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Bismut Formula for Lions Derivative of Distribution Dependent SDEs and Applications∗

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Abstract

By using Malliavin calculus, Bismut type formulas are established for the Lions derivative of $P_t f(\mu) := \mathbb{E} f(X_t^\mu)$, where $t > 0$, $f$ is a bounded measurable function, and $X_t^\mu$ solves a distribution dependent SDE with initial distribution $\mu$. As applications, explicit estimates are derived for the Lions derivative and the total variational distance between distributions of solutions with different initial data. Both degenerate and non-degenerate situations are considered. Due to the lack of the semigroup property and the invalidity of the formula $P_t f(\mu) = \int P_t f(x) \mu(dx)$, essential difficulties are overcome in the study.

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1 Introduction

The Bismut formula introduced in [4], also called Bismut-Elworthy-Li formula due to [14], is a powerful tool in characterising the regularity of distribution for SDEs and SPDEs. A plenty of results have been derived for this type formulas and applications by using stochastic analysis and coupling methods, see for instance [27] and references therein.

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On the other hand, because of crucial applications in the study of nonlinear PDEs and environment dependent financial systems, the distribution dependent SDEs (also called McKean-Vlasov or mean field SDEs) have received increasing attentions, see [12, 13, 15, 16, 20, 25, 26] and references therein. Recently, this type SDEs have been applied in [6, 11, 19, 22] to characterize PDEs involving the Lions derivative (L-derivative for short) introduced by P.-L. Lions in his lectures [7]. Moreover, Harnack inequality, gradient estimates and exponential ergodicity have been investigated in [30] and [24]. In this paper, we aim to establish Bismut type L-derivative formula for distribution dependent SDEs with possibly degenerate noise.

To introduce our main results, we first recall the L-derivative. Let \( \mathcal{P}(\mathbb{R}^d) \) be the space of all probability measures on \( \mathbb{R}^d \), and let

\[
\mathcal{P}_2(\mathbb{R}^d) = \left\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \mu(|\cdot|^2) := \int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty \right\}.
\]

Then \( \mathcal{P}_2(\mathbb{R}^d) \) is a Polish space under the Wasserstein distance

\[
\mathbb{W}_2(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu, \nu)} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \pi(dx, dy) \right)^{\frac{1}{2}}, \quad \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d),
\]

where \( \mathcal{C}(\mu, \nu) \) is the set of couplings for \( \mu \) and \( \nu \); that is, \( \pi \in \mathcal{C}(\mu, \nu) \) is a probability measure on \( \mathbb{R}^d \times \mathbb{R}^d \) such that \( \pi(\cdot \times \mathbb{R}^d) = \mu \) and \( \pi(\mathbb{R}^d \times \cdot) = \nu \). We will use \( 0 \) to denote vectors with components 0, or the constant map taking value 0.

**Definition 1.1.** Let \( f : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \), and let \( g : M \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \) for a differentiable manifold \( M \).

1. \( f \) is called L-differentiable at \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \), if the functional

\[
L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu) \ni \phi \mapsto f(\mu \circ (\text{Id} + \phi)^{-1})
\]

is Fréchet differentiable at \( 0 \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu) \); that is, there exists (hence, unique) \( \gamma \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu) \) such that

\[
\lim_{\mu(|\phi|^2) \to 0} \frac{f(\mu \circ (\text{Id} + \phi)^{-1}) - f(\mu) - \mu(\gamma, \phi)}{\sqrt{\mu(|\phi|^2)}} = 0.
\]

In this case, we denote \( D^L f(\mu) = \gamma \) and call it the L-derivative of \( f \) at \( \mu \).

2. If the L-derivative \( D^L f(\mu) \) exists for all \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \), then \( f \) is called L-differentiable. If, moreover, for every \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \) there exists a \( \mu \)-version \( D^L f(\mu)(\cdot) \) such that \( D^L f(\mu)(x) \) is jointly continuous in \((x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\), we denote \( f \in C^{1,(1,0)}(\mathcal{P}_2(\mathbb{R}^d)).\)

3. \( g \) is called differentiable on \( M \times \mathcal{P}_2(\mathbb{R}^d) \), if for any \((x, \mu) \in M \times \mathcal{P}_2(\mathbb{R}^d) \), \( g(\cdot, \mu) \) is differentiable at \( x \) and \( g(x, \cdot) \) is L-differentiable at \( \mu \). If, moreover, \( \nabla g(\cdot, \mu)(x) \) and \( D^L g(x, \cdot)(\mu)(y) \) are jointly continuous in \((x, y, \mu) \in M^2 \times \mathcal{P}_2(\mathbb{R}^d)\), where \( \nabla \) is the gradient operator on \( M \), we write \( g \in C^{1,(1,0)}(M \times \mathcal{P}_2(\mathbb{R}^d)).\)
As indicated in [22] that for any $n \geq 1$, $g \in C^1(\mathbb{R}^n)$ and $h_1, \ldots, h_n \in C^1_b(\mathbb{R}^d)$, the cylindrical function
\[
\mu \mapsto g(\mu(h_1), \cdots, \mu(h_n))
\]
is in $C^{(1,0)}(\mathcal{P}_2(\mathbb{R}^d))$ with
\[
D^L g(\mu)(x) = \sum_{i=1}^n \langle \partial_i g(\mu(h_1), \cdots, \mu(h_n)) \rangle \nabla h_i(x), \ (x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d).
\]

Obviously, if $f$ is $L$-differentiable at $\mu$, then
\[
D^L_\phi f(\mu) := \lim_{\varepsilon \downarrow 0} \frac{f(\mu \circ (\text{Id} + \varepsilon \phi)^{-1}) - f(\mu)}{\varepsilon} = \mu(D^L f(\mu)), \ \phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu).
\]

We may call $D^L_\phi$ the directional $L$-derivative along $\phi$, which was introduced in [1, 23].

When $D^L_\phi f(\mu)$ is a bounded linear functional of $\phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$, there exists a unique $\xi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ such that $D^L_\phi f(\mu) = \mu(\langle \xi, \phi \rangle)$ holds for all $\phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$.

In this case, $\phi \mapsto f(\mu \circ (\text{Id} + \phi)^{-1})$ is Gâteaux differentiable at $0$, and we say that $f$ is weakly $L$-differentiable at $\mu$, since the Gâteaux differentiability is weaker than the Fréchet one.

By (1.2), for an $L$-differentiable function $f$ on $\mathcal{P}_2(\mathbb{R}^d)$, we have
\[
(D^L f(\mu)) := \sup_{\mu(\phi^2) \leq 1} |D^L_\phi f(\mu)|.
\]

For a vector-valued function $f = (f_i)$, or a matrix-valued function $f = (f_{ij})$ with $L$-differentiable components, we write
\[
(D^L_\phi f(\mu)) = (D^L_\phi f_i(\mu)), \quad (D^L_\phi f(\mu)) = (D^L_\phi f_{ij}(\mu)), \quad \mu \in \mathcal{P}_2(\mathbb{R}^d).
\]

Let $W_t$ be a $d$-dimensional Brownian motion on the natural filtered probability space $(\Omega^0, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. To ensure that for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ there exists a random variable $X$ on $\mathbb{R}^d$ with distribution $\mu$, let $\mu^0$ be a probability measure on $\mathbb{R}^d$ which is equivalent to the Lebesgue measure, and enlarge the probability space as
\[
(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) := (\Omega^0 \times \mathbb{R}^d, \mathcal{F}^0 \times \mathcal{B}(\mathbb{R}^d), \{\mathcal{F}_t^0 \times \mathcal{B}(\mathbb{R}^d)\}_{t \geq 0}, \mathbb{P}^0 \times \mu^0).
\]

Then
\[
W_t(\omega) := W_t(\omega^0), \quad t \geq 0, \omega := (\omega^0, x) \in \Omega
\]
is a $d$-dimensional Brownian motion on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. Let $\mathcal{L}_\xi$ denote the distribution of a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In case different probability spaces are concerned, we write $\mathcal{L}_{\xi|\mathbb{P}}$ instead of $\mathcal{L}_\xi$ to emphasize the reference probability measure $\mathbb{P}$.

Consider the following distribution dependent SDE on $\mathbb{R}^d$:
\[
dX_t = b_t(X_t, \mathcal{L}_{X_t})dt + \sigma_t(X_t, \mathcal{L}_{X_t})dW_t, \quad X_0 \in L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_0, \mathbb{P}),
\]
where
\[ \sigma: [0, \infty) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^{d \otimes d}, \quad b: [0, \infty) \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d \]
are continuous such that for some increasing function \( K: [0, \infty) \to [0, \infty) \) there holds
\[
|b_t(x, \mu) - b_t(y, \nu)| + \|\sigma_t(x, \mu) - \sigma_t(y, \nu)\|
\leq K(t)(|x - y| + \mathbb{W}_2(\mu, \nu)), \quad t \geq 0, x, y \in \mathbb{R}^d, \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)
\]
and
\[
\|\sigma_t(0, \delta_0)\| + |b_t(0, \delta_0)| \leq K(t), \quad t \geq 0,
\]
where and in what follows, for \( x \in \mathbb{R}^d \) we denote by \( \delta_x \) the Dirac measure at \( x \), and \( \| \cdot \| \) is the operator norm. For any \( t \geq 0 \), let \( L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_t, \mathbb{P}) \) be the class of \( \mathcal{F}_t \)-measurable square integrable random variables on \( \mathbb{R}^d \). By (1.5) and (1.6), for any \( s \geq 0 \) and \( X_s \in L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_s, \mathbb{P}) \), (1.4) has a unique solution \((X_{s,t})_{t \geq s}\) with \( X_{s,s} = X_s \) and
\[
E \left[ \sup_{t \in [s, T]} |X_{s,t}|^2 \right] < \infty, \quad T \geq s,
\]
see, for instance [30], where gradient estimates and Harnack inequalities are also derived for the associated nonlinear semigroup. See also [18, 20] for weaker conditions ensuring the existence and uniqueness of solutions to (1.4). For any \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \) and \( s \geq 0 \), let \((X^\mu_{s,t})_{t \geq s}\) be the solution to (1.4) with \( \mathcal{L}X_{s,s} = \mu \). Denote
\[
(P^\mu_{s,t})(\mu) = (P^\mu_{s,t})(f) := \int_{\mathbb{R}^d} f d(P^\mu_{s,t}\mu) = \mathbb{E} f(X^\mu_{s,t}), \quad t \geq s, f \in \mathcal{B}_b(\mathbb{R}^d), \mu \in \mathcal{P}_2(\mathbb{R}^d).
\]
Let
\[
(P_{s,t} f)(\mu) = (P^\mu_{s,t})(f) := \int_{\mathbb{R}^d} f d(P^\mu_{s,t}\mu) = \mathbb{E} f(X^\mu_{s,t}), \quad t \geq s, f \in \mathcal{B}_b(\mathbb{R}^d), \mu \in \mathcal{P}_2(\mathbb{R}^d).
\]
Then for any \( 0 \leq s \leq t \), \( P_{s,t} \) is a linear operator from \( \mathcal{B}_b(\mathbb{R}^d) \) to \( \mathcal{B}_b(\mathcal{P}_2(\mathbb{R}^d)) \).

In this paper, we aim to establish the Bismut type formula for the \( L \)-derivative of \( P_{s,t} f \) for \( t > s \). By considering the SDE for \( \dot{X}_t := X_{t+s}, t \geq 0 \), without loss of generality we may and do assume \( s = 0 \). So, for simplicity, below we only establish the derivative formula for \( P_t f := P_{0,t} f, t > 0 \). More precisely, for any \( T > 0, \mu \in \mathcal{P}_2(\mathbb{R}^d) \) and \( \phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu) \), we aim to construct an integrable random variable \( M^\mu_{T, f} \) such that
\[
D_{\phi}^L( P_T f)(\mu) = \mathbb{E} [ f(X^\mu_T) M^{\mu, \phi}_{T} ], \quad f \in \mathcal{B}_b(\mathbb{R}^d),
\]
which in turn implies the \( L \)-differentiability of \( P_T f \). Note that the derivative formula for \( (P_T f)(x) := (P_T f)(\delta_x) \) along a vector \( v \in \mathbb{R}^d \) is derived in [3], which is the special case of (1.10) with \( \mu = \delta_x \) and \( \phi \equiv v \). Moreover, formulas of the \( L \)-derivative and integration by parts have been presented in [9] for the following de-coupled SDE:
\[
dX^{x,\mu}_t = b(t, X^{x,\mu}_t, P^\mu_t) dt + \sigma(t, X^{x,\mu}_t, P^\mu_t) dW_t, \quad X^{x,\mu}_0 = x,
\]
which is different from the original SDE (1.4) but has important applications in solving PDEs with Lions' derivatives, see [6, 19, 22] and references within.

When the SDE (1.4) is distribution independent, i.e. \( b_t(x, \mu) = b_t(x) \) and \( \sigma_t(x, \mu) = \sigma_t(x) \) do not depend on \( \mu \), the Bismut type formula

\[
\nabla P_T f(x) = \mathbb{E}[f(X_T^x) M_T^x], \quad x \in \mathbb{R}^d, f \in \mathcal{B}_b(\mathbb{R}^d)
\]

has been well studied in the literature, where \( M_T^x \) is an integrable random variable on \( \mathbb{R}^d \), which is measurable in \( x \in \mathbb{R}^d \) when it varies, see for instance [2, 17, 28, 29, 31] and references within. Since the coefficients are distribution independent, we have

\[
(P_T f)(\mu) = \int_{\mathbb{R}^d} (P_T f)(x) \mu(dx),
\]

so that \( P_T f \) is \( L \)-differentiable with \( D^L(P_T f)(\mu) = \nabla P_T f \). Hence, by (1.11) and (1.12) we obtain

\[
D^L_{\phi}(P_T f)(\mu) = \mu(\langle D^L P_T f, \phi \rangle) = \int_{\mathbb{R}^d} \mathbb{E}[f(X_T^x)(M_T^x, \phi(x))] \mu(dx)
\]

\[
= \mathbb{E}[f(X_T^x)(M_T^{X_0^\mu}, \phi(X_0^\mu))].
\]

Therefore, (1.10) holds for \( M_{T_0}^{X_0^\mu, \phi} = \langle M_T^{X_0^\mu}, \phi(X_0^\mu) \rangle \).

However, when the SDE is distribution dependent, as explained in [30] that in general (1.12) does not hold, so it is non-trivial to establish the Bismut type formula (1.10).

The remainder of the paper is organized as follows. In section 2, we state our main results on Bismut formulas of \( D^L_{\phi} P_T f \) and applications, for both non-degenerate and degenerate distribution dependent SDEs. To establish the Bismut formula using Malliavin calculus, we make necessary preparations in Section 3 concerning partial derivatives in the initial value, and Malliavin derivative for solutions of (1.4). Finally, complete proofs of the main results are addressed in Section 4.

## 2 Main results

Let \( |\cdot| \) denote the Euclidean norm in \( \mathbb{R}^d \), and \( \|\cdot\| \) denote the operator norm for matrices or more generally linear operators. We make the following assumption.

\textbf{(H)} For any \( t \geq 0, b_t, \sigma_t \in C^{1, (1,0)}(\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)) \). Moreover, there exists a continuous function \( K : [0, \infty) \to [0, \infty) \), such that (1.6) holds and

\[
\max \left\{ \|\nabla b_t(\cdot, \mu)(x)\|, \|D^L b_t(x, \cdot)(\mu)\|, \frac{1}{2} \|\nabla \sigma_t(\cdot, \mu)(x)\|^2, \frac{1}{2} \|D^L \sigma_t(x, \cdot)(\mu)\|^2 \right\} \leq K_t, \quad t \geq 0, x \in \mathbb{R}^d, \mu \in \mathcal{P}_2(\mathbb{R}^d),
\]

where as in (1.3), \( \|D^L f(\mu)\| := \|D^L f(\mu)(\cdot)\|_{L^2(\mu)} \) for an \( L \)-differentiable function \( f \) at \( \mu \).
Obviously, \((H)\) implies (1.5) and (1.6), so that the SDE (1.4) has a unique solution for any initial value \(X_0 \in L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_0, \mathbb{P})\).

In the following two subsections, we state our main results for non-degenerate and degenerate cases respectively.

### 2.1 The non-degenerate case

Due to technical reasons, the following result Theorem 2.1 only works for distribution independent \(\sigma_t\). But some other results (for instance Proposition 3.2) apply to the general setting. So, in addition to (H) we also assume

\[
\sigma_t(x, \mu) = \sigma_t(x) \quad \text{with} \quad \|\sigma_t(x)^{-1}\| \leq \lambda_t \quad \text{for some} \quad \lambda \in C([0, \infty) \to (0, \infty)).
\]

(2.1)

Let \(\mu \in \mathcal{P}_2(\mathbb{R}^d)\), and let \(X_t\) solve (1.4) for \(X_0 \in L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_0, \mathbb{P})\) with \(\mathcal{L}_{X_0} = \mu\). Given \(\phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)\), consider the following SDE for \(v_t^\phi\) on \(\mathbb{R}^d\):

\[
\begin{align*}
\frac{d}{dt} v_t^\phi &= \left\{\nabla_{\phi} b_t(\cdot, \mathcal{L}_{X_t})(X_t) + \left(\mathbb{E}\langle D^2 b_t(y, \cdot)(\mathcal{L}_{X_t})(X_t), v_t^\phi \rangle \right|_{y=X_t}\right\} dt \\
&\quad + \left\{\nabla_{\phi} \sigma_t(X_t)\right\} dW_t, \quad v_0^\phi = \phi(X_0).
\end{align*}
\]

(2.2)

By (H), this linear SDE is well-posed with \(\sup_{t \in [0, T]} E|v_t^\phi|^2 \leq C \mu(|\phi|^2)\) for some constant \(C = C(T) > 0\), see (4.21) below. Denote \(g_s = \frac{d}{ds} g_s\) for a differentiable function \(g\) on \(s \in \mathbb{R}\).

**Theorem 2.1.** Assume (H) and (2.1). Then for any \(f \in \mathcal{B}_b(\mathbb{R}^d), \mu \in \mathcal{P}_2(\mathbb{R}^d)\) and \(T > 0\), \(P_T f\) is \(L\)-differentiable at \(\mu\) such that for any \(g \in C^1([0, T])\) with \(g_0 = 0\) and \(g_T = 1\),

\[
D^L_\phi (P_T f)(\mu) = \mathbb{E}\left[f(X_T) \int_0^T \langle \zeta_t^\phi, dW_t \rangle\right], \quad \phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu),
\]

(2.3)

where \(X_t\) solves (1.4) for \(\mathcal{L}_{X_0} = \mu\), and

\[
\zeta_t^\phi := \sigma_t(X_t)^{-1}\left\{g_t^\phi v_t^\phi + \left(\mathbb{E}\langle D^2 b_t(y, \cdot)(\mathcal{L}_{X_t})(X_t), g_t v_t^\phi \rangle \right|_{y=X_t}\right\}, \quad t \in [0, T].
\]

Moreover, the limit

\[
D^L_\phi P_T^* \mu := \lim_{\varepsilon \to 0} \frac{P^*_T \mu \circ (\text{Id} + \varepsilon \phi)^{-1} - P^*_T \mu}{\varepsilon} = \psi P^*_T \mu
\]

(2.4)

exists in the total variational norm, where \(\psi\) is the unique element in \(L^2(\mathbb{R}^d \to \mathbb{R}, P^*_T \mu)\) such that \(\psi(X_T) = \mathbb{E}\left(\int_0^T \langle \zeta_t^\phi, dW_t \rangle | X_T \right), \) and \((\psi P^*_T \mu)(A) := \int_A \psi dP^*_T \mu, \) \(A \in \mathcal{B}(\mathbb{R}^d)\).

**Remark 2.1.** When \(f \in C^1_b(\mathbb{R}^d)\), (2.3) can be proved as in the distribution independent case by constructing a proper random variable \(h\) on the Cameron-Martin space such that \(D^L h X_T = \nabla_\phi X_T\). However, for the \(L\)-differentiability of \(P_T f\), one has to construct \(\gamma \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)\) such that (1.1) holds for \(P_T f\) replacing \(f\), which is non-trivial.
Moreover, comparing with the classical case where (2.3) for \( f \in C^1_b(\mathbb{R}^d) \) can be easily extended to \( f \in B_b(\mathbb{R}^d) \), there is essential difficulty to do this in the distribution dependent setting. More precisely, when \( b_t \) and \( \sigma_t \) do not depend on the distribution, we have the semigroup property \( P_T f(\mu) = P_t (P_{t,T} f)(\mu) \) for \( t \in (0, T) \), where \( P_{t,T} f(x) := P_{t,T} f(\delta_x) \) for the Dirac measure \( \delta_x \) at point \( x \). In many cases, we have \( P_{t,T} f \in C^1_b(\mathbb{R}^d) \) for \( f \in B_b(\mathbb{R}^d) \). Then for any \( f \in B_b(\mathbb{R}^d) \), one may apply the derivative formula (2.3) with \( (P_t, P_{t,T} f) \) replacing \( (P_T, f) \) to derive a derivative formula for \( P_T f \). However, in the distribution dependent case, due to the lack of (1.12) we no longer have \( P_T f(\mu) = P_t (P_{t,T} f)(\mu) \), so that this argument becomes invalid. To overcome this difficulty we will make a new approximation argument, see step (a) in the proof of Theorem 2.1 for details.

As applications of Theorem 2.1, the following result consists of estimates on the \( L^2 \)-derivative and the total variational distance between distributions of solutions with different initial data.

**Corollary 2.2.** Assume \( (H) \) and (2.1) for some increasing functions \( K \) and continuous function \( \lambda \).

1. For any \( f \in B_b(\mathbb{R}^d) \) and \( T > 0 \),
   \[
   \|D^L(P_T f)(\mu)\|^2 := \sup_{\mu(\phi) \leq 1} |D^L \phi(P_T f)(\mu)|^2 \\
   \leq \{(P_T f)(\mu) - (P_T f(\mu))^2\} \int_0^T \left( \frac{1}{T} + K_t \right)^2 \lambda_t^2 e^{8K(t)} dt.
   \]

2. For any \( T > 0 \),
   \[
   |P_T f(\mu) - P_T f(\nu)|^2 \\
   \leq \|f\|^2 \mathbb{W}_2(\mu, \nu)^2 \int_0^T \left( \frac{1}{T} + K_t \right)^2 \lambda_t^2 e^{8K(t)} dt, \quad \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d), \ f \in B_b(\mathbb{R}^d).
   \]

Consequently, for any \( T > 0 \) and \( \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d) \),
\[
\|P_T^* \mu - P_T^* \nu\|_{\text{var}}^2 := \sup_{A \in \mathcal{B}(\mathbb{R}^d)} |(P_T^* \mu)(A) - (P_T^* \nu)(A)|^2 \\
\leq \mathbb{W}_2(\mu, \nu)^2 \int_0^T \left( \frac{1}{T} + K_t \right)^2 \lambda_t^2 e^{8K(t)} dt.
\]

### 2.2 Stochastic Hamiltonian systems

Consider the following distribution dependent stochastic Hamiltonian system for \( X_t = (X^{(1)}_t, X^{(2)}_t) \) on \( \mathbb{R}^{m+d} = \mathbb{R}^m \times \mathbb{R}^d \):
\[
\begin{align*}
\text{d}X^{(1)}_t &= b^{(1)}(X_t) dt, \\
\text{d}X^{(2)}_t &= b^{(2)}(X_t, \mathcal{L}X_t) dt + \sigma_t dW_t,
\end{align*}
\]

(2.8)
where \((W_t)_{t \geq 0}\) is a \(d\)-dimensional Brownian motion as before, and for each \(t \geq 0\), \(\sigma_t\) is an invertible \(d \times d\)-matrix,

\[
b_t = (b_t^{(1)}, b_t^{(2)}): \mathbb{R}^{m+d} \times \mathcal{P}_2(\mathbb{R}^{m+d}) \to \mathbb{R}^{m+d}
\]

is measurable with \(b_t^{(1)}(x, \mu) = b_t^{(2)}(x)\) independent of the distribution \(\mu\). Let \(\nabla = (\nabla^{(1)}, \nabla^{(2)})\) be the gradient operator on \(\mathbb{R}^{m+d} = \mathbb{R}^m \times \mathbb{R}^d\), where \(\nabla^{(i)}\) is the gradient in the \(i\)-th component, \(i = 1, 2\). Let \(\nabla^2 = \nabla \nabla\) denote the Hessian operator on \(\mathbb{R}^{m+d}\). We assume

(H1) For every \(t \geq 0\), \(b_t^{(1)} \in C^2_0([\mathbb{R}^{m+d} \to \mathbb{R}^d], b_t^{(2)} \in C^{1,1}(\mathbb{R}^{m+d} \times \mathcal{P}_2(\mathbb{R}^{m+d}) \to \mathbb{R}^d)\), and there exists an increasing function \(K: [0, \infty) \to [0, \infty)\) such that (1.6) and

\[
\|\nabla b_t(\cdot, \mu)(x)\| + \|D^2 b_t^{(1)}(x, \cdot)(\mu)\| + \|\nabla^2 b_t^{(1)}(\cdot, \mu)(x)\| \leq K(t)
\]

hold for all \(t \geq 0\), \((x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)\).

Obviously, this assumption implies (H) for the SDE (2.8). We aim to establish the derivative formula of type (1.10) with \(P_t\) and \(P^*_t\) being defined by (1.8) and (1.9) for the SDE (2.8). To follow the line of [31] where the distribution independent model was investigated, we need the following assumption (H2).

For any \(s \geq 0\), let \(\{K_{t,s}\}_{t \geq s}\) solve the following linear random ODE on \(\mathbb{R}^{m \otimes m}\):

\[
d_{t}K_{t,s} = (\nabla^{(1)} b^{(1)})(X_t)K_{t,s}, \quad t \geq s, K_{s,s} = I_{m \times m},
\]

where \(I_{m \times m}\) is the \(m \times m\)-order identity matrix.

(H2) There exists \(B \in \mathcal{B}_b([0, T] \to \mathbb{R}^{m \otimes d})\) such that

\[
\langle (\nabla^{(2)} b^{(1)}_t - B_t)B^*_s a, a \rangle \geq -\varepsilon |B^*_s a|^2, \quad \forall a \in \mathbb{R}^m
\]

holds for some constant \(\varepsilon \in [0, 1)\). Moreover, there exists an increasing function \(\theta \in C([0, T])\) with \(\theta_t > 0\) for \(t \in (0, T]\) such that

\[
\int_0^t s(T - s)K_{T,s}B_sB^*_sK^*_s B^*_t ds \geq \theta_t I_{m \times m}, \quad t \in (0, T].
\]

Example 2.1. Let

\[
b_t^{(1)}(x) = Ax^{(1)} + Bx^{(2)}, \quad x = (x^{(1)}, x^{(2)}) \in \mathbb{R}^{m+d}
\]

for some \(m \times m\)-matrix \(A\) and \(m \times d\)-matrix \(B\). If the Kalman’s rank condition

\[
\text{Rank}[B, AB, \cdots, A^kB] = m
\]

holds for some \(k \geq 1\), then (H2) is satisfied with \(\theta_t = c_T t\) for some constant \(c_T > 0\), see the proof of [31, Theorem 4.2]. In general, (H2) remains true under small perturbations of this \(b_t^{(1)}\).
According to the proof of \cite[Theorem 1.1]{31}, (H2) implies that the matrices

\[ Q_t := \int_0^t s(T-s)K_{T,s}\nabla_0^{(2)}b^{(1)}(X_s)B^*_sK^*_{T,s}ds, \quad t \in (0, T] \]

are invertible with

\begin{equation}
\|Q_t^{-1}\| \leq \frac{1}{(1-\varepsilon)\theta_t}, \quad t \in (0, T].
\end{equation}

For \((X_t)_{t \in [0,T]}\) solving (2.8) with \(LX_0 = \mu\) and \(\phi = (\phi^{(1)}, \phi^{(2)}) \in L^2(\mathbb{R}^{m+d} \to \mathbb{R}^{m+d}, \mu)\), let

\begin{align*}
\alpha_t^{(2)} &= \frac{T-t}{T}\phi^{(2)}(X_0) - \frac{t(T-t)}{T}B^*_tK^*_{T,t}\int_T^T \theta_2^2 Q^{-1}_s K_{T,0}\phi^{(1)}(X_0)ds \\
&\quad - t(T-t)B^*_tK^*_{T,t}Q^{-1}_t \int_T^T \frac{T-s}{T}K_{T,s}\nabla_0^{(2)}\phi^{(2)}(X_0)b^{(1)}_s(X_s)ds, \quad t \in [0, T],
\end{align*}

and

\begin{equation}
\alpha_t^{(1)} = K_t,0\phi^{(1)}(X_0) + \int_0^t K_{t,s}\nabla_0^{(2)}b^{(1)}_s(X_s(x))\, ds, \quad t \in [0, T].
\end{equation}

Moreover, let \((h^\alpha_t, w^\alpha_t)_{t \in [0,T]}\) be the unique solution to the random ODEs

\begin{align}
\frac{dh^\alpha_t}{dt} &= \sigma_t^{-1}\{\nabla_\alpha_t b^{(2)}_t(X_t, LX_t) - (\alpha_t^{(2)})' \} \\
&\quad + (\mathbb{E}(DL^2b^{(2)}_t(y, \cdot)(LX_t)(X_t), \alpha_t + w^\alpha_t))|_{y = X_t}, \\
\frac{dw^\alpha_t}{dt} &= \nabla w^\alpha_t b_t(\cdot, LX_t)(X_t) + (0, \sigma_t(h^\alpha_t)'), \quad h^\alpha_0 = w^\alpha_0 = 0.
\end{align}

Let \((D^*, \mathcal{D}(D^*))\) be the Malliavin divergence operator associated with the Brownian motion \((W_t)_{t \in [0,T]}\), see Subsection 3.2 below for details. Then the main result in this part is the following.

**Theorem 2.3.** Assume (H1) and (H2). Then \(h^\alpha \in \mathcal{D}(D^*)\) with \(\mathbb{E}|D^*(h^\alpha)|^p < \infty\) for all \(p \in [1, \infty)\). Moreover, for any \(f \in \mathcal{B}_b(\mathbb{R}^{m+d})\) and \(T > 0\), \(P_T^*f\) is \(L\)-differentiable at \(\mu\) such that

\begin{equation}
D^*_\mu(P_T^*f)(\mu) = \mathbb{E}\left[f(X_T)\right. D^*(h^\alpha)].
\end{equation}

Consequently:

1. (2.4) holds for the unique \(\psi \in L^2(\mathbb{R}^{m+d} \to \mathbb{R}, P^*_T\mu)\) such that \(\psi(X_T) = \mathbb{E}(D^*(h^\alpha)|X_T)\).

2. There exists a constant \(c \geq 0\) such that for any \(T > 0\),

\begin{equation}
\|D^L(P_Tf)(\mu)\| \leq c\sqrt{P_T|f|^2(\mu)} - (P_Tf)^2(\mu)\frac{\sqrt{T}(T^2 + \theta_T)}{\int_0^T \theta_s^2 ds}, \quad f \in \mathcal{B}_b(\mathbb{R}^{m+d}),
\end{equation}

\begin{equation}
\|P^*_T\mu - P^*_T\nu\|_{var} \leq c\mathbb{W}_2(\mu, \nu)\frac{\sqrt{T}(T^2 + \theta_T)}{\int_0^T \theta_s^2 ds}, \quad \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d).
\end{equation}
3 Preparations

We first introduce a formula of the $L$-derivative re-organized from [7, Theorem 6.5] and [11, Proposition A.2], then investigate the partial derivatives of $X_t$ in the initial value, and the Malliavin derivatives of $X_t$ with respect to the Brownian motion $W_t$.

3.1 A formula of $L$-derivative

The following result is essentially due to [7, Theorem 6.5] for $f \in C^{(1,0)}(\mathcal{P}_2(\mathbb{R}^d))$, and [11, Proposition A.2] for bounded $X$ and $Y$. We include a complete proof for readers’ convenience.

**Proposition 3.1.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space, and let $X, Y \in L^2(\Omega \to \mathbb{R}^d, \mathbb{P})$ with $\mathcal{L}_X = \mu$. If either $X$ and $Y$ are bounded and $f$ is $L$-differentiable at $\mu$, or $f \in C^{(1,0)}(\mathcal{P}_2(\mathbb{R}^d))$, then

$$
\lim_{\varepsilon \downarrow 0} \frac{f(\mathcal{L}_{X+\varepsilon Y}) - f(\mu)}{\varepsilon} = \mathbb{E}\langle D^L f(\mu)(X), Y \rangle.
$$

Consequently,

$$
\left| \lim_{\varepsilon \downarrow 0} \frac{f(\mathcal{L}_{X+\varepsilon Y}) - f(\mu)}{\varepsilon} \right| = \left| \mathbb{E}\langle D^L f(\mu)(X), Y \rangle \right| \leq \|D^L f(\mu)\| \sqrt{\mathbb{E}|Y|^2}.
$$

\textbf{Proof.} It is easy to see that (3.2) follows from (1.3) and (3.1). Indeed, letting $\phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu)$ such that $\phi(X) = \mathbb{E}(Y|X)$, we have

$$
\left| \mathbb{E}\langle D^L f(\mu)(X), Y \rangle \right| = \left| \mathbb{E}\langle D^L f(\mu)(X), \phi(X) \rangle \right| = \left| \mu(\langle D^L f(\mu), \phi \rangle) \right|
$$

$$
\leq \|D^L f(\mu)\| \cdot \|\phi\|_{L^2(\mu)} = \|D^L f(\mu)\| \left( \mathbb{E}|\mathbb{E}(Y|X)|^2 \right)^{\frac{1}{2}} \leq \|D^L f(\mu)\| \sqrt{\mathbb{E}|Y|^2}.
$$

Below we prove (3.1) for the stated two situations respectively.

(1) Assume that $X$ and $Y$ are bounded. For any $\mathbb{R}^d$-valued random variable $\xi$, let $F(\xi) = f(\mathcal{L}_\xi)$. Next, let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless Polish probability space, and let $X \in L^2(\Omega \to \mathbb{R}^d, \mathbb{P})$ with $\mathcal{L}_X = \mu$, where $\mathcal{L}_Y$ denotes the distribution of a random variable under $\mathbb{P}$. According to [11, Proposition A.2(iii)], if

$$
\bar{F}(\bar{Y}) := f(\mathcal{L}_Y), \quad \bar{Y} \in L^2(\Omega \to \mathbb{R}^d, \mathbb{P})
$$

is Fréchet differentiable at $\bar{X}$ with derivative $D\bar{F}(\bar{X}) = D^L f(\mu)(\bar{X})$, then

$$
\lim_{\varepsilon \downarrow 0} \frac{f(\mathcal{L}_{X+\varepsilon Y}) - f(\mu) - \varepsilon \mathbb{E}\langle D^L f(\mu)(X), Y \rangle}{\varepsilon} = 0.
$$

Equivalently, (3.1) holds. Below we construct the desired $\bar{X}$ and $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ such that $D\bar{F}(\bar{X}) = D^L f(\mu)(\bar{X})$. 

10
A natural choice of \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{P}})\) is \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu)\), but to ensure the atomless property, we take \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{P}}) = (\mathbb{R}^d \times \mathbb{R}, \mathcal{B}(\mathbb{R}^d \times \mathbb{R}), \mu \times \lambda)\), where \(\lambda\) is the standard Gaussian measure on \(\mathbb{R}\). Then \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{P}})\) is an atomless Polish probability space. Let
\[\bar{X}(\omega) = x, \quad \bar{\omega} = (x, r) \in \mathbb{R}^d \times \mathbb{R}.\]

We have \(\mathcal{L}_{\bar{X}} = \mu\). Moreover, let
\[\tilde{f}(\tilde{\mu}) = f(\tilde{\mu}(\cdot \times \mathbb{R})), \quad \tilde{\mu} \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}).\]

It is easy to see that the \(L\)-differentiability of \(f\) at \(\mu\) implies that of \(\tilde{f}\) at \(\mu \times \delta_0\) with
\[
D^L \tilde{f}(\mu \times \delta_0)(x, r) = (D^L f(\mu)(x), 0), \quad (x, r) \in \mathbb{R}^d \times \mathbb{R}.
\]

Finally, on the probability space \((\Omega, \mathcal{F}, \mathcal{P})\) we have
\[
F(Y) := f(\mathcal{L}_Y) = \tilde{f}(\mathcal{L}_{\bar{Y}}), \quad \bar{Y} := (Y, 0) \in L^2(\Omega \to \mathbb{R}^d \times \mathbb{R}, \mathcal{F}, \mathcal{P}).
\]

Letting \(\bar{X} = (X, 0) \in L^2(\Omega \to \mathcal{T}^d \times \mathbb{R}, \mathcal{F}, \mathcal{P})\), by [11, Proposition A.2(iii)], the formula (3.3) holds for \((\bar{X}, \bar{Y}, \tilde{f}, \mu \times \delta_0)\) replacing \((X, Y, f, \mu)\), i.e.
\[
\lim_{\varepsilon \downarrow 0} \frac{\tilde{f}(\mathcal{L}_{\bar{X}+\varepsilon \bar{Y}}) - \tilde{f}(\mathcal{L}_{\bar{X}}) - \mathbb{E}(D^L \tilde{f}(\mu \times \delta_0), \varepsilon \bar{Y})}{\varepsilon} = 0.
\]

Combining this with (3.4) and (3.5), we prove (3.3). Therefore, (3.1) holds.

(2) Let \(f \in C^{(1,0)}(\mathcal{P}_2(\mathbb{R}^d))\) and let \(\mu \in \mathcal{P}_2(\mathbb{R}^d)\) and \(X \in L^2(\Omega \to \mathbb{R}^d, \mathcal{P})\) with \(\mathcal{L}_X = \mu\). For any \(n \geq 1\), let
\[x_n = \frac{x}{\sqrt{1 + n^{-1}|x|^2}}, \quad x \in \mathbb{R}^d.
\]

By (3.1) for bounded \(X\) and \(Y\), for any \(n \geq 1\) we have
\[
f(\mathcal{L}_{X_n + sY_n}) - f(\mathcal{L}_{X_n}) = \int_0^\varepsilon \frac{d}{ds} f(\mathcal{L}_{X_n + sY_n}) \, ds
\]
\[
= \int_0^\varepsilon \mathbb{E}(D^L f(\mathcal{L}_{X_n + sY_n})(X_n + sY_n), Y_n) \, ds.
\]

Since \(f \in C^{(1,0)}(\mathcal{P}_2(\mathbb{R}^d))\), it follows that
\[
\sup_{n \geq 1, s \in [0, \varepsilon]} \|D^L f(\mathcal{L}_{X_n + sY_n})\| < \infty, \quad \lim_{n \to \infty} \left\{ f(\mathcal{L}_{X_n + \varepsilon Y_n}) - f(\mathcal{L}_{X_n}) \right\} = f(\mathcal{L}_{X + \varepsilon Y}) - f(\mathcal{L}_X),
\]

and for any \(s \in [0, \varepsilon]\),
\[
\lim_{n \to \infty} \mathbb{E}(|X - X_n|^2 + |Y - Y_n|^2 + |D^L f(\mathcal{L}_{X_n + sY_n})(X_n + sY_n) - D^L f(\mathcal{L}_{X + sY})(X + sY)|^2) = 0.
\]

Then letting \(n \to \infty\) in (3.6) we arrive at
\[
f(\mathcal{L}_{X+\varepsilon Y}) - f(\mathcal{L}_X) = \int_0^\varepsilon \mathbb{E}(D^L f(\mathcal{L}_{X+sY})(X+sY), Y) \, ds, \quad \varepsilon > 0.
\]
This implies (3.1). More precisely, it is easy to see that \( \{L_X + sY\} \) is compact in \( \mathcal{P}_2(\mathbb{R}^d) \). So, 
\[
f \in C([0,1], \mathcal{P}_2(\mathbb{R}^d)) \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} D^\varepsilon f(L_X + sY) = Df(L_X)(X) \quad \text{weakly in} \quad L^2(\Omega \to \mathbb{R}^d, \mathbb{P}).
\]

Combining this with the continuity property of \( D^\varepsilon f \) on \( \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \), we conclude that 
\[
\lim_{\varepsilon \downarrow 0} D^\varepsilon f(L_X + sY)(X + sY) = Df(L_X)(X) \quad \text{weakly in} \quad L^2(\Omega \to \mathbb{R}^d, \mathbb{P}).
\]

In particular, 
\[
\lim_{\varepsilon \downarrow 0} \mathbb{E}(D^\varepsilon f(L_X + sY)(X + sY), Y) = \mathbb{E}(Df(L_X)(X), Y).
\]

Moreover, (3.8) implies 
\[
\sup_{s \in [0,1]} \mathbb{E}|(D^\varepsilon f(L_X + sY)(X + sY), Y)| \leq A\mathbb{E}|Y|^2 < \infty.
\]

Due to this, (3.7) and (3.9), the dominated convergence theorem gives 
\[
\lim_{\varepsilon \downarrow 0} \frac{f(L_X + sY) - f(L_X)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon \mathbb{E}(D^\varepsilon f(L_X + sY)(X + sY), Y) \, ds 
= \mathbb{E}(Df(L_X)(X), Y).
\]

\[ \square \]

### 3.2 Partial derivative in initial value

For any \( T > 0 \), let \( \mathcal{C}_T = C([0,T] \to \mathbb{R}^d) \) be the path space over \( \mathbb{R}^d \) with time interval \([0,T]\), and let \( X_0, \eta \in L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_0, \mathbb{P}) \). For any \( \varepsilon \geq 0 \), let \( (X^\varepsilon_t)_{t \geq 0} \) solve the SDE 
\[
dX^\varepsilon_t = b_t(X^\varepsilon_t, L_{X^\varepsilon_t}) \, dt + \sigma_t(X^\varepsilon_t, L_{X^\varepsilon_t}) \, dW_t, \quad X^\varepsilon_0 = X_0 + \varepsilon \eta.
\]

Obviously, \( X_t = X^0_t \) solves (1.4) with initial value \( X_0 \). Consider the following linear SDE for \( v^n_t \) on \( \mathbb{R}^d \):
\[
dv^n_t = \left\{ \nabla v^n_t b_t(\cdot, L_{X_t})(X_t) + \left( \mathbb{E}(D^\varepsilon b_t(y, \cdot)(L_{X_t})(X_t), v^n_t) \right) \bigg|_{y=X_t} \right\} \, dt 
+ \left\{ \nabla v^n_t \sigma_t(\cdot, L_{X_t})(X_t) + \left( \mathbb{E}(D^\varepsilon \sigma_t(y, \cdot)(L_{X_t})(X_t), v^n_t) \right) \bigg|_{y=X_t} \right\} \, dW_t, \quad v^n_0 = \eta.
\]

The main result of this part is the following.

**Proposition 3.2.** Assume (H). Then for any \( T > 0 \), the limit 
\[
\nabla_\eta X_t := \lim_{\varepsilon \downarrow 0} \frac{X^\varepsilon_t - X_t}{\varepsilon}, \quad t \in [0,T]
\]
exists in \( L^2(\Omega \to \mathcal{C}_T, \mathbb{P}) \). Moreover, \( (v^n_t := \nabla_\eta X_t)_{t \in [0,T]} \) is the unique solution to the linear SDE (3.11).
To prove the existence of $\nabla_{\eta}X_t$ in (3.12), it suffices to show that when $\varepsilon \downarrow 0$

\begin{equation}
(3.13) \quad \xi^\varepsilon(t) := \frac{X^\varepsilon_t - X_t}{\varepsilon}, \quad t \in [0, T]
\end{equation}

is a Cauchy sequence in $L^2(\Omega \to \mathcal{C}_T, \mathbb{P})$, i.e.

\begin{equation}
(3.14) \quad \lim_{\varepsilon, \delta \downarrow 0} \mathbb{E} \left[ \sup_{t \in [0, T]} |\xi^\varepsilon(t) - \xi^\delta(t)|^2 \right] = 0.
\end{equation}

To this end, we need the following two lemmas.

**Lemma 3.3.** Assume (H). Then

\[ \sup_{\varepsilon \in (0, 1]} \mathbb{E} \left[ \sup_{t \in [0, T]} |\xi^\varepsilon(t)|^2 \right] < \infty. \]

**Proof.** By (H), there exists a constant $C_1 > 0$ such that

\begin{align*}
d|X^\varepsilon_t - X_t|^2 &= \{2\langle b_t(X^\varepsilon_t, \mathcal{L}_{X^\varepsilon_t}) - b_t(X_t, \mathcal{L}_{X_t}), X^\varepsilon_t - X_t \rangle + \|\sigma_t(X^\varepsilon_t, \mathcal{L}_{X^\varepsilon_t}) - \sigma_t(X_t, \mathcal{L}_{X_t})\|_{HS}^2 \} dt + dM_t \\
&\leq C_1 \{ |X^\varepsilon_t - X_t|^2 + \mathbb{W}_2(\mathcal{L}_{X^\varepsilon_t}, \mathcal{L}_{X_t})^2 \} dt + dM_t,
\end{align*}

where

\[ dM_t := 2\langle X^\varepsilon_t - X_t, (\sigma_t(X^\varepsilon_t, \mathcal{L}_{X^\varepsilon_t}) - \sigma_t(X_t, \mathcal{L}_{X_t}))dW_t \rangle \]

satisfies

\begin{equation}
(3.15) \quad d\langle M \rangle_t \leq C_1^2 \{ |X^\varepsilon_t - X_t|^2 + \mathbb{W}_2(\mathcal{L}_{X^\varepsilon_t}, \mathcal{L}_{X_t})^2 \}^2 dt.
\end{equation}

Then by the Burkholder-Davis-Gundy inequality, and noting that $\mathbb{W}_2(\mathcal{L}_{\xi}, \mathcal{L}_{\eta})^2 \leq \mathbb{E}|\xi - \eta|^2$ for two random variables $\xi, \eta$, we may find out a constant $C_2 > 0$ such that

\begin{equation}
(3.16) \quad \mathbb{E} \left[ \sup_{s \in [0, t]} |X^\varepsilon_s - X_s|^2 \right] \leq \varepsilon^2|\eta|^2 + 2C_1 \int_0^t \mathbb{E}|X^\varepsilon_s - X_s|^2 ds + C_2 \mathbb{E}\sqrt{\langle M \rangle_t}.
\end{equation}

Noting that $\mathbb{W}_2(\mathcal{L}_{X^\varepsilon_s}, \mathcal{L}_{X_s})^2 \leq \mathbb{E}|X^\varepsilon_s - X_s|^2$, (3.15) yields

\begin{align*}
C_2 \mathbb{E}\sqrt{\langle M \rangle_t} &\leq C_1 C_2 \mathbb{E} \left( \int_0^t \{ |X^\varepsilon_s - X_s|^2 + \mathbb{W}_2(\mathcal{L}_{X^\varepsilon_s}, \mathcal{L}_{X_s})^2 \}^2 ds \right)^{\frac{1}{2}} \\
&\leq C_1 C_2 \mathbb{E} \left( \sup_{s \in [0, t]} \{ |X^\varepsilon_s - X_s|^2 + \mathbb{E}|X^\varepsilon_s - X_s|^2 \} \int_0^t \{ |X^\varepsilon_s - X_s|^2 + \mathbb{E}|X^\varepsilon_s - X_s|^2 \} ds \right)^{\frac{1}{2}} \\
&\leq \frac{1}{2} \mathbb{E} \left[ \sup_{s \in [0, t]} |X^\varepsilon_s - X_s|^2 \right] + \frac{C_3}{2} \int_0^t \mathbb{E}|X^\varepsilon_s - X_s|^2 ds
\end{align*}
for some constant $C_3 > 0$. Combining this with (3.16) and noting that due to (1.7)
\[
\mathbb{E} \left[ \sup_{s \in [0,t]} |X_s^\varepsilon - X_s| \right] < \infty,
\]
we arrive at
\[
\mathbb{E} \left[ \sup_{s \in [0,t]} |X_s^\varepsilon - X_s|^2 \right] \leq 2\varepsilon^2 |\eta|^2 + C_3 \int_0^t \mathbb{E} |X_s^\varepsilon - X_s|^2 \, ds, \quad t \in [0,T], \varepsilon > 0.
\]
Therefore, Gronwall’s inequality gives
\[
\sup_{\varepsilon \in (0,1]} \mathbb{E} \left[ \sup_{t \in [0,T]} |\xi^\varepsilon(t)|^2 \right] = \sup_{\varepsilon \in (0,1]} \frac{1}{\varepsilon^2} \mathbb{E} \left[ \sup_{s \in [0,T]} |X_s^\varepsilon - X_s|^2 \right] \leq 2e^{C_3T} \mathbb{E} |\eta|^2 < \infty.
\]

\[\square\]

For any differentiable (real, vector, or matrix valued) function $f$ on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, let
\[
\Xi_f^\varepsilon(t) = \frac{f(X_t^\varepsilon, \mathcal{L}X_t^\varepsilon) - f(X_t, \mathcal{L}X_t)}{\varepsilon} - \left\{ \mathbb{E} \langle D^L f(y, \cdot)(\mathcal{L}X_t)(X_t) , \xi^\varepsilon(t) \rangle \big|_{y = X_t} \right\}, \quad t \in [0,T], \varepsilon > 0.
\]

**Lemma 3.4.** Assume (H). For any (real, vector, or matrix valued) $C^{1,(1,0)}$-function $f$ on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$ with
\[
K_f := \sup_{(x,\mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)} \left( \|\nabla f(\cdot, \mu)(x)\|^2 + \|D^L f(x, \cdot)(\mu)\|^2_{L^2(\mu)} \right) < \infty,
\]
there holds
\[
\Xi_f^\varepsilon(t)^2 \leq 4K_f (\mathbb{E} |\xi^\varepsilon(t)|^2 + |\xi^\varepsilon(t)|^2) \quad \text{and} \quad \lim_{\varepsilon \downarrow 0} \mathbb{E} \Xi_f^\varepsilon(t)^2 = 0, \quad t \in [0,T].
\]

**Proof.** Let $X_t^\varepsilon(s) = X_t + s(X_{t+s}^\varepsilon - X_t)$, $s \in [0,1]$. By the chain rule and (3.1), we have
\[
\frac{f(X_t^\varepsilon, \mathcal{L}X_t^\varepsilon) - f(X_t, \mathcal{L}X_t)}{\varepsilon} = \frac{1}{\varepsilon} \int_0^1 \left\{ \frac{d}{ds} f(X_t^\varepsilon(s), \mathcal{L}X_t^\varepsilon(s)) \right\} \, ds
\]
\[
= \int_0^1 \left\{ \nabla \xi^\varepsilon(t)(\cdot, \mathcal{L}X_t^\varepsilon(s))(X_t^\varepsilon(s)) + \mathbb{E} \langle D^L f(y, \cdot)(\mathcal{L}X_t^\varepsilon(s))(X_t^\varepsilon(s)) , \xi^\varepsilon(t) \rangle \big|_{y = X_t^\varepsilon(s)} \right\} \, ds.
\]
Combining this with (3.18) we obtain
\[
|\Xi_f^\varepsilon(t)|^2 \leq 2 \int_0^1 \left| \nabla \xi^\varepsilon(t) \left\{ f(\cdot, \mathcal{L}X_t^\varepsilon(s))(X_t^\varepsilon(s)) - f(\cdot, \mathcal{L}X_t)(X_t) \right\} \right|^2 \, ds
\]
\[
+ 2 \int_0^1 \left| \left( \mathbb{E} \langle D^L f(y, \cdot)(\mathcal{L}X_t^\varepsilon(s))(X_t^\varepsilon(s)) , \xi^\varepsilon(t) \rangle \big|_{y = X_t^\varepsilon(s)} \right) - \left( \mathbb{E} \langle D^L f(y, \cdot)(\mathcal{L}X_t)(X_t) , \xi^\varepsilon(t) \rangle \big|_{y = X_t} \right) \right|^2 \, ds
\]
\[
\leq 8K_f (|\xi^\varepsilon(t)|^2 + \mathbb{E} |\xi^\varepsilon(t)|^2).
\]
So, the first inequality in (3.19) holds. Moreover, Lemma 3.3 implies
\[
\lim_{\varepsilon \downarrow 0} E \left[ \sup_{s \in [0,1]} |X_\varepsilon^{\varepsilon}(s) - X_t| \right] \leq \lim_{\varepsilon \downarrow 0} E|X_\varepsilon^\varepsilon - X_t|^2 = 0.
\]
Thus, the \(C^{1,1}(0)\)-property of \(f\); Lemma 3.3 and the first inequality in (3.20) yield that \(\Xi_\varepsilon^{\varepsilon}(t) \rightarrow 0\) in probability as \(\varepsilon \rightarrow 0\). Combining this with the first inequality in (3.19), Lemma 3.3, and using the dominated convergence theorem, we derive \(\lim_{\varepsilon \downarrow 0} E|\Xi_\varepsilon^{\varepsilon}(t)|^2 = 0\). \(\square\)

**Proof of Proposition 3.2.** Let \((\Xi^{\varepsilon}_0(t), K_{b_{\varepsilon}})\) and \((\Xi^{\varepsilon}_0(t), K_{\sigma_{\varepsilon}})\) be defined as in (3.17) and (3.18) for \(b_{\varepsilon}\) and \(\sigma_{\varepsilon}\) replacing \(f\) respectively. By (H), there exists a constant \(C_1 > 0\) such that
\[
\sup_{t \in [0,T]} (K_{b_{\varepsilon}} + K_{\sigma_{\varepsilon}}) \leq C_1 < \infty.
\]
Then Lemma 3.4 gives
\[
|\Xi_\varepsilon^{\varepsilon}(t)|^2 + |\Xi_\sigma^{\varepsilon}(t)|^2 \leq 4C(|\xi^{\varepsilon}(t)|^2 + E|\xi^{\varepsilon}(t)|^2),
\]
(3.21)
\[
\lim_{\varepsilon \downarrow 0} E(|\Xi_\varepsilon^{\varepsilon}(t)|^2 + |\Xi_\sigma^{\varepsilon}(t)|^2) = 0, \quad t \in [0,T].
\]

By (3.10), (3.13), and (3.17) for \(b_{\varepsilon}\) and \(\sigma_{\varepsilon}\) replacing \(f\), we have
\[
\xi^{\varepsilon}(t) = \int_0^t \left\{ \Xi_\varepsilon^{\varepsilon}(s) + \nabla \xi^{\varepsilon}(s)b_{\varepsilon}(\cdot, \mathcal{L}_X)(X_s) + \left( E\langle D^b b_{\varepsilon}(y, \cdot)(\mathcal{L}_X)(X_s), \xi^{\varepsilon}(s) \rangle \right|_{y = X_s} \right\} ds
+ \int_0^t \left( \nabla \xi^{\varepsilon}(s)\sigma_{\varepsilon}(\cdot, \mathcal{L}_X)(X_s) + \left( E\langle D^\sigma \sigma_{\varepsilon}(y, \cdot)(\mathcal{L}_X)(X_s), \xi^{\varepsilon}(s) \rangle \right|_{y = X_s}, dW_s \right)
\]
for \(t \in [0,T]\). So, for any \(\varepsilon, \delta \in (0,1)\), \(\xi^{\varepsilon,\delta}(t) := \xi^{\varepsilon}(t) - \xi^{\delta}(t)\) satisfies
\[
|\xi^{\varepsilon,\delta}(t)|^2 \leq 4 \int_0^t |\Xi_\varepsilon^{\varepsilon}(s) - \Xi_\delta^{\delta}(s)|^2 ds + 4 \int_0^t \left( \Xi_\sigma^{\varepsilon}(s) - \Xi_\sigma^{\delta}(s), dW_s \right)^2
+ 4T \int_0^t \left( \nabla \xi^{\varepsilon,\delta}(s)b_{\varepsilon}(\cdot, \mathcal{L}_X)(X_s) + \left( E\langle D^b b_{\varepsilon}(y, \cdot)(\mathcal{L}_X)(X_s), \xi^{\varepsilon,\delta}(s) \rangle \right|_{y = X_s} \right)^2 ds
+ 4 \int_0^t \left( \nabla \xi^{\varepsilon,\delta}(s)\sigma_{\varepsilon}(\cdot, \mathcal{L}_X)(X_s) + \left( E\langle D^\sigma \sigma_{\varepsilon}(y, \cdot)(\mathcal{L}_X)(X_s), \xi^{\varepsilon,\delta}(s) \rangle \right|_{y = X_s}, dW_s \right)^2.
\]
Combining this with (H) and using the Burkholder-Davis-Gundy inequality, we find out a constant \(C_2 > 0\) such that
\[
E \left[ \sup_{s \in [0,t]} \xi^{\varepsilon,\delta}(s) \right] \leq C_2 \int_0^t E\left( |\Xi_\varepsilon^{\varepsilon}(s) - \Xi_\delta^{\delta}(s)|^2 + |\Xi_\sigma^{\varepsilon}(s) - \Xi_\sigma^{\delta}(s)|^2 \right) ds
+ C_2 \int_0^t E|\xi^{\varepsilon,\delta}(s)|^2 ds, \quad t \in [0,T].
\]
Since Lemma 3.3 ensures that \( \mathbb{E} \left[ \sup_{s \in [0,t]} \xi^\varepsilon(s) \right] < \infty \), by Gronwall’s lemma this yields
\[
\mathbb{E} \left[ \sup_{s \in [0,t]} \xi^\varepsilon\delta(s) \right] \leq C_2 e^{C_2 T} \int_0^T \mathbb{E} \left( |\Xi^\varepsilon_0(s) - \Xi^\delta_0(s)|^2 + |\Xi^\varepsilon_\sigma(s) - \Xi^\delta_\sigma(s)|^2 \right) ds.
\]

Combining this with (3.21) and Lemma 3.3, and applying the dominated convergence theorem, we prove the first assertion in Proposition 3.2.

Finally, by (3.10), (3.12), (3.21) and (3.17) for \( b_t, \sigma_t \) replacing \( f \), we conclude that \( \nu_t^i := \nabla_\eta X_t \) solves the SDE (3.11). Since this SDE is linear, the uniqueness is trivial. Then the proof is finished. \( \square \)

### 3.3 Malliavin derivative

Consider the Cameron-Martin space
\[
\mathbb{H} = \left\{ h \in C([0,T] \to \mathbb{R}^d) : h_0 = 0, h'_t \text{ exists a.e. } t, \|h\|^2_\mathbb{H} := \int_0^T |h'_t|^2 dt < \infty \right\}.
\]

Let \( \eta \in L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_0, \mathbb{P}) \) with \( \mathcal{L}_\eta = \mu \), and let \( \mu_T \) be the distribution of \( W_{[0,T]} := \{ W_t \}_{t \in [0,T]} \), which is a probability measure (i.e. Wiener measure) on the path space \( \mathcal{C}_T := C([0,T] \to \mathbb{R}^d) \). For \( F \in L^2(\mathbb{R}^d \times \mathcal{C}_T, \mu \times \mu_T), F(\eta, W_{[0,T]}) \) is called Malliavin differentiable along direction \( h \in \mathbb{H} \), if the directional derivative
\[
D_h F(\eta, W_{[0,T]}) := \lim_{\varepsilon \to 0} \frac{F(\eta, W_{[0,T]} + \varepsilon h) - F(\eta, W_{[0,T]})}{\varepsilon}
\]
exists in \( L^2(\Omega, \mathbb{P}) \). If the map \( \mathbb{H} \ni h \mapsto D_h F \in L^2(\Omega, \mu) \) is bounded, then there exists a unique \( DF(\eta, W_{[0,T]}), h \) such that \( \langle DF(\eta, W_{[0,T]}), h \rangle_\mathbb{H} = D_h F(\eta, W_{[0,T]}) \) holds in \( L^2(\Omega, \mathbb{P}) \) for all \( h \in \mathbb{H} \). In this case, we write \( F(\eta, W_{[0,T]}) \in \mathcal{D}(D) \) and call \( DF(\eta, W_{[0,T]}) \) the Malliavin gradient of \( F(\eta, W_{[0,T]}) \). It is well known that \( (D, \mathcal{D}(D)) \) is a closed linear operator from \( L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \) to \( L^2(\Omega \to \mathbb{H}, \mathcal{F}_T, \mathbb{P}) \). The adjoint operator \( (D^*, \mathcal{D}(D^*)) \) of \( (D, \mathcal{D}(D)) \) is called Malliavin divergence. For simplicity, in the sequel we denote \( F(\eta, W_{[0,T]}) \) by \( F \). Then we have the integration by parts formula
\[
(3.22) \quad \mathbb{E}(D_h F|\mathcal{F}_0) = \mathbb{E}(FD^*(h)|\mathcal{F}_0), \quad F \in \mathcal{D}(D), h \in \mathcal{D}(D^*).
\]

It is well known that for adapted \( h \in L^2(\Omega \to \mathbb{H}, \mathbb{P}), \) one has \( h \in \mathcal{D}(D^*) \) with
\[
(3.23) \quad D^*(h) = \int_0^T \langle h'_t, dW_t \rangle.
\]

For more details and applications on Malliavin calculus one may refer to [21] and references therein.

To calculate the Malliavin derivative of \( X_t \) with \( \mathcal{L}_{X_0} = \mu \in \mathcal{P}_2(\mathbb{R}^d) \), we write \( X_t = F_t(W_s) \) as a functional of the Brownian motion \( \{ W_s \}_{s \in [0,t]} \). Then by definition, for an adapted \( h \in L^2(\Omega \to \mathbb{H}, \mathbb{P}), \)
\[
D_h X_t = \lim_{\varepsilon \downarrow 0} \frac{F_t(W_s + \varepsilon h_s) - F_t(W_s)}{\varepsilon}, \quad 0 \leq t \leq T.
\]
On the other hand, by the pathwise uniqueness of (1.4), see for instances [10, 25, 30],
\( X_{t}^{h,\varepsilon} := F_t(W_{\cdot} + \varepsilon h_{\cdot}) \) solves the SDE
\[
(3.24) \quad dX_{t}^{h,\varepsilon} = b_t(X_{t}^{h,\varepsilon}, \mathcal{L}_{X_t})dt + \sigma_t(X_{t}^{h,\varepsilon}, \mathcal{L}_{X_t})d(W_t + \varepsilon h_t), \quad X_{0}^{h,\varepsilon} = X_0,
\]
which is well-posed due to (H) and \( h' \in L^2(\Omega \times [0,T], \mathbb{P} \times dt) \). When \( \sigma_t(x,\mu) \) does not depend \( (x,\mu) \), this SDE reduces to a random ODE for \( Y_{t}^{h,\varepsilon} := X_{t}^{h,\varepsilon} - \sigma_t W_t \), which is well-posed also for non-adapted \( h \) like \( h^a \) in Theorem 2.3. The main result of this part is the following which is well known by regarding (1.4) as the classical SDE, since in (3.24) the distribution \( \mathcal{L}_{X_t} \) does not depend on the variable \( \varepsilon \).

**Proposition 3.5.** Assume (H). Let \( h \in L^2(\Omega \rightarrow \mathbb{H}, \mathbb{P}) \), which is adapted if \( \sigma_t(x,\mu) \) depends on \( x \) or \( \mu \). Then the limit
\[
(3.25) \quad D_h X_t := \lim_{\varepsilon \downarrow 0} \frac{X_{t}^{h,\varepsilon} - X_t}{\varepsilon}, \quad t \in [0,T]
\]
exists in \( L^2(\Omega \rightarrow \mathcal{C}_T, \mathbb{P}) \). Moreover, \( (w_t^h := D_h X_t)_{t \in [0,T]} \) is the unique solution to the SDE
\[
(3.26) \quad dw_t^h = \left\{ \nabla w_t^h \sigma_t(\cdot, \mathcal{L}_{X_t})(X_t) \right\} dW_t + \left\{ \nabla w_t^h b_t(\cdot, \mathcal{L}_{X_t})(X_t) + \sigma_t(\cdot, \mathcal{L}_{X_t})(X_t) h_t' \right\} dt, \quad w_0^h = 0.
\]

## 4 Proofs of main results

We first present an integration by parts formula for \( \nabla \eta X_T \) with \( \eta \in L^2(\Omega \rightarrow \mathbb{R}^d, \mathcal{F}_0, \mathbb{P}) \), then prove Theorem 2.1, Corollary 2.2 and Theorem 2.3 respectively.

### 4.1 An integration by parts formula

**Theorem 4.1.** Assume (H) and (2.1). Then for any \( f \in C^1_b(\mathbb{R}^d) \), \( \eta \in L^2(\Omega \rightarrow \mathbb{R}^d, \mathbb{P}) \), and any \( 0 \leq r < T \) and \( g \in C^1([r,T]) \) with \( g_r = 0 \) and \( g_T = 1 \),
\[
(4.1) \quad \mathbb{E}\left( \langle \nabla f(X_T) , \nabla \eta X_T \rangle | \mathcal{F}_r \right) = \mathbb{E}\left( f(X_T) \int_r^T \langle \zeta_t^n , dW_t \rangle \right| \mathcal{F}_r \)
\]
holds for
\[
\zeta_t^n := \sigma_t(X_t)^{-1} \left\{ g_t' v_t^n + (\mathbb{E}(D^L b_t(y,\cdot)(\mathcal{L}_{X_t})(X_t), g_t v_t^n)) \big|_{y = X_t} \right\}, \quad t \in [0,T].
\]

**Proof.** Having Propositions 3.2 and 3.5 in hands, the proof is more or less standard. For \( v_t^n \) solving (3.11), we take
\[
(4.2) \quad h_t = \int_{t \wedge r}^t 1_{s \geq r} \zeta_s ds, \quad t \in [0,T].
\]
By (H), (2.1), and that \( h \in L^2(\Omega \to H, \mathbb{P}) \) is adapted, Proposition 3.5 applies. Let \( \tilde{v}_t = g_tv^n_t \) for \( t \in [r, T] \). Then (3.11) and (4.2) imply

\[
\begin{align*}
    d\tilde{v}_t &= \left\{ \nabla \tilde{v}_t b_t(\cdot, \mathcal{L}X_t)(X_t) + (\mathbb{E}(D^2b_t(y, \cdot)(\mathcal{L}X_t)(X_t), \tilde{v}_t)) \bigg|_{y=X_t} + g'_tv^n_t \right\} \, dt \\
    &\quad + \left\{ \nabla \tilde{v}_t \sigma_t(\cdot, \mathcal{L}X_t)(X_t) \right\} \, dW_t \\
    &= \left\{ \nabla \tilde{v}_t b_t(\cdot, \mathcal{L}X_t)(X_t) + \sigma_t(X_t, \mathcal{L}X_t)h'_t \right\} \, dt + \left\{ \nabla \tilde{v}_t \sigma_t(X_t) \right\} \, dW_t, \quad t \geq r, \quad \tilde{v}_r = 0.
\end{align*}
\]

So, \( (\tilde{v}_t)_{t \geq r} \) solves the SDE (3.26) with \( \tilde{v}_r = 0 \). On the other hand, by (4.2) we have \( h'_t = 0 \) for \( t < r \), so that the solution to (3.26) with \( w^n_0 = 0 \) satisfies \( w^n_r = 0 \). So, the uniqueness of this SDE from time \( r \) implies \( \tilde{v}_t = w^n_t \) for all \( t \geq r \). Combining this with Propositions 3.2 and 3.5, we obtain

\[ \nabla \eta X_T = v^n_T = g_T v^n_T = \tilde{v}_T = w^n_T = D_h X_T. \]

Thus, by the chain rule and the integration by parts formula (3.22), for any bounded \( \mathcal{F}_r \)-measurable \( G \in \mathcal{D}(D) \), we have

\[
\mathbb{E} \left( G(\nabla f(X_T), \nabla \eta X_T) \right) = \mathbb{E} \left( G(\nabla f(X_T), D_h X_T) \right) = \mathbb{E} \left( GD_h f(X_T) \right) \\
= \mathbb{E} \left( D_h \{Gf(X_T)\} - f(X_T)D_h G \right) = \mathbb{E}(Gf(X_T)D^*(h)),
\]

where in the last step we have used \( D_h G = 0 \) since \( G \) is \( \mathcal{F}_r \)-measurable but \( h'_t = 0 \) for \( t \leq r \). Noting that the class of bounded \( \mathcal{F}_r \)-measurable \( G \in \mathcal{D}(D) \) is dense in \( L^2(\Omega, \mathcal{F}_r, \mathbb{P}) \), this implies

\[ \mathbb{E}(\langle \nabla f(X_T), \nabla \eta X_T \rangle | \mathcal{F}_r) = \mathbb{E}(f(X_T)D^*(h) | \mathcal{F}_r). \]

Combining this with

\[
D^*(h) = \int_r^T \langle h'_t, dW_t \rangle = \int_r^T \langle \zeta^n_t, dW_t \rangle
\]

due to (3.23) and (4.2), we prove (4.1).

\[ \square \]

### 4.2 Proof of Theorem 2.1

Let \( \mu \in \mathcal{D}_2(\mathbb{R}^d) \). We first establish (2.3) for \( f \in \mathcal{B}_b(\mathbb{R}^d) \), then construct \( \gamma \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu) \) such that

\[
\lim_{\mu(\{|\phi|^2\}) \to 0} \frac{|(P_T f)(\mu \circ (\text{Id} + \phi)^{-1}) - (P_T f)(\mu) - \mu(\langle \phi, \gamma \rangle)|}{\sqrt{\mu(\{|\phi|^2\})}} = 0,
\]

which, by definition, implies that \( P_T f \) is \( L \)-differentiable at \( \mu \) with \( D^L P_T f(\mu) = \gamma \).

(a) Proof of (2.3) for \( f \in \mathcal{B}_b(\mathbb{R}^d) \). When \( f \in C^1_b(\mathbb{R}^d) \), (2.3) follows from (4.1) for \( \eta = \phi(X_0) \). Below we extend the formula to \( f \in \mathcal{B}_b(\mathbb{R}^d) \). For \( s \in [0, 1] \), let \( X_t^{\phi,s} \) solve (1.4) for \( X_0^{\phi,s} = X_0 + s\phi(X_0) \). We have \( \mu^{\phi,s} := \mathcal{L}_{X_0^{\phi,s}} = \mu \circ (\text{Id} + s\phi)^{-1} \), and by the definition of
∇_ηX_T for η = φ(X_0),

$$\begin{align*}
(PTf)(µ^{φ,ε}) - (PTf)(µ) &= \mathbb{E}[f(X_T^{φ,ε}) - f(X_T)] = \int_0^\varepsilon \frac{d}{ds} \mathbb{E}[f(X_T^{φ,ε})] \, ds \\
&= \int_0^\varepsilon \mathbb{E}(⟨\nabla f)(X_T^{φ,ε}), \nabla φ(X_0)X_T^{φ,ε}⟩) \, ds, \ f \in C^1_b(\mathbb{R}^d).
\end{align*}$$

(4.4)

Next, let \((v_t^{φ,s})_{t∈[0,T]}\) solve (3.11) for η = φ(X_0) and \(X^s_t\) replacing \(X_t\), i.e.

$$\begin{align*}
dv_t^{φ,s} &= \left\{ \nabla_v^{φ,s}b_t(\cdot, \mathcal{L}_{X_t^{φ,s}})(X_t^{φ,s}) + \left( \mathbb{E}\langle D^Lb_t(y, \cdot)(\mathcal{L}_{X_t^{φ,s}})(X_t^{φ,s}), v_t^{φ,s} \rangle \right|_{y=X_t^{φ,s}} \right\} dt \\
&\quad + \left\{ \nabla v_t^{φ,s}σ_t(X_t^{φ,s}) \right\} dW_t, \ v_0^{φ,s} = φ(X_0).
\end{align*}$$

(4.5)

Let

$$ζ_t^{φ,s} := σ_t(X_t^{φ,s})^{-1} \left\{ g_t^{φ,s} + \left( \mathbb{E}\langle D^Lb_t(y, \cdot)(\mathcal{L}_{X_t^{φ,s}})(X_t^{φ,s}), g_t^{φ,s} \rangle \right|_{y=X_t^{φ,s}} \right\}, \ t \in [0, T].$$

Then (4.4) and (4.1) imply

$$\begin{align*}
(PTf)(µ^{φ,ε}) - (PTf)(µ) &= \int_0^\varepsilon \mathbb{E}\left[ f(X_T^{φ,ε}) \int_0^T \langle ζ_t^{φ,s}, dW_t \rangle \right] \, ds, \ f \in C^1_b(\mathbb{R}^d),
\end{align*}$$

(4.6)

By a standard approximation argument, we may extend this formula to all \(f \in \mathcal{B}_b(\mathbb{R}^d)\).

Indeed, let

$$\nu_ε(\mathcal{A}) = \int_0^\varepsilon \mathbb{E}\left[ 1_\mathcal{A}(X_T^{φ,ε}) \int_0^T \langle ζ_t^{φ,s}, dW_t \rangle \right] \, ds, \ \mathcal{A} \in \mathcal{B}(\mathbb{R}^d).$$

Then \(ν_ε\) is a finite signed measure on \(\mathbb{R}^d\) with

$$\int_{\mathbb{R}^d} f \, dν_ε = \int_0^\varepsilon \mathbb{E}\left[ f(X_T^{φ,ε}) \int_0^T \langle ζ_t^{φ,s}, dW_t \rangle \right] \, ds, \ f \in \mathcal{B}_b(\mathbb{R}^d).$$

So, (4.6) is equivalent to

$$\int_{\mathbb{R}^d} f \, dP^*_{T,ε}µ^{φ,ε} - \int_{\mathbb{R}^d} f \, dP^*_{T}µ = \int_{\mathbb{R}^d} f \, dν_ε, \ f \in C^1_b(\mathbb{R}^d).$$

(4.7)

Since \(ν_{T,ε} := P^*_{T,ε}µ^{φ,ε} + P^*_{T}µ + |ν_ε|\) is a finite measure on \(\mathbb{R}^d\), \(C^1_b(\mathbb{R}^d)\) is dense in \(L^1(\mathbb{R}^d, ν_{T,ε})\).

Hence, (4.7) holds for all \(f \in \mathcal{B}_b(\mathbb{R}^d) \subset L^1(\mathbb{R}^d, ν_{T,ε})\). Consequently, (4.6) holds for all \(f \in \mathcal{B}_b(\mathbb{R}^d)\). Thus,

$$\begin{align*}
\frac{(PTf)(µ^{φ,ε}) - (PTf)(µ)}{ε} &= \frac{1}{ε} \int_0^\varepsilon \mathbb{E}\left[ f(X_T^{φ,ε}) \int_0^T \langle ζ_t^{φ,s}, dW_t \rangle \right] \, ds, \ f \in \mathcal{B}_b(\mathbb{R}^d).
\end{align*}$$

(4.8)

It is easy to see from \((H)\) that

$$\lim_{s \to 0} \sup_{t∈[0,T]} \mathbb{E}(|X_t^{φ,s} - X_t|^2 + |v_t^{φ,s} - v_t^{φ}|^2) = 0.$$
So,

\[(4.9) \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^\epsilon \mathbb{E} \left| \int_0^T \langle \zeta_{t}^{\phi,s} - \zeta_{t}^{\phi}, dW_t \rangle \right| = 0.\]

Combining this with (4.8), we see that (2.3) for \( f \in \mathcal{B}_b(\mathbb{R}^d) \) follows from

\[(4.10) \lim_{\epsilon \downarrow 0} \mathbb{E} \left[ f(X_T^\epsilon) - f(X_T) \right] \int_0^T \langle \zeta_{t}, dW_t \rangle = 0, \quad f \in \mathcal{B}_b(\mathbb{R}^d).\]

To prove this equality, we denote

\[I_r := \int_0^r \langle \zeta_{t}^{\phi}, dW_t \rangle, \quad r \in (0, T).\]

Applying (4.1) with \( g_t := \frac{t-r}{T-r} \) for \( t \in [r, T] \) and using (H), we may find out a constant \( C(T, r) > 0 \) such that

\[
|\mathbb{E}[I_r \{f(X_T^\epsilon) - f(X_T)\}]| \leq \mathbb{E} \left[ |I_r| \cdot \int_0^\epsilon \mathbb{E}(\langle \nabla f(X_T^\epsilon), \nabla_{\phi(X_0)}X_T^\epsilon \rangle \mathbf{1}_r) ds \right],
\]

\[
\leq C(T, r) \frac{f}{T-r} \int_0^\epsilon \mathbb{E} \left[ |I_r| \left( \int_r^T |v_{t}^{\phi,s}|^2 dt \right)^{\frac{1}{2}} \right] ds, \quad f \in C^1_b(\mathbb{R}^d).
\]

By the argument extending (4.6) from \( f \in C^1_b(\mathbb{R}^d) \) to \( f \in \mathcal{B}_b(\mathbb{R}^d) \), we conclude from this that for any \( r \in (0, T) \),

\[
\lim_{\epsilon \downarrow 0} \sup_{\|f\|_{\infty} \leq 1} \mathbb{E}[I_r \{f(X_T^\epsilon) - f(X_T)\}] = 0.
\]

Therefore,

\[(4.11) \limsup_{\epsilon \downarrow 0} \sup_{\|f\|_{\infty} \leq 1} \mathbb{E} \left[ f(X_T^\epsilon) - f(X_T) \right] \int_0^T \langle \zeta_{t}^{\phi}, dW_t \rangle = 0, \quad r \in (0, T).
\]

By letting \( r \uparrow T \) we prove (4.10).

(b) For any \( f \in \mathcal{B}_b(\mathbb{R}^d) \), we intend to find out \( \gamma \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu) \) such that

\[(4.12) \mathbb{E} \left[ f(X_T) \int_0^T \langle \zeta_{t}^{\phi}, dW_t \rangle \right] = \mu(\langle \phi, \gamma \rangle), \quad \phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu).\]
When \( f \in C_b(\mathbb{R}^d) \), in step (c) we will deduce from this and (2.3) that \( \gamma = DLPTf(\mu) \). To construct the desired \( \gamma \), consider the SDE
\[
dX_t^\phi = b_t(X_t^\phi, \mathcal{L}_{X_t^\phi})dt + \sigma_t(X_t^\phi)dW_t, \quad X_0^\phi = X_0 + \phi(X_0),
\]
and let \( v_t^\phi \) solve (2.2). Since (2.2) is a linear equation for \( v_t^\phi \) with initial value \( \phi(X_0) \in L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_0, \mathbb{P}) \), the functional
\[
L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu) \ni \phi \mapsto L\phi := \mathbb{E}\left[ f(X_T) \int_0^T \langle \zeta_t^\phi, dW_t \rangle \right]
\]
is linear, and by \((H)\) and (2.1), there exists a constant \( C(T) > 0 \) such that
\[
|L\phi|^2 \leq C(T) \mathbb{E}|\phi(X_0)|^2 = C(T) \mu(|\phi|^2), \quad \phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu).
\]
Then \( L \) is a bounded linear functional on the Hilbert space \( L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu) \). By Riesz’s representation theorem, there exists a unique \( \gamma \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu) \) such that
\[
L\phi = \mu(\langle \gamma, \phi \rangle), \quad \phi \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu).
\]
Therefore, (4.12) holds.

(c) Now, for \( f \in B_b(\mathbb{R}^d) \), we intend to verify (4.3) for \( \gamma \) in (4.12), so that \( P_Tf \) is \( L \)-differentiable with \( D^L(P_Tf)(\mu) = \gamma \). By (4.8) for \( \varepsilon = 1 \), we have
\[
(P_Tf)(\mu^1) - (P_Tf)(\mu) = \int_0^1 \mathbb{E}\left[ f(X_T^{\phi,s}) \int_0^T \langle \zeta_t^{\phi,s}, dW_t \rangle \right], \quad f \in B_b(\mathbb{R}^d).
\]
For \( \mathbb{R}^d \) random variables \( X, v \), let
\[
N_t(X, v) = \sigma_t(X)^{-1}\left\{ g^t v + (\mathbb{E} D^L b_t(y, \cdot)(\mathcal{L}_X(X), g_t v)) |_{y = X} \right\}, \quad t \in [0, T].
\]
Then \( \zeta_t^{\phi,s} = N_t(X_t^{\phi,s}, v^{\phi,s}) \) and \( \zeta_t^\phi = N_t(X_t, v^\phi) \). Combining this with (4.12) and (4.13), and noting that \( \mu^1 = \mu \circ (\text{Id} + \phi)^{-1} \), we arrive at
\[
\frac{|(P_Tf)(\mu \circ (\text{Id} + \phi)^{-1}) - (P_Tf)(\mu) - \mu(\langle \phi, \gamma \rangle)|}{\sqrt{\mu(|\phi|^2)}} \leq \varepsilon_1(\phi) + \varepsilon_2(\phi) + \varepsilon_3(\phi),
\]
where
\[
\varepsilon_1(\phi) := \frac{1}{\sqrt{\mu(|\phi|^2)}} \int_0^1 \mathbb{E}\left[ \left( f(X_T^{\phi,s}) - f(X_T) \right) \int_0^T \langle \zeta_t^{\phi,s}, dW_t \rangle \right] ds,
\]
\[
\varepsilon_2(\phi) := \frac{\|f\|_\infty}{\sqrt{\mu(|\phi|^2)}} \int_0^1 \mathbb{E}\left[ \int_0^T \left| N_t(X_t^{\phi,s}, v^{\phi}) - N_t(X_t, v^\phi) \right| dW_t \right] ds,
\]
\[
\varepsilon_3(\phi) := \frac{\|f\|_\infty}{\sqrt{\mu(|\phi|^2)}} \int_0^1 \mathbb{E}\left[ \int_0^T \left| N_t(X_t^{\phi,s}, v^{\phi,s}) - N_t(X_t, v^\phi) \right| dW_t \right] ds.
\]
It is easy to deduce from \((H)\) that for any \(p \geq 2\) there exists a constant \(c(p) > 0\) such that
\[
\sup_{t \in [0,T], s \in [0,1]} \mathbb{E}\left( |X_t^{\phi,s} - X_t|^p + |v_t^{\phi,s}|^p \big| \mathcal{F}_0 \right) \leq c(p)|\phi(X_0)|^p.
\]
(4.15)
Combining this with the continuity of \(\sigma_t(x)\) in \(x\) uniformly in \(t \in [0,T]\), we conclude that
\[
\lim_{\mu(|\phi|^2) \to 0} \varepsilon_2(\phi) = 0.
\]
(4.16)
Next, by the argument deducing (2.3) from (4.8), it is easy to see that (4.15) implies
\[
\lim_{\mu(|\phi|^2) \to 0} \varepsilon_1(\phi) = 0.
\]
(4.17)
Moreover, by the SDEs for \(v_t^{\phi,s}\) and \(v_t^\phi\) we have
\[
d(v_t^{\phi,s} - v_t^\phi) = \left\{ A_t(v_t^{\phi,s} - v_t^\phi) + \tilde{A}_t v_t^{\phi,s} \right\} dt + \left\{ B_t(v_t^{\phi,s} - v_t^\phi) + \tilde{B}_t v_t^\phi \right\} dW_t,
\]
where for a square integrable random variable \(v\) on \(\mathbb{R}^d\),
\[
A_t v := \nabla v b_t(\cdot, \mathcal{L}_{X_t})(X_t) + \left( \mathbb{E}\left( D^L b_t(y, \cdot)(\mathcal{L}_{X_t})(X_t), v \right) \right)_{y = X_t},
\]
\[
\tilde{A}_t := \nabla v b_t(\cdot, \mathcal{L}_{X_t^{\phi,s}})(X_t^{\phi,s}) + \left( \mathbb{E}\left( D^L b_t(y, \cdot)(\mathcal{L}_{X_t^{\phi,s}})(X_t^{\phi,s}), v \right) \right)_{y = X_t^{\phi,s}},
\]
\[
- \nabla v b_t(\cdot, \mathcal{L}_{X_t})(X_t) - \left( \mathbb{E}\left( D^L b_t(y, \cdot)(\mathcal{L}_{X_t})(X_t), v \right) \right)_{y = X_t},
\]
\[
B_t v := \nabla v \sigma_t(X_t), \quad \tilde{B}_t v := \nabla v \sigma_t(X_t^{\phi,s}) - \nabla v \sigma_t(X_t).
\]
Combining this with (4.15) and \((H)\), there exists a constant \(c > 0\) such that
\[
d|v_t^{\phi,s} - v_t^\phi|^2 \leq c|v_t^{\phi,s} - v_t^\phi|^2 dt + c(\|\tilde{A}_t\|^2 + \|\tilde{B}_t\|^2)(|v_t^{\phi,s}|^2 + |v_t^\phi|^2) dt + dM_t, \quad |v_0^{\phi,s} - v_0^\phi| = 0
\]
holds for some martingale \(M_t\), and that
\[
\|\tilde{A}_t\|^2 + \|\tilde{B}_t\|^2 \leq c, \quad \lim_{\mu(|\phi|^2) \to 0} (\|\tilde{A}_t\|^2 + \|\tilde{B}_t\|^2) = 0, \quad t \in [0,T], s \in [0,1].
\]
(4.19)
By (4.18) and (4.15) for \(p = 4\), there exists a constant \(c' > 0\) such that
\[
\mathbb{E}(|v_t^{\phi,s} - v_t^\phi|^2 | \mathcal{F}_0)
\]
\[
\leq c \int_0^t \mathbb{E}(|v_r^{\phi,s} - v_r^\phi|^2 | \mathcal{F}_0) dr + 2c \int_0^T \sqrt{\mathbb{E}(\|\tilde{A}_t\|^4 + \|\tilde{B}_t\|^4 | \mathcal{F}_0)} \cdot \sqrt{\mathbb{E}(|v_t^{\phi,s}|^4 + |v_t^\phi|^4 | \mathcal{F}_0)} dt
\]
\[
\leq c \int_0^t \mathbb{E}(|v_r^{\phi,s} - v_r^\phi|^2 | \mathcal{F}_0) dr + c' \varepsilon(\phi)|\phi(X_0)|^2, \quad s \in [0,1], t \in [0,T],
\]
where
\[
\varepsilon(\phi) := \int_0^T \sqrt{\mathbb{E}(\|\tilde{A}_t\|^4 + \|\tilde{B}_t\|^4 | \mathcal{F}_0)} dt.
\]
Then Gronwall’s lemma and (4.19) yield
\[ \sup_{s \in [0,T]} \mathbb{E}(|v_t^\phi| - v_t^\phi|^2|\mathcal{F}_0) \leq c'e^{cT} \varepsilon(\phi)|\phi(X_0)|^2, \]
\[ \lim_{\mu(|\phi|^2) \to 0} \mathbb{E}\varepsilon(\phi) = 0. \]
Combining this with the definition of \( \varepsilon_3(\phi) \), (H), and Jensen’s inequality for the conditional expectation \( \mathbb{E}(\cdot|\mathcal{F}_0) \), we may find out constants \( C_1, C_2 > 0 \) depending on \( \|f\|_{\infty} \) and \( T \) such that
\[ \lim_{\mu(|\phi|^2) \to 0} \varepsilon_3(\phi) \leq \lim_{\mu(|\phi|^2) \to 0} \frac{C_1}{\sqrt{\mu(|\phi|^2)}} \int_0^1 \mathbb{E} \left( \int_0^T |v_t^\phi - v_t^\phi|^2 dt \right)^{\frac{1}{2}} ds \]
\[ \leq \lim_{\mu(|\phi|^2) \to 0} \frac{C_1}{\sqrt{\mu(|\phi|^2)}} \int_0^1 \mathbb{E} \left( \int_0^T \mathbb{E}( |v_t^\phi - v_t^\phi|^2 | \mathcal{F}_0) dt \right)^{\frac{1}{2}} ds \]
\[ \leq \lim_{\mu(|\phi|^2) \to 0} \frac{C_2}{\sqrt{\mu(|\phi|^2)}} \int_0^1 \mathbb{E}( |\phi(X_0)| \sqrt{\varepsilon(\phi)}) ds \]
\[ \leq \lim_{\mu(|\phi|^2) \to 0} \frac{C_2 \sqrt{\mathbb{E}( |\phi(X_0)|^2 \varepsilon(\phi))}}{\sqrt{\mu(|\phi|^2) \varepsilon(\phi)}} = \lim_{\mu(|\phi|^2) \to 0} C_2 \sqrt{\mathbb{E}\varepsilon(\phi)} = 0. \]
This, together with (4.14), (4.16) and (4.17), implies (4.3). Therefore, \( P_T f \) is \( L \)-differentiable at \( \mu \) with \( D^L(P_T f)(\mu) = \gamma \).
(d) Finally, (2.3) and (4.8) imply
\[ \left| \frac{P_T\mu \circ (\text{Id} + \varepsilon \phi)^{-1} - P_T^* \mu(f) - (\psi P_T^* \mu)(f)}{\varepsilon} \right| \]
\[ = \left| \frac{(P_T f)(\mu^{\phi,\varepsilon}) - (P_T f)(\mu)}{\varepsilon} - \mathbb{E} \left[ f(X_T) \int_0^T \langle \zeta_t^\phi, dW_t \rangle \right] \right| \]
\[ \leq \frac{\|f\|_{\infty}}{\varepsilon} \int_0^c \mathbb{E} \left| \int_0^T \langle \zeta_t^\phi - \zeta_t^\phi, dW_t \rangle \right| ds \]
\[ + \frac{1}{\varepsilon} \mathbb{E} \left[ \left\{ f(X_T^{\phi,\varepsilon}) - f(X_T) \right\} \int_0^T \langle \zeta_t^\phi, dW_t \rangle \right] \right| ds. \]
Combining this with (4.9) and (4.10) we prove (2.4).

4.3 Proof of Corollary 2.2

Proof of (1). By (H) and (2.2), there exists a martingale \( M_t \) such that
\[ d|v_t^\phi|^2 \leq 4K_t|v_t^\phi|(|v_t^\phi| + \mathbb{E}|v_t^\phi|)dt + dM_t, \quad |v_0^\phi|^2 = |\phi(X_0)|^2, \]
where \( K(t) \) is increasing in \( t \geq 0 \). Then
\[ \mathbb{E}|v_t^\phi|^2 \leq \mathbb{E}|\phi(X_0)|^2 + 4K_t \int_0^t \left\{ \mathbb{E}|v_s^\phi|^2 + (\mathbb{E}|v_s^\phi|)^2 \right\} ds \leq \mu(|\phi|^2) + 8K_t \int_0^t \mathbb{E}|v_s^\phi|^2 ds. \]
By Gronwall’s inequality this implies
\begin{equation}
\mathbb{E}|v_t|^2 \leq e^{8Kt} \mu(|\phi|^2), \quad t \in [0, T].
\end{equation}

Next, since \( \mathbb{E} \int_0^T \langle \xi_t, \ dW_t \rangle = 0 \), (2.3) is equivalent to
\[
D^L_b(\mu) = \mathbb{E} \left\{ f(X_T) - P_T f(\mu) \right\} = \int_0^T \mathbb{E} \langle \xi_t, \ dW_t \rangle.
\]

Combining this with (4.21) and using Jensen’s inequality, when \( \mu(|\phi|^2) \leq 1 \) we have
\[
|D^L_b(\mu)|^2 \leq \left\{ (P_T f^2(\mu) - (P_T f(\mu))^2) \right\} \int_0^T \mathbb{E} |\xi_t|^2 \, dt
\leq \left\{ (P_T f^2(\mu) - (P_T f(\mu))^2) \right\} \int_0^T \left( |g_t'| + K(t)|g_t| \right)^2 e^{8Kt} \, dt
\]
for any \( g \in C^1([0, T]) \) with \( g_0 = 0 \) and \( g_T = 1 \). Taking \( g_t = \frac{t}{T} \), \( t \in [0, T] \), we prove the estimate (2.5).

**Proof of (2).** Let \( f \in \mathcal{B}_b(\mathbb{R}^d) \) with \( \|f\|_\infty \leq 1 \). By Theorem 2.1, \( P_T f \) is \( L \)-differentiable. Moreover, by Theorem 4.1, \( P_T f \) is Lipschitz continuous on \( \mathcal{P}_2(\mathbb{R}^d) \). Indeed, for any \( \mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^d) \), let \( X_1, X_2 \in L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_0, \mathbb{P}) \) such that \( \mathcal{L}_{X_i} = \mu_i, 1 \leq i \leq 2 \), and \( \mathbb{E}|X_1 - X_2|^2 = \mathbb{W}_2(\mu_1, \mu_2)^2 \). Let \( X_t^s \) be the solution to (1.4) with \( X_0 = X_1 + s(X_2 - X_1), s \in [0, 1] \). Then Theorem 4.1 implies
\[
|P_T f(\mu_1) - P_T f(\mu_2)|^2 = \left| \mathbb{E} f(X_T^0) - \mathbb{E} f(X_T^1) \right|^2 = \left| \int_0^1 \frac{d}{ds} \mathbb{E} f(X_T^s) \, ds \right|^2
\leq \mathbb{E} |X_2 - X_1|^2 = c\mathbb{W}_2(\mu_1, \mu_2)^2
\]
for some constant \( c > 0 \).

To apply Proposition 3.1, we take \( \{\mu_n, \nu_n\}_{n \geq 1} \subset \mathcal{P}_2(\mathbb{R}^d) \) which have compact supports and are absolutely continuous with respect to the Lebesgue measure, such that
\begin{equation}
\lim_{n \to \infty} \{ \mathbb{W}_2(\mu, \mu_n) + \mathbb{W}_2(\nu, \nu_n) \} = 0.
\end{equation}

According to [5], see also [7, Theorem 5.8], for any \( n \geq 1 \) there exists a unique map \( \phi_n \in L^2(\mathbb{R}^d \to \mathbb{R}^d, \mu) \) such that
\begin{equation}
\nu_n = \mu_n \circ (\text{Id} + \phi_n)^{-1}, \quad \mathbb{W}_2(\mu_n, \nu_n)^2 = \mu_n(|\phi_n|^2).
\end{equation}

Let \( X_n \in L^2(\Omega \to \mathbb{R}^d, \mathcal{F}_0, \mathbb{P}) \) such that \( \mathcal{L}_{X_n} = \mu_n \). By Proposition 3.1, (2.5) and (4.23), we obtain
\[
|(P_T f)(\mu_n) - (P_T f)(\nu_n)|^2 = \left| \int_0^1 \frac{d}{ds} (P_T f)(\mathcal{L}_{X_n + s\phi_n(X_n)}) \, ds \right|^2
\]
for any
\[
\int_0^1 \mathbb{E}(D^L(P_T f)\langle \mathcal{L}_{X_n+s\phi_n(X_n)}(X_n + s\phi_n(X_n)), \phi_n(X_n) \rangle \, ds)^2
\leq \frac{\|f\|_\infty W_2(\mu_n, \nu_n)^2}{\int_0^T \lambda_t^{-2} e^{-stK_1} \, dt}.
\]

By the continuity of \( P_T f \) and (4.22), by letting \( n \to \infty \) we prove
\[
|(P_T f)(\mu) - (P_T f)(\nu)|^2 \leq \frac{W_2(\mu, \nu)^2}{\int_0^T \lambda_t^{-2} e^{-stK_1} \, dt}, \quad \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d), \quad f \in \mathcal{B}(\mathbb{R}^d), \|f\|_\infty \leq 1.
\]

Therefore, (2.6) and (2.7) hold. \( \square \)

4.4 Proof of Theorem 2.3

Let \( T > r \geq 0, \mu \in \mathcal{P}_2(\mathbb{R}^{m+d}) \) and let \( X_t \) solve (2.8) with \( \mathcal{L}_{X_0} = \mu \). To realize the procedure in the proof of Theorem 2.1 for the present degenerate setting, we first extend Theorem 4.1 using \( D^*(h_{\alpha}^\alpha) \) to replace \( \int_0^T \langle d\xi^i_t, dW_t \rangle \), where for a \( C^1([r, T] \to \mathbb{R}^{m+d}) \)-valued random variable \( \alpha = (\alpha^{(1)}, \alpha^{(2)}) \), let \( (h_{\alpha}^\alpha, w_{\alpha}^\alpha)_{t \in [r, T]} \) be the unique solution to the random ODEs

\[
\frac{dh_{\alpha}^\alpha}{dt} = \sigma_i^{-1} \left\{ \nabla_{\alpha_t} b_t^{(2)}(X_t, \mathcal{L}_{X_t}) - (\alpha^{(2)})' \right\}
+ \left( \mathbb{E}(D^2 b^{(2)}_t(y, \cdot)(\mathcal{L}_{X_t})(X_t), \alpha_t + w_{\alpha_t}^\alpha) \right|_{y=X_t} \}
\]

\[
\frac{dw_{\alpha_t}^\alpha}{dt} = \nabla_{w_{\alpha_t}^\alpha} b_t(\cdot, \mathcal{L}_{X_t})(X_t) + (0, \sigma_t(h_{\alpha_t}^\alpha)'), \quad h_{\alpha_t}^\alpha, w_{\alpha_t}^\alpha \in \mathcal{D}(D^*), \quad t \in [r, T],
\]

and \( h_{\alpha_t}^\alpha \in \mathcal{D}(D^*) \), then for any \( f \in C^1_b(\mathbb{R}^{m+d}) \),

\[
\mathbb{E}(\langle \nabla f(X_T), \nabla_{X_T} \rangle | \mathcal{F}_r) = \mathbb{E}(f(X_T) D^*(h_{\alpha}^\alpha) | \mathcal{F}_r).
\]

**Proof.** Letting \( w_t = w_{\alpha}^\alpha 1_{\{t > r\}} \), Proposition 3.5 implies that \( w_t = D_{h_{\alpha}^\alpha} X_t, \ t \in [0, T] \). By (4.24), we have

\[
w_t = \int_{t \wedge r}^T \left\{ \nabla_{w_s} b_s(\cdot, \mathcal{L}_{X_s})(X_s) + (0, \sigma_s(h_{\alpha_s}^\alpha)') \right\} \, ds, \quad t \in [0, T].
\]

Extending \( \alpha_t \) with \( \alpha_t := \nabla_{\eta} X_t \) for \( t \in [0, r] \), and letting \( v_t = w_t + \alpha_t \) for any \( t \in [0, T] \), we obtain

\[
v_t = \alpha_t + \int_{t \wedge r}^T \left\{ \nabla_{v_s} b_s(\cdot, \mathcal{L}_{X_s})(X_s) + \left( 0, \mathbb{E}(D^2 b^{(2)}_s(y, \cdot)(\mathcal{L}_{X_s})(X_s), v_s) \right|_{y=X_s} \right\}
+ (0, \sigma_s(h_{\alpha_s}^\alpha)' - \mathbb{E}(D^2 b^{(2)}_s(y, \cdot)(\mathcal{L}_{X_s})(X_s), w_s + \alpha_s) \right|_{y=X_s}) - \nabla_{\alpha_s} b_s(\cdot, \mathcal{L}_{X_s})(X_s) \right\} \, ds.
\]

25
By (4.25),
\[ \int_{t \land r}^{t} \nabla_{\alpha_s} b_s^{(1)}(\cdot, \mathcal{L}_s)(X_s) \, ds = 1_{\{t > r\}}(\alpha_t^{(1)} - \nabla_{\eta_t} X_t^{(1)}), \]
while the definition of \( h_{r,s}^\alpha \) implies
\[ \int_{t \land r}^{t} \left\{ \sigma_s(h_s^\alpha)' \right\} ds - (\mathbb{E}(D^2 b_s^{(2)}(y, \cdot)(\mathcal{L}_s)(X_s), w_s + \alpha_s)) \big|_{y = X_s} - \nabla_{\alpha_s} b_s^{(2)}(\cdot, \mathcal{L}_s)(X_s) \right\} ds \]
\[ = - \int_{t \land r}^{t} (\alpha_s^{(2)})' ds = 1_{\{t > r\}}(\nabla_{\eta_t} X_t^{(2)} - \alpha_t^{(2)}). \]

Combining these with (4.27) and Proposition 3.2 leads to
\[ v_t = \nabla_{\eta_t} X_t + \int_{t \land r}^{t} \left\{ \nabla_{v_s} b_s(\cdot, \mathcal{L}_s)(X_s) + \left( 0, \left( \mathbb{E}(D^2 b_s^{(2)}(y, \cdot)(\mathcal{L}_s)(X_s), v_s) \right) \big|_{y = X_s} \right) \right\} ds \]
\[ = \eta + \int_{0}^{t} \left\{ \nabla_{v_s} b_s(\cdot, \mathcal{L}_s)(X_s) + \left( 0, \left( \mathbb{E}(D^2 b_s^{(2)}(y, \cdot)(\mathcal{L}_s)(X_s), v_s) \right) \big|_{y = X_s} \right) \right\} ds, \quad t \in [0, T]. \]

That is, \( v_t \) solves (3.11) so that by Proposition 3.2 we obtain \( v_t := w_t + \alpha_t = \nabla_{\eta_t} X_t \). Since \( \alpha_T = 0 \), this implies \( D_{h_{r,T}^\alpha} X_T = \nabla_{\eta_T} X_T \). Thus, for any bounded \( \mathcal{F}_T \)-measurable \( G \in \mathcal{D}(D) \),
\[ \mathbb{E}[G(\nabla f(X_T), \nabla_{\eta_T} X_T)] = \mathbb{E}[GD_{h_{r,T}^\alpha} f(X_T)] \]
\[ = \mathbb{E}[D_{h_{r,T}^\alpha} \{ G f(X_T) \} - f(X_T)D_{h_{r,T}^\alpha} G] = \mathbb{E}[G f(X_T)D^r(h_{r,T}^\alpha)]; \]
where in the last step we have used the integration by parts formula (3.22) and \( D_{h_{r,T}^\alpha} G = 0 \) since \( G \) is \( \mathcal{F}_T \)-measurable but
\[ D_{h_{r,T}^\alpha} G = \int_{0}^{T} (h_{r,s}^\alpha)'(s) \cdot \{(DG)'\}(s) \, ds = 0, \]
\( (h_{r,s}^\alpha)'(s) = 0 \) for \( s \leq r \). Noting that the class of bounded \( \mathcal{F}_T \)-measurable functions \( G \in \mathcal{D}(D) \) is dense in \( L^2(\Omega, \mathcal{F}_T, \mathbb{P}) \), (4.28) implies (4.26).

\[ \square \]

**Proof of Theorem 2.3.** With Theorem 4.2 in hands, the proof is completely similar to that of Theorem 2.1. Let
\[ v_t^{(\phi)} = ((v_t^{(\phi)})^{(1)}, (v_t^{(\phi)})^{(2)}) = (\nabla_{\phi(X_0)} X_t^{(1)}, \nabla_{\phi(X_0)} X_t^{(2)}) = \nabla_{\phi(X_0)} X_t, \quad t \in [0, T]. \]

For any \( 0 \leq r < T \), let
\[ \alpha_{r,t}^{(2)} = \frac{T - t}{T - r} (v_t^{(\phi)})^{(2)} - \frac{(t - r)(T - t)}{T - r} \int_{0}^{t} \theta_s^2 Q_s^{-1} K_{T,s}(v_t^{(\phi)})^{(1)} \, ds \]
\[ = \frac{T - t}{T - r} (v_t^{(\phi)})^{(2)} - \frac{(t - r)}{T - r} \int_{0}^{t} \theta_s^2 Q_s^{-1} K_{T,s} \nabla_{\phi(X_0)} X_s^{(2)}(X_0) \, ds, \quad t \in [r, T], \]
\[ \text{and} \quad \alpha_{r,t}^{(1)} = \frac{T - t}{T - r} (v_t^{(\phi)})^{(1)} - \frac{(t - r)}{T - r} \int_{0}^{t} \theta_s^2 Q_s^{-1} K_{T,s} \nabla_{\phi(X_0)} X_s^{(1)}(X_0) \, ds, \quad t \in [r, T], \]
\[ \text{and} \quad \alpha_{r,t} = \alpha_{r,t}^{(1)} + \alpha_{r,t}^{(2)}, \quad t \in [r, T]. \]
and
\begin{equation}
(4.30) \quad \alpha^{(1)}_{r,t} = K_{t,r}(\phi^{(1)}_{t}) + \int_{r}^{t} K_{t,s} \nabla(\phi^{(2)}_{s}) b^{(1)}_{s}(X_{s}(x)) \, ds, \quad t \in [r, T].
\end{equation}

Then \( \alpha_{r,.} := (\alpha^{(1)}_{r,t}, \alpha^{(2)}_{r,t}) \) satisfies
\[ \alpha_{r,r} = \nabla_{\phi(X_{0})} X_{r}, \quad \alpha_{r,T} = 0, \]
and by (2.9) and Duhamel’s formula, (4.30) implies
\[ (\alpha^{(1)}_{r,.})'(t) = \nabla_{\alpha_{r,.}} b^{(1)}_{t}(X_{t}), \quad t \in [r, T]. \]

Moreover, let \( h^\alpha_{r,.} \) be defined in (4.24) for \( \alpha_{r,.} \), replacing \( \alpha \). Noting that \( (H1) \) and \( (H2) \)

imply [31, (H)] for \( l_{1} = l_{2} = 0 \), the proof of [31, Theorem 1.1] with \( \phi(s) := (s - r)(T - s) \) for \( s \in [r, T] \) ensures that \( h^\alpha_{r,.} \in \mathcal{D}(D^*) \) with \( D^*(h^\alpha_{r,.}) \in L^p(\mathbb{P}) \) for all \( p \in (1, \infty) \). So, by

Theorem 2.3 with \( \eta = \phi(X_{0}) \) we obtain
\begin{equation}
(4.31) \quad \mathbb{E}(\langle \nabla f(X_{T}), \nabla_{\phi(X_{0})} X_{T} \rangle | \mathcal{F}_{r}) = \mathbb{E}(f(X_{T})D^{*}(h^\alpha_{r,.}) | \mathcal{F}_{r}), \quad f \in C_{b}^{1}(\mathbb{R}^{d}), \quad r \in [0, T).
\end{equation}

In particular, taking \( r = 0 \) we obtain \( D^{*}(h) \in L^p(\mathbb{P}) \) for all \( p \in (1, \infty) \) and
\begin{equation}
(4.32) \quad D^{*}_{\phi} P_{T} f(\mu) = \mathbb{E}(\langle \nabla f(X_{T}), \nabla_{\phi(X_{0})} X_{T} \rangle) = \mathbb{E}(f(X_{T})D^{*}(h^{\alpha}) | \mathcal{F}_{r}), \quad f \in C_{b}^{1}(\mathbb{R}^{d}).
\end{equation}

Basing on these two formulas, by repeating the proof of Theorem 2.1 with \( I_{r} := \mathbb{E}(D^{*}(h^{\alpha}) | \mathcal{F}_{r}) \), we prove (2.16) and the \( L \)-differentiability of \( P_{T} f \) for \( f \in \mathcal{B}_{b}(\mathbb{R}^{m+d}) \). Finally, the estimates
(2.17) and (2.18) follows from (2.16) as in the proof of Theorem 2.1, together with the corresponding estimate on \( \mathbb{E}|D^{*}(h^{\alpha})|^{2} \) as in the proof of [31, Theorem 1.1]. For instance, below we outline the proof of (2.16).

Firstly, for \( s \in (0, 1) \) let \( X^{s}_{t} \) solve (2.8) with \( X^{\phi,s}_{0} = X_{0} + s \phi(X_{0}) \), let \( \mu^{\phi,s} = \mathcal{L}_{X^{\phi,s}_{0}} = \mu \circ (\text{Id} + \phi)^{-1} \), and let \( \alpha^{\phi,s}_{r,t} \) be defined as \( \alpha_{r,t} \) with \( X^{\phi,s}_{t} \) replacing \( X_{t} \). Then as in (4.4) and (4.7), (4.32) implies
\begin{equation}
(4.33) \quad (P_{T} f)(\mu^{\phi,e}) - (P_{T} f)(\mu) = \int_{0}^{e} \mathbb{E}[\langle \nabla f(X^{\phi,s}_{t}), \nabla_{\phi(X_{0})} X^{\phi,s}_{t} \rangle] \, ds
\end{equation}
where \( h^{\alpha^{\phi,s}} := h^{\alpha^{\phi,s}_{0,.}} \) satisfies
\begin{equation}
(4.34) \quad \lim_{s \to 0} \mathbb{E}|D^{*}(h^{\alpha^{\phi,s}}) - D^{*}(h)|^{2} = 0.
\end{equation}

By the argument leading to (4.8), (4.33) yields
\begin{equation}
(4.35) \quad (P_{T} f)(\mu^{\phi,e}) - (P_{T} f)(\mu) = \frac{1}{e} \int_{0}^{e} \mathbb{E}[f(X^{\phi,s}_{t}) D^{*}(h^{\alpha^{\phi,s}_{t}})] \, ds, \quad f \in \mathcal{B}_{b}(\mathbb{R}^{m+d}).
\end{equation}
Combining this with (4.34), we prove (2.16) provided

\[ (4.35) \quad \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^\varepsilon \mathbb{E} \left[ \{ f(X_T^{\phi,s}) - f(X_T) \} D^*(h^\alpha) \right] ds = 0. \]

For any \( r \in (0, T) \), let \( I_r = \mathbb{E}(D^*(h^\alpha)|\mathcal{F}_r). \) By (4.33) we obtain

\[ \mathbb{E} \left[ \{ f(X_T^{\phi,s}) - f(X_T) \} I_r \right] = \mathbb{E} \left[ I_r \mathbb{E}(f(X_T^{\phi,s}) - f(X_T)|\mathcal{F}_r) \right] = \mathbb{E} \left[ I_r \int_0^\varepsilon \mathbb{E}(f(X_T^{\phi,s}) D^*(h_{r_o}^\alpha)|\mathcal{F}_r) ds \right] \]

\[ = \int_0^\varepsilon \mathbb{E}[I_r f(X_T^{\phi,s}) D^*(h_{r_o}^\alpha)] ds, \quad f \in C_b^1(\mathbb{R}^d). \]

Combining this with the argument extending (4.8) from \( f \in C_b^1(\mathbb{R}^d) \) to \( f \in B_b(\mathbb{R}^d) \), we obtain

\[ \mathbb{E} \left[ \{ f(X_T^{\phi,s}) - f(X_T) \} I_r \right] = \int_0^\varepsilon \mathbb{E} \left[ I_r f(X_T^{\phi,s}) D^*(h_{r_o}^\alpha) \right] ds, \quad f \in B_b(\mathbb{R}^d). \]

Consequently,

\[ \lim_{\varepsilon \to 0} \mathbb{E} \left[ \{ f(X_T^{\phi,s}) - f(X_T) \} I_r \right] = 0, \quad f \in B_b(\mathbb{R}^d), r \in (0, T). \]

Then for any \( r \in (0, T) \),

\[ \limsup_{\varepsilon \downarrow 0} \left| \frac{1}{\varepsilon} \int_0^\varepsilon \mathbb{E} \left[ \{ f(X_T^{\phi,s}) - f(X_T) \} D^*(h^\alpha) \right] ds \right| \]

\[ = \limsup_{\varepsilon \downarrow 0} \left| \frac{1}{\varepsilon} \int_0^\varepsilon \mathbb{E} \left[ \{ f(X_T^{\phi,s}) - f(X_T) \} \cdot \{ D^*(h^\alpha) - I_r \} \right] ds \right| \]

\[ \leq 2\|f\|_\infty \mathbb{E} |D^*(h^\alpha) - \mathbb{E}(D^*(h^\alpha)|\mathcal{F}_r)|. \]

Letting \( r \uparrow T \) we derive (4.35), and hence prove (2.16) as explained above.

\[ \square \]

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