Constraints on non-Newtonian gravity from the experiment on neutron quantum states in the earth’s gravitational field

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Received 30 March 2004, in final form 26 July 2004
Published 14 September 2004
Online at stacks.iop.org/CQG/21/4557
doi:10.1088/0264-9381/21/19/005

Abstract
An upper limit to non-Newtonian attractive forces is obtained from the measurement of quantum states of neutrons in the earth’s gravitational field. It is established in a completely new way and supports the existing constraints in the nanometre range.

PACS numbers: 04.80.Cc, 28.20. −v

(Some figures in this article are in colour only in the electronic version)

1. Introduction

According to the predictions of unified gauge theories, supersymmetry, supergravity and string theory, there would exist a number of light and massless particles [1]. An exchange of such particles between two bodies gives rise to an additional force. Additional fundamental forces at short distances were intensively studied, in particular during the past few years following the hypothesis about ‘large’ supplementary spatial dimensions proposed by Arkami-Hamed, Dimopoulos and Dvali [2, 3], building on earlier ideas in [4–7]. For a review of theoretical works and recent experimental results, see [8–12] and references therein. This hypothesis could be verified using neutrons because the absence of an electric charge allows one to strongly suppress the false electromagnetic effects [13]. It was noticed in [14] that the measurement of the neutron quantum states in the earth’s gravitational field [15] is sensitive to such extra forces in the sub-micrometre range. In the case of $n = 3$ extra dimensions, the characteristic range is just in the nanometre domain [2, 13] which is accessible in this experiment. The first attempt to establish a model-dependent boundary in the range 1–10 $\mu$m was presented in [17].
An effective gravitational interaction in the presence of an additional Yukawa-type force is parametrized as

$$V_{\text{eff}}(r) = G \frac{m_1 m_2}{r} (1 + \alpha G e^{-r/\lambda}).$$

(1)

Here, $G$ is the Newtonian gravitational constant, $m_1$ and $m_2$ are interacting masses, $r$ their relative distance, $\alpha G$ and $\lambda$ are strength and characteristic range of this hypothetical interaction.

The experiment [15] consists in the measurement of the neutron flux through a slit between a horizontal mirror on bottom and a scatterer/absorber on top as a function of the slit size $\Delta h$ (see figure 1). The main aim of this experiment was to demonstrate, for the first time, the existence of the quantum states of matter in a gravitational field.

An example of the dependence of the neutron flux on the slit size $\Delta h$ is presented in figure 2 [18].

This dependence is sensitive to the presence of quantum states of neutrons in the potential well formed by the earth’s gravitational field and the mirror. In particular, the neutron flux was measured to be equal to zero within the experimental accuracy if the slit size $\Delta h$ was smaller than the characteristic spatial size (a quasiclassical turning point height) of the lowest quantum state of $\sim 15 \mu m$ in this potential well. The neutron flux at the slit size $\Delta h < 10 \mu m$ was lower by at least a factor of 200 than that for the lowest quantum state ($\Delta h \approx 20 \mu m$).

Figure 1. The general experimental scheme. From the left to right: vertical solid lines indicate two plates of the entrance collimator (1); solid arrows show classical neutron trajectories (2) between the collimators and the entrance to the slit between a mirror (3, grey rectangle on bottom) and a scatterer (4, black rectangle on top); Dashed horizontal lines show quantum motion of neutrons above the mirror (5); black box indicates a neutron detector (6). The size of the slit between the mirror and the scatterer can be finely tuned and measured.

Figure 2. A dependence of the neutron flux through a slit between the mirror and the scatterer versus the slit size. The circles show the data points, the curve is the theoretical description within the quasiclassical approach. The horizontal lines indicate the detector background and its uncertainty.
If an additional short-range force of sufficiently high strength would act between neutrons and the mirror then it would modify the quantum states parameters: an attractive force would ‘compress’ the wavefunctions towards the mirror, while a repulsive force would shift them up. In this experiment, no deviation from the expected values was observed within the experimental accuracy. This accuracy is defined by the uncertainty in the slit size which can be conservatively estimated as \( \approx 30\% \) for the lowest quantum state \([15]\).

The motion of neutrons in this system over the vertical axis \( z \) could be considered, within first and quite good approximation, as a one-dimensional problem for which the mirror provides an infinitely high potential. The interaction between neutrons and the earth is described by the first term in equation (1) and can be approximated by the usual linear potential \( (r = R + z) \):

\[
V(z) = mgz, \tag{2}
\]

with \( g = GM/R^2 \), \( R \) being the earth’s radius, \( M \) its mass, \( m \) the neutron mass.

The second term in equation (1) introduces an additional interaction. Due to the short range of this interaction, its main contribution is provided by the interaction of neutrons with a thin surface layer of the mirror and the scatterer.

In this paper, we estimate an upper limit on an additional attractive short-range force, which could be established from this experiment in a model-independent way. We show that it could not be significantly improved in an any more sophisticated model.

2. Attractive interaction

Let us first estimate the interaction of neutrons with the mirror. If the mirror’s density is constant and equal to \( \rho_m \), then an additional potential of the interaction between neutrons and the mirror is given by

\[
V'(z') = -G\alpha G m \rho_m \int_{\text{mirror}} d^3r \exp\left(-\sqrt{x^2 + y^2 + (z - z')^2}/\lambda\right). \tag{3}
\]

The volume integral is calculated over the mirror bulk: \(-\infty < x, y < \infty, z < 0\) (in fact, over the neutron’s vicinity with the size of the order of a few \( \lambda \) due to the exponential convergency of these integrals). It can be calculated analytically for small \( \lambda \):

\[
V'(z) = -U_0 e^{-z/\lambda}, \tag{4}
\]

with \( U_0 = 2\pi G\alpha G m \rho_m \lambda^2 \).

The simplest upper limit on the strength of an additional interaction follows from the condition that this additional interaction does not create itself any bound state. It is known [16] that for an exponential attractive \( (\alpha G > 0) \) potential (4) this means that

\[
\frac{U_0 m \lambda^2}{\hbar^2} < 0.72. \tag{5}
\]

This condition gives a boundary for an additional potential strength:

\[
\alpha G = 0.72 \frac{2}{\pi} \frac{\rho}{\rho_m} \frac{\hbar}{m g \lambda^2} \frac{R}{m \lambda \lambda}. \tag{6}
\]

\( \rho \) being the earth’s averaged density. In this experiment, both densities are close to each other \( \rho \approx \rho_m \), therefore their ratio \( \rho/\rho_m \) is close to 1. However, an adequate choice of the mirror material (coating) would easily allow one to gain a factor of 3–5 in the sensitivity in future
Figure 3. The constraints on $\alpha_G$ following from the experiment [15] (the solid line) in comparison with that from the measurement of the Casimir and the van der Waals forces [10] (the short dashed lines). The long dashed line shows a limit which can be easily obtained by an improvement of this experiment. The solid horizontal line represents the limit established from the atomic experiment [20].

One obtains the following numerical boundary:

$$\alpha_G = 1 \times 10^{15} \left( \frac{1 \mu m}{\lambda} \right)^4.$$  \hspace{1cm} (7)

Here, 1 $\mu$m is chosen as a natural scale for this experiment. This limit is presented in figure 3 in comparison with the limits from the Casimir-like and van der Waals force measurement experiments [10]. One can note that, in the realistic case, one has to establish a condition of non-existence of an additional bound state for the sum of (2) and (4) but not for the interaction (4) alone. The presence of the linear potential modifies slightly the critical value in (5). For instance, for $\lambda = 1$ $\mu$m it is approximately equal to 1.0 and for $\lambda = 0.1$ $\mu$m it is equal to 0.74. For smaller $\lambda$, this value is evident to tend to 0.72. It is possible to explain qualitatively why the strength of an additional interaction should be higher in the presence of the $mgz$-potential than without it. When a bound state just appears, then its wavefunction is extremely spread. If one adds a supplementary ‘external’ confining potential, it does not allow the wavefunction to be spread and thus one needs a stronger potential to create a bound state.

The range of presented $\lambda$ is 1 nm–10 $\mu$m. A deviation from a straight line in the solid curve at 1 nm is due to the finite range of increase of the mirror effective nuclear potential (impurities on the surface and its roughness). The same effect at $\lambda \approx 10$ $\mu$m is due to an ‘interference’ of the potentials (2) and (4).

It is interesting to compare this analytical limit (7) to an analogous expression obtained in [10, 11] for the Casimir force-like experiments. The simplest boundary $\alpha = \alpha(\lambda)$ following from these experiments is given by a formula

$$\alpha_C = C_C \exp(\frac{d_0}{\lambda}) \left( \frac{\lambda}{\lambda_0} \right)^3.$$  \hspace{1cm} (8)

Here $d_0$ is a gap separation and $C_C$ is a constant depending on the geometry of different experiments. This function increases exponentially when $\lambda$ tends to zero. This behaviour is
clearly seen in figure 3. In the experiment [15] we obtain
\[ \alpha_n = \frac{C_n}{\lambda^4} \]  \hspace{1cm} (9)
and \( \alpha_n \) increases only as \( 1/\lambda^4 \). This difference between (8) and (9) means that, in principle, for any \( C_C \) and \( C_n \), one could find a domain of sufficiently small but experimentally observable \( \lambda \) in which the limit obtained from our experiment is stronger than that obtained from a Casimir force-like experiment.

Up to now, we have not discussed the scatterer. As we can see from figure 3, a competitive limit could be established only for \( \lambda \) much smaller than the scatterer roughness amplitude of the order of 1 \( \mu \)m. Therefore an influence of the scatterer is negligible.

3. Repulsive interaction

Unfortunately, this experiment does not allow us to establish a competitive limit for a repulsive interaction. In this case, there could be no ‘additional’ bound state. Here, instead of the condition of ‘non-existence’ of a bound state, one could consider the critical slit size for which the first bound state appears in this system. Such an approach would be model dependent due to uncertainties in the description of the interaction of neutrons with the scatterer. Nevertheless, it is possible to obtain a simple analytical expression for small \( \lambda \) and to show explicitly a difference in sensitivity of this experiment to an attractive and to a repulsive additional interactions.

Let us remind the reader that the quasiclassical approximation for the potential composed by the linear term \( mgz \) and by the infinite mirror potential describes the problem very well. For instance, it gives the ground-state energy, for which the quasiclassical approximation is usually expected to be not valid, within 1% [16, 22]. The Bohr–Sommerfeld quantification condition in this case assumes that the quasiclassical wavefunction
\[ \Psi_{QC}(z) = \frac{C_1}{\sqrt{p(z)}} \sin \left( \frac{1}{\hbar} \int_{z}^{b} p(z') \, dz' + \frac{\pi}{4} \right), \]  \hspace{1cm} (10)
with \( p(z) = \sqrt{2m(E - mgz)} \) \( (0 < z < b, b \) being the turning point) equals to zero at \( z = 0 \) (at the impenetrable mirror). This condition, for \( n \)th quantum state, means
\[ \frac{1}{\hbar} \int_{0}^{b} p(z) \, dz + \frac{\pi}{4} = \pi(n + 1), \]  \hspace{1cm} (11)
and the corresponding energies are
\[ E_n = \left( \frac{9\pi^2}{8} \right)^{1/3} (mg^2\hbar^2)^{1/3}(n + 3/4)^{2/3}. \]  \hspace{1cm} (12)
With an additional short-range repulsive interaction \( V'(z) = U_0 \exp(-z/\lambda) \), the problem could be solved in an analogous way. One can choose such \( z_0 \) that \( \lambda \ll z_0 \ll b \). For \( 0 < z < z_0 \), the \( mgz \) potential is negligible and the problem can be solved analytically [16]. For \( z_0 < z < b \), one can use the quasiclassical approximation. At \( z = z_0 \) the quasiclassical wavefunction should be equal to the exact one:
\[ \Psi_{ex}(z_0) = C_2 \sin(kz_0 + \delta_0), \]  \hspace{1cm} (13)
with the phase shift
\[ \delta_0 = -4k\lambda \left[ C + \ln \frac{\xi}{2} + \frac{K_0(\xi)}{I_0(\xi)} \right]. \]  \hspace{1cm} (14)
Here \( k = \sqrt{2mE/\hbar^2} \), \( \xi = \sqrt{8mU_0\lambda^2/\hbar^2} \). \( C = 0.577 \) is the Euler constant, \( I_0(\xi) \) and \( K_0(\xi) \) are the modified Bessel functions of pure imaginary argument. This phase shift modifies the Bohr–Sommerfeld quantification condition and introduces a shift \( \delta E_n \) in the level’s energy. If this shift is small with respect to the non-perturbated value, one obtains the following relation between the energy shift \( \delta E_n \) and the phase shift \( \delta_0 \):

\[
\frac{1}{\hbar} \left[ \frac{2E_n^3}{mg^2} \right]^{1/2} = \delta_0. \tag{15}
\]

For very strong repulsive potential, \( \xi \gg 1 \), the phase shift is equal to

\[
\delta_0 \approx -4k\lambda \ln \frac{\xi}{2}. \tag{16}
\]

An experimental upper limit on the energy shift \( \delta E_n \) would impose an upper limit on \( \alpha_G \) for a repulsive interaction:

\[
\frac{2U_0m\lambda^2}{\hbar^2} < \exp(\lambda_0/\lambda), \tag{17}
\]

with

\[
\lambda_0 = \frac{\delta E_n}{mg}, \tag{18}
\]

or

\[
\alpha_G \leq \frac{1}{\pi} \frac{\hbar}{mg\lambda^2} \frac{\hbar}{m\lambda} \frac{R}{\lambda} \exp(\lambda_0/\lambda). \tag{19}
\]

Direct comparison of relation (19) to (5) and (6) shows that the limit (19) at small \( \lambda \) is sufficiently less restrictive than that for an attractive one (6) due to the exponential factor. On the other hand, it would be possible to achieve, for a repulsive interaction, as strict a limit as for an attractive one, if the mirror is coated with a material with negative Fermi potential.

Note that the presented approach could be applied as well for an attractive interaction \( V'(z) = -U_0 \exp(-z/\lambda) \). In this case, the phase shift is equal to

\[
\delta_0 \approx -4k\lambda \left[ C + \ln \frac{\xi}{2} - \frac{\pi}{2} \frac{N_0(\xi)}{J_0(\xi)} \right]. \tag{20}
\]

Here \( J_0(\xi) \) and \( N_0(\xi) \) are the Bessel functions. This expression is maximal in the vicinity of the first Bessel function’s zero \( \xi_0 \approx 2.40 \) where it can be developed as

\[
\delta_0 \approx -4k\lambda \frac{\pi}{2} \frac{N_0(\xi_0)}{J_1(\xi_0)} \frac{1}{\xi - \xi_0}. \tag{21}
\]

One thus obtains the relation between the strength of an attractive potential and the energy shift \( \delta E_n < 0 \):

\[
\xi \leq \xi_0 - \frac{\pi}{2} \frac{N_0(\xi_0)}{J_1(\xi_0)} \frac{\lambda}{\lambda_0}. \tag{22}
\]

with \( \lambda_0 = \frac{\hbar E_n}{mg} \), \( N_0(\xi_0) \approx 0, 51 \), \( J_1(\xi_0) \approx 0, 52 \). If one omits the second term in the latter equation, one obtains exactly relations (5) and (6) representing a condition of non-existence of an additional bound state.

Let us mention that this formula allows us to estimate the capability of this experiment to further improve an upper limit on \( \alpha_G \) in the case of an attractive interaction: in order to improve significantly (by an order of magnitude) the limit (6) in a given range of \( \lambda \) it is necessary to determine experimentally and theoretically the parameters of quantum states with a precision \( \lambda_0/\lambda \approx 1 \).
4. Occupation numbers

The considerations presented above are valid only if the neutron population in the lowest quantum state in such a system (with an additional interaction included) is sufficiently high to provide a measurable signal/noise ratio. As we have mentioned in the introduction, the experiment [15] would allow one to identify an additional quantum state if its occupation number would not be suppressed by more than a factor of 200 compared to that for other states.

In order to calculate the occupation numbers, let us start with a general expression for the probability of a rapid transition from a state \( k \) with the wavefunction \( \Psi_k(x) \) to a state \( n \) with the wavefunction \( \Phi_n(x) \), which is given by a formula [23]

\[
w_{k\rightarrow n} = \left| \int \Psi_k(x) \Phi_n^*(x) \, dx \right|^2. \tag{23}
\]

For a few initial quantum states, the probability \( w_n \) is a sum (an integral for continuous spectrum) over them:

\[
w_n = \sum_k f_k w_{k\rightarrow n}, \tag{24}
\]

with the occupation numbers \( f_k \) of initial states.

To obtain an analytical expression for the occupation numbers, let us consider a simplified model of a harmonic oscillator in a final state and a plane wave in an initial one. An explicit analytical shape of the final state wavefunction does not play a role (the only important parameter is its spatial size) and would not modify considerably the occupation numbers. The wavefunctions used are equal to

\[
\Phi_n(x) = \frac{1}{\sqrt{\pi x_0^2}} \frac{1}{\sqrt{2^n n!}} \exp\left(-\frac{x^2}{2x_0^2}\right) H_n\left(\frac{x}{x_0}\right), \tag{25}
\]

and

\[
\Psi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}. \tag{26}
\]

Here \( H_n(z) \) are the Hermite polynomials and \( x_0 \) represents a characteristic geometric size of the problem.

If initial states are populated according to the Gaussian law with a characteristic momentum \( k_0 \) then

\[
f_k = \exp\left(\frac{k^2}{k_0^2}\right), \tag{27}
\]

and all integrals (24) can be calculated analytically [24]. The occupation numbers of the final states appear to be equal to

\[
w_n = \frac{k_0 x_0}{\sqrt{1+(k_0 x_0)^2}} \frac{\left( (k_0 x_0)^2 - 1 \right)^{n/2}}{\left( (k_0 x_0)^2 + 1 \right)^{n/2}} P_n\left(\frac{(k_0 x_0)^2}{\left( (k_0 x_0)^2 + 1 \right)^{1/2}} - 1\right). \tag{28}
\]

Here \( P_n(z) \) is the Legendre polynomial. For instance, for the lowest states with \( n = 0 \) and \( n = 1 \):

\[
w_0 = \frac{k_0 x_0}{\sqrt{1+(k_0 x_0)^2}}, \quad w_1 = w_0^3. \tag{29}
\]

If \( k_0 x_0 \gg 1 \) then the occupation numbers are approximately equal for all states:

\[
w_n \approx 1. \tag{30}
\]
Let us apply these expressions to our problem. For the gravitational quantum states, $x_0 \approx 6 \mu m$; the vertical velocity distribution has a characteristic velocity of $v_0 \approx 50 \text{ cm s}^{-1}$. For these states, $k_0 x_0 \approx 50 \gg 1$ and all states should have approximately the same occupation numbers.

If an additional bound state were created by the interaction (4) then the characteristic size of such a state should be of the order of the range of the interaction $\lambda$ (or bigger). For the interaction range, for which this experiment establishes a competitive limit, one obtains $w \approx k_0 \lambda \approx 0.1$ for $\lambda = 10 \text{ nm}$ and $w \approx k_0 \lambda \approx 0.01$ for $\lambda = 1 \text{ nm}$. As we mentioned previously, if such a state exists it would be detected in this experiment.

Moreover, these numbers represent a lower estimation for the occupation numbers because they assume the state to be deeply bound. For a just appearing state, the characteristic size $x_0$ would be sufficiently bigger than the range of the interaction and would be close to that of the unperturbed states in the gravitational potential. Thus the population would not be almost suppressed.

5. A limit from exotic atoms

To complete our analysis in the nanometre range, let us note that a competitive limit follows from a recent experiment with antiprotonic atoms as well. The idea of such an analysis is the same as in [19]. An additional force between a nucleus of mass $M$ and an antiproton would change the spectrum of such an atom. The effective orbit radius $r_0$ for usually studied antiproton–nucleus atoms is about a few hundreds fm. Therefore, for the range of a hypothetical Yukawa force $\lambda \gg r_0$, an additional potential would have a $1/r$ behaviour and would change the Coulomb potential:

$$Ze^2 \rightarrow Ze^2 + mMG\alpha_G.$$  \hfill (31)

The energy spectrum $E_n = -(Ze^2)^2/2\mu$ of an atom with a reduced mass $\mu$ as well as the transition frequencies $\nu_n$ would be modified as well:

$$\nu_n \rightarrow \nu_n \left(1 + \frac{2mMG\alpha_G}{Ze^2}\right) \equiv \nu_n + \delta\nu_n.$$  \hfill (32)

Thus, the constant $\alpha_G$ is related to the shift $\delta\nu_n$ in the transition frequencies by a formula

$$\alpha_G = \frac{Ze^2}{2mMG} \frac{\delta\nu_n}{\nu_n}.$$  \hfill (33)

An upper experimental limit on the transition frequency shift $|\delta\nu_n|$ imposes an upper limit on $|\alpha_G|$. The most precise measurement of the energy spectrum of antiprotonic atoms was done for $\bar{p}\bar{3}\text{He}^+$ and $\bar{p}\bar{4}\text{He}^+$ atoms by the ASAKUSA collaboration at the antiproton decelerator at CERN [20]. In this experiment, 13 electromagnetic transitions between different levels of these atoms were investigated. No deviation was found from the values expected within the QED calculations [21]. Equation (33) allows $\alpha_G$ to be extracted. The averaged over these 13 transitions value of $\alpha_G$ is equal to

$$\alpha_G = (3 \pm 10) \times 10^{-27},$$  \hfill (34)

and is compatible with zero. A 1σ upper limit on $|\alpha_G|$ from this experiment is

$$|\alpha_G| \leq 1.3 \times 10^{-28}.$$  \hfill (35)

This value is presented by the solid horizontal line in figure 3.
6. Conclusions and perspectives

An upper limit to an additional attractive force is established from the measurement of quantum states of neutrons in the earth’s gravitational field. The relatively high sensitivity of the experiment [15] to a hypothetical additional force is due to the following factors: firstly, no ‘background’ electromagnetic interactions; secondly, the characteristic size of the neutron wavefunction in the quantum states fits well to the range of interest for the short-range forces; finally, there is a non-negligible probability of finding neutrons (quantum-mechanical object) at distances much closer to the mirror than the average value of 10 µm.

Even though this experiment was never designed to search for additional short-range forces it provides the competitive limit (7) in the nanometre range.

However, it can be easily improved in the same kind of experiment with some evident modifications; for instance, one can choose a mirror material (coating) with higher density. A more significant gain in the sensitivity could be achieved in a dedicated neutron experiment which will be presented in a forthcoming paper.

Acknowledgments

We are very grateful to P van Isaker, R Onofrio, V A Rubakov, M E Shaposhnikov and P G Tyniakov for advice and useful discussions.

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