Surface waves at the interface between two viscous fluids

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The Surface Green Function Matching analysis (SGFM) is used to study the normal modes of the interface oscillations between two non-mixed fluids by considering the difference in their densities and viscosities. The limiting case of viscous-inviscid system is used for comparison. The role of the viscosity and the density ratios on the momentum exchange and on the polarization of the surface modes is analyzed.

68.10.-m; 68.10.Cr; 68.10.Et

I. INTRODUCTION

The theory of surface waves in fluids is usually treated using the Orr-Sommerfeld equation obtained from potential method [1]. This procedure is useful to find the characteristics of the wave, such as dispersion relation and damping but becomes rather complicated when some other features are needed such as polarization and density of modes.

On the other hand, the inclusion of the viscosities of all of the media make difficult to understand the physics of the interface. Broadly speaking, the dual effects of viscosity is well known [1]: to dissipate the energy of any disturbance, but also it has the more complicated effect of diffusing momentum. At present, the theory for viscous cases is not nearly as complete or general as for inviscid cases and it provides only a partial understanding of the role of viscosity in such systems.

A suitable formalism for including all the viscosities with great ease in non-homogeneous systems studying the response function has been developed elsewhere [2,3]. This formalism, the method of Surface Green Function Matching (SGFM), has been extensively used to study various inhomogeneous problems involving surface waves at solid surfaces, both free solid surface (interface between vacuum and solid) and solid-solid interfaces [3]. It has also been used in interface involving fluids [2,4] as is the case of solid-fluid interface and even fluid-fluid interface (this last case analyzed to give an unified treatment of waves in solids and fluids which seem to be apparently unconnected problems). As far as we know, there are no previous works where the SGFM have been applied to the hydrodynamics problems as these authors suggested.

The aim of this paper is to apply the SGFM to the study of the physical characteristics (dispersion relation, damping and polarization) on the interface normal modes of two fluids at rest giving insight of the mechanisms of momentum exchange through the interface for different ratios between the viscosity and density of the two media.

In the next section a brief outline of the main points of the SGFM is given for the fluid-fluid system at rest, highlighting the considerations made in the solution of the problem. Section III is devoted to the physical analysis of the polarization of the modes and the momentum exchange across the interface. In section IV it is carried out a numerical evaluation considering the physical interpretation of the terms and the results for pair of fluids which are analyzed as illustration. Finally some conclusions are outlined.

II. SGFM FOR TWO VISCOUS FLUIDS INCLUDING SURFACE EFFECTS

The formal development of the SGFM method has been fully explained elsewhere [2,3] and in particular the treatment of matching with discontinuities [3], suitable for the case of two non mixed fluids where the interface has special effects not seen in the liquid bulk. Mathematical and formal details can be found elsewhere [2] and need not

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be repeated here. It is only necessary to add that in fluid-fluid interfaces it is better to work with the velocity of the fluid particle in agreement with the Navier-Stokes equation, instead of the fluid particle deformation, suitable when solids are present.

Consider a system formed by a fluid \( M_1 \) for \( z < 0 \) and a fluid \( M_2 \) for \( z > 0 \), both of them at rest. It has a planar interface at \( z = 0 \). Analysing first each bulk media individually to prepare its description in a suitable way for the eventual matching at the interface, the coordinate system will be choosen considering the planes \( z = \text{const} \) as those of interest. The notation will be for coordinates \( r = (\rho, z) \), \( k = (\kappa, q) \) where \( \rho \) and \( \kappa \) are 2D vectors.

As explained in [5], the SGFM start with the knowledge of the Green function G.F of the excitation studied in each bulk material constituent. Then, it is needed to analyze the physical model for the excitation to perform later matching at the interface.

Now, to know the G.F of each bulk media, the 3D differential equations of hydrodynamics are the starting point. The fluids are usually treated as incompressible and described with the Navier-Stokes equation. However, as explained in [3], it proves convenient here to give the theory for compressible fluids, even if compressibility effects are ultimately neglected. Then, the equation of mass for isentropic processes, and the momentum conservation equation that govern the fluid motion are linearized by neglecting all nonlinear terms in disturbance quantities. They may be written, respectively, as

\[
\frac{1}{c^2} \frac{\partial}{\partial t} p(r, t) + \rho \nabla \cdot V(r, t) = 0
\]

(1)

\[
\rho \frac{\partial}{\partial t} V(r, t) = -\nabla p(r, t) + \eta \nabla^2 V(r, t) + \left( \eta + \frac{\eta'}{3} \right) \nabla \cdot V(r, t)
\]

(2)

where \( c, p, \rho, \eta \) and \( \eta' \) are the velocity of sound, dynamical pressure, equilibrium density, shear and bulk viscosities respectively, all of them considered as constants in each medium. \( V(r, t) \) is the velocity of the fluid. We neglected the external forces and supposed that the perturbation is small enough to neglect the convective term for pressure in \( \nabla \cdot V(r, t) \).

All space and time dependent quantities will be Fourier transformed according to \( \exp[i(\kappa \cdot r - \omega t)] \) where \( \omega \) is a frequency. Then, for surface wave propagation, the amplitudes are functions of \( (\kappa, \omega) \) on one hand and of \( z \) on the other. This \( z \) dependence is due to the fact that there is no spatial invariance in this direction and the Fourier transform can not be accomplished. Green functions, including the ones for the bulk material constituents, are then conveniently expressed as \( G(\kappa, \omega; z, z') \) or, simply, as \( G(z, z') \), with \( (\kappa, \omega) \) understood everywhere.

Time Fourier transform will be implied now on. From eq. (2) it is obtained \( p(r, \omega) = (\rho c^2/\omega) \nabla \cdot V(r, \omega) \), which putted in eq. (1) gives rise to

\[
i \rho \omega V_i(r, \omega) + (\bar{\Gamma} - \eta) \frac{\partial}{\partial x_i} \nabla \cdot V(r, \omega) + \eta \nabla^2 V_i(r, \omega) = 0
\]

(3)

with \( i = x, y, z \) and

\[
\bar{\Gamma} = -\frac{\rho c^2}{i\omega} + \left( \eta + \frac{4}{3} \eta' \right)
\]

(4)

as the system of equations which couples the velocity components. This system must be solved as a whole as it can not be decoupled in the general case.

The actual \( G(z, z') \) of each bulk media considered separately as infinity can be obtained in different ways but using, for instance, the Fourier transform 3D, it yields for the G.F [6]:

\[
G(\kappa, \omega) = \frac{1}{i \rho \omega - \eta k^2} \left[ I + \frac{(\bar{\Gamma} - \eta) k k}{i \rho \omega - \bar{\Gamma} k^2} \right]
\]

(5)

where \( I \) is the unit matrix and \( k k \) is a diadic product of the wave vector.

Its poles

\[
q_1 = \left( \frac{i \rho \omega}{\bar{\Gamma} - \kappa^2} \right)^{1/2} \quad q_2 = \left( \frac{i \rho \omega}{\eta - \kappa^2} \right)^{1/2}
\]

(6)

describe the transverse and longitudinal modes of the infinite medium. In [6] the incompressible fluid can be considered taking \( (\bar{\Gamma} \to \infty) \) and the proper limit is achieved.
There is no physical reason for the preference of a particular direction in the $xy$-plane. This spatial symmetry of the system allows us to define, for instance, $k = (0, \kappa, q)$ without losing generality but getting simplification of the calculations.

Note that $q_1 \rightarrow i|\kappa|$ if the compressibility is neglected, see eq. (4), given rise to a vanishing longitudinal mode. So, the $q_1$ pole describes the longitudinal mode due to the compressibility of the media.

Let $G_S$ be the Green function (G.F) of the surface system just defined and $G_S$ its surface projection. Let $G_{S}^{-1}$ be the reciprocal of $G_S$ in the two-dimensional $\rho$ or $\kappa$ space. This is the central object in the SGFM analysis. In particular, knowing $G_{S}^{-1}$ it is possible to find the surface mode dispersion relation (SMDR) and the density of modes of the surface system [2]. It is important to stress that the secular equation for the SMDR, namely

$$\det G_{S}^{-1} = 0$$

expresses the continuity of the velocity and the stress components transmitted across $z = 0$. This is where the physics of the surface effects comes into the picture. These effects introduce changes in the stress components transmitted across the interface and are ultimately measured by some surface tensor $m_S$ whose physical meaning is that $m_S$, acting on the velocity field $V$, yields the extra forces per unit area transmitted across the interface.

Let us call $G_{SO}^{-1}$ to $G_{S}^{-1}$ in the absence of such surface effects, then one finds [3]

$$G_{S}^{-1} = G_{SO}^{-1} + m_S$$

Thus the problem is to find $m_S$ for the surface effects one wishes to study. It will be included in this case only the surface tension $\gamma$ according to Laplace’s Law. It can be deduced [3] that

$$m_S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{\gamma k^2}{i \omega} \end{bmatrix}$$

There is a little difference between the former expression and the expression obtained in [2] according to the fact that here the velocity of the fluid particle is considered instead the fluid particle deformation.

Then, eq. (5) expresses the continuity of the velocities and the stress components transmitted across the interface at $z = 0$. Knowing $G_{S}^{-1}$ one can find the dispersion relation of the surface modes (SMDR) through the secular equation (6)

**III. PHYSICS AND POLARIZATION OF THE SURFACE MODES**

The construction of $G_{SO}^{-1}$ is explained in [2]. The result, after adding [4], is

$$G_{S}^{-1} = \begin{bmatrix} \rho_1 \omega q_{11} + \eta_1 q_{t1} & \eta_2 q_{t2} & 0 \\ 0 & 0 & \|G_S\| \end{bmatrix}$$

where $G_S^{-1}$ is a $2 \times 2$ matrix and $0$ is the null vector $1 \times 2$. $G_S^{-1}$ has components

$$
\begin{align*}
(g_{S}^{-1})_{11} &= \frac{\rho_1 \omega q_{11}}{\kappa^2 + q_{11}^2} + \frac{\rho_2 \omega q_{12}}{\kappa^2 + q_{12}^2} \\
(g_{S}^{-1})_{22} &= \frac{\rho_1 \omega q_{12}}{\kappa^2 + q_{11}^2} + \frac{\rho_2 \omega q_{22}}{\kappa^2 + q_{22}^2} - \frac{\gamma k^2}{i \omega} \\
(g_{S}^{-1})_{12} &= -(g_{S}^{-1})_{21} = \left( \frac{\rho_1 \omega \kappa}{\kappa^2 + q_{11} q_{11}} + 2i \kappa \eta_1 \right) - \left( \frac{\rho_2 \omega \kappa}{\kappa^2 + q_{12} q_{22}} + 2i \kappa \eta_2 \right)
\end{align*}
$$

We shall refer to the modes as sagittal or S polarized with $(0, V_y, V_z)$, transverse tangent or TT$(V_T, 0, 0)$, longitudinal or L$(0, V_y, 0)$ and transverse normal or TN$(0, 0, V_z)$ modes, according to the component of the velocity they have.

Now, on using (10) in (6) the factorisation of the $(G_S^{-1})_{11}$ matrix element yields a TT mode which does not interact with the others, whose dispersion relation is

$$\eta_1 q_{t1} + \eta_2 q_{t2} = 0$$

and has $x$-axis polarization.
It is easily seen according to (11) that the TT mode has no solution but as stressed in (11), it does contribute to the density of modes and therefore plays a non trivial role in the physical properties of the interface. This mode exists but it is not a stationary one if there is other surface effects considered (13).

The rest of (11) yields the secular equation

$$\det g_s^{-1} = 0$$

(15)

It gives a sagittal mode with polarization S(0, V_y, V_z) and surface tension included. It will be analyzed in the following.

The factor \((g_s^{-1})_{11}\), see eq. (11), represents the surface movement component in y direction due to compressibility of the media while \((g_s^{-1})_{22}\) is a z direction surface movement. The factor \((g_s^{-1})_{12}\) represents a coupling between y and z movements giving rise to an S polarization mode. It means that the surface has both horizontal and vertical movements. In other words, the surface particles move in a kind of circular orbits depending of its phase difference.

On the other hand, there are no important velocities in our system, then compressibility can be neglected as described in (1) and we will discuss whether the S polarization remains or not. Putting \(q = \gamma\) movements in (10), the normal component movement still remains because of the densities of the fluids. Hence, in the case of the \(z\) axis movement both the viscosities and densities are important for the exchange of momentum.

Furthermore, the viscosity is the main cause of momentum exchange between the two media through the surface on the \(y\)-direction movement. Note that the longitudinal component movement disappears according to \((g_s^{-1})_{11} \to 0\) when \(\eta_1\) and \(\eta_2\) are neglected. On the other hand, in (13) the transverse normal movement described by \((g_s^{-1})_{22}\) exists because of the densities and viscosities of the media, (see eq. (17)). When the viscosities are neglected as in (11), the normal component movement still remains because of the densities of the fluids. Hence, in the case of the \(z\)-axis movement both the viscosities and densities are important for the exchange of momentum.
These results are in agreement to the fact that when the interface particle moves according to the longitudinal mode it remains on the plane $z = 0$ and the viscosities are the only way for the two media to interact, but when the interface particle moves according to the transverse normal mode it goes into each medium sometimes at $z > 0$ and other at $z < 0$ and then the inertial effects of the media become important according to their densities.

Expressions (15)-(18) also recover the Kelvin equation for an ideal fluid with free surface, (see references in [4]). Neglecting the viscosities, and setting $\rho_1 = 0$ it is obtained

$$\mathbf{g}_S^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & \rho_2 \omega^2 - \gamma \kappa^2 |\kappa| \end{bmatrix}$$ \tag{21}

It can be seen that our formalism not only recovers the expression for the Kelvin equation, but also recovers the $z$-polarization of that mode.

After this analysis one can return back to the problem for both viscous fluids. From (15)-(18) it is obtained the dispersion relation (22) becomes

$$\omega^2 [(\rho_1 + \rho_2)(\rho_1 q_{t2} + \rho_2 q_{t1}) - |\kappa|(\rho_1 - \rho_2)^2] + \gamma \kappa^2 |\kappa| [\rho_1(|\kappa| - q_{t2}) + \rho_2(|\kappa| - q_{t1})] +$$

$$+ 4 \kappa^2 |\kappa|(\eta_2 - \eta_1)^2(|\kappa| - q_{t1})(|\kappa| - q_{t2}) + 4\kappa \omega \eta_2 (\rho_1 |\kappa| - \rho_2 |\kappa|) = 0$$ \tag{22}

with the following new definition of $q_1 = (\kappa^2 - i \rho \omega / \eta)^{1/2}$. This expression, which we recall corresponds to two viscous non mixed incompressible fluids, can also be accomplished applying the potential method, although using that formalism it is rather difficult to obtain the polarization of the modes.

This is the equation to be used to study the modes if one includes both viscosities and surface tension effects for incompressible fluids. Expression (22) was reported in [4] to study the surface waves at the interface between a solid and a fluid. They neglected the surface tension. One of the aim of this paper is to compare this theory with the theory which just take into account only one of the viscosities. From expression (13) it is not difficult to achieve the SMDR for the viscous-inviscid fluid interface

$$- \omega^2 \rho_2 (\rho_1 + \rho_2) + \gamma \kappa^2 \rho_2 + 4 \kappa^2 \rho_2 |\kappa| \eta_2 (|\kappa| - q_{t2}) - 4 i \rho \omega \kappa^2 \eta_2 = 0$$ \tag{23}

which reduces to equation (2.5) of [4] when $\gamma = 0$ and will be evaluated in the next section along with (22) for the viscous case.

### IV. RESULTS OF THE NUMERICAL EVALUATION

In order to make a numerical study the following quantities of length and time for nondimensionalization will be taken:

$$\text{time by} \quad T_O = \frac{\eta_3}{\rho_2 \gamma^2}$$

$$\text{length by} \quad L_O = \frac{\eta_2}{\rho_2 \gamma}$$ \tag{24}

The dispersion relation (22) becomes

$$\omega^2 [(1 + Q)(q_{t1} + Q q_{t2}) - |\kappa|(1 - Q)^2] + \kappa^2 |\kappa| |\kappa|(1 + Q)q_{t1} - Q q_{t2} +$$

$$+ 4 \kappa^2 |\kappa|(1 - N)^2(|\kappa| - q_{t1})(|\kappa| - q_{t2}) + 4 \kappa \omega (1 - N) |\kappa|(1 - Q)q_{t1} - Q q_{t2} = 0$$ \tag{25}

for viscous fluids and eq. (23) gives rise to

$$- \omega^2 (1 + Q) + \kappa^2 |\kappa| + 4 \kappa^2 |\kappa|(|\kappa| - q_{t2}) - 4 i \omega \kappa^2 = 0$$ \tag{26}

for viscous-inviscid case, where $Q = \rho_1 / \rho_2$, $N = \eta_1 / \eta_2$ and

$$q_{t1} = \left( \kappa^2 - i \omega \frac{Q}{N} \right)^{1/2}$$ \tag{27}

$$q_{t2} = \left( \kappa^2 - i \omega \right)^{1/2}$$ \tag{28}
Then the characteristics of the system will be studied by its SMDR with real values of the frequency \( \omega \). Let us allow \( \kappa \) to be complex, its real part is \( 2\pi \) times the inverse of the wavelength and the imaginary part is the distance damping coefficient \( \beta \) related with the viscosities of the media. The dimensionless parameters are \( \kappa_o = 2\pi/L_o \) and \( \omega_o = 2\pi/T_o \).

Fig. 3 shows the SMDR for \( Q = 0.8 \). There is one mode which decreases its wavelength \( \lambda \) and increases its distance damping coefficient \( \beta \) with increasing frequency at a fixed value of the parameter \( N \). It is also shown that when the viscosity ratio \( N \) is increased the wavelength lightly decreases at any frequency. The curves split bigger at higher frequencies. On the other hand \( \beta \) increases with increasing \( N \). Also it was plotted the curves obtained with \( N = 0 \) from eq. (26) which means zero viscosity of the medium \( M_1 \). It can be seen that the theory which includes all the viscosities predicts small \( \lambda \) and bigger \( \beta \) for a fixed \( \omega \) with respect to the \( N = 0 \) case.

Fig. 4 shows the dependence of \( \kappa \) and \( \beta \) with respect to the variation of the density ratio \( Q \) at a fixed value of the frequency and viscosity ratio. It can be seen that \( \lambda \) decreases when \( Q \) increases at a fixed \( N \). This was deduced by Taylor in his study of the ripple formation on an infinitely thick viscous circular jet but neglecting the air viscosity. References are given in [11]. We now prove that this is also true when both viscosities are considered. Also \( \lambda \) decreases at a fixed \( Q \) when the viscosity ratio \( N \) takes higher values. So, the effect of the viscosity of medium \( M_1 \) reinforces the effect produced by the density and it can be stated that the smaller wavelength will be obtained when \( Q \) and \( N \) are both bigger. It is also plotted the curve with \( N = 0 \) corresponding to the viscous-inviscid case. It can be seen that the wavelength is always smaller in the case \( N \neq 0 \) (viscous-viscous case). The curves split bigger as \( Q \) increases and \( \beta \) grows rapidly at low values of \( Q \) for a fixed value of \( N \) and tends to saturation for higher values of \( Q \). This small variation of \( \beta \) with the variation of \( Q \) even at a fixed value of the viscosity ratio \( N \) reinforces the idea of the density as another mechanism of momentum exchange between the two media through the interface. It not only produces smaller wavelengths, but also produces lightly bigger distance damping coefficients \( \beta \).

On the other hand, the distance damping coefficient \( \beta \) also increases at higher values of the viscosity ratio \( N \) at a fixed \( Q \). It was also plotted the curves at \( N = 0 \). It is interesting to note that the theory of viscous-inviscid case predicts a small decrease of the distance damping coefficient with increasing density ratio \( Q \). This is in accordance to the fact that setting \( N = 0 \) means to neglect the momentum exchange through the interface by the viscosity and raising \( Q \) represents to increase the dynamic properties of the surface given rise to a bigger distance for the wave to travel before vanishing.

Fig. 5 shows the variation of \( \lambda \) and \( \beta \) with respect to the viscosity ratio for a fixed value of \( \omega \) at three values of \( Q \). It shows that as the viscosity ratio increases, the wavelength reduces rapidly first and tends to a limiting value for \( N \geq 1 \). The curves start in the value of \( \lambda \) corresponding to the viscous-inviscid case. Also, for a fixed value of \( N \) the wavelength decreases as \( Q \) increases, in correspondence with Fig. 2. For the coefficient \( \beta \) it is seen that it raises for increasing \( N \).

To illustrate this theory for real fluid combinations there will be used three pairs of fluids: air/water, water/aniline and water/mercury. The parameters of these fluids at room temperature are:

| Element | Density (kg/m³) | Viscosity (mPa s) | Surface Tension (mN/m) |
|---------|----------------|------------------|-----------------------|
| air     | 1.21           | 0.018            | -                     |
| water   | 998            | 0.890            | 71.99                 |
| mercury | 13500          | 1.526            | 485.48                |
| aniline | 1022           | 3.847            | 42.12                 |

Then, for the system air/water it is \( Q = 0.0012 \) and \( N = 0.0202 \). In this case the SMDR is plotted in Fig. 3. It can be seen that there is no difference of the wavelength reported by viscous-viscous and viscous-inviscid cases due to the small values of the density and viscosity ratios but there is a small increase of \( \beta \) for all frequencies when the air viscosity is considered.

However, the operating conditions in many gas turbine combustors and liquid-propellant rocket engines are such that the density and viscosity ratios are higher. It could be so also in water-oil emulsions and other problems where the interface between two fluids plays an important role. Then, the SMDR for water/aniline (\( Q = 0.977, N = 0.231 \)) and water/mercury (\( Q = 0.074, N = 0.583 \)) systems were also plotted. In the first case (water/aniline) the densities are very similar but the viscosity of the aniline is much bigger than the water viscosity. In the case of water/mercury the viscosities are near one half one another, but the density of the mercury is much bigger the density of the water. The SMDR has been plotted in Fig. 6 and 7. Note that in both cases the wavelength decreases and the distance damping coefficient increases in a visible way when the viscosity of the medium \( M_1 \) (water in both cases) is considered. The difference is bigger at high values of the frequency where the viscosity effects become important. Then, one can conclude that the inclusion of the viscosities of all media produces a substantial decrease of the wavelength if the viscosity ratio are big enough. It gives rise also to a bigger distance damping coefficient for the wave.
V. CONCLUSIONS

In the present paper the close relationship between the properties of low amplitude surface waves propagation with the viscosity and density ratios, in a system of two non-mixed incompressible fluids at rest has been set out. The SGFM method was used to accomplish the dispersion relation and the full study of wave propagation by varying different parameters of the media.

It was shown that the viscosity is a fundamental parameter for the coupling of different modes. It gives rise to an S polarization mode with $y$ and $z$ components of the movement of the particles on the surface. Also it was seen that the viscosity is the main force in producing momentum exchange in the longitudinal mode, but for the transverse normal mode both the viscosity and the density ratios are important to the momentum exchange.

When considering surface modes, it was shown that only one of them is allowed and its wavelength is smaller when considering the viscosity of both media for fixed values of the density ratio. Also it was seen a characteristic variation of the distance damping coefficient when the viscosity of all media are included. On the other hand the increasing of the density ratio also reduces the wavelength and produces a lightly increase of the distance damping coefficient, then this factor is also important in reducing the wavelength of the surface waves.

In order to see more real situations, three pair of fluids were analyzed and the importance of taking in consideration all the viscosities was shown.

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[1] Landau, L. D. and Lifshitz, E. M., Fluid Mechanics, Butterworth-Heinemann (1987).
[2] García-Moliner, F., Ann. Physique 2, 179 (1977).
[3] Velasco, V. R. and García-Moliner, F., Physica Scripta 20, 111 (1979).
[4] Platero, G., Velasco, V. R. and García-Moliner, F., Physica Scripta 23, 1108 (1981).
[5] García-Moliner, F. and Velasco, V. R., Theory of Single and Multiple Interfaces, World Scientific, Singapore, (1992).
[6] Velasco, V. R. and García-Moliner, F., Surface Sci. 67, 555 (1977).
[7] Lucassen, J. and Lucassen-Reynders, E. H., J. Colloid Interface Sci. 25, 496 (1967).
[8] Lucassen, J., Trans. Faraday Soc. 64, 2220 (1968).
[9] Lucassen-Reynders, E. H. and Lucassen, J., Advan. Colloid Interface Sci. 2, 347 (1969).
[10] Lucassen, J. and van der Tempel, M., Chem. Engin. Sci. 27, 1283 (1972).
[11] Lin, S. P., Lian, Z. W. and Creighton, B. J., J. Fluid Mech. 220, 673 (1990).

FIG. 1. Dispersion relation of the surface mode for $Q = 0.8$. The upper part gives the wavelength $\kappa/\kappa_o$ and the lower part the distance damping coefficient $\beta/\kappa_o$.

FIG. 2. Relation between wavelength and distance coefficient with respect to the density ratio at a fixed frequency for different values of viscosity ratios. The case $N = 0$ is the viscous-inviscid case.

FIG. 3. Relation between wavelength and distance coefficient with respect to the viscosity ratio at a fixed frequency for different values of density ratios.

FIG. 4. Dispersion relation of the surface mode for the air/water system. In the legend it is especificed the $M_1$ as left and $M_2$ as right in the combination $M_1/M_2$, i. e., air/water in this figure.

FIG. 5. Dispersion relation of the surface mode for the water/aniline system.
FIG. 6. Dispersion relation of the surface mode for the water/mercury system.
\[ \frac{\kappa}{\kappa_0} \times 10^{-4} \]

\[ \frac{\beta}{\kappa_0} \times 10^{-5} \]

\[ \omega/\omega_0 \times 10^{-5} \]

- Non-viscous water
- Water/aniline

\[ N = 0.231 \]

\[ Q = 0.977 \]
