On the generalized oblate spheroidal wave functions and applications

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Abstract In this paper, we introduce a new set of functions, which have the property of the completeness over a finite and infinite intervals. This family of functions, denoted for simplicity GOSWFs, are a generalization of the oblate spheroidal wave functions. They generalize also the Jacobi polynomials in some sens. The GOSWFs are nothing but the eigenfunctions of the finite weighted bilateral Laplace transform \( F_c^{(\alpha,\beta)} \). We compute this functions by two methods: In the first one we use a differential operator \( D \) which commutes with \( F_c^{(\alpha,\beta)} \). In the second one we use the Gaussian quadrature method. As an application, we use the GOSWFs to invert the finite bilateral Laplace transform. We also use the GOSWFs to approximate bilateral weighted Laplace bandlimited functions and we show that they are more advantageous then other classical basis of \( L^2((-1,1),(1-x)\alpha(1+x)\beta)dx \). Finally, we provide the reader by some numerical examples that illustrate the theoretical results.

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1 Introduction

Prolate spheroidal wave functions (PSWFs), were first known as the eigenfunctions of the following differential operator, see \[15\],

\[ L_c(\varphi)(x) = (1 - x^2) \frac{d^2 \varphi}{dx^2} - 2x \frac{d \varphi}{dx} - c^2 x^2 \varphi. \] (1)

80 years later, D. Slepian, H. Landau and H. Pollack, see \[22\] \[8\] \[9\] have shown that the previous differential operator commutes with the following integral operator

\[ F_c(f)(x) = \int_{-1}^{1} \frac{\sin c(x-y)}{\pi(x-y)} f(y)dy. \] (2)

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Also, they have given many properties of the PSWFs of high practical importance, notably the
double orthogonality over finite and infinite interval. Three years later, D. Slepian showed in [21]
that higher dimensional construction of the PSWFs is reduced to the 2D—case. Hence, the study
of the 2D—case is potentially important, see [20]. He showed the circular prolate spheroidal wave
function (CPSWFs, for short), is an eigenfunction of the finite Hankel transform. That is
\[\int_{0}^{1} \sqrt{cxy} J_{\nu}(cxy) \psi_{n,c}^{\nu}(y) dy = \gamma_{n,\nu}(c) \psi_{n,c}^{\nu}(x), \quad \nu \in \mathbb{N}, \quad x \in (0, +\infty). \tag{3}\]

For more details about these functions and their computational methods, the reader is referred to [21].
As it is done for the case of the PSWFs, Slepian has proved that the following differential
operator
\[\mathcal{L}_{c}(\varphi)(x) = (1 - x^2) \frac{d^2 \varphi}{dx^2} - 2x \frac{d \varphi}{dx} + (-c^2 x^2 + \frac{\nu^2 - 1/4}{x^2}) \varphi \tag{4}\]
commutes with the finite Hankel transform. Note here, that the differential operators \(\mathcal{L}_{c}\) is a
perturbation of the differential operator
\[\mathcal{L}_{0}(\varphi)(x) = (1 - x^2) \frac{d^2 \varphi}{dx^2} - 2x \frac{d \varphi}{dx} + (-c^2 x^2 + \frac{\nu^2 - 1/4}{x^2}) \varphi \tag{5}\]
whose eigenfunctions are nothing but
\[T_{k,\nu}(x) = \sqrt{2(2k + \nu + 1)} x^{\nu+\frac{1}{2}} P_{k}^{(\nu,0)}(1 - 2x^2), \quad k \geq 0 \tag{6}\]
where for \(\alpha, \beta > -1, P_{n}^{(\alpha,\beta)}(x)\) is the Jacobi polynomial of degree \(n\). In [13], we have extended
the CPSWFs to an arbitrary order \(\nu > -1\) instead of \(\nu \in \mathbb{N}\) and we showed that CPSWFs share the
same property as the PSWFs notably the double orthogonality over finite and infinite interval.

Recently, in [23], the authors have given a new extension of the PSWFs that generalizes the
Gegenbauer polynomials to an orthogonal system with an intrinsic tuning parameter \(c > 0\). These
generalized PSWFs denoted by GPSWFs are defined as the eigenfunctions of a Sturm-Liouville
problem \(\mathcal{D}_{x}\), that commutes with an integral operator \(\mathcal{F}_{c}^{(\alpha)}\). These operators are defined as follows,
\[\mathcal{D}_{x}u = -(1 - x^2)^{-\alpha} \frac{d}{dx} \left( (1 - x^2)^{\alpha+1} \frac{du}{dx} \right) + (c^2 x^2)u, \quad \alpha > -1, \quad c > 0, \quad x \in (-1, 1), \]
\[\mathcal{F}_{c}^{(\alpha)}(\varphi)(x) = \int_{-1}^{1} e^{ixt} \varphi(t)(1 - t^2)^{\alpha} dt. \]
The authors have shown that the GPSWFs share similar properties with the PSWFs. Also, they
have presented a number of analytic and asymptotic formulae for the GPSWFs as well as their
associated eigenvalues. They introduced efficient algorithms for their evaluations. In a recent paper
see [10], we have studied the solution of the following integral equation
\[\mathcal{G}_{\infty}^{c} \Phi_{nc}(X + c) = \int_{0}^{+\infty} Ai(X + y + 2c) \Phi_{nc}(y + c) dy = \mu_{n} \Phi_{nc}(X + c). \tag{7}\]
Here \(Ai(\cdot)\) denotes the Airy’s function of the first kind. We have showed that \(\varphi_{nc}(y + c)\) and \(\nu_{n}\)
given by
\[\int_{0}^{M} Ai(X + y + 2c) \varphi_{nc}(y + c) dy = \nu_{n} \Phi_{nc}(X + c), \tag{8}\]
well approximate respectively \(\Phi_{nc}(y + c)\) and \(\mu_{n}\). We have showed that \(\varphi_{nc}(y + c)\) share the same
properties as the PSWFs. Since in [6], the authors have showed that the following differential operator
\[L f(x) = \frac{d}{dx} \left( x \frac{df}{dx} \right) - x(x + c)f, \tag{9}\]
commutes with \( G^c_\infty \). Then we can deduce that for \( c = 0 \) the eigenfunctions of \( G^c_\infty \) coincides the Laguerre polynomials. Remark here that in the special case where \( c = 0 \) the PSWFs are nothing else the Legendre polynomials, the kth CPSWFs coincides with \( T_{k,\nu}(x) \) given by [6] and the kth GPSWFs of Wang is the kth Gegenbauer polynomial.

A natural question that can be asked is the following: If there is a kind of Slepian’s functions that generalize the Jacobi polynomials. In this paper, we answer positively to the previous question and we show that the bounded solutions of the following equation

\[
(1-x^2)\frac{\partial^2 \psi^{(\alpha,\beta)}_n(x;c)}{\partial x^2} + (\beta - \alpha - (\alpha + \beta + 2)x)\frac{\partial \psi^{(\alpha,\beta)}_n(x;c)}{\partial x} + (c^2 x^2 + cx(\beta - \alpha)) \psi^{(\alpha,\beta)}_n(x;c) = -\chi^{(\alpha,\beta)}_n(c) \psi^{(\alpha,\beta)}_n(x;c),
\]

generalizes in some sense some of the previous mentioned Slepian’s functions. In fact, the GOSWFs coincides with

- the PSWFs if \( \alpha = \beta = 0 \) and \( c = i\tilde{c} \) and therefore with Legendre polynomials if \( \alpha = \beta = c = 0 \). Here \( i^2 = -1 \) and \( \tilde{c} > 0 \).
- Wang’s GPSWFs if \( \alpha = \beta \) and \( c = i\tilde{c} \) and therefore with Gegenbauer polynomials if \( \alpha = \beta \) and \( c = 0 \).
- Jacobi polynomials if \( c = 0 \).
- Chebyshev polynomials of first kind if \( \alpha = \beta = -\frac{1}{2} \) and \( c = 0 \).
- Chebyshev polynomials of second kind if \( \alpha = \beta = \frac{1}{2} \) and \( c = 0 \).

As an application of the GOSWFs, we invert the finite bilateral Laplace transform and we show numerically that the GOSWFs are more efficient then the PSWFs as well as Wang’s GPSWFs. We also use the GOSWFs to approximate bilateral weighted Laplace bandlimited functions and we show that they are more advantageous then other classical basis of \( L^2((-1, 1), (1-x)^\alpha(1+x)^\beta dx) \) such as Jacobi polynomials.

We give also several numerical results that illustrate the theoretical ones for different values of the parameters \( \alpha, \beta \) and \( c \).

The outline of the paper is as follows: In section 2 we recall some mathematical preliminaries about Jacobi polynomials and Whittaker functions which will be used frequently later. Section 3 is devoted to define the GOSWFs and present some of their properties. The goal of section 4 is the computation of these functions and their corresponding eigenvalues by two different methods. Finally, in section 5 we give some numerical results. We use the GOSWFs to invert the weighted bilateral Laplace transform and to approximate bilateral weighted Laplace bandlimited functions.

## 2 Mathematical preliminaries

### 2.1 Jacobi polynomials

It is well known that for any two real numbers \( \alpha, \beta > -1 \), the Jacobi polynomial \( P^{(\alpha,\beta)}_n(x) \) of degree \( n \) is given by the following Rodriguez formula, see [19]

\[
P^{(\alpha,\beta)}_n(x) = \frac{(-1)^n}{2^nn!} (1-x)^{-\alpha}(1+x)^{-\beta} \frac{d^n}{dx^n} \left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right), \quad n \geq 0.
\]
Let \( k_{n}^{\alpha,\beta} \) be the leading coefficient of \( P_{n}^{\alpha,\beta}(x) \). Then \( k_{n}^{\alpha,\beta} = \frac{\Gamma(2n + \alpha + \beta + 1)}{2^n \cdot n! \Gamma(n + \alpha + \beta + 1)} \). Moreover, if

\[
a_{n}^{\alpha,\beta} = \frac{2^{(\alpha+\beta+1)} \Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{n!(\alpha + \beta + 2n + 1) \Gamma(\alpha + \beta + n + 1)}, \quad n \geq 0,
\]

then the set \( \{P_{n}^{(\alpha,\beta)}(x), \, n \in \mathbb{N}\} \) is an orthonormal basis of \( (L^{2}(-1, 1), \, d\omega_{\alpha,\beta}) \), where \( d\omega_{\alpha,\beta}(x) = (1 - x)^{\alpha}(1 + x)^{\beta} \, dx \) and \( P_{n}^{(\alpha,\beta)}(x) = \frac{1}{\sqrt{a_{n}^{\alpha,\beta}}} P_{n}^{(\alpha,\beta)}(x) \). Note here that the normalized Jacobi polynomial \( P_{k}^{(\alpha,\beta)}(x) \) satisfy the following differential equation

\[
D_{x}^{2} P_{k}^{(\alpha,\beta)}(x) = (1 - x^{2}) \frac{d^{2}}{dx^{2}} P_{k}^{(\alpha,\beta)}(x) + (\beta - \alpha - (\alpha + \beta + 2)x) \frac{d}{dx} P_{k}^{(\alpha,\beta)}(x)
\]

\[
= -\chi_{k}^{(\alpha,\beta)}(0) P_{k}^{(\alpha,\beta)}(x),
\]

where \( \chi_{k}^{(\alpha,\beta)}(0) = k(k + \alpha + \beta + 1) \). To proceed further, we recall the recurrence relation satisfied by \( P_{k}^{(\alpha,\beta)}(x) \),

\[
P_{n+1}^{(\alpha,\beta)}(x) = (A_{n}x - B_{n})P_{n}^{(\alpha,\beta)}(x) - C_{n}P_{n-1}^{(\alpha,\beta)}(x),
\]

where

\[
A_{n} = \sqrt{\frac{a_{n}^{\alpha,\beta}}{a_{n+1}^{\alpha,\beta}}} \frac{(2n + \alpha + \beta + 1)(2n + \alpha + \beta + 2)}{2(n + 1)(n + \alpha + \beta + 1)},
\]

\[
B_{n} = \sqrt{\frac{a_{n}^{\alpha,\beta}}{a_{n+1}^{\alpha,\beta}}} \frac{(\beta^{2} - \alpha^{2})(2n + \alpha + \beta + 1)}{2(n + 1)(n + \alpha + \beta + 1)(2n + \alpha + \beta)},
\]

\[
C_{n} = \sqrt{\frac{a_{n-1}^{\alpha,\beta}}{a_{n+1}^{\alpha,\beta}}} \frac{(\alpha + \beta)(\beta + n)(2n + \alpha + \beta + 2)}{(n + 1)(n + \alpha + \beta + 1)(2n + \alpha + \beta)}.
\]

Note here that (12) can be written as follows

\[
x P_{k}^{(\alpha,\beta)} = \alpha_{k} P_{k+1}^{(\alpha,\beta)} + \beta_{k} P_{k}^{(\alpha,\beta)} + \gamma_{k} P_{k-1}^{(\alpha,\beta)}.
\]

where

\[
\alpha_{k} = \frac{1}{A_{k}}, \quad \beta_{k} = \frac{B_{k}}{A_{k}}, \quad \gamma_{k} = \frac{C_{k}}{A_{k}}.
\]

By multiplying both sides of (15) by \( x \) and using (15), one gets

\[
x^{2} P_{k}^{(\alpha,\beta)} = \alpha_{k} \alpha_{k+1} B_{k+2}^{(\alpha,\beta)} + \alpha_{k} (\beta_{k+1} + \beta_{k}) B_{k+1}^{(\alpha,\beta)} + (\alpha_{k} \gamma_{k+1} + (\beta_{k})^{2} + \gamma_{k} \alpha_{k-1}) P_{k}^{(\alpha,\beta)}
\]

\[
+ \gamma_{k} (\beta_{k} + \beta_{k-1}) P_{k-1}^{(\alpha,\beta)} + \gamma_{k} \gamma_{k-1} P_{k-2}^{(\alpha,\beta)}, \quad \forall k \geq 2.
\]

2.2 Whittaker Functions

The Whittaker functions see (16) arise as solutions to the Whittaker differential equation.

\[
\frac{d^{2}u}{dz^{2}} + \left( -\frac{1}{4} + \frac{\lambda}{z} + \frac{1}{z^{2}} \right) u = 0.
\]
Two solutions are given by the Whittaker functions \( M_{\lambda,\mu}(z), W_{\lambda,\mu}(z) \), defined in terms of Kummer's confluent hypergeometric functions \( M \) and \( U \) by

\[
M_{\lambda,\mu}(z) = e^{-\frac{1}{2}z}z^{\frac{1}{2}+\mu}M\left(\frac{1}{2} + \mu - \lambda, 1 + 2\mu, z\right),
\]

\[
W_{\lambda,\mu}(z) = e^{-\frac{1}{2}z}z^{\frac{1}{2}+\mu}U\left(\frac{1}{2} + \mu - \lambda, 1 + 2\mu, z\right).
\]

The Whittaker functions \( M_{\lambda,\mu}(z) \) satisfy the following integral representations

\[
M_{\lambda,\mu}(z) = \frac{\Gamma(1 + 2\mu)}{\Gamma\left(\frac{1}{2} + \mu - \lambda\right)\Gamma\left(\frac{1}{2} + \mu + \lambda\right)} \int_{-1}^{1} e^{\frac{1}{2}zt}(1 + t)^{\mu - \frac{1}{2} - \lambda}(1 - t)^{\mu - \frac{1}{2} + \lambda} dt, \quad Re\mu + \frac{1}{2} > |Re\lambda|. \quad (18)
\]

3 Definitions and properties of the GOSWFs

3.1 Definitions of the GOSWFs

To define the GOSWFs, we need first to introduce the following operators. The first one is the differential operator given by

\[
D_x = (\omega_{\alpha,\beta}(x))^{-1}\partial_x (\omega_{\alpha+1,\beta+1}(y)\partial_x) + c^2 x^2 + cx(\beta - \alpha), \quad (19)
\]

where \( \omega_{\alpha,\beta}(x) = (1 - x)^\alpha (1 + x)^\beta \) and \( \alpha > -1, \beta > -1 \). The second one is given by

\[
\mathcal{F}^{(\alpha,\beta)}_x[\phi](x) = \int_{-1}^{1} e^{cx(1+y)}\phi(y)\omega_{\alpha,\beta}(y)dy, \quad x \in (-1, 1), c > 0. \quad (20)
\]

The following lemma borrowed from [7] will be used in sequel.

**Lemma 1.** The differential operator \( D_x \) commutes with the integral operator \( \mathcal{F}^{(\alpha,\beta)}_x \). That is, for any twice differentiable function \( f \) on \((-1, 1)\) in \( L^2((-1, 1), d\omega_{\alpha,\beta}) \), we have

\[
\mathcal{F}^{(\alpha,\beta)}_x \circ D_x f = D_x \circ \mathcal{F}^{(\alpha,\beta)}_x f. \quad (21)
\]

In the sequel we denote by \( \psi^{(\alpha,\beta)}_n(x; c) \) the \( n \)th GOSWFs, \( \chi^{(\alpha,\beta)}_n(c) \) denotes the \( n \)th eigenvalue of \( D_x \) corresponding to \( \psi^{(\alpha,\beta)}_n(x; c) \). Let also denote by \( \mu^{(\alpha,\beta)}_n(c) \) the \( n \)th eigenvalue of \( \mathcal{F}^{(\alpha,\beta)}_x \) associated to \( \psi^{(\alpha,\beta)}_n(x; c) \).

**Definition 1.** The GOSWFs are defined as:

1. the bounded eigenfunctions of the differential operator \( D_x \)
2. the eigenfunctions of \( \mathcal{F}^{(\alpha,\beta)}_x \). That is

\[
\mathcal{F}^{(\alpha,\beta)}_x[\psi^{(\alpha,\beta)}_n](x) = \int_{-1}^{1} e^{cx(1+y)}\psi^{(\alpha,\beta)}_n(y)\omega_{\alpha,\beta}(y)dy = \mu^{(\alpha,\beta)}_n(c)\psi^{(\alpha,\beta)}_n(x), \quad x \in (-1, 1), c > 0. \quad (22)
\]

3. the eigenfunctions of \( Q^{(\alpha,\beta)}_c = (\mathcal{F}^{(\alpha,\beta)}_x)^* \circ \mathcal{F}^{(\alpha,\beta)}_x \), That is

\[
Q^{(\alpha,\beta)}_c[\psi^{(\alpha,\beta)}_n](x) = \int_{-1}^{1} K_{\alpha,\beta}(x, y)\psi^{(\alpha,\beta)}_n(y)\omega_{\alpha,\beta}(y)dy = \lambda^{(\alpha,\beta)}_n(c)\psi^{(\alpha,\beta)}_n(x), \quad x \in (-1, 1), c > 0.
\]

Here \( K_{\alpha,\beta}(x, y) = e^{-2c} \int_{-1}^{1} e^{ct(1+y)}\omega_{\alpha,\beta}(t)dt = e^{-2c\alpha+\beta+1}B(\alpha+1, \beta+1) \frac{M_{\alpha-\beta, \alpha+\beta+1}(2c(x+y))}{(2c(x+y))^{\alpha+\beta+2}} \quad (23)\]

and \( \lambda^{(\alpha,\beta)}_n(c) = (\mu^{(\alpha,\beta)}_n(c))^2. \)
3.2 Properties of the GOSWFs

We summarize below some basic properties of the GOSWFs

**Proposition 1.** For any $c > 0$ and $\alpha > -1$, $\beta > -1$, we have

(i) $\{\psi_n^{(\alpha,\beta)}(x; c)\}_{n=0}^{\infty}$ form a complete orthogonal system of $L^2((-1,1), d\omega_{\alpha,\beta})$, namely,

$$\int_{-1}^{1} \psi_n^{(\alpha,\beta)}(x; c)\psi_m^{(\alpha,\beta)}(x; c)\omega_{\alpha,\beta}(x)dx = (\mu_{n}^{(\alpha,\beta)}(c))^2 \delta_{mn}. \quad (24)$$

(ii) $\{\chi_n^{(\alpha,\beta)}(c)\}_{n=0}^{\infty}$ are all real, positive, simple and ordered as

$$0 < \chi_0^{(\alpha,\beta)}(c) < \chi_1^{(\alpha,\beta)}(c) < \cdots < \chi_n^{(\alpha,\beta)}(c) < \cdots \quad (25)$$

(iii) In the special case where $\alpha = \beta$, the $n$th GOSWF $\psi_n^{(\alpha,\alpha)}(x; c)$ has the same parity as its order $n$, namely,

$$\psi_n^{(\alpha,\alpha)}(-x; c) = (-1)^n \psi_n^{(\alpha,\alpha)}(x; c), \forall x \in (-1,1). \quad (26)$$

(iv) $\psi_n^{(\alpha,\beta)}(x; c)$ has exactly $n$ real distinct zeros in the interval $(-1,1)$ and between two consecutive zeros of $\psi_n^{(\alpha,\beta)}(x; c)$ there exists exactly one zero of $\psi_n^{(\alpha,\beta)}(x; c)$.

(v) $\{\psi_n^{(\alpha,\beta)}(x; c)\}_{n=0}^{\infty}$ form a complete orthogonal system of the bilateral weighted Laplace bandlimited functions given by,

$$LB_{\omega_{\alpha,\beta}}^{\psi_{\alpha,\beta}} = \left\{ f(x) = \int_{-1}^{1} e^{c(xy-1)}g(y)\omega_{\alpha,\beta}(y)dy, \ g \in L^2((-1,1), d\omega_{\alpha,\beta}) \right\}. \quad (27)$$

(vi) The derivative of $\mu_{n}^{(\alpha,\beta)}(c)$ with respect to $c$ is given by

$$\frac{\partial \mu_{n}^{(\alpha,\beta)}(c)}{\partial c} = \frac{1}{\mu_{n}^{(\alpha,\beta)}(c)} \left( \frac{I_n(c)}{c} - 1 \right) \quad (28)$$

where

$$I_n(c) = \frac{(\psi_n^{(\alpha,\beta)}(1; c))^2 - (\psi_n^{(\alpha,\beta)}(-1; c))^2 - 1}{2} \quad (29)$$

(vii) For all $x \in (-1,1)$ we have the following inequality: $|\psi_n^{(\alpha,\beta)}(x; c)| \leq \frac{1}{P_0^{(\alpha,\beta)}(x)}$.

**Proof:** The properties (i) – (iv) can be derived from the Sturm-Liouville theory, see [1,4]. We restrict ourselves here to prove (v) and (vi).

Let $f \in LB_{\omega_{\alpha,\beta}}^{\psi_{\alpha,\beta}}$, then there exists $g \in L^2((-1,1), d\omega_{\alpha,\beta})$ such that

$$f(x) = \int_{-1}^{1} e^{c(xy-1)}g(y)\omega_{\alpha,\beta}(y)dy. \quad (27)$$

Since $g \in L^2((-1,1), d\omega_{\alpha,\beta})$, then by (i) we have

$$g(y) = \sum_{k \in \mathbb{N}} c_k \psi_k^{(\alpha,\beta)}(y; c). \quad (28)$$

Combining (27) and (28) we obtain

$$f(x) = \sum_{k \in \mathbb{N}} c_k \mu_k^{(\alpha,\beta)}(c)\psi_k^{(\alpha,\beta)}(x; c). \quad (29)$$

This achieves the proof of (v).
Proof. For the proof of (vi), we adopt the techniques used in [21] to prove a similar result for the eigenvalue of the finite Hankel transform. We differentiate both member of the following equality
\[ \int_{-1}^{1} e^{c(v-1)} \psi_n^{(\alpha,\beta)}(u) \omega_{\alpha,\beta}(u) du = \mu_n^{(\alpha,\beta)}(c) \psi_n^{(\alpha,\beta)}(v), \quad v \in (-1,1). \] (30)
with respect to \( c \), to obtain
\[ \int_{-1}^{1} (uv - 1)e^{c(uv-1)} \psi_n^{(\alpha,\beta)}(u,c) \omega_{\alpha,\beta}(u) du + \int_{-1}^{1} e^{c(uv-1)} \frac{\partial \psi_n^{(\alpha,\beta)}(u,c)}{\partial c} \omega_{\alpha,\beta}(u) du = \frac{\partial \mu_n^{(\alpha,\beta)}(c)}{\partial c} \psi_n^{(\alpha,\beta)}(v,c) + \mu_n^{(\alpha,\beta)}(c) \frac{\partial \psi_n^{(\alpha,\beta)}(v,c)}{\partial c}. \] (31)
Differentiating (30) with respect to \( v \), one gets
\[ \int_{-1}^{1} u e^{c(uv-1)} \psi_n^{(\alpha,\beta)}(u,c) \omega_{\alpha,\beta}(u) du = \frac{\mu_n^{(\alpha,\beta)}(c)}{c} \frac{\partial \psi_n^{(\alpha,\beta)}(v,c)}{\partial v}. \] (32)
Combining (31) and (32) to obtain
\[ \int_{-1}^{1} e^{c(uv-1)} \psi_n^{(\alpha,\beta)}(u,c) \omega_{\alpha,\beta}(u) du = \frac{\mu_n^{(\alpha,\beta)}(c)}{c} \frac{\partial \psi_n^{(\alpha,\beta)}(v,c)}{\partial v} - \frac{1}{c} \int_{-1}^{1} e^{c(uv-1)} \psi_n^{(\alpha,\beta)}(v,c) \omega_{\alpha,\beta}(u) du + \frac{\partial \mu_n^{(\alpha,\beta)}(c)}{\partial c} \psi_n^{(\alpha,\beta)}(v,c) + \mu_n^{(\alpha,\beta)}(c) \frac{\partial \psi_n^{(\alpha,\beta)}(v,c)}{\partial c}. \] (33)
Multiply both sides of (33) by \( \psi_n^{(\alpha,\beta)}(v) \) and integrate over \((-1,1)\). One finds
\[ \frac{\mu_n^{(\alpha,\beta)}(c)}{c} \int_{-1}^{1} \psi_n^{(\alpha,\beta)}(u,c) \frac{\partial \psi_n^{(\alpha,\beta)}(v,c)}{\partial v} \omega_{\alpha,\beta}(u) du = \mu_n^{(\alpha,\beta)}(c) \| \psi_n^{(\alpha,\beta)}(u,c) \|_2^2 \]
\[ \int_{-1}^{1} \psi_n^{(\alpha,\beta)}(v,c) \int_{-1}^{1} e^{c(uv-1)} \frac{\partial \psi_n^{(\alpha,\beta)}(u,c)}{\partial c} \omega_{\alpha,\beta}(u) \omega_{\alpha,\beta}(v) du dv = \]
\[ \frac{\partial \mu_n^{(\alpha,\beta)}(c)}{\partial c} \| \psi_n^{(\alpha,\beta)}(c) \|_2^2 + \mu_n^{(\alpha,\beta)}(c) \int_{-1}^{1} \psi_n^{(\alpha,\beta)}(v,c) \frac{\partial \psi_n^{(\alpha,\beta)}(v,c)}{\partial c} \omega_{\alpha,\beta}(v) dv. \] (34)
Using Fubini’s Theorem and (30) together with the normalization of the GOSWFs, the equality (34) can be simply written as follows
\[ \frac{\partial \mu_n^{(\alpha,\beta)}(c)}{\partial c} = \frac{1}{\mu_n^{(\alpha,\beta)}(c)} \left( I_n(c) - \frac{c}{\mu_n^{(\alpha,\beta)}(c)} \right) \] (35)
where
\[ I_n(c) = \int_{-1}^{1} \psi_n^{(\alpha,\beta)}(v,c) \frac{\partial \psi_n^{(\alpha,\beta)}(v,c)}{\partial v} \omega_{\alpha,\beta}(v) dv. \]
Integrating \( I_n(c) \) by parts one can easily check that
\[ I_n(c) = \frac{(\psi_n^{(\alpha,\beta)}(1,c))^2 - (\psi_n^{(\alpha,\beta)}(-1,c))^2 - 1}{2}. \] (36)
Thanks to (33) and (36) and the normalization of \( \psi_n^{(\alpha,\beta)}(c) \) one obtains (26).
Finally, the proof of (vii) is based on the use of Schwartz inequality and the GOSWFs normalization.

Remark here that one can check numerically that the eigenvalues \( \mu_n^{(\alpha,\beta)}(c) \) decay exponentially to 0. This statement was proved in the special cases of the PSWFs and Wang’s GPSWFs. It will be the subject of a future work.
4 Numerical computation of the eigenfunctions and the eigenvalues of the finite bilateral Laplace transform

In this section, we provide the reader by two methods for the computation of the GOSWFs. In the first method we use the differential operator $\mathcal{D}_x$, given by (19) and a series expansion of the $\psi_n^{(\alpha,\beta)}(x; c)$ with respect to Jacobi polynomial basis.

By the first definition of the GOSWFs, we have

$$\mathcal{D}_x \psi_n^{(\alpha,\beta)}(x; c) = (1-x^2) \frac{\partial^2 \psi_n^{(\alpha,\beta)}(x; c)}{\partial x^2} + (\beta - \alpha - (\alpha + \beta + 2)x) \frac{\partial \psi_n^{(\alpha,\beta)}(x; c)}{\partial x} + (c^2 x^2 + c x (\beta - \alpha)) \psi_n^{(\alpha,\beta)}(x; c) = -\lambda_n^{(\alpha,\beta)}(c) \psi_n^{(\alpha,\beta)}(x; c).$$

(37)

To compute $\psi_n^{(\alpha,\beta)}(x; c)$ as well as $\lambda_n^{(\alpha,\beta)}(c)$, we use similar technique used in [3, 21, 10] for the computation of the prolate spheroidal wave functions, the circular prolate spheroidal wave functions and also the eigenfunctions of the Airy’s transform.

The Fourier series expansion of $\psi_n^{(\alpha,\beta)}(x; c)$ with respect to $\{P_k^{(\alpha,\beta)}(x), k \in \mathbb{N}\}$, is given by

$$\psi_n^{(\alpha,\beta)}(x; c) = \sum_{k=0}^{+\infty} a_k^n P_k^{(\alpha,\beta)}(x), \quad x \in (-1, 1)$$

(38)

where $a_k^n = \int_{-1}^{1} \psi_n^{(\alpha,\beta)}(x; c) P_k^{(\alpha,\beta)}(x) \omega_{\alpha,\beta}(x) dx$. The following proposition gives a recurrence relation satisfied by the sequence $(a_k^n)_k$.

**Proposition 2.** Under the above assumptions and notations, the sequence $(a_k^n)_k$ is given by the following recurrence relation

$$c^2 a_{k-2} a_k a_{k+2} + [c^2 a_{k-1} (\beta_k + \beta_{k-1}) + c (\beta - \alpha)] a_{k-1}$$

$$+ [\lambda_k^{(\alpha,\beta)}(0) + \chi_n^{(\alpha,\beta)}(c) + c^2 (\alpha_k \gamma_{k+1} + (\beta_k)^2 + \alpha_{k-1} \gamma_k)] a_k^n$$

$$+ [c (\beta - \alpha) \gamma_k + c^2 \gamma_{k+1} (\beta_k + \beta_{k-1})] a_{k+1} + c^2 \gamma_{k+1} \gamma_k + 2 a_{k+2} = 0.$$  

(39)

**Proof:** By substituting (38) in (37), we obtain

$$\sum_{k=0}^{+\infty} a_k^n ((1-x^2) \partial_x^2 P_k^{(\alpha,\beta)} + (\beta - \alpha - (\alpha + \beta + 2)x) \partial_x P_k^{(\alpha,\beta)} + (c^2 x^2 + c x (\beta - \alpha)) \chi_n^{(\alpha,\beta)}(c)) \sum_{k=0}^{+\infty} a_k^n P_k^{(\alpha,\beta)} = 0.$$  

(40)

By combining the previous equality and (11), one gets

$$-\sum_{k=0}^{+\infty} a_k^n \lambda_k^{(\alpha,\beta)}(0) P_k^{(\alpha,\beta)} + (c^2 x^2 + c x (\beta - \alpha)) \lambda_n^{(\alpha,\beta)}(c) \sum_{k=0}^{+\infty} a_k^n P_k^{(\alpha,\beta)} = 0.$$  

(41)

or equivalently

$$\sum_{k=0}^{+\infty} a_k^n (-\lambda_k^{(\alpha,\beta)}(0) + \chi_n^{(\alpha,\beta)}(c)) P_k^{(\alpha,\beta)} + \sum_{k=0}^{+\infty} a_k^n c^2 x^2 P_k^{(\alpha,\beta)} + \sum_{k=0}^{+\infty} a_k^n c (\beta - \alpha) x P_k^{(\alpha,\beta)} = 0.$$  

(42)

Thanks to (15) and (10), the equality (42) becomes

$$\sum_{k=0}^{+\infty} a_k^n (-\lambda_k^{(\alpha,\beta)}(0) + \chi_n^{(\alpha,\beta)}(c)) P_k^{(\alpha,\beta)} + \sum_{k=0}^{+\infty} a_k^n c (\beta - \alpha) [\alpha_k P_{k+1}^{(\alpha,\beta)} + \beta_k P_k^{(\alpha,\beta)} + \gamma_k P_{k-1}^{(\alpha,\beta)}]$$

$$+ \sum_{k=0}^{+\infty} a_k^n c^2 (\alpha_k \alpha_{k+1} P_{k+1}^{(\alpha,\beta)} + \alpha_k (\beta_k + \beta_{k+1}) P_{k+1}^{(\alpha,\beta)} + (\alpha_k \gamma_{k+1} + (\beta_k)^2 + \alpha_{k-1} \gamma_k) P_k^{(\alpha,\beta)}$$

$$+ \gamma_k (\beta_k + \beta_{k-1}) P_{k-1}^{(\alpha,\beta)} + (\gamma_k \gamma_{k-1}) P_{k-2}^{(\alpha,\beta)}] = 0.$$  

(43)
By using the Stirling formulae we obtain the desired inequality.

\[ a \]

Remark 1. Note here that the equality (39) can be written in the matrix form \( M \mathbf{a}^n = -\chi_n^{(\alpha,\beta)} \mathbf{a}^n \), where \( \mathbf{a}^n \) is the vector with components \( (a_k^n)_{k \in \mathbb{N}} \) and \( M \) is the matrix with entries

\[
\begin{align*}
    a_{k,k} &= -\chi_k^{(\alpha,\beta)}(0) + c^2(\alpha_k\gamma_{k+1} + (\beta_k)^2 + \alpha_{k-1}\gamma_k), & a_{k+1,k} &= [c(\beta - \alpha)\gamma_{k+1} + c^2\gamma_{k+1}(\beta_{k+1} + \beta_{k-1})], & a_{k+2,k} &= c^2\gamma_{k+1}\gamma_{k+2}, \\
    a_{k-2,k} &= a_{k+2,k}, & a_{k-1,k} &= a_{k+1,k}, & a_{j,k} &= 0 \text{ otherwise.}
\end{align*}
\]

The above method has the advantage to provide an explicit infinite series expansion of the \( \psi_n^{(\alpha,\beta)}(x; c) \). In practice, the infinite matrix \( M \) required to compute the series expansion coefficients \( (a_k^n) \) has to be truncated to a submatrix of a sufficiently large value order \( K \). The \( n \)-th eigenvector \( (\tilde{a}_k^n)_{k \in \mathbb{N}} \) of the truncated matrix \( \tilde{M} \) is then used as the approximate values of \( (a_k^n)_{k \in \mathbb{N}} \). Numerical evidences show that for moderate values of the integer \( n \), good approximations of the true and significant values of the \( (\tilde{a}_k^n) \) are obtained by using moderate and reasonable values of the truncation order \( K \).

To study the decay rate of the sequence \( (a_k^n) \), we need the following lemma.

**Lemma 2.** For any real \( c > 0 \) and any \( y \in (-1, 1) \) and any \( k \geq 2\lfloor ec \rfloor + 1 \) we have,

\[
\left| \int_{-1}^{1} e^{cxy} P_{k}^{(\alpha,\beta)}(x) \omega(\alpha,\beta)(x) dx \right| < 2^{-k+\frac{1}{2}}. \tag{44}
\]

**Proof:** Since \( \{P_{k}^{(\alpha,\beta)}, \ k \in \mathbb{N}\} \) is an orthonormal basis of \( L^2((-1, 1), d\omega(\alpha,\beta)) \), together with the following expansion

\[
e^{cxy} = \sum_{l=0}^{+\infty} \frac{(cxy)^l}{l!}, \tag{45}
\]

we obtain

\[
J_k(y) = \int_{-1}^{1} e^{cxy} P_{k}^{(\alpha,\beta)}(x) \omega(\alpha,\beta)(x) dx
= \sum_{l=0}^{+\infty} \frac{(cxy)^l}{l!} \int_{-1}^{1} x^l P_{k}^{(\alpha,\beta)}(x) \omega(\alpha,\beta)(x) dx,
\]

Now using the Shwartz inequality we obtain

\[
|J_k(y)| \leq \sum_{l=0}^{+\infty} \frac{(c|y|)^l}{l!} \int_{-1}^{1} |P_{k}^{(\alpha,\beta)}(x)|^2 \omega(\alpha,\beta)(x) dx
\leq \sum_{l=0}^{+\infty} \frac{(c)^l}{l!} \left( \int_{-1}^{1} \omega(\alpha,\beta)(x) dx \right)^{1/2} \left( \int_{-1}^{1} |P_{k}^{(\alpha,\beta)}(x)|^2 \omega(\alpha,\beta)(x) dx \right)^{1/2}
= \frac{e^k}{k!} \left( 1 + \frac{c}{k+1} + \frac{c^2}{(k+1)(k+2)} + \frac{c^3}{(k+1)(k+2)(k+3)} + \cdots \right)
\leq \frac{e^k}{k!} \sum_{l=0}^{+\infty} \frac{c^l}{l!} = e^c \frac{e^k}{k!}.
\]

By using the Stirling formulae we obtain the desired inequality.

The following proposition gives the decay rate of the sequence \( (a_k^n) \).
Remark 2. Numerical evidences show that eigenvalues $\mu_n$ We provide the reader by several numerical results that illustrate these applications as well as of the approximation of bilateral Laplace band-limited by the use of the GOSWFs. As a second application In this section, we give two applications of the GOSWFs: The fisrt application, deals with the approximation of bilateral Laplace band-limited by the use of the GOSWFs. Theorem 1. Let $\epsilon$ be a real number satisfying $0 < \epsilon < 1$. Let $K_{\epsilon}(\alpha, \beta, c)$ be the integer defined as follows $K_{\epsilon}(\alpha, \beta, c) = \inf\{K \in \mathbb{N}, \frac{e^{2\alpha+\beta+1}K!\Gamma(K+\alpha+\beta+1)\Gamma(K+\alpha+1)\Gamma(K+\beta+1)}{2(K+\alpha+\beta+1)(\Gamma(2K+\alpha+\beta+1))^2} \leq \epsilon|\mu_n^{(\alpha, \beta)}|\}$. Then for any $k \geq \max([2\epsilon]+1, K_{\epsilon})$, we have

$$\sup_{x \in [0,1]} \left| \psi_n^{(\alpha, \beta)}(x; c) - \frac{1}{\mu_n^{(\alpha, \beta)}(c)} \sum_{j=1}^{k} \omega_j e^{(cxy_j)} \psi_n^{(\alpha, \beta)}(y_j; c) \right| < \epsilon$$

(47)

Here, $(y_j)_{1 \leq j \leq n}$ denote the different zeros of $P_n^{\alpha, \beta}(x)$. Remark 2. Numerical evidences show that $\mu_n^{(\alpha, \beta)}(c)$ decay rapidly to 0 when $n$ goes to $+\infty$.

5 Applications of the GOSWFs and numerical results

In this section, we give two applications of the GOSWFs: The fisrt application, deals with the approximation of bilateral Laplace band-limited by the use of the GOSWFs. As a second application of the GOSWFs, we use them to invert the bilateral Laplace transform of time limited functions. We provide the reader by several numerical results that illustrate these applications as well as of the eigenvalues $\mu_n^{(\alpha, \beta)}(c)$ for different values of $\alpha$, $\beta$ and $c$ and some curves of GOSWFs.
5.1 GOSWFs and approximation of bilateral Laplace band-limited signals

Similarly to what was done in [22], for the PSWFs, and in [14] for the CPSWFs, we can show that the GOSWFs solve the problem of signal concentration energy, which is an important problem from signal processing. More precisely, we assert that among the set $L^2_{\omega,\beta}$ of bilateral Laplace band-limited signals with bandwidth $c > 0$, $\psi_{0}^{(\alpha,\beta)}(;c)$ has the most concentrated energy on $(-1,1)$. More generally, for any integer $n \geq 1$, $\psi_{n}^{(\alpha,\beta)}(;c)$ is the $(n+1)$th most concentrated signal in $(-1,1)$ which is orthogonal to $\psi_{0}^{(\alpha,\beta)}(;c)$, ..., $\psi_{n-1}^{(\alpha,\beta)}(;c)$. By the properties of the GOSWFs given in Proposition 1, for any $f \in L^2((-1,1), d\omega_{\alpha,\beta})$, the GOSWFs-based expansion formula of $f$ over $(-1,1)$ is given by $f(t) = \sum_{n \geq 0} \alpha_{n} \psi_{n}^{(\alpha,\beta)}(t; c).$ since $\int_{-1}^{1} \psi_{n}^{(\alpha,\beta)}(t; c) \psi_{m}^{(\alpha,\beta)}(t;c) \omega_{\alpha,\beta}(t) dt = (\mu_{n}^{(\alpha,\beta)}(c))^{2} \delta_{nm}$, then $|\alpha_{n}| = \left| \int_{-1}^{1} f(t) \psi_{n}^{(\alpha,\beta)}(t; c) \omega_{\alpha,\beta}(t) dt \right| \leq \mu_{n}^{(\alpha,\beta)}(c) \|f\|_{2}$.

Let $f_{N}^{GOSWFs}$ denotes the N-term truncated GOSWFs series expansion of $f$, given by

$$f_{N}^{GOSWFs}(t) = \sum_{n=0}^{N} \alpha_{n} \psi_{n}^{(\alpha,\beta)}(t; c), \ t \in (-1,1).$$

Since the sequence $\mu_{n}^{(\alpha,\beta)}(c)$ decay rapidly to 0 when $n$ goes to $\infty$, and since $\|\psi_{n,c}\chi_{(-1,1)}\|_{\infty} \leq \frac{1}{\mu_{n}^{(\alpha,\beta)}(c)}$, then $f_{N}^{GOSWFs}$ converges rapidly to $f$ and $|f_{N}^{GOSWFs}(t) - f(t)| = O(\mu_{n}^{(\alpha,\beta)}(c)), \ \forall t \in (-1,1)$. Moreover, if $f \in L^2((-1,1), d\omega_{\alpha,\beta})$, then $\lim_{N \to +\infty} \|f - f_{N}^{GOSWFs}\|_{2} = 0$. In the sequel we denote by $f_{N}^{Jacobi}$ the N-term truncated Jacobi series expansion of $f$, given by

$$f_{N}^{Jacobi}(t) = \sum_{n=0}^{N} \alpha_{n} p_{n}^{(\alpha,\beta)}(t; c), \ t \in (-1,1).$$

We compare numerically, for a given $N$ and a given function $f \in L^2_{c}(-1,1), \ d\omega_{\alpha,\beta}$, which function among $f_{N}^{Jacobi}$ or $f_{N}^{GOSWFs}$ approach the best $f$.

5.2 Inversion of the bilateral Laplace transform of time limited functions

Let us present the problem first: Assuming that the finite weighted bilateral Laplace transform $F(x)$ of a function $f$ given by:

$$\int_{-1}^{1} e^{xy} f(y) \omega_{\alpha,\beta}(y) dy = F(x)$$

is known. Here $f$ is an $(L^2(-1,1), d\omega_{\alpha,\beta})$ unit norm function. The problem is to reconstruct the function $f(y)$ exactly or approximately. By applying similar technics used in [22] to invert the Laplace transform of timelimited functions and essentially timelimited functions we obtain the following solution

$$f(y) = \sum_{k \in \mathbb{N}} a_{k}(f) \psi_{k}^{(\alpha,\beta)}(y; c), \ y \in (-1,1),$$

where

$$a_{k}(f) = \frac{1}{\mu_{k}^{(\alpha,\beta)}(c)} \int_{-1}^{1} \psi_{k}^{(\alpha,\beta)}(x; c) F(y) \omega_{\alpha,\beta}(y) dy, \ k \in \mathbb{N}. \quad (50)$$

Once known $(a_{k}(f))_{k}$, we use (49) to obtain the unknown data $f$. 

11
Note here that in practice the series given by (49) is truncated to an order \( N \) to obtain the following function

\[
 f_N(y) = \sum_{k=0}^{N} a_k(f) \psi_k^{(\alpha,\beta)}(y; c), \quad y \in (-1, 1).
\] (51)

Then an error bound of the approximation of \( f \) by \( f_N \), over \((-1, 1)\) is given by the following proposition.

**Proposition 4.** Under the above notations and assumption, we have

\[
 \varepsilon_c^{(\alpha,\beta)} = ||f - f_N||_2^2 = \sum_{k=N+1}^{\infty} |a_k(f)|^2 \left( \mu_k^{(\alpha,\beta)}(c) \right)^2 \leq \left( \mu_{N+1}^{(\alpha,\beta)}(c) \right)^2.
\] (52)

As a consequence of results in [17, 18] we assert that the \( f_N \) well approximate \( f \). This is due to the rapidly decay of the eigenvalues of \( \mathcal{F}_c^{(\alpha,\beta)} \).

### 5.3 Numerical results

In this subsection we give several numerical results that illustrate the theoretical results of the previous sections. We have considered different values of the bandwidth \( c \) and the parameters \( \alpha \) and \( \beta \). Also, we have applied the Gaussian quadrature based method for the computation of the spectrum and the eigenfunctions of the finite weighted bilateral Laplace transform \( \mathcal{F}_c^{(\alpha,\beta)} \) with \( N = 40 \) quadrature points. In Table 1 we have listed the obtained eigenvalues \( \mu_k^{(\alpha,\beta)}(c) \) with different values of the parameter \( c, \alpha \) and \( \beta \).

| \( n \) | \( c = 1 \) | \( c = 1 \) | \( c = 6 \) | \( c = 6 \) |
|--------|------------|------------|------------|------------|
|        | \( (\alpha,\beta) = (0, 0) \) | \( (\alpha,\beta) = (3, 3) \) | \( (\alpha,\beta) = (6, 7) \) | \( (\alpha,\beta) = (5, 5) \) |
| 0      | 0.779836289 | 0.338455158 | 0.199353974 \( \times 10^{-2} \) | 0.211037689 \( \times 10^{-2} \) |
| 2      | 0.328060086 \( \times 10^{-1} \) | 0.305609085 \( \times 10^{-2} \) | 0.242164507 \( \times 10^{-3} \) | 0.49615392 \( \times 10^{-3} \) |
| 4      | 0.178076210 \( \times 10^{-3} \) | 0.982704157 \( \times 10^{-5} \) | 0.146846578 \( \times 10^{-4} \) | 0.312721188 \( \times 10^{-4} \) |
| 6      | 3.77194953 \( \times 10^{-7} \) | 1.57113391 \( \times 10^{-8} \) | 5.322716251 \( \times 10^{-7} \) | 0.134135193 \( \times 10^{-5} \) |
| 8      | 4.247451396 \( \times 10^{-10} \) | 1.481487170 \( \times 10^{-11} \) | 1.279499739 \( \times 10^{-8} \) | 3.673872446 \( \times 10^{-8} \) |
| 10     | 2.966038648 \( \times 10^{-13} \) | 9.151623350 \( \times 10^{-15} \) | 2.179482513 \( \times 10^{-10} \) | 6.946790778 \( \times 10^{-10} \) |
| 15     | 8.099510876 \( \times 10^{-22} \) | 2.071915188 \( \times 10^{-23} \) | 2.426475139 \( \times 10^{-15} \) | 9.333247151 \( \times 10^{-15} \) |
| 20     | 4.268133206 \( \times 10^{-31} \) | 9.817546181 \( \times 10^{-33} \) | 6.690930862 \( \times 10^{-21} \) | 2.916259632 \( \times 10^{-20} \) |

Table 1: Values of the eigenvalues \( \mu_n^{(\alpha,\beta)}(c) \) of the finite bilateral weighted Laplace transform \( \mathcal{F}_c^{(\alpha,\beta)} \) corresponding to different values of parameter \( c, \alpha \) and \( \beta \).

We remark easily that the GOSWFs are reduced to the PSWFs in the special case where the integral operator \( \mathcal{F}_c^{(\alpha,\beta)} \) is replaced by \( \mathcal{F}_c = e^{i c} \mathcal{F}_{c0}^{(0,0)} \). Here \( i^2 = -1 \). Hence the eigenvalues of the GOSWFs coincides with those of the PSWFs. In Table 2, we give some results of the eigenvalues of \( \frac{c}{2\pi} \mathcal{F}_c \circ (\mathcal{F}_c)^* \), such values was given also in [12].

12
Table 2: Values of the eigenvalues of $\frac{c}{2\pi} \tilde{F}_c \circ (\tilde{F}_c)^*$, corresponding to the different values of parameter $c$.

| $n$ | $\tilde{c} = 2$ | $\tilde{c} = 4$ | $\tilde{c} = 6$ |
|-----|----------------|----------------|----------------|
| 0   | $0.8805599223 \times 10^0$ | $0.9958854904 \times 10^0$ | $0.9999018826 \times 10^0$ |
| 5   | $1.93585252920 \times 10^{-7}$ | $0.3812917217 \times 10^{-3}$ | $0.2738716624 \times 10^{-1}$ |
| 10  | $2.1680118965 \times 10^{-19}$ | $4.5252284693 \times 10^{-13}$ | $2.2189805452 \times 10^{-9}$ |
| 15  | $1.6563615010 \times 10^{-33}$ | $3.5519079602 \times 10^{24}$ | $1.0163838373 \times 10^{-18}$ |
| 20  | $4.7105458228 \times 10^{-49}$ | $1.0352225590 \times 10^{-36}$ | $1.7132439301 \times 10^{-29S}$ |

Figure 1: Graphs of $\psi^{(\alpha,\beta)}_k(\cdot;c)$, $k = 1, 2, 3, 4$ associated to the parameter $c = 2$ and $\alpha = \beta = 3$ by the differential operator based method.

For the computation of the GOSWFs over $(-1, 1)$ by the first method, we have used the equality (38) with a maximum truncation order $K = 10$, see Figure 1. For the computation of the GOSWFs over $(-1, 1)$ by the second method, we have used the equality (47) with a maximum truncation order $K = 10$ quadrature points, see Figure 2 and Figure 3. Remark, that for $\alpha = \beta$, $\psi^{(\alpha,\beta)}_n(\cdot;c)$ has the same parity as its order $n$ contrarily to the case where $\alpha \neq \beta$.

As a numerical results of the second application: Inversion of the weighted finite bilateral Laplace transform, we give in table 3, several original time limited data $f$. The Laplace transform of the given functions are given in [5]. In the last columns of this table we give the values of the committed error given by (52) for different values of $\alpha$, $\beta$ and $c$. Note here that for the inversion we have used the GOSWFs computed with the second method with $K = 10$ quadrature points.
Figure 2: Graphs of the first four GOSWFs associated to the parameter $c = 2$ and $\alpha = 3$ and $\beta = 3$ by the Gaussian quadrature based method

Figure 3: Graphs of the first four GOSWFs associated to the parameter $c = 1$ and $\alpha = 2$ and $\beta = 3$ by the Gaussian quadrature based method
\[ f(x) = g(c(x + 1))(1 - x)^{-\alpha}(x + 1)^{-\beta} \]

| \( g_1(x) = \frac{1}{\sqrt{c}} \chi_{(0,1)}(x) \) | \( c = 7 \) | \( \alpha = 5 \) | \( \beta = 4 \) | \( \varepsilon_{(0,0)}^c \) | \( \varepsilon_{(\alpha,\alpha)}^c \) | \( \varepsilon_{(\alpha,\beta)}^c \) |
| --- | --- | --- | --- | --- | --- | --- |
| \( g_2(x) = \frac{c}{\sqrt{2cx - x^2}} \chi_{(0,2c)}(x) \) | \( c = 0.2 \) | \( \alpha = 1.5 \) | \( \beta = 2 \) | \( \varepsilon_{(0,0)}^c \) | \( \varepsilon_{(\alpha,\alpha)}^c \) | \( \varepsilon_{(\alpha,\beta)}^c \) |
| \( g_3(x) = \frac{\cos(2.5\arccos(\frac{c}{2}))}{\sqrt{4c^2x - x^2}} \chi_{(0,2c)}(x) \) | \( c = 1 \) | \( \alpha = 6 \) | \( \beta = 5 \) | \( \varepsilon_{(0,0)}^c \) | \( \varepsilon_{(\alpha,\alpha)}^c \) | \( \varepsilon_{(\alpha,\beta)}^c \) |

Table 3: Different values of the error given by (52) for different values of \( \alpha, \beta \) and \( c \) and for different timelimited functions

Remark here that for the cases where \( \alpha \neq \beta \) the reconstruction of the given functions is more accurate then the other cases. To illustrate this remark, we give in figures 4-6 the curves of the timelimited functions given in table 3 as well as the reconstructed functions by the use of the PSWFs (\( \alpha = \beta = 0 \)), the Generalized PSWFs of Gegenbauer (\( \alpha = \beta \)) and the GOSWFs for different values of \( \alpha, \beta \) and \( c \).

Figure 4: original signal \( f(x) = g_1(c(x + 1))(1 - x)^{-\alpha}(x + 1)^{-\beta} \) (line) and its reconstructed signal (point), where \( g_1(x) = x^{1/2} \chi_{(0,1)}(x) \), \( c = 7 \), \( \alpha = 0 \), \( \beta = 0 \), (left figure), \( \alpha = 5 \), \( \beta = 4 \), (left figure), \( \alpha = 5 \), \( \beta = 4 \), (left figure)

Figure 5: original signal \( f(x) = g_2(c(x + 1))(1 - x)^{-\alpha}(x + 1)^{-\beta} \) (line) and its reconstructed signal (point), where \( g_2(x) = (2cx - x^2)^{-1/2}(c-x) \chi_{(0,2c)}(x) \), \( c = 0.2 \) and \( \alpha = \beta = 0 \), (left figure), \( \alpha = \beta = 1 \), (middle figure) and \( \alpha = 2.9 \), \( \beta = 2.5 \), (right figure)

Figure 6: original signal \( f(x) = g_3(c(x + 1))(1 - x)^{-\alpha}(x + 1)^{-\beta} \) (line) and its reconstructed signal (point), where \( g_3(x) = \frac{\cos(2.5\arccos(\frac{c}{2}))}{\sqrt{4c^2x - x^2}} \chi_{(0,2c)}(x) \), \( c = 1 \) and \( \alpha = \beta = 0 \), (left figure), \( \alpha = \beta = 1 \), (middle figure) and \( \alpha = 5 \), \( \beta = 4 \), (left figure)
Figure 6: original signal $f(x) = g_3(c(x+1))(1-x)^{-\alpha}(x+1)^{-\beta}$ (line) and its reconstructed signal (point), where $g_3(x) = \cos((2\nu + 0.5)\arccos(0.5x/c))\sqrt{4c^2x-x^3}\chi_{(0,2c)}(x)$, $\nu = 0$, $c = 1$ and $\alpha = 0$, $\beta = 0$, (left figure), $\alpha = \beta = 1$, (middle figure), and $\alpha = 4$, $\beta = 3$, (right figure).

As a comparison between the approximation of functions in $LB_\omega$ by the use of the GOSWFs and Jacobi polynomials, we give the curves of three functions in $LB_\omega$, their approximation by the GOSWFs and also by the Jacobi polynomials, see figures 7-9.

Figure 7: Curves of $F(x) = \frac{e^{-cx}}{c} \left( \frac{L!}{(-x)^{L+1}} - e^{2cx} \sum_{j=0}^{L} \frac{L!}{j!} (2c)^j (-x)^{L-j+1} \right)$ (line) and its approximated signal $F_N^{GOSWFs}$ (dot), $F_N^{Jacobi}$ (dash), where $N = 3$, $c = 5$, $\alpha = 0$, $\beta = 1$, and $L = 2$. 

16
Figure 8: Curves of $F(x) = \pi I_1(-cx)$ (line) and its approximated signal $F_N^{GOSWFs}$ (dot), $F_N^{Jacobi}$ (dash), where $N = 4, c = 5, \alpha = 1, \beta = 2$.

Figure 9: Curves of $F(x) = e^{-cx} \left((-1)^\nu \sqrt{0.5\pi x I_\nu(cx)K_\nu(\nu + 0.5, cx)}\right)$ (line) and its approximated signal $F_N^{GOSWFs}$ (dot), $F_N^{Jacobi}$ (dash), where $N = 5, \nu = 2, c = 6, \alpha = 2, \beta = 1$.

References

[1] M.A. Al-Gwaiz, Sturm-Liouville Theory and Its Applications, Springer, 2007.
[2] H. B. Aouicha, M. Tahar, On the spectrum of the finite Laplace transform and applications, Applicable Analysis and Discrete Mathematics 2012 Volume 6, Issue 2, Pages: 304-316

[3] C. J. Bowkamp, On spheroidal wave functions of order zero, J. Math. Phys. 26, (1947), 79–92.

[4] E.A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955.

[5] A. Erdelyi, Tables of Integral Transforms, Vol 1, McGRAW-HILL BOOK COMPANY, INC. 1954

[6] F. A. Grunbaum, and Y. Milen, The prolate spheroidal phenomenon as a consequence of bispectrality. Superintegrability in classical and quantum systems, 301312, CRM Proc. Lecture Notes, 37, Amer. Math. Soc., Providence, RI, 2004.

[7] E.L. Ince, Ordinary Differential Equations, Dover, New York, 1956.

[8] H. J. Landau and H. O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty-III; The dimension of the space of essentially time and band-limited signals., Bell System Tech. J. 41 (1962), 1295–1336.

[9] H. J. Landau, H.O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertaintyII, Bell System Tech. J. 40 (1961), 65–84.

[10] A. Karoui, I. Mehrzi, T. Moumni, Eigenfunctions of the Airys integral transform: Properties, numerical computations and asymptotic behaviors, J. Math. Anal. Appl. 389 (2012) 9891005.

[11] A. Karoui, Uncertainty Principles, Prolate Spheroidal Wave Functions and Applications, Recent Developments in Fractals and Related Fields, 165190, Appl. Numer. Harmon. Anal., Birkhuser Boston, Inc., Boston, MA, (2010).

[12] A. Karoui and T. Moumni, New efficient methods of computing the prolate spheroidal wave functions and their corresponding eigenvalues, Appl. Comput. Harmon. Anal. 24 (2008), pp. 269-289

[13] A. Karoui and T. Moumni, Spectral Analysis of the Finite Hankel Transform Operator and Circular Prolate Spheroidal Wave Functions, Comput. Appl. Math. 233 (2009), pp. 315-333

[14] Tahar Moumni, On essentially time and Hankel band-limited functions, Integral Transforms and Special Functions Volume 23, 2012, 83-95.

[15] C. Niven, On the Conduction of Heat in Ellipsoids of Revolution, Phil. Trans. R. Soc. Lond., 171, (1880), 117-151.

[16] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, Charles W. Clark, NIST handbook of mathematical functions, Cambridge University Press, 2010

[17] J. B. Reade, Eigenvalues of positive definite kernels. SIAM J. Math. Anal. 14 (1983), no. 1, 152–157.

[18] J. B. Reade, Eigenvalues of positive definite kernels. II. SIAM J. Math. Anal. 15 (1984), no. 1, 137–142.

[19] J. Shen, T. Tang, L. Wang, Spectral Methods, Algorithms, Analysis and Applications, Springer Series in Computational Mathematics Volume 41 2011
[20] Y. Shkolnisky, Prolate spheroidal wave functions on a disc: Integration and approximation of two-dimensional bandlimited functions, Appl. Comput. Harmon. Anal. 22 (2007) 235–256.

[21] D. Slepian, Prolate spheroidal wave functions, Fourier analysis and uncertainty–IV: Extensions to many dimensions; generalized prolate spheroidal functions, Bell System Tech. J. 43 (1964), 3009–3057.

[22] D. Slepian, H.O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty I, Bell System Tech. J. 40 (1961), 43–64.

[23] L. L. Wang and J. Zhang, A new generalization of the PSWFs with applications to spectral approximations on quasi-uniform grids, Appl. Comput. Harmon. Anal. 29 (2010) 303–329.