The conventional cosmological perturbation theory has been performed under the assumption that we know the whole spatial region of the universe with infinite volume. This is, however, not the case in the actual observations because observable portion of the universe is limited. To give a theoretical prediction to the observable fluctuations, gauge-invariant observables should be composed of the information in our local observable universe with finite volume. From this point of view, we reexamine the primordial non-Gaussianity in single field models, focusing on the bispectrum in the squeezed limit. A conventional prediction states that the bispectrum in this limit is related to the power spectrum through the so-called consistency relation. However, it turns out that, if we adopt a genuine gauge invariant variable which is naturally composed purely of the information in our local universe, the leading term for the bispectrum in the squeezed limit predicted by the consistency relation vanishes.

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I. INTRODUCTION

The measurements of the cosmological fluctuations have now become a crucial tool to probe the history of the early universe. To yield the theoretical prediction of the fluctuations, which are to be compared with the observations, the gauge-invariant perturbation theory is used. It is not, however, widely recognized that the gauge invariance in the whole universe with infinite volume should be distinguished from that in the local universe with finite volume. It is widely known that the gauge-invariant cosmological perturbation can be realized by completely fixing the coordinates. In the case of the gauge invariance in the whole universe, where we request the invariance under normalizable gauge transformations, the conventional gauge fixing conditions are sufficient to uniquely determine the coordinates. By contrast, if we require that the coordinates within our local observable universe should be completely fixed only from the local information contained therein, the conventional gauge conditions are not sufficient. To fix the gauge locally, these conditions must be supplemented with appropriate boundary conditions [1]. These boundary conditions are fixed by requiring the regularity at the spatial infinity in the conventional approach. We should remind that, in the actual observations such as measurements of the Cosmic Microwave Background (CMB), we can observe only our local universe and we cannot consult the regularity at infinity. Therefore, the observable fluctuations should be calculated in the request of the invariance under both the normalizable and non-normalizable gauge transformations. We distinguish gauge-invariant quantities that can be constructed from our local observable universe, referring them as genuine gauge-invariant variables. The observable fluctuations should be such a genuinely gauge-invariant variable.

The importance of the genuine gauge invariance is also highlighted in the context of the infrared (IR) divergence problem. The adiabatic vacuum, which yields the scale-invariant spectrum, is supposed to be a natural vacuum in the inflationary universe. However, once the interaction turns on, it is not manifest whether the adiabatic vacuum is stable or unstable against the IR contributions in loop corrections. It is indeed known that the loop corrections calculated in the conventional perturbation theory yield the logarithmic divergences [2–21]. Our previous works [22, 23] indicate that this divergence is an unphysical artifact in the case of single field models of inflation. We found that the initial quantum states should satisfy certain conditions if we require that the loop correction to the two-point function of a genuine gauge invariant variable should be free from the IR divergence. Our interpretation is that these conditions are the requirements that the initial states are invariant under the residual gauge transformation. Namely, as long as the initial quantum states respect the gauge invariance in our local universe, the IR contribution to the loop correction should be suppressed. Hence, we referred to these conditions on the initial quantum states as the gauge invariance conditions.

In this paper we reexamine the primordial non-Gaussianity in single field models, taking into account the importance of genuine gauge invariance of the selected variables and the gauge invariance conditions on the initial quantum state. A conventional prediction states that the bispectrum in the squeezed limit is related to the amplitude and the spectral index of the power spec-
trum, which is often called the consistency relation. However, it will turn out that the bispectrum of a genuine gauge invariant variable, which is the counterpart of the conventional gauge invariant curvature perturbation, is more suppressed in the squeezed limit compared with the prediction widely accepted as the consistency relation.

This paper is organized as follows. In Sec. II, we review a method to introduce genuinely gauge-invariant variables and the gauge invariance conditions. The basic perturbation equations will be also provided there. In Sec. III, for illustrative purpose, we first briefly summarize the usual argument for the consistency relation, which uses the information about the region outside our observable universe. After that, we re-investigate the bispectrum in the squeezed limit, using a genuinely gauge-invariant variable and imposing the gauge invariance conditions on the initial quantum state. Section IV is devoted to the conclusion.

II. GAUGE-IN Variant Perturbations in Local Universe

Gauge invariance is guaranteed if we compute after complete gauge fixing. However, complete gauge fixing is not so straightforward as mentioned in the introduction. Let’s first impose conventional comoving gauge conditions

\[ \delta \phi = 0, \]  

and

\[ \delta \gamma^i_j = \partial_i \delta \gamma^j_i = 0, \]  

with

\[ h_{ij} = e^{2(\rho + \zeta)} \left[ e^{\delta \gamma} \right]_{ij}, \]

where we denote the scale factor by \( e^\rho \). The time coordinate is then fixed completely by the first condition (2.1). The second conditions (2.2), however, do not fix the spatial coordinates completely if we just consider the local universe and adopt gauge conditions only in this region. Namely, there are still residual gauge degrees of freedom as long as only the local metric is concerned. If we could use the information stored in the region outside our observable universe to fix the coordinates, these residual gauge degrees of freedom would disappear. In spite of this, once we accept the contamination from the outside of our observable universe to define gauge invariant variables, such quantities cannot correspond to actual observables in a strict sense. Furthermore, such contamination from outside of our observable universe may cause artificial infrared divergences. We therefore need to fix the gauge completely using only the quantities residing in our observable region, but this is technically complicated.

In our previous paper we pursued an alternative way to introduce genuine gauge-invariant variables, focusing on the scalar under three-dimensional diffeomorphism, e.g. three-curvature on the time-constant hypersurface. Such a quantity should be a genuine gauge invariant if the spatial position is specified in a coordinate independent manner. In order to specify a point in a coordinate independent manner, we used the geodesic normal coordinates spun from a representative observer. There remains the ambiguity in choosing the position of this representative observer and his/her frame. This dependence, however, should disappear in the evaluation of \( n \)-point correlation functions as long as the initial quantum state respects the translational and rotational symmetries.

We computed the two-point correlation function of thus defined genuine gauge-invariant operator at one loop level, focusing on the effect of IR modes. Although the IR contribution to the two-point function is divergent in general if the spectrum of the linear curvature perturbation is scale invariant or red, we found that, if we choose the initial quantum states appropriately, this divergences disappear. We derived the conditions for the absence of IR divergences and confirmed that such quantum states exist at least up to the second order in the slow roll expansion.

Now it is profitable to ask the question how to interpret the divergences for generic quantum states. Since Fourier modes of \( \zeta \) basically become constant in the IR limit, they can be almost absorbed by the overall scale transformation of the spatial coordinates, which is a part of the residual gauge degrees of freedom mentioned above. Although it is just a gauge, the fluctuation of this mode causes infinitely large variance of the two-point function in conventional computations. Hence, it was speculated that this divergence occurs because generic initial quantum states do not respect the expected invariance under these non-normalizable gauge transformations. Taking these in mind, we refer to the conditions necessary to avoid IR divergences in \( n \)-point functions as the gauge invariance conditions. In this section, we briefly review these gauge invariance conditions derived in our previous works [22, 23].

A. Gauge-invariant operator

The construction of genuine gauge-invariant quantities can be achieved simply by making use of scalar quantities under three-dimensional spatial diffeomorphism. In our previous works [22, 23], we picked up the scalar curvature \( sR \) as such a scalar
quantity. Although scalar quantities also vary under the gauge transformation because of the change of their arguments $x^i$, this gauge ambiguity does not appear in the $n$-point functions of these quantities if the arguments are specified in a gauge-invariant manner. In order to specify the arguments of the $n$-point functions in a gauge invariant manner, we measure distances from an arbitrary reference point to the $n$ vertices by means of the geodesic distance obtained by solving the spatial three-dimensional geodesic equation:

$$\frac{d^2 x^i}{d\lambda^2} + \Gamma^i_{jk} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda} = 0,$$

(2.4)

where $\Gamma^i_{jk}$ is the Christoffel symbol with respect to the three dimensional spatial metric on a constant time hypersurface and $\lambda$ is the affine parameter. We consider the three-dimensional geodesics whose affine parameter ranges from $\lambda = 0$ to 1 with the initial “velocity” given by

$$\frac{dx^i(X, \lambda)}{d\lambda} \bigg|_{\lambda=0} = e^{-\zeta(\lambda=0)} \left[e^{-\delta \gamma(\lambda=0)/2}\right]^j X^j.$$

(2.5)

We identify a point in the geodesic normal coordinates $X^i$ with the end point of the geodesic, $x^i(X, \lambda = 1)$.

During inflation, the scales relevant to the current observations go far outside of the inflationary horizon, which leads to the natural assumption that the observable scale is much larger than the horizon scale $1/H$. In Sec. [III] we will evaluate the fluctuation with the modes $1/L_{obs} \ll k \ll e^{\ell} H$, where $L_{obs}$ denotes the observable scale in the comoving coordinates. In this section, however, it is sufficient to restrict the expression for $x^i$ to the case with $1/L_{obs} \gg k$, as we are interested in the IR divergence. In this case $\Gamma^i_{jk}$ is suppressed since it contains a spatial differentiation. Then, the geodesic normal coordinates $X^i$ are approximately related to the global coordinates $x^i$ as

$$x^i(X) =: X^i + \delta x^i(X) \simeq e^{-\zeta} \left[e^{-\delta \gamma/2}\right]^j X^j.$$

(2.6)

We use this simple expression in this section, and we defer its generalization to the case with $1/L_{obs} \ll k$ to Sec. [III B].

Using the geodesic normal coordinates $X^i$, a genuine gauge invariant variable can be constructed as

$$\delta R(X) := \delta R(t, x^i(X)) = \sum_{n=0}^{\infty} \delta x^{i_1} \cdots \delta x^{i_n} \frac{\partial}{\partial t} \cdots \frac{\partial}{\partial x^i} \delta R(t, x^i)|_{x^i = X^i} ,$$

(2.7)

where $t$ denotes the cosmological time and we introduced the abbreviated notation $X := \{ t, X \}$.

### B. Solving non-linear perturbation

In this paper we consider a single scalar field with the canonical kinetic term, whose action takes the form

$$S = \frac{M_{pl}^2}{2} \int \sqrt{-g} \left[ R - g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 2V(\phi) \right] d^4 x ,$$

(2.8)

where $M_{pl}$ is the Planck mass and the scalar field $\phi$ was rescaled as $\phi \rightarrow \phi/M_{pl}$ to be dimensionless. We use the ADM formalism, following Ref. [24]. The metric in the ADM form is given by

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt) .$$

(2.9)

The above action is then rewritten as

$$S = \frac{M_{pl}^2}{2} \int \sqrt{h} \left[ N^4 R - 2N V(\phi) + \frac{1}{N} (E_{ij} E^{ij} - E^2) + \frac{1}{N} (\partial_i \phi - N^i \partial_\phi)^2 - N h^{ij} \partial_i \phi \partial_j \phi \right] d^4 x ,$$

(2.10)

where $^4 R$ is the three-dimensional scalar curvature and $E_{ij}$ and $E$ are defined by

$$E_{ij} = \frac{1}{2} (\partial_i h_{ij} - D_i N_j - D_j N_i) , \quad E = h^{ij} E_{ij} .$$

(2.11)

Here $D_i$ denotes the three-dimensional covariant derivative and the spatial indices are raised and lowered by $h_{ij}$. To calculate the non-linear corrections under the slow-roll approximation, it is convenient to temporally work in the flat gauge:

$$\hat{h}_{ij} = e^{2\rho} \left[ e^{\delta \gamma} \right]_{ij} ,$$

(2.12)
with the transverse-traceless conditions
\[ \delta \gamma^i_{\,i} = 0 = \partial_i \delta \gamma^i_{\,j}, \]
(2.13)
because all the interaction vertices are explicitly suppressed by the slow-roll parameters in this gauge [1, 24]. Here in this section we associate a tilde with the metric perturbations in the flat gauge to discriminate from those in the comoving gauge. The action in this gauge is given by
\[ S^{IR} \approx \frac{M_{pl}^2}{2} \int e^{3\rho} \left[ \tilde{N}^{-1} \left( -6 \dot{\rho}^2 + 4 \dot{\rho} \dot{\tilde{N}}^i + \dot{\tilde{N}}^i \right) + \tilde{N}^{-1} \left( \phi + \varphi - \tilde{N} \partial_i \varphi \right)^2 - 2 \tilde{N} \sum_{m=0} V^{(m)} \frac{m!}{m!} \varphi^m - \tilde{N} \tilde{N}^{ij} \partial_i \varphi \partial_j \varphi \right] dt \, d^3 x, \]
(2.14)
where \( V^{(m)} := d^m V / d\phi^m \). We use the symbol \( \approx^{IR} \) as in [23] to denote an equality which is valid when we neglect the terms that do not participate in the IR divergences. When we write down the Heisenberg operator in terms of the interaction picture field with differentiation.

We also adopt the slow-roll approximation for the background evolution. To characterize the deviation from the exact de Sitter evolution, we use the horizon flow functions:
\[ \varepsilon_0 := \frac{H_i}{H}, \quad \varepsilon_{m+1} := \frac{1}{\varepsilon_m} \frac{d\varepsilon_m}{d\rho} \quad \text{for} \ m \geq 0, \]
(2.15)
where \( H \) is the Hubble parameter and \( H_i \) is its value at the initial time. The horizon flow functions are related to the conventional slow-roll parameters as shown in Ref. [25]. Hereafter, assuming that the horizon flow functions \( \varepsilon_m \) with \( m \geq 1 \) are all small of \( O(\varepsilon) \), we neglect the terms of \( O(\varepsilon^3) \).

As studied in Appendix of Ref. [24], the curvature perturbation in the comoving gauge \( \zeta \) is related to the fluctuation of the dimensionless scalar field (divided by \( M_{pl} \)) in the flat gauge \( \varphi \) as
\[ \zeta^{IR} \approx \zeta_n + \frac{1}{4} \varepsilon_2 \zeta_n^2 + \zeta_n \partial_\rho \zeta_n, \]
(2.16)
where we have introduced \( \zeta_n := -(\dot{\rho} / \dot{\varphi}) \varphi \) as in Ref. [24]. In the calculation of the \( n \)-point functions, we solve the evolution equation (Heisenberg equation) for the operator \( \varphi \) iteratively to express \( \varphi \) in terms of the interaction picture field for \( \varphi \). Variation of the total action with respect to \( \varphi \) yields
\[ e^{-3\rho} \partial_\rho \left[ \frac{e^{3\rho}}{\tilde{N}} (\phi + \varphi) \right] + \tilde{N} \sum_{m=0} V^{(m+1)} \frac{m!}{m!} \varphi^m - \left( \phi + \varphi \right) \frac{1}{\tilde{N}} \partial_i \tilde{N}^i - \tilde{N} e^{-2\rho} \left( e^{\delta \gamma} \right)^{ij} \partial_i \partial_j \varphi \approx 0. \]
(2.17)
Variations with respect to the lapse function and the shift vector, respectively, yield the Hamiltonian constraint:
\[ (\tilde{N}^2 - 1)V + \tilde{N}^2 \sum_{m=1} V^{(m)} \frac{m!}{m!} \varphi^m + 2 \dot{\rho} \partial_\rho \tilde{N}^i + \dot{\varphi} \varphi + \frac{1}{2} \varphi^2 \approx 0, \]
(2.18)
and the momentum constraints:
\[ 2 \dot{\rho} \partial_\rho \tilde{N} - \tilde{N} (\dot{\varphi} \partial_\rho \varphi + \partial_i \varphi \varphi) \approx 0. \]
(2.19)
These constraint equations are solved to give
\[ \delta \tilde{N} \approx \frac{\dot{\varphi}^2}{2 \dot{\rho}^2} \zeta_n + \frac{1}{4 \dot{\rho}^2} \varphi \left( \delta \tilde{N}_1 + \dot{\varphi} \right) \approx -\varepsilon_1 \zeta_n + \frac{\varepsilon_1}{2} \left( \varepsilon_1 + \frac{\varepsilon_2}{2} \right) \zeta_n^2, \]
(2.20)
\[ \partial_i \tilde{N}^i \approx \varepsilon_1 \zeta_n - \frac{1}{2} \varepsilon_1 \varepsilon_2 \zeta_n \zeta_n. \]
(2.21)
Substituting Eqs. (2.20) and (2.21) into Eq. (2.17), the evolution equation of \( \zeta_n \) is recast into a rather compact expression,
\[ \mathcal{L} \zeta_n \approx -2 \varepsilon_1 \zeta_n \frac{e^{-2\rho}}{\dot{\rho}^2} \partial^2 \zeta_n - \varepsilon_1 \varepsilon_2 \zeta_n \partial_\rho \zeta_n - \frac{3}{4} \varepsilon_2 \varepsilon_3 \zeta_n^2 - \delta \zeta^{ij} \frac{e^{-2\rho}}{\dot{\rho}^2} \partial_i \partial_j \zeta_n, \]
(2.22)
where, changing the time coordinate from \( t \) to \( \rho \), the differential operator \( \mathcal{L} \) is given by
\[ \mathcal{L} := \partial_\rho^2 + (3 - \varepsilon_1 + \varepsilon_2) \partial_\rho - \frac{e^{-2\rho}}{\dot{\rho}^2} \partial^2. \]
(2.23)
Expanding $\zeta_n$ and $\delta \gamma_{ij}$ as

$$
\zeta_n = \psi + \zeta_{n,2} + \cdots, \quad \delta \gamma_{ij} = \delta \gamma_{ij,1} + \delta \gamma_{ij,2} + \cdots,
$$

the equation of motion $\mathcal{L}\psi = 0$ is reduced to

$$
\mathcal{L}\zeta_{n,2}^{IR} \approx -\varepsilon_1 \varepsilon_2 \psi \partial_\rho \psi - \frac{3}{4} \varepsilon_2 \varepsilon_3 \psi^2 - 2 \varepsilon_1 \psi \frac{e^{-2\rho}}{\rho^2} \partial^2 \psi - \frac{e^{-2\rho}}{\rho^2} \delta \gamma_{ij} \partial_i \partial_j \psi.
$$

(2.24)

The first-order equation (2.24) gives the mode equation for the interaction picture field and the second-order equation (2.25) is integrated to give

$$
\zeta_{n,2}(X) \approx \left(\frac{\epsilon_1}{2} + \epsilon_2\right) \psi^2 + \epsilon_1 \psi \partial_\rho \psi + \epsilon_1(\varepsilon_1 + \varepsilon_2) \psi \partial_\rho \psi + \delta \zeta_{n,2} + \lambda_2 \psi(\partial_\rho - X^i \partial_{X^i}) \psi + \frac{1}{2} \delta \gamma_{ij} X_i \partial_{X^j} \psi.
$$

(2.26)

Here, $\zeta_{n,2}$ includes the non-local term:

$$
\delta \zeta_{n,2} := -\mathcal{L}^{-1} \left[\frac{3}{4} \varepsilon_2 (2\varepsilon_1 + \varepsilon_3) \psi^2\right].
$$

(2.27)

We kept the terms with second derivatives on the right hand side of Eq. (2.24) because they also necessary to keep the terms in $\zeta_{n,2}$ which have only one interaction picture field $\psi$ with differentiation. It should be emphasized that the homogeneous solutions $\xi \psi^2$ and $\lambda_2 \psi(\partial_\rho - X^i \partial_{X^i}) \psi$, in the sense that they satisfy $\mathcal{L} \cdots \Rightarrow \mathcal{O}(\varepsilon^3)$, can be added to $\zeta_{n,2}$, where the time dependent functions $\xi_2$ and $\lambda_2$ should be $\mathcal{O}(\varepsilon^3)$ and their derivatives should be $\mathcal{O}(\varepsilon^3)$.

C. Gauge invariance conditions of initial state

Now, we address the gauge-invariance conditions to be imposed on the initial state, where initial conditions have two aspects. One is how we choose the positive frequency function for the interaction picture field $\psi$ in solving the mode equation (2.24). The other is how we construct the non-linear Heisenberg field iteratively from the interaction picture field, i.e. how we add homogeneous solutions in Eq. (2.26). The interaction picture field $\psi$ is quantized as

$$
\psi(X) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{ik \cdot X} \psi_k(\rho),
$$

(2.28)

with

$$
\psi_k(\rho) = v_k(\rho) a_k + v_k^*(\rho) a_k^\dagger,
$$

(2.29)

where the creation and annihilation operators satisfy the commutation relation

$$
[a_k, a_p^\dagger] = \delta(3)(k - p).
$$

In Ref. [23], it was shown that the regularity of the one-loop corrections to the two-point function implies that the positive frequency function should satisfy

$$
[(1 + \mu_2) \partial_\rho - (1 + \lambda_2) X^i \partial_{X^i} + \mu_1 - 2\mathcal{L}_k^{-1} \mu_3] v_k(\rho) e^{ik \cdot X} = -(1 + \lambda_2) D_k v_k(\rho) e^{ik \cdot X},
$$

(2.30)

where $D_k$ and $\mathcal{L}_k$ are defined as

$$
D_k := k \cdot \partial_k + 3/2, \quad \mathcal{L}_k := \partial_\rho^2 + (3 - \varepsilon_1 + \varepsilon_2) \partial_\rho + \frac{e^{-2\rho}}{\rho^2} k^2,
$$

(2.31)

and we introduced the time dependent functions $\mu_i$ ($i = 1, 2, 3$) written in terms of the horizon-flow functions as

$$
\mu_1 := \varepsilon_1 + \frac{1}{2} \varepsilon_2 + 2\xi_2, \quad \mu_2 := \varepsilon_1 (1 + \varepsilon_1 + \varepsilon_2) + \lambda_2, \quad \mu_3 := \frac{3}{4} \varepsilon_2 (2\varepsilon_1 + \varepsilon_2).
$$

(2.32)

The condition (2.30) should be satisfied for all wavelengths and restricts the behaviours of not only the IR modes but also the UV modes. This condition can be shown to be consistent with the mode equation (2.24). At the leading order in the slow-roll
approximation, this condition, together with the usual condition on the UV behaviour, completely determines the initial state to the Bunch-Davis vacuum. However, if we extend our argument to the higher order in the slow roll expansion, Eq. (2.30) gives a non-trivial condition on the initial state. It might be surprising that such a gauge-invariance condition (= IR regularity condition) significantly constrains the allowable initial state.

A gauge invariance condition on the coefficients of the homogeneous solutions is again derived from the regularity conditions, although it is not sufficient to completely fix both $\xi_2$ and $\lambda_2$. Imposing this gauge invariance condition, the potentially divergent terms in loop corrections add up to be total derivative terms with respect to $k$ and hence they cease to contribute to the IR divergences. As discussed in Ref. [23], the time-dependent functions $\xi_2$ and $\lambda_2$ should also satisfy another relation to keep the commutation relation between $\zeta_a$ and its conjugate momentum satisfied. Assuming that $\xi_2$ and $\lambda_2$ are appropriately determined, we do not discuss this relation any further because their explicit forms are not important for our current discussion.

### III. PRIMORDIAL NON-GAUSSIANITIES

In this section, we evaluate the primordial non-Gaussianity in the geodesic normal coordinates for the vacuum that satisfies the gauge-invariance conditions described in the preceding section. Our main focus is on the so-called consistency relation that, if we consider the genuine gauge-invariant correlation functions as defined in the preceding section, the leading term in the squeezed limit that arises in the usual computation of the bispectrum vanishes, and the well-known consistency relation does not hold.

#### A. Standard consistency relation

Now, we quickly review the derivation of the usual consistency relations for tree-level bispectra $\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle$ and $\langle \delta \gamma_{k_1}^s \zeta_{k_2} \zeta_{k_3} \rangle$. We are interested in the squeezed limit $k_1 \ll k_2 \simeq k_3$, where we treat long wavelength modes becomes most essential. In this case the scale corresponding to $k_1$ crosses the Hubble horizon much earlier than the others, so that $\zeta_{k_1}$ and $\delta \gamma_{k_1}$ become approximately constant by the time when the scales corresponding to $k_2$ and $k_3$ cross the horizon. Therefore the mode with $k_1$ affects the evolution of the other modes mostly through deformation of the background geometry. The main effect of this deformation is taken into account by shifting the wavenumbers as

$$ k_a \rightarrow \tilde{k}_a = e^{-\frac{1}{2} \zeta_{k_1}} \left( e^{\frac{1}{2} \delta \gamma_{k_1}} \right) k_a = k_a - \left( \zeta_{k_1} + \frac{1}{2} \delta \gamma_{k_1} \right) k_a + \cdots, \quad (3.1) $$

where $a = 2, 3$. This modification influences the fluctuations $\zeta_{k_2}$ and $\zeta_{k_3}$, leading to the replacement

$$ \zeta_{k_a} \rightarrow \langle \text{det } h \rangle_{k_1}^{-1/2} \zeta_{k_a} = \zeta_{k_a} - \zeta_{k_1} \left( k_a \partial_{k_a} + \frac{3}{2} \right) \zeta_{k_a} - \frac{1}{2} k_a \delta \gamma_{k_1} \partial_{k_a} \zeta_{k_a} + \cdots, \quad (3.2) $$

where the factor $\langle \text{det } h \rangle_{k_1}^{-1/2}$ takes care of the change of the volume due to the modes with $k_1$. Using the relation (3.2), we can easily evaluate the bispectrum for $\zeta$ in the squeezed limit as

$$ \langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle \simeq -P_k \left( \partial_{\log} k_2 + \partial_{\log} k_3 + 3 \right) \langle \zeta_{k_2} \zeta_{k_3} \rangle 
\simeq -P_k \delta^{(3)}(k_2 + k_3) \partial_{\log} k_2 + 3 \right) P_{k_2} \simeq -\delta^{(3)} \left( \sum_{a=1}^{3} k_a \right) (n_s - 1) P_k P_{k_2}, \quad (3.3) $$

where $P_k$ is the unnormalized amplitude of the two-point function for $\zeta_k$ defined by

$$ \langle \zeta_{k_1} \zeta_{k_2} \rangle = \delta^{(3)}(k_1 + k_2) P_{k_1}, \quad (3.4) $$

and $n_s$ is the spectral index. Thus, one can see that the three-point function in the squeezed limit is represented by $n_s$ and $P_k$. Equation (3.3) is called the consistency relation in single-field models of inflation.

In a similar way, expanding the gravitational waves as

$$ \delta \gamma_{ij,k} = \sum_{s=\pm} e^s_{ij}(k) \delta \gamma^s_{k}, \quad (3.5) $$

with the polarization tensors defined by

$$ e^{s,i}_{j}(k) = 0 = k^i e^{s}_{j}(k), \quad (3.6) $$
the three-point function for $\delta \gamma \zeta \zeta$ is given by

$$\langle \delta \gamma_{k_1} \zeta k_2 \zeta k_3 \rangle \simeq -\frac{1}{2} \langle \delta \gamma_{k_1} \delta \gamma_{k_2} \rangle (k_2 \cdot e^s - k_1 \cdot \partial k_2 + k_3 \cdot e^s - k_1 \cdot \partial k_3) \langle \zeta_{k_2} \zeta_{k_3} \rangle$$

$$\simeq \frac{4 - n_\Delta}{2} \delta^{(3)} \left( \sum_{a=1}^3 k_a \right) \frac{k_2 \cdot e^s - k_1 \cdot \partial k_2}{k_2^s} P_{k_1}^{(\delta \gamma)} P_{k_2},$$

(3.7)

where $P_{k}^{(\delta \gamma)}$ is the amplitude of the two-point function for the gravitational waves. These three-point functions were first calculated by Maldacena in Ref. [24] and these results were extended to more general single field models in Ref. [26].

B. Gauge-invariant curvature perturbation

In the following, we would like to consider the tree-level three-point function for genuine gauge-invariant variables. To compare the results with the standard computation described in the preceding subsection, it would be better to use $\zeta$ instead of $R$ as a variable, but $\zeta$ itself does not transform as a scalar on a $t$-constant hypersurface. Therefore the curvature perturbation evaluated in the geodesic normal coordinates:

$$\zeta(X) := \zeta(\rho, x^i(X)) = \zeta(X) + \delta x^i \partial_i \zeta \big|_{x^i = X^i} + \cdots$$

(3.8)

is not a genuine gauge invariant variable. However, this quantity $\zeta(X)$ is related to the genuine gauge invariant variable truncated at the required order in calculating the tree-level bi-spectrum:

$$\zeta = \zeta \rho + \delta x^i \partial_i \zeta = -2e^{-2(\rho + \zeta)} \left[ e^{-\delta \gamma} \right]^{ij} \left( 2 \delta \gamma_j + \partial_i \zeta \partial_j \zeta - \delta x^i \partial_i \left( -4e^{-2\rho} \right) \partial^2 \zeta + \cdots \right),$$

(3.9)

as

$$\zeta = -2e^{-2\rho} \left( \partial X \right)^2 \zeta + \partial X \partial X \partial_X \partial_X \zeta + \cdots.$$  

(3.10)

Here we neglected the terms quadratic in gravitational wave perturbation. For the modes with $k \gg 1/L_{\text{obs}}$, using the Fourier decomposition of the variables, spatial derivatives can be replaced with the wave number, and hence Eq. (3.10) implies that $\zeta$ is equivalently gauge-invariant as $\zeta$. Therefore, in the succeeding subsection we consider the three-point function for $\zeta$ in Fourier space.

Now, we generalize Eq. (2.6) to be consistent with our situation $1/L_{\text{obs}} \ll k \ll aH$. In spite of this effort, it turns out later that the deviation from the formula (2.6) does not contribute to the leading order terms in the squeezed limit. Below, we mainly consider Fourier components of three-point function. One may wonder if the use of Fourier components is in conflict with our philosophy that genuine gauge invariant quantities must be defined only in terms of the information contained in our local observable universe. Here the relevant modes are restricted to sufficiently short wavelength modes in the sense that $k \gg 1/L_{\text{obs}}$. In this case, we can take Fourier components after multiplying a smooth window function.

Focusing on the contribution from a single mode with the wavenumber $k$, the geodesic equation is approximated by

$$\frac{d^2x^i}{d\lambda^2} \approx -i \left[ 2(k \cdot X)X^i - X^2 k^i \right] \zeta_k e^{ik \cdot X} - i \sum_{j=1}^3 \left( k \cdot X \right) X^j \delta \gamma_j k - \frac{1}{2} k^i X^j X^k \delta \gamma_{jk} k,$$

(3.11)

where we neglected the non-linear terms, that do not contribute to the tree-level bi-spectrum. This equation is integrated form $\lambda = 0$ to 1 with the initial condition (2.5). The result of the integration (3.11) would be given in the form

$$\Delta x^i (X) \approx \frac{e^{-\zeta(X)/2}}{\Delta x^i} \left[ \frac{e^{-\delta \gamma(X)/2}}{\Delta x^i} \right] \left[ x^i + \Delta x^i \psi_k + \delta x_L \psi_k \right] \left[ X^i + \delta x_L \psi_k \right] + \delta x_T \psi_k,$$

(3.12)

where we introduced

$$\Delta x^i \psi_k = \left[ \psi_k - \psi_k(0) \right] X^i + \left[ \delta \gamma_{k,j} X^j \right] + \delta x_T \left[ \delta \gamma_{k,j} \right]\right),$$

(3.13)

The third and fourth terms in (3.12) are corrections due to the curvature perturbation and the gravitational waves contained in the Christoffel symbol, respectively. As long as the contribution from a single mode $\psi$ is concerned, $\delta x^i$ is given by

$$\delta x^i_L \psi_k = -i \psi_k \left[ 2(k \cdot X)X^i - X^2 k^i \right] \int_0^1 d\lambda \int_0^\lambda d\lambda' e^{ik \cdot X \lambda'}$$
\[\psi_k \left[ 2(\mathbf{k} \cdot \mathbf{X})X^i - X^2 k^i \right] \frac{i}{k^2} \int_0^k dk' \int_0^{k'} dk'' e^{i k'' \cdot \mathbf{x}}\]
\[= \psi_k \frac{i}{k^2} \int_0^k dk' \int_0^{k'} dk'' \left[ 2(\mathbf{k} \cdot \partial_{k''} \partial_{k''}) - k^2 \partial_{k''}^2 \right] e^{i k'' \cdot \mathbf{x}} =: \psi_k \zeta^{(L)}_{k,k,k} e^{i k'' \cdot \mathbf{x}}, \tag{3.14}\]

where \(k''\) is to be understood as \(k'' k/k\). The tensor contribution \(\delta x_T^i\) can be similarly given by

\[\delta x_T^i(\delta \gamma)(\mathbf{X}) = \sum_{s=\pm} \delta \gamma_{s,k} \frac{i}{k^2} \int_0^k dk' \int_0^{k'} dk'' \left[ (\mathbf{k} \cdot \partial_{k''}) e^{i j, s} \partial_{k''}^j - \frac{1}{2} k^l e_{jk} \partial_{k''}^j \partial_{k''}^k \right] e^{i k'' \cdot \mathbf{x}}\]
\[=: \sum_{s=\pm} \delta \gamma_{s,k} \zeta^{(T)}_{s,k,k} e^{i k'' \cdot \mathbf{x}}. \tag{3.15}\]

C. Primordial non-Gaussianity measured in the geodesic normal coordinates

In the present and succeeding subsections, we calculate the primordial non-Gaussianity measured in the geodesic normal coordinates. First, we neglect \(\Delta x, \delta x_L\) and \(\delta x_T\) in Eq. (3.12). In this case, using Eq. (2.16), the gauge-invariant curvature perturbation is given by

\[\zeta^\text{IR}_i \cong \zeta_n + \frac{1}{4} \psi \dot{c}^2 + \zeta_n (\partial \rho - \mathbf{X}^i \partial X^i) \zeta_n - \frac{1}{2} \delta \gamma_{ij} X_j \partial \zeta_n . \tag{3.16}\]

In the succeeding subsection we will extend our discussions to incorporate the effects of \(\Delta x, \delta x_L\) and \(\delta x_T\), where we will find that this extension does not alter our main conclusion. Substituting the expression of \(\zeta_n\) given in Eq. (2.26) into Eq. (3.16), we obtain

\[\zeta^\text{IR}(\mathbf{X}) \cong \psi + \frac{1}{2} \mu_1 \psi^2 + (1 + \mu_2) \psi \partial \rho \psi - \mathcal{L}^{-1} \mu_3 \psi^2 - (1 + \lambda_2) \psi \mathbf{X}^i \partial X^i \psi , \tag{3.17}\]

up to second order in perturbation, where the time-dependent coefficients \(\mu_i\) with \(i = 1, 2, 3\) are defined in Eqs. (2.32). Now, it is obvious that the contributions from the gravitational waves are cancelled in the genuine gauge-invariant curvature perturbation \(\zeta\) and therefore the contraction of \(\zeta^\text{IR}\) with \(\delta \gamma_{ij}\) trivially vanishes. The second-order perturbation in gravitational waves \(\delta \gamma_{ij}\) also contain the terms with two \(\psi\)'s that can yield non-vanishing terms in the three-point function for \(\zeta^\text{IR}\zeta^\gamma\). However, both of \(\psi\)'s in \(\delta \gamma_{ij}\) are associated with derivatives. We therefore arrive at the conclusion that the leading terms in \(\langle \zeta^\text{IR} \delta \gamma \rangle\) obtained in Eq. (3.7) do not remain, once we use \(\zeta^\text{IR}\) in stead of \(\zeta\).

Next we calculate the bispectrum of \(\zeta^\text{IR}\). Introducing the Fourier modes of \(\zeta^\text{IR}\) as

\[\zeta_{k}(\rho) = \int \frac{d^3 X}{(2\pi)^{3/2}} e^{-i k \cdot X} \zeta^\text{IR}(\rho, \mathbf{X}), \tag{3.18}\]

we consider \(\langle \zeta_{k_1}, \zeta_{k_2}, \zeta_{k_3} \rangle\) at the leading order in perturbation. Expanding \(\zeta^\text{IR}\) as \(\zeta = \zeta_2 + \cdots\), we have

\[\langle \zeta_{k_1}, \zeta_{k_2}, \zeta_{k_3} \rangle = \langle \psi_{k_1} \psi_{k_2} \psi_{k_3} \rangle + \langle \psi_{k_1} \zeta_{k_2,2} \psi_{k_3} \rangle + \langle \zeta_{k_1,2} \psi_{k_2} \psi_{k_3} \rangle . \tag{3.19}\]

Here the last term in Eq. (3.19) does not contribute to the leading term in the squeezed limit, \(k_1 \ll k_2, k_3\) because the power spectrum for the mode with \(k_1\) does not appear.

Using Eq. (3.17), the first term on the right-hand side of Eq. (3.19) is recast into

\[\langle \psi_{k_1} \psi_{k_2} \zeta_{k_3,2} \rangle = \langle \psi_{k_1} \psi_{k_2} \left( (1 + \mu_2) (\psi \partial \rho \psi)_{k_3} - (1 + \lambda_2) (\mathbf{X}^i \partial X^i \psi)_{k_3} + \frac{\mu_1}{2} (\psi^2)_{k_3} \right) \rangle - \langle \psi_{k_1} \psi_{k_2} \mathcal{L}^{-1} \mu_3 (\psi^2)_{k_3} \rangle , \]

where we defined

\[(\psi O \psi)_{k} := \int \frac{d^3 X}{(2\pi)^{3/2}} e^{-i k \cdot X} \left( \prod_{i=1,2} \int \frac{d^3 p_i}{(2\pi)^{3/2}} \right) e^{i p_1 \cdot X} \psi_{p_1} O e^{i p_2 \cdot X} \psi_{p_2} , \tag{3.20}\]

with \(O = 1, \partial \rho, \mathbf{X}^i \partial X^i\), to concisely denote the terms originating from \(\zeta_{n,2}\). The interaction picture fields \(\psi_{k_1}\) and \(\psi_{k_2}\) should be contracted with either \(\psi_{p_1}\) or \(\psi_{p_2}\) contained in \((\psi O \psi)_{k}\). Using the commutation relation for \(a_k\) and \(a_k^\dagger\), the first term of
Eq. (3.19) is rewritten as
\[
\langle \psi_k, \psi_k | g_{k_3, 2} \rangle = v_k \int \frac{d^3 p}{(2\pi)^3/2} \int \frac{d^3 X}{(2\pi)^3} \left[ e^{i(k_1 + k_3) \cdot X} \delta(3)(k_2 + p) \psi_k^* \left\{ (1 + \mu_2) \partial_\mu - (1 + \lambda_2) X^i \partial_{X^i} + \mu_1 - 2 \mathcal{L}_{\mu}^{-1} \mu_3 \right\} v_p e^{i p \cdot X} \right. \\
- 2 e^{-i K \cdot X} \delta(3)(k_2 + p) \{ \mathcal{L}_{k_3}^{-1} \mu_3 v_k^* - v_k^* \mathcal{L}_{k_3}^{-1} \mu_3 v_{k_3}^* \} \\
+ e^{-i(k_2 + k_3) \cdot X} \delta(3)(k_1 + p) \psi_{k_3}^* \left\{ (1 + \mu_2) \partial_\mu - (1 + \lambda_2) X^i \partial_{X^i} \right\} v_p e^{i p \cdot X} \right],
\tag{3.21}
\]
where we defined \( K := k_1 + k_2 + k_3 \). Here, we inserted the last terms on the first and second lines, which cancel with each other.

Following a similar calculation, the second term of Eq. (3.19) is rewritten as
\[
\langle \psi_k, g_{k_3, 2} | \psi_k \rangle = v_k \int \frac{d^3 p}{(2\pi)^3/2} \int \frac{d^3 X}{(2\pi)^3} \left[ e^{-i(k_1 + k_3) \cdot X} \delta(3)(k_3 + p) \psi_k^* \left\{ (1 + \mu_2) \partial_\mu - (1 + \lambda_2) X^i \partial_{X^i} + \mu_1 - 2 \mathcal{L}_{\mu}^{-1} \mu_3 \right\} v_p e^{i p \cdot X} \right. \\
- 2 e^{-i K \cdot X} \delta(3)(k_3 + p) \{ \mathcal{L}_{k_3}^{-1} \mu_3 v_k^* - v_k^* \mathcal{L}_{k_3}^{-1} \mu_3 v_{k_3}^* \} \\
+ e^{-i(k_2 + k_3) \cdot X} \delta(3)(k_1 + p) \psi_{k_3}^* \left\{ (1 + \mu_2) \partial_\mu - (1 + \lambda_2) X^i \partial_{X^i} \right\} v_p e^{i p \cdot X} \right].
\tag{3.22}
\]
Replacing \( X \) with \(-X\), we find that this expression is identical to the complex conjugate of Eq. (3.21) with the exchange between \( k_2 \) and \( k_3 \), which means
\[
\langle \psi_k, g_{k_3, 2} | \psi_k \rangle = \langle \psi_k, g_{k_3, 2} | \psi_k \rangle^*.
\tag{3.23}
\]
Here we noted that at superhorizon scales \( v_k \) becomes constant and commutes with the time integrals in \( \mathcal{L}_{\mu}^{-1} \) where \( i = 2, 3 \). It is, therefore, sufficient to calculate the first term of Eq. (3.19). Using the gauge-invariance condition \( \mathcal{L}_{\mu} \), the first line in Eq. (3.21) can be recast into the simple expression \[31\]

\[
(\text{First line in Eq. (3.21)}) = -(1 + \lambda_2) v_k v_{k_3} \int \frac{d^3 p}{(2\pi)^{3/2}} \int \frac{d^3 X}{(2\pi)^3} v_k e^{-i(k_1 + k_3) \cdot X} \delta(3)(k_2 + p) D_p (v_p e^{i p \cdot X}) \\
= -(1 + \lambda_2) ||v_k||^2 v_{k_3} \int \frac{d^3 p}{(2\pi)^{3/2}} \int \frac{d^3 X}{(2\pi)^3} \delta(3)(k_2 + p) \left( p \cdot \partial_p + 3/2 \right) \left( v_p^* \delta(3)(k_1 + k_3 - p) \right) \\
= - \frac{1}{2} (1 + \lambda_2) ||v_k||^2 v_{k_3} v_k^* \int \frac{d^3 X}{(2\pi)^3/2} \left[ \delta(3)(k_2 + p) \left( p \cdot \partial_p + 3/2 \right) \delta(3)(k_1 + k_3 - p) \right. \\
+ \delta(3)(k_1 + k_3 - p) \left\{ (p + k_1 - k_2 + k_3) \cdot \partial_p + 3/2 \right\} \delta(3)(k_2 + p) \right] \\
= - \frac{1}{2} (1 + \lambda_2) ||v_k||^2 v_{k_3} v_k^* \left[ (2\pi)^{-3/2} (k_1 - k_2 + k_3) \cdot \partial_p \delta(3)(K) \\
+ \frac{d^3 p}{(2\pi)^{3/2}} \partial_p p \delta(3)(p + k_1 + k_3) \delta(3)(p - k_2) \right].
\tag{3.24}
\]
In the third equality, we replaced \( p \) with \(-p - k_1 + k_2 - k_3\) in the half of the expression. The second term in the last line takes the form of total derivative and hence it does not contribute. The first term proportional to \( k_1 \cdot \partial_\mathbf{K} \delta(3)(K) \) in Eq. (3.24) vanishes in the limit \( k_1 \to 0 \). The other terms on the same line are not suppressed at this moment. Combining the contribution from the second term of Eq. (3.19), which satisfies Eq. (3.23), these terms, however, provide a factor \( \left( v_{k_2} v_{k_3}^* - v_{k_1 + k_2} v_{k_3}^* \right) \), which again vanishes in the squeezed limit, \( k_1 \to 0 \).

To evaluate the second line in Eq. (3.21), we again note that \( v_{k_1} \) commutes with \( \mathcal{L}_{\mu}^{-1} \) for \( k_1 \ll e^\mu H \). Thus, we have
\[
(\text{Second line in Eq. (3.21)}) = -2 ||v_{k_2}||^2 v_{k_3} (2\pi)^{-3/2} \delta(3)(K) \left\{ \mathcal{L}^{-1}_{k_2 + k_3} \mu_3 v_{k_2} - \mathcal{L}^{-1}_{k_2} \mu_3 v_{k_2}^* \right\}.
\tag{3.25}
\]
which vanishes in the limit \( k_1 \to 0 \).
As for the last line in Eq. (3.21), we notice that \( p \) can be replaced with \( -k \) owing to the factor \( \delta^{(3)}(k_1 + p) \). Therefore the term with the \( \rho \)-derivative acting on \( v^*_p \) is suppressed in the limit \( k_1 \to 0 \). Hence, the terms in the third line can be simplified to

\[
\text{Last line in Eq. (3.21)} = -(1 + \lambda_2)|v_{k_1}|^2|v_{k_2}|^2 \int \frac{d^3 p}{(2\pi)^{3/2}} \delta^{(3)}(k_1 + p)p \cdot \partial p \delta^{(3)}(k_2 + k_3 - p)
\]

\[
= -(1 + \lambda_2)|v_{k_1}|^2|v_{k_2}|^2 (2\pi)^{-3/2} k_1 \cdot \partial k \delta^{(3)}(k).
\]

(3.26)

This expression vanishes in the limit \( k_1 \to 0 \). Thus we find that all the leading contributions in the tree-level bispectrum Eq. (3.19) vanish in the squeezed limit.

D. Remaining contributions from the transformation to geodesic normal coordinates

Here we take into account the effects of \( \Delta x_L, \delta x_L \) and \( \delta x_T \). We calculate the three-point functions, whose three momenta are given by \( p_1, p_2 \), and \( p_3 \) without specifying which one is \( k_1 \). We first discuss the scalar contribution \( \Delta x_L^i \) to the three-point function with cubic \( \zeta \), given by

\[
\int \frac{d^3 X}{(2\pi)^3} e^{-i p_3 \cdot X} \langle \psi_{p_1} \Delta x_L^i \rangle \langle \psi_{p_2} \partial_i \psi \rangle = \int \frac{d^3 X}{(2\pi)^3} e^{-i p_3 \cdot X} \langle \psi_{p_1} \Delta x_L^i \rangle \langle \psi_{p_2} \partial_i \psi \rangle
\]

\[
= |v_{p_1}|^2|v_{p_2}|^2 \int \frac{d^3 X}{(2\pi)^3} e^{-i p_3 \cdot X} \left( e^{-i p_1 \cdot X} - 1 \right) X^i \partial_i e^{-i p_2 \cdot X}
\]

\[
= |v_{p_1}|^2|v_{p_2}|^2 \cdot \partial_p \left[ \delta^{(3)}(p_1 + p_2 + p_3) - \delta^{(3)}(p_2 + p_3) \right].
\]

(3.27)

When \( p_1 \) is identified with \( k_1 \), the factor \( \delta^{(3)}(p_1 + p_2 + p_3) - \delta^{(3)}(p_2 + p_3) \) vanishes in the limit \( k_1 \to 0 \). When \( p_2 \) is identified with \( k_1 \), there is a manifest suppression factor proportional to \( p_2 \). When \( p_3 \) is identified with \( k_1 \), the power spectrum \( |v_{k_1}|^2 \) is absent and Eq. (3.27) does not yield the dominant contribution in the squeezed limit. Therefore, for all cases, the contribution from \( \Delta x_L^i \) vanishes in the limit \( k_1 \to 0 \). A similar argument follows for the non-vanishing contributions of \( \Delta x_T \) to the three-point functions.

Next, we consider the contributions from \( \delta x_L \) and \( \delta x_T \). Again we evaluate a rather general expression:

\[
\int \frac{d^3 X}{(2\pi)^3} e^{-i p_3 \cdot X} \langle \psi_{p_1} \delta x_L \rangle \langle \psi_{p_2} \partial_i \psi \rangle = |v_{p_1}|^2|v_{p_2}|^2 \int \frac{d^3 X}{(2\pi)^3} e^{-i p_3 \cdot X} \left( \delta x^{(L)}_{p_1, k''} e^{ik'' \cdot X} \partial_i e^{-i p_2 \cdot X} \right.
\]

\[
= -|v_{p_1}|^2|v_{p_2}|^2 \int \frac{d^3 X}{(2\pi)^3} e^{-i p_3 \cdot X} \left[ 2(p_1 \cdot \partial k'') \partial_i k'' - p_1^2 \delta_{k''}^{(3)}(p_2 + p_3 - k'') \right].
\]

(3.28)

where \( k'' \) should be understood as \( -k'' p_1 / p_1 \). Suppression is manifest in the case with \( p_2 = k_1 \) or \( p_1 = k_1 \). In the former case there exists a manifest suppression factor proportional to \( p_2 \), while in the later case there is no contribution from the spectrum \( |v_{k_1}|^2 \) as before. We therefore examine the case in which \( p_1 \) is identified with \( k_1 \). In this case, it will be convenient to rewrite the last expression as

\[
\int \frac{d^3 X}{(2\pi)^3} e^{-i p_3 \cdot X} \langle \psi_{p_1} \delta x_L \rangle \langle \psi_{p_2} \partial_i \psi \rangle
\]

\[
= -|v_{p_1}|^2|v_{p_2}|^2 \int \frac{d^3 X}{(2\pi)^3} \left[ 2(p_1 \cdot \partial p_2 \partial_i p_3 - p_1^2 p_3) \right] \int \left. dk' \int dk'' \delta^{(3)}(p_2 + p_3 - k'') \right].
\]

(3.29)

Denoting \( p_2 + p_3 \) by \( p \), the part related to \( p_1 \) is then simply given in the form,

\[
\frac{|v_{p_1}|^2}{p_1} g(p_1) \int_0^{p_1} dk' \int_0^{k''} dk'' \delta^{(3)}(p - k'') = \frac{|v_{p_1}|^2}{p_1} g(p_1) \int_0^{p_1} dk' \int_0^{k''} \frac{1}{p''^2} \delta(p - k'') \delta^{(2)}(\hat{p} + \hat{p}_1)
\]

\[
= \frac{|v_{p_1}|^2}{p_1} \frac{(p_1 - p)^2}{p_1} \frac{\theta(p_1 - p)}{p_1 - p} g(p_1) \delta^{(2)}(\hat{p} + \hat{p}_1),
\]

(3.30)

where \( \hat{p} := p / p_1 \), \( \hat{p}_1 := p_1 / p_1 \), and \( g(\hat{p}) := 2(\hat{p} \cdot \partial p_2 \partial p_3) \partial_i k'' - \hat{p}^i (\partial p_2 \partial p_3) \). If we integrate the above expression for a small region \( p_1 < \epsilon = O(p) \) multiplying \( G(p_1) \), which is an arbitrary smooth function of \( p_1 \), it becomes \( O(\epsilon |v_p|^2 G(p)) \). The obtained
expression is suppressed by a factor $\epsilon$ compared with the case when we consider an integral for the simple delta function, 
\[ \int d^3p \, G(p) \left| v_{p_1} \right|^2 \delta^{(3)}(p + p_1). \]
Since this argument does not depend on the details of the function $g(p)$, a similar suppression can be easily found also in the contribution from $\delta x_T$. Now, we can understand that all the neglected terms in the preceding subsection also vanish in the squeezed limit $k_1 \to 0$.

E. Discussions

The suppression in the squeezed limit does not occur in the usual computation of the bispectrum for $\zeta$. What we have found is that the leading term in the squeezed limit, exhibited in the consistency relation (3.3), is significantly modified if we measure the gauge invariant curvature perturbation by using the geodesic normal coordinates. This can be understood from the fact that the non-Gaussianity described by the consistency relation is originating from the deformation of the local coordinates due to the IR modes that crossed the horizon much earlier than the other two modes. As shown in our previous works \[22, 23\] and also in Refs. \[27, 29\], this deformation can be absorbed by the gauge transformation in the local universe. This leads to the conclusion that there should not be the net physical effect due to these IR modes in this case. (See also Ref. \[29\], where a similar analysis is presented, while the authors seem to have the slightly different opinion about what are the actual observables.)

If we were thinking of the case with $k_1 \ll 1/L_{\text{obs}}$, this cancellation can be understood rather easily as mentioned above. By contrast, our result might be a little surprising because the longest wavelength mode $k_1$ is, in the present setup, supposed to be still within the size of our observable universe. Nevertheless, the cancellation seems to continue to work. In the actual observation what we are interested in will be the temperature fluctuation of cosmic microwave background, which is not exactly the light cone. Furthermore the observable region is not well outside the horizon even at the last scattering surface. Despite of these differences, our result suggests that for single field inflation models the leading terms of bi-spectrum in the squeezed limit may vanish also in the CMB anisotropy as long as we analyze the data by using observationally natural coordinates.

IV. CONCLUSION

In this paper we reexamined the so-called consistency relation, focusing on what are genuine observables in our local universe. If we knew the configuration of the whole universe, usual curvature perturbation $\zeta$ has gauge invariant meaning. However, the region we can observe is limited. In case $\zeta$ itself is not genuine gauge invariant.

In our previous paper we proposed a genuine gauge invariant variable that can be locally constructed, e.g. three curvature on $\phi$-constant hypersurface measured in the geodesic normal coordinates. We found that the two-point function of this quantity is free from IR divergence in single field inflation models only if the initial state satisfies what we called the gauge invariance conditions. Applying thus-obtained gauge invariance conditions, we investigated bispectrum in the squeezed limit, where one of the momenta of the arguments of the three-point function is much less than the others.

In the squeezed limit, the bispectrum is known to be expressed by the power spectrum and the spectral index. Although we are not claiming that the result of this standard computation is wrong, this conclusion is derived by evaluating the three-point function for the so-called curvature perturbation, which is not a genuine gauge invariant variable in the sense mentioned above. We therefore re-investigated the behavior of the three-point function for an alternative quantity, which is genuinely gauge invariant. As a result, we found that the leading term in the squeezed limit vanishes, i.e. the bispectrum is more suppressed in the squeezed limit than we expect from the naive calculation based on the so-called gauge-invariant curvature perturbation.

We addressed the local non-Gaussianity in single-field models with the standard kinetic term, but our argument will be easily extended to models with a non-standard kinetic term. Since the large local non-Gaussianity is reported in multi-field models, the extension to multi-field models are intriguing, too. For this extension, we need to keep in mind the different nature of the IR modes between in single-field models and in multi-field models \[30\]. In the latter case, IR modes include physical degrees of freedom as well as unphysical gauge degrees of freedom. Therefore such perfect cancellation as we observed in single field models will not be expected in a general multi-field case. We will report these issues in our forthcoming publication.

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[31] One remark is in order about the translational invariance. There arose many terms that are not accompanied by the delta function \( \delta^{(3)}(K) \).

Usually appearance of the factor \( \delta^{(3)}(K) \) is guaranteed by the momentum conservation. However, the origin of the spatial coordinates is fixed in the geodesic normal coordinates we introduced, which manifestly breaks the global translational invariance. Therefore we cannot expect that the final result should be proportional to \( \delta^{(3)}(K) \).