SCALAR EXTENSIONS OF CATEGORICAL RESOLUTIONS OF SINGULARITIES

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Abstract. Let \( X \) be a quasi-compact, separated scheme over a field \( k \) and we can consider the categorical resolution of singularities of \( X \). In this paper let \( k'/k \) be a field extension and we study the scalar extension of a categorical resolution of singularities of \( X \) and we show how it gives a categorical resolution of the base change scheme \( X_{k'} \). Our construction involves the scalar extension of derived categories of DG-modules over a DG algebra. As an application we use the technique of scalar extension developed in this paper to prove the non-existence of full exceptional collections of categorical resolutions for a projective curve of genus \( \geq 1 \) over a non-algebraically closed field.

1. Introduction

For a scheme \( X \) over a field \( k \), the derived category \( D(X) \) of quasi-coherent \( O_X \)-modules plays an important role in the study of the geometry of \( X \). In particular, a categorical resolution of singularities of \( X \) is defined to be a smooth triangulated category \( \mathcal{T} \) together with an adjoint pair \( \pi^*: D(X) \rightleftarrows \mathcal{T}: \pi_* \) which satisfies certain properties. See [7] or Definition 3.1 below for details.

On the other hand, base change techniques are also ubiquitous in algebraic geometry. In [13], a theory of scalar extensions of triangulated categories has been developed and applied to derived categories of varieties.

In this paper we define and study the scalar extension of categorical resolutions. The difficulty is to find the scalar extensions of the adjoint pair \( (\pi^*, \pi_*) \). To solve this problem we modify the definition of categorical resolution: inspired by [8], we define an algebraic categorical resolution of \( X \) to be a triple \( (A, B, T) \) where \( A \) is a differential graded (DG) algebra such that \( D(X) \cong D(A) \), \( B \) is a smooth DG algebra and \( T \) is an \( A \)-\( B \) bimodule which satisfies certain properties. See Definition 3.4 below for more details. In some important cases, which include the cases we are most interested in, these two definitions are equivalent. For the comparison of different definitions of categorical resolution see Proposition 3.4 below.

The advantage of algebraic categorical resolution is that it is compatible with base field extensions. One of the main results in this paper is the following proposition.

Proposition 1.1 (See Proposition 4.11 below). [See Proposition 4.11 below] Let \( X \) be a projective variety over a field \( k \). If \( (A, B, T) \) is an algebraic categorical resolution of \( X \), then \( (A_{k'}, B_{k'}, T_{k'}) \) is an algebraic categorical resolution of the base change variety \( X_{k'} \).

As an application we study the categorical resolution of projective curves \( X \) over a non-algebraically closed field \( k \). Using the technique of scalar extension we obtain the following theorems which generalize the main results in [17].

Theorem 1.2 (See Theorem 5.4 below). Let \( X \) be a projective curve over a field \( k \). Then \( X \) has a categorical resolution which admits a full exceptional collection if and only if the geometric genus of \( X \) is 0.

Theorem 1.3 (See Theorem 5.5 below). Let \( X \) be a projective curve with geometric genus \( \geq 1 \) over a field \( k \) and \( (\mathcal{T}, \pi^*, \pi_*) \) be a categorical resolution of \( X \). Then \( \mathcal{T}^c \) cannot have a tilting object, moreover there cannot be a finite dimensional \( k \)-algebra \( \Lambda \) of finite global dimension such that

\[
\mathcal{T}^c \cong \mathcal{D}^b(\Lambda \text{-mod}).
\]
This paper is organized as follows: In Section 2, we quickly review triangulated categories, DG categories and DG algebras. In Section 3, we review and compare different definitions of categorical resolutions. In Section 4.1, we study the scalar extension of derived categories of DG algebras and in Section 4.2, we study the scalar extension of categorical resolutions. In Section 5, we use the technique of scalar extension to study categorical resolutions of projective curves over a non-algebraically closed field.

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2. PRELIMINARIES

2.1. Review of some concepts on triangulated categories. Let $\mathcal{T}$ be a triangulated category. $\mathcal{T}$ is called cocomplete if it has arbitrary direct sums. An object $E$ of a cocomplete triangulated category $\mathcal{T}$ is called compact if the functor $\text{Hom}_\mathcal{T}(E, -)$ preserves arbitrary direct sums. Let $\mathcal{T}^c$ denote the full triangulated subcategory of $\mathcal{T}$ consisting of compact objects.

Let $I$ be a set of objects of $\mathcal{T}$. We say $I$ generates $\mathcal{T}$ if for any object $N$ of $\mathcal{T}$, $\text{Hom}_\mathcal{T}(E, N[i]) = 0$ for any $E \in I$ and $i \in \mathbb{Z}$ implies $N = 0$. We say a cocomplete triangulated category $\mathcal{T}$ is compactly generated if it is generated by a set of compact objects. An object $E$ of $\mathcal{T}$ is called a generator of $\mathcal{T}$ if the set $\{E\}$ generates $\mathcal{T}$.

We have the following well-known result.

**Lemma 2.1.** Let $\mathcal{T}$ be a cocomplete triangulated category. If a set of objects $E \subset T^c$ generates $\mathcal{T}$, then $\mathcal{T}$ coincides with the smallest strictly full triangulated subcategory of $\mathcal{T}$ consisting of compact objects. Recall a subcategory is strictly full if it is full and closed under isomorphism.

**Proof.** See the proof of [12] Theorem 4.22. □

For a scheme $X$, let $\text{D}(X)$ be the unbounded derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent cohomologies. Moreover if $X$ is noetherian let $\text{D}^{\text{perf}}(X)$ be the derived category of perfect complexes on $X$ and $\text{D}^b(\text{coh}(X))$ be the derived category of bounded complexes of $\mathcal{O}_X$-modules with coherent cohomologies.

**Remark 1.** We could also consider $\text{D}(\text{Qcoh}(X))$, the derived category of unbounded complexes of quasi-coherent sheaves on $X$. There exists a canonical triangulated functor $i : \text{D}(\text{Qcoh}(X)) \to \text{D}(X)$. For a general scheme $X$, the functor $i$ needs not to be an equivalence. However if $X$ is noetherian or quasi-compact and separated, then $i$ must be an triangulated equivalence, see [14] Tag 09T1 Proposition 35.7.3 and [2] Corollary 5.5. Since all schemes we study in this paper are either noetherian or quasi-compact and separated, we could identify $\text{D}(\text{Qcoh}(X))$ and $\text{D}(X)$ and we will restrict ourselves to $\text{D}(X)$ from now on.

We have the following result on compact objects and compact generators of $\text{D}(X)$.

**Proposition 2.2** ([3] Theorem 3.1.1). Let $X$ be a quasi-compact quasi-separated scheme. Then

1. The compact objects in $\text{D}(X)$ are precisely the perfect complexes;
2. $\text{D}(X)$ has a compact generator.

**Proof.** See the proof of [3] Theorem 3.1.1. □
2.2. **Review of some concepts on DG categories.** Most of results in this subsection could be found in [7] Section 3. For a comprehensive introduction to DG categories see [5].

Let $\mathcal{D}$ be a DG category over $k$ and let $H^0(\mathcal{D})$ denote the homotopy category of $\mathcal{D}$. Let $Z^0(\mathcal{D})$ be the category which has the same objects as $\mathcal{D}$ and whose morphisms from an object $x$ to an object $y$ are the degree 0 closed morphisms in $\mathcal{D}(x, y)$.

A right $\mathcal{D}$-module is a DG functor $\mathcal{D}^{op} \to \text{Ch}(k)$. The category of right $\mathcal{D}$-modules has a natural DG structure and let $\mathcal{M}_{d^0}(\mathcal{D})$ denote the DG category of right $\mathcal{D}$-modules. Moreover, a DG-modules $M$ over $\mathcal{D}$ is called acyclic if for any object $X \in \mathcal{D}$, the complex $M(X)$ is acyclic. Let $\text{Acycl}(\mathcal{D})$ denote the full DG subcategory of $\mathcal{M}_{d^0}(\mathcal{D})$ which consists of acyclic DG-modules over $\mathcal{D}$.

For a DG category $\mathcal{D}$, the derived category of DG-modules over $\mathcal{D}$ is defined to be the Verdier quotient

$$D(\mathcal{D}) := H^0(\mathcal{M}_{d^0}(\mathcal{D}))/H^0(\text{Acycl}(\mathcal{D})).$$

**Remark 2.** We can prove that $D(\mathcal{D})$ is a cocomplete, compactly generated triangulated category.

**Definition 2.1.** A DG-module $P$ is called $h$-projective if, for any acyclic DG-module $M$, the complex $\text{Hom}_{\mathcal{M}_{d^0}(\mathcal{D})}(P, M)$ is acyclic. Dually, a DG-module $I$ is called $h$-injective if, for any acyclic DG-module $M$, the complex $\text{Hom}_{\mathcal{M}_{d^0}(\mathcal{D})}(M, I)$ is acyclic.

We denoted the full DG subcategory of $\mathcal{M}_{d^0}(\mathcal{D})$ consisting of $h$-projective DG-modules by $\text{h-proj}(\mathcal{D})$ and the full DG subcategory of $\mathcal{M}_{d^0}(\mathcal{D})$ consisting of $h$-injective DG-modules by $\text{h-inj}(\mathcal{D})$.

**Remark 3.** We can prove that $D(\mathcal{D}) \simeq H^0(\text{h-proj}(\mathcal{D})) \simeq H^0(\text{h-inj}(\mathcal{D}))$.

There is a standard Yoneda embedding $\mathcal{D} \to \mathcal{M}_{d^0}(\mathcal{D})$ given by

$$x \mapsto h_x := \mathcal{D}(-, x).$$

**Definition 2.2.** A DG-module $M$ is called representable if it is isomorphic in $Z^0(\mathcal{M}_{d^0}(\mathcal{D}))$ to an object of the form $h_x$ for some $x \in \mathcal{D}$.

Moreover, a DG-module $M$ is called quasi-representable if it is isomorphic in $D(\mathcal{D})$ to an object of the form $h_x$ for some $x \in \mathcal{D}$.

For two DG categories $\mathcal{C}$ and $\mathcal{D}$, a $\mathcal{C}$-$\mathcal{D}$ DG-bimodule is a right DG module over $\mathcal{D}^{op} \otimes \mathcal{C}$, i.e. a DG functor $\mathcal{D} \otimes \mathcal{C}^{op} \to \text{Ch}(k)$. For a $\mathcal{C}$-$\mathcal{D}$ DG-bimodule $T$, the derived tensor product defines a functor

$$(-) \otimes^L_T : D(\mathcal{C}) \to D(\mathcal{D}).$$

**Definition 2.3.** A $\mathcal{C}$-$\mathcal{D}$ DG-bimodule $T$ is called a quasi-functor if for any $x \in \mathcal{C}$, the object $T(x, -) \in \mathcal{M}_{d^0}(\mathcal{D})$ is quasi-representable. It is clear that a quasi-functor defines a functor $H^0(\mathcal{C}) \to H^0(\mathcal{D})$.

For a DG category $\mathcal{D}$ we could consider $\mathcal{D}$-$\mathcal{D}$ DG-bimodules and in particular $\mathcal{D}$ itself could be considered as the diagonal bimodule

$$\mathcal{D}(X, Y) = \text{Hom}_{\mathcal{D}}(Y, X) \in \text{Ch}(k).$$

**Definition 2.4.** A DG category $\mathcal{D}$ is called smooth if the diagonal bimodule $\mathcal{D}$ is a perfect bimodule. In other words, if $\mathcal{D}$ is a direct summand (in the derived category of $\mathcal{D}$-$\mathcal{D}$ DG-bimodules) of a bimodule obtained from quasi-representable bimodules by finite number of shifts and cones of closed morphisms.

Moreover, a triangulated category $\mathcal{T}$ is called smooth if there exists a smooth DG category $\mathcal{D}$ such that $\mathcal{T}$ is triangulated equivalent to $D(\mathcal{D})$.

**Remark 4.** It is well known that if $X$ is a smooth variety, then the derived category $D(X)$ is a smooth triangulated category, see [16].
2.3. Review of some concepts on DG algebras. Let $A$ be a DG algebra over $k$. We could consider $A$ as a DG category with one object. Therefore most of concepts for DG categories could be defined for DG algebras without any changes. For example, we could define right DG-modules, the derived category, $h$-projective and $h$-injective modules and smoothness for a DG algebra.

For a DG algebra $A$, let $D(A)$ be the unbounded derived category of complexes of right DG $A$-modules and $\text{Perf}(A)$ be the full subcategory of perfect complexes of right $A$-modules.

For later application we recall the following characterization of compact objects in $D(A)$.

**Lemma 2.3.** ([14, Tag 09QZ] Proposition 22.27.4) Let $A$ be a DG algebra. Let $E$ be an object of $D(A)$. Then the following are equivalent

1. $E$ is a compact object;
2. $E$ is a direct summand of an object of $D(A)$ which is represented by a differential graded module $P$ which has a finite filtration $F_\bullet$ by differential graded submodules such that $F_iP/F_{i-1}P$ are finite direct sums of shifts of $A$, i.e. $E$ is an object of $\text{Perf}(A)$.

**Proof.** See the proof of [14, Tag 09QZ] Proposition 22.27.4.

The following definition plays a significant role in the constructions in this paper.

**Definition 2.5.** Let $A$ and $B$ be two DG algebras over $k$. Then we call $A$ and $B$ are DG Morita equivalent if there exists an $A$-$B$ bimodule $T$ such that the derived tensor product functor $(-) \otimes^L_AT$ induces a triangulated equivalence $D(A) \xrightarrow{\sim} D(B)$.

**Remark 5.** In a recent preprint [11], Rizzardo and Van den Bergh constructed two $A_\infty$-algebras $F$ and $F_\eta$ over a field $k$ such that $\text{Perf}(F)$ and $\text{Perf}(F_\eta)$ are triangulated equivalent but their $A_\infty$-enhancements are not $A_\infty$-equivalent. It could be deduced from this fact that there exist two DG algebras $A$ and $B$ over $k$ such that $D(A) \simeq D(B)$ but $A$ and $B$ are not DG Morita equivalent.

The following proposition is used to connect a DG category to the derived category of a DG algebra.

**Proposition 2.4** ([10] Proposition B.1 (b)). The DG categories are over an arbitrary ground ring. Let $\mathcal{C}$ be a DG category with a full pretriangulated DG subcategory $\mathcal{I}$. Let $z : P \to I$ be a closed degree zero morphism in $\mathcal{C}$ with $I \in \mathcal{I}$ such that $z^* : \text{Hom}_C(I, J) \to \text{Hom}_C(P, J)$ is an quasi-isomorphism for all $J \in \mathcal{I}$, and let $B$ be a DG algebra together with a morphism $\beta : B \to \text{End}_C(P)$ of DG algebras such that the composition $B \xrightarrow{\beta} \text{End}_C(P) \xrightarrow{z^*} \text{Hom}_C(I, P)$ is a quasi-isomorphism. If $\mathcal{I}$ has all coproducts and $I$ is a compact generator of $H^0(\mathcal{I})$, then the functor

$$\beta^* \circ \text{Hom}(P, -) : H^0(\mathcal{I}) \to D(B)$$

is an equivalence of triangulated categories.

**Proof.** See the proof of [10] Proposition B.1 (b).

3. Categorical resolution of singularities

3.1. Review of definitions of categorical resolutions. In [6] and [7] the categorical resolution of singularities has been defined and studied.

**Definition 3.1** ([6] Definition 3.2 or [7] Definition 1.3). A categorical resolution of a scheme $X$ is a smooth, cocomplete, compactly generated, triangulated category $\mathcal{T}$ with an adjoint pair of triangulated functors

$$\pi^* : D(X) \to \mathcal{T} \text{ and } \pi_* : \mathcal{T} \to D(X)$$

such that

1. $id \simeq \pi_* \circ \pi^*$;
(2) both $\pi_*$ and $\pi^*$ commute with arbitrary direct sums;
(3) $\pi_*(\mathcal{T}^c) \subset D^b(\text{coh}(X))$ where $\mathcal{T}^c$ denotes the full subcategory of $\mathcal{T}$ which consists of compact objects.

**Remark 6.** Note that Condition (1) implies that $\pi^*$ is fully faithful and Condition (2) implies that $\pi^*(D_{\text{perf}}(X)) \subset D^b(\text{coh}(X))$.

**Remark 7.** Let $X$ be a projective variety over an algebraically closed field $k$. By [12] Corollary 7.51, an object $F$ belongs to the subcategory $D^b(\text{coh}(X))$ if and only if

$$\bigoplus_{n \in \mathbb{Z}} \text{Hom}_{D(X)}(E, F[n])$$

is finite dimensional for any $E \in D_{\text{perf}}(X)$. Therefore Condition (3) in Definition 3.1 is equivalent to the following condition: For any $E \in D_{\text{perf}}(X)$ and $F \in \mathcal{T}^c$, the vector space $\bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}(E, \pi_* F[n])$ is finite dimensional. For later applications we also notice that it is equivalent to require $\bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}(\pi_* E, F[n])$ to be finite dimensional.

In [7] the existence of categorical resolutions has been proved.

**Theorem 3.1 ([7] Theorem 1.4).** Any separated scheme $Y$ of finite type over a field of characteristic 0 has a categorical resolution.

**Remark 8.** In general, the categorical resolution of a scheme is not unique.

On the other hand, we notice that in [8] there is another definition of categorical resolution.

**Definition 3.2 ([8] Definition 4.1).** Let $A$ be a DG algebra, A categorical resolution of $D(A)$ is a pair $(B, T)$ where $B$ is a smooth DG algebra and $T \in D(A^{\text{op}} \otimes B)$ such that the restriction of the functor

$$\theta(-) := (-) \otimes_A^L T; D(A) \to D(B)$$

to the subcategory $\text{Perf}(A)$ is full and faithful.

**Remark 9.** It is clear that $\theta = (-) \otimes_A^L T$ commutes with arbitrary direct sum. Moreover its right adjoint $R\text{Hom}_B(T, -)$ commutes with arbitrary direct sum if and only if $T$ is compact when considered as an object in $D(B)$. In this case it is also clear that $\theta$ maps $\text{Perf}(A)$ to $\text{Perf}(B)$ and hence by Lemma 3.2 below, $\theta : D(A) \to D(B)$ is fully faithful.

**Lemma 3.2.** Let $A$ and $B$ be DG algebras and let $F : D(A) \to D(B)$ be a triangulated functor with the following properties

1. $F(\text{Perf}(A)) \subset \text{Perf}(B)$;
2. The restriction of $F$ to $\text{Perf}(A)$ is fully faithful;
3. $F$ preserves direct sums.

Then $F$ is fully faithful.

**Proof.** See the proof of [8] Lemma 2.13. □

### 3.2. Comparison of definitions.

Conceptually Definition 3.1 and Definition 3.2 are very similar and we want to find the relation between them.

First of all, for a quasi-compact and quasi-separated scheme $X$, the derived category $D(X)$ has a compact generator $\mathcal{E}$ hence we have an equivalence of triangulated categories.

$$\Phi : D(X) \sim D(A)$$

where $A := R\text{Hom}_X(\mathcal{E}, \mathcal{E})$. We notice that $\Phi$ maps $D_{\text{perf}}(X)$ to $\text{Perf}(A)$. 
Remark 10. We know that in Definition 3.1 the triangulated category $\mathcal{T}$ is compactly generated. However, in general a triangulated category $\mathcal{T}$ is compactly generated does not imply that it has a single compact generator. For counterexamples see [4] Theorem A (2)(a).

Now we assume the triangulated category $\mathcal{T}$ in Definition 3.1 has a compact generator, then we also have a DG algebra $B$ and an equivalence
$$\Psi : \mathcal{T} \xrightarrow{\sim} D(B)$$
which maps $\mathcal{T}^c$ to Perf$(B)$.

As a result we could reformulate Definition 3.1 as follows.

**Definition 3.3 (The auxiliary definition of categorical resolution).** Let $X$ be a projective variety over a field $k$. Let $A$ be a DG algebra such that $D(X) \simeq D(A)$. Then a categorical resolution of $X$ is a smooth DG algebra $B$ with an adjoint pair of triangulated functors
$$\pi^* : D(A) \to D(B) \quad \text{and} \quad \pi_* : D(B) \to D(A)$$
such that
(1) $\pi_* \circ \pi^* \simeq id$;
(2) both $\pi_*$ and $\pi^*$ commute with arbitrary direct sums;
(3) For any $E \in \text{Perf}(A)$ and $F \in \text{Perf}(B)$, the vector space
$$\bigoplus_{n \in \mathbb{Z}} \text{Hom}_{D(B)}(\pi^* E, F[n])$$
is finite dimensional.

Remark 11. In the light of Remark 7, Condition (3) in Definition 3.3 is equivalent to Condition (3) in Definition 3.1. Moreover, by the definition of perfect complexes, Condition (3) in Definition 3.3 could be replaced by the following weaker form.

(3') The vector space
$$\bigoplus_{n \in \mathbb{Z}} \text{Hom}_{D(B)}(\pi^* A, B[n])$$
is finite dimensional.

Nevertheless we also notice that we need $X$ to be a projective variety to make sure that Condition (3) (or (3')) in Definition 3.3 could replace Condition (3) in Definition 3.1. It may fail for general schemes.

We want to find the relation between Definition 3.3 and Definition 3.2. The difficulty is to show that the functor $\pi^* : D(A) \to D(B)$ is given by a derived tensor product $(-) \otimes^L_A T$ for a $T \in D(A^{op} \otimes B)$. It is a problem because we know that there exist DG algebras which are not DG Morita equivalent but their derived categories are equivalent as triangulated categories.

Nevertheless, we can obtain another DG algebra $\tilde{A}$ with $D(A) \simeq D(\tilde{A})$ together with an $\tilde{A}$-$B$ bimodule $T$ which gives a DG Morita equivalence. Actually we have the following proposition.

**Proposition 3.3.** Let $A$ and $B$ be DG algebras and $\pi^* : D(A) \Rightarrow D(B) : \pi_*$ be an adjoint pair of triangulated functors which satisfies Condition (1),(2),(3) in Definition 3.3. Let $T := \pi^* A \in D(B)$ and $\tilde{A} := R\text{Hom}_B(T, T)$.

Then we can build a triangulated equivalence
$$\phi : D(A) \Rightarrow D(\tilde{A}).$$
Proof. The proof is essentially the same as that of [8] Proposition 2.6. We give a proof here for completeness.

First we define \( \phi \) to be the composition

\[
\phi : D(A) \xrightarrow{\pi} D(B) \xrightarrow{\text{RHom}_B(T, -)} D(\tilde{A}).
\]

\( \phi \) commutes with direct sums because \( \pi^* \) does and because \( \pi^* \) is fully faithful.

Then we prove that \( \phi \) is fully faithful. Let \( \mathcal{D}_0 \subset D(\tilde{A}) \) be the strictly full triangulated subcategory consisting of objects \( M \) such that the natural map

\[
\text{Hom}_{D(\tilde{A})}(A, M[i]) \to \text{Hom}_{D(\tilde{A})}(\phi(A), \phi(M)[i])
\]

is an isomorphism for any integer \( i \). Since \( \phi(A) = \tilde{A} \) and \( H^i(A) \cong H^i(\tilde{A}) \) for any \( i \), it is clear that \( A \in \mathcal{D}_0 \). Moreover since \( \phi \) commutes with arbitrary direct sums we see that \( \mathcal{D}_0 \) is closed under arbitrary direct sums. By Lemma [2.1] \( \mathcal{D}_0 = D(A) \). Similarly let \( \mathcal{D}_1 \subset D(A) \) be the strictly full triangulated subcategory consisting of objects \( N \) such that the natural map

\[
\text{Hom}_{D(A)}(N, M) \to \text{Hom}_{D(\tilde{A})}(\phi(N), \phi(M))
\]

is an isomorphism for any object \( M \in D(A) \). By the above reasoning \( A \in \mathcal{D}_1 \) and it is clear that \( \mathcal{D}_1 \) is closed under arbitrary direct sums. Again by Lemma [2.1] we get \( \mathcal{D}_1 = D(A) \), i.e. \( \phi \) is fully faithful.

For the essential surjectivity, since \( \phi(A) = \tilde{A} \) we know that \( \phi(D(A)) \supset \text{Perf}(\tilde{A}) \). In addition, it is clear that \( \phi(D(A)) \) is closed under direct sums. Again by Lemma [2.1] \( \phi \) is essentially surjective.

Remark 12. From the construction it is clear that \( H^i(A) \cong H^i(\tilde{A}) \) for any \( i \in \mathbb{Z} \). However, a priori there is no quasi-isomorphism between \( A \) and \( \tilde{A} \) therefore the fact that \( D(A) \cong D(\tilde{A}) \) is not trivial.

Now we give our version of definition of categorical resolution of singularities.

Definition 3.4 (Algebraic categorical resolution). Let \( X \) be a projective variety over a field \( k \). Then an algebraic categorical resolution of \( X \) is a triple \((A, B, T)\) where \( A \) is a DG algebra such that \( D(X) \cong D(A) \), \( B \) is a smooth DG algebra and \( T \) is an \( A-B \) bimodule such that

1. \( H^i(A) \to \text{Hom}_{D(B)}(T, T[i]) \) is an isomorphism for any \( i \in \mathbb{Z} \);
2. \( T \) defines a compact object in \( D(B) \);
3. \( \bigoplus_{i} \text{Hom}_{D(B)}(T, B[i]) \) is finite dimensional.

It is clear that a triple \((A, B, T)\) as in Definition 3.4 gives a triple \((\mathcal{T}, \pi^*, \pi_*)\) as in Definition 3.1. For the other direction of implication we have the following proposition.

Proposition 3.4. Let \( X \) be a projective variety over a field \( k \) and \((\mathcal{T}, \pi^*, \pi_*)\) be a categorical resolution of \( X \) in the sense of Definition 3.7. If \( \mathcal{T} \) has a compact generator, then we have an algebraic categorical resolution \((A, B, T)\) of \( X \) in the sense of Definition 3.4 such that \( D(B) \cong \mathcal{T} \).

Proof. Since \( \mathcal{T} \) has a compact generator we can find a DG algebra \( B \) such that \( D(B) \cong \mathcal{T} \). We use the \((\tilde{A}, T)\) in Proposition 3.3 as the \((A, T)\) in Definition 3.4. It is clear that they satisfy all conditions in Definition 3.4.

4. Scalar extensions of categorical resolutions of singularities

4.1. Scalar extensions of derived categories of DG algebras. In this section we study scalar extensions of categorical resolution. First we notice that the scalar extension of triangulated categories has been studied in [13]. However, we do not know whether the scalar extension as defined in [13] Definition 9 preserves fully faithful functors hence it is difficult to use that definition directly to study the scalar extension of categorical resolutions.
Nevertheless there is another approach to the scalar extension of triangulated categories which is outlined in [13] Remark 9.

**Definition 4.1.** Let $A$ be a DG algebra over a base field $k$ and we consider the derived category $\mathcal{D}(A)$. For a field extension $k'/k$, we denote $A \otimes_k k'$ by $A_{k'}$. Then we call $\mathcal{D}(A_{k'})$ the scalar extension of $\mathcal{D}(A)$.

We expect that the scalar extension depends on the triangulated category $\mathcal{D}(A)$ only. In more details we want the following conjecture to be true.

**Conjecture 1.** Let $A$ and $B$ be two DG algebras over $k$ such that we have a triangulated equivalence between derived categories $\mathcal{D}(A) \sim \mathcal{D}(B)$. Then for any field extension $k'/k$ we have a triangulated equivalence

$$\mathcal{D}(A_{k'}) \sim \mathcal{D}(B_{k'}).$$

So far we do not know whether Conjecture [1] is true. Nevertheless, we can prove it in an important special case which is sufficient for our use in algebraic geometry. First we need the following lemmas.

**Lemma 4.1.** Let $A$ be a DG algebra over $k$ and $k'/k$ be a field extension. For an object $F$ in $\mathcal{D}(A_{k'})$ the forgetful functor maps $F$ to an object in $\mathcal{D}(A)$. Moreover for any object $E$ in $\mathcal{D}(A)$ we have

$$\text{Hom}_{\mathcal{D}(A_{k'})}(E_{k'}, F) \cong \text{Hom}_{\mathcal{D}(A)}(E, F).$$

**Proof.** Since $A \to A_{k'}$ is flat, $(-) \otimes_k k' = (-) \otimes_A A_{k'}$ gives the derived tensor product functor $\mathcal{D}(A) \to \mathcal{D}(A_{k'})$ which is left adjoint to the forgetful functor $\mathcal{D}(A_{k'}) \to \mathcal{D}(A)$. □

**Lemma 4.2.** Let $A$ be a DG algebra over $k$ and $k'/k$ be a field extension. For any objects $E$ in $\text{Perf}(A)$ and $F$ in $\mathcal{D}(A)$, the natural map

$$\text{Hom}_{\mathcal{D}(A)}(E, F) \otimes k' \to \text{Hom}_{\mathcal{D}(A_{k'})}(E_{k'}, F_{k'})$$

is an isomorphism.

**Proof.** The statement is obviously true if $E = A$. Then we use the fact that any perfect complex of $A$-module can be obtained from $A$ by finite direct sums, shifts, direct summands and exact triangles. □

**Proposition 4.3.** Let $A$ and $B$ be two DG algebras over $k$ with an $A$-$B$ bimodule $T$ giving a DG Morita equivalence as in Definition 2.5. Then for any field extension $k'/k$, the $A_{k'}$-$B_{k'}$ bimodule $T_{k'}$ also gives a DG Morita equivalence between $A_{k'}$ and $B_{k'}$.

**Proof.** We need the following criterion on DG Morita equivalence.

**Lemma 4.4.** Let $A$ and $B$ be DG algebras over a field $k$ and $\theta : \mathcal{D}(A) \to \mathcal{D}(B)$ be a triangulated functor given by $(-) \otimes_A^{\mathbb{L}} T$ where $T$ is an $A$-$B$ bimodule. Then $\theta$ is a triangulated equivalence if and only if the following condition holds.

1. $T$ is compact if considered as an object in $\mathcal{D}(B)$, i.e. $T$ belongs to the full subcategory $\text{Perf}(B)$;
2. If $N$ is an object in $\mathcal{D}(B)$ and $\text{Hom}_{\mathcal{D}(B)}(T, N[i]) = 0$ for any $i \in \mathbb{Z}$, then we have $N = 0$;
3. $\text{Hom}_{\mathcal{D}(B)}(T, T[i]) \cong H^i(A)$ for any $i \in \mathbb{Z}$.

The proof of Lemma 4.4 See [14] Tag 09S5] Lemma 22.28.2. □

Now assume $T$ satisfies Condition (1), (2), (3). Then it is clear that $T_{k'}$ satisfies Condition (1). Applying Lemma 4.1 and Lemma 4.2 we see that $T_{k'}$ satisfies Condition (2) and (3). This finishes the proof of Proposition 4.3. □
We want to show that the scalar extension in Definition 4.1 is compatible with the base change of schemes. In more details, let $X$ be a quasi-compact, separated scheme over $k$ and $D(X)$ be the derived category of complexes of quasi-coherent $O_X$-modules. Let $E$ be a compact generator of $D(X)$ and $A$ be the DG algebra $R\text{Hom}_X(E,E)$. It is well-known that $D(X) \simeq D(A)$, see [13] Corollary 3.1.8 and interested readers could obtain an explicit proof using Proposition 2.4.

To study scalar extensions of schemes we first have the following lemma.

**Lemma 4.5.** Let $X$ be as above and $E \in \text{Perf}(X)$ and $F \in D(X)$. Let $k'/k$ be a field extension and $X_{k'} := X \times_k k'$ be the base change of $X$ and $p : X_{k'} \rightarrow X$ be the projection. Then the nature map

$$\text{Hom}_{D(X)}(E,F) \otimes_k k' \rightarrow \text{Hom}_{D(X_{k'})}(p^*E, p^*F)$$

is an isomorphism.

**Proof.** The proof is the same as that of Lemma 4.2. \qed

Then we have the following proposition.

**Proposition 4.6.** [See [13] Remark 9] Let $X$ and $A$ be as above and $k'/k$ be a field extension. Let $X_{k'} := X \times_k k'$ be the base change of $X$. Then we have

$$D(X_{k'}) \simeq D(A_{k'}).$$

**Proof.** Without loss of generality we can assume that $E$ is an h-injective complex of $O_X$-modules, hence $A = R\text{Hom}_X(E,E)$ is just the complex $\text{Hom}_{DG(X)}(E,E)$, where $DG(X)$ is the DG category of complexes of $O_X$-modules with quasi-coherent cohomologies.

Let $p : X_{k'} \rightarrow X$ be the natural projection and $E_{k'} = p^*E$. Then according to [3] Lemma 3.4.1, $E_{k'}$ is also a compact generator of $D(X_{k'})$. Moreover it is clear that $\text{Hom}_{DG(X_{k'})}(E_{k'}, E_{k'}) \simeq \text{Hom}_{DG(X)}(E,E) \otimes_k k' = A_{k'}$.

Let $I \subset DG(X_{k'})$ be the full pretriangulated DG subcategory consisting of h-injective objects and $z : E_{k'} \rightarrow I$ be an h-injective resolution in $DG(X_{k'})$. $I$ is a compact generator of $I$ and we want to apply Proposition 2.4 here. It is clear that $\text{Hom}_{DG(X_{k'})}(E_{k'}, J) \rightarrow \text{Hom}_{DG(X_{k'})}(I, J)$ is an isomorphism for any $J \in I$. Moreover, $H^i(\text{Hom}_{DG(X_{k'})}(E_{k'}, I)) = \text{Hom}_{DG(X_{k'})}(E_{k'}, E_{k'}[i])$ and by Lemma 4.5 the latter is isomorphic to $\text{Hom}_{DG(X)}(E, E)[i] \otimes_k k' = H^i(A_{k'})$, hence $A_{k'} \rightarrow \text{Hom}_{DG(X_{k'})}(E_{k'}, I)$ is a quasi-isomorphism. Then by Proposition 2.4

$$\text{Hom}_{DG(X_{k'})}(E_{k'}, -) : H^0(I) \rightarrow D(A_{k'})$$

is an equivalence of triangulated categories. On the other hand $H^0(I) \simeq D(X_{k'})$ and we finish the proof. \qed

Now we move on to prove that the scalar extension does not depend on the choice of $A$. To apply Proposition 4.6 we will need the following important result.

**Proposition 4.7.** Let $X$ be a projective variety and $A, B$ be two DG algebras such that $D(X) \simeq D(A)$ and $D(X) \simeq D(B)$. Then there exists an $A$-$B$ bimodule $T$ which gives a DG Morita equivalence $(-) \otimes_A^L T : D(A) \sim D(B)$.

The proof of Proposition 4.7 involves the following concepts and results.

First for a DG algebra (or more generally, a DG category) $A$, as usual we denote the DG category of (right) DG $A$-modules by $\mathcal{M}_{dg}(A)$ and we use $\text{Mod-}A$ to denote the (ordinary) category $Z^0(\mathcal{M}_{dg}(A))$. It is well-known that $\text{Mod-}A$ has a projective model structure where weak equivalences are quasi-isomorphisms of chain complexes and fibrations are degreewise epimorphisms, see [15] Definition 3.1 or [5] Theorem 3.2.

**Lemma 4.8.** For a DG algebra $A$ over a field $k$, the full DG subcategory of $\mathcal{M}_{dg}(A)$ consisting of fibrant and cofibrant objects coincides with $h\text{-proj}(A)$.
Definition that cofibrant objects are exactly h-projective modules. See [1] Proposition 1.7.

It is clear that the homotopy category $H^0(h\text{-proj}(A))$ is equivalent to $D(A)$. Now let $A$ and $B$ be as in Proposition 4.7 and we know that both $h\text{-proj}(A)$ and $h\text{-proj}(B)$ give DG enhancements of $D(X)$. Now we quote the following important fact about DG enhancement.

**Theorem 4.9.** ([9] Corollary 7.8) Let $X$ be a quasi-projective scheme and $D(X)$ be the derived category of complexes of quasi-coherent sheaves. Then $D(X)$ has a unique DG enhancement, i.e. for two DG enhancement $C$ and $D$ there exists a quasi-functor $\phi : C \to D$ which induces an equivalence between their homotopy categories.

**Proof.** See [9] Corollary 7.8. □

The proof of Proposition 4.7. Since both $h\text{-proj}(A)$ and $h\text{-proj}(B)$ give DG enhancements of $D(X)$, we obtain a quasi-functor $\phi : h\text{-proj}(A) \to h\text{-proj}(B)$ by Theorem 4.9. A priori $\phi$ is given by a $h\text{-proj}(A)$-$h\text{-proj}(B)$ DG bimodule. But since $\phi$ induces an equivalence between homotopy categories, it is continuous in the sense of [15] Section 7. Therefore by [15] Corollary 7.6, $\phi$ is given by an $A$-$B$ bimodule $T$. □

The following result, which generalizes Proposition 4.6 shows that the scalar extension in Definition 4.1 is compatible with the base change in algebraic geometry.

**Corollary 4.10.** Let $X$ be a projective variety over a field $k$ and $A$ be any DG algebra over $k$ such that $D(X) \simeq D(A)$. Then for a field extension $k'/k$ we have

$$D(X_{k'}) \cong D(A_{k'}).$$

**Proof.** It is a direct corollary of Proposition 4.6 and Proposition 4.7. □

### 4.2. Scalar extensions of categorical resolutions

In this subsection we discuss scalar extensions of categorical resolutions. First we recall our definition of algebraic categorical resolution, see Definition 3.4 above.

**Definition 4.2.** Let $X$ be a projective variety over a field $k$. Then an algebraic categorical resolution of $X$ is a triple $(A, B, T)$ where $A$ is a DG algebra such that $D(X) \simeq D(A)$, $B$ is a smooth DG algebra and $T$ is an $A$-$B$ bimodule such that

1. $H^i(A) \to \text{Hom}_{D(B)}(T, T[i])$ is an isomorphism for any $i \in \mathbb{Z}$;
2. $T$ defines a compact object in $D(B)$;
3. $\bigoplus_i \text{Hom}_{D(B)}(T, B[i])$ is finite dimensional.

Now we define the scalar extension of a categorical resolution.

**Definition 4.3.** Let $X$ be a projective variety over a field $k$ and $(A, B, T)$ be an algebraic categorical resolution. Let $k'/k$ be a field extension. Then the scalar extension of $(A, B, T)$ is given by $(A_{k'}, B_{k'}, T_{k'})$.

We need to prove that $(A_{k'}, B_{k'}, T_{k'})$ in Definition 4.3 really gives an algebraic categorical resolution of $X_{k'}$.

**Proposition 4.11.** Let $X$ be a projective variety over a field $k$. If $(A, B, T)$ is an algebraic categorical resolution of $X$, then $(A_{k'}, B_{k'}, T_{k'})$ is an algebraic categorical resolution of $X_{k'}$.

**Proof.** First of all, we know that $B_{k'}$ is smooth since $B$ is smooth. Moreover by Corollary 4.10 we know that $D(X_{k'}) \simeq D(A_{k'})$.

Then we need to show that $(A_{k'}, B_{k'}, T_{k'})$ satisfies Condition (1), (2), (3) in Definition 4.2. Condition (2) follows from Lemma 2.3 and then Condition (1) and (3) are consequences of Lemma 4.2. □
5. APPLICATION: FULL EXCEPTIONAL COLLECTIONS OF CATEGORICAL RESOLUTIONS

In [17] the following results has been proved.

**Theorem 5.1.** [17] Theorem 4.9] Let $X$ be a projective curve over an algebraically closed field $k$. Let $(\mathcal{T}, \pi^*, \pi_*)$ be a categorical resolution of $X$ (in the sense of Definition 3.7). If the geometric genus of $X$ is $\geq 1$, then $\mathcal{T}^c$ cannot have a full exceptional collection.

Actually $X$ has a categorical resolution which admits a full exceptional collection if and only if the geometric genus of $X$ is 0.

**Theorem 5.2.** [17] Theorem 4.10] Let $X$ be a projective curve over an algebraically closed field $k$ of geometric genus $\geq 1$ and $(\mathcal{T}, \pi^*, \pi_*)$ be a categorical resolution of $X$. Then $\mathcal{T}^c$ cannot have a tilting object, moreover there cannot be a finite dimensional $k$-algebra $\Lambda$ of finite global dimension such that

$$\mathcal{T}^c \simeq D^b(\Lambda\text{-mod}).$$

**Remark 13.** The proof of Theorem 5.1 depends on a careful study of the Picard group of $X$ and the Grothendieck group of the triangulated categories. In particular it involves the fact that Pic$(X)$ is not finitely generated if $X$ is a projective curve with geometric genus $\geq 1$ over an algebraically closed field $k$.

However, if the base field $k$ is not algebraically closed, then Pic$(X)$ may be finitely generated even if the geometric genus of $X$ is $\geq 1$. Therefore the proof of Theorem 5.1 in [17] does not work if $k$ is not algebraically closed. See [17] Remark 9.

In order to generalized Theorem 5.1 to curves over non-algebraically closed fields, we need to seek a different proof. The key fact is the following lemma.

**Lemma 5.3.** Let $B$ be a DG algebra over a field $k$ and assume Perf$(B)$ has a full exceptional collection $\langle E_1, \ldots, E_n \rangle$. Then for any field extension $k'/k$, $\langle (E_1)_{k'}, \ldots, (E_n)_{k'} \rangle$ is a full exceptional collection of Perf$(B_{k'})$.

**Proof.** By Lemma 4.2 we know $\text{Hom}_{D(B_{k'})}((E_i)_{k'}, (E_j)_{k'}) = \text{Hom}_{D(B)}(E_i, E_j) \otimes k'$ therefore it is clear that $\langle (E_1)_{k'}, \ldots, (E_n)_{k'} \rangle$ is an exceptional collection of Perf$(B_{k'})$.

To show that $\langle (E_1)_{k'}, \ldots, (E_n)_{k'} \rangle$ is full, it is sufficient to show that $B_{k'}$ is contained in the triangulated subcategory generated by $(E_1)_{k'}, \ldots, (E_n)_{k'}$. We know that $B$ is contained in the triangulated subcategory generated by $E_1, \ldots, E_n$, i.e. $B$ could be obtained from $E_1, \ldots, E_n$ by taking exact triangles and shifts finitely many times. Tensoring all the exact triangles with $k'$ we see that $B_{k'}$ is contained in the triangulated subcategory generated by $(E_1)_{k'}, \ldots, (E_n)_{k'}$.

Then we could generalize Theorem 5.1 to curves over non-algebraically closed fields.

**Theorem 5.4.** Let $X$ be a projective curve over a field $k$. Then $X$ has a categorical resolution which admits a full exceptional collection if and only if the geometric genus of $X$ is 0.

**Proof.** The "if" part could be obtained by an explicit construction of a categorical resolution. In fact the proof is exactly the same as that of [17] Proposition 4.1.

Now assume $X$ has geometric genus $\geq 1$ and there exists a categorical resolution $(\mathcal{T}, \pi^*, \pi_*)$ such that $\mathcal{T}^c$ admits a full exceptional collection. Then $\mathcal{T}$ has a compact generator hence by Proposition 5.4 we have an algebraic categorical resolution $(A, B, T)$ of $X$ and Perf$(B) \simeq \mathcal{T}^c$ has a full exceptional collection.

Let $\bar{k}$ be the algebraic closure of $k$ and $X_{\bar{k}}$ be the base change. It is clear that $X_{\bar{k}}$ also has geometric genus $\geq 1$. Moreover by Proposition 5.1 $(A_{\bar{k}}, B_{\bar{k}}, T_{\bar{k}})$ is a categorical resolution of $X_{\bar{k}}$. By Lemma 5.3 we know Perf$(B_{\bar{k}}) = D(B_{\bar{k}})^c$ has a full exceptional collection, which is contradictory to Theorem 5.1.

We also have the generalization of [17] Theorem 4.10 to non-algebraically closed base field.
Theorem 5.5. Let $X$ be a projective curve with geometric genus $\geq 1$ over a field $k$ and $(\mathcal{T}, \pi^*, \pi_*)$ be a categorical resolution of $X$. Then $\mathcal{T}^c$ cannot have a tilting object, moreover there cannot be a finite dimensional $k$-algebra $\Lambda$ of finite global dimension such that $\mathcal{T}^c \cong D^b(\Lambda\text{-mod})$.

Proof. The proof is similar to that of Theorem 5.4 and is left to the readers. \qed

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