Norm of Tensor Product, Tensor Norm, Cubic Power and Gelfand Limit

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Abstract

We establish two inequalities for the nuclear norm and the spectral norm of tensor products. The first inequality indicts that the nuclear norm of the square matrix is a matrix norm. We extend the concept of matrix norm to tensor norm. We show that the 1-norm, the Frobenius norm and the nuclear norm of tensors are tensor norms but the infinity norm and the spectral norm of tensors are not tensor norms. We introduce the cubic power for a general third order tensor, and show that the cubic power of a general third order tensor tends to zero as the power increases to infinity, if there is a tensor norm such that the tensor norm of that third order tensor is less than one. Then we raise a question on a possible Gelfand formula for a general third order tensor. Preliminary numerical results show that a spectral radius-like limit exists in general. We show that if such a Gelfand limit exists for one norm, then it exists for all the other norms with the same value. This limit is zero for all third order nilpotent tensors.

Key words. Tensor product, tensor norm, cubic power, Gelfand Formula.

AMS subject classifications. 15A69

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1 Introduction

The tensor completion and recovery problem aims to fill the missing or unobserved entries of partially observed tensors. It has received wide attention and achievements in areas like data mining, computer vision, signal processing, and neuroscience [10, 11]. Former approaches often proceed by unfolding tensors to matrices and then apply for matrix completion. Yuan and Zhang [11] showed that such matricization fails to exploit the structure of tensors and may lead to sub-optimality. They proposed to minimize a tensor nuclear norm directly and proved that such an approach improves the sample size requirement. This leads research enthusiasm on the tensor nuclear norm and its dual norm, i.e., the tensor spectral norm [4, 5, 6, 7, 8, 9], though this is in fact a NP-hard problem [2].

In matrix analysis [3], the matrix norm is a concept different from the vector norm. We may regard a matrix space as a vector space. A norm on that matrix space is called a vector norm. If in additional, that norm satisfies the axiom that the norm of the product of two arbitrary matrices is always not greater than the product of the norms of these two matrices, then that norm is called a matrix norm. It was shown that the 1-norm, the 2-norm and all the induced norms of matrices are matrix norms, but the infinity norm of matrices is not a matrix norm [3].

The matrix norm of a square matrix $A$ is closely linked with the spectral radius of $A$, i.e., $\rho(A)$, by the well-known Gelfand formula (1941):

$$\rho(A) = \lim_{k \to \infty} \|\|A^k\|\|^\frac{1}{k},$$

for any matrix norm $\|\| \cdot \|\|$. However, there is no "tensor norm" concept or a Gelfand formula for higher order tensors until now. In this paper, we explore this unknown territory.

In the next section, we show that the nuclear norm of the tensor product of two tensors is not greater than the product of the nuclear norms of these two tensors. As an application, we give lower bounds for the nuclear norm of an arbitrary tensor.

However, in general, the spectral norm of the tensor product of two tensors may be greater than the product of the spectral norms of these two tensors. In Section 3, we give a counterexample to illustrate this. Then, as a substitute, in that section, we show that the spectral norm of the tensor product of two tensors is not greater than the product of the spectral norm of one tensor, and the nuclear norm of another tensor. Then, we give an alternative formula for the spectral norm of a tensor. Then we introduce contraction matrices for a tensors and show that the spectral norms of the contraction matrices of a tensor are lower bounds for the product of the nuclear norm and the spectral norm of that tensor.

Then, by the result in Section 2, we conclude that the nuclear norm of the square
matrix is also a matrix norm. We study this in Section 4. Viewing the significance of
the nuclear norm of matrices in the matrix completion problem [1], and the importance
of the matrix norm in matrix analysis [3], this result may be useful in the related research.

In Section 5, we extend the concept of matrix norm to tensor norm. A real function
defined for all real tensors is called a tensor norm if it is a norm for any tensor space
with fixed dimensions, and the norm of the tensor product of two tensors is always not
greater than the product of the norms of these two tensors. We show that the 1-norm,
the Frobenius norm and the nuclear norm of tensors are tensor norms but the infinity
norm and the spectral norm of tensors are not tensor norms.

In Section 6, we introduce the cubic power for a general third order tensor. The
cubic power preserves nonnegativity, symmetry and the diagonal property. We show
that the cubic power of a general third order tensor tends to zero as the power increases
to infinity, if there is a tensor norm such that the tensor norm of that third order
tensor is less than one. A question is if a necessary and sufficient condition for such convergence exists.

Then we raise a question on a possible Gelfand formula for a general third order
tensor in Section 7. Preliminary numerical results show that a spectral radius-like limit
exists in general. We show that if such a Gelfand limit exists for one norm, then it
exists for all the other norms with the same value, and this limit is zero for all third
order nilpotent tensors.

Some final remarks are made in Section 8.

In this paper, unless otherwise stated, all the discussions will be carried out in the
filed of real numbers. We use small letters \(\lambda, x_i, u_i\), etc., to denote scalars, small bold
letters \(\mathbf{x, u, v}\), etc., to denote vectors, capital letters \(A, B, C\), etc., to denote matrices,
and calligraphic letters \(\mathcal{A, B, C}\), etc., to denote tensors, with \(\mathcal{O}\) as the zero tensor with
adequate order and dimensions.

## 2 Nuclear Norm of Tensor Product

Let \(\mathbb{N}\) be the set of positive integers, and \(\mathbb{N}\) be the set of nonnegative integers. For
\(k \in \mathbb{N}\), we use \([k]\) to denote the set \(\{1, \cdots, k\}\). For a vector \(\mathbf{u} = (u_1, \cdots, u_n)^\top\), we use
\(\|\mathbf{u}\|_2\) to denote its 2-norm. Thus,

\[
\|\mathbf{u}\|_2 := \sqrt{u_1^2 + \cdots + u_n^2}.
\]

Suppose that a \(k\)th order tensor \(\mathbf{A} = (a_{i_1 \cdots i_k}) \in \mathbb{R}^{n_{i_1} \times \cdots \times n_k}\), where \(k \in \mathbb{N}\) is called
the order of \(\mathbf{A}\), and \(n_i \in \mathbb{N}\) for \(i \in [k]\) are called the dimensions of \(\mathbf{A}\). We use \(\circ\) to
denote tensor outer product. Then for \( u^{(i)} \in \mathbb{R}^{n_i}, i \in [k], \)

\[
u^{(1)} \circ \cdots \circ u^{(k)}
\]
is a rank-one \( k \)th order tensor. The nuclear norm of \( A \) is defined \([2, 4, 7]\) as

\[
\|A\|_* := \inf \left\{ \sum_{j=1}^{r} |\lambda_j| : A = \sum_{j=1}^{r} \lambda_j u^{(1,j)} \circ \cdots \circ u^{(k,j)},\|u^{(i,j)}\|_2 = 1, u^{(i,j)} \in \mathbb{R}^{n_i}, i \in [k], j \in [r], r \in \mathbb{N} \right\}
\]

(2.1)

We have the following theorem.

**Theorem 2.1** Suppose that \( A = (a_{i_1 \cdots i_{k+p}}) \in \mathbb{R}^{n_1 \times \cdots \times n_k \times \cdots \times n_{k+p}}, \ B = (b_{i_{k+1} \cdots i_{k+p+q}}) \in \mathbb{R}^{n_{k+1} \times \cdots \times n_{k+p+q}}, \) and a tensor product of \( A \) and \( B \) is defined as \( C = (c_{i_1 \cdots i_{k+p+1} \cdots i_{k+p+q}}) \in \mathbb{R}^{n_1 \times \cdots \times n_{k+1} \times \cdots \times n_{k+p+q}} \) by

\[
c_{i_1 \cdots i_{k+p+1} \cdots i_{k+p+q}} = \sum_{i_{k+1}=1}^{n_{k+1}} \cdots \sum_{i_{k+p}=1}^{n_{k+p}} a_{i_1 \cdots i_{k+p}} b_{i_{k+1} \cdots i_{k+p+q}},
\]

for \( i_l \in [n_l], l = 1, \cdots, k, k+p+1, \cdots, k+p+q, \) with \( k, p \in \mathbb{N}, q \in \overline{\mathbb{N}}. \) Then

\[
\|C\|_* \leq \|A\|_* \|B\|_*.
\]

(2.2)

**Proof** Let \( \epsilon > 0. \) Then we have

\[
A = \sum_{j=1}^{r_1} \lambda_j u^{(1,j)} \circ \cdots \circ u^{(k+p,j)},
\]

where \( \|u^{(i,j)}\|_2 = 1 \) for \( i = 1, \cdots, k+p, \) and \( j \in [r_1], r_1 \in \mathbb{N}, \) and

\[
B = \sum_{l=1}^{r_2} \mu_l v^{(k+1,l)} \circ \cdots \circ v^{(k+p+q,l)},
\]

where \( \|v^{(i,l)}\|_2 = 1 \) for \( i = k+1, \cdots, k+p+q, \) and \( l \in [r_2], r_2 \in \mathbb{N}, \) such that

\[
\sum_{j=1}^{r_1} |\lambda_j| \leq \|A\|_* + \epsilon
\]

and

\[
\sum_{l=1}^{r_2} |\mu_l| \leq \|B\|_* + \epsilon.
\]

This implies that

\[
C = \sum_{j=1}^{r_1} \sum_{l=1}^{r_2} \lambda_j \mu_l \prod_{i=1}^{p} (u^{(k+i,j)} \circ v^{(k+i,l)} \circ u^{(1,j)} \circ \cdots \circ u^{(k,j)} \circ v^{(k+p+1,l)} \circ \cdots \circ v^{(k+p+r,l)}).
\]
Hence,
\[ \|C\|_* \leq \sum_{j=1}^{r_1} \sum_{l=1}^{r_2} |\lambda_j \mu_l| \prod_{i=1}^{p} \|u^{(k+i,j)}\|_2 \|v^{(k+i,l)}\|_2 \]
\[ = \sum_{j=1}^{r_1} \sum_{l=1}^{r_2} |\lambda_j \mu_l| \]
\[ = \left( \sum_{j=1}^{r_1} |\lambda_j| \right) \left( \sum_{l=1}^{r_2} |\mu_l| \right) \]
\[ \leq (\|A\|_* + \epsilon) (\|B\|_* + \epsilon), \]
where the first inequality is by the definition (2.1), and the second inequality is by the Cauchy inequality. Letting \( \epsilon \to 0 \), we have (2.2). \( \square \)

### 2.1 An Application: Lower Bounds for the Nuclear Norm of a Tensor

In this subsection, we present a lower bound for the nuclear norm of an arbitrary even order tensor \( A = (a_{i_1 \cdots i_k}) \in \mathbb{R}^{n_1 \times \cdots \times n_k} \) for \( k \in \mathbb{N} \) and \( k \geq 3 \).

We first assume that \( k = 3 \). Then \( A \) is a third order tensor. It is not easy to compute its nuclear norm.

**Proposition 2.2** Suppose that \( A = (a_{i_1 i_2}) \in \mathbb{R}^{n_1 \times n_2 \times n_3} \), where \( n_1, n_2, n_3 \in \mathbb{N} \). Let \( B = (b_{i_2 i_3}) \in \mathbb{R}^{n_2 \times n_3} \), and \( c = (c_1, \cdots, c_{n_1})^\top \in \mathbb{R}^{n_1} \) be defined by

\[ c_i = \sum_{i_2=1}^{n_2} \sum_{i_3=1}^{n_3} a_{i_1 i_2 i_3} b_{i_2 i_3}, \quad (2.3) \]

for \( i \in [n_1] \). Then,

\[ \|A\|_* \geq \max \{ \|c\|_* : c \text{ is calculated by (2.3), } B \in \mathbb{R}^{n_2 \times n_3}, \|B\|_* = 1 \}. \quad (2.4) \]

**Proof** Applying Theorem 2.1 with \( B = B \) and \( C = c \), \( k = 1 \), \( p = 2 \) and \( q = 0 \), we have the conclusion. \( \square \)

Here, \( B \) is a matrix, and \( c \) is a vector. Their nuclear norms are not difficult to be calculated. The tensor \( A \) in the following example is originally from [2].
**Example 2.3** Let \( k = 3 \), \( n_1 = n_2 = n_3 = 2 \), and \( A \) be a third order symmetric tensor defined by

\[
A = \frac{1}{2} \left( e^{(1)} \circ e^{(1)} \circ e^{(2)} + e^{(1)} \circ e^{(2)} \circ e^{(1)} + e^{(2)} \circ e^{(1)} \circ e^{(1)} - e^{(2)} \circ e^{(2)} \circ e^{(2)} \right),
\]

where \( e^{(1)} = (1, 0)^{\top} \) and \( e^{(2)} = (0, 1)^{\top} \). Let \( J \) be the all one matrix in \( \mathbb{R}^{2 \times 2} \). We may let \( B = J \|J\|_* \). We may calculate \( c \) by (2.4). Then by (2.4), we have

\[
\|A\|_* \geq \|c\|_* \equiv 0.6455.
\]

Actually, \( \|A\|_* = 2 \). This verifies (2.4) somehow.

We then consider the case that \( k = 4 \). Then \( A \) is a fourth order tensor. It is also not easy to compute its nuclear norm.

**Proposition 2.4** Suppose that \( A = (a_{i_1i_2i_3i_4}) \in \mathbb{R}^{n_1 \times n_2 \times n_3 \times n_4} \), where \( n_1, n_2, n_3, n_4 \in \mathbb{N} \). Let \( B = (b_{i_3i_4}) \in \mathbb{R}^{n_3 \times n_4} \), and \( C = (c_{i_1i_2}) \in \mathbb{R}^{n_1 \times n_2} \) be defined by

\[
c_{i_1i_2} = \sum_{i_3=1}^{n_3} \sum_{i_4=1}^{n_4} a_{i_1i_2i_3i_4} b_{i_3i_4},
\]

for \( i_1 \in [n_1], i_2 \in [n_2] \). Then,

\[
\|A\|_* \geq \max \{ \|C\|_* : C \text{ is calculated by (2.5), } B \in \mathbb{R}^{n_3 \times n_4}, \|B\|_* = 1 \}.
\]

**Proof** Applying Theorem 2.1 with \( B = B \) and \( C = C \), \( k = p = 2 \) and \( q = 0 \), we have the conclusion. \( \square \)

Here, \( B \) and \( C \) are matrices. Their nuclear norms are not difficult to be calculated. The following example is from [8].

**Example 2.5** Let \( k = 4 \), \( n_1 = n_2 = n_3 = n_4 = 3 \), and \( A \) be a fourth order symmetric tensor defined by

\[
A = e^{\otimes 4} - (e^{(1)})^{\otimes 4} - (e^{(2)})^{\otimes 4} - (e^{(3)})^{\otimes 4},
\]

where \( e = (1, 1, 1)^{\top} \), \( e^{(1)} = (1, 0, 0)^{\top} \), \( e^{(2)} = (0, 1, 0)^{\top} \) and \( e^{(3)} = (0, 0, 1)^{\top} \). Let \( J \) be the all one matrix in \( \mathbb{R}^{3 \times 3} \). We may let \( B = J \|J\|_* \). We may calculate \( C \) by (2.5). Then by (2.6), we have

\[
\|A\|_* \geq \|C\|_* \equiv 10.3757.
\]

Actually, \( \|A\|_* = 12 \). This verifies (2.6) somehow.

We now extend Propositions 2.2 and 2.4 to the case that \( k \geq 5 \). We have the following two propositions.
Proposition 2.6 Suppose that $A = (a_{i_1 \cdots i_k}) \in \mathbb{R}^{n_1 \times \cdots \times n_k}$, where $k = 2l+1$, $n_1, \ldots, n_k, l \in \mathbb{N}, l \geq 2$. Let $B^{(j)} = (b_{i_{2j+1}i_{2j+2}}^{(j)}) \in \mathbb{R}^{n_2 \times n_{2j+1}}$ for $j = 1, \ldots, l$, and $c = (c_1, \ldots, c_{n_1})^\top \in \mathbb{R}^{n_1}$ be defined by
\[
c_i = \sum_{i_2=1}^{n_2} \cdots \sum_{i_k=1}^{n_k} a_{i_2 \cdots i_k} b_{i_{2j+1}i_{2j+2}}^{(1)} \cdots b_{i_{2j+1}i_{2j+2}}^{(l)},
\] for $i \in [n_1]$. Then,
\[
\|A\|_* \geq \max \{\|c\|_* : c \text{ is calculated by (2.9), } B^{(j)} \in \mathbb{R}^{n_2 \times n_{2j+1}}, \|B^{(j)}\|_* = 1, j \in [l]\}.
\]

Proof Applying Theorem 2.1 repetitively, we have the conclusion. \qed

Here, $B^{(j)}$ for $j \in [l]$ are matrices, and $c$ is a vector. Their nuclear norms are not difficult to be calculated.

Proposition 2.7 Suppose that $A = (a_{i_1 \cdots i_k}) \in \mathbb{R}^{n_1 \times \cdots \times n_k}$, where $k = 2l+2$, $n_1, \ldots, n_k, l \in \mathbb{N}, l \geq 2$. Let $B^{(j)} = (b_{i_{2j+1}i_{2j+2}}^{(j)}) \in \mathbb{R}^{n_2 \times n_{2j+1}}$ for $j = 1, \ldots, l$, and $C = (c_{i_{12}}) \in \mathbb{R}^{n_1 \times n_2}$ be defined by
\[
c_{i_1i_2} = \sum_{i_3=1}^{n_3} \cdots \sum_{i_k=1}^{n_k} a_{i_1 \cdots i_k} b_{i_{2j+1}i_{2j+2}}^{(1)} \cdots b_{i_{2j+1}i_{2j+2}}^{(l)},
\] for $i_1 \in [n_1]$ and $i_2 \in [n_2]$. Then,
\[
\|A\|_* \geq \max \{\|C\|_* : C \text{ is calculated by (2.9), } B^{(j)} \in \mathbb{R}^{n_2 \times n_{2j+1}}, \|B^{(j)}\|_* = 1, j \in [l]\}.
\]

Proof Applying Theorem 2.1 repetitively, we have the conclusion. \qed

Here, $B^{(j)}$ for $j \in [l]$, and $C$ are matrices. Their nuclear norms are not difficult to be calculated.

3 Spectral Norm of Tensor Product

For $A = (a_{i_1 \cdots i_k}), B = (b_{i_1 \cdots i_k}) \in \mathbb{R}^{n_1 \times \cdots \times n_k}$, their inner product is defined as
\[
\langle A, B \rangle := \sum_{i_1=1}^{n_1} \cdots \sum_{i_k=1}^{n_k} a_{i_1 \cdots i_k} b_{i_1 \cdots i_k}.
\]

Then the spectral norm of $A$ is defined [2, 4, 7, 11] as
\[
\|A\|_* := \max \{\langle A, u^{(i)} \circ \cdots \circ u^{(k)} \rangle, u^{(i)} \in \mathbb{R}^{n_i}, \|u^{(i)}\|_2 = 1, \text{ for } i \in [k]\}.
\]

(3.11)
It is known [11] that we always have
\[ \|A\|_S \leq \|A\|_*. \]

Note that in general, we do not have
\[ \|C\|_S \leq \|A\|_S \|B\|_S, \]
if \( C \) is a tensor product of \( A \) and \( B \), as in Theorem 2.1. See the following example.

**Example 3.1** Let \( k = p = q = 2 \), and \( n_i = 2 \) for \( i = 1, \cdots, 6 \). Let \( A = (a_{i_1i_2i_3i_4}) = B = (b_{i_3i_4i_5i_6}) \) be defined by

\[
\begin{align*}
  a_{1111} &= 2, & a_{1211} &= 3, & a_{2111} &= -6, & a_{2211} &= 3, \\
  a_{1121} &= -6, & a_{1221} &= 3, & a_{2121} &= 4, & a_{2221} &= 3, \\
  a_{1112} &= 3, & a_{1212} &= 9, & a_{2112} &= 3, & a_{2212} &= -3, \\
  a_{1122} &= 3, & a_{1222} &= -3, & a_{2122} &= 3, & a_{2222} &= 15.
\end{align*}
\]

Then we have
\[
\begin{align*}
  c_{1111} &= 58, & c_{1211} &= 6, & c_{2111} &= -18, & c_{2211} &= 24, \\
  c_{1121} &= -18, & c_{1221} &= 12, & c_{2121} &= 70, & c_{2221} &= 30, \\
  c_{1112} &= 6, & c_{1212} &= 108, & c_{2112} &= 12, & c_{2212} &= -54, \\
  c_{1122} &= 24, & c_{1222} &= -54, & c_{2122} &= 30, & c_{2222} &= 1252.
\end{align*}
\]

By calculation, we have \( \|A\|_S = \|B\|_S = 16.3609 \), and \( \|C\|_S = 271.5503 \). Then \( \|A\|_S \|B\|_S = 268.6781 \), which is slightly less than \( \|C\|_S \).

However, we may establish the following theorem.

**Theorem 3.2** Suppose that \( A = (a_{i_1 \cdots i_{k+p}}) \in \mathbb{R}^{n_1 \times \cdots \times n_{k+p}} \), \( B = (b_{i_{k+1} \cdots i_{k+p+q}}) \in \mathbb{R}^{n_{k+1} \times \cdots \times n_{k+p+q}} \), and a tensor product of \( A \) and \( B \) is defined as \( C = (c_{i_1 \cdots i_{k+p+1} \cdots i_{k+p+q}}) \in \mathbb{R}^{n_1 \times \cdots \times n_k \times n_{k+p+1} \cdots \times n_{k+p+q}} \) by

\[
  c_{i_1 \cdots i_{k+p+1} \cdots i_{k+p+q}} = \sum_{i_{k+1}=1}^{n_{k+1}} \cdots \sum_{i_{k+p}=1}^{n_{k+p}} a_{i_1 \cdots i_{k+p}} b_{i_{k+1} \cdots i_{k+p+q}},
\]

for \( i_l \in [n_l], \ l = 1, \cdots, k, k+p+1, \cdots, k+p+q \), with \( k, p \in \mathbb{N}, q \in \mathbb{N} \). Then
\[
  \|C\|_S \leq \|A\|_S \|B\|_*. \tag{3.12}
\]

**Proof** Let \( \epsilon > 0 \). Then we have
\[
  B = \sum_{j=1}^{r} \mu_j v^{(k+1,j)} \circ \cdots \circ v^{(k+p+q,j)},
\]
where \( \|v^{(ij)}\| = 1 \) for \( i = k + 1, \ldots, k + p + q \), and \( j \in [r], r \in \mathbb{N} \), such that

\[
\sum_{j=1}^{r} |\mu_j| \leq \|B\|_* + \epsilon.
\]

Then

\[
b_{i_{k+1} \cdots i_{k+p+q}} = \sum_{j=1}^{r} \mu_j v_{i_{k+1}}^{(k+1,j)} \cdots v_{i_{k+p+q}}^{(k+p+q,j)},
\]

for \( i = k + 1, \ldots, k + p + q \), and \( j \in [r] \).

We have

\[
c_{i_1 \cdots i_{k+1} \cdots i_{k+p+q}} = \sum_{i_{k+1}=1}^{n_{k+1}} \cdots \sum_{i_{k+p+1}=1}^{n_{k+p}} \sum_{i_{k+p+q}=1}^{n_{k+p+q}} \mu_j a_{i_1 \cdots i_{k+p}} v_{i_{k+1}}^{(k+1,j)} \cdots v_{i_{k+p+q}}^{(k+p+q,j)},
\]

for \( i_l \in [n_l], l = 1, \ldots, k, k + p + 1, \ldots, k + p + q \), and \( j \in [r] \). This implies that

\[
\|C\|_S = \max \left\{ \left< C, u^{(1)} \circ \cdots \circ u^{(k+1)} \circ u^{(k+p+1)} \circ \cdots \circ u^{(k+p+q)} \right> : \|u^{(i)}\|_2 = 1 \right\}
\]

\[
= \max \left\{ \mathcal{A}, u^{(1)} \circ \cdots \circ u^{(k)} \circ v^{(k+1,j)} \circ \cdots \circ v^{(k+p,j)} \right) \prod_{l=k+1}^{k+p+q} \left| \langle v^{(l,j)}, u^{(l)} \rangle \right| \right\}
\]

\[
\leq \max \left\{ \mathcal{A}, u^{(1)} \circ \cdots \circ u^{(k)} \circ v^{(k+1,j)} \circ \cdots \circ v^{(k+p,j)} \right| \prod_{l=k+1}^{k+p+q} \left| \langle v^{(l,j)}, u^{(l)} \rangle \right| \right\}
\]

\[
\leq \max \left\{ \mathcal{A}, u^{(1)} \circ \cdots \circ u^{(k)} \circ v^{(k+1,j)} \circ \cdots \circ v^{(k+p,j)} \right| \prod_{l=k+1}^{k+p+q} \left| \langle v^{(l,j)}, u^{(l)} \rangle \right| \right\}
\]

\[
\leq \|A\|_S (\|B\|_* + \epsilon),
\]

where the second inequality is by the definition (3.11) and the Cauchy inequality. Letting \( \epsilon \to 0 \), we have (3.12).

### 3.1 An Alternative Formula for the Spectral Norm of a Tensor

We present an alternative formula for the spectral norm of a tensor in this subsection. This formula does not reduce the complexity of the problem, as this is impossible, but gives an alternative approach to handle the spectral norm.

**Proposition 3.3** Suppose that \( A = (a_{i_1 \cdots i_k}) \in \mathbb{R}^{n_1 \times \cdots \times n_k} \), where \( k \in \mathbb{N}, k \geq 3 \). For \( B = (b_{i_{k-1}i_k}) \in \mathbb{R}^{n_{k-1} \times n_k} \), define \( C = (c_{i_1 \cdots i_k}) \in \mathbb{R}^{n_1 \times \cdots \times n_{k-2}} \) by

\[
c_{i_1 \cdots i_k} = \sum_{i_{k-1}=1}^{n_{k-1}} \sum_{i_k=1}^{n_k} a_{i_1 \cdots i_{k-1}} b_{i_{k-1}i_k}, \tag{3.13}
\]
for $i_j \in [n_j], j = 1, \ldots, k - 2$. Then

$$\|A\|_S = \max \{ \|C\|_S : C \text{ is calculated by (3.13), } B \in \mathbb{R}^{n_{k-1} \times n_k}, \|B\|_* = 1 \}. \quad (3.14)$$

**Proof** By Theorem 3.2, we have

$$\|A\|_S \geq \max \{ \|C\|_S : C \text{ is calculated by (3.13), } B \in \mathbb{R}^{n_{k-1} \times n_k}, \|B\|_* = 1 \}. \quad (3.15)$$

On the other hand,

$$\|A\|_S = \max \{ \langle A, u^{(1)} \circ \cdots \circ u^{(k)} \rangle, u^{(i)} \in \mathbb{R}^{n_i}, \|u^{(i)}\|_2 = 1, \text{ for } i \in [k] \}
\leq \max \{ \langle A, u^{(1)} \circ \cdots \circ u^{(k-2)} \circ B \rangle, u^{(i)} \in \mathbb{R}^{n_i}, \|u^{(i)}\|_2 = 1, \text{ for } i \in [k-2], B \in \mathbb{R}^{n_{k-1} \times n_k}, \|B\|_* = 1 \}
= \max \{ \|C\|_S : C \text{ is calculated by (3.13), } B \in \mathbb{R}^{n_{k-1} \times n_k}, \|B\|_* = 1 \}. \quad (3.14)$$

This proves (3.14).

### 3.2 Lower Bounds for the Product of the Nuclear Norm and Spectral Norm of a Tensor

We now present some lower bounds for the product of the nuclear norm and the spectral norm of $A \in \mathbb{R}^{n_1 \times \cdots \times n_k}$. Denote the spectral radius of a matrix $A \in \mathbb{R}^{n \times n}$ by $\rho(A)$. We first introduce contraction matrices of $A$.

**Definition 3.4** Let $A = (a_{i_1 \cdots i_k}) \in \mathbb{R}^{n_1 \times \cdots \times n_k}$, where $n_1, \ldots, n_k, k \in \mathbb{N}$. Assume that $k \geq 2$. Let $j \in [k]$. Define a symmetric matrix $A^{(j)} = (a^{(j)}_{rs}) \in \mathbb{R}^{n_j \times n_j}$ by

$$a^{(j)}_{rs} = \sum_{i_1=1}^{n_1} \cdots \sum_{i_{j-1}=1}^{n_{j-1}} \sum_{i_{j+1}=1}^{n_{j+1}} \cdots \sum_{i_k=1}^{n_k} a_{i_1 \cdots i_{k-1} r i_{k+1} \cdots i_k} a_{i_1 \cdots i_{k-1} s i_{k+1} \cdots i_k},$$

for $r, s \in [n_j]$. We call $A^{(j)}$ the $j$th contraction matrix of $A$.

**Proposition 3.5** Suppose that $A = (a_{i_1 \cdots i_k}) \in \mathbb{R}^{n_1 \times \cdots \times n_k}$, where $n_1, \ldots, n_k, k \in \mathbb{N}$. Assume that $k \geq 2$. Let $j \in [k]$. Then,

$$\rho(A^{(j)}) \equiv \|A^{(j)}\|_S \leq \|A\|_S \|A\|_* \quad (3.15)$$

**Proof** Apply Theorem 3.2 with $B = A$ and $C = A^{(j)}$. Note that $A^{(j)} \in \mathbb{R}^{n_j \times n_j}$ is symmetric. Hence, its spectral norm is its spectral radius. We then have the conclusion.
As $A^{(j)}$ is a symmetric matrix, its spectral norm, i.e., its spectral radius is the largest absolute value of its eigenvalues. which is not difficult to be calculated. For $j = 1, \cdots, k$, this theorem gives $k$ lower bounds for the product of the nuclear norm and the spectral norm of $A$. The tensor $A$ in the following example is the same as the tensor $A$ in Example 3.2. It is originally from [2].

**Example 3.6** Let $k = 3$, $n_1 = n_2 = n_3 = 2$, and $A$ be a third order symmetric tensor defined by

$$A = \frac{1}{2} \left( e^{(1)} \circ e^{(1)} \circ e^{(2)} + e^{(1)} \circ e^{(2)} \circ e^{(1)} + e^{(2)} \circ e^{(1)} \circ e^{(1)} - e^{(2)} \circ e^{(2)} \circ e^{(2)} \right),$$

where $e^{(1)} = (1, 0)^\top$ and $e^{(2)} = (0, 1)^\top$. Then

$$A^{(1)} = A^{(2)} = A^{(3)} = 0.5I_2,$$

where $I_2$ is the identity matrix in $\mathbb{R}^{2\times2}$. Then we have $\|A\|_S = 0.5$, $\|A\|_\ast = 2$ and $\rho(A^{(1)}) = \rho(A^{(2)}) = \rho(A^{(3)}) = 0.5$. This verifies (3.15).

### 4 Matrix Norm

In matrix analysis [3], the matrix norm is a concept different from the vector norm. Consider matrices in $\mathbb{R}^{n \times n}$. Let $||| \cdot ||| : \mathbb{R}^{n \times n} \to \mathbb{R}_+$. If it is not only a vector norm in $\mathbb{R}^{n \times n}$, but it also satisfies the following additional axiom: for any $A, B \in \mathbb{R}^{n \times n}$,

$$|||AB||| \leq |||A||| \cdot |||B|||,$$

then $||| \cdot |||$ is a matrix norm in $\mathbb{R}^{n \times n}$. Otherwise, it is only a vector norm. In particular, 1-norm, 2-norm and any induced norm are matrix norms, but $\infty$-norm is only a vector norm, not a matrix norm. Matrix norms play an important role in matrix analysis. See [3] for more details on matrix norms.

By Theorem 2.1, we have the following proposition.

**Proposition 4.1** For $\mathbb{R}^{n \times n}$, the nuclear norm is a matrix norm.

As the nuclear norm plays a significant role in the matrix completion problem [1], this conclusion for matrix nuclear norm should be useful in the related research.

By Theorems 2.1 and 3.2, we have the following further results.

**Proposition 4.2** Suppose that $A \in \mathbb{R}^{n \times n}$ is invertible. Then we have

$$\|A\|_\ast \|A^{-1}\|_\ast \geq n,$$

and

$$\|A\|_\ast \|A^{-1}\|_S \geq 1.$$
Proof Apply (2.2) and (3.12) to \( A \) and \( A^{-1} \). Note that \( \|I_n\|_S = 1 \) and \( \|I_n\|_* = n \), where \( I_n \) is the identity matrix in \( \mathbb{R}^{n \times n} \). The conclusions hold.

Proposition 4.3 Suppose that \( A \in \mathbb{R}^{n \times n} \). If \( \|A\|_* \leq 1 \), then
\[
\lim_{k \to \infty} A^k = 0.
\]

Proof If \( \|A\|_* \leq 1 \), then
\[
\|A^k\|_* \leq \|A\|_*^k \to 0.
\]
The result follows.

5 Tensor Norm

We are now ready to extend the concept of matrix norms to tensor norms.

Definition 5.1 Suppose that \( \|\cdot\| \) is a function defined for all real tensors, and in any real tensor space \( \mathbb{R}^{n_1 \times \cdots \times n_k} \) of fixed dimensions with \( n_1, \ldots, n_k, k \in \mathbb{N} \), it is a vector norm. If furthermore for any two real tensors \( A \) and \( B \) such that \( A \) and \( B \) have an outer tensor product \( C \), we have
\[
\|\|C\|\| \leq \|\|A\|\| \cdot \|\|B\|\|,
\]
then \( \|\cdot\| \) is called a tensor norm.

Clearly, a tensor norm must be a matrix norm if it is restricted to \( \mathbb{R}^{n \times n} \). We have the following theorem.

Theorem 5.2 The nuclear norm, the 1-norm, the Frobenius norm are tensor norms, but the infinity norm and the spectral norm are not tensor norms.

Proof By Theorem 2.1, the nuclear norm is a tensor norm. Since the infinity norm is not a matrix norm, it is also not a tensor norm. By the counter example in Example 4.1, the spectral norm is not a tensor norm. What we need to check are the 1-norm and the Frobenius norm.

Suppose that \( A = (a_{i_1 \cdots i_{k+p}}) \in \mathbb{R}^{n_1 \times \cdots \times n_{k+p}} \), \( B = (b_{i_{k+1} \cdots i_{k+p+q}}) \in \mathbb{R}^{n_{k+1} \times \cdots \times n_{k+p+q}} \), and a tensor product of \( A \) and \( B \) is defined as \( C = (c_{i_1 \cdots i_{k+p+1} \cdots i_{k+p+q}}) \in \mathbb{R}^{n_1 \times \cdots \times n_k \times n_{k+p+1} \cdots \times n_{k+p+q}} \) by
\[
c_{i_1 \cdots i_{k+p+1} \cdots i_{k+p+q}} = \sum_{i_{k+1}=1}^{n_{k+1}} \cdots \sum_{i_{k+p}=1}^{n_{k+p}} a_{i_1 \cdots i_{k+p}} b_{i_{k+1} \cdots i_{k+p+q}}.
\]
for \( i_l \in \{n_l\}, l = 1, \ldots, k, k + p + 1, \ldots, k + p + q \), with \( k, p, q \in \mathbb{N}. \) Then,

\[
\|C\|_1 = \sum_{i_1=1}^{n_1} \cdots \sum_{i_{k+p}=1}^{n_{k+p}} \cdots \sum_{i_{k+p+q}=1}^{n_{k+p+q}} \left| \sum_{i_{k+1}=1}^{n_{k+1}} \cdots \sum_{i_{k+p}=1}^{n_{k+p}} a_{i_1 \cdots i_{k+p}} b_{i_{k+1} \cdots i_{k+p+q}} \right| \\
\leq \sum_{i_1=1}^{n_1} \cdots \sum_{i_{k+p}=1}^{n_{k+p}} \cdots \sum_{i_{k+p+q}=1}^{n_{k+p+q}} \left( \sum_{i_{k+1}=1}^{n_{k+1}} \cdots \sum_{i_{k+p}=1}^{n_{k+p}} |a_{i_1 \cdots i_{k+p}}| \right) \left( \sum_{j_{k+1}=1}^{n_{k+1}} \cdots \sum_{j_{k+p+q}=1}^{n_{k+p+q}} |b_{j_{k+1} \cdots j_{k+p+q}}| \right) \\
= \|A\|_1 \|B\|_1.
\]

This shows that the 1-norm is a tensor norm.

Let \( A, B \) and \( C \) be as defined above. Denote the Frobenius norm as \( \| \cdot \|_2 \), as it is just the 2-norm. Then

\[
\|A\|_2^2 = \sum_{i_1=1}^{n_1} \cdots \sum_{i_{k+p}=1}^{n_{k+p}} a_{i_1 \cdots i_{k+p}}^2,
\]

\[
\|B\|_2^2 = \sum_{i_{k+1}=1}^{n_{k+1}} \cdots \sum_{i_{k+p+q}=1}^{n_{k+p+q}} b_{i_{k+1} \cdots i_{k+p+q}}^2,
\]

and

\[
\|C\|_2^2 = \sum_{i_1=1}^{n_1} \cdots \sum_{i_{k+1}=1}^{n_{k+1}} \sum_{i_{k+p}=1}^{n_{k+p}} \cdots \sum_{i_{k+p+q}=1}^{n_{k+p+q}} \left( \sum_{i_{k+1}=1}^{n_{k+1}} \cdots \sum_{i_{k+p}=1}^{n_{k+p}} a_{i_1 \cdots i_{k+p}} b_{i_{k+1} \cdots i_{k+p+q}} \right)^2.
\]

These show that

\[
\|C\|_2^2 \leq \|A\|_2^2 \|B\|_2^2,
\]

i.e.,

\[
\|C\|_2 \leq \|A\|_2 \|B\|_2.
\]

Hence, the Frobenius norm is also a tensor norm. \( \square \)
6 The Cubic Power of a Third Order Tensor

Consider the third order space $\mathbb{R}^{d_1 \times d_2 \times d_3}$, where $d_1, d_2, d_3 \in \mathbb{N}$. Such a third order tensor is the model of a general higher order tensor \cite{4, 11}.

**Definition 6.1** Suppose that $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{d_1 \times d_2 \times d_3}$. Let $F : \mathbb{R}^{d_1 \times d_2 \times d_3} \rightarrow \mathbb{R}^{d_1 \times d_2 \times d_3}$ be defined as follows. We have $\mathcal{A}^3 \equiv F(\mathcal{A}) \equiv \mathcal{T} = (t_{ijk}) \in \mathbb{R}^{d_1 \times d_2 \times d_3}$, where

$$t_{ijk} = \sum_{s=1}^{d_1} \sum_{p=1}^{d_2} \sum_{q=1}^{d_3} a_{ispq} a_{sjq} a_{spk},$$

for $i \in [d_1], j \in [d_2]$ and $k \in [d_3]$. In particular, we have

$$\mathcal{A}^{m+1} = F(\mathcal{A}^m),$$

for $m \in \mathbb{N}$. If $\mathcal{A}^m = \mathcal{O}$ for some $m \in \mathbb{N}$, then $\mathcal{A}$ is said to be nilpotent.

We see that $\mathcal{A}^3$ is uniquely defined for $\mathcal{A} \in \mathbb{R}^{d_1 \times d_2 \times d_3}$.

**Example 6.2** Suppose that $d_1 = d_2 = d_3 = d$. We say that $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{d \times d \times d}$ is diagonal if $a_{ijk} = 0$ as long as $i, j$ and $k$ are not all equal. Assume that $a_{iii} = \alpha_i$ for $i \in [d]$. Then $\mathcal{A}^3 \equiv \mathcal{T} = (t_{ijk}) \in \mathbb{R}^{d \times d \times d}$ is also diagonal with $t_{iii} = \alpha_i^3$ for $i \in [d]$.

Thus, $\mathcal{A}^3$ preserves the diagonal property. The matrix power preserves nonnegativity and symmetry. We see $\mathcal{A}^3$ also preserves these.

**Proposition 6.3** Suppose that $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{d_1 \times d_2 \times d_3}$. If $\mathcal{A}$ is nonnegative, i.e., $a_{ijk} \geq 0$ for $i \in [d_1], j \in [d_2], k \in [d_3]$, then $\mathcal{A}^3$ is also nonnegative. If $d_1 = d_2 = d_3 = d$ and $\mathcal{A}$ is symmetric, i.e., $a_{ijk}$ is invariant under any index permutation, then $\mathcal{A}^3$ is also symmetric.

**Proof** It is directly from the definition that $\mathcal{A}^3$ preserves nonnegativity. Assume now $d_1 = d_2 = d_3 = d$ and $\mathcal{A}$ is symmetric. Denote $\mathcal{T} = (t_{ijk}) \equiv \mathcal{A}^3$. For $i \in [d_1], j \in [d_2]$ and $k \in [d_3]$, we have

$$t_{ijk} = \sum_{s,p,q=1}^{d} a_{jpq} a_{spq} a_{spk} = \sum_{s,p,q=1}^{d} a_{pjq} a_{s_jq} a_{spk} = t_{ijk}.$$

Then $\mathcal{T} = \mathcal{A}^3$ is also symmetric. \qed
Example 6.4  We consider the tensor $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{2 \times 2 \times 2}$ with $a_{111} = -1$, $a_{122} = a_{212} = a_{221} = 1$, and $a_{ijk} = 0$ otherwise. Then $\mathcal{A}$ is symmetric. We find that $\mathcal{A}^3 = \mathcal{O}$. Thus, $\mathcal{A}$ is nilpotent. By computation, we find that the spectral norm of $\mathcal{A}$ is $1$.

Suppose that $\mathcal{A} \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ is nilpotent, and $m$ is the smallest integer such that $\mathcal{A}^{3m} = \mathcal{O}$. Do we have $3^m \leq \max\{d_1, d_2, d_3\}$?

Coming back to tensor norms, we have the following proposition.

**Proposition 6.5** Suppose $\mathcal{A} \in \mathbb{R}^{d_1 \times d_2 \times d_3}$. If $|||\mathcal{A}||| < 1$ for a tensor norm $||| \cdot |||$, then

$$\lim_{m \to \infty} \mathcal{A}^{3m} = \mathcal{O}. \quad (6.20)$$

**Proof** For a tensor norm $||| \cdot |||$, we have

$$|||\mathcal{A}^{3m}||| \leq |||\mathcal{A}|||^{3m}. \quad (6.21)$$

The conclusion follows. □

From Theorem 5.2, $||| \cdot |||$ can be either the 1-norm, or the Frobenius norm, or the nuclear norm. By (6.21), the convergence of (6.20) should be very fast.

For a matrix $A \in \mathbb{R}^{n \times n}$,

$$\lim_{k \to \infty} A^k = 0$$

if and only if its spectral radius $\rho(A) < 1$. See Theorem 5.6.12 of [3]. Then, what is the necessary and sufficient condition for (6.20)? From Example 6.4, we see that $||A||_S < 1$ is not a necessary condition for this, as for that example, $||A||_S = 1$ but $\mathcal{A}^3 = \mathcal{O}$, which implies (6.20) immediately. Thus, $||A||_S$ cannot be a substitute of $\rho(A)$. In fact, even in the matrix case, for $A \in \mathbb{R}^{n \times n}$, $\rho(A)$ is not a norm, as $\rho(A) = 0$ if $A$ is nilpotent but not necessary a zero matrix.

Another question is on the inverse operation of the cubic power. For any $\mathcal{A} \in \mathbb{R}^{d_1 \times d_2 \times d_3}$, does there exist $\mathcal{B} \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ such that $\mathcal{A} = \mathcal{B}^3$? If such a $\mathcal{B}$ exists, is it unique? If such a $\mathcal{B}$ exists and is unique, then we may call it the cubic root of $\mathcal{A}$ and denote

$$\mathcal{B} = (\mathcal{A})^{\frac{1}{3}}.$$

7  **Gelfand Limit**

In matrix analysis, there is a well-known Gelfand formula (1941):

$$\rho(A) = \lim_{k \to \infty} |||A^k|||^{\frac{1}{k}},$$

for any matrix norm $||| \cdot |||$. See 5.6.14 of [3].
Is the following limit
\[
\lim_{m \to \infty} \| A^3_m \|^{\frac{1}{3m}}
\]
always exists for any \( A \in \mathbb{R}^{d_1 \times d_2 \times d_3} \) and any tensor norm \( \| \cdot \| \)? We have the following theorem.

**Theorem 7.1** Let \( A \in \mathbb{R}^{d_1 \times d_2 \times d_3} \), \( \| \cdot \| \) and \( \| \cdot \|' \) are two norms of \( \mathbb{R}^{d_1 \times d_2 \times d_3} \). Then

\[
\lim \sup \| A^3_m \|^{\frac{1}{3m}} = \lim \sup \| A^3_m \|'^{\frac{1}{3m}} \quad (7.22)
\]
and

\[
\lim \inf \| A^3_m \|^{\frac{1}{3m}} = \lim \inf \| A^3_m \|'^{\frac{1}{3m}}. \quad (7.23)
\]

In particular, if

\[
\lim \| A^3_m \|^{\frac{1}{3m}}
\]
exists for one norm \( \| \cdot \| \), then it exists for all the other norms with the same value.

**Proof** By the norm equivalence property in the finite dimensional space, there are positive constants \( c_1 \) and \( c_2 \) such that for any \( B \in \mathbb{R}^{d_1 \times d_2 \times d_3} \),

\[
c_1 \| B \| \leq \| B \|' \leq c_2 \| B \|.
\]
Let \( B = A^3_m \). Then

\[
c_1^{\frac{1}{3m}} \| A^3_m \|^{\frac{1}{3m}} \leq \| A^3_m \|'^{\frac{1}{3m}} \leq c_2^{\frac{1}{3m}} \| A^3_m \|^{\frac{1}{3m}}.
\]
Taking \( \lim \sup \) and \( \lim \inf \), we have (7.22) and (7.23). The last conclusion follows. \( \square \)

**Definition 7.2** Let \( A \in \mathbb{R}^{d_1 \times d_2 \times d_3} \). If

\[
\lim \| A^3_m \|^{\frac{1}{3m}}
\]
exists for one norm \( \| \cdot \| \), then it exists for all the other norms with the same value.
We call it the Gelfand limit of \( A \), and denote it as \( \rho(A) \).

Preliminary numerical experiments show that the Gelfand limit exists for randomly generated tensors in \( \mathbb{R}^{d_1 \times d_2 \times d_3} \). The followings are such two examples.

**Example 7.3** We generate a general tensor \( A = (a_{ijk}) \in \mathbb{R}^{3 \times 2 \times 2} \) randomly:

\[
\begin{pmatrix}
a_{111} & a_{112} \\
a_{211} & a_{212} \\
\end{pmatrix}
\begin{pmatrix}
a_{121} & a_{122} \\
a_{221} & a_{222} \\
\end{pmatrix}
\begin{pmatrix}
a_{311} & a_{312} \\
a_{321} & a_{322} \\
\end{pmatrix}.
\]
\[
\begin{pmatrix}
-1.123199 & 0.147831 & -0.538225 & -1.028668 \\
0.643350 & 0.481342 & 0.438291 & -2.327008 \\
0.381771 & -0.093540 & 0.587050 & -0.656072
\end{pmatrix}
\]

The sequences of \( \|\mathcal{A}^m\|_\infty \), \( \|\mathcal{A}^m\|_{1/3}^2 \), and \( \|\mathcal{A}^m\|_1^{1/3} \) are:

| \( m \) | \( \|\mathcal{A}^m\|_{1/3}^2 \) | \( \|\mathcal{A}^m\|_{1/3}^2 \) | \( \|\mathcal{A}^m\|_1^{1/3} \) |
|---|---|---|---|
| 0 | 8.44635 | 3.13265 | 2.32701 |
| 1 | 3.53543 | 2.70503 | 2.56984 |
| 2 | 2.88902 | 2.68565 | 2.64747 |
| 3 | 2.75202 | 2.68562 | 2.67285 |
| 4 | 2.70758 | 2.68562 | 2.68136 |
| 5 | 2.69292 | 2.68562 | 2.6842 |
| 6 | 2.688053936890797 | 2.685623659558486 | 2.685149555668784 |
| 7 | 2.686435077687989 | 2.685623659558486 | 2.685465615628178 |
| 8 | 2.685893581832108 | 2.685623659558486 | 2.68557097721450 |
| 9 | 2.685713630635538 | 2.685623659558486 | 2.685606098662479 |
| 10 | 2.685636495826007 | 2.685623659558486 | 2.685617805913725 |
| 11 | 2.685636561959852 | 2.685623659558486 | 2.685621708342148 |
| 12 | 2.685626991766850 | 2.685623659558486 | 2.68562309152882 |
| 13 | 2.685624770294148 | 2.685623659558486 | 2.685623442756600 |
| 14 | 2.685624029803655 | 2.685623659558486 | 2.685623587256600 |
| 15 | 2.685623782973536 | 2.685623659558486 | 2.685623651256600 |
| 16 | 2.685623700696383 | 2.685623659558486 | 2.685623659525442 |
| 17 | 2.685623673271269 | 2.685623659558486 | 2.685623659547471 |
| 18 | 2.685623664129413 | 2.685623659558486 | 2.685623659554814 |
| 19 | 2.685623661082128 | 2.685623659558486 | 2.685623659557262 |
| 20 | 2.685623659727779 | 2.685623659558486 | 2.685623659558078 |
| 21 | 2.685623659566838 | 2.685623659558486 | 2.685623659558350 |
| 22 | 2.685623659558718 | 2.685623659558486 | 2.685623659558440 |
| 23 | 2.685623659558563 | 2.685623659558486 | 2.685623659558470 |
| 24 | 2.685623659558511 | 2.685623659558486 | 2.685623659558481 |
| 25 | 2.685623659558494 | 2.685623659558486 | 2.685623659558484 |
| 26 | 2.685623659558488 | 2.685623659558486 | 2.685623659558485 |
The three sequences converge to the same limit.

**Example 7.4** We now let $d_1 = 4$ and have $A \in \mathbb{R}^{4 \times 3 \times 2}$:

$$
\begin{pmatrix}
  a_{11} & a_{12} & a_{112} & a_{122} \\
  a_{211} & a_{221} & a_{212} & a_{222} \\
  s_{311} & a_{321} & a_{312} & a_{322} \\
  s_{411} & a_{421} & a_{412} & a_{422}
\end{pmatrix}.
$$

$$
\begin{pmatrix}
-0.512159 & -0.507535 & -1.383216 & 0.203856 & -0.578312 & 1.921669 \\
0.906334 & -0.258462 & -0.982083 & 0.736707 & 0.608575 & -1.063641 \\
-0.731184 & 0.525138 & -1.347676 & -0.782006 & 0.568222 & -0.214013 \\
-0.086664 & -0.736508 & 0.474856 & 0.345770 & 0.194509 & 0.006420
\end{pmatrix}.
$$

The sequences of $\|A^m\|_\infty$, $\|A^m\|_{1/3m}$ and $\|A^m\|_{1/3m}$ are:
| $m$ | $\|A^m\|^{1/3}_1$ | $\|A^m\|^{1/3}_2$ | $\|A^m\|^{1/3}_\infty$ |
|-----|------------------|------------------|------------------|
| 0   | 15.6755          | 3.86508          | 1.92167          |
| 1   | 4.0199           | 2.70142          | 2.31202          |
| 2   | 2.82591          | 2.54596          | 2.45592          |
| 3   | 2.61624          | 2.53718          | 2.50769          |
| 4   | 2.56299          | 2.53712          | 2.52722          |
| 5   | 2.54571          | 2.53712          | 2.53382          |
| 6   | 2.53998          | 2.53711866645693 | 2.53602         |
| 7   | 2.538072064165983 | 2.53711866456933 | 2.536751440470295 |
| 8   | 2.537436425894090 | 2.537118666456933 | 2.53699625188380 |
| 9   | 2.537224581847678 | 2.537118666456933 | 2.53707860944640 |
| 10  | 2.537153971095906 | 2.537118666456933 | 2.53710506456520 |
| 11  | 2.537130434615338 | 2.537118666456933 | 2.53711432478693 |
| 12  | 2.537122589170336 | 2.537118666456933 | 2.5371152129952 |
| 13  | 2.537119974027393 | 2.537118666456933 | 2.53711862681173 |
| 14  | 2.537119102313678 | 2.537118666456933 | 2.53711862531668 |
| 15  | 2.537118811742506 | 2.537118666456933 | 2.53711862481843 |
| 16  | 2.537118714885456 | 2.537118666456933 | 2.5371186237478 |
| 17  | 2.537118682599774 | 2.537118666456933 | 2.5371186237478 |
| 18  | 2.537118671837880 | 2.537118666456933 | 2.5371186237478 |
| 19  | 2.537118668250582 | 2.537118666456933 | 2.5371186565888 |
| 20  | 2.537118667054816 | 2.537118666456933 | 2.53711866626583 |
| 21  | 2.537118666656227 | 2.537118666456933 | 2.5371186680149 |
| 22  | 2.537118666523364 | 2.537118666456933 | 2.5371186643138 |
| 23  | 2.537118666479076 | 2.537118666456933 | 2.5371186648401 |
| 24  | 2.537118666465434 | 2.537118666456933 | 2.5371186645089 |
| 25  | 2.537118666459393 | 2.537118666456933 | 2.53711866595985 |
| 26  | 2.537118666457753 | 2.537118666456933 | 2.53711866456617 |
| 27  | 2.537118666457206 | 2.537118666456933 | 2.53711866456827 |
| 28  | 2.537118666457024 | 2.537118666456933 | 2.53711866456898 |
| 29  | 2.537118666456963 | 2.537118666456933 | 2.53711866456921 |
| 30  | 2.537118666456943 | 2.537118666456933 | 2.53711866456929 |
| 31  | 2.537118666456936 | 2.537118666456933 | 2.53711866456931 |

Again, the three sequences converge to the same limit.

We now leave the question if the Gelfand limit exists for any $A \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ for a further research topic. The following conclusion is directly from the definition.

**Proposition 7.5** Suppose that $A \in \mathbb{R}^{d_1 \times d_2 \times d_3}$ is nilpotent. Then its Gelfand limit
exists and is zero, i.e.,

\[ \rho(A) = 0. \]

The question is, if \( \rho(A) = 0 \), is \( A \) nilpotent?

In matrix analysis [3], for any matrix \( A \in \mathbb{R}^{n \times n} \) and any matrix norm \( ||| \cdot ||| \), we have

\[ \rho(A) \leq |||A|||. \]

Is this true for tensors? That is, for any \( A \in \mathbb{R}^{d_1 \times d_2 \times d_3} \) and any tensor norm \( ||| \cdot ||| \), do we have

\[ \rho(A) \leq |||A|||. \]

In matrix analysis [3], for any given matrix \( A \in \mathbb{R}^{n \times n} \) and \( \epsilon > 0 \), there is a matrix norm \( ||| \cdot ||| \) such that

\[ \rho(A) \leq |||A||| \leq \rho(A) + \epsilon. \]

Is this true for tensors? That is, for any given \( A \in \mathbb{R}^{d_1 \times d_2 \times d_3} \) and \( \epsilon > 0 \), is there a tensor norm \( ||| \cdot ||| \) such that

\[ \rho(A) \leq |||A||| \leq \rho(A) + \epsilon? \]

There are various mysterious things surrounding the Gelfand limit of a third order tensor. After unveiling such mysterious things, we should understand the spectral theory of tensors more.

8 Final Remarks

In this paper, we established two inequalities for the nuclear norm and the spectral norm of tensor products. We explored some of their applications. We hope that more of their applications can be explored.

Our results show that the matrix nuclear norm is a matrix norm in the classical sense. We hope that this discovery may be useful to the related research.

Then we extended the concept of matrix norm to tensor norm. We showed that the 1-norm, the Frobenius norm and the nuclear norm of tensors are tensor norms but the infinity norm and the spectral norm of tensors are not tensor norms. We hope that these results may deepen our understanding of norms of tensors.

Finally, we introduced the cubic power for a general third order tensor, and showed that the cubic power of a general third order tensor tends to zero as the power increases to infinity, if there is a tensor norm such that the tensor norm of that third order tensor is less than one. Then we raised a question on a possible Gelfand formula for a general third order tensor. Preliminary numerical results show that such a limit exists in general. As the limit in the matrix Gelfand formula is the spectral radius of
the matrix, this may open the way for a new spectral theory for a general third order
tensor.

Our derivations are on the field of real numbers. They can be extended to the field
of complex numbers without difficulty.

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