Accumulation points of real Schur roots

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- Given $M \in \text{rep}(Q)$, we denote by $d_M \in \mathbb{Z}_{\geq 0}^n$ its dimension vector.
- We denote by $\langle - , - \rangle$ the Euler-Ringel form of $Q$, that is, $\langle d_M , d_N \rangle = \dim_k \text{Hom}(M, N) - \dim_k \text{Ext}^1(M, N)$. 
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We then call $d_M$

\[
\begin{cases}
\text{real,} & \text{if } \langle d_M, d_M \rangle = 1; \\
\text{imaginary,} & \text{if } \langle d_M, d_M \rangle \leq 0; \\
\text{isotropic,} & \text{if } \langle d_M, d_M \rangle = 0; \\
\text{strictly imaginary,} & \text{if } \langle d_M, d_M \rangle < 0;
\end{cases}
\]
Schur roots

- $\Delta_Q$ denotes the set of all rays in $\mathbb{R}^n$ in the positive orthant.
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- We denote by $[d]$ the ray of $d \in \mathbb{Z}_{\geq 0}^n$.
- A Schur root that is real or isotropic is uniquely determined by its ray.
- If $d$ is strictly imaginary, then all integral vectors in $[d]$ are strictly imaginary.
Accumulation points of real roots

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- Another paper by M. Dyer, C. Hohlweg, V. Ripoll in arXiv:1303.6710.
- A third one by C. Hohlweg, J. Préaux, V. Ripoll in arXiv:1305.0052.
Example 1

Here is an example for $\Delta_Q$ for $Q$ of type $\tilde{A}_{2,1}$
Example 1

Here is an example for $\Delta \mathbb{Q}$ for $\mathbb{Q}$ of type $\tilde{\text{A}}_2^1$. 

Derksen, Weyman
Example 2

Here is an example for $\Delta_Q$ for $Q: 1 \leftarrow 2 \leftrightarrow 3$
Example 2

Derksen, Weyman
The canonical decomposition

**Theorem (Kac)**

*Every dimension vector can be written as a positive linear combination of Schur roots $d_1, \ldots, d_r$ such that*

- The coefficient of a strictly imaginary Schur root is one.
The canonical decomposition

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- The Euler form $\langle -, - \rangle$ of $Q$. 

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- Derksen and Weyman’s algorithm can be used to find the canonical decomposition of any dimension vector. All is needed is:
  - The Euler form $\langle -, - \rangle$ of $Q$.
  - Know the canonical decomposition of quivers with two vertices.
The canonical decomposition

**Theorem**

*If the canonical decomposition of* $d \in \mathbb{Z}_{\geq 0}^n$ *involves a strictly imaginary Schur root, then there exists a small neighborhood of* $d$ *with the same property.*
Corollary

*If* $d$ *is a rational accumulation point of real Schur roots, then the canonical decomposition of* $d$ *involves pairwise orthogonal isotropic Schur roots.*
Corollary

If $d$ is a rational accumulation point of real Schur roots, then the canonical decomposition of $d$ involves pairwise orthogonal isotropic Schur roots.

Theorem

If $d$ is an isotropic Schur root, then $d$ is an accumulation point of real Schur roots.
Rational accumulation points

- The quiver $Q$ is of **weakly hyperbolic type** if the symmetrized Euler form has exactly one negative eigenvalue and the others are positive.
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- The quiver $Q$ is weakly hyperbolic (or Dynkin or Euclidean) when, for instance:
  - $|Q_0| \leq 3$. 
The quiver $Q$ is of \textit{weakly hyperbolic type} if the symmetrized Euler form has exactly one negative eigenvalue and the others are positive.

The quiver $Q$ is weakly hyperbolic (or Dynkin or Euclidean) when, for instance:

- $|Q_0| \leq 3$.
- $Q$ has a full subquiver with $n - 1$ vertices which is a union of Dynkin quivers.
Rational accumulation points

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- The quiver $Q$ is weakly hyperbolic (or Dynkin or Euclidean) when, for instance:
  - $|Q_0| \leq 3$.
  - $Q$ has a full subquiver with $n - 1$ vertices which is a union of Dynkin quivers.

**Proposition**

*If $Q$ is weakly hyperbolic, then the rational accumulation points are precisely the isotropic Schur roots of $Q$.***
Irrational accumulation points

- Assume that $Q$ is weakly hyperbolic.
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- The (dimension vectors of the) preinjective representations accumulates to $y^+$. 

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- (Ringel) $y^+, y^-$ are (irrational) eigenvectors of the Coxeter transformation.
Irrational accumulation points

- Assume that $Q$ is weakly hyperbolic.
- The (dimension vectors of the) preprojective representations accumulates to $y^-$.  
- The (dimension vectors of the) preinjective representations accumulates to $y^+$.  
- (Ringel) $y^+, y^-$ are (irrational) eigenvectors of the Coxeter transformation.  
- Are there other accumulation points?
Irrational accumulation points

For any (Generalized) Kronecker subcategory $\mathcal{C} = \text{rep}(Q')$ of $\text{rep}(Q)$, denote by $y^-_\mathcal{C}, y^+_\mathcal{C}$ the associated eigenvalues of the Coxeter transformation of $Q'$, where $y^-_\mathcal{C} = y^+_\mathcal{C}$ when $Q'$ is tame.
Irrational accumulation points

- For any (Generalized) Kronecker subcategory $\mathcal{C} = \text{rep}(Q')$ of $\text{rep}(Q)$, denote by $y_C^-, y_C^+$ the associated eigenvalues of the Coxeter transformation of $Q'$, where $y_C^- = y_C^+$ when $Q'$ is tame.

- Then $y_C^-, y_C^+$ are accumulation points in $\Delta_Q$, irrational if $\mathcal{C}$ is wild.
Irrational accumulation points

- For any (Generalized) Kronecker subcategory $\mathcal{C} = \text{rep}(Q')$ of $\text{rep}(Q)$, denote by $y_\mathcal{C}^-, y_\mathcal{C}^+$ the associated eigenvalues of the Coxeter transformation of $Q'$, where $y_\mathcal{C}^- = y_\mathcal{C}^+$ when $Q'$ is tame.

- Then $y_\mathcal{C}^-, y_\mathcal{C}^+$ are accumulation points in $\Delta_Q$, irrational if $\mathcal{C}$ is wild.

- Are there any other?
Irrational accumulation points

For any (Generalized) Kronecker subcategory $\mathcal{C} = \text{rep}(Q')$ of $\text{rep}(Q)$, denote by $y^-_{\mathcal{C}}, y^+_{\mathcal{C}}$ the associated eigenvalues of the Coxeter transformation of $Q'$, where $y^-_{\mathcal{C}} = y^+_{\mathcal{C}}$ when $Q'$ is tame.

Then $y^-_{\mathcal{C}}, y^+_{\mathcal{C}}$ are accumulation points in $\Delta_Q$, irrational if $\mathcal{C}$ is wild.

Are there any other?

Yes, $y^+, y^-$. 
Introduction

Accumulation points - Examples
Accumulation points - Canonical decomposition
Accumulation points - Rational ones
Accumulation points - Others

Irrational accumulation points

For any (Generalized) Kronecker subcategory $C = \text{rep}(Q')$ of $\text{rep}(Q)$, denote by $y_C^-, y_C^+$ the associated eigenvalues of the Coxeter transformation of $Q'$, where $y_C^- = y_C^+$ when $Q'$ is tame.

Then $y_C^-, y_C^+$ are accumulation points in $\Delta_Q$, irrational if $C$ is wild.

Are there any other?

Yes, $y^+, y^-$. 

**Theorem**

The set of accumulation points of $\Delta_Q$ is the closure of $\{y_C^-, y_C^+ \mid C \text{ Kronecker subcategory}\}$. 

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THANK YOU

Questions ?