INVASRIANCE OF PLURIGENERA FAILS IN POSITIVE AND MIXED CHARACTERISTIC

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Abstract. We construct smooth families of elliptic surface pairs with terminal singularities over a DVR of positive or mixed characteristic $(X, B) \to \text{Spec } R$, such that $P_m(X_k, B_k) > P_m(X_K, B_K)$ for all sufficiently divisible $m > 0$. In particular, this shows that invariance of all sufficiently divisible plurigenera does not follow from the MMP and Abundance Conjectures.

1. Introduction

A famous theorem by Siu ([Siu98, Siu02]) states that, if $X \to U$ is a smooth one-parameter family of complex projective varieties, the plurigenera of the fibers $P_m(X_u) := h^0(X_u, mK_{X_u})$ are independent of $u \in U$, for all $m \geq 0$. This result, and its generalizations to the log category ([BP12]), play an important role in the construction of moduli spaces for varieties of general type ([HMX18]). It is worth noticing that Siu’s argument is analytic and to this day there is no algebraic proof of [Siu02]. Notable exceptions are the cases of families of threefolds ([KM92]) and of varieties of general type ([Kaw99]). However, it has been known for a while that invariance of plurigenera follows from the Minimal Model and Abundance Conjectures ([Nak86]), at least over the complex numbers.

Much less is known when $U$ is replaced by $\text{Spec } R$, where $R$ is a DVR of positive or mixed characteristic, even for families of surfaces. By [KU85, Theorem 9.1] invariance of Kodaira dimension holds. However, invariance of all plurigenera can fail: in [Lan83, KU85] the authors constructed families of Enriques, resp. elliptic surfaces $X \to \text{Spec } R$ such that $P_1(X_k) > P_1(X_K)$. In [Suh08] it is shown that the same can happen even when $K_X$ is ample. In all these examples however, the equality $P_m(X_k) = P_m(X_K)$ holds for all sufficiently divisible $m \geq 0$. Even better, by [EH21] we have that if $(X, B) \to \text{Spec } R$ is a log smooth family of klt surface pairs, then $P_m(X_k, B_k) = P_m(X_K, B_K)$ holds for all sufficiently divisible $m \geq 0$ except, possibly, when $\kappa(K_X + B/R) = 1$ and $B_k$ is vertical with respect to the Iitaka fibration of $K_{X_k} + B_k$. It is then natural to ask whether the above equality always holds for all such $m$. We show this is not the case.

Theorem 1. For every prime $p$, there exist smooth families of minimal elliptic surface pairs of Kodaira dimension one with terminal singularities $(X, B) \to \text{Spec } R$, where $R$ is an excellent DVR, with algebraically closed residue field $k$ of characteristic $p > 0$ and fraction field $K$, such that

$$P_m(X_k, B_k) > P_m(X_K, B_K)$$

for all sufficiently divisible $m > 0$. Furthermore, for every $p$ and every sufficiently divisible $m > 0$, the difference $P_m(X_k, B_k) - P_m(X_K, B_K)$ can be arbitrarily large.
Note that, over the complex numbers, invariance of all sufficiently divisible plurigenera holds for smooth families of terminal pairs of non-negative Kodaira dimension ([P07]). When the family has relative dimension two, this can be quickly shown using the MMP and Abundance Conjecture.

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Note from the author: the previous version of this preprint claimed that Theorem 1 held even in the absence of a boundary divisor. I have been informed by Prof. János Kollár that the last part of my construction contains a mistake. We will address the boundary-free case in a future (version of this) paper.

2. Preliminaries

2.1. Notation and conventions.

- All schemes we consider will be of finite type over their bases.
- R denotes an excellent DVR with algebraically closed residue field k of characteristic p > 0, fraction field K, and uniformizer ω.
- Let F be a field: a variety is an integral and separated F-scheme. We usually assume our varieties to be normal. A family of varieties is an integral R-scheme X → Spec R whose fibers are varieties. The family is said to be smooth, resp. projective, if X is R-smooth, resp. R-projective.
- A pair (X, B) consists of a normal integral scheme X with an effective Q-divisor B such that K_X + B is Q-Cartier. We refer to [Kol13] for the various definitions of singularities of pairs. A family of pairs is a pair (X, B) → Spec R such that (X_u, B_u) is a pair for all u ∈ Spec R. The family is said to be smooth, resp. projective, if X is R-smooth, resp. R-projective.
- If (X, B) is a projective pair over a field F, we denote by P_m(X, B) := h^0(X, [m(K_X + B)]) the m-plurigenus of (X, B).

We consider \mathbb{P}^1_R with homogeneous coordinates [S : T]. We denote by \mathbb{A}^1_{R,s}, \mathbb{A}^1_{R,t} \subset \mathbb{P}^1_R the affine open sets \{ T \neq 0 \} and \{ S \neq 0 \}, respectively. The distinguished R-points \{ T = 0 \} and \{ S = 0 \} are denoted by \infty, respectively.

- Let X be an R-scheme: we denote by X_k, X_K and X_\infty, the special, the generic, and the geometric generic fiber of X → Spec R, respectively. An analogous notation will be used for sheaves over X and their sections.

2.2. Elliptic surfaces. Let F be a field: a morphism of smooth projective F-varieties f : X → C is an elliptic surface if dim(X) = 2, dim(C) = 1, f_*O_X = O_C, and a general fiber is a smooth curve of genus 1. The elliptic surface is said to be minimal if the fibers of f do not contain any (−1)-curve. Let f^*(c_i) = m_iD_i be the multiple singular fibers, where m_i is the gcd of the coefficients of the components of
Thus \( f^*(c_i) \). Since \( C \) is a smooth curve, we have a decomposition
\[
R^1 f_* \mathcal{O}_X = L \oplus T
\]
where \( L \) is a line bundle and \( T \) is torsion. We will denote by \( t \) the length of \( T \). The fibers over \( \text{Supp}(T) \) are called wild fibers; all of them are multiple ([BM77, Proposition 3]). A multiple fiber which is not wild is called tame.

**Theorem 2** ([BM77]). Let \( f : X \to C \) be a minimal elliptic surface. Then
\[
K_X \sim_{\mathbb{Q}} f^*(K_C - L + \sum_i \frac{a_i}{m_i} c_i),
\]
where \( \deg(-L) = \chi(X, \mathcal{O}_X) + t \) and \( 0 \leq a_i \leq m_i - 1 \), with \( a_i = m_i - 1 \) if and only if \( f^*(c_i) \) is tame. Moreover, \( T \) is supported precisely at those points \( c \in C \) such that \( h^1(f^{-1}(c), \mathcal{O}_{f^{-1}(c)}) > 1 \).

### 2.3. Iitaka fibration and invariance of plurigenera.

Let \( X \) be a normal, integral, and projective \( R \)-scheme, let \( D \) be an effective \( \mathbb{Q} \)-Cartier divisor, and consider the rational maps of \( R \)-schemes
\[
\phi_{[mD]} : X \to Z_m \subset \mathbb{P}H^0(X, mD)^{*},
\]
where \( Z_m \) denotes the image of \( \phi_{[mD]} \). By [Laz04, Sections 2.1.A, 2.1.B], for all \( m > 0 \) sufficiently divisible the maps \( \phi_{[mD]} \) are birational to a fixed morphism \( \phi_{\infty} : X_{\infty} \to Z_{\infty}/R \), called the Iitaka fibration of \( D \), satisfying \( \phi_{\infty*} \mathcal{O}_{X_{\infty}} = \mathcal{O}_{Z_{\infty}} \). The Iitaka dimension of \( D \) over \( R \) is defined to be \( \kappa(D/R) := \dim_R Z_{\infty}; \) note that we always have
\[
\kappa(D/R) = \kappa(D_K) \quad \kappa(D/R) \leq \kappa(D_k)
\]
by upper-semicontinuity of cohomology. If \((X, B)\) is a pair over \( R \), the Kodaira dimension of \((X, B)\) is defined to be \( \kappa(X, B/R) := \kappa(K_X + B/R) \). We will usually assume \( D \) to be semiample, i.e. \( mD \) is basepoint-free for some \( m > 0 \). The section ring \( R(D) := \bigoplus_{m \geq 0} H^0(X, mD) \) is then a finitely generated \( R \)-algebra and, for all sufficiently divisible \( m > 0 \),
\[
\phi_m = \phi_{\infty} : X \to Z := \text{Proj} R(D)
\]
is the Iitaka fibration of \( D \). Note that, if \( D \) is semiample, \( \kappa(D/R) = \kappa(D_k) \).

The following Lemma is the key to the construction of examples violating invariance of \( P_m(X_u, B_u) \) for all sufficiently divisible \( m > 0 \).

**Lemma 3.** Let \( X \to \text{Spec} R \) be a projective family of normal varieties, let \( D \) be a semiample \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor on \( X \), and let \( f : X \to Z \) be its Iitaka fibration. Then \( f_{k*} \mathcal{O}_{X_k} = \mathcal{O}_{Z_k} \) if and only if \( h^0(X_u, mD_u) \) is independent of \( u \in \text{Spec} R \) for all \( m \geq 0 \) sufficiently divisible.

**Proof.** If \( h^0(X_u, mD_u) \) is independent of \( u \in \text{Spec} R \) for all \( m \geq 0 \) sufficiently divisible, then for all such \( m \) we have surjectivity of the restriction map
\[
H^0(X, mD) \otimes k \to H^0(X_k, mD_k),
\]
thus \( f_k \) is the Iitaka fibration of \( D_k \). For the reverse implication, write \( D \sim_{\mathbb{Q}} f^* A \) for some ample \( \mathbb{Q} \)-divisor on \( Z \) and observe that for all \( m > 0 \) sufficiently divisible and all \( u \in \text{Spec} R \) we have \( h^0(X_u, mD_u) = \chi(Z_u, mA_u) \) by the projection formula and Serre vanishing. As \( Z \) is integral, it is flat over \( R \), hence \( \chi(Z_u, mA_u) \) is independent of \( u \in \text{Spec} R \).
Remark 4. Let $F$ be a field, let $f : X \to C$ be a morphism with connected fibers between smooth projective $F$-varieties, and suppose $C$ is a curve. Then we have a Stein factorization

$$f : X \xrightarrow{T} C \xrightarrow{h} C$$

where $h$ is a universal homeomorphism. If $	ext{char}(F) = 0$ then $h$ is an isomorphism by Zariski’s Main Theorem. If $	ext{char}(F) = p > 0$ then $h$ is a composition of geometric Frobenius morphisms, hence $f_*\mathcal{O}_X = \mathcal{O}_C$ if and only if a general fiber of $f$ is reduced.

3. Proof of Theorem 1

We fix an integer $n \geq 1$ and set $q := p^n$. Let $E \to \text{Spec} \, R$ be a family of elliptic curves such that $|\text{Pic}(E_K)[q]| > |\text{Pic}(E_k)[q]|$. Note that this condition is always satisfied when $R$ is of mixed characteristic, while in equicharacteristic $p$ we may consider an ordinary elliptic curve degenerating to a supersingular one. After possibly replacing $R$ by a finite extension we may then assume that there exists a non-trivial $q$-torsion line bundle $M$ on $E$ such that $M_k = \mathcal{O}_{E_k}$. Denote by $1_M \in H^0(E, M^q)$ a nowhere vanishing section.

Consider now the following commutative diagram of $R$-schemes

$$
\begin{array}{ccc}
Y & \xleftarrow{\nu} & X \\
\pi & \downarrow & \downarrow f \\
E \times_R \mathbb{P}^1_R & \xrightarrow{pr_2} & \mathbb{P}^1_R,
\end{array}
$$

where $\pi$ is the $q$-cyclic cover branched over $1_M \boxtimes TS^{q-1} \in H^0(E \times_R \mathbb{P}^1_R, M^q \boxtimes \mathcal{O}_{\mathbb{P}^1_R}(q))$, and $\nu$ is the normalization. Over $\mathbb{A}^1_{R,t}$ we have the following description

$$
\begin{array}{ccc}
\text{Spec} \frac{\mathcal{O}_E[t, \lambda]}{(\lambda^q - \varphi t)} & \xleftarrow{=} & \text{Spec} \frac{\mathcal{O}_E[t, \lambda]}{(\lambda^q - \varphi t)} \\
\downarrow & & \downarrow \\
\text{Spec} \mathcal{O}_E[t, \lambda] & \longrightarrow & \text{Spec} \, R[t],
\end{array}
$$

while over $\mathbb{A}^1_{R,s}$ we have

$$
\begin{array}{ccc}
\text{Spec} \frac{\mathcal{O}_E[s, \xi]}{(\xi^q - \psi s^{q-1})} & \xleftarrow{=} & \text{Spec} \frac{\mathcal{O}_E[s/\xi, \xi]}{(\xi - \psi(s/\xi)^{q-1})} \\
\downarrow & & \downarrow \\
\text{Spec} \mathcal{O}_E[s, \xi] & \longrightarrow & \text{Spec} \, R[s].
\end{array}
$$

Here $\varphi, \psi \in \mathcal{O}_E^*$ are local trivializations of $1_M$, hence $\varphi_k, \psi_k \in k^*$. It is straightforward to verify that $X$ is smooth over $R$. A general fiber of $f_K$ is the smooth $q$-cover $F_K \to E_K$ induced by $1_{M_K} \in H^0(E_K, M^q_K)$. By Remark 4 we then have that $f_{K,*}\mathcal{O}_{X_K} = \mathcal{O}_{F_K}$, hence $f_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^1_R}$ as Stein factorization and flat
base-change commute. From the equations we see that the function field extension induced by \( f_k \) factors as

\[
k(X_k) \leftrightarrow k(\mathbb{P}^1_k) \leftrightarrow k(\mathbb{P}^1_k),
\]

which in turn yields a non-trivial Stein factorization

\[
f_k : X_k \stackrel{\overline{k}}{\to} (\mathbb{P}^1_k)^{-n} \overset{\text{Fr}_n}{\to} \mathbb{P}^1_k.
\]

Observe that \( f_K : X_K \to \mathbb{P}^1_K \) is an isotrivial elliptic surface with multiple fibers \( qE_K \) over \( 0_K, \infty_K \). We claim both these fibers are tame: note that we can find isomorphic neighborhoods of \( f_K^{-1}(0_K) \) and \( f_K^{-1}(\infty_K) \), thus one fiber is tame if and only if the other one is. Note also that \( h^1(X_K, \mathcal{O}_{X_K}) = 1 \). By contradiction, suppose both fibers are wild. By \cite[Theorem III.12.11]{Har77} the natural map

\[
R^1 f_{K,*} \mathcal{O}_{X_K} \otimes K(b) \to H^1(X_K, \mathcal{O}_{X_K, b})
\]

is surjective for all \( b \in \mathbb{P}^1_K \) and, as \( f_{K,*} \mathcal{O}_{X_K} = \mathcal{O}_{\mathbb{P}^1_K} \), the Leray spectral sequence yields

\[
1 = h^1(X_K, \mathcal{O}_{X_K}) = h^0(\mathbb{P}^1_K, R^1 f_{K,*} \mathcal{O}_{X_K}) \geq h^0(\mathbb{P}^1_K, (R^1 f_{K,*} \mathcal{O}_{X_K})_{\text{tor}}) \geq 4.
\]

As \( \chi(X_K, \mathcal{O}_{X_K}) = 0 \), Theorem 2 implies \( K_{X_K} \sim q f_K^*(-2 + 2(q - 1)/q)H_K \), where \( H \) is an hyperplane on \( \mathbb{P}^1_R \), thus \( K_X \sim q f^*(-2 + 2(q - 1)/q)H \), as \( X_K \) is irreducible.

Let now \( z_1, \ldots, z_l \) be pairwise disjoint \( R \)-points of \( \mathbb{P}^1_R \), not intersecting 0 or \( \infty \), and let \( 0 < \epsilon \ll 1 \) be a rational number such that, setting \( B := f^*(\epsilon \sum_i z_i) \), we have that \( (X_u, B_u) \) is terminal for all \( u \in \text{Spec} R \). For such choice of \( \{z_i\}_i \) we have that \( \epsilon \) is independent of \( l \). Thus upon taking very large \( l \) we may also assume that \( K_X + B \sim q f^*A \) for some ample \( \mathbb{Q} \)-divisor on \( \mathbb{P}^1_R \) of degree \( d \). In particular, \( f \) is the Iitaka fibration of \( K_X + B \), thus \( P_m(X_u, B_u) \) will jump for all sufficiently divisible \( m > 0 \), by Lemma 3. On the special elliptic surface we have \( K_{X_k} + B_k \sim \overline{\mathbb{P}^1}(qA) \), hence the projection formula on \( f_K \) and \( \overline{\mathbb{P}^1} \) yields

\[
P_m(X_k, B_k) = h^0(\mathbb{P}^1_K, \mathcal{O}_{\mathbb{P}^1_k}^{qmd}) > h^0(\mathbb{P}^1_K, \mathcal{O}_{\mathbb{P}^1_k}^{md}) = P_m(X_K, B_K).
\]

As \( q = p^n \) we see that, for every characteristic \( p \) and every \( m \geq 1 \) divisible enough, the jump in plurigenera can be arbitrarily large, thus concluding the proof.

**Remark 5.** By taking products we can construct smooth, higher-dimensional families of terminal pairs \( (W, D) \to \text{Spec} R \) with \( \text{K}_W + D \) semiample and \( 0 < \kappa(W, D/R) < \dim R W \) such that invariance of all sufficiently divisible plurigenera fails. Consider smooth families of Abelian, resp. canonically polarized varieties, \( A \to \text{Spec} R \) and \( V \to \text{Spec} R \). Then

\[
g := f \times \text{pr}_2 : (W, D) := (X \times_R A \times_R V, B \times_R A \times_R V) \to \mathbb{P}^1_R \times_R V
\]

is the Iitaka fibration of \( K_W + D \). By construction \( g_k \) has a non-trivial Stein factorization

\[
g_k : W_k \stackrel{\overline{\text{fr}}_k \times \text{pr}_2}{\to} (\mathbb{P}^1_k)^{-n} \times V_k \overset{\text{Fr}_n \times \text{id}_V}{\to} \mathbb{P}^1_k \times V_k,
\]

hence \( P_m(W_k, B_k) - P_m(W_K, B_K) > 0 \) can be arbitrarily large for all sufficiently divisible \( m \), as in the surface case.


**References**

[BMy77] E. Bombieri and D. Mumford. Enriques' classification of surfaces in char. p. II. In *Complex analysis and algebraic geometry*, pages 23–42. 1977.

[BP12] Bo Berndtsson and Mihai Păun. Quantitative extensions of pluricanonical forms and closed positive currents. *Nagoya Math. J.*, 205:25–65, 2012.

[EH21] Andrew Egbert and Christopher D. Hacon. Invariance of certain plurigenera for surfaces in mixed characteristics. *Nagoya Math. J.*, 243:1–10, 2021.

[Har77] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[HMX18] Christopher D. Hacon, James McKernan, and Chenyang Xu. Boundedness of moduli of varieties of general type. *J. Eur. Math. Soc. (JEMS)*, 20(4):865–901, 2018.

[Kaw99] Yujiro Kawamata. Deformations of canonical singularities. *J. Amer. Math. Soc.*, 12(1):85–92, 1999.

[KM92] János Kollár and Shigefumi Mori. Classification of three-dimensional flips. *J. Amer. Math. Soc.*, 5(3):533–703, 1992.

[Kol13] János Kollár. *Singularities of the minimal model program*, volume 200 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013. With a collaboration of Sándor Kovács.

[KU85] Toshiyuki Katsura and Kenji Ueno. On elliptic surfaces in characteristic $p$. *Math. Ann.*, 272(3):291–330, 1985.

[Lan83] William E. Lang. On Enriques surfaces in characteristic $p$. *Math. Ann.*, 265(1):45–65, 1983.

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[Nak86] Noboru Nakayama. Invariance of the plurigenera of algebraic varieties under minimal model conjectures. *Topology*, 25(2):237–251, 1986.

[Pă07] Mihai Păun. Siu’s invariance of plurigenera: a one-tower proof. *J. Differential Geom.*, 76(3):485–493, 2007.

[Siu98] Yum-Tong Siu. Invariance of plurigenera. *Invent. Math.*, 134(3):661–673, 1998.

[Siu02] Yum-Tong Siu. Extension of twisted pluricanonical sections with plurisubharmonic weight and invariance of semi-positively twisted plurigenera for manifolds not necessarily of general type. In *Complex geometry (Göttingen, 2000)*, pages 223–277. Springer, Berlin, 2002.

[Suh08] Junecue Suh. Plurigenera of general type surfaces in mixed characteristic. *Compos. Math.*, 144(5):1214–1226, 2008.

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