Relativistic MHD simulations of stellar core collapse and magnetars

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Abstract. We present results from simulations of magneto-rotational stellar core collapse along with Alfvén oscillations in magnetars. These simulations are performed with the CoCoA/CoCoNuT code, which is able to handle ideal MHD flows in dynamical spacetimes in general relativity. Our core collapse simulations highlight the importance of genuine magnetic effects, like the magneto-rotational instability, for the dynamics of the flow. For the modelling of magnetars we use the anelastic approximation to general relativistic MHD, which allows for an effective suppression of fluid modes and an accurate description of Alfvén waves. We further compute Alfvén oscillation frequencies along individual magnetic field lines with a semi-analytic approach. Our work confirms previous results based on perturbative approaches regarding the existence of two families of quasi-periodic oscillations (QPOs), with harmonics at integer multiples of the fundamental frequency. Additional material is presented in the accompanying contribution by Gabler et al (2010b) in these proceedings.

1. Introduction

Understanding gravitational stellar core collapse is a long-standing problem in relativistic astrophysics (see e.g. [50] and references therein). Major progress has been accomplished through numerical modelling with increasing levels of complexity in the input physics and mathematics, regarding aspects as diverse as the treatment of the hydrodynamics, gravity, magnetic fields, nuclear matter equations of state, transport, etc. Numerical studies based upon Newtonian physics are vastly developed nowadays and state-of-the-art simulations are beginning to generate successful supernova explosions. On the other hand, relativistic approaches are becoming routine in recent years, aided by the development of conservative formulations of the general relativistic hydrodynamics equations and numerically-stable formulations of the Einstein equations (see e.g. [21] and references therein). While some of these developments also hold for the simulation of general relativistic MHD flows (GRMHD flows hereafter), their thorough numerical exploration is still ahead.

Only very recently the first GRMHD codes able to follow the evolution of matter flows in dynamical spacetimes have been developed [18, 40, 25, 1, 9]. Our numerical code, described in detail in [9], is able to handle ideal MHD flows in dynamical spacetimes in general relativity, and is particularly designed to investigate gravitational core collapse leading to neutron stars as well.
as the evolution of strongly magnetized compact objects such as magnetars. This code is based on the hydrodynamics CoCoA/CoCoNuT code (www.mpa-garching.mpg.de/hydro/COCONUT) described in [15, 16], and on its extensions discussed in [8], [11], and [12], to which interested readers are addressed for details. The Maxwell equations were already incorporated in the codes of [11] and [12], but only in the passive magnetic field approximation, where the contribution of the magnetic field to the energy-momentum tensor is neglected yielding no impact on the dynamics. In the current code, this assumption is relaxed and the effects of the magnetic field on the dynamics and the self-gravity of the fluid are incorporated following the approach laid out in [3].

In this paper we discuss two applications of our code, namely magneto-rotational collapse and Alfvén oscillations of magnetars. To date the weakest point of all existing magneto-rotational simulations is the unknown strength and geometry of the initial magnetic field in the core. If the magnetic field is initially weak, there exist several mechanisms, such as differential rotation (Ω-dynamo) and the magneto-rotational instability (MRI hereafter), which can amplify the field to a strength where it can influence the flow dynamics. Differential rotation transforms rotational energy into magnetic energy winding up any seed poloidal field into a toroidal field, while the MRI leads to an exponential growth of the field strength. The latter acts as long as the radial gradient of the angular velocity of the flow is negative, a situation which holds in core collapse simulations, as we discuss below.

The second topic considered in this paper concerns Alfvén oscillations of magnetars, neutron stars with magnetic fields larger than about $10^{14}$ G [19]. Such field strengths can be reached if efficient amplification mechanisms are operative during the first few seconds after gravitational collapse [19]. A class of these objects, the so-called Soft Gamma Repeaters (SGRs), are characterized by gamma-ray activity which sometimes intensifies as giant flares ($10^{44} - 10^{46}$ erg/s in $\sim 0.2$ s) followed by a decaying X-ray tail lasting for hundreds of seconds. Quasi-periodic oscillations have been observed in the X-ray tail of two such SGRs. The simulations reported here provide some first results in our long-term goal of understanding the link between the oscillation modes of magnetars and the observed QPO frequencies. For further results obtained with our code, the interested reader is referred to [10, 22] as well as the accompanying contribution of [23] in this volume.

2. Framework
2.1. Ideal GRMHD equations
The evolution of a magnetized fluid is determined by the conservation law of the energy-momentum, $\nabla_\mu T^{\mu\nu} = 0$, and by the continuity equation, $\nabla_\mu J^\mu = 0$, for the rest-mass current $J^\mu = \rho u^\mu$, where $\rho$ is the rest-mass density and $u^\mu$ the 4-velocity. As usual, symbol $\nabla_\mu$ is used to indicate the covariant derivative operator. In terms of the (Faraday) electromagnetic tensor $F^{\mu\nu}$, Maxwell’s equations read

$$\nabla_\nu * F^{\mu\nu} = 0, \hspace{1cm} \nabla_\nu F^{\mu\nu} = J^\mu, \hspace{1cm} (1)$$

where $F^{\mu\nu} = U^\mu E^\nu - U^\nu E^\mu - \eta^{\mu\nu\lambda\delta} U_\lambda B_\delta$, its dual $* F^{\mu\nu} = \frac{1}{2} \eta^{\mu\nu\lambda\delta} F_{\lambda\delta}$, and $\eta^{\mu\nu\lambda\delta} = \frac{1}{\sqrt{g}} [\mu\nu\lambda\delta]$, where $[\mu\nu\lambda\delta]$ is the completely antisymmetric Levi-Civita symbol. $E^\mu$ and $B^\mu$ stand for the electric and magnetic fields measured by an observer with 4-velocity $U^\mu$, and $J^\mu$ is the electric 4-current. $J^\mu = \rho q u^\mu + \sigma F^{\mu\nu} u_\nu$ where $\rho q$ is the proper charge density and $\sigma$ is the electric conductivity.

Maxwell’s equations can be simplified if the fluid is a perfect conductor. In this case the conductivity of the fluid is infinite and, to keep the current finite, the term $F^{\mu\nu} u_\nu$ must vanish, which results in $E^\mu = 0$ for a comoving observer. This case corresponds to the so-called ideal MHD condition. Under this assumption the electric field measured by the Eulerian observer has
The total stress-energy tensor is thus given by
\[ T^{\mu\nu} = \rho u^\mu u^\nu + p g^{\mu\nu} - \alpha b^\mu b^\nu, \]
and Maxwell’s equations \( \nabla_\nu F^{\mu\nu} = 0 \) reduce to the divergence-free condition plus the induction equation for the evolution of the magnetic field
\[ \frac{\partial (\sqrt{\gamma} B^i)}{\partial x^i} = 0, \]
\[ \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial t} (\sqrt{\gamma} B^i) = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^j} \{ \sqrt{\gamma} [\alpha \hat{v}^j B^i - \alpha \hat{v}^i B^j] \}. \]

In the above expressions \( \alpha \) is the spacetime lapse function, \( \gamma \) the determinant of the spatial metric, and \( \hat{v}^j = v^j - \beta^j / \alpha \), where \( \beta^j \) is the 3-velocity of the fluid and \( \beta^j \) the spacetime shift vector.

For a fluid endowed with a magnetic field the stress-energy tensor is the sum of that of the fluid and that of the electromagnetic field, \( T^{\mu\nu} = T_{\text{Fluid}}^{\mu\nu} + T_{\text{EM}}^{\mu\nu} \), with
\[ T_{\text{Fluid}}^{\mu\nu} = \rho h u^\mu u^\nu + p g^{\mu\nu} \quad \text{and} \quad T_{\text{EM}}^{\mu\nu} = F^{\mu\lambda} F_{\nu\lambda} - \frac{1}{4} g^{\mu\nu} F^{\lambda\delta} F_{\lambda\delta}, \]
where \( h \) is the enthalpy, \( p \) is the pressure, and \( g^{\mu\nu} \) is the metric tensor. In ideal MHD, the electromagnetic part of the stress-energy tensor can be rewritten as
\[ T_{\text{EM}}^{\mu\nu} = \left( u^\mu u^\nu + \frac{1}{2} g^{\mu\nu} \right) b^2 - b^\mu b^\nu, \]
where \( b^\mu \) is the magnetic field measured by the observer comoving with the fluid and \( b^2 = b^\mu b_\mu \).

The total stress-energy tensor is thus given by
\[ T^{\mu\nu} = \rho \tau u^\mu u^\nu + p^* g^{\mu\nu} - b^\mu b^\nu, \]
with \( p^* = p + b^2 / 2 \) and \( \tau^* = h + b^2 / \rho \), respectively.

Following [3] the conservation equations for the extended energy-momentum tensor, together with the continuity equation and the equation for the evolution of the magnetic field measured by the Eulerian observer, can be written as a first-order, flux-conservative, hyperbolic system.

\[ \frac{1}{\sqrt{-g}} \left[ \frac{\partial \sqrt{\gamma} U}{\partial t} + \frac{\partial \sqrt{-g} F^i}{\partial x^i} \right] = S, \]
for a state vector \( U \), flux vector \( F^i \), and source vector \( S \). In this equation \( g \) is the determinant of the 4-dimensional metric. The state vector and the vector of fluxes of the fundamental GRMHD system of equations read:
\[ U(w) = (D, S_j, \tau, B^k), \]
\[ F^i(w) = (D \hat{v}^i, S_j \hat{v}^i + p^* \delta^i_j - b_j B^i / W, \tau \hat{v}^i + p^* v^i - \alpha b^0 B^i / W, \hat{v}^i B^k - \hat{v}^k B^i), \]
where the definition of the conserved quantities is \( D = \rho W \), \( S_j = \rho h W^2 v_j - \alpha b^0 b_j \), and \( \tau = \rho h W^2 - p^* - \alpha^2 (b^0)^2 - D \), where \( W = \alpha v^0 \) is the Lorentz factor. The explicit expressions of the corresponding vector of sources can be found in [9].

The hyperbolic structure of these equations and the associated spectral decomposition of the flux-vector Jacobians, needed for their numerical solution with Riemann solvers, is given in [3, 4]. This information is required for numerically solving the system of equations using the class of high-resolution shock-capturing (HRSC) schemes implemented in our code.
2.2. Einstein equations (CFC)

In our work Einstein’s field equations are formulated and solved using the conformally flat condition (CFC hereafter), introduced by [28]. In this approximation, the 3-metric in the ADM gauge is assumed to be conformally flat, \( \gamma_{ij} = \phi^4 \hat{\gamma}_{ij} \). Under the CFC assumption the gravitational field equations can be written as a system of five nonlinear elliptic equations for the set of variables \((\phi, \alpha\phi, \beta^i)\), namely

\[
\hat{\Delta}\phi = -2\pi\phi^5 \left( \rho W^2 - p + \frac{K_{ij}K^{ij}}{16\pi} \right), \tag{11}
\]

\[
\hat{\Delta}(\alpha\phi) = 2\pi\alpha\phi^5 \left( \rho(3W^2 - 2) + 5p + \frac{7K_{ij}K^{ij}}{16\pi} \right), \tag{12}
\]

\[
\hat{\Delta}\beta^i = 16\pi\alpha\phi^4 S^i + 2\phi^{10}K^{ij}\hat{\nabla}_j \left( \frac{\alpha}{\phi^6} \right) - \frac{1}{3}\hat{\nabla}^i\hat{\nabla}_k\beta^k, \tag{13}
\]

where \(\hat{\nabla}_i\) and \(\hat{\Delta}\) are the flat space Nabla and Laplace operators, respectively, \(K_{ij}\) is the extrinsic curvature, and \(S^i\) is the \(i\)-component of the momentum.

The simplified Einstein’s equations introduced by the conformally-flat condition (CFC) are adequate in situations for which the deviations from spherical symmetry are not extreme. They have been used to simulate rotational core collapse in [15, 16, 17]. Modifications of the original CFC equations to account for additional post-Newtonian terms or to allow for black hole formation following core collapse are presented in [8, 14].

2.3. Numerical code

Our numerical code solves the coupled time evolution of the equations governing the dynamics of the spacetime, the fluid, and the magnetic field in general relativity. The equations are implemented using spherical polar coordinates \(\{t, r, \theta, \phi\}\) and axisymmetry and equatorial plane symmetry are assumed. The evolution of the matter fields is handled with a HRSC scheme which updates the variables \((D, S_i, \tau)\). Various cell-reconstruction procedures are available in the code, which are either second-order or third-order accurate in space, namely minmod, MC, and PHM (see [47] for definitions). The time update of the state vector \(U\) relies on the method of lines in combination with a second-order accurate Runge–Kutta scheme. The numerical fluxes at cell interfaces are obtained using either the HLL single-state solver of [26] or the symmetric scheme of [30]. The solenoidal condition of the magnetic field is ensured with the use of the flux constraint transport method, which has been adapted to the spherical polar coordinates used in the code, and uses cell interface-centered poloidal and (because of the assumption of axisymmetry) cell-centered toroidal magnetic field components. The time discretization of the induction equation is done in the same way as for the fluid equations. Correspondingly, we use a fix-point iteration scheme in combination with a linear Poisson solver to solve the CFC nonlinear elliptic equations (for further details see [8] and [15]). Furthermore, the code incorporates several equations of state, ranging from simple analytical expressions to tabulated microphysical equations of state.

The interested reader is addressed to [9] for details on the various tests passed by the code. We here simply point out that the test calculations performed demonstrate the ability of the code to properly handle all aspects appearing in the astrophysical scenarios the code is intended for, namely relativistic shocks, strongly magnetized fluids, and equilibrium configurations of magnetized neutron stars.

For the core collapse simulations we discuss below we consider two different initial models, namely a weakly magnetized model B10 with a central magnetic field of \(|B_c| = 10^{10} \sqrt{4\pi}\) Gauss, and a strongly magnetized model B12 with \(|B_c| = 10^{12} \sqrt{4\pi}\) Gauss. Correspondingly, for our study of magnetars we construct equilibrium magnetar models using the self-consistent solution
of magnetized stars with a dipolar magnetic field presented by [6], where all the effects of the magnetic field on the matter and spacetime are taken into account. These equilibrium models are computed using the LORENE library (www.lorene.obspm.fr). We construct non-rotating polytropic equilibrium models with $\Gamma = 2$ and $K = 1.455 \times 10^5$ (cgs units). The specific central enthalpy is chosen to be $h_c = 1.256$ and the form of the current density which gives rise to the dipolar magnetic field is given in [6]. Here, we use the equilibrium model MNS2 of [10] as a reference model, with a mass of $1.4 M_\odot$ and a magnetic field strength at the pole equal to $B_{\text{pole}} = 6.5 \times 10^{15}$ G. The interested reader is addressed to [10] for results with a larger sample of initial models.

3. Magneto-rotational core collapse simulations

The initial models used for the magneto-rotational core collapse simulations are evolved with the tabulated equation of state of [39] and an approximate deleptonization scheme [33] as described in [17] and [12]. Further discussion on the results reported next can be found in [9]. As the collapse proceeds both the density and the magnetic energy grow very similarly for both models, because even in the highly magnetized progenitor model B12 the strength of the magnetic field is not large enough to affect the collapse dynamics. After core bounce, however, both models behave differently. On the one hand, for model B10 the magnetic field is far from saturation and grows linearly with time at the end of the simulation, while model B12 shows a saturation of the magnetic field energy shortly after core bounce. Its central density continues to grow beyond bounce, and the model eventually approaches an equilibrium configuration with a central density about 10% larger than in the weakly magnetized model.

The behavior of the central density can be understood by analyzing the profiles of the angular velocity $\Omega$, which are plotted in Figure 1. At the time of bounce those profiles are very similar for all models, since the magnetic field is still unimportant for the dynamics: the innermost 10 km of the core rotate rigidly, while further out $\Omega$ follows a power law with an exponent $\sim -1.2$. During the subsequent evolution the central region spins down, and the central density rises, the effect being more prominent in the strongest magnetized model B12 (right panel in Fig. 1). In the region $10 \text{ km} \leq r \leq 30 \text{ km}$ the magnetic field is strongest as differential rotation winds up the magnetic field more efficiently there [12]. On a time scale of about 50 ms, the angular velocity decreases by about a factor 10, and the innermost few kilometers of the core even acquire retrograde rotation, an effect already observed in the Newtonian simulations of [35] (see also [34]).
The spin down of the core may be understood by means of the magneto-rotational instability (MRI hereafter; see [5]). In unstable regions the MRI grows exponentially for all length-scales larger than a critical length-scale $\lambda_{\text{crit}} \sim 2\pi c_A/\Omega$, where $c_A$ is the Alfvén speed. The fastest growing MRI mode develops at length-scales near $\lambda_{\text{crit}}$ on a typical time scale of $\tau_{\text{MRI}} = 4\pi[\varpi_{\theta}\varpi_{\Omega}]^{-1}$. Therefore, one needs to resolve length-scales of about the size of $\lambda_{\text{crit}}$ in order to numerically capture the MRI. For model B12 such critical length-scale at bounce is $\lambda_{\text{crit}} \sim 1-5$ km inside the unstable region ($10\text{ km} \leq r \leq 30\text{ km}$). This region is covered with 60 radial and 30 angular zones, which corresponds to a resolution $(\Delta r, r\Delta \theta)$ of $125\text{ m} \times 500\text{ m}$ at $r = 10\text{ km}$, and $900\text{ m} \times 1500\text{ m}$ at $r = 30\text{ km}$. This resolution is marginally sufficient to resolve the length-scale of the fastest growing mode of the MRI at bounce ($5-10$ radial zones, and $2-3$ angular zones). The strong redistribution of the angular momentum observed for model B12 might therefore be caused by the MRI. On the other hand, for model B10 the critical length-scale at bounce is about a factor of 100 smaller, i.e. $\lambda_{\text{crit}} \sim 10-50$ m, and thus the fastest growing mode of the MRI cannot be resolved with our grid resolution.

In addition, we show in Fig. 2 the magnetic field topology for both models at the end of the simulation. In the low magnetized model B10 (left panel) the (transient) prompt convection developing after bounce twists the magnetic field outside the so-called neutrino-sphere, which is assumed to be located at about 30 km. In model B12 the magnetic field grows to values near equipartition, and a distinctive, strongly magnetized outflow propagates along the axis behind the shock front. Between $10\text{ km} < r < 30\text{ km}$, where the MRI is predominantly growing, axisymmetric channel flows form, which are morphologically similar to the flows found in the accretion disk simulations of [27].

4. Simulations of Alfvén QPOs in magnetars

4.1. Semi-analytic approach

Before discussing the results from our numerical simulations of Alfvén QPOs in magnetars, we consider a semi-analytic model which provides an alternative approach to this study. This model, described in full detail in [10], is based on the computation of Alfvén wave travel times along individual magnetic field lines (assuming a short-wavelength approximation). It only requires information on the initial equilibrium model and on the magnetic field configuration. Torsional Alfvén oscillations can be treated as axial perturbations traveling along individual magnetic field lines at the local Alfvén speed $v_a$. The path of an Alfvén wave inside the star can be obtained
by integrating the equation
\[ \frac{dx}{dt} = v_a(x), \]  
starting from a given initial location \( x_0 \) inside the star.

Instead of using polar coordinates \((r, \theta)\) we employ for convenience dimensionless coordinates adapted to the magnetic field lines \((\chi, \xi)\), which give the position of the magnetic field line and the location of points along an individual line, respectively. These are defined as \( \chi \equiv r/r_c \) and \( \xi \equiv t_a(r, \chi)/t_{tot}(\chi) - 1/2 \), where \( r_c \) labels the radial location in the equatorial plane where closed magnetic field lines have zero length, \( t_a \) is the arrival time of an Alfvén wave at the selected location and \( t_{tot}(\chi) \) is twice the total travel time of an Alfvén wave traveling along a magnetic field line. Let \( Y(\xi, \chi; t) \) be the displacement describing traveling waves along a specific magnetic field line \( \chi \), which satisfies the wave equation
\[ \frac{\partial^2 Y(\xi, t)}{\partial t^2} = \sigma^2 \frac{\partial^2 Y(\xi, t)}{\partial \xi^2}, \]  
where \( \sigma = 1/t_{tot} \) is the wave speed. To find a correspondence with QPOs we are interested in solutions of this equation in the form of standing waves \( Y(\xi, t) = A(\xi) \cos(2\pi ft + \varphi) \), where \( f \) is the oscillation frequency and \( \varphi \) is a phase. The spatial dependence can be written as \( A(\xi) = a \sin(\kappa \xi + \phi) \), where \( \kappa \) is the wavenumber, \( \phi \) a spatial phase, and \( a \) the amplitude of the oscillation. The boundary condition at the surface is chosen such that it mimics the presence of a solid crust. We therefore assume that the parts of the traction related to the crust have to vanish at the surface of the star. In practice, this condition leads to zero radial derivative of the displacement at this position. Such zero-traction boundary condition applied to a standing-wave yields the following equations:
\[ \left(2\pi f \frac{\partial f}{\partial \chi} + \frac{\partial \varphi}{\partial \chi} \right) \frac{\partial \chi}{\partial r} = 0, \quad \frac{\partial a}{\partial \chi} \frac{\partial \chi}{\partial r} = 0, \]  
which are fulfilled only in two cases. First, if \( \frac{\partial f}{\partial \chi} = 0, \frac{\partial \varphi}{\partial \chi} = 0, \frac{\partial a}{\partial \chi} = 0 \), i.e. if \( f(\chi), \varphi(\chi) \) and \( a(\chi) \) have a common turning point, or if they are constant. The second case is when \( \frac{\partial a}{\partial \chi} = 0 \), i.e. for those magnetic field lines that cross the stellar surface perpendicular to it, a condition which always holds in axisymmetry for the magnetic field line along the pole.

The time evolution of an initial perturbation will remain close to a standing wave solution as long as \( 2\pi f t \ll 1, \frac{\partial \varphi}{\partial \chi} \sim 0 \) and \( \frac{\partial a}{\partial \chi} \sim 0 \), which provides the characteristic timescale on which phase mixing occurs and a global standing wave is damped, namely \( \tau_d = \frac{1}{2\pi} \left( \frac{\partial f}{\partial \chi} \right)^{-1} \).

For timescales much shorter than the damping timescale a standing wave continuum of magnetic field lines throughout the star can be defined for each overtone with frequency \( f(\chi) = \frac{n}{2\pi t_{tot}(\chi)} \), \( n = 1, 2, 3, \cdots \). QPO overtones thus appear at exact integer multiples of the corresponding fundamental frequency.

Figure 3 displays the different branches of the standing-wave continuum of frequencies for our magnetar model, in a frequency vs. magnetic field line plot. For each overtone, the branch which corresponds to symmetric standing waves in the region of open field lines \((0 < \chi < 0.66)\) smoothly joins to the two overlapping branches of symmetric and antisymmetric standing waves in the region of closed magnetic field lines \((0.66 < \chi < 1.0)\). For the chosen magnetic field configuration, the standing-wave frequencies have a local maximum at \( \chi = 0 \) (i.e. at the pole) where the first set of conditions given by Eq. (16) are fulfilled. The predicted frequencies of these QPOs at the pole (the “upper” QPOs) are shown in Fig. 3 and labeled as \( U_n^{(+)} \) and \( U_n^{(-)} \), for symmetric and antisymmetric standing waves, respectively. Correspondingly, in the region of closed field lines, the standing-wave continuum for each overtone features a local minimum.
These local minima also give rise to long-lived “lower” QPOs which are labeled as $L_n^{(\pm)}$ in Fig. 3 (the frequency of symmetric and antisymmetric standing waves coincide).

4.2. Alfvén QPO model

We have also performed numerical simulations of Alfvén QPOs which help to validate, and extend, the picture just described obtained with our semi-analytic approach. These simulations have been carried out using the same CoCoA/CoCoNuT code used for the work on magneto-rotational core collapse presented in the preceding section. For the simulations of Alfvén QPOs we adopt the Cowling approximation in which the dynamics of the spacetime metric is neglected and only consider small-amplitude torsional oscillations. Moreover, our numerical approach is based on the relativistic anelastic approximation [7] in which sound waves are neglected. This is a central approximation in our work as it reduces the time-step restrictions of numerical codes based on time-explicit schemes. These restrictions are imposed by the dependence of the eigenvalues of the hyperbolic system on the speed of sound (see [3]). We note that we can use the anelastic approximation since we deal with low-amplitude torsional oscillations of an equilibrium star. They have axial parity and couple weakly to density perturbations (density and pressure can be considered constant). As a result of our assumptions there only remain evolution equations for the azimuthal components of the momentum and of the magnetic field, $S^\varphi$ and $B^\varphi$ (see [10] for details).

The fundamental variable in our simulations is the 3-velocity component $v^\varphi$ and the corresponding displacement $Y$, which we define as $\dot{Y} = \alpha v^\varphi$, where $\alpha$ is the spacetime lapse function and a dot denotes a time derivative. We choose an initial perturbation of the form

$$Y(t = 0) = 0, \quad \dot{Y}(t = 0) = f(r) b(\theta),$$

with

$$f(r) = \sin\left(\frac{3\pi r}{2r_s}\right), \quad b(\theta) = b_2 \frac{1}{3} \partial_\theta P_2(\cos \theta) + b_3 \frac{1}{6} \partial_\theta P_3(\cos \theta),$$
Table 1. QPO frequencies in the anelastic approximation. In parenthesis we give the relative difference with respect to the semi-analytic approach.

| upper QPOs | f (Hz) | lower QPOs | f (Hz) |
|------------|--------|------------|--------|
| $U_0^{-}$  | 35.1 (4.%)  | $L_0^{±}$   | 41.30 (1.0%)  |
| $U_0^{(+)}$ | 72.6 (0.05%) | $L_1^{±}$   | 83.85 (0.6%)  |
| $U_1^{-}$  | 110.2 (1.1%) | $L_2^{±}$   | 126.4 (1.0%)  |
| $U_1^{(+)}$ | 147.8 (1.8%) | $L_3^{±}$   | 170.2 (2.0%)  |
| $U_2^{-}$  | 182.8 (0.7%) | $L_4^{±}$   | 210.3 (0.9%)  |
| $U_2^{(+)}$ | 220.4 (1.2%) | $L_5^{±}$   | 251.6 (0.6%)  |
| $U_3^{-}$  | 256.7 (1.0%) | $L_6^{±}$   | 295.4 (1.2%)  |
| $U_3^{(+)}$ | 294.3 (1.3%) | $L_7^{±}$   | 336.7 (0.9%)  |

where $P_2$ and $P_3$ are the corresponding Legendre polynomials and the coefficients $b_2$ and $b_3$ are chosen to fix the symmetry of the perturbation with respect to the equatorial plane.

Our simulations predict strong QPOs both, near the magnetic axis and within the region of closed magnetic field lines, and precisely at the frequencies obtained with our semi-analytic model. For overtones, the number of nodes also agrees with the predictions of the semi-analytic model. The corresponding frequencies of our magnetar model (model MNS2 of [10]) are summarized in Table 1, along with their relative difference (in parenthesis) when compared to the predictions of the semi-analytic model. The frequency of the fundamental QPO agrees to within 4% between the two approaches, while the agreement in all other frequencies is at the 1-2% level. The larger discrepancy for the fundamental upper QPO reflects the inaccuracy of the semi-analytic (short-wavelength) approach for this particular mode since the wavelength of the corresponding standing-wave is twice the size of the star itself.

The simulations performed in our study reported in [10] include models with either different central current density or central density with respect to our reference magnetar model. When varying the strength of the magnetic field we find that, as expected from the results of [43], the QPO frequencies change linearly with $B$. Also, the effect of changing the stellar mass is consistent (within a few percent) with the expansion in terms of the compactness, $M/R$, used in the empirical relations presented by [43]. This agreement allows us to construct empirical relations for all QPO frequencies, taking advantage of the $M/R$ expansion in [43], which was constructed for a set of different EOSs (while here we only consider a polytropic EOS). These relations, which can be found in [10], allow us to provide constraints on the amplitude of the magnetic field for SGR 1806-20 and SGR 1900+14. In the former case the dipolar component of the magnetic field varies from $5 \times 10^{15}$ G to $1.2 \times 10^{16}$ G, while for the latter it is between $4.7 \times 10^{15}$ G and $1.12 \times 10^{16}$ G. These values give a mean surface magnetic field strength of about $3 - 8 \times 10^{15}$ G (as the mean value at the surface is $\sim 2/3$ of the value at the pole).

The interested reader is also addressed to the more recent results discussed by [23] in this volume (see also [22]) where the first numerical simulations of axisymmetric, torsional Alfvén oscillations in magnetars including an extended crust were presented. In the limit of very strong magnetic fields the results of [22] agree with previous studies of Alfvén oscillations in the absence of a solid crust [43, 10, 13], while for very weak magnetic fields crustal shear oscillations are recovered. In the intermediate regime, $5 \times 10^{13}$ Gauss to $1 \times 10^{15}$ Gauss, however, [22] find strong
resonant absorption of crustal shear modes by the Alfvén continuum of the core, while the Alfvén oscillations are influenced by the shear modulus and become magneto-elastic oscillations.

5. Summary

Results from numerical simulations of magneto-rotational stellar core collapse and Alfvén oscillations in magnetars have been presented. These simulations have been performed with a numerical code, the CoCoA/CoCoNuT code, able to handle ideal MHD flows in dynamical spacetimes in general relativity. This code is based on high-resolution shock-capturing schemes to solve the flux-conservative hyperbolic GRMHD equations, and the flux constraint transport method to ensure the solenoidal condition of the magnetic field. The Einstein equations are formulated in the conformal flatness condition approximation, and the resulting elliptic equations are solved using a linear Poisson solver.

The first topic we have discussed has been the simulation of general relativistic magneto-rotational core collapse using a realistic stellar progenitor model and a microphysical equation of state. We have shown that the cores of collapse progenitors with unrealistically large values of the initial magnetic field, suffer a rapid spin down, an effect which may be understood resorting to the magneto-rotational instability.

The second topic has been the study of Alfvén oscillations in magnetars using both a semi-analytic approach and nonlinear MHD simulations in the anelastic approximation. Our results show that the Alfvén oscillations form a continuum in the core of magnetars, whose turning points correspond to QPOs. When an extended crust is included in the magnetar model, strong resonant absorption of crustal shear modes by the Alfvén continuum of the core is found for intermediate values of the magnetic field (see [22, 23]).

Acknowledgments

This work has been supported by a GSRT Greece-Spain bilateral grant, a DAAD Germany-Greece bilateral grant, the EU network ILIAS, the Spanish Ministry of Education and Science (AYA 2007-67626-C03-01), the Collaborative Research Center on Gravitational Wave Astronomy of the Deutsche Forschungsgesellschaft (DFG SFB/Transregio 7), and the European research community Compstar. One of the authors (MG) also acknowledges support through an individual DAAD exchange grant.

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