariants (the simplest is the Ricci scalar $R$) which well represent the spacetime dynamics. Observations and experiments could, in principle, help us to reconstruct the effective form of the theory by solving inconsistencies and shortcomings at various scales.

However, any theory of physics is viable if the initial value problem is correctly formulated. As a consequence, the following dynamical evolution is uniquely determined and agrees with
causality requests. Besides, also if the initial value problem is well formulated, we need further properties which are: (i) small perturbations in the initial data have to produce small perturbations in the subsequent dynamics over all the spacetime where it is defined, (ii) changes in the initial data region have to preserve the causal structure of the theory. If both these requirements are satisfied, the initial value problem of the theory is also well posed.

General relativity has a well-formulated and well-posed initial value problem, and then it results in a stable theory with a robust causal structure [5–8]. Such a theoretical structure should also be achieved for \( f(R) \)-gravity, if one wants to consider it a viable extension of the Einstein theory.

The debate on the self-consistency of \( f(R) \)-gravity cannot leave out of consideration the Cauchy problem which gives some concerns coming from higher-order derivatives in the field equations and the increased number of degrees of freedom. Such a point is crucial with respect to general relativity, since the further degrees of freedom could lead to an ill-formulated initial value problem. In fact, their role has to be clearly understood in order to discriminate among ghost modes, standard massless modes or further massive gravitational modes. To be more specific, from a fundamental physics viewpoint, higher-order theories of gravity can admit massive gravitons but such a feature strictly depends on the parameters and on the specific forms of the models (see, e.g., [9, 10]). In general, such an analysis is addressed by conformally transforming the \( f(R) \)-gravity (or the various extension of general relativity) from the Jordan frame to the Einstein frame. The extra degrees of freedom result in auxiliary scalar fields minimally coupled to the standard gravitational Hilbert–Einstein action and the analysis can result simplified. However, it is, up to now, unclear if the prolongation of standard methods can be used, in general, to tackle the initial value problem for any \( f(R) \)-theory. Hence it is doubtful that the Cauchy problem could be properly addressed, if one simply takes into account the results already obtained for fourth-order theories stemming from quadratic Lagrangians [12, 13].

On the other hand, being \( f(R) \)-gravity, like general relativity, a gauge theory, the initial value formulation could depend on suitable constraints and on suitable ‘gauge choices’ that mean a choice of coordinates so that the Cauchy problem results in being well formulated and, possibly, well posed. In [12, 14], the initial value problem was studied for quadratic Lagrangians in the metric approach with the conclusion that it is well posed. On the other hand, in [15], the Cauchy problem for generic \( f(R) \)-models has been studied in metric, and Palatini approaches using the dynamical equivalence between these theories and the Brans–Dicke gravity. The result was that the problem is well formulated for the metric approach in the presence of matter and also well posed in vacuo. For the Palatini approach, instead, the Cauchy problem is not well formulated and well posed since, considering the 3 + 1 ADM formulation of the equivalent scalar-tensor theory, the Brans–Dicke parameter

\[
\omega = -\frac{3}{2}
\]

seems to lead to a non-dynamical field \( \phi \) and to the impossibility of a first-order formulation of the problem. However, the role of the scalar field, also in cases where it could appear without dynamics, has been clarified in several papers where it has been shown that the related self-interacting potential is a key feature in recovering the underlying dynamics\(^3\) (see, e.g., [16]. In any case, the debate on the well formulation and the well posedness of the Cauchy problem of \( f(R) \)-theories in the Palatini approach has recently been discussed and improved in [17, 18].

A different approach is adopted in [19]: it is possible to show that the Cauchy problem of metric-affine \( f(R) \)-gravity is well formulated and well posed in vacuo, while it can be, at least,
well formulated for various forms of matter fields. The reason of the apparent contradiction with respect to the results in [15] lies on the above-mentioned gauge choice. Following [7], Gaussian normal coordinates have been adopted. Such a choice, introducing further constraints on the Cauchy data surface, results more suitable to set the initial value problem in such a way that the well formulation can easily be achieved. As a general remark, we can say that the well position cannot be achieved for any metric-affine $f(R)$-gravity theory but it has to be formulated specifying, case by case, the source term in the field equations. However, it is straightforward to demonstrate that, in the vacuum case, as well as for electromagnetic and generic Yang–Mills fields acting as sources, the Cauchy problem always results in being well formulated and well posed since it is possible to demonstrate that $f(R)$-gravity reduces to $R + \Lambda$, that is the general relativity plus a cosmological constant [19].

In this paper, we want to face the Cauchy problem for metric-affine $f(R)$-gravity, in the Palatini approach and with torsion, assuming perfect-fluid matter as a source. Performing the conformal transformation from the Jordan to the Einstein frame and following the approach by Bruhat adopted for general relativity [5, 6], it is possible to show that the initial value problem results in being well formulated and well posed in this case. The layout of the paper is the following. In section 2, we derive the field equations for a generic $f(R)$-gravity theory discussing the differences and the analogies between the approach in the manner of Palatini and with torsion. Both theories can be dealt under the same standard by defining a suitable scalar field. A main role, as we will show, is played by the conservation laws. Section 3 is devoted to the conformal transformation from the Jordan frame to the Einstein frame, while the well-posedness of the Cauchy problem in the presence of perfect-fluid matter is discussed in section 4. The key of the demonstration is due to the fact that it is possible to achieve a perfect-fluid form for the stress–energy tensor in both frames and the conservation laws are preserved under the conformal transformation. This fact, as we will show, allows the well-posedness. The relevant example, $f(R) = R + \alpha R^2$, is discussed in section 5. Conclusions are drawn in section 6.

2. The field equations of $f(R)$-gravity in the metric-affine formulation

In the metric-affine formulation of $f(R)$-gravity, the (gravitational) dynamical fields are pairs $(g, \Gamma)$ consisting of a pseudo-Riemannian metric $g$ and a linear connection $\Gamma$ on the spacetime manifold $M$ [20]. In the Palatini approach, the connection $\Gamma$ is torsionless but it is not requested to be metric compatible, instead, in the approach with torsion, the dynamical connection $\Gamma$ is forced to be metric but with torsion different from zero.

The field equations are derived from an action functional of the form

$$ A(g, \Gamma) = \int (\sqrt{|g|} f(R) + \mathcal{L}_m) \, ds, $$

where $f(R)$ is a real function, $R(g, \Gamma) = g^{ij} R_{ij}$ (with $R_{ij} := R^h_{ij}$) is the scalar curvature associated with the connection $\Gamma$, and $\mathcal{L}_m \, ds$ is a suitable material Lagrangian.

Assuming that the material Lagrangian does not depend on the dynamical connection, the field equations are

$$ f'(R) R_{ij} - \frac{1}{2} f(R) g_{ij} = \Sigma_{ij}, \tag{2a} $$

$$ T^{h}_{ij} = - \frac{1}{2 f'(R)} \frac{\partial f'(R)}{\partial x^\nu} \left( \delta^\nu_i \delta^h_j - \delta^\nu_j \delta^h_i \right), \tag{2b} $$

3
for \( f(R) \)-gravity with torsion, and

\[
\nabla_k (f'(R) g_{ij}) = 0, \quad (3b)
\]

for \( f(R) \)-gravity in the Palatini approach [21]. In equations (2a) and (3a), the quantity

\[
\Sigma_{ij} := -\frac{1}{2\sqrt{|g|}} \delta_{ij} \frac{\delta L}{\delta g_{ij}}
\]

plays the role of the stress–energy tensor, the source of the field equations.

Considering the trace of equations (2a) and (3a), we obtain a relation between the curvature scalar \( R \) and the trace of the stress–energy tensor \( \Sigma := g^{ij} \Sigma_{ij} \). Indeed, we have

\[
f'(R) R - 2f(R) = \Sigma. \quad (4)
\]

From now on, we shall suppose that relation (4) is invertible as well as that \( \Sigma \neq \text{const} \) (this implies, for example, \( f(R) \) is different from \( \alpha R^2 \) which is only compatible with \( \Sigma = 0 \)). Under these hypotheses, the curvature scalar \( R \) can be expressed as a suitable function of \( \Sigma \), namely

\[
R = F(\Sigma). \quad (5)
\]

If \( \Sigma = \text{const} \), general relativity plus the cosmological constant is recovered [21]. Starting from equation (5) and defining the scalar field

\[
\psi := f'(F(\Sigma)),
\]

we can put the Einstein-like field equations of both in the manner of Palatini and with torsion theories in the same form [21, 22], that is

\[
\tilde{R}_{ij} - \frac{1}{2} \tilde{R} g_{ij} = \frac{1}{\psi} \Sigma_{ij} + \frac{1}{\psi^2} \left( \phi \nabla_i (\phi \nabla_j) + \frac{3}{2} \partial_i \phi \partial_j \phi + \frac{3}{4} \partial_i \phi \partial_j \phi \right) g_{ij}
\]

\[
- \phi \nabla_i (\phi \nabla_j g_{ij}) = V(\psi) g_{ij}, \quad (7)
\]

where we have introduced the effective potential

\[
V(\psi) := \frac{1}{4}[\phi F^{-1} ((f')^{-1}(\psi)) + \phi^2 (f')^{-1}(\psi)].
\]

for the scalar field \( \psi \). In equation (7), \( \tilde{R}_{ij}, \tilde{R} \) and \( \nabla \) denote, respectively, the Ricci tensor, the scalar curvature and the covariant derivative associated with the Lévi-Civitá connection of the dynamical metric \( g_{ij} \).

Therefore, if the dynamical connection \( \Gamma \) is not coupled with matter, both the theories (with torsion and Palatini-like) generate identical Einstein-like field equations. In contrast, the field equations for the dynamical connection are different and (in general) give rise to different solutions. In fact, the connection \( \Gamma \) solution of equations (2b) is

\[
\Gamma^h_{ij} = \tilde{\Gamma}^h_{ij} + \frac{1}{\phi} \left( \frac{\partial \phi}{\partial x^j} \phi^h - \frac{1}{2\phi} \frac{\partial \phi}{\partial x^h} g^h g_{ij} \right), \quad (9)
\]

where \( \tilde{\Gamma}^h_{ij} \) denote the coefficients of the Lévi-Civitá connection associated with the metric \( g_{ij} \), while the connection \( \tilde{\Gamma} \) solution of equations (3b) is

\[
\tilde{\Gamma}^h_{ij} = \tilde{\Gamma}^h_{ij} + \frac{1}{\phi} \left( \frac{\partial \phi}{\partial x^j} \phi^h - \frac{1}{2\phi} \frac{\partial \phi}{\partial x^h} g^h g_{ij} + \frac{1}{2\phi} \frac{\partial \phi}{\partial x^j} \phi^h \right), \quad (10)
\]

and coincides with the Lévi-Civitá connection induced by the conformal metric \( \tilde{g}_{ij} := \psi g_{ij} \). By comparison, the connections \( \Gamma \) and \( \tilde{\Gamma} \) satisfy the relation

\[
\tilde{\Gamma}^h_{ij} = \Gamma^h_{ij} + \frac{1}{2\phi} \frac{\partial \phi}{\partial x^j} \phi^h. \quad (11)
\]
Of course, the Einstein-like equations (7) are coupled with the matter field equations. In this respect, it is worth pointing out that equations (7) imply the same conservation laws holding in general relativity. We have, in fact,

**Proposition 2.1.** Equations (6), (7) and (8) imply the standard conservation laws \( \tilde{\nabla}^j \Sigma_{ij} = 0 \).

**Proof.** First of all, we recall that equations (6) and (8) are equivalent to the relation

\[
\Sigma = \frac{6}{\varphi} V(\varphi) + 2V'(\varphi) = 0
\]

(12)

(see [21] for the proof). After that, taking the trace of equation (7) into account, we obtain

\[
\Sigma = -\varphi \tilde{R} + \frac{3}{2 \varphi} \varphi_i \varphi^i + 3 \tilde{\nabla}_i \varphi^i + \frac{4}{\varphi} V(\varphi),
\]

(13)

where, for the sake of simplicity, we have defined \( \varphi_i := \frac{\partial \varphi}{\partial x^i} \). Substituting equation (13) in equation (12), we obtain

\[
\tilde{R} + \frac{3}{2 \varphi} \varphi_i \varphi^i - \frac{3}{\varphi} \tilde{\nabla}_i \varphi^i + \frac{2}{\varphi^2} V(\varphi) - \frac{2}{\varphi} V'(\varphi) = 0.
\]

(14)

We rewrite equation (7) in the form

\[
\varphi \tilde{R}_{ij} - \frac{\varphi}{2} \tilde{R} g_{ij} = \Sigma_{ij} + \frac{1}{\varphi} \left( -\frac{3}{2} \varphi_i \varphi_j + \varphi \tilde{\nabla}_j \varphi_i + \frac{3}{4} \varphi h \varphi^h g_{ij} - \varphi \tilde{\nabla}^h \varphi_h g_{ij} - V(\varphi) g_{ij} \right).
\]

(15)

The covariant divergence of (15) yields

\[
(\tilde{\nabla}^i \varphi) \tilde{R}_{ij} + \varphi \tilde{\nabla}^i \tilde{G}_{ij} - \frac{1}{2} \tilde{\nabla}^i \tilde{\nabla}_j \varphi = \tilde{\nabla}^j \Sigma_{ij} + \left( \tilde{\nabla}^i \tilde{\nabla}_j - \tilde{\nabla}_j \tilde{\nabla}^i \right) \varphi
\]

\[
+ \tilde{\nabla}^j \left[ \frac{1}{\varphi} \left( -\frac{3}{2} \varphi_i \varphi_j + \frac{3}{4} \varphi h \varphi^h g_{ij} - V(\varphi) g_{ij} \right) \right].
\]

(16)

By definition, the Einstein and the Ricci tensor satisfy \( \tilde{\nabla}^j \tilde{G}_{ij} = 0 \) and \( (\tilde{\nabla}^i \varphi) \tilde{R}_{ij} = \left( \tilde{\nabla}^i \tilde{\nabla}_j - \tilde{\nabla}_j \tilde{\nabla}^i \right) \varphi \). Then equation (16) reduces to

\[
-\frac{1}{2} \tilde{\nabla}^i \varphi = \tilde{\nabla}^j \Sigma_{ij} + \tilde{\nabla}^j \left[ \frac{1}{\varphi} \left( -\frac{3}{2} \varphi_i \varphi_j + \frac{3}{4} \varphi h \varphi^h g_{ij} - V(\varphi) g_{ij} \right) \right].
\]

(17)

Finally, making use of equation (14) it is easily seen that

\[
-\frac{1}{2} \tilde{\nabla}^i \varphi = \tilde{\nabla}^j \left[ \frac{1}{\varphi} \left( -\frac{3}{2} \varphi_i \varphi_j + \frac{3}{4} \varphi h \varphi^h g_{ij} - V(\varphi) g_{ij} \right) \right],
\]

(18)

from which the conclusion \( \tilde{\nabla}^j \Sigma_{ij} = 0 \) follows.

This result will be particularly useful for the following considerations.

### 3. From the Jordan to the Einstein frame

The field equations (7) can be simplified by passing from the Jordan to the Einstein frame. Indeed, performing the conformal transformation \( \tilde{g}_{ij} = \varphi g_{ij} \), equations (7) assume the simpler form (see, for example, [21, 22]):

\[
R_{ij} - \frac{1}{2} R \tilde{g}_{ij} = \frac{1}{\varphi} \Sigma_{ij} - \frac{1}{\varphi^3} V(\varphi) \tilde{g}_{ij},
\]

(19)

where \( R_{ij} \) and \( R \) are, respectively, the Ricci tensor and the curvature scalar derived from the conformal metric \( \tilde{g}_{ij} \). It is worth noting that the conformal transformation is working if the
trace $\Sigma$ of the stress–energy tensor is independent of the metric $g_{ij}$. Only in this case, in fact, equations (19) depend exclusively on the conformal metric $\tilde{g}_{ij}$ and the other matter fields.

Again, equations (19) have to be considered together with the matter field equations. The latter are usually written in the Jordan frame, so involving the connection $\tilde{\Gamma}_{ij}^l$ (or, also, $\Gamma_{ij}^l$). Then, making use of relations (10) and (11), one can easily express the matter field equations in terms of the connection $\tilde{\Gamma}_{ij}^l$ instead of $\Gamma_{ij}^l$ (or $\Gamma_{ij}^l$).

Moreover, for further applications, it is useful to show the relations existing between the conservation laws in the Jordan frame and in the Einstein frame. To this end, defining

$$T_{ij} := \frac{1}{\varphi} \Sigma_{ij} - \frac{1}{\varphi^3} \psi V(\varphi)\tilde{g}_{ij},$$  

(20)
the quantity appearing in the right-hand side of equations (19), we can state the following:

**Proposition 3.1.** Given the Lévi-Civita connection $\tilde{\Gamma}_{ij}^l$, derived from the conformal metric tensor $\tilde{g}$, and given the associated covariant derivative $\tilde{\nabla}_{ij}$, the condition $\tilde{\nabla}_{ij} T_{ij} = 0$ is equivalent to the condition $\tilde{\nabla}_{ij} \Sigma_{ij} = 0$.

**Proof.** Let us develop the divergence

$$\tilde{\nabla}^l T_{lj} = \frac{1}{\varphi} g^{lj} \tilde{\nabla}_{ij} T_{ij} = \frac{1}{\varphi} g^{lj} \left[ \tilde{\nabla}_j T_{ij} - \frac{1}{2\varphi} \left( \frac{\partial \varphi}{\partial x^l} \delta^q_i + \frac{\partial \varphi}{\partial x^l} \delta^q_j - \frac{\partial \varphi}{\partial x^l} g^{qk} g_{kl} \right) T_{qj} \right].$$

(21)

We have separately

$$\frac{1}{\varphi} g^{lj} \tilde{\nabla}_{ij} T_{ij} = \frac{1}{\varphi} g^{lj} \tilde{\nabla}_{ij} \left( \frac{1}{\varphi} \Sigma_{ij} - \frac{1}{\varphi^3} V(\varphi) g_{ij} \right)$$

$$= \frac{1}{\varphi} \tilde{\nabla}^l \Sigma_{ij} - \frac{1}{\varphi^3} \tilde{\nabla}^l \left( \frac{1}{\varphi} \psi V(\varphi) \right) \delta^l_i,$$

(22)

$$\frac{1}{\varphi} g^{lj} \frac{1}{2\varphi} \left( \frac{\partial \varphi}{\partial x^l} \delta^q_i + \frac{\partial \varphi}{\partial x^l} \delta^q_j - \frac{\partial \varphi}{\partial x^l} g^{qk} g_{kl} \right) T_{qj}$$

$$= \frac{1}{\varphi} g^{lj} \frac{1}{2\varphi} \left( \frac{\partial \varphi}{\partial x^l} \delta^q_i + \frac{\partial \varphi}{\partial x^l} \delta^q_j - \frac{\partial \varphi}{\partial x^l} g^{qk} g_{kl} \right) \left( \frac{1}{\varphi} \Sigma_{qj} - \frac{1}{\varphi^3} V(\varphi) g_{qj} \right)$$

$$= \frac{1}{\varphi^3} g^{lj} \left( \frac{\partial \varphi}{\partial x^l} \Sigma_{ij} - \frac{\partial \varphi}{\partial x^l} \Sigma_{ij} - \frac{\partial \varphi}{\partial x^l} g_{ij} \Sigma^l \right)$$

$$- \frac{1}{\varphi^3} g^{lj} \left( \frac{\partial \varphi}{\partial x^l} \psi V(\varphi) g_{ij} + \frac{\partial \varphi}{\partial x^l} V(\varphi) g_{ij} - \frac{\partial \varphi}{\partial x^l} V(\varphi) \delta^l_i g_{ij} \right)$$

$$= \frac{1}{\varphi^3} g^{lj} \frac{1}{2\varphi} \frac{\partial \varphi}{\partial x^l} \Sigma - \frac{2}{\varphi^3} \frac{\partial \varphi}{\partial x^l} V(\varphi),$$

(23)

$$\frac{1}{\varphi^3} g^{lj} \frac{1}{2\varphi} \left( \frac{\partial \varphi}{\partial x^l} \delta^q_i + \frac{\partial \varphi}{\partial x^l} \delta^q_j - \frac{\partial \varphi}{\partial x^l} g^{qk} g_{kl} \right) T_{qj}$$

$$= \frac{1}{\varphi^3} g^{lj} \frac{1}{2\varphi} \left( \frac{\partial \varphi}{\partial x^l} \delta^q_i + \frac{\partial \varphi}{\partial x^l} \delta^q_j - \frac{\partial \varphi}{\partial x^l} g^{qk} g_{kl} \right) \left( \frac{1}{\varphi} \Sigma_{qj} - \frac{1}{\varphi^3} V(\varphi) g_{qj} \right)$$

$$= \frac{1}{\varphi^3} g^{lj} \left( \frac{\partial \varphi}{\partial x^l} \Sigma_{ij} - \frac{\partial \varphi}{\partial x^l} \Sigma_{ij} - \frac{\partial \varphi}{\partial x^l} g_{ij} \Sigma^l \right)$$

$$- \frac{1}{\varphi^3} g^{lj} \left( \frac{\partial \varphi}{\partial x^l} \psi V(\varphi) g_{ij} + \frac{\partial \varphi}{\partial x^l} V(\varphi) g_{ij} - \frac{\partial \varphi}{\partial x^l} V(\varphi) \delta^l_i g_{ij} \right).$$
Collecting equations (22), (23) and (24), we have then
\[
\bar{\nabla}^j T_{ij} = \frac{1}{\varphi^2} \bar{\nabla}^j \Sigma_{ij} + \frac{1}{\varphi^3} \frac{\partial \varphi}{\partial x^i} \left[ -\frac{1}{2} \Sigma + \frac{3}{\varphi} V(\varphi) - V'(\varphi) \right] = \frac{1}{\varphi^2} \bar{\nabla}^j \Sigma_{ij},
\] (25)
because the identity \(-\frac{1}{2} \Sigma + \frac{3}{\varphi} V(\varphi) - V'(\varphi) = 0\) holds identically, being equivalent to the definition \(\varphi = f'(F(\Sigma))\) [21]. □

From proposition 3.1, we conclude that the quantity (20) plays the role of the effective stress–energy tensor for the conformally transformed theory (for a discussion on the covariant conservation of the energy–momentum tensor in modified gravity, see also [25]).

4. The well posedness of the Cauchy problem

Let us consider now the \(f(R)\)-gravity, in the metric-affine formalism, coupled with perfect-fluid matter acting as a source in the field equations. We are going to demonstrate that, in the Einstein frame, the analysis of the related Cauchy problem can be carried out by following the same arguments developed in [5, 6]. To see this point, we start with looking for a metric \(g_{ij}\) of signature \((-+++\)) in the Jordan frame. The stress–energy tensor of perfect-fluid matter is of the form
\[
\Sigma_{ij} = (\rho + p)U_i U_j + p g_{ij},
\] (26a)
giving rise to corresponding matter field equations:
\[
\bar{\nabla}^j \Sigma_{ij} = 0.
\] (26b)
In equations (26), the scalars \(\rho\) and \(p\) denote, respectively, the matter–energy density and the pressure of the fluid, while \(U_i\) indicates the 4-velocity of the fluid, satisfying the obvious condition \(g^{ij} U_i U_j = -1\).

After performing the conformal transformation \(\bar{g}_{ij} = \varphi g_{ij}\), we can express the field equations in the Einstein frame as
\[
R_{ij} - \frac{1}{2} R \bar{g}_{ij} = T_{ij}
\] (27a)
and
\[
\bar{\nabla}^j T_{ij} = 0,
\] (27b)
where
\[
T_{ij} = \frac{1}{\varphi} (\rho + p) U_i U_j + \left( \frac{p}{\varphi^2} - \frac{V(\varphi)}{\varphi^3} \right) \bar{g}_{ij}
\] (28)
is the effective stress–energy tensor. In view of proposition 3.1, equations (27b) are equivalent to equations (26b). As we shall see, this is a key point of our discussion, allowing us to apply to the present case the results achieved in [5, 6]. Moreover, we shall suppose for the moment that the scalar field \(\varphi\) is positive, that is \(\varphi > 0\). The opposite case \(\varphi < 0\), differing from the former only for some technical aspects, will be briefly discussed below.

Under the stated assumptions, the 4-velocity of the fluid in the Einstein frame can be expressed as \(U_i = \sqrt{\varphi} U_i\). In view of this, the stress–energy tensor (28) can be rewritten in terms of the 4-velocity \(U_i\) as
\[
T_{ij} = \frac{1}{\varphi} (\rho + p) U_i U_j + \left( \frac{p}{\varphi^2} - \frac{V(\varphi)}{\varphi^3} \right) \bar{g}_{ij},
\] (29)
Furthermore, introducing the effective mass–energy density
\[ \bar{\rho} := \frac{\rho}{\varphi^2} + \frac{V(\varphi)}{\varphi^3}, \]  
(30a)
and the effective pressure
\[ \bar{p} := \frac{p}{\varphi^2} - \frac{V(\varphi)}{\varphi^3}, \]  
(30b)
the stress–energy tensor (29) assumes the final standard form,
\[ T_{ij} = (\bar{\rho} + \bar{p}) \bar{U}_i \bar{U}_j + \bar{p} \bar{g}_{ij}. \]  
(31)

It is worth noting that, starting from an equation of state of the form \( \rho = \rho(p) \) and assuming that relation (30b) is invertible \( (p = p(\bar{p})) \), by composition with equation (30a) we derive an effective equation of state \( \bar{\rho} = \bar{\rho}(\bar{p}) \). In addition, we recall that the explicit expression of the scalar field \( \varphi \), as well as of the potential \( V(\varphi) \), is directly related to the particular form of the function \( f(R) \). Then, the requirement of invertibility of relation (30b) together with the condition \( \varphi > 0 \) (or, equivalently, \( \varphi < 0 \)) become criteria for the viability of the functions \( f(R) \). In other words, they provide us with precise rules of selection for the admissible functions \( f(R) \).

From now on, the treatment of the Cauchy problem can proceed step by step as in [5, 6]. We do not repeat Bruhat’s analysis here, referring the reader to her well-known papers. We only recall the conclusion stated in [5, 6], where it is proved that the Cauchy problem for the system of differential equations (27), with the stress–energy tensor given by equation (31) and equation of state \( \bar{\rho} = \bar{\rho}(\bar{p}) \), is well posed if the condition
\[ \frac{d\bar{\rho}}{d\bar{p}} \geq 1, \]  
(32)
is satisfied. We stress that, in order to check the requirement (32), we do not need to invert explicitly relation (30b), but more simply, we can verify
\[ \frac{d\bar{\rho}}{d\bar{p}} = \frac{d\bar{\rho}/dp}{d\bar{p}/dp} \geq 1 \]  
(33)
directly from expressions (30) and the equation of state \( \rho = \rho(p) \). Once again, condition (33), depending on the peculiar expressions of \( \varphi \) and \( V(\varphi) \), is strictly related to the particular form of the function \( f(R) \). Then, condition (33) represents a further criterion for the viability of \( f(R) \)-models whose initial value problem is well formulated and well posed.

For completeness, we outline the case \( \varphi < 0 \). We still suppose that the signature of the metric in the Jordan frame is \((-++++\)) and the 4-velocity of the fluid in the Einstein frame will be \( \bar{U}_i = \sqrt{-\varphi} U_i \).

The effective stress–energy tensor is given now by
\[ T_{ij} = -\frac{1}{\varphi^2}(\rho + p)\bar{U}_i \bar{U}_j + \left( \frac{p}{\varphi^2} - \frac{V(\varphi)}{\varphi^3} \right) \bar{g}_{ij} = (\bar{\rho} + \bar{p}) \bar{U}_i \bar{U}_j - \bar{p} \bar{g}_{ij}. \]  
(34)

where we have introduced the quantities
\[ \bar{\rho} := -\frac{\rho}{\varphi^2} - \frac{V(\varphi)}{\varphi^3} \]  
(35a)
and
\[ \bar{p} := -\frac{p}{\varphi^2} + \frac{V(\varphi)}{\varphi^3}, \]  
(35b)
representing, as above, the effective energy-density and the effective pressure. At this point, everything will proceed again as in [5, 6], except for a technical aspect. The quantity
$r := \bar{\rho} + \bar{p} = -\frac{\rho + p}{\phi^2}$ is now negative (if, as usual, $\rho$ and $p$ are assumed to be positive).

Therefore, instead of using the function $\log(f^{-2}r)$ as in [5, 6] (where $f$ denotes the index of the fluid [11]), we need to take into account $\log(-f^{-2}r)$. The reader can easily verify that, with this choice, Bruhat’s arguments apply equally well.

5. An example

As an illustrative example of the above demonstration, let us take into account the $f(R) = R + \alpha R^2$ theory coupled with dust which is a particular case of perfect-fluid matter. The matter stress–energy tensor in the Jordan frame is $\Sigma_{ij} = \rho U_i U_j$ being $p = 0$.

From the trace of the field equations (2a), we derive the relation

$$(1 + 2\alpha R)R - 2R - 2\alpha R^2 = -\rho \quad \leftrightarrow \quad R = \rho,$$

so that the scalar field (6) becomes

$$\varphi(\rho) = f'(R(\rho)) = 1 + 2\alpha \rho.$$ 

(37)

For small values of the density $\rho \ll 1$ (for example, the present cosmological density) and values of $|\alpha|$ not comparable with $1/\rho$, we can reasonably suppose $\varphi > 0$, independently of the sign of the parameter $\alpha$.

Let us consider now the potential (8):

$$V(\varphi) = \frac{1}{4}[\varphi F^{-1}((f')^{-1}(\varphi)) + \varphi^2 (f')^{-1}(\varphi)].$$

(38)

Being $(f')^{-1}(\varphi) = \rho$, one has

$$\frac{1}{4}\varphi^2 (f')^{-1}(\varphi) = \frac{1}{4}(1 + 2\alpha \rho)^2 \frac{1}{2\alpha}(1 + 2\alpha \rho - 1) = \frac{1}{4}(1 + 2\alpha \rho)^2 \rho$$

(39)

and considering the relation $F^{-1}(Y) = f'(Y) K - 2f(Y)$, it is

$$\frac{1}{4} F^{-1}((f')^{-1}(\varphi)) = \frac{1}{4} F^{-1}(\rho) = -\rho.$$ 

(40)

We have also

$$\frac{1}{4} \varphi F^{-1}((f')^{-1}(\varphi)) = -\frac{(1 + 2\alpha \rho)\rho}{4},$$

(41)

and then we conclude that

$$V(\varphi(\rho)) = \frac{\alpha \rho^2(1 + 2\alpha \rho)}{2}.$$ 

(42)

In the Einstein frame, the stress–energy tensor is expressed as

$$T_{ij} = \frac{\rho}{\phi^2} \bar{U}_i \bar{U}_j - \frac{V(\varphi)}{\phi^3} \bar{g}_{ij}.$$ 

(43)

The latter can be seen as the stress–energy tensor of a perfect fluid with density and pressure given, respectively, by

$$\bar{\rho} := \frac{\rho}{\phi^2} + \frac{V(\varphi)}{\phi^3} = \frac{2\rho + \alpha \rho^2}{2(1 + 2\alpha \rho)^2}$$

(44a)

and

$$\bar{p} := -\frac{V(\varphi)}{\phi^3} = -\frac{\alpha \rho^2}{2(1 + 2\alpha \rho)^2}.$$ 

(44b)

Under the stated assumptions, the function (44b) is invertible, indeed, for $\rho > 0$,

$$\frac{d\bar{p}}{d\rho} = -\frac{4\alpha \rho}{4(1 + 2\alpha \rho)^3} \neq 0.$$ 

(45)
Moreover, we have
\[
\frac{d\bar{\rho}}{d\rho} = \frac{4 - 4\alpha\rho}{4(1 + 2\alpha\rho)^{\frac{3}{2}}}
\]  
(46)
and then
\[
\frac{d\bar{\rho}}{d\bar{p}} = \frac{d\bar{\rho}/d\rho}{d\bar{p}/d\rho} = \frac{-1 + \alpha\rho}{\alpha\rho} \geq 1 \iff \alpha < 0.
\]  
(47)

This last result allows us to select the form of the physically viable theories. In fact, the parameter \(\alpha\), negatively defined, allows stable cosmological solutions and positively defined massive states (see [26] and references therein). This means that the model is physically viable in agreement with the Cauchy problem which is well formulated and well posed in this case. Such a situation has to be discussed in details. As stated above, the crucial feature is the invertibility of the state equation, and the further requirement is that the solution of the above system of equations (44\(a\)), (44\(b\)) and (45) does not result in an empty set of viable \(f(R)\)-theories. The effective equation of state, derived in the Einstein frame, assumes physical meaning if and only if it is capable of selecting viable models as that discussed here after the inverse transformation in the Jordan frame has been performed. Then only \textit{a posteriori}, i.e. once the model has been selected, we can say if the approach has produced significant results.

6. Conclusions

Following the prescriptions in [5, 6], it is possible to show that the Cauchy problem for metric-affine \(f(R)\)-theories in the manner of Palatini and with torsion, in the presence of perfect-fluid matter acting as a source, is well formulated and well posed. Besides, the procedure allows us to select physically viable models which have to constitute a non-empty set of solutions. The key points of the demonstration are: (i) the conservation laws (the contracted Bianchi identities) are preserved under the conformal transformation from the Jordan to the Einstein frame, (ii) the Bruhat arguments can be applied if it is possible to recast the stress-energy tensor in the perfect-fluid form, in both frames, (iii) the condition \(\frac{d\bar{\rho}}{d\bar{p}} \geq 1\), specifically one of the Bruhat conditions, allows us to select suitable \(f(R)\)-models and to formulate the energy theorems for this kind of theories [27].

It is worth noting the role of the perfect-fluid which allows both the well formulation [19] and the well posedness of the problem. Besides, it acts also as a sort of ‘selection rule’ since not any \(f(R)\)-model is consistent with the well posedness of the initial value problem but only those where the condition \(\frac{d\bar{\rho}}{d\bar{p}} \geq 1\) holds, as we have seen for the \(f(R) = R + \alpha R^2\) case. A main role to get this result is played by the conformal transformations since, in the Einstein frame, it is always possible to disentangle the extra gravitational degrees of freedom.

As a concluding remark, we can say that the Cauchy problem results in being, in general, well formulated for \(f(R)\)-gravity in the metric-affine formalism [19] as well as in the metric formalism [15]. On the other hand, the well posedness, which always holds in vacuum (and also in the case of coupling with an electromagnetic field or with Yang–Mills fields) [19], strictly depends on the source and the parameters of the theory (e.g. \(\alpha\) has to be negatively defined in the above example. In the case of perfect-fluid matter, it works, essentially, because the problem can be reduced to the Einstein frame by a conformal transformation. However, due to the fact that \(f(R)\)-gravity presents further degrees of freedom, the analogy with general relativity, achieved considering the Bruhat arguments, has to be tested, case by case, by asking for the invertibility of the effective equation of state, from one side, and, \textit{a posteriori}, for the physical viability of the \(f(R)\)-model. In other words, the limits of the approach strictly depend on the fluids adopted as sources and on the intrinsic parameters of the \(f(R)\)-models consistent
with the Cauchy problem. In conclusion, one cannot say that the initial value problem is always well posed also if, in general, it can be well formulated in the case of perfect-fluid matter.

The Cauchy problem for other forms of sources will be the arguments of future investigations.

References

[1] Schmidt H-J 2004 Lectures in mathematical cosmology arXiv:gr-qc/0407095
[2] Nojiri S and Odintsov S D 2007 Int. J. Geom. Methods Mod. Phys. 4 115
[3] Capozziello S and Francaviglia M 2008 Gen. Rel. Grav. 40 357
[4] Sotiriou T P and Faraoni V 2008 arXiv:0805.1726 [gr-qc]
[5] Fouré–Bruhat Y 1958 Bull. de la S. M. F. 86 155
[6] Choquet–Bruhat Y 1962 Cauchy Problem, in Gravitation: An Introduction to Current Research ed L Witten (New York: Wiley)
[7] Synge J L 1971 Relativity: The General Theory (Amsterdam: North–Holland)
[8] Wald R M 1984 General Relativity (Chicago: The University of Chicago Press)
[9] van Dam H and Veltman M 1970 Nucl. Phys. B 22 397
[10] Stelle K 1978 Gen. Rel. Grav. 9 353
[11] Lichnerowicz A 1955 Théories Relativistes de la Gravitation et de l’Électromagnetisme (Paris: Masson)
[12] Teyssandier P and P Tourrenc P 1983 J. Math. Phys. 24 2793
[13] Durrisseau J P and Kerner R 1983 Gen. Rel. Grav. 15 797
[14] Noakes D R 1983 J. Math. Phys. 24 1846
[15] Lanahan–Tremblay N and Faraoni V 2007 Class. Quantum Grav. 24 5667
[16] Tsujikawa S, Uddin K and Tavakol R 2008 Phys. Rev. D 77 043007
[17] Capozziello S and Vignolo S 2009 Class. Quantum Grav. 26 168001
[18] Faraoni V 2009 Class. Quantum Grav. 26 168002
[19] Capozziello S and Vignolo S 2009 Int. J. Geom. Methods Mod. Phys. at press (arXiv:0901.3136 [gr-qc])
[20] Magnano G, Ferraris M and Francaviglia M 1990 Class. Quantum Grav. 7 557
[21] Capozziello S, Cianci R, Stornaiolo C and Vignolo S 2007 Class. Quantum Grav. 24 6417
[22] Olmo G J 2005 Phys. Rev. D. 72 083505
[23] Capozziello S, Cianci R, Stornaiolo C and Vignolo S 2008 Int. J. Geom. Methods Mod. Phys. 5 765
[24] Capozziello S, Cianci R, Stornaiolo C and Vignolo S 2008 Phys. Scr. 78 065010
[25] Kovisto T 2006 Class. Quantum Grav. 23 4289
[26] Barrow J D and Ottewill A C 1983 J. Phys. A: Math. Gen. 16 2757
[27] Magnano G and Sokolowski L M 1994 Phys. Rev. D 50 5039