Quantum Ergodic Restriction Theorems, I: Interior Hypersurfaces in Domains with Ergodic Billiards

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Abstract. Quantum ergodic restriction (QER) is the problem of finding conditions on a hypersurface $H$ so that restrictions $\varphi_j|_H$ to $H$ of $\Delta$-eigenfunctions of Riemannian manifolds $(M, g)$ with ergodic geodesic flow are quantum ergodic on $H$. We prove two kinds of results: First (i) for any smooth hypersurface $H$, the Cauchy data $(\varphi_j|_H, \partial \varphi_j|_H)$ is quantum ergodic if the Dirichlet and Neumann data are weighted appropriately. Secondly (ii) we give conditions on $H$ so that the Dirichlet (or Neumann) data is individually quantum ergodic. The condition involves the almost nowhere equality of left and right Poincaré maps for $H$. The proof involves two further novel results: (iii) a local Weyl law for boundary traces of eigenfunctions, and (iv) an ‘almost-orthogonality’ result for Fourier integral operators whose canonical relations almost nowhere commute with the geodesic flow.

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1. Introduction

This is the first in a series of articles on “quantum ergodic restriction” (or QER) theorems. Quantum ergodicity (in the Riemannian setting) is the study of asymptotics of eigenfunctions of the Laplacian $\Delta_g$ on Riemannian manifolds $(M, g)$ with ergodic geodesic flow. The QER question is whether restrictions of $\Delta_g$-eigenfunctions to hypersurfaces $H \subset M$ are quantum ergodic along the hypersurface. More precisely, the aim is to find conditions on $H$ so that...

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the Dirichlet data $\varphi_\lambda|_H$, or Neumann data $\partial^H\varphi_\lambda|_H$ or full Cauchy data $(\varphi_\lambda|_H, \partial^H_\nu \varphi_\lambda|_H)$ is quantum ergodic along $H$. Here, $\partial^H_\nu$ is a fixed choice of unit normal to $H$.

In this article, we consider the case of smooth hypersurfaces $H \subset \Omega$ in piecewise smooth Euclidean domains $\Omega \subset \mathbb{R}^N$ with ‘totally’ ergodic billiards, i.e. such that all powers $\beta^k : B^*\partial\Omega \to B^*\partial\Omega$ of the billiard map are ergodic. We first prove that in a suitably weighted sense, the Cauchy data is quantum ergodic for any hypersurface. This is a general phenomenon that holds for all Riemannian manifolds with boundary (further results are proved in [CHTZ]). We then consider the subtler question of when the Dirichlet data (or Neumann data) is itself quantum ergodic. It is simple to find examples of $H$ for which $\varphi_\lambda|_H$ are not quantum ergodic. But in Theorem 4 we give a generic sufficient condition on terms of a ‘dynamical’ condition on the ‘transfer map $\tau$ to the boundary (see Definitions 2 and 3). The proof is based on a seemingly new principle in quantum dynamics: the almost orthogonality, $\langle F(\lambda_j)\varphi_j, \varphi_j \rangle \to 0$, of $\Delta$-eigenfunctions $\varphi_\lambda$ (and their Cauchy data) to their images $F(\lambda)\varphi_\lambda$ under a semi-classical Fourier integral operator whose underlying symplectic map almost nowhere commutes with the geodesic flow (or billiard map).

To our knowledge, the only prior quantum ergodic restriction theorem concerns the Cauchy data along the boundary of eigenfunctions of the Laplacian on bounded domains satisfying Dirichlet or generalized Neumann boundary conditions [GL, HZ, B]. However, heuristic and numerical work of Hejhal-Rackner [HR] appears to show that restrictions of eigenfunctions on finite area hyperbolic surfaces to closed horocycles are quantum ergodic. We restrict to Euclidean domains in this article because there are special techniques available for them and there are many interesting examples. At the end of the introduction, we discuss related work (and work in progress) on quantum ergodic restriction theorems in more general settings.

To state our results, we introduce some notation (See §[10] for a notational index). We consider the eigenvalue problem in a piecewise smooth bounded Euclidean domain $\Omega \subset \mathbb{R}^n$ with Neumann (resp. Dirichlet) boundary conditions:

$$
\begin{cases}
-\Delta \varphi_\lambda = \lambda_j^2 \varphi_\lambda & \text{in } \Omega, \\
\partial^\nu_\nu \varphi_\lambda = 0 \ (\text{resp. } \varphi_\lambda = 0) & \text{on } \partial\Omega.
\end{cases}
$$

Here, $\Delta$ is the Euclidean Laplacian and $\partial^\nu$ is the interior unit normal to $\Omega$. We denote by $\{\varphi_\lambda\}$ an orthonormal basis of eigenfunctions of the boundary value problem corresponding to the eigenvalues $\lambda_0^2 < \lambda_1^2 \leq \lambda_2^2 \cdots$ enumerated according to multiplicity. We denote by $\beta : B^*\partial\Omega \to B^*\partial\Omega$ the billiard map of $\Omega$ (see §[2.1]).

The quantum ergodic restriction problem for Dirichlet data is to determine the limits of the diagonal matrix elements

$$
\rho^H_\lambda (O^H_{\varphi}(a)) := \langle O^H_{\varphi}(a) \varphi_\lambda|_H, \varphi_\lambda|_H \rangle_{L^2(H)}, \quad O^H_{\varphi}(a) \in \Psi^{0}_sc(H)
$$

of (semi-classical) pseudo-differential operators $O^H_{\varphi}(a)$ along $H$. That is, $a \in S^{0,0}_c(T^*H \times (0, \lambda_0^{-1}])$ lies in the class of zeroth-order semiclassical symbols with polyhomogeneous expansions in $\lambda^{-1}$ with $\lambda \in [\lambda_0, \infty)$ We recall that [DS]

$$
S^{0,0}_c(T^*H \times (0, \lambda_0^{-1}]) := \{a(s, \tau; \lambda) \in C^\infty(T^*H \times (0, \lambda_0^{-1})); a \sim_h \sum_j a_j \lambda^{-j},
$$

$$
|\partial^\nu_\nu \partial^\nu a_j(s, \tau)| \leq (2^j_{(j, \alpha, \beta)}(\tau)^{-j-|\beta|}.
$$
For simplicity, we denote the corresponding space of semiclassical pseudodifferential operators by 
\[ \Psi_{sc}^0(H) := Op_{\lambda}(S^0,0(T^*H \times [0, \lambda^{-1}])). \]

The sequence of Dirichlet data \( \varphi_{\lambda_j}|_H \) is called quantum ergodic on \( L^2(H) \) if almost all of the matrix elements (2) tend to the integral
\[ \omega_{H}^{\infty}(a) = \int_{B^* H} a_{0} d\mu_{\infty} \]
where \( d\mu_{\infty} \) is the image under the transfer map \( \tau^{\Omega}_{H} \) (see Definition [2]) to \( B^* H \) of the limit measure \( d\mu_{B} \) of \( [H] \) with boundary conditions \( B \). The quantum ergodic restriction problem for Neumann data is similar, while that for Cauchy data is to study the matrix elements
\[ \langle A_{2} C\rho_{H}(\varphi_{\lambda_j}), C\rho_{H}(\varphi_{\lambda_j}) \rangle, \]
where \( A_{2} \) is a \( 2 \times 2 \) matrix of pseudo-differential operators on \( H \). As can be seen in the table below, we need to weight the Dirichlet and Neumann data in a special way and to choose special \( A_{2} \) to obtain an almost unique weak limit.

To put the quantum ergodic restriction problem into context, we recall that the Dirichlet or Neumann eigenfunctions are quantum ergodic on \( M \) (i.e. in the ‘interior’) whenever the geodesic flow of \( (M, g) \) (resp. billiard flow, if \( \partial M \neq \emptyset \)) is ergodic. In this article we restrict our attention to bounded domains in \( \mathbb{R}^n \), and refer to [GL, ZZ] for the relevant interior quantum ergodic theorems. It is shown in [GL, HZ, Bu] that \( (M, \partial M) \) satisfies QER if the classical billiard flow of \( M \) is ergodic, with \( \omega^{H}_{\infty} \) depending on the choice of boundary conditions (which we suppress in the notation). The boundary quantum ergodic limit measures are as follows [HZ]:

| Boundary traces and quantum limits |
|----------------------------------|
| \( B \) | \( Bu \) | \( u' \) | \( d\mu_{B} \) |
| Dirichlet | \( u|_{Y} \) | \( \lambda^{-1} \partial_{\nu} u|_{Y} \) | \( \gamma(q) d\sigma \) |
| Neumann | \( \partial_{\nu} u|_{Y} \) | \( u|_{Y} \) | \( \gamma(q)^{-1} d\sigma \) |

Here, we denote by \( d\sigma = dq d\eta \) the natural symplectic volume measure on \( B^* \partial \Omega \). The standard cotangent projection is denoted by \( \pi : T^* \partial \Omega \rightarrow \partial \Omega \). We also define the function \( \gamma(q, \eta) \) on \( B^* \partial \Omega \) by
\[ \gamma(q, \eta) = \sqrt{1 - |\eta|^2}. \] (4)

The set \( S^*M|_H \) of unit vectors to \( M \) along \( H \) is a kind of cross-section to the geodesic or billiard flow and one might expect that on both the classical and quantum levels, the ambient ergodicity induces ergodicity on the cross section. However, this heuristic idea is based on the assumption that the Cauchy data (or Dirichlet data) on a hypersurface is a suitable notion of quantum cross section. In effect we are investigating the extent to which that is true (the abstract microlocal notion of quantum cross section is studied in [CHTZ2]).

1.1. Quantum ergodic restriction theorem for Cauchy data. The difference in the quantum ergodic restriction problems for Cauchy data versus Dirichlet data is visible in the one-dimensional case where \( M = S^1 \) or where \( M = [0, \pi] \). Modulo time-reversal, the classical systems are obviously ergodic and consist of just one orbit. The real-valued eigenfunctions \( \sin kx, \cos kx \) do not individually have quantum ergodic restrictions to a hypersurface (i.e. a point) \( x = x_0 \). Indeed, they oscillate as \( k \rightarrow \infty \). On the other hand, if we weight the real Neumann data by \( \frac{1}{k} \) then the formula \( \cos^2 kx + \sin^2 kx = 1 \) shows that the Cauchy data is quantum ergodic.
A somewhat more revealing example is that of eigenfunctions of semi-classical Schrödinger operators $\hbar^2 D^2 + V$ in one dimension and with connected level sets of the energy $E^2 + V(x) = E$. The real valued eigenfunctions have the WKB form $A_h \cos(\frac{1}{\hbar} S(x))$ and we see that again the Dirichlet data at a point fails to be quantum ergodic but that the Cauchy data is quantum ergodic if the Dirichlet data is weighted by $S'(x)$ and the Neumann data by $\hbar$.

As these examples show, we should study the normalized Cauchy data

$$CD_H(\varphi_\lambda) : (\varphi_\lambda|_H, \lambda^{-1} \partial_\nu \varphi_\lambda|_H).$$

Henceforth, Cauchy data will always refer to this normalization.

To state our result, we introduce some further notation. We assume that $H \subset \Omega^o$, although with some extra technical complications, we could allow $H$ to intersect $\partial \Omega$ (e.g. the midline of the Bunimovich stadium). We denote by $s \mapsto q_H(s) \in H$ a smooth parameterization of the hypersurface $H \subset \Omega$ with $s \in U \subset \mathbb{R}^{n-1}$ an open set and $U \subset y \mapsto q(y) \in \partial \Omega$ a local smooth parametrization of the boundary of $\Omega$. The dual coordinates are $\tau \in T^*_s(H)$ and $\eta \in T^*_y(\partial \Omega)$ respectively. We sometimes abuse notation and use $(s, \tau)$ as coordinates on $B^*H$, resp. $(y, \eta)$ as coordinates on $B^*\partial \Omega$.

The examples above indicate that we can only expect QER for rather special matrices $A_2$ of pseudo-differential operators. We say that $A_2(h)$ is a normalized scalar matrix of pseudo-classical pseudo-differential operators if,

- (i) $A_{21}(h) = A_{12}(h) = 0$;
- (ii) $\sigma_{A_{11}}(s, \tau) = (1 - |\tau|^2)\sigma_{A_{22}}(s, \tau)$ for all $(s, \tau) \in B^*H$.

**Theorem 1.** Assume that $\Omega \subset \mathbb{R}^n$ is a piecewise smooth Euclidean plane domain with totally ergodic billiards, and let $H \subset \Omega^o$ be any smooth hypersurface. Also, let $A$ be a normalized scalar matrix of zeroth order pseudo-differential operators on $H$. Then, there exists a density-one subset $S$ of $\mathbb{N}$ such that the (normalized) Cauchy data of Dirichlet or Neumann eigenfunctions satisfies

$$\lim_{\lambda_j \to \infty, j \in S} \langle A_2 CD_H(\varphi_{\lambda_j}), CD_H(\varphi_{\lambda_j}) \rangle = \int_{B^*H} (\sigma_{A_{11}} + \sigma_{A_{22}})d\mu_\infty,$$

where $d\mu_\infty = (\tau^H_{\partial \Omega})_s d\mu_B$ is the push-forward by $\tau^H_{\partial \Omega}$ (Definition 2) to $B^*H$ of the limit measure $d\mu_B$ in the table above.

In other words, there exists a measure $d\mu_\infty$ on $B^*H$ so that along a subsequence of eigenvalues of density one we have,

$$\langle Op_\lambda((1 - |\tau|^2)a(s, \tau))\varphi_\lambda|_H, \varphi_\lambda|_H \rangle + \lambda^{-2}\langle Op_\lambda(a(s, \tau))\partial_\nu \varphi_\lambda|_H, \partial_\nu \varphi_\lambda|_H \rangle$$

$$\to \int_{B^*H} 2(1 - |\tau|^2)ad\mu_\infty.$$

When the correspondence $\tau^H_{\partial \Omega}$ has two branches, so that $\Gamma^H_{\partial \Omega}, \Gamma^{H,1}_{\partial \Omega} \cup \Gamma^{H,2}_{\partial \Omega}$ where $\tau^{H,j}_{\partial \Omega} : j = 1, 2$, are single-valued maps (see section 3.2), the push-forward is given by

$$(\tau^H_{\partial \Omega})_* := (\tau^{H,1}_{\partial \Omega})_* + (\tau^{H,2}_{\partial \Omega})_*.$$
that this theorem can be naturally interpreted as a quantum ergodicity result for the quantum flux norm of the Cauchy data along the hypersurface, \( H \subset \Omega \). Moreover, the latter argument is insensitive to whether or not \( \partial \Omega \neq 0 \). Thus, our result in \([\text{CHTZ2}]\) extends Theorem 1 to interior hypersurfaces \( H \subset M \) of arbitrary compact ergodic manifolds \((M,g)\) with or without boundary.

1.2. Quantum ergodic restriction for Dirichlet data. The one-dimensional case raises doubts that there can exist a quantum ergodic restriction theorem for the Dirichlet data alone. However, it is not hard to show that quantum ergodic restriction does hold for random waves on general smooth hypersurfaces in dimensions \( n \geq 2 \). In this section, we give a sufficient condition on \( H \) so that QER is valid.

To motivate the condition, we consider another special case of the QER problem: i.e. the case of Neumann eigenfunctions of a Bunimovich stadium \( S \subset \mathbb{R}^2 \), with the curve \( H \) given by the vertical midline of \( S \). We note that \( S \) has a left-right symmetry \( \sigma_L \) (and an up-down symmetry). The midline \( H \) is pointwise fixed by \( \sigma_L \). ‘Half’ of the Neumann eigenfunctions are even with respect to \( \sigma_L \) and half are odd (in the sense of spectral density). Obviously, the odd eigenfunctions restrict to zero on \( H \). On the other hand, the even ones are quantum ergodic on \( H \): in the quotient domain \( S/\mathbb{Z}_2 \) by \( \sigma_L \), \( H \) becomes a boundary component and even Neumann eigenfunctions of \( S \) are Neumann eigenfunctions of \( S/\mathbb{Z}_2 \). Hence a full density subsequence of their restrictions to \( H \) have the boundary quantum limit \( d\mu_B \) given in the table above. Thus, there exist two subsequences of eigenfunctions of density \( 1/2 \) with different quantum limits, and therefore QER (quantum ergodic restriction) of the Dirichlet fails for \((S,H)\). Consistently with Theorem 1, the Cauchy data is quantum ergodic. Indeed, the Neumann data is quantum ergodic on the midline by the arguments of \([\text{HZ}, \text{Bu}]\) applied to the half-stadium with Neumann boundary conditions on all but the midline and with Dirichlet boundary conditions on the midline. A more general example of the same kind is a curve \( H \) which intersects the midline in a segment of the midline. The same kind of example exists on the fixed point set of any manifold (with or without boundary) that possesses a \( \mathbb{Z}_2 \) symmetry.

We now give a sufficient condition for restricted quantum ergodicity of the Dirichlet data which, while difficult to verify in many cases, is generic for smooth hypersurfaces, at least for some classes of domains. As above, we let \( H \subset \partial \Omega \) (the interior of \( \Omega \)) denote a smooth hypersurface. The key objects associated to \( H \) are the transfer maps \( \tau^H_{\partial \Omega} \) and \( \tau^\Omega_H \). They are somewhat complicated because of the possible number of intersection points of a billiard trajectory with \( H \). In \([3.1]\) we will give a more detailed definition of the transfer maps (see Definition 3.5).

Let \((y, \eta) \in B^*\partial \Omega \) and let \( q(y) + t\zeta(y, \eta) \) denote its billiard trajectory where (see \([2.1]\))

\[
\zeta(y, \eta) := \eta + \sqrt{1 - |\eta|^2} \nu_y
\]

is the lift of \((y, \eta)\) to an interior unit vector to \( \Omega \) at \( y \).

\textbf{Definition 2.} The transfer map \( \tau^H_{\partial \Omega} : B^*\partial \Omega \to B^*H \) is defined as follows: We say that \((y, \eta)\) is in the domain of \( \tau^H_{\partial \Omega} \), (ie. \((y, \eta) \in \mathcal{D}(\tau^H_{\partial \Omega})\)) if its billiard trajectory intersects \( H \). Denote by \( E(t_H(y, \eta), y, \eta) \in H \) any intersection point (see section \([3.2]\)) and by \( \pi^T_{E(t_H(y, \eta), y, \eta)} \zeta(y, \eta) \) the tangential projection of the terminal velocity at \( E(t_H(y, \eta), y, \eta) \) to
$T_{E(t_H(y, η), y, η)}H$. The graph of transfer map is by definition the canonical relation,

$$\Gamma_{\partial H} := \{(y, η, s(y, η), π_{E(t_H(y, η), y, η)}^T ζ(y, η)) ∈ B^*∂Ω × B^*H, (y, η) ∈ D(τ_{\partial H}^H), q_H(s(y, η)) = E(t_H(y, η), y, η)\}. \quad (8)$$

In the inverse direction, we have the double-valued transfer map (or correspondence)

$$\tau_{\partial Ω}^H : B^*H → B^*∂Ω,$$ \quad (9)

defined by taking $(s, τ) ∈ B^*H$ to the two unit covectors $ξ_±(s, τ) ∈ S^*_ν(s, τ) \Omega$ which project to it, following each of their trajectories until they hit the boundary and then projecting each terminal velocity vector to $B^*∂Ω$. The graph of $\tau_{\partial Ω}^H$ is the canonical relation,

$$\Gamma_{\tau_{\partial Ω}^H} := \{(s, τ, y(s, τ), π_{t_H(y(s, τ))}^T ζ(s, τ)) ∈ B^*H × B^*∂Ω\}. \quad (10)$$

We emphasize that $\tau_{\partial Ω}^H$ is double-valued due to the fact that each $(s, τ) ∈ B^*H$ lifts to two unit covectors

$$ξ_±(s, τ) = τ + \sqrt{1 - |τ|^2} ν_± ∈ S^*_ν Ω \quad (11)$$

which project to $τ$. More precisely, the normal bundle $N^*H$ to an orientable hypersurface $H$ decomposes into two $\mathbb{R}_+$ bundles $N^*_±$, which we view as the two infinitesimal sides of $N^*H$. Then $ξ_±(s, τ) ∈ N^*_± H$ are the lifts of $τ$ to unit covectors on the two sides. The lifts are analogous to (7) but only the interior side of the boundary is relevant in (7). The two lifts in (11) give rise to two branches

$$\tau_± : B^*H → B^*∂Ω \quad (12)$$

of $\tau_{\partial Ω}^H$ by taking $τ_±(s, τ)$ to be the element of $\tau_{\partial Ω}^H(s, τ)$ coming from the trajectory defined by the unit vector $ξ_±(s, τ) \in S^*_ν y(s, τ) Ω$. This double-valued-ness is independent of the shape of $H$ and is an essential feature of the quantum restriction problem.

On the other hand, $\tau_{\partial Ω}^H$ is multi-valued due to the shape of $H$, specifically to the many possible intersection points of a billiard trajectory with $H$. This is an unavoidable feature of the QER problem which causes notational inconveniences more than essential analytical problems. For expository simplicity, we will assume that $H$ is weakly convex, so that a generic line intersects $H$ in at most two points. This saves us from tedious notations and indices for the branches of $\tau_{\partial Ω}^H$ caused by the multiple intersection points. In the course of the proof we indicate the modifications necessary to deal with general smooth hypersurfaces. In Definition 3.5 we introduce notation for the multiple branches.

Another inevitable and tedious complication is that the correspondences have singularities at directions where billiard trajectories intersect $H$ or $∂Ω$ tangentially or which run into a corner of $∂Ω$. It is not apriori clear that such tangential rays cause only technical complications. Without a detailed analysis, it is possible that eigenfunction mass could get concentrated microlocally in tangential directions to $H$. In the case $H = \partial Ω$ it was proved in [HZ] that no such concentration occurs. The same method shows that no such concentration occurs on $H$ either. In §8.3 we give a self-contained pointwise Weyl law argument to show that for a full density of $u^H_A$’s, no such tangential mass concentration occurs. This method works equally well for manifolds, $M$, with or without boundary.
Another important symplectic correspondence associated to $H$ is the once-broken transmission billiard map (or correspondence) through $H$, $\beta_H : B^*\partial\Omega \to B^*\partial\Omega$, defined by

$$\beta_H(y, \eta) = \begin{cases} 
\tau_-^{-1}\tau_+^{-1}(y, \eta), & (y, \eta) \in \text{range } \tau_+, \\
\tau_+^{-1}\tau_-^{-1}(y, \eta), & (y, \eta) \in \text{range } \tau_.
\end{cases} \tag{13}$$

Equivalently, $\beta_H$ maps the two endpoints of the rays through $\xi_{\pm}(s, t)$ with $(s, t) \in B^*H$ to each other. Thus, $\beta_H$ follows the trajectory of $(y, \eta) \in B^*\partial\Omega$ backward to its intersection point $q_H(s(y, \eta)) \in H$ (assuming it intersects $H$), breaks at $q_H(s(y, \eta))$ by the law of equal angles, and then proceeds along the second link in the forward direction to $\partial\Omega$. Since the trajectory defined by $(s, \tau) \in B^*H$ may have multiple intersection points with $H$, this map is also a multi-valued correspondence. With the simplifying assumption that $H$ is weakly convex, a generic trajectory intersects $H$ in at most two points, and we denote the corresponding branches of $\beta_H$ by $\beta^k_H; k = 1, 2$. See [3] for further details. Note that the trajectory breaks only once if the trajectory intersects $H$ twice.

**Remark:** $\beta_H$ is not the same as the broken billiard flow with a break on $H$ because the trajectory travels backwards on the first link. Starting at $(y, \eta) \in B^*\partial\Omega$, the map first time reverses the $\eta \to -\eta$, then proceeds on the broken trajectory and then projects the terminal velocity. Hence, $\beta_H(y, \eta)$ differs by a time-reversal from the usual broken billiard trajectory. As a check, let us note that in the case of the midline of the stadium, the two unit vectors projecting to $v \in B^*_xH$ are related by $\sigma_L$. Hence their rays, and the projections of the terminal velocity vectors, are related by $\sigma_L$. But the two-link ray starting at $v \in T_y\partial\Omega$ between them is not so related: the terminal velocity is the time reversal of $\sigma_L(v)$.

Given $A \subset B^*\partial\Omega$ in the following we denote the symplectic volume $|dyd\eta|$- measure of $A$ by $|A|$.

**Definition 3.** We say that $(\beta_H, \beta)$ almost nowhere commute if the (commutation) sets

$$\mathcal{CO}_{p,k} := \{(y, \eta) \in B^*\partial\Omega : \beta_H \beta^k(y, \eta) = \beta^p \beta_H(y, \eta)\}$$

have symplectic volume measure zero, $|\mathcal{CO}_{p,k}| = 0$, for all $p, k = 1, 2, 3, \ldots$.

The almost nowhere commutativity (ANC) condition may be re-formulated in terms of ‘left’ and ‘right’ return maps to $B^*H$. Given covector $(s, \tau) \in B^*H$, one may lift it in two ways (11) to a unit covector $(s, \xi_{\pm}) \in S^*_H\Omega$ which projects to $\tau$. We then follow the two geodesics $\gamma_{\pm}(t)$ with initial conditions $(s, \xi_{\pm})$ until they return to $H$ on the same side as $(s, \xi_{\pm})$ and then project the terminal co-vector back to $B^*H$. The almost nowhere commutativity condition is equivalent to the condition that these two oriented return maps do not coincide on a set of positive measure in $B^*\partial\Omega$. The equivalence of the definitions is easily seen by re-writing the equation $\beta_H \beta^k(y, \eta) = \beta^p \beta_H(y, \eta)$ (see (13)) as

$$\tau_-^{-1}\tau_+^{-1}\beta^k(y, \eta) = \beta^p \tau_-^{-1}\tau_+^{-1}(y, \eta) \iff \tau_+^{-1}\beta^k\tau_+(s, \tau) = \tau_-^{-1}\beta^p\tau_-(s, \tau),$$

with $(y, \eta) = \tau_+(s, \tau)$. We observe that $\tau_{\pm}^{-1}\beta^k\tau_{\pm}(s, \tau)$ is the $\pm \to \pm$ sided return map to $B^*H$ after $k$ bounces off $\partial\Omega$. The union over $k$ gives the full one-sided return map. The fact that $k$ and $p$ may differ in Definition 3 then seems rather natural, since the equality of the one-sided return maps should not depend on the number of bounces off $\partial\Omega$.

Our main result on Dirichlet data is:
Theorem 4. Let $\Omega \subset \mathbb{R}^n$ be a piecewise-smooth billiard with totally ergodic billiard flow and let $H \subset \text{int}(\Omega)$ be a smooth interior hypersurface. Let $\varphi_{\lambda_j}; j = 1, 2, \ldots$ denote the $L^2$-normalized Neumann eigenfunctions in $\Omega$. Then, if $(\beta_H, \beta)$ almost nowhere commute, there exists a density-one subset $S$ of $\mathbb{N}$ such that for $\lambda_0 > 0$ and $a(s, \tau; \lambda) \in S_{cl}^{0,0}(T^*H \times (0, \lambda_0^{-1}))$,

$$\lim_{\lambda_j \to \infty, j \in S} (\text{Op}_{\lambda_j}(a_0) \varphi_{\lambda_j}|_H, \varphi_{\lambda_j}|_H)_{L^2(H)} = c_n \int_{B^*H} a(s, \tau) \rho_{\partial\Omega}^H(s, \tau) \, dsd\tau.$$  

Here, $c_n = \frac{\text{vol}(S^n-1)}{\text{vol}(U)}$ and $\rho_{\partial\Omega}^H \, dsd\tau := (\tau_{\partial\Omega}^H)_* (\gamma^{-1} \, dyd\eta)$. Similarly for Dirichlet data on $H$ of Dirichlet eigenfunctions, except that $\gamma(y, \eta) = (1 - |\eta|^2)^{-1/2}$, and for Neumann data if one multiplies the above by $(1 - |\tau|^2)$.

We summarize the conclusion in the following table.

| B        | Trace on H | $u^H$         | $d\mu_\infty$ |
|----------|------------|---------------|---------------|
| Dirichlet| Dirichlet  | $u|_H$        | $(\tau_{\partial\Omega}^H)_* \gamma(q) \, d\sigma$ |
| Dirichlet| Neumann    | $\lambda^{-1} \partial_{\eta^k} u|_H$ | $(1 - |\tau|^2)(\tau_{\partial\Omega}^H)_* \gamma(q) \, d\sigma$ |
| Neumann  | Dirichlet  | $u|_H$        | $(\tau_{\partial\Omega}^H)_* \gamma(q) \, d\sigma$ |
| Neumann  | Neumann    | $\lambda^{-1} \partial_{\eta^k} u|_H$ | $(1 - |\tau|^2)(\tau_{\partial\Omega}^H)_* \gamma(q) \, d\sigma$ |

There is a stronger but more technical version of this result which uses a quantitative refinement of the almost nowhere commuting condition (see the definition of quantitative almost nowhere commutation (QANC) condition in Definition 4 of subsection 8.2.3).

Theorem 5. Theorem 4 is valid as long as $(\beta_H, \beta)$ quantitatively almost nowhere commute.

To illustrate the result, we reconsider the midline $H$ of the Bunimovich stadium $S$. In this case, for $(y, \eta) \in B^*_{\text{cl}(y)} H$, the two unit vectors $\xi_{\pm}(y, \eta) \in S^*_\Omega\Omega$ projecting to $\eta$ are $\sigma_L$-related, i.e. images of each other under $\sigma_L$. Hence their trajectories, and the projections to $T\partial\Omega$ of their terminal velocities, are $\sigma_L$ related. It follows that $\beta_H(y, \eta) = \sigma_L(y, \eta)$. Hence $\beta^k \sigma_L = \sigma_L \beta^k$ for all $k$, and $(S, H)$ violates the condition of quantitative almost nowhere vanishing. Thus, our condition rules out this ‘counterexample’ to QER and the original ANC condition. At this time, we are not aware of any other kind of ‘counter-example’. Indeed, the equality of the left and right sided return maps is a kind of left/right symmetry condition on $H$. It might be equivalent to the existence of an involutive isometry $\sigma : M \to M$ fixing a positive measure of $H$, but at this time we do not know how to prove that.

Unfortunately, it is often difficult to check whether a given hypersurface $H$ in the billiard setting satisfies the almost nowhere commutativity condition. The next result shows that the condition holds for generic $H$ in a reasonably broad billiard setting. In [9] we prove:

Proposition 6. Let $\Omega$ be a two-dimensional dispersing billiard table (i.e. with hyperbolic billiards). Then for generic convex curves, $H \subset \Omega$, $(\beta_H, \beta)$ almost never commute.

This result is just an example of the genericity of the condition. We only use hyperbolicity of the billiards to prevent the existence of a positive measure of self-conjugate directions, and we restrict to dimension two to simplify one step in the argument. A much more general result on genericity of the condition should be possible, and probably could be proved by the methods of [PS1, PS2]. The non-commutativity condition is also similar to the condition.
that Poincaré maps of periodic reflecting rays of generic domains do not have roots of unity as eigenvalues; see [PS1, PS2, S]. But that would take us too far afield in this article and we only give the simpler result to convince the reader that our non-commutativity condition is far from vacuous. In the boundaryless case, we believe that the condition can be proved to hold for closed geodesics of hyperbolic surfaces which are not fixed by isometric involutions [TZ2] and possibly for closed horocycles.

We also remark that the ‘total ergodicity assumption’ is not overly restrictive: all of the known examples of ergodic billiards that we are aware of are totally ergodic. These include: the Bunimovitch stadium, the cardioid and generic polygonal billiards [CFS, P]. The first two examples are known to satisfy stronger mixing conditions (the stadium is in fact Bernoullii [B]), and Troubetzkoy [Tr] has recently proved that these billiards are also totally ergodic.

1.3. Convexity of $H$ and QER. As mentioned above, we assume $H$ is convex for simplicity of exposition, but the assumption is not necessary for the quantum ergodic restriction to hold. Indeed, we may localize (and microlocalize) the problem by considering symbols supported on small regions of $H$. On a sufficiently small region of $B^*H$, concave or convex hypersurfaces are essentially the same. Flat regions are different, but since tangential intersections only occur for a zero measure of rays, it suffices to prove that only a sparse (density zero) subsequence of eigenfunctions can concentrate microlocally on tangential rays. We prove such a result in Lemma 8.5 using a pointwise local Weyl law. In [HZ] the analogous proof is in Section 7 (see Step 2 and Lemma 7.1), Section 9 (see Lemma 9.2) and in Appendix 12 (see especially Lemma 12.5 in the Dirichlet case).

The fact that QER is independent of the curvature of the hypersurface may seem surprising in view of the results of Burq-Gérard-Tzetkov [BGT] (see also [R, KTZ, To, So]) on $L^p$ norms of restrictions of eigenfunctions. In these results, the $L^p$ norm of the restricted eigenfunctions depends on whether the curve (or hypersurface) is geodesically convex or not. There is a dramatic difference, for example, between norms of restrictions to distance circles and to closed geodesics.

However, these results do not make use of the global assumption of ergodicity of the geodesic flow. They are sharp in the case of the round sphere, but need not be sharp for negatively curved surfaces (for instance). The results of [HZ, Bu] amply indicate that geodesic curvature of $H$ is irrelevant to quantum ergodic restriction theorems: Indeed, when $H = \partial \Omega$, QER is valid for non-convex $H$ and for $H$ with flat sides (e.g. the Bunimovitch stadium).

Since the validity of the results for general smooth hypersurfaces is possibly surprising, and since we assume convexity at some points to simplify the notation, we signal in each section where and when we use the convexity assumption.

1.4. Applications. Under the assumptions of Theorem 4, the $L^2$-restriction bounds along $H \subset \Omega$ of a quantum ergodic sequence of eigenfunctions are uniformly bounded above and below. Indeed, choosing $a(s, \tau) = 1$ in Theorem 4 gives the following

**Corollary 7.** Under the same assumptions as in Theorem 4,

$$\lim_{\lambda_j \to \infty, \lambda_j \in S} \left\| \varphi_{\lambda_j} \right\|^2_{L^2(H)} = c_n \int_{B^*H} \rho^H_{\partial \Omega}(s, \tau) ds d\tau.$$
Except for possible exceptional sparse subsequences of eigenfunctions, in the ergodic case, the asymptotics in Corollary 7 substantially improve on some recent bounds of $L^p$ norms of restrictions of eigenfunctions to submanifolds in the articles [11 BCT, 10 I T So BR].

In subsection 8.5, we give an application of Theorem 4 to the study of asymptotic nodal structure of eigenfunctions of ergodic planar billiards $\Omega \subset \mathbb{R}^2$. Consider either Dirichlet or Neumann eigenfunctions $\varphi_\lambda$ with $-\Delta \varphi_\lambda = \lambda^2 \varphi_\lambda$. In [TZ], we show that for general piecewise-analytic domains, the eigenfunction nodal set $\mathcal{N}_{\varphi_\lambda} = \{x \in \Omega; \varphi_\lambda(x) = 0\}$ (where we omit the boundary itself) for both Neumann and Dirichlet boundary conditions satisfies

$$\text{card} (\mathcal{N}_{\varphi_\lambda} \cap \partial \Omega) = O(\lambda). \quad (14)$$

Here, (14) requires no dynamical assumptions. In Theorem 6 of [TZ], we proved a similar bound for intersections of nodal lines with interior analytic curves under an additional technical assumption of ‘goodness’ of the curve $C$ (see the proof of the corollary). The quantum ergodicity result in Theorem 4 yields the following

**Corollary 8.** Let $\Omega \subset \mathbb{R}^2$ be a piecewise-smooth, totally ergodic planar billiard and $C \subset \Omega$ a real-analytic interior curve. Assume $(\beta_C, \beta)$ almost never commute. Then, for a full-density of (Dirichlet or Neumann) eigenfunctions,

$$\text{card} (\mathcal{N}_{\varphi_\lambda} \cap C) = O(\lambda).$$

1.5. **Sketch of proof.** We now outline the proofs of Theorems 1 and 4. This outline is an integral part of the proofs of the Theorems; later sections fill in the considerable number of technical details, but rigidly adhere to the following outline.

The first step is to use the ‘quantization’ of the transfer map $\tau_{\partial \Omega}^H : B^* \partial \Omega \to B^* H$ to transfer the Dirichlet data $u_\lambda^H := \varphi_\lambda|_H$ of the eigenfunctions on $H$ to the Dirichlet data $u_\lambda^b := \varphi_\lambda|_{\partial \Omega}$ on $\partial \Omega$. As in [HZ, TZ], the semi-classical quantization of the transfer map is a layer potential integral operator. By Green’s formula, we have

$$u_\lambda^H(q_H) = [N_{\partial \Omega}^H(\lambda)u_\lambda^b](q_H) := \int_{\partial \Omega} N_{\partial \Omega}^H(\lambda, q_H, q) u_\lambda^b(q)d\sigma(q), \quad (15)$$

where $q \in \partial \Omega$, $q_H \in H$ and where $N_{\partial \Omega}^H(\lambda)$ is a boundary integral operator defined in Definition 3.1 and (30). There is a similar operator $N_{\partial \Omega}^{H, \nu}$ that transfers the Dirichlet data on the boundary to Neumann data $u_\lambda^{H, \nu} = \partial_\nu \varphi_\lambda|_H$ on $H$. By a modification of a similar argument in [HZ] (see 3), we show that the operators $N_{\partial \Omega}^H(\lambda), N_{\partial \Omega}^{H, \nu}(\lambda) : C^\infty(\partial \Omega) \to C^\infty(H)$ are semiclassical Fourier Integral operators quantizing $\tau_{\partial \Omega}^H : B^* \partial \Omega \to B^* H$.

We then consider matrix elements (2) of semi-classical pseudo-differential operators on $H$, i.e. operators with symbols $a \in S_{cl}^{0,0}(T^* H)$. From (15) it follows that for any $a \in S_{cl}^{0,0}(T^* H)$,

$$\langle Op_\lambda(a)u_\lambda^H, u_\lambda^H \rangle_{L^2(H)} = \langle N_{\partial \Omega}^H(\lambda)^* Op_\lambda(a)N_{\partial \Omega}^H(\lambda)u_\lambda^b, u_\lambda^b \rangle_{L^2(\partial \Omega)}. \quad (16)$$

Similarly for Neumann data. The formula (16) is an advantage to working in the boundary setting; it allows us to use results of [HZ] to avoid many technicalities.

At first sight, it appears that this transfer operator fully reduces our problem to the known results in [HZ] on $\partial \Omega$. However, this is not the case. The key point is that $N_{\partial \Omega}^H(\lambda)^* Op_\lambda(a)N_{\partial \Omega}^H(\lambda)$ is not a pseudo-differential operator. Rather, in (14) we show that

$$N_{\partial \Omega}^H(\lambda)^* Op_\lambda(a)N_{\partial \Omega}^H(\lambda) = Op_\lambda((\tau_{\partial \Omega}^H)^* a \cdot r_{\partial \Omega}^H) + F_2(\lambda), \quad (17)$$
where $\rho_{\partial \Omega}^H \in C^\infty(B^*H)$ is a smooth density and where $F_2(\lambda)$ is a semi-classical Fourier integral operator which quantizes the transmission map \cite{HZ}. Thus, one of the main contributions of the present article is the analysis of the matrix elements of $F_2(\lambda)$.

The proof of Theorem \ref{thm:main} is then rather simple: The Neumann data produces a second Fourier integral operator $F_2''(\lambda)$. The key observation is that the composition of $F_2(\lambda)$ on the left by an operator with symbol $1 - |\eta|^2$ cancels $F_2''(\lambda)$ in the sum of the matrix elements with respect to both the Dirichlet and Neumann data. Thus, the Fourier integral operator terms cancel, leaving only a pseudo-differential matrix element whose limit was calculated in \cite{HZ}.

For the Dirichlet data alone, there is nothing to cancel $F_2(\lambda)$. Therefore, an important step of the proof of Theorem \ref{thm:main} is to determine the weak* limits of the matrix elements $\langle F_2(\lambda)u^b_\lambda, u^b_\lambda \rangle$. This is a special case of the more general problem of determining limits of matrix elements of Fourier integral operators relative to eigenfunctions, which is discussed in \cite{Z2} and will be investigated more thoroughly in \cite{TZ3}. A significant additional complication is that the matrix elements are relative to boundary restrictions of eigenfunctions rather than to the eigenfunctions themselves.

In the case of pseudo-differential matrix elements, the first step in obtaining the limits of matrix elements relative to $\Delta$-eigenfunctions is to show that they are invariant measures for the billiard flow. In the case of matrix elements of Fourier integral operators, it is not clear what kind of invariance properties the limit has, viewed as a functional of the symbol of $F_2(\lambda)$, when the canonical relation of $F$ fails to be invariant under the billiard flow. Under the condition of Definition \ref{def:almost}, the canonical relation is, in fact, almost nowhere invariant under the billiard flow. We turn this to our advantage by using a rather surprising averaging argument to show that the limits must be zero under the almost nowhere commuting condition.

The starting point of this argument is the classical boundary integral equation

$$u^b_\lambda(q) = \int_{\partial \Omega} N_{\partial \Omega}^\Omega(\lambda)(q, q') u^b_{\lambda}(q')d\sigma(q')$$

for the Dirichlet data of $\varphi_\lambda$ on $\partial \Omega$. The operator $N_{\partial \Omega}^\Omega(\lambda)$ is studied in detail in \cite{HZ} as a semi-classical Fourier integral operator quantizing the billiard map $\beta : B^*\partial \Omega \to B^*\partial \Omega$. The classical formula, $N_{\partial \Omega}^\Omega(\lambda)u^b_\lambda = u^b_\lambda$ was the key input to the quantum ergodicity result of \cite{HZ}. In particular, it shows that when $H = \partial \Omega$, the weak* limits of (2) are invariant under $\beta$.

We apply this quantum invariance to the $F_2(\lambda)$ term. For all $m$, we have

$$\langle Op_\lambda(a)u^H_\lambda, u^H_\lambda \rangle_{L^2(H)} = \langle N_{\partial \Omega}^\Omega(\lambda)^m N_{\partial \Omega}^H(\lambda)^*Op_\lambda(a)N_{\partial \Omega}^H(\lambda)N_{\partial \Omega}^\Omega(\lambda)^m u^b_\lambda, u^b_\lambda \rangle_{L^2(\partial \Omega)}. \quad (19)$$

At this point, we cannot apply Egorov’s theorem to the conjugation because of the almost nowhere commuting condition in Definition \ref{def:almost}. Under that condition, the canonical relation underlying $N_{\partial \Omega}^\Omega(\lambda)^m N_{\partial \Omega}^H(\lambda)^*Op_\lambda(a)N_{\partial \Omega}^H(\lambda)N_{\partial \Omega}^\Omega(\lambda)^m$ changes with each $m$, and it appears that this destroys any invariance property of the quantum limits. What we show is that it forces the matrix element $\langle F_2(\lambda)u^b_\lambda, u^b_\lambda \rangle \to 0$ on average as $\lambda \to \infty$. Thus, $F_2(\lambda_j)u^b_{\lambda_j}$ is ‘almost orthogonal’ to $u^b_\lambda$ under the almost nowhere commuting condition.

The proof is based on a special case of a novel boundary local Weyl law for semi-classical Fourier integral operators.
THEOREM 9. Let $\Omega \subset \mathbb{R}^n$ be a Euclidean domain with piecewise-smooth boundary and $u^b_{\lambda_j} = \varphi_{\lambda_j}|_{\partial \Omega}$ (resp. $u^\lambda_{\lambda_j} = \frac{1}{\lambda_j} \partial_\nu \varphi_{\lambda_j}|_{\partial \Omega}$) be the Neumann (resp. Dirichlet) eigenfunction boundary traces. Let $F(\lambda) : C^\infty(\partial \Omega) \to C^\infty(\partial \Omega)$ be a zeroth-order $\lambda$-FIO with canonical relation $\Gamma_{F(\lambda)} := \text{graph}(\kappa_F) \subset B^*(\partial \Omega) \times B^*(\partial \Omega)$ and let

$$\Sigma_{F(\lambda)} := \bigcup_{m \in \mathbb{Z}} \{(y, \eta) \in B^*(\partial \Omega); \beta^m(y, \eta) = \kappa_{F(y, \eta)}\}.$$

Then, under the assumption that $|\Sigma_{F(\lambda)}| = 0$,

$$\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle F(\lambda_j)u^b_{\lambda_j}, u^b_{\lambda_j} \rangle = o(1).$$

In [TZ3], more general local Weyl laws for semi-classical Fourier integral operators are proved. The general boundary local Weyl law is that there exists a density $\rho_F \in C^\infty(B^*(\partial \Omega))$ such that

$$\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle F(\lambda_j)u^b_{\lambda_j}, u^b_{\lambda_j} \rangle = \int_{\Sigma_{F(\lambda)}} \sigma(F(\lambda))(y, \eta) \cdot \rho_F(y, \eta) \, dyd\eta. \quad (20)$$

Moreover, $\rho_F \, dyd\eta$ can be explicitly computed as a sum of certain measures associated with the various multi-link contributions to the fixed-point set, $\Sigma_{F(\lambda)}$. The proof is valid for arbitrary Riemannian manifolds with piecewise-smooth boundaries. We present only the narrow version above since it is the only statement we need for Theorem 4 and since the proof of the general result is lengthy. Indeed, the proof of the special case in Theorem 9 is already the lengthiest part of the proof of Theorem 4 due to the complicated nature of the wave group on a domain with boundary.

Using Theorem 9, we reduce Theorem 4 to the following almost orthogonality theorem, which has an independent interest.

THEOREM 10. With the same notation as in Theorem 9, assume that $F(\lambda)$ is a zeroth order $\lambda$-FIO and assume that its underlying symplectic map $\kappa_F$ almost nowhere commutes with $\beta$ in the sense of Definition 3. Then

$$\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle F(\lambda_j)u^b_{\lambda_j}, u^b_{\lambda_j} \rangle|^2 = o(1) \quad \text{as} \ \lambda \to \infty.$$

In Theorem 8.1 we restate the result in the form we need for the proof of Theorem 4. However, the proof is valid with no essential modification in the general case. An interesting point is that the result does not use ergodicity of the billiard map. It is a completely general almost orthogonality result for pairs of operators satisfying the almost nowhere commutativity condition.

The proof of Theorem 10 is based on a somewhat curious averaging argument. By quantum invariance, we may (and do) replace $F(\lambda)$ by the operator

$$\langle F(\lambda) \rangle_M := \frac{1}{M} \sum_{m=0}^M [N_{\partial \Omega}^m(\lambda)]^m F(\lambda) [N_{\partial \Omega}^m(\lambda)]^m. \quad (21)$$
Then, by the Schwartz inequality,
\[
\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle F(\lambda_j) u_{\lambda_j}^b, u_{\lambda_j}^b \rangle|^2 \leq \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle F(\lambda_j) \rangle_M u_{\lambda_j}^b, u_{\lambda_j}^b \rangle|^2 \leq \frac{1}{N(\lambda)} \langle F_M(\lambda_j) u_{\lambda_j}^b, u_{\lambda_j}^b \rangle, \tag{22}
\]
where,
\[
F_M(\lambda) := \langle F(\lambda) \rangle_M^* \langle F(\lambda) \rangle_M. \tag{23}
\]
Under the almost nowhere commuting assumption, these operators \( F_M(\lambda) \) satisfy the assumptions of Theorem 9. The almost nowhere commuting condition (and its quantitative refinements) arises because the fixed point set of the canonical relation of (23) is
\[
\Sigma_{F_M}(\lambda) = \bigcup_{k=-M}^M \bigcup_{p \in \mathbb{Z}} \{ \zeta \in B^* \partial \Omega; \ \beta^p \beta^H_{\partial \Omega}(\zeta) = \beta^H_{\partial \Omega} \beta^k(\zeta) \} \tag{24}
\]
and the quantitative almost-nowhere commutativity condition (see Definition 13) implies that
\[
\limsup_{M \to \infty} \frac{1}{M} |\Sigma_{F_M}(\lambda)| = 0. \tag{25}
\]
This is sufficient to imply that the average over eigenvalues \( \lambda_j \) of (23) is \( O\left(\frac{1}{M}\right) \) and hence the almost orthogonality statement in Theorem 10.

Quite a lot might be lost in the use of the Schwartz inequality in (23). As remarked in §8.2, the almost nowhere commutativity condition is the condition under which averages over the spectrum of the norms \( ||\langle F_2(\lambda_j; a, \varepsilon) \rangle_M u_{\lambda_j}^b || \) tend to zero. It is another curious feature that we obtain a ‘subsequence of density one’ along which the almost orthogonality holds. It is not clear whether (as for quantum ergodicity) this is an essential feature or a defect of the proof.

1.6. Final introductory remarks. We close the introduction by posing the quantum ergodic restriction problem in a more general context, and by describing some continuing work on the general problem.

The quantum ergodic restriction problem concerns the linear functionals (2) defined on the space \( \Psi_{sc}^0(H) \) of zeroth order semi-classical pseudo-differential operators on \( H \). As we have seen, there may exist large subsequences of eigenfunctions which restrict to zero; on the other hand, the known restriction estimates (see [BGT, KTZ]) do not give uniform bounds on the \( L^2(H) \) norms of the restrictions. Hence the family \( \{ \rho_H^j \} \) (2) of functionals corresponding to Dirichlet (or Neumann) data of \( \Delta \)-eigenfunctions along \( H \) is not apriori a bounded family of functionals, hence not necessarily compact in the weak* topology. Nevertheless, we may pose the

**Problem 11.** Let \( H \subset M \) be a hypersurface. Determine the set \( \mathcal{Q}_H \) of ‘restricted quantum limits’, i.e. weak* limit points of the sequence of the (un-normalized) semi-positive states \( \{ \rho_H^j \} \) corresponding to the Dirichlet (or Neumann) data of eigenfunctions on \( H \). Similarly, determine the restricted quantum limits for the functionals defined (as in Theorem 4) by the full Cauchy data (5).

The analogue of an interior quantum ergodic sequence of eigenfunctions is given by:
Definition 12. The pair \((M, H)\) satisfies QER (quantum ergodic restriction) for Dirichlet data if \(Q_H\) contains a limit state \(\omega^H_\infty\) so that a full density subsequence of \(\rho^H_j \to \omega^H_\infty\) in the weak* sense. Similarly for Neumann and full Cauchy data.

In [TZ2], we prove a quantum ergodic restriction theorem for a general Riemannian manifold \((M, g)\) without boundary. In this setting, there do not exist transfer maps and broken billiard maps, and we base the proof on the almost nowhere equality of the left and right return maps to \(H\). We believe the condition is satisfied for closed geodesics of negatively curved surfaces which are not fixed by isometric involutions.

In [CHTZ] we use somewhat different methods from this article on manifolds with or without boundary to prove QER for the Cauchy data when the billiard flow is ergodic. We use two methods: one method (close to that of this article) to study the (matrix) Calderon projector \(C(\lambda) : \varphi_\lambda \to (\varphi_\lambda|_H, \partial_\nu \varphi_\lambda|_H)\) as a semi-classical Fourier integral operator quantizing a certain first return map to \(S^*_HM\) (the unit cotangent bundle of \(M\) along \(H\)). Our second method is a modification of the approach of [Bu, GL] in directly relating interior and restricted quantum ergodicity.

In further work in progress [CHTZ2] we are studying a more general microlocal setting in which \(H\) (more precisely, \(S^*_HM\)) is replaced by a general hypersurface \(T \subset S^*M\) which is (roughly speaking) a cross-section to the geodesic flow. Quantum ergodicity of Cauchy data should hold if the first return map for \(S^*_HM\) is ergodic. The relevant notion of Cauchy data is provided by a microlocal restriction operator to \(T\) that arises in Grushin reductions. It is then an interesting question to relate this abstract reduction and the conditions for QER to those obtained in this article and in [CHTZ] for the cross section defined by \(S^*_H\Omega\), the unit cotangent vectors with footpoints on \(H\).

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2. Classical billiards on a domain with corners

This section is a compilation of rather standard facts and definitions regarding billiards on domains with corners. The discussion does not involve \(H\) or any objects specific to our QER problem. However, we go over the basic definitions in some detail for the sake of clarity and to introduce notation that will be used in the rest of the article.

We assume throughout that \(\Omega \subset \mathbb{R}^m\) is a smooth domain with corners. We define a smooth domain with corners and with \(M\) boundary faces (hypersurfaces) to be a set of the form \(\{x \in \mathbb{R}^m : \rho_j \leq 0, j = 1, \ldots, M\}\), where the defining functions \(\rho_j\) are smooth in a neighborhood of \(\Omega\), and are independent in the following sense: at each \(p\) such that \(\rho_j(p) = 0\) for some finite subset \(j \in I_p\) of indices, the differentials \(d\rho_j\) are linearly independent for all \(j \in I_p\). A boundary hypersurface \(H_j\) is the intersection of \(\Omega\) with one of the hypersurfaces \(\{\rho_j = 0\}\). The boundary faces of codimension \(k\) are the components of \(\rho_{j_1} = \cdots = \rho_{j_k} = 0\) for some subset \(\{j_1, \ldots, j_k\}\) of the indices; each is a manifold with corners. This definition is extrinsic in the sense of [M] and is not the most general one possible.
It is essential to allow $\Omega$ to have corners since domains with ergodic billiard flow in $\mathbb{R}^n$ have corners. The singularities play a very significant role in the dynamics. They occur at corners, at trajectories which become tangent to the boundary, and in our setting of domains with obstacles, at trajectories which run tangent to the obstacle.

We denote the smooth part of $\partial \Omega$ by $(\partial \Omega)^o$. Here, and throughout this article, we denote by $W^o$ the interior of a set $W$ and, when no confusion is possible, we also use it to denote the regular set of $\partial \Omega$. Thus, $\partial \Omega = (\partial \Omega)^o \cup \Sigma$, where $\Sigma = \bigcup_{i\neq j} (H_i \cap H_j)$ is the singular set. When $\dim \Omega = 2$, the singular set is a finite set of points and the $H_i$ are smooth curves. In higher dimensions, the $H_i$ are smooth hypersurfaces; $H_i \cap H_j$ is a stratified smooth space of co-dimension one, and in particular $\Sigma$ is of measure zero. We denote by $S^*_x \Omega$ the set of unit vectors to $\Omega$ based at points of $\Sigma$. We also define $C^\infty(\partial \Omega)$ to be the restriction of $C^\infty(\mathbb{R}^n)$ to $\partial \Omega$.

We define the open unit ball bundle $B^*(\partial \Omega)^o$ to be the projection to $T^*\partial \Omega$ of the inward pointing unit vectors to $\Omega$ along $(\partial \Omega)^o$. We leave it undefined at the singular points.

To simplify the exposition, we describe $B^*\partial \Omega$ as if it consisted of tangent vectors rather than cotangent vectors, using the Euclidean metric to identify the two. We denote the standard projection by $\pi : B^*\partial \Omega \to \partial \Omega$.

2.1. **Billiard map.** At smooth points of $\partial \Omega$, the billiard map $\beta$ is defined on $B^*\partial \Omega$ as follows: given $(y, \eta) \in B^*\partial \Omega$, i.e. with $|\eta| < 1$ we let $(y, \zeta) \in S^*\Omega$ be the unique inward-pointing unit covector at $y$ which projects to $(y, \eta)$ under the map $T^*\partial \Omega \to T^*\partial \Omega$ (see (11)). Then we follow the geodesic (straight line)

\[(y, \eta) \in B^*(\partial \Omega)^o \to E(t, y, \eta) := q(y) + t\zeta(y, \eta) \in \Omega \subset \mathbb{R}^n \quad (26)\]

to the first place it intersects the boundary again; let $y' \in \partial \Omega$ denote this first intersection. Thus, we find the least $t = t(y, \eta)$ so that $\rho_j(E(t(y, \eta), y, \eta)) = 0$ for some $j$ and let $y' := E(y, \eta)$ solve the equation

\[q(y') = E(t(y, \eta), y, \eta).\]

The notation $E(t, y, \eta)$ is intended to suggest ‘exponential map’, i.e. the position in $\overline{\Omega}$ reached by the billiard trajectory at time $t$.

Denoting the inward unit normal vector at $y'$ by $\nu_{y'}$, we let $\zeta' = \zeta + 2(\zeta \cdot \nu_{y'})\nu_{y'}$ be the direction of the geodesic after elastic reflection at $y'$, and let $\eta'$ be the projection of $\eta'$ to $T^*_{y'}\partial \Omega$. Then we define explicitly,

\[\beta(y, \eta) = (y', \eta'). \quad (27)\]

We call each (linear) segment of a billiard trajectory a ‘link’, so that $\beta$ relates the initial and terminal directions of a link. We denote the first link of the billiard trajectory determined by $(y, \eta) \in B^*\partial \Omega$ by

\[(y, \eta). \quad (28)\]

We also denote by

\[G^r : S^*\Omega \to S^*\Omega \quad (29)\]
the (partially defined) broken geodesic flow in the full domain $S^*\Omega$. The trajectory $G^r(x, \xi)$ consists of the straight line motion between impacts with the boundary, and by the usual reflection law at the boundary. Thus, $\beta$ is the reduction to the boundary of $G^r$. We refer to [PS, SV] for background.
We now address the complications caused by tangent directions and singular points. The billiard map is not well-defined on initial directions which are tangent to \( \partial \Omega \), i.e. \((y, \eta) \in S^*\partial \Omega \) with \(|\eta| = 1\). If the domain is convex at \( y \), then the line \( q(y) + t\eta \) immediately exits the domain. On the other hand, if it is concave at \( y \), then the trajectory remains in the interior of \( \Omega \) and defines a link. If the domain is a ‘saddle surface’ at \( y \), then some tangential links stay in \( \Omega \) and others exit. We introduce some terminology to distinguish these cases.

**Definition 2.1.** Let \((y, \eta) \in S^*\partial \Omega \) be a unit tangent vector.

1. We say that \((y, \eta)\) is an exit direction, if the oriented ray \( q(y) + t\eta \) exits \( \Omega \) for \( t > 0 \). We denote the set of exit directions in \( B^*_y \partial \Omega \) by \( \mathcal{E}(y) \) and define \( \beta(y, \eta) = (y, \eta) \) for such exit directions.
2. We say that \((y, \eta)\) is an interior tangent direction, if \( q(y) + t\eta \) remains in \( \Omega \) for sufficiently small \( t > 0 \). We define \( \beta(y, \eta) = (y, \eta') \) by following \( q(y) + t\eta \) for \( t > 0 \) until its first point of intersection \( y' \in \partial \Omega \) and let \( \eta' \) be the tangential projection of \( \eta \) at \( y' \).

It is common to puncture out all tangent directions from \( B^* \partial \Omega \), i.e. all \( \eta \in S^*\partial \Omega \) with \(|\eta| = 1\) and only define billiard maps on the open unit ball. But it makes sense to retain unit tangent directions as long as the straight-line trajectory remains well defined in \( \Omega \). Unit tangent vectors to \( \partial \Omega \) at convex points can never be terminal velocity vectors of incoming trajectories and are therefore irrelevant to the definition of the billiard flow. However, when \( \dim \Omega \geq 3 \), as in the case where the boundary is a monkey saddle, it is possible for an incoming trajectory to arrive at an exit direction and terminate there.

We will also need to distinguish initial directions whose trajectories terminate in tangent vectors to \( \partial \Omega \).

**Definition 2.2.** Let \( \dim \Omega \geq 3 \). We denote by \( \mathcal{G}(y) \subset B^*(\partial \Omega)^o \) the set of grazing directions \( \eta \) such that \( \beta(y, \eta) \in S^*\partial \Omega \), i.e. such that the terminal vector of the link is tangent to \( \partial \Omega \).

No such exit directions exist when \( \dim \Omega = 2 \). We now consider the regularity properties of \( \beta \) and the impact of the corner set.

**Definition 2.3.** We denote by \( B^*_S \partial \Omega \) or equivalently by \( S^*_S \Omega \) the set of unit vectors to \( \Omega \) at points of the singular set. We define

\[
\mathcal{R}_1 = B^* (\partial \Omega)^o \setminus \mathcal{G},
\]

i.e. the set of smooth directions of links whose next collision with \( \partial \Omega \) is neither at \( \Sigma \) nor tangent to \( \partial \Omega \).

**Lemma 2.4.** The domain of \( \beta \) is \( \mathcal{R}_1 \cup \mathcal{G} \). It is a smooth symplectic map on the interior of \( \mathcal{R}_1 \), which is smooth domain with corners in \( B^*(\partial \Omega) \).

**Proof.** It is clear from the definition that \( \beta \) is well-defined on \( \mathcal{R}_1 \cup \mathcal{G} \) and on no larger domain. It is standard that \( \beta \) is symplectic where it is defined; see [CFS]. A simple way to prove it is that \( \beta \) has a generating function, namely the boundary distance function \( d_\Omega(y, y') = |q(y) - q(y')| \) on \( \partial \Omega \times \partial \Omega \). We have punctured out \( \mathcal{G} \) since \( \beta \) has jumps along this set, i.e. is discontinuous.

Regarding regularity, it is clear from the formulae [26]-[27] that \( E(t, y, \eta) \) and \( \beta(y, \eta) \) are smooth wherever \( \nu_y \) is smooth and \( t(y, \eta) \) is smooth. Of course, \( \nu_y \) is smooth away from the
corners. Clearly, \( t(y, \eta) \) is smooth as long as \( \nu_y \) is smooth, \( E(t(y, \eta), y, \eta) \notin \Sigma \) and as long as \( (y, \eta) \notin G \); at the latter points, \( t(y, \eta) \) is dis-continuous. Hence, \( \beta(y, \eta) \) is smooth on \( R^1 \) and on no larger set.

We observe that \( E^{-1}(\Sigma) \subset B^*\partial \Omega \) is locally defined as the simultaneous zero set of two smooth functions. Indeed, it the union of the sets \( \{(y, \eta) : E(y, \eta) \in H_i \cap H_j\} \). We may extend the hypersurfaces \( H_j \) beyond \( \Sigma \) and consider the travel time \( t_j \) from \( (y, \eta) \) to these hypersurfaces. I.e. we may extend the time functions \( t_j(y, \eta) \) as the smallest value of \( t \) so that \( \rho_j(E(t, y, \eta)) = 0 \). Then \( E^{-1}(H_i \cap H_j) \) is defined by \( t_i(y, \eta)t_j(y, \eta) \). Similarly for multiple intersections.

\[ \Box \]

2.2. Iterates of the billiard map. We now consider the inverse \( \beta^{-1} \) of \( \beta \) and their iterates. We note that the billiard flow is ‘time-reversal invariant’, i.e. \( \beta^{-1}(y, \eta) = -\beta(y, -\eta) \). Hence \( \beta^{-1} \) is well defined on the set \( R^{-1} \) of directions in \( B^*(\partial \Omega)^o \) such that \( E(y, -\eta) \notin \Sigma \). With a slight abuse of notation, we let \( \beta^{-1}(B^*_\Sigma) = E^{-1}(\Sigma) \), i.e. it is the set of directions which map to the corner set; strictly speaking, there we do not define the image directions there.

We now define open sets \( R^k \) such that \( R^1 \supset R^2 \supset \ldots, R^{-1} \supset R^{-2} \supset \ldots \) where \( \beta(k) \) is defined, for any integer \( k \).

**Definition 2.5.** For \( k \geq 1 \), we define the set \( R^k \) of \( k \)-regular directions such that the first \( k \) links of its trajectory stays in \( B^*(\partial \Omega)^o \backslash G \). For \( k \leq -1 \), we define it similarly for \( \beta^{-1} \).

The coball singular set is the set \( \Sigma^* := \bigcup_k \beta^{-k}(B^*_\Sigma \partial \Omega) \) and the \( N \)-th order coball singular set is \( \Sigma_N^* := \bigcup_{|k| \leq N} \beta^{-k}(B^*_\Sigma \partial \Omega) \).

**Definition 2.6.** We define the generalized grazing set to be

\[ G^*_k = \bigcup_{k \in \mathbb{Z}} \beta_k(S^* \partial \Omega). \]

Similarly, the \( N \)-th grazing set is denoted by

\[ G^*_N = \bigcup_{|k| \leq N} \beta_k(S^* \partial \Omega). \]

By repeating the proof of Lemma [2.4] we have:

**Lemma 2.7.** The domain of \( \beta^k \) is \( R^k \). It is a smooth symplectic map on the interior of \( R^k \).

The regularity of \( \beta^k \) near \( (y, \eta) \) is essentially the issue of the regularity of the time \( t^{(k)}(y, \eta) \) at which the billiard trajectory defined by \( (y, \eta) \) hits \( \partial \Omega \) for the \( k \)th time. It fails to be defined if the trajectory runs into \( \Sigma \) or becomes an exit direction before this time, and additionally fails to be smooth if the trajectory becomes tangent to \( \partial \Omega \) at time \( t^{(k)}(y, \eta) \).

Since the corner set and its images and pre-images under \( \beta \) is a countable union of stratified smooth hypersurfaces, \( \beta \) is a measure-preserving map on a full-measure subset of \( B^*(\partial \Omega)^o \).

3. Hypersurfaces \( H \): classical and quantum transfer maps

In this section, we define the quantum transfer operators \( N_{\delta \Omega}^H(\lambda) \) (resp. \( N_{\partial \Omega}^H(\nu) \) (see [13]), and show that they are semi-classical Fourier integral operators associated to the billiard relation given by transfer maps \( \tau_{H}^\partial \) (resp. \( \tau_{H}^{\partial \Omega} \)). As discussed in the introduction, the
quantum transfer operators are fundamental in our approach, which is based on transferring the expected value \( \langle Op_\lambda(a)u^H_\lambda, u^H_\lambda \rangle_{L^2(H)} \) to the boundary by the identity \([16]\).

The operator \( N^H_{\partial \Omega} (\lambda) : C^\infty (\partial \Omega) \rightarrow C^\infty (H) \) is defined as follows: By Green’s theorem, if \( \varphi_{\lambda_j} \) is a Laplace eigenfunction, then for any \( x \in \Omega^o \),

\[
\varphi_{\lambda_j} (x) = \int_{\partial \Omega} \left[ \partial_{\nu_q} G_0(x, q, \lambda_j) u^b_{\lambda_j} (q) - G_0(x, q, \lambda_j) \partial_{\nu_q} \varphi_{\lambda_j} (q) \right] d\sigma(q),
\]

where

\[
G_0(x, x', \lambda) = \frac{i}{4} \lambda^{n-2} (2\pi \lambda |x - x'|)^{-(n-2)/2} H_{a/2-1}(\lambda |x - x'|)
\]

is the free outgoing Green function on \( \mathbb{R}^n \). The Neumann (resp. Dirichlet) boundary conditions eliminate the second (resp. first) term, and by restricting \( z \) to the hypersurface in \([30]\), one gets the boundary integral equation

\[
u^H_{\lambda_j} = N^H_{\partial \Omega} (\lambda_j) u^b_{\lambda_j}.
\]

The transfer operator from boundary data to Dirichlet data along \( H \) is defined as follows:

**Definition 3.1.** \( N^H_{\partial \Omega} (\lambda) : C^\infty (\partial \Omega) \rightarrow C^\infty (H) \) is the operator whose Schwartz kernel with respect to the hypersurface volume element \( d\sigma(q) \) on \( (\partial \Omega)^o \) is given by,

\[
N^H_{\partial \Omega} (q_h, q, \lambda_j) = \begin{cases} 
\partial_{\nu_q} G_0(q_h, q, \lambda_j), & \text{Neumann} \\
G_0(q_h, q, \lambda_j), & \text{Dirichlet}
\end{cases}
\]

Similarly, the transfer operator from boundary data to Neumann data along \( H \) is given in:

**Definition 3.2.** With the same notation and assumptions, \( N^H_{\partial \Omega} (\lambda) : C^\infty (\partial \Omega) \rightarrow C^\infty (H) \) is the operator with Schwartz kernel

\[
N^H_{\partial \Omega} (q_h, q, \lambda_j) = \begin{cases} 
\partial_{\nu_q} \partial_{\nu_q} G_0(q_h, q, \lambda_j), & \text{Neumann} \\
\partial_{\nu_q} G_0(q_h, q, \lambda_j), & \text{Dirichlet}
\end{cases}
\]

We now prove that \( N^H_{\partial \Omega} (\lambda_j) \), \( N^H_{\partial \Omega} (\lambda_j) \) are semi-classical Fourier integral operators (with singularities) quantizing the partially defined multi-valued transfer map \( \tau^H_{\partial \Omega} \) between \( B^* \partial \Omega \) and \( B^* H \). This is the first place where the convexity of \( H \) is relevant.

We recall that a semiclassical (i.e. \( \lambda \))-Fourier integral operator, \( F(\lambda) : C^\infty (\partial \Omega) \rightarrow C^\infty (\partial \Omega) \), is an oscillatory integral operator whose Schwartz kernel can be written locally in the form

\[
F(y, y' ; \lambda) = (2\pi \lambda)^{n-1} \int_{\mathbb{R}^{n-1}} e^{i \lambda \Phi (q(y), q(y'), \theta)} A(q(y), q(y'), \theta, \lambda) d\theta
\]

where the amplitude is a semi-classical (polyhomogeneous) symbol in \( \lambda \), \( A(q(y), q(y'), \theta; \lambda) \sim \sum_{k=0}^{\infty} A_k (q(y), q(y'), \theta) \lambda^{n-k} \) with \( A_k \in C^\infty (\partial \Omega \times \partial \Omega \times \mathbb{R}^{n-1}) \), \( k = 0, 1, 2, \ldots \). The associated canonical relation is defined by

\[
\Gamma_F = \{ (y, \nabla_y \Phi, y', -\nabla_{y'} \Phi : \nabla_\theta \Phi = 0) \subset T^* \partial \Omega \times T^* \partial \Omega \}.
\]
PROPPOSITION 3.3. \( N_{\partial \Omega}^H(\lambda_j) \), resp. \( \lambda_j^{-1} N_{\partial \Omega}^H(\lambda_j) \), are semi-classical Fourier integral operators of order zero with canonical relation \( \mathcal{F} \). The principal symbol of \( N_{\partial \Omega}^H \) is the half density on the graph of \( \tau_{\partial \Omega}^H \) given at a point \( ((y, \eta), \tau_{\partial \Omega}^H(y, \eta)) \in \Gamma_{\partial \Omega}^H \) with \( (y, \eta) \in B^* \partial \Omega \) by

\[
\sigma(N_{\partial \Omega}^H(\lambda))(y, \eta) = \left( \frac{\gamma(y, \eta)}{\gamma(\tau_{\partial \Omega}^H(y, \eta))} \right)^{1/2} |dyd\eta|^{1/2}
\]

in the symplectic coordinates \((y, \eta)\) of \( B^* \partial \Omega \). Also,

\[
\sigma(\lambda^{-1} N_{\partial \Omega}^H(\lambda))(y, \eta) = i \left( \gamma(y, \eta) \gamma(\tau_{\partial \Omega}^H(y, \eta)) \right)^{1/2} |dyd\eta|^{1/2}.
\]

Remark: We observe that the symbol of \( N_{\partial \Omega}^H(\lambda) \) is infinite on the set where \( \gamma(\tau_{\partial \Omega}^H(y, \eta)) = 0 \). This is also true in the case \( H = \partial \Omega \) in [HZ]. We will cut off away from this set in the proof of the main theorems (see Proposition 4.2). The key point is that for a quantum ergodic sequence, eigenfunction mass cannot concentrate in this set (see Lemma 8.5 in subsection 8.4).

Proof
The proof of the Proposition consists of a sequence of Lemmas. We first observe that the quantum transfer maps are Fourier integral operators with phase equal to the distance function \(|q_H - q|\) with \( q_H \in H, q \in \partial \Omega \). We then discuss the corresponding canonical relation. In section 3.3 (see Lemma 3.6) we calculate the principal symbols.

Under our assumption that \( H \subset \Omega^o \), we have \(|q_H - q| \geq \text{dist}(H, \partial \Omega) > 0 \), and we may use the asymptotics of the Hankel function \( (31) \) to obtain,

\[
G_0(q_H(s), q(y); \lambda_j) \sim_{\lambda_j \to \infty} (2\pi \lambda_j)^{-\frac{n-3}{2}} e^{i\lambda_j |q_H(s)-q(y)|} \sum_{k=0}^{\infty} b_k(q_H(s), q(y)) \lambda_j^{-k}
\]

with \( b_k \in C^\infty(H \times \partial \Omega) \) and \( b_0 > 0 \). The expansions for \( \partial_{q_H} G_0(q_H(s), q(y); \lambda_j) \) and of \( \partial_{q_H} \partial_{q_H} G_0(q_H(s), q(y); \lambda_j) \) are of the same form and are computed by taking the normal derivatives of the expansion in \((36)\). Thus, the operators in Definitions \( 3.1-3.2 \) are semi-classical Fourier integral operators with phase equal to the distance function \(|q_H(s) - q(y)|\) between a point of \( H \) and a point of \( \partial \Omega \). The canonical relation is given by

\[
WF_\lambda(N_{\partial \Omega}^H) \subset \{(s, \pi_{q_H(s)}^T r(q_H(s), q(y)), y, \pi_{q(y)}^T r(q_H(s), q(y)))\} \subset B^* H \times B^* \partial \Omega,
\]

where,

\[
r(q_H, q) := \frac{q_H - q}{|q_H - q|}.
\]

The canonical relation in \((37)\) is easily seen to be the same as \((8)\).

Remark: This statement is simpler than the one for the boundary integral operator \( N_{\partial \Omega}^H(\lambda) \) \((18)\) in [HZ]. In that case, the phase and amplitude are singular on the diagonal and have to be regularized. Here, the operator kernel is smooth away from the corners of \( \partial \Omega \). The singularities at the corners are only a technical inconvenience in the quantum ergodic theorems, and we refer to [HZ] for a more detailed discussion of them.
3.1. Details on the transfer map $\tau_{\partial \Omega}^H$. We now describe in more detail the canonical relation (37) and the transfer map $\tau_{\partial \Omega}^H$ of Definition 2. As mentioned in the introduction, the definition of this map involves several technicalities:

- The transfer map $\tau_{\partial \Omega}^H$ is only partially defined on $B^* \partial \Omega$, namely on the range of the adjoint transfer map $\tau_{\partial \Omega}^{\partial H} : B^* H \to B^* \partial \Omega$.
- $\tau_{\partial \Omega}^H$ is multi-valued since the trajectory $(y, \eta)$ defined by $(y, \eta) \in B^* \partial \Omega$ may intersect $H$ generically in multiple points.
- It has singularities when the trajectory $(y, \eta)$ from $(y, \eta) \in B^* \partial \Omega$ is tangent to $H$ or when the trajectory is tangent to $\partial \Omega$. Such a trajectory may go on to intersect $H$ again.
- The adjoint transfer map $\tau_{\partial \Omega}^{\partial H} : B^* H \to B^* \partial \Omega$ is singular when a ray from $B^* H$ runs into a corner.

We now introduce notation to deal with these issues. At this point, we assume that $H$ is convex. With this assumption, a generic line intersects $H$ at most twice (indeed, it only intersects it more than twice if in intersects it in a flat part).

**Definition 3.4.** The domain $\mathcal{C}$ of $\tau_{\partial \Omega}^H$ is the set of $(y, -\eta) \in B^*(\partial \Omega)^o$ such that the link $(y, -\eta)$ (i.e. $E(t, y, -\eta)$) intersects $H$ (see (28)). Let $\mathcal{C}^o \subset \mathcal{C}$ denote the open dense subset such that $(y, -\eta) \cap H$ consists of two distinct points $\{q_1^H(y, -\eta), q_2^H(y, -\eta)\}$, ordered so that $|q_1^H(y, -\eta) - y| < |q_2^H(y, -\eta) - y|$, i.e. so that the line intersects $H$ first at $q_1^H(y, -\eta)$ and second at $q_2^H(y, -\eta)$.

Let $\mathcal{T} \subset \mathcal{C}$ be the set of directions $\eta$ such that $(y, -\eta)$ is tangent to $H$ at its points of intersection.

If $(y, -\eta) \in \mathcal{C}^o = \mathcal{C} \setminus \mathcal{T}$, let $t_1^H(y, \eta) < t_2^H(y, \eta)$ be the times of intersection with $H$.

The reason for the minus sign in $-\eta$ is that the transfer map follows the trajectory defined by $(y, \eta)$ in the backwards direction (see (13)). The time reversal is the map $\eta \to -\eta$. Since $H$ is smooth, the time functions $t_j^H(y, \eta)$ are smooth functions of their arguments in $\mathcal{C}^o$.

3.2. Formulae for the branches of the transfer map. In this section, we give explicit formulae for the branches of the transfer map away from $\mathcal{T}$. They are needed in [33] in the proof of the genericity of the condition on $H$.

We define branches $\tau_{\partial \Omega}^{H,1}(y, \eta)$ and $\tau_{\partial \Omega}^{H,2}(y, \eta)$ of $\tau_{\partial \Omega}^H$ for $(y, \eta) \notin \mathcal{T}$ corresponding to the two intersection points with $H$. For $(y, \eta) \in \mathcal{C}^o$, we define the once $H$-broken trajectories $E_{H,j}(t, y, \eta)$ with breaks at $q_j^H(y, \eta)$ and intersection times $t_j^H(y, \eta)$ by

$$E_{H,j}(t, y, \eta) := \begin{cases} E(t, y, -\eta), & t \leq t_j^H(y, \eta); \\ E(t_j^H(y, \eta), y, \eta) + (t - t_j^H(y, \eta)) \zeta_j^H(y, -\eta), & t \geq t_j^H(y, \eta) \end{cases}$$

where

$$\zeta_j^H(y, -\eta) = -\zeta(y, -\eta) + 2(\zeta \cdot \nu_{q_j^H(y, \eta)}) \nu_{q_j^H(y, \eta)}.$$  \hspace{1cm} (38)

As above, the $-\eta$ is needed since the trajectory proceeds backwards from the initial conditions. The unit normal is understood to be on the $+\text{ side of } N^* H$.

After the break, the trajectory proceeds on the straight line $E(t_j^H(y, \eta), y, \eta) + (t - t_j^H(y, \eta)) \zeta_j^H(y, \eta)$ until it intersects the boundary again at a point $q_{H,\partial \Omega}^j(y, \eta)$. We denote
by $t^j_{H,\partial \Omega}(y,\eta)$ the time at which this occurs. The following is a more detailed version of Definition 2.

**Definition 3.5.** Define (see (38)) $q^j_H(y,\eta)$ to be the solution of the equation

$$q_H(q^j_H(y,\eta)) = E(t^j_H(y,\eta), y, \eta) + (t^j_{H,\partial \Omega}(y, \eta) - t^j_H(y, \eta))\zeta^j_H(y, \eta).$$

If $q^j_{H,\partial \Omega}(y, \eta) \notin \Sigma$, put

$$\tau^j_{\partial \Omega}(y, \eta) = (\eta, \pi_{\Omega}(q^j_{H,\partial \Omega}(y, \eta)), r(q^j_{H,\partial \Omega}(y, \eta)), E(t^j_H(y, \eta), y, -\eta)).$$

Also define the correspondence

$$\tau^j_{\partial \Omega}(y, \eta) = \{\tau^{H,1}_{\partial \Omega}(y, \eta), \tau^{H,2}_{\partial \Omega}(y, \eta)\}.$$

In the notation of [Z], this double-valued correspondence would be written in the additive notation,

$$\tau^j_{\partial \Omega}(y, \eta) = \tau^{H,1}_{\partial \Omega}(y, \eta) + \tau^{H,2}_{\partial \Omega}(y, \eta).$$

(39)

If $(y, \eta) \in \mathcal{T}$, then its trajectory runs tangent to $H$, and it proceeds on a straight line to $\partial \Omega$. We then define $\tau^{H,1}_{\partial \Omega}(y, \eta) = \tau^{H,2}_{\partial \Omega}(y, \eta) = \beta(y, \eta)$, the usual billiard map. Thus, the graph $\Gamma^{H,j}_{\partial \Omega} \subset B^*\partial \Omega \times B^*\partial \Omega$ of the indicated partially defined symplectic map is given by

$$\Gamma^{H,j}_{\partial \Omega} = \Gamma^{H,1}_{\partial \Omega} \cup \Gamma^{H,2}_{\partial \Omega},$$

(40)

where the two graphs on the right intersect along their common values for $(y, \eta) \in \mathcal{T}$.

As in the discussion of $\beta$, we note that $\tau^{H,1}_{\partial \Omega}$ and $\tau^{H,2}_{\partial \Omega}$ are piecewise smooth maps. It follows easily that the domain of definition of $\tau^{H,j}_{\partial \Omega}$ is $C\backslash E^{-1}_{H,j}(\Sigma)$ and each map is smooth on the interior of its domain.

In the case of more general hypersurfaces, billiard trajectories from $\partial \Omega$ may intersect $H$ in more points, and may intersect both tangentially and non-tangentially. As mentioned above, it is unnecessary to deal with such complications since the QER theorem may be proved for symbols $a$ with arbitrarily small microsupport on $B^*H$.

3.3. **Symbols of $N^H_{\partial \Omega}$ and $N^{H,\nu}_{\partial \Omega}$.** We first recall the definition of the symbol of a semi-classical Fourier integral operator $F(\lambda) \in I^0(M \times M, C)$ where $\lambda \geq \lambda_0$ is the inverse semi-classical parameter. It is a smooth section of the bundle $\Omega_{1/2} \otimes L$ of half-densities on $C$ and the Maslov line bundle.

In a local Fourier integral representation $F(\lambda)(x, y) = (2\pi \lambda)^{-n} \int_{\mathbb{R}^n} e^{-i\lambda \varphi(x,y,\xi)} a(x, y, \xi, \lambda) d\xi$ with $a(x, y, \xi, \lambda) \sim_{\lambda \to \infty} \sum_{j=0}^{\infty} a_j(x, y, \xi) \lambda^{-j}$, $a_j(x, y, \xi) \in S^{-j}_{1,0}$, the symbol is transported from the critical set

$$C_\varphi = \{(x, y, \xi) : d_\xi \varphi(x, y, \xi) = 0\} \subset M \times M \times \mathbb{R}^n$$

by the Lagrange immersion $i_\varphi$

$$i_\varphi : C_\varphi \to C, \ (x, y, \xi) \to (x, \varphi_x, y, -\varphi_y).$$

On $C_\varphi$ we form the Leray density

$$d_{C_\varphi} = \frac{dx \wedge dy \wedge d\xi}{dt(\frac{d\xi}{\xi})}$$
At a point \((x_0, \xi_0, y_0, \eta_0) \in C\), the symbol of \(F\) is defined as
\[
\sigma_F(x_0, \xi_0, y_0, \eta_0) = i r * a_0 \sqrt{d_{C_r}}.
\]

**Lemma 3.6.** The principal symbol of \(N^H_{\partial\Omega}\) is the half density on the graph of \(\tau^H_{\partial\Omega}\) given at a point \(((y, \eta), \tau^H_{\partial\Omega}(y, \eta)) \in \Gamma^H_{\partial\Omega}\) with \((y, \eta) \in B^* \partial\Omega\) by
\[
\sigma(N^H_{\partial\Omega}(\lambda))(y, \eta) = \left( \frac{\gamma(y, \eta)}{\gamma(\tau^H_{\partial\Omega}(y, \eta))} \right)^{1/2} |dyd\eta|^{1/2}
\]
in the symplectic coordinates \((y, \eta)\) of \(B^* \partial\Omega\).

Also,
\[
\sigma(\lambda^{-1} N^H_{\partial\Omega}(\lambda))(y, \eta) = i \left( \frac{\gamma(y, \eta)}{\gamma(\tau^H_{\partial\Omega}(y, \eta))} \right)^{1/2} |dyd\eta|^{1/2}.
\]

*Proof.* The calculation of the symbol of \(N^H_{\partial\Omega}(\lambda)\) is similar to that of \(N^H_{\partial\Omega}(\lambda)\) in Proposition 6.1 of [HZ], and we follow that calculation closely, referring there for some of the details and emphasizing only the steps where something new occurs. From the asymptotics of the Hankel function,
\[
\mathbf{H}_{n/2-1}(t) \sim e^{-i(n-1)\pi/4} e^{it} \sum_{j=0}^{\infty} a_j t^{-1/2-j},
\]
the principal symbol of \(N^H_{\partial\Omega}(\lambda)\) at the billiard Lagrangian is the same (up to an eighth root of unity) as that of
\[
(2\pi)^{-(n-1)/2} (n-1)/2 e^{i\lambda q(y) - q_H(s)} |q(y) - q_H(s)|^{-(n-1)/2} \partial_{\nu_y} |q(y) - q_H(s)|,
\]
where \(q(y) \in \partial\Omega, q_H(s) \in H\), and where \(\partial_{\nu_y} |q(y) - q_H(s)|\) is the directional derivative taken in \(\mathbb{R}^n \times \mathbb{R}^n\). Hence the symbol is given by
\[
|q(y) - q_H(s)|^{-(n-1)/2} \partial_{\nu_y} |q(y) - q_H(s)| |dyds|^{1/2},
\]
where we use \((y, s) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}\) as coordinates on the graph of \(\tau^H_{\partial\Omega}\).

To express the graph half-density in terms of \(d\eta d\eta\) we need to express \(ds\) in terms of \(d\eta\), keeping \(y\) fixed. Along the graph of \(\tau^H_{\partial\Omega}\), \(\eta = d_y |q(y) - q_H(s)|\) where now the derivative is only along \(\partial\Omega\). Hence,
\[
|d\eta| = \det \left( \frac{\partial^2}{\partial Y_i \partial S_j} [(q(y) + Y_i e_i) - (q_H(s) + S_i e_i')] \right)_{S=Y=0} |ds|,
\]
where \(e_i\) is an orthonormal basis for \(T_y \partial\Omega\), and \(e'_i\) an orthonormal basis for \(T_s H\). This is the same type of determinant as in [HZ], Proposition 6.1, except that \(q_H(s) \in H\) rather than in \(\partial\Omega\). As in (loc. cit.) we first compute the determinant in the two dimensional case. We choose coordinates so that \(q(y) = (0,0), q_H(s) = (0, r), e_1 = (\cos \alpha, \sin \alpha)\) and \(e_2 = (\cos \beta, \sin \beta)\). Then the determinant is
\[
\frac{\partial^2}{\partial Y \partial S} |(Y \cos \alpha - S \cos \beta, Y \sin \alpha - r - S \sin \beta)|_{S=Y=0},
\]
which equals
\[
r^{-1} \cos \alpha \cos \beta = |q(y) - q_H(s)|^{-1} \partial_{\nu_y} |q(y) - q_H(s)| \partial_{\nu_x} |q(y) - q_H(s)|.
\]
Taking this number to the power $-1/2$ and substituting in (41), we find that the factors of $|q(y) - q_H(s)|^{-1/2}$ cancel, and that there is a half power cancellation in $\partial_{\nu_s}|q(y) - q_H(s)|$, so that the symbol is given by

$$
\left(\frac{d_{\nu_s}|q(y) - q_H(s)|}{|d_{\nu_s}|q(y) - q_H(s)|}\right)^{1/2} |d\gamma|^{1/2} = \left(\frac{\gamma(y, \eta)}{\gamma(\tau_{\partial\Omega}^H(y, \eta))}\right)^{1/2} |d\gamma|^{1/2}. \quad (42)
$$

To complete the calculation, we observe that $\partial_{\nu_s}|q(y) - q_H(s)| = \gamma(s, \tau)$, and $\partial_{\nu_s}|q(y) - q_H(s)| = \gamma(y, \eta)$, and this gives the symbol of $N_{\partial\Omega}^H$ in dimension two.

In higher dimensions, as in [HZ], we use the subspace $T_{q(y)}\partial\Omega \cap T_{q_H(s)}H \cap \ell_{q,s}$, where $\ell_{q,s}$ is the line joining $q(y)$ and $q_H(s)$. We then introduce the same coordinates as in [HZ] and find that in the $n$-dimensional case, the determinant equals $r^{-(n+1)} \cos \alpha \cos \gamma$ where $e_1 = (\cos \alpha, 0, \sin \alpha), e_2 = (0, 1, 0)$ are the first two elements of an orthonormal basis of $T_{q(y)}\partial\Omega$ and $e'_1 = (\cos \gamma \cos \beta, \cos \gamma \sin \beta, \sin \gamma), e'_2 = (-\sin \beta, \cos \beta, 0)$ are the first two elements of an orthonormal basis of $T_{q_H(s)}H$. The factors of $|q(y) - q_H(s)|$ again cancel out due to the factor of $r^{-n+1}$. The cosine factors also produce half-cancellations as in the two-dimensional case, leaving the stated symbol.

The calculation of the symbol of $N_{\partial\Omega}^{H\nu}$ is similar except that the kernel has one further derivative, and the leading term arises by differentiating the exponent in $\nu$. So its symbol is that of $N_{\partial\Omega}^H$ multiplied by $i\lambda \partial_{\nu_s}|q_H(s) - q(y)|$, where $s \in H, y \in \partial\Omega$ and the derivative is computed in $\mathbb{R}^n \times \mathbb{R}^n$. In terms of the symplectic coordinates $(y, \eta)$ on the boundary, the symbol of $N_{\partial\Omega}^{H\nu}$ is the product of the symbol of $N_{\partial\Omega}^H$ by the additional factor of $i\lambda \partial_{\nu_s}|q(y) - q_H(s)|$.

\[ \square \]

This completes the proof of Proposition 3.3. Q.E.D.

4. $F(\lambda) = N_{\partial\Omega}^H(\lambda)^*Op_\lambda(a)N_{\partial\Omega}^H(\lambda)$ and $F^\nu(\lambda) = N_{\partial\Omega}^{H\nu}(\lambda)^*Op_\lambda(a)N_{\partial\Omega}^{H\nu}(\lambda)$

The purpose of this section is to prove that the compositions

$$
F(\lambda) := N_{\partial\Omega}^H(\lambda)^*Op_\lambda(a)N_{\partial\Omega}^H(\lambda), \quad \text{resp.} \quad F^\nu(\lambda) := N_{\partial\Omega}^{H\nu}(\lambda)^*Op_\lambda(a)N_{\partial\Omega}^{H\nu}(\lambda)
$$

have the decomposition as described in (17). The precise statement is given in Proposition 4.2. The details and statements are the same for $F(\lambda)$ and $F^\nu(\lambda)$ except for the calculation of the symbols. Hence, we sometimes give them only for $F(\lambda)$.

By Proposition 3.3 and the calculus of Fourier integral operators, $F(\lambda)$ would be a Fourier integral operator associated to the composite canonical relation $(\Gamma_{\beta}^H)^* \circ \Gamma_{\beta}^H$ if the composition were non-degenerate (or clean); see [DS]. The composition is degenerate along the tangential set, and so in analogy with the boundary case in [HZ] we must take care to avoid both the tangential and singular sets. We handle these problems by using appropriate cutoffs to remove these thin subsets. Thus, we first consider $\lambda$-pseudodifferential operators $Op_\lambda(a) \in Op_\lambda(S^{0,0}_{\partial\Omega}(T^*H \times [0, \lambda_0^{-1}])))$ with supp $a \cap \tau_{\partial\Omega}^H(T \cup \Sigma \cup G) = \emptyset$ for $k = 1, 2$; here as above, $\tau_{\partial\Omega}^H : B^*\partial\Omega \to B^*H$ is the transfer billiard map. As in [HZ], a density argument shows that such operators suffice to prove Theorem 4 in the general case. With this in mind, we
henceforth assume that the symbol $a \in S^{0,0}_c(T^* H \times [0, \lambda_0^{-1}])$ satisfies the following support condition: For arbitrarily small fixed $\varepsilon > 0$ and any $(s_0, \tau_0) \in \tau^H_{\partial \Omega}(T \cup G \cup B^*_2 \partial \Omega)$,

$$a(s, \tau; \lambda) = 0,$$

for all $(s, \tau) \in B^* H$ with $|s - s_0|^2 + |\tau - \tau_0|^2 < \varepsilon^2$ and $\lambda \geq \lambda_0$.

4.1. **The canonical relation of** $F(\lambda)$. Temporarily ignoring the complications involving singular and tangential rays, the phase of the composition is $\Psi(s, y, y') = |q(y) - q_H(s)| - |q_H(s) - q(y')|$ with $q_H(s) \in H, q(y), q(y') \in \partial \Omega$. Hence the critical set of the phase $\Psi$ is

$$C_{\Psi} = \{(s, y, y'); \nabla_s (|q(y) - q_H(s)| - |q_H(s) - q(y')|) = 0\}$$

$$= \{(s, y, y'); \langle r(q_H(s), q(y)), T_s \rangle = \langle r(q_H(s), q(y')), T_s \rangle\}.$$  

The canonical relation of $F(\lambda)$ is then

$$\Gamma_{F(\lambda)} = \{(y, \pi^y y r(q(y), q_H(s)), y', -\pi^y y r(q(y'), q_H(s)); (s, y, y') \in C_{\Psi}\}.$$  

One component of $\Gamma_{F(\lambda)}$ consists of the diagonal $\Delta_{\mathcal{C} \times \mathcal{C}} \subset \mathcal{C} \times \mathcal{C}$ (see Definition 3.4). Under the convexity assumption on $H$, this component occurs with multiplicity two since there are two points of $H$ where the ray my intersect $H$ and then reflect to its initial position. Two other ‘components’ of $\Gamma_{F(\lambda)}$ (which meet along rays tangential to $H$) come from once-broken trajectories from $(y, \eta) \in B^* \partial \Omega$ to $(y', \eta') \in \partial \Omega$ with the break at $q_H(s) \in H$. There exist (at most) two possible break points corresponding to the (at most) two possible intersection points of a line with $H$.  

We denote the partial symplectic maps defined by the once-broken trajectories by $\beta_1^H$, resp. $\beta_2^H$, where the index indicates the first, resp. second point of intersection with $H$. As in the introduction (see (13)) we refer to these partial symplectic maps as transmission maps. Summing up, we have

**Lemma 4.1.** Assuming $H$ is convex, the canonical relation underlying $N^H_{\partial \Omega}(\lambda)^* Op(a) N^H_{\partial \Omega}(\lambda)$ (and of $N^H_{\partial \Omega}(\lambda)^* Op(a) N^H_{\partial \Omega}(\lambda)$) equals

$$\Gamma_{\pi^\mu_{\partial \Omega}} \circ \Gamma_{\tau^H_{\partial \Omega}} = \Delta_{\mathcal{C} \times \mathcal{C}} \cup \Gamma_1^H \cup \Gamma_2^H$$

which consists of three branches:

- The identity branch on $\mathcal{C}$ (with multiplicity two);
- The graphs of $\beta_1^H$ and $\beta_2^H$; the index specifies the point of intersection with $H$.

The case of general smooth $H$ is of a similar nature but with more branches. We therefore omit the details.

4.2. **The decomposition of** $F(\lambda)$. We now give a precise proof that deals with the $\lambda$-pseudodifferential factor, the tangential degeneracy and the singular points. We continue to assume that $H$ is convex, but again, this is for notational simplicity. The $\lambda$-FIO contribution to $N^H_{\partial \Omega}(\lambda)^* Op_{\lambda}(a) N^H_{\partial \Omega}(\lambda)$ is discussed in subsection 4.2.1.

Let $\delta(\varepsilon) > 0$ be small (to be specified later on) and let $\chi \in C_0^\infty(\mathbb{R})$ with $\chi(x) = 1$ when $|x| \leq 1/2$ and $\chi(x) = 0$ for $|x| \geq 1$. We define the rescaled cutoff function $\chi_{\delta(\varepsilon)}(x) := \chi(\frac{x}{\delta(\varepsilon)})$. The point of the next Proposition is to show that by choosing symbols $a$ satisfying (41) (ie. with support disjoint from the image under the transfer map of the generalized grazing and singular sets), the corresponding $F_1(\lambda; a, \varepsilon)$-operator in the decomposition (48)
is, modulo a residual operator, a $\lambda$-pseudodifferential operator of order zero with principal symbol $(\tau_{\partial\Omega}^{\lambda})^*a_0\cdot|\sigma(N_{\partial\Omega}^{\lambda})|^2$. Following \cite{HZ} we say that a $\lambda$-dependent operator $R$ is residual provided it is smoothing and $|\partial_x^j\partial_y^j R(x, y; \lambda)| = O_{\alpha, \beta}(\lambda^{-\infty})$. The $F_2(\lambda; a, \varepsilon)$-operator in the decomposition \cite{HN} is studied subsequently in subsection 4.2.1. It is a sum of two zeroth-order $\lambda$-Fourier integral operators.

**Proposition 4.2.** Suppose that $H$ is convex and that $a \in S^{0,0}(T^*H \times (0, \lambda_{-1}^{-1}])$ with $a \sim \sum_{j=0}^{\infty} a_j\lambda^{-j}$ satisfies the support condition \cite{[44]}44]. Then, for any $\varepsilon > 0$

$$(i) \quad N_{\partial\Omega}^{H}(\lambda)^*Op_{\lambda}(a)N_{\partial\Omega}^{H}(\lambda) = Op_{\lambda}(a_0(\tau_{\partial\Omega}^{H}(y, \eta)) \cdot \gamma(y, \eta) \cdot \gamma^{-1}(\tau_{\partial\Omega}^{H}(y, \eta)))$$

$$+ F_{21}(\lambda; a, \varepsilon) + F_{22}(\lambda; a, \varepsilon) + R(\lambda; \varepsilon),$$

$$(ii) \quad \lambda^{-2}N_{\partial\Omega}^{H}(\lambda)^*Op_{\lambda}(a)N_{\partial\Omega}^{H}(\lambda) = Op_{\lambda}(a_0(\tau_{\partial\Omega}^{H}(y, \eta)) \cdot \gamma(y, \eta) \cdot \gamma(\tau_{\partial\Omega}^{H}(y, \eta)))$$

$$+ F_{21}'(\lambda; a, \varepsilon) + F_{22}'(\lambda; a, \varepsilon) + R'(\lambda; \varepsilon). \quad (46)$$

In both cases $F_{21}(\lambda; a, \varepsilon)$ (resp. $F_{21}'(\lambda; a, \varepsilon)$) and $F_{22}(\lambda; a, \varepsilon)$ (resp. $F_{22}'(\lambda; a, \varepsilon)$) are $\lambda$-Fourier integral operators of order zero with $\kappa(F_{21}(\lambda; a, \varepsilon)) = \kappa(F_{21}'(\lambda; a, \varepsilon)) = \beta_H$ and $\kappa(F_{22}(\lambda; a, \varepsilon)) = \kappa(F_{22}'(\lambda; a, \varepsilon)) = \beta_H$ and max $(\|R(\lambda; \varepsilon)\|_{L^2 \rightarrow L^2}, \|R'(\lambda; \varepsilon)\|_{L^2 \rightarrow L^2}) = O_{\varepsilon}(\lambda^{-1}).$

**Remark:** The case of a general smooth $H$ is similar but with a number of branches depending on $(y, \eta)$. We omit the details.

**Proof.** Consider the operator $N_{\partial\Omega}^{H}(\lambda)^*Op_{\lambda}(a)N_{\partial\Omega}^{H}(\lambda)$ with Schwartz kernel $N(q_H, q; \lambda) = \partial_{q_H}G_0(\eta_H, q; \lambda)$. By an integration by parts argument, it follows that

$$WF_{\lambda}(N_{\partial\Omega}^{H}(\lambda)) \subset B^{*}\partial\Omega \times B^{*}H. \quad (47)$$

Using $\chi_{\delta(\varepsilon)}$ we decompose the Schwartz kernel of $N_{\partial\Omega}^{H}(\lambda)^*Op_{\lambda}(a)N_{\partial\Omega}^{H}(\lambda)$ into near-diagonal and complimentary terms:

$$N_{\partial\Omega}^{H}(\lambda)^*Op_{\lambda}(a)N_{\partial\Omega}^{H}(\lambda)(q, q') = F_1(\lambda; a, \varepsilon)(q, q') + F_2(\lambda; a, \varepsilon)(q, q'); \quad (q, q') \in \partial\Omega \times \partial\Omega, \quad (48)$$

where,

$$F_1(\lambda; a, \varepsilon)(q, q') := N_{\partial\Omega}^{H}(\lambda)^*Op_{\lambda}(a)N_{\partial\Omega}^{H}(\lambda)(q, q') \chi_{\delta(\varepsilon)}(|q - q'|) \quad (49)$$

and

$$F_2(\lambda; a, \varepsilon)(q, q') := N_{\partial\Omega}^{H}(\lambda)^*Op_{\lambda}(a)N_{\partial\Omega}^{H}(\lambda)(q, q') \chi_{\delta(\varepsilon)}(|q - q'|). \quad (50)$$

We also introduce a new cutoff $\Xi_{\varepsilon} \in C_0^\infty(B_{1+\varepsilon}^{*}H)$ with $\Xi_{\varepsilon}(y, \eta) = 1$ for $(y, \eta) \in B_{1+\varepsilon/2}^{*}H$. Then by the calculus of wave-front sets, we have

$$F_1(\lambda; a, \varepsilon)(q, q') = N_{\partial\Omega}^{H}(\lambda)^*Op_{\lambda}(\Xi_{\varepsilon})Op_{\lambda}(a)N_{\partial\Omega}^{H}(\lambda)(q, q') \chi_{\delta(\varepsilon)}(|q - q'|) + O_{\varepsilon}(\lambda^{-\infty})$$

uniformly for $(q', q) \in \partial\Omega \times \partial\Omega$. By \cite{[14]}14 it suffices to assume that $a \in C_0^\infty(B_{1+\varepsilon}^{*}H)$ and satisfies the support condition \cite{[44]}44] and we will do so from now on.

From Proposition 3.3 we know that $N_{\partial\Omega}^{H}(\lambda)$ is a zeroth-order $\lambda$-Fourier integral operator with canonical relation $\Gamma_{\tau_{\partial\Omega}^{H}}^{H}$. Differentiation of the phase function $\Psi(y, y', s) = |q_H(s) - q(y)| - |q_H(s) - q(y')|$ in $s$ and a repeated integration by parts implies that for some constant $C > 0$,

$$WF_{\lambda}(F_1(\lambda; a, \varepsilon)) \subset \{(y, \xi, y', \eta) \in B^{*}\partial\Omega \times B^{*}\partial\Omega; |y - y'| \leq \delta(\varepsilon), |\xi - \eta| \leq C\delta(\varepsilon)\}.$$
Since (see section 4.1),
\[(\Gamma_{\beta_H^1} \cup \Gamma_{\beta_H^2}) \cap \Delta_{c \times c} \subset \mathcal{T} \cup \mathcal{G},\]
and \(\text{dist}(\text{supp} \, a, \tau^H_{\partial\Omega} (\Sigma \cup \mathcal{T} \cup \mathcal{G})) \geq \varepsilon,\) it follows that by choosing \(\delta(\varepsilon) > 0\) sufficiently small,
\[WF_{\lambda}^\varepsilon(F_1(\lambda; a, \varepsilon)) \cap (\Gamma_{\beta_H^1} \cup \Gamma_{\beta_H^2}) = \emptyset.\]
Thus, in view of Lemma 4.1,
\[WF_{\lambda}^\varepsilon(F_1(\lambda; a, \varepsilon)) \subset \Delta_{c \times c^*}.\]  
(51)

By the semiclassical Egorov theorem [DS], \(F_1(\lambda; \varepsilon) \in \Psi^0_{sc}(\partial\Omega),\) and moreover,
\[F_1(\lambda; a, \varepsilon) = Op_{\lambda}( (\tau^H_{\partial\Omega} \ast a_0 \cdot |\sigma(N^H_{\partial\Omega}(\lambda))|^2) + O_{\varepsilon}(\lambda^{-1})_{L^2 \rightarrow L^2}. \]  
(52)

From Lemma 3.6 \(|\sigma(N^H_{\partial\Omega}(\lambda)(y, \eta))|^2 = \gamma(y, \eta) \times \gamma^{-1}(\tau^H_{\partial\Omega}(y, \eta))\) and so the formula in (52) implies that
\[F_1(\lambda; a, \varepsilon) = Op_{\lambda}( a_0(\tau^H_{\partial\Omega}(y, \eta)) \cdot \gamma(y, \eta) \cdot \gamma^{-1}(\tau^H_{\partial\Omega}(y, \eta))) + O_{\varepsilon}(\lambda^{-1})_{L^2 \rightarrow L^2}. \]  
(53)

In case (ii) one makes the analogous decomposition
\[N^H_{\partial\Omega}(\lambda) \ast Op_{\lambda}(a)N^H_{\partial\Omega}(\lambda)(q, q')F_1^\nu(\lambda; a, \varepsilon)(q, q') + F_2^\nu(\lambda; a, \varepsilon)(q, q') ; \quad (q, q') \in \partial\Omega \times \partial\Omega \]  
(54)
where,
\[F_1^\nu(\lambda; a, \varepsilon)(q, q') := N^H_{\partial\Omega}(\lambda) \ast Op_{\lambda}(a)N^H_{\partial\Omega}(\lambda)(q, q') \times \chi_{\delta(\varepsilon)}(|q - q'|) \]  
(55)
and
\[F_2^\nu(\lambda; a, \varepsilon)(q, q') := N^H_{\partial\Omega}(\lambda) \ast Op_{\lambda}(a)N^H_{\partial\Omega}(\lambda)(q, q') \times (1 - \chi_{\delta(\varepsilon)}(|q - q'|)). \]  
(56)

The same reasoning as in case (i) implies that
\[F_1^\nu(\lambda; a, \varepsilon) = Op_{\lambda}( a_0(\tau^H_{\partial\Omega}(y, \eta)) \cdot \gamma(\tau^H_{\partial\Omega}(y, \eta)) \cdot \gamma(y, \eta)) + O_{\varepsilon}(\lambda^{-1})_{L^2 \rightarrow L^2}\]  
(57)
and by the symbol computation in Lemma 3.6 one gets
\[F_1^\nu(\lambda; a, \varepsilon) = Op_{\lambda}( a_0(\tau^H_{\partial\Omega}(y, \eta)) \cdot \gamma(\tau^H_{\partial\Omega}(y, \eta)) \cdot \gamma(y, \eta)) + O_{\varepsilon}(\lambda^{-1})_{L^2 \rightarrow L^2}. \]  
(58)

4.2.1. Analysis of the \(F_2(\lambda; \varepsilon)\)-operator. In this section, we prove:

**Lemma 4.3.** \(F_{21}(\lambda; a, \varepsilon)\) (resp. \(F_{22}(\lambda; a, \varepsilon)\)) are \(\lambda\)-FIO’s of order zero with \(\kappa(F_{21}(\lambda; a, \varepsilon)) = \beta_H^{1}\) (resp. \(\kappa(F_{22}(\lambda; a, \varepsilon)) = \beta_H^{2}\)). Their principal symbol (on the respective branch of the correspondence) is given by
\[
a \left( d_{r \psi_1} |q(y_1) - q_H(s)| \right)^{1/2} \left( d_{r \psi_2} |q(y_2) - q_H(s)| \right)^{1/2} \mid_{s = q_{H,k}(y_1, y_2)} = a(\tau^H_{\partial\Omega}(y_2, \eta_2)) \left( \gamma(y_1, \eta_1) \right)^{1/2} \left( \gamma(y_2, \eta_2) \right)^{1/2} ; \quad (y_2, \eta_2) = \beta^k_H(y_1, \eta_1), \; k = 1, 2. \]  
(59)

Here, \(s = q_{H,k}(y, y') \in H; k = 1, 2\) is the point on \(H\) joining \(y, y' \in \partial\Omega\) under the \(H\)-broken billiard map \(\beta^k_H; k = 1, 2,\) (see section 3).
Proof. Because of the cutoff \((1 - \chi_{\delta(c)})(|q(y) - q(y')|)\) in the amplitude of the \(F_{2}(\lambda; a, \varepsilon)\)-kernel the canonical relation corresponding to \(F_{2}(\lambda; a, \varepsilon)\) does not intersect the diagonal \(\Delta_{B^{*} \partial \Omega \times B^{*} \partial \Omega}\) and so, by the argument in section 3 it must consist of the reflection canonical relations \(\beta_{H_{\nu}}^{1}\) (resp. \(\beta_{H_{\nu}}^{2}\)). The corresponding quantizations are the \(\lambda\)-Fourier integral operators \(F_{21}(\lambda; a, \varepsilon)\) (resp. \(F_{22}(\lambda; a, \varepsilon)\)). Modulo pointwise uniform \(O_{\varepsilon}(\lambda^{-\infty})\)-errors, the Schwartz kernel is locally given by the formula

\[
F_{2k}(\lambda; a, \varepsilon)(y, y') = (2\pi \lambda)^{-1} \int_{H} e^{i\lambda[q_{H}(s) - q(y) - q(y')]} c(y, y', s; \lambda) (1 - \chi_{\delta(c)})(|y - y'|) \times \chi_{\varepsilon}(q_{H}(s) - q_{H,k}(y, y')) d\sigma_{H}(s) \tag{60}
\]

where, \(c \sim \sum_{j=0}^{\infty} c_{j} \lambda^{-j}\) with

\[
c_{0}(y, y', s) = a_{0}(q_{H,j}(s), d_{s}q_{H,j}(s)r(q_{H,j}(s), q(y'))) \times \langle \nu_{y}, r(q_{H}(s), q(y)) \rangle \times \langle \nu_{y'}, r(q_{H}(s), q(y')) \rangle \cdot b_{0}(q(y), q_{H,j}(s)) \cdot b_{0}(q_{H,j}(s), q(y')) \tag{61}
\]

where, \(b_{0}(q_{H}(s), q(y)) = |q_{H}(s) - q(y)|^{-\frac{m_{\nu}}{2}}\).

The symbol is computed by taking the product of the principal symbols of the factors given in Lemma 3.6.

Combining (62) with Lemma 4.3 completes the proof of Proposition 4.2 in the case of the \(N_{\partial \Omega}^{H}(\lambda)^{*} Op_{\lambda}(a)N_{\partial \Omega}^{H}(\lambda)\)- operator.

The following subsection completes the proof of the Proposition.

4.2.2. Modification for \(N_{\partial \Omega}^{H_{\nu}}(\lambda)^{*} Op(a)N_{\partial \Omega}^{H_{\nu}}(\lambda)\).

Lemma 4.4. \(F_{21}^{\nu}(\lambda; a, \varepsilon)\) (resp. \(F_{22}^{\nu}(\lambda; a, \varepsilon)\)) are \(\lambda\)-Fourier integral operators of order one with \(\kappa(F_{21}(\lambda; a, \varepsilon)) = \beta_{\partial \Omega}^{1}\) (resp. \(\kappa(F_{22}(\lambda; a, \varepsilon)) = \beta_{\partial \Omega}^{2}\)). Its symbol

\[
-\left( d_{\nu_{y}} |q(y_{1}) - q_{H}(s)| d_{\nu_{s}} |q(y_{1}) - q_{H}(s)| \right)^{1/2} \times \left( d_{\nu_{y}} |q(y_{2}) - q_{H}(s)| d_{\nu_{s}} |q(y_{2}) - q_{H}(s)| \right)^{1/2} \mid s = q_{H,k}(y_{1}, y_{2})
\]

is that of \(F_{2j}^{\nu}\) multiplied by

\[
- (i\lambda \partial_{\nu_{s}} |q(y_{1}) - q_{H}(s)|)(-i\lambda \partial_{\nu_{s}} |q(y_{2}) - q_{H}(s)|) \mid s = q_{H,k}(y_{1}, y_{2}).
\]

In symplectic coordinates, it is given by

\[
- \left( \gamma(y_{1}, \eta_{1}) \gamma(\tau_{H}(y_{1}, \eta_{1})) \right)^{1/2} \left( \gamma(y_{2}, \eta_{2}) \gamma(\tau_{H}(y_{2}, \eta_{2})) \right)^{1/2}; (y_{2}, \eta_{2}) = \beta_{H}^{k}(y_{1}, \eta_{1}), k = 1, 2.
\]

The proof is the same as for Lemma 4.3 and is therefore omitted. This completes the proof of the Proposition.

5. Proof of Theorem 1

It is important to observe the sign difference between the symbol of the FIO part of \(N_{\partial \Omega}^{H}(\lambda)^{*} Op(a)N_{\partial \Omega}^{H}(\lambda)\) and its \(\nu\)-partner. In the calculation of the first symbol, the normal derivative with respect to \(s \in H\) comes about when we change from \((s, s')\) coordinates to \((s, \tau)\) coordinates. So this normal is symmetric with respect to the two sides of \(H\) (at each intersection point). But in the \(N_{\partial \Omega}^{H_{\nu}}\) operator it is assymetrical. This explains why the \(F_{2}\) operators have opposite signs in the two \(N^{*}N\) terms.
We note also that in the case of the \( \lambda \)-pseudodifferential \( F^\nu (\lambda; a, \varepsilon) \)-term, \( y_1 = y_2 \) and so, \( y_1 \) and \( y_2 \) trivially lie on the same side of \( H \) relative to the normal vector \( \nu_{s=q_H,\lambda}(y_1,y_2) \). On the other hand, in the \( \lambda \)-Fourier integral case of \( F^2_2 (\lambda; a, \varepsilon) \), \( y_1 \) and \( y_2 \) lie on opposite sides of \( H \) relative to the normal vector \( \nu_{s=q_H,\lambda}(y_1,y_2) \). This creates the extra minus sign in Lemma 4.4 which allows for the cancellation between the (appropriately weighted) \( F_2 \) and \( F^2_2 \)-terms.

We use these observations to reduce the proof of Theorem 1 to that of [HZ]:

**Lemma 5.1.** The principal symbol of the Fourier integral operator part of

\[
N^H_{\partial \Omega}(\lambda)^* \text{Op}_\lambda((1 - |\tau|^2)a)N^{H}_{\partial \Omega}(\lambda) + \lambda^{-2}N^{H\nu}_{\partial \Omega}(\lambda)^* \text{Op}(a)N^{H\nu}_{\partial \Omega}(\lambda)
\]

equals zero.

**Proof.** The principal symbols of the \( F_2 (\lambda) \) terms of the two operators are calculated respectively in Lemmas 4.3 and 4.4. As discussed in the proofs, the second (Neumann) symbol equals the first (Dirichlet) multiplied by the factor of \( \partial_{\nu_\sigma}|q(y_1) - q_H(s)||\partial_{\nu_\sigma}|q(y_2) - q_H(s)||_{s=q_H,\lambda}(y_1, y_2) \). By definition, \( q(y_2) - q_H(s) \) and \( q(y_1) - q_H(s) \) have the same tangential projection to \( H \). But they lie on opposite sides of \( H \) and have opposite normal projections. So this factor equals \( -\cos \vartheta(y_1, s)^2 \) where \( \vartheta \) is the angle to the normal. In symplectic coordinates of the projection \( (s, \tau) \) on \( B^*H \), \( \cos \vartheta(y_1, s)^2 = 1 - |\tau|^2 \). Hence, if we take a diagonal pseudo-differential operator \( A_2 \) with \( \sigma_{A_{11}}(s, \tau) = (1 - |\tau|^2)a(s, \tau) \) and \( \sigma_{A_{22}}(s, \tau) = a(s, \tau) \) for all \( (s, \tau) \in B^*H \), and if we normalize the second term by dividing the Neumann data by \( \lambda \), then the Fourier integral parts of the \( N^H_{\partial \Omega}(\lambda)^* \text{Op}_\lambda(a)N^{H}_{\partial \Omega}(\lambda) \) compositions cancel each other. Then we are left with only the pseudo-differential parts of each composition, completing the proof of the Lemma. \( \square \)

Thus, by Lemma 5.1 and Proposition 4.2

\[
\langle A_{11}(\lambda)u^H_{\lambda}, u^H_{\lambda}\rangle_{L^2(H)} + \lambda^{-2}\langle A_{22}(\lambda)u^H_{\lambda}, u^H_{\lambda}\rangle_{L^2(H)} = 2\langle \text{Op}_\lambda((\tau^H_{\partial \Omega})^*a \cdot (\tau^H_{\partial \Omega})^*\gamma \cdot \gamma)u^b_{\lambda}, u^b_{\lambda}\rangle_{L^2(\partial \Omega)} + \mathcal{O}(\lambda^{-1})\|u^b_{\lambda}\|^2_{L^2(\partial \Omega)} \quad (62)
\]

and the statement of Theorem 1 then follows immediately from (62) and an application of the boundary quantum ergodicity result of [HZ]. \( \square \)

### 6. Boundary Weyl law: Proof of Theorem 9

We now prove a key result underlying Theorem 1 on quantum ergodic restriction for Dirichlet data. The proof hinges on an averaging argument in [8] and on a boundary local Weyl law. The purpose of this and the next section is to prove a weak version of a boundary Weyl law for \( \lambda \)-Fourier integral operators. The presence of the boundary causes a number of technical complications, but the result seems to us of independent interest. This section is independent of the choice of \( H \).

To prepare for the boundary result, let us first recall the case of manifolds without boundary and eigenfunctions of a Laplacian [Z]. The Weyl law for classical Fourier integral operators \( F : C^\infty(M) \to C^\infty(M) \) of order zero says that

\[
\frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \langle F \varphi_{\lambda_j}, \varphi_{\lambda_j}\rangle \to \int_{\Gamma_F \cap \Delta_{2*}M} \sigma_\Delta(F) d\mu, \quad (63)
\]
where the integral is over the intersection of the canonical relation \( \Gamma_F \) of \( F \) with the diagonal of \( T^*M \times T^*M \), i.e. the fixed-point set of the symplectic correspondence underlying \( F \). The right side is zero unless the fixed-point set has dimension \( m = \dim M \), the leading order trace sifts out the ‘pseudo-differential part’ of \( F \). The fixed point set is assumed to be almost-clean in the sense defined above Theorem 9.

The purpose of this section is to prove an analogous result for semi-classical Fourier integral operators acting on boundary values of eigenfunctions. Our goal is to prove (Theorem 9) that

\[
N_F(\lambda) := \sum_{j : \lambda_j \leq \lambda} \langle F(\lambda_j)u^b_{\lambda_j}, u^b_{\lambda_j} \rangle = o(1) \quad (64)
\]

if \( |\Sigma_F(\lambda)| = 0 \) or if the quantitative almost nowhere commuting condition holds.

The study of these \( \lambda \)-FIO Weyl sums introduces two new aspects to the local Weyl law. First is the semi-classical aspect, which would also arise in local Weyl laws for matrix elements \( \langle F(\lambda_j)\varphi_{\lambda_j}, \varphi_{\lambda_j} \rangle \) of semi-classical FIO’s relative to Laplace eigenfunctions. As is usual with semi-classical FIO’s, we ‘homogenize’ the traces by taking the Fourier transform in the semiclassical parameter \( \lambda \) and integrating over the dual frequency variable.

The second novelty is the restriction to the boundary. The semi-classical parameter is the interior eigenvalue, not the eigenvalue of an operator on the boundary. The restriction operator to the boundary resembles a Fourier integral operator, except for the complications of grazing rays, corners and so on. These play a minor role in the trace since they occur on sets of measure zero and are handled as in [HZ] by introducing appropriate cutoff operators. We omit some of these technical details for the sake of brevity, since they are the same as in [HZ]. In the subsequent article [TZ3], we prove a more complete boundary Weyl law for \( \lambda \)-FIO’s without assuming measure zero conditions.

Before stating our result, we introduce some further notation and assumptions. We assume that there is an covering \( \bigcup_{j=1}^K q^{(j)}(U) = \partial \Omega \) with \( U \subset \mathbb{R}^{n-1} \) open, such that in terms of the local parametrizations \( y \mapsto q^{(j)}(y) ; j = 1, \ldots, K \), the phase function has the form

\[
\Phi(q^{(j)}(y), q^{(j)}(y'), \theta) = \langle y, \theta \rangle - \psi^{(j)}(y', \theta), \quad \psi^{(j)} \in C^\infty(U \times \mathbb{R}^{n-1}). \quad (65)
\]

The associated canonical relation \( (35) \) is the graph of a symplectic correspondence

\[
\kappa_F : B^*\partial \Omega \rightarrow B^*\partial \Omega. \quad (66)
\]

This is sufficient for our purposes since the semi-classical FIO’s of interest here are the pieces of \( F(\lambda)N^H_{\partial \Omega}(\lambda)^*O_p(\alpha)N^H_{\partial \Omega}(\lambda) \). We denote by \( r_{\partial \Omega} : S^*\Omega_{\partial \Omega} \rightarrow S^*\Omega_{\partial \Omega} \) the reflection of inward pointing covectors at the boundary with respect to the interior unit normal given by \( r_{\partial \Omega}(\zeta) = \zeta + 2\langle \zeta, \nu_\partial \rangle \nu_\partial ; \ q \in \partial \Omega \). Also, the map \( \pi_T : S^*_{\partial \Omega} \Omega \rightarrow B^*\partial \Omega \) denotes the canonical tangential projection along \( \partial \Omega \) and we recall that \( G^r \) denotes the broken geodesic (billiard) flow \( (29) \). From now on, we drop the \( (j) \)-superscripts in the phase functions \( (65) \) with the understanding that computations are local.

6.1. **Proof of Theorem 9.**

*Proof.* In the following, we assume that \( \partial \Omega \) is strictly convex. We treat the general non-convex case in subsection 6.7.
Following the strategy of the Fourier Tauberian theorem [SY], we calculate the principal term of the Weyl asymptotics of order \( \lambda^n \) by computing the singularity at \( t = 0 \) of order \((t + i0)^{-n}\) of

\[
I^b(t) := \sum_j e^{it\lambda_j} \langle F(\lambda_j)u^b_{\lambda_j}, u^b_{\lambda_j} \rangle. \tag{67}
\]

We note that by adding a constant multiple of the identity operator, we may assume that the Weyl sums are monotonically increasing, hence the Fourier Tauberian theorem applies (see [Z] for the parallel statement in the boundaryless case).

To calculate the principal singularity of \( \langle b_{\lambda} \rangle \) at \( t = 0 \), we write

\[
F(\lambda) = \frac{1}{2\pi} \int_\mathbb{R} e^{i\lambda s} \hat{F}(s) ds,
\]

where

\[
\hat{F}(s) = (\mathcal{F}_{\mu \to s} F)(s) = \int_\mathbb{R} e^{-i\mu s} F(\mu) d\mu.
\]

Then \( \hat{F}(s) \) is a homogeneous Fourier integral operator with Schwartz kernel,

\[
\hat{F}(s)(q, q') = (2\pi \mu)^{n-1} \int_{\mathbb{R}^+} \int_{\mathbb{R}^{n-1}} e^{i\mu(q'q')^{-s}} A(q, q', \theta, \mu) d\theta d\mu, \tag{68}
\]

where \( A(q, q', \theta, \mu) \sim \sum_{j=0}^{\infty} A_j(q, q', \theta) \mu^{-j} \), with \( A_j \in C_0^\infty(\partial \Omega \times \partial \Omega \times \mathbb{R}^{n-1}) \). We then have,

\[
\sum_j e^{it\lambda_j} \langle F(\lambda_j)u^b_{\lambda_j}, u^b_{\lambda_j} \rangle = \int_\mathbb{R} \sum_j e^{i\lambda_j(s+t)} \langle \hat{F}(s)u^b_{\lambda_j}, u^b_{\lambda_j} \rangle ds,
\]

or equivalently,

\[
I^b(t) := Tr \int_\mathbb{R} \gamma^s_{\partial \Omega} \hat{F}(s) \gamma_{\partial \Omega} U(t + s) ds. \tag{69}
\]

The presence of the boundary restriction operator \( \gamma_{\partial \Omega} \) has the effect of restricting kernels to the boundary. To see this, we note that if \( \gamma_H \) denotes the restriction operator to \( H \), \( \gamma_H f = f|_H \), then \( \gamma_H f = f|_H \), since \( \langle \gamma_{\partial H} f, g \rangle \int_H fg ds. \) Hence, in the Neumann case, writing \( U^b(s) = \gamma_{\partial \Omega} U(s) \gamma_{\partial \Omega}; s \in \mathbb{R} \) for the boundary restriction of the operator \( U(s) \),

\[
I^b(t) = \int_\mathbb{R} \int_{\partial \Omega} \int_{\partial \Omega} U^b(t + s, q, q') \hat{F}(s, q', q) ds d\sigma(q') d\sigma(q). \tag{70}
\]

In the Dirichlet case, restriction first takes normal derivatives. In the following, we wish to compute the coefficient of the large \((t + i0)^{-n}\)-singularity of \( I^b(t) \) as \( t \to 0 \).

6.2. Microlocal cutoffs. To simplify the trace we introduce some microlocal cutoffs: Since both the singular set, \( \Sigma \), and the grazing set, \( \mathcal{G} \), are closed and of measure zero, by the \( C^\infty \) Urysohn lemma, for \( \varepsilon > 0 \) arbitrarily small, we let \( \chi_\varepsilon \in C^\infty_0(B^* \partial \Omega) \) be supported an \( \varepsilon \)-neighbourhood of \( \Sigma \cup \mathcal{G} \). Such cutoffs are necessary but quite standard [HZ].

In addition, we will also insert a cutoff in the integration in \( s \) in (70) near \( s \sim \Phi \) and localize the trace in time. This cutoff is more novel and is also important since it allows us to ultimately replace \( U^b(t + s) \) by the corresponding finite-time reflection (ie. Chazarain) parametrix in the formula (70) for the boundary trace, \( I^b(t) \). We define the \( \lambda \)-microlocally cutoff operators \( F^\varepsilon(\lambda) := Op_\lambda(1 - \chi_\varepsilon)F(\lambda)Op_\lambda(1 - \chi_\varepsilon) \) and \( U^b(t + s) := Op_\lambda(1 - \chi_\varepsilon)U^b(t + s)Op_\lambda(1 - \chi_\varepsilon) \). By standard composition calculus, one can write the
Schwartz kernel $F_\varepsilon(\lambda)(q(y),q(y')) = (2\pi\lambda)^{-n} \int_{\mathbb{R}^{n-1}} e^{i\lambda \Phi(q(y),q(y'),\theta)} A_\varepsilon(q(y),q(y'),\theta) d\theta$ where $A_\varepsilon(q(y),q(y'),\theta) = A(q(y),q(y'),\theta)(1-\varepsilon)(q(y),d_y \Phi) \cdot (1-\chi)(q(y'),d_y \Phi) + O(\lambda^{-1})$. The precise statement of the $\lambda$-microlocalization of the boundary trace is given in the following

**Lemma 6.1.** Let $\chi \in C^\infty_0(\mathbb{R})$ with $\chi(s) = 1$ near $s = 0$ and for any $\varepsilon > 0$ let $\chi_\varepsilon \in C^\infty_0(B^* \partial \Omega)$ be supported in an $\varepsilon$-neighbourhood of $B^*_\varepsilon \partial \Omega \cup G$. Then,

$$I^b(t) = \int_{\Omega} \int_{\partial \Omega} \int_{\partial \Omega} U^b(t + s, q, q') e^{i\mu(\Phi(q',q,\theta)-s)} A_\varepsilon(q', q, \theta, \mu) \times \chi(s - \Phi(q, q', \theta))(2\pi \mu)^{n-1} d\sigma(q') d\sigma(q) \mu d\sigma d\mu + R(t) + R_\varepsilon(t),$$

where $R(t) \in C^\infty(\mathbb{R})$ and $R_\varepsilon(t) = O(\varepsilon) t^{-n}$ for $|t|$ small.

**Proof.** From $L^2$-boundedness of $F(\lambda)$ and the boundary Weyl law (see [HZ] Lemmas 7.1, 9.2, Appendix 12 as well as Lemma 8.3 in section 8.4 of this paper), it follows that

$$\frac{1}{N(\lambda)} \left| \sum_{\lambda_j \leq \lambda} \langle Op_{p_\lambda}(\chi_\varepsilon) F(\lambda_j) Op_{p_\lambda}(\chi_\varepsilon) u^b_{\lambda_j}, u^b_{\lambda_j} \rangle \right| \leq \frac{C}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle Op_\lambda(\chi_\varepsilon) * Op_\lambda(\chi_\varepsilon) u^b_{\lambda_j}, u^b_{\lambda_j} \rangle = O(\varepsilon).$$

Similarly, by applying Cauchy-Schwartz and an argument like the one above it follows that the composite operators $Op_{p_\lambda}(\chi_\varepsilon) F(\lambda_j) Op_{p_\lambda}(\chi_\varepsilon)(1 - \chi_\varepsilon)$ and $Op_{p_\lambda}(\chi_\varepsilon) F(\lambda_j) Op_{p_\lambda}(\chi_\varepsilon)$ each have $O(\varepsilon)$ boundary traces (see also [Z MO]).

Thus, substitution of the the integral formula (68) for $\hat{F}(s, q, q')$ in the formula for $I^b(t)$ gives

$$I^b(t) = \int_{\mathbb{R}} \int_{\partial \Omega} \int_{\partial \Omega} U^b(t + s, q, q') \hat{F}(s, q', q) ds d\sigma(q') d\sigma(q)$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\partial \Omega} \int_{\partial \Omega} U^b(t + s, q, q') e^{i\mu(\Phi(q',q,\theta)-s)} A_\varepsilon(q', q, \theta, \mu)(2\pi \mu)^{n-1} d\sigma(q') d\sigma(q) \mu d\sigma d\mu + R(t) + R_\varepsilon(t).$$

We now cutoff in the $s$-time variable. First, note that since $A_\varepsilon$ is symbolic in $\mu$,

$$\partial_\mu^\alpha A_\varepsilon(q, q', \theta, \mu) = O_{\alpha, \varepsilon}(\mu^{-\alpha})$$

uniformly in the $(q,q',\theta)$-variables.

We make the following decomposition

$$I^b(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\partial \Omega} \int_{\partial \Omega} U^b(t + s, q, q') e^{i\mu(\Phi(q',q,\theta)-s)} A_\varepsilon(q', q, \theta, \mu) \times \chi(s - \Phi(q, q', \theta))(2\pi \mu)^{n-1} d\sigma(q') d\sigma(q) \mu d\sigma d\mu + R(t) + R_\varepsilon(t),$$

where, $\int_0^\infty e^{i\mu t} R_\varepsilon(t) dt = O(\varepsilon) \mu^{-n}$ and

$$R(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \int_{\partial \Omega} \int_{\partial \Omega} U^b(t + s, q, q') e^{i\mu(\Phi(q',q,\theta)-s)} A_\varepsilon(q', q, \theta, \mu) \times (1 - \chi)(s - \Phi(q, q', \theta))(2\pi \mu)^{n-1} d\sigma(q') d\sigma(q) \mu d\sigma d\mu.$$
\[ \times (1 - \chi)(s - \Phi(q, q', \theta)) \times O_N(|s|^{-N} \mu^{n-1-N}) \, d\sigma(q') d\sigma(q) d\theta d\sigma \mu. \]

Since the integrand is absolutely convergent, one can take the \( t \to 0 \) limit inside the integral and this shows that \( R(t) \) is continuous near \( t = 0 \). Continuity for the derivatives of \( R \) follows a similar fashion: Since \( \partial_t U^b_\varepsilon(t + s) = \partial_t U^b_\varepsilon(t + s) \), one integrates by parts in \( s \). Each \( s \)-derivative of the exponential \( e^{i\mu(s-\Phi)} \) creates a power of \( \mu \). However, this is killed by the \( O_N(|s|^N \mu^{m-N}) \)-factor since \( N > 0 \) is arbitrary. The result is an absolutely convergent integral. By iterating this argument, one can differentiate to any order in \( t \) inside the integral and the result is absolutely convergent. It follows that \( R(t) \in C^\infty \) for \( |t| \) small and this finishes the proof of the lemma. \( \square \)

Thus, it is enough to consider from now on the principal part of the trace, \( I^b(t) \), which by Lemma 6.1 is given by

\[ I^b_{\text{sing}}(t) := \int_{\mathbb{R}_+} \int_{\mathbb{R}^{n-1}} \int_{\partial \Omega} \int_{\partial \Omega} U^b_\varepsilon(t + s, q, q') e^{i\mu(\Phi(q', \theta) - s)} A_\varepsilon(q', q, \theta, \mu) \]

\[ \times \chi(s - \Phi(q, q', \theta)) (2\pi \mu)^{-n} d\sigma(q') d\sigma(q) d\theta d\sigma \mu. \]  

(73)

The remainder of the proof consists of carrying out a singularity analysis of the integral for \( I^b_{\text{sing}}(t) \) in (73) near \( t = 0 \). It is useful to note that since it suffices here to choose \( |t| \leq \varepsilon_0 \) with \( \varepsilon_0 > 0 \) fixed arbitrarily small, \( |t + s| \leq \varepsilon_0 + \sup \Phi(q, q', \theta) < \infty \) for \( (q, q', \theta) \in \text{supp} \ A_\varepsilon \) since \( \text{supp} \ A_\varepsilon \) is compact. Thus, an immediate consequence of Lemma 6.1 is the identity

\[ I^b(t) = I^b_{\text{sing}}(t) + R_\varepsilon(t) + R(t), \]  

(74)

Here, we recall that

\[ F_\varepsilon(q, q'; \mu) := (2\pi \mu)^{-n} \int_{\mathbb{R}^{n-1}} e^{i\mu\Phi(q, q', \theta)} A_\varepsilon(q, q', \theta; \mu) \chi(s - \Phi(q, q', \theta)) \, d\theta \]  

(75)

and for \( \varepsilon > 0 \) arbitrarily small, \( R_\varepsilon(t) = O(\varepsilon t^{-n}, R(t) \in C^\infty(\mathbb{R}_+) \) for \( |t| \) small and \( I := [-||\Phi||_{L^\infty} - \varepsilon_0, ||\Phi||_{L^\infty} + \varepsilon_0 \).

6.3. Chazarain parametrix. Since \( U^b_\varepsilon(t + s) \) is \( \lambda \)-microlocalized away from the set \( (B^* \partial \Omega \times [\mathcal{G} \cup B^*_\varepsilon \partial \Omega]) \cup ((\mathcal{G} \cup B^*_\varepsilon \partial \Omega) \times B^* \partial \Omega) \), this allows us to substitute the boundary trace of the Chazarain parametrix for \( U^b_\varepsilon(t + s) \) in the formula for \( I^b_{\text{sing}}(t) \) in (73). Here, we use the crucial fact that by Lemma 6.1 (see also (74)), one need only consider finite times with \( |s + t| \leq \sup \Phi + \varepsilon_0 \). This parametrix construction is well-known [GM] [PS]. However, as far as we know, the properties vis-a-vis boundary restriction are not available in the literature. Thus, we review here the aspects that are most relevant to the analysis of the boundary traces.

It suffices to assume that \( \partial \Omega \) is \( C^\infty \) since \( W F_\varepsilon(U^b_\varepsilon(t + s) \circ \hat{F}_\varepsilon(s)) \cap ((B^*_\varepsilon \partial \Omega \cup \mathcal{G}) \times B^* \partial \Omega \cup B^* \partial \Omega \times [B^*_\varepsilon \partial \Omega \cup \mathcal{G}]) = \emptyset \). Let \( \partial \Omega = \{x \in \mathbb{R}^n : f(x) = 0 \text{ with } df(x) \neq 0 \text{ for } x \in \partial \Omega \}. \) Given \( (q, \omega) \in S^*_+(\partial \Omega) \), we let \( 0 < t_1 < ... < t_3 < ... < t_j \in C^\infty(S^*_+(\partial \Omega)) \) be the ordered intersection points of the bicharacteristic starting at \( (q, \omega) \). Thus, the \( t_j, j = 1, 2, ... \) satisfy the defining equation

\[ f(t_j(q, \omega)) = 0; \quad j = 1, 2, 3, ... \]
The remainder of the proof of Theorem 9 consists of substituting the boundary trace of the explicit Chazarain parametrix (which we describe below) in (74) and then carrying out a singularity analysis of $I^b(t)$ near $t = 0$.

To describe the parametrix boundary trace, we decompose $S^*_+ \partial \Omega = \bigcup_k \Gamma_k$ where the $\Gamma_k$ are sufficiently small open sets and let $t_j^{(k)} = \min_{(q, \omega) \in \Gamma_k} t_j(q, \omega)$ and $T_j^{(k)} = \max_{(q, \omega) \in \Gamma_k} t_j(q, \omega)$. We choose $\Gamma_k$ small enough so that

$$t_1^{(k)} < T_1^{(k)} < t_2^{(k)} < T_2^{(k)} < \cdots ; \quad k = 1, 2, 3, \ldots, N. \quad (76)$$

Let $\chi_k \in C^\infty(S^* \partial \Omega); k = 1, \ldots, N$, be a partition of unity subordinate to the covering by $\Gamma_k; k = 1, \ldots, N$.

One constructs a finite-time microlocal parametrix for the wave operator, $U(s)$, on the sets $\Gamma_k; k = 1, \ldots, N$. To describe the construction, consider first the case where $s \in (-2\varepsilon_0, t_1^{(k)} - \varepsilon_0)$ where $t_1^{(k)} := \min_{(q, \omega) \in \Gamma_k} t_1(q, \omega)$ and $\varepsilon_0 > 0$ is a fixed small constant. We let

$$U_0^{(k),+}(s, x, y; \lambda) := U_0 \circ Op_\lambda(\chi_k)(s, x, y; \lambda) = (2\pi \lambda)^n \int_{\mathbb{R}^n} e^{i\lambda\varphi(s,x,y,\xi)} B_0^{(k)}(s, y, \xi, \lambda) d\xi,$$

where $U_0$ is the free-space solution of the wave equation with $\varphi(s, x, y, \xi) := \langle x - y, \xi \rangle - s|\xi|$. Let $x^*(x) \in \mathbb{R}^n - \Omega$ be the geodesic reflection of $x \in \Omega$ in the boundary $\partial \Omega$ and put

$$U_0^{(k),-}(s, x, y; \lambda) := U_0 \circ Op_\lambda(\chi_k)(s, x^*(x), y; \lambda) = (2\pi \lambda)^n \int_{\mathbb{R}^n} e^{i\lambda\varphi(s,x^*(x),y,\xi)} B_0^{(k)}(s, y, \xi, \lambda) d\xi$$

and for $(s, x, y) \in (-\varepsilon_0, t_1^{(k)} + \varepsilon_0) \times \Omega \times \Omega$, define

$$2U_1^{(k)}(s, x, y; \lambda) := U_0^{(k),+}(s, x, y; \lambda) + U_0^{(k),-}(s, x, y; \lambda). \quad (77)$$

Clearly, for all $(s, x, y) \in (-2\varepsilon_0, t_1^{(k)} - \varepsilon_0) \times \Omega \times \Omega$, $U_1^{(k)}$ solves the equation $\frac{1}{i} \partial_s U_1^{(k)} + \sqrt{\Delta} U_1^{(k)} = 0$ and since $x^* \in \mathbb{R}^n - \Omega$, by integration by parts in $\xi$ it follows that for $(x, y) \in \Omega \times \Omega$,

$$U_1^{(k)}(0, x, y; \lambda) = Op_\lambda(\chi_k)(x, y) + R(x, y; \lambda), \quad (78)$$

where $R(x, y; \lambda) \in C^\infty$ with $|\partial_x^a \partial_y^b R(x, y; \lambda)| = \mathcal{O}_{a,b}(\lambda^{-\infty})$. Following [HZ], we call such kernels residual. Similiarly, since $\partial_\nu [\varphi(s, x^*, y, \xi) + \varphi(s, x, y, \xi)]|_{x=x^*=y=\partial \Omega} = 0$, $U_1^{(k)}$ satisfies the approximate Neumann boundary condition

$$\partial_\nu U_1^{(k)}(s, q, y; \lambda) = R_1(s, q, y; \lambda), \quad (s, q, y) \in (-\varepsilon_0, t_1^{(k)} + \varepsilon_0) \times \partial \Omega \times \Omega \quad (79)$$

with $R_1(s, q, y; \lambda) = \mathcal{O}(\lambda^{-\infty})$ residual. Also, it is clear that

$$[U_1^{(k)}]_b(s, q, q'; \lambda) = [U_0^{(k),+}]_b(s, q, q'; \lambda); \quad (q, q') \in \partial \Omega \times \partial \Omega.$$

We microlocally cutoff $U_1^{(k)}$ away from grazing and singular directions and define the microlocal parametrix

$$U_{1,\varepsilon}^{(k)}(s; \lambda) := (Id - Op_\lambda(\chi_\varepsilon)) \circ U_1^{(k)}(s; \lambda) \circ (Id - Op_\lambda(\chi_\varepsilon)).$$

The analogous, microlocally-cutoff wave operator is

$$U_{\varepsilon}^{(k)}(s; \lambda) := (Id - Op_\lambda(\chi_\varepsilon)) \circ U(s) \circ Op_\lambda(\chi_k) \circ (Id - Op_\lambda(\chi_\varepsilon)).$$
From (78) and (79), both $U^{(k)}_{1,\varepsilon}(s; \lambda)$ and $U^{(k)}_{1,\varepsilon}(s; \lambda)$ are microlocal parametrices for the wave operator on $\text{supp} \chi_k(1 - \chi_\varepsilon)$. Thus, $U^{(k)}_{1,\varepsilon}(s; \lambda) - U^{(k)}_{1,\varepsilon}(s; \lambda)$ is residual for $s \in (-2\varepsilon_0, t^{(k)}_1 - \varepsilon_0)$ and in particular, for $(t + s, q, q') \in (-2\varepsilon_0, t^{(k)}_1 - \varepsilon_0) \times \partial \Omega \times \partial \Omega$,

$$[U^{(k)}_{\varepsilon}]^b(t + s, q, q'; \lambda) = [U^{(k)}_{1,\varepsilon}]^b(t + s, q, q'; \lambda) + R(t + s, q, q'; \lambda)$$  \hspace{1cm} (82)

where, $R(\cdot, \cdot, \cdot, \cdot; \lambda) \in C^\infty((-2\varepsilon_0, t^{(k)}_1 - \varepsilon_0) \times \partial \Omega \times \partial \Omega)$ is residual for $\lambda \geq \lambda_0$.

To construct parametrices up to finite, multiple reflections at the boundary, we choose $M = M(k, \varepsilon)$ large enough so that the finite time-interval, $I$ of integration in (74) satisfies $I \subset \bigcup_{j=1}^M (t^{(k)}_{j-1} - 2\varepsilon_0, t^{(k)}_j - \varepsilon_0) \cup_{j=1}^M (-t^{(k)}_j - 2\varepsilon_0, -t^{(k)}_{j-1} - \varepsilon_0)$ with $t^{(k)}_0 = 0$ and we let $\chi_{\pm j}(\cdot) \in C^\infty_0(\mathbb{R})$: $j = 1, \ldots, M$ be a partition of unity subordinate to this covering. In view of (82), the parametrix for $s + t \in (-2\varepsilon_0, t^{(k)}_1 - \varepsilon_0)$ is $U^{(k)}_{1,\varepsilon}(s + t)$. For times $t + s \in (t^{(k)}_1 - 2\varepsilon_0, t^{(k)}_2 - \varepsilon_0)$, one puts

$$U^{(k)}_{2,\varepsilon}(s + t, x, q; \lambda) = \int_{\Omega} U^{(k)}_{1,\varepsilon}(s + t - \tau, x, z; \lambda) \cdot U^{(k)}_{1,\varepsilon}(\tau, z, y; \lambda) \, dz, \hspace{1cm} \tau := |z - q(y)|$$  \hspace{1cm} (83)

Clearly, $\frac{1}{2} \partial_{t} U^{(k)}_{2,\varepsilon} + \sqrt{\sum} U^{(k)}_{2,\varepsilon} = 0$ and by (82), $U^{(k)}_{2,\varepsilon}(s + t)$ satisfies the Neumann boundary condition.

Moreover, iteration of this construction gives the formula

$$U^{(k)}_{j,\varepsilon}(s + t, x, y; \lambda) = \int_{\Omega} U^{(k)}_{1,\varepsilon}(s + t - \tau, x, z; \lambda) \cdot U^{(k)}_{j-1,\varepsilon}(\tau, z, y; \lambda) \, dz$$  \hspace{1cm} (84)

$$s + t \in (t^{(k)}_{j-1} - 2\varepsilon_0, t^{(k)}_j - \varepsilon_0).$$

Just as in the $j = 1$ case, it follows that for any $j = 1, \ldots, M$, and $s + t \in (t^{(k)}_{j-1} - 2\varepsilon_0, t^{(k)}_j - \varepsilon_0)$,

$$[U^{(k)}_{j,\varepsilon}]^b(s + t, q, q'; \lambda) = U^{(k)}_{\varepsilon}(s + t, q, q'; \lambda) + R_j(q, q'; \lambda); \hspace{1cm} (q, q') \in \partial \Omega$$  \hspace{1cm} (85)

with $R_j$ residual. We summarize the parametrix approximation of the boundary trace in the following

**Lemma 6.2.** Given $\chi_{j}^{(k)} \in C^\infty_0(\mathbb{R})$: $j = \pm 1, \ldots, \pm M$ as above, it follows from (74) and (82) that for $|t|$ sufficiently small, $I^b_{\text{sing}}(t) = \sum_{j,k} [I^{(k)}_{j}]^b(t)$ where,

$$[I^{(k)}_{j}]^b(t) := \text{Tr} \int_{I} \chi_{j}^{(k)}(s + t) \left( [U^{(k)}_{\varepsilon}]^b(s + t) \circ \hat{F}_\varepsilon(s) \right) \, ds$$

$$= \text{Tr} \int_{I} \chi_{j}^{(k)}(s + t) \left( [U^{(k)}_{j,\varepsilon}]^b(s + t) \circ \hat{F}_\varepsilon(s) \right) \, ds,$$

where $U^{(k)}_{j,\varepsilon}$ is the microlocal parametrix with Schwartz kernel defined in (84).

To carry out our analysis of the principal singularity of $I^b_{\text{sing}}(t)$ we use the following generalized stationary phase lemma ([SV] Proposition 4.1.16):

**Lemma 6.3.** Given $\Psi(x) \in C^\infty(\mathbb{R}^n)$ with critical set $\text{Crit}(\Psi) := \{x \in \mathbb{R}^n : \nabla_2 \Psi(x) = 0\}$ and $\alpha(x) \in C^\infty_0(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} e^{i\lambda \Psi(x)} \alpha(x) \, dx = \int_{\text{Crit}(\Psi)} e^{i\lambda \Psi(x)} \alpha(x) \, dx + o(1).$$  \hspace{1cm} (86)
The computation of the principal singularity of $I^b(t)$ amounts to a detailed stationary phase analysis of the explicit boundary trace of the $\lambda$-microlocalized Chazarain parametrix given in Lemma 6.2 combined with multiple applications of (86). The analysis is somewhat lengthy since there are many critical points of the phases in the parametrix coming from various multilink bicharacteristics. We thus carry out the analysis systematically for the various links indexed by the the number of boundary reflections.

6.4. **Single-link contribution to $I^b_{\text{sing}}(t)$**. We start here with the computation of $[I_1]^b(t)$ for small $t$ with $t \in (-2\varepsilon_0, \min_e \xi^{(k)}_1]$. This is the part of the trace that comes from single-link contributions. These are rays that are either trivial (ie. ones that intersect the boundary at only the initial point) or ones that intersect twice, with the initial and terminal points lying on the boundary. The main result of this section is

**Lemma 6.4.** Under the assumptions in Theorem 2, the single-link contribution to the boundary wave trace is

$$I^b_1(t) = o(t^{-n})$$

for $|t|$ sufficiently small.

In view of Lemma 6.2, substitution of the restricted parametrix $[U^{(k)}_1]_b$ in $[I_1^{(k)}]^b(t)$ combined with the polar variables decomposition $\xi = r\omega$ where $(r, \omega) \in [0, \infty) \times S^{n-1}$ and the semiclassical rescaling $\xi \mapsto \mu \xi$, gives the formula

$$[I_1^{(k)}]^b(t) = \int_{\mathbb{R}_+} \int_{I} \int_{\mathbb{R}^{n-1}} \int_{\partial \Omega} \int_{\partial \Omega} \int_{S^{n-1}_+} e^{i\mu [r\varphi(s+t,q,q',\omega) + \Phi(q', q, \theta) - s]} A_\varepsilon(q', q, \theta, \mu) B_\varepsilon^{(k)}(q, q', r\omega, \mu)$$

$$\times \chi(s - \Phi(q', q, \theta)) \chi(r - 1) r^{n-1} dr d\omega \chi_1(s + t) d\theta d\sigma(q') d\sigma(q) ds (2\pi \mu)^{2n-1} d\mu + R(t). \quad (87)$$

Here, $R(t) \in C^\infty(\mathbb{R})$ for $|t|$ small and the cutoff $\chi(r - 1)$ is inserted in (87) modulo smooth error by integration by parts in $s$ using the eikonal equation $\partial_s \varphi = r + O(t)$ with $|t| \ll 1$. Also, the integration in $\omega$ is over the hemisphere $S^{n-1}_+$ since $WF\chi(U^{(k)}_1(s + t)) \subset S^{*}_{\partial \Omega, +} \times S^{*}_{\partial \Omega, +} \Omega$ for all $s + t \in I$. From (87) it is clear that $[I_1^{(k)}]^b(t)$ is a semi-classical (in $\mu$) oscillatory integral with phase function

$$\Psi(q, q', r, \omega, \theta, s; t) = r\varphi(t + s, q, q', \omega) + \Phi(q', q, \theta) - s.$$

We apply stationary phase to (87). Since the integrand in (87) is now compactly supported in $(s, r)$, one can apply stationary phase in those variables first. Since

$$\varphi(t + s, q, q', \omega) = \langle q - q', \omega \rangle + (s + t),$$

the critical point equations are

$$\begin{cases}
  d_r \Psi = \varphi(t + s, q, q', \omega) = 0, \\
  d_s \Psi = r \partial_s \varphi(t + s, q, q', \omega) - 1 = 0.
\end{cases}$$

The non-degeneracy of the phase is clear from the identity

$$\partial_r \partial_s \Psi(q, q', r, \omega, \theta, s; t) = \partial_s \varphi(t + s, q, q', \omega) = 1.$$
which follows from the eikonal equation. The critical points are
\[
\begin{align*}
  r &= 1, \\
  s(q, q', \omega) &= -\langle q - q', \omega \rangle - t.
\end{align*}
\]

The result is that
\[
\begin{align*}
 [I_1^{(k)}]^{b}(t) &= \int_{\mathbb{R}^+} \int_{\mathbb{R}^{n-1}} \int_{\partial \Omega} \int_{\mathbb{R}^{n-1}} e^{i\mu t} e^{i\mu [\Phi(q',q,\theta) + \langle q-q',\omega \rangle]} B_0^{(k)}(-\langle q-q',\omega \rangle, q, q', \omega) \\
 &\times A_\varepsilon(q, q', \theta, \mu) \chi(-\langle q-q',\omega \rangle - \Phi(q', q, \theta)) \chi_1(-\langle q-q',\omega \rangle) d\omega d\theta d\sigma(q') d\sigma(q) \\
 &= \int_{\mathbb{R}^+} e^{i\mu s} c_1^{(k)}(\mu) \mu^{2(n-1)} d\mu + R(t).
\end{align*}
\]

We put \( c_1 := \sum_k c_1^{(k)} \) and then from the last line of (88) it follows that
\[
\begin{align*}
 c_1(\mu) &= \int_{\mathbb{R}^n} \int_{\partial \Omega} \int_{\mathbb{R}^{n-1}} e^{i\mu [\Phi(q',q,\theta) + \langle q-q',\omega \rangle]} A_{\varepsilon,0}(q, q', \theta) \chi_1(-\langle q-q',\omega \rangle) d\omega d\theta d\sigma(q') d\sigma(q) \\
 &= A_\varepsilon(q, q', \theta, \mu) \sum_{m=0}^{\infty} A_{\varepsilon,m}(q, q, \theta) \mu^{-m} \quad \text{and} \quad R(t) \text{ is only singular of order } \mathcal{O}(t^{-n+1}).
\end{align*}
\]

Here, we have also used that \( B_0 = 1 \).

We further decompose \( c_1(\mu) \) into diagonal and off-diagonal parts by introducing a further microlocal cutoff. We write
\[
\begin{align*}
 c_1(\mu) &= \int_{\mathbb{R}^n} \int_{\partial \Omega} \int_{\mathbb{R}^{n-1}} e^{i\mu [\Phi(q',q,\theta) + \langle q-q',\omega \rangle]} A_{\varepsilon,0}(q, q', \theta) \chi(-\langle q-q',\omega \rangle) d\omega d\theta d\sigma(q') d\sigma(q) \\
 &= A_\varepsilon(q, q', \theta, \mu) \sum_{m=0}^{\infty} A_{\varepsilon,m}(q, q, \theta) \mu^{-m} \quad \text{and} \quad R(t) \text{ is only singular of order } \mathcal{O}(t^{-n+1}).
\end{align*}
\]

6.4.1. **Diagonal term.** The term \( c_1^\Lambda(\mu) \) sifts out the \( \lambda \)-pseudodifferential piece of \( F(\lambda) \). Indeed, since \( |q - q'| \leq \varepsilon \), by Taylor expansion,
\[
\begin{align*}
 \langle q(y) - q(y'), \omega \rangle &= \langle T_{y'}, \omega \rangle (y - y') + \mathcal{O}(|y - y'|^2),
\end{align*}
\]
and so, by making first the change of variables \( r : \mathbb{R}^{n-1}_+ \to B^{n-1} \), given by \( r(\omega) = \omega - \langle \nu_{y'}, \omega \rangle \nu_{y'} =: \eta \) followed by another of the form \( \eta \mapsto \eta(1 + \mathcal{O}(|y - y'|)) \), it follows that
\[
\begin{align*}
 c_1^\Lambda(\mu) &= \int_{\mathbb{R}^n} \int_{\partial \Omega} \int_{\mathbb{R}^{n-1}} e^{i\mu y \theta - \psi(y,\theta) + \langle y-y',\eta \rangle} A_{\varepsilon,0}(q(y), q(y'), \theta) (1 + \mathcal{O}(|y - y'|)) \\
 &\quad \times \chi(\varepsilon^{-1}|y - y'|) \gamma(\eta)^{-1} d\eta d\theta d\sigma(y') d\sigma(y).
\end{align*}
\]
Here, we recall that $\gamma(\eta) = \sqrt{1 - |\eta|^2}$. An application of stationary phase in $(y', \theta)$ implies

\[
(2\pi \mu)^{-(n-1)} \int_{\partial \Omega} \int_{\mathbb{R}^{n-1}} e^{i\mu [\langle y, \eta \rangle - \psi(y, \eta)]} A_{\varepsilon,0}(q(y), q(\nabla_\eta \psi), \eta) (1 + \mathcal{O}(|y - \nabla_\eta \psi(y, \eta)|)) \gamma(\eta)^{-1} dydn.
\]

(92)

The assumption $|\Sigma_F(\lambda)| = 0$ in Theorem [9] implies in particular that $|\Gamma_{\kappa_F} \cap \Delta| = 0$. Consequently, the generalized stationary phase estimate (86) applied to the integral in (92) implies that for the diagonal piece, $c^\Delta_1(\mu)$, of the single-link contribution to $I^b(t)$,

\[
c^\Delta_1(\mu) = o(\mu^{-(n-1)}).
\]

(93)

Thus, it follows that the contribution of the diagonal to the trace $I^b_{\text{sing}}(t)$ is

\[
\int_0^\infty e^{-i\mu c^\Delta_1(\mu) \mu^{2(n-1)}} d\mu = o(t^{-n}) \quad \text{for } |t|\text{-small.}
\]

6.4.2. Computation of $c^\Delta_1(\mu)$: This piece sifts-out the contribution to $c_1(\mu)$ of single links with initial and terminal points on the boundary. To compute the asymptotics of $c^\Delta_1(\mu)$, we apply stationary phase in $\omega \in \mathbb{S}_+^{n-1}$. The solution of the critical point equation

\[
d_\omega \Psi = d_\omega \langle q' - q', \omega \rangle = 0
\]

is the non-degenerate critical point $\omega(q, q') = -\frac{q - q'}{|q - q'|}$ (we note here that the integrand in $c_1(\mu)$ is supported on the set where $|q - q'| \geq \varepsilon > 0$). There is only one critical point, since the integration is only over $\omega \in \mathbb{S}_+^{n-1}$ corresponding to directions pointing into the domain $\Omega$. The result is that

\[
c^\Delta_1(\mu) = (2\pi \mu)^{-(n-1)/2} \int_{\mathbb{R}^{n-1}} \int_{\partial \Omega} e^{i\mu \Phi(q(y'), q(y), \theta) - |\langle q(y') - q(y) \rangle|} A_{\varepsilon,0}(q(y), q(y'), \theta) (1 + \mathcal{O}(\mu^{-1}))
\]

\[
\times (1 - \chi_1(\varepsilon |y - y'|)) d\sigma(y') d\sigma(y) d\theta,
\]

where, $\Phi(q(y'), q(y), \theta) = \langle y', \theta \rangle - \psi(y, \theta).

Stationary phase in the $y'$-variables gives the critical point equation

\[
d_{y'} \Psi = \langle T_{y'}, \frac{q(y') - q(y)}{|q(y) - q(y')|} \rangle = d_{y'} \Phi = \langle T_{y'}, \frac{q(y') - q(y)}{|q(y) - q(y')|} \rangle - \theta = 0.
\]

The non-degeneracy follows from the identity

\[
|\det \nabla_{y'}^2 [y'\theta - \psi(y, \theta) - |q(y') - q(y)|]| = |\det \nabla_{y'}^2(|q(y') - q(y)|)|
\]

\[
= \left| \det \nabla_{y'} \langle T_{y'}, \frac{q(y') - q(y)}{|q(y') - q(y)|} \rangle \right| \geq C_{\partial \Omega} \langle \nu_{y'}, r(q(y'), q(y)) \rangle \left( 1 + \frac{\langle \nu_{y'}, r(q(y'), q(y)) \rangle}{|q(y') - q(y)|} \right)
\]

where $C_{\partial \Omega} > 0$. The Hessian matrix is non-degenerate since the integral formula for $c(\mu)$ is microlocally cutoff away from grazing directions, ghost directions and unclean points and so, in particular, $\langle \nu_{y'}, r(q(y'), q(y)) \rangle \geq C(\varepsilon) > 0$. We denote the critical point by $y'_c(y, \theta)$ and let $q'_c = q(y'_c(y, \theta))$. Then,

\[
c^\Delta_1(\mu) = (2\pi \mu)^{-(n+1)} \int_{\mathbb{R}^{n-1}} \int_{\partial \Omega} e^{i\mu [\langle y, \eta \rangle - \psi(y, \eta) + \Phi(q_c, q(y), \theta)]} \frac{A_{\varepsilon,0}(q(y), q_c, |q(y) - q'_c|)}{|q(y) - q'_c|^{1/2}}
\]

\[
\times \langle \nu_{y'}, r(q(y), q_c) \rangle \chi_1(|y - y'|) d\sigma(y) d\theta + R(\mu),
\]

(94)
where, $R(\mu) = \mathcal{O}(\mu^{-n})$ as $\mu \to \infty$. The critical set of the phase function $\Psi_1(q(y), \theta) = -|q(y) - q_k'| + \Phi(q_k', q(y), \theta)$ in (94) is

$$\text{Crit}(\Psi_1) := \{(y, \theta) \in \partial \Omega \times \mathbb{R}^{n-1}; \theta = \langle T_y q', q(y) - q(y') \rangle, \ d_y \Phi = 0, \ \langle T_y q(y) - q(y') \rangle = d_y \Phi \}.$$  

The assumption $|\Sigma_{F(\lambda)}| = 0$ in Theorem 3 implies that

$$|\text{Crit}(\Psi_1)| = 0. \quad (95)$$

The measure zero condition in (95) follows by relating $\text{Crit}(\Psi_1)$ to the Lagrange immersion of the critical set $C_\Phi = \{d_\theta \Phi = 0\}$ (resp. $C_{\phi_{\tau}} = \{d_{\omega} \phi_{\tau} = 0\}$), where $\phi_{\tau}(q, q', \omega) := \varphi(\tau, q, q', \omega)$. The Lagrangian immersions are given by

$$\iota_\Phi : C_\Phi \to \Gamma_F,$$

$$(y, y', \theta) \mapsto (y', d_y \Phi, y, -d_y \Phi) = (y', \theta; y, -d_y \psi), \ d_\theta \Phi = 0$$

and

$$\iota_{\phi_{\tau}} : C_{\phi_{\tau}} \to \Gamma_\beta,$$

$$(y, y', \omega) \mapsto \left(\left(\langle T_y q', \frac{q(y') - q(y)}{|q(y') - q(y)|} \rangle, y, -\langle T_y q(y') - q(y) \rangle \right), \ y' = \pi G^\tau(y, \omega) \right).$$

It then follows that under the above identification, $\text{Crit}(\Psi_1)$ corresponds to the set

$$\Sigma^{(1)}_{F(\lambda)} = \{(y, \eta) \in B^* \partial \Omega; \ \beta(y, \eta) = \kappa_F(y, \eta) \}. \quad (96)$$

By assumption, $|\Sigma^{(1)}_{F(\lambda)}| = 0$ and so $|\text{Crit}(\Psi_1)| = 0$ also. Thus from (94) and (96) it follows that $c_1^{\Delta_{c}}(\mu) = o(\mu^{-n-1})$ as $\mu \to \infty$ and so, $\int_0^\infty e^{it\mu} \mu^{2(n-1)} c_1^{\Delta_{c}}(\mu) d\mu = o(t^{-n})$ for $|t|$ small.

### 6.5. Multiple-link contribution to $I^b(t)$.

The main result of this section is the computation of the contribution of higher-order multilinks to $I^b(t)$. We prove here

**Lemma 6.5.** Under the assumption that $|\Sigma_{F(\lambda)}| = 0$,

$$I_j^b(t) = o(t^{-n}) \quad \text{for all } j = 1, \ldots, M.$$

**Proof.** We first compute the contribution to $I^b(t)$ given by double links with initial and terminal points pinned on the boundary. The computation for higher-order multilinks is carried out in the same way as for double links. In view of Lemma 6.2 the contribution of double links amounts to computing the small $t$ asymptotics of

$$I^b_2(t) = \sum_{k=1}^M \int_{\partial \Omega} \int_I \chi_2^{(k)}(s + t) \left([U_{2, \epsilon}^{(k)}(s + t) \circ \hat{F}_\epsilon(s)](q, q) \right) ds \ d\sigma(q).$$

From (82), writing $\tau = |z - q(y')|$ the reflected parametrix

$$[U_{2, \epsilon}^{(k)}(s + t, q(y), q(y'))] = \int_{\Omega} U_{1, \epsilon}^{(k)}(s + t - \tau, q(y), z) \circ U_{1, \epsilon}^{(k)}(\tau, z, q(y')) \ dz$$

$$= (2\pi \mu)^{2n} \int_\Omega \int_{S^{n-1}} \int_{S^{n-1}} \int_0^\infty \int_0^\infty e^{i\lambda \Psi(y, y', r_1 \omega_1, r_2 \omega_2, z, \tau)} B_\epsilon^{(k)}(q(y), q(y')), z, r_1 \omega_1, r_2 \omega_2) \times \chi_k(q(y'), r_2 \omega_2) \chi_k(q(y), r_1 \omega_1) r_1^{n-1} r_2^{n-1} dr_1 dr_2 d\omega_1 d\omega_2 \ dz$$

where,

$$\Psi(y, y', r_1 \omega_1, r_2 \omega_2, z, \tau) := r_1 \langle q(y) - z, \omega_1 \rangle + r_2 \langle z - q(y'), \omega_2 \rangle - (s + t - \tau) r_1 - \tau r_2.$$
In analogy with the single-link case, one applies stationary phase in the spherical variables $(\omega_1, \omega_2) \in S^{n-1}_+ \times S^{n-1}_+$. The result is that
\[
I^{(k)}_2(t) = \int_0^\infty (2\pi \mu e^{i\mu t} c^{(k)}_2(\mu) d\mu,
\]
\[
c^{(k)}_2(\mu) := \int_O \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \int_0^\infty \int_0^\infty e^{i\mu |r_1 q(y)-z|+|r_2 q(y')-z|} A^{(k)}_{\epsilon,0}(q(y), q(y'), \theta)\times B^{(k)}_{\epsilon,0}(q(y), q(y'), z, r_1 r(q(y), z), r_2 r(q(y'), z))\chi_{1}^{(k)}(s+t) r_1^{-1} r_2^{-1} dsdr_1 dr_2 d\sigma(y)d\sigma(y') d\theta dz.
\]
(98)

Now, stationary phase in $(s, r_1)$ gives the critical points $r_1 = 1, s = |q(y) - z| + \tau = |q(y') - z| + |q(y') - z'|$ and so, the result is that
\[
c^{(k)}_2(\mu) = (2\pi \mu)^{-1} \int_O \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i\mu |q(y) - z|+|q(y') - z|} A^{(k)}_{\epsilon,0}(q(y), q(y'), z, \theta)\times \chi^{(k)}_2(|q(y) - z| + |q(y') - z|) d\sigma(y) d\sigma(y') d\theta dz,
\]
(99)
with $A^{(k)}_{\epsilon,0} \in C^\infty(\partial O \times \partial O \times \partial O \times \mathbb{R}^{n-1})$. We expand the $z$-dependent part of the phase function,
\[
\tilde{\Psi}(z, y, y') = |q(y) - z| + |z - q(y)|; \quad z \in O,
\]
(100)
in normal coordinates $(z', z_1)$ near the boundary, so that $z_1 = 0$ on $O$ and $z_1 > 0$ in the interior of $O$. Writing $z = q(z') + z_1 \nu_{z'}$, one makes a Taylor expansion around $z_1 = 0$
\[
\tilde{\Psi}(z, y, y') = |q(y) - z| + |q(y') - z| \]
\[
= |q(y) - q(z')| + |q(y') - q(z')| + \langle \nu_{z'}, r(q(y), q(z')) + r(q(y'), q(z')) \rangle \cdot z_1 + O(z_1^2).
\]
(101)
Since the parametrix is constructed so that the operator wave front propagates along reflected, non-tangential bicharacteristics, it follows that there exists a constant $C(\epsilon) > 0$ such that
\[
\langle \nu_{z'}, r(q(y), q(z')) + r(q(y'), q(z')) \rangle \geq C(\epsilon); \quad (q(y), q(y'), z) \in \text{supp } A_{\epsilon,0}(\cdot, \theta).
\]
(102)
By Taylor expansion of the amplitude, $A^{(k)}_{\epsilon,0}$, in (99) around $z_1 = 0$ and using the inequality (102), one integrates out the $z_1$ normal variable to the boundary in (99). This creates an extra power of $\mu^{-1}$ and the result is that
\[
c^{(k)}_2(\mu) = (2\pi \mu)^{-2} \int_{\partial O} \int_{\partial O} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i\mu |q(y) - q(z')|+|q(y') - q(z')|+\Phi(q(y'), q(y), \theta)} A^{(k)}_{\epsilon,0}(q(y), q(y'), q(z'), \theta)\times a^{(k)}_{\epsilon,0}(q(y), q(y'), q(z'), \theta)\times \chi^{(k)}_2(|q(y) - q(z')| + |q(y') - q(z')|) d\sigma(y) d\sigma(y') d\theta dz',
\]
(103)
with $a^{(k)}_{\epsilon,0} \in C^\infty(\partial O \times \partial O \times \partial O \times \mathbb{R}^{n-1})$. One then applies stationary phase in the $(y', z')$-variables. Strict convexity ensures non-degeneracy of the Hessian. The critical point equations are:
where the assumption that $|I_o| = 0$ implies and $|\text{Crit}(\Psi_2)| = 0$. Summation over the microlocal partition (ie. summing in $k = 1, ..., M$) and an application of (86) then implies that $c_2(\mu) = o(\mu^{-n-1})$ and so, $I_2^b(t) = \int_0^\infty e^{iut}(2\pi \mu)^{2n}c_2(\mu)d\mu = o(t^{-n})$ as $t \to 0$.

The computation for arbitrary multilinks follows in the same way as for double links and this finishes the proof of Lemma 6.5. □

Since $I^b(t) = \sum_{j=1}^M I_j^b(t) + R_\varepsilon(t) + R(t)$ where $R(t)$ is locally smooth and $R_\varepsilon(t) = O(\varepsilon)t^{-n}$ near $t = 0$. Since $\varepsilon > 0$ in Lemma 6.1 can be taken arbitrarily small, by letting $\varepsilon \to 0^+$ and applying the Fourier Tauberian theorem, this completes the proof of Theorem 1 under the assumption that $\partial \Omega$ is strictly-convex.
6.6. WKB Case. In our application to QER, the semi-classical Fourier integral operators $F(\lambda)$ which arise are naturally represented in a somewhat different form with Schwartz kernels of WKB-type. We now modify the proof of Theorem 9 to apply more directly to operators of this form. The proof is very similar to the one given in Theorem 9 but simplifies slightly.

Consider the special case where $F(\lambda)$ has Schwartz kernel

$$F(\lambda)(q(y), q(y')) = \lambda^{\frac{n}{2}} e^{i\lambda \psi(y, y')} B(q(y), q(y'), \lambda),$$

where $B(q(y), q(y'), \lambda) \sim_{\lambda \to \infty} \sum_{k=0}^{\infty} B_k(q(y), q(y')) \lambda^k$ with $B_k \in C^\infty(\partial \Omega \times \partial \Omega)$ and supp $B_k \cap \Delta_{\partial \Omega \times \partial \Omega} = \emptyset$, $k = 0, 1, 2, \ldots$. The Fourier transform

$$\hat{F}(s)(q(y), q(y')) = \int \mu \frac{n}{2} e^{iu \psi(y, y') - s} B(q(y), q(y'), \mu) d\mu.$$

To determine the leading $(t + i0)^{-n}$-singularity of $I^b(t) = Tr \int_{\mathbb{R}} U^b(t + s) \hat{F}(s) ds$, one repeats the argument in Theorem 9, except that there is no $\theta$-integration at the end and one does not carry out stationary phase in $y'$. Rather, one directly computes that the single link contribution to the $(t + i0)^{-n}$-singularity is given by $\int_0^\infty e^{iut} c_1(\mu) d\mu$ where,

$$c_1(\mu) = (2\pi \mu)^{-n-1} \int_{\partial \Omega} \int_{\partial \Omega} e^{i\mu \psi(y, y')} |a(y) - a(y')| B_0(q(y), q(y')) \frac{\partial G_0(q(y), q(y'))}{\partial y} d\sigma(y) d\sigma(y') + O(\mu^{n-2})$$

$$= \int_{\partial \Omega} \int_{\partial \Omega} F(\mu)(q(y), q(y')) [(2\pi \mu) G_0(\mu)(q(y'), q(y))] d\sigma(y) d\sigma(y') + O(\mu^{n-2}),$$

(108)

where, $G_0(\mu)$ is the free Greens function in (31).

The $k$-link contributions for $|k| \geq 2$ are computed in a similar way by inserting the reflection parametrix terms. The result is that

$$c_k(\mu) = \int_{\partial \Omega} \int_{\partial \Omega} F(\mu)(q(y), q(y')) [(2\pi \mu) G_0(\mu)]^k(q(y'), q(y)) d\sigma(y) d\sigma(y') + O(\mu^{n-2}).$$

(109)

As in the proof of Theorem 9, one shows that the critical point equations in $(y, y')$ are equivalent to the fixed point equations $\beta^\delta(y, \eta) = \kappa_F(y, \eta)$; $(y, \eta) \in B^* \partial \Omega$ where $(y, \eta; \kappa_F(y, \eta)) = (y, d_y \psi(q(y), q(y')); y', -d_{y'} \psi(q(y), q(y')))$. It follows from (86) that in the sense of distributions,

$$I^b(t) = o(t^{-n})$$

(110)

under the measure zero assumption, $|\Sigma_{F(\lambda)}| = 0$.

6.7. Non-convex boundary. In the non-convex case, the canonical relation, $\Gamma_{N^0_{\Omega}}$, of the boundary jumps operator is larger than graph of the billiard map, $\Gamma_\beta$. This is due to "spurious" geodesics that exit $\Omega$ at a boundary point $y \in \partial \Omega$, only to re-enter at another point $y' \in \partial \Omega$. Following [HZ], we say that $(y, \eta, y', \eta') \in B^* \partial \Omega \times B^* \partial \Omega$ with $\eta = d_y q(y) - q(y')$ and $\eta' = -d_{y'} q(y) - q(y')$ is a spurious point of $WF_\lambda(N_{\Omega}^0(\lambda))$ if the line $q(y)q(y')$ leaves $\Omega$. It is shown in [HZ] that these geodesics can be ignored for the purposes for spectral asymptotics of boundary traces of eigenfunctions. We briefly recall the argument here. Further details can be found in [HZ] section 11.

Let $b$ be a smooth, compactly-supported non-negative function on $\mathbb{R}^n$ which vanishes on $\Omega$. Consider the metric

$$g_s = (1 + sb)g_{\text{Euclidean}}; \quad s \in [0, \delta],$$

where $g_{\text{Euclidean}}$ is the Euclidean metric on $\mathbb{R}^n$. The semi-classical Fourier integral operators $F(\lambda)$ associated with $g_s$ are then of WKB-type and the leading $(t + i0)^{-n}$-singularity of $I^b(t) = Tr \int_{\mathbb{R}} U^b(t + s) \hat{F}(s) ds$ is given by $\int_0^\infty e^{iut} c_1(\mu) d\mu$ where

$$c_1(\mu) = (2\pi \mu)^{-n-1} \int_{\partial \Omega} \int_{\partial \Omega} e^{i\mu \psi(y, y')} |a(y) - a(y')| B_0(q(y), q(y')) \frac{\partial G_0(q(y), q(y'))}{\partial y} d\sigma(y) d\sigma(y') + O(\mu^{n-2})$$

$$= \int_{\partial \Omega} \int_{\partial \Omega} F(\mu)(q(y), q(y')) [(2\pi \mu) G_0(\mu)(q(y'), q(y))] d\sigma(y) d\sigma(y') + O(\mu^{n-2}),$$

(108)

where, $G_0(\mu)$ is the free Greens function in (31).
with $\delta > 0$ small and the associated resolvent kernel $G_s^\chi(z, z'; \lambda) = (\Delta_s - (\lambda + i0)^2)^{-1}(z, z')$ of the $g_s$-Laplacian on $\mathbb{R}^n$. The associated boundary jumps operator is

$$[N_s]_{\partial \Omega}^\chi(\lambda)(q, q') = 2 \frac{\partial}{\partial \nu_q} G_s(q, q', \lambda); \quad q, q' \in \partial \Omega,$$

and one forms the averaged operator

$$\tilde{N}_{\partial \Omega}^\chi(\lambda) := \int_0^1 \chi(s)[N_s]_{\partial \Omega}^\chi(\lambda) \, ds, \quad (111)$$

where $\chi_0((0, \delta))$ with $\chi \geq 0$ and $\int_0^{\delta} \chi = 1$. The averaged operator, $\tilde{N}_{\partial \Omega}^\chi(\lambda)$ satisfies two key properties (see [HZ] section 11):

$$\begin{align*}
(i) \tilde{N}(\lambda)u^b_{\lambda} &= u^b_{\lambda}, \\
(ii) W F^\chi_{\lambda}(\tilde{N}(\lambda)) &\subset \Gamma_{\beta}.
\end{align*} \quad (112)$$

The first condition follows from Greens formula since the metric $g_s$ stays Euclidean in $\Omega$ and the second, by an integration by parts argument in $s \in [0, \delta]$ (see [HZ] section 11). Now, let $\chi^sp_\epsilon \in C^\infty_0(B^* \partial \Omega)$ with $\chi^sp_\epsilon = 0$ in an $\epsilon$-neighbourhood of set of spurious directions and $\chi^sp_\epsilon = 1$ outside a $2\epsilon$-neighbourhood of the same set. Then, from (112)

$$\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} (F(\lambda_j)u^b_{\lambda_j}, u^b_{\lambda_j}) = \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle \tilde{N}(\lambda_j)F(\lambda_j)\tilde{N}(\lambda_j)u^b_{\lambda_j}, u^b_{\lambda_j} \rangle = \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle \tilde{N}(\lambda_j)Op_{\lambda_j}(\chi^sp_\epsilon)F(\lambda_j)Op_{\lambda_j}(\chi^sp_\epsilon)\tilde{N}(\lambda_j)u^b_{\lambda_j}, u^b_{\lambda_j} \rangle + O(\epsilon) \quad (113)$$

In the second last line of (113), one uses boundary ergodicity and Cauchy-Schwartz to get the $O(\epsilon)$ error term. We replace $F(\lambda_j)$ by $Op_{\lambda_j}(\chi^sp_\epsilon)F(\lambda_j)Op_{\lambda_j}(\chi^sp_\epsilon)$ in the Weyl sum and repeat the argument in the strictly convex case. Finally, we recall that by removing the cutoff $\chi_\epsilon$ supported in $\epsilon$-tubes around the grazing and singular set $G \cup B^{*}_{\epsilon} \partial \Omega$, we obtain an $O(\epsilon)$-error. As in [Z1, MO] we can let $\epsilon \to 0$ to complete the proof of Theorem 9. \hfill \Box

Remark: We may also consider the singularity at any other value $t = t_0$ of

$$\sum_j e^{it\lambda_j} \langle F(\lambda_j)u^b_{\lambda_j}, u^b_{\lambda_j} \rangle.$$

It gives the growth rate of

$$\sum_{j: \lambda_j \leq \lambda} e^{it_0\lambda_j} \langle F(\lambda_j)u^b_{\lambda_j}, u^b_{\lambda_j} \rangle$$

As an example, when $F(\lambda) = N_{\partial \Omega}^\chi(\lambda)$ and under the non-looping assumption, it follows by the argument in Theorem 9 that for $t_0 \neq 0$, $|\Sigma_{F(\lambda)}| = 0$ and so,

$$\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} e^{it_0\lambda_j} u^b_{\lambda_j} \| u^b_{\lambda_j} \|_{L^2(\partial \Omega)}^2 = o(1) \text{ as } \lambda \to \infty.$$
7. LOCAL WEYL LAW ON $H$: CALCULATION OF $\omega_\infty$

We use the local Weyl law for Fourier integral operators to calculate the limit state $\omega_\infty(a)$. It is provisionally defined on $C^\infty(B^*H)$ by

$$\omega_\infty(a) = \lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{j: \lambda_j \leq \lambda} \langle Op_{\lambda_j}(a) \varphi_{\lambda_j} | H, \varphi_{\lambda_j} | H \rangle_{L^2(H)}.$$  \hfill (114)

We now show that the limit exists and we calculate it. We use the notation,

$$\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle Op_{\lambda_j}(a)u_{\lambda_j}^b, u_{\lambda_j}^b \rangle_{\omega_\infty^\partial\Omega}(a)$$

for the limit measure in the boundary case. This section of course depends on the choice of $H$, but the cutoff estimates above show that the singular and tangential sets, hence the convexity of $H$, is irrelevant to the result.

**Lemma 7.1.** The limit state $\omega_\infty$ is well defined and is given by

$$\omega_\infty(a) = \omega_\infty^{\partial\Omega}(\tau_H^\partial\Omega)^* a\rho_H^\infty.$$  

*Proof.* We prove the Lemma by reducing the calculation to the local Weyl law along the boundary in Lemma 1.2 of [HZ].

By (16), we have

$$\omega_H^\infty(a) = \lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle N^H_{\partial\Omega}(\lambda_j)^* Op_{\lambda_j}(a)N^H_{\partial\Omega}(\lambda_j) u_{\lambda_j}^b, u_{\lambda_j}^b \rangle_{L^2(\partial\Omega)}. \hfill (115)$$

By (17), the limit in (115) is the sum of a pseudo-differential Weyl law term and a Fourier integral term. By Egorov’s theorem (see Proposition 4.2), the limit of the pseudo-differential term is

$$\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle Op_{\lambda_j}(\tau_H^{\partial\Omega})^* a\rho_H^\infty u_{\lambda_j}^b, u_{\lambda_j}^b \rangle \to \omega_\infty^{\partial\Omega}(\tau_H^{\partial\Omega})^* a\rho_H^\infty.$$  

To complete the proof, we need to calculate the limit of the Fourier integral term. We claim that

$$\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle F_2(\lambda_j; a, \varepsilon) u_{\lambda_j}^b, u_{\lambda_j}^b \rangle \to 0.$$  

By Theorem [2], it suffices to show that the set of points where the broken billiard map $\beta_H$ equals a power of $\beta$ has measure zero. However, this is special case of our non-commutativity assumption: If $\beta_H(\zeta) = \beta^k(\zeta)$ then $\beta^l\beta_H(\zeta) = \beta^{k+l}(\zeta)$. Also, if $\beta_H(\beta^l\zeta) = \beta^k(\beta^l\zeta)$, then both equal $\beta^{k+l}(\zeta)$. Hence, $\beta^{l}\beta_H(\zeta) = \beta_{\partial\Omega}(\zeta)^* \beta^l$ on a set of positive measure if $\beta_H(\zeta) = \beta^k(\zeta)$ on a set of positive measure and if, for some $\ell \neq 0$, $\beta_H(\beta^l\zeta) = \beta^k(\beta^l\zeta)$ on a set of positive measure. We claim that the first condition implies the second: let the first set be called $B$. Then $B \cap B$ must have positive measure for some $\ell$ by Poincaré’s recurrence theorem or by ergodicity of $\beta$.  

$\square$
8. Quantum ergodic restriction: Proof of Theorem 4

We now have the ingredients to prove the main result. As mentioned in section 4, we prove Theorem 4 first for \( \lambda \)-pseudodifferentials \( Op_\lambda(a) \in Op_\lambda(S_{cl}^{0,0}(T^*H \times [0, \lambda_0^{-1}])) \) with supp \( a \cap \tau_{\partial\Omega}(T \cup B_{r_0}^c \partial\Omega \cup \mathcal{G}) = \emptyset; k = 1, 2. \) In \([8, 4]\) we show that for a full-density of eigenfunctions, \( L^2 \)-mass is not concentrated on the set \( \tau_{\partial\Omega}(B_{r_0}^c \partial\Omega \cup T \cup \mathcal{G}) \) (see \([HZ]\) for an alternative proof of this fact using the boundary trace of the Neumann heat kernel). Using this fact, in \([8, 4]\) we extend the proof of Theorem 4 to arbitrary symbols \( a \in S_{cl}^{0,0}(T^*H \times [0, \lambda_0^{-1}]). \) We therefore assume here that the symbol \( a \in S_{cl}^{0,0}(T^*H \times [0, \lambda_0^{-1}]) \) satisfies the following support condition \([41]\) with arbitrarily small but fixed \( \varepsilon > 0. \)

We now reduce the proof of Theorem 4 to Theorem 10.

8.1. Reduction to Theorem 10. To prove quantum ergodicity for the eigenfunction restrictions \( u_{\lambda_j}^H = \varphi_{\lambda_j} \big|_{H}, j = 1, 2, \ldots \) we put \( F(\lambda) = N_{\partial\Omega}^H(\lambda)^* Op_\lambda(a) N_{\partial\Omega}^H(\lambda) \) and use \((17)\) to compute the variance for the eigenfunction restrictions, \( u_{\lambda_j}^b. \)

\[
\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle Op_\lambda(a) \varphi_{\lambda_j} |_H, \varphi_{\lambda_j} |_H \rangle - \omega_\infty(\sigma(F))|^2
\]

\[
= \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle [F(\lambda_j) - \omega_\infty(\sigma(F))] u_{\lambda_j}^b, u_{\lambda_j}^b \rangle|^2 + o(1)
\]

\[
= \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle [F_1(\lambda_j; a, \varepsilon) - \omega_\infty(\sigma(F)) + F_2(\lambda_j; a, \varepsilon)] u_{\lambda_j}^b, u_{\lambda_j}^b \rangle|^2 + o(1).
\]

\[
\leq \frac{2}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle [F_1(\lambda_j; a, \varepsilon) - \omega_\infty(\sigma(F))] u_{\lambda_j}^b, u_{\lambda_j}^b \rangle|^2 + \frac{2}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle F_2(\lambda_j; a, \varepsilon) u_{\lambda_j}^b, u_{\lambda_j}^b \rangle|^2 + o(1).
\]

(116)

where the limiting state \( \omega_\infty(\sigma(F)) \) was determined in \((17)\).

The second line in \((116)\) follows from the special case of the boundary Weyl law of \([HZ]\) that says that \( \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \| u_{\lambda_j} \|_{L^2} \sim 1. \) From the same asymptotic formula, it follows by the Cauchy-Schwartz inequality that the first term in the last line of \((116)\) is bounded by

\[
\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle [F_1(\lambda_j; a, \varepsilon) - \omega_\infty(\sigma(F))]* [F_1(\lambda_j; a, \varepsilon) - \omega_\infty(\sigma(F))] u_{\lambda_j}, u_{\lambda_j} \rangle + o(1). \]

(117)

We know by Proposition 3.2 that for \( \varepsilon > 0 \) sufficiently small,

\[
F_1(\lambda; a, \varepsilon) \in Op_\lambda(S_{cl}^{0,0}(T^*\partial\Omega)).
\]

By Theorem 9 and boundary quantum ergodicity for \( \lambda \)-pseudodifferentials \([HZ]\), it follows that for \( \varepsilon > 0 \) small,

\[
\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle [F_1(\lambda_j; a, \varepsilon) - \omega_\infty(\sigma(F))] u_{\lambda_j}^b, u_{\lambda_j}^b \rangle| = o_\lambda(1). \]

(118)
As for the second term in (110), by making the further decomposition $F_2(\lambda_j; a, \varepsilon) = F_{21}(\lambda_j; a, \varepsilon) + F_{22}(\lambda_j; a, \varepsilon)$ with $\Gamma_{F_{2k}(\lambda; \varepsilon)} = \text{graph} \beta^{k}_H$: $k = 1, 2$, it follows from the inequality $|a + b|^2 \leq 2(a^2 + b^2)$ that
\[
\frac{2}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle F_2(\lambda_j; a, \varepsilon)u^b_{\lambda_j}, u^b_{\lambda_j} \rangle|^2 
\leq \frac{4}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle F_{21}(\lambda_j; a, \varepsilon)u^b_{\lambda_j}, u^b_{\lambda_j} \rangle|^2 + \frac{4}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle F_{22}(\lambda_j; a, \varepsilon)u^b_{\lambda_j}, u^b_{\lambda_j} \rangle|^2.
\]

The remaining step is the proof of Theorem 10.

8.2. **Proof of Theorem 10.** We now prove a special case of Theorem 10 where $\kappa_F = \beta^1_H$ (resp. $\beta^2_H$), and where $F(\lambda) = F_{21}(\lambda; a, \varepsilon)$ (resp. $F_{22}(\lambda; a, \varepsilon)$). We re-state it for the sake of clarity in the special case.

**Theorem 8.1.** Let $F_{21}(\lambda; a, \varepsilon)$ be a semi-classical Fourier integral operator associated to the canonical relation $\beta^1_H$. Assume that $(\beta_H, \beta)$ almost never commute. Then

\[
\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle F_{21}(\lambda_j; a, \varepsilon)u^b_{\lambda_j}, u^b_{\lambda_j} \rangle|^2 = 0.
\]

Similarly for $F_{22}(\lambda; a, \varepsilon)$.

The proof of the general case is essentially the same. As discussed in the introduction, the proof is based on time-averaging as in [21].

**Proof.** In analogy with the boundary case treated in [HZ] we define the $M$-th order time-averaged operator:
\[
\langle F_{2k}(\lambda; a, \varepsilon) \rangle_M := \frac{1}{M} \sum_{m=1}^{M} [N_{\partial\Omega}^{\beta^{k}_H}(\lambda)]^m F_{2k}(\lambda; a, \varepsilon)[N_{\partial\Omega}^{\beta^{k}_H}(\lambda)]^m; \quad k = 1, 2,
\]

From the boundary jump equation $N_{\partial\Omega}^{\beta^{k}_H}(\lambda)u^b_{\lambda_j} = u^b_{\lambda_j}$, we have for $k = 1, 2$,
\[
\frac{4}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle F_{2k}(\lambda_j; a, \varepsilon)u^b_{\lambda_j}, u^b_{\lambda_j} \rangle|^2 = \frac{4}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle (\langle F_{2k}(\lambda_j; a, \varepsilon) \rangle_M u^b_{\lambda_j}, u^b_{\lambda_j} \rangle|^2.
\]

By the Cauchy-Schwartz inequality (combined with the local Weyl law $\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} ||u^b_{\lambda_j}||^2 \sim_{\lambda \to \infty} 1$), we have
\[
\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle F_{2k}(\lambda_j; a, \varepsilon) \rangle_M u^b_{\lambda_j}, u^b_{\lambda_j} \rangle|^2 \leq \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} |\langle F_{2k}(\lambda_j; a, \varepsilon) \rangle_M u^b_{\lambda_j}, u^b_{\lambda_j} \rangle| + o(1).
\]

**Remark:** This use of the Cauchy-Schwartz inequality may seem rather un-natural. We are trying to prove that $\langle F_{2k}(\lambda_j; a, \varepsilon) \rangle_M u^b_{\lambda_j}, u^b_{\lambda_j} \rangle \to 0$ as $\lambda_j \to \infty$, i.e. that $F_{2k}(\lambda_j; a, \varepsilon) \rangle_M u^b_{\lambda_j}$ is asymptotically orthogonal to $u^b_{\lambda_j}$. Use of the Cauchy-Schwartz inequality converts the orthogonality condition to the condition that the average over the spectrum of the norms $||\langle F_{2k}(\lambda_j; a, \varepsilon) \rangle_M u^b_{\lambda_j}|| \to 0$ is of order $O(\frac{1}{M})$. 

We deal with the case of general symbols at the end in section 8.3. Assume that the symbol \( a \in \mathcal{S}_f^0 \) satisfies the somewhat stronger support assumption

\[
\text{dist} (\text{supp } a, \tau^H_{\partial \Omega} (G_{C(M, \varepsilon)}^* \cup \Sigma_{C'M}^* \cup \mathcal{T}) ) \geq \varepsilon > 0.
\]

(125)

We deal with the case of general symbols at the end in section 8.3.

It suffices to bound the averaged matrix-elements on the right side of (120). Writing out these terms explicitly, one gets

\[
\frac{1}{N(M)} \sum_{\lambda_j \leq \lambda} \langle F_{2k}(\lambda_j; a, \varepsilon) \rangle^*_M \langle F_{2k}(\lambda_j; a, \varepsilon) \rangle_M u^k_{\lambda_j}, u^k_{\lambda_j} \rangle
\]

\[
= \frac{1}{M^2 N(M)} \sum_{m,n=0}^M \sum_{\lambda_j \leq \lambda} \langle \langle F_{2k}(\lambda_j; a, \varepsilon) \rangle^*[N^j_{\partial \Omega} \lambda_j])^m [N^j_{\partial \Omega} \lambda_j] (a, \varepsilon)^n F_{2k}(\lambda_j; a, \varepsilon) [N^j_{\partial \Omega} \lambda_j] n u^b_{\lambda_j}, u^b_{\lambda_j} \rangle
\]

\[
= \frac{1}{M^2} \sum_{m\neq n, m,n=0} \text{same} + \frac{1}{M^2} \sum_{m=n=0} \text{same}
\]

(121)

In (121), we have broken up the double sum into the diagonal and off-diagonal terms.

Due to the time-averaging over \( m, n \) in the sums in (121) it is useful to first slightly refine the microlocal cutoff condition in (44). We first prove (128) and Proposition 8.2 for this somewhat more restrictive class of symbols and then apply a density argument using the pointwise Weyl law argument in Lemma 8.5 to deal with the general case.

We assume that \( |m-n| \leq 2M; m,n \in \mathbb{Z} \) and fix \( \varepsilon > 0 \). We say that a link is \( \varepsilon \)-transversal at the boundary if it makes an angle of at least \( \varepsilon \) to the tangent plane at each point of intersection with \( \partial \Omega \). We define the extremal link-lengths:

\[
L_H(M, \varepsilon) := \max_{|m-n| \leq 2M} \sup_{(y, \eta) \in \Omega^M} \left( |\pi \beta^{-1}_H \beta_{m-n}(y, \eta) - \pi \beta^{m-n}(y, \eta)| \right.
\]

\[
+ \sum_{k=1}^{m\neq n} |\pi \beta^k \beta_{m-n}(y, \eta) - \pi \beta^{k-1}(y, \eta)| + |\pi \beta(y, \eta) - y| \right),
\]

(122)

where, \( \Omega^M := \{(y, \eta); \text{dist}((y, \eta), \Sigma^\varepsilon_{2M+1} \cup G^*_{2M+1} \cup \mathcal{T}) \geq \varepsilon \} \). Thus, \( L_H(M, \varepsilon) \) is the maximum length of \( H \)-broken and \( \varepsilon \)-transversal geodesics with at most \( 2M \)-reflections at the boundary, \( \partial \Omega \). Similarly, we define

\[
L(p, \varepsilon) := \inf \left( \sum_{(y, \eta) \in \Omega^p} |\pi \beta^k(y, \eta) - \pi \beta^{k-1}(y, \eta)| \right),
\]

(123)

Thus, \( L(p, \varepsilon) \) is the minimum length of \( \varepsilon \)-transversal geodesics in \( \Omega \) with \( p \)-reflections at the boundary, \( \partial \Omega \). We define the constant

\[
C(M, \varepsilon) = 2M + 1 + \max \{|p|; L(p, \varepsilon) \leq L_H(M, \varepsilon) + 1\}.
\]

(124)

To estimate the time-averaged matrix elements in (121), in the next two sections, we require that the symbol \( a \in \mathcal{S}_f^0 \) \((T^* H \times (0, \lambda_0])\) satisfy the somewhat stronger support assumption

\[
\text{dist} (\text{supp } a, \tau^H_{\partial \Omega} (G_{C(M, \varepsilon)}^* \cup \Sigma_{C'M}^* \cup \mathcal{T}) ) \geq \varepsilon > 0.
\]

We deal with the case of general symbols at the end in section 8.3.
8.2.1. Analysis of the diagonal sum. The first important point is that the diagonal sum is bounded by \( O\left(\frac{1}{\lambda^2}\right) \). Indeed, the operators
\[
A_{\lambda,n}(\lambda; a, \varepsilon) := \langle [N_{\partial \Omega}^{0,\lambda}^{(n)}](\lambda_j)^n F_{2k}(\lambda_j; a, \varepsilon)^*[N_{\partial \Omega}^{0,\lambda}^{(n)}](\lambda_j)^n \rangle
\]
in the diagonal terms are \( \lambda \)-pseudodifferentials; i.e. for each \( n = 1, \ldots, M \),
\[
A_{\lambda,n}(\lambda; a, \varepsilon) \in \text{Op}_\lambda(S_{\text{cl}}^{0,0}(T^*\partial\Omega)).
\]
By the \( \lambda \)-Egorov theorem for boundary \( \lambda \)-pseudodifferential operators (see [HZ] Lemma 1.4), it follows that with \( \zeta := (y, \eta) \),
\[
A_{\lambda,n}(\lambda; a, \varepsilon) = \mathcal{O}_\lambda \left( \frac{\gamma(\zeta)}{\gamma(\beta^n \zeta)} \times \sigma(F_{2k}(\lambda; a, \varepsilon)F_{2k}(\lambda; a, \varepsilon)) \right) + \mathcal{O}_{n,\varepsilon}(\lambda_j^{-1})_{L^2 \rightarrow L^2} \text{ as } \lambda_j \to \infty.
\]
Substitution of (127) in (121) together with the symbol computation for \( F_{2k}(\lambda; a, \varepsilon); k = 1, 2 \) in Lemma 4.3 and an application of the boundary local Weyl-law for \( \lambda \)-pseudodifferential operators [HZ] gives
\[
\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle A_{\lambda,n}(\lambda; a, \varepsilon) u_{\lambda_j}^b, u_{\lambda_j}^b \rangle = \int_{B^{\ast}{}\partial \Omega} \frac{\gamma(\zeta)}{\gamma(\beta^n \zeta)} \times \sigma(F_{2k}(\lambda; a, \varepsilon)) (\beta^n \zeta)^{2 \gamma^{-1}(\zeta)} dyd\eta \]
\[
= \int_{B^{\ast}{}\partial \Omega} \gamma^{-1}(\beta^n \zeta) \times \sigma(F_{2k}(\lambda; a, \varepsilon)) (\beta^n \zeta)^{2} dyd\eta \]
\[
= \int_{B^{\ast}{}\partial \Omega} \gamma^{-1}(\zeta) \times \sigma(F_{2k}(\lambda; a, \varepsilon)) (\zeta)^{2} dyd\eta \]
\[
= \int_{B^{\ast}H} a(s, \tau; \varepsilon) \rho_{\partial \Omega}^H(s, \tau) dsd\tau.
\]
(128)
since \( \beta \) is symplectic. Summing over the \( M \) terms and dividing by \( \frac{1}{M^2} \) gives the following formula for the diagonal sum
\[
\frac{1}{M^2} \sum_{n=1}^{M} \lim_{\lambda \to \infty} \left( \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \langle A_{\lambda,n}(\lambda; a, \varepsilon) u_{\lambda_j}, u_{\lambda_j} \rangle \right) = \frac{1}{M} \int_{B^{\ast}H} a(s, \tau; \varepsilon) \rho_{\partial \Omega}^H(s, \tau) dsd\tau = \mathcal{O}(M^{-1} \|a\|_{L^\infty}).
\]
(129)

8.2.2. Analysis of the off-diagonal sum. The next step is to apply Theorem 9 to the averaged operator \( F_M(\lambda) \) in (23). The non \( \lambda \) pseudodifferential pieces of this operator can be written as a sum of the \( \lambda \)-FIO’s \( F_{m,n}(\lambda; a, \varepsilon) \) where \( m \neq n \) and \( WF(\lambda a, \varepsilon)) \subset \text{graph} [(\beta_H^m)^*(\beta^n)(\beta_H)] \) (see (130)). One can locally write these canonical graphs in terms of generating functions and so the operators \( F_{m,n}(\lambda) \) can be written as a sum of \( \lambda \)-FIO’s with phase functions of the form (65). Thus, Theorem 9 applies to these operators.

We first observe that we can remove the outer factors of \( N_{\partial \Omega}^{0,\lambda} \) since they have eigenvalue 1 on \( u_{\lambda_j}^b \). For \( a \in S_{\text{cl}}^{0,0}(T^*H \times (0, \lambda_0]) \) satisfying the support assumption in (125), we define
\[
F_{m,n}(\lambda; a, \varepsilon) := F_{2k}(\lambda; a, \varepsilon)^*[N_{\partial \Omega}^{0,\lambda}^{(n)}](\lambda_j)^n \langle [N_{\partial \Omega}^{0,\lambda}^{(n)}](\lambda_j)^n \rangle F_{2k}(\lambda; a, \varepsilon), \ m \neq n.
\]
(130)
The remaining step in Theorem 9 is the following
Proposition 8.2. Let \( a \in S_{cl}^{0,0}(T^*H \times (0, \lambda_0]) \) satisfy the support assumption in (126). Then, under the quantitative almost commutativity assumption in Definition 9, it follows that

\[
\lim_{M \to \infty} \lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \frac{1}{M^2} \sum_{m,n=0; m \neq n}^{M} \sum_{\lambda_j \leq \lambda} \langle F_{m,n}(\lambda_j; a, \varepsilon) u^{b}_{\lambda_j}, u^{b}_{\lambda_j} \rangle = 0. \tag{131}
\]

Proof. In the sum in (131), the quantum observables are non-pseudo-differential \( \lambda \)-Fourier integral operators, i.e. their canonical relations are graphs of non-identity maps or correspondences. We apply the Fourier integral operator local Weyl law of Theorem 9 and the quantitative almost nowhere commuting condition in (3) to prove (131).

For \( m \neq n \), we denote the canonical relation of \( F_{m,n}(\lambda_j; a, \varepsilon) \) by \( \Gamma F_{2k}(\lambda_j; \varepsilon) \). By the composition theorem for semi-classical Fourier integral operators, this canonical relation is the union of the two branches \((k = 1, 2)\),

\[
\text{graph} \left[ (\beta^k_H)^*(\beta^m_H)^*(\beta^k_H) \right] \subset B^* \partial \Omega \times B^* \partial \Omega,
\]

where \( \beta : B^* \partial \Omega \to B^* \partial \Omega \) is the usual billiard map of the domain \( \Omega \) and \( \beta^k_H : C \setminus E_k^{-1}(\Sigma) \to B^* \partial \Omega \) are the branches of billiard maps defined in (13) and in Lemma 411. Equivalently, the canonical relation is the graph of

\[
(\beta^k_H)^{-1} \beta^m_H \beta^k_H; \ k = 1, 2.
\]

Note that there are two factors of \( N_H \partial \Omega \wedge N_H \partial \Omega \), which accounts for the two factors of \( (\beta^k_H) \).

By Theorem 9, the fixed point set whose measure we need to calculate is the fixed point set

\[
(\beta^k_H)^{-1} \beta^m_H \beta^k_H (y, \eta) = (y', \eta'), \ G^{-\tau}(y', \eta') = (y, \eta), \ \tau = \psi_{m-n}(q(y), q(y')), \tag{132}
\]

where \( \psi_{m-n} \) is the phase function of \( F_{m,n}(\lambda; a, \varepsilon) \). Clearly, \( \psi_{m-n}(q(y), q(y')) \) is the length of the broken trajectory which starts at \( q \) in the direction \( \eta + \sqrt{1 - |\eta|^2} \nu_y \), breaks on \( H \), then proceeds on a broken geodesic until it has made \( m - n \) reflections at the boundary. Finally, for the last two links the trajectory breaks again at \( H \). Equivalently,

\[
\Sigma_{F_{m,n}(\lambda; a, \varepsilon)} = \bigcup_{p \in \mathbb{Z}} \mathcal{C}O_{p,m-n},
\]

\[
\mathcal{C}O_{p,m-n} = \{(y, \eta) \in B^* \partial \Omega : \beta^m_H \beta^k_H (y, \eta) = \beta^k_H \beta^p (y, \eta) \}. \tag{133}
\]

As this shows, \( \Sigma_{F_{m,n}(\lambda; a, \varepsilon)} \) depends only on \( m - n \).

We compute

\[
\lim_{M \to \infty} \lim_{\lambda \to \infty} \frac{1}{N(\lambda)} M^2 \left[ \sum_{m,n=0; m \neq n}^{M} \sum_{\lambda_j \leq \lambda} \langle F_{m,n}(\lambda_j; a, \varepsilon) u^{b}_{\lambda_j}, u^{b}_{\lambda_j} \rangle \right]
\leq \lim_{M \to \infty} \frac{1}{M} \sum_{m+n=1}^{2M} \sum_{m-n=1}^{2M} \lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \left[ \sum_{\lambda_j \leq \lambda} \langle F_{mn}(\lambda_j; a, \varepsilon) u^{b}_{\lambda_j}, u^{b}_{\lambda_j} \rangle \right]
\leq C \lim_{M \to \infty} \frac{1}{M} \sum_{m+n=1}^{2M} \sum_{m-n=1}^{2M} \lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \left[ \sum_{\lambda_j \leq \lambda} \langle F_{mn}(\lambda_j; a, \varepsilon) u^{b}_{\lambda_j}, u^{b}_{\lambda_j} \rangle \right]. \tag{134}
\]

The final step in the proof of Proposition 8.2 reduces to estimation of the Weyl sum in the last line of (133) in terms of the quantitative ANC condition in Definition 3. First, we need to define the relevant \( \mu_{p,k}^{\varepsilon} \)-measures (see Definition 3). To do this, we need to...
further $\lambda$-microlocalize the $F_{m,n}(\lambda; a, \varepsilon)$-operators away from large-time iterates of the grazing and singular sets. When $a \in S_{cl}^{0,0}(T^*H \times (0, \lambda_0])$ satisfies \([125]\), by semiclassical wavefront calculus,

$$WF_\lambda(F_2(\lambda; a, \varepsilon)) \subset \iota_\Delta \left( \cap_{|k| \leq C(M, \varepsilon)} \{(y, \eta) \in B^*\partial\Omega; \text{dist}(\beta_k(y, \eta), S^*\partial\Omega) \geq \varepsilon, \text{dist}(\beta_k(y, \eta), \Sigma) \geq \varepsilon \} \right),$$

where, $\iota_\Delta : B^*\partial\Omega \to B^*\partial\Omega \times B^*\partial\Omega$ with $\iota_\Delta(y, \eta) = (y, \eta, y, \eta)$. Consider the closed set $G^*_{C(M, \varepsilon)} \cup \Sigma^*_{C(M, \varepsilon)} \cup \mathcal{T}$ with $|G^*_{C(M, \varepsilon)} \cup \Sigma^*_{C(M, \varepsilon)} \cup \mathcal{T}| = 0$ and the disjoint open set

$$U^M_{\varepsilon} := \cap_{|k| \leq C(M, \varepsilon)} \{(y, \eta) \in B^*\partial\Omega; \text{dist}(\beta_k(y, \eta), S^*\partial\Omega) > \varepsilon, \text{dist}(\beta_k(y, \eta), \Sigma) > \varepsilon \}.$$ 

By the $C^\infty$ Urysohn lemma there exists $\chi^M_\varepsilon \in C^\infty_0(T^*\partial\Omega)$ with the property that $\chi^M_\varepsilon(y, \eta) = 1$ when $(y, \eta) \in G^*_{C(M, \varepsilon)} \cup \Sigma^*_{C(M, \varepsilon)} \cup \mathcal{T}$ and $\chi^M_\varepsilon(y, \eta) = 0$ for $(y, \eta) \in U^M_{\varepsilon}$. To simplify the writing in the following, we abuse notation somewhat and write $\chi^M_\varepsilon$ instead of $Op_\lambda(\chi^M_\varepsilon)$.

Clearly, $\beta_k(U^M_{\varepsilon}) \subset U^M_{\varepsilon, k}$ where for $|k| < C(M, \varepsilon),$

$$U^M_{\varepsilon, k} := \bigcap_{|k| \leq C(M, \varepsilon) - |k|} \{(y, \eta) \in B^*\partial\Omega; \text{dist}(\beta_k(y, \eta), S^*\partial\Omega) > \varepsilon, \text{dist}(\pi\beta_k(y, \eta), \Sigma) > \varepsilon \}.$$

Let $\chi^M_{\varepsilon, k} \in C^\infty_0(\partial\Omega)$ be a smooth cutoff equal to 1 on $G^*_{C(M, \varepsilon)} \cup \Sigma^*_{C(M, \varepsilon)} \cup \mathcal{T}$ and equal to 0 on $U^M_{\varepsilon, k}$. Since $WF_\lambda([\chi^M_{\varepsilon} G_0^\varepsilon(\lambda)]^k) \subset \text{graph } \beta_k$ and $C(M, \varepsilon) \geq 2M + 1 \geq |m - n| + 1$, it then follows by semiclassical wave front calculus that

$$F_{m,n}(\lambda; a, \varepsilon) = \tilde{F}_{m,n}(\lambda; a, \varepsilon) + \mathcal{O}_{\varepsilon, m-n}(\lambda^{-\infty})_{L^2 \to L^2}, \text{ where,}$$

$$\tilde{F}_{m,n}(\lambda; a, \varepsilon) := (1 - \chi^M_{\varepsilon}) F_2(\lambda; a, \varepsilon)^*(1 - \chi^M_{\varepsilon}) \cdot N^\Theta_{\partial\Omega}(\lambda)(1 - \chi^M_{\varepsilon, m-n}) \cdot N^\Theta_{\partial\Omega}(\lambda)(1 - \chi^M_{\varepsilon, |m-n|-1}) \cdots (1 - \chi^M_{\varepsilon, z}) N^\Theta_{\partial\Omega}(\lambda)(1 - \chi^M_{\varepsilon, 1}) F_2(\lambda; a, \varepsilon)(1 - \chi^M_{\varepsilon}) + \mathcal{O}_{\varepsilon, m-n, M}(\lambda^{-\infty})_{L^2 \to L^2}; \ |m - n| \leq 2M. \quad (137)$$

Similarily, we define the microlocalized operator with Schwartz kernel

$$G^\varepsilon_0(q(y), q(y')) := (1 - \chi^M_{\varepsilon}) G_0(\lambda)(1 - \chi^M_{\varepsilon})(q(y), q(y')), \quad (138)$$

where $G_0(\lambda)$ is the free Greens operator in \([31]\). For each $p \in \mathbb{Z}$, the composite operator $F_{m,n}(\lambda; a, \varepsilon)[\lambda G_0^\varepsilon(\lambda)]^p$ is a $\lambda$-Fourier integral operator of order zero in the sense of \([31]\). Consequently, for $(q(y), q(y')) \in \partial\Omega \times \partial\Omega,$

$$\rho^\varepsilon_{p,m-n}(q(y), q(y')) := \limsup_{\lambda \to \infty} \frac{1}{\chi_{n-1}} |\tilde{F}_{m,n}(\lambda; a, \varepsilon)(q(y), q(y'))| \times ||\lambda G_0^\varepsilon(\lambda)||^p(q(y'), q(y))| < \infty.$$

### 8.2.3. The QANC-condition

**Definition 8.3.** Given the map $\omega : \partial\Omega \times \partial\Omega - \Delta_{\partial\Omega \times \partial\Omega} \to B^*\partial\Omega$ with $w(y, y') = (y, d_y|q(y) - q(y')|),$ we define the measures (see Definition \([13])$

$$\mu^\varepsilon_{p,m-n} := w_* (\rho^\varepsilon_{p,m-n}(q(y), q(y'))d\sigma(y)d\sigma(y')).$$

We can now define the QANC condition in terms of the measures $\mu^\varepsilon_{p,m-n}$. 

Here, for \( m > n \), \( \psi \) quantitatively almost never commute if
\[
\lim_{M \to \infty} \frac{1}{M} \sum_{|k| \leq M} \sum_{p \in \mathbb{Z}} \limsup_{\epsilon \to 0^+} \mu_{p,k}(\mathcal{C}O_{p,k} \cap U_{\epsilon}^M) = 0.
\]

Remark: For general piecewise smooth domains, the measures \( \mu_{p,k}^\varepsilon \) are given by formulas that are quite complicated because of multiple-reflection solutions to the commutator equations. However, when \( \partial\Omega \) is convex, the measures \( \mu_{p,m-n}^\varepsilon \) simplify and satisfy the estimates in Lemma 8.4(b) below.

**Lemma 8.4.** Fix \( \varepsilon > 0 \) and let \( a \in S_{cl}^{0,0}(T^*H \times [\lambda_0, \infty)) \) satisfying the support condition in (129).

(a) For \( m, n \in \mathbb{Z} \) with \( 1 \leq |m - n| \leq 2M \),
\[
\lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} (F_{m,n}(\lambda_j; a_M, \varepsilon) u_b^{\lambda_j} u_b^{\lambda_j}) \leq \sum_{p; L(p, \varepsilon) \leq L_H(|m-n|, \varepsilon)+1} \mu_{p,m-n}^\varepsilon(\mathcal{C}O_{p,m-n}) \quad (139)
\]
where, the measures \( \mu_{p,m-n}^\varepsilon \) are defined in Definition 8.3.

(b) When \( \partial\Omega \) is convex and \( \Lambda \subset (B^*\partial\Omega \cap U_{\varepsilon}^M) \) is measurable, for all \( k \) with \( |k| \leq 2M \),
\[
\mu_{p,k}^\varepsilon(\Lambda) \leq \int_\Lambda L_p(y, \eta) \frac{\omega^{\varepsilon}}{\omega^{\varepsilon}} L_k(\beta(\eta, \varepsilon)) \prod_{\ell=1}^p \gamma^{-2}(\beta^\varepsilon(y, \eta)) \, dyd\eta,
\]
with \( L_p(y, \eta) := |\pi \beta^p(y, \eta) - \pi \beta^{p-1}(y, \eta)| + |\pi \beta^{p-1}(y, \eta) - \pi \beta^{p-2}(y, \eta)| \).

**Proof.** Part (a) follows from the formulas in section 6.6. Indeed by (109),
\[
\limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} F_{m,n}(\lambda_j; a, \varepsilon) u_b^{\lambda_j} u_b^{\lambda_j} = \limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} F_{m,n}(\lambda_j; a, \varepsilon) u_b^{\lambda_j} u_b^{\lambda_j}
\]
\[
\leq \sum_{p; L(p, \varepsilon) \leq L_H(|m-n|, \varepsilon)+1} \int_{\mathcal{C}O_{p,m-n}} w_s(|[\lambda G^\varepsilon_0(\lambda)]^p(q(y), q(y'))| \times |F_{m,n}(\lambda; a, \varepsilon)(q(y'), q(y))| \, dq(y) dq(y'))
\]
\[
= \sum_{p; L(p, \varepsilon) \leq L_H(|m-n|, \varepsilon)+1} \mu_{p,m-n}^\varepsilon(\mathcal{C}O_{p,m-n}). \quad (140)
\]

As for part (b), when \( \partial\Omega \) is convex, we prove the estimate in (139) by substituting the explicit WKB formulas for the operators \( N^H_{\partial\Omega}(\lambda) \) and \( N^\varepsilon_{\partial\Omega}(\lambda) \) suitably \( \lambda \)-microlocalized away from the generalized grazing and singular sets. Since \( \Delta \cap WF_\Lambda((1 - \chi_{1,1}^M)N_{\partial\Omega}^\varepsilon(\lambda)(1 - \chi_{1,1}^M)) = \emptyset \) for all \( k = \pm 1, \ldots, \pm 2M \), from (60), the WKB formula for the \( (1 - \chi_{1,1}^M)N_{\partial\Omega}^\varepsilon(\lambda)(1 - \chi_{1,1}^M) \) and a standard stationary phase argument, it follows that
\[
F_{m,n}(\lambda; \varepsilon)(q(y), q(y')) \sim_{\lambda \to \infty} (2\pi \lambda)^{\frac{1}{2}-1} e^{i\psi_{m-n}^H(q(y), q(y'))} [B_{m-n,\varepsilon}^{(0)}(q(y), q(y')) + \lambda^{-1} B_{m-n,\varepsilon}^{(1)}(q(y), q(y')) + \ldots]. \quad (141)
\]
Here, for \( m > n \), \( \psi_{m-n}^H(q(y), q(y')) \) is the length of the locally unique \( H \)-broken, \( \varepsilon \)-transversal \( m-n \) link joining \( q(y) \) and \( q(y') \). For \( m < n \) it is minus the length of the link. In the same
way, for all $p \in \mathbb{Z}$ satisfying $L(p, \varepsilon) \leq L_H(M, \varepsilon) + 1$ (see (124)) it follows by wavefront calculus and repeated applications of stationary phase that

$$
[(1 - \chi^M) \lambda G_0(\lambda)(1 - \chi^M)]^p(q(y), q(y')) \\
\sim_{\lambda \to \infty} (2\pi \lambda)^{-\frac{n-1}{2}} e^{i\lambda \psi_p(q(y), q(y'))} [A_{p, \varepsilon}^{(0)}(q(y), q(y')) + \lambda^{-1} A_{p, \varepsilon}^{(1)}(q(y), q(y')) + \cdots].
$$

(142)

In (142), for $p > 0$ the phase $\psi_p(q, q')$ is the length of the locally unique $p$-link joining $q(y)$ and $q(y')$. For $p < 0$ it is negative of the length. In the following, it will be useful to define the sum of these phase functions

$$
\Psi_{m-n,p}(q(y), q(y')) := \psi_{m-n}^H(q(y), q(y')) + \psi_p(q(y'), q(y)).
$$

(143)

It is clear from (123) and (122) that for $|m - n| \leq 2M$,

$$
\inf_{q, q'} |\Psi_{m-n,p}(q(y), q(y'))| \geq L(p, \varepsilon) - L_H(M, \varepsilon).
$$

(144)

We write $y_k = \pi \beta^k(y, \eta); k = 1, \ldots, m - n - 1$ for the intermediate reflection points of the geodesic joining $y_0 = y$ and $y_{m-n} = y'$ and put $L(q(y_p), q(y_{p-2})) = |q(y_p) - q(y_{p-1})| + |q(y_{p-1}) - q(y_{p-2})|$ (see Definition 3). Using the convexity of $\partial \Omega$, it follows that for $p \geq 2$,

$$
|A_{p, \varepsilon}^{(0)}(q(y), q(y'))| \leq \prod_{k=1}^{p-2} \left( \frac{|q(y_{k+1}) - q(y_k)|}{|q(y_{k+1}) - q(y_k)| + |q(y_k) - q(y_{k-1})|} \right)^{\frac{n-1}{2}} \times \prod_{k=1}^{p-1} \left( \frac{1}{|q(y_k) - q(y_{k-1})| + |q(y_k) - q(y_{k+1})|} \right)^{\frac{n-1}{2}}
$$

$$
\times \prod_{k=1}^{p-1} (\nu_{y_k}, r(q(y_k), q(y_{k+1})))^{-1} \times \left( \frac{1}{|q(y_k) - q(y_{k-1})| + |q(y_k) - q(y_{k+1})|} \right)^{\frac{n-1}{2}}
$$

$$
\leq L(q(y_p), q(y_{p-2}))^{-\frac{n-1}{2}} \prod_{k=1}^{p-1} (\nu_{y_k}, r(q(y_k), q(y_{k+1})))^{-1} \times \prod_{k=1}^{p-1} (\nu_{y_k}, r(q(y_k), q(y_{k-1}))) + r(q(y_k), q(y_{k+1}))^{-1}
$$

$$
= 2^{-(p-1)} L(q(y_p), q(y_{p-2}))^{-\frac{n-1}{2}} \prod_{k=1}^{p-1} (\nu_{y_k}, r(q(y_k), q(y_{k+1})))^{-2}.
$$

(145)

In the last line of (145) we have used that $\langle \nu_{y_k}, r(q(y_k), q(y_{k-1})) \rangle = \langle \nu_{y_k}, r(q(y_k), q(y_{k+1})) \rangle$.

Similarly, we let $y_0^H = \pi \beta_H(y, \eta), y_{[m-n+1]}^H = \pi \beta_H^{-1} \beta^{m-n} \beta_H(y, \eta)$ and $y_k^H = \pi \beta_k \beta_H(y, \eta), k = 1, \ldots, |m - n|$. It follows that with a constant $C_{H, \Omega} > 0$,

$$
|B_{m-n, \varepsilon}^{(0)}(q(y), q(y'))| \leq 2^{-|m-n|-1} C_{H, \Omega} L(q(y_{m-n}^H), q(y_{m-n-2}^H))^{-\frac{n-1}{2}}.
$$

(146)

The estimate in (146) follows in the same way as in (145) noting that $N_{H, \Omega}^{(0)}(\lambda)(q(y), q(y'))$-kernel has the additional $\langle \nu_y, r(q(y), q(y')) \rangle$-term in the numerator as compared with the Greens function $G_0(\lambda)(q(y), q(y'))$. This accounts for the absence of the additional $\langle \nu_{y_k}, r(q(y_k), q(y_{k-1})) \rangle$-terms in the denominator of (146). It then follows from Theorem 9 (see also the argument
in subsection [6.6], the time-cutoff Lemma [6.1] and (144) that

\[
\limsup_{\lambda \to \infty} \left| \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} F_{m,n}(\lambda_j; a, \varepsilon) u_{\lambda_j}^b, u_{\lambda_j}^b \right| = \limsup_{\lambda \to \infty} \left| \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} (\bar{F}_{m,n}(\lambda_j; a, \varepsilon) u_{\lambda_j}^b, u_{\lambda_j}^b) \right|
\]
\[
\leq \limsup_{\lambda \to \infty} \frac{C}{N(\lambda)} \sum_{p: L(p, \varepsilon) \leq L_H(|m-n|, \varepsilon) + 1} \left| \int_{\text{Crit}(\Psi_{m-n,p})} \left[ \lambda G_0^p(\lambda) \right]^p(q(y), q(y')) \times \bar{F}_{m,n}(\lambda; a, \varepsilon)(q(y'), q(y)) \ dq(y) dq(y') \right|
\]
\[
\leq \limsup_{\lambda \to \infty} C \sum_{p: L(p, \varepsilon) \leq L_H(|m-n|, \varepsilon) + 1} \left| \int_{\text{Crit}(\Psi_{m-n,p})} A_{0, \varepsilon}^p(q(y), q(y')) B_{m-n, \varepsilon}^p(q(y'), q(y)) \right| \ d\sigma(y) d\sigma(y').
\]

Substitution of the estimates (145) and (146) in (147) gives

\[
\limsup_{\lambda \to \infty} \left| \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} F_{m,n}(\lambda_j; a, \varepsilon) u_{\lambda_j}^b, u_{\lambda_j}^b \right|
\]
\[
\leq \sum_{p: L(p, \varepsilon) \leq L_H(|m-n|, \varepsilon) + 1} 2^{-(p+|m-n|+2)} \left| \int_{\text{Crit}(\Psi_{p,m-n})} L(q(y_H^H), q(y_H^m-n-2)) \right| \frac{2^{-1}}{n-1} \times \left| \int_{\text{Crit}(\Psi_{p,m-n})} L(q(y_H^H), q(y_H^m-n-2)) \right| \frac{2^{-1}}{n-1} \times \left| \int_{\text{Crit}(\Psi_{p,m-n})} L(q(y_H^H), q(y_H^m-n-2)) \right|
\]
\[
\times L(q(y_p), q(y_{p-2})) \frac{2^{-1}}{n-1} \prod_{k=1}^{n-1} \left| v_{y_k}, r(q(y_k), q(y_{k+1})) \right|^{-2} d\sigma(y) d\sigma(y').
\]

We rewrite (147) by making the change of variables \( w : (y, y') \mapsto (y, \pi_T y'(q(y), q(y'))) \): (\( y, \eta \)). The result is that

\[
\limsup_{\lambda \to \infty} \left| \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} F_{m,n}(\lambda_j; a, \varepsilon) u_{\lambda_j}^b, u_{\lambda_j}^b \right|
\]
\[
\leq \sum_{p: L(p, \varepsilon) \leq L_H(|m-n|, \varepsilon) + 1} \left| \int_{\text{Crit}(\Psi_{p,m-n})} L(q(y_H^H), q(y_H^m-n-2)) \right| \frac{2^{-1}}{n-1} \times \left| \int_{\text{Crit}(\Psi_{p,m-n})} L(q(y_H^H), q(y_H^m-n-2)) \right|
\]
\[
\times L(q(y_p), q(y_{p-2})) \frac{2^{-1}}{n-1} \prod_{k=1}^{n-1} \left| v_{y_k}, r(q(y_k), q(y_{k+1})) \right|^{-2} d\sigma(y) d\sigma(y').
\]

This finishes the proof of part (b).

\[
\square
\]

In view of the quantitative almost commutativity assumption, substitution of the bound (139) in the last line of (134) concludes the proof of the Proposition [8.2]

8.3. Completion of the proof of Theorem 4: general symbols. Given \( a \in S_{r_0}^{0,0}(T^*H \times (0, \lambda_0]) \) we let \( \chi_\varepsilon^M \in C_0^\infty(T^*\partial\Omega) \) as above and write \( a = (T_{\partial\Omega}^H)^* \chi_\varepsilon^M a + [1 - (T_{\partial\Omega}^H)^* \chi_\varepsilon^M] a \). From
Proposition 8.2 and the diagonal step, it follows that
\[
\limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda \leq \lambda} |(Op_{\lambda_j}(a) \varphi_{\lambda_j}|_{H}, \varphi_{\lambda_j}|_{H}) - \int_{B^*H} a \rho_{\Omega^1} dsdt|^2 \\
\leq 3 \limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda \leq \lambda} |(Op_{\lambda_j}([1 - (\tau_{\Omega^1})^n H^* \chi_{\lambda}^M]a) \varphi_{\lambda_j}|_{H}, \varphi_{\lambda_j}|_{H}) \\
- \int_{B^*H} (1 - \chi_{\lambda}^M)(s, \tau)a(s, \tau) \rho_{\Omega^1} dsdt|^2 \\
+ 3 \left| \int_{B^*H} \chi_{\lambda}^M(s, \tau)a(s, \tau) \rho_{\Omega^1} dsdt \right|^2 \\
+ 3 \limsup_{\lambda \to \infty} \frac{2}{N(\lambda)} |(Op_{\lambda_j}([\tau_{\Omega^1}^H]^* \chi_{\lambda}^M]a) \varphi_{\lambda_j}|_{H}, \varphi_{\lambda_j}|_{H})|^2.
\]

From Lemma 8.5 it follows that
\[
\limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} |(Op_{\lambda_j}([\tau_{\Omega^1}^H]^* \chi_{\lambda}^M]a) \varphi_{\lambda_j}|_{H}, \varphi_{\lambda_j}|_{H})|^2 = O(M(\varepsilon)),
\]
and it is clear that
\[
\left| \int_{B^*H} \chi_{\lambda}^M(s, \tau)a(s, \tau) \rho_{\Omega^1} dsdt \right| = O(M(\varepsilon)).
\]

Finally, since dist( supp( a - [(\tau_{\Omega^1}^H)^* \chi_{\lambda}^M]a ), \tau_{\Omega^1}^H(G_{C(M, \varepsilon)}^* \cup \Sigma_{C(M, \varepsilon)}^* \cup T) ) \geq \varepsilon, it follows from (128) and Proposition 8.2 that
\[
\limsup_{\lambda \to \infty} \frac{2}{N(\lambda)} \sum_{\lambda \leq \lambda} |(Op_{\lambda_j}(a - [(\tau_{\Omega^1}^H)^* \chi_{\lambda}^M]a) \varphi_{\lambda_j}|_{H}, \varphi_{\lambda_j}|_{H}) - \int_{B^*H} (1 - \chi_{\lambda}^M)(s, \tau)a(s, \tau) \rho_{\Omega^1} dsdt|^2 \\
\leq \frac{1}{M^2} \sum_{n=1}^M \left( \limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda \leq \lambda} |Op_{\lambda_j}(a - [(\tau_{\Omega^1}^H)^* \chi_{\lambda}^M]a)_{n,n}(\lambda_j, a, \varepsilon)u_{\lambda_j}^b, u_{\lambda_j}^b) \right) \\
+ \frac{1}{M} \sum_{|m-n|=1} \sum_{p \in \mathbb{Z}} \mu_{p, m-n}^\varepsilon (\mathcal{C} O_{p, m-n}) \\
= \frac{1}{M} \int_{B^*H} (1 - \chi_{\lambda}^M)a \rho_{\Omega^1} dsdt + o(1) \leq \frac{C}{M} \|a\|_{L^\infty} + o(1)
\]
as \(M \to \infty\). By the quantitative almost commutation condition in Definition 13 for the last term on the RHS of (153), \(\limsup_{\varepsilon \to 0^+} o(1) = o(1)\) as \(M \to \infty\). The constant \(C = C(\Omega) > 0\) in the last line of (153) is uniform in \(\varepsilon\) and \(M\). Letting \(\varepsilon \to 0^+\) kills the first two terms in (151) and (152) and then taking \(M \to \infty\) kills the last term in (153). This completes the proof of Theorem 1.

\[\Box.\]

Remark: As a check on the midline of the stadium, we note that in that case \(|\Sigma_{m,n}| = 1\) for all \(m, n\) since \(\beta_{\Omega^1}^H\) commutes with the transmission map \(\beta_{\Omega^1}^{H,k}\) (which equals \(\sigma\)). Thus, the sets in the union in (133) are empty except when \(m - n = p\) and that one has full measure. It is simple to check that the limit is non-zero.

8.4. Extensions and generalizations. We now observe that the proof so of Theorems 1 and 3 extend to general symbols and to Dirichlet boundary conditions.
8.4.1. Eigenfunction mass along tangential and singular sets. The extension of Theorems \[1\] and \[3\] in section 8.3 (see (151)) to general symbols \(a \in S^{0,0}(T^*H \times (0,\lambda_0^{-1}])\) relies on showing that for a full-density of eigenfunctions, mass does not concentrate along tangential or singular sets.

The point of this subsection is to establish this fact (see Lemma 8.5) by using a pointwise microlocal Weyl law argument to show that there is no such mass concentration on any closed set of measure zero. This method has the advantage of working whether or not \(\Omega\) has a boundary. An alternative method is to use the potential layer \(N^H_{\partial\Omega}(\lambda)\) to reduce the problem to \(\partial\Omega\) and then apply the argument in \([HZ]\) (see Lemmas 7.1, 9.2 and Appendix 12) which uses the Karamata Tauberian argument for the restriction of the Neumann heat kernel to the boundary diagonal.

In the following we let \((x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}\) denote normal coordinates near the interior hypersurface \(H \subset \Omega\) defined by \(\Omega \ni x = q_H(x')+x_n\nu_e'\) and write \((\xi', \xi_n)\) for the corresponding fiber coordinates defined by \(T_x^*\Omega \ni \xi = \xi' + \xi_n\nu_e\). In the following, we denote the restriction operator to \(H\) by \(\gamma_H : f \mapsto f|_H\) and continue to write \(\zeta : B^*_H \to S_H^*\Omega\) for the map \(\zeta : (x', \xi') \mapsto (x', \xi' + \sqrt{1 - |\xi'|^2}\nu_e').\)

Let \(Z \subset B^*_H\) be a closed subset with \(|Z| = 0\). Then, by the \(C^\infty\) Urysohn lemma, we can choose \(\chi_{\varepsilon,Z}(x', \xi)\) smooth and positive homogeneous of degree zero with the property that for \((x', \xi) \in S_H^*\Omega\),

\[
\begin{align*}
\chi_{\varepsilon,Z}(x', \xi) &= 1 \quad \text{when} \quad \text{dist} \left((x', \xi), \zeta(Z)\right) \leq \varepsilon, \\
\chi_{\varepsilon,Z}(x', \xi) &= 0 \quad \text{when} \quad \text{dist} \left((x', \xi), \zeta(Z)\right) \geq 2\varepsilon.
\end{align*}
\]  

(154)

We define

\[
\tilde{\chi}_{\varepsilon,Z}(x, \xi) = \chi_{\varepsilon,Z}(x', \xi) \cdot \chi(x_n),
\]

where, \(\chi(u) \in C^\infty_0(\mathbb{R})\) equal to 1 near \(u = 0\) and the corresponding restriction \(\chi_{\varepsilon,Z}^H \in C^\infty_0(T^*H)\) given by

\[
\chi_{\varepsilon,Z}^H(x', \xi) : = \chi_{\varepsilon,Z}(x'; \xi', \xi_n = 0).
\]  

(155)

Consider the operator \(\text{Op}(\tilde{\chi}_{\varepsilon,Z}) : C^\infty(\Omega) \to C^\infty(\Omega)\) and let \(U(t) = e^{it\sqrt{\Delta_N}}\) denote the Neumann wave operator on \(\Omega\). We wish to compute the small time asymptotics for \(\sum_j e^{i\lambda_j t}\|\text{Op}_{\lambda_j}(\chi_{\varepsilon,Z}^H)u_{\lambda_j}^H\|_{L^2(H)}^2\)

and then apply the usual Fourier Tauberian theorem to get large \(\lambda\) asymptotics for the \(\lambda\)-microlocalized Weyl sum \(\frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \|\text{Op}_{\lambda_j}(\chi_{\varepsilon,Z}^H)u_{\lambda_j}^H\|_{L^2(H)}^2\).

For \(|t| < \text{dist}(x', \partial\Omega)\), we have that

\[
U_{\varepsilon,Z}(t, x', x') := \sum_j e^{i\lambda_j t} \left[\text{Op}(\chi_{\varepsilon,Z}^H)u_{\lambda_j}^H\right](x')^2
\]  

(156)

\[
= [\gamma_H \text{Op}(\tilde{\chi}_{\varepsilon,Z}) U(t) \text{Op}(\tilde{\chi}_{\varepsilon,Z})^* \gamma_H^H](t, x', x'), \quad x' \in H.
\]

Since \(H\) is an interior hypersurface, for \(|t| < \text{dist}(x', \partial\Omega)\), bicharacteristics starting from \(x' \in H\) do not intersect the boundary, \(\partial\Omega\). Therefore, by propagation of singularities, modulo a \(C^\infty\)-error, we can substitute the interior parametrix \(U_0(t, x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i|x-y,\xi|} a(x, y, \xi) d\xi\)

for \(U(t, x, y)\) in (156). By the usual composition calculus for Fourier integral and pseudodifferential operators, it follows from (156) that

\[
U_{\varepsilon,Z}(t, x', x') = (2\pi)^{-n} \int_{T^*\Omega} e^{-i|\xi|} |\chi_{\varepsilon,Z}(x', \xi)|^2 d\xi + \cdots,
\]  

(157)
where the dots denote terms with lower-order singularities at \( t \to 0 \). By making a polar coordinates decomposition in the fiber variables in (157) and using that \( \chi_{\varepsilon, Z} \) is homogeneous of degree zero, it follows that as \( t \to 0 \),

\[
U_{\varepsilon, Z}(t, x', x') = (2\pi)^{-n} \left( \int_{S^*_r, \Omega} |\chi_{\varepsilon, Z}(x', \omega)|^2 d\omega \right) (t + i0)^{-n} + o(t^{-n})
\]

(158)

Finally, an application of the Fourier Tauberian theorem, integration over \( x' \in H \) on both sides of (158) and using that \( Op(\chi_{\varepsilon, Z}) = Op\lambda_j(\chi_{\varepsilon, Z}) \) (since \( \chi_{\varepsilon, Z} \) is homogeneous of order zero) proves the following lemma:

**Lemma 8.5.** Let \( Z \subset B^*H \) be closed with \( |Z| = 0 \) and for arbitrary \( \varepsilon > 0 \) let \( \chi_{\varepsilon, Z} \in C^\infty_0(T^*H) \) be the symbol in (157). Then,

\[
\limsup_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_j \leq \lambda} \|Op\lambda_j(\chi_{\varepsilon, Z})u_{\lambda_j})\|_{L^2(H)}^2 = (2\pi)^{-n} \left( \int_{B^*H} |\chi_{\varepsilon, Z}(x', \xi')|^2 \gamma_{-1}(x', \xi') dx'd\xi' \right) = O(\varepsilon).
\]

Thus, for a full-density of eigenfunctions, mass does not concentrate on measure-zero closed subsets \( Z \subset B^*H \).

In subsection 8.3 (see (151)), we apply Lemma 8.5 with \( Z = \tau_{\partial\Omega}(\Sigma_{C^\infty(M, \varepsilon)} \cup G^*_{C^\infty(M, \varepsilon)} \cup T) \) to extend Theorem 4 to arbitrary symbols \( \sigma \in S^{0,0} \).

**8.4.2. The Dirichlet Case.** The proof in the Dirichlet case is very similar to the Neumann case. There are a few minor changes: the relevant density is \( \rho_H(s, \eta) = (1 - |\eta|^2)^{1/2} \) for Dirichlet and this follows from the result in [HZ] for boundary ergodicity. Also, the \( N_{\partial\Omega}^H \) operator gets replaced by the operator with kernel \( G_0(q, q_H; \lambda) \) and this operator has the same microlocal properties as \( N_{\partial\Omega}^H \). Finally, of course the boundary traces of Dirichlet eigenfunctions are \( v^H_{\lambda} : \partial_\nu \varphi_{\lambda} \).

**8.5. Nodal intersections with interior curves: Proof of Corollary 8.**

**Proof.** Let \( \Omega \subset \mathbb{R}^2 \) be a piecewise analytic planar domain. The corollary follows from Theorem 6 in [IZ] provided we establish the following "goodness" condition on \( C \subset \Omega \):

\[
\frac{\|u^{C,\nu}_{\lambda}\|_{L^2(C)}}{\|u^{C}_{\lambda}\|_{L^2(C)}} = O(e^{a\lambda}).
\]

(159)

Here, we write \( u^{C}_{\lambda} = \varphi_{\lambda}|_C \) and \( u^{C,\nu}_{\lambda} = \partial_{\nu C} \varphi_{\lambda}|_C \) and \( a > 0 \) is an arbitrary positive constant. Assume the eigenfunctions \( \varphi_{\lambda} \) are \( L^2 \)-normalized in \( \Omega \) so that \( \int_{\Omega} |\varphi_{\lambda}|^2 dx = 1 \). Then, by Theorem 4 it follows that for an ergodic sequence of the \( u^{C,\nu}_{\lambda} \)s,

\[
\|u^{C}_{\lambda}\|_{L^2(C)} \sim 1
\]

as \( \lambda \to \infty \). By standard sup-estimates, we also have that

\[
\sup_{x \in \Omega} |\partial_\nu \varphi_{\lambda}(x)| = O(\lambda^{3/2}).
\]

Corollary 8 then follows from (159) since clearly \( \lambda^{3/2} \ll e^{a\lambda} \) for any \( a > 0 \).
9. Generic interior hypersurfaces $H \subset \Omega$: Proof of Proposition 6

In this section, we prove the genericity of the condition of almost never commutativity stated in Proposition 6. As mentioned in the introduction, the non-commuting condition should be generic in a much wider sense. We present only a simple result here for the sake of brevity; we hope to give a more general one elsewhere.

There are at least two standard ways to approach generic domains. Following [CPS, PS1, PS2, S], we may consider the Frechet space $C^\infty(S^1, \Omega)$ of smooth embeddings $f: S^1 \to \Omega^\circ$ (the interior of $\Omega$); one could relax the smoothness to obtain a Banach space $C^k(H, \Omega)$ for some $k \geq 2$. A second approach (see [U, F0]) is to fix a hypersurface $H$ (here, a curve) and to parameterize nearby hypersurfaces by flowing out $H$ under vector fields along $H$, extended smoothly to a neighborhood of $H$. In either definition, an infinitesimal variation of $H$ is given by a smooth vector field $X$ along $H$. We say that a property is generic if it holds for a residual subset of the relevant space, i.e. a subset containing a countable intersection of dense open sets.

The main idea is to prove that the sets $CO_{k,p}$ of Definition 3 are smooth hypersurfaces away from the corners. To do this we use a well-known transversality theorem. We recall that if $F: M \to N$ is a $C^1$ map of Banach manifolds, then $x \in M$ is a regular point if $DF_x: T_x M \to T_{F(x)} N$ is surjective. Also, $y \in N$ is a regular value if every point of $F^{-1}(y)$ is regular.

**Transversality theorem** [AR, O, U] Let $\Phi: \mathcal{H} \times B \to E$ be a $C^k$ map of Banach manifolds with $E, \mathcal{H}$ separable. If 0 is a regular value of $\Phi$, and if $\Phi_H = \Phi(H, \cdot)$ is a Fredholm map of index $< k$ then the set $\{H \in \mathcal{H}: 0$ is a regular value of $\Phi_H\}$ is residual in $\mathcal{H}$.

We let $X = S^1$ and $Y = \Omega$. The Fredholm condition is then trivial. We denote an embedding by $f: S^1 \to \Omega$ and let $H = f(S^1)$. The properties of $f$ which concern us are properties only of the image $H$. The principal property is that the measure of $CO_{p,k}$ is positive. We introduce the partially defined symplectic correspondence,

$$\Phi_{j,p}: \mathcal{H} \times B^*\partial \Omega \to B^*\partial \Omega, \quad \Phi_{k,p}(H, \zeta) = \beta^{-p} \beta_H^{k*} \beta^j \beta_H^k(\zeta).$$

Here, $\beta_H^{k*}$ denotes the (partially defined) inverse to $\beta_H^k$, which makes sense since $\beta_H^k$ is symplectic on its domain of definition. More precisely, we restrict $\zeta$ to the domain of $\beta^{-p} \beta_H^{k*} \beta^j \beta_H^k$, which depends (slightly) on $H$. Our aim is to prove that, for generic (i.e. a residual set of) $H$, the set $\{\zeta \in B^*\partial \Omega : \beta^{-p} \beta_H^{k*} \beta^j \beta_H^k(\zeta) = \zeta\}$ is a hypersurface (i.e. curve) away from the corner set in $B^*\partial \Omega$ and hence of Minkowski content zero. It suffices to prove that 0 is a regular value of the map $\Phi_{j,p}(H, \zeta) - \zeta$, i.e. that $D(\Phi_{j,p}(H, \zeta) - \zeta)$ is surjective for each $(H, \zeta)$ such that $\Phi_{j,p}(H, \zeta) - \zeta = 0$. If $\Phi_{j,p}(H, \zeta) \zeta$ then $D_\zeta \Phi_{j,p}(H, \zeta) : T_\zeta B^*\partial \Omega \to T_\zeta B^*\partial \Omega$. Surjectivity of

$I - D_\zeta \Phi_{j,p}(H, \zeta): T_\zeta B^*\partial \Omega \to T_\zeta B^*\partial \Omega$

holds if and only if 1 is not an eigenvalue of this kind of ‘Poincaré map’ for generic $H$ except for a possible curve in $B^*\partial \Omega$. Note that if $CO_{p,j}$ had positive measure, then all eigenvalues of this Poincaré map would equal one at any point of density $(y, \eta)$ of this set. In [PS1, PS2, S] the somewhat analogous result is proved that for generic domains, the spectrum of the Poincaré map for every periodic reflecting ray omits 1 from its spectrum. It is possible that the multi-jet transversality approach of that paper could be adapted to our problem. Here we opt for a simpler proof when the billiard flow is hyperbolic.
Proof. We tacitly view the equation $\beta_H^j \beta_p(y, \eta) - \beta_H \beta_H^j(y, \eta) = 0$ as written in local coordinates in $B^* \partial \Omega$. We henceforth write $\beta_H^j = \beta_H$ and suppress the fact that $\beta_H$ is double-valued since we can separately consider the four branches of $\Phi_{j,p}$. We would like to show that the derivative in $H$ of the vector valued function $\beta_H^j \beta_p(y, \eta) - \beta_H \beta_H^j(y, \eta)$ spans the tangent space $T_{\beta_H^j \beta_p(y, \eta)} B^* \partial \Omega$ at each point where the equation holds.

The variation $\delta \beta_H(y, \eta)$ is defined as follows: Let $H_\epsilon$ be a smooth curve of hypersurfaces through $H_0 = H$ with variational vector field $X$ along $H$. Then,

$$\delta \beta_H(\zeta) := \frac{d}{d\epsilon}|_{\epsilon=0} \beta_{H_\epsilon}(\zeta).$$

It is a tangent vector to a curve through $\beta_H(\zeta)$ depending on the vector field $X$ on $H$ defining the variation. i.e. $\delta \beta_H(\zeta) : T_H \mathcal{H} \rightarrow T_{\beta_H(\zeta)} B^* \partial \Omega$. To see the dependence on $X$ we give an explicit formula for $\beta_H$ similar to that in §3.1

$$\beta_H(y, \eta) = E(t_H(y, -\eta), y, -\eta) - (t^2_H(y, \eta) - t_H^1(y, -\eta)) \zeta_H^j(y, -\eta).$$

(160)

Here, $t_H^j(y, -\eta)$ is the time at which the reflected ray hits $\partial \Omega$ and $\zeta_H^j(y, \eta)$ is defined in (38). This vector is smooth in the unit normal $\nu_{q_H^j(y, \eta)}$. The variation thus consists of two types of terms: ones where we differentiate the hitting times and one where we vary the normal direction:

$$\delta \zeta_H^j(y, -\eta) = 2 \delta(\zeta \cdot \nu_{q_H^j(y, -\eta)}) \nu_{q_H^j(y, -\eta)}. $$

(161)

It is clear that both terms depend on (and only on) $X_{q_H^j(y, -\eta)}$ (we recall that $q_H^j(y, -\eta)$ is the point where the trajectory from $(y, \eta)$ intersects $H$ (see Definition 3.4). Thus, $\delta \beta_H(0, y, -\eta)$ depends on (and only on) the value of $X$ at the point $q_H^j(y, -\eta)$.

The variation of the equation has the form $\delta \beta_H(\beta_p(y, -\eta)) = D^{\beta_H}(\delta \beta_H(y, -\eta))$. As just noted, $\delta \beta_H(y, -\eta)$ depends on (and only on) the value of $X$ at $q_H^j(y, -\eta)$. On the other hand, $\delta \beta_H(\beta_p(y, -\eta))$ depends only on the point $q_H^j(y, -\eta)$ where the trajectory intersects $H$ after $p$ bounces. We now prove that these points must be different.

**Lemma 9.1.** Let $\Omega$ be a hyperbolic planar billiard table. Let $H \subset \Omega$ be a convex curve. Then for almost all $(y, \eta) \in B^* \partial \Omega$, the set of $\leq 2$ points where the line segment $\overline{y, \eta}$ hits $H$ is disjoint from the set where the line segment $\overline{\beta_H(y, \eta)}$ hits $H$.

**Proof.** The statement is equivalent to saying that the ‘probability’ in $\partial \Omega \times H$ of pairs $(x, q)$ such that the trajectory defined by $(x, -\frac{x-q}{|x-q|})$ has the property that the point $q$ where it first hits $H$ is the same as the point $q'$ where it hits $H$ after $p$ bounces from $\partial \Omega$. We may think of the orbit as beginning at $q$ rather than $x$ with the same initial direction $-\frac{x-q}{|x-q|}$ and proceeding along the billiard flow of $\Omega$ until it has bounced off $\partial \Omega$ $p$ times. We then terminate the trajectory when it hits $H$ for the first time after hitting at the $p$th bounce point $x_p$. If $q' = q$ for a set of positive measure in $x$, then $q$ is a self-conjugate point in the sense that there exists a normal Jacobi field along the broken geodesic which vanishes at $t = 0$ (at the point $q$) and also at a later time when it returns to $q$. In fact, it is a self-conjugate point in the stronger sense that there exists a positive measure set of directions in $S_q^* \Omega$ for which the billiard trajectory returns to $q$ (and additionally so that there exist exactly $p$ bounces before the return to $q$). This is trivially impossible for hyperbolic billiards.
since the (broken) Jacobi fields defined by the billiard trajectories have no zeros and hence there are no self-conjugate points (see [W]).

Although we did not state it, the argument is valid in all dimensions. The assumption of hyperbolicity is more than necessary; for the Proposition, we only use that the set \((q, \eta)\) of directions of tangential conjugate points is a set of content zero in \(B^*\partial \Omega\).

Henceforth we assume \(y \neq y'\). Since \(X\) ranges over all vector fields, we may assume that \(X\) is zero at one of these points and non-zero at the other. Since \(D\beta^{-\nu}\) is symplectic (hence an isomorphism), the following Lemma is sufficient to prove surjectivity of the differential away from curve of \((y, \eta)\in B^*\partial \Omega\) where \(\beta_H(y, \eta)(y, \eta)\).

**Lemma 9.2.** If \(\dim \Omega = 2\), then for all \(\zeta \in B^*\partial \Omega\), \(\{\delta \beta_H(q, \eta)\}\) spans \(T_{\beta_H(q, \eta)} B^* \Omega\) as the variation runs over vector fields on \(H\), except in the case where \(\beta_H(q, \eta) = (q, \eta)\), i.e. where \(q + t\zeta(q, \eta)\) intersects \(H\) orthogonally.

**Proof.** We choose the variations so that the variation vector fields \(X\) vanish at \(q_H^\prime(H^\beta \beta_H(\zeta))\) and span \(T_{q_H^\prime}(\partial \Omega)\). We want to prove that \(D\beta^{\delta H} \beta_H(q, \eta)\) spans \(T_{\beta_H(q, \eta)} B^* \partial \Omega\). It is helpful to use the following notation: a tangent vector \(v\) to \(B^* \partial \Omega\) is said to be vertical \((v \in VB^* \Omega)\) if it is tangent to the fibers of the natural projection to \(\Omega\); it is horizontal \((v \in HB^* \Omega)\) if the momentum variable is fixed while the position variable changes.

We first observe that \(q(y) + t\zeta(y, -\eta)\) is a fixed ray independent of \(H\). We note that the reflection from \(H\) depends only on the tangent line to \(H\) at the point of impact. In varying \(H\), we can vary (and only vary) the tangent line at the point of impact and the distance along the ray to the tangent line. These parameters correspond to two variations of \(H\): in the first, we fix the point of impact and deform \(H\) so that the tangent line at the point of impact varies; while in the second, we hold the tangent line fixed and move the point of impact along the ray \(q(y) + t\zeta(y, -\eta)\).

The possible reflected rays thus have the form

\[
Y(t, \nu_\theta) = q(y) + t_2 \zeta(y, -\eta) - 2(t_2 - t)(\nu_\theta, \zeta(y, -\eta))\nu_\theta,
\]

where \(t_2 = t_2(t, \theta)\) is the time at which the reflected ray hits \(\partial \Omega\) and where \(\nu_\theta = (\cos \theta, \sin \theta)\). The reflected direction is \(=(Y, \zeta(y, -\eta) - 2(\nu_\theta, \zeta(y, -\eta))\nu_\theta)\).

We note that the direction is independent of \(t\). So for fixed \(\theta\),

\[
\frac{\partial \beta_H}{\partial t} = \frac{\partial Y}{\partial t} \oplus 0 \in H_{\beta_H(y, \eta)} B^* \partial \Omega.
\]

Here,

\[
\frac{\partial Y}{\partial t} = \frac{\partial t_2}{\partial t} \zeta - 2(\nu_\theta, \zeta)\nu_\theta.
\]

Further,

\[
\frac{\partial \beta_H}{\partial \theta} = \frac{\partial Y}{\partial \theta} \oplus \frac{\partial}{\partial \theta} (-2(\nu_\theta, \zeta)\nu_\theta).
\]

We now check that these two vectors span \(T_{\beta_H(y, \eta)} B^* \partial \Omega\). We have, \(\frac{\partial}{\partial \nu_\theta}(\nu_\theta, \zeta)\nu_\theta) = (\frac{\partial}{\partial \nu_\theta} \nu_\theta, \zeta)\nu_\theta + (\nu_\theta, \zeta) \frac{\partial}{\partial \theta} \nu_\theta\). The two terms are orthogonal, so the vector cannot vanish unless both coefficients vanish, but that is also impossible by orthogonality. This tangent vector is vertical while the \(t\) derivative is horizontal so to complete the proof it suffices to consider the case
where \( \frac{\partial \beta}{\partial r} \zeta - 2(\nu, \zeta) \nu = 0 \). The sum vanishes at \( \theta = 0 \) if and only \( \zeta = \nu \), the normal to \( H \)
at the point of impact. But in that case, \( \beta_H(\zeta) = \zeta \).

\[ \square \]

10. Notational Appendix

(1) To notationally distinguish points of \( \partial \Omega \) and points of \( H \), points of \( \partial \Omega \) are usually denoted by \( y \) or by \( q \), while points of \( H \) are denoted by \( q_H \). By a slight abuse of notation, we also write \( y \mapsto q(y) \in \partial \Omega \) for the parameter of a local parameterization, and \( s \mapsto q_H(s) \in H \) for a local parameterization of \( H \); we sometimes use \( s \) to denote a point of \( H \). The dual coordinates are \( \eta \in T^*_y(\partial \Omega) \) and \( \tau \in T^*_\partial \Omega(H) \).

(2) \( \pi^T_\partial(\xi) \) is the projection onto the tangent space at \( q \) of \( \xi \in S^*_\partial \Omega \); we use the same notation for \( \partial \Omega \) and for \( H \).

(3) \( r(q_H, q) = \frac{q_H - q}{|q_H - q|} \): the direction vector joining \( q \in \partial \Omega \) and \( q_H \in H \).

(4) \( \gamma_H \) resp. \( \gamma_{\partial \Omega} \) are the restriction operators to \( H \), resp. \( \partial \Omega \).

(5) \( u^H_\lambda \): boundary traces of eigenfunctions (see table). Traces on \( H \): \( u^H_\lambda = \varphi|_H \); \( u^{H,\nu}_\lambda = \frac{1}{\nu} \partial_{\nu} \varphi|_H \).

(6) \( \gamma(y, \eta) = \sqrt{1 - |\eta|^2} \) (see (41)).

(7) The length of successive double links is \( L_p(y, \eta) := |\pi \beta^p(y, \eta) - \pi \beta^{p-1}(y, \eta)| + |\pi \beta^{p-2}(y, \eta)| \).

(8) \( (y, \eta) \) is the single-link billiard trajectory starting at \( y \in \partial \Omega \) with direction \( \zeta(y, \eta) = \eta + \sqrt{1 - |\eta|^2} \nu_y \) and endpoint on \( \partial \Omega \).

(9) \( \tau^H_{\partial \Omega} \): the transfer map from \( B^* \partial \Omega \) to \( B^*H \) (Definition 2).

(10) \( \tau^H_{\partial \Omega} \): the transfer map from \( B^*H \) to \( B^*\partial \Omega \) (9).

(11) \( \mathcal{C} \): The domain of \( \tau^H_{\partial \Omega} \) (Definition 3.4).

(12) \( \mathcal{G} \): The grazing set. Generalized grazing sets: \( \mathcal{G}^* := \cup_{k \in \mathbb{Z}} \beta^k(S^*\partial \Omega) \) and \( \mathcal{G}^*_N := \cup_{|k| \leq N} \beta^k(S^*\partial \Omega) \).

(13) \( \Sigma \): The singular set. Generalized singular sets: \( \Sigma^* := \cup_{k \in \mathbb{Z}} \beta^k(B^*_\Sigma \partial \Omega) \) and \( \Sigma^*_N := \cup_{|k| \leq N} \beta^k(B^*_\Sigma \partial \Omega) \).

(14) From Definition 2, Definition 3.5 and (40): The graph of the transfer map is denoted by \( \Gamma^H_{\partial \Omega} = \Gamma^H_{\partial \Omega, 3} \cup \Gamma^H_{\partial \Omega, 2} \).

(15) The \( \varepsilon \)-regular set: \( U_\varepsilon := \{(y, \eta) \in B^*\partial \Omega; \text{dist}(\tau^H_{\partial \Omega}(B^*_\Sigma \partial \Omega \cup \mathcal{G} \cup \mathcal{T}) > \varepsilon) \} \).

(16) \( N^H_{\partial \Omega}(\lambda) \) is the quantum transfer operator of (15).

(17) \( N^B_{\partial \Omega}(\lambda) \) is the boundary integral operator of (18).

(18) \( \beta_H \) (4.1 and 13)): the once broken transmission map (or correspondence) on \( B^*\partial \Omega \to B^*\partial \Omega \). It has two branches \( \beta^*_H(k = 1, 2) \).

(19) \( F(\lambda) := N^H_{\partial \Omega}(\lambda)^*Op_\lambda(a)N^H_{\partial \Omega}(\lambda) \) (see 43).
From Lemma 4.1: The canonical relation of $N^H_{\partial \Omega}(\lambda)^*Op(a)N^H_{\partial \Omega}(\lambda)$ equals
\[
\Gamma_{\partial \Omega}^H \circ \Gamma_{\partial \Omega}^H = \Delta_{C \times C} \cup \Gamma_{\beta_1^H} \cup \Gamma_{\beta_2^H}.
\]

$F_{ij}^\nu(\lambda)$: terms in the decomposition of $F(\lambda)$ (see Proposition 4.2);

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