A Simple Theory for Cathode Jets in Plasma Arcs

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Abstract

In this paper, we propose a simple and practical method for the modeling of the local and total magnetic force generated at the cathode. We prove that, if the current density profile obeys the same distribution along the axial-direction, the axial force depends only on the total current and the ratio of the arc spot radius to the radius of the arc column. We also provide an estimate for that the radial force. The power generated by Joules heating is also inserted as a term source in the energy balance equation. The use of source terms is very attractive for large scale industrial applications as it prevents from solving Maxwell’s equations that require fine meshing in the arc region and imprecise boundary conditions. It also helps to circumvent the extra complications in computation that are related to the cathode and anode falls. In the case of a mono-phase or 3-phase arc, if steady-state simulations are desired, a time-average is calculated.

Keywords: cathode-jets, thermal torches, plasma reactors

1. Introduction

For a complete self-consistent numerical simulation of the arc, it is inevitable to include the very small arc region in the rest of the computational domain and to solve the Maxwell’s equations. However, in large scale industrial plasma reactors, the region where the arc is generated is significantly small compared to the body of the reactor. Consequently, this requires a fine mesh at the arc region which substantially increases the computational cost and time. To avoid such problem, we propose to analytically model the most important features of the arc and to include them in the numerical simulation as source terms. The arc is perceived as an energy and momentum source. Analytical expressions can be found for both of them. The major contribution to the arc momentum is mainly due to the cathode jet, which results from the magnetic pinch in the vicinity of the cathode tip. In the AC transient arc, the computational cost turns out to be considerably high if one resorts to an unsteady scheme. A time-average of the momentum source term comes in too handy to avoid tedious complicated expressions. A sketch of the magnetic forces is shown in figure 1.

2. Mathematical model

The magnetic force can be written as $F = j \times B$. Assuming axisymmetry, the local axial and radial components of this force ($f_z$ and $f_r$ respectively) are expressed as:

\[ f_z = j_r B_\theta \quad ; \quad f_r = -j_z B_\theta \]  

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From the Maxwell equation $\nabla \times \mathbf{B} = \mu_0 j$, we deduce that:

$$\frac{1}{r} \frac{\partial (r B_r)}{\partial r} = \mu_0 j_z \quad \Rightarrow \quad B_r = \frac{\mu_0}{r} \int_0^r j_z \xi d\xi$$

In order to find an expression for $j_r$, we will make use of the equation of the conservation of charge, i.e. $\nabla \cdot j = 0$. In an axisymmetric configuration, we formulate it as:

$$\frac{1}{r} \frac{\partial (r j_r)}{\partial r} = - \frac{\partial j_z}{\partial z} \quad \Rightarrow \quad j_r = - \frac{1}{r} \int_0^r \frac{\partial j_z}{\partial z} \xi d\xi$$

Inserting (2) and (3) in (1), we find:

$$f_z = \frac{\mu_0}{r} \left( \int_0^r \frac{\partial j_z}{\partial z} \xi d\xi \right) \left( \int_0^r j_z \xi^2 d\xi \right) ; \quad f_r = \frac{\mu_0}{r} \int_0^r j_z \xi^2 d\xi$$

We denote by $R(z)$ the arc radius at every axial position $z$. If $l$ is the total current, then $\bar{f}_z(R(z))$ and its derivative with respect to $R(z)$ can be written as:

$$\bar{f}_z(R(z)) = \frac{1}{\pi R^2(z)} ; \quad \frac{\partial \bar{f}_z}{\partial R} = - \frac{2l}{\pi R^3(z)} = - \frac{2}{R(z)}$$

A volume integral must be carried out in order to obtain the total forces along $z$ and $r$. Using axisymmetry, we find:

$$F_z = \int_0^H \int_0^{R(z)} 2\pi r f_z dr dz ; \quad F_r = \int_0^H \int_0^{R(z)} 2\pi r f_r dr dz$$

### 3. General profile

In this section, we prove that the total force $F_z$ is both independent of the choice of the current-density profile and the expansion $R(z)$. We provide also bounds for the force $F_r$. To do so, we first set: $e = r/R(z) ; \quad \chi_1 = \xi_1 / R(z) ; \quad \chi_2 = \xi_2 / R(z)$. In the following, we use $R$ instead of $R(z)$ for simplicity. The magnetic field from expression (2) is now written as:

$$B_r = \frac{\mu_0 R^2}{r} \int_0^e j_z \chi_1 d\chi_1 = \frac{\mu_0 R}{e} \int_e^c j_z \chi_1 d\chi_1$$

Instead of expressing the current density as $j_z(r,R)$, we express it here as $j_z(e,\bar{R})$. More precisely, we write it as $j_z = C j_z F(e)$. Here $C$ is an adjusting constant that depends on the profile distribution such that $\int_0^e 2\pi r j_z dr = l$. An easy computation yields $C = \left( \frac{1}{\int_0^1 e F(e) de} \right)^{-1}$. For example, in the case of a power-law distribution, i.e, $F(e) = 1 - e^a$, $C$ is found to be equal to $(n+2)/n$ ($n \to \infty$ corresponds to the uniform distribution). We also assume that $F(e)$ is a non-increasing and non-negative function on the interval $[0,1]$ with $F(0) = 1$ and $F(1) = 0$.

#### 3.1. Expression of $F_z$

Using (5) and the fact that $j_z = j_z(e,\bar{R})$, we obtain:
\[
\frac{\partial j_z}{\partial z} = \frac{\partial R}{\partial z} \frac{\partial j_z}{\partial R} = \frac{\partial R}{\partial z} \left( \frac{\partial j_z}{\partial R} + \frac{\partial j_z}{\partial j_z} \frac{\partial j_z}{\partial R} \right) = \frac{\partial R}{\partial z} \left( \frac{\partial j_z}{\partial j_z} + 2 \frac{j_z}{R} \right)
\]  

(9)

It’s worthwhile noting here that we used relation (6) and:

\[
\frac{\partial e}{\partial R} = -\frac{r}{R^2} = -\frac{e}{R} ; \quad \frac{\partial j_z}{\partial j_z} - \frac{j_z}{j_z}
\]

Upon plugging (9) in (3) and noting that \(\frac{\partial(\chi^2 j_z)}{\partial \chi} = \chi^2 \frac{\partial j_z}{\partial \chi} + 2 \chi j_z\), we find:

\[
j_r = R \frac{\partial R}{\partial d} \int_0^\epsilon \left( \chi^2 \frac{\partial j_z}{\partial \chi} + 2 \chi j_z \right) d\chi = \frac{\partial R}{\partial d} \epsilon j_z
\]

Replacing (8) and (10) in (4) and then in (7) for \(j_z\) and \(F_r\), gives:

\[
F_z = \int_0^H 2\pi \int_0^{R_0} R j_z j_z \frac{\partial R}{\partial d} \left( \int_0^\epsilon j_z \chi d\chi \right) dR dz = \int_0^H 2\pi \int_0^{R_0} R \left( \int_0^\epsilon j_z \chi d\chi \right) dR dz
\]

\[
= \int_0^H \pi R^2 \int_0^\epsilon j_z \chi d\chi \frac{\partial R}{\partial d} \left( \int_0^\epsilon j_z \chi d\chi \right) dz = \int_0^H \pi R^2 \left( \int_0^\epsilon j_z \chi d\chi \right) dR dz
\]

(11)

In the above equation, we used the fact that \(\frac{d}{de} \left( \int_0^\epsilon j_z \chi d\chi \right)^2 = 2e j_z \left( \int_0^\epsilon j_z \chi d\chi \right)\) and that \(\int_0^1 j_z \chi d\chi = 12\). Hence the result is proven. It’s worthwhile noting that the same result was found in [1]. As a general rule, if \(j_z\) can be written as: \(j_z = j_z(\mathcal{J}, e)\) where \(e\) is defined as a above and \(\mathcal{J}\) is any function that is proportional to \(R^{-2}\) and depends only on it (it could be \(\mathcal{J}\) or \(j_{max}\) in the exponential profile), the same result is obtained as in (11). This result certainly does not hold when \(j_z\) obeys different distribution profiles along \(z\).

3.2. Expression of \(F_r\)

Inserting (8) in (4) and then in (7) for \(F_r\) and \(F_z\), leads to:

\[
F_r = \int_0^H \int_0^{R_0} 2\pi \int_0^{R_0} j_z \left( \int_0^\epsilon j_z \chi d\chi \right) dR dz = \int_0^H \pi R^2 \left( \int_0^\epsilon j_z \chi d\chi \right) dR dz
\]

(12)

Here again, \(\int_0^1 j_z \chi d\chi = \frac{12}{2}\). Given the fact that \(F(e)\) is non-increasing and non-negative on \([0,1]\), we obtain the following upper bound using Chebyshev’s inequality:

\[
\frac{1}{2} \int_0^\epsilon j_z \chi d\chi = \frac{1}{2} \int_0^\epsilon j_z \chi d\chi = C \int_0^\epsilon F(e) d\chi \leq C \int_0^\epsilon F(e) d\chi \leq \frac{1}{2} \Rightarrow \int_0^\epsilon \left( \frac{1}{2} j_z \chi d\chi \right)^2 d\chi \leq C \frac{1}{2}
\]

Hence, a lower and an upper bound for \(F_r\), can be obtained:
Assuming that the anode jet is much weaker, it is considered negligible. Therefore the local and total forces are given by

\[ F_z \leq \int_0^H \pi \mu_0 \frac{z^2}{4} dz \leq F_t \leq \int_0^H \pi \mu_0 \frac{z^2}{4} (1 + \frac{C^2}{3}) dz \]

Furthermore, setting as before \( R(0) = R_c \) and \( R(H) = R_a \), and assuming that \( R_c < R(z) < R_a \) for \( z \in [0, H] \) gives:

\[ \frac{\mu_0 I^2 H}{4\pi R_a} \leq F_t \leq \frac{\mu_0 I^2}{4\pi} (1 + \frac{C^2}{3}) \frac{H}{R_c} \]

We notice that \( F_t \) is not negligible even in the arc column if the height \( H \) is extended to include the rest of the arc column. \( F_z \) however is only present in the near cathode expansion.

4. Time dependent AC current

For a mono-phase AC, each electrode acts as cathode during half of the period. Assuming that the anode jet is much weaker, it is considered negligible. Assuming quasi-neutrality, the equation of the conservation of the current, i.e. \( \nabla \cdot j = 0 \) holds. The current displacement equation \( \nabla \times B = \mu_0 j \) holds as well. As a consequence, the expressions of the local force \( j_z \) and the total force \( F_z \) are still valid. This time, however, the axial current density \( j_z \) and the total current \( I \) are functions of time, involving a sine wave. For a stationary simulation, time-averaging is required. For an AC current with frequency of \( \omega / 2\pi \), we have \( j_z(t) = j_z^M \sin(\omega t) \) and \( I(t) = I^M \sin(\omega t) \). Therefore the local and total forces are given by \( F_z(t) = F_z^M \sin^2(\omega t) \) and \( F_t(t) = F_t^M \sin^2(\omega t) \). \( M \) denotes the maximal value. For simplicity, time-averaging is carried out only for \( F_z(t) \). The result applies also to \( f_z(t), f_z(t) \) and \( f_z(t) \). \( F_z^eff \) designates the time-average of \( F_z(t) \). Knowing that the efficient current is given by \( I^{eff} = I^M / \sqrt{2} \), we find:

\[ F_z^{eff} = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} F_z^M \sin^2(\omega t) dt = \frac{\omega}{2\pi} \int_0^{\pi/\omega} F_z^M \sin^2(\omega t) dt + \int_{\pi/\omega}^{2\pi/\omega} F_z^M \sin^2(\omega t) dt = \frac{\mu_0 (I^{eff})^2}{8\pi} \ln \left( \frac{R_a}{R_c} \right) \]

For a 3-phase setting, the same result is expected per electrode as each electrode acts as cathode for half the period but are dephased by an angle of \( 2\pi/3 \). Although the voltage signals do not have a wave-form, we assume they do for simplicity. The phase voltages between each pair of electrodes in a balanced \( \Delta \)-wiring configuration are expressed as:

\[ u_{ab}(t) = U_0 \cos(\omega t + 2\pi/3) \quad u_{bc}(t) = U_0 \cos(\omega t + 4\pi/3) \quad u_{ca}(t) = U_0 \cos(\omega t - 2\pi/3) \]

Figure 2 depicts a sketch of a 3-phase \( \Delta \)-connection. \( a, b \) and \( c \) represent the electrodes. The phase voltages are shown in figure 3a. \( u_a, u_b \) and \( u_c \) are the line currents feeding each electrode. From experiments, it has been observed that the arc burns instantaneously between the electrodes subjected to the highest phase voltage. Although there is some overlapping, we assume that the arc burns between two electrodes at once. The discontinuous yellow line when it coincides with curve \( u_{ij} \) indicates that the arc is burning between electrodes \( i \) and \( j \), with \( i \) acting as cathode when \( u_{ij} \) is positive and \( j \) acting cathode when \( u_{ij} \) is negative. The remaining electrode is neutral during this time interval.
Fz

The forces profile, similar to the one used in [2] given by

\[ F_{z}(t) = \frac{\omega}{2\pi} \int_{0}^{2\pi/\omega} F_{z}(\theta) d\theta = \frac{\omega}{2\pi} \left[ \int_{0}^{\pi/6\omega} F_{z}(\theta) d\theta + \int_{5\pi/6\omega}^{\pi} F_{z}(\theta) d\theta + \int_{\pi/6\omega}^{\pi/2} F_{z}(\theta) d\theta \right] \]

\[ = \frac{F_{M}}{6} = \frac{n_{0}(v'f)^{2}}{12\pi} \ln \left( \frac{R_{a}}{R_{c}} \right) \]

\[ (17) \]

5. Recommendations for numerical modeling and some simulation results

The forces \( F_{z} \) and \( F_{r} \) as source term in the momentum equations for numerical modeling. In order to estimate these forces, it is indispensable to determine the ratio \( R_{a}/R_{c} \) for the computation of \( F_{z} \) and the height of the arc cone expansion for the determination of \( F_{r} \). Empirical data can be found in [1]: \( R_{a}/R_{c} \approx 4 \). The ratio \( R_{a}/R_{c} \) actually depends on the current profile and on the spot radius \( R_{c} \). In fact, the sharper and greater (higher peaks at the centre like the exponential profile) the profile of the current density \( j_{r} \) is, the higher is the ratio as the \( j_{r} \) would become higher. Another key factor in the determination of \( R_{a}/R_{c} \) is thermal convection; if the arc is burning freely without any flow rate of gas injection, the arc would tend to expand by diffusion and therefore \( R_{a}/R_{c} \) becomes greater. If, on the other hand, there is a forced convection due to gas injection along the electrode, then this ratio tends to be lower.

For large scale reactors in which the arc occupies only a very small region, the total forces are calculated and divided by the volume of the plasma expansion cone near the cathode and inserted as source terms in the momentum equations. The power density is also distributed uniformly in the area occupied by the arc between the electrode. The obtained results are quite satisfactory. At the level of the arc however, more accuracy is desired (e.g., if this method is adopted for the simulation of the axisymmetric free-burning arc). Therefore suitable assumptions are to be made. For example, instead of employing a uniform distribution for the power and the forces inside the volume sources, a profile distribution is more preferable. We first model a welding arc using the source domain approach and we compare the profile of the current density \( j_{r} \) is inserted linearly with respect to \( z \) in a shape of a cone (its radius at the cathode is 0.8 mm and at the anode its radius is 4 mm), with the highest power density near the tip of the cathode. The axial force is inserted uniformly according
EM resolution (MHD) | Source terms
---|---
![Temperature Contours](image1.png) | ![Temperature Contours](image2.png)
![Velocity vector](image3.png) | ![Velocity vector](image4.png)

**FIG. 4:** A comparison between temperature profiles (up) and velocity vectors (down) obtained by solving the EM equations for 200 A Argon arc at 1 bar (left) and by means of source terms (right).

to 11 in a domain source defined by cylinder underneath the cathode. It has the same radius of the cathode and has a 1.5 mm height. The radial force is inserted along all the arc column according to 12. The good agreement between both models is remarkable (see figure 4). This proofs that even on the scale of the arc, the source domain method is somewhat surprisingly accurate. The source term approach yields, however, slightly lower velocities (with a maximum of the order of 200 m·s⁻¹) in comparison with the MHD method (with a maximum of the order of 250 m·s⁻¹). This could be attributed to the fact that the force source term (axial and radial) corresponding to the magnetic pinch is distributed uniformly in the chosen domain source. For the industrial scale, when the flow in a plasma reactor is modeled, few details are needed at level of the small arc region. The power is distributed uniformly in a toroidal volume (in orange in figure 5) underneath the electrodes and the forces in cylinders (in green in figure 5). Figure 5 shows an example of a 3-phase plasma reactor modeled using this method. The forces are computed as per expression 17. The velocity profiles showing the cathode jets immediately under the electrode and the temperature distribution in the reactor are also displayed in figure 5. They seem to be in good agreement with experimental results.

### 6. Conclusion

We have presented in this paper a simplified method for numerical simulation that allows one to separate the physics of arc-scale phenomena governed by electromagnetic effects and the physics of the flow on a much larger scale (e.g. an industrial reactor). The electromagnetic effects such as Lorentz force and the Joules heating are modeled analytically by means of source terms inserted in domain source. Agreement between this approach and an MHD approach is very good. It is noteworthy to mention that this approach could be applied as it is (with of course assumptions on the source domains and source terms), in any kind of industrial applications that involve relatively slow moving arcs with almost fixed spots. However, when it comes to moving arcs as in circuit breakers or in a moving anode spot as in spraying torches, we believe that the approach still works, but extra information is needed (possibly empirical data) concerning
the speed and position of the arc and its spots, so that the source domains and the source terms could be modified or adjusted accordingly at each time step.

**FIG. 5:** Numerical simulation with source terms carried out for an industrial reactor: Sketch of the geometry (left) and temperature profiles, velocity vectors and BC (right)
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