A CATEGORY OF KERNELS FOR GRADED MATRIX FACTORIZATIONS AND ITS IMPLICATIONS FOR HODGE THEORY

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Abstract. We provide a matrix factorization model for the internal Hom (continuous), in the homotopy category of \( k \)-linear dg-categories, between dg-categories of graded matrix factorizations. This description is used to calculate the derived natural transformations between twists functors on categories of graded matrix factorizations. Furthermore, we combine our model with a theorem of Orlov to establish a geometric picture related to Kontsevich’s Homological Mirror Symmetry Conjecture. As applications, we obtain new cases of a conjecture of Orlov concerning the Rouquier dimension of the bounded derived category of coherent sheaves on a smooth variety and a proof of the Hodge conjecture for \( n \)-fold products of a \( K3 \) surface closely related to the Fermat cubic fourfold. We also introduce Noether-Lefschetz spectra as a new Morita invariant of dg-categories. They are intended to encode information about algebraic classes in the cohomology of an algebraic variety.

1. Introduction

The subject of matrix factorizations and categories of singularities has generated an exciting variety of articles both historically and now quite actively. The current line of investigation involves ideas from commutative algebra, dg-categories, physics, and algebraic geometry, creating a promising opportunity for interdisciplinary collaboration. Our contribution in this paper begins with a thorough investigation of categories of graded singularities, providing a model of the dg functor category and using it to compute Hochschild homology and cohomology in the graded case. The main novelty is the range of applications, not only internally to category theory, but also to classical problems in algebraic geometry and Hodge theory. Specifically, for categorical applications, we describe a conjectural framework to understand Homological Mirror Symmetry for products and hypersurfaces of general type (extending a conjecture of the third named author) and a categorical covering theory for calculating generation time, Rouquier dimension, and Orlov spectra. In terms of classical applications, we apply our picture of categorical covers to compare algebraic classes between cohomology groups of algebraic varieties. As an example we prove a new case of the Hodge conjecture. Before we delve into detailed statements, let us try to set the scene.

Perhaps the simplest class of singular rings is hypersurface singularities, i.e. a ring that is the quotient of a regular ring by a single element (also called hypersurface rings). In the foundational paper, \([\text{Eis80}]\), D. Eisenbud defined matrix factorizations and demonstrated their precise relationship with maximal Cohen-Macaulay (MCM) modules over a hypersurface singularity. Building on Eisenbud’s description, R.O. Buchweitz introduced the proper categorical framework in \([\text{Buc86}]\). Buchweitz showed that the homotopy category of matrix factorizations, the stable category of MCM modules over the associated hypersurface ring, and the stable derived category of the associated hypersurface ring are all equivalent descriptions of the same triangulated category.
Outside of commutative algebra, interest in matrix factorizations grew due to intimate connections with physics; physical theories with potentials, called Landau-Ginzburg models, are ubiquitous in (theoretical) physics. Building on the large body of work on Landau-Ginzburg models without boundary, (see, for example, C. Vafa’s computation, [Vaf91], of the closed string topological sector as the Jacobian algebra of the potential), Kontsevich proposed matrix factorizations as the appropriate category of D-branes for the topological B-model in the presence of a potential.

In physics, A. Kapustin and Y. Li confirmed Konstevich’s prediction and gave a mathematically conjectural description of the Chern character map and Hochschild homology pairing for the category of singularities of a hypersurface, [KL03a] [KL03b].

In mathematics, several foundational papers by D. Orlov soon followed, [Orl04, Orl06, Orl09a]. In particular, Orlov gave a global model for the stable (bounded) derived category of a Noetherian scheme with enough locally-free sheaves (the category of singularities), proved that the category of B-branes for an LG-model is equivalent to the coproduct of the category of singularities of the fibers, and described a tight relationship between the bounded derived categories of coherent sheaves on a projective hypersurface and the graded category of singularities of its affine cone.

In another early development, signaling the fertility of the marriage of physical inspiration to matrix factorizations, M. Khovanov and L. Rozansky categorified the HOMFLY polynomial using matrix factorizations, [KR08a, KR08b]. Khovanov and Rozansky also introduced several important ideas to the study of matrix factorizations. Central to this work is a natural construction to go from a matrix factorization to a functor between categories of matrix factorizations. A strong analogy exists between Khovanov and Rozansky’s construction and the calculus of kernels of integral transforms between derived categories of coherent sheaves on algebraic varieties. This analogy motivates the terminology in the title of this paper.

The introduction of D-branes into Landau-Ginzburg models fit squarely into the framework of Homological Mirror Symmetry (HMS), a profound insight of Konstevich, [Kon95]. Today, HMS plays a central role in geometry, and extensive progress has been made applying matrix factorizations to HMS, including a proof of HMS for higher genus curves by P. Seidel [Sei08b] and A. Efimov [Efi09] and for punctured spheres by M. Abouzaid, D. Auroux, Efimov, the third author, and Orlov [AAEKO11]. In [KKOY09], Kapustin, the third author, Orlov, and M. Yotov propose a generalization of HMS to varieties of general type by way of categories of singularities.

In addition to new formulations and proofs of HMS, mathematicians continued to develop the Hodge theory of matrix factorizations. In [KKP08], the third author, Kontsevich, and T. Pantev give explicit constructions describing the Hodge theory associated to the category of matrix factorizations. For the case of an isolated local hypersurface singularity, T. Dyckerhoff proved, in [Dyc09], that the category of kernels, introduced in [KR08a], is the correct one from the perspective of [Toe07]. More precisely, the dg-category of kernels from [KR08a] and [Dyc09] is quasi-equivalent to the internal homomorphism dg-category in the homotopy category of dg-categories. Using this result, Dyckerhoff rigourously established Kapustin and Li’s description of the Hochschild homology. D. Murfet gave a mathematical derivation of the Kapustin-Li pairing [Muf09] which subsequently was expanded in [DM10]. In addition, E. Segal gave a description of the Kapustin-Li package in [Seg09].

Following this lead, several groups of authors extended Dyckerhoff’s results. A. Polishchuk and A. Vaintrob gave a description of the Chern character, the bulk-boundary map, and
proved an analog of Hirzebruch-Riemann-Roch in the case of the category of $G$-equivariant
singularities of a local isolated hypersurface ring when $G$ is a finite group \cite{PV10a}. Orlov
defined a category of matrix factorizations in the case of a non-affine scheme with a global
regular function and proved it is equivalent to the category of singularities of the associated
hypersurface in the case when the ambient scheme is regular \cite{Orl11}. K. Lin and D. Pomer-
leano tackled the case of non-affine matrix factorizations \cite{LP11}. Contemporaneously, A.
Preygel, using genuinely new ideas rooted in derived algebraic geometry, handled the case
of matrix factorizations on derived schemes, \cite{Pre11}.

Extending from the case of a global hypersurface to a local hypersurface, i.e. using a
section of a line bundle instead of a global regular function, is still a work in progress. The
first such results were obtained by A. Căldăraru and J. Tu. in \cite{CT10}. Căldăraru and
Tu defined a curved $A\infty$-algebra associated to a hypersurface in projective space and
computed the Borel-Moore homology of the curved algebra. Furthermore, in \cite{Tu10}, Tu clarified
the relationship between Borel-Moore homology and Hochschild homology. Polishchuk and
Vaintrob gave a definition of a category of matrix factorizations on a stack satisfying appropri-
ate conditions and proved that their category of matrix factorizations coincided with the
category of singularities of the underlying hypersurface, \cite{PV10b}. L. Positselski, using his
work on co- and contra-derived categories of curved dg-modules over a curved dg-algebra
\cite{Pos09}, defined an enlargement of the category of matrix factorizations in the case of a
section of line bundle. He also defined a relative singularity for an embedding of $Y$ in $X$
and proved that the relative singularity category of the hypersurface defined by a section of
a line bundle coincides with his category of factorizations even if the ambient scheme is not
regular, \cite{Pos11}.

This paper can be viewed as a continuation of the study of matrix factorizations in the case
of a local hypersurface. The results in Sections \ref{sect4} through \ref{sect6} concern matrix factorizations
associated to a line bundle on a particular class of stacks. However, we mostly stick to
algebraic language to describe the situation. The first main result of our paper provides an
internal description of the functor category between categories of graded matrix factorizations
i.e. as yet another category of graded matrix factorizations.

Let us be more precise. Let $k$ be an algebraically closed field of characteristic zero and
let $M$ and $N$ be finitely-generated Abelian groups. Let $A$ be the symmetric algebra on an
$M$-graded vector space with its induced $M$-grading and let $B$ be the symmetric algebra on
an $N$-graded vector space. Take nonzero $w \in A_d$ and $v \in B_e$ satisfying: $w \in (dw)$, $v \in (dv)$,
$d$ is not torsion in $M$, and $e$ is not torsion in $N$. Let $\text{MF}(A, w, M)$ and $\text{MF}(B, w, N)$ denote
the dg-categories of graded matrix factorizations.

**Theorem 1.1.** In the homotopy category of $k$-linear dg-categories, there is an equivalence

\[ \text{RHom}_c(\text{MF}(\widehat{A}, w, M), \text{MF}(\widehat{B}, w, N)) \cong \text{MF}(A \otimes_k B, -w \oplus v, M \sqcup N). \]

Here, $-w \oplus v = -w \otimes_k 1 + 1 \otimes_k v$ and $M \sqcup N = M \oplus N/(d, -e)$. We use this theorem
to compute the graded ring of derived natural transformations $\bigoplus_{m \in M} \text{RHom}(\text{Id}, (m))$ in
$\text{MF}(A, w, M)$. The computation, in the case $M/(d)$ is finite, is along the lines of \cite{PV10a}.
Recall that $A$ is the symmetric algebra of an $M$-graded vector space, which we shall now
call $V$.

**Corollary 1.2.** Assume in addition that $M/(d)$ is finite and let $G$ be the group whose
characters are $M/(d)$. Let $w_g$ be the restriction of $w$ to the fixed locus of $g$ on $V$ and
general type defined respectively by polynomials different but a bit more involved to discuss). Let,

Conjecture 2. Suppose $X \subseteq \mathbb{P}^n$ is a smooth hypersurface of general type defined by a polynomial $f$ of degree $d$. Let,

$$D := \{(\alpha, \beta, \gamma) | \alpha^d = \beta^2 = \gamma^2 \} \subseteq G^3_m,$$

act on $k[x_0, \ldots, x_n, u, v]$ by multiplication by $\alpha$ on the $x_i$, multiplication by $\beta$ on $u$, and multiplication by $\gamma$ on $v$. Denote by $Z$ the affine variety defined by the vanishing of $f + u^2 + v^2$. The mirror to $X$ can be realized as the Landau-Ginzburg mirror to the global quotient stack $[Z/D]$ with all singular fibers away from the origin removed.

Our framework also works very nicely for products. Generalizing the conjecture above, we propose:

Conjecture 2. Suppose $X_i \subseteq \mathbb{P}^{m_i}$ are smooth varieties which are either Calabi-Yau or of general type defined respectively by polynomials $f_i$ of degree $d_i$ (the Fano case is not much different but a bit more involved to discuss). Let,

$$D := \{(\lambda_1, \ldots, \lambda_s, \beta, \gamma) | \lambda_i^{d_i} = \beta^2 = \gamma^2 \forall i, j \} \subseteq G^s_m$$
act on $k[x_0, \ldots, x_{n_1+\ldots+n_s+s}, u, v]$ by multiplication by $\lambda_i$ on the coordinates, $x_{d_1+\ldots+d_i}$ through $x_{d_1+\ldots+d_i}$, multiplication by $\beta$ on $u$, and multiplication by $\gamma$ on $v$. Let $Y$ denote the hypersurface in $\mathbb{A}^{n_1+\ldots+n_s+s+2}$ defined by the zero locus of $f_1+\cdots+f_s+u^2+v^2$. The mirror to $X$ can be realized as the Landau-Ginzburg mirror to the global quotient stack $[Y/D]$ with all singular fibers away from the origin removed.

In physics, the above process of adding a quadratic term is often referred to as adding boson mass terms. A proposal of Sethi [Set94], suggests adding fermionic terms as well. This super commutative case can be handled with some slight adjustments and allows one to subtract from the Gorenstein parameter as well. We leave this to future work, see [BFK11b].

Our geometric picture fits perfectly with various examples in the literature. For example, it predicts the relationship between a quartic double solid and an Enriques surface described by C. Ingalls and A. Kuznetsov [IK10] and the relationship between $K3$ surfaces and cubic fourfolds described by Kuznetsov in [Kuz09b]. Furthermore, in addition to predictions in mirror symmetry, our results conjecturally have nontrivial implications towards decompositions of motives, see [Orl05]. For example, although the group action is slightly different, we get a similar motivic decomposition to that of V. Guletskii and C. Pedrini [GP02] for Godeaux surfaces. We also believe that in the case of a curve, graded Kn"orrer periodicity should be related to the classical Pyrm variety construction of Mumford [Mum74] for Jacobians of conic bundles by work of Kuznetsov in [Kuz05, Kuz09a]. Building on the seminal work by Voisin, see [Vos00], this picture is also readily applicable to the study of Griffiths groups, see [FIK11].

In the context of triangulated categories, the precise relationship between the quotients arising in the HMS picture above can be expressed in the language of orbit categories introduced by B. Keller in [Kel05]. Inspired further by the ideas of Keller, Murfet, and M. Van den Bergh in [KMV08], we provide this description in Section 8.

This categorical/geometric picture is readily applicable to the notions of Rouquier dimension, Orlov spectra, and generation time - the impetus of this work. Roughly, given a strong generator of a triangulated category, the generation time is the number of exact triangles necessary to reconstruct the category after closing under sums, shifts, and summands. The Orlov spectrum of a triangulated category is the set of all such generation times, and the Rouquier dimension is the minimum of the Orlov spectrum. Orlov has conjectured that this provides a categorical notion of dimension in [Orl09b], where he proves the following for smooth algebraic curves:

**Conjecture 1.3 (Orlov).** For a smooth algebraic variety, $X$, the Krull dimension of $X$ and the Rouquier dimension of $D^b(\text{coh } X)$ are equal.

Section 9 provides an overview of these notions and works out the case of weighted Fermat hypersurfaces in full detail. As a consequence, we obtain various special cases of the above conjecture. In addition, we discuss a relationship between Orlov spectra of triangulated categories and their orbit categories. Our work is a continuation of the work done in [BFK10] which connects Orlov spectra with classical questions of rationality for algebraic varieties, an idea stemming from A. Bondal, M. Kapranov, Kuznetsov, and Orlov. In addition, our work is foundational to providing a link between Orlov spectra and wall crossing formulas, see [KST11].
Finally in Section 10, we continue to encourage the relationship between Orlov spectra and noncommutative Hodge theory, as in [BFK10]. Specifically, we establish a fundamental connection between generation time and algebraic classes in Hochschild homology. For categories of graded matrix factorizations, we are able to combine the geometric picture of Section 7 i.e. the results of Orlov in [Orl09a] with work of Kuznetsov [Kuz09a, Kuz09b] to prove the Hodge conjecture for \( n \)-fold products of a \( K3 \) surface of Picard rank 20 and polarization of degree 14. We also introduce the notion of Noether-Lefschetz spectra - a refinement of Orlov spectra (for saturated dg-categories) and compute some examples.

Saturated dg/\( A_\infty \)-categories provide a natural framework for Orlov and Noether-Lefschetz spectra. Specifically, in forthcoming work of G. Kerr, generation time is established as a type of homology theory for \( A_\infty \)-categories. We provide some preliminaries on dg-categories in Appendix A.1.

In Appendix A.2 we extend the work of C˘ ald˘ araru and S. Willerton on the Hochschild structures of smooth and projective varieties (or, more generally, spaces in the terminology of [CW10]) to saturated dg-categories over a field. We believe the results of this appendix may be of independent interest.

We expect that, with this work, we have only dipped our toes into the stream; the methods involved are clearly far more general. Indeed, the connections between matrix factorizations, Orlov/Noether-Lefschetz spectra, and classical Hodge theory seem to be extremely deep. Due to the concrete nature of Orlov/Noether-Lefschetz spectra as a type of derived homological dimension, we hope that in the long-term, these ideas provide a powerful computational tool.

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2. Notations and conventions

Given two dg-categories, \( A, B \), over \( k \), the category \( A \otimes_k B \), denotes the dg-category whose objects are pairs of objects \([a, b]\) with \( a \in A \) and \( b \in B \), and morphisms are the tensor product of the chain complexes: \( \text{Hom}_{A \otimes_k B}([a, b], [a', b']) := \text{Hom}_A(a, a') \otimes_k \text{Hom}_B(b, b') \). We denote by \( \hat{A} \) the full subcategory of fibrant and cofibrant objects in the model category of right dg-modules over \( A \) and \( \hat{A}_{pe} \) is the full subcategory of perfect objects in \( \hat{A} \). For notational purposes we denote \( \hat{A} \otimes_k B \) by \( A \hat{\otimes}_k B \) as well. More information on dg-categories is provided in Appendix A.1.
3. Preliminaries on graded rings

Let $k$ be an algebraically-closed field of characteristic zero and let $M$ be a finitely-generated Abelian group. An $M$-graded $k$-algebra, $A$, is a commutative Noetherian $k$-algebra with a decomposition, $A = \bigoplus_{m \in M} A_m$, over $k$, so that $A_m A_{m'} \subset A_{m+m'}$. A graded left $A$-module, $E$, is a left $A$-module with a decomposition, $E = \bigoplus_{m \in M} E_m$, over $k$, so that $A_m E_{m'} \subset E_{m+m'}$. Graded left $A$-modules naturally form an abelian category where morphisms are degree zero $A$-modules morphisms. Denote this category by $\text{GrMod}(A,M)$. There is also an abelian category of finitely-generated $M$-graded $A$-modules, denoted $\text{grmod}(A,M)$.

We will need to be slightly more restrictive. We will say an $M$-graded $k$-algebra, $A$, satisfies $\dagger$ if $A$ is isomorphic to $\text{Sym} V/I$ and

- $V$ is an $M$-graded vector space inducing a grading on $\text{Sym} V$.
- the grading on $A$ is induced by the grading on $\text{Sym} V$ via quotienting by the homogeneous ideal $I$.

As an additional piece of notation, we will say that a graded algebra is a polynomial algebra if $A$ is isomorphic to $\text{Sym} V$ for a graded vector space $M$.

We have the following Jacobian criterion for smoothness, see Theorem 16.19 of [Eis95].

**Proposition 3.1.** Let $R = k[x_1, \ldots, x_n]/I$ be an affine ring over $k$ and suppose that $I$ has pure codimension $c$. Suppose that $I = (f_1, \ldots, f_c)$. If $J$ is the ideal of $R$ generated by the $c \times c$ minors of the Jacobian matrix $(\partial f_i/\partial x_i)$, then $J$ defines the singular locus of $R$ in the sense that a prime $\mathfrak{p}$ of $R$ contains $J$ if and only if $R_\mathfrak{p}$ is not a regular local ring.

**Lemma 3.2.** If $A$ satisfies $\dagger$, then the singular locus is equal, as subset of $\text{Spec}(A)$, to the Zariski closed subset corresponding to a homogeneous ideal of $A$.

**Proof.** We apply Proposition 3.1 and note that since $f_i$ and $x_j$ are homogeneous, so is $\partial f_i/\partial x_j$. \qed

We will also want to take radicals. However, for a general $M$, there is no guarantee that radicals of homogeneous ideals are homogeneous. If $M$ is torsion-free, then we have a total ordering defined by using lexicographic order. The following proposition can be proven using the same arguments as in the case $M = \mathbb{Z}$, for an outline see Exercise 3.5 of [Eis95].

**Proposition 3.3.** Let $R$ be a Noetherian $M$-graded commutative ring where $M$ is totally-ordered. Then, the following hold:

1. Radicals of homogeneous ideals are homogeneous.
2. Any associated prime of a graded module is homogeneous.

As a corollary of these facts, we have

**Corollary 3.4.** Given an $M$-graded ring $R$ satisfying $\dagger$. If $M$ is torsion-free, then $R/\sqrt{0}$ and the irreducible components of $R$ satisfy $\dagger$.

Let $A$ be an $M$-graded $k$-algebra and let $\phi : M \to M'$ be a homomorphism of Abelian groups. $A$ is naturally $M'$-graded where $A_{m'} = \bigoplus_{\phi(m) = m'} A$. $\phi$ induces an adjoint pair of functors described below.
Define
\[ \phi_* : \text{GrMod}(A, M) \to \text{GrMod}(A, M') \]
\[ (\phi_* E)_{m'} := \bigoplus_{\phi(m) = m'} E_m \]
and
\[ \phi^* : \text{GrMod}(A, M') \to \text{GrMod}(A, M) \]
\[ (\phi^* F)_m := F_{\phi(m)}. \]

**Lemma 3.5.** \( \phi_* \) is left adjoint to \( \phi^* \). For any \( E \in \text{GrMod}(A, M) \), there is a natural isomorphism \( \phi^* \phi_* E \cong \bigoplus_{l \in \text{Ker} \phi} E(l) \). For any \( F \in \text{GrMod}(A, M') \), there is a natural isomorphism \( \phi_* \phi^* F \cong \bigoplus_{l \in \text{Ker} \phi} F \). For any \( m \in M \) and any \( F \in \text{GrMod}(A, M') \), there is a natural isomorphism \( \phi^*(F(\phi(m))) \cong (\phi^* E)(m) \).

**Remark 3.6.** This adjunction is really between \( M \)-graded vector spaces and \( N \)-graded vectors spaces. Algebra objects, modules over those algebra objects, etc... then get passed back and forth.

An \( M \)-algebra, \( A \), is a \( k \)-algebra with a decomposition \( A = \bigoplus_{m, m' \in M} A_{m, m'} \) so that \( A_{m, l} A_{l, m'} = 0 \) if \( l \neq -l' \) and is contained in \( A_{m, m'} \) if \( l = -l' \). We also require that each \( A_{m, -m} \) has a unit \( e_{m, -m} := e_m \).

Given an \( M \)-graded algebra, \( A \), one can naturally construct an \( M \)-algebra, \( A \), with \( A_{m, m'} = A_{m + m'} \).

**Remark 3.7.** Given a graded algebra \( A \), the associated \( M \)-algebra, \( A \), will be standing notation.

For any left \( A \)-module, \( E \), we set \( E_m := e_m E \) and say that \( m \) is complete if \( E = \bigoplus_m E_m \). Let \( \text{Mod}_{\text{comp}}(A) \) denote the category of complete left \( A \)-modules.

**Lemma 3.8.** Let \( A \) be an \( M \)-graded \( k \)-algebra and let \( A \) be the associated \( M \)-algebra. There is an equivalence of categories between \( \text{GrMod}(A, M) \cong \text{AMod}_{\text{comp}} \) between \( \text{GrMod}(A^{op}, M) \) and \( \text{Mod}_{\text{comp}} A \), and between \( \text{BiGrMod}(A, M) \) and \( \text{BMod}_{\text{comp}}(A) \).

**Proof.** Given any element \( E \in \text{GrMod}(A, M) \), we can construct \( E \) in \( A \text{Mod} \) by setting \( E_{i, j} = E_{i+j} \) similar to above. Given an element \( E \) of \( A \text{Mod} \) that is complete we set \( E_i = e_i E \) for the idempotent \( e_i \). It is easy to check that \( A_i E_i \subseteq E_{i+j}. \)

For right modules, we again send an graded right \( A \)-module, \( E \), to \( E_{i, j} = E_{i+j} \). However, to get compositions to work out we need to take \( E_i = E_{e_i} \) to go from complete right \( A \)-modules to graded right modules.

The proof for bimodules in now clear. \( \square \)

Identifying via this equivalences, any bi-graded \( A, B \)-bimodule, \( K \), induces a functor,
\[ \otimes_A K : \text{GrMod}(A, M) \to \text{GrMod}(B, N), \]
with \( A \) being the kernel of the identity functor.

By an \( M \)-graded scheme, \( X \), we will mean a Noetherian scheme \((X, \mathcal{O}_X)\) where \( \mathcal{O}_X \) is a sheaf of \( M \)-graded rings. In this paper, we will mainly be interested in graded \( k \)-algebras and open subschemes of graded \( k \)-algebras that are complements of homogeneous ideals. We have categories of \( M \)-graded quasi-coherent \( \mathcal{O}_X \) modules, \( \text{Qcoh}(X, M) \), and of \( M \)-graded
coherent \( \mathcal{O}_X \)-modules, \( \text{coh}(X, M) \). Recall that the global dimension of an abelian category, \( \mathcal{C} \), is maximal \( n \) so that \( \text{Ext}^n_{\mathcal{C}}(E, F) \) is nonzero for some pair of objects, \( E \) and \( F \), of \( \mathcal{C} \).

**Lemma 3.9.** The global dimension of \( \text{Qcoh}(X, M) \) is equal to the global dimension of \( \text{Qcoh}(X) \).

**Proof.** For any any coherent graded module, \( F \), the natural map 
\[
\bigoplus_{m \in M} \text{Hom}_{\text{GrQcoh}(X, M)}(F, \mathcal{E}(m)) \to \text{Hom}_{\text{Qcoh}(X)}(F, \mathcal{E})
\]
is an isomorphism. As a consequence, taking injective resolutions, we get that the natural map 
\[
\bigoplus_{m \in M} \text{Ext}^i_{\text{GrQcoh}(X, M)}(F, \mathcal{E}(m)) \to \text{Ext}^i_{\text{Qcoh}(X)}(F, \mathcal{E})
\]
is an isomorphism for each \( i \). Since the injective dimension of a graded quasi-coherent \( \mathcal{O}_X \)-module can be determined by testing on coherent submodules, using an analog of Baer’s criteria, we see that the global dimension of \( \text{Qcoh}(X, M) \) equals the global dimension of \( \text{Qcoh}(X) \). \( \square \)

One can sheafify the passage from an \( M \)-graded ring to an \( M \)-algebra to get a topological space with a sheaf of \( M \)-algebras from an \( M \)-graded scheme \( X \). The following lemma can be proved in exactly the same manner as Lemma 3.8.

**Lemma 3.10.** Let \( X \) be an \( M \)-graded scheme and let \( \mathcal{X} \) be the associated space equipped with a sheaf of \( M \)-algebras. There is an equivalence of categories between \( \text{Qcoh}(X, M) \cong \mathcal{X} \text{Mod}_{\text{comp}} \), between \( \text{Qcoh}(X, M) \) and \( \text{Mod}_{\text{comp}} \mathcal{X} \), and between \( \text{Qcoh}(X \times X, M \oplus M) \) and \( \text{BiMod}_{\text{comp}}(\mathcal{X}) \).

### 4. Generation of graded singularity categories

Let us recall some notions of generation, for more details see subsection 9.1. Given a triangulated category, \( \mathcal{T} \), we say that a subcategory \( \mathcal{S} \) generates \( \mathcal{T} \), if the smallest full triangulated subcategory of \( \mathcal{T} \) containing \( \mathcal{S} \), and closed under finite coproducts and summands, is equivalent to \( \mathcal{T} \) itself. We say that \( \mathcal{S} \) generates \( \mathcal{T} \) up to infinite coproducts if the smallest full triangulated subcategory of \( \mathcal{T} \) containing \( G \), and closed under arbitrary coproducts and summands, is equivalent to \( \mathcal{T} \) itself.

In addition, recall the an object \( C \) of \( \mathcal{T} \) is compact if \( \text{Hom}_{\mathcal{T}}(C, \bullet) \) commutes with coproducts. The following statement from [BV03] relates the two notions of generation above. See also Corollary 3.13 of [Rou08].

**Lemma 4.1.** Let \( \mathcal{C} \) be a subcategory of compact objects of \( \mathcal{T} \). If \( \mathcal{C} \) generates \( \mathcal{T} \) up to infinite coproducts, then \( \mathcal{C} \) generates the subcategory of compact objects.

We focus on graded rings, \( (A, M) \), where \( A \) satisfies \( \dagger \) is Section 3. We record some generation results about the category \( \text{D}_{\text{sg}}(A, M) \). Their statements and proofs are in the style of Rouquier, [Rou08], see also the arguments in [LP11].

**Lemma 4.2.** If \( G \) is a set of objects of \( \text{D}^b(\text{grmod}(A, M)) \) which generate \( \text{D}^b(\text{GrMod}(A, M)) \) up to infinite coproducts, then \( G \) is a set of generators of \( \text{D}^b(\text{grmod}(A, M)) \).

**Proof.** We apply Lemma 4.1. The compact objects of \( \text{D}^b(\text{GrMod}(A, M)) \) are exactly the objects of \( \text{D}^b(\text{grmod}(A, M)) \) by Proposition 6.15 of [Rou08]. \( \square \)
Decompose $M = M_{\text{free}} \oplus M_{\text{tors}}$. Let $f : \text{GrMod}(A, M) \to \text{GrMod}(A, M_{\text{free}})$ be the forgetful functor and let $f^* : \text{GrMod}(A, M_{\text{free}}) \to \text{GrMod}(A, M)$ denote its right adjoint. Note that $f^*$ preserves the subcategories of finitely-generated modules and graded projective modules. Abusing notation, we will also use $f$ and $f^*$ to denote the functors on associated derived categories of modules.

**Lemma 4.3.** The objects of the form $f^* M$ for $M$ in the category $D^b(\text{GrMod}(A, M_{\text{free}}))$ generate $D^b(\text{GrMod}(A, M))$.

**Proof.** Note that $f^* : D^b(\text{GrMod}(A, M_{\text{free}})) \to D^b(\text{GrMod}(A, M))$ is dense as we are in characteristic zero. Similarly, its restriction to the bounded derived categories of finitely-generated modules is also dense. □

Lemma 4.3 allows us to reduce to the case of a torsion-free grading group any argument involving generation.

**Lemma 4.4.** Assume $M$ is torsion-free. Let $A_{\text{red}}$ be the ring $A/\sqrt{0}$. The objects of $D^b(\text{GrMod}(A_{\text{red}}, M))$ generate $D^b(\text{GrMod}(A, M))$ up to infinite coproducts.

**Proof.** Any finitely-generated module admits a finite filtration by power of $\sqrt{0}$. Any graded module is a colimit of its finitely-generated submodules. This colimit can be expressed as a cone over a morphism between the coproduct of its finitely-generated submodules. Consequently, we can generate any graded module from those pulled back from the category, $D^b(\text{GrMod}(A_{\text{red}}, M))$, up to infinite coproducts. □

In analogy with the ungraded setting, if we can write $A \cong A_1 \otimes_k A_2$ with $A_1$ and $A_2$ $M$-graded $k$-algebras, we will say that $A_1$ and $A_2$ are components of $A$. Thanks to our assumptions the components of $A$ as an ungraded $k$-algebra are naturally $M$-graded, if $M$ is torsion-free.

**Lemma 4.5.** Assume that $M$ is torsion-free. Let $A_1$ and $A_2$ be components of $A$. Objects of $D^b(\text{GrMod}(A_1, M))$ and $D^b(\text{GrMod}(A_2, M))$ generate $D^b(\text{GrMod}(A, M))$.

**Proof.** Let $I_1$ and $I_2$ be the ideals corresponding to $A_1$ and $A_2$ respectively. Consider the graded local cohomology functor $R\Gamma_{I_1}$. For each graded $A$-module, $M$, there is an exact triangle of graded modules

$$R\Gamma_{I_1} M \to M \to RQ_{I_1} M$$

with the support of $RQ_{I_1} M$ lying outside $I_1$ and the support of $R\Gamma_{I_1} M$ lying inside $I_1$. $R\Gamma_{I_1} M$ lies in $D^b(\text{GrMod}(A_1, M))$ and $RQ_{I_2} M$ lies in $D^b(\text{GrMod}(A_2, M))$. □

**Lemma 4.6.** Assume that $A$ is reduced and irreducible and $M$ is torsion-free. Let $J$ denote the radical ideal of the singular locus $A$. If we have a set of generators, $G$, for $D^b(\text{GrMod}(A/J, M))$ up to infinite coproducts, then $G$ together with all twists of $A$ form a set of generators, up to infinite coproducts, for $D^b(\text{GrMod}(A, M))$.

**Proof.** We, again, have the derived local cohomology functor, $R\Gamma_J$. It fits into an exact triangle,

$$R\Gamma_J \to \text{Id} \to RQ_J,$$

of exact functors. Another way to view it is $Q_J$ is the restriction functor to the graded scheme that is the complement of zero locus of $J$. Let us denote this graded scheme by $X$. Note that $X$ is smooth as a scheme by the Jacobian criteria, Proposition 3.1.
By Lemma \[\text{[3.9]}\] the global dimension of the category of \(M\)-graded quasi-coherent \(\mathcal{O}_X\)-modules is the same as the category of quasi-coherent \(\mathcal{O}_X\)-modules, which is finite. The quotient functor \(\pi : \text{GrMod}(A, M) \to \text{Qcoh}(\mathcal{O}_X, M)\) is exact. Take a graded free resolution of a module \(E\),

\[
\cdots \to F_i \to \cdots \to F_0 \to E \to 0
\]

and gently truncate it to a long exact sequence of length greater than the global dimension of \(X\),

\[
0 \to C \to F_i \to \cdots \to E \to 0.
\]

Applying \(\pi\) yields an exact sequence in \(\text{Qcoh}(A, M)\), which splits as \(-i\) is greater than the global dimension of \(X\). Thus, \(\pi E\) can be generated by the \(\pi A(m)\) using only finitely many cones. Applying the right adjoint \(R \omega : \text{D}^b(\text{Qcoh}(A, M)) \to \text{D}^b(\text{GrMod}(A, M))\), we get that \(RQ_i E\) can be generated by \(RQ_i A(m)\) using only finitely many cones. Thus, for any object \(E\) of \(\text{D}^b(\text{GrMod}(A, M))\), \(RQ_i(E)\) again lies in \(\text{D}^b(\text{GrMod}(A, M))\). Since \(E\) also clearly lies in \(\text{D}^b(\text{GrMod}(A, M))\), so does \(R\Gamma_j(E)\).

\(R\Gamma_j(E)\) lies in the subset of objects \(F\) of \(\text{D}^b(\text{GrMod}(A, M))\) such that every \(f \in F\) there is some power \(l\) of \(J\) with \(J^l f = 0\). This is the subcategory generated, up to cones, shifts, infinite coproducts, and summands, by the image of \(A/J\)-modules in \(\text{D}^b(\text{GrMod}(A, M))\). Indeed, any module is the colimit of its finitely-generated submodules, each of which are annihilated by a power of \(J\). Using the filtration by \(J\), each finitely-generated \(J\)-torsion module is generated by \(A/J\)-modules using cones, shifts, finite coproducts, and summands. The colimit is isomorphic to a cone over a map between coproducts of finitely-generated modules in the derived category.

So we can generate \(R\Gamma_j(E)\) using the set \(G\) up to cones, shifts, infinite coproducts, and summands. We are just left with \(RQ_j(E)\) which we have seen can be generated using twists of \(RQ_j(A)\). Since we can get any twist of \(RQ_j(A)\) from \(J\)-torsion modules and the twists of \(A\), we conclude that the set of objects \(A(m), m \in M\) and \(G\) generate \(\text{D}^b(\text{GrMod}(A, M))\). \(\square\)

**Lemma 4.7.** Given two graded algebras, \(A\), and \(B\), one can choose objects

\[
G_i \in \text{D}^b(\text{GrMod}(A, M_{\text{free}}))
\]

and

\[
H_j \in \text{D}^b(\text{GrMod}(B, N_{\text{free}}))
\]

with \(i \in I\) and \(j \in J\) of compact generators up to coproducts such that \(f^* G_i \otimes_k f^* H_j, i \in I, j \in J\) is a set of compact generators for \(\text{D}^b(\text{GrMod}(A \otimes_k B, M \oplus N))\).

**Proof.** Again, it suffices to assume \(M\) and \(N\) are torsion-free. We proceed by induction on the quantity \(\dim A + \dim B\). The case of \(\dim = \dim B = 0\) is clear.

Let assume that the result is true for \(\dim A + \dim B < n\) and consider the case \(\dim A + \dim B = n\). We can assume that \(A\) and \(B\) are reduced and irreducible. The singular locus of \(A \otimes_k B\) is decomposable. The first component corresponds to the ring \(A/J_A \otimes_k B\) where \(J_A\) is the radical ideal of the singular locus of \(A\). The second component corresponds to \(A \otimes_k B/J_B\) where \(J_B\) is the radical ideal of the singular locus of \(B\). Each component has dimension less that \(\dim A + \dim B\) so we can apply the induction hypothesis. There is a set of compact generators up to infinite coproducts of \(A/J_A \otimes_k B\) that are exterior products of generators of \(A/J_A\) and \(B\). A similar statement holds for \(A \otimes_k B/J_B\). Taking these sets and including all twists of \(A \otimes_k B\), Lemma \[\text{[4.6]}\] says we have a set of generators for \(\text{D}^b(\text{GrMod}(A \otimes_k B, M_{\text{free}} \oplus N_{\text{free}}))\). Applying Lemma \[\text{[4.3]}\] gives the result. \(\square\)
5. Graded matrix factorizations

5.1. Preliminaries. Let $k$ be a field and let $M$ be a finitely-generated Abelian group. Let $A$ be an $M$-graded $k$-algebra. Choose a homogeneous element $w \in A_d$. The category of graded matrix factorizations of $(A, w)$ has as objects pairs $(E_{-1}, E_0)$ of finitely-generated graded projective $A$-modules with morphisms

$$
\begin{array}{ccc}
E_{-1} & \xrightarrow{\phi_{-1}^E} & E_0 \\
\downarrow{\phi_{-1}^E} & & \downarrow{\phi_{-1}^E} \\
E_{-1} & \xrightarrow{\phi_{-1}^E} & E_0
\end{array}
$$

with the degree of $\phi_0^E$ being 0, the degree of $\phi_{-1}^E$ being $d$, and satisfying $\phi_{-1}^E(d) \circ \phi_{-1}^E = w$ and $\phi_{-1}^E \circ \phi_{-1}^E = w$. Often the matrix factorization, $(E_{-1}, E_0, \phi_{-1}^E, \phi_0^E)$, will be denoted by $E$. A morphism, $f : E \to F$ of matrix factorizations is a pair of degree zero $A$-module morphisms $f_0 : E_0 \to F_0$ and $f_{-1} : E_{-1} \to F_{-1}$ making the squares in the diagram

$$
\begin{array}{ccc}
E_0(-d) & \xrightarrow{\phi_{-1}^E} & E_{-1} & \xrightarrow{\phi_0^E} & E_0 \\
\downarrow{f_0(-d)} & & \downarrow{f_{-1}} & & \downarrow{f_0} \\
F_0(-d) & \xrightarrow{\phi_{-1}^E} & F_{-1} & \xrightarrow{\phi_0^E} & F_0
\end{array}
$$

commute. We denote the category of graded matrix factorizations of $(A, w)$ by $\text{MF}(A, w, M)$. It is useful to enlarge this category.

5.2. The absolute derived category. As before, let $A$ be an $M$-graded ring and $w \in A_d$. Following L. Positselski [Pos09, Pos11], we induce the following dg-category, denoted by $\text{GrMod}(A, w, M)$. The objects are pairs

$$
\begin{array}{ccc}
E_{-1} & \xrightarrow{\phi_{-1}^E} & E_0 \\
\downarrow{\phi_{-1}^E} & & \downarrow{\phi_{-1}^E} \\
E_{-1} & \xrightarrow{\phi_{-1}^E} & E_0
\end{array}
$$

of morphisms in $\text{GrMod}(A, w, M)$ with degree of $\phi_{-1}^E$ being zero, degree of $\phi_{-1}^E$ being $d$, and satisfying $\phi_{-1}^E(d) \circ \phi_{-1}^E = w$ and $\phi_{-1}^E \circ \phi_{-1}^E = w$. We denote the object by $(E_{-1}, E_0, \phi_{-1}^E, \phi_0^E)$ or just by $E$ when there is no confusion. The morphism complex between two objects, $E$ and $F$, as a graded vector space, can be described as follows. For $n = 2l$, we have

$$
\text{Hom}^{n}_{\text{GrMod}(A, w, M)}(E, F) = \text{Hom}_{\text{GrMod}(A, M)}(E_{-1}, F_{-1}(dl)) \oplus \text{Hom}_{\text{GrMod}(A, M)}(E_0, F_0(dl))
$$

and for $n = 2l + 1$, we have

$$
\text{Hom}^{n}_{\text{GrMod}(A, w, M)}(E, F) = \text{Hom}_{\text{GrMod}(A, M)}(E_0, F_{-1}(d(l + 1))) \oplus \text{Hom}_{\text{GrMod}(A, M)}(E_{-1}, F_0(dl))
$$

The differential is defined, as usual, by the graded commutator with the morphisms defining $E$ and $F$. Note that in the case that the subcategory of factorizations $E$ with $E_0$ and $E_{-1}$ being finitely-generated modules is simply the category of graded matrix factorizations of $w$.

From the dg-category, we can take the homotopy category $\text{Ho}(\text{GrMod}(A, M, w))$. Following Orlov [Orl11], we denote this as $\text{Pair}(A, w, M)$.
There is a natural shifting operation, \([1]\), which sends \(E\) to \((E_{-1}(d), E_0, -\phi_0^E, -\phi_{-1}^E(d))\). For any closed morphism, \(f : E \to F\), in \(\text{GrMod}(A, w, M)\), we can define the cone factorization, \(C(f)\), given by the pair of morphisms

\[
C(f)_{-1} = E_0 \oplus F_{-1} \quad \to \quad E_{-1}(d) \oplus F_0 = C(f)_0
\]

and

\[
C(f)_0(-d) \quad \to \quad C(f)_{-1}
\]

There are morphisms, \(F \to C(f)\) and \(C(f) \to E[1]\), and we declare the following sequence of morphisms

\[
E \xrightarrow{f} F \to C(f) \to E[1]
\]

to be a standard triangle. We define the class of distinguished triangles in \(\text{Pair}(A, w, M)\) to be those isomorphic to standard triangles. The proof of the following proposition is completely analogous to proof that the homotopy category of chain complexes of an abelian category is triangulated.

**Proposition 5.1.** \(\text{Pair}(A, w, M)\) equipped with the class of distinguished triangles described above is a triangulated category.

In addition, \(\text{Pair}(A, w, M)\) is also an abelian category where the monomorphisms and the epimorphisms are defined component wise.

Given a sequence of morphisms in \(\text{Pair}(A, w, M)\),

\[
E^l \xrightarrow{f_l} E^{l-1} \xrightarrow{f_{l-1}} \cdots \xrightarrow{f_1} E_0,
\]

we construct a totalization, \(T\), defined as follows.

\[
T_{-1} = \bigoplus_{i=2l} E^i_{-1}(dl) \oplus \bigoplus_{i=2l+1} E^i_0(dl)
\]

\[
T_0 = \bigoplus_{i=2l} E^i_0(dl) \oplus \bigoplus_{i=2l+1} E^i_{-1}(dl + d)
\]

We can view \(T\) as \(\bigoplus_i E^i[i]\) and the morphisms are the sum of morphisms from the \(E^i[i]\) and the components of the \(f_j\).

As defined by Positselski, let \(\text{Acyl}(A, w, M)\) denote the full subcategory of objects of \(\text{Pair}(A, w, M)\) consisting of totalizations of exact complexes. The absolute derived category, \(\text{D}^{\text{abs}}(\text{Pair}(A, w, M))\), is the Verdier quotient of \(\text{Pair}(A, w, M)\) by \(\text{Acyl}(A, w, M)\).

Let us recall some useful facts about \(\text{D}^{\text{abs}}(\text{Pair}(A, w, M))\). Inside \(\text{Pair}(A, w, M)\), we have the full subcategory consisting of \(M\) such that \(E_{-1}\) and \(E_0\) are graded projective modules over \(A\). There is a natural dg-category structure underlying this subcategory which we denote \(\text{Proj}(A, w, M)\). The image is \(\text{Ho}(\text{Proj}(A, w, M))\).

**Proposition 5.2.** The composition,

\[
\text{Ho}(\text{Proj}(A, w, M)) \to \text{Pair}(A, w, M) \to \text{D}^{\text{abs}}(\text{Pair}(A, w, M))
\]

is an equivalence of categories in the case where the graded ring, \((A, M)\), has finite global dimension.
Proof. This a version of Theorem 3.6 of Positselski \cite{Pos09}. We first note that \( \text{Pair}(A, w, M) \) can be viewed as the category of modules over a (bi)graded CDG \( k \)-algebra whose \((l, j)\)-degree terms are zero for \( j \) odd and whose degree \((l, 2i)\) term is \( A(d_i) \) with \( w \in A(d) \) being the curvature. As noted in [Pos11] in the non-affine case which is similar, the abelian category \( \text{Pair}(A, w, M) \) is equivalent to \( \mathbb{Z}/2\mathbb{Z} \)-graded modules over \((A, M)\). The global dimension of this category is finite as \((A, M)\) is concentrated in even degree with respect to the \( \mathbb{Z}/2\mathbb{Z} \)-grading. One can now apply the method of proof of Theorem 3.6 of [Pos09], while making sure the extra grading is accounted for, and reach the desired conclusion. □

Proposition 5.3. The objects of \( D^{\text{abs}}(\text{Pair}(A, w, M)) \) which are isomorphic to objects of \( \text{Pair}(A, w, M) \) whose underlying modules are finitely-generated graded \( A \)-modules form a set of compact generators for \( D^{\text{abs}}(\text{Pair}(A, w, M)) \).

Proof. The proof is a repetition of Theorem 2 of section 3.11 of [Pos09] while, again, making sure all the morphisms are of degree zero. □

Remark 5.4. While Proposition 5.3 tells us that we have a conceptually simple description of a set of compact generators, determining whether they are all compact objects is far more subtle.

6. Bimodule and functor categories for graded matrix factorizations

Graded matrix factorizations also admit a more geometric description due to Orlov \cite{Orl09a} and Polishchuk-Vaintrob \cite{PV10b}. For any graded matrix factorization, \( E \), we can form the \( M \)-graded \( A/(w) \)-module, \( \text{cok} \varphi_0 \).

Theorem 6.1. Suppose the graded ring \((A, M)\) has finite homological dimension and \( w \) is nonzero. The composition \( \text{Ho}(\text{MF}(A, w, M)) \xrightarrow{\text{cok}} D^b(\text{grmod}(A/(w), M)) \rightarrow D_{\text{sg}}(A/(w), M) \) is an equivalence of categories.

Proof. One can either note that the use of projectives circumvents the need for connectedness in Theorem 3.10 of [Orl09a] or consider the stack \([A/M]\) and use Theorem 3.14 of [PV10b]. □

As a corollary to the previous results in Section 4, we get the following generation statement.

Lemma 6.2. Let \((A, M)\) and \((B, N)\) be graded polynomial rings. Assume that \( w \in A_d \) and \( v \in N_e \) are nonzero homogeneous elements satisfying \( w \in (dw) \) and \( v \in (dv) \). There exists a set of generators \( C_i \in D^b(\text{grmod}(A/(w), M)), i \in I \) and \( D_j \in D^b(\text{grmod}(B/(v), N)), j \in J \) whose Künneth products, \( C_i \otimes D_j \) generate \( D_{\text{sg}}(A \otimes_k B/(w \oplus v), M \boxtimes N) \).

Proof. Let \( C = A \otimes_k B/(w \oplus v) \). From Lemma 4.6 we can generate \( D_{\text{sg}}(C, M \boxtimes N) \) if we can generate \( D^b(\text{grmod}(C/J_C, M_{\text{free}} \boxtimes N_{\text{free}})) \) where \( J_C \) is the radical ideal of the singular locus of \( C \). From our assumption that \( w \in (dw) \) and \( v \in (dv) \), \( C/J_C \cong A/J_A \otimes_k B/J_B \).
From Lemma 4.7 we can choose $C_i$ and $D_i$ generators of $D^b(\text{grmod}(A/J_A, M_{\text{free}}))$ and $D^b(\text{grmod}(B/J_B, N_{\text{free}}))$ respectively so that $C_i\otimes D_j$ generates $D^b(\text{grmod}(C/J_C, M_{\text{free}} \oplus N_{\text{free}}))$. Reducing grading from $M \oplus N$ to $M \boxplus N$ gives the statement. \hfill \blacksquare

**Remark 6.3.** There exist weaker conditions to guarantee that the conclusion of Lemma 6.2 holds. We choose $w \in (dw)$ and $v \in (dv)$ as a hypothesis because it seems fairly natural in the case of graded modules, e.g. Euler’s formula.

We now record a definition of Künneth products for objects of $\text{GrMod}(A, w, M)$ and $\text{GrMod}(B, v, N)$. We have a dg-functor

$$\boxtimes : \text{GrMod}(A, w, M) \otimes_k \text{GrMod}(B, v, N) \to \text{GrMod}(A \otimes_k B, w \boxplus v, M \boxplus N)$$

defined as

$$(E \boxtimes F)_{-1} := E_{-1} \otimes_k F_0 \oplus E_{-1} \otimes_k F_0$$

$$(E \boxtimes F)_0 := E_0 \otimes_k F_0 \oplus E_{-1} \otimes_k F_{-1}(d, 0)$$

$$\phi^{E \otimes F}_{-1} := \begin{pmatrix}
\text{Id}_{E_0} \otimes_k \phi_{-1}^F & \phi_0^E \otimes_k \text{Id}_{F_{-1}} \\
-\phi_{-1}^E \otimes_k \text{Id}_{F_0} & \text{Id}_{E_{-1}} \otimes_k \phi_{-1}^F
\end{pmatrix}$$

$$\phi^{E \otimes F}_0 := \begin{pmatrix}
\text{Id}_{E_0} \otimes_k \phi_0^F & -\phi_0^E \otimes_k \text{Id}_{F_0} \\
\phi_{-1}^E (d) \otimes_k \text{Id}_{F_{-1}} & \text{Id}_{E_{-1}} \otimes_k \phi_0^F
\end{pmatrix}$$

on objects. Given two morphisms, $f : E_1 \to E_2$ and $g : F_1 \to F_2$, the map, $f \boxtimes g : E_1 \boxtimes F_1 \to E_2 \boxtimes F_2$, has components

$$(f \boxtimes g)_{-1} := \begin{pmatrix} f_0 \otimes_k g_{-1} & 0 \\
0 & f_{-1} \otimes_k g_0 \end{pmatrix}$$

$$(f \boxtimes g)_0 := \begin{pmatrix} f_0 \otimes_k g_0 & 0 \\
0 & f_{-1} \otimes_k g_{-1}(d, 0) \end{pmatrix}.$$
where $K$ is the kernel of the homomorphism $M \oplus N \to M \square N$. Since $(d) = [2]$ in $\text{GrMod}(A,w,M)$ and $(e) = [2]$ in $\text{GrMod}(B,v,N)$ and we have assumed that $d$ and $e$ are not torsion, one inspects the components of $\text{Hom}_{\text{GrMod}(A \otimes_k B, w \oplus v, M \square N)}(E_1 \boxtimes F_1, E_2 \boxtimes F_2)$ and recognizes the image of $\text{Hom}_{\text{GrMod}(A,w,M)}(E_1, E_2) \otimes_k \text{Hom}_{\text{GrMod}(B,v,N)}(F_1, F_2)$ is everything. □

Proposition 6.5. Let $(A, M)$ and $(B, N)$ be graded polynomial algebras. Assume that $w \in A_d$ and $v \in N_e$ are nonzero homogeneous elements satisfying $w \in (dv)$ and $v \in (dv)$. There exists a set of compact generators, $C_i$, of $D_{\text{abs}}(\text{Pair}(A, w, M))$ and a set of compact generators, $D_j$, of $D_{\text{abs}}(\text{Pair}(B, v, N))$ such that $C_i \boxtimes D_j$ are a set of compact generators $D_{\text{abs}}(\text{Pair}(A \otimes_k B, w \oplus v, M \square N))$.

Proof. As the absolute derived categories are compactly-generated triangulated categories, to check that a set of compact objects are compact generators it suffices to check that they generate all compact objects. To do this, we can pass from matrix factorizations to singularity categories using Theorem 6.1. Applying Lemma 6.2, we can choose generators for $D_{\text{sg}}(A/w, M)$ and $D_{\text{sg}}(B/v, N)$ whose Künneth products generate $D_{\text{sg}}(A \otimes_k B/(w \oplus v), M \square N)$. We are, then, left with checking that the Künneth product of matrix factorizations corresponds to the Künneth products of their cokernels in the singularity category.

Given objects $S \in D_{\text{sg}}(A/(w), M)$, $T \in D_{\text{sg}}(B/(v), N)$, $E \in \text{MF}(A, w, N)$, and $F \in \text{MF}(B, v, N)$ with $\text{cok}E = S$ and $\text{cok}F = T$, we note there are two triangles:

\[ A_1 \to E \to S \]

in $\text{Pair}(A, w, M)$ and

\[ A_2 \to F \to T \]

in $\text{Pair}(B, v, N)$ with $A_1$ and $A_2$ being acyclic objects. The map $E \boxtimes F \to \text{cok}E \boxtimes \text{cok}F$ fits into a triangle

\[ K \to E \boxtimes F \to \text{cok}E \boxtimes \text{cok}F \]

in $\text{Pair}(A \otimes_k B, w \oplus v, M \square N)$. It also factors as $E \boxtimes F \to E \boxtimes \text{cok}F \to \text{cok}E \boxtimes \text{cok}F$ so $K$ fits in the triangle

\[ E \boxtimes A_2[1] \to K \to A_1[1] \boxtimes \text{cok}F \]

in $\text{Pair}(A \otimes_k B, w \oplus v, M \square N)$. As both functors, $E \boxtimes \bullet : \text{Pair}(B, v, N) \to \text{Pair}(A \otimes_k B, w \oplus v, M \square N)$ and $\bullet \boxtimes \text{cok}F : \text{Pair}(A, w, M) \to \text{Pair}(A \otimes_k B, w \oplus v, M \square N)$ are exact on the underlying abelian categories, $E \boxtimes A_2$ and $A_1 \boxtimes \text{cok}F$ are trivial in $D_{\text{abs}}(\text{Pair}(A \otimes_k B, w \oplus v, M \square N))$. Thus, $E \boxtimes F$ is isomorphic to $cokE \boxtimes F$ in $D_{\text{abs}}(\text{Pair}(A \otimes_k B, w \oplus v, M \square N))$. The argument of Proposition 3.2 of [Orl11] easily extends to the graded (and coherent) case, see also the proof of the Theorem of section 2 of [Pos11], to show that $\text{cok}$ is a well-defined exact functor from $D_{\text{abs}}(\text{Pair}(A \otimes_k B, w \oplus v, M \square N)) \to D_{\text{sg}}(A \otimes_k B/(w \oplus v, M \square N))$ extending the usual functor $\text{Ho}(\text{MF}(A \otimes_k B, w \oplus v, M \square N)) \to D_{\text{sg}}(A \otimes_k B/(w \oplus v, M \square N))$. Note here the subscript $\text{fg}$ means the components of the factorization are finitely-generated as modules over $A \otimes_k B$. □

Now consider the dg-categories $\text{Proj}(A, w, M)$, $\text{Proj}(B, v, N)$, and $\text{Proj}(A \otimes_k B, w \oplus v, M \square N)$. Inside $\text{Proj}(A, w, M)$ we have the full subcategory of objects, $C_i$. Let us denote that by $\mathcal{C}$. Similarly let us denote the full subcategory of the $D_i$ inside $\text{Proj}(B, v, N)$ by $\mathcal{D}$.

Lemma 6.6. The full subcategory consisting of the set of objects $C_i \boxtimes D_i$ in $\text{Proj}(A \otimes_k B, w \oplus v, M \square N)$ is quasi-equivalent as a dg-category to $\mathcal{C} \otimes_k \mathcal{D}$. 
Proof. This is a consequence of Lemma 6.4.

Applying Theorem 5.2 of [Dyc09], we get the following statement:

**Proposition 6.7.** In \( \text{Ho}(\text{dg-cat}_k) \), there are isomorphisms

1. \( \text{Proj}(A, w, M) \cong \hat{C} \).
2. \( \text{Proj}(B, v, N) \cong \hat{D} \).
3. \( \text{Proj}(A \otimes_k B, w \boxplus v, M \boxminus N) \cong \hat{C} \otimes_k \hat{D} \).

**Proof.** The first and second statements are direct consequences of Theorem 5.2 of loc. cit. as we have chosen a set of compact generators for \( \text{Proj}(A, w, M) \) and \( \text{Proj}(B, v, N) \). To verify the third statement, we use Lemma 6.6 to see that \( \hat{C} \otimes_k \hat{D} \) is quasi-equivalent to a subcategory of \( \text{Proj}(A \otimes_k B, w \boxplus v, M \boxminus N) \) and then apply Proposition 6.5 to conclude that every object of \( \text{Proj}(A \otimes_k B, w \boxplus v, M \boxminus N) \) is homotopic to a colimit of objects from \( \hat{C} \otimes_k \hat{D} \).

**Remark 6.8.** A description of the \( k[t, t^{-1}] \)-linear functor category of matrix factorizations on derived schemes over \( B \mathbb{G}_m \) whose potentials are sections of line bundles pulled back from \( B \mathbb{G}_m \) appears in Section 5.4.7 of [Pre11]. It seems that there are material differences with the description above so it would be interesting to compare and contrast the two.

We can dualize a graded module, \( E^\vee = \bigoplus_{m \in M} \text{Hom}_{\text{GrMod}(A, M)}(E, A(m)) \). This extends to a dualization functor

\[
\bullet^\vee : \text{MF}(A, w, M) \to \text{MF}(A, -w, M)
\]

\((E_{-1}, E_0, \phi_{-1}, \phi_0) \mapsto (E_0^\vee, E_{-1}(d)^\vee, \phi_0(d)^\vee, -\phi_{-1}(d)^\vee)\).

**Remark 6.9.** Note that the autoequivalence \((m)\) on \( \text{MF}(A, w, M) \) corresponds to \((-m)\) on \( \text{MF}(A, -w, M) \) under \( \bullet^\vee \).

Applying Theorem 5.2 of [Dyc09], we can note that \( \text{MF}(A, -w, M) \) is quasi-equivalent to \( \hat{C}^{\text{cop}} \). Applying Corollary 7.6 of [To¨e07], we can conclude the following:

**Corollary 6.10.** Assume that \((A, M)\) and \((B, N)\) are graded polynomial algebras, \( w \in A_d \) and \( v \in N_e \) are nonzero with \( w \in (dw) \), \( v \in (dv) \) and \( d \in M, e \in N \) both non-torsion. In \( \text{Ho}(\text{dg-cat}_k) \), there is an isomorphism

\[
\text{RHom}_k(\text{Proj}(A, w, M), \text{Proj}(B, v, N)) \cong \text{Proj}(A \otimes_k B, -w \boxplus v, M \boxminus N).
\]

**Remark 6.11.** The reader may feel the inkling that the above construction is not the most natural one. The assumptions \( w \in (dw) \) and \( v \in (dv) \) and \( d, e \) being non-torsion should be unnecessary. Indeed, there is more natural construction of the bimodule and functor categories utilizing the Cayley trick. This will be explored in the upcoming paper [BFK11a].

This construction is closer to the \( k[t, t^{-1}] \)-linear functor category. Indeed, if one sets \( M \) and \( N \) to zero, it is easy to check that tensor product as \( \mathbb{Z} \)-graded complexes in Lemma 6.4 needs to be replaced with a \( \mathbb{Z}/2\mathbb{Z} \)-graded tensor product of two-periodic complexes. The generation problems for these categories already appear in [Dyc09, LP11, Pre11]. An explicit description of the image of K"unneth objects as the subcategory objects with support in \( A \otimes_k B/(u, v) \) is given in [Pre11]. It should be straightforward to prove an analogous description if we remove the assumptions \( w \in (dw) \) and \( v \in (dv) \) using the singularity categories.
6.1. Integral transforms. From Corollary [6.10] we see that we can compute the derived
natural transformations from the identity functor to \((m)\) on \(\text{MF}(A, w, M)\) by computing
in \(\text{Proj}(A \otimes_k B, -w \boxplus v, M \boxplus N)\). We do this next. While it is simple to identify each \((m)\)
without discussing the notion of integral transforms, we feel that integral transforms provide
a more holistic answer.

Given an object \(K\) of \(\text{GrMod}(A^{op} \otimes_k B, -w \boxplus v, M \boxplus N)\) we have the following dg-functor,
\(\Phi_{K}^{A \rightarrow B} : \text{GrMod}(A, w, M) \rightarrow \text{GrMod}(B, v, N)\) defined by:

\[
\left(\Phi_{K}^{A \rightarrow B}(E)\right)_1 := \pi^*K_0 \otimes_A E_{-1} \oplus \pi^*K_{-1} \otimes_A E_0
\]

\[
\left(\Phi_{K}^{A \rightarrow B}(E)\right)_0 := \pi^*K_0 \otimes_A E_0 \oplus \pi^*K_{-1} \otimes_A E_{-1}(e)
\]

\[
\phi_{-1}^{A \rightarrow B}(E) := \begin{pmatrix}
\pi^*\phi_{-1}^K \otimes_A \phi_{-1}^E & \pi^*\phi_0^K \otimes_A \phi_{-1}^E \\
-\pi^*\phi_{-1}^K \otimes_A \phi_0^E & \pi^*\phi_0^K \otimes_A \phi_0^E
\end{pmatrix}
\]

\[
\phi_0^{A \rightarrow B}(E) := \begin{pmatrix}
\pi^*\phi_0^K \otimes_A \phi_{-1}(d, 0) \otimes A \phi_{-1}(d) & -\pi^*\phi_0^K \otimes_A \phi_0^E \\
\phi_{-1}(d, 0) \otimes A \phi_{-1}(d) & \pi^*\phi_0^K \otimes_A \phi_0^E
\end{pmatrix}
\]

where \(\pi : M \oplus N \rightarrow M \boxplus N\) is the projection. The action on morphisms is clear. The
assignment \(K \rightarrow \Phi_K\) extends to a dg-functor from \(\text{GrMod}(A^{op} \otimes_k B, -w \boxplus v, M \boxplus N)\) to the
category of dg-functors from \(\text{GrMod}(A, w, M)\) to \(\text{GrMod}(B, v, N)\) in a straightforward
manner.

Note that \(A\) itself cannot be viewed as an object of \(\text{GrMod}(A \otimes_k A, -w \boxplus w, M \boxplus M)\) for
the simple reason that the grading by \(M \boxplus M\) is not well-defined. To remedy this, consider the
summing map \(s : M \boxplus M \rightarrow M\). The kernel is isomorphic to \(M/(d)\). As a replacement
for \(A\) we will use \(s^*A\). More precisely, we will consider the factorization

\[
0 \xleftarrow{0} s^*A \xrightarrow{0} 0
\]

and, by abuse of notation, also refer to it as \(s^*A\).

Lemma 6.12. The integral transform, \(\Phi_{s^*A}\), is the identity functor on \(\text{GrMod}(A, w, M)\). In
addition, the integral transform, \(\Phi_{s^*A(m)}\), is isomorphic to \((m)\).

Proof. Consider the total sum map \(S : M \oplus M \rightarrow M\). \(S^*A\) is a \(M \oplus M\)-graded module is
isomorphic to the module obtained from \(A\) under the equivalence of categories of Lemma 3.8
\(S^*A\) is then the identity functor. Now, note that \(S^*A = \pi^*s^*A\).

Let \(A(m)\) be the \(A\)-bimodule with \(A(m)_{i,j} = A_{i+j+m}\). A quick computation shows that
\(A(m)\) is the kernel of \((s)\). Now, note \(S^*(A(m)) \cong A(m)\) as bigraded bimodules. \(\square\)

Each object, \(K\), of \(\text{Proj}(A \otimes_k B, -w \boxplus v, M \boxplus N)\) provides a dg-module over \(C \otimes_k D^{op}\). This
is simply \(\text{Hom}_{\text{Proj}(A \otimes_k B, -w \boxplus v, M \boxplus N)}(\cdot \boxplus -K)\). The following lemma is straightforward.

Lemma 6.13. For any object \(E\) of \(\text{Proj}(A, w, M)\) and any object \(F\) of \(\text{Proj}(B, v, N)\), there
are natural maps

\[
\text{Hom}_{\text{Proj}(B, v, N)}(F, \Phi_{K}^{A \rightarrow B}(E)) \rightarrow \text{Hom}_{\text{Proj}(A \otimes_k B, -w \boxplus v, M \boxplus N)}(E^\vee \boxplus F, K).
\]

which are isomorphisms for \(E \in \text{MF}(A, w, M)\).
Proof. Let us compute the degree $i = 2t$ portions of each of side. The morphism set, $\text{Hom}_{\text{Proj}(A \otimes_k B, -u \oplus v, M \otimes N)}(E^v \otimes F, K)$, is isomorphic to:

$$\text{Hom}(E_0^v \otimes_k F_0, K_0(dt, 0)) \oplus \text{Hom}(E_{-1}^v(d) \otimes_k F_{-1}, K_0(dt - d, 0))$$
$$\oplus \text{Hom}(E_0^v \otimes_k F_{-1}, K_{-1}(dt, 0)) \oplus \text{Hom}(E_{-1}^v(d) \otimes_k F_0, K_{-1}(dt, 0)).$$

And, $\text{Hom}_{\text{Proj}(B, v, N)}(F, \Phi_K^{A \to B}(E))$ is isomorphic to

$$\text{Hom}(F_0, \pi^*K_0 \otimes_A E_0(0)) \oplus \text{Hom}(F_0, \pi^*K_{-1} \otimes_A E_{-1}(t(e + 1)))$$
$$\oplus \text{Hom}(F_{-1}, \pi^*K_0 \otimes_A E_{-1}(te)) \oplus \text{Hom}(F_{-1}, \pi^*K_{-1} \otimes_A E_0(te)).$$

Given an $M$-graded $A$-module, $G$, and an $N$-graded $B$-modules, $H, G \otimes_k H$ is naturally an $M \oplus N$ graded $A \otimes_k B$ module. Above, we have reduced the grading group to $M \boxplus N$ on $E_i \oplus F_j$ via $\pi, i, j = 0, -1$. Adjunction then gives natural isomorphisms,

$$\text{Hom}(E_0^v \otimes_k F_0, K_0(dt, 0)) \cong \text{Hom}(E_0^v \otimes_k F_0, \pi^*K_0(dt, 0))$$
$$\text{Hom}(E_{-1}^v(d) \otimes_k F_{-1}, K_0(dt + d, 0)) \cong \text{Hom}(E_{-1}^v \otimes_k F_{-1}, \pi^*K_0(dt, 0))$$
$$\text{Hom}(E_0^v \otimes_k F_{-1}, K_{-1}(dt, 0)) \cong \text{Hom}(E_0^v \otimes_k F_{-1}, \pi^*K_{-1}(dt, 0))$$
$$\text{Hom}(E_{-1}^v(d) \otimes_k F_0, K_{-1}(dt, 0)) \cong \text{Hom}(E_{-1}^v \otimes_k F_0, \pi^*K_{-1}(dt + d, 0)).$$

$K_i(dt, 0)$ is isomorphic to $K_i(0, et)$ as an $M \boxplus N$ graded bimodule so we are free to shift the twists from one grading to the other. Performing this and using $\text{Hom}$-tensor adjunction, we get natural isomorphisms,

$$\text{Hom}(E_0^v \otimes_k F_0, \pi^*K_0(0, et)) \cong \text{Hom}(F_0, \text{Hom}(E_0^v, \pi^*K_0(0, et)))$$
$$\text{Hom}(E_{-1}^v \otimes_k F_{-1}, \pi^*K_0(0, et)) \cong \text{Hom}(F_{-1}, \text{Hom}(E_{-1}^v, \pi^*K_0(0, et)))$$
$$\text{Hom}(E_0^v \otimes_k F_{-1}, \pi^*K_{-1}(0, et)) \cong \text{Hom}(F_{-1}, \text{Hom}(E_0^v, \pi^*K_{-1}(0, et)))$$
$$\text{Hom}(E_{-1}^v \otimes_k F_0, \pi^*K_{-1}(0, et + e)) \cong \text{Hom}(F_0, \text{Hom}(E_{-1}^v, \pi^*K_0(0, et + e))).$$

There are natural maps,

$$\pi^*K_j \otimes_A E_i \to \text{Hom}_{\text{GrMod}(A, M)}(E_i^v, \pi^*K_j),$$

which are isomorphisms for $E_i$ finitely-generated projective. Applying these, we finally recognize $\text{Hom}_{\text{Proj}(B, v, N)}(F, \Phi_K^{A \to B}(E))$. A similar argument shows that there are natural isomorphisms when $i = 2t + 1$. It is straightforward to verify that these commute with the differentials. \hfill \Box

As a corollary we get the following:

**Corollary 6.14.** Under the isomorphisms of Corollary 6.10, the dg-functor isomorphic to $K$ in $\text{RHom}_c(\text{Proj}(A, w, M), \text{Proj}(B, v, N))$ is $\Phi_K^{A \to B}$.

**Remark 6.15.** One can also verify that, for $K_1 \in \text{GrMod}(B^{op} \otimes_k C, -v \oplus u, N \boxplus L), K_2 \in \text{GrMod}(A^{op} \otimes_k B, -w \oplus v, M \boxplus N)$, the composition, $\Phi_{K_1} \circ \Phi_{K_2}$, is isomorphic to $\Phi_{K_1 \boxplus B K_2}$ where $K_1 \boxplus B K_2 \in \text{GrMod}(A^{op} \otimes_k C, -w \oplus u, M \boxplus L)$ is $\pi^*K_1 \boxplus B \pi^*K_2$ with the induced $M \boxplus L$ grading. We have abused notation to use $\pi$ to stand for both projections $M \oplus N \to M \boxplus N$ and $N \oplus L \to N \boxplus L$. 

6.2. Derived graded natural transformations. In this section, we compute the the graded ring of derived natural transformations from Id to $(m)$ for any $m \in M$. From Corollary 6.10, we only need to describe these functors as objects in $\text{Proj}(A \otimes_k A, -w \boxplus w, M \boxplus M)$ and simply compute homotopy classes of morphisms.

Recall that we assume $A$ is a polynomial algebra over $k$. Thanks to this assumption, the computation of derived graded natural transformations becomes very explicit.

First, we construct an explicit matrix factorization to replace the curved complex $s^*(A(m))$ in the category $\text{Proj}(A \otimes_k A, -w \boxplus w, M \boxplus M)$. Let us write $A = k[x_1, \ldots, x_n]$ and $A \otimes_k A = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ to set notation.

In $A \otimes_k A$, we can write

$$w \otimes_k 1 - 1 \otimes_k w = \sum_{i=1}^{n} \Delta_i(x_i - y_i).$$

The factor, $\Delta_i$, can be taken to be homogeneous with respect to the diagonal grading. If we grade by $M \boxplus M$, then $\Delta_i$ decomposes into homogeneous pieces,

$$\Delta_i = \sum_{m \in M/(d)} \Delta_{i,m},$$

where the degree of $\Delta_{i,m}$ is $(-m + d - d_i, m)$. Decomposing the equation,

$$w \otimes 1 - 1 \otimes w = \sum_{i=1}^{n} \Delta_i(x_i - y_i),$$

into homogeneous pieces we get

$$\sum_{i=1}^{n} \Delta_{i,m} w_i - \Delta_{i,m-d_i} y_i = \begin{cases} 0 & \text{if } m \neq 0 \\ w \otimes 1 - 1 \otimes w & \text{if } m = 0. \end{cases}$$

Let $V = \bigoplus_{i=1}^{n}(A \otimes_k A)e_i$ with degree of $e_i$ being $(d_i, 0)$ and let $W_j$ be the one dimensional $M \boxplus M$-graded vector space with basis vector, $f_j$, of degree $(-j, j)$ for $j \in M/(d)$ and set $W = \bigoplus_{j \in M/(d)} W_j$. Let $t_i : W \to W$ be the $k$-linear map that sends $f_j$ to $f_{j+i}$.

Let $K_{-p} = \Lambda^p V \otimes_k W = \bigoplus_{j \in M/(d)} \Lambda^p V f_j$. We have a differential

$$d : K_{-p} \to K_{-p+1}$$

$$\omega f_j \mapsto \sum_{i=1}^{n} \omega \downarrow (x_i e_i^\vee) f_{j} - \omega \downarrow (y_i e_i^\vee) f_{j-d_i}.$$

Let $v = \sum_{i=1}^{n} (x_i t_0 - y_i t_{-d_i}) e_i^\vee$. We write $d = \downarrow v$. We have a surjective map from $K_0$ to $s^*A$ given by sending $f_j \mapsto 1 \in (s^*A)_{-j-j}$.

**Lemma 6.16.** The complex $(K, d)$ is quasi-isomorphic to $s^*A$.

**Proof.** The complex $(K, d)$ is isomorphic to a Koszul complex over the group ring $A \otimes_k A[M/(d)]$. Indeed, $A \otimes_k A \otimes_k W$ is isomorphic as a graded vector space to $A \otimes_k A[M/(d)]$. The $t_i : W \to W$ then get identified with multiplication in the group ring. $(K, d)$ is then the tensor product of the Koszul complexes for $x_i t_0 - y_i t_{-d_i}$. As we are working with coproducts, it is easy to check that multiplication by $x_i t_0 - y_i t_{-d_i}$ has no kernel. A standard argument then gives that $(K, d)$ is exact. \[\square\]
We have an additional differential.

\[ h : K_{-p} \rightarrow K_{-p-1} \]

\[ \omega f_j \mapsto \sum_{l \in M/(d)} \omega \wedge \left( \sum_{i=1}^{n} \Delta_{i,j-l} e_i \right) f_l. \]

Let \( \delta = \sum_{i=1}^{n} \sum_{s \in M/(d)} \Delta_{i,s} e_i t^{-s} \). We write \( h = -\wedge \delta \).

We consider the factorization, \( \Delta \),

\[ \Delta_{-1} = \bigoplus_{p=2q+1} K_{-p}(-dq), \quad \Delta_0 = \bigoplus_{p=2q} K_{-p}(-dq) \]

with the maps \( d + h \).

We have a morphism, \( \epsilon : \Delta \rightarrow s^*A \), which is only nonzero on \( K_0 \). On \( K_0 \), \( \epsilon \) is the surjection \( K_0 \rightarrow s^*A \).

**Proposition 6.17.** The map, \( \epsilon \), yields an isomorphism in \( \text{D}^{\text{abs}}(\text{GrMod}(A \otimes_k A, -w \boxplus w, M \boxminus M)) \).

**Proof.** Let us set some notation. Let \( Q_0 = s^*A \) as a graded module and inductively define \( Q_i \) as fitting into the exact sequences

\[ 0 \rightarrow Q_i \rightarrow K_i \rightarrow Q_{i+1} \rightarrow 0. \]

We wish to show that \( C(\epsilon) \) is trivial in \( \text{D}^{\text{abs}}(\text{GrMod}(A \otimes_k A, -w \boxplus w, M \boxminus M)) \). Consider the exact sequences

\[ 0 \rightarrow K_{-n} \rightarrow K_{-n+1} \rightarrow \cdots \rightarrow K_{-l} \rightarrow Q_{-l} \rightarrow 0. \]

\( -\wedge h \) extends to a null-homotopy of multiplication of \( -w \boxplus w \) on each of these sequences. Let \( T^l \) denote the curved complex with

\[ T^l_{-1} = \bigoplus_{p=2q+1, -n \leq p \leq -l} K_{-p}(dq) \oplus \alpha_l Q_{-l}(dt) \]

\[ T^l_0 = \bigoplus_{p=2q, -n \leq p \leq -l} K_{-p}(dq) \oplus \beta_l Q_{-l}(dt) \]

where if \( l \) is odd \( \alpha_l = 1, \beta_l = 0 \) and \( 2t + 1 = l \) and if \( l \) is even \( \alpha_l = 0, \beta_l = 1 \) and \( 2t = l \).

There are short exact sequences of curved modules

\[ 0 \rightarrow T^l \rightarrow T^{l+1} \xrightarrow{\lambda_l} S^l \rightarrow 0 \]

where \( S^l \) is the curved modules with \( S^l_{-1}, S^l_0 = Q_l \) with the pair of maps being the identity and multiplication by \( w \). Thus, \( C(\lambda_l) \) is isomorphic to \( T^l \) in \( \text{D}^{\text{abs}}(\text{GrMod}(A \otimes_k A, -w \boxplus w, M \boxplus M)) \). But, \( S^l \) is trivial \( \text{Ho}(\text{GrMod}(A \otimes_k A, -w \boxplus w, M \boxminus M)) \). So \( C(\lambda_l) \cong T^{l+1}[1] \) in \( \text{D}^{\text{abs}}(\text{GrMod}(A \otimes_k A, -w \boxplus w, M \boxminus M)) \). \( T^l \) is zero for \( l > n \) so \( T^l \) is trivial for \( l \geq 0 \). In particular, \( T^0 = C(\epsilon) \) is zero. \( \square \)

**Corollary 6.18.** The object, \( \Delta(0, m) \), is isomorphic to \( s^*A(m) \) in \( \text{D}^{\text{abs}}(\text{GrMod}(A \otimes_k A, -w \boxplus w, M \boxplus M)) \).

**Proof.** This follows from Proposition 6.17 by twisting both objects by \( (m) \). \( \square \)
Thus, to compute
\[ \bigoplus_{l \in \mathbb{Z}} \bigoplus_{m \in \mathbb{M}} \text{Hom}_{\mathbb{D}_{\text{abs}}(\text{GrMod}(A \otimes_k A, - \otimes \mathbb{K}(\mathbb{M})))(s^* A, s^* A(m)[l])} \]
we need to compute
\[ \bigoplus_{l \in \mathbb{Z}} \bigoplus_{m \in \mathbb{M}} \text{Hom}_{\mathbb{H}(\text{Proj}(A \otimes_k A, - \otimes \mathbb{K}(\mathbb{M})))(\Delta, \Delta(0, m)[l])}. \]
As \( \Delta \) is left orthogonal to acyclic objects, we can replace \( \Delta(0, m) \) with \( s^* A(m) \) and compute
\[ \bigoplus_{l \in \mathbb{Z}} \bigoplus_{m \in \mathbb{M}} \text{Hom}_{\mathbb{Pair}(A \otimes_k A, - \otimes \mathbb{K}(\mathbb{M}))}(\Delta, s^* A(m)[l]) \]
We do this now in the case that \( M/(d) \) is finite. The computation is quite similar to [PV10a]. Applying \( \text{Hom}_{\mathbb{GrMod}(A \otimes_k A, - \otimes \mathbb{K}(\mathbb{M}))}(\Delta, s^* A(m)[l]) \) to \( \Delta \) gives a double complex, which we shall denote as \( C \). The terms of \( C \) are
\[ C_t = \left\{ \bigoplus_{n \in M/(d)} \bigoplus_{p=2q} \left( \Lambda^p \tilde{V} \otimes_k A \right)_{m-dq+dt} f_{n} \mid t = 2l \right\} \]
\[ + \left\{ \bigoplus_{n \in M/(d)} \bigoplus_{p=2q+1} \left( \Lambda^p \tilde{V} \otimes_k A \right)_{m-dq+dt} f_{n} \mid t = 2l + 1 \right\} \]
Here \( \tilde{V} \) is the graded vector space \( k\tilde{e}_1 \oplus \cdots \oplus k\tilde{e}_n \) with deg \( \tilde{e}_i = -d_i \) and \( f_n \) has no \( M \)-degree attached. It is simply a placeholder for sum over \( M/(d) \) and allow us to continue to use the notation for the operators \( t_m \). Note that under the identification \( \tilde{e}_i \leftrightarrow x_i \) \( A \) is isomorphic to \( \text{Sym} \tilde{V}^\vee \). We will implicitly use this identification.

The differentials on \( C \) are given by,
\[ d = \bullet \wedge \sum_{i=1}^{n} x_i \tilde{e}_i (t_0 - t_{-d_i}), \]
and
\[ h = \bullet \cup \sum_{i=1}^{n} \sum_{m \in M/(d)} \Delta_{i,m} \tilde{e}_i \tilde{e}_i^{\vee} t_{-m}. \]

Let \( G \) be the group whose characters are \( M/(d) \) and let \( f_g = \sum_{\chi \in M/(d)} \chi(g) f_{\chi} \). \( C \) splits into the sum of complexes \( C = \bigoplus_{g \in G} C_g \) corresponding to the spans of the \( f_g \). The induced differentials on \( C_g \) are \( \bullet \wedge \sum_{i=1}^{n} (x_i - g^*(x_i)) \tilde{e}_i \) and \( \bullet \cup \sum_{i=1}^{n} g^*(\Delta_i) \tilde{e}_i^{\vee} \). \( \tilde{V} \) is an \( M \)-graded vector space \( \tilde{V} = \bigoplus_{m \in \mathbb{M}} \tilde{V}_m \). Let \( \tilde{V}^g = \bigoplus_{m \in \text{ker } g} \tilde{V}_m \) and \( \tilde{V}^g = \bigoplus_{m \in \text{ker } g} \tilde{V}_m \). Here we view \( g \) as function on \( M \) by passing to the quotient \( M/(d) \).

Let \( c_g = \dim \tilde{V}^g \). Note that the splitting \( \tilde{V} = \bigoplus_{g \in G} \tilde{V}^g \) determines maps \( \text{Sym} \tilde{V}^g \hookrightarrow A \rightarrow \text{Sym} \tilde{V}^g \). The last map is the quotient by the ideal given by \( \text{Sym} \tilde{V}_g \) in \( A \). Set \( A^g = \text{Sym}(\tilde{V}^g)^\vee \).

Let us consider another complex, denoted \( D_g \).
\[ (D_g)_t = \left\{ \bigoplus_{n \in M/(d)} \bigoplus_{p=2q} \left( \Lambda^p \tilde{V}^g \otimes_k A^g \right)_{m-dq+dl} f_{n} \mid t = 2l \right\} \]
\[ + \left\{ \bigoplus_{n \in M/(d)} \bigoplus_{p=2q+1} \left( \Lambda^p \tilde{V}^g \otimes_k A^g \right)_{m-dq+dl} f_{n} \mid t = 2l + 1 \right\} \]
whose differential \( \bullet \cup \sum_{i=1}^{n} \partial_i v_i \tilde{e}_i \tilde{e}_i^{\vee} \) where \( v_i \) is a basis of \( \tilde{V}^g \).

There is embedding of \( D_g \) into \( C_g \) given by the inclusion \( \Lambda^{p-c_0} \tilde{V}^g(-v_g) \rightarrow \Lambda^p \tilde{V} \) and the map \( \text{Sym} \tilde{V}^g \rightarrow A \). Here \( v_g \) is the degree of a volume form on \( \tilde{V}_g \). It is clear that \( d \) vanishes on the image. It is easy to check that \( g^* \Delta_i \), restricted to diagonal is isomorphic to
∂_i w_g where w_g is the image of w under the projection A → A^g. Thus, ∑_i=1^n-c_g ∂_i w_g \tilde{v}_i agrees with h restricted to the image.

**Lemma 6.19.** The inclusion D_g → C_g is a quasi-isomorphism.

**Proof.** If we take the spectral sequence associated to the double complex with first differential given d, we see that the inclusion induces a map of spectral sequences where D_g is considered as double complex with a zero extra differential. The map on the E_1 is an isomorphism. □

The cohomology of D_g is essentially the cohomology of the Koszul complex of the Jacobian ideal of w_g. To be precise, let H^i(dw_g; A^g) denote the Koszul cohomology of the ideal (dw_g) in A^g.

**Lemma 6.20.** We have

\[ H^i(C_g)_m = \begin{cases} \bigoplus_{p=2q} H^{p-c_g}(dw_g; A^g)_{m+d(l-q)-v_g} & \text{if } t = 2l \\ \bigoplus_{p=2q+1} H^{p-c_g}(dw_g; A^g)_{m+d(l-q)-v_g} & \text{if } t = 2l + 1. \end{cases} \]

**Proof.** It suffices to compute this for D_g where the computation is immediate. □

This gives the final computation.

**Theorem 6.21.**

\[ \text{Hom}_{\text{Ho}(\text{REnd}_{\mathcal{A}}(\text{Proj}(A, w, M)))}(\text{Id}, (m)[l]) = \begin{cases} \bigoplus_{g \in G} \bigoplus_{p=2q} H^{p-c_g}(dw_g; A^g)_{m+d(l-q)-v_g} & \text{if } t = 2l \\ \bigoplus_{g \in G} \bigoplus_{p=2q+1} H^{p-c_g}(dw_g; A^g)_{m+d(l-q)-v_g} & \text{if } t = 2l + 1. \end{cases} \]

We will give some examples in the following section.

6.3. **Examples.** Let us start with the simplest example.

**Example 6.22.** Let us give a thorough computation in the case of Z-graded matrix factorizations of the potential x^d in k[x]. The factorization of the diagonal is

\[
\begin{pmatrix}
 x & -y & 0 & \cdots & 0 \\
 0 & x & -y & \cdots & 0 \\
 0 & 0 & x & \cdots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 -y & 0 & 0 & \cdots & x \\
\end{pmatrix}
\]

\[ \bigoplus_{i=0}^{d-1} k[x, y](−i − 1, i) \quad \bigoplus_{i=0}^{d-1} k[x, y](−i, i) \]

\[
\begin{pmatrix}
 x^d & x^{d-1}y & x^{d-2}y^2 & \cdots & y^d \\
 y^{d-1} & x^{d-1} & x^{d-2}y & \cdots & xy^{d-2} \\
 xy^{d-2} & y^{d-1} & x^{d-1} & \cdots & x^2y^{d-3} \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 x^{d-2}y & x^{d-3}y^2 & x^{d-4}y^3 & \cdots & x^{d-1} \\
\end{pmatrix}
\]

Let us apply Hom_{MF}(k[x,y], x^d−y^d, Z)(−, Δ) to get
of all torsion elements. For the degree map, deg : \( M \) of degree one. Let 
\( \mathcal{B} \) algebra with 
\( \bigoplus \) abelian group of rank one. Let us gather the setup:

\[ \text{Example 6.23.} \]

Let \( w \in k[x_0, \ldots, x_n] \) be of degree \( d \) where \( k[x_0, \ldots, x_n] \) carries the usual \( \mathbb{Z} \)-grading and let us compute the graded ring of derived natural transformations \( \text{Id} \to (i) \) for \( i \in \mathbb{Z} \) under the assumption that \( w \) has an isolated singularity at 0. The Hochschild homology of the category \( \text{MF}(k[x_0, \ldots, x_n], w, \mathbb{Z}) \) has already been computed by Căldăraru-Tu [CT10 Tu10] and we get their answer as a consequence of this computation. Indeed, the Serre functor on \( \text{MF}(k[x_0, \ldots, x_n], x^d, \mathbb{Z}) \) is \( (1 - d)[-1] \) (see [KMV08]). Computing the morphism space, \( \text{Hom}_{\text{MF}(k[x,y], x^d - y^d, \mathbb{Z})}(\Delta, S) \), gives

\[
\begin{pmatrix}
  z & -z & 0 & \cdots & 0 \\
  0 & z & -z & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & z & \cdots & 0 \\
\end{pmatrix}
\]

which gives \( \text{HH}^0(\text{MF}(k[x], x^d, \mathbb{Z})) = k \) and \( \text{HH}^i(\text{MF}(k[x], x^d, \mathbb{Z})) = 0 \) for \( i \neq 0 \), as expected.

As \( w \) is isolated, there is no higher Koszul cohomology, only the graded Jacobian ring. In addition, for \( g \in \mathbb{Z}/d\mathbb{Z} \) that is not zero, the fixed locus of \( g \) is zero. Thus, the twisted sections contribute \( d - 1 \) copies of \( k \) for \( \text{Hom}(\text{Id}, (d - (n + 1))[n - 1]) \). The untwisted sector gives a copy of the Jacobian ring inside \( \bigoplus_{i \in \mathbb{Z}} \text{RHom}(\text{Id}, (i)) \).

The computation in this case gives a natural extension of Griffiths’ residue computation, [Gri69], for the middle dimensional cohomology of an even dimensional projective hypersurface.

7. A theorem of Orlov and a geometric description of the functor category

In [Orl09a], Orlov demonstrates a beautiful correspondence between derived categories of coherent sheaves and triangulated categories of singularities involving semi-orthogonal decompositions (see Section 9.1 for a definition). His equivalence also generalizes to a statement on the level of dg-categories, see [CT10]. In what follows we will need a very slight generalization, from \( \mathbb{Z} \)-graded algebras to \( M \)-graded algebras, where \( M \) is a finitely generated abelian group of rank one. Let us gather the setup:

Let \( M \) be a finitely generated abelian group of rank one, and \( H \) be the subgroup consisting of all torsion elements. For the degree map, \( \deg : M \to M/H \), choose \( \delta \in M \), an element of degree one. Let \( B \) be a finitely generated connected \( M \)-graded algebra over \( k \) of finite homological dimension, \( n \), and \( W \) a homogeneous central element. Let \( C \) be the \( \mathbb{Z} \)-graded algebra with \( B_r := \bigoplus_{h \in H} B_{r\delta + h} \) and the same algebra structure as \( B \).
Although Orlov only writes up the proof in the $\mathbb{Z}$-graded case, in the setup above, one can follow the proof in [Orl09a] verbatim, replacing all categories in question with their orbit under the action of $H$. Hence the following theorem is certainly due to Orlov:

**Theorem 7.1.** With the notation above, suppose $C$ is Gorenstein with Gorenstein parameter $a$.

i) If $a > 0$, there is a semi-orthogonal decomposition,

$$\text{D}^b(\text{coh } (\mathbb{Z}^s + n_1 + \cdots + n_s)) \cong \left( \bigoplus_{h \in H} A((-a+1)\delta + h), \bigoplus_{h \in H} A(h), \text{MF}(B, W, M) \right).$$

ii) If $a = 0$, there is an equivalence of triangulated categories,

$$\Phi : \text{D}^b(\text{coh } (\mathbb{Z}^s + n_1 + \cdots + n_s)) \rightarrow \text{MF}(B, W, M).$$

iii) If $a < 0$, there is a semi-orthogonal decomposition,

$$\text{MF}(B, W, M) \cong \left( \bigoplus_{h \in H} k(h), \bigoplus_{h \in H} k((a + 1)\delta + h), \text{D}^b(\text{coh } (\mathbb{Z}^s + n_1 + \cdots + n_s)) \right).$$

Combining the above with Corollary 6.10 allows for a geometric description of the tensor product of matrix factorization categories. Consider a collection of hypersurfaces, $X_i \subseteq \mathbb{P}^{n_i}$ defined by polynomials $f_i$ of degree $d_i$ for $1 \leq i \leq s$. Let $R_i$ be the coordinate rings of the $\mathbb{P}^{n_i}$. Consider the free abelian group of rank $s$, $\mathbb{Z}^s$, with basis $e_i$, $1 \leq i \leq s$. Let $L$ be the subgroup generated by $d_i e_i = d_j e_j$ and $M := \mathbb{Z}^s / L$. Denote by $H$ the torsion subgroup of $M$. Explicitly, letting $d_{ij}$ be the greatest common divisor of $d_i$ and $d_j$, $H$ is the finite subgroup of $M$ generated by the images of $\frac{d_i}{d_{ij}} e_i - \frac{d_j}{d_{ij}} e_j$. One has $M / H \cong \mathbb{Z}$. Let $m$ be the least common multiple of the $d_i$. In this setting the degree map $\text{deg} : M \rightarrow \mathbb{Z}$ can be identified with the mapping which takes $e_i$ to $\frac{d_i}{d}$. Let $\delta$ be an element of degree 1.

The dual group to $M$ can be identified with the set, $D := \{ (\lambda_1, \ldots, \lambda_s) | \lambda_i^d_i = \lambda_j^d_j \forall i, j \} \subseteq (k^s)^s$ and acts on $\mathbb{A}^{n_1 + \cdots + n_s}$ by multiplication by $\lambda_i$ on the coordinates, $x_{d_1 + \cdots + d_i - 1}$ through $x_{d_1 + \cdots + d_i}$. Let $Y$ denote the hypersurface in $\mathbb{A}^{n_1 + \cdots + n_s}$ defined by the zero locus of $f_1 + \cdots + f_s$ and consider the quotient stack, $Z := [Y / D]$.

**Lemma 7.2.** The $M$-graded coordinate ring of $Z$ has Gorenstein parameter, $\sum_{i=1}^{d_1 + \cdots + d_s} e_i - d_1 e_1$. The degree of this parameter is $a := m(-1 + \sum_{i=1}^{s} \frac{1}{d_i})$.

**Proof.** The $M$-graded polynomial ring has Gorenstein parameter $\sum_{i=1}^{d_1 + \cdots + d_s} e_i$, see [GLS7]. The statement follows from adjunction.

**Theorem 7.3.** With the notation above,

i) If $a > 0$, there is a semi-orthogonal decomposition,

$$\text{D}^b(\text{coh } Z) = \left( \bigoplus_{h \in H} \mathcal{O}_Z((-a+1)\delta h), \bigoplus_{h \in H} \mathcal{O}_Z(h), (\text{MF}(R_1, f_1, Z) \otimes_k \cdots \otimes_k \text{MF}(R_s, f_s, Z))_{\text{pe}} \right).$$

ii) If $a = 0$, there is an equivalence of triangulated categories,

$$\text{D}^b(\text{coh } Z) \cong (\text{MF}(R_1, f_1, Z) \otimes_k \cdots \otimes_k \text{MF}(R_s, f_s, Z))_{\text{pe}}.$$
iii) If $a < 0$, there is a semi-orthogonal decomposition,

$$(\text{MF}(R_1, f_1, Z) \otimes_k \cdots \otimes_k \text{MF}(R_s, f_s, Z))_{\text{pe}} \cong \left( \bigoplus_{h \in H} k(h), \ldots, \bigoplus_{h \in H} k((a + 1)\delta h), D^b(\text{coh } Z) \right).$$

**Proof.** Applying Theorem 7.1 and Lemma 7.2 we obtain the semi-orthogonal decompositions above, with $\text{MF}(R_1 \otimes_k \cdots \otimes_k R_s, f_1 + \cdots + f_s, M)$ as the matrix factorization component. This category is dense in $\text{MF}(R_1 \otimes_k \cdots \otimes_k R_s, f_1 + \cdots + f_s, M)$ by iterated application of Corollary 6.10. To get an equivalence, we observe that $\text{MF}(R_1 \otimes_k \cdots \otimes_k R_s, f_1 + \cdots + f_s, M)$ is Karoubi complete. In fact, we prove that it is saturated and therefore, in particular, it is Karoubi complete. There are two cases; in the case where the degree of the Gorenstein parameter is non-negative, Theorem 7.1 tells us that $\text{MF}(R_1 \otimes_k \cdots \otimes_k R_s, f_1 + \cdots + f_s, M)$ is an admissible subcategory of a saturated triangulated category. Therefore by Proposition 2.8 of [BK89], it is also saturated. In the case where the Gorenstein parameter is negative, Theorem 7.1 tells us that $\text{MF}(R_1 \otimes_k \cdots \otimes_k R_s, f_1 + \cdots + f_s, M)$ has a semi-orthogonal decomposition whose components are saturated triangulated categories. The result then follows from Theorem 2.10 of loc. cit. \qed

If $X_i$ is of general type for some $i$, then by Theorem 7.1 one has a semi-orthogonal decomposition of $\text{MF}(R_i, f_i, Z)$. Hence,

$$(\text{MF}(R_1, f_1, Z) \otimes_k \cdots \otimes_k \text{MF}(R_s, f_s, Z))_{\text{pe}},$$

can also be decomposed further by iterated application of the following simple lemma:

**Lemma 7.4.** Let $A$ and $B$ be triangulated dg-categories over $k$. Suppose their are semi-orthogonal decompositions: $A = \langle A_0, \ldots, A_s \rangle$ and $B = \langle B_0, \ldots, B_t \rangle$. Then there is a semi-orthogonal decomposition:

$$(A \otimes_k B)_{\text{pe}} = \langle \coprod_{i+j=0} (A_i \otimes_k B_j)_{\text{pe}}, \ldots, \coprod_{i+j=s+t} (A_i \otimes_k B_j)_{\text{pe}} \rangle.$$

**Proof.** This is an immediate consequence of the definitions. \qed

Using this observation, one sees a general picture for creating Landau-Ginzburg mirrors to products of hypersurfaces, $X_i \subseteq \mathbb{P}^{p_i},$ defined by polynomials, $f_i$, of degree $d_i$ for $1 \leq i \leq s$. For simplicity, we will describe the case where all of the $X_i$ are either Calabi-Yau or of general type. However, we want to first ensure that when we product these varieties the resulting Gorenstein parameter is positive, as the mirror symmetry constructions in this case have been explored in much greater depth. To do so, we simply employ a graded version of Knörrer periodicity. Namely, it was proven by Knörrer in [Knö87] that there is an equivalence of categories $\text{MF}(R, f) \cong \text{MF}(R, f + x^2 + y^2)$. In the graded case, this manifests similarly by first adding the potential $x^2$ alone. One can see by inspection that $\text{MF}(k[x], x^2, Z)$ is quasi-isomorphic to the dg-category of chain complexes of vector spaces (see also [Orl09a]). Therefore given $f \in R_m$, we have:

$$\text{MF}(R, f, M) \cong (\text{MF}(k[x], x^2, Z) \otimes \text{MF}(R, f, M))_{\text{pe}} \cong \text{MF}(R[x], f + x^2, M \oplus Z/(m, -2)).$$

Similarly we have:

$$\text{MF}(R, f, M) \cong \text{MF}(R[x, y], f + x^2 + y^2, M \oplus Z \oplus Z/\langle (m, -2, 0), (m, 0, -2) \rangle).$$
Combining this with Theorem 7.3, we observe that the Gorenstein parameter of the $M \oplus \mathbb{Z} \oplus \mathbb{Z}/\langle (m, -2, 0), (m, 0, -2) \rangle$-graded ring $R[x, y]/f + x^2 + y^2$, always has positive degree. Therefore, while, we may start with say, some hypersurface of general type, we can nevertheless embed our category as an admissible subcategory of the derived category of coherent sheaves on a certain stack. This should be viewed as a generalization of the conic bundle construction for varieties of general type observed by the third named author and carried to fruition in [Sci08b, EHi09, KKOY09]. Specifically let,

$$D := \{(\lambda_1, \ldots, \lambda_s, \beta, \gamma)|\lambda_i^{d_i} = \lambda_j^{d_j} = \beta^2 = \gamma^2 \forall i, j \subseteq (k^*)^s$$

act on $k[x_0, \ldots, x_{n_1+\cdots+n_s+s}, u, v]$ by multiplication by $\lambda_i$ on the coordinates, $x_{d_1+\cdots+d_s+1}$ through $x_{d_1+\cdots+d_s}$, multiplication by $\beta$ on $u$, and multiplication by $\gamma$ on $v$. Let $Y$ denote the hypersurface in $\mathbb{A}^{n_1+\cdots+n_s+s+2}\setminus 0$ defined by the zero locus of $f_1 + \cdots + f_s + u^2 + v^2$ and $Z := [Y/D]$ be the global quotient stack. Now for each $X_i$, by Orlov’s theorem, one has a semi-orthogonal decomposition where $D^b(\text{coh} X_i)$ is a component of $\text{MF}(R_i, f_i, Z)$. If we employ our conic bundle construction, the other components of the semi-orthogonal decomposition correspond to singular fibers away from the origin. As it is well known that,

$$(D^b(\text{coh} X_1) \hat{\otimes}_k \cdots \hat{\otimes}_k D^b(\text{coh} X_t))_{pe}$$

is equivalent to $D^b(\text{coh} X_1 \times \cdots \times X_t)$, which by our Theorem 7.3 and Lemma 7.4 is a component in the semi-orthogonal decomposition of $D^b(\text{coh} Z)$, in the mirror, this category corresponds to the deep singular fiber at the origin (by [LO10], we need not be careful about the choice of dg-structure on $D^b(\text{coh} X_i)$). We arrive at the following conjecture (also Conjecture 2 in the introduction):

**Conjecture 7.5.** The mirror to $X$ can be realized as the Landau-Ginzburg mirror to $Z$ with all singular fibers away from the origin removed.

Perhaps the simplest example of our result is the case of hypersurfaces in weighted projective space which are weighted projective lines. These examples are very similar to the ones found in [KST07], especially Appendix A1 due to Ueda. The main difference is that we leave the grading group as it is without reducing to a $\mathbb{Z}$-grading and hence look at categories of matrix factorizations as opposed to more general categories of singularities. As it will be useful to us later on, we shall treat the case where the Gorenstein parameter of the coordinate ring is positive, like in the appendix of loc. cit. A much more general case is handled in subsection 9.2, where weighted Fermat hypersurfaces are discussed with arbitrary Gorenstein parameter. Comparing our results with theirs, the affect is that we get different $ADE$ quivers. The explanation is that by applying Theorem 7.4 when the grading group has torsion, our category is orthogonal to a larger exceptional collection, and hence we get an admissible subcategory of theirs.

We now recall the definition of a weighted projective line, which was introduced in [GL87]. Differing slightly from the way things are presented in [GL87], we will present our definition ultimately in the language of stacks. A triple, $(p, q, r)$, is called a weight sequence (more generally we will use this term for any $n$-tuple of integers later on). To each weight sequence, we attach a group, $M := \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/\langle (p, -q, 0), (p, 0, -r) \rangle$, and let $H$ be the torsion subgroup of $M$. The dual group, $D := \{\alpha, \beta, \gamma|\alpha^p = \beta^q = \gamma^r \subseteq (k^*)^3$, acts on $\mathbb{A}^3/0 = \text{Spec} k[x, y, z]$ by multiplication by $\alpha$ on $x$, $\beta$ on $y$, and $\gamma$ on $z$. Let $Y$ denote the hypersurface in $\mathbb{A}^3/0$ defined by the zero locus of $x^p + y^q + z^r$. The weighted projective line $\mathbb{P}^1(p : q : r)$, in modern language, is the quotient stack, $[Y/D]$. In other words, in the setup of Theorem 7.3
\( \mathbb{P}^1(p : q : r) \) is nothing more than the stack \( Z \) related by a semi-orthogonal decomposition to,

\[
(\text{MF}(k[x], x^p, \mathbb{Z}) \otimes_k \text{MF}(k[y], y^q, \mathbb{Z}) \otimes_k \text{MF}(k[z], z^r, \mathbb{Z}))_{pe}.
\]

Now for a Dynkin diagram of type \( ADE \), create a quiver by attaching arrows from left to right (it is really inconsequential how we attach the arrows, the derived category will only depend on the underlying graph). By abuse of notation, we denote this quiver the same way as the underlying graph. For example, we write \( \text{mod} - \mathcal{A}_{p-1} \) as the category of left modules over the path algebra of the quiver whose underlying graph is \( \mathcal{A}_{p-1} \). Now, \( \text{MF}(k[x], x^p, \mathbb{Z}) \) is equivalent to \( \text{D}^b(\text{mod} - \mathcal{A}_{p-1}) \) (see [Orl09a]). Therefore,

\[
(\text{MF}(k[x], x^p, \mathbb{Z}) \otimes_k \text{MF}(k[y], y^q, \mathbb{Z}) \otimes_k \text{MF}(k[z], z^r, \mathbb{Z}))_{pe} \\
\cong ((\text{D}^b(\text{mod} - \mathcal{A}_{p-1})) \otimes_k \text{D}^b(\text{mod} - (A_{q-1}))) \otimes_k \text{D}^b(\text{mod} - (A_{r-1})))_{pe} \\
\cong \text{D}^b(\text{mod} - \mathcal{A}_{p-1} \otimes_k A_{q-1} \otimes_k A_{r-1}).
\]

Therefore, using Corollary 6.10, we are able to compare \( \text{D}^b(\text{mod} - \mathcal{A}_{p-1} \otimes_k A_{q-1} \otimes_k A_{r-1}) \) to \( \text{D}^b(\text{coh} \mathbb{P}^1(p : q : r)) \). Let \( m \) be the least common multiple of \( p, q, r \). The degree of the Gorenstein parameter for the coordinate ring of \( \mathbb{P}^1(p : q : r) \) is \( a = m\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1\right) \).

In the case when \( a > 0 \), the weight sequence \((p, q, r)\) is called of Dynkin type. In this case, we have the following semi-orthogonal decomposition of \( \text{D}^b(\text{coh} \mathbb{P}^1(p : q : r)) \):

\[
\bigoplus_{h \in H} \mathcal{O}_Z((-mt + 1)\delta + h), \ldots, \bigoplus_{h \in H} \mathcal{O}_Z(h), \text{D}^b(\text{mod} - (A_{p-1} \otimes_k A_{q-1} \otimes_k A_{r-1}))).
\]

The possible weight sequences of Dynkin type, match up precisely with the simple singularities. One has, \( A_{p+q} \) for \((1, p, q)\), \( D_{l+2} \) for \((2, 2, l)\), \( E_6 \) for \((2, 3, 3)\), \( E_7 \) for \((2, 3, 4)\), and \( E_8 \) for \((2, 3, 5)\). In [KST07], they show that for each such weight sequence, there is a certain \( \mathbb{Z}\)-graded category of singularities coming from a subring of the coordinate ring of the weighted projective lines for this sequence, which is equivalent to the derived category of representations of the corresponding \( ADE \) quiver.

In our setup, given a weight sequence of Dynkin type \((p, q, r)\), the category,

\[
\text{D}^b(\text{mod} - (A_{p-1} \otimes_k A_{q-1} \otimes_k A_{r-1})),
\]

is an admissible subcategory of the category of representations of the quiver corresponding to the type of simple singularity. In the case, \((1, p, q)\), the variety is smooth and we get the zero category as our category of matrix factorizations. In the case, \((2, 2, l)\), the category of matrix factorizations is equivalent to:

\[
\text{D}^b(\text{mod} - (A_1 \otimes_k A_1 \otimes_k A_{l-1})) \cong \text{D}^b(\text{mod} - (A_{l-1})). \tag{7.1}
\]

In the case, \((2, 3, 3)\), the category of matrix factorizations is equivalent to:

\[
\text{D}^b(\text{mod} - (A_1 \otimes_k A_2 \otimes_k A_2)) \cong \text{D}^b(\text{mod} - (A_2 \otimes_k A_2)) \cong \text{D}^b(\text{mod} - (D_4)). \tag{7.2}
\]

In the case, \((2, 3, 4)\), the category of matrix factorizations is equivalent to:

\[
\text{D}^b(\text{mod} - (A_1 \otimes_k A_2 \otimes_k A_3)) \cong \text{D}^b(\text{mod} - (A_2 \otimes_k A_3)) \cong \text{D}^b(\text{mod} - (E_6)). \tag{7.3}
\]

In the case, \((2, 3, 5)\), the category of matrix factorizations is equivalent to:

\[
\text{D}^b(\text{mod} - (A_1 \otimes_k A_2 \otimes_k A_4)) \cong \text{D}^b(\text{mod} - (A_2 \otimes_k A_4)) \cong \text{D}^b(\text{mod} - (E_8)). \tag{7.4}
\]
These exceptional equivalences are well known and will be returned to later, and can also be proven using the results in [KST07] combined with some mutations of the exceptional collections involved.

Let us provide another example to illustrate that this picture should conjecturally be related to motives,

**Example 7.6.** Consider the stack $Z$ which is the quotient of a Fermat quintic surface by the $\mathbb{Z}_5$ action given by a fifth root of unity on the first two coordinates and its inverse on the last two coordinates. We have:

$$D^b(\text{coh } Z) \cong (\text{MF}(k[x, y], x^5 + y^5, \mathbb{Z}) \otimes_k \text{MF}(k[x, y], x^5 + y^5, \mathbb{Z}))_{\text{pe}}.$$  

The function $x^5 + y^5$ defines 5 points. Therefore by Orlov’s theorem, $\text{MF}(k[x, y], x^5 + y^5, \mathbb{Z})$ has an exceptional collection. It follows that $D^b(\text{coh } Z)$ has an exceptional collection as well.

Conjecturally (see [Orl05]), this implies that the motive of $Z$ is a direct sum of Lefschetz motives. With a slightly different group action, this was proven by V. Guletskii and C. Pedrini in [GP02].

8. **Different Gradings and Orbit Categories**

We would now like to establish a geometric picture by varying the grading group for a fixed ring with potential. We use the notion of an orbit category, defined by B. Keller in [Kel05]. We only review the definitions, for a more complete treatment see loc. cit. Once again, rather than work over $\mathbb{Z}$ as in loc. cit., we define this notion for a finitely generated abelian group of rank one.

**Definition 8.1.** Let $\Gamma$ be a finitely generated abelian group of rank at most one which is a subgroup of the automorphism group of a triangulated category $\mathcal{T}$. The orbit category of $\mathcal{T}$ by $\Gamma$, denoted $\mathcal{T}/\Gamma$ has the same objects as $\mathcal{T}$ with morphisms from $A$ to $B$ given by

$$\text{Hom}_{\mathcal{T}/\Gamma}(A, B) = \bigoplus_{g \in \Gamma} \text{Hom}_\mathcal{T}(A, g(B)).$$

Composition of morphisms is defined in the obvious way.

In general, the orbit category is not triangulated, however, in loc. cit., Keller provides of a way of rectifying this when $\mathcal{T}$ is the homotopy category of a dg-category, $T$, and $\Gamma$ lifts to an action on $T$. In this case, one can take the homotopy category of the dg orbit category to get a triangulated category. This will be called the dg orbit category for distinction.

**Definition 8.2.** Let $\mathcal{T}$ and $\mathcal{S}$ be triangulated categories and $\Gamma$ be a group of triangulated automorphisms of $\mathcal{T}$. We say that $\mathcal{T}$ is a $\Gamma$-cover of $\mathcal{S}$ if there is a fully faithful functor,

$$F : \mathcal{T}/\Gamma \to \mathcal{S},$$

such that every object in $\mathcal{S}$ is a summand of the essential image of $F$.

The following is a special case of Propositions 1.4 and 1.5 in [KMV08].

**Proposition 8.3** (Keller, Murfet, van den Bergh). Let $R$ be a finitely generated commutative $\mathbb{Z}$-graded ring and $w$ a homogeneous element with isolated singularities. The the dg orbit category of $\text{MF}(R, w, \mathbb{Z})$ by $\mathbb{Z}$ is equivalent to $\text{MF}(R, w)$.

The above proposition serves as inspiration for:
Proposition 8.4. Let $M$ be a finitely generated abelian group and $L$ be a finite subgroup of $M$ of order $n$. Let $S$ be an $M$-graded ring and assume that $n$ is a unit in $S$. Denote by $T$ the ring $S$ with the $M/L$ grading given by $S_{[m]} := \bigoplus_{l \in L} S_{lm}$. The category, $D^\grg(S)$ is an $L$-cover of $D^\grg(T)$.

Proof. Consider the functor $\text{Res}: D^b(\text{grmod} - S) \to D^b(\text{grmod} - T)$ induced by the exact functor which takes an $M$-graded $S$-module, $A$, to the $M/L$-graded $T$-module, $\hat{A}$, where $\hat{A}_{[m]} := \bigoplus_{l \in L} A_{lm}$. Notice that for any $m \in M$, $\text{Res}(S(m)) = T([m])$, hence $\text{Res}$ induces a morphism $\text{Res}_{\text{sg}}: D^\grg(S) \to D^\grg(T)$.

Let us compute $\text{Hom}(\text{Res}(X), \text{Res}(Y))$ for $X, Y \in D^b(\text{grmod} - S)$. Notice that for any $l \in L$ there is a natural isomorphism: $\phi_l : \text{Res}(Y(l)) \to \text{Res}(Y)$. Hence, any morphism, $f : X \to Y(l)$, induces a morphism $A(f) : \text{Res}(X) \to \text{Res}(Y)$ where $A(f) = \phi_l \circ \text{Res}(f)$. Hence we have a morphism:

$$\psi : \text{Hom}(X, \bigoplus_{l \in L} Y(l)) \to \text{Hom}(\text{Res}(X), \text{Res}(Y))$$

$$\bigoplus_{l \in L} f_l \mapsto \sum_{l \in L} A(f_l).$$

Conversely, we can split any morphism from $g : \text{Res}(X) \to \text{Res}(Y)$ into $L$-graded components in the complexes $X$ and $Y$. This gives a bijection of sets, and induces a fully faithful functor,

$$F : D^b(\text{grmod} - S)/L \to D^b(\text{grmod} - T),$$

which sends the orbit of $X$ to $\text{Res}(X)$. Notice that in particular, the orbit of $S(m)$ is sent to $T([m])$. Hence, $F$ restricts to a fully faithful functor from $\text{perf}(S)/L$ to $\text{perf}(T)$. Taking quotients, one obtains a fully faithful functor:

$$F_{\text{sg}} : D^\grg(S)/L \to D^\grg(T).$$

To finish, it suffices to show that every object $Z \in D^\grg(T)$ is a summand of the essential image of $F_{\text{sg}}$. More generally, we demonstrate that every object in $D^b(\text{grmod} - T)$ is a summand of the essential image of $F$. To see this, consider the exact functor $\text{Inf} : \text{grmod} - T \to \text{grmod} - S$, that takes an $M/L$-graded $T$-module, $A$ to the $M$-graded $S$-module with $\text{Inf}(A)_{lm} = A_{[m]}$. Denote the derived functor by the same notation. The functor $\text{Res}$ is the left adjoint to $\text{Inf}$ with adjunction morphism in $D^b(\text{grmod} - T)$ defined by:

$$\sum_{l \in L} \phi_l : \text{Inf}(\text{Res}(B)) \to B.$$

The map,

$$\frac{1}{n} \bigoplus_{l \in L} \phi_l^{-1} : B \to \text{Inf}(\text{Res}(B)),$$

provides a splitting of the map above. Therefore, $B$ is a summand of $\text{Inf}(\text{Res}(B))$ as required.

\[\square\]

Corollary 8.5. Suppose, $R$ and $R'$ are $\mathbb{Z}$-graded rings over $k$ with $\text{deg} w = d$ and $\text{deg} w' = d'$ with $w \in dw$ and $w' \in dw'$. Let $m$ be the least common multiple of $d$ and $d'$. Equip $R \otimes_k R'$ with the $\mathbb{Z}$ grading $(R \otimes_k R')^s := \bigoplus_{d_i + d_j = s} R_i \otimes R_j$. The category,

$$(\text{MF}(R, w, \mathbb{Z}) \hat{\otimes}_k \text{MF}(R', w', \mathbb{Z}))_{\text{pe}},$$

is a $\mathbb{Z}_m$-cover of $\text{MF}(R \otimes_k R', w \otimes 1 + 1 \otimes w', \mathbb{Z})$. 
Proof. By Corollary 6.10, $(\text{MF}(R, w, Z) \otimes_k \text{MF}(R', w', Z))_{\text{pe}}$ is equivalent to $\text{MF}(R \otimes_k R', w \otimes 1+1 \otimes w', Z \times Z/(d, -d'))$. The element, $(\frac{d}{m}, -\frac{d'}{m})$, generates a cyclic subgroup of $Z \times Z/(d, -d')$ of order $m$. When one quotient this by this cyclic subgroup, it induces the $Z$ action described above. Hence by Proposition 8.4 we obtain the result. \hfill \square

Example 8.6. Let $f(x, y, z) = x(x - z)(x - \lambda z) - zy^2$ and $g(u, v, w) = u(u - w)(u - \gamma w) - vw^2$ define two smooth elliptic curves, $E$ and $F$ respectively. Then $f + g$ defines a smooth cubic fourfold containing at least three planes, $P, Q, R$, with $P + Q + R = H^2$ by setting $z = w = 0$. Hence combining Theorem 7.1 with the results in [Kuz09b] on cubics which contain planes, the category $\text{MF}(k[x, y, z, u, v, w], f + g, Z)$ is equivalent to the derived category of a gerbe on certain $K3$ surface, $(Y, \beta)$, see [Kuz09b] for details (the gerbe is trivial if and only if there exists a 2-dimensional cycle, $T$, such that $T \cdot H^2 - T \cdot P$ is odd). On the other hand, letting $M = Z \oplus Z/(3, -3)$ with $x, y, z$ in degree $(1, 0)$ and $u, v, w$ in degree $(0, 1)$, we have $\text{MF}(k[x, y, z, u, v, w], f + g, M) \cong (\text{MF}(k[x, y, z], f, Z) \otimes_k \text{MF}(k[u, v, w], g, Z))_{\text{pe}}$. From Theorem 7.1 we have,

$$\text{MF}(k[x, y, z], f, Z) \cong D^b(\text{coh} E) \text{ and } \text{MF}(k[u, v, w], g, Z) \cong D^b(\text{coh} F).$$

Hence,

$$\text{MF}(k[x, y, z, u, v, w], f + g, M) \cong D^b(\text{coh} E \times_k F).$$

As in the corollary above, $D^b(\text{coh} E \times_k F)$ is a $Z_3$-cover of $D^b(\text{coh}(Y, \beta))$.

Furthermore, on each elliptic curve, $E, F$ the autoequivalence (1) is a composition of Dehn twists (see [BFK10] [KMV08]). Hence this autoequivalence can be viewed as a symplectic automorphism of the mirror. The action of $Z_3$ on $D^b(\text{coh} E \times_k F)$ is given by $(1, -1)$. This can therefore be considered as a product of symplectic automorphisms of the product of the two mirrors. The relationship between the (gerby) surfaces $E \times_k F$ and $(Y, \beta)$ can then be seen by viewing the mirror of $E \times_k F$ as a three to one symplectic cover of the mirror of $(Y, \beta)$. A similar observation appears in [KP09].

Example 8.7. Consider a quartic $K3$ surface, $Y$, defined by $f(x, y, z, w)$ in $\mathbb{P}^3$. Let $Q$ be the quartic double solid defined by the equation, $t^2 - f$, in weighted projective space $\mathbb{P}(2 : 1 : 1 : 1 : 1)$. As mentioned above, $\text{MF}(k[t], t^2, Z)$ is equivalent to the derived category of vector spaces. Hence a graded analog of Knörrer periodicity yields:

$$(\text{MF}(k[t], t^2, Z) \otimes_k \text{MF}(k[x, y, z, w], f, Z))_{\text{pe}} \cong \text{MF}(k[x, y, z, w], f, Z) \cong D^b(\text{coh} Y).$$

Therefore, $D^b(\text{coh} Y)$ is a $Z_2$-cover of $\text{MF}(k[t, x, y, z, w], t^2 - f, Z)$, an admissible subcategory of $D^b(\text{coh} Q)$. This category, in certain special cases, is conjecturally related to the Enriques surface obtained from the $K3$ by quotienting by the $Z_2$ action, see [IK10] [IKPT1] for details.

Example 8.8. Let $A$ be the free abelian group generated by $e_0, \ldots, e_n$ and $B$ be the subgroup generated by $de_i - de_j$ for all $i, j$. Let $M := A/B$. Let $R$ be the polynomial algebra $k[x_0, \ldots, x_n]$ with its usual $Z$ grading, and let $f := x_0^d + \cdots + x_n^d$ be the Fermat polynomial. We have:

$$D^b(\text{mod} - (A_{d-1})^{\otimes n+1}) \cong (D^b(\text{mod} - A_{d-1}))^{\otimes n+1} \cong (\text{MF}(k[x], x^d, Z)^{\otimes n+1})_{\text{pe}} \cong \text{MF}(R, f, M).$$

The first equivalence is standard, the second can be found in [Orl09a], the third comes from iterated application of Corollary 6.10.

Let $C \cong Z_d^{\otimes n}$ be the subgroup generated by $e_i - e_j$ for all $i, j$. Then $(A/B)/C \cong Z$. Hence one realizes $D^b(\text{mod} - (A_{d-1})^{\otimes n+1})$ as a $Z_d^{\otimes n}$-cover of $\text{MF}(R, f, Z)$. More generally,
consider a partition, \( \mathcal{P} = \{0, \ldots, i_0\} \cdots \{i_{m-1} + 1, \ldots, i_m\} \) of the set \( \{0, \ldots, n\} \) into \( m \) parts. Let \( N_\mathcal{P} \cong \mathbb{Z}_d^{\oplus n+1-m} \) be the subgroup generated by \( e_i - e_j \) for all \( i, j \) in the same part of the partition and \( M_\mathcal{P} := M/N_\mathcal{P} \). One obtains, \( D^b(\text{mod} - (A_{d-1})^\oplus n+1) \) as a \( \mathbb{Z}_d^{\oplus n+1-m} \)-cover of \( \text{MF}(R, f, M_\mathcal{P}) \) which is equivalent to
\[
(\text{MF}(k[x_0, \ldots, x_i, x^d_0 + \cdots + x^d_{i_0}, Z]) \otimes_k \cdots \otimes_k \text{MF}(k[x_{i_0+1}, \ldots, x_{i_1}], x^d_{i_1+1} + \cdots + x^d_{i_2}), Z))_{\text{pe}},
\]
and related to various stacks by Theorem 7.3. Notice that varying the partitions, one gets a partially ordered collection of covers with maximal element \( D^b(\text{mod} - (A_{d-1})^\oplus n+1) \) and minimal element, \( \text{MF}(R, f, Z) \). Using Proposition 8.3 one can also quotient the minimal element by \( Z \) to get \( \text{MF}(R, f) \) as a further quotient but only in the dg orbit category sense. The general case of weighted Fermat hypersurfaces will be studied in detail in subsection 9.2.

9. Applications to generation time

9.1. Preliminaries. We recall the following definitions. For a more complete treatment see, \cite{BFK10, Rou08}. Let \( \mathcal{T} \) be a triangulated category. For a full subcategory, \( \mathcal{I} \), of \( \mathcal{T} \) we denote by \( \langle \mathcal{I} \rangle \) the full subcategory of \( \mathcal{T} \) whose objects are isomorphic to summands of finite coproducts of shifts of objects in \( \mathcal{I} \). In other words, \( \langle \mathcal{I} \rangle \) is the smallest full subcategory containing \( \mathcal{I} \) which is closed under isomorphisms, shifting, and taking finite coproducts and summands. For two full subcategories, \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \), we denote by \( \mathcal{I}_1 \ast \mathcal{I}_2 \) the full subcategory of objects, \( B \), such that there is a distinguished triangle, \( B_1 \to B \to B_2 \to B_1[1] \), with \( B_i \in \mathcal{I}_i \). Set \( \mathcal{I}_1 \diamond \mathcal{I}_2 := \langle \mathcal{I}_1 \ast \mathcal{I}_2 \rangle \), \( \langle \mathcal{I} \rangle_0 := \langle \mathcal{I} \rangle \), and inductively define
\[
\langle \mathcal{I} \rangle_n := \langle \mathcal{I} \rangle_{n-1} \diamond \langle \mathcal{I} \rangle.
\]
Similarly we define
\[
\langle \mathcal{I} \rangle_\infty := \bigcup_{n \geq 0} \langle \mathcal{I} \rangle_n.
\]
For an object, \( E \in \mathcal{T} \), we notationally identify \( E \) with the full subcategory consisting of \( E \) in writing, \( \langle E \rangle_n \). The reader is warned that, in some of the previous literature, \( \langle \mathcal{I} \rangle_0 := 0 \) and \( \langle \mathcal{I} \rangle_1 := \langle \mathcal{I} \rangle \). We follow the notation in \cite{BF09, BFK10}. With our convention, the index equals the number of cones allowed.

**Definition 9.1.** Let \( E \) be an object of a triangulated category, \( \mathcal{T} \). If there is an \( n \) with \( \langle E \rangle_n = \mathcal{T} \), we set
\[
\Theta_\mathcal{T}(E) := \min \left\{ n \geq 0 \mid \langle E \rangle_n = \mathcal{T} \right\}.
\]
Otherwise, we set \( \Theta_\mathcal{T}(E) := \infty \). We call \( \Theta_\mathcal{T}(E) \) the **generation time** of \( E \). When, \( \mathcal{T} \) is clear from context, we omit it and simply write \( \Theta(E) \). If \( \langle E \rangle_\infty = \mathcal{T} \), we say that \( E \) is a **generator**. If \( \Theta(E) \) is finite, we say that \( E \) is a **strong generator**. The **Orlov spectrum** of \( \mathcal{T} \), denoted \( \text{OSpec} \mathcal{T} \), is the set
\[
\{ \Theta(G) \mid G \in \mathcal{T}, \Theta(G) < \infty \} \subset \mathbb{Z}_{\geq 0}.
\]
The **Rouquier dimension** of \( \mathcal{T} \), denoted \( \text{rdim} \mathcal{T} \), is the infimum of the set of all generation times.

The results from the previous sections can be applied to the study of Orlov spectra and Rouquier dimension. In particular, in some special cases we will get a positive answer to Conjecture 1.3 (due to Orlov).
Definition 9.2. Let $\mathcal{T}$ be a triangulated category, $f$ be a morphism, and $\mathcal{I}$ be a full subcategory. We say that $f$ is $\mathcal{I}$ ghost if, for all $I \in \mathcal{I}$, the induced map, $\text{Hom}_\mathcal{T}(I, X) \to \text{Hom}_\mathcal{T}(I, Y)$, is zero. We say that $f$ is $\mathcal{I}$ co-ghost if, for all $I \in \mathcal{I}$, the induced map, $\text{Hom}_\mathcal{T}(Y, I) \to \text{Hom}_\mathcal{T}(Y, I)$, is zero. If $G$ is an object of $\mathcal{T}$, we will say that $f$ is $G$ ghost if $f$ is $\langle G \rangle_0$ ghost and $f$ is $G$ co-ghost if $f$ is $\langle G \rangle_0$ co-ghost.

The following lemmas are standard tools in the study of generation time:

Lemma 9.3 ((Co-)ghost Lemma). Let $\mathcal{T}$ be a triangulated category and let $\mathcal{I}$ be a full subcategory of $\mathcal{T}$. If there exists a sequence of morphisms, $f_i : X_{i-1} \to X_i$, $1 \leq i \leq t$, in $\mathcal{T}$ where each $f_i$ is $\mathcal{I}$ ghost (resp. co-ghost) and $f_t \circ \cdots \circ f_1 \neq 0$, then $X_0 \notin \langle \mathcal{I} \rangle_{t-1}$ (resp. $X_t \notin \langle \mathcal{I} \rangle_{t-1}$).

For many “nice” triangulated categories, the converse also holds. The following converse to the ghost lemma can be found in the thesis of Oppermann [Opp08], the proof for the co-ghost version can be seen by taking opposite categories:

Lemma 9.4. Suppose $\mathcal{T}$ is a full subcategory of the category of compact objects of a co-complete triangulated category $\mathcal{S}$. Let $\mathcal{I}$ be a full subcategory, such that for any object $B \in \mathcal{T}$, $\text{Hom}(A, B) = 0$ for all but finitely many $A \in \mathcal{I}$. Then for any object $X_1$ (resp. $X_t$), $X_1 \in \langle \mathcal{I} \rangle_{t-1}$ (resp. $X_t \in \langle \mathcal{I} \rangle_{t-1}$) if and only if there exists a sequence of morphisms, $f_i : X_{i-1} \to X_i$, $1 \leq i \leq t$, in $\mathcal{T}$ where each $f_i$ is $\mathcal{I}$ ghost (resp. co-ghost) and $f_t \circ \cdots \circ f_1 \neq 0$ and no such sequence exists for $t + 1$ objects.

Definition 9.5. Let $\alpha : \mathcal{A} \to \mathcal{T}$ be the inclusion of a full triangulated subcategory of $\mathcal{T}$. The subcategory, $\mathcal{A}$, is called right admissible if the inclusion functor, $\alpha$, has a right adjoint $\alpha^!$ and left admissible if it has a left adjoint $\alpha^*$. A full triangulated subcategory is called admissible if it is both right and left admissible.

Let $\mathcal{T}$ be a triangulated category and $\mathcal{I}$ a full subcategory. Recall that the left orthogonal, $\perp \mathcal{I}$, is the full subcategory $\mathcal{T}$ consisting of all objects, $T \in \mathcal{T}$, with $\text{Hom}_\mathcal{T}(T, I) = 0$ for any $I \in \mathcal{I}$. The right orthogonal, $\mathcal{I} \perp$, is defined similarly.

Definition 9.6. A semi-orthogonal decomposition of a triangulated category, $\mathcal{T}$, is a sequence of full triangulated subcategories, $\mathcal{A}_1, \ldots, \mathcal{A}_m$, in $\mathcal{T}$ such that $\mathcal{A}_i \subset \mathcal{A}_j^\perp$ for $i < j$ and, for every object $T \in \mathcal{T}$, there exists a diagram:

$$
\begin{array}{cccccc}
0 & \rightarrow & T_{m-1} & \rightarrow & \cdots & \rightarrow & T_2 & \rightarrow & T_1 & \rightarrow & T \\
& & & & & & & & & \\
& & & & & & \downarrow & & & & \\
& & & & & & A_m & & & & \\
& & & & & & \downarrow & & & & \\
& & & & & & A_{m-1} & & & & \\
\end{array}
$$

where all triangles are distinguished and $A_k \in \mathcal{A}_k$. We shall denote a semi-orthogonal decomposition by $\langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle$. When we speak about semi-orthogonal decompositions of triangulated dg-categories we shall always implicitly mean their homotopy categories but suppress this from the discussion, see the appendix to [CT10] for details on this.

In [BFK10], an in depth analysis of the relationship between generation time and semi-orthogonal decompositions is provided, however, we will only need some basic results concerning these definitions in what follows.

The following lemma is clear from the definitions (see also [BFK10]):
Lemma 9.7. Let $\mathcal{T} = A_1, \ldots, A_s$ be a semi-orthogonal decomposition. For each $i$, let $G_i$ be a strong generator of $A_i$. Then $\bigoplus_{i=1}^s G_i$ is a strong generator of $\mathcal{T}$ and $\mathcal{O}(\bigoplus_{i=1}^s G_i) \leq \sum_{i=1}^s \mathcal{O}(G_i)$. Furthermore if $A$ is an admissible subcategory of $\mathcal{T}$ then the Rouquier dimension of $A$ is at most the Rouquier dimension of $\mathcal{T}$.

Proposition 9.8. Let $\mathcal{T}$ be a full triangulated subcategory of the category of compact objects of a cocomplete triangulated category and $\Gamma$ be a group of automorphisms of $\mathcal{T}$. Suppose the orbit category $\mathcal{T}/\Gamma$ has a triangulated structure such that the natural functor $\pi : \mathcal{T} \to \mathcal{T}/\Gamma$ is triangulated. Let $\hat{\Gamma}$ denote the quotient of $\Gamma$ by its intersection with the shift functor and suppose further that $\hat{\Gamma}$ is finite. For each $\hat{h} \in \hat{\Gamma}$ choose a representative $h \in \Gamma$. If $\pi(A)$ is a classical generator of $\mathcal{T}/\Gamma$ then $\bigoplus_{\hat{h} \in \hat{\Gamma}} h(A)$ is a classical generator of $\mathcal{T}$. Furthermore, if $\pi(A)$ is a strong generator then so is $\bigoplus_{\hat{h} \in \hat{\Gamma}} h(A)$ and $\mathcal{O}(\bigoplus_{\hat{h} \in \hat{\Gamma}} h(A)) = \mathcal{O}(A)$. If $\mathcal{T}/\Gamma$ is not triangulated, but instead is a dg orbit category, one still has an inequality, $\mathcal{O}(\bigoplus_{\hat{h} \in \hat{\Gamma}} h(A)) \leq \mathcal{O}(A)$.

Proof. If $\pi(A)$ is a classical generator, then for any $B \in \mathcal{T}$, there exists an $n$ such that $\pi(B) \in \langle \pi(A) \rangle_n$. Suppose $B \notin \langle \bigoplus_{\hat{h} \in \hat{\Gamma}} h(A) \rangle_n$.

By Lemma 9.4, there exists a nonzero co-ghost sequence for $\bigoplus_{\hat{h} \in \hat{\Gamma}} h(A)$,

$$X_1 \to \cdots \to X_n.$$  

As being co-ghost is closed under shift, the sequence is also co-ghost for $\bigoplus_{\hat{h} \in \hat{\Gamma}} h(A)$. By applying $\pi$, we obtain a sequence of morphisms in the orbit category,

$$\pi(X_1) \to \cdots \to \pi(X_n).$$

The total composition is nonzero as $\pi$ is faithful. Furthermore, all maps in the sequence are co-ghost for $\pi(A)$. This follows from the isomorphism of functors:

$$\text{Hom}_{\mathcal{T}/\Gamma}(-, A) \circ \pi = \text{Hom}_{\mathcal{T}}(-, \bigoplus_{\hat{h} \in \hat{\Gamma}} h(A)) = \bigoplus_{\hat{h} \in \hat{\Gamma}} \text{Hom}_{\mathcal{T}}(-, h(A)).$$

Hence by Lemma 9.3, $\pi(B) \notin \langle \pi(A) \rangle_n$. This yields a contradiction. It follows that $\bigoplus_{\hat{h} \in \hat{\Gamma}} h(A)$ is a classical generator. When $\pi(A)$ is a strong generator, then by definition $n$ is uniform. Hence, $\mathcal{O}(\bigoplus_{\hat{h} \in \hat{\Gamma}} h(A)) \leq \mathcal{O}(A)$.

Conversely, the functor $\pi : \mathcal{T} \to \mathcal{T}/\Gamma$ is triangulated and essentially surjective (in the case where we take the regular orbit category as opposed to the dg orbit category) and $\pi(\bigoplus_{\hat{h} \in \hat{\Gamma}} h(A)) = A$. Hence, $\mathcal{O}(\bigoplus_{\hat{h} \in \hat{\Gamma}} h(A)) \geq \mathcal{O}(A)$.

\[ \square \]

Remark 9.9. If $\hat{\Gamma}$ is not finite, one could assume instead that for any object $B \in \mathcal{T}$, $\text{Hom}(g(A), B) = 0$ for all but finitely many $g \in \Gamma$. In this case the full subcategory consisting of all objects $g(A)$, for all $g \in \Gamma$ still generates $\mathcal{T}$ in equal amount of time to $A$ but can’t be summed to form an object of the category.

Corollary 9.10. Let $\mathcal{T}$ be a full triangulated subcategory of the category of compact objects of a cocomplete triangulated category and $\Gamma$ be a group of automorphisms of $\mathcal{T}$. Suppose the orbit category $\mathcal{T}/\Gamma$ has a triangulated structure such that the natural functor $\pi : \mathcal{T} \to \mathcal{T}/\Gamma$ is triangulated. Let $\mathcal{T}, \mathcal{S}$ be triangulated categories with $\mathcal{T}$ a $\Gamma$-cover of $\mathcal{S}$. Suppose $\hat{\Gamma}$, the quotient of $\Gamma$ by its intersection with the shift functor, is finite. Then $\mathcal{T}$ and $\mathcal{S}$ have the same Rouquier dimension.
Proof. This follows from Proposition 9.8 coupled with the fact that a triangulated category has the same Rouquier dimension as its Karoubian closure.

**Corollary 9.11.** Let $M, M'$ be finitely generated abelian groups of rank one. Let $R$ be a $\mathbb{Z}$-graded ring which can be extended to an $M$ and $M'$ grading. Suppose $f \in R$ is homogeneous with respect to all gradings. The Rouquier dimension of $MF(R, f, M)$ is equal to the Rouquier dimension of $MF(R, f, M')$.

Proof. This follows directly from Corollary 9.10, Proposition 8.4, and Proposition 8.3.

**Corollary 9.12.** Let $\mathcal{T}$ be a full triangulated subcategory of the category of compact objects of a cocomplete triangulated category and $\Gamma$ be a group of automorphisms of $\mathcal{T}$. Suppose the orbit category $\mathcal{T}/\Gamma$ has a triangulated structure such that the natural functor $\pi : \mathcal{T} \to \mathcal{T}/\Gamma$ is triangulated. The Orlov spectrum of $\mathcal{T}/\Gamma$ is a subset of the Orlov spectrum of $\mathcal{T}$.

Proof. Again, this follows directly from Corollary 9.10, Proposition 8.4, and Proposition 8.3.

In general, the orbit category $\mathcal{T}/\Gamma$ is not Karoubi closed, and so with the assumptions above we have two separate containments:

$$OSpec(\mathcal{T}/\Gamma) \subseteq OSpec(\overline{\mathcal{T}/\Gamma})$$

$$OSpec(\mathcal{T}/\Gamma) \subseteq OSpec(\mathcal{T})$$

where $\overline{\mathcal{T}/\Gamma}$ is the Karoubi completion of $\mathcal{T}/\Gamma$. So in general, we can not directly compare the Orlov spectra of $\Gamma$-covers.

### 9.2. Rouquier dimension of weighted Fermat hypersurfaces

In this section, we would like to apply our results to estimate the Rouquier dimension of products of quivers of type $ADE$ and their corresponding weighted Fermat hypersurfaces. An upper bound will come from the interpretation of this category as an algebra. We will need the following Lemma due to Rouquier (see [Rou08]):

**Lemma 9.13** (Rouquier). Let $S$ be a finite dimensional $k$-algebra of Loewy length $r$ and $Q$ be a quiver whose underlying graph is Dynkin of type $ADE$. The Rouquier dimension of $D^b(\text{mod-} S \otimes_k kQ)$ is at most $r - 1$.

Proof. Let $N$ be the nilradical of $S$ and set $I = N \otimes_k kQ$ in Lemma 7.35 of [Rou08]. Furthermore, notice that $S/I$ is isomorphic to a finite direct sum of algebras $kQ$, hence $D^b(\text{mod-} S/I)$ has Rouquier dimension zero. The conclusion follows.

For a lower bound we will need the following slight generalization of another result of Rouquier in loc. cit., the proof can be found in [BF09].

**Lemma 9.14.** Let $Z$ be a tame Deligne-Mumford stack with a reduced and separated coarse moduli space. The Rouquier dimension of $D^b(\text{coh } Z)$ is at least the dimension of $Z$.

Given a collection, $S_1, \ldots, S_r$, of path algebras of quivers whose underlying graph is of type $ADE$, we would like to find an admissible subcategory of $D^b(\text{mod-} S_1 \otimes_k \cdots \otimes_k S_r)$ which is equivalent to the derived category of a weighted Fermat hypersurface. First we simplify the situation a bit by noticing that $D^b(\text{mod-} A_s^{-1})$ is an admissible subcategory of both $D^b(\text{mod-} D_s)$ and $D^b(\text{mod-} E_s)$. Using this simple observation, given, $S$, the path algebra of a quiver whose underlying graph is of type $ADE$, define:

$$e(S) := \begin{cases} s + 1 & \text{if } S \text{ is of type } A_s \\ s & \text{if } S \text{ is of type } D_s \text{ or } E_s. \end{cases}$$
Applying Lemma 9.13 we arrive at the upper bound.

For any $\mathfrak{p} \in \mathcal{P}$, let $a_{\mathfrak{p}} := -1 + \sum_{i \in \mathfrak{p}} \frac{1}{e_{S_i}}$ and $|\mathfrak{p}|$ denote the cardinality of the part. Let $\mathcal{P}^{\leq 0}$ be the subset of the partition consisting of all its parts, $\mathfrak{p}$, with $a_{\mathfrak{p}} \leq 0$. Let $|\mathcal{P}^{\leq 0}|$ denote the cardinality of the set partitioned, and $\alpha(\mathcal{P}^{\leq 0})$ denote the number of parts in $\mathcal{P}^{\leq 0}$.

**Theorem 9.15.** Let $S_1, \ldots, S_r$ be path algebras of quivers whose underlying graph is of type ADE. The Rouquier dimension of $\mathcal{D}^{b}(\text{mod} - (S_1 \otimes_k \cdots \otimes_k S_r))$ is at most $r - 1$. If the characteristic of the base field does not divide any of the $e(S_i)$ for any $i$ then the Rouquier dimension is at least $\max_{\mathfrak{p} \in \mathcal{I}}(|\mathcal{P}^{\leq 0}| - 2\alpha(\mathcal{P}^{\leq 0}))$.

**Proof.** For any $i$, the simple modules in $\text{mod} - (S_i)$, after appropriate shifting, form a strong exceptional collection whose endomorphism algebra has Loewy length 2. Hence, after shifting, the simple modules of $\text{mod} - (S_1 \otimes_k \cdots \otimes_k S_r)$ form an exceptional collection whose Loewy length is $r$. Let $S$ be the endomorphism algebra of the simple modules of $\text{mod} - (S_1 \otimes_k \cdots \otimes_k S_r)$. By tensoring these simple modules over $k$ with the projective modules in $A$, we get a tilting object in $\mathcal{D}^{b}(\text{mod} - S)$. This induces an equivalence,

$$\mathcal{D}^{b}(\text{mod} - (S_1 \otimes_k \cdots \otimes_k S_r)) \cong \mathcal{D}^{b}(\text{mod} - S \otimes_k S_r).$$

Applying Lemma 9.13 we arrive at the upper bound.

For the lower bound, let $\mathcal{P}$ be a partition of the set $\{1, \ldots, r\}$. For each part, $\mathfrak{p} \in \mathcal{P}$, with $a_{\mathfrak{p}} \geq 0$, let $m_\mathfrak{p}$ be the least common multiple of the $e(S_i)$ for all $l \in \mathfrak{p}$. Consider the free commutative polynomial ring, $R_\mathfrak{p}$, in $|\mathfrak{p}|$ variables, $x_l$ for each $l \in \mathfrak{p}$, with weights $\frac{m_\mathfrak{p}}{e(S_l)}$ and let $f_\mathfrak{p} := \sum_{l \in \mathfrak{p}} x_l^{e(S_l)}$ and $Z_\mathfrak{p}$ be the stacky hypersurface defined by $f_\mathfrak{p}$. When $\mathfrak{p} \in \mathcal{P}^{\leq 0}$, by Theorem 7.4, the category $\mathcal{D}^{b}(\text{coh} Z_\mathfrak{p})$ is an admissible subcategory of $\text{MF}(R_\mathfrak{p}, f_\mathfrak{p}, \mathbb{Z})$. Now by Lemma 7.4, the category,

$$\left(\bigotimes_{\mathfrak{p} \in \mathcal{P}^{\leq 0}} \mathcal{D}^{b}(\text{coh} Z_\mathfrak{p})\right)_{\mathfrak{p}^e} \cong \mathcal{D}^{b}(\text{Coh} - \prod_{\mathfrak{p} \in \mathcal{P}^{\leq 0}} Z_\mathfrak{p}),$$

is an admissible subcategory of,

$$\left(\bigotimes_{\mathfrak{p} \in \mathcal{P}^{\leq 0}} \text{MF}(R_\mathfrak{p}, f_\mathfrak{p}, \mathbb{Z})\right)_{\mathfrak{p}^e}.$$

Now, by the condition on the base field, $\prod_{\mathfrak{p} \in \mathcal{P}^{\leq 0}} Z_\mathfrak{p}$ a tame Deligne-Mumford stack with a reduced and separated coarse moduli space of dimension, $\sum_{\mathfrak{p} \in \mathcal{P}^{\leq 0}} (|\mathfrak{p}| - 2)$, therefore by Lemma 7.14 the Rouquier dimension of $\mathcal{D}^{b}(\text{Coh} - \prod_{\mathfrak{p} \in \mathcal{P}^{\leq 0}} Z_\mathfrak{p})$ is at least $\sum_{\mathfrak{p} \in \mathcal{P}^{\leq 0}} (|\mathfrak{p}| - 2) = |\mathcal{P}^{\leq 0}| - 2\alpha(\mathcal{P}^{\leq 0})$. Therefore, the same lower bound applies to,

$$\left(\bigotimes_{\mathfrak{p} \in \mathcal{P}^{\leq 0}} \text{MF}(R_\mathfrak{p}, f_\mathfrak{p}, \mathbb{Z})\right)_{\mathfrak{p}^e},$$

as the category, $\mathcal{D}^{b}(\text{Coh} - \prod_{\mathfrak{p} \in \mathcal{P}^{\leq 0}} Z_\mathfrak{p})$, is an admissible subcategory. Applying Corollary 9.11 we obtain the desired result (see Example 8.8). \hfill \qed

At the end of the proof above, we applied Corollary 9.11 to compare the Rouquier dimension of our algebraic category, to a geometric one and use geometric lower bounds. On the other hand, we can use the algebraic upper bound and Corollary 9.11 to deduce upper bounds on the Rouquier dimension of geometric categories.
Let $X$ be a weighted Fermat hypersurface defined by $f = x_0^{d_0} + \cdots + x_n^{d_n}$ in $\mathbb{P}(\frac{m}{d_1} : \cdots : \frac{m}{d_n})$, where $m$ is the least common multiple of the $d_i$. Let $R$ be the coordinate ring of $\mathbb{P}(\frac{m}{d_1} : \cdots : \frac{m}{d_n})$. To use the theorem above, we want to partition the variables so that we can compare the Rouquier dimension of $R$ to a product of path algebras whose underlying graph is of type $ADE$. With this in mind, denote by $I_{ADE}$ the set of partitions, $P$, of $\{0, \ldots, n\}$, such that for each part, $\mathcal{P}$, the corresponding set with repetition, $\bigcup_{l \in \mathcal{P}} \{d_l\}$ is one of the following forms: \{2, ..., 2, a\}, \{2, ..., 2, 3, 3\}, \{2, ..., 2, 3, 4\}, or \{2, ..., 2, 3, 5\} (we allow there to be no twos in these forms as well).

**Theorem 9.16.** Let $X$ be a weighted Fermat hypersurface defined by $f = x_0^{d_0} + \cdots + x_n^{d_n}$. The Rouquier dimension of $MF(R, f, \mathbb{Z})$ is at most $\min_{P \in I_{ADE}}(|P|) - 1$, where $|P|$ is the number of parts in $P$.

**Proof.** As in Example 8.8, for any partition $P$ of \{0, ..., n\}, the Rouquier dimension of $MF(R, f, \mathbb{Z})$ is equal to the Rouquier dimension of, $$(\bigotimes_{\mathcal{P} \in \mathcal{P}} MF(R_{\mathcal{P}}, f_{\mathcal{P}}, \mathbb{Z}))_{pe},$$ Now, consider a partition, $P \in I_{ADE}$. For any part, $\mathcal{P} \in P$, first notice, that we may assume $2 \notin \mathcal{P}$. This is seen by partitioning off the 2, and the fact that $MF(k[x], x^2, \mathbb{Z})$ is equivalent to the derived category of vector spaces over the base field. Now, by equations 7.2 and 7.4 for any part $\mathcal{P} \in P$, $MF(R_{\mathcal{P}}, f_{\mathcal{P}}, \mathbb{Z})$ is equivalent to the derived category of modules of a path algebra of a quiver whose underlying graph is of type $ADE$.

Hence, the Rouquier dimension of $MF(R, f, \mathbb{Z})$ is equal to the Rouquier dimension of $D^b(\text{mod} -(S_1 \otimes_k \cdots \otimes_k S_{|P|}))$, where each $S_i$ is the path algebra of a quiver whose underlying graph is of type $ADE$. The theorem then follows from Theorem 9.15. □

**Definition 9.17.** Let $(d_0, \ldots, d_n)$ be a weight sequence. We say that $(d_0, \ldots, d_n)$ is **negative**, **neutral**, **positive**, **nonnegative**, or **nonpositive** if $-1 + \sum_{i=1}^{n} \frac{1}{d_i}$ is negative, neutral, positive, nonnegative, or nonpositive respectively. We say that a weight sequence is **minimizing** if it is nonpositive and contains either \{2\}, \{3, 3\}, \{3, 4\}, or \{3, 5\}.

**Corollary 9.18.** Let $(d_0, \ldots, d_n)$ be a weight sequence. Let $P$ be a partition of the set \{0, ..., n\} such that the weight sequence corresponding to each part is nonpositive. Then the Rouquier dimension of $\prod_{\mathcal{P} \in P} Z_{\mathcal{P}}$ is at most $\min_{P \in I_{ADE}}(|P|) - 1$. In particular, if $(d_0, \ldots, d_n)$ is minimizing and the characteristic of the base field does not divide any of the $d_i$ then Conjecture 1.3 holds for $X$.

**Proof.** Let $P$ be a partition of the set \{0, ..., n\} such that the weight sequence of every part, $\mathcal{P}$, is nonpositive. By Theorem 7.1, $D^b(\text{coh} Z_{\mathcal{P}})$ is an admissible subcategory of $MF(R_{\mathcal{P}}, f_{\mathcal{P}}, M_{\mathcal{P}})$. Corollary 9.11 tells us that the Rouquier dimension of $MF(R_{\mathcal{P}}, f_{\mathcal{P}}, M_{\mathcal{P}})$ is equal to that of $MF(R_{\mathcal{P}}, f_{\mathcal{P}}, \mathbb{Z})$. By Theorem 9.16, this is bounded above by $\min_{P \in I_{ADE}}(|P|) - 1$.

When the weight sequence is minimizing this implies that, $\min_{P \in I_{ADE}}(|P|) - 1 = n - 1 = \dim(X)$. Further, by the assumption on the base field, we know that the Rouquier dimension of $D^b(\text{coh} X)$ is at least $\dim(X)$ by Lemma 9.14. □
Example 9.19. Consider the weight sequence $(3, 3, 3)$. The partition $\{3\}, \{3, 3\}$ is in $\mathcal{I}_{ADE}$ and yields an upper bound of 1, implying that Conjecture 1.3 holds for the Fermat elliptic curve and similarly for the weighted projective line corresponding to this weight sequence. In fact it is known that the conjecture holds for curves in general, see [Orl09b], and for tubular weighted projective lines [Opp10].

Example 9.20. Consider the weight sequence $(3, 3, 3, 3, 4, 4, 4, 4)$. Let $X_3$ be the Fermat elliptic curve and $X_4$ be the Fermat K3 surface. One the one hand, this can be partitioned into $\{3, 3, 3\}, \{3, 3, 4\}, \{4, 4, 4\}$ yielding an equality between the Rouquier dimension of $D^b(\text{coh } X_3 \times X_3 \times X_4)$ and $\text{MF}(R, f, \mathbb{Z})$, where $f$ is the Fermat polynomial corresponding to this weight sequence. On the other hand, the partition $\{3, 3\}, \{3, 4\}, \{3, 4\}, \{3, 4\}$ is in $\mathcal{I}_{ADE}$ and yields an upper bound of 4, implying that Conjecture 1.3 holds for $X_3 \times X_3 \times X_4$.

Example 9.21. Let $(2, d_1, \ldots, d_n)$ be a weight sequence with $\sum_{i=1}^{n} \frac{1}{d_i} \leq \frac{1}{3}$. This sequence is minimizing, hence Conjecture 1.3 holds for the corresponding weighted Fermat hypersurface, $X$. Moreover, the Rouquier dimension of $\text{MF}(k[x_0, \ldots, x_n], x_0^2 + x_1^{d_1} + \cdots + x_n^{d_n}, \mathbb{Z})$ is equal to $n - 1$. But as noted in the proofs above, the quadratic part can be eliminated, hence the Rouquier dimension of $\text{MF}(k[x_1, \ldots, x_n], x_1^{d_1} + \cdots + x_n^{d_n}, \mathbb{Z})$ is also $n - 1$. Furthermore, this is equal to the Rouquier dimension of $D^b(\text{mod } -(A_{d_1-1} \otimes_\mathbb{K} \cdots \otimes_\mathbb{K} A_{d_n-1}))$. As mentioned above, for any smooth algebraic curve, the bounded derived category of coherent sheaves has Rouquier dimension one, [Orl09b], however, the above argument shows that the category of matrix factorizations of a Fermat curve of sufficiently high genus has Rouquier dimension two.

Remark 9.22. This is not an exhaustive list. By applying the theorem, one can obtain many other examples which provide new valid cases of Conjecture 1.3.

10. Generation time and algebraic classes

In this section we outline a relationship between the existence of algebraic classes in the deRham cohomology of a smooth variety, $X$, over $\mathbb{C}$ and the generation time of certain objects in the bounded derived category of coherent sheaves on $X$. In the background, is always our long-term ambition of realizing the Orlov spectrum as both an algebraic and symplectic invariant of monodromy, see [BFK10, KNS10]. We begin with the general relationship in subsection 10.1 and proceed with the hypersurface case in subsection 10.2 by applying our work from the previous sections. In subsection 10.3 we analyze the relationship between algebraic classes under grading changes between categories of matrix factorizations. In subsection 10.4 we apply our theory to prove some very special cases of the Hodge conjecture. The examples are meant to be illustrative and in future work we hope to provide a far larger class of examples, delving deeper into the relationship between the categorical constructions here and the classical geometric constructions. Finally in subsection 10.5 we introduce an enhancement of the Orlov spectrum which we call the Noether-Lefschetz spectrum, in reference to the classical Noether-Lefschetz loci. We expect this spectrum to encode all relevant information about the $\mathbb{C}$-linear span of the algebraic classes in deRham cohomology.

10.1. The boundary-bulk map, bulk-boundary map, natural transformations, and generation time. We begin with a brief discussion of the boundary-bulk and bulk-boundary maps following [CW10, PV10a]. See these works for a much more complete treatment. This subsection is mostly a quick overview of the these works and [Cal05] together with some
corollaries of the results found therein. The new contribution in this subsection is at the end where we describe a relationship with generation time.

Let $T$ be a saturated dg-category over $k$ (see Appendix A.1 for a definition). The Hochschild structure of $T$ is a Morita invariant of the category consisting of the following data:

- the graded Hochschild cohomology ring of $T$, $\text{HH}^*(T)$,
- the graded left $\text{HH}^*(T)$-module, $\text{HH}_*(T)$,
- a non-degenerate graded pairing $\langle -, - \rangle_M$ on $\text{HH}_*(T)$ called the generalized Mukai pairing.

When $X$ is a smooth proper algebraic variety over $\mathbb{C}$, the graded ring of polyvector fields,

$$\bigoplus_{p+q=i} H^p(X, \wedge^q T_X),$$

the graded ring of holomorphic forms,

$$\bigoplus_{q-p=i} H^p(X, \Omega^q_X),$$

and a modified cup product pairing, $\langle -, - \rangle_M$, (see [Cal05]) amount to the same data as the Hochschild structure on a dg-category $T$, as above. Furthermore, when $T = \text{D}^b(\text{coh} X)$, there are Hochschild-Kostant-Rosenberg (see [HKR62]) isomorphisms,

$$I^{\text{HKR}} : \text{HH}^i(\text{D}^b(\text{coh} X)) \to \bigoplus_{p+q=i} H^p(X, \wedge^q T_X)$$

and

$$I^{\text{HKR}} : \text{HH}_i(\text{D}^b(\text{coh} X)) \to \bigoplus_{q-p=i} H^p(X, \Omega^q_X).$$

The Hochschild structure on $\text{D}^b(\text{coh} X)$ is isomorphic to the data given by the polyvector fields and forms, after modifying the $\text{HKR}$ maps by the square root of the todd class. This was first observed by Kontsevich in [Kon97], where he demonstrated that this modification of $I^{\text{HKR}}$ gives a ring homomorphism as a result of his proof of the formality conjectures. The equivalence of the data discussed above was conjectured by Căldăraru in [Cal05], and ultimately assembled in parts by N. Markarian, A. Ramadoss, and D. Calaque, C. Rossi, and M. Van den Bergh, [Mar08, Ram10, CRV09]. It was also proven in the proper Calabi-Yau case by D. Huybrechts and M. Nieper-Wiikirchen in [HN10].

The $\text{HKR}$ isomorphisms allow one to study deRham cohomology by means of category theory. Specifically, if we wish to understand algebraic classes in a categorical setting, we will need to understand the image of the Chern character map under the $\text{HKR}$ isomorphism. This was handled once again in [CW10] and [Cal05]. In [CW10], they work in the context of categories of “spaces”. This essentially means a monodial 2-category whose objects are triangulated categories with a Serre functor given by a 1-morphism such that the product of the Serre functors is the Serre functor of the product (this is their Proposition 1) and the 1-morphisms are what they call reflexively polite. At least, this is what is needed to prove their Propositions 5 and 11, which we will use. Our main goal is to apply this work to categories of graded matrix factorizations, which are “spaces” in the sense above. The other results we will use from loc. cit. are Theorem 6 for pushforwards, which is also Lemma 1.2.1 of [PV10a] stated for perfect objects of a dg-category, and the so called Baggy Cardy condition, Theorem
16 of [CW10] for spaces, and Theorem 1.3.1 of [PV10a] for dg-categories whose perfect objects are saturated. To justify the use of the results in [CW10], in Appendix A.2, we demonstrate that the bicategory of saturated small dg-categories over a field, $k$, is a category of spaces in the sense above i.e. that is satisfies sufficient properties to apply the work in loc. cit. For the remainder of this paper therefore, we shall take $T$ to be a saturated essentially small dg-category over $k$. We should also mention that along these lines, Shylkarov, in [Shk07b], gives a description of the Chern character and the Hochschild pairing for dg-algebras and proves a version of Hirzebruch-Riemann-Roch for perfect modules over a dg-algebra. These results of Skylkarov are comparable but will not be used.

Our main application will be to categories of graded matrix factorizations like those appearing in Section 7. In fact, as observed there, by a graded form of Knörrer periodicity, any such category is an admissible subcategory of the derived category of coherent sheaves on a DM stack. In the isolated case, the reader if they so prefer, can therefore view our category of spaces as a sort of idempotent completion of the 2-category of smooth proper DM stacks.

A more uniform approach would be to treat the entire theory from the perspective of graded matrix factorizations, perhaps regarding the dg-categories themselves as having particular grading groups. This was done by Polishchuk and Vaintrob for matrix factorizations with a $\mathbb{Z}_2$-graded dg enhancement in [PV10a]. In particular, they derive a precise formula, known as the Kapustin-Li formula, for the boundary-bulk map and the trace pairing described below. This was also done in independently by Dyckerhoff and Murfet in [DM10]. Furthermore, recent progress towards a uniform approach has been made by Polishchuk and Vaintrob in [PV11], see sections 2.5 and 2.7. Indeed, the boundary-bulk map and trace pairing appearing in [CW10] applied to the category of graded matrix factorizations should, in the long-term (or perhaps not so long-term), be revealed explicitly in terms of a graded Kapustin-Li package.

For any object $E \in T$, the boundary-bulk map, $\alpha_E$, is a generalization of the Chern character map thought of as a morphism,

$$\alpha_E : \text{Hom}(E, E[i]) \to \text{HH}_i(T).$$

In the context of dg-categories the Chern character of $E$, $\text{ch}(E)$, is often defined as $\alpha_E(\text{Id}_E)$. Regarding the base field $k$ as a algebra, let $S$ denote the dg-category of perfect modules over $k$ i.e. the category of complexes of finite dimensional graded $k$-vector spaces. For any object $E \in T$ the functor $\otimes_k E : S \to T$ induces a map on Hochschild homology $(- \otimes_k E)_* : \text{HH}_0(S) \to \text{HH}_0(T)$. The Chern character of $E$ agrees with $(- \otimes_k E)_*(k)$. The relationship with the usual Chern character map is made precise by the following theorem proven by Căldăraru in [Cal05]:

**Theorem 10.1.** For any $E \in \text{D}^b(\text{coh} \ X)$, $I_{\text{HKR}}(\alpha_E(\text{Id}))$ agrees with the usual Chern character map.

The “image” of the boundary-bulk map, $W(T)$, is the graded left $\text{HH}^*(T)$-submodule of $\text{HH}_*(T)$ generated by all $w \in \text{HH}_*(T)$ such that $w = \alpha_E(f)$ for some object, $E$, and some morphism, $f$. In particular by the theorem above, the image of $W(\text{D}^b(\text{coh} \ X))$ under the HKR isomorphism contains all algebraic classes.

In the other direction, there is a bulk-boundary map, $\beta_E$. This is a morphism,

$$\beta_E : \text{HH}_i(T) \to \text{Hom}(E, S(E)[i]).$$
Furthermore, for any \( v \in \text{HH}_i(T) \) the collection of morphisms \( \beta_E(v) \) can be packaged into a natural transformation, \( \beta(v) : \text{Ho} T \rightarrow \text{Ho} T \) i.e. we have a map:

\[
\beta : \text{HH}_i(T) \rightarrow \text{Nat}(\text{Id}, S[i]).
\]

With Theorem [10.1] in mind, we define,

**Definition 10.2.** A class \( v \in \text{HH}_0(T) \) is called **algebraic** if \( v = \alpha_E(\text{Id}) \) for some \( E \in T \).

A class, \( v \in \text{HH}_0(T) \), is called **antialgebraic** if it pairs trivially with any algebraic class. A class \( v \in \text{HH}_i(T) \) is called **nilpotent** if \( v = \alpha_E(f) \) for some \( E \in T \) and nilpotent morphism, \( f : E \rightarrow E[i] \), it is called **0-nilpotent** if it lies in \( \text{HH}_0(T) \).

Let us mention that the Lefschetz Standard Conjecture generalizes to our context:

**Conjecture 10.3** (Noncommutative Lefschetz Standard). Let \( T \) be a saturated dg-category. The trace pairing restricted to the algebraic classes is nondegenerate. In particular, any antialgebraic class is not algebraic.

**Proposition 10.4.** If \( f : F \rightarrow F \) is nilpotent, then \( \alpha_F(f) \) is antialgebraic.

The following theorem of Kuznetsov (see [Kuz09a] Theorem 7.3), allows us to analyze algebraic classes in the presence of a semi-orthogonal decomposition. Though we state the theorem in a slightly different context, the proof is the same (for the applications in subsection [10.3], we actually use Kuznetsov’s statement exactly, similarly we could always assume that we are an admissible subcategory of a certain stack as described in Section [7]).

**Theorem 10.5.** Let \( T \) be a saturated dg-category and \( \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle \) be a semi-orthogonal decomposition of \( \text{Ho} T \). Let \( A_i \) denote the full subcategory of \( T \) lie in \( \mathcal{A}_i \) in \( \text{Ho} T \). Suppose that the inclusion, \( \gamma_i : \mathcal{A}_i \rightarrow \text{Ho} T \), and projection, \( \pi_i : \text{Ho} T \rightarrow \mathcal{A}_i \), functors are induced by dg functors \( \Gamma_i : A_i \rightarrow T \) and \( \Pi_i : T \rightarrow A_i \). One has a decomposition, 

\[
\text{HH}_i(T) \cong \text{HH}_i(\mathcal{A}_1) \oplus \cdots \oplus \text{HH}_i(\mathcal{A}_m),
\]

induced by \( \Gamma_{i*} \) and \( \Pi_{i*} \). Furthermore, the image of the Chern character map respects this decomposition.

**Proof.** The first part is as in the proof of Theorem 7.3 in see [Kuz09a] Theorem 7.3. The second part follows from Proposition 5 and Theorem 6 in [CW10]/Lemma 1.2.1 of [PV10a].

**Proposition 10.6.** For any \( v \in \text{HH}_i(T) \), \( \beta(v) = 0 \) if and only if \( \langle v, W(T) \rangle = 0 \). In particular if \( \beta(v) = 0 \) and \( i = 0 \) then \( v \) is antialgebraic.

**Proof.** For any object \( E \), any \( i \), and any morphism \( e : E \rightarrow E[-i] \), consider \( \alpha_E(e) \in \text{HH}_{-i}(T) \). We have,

\[
\langle \alpha_E(e), v \rangle_M = \langle e, \beta(v)(E) \rangle = 0,
\]

where the latter equality comes from adjunction (see loc. cit., Proposition 11). By nondegeneracy of the Serre pairing, \( \beta(v)(E) \) pairs trivially with all endomorphisms, \( e : E \rightarrow E[-i] \), if and only if \( \beta(v)(E) = 0 \). Hence \( \beta(v) \) pairs trivially with \( W(T) \) if and only if \( \beta(v) = 0 \).

**Corollary 10.7.** The positive Hochschild homology, \( \text{HH}_{>0}(T) \), is a submodule of \( W(T) \) if and only if for any \( v \in \text{HH}_{<0}(T), \beta(v) \neq 0 \). Similarly, if \( W(T) \) does not intersect \( \text{HH}_{<0}(T) \), then for any \( v \in \text{HH}_{>0}(T), \beta(v) = 0 \).
Proposition 10.8. Let $T$ be a saturated dg-category over $\mathbb{C}$. The image of the boundary-bulk map, $\mathcal{W}(T)$, is spanned by the algebraic and nilpotent classes.

Proof. Since $T$ is smooth and proper over $\mathbb{C}$, $\text{End}(E)$ is an artinian ring. Quotienting out by the nilradical, we obtain a direct sum of division rings which are a central extensions of $\mathbb{C}$. Therefore we get a direct sum of copies of $\mathbb{C}$. Therefore $\text{End}(E)$ is spanned as a $\mathbb{C}$-vector space by idempotent and the nilpotent elements. The idempotent morphisms induces algebraic classes, and the nilpotent ones induce nilpotent classes. \hfill $\square$

Proposition 10.9. For any nilpotent class $v \in \text{HH}_0(T)$, $\langle v, \mathcal{W}(T) \rangle = 0$. In particular, $\beta(v) = 0$ and any 0-nilpotent class is antialgebraic.

Proof. Let $v = \alpha_F(f)$ be a nilpotent class. For any object $E$ and any morphism $e : E \to E[-i]$, consider $\alpha_F(e) \in \text{HH}_{-i}(T)$. Let $f_{m_e} : \text{Hom}^*(E,F) \to \text{Hom}^*(E,F)$ be the morphism of graded vector spaces obtained by pre-composition with $e$ and post composition with $f$. By the Baggy Cardy condition, (see [CW10], Theorem 16/Theorem 1.3.1 of [PV10a]) we have,
\[
\langle \alpha_F(e), \alpha_F(f) \rangle_M = \text{Tr} f_{m_e},
\]
where the latter trace is the ordinary super trace of graded vector spaces. This trace is zero as $f$ is nilpotent and hence $f_{m_e}$ is also nilpotent. Therefore,
\[
0 = \langle \alpha_F(e), \alpha_F(f) \rangle_M = \langle e, \beta(\alpha_F(f))(E) \rangle_M,
\]
where the latter equality comes from adjunction (see loc. cit., Proposition 11). Rephrasing, this is nothing more than $\langle \alpha_F(f), \mathcal{W}(T) \rangle = 0$. The rest of the statement is Proposition 10.6 \hfill $\square$

There is also a partial converse to Proposition 10.6.

Proposition 10.10. Let $T$ be a saturated dg-category over $\mathbb{C}$. Assume there are no nontrivial 0-nilpotent classes. Then for any $v \in \text{HH}_0(T)$, $\beta(v) = 0$ if and only if $v$ is antialgebraic.

Proof. Suppose $v$ is antialgebraic i.e. pairs trivially with all algebraic classes. By Proposition 10.8 $\mathcal{W}(T) \cap \text{HH}_0(T)$ is spanned by algebraic and 0-nilpotent classes, as there are no 0-nilpotent classes, $v$ pairs trivially with $\mathcal{W}(T)$. The result follows from Proposition 10.6. \hfill $\square$

Proposition 10.11. The positive Hochschild homology, $\text{HH}_{>0}(T)$, is a submodule of $\mathcal{W}$ if and only if for any $v \in \text{HH}_{<0}$, $\beta(v) \neq 0$.

Proof. By Proposition 10.6 $\beta(v) = 0$ if and only if it pairs trivially with $\mathcal{W}$. When $v \in (\text{HH})_i(T)$ with $i < 0$

Suppose $\text{HH}_{>0}(T) \subseteq \mathcal{W}$ and consider $v \in \text{HH}_i$ with $i < 0$. By nondegeneracy of the Serre pairing, there exists $w \in \text{HH}_{>0}(T)$ such that $\langle w, v \rangle_M \neq 0$. By assumption, there exists an object $F$ and a morphism $f : F \to F[i]$ such that $w = \alpha_F(f)$. One has,
\[
0 \neq \langle w, v \rangle_M = \langle \alpha_F(f), v \rangle_M = \langle f, \beta(v)(F) \rangle_M.
\]
Hence the natural transformation $\beta(v)$ is nonzero.

Conversely, if $\beta(v) \neq 0$, there exists an $F \in T$ such that $\beta(v)(F) \neq 0$. Therefore, by nondegeneracy of the Serre pairing, there exists a morphism $f : F \to F[-i]$ such that $\langle f, \beta(v)(F) \rangle_M \neq 0$. Hence,
\[
\langle \alpha_F(f), v \rangle_M = \langle f, \beta(v)(F) \rangle_M \neq 0.
\]
\hfill $\square$
Proposition 10.12. Suppose that the Lefschetz Standard Conjecture holds and that for any \( j \in \text{HH}_{<0}(T) \), the submodule generated by \( j \) intersects the image of the Chern character map, then for any \( v \in \text{HH}_{>0}(T) \), \( \beta(v) = 0 \).

Proof. Suppose \( j \in W(T) \cap \text{HH}_{<0}(T) \). By taking a basis, we may assume that \( j = \alpha_f(f) \), where \( F \) is an indecomposable object, and \( f : F \to F[i] \) is a morphism with \( i < 0 \). By assumption, there exists \( x \in \text{HH}^{>0}(T) \) such that \( xj \) is equal to the Chern character of some object, \( E \). Furthermore, we have \( xj = x\alpha_f(f) = \alpha_f(x(F) \circ f) \). As \( F \) is indecomposable, the morphisms \( x(F) \) and \( f \) are graded, and the total \( \mathbb{Z} \)-graded morphism space is finite dimensional, \( x(F) \circ f \) is nilpotent. By Proposition 10.9, \( \langle \alpha_f(f), W(T) \rangle = 0 \). By the Lefschetz Standard Conjecture, we have \( W(T) \cap \text{HH}_{<0}(T) = 0 \). The result follows from Corollary 10.7. 

Proposition 10.13. Let \( T \) be a saturated dg-category over \( \mathbb{C} \). The \( \mathbb{C} \)-linear span of the algebraic classes is equal to \( \text{HH}_0(T) \) if and only if \( \beta(v) \neq 0 \) for any nonzero \( v \in \text{HH}_0(X) \).

Proof. By Proposition 10.6, \( \beta(v) = 0 \) if and only if it pairs trivially with \( W(T) \). By nondegeneracy of the Serre pairing, \( \beta(v) \neq 0 \) for all \( v \in \text{HH}_0(T) \) if and only if \( W(T) \cap \text{HH}_0(T) = \text{HH}_0(T) \). By Proposition 10.8, \( W(T) \cap \text{HH}_0(T) \) is spanned by algebraic and 0-nilpotent classes. However, all nilpotent classes act trivially as natural transformations by Proposition 10.9. Therefore \( W(T) \cap \text{HH}_0(T) = \text{HH}_0(T) \) if and only if \( \text{HH}_0(T) \) is spanned by algebraic classes. 

We now relate the property of being (anti)algebraic to generation time of certain categories. Let \( \gamma : \text{HH}^i(T) \to \text{Nat}(\text{Id}, [i]) \) be the natural map which sends an element of Hochschild cohomology to its associated natural transformation.

Definition 10.14. Let \( S \) be a collection of natural transformations from the identity functor to another endofunctor of a triangulated category \( T \). The category vanishing on \( S \), denoted \( T(S) \), is the full subcategory of objects in \( T \) on which the natural transformation vanishes. If \( T \) is a triangulated category that is homotopy category of a saturated dg-category and \( J \) is a ideal in the Hochschild cochain complex, \( \text{HH}^*(T) \), the category vanishing on \( J \), denoted \( T(J) \), is \( T(\gamma(J)) \), which by abuse of notation we shall also denote as \( T(J) \).

Lemma 10.15. Let \( J \) be a dg-ideal in a dg-algebra, \( A \), generated by cycles of even degree in the center of \( H^*(A) \). For any \( n \), the cone of the inclusion \( J^{n+1} \subseteq J^n \) is annihilated by even cocycles of \( J \) in the homotopy category. Similarly, for cocycle \( j \in A \) the cone of the action of \( j \) on any object \( M \) is annihilated by \( j \).

Proof. For any element \( j \in J \subseteq A \) of even degree, \( 2s \), there exists a morphism \( \alpha : J^n \to J^{n+1}[2s] \) such that multiplication by \( j \) from \( J^n \) to \( J^n \) factors as \( i \circ \alpha \) where \( i : J^{n+1} \to J \) is the inclusion. Consider the matrices:

\[
h := \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}, \quad d := \begin{bmatrix} -d_{J^{n+1}} & 0 \\ \gamma & d_J \end{bmatrix}.
\]
with \( h : J_{i+1}^{n+1} \oplus J_{i+1}^{n} \to J_{i-1+2s}^{n+1} \oplus J_{i+2s}^{n} \) and \( d \) is the differential of the cone of \( j : J^{n+1} \to J^{n} \).

We have a homotopy:

\[
\begin{array}{ccccccccccc}
\cdots & d & J_{i-1}^{n+1} \oplus J_{i}^{n} & d & J_{i}^{n+1} \oplus J_{i+1}^{n} & d & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\cdots & h & J_{i-1+2s}^{n+1} \oplus J_{i+2s}^{n} & h & J_{i+2s}^{n+1} \oplus J_{i+1+2s}^{n} & h & \cdots \\
\end{array}
\]

Therefore the action of \( J \) on the cone in the homotopy category is generated by natural transformations associated to cocycles acting by zero. There is a similar homotopy for the cone of the action of \( j \) on any object \( M \).

Given an ideal of \( \mathcal{HH}^*(T) \) generated by even elements, we can lift it to an ideal of a dga, \( \mathcal{HH}(T) \), computing \( \mathcal{HH}^*(T) \) by choose cocycles lifting of each of the generators and taking the ideal, \( \mathcal{J} \), generated by these cocycles in the \( \mathcal{HH}(T) \). In general, there will be a discrepancy between the \( J^n \) and \( H^*(\mathcal{J}^n) \). Moreover, order of vanishing of \( H^*(\mathcal{J}^n) \) is not necessarily a Morita invariant. However, it will allow use to give an upper bound on generation by \( \mathcal{T}(\mathcal{J}) \).

**Proposition 10.16.** Let \( T \) be the homotopy category of a saturated dg-category and \( J \) be an ideal in \( \mathcal{HH}^{2*}(T) \). The generation time of \( \mathcal{T}(J) \) is bounded above by the minimal \( n \) such that \( H^*(\mathcal{J}^n) \) vanishes. When \( J \) is a principal ideal generated by an element of even degree, \( f \), then the generation time of \( \mathcal{T}((f)) \) is equal to the smallest \( n \) such that \( \gamma(f^{n+1}) = 0 \).

**Proof.** The dg-category, \( T \), is Morita equivalent to the homotopy category of perfect modules over a smooth and proper dga, \( A \). To get a dga-computing \( \mathcal{HH}^*(T) \), one can resolve \( A \) as an \( A^n \)-bimodule. Denote a choice of resolution by \( R \). Then \( \mathcal{HH}(T) \) can be taken to be \( \text{Hom}_{A^n}(R, R) \). Lift \( J \) to as described above to get a sequence of bimodule morphisms \( \mathcal{J}^{n+1} \to \mathcal{J}^n \) for which multiplication by an cocycle of \( \mathcal{J} \) is null-homotopic. For any module, \( M \), over \( A \), \( R \otimes_A M \) is quasi-isomorphic to \( M \). Now, tensor \( \mathcal{J}^{n+1} \to \mathcal{J}^n \) with \( R \otimes_A M \) over \( \mathcal{HH}(T) \) to get a filtration of \( M \) where the cone over each morphism is annihilated by \( J \) and \( \mathcal{J}^n \otimes_{\mathcal{HH}(T)} R \otimes_A M \) is acyclic once \( \mathcal{J}^n \) is acyclic.

When \( J = (f) \) with degree \( 2t \), we can apply the argument above to show that the cone over \( f \) must be annihilated by \( f \). If \( f^{s+1} = 0 \), consider the composition

\[
M \xrightarrow{f} \cdots \xrightarrow{f} M[(s+1)(2t)].
\]

Since the total morphism is zero, by a standard argument involving iterated application of the octahedral axiom (for example see section I.5e in [Sei08a]) one sees that the order of nilpotence of \( f \) bounds the generation time of \( \mathcal{T}((f)) \). The bound in the other direction is the Lemma 9.3.

As observed above, the following is a corollary of Lemma 9.3.

**Corollary 10.17.** Suppose there exists a strong generator of \( T \), \( G \), contained in \( \mathcal{T}(J) \) for some ideal \( J \in \mathcal{HH}^{2*}(T) \). Then \( \gamma(J) \) is nilpotent and if \( \gamma(J)^n \neq 0 \), then \( \mathcal{G}(G) \geq n \).
10.2. Hypersurfaces, generation time, and algebraic classes. Let $X$ be a smooth hypersurface of degree $d$ in $\mathbb{P}^n$ defined by $f$. Fix $R := k[x_0, \ldots, x_n]$. The goal of this subsection is to relate generation times of certain categories in $D^b(\text{coh } X)$ to algebraic classes in $H^*(X, \mathbb{C})$. Although many of the concepts in the previous subsection should, in principle, apply to far more general situations, this special case, for the moment, is more approachable. This is due to the combination of the calculations of Hochschild homology appearing in Section 6 and Orlov’s comparison between categories of graded matrix factorizations and the bounded derived category of coherent sheaves on a hypersurface (see [Orl09a] and Section 7).

First, let us recall a famous theorem of Griffiths which describes the deRham cohomology of a hypersurface, [Gri69]. An element of $H^{n-1-k}(X)$ is called primitive if it cups trivially with $H^k$, where $H$ is the class of a hyperplane section. In our context, by the Lefschetz Hyperplane Theorem, all primitive cohomology classes lie in the middle cohomology, $H^{n-1}(X)$. Furthermore, all elements are primitive $H^{n-1}(X)$ when $n$ is even. When $n$ is odd, all classes in $H^{n-i,i}(X)$ with $i \neq \frac{n-1}{2}$ are primitive, while the primitive classes in $H^{\frac{n+1}{2},\frac{n+1}{2}}(X)$ are just those lying in the kernel of the cup product with $H$.

**Theorem 10.18** (Griffiths). There is an isomorphism between primitive cohomology classes in $H^{n-i,i}(X)$ and polynomials of degree $(i-1)d + n - 1$ in the jacobian ring, $J_f$, which takes a polynomial, $p$, of degree $(i-1)d + n - 1$, to the image of the holomorphic form $\frac{p}{f}\Omega$ in $H^*(\mathbb{P}^n \setminus X, \mathbb{C})$ under the residue map.

This description matches perfectly with the calculations of Hochschild homology in Section 6 combined with Orlov’s Theorem 7.1 and Kuznetsov’s Theorem 10.5. Namely, the Hochschild homology of $\text{MF}(R, f, \mathbb{Z})$ consists of certain graded pieces of the Jacobian as above and twisted sectors. Under Theorems 7.1 and 10.5, when $X$ is Fano, $HH_*(\text{MF}(R, f, \mathbb{Z}))$ is the orthogonal to the Chern characters of $O, \ldots, O(d-n-2)$. While when $X$ is of general type, $HH_*(D^b(\text{coh } X))$ is the orthogonal to the Chern characters of $k, \ldots, k(n+2-d)$.

Since the todd class of $X$ pairs trivially with the primitive classes, the $HKR$ isomorphism should take a Hochschild class corresponding to a polynomial $p \in J_f$, of degree $(i-1)d + n - 1$, to the image of the holomorphic form $\frac{p}{f}\Omega$ as above. We leave this observation to future work and content ourselves with the fact by Theorem 10.1 the rank of the lattice of algebraic classes in $HH_0(D^b(\text{coh } X))$ agrees with the rank of the algebraic lattice in $H^*(X, \mathbb{C})$. Furthermore we have:

**Lemma 10.19.** Let $Z$ be a smooth stack as defined in Section 7, $\text{MF}(R, f, M)$, and $a$ be the Gorenstein parameter of $R/f$ as a $\mathbb{Z}$-graded ring. Let $r$ be the rank of the lattice of algebraic classes in $HH_0(D^b(\text{coh } Z))$. The rank of the lattice of algebraic classes in $\text{MF}(R, f, M)$ is equal to $r - a|H|$ (assuming $r$ is finite). Furthermore, for any algebraic variety, $X$, the rank of the lattice of algebraic classes in $HH_0(D^b(\text{coh } X))$ is equal to the rank of the algebraic classes in $H^*(X, \mathbb{C})$. Furthermore, the same statements hold for the dimension of the $\mathbb{C}$-linear span of these lattices.

**Proof.** This follows directly by combining Theorem 7.1, Theorem 10.5, and Theorem 10.1. 

With the above lemma in mind, from now on we study the algebraic lattices in $\text{MF}(R, f, \mathbb{Z})$. Our goal is to apply Proposition 10.16 to compare generation time with algebraic classes in this hypersurface setting.
Now notice that, for the category of matrix factorizations (with its \( \mathbb{Z}/2\mathbb{Z} \)-graded dg structure), \( \text{HH}_2(\text{MF}(R, f)) \cong J_\mathcal{f} \) see [Dyc09]. Furthermore, given an ideal \( J \subseteq J_\mathcal{f} \), the notion of a matrix factorization, \( A \), lying in \( \text{MF}(R, f)(J) \) is equivalent to the matrix factorization being scheme theoretically supported on \( J \) in the following sense:

**Definition 10.20.** For a (graded) matrix factorization, \( A \) of \( R, f \), the scheme theoretic support of \( A \) is the ring \( R/J \) where \( J \) is the ideal which annihilates \( A \) in the homotopy category.

For simplicity, we denote by \( \text{MF}(J) \) the full subcategory of the category of matrix factorizations scheme theoretically supported on \( J \) when the ring, function, and grading are implicit.

**Lemma 10.21.** Let \( \mathfrak{m} \) be the maximal ideal of \( J_\mathcal{f} \). Let \( M \) be a finitely generated abelian group of rank one. The category of matrix factorizations scheme theoretically supported on \( \mathfrak{m} \) is equal to the additive category generated by the residue field by \( k(\mathfrak{m}) \) for all \( m \in M \).

**Proof.** When there is no grading, this follows from the proof of Lemma 4.18 in [BFK10]. Since the forgetful functor, \( \text{For} : \text{MF}(R, w, M) \rightarrow \text{MF}(R, w) \), is faithful, and the action of \( \mathfrak{m} \) on \( \text{MF}(R, w) \) agrees with forgetting the action of \( \text{MF}(R, w, \mathbb{Z}) \), it follows that any graded matrix factorization, \( A \), which is annihilated by \( \mathfrak{m} \), is in the additive category generated by \( k \) once the grading is forgotten. This is precisely the category described. \[ \square \]

**Theorem 10.22.** Let \( R = k[x_0, \ldots, x_n] \) and \( f \) be an isolated singularity at the origin. Let \( I \) be the annihilator of \( W(\text{MF}(R, f)) \). For any ideal \( J \subseteq J_\mathcal{f} \), the generation time of \( \text{MF}(J) \) is bounded below by one less than the nilpotent order of \( J \) in of \( J_\mathcal{f}/I \), with equality when \( J \) is principal.

**Proof.** By Proposition 10.16, the generation time of \( \text{MF}(J) \) is bounded below by the largest \( n \) such that \( \beta(J^n) \neq 0 \) with equality when \( J \) is principal. This is equivalent by Proposition 10.6 to \( \langle J^n, W \rangle \neq 0 \). As the pairing is given by composition, which in this case is just multiplication of polynomials, this is equivalent to \( J^n \) being annihilated by \( W \). \[ \square \]

**Theorem 10.23.** Let \( X \) be a smooth hypersurface in \( \mathbb{P}^n \), defined by a homogeneous polynomial, \( f \), of degree \( d \), \( k \in \text{MF}(R, f, \mathbb{Z}) \) be the residue field, and \( I \) be the ideal of polynomials in \( J_\mathcal{f} \) which are homotopic to zero for all matrix factorizations in \( \text{MF}(R, f, \mathbb{Z}) \). For any homogeneous ideal \( J \subseteq J_\mathcal{f} \), the generation time of \( \text{MF}(J) \), the category of \( \mathbb{Z} \)-graded matrix factorizations scheme theoretically supported on \( J \), is bounded below by one less than the nilpotent order of \( J \) in \( J_\mathcal{f}/I \) with equality if \( J \) is principal.

**Proof.** Notice that we are in the setup of Proposition 9.8 where \( \text{MF}(R, f) \) is the dg orbit category of \( \text{MF}(R, f, \mathbb{Z}) \) by \( \mathbb{Z} \). Let \( \text{For} : \text{MF}(R, f, \mathbb{Z}) \rightarrow \text{MF}(R, f) \) be the forgetful functor. The image of this functor can be identified with the orbit category before taking the triangulated closure (taking the triangulated closure yields the dg orbit category). As in the proof of Proposition 9.8, for any graded matrix factorization, \( A \in \text{MF}(R, f, \mathbb{Z}) \), we have \( A \in \langle \bigoplus_{i=0}^{d-1} k(i) \rangle_s \) if and only if \( \text{For}(A) \in \langle k \rangle_s \). Meaning that the generation time of \( \text{MF}(J) \) is equal to the time it takes for the image of \( \text{MF}(J) \) to generate the image of the forgetful functor. As in the proof of the above theorem, the generation time of \( \text{MF}(J) \) is therefore bounded below by the nilpotent order of \( J \) in \( J_\mathcal{f}/I \) with equality when \( J \) is principal. \[ \square \]
Corollary 10.24. If $\bigotimes_{i=0}^{d-1} k(i) < \frac{(n+1)(d-2)}{2}$, then $X$ has no primitive algebraic classes.

Proof. This follows from the theorem and Lemma [10.21] \qed

Recall that a lattice contained in a vector space is called a full sublattice if it spans the vector space.

Corollary 10.25. Let $X$ be a smooth hypersurface of even dimension in $\mathbb{P}^n$. Suppose the algebraic classes form a full sublattice of $H^{\frac{n+1}{2}, \frac{n-1}{2}}(X, \mathbb{C})$ i.e. there is a basis of the lattice which forms a $\mathbb{C}$-basis of $H^{\frac{n+1}{2}, \frac{n-1}{2}}(X, \mathbb{C})$. For any ideal $J \subseteq J_f$, generated by homogeneous polynomials of degree $i$, the generation time of $MF(J)$ is bounded below by $\left\lfloor \frac{(n+1)(d-2)}{2i} \right\rfloor$ in both $MF(R, f)$ and $MF(R, f, \mathbb{Z})$ with equality when $J$ is principal.

Proof. Since the algebraic classes form a full sublattice in $H^{\frac{n+1}{2}, \frac{n-1}{2}}(X, \mathbb{C})$, the same is true for $MF(R, f, \mathbb{Z})$ by Lemma [10.19]. Therefore, given any polynomial of degree at least $\frac{(n+1)(d-2)}{2}$ we can multiply it by some polynomial to get an algebraic class. The resulting polynomial, as a natural transformation, has to be nonzero by nondegeneracy of the Serre pairing. Hence, all polynomials of degree at least $\frac{(n+1)(d-2)}{2}$ are nonzero as natural transformations in $MF(R, f)$ by faithfulness of the forgetful functor. Conversely, given a monomial, $p$, of degree greater than $\frac{(n+1)(d-2)}{2}$. We can write $p = ba$ where $a$ is algebraic. Therefore, $a = \alpha E(Id)$ for some $E \in MF(R, f, \mathbb{Z})$. Now in $MF(R, f)$ we have $p = b\alpha E(Id) = \alpha E(b)$ since $b$ can be considered as the action of a Hochschild cohomology class. As $b$ is nilpotent, $p$ induces the zero natural transformation on $MF(R, f)$ by Proposition [10.9]. The result follows from Theorems [10.22 and 10.23] \qed

10.3. Grading changes and algebraic classes. In this section we examine how algebraic classes behave under grading changes. Let $R = \mathbb{C}[x_0, \ldots, x_n]$ be graded by a finitely generated abelian group of rank one, $M$, with the $x_i$ homogeneous. In [PVII], Polishchuk and Vaintrob identify the trace for $MF(R, w, M)$ where $w \in R_d$ is an isolated singularity in a graded polynomial ring where the grading group, $M$, has rank one and $M/d$ is finite, and they compute the Hochschild homology using the procedure outlined above.

The answer, as in computations in Section 6.2, is in terms of the dual group to $M/d$. Let us denote that group by $G$. Recall that $w_g$ is the restriction of $w$ to the fixed locus of $g$. Let $n_g$ be the dimension of the fixed locus of $g$, and $d_g$ be the degree of the $a$ volume form on the fixed locus. The following is Theorem 2.6.1 of [PVII]:

Theorem 10.26. (i) The Hochschild homology of $MF(R, w, M)$ is

$$HH_i(MF(R, w, M)) = \begin{cases} \oplus_{n_g=2l} \text{Jac}(w_g)_{d(q_a-t)-d_g} & i = 2l \\ \oplus_{n_g=2l+1} \text{Jac}(w_g)_{d(q_a-t)-d_g} & i = 2l + 1. \end{cases}$$

(ii) The action of $(m)_*: HH_*(MF(R, w, M)) \to HH_*(MF(R, w, M))$ is multiplication by $m(g)^{-1}$ on the piece corresponding to $\text{Jac}(w_g)$.

(iii) Let $\pi: M \to L$ be a quotient map with $\pi(d)$ non-torsion. We have an associated functor

$$\text{Res} : MF(R, w, M) \to MF(R, w, L).$$

with a map on Hochschild homology, $\text{Res}_*: HH_*(MF(R, w, M)) \to HH_*(MF(R, w, L))$. Let $G'$ be the dual group to $L/d$. There is a natural inclusion $G' \subset G$. $\text{Res}_*$ acts by the natural inclusion on $\text{Jac}(w_g)_{d(q_a-t)-d_g}$ if $g \in G'$ and acts by zero otherwise.
Remark 10.27. Using Polishchuk and Vaintrob’s identification of the trace for graded matrix factorizations, one can extend Polishchuk and Vaintrob’s Hochschild homology computations to the case where \( w \) is not an isolated singularity using the computations of Section 6. We leave this more general statement to future work as it will not be needed for current applications.

Using the theorem above, the induction functor, and functoriality of the Chern character map, one can precisely relate the algebraic classes obtained from one another under grading changes. We have the following corollary,

**Corollary 10.28.** Consider the inflation functor,

\[
\Ind : \text{MF}(R, f, L) \to \text{MF}(R, f, M),
\]

and the induced map,

\[
\Ind_* : \text{HH}_*(\text{MF}(R, f, L)) \to \text{HH}_*(\text{MF}(R, f, M)).
\]

The composition,

\[
\Ind_* \circ \Res_* = (\Ind \circ \Res)_* : \text{HH}_*(\text{MF}(R, w, M)) \to \text{HH}_*(\text{MF}(R, w, M)),
\]

acts by multiplication by the order of the kernel of \( \pi : M \to L \) on the factor \( \text{Jac}(w_g)_{d(q_g - l) - d_g} \) if \( g \in G' \) and acts by zero otherwise.

**Proof.** Let \( K \) denote the kernel of \( \pi : M \to L \). Note that \( k(g) = 1 \) if and only if \( g \in G' \). We have an isomorphism of functors, \( \Ind \circ \Res \cong \bigoplus_{k \in K} (k) \). Note that \( \Ind \circ \Res \) can be factored as a composition

\[
\text{MF}(R, w, M) \xrightarrow{\kappa} \bigsqcup_{k \in K} \text{MF}(R, w, M) \xrightarrow{\oplus} \text{MF}(R, w, M)
\]

where \( \kappa \) maps to the factor corresponding to \( k \) by \( (k) \) and \( \oplus \) is the functor that takes \( \bigsqcup E_k \) to \( \oplus E_k \). Note here that \( \bigsqcup_{k \in K} \text{MF}(R, w, M) \) denotes the category whose objects are \( |K| \)-tuples of objects from \( \text{MF}(R, w, M) \) and whose morphisms are \( |K| \)-tuples of morphisms \( \text{MF}(R, w, M) \). One can denote an object of \( \bigsqcup_{k \in K} \text{MF}(R, w, M) \) by \( \oplus_{k \in K} E_k \varepsilon_k \) where we think of \( \varepsilon_k \) as orthogonal idempotents.

Choose a generator, \( G \), for \( \text{MF}(R, w, M) \) and let \( A \) denote its endomorphism complex. If we take \( \oplus G \varepsilon_k \) as our generator of \( \bigsqcup_{k \in K} \text{MF}(R, w, M) \), we see its endomorphism complex is \( \tilde{A} = A \varepsilon_1 \oplus \cdots \oplus A \varepsilon_k \) where now \( \varepsilon_k \) are really (closed) orthogonal idempotents. It is easy to see that \( \tilde{A} \otimes \tilde{A} \cong \bigoplus_{k \in K} A \otimes A \varepsilon_k \). Thus, \( \text{HH}_*(\bigsqcup_{k \in K} \text{MF}(R, w, M)) \) is isomorphic to \( \bigoplus_{k \in K} \text{HH}_*(\text{MF}(R, w, M)) \varepsilon_k \).

Part (ii) of Theorem 10.26 says that action on the component of \( \text{HH}_*(\bigsqcup_{k \in K} \text{MF}(R, w, M)) \) corresponding to \( \text{Jac}(w_g)_{d(q_g - l) - d_g} \varepsilon_k \) is multiplication by \( k(g)^{-1} \).

In terms of \( \tilde{A} \) and \( A \), \( \oplus : \bigsqcup_{k \in K} \text{MF}(R, w, M) \to \text{MF}(R, w, M) \) corresponds the summing map \( \tilde{A} \to A \) which takes \( \oplus a_k \varepsilon_k \) to \( \sum a_k \). It is easy to see the induced action on Hochschild homology is again summation.

Now, we see that if \( g \in G' \), then each \( k \) acts trivially and the component corresponding to \( \text{Jac}(w_g)_{d(q_g - l) - d_g} \) gets multiplied by \( |K| \). If \( g \notin G' \), then \( (\Ind \circ \Res)_* \) acts by multiplication by a sum over full sets of roots of unity - which is zero.

\[\square\]
10.4. The Hodge conjecture: Fermat hypersurfaces and a certain K3 surface.

In this subsection, we apply the techniques of the previous sections to prove the Hodge conjecture for known cases of Fermat hypersurfaces and a certain K3 surface closely related to the Fermat cubic fourfold. The latter case appears to be new. For Fermat hypersurfaces of prime degree, Z. Ran proved in [Ran80] that all Hodge classes are obtained inductively by taking a cone construction over algebraic classes coming from two Fermat hypersurfaces of lower degree. This method was proven to work more generally for Fermat hypersurfaces of degree \( d \) where every prime divisor of \( d \) is at least three greater than the dimension of the hypersurface by N. Aoki in [Aok83]. In the language of matrix factorizations, this cone construction is nothing more than the tensor product followed by the forgetful functor which returns you to the \( \mathbb{Z} \)-grading. Hence we are able to recover the described work of Ran and Aoki. It should also be mentioned, that Aoki’s work relies on that of T. Shioda in [Shi79]. In loc. cit., Shioda proves that the Hodge conjecture holds for Fermat hypersurfaces as long as a certain arithmetic condition is satisfied. He proceeds by verifying that this condition holds when the degree of the hypersurface is at most twenty. Furthermore, he recovers Ran’s work in this way, by verifying this arithmetic condition for primes. Aoki then studies the arithmetic condition of Shioda directly. As a consequence of Shioda’s work and the arithmetic theorem Aoki proves, he arrives at the conclusion discussed above.

Using work of Kuznetsov and Orlov, algebraic classes on \( n \)-fold products of a certain \( K3 \) surface are intimately related to algebraic classes on Fermat cubics. Using the positivity of the Hodge conjecture for Fermat cubics, we thus obtain positivity of the Hodge conjecture for these \( n \)-fold products of this \( K3 \) surface as well. Other new cases should follow similarly whenever one has an equivalence between the derived category of coherent sheaves on a space and the category of graded matrix factorizations of a polynomial for which the corresponding hypersurface satisfies the Hodge conjecture.

**Lemma 10.29.** The algebraic classes in \( \text{MF}(k[x, y], x^d + y^d, \mathbb{Z}) \) form a full sublattice of \( \text{HH}_0(\text{MF}(k[x, y], x^d + y^d, \mathbb{Z})) \).

*Proof.* By Lemma 10.19 this reduces to the same statement for \( d \) points. \( \square \)

**Theorem 10.30.** The Hodge conjecture over \( \mathbb{C} \) holds for Fermat hypersurfaces of degree \( d \) when \( d \) is prime, \( d = 4 \), or every prime divisor of \( d \) is greater than \( m + 1 \). Furthermore, for Fermat cubics and quartics, the algebraic classes form a full sublattice of the \((p, p)\) classes.

*Proof.* Let \( X \) be the Fermat hypersurface defined by \( f = x_0^d + \cdots + x_{2m-1}^d \) and \( R \) be the coordinate ring of \( \mathbb{P}^{2m-1} \). We prove that, for prime degree, the dimension of \( \mathbb{C} \)-linear span of the algebraic lattice in \( \text{MF}(R, f, \mathbb{Z}) \) is equal to the dimension of the \( \mathbb{C} \)-linear span of the space of Hodge classes in \( X \) up to expected difference in Lemma 10.19. We may assume that our hypersurface is even dimensional as the odd dimensional case is trivial. The algebraic classes of \( \text{MF}(R, f, \mathbb{Z}) \) orthogonal to the twisted sectors are given by polynomials of degree \( m(d - 2) \) in \( J_f \). Let us consider the basis given by the monomials, \( x_0^{e_0} \cdots x_{2m-1}^{e_{2m-1}} \), with \( m(d - 2) = e_0 + \cdots + e_{2m-1} \) and \( e_i \leq d - 2 \) for all \( i \).
Consider any partition, \( \mathcal{P} \), of the set \( \{0, \ldots, 2m-1\} \) into parts, \( \mathfrak{P} \), of cardinality two. By the Künneth isomorphism we have an inclusion:

\[
\Phi : \bigotimes_{\mathfrak{P} \in \mathcal{P}} \HH_0(\MF(R_{\mathfrak{P}}, f_{\mathfrak{P}}, \mathbb{Z})) \to \HH_0(\bigotimes_{\mathfrak{P} \in \mathcal{P}} \MF(R_{\mathfrak{P}}, f_{\mathfrak{P}}, \mathbb{Z}))_{pe}
\]

\[
\bigotimes_{\mathfrak{P} \in \mathcal{P}} \text{ch}(E_{\mathfrak{P}}) \mapsto \text{ch}(\bigotimes_{\mathfrak{P} \in \mathcal{P}} E_{\mathfrak{P}}).
\]

Now for each part, \( \{a, b\} \), \( x_a^i x_b^{d-2-i} \) is in the \( \mathbb{C} \)-linear span of the algebraic classes in \( \MF(k[x_a, x_b], x_a^d + x_b^d, \mathbb{Z}) \) by Lemma \[10.29\]. Composing the Künneth isomorphism with the grading change to \( \mathbb{C} \) as in Theorem \[10.26\] we obtain all monomials, \( x_0^{e_0} \cdots x_{2m-1}^{e_{2m-1}} \), in the \( \mathbb{C} \)-linear span of the algebraic classes. These monomials have the following properties, \( m(d-2) = e_0 + \cdots + e_{2m-1} \) and \( e_i \leq d-2 \) for all \( i \) and there exists a partition of \( \{0, \ldots, 2m-1\} \) with parts of cardinality at most two with the property that \( e_a + e_b = d-2 \) for each part, \( \{a, b\} \). In the notation of \[Aok83\], these span the Hodge classes corresponding to \( \mathcal{D}^0_0 \). By Theorem A of loc. cit., these span all of the Hodge classes when \( d \) is prime, \( d = 4 \), or every prime divisor of \( d \) is greater than \( m+1 \) (in fact this is a necessary and sufficient condition for these to be all Hodge classes). Furthermore, observe that for \( d = 3 \) and \( d = 4 \) this is all monomials in \( J_f \) of degree \( m(d-2) \).

Let \( Y \) be the \( K3 \) surface such that \( \mathcal{D}^b(\text{coh } Y) \cong \MF(k[x_0, \ldots, x_5], x_0^3 + \cdots + x_5^3, \mathbb{Z}) \), such a \( K3 \) surface exists from work of Kuznetsov in \[Kuz06, Kuz09b\] and Orlov in \[Orl09a\] (in this case the cubic is Pfaffian).

**Theorem 10.31.** *Let \( Y \) be the \( K3 \) surface obtained by the Pfaffian construction from the Fermat cubic fourfold. The Hodge conjecture over \( \mathbb{Q} \) holds for \( n \)-fold products of \( Y \).*

**Proof.** Let \( R = k[x_0, \ldots, x_5] \) and \( f = x_0^3 + \cdots + x_5^3 \). Since \( \mathcal{D}^b(\text{coh } Y) \cong \MF(R, f, \mathbb{Z}) \), we have:

\[
\mathcal{D}^b(\text{coh } Y^n) \cong (\mathcal{D}^b(\text{coh } Y) \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{D}^b(\text{coh } Y))_{pe}
\]

\[
\cong (\MF(R, f, \mathbb{Z}) \widehat{\otimes} \cdots \widehat{\otimes} \MF(R, f, \mathbb{Z}))_{pe}
\]

\[
\cong \MF(k[x_0, \ldots, x_{5n-1}], g_n, M_n),
\]

with \( g_n \) the Fermat cubic in \( 6n-1 \) variables and \( M_n \) the free abelian group generated by \( e_i \) module \( 3e_i = 3e_j \) for \( i, j = 1, \ldots, n \). The first isomorphism is standard, the second is the equivalence in the discussion above the statement of the theorem (actually we use the dg version in \[CT10\]), the third is discussed in Example \[3.8\]. We prove that the algebraic classes form a full sublattice of \( \HH_0(\MF(k[x_0, \ldots, x_{5n-1}], g_n, M_n)) \).

We proceed by induction on \( n \), assuming that the algebraic classes,

\[
v \in \HH_0(\MF(k[x_0, \ldots, x_{(6n-1)-1}], g_{n-1}, M_{n-1})),
\]

form a full sublattice. The base case, \( n = 1 \), is the fact that the Picard rank is 20 which follows directly from Lemma \[10.19\].

Now from the Künneth formula for an \( n \)-fold product, the classes in \( \HH_0(\mathcal{D}^b(\text{coh } Y^n)) \) which are a tensor product of \( n \) classes, \( v_1 \otimes \cdots \otimes v_n \), with \( v_i \in \HH_{j(i)}(\mathcal{D}^b(\text{coh } Y)) \) and \( j(i) \in \{ -2,0,2 \} \) and \( \sum_{i=1}^n j(i) = 0 \) form a \( \mathbb{C} \)-linear basis. Partition the set \( \{1, \ldots, n\} \) into three parts, \( \mathfrak{P}^+, \mathfrak{P}^-, \mathfrak{P}^0 \), corresponding to when the \( j(i) \) are positive, negative, or zero respectively.
Suppose that \( \mathfrak{P}^0 \) is nonempty. As the cardinality of \( \mathfrak{P}^+ \) is equal to the cardinality of \( \mathfrak{P}^- \), i.e. \( |\mathfrak{P}^+| = |\mathfrak{P}^-| := s \), their tensor product, \( \otimes_{i \in \mathfrak{P}^\pm} v_i \), forms a class in \( \text{HH}_0(\text{D}^b(\text{coh} Y^{2s})) \). As \( 2s < n \) (we assumed \( \mathfrak{P}^0 \neq \emptyset \)), any such class is in the \( \mathbb{C} \)-linear span of the algebraic classes by the induction hypothesis and similarly for the class, \( \otimes_{i \in \mathfrak{P}^0} v_i \in \text{HH}_0(\text{D}^b(\text{coh} Y^{n-2s})) \). Hence, the total class, \( v_1 \otimes \cdots \otimes v_n \), is in the \( \mathbb{C} \)-linear span of the algebraic classes in \( \text{HH}_0(\text{D}^b(\text{coh} Y^n)) \).

Now when \( \mathfrak{P}^0 \) is empty, consider the decomposition of part (i) in Polishchuk and Vaintrob’s Theorem 10.26. We get a class lying in the component for \( g = 1 \). The result follows from Corollary 10.28 and Theorem 10.30.

10.5. Noether-Lefschetz Spectra. In this subsection we introduce a new Morita invariant of a dg-category called the Noether-Lefschetz Spectrum of the homotopy category, \( T \), of a saturated dg-category. In the case of a smooth proper variety, the Noether-Lefschetz Spectrum is intended to encode all relevant information about the \( \mathbb{C} \)-linear span of the algebraic classes.

**Definition 10.32.** Let \( J \in \text{HH}^*(T) \) be a ideal which is nilpotent with respect to its action by natural transformations. The Orlov spectrum of \( T \) along \( J \) with respect to \( S \), \( \text{OSpec}_S(T, J) \), is the set of numbers, \( s \), such that there exists an object \( G \in T(J) \), with \( S \subseteq \langle G \rangle_s \) \( S \not\subseteq \langle G \rangle_s \). The Rouquier dimension of \( T \) along \( J \) with respect to \( S \) is the minimum of \( \text{OSpec}_S(T, J) \). The ultimate dimension of \( T \) along \( J \) with respect to \( S \) is the maximum of \( \text{OSpec}_S(T, J) \). The extended Orlov spectrum of \( T \) along \( J \) with respect to \( S \), \( \text{NLSpec}_S(T, J) \), is the same notion with objects, \( G \), replaced by full subcategories of \( T(J) \).

If \( I \subseteq J \) then as \( T(J) \subseteq T(I) \) there is a natural inclusion \( \text{OSpec}(T, I) \subseteq \text{OSpec}(T, J) \) and similarly, \( \text{NLSpec}(T, I) \subseteq \text{NLSpec}(T, J) \). Let \( \mathcal{I} \) be the category whose objects are nilpotent ideals in \( \text{HH}^*(T) \) and whose morphisms are inclusions of ideals.

**Definition 10.33.** Let \( S \) be a full subcategory of a saturated dg-category, \( T \). The Noether-Lefschetz spectrum of \( T \) with respect to \( S \) is the functor:

\[
\text{NLSpec}_S(T, -) : \mathcal{I} \to \text{Sets}.
\]

The Noether-Lefschetz dimension of \( T \) with respect to \( S \) is the function:

\[
\text{NLdim}_S(J) := \min(\text{NLSpec}_S(T, J)).
\]

The Noether-Lefschetz ultimate dimension of \( T \) with respect to \( S \) is the function:

\[
\text{NLudim}_S(J) := \max(\text{NLSpec}_S(T, J)).
\]

When \( S = T \) we drop the subscript and the “with respect to”.

The Noether-Lefschetz spectrum and dimension, allow us to repackage the ideas of the previous subsections.

For example, Proposition \([10.16]\) becomes:

**Proposition 10.34.** Let \( T \) be the homotopy category of a saturated Calabi-Yau dg-category and \( I \) be the kernel of the bulk-boundary map, \( \beta \), intersected with \( \text{HH}^{2*}(T) \). The Noether-Lefschetz dimension restricted to principal ideals is the function which takes \( J \) to the nilpotent order of \( J \) in \( \text{HH}^{2*}(T)/I \).

While Theorem \([10.22]\) becomes:
Theorem 10.35. Let $R = k[x_0, \ldots, x_n]$ and $f$ be an isolated singularity at the origin. Let $I$ be the annihilator of $W(MF(R, f))$. The Noether-Lefschetz dimension of $J$ restricted to principal ideals is the function which takes $J$ to the nilpotent order of $J$ in of $J_f/I$ minus one.

For an interpretation of Theorem 10.23, let $S$ be the category of $\mathbb{Z}$-gradable matrix factorizations i.e. the essential image of the forgetful functor from $MF(R, f, \mathbb{Z})$ to $MF(R, f)$.

Theorem 10.36. Let $X$ be a smooth hypersurface in $\mathbb{P}^n$, defined by a homogeneous polynomial, $f$, of degree $d$, $k \in MF(R, f, \mathbb{Z})$ be the residue field, and $I$ be the ideal of polynomials in $J_f$ which are homotopic to zero for all matrix factorizations in $MF(R, f, \mathbb{Z})$. The Noether-Lefschetz dimension of $MF(R, f, \mathbb{Z})$ takes principal homogeneous ideals coming from polynomials, $J \subseteq HH_{2*}(MF(R, f, \mathbb{Z}))$, to the nilpotent order of $J$ in $J_f/I$ minus one (denote this nilpotent order by $s$). Furthermore, we can analyze any principal homogeneous ideal in $J_f$ with the same formula by:

$$NLdim_s(MF(R, f), J) = s - 1$$

Corollary 10.25 becomes:

Corollary 10.37. Let $X$ be a smooth hypersurface of even dimension in $\mathbb{P}^n$. Suppose the algebraic classes form a full sublattice of $H^{n-1, n-1}(X, \mathbb{C})$. The Noether-Lefschetz dimension takes a principal homogeneous ideal of degree $i$ to $\lfloor \frac{(n+1)(d-2)}{2i} \rfloor$ in both $MF(R, f)$ and $MF(R, f, \mathbb{Z})$.

Let us also comment on some relationships between the extended Orlov spectra along different ideals.

Proposition 10.38. Let $R = \text{Sym}^*V$ be a $M$-graded $k$-algebra with on $V$ a $M$-graded vector space. Let $J, J'$ be nilpotent ideals in $HH^*(MF(R, f, M))$ generated by elements of even degree. Suppose that the ultimate dimension of $MF(R, f, M)$ is finite. One has,

$$NLdim(J') \leq (s + 1)(NLdim(J) + 1) - 1,$$

where $s$ is the dimension of $J/I(J', J^2)$ as a $k$-vector space.

Proof. Let $x_1, \ldots, x_s$ be a generating set for $J/I(J', J^2)$ and lift them to $R$. Consider the Koszul complex, $K(x_1, \ldots, x_s)$, in $R$. Suppose $G$ is any generator in $T(J')$. Then $K(x_1, \ldots, x_s; G)$ belongs to $T(J)$ by Lemma 10.15 and lies in $\langle G \rangle_s$ by construction.

Furthermore, we claim that $K(x_1, \ldots, x_s; G)$ is a strong generator, as $G$ lies in the triangulated closure of $K(x_1, \ldots, x_s; G)$. This is seen by induction on $s$. Namely, as each $x_i$ is nilpotent, one can consider the sequence,

$$K(x_1, \ldots, x_{i-1}; G) \xrightarrow{x_i} \cdots \xrightarrow{x_i} K(x_1, \ldots, x_{i-1}; G).$$

The cones of this sequence are $K(x_1, \ldots, x_i; G)$, and it is eventually zero. By a standard argument involving iterated application of the octahedral axiom (for example see section I.5e in [Sei08a]), $K(x_1, \ldots, x_{i-1}; G)$ lies in the triangulated closure of $K(x_1, \ldots, x_i; G)$. The claim follows.

Since $K(x_1, \ldots, x_s; G)$ belongs to $T(J)$, we have

$$\Theta(K(x_1, \ldots, x_s; G) \leq NLdim(J).$$
Therefore, as $K(x_1, \ldots, x_s; G) \in (G)_s$, 
\[ \Theta(G) \leq (s + 1)(\text{NLudim}(J) + 1) - 1. \]

\[ \square \]

**Remark 10.39.** This inequality is a direct generalization of the bounds found in [BFK10]. In principle, the proof should extend to Noether-Lefschetz spectra in general perhaps with some slight modification, though for this one would have to lift the Koszul complexes to the Hochschild complex.

Let us now discuss the Noether-Lefschetz spectrum of the bounded derived category of coherent sheaves, $\text{D}^b(\text{coh}
 X)$, of a Calabi-Yau hypersurface, $X$, in $\mathbb{P}^n$ defined by $f$. Equivalently by Orlov’s Theorem [ST01] this is a discussion of the Noether-Lefschetz spectrum of the category of graded matrix factorizations of the affine cone, $\text{MF}(R, f, \mathbb{Z})$. Consider the autoequivalence of $\text{D}^b(\text{coh}
 X)$ given by $\{1\} := L_O \circ T_{O(1)}$ where $L_O$ is the left twist by the spherical object, $O$, and $T_{O(1)}$ is tensor with the line bundle $O(1)$ as in [ST01]. Under an appropriate choice of equivalence using Orlov’s Theorem [ST01], the functor $\{1\}$ corresponds to the grading shift of modules (1), see [BFK10] [KM08]. Now, any homogeneous polynomial of degree $i$, $p \in k[x_0, \ldots, x_n]/f$, can be considered both as a section of $O(i)$ and as a natural transformation $\text{Id} \to \{i\}$. By the relations in [ST01], we have a isomorphism of functors, $\{i\} \cong L_O \circ \cdots \circ L_{O(i)}T_{O(i)}$. Now, for any spherical twist by $S$, by construction, there is a canonical natural transformation, $\zeta(S) : \text{Id} \to L_S$. Let $s$ be the natural transformation corresponding to multiplication by the section of $O(i)$ given by $p$. One can easily verify that $p : \text{Id} \to \{i\}$, corresponds to the convolution $\zeta(O) \ast \cdots \ast \zeta(O(i)) \ast s$. Now let $Z$ be the scheme-theoretic support of the section of $O(i)$ corresponding to $p$ and consider the push-forward, $j_* : \text{D}^b(\text{coh}
 Z) \to \text{D}^b(\text{coh}
 X)$. Notice that $s$ vanishes on all objects in the essential image of $j_*$. Therefore we have:

**Proposition 10.40.** Let $X$ be a Calabi-Yau hypersurface in $\mathbb{P}^n$ defined by $f$. Under an appropriate equivalence with $\text{D}^b(\text{coh}
 X)$, the category of graded matrix factorizations scheme-theoretically supported on a homogeneous polynomial of degree $i$, $p \in k[x_0, \ldots, x_n]/f$, contains the essential image of $j_* : \text{D}^b(\text{coh}
 Z) \to \text{D}^b(\text{coh}
 X)$ where $Z$ is the scheme theoretic support of the corresponding section. In particular for any object $A \in \text{D}^b(\text{coh}
 Z)$, the object $j_*A \oplus \cdots \oplus j_*(A)\{n\}$ is a generator lying in $\text{MF}(p)$. Hence, its generation time produces an element of $\text{NLSpec}(\text{D}^b(\text{coh}
 X), (p))$ (in particular its generation time is bounded below by the nilpotent order of $p$ as a natural transformation).

We end this section by displaying the geometry encoded in the Noether-Lefschetz spectrum in the simple case of an elliptic curve. Specifically, let $f$ be a cubic polynomial in $R := k[x, y, z]$ defining a smooth curve in $\mathbb{P}^2$. In the spirit of Theorem [10.36] we compute the Noether-Lefschetz spectrum of $\text{MF}(R, f)$ along the gradable matrix factorizations, $S$. This calculation relies of the techniques found in [BFK10] [Opp10] [Orl09b] to compute the Orlov spectrum of $\text{D}^b(\text{coh}
 E)$, combined with the fact that $\text{D}^b(\text{coh}
 E)$ is equivalent to $\text{MF}(R, f, \mathbb{Z})$ by Theorem [7.1] and Proposition [9.8].

The following lemma is a simple calculation using the Riemann Roch formula:

**Lemma 10.41.** Let $V$ be a vector bundle of slope $\frac{a}{b}$, then the slope $V\{1\}$ is equal to $-\frac{a+3b}{2b+a}$ and the slope of $V\{2\}$ is equal to $-\frac{3b+2a}{b+a}$.
Lemma 10.42. Let $l \in k[x, y, z]$ be a linear polynomial. Let $D$ be the divisor defined by the intersection of this line with the elliptic curve in $\mathbb{P}^2$. Suppose that $V$ is an indecomposable object of $D^b(\text{coh} E)$ which is annihilated by the natural transformation $l : \text{Id} \to \{1\}$. Then, up to homological shift, $[i]$, and grading shift, $\{j\}$, $V$ is isomorphic to $O$ or $O_E$ where $E + E' = D$ for some effective divisors $E, E'$.

Proof. First we remind the reader that, as discussed above, the natural transformation $l$, under an appropriate choice of Orlov’s equivalence, corresponds to the composition of the natural transformation from $\text{Id}$ to $T_{\mathcal{O}(1)}$ induced by multiplication by $l$ thought of as a section convolved with the natural transformation, $\zeta : \text{Id} \to L_\mathcal{O}$, arising from the definition of a spherical twist.

As a consequence of Lemma 10.41 one can view $(-1, \infty]$ as a fundamental domain for the “slope” of an orbit of a vector bundle under the action of $\{1\}$. Up to shift, any indecomposable object in $D^b(\text{coh} E)$ is a semistable vector bundle (we consider torsion sheaves to be semistable vector bundles of slope $\infty$). Let $\mu$ be the slope of $V$. The morphism, $l(V)$ vanishes if and only if the section, $l$, as a map from $V$ to $V(1)$ factors through $O_{\mathcal{O}'}$ for some $r$. Therefore, $l(V) \neq 0$ as long as $\text{Hom}(V, \mathcal{O}) = 0$, covering the case of $\mu \in (0, \infty)$. For $\mu \in (-1, 0)$, shifting by $\{2\}$, we obtain the region $\mu \in (\infty, -3)$. Looking in this region instead we have, $\text{Hom}(\mathcal{O}, V(1)) = 0$, hence once again $l(V) \neq 0$. Similarly, when $\mu = 0$ and $\text{Hom}(\mathcal{O}, V) = 0$ then $l(V) \neq 0$. Now when $\mu = 0$ and $\text{Hom}(\mathcal{O}, V) \neq 0$, $V$ is an iterated extension of $\mathcal{O}$. Since the map $V \to V(1)$ is injective, supposing that $l(V) = 0$, it follows that $V$ is a subsheaf of $O_{\mathcal{O}'}$ for some $r$, which occurs if and only if $V \cong O$. Finally, in the case that $\mu = \infty$ i.e. when $V$ is torsion, then clearly the map induced by the section from $V$ to $V(1)$ can not factor through $O_{\mathcal{O}'}$ for any $r$. However, since $V$ is torsion, the map induced by the section itself may be zero which occurs precisely when $V$ is scheme theoretically supported on the section. The result follows.

Lemma 10.43. Let $f$ be a cubic polynomial in $R := k[x, y, z]$ defining a smooth curve, $E$, in $\mathbb{P}^2$, in $\text{MF}(R, f, \mathbb{Z})$ one has $\mathcal{O}(k \oplus (k(1) \oplus k(2))) = 1$. Equivalently, $S \subseteq \langle k \rangle_1$. Therefore, any polynomial of degree at least 2 in $J_f = HH^*(\text{MF}(R, f))$, vanishes as a natural transformation on $\text{MF}(R, f, \mathbb{Z})$.

Proof. As in the proof of Theorem 10.23 the first two statements are equivalent. The final statement them comes from Theorem 10.23.

Choosing the equivalence of Theorem 7.1 appropriately, $k \in \text{MF}(R, f, \mathbb{Z})$ corresponds to $\mathcal{O}_E \in D^b(\text{coh} E)$ and the functor $\{1\}$ on $\text{MF}(R, f, \mathbb{Z})$ corresponds to $L_\mathcal{O} \circ T_{\mathcal{O}(1)}$, see [BFK10] [KMV08]. It follows that $k(2) \cong k(-1)[2]$ corresponds to $\mathcal{O}(-1)$. Using Proposition 4.3 of [Opp10] with $\mathcal{F}_1 = \mathcal{O}(-1)$ and $\mathcal{F}_2 = \mathcal{O}$, we see that all semistable vector bundles of slope, $-2 < \mu \leq -1$, are contained in $\langle \mathcal{O}(-1) \oplus \mathcal{O} \rangle_1$ (our indexing is one less than the one found there) from part (3). This is a fundamental domain for the slopes of the orbits.

Lemma 10.44. Let $D$ be a divisor on $E$. If the degree of $D$ is less than three, then the generation time of the orbit of $\mathcal{O}_D$ is three. If the degree of $D$ is greater than or equal to three, then the generation time of the orbit of $\mathcal{O}_D$ is two.

Proof. Let $G = \mathcal{O}_D \oplus \mathcal{O}_D \{1\} \oplus \mathcal{O}_D \{2\}$.

If the degree of $D$ is less than or equal to two then, consider a point $q$ with $q$ linearly equivalent to $D - q$ (in the degree two case) or $2D - q$ (in the degree one case). The sequence of maps,

$$0 \to \mathcal{O}_q \to \mathcal{O}_D \{1\}(-q)[1] \to \mathcal{O}_D \{1\}(D - q)[1] \to \mathcal{O}_q[1],$$
is a nonzero ghost sequence of length three for \( G \). If the degree of \( D \) is greater than or equal to three then, consider any point \( q \) with which does not intersect \( D \). The sequence of maps,
\[
\mathcal{O}_q \to \mathcal{O}_D\{1\}(-q)[1] \to \mathcal{O}_q[1],
\]
is a nonzero ghost sequence of length two for the orbit of \( \mathcal{O}_D \). These ghost sequences give the lower bounds.

For the upper bounds, notice that \( \mathcal{O}_D\{1\}(\pm nD), \mathcal{O}_D\{1\}(\pm nD) \in \langle G \rangle_n \). When the degree of \( nD \) is at least 3, one can apply Proposition 4.3 of [Opp10]. Applying part (2) to \( F_1 = \mathcal{O}_D\{1\}(-nD) \) and \( F_2 = \mathcal{O}_D\{1\}(D) \) all semistable vector bundles of slope, \( \mu > 0 \) belong to \( \langle G \rangle_{n+1} \) (one can also obtain all torsion sheaves using these two, see [Ori09b, Opp10]). For \(-1 < \mu \leq 0 \), we may use part (3) applied to \( F_1 = \mathcal{O}_D\{2\}(-nD) \) and \( F_2 = \mathcal{O}_D\{2\} \). We have obtained all slopes in the fundamental domain \((-1, \infty)\). The upper bounds follow. \( \square \)

**Lemma 10.45.** The generation time of any orbit of \( \text{D}^b(\text{coh} \ E) \) is at most three.

*Proof.* Let \( G \) the orbit of \( V \in \text{D}^b(\text{coh} \ E) \), we may assume \( V \) is not torsion nor is any element of its orbit (equivalently the slope of \( V \) is not \(-1 \) or \(-2 \)) as we have already calculated this in the previous lemma. For any linear polynomial, \( l : V \to V\{1\} \) is annihilated by \( l \). By Lemma 10.42 the cone either contains \( \mathcal{O} \) as a summand or is a torsion sheaf scheme theoretically supported on the intersection of the line with the elliptic curve. Supposing that \( \mathcal{O} \notin \langle G \rangle_1 \), we see that there are infinitely many nonisomorphic torsion sheaves in \( \langle G \rangle_1 \). Therefore, for arbitrary degree, \( d \), there exists a divisor \( D \) of degree \( d \) such that \( V(D) \in \langle G \rangle_2 \). Let \( X \) be a summand of \( G \) whose slope lies in the domain \((-3, -2) \cup [0, \infty) \) (this is a fundamental domain up to the end points whose case we have already covered). If the slope of \( X \) lies in \([0, \infty) \), we obtain a fundamental domain in \( \langle G \rangle_3 \) by applying Proposition 4.3 of [Opp10] part (1) to \( F_1 = X, F_2 = X(D) \) with the degree of \( D \) sufficiently large. If the slope of \( X \) lies in \((-3, -2) \), we obtain a fundamental domain in \( \langle G \rangle_3 \) by applying Proposition 4.3 of [Opp10] part (3) to \( F_1 = X, F_2 = X(D) \) with the degree of \( D \) sufficiently large.

\( \square \)

Let \( J \) be a nilpotent ideal in \( J_f \). By the Lemma 10.42 the category \( \text{MF}(R, f, Z)(J) \) only depends on \( J := J \cap m/m^2 \). Hence, \( \text{NLSpec}_S(\text{MF}(R, f), J) \) only depends on \( J \). The vanishing locus of \( J \) defines a linear space, \( A \subseteq \mathbb{P}^2 \), let \( Z := A \cap E \). Notice that by Lemma 10.42 \( T(J) \) and therefore, \( \text{NLSpec}(T, J) \), depends only on \( Z \). Specifically we have the following:

**Proposition 10.46.** Let \( f \) be a cubic polynomial in \( R := k[x, y, z] \) defining a smooth curve in \( \mathbb{P}^2 \). Let \( J \) be a nilpotent ideal in \( J_f = HH^2(T)(\text{MF}(R, f, )) \). With the notation above,

\[
\text{NLSpec}_S(\text{MF}(R, f), J) = \begin{cases}
\{1\} & \text{if } Z = E, \\
\{1, 2, 3\} & \text{if } Z \text{ is a divisor of degree three}, \\
\{1, 3\} & \text{if } Z \text{ is a divisor of degree one}, \\
\{1, 2, 3\} & \text{if } Z = \emptyset.
\end{cases}
\]

*Proof.* If \( Z = E \), then \( T(J) \) is just the additive category generated by shifts of the residue field by Lemma 10.21. Therefore, there is really only one object to consider and its generation time is one by Lemma 10.44.

If \( Z \) is a divisor of degree one, then there are three possibilities: the orbit of \( \mathcal{O}_Z \), the orbit of \( \mathcal{O} \), and the orbit of \( \mathcal{O} \oplus \mathcal{O}_Z \). By Lemma 10.43 the orbit of \( \mathcal{O} \) and \( \mathcal{O} \oplus \mathcal{O}_Z \) have generation time one while the orbit of \( \mathcal{O}_Z \) has generation time three by Lemma 10.44.
If \( Z \) is a divisor of degree three, we have the orbit of \( \mathcal{O}_Z \) with generation time two, the orbit of \( \mathcal{O}_p \) where \( p + E' = Z \) for some effective divisor \( E' \) with generation time three, and the orbit of \( \mathcal{O} \) with generation time one. All of these numbers are computed in Lemma \[10.44\].

If \( Z = \emptyset \), then \( T(J) = \mathcal{D}^b(\text{coh } E) \) hence, \( \text{NLSpec}(T, J) = \text{OSpec}(\text{MF}(R, f)) = \{1, 2, 3\} \) by Lemma \[10.44\] and Lemma \[10.45\]. \[\square\]

**Appendix A. Hochschild structures on dg-categories**

A.1. Preliminaries on dg-categories. Let us recall some notation and results of \[\text{To"e07}\].

A dg-category, \( \mathcal{C} \), over \( k \) is a category enriched over \( C(k) \). Specifically, \( \mathcal{C} \) is a category where each Hom-set \( \text{Hom}_\mathcal{C}(x, y) \) is a chain complex of \( k \)-modules. If \( \mathcal{C} \) and \( \mathcal{D} \) are dg-categories, we can form the category \( \mathcal{C} \otimes_k \mathcal{D} \) whose objects are pairs \( x \in \mathcal{C}, y \in \mathcal{D} \) with Hom-sets,

\[
\text{Hom}((x_1, y_1), (x_2, y_2)) := \text{Hom}_\mathcal{C}(x_1, x_2) \otimes_k \text{Hom}_\mathcal{D}(y_1, y_2),
\]

where the tensor product is as \( \mathbb{Z} \)-graded complexes. The functor, \( \otimes_k \), gives a closed symmetric monoidal structure on \( \text{dgcat} \). The dg-category, \( \mathcal{C} \text{-Mod} \), is the dg-category of dg-functors \( \mathcal{C} \to C(k) \). There is a Yoneda functor, \( \mathbf{h}_\mathcal{C} : \mathcal{C} \to \mathcal{C}^{\text{op}} \text{-Mod} \). From \( \mathcal{C} \), we can produce an ordinary category, \( [\mathcal{C}] \), known as the homotopy category of \( \mathcal{C} \), whose objects are the same as \( \mathcal{C} \) but with \( \text{Hom}_{[\mathcal{C}]}(x, y) := H^0(\text{Hom}_\mathcal{C}(x, y)) \).

A morphism, \( f : \mathcal{C} \to \mathcal{D}, \) of dg-categories is called quasi-fully-faithful if the induced morphism, \( f_* : \text{Hom}_\mathcal{C}(x, y) \to \text{Hom}_\mathcal{D}(f(x), f(y)) \), is a quasi-isomorphism for each pair of objects \( x, y \in \mathcal{C} \). A morphism, \( f \), is called quasi-essentially-surjective if \([f] : [\mathcal{C}] \to [\mathcal{D}] \) is essentially surjective. We say \( f \) is a quasi-equivalence if it is quasi-fully-faithful and quasi-essentially-surjective. As most questions of geometric interest factor through quasi-equivalence, one is naturally led to consider the localization of the category of dg-categories, \( \text{dg-cat} \), by the class of quasi-equivalences. (To take the proper set-theoretic precautions we assume that the dg-categories, \( \mathcal{C} \) and \( \mathcal{D} \), are essentially-small. For the general solution, we refer the reader to loc. cit.) We denote this by \( \text{Ho}(\text{dg-cat}) \), and following Toën, we denote the Hom-set from \( \mathcal{C} \) to \( \mathcal{D} \) in \( \text{Ho}(\text{dg-cat}) \) by \([\mathcal{C}, \mathcal{D}]\).

One common method to assert some control over such a localization problem is to equip the initial category with a model category structure, whose weak equivalences are quasi-equivalences. This is was done by G. Tabuada in \[\text{Tab05}\]. For any dg-category, \( \mathcal{C} \), we also have a model category structure on \( \mathcal{C}^{\text{op}} \text{-Mod} \), defined in \[\text{To"e07}\]. The derived category of \( \mathcal{C} \), \( \mathcal{D}(\mathcal{C}) \), is defined to be the localization of \( \mathcal{C} \text{-Mod} \) by the weak equivalences of this model structure. One can check that the image of the Yoneda functor consists of fibrant and cofibrant \( \mathcal{C}^{\text{op}} \)-modules and induces a fully-faithful functor \([\mathcal{C}] \to \mathcal{D}(\mathcal{C}) \). One can check that these model structures are \( C(k) \)-model structures.

In general, with the additional structure of a model category, one can attempt to describe the morphisms spaces in \( \text{Ho}(\text{dg-cat}) \) in more tractable terms. With a symmetric monoidal model structure, the internal Hom can be derived to provide an internal Hom on the localization. However, \( \otimes_k \) does not provide dg-cat with the structure of a symmetric monoidal model structure forcing Toën to work harder in \[\text{To"e07}\]. \( \otimes_k \) does provide a symmetric monoidal model structure on a closely-related, Quillen-equivalent category, \[\text{Tab10}\].) Instead, Toën describes the internal Hom in \( \text{Ho}(\text{dg-cat}) \) as follows: given a dg-category, \( \mathcal{C} \), we say a module \( F \in \mathcal{C}^{\text{op}} \text{-Mod} \) is quasi-representable if it is isomorphic in \( \mathcal{D}(\mathcal{C}) \) to \( \mathbf{h}_\mathcal{C} \) for some \( x \in \mathcal{C} \). Given two dg-categories, \( \mathcal{C} \) and \( \mathcal{D} \), we say a bimodule \( Q \in \mathcal{C} \otimes_k \mathcal{D}^{\text{op}} \text{-Mod} \) is right quasi-representable if \( Q(x, \bullet) \in \mathcal{D}^{\text{op}} \text{-Mod} \) is quasi-representable for all \( x \in \mathcal{C} \). Let us denote the
Theorem A.1. \( \mathbf{R} \text{Hom}(C, D) \) exists and is naturally isomorphic to \( \text{Int}(C \otimes_k D^{\text{op}}\text{-Mod}^{\text{rf}}) \) where \( C \otimes_k D^{\text{op}}\text{-Mod}^{\text{rf}} \) is the full subcategory of right quasi-representable bimodules and \( \text{Int} \) denotes the full subcategory of fibrant and cofibrant objects.

Given a dg-category, \( C \), we let \( \widehat{C} \) be the dg-category, \( \text{Int}(C^{\text{op}}\text{-Mod}) \). The dg-category, \( \widehat{C} \), provides an enhancement of \( D(C^{\text{op}}) \) as \( \widehat{C} \cong D(C^{\text{op}}) \). Furthermore, \( D(C^{\text{op}}) \) is naturally a compactly-generated triangulated category. The category, \( \widehat{C}_{\text{pe}} \), denotes the subcategory of \( \widehat{C} \) which are isomorphic to compact objects in \( D(C^{\text{op}}) \). In a certain sense, \( \widehat{C}_{\text{pe}} \) is a “triangulated and idempotent completion” of \( \widehat{C} \). Let \( \text{Perf} C \) denote the category \( \text{Ho}(\widehat{C}_{\text{pe}}) \) and call \( \text{Perf} C \) the perfect derived category of \( C \).

Using the description of internal Homs of \( \text{Ho}(\text{dg-cat}) \) from Theorem A.1, Toën proves a derived Morita statement for dg-categories. Let \( \mathbf{R} \text{Hom}_c(C, D) \) denote the full subcategory of \( \mathbf{R} \text{Hom}(C, D) \) consisting of \( u \in \mathbf{R} \text{Hom}(C, D) \) where \( [u] : [C] \to [D] \) commutes with direct sums. The following is Corollary 7.6 of loc. cit. and serves as the analog of the Morita theorem for dg-categories.

Corollary A.2. In \( \text{Ho}(\text{dg-cat}) \), there are natural isomorphisms

\[
\mathbf{R} \text{Hom}_c(\widehat{C}, \widehat{D}) \cong C^{\text{op}} \otimes_k D.
\]

Recall that, for a dg-category, \( C \), we have a natural trace functor,

\[
\text{Tr}_C : C^{\text{op}} \otimes_k C \to C(k),
\]

which is uniquely determined by \( \text{Tr}_C(c \otimes_k c') = \text{Hom}_C(c, c') \). The Hochschild homology of \( C \) is the cohomology of \( \text{Tr}_C(\Delta_C) \), computed in \( D(C^{\text{op}} \otimes_k C) \), where \( \Delta_C \) is the \( C \)-bimodule,

\[
\Delta_C(c, c') = \text{Hom}_C(c', c),
\]

corresponding to the identity functor. From this characterization, it is clear that Hochschild homology is invariant under Morita equivalence. In the case, \( C = A \) for some dg-algebra, \( A \), it easy to identify the trace with \( A^L \otimes_{A^{\text{op}} \otimes_k A} \bullet \). Thus, one sees that the usual definition of Hochschild homology, in terms of derived Tor’s of the diagonal, agrees with the definition via trace above.

Another definition of the Hochschild homology of a small dg-category is given by the cohomology of a standard bar complex, see [Ke09]. As the standard bar complex comes from a resolution of \( \Delta_C \) via representable functors, which are fibrant and cofibrant, Polishchuk and Vaintrob note that the definitions via derived trace and the standard bar complex coincide in [PV11]. The final common definition of Hochschild homology is as derived natural transformations from the anti-Serre kernel to the identity kernel, see [CW10]. When \( C \) is nice enough to admit an anti-Serre kernel, we show that this definition coincides with the other two.

The final definition has the advantage that it is internal to a particular bi-category. Meaning, one can manipulate string diagrams to deduce nontrivial relations as in loc. cit.

The following definition is a strengthening of the notion of a (weak) Serre functor.
Remark A.5. It is clear how to extend the definitions in the case that $\text{Hom}$ to show that $\Delta^*$

Example A.4. Let $S_{\text{perf}}$ be an essentially-small dg-category over a field $k$. The trace functor for any dg-category underlying $D(k)$ is quasi-representable in $\text{Ho}(\text{Dense DG Cat})$. We leave this for later work.

Proposition A.6. A Serre kernel for $C$ always exists. An anti-Serre kernel for $C$ exists if and only if $\text{Tr}_C$ is co-continuous. Serre kernels and anti-Serre kernels are unique.

Proof. $S$ a cohomological functor that sends coproducts to products as $\text{Tr}_C$ is continuous. Brown representability, see [Nee96], for $D(C^{\text{op}} \otimes_k C)$ assures that $S$ exists. Brown representability for the dual, [Kra00], states that $\text{Tr}_C$ is representable in $D(C^{\text{op}} \otimes_k C)$ if and only if it commutes with products.

As $S$ and $S^{-1}$ are solutions to representability problems, they are unique if they exist. □

Given an element of $K \in C^{\text{op}} \otimes_k D$, we get a dg-functor $\Phi_K : C \to \hat{D}$ which assigns the module, $K(c, \bullet) : D^{\text{op}} \to C(k)$, to an object $c \in C$. The following lemma justifies the notation:

Lemma A.7. If $S$ is the Serre kernel for $C$, then there are natural isomorphisms

$$\text{Hom}_{D(C)}(c', \Phi_S(c)) \cong \text{Hom}_{D(C)}(c, c')^*.$$  

for $c \in C$ and $c' \in \hat{C}$.

Proof. If $S$ is quasi-representable, then there is an element $S \in C^{\text{op}} \otimes_k C$ such that the functor, $\text{Hom}_{C^{\text{op}} \otimes_k C}(\bullet, S)$, is isomorphic to $\Psi$ in $\text{Ho}(C^{\text{op}} \otimes_k C)$. Evaluate at $h_{\text{cop}} \otimes_k c'$ and apply the Yoneda lemma to verify the assertion. □

A.2. Polite duality for dg-cat. The purpose of this subsection of the appendix is to extend the results of Căldăraru and Willerton in [CW10] to the bicategory of saturated dg-categories over a field, $k$. To do so, one needs to demonstrate a couple of basic properties of this bicategory, perhaps the most technical of which, is the notion of a polite duality between its 1-morphisms (definition to follow). The signs have been suppressed, although one can easily verify that all the diagrams indeed commute (using the Koszul sign convention).

A bicategory, also called a weak 2-category, is a category weakly enriched over $\text{Cat}$, the category of small categories. Similar to a 2-category, see loc. cit., the 1-morphisms of a bicategory are themselves categories, the difference being that the associativity and unity laws of enriched categories hold only up to coherent isomorphism. This is relatively inconsequential as the various notions of 2-categories in loc. cit. produce the same theories as the notion of a bicategory.
Recall from [TV07], that a saturated dg-category, $\mathcal{C}$, is, by definition, a dg-category satisfying the following conditions:

- $\mathcal{C}$ is proper: for any pair of objects $x, y$ in $\mathcal{C}$ $\text{Hom}_C(x, y)$ has bounded and finite dimensional cohomology and the homotopy category of $\mathcal{C}$-modules has a compact generator,
- $\mathcal{C}$ is smooth: $\mathcal{C}$, considered as $\mathcal{C}$-bimodule, is a perfect object of $\mathcal{C}^{\text{op}} \otimes_k \mathcal{C}$.
- $\mathcal{C}$ is triangulated: the Yoneda embedding $\mathcal{C} \to \widehat{\mathcal{C}}_{\text{pe}}$ is a quasi-equivalence.

Recall that $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent if $\widehat{\mathcal{C}}_{\text{pe}}$ and $\widehat{\mathcal{D}}_{\text{pe}}$ are quasi-equivalent. As a saturated dg-category, $\mathcal{C}$, admits a compact generator by definition, $\mathcal{C}$ is always Morita equivalence to dg-category of perfect modules over some dg-algebra, $A$. Properness implies that $A$ has bounded and finite-dimensional cohomology. Smoothness implies that $A$, as an $A^{\text{op}} \otimes_k A$-module, is perfect.

The following lemma is Lemma 2.8.3 of [TV07].

**Lemma A.8.** Let $\mathcal{C}$ be a smooth and proper dg-category. A bimodule $K \in \mathcal{C}^{\text{op}} \otimes_k \mathcal{D}$ is perfect if and only the dg-functor $\Phi_K : \mathcal{C}^{\text{op}} \to \mathcal{D}$ factorizes

$$
\begin{array}{ccc}
\mathcal{C}^{\text{op}} & \xrightarrow{\Phi_K} & \mathcal{D} \\
\downarrow & & \downarrow
\end{array}
$$

in $\text{Ho}(\text{dgcat}_k)$.

**Corollary A.9.** Let $\mathcal{C}$ be a smooth and proper dg-category. In $\text{Ho}(\text{dgcat}_k)$, there is a natural isomorphism

$$
\text{RHom}(\widehat{\mathcal{C}}_{\text{pe}}, \widehat{\mathcal{D}}_{\text{pe}}) \cong \left( \mathcal{C}^{\text{op}} \otimes_k \mathcal{D} \right)_{\text{pe}}.
$$

**Proof.** We have the following collection of natural morphisms:

$$
\begin{array}{ccc}
\text{RHom}(\widehat{\mathcal{C}}_{\text{pe}}, \widehat{\mathcal{D}}_{\text{pe}}) & \xrightarrow{1} & \text{RHom}(\mathcal{C}, \widehat{\mathcal{D}}_{\text{pe}}) \\
\downarrow & & \downarrow
\end{array}
$$

\begin{array}{ccc}
\text{RHom}(\mathcal{C}, \widehat{\mathcal{D}}_{\text{pe}}) & \xrightarrow{2} & \text{RHom}(\mathcal{C}, \widehat{\mathcal{D}}) \\
\downarrow & & \downarrow
\end{array}
$$

\begin{array}{ccc}
\text{RHom}(\mathcal{C}, \widehat{\mathcal{D}}) & \xrightarrow{\phi} & \mathcal{C}^{\text{op}} \otimes_k \mathcal{D}
\end{array}
$$

where 1 is the induced by the inclusion $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}_{\text{pe}}$ and 2 is induced by the inclusion $\widehat{\mathcal{D}}_{\text{pe}} \hookrightarrow \widehat{\mathcal{D}}$. The category, $\text{RHom}(\mathcal{C}, \widehat{\mathcal{D}})_{\text{pe}}$, is full subcategory of $\text{RHom}(\mathcal{C}, \widehat{\mathcal{D}})$ consisting of morphisms factorized by $\widehat{\mathcal{D}}_{\text{pe}}$ in $\text{Ho}(\text{dgcat}_k)$. The functor, 1, is a quasi-equivalence by Theorem 7.2
Lemma A.8 implies that the restriction of $\Phi$ to $C^{op} \otimes_k D_{pe}$ is a quasi-equivalence with $R\text{Hom}(C, \hat{D})_{pspe}$.

Thus, to verify the claim of the corollary, we need to check that $2$ is quasi-essentially surjective. Let $\Psi : C \to \hat{D}$ be an object of $R\text{Hom}(C, \hat{D})_{pspe}$ that factors through $i : \hat{D}_{pe} \to \hat{D}$ in $\text{Ho}(dgcat_k)$ via a map $\phi : C \to \hat{D}_{pe}$. As there is bijection between the isomorphism classes of objects of $[R\text{Hom}(C, \hat{D}_{pe})]$ and $[C, \hat{D}_{pe}]$, there is an object $\tilde{\phi}$ of $R\text{Hom}(C, \hat{D}_{pe})$ isomorphic to $\phi$ in $[R\text{Hom}(C, \hat{D}_{pe})]$. Consequently, $[\Phi]$ is isomorphic to $[i \circ \tilde{\phi}]$ in $[C, \hat{D}] \cong [R\text{Hom}(C, \hat{D})]$. □

**Definition A.10.** Let $DGSatk$ be the bicategory whose objects are essentially-small saturated dg-categories over $k$. The 1-morphisms from $C$ to $D$ are objects of the derived category of bimodules, $[R\text{Hom}(\hat{C}_{pe}, \hat{D}_{pe})]$. The 2-morphisms are the morphisms in $[R\text{Hom}(\hat{C}_{pe}, \hat{D}_{pe})]$. Composition of 1-morphisms and horizontal composition of 2-morphisms given by composition of functors and vertical composition of 2-morphisms just being the usual composition in $[R\text{Hom}(\hat{C}_{pe}, \hat{D}_{pe})]$. We call $DGSatk$ the **bicategory of saturated dg-categories**.

We first reduce from $DGSatk$ to an equivalent bicategory. Let $DGsalg_k$ be the sub-bicategory whose objects are the dg-categories of perfect left modules over smooth and proper dg-algebras over $k$. To set notation, we let $\text{Mod} A$ denote the dg-category of left $A$-modules and we let $\text{Perf} A$ be the derived category of perfect $A$-modules.

**Proposition A.11.** The inclusion of the full sub-bicategory $DGsalg_k$ into $DGSatk$ induces a biequivalence of bicategories.

**Proof.** Recall that $F : B_1 \to B_2$ is a biequivalence of bicategories if it is a morphism of bicategories and there exists another morphism of bicategories, $G : B_2 \to B_1$, such that $FG$ and $GF$ are equivalent to the identities in their respective functor bicategories. Similar to the criteria for equivalences of categories, $F$ is a biequivalence if and only if $F$ is biessentially surjective and $F_{xy} : B_1(x, y) \to B_2(Fx, Fy)$ is an equivalence of the categories for each pair of objects $x, y$ of $B_1$, see Lemma 3.1 of [Gur11] for a proof.

As a full sub-bicategory, the inclusion is, by definition, a morphism of bicategories which is equivalence for the categories $[R\text{Hom}(\hat{A}_{pe}, \hat{B}_{pe})]$ where $A$ and $B$ are smooth and proper dg-algebras. We are left with showing that for any saturated dg-category, $C$, there is a smooth and proper dg-algebra, $A$, and 1-morphisms, $f \in [R\text{Hom}(\hat{A}_{pe}, \hat{C}_{pe})], g \in [R\text{Hom}(\hat{C}_{pe}, \hat{A}_{pe})]$ so that $f \circ g$ and $g \circ f$ are isomorphic to the identity functors in their respective categories. By assumption, $\hat{C}$ has a generator, $G$. We set $A := R\text{Hom}(G, G)$. By Lemma 6.1 of [Kel94], we have induced quasi-equivalences, $R\text{Hom}(G, \bullet) : \hat{C}_{pe} \to \hat{A}_{pe}, \bullet \otimes_A G : \hat{A}_{pe} \to \hat{C}_{pe}$. We set $f := R\text{Hom}(G, \bullet)$ and $g := \bullet \otimes_A G$. □

Let us now introduce a slightly different bicategory. $DGsalg_k$ has as objects smooth and proper dg-algebras over $k$. The space of 1-morphisms between $A$ and $B$ is $\text{Perf} A^{op} \otimes_k B$. 2-morphisms are morphisms in $\text{Perf} A^{op} \otimes_k B$. Composition of 1-morphisms and horizontal composition of 2-morphisms given by the derived tensor product of bimodules and vertical composition of 2-morphisms just being the usual composition. There is functor, $\Phi : DGsalg_k \to DGsalg_k$, which is the identity on objects and sends a bimodule $K$ to the functor $\Phi_K$.

**Proposition A.12.** $\Phi$ is a biequivalence.
Proof. It is clear that \( \Phi \) is biessentially surjective. Equivalences of the morp hism categories is an immediate consequence of Toën's results, Corollary 7.6 of [Toë07].

Having reduced to the case of smooth and proper dg algebras, \( \text{DGSalg}_k \), we now recall some basic adjunctions between functors determined by bimodules. We will use \( A, B, \) and \( C \) to stand for smooth and proper dg-algebras. Given a dg-algebra, \( A, A^* \) will denote the bimodule, \( \text{Hom}_k(A, k) \). Let \( P \in \text{Mod} A^{\text{op}} \otimes_k B \). We get an induced functor, \( P \otimes_A - : \text{Mod} A \to \text{Mod} B \). We let \( P^\nu \) be the element of \( \text{Mod} B^{\text{op}} \otimes_k A \) given by \( \text{Hom}_{A^{\text{op}} \otimes_k B}(P, A \otimes_k B) \).

The first necessity is the standard tensor-Hom adjunction.

**Lemma A.13.** For objects, \( M \in \text{Mod} A, N \in \text{Mod} B \), there are natural isomorphisms of chain complexes

\[
\theta_P : \text{Hom}_B(P \otimes_A M, N) \to \text{Hom}_A(M, \text{Hom}_B(P, N))
\]

**Lemma A.14.** For objects, \( R \in \text{Mod} C^{\text{op}} \otimes_k A, P \in Q \in \text{Mod} C^{\text{op}} \otimes_k B \), there are natural morphisms of chain complexes

\[
\nu_1^P : \text{Hom}_{C^{\text{op}} \otimes_k B}(Q, P \otimes_A R) \to \text{Hom}_{C^{\text{op}} \otimes_k A}(P^\nu \otimes_B B^* \otimes_B R)
\]

\[
\nu_2^P : \text{Hom}_{C^{\text{op}} \otimes_k A}(R, A^* \otimes_A P^\nu \otimes_B Q) \to \text{Hom}_{C^{\text{op}} \otimes_k B}(P \otimes_A R, Q)
\]

\[
\nu_3^P : \text{Hom}_{C^{\text{op}} \otimes_k A}(R, P^\nu \otimes_B Q) \to \text{Hom}_{C^{\text{op}} \otimes_k B}(P \otimes_A A^* \otimes_A R, Q)
\]

\[
\nu_4^P : \text{Hom}_{C^{\text{op}} \otimes_k B}(Q, B^* \otimes_B P \otimes_A R) \to \text{Hom}_{C^{\text{op}} \otimes_k A}(P^\nu \otimes_B R, Q)
\]

which induce adjunctions,

\[
P^\nu \otimes_B B^* \otimes_B \bullet 
\]

\[
\gamma_1 P \otimes_A \bullet 
\]

\[
\alpha_2 A^* \otimes_A P^\nu \otimes_B \bullet 
\]

\[
\gamma_3 P^\nu \otimes_B \bullet 
\]

\[
\gamma_4 B^* \otimes_B P \otimes_A \bullet 
\]

in the derived category.

**Proof.** Let us start with \( \nu_1^P \). There is a natural map

\[
\alpha_P : P \otimes_A R \to \text{Hom}_A(P^\nu \otimes_B B^*, R)
\]

\[
\alpha_P(p \otimes r)(\phi \otimes \lambda) := \lambda(\phi(p))r.
\]

where \( \lambda \) operates on an element of \( A \otimes_k B \) by contraction with the \( B \) term. The morphism of chain complexes, \( \nu_1^P \) is then defined as the composition making the following diagram commute.

**Next, we define \( \nu_2^P \). There is similarly a natural map**

\[
\beta_P : A^* \otimes_A P^\nu \otimes_B Q \to \text{Hom}_A(P, Q)
\]

\[
\beta_P(\lambda \otimes \phi \otimes q)(p) := \lambda(\phi(p))q.
\]

Now, \( \nu_2^P \) is defined as the composition making the following diagram commute.
Similarly, there is a natural map,
\[
\gamma_P : P^\vee \otimes_B Q \to \Hom_B(P, \Hom_k(A^*, Q))
\]
\[
\gamma_P(\phi \otimes q)(p)(\lambda) := \lambda(\phi(p))q,
\]
and \(\nu_3^P\) is the composition which makes the following diagram commute.

Finally, using the natural map,
\[
\delta_P : B^* \otimes_B P \otimes_A R \to \Hom_{A^{op}}(P^\vee, R)
\]
\[
\delta_P(\lambda \otimes p \otimes r)(\phi) := \lambda(\phi(p))r,
\]
we define \(\nu_4^P\) as the composition which makes the following diagram commute.

To check that each \(\nu_i\) induces an adjunction on the derived category, it suffices, by a standard argument, to check that statements for \(P = A \otimes_k B, Q = B \otimes_k C, R = A \otimes_k C\) where it is clear. \(\Box\)

**Lemma A.15.** There are natural morphisms of chain complexes,
\[
\Hom_A(M, A^* \otimes_A N) \to \Hom_A(M, \Hom_k(\Hom_A(N, A), k)) \leftarrow \Hom_k(\Hom_A(N, M), k),
\]
such that the resulting maps in the derived category are isomorphisms.

**Proof.** We have the natural map
\[
\epsilon_A : A^* \otimes_A N \to \Hom_k(\Hom_A(N, A), k)
\]
\[
\epsilon_A(\lambda \otimes n)(\mu) := \lambda(\mu(n)).
\]
The first map in the sequence is
\[ \epsilon \circ - : \text{Hom}_A(M, A^* \otimes_A N) \to \text{Hom}_A(M, \text{Hom}_k(\text{Hom}_A(N, A), k)). \]

We also have the natural map
\[ \rho_A : \text{Hom}_A(N, A) \otimes_A M \to \text{Hom}_A(N, M) \]
\[ \rho_A(\mu \otimes m)(n) = \mu(n)m. \]

The second map is the composition
\[ \text{Hom}_k(\text{Hom}_A(N, M), k) \to \text{Hom}_A(M, \text{Hom}_k(\text{Hom}_A(N, A), k)) \]
\[ \rho_A \circ - \theta_{\text{Hom}_A(N, A)} \to \text{Hom}_A(\text{Hom}_A(N, A) \otimes_A M, k) \]

To check the maps are isomorphisms in the derived category it suffices to consider the case
\[ M = N = A \] where the result is clear. \[ \square \]

As demonstrated by D. Shklyarov [Shk07a], this shows that the bimodule, \( A^* \otimes_A \bullet \cong \text{Hom}_k(\text{RHom}_A(\bullet, A), k) \), is the Serre functor on Perf \( A \). Let us identify the Serre kernel and anti-Serre kernel.

Lemma A.16. Let \( A \) be a smooth and proper dg-algebra over \( k \). \( A^* \) is the Serre kernel and \( A^{R^\vee} \) is the anti-Serre kernel for \( A \).

Proof. The natural morphisms in Lemma A.15 give quasi-isomorphisms between the functors, \( S \) and \( \text{Hom}(\bullet, A^*) \), on the full dg-subcategory consisting of box products. This isomorphism naturally extends to all perfect complexes.

We have natural maps
\[ \text{RHom}_{A^\text{op} \otimes_k A}(A^{R^\vee}, M) \to \text{RHom}_{A^\text{op} \otimes_k A}(A^{R^\vee}, M) \leftarrow A^{R^\vee R^\vee} \otimes_{A^\text{op} \otimes_k A} M \leftarrow A \otimes_{A^\text{op} \otimes_k A} M \]
where the first map comes from adjunction, the second from evaluation, and the third from the quasi-isomorphism, \( A \to A^{R^\vee R^\vee} \). These are quasi-isomorphisms whenever \( M \) is perfect. \[ \square \]

Now that we have identified the Serre kernel, we need to verify the following three facts:

- The Serre kernel on Perf \( A^\text{op} \otimes_k B \) is \( A^* \otimes_k B^* \).
- The \( \nu_i \) give a polite duality.
- The \( \nu_i \) give a reflexively polite duality.

The above list then provides sufficient axioms to apply the results of [CW10] to DGSat\( k \), i.e. DGSat\( k \) is a category of spaces in the sense of loc. cit.

The first condition is obvious from Lemma A.15 so we simply state it as a result.

Lemma A.17. The Serre kernel for \( A^\text{op} \otimes_k B \) is \( A^* \otimes_k B^* \).

Now let us recall the definition of a polite duality from [CW10] in our context.
**Definition A.18.** A polite duality between $P \in \text{Perf}^{\text{op}} \otimes_k B$ and $P^\dagger \in \text{Perf}^{\text{op}} \otimes_k A$, denoted $P \leftrightarrow P^\dagger$, consists of adjunctions

$$P^\dagger \otimes_B B^* \otimes_B \bullet \dashv_1 P \otimes_A \bullet \dashv_2 A^* \otimes_A P^\dagger \otimes_B \bullet$$

$$P \otimes_A A^* \otimes_A \bullet \dashv_3 P^\dagger \otimes_B B^* \otimes_B P \otimes_A \bullet$$

such that the following compatibilities hold:

1. For $Q \in \text{Perf}^{\text{op}} \otimes_k C$, $R \in \text{Perf}^{\text{op}} \otimes_k B$, the following diagram of isomorphisms commutes:

   $\text{Hom}_{C^{\text{op}} \otimes_k B}(Q, P \otimes_A R) \xrightarrow{1} \text{Hom}_{C^{\text{op}} \otimes_k A}(P^\dagger \otimes_B B^* \otimes_B Q, R)$

   $\xrightarrow{\text{Serre}} \text{Hom}_{C^{\text{op}} \otimes_k A}(R, A^* \otimes_A P^\dagger \otimes_B B^* \otimes_B Q \otimes C C^*)$.

2. The composition,

$$\text{Hom}_{C^{\text{op}} \otimes_k B}(P \otimes_A R, B^* \otimes_B Q \otimes C C^*)^* \xrightarrow{2} \text{Hom}_{C^{\text{op}} \otimes_k A}(R, A^* \otimes_A P^\dagger \otimes_B B^* \otimes_B Q \otimes C C^*)^*$$

is equal to application of $A^* \otimes_A \bullet$.

3. The same as (1+4) with 3 in place of 1 and 2 in place of 4.

4. The same as (2+3) with 1 in place of 3 and 4 in place of 2.

The following fact is essentially the content of Proposition 19 of [CW10].

**Proposition A.19.** Any of the adjunctions in a polite duality determine the others.

Let us verify the conditions of a polite duality in a sequence of two lemmas.

**Lemma A.20.** The adjunctions defined in Lemma A.14 satisfy (1+2) and (3+4) in the definition of a polite duality.

**Proof.** Verification of (1+2) is equivalent to checking commutativity of the following complex:

$$\text{Hom}_{C^{\text{op}} \otimes_k B}(P \otimes_A R, B^* \otimes_B Q \otimes C C^*) \xleftarrow{\nu_1^\dagger} \text{Hom}_{C^{\text{op}} \otimes_k A}(R, A^* \otimes_A P^\dagger \otimes_B B^* \otimes_B Q \otimes C C^*)$$

$$\xrightarrow{\text{Hom}_{C^{\text{op}} \otimes_k B}(P \otimes_A R, \text{Hom}_k(Q^\vee, k))} \text{Hom}_{C^{\text{op}} \otimes_k A}(R, \text{Hom}_k(P^\dagger \otimes_B B^* \otimes_B Q^\vee, k))$$

$$\xrightarrow{(\nu_2^\dagger)^*} \text{Hom}_k(\text{Hom}_{C^{\text{op}} \otimes_k B}(Q, P \otimes_A R), k) \xrightarrow{(\nu_1^\dagger)^*} \text{Hom}_k(\text{Hom}_{C^{\text{op}} \otimes_k A}(P^\dagger \otimes_B B^* \otimes_B Q, R), k)$$

To make sense of commutativity, we first show that we can fill in the middle row with a natural morphism. Note that we have a natural morphism

$$\tau : Q^\vee \otimes_B P \to (P^\dagger \otimes_B B^* \otimes_B Q)^\vee$$

$$\tau(\psi \otimes p)(\phi \otimes \lambda \otimes q) = \lambda(\psi(q)\phi(p))$$

where we have contracted the $B$ components of the tensor and then applied $\lambda$. Using $\tau$, we get a natural morphism
Now, we just need to check that the following diagram commutes.

\[
\begin{array}{ccc}
\text{Hom}_{C^\text{op} \otimes A}(P \otimes_A R, B^* \otimes_B Q \otimes_C C^*) & \xrightarrow{\nu_2^p} & \text{Hom}_{C^\text{op} \otimes A}(R, A^* \otimes_A P^\text{op} \otimes_B B^* \otimes_B Q \otimes_C C^*) \\
\text{Hom}_{C^\text{op} \otimes A}(P \otimes_A R, \text{Hom}_k(Q^\text{op} \otimes B, k)) & \xrightarrow{\xi^p} & \text{Hom}_{C^\text{op} \otimes A}(R, \text{Hom}_k(P, \text{Hom}_k(Q^\text{op}, k))) \\
\text{Hom}_k(\text{Hom}_{C^\text{op} \otimes A}(Q, P \otimes_A R, k)) & \xrightarrow{(\nu_2^p)^*} & \text{Hom}_k(\text{Hom}_{C^\text{op} \otimes A}(P^\text{op} \otimes_B B^* \otimes_B Q, R, k))
\end{array}
\]

For the top square, take \( f : R \to A^* \otimes_A P^\text{op} \otimes_B B^* \otimes_B Q \otimes_C C^* \), and set \( f(r) = \lambda_r \otimes \phi_r \otimes \mu_r \otimes q_r \otimes \kappa_r \) for notational simplicity. Tracing through either path gives the morphism that sends \( p \otimes r \) to the morphism that sends to \( \psi \) to \( \lambda_r \otimes_k \mu_r \otimes_k \kappa_r (\phi_r(p) \psi(q_r)) \in k \).

For the bottom square, given \( g : \text{Hom}_{A^\text{op} \otimes A}(P^\text{op} \otimes_B B^* \otimes_B Q, R, k) \) to \( k \), tracing along either path gives a map \( h : P \otimes_R \to \text{Hom}_k(Q^\text{op}, k) \) so that \( h(p \otimes r)(\psi) = g(p \otimes \psi \otimes r) \) where we view \( p \otimes \psi \otimes r \) as the map that sends \( \phi \otimes \mu \otimes q \to \mu(\phi(p) \psi(q))r \).

Verification of \((3+4)\) follows via similar arguments. \( \square \)

**Lemma A.21.** The adjunctions defined in Lemma [A.14] satisfy conditions \((2+3)\) and \((1+4)\) in the definition of a polite duality.

**Proof.** We prove \((2+3)\) as \((1+4)\) is similar. We must check the following diagram commutes

\[
\begin{array}{ccc}
\text{Hom}_{C^\text{op} \otimes A}(R, P^\text{op} \otimes_B Q) & \xrightarrow{\nu_2^p} & \text{Hom}_{C^\text{op} \otimes A}(A^* \otimes_A R, A^* \otimes_A P^\text{op} \otimes_A Q) \\
\text{Hom}_{C^\text{op} \otimes A}(P \otimes_A A^* \otimes A R, Q) & \xrightarrow{\nu_1^p} & \text{Hom}_{C^\text{op} \otimes A}(P \otimes_A A^* \otimes A R, Q)
\end{array}
\]

Given \( g : R \to P^\text{op} \otimes_B Q \), let \( g(r) = \phi_r \otimes q_r \) for simplicity. Tracing along either path, gives the map from \( P \otimes_A A^* \otimes_A R \) to \( Q \) which sends \( p \otimes \lambda \otimes r \) to \( \lambda(\phi_r(p))q_r \). \( \square \)

Given the polite duality, \( P \leftrightarrow P^\text{op} \), we can switch the adjunctions to get another polite duality, \( P^\text{op} \leftrightarrow P \). From Lemmas [A.20] and [A.21], we also have a polite duality, \( P^\text{op} \leftrightarrow P^{\text{opp}} \). We say that \( P \leftrightarrow P^\text{op} \) is a **reflexively polite duality** if these two polite dualities coincide under the natural identification \( P \cong P^{\text{opp}} \). By Proposition [A.19] to check that two polite dualities coincide, it suffices to check just one adjunction, of the four, agrees between the two polite dualities.
Consequently, a reflexively polite duality between \( P \) and \( P^\vee \) is equivalent to the following diagram commuting:

\[
\begin{array}{ccc}
\text{Hom}_{C^{\text{op}} \otimes k} \left( R, P^\vee \otimes_B Q \right) & \xrightarrow{\nu^P_3} & \text{Hom}_{C^{\text{op}} \otimes A} \left( P \otimes_A A^* \otimes_A R, Q \right) \\
\downarrow{\nu^P_1} & & \downarrow{\nu^{P^\vee}_1} \\
\text{Hom}_{C^{\text{op}} \otimes A} \left( P^{\vee^\vee} \otimes_A A^* \otimes_A R, Q \right) & & \\
\end{array}
\]

where the vertical arrow is induced by the natural map \( P \to P^{\vee^\vee} \).

**Lemma A.22.** The diagram above commutes.

**Proof.** This is a straightforward computation. \( \square \)

**Remark A.23.** Instead of reducing to smooth and proper dg-algebras, one could start with the bicategory of saturated dg-categories and replace functor by bimodules and composition with tensor product. All the previous adjunction results continue to hold when \( A, B, \) and \( C \) are replaced by essentially small dg-categories, \( A, B, \) and \( C \).

The consequences for the structure of Hochschild homology are discussed in Section 10.

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