Evidence for Internal Topological Constraints on Neutrino Mixtures

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Abstract

A new internal description of fundamental fermions (quarks and leptons), based on a matrix-generalization (F) of the scalar fermion-number f, predicts that only three families of quarks and leptons, and their associated neutrino flavors (νe, νµ, and ντ), exist. Moreover, this description appears to place important topological constraints on neutrino mixing. For example, with respect to F, the topology of the νe (νµ or ντ) neutrino flavor is found to be that of a cylinder (Möbius strip). Assuming that neutrino-neutrino transitions exhibiting topology change (i.e., νe → νµ, ντ, or νµ, ντ → νe) are suppressed, while neutrino-neutrino transitions without topology-change (i.e., νe → νe, νµ → νµ, ντ → ντ and νµ → ντ or ντ → νµ) are relatively enhanced, one may have an explanation for recent short-distance (i.e., “atmospheric”) observations of (nearly) maximal νµ-ντ mixing. To test this idea, I was able to use simple topological arguments to uniquely determine the (symmetric) matrix of time-average probabilities Mαβ = ⟨Pνα→νβ⟩ describing long-distance neutrino mixtures, which is identical to that proposed by Georgi and Glashow on different grounds. In particular, using a conventional parameterization of the CKM-like neutrino mixing matrix U, and the proposed topological constraints, I predict the following mixing angle parameters: s22 = c22 = s23 = c23 = 1/2 and s21 = 1 or c21 = 0, with a Dirac-type CP-noninvariant phase factor eiδ, weakly constrained by sin2δ̸= 1 or cos2δ̸= 0 (i.e., CP violation is not maximal). Then Mαβ is predicted to be Mαβ = (M11 = 1/2, M12 = M13 = 1/4, M22 = M23 = M33 = 3/8). Experimental verification of these predictions would provide strong circumstantial evidence in support of the new description of fundamental fermions, which requires, among other things, that the νe and (νµ or ντ) neutrino flavors start and end “life” as topologically-distinct quantum objects.
1.0 Introduction

Except where explicitly prevented by some “absolute” conservation law (e.g., the conservation of electric charge or spin angular momentum), quantum mechanics generally permits transitions between states having different topologies [1]. While a change in topology may be energetically (or otherwise) inhibited, unavoidable quantum fluctuations are expected to catalyze such processes. Hence, there is always the possibility of mixing between otherwise similar quantum states having distinct topologies [2].

Recently, a new internal description of fundamental fermions (quarks and leptons) was proposed [3]. The new description is based on a matrix-generalization $F$ of the scalar fermion number $f$. One of the main predictions of the new description is that only three families of quarks and leptons exist—hence that there are only three low-mass neutrino flavors, namely, the $\nu_e$, $\nu_\mu$ and $\nu_\tau$ neutrinos. Moreover, because of the way fundamental fermions are represented by certain geometric objects in the space on which the matrix transformation $F$ acts (See Appendix A, and Ref. 3, p. 57, pp. 85–87, and Ref. 4, pp. 244–255), the $\nu_e$ and ($\nu_\mu$ or $\nu_\tau$) neutrino flavors are found to have different topologies with respect to $F$. This fact could have important implications for neutrino mixing [5].

With respect to the matrix transformation $F$, the topology of both the $\nu_\mu$ and $\nu_\tau$ is found to be that of a M"obius strip (Ref. 4, p. 143). By contrast, the topology of the $\nu_e$ (with respect to $F$) is that of a cylinder. And because we assume that a change in topology during transitions tends to be suppressed, the foregoing topological distinctions between neutrinos may help explain recent observations of (nearly) maximal $\nu_\mu$-$\nu_\tau$ mixing [6, 7]. However, it is important to stress at the outset that there are good reasons for believing that similar topological distinctions among quarks do not play an important role in $d, s, b$ quark mixing (See Ref. 8 and Sec. A.4.2). The principal purpose of this paper then, will be to focus attention almost exclusively on neutrinos, and to determine the constraints that the proposed topological distinctions place on conventional $\nu_e, \nu_\mu, \nu_\tau$ neutrino mixtures.

2.0 Conventional Description of Neutrino Mixing

At “birth” via weak decays, or upon detection via weak capture interactions, neutrinos have a definite flavor and topology [9]. However, between birth and detection they are in a mixed state having no definite flavor or topology. In this intermediate region the probability of flavor (or topology) maintenance and/or change oscillates. Only at great distances from the neutrino source do these oscillations finally “damp out.”

In the conventional description of three-flavor neutrino mixing flavor eigenstates are related to neutrino mass eigenstates (states of definite mass) via a unitary Cabibbo-Kobayashi-Maskawa (CKM)-like “mixing” matrix $U_{ai}$, as follows [10, 11]

$$\nu_\alpha = \sum_{i=1}^{3} U_{ai} \nu_i.$$  \hfill (1)
Here, $\nu_i = \nu_1, \nu_2$ and $\nu_3$ are mass eigenstates with mass eigenvalues $m_1, m_2$ and $m_3$, respectively, while $\nu_\alpha = \nu_e, \nu_\mu$ and $\nu_\tau$ are flavor eigenstates. Using a conventional parameterization for $U_{\alpha i}$, (1) becomes [12]

$$
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix} =
\begin{pmatrix}
c_1 & s_1 c_3 & s_1 s_3 \\
-s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\
-s_1 s_2 & c_1 s_2 c_3 + s_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta}
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix},
$$

(2)

where $c_i \equiv \cos \theta_i$ and $s_i \equiv \sin \theta_i$, and $\delta$ is associated with a Dirac-type CP-noninvariant phase factor $e^{i\delta}$.

Using (2) it can be shown that the probability of detecting a neutrino of flavor type $\beta$ at a distance $X$ from a source of neutrinos of flavor type $\alpha$ is given by [11]

$$
P_{\nu_\alpha \rightarrow \nu_\beta} = \sum_{i=1}^{3} |U_{\alpha i}|^2 |U_{\beta i}|^2 + \sum_{i \neq j}^{3} U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j} \cos \left( \frac{2\pi X}{l_{ij}} \right).
$$

(3)

Here, the so-called oscillation lengths $l_{ij}$ are given by $l_{ij} = 2\pi/(E_i - E_j)$, where the total relativistic energy differences in a beam of neutrinos having fixed momentum $p$ are $E_i - E_j = (m_i^2 - m_j^2)/2p$.

Examination of (3) shows that at the neutrino source ($X = 0$, and $t = 0$) the probability $P_{\nu_\alpha \rightarrow \nu_\beta}$ reduces, as expected, to the $3 \times 3$ identity matrix [13]

$$
P_{\nu_\alpha \rightarrow \nu_\beta} \bigg|_{X=0} = I_3,
$$

(4)

while at “intermediate” distances from the neutrino source ($X \approx l_{ij}$), the probability $P_{\nu_\alpha \rightarrow \nu_\beta}$ undergoes oscillations. Finally, at great distances from the neutrino source ($X \gg l_{ij}$), all time- or distance-dependent oscillations “damp out,” and we are left with a $3 \times 3$ matrix of time-average probabilities [11], namely,

$$
\langle P_{\nu_\alpha \rightarrow \nu_\beta} \rangle = \sum_{i=1}^{3} |U_{\alpha i}|^2 |U_{\beta i}|^2.
$$

(5)

From (5) it is clear that this matrix ($\alpha =$ row index, $\beta =$ column index), which describes long-distance neutrino mixtures, is symmetric. Keeping in mind this symmetry, and calling this matrix $M$, one has

$$
M = \begin{pmatrix}
\langle P_{\nu_e \rightarrow \nu_e} \rangle, & \langle P_{\nu_e \rightarrow \nu_\mu} \rangle, & \langle P_{\nu_e \rightarrow \nu_\tau} \rangle \\
\langle P_{\nu_\mu \rightarrow \nu_e} \rangle, & \langle P_{\nu_\mu \rightarrow \nu_\mu} \rangle, & \langle P_{\nu_\mu \rightarrow \nu_\tau} \rangle \\
\langle P_{\nu_\tau \rightarrow \nu_e} \rangle, & \langle P_{\nu_\tau \rightarrow \nu_\mu} \rangle, & \langle P_{\nu_\tau \rightarrow \nu_\tau} \rangle
\end{pmatrix}.
$$

(6)

Note that all rows and columns of $M$ must sum to unity (total probability 1).
Using $M$ we can describe the expected neutrino flavor content at a great distance from a neutrino source (e.g., a supernova) as follows

$$\{D_e, D_\mu, D_\tau}\ =\ M\{B_e, B_\mu, B_\tau\},$$

(7)

where $\{ \}$ signifies column vectors, and $D_\alpha$ is the number of detected neutrinos of definite flavor $\nu_\alpha$, and $B_\alpha$ their number at “birth” at some distant neutrino source. Note that because neutrinos are assumed to be conserved, the total number of neutrinos at birth equals their number upon “detection,” namely,

$$B_e + B_\mu + B_\tau = D_e + D_\mu + D_\tau.$$

(8)

### 3.0 Proposed Topological Constraints on Neutrino Mixing

In this paper we will show, among other things, that certain proposed qualitative topological constraints on the matrix of time-average probabilities $\langle P_{\nu_\alpha \to \nu_\beta} \rangle$ or $M$, result in a quantitative determination of certain ($U$-matrix) neutrino mixing-parameters [14]. These mixing parameters, in turn, serve to provide a unique determination of the matrix $\langle P_{\nu_\alpha \to \nu_\beta} \rangle$ or $M$.

Given that the $\nu_e$ ($\nu_\mu$ or $\nu_\tau$) neutrino flavor has the topology of a cylinder (Möbius strip) with respect to the internal transformation $F$, and assuming that topological constraints are the primary determinants of the matrix $M$ describing long-distance neutrino mixtures, the form of $M$ is easily determined. Moreover, these same topological constraints can be used to place numerical bounds on the components of $M$. To accomplish these results we need only apply the following very general principle to neutrino-neutrino transitions:

*All other things being equal, any neutrino flavor $\nu_\alpha$ (i.e., $\nu_e$, $\nu_\mu$ or $\nu_\tau$), which undergoes neutrino-neutrino transitions that change neutrino topology, will tend to be suppressed, while neutrino-neutrino transitions that maintain neutrino topology will tend to be (relatively) enhanced* [5].

To this principle we add the following corollary:

*All other things being equal, because the $\nu_\mu$ and $\nu_\tau$ neutrinos have the same topology, they will act the same way in all neutrino-neutrino transitions (involving long-distance neutrino mixtures).*

Given these principles, and assuming as stated previously, that topological constraints are the primary determinants of the matrix $M$, we immediately have the following topological constraints on long-distance neutrino mixtures:

A. No matter what neutrino flavor ($\nu_\alpha$) and topology one starts with at some distant source (say a supernova), by the time the neutrino mixture reaches its “equilibrium” state (where all time-dependent oscillations have “damped out”), it should contain equal fractions of $\nu_\mu$ and $\nu_\tau$, because these neutrinos have the same topology.
B. Because the $\nu_\mu$ and $\nu_\tau$ neutrinos have the same topology, if one starts out with either a pure $\nu_\mu$ or a pure $\nu_\tau$ source, one should end up with the same long-distance equilibrium mixture of $\nu_e$, $\nu_\mu$ and $\nu_\tau$.

C. If topology is the controlling factor in describing long-distance neutrino mixtures, then there should be absolutely no (effective) difference between the two functions (of $U$-matrix mixing parameters), which describe $\langle P_{\nu_e \rightarrow \nu_\mu} \rangle$ and $\langle P_{\nu_e \rightarrow \nu_\tau} \rangle$, because the $\nu_\mu$ and $\nu_\tau$ neutrinos have the same topology. Very loosely speaking we are assuming that these mathematical functions are effectively “topological invariants” with respect to the exchange of flavor indices $\mu$ and $\tau$ (See Ref. 4, pp. 20 and 21). That is, not only are the two functions $\langle P_{\nu_e \rightarrow \nu_\mu} \rangle$ and $\langle P_{\nu_e \rightarrow \nu_\tau} \rangle$ required to be equal, but they are also required to be equal, term-by-term (i.e., they are required to be term-wise equal). Similarly the three functions $\langle P_{\nu_\mu \rightarrow \nu_\mu} \rangle$, $\langle P_{\nu_\mu \rightarrow \nu_\tau} \rangle$ and $\langle P_{\nu_\tau \rightarrow \nu_\tau} \rangle$ are required to be term-wise equal.

3.1 The form of the matrix $M$

Constraints A), B) and C) in the previous section, together with Eqs. (7) and (8), dictate that the symmetric matrix $M$ describing long-distance neutrino mixtures must have the form expressed by

$$
\begin{pmatrix}
D_e \\
D_\mu \\
D_\tau
\end{pmatrix}
= 
\begin{pmatrix}
a & b & b \\
b & c & c \\
b & c & c
\end{pmatrix}
\begin{pmatrix}
B_e \\
B_\mu \\
B_\tau
\end{pmatrix}.
$$

(9)

Moreover, the proposed topological constraints place numerical bounds on the matrix elements $a$, $b$ and $c$. For example, because we are assuming that the $\nu_e$ is inhibited, but because of quantum fluctuations not prevented, by its topology from turning into a $\nu_\mu$ or $\nu_\tau$, we require the inequalities

$$a > b > 0.\quad (10)$$

Similarly, because we are assuming that the $\nu_\mu$ and/or the $\nu_\tau$ are inhibited by their topology from turning into a $\nu_e$, we require the inequality

$$c > b.\quad (11)$$

Equations (10) and (11), together with the requirement that all rows and columns of the matrix $M$ in (7) and (8) sum to unity (i.e., $a+2b=b+2c=1$), further leads to the following inequalities

$$\left( \frac{1}{3} < a < 1 \right), \quad \left( 0 < b < \frac{1}{3} \right), \quad \left( \frac{1}{3} < c < \frac{1}{2} \right).\quad (12)$$
Moreover, since \( a + 2b = b + 2c \), the arithmetic mean of \( a \) and \( b \) is \( c \), i.e., \( (a + b)/2 = c \), which means that \( c \) lies half-way between \( a \) and \( b \). Hence, the matrix elements \( a, b \) and \( c \) are subject to the combined inequalities

\[
a > c > b > 0. \tag{13}
\]

Equations (9), (12) and (13), together with items A), B) and C) above, constitute the proposed topological constraints on the matrix \( M \). It happens that these constraints on \( M \) are very restrictive. In particular, we will show in the next section (See also Appendix B) that these constraints, together with the assumption that \( \sin^2 \delta \neq 1 \) or \( \cos^2 \delta \neq 0 \) (i.e., CP violation is not maximal), admit of only one solution for \( M \!\!\!\). 

### 4.0 Determination of the Matrix \( M \)

When the proposed topological constraints of Section 3.0 are applied to the conventional description of neutrino mixing (3), the matrix \( M \) in (6), (7) and (9), is uniquely determined. To see how this happens consider the following time-average probabilities (See Appendix B for details)

\[
\langle P_{\nu_e \rightarrow \nu_\mu} \rangle = 2c_1^2s_1^2c_2^2 + 2s_1^2s_2^2c_3(s_2^2 - c_1^2c_2^2) + 2s_1^2s_2s_3c_1c_2c_3 \cos \delta(s_3^2 - c_3^2), \tag{14}
\]

and

\[
\langle P_{\nu_e \rightarrow \nu_\tau} \rangle = 2c_1^2s_1^2s_2^2 + 2s_1^2s_2^2c_3^2(c_2^2 - c_1^2s_2^2) - 2s_1^2s_2s_3c_1c_2c_3 \cos \delta(s_3^2 - c_3^2). \tag{15}
\]

According to the proposed topological constraints of Section 3.0, these two time-average probabilities must be term-wise equal, and nonzero. These requirements place three constraints on the \( U \)-matrix mixing parameters, namely, \( s_1^2 > 0, s_2^2 = c_2^2 = \frac{1}{2} \), and

\[
s_1^2s_2s_3c_1c_2c_3 \cos \delta(s_3^2 - c_3^2) = 0. \tag{16}
\]

Note that (14) and (15) are term-wise equal as required, even with the minus sign preceeding the last term of (15) because this term is of zero magnitude.

Next, the topological constraint of Section 3.0, namely, \( \langle P_{\nu_\mu \rightarrow \nu_\mu} \rangle = \langle P_{\nu_\tau \rightarrow \nu_\tau} \rangle \) can be realized provided the \( U \)-matrix mixing parameters are further constrained by \( s_3^2 = c_3^2 = \frac{1}{2} \), which also happens to satisfy (16). That is, given \( s_2^2 = c_2^2 \) and \( s_3^2 = c_3^2 \) one has (See Appendix B for details)

\[
\langle P_{\nu_\mu \rightarrow \nu_\mu} \rangle = s_1^4c_2^4 + 2c_2^4c_3^4(c_1^2 + 1)^2 + 8c_1^2c_2^4c_3^4 \cos^2 \delta, \tag{17}
\]

and

\[
\langle P_{\nu_\tau \rightarrow \nu_\tau} \rangle = s_1^4c_2^4 + 2c_2^4c_3^4(c_1^2 + 1)^2 + 8c_1^2c_2^4c_3^4 \cos^2 \delta. \tag{18}
\]
Now the topological constraints of Section 3.0 also require that $\langle P_{\nu_\mu \rightarrow \nu_\tau} \rangle$ be equal to (17) and (18). This places additional constraints on $M$, and the $U$-matrix mixing parameters. Comparing (17) and (18) with the following expression from Appendix B for $\langle P_{\nu_\nu \rightarrow \nu_\tau} \rangle$, namely,

$$\langle P_{\nu_\nu \rightarrow \nu_\tau} \rangle = s_4^4 c_2^4 + 2c_2^2 c_3^4 (c_1^2 + 1)^2 - 8c_1^2 c_2^2 c_3^4 \cos^2 \delta,$$

we see that the requisite equality $\langle P_{\nu_\mu \rightarrow \nu_\nu} \rangle = \langle P_{\nu_\tau \rightarrow \nu_\tau} \rangle = \langle P_{\nu_\mu \rightarrow \nu_\tau} \rangle$ leads to the constraint (note that $c_2^4 = c_3^4 > 0$)

$$c_1^2 \cos^2 \delta = 0. \quad (20)$$

Assuming [15, 16] that $\sin^2 \delta \neq 1$ or $\cos^2 \delta \neq 0$ (i.e., CP violation is not maximal), we further determine from (20) that

$$c_1^2 = 0 \text{ and } s_1^2 = 1. \quad (21)$$

Note that (17), (18) and (19) exhibit the requisite term-wise equality [See item C) in Sec. 3.0] even with the minus sign preceding the third term of (19), because this term is of zero magnitude.

Gathering together the predicted mixing-parameter constraints, namely, $c_2^4 = s_2^4$, $c_3^4 = s_3^4$, $s_1^2 = 1$ and $c_1^2 = 0$, we can express six of the time-average probabilities associated with the (symmetric) matrix $M$ as follows (Use Eqs. B6, B4, B3 and B1, B2)

$$\langle P_{\nu_e \rightarrow \nu_e} \rangle = 1 - 2c_3^4, \quad (22)$$

$$\langle P_{\nu_e \rightarrow \nu_\mu} \rangle = \langle P_{\nu_e \rightarrow \nu_\tau} \rangle = 2c_2^2 c_3^4, \quad (23)$$

and

$$\langle P_{\nu_\mu \rightarrow \nu_\mu} \rangle = \langle P_{\nu_\tau \rightarrow \nu_\tau} \rangle = \langle P_{\nu_\mu \rightarrow \nu_\tau} \rangle = c_2^4 (1 + 2c_3^4). \quad (24)$$

But, the previous arguments have shown that the $U$-matrix mixing parameters satisfy $c_2^4 = c_3^4 = \frac{1}{2}$, which leads to the following specific numerical predictions

$$\langle P_{\nu_e \rightarrow \nu_e} \rangle = \frac{1}{2} = a, \quad (25)$$

$$\langle P_{\nu_e \rightarrow \nu_\mu} \rangle = \langle P_{\nu_e \rightarrow \nu_\tau} \rangle = \frac{1}{4} = b, \quad (26)$$

and

$$\langle P_{\nu_\mu \rightarrow \nu_\mu} \rangle = \langle P_{\nu_\tau \rightarrow \nu_\tau} \rangle = \langle P_{\nu_\mu \rightarrow \nu_\tau} \rangle = \frac{3}{8} = c. \quad (27)$$
Given that $M$ is symmetric, note that (25), (26) and (27) are the only matrix elements, which could be consistent with the proposed topological constraints of Section 3.0, and the requirement that $\sin^2 \delta \neq 1$ or $\cos^2 \delta \neq 0$ (See Ref. 15 and 16).

Employing Eqs. (9) and (25–27), one finally has the prediction [6, 16] 

$$M = \frac{1}{8} \begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{pmatrix}. \tag{28}$$

To summarize, the proposed (qualitative) topological constraints on the matrix $M$ (See Sec. 3.0) result in quantitative constraints on the $U$-matrix mixing parameters (See Eq. 3), namely, $(s_2^2 = c_2^2 = c_3^2 = s_3^2 = \frac{1}{2}, s_1^2 = 1$ or $c_1^2 = 0$ with $\sin^2 \delta \neq 1$ or $\cos^2 \delta \neq 0$ assumed), which in turn result in a unique quantitative determination of the matrix $M$ (See Eq. 28).

5.0 Topology-Maintaining and Topology-Changing Influences in Equilibrium?

Not only is the matrix $M$ in (28) a unique solution to the proposed topological constraints of Section 3.0 with $\sin^2 \delta \neq 1$ or $\cos^2 \delta \neq 0$, but it also has some very special properties that may eventually help reveal the deeper dynamical significance of this matrix. Note that (6), (9), and (28) have the very special property

$$\langle P_{\nu_e \rightarrow \nu_e} \rangle = \langle P_{\nu_e \rightarrow \nu_\mu} \rangle + \langle P_{\nu_e \rightarrow \nu_\tau} \rangle. \tag{29}$$

This equation says that the time-average probability that the $\nu_e$ topology doesn’t change, namely,

$$P_{NC} = \langle P_{\nu_e \rightarrow \nu_e} \rangle = a, \tag{30}$$

and the time-average probability that the $\nu_e$ topology does change, namely,

$$P_C = \langle P_{\nu_e \rightarrow \nu_\mu} \rangle + \langle P_{\nu_e \rightarrow \nu_\tau} \rangle = (1 - a), \tag{31}$$

are equal, namely,

$$P_C = P_{NC}. \tag{32}$$

Now this equality looks very much like an “equilibrium” condition between those underlying physical influences that would act to change the $\nu_e$ topology (quantum fluctuations), and those underlying physical influences that would act to maintain the $\nu_e$ topology (e.g., energy “barriers”).
We will now provide further support for this proposal. In particular, consider the “joint” probability [17]

\[ P = P_C \cdot P_{NC}, \]  

and notice that

\[ \frac{dP}{da} = P_C \frac{dP_{NC}}{da} + P_{NC} \frac{dP_C}{da}. \]  

From (30) and (31) this last equation reduces to

\[ \frac{dP}{da} = P_C - P_{NC}. \]  

And, taking the second derivative, we also find

\[ \frac{d^2P}{da^2} = -2 < 0. \]

Therefore, when \( P_C = P_{NC} \) we discover that the joint probability \( P = P_C \cdot P_{NC} \) is a maximum, namely, it characterizes some most probable condition or “state.” And, this of course is the very essence of an “equilibrium” condition [7]. However, it must be understood that this hypothetical (long-distance) “equilibrium” between topology-changing, and topology-maintaining physical influences, is only a (cumulative) result of deeper, and largely unknown (short-distance), dynamical processes in the vacuum, which first begin to act on neutrinos at their source—thereby, eventually establishing the equilibrium condition \( P_C = P_{NC} \)—on time scales very much shorter than the time it takes for the time-dependent oscillations in neutrino mixtures to “damp out.” This is an essential requirement if these hypothetical short-distance processes are to be responsible for “selecting” the (constant) \( U \)-matrix mixing parameters prior to neutrino mixing.

Finally, while the unique nature of (28) has been demonstrated, and while we see that this result seems to be closely connected with (short-distance) physical processes involving topological constraints that select the neutrino mixing parameters, it is also clear that we do not possess a first-principles formulation of these ideas. As such, they are only promising conjectures.

6.0 Conclusions

Topological constraints could play a major role in determining the nature of neutrino mixtures. Given that the \( \nu_e \) and \( (\nu_\mu \text{ or } \nu_\tau) \) neutrino flavors have distinct topologies, and assuming that topology changes are suppressed in neutrino-neutrino transitions, while neutrino-neutrino transitions without topology-change are relatively enhanced—one easily determines both the form of the matrix describing long-distance neutrino mixtures, and uniquely determines the numerical magnitudes of the individual matrix elements. If the predicted matrix
(28), is eventually verified by observations of neutrinos from distant astronomical sources (e.g., supernovae), and/or in long-baseline terrestrial experiments, this will provide qualitative support for the new description of fundamental fermions, which requires, among other things, that the $\nu_e$ and ($\nu_\mu$ or $\nu_\tau$) neutrino flavors start, and end, “life” as topologically distinct quantum objects (See Appendix A and Refs. 3–5). However, while experiments could certainly falsify (28), (9), (12) and (13), together with items A), B) and C) in Section 3.0, and the assumption [15, 16] that CP violation is not maximal (i.e., $\sin^2 \delta \neq 1$ or $\cos^2 \delta \neq 0$), experimental verification of these things would not confirm the proposed topological constraints of Section 3.0. A much better theoretical understanding of the dynamical significance (if any) of the new 2-space description of quarks and leptons will be required before one can be confident that, if verified, (28) really is a reflection of underlying topological constraints [18].

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Appendix A. A new Internal 2-Space Description of Quarks and Leptons

A.1 Background

In spite of the many successes of the standard model of particle physics, the observed proliferation of matter-fields, in the form of “replicated” generations or families, is a major unsolved problem. In [3, 19 and 20] a new organizing principle for fundamental fermions was proposed, i.e., a minimalistic “extension” of the standard model based, in part, on the Cayley-Hamilton theorem for matrices. To introduce (internal) global degrees of freedom that are capable of distinguishing all observed flavors, the Cayley-Hamilton theorem was used to generalize the familiar standard-model concept of scalar fermion-numbers \( f \) (i.e., \( f_m = +1 \) for all fermions and \( f_a = -1 \) for all antifermions). This theorem states that every (square) matrix satisfies its characteristic equation. Hence, if \( f_m \) and \( f_a \) are taken to be the eigenvalues of some real matrix \( F \) (a “generalized fermion number”), it follows from this theorem that both \( f \) and \( F \) are square-roots of unity. Assuming further that the components of both \( F \) and its eigenvectors are global charge-like quantum observables, and that \( F \) “acts” on a (real) internal vector 2-space, both the form of \( F \) and the 2-space metric are determined. One finds that the 2-space has a non-Euclidean or “Lorentzian” metric, and that various associated 2-scalars serve as global flavor-defining “charges,” which can be identified with Lorentz 4-scalar charges such as strangeness, charm, baryon and lepton numbers.

Because these global charges are essentially the global charges associated with the so-called “accidental symmetries” of the Lagrangian describing strong and electroweak interactions [19–21], they can be used to describe individual flavors (i.e., flavor eigenstates), flavor doublets and families. Moreover, because of the aforementioned non-Euclidean constraints, and certain standard-model constraints, one finds that these global charges are effectively “quantized” in such a way that families are replicated. Finally, because these same constraints dictate that there are only a limited number of values these charges can assume, one finds that families, and their associated neutrinos, always come in “threes.”

A.2 Representing Flavor Eigenstates

The eigenvectors \( Q \) of \( F \) (i.e., \( FQ = fQ \) where \( f \) is the scalar fermion-number), together with certain pairs of linearly independent vectors (\( U \) and \( V \)) that resolve \( Q \) (i.e., \( Q = U + V \)), namely, various non-Euclidean vector “triads” (\( Q, U, V \))—these are the analogs of Euclidean triangles—serve to represent flavor-doublets in terms of a pair of quark or lepton flavor-eigenstates as follows:

\[
|\text{“up”}\rangle = |q_1, u_1, v_1, Q^2, U^2, 2U \cdot V\rangle \tag{A1}
\]

and

\[
|\text{“down”}\rangle = |q_2, u_2, v_2, Q^2, U^2, 2U \cdot V\rangle. \tag{A2}
\]
Here, \( Q = \{q_1, q_2\} \), \( U = \{u_1, u_2\} \) and \( V = \{v_1, v_2\} \) are column-vectors and their components \( q_1, q_2, u_1, u_2, v_1 \) and \( v_2 \), together with the non-Euclidean scalar products \( Q^2 \), \( U^2 \), \( V^2 \) and \( U \cdot V \), are various global mutually-commuting flavor-defining charge-like quantum numbers (e.g., \( q_1 \) and \( q_2 \) are electric charges carried by “up”- and “down”-type flavors in a flavor doublet).

A.3 Topology of Vector Triads

When we refer to the “topology” of a particular neutrino flavor-eigenstate (e.g., the \( \nu_e \)), we are referring to the topology of the corresponding vector triad \((Q, U, V)\) with respect to the internal transformation \( F \). And, because \( F \) generates the Möbius group \( Z_2 \) (i.e., \( F^2 = I_2 \)), [i.e., symbolically \( F(v)(Q, U, V) \equiv (Q, U, V) \)], have the (abstract) topology of a cylinder, while vector triads that are changed by \( F \) [i.e., \( F(v)(Q, U, V) \not\equiv (Q, U, V) \)], but obviously not changed by the identity \( F^2 = I_2 \), have the (abstract) topology of a Möbius strip. And, as it turns out, the neutrino flavor \( \nu_e \) (\( \nu_\mu \) or \( \nu_\tau \)) corresponds to a vector triad having the topology of a cylinder (Möbius strip) with respect to \( F \). See the qualifying remarks in the next section.

A.4 Qualifying Remarks

With the possible exception of one family, all quarks and leptons within a given family are found to exhibit the same topology with respect to \( F \). This is certainly the case for the first and third families. And, because of these facts, together with the requirement of quark-lepton “universality,” at least within any given family, we naturally expect this to be the case for the second family as well. It happens that the second-family \( c \) and \( s \) quarks exhibit Möbius topology with respect to \( F \). However, strictly speaking, the second-family leptons, namely, the muon and its associated neutrino (and associated antiparticles) exhibit the requisite Möbius topology only if the components of the associated \( V \)-vectors are not exactly zero (See p. 57, and Tables II and IV in Ref. 3 where the muon \( V \)-vectors, which are functions of the strong-color multiplicity \( M_c \), are shown to be \( V \equiv 0 \) in the limit where \( M_c \equiv 1 \)). Therefore, if we wish to maintain a kind of quark-lepton “universality” within families, the \( V \)-vector components associated with muons, though very small, must be nonzero. And, according to the 2-space mathematical description of \( V \)-vectors, this is always the case whenever \( M_c \gtrsim 1 \).

A.4.1 \( \nu_\mu \) neutrinos with \( M_c \gtrsim 1 \).

It has been shown (See Ref. 3, pp. 52–55) that the quark and lepton electric charges are the “up”-“down” components of the eigenvectors of the matrix \( F(v) \), where \( v = \ln M_c \). In particular, the quark charges are given by \((M_c = 3)\)

\[
q_1(f) = \frac{(M_c^2 - 1)}{2M_c(M_c - f)} = \begin{cases} 
+\frac{2}{3} & \text{for } f = +1 \\
+\frac{1}{3} & \text{for } f = -1,
\end{cases}
\]

\[
q_2(f) = q_1(f) - 1,
\]

\( A3 \)

\( A4 \)
where the baryon number for quarks is \( B = q_1^2(f) - q_2^2(f) = \pm \frac{1}{3} \) for \( f = \pm 1 \). Similarly, the lepton electric charges are given by \( (M_c = 1) \)

\[
q_1'(f) = \frac{-(M_c^2 - 1)}{2M_c(M_c - f)} = -1 \text{ for } f = +1 \text{ and } 0 \text{ for } f = -1, \quad \text{(A5)}
\]

\[
q_2'(f) = q_1'(f) + 1, \quad \text{(A6)}
\]

where the lepton number for leptons is \( L = [q_1'(f)]^2 - [q_2'(f)]^2 = \pm 1 \) for \( f = \pm 1 \).

Now, if the eigenvector \( Q \) of \( \mathbf{F}(v) \) is \( Q = \{ q_1'(f), q_2'(f) \} \), and \( f = +1 \) for leptons where \( Q = U + V \), we have for leptons

\[
V = Q - U, \quad \text{(A7)}
\]

where the 2-vectors \( U, V \) are appropriate to a description of leptons. According to the 2-space description, \( Q \) and \( V \) are both functions of \( M_c \), while \( U \) is not. Only in the case of the muons (\( \mu^- \) and \( \nu_\mu \)) with \( M_c \equiv 1 \) is \( Q \equiv U \equiv \{-1, 0\} \) and \( V \equiv 0 \). In this limiting case, the vector triad \( (Q, U, V) \) does not possess the requisite second-family Möbius topology. In fact, it transforms under \( \mathbf{F} \) like a cylinder. So, a way to insure that \( V \neq 0 \), which lies completely within the context of the existing 2-space mathematical description, is to treat \( M_c \) as a continuous variable, and let \( M_c \) closely approach, but never actually equal one in \( (A5), (A6), \) and \( (A7) \) while maintaining \( U = \{-1, 0\} \). In this case, \( V \neq 0 \), which leads to the requisite second-family Möbius topology for the vector triad \( (Q, U, V) \), and hence the \( \nu_\mu \) neutrino. Of course, when \( M_c \gtrsim 1 \), neutrinos, including the \( \nu_\mu \), would carry an infinitesimal electric charge \( q_2'(f) \gtrsim 0 \). However, from a “practical” physical standpoint this is not a serious problem because \( q_2'(f) \) can always be made arbitrarily small, while always maintaining the requisite second-family Möbius topology of the vector triad associated with the \( \nu_\mu \) neutrino.

It is clear from the foregoing discussion that the current mathematical and physical description of the 2-space is simply not “robust” enough to demonstrate, without a doubt, that \( V \neq 0 \) for the \( \nu_\mu \) neutrino. Nevertheless, for various reasons outlined at the beginning of this subsection (A.4), and because of the foregoing mathematical justification, we will assume throughout this paper that this is indeed the case, and that, as a consequence, the vector triad associated with the \( \nu_\mu \) neutrino does possess the requisite second-family Möbius topology with respect to \( \mathbf{F} \).

A.4.2  Topological influences in quark mixing.

As described above, all quarks and leptons within a given family exhibit the same topology with respect to the internal transformation \( \mathbf{F} \). However, as described in [8], owing to the uncertainty principle, and the relatively large mass differences between quarks, mixing among the \( d, s, b \) quarks (inside quark composites) is not expected to be controlled by topology. By contrast, \( \nu_e, \nu_\mu, \nu_\tau \) neutrino mixing is expected to be controlled by topology, owing to the small value, and near degeneracy, of neutrino masses.
Appendix B. Determination of $\langle P_{\nu_\alpha \rightarrow \nu_\beta} \rangle$.

The standard description of three-flavor neutrino mixtures [10, 11] involves a CKM-like unitary matrix $U_{\alpha i}$ that transforms mass eigenstates $\nu_i = (\nu_1, \nu_2, \nu_3)$ into flavor eigenstates $(\nu_\text{e}, \nu_\mu, \nu_\tau)$ as follows (See Eqs. 1 and 2 in the main text)

$$\nu_\alpha = \sum_{i=1}^{3} U_{\alpha i} \nu_i. \tag{B1}$$

Using this formulation one can calculate the time-dependent probability $P_{\nu_\alpha \rightarrow \nu_\beta}$ that a neutrino of some definite flavor $\alpha$ (at “birth”) will transform into a neutrino of a generally different flavor $\beta$, upon detection. The result is (3) in the main text.

At sufficiently great distances from the source of neutrinos, the time-dependent part of (3) “damps out,” and we are left with the time-independent or time-average probability

$$\langle P_{\nu_\alpha \rightarrow \nu_\beta} \rangle = \sum_{i=1}^{3} |U_{\alpha i}|^2 |U_{\beta i}|^2. \tag{B2}$$

Using (B2), and the $U$-matrix given by (2) in the main text, we can calculate all $\langle P_{\nu_\alpha \rightarrow \nu_\beta} \rangle$. For example, we find (Note that $e$, $\mu$ and $\tau$ refer to $U$-matrix row indices 1, 2 and 3, respectively)

$$\langle P_{\nu_\alpha \rightarrow \nu_e} \rangle = \sum_{i=1}^{3} |U_{1i}|^4 = |U_{11}|^4 + |U_{12}|^4 + |U_{13}|^4 \tag{B3}$$

$$= c_1^4 + s_1^4 c_3^4 + s_1^4 s_3^4 \tag{B4}$$

$$= c_1^4 + s_1^4 (1 - 2c_3^2 s_3^2), \tag{B5}$$

$$\langle P_{\nu_\alpha \rightarrow \nu_e} \rangle = 1 - 2c_1^2 s_1^2 - 2s_1^2 s_3^2 c_3^2. \tag{B6}$$
Similarly, we have

\[
\langle P_{\nu_e \to \nu_\mu} \rangle = \sum_{i=1}^{3} |U_{1i}|^2 |U_{2i}|^2 = |U_{11}|^2 |U_{21}|^2 + |U_{12}|^2 |U_{22}|^2 + |U_{13}|^2 |U_{23}|^2,
\]

(B7)

\[
= c_1^2 s_1^2 s_2^2 + s_1^2 c_3^2 c_1 c_2 c_3 - s_2 s_3 \cos \delta - i s_2 s_3 \sin \delta|^2
+ s_1^2 s_3^2 c_1 c_2 s_3 + s_2 c_3 \cos \delta + i s_2 c_3 \sin \delta|^2,
\]

(B8)

\[
= s_1^2 (c_1^2 c_2^2 + c_1^2 c_3^2 c_2^2)\cos \delta + s_2^2 s_3^2 \sin \delta
+ s_3^2 [(c_1 c_2 s_3 + s_2 c_3) \cos \delta + s_2^2 c_3^2 \sin \delta],
\]

(B9)

\[
= s_1^2 (c_1^2 c_2^2 + c_1^2 c_3^2 c_2^2 - 2 c_1 c_2 c_3 s_2 s_3 \cos \delta + s_2^2 s_3^2 \cos \delta + s_2^2 s_3^2 \sin \delta)
+ s_3^2 [c_1^2 c_2^2 s_3^2 + 2 c_1 c_2 c_3 s_2 s_3 \cos \delta + s_2^2 c_3^2 \cos \delta + s_2^2 c_3^2 \sin \delta],
\]

(B10)

\[
= s_1^2 (c_1^2 c_2^2 + c_1^2 c_3^2 (c_3^2 + s_3^2) - 2 c_1 c_2 c_3 s_2 s_3 \cos \delta (c_3^2)
+ 2 c_1 c_2 c_3 s_2 s_3 \cos \delta s_3^2 + 2 c_3^2 s_2^2 s_3^2),
\]

(B11)

\[
= s_1^2 (c_1^2 c_2^2 + c_1^2 c_3^2 (1 - 2 c_3^2 s_2^2) + 2 c_3^2 s_2^2 s_3
+ 2 c_1 c_2 c_3 s_2 s_3 \cos \delta (s_3^2 - c_3^2)),
\]

(B12)

\[
= s_1^2 (2 c_1^2 c_2^2 + 2 c_3^2 s_2^2 s_3 - 2 c_1^2 c_3^2 s_3^2
+ 2 c_1 c_2 c_3 s_2 s_3 \cos \delta (s_3^2 - c_3^2)),
\]

(B13)

\[
\langle P_{\nu_e \to \nu_\mu} \rangle = 2 c_1^2 s_1^2 c_2^2 + 2 s_1^2 s_3^2 s_3^2 (s_2^2 - c_2^2 c_3^2) + 2 s_2^2 s_3^2 s_3 c_1 c_2 c_3 \cos \delta (s_3^2 - c_3^2).
\]

(B14)

\[
\langle P_{\nu_e \to \nu_\tau} \rangle = \sum_{i=1}^{3} |U_{1i}|^2 |U_{3i}|^2 = |U_{11}|^2 |U_{31}|^2 + |U_{12}|^2 |U_{32}|^2 + |U_{13}|^2 |U_{33}|^2,
\]

(B15)

\[
= c_1^2 s_1^2 s_2^2 + s_1^2 c_3^2 c_1 s_2 c_3 + c_2 s_3 \cos \delta + i c_2 s_3 \sin \delta|^2
+ s_1^2 s_3^2 [c_1 s_2 c_3 - c_2 c_3] \cos \delta - i c_2 c_3 \sin \delta|^2,
\]

(B16)

\[
= s_1^2 (c_1^2 s_2^2 + c_1^2 c_3^2 (c_2^2 s_3^2) + 2 c_1 c_2 c_3 s_2 s_3 \cos \delta + s_2^2 s_3^2 \cos \delta + s_2^2 s_3^2 \sin \delta)
+ s_3^2 [(c_1 s_2 c_3 - c_2 c_3) \cos \delta + s_2^2 c_3^2 \sin \delta],
\]

(B17)

\[
= s_1^2 (c_1^2 s_2^2 + c_1^2 c_3^2 s_3^2 + 2 c_1 c_2 c_3 s_2 s_3 \cos \delta + s_2^2 s_3^2 \cos \delta + s_2^2 s_3^2 \sin \delta)
+ s_3^2 [c_1^2 c_3^2 s_3^2 - 2 c_1 c_2 c_3 s_2 s_3 \cos \delta + c_2^2 c_3^2 \cos \delta + c_2^2 c_3^2 \sin \delta],
\]

(B18)

\[
= s_1^2 (c_1^2 s_2^2 + c_1^2 c_3^2 (c_3^2 - s_3^2) - 2 c_1 c_2 c_3 s_2 s_3 \cos \delta + c_2^2 c_3^2 \cos \delta + c_2^2 c_3^2 \sin \delta),
\]

(B19)

\[
= s_1^2 (c_1^2 s_2^2 + c_1^2 c_3^2 (c_3^2 - s_3^2) + 2 c_1 c_2 c_3 s_2 s_3 \cos \delta (c_3^2)
- 2 c_1 c_3 c_3 s_3 s_3 \cos \delta s_3^2 + 2 c_3^2 s_2^2 s_3^2),
\]

(B20)

\[
= s_1^2 (c_1^2 s_2^2 + c_1^2 c_3^2 (1 - 2 c_3^2 s_2^2) + 2 c_3^2 s_2^2 s_3 \cos \delta (s_3^2 - c_3^2)),
\]

(B21)

\[
= s_1^2 (2 c_1^2 s_2^2 + 2 c_3^2 s_2^2 s_3 - 2 c_1^2 c_3^2 s_3^2 - 2 c_1 c_2 c_3 s_2 s_3 \cos \delta (s_3^2 - c_3^2)),
\]

(B22)

\[
= s_1^2 (2 c_1^2 s_2^2 + 2 c_3^2 s_3^2 (s_2^2 - c_2^2 c_3^2) - 2 c_1 c_2 c_3 s_2 s_3 \cos \delta (s_3^2 - c_3^2)),
\]

(B23)

\[
\langle P_{\nu_e \to \nu_\tau} \rangle = 2 c_1^2 s_1^2 s_2^2 + 2 s_1^2 s_3^2 s_3^2 (s_2^2 - c_2^2 c_3^2) - 2 s_2^2 s_3^2 s_3 c_1 c_2 c_3 \cos \delta (s_3^2 - c_3^2).
\]

(B24)

Now, according to the proposed topological constraints of Section 3.0 in the main text, (B15) and (B24) must be term-wise equal and nonzero. These conditions yield three constraints.
on the mixing parameters, namely,

\[ s_1^2 > 0, \quad \text{(B25)} \]
\[ s_2^2 = c_2^2 = \frac{1}{2}, \quad \text{(B26)} \]

and

\[ s_1^2 s_2 s_3 c_1 c_2 c_3 \cos \delta (s_3^2 - c_3^2) = 0. \quad \text{(B27)} \]

Next, according to the proposed topological constraints of Section 3.0 in the main text, the matrix elements \( \langle P_{\nu_\mu \rightarrow \nu_\mu} \rangle \) and \( \langle P_{\nu_e \rightarrow \nu_e} \rangle \) must be equal. As we will now demonstrate, this condition yields another constraint on the mixing parameters, namely,

\[ s_3^2 = c_3^2 = \frac{1}{2}, \quad \text{(B28)} \]

which also happens to satisfy \( \text{(B27)} \).

\[
\langle P_{\nu_\mu \rightarrow \nu_\mu} \rangle = \sum_{i=1}^{3} |U_{2i}|^4 = |U_{21}|^4 + |U_{22}|^4 + |U_{23}|^4,
\]

\[
= s_1^4 c_2^4 + [(c_1 c_2 c_3 - s_2 s_3 \cos \delta)^2 + s_2^2 s_3^2 \sin^2 \delta]^2
+ [(c_1 c_2 s_3 + s_2 c_3 \cos \delta)^2 + s_2^2 c_3^2 \sin^2 \delta]^2,
\]

\[
= s_1^4 c_2^4 + [c_1^2 c_2^2 c_3^2 - 2 c_1 c_2 c_3 s_2 s_3 \cos \delta + s_2^2 s_3^2 \cos^2 \delta + s_2^2 s_3^2 \sin^2 \delta]^2
+ [c_1^2 c_2^2 s_3^2 + 2 c_1 c_2 s_3 s_2 c_3 \cos \delta + s_2^2 c_3^2 \cos^2 \delta + s_2^2 c_3^2 \sin^2 \delta]^2.
\]

Using \( s_2^2 = c_2^2 \) and \( c_3^2 = s_3^2 \)

\[
\langle P_{\nu_\mu \rightarrow \nu_\mu} \rangle = s_1^4 c_2^4 + [c_1^2 c_2^2 c_3^2 - 2 c_1 c_2 c_3 s_2 s_3 \cos \delta + c_2^2 c_3^2 \cos^2 \delta + c_2^2 c_3^2 \sin^2 \delta]^2
+ [c_1^2 c_2^2 s_3^2 + 2 c_1 c_2 c_3 s_2 s_3 \cos \delta + c_2^2 c_3^2 \cos^2 \delta + c_2^2 c_3^2 \sin^2 \delta]^2,
\]

\[
= s_1^4 c_2^4 + [(c_1 c_2 c_3 + s_2 c_3 \cos \delta)^2 + s_2^2 c_3^2 \sin^2 \delta]^2
+ [(c_1 c_2 s_3 + c_2 c_3 \cos \delta)^2 + c_1^2 c_2 c_3 c_2 s_3 \cos \delta]^2,
\]

\[
= s_1^4 c_2^4 + 2(c_1^2 c_2^2 c_3^2 + c_1 c_2 c_3^2)^2 + 2(2 c_1 c_2 c_3 s_2 s_3 \cos \delta)^2,
\]

\[
= s_1^4 c_2^4 + 2 c_1^2 c_2^4 (c_1^2 + 1)^2 + 8 c_1 c_2 c_3^2 s_2 s_3 \cos^2 \delta,
\]

\[
\langle P_{\nu_\mu \rightarrow \nu_\mu} \rangle = s_1^4 c_2^4 + 2 c_1^2 c_2^4 (c_1^2 + 1)^2 + 8 c_1 c_2 c_3^4 \cos^2 \delta.
\]
Similarly, with a few steps removed, and once again using \( s_2^2 = c_2^2, \ s_3^2 = c_3^2 \)

\[
\langle P_{\nu \rightarrow \nu_r} \rangle = \sum_{i=1}^{3} |U_{3i}|^4 = |U_{31}|^4 + |U_{32}|^4 + |U_{33}|^4, \tag{B37}
\]

\[
= s_1^4 s_2^2 + [(c_1 s_2 c_3 + c_2 s_3 \cos \delta)^2 + (c_2 s_3 \sin \delta)^2]^2 \\
+ [(c_1 s_2 s_3 - c_2 c_3 \cos \delta)^2 + (c_2 c_3 \sin \delta)^2]^2, \tag{B38}
\]

\[
= s_1^4 c_2^4 + [c_1^2 c_2^2 c_3^2 + c_2 c_3 \cos \delta + c_2^2 c_3^2 \cos^2 \delta + c_2^2 c_3^2 \sin^2 \delta]^2 \\
+ [c_1^2 c_2^2 c_3^2 - 2 c_1 c_2 c_3 \cos \delta + c_2^2 c_3^2 \cos^2 \delta + c_2^2 c_3^2 \sin^2 \delta]^2, \tag{B39}
\]

\[
= s_1^4 c_2^4 + [(c_2^2 c_3^2 + c_2^2 c_3^2 + (2 c_1 c_2 c_3^2 \cos \delta))^2 \\
+ [c_1^2 c_2^2 c_3^2 + c_2^2 c_3^2 - 2 c_1 c_2 c_3 \cos \delta]^2], \tag{B40}
\]

\[
= s_1^4 c_2^2 + [2(c_1^2 c_2^2 c_3^2 + c_2^2 c_3^2) + 2(2 c_1 c_2 c_3^2 \cos \delta)^2], \tag{B41}
\]

\[
\langle P_{\nu \rightarrow \nu_r} \rangle = s_1^4 c_2^4 + 2 c_2^4 c_3^4(c_1^2 + 1)^2 + 8 c_2^4 c_3^4 \cos^2 \delta. \tag{B42}
\]

Clearly, (B42) and (B36) are properly term-wise equal, as required by the topological constraints of Section 3.0.

Thus far, we have established the following constraints on the mixing parameters \( s_2^2 = c_2^2, \ s_3^2 = c_3^2 \) and \( s_1^2 > 0 \). Using these constraints we will now calculate \( \langle P_{\nu \rightarrow \nu_r} \rangle \).

\[
\langle P_{\nu \rightarrow \nu_r} \rangle = \sum_{i=1}^{3} |U_{2i}|^2 |U_{3i}|^2 = |U_{21}|^2 |U_{31}|^2 + |U_{22}|^2 |U_{32}|^2 + |U_{23}|^2 |U_{33}|^2, \tag{B43}
\]

\[
= s_1^4 c_2^4 + |c_1 c_2 c_3 - s_2 s_3 e^{i\delta}|^2|c_1 s_2 c_3 + c_2 s_3 e^{i\delta}|^2 \\
+ |c_1 s_2 c_3 + s_2 s_3 e^{i\delta}|^2|c_1 s_2 c_3 - c_2 s_3 e^{i\delta}|^2, \tag{B44}
\]

\[
= s_1^4 c_2^2 + [c_1 c_2 c_3 - s_2 s_3 \cos \delta - i s_2 s_3 \sin \delta]^2|c_1 s_2 c_3 + c_2 s_3 \cos \delta + i c_2 s_3 \sin \delta|^2 \\
+ |c_1 s_2 c_3 + s_2 s_3 \cos \delta + i s_2 s_3 \sin \delta|^2|c_1 s_2 c_3 - c_2 s_3 \cos \delta - i c_2 s_3 \sin \delta|^2, \tag{B45}
\]

\[
= s_1^4 c_2^4 + [c_1^2 c_2^2 c_3^2 - 2 c_1 c_2 c_3^2 \cos \delta + c_2^2 c_3^2 \cos^2 \delta + c_2^2 c_3^2 \sin^2 \delta] \\
\times [c_1^2 c_2^2 c_3^2 + 2 c_1 c_2 c_3^2 \cos \delta + c_2^2 c_3^2 \cos^2 \delta + c_2^2 c_3^2 \sin^2 \delta] \\
+ [c_1^2 c_2^2 c_3^2 + 2 c_1 c_2 c_3^2 \cos \delta + c_2^2 c_3^2 \cos^2 \delta + c_2^2 c_3^2 \sin^2 \delta] \\
\times [c_1^2 c_2^2 c_3^2 - 2 c_1 c_2 c_3^2 \cos \delta + c_2^2 c_3^2 \cos^2 \delta + c_2^2 c_3^2 \sin^2 \delta], \tag{B46}
\]

\[
= s_1^4 c_2^4 + [c_1^2 c_2^2 c_3^2 + c_2^2 c_3^2 - 2 c_1 c_2 c_3^2 \cos \delta][c_1^2 c_2^2 c_3^2 + c_2^2 c_3^2 + 2 c_1 c_2 c_3^2 \cos \delta] \\
+ [c_1^2 c_2^2 c_3^2 + c_2^2 c_3^2 + 2 c_1 c_2 c_3^2 \cos \delta][c_1^2 c_2^2 c_3^2 + c_2^2 c_3^2 - 2 c_1 c_2 c_3^2 \cos \delta], \tag{B47}
\]

\[
= s_1^4 c_2^4 + 2 c_2^2 c_3^2(c_1^2 + 1) - 2 c_1 c_2 c_3^2 \cos \delta][c_2^2 c_3^2(c_1^2 + 1) + 2 c_1 c_2 c_3^2 \cos \delta], \tag{B48}
\]

\[
= s_1^4 c_2^4 + 2 c_2^4 c_3^4[(c_1^2 + 1)^2 - 2 c_1 c_2 \cos \delta][c_1^2 + 1] + 2 c_1 c_2 \cos \delta], \tag{B49}
\]

\[
= s_1^4 c_2^4 + 2 c_2^4 c_3^4[(c_1^2 + 1)^2 - 4 c_1^2 \cos^2 \delta], \tag{B50}
\]

\[
\langle P_{\nu \rightarrow \nu_r} \rangle = s_1^4 c_2^4 + 2 c_2^4 c_3^4(c_1^2 + 1)^2 - 8 c_1^2 c_2^4 c_3^4 \cos^2 \delta. \tag{B51}
\]

Now according to the topological constraints of Section 3.0 in the main text, (B51), (B42) and (B36) must be equal. These conditions yield the last constraint on the mixing parameters,
namely,
\[ c_2^2 c_3^4 \cos^2 \delta = 0. \]  
(B52)

Since \( c_2^2 = s_2^2 = \frac{1}{2} \) and \( c_3^2 = s_3^2 = \frac{1}{2} \) are nonzero, and since it is reasonable to assume \([15, 16]\) that the Dirac-type CP-noninvariant phase factor \( e^{i\delta} \) is consistent with the requirement that \( \sin^2 \delta \neq 1 \) or \( \cos^2 \delta \neq 0 \) (i.e., CP violation is not maximal), we find that \( c_2^2 = 0 \) or \( s_2^2 = 1 \). From this point on, the determination of \( M \) proceeds exactly as described in the main text (See Eqs. 21–28).

References and Footnotes

[1] A. P. Balachandran, “Bringing Up a Quantum Baby,” [arXiv:quant-ph/9702055].

[2] G. Holzwarth, “Formation of Extended Topological-Defects During Symmetry-Breaking Phase Transitions in \( O(2) \) and \( O(3) \) Models”, [arXiv:hep-ph/9901296]. Analogous examples of fluctuation-induced topology-change in macroscopic objects (e.g., destruction of topological objects due to thermal fluctuations at phase transitions) abounds. For example, otherwise persistent (“conserved”) topological defects in crystals can be destroyed by raising the temperature sufficiently (i.e., by melting the crystal). Similarly, otherwise persistent (“conserved”) magnetic flux-tubes in Type II superconductors and/or vortices in a superfluid, can both be destroyed by raising the temperature above the critical temperature \( T_c \).

And, conversely, topological defects are always created when such macroscopic systems first condense (or crystalize) as the temperature is lowered. We imagine that something roughly similar can happen when quantum-fluctuations (vacuum fluctuations) act on otherwise very similar quantum objects (i.e., same electric charge, spin, lepton number, and nearly identical mass) that also happen to start “life” as topologically-distinct quantum objects. That is, we are assuming that, if not prevented by some absolute conservation law, transitions between such states (e.g., \( \nu_e \) and \( \nu_\tau \) neutrinos) will be catalyzed by quantum fluctuations.

[3] Gerald L. Fitzpatrick, The Family Problem-New Internal Algebraic and Geometric Regularities, Nova Scientific Press, Issaquah, Washington, 1997. Additional information: [http://physicsweb.org/TIPTOP/](http://physicsweb.org/TIPTOP/) or [http://www.amazon.com/exec/obidos/ISBN=0965569500](http://www.amazon.com/exec/obidos/ISBN=0965569500).

[4] C. Nash and S. Sen, Topology and Geometry for Physicists, Academic Press, New York, 1983.

[5] D. J. Thouless, Topological Quantum Numbers in Nonrelativistic Physics, World Scientific, Singapore (1998). Numerous examples from physics and mathematics could be cited to establish the following very general principle:

In physical systems characterized by a well defined topology, topology-change tends to be suppressed, relative to the condition of topology-maintenance.

The basis for this principle is that topology change involves either an energy “expense” (an energy “barrier” must be overcome) or a violation of some “topological charge” conservation
law, or both. In general, we expect this principle to apply to any discontinuous operation such as the abstract equivalent of “tearing,” and subsequently “glueing” surfaces back together to form states with new topologies. For example, one cannot continuously deform a doughnut into a sphere. It must first be “cut” and “glued” back together in a new way to achieve such a transformation.

Now, given the “facts” regarding the topology of neutrinos with respect to the (abstract) internal transformation $F$, and assuming that there are no “energy barriers” associated with transitions between neutrinos having the same topology, the foregoing principle suggests that these facts could have the following dynamical significance:

Topology change in neutrino-neutrino transitions ($\nu_\alpha \rightarrow \nu_\beta$) is suppressed by topological energy and/or topological charge “barriers,” while topology maintenance in neutrino-neutrino transitions is relatively enhanced.

It should be emphasized that even though the $\nu_\mu$ and $\nu_\tau$ neutrinos have the same topology with respect to $F$, they are quite distinct in other respects. For example, in conventional weak decays or weak capture reactions involving these particles, mu- and tau-numbers are separately conserved.

[6] The Super-Kamiokande, Kamiokande Collaboration, \texttt{arXiv:hep-ex/9810001}, and the SNO Collaboration, \texttt{arXiv:nucl-ex/0106015}. While the matrix proposed to describe long-distance neutrino mixtures (See Eq. 28 in the present paper), is not in conflict with current experimental observations (e.g., $M_{11} = \frac{1}{2}$ is in good agreement with observations of the solar neutrino deficit), until long-distance neutrino mixtures (e.g., those from a supernova source) are fully characterized, the status of Eq. (28), as a prediction, will not be known.

[7] G. L. Fitzpatrick, “Topological Constraints on Long-Distance Neutrino Mixtures,” \texttt{arXiv:physics/0007039}, 13 July 2000. The proposal for such an equilibrium condition was first put forward in this paper, which is a “precursor” to the present, more comprehensive, paper.

[8] M. Gronau, “Patterns of Fermion Masses, Mixing Angles and CP Violation,” in \textit{The Fourth Family of Quarks and Leptons, First International Symposium}, edited by: D. B. Cline and Amarjit Soni, Annals of The New York Academy of Sciences, New York, New York, Volume 518, 1987, p. 190. According to the scheme described in Appendix A of the present paper, the fundamental fermions $d, \nu_e(s, b, \nu_\mu, \nu_\tau)$ have the topology of a cylinder (Möbius strip) with respect to the internal transformation $F$. Hence, we imagine that if, in some imaginary world, quark couplings to the Higgs mass-producing fields were nearly “switched off,” the resulting “low” mass $d, s$ and $b$ quark mixtures (located inside quark composites) could conceivably look very much like the $\nu_e, \nu_\mu$ and $\nu_\tau$ low mass neutrino mixtures—owing to topological influences of exactly the kind proposed in the present paper. In particular, in this imaginary world, we would expect the $s$ and $b$ quarks (like the $\nu_\mu$ and $\nu_\tau$ neutrinos, respectively) to exhibit bimaximal mixing. However, because of uncertainty-principle considerations, and because the $b$ quark mass is, in the “real” world, very much larger than the $d$ and $s$ quark masses, very little mixing with the $b$ quark occurs in $d, s, b$ quark mixtures in spite of topological influences that would tend to encourage this. Thus, in
the case of strongly-interacting quarks, it seems very likely that topological constraints of the kind considered in the present paper can, at most, play a minor role in determining such things as CKM-type matrix elements. For example, in the case of d and s quarks, Gronau shows that the mixing angle $\theta_c$ depends on quark masses since the matrix element $V_{12}$, and the d, s quark masses $m_d$ and $m_s$, respectively, are known to be empirically related via $V_{12} = \sin \theta_c = \sqrt{\frac{m_d}{m_s}}$, where $m_s > m_d \neq 0$. This is nothing like the corresponding matrix element for $\nu_e$ and $\nu_\mu$ neutrinos found in the present paper. Thus, although a detailed understanding of these matters is far from being achieved, it appears that topological influences are allowed to play a major role in neutrino, but not quark mixing, because of the small value, and near degeneracy of neutrino masses in comparison to the very large differences between d, s and b quark masses.

[9] Although conventional treatments of neutrino mixing do not involve topological considerations, they do describe mixing among neutrinos that initially have a definite flavor. And, because we argue that a neutrino with a definite flavor also has a definite topology, hereinafter we will often use these terms together as in “... neutrinos have a definite flavor and topology.”

[10] O. Nachtmann, *Elementary Particle Physics — Concepts and Phenomena*, Springer-Verlag, Berlin, 1990, p. 365.

[11] Ta-Pei Cheng and Ling-Fong Li, *Gauge Theory of Elementary Particle Physics*, Clarendon Press, Oxford, 1984, pp. 409–414. While this reference provides a good summary of the formalism involved in neutrino mixing, the reader should be cautioned that there are several “typographical” errors that could cause confusion. For example, in Eq. (13.22) the first term in the expression for the matrix element $U_{32}$ should read $c_1 s_2 c_3$ not $c_1 s_2 s_3$. In Eq. (13.31) the last term in the expression for the time-average probability $\langle \nu_\alpha \to \nu_\beta \rangle$ should be preceded by a minus sign, not a plus sign. Finally, the left hand side of Eq. (13.33) should read $P_{\nu_\alpha \to \nu_\beta}$ not $\langle P_{\nu_\alpha \to \nu_\beta} \rangle$.

Note that the time-dependent probability of detecting a neutrino of definite flavor $\beta$ at time $t > 0$ after emission at time $t = 0$ from a source of neutrinos of definite flavor $\alpha$, is

$$P_{\nu_\alpha \to \nu_\beta}(t) = |\langle \nu_\beta(t) | \nu_\alpha \rangle|^2,$$

where the “scalar product” $\langle \nu_\beta(t) | \nu_\alpha \rangle$ is the “probability amplitude” for the process described. For example, if $\alpha = \beta = e$, one has ($\nu_1, \nu_2$ and $\nu_3$ are neutrino mass eigenstates) the “column” or “ket” vectors

$$|\nu_e(t)\rangle = c_1 e^{-iE_1 t} |\nu_1\rangle + c_1 c_3 e^{-iE_2 t} |\nu_2\rangle + s_1 s_3 e^{-iE_3 t} |\nu_3\rangle,$$

and

$$|\nu_e(0)\rangle = |\nu_e\rangle = c_1 |\nu_1\rangle + s_1 c_3 |\nu_2\rangle + s_1 s_3 |\nu_3\rangle.$$
Then, given \( \langle \nu_i | \nu_j \rangle = \delta_{ij} \), and the “row” or “bra” vector

\[
\langle \nu_e(t) | = c_1 e^{+E_1 t} \langle \nu_1 | + s_1 c_3 e^{+E_2 t} \langle \nu_2 | + s_1 s_3 e^{+E_3 t} \langle \nu_3 |
\]

\[
P_{\nu_e \rightarrow \nu_e(t)} = |\langle \nu_e(t) | \nu_e \rangle|^2
\]

\[
= |(c_1^2 \cos E_1 t + s_1^2 c_3^2 \cos E_2 t + s_1^2 s_3^2 \cos E_3 t) + i(c_1^2 \sin E_1 t + s_1^2 c_3^2 \sin E_2 t + s_1^2 s_3^2 \sin E_3 t)|^2
\]

\[
= (c_1^2 \cos E_1 t + s_1^2 c_3^2 \cos E_2 t + s_1^2 s_3^2 \cos E_3 t)^2
\]

\[
+ (c_1^2 \sin E_1 t + s_1^2 c_3^2 \sin E_2 t + s_1^2 s_3^2 \sin E_3 t)^2
\]

After expansion, and further simplification, the previous expression becomes [use \( \cos(A - B) = \cos A \cos B + \sin A \sin B \)]

\[
P_{\nu_e \rightarrow \nu_e(t)} = (1 - 2c_1^2 s_1^2 - 2s_1^4 c_3^2 s_3^2) + 2c_1^2 s_1^2 c_3^2 \cos(E_1 - E_2) t
\]

\[
+ 2c_1^2 s_1^2 \cos(E_1 - E_3) t
\]

\[
+ 2s_1^4 c_3^2 s_3^2 \cos(E_2 - E_3) t.
\]

In general, at time \( t \) or distance \( X \) from the neutrino source, one has the expression

\[
P_{\nu_\alpha \rightarrow \nu_\beta(t)} = \sum_{i=1}^{3} |U_{\alpha i}|^2 |U_{\beta i}|^2 + \sum_{i \neq j}^{3} U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j} \cos \left( \frac{2\pi X}{l_{ij}} \right).
\]

where \( l_{ij} = 2\pi/(E_i - E_j) \), and \( E_i - E_j = (m_i^2 - m_j^2)/2p \).

Clearly, the general time-average probability, when \( X \gg l_{ij} \) is given by (note that time-averages of each of the three cosine terms above produce “sinc” functions varying as \( \frac{\sin kX}{kX} \), each of which approaches zero as the magnitude of \( X \) increases)

\[
\langle P_{\nu_\alpha \rightarrow \nu_\beta(t)} \rangle = \sum_{i=1}^{3} |U_{\alpha i}|^2 |U_{\beta i}|^2.
\]

For example, referring to the expression for \( P_{\nu_e \rightarrow \nu_e(t)} \) worked out above, one sees that

\[
\langle P_{\nu_e \rightarrow \nu_e(t)} \rangle = \sum_{i=1}^{3} |U_{\epsilon i}|^2 |U_{\epsilon i}|^2,
\]

or

\[
\langle P_{\nu_e \rightarrow \nu_e(t)} \rangle = (1 - 2c_1^2 s_1^2 - 2s_1^4 c_3^2 s_3^2).
\]

Note that \( P_{\nu_\alpha \rightarrow \nu_\beta(t)} \) and \( \langle P_{\nu_\alpha \rightarrow \nu_\beta(t)} \rangle \) are also commonly written as \( P_{\nu_\alpha \rightarrow \nu_\beta} \) and \( \langle P_{\nu_\alpha \rightarrow \nu_\beta} \rangle \), respectively.

[12] H. Fritzsch and Z. Xing, “How to Describe Neutrino Mixing and CP Violation,” arXiv:hep-ph/0103242. CP- and T-violating asymmetries in “normal” (lepton-number conserving) neutrino oscillations depend only on the Dirac-type phase \( \delta \). In particular, they have nothing to
do with the Majorana-type CP-violating phases $\rho$ and $\sigma$. Moreover, because lepton-number violating processes (e.g., neutrinoless double beta decay) are known from experiment to be strongly suppressed, the issue of whether neutrinos are Dirac particles or Majorana particles need not be addressed in the present paper. However, it should be understood that if neutrinos are Majorana particles, lepton-number violating processes would be possible (if not probable!), and the matrix $U_{\alpha i}$ in Eqs. (1) and (2) in the main text would have to be multiplied by a diagonal matrix such as

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\rho} & 0 \\
0 & 0 & e^{i\sigma}
\end{pmatrix},
$$

where $\rho$ and $\sigma$ are phases associated with the Majorana-type CP-violating phase factors $e^{i\rho}$ and $e^{i\sigma}$, respectively.

[13] By definition, mixing does not occur at the source of neutrinos. Hence, Eq. (4) in the main text is obviously true. For example, at the source, neutrinos are pure flavor eigenstates $\nu_\alpha$, hence

$$
P_{\nu_\alpha \to \nu_\beta} \bigg|_{X=0} = |\langle \nu_\beta | \nu_\alpha \rangle|^2 = |\delta_{\beta \alpha}|^2,
$$

where $\delta_{\beta \alpha} = 1$ when $\alpha = \beta$, and $\delta_{\beta \alpha} = 0$ when $\alpha \neq \beta$. However, it is instructive to demonstrate the validity of Eq. (4) for at least one matrix element, starting with Eq. (3) in the main text. In particular, let us demonstrate that when $X = 0$, $P_{\nu_e \to \nu_e} = 1$. The other matrix elements can be found in a similar way, which demonstrates the validity of Eq. (4).

When $X = 0$, Eq. (3) becomes (Use Eq. 5)

$$
P_{\nu_e \to \nu_e} = \langle P_{\nu_e \to \nu_e} \rangle + \sum_{i,j} U_{i1} U_{i1}^* U_{1j} U_{1j}
$$

$$
= \langle P_{\nu_e \to \nu_e} \rangle + U_{11} U_{11}^* U_{12} U_{12} + U_{11} U_{11}^* U_{13} U_{13} +
U_{12} U_{12}^* U_{11} U_{11} + U_{12} U_{12}^* U_{13} U_{13} +
U_{13} U_{13}^* U_{11} U_{11} + U_{13} U_{13}^* U_{12} U_{12}
$$

$$
= \langle P_{\nu_e \to \nu_e} \rangle + |U_{11}|^2 |U_{12}|^2 + |U_{11}|^2 |U_{13}|^2 + |U_{12}|^2 |U_{11}|^2
+ |U_{12}|^2 |U_{13}|^2 + |U_{13}|^2 |U_{11}|^2 + |U_{13}|^2 |U_{12}|^2
$$

$$
= \langle P_{\nu_e \to \nu_e} \rangle + |U_{11}|^2 \{ |U_{12}|^2 + |U_{13}|^2 \} + |U_{12}|^2 \{ |U_{11}|^2 + |U_{13}|^2 \}
+ |U_{12}|^2 \{ |U_{11}|^2 + |U_{13}|^2 \}
$$

$$
P_{\nu_e \to \nu_e} = \langle P_{\nu_e \to \nu_e} \rangle + 2 |U_{11}|^2 \{ |U_{12}|^2 + |U_{13}|^2 \} + 2 |U_{12}|^2 |U_{13}|^2.
$$

Given (B6) and Eq. (2) in the present paper, one has

$$
P_{\nu_e \to \nu_e} = 1 - 2 c_1^2 s_1^2 - 2 s_1^4 s_3^2 c_3^2 + 2 c_1^2 s_1^2 (c_3^2 + s_3^2) + 2 s_1^4 s_3^2 c_3^2
$$
or \( P_{\nu_e \to \nu_e} = 1 \). Proceeding in this way with the calculation of the other matrix elements, one establishes the intuitively obvious result

\[
P_{\nu_\alpha \to \nu_\beta} \bigg|_{X=0} = I_3.
\]

[14] The proposed topological constraints are of course assumed. We imagine that they are to be justified by physics at some deeper level than the conventional description of neutrino mixing. While we have no way to prove these qualitative assumptions regarding topological constraints, their adoption certainly leads to testable predictions regarding mixing parameters, and the matrix of time-average probabilities \( \langle P_{\nu_\alpha \to \nu_\beta} \rangle \).

[15] E. Rodriguez-Jauregui, “Implications of Maximal Jarlskog Invariant and Maximal CP-Violation,” arXiv: hep-ph/0104092. While the author of this reference argues that maximal CP-violation may occur in both the quark and lepton sectors, these arguments do not seem to be compelling. Instead, I find it necessary in the present paper to assume the contrary, namely, that CP-violation is not maximal, at least in the lepton sector.

According to Eqs. (20) and (B52) in the present paper, the topological constraints of Section 3.0 have forged a kind of “linkage” between the mixing angle \( \theta_1 \), and the angle \( \delta \) associated with the Dirac-type CP-noninvariant phase factor \( e^{i\delta} \), namely,

\[
c_1^2 \cos^2 \delta = 0.
\]

Even though we do not know the physical origin of CP-noninvariance, it is clear that if we make what seems to be a reasonable assumption, namely, that CP-violation is not maximal (e.g., \( \delta \) is not equal to \( \pm \frac{\pi}{2} \)), \( \delta \) will be subject to the constraint \( \sin^2 \delta \neq 1 \) or \( \cos^2 \delta \neq 0 \), in which case \( \theta_1 \) will be constrained by \( c_1^2 = 0 \) or \( c_1 = 0 \).

In this case, Eq. (2) yields, for example (Use \( s_1 = 1 \))

\[
\nu_e = s_1 c_3 \nu_2 + s_1 s_3 \nu_3.
\]

But, the topological constraints of Section 3.0 have also yielded the constraint \( c_3^2 = s_3^2 = \frac{1}{2} \), which means that (For the sake of argument assume \( \theta_3 \) is a first-quadrant angle)

\[
\nu_e = \frac{\sqrt{2}}{2} \nu_2 + \frac{\sqrt{2}}{2} \nu_3.
\]

Clearly, this last equation describes “bi-maximal” mixing of the mass eigenstates \( \nu_2 \) and \( \nu_3 \).

[16] H. Georgi and S. L. Glashow, “Neutrinos on Earth and in the Heavens,” arXiv: hep-ph/9808293, and hep-ph/9808293v2, page 5, Eq. (20). It is interesting, and probably significant that Georgi and Glashow independently arrived at Eq. (28) in the main text of the present paper starting from six experimental neutrino “facts,” some of which are completely different than the facts and assumptions employed in the present paper. In particular, these authors began by assuming
1. There are just three chiral neutrino states having Majorana masses.

2. Atmospheric neutrinos rarely oscillate into electron neutrinos.

3. Atmospheric muon neutrinos suffer maximal, or nearly maximal, two flavor oscillations into tau neutrinos.

4. & 5. Two experimentally (and theoretically) motivated assumptions regarding the order of magnitude of individual neutrino masses, and various mass squared differences.

and

6. Neutrinoless double beta decay probably does not occur.

Given these six “facts” these authors derived both the neutrino mass matrix, and partially determined the associated CKM-like unitary matrix $U$, which describes neutrino mixtures. From $U$ they then derived Eq. (28) in the present paper. In closing, these authors also noted that CP-violation in this situation could be “superweak.” Clearly, this would be consistent with the assumption in the present paper that CP-violation is not maximal, i.e., $\sin^2 \delta \neq 1$ or $\cos^2 \delta \neq 0$.

While the present approach, and that of Georgi and Glashow in deriving Eq. (28), are quite different they are, nevertheless, expected to be compatible. In particular, we anticipate that such things as neutrino masses and mixing parameters, ultimately owe their existence to physics at a deeper level, where topological considerations, of the kind proposed in the present paper, are expected to play an important role.

[17] Note that for all practical purposes, time-dependent quantum probabilities $P_{\nu_\alpha \rightarrow \nu_\beta}$ become time-average or classical probabilities $\langle P_{\nu_\alpha \rightarrow \nu_\beta} \rangle$ when $t$ is sufficiently large. For this reason, it makes sense to consider classical probability measures such as the “joint” probability $P = P_C P_{NC}$, where $P_C$ and $P_{NC}$ are time-averages of quantum probabilities.

[18] Even if the matrix $M$ in Eq. (28) in the main text is verified by experiment, we could not be certain that the constraints expressed by Eqs. (9), (12) and (13), together with items A), B) and C) in Section 3.0, are topological constraints related to the 2-space description of quarks and leptons. There are three basic reasons for this assertion.

First, we do not know why the topology of vector triads (with respect to the internal transformation $F$) should be relevant to neutrino mixing. Second, we do not have any detailed understanding of the hypothetical mechanism that supplies the “energy barriers” or “topological charge conservation laws” that serve to inhibit topology change in neutrino mixing. Third, as indicated in Appendix A (Sec. A.4), we cannot be absolutely certain that the $\nu_\mu$ neutrino has the requisite second-family Möbius topology with respect to $F$.

What we do know with certainty is that, regardless of their physical origins, the constraints of Section 3.0, together with the additional assumption $\sin^2 \delta \neq 1$ or $\cos^2 \delta \neq 0$, definitely determine $M$ uniquely.

[19] G. L. Fitzpatrick, “Continuation of the Fermion-Number Operator and the Puzzle of Families,” arXiv:physics/0007038, 13 July 2000.
[20] G. L. Fitzpatrick, “Electric Charge as a Vector Quantity,” arXiv:physics/0011073, 30 November 2000.

[21] S. Weinberg, The Quantum Theory of Fields, Vol. I, Foundations, Cambridge University Press, New York, NY (1995), pp. 529–531; The Quantum Theory of Fields, Vol. II, Modern Applications, Cambridge University Press, New York, NY (1996), p. 155.