UNIVERSAL PADÉ APPROXIMATION

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Abstract

In transferring some results from universal Taylor series to the case of Padé approximants we obtain stronger results, such as, universal approximation on compact sets of arbitrary connectivity and generic results on planar domains of any connectivity and not just on simply connected domains.

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1 Introduction

Professor C. Brezinski in a colloquium talk at the University of Athens held on September 22, 2010, presented an overview on Padé approximants. In his thesis, under Charles Hermite, Henri Padé arranged these approximants in a double array now known as the Padé table of the formal power series

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n = (a_0, a_1, \ldots). \]

In particular, for \( q = 0 \) the Padé approximant \( \left[ \frac{p}{q} \right]_f \) (\( p = 0, 1, 2, \ldots \)) is a polynomial and coincides with a partial sum of \( f \), while for \( q \geq 1 \) the Padé approximant \( \left[ \frac{p}{q} \right]_f \) is a rational function with some poles in general.

Padé approximants have been applied in proofs of irrationality and transcendence in number theory, in practical computation of special functions, and
in the analysis of different schemes for numerical solution of ordinary or partial
differential equations. For a short history on Padé approximation we refer to
the review-research paper [18].

In addition to their wide variety of applications, Padé approximants are
also connected with continued fraction expansions ([29], [30], [46], and [51]),
orthogonal polynomials ([9], [22], [60], and [53]), moment problems ([8] and
[44]), the theory of quadrature ([12] and [13]) and convergence acceleration
methods ([10], [11], and [53]). However, the application which brought them to
prominence in the 1960’s and 1970’s, was localizing the singularities of functions:
in various problems, for example in inverse scattering theory, one would have
a means for computing the coefficients of a power series \( f \). One could use
these coefficients to compute a Padé approximant to \( f \), and use the poles of the
approximant as predictors of the location of poles or other singularities of \( f \).
Under certain conditions on \( f \), which were often satisfied in physical examples,
this process could be theoretically justified.

Furthermore, since 1965 a growing interest for Padé approximants appeared
in theoretical physics, chemistry, electronics and numerical analysis; see, for
example, the books [2], [4], [13], and [26] and the international conferences
proceedings [20], [48], and [54].

One of the fascinating features of Padé approximants is the complexity of
their asymptotic behaviour ([1], [2], [3], [5], [7], [15], [19], [21], [23], [24], [30], [34],
[45], [47], [52], and [56]). The convergence problem for Padé approximants can
be stated as follows. Given a power series \( f = \sum_{\nu=0}^{\infty} a_{\nu} z^{\nu} = (a_{0}, a_{1}, \cdots) \) examine
the convergence of subsequences \( \left[ \frac{p_{n}/q_{n}}{f} \right]_{n=0,1,2,\ldots} \) extracted from the Padé
table as \( n \to +\infty \). If \( f \) is the Taylor development of an entire function, then it is
known that generically there exists a sequence \( (p_{n}, q_{n})_{n=0,1,2,\ldots} \) \( q_{n} > 0 \), so
that \( \lim_{n \to \infty} \left[ p_{n}/q_{n} \right]_{f} = \tilde{f} \) ([8]).

In the present paper we investigate all the possible limits of sequences
\( \left[ \frac{p_{n}/q_{n}}{f} \right]_{n=0,1,2,\ldots} \) on compact subsets \( K \) of \( \mathbb{C} \setminus \{0\} \) or compact sets dis-
joint from the domain of definition of the holomorphic function \( f \). We show that
generically all functions holomorphic in a neighborhood of \( K \) are such limits.
Thus, we have formal power series (or holomorphic functions on a domain \( \Omega \))
with universal Padé approximants. The particular case \( q = 0 \) is the well known
case of universal Taylor series ([6], [16], [32], [33], [35], [37], [41], [12], and [49])
where the approximation is realized by the partial sums. However, now we
impose several conditions on the approximating integers \( (p_{n}, q_{n}) \) and \( q_{n} \) may
be different from 0; in particular, we can have universal approximations with
\( p_{n} = q_{n} \), or \( \lim_{n \to \infty} p_{n} = \lim_{n \to \infty} q_{n} = +\infty \) or \( \lim_{n \to \infty} (p_{n} - q_{n}) = +\infty \) and
others.

It was during the inspiring talk of Professor Brezinski that we got the idea
that the results on universal Taylor series may be transferred to the case of
Padé approximants. Thus, we obtain the universal approximation by Padé
approximants on compact sets $K$ with connected complement $K^c$. However, the fact that Padé approximants may also have poles allows us to do approximation on compact sets $K$ of arbitrary connectivity and these results are generic on spaces of holomorphic functions defined on arbitrary planar domains and not just on simply connected domains. This is not possible in the case of universal Taylor series where the approximation is realized by polynomials (the partial sums). As methods of proofs we use Baire’s Category Theorem combined with Runge’s or Mergelyan’s Theorems. For the role of Baire’s Category Theorem in analysis we refer to [28] and [31].

2 PRELIMINARIES

If $f = \sum_{v=0}^{\infty} a_v z^v \in \mathbb{C}^{N_0}$ and $p, q$ are non negative integers, we denote by

$$\left[\frac{p}{q}\right]_f (z)$$

a rational function of the form

$$\frac{\sum_{v=0}^{p} n_v z^v}{\sum_{v=0}^{q} d_v z^v}, \quad d_0 = 1, \quad n_p d_q \neq 0$$

such that its Taylor development $\sum_{v=0}^{\infty} b_v z^v$ coincides with $\sum_{v=0}^{\infty} a_v z^v$ up to the first $p + q + 1$ terms:

$$b_v = a_v \text{ for all } v = 0, 1, \ldots, p + q.$$

Such a rational function does not always exist (see, for instance [2] and [14]). It may also happen that there exist several such rational functions ([22] and [26]). When such a rational fraction exists, it is called a Padé approximant of type $(p, q)$ to the series $f$. A necessary and sufficient condition for existence and uniqueness is that the determinant of the Hankel matrix $H_q^{(f)}(a_{p-q+1})$ of order $q$ at $a_{p-q+1}$ is different from zero:

$$\det \left( H^f_q (a_{p-q+1}) \right) := \det \begin{pmatrix} a_{p-q+1} & a_{p-q+2} & a_{p-q+3} & \cdots & a_p \\ a_{p-q+2} & a_{p-q+3} & a_{p-q+4} & \cdots & a_{p+1} \\ a_{p-q+3} & a_{p-q+4} & a_{p-q+5} & \cdots & a_{p+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_p & a_{p+1} & a_{p+2} & \cdots & a_{p+q-1} \end{pmatrix} \neq 0.$$

Then we write $f \in \mathcal{D}_{p,q}$. If $f \in \mathcal{D}_{p,q}$, then the Jacobi explicit formula for $\left[\frac{p}{q}\right]_f (z)$ involves polynomial expressions on a finite number of the coefficients.
\( a_v \) of \( f \) and its partial sums (14):

\[
[p/q]_f(z) = \frac{\det \begin{pmatrix}
  z^q S_{p-q}(z) & z^{q-1} S_{p-q+1}(z) & \cdots & S_p(z) \\
  a_{p-q+1} & a_{p-q+2} & \cdots & a_{p+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_p & a_{p+1} & \cdots & a_{p+q}
\end{pmatrix}}{\det \begin{pmatrix}
  z^q & z^{q-1} & \cdots & 1 \\
  a_{p-q+1} & a_{p-q+2} & \cdots & a_{p+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_p & a_{p+1} & \cdots & a_{p+q}
\end{pmatrix}}
\]

with

\[
S_k(z) = \begin{cases}
  \sum_{v=0}^{k} a_v z^v & \text{for } k \geq 0, \\
  0 & \text{for } k < 0.
\end{cases}
\]

Thus, when we restrict our attention to \( f \)'s in \( \mathcal{D}_{p,q} \), which is an open dense subset of the vector space \( C^{S_0} \) endowed with the Cartesian topology, then \([p/q]_f(z)\) varies continuously with \( f \).

We shall need the following lemmas.

**Lemma 2.1** Let \( F(z) \) and \( G(z) \) be two complex valued functions on a set \( E \). We assume that \( \sup_{z \in E} |F(z)| < +\infty, \inf_{z \in E} |G(z)| > 0 \) and \( \sup_{z \in E} |G(z)| < +\infty \). Let also \( \Phi(z,x) \) and \( W(z,x) \) be two complex valued functions defined on \( E \times (\mathbb{C}^n \setminus \{0\}) \) such that \( \Phi(z,x) \to F(z) \) and \( W(z,x) \to G(z) \) uniformly on \( E \) as \( x \to 0 \). Then, for every \( a > 0 \) there exists a \( \delta > 0 \) so that

\[
\sup_{z \in E} \frac{\Phi(z,x)}{W(z,x)} - \frac{F(z)}{G(z)} < a
\]

whenever \( x \in \mathbb{C}^n, \ 0 < ||x|| < \delta \).

The proof is elementary and is omitted.

**Lemma 2.2** Let \( M = (a_{i,j}(d))_{i,j=1}^q \) be a quadratic matrix with entries \( a_{i,j} = a_{i,j}(d) \) depending linearly on a parameter \( d \), in the sense that \( a_{i,j}(d) = c_{i,j} d + \tau_{i,j} \). Assume that

\[
\begin{cases}
  i + j < q & \Rightarrow c_{i,j} = 0 \\
  i + j = q & \Rightarrow c_{i,j} = c \neq 0 \text{ and } \tau_{i,j} = \tau.
\end{cases}
\]

Then the determinant \( \det(M) \) of \( M \) is a polynomial in \( d \) of degree \( q \) with leading coefficient \((-1)^q \cdot q! \neq 0 \).

The proof is elementary and is omitted.

**Lemma 2.3** Let \( P(z) = \sum_{v=0}^{N} \varepsilon_v z^v \) be a non-zero polynomial, \( K \subset \subset \mathbb{C} \) be a compact set and \( a > 0 \). Suppose \( p \) and \( q \) are positive integer numbers such that \( p > \deg P + q \). Then there exist \( c, d \in \mathbb{C} \setminus \{0\} \) such that the rational function

\[
R(z) = P(z) + \frac{dz^p}{1 - (c z)^q}
\]

is a real analytic function on \( \mathbb{C} \).
satisfies the following

i. \( 1 - (cz)^v \neq 0 \), for all \( z \in K \).

ii. \( \sup_{z \in K} |P(z) - R(z)| < a \).

iii. The Taylor expansion \( R(z) = \sum_{v=0}^{\infty} \beta_v z^v \) of \( R(z) \) around 0 satisfies

\[
\beta_v = \varepsilon_v \quad \text{for all } v \leq \deg P.
\]

iv. \( R(z) \in \mathcal{D}_{p,q} \) and the Padé approximant \( [p/q]_R(z) \) coincides with the rational function \( R(z) \).

**Proof** Let \( M < +\infty \) be such that \( |z| \leq M \) for all \( z \in K \). Then for \( z \in K \) we have \( |(cz)^v| \leq |c|^v M^v < 1 \), provided \( |c| \leq (1/M) \). This implies

i. Application of Lemma 2.1 for \( x = (c,d) \in \mathbb{C}^2 \), \( F(z) \equiv 0 \), \( G(z) \equiv 1 \), \( \Phi(z, x) \equiv dz^p \) and \( W(z, x) \equiv 1 - (cz)^q \) shows that there is \( \delta \) such that

\[
\sup_{z \in E} \left| \frac{\Phi(z, x)}{W(z, x)} - \frac{F(z)}{G(z)} \right| = \sup_{z \in K} \left| \frac{dz^p}{1 - (cz)^q} \right| = \sup_{z \in K} |P(z) - R(z)| < a
\]

whenever \( \|(c,d)\| < \delta \). This proves ii. To obtain iii, notice that the identity

\[
\frac{1}{1 - (cz)^q} = 1 + (cz)^q + (cz)^{2q} + \ldots
\]

implies that the Taylor development \( \sum_{v=0}^{\infty} \beta_v z^v \) of \( R(z) \) around 0 is \( R(z) = P(z) + dz^p \left( 1 + (cz)^q + (cz)^{2q} + \ldots \right) \). Since \( p > \deg P \), \( P(z) \) is a partial sum of \( R \)’s Taylor expansion, and \( \beta_v = \varepsilon_v \) for all \( v \leq \deg P \). Finally, to show iv, observe that the Taylor coefficient \( \beta_v \) does not depend on the parameter \( d \) whenever \( v < p \). We also have \( \beta_p = d \). Further, in the expression of \( \beta_v \), for \( v > p \), we have \( \beta_v = d \tau_v \) where \( \tau_v \) is independent of \( d \). By Lemma 2.2, the Hankel determinant \( H_q^{(R)}(\beta_{p-q+1}) = 0 \) is a polynomial in \( d \) of degree \( q \) with leading coefficient \((-1)^{q+1}\). Hence, equation \( H_q^{(R)}(\beta_{p-q+1}) = 0 \) with unknown \( d \) has a finite number of solutions. Choosing a \( d \in \mathbb{C} \) so that \( H_q^{(R)}(\beta_{p-q+1}) \neq 0 \), we infer \( R(z) \in \mathcal{D}_{p,q} \). Since the constant term in \( R(z)’s \) denominator equals 1, the uniqueness of the Padé approximant of type \((p,q)\) to the Taylor series

\[
P(z) + dz^p \left( 1 + (cz)^q + (cz)^{2q} + \ldots \right)
\]

guarantees that the Padé approximant \( [p/q]_R(z) \) coincides with the rational function \( R(z) \). The proof is complete. \( \blacksquare \)

**Lemma 2.4** Let \( \tilde{P}(z) = \sum_{v=0}^{N} \varepsilon_v z^v \) be a non-zero polynomial, \( K \subset \subset \mathbb{C} \) be a compact set and \( a > 0 \). Suppose \( p \) and \( q \) are positive integer numbers such that \( p, q > \deg P \). Then there exist \( c, d \in \mathbb{C} \setminus \{0\} \) such that the rational function

\[
R(z) = \frac{\tilde{P}(z) + dz^p}{1 - (cz)^q}
\]
satisfies the following:

i. $1 - (cz)^q \neq 0$ for all $z \in K$.

ii. $\sup_{z \in K} | P(z) - R(z) | < a$.

iii. The Taylor expansion $R(z) = \sum_{v=0}^{\infty} \beta_v z^v$ of $R(z)$ around 0 satisfies

$$\beta_v = \varepsilon_v \text{ for all } v \leq \deg \tilde{P}.$$

iv. $R(z) \in \mathcal{D}_{p,q}$ and the Padé approximant $[p/q]_R(z)$ coincides with the rational function $R(z)$.

Proof Let $M < +\infty$ be such that $|z| \leq M$ for all $z \in K$. Then for $z \in K$ we have $|(cz)^q| \leq c |z|^q < 1$, provided $|c| \leq (1/M)$. This implies i. To prove ii, it is enough to apply Lemma 2.1 for $z \in E = K$, $x = (c,d) \in \mathbb{C}^2, F(z) \equiv 0$, $G(z) \equiv 1$, $\Phi(z,x) \equiv c^q z^q \tilde{P}(z) + d z^p$ and $W(z,x) \equiv 1 - (cz)^q$. It follows that there exists $s \delta > 0$ such that

$$\sup_{z \in K} \left| \frac{\Phi(z,x)}{W(z,x)} - \frac{F(z)}{G(z)} \right| = \sup_{z \in K} \left| c^q z^q \tilde{P}(z) + d z^p \right| = \sup_{z \in K} \left| \tilde{P}(z) - R(z) \right| < a$$

whenever $||(c,d)|| < \delta$ and ii is proved. To obtain iii, notice that the Taylor development $\sum_{v=0}^{\infty} \beta_v z^v$ of $R(z)$ around 0 is

$$R(z) = \tilde{P}(z) + d z^p + (cz)^q \tilde{P}(z) + (cz)^q d z^p + (cz)^{2q} d z^p + \ldots.$$

Since $p, q > \deg \tilde{P}$, $\tilde{P}(z)$ is a partial sum of $R$'s Taylor expansion. Thus, $\beta_v = \varepsilon_v$ for all $v \leq \deg \tilde{P}$. Finally, to show iv, observe that

$$\beta_v = \begin{cases} \lambda_v d + \tau_v, & \text{with } \lambda_v \text{ and } \tau_v \text{ independent of } d \\ \tau_v, & \text{with } \tau_v \text{ independent of } d . \end{cases}$$

For $v < p$, it holds $\beta_v = \tau_v$ and therefore, the coefficient $\beta_v$ is independent of $d$. Further, if $q > 0$ and $v = p$, then $\beta_v = d + \tau_v$; and if $v > p$ and $v < p + q - 1$, we have $\beta_v = \tau_v + \lambda_v d$ where $\tau_v$ is independent of $d$. Thus, by Lemma 2.2, the Hankel determinant $H_q^{(R)} (\beta_{p-q+1})$ is a polynomial in $d$ of degree $q$ with leading coefficient $(-1)^{q+1}$. Hence, equation $H_q^{(R)} (\beta_{p-q+1}) = 0$ with unknown $d$ has a finite number of solutions and we can choose $d$ so that $H_q^{(R)} (\beta_{p-q+1}) \neq 0$. We infer $R(z) \in \mathcal{D}_{p,q}$. Since the constant term in $R(z)$'s denominator equals 1, the uniqueness of the Padé approximant of type $(p,q)$ to the Taylor series $P(z) + d z^p + (cz)^q \tilde{P}(z) + (cz)^q d z^p + (cz)^{2q} \tilde{P}(z) + (cz)^{3q} d z^p + \ldots$ guarantees that the Padé approximant $[p/q]_R(z)$ coincides with the rational function $R(z)$. The proof is complete. \[\]

Lemma 2.5 Let $A(z), B(z)$ and $P(z) = \sum_{\nu=0}^{N} \varepsilon_{\nu} z^\nu$ be nonzero polynomials. Let also $K \subseteq \mathbb{C}$ be a compact set, $\lambda$ be a positive integer such that $\lambda > \deg P$
and $a > 0$. Assume that $B(z) \neq 0$ for all $z \in K$ and $B(0) = 1$. Then, for any $q \geq \deg B$ and $p > \max \{\lambda + \deg A, q + \deg P\}$, there exist two non zero polynomials $A(z)$ and $B(z)$, with degrees $p = \deg A$ and $q = \deg B$ respectively, such that the rational function

$$R(z) = \frac{\hat{A}(z)}{B(z)}$$

satisfies the following.

i. $\hat{B}(z) \neq 0$, for all $z \in K \cup \{0\}$.

ii. $\sup_{z \in K} |R(z) - [P(z) + z^\lambda A(z)/B(z)]| < a$.

iii. The Taylor expansion $R(z) = \sum_{\nu=0}^{\infty} \beta_\nu z^\nu$ of $R(z)$ around $0$ satisfies

$$\beta_\nu = \varepsilon_\nu, \text{ for all } \nu \leq \deg P.$$

iv. $R(z) \in \mathcal{D}_{p,q}$, and the Padé approximant $[p/q]_R(z)$ coincides with the rational function $\hat{R}(z)$.

Proof Let $M < +\infty$ be such that $|z| \leq M$ for all $z \in K$. Put

$$\hat{A}(z) \equiv [B(z) + cz^q] P(z) + z^\lambda A(z) + dz^p, \hat{B}(z) \equiv B(z) + cz^q$$

and

$$R(z) = \frac{\hat{A}(z)}{B(z)}.$$

Then for $z \in K$ we have $|\hat{B}(z)| \geq \inf_{z \in K \cup \{0\}} |B(z)| - |c| M^q > 0$, provided $|c| < (\inf_{z \in K \cup \{0\}} |B(z)|)/M^q).$ This implies i. Further, application of Lemma 2.1 for $z \in E = K$, $x = (c, d) \in \mathbb{C}^2$, $F(z) \equiv 0$, $G(z) \equiv B^2(z)$, $\Phi(z, x) \equiv -c^q z^\lambda A(z) + dz^p B(z)$ and $W(z, x) \equiv [B(z)] [B(z) + cz^q]$ shows that there is a $\delta$ such that

$$\sup_{z \in E} \left| \frac{\Phi(z, x)}{W(z, x)} - \frac{F(z)}{G(z)} \right| = \sup_{z \in K} \left| \frac{z^\lambda A(z)}{B(z) + cz^q} - \frac{z^\lambda A(z)}{B(z)} + \frac{dz^p}{B(z) + cz^q} \right| = \sup_{z \in K} \left| R(z) - \left[ P(z) + z^\lambda A(z) B(z) \right] \right| < a,$$

whenever $\|(c, d)\| < \bar{\delta}$. This proves ii. To prove iii, we note that

$$[B(z) + cz^q]^{-1} = [B(0)]^{-1} + \tau_1 z + \tau_2 z^2 + \ldots$$

and therefore the Taylor development $\sum_{\nu=0}^{\infty} \beta_\nu z^\nu$ of $R(z)$ around $0$ is

$$R(z) = P(z) + \left( \frac{z^\lambda A(z)}{B(0)} + \tau_1 z^{\lambda+1} A(z) + \ldots \right) + \left( \frac{dz^p}{B(0)} + \tau_1 dz^{p+1} + \tau_2 dz^{p+2} + \ldots \right).$$
Since \( \lambda > \deg P \), the polynomial \( P(z) = \sum_{\nu=0}^{N} \varepsilon_\nu z^\nu \) is a partial sum of \( R \)'s Taylor expansion. Thus, \( \beta_\nu = \varepsilon_\nu \) whenever \( \nu \leq \deg P \). It remains to show iv. To do so, remind that

\[
R(z) = P(z) + \left( [B(0)]^{-1} z^\lambda A(z) + \tau_1 z^{\lambda+1} A(z) + \tau_2 z^{\lambda+2} A(z) + \ldots \right) + \left( [B(0)]^{-1} dz^p + \tau_1 dz^{p+1} + \tau_2 dz^{p+2} + \ldots \right).
\]

So, for any \( \nu < p \), the parameter \( d \) does not appear in \( R \)'s Taylor coefficient \( \beta_\nu \). For \( \nu = p \), it holds, \( \beta_p = ([d/B(0) + w_p]) \), with \( w_p \) independent of \( d \). If \( \nu > p \), the Taylor coefficient \( \beta_\nu \) depends at most linearly on \( d \). From Lemma 2.2, it follows that the Hankel determinant \( H_q^R(\beta_{p-q+1}) \) is a polynomial in \( d \) of degree \( q \) with leading coefficient \((-1)^{q+1}\). This determinant vanishes on a finite set of values of \( d \). We can avoid these values and choose \( d \) so that \( H_q^R(\beta_{p-q+1}) \neq 0 \). We infer \( R(z) \in D_{p,q} \). Since the constant term in \( R(z) \)'s denominator equals 1, the uniqueness of the Padé approximant of type \( (p,q) \) to the above Taylor series guarantees that the Padé approximant \( [p/q]_R(z) \) coincides with the rational function \( R(z) \). The proof is complete.\( \blacksquare \)

**Remark 2.6** In the above lemmas the rational function \( R(z) \) is always a quotient of a polynomial of degree \( p \) as numerator and of a polynomial of degree \( q \) as denominator. With a little more effort, we can guarantee that these polynomials don’t have common zeros in \( \mathbb{C} \).

**Lemma 2.7** Let \( K \) be a compact subset of \( \mathbb{C} \) such that \( 0 \notin K \) and the complement \( K^c \) of \( K \) is connected. Let \( \bar{D} \) be a closed disk centered at 0 with radius \( r > 0 \) and such that \( \bar{D} \cap K = \emptyset \). Let also \( P(z) \) and \( Q(z) \) be two analytic polynomials. Then, for any \( \varepsilon > 0 \), there exist a polynomial \( \bar{P}(z) \) such that

i. \( \sup_{z \in \bar{D}} \left| \bar{P}(z) - P(z) \right| < \varepsilon \).

ii. \( \sup_{z \in K} \left| \bar{P}(z) - Q(z) \right| < \varepsilon \).

iii. A partial sum of \( \bar{P}(z) \) is exactly the polynomial \( P(z) \).

**Proof** Let \( \lambda > \deg P \). We are looking for \( \bar{P}(z) = P(z) + z^\lambda \Pi(z) \), where \( \Pi(z) \) is a polynomial suitably chosen. To do so, it is enough to have

\[
\sup_{z \in K} \left| \Pi(z) - \frac{Q(z) - P(z)}{z^\lambda} \right| < \frac{\varepsilon}{M^\lambda} \text{ and } \sup_{z \in \bar{D}} |\Pi(z)| < \frac{\varepsilon}{r^\lambda},
\]

where \( M = \sup_{z \in K} |z| \). Indeed, it suffices to approximate on \( K \cup \bar{D} \) the function \( f \) defined by

\[
f(z) = \begin{cases} 
0 & \text{on } \bar{D} \\
\frac{Q(z) - P(z)}{z^\lambda} & \text{on } K
\end{cases}
\]

This is possible by Runge’s theorem. The proof is complete.\( \blacksquare \)
3 SELEZNEV- PADÉ UNIVERSAL APPROXIMANTS

Definition 3.1 Let $\mathfrak{F} \subset \mathbb{N}_0^2 = \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\}$.

(i) We say that $\mathfrak{F}$ satisfies condition $(\mathcal{H})$, if it contains a sequence $(p_n, q_n)_{n=1,2,\ldots} \in \mathfrak{F}$ satisfying at least one of the following conditions:

$$(\mathcal{H}_1) \lim_{n \to \infty} p_n = \lim_{n \to +\infty} q_n = +\infty.$$  

$$(\mathcal{H}_2) \lim_{n \to \infty} (p_n - q_n) = +\infty.$$  

(ii) We denote by 

$$\mathcal{U}$$

the class of all formal power series $f = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ satisfying the following condition:

For every compact set $K \subset \mathbb{C} \setminus \{0\}$ with connected complement $K^c$ and every function $h : K \to \mathbb{C}$ which is continuous on $K$ and holomorphic in the interior $intK$ of $K$ ($h \in A(K)$), there exists a sequence $(\tilde{p}_n, \tilde{q}_n)_{n=1,2,\ldots} \in \mathfrak{F}$ such that

(a) $f \in \mathcal{D}_{\tilde{p}_n, \tilde{q}_n}$ for all $n = 1, 2, \ldots$ and $h$ every polynomial. This is due to the fact that $\mathbb{C} \setminus \{0\}$ is open; see [17].

Remark 3.2 In the above definition of $\mathcal{U}$, condition (b) is equivalent to require

$$\lim_{n \to \infty} \left( d^l \frac{\tilde{p}_n/\tilde{q}_n}{d z^l} \right)(z) = h^{(l)}(z)$$

uniformly on $K$, for any $l = 0, 1, 2, \ldots$ and $h$ every polynomial. This is due to the fact that $\mathbb{C} \setminus \{0\}$ is open; see [17].

Remark 3.3 If the sequence $(\tilde{p}_n, \tilde{q}_n)_{n=1,2,\ldots} \in \mathfrak{F}$ takes infinitely many values, then we can pass to a subsequence which takes every value at most once. If the sequence $(\tilde{p}_n, \tilde{q}_n)_{n=1,2,\ldots} \in \mathfrak{F}$ takes finitely many values, then

$$h(z) = [p/q]_f(z) \text{ for some } (p, q) \in \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\}. $$

The set of all these $h$'s is denumerable; so, there is a sequence $(\varepsilon_n)_{n=1,2,\ldots}$ such that $\lim_{n \to \infty} \varepsilon_n = 0$ and $h + \varepsilon_n \neq [p/q]_f$ for every $n = 1, 2, \ldots$ and $(p, q) \in \mathbb{N}_0^2$. Thus, if $(\tilde{p}_1, \tilde{q}_1), (\tilde{p}_2, \tilde{q}_2), \ldots, (\tilde{p}_{n-1}, \tilde{q}_{n-1})$ are already defined, we can choose $(\tilde{p}_n, \tilde{q}_n) \in \mathfrak{F}$ different from $(\tilde{p}_1, \tilde{q}_1), (\tilde{p}_2, \tilde{q}_2), \ldots, (\tilde{p}_{n-1}, \tilde{q}_{n-1})$ so that

$$\sup_{z \in K} \left| \frac{\tilde{z}}{\tilde{q}_n} f'(z) - (h(z) + \varepsilon_n) \right| < \frac{1}{n}.$$ 

It follows that in Definition 3.1.(ii) it is equivalent if we require in addition

$$(\tilde{p}_n, \tilde{q}_n) \neq (\tilde{p}_m, \tilde{q}_m) \text{ for all } n, m \text{ with } n \neq m.$$ 

We are now in position to formulate the first main result of the section.

Theorem 3.4 The class $\mathcal{U}$ is dense and $G_\delta$ in the space $\mathbb{C}^{\mathbb{N}_0}$ of all formal power series endowed with the Cartesian topology, provided $\mathfrak{F}$ satisfies condition $(\mathcal{H})$.

The proof of Theorem 3.4 requires the following well known result.
Lemma 3.5 ([35], [37] and [49]) There exists a sequence \( \{K_m \subset C \setminus \{0\}\}_{m=1,2,...} \) of compact sets with connected complement \( K_m^c \) such that the following holds:

For every compact subset \( K \) of \( C \setminus \{0\} \) with connected complement \( K^c \), there exists an \( m \in \{1,2,...\} \) such that \( K \subset K_m \). \( \blacksquare \)

Proof of Theorem 3.4 Regarding Definition 3.1 it is equivalent to consider approximations only on the compact sets \( K_m \). Because if \( h \in A(K) \) and \( \varepsilon > 0 \) are given, then by Mergelyan’s Theorem, we can find a polynomial \( Q(z) \) such that

\[
\sup_{z \in K} |Q(z) - h(z)| < \frac{\varepsilon}{2}.
\]

Let \( m \in \{1,2,...\} \) be such that \( K \subset K_m \) (see Lemma 3.5). Then, \( Q(z) \) being a polynomial belongs to \( A(K_m) \) and can be approximated on \( K_m \) by a Padé approximant \( \left[ \frac{p}{q} \right] f(z) \), with \( (p, q) \in \mathbb{F} \):

\[
\sup_{z \in K_m} \left| \frac{p}{q} f(z) - Q_j(z) \right| < \frac{\varepsilon}{2}.
\]

Thus, the triangle inequality implies:

\[
\sup_{z \in K} \left| \frac{p}{q} f(z) - h(z) \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Further, by Mergelyan’s Theorem, it suffices to approximate not all functions \( h \in A(K_m) \) but only the polynomials, with coefficients in \( \mathbb{Q} + i\mathbb{Q} \); the set of these polynomials is countable and can be represented as a sequence

\[
(Q_j)_{j=1,2,...}.
\]

Next, we define the sets

\[
\mathbb{E}(m, j, s, (p, q)) = \left\{ f \in \mathbb{C}^N_0 : f \in D_{p,q} \text{ and } \sup_{z \in K_m} \left| \frac{p}{q} f(z) - Q_j(z) \right| < \frac{1}{2} \right\}.
\]

It is easily seen that

\[
\mathcal{U} = \bigcap_{m,j,s=1,2,...} \bigcup_{(p,q)\in \mathbb{F}} \mathbb{E}(m, j, s, (p, q)).
\]

(See [37] and [41]).

To complete the proof of Theorem 3.4, we also need the following.

Lemma 3.6 For any \( m, j, s \in \{1,2,...\} \) and any \( (p, q) \in \{0,1,2,...\} \times \{0,1,2,...\} \), the set \( \mathbb{E}(m, j, s, (p, q)) \) is open in \( \mathbb{C}^N_0 \).

Proof This follows from the fact that \( D_{p,q} \) is open in \( \mathbb{C}^N_0 \) and that, by Jacobi explicit formula, the coefficients of the numerator and the denominator of \( \frac{p}{q} f(z) \) vary continuously with \( f \in D_{p,q} \). \( \blacksquare \)

Proposition 3.7 For every \( m, j, s \in \{1,2,...\} \), the set \( \bigcup_{(p,q)\in \mathbb{F}} \mathbb{E}(m, j, s, (p, q)) \) is open and dense in \( \mathbb{C}^N_0 \).
**Proof** Lemma 3.6 implies that the set is open as a union of open sets. In order to prove density, let $P$ be any non-zero polynomial. It suffices to find $(p, q) \in \mathfrak{F}$ and $f \in E(m, j, s, (p, q))$ such that $P(z)$ is a partial sum of $f(z)$. To do so, let us recall that $\mathfrak{F}$ contains condition $(\mathcal{H})$: that is $\mathfrak{F}$ contains a sequence $(p_n, q_n)_{n=1,2,...} \in \mathbb{N}^2$ satisfying condition $(\mathcal{H}_1)$ or condition $(\mathcal{H}_2)$. Assume that $(\mathcal{H}_1)$ holds. Since $\lim_{n \to +\infty} p_n = \lim_{n \to +\infty} q_n = +\infty$, there exists an $n_0$ so that $p_{n_0}, q_{n_0} > \deg P$. We set

$$p = p_{n_0}, \quad q = q_{n_0}, \quad K = K_m \quad \text{and} \quad a = \frac{1}{s}$$

and we apply Lemma 2.4. To do this, we consider a small closed disk $\bar{D}$ centered at zero disjoint from $K_m$. We also consider the function $w$ defined by $w(z) = P(z)$ on $\bar{D}$ and $w(z) = Q_j(z)$ on $K_m$. By Lemma 2.7 we find a polynomial $\tilde{P}(z)$ approximating $w(z)$ on $K_m \cup \bar{D}$ and such that a partial sum of $\tilde{P}(z)$ is $P(z)$.

Then we find a rational function $R(z)$ given by Lemma 2.4. Letting $f = R(z)$, we infer that $f \in E(m, j, s, (p, q)), (p, q) \in \mathfrak{F}$ and a partial sum of $f(z)$ is $P(z)$ as required. Assume now that $(\mathcal{H}_2)$ holds. Since $\lim_{n \to +\infty} (p_n - q_n) = +\infty, q_n \geq 0$, we find a $n_0$ so that $p_{n_0} - q_{n_0} > \deg P$. Therefore, we apply Lemma 2.3 and we have the result as in the previous case. This completes the proof of Proposition 3.7.

**End of Proof of Theorem 3.4** Since $\mathbb{C}^{\mathbb{N}}$ is a complete metric space, Baire’s Category Theorem combined with Proposition 3.7 implies that the class $\mathcal{U}$ is dense and $G_\delta$ in $\mathbb{C}^{\mathbb{N}}$. This completes the proof of Theorem 3.4.

**Remark 3.8** It is an open question if $\mathcal{U}$ contains a dense vector space or closed infinite subspace except 0 in $\mathbb{C}^{\mathbb{N}}$.

**Remark 3.9** A careful examination of the proof of Theorem 3.4 shows that it remains also valid if we replace the Cartesian topology with the topology of the ring $\mathbb{C}[[z]]$ of formal power series over $\mathbb{C}$. Recall that the distance between two distinct sequences $(a_n)_{n \in \mathbb{N}} \in \mathbb{C}[[z]]$ and $(b_n)_{n \in \mathbb{N}} \in \mathbb{C}[[z]]$ in this topology is defined to be $dist((a_n), (b_n)) = 2^{-\kappa}$, where $\kappa$ is the smallest natural number such that $a_\kappa \neq b_\kappa$. The distance between two equal sequences is, of course, zero.

**Remark 3.10** In the case $\mathfrak{F} = \{(p, 0) : p = 0, 1, 2, \ldots\}$ the result of Theorem 3.4 is a well known result of Selznev (see [6] and [19]).

**Remark 3.11** The case $(p_n)_{n=1,2,...}$ is bounded is not covered by Theorem 3.4. However for $p_n = 0, q_n = n (n = 1, 2, \ldots)$, the function $Q(z) = z - 5$ can not be uniformly approximated on the compact set $K = \{z : |z - 5| \leq 2\}$ by functions of the form $1/P_n(z)$ ($n = 1, 2, \ldots$) where $P_n$ are analytic polynomials. For, otherwise we would have

$$\frac{1}{2} \leq \left| \frac{1}{P_n(z)} \right| \leq \frac{3}{2} \quad \text{for} \quad n \geq n_0 \quad \text{on} \quad \{z : |z - 5| = 1\}.$$ 

This implies $|P_n(z)| \leq 2$ for all $n \geq n_0$ on $\{z : |z - 5| = 1\}$. The maximum principle implies $|P_n(5)| \leq 2$ for all $n \geq n_0$. Thus,

$$\left| \frac{1}{P_n(5)} \right| \geq \frac{1}{2}, \quad \text{for all} \quad n \geq n_0.$$
which contradicts \( \lim_{n \to \infty} q_n = 0 \).

**Remark 3.12** Cases covered by Theorem 3.4 are the following:

(i). \( l > 0 \) and \( p_n = lq_n, \ q_n = n \),

(ii). \( l \in \mathbb{Q} \), \( l > 0 \), and \( p_n = lq_n \), with \( q_n \to +\infty \).

(iii). \( q_n \leq M \) and \( p_n \to +\infty \). (The case \( M = 0 \) corresponds to Seleznev’s result).

Especially, by (ii) of Remark 3.12 we see that we can also work on the diagonal \( p = q \) of the Padé table.

4 Approximation on arbitrary compact sets in \( \mathbb{C} \setminus \{0\} \)

Lemma 2.5 allows approximating holomorphic functions in a neighborhood of a compact set \( K \subset \mathbb{C} \) with arbitrary connectivity.

**Definition 4.1** A set \( \mathcal{S} \subset \mathbb{N}_0^2 = \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\} \) satisfies condition \( (\mathcal{H}) \) if it contains a sequence \( (p_n, q_n)_{n=1,2,\ldots} \in \mathcal{S} \) such that:

\[
\lim_{n \to \infty} q_n = \lim_{n \to \infty} (p_n - q_n) = +\infty .
\]

Let \( (K_m)_{m=1,2,\ldots} \) be a fixed sequence of compact subsets of \( \mathbb{C} \setminus \{0\} \).

**Definition 4.2** Let \( \mathcal{S} \subset \mathbb{N}_0^2 = \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\} \) and \( (K_m)_{m=1,2,\ldots} \) be as above. We denote by

\[ \widetilde{U} \]

the class of all formal power series \( f = \sum_{v=0}^{\infty} a_v z^v \in \mathbb{C}^{\mathbb{N}_0} \) such that for every \( m = 1, 2, \ldots \) and every function \( Q(z) \) holomorphic in a neighborhood of \( K_m \) the following holds.

There exists a sequence \( (\tilde{p}_n, \tilde{q}_n)_{n=1,2,\ldots} \in \mathcal{S} \) such that

(i). \( f \in \mathcal{D}_{\tilde{p}_n, \tilde{q}_n} \) for all \( n = 1, 2, \ldots \) and

(ii). \( \lim_{n \to \infty} [\tilde{p}_n/\tilde{q}_n] f(z) = Q(z) \) uniformly on \( K_m \).

It is easy to see that \( \tilde{U} \) remains unchanged if we require in addition

\( \tilde{p}_n, \tilde{q}_n \neq \tilde{p}_l, \tilde{q}_l \) for \( n \neq l \).

**Remark 4.3** Since \( \mathbb{C} \setminus \{0\} \) is open, according to \( [17] \) condition (ii) in the above Definition 4.2 is equivalent to the requirement \( \lim_{n \to \infty} \left( d^l \left[ \tilde{p}_n/\tilde{q}_n \right] f / dz^l \right)(z) = Q^{(l)}(z) \) uniformly on \( K_m \), for any \( l = 0, 1, 2, \ldots \).

**Proposition 4.4** If \( \mathcal{S} \) satisfies condition \( (\mathcal{H}) \) and \( (K_m)_{m=1,2,\ldots} \) are as above, then the class \( \tilde{U} \) is dense and \( G_\delta \) in the space \( \mathbb{C}^{\mathbb{N}_0} \) of all formal power series endowed with the Cartesian topology.

**Proof** By Runge’s Theorem, it suffices to approximate on each \( K_m \) all rational functions with poles off \( K_m \cup \{0\} \). By choosing all coefficients of the numerator and the denominator from \( \mathbb{Q} + i\mathbb{Q} \), it is easy to see that it is enough to approximate on \( K_m \) a denumerable set of rational functions \( Q_{j,m} (j = 1, 2, \ldots) \) with
poles off $K_m \cup \{0\}$. Fix the functions

$$Q_{j,m} \ (j = 1, 2, \ldots \text{ and } m = 1, 2, \ldots)$$

and consider the sets

$$E(m, j, s, (p, q)) = \left\{ f \in \mathbb{C}^{\mathbb{N}_0} : f \in \mathbb{D}_{p,q} \text{ and } \sup_{z \in K_m} \left| \frac{p}{q} f(z) - Q_{j,m}(z) \right| < \frac{1}{s} \right\}$$

It is easily seen that

$$\tilde{U} = \bigcap_{m,j,s=1,2,\ldots} \bigcup_{(p,q) \in \Im} E(m, j, s, (p, q)).$$

Next, each set $E(m, j, s, (p, q))$ is open. The justification is the same as that of Lemma 3.6. In order to apply Baire’s Category Theorem it remains to show $\bigcup_{(p,q) \in \Im} E(m, j, s, (p, q))$ is dense in $\mathbb{C}^{\mathbb{N}_0}$. This can be done in a similar way with the proof of Proposition 3.7. The only difference is that we use Lemma 2.5 instead of Lemmas 2.3 and 2.4. In order to do this, it is enough to approximate $Q_{j,m}(z)$ on $K_m$ by a function of the form

$$P(z) + z^\lambda \frac{A(z)}{B(z)}$$

with

- $P$ an arbitrary polynomial,
- $\lambda > \deg P$,
- $A, B$ arbitrary polynomials and
- $B(z) \neq 0$ on $K_m \cup \{0\}$.

For this purpose, it suffices that $A(z)/B(z)$ approximates $[Q_{j,m}(z) - P(z)]/z^\lambda$ on $K_m$. This is assured by Runge’s Theorem. Since we easily obtain $B(z) \neq 0$ on $K_m \cup \{0\}$, the result follows. ■

**Remark 4.5** Cases covered by Theorem 4.4 are the following:

(i). $p_n \geq lq_n$ for all $n, l > 1 \text{ and } q_n \rightarrow +\infty$.

(ii). $l \in \mathbb{Q}, \ l > 1 \text{ and } p_n = lq_n \text{ with } q_n \rightarrow +\infty$.

A naturally posed question is can we have $p_n = q_n$?

We consider a non-empty finite union of open disks with centres in $\mathbb{Q} + i\mathbb{Q}$ and rational radii and such that one of these disks contains 0. The set of all such unions is denumerable. We denote by $(K_m)_{m=1,2,\ldots}$ the sequence of complements of these unions in $\{z \in \mathbb{C} : |z| \leq n\}$, where $n$ varies in the set of natural numbers. Then if a formal power series $f = \sum_{v=0}^{\infty} a_v z^v \in \mathbb{C}^{\mathbb{N}_0}$ belongs

\[
\text{13}
\]
to the class $\tilde{U}$ (with respect to the sequence $(K_n)_{m=1,2,...}$), it also has the following property, provided $\exists$ satisfies condition $(\tilde{\mathcal{H}})$: For every compact set $K \subset \mathbb{C} \setminus \{0\}$ and every function $Q$ holomorphic in a neighborhood of $K$, the following holds. There exists a sequence $(\tilde{p}_n, \tilde{q}_n)_{n=1,2,...} \in \exists$ such that

(i). $f \in \mathcal{D}_{p_n, q_n}$ for all $n = 1, 2,...$ and

(ii). $\lim_{n \to \infty} [\tilde{p}_n/\tilde{q}_n]_f (z) = Q(z)$ uniformly on $K$.

The reason is that by Runge’s Theorem we can approximate $Q(z)$ on $K$ by a rational function $\tilde{Q}(z)$ with poles $r_1, r_2, \ldots, r_N$ in $([Q+iQ] \setminus K)$. Then we find $m = 1, 2, \ldots$ so that $K \subset K_m$ and $r_1, r_2, \ldots, r_N \in K_m^c$. Thus, $\tilde{Q}$ is holomorphic in a neighborhood of $K_m$ and can be approximated on $K_m$ by $[p/q]_f (z)$ as in Definition 4.2. Thus, we have proved the following.

**Theorem 4.6** Let $\exists \subset \mathbb{N}_0^2 = \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\}$ satisfies condition $\tilde{\mathcal{H}}$. Then there exists a formal power series $f \in \mathbb{C}^{\tilde{\mathcal{H}}}$ such that the following holds. For every compact set $K \subset \mathbb{C} \setminus \{0\}$ and every function $Q(z)$ holomorphic in a neighborhood of $K$, there exists a sequence $(\tilde{p}_n, \tilde{q}_n)_{n=1,2,...} \in \exists$ so that

(i). $f \in \mathcal{D}_{p_n, q_n}$ for all $n = 1, 2,...$ and

(ii). $\lim_{n \to \infty} [\tilde{p}_n/\tilde{q}_n]_f (z) = Q(z)$ uniformly on $K$.

The collection of all such formal series $f$ is a dense $G_\delta$ subset of $\mathbb{C}^{\tilde{\mathcal{H}}}$ endowed with the Cartesian topology. It is also dense and $G_\delta$ in $\mathbb{C}[[Z]]$ endowed with the ring topology. $\blacksquare$

## 5 SIMPLY CONNECTED DOMAINS

Consider a simply connected domain $\Omega \neq \mathbb{C}$ containing 0. Let $f$ be a holomorphic function in $\Omega$ ($f \in \mathcal{O}(\Omega)$) and let the Taylor development $\sum_{v=0}^{\infty} a_v z^v$ of $f$ around 0. Let finally $p$ and $q$ be two non negative integers. As it is already pointed out in §2, $f \in \mathcal{D}_{p,q}$ if and only if $H_{q}(f) (a_{p-q+1}) \neq 0$. Cauchy’s estimates imply that $\mathcal{D}_{p,q}$ is an open subset of $\mathcal{O}(\Omega)$ if $\mathcal{O}(\Omega)$ is endowed with the topology of uniform convergence on compact subsets of $\Omega$.

Assume that $(L_k)_{k=1,2,...}$ is an increasing sequence of compact subsets of $\overline{\Omega}$ such that

- $0 \in int (L_1),$
- $\overline{L_k \cap \Omega} = L_k$ whenever $k = 1, 2,$..., 
- $L_k^c$ is connected for all $k = 1, 2,$... and
- every compact set $T \subset \Omega$ is contained in some $L_k$.

Let us consider the space $\mathcal{O}(\Omega, \{L_k\})$ of holomorphic functions $f \in \mathcal{O}(\Omega)$ such that, for each derivative $f^{(l)} (l = 1, 2,$...) and each $L_k$ ($k = 1, 2,$...) the restriction $f^{(l)} | L_k \cap \Omega$
is uniformly continuous on $L_k \cap \Omega$. Therefore, it extends continuously on $L_k \cap \Omega = L_k$. The space $\mathcal{O}(\Omega, \{L_k\})$ endowed with the seminorms

$$
\sup_{z \in L_k} \left| f^{(l)}(z) \right| \ (l = 1, 2, \ldots \text{ and } k = 1, 2, \ldots)
$$

becomes a Fréchet space containing the polynomials. Since we do not know if the polynomials are dense in $\mathcal{O}(\Omega, \{L_k\})$, we consider the closure $\overline{\mathcal{P}}(\Omega, \{L_k\})$ of the set of polynomials in $\mathcal{O}(\Omega, \{L_k\})$. Then $\mathcal{O}(\Omega, \{L_k\})$ and $\overline{\mathcal{P}}(\Omega, \{L_k\})$ are again Fréchet spaces and Baire’s Category Theorem is at our disposal. Let now $(K_m)_{m=1,2,\ldots}$ be a sequence of compact sets with

- connected complement $K_m^c$ and
- $K_m \cap L_k = \emptyset$ whenever $(m = 1, 2, \ldots \text{ and } k = 1, 2, \ldots)$.

We are in position to prove the following:

**Theorem 5.1** Suppose $\mathcal{H} \subset \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\}$ satisfies condition $(\mathcal{H})$. Under the above assumptions, there exists a function $f \in \overline{\mathcal{P}}(\Omega, \{L_k\})$ such that for every polynomial $Q(z)$ and every $m = 1, 2, \ldots$ there is a sequence $(p_n, q_n)_{n=1,2,\ldots} \in \mathcal{H}$ so that $f \in \mathcal{D}_{p_n, q_n} \cap \overline{\mathcal{P}}(\Omega, \{L_k\})$ for all $n$ and the following hold.

For every $l = 0, 1, 2, \ldots$ we have

- $\sup_{z \in L_k} \left| \frac{d^l}{dz^l} [p_n/q_n] f(z) - \frac{d^l}{dz^l} f(z) \right| \to 0$ as $n \to \infty$ for all $k = 1, 2, \ldots$

and

- $\sup_{z \in K_m} \left| \frac{d^l}{dz^l} [p_n/q_n] f(z) - \frac{d^l}{dz^l} Q(z) \right| \to 0$ as $n \to \infty$.

The set of these functions $f$ is dense and $G_\delta$ in the space $\overline{\mathcal{P}}(\Omega, \{L_k\})$.

**Proof** It is easy to see that it suffices to approximate all polynomials $Q_j(z)$ ($j = 1, 2, \ldots$) with coefficients in $\mathbb{Q} + i\mathbb{Q}$ and their derivatives on the compact sets $K_m$ ($m = 1, 2, \ldots$). We consider the sets

$$
\hat{\mathcal{E}}_\Omega(m, j, s, k, (p, q)) = \left\{ f \in \overline{\mathcal{P}}(\Omega, \{L_k\}) \cap \mathcal{D}_{p,q} : \begin{array}{l}
\sup_{z \in K_m} \left| \frac{d^l}{dz^l} [p/q] f(z) - \frac{d^l}{dz^l} Q_j(z) \right| < \frac{1}{k} \text{ and } \\
\sup_{z \in L_k} \left| \frac{d^l}{dz^l} [p_n/q_n] f(z) - \frac{d^l}{dz^l} f(z) \right| < \frac{1}{s} \\
\text{ for } l = 0, 1, 2, \ldots \end{array} \right\}.
$$

Obviously, the set of functions satisfying the hypotheses of the Theorem can be written as

$$
\Omega(\Omega, \{L_k\}, \{K_m\}) := \bigcap_{m,j,s,k=1, (p,q) \in \mathcal{H}} \bigcup_{m,j,s,k=1, (p,q) \in \mathcal{H}} \hat{\mathcal{E}}_\Omega(m, j, s, k, (p, q)).
$$

Cauchy estimates and the continuity of the operator $\mathcal{D}_{p,q} f \mapsto [p/q] f$ imply that $\hat{\mathcal{E}}_\Omega(m, j, s, k, (p, q))$ is open in $\overline{\mathcal{P}}(\Omega, \{L_k\})$. It remains to show
That \( \bigcup_{(p,q) \in \mathbb{N}} \hat{E}_{\Omega}(m, j, s, k, (p,q)) \) is dense in \( \hat{F}(\Omega, \{L_k\}) \). For this purpose fix any \( L = L_k \) \((k = 1, 2, \ldots)\) and consider a polynomial \( \varphi(z) \), \( \varepsilon > 0 \) and \( N \in \mathbb{N} \equiv \{1, 2, \ldots\} \). We are looking for a function \( f \in \hat{E}_{\Omega}(m, j, s, k, (p,q)) \) and a \((p,q) \in \mathbb{N}\) such that

\[
\begin{align*}
\bullet & \quad \sup_{z \in L} \left| \frac{d^l}{dz^l} f(z) - \frac{d^l}{dz^l} \varphi(z) \right| < \varepsilon, \text{ for } l = 1, 2, \ldots, N \\
\bullet & \quad \sup_{z \in K_m} \left| \frac{d^l}{dz^l} \left( \frac{p}{q} f(z) - \frac{d^l}{dz^l} Q_j(z) \right) \right| < \frac{1}{n} \text{ and} \\
\bullet & \quad \sup_{z \in L} \left| \frac{d^l}{dz^l} \left( \frac{p}{q} f(z) - \frac{d^l}{dz^l} f(z) \right) \right| < \frac{1}{s} \text{ for all } l = 1, 2, \ldots, s.
\end{align*}
\]

By \([17]\) and \([27]\), there exist two simply connected open sets

\[
\begin{align*}
V \text{ with } L \subset V \subset \Omega \text{ and } W \text{ with } K_m \subset W
\end{align*}
\]

such that \( V \cap W = \emptyset \). Considering an exhausting family of compact sets for \( W \), we find a compact set \( S \subset W \) such that \( K_m \subset int(S) \) and the complement \( S^c \) is connected. Similarly, we find a compact set \( T \) such that \( L \subset int(T) \subset T \subset V \).

We consider the function \( w(z) = \varphi(z) \) on \( V \) and the function \( w(z) = Q_j(z) \) on \( W \). Runge’s Theorem gives a polynomial \( P_n(z) \) such that

\[
\sup_{z \in T \cup S} |P_n(z) - w(z)| < \frac{1}{n} \quad (n = 1, 2, \ldots).
\]

Using Lemmas 2.3 and 2.4 we find rational functions \( R_n(z) \) with poles outside \( \Omega \cup S \) (close to \( \infty \)) such that for some \((p,q) \in \mathbb{N}\) depending on \( n \)

\[
\begin{align*}
\bullet & \quad \sup_{z \in T \cup S} |R_n(z) - P_n(z)| < \frac{1}{n}, \\
\bullet & \quad R_n(z) \in \mathcal{O}_{p,q} \text{ and} \\
\bullet & \quad [p/q]_{R_n}(z) = R_n(z).
\end{align*}
\]

Since \( \lim_{n \to \infty} R_n(z) = Q_j(z) \) uniformly on the open set \( int(S) \cup int(T) \), Weierstrass Theorem implies that

\[
\lim_{n \to \infty} \left( \frac{d^l}{dz^l} R_n(z) / dz^l \right) = \left( \frac{d^l}{dz^l} w(z) / dz^l \right)
\]

uniformly on \( K_m \cup L \) \((l = 0, 1, 2, \ldots)\). Choosing \( f(z) \) to be one of the functions \( R_n(z) \) \((n \text{ big})\) and \((p,q) \in \mathbb{N}\) as above, we find

\[
\begin{align*}
\bullet & \quad \sup_{z \in L} \left| \frac{d^l}{dz^l} f(z) - \frac{d^l}{dz^l} \varphi(z) \right| < \varepsilon \quad (l = 1, 2, \ldots, N), \\
\bullet & \quad f \in \mathcal{O}(\Omega) \cap \mathcal{O}_{p,q} \text{ and} \\
\bullet & \quad \sup_{z \in K_m} \left| \frac{d^l}{dz^l} f(z) - \frac{d^l}{dz^l} Q_j(z) \right| < \frac{1}{s} \quad (l = 1, 2, \ldots, s).
\end{align*}
\]
Since \( f(z) - \left[p/q\right] f(z) = 0 \), Baire’s Theorem completes the proof. ■

To give a first application of Theorem 5.1, let us consider

- a simply connected domain \( \Omega \) containing 0 and
- an exhausting sequence \((L_k)_{k=1,2,...}\) of compact subsets of \( \Omega \).

The sequence \((K_m)_{m=1,2,...}\) is chosen to satisfy the following.

**Lemma 5.2** (34, 35) Let \( \Omega \) be a domain of \( C \). There exists a sequence \((K_m)_{m=1,2,...}\) of compact subsets of \( C \) with \( \bigcap K_m = \emptyset \) \( K_m^c \) connected, and such that the following holds.

- For every compact set \( K \subset C \), with \( \bigcap K = \emptyset \) and \( K^c \) connected, there exists a \( m \in \{1,2,\ldots\} \) so that \( K \subset K_m \). ■

Then we obtain the following special case of Theorem 5.1.

**Theorem 5.3** Let \( \Omega \) be a simply connected domain of \( C \) containing 0. Let also \( \exists \subset \{0,1,2,\ldots\} \times \{0,1,2,\ldots\} \) satisfying condition (H). There exists a holomorphic function \( f(z) \in O(\Omega) \) such that

- For every compact set \( K \subset C \), such that \( \bigcap K = \emptyset \) and \( K^c \) connected, and every polynomial \( \varphi(z) \), there exists a sequence \((p_n, q_n)_{n=1,2,...} \in \exists \) so that
  1. \( f \in D_{P_n,q_n} \) for all \( n = 1,2,\ldots \)
  2. \( \lim_{n \to \infty} \frac{d^l}{dz^l} [p_n/q_n] f(z) = \frac{d^l}{dz^l} \varphi(z) \) uniformly on \( K \) \( (l = 0,1,2,\ldots) \)
  3. \( \lim_{n \to \infty} [p_n/q_n] f(z) = f(z) \), uniformly on each compact subset of \( \Omega \)
  4. In particular, for every \( h(z) \in A(K) \) we have \( \lim_{n \to \infty} [p_n/q_n] f(z) = h(z) \), uniformly on \( K \).

The set of all such \( f \)'s is a dense \( G_\delta \) subset of \( O(\Omega) \) endowed with the topology of uniform convergence on compacta. ■

To give a second application of Theorem 5.1, we consider an exhausting sequence \((L_k)_{k=0,1,2,...}\) of compact subsets of \( \Omega \), with \( 0 \in \text{int}(L_0) \). The sequence \((K_m)_{m=1,2,...}\) is given by the following.

**Lemma 5.4** (36, 37, 38 and 39) Let \( \Omega \) be a domain of \( C \). There exists a sequence \((K_m)_{m=1,2,...}\) of compact subsets of \( C \) with \( \bigcap K_m = \emptyset \) \( K_m^c \) connected, and such that the following holds.

- For every compact set \( K \subset C \), with \( \bigcap K = \emptyset \) and \( K^c \) connected, there exists a \( m \in \{1,2,\ldots\} \) so that \( K \subset K_m \). ■

Then we obtain the following special case of Theorem 5.1.

**Theorem 5.5** Let \( \Omega \) be a simply connected domain of \( C \) containing 0. Let also \( \exists \subset \{0,1,2,\ldots\} \times \{0,1,2,\ldots\} \) satisfying condition (H). Then there exists a holomorphic function \( f(z) \in O(\Omega) \) such that
For every compact set $K \subset \mathbb{C}$, such that $K \cap \overline{\Omega} = \emptyset$ and $K^c$ connected, and every function $h(z) \in A(K)$, there exists a sequence $(p_n, q_n)_{n=1,2,...} \in \mathfrak{S}$ so that

(i). $f \in D_{p_n, q_n}$ for all $n = 1, 2, \ldots$

(ii). $\lim_{n \to \infty} [p_n/q_n]_f(z) = h(z)$ uniformly on $K$ and

(iii). $\lim_{n \to \infty} [p_n/q_n]_f(z) = f(z)$, uniformly on each compact subset of $\Omega$.

The set of all such $f$’s is a dense $G_δ$ subset of $O(\Omega)$ endowed with the topology of uniform convergence on compacts. ■

Remark 5.6 In the above theorem it is equivalent to require convergence of all order derivatives because $\Omega$ and $\overline{\Omega}^c$ are open sets ($\overline{\Omega}^c$).

To give a third application of Theorem 5.1, we consider $\Omega \subset \mathbb{C}$ to be a simply connected domain containing 0, such that $\{\infty\} \cup [\mathbb{C} \setminus \overline{\Omega}]$ is connected.

We set $L_k = \{z \in \overline{\Omega} : |z| < n\}$. The sequence $(K_m)_{m=1,2,...}$ is given by Lemma 5.4. Now the universal functions are smooth on $\partial \Omega$, in fact they belong to the closure of the set of polynomials in $A^\infty(\Omega)$. We remind that $A^\infty(\Omega)$ is the class of all holomorphic functions $f(z) \in O(\Omega)$, such that every derivative $(d^l f(z))/(d z^l)) (l = 0, 1, 2, \ldots)$ extends continuously on $\overline{\Omega}$ (where the closure is taken in $\mathbb{C}$). The natural topology on $A^\infty(\Omega)$ is that of uniform convergence of all orders derivatives on each compact subset of $\overline{\Omega}$.

Theorem 5.7 Let $\Omega$ be a simply connected domain of $\mathbb{C}$ containing 0, such that $\{\infty\} \cup [\mathbb{C} \setminus \overline{\Omega}]$ is connected. Let $\mathcal{F}(A^\infty(\Omega))$ denote the closure of polynomials in $A^\infty(\Omega)$. Let also $\mathfrak{S} \subset \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\}$ satisfying condition ($\mathcal{H}$). Then there exists a holomorphic function $f(z) \in \mathcal{F}(A^\infty(\Omega))$ such that

- For every compact set $K \subset \mathbb{C}$, such that $K \cap \overline{\Omega} = \emptyset$ and $K^c$ connected, and every polynomial $\varphi(z)$, there exists a sequence $(p_n, q_n)_{n=1,2,...} \in \mathfrak{S}$ so that

  (i). $f \in D_{p_n, q_n}$ for all $n = 1, 2, \ldots$

  (ii). $\lim_{n \to \infty} \frac{d}{dz^l} [p_n/q_n]_f(z) = \frac{d^l}{dz^l} \varphi(z)$, uniformly on $K$ ($l = 0, 1, 2, \ldots$) and

  (iii). $\lim_{n \to \infty} \frac{d}{dz^l} [p_n/q_n]_f(z) = \frac{d^l}{dz^l} f(z)$, uniformly on each compact subset of $\overline{\Omega}$.

The set of all these functions is a dense $G_δ$ subset of $\mathcal{F}(A^\infty(\Omega))$. ■

Remark 5.8 If $\Omega$ is a Jordan domain with rectifiable boundary, then $\mathcal{F}(A^\infty(\Omega)) = A^\infty(\Omega)$ ($\mathcal{R}$). ■

Example 3.4 in $\mathcal{R}$ section 3 may be transferred to our case. In this example $\Omega$ is the unit disk. Several properties on universal Taylor series ($\ast : q = 0$) have been established in the literature especially in the case of the unit disk. One wonder is if they remain valid in our case. For instance is every universal function non extendable? Even simpler it is to ask if the radius of convergence of the Taylor development of a universal function $f$ is exactly equal to the distance of 0 from the boundary $\partial \Omega$ of $\Omega$. 

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In the case $q = 0$, it has been examined if we can have universal Taylor series with respect to several centres simultaneously ([6, 36, 37] and [10]). In our case instead of developing with centre 0 we can develop with respect to another centre $z_0$ and obtain a formal power series $f \equiv f_{z_0} = \sum_{v=0}^{\infty} a_v (z - z_0)^v$ with centre $z_0$. If $p \in \mathbb{N}_0$ and $q \in \mathbb{N}_0$ are given ($\mathbb{N}_0 = \{0, 1, 2, \ldots\}$), then

$$[p/q]_{f,z_0}(z)$$

will denote a rational function

$$\frac{\sum_{v=0}^{p} n_v (z - z_0)^v}{\sum_{v=0}^{q} d_v (z - z_0)^v}, \quad d_0 = 1, \quad n_p d_q \neq 0$$

such that its Taylor development $\sum_{v=0}^{\infty} b_v (z - z_0)^v$ with centre $z_0$ will have the same $p + q + 1$ first coefficients with $f \equiv f_{z_0}$ that is $b_v = a_v$ for all $v = 0, 1, \ldots, p + q$. This rational function may exist or not and it is not necessarily unique. When such a rational fraction exists is called a Padé form of type $(p, q)$ to the series $f$. A necessary and sufficient condition for existence and uniqueness is that

$$\det \left( H^{(f)}_{q} (a_{p-q+1}) \right) := \det \left( \begin{array}{cccccc} a_{p+q+1} & a_{p+q+2} & a_{p+q+3} & \cdots & a_p \\ a_{p+q+2} & a_{p+q+3} & a_{p+q+4} & \cdots & a_{p+1} \\ a_{p+q+3} & a_{p+q+4} & a_{p+q+5} & \cdots & a_{p+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_p & a_{p+1} & a_{p+2} & \cdots & a_{p+q-1} \\ \end{array} \right) \neq 0,$$

Then we write

$$f \in \mathcal{D}_{p,q}^{(z_0)}$$

and the Padé form is said to be a **Padé approximant of type** $(p, q)$ to the series $f$.

We are now looking for holomorphic functions $f(z) \in \mathcal{O}(\Omega)$ in a simply connected domain $\Omega$ of $\mathbb{C}$ so that for every compact set $K \subset \mathbb{C}$, with $K \cap \Omega = \emptyset$ and $K^c$ connected, and every function $h(z) \in A(K)$, there exists a sequence $(p_n, q_n)_{n=1,2,\ldots} \in \mathbb{N}$ such that

- $[p_n/q_n]_{f,z_0}$ exists as a Padé form for all $z_0 \in \Omega$ and all $n = 1, 2, \ldots$,
- $\lim_{n \to \infty} [p_n/q_n]_{f,z_0} (z) = h(z)$, uniformly on compact subsets of $(z_0, z) \in \Omega \times K$ and
- $\lim_{n \to \infty} [p_n/q_n]_{f,z_0} (z) = f(z)$, uniformly on each compact subset of $(z_0, z) \in \Omega \times \Omega$. 

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A first question is if there exist universal functions with respect to all centres. A second question that arises naturally is the following: does the class of universal functions with respect to one centre coincide with that with respect to all centres? (See [37] and [39]).

We also mention that universal Taylor series do not exist in some unbounded non-simply connected domains ([25]). What about universality of Padé approximants?

Finally, we mention that in Theorem 5.1 one can replace the assumption "\(K_m^c\) connected" by the assumption that "0 belongs to the unbounded component of \(K_m^c\)" and then \(Q\) will not be anymore a polynomial, but any holomorphic function in the neighbourhood of \(K_m\) (\(m \in \{1, 2, \ldots\}\) being fixed). Then the same result holds provided that \(\exists \subset \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\}\) satisfies condition \(\left(\tilde{H}\right)\) instead of \((H)\). A difference in the proof is that \(P_n\) given by Runge’s Theorem will not be any more a polynomial, but it will be a rational function with poles outside \(\Omega \cup S\). Then instead of Lemma 2.3 or 2.4, we can use a variant of Lemma 2.5. In this variant we do not care about the first coefficients of the Taylor development of the rational function \(R\); thus, the condition \(\lim_{n \to \infty} p_n = \lim_{n \to \infty} q_n = +\infty\) is sufficient. Combining the above with the following Lemma, we obtain Theorem 5.10 below.

**Lemma 5.9** Let \(\Omega\) be a simply connected domain in \(\mathbb{C}\) containing 0. There is a sequence \((K_m)_{m=1, 2, \ldots}\) of compact subsets of \(\Omega^c\) where the complement \(\tilde{K}_m^c\) has a finite number of components and 0 belongs to the unbounded component of \(\tilde{K}_m^c\), such that the following holds.

- For every compact set \(K \subset \Omega^c\) such that \(K^c\) has a finite number of components and 0 belongs to the unbounded component of \(K^c\), there exists a \(m = 1, 2, \ldots\) so that
  
  (i). \(K \subset \tilde{K}_m\) and

  (ii). every component of \(K^c\) contains a component of \(\tilde{K}_m^c\).

**Proof** To construct the sequence \((\tilde{K}_m)_{m=1, 2, \ldots}\), we start with the sequence \((K_m)_{m=1, 2, \ldots}\) given by Lemma 5.2. From each \(K_m\), we take out a finite union of disjoint open discs with centers in \(\mathbb{Q} + i\mathbb{Q}\) and rational radii which are included in the interior \(\text{int}(K_m)\) of \(K_m\). If \(\text{int}(K_m) = \emptyset\) then we consider only \(K_m\) without taking out any disk. Any way even if \(\text{int}(K_m) \neq \emptyset\) we also keep \(K_m\) itself considering that we took off the empty union of discs. The resulting compact sets are denumerable. An enumeration gives the family \((\tilde{K}_m)_{m=1, 2, \ldots}\). One can easily verify that it has the required property.

**Theorem 5.10** Let \(\Omega\) be a simply connected domain of \(\mathbb{C}\) containing 0. Let also \(\exists \subset \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\}\) containing a sequence \((p_n, q_n)_{n=1, 2, \ldots}\) satisfying \(\lim_{n \to \infty} p_n = \lim_{n \to \infty} q_n = +\infty\). Then there exists a holomorphic function \(f(z) \in \mathcal{O}(\Omega)\) such that

- For every compact set \(K \subset \mathbb{C}\), such that \(K \cap \Omega = \emptyset\) and \(K^c\) having a finite number of components with 0 being in the unbounded component of \(K^c\).
and every function $h(z)$ holomorphic in a neighborhood of $K$, there exists a sequence $(p_n, q_n)_{n=1,2,...} \in \mathcal{S}$ such that the following hold.

(i). $f \in \mathcal{D}_{p_n, q_n} \cap \mathcal{O}(\Omega)$ for all $n = 1, 2, \ldots$

(ii). $\lim_{n \to \infty} \frac{d^l}{dz^l} [p_n/q_n] f(z) = \frac{d^l}{dz^l} h(z)$, uniformly on $K$ ($l = 0, 1, 2, \ldots$) and

(iii). $\lim_{n \to \infty} \frac{d^l}{dz^l} [p_n/q_n] f(z) = \frac{d^l}{dz^l} f(z)$, uniformly on each compact subset of $\Omega$ ($l = 0, 1, 2, \ldots$).

The set of all these functions is dense and $G_\delta$ in $\mathcal{O}(\Omega)$ endowed with the topology of uniform convergence on compacts. $\blacksquare$

6 THE CASE OF PLANAR DOMAINS WITH ARBITRARY CONNECTIVITY

Let $\mathcal{S} \subset \mathbb{N}^2_0 = \{0, 1, 2, \ldots \} \times \{0, 1, 2, \ldots \}$ containing a sequence $(p_n, q_n)_{n=1,2,...} \in \mathcal{S}$ with $\lim_{n \to \infty} p_n = \lim_{n \to \infty} q_n = +\infty$.

Let $\Omega$ be a bounded domain in $\mathbb{C}$ containing 0. As usually, $A(\Omega)$ denotes the space of all functions $\phi : \overline{\Omega} \to \mathbb{C}$ continuous on $\overline{\Omega}$ and holomorphic on $\Omega$. We suppose that $A(\Omega)$ is endowed with the supremum norm. Further, $X$ denotes the closure in $A(\Omega)$ of the set of functions holomorphic in some (varying) neighborhood of $\Omega$.

**Lemma 6.1** Let $\Omega$, $X$ and $\mathcal{S}$ be as above. Let also $K \subset \mathbb{C}$ be a compact set with $K \cap \overline{\Omega} = \emptyset$. Then there exists a $\phi(z) = \sum_{n=0}^{\infty} \varepsilon_n z^n \in X$, such that, for every function $Q(z)$ holomorphic in a neighborhood of $K$, there exists a sequence $(\tilde{p}_n, \tilde{q}_n)_{n=1,2,...} \in \mathcal{S}$, such that

(i). $\phi(z) \in \mathcal{D}_{\tilde{p}_n, \tilde{q}_n}$ for all $n = 1, 2, \ldots$ and

(ii). $\lim_{n \to \infty} \frac{\tilde{p}_n/\tilde{q}_n}{Q(z)} (z) = Q(z)$ uniformly on $K$.

The set of all such functions $\phi \in X$ is dense and $G_\delta$ in $X$.

**Proof** For the proof, we may use Baire’s Theorem. The essential step is to prove the density of the set

$$\bigcup_{(p,q)\in \mathcal{S}} \left\{ g(z) \in X : g \in \mathcal{D}_{p,q} \text{ and } \sup_{z \in K} \left| \frac{p/q}{Q_j} (z) - Q_j (z) \right| < \frac{1}{2} \right\},$$

where $Q_j$, $j = 1, 2, \ldots$ is an enumeration of rational functions with poles in $(\mathbb{Q} + i\mathbb{Q}) \cap K^c$ such that all coefficients of their numerator and denominator belong to $(\mathbb{Q} + i\mathbb{Q})$. Let $w(z)$ be a holomorphic function on a neighborhood of $\overline{\Omega}$ and $\varepsilon > 0$ be given. Let also $L = \overline{\Omega}$. Using Runge’s Theorem, we find two polynomials $A(z)$ and $B(z)$, with $B(z) \neq 0$ on $K \cup L$ (in particular $B(0) \neq 0$) and such that

$$\sup_{z \in K} \left| \frac{A(z)}{B(z)} - Q_j (z) \right| < \frac{1}{2s} \quad \text{and} \quad \sup_{z \in L} \left| \frac{A(z)}{B(z)} - w(z) \right| < \frac{\varepsilon}{2}.$$

Choose a $(p,q) \in \mathcal{S}$ so that $\deg A < p$ and $\deg B < q$. We consider

$$A(z) = A(z) + dz^p \quad \text{and} \quad B(z) = B(z) + (cz)^q,$$
where the constants \( d, c \in \mathbb{C} \setminus \{0\} \) will be chosen later on. According to Lemma 2.1, \( c \) and \( d \) may be chosen close to zero so that

\[
\sup_{z \in K \cup L} \left| \frac{\tilde{A}(z)}{\tilde{B}(z)} - \frac{A(z)}{B(z)} \right| < a,
\]

where \( 0 < a < \min \left\{ \left( \frac{\varepsilon}{q} \right), \left( \frac{\varepsilon}{2q} \right) \right\} \). For \( c \) fixed, the Hankel matrix \( H_{q}^{(\tilde{A}/\tilde{B})}(\delta_{p-q+1}) \) of order \( q \) at \( \delta_{p-q+1} \) for the series

\[
\frac{\tilde{A}(z)}{\tilde{B}(z)} = A(z) \left[ \frac{1}{B(0)} + \cdots \right] + dz^p \left[ \frac{1}{B(0)} + \cdots \right] = \sum_{v=0}^{\infty} \delta_v z^v
\]

depends linearly on \( d \). For \( v < p \), the coefficients \( \delta_v \) are independent of \( d \). For \( v = p \), the coefficient \( \delta_p \) has the form \( \delta_p = d(1/B(0)) + c_p \), where \( c_p \) is independent of \( d \) and \( (1/B(0)) \neq 0 \). Thus, following to Lemma 2.2, the determinant \( \det \left( H_{q}^{(\tilde{A}/\tilde{B})}(\delta_{p-q+1}) \right) \) is a polynomial in \( d \) of degree \( q \) with leading coefficient \( (1/B(0))^q \neq 0 \). The zeros of such a polynomial are finite and we can avoid them by choosing \( d \) close to zero. We infer

\[
\frac{\tilde{A}(z)}{\tilde{B}(z)} \in \mathcal{D}_{p,q} \quad \text{and} \quad [p/q]_{\tilde{A}/\tilde{B}}(z) = \frac{\tilde{A}(z)}{\tilde{B}(z)}.
\]

Thus, it suffices to set

\[
g(z) = \frac{\tilde{A}(z)}{\tilde{B}(z)}.
\]

This gives the result. ■

We now consider an exhausting sequence of compact subsets of \( \mathbb{C} \setminus \overline{\Omega} \). From each member of this sequence, we take out all possible finite unions of open disks centered at points of \( \mathbb{Q} + i\mathbb{Q} \) and with rational radii. Thus, we obtain a sequence \( (K_m)_{m=1,2,\ldots} \) of compact sets. For any \( K \) and every rational function with poles off \( K \), there exists an \( m \) so that \( K \subset K_m \) and the function is holomorphic in \( K_m \). Then using Baire’s Theorem once more, we obtain the following.

**Theorem 6.2** Let \( \exists \subset \mathbb{Q}^3 \equiv \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\} \) containing a sequence \( (p_n, q_n)_{n=1,2,\ldots} \in \exists \) such that \( \lim_{n \to \infty} p_n = \lim_{n \to \infty} q_n = +\infty \). Let \( \Omega \) be a bounded domain of \( \mathbb{C} \) containing 0. Let also \( X \) denote the closure in \( A(\Omega) \) of the set of functions holomorphic in a (varying) neighborhood of \( \Omega \). Then there exists a \( \phi(z) \in X \), such that the following holds.

- For all compact sets \( K \subset \mathbb{C} \), such that \( K \cap \overline{\Omega} = \emptyset \) and all functions \( Q(z) \) holomorphic in a neighborhood of \( K \) there exists a sequence \( (p_n, q_n)_{n=1,2,\ldots} \in \exists \) so that
  
  (i). \( \phi \in \mathcal{D}_{p_n,q_n} \) for all \( n = 1,2,\ldots \) and
  
  (ii). \( \lim_{n \to \infty} [p_n/q_n]_{\phi}(z) = Q(z) \) uniformly on \( K \).
The set of all such $\phi$'s is a dense $G_\delta$ subset of $X$.

**Remark 6.3** In Theorem 6.2 one can also require

\[ \lim_{n \to +\infty} |p_n/q_n| \phi(z) = \phi(z) \text{ in } X. \]

The reason is that in the proof of Lemma 6.1 we automatically have $\phi - [p/q] \phi \equiv 0$ because $\phi(z) = A(z)/B(z) \in \mathfrak{D}_{p_n,q_n}$ is a rational function. We see therefore that it is a generic property of holomorphic functions in a domain to be limits of some Padé approximants of them; see also (8).

**Remark 6.4** With some more effort one can prove a version of Theorem 6.2 where $X$ is replaced by $X^{\infty}$ the closure in $A^\infty(\Omega)$ of the set of functions holomorphic in some (varying) neighborhood of $\overline{\Omega}$. The set $A^\infty(\Omega)$ is the space of all holomorphic functions $f(z)$ in $\Omega$ such that every derivative $f^{(l)}(z)$ ($l = 0, 1, 2, \ldots$) extends continuously on $\Omega$. The topology of $A^\infty(\Omega)$ is defined by the sequence of seminorms $\|f\|_l = \sup_{z \in \Omega} |f^{(l)}(z)|$, provided $\Omega$ is bounded.

**Remark 6.5** The assumption in Theorem 6.2 that $\Omega$ is bounded is not essential. In fact, if $L = \{z \in \Omega : |z| < n\}$ is a compact subset of $\overline{\Omega}$ and a pole $z_0$ of the approximating rational function belongs to $\Omega \setminus L$, then $z_0$ lies in the same component of $\mathbb{C} \cup \{\infty\} \setminus (L \cup K)$ with some point $z_1 \in (\mathbb{C} \cup \{\infty\}) \setminus (\overline{\Omega} \cup K)$. This is possible since $\text{dist}(\overline{\Omega}, K) > 0$. Thus, the approximating rational function is finite on $\Omega$.

**References**

[1] G.A. Baker, Jr.: The existence and convergence of subsequences of Padé approximants, Journal of Mathematical Analysis and Applications, Volume 43, Issue 2, August 1973, Pages 498-528.

[2] G.A. Baker, Jr: Essentials of Padé Approximants, Academic Press, New York, 1975. Also, A. Magnus: Review of the book Essentials of Padé Approximants, Bulletin of the American Mathematical Society, Volume 82, Issue 2, March 1976, Pages 243-246.

[3] G. A. Baker and P.R. Graves-Morris: Convergence of the rows of the Padé table, Journal of Mathematical Analysis and Applications, Volume 57, 1977, Pages 323-339.

[4] G. A. Baker and P.R. Graves-Morris: Padé Approximants, Vol. 1 and 2, Encyclopedia of Mathematics and its Applications, Addison Wesley, Reading, Mass., Vols. 13 and 14, 1981.

[5] G.A. Baker, Jr. and P. R. Graves-Morris: The convergence of sequences of Padé approximants, Journal of Mathematical Analysis and Applications, Volume 87, Issue 2, June 1982, Pages 382-394.

[6] F. Bayart, K.G. Grosse-Erdman, V. Nestoridis and C. Papadimitropoulos: Abstract theory of universal series and applications, Proc. London Math. Soc., Volume 96, Issue 3, 2008, Pages 417-463.
[7] A.G. Beardon: On the location of poles of Padé approximants, Journal of Mathematical Analysis and Applications, Volume 21, 1968, Pages 469-474.

[8] P.B. Borwein: The usual behaviour of rational approximants, Canadian Mathematical Bulletin, Volume 26, Issue 3, 1983, Pages 317-323.

[9] C. Brezinski: Padé approximants and orthogonal polynomials, in Padé and Rational Approximation, E.B. Saff and R.S. Varga eds., Academic Press, New York, 1977.

[10] C. Brezinski: Accélération de la Convergence en Analyse Numérique, Lecture Notes in Mathematics 584, Springer Verlag, Heidelberg, 1977.

[11] C. Brezinski: Algorithmes d’Accélération de la Convergence. Etude Numérique, Editions Technip, Paris, 1978.

[12] C. Brezinski: Rational approximation to formal power series, Journal of Approximation Theory, Volume 25, 1979, Pages 295-317.

[13] C. Brezinski: Padé-Type Approximants and General Orthogonal Polynomials, International Series in Numerical Mathematics, Birkhäuser, Basel, 1980.

[14] C. Brezinski: Padé approximants: old and new, Jahrbuch Überblicke Mathematik, 1983, Pages 37-63.

[15] V.I. Buslaev, A.A. Gončar and S.P. Suetin: On convergence of subsequences of the nth row of the Padé table, Mathematical Sbornik, Volume 120, 1983, Page 162.

[16] C. Chui and M.N. Parnes: Approximation by overconvergence of power series, Journal of Mathematical Analysis and Applications, Volume 36, 1971, Pages 693-696.

[17] G. Costakis: Some remarks on universal functions and Taylor series, Math. Proc. Camb. Philos. Soc., Volume 128, 2000, Pages 157-175.

[18] N.J. Daras: Padé and Padé - type approximants for 2π-periodic Lp functions, Acta Applicandae Mathematicae, Volume 62, Issue 3, July 2000, Pages 245-343.

[19] M.G. De Bruin: Some classes of Padé tables whose upper halves are normal, Nieuw Archief voor Wiskunde, Volume 25, 1977, Pages 148-160.

[20] M.G. De Bruin and H. Van Rossum eds: Padé Approximation and its Applications, Amsterdam 1980, Lecture Notes in Mathematics 888, Springer Verlag, Heidelberg, 1981.

[21] R. De Montessus de Ballorre: Sur les fractions continues algébriques, Bull. Soc. Math. France, Volume 30, 1902, Pages 28-36.
[22] A. Draux: Polynômes Orthogonaux Formels, Applications, Lectures Notes in Mathematics 974, Springer-Verlag, Heidelberg, 1983.

[23] A. Edrei: The Padé table of functions having a finite number of essential singularities, Pacific J. Math., Volume 56, 1975, Pages 429-453.

[24] A. Edrei: The Padé tables of entire functions, Journal of Approximation Theory, Volume 28, 1980, Pages 54-82.

[25] W. Gehlen W. Luh and J. Müller: On the existence of O-universal functions, Complex Variables, Volume 41, 2000, Pages 81-90.

[26] J. Gilewicz: Approximants de Padé, Lectures Notes in Mathematics 667, Springer-Verlag, Heidelberg, 1979.

[27] K.-G. Grosse-Erdmann: Holomorphe Monster und universelle Funktionen, Mitt. Math. Sem. Giessen, Volume 176, 1987, iv+84 Pages.

[28] K.-G. Grosse-Erdmann: Universal families and hypercyclic operators, Bull. Amer. Math. Soc., Volume 36, 1999, Pages 345-381.

[29] P. Henrici: Applied and Computational Complex Analysis, John Wiley, New York, Vol. 11, 1974 & Vol. 2, 1976.

[30] W. B. Jones and W. J. Thron: Continued Fractions. Analytic Theory and Applications, Addison Wesley, Reading, Mass., Vol. 11, 1980.

[31] J.-P. Kahane: Baire’s Category theorem and trigonometric series, J. Anal. Math., Volume 80, 2000, Pages 143-182.

[32] Ch. Kariofillis, Ch. Konstandaki and V. Nestoridis: Smooth universal Taylor series, Monatsh. Math., Volume 147, Issue 3, 2006, Pages 249-257.

[33] Ch. Kariofillis and V. Nestoridis: Universal Taylor series in simply connected domains, Computational Methods and Function Theory, Volume 6, Issue 2, 2006, Pages 437-446.

[34] D.S. Lubinsky: Rogers-Ramanujan and the Baker-Gammel-Wills conjecture, Annals of Mathematics, Volume 157, 2003, Pages 847-889.

[35] W. Luh: Approximation analytischer Funktionen durch überkonvergente Potenzreihen und deren Matrix-Transformierten, Mitt. Math. Sem. Giessen, Volume 88, 1970, Pages 1-56.

[36] W. Luh: Universal approximation properties of overconvergent power series on open sets, Analysis, Volume 6, 1986, Pages 191-207.

[37] A. Melas and V. Nestoridis: Universality of Taylor series as a generic property of holomorphic functions, Adv. Math., Volume 157, 2001, Pages 138-176.

[38] A. Melas and V. Nestoridis: On various types of universal Taylor series, Complex Variables Theory, Volume 44, Issue 3, 2001, Pages 245-258.
[39] J. Müller, V. Vlachou and A. Yavrian: Universal overconvergence and ostrowski-gaps, Bull. London Math. Soc., Volume 38, Issue 4, 2006, Pages 597-606.

[40] V. Nestoridis: An extension of the notion of universal Taylor series, in N. Papamichael, S. Ruscheweyh, E. B. Saff (eds) Proceedings of the 3rd CMFT Conference on Computational Methods and Function Theory, 1997, Nicosia, Cyprus, October 13-17, 1997, World Scientific Ser. Approx. Decompos. 11(1999), Page 421-430.

[41] V. Nestoridis: Universal Taylor series, Annales de l’Institut Fourier, Volume 46, 1996, Pages 1293-1306.

[42] V. Nestoridis: A strong notion of universal Taylor series, J. London Math. Soc., Volume 68, Issue 2, 2003, Pages 712-724.

[43] O. Njåstad: Unique solvability of an extended Hamburger moment problem, Journal of Mathematical Analysis and Applications, Volume 124, 1987, Pages 502-519.

[44] O. Njåstad: An extended Hamburger moment problem, Proc. Edinburg Math. Soc., Volume 28, 1995, Pages 167-183.

[45] J. Nuttall: Convergence of Padé approximants of meromorphic functions, Journal of Mathematical Analysis and Applications, Volume 31, 1970, Pages 147-153.

[46] O. Perron: Die Lehre von den Kettenbrüchen, Chelsea, New York, 1957.

[47] C. Pommerenke: Padé approximants and convergence in capacity, Journal of Mathematical Analysis and Applications, Volume 41, 1973, Pages 775-780.

[48] E.B. Saff and R.S. Varga eds: Padé and Rational Approximation, Academic Press, New York, 1977.

[49] A.I. Seleznev: On universal power series (Russian), Mat.Sbornik (N.S.), Volume 28, 1951, Pages 453-460.

[50] H. Van Rossum: A Theory of Orthogonal Polynomials based on the Padé Table, Thesis, University of Utrecht, Van Gorcum, Assen, 1983.

[51] H.S. Wall: The Analytic Theory of Continued Fractions, Van Nostrand, New York, 1948.

[52] H. Wallin: On the convergence theory of Padé approximants, in “Linear operators and approximation” (Proceedings of the Conference Oberwolfach, 1971), International Series Numerical Mathematics, Birkhäuser, Basel, Volume 20, 1972, Pages 461-469.

[53] P. Wimp: Sequences Transformations and Their Applications, Academic Press, New York, 1981.
[54] L. Wuytack ed: Padé Approximation and its Applications, Lecture Notes in Mathematics 765, Springer Verlag, Heidelberg, 1979.

[55] P. Wynn: A general system of orthogonal polynomials, Quart. J. Math. Oxford, Volume 18, Issue 2, 1967, Pages 81-96.

[56] J. Zinn-Justin: Convergence of Padé approximants in the general case, Colloquium on Advanced Computing Methods in Theoretical Physics, A. Visconti (ed.), C.N.R.S., Marseille, 1971, Pages 88-102.