On Integer Additive Set-Sequential Graphs

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Abstract

A set-labeling of a graph $G$ is an injective function $f : V(G) \to \mathcal{P}(X)$, where $X$ is a finite set of non-negative integers and a set-indexer of $G$ is a set-labeling such that the induced function $f^\oplus : E(G) \to \mathcal{P}(X) - \{\emptyset\}$ defined by $f^\oplus(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective. A set-indexer $f : V(G) \to \mathcal{P}(X)$ is called a set-sequential labeling of $G$ if $f^\oplus(V(G) \cup E(G)) = \mathcal{P}(X) - \{\emptyset\}$. A graph $G$ which admits a set-sequential labeling is called a set-sequential graph.

An integer additive set-labeling is an injective function $f : V(G) \to \mathcal{P}(\mathbb{N}_0)$, where $\mathbb{N}_0$ is the set of all non-negative integers and an integer additive set-indexer is an integer additive set-labeling such that the induced function $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is also injective. In this paper, we extend the concepts of set-sequential labeling to integer additive set-labelings of graphs and provide some results on them.

Key words: Integer additive set-indexers, set-graceful graphs, set-sequential graphs, integer additive set-graceful labeling, integer additive set-sequential labeling, integer additive set-sequential graphs.

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1 Introduction

For all terms and definitions, not defined specifically in this paper, we refer to [17] and for more about graph labeling, we refer to [13]. Unless mentioned otherwise, all graphs considered here are simple, finite and have no isolated vertices.
All sets mentioned in this paper are finite sets of non-negative integers. We denote the cardinality of a set $A$ by $|A|$. We denote, by $X$, the finite ground set of non-negative integers that is used for set-labeling the elements of $G$ and cardinality of $X$ by $n$.

The research in graph labeling commenced with the introduction of $\beta$-valuations of graphs in [19]. Analogous to the number valuations of graphs, the concepts of set-labelings and set-indexers of graphs are introduced in [1] as follows.

Let $G$ be a $(p,q)$-graph. Let $X$, $Y$ and $Z$ be non-empty sets and $\mathcal{P}(X)$, $\mathcal{P}(Y)$ and $\mathcal{P}(Z)$ be their power sets. Then, the functions $f : V(G) \to \mathcal{P}(X)$, $f : E(G) \to \mathcal{P}(Y)$ and $f : V(G) \cup E(G) \to \mathcal{P}(Z)$ are called the set-assignments of vertices, edges and elements of $G$ respectively. By a set-assignment of a graph, we mean any one of them. A set-assignment is called a set-labeling or a set-valuation if it is injective.

A graph with a set-labeling $f$ is denoted by $(G,f)$ and is referred to as a set-labeled graph or a set-valued graph. For a $(p,q)$-graph $G = (V,E)$ and a non-empty set $X$ of cardinality $n$, a set-indexer of $G$ is defined as an injective set-valued function $f : V(G) \to \mathcal{P}(X)$ such that the function $f^{\oplus} : E(G) \to \mathcal{P}(X) - \{\emptyset\}$ defined by $f^{\oplus}(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective, where $\mathcal{P}(X)$ is the set of all subsets of $X$ and $\oplus$ is the symmetric difference of sets.

**Theorem 1.1.** [1] Every graph has a set-indexer.

Analogous to graceful labeling of graphs, the concept of set-graceful labeling and set-sequential labeling of a graph are defined in [1] as follows.

Let $G$ be a graph and let $X$ be a non-empty set. A set-indexer $f : V(G) \to \mathcal{P}(X)$ is called a set-graceful labeling of $G$ if $f^{\oplus}(E(G)) = \mathcal{P}(X) - \{\emptyset\}$. A graph $G$ which admits a set-graceful labeling is called a set-graceful graph.

Let $G$ be a graph and let $X$ be a non-empty set. A set-indexer $f : V(G) \to \mathcal{P}(X)$ is called a set-sequential labeling of $G$ if $f^{\oplus}(V(G) \cup E(G)) = \mathcal{P}(X) - \{\emptyset\}$. A graph $G$ which admits a set-sequential labeling is called a set-sequential graph.

Let $\mathbb{N}_0$ be the set of all non-negative integers. An integer additive set-labeling (IASL, in short) is an injective function $f : V(G) \to \mathcal{P}(\mathbb{N}_0)$. A graph $G$ which admits an IASL is called an IASL graph. An integer additive set-labeling $f$ is an integer additive set-indexer (IASI, in short) if the induced function $f^+ : E(G) \to \mathcal{P}(\mathbb{N}_0)$ defined by $f^+(uv) = f(u) + f(v)$ is injective. A graph $G$ which admits an IASI is called an IASI graph.

The cardinality of the set-label of an element (vertex or edge) of a graph $G$ is called the set-indexing number of that element. An IASL (or an IASI) is said to be a $k$-uniform IASL (or $k$-uniform IASI) if $|f^+(e)| = k \forall e \in E(G)$. The vertex set $V(G)$ is called $l$-uniformly set-indexed, if all the vertices of $G$ have the set-indexing number $l$.

**Definition 1.2.** Let $G$ be a graph and let $X$ be a non-empty set. An integer additive set-indexer $f : V(G) \to \mathcal{P}(X) - \{\emptyset\}$ is called a integer additive set-graceful labeling (IASGL, in short) of $G$ if $f^+(E(G)) = \mathcal{P}(X) - \{\emptyset, \{\emptyset\}\}$. A graph $G$ which admits an integer additive set-graceful labeling is called an integer additive set-graceful graph (in short, iasg-graph).
In this paper, we extend the concepts of set-sequential labelings of graphs to integer additive set-sequential labelings and establish some results on them.

2 Integer Additive Set-Sequential Graphs

First, note that under an integer additive set-labeling, no element of a given graph can have $\emptyset$ as its set-labeling. Hence, we need to consider only non-empty subsets of $X$ for set-labeling the elements of $G$.

Let $f$ be an integer additive set-indexer of a graph $G$. Therefore, $f : V(G) \rightarrow \mathcal{P}(X)$ be an injective function defined on $G$ such that the associated function $f^+ : E(G) \rightarrow \mathcal{P}(X) - \{\emptyset\}$, defined by $f^+(uv) = f(u) + f(v) \forall u, v \in V(G)$, is also injective. Let $x$ be an arbitrary element (a vertex or an edge) of a graph $G$. Define a function $f^* : V(G) \cup E(G) \rightarrow \mathcal{P}(X) - \{\emptyset\}$ as follows.

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in V(G) \\ f^+(x) & \text{if } x \in E(G) \end{cases}$$  \hspace{1cm} (2.1)

Clearly, $f^*$ is an extension of both $f$ and $f^+$ of $G$. Throughout our discussions in this paper, the function $f^*$ is as per the defined in Equation (2.1).

**Proposition 2.1.** Let $G$ be a connected graph. If the function $f^*$ defined above is an injective function, then no vertex of $G$ can have a set-label $\{0\}$.

**Proof.** If possible let a vertex, say $v$, has the set-label $\{0\}$. Since $G$ is connected, $v$ is adjacent to at least one vertex in $G$. Let $u$ be an adjacent vertex of $v$ in $G$ and $u$ has a set-label $A \subset X$. Then, $f^*(u) = f(u) = A$ and $f^*(uv) = f^+(uv) = A$, which is a contradiction to the hypothesis that $f^*$ is injective.

In view of Observation 2.1, we notice the following points.

**Remark 2.2.** Suppose that the function $f^*$ defined in (2.1) is injective. Then, if one vertex $v$ of $G$ has the set label $\{0\}$, then $v$ is an isolated vertex of $G$.

**Remark 2.3.** If the function $f^*$ defined in (2.1) is injective, then no edge of $G$ can also have the set label $\{0\}$.

Invoking the observations and remarks made above, we introduce the notion of integer additive set-sequential labeling of a graph as follows.

**Definition 2.4.** Let $G$ be a graph and let $X$ be a non-empty set of non-negative integers. An integer additive set-indexer $f : V(G) \rightarrow \mathcal{P}(X) - \{\emptyset, \{0\}\}$ is called a integer additive set-sequential labeling (IASSL, in short) of $G$ if $f^*[V(G) \cup E(G)] = \mathcal{P}(X) - \{\emptyset, \{0\}\}$, where $f^*$ is defined as in (2.1). A graph $G$ which admits an integer additive set-graceful labeling is called an integer additive set-sequential graph (in short, IASS-graph).

In this paper, we write $f^*(G)$ instead of $f^*[V(G) \cup E(G)]$. We say that two sets $A$ and $B$ are of same parity if their cardinalities are simultaneously odd or simultaneously even. From Definition 2.4, we make the following theorem.
Theorem 2.5. If $G$ is an integer additive set-sequential graph, then its order and size are either simultaneously odd or simultaneously even. That is, $V(G)$ and $E(G)$ are of same parity.

Proof. Let $f$ be a integer additive set-sequential labeling of a given graph $G$. Then, $f^*(G) = P(X) - \{0, \{0\}\}$. Therefore, $|f^*(G)| = 2^{|X|} - 2$, which is an even number. Since $f^*$ is injective, we have

$$|f^*(G)| = |V(G) \cup E(G)| = |V(G)| + |E(G)|.$$

$$\therefore |V(G)| + |E(G)| = 2^{|X|} - 2.$$ 

Hence, $|V(G)| + |E(G)|$ is an even integer. Therefore, $|V(G)|$ and $|E(G)|$ must be either simultaneously odd or simultaneously even.

In view of Theorem 2.5, we have the following corollary.

Corollary 2.6. If $G$ is an integer additive set-sequential graph, then $G$ has always an even number of elements.

For a given graph $G$, the choice of a ground set $X$ is also very important to have an integer additive set-sequential labeling. There are certain other restrictions in assigning set-labels to the elements of $G$. We explore the properties of a graph $G$ that admits an IASSL with respect to a given ground set $X$. As a result, we have the following observations.

Proposition 2.7. Let $G$ be a connected integer additive set-sequential graph with respect to a ground set $X$. Let $x_1$ and $x_2$ be the two minimal non-zero elements of $X$. Then, no edges of $G$ can have the set-labels $\{x_1\}$ and $\{x_2\}$.

Proof. In any IASL-graph $G$, the set-label of an edge is the sumset of the set-labels of its end vertices. Therefore, a subset $A$ of the ground set $X$, that is not a sumset of any two subsets of $X$, can not be the set-label of any edge of $G$. Since $x_1$ and $x_2$ are the minimal non-zero elements of $X$, $\{x_1\}$ and $\{x_2\}$ can not be the set-labels of any edge of $G$.

Proposition 2.8. Let $G$ be a connected integer additive set-sequential graph with respect to a ground set $X$. Then, any subset of $X$ that contains the maximal element of $X$ can not be the set-label of a vertex of $G$.

Proof. Let $x_n$ be the maximal element in $X$ and let $A$ be a subset of $X$ that contains the element $x_n$. If possible, let $A$ be the set-label of a vertex, say $v$, in $G$. Since $G$ is a connected graph, there exists at least one vertex in $G$ that is adjacent to $v$. Let $u$ be an adjacent vertex of $v$ in $G$ and let $B$ be its set-label. Then, the edge $uv$ has the set-label $A + B$. As $B \neq \{0\}$, there exists at least one element $x_i \neq 0$ in $B$ and hence $x_i + x_n \in A + B$, which is a contradiction to the fact that $G$ is an IASS-graph.

As all graphs, in general, do not admit an integer additive set-sequential labeling, in the following section, we check the admissibility of different graph classes.
3 Properties of Certain IASS-Graphs

In this section, we discuss certain properties of some graph classes which admit integer additive set-sequential labeling. We begin with trees, one of the simplest class among different graph classes.

**Theorem 3.1.** No trees admit an IASSL.

**Proof.** Let $G$ be a tree on $n$ vertices. If possible, let $G$ admits an IASSL. Then, $|E(G)| = n - 1$. Therefore, $|V(G)| + |E(G)| = n + n - 1 = 2n - 1$. But by Theorem 2.5, $2^{|V|} - 2 = 2n - 1$, which is a contradiction. \hfill \square

**Theorem 3.2.** If a cycle $C_n$ admits an IASSL, then $C_n$ is an odd cycle.

**Proof.** Let the cycle $C_n$ admits an IASSL. Then, by Theorem 2.5, we have

$$|V(C_n)| + |E(C_n)| = 2^{|V|} - 2$$
$$\Rightarrow 2n = 2^{|V|} - 2$$
$$\Rightarrow n = 2^{|V|-1} - 1.$$  

That is, $C_n$ is an odd cycle. \hfill \square

From Theorem 3.2, we notice that an even cycle does not have an IASSL. It can be seen that all odd cycles need not admit an IASSL. By Theorem 3.2, we observe that the necessary condition for an odd cycle $C_n$ to admit an IASSL is that $n + 1$ is a power of 2.

A natural question that arises in this context is about the characteristics of bipartite graphs which admit IASSL. The following proposition establishes the nature of the edge set and the vertex set of a bipartite graph that admits an IASSL.

**Proposition 3.3.** Let a bipartite graph $G$, with a bipartition $(V_1, V_2)$, admits an integer additive set-sequential labeling. Then, the size of $G$ is even if and only if both $V_1$ and $V_2$ are of the same parity.

**Proof.** Since $G$ is bipartite, $|V(G)| = |V_1| + |V_2|$. Since $G$ admits an IASSL, by Theorem 2.5, both $E(G)$ and $V(G)$ are of the same parity. Assume that $|E(G)|$ is even. Then, $|V(G)| = |V_1| + |V_2|$ is also even. Therefore, $|V_1|$ and $|V_2|$ are either simultaneously odd or simultaneously even. \hfill \square

**Remark 3.4.** From Proposition 3.3, it can be noted that the size of $G$ is odd if and only if both $V_1$ and $V_2$ are of different parity.

**Theorem 3.5.** If a complete bipartite graph $K_{m,n}$ admits an integer additive set-sequential labeling, then both $m$ and $n$ are even.

**Proof.** Assume that $K_{m,n}$ admits an IASSL, say $f$ with respect to a ground set $X$. Then, by Theorem 2.5

$$|V(K_{m,n})| + |E(K_{m,n})| = 2^{|X|} - 2$$
$$m + n + mn = 2^{|X|} - 2$$
$$(m + 1)(n + 1) = 2^{|X|} - 1.$$  

Therefore, $(m + 1)(n + 1)$ is an odd integer. That is, both $m$ and $n$ are even. \hfill \square
The following theorem discusses the characteristics of the vertex set and edge set of a complete graph that admits an IASSL.

**Theorem 3.6.** If a complete graph $K_n$ has an integer additive set-sequential labeling, then $n \equiv 0, 3 \pmod{4}$.

**Proof.** For $K_n$, $|V(G)| + |E(G)| = n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$. By Theorem 2.5 we have,

$$2^{|X|} - 2 = \frac{n(n + 1)}{2}.$$ 

That is, $2^{|X|} - 1 = \frac{n(n + 1)}{4}$.

Since $2^{|X|} - 1$ is an integer, $n(n + 1)$ is divisible by 4. That is, either $n$ or $n + 1$ is divisible by 4. Hence, $n \equiv 0, 3 \pmod{4}$. □

The results, we have discussed so far, are the necessary conditions for the graphs concerned to admit an integer additive set-sequential labeling. The question about the corresponding sufficient conditions are really worth studying. The sufficiency of the above results depend on the nature of the ground set $X$ also. Hence, we have

**Theorem 3.7.** A graph $G$ on $n$ vertices and $m$ edges admits an integer additive set-sequential labeling if and only if there exists a ground set $X$ such that $m + n = 2^{|X|} - 2$ such that at least $m$ subsets of $X$, other than $\emptyset$ and $\{0\}$, are sumsets of the other subsets of $X$, other than $\emptyset$ and $\{0\}$.

### 4 Conclusion

In this paper, we have discussed an extension of set-sequential labeling of graphs to sum-set labelings and have studied the properties of certain graphs that admit IASSL. Certain problems regarding the complete characterisation of IASSL-graphs are still open.

We note that the admissibility of integer additive set-indexers by the graphs depends upon the nature of elements in $X$. A graph may admit an IASSL for some ground sets and may not admit an IASSL for some other ground sets. Hence, choosing a ground set is very important to discuss about IASS-graphs.

Some of the areas which seem to be promising for further studies are listed below.

**Problem 1.** Characterise different graph classes which admit integer additive set-sequential labelings.

**Problem 2.** Verify the existence of integer additive set-sequential labelings for different graph operations, graph products and graph products.

The integer additive set-indexers under which the vertices of a given graph are labeled by different standard sequences of non negative integers, are also worth studying. All these facts highlight a wide scope for further studies in this area.
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