Recursive projection-aggregation decoding of Reed-Muller codes

Min Ye  Emmanuel Abbe

Abstract

We propose a new class of efficient decoding algorithms for Reed-Muller (RM) codes over binary-input memoryless channels. The algorithms are based on projecting the code on its cosets, recursively decoding the projected codes (which are lower-order RM codes), and aggregating the reconstructions (e.g., using majority votes). We further provide list-decoding and code concatenation extensions of the algorithms.

We run our main algorithm for AWGN channels and Binary Symmetric Channels at the short code length ($\leq 1024$) and low code rate ($\leq 0.5$) regime. Simulation results show that the new algorithm not only outperforms the previous decoding algorithms for RM codes, it also outperforms the optimal decoder for polar codes (SCL+CRC) with the same parameters by a wide margin. The performance of the new algorithm for RM codes in those regimes is in fact close to that of the maximal likelihood decoder. Finally, the new decoder naturally allows for parallel implementations.

I. INTRODUCTION

Reed-Muller (RM) codes are among the oldest families of error-correcting codes [1]. The recent breakthrough of polar codes [2] has brought the attention back to RM codes, due to the closeness of the two codes. RM codes have in particular the advantage of having a simple and universal code construction, and promising performances were demonstrated in several works [3], [4], with a scaling law conjectured to be comparable of that of random codes.

RM codes do not possess yet the generic analytical framework of polar codes (i.e., polarization theory). It was recently shown that RM codes achieve capacity on the Binary Erasure Channel (BEC) at constant rate [5], as well as for extremal rates for BEC and Binary Symmetric Channels (BSC) [6], but obtaining such results for a broader class of communication channels and rates remains open. Recent progress was made on these questions with a polarization approach to RM codes shown in [7].

An important missing component for RM codes is an efficient decoding algorithm that competes with that of polar codes. Various decoding algorithms have been proposed for RM codes, starting with Reed algorithm [1], [8], and three important more recent line of works based on recursive list-decoding [9]–[11], a new Berlekamp-Welch type of algorithm [12], [13], and a new algorithm utilizing minimum-weight parity checks [14]. In particular, [9]–[13] give fairly powerful theoretical guarantees for efficient decoding of RM codes in specific regimes. However, there appears to be no known algorithm for RM codes competing with the performance of polar codes in the low rate/blocklength regime [15].

In this paper, we propose a new class of decoding algorithms for Reed-Muller codes over any binary-input memoryless channels. The new algorithms are based on recursive projections and aggregations of cosets decoding, exploiting the self-similarity of RM codes, and are extended with list-decoding procedures and with outer-code concatenations. The main algorithm significantly improves on the existing algorithms for RM codes [1], [9]–[13] and on the best known decoding algorithm for polar codes [15] for the regimes of interest, i.e., short code length ($\leq 1024$) and low code rate ($\leq 0.5$) regime. This is the type of regime where polar codes are planned to enter the 5G standards [16] as well as relevant regimes for applications in the Internet of Things (IoT).

E. Abbe is with the Mathematics Institute and the School of Computer and Communication Sciences at EPFL, Switzerland, and the Program in Applied and Computational Mathematics and the Department of Electrical Engineering in Princeton University, USA. M. Ye is with Department of Electrical Engineering, Princeton University, USA; email: yeemmi@gmail.com.
More specifically, we compare our new algorithm for RM codes with the Successive Cancellation List (SCL) decoder for CRC-aided polar codes [15], where we set the list and CRC size to take optimal values. This gives essentially the optimal decoding error probability for polar codes. For AWGN channels, our new algorithm has about 0.75dB gain (more in some cases) over polar codes in the short code length ($\leq 1024$) and low code rate ($\leq 0.5$) regime, and similar improvements are also obtained for BSC channels.

In the above regimes, the decoding error probability of our new algorithm is in fact shown to be close to that of the Maximal Likelihood decoder on RM codes. Some extensions and variants to potentially further improve the performance are also discussed, as well as possible extensions of the projection-aggregation and list-decoding procedures to other families of codes.

In Section II we give a high level description of the new type of algorithms. In Section III we present decoding algorithm for BSC channels. In Section IV we generalize the algorithms to decode RM codes over any binary-input channel. Finally, in Section V we present simulation results. In addition to the previously mentioned improvements over polar codes, we also empirically validate the improved scaling-law of RM codes over polar codes on BSC channels [17].

II. A HIGH-LEVEL DESCRIPTION OF THE NEW ALGORITHMS

We begin with some notation and background on RM codes. In this paper, we use $\oplus$ to denote sums over $\mathbb{F}_2$. Let us consider the polynomial ring $\mathbb{F}_2[Z_1, Z_2, \ldots, Z_m]$ of $m$ variables. Since $Z^2 = Z$ in $\mathbb{F}_2$, the following set of $2^m$ monomials forms a basis of $\mathbb{F}_2[Z_1, Z_2, \ldots, Z_m]$: $$ \{ \prod_{i \in A} Z_i : A \subseteq [m] \}, \text{ where } \prod_{i \in \emptyset} Z_i := 1. $$

Next we associate every subset $A \subseteq [m]$ with a row vector $v_m(A)$ of length $2^m$, whose components are indexed by a binary vector $z = (z_1, z_2, \ldots, z_m) \in \{0, 1\}^m$. The vector $v_m(A)$ is defined as follows:

$$ v_m(A, z) = \prod_{i \in A} z_i, \quad (1) $$

where $v_m(A, z)$ is the component of $v_m(A)$ indexed by $z$, i.e., $v_m(A, z)$ is the evaluation of the monomial $\prod_{i \in A} Z_i$ at $z$. For $0 \leq r \leq m$, the set of vectors

$$ \{ v_m(A) : A \subseteq [m], |A| \leq r \} $$

forms a basis of the $r$-th order Reed-Muller code $\mathcal{R}M(m, r)$ of length $n := 2^m$ and dimension $\sum_{i=0}^{r} \binom{m}{i}$.

**Definition 1.** The $r$-th order Reed-Muller code $\mathcal{R}M(m, r)$ code is defined as the following set of binary vectors

$$ \mathcal{R}M(m, r) := \left\{ \sum_{A \subseteq [m], |A| \leq r} u(A)v_m(A) : u(A) \in \{0, 1\} \text{ for all } A \subseteq [m], |A| \leq r \right\}. $$

In other words, each vector $v_m(A)$ consists of all the evaluations of the monomial $\prod_{i \in A} Z_i$ at all the points in the vector space $E := \mathbb{F}_2^n$, and each codeword $c \in \mathcal{R}M(m, r)$ corresponds to an $m$-variate polynomial with degree at most $r$. The coordinates of the codeword $c$ are also indexed by the binary vectors $z \in E$, and we write $c = (c(z), z \in E)$. Let $B$ be an $s$-dimensional subspace of $E$, where $s \leq r$. The quotient space $E/B$ consists of all the cosets of $B$ in $E$, where every coset $T$ has form $T = z + B$ for some $z \in E$. For a binary vector $y = (y(z), z \in E)$, we define its projection on the cosets of $B$ as

$$ y/B = \text{Proj}(y, B) := \left( y/B(T), T \in E/B \right), \text{ where } y/B(T) := \bigoplus_{z \in T} y(z) \quad (2) $$

$^1$The optimal CRC size depends on the choice of code length and rate.
Algorithm 1 The RPA_RM decoding function for BSC

**Input:** The corrupted codeword \( y = (y(z), z \in E) \); the parameters of the Reed-Muller code \( m \) and \( r \); the maximal number of iterations \( N_{\text{max}} \)

**Output:** The decoded codeword \( \hat{c} \)

1: for \( i = 1, 2, \ldots, N_{\text{max}} \) do
2: \( y/B_i \leftarrow \text{Proj}(y, B_i) \) for \( i = 1, 2, \ldots, 2^m - 1 \) \( \triangleright \) Projection
3: \( \hat{y}/B_i \leftarrow \text{RPA_RM}(y/B_i, m - 1, r - 1, N_{\text{max}}) \) for \( i = 1, 2, \ldots, 2^m - 1 \) \( \triangleright \) Recursive decoding
4: \( \triangleright \) If \( r = 2 \), then we use the Fast Hadamard Transform to decode the first-order RM code [8]
5: \( \hat{y} \leftarrow \text{Aggregation}(y, \hat{y}/B_1, \hat{y}/B_2, \ldots, \hat{y}/B_{n-1}) \) \( \triangleright \) Aggregation
6: if \( y = \hat{y} \) then
7: break \( \triangleright y = \hat{y} \) means that the algorithm already converges to a fixed (stable) point
8: end if
9: \( y \leftarrow \hat{y} \)
10: end for
11: \( \hat{c} \leftarrow \hat{y} \)
12: return \( \hat{c} \)

is the binary vector obtained by summing up all the coordinates of \( y \) in each coset \( T \in E/B \). Here the sum is over \( \mathbb{F}_2 \) and the dimension of \( y/B \) is \( n/|B| \).

In the next section, we will show that if \( c \) is a codeword of \( \mathcal{RM}(m, r) \), then \( c/B \) is a codeword of \( \mathcal{RM}(m - s, r - s) \), where \( s \) is the dimension of \( B \). Our new decoding algorithm makes use of the case \( s = 1 \), namely, the one-dimensional subspaces. More precisely, let \( y = (y(z), z \in E) \) be the output vector of transmitting a codeword of \( \mathcal{RM}(m, r) \) over some BSC channel. Our decoding algorithm is defined in a recursive way: For every one-dimensional subspace \( B \), we first obtain the projection \( y/B \), and then we use the decoding algorithm for \( \mathcal{RM}(m - 1, r - 1) \) to decode \( y/B \), where the decoding result is denoted as \( \hat{y}/B \). Since every one-dimensional subspace of \( E \) consists of 0 and a non-zero element, there are \( n - 1 \) such subspaces in total. After the projection and recursive decoding steps, we obtain \( n - 1 \) decoding
results $\hat{y}_1/B_1, \hat{y}_2/B_2, \ldots, \hat{y}_n/B_n$. Next we use a majority voting scheme to aggregate these decoding results together with $y$ to obtain a new estimate $\hat{y}$ of the original codeword. Finally we update $y$ as $\hat{y}$, and iterate the whole procedure for up to $N_{\text{max}}$ rounds. Notice that if $y = \hat{y}$ (see line 6), then $y$ is a fixed (stable) point of this algorithm and will remain unchanged for the next iterations. In this case we should exit the for loop on line 1 (see line 6–8). In practice we set the maximal number of iterations $N_{\text{max}} = \lceil m/2 \rceil$ to prevent the program from running into an infinite loop, and typically $\lceil m/2 \rceil$ iterations are enough for the algorithm to converge to a stable $y$. This high-level description is summarized in Fig. 1 and Algorithm 1.

While this description focuses on the decoding algorithm over BSC, a natural extension of this algorithm bases on log-likelihood ratios (LLRs) allows us to decode RM codes over any binary-input memoryless channels, including the AWGN channel; see Section IV for details.

### A. List decoding procedure and code concatenation

Here we describe a simple and generic list decoding procedure and a code concatenation method that can be composed with any unique decoding method for any code family in order to decrease the decoding error probability. Suppose that we have a unique decoding algorithm $\text{decode}_C$ for some code $C$ over some binary-input memoryless channel $W : \{0, 1\} \to \mathcal{W}$. Without loss of generality, assume that $\text{decode}_C$ is based on the LLR vector of the channel output, where the LLR of an output symbol $x \in \mathcal{W}$ is defined as

$$\text{LLR}(x) := \ln \left( \frac{W(x|0)}{W(x|1)} \right).$$

Clearly, if $|\text{LLR}(x)|$ is small, then $x$ is a noisy symbol, and if $|\text{LLR}(x)|$ is large, then $x$ is relatively noiseless.

Our list decoding procedure works as follows. Suppose that $y = (y_1, y_2, \ldots, y_n)$ is the output vector when we send a codeword of $C$ over the channel $W$. We first sort $|\text{LLR}(y_i)|, i \in [n]$ from small to large. Without loss of generality, let us assume that $|\text{LLR}(y_1)|, |\text{LLR}(y_2)|, |\text{LLR}(y_3)|$ are the three smallest components in the LLR vector, meaning that $y_1, y_2$ and $y_3$ are the three most noisy symbols in the channel outputs (we take three arbitrarily). Next we enumerate all the possible cases of the first three bits of the codeword $c = (c_1, c_2, c_3)$. The first three bits $(c_1, c_2, c_3)$ can be any vector in $\mathbb{F}_2^3$, so there are 8 cases in total, and for each case we change the value of $\text{LLR}(y_1), \text{LLR}(y_2), \text{LLR}(y_3)$ according to the values of $c_1, c_2, c_3$. More precisely, we set $\text{LLR}(y_i) = (-1)^{c_i}L_{\text{max}}$ for $i = 1, 2, 3$, where $L_{\text{max}}$ is some large real number. In practice, we can choose $L_{\text{max}} := \max(|\text{LLR}(y_i)|, i \in [n])$ or $L_{\text{max}} := 2\max(|\text{LLR}(y_i)|, i \in [n])$. For each of these 8 cases, we use $\text{decode}_C$ to obtain a decoded codeword, and we denote them as $\hat{c}^{(1)}, \hat{c}^{(2)}, \ldots, \hat{c}^{(8)}$. Finally, we calculate the posterior probability of $W^n(y | \hat{c}^{(i)}), 1 \leq i \leq 8$, and choose the largest one as the final decoding result, namely, we perform a maximal likelihood decoding among the 8 candidates in the list.

When we apply this list decoding procedure together with Algorithm 1 to decode RM codes, the decoding error probability is typically close to that of the Maximal Likelihood decoder.

The list decoding procedure can be further composed with the code concatenation method. More precisely, we first use a (high rate) outer code $C_{\text{out}}$ to encode the information bits, and then we encode the codeword of $C_{\text{out}}$ by $C$. In the list decoding procedure above, after obtaining $\hat{c}^{(1)}, \hat{c}^{(2)}, \ldots, \hat{c}^{(8)}$, we need to check whether $\hat{c}^{(i)}$ is a codeword of $C_{\text{out}}$ or not and only keep it in the list if it belongs to $C_{\text{out}}$. In this way, we can further shrink the list and decrease the decoding error probability. At the same time, this method will also decrease the code rate. In practice, the number of parities (i.e., the difference between code length and dimension) in $C_{\text{out}}$ is set to be 1 or 2 in order to obtain best performance, and we can use random codes as $C_{\text{out}}$, i.e., we generate the parity check matrix as i.i.d. Bernoulli-1/2 random variables.
III. DECODING ALGORITHM FOR BSC

We begin with the definition of the quotient code. Then we show that the quotient code of an RM code is also an RM code.

Definition 2. Let \( s \leq r \leq m \) be integers, and let \( \mathbb{B} \) be an \( s \)-dimensional subspace of \( \mathbb{E} := \mathbb{F}_2^m \). We define the quotient code \( Q(m, r, \mathbb{B}) := \{c/\mathbb{B} : c \in \mathcal{R}M(m, r)\} \).

Lemma 1. Let \( s \leq r \leq m \) be integers, and let \( \mathbb{B} \) be an \( s \)-dimensional subspace of \( \mathbb{E} := \mathbb{F}_2^m \). The code \( Q(m, r, \mathbb{B}) \) is the Reed-Muller code \( \mathcal{R}M(m - s, r - s) \).

The proof of this Lemma is straightforward and can be found in Appendix A.

Note that Reed’s algorithm [1] relies on the special case of \( s = r \) in Lemma 1 and our new decoding algorithm makes use of the case \( s = 1 \) in Lemma 1 (in addition to using all subspaces and adding an iterative process). The \( \text{RPA}_{\mathcal{R}M} \) decoding function is already presented in the previous section. Here we fill in the only missing component, namely the Aggregation function; see Algorithm 2 below. Both \( y/\mathbb{B}_i = (y/\mathbb{B}_i(T), T \in \mathbb{E}/\mathbb{B}_i) \) and \( \hat{y}/\mathbb{B}_i = (\hat{y}/\mathbb{B}_i(T), T \in \mathbb{E}/\mathbb{B}_i) \) are indexed by the cosets \( T \in \mathbb{E}/\mathbb{B}_i \), and we use \( [z + \mathbb{B}_i] \) to denote the coset containing \( z \) (see line 3).

Algorithm 2 The Aggregation function for BSC

Input: \( y, \hat{y}/\mathbb{B}_1, \hat{y}/\mathbb{B}_2, \ldots, \hat{y}/\mathbb{B}_{n-1} \)

Output: \( \hat{y} \)

1. Initialize \( \text{changevote}(z), z \in \{0, 1\}^m \) as an all-zero vector indexed by \( z \in \{0, 1\}^m \)
2. \( n \leftarrow 2^m \)
3. \( \text{changevote}(z) \leftarrow \sum_{i=1}^{n-1} 1[y/\mathbb{B}_i([z + \mathbb{B}_i]) \neq \hat{y}/\mathbb{B}_i([z + \mathbb{B}_i])] \) for each \( z \in \{0, 1\}^m \)
4. \( y(z) \leftarrow y(z) \oplus 1[\text{changevote}(z) > \frac{n-1}{2}] \) for each \( z \in \{0, 1\}^m \) \( \triangleright \) Here addition is over \( \mathbb{F}_2 \)
5. \( \hat{y} \leftarrow y \)
6. return \( \hat{y} \)

From line 3, we can see that the maximal possible value of \( \text{changevote}(z) \) for each \( z \in \mathbb{E} \) is \( n - 1 \). Therefore the condition \( \text{changevote}(z) > \frac{n-1}{2} \) on line 4 can indeed be viewed as a majority vote. As discussed in Section III-A this algorithm can be viewed as one step of the power iteration method to find the eigenvector of a matrix built from the quotient code decoding.

In Appendix C we write the pseudo codes in a mathematical fashion for the ease of understanding. In Appendix C we present another version of the \( \text{RPA}_{\mathcal{R}M} \) function in a program language fashion.

Proposition 1. The complexity of Algorithm 1 is \( O(n^r \log n) \) in sequential implementation and \( O(n^2) \) in parallel implementation with \( O(n^r) \) processors.

In Section V-A we further discuss options to reduce the computation time by using fewer subspaces in the projection step.

Proof. We prove by the induction on the order of the RM code. To establish the base case, observe that the complexity of decoding first-order RM codes using Fast Hadamard Transform (FHT) [8] is \( O(n \log n) \). Now we assume the proposition holds for decoding \( (r - 1) \)-th order RM codes and prove the inductive step. Clearly, the complexity of Algorithm 1 is determined by the complexity of the recursive decoding step on line 3. By induction hypothesis, the complexity of decoding each \( y/\mathbb{B}_i \) is \( O(n^{r-1} \log n) \). Since there are \( n - 1 \) one-dimensional subspaces \( \mathbb{B}_1, \mathbb{B}_2, \ldots, \mathbb{B}_{n-1} \), the complexity of Algorithm 1 is indeed \( O(n^r \log n) \).

\footnote{While we did not find a reference to provide for such a result, the result is unlikely to be new.}
In the next proposition, we show that whether Algorithm 1 outputs the correct codeword or not is independent of the transmitted codeword and only depends on the error pattern imposed by the BSC channel.

**Proposition 2.** Let \( c \in \mathcal{RM}(m, r) \) be a codeword of the RM code. Let \( e = (e(z), z \in \mathcal{E}) \) be the error vector imposed on \( c \) by the BSC channel, and the output vector of the BSC channel is \( y = c + e \). Denote the decoding result as \( \hat{c} = \text{RPA}_\text{RM}(y, m, r, N_{\text{max}}) \). Then the indicator function of decoding error \( 1[\hat{c} \neq c] \) is independent of the choice of \( c \) and only depends on the error vector \( e \).

Notice that we use maximal likelihood decoder for first-order RM code, and the proposition can be proved by induction on the order of the RM code\(^3\). This proposition is useful for simulations because we can simply transmit the all-zero codeword over the BSC channel to measure the decoding error probability.

### A. Spectral interpretations of Algorithm 4

Algorithm 4 can be viewed as a one-step power iteration of a spectral algorithm. More precisely, observe that \( \hat{y}/b_1, \hat{y}/b_2, \ldots, \hat{y}/b_{n-1} \) contain the estimates of \( c(z) \oplus c(z') \) for all \( z \neq z' \), where \( c = (c(z), z \in \mathcal{E}) \) is the transmitted codeword. Suppose for the moment that we want to find a \( \hat{y} = (\hat{y}(z), z \in \mathcal{E}) \in \{0, 1\}^n \) to agree with as many estimates of these sums as possible. This is in fact a reconstruction problem on the following graph model: Consider the complete graph whose vertices are all the points in \( \mathcal{E} \). Each vertex \( z \in \mathcal{E} \) is assigned a value \( \hat{y}(z) \). On each edge, we have an estimate of the sum of the values on its two endpoints, so we can build the adjacency matrix \( A \) whose dimension is \( n \times n \). The rows and columns of \( A \) are indexed by \( z \in \mathcal{E} \), and the entry \( A_{xz} \) is the estimate of \( c(z) \oplus c(z') \). Now we map all the \( \{0, 1\} \) in this problem to \( \{1, -1\} \). Ignoring what was the received word \( y \), one would like to find a reconstruction that agrees with the maximal number of constraints, i.e.,

\[
\arg\max_{\hat{y} \in \{1, -1\}^n} \hat{y}^T A \hat{y}.
\]

If no errors perturb the codeword, then we can satisfy all the constraints by taking any spanning tree of the graph (such as connecting one vertex to all others) and we obtain two solutions: the codeword and its complement. Without a perfect solution, it corresponds to a min-cut problem (on a censored block model\(^4\)) with the following classical spectral relaxation:

\[
\arg\max_{\hat{y} \in \mathcal{E}^n, \|\hat{y}\|^2 = n} \hat{y}^T A \hat{y}.
\]

The solution to this problem is the eigenvector corresponding to the largest eigenvalue. One way to find this eigenvector is to use the power iteration method: pick some vector \( v \) (e.g., at random), then \( A^tv \) converges to this eigenvector when \( t \) is large enough.\(^4\) Then one can look at the sign of each coordinate of \( A^tv \) to determine our final output \( \hat{y} \) (picking the global sign flip so that we are closer to the received word \( y \)). In our case, we also want to find a reconstruction that is close to the received word (MAP looks for the closest codeword), so a possibility is to start the power-iteration at \( y \) itself. Then, one-step of this iteration gives precisely the voting algorithm described previously.

We tried to use the power-iteration method in the **Aggregation function** based on majority vote. This is because in the spectral method above we tried our best to agree with \( \hat{y}/b_1, \hat{y}/b_2, \ldots, \hat{y}/b_{n-1} \), ignoring the original channel output \( y \), and many of these are very noisy measurements.

\(^3\)See the proof of Proposition 4 for a rigorous argument. The ideas of the proofs of these two propositions are exactly the same.

\(^4\)Assume the largest eigenvalue has largest magnitude.
Another possibility would be to use an SDP relaxation, or modify the graph operator as done for community detection algorithms [18].

IV. Decoding Algorithm for General Binary-Input Memoryless Channels

The decoding algorithm in the previous section only works for the BSC. In this section, we will present a natural extension of Algorithm 1 that works for any binary-input memoryless channels, and this new algorithm is based on LLRs (see [3]). Similarly to Algorithm 1, this new algorithm is also defined recursively, i.e., we first assume that we know how to decode \((r-1)\)-th order Reed-Muller code, and then we use it to decode the \(r\)-th order Reed-Muller code. To begin with, we need to show how to decode the first order RM code efficiently. We will show that based on LLR, we can also use the Fast Hadamard Transform to implement the Maximum Likelihood (ML) decoder for general binary-input channels, and the complexity is \(O(n \log n)\), as opposed to the naive implementation of the ML decoder, whose complexity is \(O(n^2)\).

We still use \(c = (c(z), z \in E)\) to denote the transmitted (true) codeword and \(y = (y(z), z \in E)\) to denote the corresponding channel output. Given the output vector \(y\), the ML decoder for first order RM codes aims to find \(c \in RM(m, 1)\) to maximize \(\prod_{z \in E} W(y(z)|c(z))\). This is equivalent to maximizing the following quantity:

\[
\prod_{z \in E} \frac{W(y(z)|c(z))}{\sqrt{W(y(z)|0)W(y(z)|1)}},
\]

which is further equivalent to maximizing

\[
\sum_{z \in E} \ln \left( \frac{W(y(z)|c(z))}{\sqrt{W(y(z)|0)W(y(z)|1)}} \right),
\]

which is further equivalent to maximizing

\[
\sum_{z \in E} \ln \left( \frac{W(y(z)|c(z))}{\sqrt{W(y(z)|0)W(y(z)|1)}} \right),
\]

Notice that the codeword \(c\) is a binary vector. Therefore,

\[
\ln \left( \frac{W(y(z)|c(z))}{\sqrt{W(y(z)|0)W(y(z)|1)}} \right) = \begin{cases} 
\frac{1}{2} \text{LLR}(y(z)) & \text{if } c(z) = 0 \\
-\frac{1}{2} \text{LLR}(y(z)) & \text{if } c(z) = 1
\end{cases}.
\]

From now on we will use the shorthand notation

\[L(z) := \text{LLR}(y(z)),\]

and the formula in (4) can be written as

\[
\frac{1}{2} \sum_{z \in E} \left( (-1)^{c(z)} L(z) \right),
\]

so we want to find \(c \in RM(m, 1)\) to maximize this quantity.

By definition, every \(c \in RM(m, 1)\) corresponds to a polynomial in \(\mathbb{F}_2[Z_1, Z_2, \ldots, Z_m]\) of degree one, so we can write every codeword \(c\) as a polynomial \(u_0 + \sum_{i=1}^m u_i Z_i\). In this way, we have \(c(z) = u_0 + \sum_{i=1}^m u_i z_i\), where \(z_1, z_2, \ldots, z_m\) are the coordinates of the vector \(z\). Now our task is to find \(u_0, u_1, u_2, \ldots, u_m \in \mathbb{F}_2\) to maximize

\[
\sum_{z \in E} \left( (-1)^{u_0 + \sum_{i=1}^m u_i z_i} L(z) \right) = (-1)^{u_0} \sum_{z \in E} \left( (-1)^{\sum_{i=1}^m u_i z_i} L(z) \right).
\]

For a binary vector \(u = (u_1, u_2, \ldots, u_m) \in E\), we define

\[\hat{L}(u) := \sum_{z \in E} \left( (-1)^{\sum_{i=1}^m u_i z_i} L(z) \right).\]

Clearly, to find the maximizer of (6), we only need to calculate \(\hat{L}(u)\) for all \(u \in E\), but the vector \((\hat{L}(u), u \in E)\) is exactly the Hadamard Transform of the vector \((L(z), z \in E)\), so it can be calculated using
the Fast Hadamard Transform with complexity $O(n \log n)$. Once we know the values of $(\hat{L}(u), u \in \mathbb{E})$, we can find $u^* = (u_1^*, u_2^*, \ldots, u_m^*) \in \mathbb{E}$ that maximizes $|\hat{L}(u)|$. If $\hat{L}(u^*) > 0$, then the decoder outputs the codeword corresponding to $u_0^* = 0, u_1^*, u_2^*, \ldots, u_m^*$. Otherwise, the decoder outputs the codeword corresponding to $u_0^* = 1, u_1^*, u_2^*, \ldots, u_m^*$. This completes the description of how to decode the first order RM codes for general channels.

The next problem is how to extend (2) in the general setting. The purpose of (2) is mapping two output symbols $(y(z), z \in T)$ whose indices are in the same coset $T \in \mathbb{E}/\mathbb{B}$ to one symbol. In this way, we reduce the $r$-th order RM code to an $(r-1)$-th order RM code. For BSC, this mapping is simply the addition in $\mathbb{F}_2$. The sum $y_{\beta}(T)$ can be interpreted as an estimate of $c_{/B}(T)$, where $c$ is the transmitted (true) codeword. In other words,

$$\Pr(Y_{/B}(T) = c_{/B}(T)) > \Pr(Y_{/B}(T) = c_{/B}(T) \oplus 1),$$

where $Y$ is the channel output random vector.

For general channels, we also want to estimate $c_{/B}(T)$ based on the LLRs $(L(z), z \in T)$. More precisely, given $(y(z), z \in T)$, or equivalently given $(L(z), z \in T)$, we would like to calculate the following LLR:

$$L_{/B}(T) := \ln \left( \frac{\Pr(Y = y(z), z \in T|c_{/B}(T) = 0)}{\Pr(Y = y(z), z \in T|c_{/B}(T) = 1)} \right).$$

We use the following simplified model to calculate this LLR: Suppose that $S_1$ and $S_2$ are i.i.d. Bernoulli-1/2 random variables, and we transmit them over two independent copies of the channel $W : \{0, 1\} \rightarrow \mathbb{W}$. The corresponding channel output random variables are denoted as $X_1$ and $X_2$, respectively. Then for $x_1, x_2 \in \mathbb{W}$,

$$\ln \left( \frac{\Pr(X_1 = x_1, X_2 = x_2|S_1 + S_2 = 0)}{\Pr(X_1 = x_1, X_2 = x_2|S_1 + S_2 = 1)} \right) = \ln \left( \frac{\Pr(X_1 = x_1, X_2 = x_2, S_1 + S_2 = 0)}{\Pr(X_1 = x_1, X_2 = x_2, S_1 + S_2 = 1)} \right)$$

$$= \ln \left( \frac{1}{2} \Pr(X_1 = x_1) \Pr(X_2 = x_2) + \frac{1}{2} \Pr(X_1 = x_1, X_2 = x_2) \right) = \ln \left( \frac{W(x_1|0)W(x_2|0) + W(x_1|1)W(x_2|1)}{W(x_1|0)W(x_2|0) + W(x_1|1)W(x_2|1)} \right)$$

$$= \ln \left( \exp(\text{LLR}(x_1)) + \exp(\text{LLR}(x_2)) \right) - \ln \left( \exp(\text{LLR}(x_1)) + \exp(\text{LLR}(x_2)) \right) - \ln \left( \exp(\sum_{z \in T} L(z)) \right).$$

Now replacing $x_1, x_2$ with $(y(z), z \in T)$, we obtain that

$$L_{/B}(T) = \ln \left( \exp \left( \sum_{z \in T} L(z) \right) + 1 \right) - \ln \left( \sum_{z \in T} \exp(L(z)) \right).$$

In fact, this model approximates RM codes fairly well because the marginal distribution of each coordinate of the codeword is indeed Bernoulli-1/2 if we pick the codeword uniformly, and intuitively, the correlation between two coordinates is small when the code length is large.

Now we are ready to present the decoding algorithm for general binary-input channels. In Algorithms 3–4 below, we still denote the decoding result of the $(r-1)$-th order RM code as $\hat{y}_{/B}$ (see line 7 of Algorithm 3), where $\hat{y}_{/B} = (\hat{y}_{/B}(T), T \in E/\mathbb{B})$ are indexed by the cosets $T \in E/\mathbb{B}$, and we use $[z + \mathbb{B}]$ to denote the coset containing $z$ (see line 3 of Algorithm 3).

Algorithm 3 is very similar to Algorithm 1. From line 8 to line 10, we compare $\hat{L}(z)$ with the original $L(z)$. If the relative difference between these two is below the threshold $\theta$ for every $z \in E$, then the values of $L(z), z \in E$ change very little in this iteration, and the algorithm reaches a “stable” state, so we can exit the for loop on line 2. In practice, we find that $\theta = 0.05$ works fairly well, and we still set the maximal number of iterations $N_{\text{max}} = m/2$, which is the same as in Algorithm 1. On line 13, the algorithm simply produces the decoding result according to the LLR at each coordinate.
A few explanations of Algorithm 4. On line 3, we set \( \text{cumuLLR}(z) = \sum_{z' \neq z} \alpha(z, z')L(z') \), where the coefficients \( \alpha(z, z') \) can only be 1 or -1. More precisely, \( \alpha(z, z') = 1 \) if the decoding result of the corresponding \( (r - 1) \)th order RM code at the coset \( \{z, z'\} \) is 0, and \( \alpha(z, z') = -1 \) if the decoding result at the coset \( \{z, z'\} \) is 1. The reason behind this assignment is simple: The decoding result at the coset \( \{z, z'\} \) is an estimate of \( c(z) \oplus c(z') \). If \( c(z) \oplus c(z') \) is more likely to be 0, then the sign of \( L(z) \) and \( L(z') \) should be the same. Here \( \text{cumuLLR}(z) \) serves as an estimate of \( L(z) \) based on all the other \( L(z'), z' \neq z \), so we assign the coefficient \( \alpha(z, z') \) to be 1. Otherwise, if \( c(z) \oplus c(z') \) is more likely to be 1, then the sign of \( L(z) \) and \( L(z') \) should be different, so we assign the coefficient \( \alpha(z, z') \) to be -1.

**Algorithm 3** The RPA_RM decoding function for general binary-input memoryless channels

**Input:** The LLR vector \( (L(z), z \in \{0, 1\}^m) \); the parameters of the Reed-Muller code \( m \) and \( r \); the maximal number of iterations \( N_{\text{max}} \); the exiting threshold \( \theta \)

**Output:** The decoded codeword \( \hat{c} \)

1: \( \mathbb{E} := \{0, 1\}^m \)
2: for \( i = 1, 2, \ldots, N_{\text{max}} \) do
3: \( L_{/B_i} \leftarrow (L_{/B_i}(T), T \in \mathbb{E}/B_i) \) for \( i = 1, 2, \ldots, 2^m - 1 \) \hspace{1cm} \( \triangleright \) Projection
4: \( \triangleright L_{/B_i}(T) \) is calculated from \( (L(z), z \in \mathbb{E}) \) according to (7)
5: \( \hat{y}_{/B_i} \leftarrow \text{RPA_RM}(L_{/B_i}, m - 1, r - 1, N_{\text{max}}, \theta) \) for \( i = 1, 2, \ldots, 2^m - 1 \) \hspace{1cm} \( \triangleright \) Recursive decoding
6: \( \triangleright \) If \( r = 2 \), then we use the Fast Hadamard Transform to decode the first-order RM code
7: \( \hat{L} \leftarrow \text{Aggregation}(L_{/B_1}, \hat{y}_{/B_1}, \hat{y}_{/B_2}, \ldots, \hat{y}_{/B_{n-1}}) \) \hspace{1cm} \( \triangleright \) Aggregation
8: if \( |\hat{L}(z) - L(z)| \leq \theta |L(z)| \) for all \( z \in \mathbb{E} \) then \hspace{1cm} \( \triangleright \) The algorithm reaches a stable point
9: \( \triangleright \) break
10: end if
11: \( \hat{c}(z) \leftarrow 1[\hat{L}(z) < 0] \) for each \( z \in \mathbb{E} \)
12: end for
13: return \( \hat{c} \)

**Algorithm 4** The Aggregation function for general binary-input memoryless channels

**Input:** \( L, \hat{y}_{/B_1}, \hat{y}_{/B_2}, \ldots, \hat{y}_{/B_{n-1}} \)

**Output:** \( \hat{L} \)

1: Initialize \( (\text{cumuLLR}(z), z \in \{0, 1\}^m) \) as an all-zero vector indexed by \( z \in \{0, 1\}^m \)
2: \( n \leftarrow 2^m \)
3: \( \text{cumuLLR}(z) \leftarrow \sum_{i=1}^{n-1} ((1 - 2\hat{y}_{/B_i}([z + B_i]))L(z \oplus z_i)) \) for each \( z \in \{0, 1\}^m \)
4: \( \triangleright \) \( z_i \) is the nonzero element in \( B_i \)
5: \( \triangleright \) \( \hat{y}_{/B_i} \) is the decoded codeword, so \( \hat{y}_{/B_i}([z + B_i]) \) is either 0 or 1
6: \( \hat{L}(z) \leftarrow \frac{\text{cumuLLR}(z)}{n-1} \) for each \( z \in \{0, 1\}^m \)
7: return \( \hat{L} \)

In Algorithms 3–4 we write the pseudo codes in a mathematical fashion for the ease of understanding. In Appendix D, we present another version of the RPA_RM function in a program language fashion. Following the same proof of Proposition 1, we have the following result:

**Proposition 3.** The complexity of Algorithm 3 is \( O(n^r \log n) \) in sequential implementation and \( O(n^2) \) in parallel implementation with \( O(n^r) \) processors.

In Section V-A, we further discuss options to reduce the computation time by using fewer subspaces in the projection step. Similarly to Proposition 2, we can also show that the decoding error probability of Algorithm 3 is independent of the transmitted codeword for binary-input memoryless symmetric (BMS) channels.
Definition 3 (BMS channel). We say that a memoryless channel \( W : \{0, 1\} \rightarrow \mathcal{W} \) is a BMS channel if there is a permutation \( \pi \) of the output alphabet \( \mathcal{W} \) such that \( \pi^{-1} = \pi \) and \( W(x|1) = W(\pi(x)|0) \) for all \( x \in \mathcal{W} \).

Proposition 4. Let \( W : \{0, 1\} \rightarrow \mathcal{W} \) be a BMS channel. Let \( c_1 \) and \( c_2 \) be two codewords of \( \mathcal{R}_M(m, r) \). Let \( Y_1 \) and \( Y_2 \) be the (random) channel outputs of transmitting \( c_1 \) and \( c_2 \) over \( n = 2^m \) independent copies of \( W \), respectively. Let \( L^{(1)} \) and \( L^{(2)} \) be the LLR vectors corresponding to \( Y_1 \) and \( Y_2 \), respectively. Then for any \( c_1, c_2 \in \mathcal{R}_M(m, r) \), we have
\[
\mathbb{P}(\text{RPA}_{\text{RM}}(L^{(1)}, m, r, N_{\text{max}}, \theta) \neq c_1) = \mathbb{P}(\text{RPA}_{\text{RM}}(L^{(2)}, m, r, N_{\text{max}}, \theta) \neq c_2).
\]

The proof is given in Appendix B. Similarly to Proposition 2, this proposition is also very useful for simulations because we can simply transmit the all-zero codeword over the BMS channel \( W \) to measure the decoding error probability.

In the last part of this section, we present the list decoding and code concatenation version of the \( \text{RPA}_{\text{RM}} \) function. The main idea is already explained in Section II-A. Here we only write down the pseudo code of the list decoding version. Note that the purpose of line 8 is to make sure that \( \hat{c}(u) \) is a codeword of RM code, which is not always true for the decoding result of the \( \text{RPA}_{\text{RM}} \) function. The list decoding+code concatenation version is rather similar, and we put the pseudo code in Appendix E.

Algorithm 5 The \( \text{RPA}_{\text{LIST}} \) decoding function for general binary-input memoryless channels

Input: The LLR vector \((L(z), z \in \{0, 1\}^m)\); the parameters of the Reed-Muller code \( m \) and \( r \); the maximal number of iterations \( N_{\text{max}} \); the exiting threshold \( \theta \); the list size \( 2^t \)

Output: The decoded codeword \( \hat{c} \)

1: \( \hat{L} \leftarrow L \)
2: \((z_1, z_2, \ldots, z_t) \leftarrow \) indices of the \( t \) smallest entries in \((|L(z)|, z \in \{0, 1\}^m)\)
3: \( L_{\text{max}} \leftarrow 2 \max(|L(z)|, z \in \{0, 1\}^m) \)
4: \text{for each } u \in \{L_{\text{max}}, -L_{\text{max}}\}^t \text{ do}
5: \quad (L(z_1), L(z_2), \ldots, L(z_t)) \leftarrow u
6: \quad \hat{c}(u) \leftarrow \text{RPA}_{\text{RM}}(L, m, r, N_{\text{max}}, \theta)
7: \quad \hat{c}(u) \leftarrow \text{Reedsdecoder}(\hat{c}(u)) \quad \triangleright \text{Reedsdecoder is the classical decoding algorithm in [1]}
8: \text{end for}
9: u* \leftarrow \arg\max_u \sum_{z \in \{0, 1\}^m} \left((-1)^{\hat{c}(u)(z)} \hat{L}(z)\right)
10: \triangleright \text{This follows from [5]. Maximization is over } u \in \{L_{\text{max}}, -L_{\text{max}}\}^t
11: \hat{c} \leftarrow \hat{c}(u^*)
12: \text{return } \hat{c}

V. SIMULATION RESULTS

We run our decoding algorithm for second and third order Reed-Muller codes with code length 256, 512 and 1024 over AWGN channels and BSCs, and we compare its performance with the recent algorithms for polar codes with the same length and dimension. We compare to two versions of polar codes: Polar codes with optimal CRC size and polar codes without CRC, and we use the Successive Cancellation List (SCL) decoder introduced by Tal and Vardy [15] as the decoder, where we set list size to be 32. In [15], it was shown that SCL decoder with list size 32 has essentially the same performance as the Maximum Likelihood (ML) decoder for polar codes, so we compared with the optimal decoder for polar codes.

The simulation results for AWGN channels are plotted in Figure 2 where the number of Monte Carlo trials is \( 10^5 \). We can see that our decoding algorithm for RM codes has about 0.75dB gain (sometimes

\(^5Y_1 \) and \( Y_2 \) are random vectors, and the randomness comes from the channel noise. As a result, \( L^{(1)} \) and \( L^{(2)} \) are also random vectors.
more) over CRC-aided polar codes with SCL decoder, which has the best known performance among various versions of polar codes. Moreover, for certain cases the list decoding version of RPA decoding algorithm has almost the same performance as the Maximal Likelihood (ML) decoder for RM codes. The performance improvement is thus in agreement with the advantages of RM codes over polar codes under ML decoding \cite{4}. We also make comparisons with Dumer’s recursive decoding algorithm \cite{9}\textendash\cite{11}. To the best of our knowledge, this is the decoder for RM codes affording best numerical performance over AWGN channels. Due to complexity constraints, Dumer’s decoding algorithm can only be applied to the cases in (a),(c),(f) of Fig. 2 and its performance is very close to that of polar codes with optimal CRC size. Note also that the algorithm in \cite{14} only applies to codes with very short code length (no larger than 128) due to complexity constraints.

For the BSC channel, the simulation results are plotted in Figure 3. The number of Monte Carlo trials is $10^5$. We also tested in this case all the previous decoding algorithms known for RM codes, including Reed’s algorithm \cite{1}, Dumer’s recursive decoding algorithm \cite{9}\textendash\cite{11} as well as the algorithm from Saptharishi-Shpilka-Volk \cite{12}. For all these algorithms, the decoding error probability exceeds 0.1 for the tested parameters, so we did not include them in Figure 3 as they would not fit. From Figure 3 we can clearly see that the new decoding algorithm for RM codes significantly outperforms the SCL decoder for polar codes: On average the decoding error probability of our decoder is typically between 100 or 1000 times smaller than that of SCL decoder for polar codes.

We also compare the running time of our decoder and the SCL decoder for polar codes. Note that we use here the most naive implementation of our algorithm, without boosting (i.e., using a subset of the subspaces $B_1, B_2, \ldots, B_{n-1}$ in the RPA algorithms) and parallelization. The results are listed in Table I. We can see that for second order RM codes, our decoder is always faster than the SCL decoder for polar codes with the same parameters. However, for third order RM codes, our decoder is slower than the SCL decoder. This is because when we increase the order of RM code by 1, the decoding complexity roughly grows by a factor of $n$ (see Proposition \cite{1}) while the decoding complexity of polar codes does not change too much if we only increase the dimension of the code and keep the code length the same. Reducing the number of subspaces would potentially help here. More importantly, the algorithm should be run in parallel fashion as discussed next.

| RM(8, 2)  | P(8, 2)  | RM(9, 2)  | P(9, 2)  | RM(10, 2) | P(10, 2)  |
|-----------|----------|-----------|----------|-----------|-----------|
| 4.3ms     | 102.7ms  | 18.2ms    | 170.2ms  | 76.7ms    | 232.3ms   |

| RM(8, 3)  | P(8, 3)  | RM(9, 3)  | P(9, 3)  |
|-----------|----------|-----------|----------|
| 0.41s     | 0.12s    | 3.37s     | 0.31s    |

TABLE I: Comparison of decoding time between RM codes and polar codes. $P(m, r)$ denotes polar codes with the same length and dimension as $RM(m, r)$. We use the same decoders as in Figure 3.

A. **Parallelization and acceleration**

Another important advantage of the new decoding algorithm for RM codes over the SCL decoder for polar codes is that our algorithm naturally allows parallel implementation while the SCL decoder is not parallelizable. The key step in our algorithm for decoding a codeword of RM($r$, $m$) is to decode the quotient space codes which are in RM($r-1$, $m-1$) codes, and each of these can be decoded in parallel. Such a parallel structure is crucial to achieving high throughput and low latency.

Another way to accelerate the algorithm is to use only certain “voting sets”: In the projection step, we can take a subset of one-dimensional subspaces instead of all the one-dimensional subspaces. Then we still use recursive decoding followed by the aggregation step. In this way, we decode fewer RM($r-1$, $m-1$)

---

\footnote{We use the method in \cite{15} to measure ML lower bound: Whenever our decoder outputs a wrong codeword, we compare the posterior probability of the decoded word and that of the correct codeword. Most of the time the posterior probability of the decoded word is larger, which means that even an ML decoder will make a mistake in this case.}
B. Optimal scaling and sharp threshold of Reed-Muller codes over BSC channels

Recently, Hassani et al. gave theoretical results backing the conjecture that RM codes have an almost optimal scaling-law over BSC channels under ML decoding [17], where optimal scaling-law means that for a fixed linear code, the decoding error probability of ML decoder transitions from 0 to 1 as a function of the crossover probability of the BSC channel in the sharpest manner (i.e., comparable to random codes). In particular, this implies that RM codes have sharper transition than polar codes under ML decoding (if capacity achieving). In this section we give simulation results that show that for BSC channels, Reed-Muller codes under the RPA decoder also have sharper transition than polar codes under SCL+list decoder.

In Figure 4, we plot the decoding error probability of RM codes and polar codes over BSC channels as a function of the channel crossover probability, where for RM codes we use the RPA decoder in Algorithm [3] and for polar codes we use SCL decoder with list size 32. We can see that in all 4 cases, the transition in the curve of RM codes is sharper than the transition in the curve of polar codes. To further quantify the transition width, we introduce the following common notation: Let us denote the
channel crossover probability as $\epsilon$. For a given code and a corresponding decoding algorithm, we write its decoding error probability over BSC($\epsilon$) as $P_e(\epsilon)$. For $0 < \delta < 1/2$, we define the transition width\footnote{Typically $P_e(\epsilon)$ is an increasing function of $\epsilon$, so the inverse function exists.}

$$w(\delta) := P_e^{-1}(1 - \delta) - P_e^{-1}(\delta).$$

Clearly, $w(\delta)$ is a decreasing function. For a fixed value of $\delta$, smaller $w(\delta)$ means sharper transition and better scaling of the code and the corresponding decoder.

In Figure 5, we compare $w(0.1)$ and $w(0.01)$ between RM codes and polar codes with the same parameters, where we use the same decoders as above. We can see that RM codes always have smaller transition width than polar codes. Moreover, within the same code family, the transition width $w(0.1)$ and $w(0.01)$ both decrease with the code length, meaning that the transition becomes sharper as the code length increases. This phenomena has already been proved for ML decoders in [19] and [17].

VI. EXTENSIONS

Here we mention a few possible extensions of the decoding algorithms.
Fig. 4: Decoding error probability over BSC channels as a function of the channel crossover probability

Fig. 5: Comparison of transition width $w(0.1)$ and $w(0.01)$ between different codes. $R(m, r)$ refers to Reed-Muller codes, and $P(m, r)$ refers to polar codes with the same length and dimension as $R(m, r)$. 
1. The “voting sets” idea to further accelerate the RPA decoding, as discussed in Section [V-A].

2. Our new algorithms make use of one-dimensional subspace reduction. In practice, we can change the $\mathbb{B}_1, \ldots, \mathbb{B}_{n-1}$ in the RPA decoding algorithms to any of the $s$-dimensional subspaces, with different combinations possible.

3. The RPA decoding algorithms can also be used to decode other codes that are supported on a vector space, or any code that has a well-defined notion of “code projection” that can be iteratively applied to produce eventually a trivial code (that can be decoded efficiently). In the case of RM codes, the quotient space projection has the specificity of producing again RM codes, and the trivial code is the Hadamard code that can be decoded using the FHT.

4. As discussed in Section [III-A], we can use spectral decompositions or other relaxations in the Aggregation step instead of the majority voting, and depending on the regimes, one may take multiple iteration of the power-iteration method.

REFERENCES

[1] I. Reed, “A class of multiple-error-correcting codes and the decoding scheme,” Transactions of the IRE Professional Group on Information Theory, vol. 4, no. 4, pp. 38–49, 1954.

[2] E. Arıkan, “Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels,” IEEE Transactions on Information Theory, vol. 55, no. 7, pp. 3051–3073, 2009.

[3] ———, “A performance comparison of polar codes and Reed-Muller codes,” IEEE Communications Letters, vol. 12, no. 6, 2008.

[4] M. Mondelli, S. H. Hassani, and R. L. Urbanke, “From polar to Reed-Muller codes: A technique to improve the finite-length performance,” IEEE Transactions on Communications, vol. 62, no. 9, pp. 3084–3091, 2014.

[5] S. Kudekar, S. Kumar, M. Mondelli, H. D. Pfister, E. Şaşoğlu, and R. Urbanke, “Reed–Muller codes achieve capacity on erasure channels,” IEEE Transactions on Information Theory, vol. 63, no. 7, pp. 4298–4316, 2017.

[6] E. Abbe, A. Shpilka, and A. Wigderson, “Reed–Muller codes for random erasures and errors,” IEEE Transactions on Information Theory, vol. 61, no. 10, pp. 5229–5252, 2015.

[7] E. Abbe and M. Ye, “Reed-Muller codes polarize,” 2019, arXiv:1901.11533.

[8] F. J. MacWilliams and N. J. A. Sloane, The theory of error-correcting codes, Elsevier, 1977.

[9] I. Dumer, “Recursive decoding and its performance for low-rate Reed-Muller codes,” IEEE Transactions on Information Theory, vol. 50, no. 5, pp. 811–823, 2004.

[10] ———, “Soft-decision decoding of Reed-Muller codes: A simplified algorithm,” IEEE transactions on information theory, vol. 52, no. 3, pp. 954–963, 2006.

[11] I. Dumer and K. Shabunov, “Soft-decision decoding of Reed-Muller codes: Recursive lists,” IEEE transactions on information theory, vol. 52, no. 3, pp. 1260–1266, 2006.

[12] R. Saptharishi, A. Shpilka, and B. L. Volk, “Efficiently decoding Reed–Muller codes from random errors,” IEEE Transactions on Information Theory, vol. 63, no. 4, pp. 1954–1960, 2017.

[13] O. Sberlo and A. Shpilka, “On the performance of Reed-Muller codes with respect to random errors and erasures,” arXiv:1811.12447, 2018.

[14] E. Santi, C. Häger, and H. D. Pfister, “Decoding Reed-Muller codes using minimum-weight parity checks,” 2018, arXiv:1804.10319.

[15] I. Tal and A. Vardy, “List decoding of polar codes,” IEEE Transactions on Information Theory, vol. 61, no. 5, pp. 2213–2226, 2015.

[16] “Final report of 3GPP TSG RAN WG1 #87 v1.0.0,” http://www.3gpp.org/ftp/tpg_ran/WG1_RL1/TSGR1_87/Report/.

[17] H. Hassani, S. Kudekar, O. Ordentlich, Y. Polyanskiy, and R. Urbanke, “Almost optimal scaling of Reed-Muller codes on BEC and BSC channels,” 2018, arXiv:1801.09481.

[18] E. Abbe, “Community detection and stochastic block models,” Foundations and Trends in Communications and Information Theory, vol. 14, no. 1-2, pp. 1–162, 2018. [Online]. Available: http://dx.doi.org/10.1561/0100000067

[19] J. P. Tillich and G. Zémor, “Discrete isoperimetric inequalities and the probability of a decoding error,” Combinatorics, Probability and Computing, vol. 9, no. 5, pp. 465–479, 2000.

APPENDIX A

PROOF OF LEMMA 11

Let $b_1, b_2, \ldots, b_m$ be a basis of $\mathbb{B}$ over $\mathbb{F}_2$ such that the first $s$ vectors $b_1, b_2, \ldots, b_s$ form a basis of $\mathbb{B}$. Let $e_1, e_2, \ldots, e_m$ be the standard basis of $\mathbb{B}$, i.e., all but the $i$-th coordinate of $e_i$ are 0. Then there is an $m \times m$ invertible matrix $M$ such that

$$(b_1, b_2, \ldots, b_m)^T = M(e_1, e_2, \ldots, e_m)^T.$$
Let \((z_1, z_2, \ldots, z_m)\) be the coordinates of a point in \(E\) under the standard basis \((e_1, e_2, \ldots, e_m)\), and let \((z'_1, z'_2, \ldots, z'_m)\) be the coordinates of the same point under the basis \((b_1, b_2, \ldots, b_m)\). Then
\[
(z'_1, z'_2, \ldots, z'_m) = (z_1, z_2, \ldots, z_m)M^{-1}.
\]

Notice that \(B = \{z : (z'_1, z'_2, \ldots, z'_m) \in F_2^m, z'_{s+1} = \cdots = z'_m = 0\}\). Therefore for every coset \(T \in E/B\), the last \(m - s\) coordinates under the basis \((b_1, b_2, \ldots, b_m)\) are the same for all the points in \(T\). As a result, we can use binary vectors of length \(m - s\) to label the cosets, i.e.,
\[
[a_1, a_2, \ldots, a_{m-s}] := \{z : (z'_1, z'_2, \ldots, z'_m) \in F_2^m, z'_{s+1} = a_1, z'_{s+2} = a_2, \ldots, z'_m = a_{m-s}\}.
\]

Next we associate every subset \(A \subseteq [m]\) with another row vector \(v'_m(A)\) of length \(2^m\), whose components are indexed by \(z = (z_1, z_2, \ldots, z_m) \in E\). The vector \(v'_m(A)\) is defined as follows:
\[
v'_m(A, z) = \prod_{i \in A} z'_i,
\]
where \(v'_m(A, z)\) is the component of \(v'_m(A)\) indexed by \(z\), i.e., \(v'_m(A, z)\) is the evaluation of the polynomial \(\prod_{i \in A} z'_i\) at \(z\), where \((z'_1, z'_2, \ldots, z'_m) = (z_1, z_2, \ldots, z_m)M^{-1}\). Since all the invertible linear transforms belong to the automorphism group of Reed-Muller codes [8], we have the following alternative characterization of RM codes
\[
\mathcal{RM}(m, r) := \left\{ \sum_{A \subseteq [m], |A| \leq r} u'(A)v'_m(A) : u'(A) \in \{0, 1\} \text{ for all } A \subseteq [m], |A| \leq r \right\}.
\]

It is easy to check that for every coset \(T = [z'_{s+1}, z'_{s+2}, \ldots, z'_m] \in E/B\), if \(s \subseteq A\) then \(\sum_{z \in T} v'_m(A, z) = \prod_{i \in (A \setminus [s])} z'_i\), and if \(s \not\subseteq A\) then \(\sum_{z \in T} v'_m(A, z) = 0\). Now let \(c\) be a codeword of \(\mathcal{RM}(m, r)\), then it can be written as \(c = \sum_{A \subseteq [m], |A| \leq r} u'(A)v'_m(A)\), and for every coset \(T = [z'_{s+1}, z'_{s+2}, \ldots, z'_m] \in E/B\), we have
\[
\sum_{z \in T} c(z) = \sum_{A \subseteq [m], |A| \leq r} u'(A) \prod_{i \in (A \setminus [s])} z'_i = \sum_{A \subseteq [m], |A| \leq r-s} u'(A) \prod_{i \in A} z'_i.
\]

Therefore every codeword in \(Q(m, r, B)\) corresponds to an \((m-s)\)-variational polynomial in \(F_2[Z'_{s+1}, Z'_{s+2}, \ldots, Z'_m]\) with degree at most \(r-s\), and this is exactly the definition of the \((r-s)\)-th order Reed-Muller code \(\mathcal{RM}(m-s, r-s)\).

**APPENDIX B**

**PROOF OF PROPOSITION**

We need the following technical lemma to prove Proposition [4]

**Lemma 2.** Let \(c_0 = (c_0(z), z \in E)\) be a codeword of \(\mathcal{RM}(m, r)\). Let \(L^{(1)} = (L^{(1)}(z), z \in E)\) and \(L^{(2)} = (L^{(2)}(z), z \in E)\) be two LLR vectors such that
\[
L^{(2)}(z) = (-1)^{c_0(z)}L^{(1)}(z) \quad \forall z \in E.
\]

Denote \(\hat{c}_1 = \text{RPA}_{\mathcal{RM}}(L^{(1)}, m, r, N_{\max}, \theta)\) and \(\hat{c}_2 = \text{RPA}_{\mathcal{RM}}(L^{(2)}, m, r, N_{\max}, \theta)\). Then \(\hat{c}_1 = \hat{c}_2 + c_0\).

**Proof.** We prove by induction on \(r\). For the base case \(r = 1\), we use the ML decoder as described at the beginning of this section. More precisely, according to [5], \(\hat{c}_2 = \text{RPA}_{\mathcal{RM}}(L^{(2)}, m, 1, N_{\max}, \theta)\) is the codeword in \(\mathcal{RM}(m, 1)\) that maximizes
\[
\sum_{z \in E} \left((-1)^{c(z)}L^{(2)}(z)\right),
\]
i.e.,
\[
\sum_{z \in E} \left((-1)^{c(z)}L^{(2)}(z)\right) \geq \sum_{z \in E} \left((-1)^{c(z)}L^{(2)}(z)\right) \quad \forall c \in \mathcal{RM}(m, 1).
\]

By (8), we have
\[
\sum_{z \in E} \left( (-1)^{\hat{c}_2(z) \oplus c_0(z)} L^{(1)}(z) \right) \geq \sum_{z \in E} \left( (-1)^{c(z) \oplus c_0(z)} L^{(1)}(z) \right) \quad \forall c \in \mathcal{M}(m, 1).
\]
Since \( c_0 \) is a codeword of \( \mathcal{M}(m, 1) \), we have: \( c_0 + \mathcal{M}(m, 1) = \mathcal{M}(m, 1) \). As a result,
\[
\sum_{z \in E} \left( (-1)^{\hat{c}_2(z) \oplus c_0(z)} L^{(1)}(z) \right) \geq \sum_{z \in E} \left( (-1)^{c(z)} L^{(1)}(z) \right) \quad \forall c \in \mathcal{M}(m, 1).
\]
Therefore, \( \hat{c}_2 \oplus c_0 \) is the codeword in \( \mathcal{M}(m, 1) \) that maximizes
\[
\sum_{z \in E} \left( (-1)^{c(z)} L^{(1)}(z) \right).
\]
Thus we conclude that \( \hat{c}_1 = \hat{c}_2 \oplus c_0 \). This establishes the base case.

For the inductive step, let us assume that the lemma holds for \( r - 1 \) and prove it for \( r \). Notice that in Algorithm 3, \( \hat{c}(z) \) is simply determined by the sign of \( L(z) \). It is easy to see that if in Algorithm 4 the updated LLR vectors \( \hat{L}^{(1)} \) and \( \hat{L}^{(2)} \) always satisfy (8), then \( \hat{c}_1 = \hat{c}_2 \oplus c_0 \). Therefore, we only need to prove (8) for the updated LLR vectors \( \hat{L}^{(1)} \) and \( \hat{L}^{(2)} \).

Assuming that \( L^{(1)} \) and \( L^{(2)} \) satisfy (8), our task is to show that \( \hat{L}^{(2)}(z) = (-1)^{c_0(z)} \hat{L}^{(1)}(z) \) for all \( z \in E \). From the analysis in Section IV, we know that
\[
\hat{L}^{(i)}(z) = \frac{1}{n - 1} \sum_{z' \neq z} \alpha_i(z, z') L^{(i)}(z') \quad \text{for } i = 1, 2.
\]
The coefficient \( \alpha_i(z, z') \) is 1 if the decoding result of the corresponding \( (r - 1) \)th order RM code at the coset \( \{z, z'\} \) is 0, and \( \alpha_i(z, z') \) is \(-1\) if the decoding result at the coset \( \{z, z'\} \) is 1 (see line 3 of Algorithm 4).

Next we will show that \( \alpha_2(z, z') = (-1)^{c_0(z) \oplus c_0(z')} \alpha_1(z, z') \). Note that \( \alpha_i(z, z') \) is determined by the decoding result \( y_B^{(i)} = \text{RFA}_\text{RM}(L_B^{(i)}, m - 1, r - 1, N, \theta) \), where \( B = \{0, z \oplus z'\} \). By (7), we have
\[
L_B^{(2)}(T) = \ln \left( \exp \left( \sum_{z \in T} L_B^{(2)}(z) \right) + 1 \right) - \ln \left( \sum_{z \in T} \exp(L_B^{(2)}(z)) \right)
= \ln \left( \exp \left( \sum_{z \in T} (-1)^{c_0(z)} L_B^{(1)}(z) + 1 \right) \right) - \ln \left( \sum_{z \in T} \exp((-1)^{c_0(z)} L_B^{(1)}(z)) \right)
= (-1)^{\Theta \in T} c_0(z) L_B^{(1)}(T).
\]

Let us write \( c_0(T) := \bigoplus_{z \in T} c_0(z) \). Then \( L_B^{(2)}(T) = (-1)^{c_0(T)} L_B^{(1)}(T) \) for all \( T \in E \setminus B \). Moreover, since \( c_0 \) is a codeword of \( \mathcal{M}(m, r) \) and \( B \) is a one-dimensional subspace of \( E \), by Lemma 1, we know that \( (c_0(T), T \in E \setminus B) \) is a codeword of \( \mathcal{M}(m - 1, r - 1) \). Therefore, the codeword \( (c_0(T), T \in E \setminus B) \) and the two LLR vectors \( (L_B^{(1)}(T), T \in E \setminus B) \) and \( (L_B^{(2)}(T), T \in E \setminus B) \) satisfy the conditions of this lemma. By the induction hypothesis, \( y_B^{(2)}(T) = y_B^{(1)}(T) \oplus c_0(T) \) for all \( T \in E \setminus B \). As a result, we have \( \alpha_2(z, z') = (-1)^{c_0(z) \oplus c_0(z')} \alpha_1(z, z') \). Taking this into (9), we conclude that for all \( z \in E \),
\[
\hat{L}^{(2)}(z) = \frac{1}{n - 1} \sum_{z' \neq z} \alpha_2(z, z') L^{(2)}(z')
= \frac{1}{n - 1} \sum_{z' \neq z} \left( (-1)^{c_0(z) \oplus c_0(z')} \alpha_1(z, z') L^{(1)}(z') \right).
\]
Since the random channel outputs when transmitting $L$
Recall that in Proposition 4, we use outputting $L$
This completes the proof of the inductive step and establishes the lemma.

Proof of Proposition 4 Since $W$ is a BMS channel, there is a permutation $\pi$ of the output alphabet $\mathcal{W}$ satisfying the two conditions in Definition 3. Since both $c_1$ and $c_2$ are codewords of $\mathcal{RM}(m, r)$, $c_0 := c_1 + c_2$ is also a codeword of $\mathcal{RM}(m, r)$. Clearly, both channel output vectors $Y_1$ and $Y_2$ belong to $\mathcal{W}_m$. Now we define a permutation $\pi^{c_0}$ on $\mathcal{W}_m$: For any $y = (y(z), z \in E) \in \mathcal{W}_m$,

$$\pi^{c_0}(y) := (\pi^{c_0}(z)(y(z)), z \in E).$$

Notice that $c_0(z)$ is either 0 or 1, and $\pi^0$ is the identity map. Since $\pi$ is a permutation on $\mathcal{W}$, $\pi^{c_0}$ is clearly a permutation on $\mathcal{W}_m$. For a given $y = (y(z), z \in E) \in \mathcal{W}_m$, we denote the LLR vector corresponding to $y$ as $L_y^{(1)} := (L_y^{(1)}(z), z \in E)$, i.e., $L_y^{(1)}(z) = LLR(y(z))$ for all $z \in E$, and we denote the LLR vector corresponding to $\pi^{c_0}(y)$ as $L_y^{(2)} := (L_y^{(2)}(z), z \in E)$, i.e., $L_y^{(2)}(z) = LLR(\pi^{c_0}(y(z)))$ for all $z \in E$. By the property of $\pi$ (see Definition 3), we have

$$L_y^{(2)}(z) = (-1)^{c_0(z)} L_y^{(1)}(z) \quad \forall z \in E.$$

Since $c_0 \in \mathcal{RM}(m, r)$, by Lemma 2 we know that

$$\text{RPA}_{_\text{RM}}(L_y^{(1)}, m, r, N_{max}, \theta) = \text{RPA}_{_\text{RM}}(L_y^{(2)}, m, r, N_{max}, \theta) + c_0.$$

As a result, $\text{RPA}_{_\text{RM}}(L_y^{(1)}, m, r, N_{max}, \theta) \neq c_1$ if and only if $\text{RPA}_{_\text{RM}}(L_y^{(2)}, m, r, N_{max}, \theta) \neq c_2$.

For a vector $y \in \mathcal{W}_m$ and a codeword $c \in \mathcal{RM}(m, r)$, we use $W^n(y|c)$ to denote the probability of outputting $y$ when the transmitted codeword is $c$. Again by the property of $\pi$, it is easy to see that

$$W^n(y|c_1) = W^n(\pi^{c_0}(y)|c_2) \quad \forall y \in \mathcal{W}_m.$$

Recall that in Proposition 4, we use $L^{(1)}$ and $L^{(2)}$ to denote the random LLR vectors corresponding to the random channel outputs when transmitting $c_1$ and $c_2$, respectively. Therefore,

$$\mathbb{P}(\text{RPA}_{_\text{RM}}(L^{(1)}, m, r, N_{max}, \theta) \neq c_1) = \sum_{y \in \mathcal{W}_m} W^n(y|c_1) \mathbb{1}[\text{RPA}_{_\text{RM}}(L_y^{(1)}, m, r, N_{max}, \theta) \neq c_1]$$

$$= \sum_{y \in \mathcal{W}_m} W^n(\pi^{c_0}(y)|c_2) \mathbb{1}[\text{RPA}_{_\text{RM}}(L_y^{(2)}, m, r, N_{max}, \theta) \neq c_2]$$

$$= \mathbb{P}(\text{RPA}_{_\text{RM}}(L^{(2)}, m, r, N_{max}, \theta) \neq c_2).$$

This completes the proof of Proposition 4. \qed
Algorithm 6 The RPA\_RM decoding function for BSC

**Input:** The corrupted codeword $y = (y(z), z \in \{0, 1\}^m)$; the parameters of the Reed-Muller code $m$ and $r$; the maximal number of iterations $N_{\text{max}}$

**Output:** The decoded codeword $\hat{c}$

1: for $i = 1, 2, \ldots, N_{\text{max}}$ do
2:   Initialize $(\text{changevote}(z), z \in \{0, 1\}^m)$ as an all-zero vector indexed by $z \in \{0, 1\}^m$
3:   for each non-zero $z_0 \in \{0, 1\}^m$ do
4:     Set $B = \{0, z_0\}$
5:     $\hat{y}/B \leftarrow \text{RPA\_RM}(y/B, m - 1, r - 1, N_{\text{max}})$
6:     $\triangleright$ If $r = 2$, then we use the Fast Hadamard Transform to decode the first-order RM code [8]
7:     for each $z \in \{0, 1\}^m$ do
8:       if $y/B([z + B]) \neq \hat{y}/B([z + B])$ then
9:         $\text{changevote}(z) \leftarrow \text{changevote}(z) + 1$ $\triangleright$ Here addition is between real numbers
10:     end if
11:   end for
12: end for
13: numofchange $\leftarrow 0$
14: $n \leftarrow 2^m$
15: for each $z \in \{0, 1\}^m$ do
16:   if $\text{changevote}(z) > \frac{n - 1}{2}$ then
17:     $y(z) \leftarrow y(z) \oplus 1$ $\triangleright$ Here addition is over $F_2$
18:     numofchange $\leftarrow$ numofchange $+ 1$ $\triangleright$ Here addition is between real numbers
19:   end if
20: end for
21: if numofchange = 0 then
22:   break $\triangleright$ Exit the first for loop of this function
23: end if
24: end for
25: $\hat{c} \leftarrow y$
26: return $\hat{c}$
Algorithm 7 The RPA_RM decoding function for general binary-input memoryless channels

Input: The LLR vector \((L(z), z \in \{0, 1\}^m)\); the parameters of the Reed-Muller code \(m\) and \(r\); the maximal number of iterations \(N_{\text{max}}\); the exiting threshold \(\theta\)

Output: The decoded codeword \(\hat{c} = (\hat{c}(z), z \in \{0, 1\}^m)\)

1: \(E := \{0, 1\}^m\)
2: for \(i = 1, 2, \ldots, N_{\text{max}}\) do
3:   Initialize \((\text{cumuLLR}(z), z \in E)\) as an all-zero vector indexed by \(z \in E\)
4:   for each non-zero \(z_0 \in E\) do
5:     Set \(B = \{0, z_0\}\)
6:     \(L_B \leftarrow (L_B(T), T \in E/B)\) \(\triangleright L_B(T)\) is calculated from \((L(z), z \in E)\) according to \((7)\)
7:     \(\hat{y}_B \leftarrow \text{RPA}_\text{RM}(L_B, m - 1, r - 1, N_{\text{max}}, \theta)\)
8:     \(\triangleright\) If \(r = 2\), then we use the Fast Hadamard Transform to decode the first-order RM code
9:     for each \(z \in E\) do
10:    if \(\hat{y}_B([z + B]) = 0\) then
11:      \(\text{cumuLLR}(z) \leftarrow \text{cumuLLR}(z) + L(z \oplus z_0)\)
12:    else \(\triangleright\) \(\hat{y}_B\) is the decoded codeword, so \(\hat{y}_B([z + B])\) is either 0 or 1
13:      \(\text{cumuLLR}(z) \leftarrow \text{cumuLLR}(z) - L(z \oplus z_0)\)
14:    end if
15:  end for
16: end for
17: numofchange \leftarrow 0
18: \(n = 2^m\)
19: for each \(z \in E\) do
20:    \(\text{cumuLLR}(z) \leftarrow \frac{\text{cumuLLR}(z)}{n-1}\)
21:    if \(|\text{cumuLLR}(z) - L(z)| > \theta|L(z)|\) then
22:      numofchange \leftarrow numofchange + 1 \(\triangleright\) Here addition is between real numbers
23:    end if
24:    \(L(z) \leftarrow \text{cumuLLR}(z)\)
25: end for
26: if numofchange = 0 then
27:    break \(\triangleright\) Exit the first for loop of this function
28: end if
29: end for
30: for each \(z \in E\) do
31:    if \(L(z) > 0\) then
32:      \(\hat{c}(z) \leftarrow 0\)
33:    else
34:      \(\hat{c}(z) \leftarrow 1\)
35:    end if
36: end for
37: return \(\hat{c}\)
APPENDIX E

LIST DECODING+CODE CONCATENATION VERSION OF THE RPA_RM FUNCTION

In the implementation of the following algorithm, we only need to generate the (random) parity check matrix of the outer code \( C_{\text{out}} \). On line 11, we only need to check whether the product of the parity check matrix and the information bits of \( \hat{c}(u) \) is 0 or not.

Algorithm 8 The \texttt{RPA\_LIST\_CONCATENATION} decoding function for general binary-input memoryless channels

\textbf{Input:} The LLR vector \((L(z), z \in \{0, 1\}^m)\); the parameters of the Reed-Muller code \( m \) and \( r \); the maximal number of iterations \( N_{\text{max}} \); the exiting threshold \( \theta \); the list size \( 2^t \); the outer code \( C_{\text{out}} \)

\textbf{Output:} The decoded codeword \( \hat{c} \)

1: \( \tilde{L} \leftarrow L \)
2: \((z_1, z_2, \ldots, z_t) \leftarrow \) indices of the \( t \) smallest entries in \(|L(z)|, z \in \{0, 1\}^m\) \( \triangleright z_i \in \{0, 1\}^m \) for all \( i = 1, 2, \ldots, t \)
3: \\( \text{sumL} \leftarrow - \sum_{z \in \{0, 1\}^m} |L(z)| \)
4: \( \text{flag} \leftarrow 0 \)
5: \( L_{\text{max}} \leftarrow 2 \max(|L(z)|, z \in \{0, 1\}^m) \)
6: \text{for each} \( u \in \{L_{\text{max}}, -L_{\text{max}}\}^t \) \text{do}
7: \( (L(z_1), L(z_2), \ldots, L(z_t)) \leftarrow u \)
8: \( \hat{c}(u) \leftarrow \text{RPA\_RM}(L, m, r, N_{\text{max}}, \theta) \)
9: \( \text{if} \ I(\hat{c}(u)) \in C_{\text{out}} \text{ then} \)
10: \( \triangleright I(\hat{c}(u)) \) denotes the information bits of the codeword \( \hat{c}(u) \)
11: \( \text{flag} \leftarrow 1 \)
12: \( \text{end if} \)
13: \( \text{if} \ \text{sumL} < \sum_{z \in \{0, 1\}^m} (-1)^{I(\hat{c}(u))} \tilde{L}(z) \) \text{ then}
14: \( \text{sumL} \leftarrow \sum_{z \in \{0, 1\}^m} (-1)^{I(\hat{c}(u))} \tilde{L}(z) \)
15: \( u^* \leftarrow u \)
16: \( \text{end if} \)
17: \( \text{end if} \)
18: \( \text{end for} \)
19: \( \text{if} \ \text{flag}=0 \text{ then} \)
20: \( \triangleright \) In this case, no codeword in the list belong to the concatenated code, so we output the all 0 codeword (or a random codeword)
21: \( \text{return} \ 0 \)
22: \( \text{else} \)
23: \( \hat{c} \leftarrow \hat{c}(u^*) \)
24: \( \text{return} \ \hat{c} \)
25: \( \text{end if} \)