Hidden symmetries and large $N$ factorisation for permutation invariant matrix observables

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ABSTRACT: Permutation invariant polynomial functions of matrices have previously been studied as the observables in matrix models invariant under $S_N$, the symmetric group of all permutations of $N$ objects. In this paper, the permutation invariant matrix observables (PIMOs) of degree $k$ are shown to be in one-to-one correspondence with equivalence classes of elements in the diagrammatic partition algebra $P_k(N)$. On a 4-dimensional subspace of the 13-parameter space of $S_N$ invariant Gaussian models, there is an enhanced $O(N)$ symmetry. At a special point in this subspace, is the simplest $O(N)$ invariant action. This is used to define an inner product on the PIMOs which is expressible as a trace of a product of elements in the partition algebra. The diagram algebra $P_k(N)$ is used to prove the large $N$ factorisation property for this inner product, which generalizes a familiar large $N$ factorisation for inner products of matrix traces invariant under continuous symmetries.

KEYWORDS: $1/N$ Expansion, Discrete Symmetries, Matrix Models, Gauge-Gravity Correspondence

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1 Introduction

The simplifications of large $N$ in matrix quantum field theories in diverse dimensions with continuous gauge symmetries such as $U(N), O(N), Sp(N)$ discovered in [1] have played a major role in the development of gauge-string duality in subsequent years. This includes low-dimensional non-critical string theories dual to zero-dimensional QFTs (matrix models) [2–4], the string dual of two-dimensional Yang-Mills theories [5], and the generalization to higher dimensions in the AdS/CFT correspondence [6]. In theories with continuous symmetries containing adjoint fields, the space of gauge invariants is generated by traces of matrices. An important aspect of simplicity in the large $N$ limit is “large $N$ factorisation”. In the context of AdS/CFT, large $N$ factorisation for two-point functions involving gauge invariants built from a complex matrix is an expression of orthogonality for distinct trace structures [7]. This plays an important role in the connection between multi-traces constructed from a small number of matrices and perturbative gravitons in the AdS dual [8, 9]. The breakdown of this orthogonality when the number of matrices becomes comparable to $N$ guides the identification of CFT duals [7, 10, 11] for giant gravitons [12–14]. Large $N$ factorisation also enters the construction of gauge-string duals in collective field theory [15–18], which gives useful insights into the emergence of classical limits at large $N$. It is also
employed in the Master field approach to large $N$ [19] and loop equations [20] (see for example [21–23] for recent developments of these themes). In the geometrical construction of gauge-string duality, based on Schur-Weyl duality and branched covers, in instances such as large $N$ 2d Yang Mills [5, 24–31] or the simple toy model of Gaussian Hermitian matrix theory [32–35], trace structures of matrix invariants correspond to branching structures of covering maps from string worldsheets.

In this paper, we will develop the theme of large $N$ factorisation for permutation invariant matrix models [36–39]. Permutation invariance was motivated in [36] in the context of matrix data arising in computational linguistics [40–43]. The formulation of large $N$ factorisation we will use is similar to the one in [7]. We will use the simplest inner product on the space of permutation invariant matrix observables (PIMOs). It comes from a special point on the moduli space of $S_N$ invariant Gaussian matrix models where the action has $O(N)$ symmetry. This is the first sense in which hidden symmetries appear in this paper. Permutation invariant random matrix distributions have also been studied from the point of view of mathematical statistics, using partition algebra diagrams [44–46].

The second kind of hidden symmetry appearing in this paper is based on Schur-Weyl duality. Observables invariant under the action of a symmetry group $G$, as we will discuss in this paper, are organized by algebras dual to $G$. For the case of $U(N)$ symmetry the dual algebras are based on the standard Schur-Weyl duality [47] between $U(N)$ and $S_k$ in the $k$-fold tensor product of $V^\otimes k$ of the fundamental representation $V$ of $U(N)$. Applications of Schur-Weyl duality to the computation of correlators in matrix models with $U(N)$ symmetry are developed in [10, 48–60] and short reviews are [61, 62]. The $U(N)$ case serves as a powerful source of useful analogies throughout the paper. When $U(N)$ is replaced by $S_N$ as the invariance of interest, the Schur-Weyl dual algebras are diagrammatic partition algebras $P_k(N)$. These algebras have been studied in statistical physics and representation theory [63–65]. The algebras $P_k(N)$ are finite dimensional and have a basis which can be labelled by diagrams, corresponding to set partitions of a set of $2k$ objects. A set partition of a set $S$ is a collection of non-empty subsets of $S$, such that any pair of subsets has zero intersection, and the union of the subsets is the set $S$. Equivalently, every element of the set is included in exactly one of the subsets (see for example [66] for further information on set partitions).

The paper is organised as follows. In section 2 we review the construction of the permutation invariant Gaussian 1-matrix model, and the counting of invariant matrix observables developed in [36, 37]. Here we give a new description of the counting, which emphasizes the underlying hidden partition algebra symmetry arising as a consequence of Schur-Weyl duality. We end the section with a derivation of the $O(N)$ symmetric point in the moduli space of $S_N$ invariant 1-matrix models.

Section 3 is dedicated to the construction of PIMOs by means of partition algebras. We give a brief description of the partition algebras. In particular we present the diagram basis and describe how the product is computed by using a composition of diagrams. The construction of $U(N)$ invariants using symmetric group algebras is reviewed as a warm-up exercise. This is generalised to give a map from partition algebra elements to PIMOs (equation (3.13)), leading to a correspondence between PIMOs and equivalence classes of
partition algebra elements. These equivalence classes are defined in equation (3.16). The simplest $O(N)$ invariant action is used to define an inner product on the space of PIMOs in terms of a trace of partition algebra elements (equation (3.18)).

Section 4 proves the large $N$ factorization of the inner product on PIMOs thus defined. That is, we show that

$$\langle \hat{O}_i \hat{O}_j \rangle = \delta_{ij} + O(1/\sqrt{N}), \quad (1.1)$$

where $\hat{O}_i, \hat{O}_j$ are normalized PIMOs labelled by indices $i, j$ running over equivalence classes of partition algebra elements. The proof of large $N$ factorization relies on the existence of a partial ordering on the diagram basis for the partition algebra. The partial ordering is related to an inclusion of diagrams, and can itself be described by another diagram of diagrams called a Hasse diagram [67]. We end the section by extending the proof to multi-matrix observables.

2 Hidden symmetries in permutation invariant Gaussian matrix models

In [37] a 13-parameter family of Gaussian matrix models consistent with permutation invariance was constructed, by using a transformation from the matrix variables $M_{ij}$ to variables labelled by irreducible representations of $S_D$. The expectation values of linear and quadratic permutation invariant polynomials in $M_{ij}$ were given in terms of the representation theoretic parameters. Expectation values for a sample of cubic and quartic invariant polynomials were computed using Wick’s theorem. Additional examples were computed in [38]. The results were generalized to the 2-matrix case in [39]. Computer code for expectation values of invariant polynomials in the 1-matrix and 2-matrix cases is available as part of [39].

The schematic form of the permutation invariant Gaussian matrix model (PIGMM) is

$$\int dM \exp(-S(M)) = \int dM \exp \left( -\sum_{i=1}^2 \mu_i L_i(M) - \frac{1}{2} \sum_{i=1}^{11} g_i Q_i(M) \right). \quad (2.1)$$

The action $S(M)$ contains two linear terms: $L_1, L_2$; and eleven quadratic terms $Q_1, \ldots, Q_{11}$. It is the most general quadratic action invariant under the following group action of $S_N$ (the symmetric group on $N$ objects),

$$S(M_{\sigma(i)\sigma(j)}) = S(M_{ij}), \quad \text{for all } \sigma \in S_N. \quad (2.2)$$

The permutations $\sigma \in S_N$ are invertible maps $\sigma : \{1, \cdots, N\} \to \{1, \ldots, N\}$. The product of two permutations $\sigma_1, \sigma_2$ is defined by composing the maps: $\sigma_1 \sigma_2(i) = \sigma_2(\sigma_1(i))$. As an example, consider the following two permutations in $S_3$:

$$\sigma_1 : 1 \mapsto 2, \ 2 \mapsto 3, \ 3 \mapsto 1,$$

$$\sigma_2 : 1 \mapsto 2, \ 2 \mapsto 1, \ 3 \mapsto 3. \quad (2.3)$$

In this case

$$\sigma_1 \sigma_2 : 1 \mapsto 1, \ 2 \mapsto 3, \ 3 \mapsto 2. \quad (2.4)$$
Let $V_N$ be an $N$-dimensional vector space with orthonormal basis $e_i,$ $i = 1, \ldots, N$. The defining representation $\rho_N : S_N \to \text{GL}(V_N)$ of $S_N$ assigns the following linear operator to every permutation

$$\rho_N(\sigma)e_i = e_{\sigma^{-1}(i)}.$$  

(2.5)

From equation (2.2), we see that the vector space spanned by $M_{ij}$ is acted on by $S_N$ in the same way as $V_N \otimes V_N$. We have the identification

$$M_{ij} \longleftrightarrow e_i \otimes e_j.$$  

(2.6)

This is not an irreducible representation, it decomposes into several irreducible components

$$V_N \otimes V_N \cong 2V_{[N]} \oplus 3V_{[N-1,1]} \oplus V_{[N-2,1,1]} \oplus V_{[N-2,2]}.$$

(2.7)

Here $V_{[N]}$ corresponds to the trivial one-dimensional representation of $S_N$. The representations $V_{[N-1,1]}, V_{[N-2,1,1]}, V_{[N-2,2]}$ are non-trivial irreducible representations, labeled by integer partitions of $N$. Their dimensions are respectively

$$N - 1, (N - 1)(N - 2)/2, N(N - 3)/2.$$  

(2.8)

We will use the index $\Lambda_1$ ranging over the labels for irreducible representations

$$\Lambda_1 \in \{[N],[N-1,1],[N-2,1,1],[N-2,2]\}$$

(2.9)

and we refer to the corresponding irreducible representations of $S_N$ as $V_{\Lambda_1}^{S_N}$. The above decomposition can be deduced using

$$V_N \cong V_{[N]} \oplus V_{[N-1,1]},$$

(2.10)

together with the tensor product rule described in section 7.13 of [68]. See also [69] for a dedicated treatment of symmetric group representation theory.

Note that the multiplicity of $V_{[N]}$ in (2.7) is exactly why there are two linear terms $L_1, L_2$ in the action (2.1). Furthermore the isomorphism in equation (2.7) implies that there exists a set of linear combinations of matrix elements labelled by $\Lambda_1$

$$S_{a,\Lambda_1,\alpha} = \sum_{i,j} C_{a,ij}^{\Lambda_1,\alpha} M_{ij}.$$  

(2.11)

The index $a$ is a state index for the irreps, while $\alpha$ is a multiplicity index

$$a \in \{1, \ldots, \text{Dim} V_{\Lambda_1}^{S_N}\},$$

$$\alpha \in \{1, \ldots, \text{Mult}(V_N \otimes V_N \to V_{\Lambda_1}^{S_N})\},$$

(2.12)

where $\text{Mult}(V_N \otimes V_N \to V_{\Lambda_1}^{S_N})$ is the multiplicity of $V_{\Lambda_1}^{S_N}$ in $V_N \otimes V_N$. The change of basis is given by the Clebsch-Gordan coefficients $C_{a,ij}^{\Lambda_1,\alpha}$. They have the property

$$\sum_{i,j} C_{a,ij}^{\Lambda_1,\alpha} M_{\sigma^{-1}(i)\sigma^{-1}(j)} = \sum_b D_{ab}^{\Lambda_1} (\sigma) S_b^{\Lambda_1,\alpha},$$

(2.13)
where the matrices \( D_{ab}^{\Lambda_1}(\sigma) \) are irreducible matrix representations of \( S_N \) (background on the Clebsch-Gordan coefficients for symmetric groups is available in [68]). Without loss of generality, we can assume that the Clebsch-Gordan coefficients define an orthonormal basis with respect to the inner product

\[
(M_{ij}, M_{kl}) = \delta_{ik}\delta_{jl}. \tag{2.14}
\]

Equivalently, the representation theoretic variables satisfy

\[
(S_a^{\Lambda_1,\alpha}, S_b^{\Lambda_1,\beta}) = \delta_{ab}\delta^{\Lambda_1}\delta^{\alpha\beta}. \tag{2.15}
\]

Together with the fact that the inner product (2.14) is \( S_N \) invariant, it follows that

\[
D_{ab}^{\Lambda_1}(\sigma^{-1}) = D_{ba}^{\Lambda_1}(\sigma). \tag{2.16}
\]

Using the above basis it immediately follows that the quadratic combination

\[
\sum_a S_a^{\Lambda_1,\alpha} S_a^{\Lambda_1,\beta} = \sum_{i,j,k,l} M_{ij} Q_{ijkl}^{\Lambda_1,\alpha\beta} M_{kl} \tag{2.17}
\]

is an invariant polynomial, where

\[
Q_{ijkl}^{\Lambda_1,\alpha\beta} = \sum_a C_{a,ij}^{\Lambda_1,\alpha} C_{a,kl}^{\Lambda_1,\beta}. \tag{2.18}
\]

A useful observation is that, while the Clebsch-Gordan coefficients depend on a choice of basis for every irreducible component in (2.7), the tensors \( Q_{ijkl}^{\Lambda_1,\alpha\beta} \) do not. For all four \( \Lambda_1 \), they can be constructed by using only the explicit bases for the subspaces \( V_{[N]} \) and \( V_{[N-1,1]} \) in (2.7) [37].

We may associate a unique parameter to every invariant. Since

\[
\sum M_{ij} Q_{ijkl}^{\Lambda_1,\alpha\beta} M_{kl} = \sum M_{ij} Q_{ijkl}^{\Lambda_1,\beta\alpha} M_{kl}, \tag{2.19}
\]

there is a symmetric matrix of dimension \( \text{Mult}(V_N \otimes V_N \rightarrow V_{S_N}^{\Lambda_1}) \) parametrising the quadratic part of the action, for every choice of \( \Lambda_1 \). Using the multiplicities in the decomposition (2.7), we have

\[
11 = \frac{2 \cdot 3}{2!} + \frac{3 \cdot 4}{2!} + \frac{1 \cdot 2}{2!} + \frac{1 \cdot 2}{2!}, \tag{2.20}
\]

independent parameters. The two linear terms are given by

\[
\mu_1 L_1 = \mu_1 S^{[N],1} \quad \text{and} \quad \mu_2 L_2 = \mu_2 S^{[N],2}. \tag{2.21}
\]

The quadratic part is

\[
\sum_{\Lambda_1,\alpha,\beta} S_a^{\Lambda_1,\alpha} g_{a,\beta\alpha} S_a^{\Lambda_1,\beta} = \sum_{i,j,k,l} g_{\alpha,\beta} M_{ij} Q_{ijkl}^{\Lambda_1,\alpha\beta} M_{kl}, \tag{2.22}
\]
where the matrices $g_{\alpha \beta}^{A_1}$ are parameters of the model. In this basis the partition function is,

$$
\int dM \exp(-S(M)) = \int dS \exp\left(-\mu_1 S^{[N],1} - \mu_2 S^{[N],2} - \sum_{\Lambda_1,a} S_{\alpha}^{\Lambda_1,\alpha} g_{\alpha \beta}^{\Lambda_1} g_{\alpha \beta}^{\Lambda_1} \right)
$$

(2.23)

The matrices $g_{\alpha \beta}^{A_1}$ must have non-negative eigenvalues to define a convergent integral.

Note that the parameters in the quadratic part of the action in [37] are related to the parameters in this paper as

$$
(\Lambda_0^V)_{\alpha \beta} \leftrightarrow g_{\alpha \beta}^{[N]},
(\Lambda_{VH})_{\alpha \beta} \leftrightarrow g_{\alpha \beta}^{[N-1,1]},
(\Lambda_1) \leftrightarrow g^{[N-2,2]},
(\Lambda_2) \leftrightarrow g^{[N-2,1,1]}.
$$

(2.24)

The slight shift of notation makes the connection between the parameters and the decomposition (2.7) more manifest.

### 2.1 Counting matrix observables using partition algebras

Permutation invariant matrix polynomials, which have been studied as the natural class of observables in the matrix model with permutation invariant measure and action, are defined to obey

$$
\mathcal{O}(M_{\sigma(i)\sigma(j)}) = \mathcal{O}(M_{ij}) \text{ for all } \sigma \in S_N.
$$

(2.25)

These permutation invariant matrix observables (PIMOs) can be organized by their degree. At degree $k$, the matrix monomials

$$
M_{i_1 i_1'} M_{i_2 i_2'} \ldots M_{i_k i_k'},
$$

(2.26)

form a basis for a vector space isomorphic to $\text{Sym}^k(V_N \otimes V_N)$. The symmetric group $S_k$ acts on $(V_N \otimes V_N)^{\otimes k}$ by permuting the $k$ tensor factors. The subspace $\text{Sym}^k(V_N \otimes V_N)$ is the subspace of $S_k$ invariants in $(V_N \otimes V_N)^{\otimes k}$. This $S_k$ invariance is imposed because of the bosonic symmetry of the matrix variables $M_{ij}$. The PIMOs form the $S_N \times S_k$ invariant subspace of $(V_N \otimes V_N)^{\otimes k}$:

$$
\text{Matrix polynomials of degree } k \text{ invariant under } S_N = \text{Invariants}_{S_N \times S_k} (V_N \otimes V_N)^{\otimes k} \equiv [(V_N \otimes V_N)^{\otimes k}]_{S_N \times S_k}
$$

$$
= \{ v \in (V_N \otimes V_N)^{\otimes k} : \sigma v = v, \tau v = v \ | \ \text{for all } \sigma \in S_N, \tau \in S_k \}.
$$

(2.27)

Note that the action of $\tau \in S_k$ on $(V_N \otimes V_N)^{\otimes k}$ commutes with the action of $\sigma \in S_N$. This follows since the same $\sigma$ is applied to all tensor factors.
In [36] the dimension of the space of independent PIMOs for matrices of size \( N \) and polynomial degree \( k \) was obtained as

\[
\mathcal{N}(N,k) = \frac{1}{N!k!} \sum_{p \vdash N} \sum_{q \vdash k} \frac{N!}{\prod_{i=1}^{N} p_i !} \frac{k!}{\prod_{i=1}^{k} q_i !} \left( \sum_{l|i} l p_i \right)^{2q_i}.
\]

(2.28)

The initial sums run over integer partitions (Young diagrams) \( p \) of \( N \), and integer partitions \( q \) of \( k \) while the final sum is over the integer divisors \( l \) of \( i \). The equation (2.28) computes the multiplicity of the trivial representation of \( S_N \times S_k \) in the decomposition of \( (V_N \otimes V_N) \otimes^k \), which is the dimension of \([ (V_N \otimes V_N) \otimes^k ]_{S_N \times S_k} \). There exists an isomorphism

\[
(V_N \otimes V_N) \otimes^k \cong \bigoplus_{\Lambda_1,\Lambda_2} V_{\Lambda_1}^{S_N} \otimes V_{\Lambda_2}^{S_k} \otimes V_{\Lambda_1 \Lambda_2},
\]

(2.29)

organizing the space into irreducible representations of \( S_N \times S_k \), with multiplicities \( V_{\Lambda_1 \Lambda_2} \). Let \( V_N \otimes V_k \) denote the trivial representation of \( S_N \times S_k \) with multiplicity space \( V_{N[k]} \), then the dimension of \( S_N \times S_k \) invariants is given by

\[
\mathcal{N}(N,k) = \text{Dim } V_{N[k]}.
\]

(2.30)

The generalization to multi-matrix observables and a proof of their correspondence with colored directed graphs was developed in [39]. The approach in this paper is based on a new way of counting PIMOs, utilising the connection between dual algebras and matrix invariants.

We begin by reviewing this connection in the case of \( U(N) \) invariants. Tensor products of the defining representation \( V \) of \( U(N) \) have a multiplicity free decomposition into irreducible representations of \( U(N) \times S_k \) labelled by Young diagrams

\[
V \otimes^k \cong \bigoplus_{\Lambda, l(\Lambda) \leq N} V_{\Lambda}^{U(N)} \otimes V_{\Lambda}^{CS_k}.
\]

(2.31)

The sum runs over Young diagrams \( \Lambda \) with \( k \) boxes, and for \( k > N \) is restricted such that the number of rows \( l(\Lambda) \) in the Young diagram \( \Lambda \) is not greater than \( N \). In the remainder of this paper we will assume \( N \geq k \) for discussions of the unitary group. This result is known as Schur-Weyl duality (see chapter 6 in [47]). On the left-hand side of this equation we have a basis \( \epsilon_{i_1} \otimes \epsilon_{i_2} \otimes \cdots \otimes \epsilon_{i_k} \) with each index \( i \) running from 1 to \( N \). On the right-hand side we have a basis \( E_{Mm}^{\Lambda} \) with

\[
m \in \{1,\ldots, \text{Dim } V_{\Lambda}^{CS_k} \},

M \in \{1,\ldots, \text{Dim } V_{\Lambda}^{U(N)} \}.
\]

(2.32)

For a fixed Young diagram \( \Lambda \) and a fixed state \( M \) in \( V_{\Lambda}^{U(N)} \), there is a multiplicity of \( \text{Dim } V_{\Lambda}^{CS_k} \). That is, we have

\[
\text{Mult}(V \otimes^k \rightarrow V_{\Lambda}^{U(N)}) = \text{Dim } V_{\Lambda}^{CS_k}.
\]

(2.33)
It is well-known that $\text{U}(N)$ invariant matrix observables have a basis of multi-traces. These traces can be parameterised by conjugacy classes of permutations. A description of the connection between gauge invariant observables and equivalence classes of permutations for single matrix as well as multi-matrix problems, with applications to AdS/CFT is given in [62]. We review the connection here with an emphasis on Schur-Weyl duality from the outset. This framework, as explained in [62], can be used to give a description of finite $N$ effects on the counting and construction of gauge invariant observables, but we will focus here, as previously mentioned, on the case $N \geq k$. For the unitary group the matrix elements $M_{ij}$ are isomorphic to $V \otimes V^*$, where $V^*$ is the complex conjugate representation of $V$. In other words, $U \in \text{U}(N)$ acts on $M$ by conjugation,

$$M \mapsto UMU^\dagger.$$   \hfill (2.34)

Since $(V \otimes V^*)^\otimes k \cong V^\otimes k \otimes (V^*)^\otimes k$, we have

$$\left(V \otimes V^*\right)^\otimes k \cong \bigoplus_{\Lambda \vdash k} \left(V^U_{\Lambda} \otimes V^{CS_k}_{\Lambda}\right) \otimes \left(V^{U(N)}_{\Lambda^\dagger} \otimes V^{CS_k}_{\Lambda^\dagger}\right).$$ \hfill (2.35)

$\text{U}(N)$ invariants appear in a tensor product $V^U_{\Lambda} \otimes (V^*)^U_{\Lambda^\dagger}$ (with multiplicity 1) if and only if $\Lambda = \Lambda^\dagger$:

$$\text{Dim}[V^U_{\Lambda} \otimes (V^*)^U_{\Lambda^\dagger}] = \delta_{\Lambda \Lambda^\dagger}. \quad (2.36)$$

We are using $[W]^U_{\text{U}(N)}$ to refer to the $\text{U}(N)$ invariant subspace of the representation $W$. We have

$$[\left(V \otimes V^*\right)^\otimes k]^U_{\text{U}(N)} \cong \bigoplus_{\Lambda \vdash k} \left[V^U_{\Lambda} \otimes (V^*)^U_{\Lambda^\dagger}\right] \otimes V^{CS_k}_{\Lambda} \otimes V^{CS_k}_{\Lambda^\dagger}$$

\begin{align*}
&\cong \bigoplus_{\Lambda \vdash k} \delta_{\Lambda \Lambda^\dagger} V^{CS_k}_{\Lambda} \otimes V^{CS_k}_{\Lambda^\dagger} \\
&\cong \bigoplus_{\Lambda \vdash k} V^{CS_k}_{\Lambda} \otimes V^{CS_k}_{\Lambda^\dagger}, \quad (2.37)
\end{align*}

where the second line follows from Schur’s Lemma which implies equation (2.36). Since we are looking for $\text{U}(N)$ invariant polynomials of degree $k$ in $M_{ij}$, the counting is given by the $\text{U}(N)$ invariant subspace of $\text{Sym}^k(V_N \otimes V_N^*)$. Equivalently this is the space $[(V_N \otimes V_N^*)^\otimes k]^{U(N) \times S_k}$. There is a one-dimensional space of $S_k$ invariants in $V^{CS_k}_{\Lambda} \otimes V^{CS_k}_{\Lambda^\dagger}$ for each $\Lambda$. Hence the counting is given by

$$\text{Dimension of the space of} \; \text{U}(N) \; \text{invariant polynomials of degree} \; k \; \text{in} \; M_{ij}$$

$$= \sum_{\Lambda \vdash k} 1$$

$$= \text{Number of integer partitions of} \; k$$

$$= \text{Number of multi-trace structures with} \; k \; \text{copies of} \; M. \quad (2.38)$$

Thus the counting of $\text{U}(N)$ invariants is controlled by the symmetric group algebra, which appeared through Schur-Weyl duality.
Similarly in the case of $S_N$ invariant observables there is a dual algebra at play. The dual algebra for the defining representation of $S_N$ is called the partition algebra, denoted $P_k(N)$ [63, 64]. The representations of the partition algebra determine the multiplicities of $S_N$ irreducible representations in the decomposition (see section 2.5 in [70])

$$V_N^\otimes k \cong \bigoplus_{l=0}^{k} V^{S_N}_{[N-l, \Lambda^k_l]} \otimes V^{P_k(N)}_{[N-l, \Lambda^k_l]}.$$  \hfill (2.39)

The Young diagram $\Lambda_1 = [N-l, \Lambda^k_l]$, which is an integer partition of $N$, is constructed by placing the diagram $\Lambda^k_l$ below a first row of $N-l$ boxes. Requiring $\Lambda_1$ to be a valid Young diagram imposes some constraints on $\Lambda^k_l$, which are not manifest in (2.39). This occurs for $N < 2k$ as we explain, while it does not occur for $N \geq 2k$. The latter is called the stable limit. To understand this, we denote the first row length of $\Lambda^k_l$ by $r_1(\Lambda^k_l)$. For $N \geq 2k$, all values of $l$ and all choices of $\Lambda^k_l$ give valid Young diagrams $\Lambda_1$, since $N - l \geq r_1(\Lambda^k_l)$. Indeed writing $N = 2k + a$ for $a \geq 0$, we have

$$N - l = 2k + a - l \geq k + a.$$  \hfill (2.40)

The inequality follows since $l \leq k$ in equation (2.39). We also have

$$k + a \geq r_1(\Lambda^k_l).$$  \hfill (2.41)

This follows because $\Lambda^k_l$ has no more than $k$ boxes. For $N < 2k$, the condition $N - l \geq r_1(\Lambda^k_l)$ imposes a non-trivial $N$-dependent restriction on $\Lambda^k_l$. Indeed let $N = 2k - a$ for $a > 0$, then the condition $N - l \geq r_1(\Lambda^k_l)$ becomes

$$k - a \geq r_1(\Lambda^k_l).$$  \hfill (2.42)

This is non-trivial condition since $\Lambda^k_l$ can have up to $k$ boxes.

Note that the symmetric group algebra $\mathbb{C}S_k$ is a subalgebra of $P_k(N)$ (permutations of the tensor factors commute with the action of $S_N$ on $V_N^\otimes k$). We can restrict any representation $V^{P_k(N)}_{\Lambda_1}$ to $\mathbb{C}S_k$ to give a decomposition of the form

$$V^{P_k(N)}_{\Lambda_1} \cong \bigoplus_{\Lambda_2 \geq k} V^{S_k}_{\Lambda_2} \otimes V^{P_k(N) \to \mathbb{C}S_k}_{\Lambda_1 \Lambda_2}.$$  \hfill (2.43)

The dimension of $V^{P_k(N) \to \mathbb{C}S_k}_{\Lambda_1 \Lambda_2}$ is the branching multiplicity

$$\text{Dim} \left( V^{P_k(N) \to \mathbb{C}S_k}_{\Lambda_1 \Lambda_2} \right) = \text{Mult} \left( V^{P_k(N)}_{\Lambda_1} \to V^{\mathbb{C}S_k}_{\Lambda_2} \right).$$  \hfill (2.44)

Since $(V_N \otimes V_N)^\otimes k \cong V_N^\otimes k \otimes V_N^\otimes k$ we have

$$(V_N \otimes V_N)^\otimes k \cong \left( \bigoplus_{\Lambda_1, \Lambda_2} V^{S_N}_{\Lambda_1} \otimes V^{S_N}_{\Lambda_2} \otimes V^{P_k(N) \to \mathbb{C}S_k}_{\Lambda_1 \Lambda_2} \right) \otimes \left( \bigoplus_{\Lambda_1', \Lambda_2'} V^{S_N}_{\Lambda_1'} \otimes V^{S_N}_{\Lambda_2'} \otimes V^{P_k(N) \to \mathbb{C}S_k}_{\Lambda_1' \Lambda_2'} \right).$$  \hfill (2.45)
There is a single $S_N$ invariant state in every tensor product $V_{\Lambda_1} \otimes V_{\Lambda_1}'$ if and only if $V_{\Lambda_1} \cong V_{\Lambda_1}'$, and similarly for $S_k$. Therefore

$$[(V_N \otimes V_N)^\otimes k]^{S_N \times S_k} \cong \bigoplus_{\Lambda_1 \vdash N, \Lambda_2 \vdash k} V_{\Lambda_1}^{P_k(N) \rightarrow C S_k} \otimes V_{\Lambda_1}^{P_k(N) \rightarrow C S_k}$$

(2.46)

and considering the dimension of this subspace of $S_N \times S_k$ invariants, $V_{[N][k]}$, we find

$$\text{Dim } V_{[N][k]} = N(N, k) = \sum_{\Lambda_1 \vdash N} \sum_{\Lambda_2 \vdash k} \text{Mult} \left( V_{\Lambda_1}^{P_k(N) \rightarrow C S_k} \right)^2.$$  

(2.47)

The sum of squares is indicative of a matrix (Artin-Wedderburn) decomposition [71, 72] of a hidden algebra parametrising PIMOs (we found the exposition of the Artin-Wedderburn decomposition in [73] to be useful). We will turn to an explicit construction of PIMOs using partition algebra elements in line with the counting (2.47) in section 3. This sum of squares form in counting invariants, and their connection to the Artin-Wedderburn structure of algebras, have been used in a number of multi-matrix and tensor model applications, e.g. [74–78].

2.2 Enhanced $O(N)$ symmetry in parameter space

The quadratic GOE (Gaussian Orthogonal Ensemble) is determined by the probability density function

$$\exp(-S(M)) = \exp\left( - \text{Tr}\left( M M^T \right) \right),$$

(2.48)

on the space of real symmetric matrices (see definition 2.3.1. in [79]). The matrix elements $M_{ij}$ for $i \leq j$ in this ensemble of matrices are statistically independent. There are no mixing terms. Here we consider the underlying space to be the space of real matrices, with no symmetry constraint. There is a 4-parameter family of $O(N)$ invariant quadratic actions

$$S(M) = N \epsilon \text{Tr}(M) - \left( N \alpha \text{Tr}\left( M M^T \right) + N \beta \text{Tr}(M M) + \gamma \left( \text{Tr} M \right)^2 \right).$$

(2.49)

In this model, the matrix elements are not statistically independent, but the linear and quadratic moments are readily solvable, as we now show. Higher moments can be obtained using Wick’s theorem.

This 4-parameter family is a special case of recently studied [36–39] more general Gaussian matrix models, with permutation symmetry. We now solve for the second moments of matrix variables for the model in (2.49) and compare with the second moments for the permutation invariant Gaussian 1-matrix model. This gives a system of linear equations for the parameters in the permutation invariant Gaussian 1-matrix model in terms of the parameters $\alpha, \beta, \gamma$. See appendix A for an algorithm and computer code to reproduce these results.

We begin by rewriting the action:

$$S(M) = N \epsilon \sum_i M_{ii} - N(\alpha + \beta) \sum_i M_{ii}^2 - N \alpha \sum_{i \neq j} M_{ij}^2 - N \beta \sum_{i \neq j} M_{ij} M_{ji} - \gamma \sum_{i,j} M_{ii} M_{jj}.$$

(2.50)
Let
\[ z = (M_{11}, M_{22}, \ldots, M_{NN}, M_{12}, M_{21}, M_{13}, M_{31}, \ldots, M_{N-1N}, M_{NN-1}), \] (2.51)
then the action can be expressed as
\[ S(z) = z\mu - zGz^T. \] (2.52)

The vector \( \mu \) is
\[ \mu = \begin{pmatrix} N\epsilon \\ \vdots \\ N\epsilon \\ 0 \\ \vdots \\ 0 \end{pmatrix} \] (2.53)
with the first \( N \) terms equal to \( \epsilon \) and the rest 0 and
\[ G = \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_2 \end{pmatrix}, \quad G_1 = N \begin{pmatrix} \alpha + \beta \\ \alpha + \beta + \gamma \\ \alpha + \beta + \gamma \end{pmatrix}, \quad G_2 = N\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}. \] (2.54)

The inverse of \( G_2 \) is
\[ (G_2)^{-1} = \frac{1}{N(\alpha^2 - \beta^2)} \begin{pmatrix} \alpha & -\beta \\ -\beta & \alpha \end{pmatrix} \] (2.55)
while the inverse of \( G_1 \) is given by
\[ (G_1)^{-1} = \begin{cases} \frac{1}{N^2} \left( \frac{N-1}{\alpha + \beta} + \frac{1}{\alpha + \beta + \gamma} \right), & \text{if } i = j \\ -\frac{1}{N^2} \left( \frac{\gamma}{(\alpha + \beta)(\alpha + \beta + \gamma)} \right), & \text{if } i \neq j \end{cases} \] (2.56)

From the form of these inverse matrices we can write down the connected two-point function
\[ \langle M_{ij} M_{kl} \rangle = \delta_{ij} \delta_{kl} \frac{1}{N^2} \left( \frac{N-1}{\alpha + \beta} + \frac{1}{\alpha + \beta + \gamma} \right) - \left( \delta_{ij} \delta_{kl} - \delta_{ij} \delta_{kl} \delta_{ij} \right) \frac{1}{N^2 (\alpha + \beta)(\alpha + \beta + \gamma)} \gamma \\
+ \left( \delta_{ik} \delta_{jl} - \delta_{ik} \delta_{jl} \delta_{ij} \right) \frac{1}{N^2 \alpha^2 - \beta^2} - \left( \delta_{il} \delta_{kj} - \delta_{il} \delta_{kj} \delta_{ij} \right) \frac{1}{N^2 \alpha^2 - \beta^2}. \] (2.57)

Defining
\[ a = \frac{1}{N^2} \left( \frac{N-1}{\alpha + \beta} + \frac{1}{\alpha + \beta + \gamma} \right), \quad b = \frac{1}{N^2 (\alpha + \beta)(\alpha + \beta + \gamma)} \gamma, \]
\[ c = \frac{1}{N^2 \alpha^2 - \beta^2}, \quad d = \frac{1}{N^2 \alpha^2 - \beta^2}, \] (2.58)
and collecting like terms we are left with the following expression for the two-point function

$$\langle M_{ij} M_{kl} \rangle = \delta_{ij} \delta_{kl} \delta_{il} \delta_{jk} c - \delta_{ij} \delta_{kl} b + \delta_{ik} \delta_{jl} b c - \delta_{il} \delta_{jk} d. \quad (2.59)$$

The parameters \( a, b, c, d \) satisfy \( a + b + d = c \) and therefore the fully simplified two-point function is given by

$$\langle M_{ij} M_{kl} \rangle = -\delta_{ij} \delta_{kl} b + \delta_{ik} \delta_{jl} (a + b + d) - \delta_{il} \delta_{jk} d. \quad (2.60)$$

Comparing this to the two-point function of the permutation invariant matrix model (equation (3.6) in [37]) we find that it is reproduced in the following parameter limit

$$\begin{align*}
(g^{-1}_{[N]}{)}_{11} & = a \\
(g^{-1}_{[N]}{)}_{22} & = a - (N - 2)b \\
(g^{-1}_{[N]}{)}_{12} & = -\sqrt{N - 1}b \\
(g^{-1}_{[N-1,1]}{)}_{11} & = a + b + d \\
(g^{-1}_{[N-1,1]}{)}_{22} & = a + b + d \\
(g^{-1}_{[N-1,1]}{)}_{33} & = a + b \\
(g^{-1}_{[N-1,1]}{)}_{12} & = -d \\
(g^{-1}_{[N-1,1]}{)}_{13} & = 0 \\
(g^{-1}_{[N-1,1]}{)}_{23} & = 0 \\
(g^{-1}_{[N-2,2]}{)} & = a + b \\
(g^{-1}_{[N-2,1,1]}{)} & = a + b + 2d
\end{align*} \quad (2.61)$$

where we have again written \( g \) instead of \( \Lambda \) for our quadratic couplings, labelling them using integer partitions under the identification given in (2.24).

There is a special point in this limit that recovers the two-point function for the simple \( O(N) \) model with action

$$S(M) = \text{Tr} \left( M M^T \right). \quad (2.62)$$

Setting \( \varepsilon = \beta = \gamma = 0 \) in equation (2.49), we find that the relevant limit of the permutation invariant Gaussian model is found by taking \( a = 1 \) and \( b = d = 0 \) in (2.61) which gives us

$$\begin{align*}
(g^{-1}_{[N]}{)}_{11} & = (g^{-1}_{[N]}{)}_{22} = (g^{-1}_{[N-1,1]}{)}_{11} = (g^{-1}_{[N-1,1]}{)}_{22} = (g^{-1}_{[N-1,1]}{)}_{33} = (g^{-1}_{[N-2,2]}{)} = (g^{-1}_{[N-2,1,1]}{)} = 1
\end{align*} \quad (2.63)$$

as the only non-zero parameters.

A quick check on the above computation is the following. Using Clebsch-Gordan coefficients we have

$$\begin{align*}
\text{Tr} \left( M M^T \right) & = \sum_{ij} M_{ij} M_{ij} = \sum_{ij} \sum_{a,b,\Lambda_1,\Lambda_1',\alpha,\beta} C_{a,i,j}^{\Lambda_1,\alpha} C_{b,j,i}^{\Lambda_1',\beta} S_a^{\Lambda_1,\alpha} S_b^{\Lambda_1',\beta} \\
& = \sum_{a,b,\Lambda_1,\Lambda_1',\alpha,\beta} \delta_{ab} \delta_{\Lambda_1 \Lambda_1'} \delta_{\alpha \beta} S_a^{\Lambda_1,\alpha} S_b^{\Lambda_1',\beta} = \sum_{a,\Lambda_1,\alpha} S_a^{\Lambda_1,\alpha} S_a^{\Lambda_1,\alpha},
\end{align*} \quad (2.64)$$
where the second line uses orthogonality of the Clebsch-Gordan coefficients. Comparing with equation (2.23) recovers the parameter limit (2.63).\footnote{Immediate comparison gives a parameter limit for the coupling matrices, as opposed to their inverses as in equation (2.63). In this case, they are identical.}

3 Permutation Invariant Matrix Observables (PIMOs)

We will now describe the partition algebra and how the PIMOs are constructed from the $S_k$ invariant subalgebra of $P_k(N)$. Properties of the partition algebra \cite{63–65, 80} will allow us to prove large $N$ factorisation of PIMOs in the $O(N)$ symmetric matrix model.

The partition algebra $P_k(N)$ is a diagram algebra. It has a finite basis, labelled by diagrams, where multiplication is a type of composition of diagrams. A diagram in $P_k(N)$ has $2k$ labelled vertices arranged into two rows, with $k$ vertices in each row. Any set of edges between the vertices are allowed. We use the convention in which the bottom vertices are labelled (from left to right) by $1, \ldots, k$ and the top vertices by $1', \ldots, k'$. For example, $P_2(N)$ has a basis of 15 diagrams. Among these are,

\begin{align}
&1' \quad 2'
\quad 1 \quad 2,
&1' \quad 2'
\quad 1 \quad 2,
&1' \quad 2'
\quad 1 \quad 2
\end{align}

In general, the dimension of $P_k(N)$ is the number of set partitions of $2k$ (also known as Bell numbers).

The underlying set for this basis of the partition algebra is the set of set partitions of the $2k$ labelled vertices. There is a redundancy in the diagram picture. The redundancy arises from the fact that we are free to choose any set of edges, as long as every vertex in a subset of the set partition can be reached from any other vertex in the same subset, by a path along the edges. For example, the following pair of diagrams correspond to the same element.

\begin{align}
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1.5,0);
\draw (0,0.5) -- (1.5,0.5);
\end{tikzpicture}
\end{array} = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1.5,0);
\draw (0,0.5) -- (1.5,1.5);
\end{tikzpicture}
\end{array}.
\end{align}

The product in $P_k(N)$ is independent of the choice of representative diagram.

Let $d_1$ and $d_2$ be two diagrams in $P_k(N)$. The composition $d_3 = d_1 d_2$ is constructed by placing $d_1$ above $d_2$ and identifying the bottom vertices of $d_1$ with the top vertices of $d_2$. The diagram is simplified by following the edges connecting the bottom vertices of $d_2$ to the top vertices of $d_1$. Any connected components within the middle rows are removed and we multiply by $N^c$, where $c$ is the number of these components removed. For example,

\begin{align}
\begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1.5,0);
\draw (0,0.5) -- (1.5,0.5);
\end{tikzpicture}
\end{array} = N \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1.5,0);
\draw (0,0.5) -- (1.5,1.5);
\end{tikzpicture}
\end{array} \quad \text{and} \quad \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1.5,0);
\draw (0,0.5) -- (1.5,1.5);
\end{tikzpicture}
\end{array} = \begin{array}{c}
\begin{tikzpicture}
\draw (0,0) -- (1.5,0);
\draw (0,0.5) -- (1.5,1.5);
\end{tikzpicture}
\end{array}.
\end{align}
where the factor of $N$ in the first equation comes from removing the middle component at vertex 1 and 2. For linear combinations of diagrams, multiplication is defined by linear extension.

The subset of diagrams with $k$ edges, each connecting a vertex at the top to a vertex at the bottom and where every vertex has exactly one incident edge, span a subalgebra. This subalgebra is isomorphic to the symmetric group algebra $\mathbb{C}S_k$. For example, there is a one-to-one correspondence between permutations in $S_3$ and the following set of diagrams,

$$
\begin{array}{c}
| \quad | \quad | \\
\times & | & \times \\
| & | & |
\end{array}
$$

(3.4)

### 3.1 Construction of PIMOs

We will construct degree $k$ PIMOs from elements $d \in P_k(N)$. As a warm-up, we recap the construction of $U(N)$ invariants using elements in $\mathbb{C}S_k$. See [62] for a review of the background literature.

For this construction it will be useful to rewrite $M_{ij}$ as $M^i_j$ and think of these as the matrix elements of a linear operator acting on $V$, the defining representation of $U(N)$. Define $M$ to be the linear operator $M : V \rightarrow V$ with matrix elements

$$
M e_i = \sum_j M^i_j e_j,
$$

(3.5)
in a basis $e_i$ for $V$. In diagram notation the linear operator $M$ is represented by a box labelled $M$, with one incoming and one outgoing edge,

$$
M^j_i = \begin{array}{c}
| \\
\uparrow \\
\downarrow \\
M \\
\downarrow \\
\uparrow \\
i
\end{array}
$$

(3.6)

The operator $M^\otimes k$ acts on $V^\otimes k$ as

$$
M^\otimes k e_{i_1} \otimes \cdots \otimes e_{i_k} = M e_{i_1} \otimes \cdots \otimes M e_{i_k}.
$$

(3.7)

Diagrammatically, tensor products of operators are represented by horizontally composing the diagrams,

$$
(M^\otimes k)^{j_1 \cdots j_k}_{i_1 \cdots i_k} = M^{j_1}_{i_1} \cdots M^{j_k}_{i_k} = \begin{array}{c}
| \\
\uparrow \\
\downarrow \\
M \\
\downarrow \\
\uparrow \\
i_1 \\
\uparrow \\
\downarrow \\
M \\
\downarrow \\
\uparrow \\
j_1 \\
\uparrow \\
\downarrow \\
j_k
\end{array}
$$

(3.8)

When viewed as a matrix polynomial, the trace

$$
O_\tau = \text{Tr}_{V^\otimes k}(M^\otimes k \tau) = \sum_{i_1 \cdots i_k}^{i_1' \cdots i_k'} (\tau)^{i_1' \cdots i_k'}_{i_1 \cdots i_k} M^{i_1}_{i_1} \cdots M^{i_k}_{i_k} = \begin{array}{c}
| \\
\uparrow \\
\downarrow \\
M \\
\downarrow \\
\uparrow \\
\tau \\
\downarrow \\
\uparrow \\
\cdots \\
\uparrow \\
\downarrow \\
M
\end{array}
$$

(3.9)
is a unitary invariant of degree $k$. The matrix elements of the permutation $\tau$ as a linear operator on $V^\otimes k$ are

$$
(\tau)_{i_1 \ldots i_k}^{i_1' \ldots i_k'} = \delta_{i_1'}^{i_{r(1)}} \ldots \delta_{i_k'}^{i_{r(k)}}.
$$

(3.10)

The diagram representing $\tau$ is obtained by associating an edge with every Kronecker delta. For example, for $\tau = (12)$ we have the diagram

$$
\begin{array}{c}
\begin{tikzpicture}
  \node (i1) at (0,0) {$i_1$};
  \node (i2) at (0,-1) {$i_2$};
  \node (i1') at (0.5,0) {$i_1'$};
  \node (i2') at (0.5,-1) {$i_2'$};
  \draw (i1) -- (i1');
  \draw (i2) -- (i2');
\end{tikzpicture}
\end{array}
= \delta_{i_1'}^{i_1} \delta_{i_2'}^{i_2}.
$$

(3.11)

The horizontal lines in equation (3.9) are used to indicate that the incoming and outgoing edges are identified, as expected from a trace.

Invariance under the action of $U(N)$ follows because $\tau \in S_k$ commutes with any $U(N)$ acting on $V^\otimes k$. The correspondence between gauge invariant operators and permutations has a redundancy given by,

$$
O_{\gamma \tau \gamma^{-1}} = O_\tau, \quad \text{for all } \gamma \in S_k.
$$

(3.12)

This follows because $\gamma^{-1} M^{\otimes k} \gamma = M^{\otimes k}$. Therefore, a basis of multi-trace observables is in one-to-one correspondence with conjugacy classes of $S_k$, as expected from the counting in equation (2.38).

The construction of degree $k$ PIMOs from elements of $P_k(N)$ is identical. For any $d \in P_k(N)$, the matrix polynomial

$$
O_d = \text{Tr}_{V^\otimes k} (M^{\otimes k} d) = \sum_{i_1 \ldots i_k} (d)_{i_1 \ldots i_k}^{i_1' \ldots i_k'} M_{i_1}^{i_1'} \ldots M_{i_k}^{i_k'} = \begin{array}{c}
\begin{tikzpicture}
  \node (d) at (0,0) {$d$};
  \node (M) at (0,-1) {$\ldots$};
  \node (M') at (1.5,-1) {$\ldots$};
  \node (M''') at (3,0) {$\ldots$};
\end{tikzpicture}
\end{array},
$$

(3.13)

is a PIMO, because $d$ commutes with the action of $S_N$ acting on $V^\otimes k$. The matrix elements $(d)_{i_1 \ldots i_k}^{i_1' \ldots i_k'}$ also correspond to the diagram representation by associating every Kronecker delta to an edge connecting a pair of vertices. For example,

$$
\begin{array}{c}
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (2) at (0,-1) {$2$};
  \node (1') at (0.5,0) {$1'$};
  \node (2') at (0.5,-1) {$2'$};
  \draw (1) -- (1');
  \draw (2) -- (2');
\end{tikzpicture}
\end{array}
= \delta_{i_1 i_2} \delta_{i_2'}^{i_1} \delta_{i_2}^{i_1'}, \quad \text{and} \quad
\begin{array}{c}
\begin{tikzpicture}
  \node (1) at (0,0) {$1$};
  \node (2) at (0,-1) {$2$};
  \node (1') at (0.5,0) {$1'$};
  \node (2') at (0.5,-1) {$2'$};
  \draw (1) -- (1');
  \draw (2) -- (2');
\end{tikzpicture}
\end{array}
= \delta_{i_1 i_2} \delta_{i_1'}^{i_1}.
$$

(3.14)

As before, for any $\gamma \in S_k$ we have

$$
O_{\gamma d \gamma^{-1}} = O_d.
$$

(3.15)

Degree $k$ PIMOs are in one-to-one correspondence with the $S_k$ invariant subalgebra of $P_k(N)$. A basis is given by the set of distinct equivalence classes

$$
[d] = \{ \gamma d \gamma^{-1} \mid \forall \gamma \in S_k \}.
$$

(3.16)
3.2 Inner product on PIMOs

The simplest $O(N)$ invariant matrix model has the quadratic expectation value

\begin{equation}
\langle M^i_j M^k_l \rangle = \delta^{ik} \delta_{jl}. \tag{3.17}
\end{equation}

Let $d_1, d_2 \in P_k(N)$, and define the two-point function of PIMOs $O_{d_1}, O_{d_2}$ by using Wick’s theorem and the equation (3.17), keeping only Wick contractions between the two observables i.e. we are treating them as “normal-ordered”. There is an expression for this two-point function

\begin{equation}
\langle O_{d_1} O_{d_2} \rangle = \sum_{\gamma \in S_k} \text{Tr}_{V_N} (d_1 \gamma d_2^T \gamma^{-1} \gamma^{-1}), \tag{3.18}
\end{equation}

where $d^T$ is the transpose of the diagram $d$, obtained from $d$ by flipping the top and bottom vertices. The permutations $\gamma$ parameterize the Wick contractions. The proof of (3.18) goes as follows. Note that the quadratic expectation value (3.17) diagrammatically corresponds to the replacement

\begin{equation}
\begin{pmatrix}
  \begin{bmatrix} i & k \\ j & l \end{bmatrix} \\
  \begin{bmatrix} i & k \\ j & l \end{bmatrix}
\end{pmatrix} = \begin{pmatrix}
  \begin{bmatrix} i & k \\ j & l \end{bmatrix} \\
  \begin{bmatrix} i & k \\ j & l \end{bmatrix}
\end{pmatrix}, \tag{3.19}
\end{equation}

where the Kronecker deltas have been replaced by edges. The two-point function in equation (3.18) can be represented by the diagram in the first line below

\begin{equation}
\begin{pmatrix}
  \begin{bmatrix} d_1 \\ M \end{bmatrix} \\
  \begin{bmatrix} d_2 \\ M \end{bmatrix}
\end{pmatrix} = \sum_{\gamma \in S_k} \text{Tr}_{V_N} (d_1 \gamma d_2^T \gamma^{-1} \gamma^{-1}), \tag{3.20}
\end{equation}

The second line is the sum over Wick contractions parameterized by $\gamma \in S_k$. The last equality comes from straightening the diagram. By following the lines and recording the
operators encountered on the way, we recognize the last diagram as the representation of $\text{Tr}_{V_N^k}(d_1 \gamma d_2^T \gamma^{-1})$.

The symmetry of the two-point function is proved by observing that

$$\sum_{\gamma \in S_k} \text{Tr}_{V_N^k}(d_1 \gamma d_2^T \gamma^{-1}) = \sum_{\gamma \in S_k} \text{Tr}_{V_N^k}(\gamma d_2 \gamma^{-1} d_1^T) = \sum_{\gamma \in S_k} \text{Tr}_{V_N^k}(d_2 \gamma d_1^T \gamma^{-1}).$$

(3.21)

We have used the invariance of the trace under transposition, cyclicity of the trace and a re-labelling of $\gamma \rightarrow \gamma^{-1}$. The non-degeneracy of the two-point function at large $N$ follows from the factorization property in the next section. The non-degeneracy at all orders in $1/\sqrt{N}$ is proved in the companion paper by exhibiting an orthogonal basis constructed using representation theory data [81]. This shows that the two-point function defines an inner product.

4 Large $N$ factorisation

In this section, we will show that the normalized PIMOs

$$\hat{O}_d = \frac{O_d}{\sqrt{\langle O_d O_d \rangle}},$$

(4.1)

factorize for large $N$

$$\langle \hat{O}_{d_1} \hat{O}_{d_2} \rangle = \begin{cases} 1 + O(1/\sqrt{N}) & \text{if } [d_1] = [d_2], \\ 0 + O(1/\sqrt{N}) & \text{if } [d_1] \neq [d_2]. \end{cases}$$

(4.2)

To prove large $N$ factorization we will study the powers of $N$ appearing in

$$\text{Tr}_{V_N^k}(d_1 \gamma d_2^T \gamma^{-1}),$$

(4.3)

or equivalently, the r.h.s. of equation (3.18) for the two-point function.

It is useful to consider the simpler case

$$\text{Tr}_{V_N^k}(d_1 d_2^T).$$

(4.4)

This trace can be computed in terms of the number of connected components in the diagram $d_1 \lor d_2$, given by a diagram with all the edges of $d_1$ and $d_2$. In the mathematics literature, this operation is called the join on the partition lattice (see [67]). It is given by

$$\text{Tr}_{V_N^k}(d_1 d_2^T) = N^{c(d_1 \lor d_2)}. \tag{4.5}$$

where $c(d)$ is the number of connected components in the diagram $d$. Examples of the join operation are

$$\begin{array}{c} \text{\includegraphics[width=1cm]{d1}} \lor \text{\includegraphics[width=1cm]{d2}} = \text{\includegraphics[width=2cm]{d1d2}}, & \text{and} & \text{\includegraphics[width=1cm]{d1}} \lor \text{\includegraphics[width=1cm]{d2}} = \text{\includegraphics[width=2cm]{d1d2}}. \end{array}$$

(4.6)

Examples of $c(d)$ are

$$c\left(\begin{array}{c} \text{\includegraphics[width=1cm]{d1}} \\ \text{\includegraphics[width=1cm]{d2}} \end{array}\right) = 2, \quad c\left(\begin{array}{c} \text{\includegraphics[width=1cm]{d1}} \end{array}\right) = 3. \tag{4.7}$$
To illustrate equation (4.5) consider the following pair of diagrams

\[
\begin{align*}
\left(\begin{array}{ccc}
\vdots \\
\end{array}\right)_{i_1 i_2}^{i_1' i_2'} = \delta_{i_1 i_1'}, \quad \left(\begin{array}{ccc}
\vdots \\
\end{array}\right)_{i_1 i_2}^{i_1' i_2'} = \delta_{i_2 i_2'}. 
\end{align*}
\] (4.8)

The join is given by

\[
\begin{align*}
\left(\begin{array}{ccc}
\vdots \\
\end{array}\right)_{i_1 i_2}^{i_1' i_2'} \vee \left(\begin{array}{ccc}
\vdots \\
\end{array}\right)_{i_1 i_2}^{i_1' i_2'} = \left(\begin{array}{ccc}
\vdots \\
\end{array}\right)_{i_1 i_2}^{i_1' i_2'} = \delta_{i_1 i_1'} \delta_{i_2 i_2'}. 
\end{align*}
\] (4.9)

The diagram multiplication gives

\[
\begin{align*}
\text{Tr}_{V_N^\otimes 2} \left(\begin{array}{ccc}
\vdots \\
\end{array}\right)^T_{i_1 i_2}^{i_1' i_2'} = \text{Tr}_{V_N^\otimes 2} \left(\begin{array}{ccc}
\vdots \\
\end{array}\right)_{i_1 i_2}^{i_1' i_2'} = N^2, 
\end{align*}
\] (4.10)

while the corresponding expression using the join gives

\[
\begin{align*}
\text{Tr}_{V_N^\otimes 2} \left(\begin{array}{ccc}
\vdots \\
\end{array}\right)^T_{i_1 i_2}^{i_1' i_2'} = N^c \left(\begin{array}{ccc}
\vdots \\
\end{array}\right)^T_{i_1 i_2}^{i_1' i_2'} = N^c \left(\begin{array}{ccc}
\vdots \\
\end{array}\right)_{i_1 i_2}^{i_1' i_2'} = N^2. 
\end{align*}
\] (4.11)

To prove this, recall that every edge in a diagram corresponds to a Kronecker delta when acting on $V_N^\otimes k$ (see examples in (3.14)). Consequently

\[
\begin{align*}
(d_1 \vee d_2)_{i_1 \ldots i_k}^{i_1' \ldots i_k'} &= (d_1)_{i_1 \ldots i_k}^{i_1' \ldots i_k'} (d_2)_{i_1 \ldots i_k}^{i_1' \ldots i_k'}. 
\end{align*}
\] (4.12)

It follows that

\[
\begin{align*}
\text{Tr}_{V_N^\otimes k} (d_1 d_2^T) &= \sum_{i_1, \ldots, i_k} (d_1)_{i_1 \ldots i_k}^{i_1' \ldots i_k'} (d_2^T)_{i_1 \ldots i_k}^{i_1' \ldots i_k'} = \sum_{i_1, \ldots, i_k} (d_1)_{i_1 \ldots i_k}^{i_1' \ldots i_k'} (d_2)_{i_1 \ldots i_k}^{i_1' \ldots i_k'} \\
&= \sum_{i_1, \ldots, i_k} (d_1 \vee d_2)_{i_1 \ldots i_k}^{i_1' \ldots i_k'}. 
\end{align*}
\] (4.13)

Equivalently, the diagrammatic representation of a trace identifies the bottom vertices with the top vertices,

\[
\begin{align*}
\text{Tr}_{V_N^\otimes k} (d_1 d_2^T) &= \begin{array}{c}
\hline
\quad d_1 \\
\hline
\end{array} \\
\begin{array}{c}
\hline
\quad d_2^T \\
\hline
\end{array}
\end{align*}
\] (4.14)

Taken literally, this means that we identify the bottom vertices of $d_2^T$ with the top vertices of $d_1$, and the top vertices of $d_2^T$ with the bottom vertices of $d_1$. The diagram constructed in this manner has all the edges of $d_1$ together with all the edges of $d_2$, which is precisely equal to $d_1 \vee d_2$. See figure 1 for an illustration.

To complete the proof we show that

\[
\begin{align*}
\text{Tr}_{V_N^\otimes k} (d_1 d_2^T) &= \begin{array}{c}
\hline
\quad d_1 \\
\hline
\end{array} \\
\begin{array}{c}
\hline
\quad d_2^T \\
\hline
\end{array} \\
&= \sum_{i_1, \ldots, i_k} (d_1 \vee d_2)_{i_1 \ldots i_k}^{i_1' \ldots i_k'} = N^c (d_1 \vee d_2). 
\end{align*}
\] (4.15)
Figure 1. By identifying the bottom vertices of $d_T^2$ with the top vertices of $d_1$, and the top vertices of $d_T^2$ with the bottom vertices of $d_1$, we have constructed a diagram with all the edges of $d_1$ together with all the edges of $d_2$. This is equal to the diagram $d_1 \lor d_2$. This figure is a central koan/insight of the factorization proof.

Let $b_1, \ldots, b_l$ be sets containing the vertices of connected components of $d_1 \lor d_2$. Then,

$$
\sum_{i_1, \ldots, i_k} (d_1 \lor d_2)_{i_1', \ldots, i_k'} = \left( \sum_{b_1} 1 \right) \left( \sum_{b_2} 1 \right) \cdots \left( \sum_{b_l} 1 \right) = N^{c(d_1 \lor d_2)},
$$

where the sums over connected components correspond to sums where the indices in each component are set equal. For example, if $b_1 = \{1, 3, 5', 8\}$ then

$$
\sum_{b_1} 1 \equiv \sum_{i_1, i_3, i_5', i_8} \delta_{i_1 i_3} \delta_{i_3 i_5'} \delta_{i_5' i_8} = \sum_{i_1 = i_3 = i_5' = i_8} 1 = N.
$$

4.1 Factorization for trace form on $P_k(N)$

The proof of the following version of factorization

$$
\frac{\text{Tr}_{V_N^{\otimes k}}(d_1 d_2^T_i) \sqrt{\text{Tr}_{V_N^{\otimes k}}(d_1 d_2^T_i) \text{Tr}_{V_N^{\otimes k}}(d_2 d_2^T)}}{\sqrt{\text{Tr}_{V_N^{\otimes k}}(d_1 d_2^T_i) \text{Tr}_{V_N^{\otimes k}}(d_2 d_2^T)}} = \begin{cases} 1 + O(1/\sqrt{N}) & \text{if } d_1 = d_2, \\ 0 + O(1/\sqrt{N}) & \text{if } d_1 \neq d_2, \end{cases}
$$

contains most of the essential ingredients necessary for the 1-matrix case. This is a useful warm-up exercise and, as we will see in section 4.2, a special case of factorization in multi-matrix models. This equation (4.18) is related to the properties of the distance function defined in proposition 3.1 of [44].

\footnote{We thank Franck Gabriel for this observation.}
The factorization in equation (4.18) is a consequence of the following
\[
2c(d_1 \lor d_2) = c(d_1 \lor d_1) + c(d_2 \lor d_2) = c(d_1) + c(d_2) \quad \text{if} \quad d_1 = d_2,
\]
\[
2c(d_1 \lor d_2) < c(d_1 \lor d_1) + c(d_2 \lor d_2) = c(d_1) + c(d_2) \quad \text{if} \quad d_1 \neq d_2,
\]
where we have used \(c(d_1 \lor d_1) + c(d_2 \lor d_2) = c(d_1) + c(d_2)\) since \(d \lor d = d\). We will prove (4.19) by separating the general pairs \(d_1, d_2\) into three distinct cases:

1. If \(d_1\) only contains edges that are also contained in \(d_2\), but \(d_1 \neq d_2\), we write \(d_1 < d_2\). For example,
\[
\begin{array}{c}
\circlearrowright < \bigcirc, \quad \text{and} \quad \cdots < \bigcirc.
\end{array}
\]
In this case, \(d_1 \lor d_2 = d_2\) and it follows that,
\[
c(d_1 \lor d_2) = c(d_2).
\]
Note that \(d_1 < d_2\) implies \(c(d_1) > c(d_2)\). Therefore,
\[
2c(d_1 \lor d_2) = c(d_2) + c(d_2) < c(d_1) + c(d_2).
\]
Since the l.h.s. and r.h.s. are symmetric under exchanging \(d_1 \leftrightarrow d_2\), the inequality \(2c(d_1 \lor d_2) < c(d_1) + c(d_2)\) holds for \(d_2 < d_1\) as well.

2. Suppose \(d_1 \neq d_2\) and that there is no set of edges that can be added to \(d_1\) to turn it into \(d_2\), nor is there a set of edges that can be added to \(d_2\) to obtain \(d_1\). Then, we say that \(d_1\) and \(d_2\) are incomparable. We denote this by \(d_1 \not\leq d_2\). The following diagrams are examples of incomparable diagrams
\[
\begin{array}{c}
\cdots \not\leq \bigcirc, \quad \text{and} \quad \cdots \not\leq \bigcirc.
\end{array}
\]
In this incomparable case, we have
\[
c(d_1 \lor d_2) < c(d_1) \quad \text{and} \quad c(d_1 \lor d_2) < c(d_2)
\]
since the forming of the join involves adding to \(d_1\), additional edges creating connections which did not exist in \(d_1\), or alternatively adding to \(d_2\) additional edges that did not exist in \(d_2\). Consequently we have the inequality
\[
2c(d_1 \lor d_2) < c(d_1) + c(d_2).
\]

3. If \(d_1 = d_2\) we have
\[
c(d_1 \lor d_2) = c(d_1 \lor d_1) = c(d_1) = c(d_2),\]
and therefore,
\[
2c(d_1 \lor d_2) = c(d_1) + c(d_2).
\]

To summarize, \(2c(d_1 \lor d_2) \leq c(d_1) + c(d_2)\) with equality if and only if \(d_1 = d_2\).

As a corollary of the above discussion, which will be useful in the next sub-section, note that if we consider a fixed diagram \(d_1\) and a family of diagrams \(d_3\) with fixed \(c(d_3)\) such that \(c(d_1) > c(d_3)\), then we have for each \(d_3\) in the family one of the following
\[
c(d_1 \lor d_3) < c(d_3) \quad \text{if} \quad d_1 \not\leq d_3,
\]
\[
c(d_1 \lor d_3) = c(d_3) \quad \text{if} \quad d_1 < d_3.
\]
This follows from (4.21) and (4.24).
4.2 Factorization for PIMOs

The 1-matrix connected two-point function (3.18) includes a sum over $\gamma \in S_k$,

$$\langle \hat{O}_{d_1} \hat{O}_{d_2} \rangle = \frac{\sum_{\gamma_1 \in S_k} N^{c(d_1 \vee \gamma_1 d_2 \gamma_1^{-1})}}{\sqrt{\sum_{\gamma_2 \in S_k} N^{c(d_1 \vee \gamma_2 d_1 \gamma_2^{-1})} \sum_{\gamma_3 \in S_k} N^{c(d_2 \vee \gamma_3 d_2 \gamma_3^{-1})}}}.$$  \hfill (4.29)

Large $N$ factorization of PIMOs follows from the inequalities

$$\begin{align*}
2 \max_{\gamma_1} c(d_1 \vee \gamma_1 d_2 \gamma_1^{-1}) &= \max_{\gamma_2} c(d_1 \vee \gamma_2 d_1 \gamma_2^{-1}) + \max_{\gamma_3} c(d_2 \vee \gamma_3 d_2 \gamma_3^{-1}) \quad \text{if} \quad [d_1] = [d_2], \\
2 \max_{\gamma_1} c(d_1 \vee \gamma_1 d_2 \gamma_1^{-1}) &< \max_{\gamma_2} c(d_1 \vee \gamma_2 d_1 \gamma_2^{-1}) + \max_{\gamma_3} c(d_2 \vee \gamma_3 d_2 \gamma_3^{-1}) \quad \text{if} \quad [d_1] \neq [d_2].
\end{align*}$$  \hfill (4.30)

The first step in proving equation (4.30) is to simplify the terms on the r.h.s. The inequalities in equation (4.19) imply that $c(d \vee \gamma d \gamma^{-1})$ is maximized when $d = \gamma d \gamma^{-1}$. Of course, the identity permutation always satisfies this equality. Therefore,

$$\max_{\gamma} c(d \vee \gamma d \gamma^{-1}) = c(d).$$  \hfill (4.31)

We are left with proving

$$\begin{align*}
2 \max_{\gamma} c(d_1 \vee \gamma d_2 \gamma^{-1}) &= c(d_1) + c(d_2) \quad \text{if} \quad [d_1] = [d_2], \\
2 \max_{\gamma} c(d_1 \vee \gamma d_2 \gamma^{-1}) &< c(d_1) + c(d_2) \quad \text{if} \quad [d_1] \neq [d_2].
\end{align*}$$  \hfill (4.32)

We employ the same strategy as before, and consider the three distinct cases.

1. Suppose $c(d_1) > c(d_2)$, and consider the diagrams $\gamma d_2 \gamma^{-1}$ for $\gamma \in S_k$. We have $c(d_1) > c(\gamma d_2 \gamma^{-1}) = c(d_2)$. Assume $d_1, d_2$ are such there exists some $\gamma^*$ such that $d_1 < \gamma^* d_2 (\gamma^*)^{-1}$. For any such $\gamma^*$, the equality in (4.28) implies that

$$2c(d_1 \vee \gamma^* d_2 (\gamma^*)^{-1}) = 2c(d_2) < c(d_1) + c(d_2).$$  \hfill (4.33)

Any $\gamma$ not satisfying this condition leads to $d_1 \nleq \gamma d_2 \gamma^{-1}$, and the inequality in (4.28) implies that

$$2c(d_1 \vee \gamma d_2 \gamma^{-1}) < 2c(d_2).$$  \hfill (4.34)

This implies that

$$2 \max_{\gamma} c(d_1 \vee \gamma d_2 \gamma^{-1}) = 2c(d_1 \vee \gamma^* d_2 (\gamma^*)^{-1}) = 2c(d_2) < c(d_1) + c(d_2).$$  \hfill (4.35)

The pair

$$d_1 = \begin{array}{c} \gamma \end{array}, \quad d_2 = \begin{array}{c} \gamma \end{array},$$  \hfill (4.36)

is an example of this case since

$$\begin{array}{c} \gamma \end{array} \triangleright \begin{array}{c} \gamma \end{array} = \begin{array}{c} \gamma \end{array}.$$  \hfill (4.37)
The argument is identical for the case where $c(d_1) < c(d_2)$, and there exists some $\gamma^* \in S_k$ such that $d_2 \leq \gamma^* d_1 (\gamma^*)^{-1}$. In this case, by renaming $d_1 \rightarrow d_2$ in (4.35), we have

$$2 \max_\gamma c(d_2 \vee \gamma d_1 \gamma^{-1}) = 2 c(d_2 \vee \gamma^* d_1 (\gamma^*)^{-1}) = 2 c(d_1) < c(d_1) + c(d_2).$$

Using the symmetry of the inner product (3.21) it follows

$$2 \max_\gamma c(d_1 \vee \gamma d_2 \gamma^{-1}) < c(d_1) + c(d_2).$$

2. Secondly, consider the case of incomparability,

$$d_1 \not\leq \gamma d_2 \gamma^{-1} \ \forall \gamma \in S_k.$$  

Recall that for incomparable diagrams (4.25),

$$2 c(d_1 \vee \gamma d_2 \gamma^{-1}) < c(d_1) + c(\gamma d_2 \gamma^{-1}) = c(d_1) + c(d_2),$$

where the last equality follows because conjugation by a permutation does not change the number of connected components. Therefore

$$2 \max_\gamma c(d_1 \vee \gamma d_2 \gamma^{-1}) < c(d_1) + c(d_2),$$

in this case as well.

3. When $d_1 = \gamma d_2 \gamma^{-1}$ for some $\gamma \in S_k$, the bound is saturated and

$$2 \max_\gamma c(d_1 \vee \gamma d_2 \gamma^{-1}) = 2 c(d_1).$$

The condition $d_1 = \gamma d_2 \gamma^{-1}$ implies $[d_1] = [d_2]$. We have proven the inequalities in equation (4.30). As a consequence, we have large $N$ factorization of permutation invariant matrix observables.

### 4.3 Factorization for multi-matrix observables

The above argument generalizes to multi-matrix models. Let $M^{(f)}$ be $n$ matrices with flavour label $f = 1, \ldots, n$ and second moment

$$\left(\langle (M^{(f)})^i_j (M^{(f)})^k_l \rangle \right) = \delta^{il} \delta^{jk} \delta_{ij}.$$

Permutation invariant multi-matrix observables of degree $k = k_1 + k_2 + \cdots + k_n$, where $k_f$ is the degree of matrix $M^{(f)}$, are constructed using partition algebra elements. Multi-matrix observables are labelled by $\vec{k} = (k_1, \ldots, k_n)$ and $d \in P_k(N)$

$$O_{\vec{k},d} = \text{Tr}_{V_N^\otimes k}((M^{(1)})^{k_1} \otimes \cdots \otimes (M^{(n)})^{k_n} d).$$

As before, we have bosonic symmetry. For any $\gamma \in S_k \equiv S_{k_1} \times \cdots \times S_{k_n}$ observables are invariant

$$O_{\vec{k},\gamma d \gamma^{-1}} = \text{Tr}_{V_N^\otimes k}((M^{(1)})^{k_1} \otimes \cdots \otimes (M^{(n)})^{k_n} \gamma d \gamma^{-1})$$

$$= \text{Tr}_{V_N^\otimes k}((M^{(1)})^{k_1} \otimes \cdots \otimes (M^{(n)})^{k_n} d) = O_{\vec{k},d}.$$
Multi-matrix observables are in one-to-one correspondence with partition algebra equivalence classes
\[ [d] = \{ \gamma d \gamma^{-1} | \gamma \in S_k \}. \quad (4.47) \]

Wick contractions vanish unless the flavour indices match, and the sum over \( \gamma \in S_k \) reduces to a sum over \( \gamma \in S_k \)
\[ \langle \mathcal{O}_{k,d_1} \mathcal{O}_{k',d_2} \rangle = \delta_{kk'} \sum_{\gamma \in S_k} \text{Tr}_{V^N_k} (d_1 \gamma d_2 \gamma^{-1}) = \delta_{kk'} \sum_{\gamma \in S_k} N^{c(d_1 \lor d_2 \gamma \gamma^{-1})}. \quad (4.48) \]
The same argument holds for the inequality
\[ 2 \max_{\gamma} c(d_1 \lor d_2 \gamma \gamma^{-1}) \leq \max_{\gamma} c(d_1 \lor d_2 \gamma^{-1}) = c(d_1) + c(d_2). \quad (4.49) \]
It is saturated if and only if there exists a \( \gamma \in S_k \) such that \( d_1 = \gamma d_2 \gamma^{-1} \). That is, if and only if \( [d_1] = [d_2] \) or
\[ \mathcal{O}_{k,d_1} = \mathcal{O}_{k,d_2}. \quad (4.50) \]
To summarize we have
\[ \langle \hat{\mathcal{O}}_{k,d_1} \hat{\mathcal{O}}_{k',d_2} \rangle = \delta_{kk'} \times \begin{cases} 1 + O(1/\sqrt{N}) & \text{if } [d_1] = [d_2], \\ 0 + O(1/\sqrt{N}) & \text{if } [d_1] \neq [d_2], \end{cases} \quad (4.51) \]
for permutation invariant multi-matrix observables in the above Gaussian \( O(N) \) model.

Note that in the case \( n = k, k_f = 1 \) (all matrices distinct), we have
\[ S_k = S_1 \times \cdots \times S_1. \quad (4.52) \]
Therefore, the sum over Wick contractions reduces to a single element (the identity element). The corresponding two-point function is the first case we considered (equation (4.18)).

Finally, we observe that the proof of factorization presented here for general observables labelled by partition algebra diagrams specializes to a new way of thinking about factorization in the case of matrix invariants with continuous symmetry, where the partition algebra diagrams specialize to permutations. The previously known proof based on permutation products can be understood, in the one-matrix case, from the equation
\[ \langle \mathcal{O}_{\sigma_1}(Z) \mathcal{O}_{\sigma_2}(Z^\dagger) \rangle = \frac{k!}{|T_1||T_2|} \sum_{\sigma'_1 \in T_1} \sum_{\sigma'_2 \in T_2} \sum_{\sigma_3 \in S_n} \delta(\sigma'_1 \sigma'_2 \sigma_3) N^{C_{\sigma_3}} \quad (4.53) \]
This equation is derived and explained as eq. (2.12) in [62] (multi-matrix generalisations are discussed in references therein). Gauge invariant operators are labelled by permutations \( \sigma_1, \sigma_2 \) in conjugacy classes \( T_1, T_2 \), while \( |T_1|, |T_2| \) are the sizes of these conjugacy classes. Large \( N \) factorisation follows from the fact that the largest power of \( N \) comes from the case where \( \sigma_3 \) is the identity and this only occurs when \( T_1 = T_2 \). In the present way of looking at permutations as special cases of partition algebra diagrams, permutations belonging to distinct conjugacy classes are always incomparable in the partial order on set partitions associated to the diagrams. This corresponds to Case 2 in of the proofs in sections 4.1 and 4.2.
5 Discussion

In this paper we considered $S_N$ invariant matrix models. These can be viewed as generalisations of their more familiar cousins invariant under continuous symmetries. The most general $S_N$ invariant Gaussian matrix model is specified by a 13-dimensional parameter space. We have shown that there exists a four-dimensional subspace of the 13-dimensional parameter space in which the $S_N$ symmetry is enhanced to $O(N)$. The parameter limit in which this enhancement takes place is given by equation (2.61). The special case of the simplest $O(N)$ invariant Gaussian (2.62) arises at the parameters given in (2.63).

The factorisation property of multi-trace matrix observables invariant under continuous symmetries such as $U(N)$ in the large $N$ limit is a well known result. We have shown that this continues to hold for $S_N$ invariant observables. In the $U(N)$ case this can be seen using properties of the symmetric group by exploiting the Schur-Weyl duality of $U(N)$ and $S_k$ in order to establish a correspondence between observables and conjugacy classes of $S_k$. Analogously, we gave a description of the permutation invariant matrix polynomial functions in terms of a diagram basis for partition algebras. We used the inner product on the permutation invariant polynomials arising from the simplest $O(N)$ invariant action, and proved large $N$ factorisation. The partial order on the diagram basis elements, which can itself be described by a Hasse diagram, plays a role in the proof of factorisation.

An interesting future direction for this work would be to investigate the large $N$ factorisation properties of the inner product of permutation invariant observables arising from the most general $S_N$ invariant action. Progress in this direction has already been made in that the $S_N$ invariant two-point function of the fundamental fields $M_{ij}$ is known from earlier work (see section 3 of [37]). In this paper the form of the simplest $O(N)$ invariant two-point function of the fundamental fields, given in (3.17), allowed us to write a simple expression for the associated two-point function of PIMOs of general order $k$, equation (3.18). In contrast, the form of the $S_N$ invariant two-point function of the fundamental fields involves many more terms and is much more complicated. Dealing with this complication goes beyond the techniques of the current paper.

As explained in the introduction, there are two guiding principles in this paper: the analogies between results for $U(N)$ invariant matrix models and $S_N$ invariant models, and the Schur-Weyl dual algebras of these respective invariances. These principles can be exploited in a number of natural generalisations of the results in this paper. For example, they are applicable to one-dimensional quantum mechanics models of matrices (see our companion paper [81]). They are also applicable to tensor models: this is being developed in [82]. Permutation invariant random matrix distributions have been considered using techniques from probability theory [83]. It would be interesting to investigate the implications of the factorisation results presented here in that context.

The $1/N$ expansion of simple correlators in $U(N)$ invariant matrix models has a geometrical interpretation in terms of Belyi maps, which are branched covers of the sphere with three branch points [32, 33]. This has an interpretation within topological A-model strings [34, 35]. The links with tau functions of integrable models are developed in [84]. Matrix model formulations of general Hurwitz space problems are developed in [85]. The
present paper shows that the large $N$ simplicity of the trace structures of $U(N)$ theories extends to the large $N$ simplicity of permutation invariant observables. This suggests that there may well be a rich analogous geometrical story in the large $N$ expansion of permutation invariant models. $U(N)$ invariant models are related to two-dimensional topological field theories based on lattice gauge theories constructed from symmetric group algebras [54, 86]. We expect analogous developments for $S_N$ invariant models involving topological field theories based on partition algebras. The partition functions of $U(N)$ matrix models display rich large $N$ phase structures which should have interesting parallels in the $S_N$ invariant case [87–96]. A recent study of $S_N$ lattice gauge theory partition functions is in [97].

Many results in $U(N), SO(N), Sp(N)$ matrix models have been developed in the physical context of gauge-string duality. A natural question which encompasses many of the above technical directions is whether there is a gauge-string dual interpretation for the correlators of permutation invariant observables in the simplest $O(N)$ invariant model, where we have established large $N$ factorisation. This permutation invariant sector is one which goes beyond singlets of the continuous symmetry. Non-singlet sectors have been organised according to more general representations of the continuous symmetry and discussed in gauge-string duality in connection with low dimensional models of stringy black hole physics [98–100]. It would be interesting to explore the implications of the large $N$ factorisation we have described in terms of space-time duals of this form, in particular whether there is some generalization of the connection between multi-particle states in a dual background and the factorization property along the lines of [7]. Double scaled matrix models, which have returned to current interest (see e.g. [101–104]) should provide interesting settings for the investigation of permutation invariant observables in models with actions invariant under continuous symmetries. Results in holography for $O(N)$ vector models may also provide useful background in the search for the dual of the present models [105].

Finally it is worth noting that $S_N$ symmetry has been considered in the context of finite deformations of quantum mechanics in [106–109]. The mathematics of permutation invariant matrix models should have interesting interfaces with these deformations.

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A Parameter limits where $S_N$ invariant Gaussian models have enhanced $O(N)$ symmetry

In this appendix we briefly explain the idea behind the Sage code used to find parameter limits in section 2. The code can be found together with the arXiv version of the paper.

The permutation invariant Gaussian 1-matrix model defines a 11-parameter second moment
\[ \langle M_{ij}M_{kl} \rangle_{\text{PIGMM}}, \]  
and the $O(N)$ model defines a 4-parameter second moment
\[ \langle M_{ij}M_{kl} \rangle_{O(N)}. \]  
The solution to the following set of $N^4$ linear equations,
\[ \langle M_{ij}M_{kl} \rangle_{\text{PIGMM}} = \langle M_{ij}M_{kl} \rangle_{O(N)}, \]  
gives the parameter limit in the PIGMM which reconstructs the $O(N)$ model.

The above set of equations are linearly dependent. The maximal number of linearly independent equations is 11 (the number of parameters in the PIGMM). Such a set can be constructed as follows. There are 11 independent choices of $i,j,k,l$ in the second moment for the PIGMM, one for every inequivalent choice of setting $i,j,k,l$ equal or unequal. Two choices are equivalent if they are related by a swap $i \leftrightarrow k, j \leftrightarrow l$, or relabeling $i,j,k,l \mapsto \sigma(i),\sigma(j),\sigma(k),\sigma(l)$ for $\sigma \in S_N$. The Sage code uses the following values for $i,j,k,l$:
\[
\begin{array}{cccc}
i & j & k & l \\
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
2 & 1 & 2 & 2 \\
1 & 2 & 1 & 2 \\
1 & 1 & 2 & 2 \\
1 & 2 & 2 & 1 \\
1 & 2 & 3 & 4 \\
1 & 2 & 1 & 3 \\
1 & 2 & 3 & 2 \\
1 & 2 & 2 & 3 \\
1 & 2 & 3 & 3 \\
\end{array}
\]  

We repeat the same procedure for the simplest $O(N)$ model to find the parameter limit specified in equation (2.63).

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