ON THE NON-GENERIC REPRESENTATION THEORY OF THE SYMPLECTIC BLOB ALGEBRA

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Abstract. This paper reports advances in the study of the symplectic blob algebra. We find a presentation for this algebra. We find a minimal poset for this as a quasi-hereditary algebra. We discuss how to reduce the number of parameters defining the algebra from 6 to 4 (or even 3) without loss of representation theoretic generality. We then find some non-semisimple specializations by calculating Gram determinants for certain cell modules (or standard modules) using the good parametrization defined. We finish by considering some quotients of specializations of the symplectic blob algebra which are isomorphic to Temperley–Lieb algebras of type $A$.

Introduction

One of the topics considered in a previous paper [13] was a new diagram algebra, the symplectic blob algebra. In that paper we investigated its generic representation theory and proved various important properties of the algebra, for instance that it has a cellular basis, that it is generically semi-simple, and that it is a quotient of the Hecke algebra of type $\tilde{C}$. The problems of determining a presentation for the algebra, and of determining its non-generic representation theory, were left open. In this paper we prove (in section 2) an isomorphism with an algebra defined by a presentation, and begin to classify the cases when the algebra is not semi-simple.

With the Temperley–Lieb [17] and blob algebras [15], the symplectic blob algebra (or isomorphically, the affine symmetric Temperley–Lieb algebra, $b_{2n}^{\phi}$, also defined in [13]) belongs to an intriguing class of diagram realisations of Hecke algebra quotients. The first two have representation theories beautifully and efficiently described in alcove geometrical language, where the precise geometry (the realisation of the reflection group in a weight space) is determined by the parameters of the algebra. In these first two algebras the “good” parametrisation appropriate to reveal this structure is not that in which the algebras were first described. Rather, it was discovered during efforts to put the low rank data on non-semisimple manifolds in parameter space in a coherent format. The determination of the representation theory of $b_{2n}^{\phi}$ in the non-semisimple cases is the next important problem in the programme initiated in [13].

The paper is structured as follows. We first review the various objects and notations and some of the basic properties of the symplectic blob algebra that will be used in this paper. This is followed by a statement and proof of a presentation for the algebra. We then discuss an analogue...

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of the blob good parametrisation, and show how by sacrificing integrality we can reduce the number of parameters defining the algebra from 6 to 4 (or even 3). Following this, we use globalisation and localisation functors to find a minimal labelling poset for this quasi-hereditary algebra, as a first step to finding alcove geometry or a “linkage principle” (a geometrical block statement [10]).

We then begin to tackle the question of non-semisimplicity by calculating Gram determinants for certain cell modules (or standard modules) using the good parametrisation above (this complements work of De Gier and Nichols [3], who effectively compute Gram determinants for a particular and distinct kind of cell module). We finish by considering some quotients of specialisations of the symplectic blob algebra which are isomorphic to Temperley–Lieb algebras of type $A$. (Generically there is no such quotient, so these constructions provide another way of detecting non-semisimple structure.)

1. Review

We first review the objects and notations that will be used in this paper.

1.1. The symplectic blob algebra. Fix $n, m \in \mathbb{N}$, with $n + m$ even, and $k$ a field. A Brauer $(n,m)$-partition is a partition of the set $V \cup V'$ into pairs, where $V = \{1, 2, \ldots, n\}$ and $V' = \{1', 2', \ldots, m'\}$. Following Weyl [18] we will think of these as Brauer $(n,m)$-diagrams by taking a rectangle with $n$ vertices labelled 1 through to $n$ on the top and $m$ vertices labelled $1'$ through to $m'$ on the bottom and connecting the two vertices $a$, and $b$ with an arbitrary line embedded in the plane of the rectangle, if the set $\{a, b\}$ occurs in the partition of $V \cup V'$. For example: Take $n = m = 5$ and the partition $\{\{1, 2\}, \{1', 3\}, \{2', 3'\}, \{4, 5'\}, \{4', 5\}\}$ and we obtain the diagram:

\[
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
\end{array}
\]

\[
\begin{array}{ccccc}
1' & 2' & 3' & 4' & 5' \\
\end{array}
\]

Obviously each line can be deformed isotopically inside the rectangle without changing the $(n,m)$-partition. Thus any two diagrams coding the same set partition are regarded as the same diagram.

Diagrams that can be deformed isotopically within the rectangle to obtain a diagram with no lines crossing are known as Temperley–Lieb diagrams. Thus
are Temperley–Lieb diagrams while the following diagram

\[
\text{\begin{tikzpicture}
\draw [thick] (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw [thick] (0.5,0) -- (0.5,1);
\end{tikzpicture}}
\]

is \textit{not}.

Our first objective is to define a certain diagram category, that is a \(k\)-linear category whose hom-sets each have a basis consisting of diagrams, and where multiplication is defined by diagram concatenation, and simple straightening rules (when the concatenated object is not formally a diagram). For example in our case:

\[
\text{\begin{tikzpicture}
\draw [thick] (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw [thick] (0.5,0) -- (0.5,1);
\end{tikzpicture}} \quad \equiv \quad \text{\begin{tikzpicture}
\draw [thick] (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\end{tikzpicture}}
\]

by isotopy. When we concatenate diagrams we may get loops, for example:

\[
\text{\begin{tikzpicture}
\draw [thick] (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw [thick] (0.5,0) -- (0.5,1);
\end{tikzpicture}} \quad \equiv \quad \text{\begin{tikzpicture}
\draw [thick] (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\draw [thick] (0.5,0) -- (0.5,1);
\end{tikzpicture}}
\]

(1)

A straightening rule is a way of expressing such products in the span of basis diagrams.

The resulting diagram in (1) is an example of a \textit{pseudo Temperley–Lieb diagram}. The \textit{pseudo Temperley–Lieb diagrams} include all the Temperley–Lieb diagrams, but we also allow diagrams with loops, which may appear anywhere in the diagram although still with no crossing lines. Loops that can be deformed isotopically into other loops without crossing a line are equivalent. We will impose a relation on the \(k\)-space spanned by pseudo Temperley–Lieb diagrams that will remove the loop.

\textbf{Definition 1.1.1.} The \textit{Temperley–Lieb algebra} \(TL_n\) is the \(k\)-algebra with \(k\)-basis the Temperley–Lieb \((n,n)\)-diagrams and multiplication defined by concatenation. We impose the relation: each loop that may arise when multiplying is replaced by \(\delta, \delta \in k\) a parameter.

We can also have \textit{decorated Temperley–Lieb} diagrams where we put elements of a monoid on the strings. When diagrams are concatenated and words in the monoid elements form then we multiply in the monoid. This gives us a well defined associative diagram calculus — see section 3 of [13] for
a detailed discussion and proof of this. Of course, we also have decorated pseudo Temperley–Lieb diagrams which can have decorated loops.

We will focus on a particular set of decorated Temperley–Lieb diagrams — the ones that define the symplectic blob algebra. Here our monoid that decorates the diagrams is the non-commutative free monoid on two generators: a “left” blob, $L$, (usually a black filled-in circle on the diagrams) and a “right” blob, $R$, (usually a white filled-in circle on the diagrams).

A line in a (pseudo) Temperley–Lieb diagram is said to be $L$-exposed (respectively $R$-exposed) if it can be deformed to touch the left hand side (respectively right hand side) of the diagram without crossing any other lines.

A left (respectively right) blob pseudo-diagram is a diagram obtained from a pseudo Temperley–Lieb diagram, in which only $L$-exposed lines (respectively $R$-exposed lines) are allowed to be decorated with left (respectively right) blobs. A left-right blob pseudo-diagram is a diagram obtained from a pseudo Temperley–Lieb diagram by allowing left and right blob decorations, with the further constraint that it must be possible to deform decorated strings so as to take left blobs to the left and right blobs to the right simultaneously.

Concatenating diagrams cannot change a $L$-exposed line to a non-$L$-exposed line, and similarly for $R$-exposed lines. Thus the set of left-right blob pseudo-diagrams is closed under diagram concatenation. (See [13, proposition 6.1.2].)

The set of left-right blob pseudo-diagrams is infinite: various features may appear. To define a finite dimensional algebra, as for the blob algebra (see [16, section 1.1] for a definition) and the Temperley–Lieb algebra (defined above), we will straighten by replacing certain features with other features (possibly none) multiplied by parameters from a field, $k$.

We define $B^x_{n,m}$ to be the set of left-right blob pseudo-diagrams with $n$ vertices at the top and $m$ at the bottom of the diagram that do not have features from the following table.

| Feature | $\delta$ | $\delta_L$ | $\delta_R$ | $\kappa_L$ | $\kappa_R$ | $\kappa_{LR}$ | $k_L$ | $k_R$ |
|---------|----------|------------|------------|------------|------------|---------------|-------|-------|

Table 1. Table encoding most of the straightening relations for $b^x$. 
The set $B_{n,m}^{x'}$ is finite and we call its elements left-right blob diagrams. Define $B_{n}^{x'} = B_{n,n}^{x'}$ and $B^{x'} = \bigcup_{n} B_{n,n}^{x'}$.

Now define a relation on the $k$-span of all left-right blob pseudo-diagrams by $d \sim xd'$ if diagram $d'$ differs from $d$ by a substitution from left to right in the table (and extend $k$-linearly).

A moment’s thought makes it clear that to obtain a consistent set of relations we need $RRLR = kRRL = kLRL$, i.e., that $kL = kR$.

Another (perhaps longer) moment’s thought reveals that the $k_L$ relation is only needed for $n$ odd and the $\kappa_{LR}$ relation is only needed when $n$ is even. It turns out to be convenient to set $\kappa_{LR} = kL = kR$.

**Proposition 1.1.2** ([13, section 6.3]). The above relations on the $k$-span of left-right blob pseudo-diagrams define a finite dimensional algebra, $b_{n}^{x}$, which has a diagram basis $B_{n}^{x'}$.

We study this algebra by considering the quotient by the “topological relation”:

$$\kappa_{LR}$$

where each labelled shaded area is shorthand for subdiagrams that do not have propagating lines and where a line is called *propagating* if it joins a vertex on the top of the diagram to one on the bottom of the diagram. (Note that there is no freedom in choosing the scalar multiple, once we require a relation of this form.)

We define $B_{n}^{x}$ to be the subset of $B_{n}^{x'}$ that does not contain diagrams with features as in the right hand side of relation (2).

**Definition 1.1.3.** We define the symplectic blob algebra, $b_{n}^{x}$ (or $b_{n}^{x}(\delta, \delta_{L}, \delta_{R}, \kappa_{L}, \kappa_{R}, \kappa_{LR})$ if we wish to emphasise the parameters) to be the $k$-algebra with basis $B_{n}^{x}$, multiplication defined via diagram concatenation and relations as in the table above (with $\kappa_{LR} = kL = kR$) and with relation (2).

That these relations are consistent and that we do obtain an algebra with basis $B_{n}^{x}$ is proved in [13, section 6.5].

We have the following (implicitly assumed in [13]):

**Proposition 1.1.4.** The symplectic blob algebra, $b_{n}^{x}$, is generated by the following diagrams

$$e := \cdots, e_{1} := \cdots, e_{2} := \cdots, \cdots,$$

$$e_{n-1} := \cdots, f := \cdots.$$
Proof. We may argue in a similar fashion as in appendix A of [13] but by now inducting on the number of decorations. If a diagram $d$ has no decorations then the diagram is a Temperley–Lieb diagram and the result follows.

So now assume that we have a diagram $d$ with $m$ decorations and that (for the sake of illustration) there is a left blob — we would use the dual reduction in the case of a right blob. We claim that we may use the same procedure as in the $l = 0$ case of [13, appendix A]. If there is a decorated line starting in the first position, then we can decompose the diagram into a product of $e$ then a diagram with one fewer decoration. If there is no such line then take the first line decorated with a black blob and do the same reduction as in [13, appendix A].

The white blobs can either be moved into the shaded regions or above or below the horizontal dotted lines. The middle region (after “wiggling” the line enough times) is then the product $e_1ee_2e_1$. The outside diagrams have strictly fewer than $m$ decorations and hence the result follows by induction. □

We now quote two results about this algebra.

**Proposition 1.1.5** ([13, proposition 6.5.4]). If $\delta_L$ is invertible then setting $e' = \frac{e}{\delta_L}$ we have

$$b^n(\delta, \delta_L, \delta_R, \kappa_L, \kappa_R, \kappa_{LR}) \cong e' b^{n+1}(\delta, \kappa_L, \delta_R, \delta_L, \kappa_R, \kappa_{LR})e'.$$

Similarly we have:

**Proposition 1.1.6.** If $\delta_R$ is invertible then setting $f' = \frac{f}{\delta_L}$ we have

$$b^n(\delta, \delta_L, \delta_R, \kappa_L, \kappa_R, \kappa_{LR}) \cong f' b^{n+1}(\delta, \delta_L, \kappa_R, \kappa_L, \delta_R, \kappa_{LR})f'.$$

Note the swapping of parameters required to obtain the above isomorphisms.
1.2. The affine symmetric Temperley–Lieb algebra. We now turn to defining the affine symmetric Temperley–Lieb algebra. We may obtain (undecorated) annular Temperley–Lieb diagrams by enforcing non-crossing in a cylinder or an annulus rather than a planar rectangle:

![Diagram of annular Temperley–Lieb diagram]

We define the annular Temperley–Lieb category by using diagram concatenation, isotopy, multiplication and relations of form “loop=parameter” as before. We distinguish (contractible) loops and non-contractible loops:

![Diagram showing contractible and non-contractible loops]

Thus we will also need a relation for these non-contractible loops to obtain finite dimensional algebras.

(Another way to think of this is that the one loop parameter category admits a deformation in which the loop parameter is deformed for non-contractible loops.)

A symmetric annular Temperley–Lieb diagram is an annular Temperley–Lieb diagram that is symmetric about a line of symmetry — where we now only consider diagrams with $2n$ vertices on the top and $2m$ on the bottom and there are $n$ (respectively $m$) vertices on either side of the line, for example:

![Diagram of symmetric annular Temperley–Lieb diagram]
The *pseudo annular Temperley–Lieb* diagrams include all the annular Temperley–Lieb diagrams, but we also allow closed loops, including non-contractible loops. The *symmetric pseudo annular Temperley–Lieb* diagrams are the subset of the pseudo annular Temperley–Lieb diagrams that are symmetric. However, we will insist on the isotopies being “symmetric”, so that the following two diagrams are not equivalent:

This will allow us to admit a further deformation (see [13, appendix B] for a more detailed discussion of this).

We denote the set of symmetric annular pseudo Temperley–Lieb diagrams by $D_{2n,2m}^\phi$, and with $n = m$ by $D_{2n}^\phi$. (Although this is not identical with the previous paper [13] — where $D_{2n}^\phi$ were the left-right symmetric periodic pseudo Temperley–Lieb diagrams of period $2n$ — this set is equivalent to ours by a trivial unfolding map.)

We can (and often will) colour the annular diagrams with two colours, black and white, such that two adjacent regions are different colours and the marked corner (top of the 0-reflection line) is always white. We also split the line of symmetry into two parts — the 1-reflection line and 0-reflection line as marked:

If both the top and the bottom corner of the 0-reflection line are white then we say the pseudo diagrams are *colouring composable* (or CC for short). (The above diagram is not colouring composable.)

The subset of $D_{2n}^\phi$ consisting of CC diagrams will be denoted by $CC_{2n}$.

The set $CC_{2n}$ is closed under diagram concatenation as the following example illustrates:
Example 1.2.1.

The set $CC_{2n}$ is not finite, so to produce a finite dimensional diagram algebra we will need some relations.

We take $B^\phi_{2n} \subset CC_{2n}$ to be those elements of $D^\phi_{2n}$ that do not have features on the LHS of Table 2. The relations will be that the features on the LHS are replaced by the parameter on the RHS.

The set $B^\phi_{2n}$ is finite.

Definition 1.2.2. The affine symmetric Temperley–Lieb algebra, $b^\phi_{2n}$, is the $k$-algebra with basis $B^\phi_{2n}$, multiplication defined via diagram concatenation and relations as in the table above.

The reason why we introduce an odd-even dependency of the parameters for the affine symmetric Temperley–Lieb algebra is the following result:

Proposition 1.2.3 ([13, proposition 7.2.4]). The symplectic blob algebra and the affine symmetric Temperley–Lieb algebra are isomorphic, with the obvious identification of parameters.

$$b^\phi_n(\delta, \delta_L, \delta_R, \kappa_L, \kappa_R, \kappa_{LR}) \cong b^\phi_{2n}(\delta, \delta_L, \delta_R, \kappa_L, \kappa_R, \kappa_{LR}).$$

We do not stress the explicit isomorphism here as it is not needed in the sequel. But as a corollary we obtain the following:
Proposition 1.2.4. The affine symmetric Temperley–Lieb algebra, $b^\phi_{2n}$, is generated by the following diagrams:

\[
e := \begin{array}{c}
\includegraphics{diagram1.png}
\end{array},
\]

\[
e_1 := \begin{array}{c}
\includegraphics{diagram2.png}
\end{array},
\]

\[
e_2 := \begin{array}{c}
\includegraphics{diagram3.png}
\end{array},
\]

\[
\ldots,
\]

\[
e_{n-1} := \begin{array}{c}
\includegraphics{diagram4.png}
\end{array},
\]

\[
f := \begin{array}{c}
\includegraphics{diagram5.png}
\end{array}.
\]

Here we have abused notation slightly and called the generators by the same names on both sides of the isomorphism. This above identification would also define the isomorphism from $b^x_n$ to $b^\phi_{2n}$ if we knew a priori that $b^\phi_{2n}$ was generated by the images of the generators for $b^x_n$.

2. Presenting the symplectic blob algebra

The two boundary Temperley–Lieb algebra (2BTL) is (a certain parametrisation of) an infinite-dimensional quotient of the Temperley–Lieb algebra of type $\tilde{C}$. In this section we show that the symplectic blob algebra has a presentation consisting of the relations for the 2BTL, together with two additional relations that make it finite dimensional.

Presently little is known, even in the generic case, about the larger algebras, so attacking non-trivial but tractable finite dimensional quotients is an effective approach to their study. Our result shows that the symplectic blob algebra is a good tool for this study.

We start by giving the definition of 2BTL in a suitable form. We must then assemble a large number of preparatory lemmas, before finally approaching the proof of the theorem.

2.1. A presentation.

Definition 2.1.1. Fix $n$. Let $S = \{E_0, E_1, \ldots, E_n\}$, and let $S^*$ be the free monoid on $S$. Define the commutation monoid $M$ to be the quotient of $S^*$ by the relations

$$E_iE_j \equiv E_jE_i \text{ for all } 0 \leq i, j \leq n \text{ with } |i - j| > 1.$$ 

Definition 2.1.2. Let $A_n$ be the quotient of the $k$-monoid-algebra of $M$ by the following relations:

\[
E_0^2 = \delta_L E_0 \quad \quad E_1 E_0 E_1 = \kappa_L E_1
\]

\[
E_i^2 = \delta E_i \quad \text{for } 1 \leq i \leq n - 1 \quad \quad E_i E_{i+1} E_i = E_i \quad \text{for } 1 \leq i \leq n - 2
\]

\[
E_n^2 = \delta_R E_n \quad \quad E_{i+1} E_i E_{i+1} = E_{i+1} \quad \text{for } 1 \leq i \leq n - 2
\]

\[
E_{n-1} E_n E_{n-1} = \kappa_R E_{n-1}
\]

\[
IJI = \kappa_{LR} I \quad \quad JIJ = \kappa_{LR} J
\]
where
\[
I = \begin{cases} 
E_1 E_3 \cdots E_{2m-1} & \text{if } n = 2m \\
E_1 E_3 \cdots E_{2m-1} E_{2m+1} & \text{if } n = 2m + 1
\end{cases}
\]
and
\[
J = \begin{cases} 
E_0 E_2 \cdots E_{2m-2} E_{2m} & \text{if } n = 2m \\
E_0 E_2 \cdots E_{2m} & \text{if } n = 2m + 1
\end{cases}
\]

Note \(I = E_1\) and \(J = E_0\) if \(n = 1\). We will sometimes write \(E\) for \(E_0\) and \(F\) for \(E_{n-1}\).

The algebra \(A_n\) is the quotient of the two boundary Temperley–Lieb algebra (or the Temperley–Lieb algebra of type \(\tilde{C}\)) by the additional relations \(IJI = \kappa_{LR} I\) and \(JIJ = \kappa_{LR} J\).

**Theorem 2.1.3.** The symplectic blob algebra \(b^x_n\) is isomorphic to the algebra \(A_n\) via an isomorphism
\[
\phi : A_n \to b^x_n
\]
induced by \(E \mapsto e, E_1 \mapsto e_1, \ldots, E_{n-1} \mapsto e_{n-1}\) and \(F \mapsto f\).

It is straightforward to check that the generators already given for both the affine symmetric Temperley–Lieb algebra and the symplectic blob algebra satisfy the \(A_n\) relations. Thus the map \(\phi\) in the theorem is a surjective homomorphism and hence we need only to prove injectivity. The rest of this section is devoted to proving this theorem.

### 2.2. Definitions associated to the monoid \(M\).

Two monomials \(u, u'\) in the generators \(S\) are said to be **commutation equivalent** if \(u \equiv u'\) in \(M\). The commutation class, \(\overline{u}\), of a monomial \(u\) consists of the monomials that are commutation equivalent to it.

The **left descent set** (respectively, **right descent set**) of a monomial \(u\) consists of all the initial (respectively, terminal) letters of the elements of \(\overline{u}\). We denote these sets by \(L(u)\) and \(R(u)\), respectively.

**Definition 2.2.1.** A **reduced monomial** is a monomial \(u\) in the generators \(S\) such that no \(u' \in \overline{u}\) can be expressed as a scalar multiple of a strictly shorter monomial using the relations in Definition 2.1.2.

If we have \(u = u_1s u_2 s u_3\) for some generator \(s\), then the occurrences of \(s\) in \(u\) are said to be **consecutive** if \(u_2\) contains no occurrence of \(s\).

**Definition 2.2.2.** Two monomials in the generators, \(u\) and \(u'\), are said to be **weakly equivalent** if \(u\) can be transformed into a nonzero multiple of \(u'\) by applying finitely many relations.

In this situation, we also say that \(D\) and \(D'\) are weakly equivalent, where \(D\) and \(D'\) are the diagrams equal to \(\phi(u)\) and \(\phi(u')\), respectively. If \(P\) is a property that diagrams may or may not possess, then we say \(P\) is **invariant under weak equivalence** if, whenever \(D\) and \(D'\) are weakly equivalent diagrams, then \(D\) has \(P\) if and only if \(D'\) has \(P\).
Definition 2.2.3. Let $D$ be a diagram. For $g \in \{L, R\}$ and

$$k \in \{1, \ldots, n, 1', \ldots, n'\},$$

we say that $D$ is $g$-decorated at the point $k$ if (a) the edge $x$ connected to $k$ has a decoration of type $g$, and (b) the decoration of $x$ mentioned in (a) is closer to point $k$ than any other decoration on $x$.

In the sequel, we will sometimes invoke Lemma 2.2.4 without explicit comment.

Lemma 2.2.4. The following properties of diagrams are invariant under weak equivalence:

(i) the property of being $L$-decorated at the point $k$;
(ii) the property of being $R$-decorated at the point $k$;
(iii) for fixed $1 \leq i < n$, the property of points $i$ and $(i + 1)$ being connected by an undecorated edge;
(iv) for fixed $1 \leq i < n$, the property of points $i'$ and $(i + 1)'$ being connected by an undecorated edge.

Proof. It is enough to check that each of these properties is respected by each type of diagrammatic reduction, because the diagrammatic algebra is a homomorphic image of the algebra given by the monomial presentation. This presents no problems, but notice that the term “undecorated” cannot be removed from parts (iii) and (iv), because of the topological relation. □

Elements of the commutation monoid $M$ have the following normal form, established in [1].

Proposition 2.2.5 (Cartier–Foata normal form). Let $s$ be an element of the commutation monoid $M$. Then $s$ has a unique factorization in $M$ of the form

$$s = s_1 s_2 \cdots s_p$$

such that each $s_i$ is a product of distinct commuting elements of $S$, and such that for each $1 \leq j < p$ and each generator $t \in S$ occurring in $s_{j+1}$, there is a generator $s \in S$ occurring in $s_j$ such that $st \neq ts$.

Remark 2.2.6. The Cartier–Foata normal form may be defined inductively, as follows. Let $s_1$ be the product of the elements in $L(s)$. Since $M$ is a cancellative monoid, there is a unique element $s' \in M$ with $s = s_1 s'$. If

$$s' = s_2 \cdots s_p$$

is the Cartier–Foata normal form of $s'$, then

$$s_1 s_2 \cdots s_p$$

is the Cartier–Foata normal form of $s$. 
Remark 2.2.7. The monoid $M$ is useful for our purposes because the symplectic blob algebra is a quotient of the monoid algebra of $M$, where we identify $E_0$ with $e$, $E_i$ with $e_i$ and $E_n$ with $f$, as usual.

Definition 2.2.8. Let $u$ be a reduced monomial in the generators $E_0, \ldots, E_n$. We say that $u$ is left reducible (respectively, right reducible) if it is commutation equivalent to a monomial of the form $u' = stv$ (respectively, $u' = vts$), where $s$ and $t$ are noncommuting generators and $t \not \in \{E, F\}$. In this situation, we say that $u$ is left (respectively, right) reducible via $s$ to $tv$ (respectively, to $vt$).

2.3. Preparatory lemmas. The following result is similar to [9, Lemma 5.3], but we give a complete argument here because the proof in [9] contains a mistake (we thank D.C. Ernst for pointing this out).

Lemma 2.3.1. Suppose that $s \in M$ corresponds to a reduced monomial, and let $s_1s_2\cdots s_p$ be the Cartier–Foata normal form of $s$. Suppose also that $s$ is not left reducible. Then, for $1 \leq i < p$ and $0 \leq j \leq n$, the following hold:

(i) if $E_0$ occurs in $s_{i+1}$, then $E_1$ occurs in $s_i$;
(ii) if $E_n$ occurs in $s_{i+1}$, then $E_{n-1}$ occurs in $s_i$;
(iii) if $j \not \in \{0, n\}$ and $E_j$ occurs in $s_{i+1}$, then both $E_{j-1}$ and $E_{j+1}$ occur in $s_i$.

Proof. The assertions of (i) and (ii) are immediate from properties of the normal form, because $E_1$ (respectively, $E_{n-1}$) is the only generator not commuting with $E_0$ (respectively, $E_n$). We will now prove (iii) by induction on $i$. Suppose first that $i = 1$.

Suppose that $j \not \in \{0, n\}$ and that $E_j$ occurs in $s_2$. By definition of the normal form, there must be a generator $s \in s_1$ not commuting with $E_j$. Now $s$ cannot be the only such generator, or $s$ would be left reducible via $s$. Since the only generators not commuting with $E_j$ are $E_{j-1}$ and $E_{j+1}$, these must both occur in $s_1$.

Suppose now that the statement is known to be true for $i < N$, and let $i = N \geq 2$. Suppose also that $j \not \in \{0, n\}$ and that $E_j$ occurs in $s_{N+1}$. As in the base case, there must be at least one generator $s$ occurring in $s_N$ that does not commute with $E_j$.

Let us first consider the case where $j \not \in \{1, n-1\}$, and write $s = E_k$ for some $1 \leq k \leq n - 1$. The restrictions on $j$ mean that we cannot have $E_jE_kE_j$ occurring as a subword of any reduced monomial. However, $E_j$ occurs in $s_{N-1}$ by the inductive hypothesis, and this is only possible if there is another generator, $s'$, in $s_N$ that does not commute with $E_j$. This implies that $\{s', E_k\} = \{E_{j-1}, E_{j+1}\}$, as required.

Now suppose that $j = 1$ (the case $j = n - 1$ follows by a symmetrical argument). If both $E_0$ and $E_2$ occur in $s_N$, then we are done. If $E_2$ occurs in $s_N$ but $E_0$ does not, then the argument of the previous paragraph applies. Suppose then that $E_0$ occurs in $s_N$ but $E_2$ does not. By statement (i), $E_1$ occurs in $s_{N-1}$, but arguing as in the previous paragraph, we find this cannot happen, because
it would imply that \( s \) was commutation equivalent to a monomial of the form \( v'E_1E_0E_1v'' \), which is incompatible with \( s \) being reduced. This completes the inductive step. \( \square \)

The following is a key structural property of reduced monomials.

**Proposition 2.3.2.** Suppose that \( s \in M \) corresponds to a reduced monomial, and let \( s_1s_2 \cdots s_p \) be the Cartier–Foata normal form of \( s \), where \( s_p \) is nonempty. Suppose also that \( s \) is neither left reducible nor right reducible. Then either (i) \( p = 1 \), meaning that \( s \) is a product of commuting generators or (ii) \( p = 2 \) and either \( s = IJ \) or \( s = JI \).

**Proof.** If \( p = 1 \), then case (i) must hold, so we will assume that \( p > 1 \).

A consequence of Lemma 2.3.1 is that if \( s_{i+1} = I \) then \( s_i = J \), and if \( s_{i+1} = J \) then \( s_i = I \). It follows that if \( s_p \in \{ I, J \} \) (as algebra elements), then \( s \) must be an alternating product of \( I \) and \( J \). Since \( s \) is reduced, this forces \( p = 2 \) and either \( s = IJ \) or \( s = JI \). We may therefore assume that \( s_p \notin \{ I, J \} \).

Since \( s_p \notin \{ I, J \} \) and \( s_p \) is a product of commuting generators, at least one of the following two situations must occur.

(a) For some \( 2 \leq i \leq n \), \( s_p \) contains an occurrence of \( E_i \) but not an occurrence of \( E_{i-2} \).

(b) For some \( 0 \leq i \leq n - 2 \), \( s_p \) contains an occurrence of \( E_i \) but not an occurrence of \( E_{i+2} \).

Suppose we are in case (a). In this case, Lemma 2.3.1 means that there must be an occurrence of \( E_{i-1} \) in \( s_{p-1} \); furthermore, \( E_{i-1} \notin \{ E, F \} \), because \( e_{i-1} \) fails to commute with two other generators (\( E_i \) and \( E_{i-2} \)). However, one of these generators, \( E_{i-2} \) does not occur in \( s_p \). It follows that \( s \) is right reducible (via \( E_i \)), which is a contradiction. Case (b) leads to a similar contradiction, again involving right reducibility, which completes the proof. \( \square \)

**Lemma 2.3.3.** Let \( u = u_1su_2su_3 \) be a reduced word in which the occurrences of the generator \( s \) are consecutive, and suppose that every generator in \( u_2 \) not commuting with \( s \) is of the same type, \( t \) say. Then \( u_2 \) contains only one occurrence of \( t \), and \( s \in \{ E, F \} \).

**Proof.** The proof is by induction on the length, \( l \), of the word \( u_2 \). Note that \( u_2 \) must contain at least one generator not commuting with \( s \), or after commutations, we could produce a subword of the form \( ss \). This means that the case \( l = 0 \) cannot occur.

If \( u_2 \) contains only one generator not commuting with \( s \), then after commutations, \( u \) contains a subword of the form \( sts \). This is only possible if \( s \in \{ E, F \} \), and this establishes the case \( l = 1 \) as a special case.

Suppose now that \( l > 1 \). By the above paragraph, we may reduce to the case where \( u_2 = u_4tu_5u_6 \), and the indicated occurrences of \( t \) are consecutive. Since \( u_5 \) is shorter than \( u_2 \), we can apply the inductive hypothesis to show that \( t \in \{ E, F \} \) and \( u_5 \) contains only one generator, \( u \), that does not commute with \( t \). We cannot have \( u = s \), or the original occurrences of \( s \) would not
be consecutive. This means that \( t \) fails to commute with two different generators, contradicting the fact that \( t \in \{E,F\} \) and completing the proof.

**Lemma 2.3.4.** Let \( u \) be a reduced monomial.

(i) Between any two consecutive occurrences of \( E \) in \( u \), there is precisely one letter not commuting with \( E \) (i.e., an occurrence of \( E_1 \)).

(ii) Between any two consecutive occurrences of \( F \) in \( u \), there is precisely one letter not commuting with \( F \) (i.e., an occurrence of \( E_{n-1} \)).

(iii) Between any two consecutive occurrences of \( E_i \) in \( u \), there are precisely two letters not commuting with \( E_i \), and they correspond to distinct generators.

**Proof.** To prove (i), we apply Lemma 2.3.3 with \( s = E \); the hypotheses are satisfied as we necessarily have \( t = E_1 \). The proof of (ii) is similar.

To prove (iii), write \( u = u_1su_2su_3 \) for consecutive occurrences of the generator \( s = E_i \). Since \( s \not\in \{E,F\} \), the hypotheses of Lemma 2.3.3 cannot be satisfied, so \( u_2 \) must have at least one occurrence of each of \( t_1 = E_{i-1} \) and \( t_2 = E_{i+1} \). Suppose that \( u_2 \) contains two or more occurrences of \( t_1 \). The fact that the occurrences of \( s \) are consecutive means that two consecutive occurrences of \( t_1 \) cannot have an occurrence of \( s \) between them. Applying Lemma 2.3.3, this means that there is precisely one generator \( u \) between the consecutive occurrences of \( t_1 \) such that \( t_1u \neq ut_1 \), and furthermore, that \( t_1 \in \{E,F\} \). This is a contradiction, because \( t_1 \) fails to commute with two different generators (\( s \) and \( u \)).

One can show similarly that \( u_2 \) cannot contain two or more occurrences of \( t_2 \). We conclude that each of \( t_1 \) and \( t_2 \) occurs precisely once, as required.

![Figure 1. Direction reversal of an arc](image)

2.4. The map \( \phi \).

**Lemma 2.4.1.** Let \( D \) be a diagram representing a reduced monomial (i.e., it is \( \phi \) applied to a reduced monomial). Then the only way an arc of \( D \) may change direction from left to right or vice versa is as shown in Figure 1 or its mirror image: the turn is performed with a vertical decorated section, and the turn is tight in the sense that the two horizontal arcs connected by a thin dotted line in the diagram arise from the same letter of the monomial.

**Proof.** This is a restatement of Lemma 2.3.4 (iii).
Lemma 2.4.2. Let $D$ be a diagram representing a reduced monomial $u$ (i.e., $D = \phi(u)$).

(i) The diagram $D$ is $L$-decorated at 1 (respectively, $1'$) if and only if the left (respectively, right) descent set of $u$ contains $E$.

(ii) The diagram $D$ is $R$-decorated at $n$ (respectively, $n'$), if and only if the left (respectively, right) descent set of $u$ contains $F$.

(iii) Suppose that $1 \leq i < n$. Then points $i$ and $i + 1$ (respectively, $i'$ and $(i + 1)'$) in $D$ are connected by an undecorated edge if and only if the left (respectively, right) descent set of $u$ contains $E_i$.

Proof. In all three cases, the “if” statements follow easily from diagram calculus considerations, so we only prove the “only if” statements.

Suppose for a contradiction that $D$ is $L$-decorated at 1, but that the left descent set of $u$ does not contain $E$. For this to happen, the arc leaving point 1 must eventually encounter an $L$-decoration, but must first encounter a horizontal arc corresponding to an occurrence of the generator $E_1$. The only way this can happen and be consistent with Lemma 2.4.1 is for the arc to then travel to the eastern edge after encountering $E_1$, then change direction and then travel back to the western edge, as shown in Figure 2. (Note that this can only happen if $n$ is odd, and that as before, the thin dotted lines indicate pairs of horizontal edges that correspond to the same generator.)

By Lemma 2.2.4, this implies that the arc connected to point 1 is $R$-decorated, which in turn means that it cannot also be $L$-decorated. This proves that if $D$ is $L$-decorated at 1 then the left descent set of $u$ contains $E$, and the claim regarding $1'$ and the right descent set is proved similarly. This completes the proof of (i), and the proof of (ii) follows by modifying the above proof in the obvious way.

We now turn to (iii). Suppose for a contradiction that points $i$ and $i + 1$ in $D$ are connected by an undecorated edge, but that the left descent set of $u$ does not contain $E_i$. For this to happen, it must be the case that either (a) it is not the case that the arc leaving point $i$ encounters the northernmost occurrence of $E_i$ before any other generator or (b) it is not the case that the arc leaving point $i + 1$ encounters the northernmost occurrence of $E_i$ before any other generator. (We
allow the possibility that (a) and (b) could both occur. Notice that since the arc crosses the line $x = i + 1/2$, it must encounter a generator $E_i$ at some stage.) We deal with case (a); the treatment of case (b) follows by a symmetrical argument. The only way case (a) can occur consistently with Lemma 2.4.1 is for the situation in Figure 3 to occur, and even this is impossible unless $i$ is odd.

This is a contradiction by Lemma 2.2.4, because it implies that the arc connected to point $i$ is $L$-decorated, and we assumed that it was undecorated. This proves that if points $i$ and $i + 1$ of $D$ are connected by an undecorated edge, then the left descent set of $u$ contains $E_i$, and the claim regarding the right descent set is proved similarly. This completes the proof of (iii).

Lemma 2.4.3. Let $u$ and $u'$ be reduced monomials that map to the same diagram $D$ under $\phi$.

(i) If $u'$ is a product of commuting generators, then $u$ and $u'$ are equal as algebra elements.
(ii) If $u' = IJ$ or $u' = JI$, then $u$ and $u'$ are equal as algebra elements.

Proof. We first prove (i). By Lemma 2.4.2, we must have

$$L(u) = R(u) = L(u') = R(u'),$$

because $u$ and $u'$ are represented by the same diagram.

Suppose that $u'$ contains an occurrence of the generator $E$. This implies (a) that $u$ must contain an occurrence of $E$, because $u$ and $u'$ are represented by the same diagram, and (b) $E \in L(u) \cap R(u)$, by Lemma 2.4.2. Suppose also (for a contradiction) that $u$ contains two occurrences of the generator $E = E_0$. By Lemma 2.3.4, there must be an occurrence of $E_1$ between the first (i.e., leftmost or northernmost) two occurrences of $E_0$.

Since points 1 and 1' of $D$ are connected by an $L$-decorated edge, there must be an occurrence of $E_2$ immediately above the aforementioned occurrence of $E_1$ in order to prevent the edge emerging from point 1 from exiting the box at point 2. (“Immediately above” means that there are no other occurrences of $E_1$ or $E_2$ between the two occurrences mentioned.) In turn, we must have an occurrence of $E_3$ immediately below the aforementioned occurrence of $E_2$ in order to prevent the edge from exiting the box at point 3'. This procedure is only sustainable with respect to Lemma 2.4.1 if $n$ is odd, and in this case, the situation is as shown in Figure 2, with the extra condition that the vertical edge in the top left of the picture is $L$-decorated. There are two ways
this picture can continue to the bottom left consistently with Lemma 2.4.1: either the edge exits
the box at point $1'$ without encountering further generators, or the edge encounters an occurrence
of $E_1$. The first situation cannot occur because it contradicts Lemma 2.4.2 and the hypothesis that
$E \in \mathcal{R}(u)$. The second situation cannot occur because it shows that $u$ is commutation equivalent
to a monomial of the form $vJIJv'$, which contradicts the hypothesis that $u$ be reduced.

We conclude that $u$ contains precisely one occurrence of $E$. By Lemma 2.4.2 and the fact that
$E \in \mathcal{L}(u) \cap \mathcal{R}(u)$, this can only happen if $u$ contains no occurrences of $E_1$.

A similar argument shows that if $u'$ contains an occurrence of the generator $F$, then $u$
contains at most one occurrence of $F$, and it can only contain $F$ if it contains no occurrences of $E_{n-1}$.

It follows that at least one of the four situations must occur:

(a) $u'$ contains $E$ and $u = ED_E$, where $D_E$ contains no occurrences of $E$ or $E_1$;
(b) $u'$ contains $F$ and $u = D_F F$, where $D_F$ contains no occurrences of $E_{n-1}$ or $F$;
(c) $u'$ contains neither $E$ nor $F$.

In cases (a) and (b), there is a corresponding factorization of $u'$, and the result claimed now
follows from the faithfulness of the blob algebra as a diagram calculus for the type-$B$ TL algebra
[2, 11]. For example, in case (a), we have $u' = ED_E'$, and the fact that $E$ commutes with each
generator in each of $D_E$ and $D_E'$ implies that $D_E$ and $D_E'$ map to the same blob diagram, where
the blob in this case is identified with $F$.

Suppose that we are in case (c), but that $u$ contains an occurrence of $E$ or $F$. Because the
diagram $D$ corresponds to $u'$, it cannot have loops, so it must be the case that $u$ is either $L$-
decorated at some point, or $R$-decorated at some point. This contradicts the hypotheses on $u'$,
using Lemma 2.2.4. Since neither $u$ nor $u'$ contains $E$ or $F$, the result follows by the faithfulness
of the diagram calculus for the Temperley–Lieb algebra [12, §6.4]. This completes the proof of (i).

We now prove (ii) in the case where $u' = IJ$; the case $u' = JI$ follows by a symmetrical
argument. Thus, $u$ maps to the same diagram as $IJ$. The fact that $\mathcal{L}(u)$ is the set of generators
in $I$ and $\mathcal{R}(u)$ is the set of generators in $J$ means that $u$ cannot be left or right reducible. By
Proposition 2.3.2 (ii), this immediately means that $u = IJ$. \hfill \Box

2.5. Proof of the theorem.

Lemma 2.5.1. Let $u$ be a reduced monomial and let $D$ be the corresponding diagram. Then $D$
avoids all the features on the left hand sides of Table 1. Furthermore, $D$ contains at most one arc
with more than one decoration.

Proof. The proof is by induction on the length of $u$. If $u$ is a product of commuting generators, or
$u = IJ$, or $u = JI$, the assertions are easy to check, so we may assume that this is not the case.
(This covers the base case of the induction as a special case.)
By Proposition 2.3.2, \( u \) must either be left reducible or right reducible. We treat the case of left reducibility; the other follows by a symmetrical argument.

By applying commutations to \( u \) if necessary, we may now assume that \( u = stv \), where \( s \) and \( t \) are noncommuting generators, and \( t \not\in \{E,F\} \). By induction, we know that the reduced monomial \( tv \) corresponds to a diagram \( D' \) with none of the forbidden features and at most one edge with two decorations.

Suppose that \( t = E_1 \) and \( s = E \). By Lemma 2.4.2, points 1 and 2 of \( D' \) must be connected by an undecorated edge, and the effect of multiplying by \( E \) is simply to decorate this edge. This does not introduce any forbidden features, nor does it create an edge with two decorations, and this completes the inductive step in this case.

The case where \( t = E_{n-1} \) and \( s = F \) is treated similarly to the above case, so we may now assume that \( s, t \not\in \{E,F\} \). We must either have \( s = E_i \) and \( t = E_{i+1} \), or vice versa.

Suppose that \( s = E_i \) and \( t = E_{i+1} \). By Lemma 2.4.2, this means that points \( i+1 \) and \( i+2 \) of \( D' \) are connected by an undecorated edge. The effect of multiplying by \( s \) is then (a) to remove this undecorated edge, then (b) to disconnect the edge emerging from point \( i \) of \( D' \) and reconnect it to point \( i+2 \), retaining its original decorated status, then (c) to install an undecorated edge between points \( i \) and \( i+1 \). This procedure does not create any forbidden features, nor does it create a new edge with more than one decoration.

The case in which \( s = E_{i+1} \) and \( t = E_i \) is treated using a parallel argument, and this completes the inductive step in all cases. \( \square \)

**Lemma 2.5.2.** Let \( u \) be a reduced monomial with corresponding diagram \( D \).

(i) If points 1 and 2 (respectively, \( 1' \) and \( 2' \)) are connected in \( D \) by an edge decorated by \( L \) but not \( R \), then \( u \) is equal (as an algebra element) to a word of the form \( u' = E_1v \) (respectively, \( u' = vE_1E \)).

(ii) If points \( n-1 \) and \( n \) (respectively, \( (n-1)' \) and \( n' \)) are connected in \( D \) by an edge decorated by \( R \) but not \( L \), then \( u \) is equal (as an algebra element) to a word of the form \( u' = F_{n-1}v \) (respectively, \( u' = vE_{n-1}F \)).

**Proof.** We first prove the part of (i) dealing with points 1 and 2. By Lemma 2.4.2, we have \( E \in \mathcal{L}(u) \), so \( u = Ev' \). Now \( v' \) is also a reduced monomial, and by Lemma 2.5.1, \( v' \) corresponds to a diagram \( D' \) with no forbidden features. Since multiplication by \( e \) does not change the underlying shape of a diagram (ignoring the decorations), it must be the case that points 1 and 2 of \( D' \) are connected by some kind of edge. Since \( D \) has no forbidden features and the corresponding edge in \( D \) has no \( R \)-decoration, the only way for this to happen is if the edge connecting points 1 and 2 in \( D' \) is undecorated. By Lemma 2.4.2, this means that \( v' \) is equal as an algebra element to a monomial of the form \( E_1v \), and this completes the proof of (i) in this case.

The other assertion of (i) and the assertions of (ii) follow by parallel arguments. \( \square \)
Proof of Theorem 2.1.3. It is enough to prove the statement using the rescaling in which \( \kappa_L = \kappa_R = 1 \) (see section 4 for details).

It is clear from the generators and relations that the reduced monomials are a spanning set, and that the diagram algebra is a homomorphic image of the abstractly defined algebra. By Lemma 2.5.1, all reduced monomials map to basis diagrams. The only way the homomorphism could fail to be injective is therefore for two reduced monomials \( u \) and \( u' \) to map to the same diagram \( D \), and yet to be distinct as algebra elements.

It is therefore enough to prove that if \( u \) and \( u' \) are reduced monomials mapping to the same diagram, then they are equal as algebra elements. Without loss of generality, we assume that \( \ell(u) \leq \ell(u') \) (where \( \ell \) denotes length).

We proceed by induction on \( \ell(u) \). If \( \ell(u) \leq 1 \), or, more generally, if \( u \) is a product of commuting generators, then Lemma 2.4.3 shows that \( u = u' \). Similarly, if \( u = IJ \) or \( u = JI \), then \( u = u' \), again by Lemma 2.4.3. In particular, this deals with the base case of the induction.

By Proposition 2.3.2, we may now assume that \( u \) is either left or right reducible. We treat the case of left reducibility, the other being similar. By applying commutations if necessary, we may reduce to the case where \( u = stv \), \( s \) and \( t \) are noncommuting generators, and \( t \notin \{ E, F \} \).

Suppose that \( s = E \), meaning that \( t = E_1 \). In this case, points 1 and 2 of \( D \) are connected by an edge decorated by \( L \) but not \( R \). By Lemma 2.5.2 (i), this means that we have \( u' = stv' \) as algebra elements. Since \( u \) and \( u' \) share a diagram, the (non-reduced) monomials \( tu \) and \( tu' \) must also share a diagram. Since \( tst = \kappa_L t = t \), the (reduced) monomials \( tv \) and \( tv' \) also map to the same diagram, \( D' \). However, \( tv \) is shorter than \( u \), so by induction, \( tv = tv' \), which in turn implies that \( u = u' \).

Suppose that \( s = F \), meaning that \( t = E_{n-1} \). An argument similar to the above, using Lemma 2.5.2 (ii), establishes that \( u = u' \) in this case too.

We are left with the case where \( s = E_i \) and either \( t = E_{i+1} \) or \( t = E_{i-1} \) (where \( t \notin \{ E, F \} \)). We will treat the case where \( t = E_{i+1} \); the other case follows similarly. In this case, we have \( tst = t \), and so \( tu = tsv = tv \). It is not necessarily true that \( tu' \) is a reduced monomial, but it maps to the same diagram as \( tv \), which is reduced. After applying algebra relations to \( tu' \), we may transform it into a scalar multiple of a reduced monomial, \( r \). Since reduced monomials map to basis diagrams (Lemma 2.5.1), the scalar involved must be 1. Now the reduced monomials \( tv \) and \( r \) map to the same basis diagram, and \( tv \) is shorter than \( u \), so by induction, we have \( tv = r \) as algebra elements.

Since \( s \in \mathcal{L}(u) \), we have \( s \in \mathcal{L}(u') \) by Lemma 2.4.2, so that \( u' = sv'' \) for some reduced monomial \( v'' \). Since \( sts = s \), we have \( s(tu') = u' \). We have shown that \( tu' = r = tv \), so we have

\[
u' = stu' = stv = u,\]

which completes the proof. □
3. A cellular basis

In what follows we shall make repeated use of the cellular basis for the symplectic blob algebra (or equivalently the affine symmetric Temperley–Lieb algebra). In this section we review the construction of this cellular basis.

Let \( d \in B_{2n}^\phi \) and define \( \#(d) \) to be the number of propagating lines in \( d \). We extend this notation to scalar multiples of \( d \).

Suppose \( \#(d) \geq 2 \). Then there is a unique pair of propagating lines that can be simultaneously deformed to touch the 0-reflection line:

the “closest” propagating lines to the 0-reflection line. This defines a unique inner region that will be black or white when the diagram is coloured.

We define:

\[
\begin{align*}
c(d) &= \begin{cases} 
b & \text{if the inner region is black} \\
w & \text{if the inner region is white} 
\end{cases}
\end{align*}
\]

**Lemma 3.0.3** ([13, lemma 8.2.1]). For all \( d, d' \in B_{2n}^\phi \) we have:

(i) \( \#(dd') \leq \#(d) \)

(ii) If \( \#(dd') = \#(d) \) and \( \#(d) \neq 0 \) then \( c(dd') = c(d) \).

Inspired by this we define for \( 0 < l \leq n \):

\[
B_{2n}^\phi [l] = \{ d \in B_{2n}^\phi \mid \#(d) = 2l \text{ and } c(d) = b \}
\]

\[
B_{2n}^\phi [-l] = \{ d \in B_{2n}^\phi \mid \#(d) = 2l \text{ and } c(d) = w \}
\]

and

\[
B_{2n}^\phi [0] = \{ d \in B_{2n}^\phi \mid \#(d) = 0 \}.
\]

We define for \( -n \leq l \leq n - 1 \)

\[
B_{2n}^\phi (l) = B_{2n}^\phi [l] \cup \bigcup_{-|l| < a < |l|} B_{2n}^\phi [a]
\]

and \( I_{2n}^\phi (l) \) to be the ideal of \( B_{2n}^\phi \) generated by \( B_{2n}^\phi [l] \)

**Proposition 3.0.4** ([13, proposition 8.2.2]). The ideal \( I_{2n}^\phi (l) \) has basis \( B_{2n}^\phi (l) \).
We have set inclusions:

\[
\begin{align*}
& B_{2n}(n-1) \leftarrow \cdots \leftarrow B_{2n}^{0} \leftarrow B_{2n}^{1} \leftarrow B_{2n}^{2} \leftarrow \cdots \\
& B_{2n}^{0}(n) \leftarrow \cdots \leftarrow B_{2n}^{0}(-1) \leftarrow B_{2n}^{0}(-2) \leftarrow \cdots \\
& B_{2n}^{0}(-n) \leftarrow \cdots \leftarrow B_{2n}^{0}(-(n-1)) \leftarrow B_{2n}^{0}(-(n-2)) \leftarrow \cdots \\
& B_{2n}^{0}(-n) \leftarrow \cdots \leftarrow B_{2n}^{0}(-(n-1)) \leftarrow B_{2n}^{0}(-(n-2)) \leftarrow \cdots
\end{align*}
\]

and this passes to a subideal structure ([13, proposition 8.2.2]). We can use the ideals $I_{2n}^{0}(l)$ to define a cellular structure on $b_{2n}^{0}$. We define $S_{l} = I_{2n}^{0}(-l)$ and $T_{l} = S_{l} + I_{2n}^{0}(l)$. We have a chain of ideals:

\[
b_{2n}^{0} = S_{n} \supseteq T_{n-1} \supseteq S_{n-1} \supseteq T_{n-2} \supseteq \cdots \supseteq T_{1} \supseteq S_{1} \supseteq S_{0}.
\]

**Theorem 3.0.5** ([13, theorem 8.2.8]). The above chain of ideals is a cellular chain for $b_{2n}^{0}$ and the set $B_{2n}^{0}$ is a cellular basis for $b_{2n}^{0}$.

We define for $l \in \{-n, -n+1, \ldots, n-1\}$ and $d \in B_{2n}^{0}[l]$:

\[
S_{2n}^{d}(l) := \frac{b_{2n}^{0}d + T_{[l]-1}}{T_{[l]-1}}.
\]

These are the cell modules for $b_{2n}^{0}$ (which have the flavour of Specht modules).

**Theorem 3.0.6** ([13, theorem 8.2.9]). If $\delta, \delta_L, \delta_R, \kappa_L, \kappa_R,$ and $\kappa_{LR}$ are all units then $b_{2n}^{0}$ is quasi-hereditary with heredity chain as above.

Thus in the case where all the parameters are units then the cell modules defined above are standard modules which depend only on $l$ (and not on $d$) and there are exactly $2n$ simple modules for $b_{2n}^{0}$.

We may label the simple modules with the following poset:
where 0 is the maximal element and \(-n\) is the minimal element. When the algebra is quasi-hereditary with the above poset, this has important consequences for the representation theory of the algebra.

**Remark 3.0.7.** The parameter \(\kappa_{LR}\) can only appear in expressions where there are diagrams with no propagating lines. Thus, it has no effect on the action of the algebra on standard modules whose basis has at least one propagating line. Thus, to study standard modules with non-zero label we could safely ignore the value of \(\kappa_{LR}\) and not affect the representation theory. We have several successive quotients:

\[
H(\tilde{C}) \twoheadrightarrow 2BTL \twoheadrightarrow b^\phi_{2n} \twoheadrightarrow \frac{b^\phi_{2n}}{I^\phi_{2n}(0)}
\]

where \(H(\tilde{C})\) is the Hecke algebra of type \(\tilde{C}_n\), \(I^\phi_{2n}(0)\) is the ideal in \(b^\phi_{2n}\) generated by elements with no propagating lines, and \(2BTL\) is defined in section 2. The algebra \(\frac{b^\phi_{2n}}{I^\phi_{2n}(0)}\) is independent of \(\kappa_{LR}\).

The \(b^\phi_{2n}\) version of proposition 1.1.5 is the following:

**Proposition 3.0.8** ([13, proposition 8.1.1]). Suppose \(\delta_L \neq 0\), and set \(e' = \frac{e}{\delta_L}\). Then we have

\[
e' b^\phi_{2n}(\delta, \delta_L, \delta_R, \kappa_L, \kappa_R, \kappa_{LR})e' \cong b^\phi_{2(n-1)}(\delta, \kappa_L, \delta_R, \kappa_L, \kappa_R, \kappa_{LR})\]

We also have a “dual” version of proposition 3.0.8 (the \(b^\phi_{2n}\) version of proposition 1.1.6).

**Proposition 3.0.9.** Suppose \(\delta_R \neq 0\), and set \(f' = \frac{f}{\delta_R}\). Then we have

\[
f' b^\phi_{2n}(\delta, \delta_L, \delta_R, \kappa_L, \kappa_R, \kappa_{LR})f' \cong b^\phi_{2(n-1)}(\delta, \delta_L, \kappa_R, \kappa_L, \delta_R, \kappa_{LR})\]

We can thus use localisation and globalisation functors to inductively prove many properties for \(b^\phi_{2n}\). (Our choice of relations was partly designed to make this proposition work, but still obtain an accessible quotient of the Hecke algebra of type \(\tilde{C}\).) These functors will be introduced later in section 5.

### 4. Good parametrisations

The symplectic blob algebra is a quotient of the Hecke algebra of type \(\tilde{C}\), which has three parameters (itself a quotient of the group algebra of a Coxeter–Artin system). Yet our definition has six. One of the parameters, \(\kappa_{LR}\), is used to make the quotient ‘smaller’ than the Hecke algebra (indeed finite dimensional), just as the three Hecke parameters make the Hecke algebra smaller than the braid group. Provided the appropriate parameters are units, two of the other parameters can be scaled away, as we shall see, leaving a parameter set corresponding to that of the Hecke algebra.

However, our aim here is to determine the exceptional representation theory of the symplectic blob (and hence part of the Hecke/braid representation theory). Therefore we are interested in
working over rings from which we can base change to the exceptional cases. Clearly this requires some up-front knowledge of what the exceptional cases are. This bootstrap problem can be solved in significant part by looking at Gram matrices for standard modules, exactly as in [15, 16]. We give details in Section 4.1, but the good blob algebra parametrisation [14] leads us to consider four parameters $q, l, r$ and $\kappa_{LR}$, determining $\delta = [2]; \delta_L = [l]; \delta_R = [r]; \kappa_L = [l - 1]; \kappa_R = [r - 1]$. Here the notation $[n]$ is used to denote the $q$-number $\frac{q^n - 1}{q - 1}$, with $q \in k$. (If $k = \mathbb{C}$ then it is usual to take $q = \exp(i\pi/m)$, thus defining an equivalent parameter $m$.)

The new parameters $q, l, r$ may not even be real, if one wishes to work in the complex setting. But if $l$ and $r$ are integral then we can work in the ring of Laurent polynomials (and the integral cases are the most singular, as flagged by the Gram determinants).

4.1. Gram matrices. Given an algebra $A$ with an involutive antiautomorphism, and an $A$-module with a simple head and a natural inner product, then a condition for semisimplicity of a specialisation is that this inner product be non-degenerate. Conversely, for non-semisimple cases, we look for conditions under which the Gram determinant vanishes. Depending on the number and incarnation of parameters, this vanishing may appear to describe a complicated variety. However experience shows that for a good choice of incarnation the non-semisimplicity condition can often be stated simply. For an initial illustrative example, consider Temperley–Lieb cell modules in the ‘upper half-diagram’ bases, such as the following $n = 5, l = 3$ case. The basis (acted on by diagrams from above, ignoring irrelevant arcs below) may be written

$$H^3_3(5, 3):$$

whereupon the inner product is computed using the array in Figure 4 (via the usual Temperley–Lieb diagram inversion antiautomorphism). This gives immediately the Gram matrix
\[
M_{TL}^5(3) = \begin{pmatrix}
\delta & 1 & 0 & 0 \\
1 & \delta & 1 & 0 \\
0 & 1 & \delta & 1 \\
0 & 0 & 1 & \delta
\end{pmatrix}
\]

(note the slight difference in the way diagrams act on the basis, cf. usual diagram multiplication). The generalisation to \(M_{TL}^n(n-2)\) will be obvious. Evaluation of the determinant is also straightforward. Indeed, more generally still, writing \(M' \oplus_1 M''\) for the almost block diagonal matrix

\[
M' \oplus_1 M'' := \begin{pmatrix}
M' & 0 \\
1 & \\
0 & M''
\end{pmatrix}
\]

and \(\mu_n(M) = M \oplus_1 (\delta) \oplus_1 (\delta) \oplus_1 (\delta) \cdots (\delta) \cdots (\delta) (n+1\, \text{terms})\) for any initial matrix \(M\) so, for example, that

\[M_{TL}^n(n-2) = \mu_{n-2}(\delta)\],

we have

\[
det(\mu_n(M)) = \delta \ det(\mu_{n-1}(M)) - det(\mu_{n-2}(M)) \quad (n > 0)
\]

where \(\mu_{-1}(M) = M^{dd}\) (the matrix \(M\) with the last row and column removed).

As is well known the recurrence

\[M(n) = [2]M(n-1) - M(n-2)\]

is solved by \(M(n) = \alpha [s+n]\) for any constants \(s, \alpha\). (Two pieces of initial data, such as \(M(0), M(1)\), fix them via \(M(0) = \alpha [s], M(1) = \alpha [s+1]\).)

Comparing (4) with (3) then leads us to the parametrisation \(\delta = [2]\), which makes \(det(M_{TL}^n(n-2)) = [n]\). The vanishing of this form is well understood, requiring \(q\) to be a root of 1. (Although this only gives one Gram determinant per algebra, abstract representation theory tells us that it gives a complete picture in the Temperley–Lieb case.)

A similar analysis proceeds for the ordinary and symplectic blob algebras. For example a basis for one of our symplectic cell modules is the left-most (labelling) column of the array in Figure 5. In fact this picture simultaneously encodes three of our \(n = 4\) cell modules \(S_{2n}(l)\) (those with label \(l = -2, 1, -1\), depending on how blobs are understood to act on the two propagating lines (the leftmost of which is marked with a \(\times\) to flag this choice). Indeed, by omitting the last row and column we get the corresponding array for (two cell modules of) the ordinary blob algebra.
In the blob case, choosing a parametrisation in which $e$ is idempotent, for arithmetic simplicity, we thus have Gram matrices $M_n^b(n-2) = \mu_n(B_+)$, and $M_n^b(-(n-2)) = \mu_{n-1}(B_-)$, where

$$B_+ = \begin{pmatrix} \kappa_L & \kappa_L \\ \kappa_L & \delta \end{pmatrix}, \quad B_- = \begin{pmatrix} \kappa_L & \kappa_L & 1 \\ \kappa_L & \delta & 1 \\ 1 & 1 & \delta \end{pmatrix}$$

In other words we have the same bulk recurrence as for Temperley–Lieb, but more interesting initial conditions. The idea is to try to parametrise $\kappa_L$ so that the initial conditions conform to the natural form $M(n) = \alpha [s + n]$, for some choice of $\alpha, s$, in each case. We have

$$(B_+) : \quad \kappa_L = \alpha[s + 1], \quad \kappa_L([2] - \kappa_L) = \alpha[s + 2],$$

Eliminating $\alpha$, one sees that the parametrisation $\kappa_L = \frac{[l]}{[l+1]}$ is indicated (or equivalently $\kappa_L = \frac{[l]}{[l-1]}$ by exchanging $l$ and $-l$). We have chosen the symbol $l$ for the $s$-parameter in $\kappa_L$ for obvious reasons. The same parametrisation works for $B_-$. This gives $\det(M_n^b(n-2)) = \frac{[l]}{[l+1]}[n+l]$ and $\det(M_n^b(-(n-2))) = \frac{[l+2]}{[l+1]}[2-n+l]$. Once again this is in a convenient form to simply characterise the singular cases for all $n$. The result (as is well known [14]) is a generalised form of the kind of alcove geometry that occurs in Lie theory (or more precisely in quantum group representation theory when $q$ is an $m$-th root of 1). The blocks of the algebra are described by orbits of an affine reflection group, with the separation of affine walls determined by $m$, and the ‘$\rho$-shift’ of the origin determined by $l$. 
Returning finally to the symplectic case, choosing a normalisation in which e and f are idempotent, the most complicated of the three cell choices gives (in the obvious generalisation to any $n$, and $n-2$ propagating lines) the $(n+1) \times (n+1)$ matrix:

\[
M'(n, \kappa_L, \kappa_R) = \begin{pmatrix}
\kappa_L & \kappa_L & 1 & 0 & 0 & 0 & \cdots & 0 \\
\kappa_L & [2] & 1 & 0 & 0 & 0 & \cdots & 0 \\
1 & 1 & [2] & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & [2] & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & 1 & [2] & 1 & 0 \\
0 & \cdots & 0 & 0 & 0 & 1 & [2] & \kappa_R \\
0 & \cdots & 0 & 0 & 0 & 0 & \kappa_R & \kappa_R
\end{pmatrix}
\]

(5)

This is clearly just another variant of the Temperley–Lieb Gram matrix, with yet more interesting ‘boundary conditions’. Laplace expanding we have:

\[
\det(M'(n, \kappa_L, \kappa_R)) = \kappa_R \frac{[l + 2]}{[l + 1]^2} ([2 - n + l] - \kappa_R[3 - n + l]).
\]

Again one seeks to parametrise $\kappa_R$ by a parameter $r$ such that, for every $n$, this determinant becomes a simple product of quantum numbers, quantum-integral when $r$ is integral. Clearly $\kappa_R = \frac{[r]}{[r+1]}$ does this (the precise form is chosen for symmetry with $\kappa_L$). One obtains:

\[
\det(M'(n, \kappa_L, \kappa_R)) = \frac{[r][l + 2]}{[r + 1][l + 1]} \frac{[l - (r + n - 2)]}{[r + 1][l + 1]}
\]

(and similarly for the others — we give complete results for these cases and more, and demonstrate that our normalisation choice was without loss of generality, in Section 6). Once again then, the most singular cases are when $q$ is a root of 1 and $l$ and $r$ are integral. It is intriguing to speculate on the corresponding alcove geometry (cf. the ordinary blob case above) — see later. However there is a significant difference in this symplectic case, in that here abstract representation theory does not allow us simply to reconstruct all other Gram determinants from this subset. It is for this reason that we return to report some less straightforward Gram determinant results in Section 6.
4.2. Four rescalings. We tabulate the parameter shifts associated to four choices of generator rescalings, such that only four free parameters remain in each case:

| way | gen. shifts | parameter shifts |
|-----|-------------|------------------|
| 1   | $e \mapsto \frac{f}{\kappa_L}$ | $\delta \mapsto \delta_L \mapsto \delta_R \mapsto \kappa_L \mapsto \kappa_R \mapsto \kappa_{LR}$ |
| 2   | $\frac{e}{\kappa_L} \mapsto \frac{f}{\kappa_R}$ | $\delta \mapsto \frac{\delta_L}{\kappa_L} \mapsto \frac{\delta_R}{\kappa_R} \mapsto 1 \mapsto 1 \mapsto \kappa_{LR}/(\delta_R \kappa_L)$ |
| 3   | $\frac{e}{\kappa_L} \mapsto \frac{f}{\delta_R}$ | $\delta \mapsto \frac{\delta_L}{\kappa_L} \mapsto 1 \mapsto 1 \mapsto \frac{\delta_R}{\kappa_R} \mapsto \kappa_{LR}/(\delta_R \kappa_L)$ |
| 4   | $\frac{e}{\kappa_L} \mapsto \frac{f}{\delta_R}$ | $\delta \mapsto 1 \mapsto \frac{\delta_R}{\kappa_R} \mapsto \frac{\delta_L}{\kappa_L} \mapsto 1 \mapsto \kappa_{LR}/(\delta_L \kappa_R)$ |

This is easy to check. For instance, we know that the symplectic blob has generators that satisfy the relations given in the presentation in section 2, and since the parameter change is given by rescaling the generators, it is easy to check what the new values of the parameters should be.

Note, however that only using four parameters obscures the swapping of the parameters induced by the globalisation and localisation functors $G, G', F$ and $F'$ which will be introduced later.

4.3. Two more ways to reparametrise. The ‘good’ parametrisation is recalled in the row labelled blob in our table. Scaled in “Way 2”, this is similar to the parametrisation used by De Gier and Nichols in [3] (row DN). They in effect have four parameters, $q, \omega_1, \omega_2$ and $b$, but significantly they reparametrise $b$ in terms of box numbers and $\theta$. We will use the parametrisation in row GMP with

$$\kappa_{LR} = \begin{cases} 
\frac{[w_1 + w_2 + \theta + 1]}{2} & \text{if } n \text{ even} \\
-\frac{[w_1 - w_2 + \theta]}{2} & \text{if } n \text{ odd.}
\end{cases}$$

which is connected to the parametrisation of De Gier and Nichols via “Way 2”. We use GMP to avoid too many terms in the denominator for the Gram determinants calculated later.

5. On exceptional cases

Throughout this section we will assume that all the parameters are units, so that $b_{2n}^\phi$ is quasi-hereditary.

5.1. Globalisation and localisation functors. We now define various globalisation and localisation functors. Their construction is very general and further information about their properties may be found in [13, section 2], [8] or [14].
Recall that $e' = \frac{e}{d_L}$ and $f' = \frac{f}{d_R}$. Let

$$G : e'b_{2n}^\phi e' \mod \to b_{2n}^\phi \mod$$

$$M \mapsto b_{2n}^\phi e' \otimes e'b_{2n}^\phi M$$

be the globalisation functor with respect to $e$ and

$$G' : f'b_{2n}^\phi f' \mod \to b_{2n}^\phi \mod$$

$$M \mapsto b_{2n}^\phi f' \otimes f'b_{2n}^\phi M$$

be the globalisation functor with respect to $f$. The functors $G$ and $G'$ are both right exact and $G \circ G' = G' \circ G$.

We also have localisation functors:

$$F : b_{2n}^\phi \mod \to e'b_{2n}^\phi \mod$$

$$M \mapsto e'M$$

$$F' : b_{2n}^\phi \mod \to f'b_{2n}^\phi \mod$$

$$M \mapsto f'M.$$ 

The functors $F$ and $F'$ are both exact. They also map simple modules to simple modules (or zero) [8]. We have $F \circ G = \text{id}$ and $F' \circ G' = \text{id}$.

We will abuse notation slightly and identify a module for the algebra $e'b_{2n}^\phi e'$ with its image via the isomorphism of Proposition 3.0.8 as a module for $b_{2n-2}$ and similarly for $f'b_{2n}^\phi f'$. The functors applied will make it clear which parameter choice we are making.

Standards behave well with respect to globalisation [13, proposition 8.2.10]:

$$GS_{2n-2}(l) = S_{2n}(-l)$$

$$G'S_{2n-2}(l) = S_{2n}(l).$$

If we have a standard module $S_{2n}(l)$ that is not simple then there is a simple module $L_{2n}(m)$ in the socle of $S_{2n}(l)$ with $m < l$, because $S_{2n}(l)$ is a standard module (and using the definition of standard modules for quasi-hereditary algebras). Since $L_{2n}(m)$ is the head of $S_{2n}(m)$ there thus exists an $m < l$ such that there is a non-zero map

$$S_{2n-2}(m) \xrightarrow{\psi} S_{2n-2}(l).$$

Globalising then gives us:

$$S_{2n}(-m) \xrightarrow{G\psi} S_{2n}(-l)$$

$$S_{2n}(m) \xrightarrow{G'\psi} S_{2n}(l)$$

with $G\psi$ and $G'\psi$ both non-zero. Note that the non-zero map is non-zero on the simple head of the standard module, and hence the head, $L_{2n}(m)$, must be a composition factor of the image in
$S_{2n}(±l)$. As $m < l$, this factor is not equal to $L_{2n}(l)$ or $L_{2n}(-l)$, which implies that $S_{2n}(l)$ and $S_{2n}(-l)$ are also not simple. Thus parameter choices that give non-simple standards propagate. (We must take care with the parameter swapping effect of $G$ and $G'$, however.)

We set $L_{2n}(l)$ to be the irreducible head of the standard module $S_{2n}(l)$. We have the following proposition.

**Proposition 5.1.1.** We have

$$FL_{2n}(l) = \begin{cases} 0 & \text{if } l = -n \text{ or } l = -n + 1, \\ L_{2n-2}(l) & \text{otherwise}; \end{cases}$$

$$F'L_{2n}(l) = \begin{cases} 0 & \text{if } l = -n \text{ or } l = n - 1, \\ L_{2n-2}(l) & \text{otherwise}; \end{cases}$$

$$FS_{2n}(l) = S_{2n-2}(l) \text{ for } l \neq -n \text{ and } l \neq -n + 1;$$

$$F'S_{2n}(l) = S_{2n-2}(l) \text{ for } l \neq -n \text{ and } l \neq n - 1.$$

**Proof.** As $e$ or $f$ may be taken as part of a heredity chain we obtain using [14, proposition 3] or [5, appendix A1] the result for standard modules. We may use [13, proposition 2.0.1] and the above result for the globalisation functor to determine which simple modules $F$ or $F'$ maps to zero. □

5.2. Some simple standard modules. By counting the number of diagrams in $B_{2n}[l]$ for $l = -n, -n + 1 \neq 0, n - 1 \neq 0$ (and $n - 2 \neq 0$) and considering the action of $b_{2n}^\phi$ it is clear that

**Lemma 5.2.1.** Suppose that $n \geq 2$, then $S_{2n}(-n)$, $S_{2n}(-n + 1)$ and $S_{2n}(n - 1)$ are all one-dimensional and hence are irreducible. If further $n \geq 3$ then $S_{2n}(n - 2)$ is one-dimensional and hence irreducible.

When $b_{2n}^\phi$ is quasi-hereditary and if $n \geq 0$ then $S_{2n}(-n) = L_{2n}(-n)$ is the trivial module, i.e. the one-dimensional module where all the generators of the algebra act as zero. If $n \geq 2$ then the module $S_{2n}(-n + 1) = L_{2n}(-n + 1)$ is the one-dimensional module where $f$ acts as multiplication by $\delta_R$ and all the other generators of the algebra act as zero. Similarly, if $n \geq 2$ then the module $S_{2n}(n - 1) = L_{2n}(n - 1)$ is the one-dimensional module where $e$ acts as multiplication by $\delta_L$ and all the other generators of the algebra act as zero.

We may similarly note that if $n \geq 3$ then the module $S_{2n}(n - 2) = L_{2n}(n - 2)$ is the one-dimensional module where $e$ acts as multiplication by $\delta_L$, $f$ acts as multiplication by $\delta_R$ and all the other generators of the algebra act as zero.

We use $\text{Ext}^i(-, -)$ to denote the right derived functors of $\text{Hom}(-, -)$ which may be defined in the usual way in mod $b_{2n}^\phi$ as $b_{2n}^\phi$ is quasi-hereditary (and so there are enough projectives and injectives).
We also use \([M : L]\) to denote the multiplicity of \(L\), a simple module as a composition factor of a (finite dimensional) module \(M\).

We also note that the algebra \(b_{2n}^\phi\) has a simple-preserving duality — namely the one induced by the algebra anti-automorphism that turns diagrams upside down.

We refer the reader to [5, appendix A] or [4] for the definition and general properties of quasi-hereditary algebras.

**Lemma 5.2.2.** If \(b < a\) and \([S_{2n}(a) : L_{2n}(b)] = 0\), then
\[
\text{Ext}^1(L_{2n}(a), L_{2n}(b)) = \text{Ext}^1(L_{2n}(b), L_{2n}(a)) = 0.
\]

**Proof.** Assume for a contradiction that there is a non-split extension of \(L_{2n}(b)\) by \(L_{2n}(a)\) for \(a > b\) and \([S_{2n}(a) : L_{2n}(b)] = 0\). Now note that \(\text{Ext}^1(L_{2n}(a), L_{2n}(b)) \cong \text{Ext}^1(L_{2n}(b), L_{2n}(a))\) as \(b_{2n}^\phi\) has a simple-preserving duality. Thus we take \(E\) to be the non-split extension defined by the short exact sequence
\[
0 \rightarrow L_{2n}(b) \rightarrow E \rightarrow L_{2n}(a) \rightarrow 0.
\]
Now as \(S_{2n}(a)\) is a standard module, it is the largest \(b_{2n}^\phi\) module with simple head \(L_{2n}(a)\) and all other composition factors having labels less than \(a\). Thus there must be a surjection \(S_{2n}(a) \rightarrow E\). This implies that \([S_{2n}(a), L_{2n}(b)] \neq 0\), the desired contradiction. \(\square\)

Thus, for \(n \geq 3\), by quasi-heredity, and the previous lemma there are no non-split extensions between the modules \(L_{2n}(-n), L_{2n}(n - 1), L_{2n}(-n + 1)\) and \(L_{2n}(n - 2)\).

As \(b_{2n}^\phi\) has a simple-preserving duality, when we are coarsening the quasi-hereditary order we need only consider the composition factors of the standard modules. In other words, if two adjacent labels \(a > b\) in our original poset satisfy \([S_{2n}(a), L_{2n}(b)] = 0\), then the relation \(a > b\) can be removed from the poset and these labels do not need to be comparable in a poset giving a quasi-hereditary order. This procedure is discussed in greater detail after Lemma 1.1.1 in [6].

### 5.3. Some composition multiplicities and coarsening the labelling poset.

**Proposition 5.3.1.** Suppose \(l \geq 2\) and \(n \geq 2\). Then we have
\[
[S_{2n}(l - 1) : L_{2n}(\pm l)] = 0
\]
and
\[
[S_{2n}(-l + 1) : L_{2n}(\pm l)] = 0.
\]
Suppose further that \(l \geq 3\) and \(n \geq 3\). Then we have
\[
[S_{2n}(l - 2) : L_{2n}(\pm l)] = 0,
\]
and
\[
[S_{2n}(-l + 2) : L_{2n}(l)] = 0.
\]
(We interpret \([M : L_{2n}(n)]\) to mean zero.)

**Proof.** We first prove that \([S_{2n}(\pm(l - 1)) : L_{2n}(-l)] = 0\) by induction on \(k = n - l\). The base case, \(k = 0\), follows from Lemma 5.2.1. This case also contains the case \(n = 2\), so we may assume that \(n > 2\) and \(k > 0\). Proposition 5.1.1 now shows that

\([S_{2n}(\pm(l - 1)) : L_{2n}(-l)] = [F' S_{2n}(\pm(l - 1)) : F' L_{2n}(-l)] = [S_{2n-2}(\pm(l - 1)) : L_{2n-2}(-l)]\)

which completes the inductive step. The same line of argument proves the third assertion, namely that \([S_{2n}(l - 2) : L_{2n}(-l)] = 0\) if \(n \geq 3\) and \(l \geq 3\).

Next, we prove that \([S_{2n}(\pm(l - 1)) : L_{2n}(l)] = 0\) by induction on \(k = n - l\). The case \(k = 0\) follows from Lemma 5.2.1. If \(k > 0\), then Proposition 5.1.1 shows that

\([S_{2n}(\pm(l - 1)) : L_{2n}(l)] = [FS_{2n}(\pm(l - 1)) : FL_{2n}(l)] = [S_{2n-2}(\mp(l - 1)) : L_{2n-2}(-l)]\)

which reduces to a previous case and completes the proof of the first two assertions. The same line of argument also reduces the fourth assertion (that \([S_{2n}(-l + 2) : L_{2n}(l)] = 0\) if \(n \geq 3\) and \(l \geq 3\)) to previously proved assertions.

\(\square\)

The ultimate aim would be to find some “alcove like” combinatorics for the labelling poset for the simple modules. In other words, as in the Temperley–Lieb case where the representation theory is controlled by \(\hat{A}_1\) type alcove combinatorics, we should be able to find a labelling poset that gives us alcove combinatorics for some affine Weyl group, mirroring the fact that this algebra is a quotient of a Hecke algebra of \(\hat{C}\) type. Since the above result on composition factors is true for any specialisation of the parameters, the intriguing consequence of this result is that the two labels \(\pm l\) and \(\pm(l - 1)\) need never be comparable (providing \(l \geq 2\)).

Another consequence of this proposition using Lemma 5.2.2 is that there are no non-split extensions between the simple modules \(L_{2n}(\pm l)\) and \(L_{2n}(\pm(l - 1))\), provided \(l \geq 2\).

**Lemma 5.3.2.** We have \([S_{2n}(l - 4) : L_{2n}(-l)] = 0\) for \(5 \leq l \leq n\) and \([S_{2n}(-(l - 4)) : L_{2n}(l)] = 0\) for \(5 \leq l \leq n - 1\).

**Proof.** If \(n > l\), we have

\([S_{2n}(l - 4) : L_{2n}(-l)] = [F' S_{2n}(l - 4) : F' L_{2n}(-l)] = [S_{2n-2}(l - 4) : L_{2n-2}(-l)]\)

and

\([S_{2n}(-(l + 4)) : L_{2n}(l)] = [FS_{2n}(-(l + 4)) : FL_{2n}(l)] = [S_{2n-2}(l - 4) : L_{2n-2}(-l)].\)

The result will then follow by induction on \(n - l\) if we can show that \([S_{2n}(n - 4) : L_{2n}(-n)]\) is zero.

Since all possible composition factors of \(S_{2n}(n - 4)\) (apart from the simple head \(L_{2n}(n - 4)\)) cannot extend each other, it follows that if \(L_{2n}(-n)\) is a composition factor of \(S_{2n}(n - 4)\) then it
must embed in $\mathcal{S}_{2n}(n-4)$. Thus it is enough to show that there is no embedding of $L_{2n}(-n)$ into $\mathcal{S}_{2n}(n-4)$ when the parameters are invertible.

Now, $\mathcal{S}_{2n}(n-4)$ is generated by $ee_2f$ (modulo $I_{2n}(n-5) + I_{2n}(-n+5)$) and so has basis given by

$$\{ee_1ee_2f, e_1ee_2f, ee_2f, ee_3ee_2f, \ldots, e_{n-1}\cdots e_4ee_2f, fe_{n-1}\cdots e_4ee_2f\}.$$ 

Let $v = (a_0, a_1, \ldots, a_n) \in \mathcal{S}_{2n}(n-4)$ with respect to this basis. If $v$ generates a one-dimensional submodule of $\mathcal{S}_{2n}(n-4)$ isomorphic to the trivial module ($L_{2n}(-n)$) then $e$, $f$, and $e_i$ must all act trivially on $v$. Thus

$$0 = ev = (\delta_L a_0 + a_1, 0, \delta_L a_2, \ldots, \delta_L a_n)$$

and so $a_2 = a_3 = \cdots = a_n = 0$ as $\delta_L \neq 0$ and $\delta_L a_0 + a_1 = 0$. So $v = (a_0, -\delta_L a_0, 0, 0, \ldots, 0)$. We also need

$$0 = fv = (\delta_R a_0, -\delta_R \delta_L a_0, 0, 0, \ldots, 0)$$

and so $a_0 = a_1 = 0$ as $\delta_R \neq 0$. Thus $v = 0$ and there is no submodule of $\mathcal{S}_{2n}(n-4)$ isomorphic to the trivial module. \qed

We may now produce a poset that works for all parametrisations for which the parameters are units. The next section will show that this poset cannot be coarsened further for all unit parametrisations. Thus, we can begin to form a picture of what our “alcove geometry” must look like.

**Proposition 5.3.3.** The affine symmetric Temperley–Lieb algebra is quasi-hereditary with the same standards, $\mathcal{S}_{2n}(l)$, and with poset:
It is possible to draw a more planar version of the above at the cost of not having elements that are lower down in the order, lower down on the page:

6. SOME GENERAL GRAM DETERMINANTS

The Gram determinant of $S_{2n}(0)$ has been determined in [3] using a particular basis of the standard module $S_{2n}(0)$, also known as the “spine module” for a particular parametrisation. In this section we calculate the Gram determinant for the $n + 1$ dimensional standard modules, and give some empirical justification for our chosen parametrisation. A rigorous definition of a Gram determinant may be found in [13, equation (40), section 8]. In essence what we do is define a suitable inner product on the module and then take the determinant of the square matrix formed by the inner products between all the basis elements for the module.

Using the definition of $S_{2n}(l)$ we can always find a monomial basis for $S_{2n}(l)$, by letting the generators of $b_{2n}^d$ act on $d$, the diagram that generates $S_{2n}(l)$. If $b_1$, $b_2$ are two such monomial basis elements then they have the same lower half diagram, which is the same as the lower half diagram of $d$, denoted $\langle d \rangle$. We let $b_2^d$ be the element obtained from $b_2$ by turning all its component diagrams upside down. When $b_1$ is multiplied by $b_2^d$ we obtain a scalar multiple of $\langle d \rangle \langle d \rangle$, possibly zero if $d_2^d = 1$ has fewer than $|2l|$ propagating lines. This scalar is then the value of the inner product of $b_2$ by $b_1$.

Since for this section $l \neq 0$, we need not worry about powers of $\kappa_{LR}$ and the Gram determinant may be calculated in this way. We continue to use $\Gamma_{2n}(l)$ to denote the Gram determinant of $S_{2n}(l)$.

6.1. SOME LOW RANK CALCULATIONS. We first do some low rank calculations with the original parametrisation introduced in section 1.

Case $m = 1$:

We see from [13, figure 3] that $b_2^d$ has dimension $5 = 2^2 + 1$, and two standard modules. The first
one is simple: \( S_2(-1) = L_2(-1) \) and the second \( S_2(0) \), is two dimensional. Either \( S_2(0) = L_2(0) \) and \( b_2^0 \) is semi-simple or \( \dim L_2(0) = 1 \) and \( b_2^0 \) is not semi-simple. Now \( S_2(0) \cong b_2^0 e \cong b_2^0 f \), and so it has basis \( \{ e, fe \} \) and Gram matrix

\[
\begin{pmatrix}
\delta_L & \kappa_{LR} \\
\kappa_{LR} & \delta_R \kappa_{LR}
\end{pmatrix}
\]

which gives the Gram determinant (up to a \( \pm \) sign)

\[
\Gamma_2(0) = \kappa_{LR}(\delta_L \delta_R - \kappa_{LR})
\]

\[
= -\left[ \frac{w_1 - w_2 + \theta}{2} \right] \left[ \frac{w_1 - w_2 - \theta}{2} \right] \left[ \frac{w_1 - w_2 + \theta}{2} \right] \left[ \frac{w_1 - w_2 - \theta}{2} \right]
\]

\[
= -\left[ \frac{w_1 - w_2 + \theta}{2} \right] \left[ \frac{w_1 - w_2 - \theta}{2} \right] \left[ \frac{w_1 + w_2 + \theta}{2} \right] \left[ \frac{w_1 + w_2 - \theta}{2} \right]
\]

where the last line gives this determinant in terms of the parametrisation introduced at the end of subsection 4.3. (NB: to obtain this, we used the quantum number identity \( [a][b] = [a + b - 1] + [a + b - 3] + \cdots [a - b + 1] \).)

Case \( n = 2 \):

We see from [13, figure 3] that \( \dim b_4^0 = 10 = 4^2 + 1 + 1 + 1 \). We have \( S_4(-2) = L_4(-2) \), \( S_4(-1) = L_4(-1) \) and \( S_4(1) = L_4(1) \) as noted in Proposition 5.2.1. The dimension of the remaining standard module \( S_4(0) \) is 4. It has Gram determinant (up to a \( \pm \) sign)

\[
\Gamma_4(0) = \kappa_{LR}(\kappa_{LR} - \kappa_L \delta_R)(\kappa_{LR} - \delta_L \kappa_R)(\kappa_{LR} - \delta_L \kappa_R - \kappa_L \delta_R + \delta \kappa_{LR} \kappa_R)
\]

\[
= \left[ \frac{w_1 + w_2 + \theta + 1}{2} \right] \left[ \frac{w_1 + w_2 - \theta + 1}{2} \right] \left[ \frac{w_1 - w_2 + \theta - 1}{2} \right] \left[ \frac{w_1 - w_2 - \theta - 1}{2} \right]
\]

\[
\times \left[ \frac{-w_1 + w_2 + \theta - 1}{2} \right] \left[ \frac{-w_1 + w_2 - \theta - 1}{2} \right] \left[ \frac{w_1 + w_2 + \theta + 3}{2} \right] \left[ \frac{w_1 + w_2 - \theta + 3}{2} \right]
\]

Here we have suppressed the details of the calculation — but we note that the factorising of the determinant was aided by using globalisation to get the factors of \( \kappa_{LR} - \kappa_L \delta_R \) and \( \kappa_{LR} - \delta_L \kappa_R \) and using Gap4 [7].

6.2. Some more general Gram determinants. Consider \( S_{2n}(-(n-2)) \) with \( n \geq 3 \). One basis for \( S_{2n}(-(n-2)) \), generated by \( e_1 \), (modulo \( I_{2n}(n-3) + I_{2n}(-n+3) \)) is

\[
\{ ee_1, e_1, e_2 e_1, e_3 e_2 e_1, e_4 e_3 e_2 e_1, \ldots, e_{n-1} e_{n-2} \cdots e_1, fe_{n-1} e_{n-2} \cdots e_1 \}.
\]
Using this basis the Gram matrix for $S_{2n}(-(n-2))$ is

$$
\begin{pmatrix}
\delta_L \kappa_L & \kappa_L & 0 & 0 & 0 & 0 & \cdots & 0 \\
\kappa_L & \delta & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & \delta & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \delta & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \delta & 1 & \delta & \kappa_R \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \kappa_R & \delta R \kappa_R
\end{pmatrix}.
$$

Consider $S_{2n}(n-3)$ with $n \geq 4$. One basis for $S_{2n}(n-3)$, generated by $ee_2$, (modulo $I_{2n}(n-4) + I_{2n}(-n+4)$) is

$$\{ee_1ee_2, e_1ee_2, ee_2, e_3ee_2, e_4e_3ee_2, e_4e_3e_2e_1, \ldots, e_{n-1}e_{n-2}e_3ee_2, fe_{n-1}e_{n-2}e_3ee_2\}.$$

Using this basis the Gram matrix for $S_{2n}(n-3)$ is

$$
\begin{pmatrix}
\delta_L^2 \kappa_L & \delta_L \kappa_L & \delta_L^2 & 0 & 0 & 0 & \cdots & 0 \\
\delta_L \kappa_L & \delta_L \delta & \delta_L & 0 & 0 & 0 & \cdots & 0 \\
\delta_L^2 & \delta_L & \delta_L \delta & \delta_L & 0 & \cdots & 0 \\
0 & 0 & \delta_L & \delta_L \delta & \delta_L & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \delta_L & \delta_L \delta & \delta_L & \kappa_R \\
0 & 0 & \cdots & 0 & 0 & \delta_L & \delta_L \delta & \delta_L \kappa_R \\
0 & 0 & \cdots & 0 & 0 & 0 & \delta_L \kappa_R & \delta_L \delta R \kappa_R
\end{pmatrix}.
$$

Consider $S_{2n}(-(n-3))$ with $n \geq 4$. One basis for $S_{2n}(-(n-3))$, generated by $e_1f$ (modulo $I_{2n}(n-4) + I_{2n}(-n+4)$) is

$$\{ee_1f, e_1f, e_2e_1f, e_3e_2e_1f, \ldots, e_{n-1}e_{n-2}e_1f, fe_{n-1}e_{n-2}e_1f\}.$$

Using this basis the Gram matrix for $S_{2n}(-(n-3))$ is

$$
\begin{pmatrix}
\delta_L \delta R \kappa_L & \delta_R \kappa_L & 0 & 0 & 0 & 0 & \cdots & 0 \\
\delta_R \kappa_L & \delta_R \delta & \delta_R & 0 & 0 & 0 & \cdots & 0 \\
\delta_L \kappa_L & \delta_R \delta & \delta_R & 0 & 0 & \cdots & 0 \\
0 & \delta_R & \delta_R \delta & \delta_R & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \delta_R & \delta_R \delta & \delta_R & \delta_R^2 \\
0 & 0 & \cdots & 0 & 0 & \delta_R & \delta_R \delta & \delta_R \kappa_R \\
0 & 0 & \cdots & 0 & 0 & 0 & \delta_R^2 & \delta_R \kappa_R \delta_R^2
\end{pmatrix}.
$$
Proposition 6.2.1. For $n \geq 3$, we have (up to a \pm sign)

$$\Gamma_{2n}(-(n-2)) = [w_1 + 1][w_2 + 1][w_1 + w_2 - n + 2].$$

Similarly for $n \geq 4$, we have

$$\Gamma_{2n}(-(n-3)) = [w_1]^{n+1}[w_1 - 1][w_2 + 1][-w_1 + w_2 + n - 2]$$

and

$$\Gamma_{2n}(n-3) = [w_2]^{n+1}[w_2 - 1][w_1 + 1][w_1 - w_2 + n - 2],$$

and for $n \geq 5$, we have

$$\Gamma_{2n}(n-4) = [w_1]^{n+1}[w_2]^{n+1}[w_1 - 1][w_2 - 1][w_1 + w_2 + n - 2].$$
Proof. We prove the fourth statement; the remaining ones are similar. Removing the factor of \( \delta_L \delta_R \) from each line and expanding the determinant out along the first row we have:

\[
\begin{align*}
\delta_L^{n+1} \delta_R^{-n+1} & \Gamma_{2n}(n - 4) \\
& = \begin{vmatrix}
\delta_L \kappa_L & \delta_L & 0 & 0 & 0 & \cdots & 0 \\
\kappa_L & \delta & 1 & 0 & 0 & 0 & \cdots & 0 \\
\delta_L & 1 & \delta & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \delta & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 & \delta & 1 & \delta_R \\
0 & 0 & \cdots & 0 & 0 & 1 & \delta & \kappa_R \\
0 & 0 & \cdots & 0 & 0 & \delta_R & \kappa_R & \delta_R \kappa_R |_{n+1} \\
\end{vmatrix} \\
& = \delta_L \kappa_L \\
& = \begin{vmatrix}
\delta & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & \delta & 1 & 0 & \cdots & 0 \\
1 & \delta & 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & \delta & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 & \delta & 1 & \delta_R \\
0 & 0 & \cdots & 0 & 0 & 1 & \delta & \kappa_R \\
0 & 0 & \cdots & 0 & 0 & \delta_R & \kappa_R & \delta_R \kappa_R |_{n-1} \\
\end{vmatrix} \\
& \quad - \kappa_L^2 \\
& \quad + (2\delta_L \kappa_L - \delta_L^2 \delta) \\
\end{align*}
\]

where the subscripts on the square matrices keep note of their size. Now the determinant of each submatrix is given by

\[
\begin{align*}
& \begin{vmatrix}
\delta & 1 & 0 & 0 & 0 & \cdots & 0 \\
1 & \delta & 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & \delta & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & \delta & 1 & \delta_R \\
0 & \cdots & 0 & 0 & 1 & \delta & \kappa_R \\
0 & \cdots & 0 & 0 & \delta_R & \kappa_R & \delta_R \kappa_R |_{n} \\
\end{vmatrix} \\
& = \delta_R \kappa_R \\
& = \begin{vmatrix}
\delta & 1 & \cdots & 0 \\
1 & \delta & \cdots & 0 \\
0 & \cdots & \ddots & \cdots \\
0 & \cdots & 0 & \delta_R \\
0 & \cdots & 0 & 0 & \delta_R & \kappa_R & \delta_R \kappa_R |_{n-1} \\
\end{vmatrix} \\
& = \delta_R \kappa_R |_{n} - \kappa_R^2 |_{n-1} + (2\delta_R \kappa_R - \delta_R^2) |_{n-2} \\
& = |w_2| |w_2 + 1| |n| - |w_2 + 1|^2 |n - 1| + \left( 2|w_2| |w_2 + 1| - |2| |w_2|^2 \right) |n - 2|.
\end{align*}
\]

Using the quantum number identities

\[
[a + 1][b + 1] - [a][b] = [a + b + 1] \quad \text{and} \quad [a + 2][b + 2] - [a][b] = [2][a + b + 2]
\]
we obtain the following for the above determinant:

\[
\begin{align*}
&= [w_2][w_2 + 1][n] - [w_2 + 1]^2[n - 1] + [w_2]([w_2 + 1] - [w_2 - 1])[n - 2] \\
&= [w_2]([w_2 + 1][n] - [w_2 - 1][n - 1]) - [w_2 + 1]([w_2 + 1][n - 1] - [w_2][n - 1]) \\
&= [w_2][2][w_2 + n - 1] - [w_2 + 1][w_2 + n - 1] \\
&= [w_2 + n - 1][w_2 - 1],
\end{align*}
\]

Thus

\[
[w_1]^{-n+1}[w_2]^{-n+1}\Gamma_{2n}(n - 4)
= [w_1][w_1 + 1][w_2 + n - 1][w_2 - 1] - [w_1 + 1]^2[w_2 + n - 2][w_2 - 1]
+ \left(2[w_1][w_1 + 1] - [w_1]^2[2]\right)[w_2 + n - 3][w_2 - 1]
= [w_2 - 1]\left([w_1][w_1 + 1][w_2 + n - 1] - [w_1 + 1]^2[w_2 + n - 2]
+ [w_1]([w_1 + 1] - [w_1 - 1])[w_2 + n - 3]\right).
\]

Noting that the expression in large brackets is the same as equation for the submatrix only with
\(w_2\) replaced with \(w_1\) and \(n\) replaced with \(w_2 + n - 1\), we obtain

\[
\Gamma_{2n}(n - 4) = [w_1]^{n-1}[w_2]^{n-1}[w_2 - 1][w_1 - 1][w_1 + 1 + w_2 + n - 2],
\]

as required. □

In order to globalise the factors in these determinants, we need to know what composition factors correspond to the factors of the Gram determinant vanishing. Since \(-n + 2\), \(-n + 3\), and \(n - 4\) are almost minimal in the poset order, the only possible other composition factors of these standard modules are \(L_{2n}(-n)\), \(L_{2n}(-n + 1)\), \(L_{2n}(n - 1)\), and \(L_{2n}(n - 2)\), the one dimensional modules mentioned in the previous section. Thus it is possible to find “by hand” embeddings of these one dimensional modules into the standard ones.

**Proposition 6.2.2.** We have maps

\[
\begin{align*}
S_{2n}(-n) &\hookrightarrow S_{2n}(-(n - 3)) \quad \text{and} \quad S_{2n}(-(n - 1)) \hookrightarrow S_{2n}(n - 4) \quad \text{for} \quad [w_1 - 1] = 0, \\
S_{2n}(-n) &\hookrightarrow S_{2n}(n - 3) \quad \text{and} \quad S_{2n}(n - 1) \hookrightarrow S_{2n}(n - 4) \quad \text{for} \quad [w_2 - 1] = 0, \\
S_{2n}(-n) &\hookrightarrow S_{2n}(-(n - 2)) \quad \text{for} \quad [w_1 + w_2 - n + 2] = 0, \\
S_{2n}(-n + 1) &\hookrightarrow S_{2n}(-(n - 3)) \quad \text{for} \quad [-w_1 + w_2 + n - 2] = 0, \\
S_{2n}(n - 1) &\hookrightarrow S_{2n}(n - 3) \quad \text{for} \quad [w_1 - w_2 + n - 2] = 0,
\end{align*}
\]
$${\mathcal S}_{2n}(n - 2) \hookrightarrow {\mathcal S}_{2n}(n - 4) \text{ for } [w_1 + w_2 + n - 2] = 0.$$ 

**Proof.** In each case, it is a matter of either finding explicit elements of the larger module that generate a one-dimensional submodule with the appropriate action (see section 5.2), or showing that the required one-dimensional submodule must exist.

Consider the module $${\mathcal S}_{2n}(n - 3)$$. We use the basis already introduced for this module. To show that $${\mathcal S}_{2n}(n - 3)$$ can be embedded into $${\mathcal S}_{2n}(n - 3)$$, we need a non-zero vector $v \in {\mathcal S}_{2n}(n - 3)$ such that all elements of $${b^n_{2n}}$$ act trivially on it. Let $v = (a_0, a_1, \ldots, a_n) \in {\mathcal S}_{2n}(n - 3)$ with respect to the basis given above; note that the coordinates of $v$ are numbered starting at zero. We have

$$ev = (\delta_L a_0 + a_1, 0, \delta_L a_2, \ldots, \delta_L a_n)$$

$$fv = (0, 0, \ldots, 0, a_{n-1} + \delta_R a_n)$$

$$e_1 v = (0, \kappa_L a_0 + \delta a_1 + a_2, 0, 0, \ldots, 0)$$

$$e_2 v = (0, 0, \delta_L a_0 + a_1 + \delta a_2 + a_3, 0, \ldots, 0)$$

for $3 \leq i \leq n - 2$ $e_i v = (0, 0, \ldots, 0, a_{i-1} + \delta a_i + a_{i+1}, 0, \ldots, 0)$ ($i$-th position)

$$e_{n-1} v = (0, 0, \ldots, 0, a_{n-2} + \delta a_{n-1} + \kappa_L a_n, 0).$$

It follows that if $ev = 0$ then $a_2 = a_3 = \cdots = a_n = 0$, as $\delta_L \neq 0$ and $\delta_L a_0 + a_1 = 0$. So $v = (a_0, -\delta_L a_0, 0, 0, \ldots, 0)$, and thus $fv$ and $e_i v$ are also zero for $i \geq 2$. We also need $e_1 v = 0$, which gives us that $\kappa_L a_0 + \delta a_1 = 0$. Thus to have a consistent set of equations, we need $\kappa_L a_0 - \delta_L a_0 = 0$. So either $a_0 = 0$ (and hence $v = 0$) or $\kappa_L - \delta_L = 0 = [w_1 + 1] - |w_1| = -|w_1 - 1|$. Thus either $[w_1 - 1] \neq 0$ and $S_{2n}(-n)$ does not embed in $S_{2n}(n - 3)$ or $[w_1 - 1] = 0$ and we do have an embedding of $S_{2n}(-n)$ into $S_{2n}(n - 3)$.

Similarly, $S_{2n}(n-1)$, the module where $b^n_{2n}$ acts trivially except for $e$, which acts as multiplication by $\delta_L$, embeds in $S_{2n}(n - 3)$ if the following set of equations are satisfied:

$$\delta_L a_0 + a_1 = \delta_L a_0, \quad a_{n-1} + \delta_R a_n = 0,$$

$$\kappa_L a_0 + \delta a_1 + a_2 = 0, \quad \delta_L a_0 + a_1 + \delta a_2 + a_3 = 0,$$

$$a_{i-1} + \delta a_i + a_{i+1} = 0, \text{ for } 3 \leq i \leq n - 2, \quad a_{n-2} + \delta a_{n-1} + \kappa_R a_n = 0.$$
The system of linear equations in the \( a_i \) has \( n + 1 \) by \( n + 1 \) size coefficient matrix:

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
\kappa_L & \delta & 1 & 0 & 0 & 0 & 0 & \cdots \\
\delta_L & 1 & \delta & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & \delta & 1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 1 & \delta & 1 & 0 & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 & \delta & 1 & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & \delta & \kappa_R \\
0 & 0 & \cdots & 0 & 0 & 0 & 1 & \delta_R
\end{pmatrix}
\]

Thus to get non-trivial solutions to the above set of equations we need the matrix to have zero determinant. This matrix has determinant

\[
-\delta_R \kappa_L [n - 1] + \delta_L \delta_R [n - 2] + \kappa_L \kappa_R [n - 2] - \delta_L \kappa_R [n - 3] = [w_1 - w_1 + n - 2].
\]

Thus \( S_{2n}(n - 1) \) can only embed in \( S_{2n}(n - 3) \) if \([w_1 - w_1 + n - 2] = 0\).

The other maps are constructed in a similar fashion.

\[\square\]

**Proposition 6.2.3.** We have maps for \( n \geq 3 \) and \( n - m \) even

\[
S_{2n}(-m) \rightarrow S_{2n}(-(m - 3)) \quad \text{and} \quad S_{2n}(-(m - 1)) \rightarrow S_{2n}(m - 4) \quad \text{for} \ [w_1 - 1] = 0,
\]

\[
S_{2n}(-m) \rightarrow S_{2n}(m - 3) \quad \text{and} \quad S_{2n}(m - 1) \rightarrow S_{2n}(m - 4) \quad \text{for} \ [w_2 - 1] = 0,
\]

\[
S_{2n}(-m) \rightarrow S_{2n}(-(m - 2)) \quad \text{for} \ [w_1 + w_2 - m + 2] = 0,
\]

\[
S_{2n}(-m + 1) \rightarrow S_{2n}(-(m - 3)) \quad \text{for} \ [-w_1 + w_2 + m - 2] = 0,
\]

\[
S_{2n}(m - 1) \rightarrow S_{2n}(m - 3) \quad \text{for} \ [w_1 - w_2 + m - 2] = 0,
\]

\[
S_{2n}(m - 2) \rightarrow S_{2n}(m - 4) \quad \text{for} \ [w_1 + w_2 + m - 2] = 0.
\]

We have maps for \( n \geq 3 \) and \( n - m \) odd

\[
S_{2n}(-m) \rightarrow S_{2n}(-(m - 3)) \quad \text{for} \ [w_1 - 1] = 0,
\]

\[
S_{2n}(-(m - 1)) \rightarrow S_{2n}(m - 4) \quad \text{for} \ [w_1 - 1] = 0,
\]

\[
S_{2n}(+m) \rightarrow S_{2n}(-m + 3) \quad \text{for} \ [w_2 - 1] = 0,
\]

\[
S_{2n}(-m + 1) \rightarrow S_{2n}(-m + 4) \quad \text{for} \ [w_2 - 1] = 0.
\]
Proof. To prove this, it is enough to globalise the maps from the previous proposition, taking care with the swapping of the parameters. If we assume that \( n - m \) is even then we may apply either \( G \circ G \) or \( G' \circ G' \) to the maps — the double application of \( G \) or \( G' \) ensures that the swapped parameters are swapped back.

For example: take the map \( S_{2n}(-m) \rightarrow S_{2n}(-(m-3)) \) that exists when \( [w_1 - 1] = 0 \). Applying \( G' \) once gives a map \( S_{2n+2}(-m) \rightarrow S_{2n+2}(-(m-3)) \) for \( [w_1 + 1] = 0 \). (Note that \( G' \) has no effect on \( \delta_L \) and \( \kappa_L \) and hence no effect on \( w_1 \).) Applying \( G' \) again gives: \( S_{2n+4}(-m) \rightarrow S_{2n+4}(-(m-3)) \) for \( [w_1 + 1] = 0 \). Alternatively, we can apply \( G \circ G \) to the original map and get the map above. □

Corollary 6.2.4. The poset in proposition 5.3.3 cannot be coarsened further and still be a poset for which \( b_{2n}^\phi \) is quasi-hereditary for all parametrisations.

Proof. We need to exhibit parametrisations for which each link in the poset in proposition 5.3.3 is necessary. To do this it is enough to use the maps from the previous proposition for links not involving \( 0 \). For the links involving \( 0 \) we use [13, section 9.2]. □

7. Quotients of the symplectic blob

The appearance of \( TL_n \) as a subalgebra of \( b_n \), although constructively natural, tells us relatively little about their representation theory. More significant is the appearance of \( TL_n \) as a quotient of \( b_n \) for certain special values of the blob parameter [15]. Indeed this is part of the original paradigm for the generalised alcove geometric approach [14], which we aim eventually to generalise further to include the symplectic case. Further, one knows by elementary combinatorial arguments that \( TL_n \) cannot appear as a quotient of \( b_n \) “generically”, so that, when it does so, we are guaranteed to be studying the non-generic sector. By analogy, the study of quotient algebras of the symplectic blob algebra (with known representation theory) provides another tool with which to investigate its representation theory. Again one knows that \( TL_n \) and \( b_n \) are not generic quotients, so any such map would embed these known structures into the non-generic sector. The study of quotients of the symplectic blob algebra is more complicated than for the usual blob algebra because of the “topological relation”.

7.1. Some quotients.

Proposition 7.1.1. If \( \delta = \kappa_L = \kappa_R, \delta_L = \delta_R = \kappa_{LR} = 1 \) and \( n \geq 3 \) is odd then \( b_n^\phi / I \cong TL_n \), where \( I \) is the ideal generated by \( e - 1 \) and \( f - 1 \) and \( TL_n \) has \( q + q^{-1} = \delta \).

Proof. Define a map from \( b_n^\phi \rightarrow TL_n \) that takes the diagram elements and removes any blobs. Note that this map does not change the underlying structure of any diagram and lifts to a homomorphism on multiplication of the underlying diagrams (ignoring the decorations). This map clearly has kernel equal to \( I \) above.
By considering the relations on the diagrams in $b^x_n$, we see that to make this map a homomorphism from $b^x_n$, we need $\delta = \kappa_L = \kappa_R$ and $\delta_L = \delta_R = 1$. (This is by considering the three loop relations and the two blobs = one blob relations.) This leaves the “three blobs” relation — which gives us that $\kappa_{LR} = 1$. As $n$ is odd we do not have the “topological relation” and we do not have the loop with a black and a white blob replaced by the $\kappa_{LR}$ relation.

The quotient clearly has basis equal to the diagram basis for $TL_n$ and also the same multiplication so we are done.

□

**Proposition 7.1.2.** If $\delta = \kappa_L = \delta_R$, $\delta_L = \kappa_R = \kappa_{LR} = 1$ and $n$ is even (and not zero) then there is a quotient of $b^x_n$ that is isomorphic to $TL_{n+1}$, where $q + q^{-1} = \delta$.

**Proof.** We will distinguish the “$f$” for $b^x_{n+1}$ with a subscript $n+1$. Consider the composition of maps

$$b^x_n \xrightarrow{\phi} f' b^x_{n+1} \xrightarrow{\iota} b^x_{n+1} \xrightarrow{\psi} TL_{n+1},$$

where $\phi$ is the isomorphism of (the blob version of) Proposition 3.0.9, $f' = \delta^{-1}_R f_{n+1}$, $\phi$ is the inverse map to the map $\rho$ of [13] (with no scalar factor as $\kappa_R = 1$), $\iota$ is the natural embedding and $\psi$ is the quotient map from the previous proposition.

We claim that $\xi := \psi \circ \iota \circ \phi$ is onto. For this, note that $\xi(e_i)$ for $1 \leq i \leq n-1$ is the diagram $U_i$ for $TL_{n+1}$. Also, $\xi(f) = \psi(f_n e_n f_{n+1}) = U_n$. Thus since $\im \xi$ contains all the generators for $TL_{n+1}$, and it is a homomorphism, it must be onto.

□

Note that for these propositions, we start with a non-semi-simple symplectic blob algebra. Consider $n$ odd. Now the $b^\delta_{2n}$ considered has $\delta_L \delta_R - \kappa_{LR} = 1 - 1 = 0$ on one hand. On the other hand, $\delta_L \delta_R - \kappa_{LR} = \left[\frac{w_1+w_2}{2}\right] \left[\frac{w_1+w_2-\delta}{2}\right]$. Thus, we have some singular Gram determinants and at least one of the standard modules is not simple and so this algebra is not semi-simple. Hence the quotient, which is isomorphic to the Temperley–Lieb algebra, for which we know all decomposition numbers, gives us information about the possible form of decomposition numbers of the standard modules for the symplectic blob.

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