COMPACT EMBEDDING DERIVATIVES OF HARDY SPACES INTO LEBESGUE SPACES

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(Communicated by Pamela B. Gorkin)

Abstract. We characterize the positive Borel measures such that the differentiation operator of order \( n \in \mathbb{N} \cup \{0\} \) is compact from the Hardy space \( H^p \) into \( L^q(\mu) \), \( 0 < p, q < \infty \).

1. Introduction

Let \( \mathbb{D} \) denote the open unit disc of the complex plane and let \( \mathbb{T} \) denote the unit circle. Also, let \( H^p \), \( 0 < p < \infty \) denote the standard Hardy space of analytic functions in \( \mathbb{D} \).

The aim of this paper is to characterize the positive Borel measures \( \mu \) on the unit disc \( \mathbb{D} \) such that the differentiation operator \( D^{(n)}(f) = f^{(n)} \) is compact from \( H^p \) into \( L^q(\mu) \), \( n \in \mathbb{N} \cup \{0\} \) and \( 0 < p, q < \infty \).

The analogous problem for the standard Bergman spaces \( A^p_\alpha \) has been solved [9,11,14]. The formula

\[
\|f\|_{A^p_\alpha} \asymp \sum_{j=0}^{n-1} |f^{(j)}(0)| + \|f^{(n)}\|_{A^{p+n,p}_\alpha}
\]

implies that for these spaces the question of when the differentiation operator \( D^{(n)} \) from \( A^p_\alpha \) into \( L^q(\mu) \) is bounded or compact can be answered once the case \( n = 0 \) is solved. However, this method does not work for Hardy spaces, because such a Littlewood-Paley formula does not exist for \( p \neq 2 \). Nevertheless, an equivalent \( H^p \)-norm in terms of the \( n \)th derivative can be given by using the square functions

\[
S_{\sigma,n}f(\zeta) = \left( \int_{\Gamma_\sigma(\zeta)} |f^{(n)}(z)|^2 (1 - |z|)^{2n-2} \, dA(z) \right)^{1/2}
\]

where \( \Gamma_\sigma(\zeta) = \{ z \in \mathbb{D} : |\arg \zeta - \arg z| < \sigma (1 - |z|) \} \) denotes the Stolz angle (lens type region) with vertex at \( \zeta \in \mathbb{T} \) and aperture \( \sigma > 0 \). Precisely, for \( 0 < p < \infty \)

\[
\|f\|_{H^p} \asymp \sum_{j=0}^{n-1} |f^{(j)}(0)|^p + \int_{\mathbb{T}} S^p_{\sigma,n}f(\zeta) \, dm(\zeta).
\]

Here and throughout in what follows \( m \) denotes the arclength measure on \( \mathbb{T} \). In view of (1.1) and the \( H^p \) characterization through the non-tangential maximal

Received by the editors December 8, 2014 and, in revised form, February 12, 2015.

2010 Mathematics Subject Classification. Primary 30H10.

Key words and phrases. Hardy spaces, tent spaces, Carleson measures, differentiation operator, compact operators.

The author was supported in part by the Ramón y Cajal program of MICINN (Spain), Ministerio de Educación y Ciencia, Spain, (MTM2011-25502 and MTM2014-52865-P), from La Junta de Andalucía, (FQM210) and (P09-FQM-4468).

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function, it is natural that Luecking [10,11] employed tent spaces to describe the positive Borel measures such that \( D^{(n)} : H^p \to L^q(\mu) \) is bounded. It is worth noticing that Coifman, Meyer and Stein [6] introduced the theory of tent spaces in an harmonic analysis context. It was further extended by Cohn and Verbitsky [4] and has become a very useful tool in operator theory on Hardy spaces [4,5].

Let us now recall some definitions that will enable us to state the solution to the primary question of this paper. Let us write \( \Gamma \) as the related interval. The Carleson square \( S(I) \) based on an interval \( I \subset \mathbb{T} \) is the set \( S(I) = \{ re^{it} : e^{it} \in I, 1 - |I| \leq r < 1 \} \), where \(|E|\) denotes the Lebesgue measure of \( E \subset \mathbb{T} \).

If \(|I| < \frac{\pi}{4}\), the tent \( T(I) \) is the open subset of \( \mathbb{D} \) bounded by the arc \( I \subset \mathbb{T} \) and two straight lines through the endpoints of \( I \) forming with \( I \) an angle of \( \frac{\pi}{4} \). If \(|I| \geq \frac{\pi}{4}\), we set \( T(I) = \bigcup_{J \subset I, |J| < 1} T(J) \cup \{ 0 \} \). For each \( a \in \mathbb{D} \), let \( S(a) = S(I(a)) \) and \( T(a) = T(I(a)) \).

For \( 0 < q < \infty \) and a positive Borel measure \( \nu \) on \( \mathbb{D} \), finite on compact sets, denote \( A^q_{q,\nu}(f)(\zeta) = \int_{\Gamma(\zeta)} |f(z)|^q \, d\nu(z) \) and \( A_{\infty,\nu}(f)(\zeta) = \nu-\text{ess sup}_{z \in \Gamma(\zeta)} |f(z)| \).

For \( 0 < p < \infty \), \( 0 < q \leq \infty \) the tent space \( T^p_q(\nu) \) consists of the \( \nu \)-equivalence classes of \( \nu \)-measurable functions \( f \) such that \( \|f\|_{T^p_q(\nu)} = \|A_{q,\nu}(f)\|_{L^p_T(\mathbb{T},\mu)} \) is finite. For \( 0 < q < \infty \), define \( C^q_{q,\nu}(f)(\zeta) = \sup_{a \in \Gamma(\zeta)} \frac{1}{|I(a)|} \int_{T(a)} |f(z)|^q (1 - |z|) \, d\nu(z), \quad \zeta \in \mathbb{T} \).

A quasi-norm in the tent space \( T^p_q(\nu) \) is defined by \( \|f\|_{T^p_q(\nu)} = \|C^q_{q,\nu}(f)\|_{L^\infty(\mathbb{T},\mu)} \).

The following result gives a description of the dual of \( T^p_q(\nu) \) [6,10]. We also refer to [13] Theorem 4 where an analogue was proved for a family of weighted tent spaces on the unit disc.

**Theorem A.** Let \( 1 \leq p, q < \infty \) with \( p + q \neq 2 \) and let \( \nu \) be a positive Borel measure on \( \mathbb{D} \), finite on compact sets of \( \mathbb{D} \). Then the dual of \( T^p_q(\nu) \) (up to an equivalence of norms) under the pairing

\[
\langle f, g \rangle_{T^p_q(\nu)} = \int_{\mathbb{D}} f(z) \overline{g(z)} (1 - |z|) \, d\nu(z).
\]

For the sake of completeness, and because it is a key to describe those positive Borel measures such that \( D^{(n)} : H^p \to L^q(\mu) \) is compact, we shall prove in Section 2 that each \( g \in T^p_q(\nu) \) induces a bounded linear functional on \( T^p_q(\nu) \). In the proof for \( p = 1 \), a stopping time involving \( A_{q,\nu}(f) \) and \( C_{q,\nu}(f) \) is a fundamental step (see also [6]). Let \( \Delta(a,r) \) and \( D(a,r) \) respectively denote the pseudohyperbolic and Euclidean discs of center \( a \) and radius \( r \). Our main result is the following.

**Theorem 1.** Let \( 0 < p, q < \infty \), \( n \in \mathbb{N} \) and \( \mu \) be a positive Borel measure on \( \mathbb{D} \). Further, let \( dh(z) = dA(z)/(1 - |z|^2)^2 \) denote the hyperbolic measure.

(a) If \( p \geq q \), \( D^{(n)} : H^p \to L^q(\mu) \) is compact if and only if, for any fixed \( r \in (0,1) \), the function

\[
\Phi_\mu(z) = \frac{\mu(\Delta(z,r))}{(1 - |z|)^{1+qn}}, \quad z \in \mathbb{D},
\]

satisfies that

(i) \( \Phi_\mu \in T^{\frac{p}{q}}_{\frac{p}{q}}(h) \), if \( q < \min\{2, p\} \);
(ii) \( \lim_{|a| \to 1-} \frac{1}{|a|} \int_{T(a)} |\Phi_\mu(z)|^\frac{2}{p} \frac{dA(z)}{1-|z|} = 0, \) if \( q = p < 2; \)

(iii) \( \lim_{r \to 1} \int_{T} \left( \sup_{z \in \Gamma \setminus D(0, R)} |\Phi_\mu(z)| \right)^\frac{p}{q} \frac{dm(\zeta)}{r^q} = 0, \) if \( 2 \leq q < p. \)

(b) If either \( q > p \) or \( 2 \leq q = p, \) the following conditions are equivalent:

(i) \( D^{(n)} : H^p \to L^q(\mu) \) is compact;

(ii) \( \lim_{|z| \to 1-} \frac{\mu(S(z))}{(1-|z|)^\frac{2}{p} + nq} = 0; \)

(iii) \( \lim_{|z| \to 1-} \frac{\mu(\Delta(z, r))}{(1-|z|)^\frac{2}{p} + nq} = 0 \) for any fixed \( r \in (0, 1). \)

As for \( n = 0, \) \( I_d : H^p \to L^q(\mu) \) is compact if and only if \( \lim_{|z| \to 1-} \frac{\mu(S(z))}{(1-|z|)^\frac{2}{p}} = 0, \) whenever \( 0 < p \leq q < \infty \) [3]. In the previous condition \( \mu(S(z)) \) may be replaced by \( \mu(\Delta(z, r)) \) if \( p < q. \) In the triangular case \( 0 < q < p < \infty, \) \( I_d : H^p \to L^q(\mu) \) is bounded if and only if the function \( B_\mu(\zeta) = \int_{\Gamma} \frac{d\mu(z)}{1-|z|^2} \) belongs to \( L_{p^\frac{1}{q}}(\mathbb{T}, m) \) [10]. For this last range of values it is probably known, at least to experts working in the field, that \( I_d : H^p \to L^q(\mu) \) is compact if and only if it is bounded. Since we were not able to find a proof in the existing literature, we include a proof here.

**Theorem 2.** Let \( 0 < q < p < \infty \) and let \( \mu \) be a positive Borel measure on \( \mathbb{D}. \) Then the following conditions are equivalent:

(i) \( I_d : H^p \to L^q(\mu) \) is compact;

(ii) \( I_d : H^p \to L^q(\mu) \) is bounded;

(iii) the function \( B_\mu(\zeta) = \int_{\Gamma} \frac{d\mu(z)}{1-|z|^2} \) belongs to \( L_{p^\frac{1}{q}}(\mathbb{T}, m). \)

Throughout the paper \( \frac{1}{p} + \frac{1}{q} = 1. \) Further, the letter \( C = C(\cdot) \) will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation \( a \lesssim b \) if there exists a constant \( C = C(\cdot) > 0 \) such that \( a \leq Cb, \) and \( a \gtrsim b \) is understood in an analogous manner. In particular, if \( a \lesssim b \) and \( a \gtrsim b, \) then we will write \( a \asymp b. \)

2. Preliminary results

**Proposition B.** Let \( 1 \leq p, q < \infty \) with \( p + q \neq 2 \) and let \( \nu \) be a positive Borel measure on \( \mathbb{D}, \) finite on compact sets of \( \mathbb{D}. \) Then, there exists a positive constant \( C \) such that

\[
|\langle f, g \rangle_{T^p_q(\nu)}| \leq C \|f\|_{T^p_q(\nu)} \|g\|_{T^p_q(\nu)}
\]

for any \( f \in T^p_q(\nu) \) and \( g \in T^p_q(\nu). \)

**Proof.** If \( 1 < p, q < \infty, \) then Fubini’s theorem and two applications of Hölder’s inequality give

\[
|\langle f, g \rangle_{T^p_q(\nu)}| \leq \int_{\mathbb{T}} A_{q, \nu}(f)(\zeta) A_{q', \nu}(g)(\zeta) \, dm(\zeta) = \|f\|_{T^p_q(\nu)} \|g\|_{T^{p'}_{q'}(\nu)}.
\]

If \( q = 1 \) and \( 1 < p < \infty, \) then Hölder’s inequality yields

\[
|\langle f, g \rangle_{T^p_q(\nu)}| \leq \int_{\mathbb{T}} A_{1, \nu}(f)(\zeta) A_{1, \nu}(g)(\zeta) \, dm(\zeta) = \|f\|_{T^1_q(\nu)} \|g\|_{T^1_{q'}(\nu)}.
\]

Now let \( p = 1 \) and \( 1 < q < \infty. \) For \( \zeta \in \mathbb{T} \) and \( 0 \leq h \leq \infty, \) let

\[
\Gamma^h(\zeta) = \Gamma(\zeta) \setminus D\left(0, \frac{1}{1 + h}\right) = \left\{ z \in \mathbb{D} : |\arg z - \arg \zeta| < \frac{1 - |z|}{2} < \frac{h}{2(1 + h)} \right\}
\]
and
\[ A^{q'}_{q',\nu}(g|h)(\zeta) = \int_{\Gamma^h(\zeta)} |g(z)|^{q'} \, d\nu(z), \quad \zeta \in \mathbb{T}. \]

For every \( g \in T^\infty_2(\nu) \) and \( \zeta \in \mathbb{T} \), define the stopping time by
\[ h(\zeta) = \sup \{ h : A^{q',\nu}(g|h)(\zeta) \leq C_1 C^{q',\nu}(g)(\zeta) \}, \]
where \( C_1 > 0 \) is a large constant to be determined later. Assume for a moment that there exists a constant \( C_2 > 0 \) such that
\[ \int_0^1 |g(z)(1 - |z|)| \, d\nu(z) \leq C_2 \int_0^1 \left( \int_{\Gamma^h(\zeta)} k(z) \, d\nu(z) \right) \, dm(\zeta) \]
for all \( \nu \)-measurable non-negative functions \( k \). Then, applying Hölder’s inequality
\[ |\langle f, g \rangle_{T^2_2(\nu)}| \leq C_2 \int_0^1 \left( \int_{\Gamma^h(\zeta)} |f(z)||g(z)| \, d\nu(z) \right) \, dm(\zeta) \]
\[ \leq C_1 C_2 \int_0^1 A_{q,\nu}(f)(\zeta) C_{q',\nu}(g)(\zeta) \, dm(\zeta) \]
\[ \leq \|f\|_{T^2_2(\nu)} \|g\|_{T^{q'}_q(\nu)}. \]

Now let us prove (2.3). Fubini’s theorem yields
\[ \int_{\mathbb{T}} \left( \int_{\Gamma^h(\zeta)} k(z) \, d\nu(z) \right) \, dm(\zeta) = \int_{\mathbb{D}} |I(z) \cap H(\zeta)| k(z) \, d\nu(z), \]
where \( H(\zeta) = \{ \zeta \in \mathbb{T} : \frac{1}{z + h(\zeta)} \leq |z| \} \), so it suffices to show that
\[ \frac{|I(z) \cap H(\zeta)|}{|I(z)|} \geq \frac{1}{C_2} \]
for all \( z \in \mathbb{D} \). We will prove this only for \( z \) close enough to the boundary \( \mathbb{T} \); the proof for other values of \( z \) follows from this reasoning with appropriate modifications. For \( |z| \geq 1 - \frac{1}{n} \), set \( z' = (1 - n(1 - |z|)/|z|)z/|z| \) and \( x = n/|z| - 1 \), where \( n \) is a natural number \( \geq 2 \) chosen such that \( I(z) \cap I(u) = \emptyset \) if \( u \notin T(z') \cup \left( \mathbb{D} \setminus \overline{D(0,|z|)} \right) \). This together with Fubini’s theorem gives
\[ \frac{1}{|I(z)|} \int_{I(z)} \left( \int_{\Gamma^z(x)} |g(u)|^{q'} \, d\nu(u) \right) \, dm(\zeta) \]
\[ = \frac{1}{|I(z)|} \int_{\{|z|<|u|<1\}} |I(z) \cap I(u)||g(u)|^{q'} \, d\nu(u) \]
\[ \leq \frac{1}{|I(z)|} \int_{T(z')} |I(z) \cap I(u)||g(u)|^{q'} \, d\nu(u) \]
\[ \leq \frac{C_3}{|I(z')|} \int_{T(z')} |g(u)|^{q'} (1 - |u|) \, d\nu(u) \leq C_3 \inf_{v \in I(z)} C_{q',\nu}(g)(v), \]
where the last inequality is valid because
\[ \frac{1}{|I(z')|} \int_{T(z')} |g(u)|^{q'} (1 - |u|) \, d\nu(u) \leq \sup_{a \in T(v)} \frac{1}{|I(a)|} \int_{T(a)} |g(u)|^{q'} (1 - |u|) \, d\nu(u) \]
for all $v \in I(z)$. Denote $E(z) = \mathbb{T} \setminus H(z) = \{ \zeta \in \mathbb{T} : (1 + h(\zeta))|z| < 1 \}$. By the definition of $h(\zeta)$ and (2.6), and by choosing $C_1$ sufficiently large so that $C_1^{q'} > 2C_3$, we deduce

$$|I(z) \cap E(z)| \leq \int_{I(z)} \frac{A_{q',\nu}^\prime (g|x)(\zeta)}{C_1^{q'}} dm(\zeta)$$

$$\leq \frac{1}{C_1^{q'}} \inf_{v \in I(z)} \frac{A_{q',\nu}^\prime (g)(v)}{C_1^{q'}} \int_{I(z)} A_{q',\nu}^\prime (g|x)(\zeta) dm(\zeta)$$

$$\leq \frac{C_3|I(z)|}{C_1^{q'}} < \frac{1}{2}|I(z)|.$$

Therefore,

$$\frac{|I(z) \cap E(z)|}{|I(z)|} = 1 - \frac{|I(z) \cap E(z)|}{|I(z)|} \geq \frac{1}{2}, \quad |z| \geq 1 - \frac{1}{n},$$

and the inequality (2.5) follows. \hfill \Box

The reverse implication of Theorem A can be proved by using geometric ideas, the boundedness of maximal functions and interpolation theorems on $L^p L^q(\nu, m)$ due to Benedek and Panzone.

The analogue of the following result on $\mathbb{R}^n \times (0, \infty)$ was proved in [11] Proposition 1. See also [13, Lemma 4].

Lemma C. Let $0 < p < \infty$ and let $\nu$ be a positive Borel measure on $\mathbb{D}$, finite on compact sets. Then there exists $\lambda_0 = \lambda_0(p) \geq 1$ such that

$$\int_{\mathbb{T}} \left( \int_{\mathbb{D}} \left( \frac{1 - |z|}{|1 - \zeta z|} \right)^\lambda d\nu(z) \right)^p dm(\zeta) \asymp \int_{\mathbb{T}} (\nu(\Gamma(\zeta)))^p dm(\zeta) + \nu(\{0\})$$

for each $\lambda > \lambda_0$.

We defined the tent space $T^p_q(\nu)$ by using the lenses $\Gamma(\zeta)$. Different types of nontangential approach regions could be used and they would induce the same spaces. In particular, the proof of Lemma C shows that we may replace $\Gamma(\zeta)$ by $\Gamma_\alpha(\zeta)$ for any $\alpha \in (0, \pi)$ in (2.7), and consequently the space $T^p_q(\nu)$ is independent of the aperture of the lens appearing in the definition, and the quasi-norms obtained for different lenses are equivalent.

Recall that $Z = \{z_k\}_{k=0}^\infty \subset \mathbb{D}$ is called a separated sequence if it is separated in the pseudohyperbolic metric, it is an $\varepsilon$-net if $\mathbb{D} = \bigcup_{k=0}^\infty \Delta(z_k, \varepsilon)$, and finally it is a $\delta$-lattice if it is a $5\delta$-net and separated with constant $\gamma = \delta/5$. If $\nu = \sum_k \delta_{z_k}$, then we write $T^p_q(\nu) = T^p_q(\{z_k\})$. The next result [11] Theorem 2] (see also [13, Lemma 6]) plays an essential role in the proof of Theorem A.

Lemma D. Let $0 < p < \infty$ and let $\{z_k\}$ be a separated sequence. Define

$$S_\lambda(f)(z) = \sum_k f(z_k) \left( \frac{1 - |z_k|}{1 - z_k z} \right)^\lambda, \quad z \in \mathbb{D}.$$

Then $S_\lambda : T^p_q(\{z_k\}) \to H^p$ is bounded for all $\lambda > \lambda_0$, where $\lambda_0 = \lambda_0(p) \geq 1$ is that of Lemma C.

We shall also use the following inequality. Here and in the following $\triangle$ denotes the Laplacian.
Lemma 3. If $q \geq 2$ and $0 < r < 1$, there is a constant $C(q, r) > 0$ such that

$$|f'(z)|^q (1 - |z|^2)^q \leq C(q, r) \int_{\Delta(z, r)} |f|^q(\zeta) \, dA(\zeta), \quad z \in \mathbb{D}.$$  \hspace{1cm} (2.8)

Proof. Let $r \in (0, 1)$ be fixed. The classical Hardy-Stein-Spencer identity $\|f\|_{H^q}^q = |f(0)|^q + \frac{1}{2} \int_{\Delta} \Delta|f(z)|^q \log \frac{1}{|z|} \, dA(z)$ and the fact that the Laplacian $\Delta|f|^q$ is sub-harmonic when $q \geq 2$ give

$$|f'(0)|^q \leq \left(\|f\|_{H^q}^q - |f(0)|^q\right)^2 \leq \|f\|_{H^q}^q - |f(0)|^q \leq \|f\|_{H^q}^q - |f(0)|^q \leq \frac{1}{2} \int_{\Delta} \Delta|f|^q(z) \log \frac{1}{|z|} \, dA(z) \leq C(q) \int_{\Delta} |f|^q(z)(1 - |z|) \, dA(z).$$

An application of this inequality to the function $f(rz)$ gives

$$|f'(0)|^q \leq C(q, r) \int_{\Delta(0, r)} |f|^q(z) \left(1 - \frac{|z|}{r}\right) \, dA(z).$$

Now replace $f$ by $f \circ \varphi_z$ to obtain

$$|f'(z)|^q (1 - |z|^2)^q \leq C(q, r) \int_{\Delta(z, r)} |f|^q(\zeta) \left(1 - \frac{\varphi_z(\zeta)}{r}\right) \, dA(\zeta) \leq C(q, r) \int_{\Delta(z, r)} |f|^q(\zeta) \, dA(\zeta), \quad z \in \mathbb{D}. \hspace{1cm} \Box$$

3. Proof of main results

We begin with proving Theorem 1(a).

Theorem 4. Let $0 < q \leq p < \infty$, $n \in \mathbb{N}$ and let $\mu$ be a positive Borel measure on $\mathbb{D}$. Then $D^{(n)} : H^p \to L^q(\mu)$ is compact if and only if, for any fixed $r \in (0, 1)$, the function

$$\Phi_\mu(z) = \frac{\mu(\Delta(z, r))}{(1 - |z|)^{1+qn}}$$

satisfies that

(i) $\Phi_\mu \in T(\frac{p}{2-n}, h)$, if $q < \min\{2, p\};$

(ii) $\lim_{|a| \to 1^-} \frac{1}{|I(a)|} \int_{I(a)} |\Phi_\mu(z)|^{\frac{2-n}{2}} \, dA(z) = 0$, if $q = p < 2$;

(iii) $\lim_{R \to 1^-} \int_\mathbb{T} \left(\sup_{z \in \Gamma(\zeta) \setminus D(0, R)} |\Phi_\mu(z)|\right)^{\frac{p}{2-q}} \, dm(\zeta) = 0$, if $2 \leq q < p$.

Proof. Recall the known estimate [9] Lemma 2.1]

$$|f^{(n)}(z)|^s \lesssim \frac{1}{(1 - |z|)^{2+n^2}} \int_{\Delta(z, r)} |f(\zeta)|^s \, dA(\zeta), \quad z \in D, \, s > 0. \hspace{1cm} (3.1)$$

Then, the above inequality and Fubini's theorem give

$$\|f^{(n)}\|_{L^q(\mu)}^q \lesssim \int_{\mathbb{D}} \frac{1}{(1 - |z|)^{2+(n-1)s}} \int_{\Delta(z, r)} |f'(w)|^q \, dA(w) \, d\mu(z) \hspace{1cm} (3.2)
\times \int |f'(w)|^q \frac{\mu(\Delta(w, r))}{(1 - |w|)^{3+(n-1)q}} |I(w)| \, dA(w)
\times \int |f'(w)|(1 - |w|)^q |\Phi_\mu(w)| |I(w)| \, dh(w).$$
Let \( \{f_k\}_{k=1}^{\infty} \) such that \( \sup_k \|f_k\|_{H^p} < \infty \). Then, there is subsequence \( \{f_{n_k}\}_{k=1}^{\infty} \) which converges uniformly on compact subsets of \( \mathbb{D} \) to an analytic function \( f \). Let us denote \( g_k = f_{n_k} - f \), \( G_k(w) = |g'_k(w)|^q(1 - |w|^q) \) and \( dh_R = dh\chi_{\{|z|<1\}} \), \( 0 \leq R < 1 \). Now, we shall show that conditions (i)-(iii) are sufficient.

(i). Fix \( \varepsilon > 0 \). Since \( \Phi_{\mu} \in T_{\frac{p}{q},\frac{p}{q}}^q(h) \), by the dominated convergence theorem there is \( R_0 \) such that

\[
\sup_{R \geq R_0} \|\Phi_{\mu}\|_{T_{\frac{p}{q},\frac{p}{q}}^q(h, R)} < \varepsilon^q.
\]

Next, choose \( k_0 \) with \( |g_k(z)| < \varepsilon \) for any \( k \geq k_0 \) and \( |z| \leq R_0 \). Then, bearing in mind (3.2) and (2.1) and the inequality \( \|G_k\|_{T_{\frac{p}{q},\frac{p}{q}}^q(h, R)} \leq \|g_k\|_{H^p} \) (see (1.1))

\[
\|g_k^{(n)}\|_{L^q(\mu)} \lesssim \varepsilon^q \int_{|w| \leq R_0} (1 - |w|)^q |\Phi_{\mu}(w)| I(w) \, dh(w) + \int_{\mathbb{D}} G_k(w) |\Phi_{\mu}(w)| I(w) \, dh_{R_0}(w)
\]

\[
= \varepsilon^q \langle (1 - |w|)^q, \Phi_{\mu} \rangle_{T_{\frac{p}{q},\frac{p}{q}}^q(h)} + \langle G_k, \Phi_{\mu} \rangle_{T_{\frac{p}{q},\frac{p}{q}}^q(h, R_0)}
\]

\[
\lesssim \varepsilon^q \|G_k\|_{T_{\frac{p}{q},\frac{p}{q}}^q(h, R_0)} \lesssim \varepsilon^q,
\]

so \( D^{(n)} : H^p \to L^q(\mu) \) is compact. This together with [10, Theorem 1(i)] proves (i).

(ii). A standard argument (see [3, Theorem 3.4] for details) gives that \( \lim_{|a| \to 1-} \frac{1}{|I(a)|} \int_{T(a)} |\Phi_{\mu}(z)| (\frac{\varepsilon}{2})^q (1 - |z|) \, dh(z) = 0 \) if and only if

\[
\lim_{R \to 1-} \sup_{a \in \mathbb{D}} \frac{1}{|I(a)|} \int_{T(a)} |\Phi_{\mu}(z)| (\frac{\varepsilon}{2})^q (1 - |z|) \, dh_R(z)
\]

\[
= \lim_{R \to 1-} \|\Phi_{\mu}\|_{T_{\frac{p}{q},\frac{p}{q}}^q(h, R)} = 0.
\]

So fixed \( \varepsilon > 0 \), there is \( R_0 \) such that

\[
\sup_{a \in \mathbb{D}, R \geq R_0} \frac{1}{|I(a)|} \int_{T(a)} |\Phi_{\mu}(z)| (\frac{\varepsilon}{2})^q (1 - |z|) \, dh_R(z) = \sup_{R \geq R_0} \|\Phi_{\mu}\|_{T_{\frac{p}{q},\frac{p}{q}}^q(h, R)} < \varepsilon^p.
\]

Let \( k_0 \) be such that \( \sup_{k \geq k_0, |z| \leq R_0} |g_k(z)| < \varepsilon \). Then, by (3.2), (2.1) and (1.1)

\[
\|g_k^{(n)}\|_{L^p(\mu)} \lesssim \varepsilon^p \langle (1 - |w|)^p, \Phi_{\mu} \rangle_{T_{\frac{p}{q},\frac{p}{q}}^q(h)} + \langle G_k, \Phi_{\mu} \rangle_{T_{\frac{p}{q},\frac{p}{q}}^q(h, R_0)}
\]

\[
\lesssim \varepsilon^p \langle (1 - |w|)^p, \Phi_{\mu} \rangle_{T_{\frac{p}{q},\frac{p}{q}}^q(h)} + \|G_k\|_{T_{\frac{p}{q},\frac{p}{q}}^q(h, R_0)} \lesssim \varepsilon^p,
\]

which implies that \( D^{(n)} : H^p \to L^p(\mu) \) is compact.
(iii). Let us observe that (2.8) and Fubini’s theorem give
\[
\|G_k\|_{L^q_T}^\frac{q}{r} \geq \int_T \int_{\Gamma(\zeta)} |g_k'(w)|^q (1 - |w|)^q \, dh(w) \, dm(\zeta)
\]
\[\leq \int_T \left( \int_{\Gamma(\zeta)} \Delta g_k^q(z) \, dA(z) \, dh(w) \right) \, dm(\zeta)
\]
\[\leq \int_T \left( \int_{\Gamma'(\zeta)} \triangle g_k^q(z) \, dA(z) \right) \, dm(\zeta)
\]
where \(\Gamma'(\zeta) = \{ z : \Gamma(\zeta) \cap \Delta(z, r) \neq \emptyset \} \). Using Lemma\[C\] and a result by Calderón \[12\] Theorem 1.3, we get \(G_k \in T^p_1(h)\) with \(\|G_k\|_{L^q_T}^\frac{q}{r} \lesssim \|g_k\|_{H^p}^q\). From now on, the proof is analogous to both previous cases, so it will be omitted.

Reciprocally, assume that \(D^{(n)} : H^p \to L^q(\mu)\) is compact. Let \(\{ z_k \}\) be a \(\delta\)-lattice such that \(z_k \neq 0\) for all \(k\) and let \(CT^p_2(\{ z_k \}) = \{ f \in T^p_2(\{ z_k \}) : ||f||_{T^p_2(\{ z_k \})} = 1 \}\). For each \(R \in [0, 1)\) and \(\lambda > \lambda_0\) (\(\lambda_0\) is that of Lemma\[D\]) consider the operator
\[
S_{\lambda, R}(f)(z) = \sum_{|z_k| \geq R} f(z_k) \left( \frac{1 - |z_k|}{1 - z_k z} \right) ^\lambda, \quad z \in \mathbb{D}.
\]
Let us observe that \(S_{\lambda, 0}(f) = S_{\lambda}(f)\). By Lemma\[D\] there exists \(C > 0\) such that
\[
\|S_{\lambda, R}(f)\|_{H^p} \leq C \|f\|_{T^p_2(\{ z_k \})}, \quad \text{for each } R \in [0, 1).
\]
So by the assumption the closure of the set \(\{ D^{(n)} \circ S_{\lambda, R}(CT^p_2(\{ z_k \})) \}_{R \in [0, 1)}\) is compact in \(L^q(\mu)\). So, fixed \(\varepsilon > 0\), standard arguments assert that there is \(\rho\) such that
\[
\int_{|z| < 1} |D^{(n)} \circ S_{\lambda, R}(f)(z)|^q \, d\mu(z) < \varepsilon^q \quad \text{for any } R \in [0, 1) \text{ and } f \in CT^p_2(\{ z_k \}).
\]
Since \(\{ z_k \}\) is separated and \(\lambda > 1\), there is \(R_0\) such that for any \(R \geq R_0\)
\[
\sum_{|z_k| \geq R} (1 - |z_k|)^\lambda < \varepsilon^2.
\]
Joining this with Lemma\[C\] we get
\[
|D^{(n)} \circ S_{\lambda, R}(f)(z)| \leq C(\rho, n) \sum_{|z_k| \geq R} |f(z_k)| (1 - |z_k|)^\lambda
\]
\[\leq C\rho \left( \sum_{|z_k| \geq R} |f(z_k)|^2 (1 - |z_k|)^\lambda \right) ^\frac{1}{2} \varepsilon
\]
\[\leq C(\rho, n)\varepsilon \inf_{\zeta \in T} \left( \sum_k |f(z_k)|^2 \left( \frac{1 - |z_k|}{1 - z_k \zeta} \right)^\lambda \right) ^\frac{1}{2}
\]
\[\leq C(\rho, n)\varepsilon \int_T \left( \sum_k |f(z_k)|^2 \left( \frac{1 - |z_k|}{1 - z_k \zeta} \right)^\lambda \right) ^\frac{1}{2} \, dm(\zeta)
\]
\[\leq C(\rho, n)\varepsilon \|f\|_{T^p_2(\{ z_k \})}^\frac{p}{r}, \quad \text{for any } |z| \leq \rho \text{ and } R \geq R_0,
\]
which together with (3.3) gives that

$$(3.4) \quad \|D^{(n)} \circ S_{x,R}(f)\|_{L^q(\mu)} \lesssim \|f\|_{T_2^q(\{z_k\})}, \quad \text{for all } R \geq R_0 \text{ and } f \in T_2^q(\{z_k\}).$$

That is,

$$\int_{\mathbb{D}} \left| \sum_{|z_k| \geq R} f(z_k) \frac{(1 - |z_k|)^{\lambda}}{(1 - \overline{z}_k z)^{\lambda + n}} \right|^q d\mu(z) \lesssim q^q \|f\|^q_{T_2^q(\{z_k\})},$$

for all $R \geq R_0$ and $f \in T_2^q(\{z_k\})$. Now replace $f(z_k)$ by $f(z_k)r_k(t)$, where $r_k$ denotes the $k$th Rademacher function, and integrate with respect to $t$ to obtain

$$\int_0^1 \int_{\mathbb{D}} \left| \sum_{|z_k| \geq R} f(z_k) \frac{(1 - |z_k|)^{\lambda}}{(1 - \overline{z}_k z)^{\lambda + n}} r_k(t) \right|^q d\mu(z) dt \lesssim q^q \|f\|^q_{T_2^q(\{z_k\})},$$

from which Fubini’s theorem and an application of Khinchine’s inequality yield

$$I = \int_{\mathbb{D}} \left( \sum_{|z_k| \geq R} |f(z_k)|^2 \frac{(1 - |z_k|)^{2\lambda}}{(1 - \overline{z}_k z)^{2\lambda + 2n}} \right)^{\frac{q}{2}} d\mu(z) \lesssim q^q \|f\|^q_{T_2^q(\{z_k\})}.$$

Now, for any fixed $r \in (0, 1),$

$$I \gtrsim \sum_{|z_j| \geq R} \int_{\Delta(z_j, r)} \left( \sum_{|z_k| \geq R} |f(z_k)|^2 \frac{(1 - |z_k|)^{2\lambda}}{(1 - \overline{z}_k z)^{2\lambda + 2n}} \right)^{\frac{q}{2}} d\mu(z)$$

$$\gtrsim \sum_{|z_j| \geq R} \left( |f(z_j)|^2 \frac{(1 - |z_j|)^{2\lambda}}{(1 - |z_j|)^{2\lambda + 2n}} \right)^{\frac{q}{2}} \mu(\Delta(z_j, r))$$

$$= \sum_{|z_j| \geq R} \frac{|f(z_j)|^q}{(1 - |z_j|)^{qn}} \mu(\Delta(z_j, r)), $$

and hence

$$\sum_{|z_j| \geq R} \frac{|f(z_j)|^q}{(1 - |z_j|)^{qn}} \mu(\Delta(z_j, r))$$

$$= \sum_{|z_j| \geq R} |f(z_j)|^q \left( \frac{\mu(\Delta(z_j, r))}{(1 - |z_j|)^{1+qn}} \right) (1 - |z_j|) \lesssim q^q \|f\|^q_{T_2^q(\{z_k\})}$$

$$(3.5) \quad = \varepsilon^q \left( \int_{\mathbb{T}} \left( \sum_{z_k \in \Gamma(\zeta)} |f(z_k)|^2 \right)^{\frac{q}{p}} \, dm(\zeta) \right)^{\frac{q}{q}}$$

$$= \varepsilon^q \left( \int_{\mathbb{T}} \left( \left( \sum_{z_k \in \Gamma(\zeta)} (|f(z_k)|^q)^{\frac{2}{q}} \right)^{\frac{q}{2}} \right)^{\frac{q}{q}} \, dm(\zeta) \right)^{\frac{q}{p}}.$$
(ii) If \( q = p < 2 \), then \( s = \frac{p}{q} = 1 \) and \( v = 2/p > 1 \), so by Theorem \( \text{(A)} \)
\((T^p_\ast(\{z_k\})) \ast T^\infty_\nu(\{z_k\}))\) with equivalence of norms. Therefore \((3.5)\) yields
\[
\sup_{a \in \mathbb{D}} \frac{1}{|I(a)|} \sum_{z_k \in T(a), |z_k| \geq R} \left( \frac{\mu(\Delta(z_k, r))}{(1 - |z_k|)^{1+p\eta}} \right)^{\frac{2}{p-1}} |I(z_k)| \lesssim \varepsilon^p
\]
for all \( R \geq R_0 \). The above inequality is a discrete version of
\[
\sup_{a \in \mathbb{D}, R \geq R_0} \frac{1}{|I(a)|} \int_{T(a) \cap \{R < |z| < 1\}} |\Phi(\mu(z))|^{\frac{2}{p-1}} \frac{dA(z)}{1 - |z|} \lesssim \varepsilon^p.
\]
So, \( \lim_{R \to 1^-} \left( \sup_{a \in \mathbb{D}} \frac{1}{|I(a)|} \int_{T(a) \cap \{R < |z| < 1\}} |\Phi(\mu(z))|^{\frac{2}{p-1}} \frac{dA(z)}{1 - |z|} \right) = 0 \), which is equivalent to \( \lim_{|a| \to 1^-} \frac{1}{|I(a)|} \int_{T(a)} |\Phi(\mu(z))|^{\frac{2}{p-1}} \frac{dA(z)}{1 - |z|} = 0 \).

(iii) If \( 2 < q < p \), then \( s = \frac{p}{q} > 1 \) and \( v = \frac{q}{q} < 1 \), and hence \([10] \) Proposition 3 yields
\[
\int_{\mathbb{T}} \left( \sup_{z_k \in \Gamma(\zeta) \setminus D(0, R)} \frac{\mu(\Delta(z_k, r))}{(1 - |z_k|)^{1+\eta}} \right)^{\frac{p}{p-q}} \frac{dA(z)}{1 - |z|} \lesssim \varepsilon^p,
\]
for all \( R \geq R_0 \), from which the assertion follows. The case \( q = 2 \) is proved similarly by using Theorem \( \text{(A)} \) instead of \([10] \) Proposition 3. This finishes the proof. \( \square \)

Now we deal with the second part of Theorem \( \text{II(b)} \).

**Theorem 5.** Let either \( 0 < p < q < \infty \) or \( 2 \leq p = q < \infty \) and \( n \in \mathbb{N} \), and let \( \mu \) be a positive Borel measure on \( \mathbb{D} \). Then the following conditions are equivalent:
(1) \( D^{(n)} : H^p \to L^q(\mu) \) is compact;
(2) \( \lim_{|z| \to 1^-} \frac{\mu(\delta(z))}{(1 - |z|)^{n+\frac{2}{n}}} = 0 \);
(3) \( \lim_{|z| \to 1^-} \frac{\mu(\Delta(z, r))}{(1 - |z|)^{n+\frac{2}{n}}} = 0 \) for any fixed \( r \in (0, 1) \).

**Proof.** A proof of (i) \( \Rightarrow \) (ii) and (i) \( \Rightarrow \) (iii) can be obtained by using the test functions
\( f_a(z) = \left( \frac{1 - |a|^2}{1 - |z|^2} \right)^{1/p} \), \( a \in \mathbb{D} \), and a regular reasoning, so it is omitted.

It is enough to prove (iii) \( \Rightarrow \) (i). We shall split the proof into two cases.

**Case 0 < p < q < \infty.** The argument follows ideas from the proof of \([9] \) Theorem 3.1. Let \( r \in (0, 1) \) be fixed, choose \( s \in (p, q) \) and denote
\[
d\mu^s(\zeta) = (1 - |\zeta|)^{s-2} dA(\zeta), \quad \zeta \in \mathbb{D}.
\]
Let \( \{f_k\}_{k=1}^\infty \) such that \( \sup_k ||f_k||_{H^p} < \infty \). Then, there is subsequence \( \{f_{n_k}\}_{k=1}^\infty \) which converges uniformly on compact subsets of \( \mathbb{D} \) to an analytic function \( f \). Let us denote \( g_k = f_{n_k} - f \). Fix \( \varepsilon > 0 \). By hypothesis there is \( \rho \) such that
\[
\frac{\mu(\Delta(z, r))}{(1 - |z|)^{n+\frac{2}{n}}} < \varepsilon^q, \quad \text{if } \rho < |z| < 1.
\]
On the other hand, there is $k_0$ such that $|g_k(z)| < \varepsilon$ for any $k \geq k_0$ and $|z| \leq \rho$. So, bearing in mind (3.1) and Minkowski’s inequality in continuous form

$$\|g_k^{(n)}\|_{L^q(\mu)}^q \lesssim \int_D \left( \frac{1}{(1 - |z|)^{2 + ns}} \int_{\Delta(z,r)} |g_k(\zeta)|^s \, dA(\zeta) \right)^{\frac{q}{2}} \, d\mu(z)$$

$$\lesssim \left( \int_D |g_k(\zeta)|^s \frac{(\mu(\Delta(z,r)))^\frac{s}{q}}{(1 - |\zeta|)^{ns}} \, d\mu(\zeta) \right)^{\frac{q}{2}}$$

$$\lesssim \varepsilon^q \left( \int_{|\zeta| \leq \rho} \frac{(\mu(\Delta(z,r)))^\frac{s}{q}}{(1 - |\zeta|)^{ns}} \, d\mu(\zeta) \right)^{\frac{q}{2}} + \varepsilon^q \left( \int_{|\rho < |\zeta| < 1} |g_k(\zeta)|^s \, d\mu^*(\zeta) \right)^{\frac{q}{2}}$$

$$\lesssim \varepsilon^q (\mu^*(D))^\frac{q}{2} + ||g_k||_{L^q(\mu^*)}^q.$$

Next, since $\mu^*(S(a)) \lesssim (1 - |a|)^{\frac{n}{2}}$ for $a \in D$, by [7, Theorem 9.4] $||g_k||_{L^q(\mu^*)} \lesssim ||g_k||_{H^p}$, which together with the above inequalities implies that $D^{(n)} : H^p \to L^q(\mu)$ is compact.

**Case $q = p \geq 2$.** By (3.1), (2.8) and Fubini’s theorem

$$\|f^{(n)}\|_{L^p(\mu)}^p \lesssim \int_D \frac{1}{(1 - |z|)^{2 + (n-1)p}} \left( \int_{\Delta(z,\rho)} |f'(\zeta)|^p \, dA(\zeta) \right) \, d\mu(z)$$

$$\lesssim \int_D \frac{1}{(1 - |z|)^{2 + np}} \left( \int_{\Delta(z,\rho)} \left( \int_{\Delta(z,s)} \Delta|f|^p(u) \, dA(u) \right) \, dA(\zeta) \right) \, d\mu(z)$$

$$\lesssim \int_D \frac{1}{(1 - |z|)^{np}} \left( \int_{\Delta(z,r)} \Delta|f|^p(u) \, dA(u) \right) \, d\mu(z)$$

$$\times \int_D \frac{\Delta|f|^p(u)}{(1 - |u|)^{np}} \mu(\Delta(u, r)) \, dA(u),$$

where $\rho, s \in (0, 1)$ are chosen sufficiently small depending only on $r$. Putting together this inequality with the Hardy-Stein-Spencer identity $||f||_{H^p}^p = |f(0)|^p + \frac{1}{2} \int_D \Delta|f|^p \log \frac{1}{|z|} \, dA(z)$, the proof can be finished as in the previous case. \(\square\)

The following result contains Theorem 2.

**Theorem 6.** Let $0 < q < p < \infty$ and let $\mu$ be a positive Borel measure on $D$. Then the following conditions are equivalent:

(i) $I_d : H^p \to L^q(\mu)$ is compact;

(ii) $I_d : H^p \to L^q(\mu)$ is bounded;

(iii) The function

$$\Psi_{\mu}(\zeta) = \int_D \left( \frac{1 - |z|}{|1 - \zeta z|} \right)^{\lambda} \frac{d\mu(z)}{1 - |z|}$$

belongs to $L^{\frac{p}{p-q}}(T, m)$ for all $\lambda > 0$ large enough;

(iv) the function $B_{\mu}(\zeta) = \int_{\Gamma(\zeta)} \frac{d\mu(z)}{1 - |z|}$ belongs to $L^{\frac{p}{p-q}}(T, m)$.
for each $0 < r < 1$, the function
\[
\zeta \mapsto \int_{\Gamma(\zeta)} \frac{\mu(\Delta(z, r))}{(1 - |z|)^3} \, dA(z)
\]
belongs to $L^{p-q}(\mathbb{T}, m)$;

(vi) $M(\mu)(z) = \sup_{z \in S(\alpha)} \frac{\mu(S(\alpha))}{1 - |z|} \in L^{\frac{p}{p-q}}(\mathbb{T}, m)$.

**Proof.** The equivalences (ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv)$\Leftrightarrow$(vi) follow from [10, Section D]. Next, fixed $0 < r < 1$, let us observe that for a positive measure $\nu$

\[
\int_D \left( \frac{1 - |\zeta|}{1 - |\zeta u|} \right)^\lambda \, d\nu(z) = \int_D \left( \int_{\Delta(u, r)} \left( \frac{1 - |\zeta|}{1 - |\zeta z|} \right)^\lambda \frac{1}{|\Delta(z, r)|} \, d\nu(z) \right) \, dA(u)
\]

(3.6)

\[
\times \int_D \left( \frac{1 - |u|}{1 - |\zeta u|} \right)^\lambda \nu(\Delta(u, r)) \, dh(u), \quad \text{for all } \zeta \in \mathbb{T}.
\]

So choosing $d\nu(z) = \frac{m(\zeta)}{1 - |\zeta|^2}$ and applying Lemma [3], we have that (iv)$\Leftrightarrow$(v).

Finally, let us see (iv)$\Rightarrow$(i). By hypothesis and (3.6)

\[
\int_T \left( \int_D \left( \frac{1 - |u|}{1 - |\zeta u|} \right)^\lambda \frac{\mu(\Delta(u, r))}{(1 - |u|)^3} \, dA(u) \right) \, d\mu(z) \ll \int_T \int_D \frac{m(\zeta)}{1 - |\zeta|^2} \, d\nu(z) \, dm(\zeta) < \infty.
\]

Then, by dominated convergence theorem and (3.6),

\[
0 = \lim_{R \to 1^-} \int_T \left( \int_{\{R < |z| < 1\}} \left( \frac{1 - |\zeta|}{1 - |\zeta z|} \right)^\lambda \frac{d\mu(z)}{1 - |z|} \right) \, d\mu(z)
\]

\[
\geq \lim_{R \to 1^-} \int_T \left( \int_{\Gamma(\zeta) \setminus D(0, R)} \frac{d\mu(z)}{1 - |z|} \right)^\frac{p}{p-q} \, dm(\zeta).
\]

This together with the equivalence $||f||_{H^p} \asymp \int_T \left( \sup_{z \in \Gamma(\zeta)} |f(z)| \right)^p \, dm(\zeta)$ and standard arguments, yields that $I_d : H^p \to L^q(\mu)$ is compact. \hfill $\Box$

**References**

[1] Patrick Ahern and Joaquim Bruna, *Maximal and area integral characterizations of Hardy-Sobolev spaces in the unit ball of $C^n$,* Rev. Mat. Iberoamericana 4 (1988), no. 1, 123–153, DOI 10.4171/RMI/66. MR1009122 (90h:32011)

[2] A. Benedek and R. Panzone, *The space $L^p$ with mixed norm,* Duke Math. J. 28 (1961), 301–324. MR0126155 (23 #A3451)

[3] Oscar Blasco de la Cruz and Hans Jarchow, *A note on Carleson measures for Hardy spaces,* Acta Sci. Math. (Szeged) 71 (2005), no. 1-2, 371–389. MR2160373 (2006d:42047)

[4] W. S. Cohn and I. E. Verbitsky, *Factorization of tent spaces and Hankel operators,* J. Funct. Anal. 175 (2000), no. 2, 308–329, DOI 10.1006/jfan.2000.3589. MR1780479 (2001g:42047)

[5] William Cohn, Sarah H. Ferguson, and Richard Rochberg, *Boundedness of higher order Hankel forms, factorization in potential spaces and derivations,* Proc. London Math. Soc. (3) 82 (2001), no. 1, 110–130, DOI 10.1112/S0024611500012727. MR1794259 (2001k:47036)

[6] R. R. Coifman, Y. Meyer, and E. M. Stein, *Some new function spaces and their applications to harmonic analysis,* J. Funct. Anal. 62 (1985), no. 2, 304–335, DOI 10.1016/0022-1236(85)90007-2. MR791851 (86j:46029)

[7] Peter L. Duren, *Theory of $H^p$ spaces,* Pure and Applied Mathematics, Vol. 38, Academic Press, New York-London, 1970. MR0268655 (42 #3552)

[8] C. Fefferman and E. M. Stein, *$H^p$ spaces of several variables,* Acta Math. 129 (1972), no. 3-4, 137–193. MR0447953 (56 #6283)
[9] Daniel H. Luecking, *Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives*, Amer. J. Math. **107** (1985), no. 1, 85–111, DOI 10.2307/2374458. MR778090 (86g:30002)

[10] Daniel H. Luecking, *Embedding derivatives of Hardy spaces into Lebesgue spaces*, Proc. London Math. Soc. (3) **63** (1991), no. 3, 595–619, DOI 10.1112/plms/s3-63.3.595. MR1127151 (92k:42030)

[11] Daniel H. Luecking, *Embedding theorems for spaces of analytic functions via Khinchine’s inequality*, Michigan Math. J. **40** (1993), no. 2, 333–358, DOI 10.1307/mmj/1029004756. MR1226835 (94e:46046)

[12] Miroslav Pavlović, *On the Littlewood-Paley $g$-function and Calderón’s area theorem*, Expo. Math. **31** (2013), no. 2, 169–195, DOI 10.1016/j.exmath.2013.01.006. MR3057123

[13] José Ángel Peláez and Jouni Rättyä, *Embedding theorems for Bergman spaces via harmonic analysis*, Math. Ann. **362** (2015), no. 1-2, 205–239, DOI 10.1007/s00208-014-1108-5. MR3343875

[14] K. Zhu and R. Zhao, *Theory of Bergman spaces in the unit ball*, Memoires de la SFM 115 (2008).

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