On a fundamental system of solutions of a certain hypergeometric equation

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Abstract

We study the linear Pfaffian systems satisfied by a certain class of hypergeometric functions, which includes Gauß’s $2F_1$, Thomae’s $\text{LF}_{L-1}$ and Appell–Lauricella’s $F_D$. In particular, we present a fundamental system of solutions with a characteristic local behavior by means of Euler-type integral representations. We also discuss how they are related to the theory of isomonodromic deformations or Painlevé equations.

1 Introduction

1.1 Hypergeometric function $F_{L,N}$

Fix integers $L \geq 2$ and $N \geq 1$. We consider the hypergeometric function $F_{L,N} = F_{L,N}(\alpha, \beta, \gamma; x)$ in $N$ variables $x = (x_1, \ldots, x_N)$ defined by means of the power series

$$F_{L,N}(\alpha, \beta, \gamma; x) = \sum_{m \geq 0} \frac{(\alpha)_m \cdots (\alpha_{L-1})_m (\beta)_m \cdots (\beta_N)_m}{(\gamma_1)_m \cdots (\gamma_{L-1})_m (1)_m \cdots (1)_m} x_1^{m_1} \cdots x_N^{m_N} \tag{1.1}$$

convergent in the polydisc $D_0 = \{|x_1| < 1, \ldots, |x_N| < 1\} \subset \mathbb{C}^N$. Here $|m| = m_1 + \cdots + m_N$ and $(a)_n = \Gamma(a + n)/\Gamma(a)$ is the Pochhammer symbol. The $2L + N - 2$ parameters

$$(\alpha, \beta, \gamma) = (\alpha_1, \ldots, \alpha_{L-1}, \beta_1, \ldots, \beta_N, \gamma_1, \ldots, \gamma_{L-1})$$

are complex constants such that $\gamma_n \not\in \mathbb{Z}_{<0}$. Note that, if $(L, N) = (2, 1)$, $(L, 1)$ and $(2, N)$, then the hypergeometric function $F_{L,N}$ reduces to Gauß’s $2F_1$, Thomae’s $\text{LF}_{L-1}$ and Appell–Lauricella’s $F_D$, respectively.

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It is straightforward to verify from the power series (1.1) that \( F_{L,N} \) solves the system of linear differential equations

\[
\begin{cases}
  x_i (\beta_i + \delta_i) \prod_{k=1}^{L-1} (\alpha_k + D) - \delta_i \prod_{k=1}^{L-1} (\gamma_k - 1 + D) y = 0 \quad (1 \leq i \leq N)
\end{cases}
\]

where

\[
\delta_i = x_i \frac{\partial}{\partial x_i} \quad \text{and} \quad D = \sum_{i=1}^{N} \delta_i.
\]

Moreover, \( F_{L,N} \) possesses an Euler-type integral representation (see [7, Proposition 2.1])

\[
F_{L,N} = \prod_{k=1}^{L-1} \frac{\Gamma(\gamma_k)}{\Gamma(\alpha_k) \Gamma(\gamma_k - \alpha_k)} \times \int_{\Delta_0} U(t) \varphi_0
\]

with the domain \( \Delta_0 \) being an \((L - 1)\)-simplex

\[
\Delta_0 = \{0 \leq t_{L-1} \leq \cdots \leq t_2 \leq t_1 \leq 1\} \subset \mathbb{R}^{L-1}
\]

and the integrand \( U(t) \varphi_0 \) given by (1.4) and (1.5) below. Starting from the above integral representation, we have a certain linear Pfaffian system (see [7, Theorem 2.2]) equivalent to (1.2), which is the main object studied in this paper. We shall briefly introduce this linear Pfaffian system, denoted by \( \mathcal{P}_{L,N} \), in the following Sect. 1.2.

### 1.2 Linear Pfaffian system \( \mathcal{P}_{L,N} \) (the hypergeometric equation)

Let

\[
\zeta_k = \alpha_k - \gamma_{k+1} \quad (\gamma_L = 1), \quad \eta_k = \gamma_k - \alpha_k, \quad \theta_i = -\beta_i
\]

and consider a multi-valued function

\[
U(t) = \prod_{k=1}^{L-1} t_k^{\zeta_k} (t_{k-1} - t_k)^{\eta_k} \prod_{i=1}^{N} (1 - x_i t_{L-1})^{\theta_i}
\]

in \( t = (t_1, t_2, \ldots, t_{L-1}) \) with \( t_0 = 1 \). Consider the rational \((L - 1)\)-forms

\[
\varphi_0 = \frac{dt}{\prod_{k=1}^{L-1} (t_{k-1} - t_k)}, \quad \varphi_n^{(i)} = \frac{dt}{(x_i t_{L-1} - 1) \prod_{k=1}^{L-1} (t_{k-1} - t_k)} \left( \begin{array}{c} 1 \leq i \leq N \\ 1 \leq n \leq L - 1 \end{array} \right)
\]

where

\[
dt = dt_1 \wedge \cdots \wedge dt_{L-1}.
\]

Now we define the vector-valued function

\[
y = y(x; \Delta) = \begin{pmatrix}
y_0, y_1^{(1)}, \ldots, y_{L-1}^{(1)}, y_1^{(2)}, \ldots, y_{L-1}^{(2)}, \ldots, y_1^{(N)}, \ldots, y_{L-1}^{(N)}
\end{pmatrix}
\]
by the integrals

\[ y_0 = \int_\Delta U(t)\varphi_0, \quad y_n^{(i)} = \int_\Delta U(t)\varphi_n^{(i)} \]  

(1.7)

over a suitable domain \( \Delta \). The function \( y \) then satisfies the linear Pfaffian system

\[
\text{dy} = \left\{ \sum_{i=1}^N \left( E_i d \log x_i + F_i d \log(1 - x_i) \right) + \sum_{1 \leq i < j \leq N} G_{ij} d \log(x_i - x_j) \right\} y
\]  

\((P_{L,N})\)

of rank \( N(L - 1) + 1 \). Here, the coefficient matrices are linear functions in the constant parameters \((\alpha, \beta, \gamma)\) given by

\[
E_i = \begin{bmatrix}
0 & 1 & 2 & \cdots & i & N \\
\alpha_1 & b_{i,1} & a_1 & a_2 & b_{i,2} & \cdots & a_l & a_1 & a_2 & a_3 & b_{i,3} & \cdots & a_l & \cdots & a_l & b_{i,L-1} \\
-a_1 & -a_1 & -a_1 & \cdots & -a_1 \\
-a_2 & -a_2 & -a_2 & \cdots & -a_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-a_{l-1} & -a_{l-1} & -a_{l-1} & \cdots & -a_{l-1}
\end{bmatrix}
\]

and \( a_n = \alpha_n - \gamma_n \) and \( b_{i,n} = \sum_{j \neq i} \beta_j - \gamma_n \). The symbol \( I_{L-1} \) denotes the identity matrix of size \( L - 1 \).

We wrote a square matrix \( M \) of size \( N(L - 1) + 1 \) with separating it into \((N + 1)^2\) blocks as

\[
M = \begin{bmatrix}
M_{00} & M_{01} & \cdots & M_{0N} \\
M_{10} & M_{11} & \cdots & M_{1N} \\
\vdots & \vdots & \ddots & \vdots \\
M_{N0} & M_{N1} & \cdots & M_{NN}
\end{bmatrix}
\]

where \( M_{ij} (i, j \neq 0) \) is a square matrix of size \( L - 1 \) and thus \( M_{00} \) is a scalar, \( M_{0j} (j \neq 0) \) and \( M_{i0} (i \neq 0) \) are row and column \((L - 1)\)-vectors, respectively.

The singular locus of the system \( P_{L,N} \) is a union of hyperplanes

\[
\Xi = \bigcup_{i=1}^N (\{x_i = 0\} \cup \{x_i = 1\} \cup \{x_i = \infty\}) \cup \bigcup_{1 \leq i < j \leq N} \{x_i = x_j\}. \]  

(1.8)
Therefore, the holomorphic function $F_{L,N}$ at $x = 0$ can be analytically continued along any path outside $\Xi$. The characteristic exponents at each divisor, i.e. the eigenvalues of each residue matrix $E_i, F_i$ or $G_{ij}$ are listed in the following table (Riemann scheme):

| Divisor | Characteristic exponents |
|---------|-------------------------|
| $x_i = 0$ | $(b_{i,1}, b_{i,2}, \ldots, b_{i,L-1}, 0, \ldots, 0)$ |
| $x_i = 1$ | $(-\beta_i - \sum_{n=1}^{L-1} a_n, 0, \ldots, 0)$ |
| $x_i = \infty$ | $(\alpha_1, \alpha_2, \ldots, \alpha_{L-1}, \beta_i, \ldots, \beta_j)$ |
| $x_i = x_j (i \neq j)$ | $(-\beta_i - \beta_j, \ldots, -\beta_i + \beta_j, 0, \ldots, 0)$ |

(1.9)

**Remark 1.1.** If we regard one specific variable $x_i$ as an independent variable and all other $x_j$’s ($j \neq i$) as fixed constants, then $P_{L,N}$ is a Fuchsian system of ordinary differential equations with respect to $x_i$. Notice that the sum of characteristic exponents at $x_i = 0, 1, \infty, x_j (j \neq i)$ is equal to zero; i.e. the Fuchsian relation holds. The *spectral type* of this Fuchsian system is given by the $(N + 2)$-tuple

$$1, 1, \ldots, 1, (N - 1)(L - 1) + 1 \quad \text{at } x_i = 0, \infty,$$

$$1, N(L - 1) \quad \text{at } x_i = 1,$$

$$L - 1, (N - 1)(L - 1) + 1 \quad \text{at } x_i = x_j (j \neq i)$$

of partitions of $N(L - 1) + 1$, which indicates how the characteristic exponents overlap at each of the singularities. We know, by counting Katz’s index [3], that this Fuchsian system is rigid in the sense that its global monodromy is determined only from its local monodromy, or from its characteristic exponents, at each singularity.

**Remark 1.2.** The linear Pfaffian system $P_{L,N}$ can be derived from the integral representations (1.7), with the aid of twisted de Rham theory [1]. For details refer to [7] (in which the symbols $U(t), \varphi_0$ and $\varphi^{(i)}_n$ are slightly different from the present ones).

### 1.3 Holomorphic solution at the origin

If the domain $\Delta$ of integration is chosen to be the $(L - 1)$-simplex $\Delta_0$ (see (1.3)), then $y(x; \Delta_0)$ becomes holomorphic at the origin $x = 0 \in \mathbb{C}^N$. This is the unique solution of $P_{L,N}$ holomorphic at $x = 0$ (up to multiplication by constants), and is expressible in terms of the hypergeometric function $F_{L,N}(\alpha, \beta, \gamma; x)$ as

$$y_0 = cF_{L,N}, \quad y_1^{(i)} = \frac{\alpha_1 - \gamma_1}{\gamma_1} cF_{L,N}(\beta_i + 1, \gamma_1 + 1),$$

$$y_2^{(i)} = \frac{\alpha_1(\alpha_2 - \gamma_2)}{\gamma_1 \gamma_2} cF_{L,N}(\alpha_1 + 1, \beta_i + 1, \gamma_1 + 1, \gamma_2 + 1), \quad \ldots$$

$$y_n^{(i)} = \frac{\alpha_1 \cdots \alpha_{n-1}(\alpha_n - \gamma_n)}{\gamma_1 \cdots \gamma_n} cF_{L,N}(\alpha_1 + 1, \ldots, \alpha_{n-1} + 1, \beta_i + 1, \gamma_1 + 1, \ldots, \gamma_n + 1), \quad \ldots$$
where \( c = \prod_{k=1}^{L-1} \Gamma(\alpha_k) \Gamma(\gamma_k - \alpha_k)/\Gamma(\gamma_k) \). For notational simplicity, we used the abbreviation \( F_{L,N}(\beta_1 + 1, \gamma_1 + 1) \) to mean that among the parameters \((\alpha, \beta, \gamma)\) only the indicated ones \( \beta_i \) and \( \gamma_i \) are shifted by one, and so forth. We note that the first element \( y = y_0 \) of \( y \) solves (1.2) indeed.

### 1.4 Main result: fundamental system of solutions of \( \mathcal{P}_{L,N} \)

Let \( D_0 \) and \( D^{(j)} \) \((j = 1, \ldots, N)\) be the domains in \( \mathbb{C}^N \) given by

\[
D_0 = \{ |x_1| < 1, \ldots, |x_N| < 1 \},
\]

\[
D^{(j)} = \{ |x_j| < 1 \} \cap \bigcap_{i \neq j} \{ x_i \neq x_j \}.
\]

According to the Riemann scheme (1.9), a fundamental system of solutions of \( \mathcal{P}_{L,N} \) of the following form can be expected.

**Theorem 1.3.** There exists a fundamental system \( Y(x) \) of solutions of \( \mathcal{P}_{L,N} \) such that

\[
Y(x) = \Phi(x) \text{ diag } \begin{bmatrix} 1, x_1^{b_1,1}, \ldots, x_1^{b_{1,L-1}}, x_2^{b_2,1}, \ldots, x_2^{b_{2,L-1}}, \ldots, x_N^{b_{N,1}}, \ldots, x_N^{b_{N,L-1}} \end{bmatrix},
\]

\[
\Phi(x) = \begin{bmatrix}
** & ** & \cdots & ** \\
* & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & * \\
\end{bmatrix}
\]

\[
+ \begin{bmatrix} R_0(x) \mid x_1 R^{(1)}(x) \mid x_2 R^{(2)}(x) \mid \cdots \mid x_N R^{(N)}(x) \end{bmatrix},
\]

(1.10)

where \( R_0 \) is a column vector and other \( R^{(j)} \)'s \((j = 1, 2, \ldots, N)\) are matrices with \( L - 1 \) columns; \( R_0 \) is holomorphic on \( D_0 \) and \( R_0(0) = 0 \) and each \( R^{(j)} \) is holomorphic on \( D^{(j)} \).

To be more precise, if \( \Phi_0 \) denotes the first term of the right-hand side of (1.10), each element of the 0th column of \( \Phi_0 \) is a nonzero constant. Concerning the \( j \)th block \((j = 1, 2, \ldots, N)\), \((*)\) represents a holomorphic function on \( D^{(j)} \) whose restriction to \( \{ x_j = 0 \} \) is not identically zero. In the rest of this paper, we shall prove this theorem. Notice first that the holomorphic solution at the origin given in Sect. 1.3 provides the 0th column of \( Y(x) \) or of \( \Phi(x) \). Others will be explicitly constructed by a systematic use of Euler-type integral representations together with an iteration of ‘cyclic’ transformations; see Theorems 3.2 and 3.3 for \( N = 1 \) case and Theorem 4.4 for general \((L,N)\) case.

In Sect. 2 we prepare a cyclic expression of the hypergeometric integrals (1.7), i.e. the general solution of \( \mathcal{P}_{L,N} \). We then construct a fundamental system of solutions, having the desired local
behavior, for \( N = 1 \) case in Sect. 3 and for general \((L, N)\) case in Sect. 4. We also indicate a connection between the hypergeometric functions and isomonodromic deformations of a certain Fuchsian system in Sect. 5.

2 A cyclic expression of hypergeometric integrals

Let us first homogenize the integrands of the hypergeometric integrals (1.7) by introducing a set of \( L \) variables \( \tau = (\tau_0, \tau_1, \ldots, \tau_{L-1}) \) with \( t_n = \tau_n/\tau_0 \). Observe that

\[
\frac{dt}{t_0} = \tau_0^{-1} \omega, \quad \text{where} \quad \omega = \sum_{n=0}^{L-1} (-1)^n t_n d\tau_0 \wedge \cdots \wedge \hat{d}\tau_n \wedge \cdots \wedge d\tau_{L-1},
\]

and thereby

\[
\varphi_0 = \frac{\omega}{\tau_0 \prod_{k=1}^{L-1} (\tau_{k-1} - \tau_k)},
\]

\[
\varphi^{(i)}_n = \frac{\omega}{\tau_0 (x_i \tau_{L-1} - \tau_0) \prod_{k=1}^{L-1, k \neq n} (\tau_{k-1} - \tau_k)} \left( \begin{array}{c}
1 \leq i \leq N \\
1 \leq n \leq L - 1
\end{array} \right).
\]

Cf. (1.5) and (1.6) in Sect. 1.2. The multi-valued function \( U(t) \) is rewritten as

\[
U(t) = \prod_{k=1}^{L-1} \left( \frac{\tau_k}{\tau_0} \right)^{\zeta_k} \left( \frac{\tau_{k-1} - \tau_k}{\tau_0} \right)^{\eta_k} \prod_{i=1}^N \left( 1 - \frac{x_i \tau_{L-1}}{\tau_0} \right)^{\theta_i}
\]

\[
= \tau_0^{-\zeta_0 + 1} \prod_{k=1}^{L-1} \tau_k^{\zeta_k} (\tau_{k-1} - \tau_k)^{\eta_k} \prod_{i=1}^N (\tau_0 - x_i \tau_{L-1})^{\theta_i},
\]

where

\[
\zeta_0 = -1 - \sum_{k=1}^{L-1} (\zeta_k + \eta_k) - \sum_{i=1}^N \theta_i.
\]

Now, we shall be concerned with the multi-valued function

\[
V(\tau) = \tau_0^{-1} U(t)
\]

\[
= \tau_0^{-\zeta_0} \prod_{k=1}^{L-1} \tau_k^{\zeta_k} (\tau_{k-1} - \tau_k)^{\eta_k} \prod_{i=1}^N (\tau_0 - x_i \tau_{L-1})^{\theta_i} \tag{2.1}
\]

and the rational \((L - 1)\)-forms

\[
\psi_0 = \tau_0 \varphi_0 = \frac{\omega}{\prod_{k=1}^{L-1} (\tau_{k-1} - \tau_k)},
\]

\[
\psi^{(i)}_n = \tau_0 \varphi^{(i)}_n = \frac{\omega}{(\tau_{L-1} \tau_0) \prod_{k=1}^{L-1, k \neq n} (\tau_{k-1} - \tau_k)} \left( \begin{array}{c}
1 \leq i \leq N \\
1 \leq n \leq L - 1
\end{array} \right) \tag{2.2}
\]
in \( \tau = (\tau_0, \ldots, \tau_{L-1}) \). Accordingly, the integrals (1.7) can be expressed as

\[
y_0 = \int_\Delta V(\tau)\psi_0, \quad y_n^{(n)} = \int_\Delta V(\tau)\psi_n^{(n)}
\]
since \( U(\tau)\psi_0 = V(\tau)\psi_0 \) and \( U(\tau)\psi_n^{(n)} = V(\tau)\psi_n^{(n)} \). Without fear of repetition, we summarize the correspondence

\[
\zeta_k = \alpha_k - \gamma_{k+1}, \quad \eta_k = \gamma_k - \alpha_k, \quad \theta_i = -\beta_i
\]
of constant parameters, where

\[
\alpha_0 = \sum_{i=1}^N \beta_i, \quad \gamma_L = 1.
\]

Next, let us introduce a ‘cyclic’ transformation \( \pi_i \) of the variables \( \tau = (\tau_0, \ldots, \tau_{L-1}) \) defined by

\[
\pi_i : \tau_k \mapsto \begin{cases} \tau_{k+1} & (0 \leq k \leq L-2) \\ \tau_0/x_i & (k = L-1) \end{cases}
\]
for each \( i = 1, 2, \ldots, N \). We then have

\[
\pi_i^n \left[ \prod_{k=1}^{L-1} (\tau_{k-1} - \tau_k) \right] = \prod_{k=1}^{L-1} (\tau_{k+n-1} - \tau_{k+n}) = (\tau_n - \tau_{n+1}) \cdots (\tau_{L-2} - \tau_{L-1})(\tau_{L-1} - \tau_L) \cdots (\tau_{L+n-2} - \tau_{L+n-1})
\]

\[
= x_i^{-n}(x_i^L - \tau_0) \prod_{k=1, k\neq n}^{L-1} (\tau_{k-1} - \tau_k),
\]
where we regard \( \tau_k \) for \( k \geq L \) as \( \tau_k = \tau_{k-L}/x_i \) tentatively. Combining this with \( \pi_i(\omega) = (-1)^{L-1}\omega/x_i \), we have

\[
\pi_i^n(\psi_0) = (-1)^{n(L-1)}\psi_n^{(n)}.
\]
\[ (2.3) \]

Note that the suffix \( n \) of \( \psi_n^{(n)} \) can be extended to be any \( n \in \mathbb{Z} \) by the conditions

\[
\psi_n^{(n)} = \frac{\psi_n^{(n)}}{x_i} \quad \text{and} \quad \psi_0^{(n)} = \psi_0.
\]

Hence we arrive at a ‘cyclic’ expression

\[
y_0 = \int_\Delta V(\tau)\psi_0, \quad y_n^{(n)} = (-1)^{n(n-1)} \int_\Delta V(\tau)\pi_i^n(\psi_0)
\]
of the hypergeometric integrals (1.7). Concerning the domain of integration, the \((L-1)\)-simplex \( \Delta_0 \) can be written as

\[
\Delta_0 = \{ 0 \leq \tau_{L-1} \leq \cdots \leq \tau_1 \leq \tau_0 \} \subset \mathbb{R}^L
\]
for instance; cf. (1.3).
3 Case $N = 1$: Thomae’s $L F_{L-1}$

This section is devoted to the case where $N = 1$, i.e., the hypergeometric function $F_{L,N}$ reduces to Thomae’s $L F_{L-1}$. The linear Pfaffian system under consideration is of the form

$$\frac{dy}{dx} = \left\{ \frac{1}{x} \begin{bmatrix} 0 & b_1 \\ a_1 & b_1 & b_2 \\ \vdots & \vdots & \vdots \\ a_{L-1} & a_{L-1} & \cdots & a_{L-1} & b_{L-1} \end{bmatrix} \right\} + \frac{1}{1-x} \begin{bmatrix} \beta & \beta & \cdots & \beta \\ a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{L-1} & a_{L-1} & \cdots & a_{L-1} \end{bmatrix} y, \quad (P = P_{L,1})$$

where $a_n = \alpha_n - \gamma_n$ and $b_n = -\gamma_n$ ($n = 1, 2, \ldots, L - 1$), and its characteristic exponents at $x = 0$ read $(0, b_1, \ldots, b_{L-1})$.

Our aim here is to write down the fundamental system of solutions $Y(x)$ of $P$ having such a local power series expansion as

$$Y(x) = \Phi(x) \text{ diag } (1, x^{b_1}, \ldots, x^{b_{L-1}}),$$

$$\Phi(x) = \begin{bmatrix} * \\ \vdots & \ddots \\ * & \cdots & * \end{bmatrix} + O(x) = (f_0, f_1, \ldots, f_{L-1}), \quad f_n \in \mathbb{C}[\llbracket x \rrbracket], \quad (3.1)$$

$\Phi(0)$: invertible

around the origin $x = 0$. For instance, we can take $f_0 = y(x; \Delta_0)$ for $\Delta_0 = \{0 \leq t_{L-1} \leq \cdots \leq t_2 \leq t_1 \leq 1\} \subset \mathbb{R}^{L-1}$.

First we observe that the $n$th column $f_n$ ($n = 1, 2, \ldots, L - 1$) satisfies the equation

$$\frac{df_n}{dx} = K_n f_n, \quad (3.2)$$

where

$$K_n = \frac{1}{x} \begin{bmatrix} -b_n \\ a_1 & b_1 - b_n \\ a_2 & a_2 & b_2 - b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{L-1} & a_{L-1} & \cdots & a_{L-1} & b_{L-1} - b_n \end{bmatrix} + \frac{1}{1-x} \begin{bmatrix} \beta & \beta & \cdots & \beta \\ a_1 & a_1 & \cdots & a_1 \\ a_2 & a_2 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{L-1} & a_{L-1} & \cdots & a_{L-1} \end{bmatrix}.$$ 

Next we introduce a rotational matrix

$$\Lambda = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ x^{-1} & & & 0 \end{bmatrix}$$

and consider the equation satisfied by $g = \Lambda^n f_n$, namely,

$$\frac{dg}{dx} = \left( \Lambda^n K_n \Lambda^{-n} + \frac{d\Lambda^n}{dx} \Lambda^{-n} \right) g.$$

It is easy to verify the following.
Lemma 3.1. (1) For a square matrix \( M \) of size \( L \), it holds that
\[
\Lambda^n M \Lambda^{-n} = \Lambda^n \begin{bmatrix} N \mid E \\ \tilde{W} \mid S \end{bmatrix} \Lambda^{-n} = \begin{bmatrix} S' \mid x'W' \\ \tilde{E}' \mid N' \end{bmatrix}
\]
with \( N \) being a square matrix of size \( n \).
(2) It holds that
\[
\frac{d\Lambda^n}{dx} \Lambda^{-n} = -\frac{1}{x} \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix},
\]
where \( I_n \) denotes the identity matrix of size \( n \).

If we write the coefficient matrix \( K_n \) of (3.2) as
\[
K_n = \frac{1}{x} \begin{bmatrix} N \mid E \\ \tilde{W} \mid S \end{bmatrix} + \frac{1}{1-x} \begin{bmatrix} N' \mid E' \\ \tilde{W}' \mid S' \end{bmatrix}
\]
then, by virtue of Lemma 3.1 and \( E = 0 \) and \( \tilde{W} = \tilde{W}' \), we have
\[
\bar{K}_n = \Lambda^n K_n \Lambda^{-n} + \frac{d\Lambda^n}{dx} \Lambda^{-n}
\]
\[
= \frac{1}{x} \begin{bmatrix} S \mid x \tilde{W} \\ \tilde{E}' \mid N \\ 1-x \end{bmatrix} + \frac{1}{1-x} \begin{bmatrix} S' \mid x \tilde{W}' \\ \tilde{E}' \mid N' \end{bmatrix} - \frac{1}{x} \begin{bmatrix} 0 & 0 \\ 0 & I_n \end{bmatrix}
\]
\[
= \frac{1}{x} \begin{bmatrix} S \mid 0 \\ \tilde{E}' \mid N-I_n \end{bmatrix} + \frac{1}{1-x} \begin{bmatrix} S' \mid \tilde{W}' \\ \tilde{E}' \mid N' \end{bmatrix}.
\]

Hence, we obtain
\[
\bar{K}_n = \frac{1}{x} \begin{bmatrix} a_{n+1} & b_{n+1} - b_n \\ a_{n+2} & a_{n+2} - b_n \\ \vdots & \vdots & \ddots & \ddots \\ a_{L-1} & a_{L-1} & \cdots & a_{L-1} & b_{L-1} - b_n \\ \beta & \beta & \cdots & \beta & \beta \\ a_1 & a_1 & \cdots & a_1 & a_1 \\ \vdots & \vdots & \ddots & \vdots & \ddots \\ a_{n-1} & a_{n-1} & \cdots & a_{n-1} & a_{n-1} \\ \end{bmatrix} - b_n - 1
\]
\[
+ \frac{1}{1-x} \begin{bmatrix} a_n & a_n & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_{L-1} & a_{L-1} & \cdots & a_{L-1} \\ \beta & \beta & \cdots & \beta \\ a_1 & a_1 & \cdots & a_1 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-1} & \cdots & a_{n-1} \\ \end{bmatrix}.
\]

For convenience, we shall extend the suffixes \( k \) of constant parameters to be any integers by \( a_{k+L} = a_k, b_{k+L} = b_k - 1, a_0 = \beta \) and \( b_0 = 0 \). We therefore find that \( \bar{K}_n \) is of the same form as the coefficient matrix of the original \( P \) except the replacement
\[
a_k \mapsto a_{k+n}, \quad b_k \mapsto b_{k+n} - b_n, \quad \beta \mapsto a_n
\]
of constant parameters. In terms of the parameters \((\alpha, \beta, \gamma)\), this replacement amounts to the transformation

\[ T_n : \alpha_k \mapsto \alpha_{k+n} - \gamma_n, \quad (\alpha_0 = \beta) \mapsto \alpha_n - \gamma_n, \quad \gamma_k \mapsto \gamma_{k+n} - \gamma_n, \]

where \(\alpha_{k+L} = \alpha_k + 1, \gamma_{k+L} = \gamma_k + 1, \alpha_0 = \beta\) and \(\gamma_0 = 0\).

Applying \(T_n\) to the multi-valued function

\[ U_0 = U(t) = (-1)^{-\beta} \prod_{k=0}^{L-1} t_k^{\alpha_k - \gamma_{k+1}} (t_{k+1} - t_k)^{\gamma_k - \alpha_k} \]

shows that

\[ U_n = T_n(U) = (-1)^{\gamma_n - \alpha_n} \prod_{k=0}^{L-1} t_k^{\alpha_{k+n} - \gamma_{k+n+1}} (t_{k+1} - t_k)^{\gamma_{k+n} - \alpha_{k+n}}. \]

We shall write

\[ \varphi_n = \varphi_n^{(1)} = \frac{dt}{\prod_{k=0}^{L-1} (t_{k+1} - t_k)} \]

with \(t_0 = 1\) and \(t_{-1} = xt_{L-1}\), and extendedly use the symbol \(\varphi_n\) for any integer \(n\) by the quasi-periodicity \(\varphi_{n+L} = \varphi_n/x\). Now, the result can be stated as follows.

**Theorem 3.2.** Let

\[ \Phi(x) = \int_{\Delta_0} \begin{bmatrix} \varphi_0 & x\varphi_{L-1} & x\varphi_{L-2} & \cdots & x\varphi_1 \\ \varphi_1 & \varphi_0 & x\varphi_{L-1} & \cdots & x\varphi_2 \\ \varphi_2 & \varphi_1 & \varphi_0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & x\varphi_{L-1} \\ \varphi_{L-1} & \varphi_{L-2} & \cdots & \cdots & \varphi_1 & \varphi_0 \end{bmatrix} \text{diag} (U_0, U_1, \ldots, U_{L-1}) \]

\[ = \int_{\Delta_0} (U_n\varphi_{m-n})_{0 \leq m,n \leq L-1}. \]

Then, \(Y(x) = \Phi(x)\text{diag}(1, x^{-\gamma_1}, \ldots, x^{-\gamma_{L-1}})\) is a fundamental system of solutions of \(\mathcal{P}(= \mathcal{P}_{L,1})\) that fulfills the requirement \((3.1)\).

We used only integrals over a single domain \(\Delta_0 = \{0 \leq t_{L-1} \leq \cdots \leq t_2 \leq t_1 \leq 1\}\) in the above theorem.

Furthermore, let us present an alternative expression of the same solution. Introduce new variables \(\tau = (\tau_0, \ldots, \tau_{L-1})\) and set \(t_n = \tau_n/\tau_0\). Recall Sect.2. We thus have \(\Delta_0 = \{0 \leq \tau_{L-1} \leq \cdots \leq \tau_1 \leq \tau_0\}\). Define

\[ \Delta_n = \pi^n(\Delta_0) = \{0 \leq \tau_{L-1+n} \leq \cdots \leq \tau_{1+n} \leq \tau_n\} \quad \text{for } n \in \mathbb{Z} \]

by using the ‘cyclic’ transformation \(\pi : \tau_k \mapsto \tau_{k+1}\) with \(\tau_{k+L} = \tau_k/x\).

**Theorem 3.3.** The \(L \times L\) matrix function

\[ Y(x) = \left( \int_{\Delta_n} U(t)\varphi_m \right)_{0 \leq m,n \leq L-1} \]

is a fundamental system of solutions of \(\mathcal{P}\) that fulfills the requirement \((3.1)\).
Proof. Theorem 3.2 implies that \( Y(x) \) can be expressed as

\[
\int_{\Delta_0} (U_n \varphi_{m-n} x^{-\gamma_0})_{0 \leq n, m \leq L-1} \times C,
\]

(3.3)

where \( \gamma_0 = 0 \) and \( C \) is a constant diagonal matrix. We set

\[
V(\tau) = \tau_0^{-1} U(t) = (-1)^{-\beta} \prod_{k=0}^{L-1} \tau_k \psi_0 = \frac{\omega}{\prod_{k=0}^{L-1} (\tau_{k-1} - \tau_k)}
\]

and

\[
\omega = \sum_{n=0}^{L-1} (-1)^n \tau_n dt \wedge \cdots \wedge d\tau_L \wedge \cdots \wedge d\tau_{L-1}
\]

as in Sect. 2. It holds that \( \pi^{-n}(V) = x^{-\gamma_0}(-1)^{n-\gamma_0-\beta} \tau_0^{-1} U_n \) and \( \pi^{-n}(\psi_m) = (-1)^{n(m-1)} \psi_{m-n} \), where \( \psi_{k+L} = \psi_k/x \). Therefore,

\[
\pi^{-n}(U \varphi_m) = \pi^{-n}(V \psi_m) = x^{-\gamma_0}(-1)^{n(m-1)+\alpha_0 - \gamma_0} U_n \varphi_{m-n},
\]

which completes the proof in view of (3.3).

Example 3.4 (Gauß’s \( zF_1 \)). Let us restrict ourselves to the case where \( (L, N) = (2, 1) \), i.e. the hypergeometric equation \( P_{L, N} \) thus becomes

\[
\frac{dy}{dx} = \left[ \begin{array}{cc} 1 & 0 \\ 0 & -\gamma \end{array} \right] + \left[ \begin{array}{cc} 1 & \beta \\ \alpha - \gamma & \alpha \end{array} \right] y.
\]

(3.4)

The multi-valued function \( U = U(t) \) in a single variable \( t = t_1 \) and the rational 1-forms \( \varphi_n \) read

\[
U(t) = t^{\alpha-1} (1-t)^{\gamma_0-\alpha} (1-xt)^{-\beta}, \quad \varphi_0 = \frac{dt}{1-t}, \quad \varphi_1 = \frac{dr}{xt-1}.
\]

The domains \( \Delta_0 = [0 \leq t_1 \leq \tau_0] \) and \( \Delta_1 = [0 \leq t_2 \leq \tau_1] \) of integration are translated as

\[
\Delta_0 = [0 \leq t \leq 1] \quad \text{and} \quad \Delta_1 = \{1/x \leq t \leq \infty\}
\]

by the correspondence \( t_n = \tau_n/\tau_0 \) and \( t_{n+2} = \tau_n/x \). Hence, it follows from Theorem 3.3 that the hypergeometric integrals

\[
Y_{00}(x) = \int_0^1 t^{\alpha-1} (1-t)^{\gamma_0-\alpha-1} (1-xt)^{-\beta} dt, \quad Y_{01}(x) = \int_{1/x}^0 t^{\alpha-1} (1-t)^{\gamma_0-\alpha-1} (1-xt)^{-\beta} dt,
\]

\[
Y_{10}(x) = -\int_0^1 t^{\alpha-1} (1-t)^{\gamma_0-\alpha} (1-xt)^{-\beta-1} dt, \quad Y_{11}(x) = -\int_{1/x}^0 t^{\alpha-1} (1-t)^{\gamma_0-\alpha} (1-xt)^{-\beta-1} dt
\]

provide a fundamental system of solutions of (3.4) with the local behavior

\[
Y(x) = \begin{bmatrix} Y_{00}(x) & Y_{01}(x) \\ Y_{10}(x) & Y_{11}(x) \end{bmatrix} = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix} + O(x) \operatorname{diag} (1, x^{-\gamma})
\]

near \( x = 0 \).
4 General \((L, N)\) case

Let us consider the domains

\[
\Delta_n^{(i)} = \pi_i^n(\Delta_0) \quad \begin{pmatrix} 1 \leq i \leq N \\ 1 \leq n \leq L - 1 \end{pmatrix}
\]

doing integration, defined by applying the ‘cyclic’ transformation

\[
\pi_i : \tau_k \mapsto \begin{cases} 
\tau_{k+1} \\ x_i^{-1} \tau_0 \end{cases} \quad (0 \leq k \leq L - 2)
\]

\(k = L - 1\)

to \(\Delta_0 = \{0 \leq \tau_{L-1} \leq \cdots \leq \tau_1 \leq \tau_0 \} \subset \mathbb{R}^L\).

**Theorem 4.1.** The \((N(L - 1) + 1) \times (N(L - 1) + 1)\) matrix function

\[
Y(x) = \left( \int_{\Delta_0^{(j)}} U\psi_m^{(j)} \right)_{1 \leq i, j \leq N \atop 0 \leq m, n \leq L-1}
\]

is a fundamental system of solutions of \(\mathcal{P}_{L,N}\) having the local behavior stated in Theorem 1.3.

**Proof.** Fix \(j \in \{1, \ldots, N\}\) and \(n \in \{1, \ldots, L - 1\}\). We shall examine the local behavior of the column vector

\[
y(x; \Delta_n^{(j)}) = \left( \int_{\Delta_n^{(j)}} U\psi_m^{(j)} \right)_{1 \leq i \leq N \atop 0 \leq m \leq L-1}
\]

of hypergeometric integrals, which belongs to the \(j\)th block of the matrix function \(Y(x)\). To this end, we first rewrite its element as an integral over the simplex \(\Delta_0\), namely

\[
\int_{\Delta_n^{(j)}} U\psi_m^{(j)} = \int_{\Delta_n^{(j)}} V\psi_m^{(j)} = \int_{\Delta_0} \pi_j^{-n} (V\psi_m^{(j)}).
\]

Recall (2.1) and (2.2) for notations. Here, we mention that \(y(x; \Delta_n^{(j)})\) certainly solves \(\mathcal{P}_{L,N}\) since \(\Delta_n^{(j)}\) is a chamber framed by the hyperplanes which are the singular loci of the multi-valued function \(U = U(t)\); cf. [7].
Applying $\pi_j^{-n}$ to $V = V(\tau)$ yields

$$
\pi_j^{-n}(V) = \tau^{-n} \prod_{k=1}^{L-1} \tau_{k-n}^{\zeta_0} (\tau_{k-n-1} - \tau_{k-n})^\eta_i \prod_{i=1}^{N} \left( (\tau_n - x_i \tau_{L-n-1})^{\theta_i} \right)
$$

$$
= x_j^{\zeta_0 + \sum_{i=1}^{n-1} (\xi_i + \eta_i) + \theta_j} \tau_{L-n}^{\zeta_0} \prod_{k=1}^{n-1} \tau_{L+k-n}^{\zeta_0} (\tau_{L+k-n-1} - \tau_{L+k-n})^\eta_i
$$

$$
\times \prod_{k=n}^{L-1} \tau_{k-n}^{\zeta_0} (\tau_{k-n-1} - \tau_{k-n})^\eta_i
$$

$$
\times (\tau_{L-n} - \tau_{L-n-1})^\theta_i \prod_{i \neq j} (x_j \tau_{L-n} - x_i \tau_{L-n-1})^\theta_i
$$

$$
=: x_j^{b_{ij}} g(x, \tau)
$$

by the use of the quasi-periodicity $\tau_{k+L} = \tau_k / x_j$ and $b_{ij} = \sum_{i \neq j} \beta_i - \gamma_n = \zeta_0 + \sum_{k=1}^{n-1} (\xi_k + \eta_k) + \theta_j$. Observe that $g(x, \tau)$ is holomorphic on the interior $\text{Int}(\Delta_0)$ of $\Delta_0$ provided

$$
|x_i| > |x_j| \quad \text{for} \quad i \neq j.
$$

(4.2)

It readily follows from (2.3) that

$$
\pi_j^{-n}(\psi_m^{(j)}) = \begin{cases} 
(\xi^+(L-1)) x_j \psi_{L+m-n}^{(j)} & (m < n) \\
(\xi^+(L-1)) \psi_0 & (m = n) \\
(\xi^+(L-1)) \psi_{m-n}^{(j)} & (m > n)
\end{cases}
$$

and thus $\pi_j^{-n}(\psi_m^{(j)})$ is holomorphic on $\text{Int}(\Delta_0)$ as long as $|x_j| < 1$. If we remember that any solution of the linear Pfaffian system $\mathcal{P}_{L,N}$ is holomorphic outside its singular locus $\Xi$ (see (1.8)) and notice that (4.2) is an open condition and thereby removable via the identity theorem, then we can conclude that

$$
\int_{\Delta_0} \pi_j^{-n}(V\psi_m^{(j)}) = \begin{cases} 
x_j^{b_{ij}+1} \times \text{(holomorphic on } \mathcal{D}^{(j)} \text{)} & (m < n) \\
x_j^{b_{ij}} \times \text{(holomorphic on } \mathcal{D}^{(j)} \text{)} & (m \geq n)
\end{cases}
$$

where $\mathcal{D}^{(j)} = \{|x_i| < 1\} \cap \bigcap_{i \neq j} \{x_i \neq x_j\} \subset \mathbb{C}^N$.

Next we deal with the case where $i \neq j$. Applying $\pi_j^{-n}$ to the denominator and numerator of
yields

\[ \pi_j^{-n} \left( (x_i\tau_{L-1} - \tau_0) \prod_{k=1}^{L-1} (\tau_{k-1} - \tau_k) \right) \]

\[ = (x_i\tau_{L-n-1} - \tau_{-n}) \prod_{k=1}^{L-1} (\tau_{k-n-1} - \tau_{k-n}) \]

\[ = (x_i\tau_{L-n-1} - x_j\tau_{L-n}) (\tau_{-n} - \tau_{1-n}) \cdots (\tau_{2-n} - \tau_{1-n}) (\tau_{-n} - \tau_0) \cdots (\tau_{L-n-2} - \tau_{L-n-1}) \]

\[ \times \frac{1}{\tau_{m-n-1} - \tau_{m-n}} \]

\[ x_j^{n-2} (x_i\tau_{L-n-1} - x_j\tau_{L-n}) (x_j\tau_{L-n} - \tau_0) \prod_{k=1}^{L-1} (\tau_{k-1} - \tau_k) \quad (m < n) \]

\[ x_j^{n-1} (x_i\tau_{L-n-1} - x_j\tau_{L-n}) \prod_{k=1}^{L-1} (\tau_{k-1} - \tau_k) \quad (m = n) \]

\[ x_j^{n-1} (x_i\tau_{L-n-1} - x_j\tau_{L-n}) (x_j\tau_{L-n} - \tau_0) \prod_{k=1}^{L-1} (\tau_{k-1} - \tau_k) \quad (m > n) \]

and \( \pi_j^{-n}(\omega) = (-1)^{n(L-1)} x_j^n \omega \). Therefore, taking an integral of \( \pi_j^{-n}(V\psi_m^{(i)}) \) over \( \Delta_0 \) shows that

\[ \int_{\Delta_0} \pi_j^{-n}(V\psi_m^{(i)}) = \begin{cases} x_j^b \times (\text{holomorphic on } D^{(j)}) & (m < n) \\ x_j^{b+1} \times (\text{holomorphic on } D^{(j)}) & (m \geq n) \end{cases} \]

for \( i \neq j \) in the same manner as above.

We have verified that (4.1) certainly possesses the characteristic behavior specified in Theorem 1.3. \( \square \)

5 From hypergeometric equation \( \mathcal{P}_{L,N+1} \) to isomonodromic deformations

The subject of this section is a connection between the hypergeometric equation and isomonodromic deformations of a certain Fuchsian system, from which the hypergeometric solution of the Painlevé equation naturally arises as a by-product; cf. [7].

Consider the linear Pfaffian system \( \mathcal{P}_{L,N+1} \) of rank \( (N+1)(L-1) + 1 \). Suppose \( \beta_{N+1} = 0 \). It is then obvious that \( U = U(t) \) does not depend on \( x_{N+1} \); see (4.4). Accordingly, the \( N(L-1) + 1 \) functions

\[ y_0 = \int_\Delta U(t)\varphi_0, \quad y_n^{(i)} = \int_\Delta U(t)\varphi_n^{(i)} \quad \left( \begin{array}{c} 1 \leq i \leq N \\ 1 \leq n \leq L - 1 \end{array} \right) \]

do not depend on \( x_{N+1} \) and thus constitute a solution of \( \mathcal{P}_{L,N} \) if the domain \( \Delta \) of integration is suitably chosen. We are now interested in how the other \( L-1 \) functions \( y_n^{(N+1)} \) \( (1 \leq n \leq L-1) \) depend on \( x_{N+1} \).

We take the change of variables

\[ x_i = \frac{1}{u_i} \quad (1 \leq i \leq N), \quad x_{N+1} = \frac{1}{z} \]
and rewrite the constant parameters as
\[
\alpha_n = e_n - e_0, \quad \beta_i = -\theta_i, \quad \gamma_n = e_n - e_0 - \kappa_n \quad \left(\begin{array}{c}
1 \leq i \leq N \\
1 \leq n \leq L - 1
\end{array}\right)
\] (5.1)
for the sake of convenience. Let
\[
f_0 = 1, \quad f_n = -y_n^{(N+1)} \prod_{j=1}^{N} u_j^{\theta_j} \quad (1 \leq n \leq L - 1).
\]
Then, we verify from \(\mathcal{P}_{L,N+1}\) that \(f = (f_0, f_1, \ldots, f_{L-1})\) satisfies the Fuchsian system
\[
\frac{df}{dz} = A'f = \sum_{i=0}^{N+1} \frac{A_i}{z - u_i} f \quad (u_0 = 1, \ u_{N+1} = 0)
\] (5.2)
of ordinary differential equations with respect to \(z = 1/x_{N+1}\), whose coefficients are given by
\[
A'_0 = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
\kappa_1 h & -\kappa_1 & \cdots & -\kappa_1 \\
\vdots & \vdots & \ddots & \vdots \\
\kappa_{L-1} h & -\kappa_{L-1} & \cdots & -\kappa_{L-1}
\end{bmatrix}, \quad A'_i = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
\theta_i & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\theta_i & \cdots & \cdots & \cdots
\end{bmatrix}
\] (1 \leq i \leq N)
\[
A'_{N+1} = \begin{bmatrix}
0 & 0 & \cdots & 0 \\
e_1 - e_0 & \kappa_1 & \cdots & \cdots \\
e_2 - e_0 & \kappa_2 & \cdots & \cdots \\
e_3 - e_0 & \cdots & \cdots & \cdots \\
e_{L-1} - e_0 & \cdots & \cdots & \cdots
\end{bmatrix}
\]
with
\[
h = y_0 \prod_{j=1}^{N} u_j^{\theta_j}, \quad b^{(i)} \prod_{j=1}^{N} u_j^{\theta_j} \quad \left(\begin{array}{c}
1 \leq i \leq N \\
1 \leq n \leq L - 1
\end{array}\right).
\]
It goes without saying that \(5.2\) is reducible. More properly, its solution space turns out to be a direct sum of \(\mathbb{C}f\) and the solution space of \(\mathcal{P}_{L-1,1}\); see Remark 5.2 below.

Let \(Y'\) denote the fundamental system of solutions of \(5.2\). Its gauge transformation
\[
Y = \left(\varepsilon_0 \prod_{i=1}^{N} (z - u_i)^{-\theta_i} u_i^{\theta_i/L} \right) Y'
\]
then satisfies the Fuchsian system
\[
\frac{dY}{dz} = AY = \sum_{i=0}^{N+1} \frac{A_i}{z - u_i} Y
\] (5.3)
whose coefficients are given by $A_0 = A_0'$, $A_i = A'_i - \theta_i I_L$ ($1 \leq i \leq N$) and $A_{N+1} = A'_{N+1} + e_0 I_L$. We know a priori that the monodromy of $Y(z)$ is independent of variables $u_i$ $(1 \leq i \leq N)$; see Remark 1.1. Actually, we verify again from $\mathcal{P}_{L,N+1}$ that $Y$ satisfies the extended system

$$\frac{\partial Y}{\partial u_i} = B_i Y \quad (1 \leq i \leq N)$$

(5.4)

of linear differential equations with respect to $u_i$, whose coefficients are given by

$$B_i = \frac{1}{u_i - z} \begin{bmatrix} -\theta_i & 0 & \cdots & 0 \\ b_1^{(i)} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_L^{(i)} & \cdots & \cdots & 0 \\ \end{bmatrix} - \frac{1}{u_i} \begin{bmatrix} -\frac{\theta_i}{L} & \frac{\theta_i}{L} \\ b_1^{(i)} & \cdots & \cdots & \frac{\theta_i}{L} \\ \vdots & \vdots & \ddots & \vdots \\ b_L^{(i)} & \cdots & \cdots & \frac{\theta_i}{L} \\ \end{bmatrix}.$$

The coefficient $B_i$ is a rational function in $z$ and, thus, (5.4) describes the isomonodromic family of (5.3) along the deformation parameters $u_i$. Namely, (5.4) guarantees the existence of a fundamental system of solutions of (5.3) whose monodromy matrices do not depend on $u_i$; see [5].

The Riemann scheme of (5.3) is written as follows:

| Singularity | Characteristic exponents |
|-------------|--------------------------|
| $z = 0$     | $(e_0, e_1, \ldots, e_{L-1})$ |
| $z = \infty$| $(\kappa_0 - e_0, \kappa_1 - e_1, \ldots, \kappa_{L-1} - e_{L-1})$ |
| $z = u_0 = 1$| $(-\sum_{n=1}^{L-1} \kappa_n, 0, \ldots, 0)$ |
| $z = u_i$ $(1 \leq i \leq N)$ | $(-\theta_i, 0, \ldots, 0)$ |

(However, the relation $\kappa_0 = \sum_{i=1}^{N} \theta_i$ holds.) Accordingly, its spectral type reads

$$1, 1, \ldots, 1 \quad \text{at} \quad z = 0, \infty,$$

$$1, L - 1 \quad \text{at} \quad z = u_i \quad (0 \leq i \leq N).$$

In the general case, a Fuchsian system with the above spectral type is non-rigid (cf. Remark 1.1); in fact, it is equipped with $2N(L - 1)$ accessory parameters. The coefficients of such a Fuchsian system

$$\frac{dY}{dz} = \sum_{i=0}^{N+1} \mathcal{A}_i (z - u_i) Y$$

($\mathcal{L}_{L,N}$)

can be parametrized as

$$\mathcal{A}_i = \mathcal{T} \begin{pmatrix} b_1^{(i)} & \cdots & b_L^{(i)} \\ e_0 & \cdots & e_{L-1} \\ \end{pmatrix} \mathcal{T}^{-1} \begin{pmatrix} c_0^{(i)} & \cdots & c_{L-1}^{(i)} \\ e_0 & \cdots & e_{L-1} \\ \end{pmatrix} \quad (0 \leq i \leq N),$$

$$\mathcal{A}_{N+1} = \begin{pmatrix} e_0 & w_{0,1} & \cdots & w_{0,L-1} \\ e_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & w_{L-2,L-1} \\ e_{L-1} & \cdots & \cdots & 0 \\ \end{pmatrix}$$

with $w_{m,n} = -\sum_{i=0}^{N} b_m^{(i)} c_n^{(i)}$.

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where \(c_0^{(i)} = 1\), \(\text{tr} \mathcal{A}_i = \sum_{n=0}^{L-1} b_n^{(i)} c_n^{(i)} = -\theta_i\) and \(\sum_{n=0}^{N} b_n^{(i)} c_n^{(i)} = -\kappa_n\). We can and will normalize the characteristic exponents by

\[
\text{tr} \mathcal{A}_{N+1} = \sum_{n=0}^{L-1} e_n = \frac{L - 1}{2} \tag{5.5}
\]

without loss of generality. Assume the Fuchsian relation

\[
\sum_{n=0}^{L-1} \kappa_n = \sum_{i=0}^{N} \theta_i \tag{5.6}
\]

holds. As shown in [3], the isomonodromic deformations of \(\mathcal{L}_{L,N}\) are governed by the Hamiltonian system

\[
\frac{\partial q_n^{(i)}}{\partial x_j} = \frac{\partial H_j}{\partial p_n^{(i)}}, \quad \frac{\partial p_n^{(i)}}{\partial x_j} = -\frac{\partial H_j}{\partial q_n^{(i)}} \quad \left( 1 \leq i, j \leq N \right), \quad \left( 1 \leq n \leq L - 1 \right) \tag{\mathcal{H}_{L,N}}
\]

of partial differential equations with respect to variables \(x_i = 1/u_i\), whose Hamiltonian function \(H_i\) is defined by

\[
x_i H_i = \sum_{n=0}^{L-1} e_n q_n^{(i)} p_n^{(i)} + \sum_{j=0}^{N} \sum_{0 \leq m < n \leq L-1} q_m^{(i)} p_m^{(i)} q_n^{(i)} p_n^{(i)} + \sum_{j=0}^{N} \frac{x_j}{x_i - x_j} \sum_{m,n=0}^{L-1} q_m^{(i)} p_m^{(i)} q_n^{(i)} p_n^{(i)}
\]

with \(x_0 = q_0^{(0)} = 0, p_0^{(0)} = \kappa_n - \sum_{i=1}^{N} q_i^{(i)} p_i^{(i)}\) and \(p_0^{(i)} = \theta_i - \sum_{n=1}^{L-1} q_n^{(i)} p_n^{(i)}\). Thus, \(H_i\) becomes a polynomial in the \(2N(L - 1)\) unknowns (canonical coordinates)

\[
q_n^{(i)} = \frac{c_n^{(i)}}{c_n^{(0)}} \quad \text{and} \quad p_n^{(i)} = -b_n^{(i)} c_n^{(0)} \quad \left( 1 \leq i \leq N \right), \quad \left( 1 \leq n \leq L - 1 \right). \tag{5.7}
\]

The Hamiltonian system \(\mathcal{H}_{L,N}\) contains constant parameters

\[
(e, \kappa, \theta) = (e_0, \ldots, e_{L-1}, \kappa_0, \ldots, \kappa_{L-1}, \theta_0, \ldots, \theta_N),
\]

but their number is essentially \(2L + N - 1\) on account of (5.5) and (5.6). Note that \(\mathcal{H}_{L,N}\) is an extension of the sixth Painlevé equation, as it literally recovers the original if \((L, N) = (2, 1)\).

Interestingly enough, the previous system (5.3) coincides with a particular case of \(\mathcal{L}_{L,N}\) such that

\[
\kappa_0 = \sum_{i=1}^{N} \theta_i \quad \text{and} \quad b_0^{(i)} = 0, \quad b_0^{(i)} = -\theta_i, \quad c_0^{(i)} = \frac{-1}{h}, \quad c_n^{(i)} = 0 \quad \left( 1 \leq i \leq N \right), \quad \left( 1 \leq n \leq L - 1 \right).
\]

Hence, if we remember the algebraic relations (5.7) between the canonical coordinates of \(\mathcal{H}_{L,N}\) and the coefficients of its associated Fuchsian system \(\mathcal{L}_{L,N}\), then we derive directly from the above argument an \(N(L - 1)\)-parameter family of particular solutions of \(\mathcal{H}_{L,N}\); cf. [7, Theorem 3.2].

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Theorem 5.1. If $\kappa_0 = \sum_{i=1}^{N} \theta_i$, then the Hamiltonian system $H_{L,N}$ has a particular solution

$$q_n^{(i)} = 0 \quad \text{and} \quad p_n^{(i)} = \frac{y_n^{(i)}}{y_0},$$

where $\{y_0, y_0^{(i)}\}$ is an arbitrary solution of the linear Pfaffian system $P_{L,N}$ with (5.1).

Remark 5.2. A fundamental system $Y'$ of solutions of (5.2) can be taken of the form

$$Y' = \begin{bmatrix} f_0 & 0 & \cdots & 0 \\ f_1 & & & \\ \vdots & & & W \\ f_{L-1} & & & \end{bmatrix}.$$

The gauge transformation $Z = (z^{e_1-e_1+e_0} \prod_{i=1}^{N} (z-u_i)^{-\theta_i}) W$ satisfies the Fuchsian system

$$\frac{dZ}{dw} = \frac{1}{w} \begin{bmatrix} 0 & \kappa_2 & \kappa_2 - \kappa_1 - e_2 + e_1 & \cdots & \kappa_{L-1} - \kappa_{L-1} - e_{L-1} + e_1 \\ \kappa_3 & \kappa_3 & \kappa_3 - \kappa_1 - e_3 + e_1 & \cdots & \kappa_{L-1} - \kappa_{L-1} - e_{L-1} + e_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \kappa_{L-1} & \kappa_{L-1} & \cdots & \kappa_{L-1} - \kappa_{L-1} - e_{L-1} + e_1 \\ \end{bmatrix} Z + \frac{1}{1-w} \begin{bmatrix} \kappa_1 & \cdots & \kappa_1 \\ \vdots & \ddots & \vdots \\ \kappa_{L-1} & \cdots & \kappa_{L-1} \\ \end{bmatrix} Z$$

with $w = 1/z$, which is exactly the hypergeometric equation $P_{L-1,1}$ (for Thomae’s $L-1F_{L-2}$) under the correspondence of constant parameters as

$$\alpha_1 = \kappa_1 - e_1 + e_2, \quad \alpha_2 = \kappa_1 - e_1 + e_3, \quad \ldots, \quad \alpha_{L-2} = \kappa_1 - e_1 + e_{L-1}, \quad \beta = \kappa_1,$$

$$\gamma_1 = \kappa_1 - \kappa_2 - e_2 + e_2, \quad \gamma_2 = \kappa_1 - \kappa_3 - e_3 + e_3, \quad \ldots, \quad \gamma_{L-2} = \kappa_1 - \kappa_{L-1} - e_1 + e_{L-1}.$$

Cf. Sect. [1.2]

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