THE SIMILARITY DEGREE OF SOME C*-ALGEBRAS

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Abstract

We define the class of weakly approximately divisible unital C*-algebras and show that this class is closed under direct sums, direct limits, any tensor product with any C*-algebra, and quotients. A nuclear C*-algebra is weakly approximately divisible if and only if it has no finite-dimensional representations. We also show that Pisier’s similarity degree of a weakly approximately divisible C*-algebra is at most five.

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1. Introduction

One of the most famous and oldest open problems in the theory of C*-algebras is Kadison’s similarity problem [12], which asks whether every bounded unital homomorphism \( \rho \) from a C*-algebra \( \mathcal{A} \) into the algebra \( B(H) \) of operators on a Hilbert space \( H \) must be similar to a \(*\)-homomorphism, that is, does there exist an invertible \( S \in B(H) \) such that \( \pi(A) = S \rho(A)S^{-1} \) defines a \(*\)-homomorphism? One measure of the quality of a good problem is the number of interesting equivalent formulations. In this regard Kadison’s problem gets high marks.

(1) Inner derivation problem [4, 13]: if \( \mathcal{M} \subseteq B(H) \) is a von Neumann algebra and \( \delta : \mathcal{M} \to B(H) \) is a derivation, does there exist a \( T \in B(H) \) such that, for every \( A \in \mathcal{M} \),

\[ \delta(A) = AT - TA \]

(2) Hyperreflexivity problem [4, 13]: if \( \mathcal{M} \subseteq B(H) \) is a von Neumann algebra, does there exist a \( K, 1 \leq K < \infty \), such that, for every \( T \in B(H) \),

\[ \text{dist}(T, \mathcal{M}) \leq K \sup\{\|PT - TP\| : P \in \mathcal{M}', \ P = P^* = P^2\} \]

(3) Dixmier’s invariant operator range problem [6] (Foiaş [7], Pisier [21, Theorem 10.5], see also [10]): if \( \mathcal{M} \subseteq B(H) \) is a von Neumann algebra, \( A \in B(H) \) and \( T(A(H)) \subseteq A(H) \) for every \( T \in \mathcal{M} \), then does there exist \( D \in \mathcal{M}' \) such
that $A(H) = D(H)$? Paulsen [16] proved that an affirmative answer is equivalent to the assertion that the range of $A \oplus A \oplus \cdots$ is invariant for $M \otimes \mathcal{K}(\ell^2)$.

In [8] Haagerup proved that Kadison’s question has an affirmative answer whenever the representation $\rho$ has a cyclic vector, a result that is independent of the structure of the algebra $\mathcal{A}$. Haagerup [8] also showed that a homomorphism $\rho$ is similar to a $*$-homomorphism if and only if $\rho$ is completely bounded. (See also [3]; see the union of [9] and [26] for another proof; see [16, 17] for a lovely exposition of these ideas.) In [18] Pisier proved that, for a fixed $C^*$-algebra $\mathcal{A}$, every bounded homomorphism of $\mathcal{A}$ is similar to a $*$-homomorphism if and only if $\rho(\mathcal{A})$ is bounded, $\rho(\mathcal{T}) = D^{-1}TD = (2^{j-i}A_{ij})$ is bounded, then Kadison’s similarity problem has an affirmative answer if and only if, for every unital $C^*$-subalgebra $\mathcal{A}$ of $\mathcal{S}$, the homomorphism $\rho|_{\mathcal{A}}$ is similar to a $*$-homomorphism.

Our main focus in this paper is another amazing result of Pisier [18] where he shows that, for a unital $C^*$-algebra $\mathcal{A}$, Kadison’s similarity property holds for $\mathcal{A}$ if and only if there is a positive number $d$ for which there is a positive number $K$ such that

$$
\|\rho\|_{cb} \leq K\|\rho\|^d
$$

for every bounded unital homomorphism $\rho$ on $\mathcal{A}$. Pisier proved that the smallest such $d$ is an integer which he calls the similarity degree $d(\mathcal{A})$ of $\mathcal{A}$. Here are a few results on the similarity degree.

1. $\mathcal{A}$ is nuclear if and only if $d(\mathcal{A}) = 2 [2, 4, 22]$;
2. if $\mathcal{A} = \mathcal{B}(\mathcal{H})$, then $d(\mathcal{A}) = 3 [20]$;
3. $d(\mathcal{A} \otimes \mathcal{K}(\mathcal{H})) \leq 3$ for any $C^*$-algebra $\mathcal{A} [8, 19]$;
4. if $\mathcal{M}$ is a factor of type $II_1$ with property $\Gamma$, then $d(\mathcal{M}) = 3 [5]$;
5. if $\mathcal{A}$ is an approximately divisible $C^*$-algebra [1], then $d(\mathcal{A}) \leq 5 [14, 15]$;
6. if $\mathcal{A}$ is nuclear and contains unital matrix algebras of any order, then $d(\mathcal{A} \otimes \mathcal{B}) \leq 5$ for any unital $C^*$-algebra $\mathcal{B} [23]$;
7. if $\mathcal{A}$ is nuclear and contains finite-dimensional $C^*$-subalgebras of arbitrarily large subrank (see the definition below), then $d(\mathcal{A} \otimes \mathcal{B}) \leq 5$ for any unital $C^*$-algebra $\mathcal{B} [14]$;
8. if $\mathcal{A}$ is nuclear and contains homomorphic images of certain dimension-drop $C^*$-algebras $\mathcal{Z}_{p,q}$ for all relatively prime integers $p, q$ (for example, $\mathcal{A}$ contains a copy of the Jiang–Su algebra), then $d(\mathcal{A} \otimes \mathcal{B}) \leq 5$ for any unital $C^*$-algebra $\mathcal{B} [11]$.

In this paper we define the class of weakly approximately divisible $C^*$-algebras and show that this class is closed under unital $*$-homomorphisms, arbitrary tensor products
and direct limits. We also define the class of tracially nuclear $C^*$-algebras that properly contains the class of nuclear $C^*$-algebras, and we show that a tracially nuclear $C^*$-algebra is weakly approximately divisible if and only if it has no finite-dimensional representations. We prove that if $\mathcal{A}$ is weakly approximately divisible, then $d(\mathcal{A}) \leq 5$. We extend the results (6)–(8) above to the case when $\mathcal{A}$ is tracially nuclear and has no finite-dimensional representations, and the tensor product is with respect to any $C^*$-crossnorm.

2. Weakly approximately divisible algebras

If $\tau$ is a tracial state on $\mathcal{M}$, we let $\| \cdot \|_\tau$ denote the seminorm on $\mathcal{M}$ defined in the Gelfand–Naimark–Segal (GNS) construction by

$$\|a\|_\tau^2 = \tau(a^*a).$$

Let $\mathcal{B}$ be a finite-dimensional unital $C^*$-subalgebra of a unital $C^*$-algebra $\mathcal{A}$. First, we know that $\mathcal{B}$ is $*$-isomorphic to $M_{k_1}(\mathbb{C}) \oplus \cdots \oplus M_{k_m}(\mathbb{C})$ and its subrank, $\text{subrank}(\mathcal{B})$, is defined to be $\min(k_1, \ldots, k_m)$. Note that if $\pi : \mathcal{B} \to \mathcal{D}$ is a unital $*$-homomorphism, then

$$\text{subrank}(\mathcal{B}) \leq \text{subrank}(\pi(\mathcal{B})).$$

If $P_1 = 1 \oplus 0 \oplus \cdots \oplus 0$, $P_2 = 0 \oplus 1 \oplus 0 \oplus \cdots \oplus 0, \ldots, P_m = 0 \oplus \cdots \oplus 1$ are the minimal central projections of $\mathcal{B}$, then, for $1 \leq s \leq m$, we have $P_s \mathcal{A} P_s$ is isomorphic to $M_{k_s}(\mathbb{C}) \otimes \mathcal{A}_s = M_{k_s}(\mathcal{A}_s)$ for some algebra $\mathcal{A}_s$. The relative commutant of $M_{k_s}(\mathbb{C})$ in $M_{k_s}(\mathcal{A}_s)$ is

$$D_s = \left\{ \begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{pmatrix} : A \in \mathcal{A}_s \right\},$$

and the relative commutant of $\mathcal{B}$ in $\mathcal{A}$ is $D_1 \oplus \cdots \oplus D_m$. Suppose that $T \in \mathcal{A}$, and $P_s T P_s = (a_{ij,s})_{1 \leq i, j \leq k_s}$. Let $D_s = \text{diag}(c, \ldots, c)$ where $c = (1/k_s a_{11,s} + \cdots + a_{k_s,s})$. The map $E_{\mathcal{B}} : \mathcal{A} \to \mathcal{B}' \cap \mathcal{A}$ sending $T$ to $D_1 \oplus \cdots \oplus D_m$ is called the conditional expectation from $\mathcal{A}$ to $\mathcal{B}' \cap \mathcal{A}$ and is a completely positive unital idempotent. For $1 \leq s \leq m$, let $G_s$ be the group of all matrices in $M_{k_s}(\mathbb{C})$ such that the only nonzero entry in each row and each column is $1$ or $-1$, and let $G = G_1 \oplus \cdots \oplus G_m \subseteq \mathcal{B}$. Then

$$E_{\mathcal{B}}(T) = \frac{1}{\text{Card } G} \sum_{U \in G} U T U^*.$$

Moreover, if $S \in \mathcal{B}' \cap \mathcal{A}$ and $T \in \mathcal{A}$, then

$$E_{\mathcal{B}}(ST) = SE_{\mathcal{B}}(T) \quad \text{and} \quad E_{\mathcal{B}}(TS) = E_{\mathcal{B}}(T)S.$$

Furthermore, if $\tau$ is a tracial state on $\mathcal{A}$, then, for every $A \in \mathcal{A}$,

$$\|E_{\mathcal{B}}(A)\|_\tau \leq \|A\|_\tau.$$
Suppose that $\mathcal{M}$ is a von Neumann algebra and $\{v_i : i \in I\} \subseteq \mathcal{M}$ is a family satisfying $\sum_{i \in I} v_i^* v_i = 1$ (convergence is in the weak* topology). Then $\varphi(T) = \sum_{i \in I} v_i^* T v_i$ defines a unital completely positive map from $\mathcal{M}$ to $\mathcal{M}$. Let us call such a map "internally spatial," and call a unital completely positive map "internal" if it is a convex combination of internally spatial maps on $\mathcal{M}$.

**Remark 2.1.** There are two key properties of internal maps.

1. They can be pushed forward through normal unital $\ast$-homomorphisms between von Neumann algebras. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are von Neumann algebras and $\rho : \mathcal{M} \to \mathcal{N}$ is a unital weak*-weak*-continuous unital $\ast$-homomorphism, and suppose that $\{v_i : i \in I\} \subseteq \mathcal{M}$ with $\sum_{i \in I} v_i^* v_i = 1$ and $\varphi(T) = \sum_{i \in I} v_i^* T v_i$. Then $\{\pi(v_i) : i \in I\} \subseteq \mathcal{N}$ and

$$
1 = \pi(1) = \pi\left( \sum_{i \in I} v_i^* v_i \right) = \sum_{i \in I} \pi(v_i)^* \pi(v_i).
$$

We define $\varphi^\pi(S) = \sum_{i \in I} \pi(v_i)^* S \pi(v_i)$, and we have, for every $a \in \mathcal{M}$,

$$
\varphi^\pi(\pi(a)) = \pi(\varphi(a)).
$$

So if $b \in \pi(\mathcal{A})$ and $b = \pi(a)$, then $\varphi^\pi(b) = \pi(\varphi(a))$, which is independent of $a$. For a general $\varphi$ this only makes sense when $\varphi(\ker \pi) \subseteq \ker \pi$. It follows that $\varphi^\pi$ makes sense when $\varphi$ is an internal map, and in this case, $\varphi^\pi$ is an internal map on $\mathcal{N}$.

2. If $\varphi(T) = \sum_{i \in I} v_i^* T v_i$ and $T$ commutes with each $v_i$, then, for every $S$,

$$
\varphi(S T) = \varphi(S) T.
$$

Hence if $\psi$ is a convex combination of spatially internal maps defined in terms of elements commuting with an operator $T$, we have $\psi(S T) = \psi(S) T$.

**Definition 2.2.** We say that a unital $C^*$-algebra $\mathcal{A}$ is weakly approximately divisible if and only if, for every finite subset $\mathcal{F}$ of $\mathcal{A}$, there is a net $\{(\mathcal{B}_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ where each $\mathcal{B}_\lambda$ is a finite-dimensional unital $C^*$-subalgebra of $\mathcal{A}^{\#\#}$ and $\varphi_\lambda$ is an internal completely positive map such that:

1. $\lim_{\lambda} \text{subrank}(\mathcal{B}_\lambda) = \infty$;
2. $\varphi_\lambda : \mathcal{A} \to \mathcal{B}_\lambda' \cap \mathcal{A}^{\#\#}$;
3. for every $a \in \mathcal{F}$, $\varphi_\lambda(a) \to a$ in the weak* topology on $\mathcal{A}^{\#\#}$.

**Remark 2.3.** Suppose that $n$ is a positive integer and let $\mathcal{V}_n$ be the set of $n$-tuples $(a_1, \ldots, a_n)$ of elements in $\mathcal{A}$ such that the conditions in Definition 2.2 hold when $\mathcal{F} = \{a_1, \ldots, a_n\}$. Suppose that $U_k$ is a weak* neighbourhood of $a_k$ in $\mathcal{A}^{\#\#}$ for $1 \leq k \leq n$. Since addition on $\mathcal{A}^{\#\#}$ is weak*-continuous, there is a weak* neighbourhood $\mathcal{V}_k$ of $a_k$ and a weak* neighbourhood $\mathcal{E}$ of 0 such that

$$
\mathcal{V}_k + \mathcal{E} \subseteq U_k
$$
for \( 1 \leq k \leq n \). Suppose that \((b_1, \ldots, b_n)\) is in the norm closure of \( V_n \) and that \( U_k \) is a weak* neighbourhood of \( b_k \) in \( \mathcal{A}^{\#\#} \) for \( 1 \leq k \leq n \). Since addition on \( \mathcal{A}^{\#\#} \) is weak*-continuous, there is a weak* neighbourhood \( V_k \) of \( b_k \) and a weak* neighbourhood \( E \) of 0 such that

\[
V_k + E \subseteq U_k
\]

for \( 1 \leq k \leq n \). Since \( 0 \in E \) and \( E \) is weak*-open, there is an \( \varepsilon > 0 \) such that \( \{ x \in \mathcal{A}^{\#\#} : \| x \| < \varepsilon \} \subseteq E \). Now choose \((a_1, \ldots, a_n)\) in \( V_n \) so that \( a_k \in V_k \) and \( \| a_k - b_k \| < \varepsilon \) for \( 1 \leq k \leq n \). Next suppose that \( m \) is a positive integer. It follows from the definition of \( V_n \) that there is a finite-dimensional \( C^*\)-subalgebra \( \mathcal{B} \) of \( \mathcal{A}^{\#\#} \) and a completely positive unital map \( \varphi : \mathcal{A} \to \mathcal{B} \cap \mathcal{A}^{\#\#} \) such that \( \text{subrank}(\mathcal{B}) \geq m \) and such that \( \varphi(a_k) \in V_k \) for \( 1 \leq k \leq n \). It follows that \( \varphi(b_k) - \varphi(a_k) = \varphi(b_k - a_k) \in E \) for \( 1 \leq k \leq n \), so

\[
\varphi(b_k) \in V_k + E \subseteq U_k
\]

for \( 1 \leq k \leq n \). Hence \((b_1, \ldots, b_n) \in V_n \). Thus \( V_n \) is norm closed. It is also clear that \( V_n \) is a linear space. Hence, to verify that \( \mathcal{A} \) is weakly approximately divisible, it is sufficient to show that the conditions of Definition 2.2 hold for all finite subsets \( \mathcal{F} \) of a set \( W \) whose norm closed linear span \( \overline{\text{sp}}(W) \) is \( \mathcal{A} \).

Recall [25] that a \( C^*\)-algebra \( \mathcal{A} \) is **nuclear** if, for every Hilbert space \( H \) and every unital \(*\)-homomorphism \( \pi : \mathcal{A} \to \mathcal{B}(H) \), we have that \( \pi(\mathcal{A})'' \) is a hyperfinite von Neumann algebra. We say that \( \mathcal{A} \) is **tracially nuclear** if, for every tracial state \( \tau \) on \( \mathcal{A} \) with GNS representation \( \pi_\tau \), we have that \( \pi_\tau(\mathcal{A})'' \) is a hyperfinite von Neumann algebra. As a flip side of the notion of residually finite-dimensional (RFD) \( C^*\)-algebras, we say that a unital \( C^*\)-algebra \( \mathcal{A} \) is **NFD** if \( \mathcal{A} \) has no unital finite-dimensional representations.

**Theorem 2.4.** Suppose that \( \mathcal{A} \) and \( \mathcal{D} \) are unital \( C^*\)-algebras. Then the following statements hold.

1. If \( \mathcal{A} \) is approximately divisible, then \( \mathcal{A} \) is weakly approximately divisible.
2. If \( \mathcal{A} \) is weakly approximately divisible and \( \pi : \mathcal{A} \to \mathcal{D} \) is a surjective unital \(*\)-homomorphism, then \( \mathcal{D} \) is weakly approximately divisible.
3. If \( \mathcal{A} \) is weakly approximately divisible, then \( \mathcal{A} \) has no finite-dimensional representations.
4. If \( \mathcal{A} \) is weakly approximately divisible, then \( \mathcal{A} \otimes_{\text{max}} \mathcal{D} \) is weakly approximately divisible.
5. A finite direct sum \( \sum_{1 \leq k \leq n} \mathcal{A}_k \) of unital \( C^*\)-algebras is weakly approximately divisible if and only if each summand \( \mathcal{A}_k \) is weakly approximately divisible.
6. If \( n \) is a positive integer, then \( \mathcal{A} \otimes \mathcal{M}_n(\mathbb{C}) \) is weakly approximately divisible if and only if \( \mathcal{A} \) is.
7. A direct limit of weakly approximately divisible \( C^*\)-algebras is weakly approximately divisible.
8. If \( \mathcal{A} \) is an NFD \( C^*\)-algebra and \( \mathcal{M} \) is the type \( II_1 \) direct summand of \( \mathcal{A}^{\#\#} \) and \( \gamma : \mathcal{A} \to \mathcal{M} \) is the inclusion into \( \mathcal{A}^{\#\#} \) followed by the projection map, then \( \mathcal{A} \) is...
weakly approximately divisible if and only if, for every finite subset $\mathcal{F} \subseteq \mathcal{A}$ there is a net $((\mathcal{B}_i, \varphi_i))$ where $\mathcal{B}_i$ is a finite-dimensional $C^*$-subalgebra of $\mathcal{M}$, $\varphi_i$ is an internal map on $\mathcal{M}$ and

$$\varphi_i(\pi(a)) \rightarrow \gamma(a)$$

in the weak* topology for every $a \in \mathcal{F}$.

(9) If $\mathcal{A}$ is tracially nuclear, then $\mathcal{A}$ is weakly approximately divisible if and only if $\mathcal{A}$ is NFD.

(10) If $\mathcal{A}$ is nuclear, then $\mathcal{A}$ is weakly approximately divisible if and only if $\mathcal{A}$ is NFD.

**Proof.** (1) This follows immediately from the definitions.

(2) If $\pi : \mathcal{A} \rightarrow \mathcal{D}$ is a surjective unital $*$-homomorphism, then $\pi$ extends to a weak*-weak*-continuous surjective unital $*$-homomorphism $\rho : \mathcal{A}^{##} \rightarrow \mathcal{D}^{##}$. Given $d_1, \ldots, d_n \in \mathcal{D}$, choose $a_1, \ldots, a_n \in \mathcal{A}$ so that $\pi(a_k) = d_k$ for $1 \leq k \leq n$. Choose a net $((\mathcal{B}_k, \varphi_k))$ according to Definition 2.2 with $\mathcal{F} = \{a_1, \ldots, a_n\}$. It follows that $\varphi_k^{\prime}$ is an internal completely positive map on $\mathcal{D}^{##}$ and

$$\varphi_k(\mathcal{D}) = \varphi_k^{\prime}(\rho(\mathcal{A})) = \rho(\varphi_k(\mathcal{A})) \subseteq \rho(\mathcal{B}_k \cap \mathcal{A}^{##}) \subseteq \rho(\mathcal{B}_k) \cap \mathcal{D}^{##}.$$ 

Further, for each $d_k$,

$$\text{w*-lim}_A \varphi_k^{\prime}(d_k) = \text{w*-lim}_A \rho(\varphi_k(a_k)) = \rho(a_k) = d_k,$$

since $\rho$ is weak*-weak*-continuous. Since $\text{subrank}(\mathcal{B}_k) \leq \text{subrank}(\rho(\mathcal{B}_k))$, we conclude that $\mathcal{D}$ is weakly approximately divisible.

(3) This follows from (2) and the obvious fact that no finite-dimensional $C^*$-algebra is weakly approximately divisible.

(4) Let $\rho : \mathcal{A} \otimes_{\text{max}} \mathcal{D} \rightarrow (\mathcal{A} \otimes_{\text{max}} \mathcal{D})^{##}$ be the natural inclusion map. We can assume $(\mathcal{A} \otimes_{\text{max}} \mathcal{D})^{##} \subseteq B(H)$ for some Hilbert space $H$ so that, on bounded subsets of $(\mathcal{A} \otimes_{\text{max}} \mathcal{D})^{##}$, the weak* topology coincides with the weak-operator topology. If $\rho : \mathcal{A} \rightarrow \mathcal{A} \otimes 1 \subseteq \mathcal{A} \otimes_{\text{max}} \mathcal{D}$ is the inclusion map, then there is a weak*-weak*-continuous unital $*$-homomorphism $\sigma : \mathcal{A}^{##} \rightarrow (\mathcal{A} \otimes_{\text{max}} \mathcal{D})^{##}$ such that the restriction of $\sigma$ to $\mathcal{A}$ is $\rho$. Let $W = \{a \otimes b : a \in \mathcal{A}, b \in \mathcal{B}\}$. Clearly, $\overline{\text{sp}} W = \mathcal{A} \otimes_{\text{max}} \mathcal{B}$ (where the closure is with respect to $\|\|_{\text{max}}$). Suppose that $a_1 \otimes b_1, \ldots, a_n \otimes b_n \in W$. Since $\mathcal{A}$ is weakly approximately divisible, we can choose a net $((\mathcal{B}_k, \varphi_k))$ as in Definition 2.2. We know that $\{\varphi_k^{\prime}\}$ is a net of internal maps on $(\mathcal{A} \otimes_{\text{max}} \mathcal{D})^{##}$ and

$$\varphi_k^{\prime}(a_k \otimes 1) = \varphi_k^{\prime}(\sigma(a_k)) = \sigma(\varphi_k(a_k)) \rightarrow \sigma(a_k) = a_k \otimes 1$$

in the weak* topology for $1 \leq k \leq n$. On the other hand, each $\varphi_k$ is a convex combination of spatially internal maps defined by partial isometries in $\mathcal{A}^{##}$, so each $\varphi_k^{\prime}$ is a convex combination of spatially internal maps defined by partial isometries in $\sigma(\mathcal{A}^{##})$ which is contained in $(\mathcal{A} \otimes_{\text{max}} \mathcal{D})^{##} \cap (1 \otimes \mathcal{D})^\prime$. Hence, for every $S \in (\mathcal{A} \otimes_{\text{max}} \mathcal{D})^{##}$ and every $d \in \mathcal{D}$,

$$\varphi_k^{\prime}(S(1 \otimes d)) = \varphi_k^{\prime}(S)(1 \otimes d).$$
Hence, for $1 \leq k \leq n$,

$$\varphi^T_\lambda (a_k \otimes d_k) = \varphi^T_\lambda ((a_k \otimes 1)(1 \otimes d_k)) = \varphi^T_\lambda (a_k \otimes 1) (1 \otimes d_k).$$

But $\varphi^T_\lambda (a_k \otimes 1) \to a_k \otimes 1$ in the weak* topology. Hence

$$\varphi^T_\lambda (a_k \otimes d_k) \to a_k \otimes d_k$$

in the weak* topology on $(\mathcal{A} \otimes_{\text{max}} \mathcal{B})^{\#\#}$ for $1 \leq k \leq n$. Since, for every $\lambda$,

$$\text{subrank}(\mathcal{B}_\lambda) \leq \text{subrank}(\sigma(\mathcal{B}_\lambda)),$$

we see that $\mathcal{A} \otimes_{\text{max}} \mathcal{B}$ is weakly approximately divisible.

(5) This easily follows from the fact that $(\sum_{1 \leq k \leq n} \mathcal{A}_k)^{\#\#} = \sum_{1 \leq k \leq n} \mathcal{A}_k^{\#\#}$.

(6) This is clear, since $(\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C}))^{\#\#}$ is isomorphic to $\mathcal{A}^{\#\#} \otimes \mathcal{M}_n(\mathbb{C})$.

(7) Suppose that $\{\mathcal{A}_i : i \in I\}$ is an increasingly directed family of $C^*$-subalgebras of $\mathcal{A}$ such that $W = \bigcup_{i \in I} \mathcal{A}_i$ is dense in $\mathcal{A}$. Suppose that $\mathcal{F} \subseteq W$ is finite. Then there is an $i \in I$ such that $\mathcal{F} \subseteq \mathcal{A}_i$. If $\rho : \mathcal{A}_i \to \mathcal{A}$ is the inclusion map, there is a unital weak*–weak*-continuous unital $*$-homomorphism $\sigma : \mathcal{A}_i^{\#\#} \to \mathcal{A}^{\#\#}$ whose restriction to $\mathcal{A}_i$ is $\rho$. The rest follows as in the proof of (2).

(8) If $\mathcal{A}$ is weakly approximately divisible, then for a finite subset $\mathcal{F} \subseteq \mathcal{A}$ we can find a net $\{(\mathcal{B}_\lambda, \varphi_\lambda)\}$ as in Definition 2.2 that works in $\mathcal{A}^{\#\#}$, and if we project all of this onto $\mathcal{M}$, we get the desired net. Now suppose that $\mathcal{A}$ satisfies the condition in (8). We can write $\mathcal{A}^{\#\#} = \mathcal{M} \oplus \mathcal{N}$, and since $\mathcal{A}$ has no finite-dimensional representations, $\mathcal{N}$ is the direct sum of a type $I_\infty$ algebra, a $II_\infty$ and a type $III$ algebra. In particular, this means that there is an orthogonal sequence $\{P_n\}$ of pairwise Murray–von Neumann equivalent projections whose sum is $1$. Suppose that $N$ is a positive integer, and let $Q_k = \sum_{j=(k-1)N+1}^{kN} P_j$. Then $\{Q_n\}$ is an orthogonal sequence of pairwise equivalent projections whose sum is $1$. We can construct a system of matrix units $\{E_{ij}\}_{1 \leq i, j < \infty}$ so that $E_{kk} = Q_k$ for all $k \geq 1$. Then every $T \in \mathcal{N}$ has an infinite operator matrix $T = (T_{ij})$. The map

$$\psi_N(T) = \text{diag}(T_{11}, T_{11}, \ldots) = \sum_{j=1}^{\infty} E_{j1} T E_{j1}^*$$

is spatially internal and, for every $T$,

$$\left( \sum_{k=1}^{N} P_k \right) \psi_N(T) \left( \sum_{k=1}^{N} P_k \right) = \left( \sum_{k=1}^{N} P_k \right) T \left( \sum_{k=1}^{N} P_k \right) \to T$$

in the weak* topology. Hence $\psi_N(T) \to T$ in the weak* topology. Moreover, $\mathcal{N} \cap \psi_N(\mathcal{N})$ contains full matrix algebras of all orders. Next suppose that $\mathcal{F} \subseteq \mathcal{A}$ is finite. For each $A \in \mathcal{F}$ we write $A = \gamma(A) \oplus T_A$ relative to $\mathcal{A}^{\#\#} = \mathcal{M} \oplus \mathcal{N}$. Given the net $\{(\mathcal{B}_\lambda, \varphi_\lambda)\}$ in $\mathcal{M}$ based on our assumption on $\mathcal{A}$, we let $N_\lambda = \text{subrank}(\mathcal{B}_\lambda)$ and choose a full $N_\lambda \times N_\lambda$ matrix algebra $C_\lambda$ in $\mathcal{N} \cap \psi_N(\mathcal{N})$. Then $\tau_\lambda(S \oplus T) = \varphi_\lambda(S) \oplus \psi_N(T)$ is an internal map on $\mathcal{A}^{\#\#}$ whose range is in $(\mathcal{B}_1 \oplus C_\lambda)' \cap \mathcal{A}^{\#\#}$ such that

$$\tau_\lambda(A) \to A$$
in the weak* topology for every \( A \in \mathcal{F} \). Hence \( \mathcal{A} \) is weakly approximately divisible.

(9) Let \( \mathcal{M} \) and \( \gamma \) be as in (8). Let \( \Lambda \) be the set of all triples \( \lambda = (\mathcal{F}_\lambda, \mathcal{T}_\lambda, k_\lambda) \) where \( \mathcal{F}_\lambda \subseteq \mathcal{A} \) is finite, \( \mathcal{T}_\lambda \) is a finite set of normal tracial states on \( \mathcal{M} \), and \( k_\lambda \in \mathbb{N} \). With the ordering \( (\subseteq, \subseteq, \leq) \) we see that \( \Lambda \) is a directed set. If \( \tau \) is a tracial state on \( \mathcal{M} \), we let \( \| \cdot \|_\tau \) denote the seminorm on \( \mathcal{M} \) defined by

\[
\| A \|_\tau = \tau(A^* A)^{1/2}.
\]

Suppose that \( \lambda \in \Lambda \). There is a central projection \( P \in \mathcal{M} \) so that \( \mathcal{M} = \mathcal{M}_a \oplus \mathcal{M}_s \) (\( \mathcal{M}_a = \mathcal{P} \mathcal{M} \)) and so that \( \gamma = \gamma_a \oplus \gamma_s \) and such that \( \gamma_a \ll \sum_{\tau \in \mathcal{T}_\lambda} \pi_\tau \) and \( \gamma_s \) is disjoint from \( \sum_{\tau \in \mathcal{T}_\lambda} \pi_\tau \). Also, by assumption, \( (\sum_{\tau \in \mathcal{T}_\lambda} \pi_\tau)(\mathcal{A})' = \mathcal{M}_a \) is hyperfinite. Hence, there is a finite-dimensional unital subalgebra \( \mathcal{D}_\lambda \) of \( \mathcal{M}_a \) and a contractive map \( \eta : \mathcal{F}_\gamma \to \mathcal{D}_\lambda \) such that

\[
\max_{\tau \in \mathcal{T}_\lambda, \lambda \in \mathcal{F}_\lambda} \| P \gamma(A) - \eta(A) \|_\tau < \frac{1}{k_\lambda}.
\]

Note that \( \| T \|_\tau = \| PT \|_\tau \) for every \( T \in \mathcal{M} \) and every \( \tau \in \mathcal{T}_\lambda \). The relative commutant \( \mathcal{D}_\lambda' \cap \mathcal{M}_a \) is also a \( II_1 \) von Neumann algebra, so there are \( k_\lambda \) mutually orthogonal unitarily equivalent projections in \( \mathcal{D}_\lambda' \cap \mathcal{M}_a \) whose sum is 1. Hence \( \mathcal{D}_\lambda' \cap \mathcal{M}_a \) contains a unital subalgebra \( \mathcal{E}_\lambda \) that is isomorphic to \( \mathcal{M}_{k_{\lambda}}(\mathbb{C}) \). Similarly, \( \mathcal{M}_s \) (if it is not 0) is a \( II_1 \) von Neumann algebra and contains an isomorphic copy \( \mathcal{G}_\lambda \) of \( \mathcal{M}_{k_{\lambda}}(\mathbb{C}) \). Then \( \mathcal{B}_\lambda = \mathcal{E}_\lambda \oplus \mathcal{G}_\lambda \) is finite-dimensional and \( \text{subrank}(\mathcal{B}_\lambda) = k_\lambda \). Define \( \varphi_\lambda = E_{\mathcal{B}_\lambda} \). For every \( A \in \mathcal{F}_\lambda \) and \( \tau \in \mathcal{T}_\lambda \),

\[
\| A - \varphi_\lambda(A) \|_\tau = \| PA - P \varphi_\lambda(A) \|_\tau \leq \| PA - \eta(A) \|_\tau + \| \eta(A) - E_{\mathcal{E}_\lambda}(PA) \|_\tau
\]

\[
= \| PA - \eta(A) \|_\tau + \| E_{\mathcal{E}_\lambda}(\eta(A)) - E_{\mathcal{E}_\lambda}(PA) \|_\tau
\]

\[
\leq 2 \| PA - \eta(A) \|_\tau \leq \frac{2}{k_\lambda}.
\]

Clearly,

\[
\lim_{\lambda} \text{subrank}(\mathcal{B}_\lambda) = \infty,
\]

and, since there are sufficiently many tracial states on \( \mathcal{M} \) [24], we have, for every \( A \in \mathcal{A} \),

\[
\varphi_\lambda(a) \to A
\]

in the ultrastrong topology on \( \mathcal{M} \). By assumption \( \mathcal{A} \) has no finite-dimensional representations, so it follows from (8) that \( \mathcal{A} \) is weakly approximately divisible.

(10) This follows immediately from (9) since the nuclearity of \( \mathcal{A} \) is equivalent to the hyperfiniteness of \( \pi(\mathcal{A})'' \) for every representation \( \pi \) of \( \mathcal{A} \). \( \Box \)

3. Similarity degree

**Theorem 3.1.** If \( \mathcal{A} \) is weakly approximately divisible, then the similarity degree of \( \mathcal{A} \) is at most five.
Proof. Suppose that $H$ is a Hilbert space and $\rho : A \to B(H)$ is a bounded unital homomorphism. Then $\rho$ extends uniquely to a normal homomorphism $\overline{\rho} : A^{\text{\#}} \to B(H)$. Suppose that $A = (a_{ij}) \in M_n(A)$. Since $A$ is weakly approximately divisible, we can choose a net $\{(B_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ as in Definition 2.2 corresponding to $F = \{a_{ij} : 1 \leq i, j \leq n\}$. We know that

$$\overline{\rho}_n(\varphi_\lambda(a_{ij})) = (\overline{\rho}(\varphi_\lambda(a_{ij}))) \to (\overline{\rho}(a_{ij})) = \rho_n(A),$$

where the convergence is in the weak* topology. Moreover, since $\varphi_\lambda$ is completely contractive,

$$\|\varphi_\lambda(a_{ij})\| \leq \|A\|,$$

so

$$\lim_{\lambda} \|\varphi_\lambda(a_{ij})\| = \|A\|,$$

and

$$\|\rho_n(A)\| \leq \limsup_{\lambda} \|\overline{\rho}_n(\varphi_\lambda(a_{ij}))\|.$$

However, $\varphi_\lambda(a_{ij}) \in B_\lambda'$ for $1 \leq i, j \leq n$ and $\lim_{\lambda} \text{subrank}(B_\lambda) = \infty$. So the remainder of the proof follows from [14, Lemma 3.1].

In [23] Pop proved that if $A$ is a nuclear $C^*$-algebra containing copies of $M_n(\mathbb{C})$ for arbitrarily large values of $n$, then the similarity degree of $A \otimes B$ is at most five for every unital $C^*$-algebra $B$. In [14] the second author showed that this result remains true if $A$ is nuclear and contains finite-dimensional algebras with arbitrarily large subrank. It was shown by [11] that if $A$ is nuclear and contains homomorphic images of certain dimension-drop $C^*$-algebras $\mathbb{Z}_{p,q}$ for all relatively prime integers $p, q$ (for example, $A$ contains a copy of the Jiang–Su algebra), then, for every unital $C^*$-algebra $B$, the similarity degree of $A \otimes B$ is at most five. The following corollary includes all of these results.

Corollary 3.2. If $A$ is a unital tracially nuclear NFD $C^*$-algebra, then, for every unital $C^*$-algebra $B$, the similarity degree of $A \otimes B$ is at most five.

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