Greedy algorithms and poset matroids

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Abstract

We generalize the matroid-theoretic approach to greedy algorithms to the setting of poset matroids, in the sense of Barnabei, Nicoletti and Pezzoli (1998) [BNP]. We illustrate our result by providing a generalization of Kruskal algorithm (which finds a minimum spanning subtree of a weighted graph) to abstract simplicial complexes.

1 Introduction

An independence system is a pair \((E, F)\) such that \(E\) is a finite set and \(F\) is a down-set of the Boolean algebra \(\wp(E)\). A matroid is an independence system satisfying the following axiom: for any \(A, B \in F\) such that \(|B| = |A| + 1\), there exists \(b \in B \setminus A\) such that \(A \cup \{b\} \in F\).

In the paper [BNP] the authors propose a generalization of the notion of matroid where the ground set is equipped with a partial order. The central definition of their work is the following: a poset matroid is a pair \((P, I)\) where \(P\) is a finite partially ordered set and \(I\) is a nonempty family of up-sets of \(P\) satisfying the following properties:

(i) if \(X, Y\) are up-sets of \(P\) such that \(Y \in I\) and \(X \subseteq Y\), then \(X \in I\);

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(ii) for every $X, Y \in \mathcal{I}$ with $|X| < |Y|$, there exists $y \in \text{Max}(Y \setminus X)$ such that $X \cup \{y\} \in \mathcal{I}$.

The elements of $\mathcal{I}$ are called the independent sets of the poset matroid. To understand the above definition, we recall that an up-set (resp., down-set) of a poset $P$ is a subset $S$ of $P$ such that, if $x \in S$ and $x \leq y$ (resp. $x \geq y$), then $y \in S$. Moreover, for any $S \subseteq P$, we denote by $\text{Max}(S)$ the set of maximal elements of $S$.

We remark that the definition given here differs from the original one in [BNP], which is given in terms of the notion of basis. However, the two definitions are clearly equivalent, as it is shown in [BNP].

Given this generalized notion of matroid, it is natural to try to generalize notions and results of matroid theory to the context of poset matroids. Among the open problems proposed in [BNP], the last one is the following: is it possible to generalize the generic greedy algorithm to the setting of poset matroids? To better understand this problem, recall that there is a strong relationship between greedy algorithms and the notion of matroid, which we will briefly summarize below.

Given a weight function $w : E \to \mathbb{R}^+$, we consider the following problem:

**Input:** an independence system $(E, \mathcal{F})$ and a weight function $w : E \to \mathbb{R}^+$.

**Output:** a set $M \in \mathcal{F}$ such that $w(M) = \sum_{x \in M} w(x)$ is maximum.

The greedy algorithm for the independence system $(E, \mathcal{F})$ attempts to solve the above problem, and consists of the following
procedure:

| Algorithm 1: GREEDY((E, F), w) |
|--------------------------------|
| S := ∅; |
| Q := E; |
| while Q ≠ ∅ do |
| find m ∈ Q having maximum weight; |
| Q := Q \ {m}; |
| if S ∪ {m} ∈ F then |
| S := S ∪ {m}; |
| return S; |

The procedure GREEDY tries to find a global solution by making the local best choice at each step. Unfortunately, such an algorithm is not always correct (that is, it does not solve the above problem in general). The following theorem by Edmonds and Rado [E, R] tells us in which cases it works.

Theorem 1.1 Given an independence system (E, F), the following statements are equivalent:

a) for any weight function w, GREEDY is correct on input (E, F), w;

b) (E, F) is a matroid.

In the next section we will generalize the Edmonds-Rado theorem to the setting of poset matroid. In section 3 we will see how our generalization can be used to find an analog of Kruskal algorithm, which determines a minimum spanning subtree of a weighted graph, where the graph is replaced by an abstract simplicial complex. Finally, in the last section we will give some hints to relate our work with previous approaches on the same (or perhaps similar) subject.

2 The Edmonds-Rado theorem for poset matroids

Given a poset \( P \), let \( \mathcal{I} \) be a family of up-sets of \( P \) satisfying condition (i) in the definition of poset matroid (i.e. \( \mathcal{I} \) is a down-set of up-sets of \( P \)). Call such a pair \((P, \mathcal{I})\) a po-independence system.
Consider the following problem:

**input:** a po-independence system \((P, \mathcal{I})\) and a weight order-preserving function \(w : P \to \mathbb{R}^+\).

**output:** an up-set \(M \in \mathcal{I}\) such that \(w(M) = \sum_{x \in M} w(x)\) is maximum.

To solve it we try to adapt the greedy algorithm as follows:

**Algorithm 2: PGREEDY\(((P, \mathcal{I}), w)\)**

\[
\begin{align*}
S &:= \emptyset; \\
Q &:= P; \\
\text{while } Q \neq \emptyset &\text{ do} \\
&\quad \text{find a maximal element } m \in Q \text{ having maximum weight;} \\
&\quad Q := Q \setminus \{m\}; \\
&\quad \text{if } S \cup \{m\} \in \mathcal{I} \text{ then} \\
&\quad \quad S := S \cup \{m\}; \\
\text{return } S;
\end{align*}
\]

Our main result is the following generalization of the Edmonds-Rado theorem for poset matroids.

**Theorem 2.1** Given a po-independence system \((P, \mathcal{I})\), the following statements are equivalent:

a) for any weight order-preserving function \(w\), PGREEDY is correct on input \((P, \mathcal{I}), w\);

b) \((P, \mathcal{I})\) is a poset matroid.

**Proof.** a) \(\Rightarrow\) b) Suppose that \((P, \mathcal{I})\) is not a poset matroid. This means that there exist \(A, B \in \mathcal{I}\), with \(|A| = k\) and \(|B| = k + 1\), such that, for all \(b \in \text{Max}(B \setminus A)\), \(A \cup \{b\}\) is an up-set but \(A \cup \{b\} \notin \mathcal{I}\). Consider the weight function \(w : P \to \mathbb{R}^+\) defined as follows:

\[
w(x) = \begin{cases} 
\alpha (> 1) & x \in A \\
1 & x \in B \setminus A \\
0 & x \notin A \cup B
\end{cases}
\]

We start by observing that \(w\) is order preserving. Indeed, let \(x, y \in P\) such that \(x \leq y\). If \(x \in A\), then also \(y \in A\) (since \(A \]
is an up-set), whence trivially \( w(x) = w(y) = \alpha \). If \( x \in B \setminus A \), then clearly \( y \in B \) (since \( B \) is an up-set); now, if \( y \in A \), then \( w(x) = 1 < \alpha = w(y) \), whereas, if \( y \notin A \), then \( w(x) = 1 = w(y) \). Finally, if \( x \notin A \cup B \), then trivially \( w(x) = 0 \leq w(y) \).

Now let \( S \in \mathcal{I} \) be the solution provided by \textsc{pgreedy}. Depending on its cardinality, \( S \) is a subset of \( A \) or it contains all elements of \( A \) and some elements not in \( B \setminus A \). In fact, the elements of \( A \) are the first ones that are chosen by \textsc{pgreedy}, since they have maximum weight (at each step, \textsc{pgreedy} will select a maximal element among the remaining ones in \( A \)). In case all the elements of \( A \) have already been selected, it is possible that some (possibly all) of the elements of \( P \) not belonging to \( B \setminus A \) are included in \( S \). Denote by \( C \) the set of these elements (\( C \) may also be empty). Observe that \textsc{pgreedy} cannot choose other elements, since, by hypothesis, \( A \cup \{b\} \notin \mathcal{I} \), for all \( b \in \text{Max}(B \setminus A) \) (and so \textsc{pgreedy} never enters \( B \setminus A \)). Now, set \( t = |A \cap B| \), we have

\[
\begin{align*}
    w(S) & \leq w(A \cup C) = w(A) + w(C) = \alpha \cdot |A| = \alpha \cdot k \\
    w(B) & = w(B \setminus A) + w(A \cap B) = (k + 1 - t) + \alpha \cdot t.
\end{align*}
\]

Thus, if we choose \( 1 < \alpha < 1 + \frac{1}{k-t} \), we get \( w(S) < w(B) \), that is \( S \) has not maximum weight, whence \textsc{pgreedy} is not correct in this case.

\( b) \Rightarrow a) \) Let \( S = \{b_1, b_2, \ldots, b_n\} \) be the solution provided by \textsc{pgreedy} on input \((P, \mathcal{I}), w\), and suppose that \( w(b_1) \geq w(b_2) \geq \cdots \geq w(b_n) \). Now consider \( A = \{a_1, a_2, \ldots, a_m\} \in \mathcal{I} \), with \( w(a_1) \geq w(a_2) \geq \cdots \geq w(a_m) \).

We start by observing that \( m \leq n \). Indeed, suppose \( n < m \); then (since \((P, \mathcal{I})\) is a poset matroid) there would exist \( a_j \in \text{Max}(A \setminus S) \) such that \( S \cup \{a_j\} \notin \mathcal{I} \). Moreover, for every up-set \( R \subseteq S \cup \{a_j\} \), we would obviously have \( R \in \mathcal{I} \), so \( a_j \) should belong to \( S \), which is not.

Next we will prove that \( w(a_i) \leq w(b_i) \), for all \( i = 1, \ldots, m \). Suppose it is not, and let \( k \) be the minimum index for which \( w(a_k) > w(b_k) \). Notice that \( D = \{b_1, \ldots, b_{k-1}\} \in \mathcal{I} \), up to rearranging the elements of \( S \). This can be achieved without losing the property \( w(b_1) \geq w(b_2) \geq \cdots \geq w(b_n) \), since \( w \) is order-preserving. The same argument also shows that \( \{a_1, \ldots, a_k\} \in \mathcal{I} \). Now, since \( |D| + 1 = |\{a_1, \ldots, a_k\}| \), there exists \( a_j \in \text{Max}(\{a_1, \ldots, a_k\} \setminus D) \) such that
$D \cup \{a_j\} \in I$. But $w(b_k) \geq w(a_j)$ (since at the $k$-th step PGREEDY chooses the element having maximum weight among the remaining maximal ones) and $w(a_j) \geq w(a_k)$ (since $j \leq k$), whence $w(b_k) \geq w(a_k)$, which is contrary to the assumption.

The two above facts implies that $w(A) \leq w(S)$, and so that $S$ is indeed the correct solution, as desired. ■

3 Acyclic subcomplexes of an abstract simplicial complex

In order to illustrate our generalization of Edmonds-Rado theorem to poset matroids, we propose a generalization of the well-known Kruskal algorithm, which constructs a minimum spanning sub tree of a weighted graph.

Recall that an abstract simplicial complex on a finite set $X$ is a family $C$ of subsets of $X$ such that, if $F \in C$ and $G \subseteq F$, then $G \in C$ (i.e., a down-set of the powerset of $X$ partially ordered by containment). Given $F \in C$, we say that $F$ is a face of dimension $i$ of $C$ when $|F| = i$. The set of all faces of dimension $i$ of $C$ will be denoted $F_i$. Therefore, if the maximum dimension of a face of $C$ is $k$ (also called the dimension of $C$), then $C = \bigcup_{i=0}^{k} F_i$. Moreover, given $D \subseteq C$, we say that $D$ is a subcomplex of $C$ when it is itself an abstract simplicial complex.

The faces of an abstract simplicial complex can be partially ordered in a natural way by containment. However, to be consistent with the theory we have developed in the previous sections, we rather need to consider the dual order. Thus, given $F, G \in C$, we define $F \leq G$ whenever $G \subseteq F$. Observe that a subcomplex of $C$ is an up-set of $(C, \leq)$.

Suppose that $C$ is an abstract simplicial complex of dimension $k$. For any $2 \leq h \leq k$, we say that $D \subseteq C$ is an $h$-cycle of $C$ when:

- $D \subseteq F_h$;
• for every \( F \in \mathcal{D} \) and for every \( x \in F \), there exists precisely one face \( H \in \mathcal{D} \) such that \( F \cap H = F \setminus \{x\} \).

When an abstract simplicial complex does not have any \( h \)-cycles it will be called \( h \)-acyclic. Observe that an \( h \)-cycle of a complex \( \mathcal{C} \) is not a subcomplex of \( \mathcal{C} \). Moreover, given a face \( F \in \mathcal{C} \) of dimension \( h + 1 \), the set of all faces of \( F \) of dimension \( h \) is an \( h \)-cycle, which will be denoted \(< F >\).

The following key lemma is central in the proof of our final result.

**Lemma 3.1** Let \( \mathcal{D}_1, \mathcal{D}_2 \) be two \( h \)-cycles of the abstract simplicial complex \( \mathcal{C} \) such that \( \mathcal{D}_1 \cap \mathcal{D}_2 \neq \emptyset \). Then \( \mathcal{D} = (\mathcal{D}_1 \cup \mathcal{D}_2) \setminus (\mathcal{D}_1 \cap \mathcal{D}_2) \) is an \( h \)-cycle of \( \mathcal{C} \) as well.

**Proof.** Obviously \( \mathcal{D} \subseteq \mathcal{F}_h \). Now take \( F \in \mathcal{D} \) and \( x \in F \); suppose moreover (w.l.o.g.) that \( F \in \mathcal{D}_1 \setminus \mathcal{D}_2 \subseteq \mathcal{D}_1 \). Since \( \mathcal{D}_1 \) is an \( h \)-cycle, there exists a unique \( H \in \mathcal{D}_1 \) such that \( F \cap H = F \setminus \{x\} \). Moreover, it is clearly \( F \setminus \{x\} = H \setminus \{y\} \), for some \( y \in H \). If \( H \notin \mathcal{D}_1 \cap \mathcal{D}_2 \), then there is nothing else to prove. Otherwise, if \( H \in \mathcal{D}_1 \cap \mathcal{D}_2 \), then in particular \( H \in \mathcal{D}_2 \), whence there is a unique \( G \in \mathcal{D}_2 \) such that \( H \cap G = H \setminus \{y\} \). Once again, we also have that \( H \setminus \{y\} = G \setminus \{z\} \), for some \( z \in G \). Observe that \( G \notin \mathcal{D}_1 \), since otherwise there would exist two distinct faces in \( \mathcal{D}_1 \) whose intersection with \( H \) equals \( H \setminus \{y\} \) (namely \( F \) and \( G \)). Thus, in particular, \( G \in \mathcal{D} \), and we have:

\[
F \cap G = ((F \setminus \{x\}) \cup \{x\}) \cap G = ((F \setminus \{x\}) \cap G) \cup (\{x\} \cap G)
\]

\[
(H \setminus \{y\}) \cap G = H \cap G = H \setminus \{y\} = F \setminus \{x\}.
\]

Finally, observe that \( G \) is the unique face in \( \mathcal{D} \) having the above property, since otherwise there would exist two distinct faces in \( \mathcal{D}_2 \) whose intersection with \( H \) equals \( H \setminus \{y\} \). \( \blacksquare \)

We will also use a result of [BNP], which asserts that property (ii) in the definition of a poset matroid can be replaced by a sort of “local version”. We report the precise statement in the next lemma.

**Lemma 3.2** ([BNP]) Let \( \mathcal{I} \) a nonempty family of filters of a poset \( P \) satisfying property (i) in the definition of poset matroid. Then the following are equivalent:
(ii) for every $X, Y \in \mathcal{I}$ with $|X| < |Y|$, there exists $y \in \text{Max}(Y \setminus X)$ such that $X \cup \{y\} \in \mathcal{I}$;

(ii') for every $X, Y \in \mathcal{I}$ with $|Y| = 1 + |X|$ and $|X| = 1 + |X \cap Y|$, there exists $y \in \text{Max}(Y \setminus X)$ such that $X \cup \{y\} \in \mathcal{I}$.

Given $2 \leq h \leq k$, define $\mathcal{J}_h = \{D \subseteq C \mid D$ is a subcomplex of $C$ and does not contain $h$-cycles$\}$.  

Proposition 3.1 For any given $h$, $(C, \mathcal{J}_h)$ is a poset matroid.

Proof. First of all, it is clear that, if $D \in \mathcal{J}_h$ and $\tilde{D} \subseteq D$ is a subcomplex, then $\tilde{D} \in \mathcal{J}_h$ as well.

To conclude the proof it will be enough to show that property (ii') of the above lemma holds. So let $D_1, D_2 \in \mathcal{J}_h$ such that $|D_2| = 1 + |D_1|$ and $|D_1| = 1 + |D_1 \cap D_2|$. Observe that, in this situation, it is $|D_1 \setminus D_2| = 1$ and $|D_2 \setminus D_1| = 2$. Suppose that $D_1 \setminus D_2 = \{x\}$ and $D_2 \setminus D_1 = \{y_1, y_2\}$. There are of course two distinct possibilities concerning $y_1$ and $y_2$. Suppose first that $y_1$ and $y_2$ are incomparable. By way of contradiction, suppose there exist $h$-cycles $Z_1, Z_2$ such that $Z_i \subseteq D_1 \cup \{y_i\}$, for $i = 1, 2$. Since $D_2 \in \mathcal{J}_h$, there are $x_1, x_2 \in D_1 \setminus D_2$ such that $x_i \in Z_i$, for $i = 1, 2$. However, our hypotheses imply that $x_1 = x_2 = x$, whence we would have (from lemma 3.1) that $(Z_1 \cup Z_2) \setminus \{x_1\} \subseteq D_2$ contains an $h$-cycle, which is forbidden since $D_2 \in \mathcal{J}_h$. Finally, suppose that $y_1 < y_2$. Once again, we argue by contradiction, supposing that there exists an $h$-cycle $Z \subseteq D_1 \cup \{y_2\}$. This implies that $y_2$ is a face of dimension $h$, and so $y_1 \in D_2$ has dimension $h + 1$. Observe that all the faces of $y_1$ but $y_2$ must be both in $D_2$ (since $D_2$ is an up-set) and in $D_1$ (since $D_2 \setminus D_1 = \{y_1, y_2\}$), whence $(Z \cup y_1 >) \setminus \{y_2\} \subseteq D_1$. Moreover $Z$ and $y_1 >$ are $h$-cycles both containing $y_2$, hence, by lemma 3.1 $(Z \cup y_1 >) \setminus \{y_2\}$ contains an $h$-cycle, which is impossible.  

We are now in a position to provide a Kruskal-like algorithm to find a maximum spanning subcomplex of an abstract simplicial complex with respect to a suitable weight function of its faces. A spanning subcomplex of a complex $C$ is a subcomplex of $C$ containing all its 0-dimensional faces. The weight of a (sub)complex is simply the sum of the weights of its faces.
Theorem 3.1 Let \( C \) be an abstract simplicial complex, and let \( w : C \to \mathbb{R}^+ \) be an order-reversing function (given that \( C \) is partially ordered by containment). Then the algorithm \textsc{PGreedy} is correct on input \( (\langle C, \mathcal{I}_h \rangle, w) \), and returns a spanning \( h \)-acyclic subcomplex of \( C \) having maximum weight.

4 Conclusions

In this note we have extended the classical Edmonds-Rado theorem to the more general setting of poset matroids described in [BNP]. We have illustrated our result by generalizing a classical algorithm on graphs due to Kruskal to the setting of abstract simplicial complexes. Of course, lots of other possible applications can be considered. One of the most interesting is perhaps the generalization of the greedy solution of the classical task scheduling problem presented, for instance, in [CLRS]. The obvious modification of this very well-known application of Edmonds-Rado theorem consists of introducing a priority between tasks, which can be naturally formalized as a partial order relation. However, our attempts to find a correct analog of this problem (and its solution) in the context of poset matroids have been unsuccessful, so it would be very interesting to have some results in this direction.

We remark that the extension of the concept of matroid on finite sets to posets considered in the present paper is not the only one that can be found in the literature. Another well known approach is through the theory of geometries on partially ordered sets due to Faigle [F2], which is however intimately related to the one proposed in [BNP].

Even more interestingly, in [F1] Faigle finds a necessary and sufficient condition for a generic greedy algorithm to be correct in a setting that is extremely similar to ours. Apart from the fact that he considers independent set to be down-set rather than up-sets, which is an immaterial difference (it just consists of dualizing all the definitions given here), the analogies with our results are really striking. However, the conditions found by Faigle (which are condensed in what he calls a “generating set”) are slightly different from our, and it is not immediately evident how to relate the two approaches.
Another well-known generalization of matroid theory, which is more oriented towards greedy algorithms, is the theory of greedoids introduced by Korte and Lovász in [KL]. In [LZ] the authors try to merge the notions of poset matroid and of greedoid by developing the theory of poset greedoids. It would be interesting to have a generalization of our results to the setting of poset greedoids.

We conclude by recalling that in [S] the author proves that the correctness of a general greedy algorithm for a hereditary system is equivalent to the fact that such system is a so-called strict cg-matroid. It is likely that there is a relationship between the results of the present paper and those of [S], but it is not clear to us how to make it explicit.

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