On the form of local conservation laws for some relativistic field theories in 1+1 dimensions.

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Abstract

We investigate the possible form of local translation invariant conservation laws associated with the relativistic field equations $\partial \overline{\partial} \phi_i = -v_i(\phi)$ for a multicomponent field $\phi$. Under the assumptions that (i) the $v_i$’s can be expressed as linear combinations of partial derivatives $\partial w_j/\partial \phi_k$ of a set of functions $w_j(\phi)$, (ii) the space of functions spanned by the $w_j$’s is closed under partial derivations, and (iii) the fields $\phi$ take values in a simply connected space, the local conservation laws can either be transformed to the form $\partial \overline{\partial} \equiv \partial \overline{\partial} = \partial \sum_j w_j Q_j$ (where $P$ and $Q_j$ are homogeneous polynomials in the variables $\partial \phi_i$, $\partial^2 \phi_i$, . . .), or to the parity transformed version of this expression $\partial \equiv (\partial_t + \partial_x)/\sqrt{2} \Leftrightarrow \overline{\partial} \equiv (\partial_t - \partial_x)/\sqrt{2}$. 
1 Introduction

As has been known at least since the days of Liouville\cite{1} there exist exactly solvable non-linear field theories in 1+1 space-time dimensions. During the last few decades the knowledge of such models has increased dramatically. Examples of current interest are the Toda and affine Toda field theories\cite{2, 3}. These are (multi-component) field models with relativistic (and additional) symmetries. There is essentially one such model for each Lie algebra\cite{4} or affine Lie algebra\cite{5}. The ordinary Toda field theories are conformally invariant, while each affine Toda field theory may be viewed as a perturbation of an ordinary model away from its conformal limit. The simplest examples of such models are respectively the Liouville equation\cite{1},

\[ \partial \bar{\partial} \phi = -e^\phi, \]  

and the sinh-Gordon equation,

\[ \partial \bar{\partial} \phi = -e^\phi + \varepsilon e^{-\phi}, \]  

which formally can be viewed as a perturbation of the Liouville equation. (After the transformations $\phi \to \phi + \frac{1}{2} \log \varepsilon$, $(\partial, \bar{\partial}) \to 2\sqrt{\varepsilon} (\partial, \bar{\partial})$ equation (2) acquires the conventional form $\partial \bar{\partial} \phi = -\sinh \phi$.)

Since two-dimensional statistical models (with local interactions) can be described by conformal field theories at their critical points\cite{6, 7}, the (affine) Toda field theories may be useful for modelling the relevant degrees of freedom of many 2d statistical models at, and in the vicinity of, their critical points. It has been shown that the Toda field theories can be chosen to have critical exponents in agreement with the known rational exponents of a large class of such models\cite{8, 9}. This, of course, requires quantization or functional integral formulation, followed by a Wick rotation to Euclidean space. The (affine) Toda field theories also provide interesting examples of interacting relativistic quantum field theories.

Consider now the classical theories. As an integrable system each (affine) Toda field theory has an infinite set of independent conserved quantities in involution (so that the classical Poisson bracket of any two members of the set vanishes). Since the field theory is local, one expects the conserved quantities to be expressible as the integrated charges

\[ Q = \int dx \left( X - \bar{X} \right) \]

of a set of local conservation laws,

\[ \bar{\partial} X = \partial \bar{X}. \]  

Olive and Turok\cite{10} have found a method for calculating an infinite set of local conservation laws for most of the affine Toda field theories, and have shown that
the corresponding charges are in involution. It is a more subtle question to decide whether the set generated by this method is complete. Provided it is, it might be used to construct a complete set of action variables for the model. (For a complete description the canonically conjugate angle variables would also have to be found.)

It is believed that the integrability of the Toda field theories in some sense survives quantization \[11\] (although it in general is rather unclear what should be meant by ‘quantum integrability \[12\]). In the process of quantizing these models the problems of renormalization and operator ordering are encountered. In order to investigate these problems we felt it would be useful to have explicit expressions for some conserved currents beyond the energy-momentum tensor. For this purpose we first attempted to use the method of Olive and Turok \[10\] for explicit computations. However, we found it very cumbersome to proceed beyond the energy-momentum tensor. (This may perhaps be entirely due to our ineptness, but somewhat to our comfort there does not seem to be anyone else in the literature who has succeeded either.) In the end we found it simpler to use a rather pedestrian direct method, where we by systematic algebraic reduction narrow the possible forms of the conservations laws (of a given class) until a (possibly empty) set of solutions is found. The prospects for proving the existence of infinitely many conservation laws with such a procedure are of course rather meagre, but the method works quite satisfactory for the first few conservation laws—in particular with the assistance of an algebraic manipulation program. The procedure should be viewed as supplementary to the powerful method in reference \[10\], whose results also provides great encouragement for the searching process—since the sought-for quantities are already known to exist. But our method can also be applied to cases which are not covered by the method \[10\], and we have indeed discovered additional conservation laws through it.

The purpose of this paper is to perform the first step of the reduction process mentioned above, for a quite general class of models. This reduction is based on the fact that on the classical level there are many essentially equivalent ways of writing a conservation law (3), by modifications of the form

\[
X \rightarrow X + \partial Y, \quad \bar{X} \rightarrow \bar{X} + \bar{\partial} Y.
\]

We shall refer to this process as the addition of trivial conservation laws (although some of these ‘trivial’ laws in some Toda field theories may integrate to a non-zero topological charge).

Using this freedom alone we are able to restrict the form of the conservation laws to the extent stated in the Abstract. The remainder of this paper is devoted the proof of these results. The next section is mainly of illustrative purpose; here we first apply our procedure to the Liouville model, and next show how the obtained results can be generalized to a much wider class of one-component field theories. Following this discussion, the general case of multi-component fields is treated in the section 4.
We have used the results of this paper as the starting point for finding explicit expressions for the conservation laws in the ordinary and affine Toda field theories [13, 14].

2 Example: The Liouville Model

We shall in this section introduce our notation and basic ideas in the simplest example of a Toda field theory, i.e. the Liouville model—defined by a one-component field \( \phi \) satisfying the equation of motion (1), where \( \partial \equiv 2^{-1/2}(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}) \) and \( \bar{\partial} \equiv 2^{-1/2}(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}) \). A priori, a local translation invariant conservation law for this model must be of the form (3), where \( X \) and \( \bar{X} \) depend on \((t, x)\) through the variables \( u \equiv (\partial \phi, \partial^2 \phi, \partial^3 \phi, \ldots) \), \( \bar{u} \equiv (\bar{\partial} \phi, \bar{\partial}^2 \phi, \bar{\partial}^3 \phi, \ldots) \), and \( v \). Variables involving mixed derivatives, \( \partial^m \bar{\partial}^n \phi \) with \( mn \neq 0 \), can be eliminated by (repeated) use of the equation of motion (1). We consider the case that \( X \) and \( \bar{X} \) are finite polynomials in \( u, \bar{u}, \) and \( v \). Since equation (1) is invariant under the rescalings, \( \partial \rightarrow \Lambda \partial, \bar{\partial} \rightarrow \bar{\Lambda} \bar{\partial}, v \rightarrow \Lambda \bar{\Lambda} v \), (5) we may restrict our attention to currents which transform irreducibly under these, i.e. pairs \((X, \bar{X})\) such that

\[
X \rightarrow \Lambda^h \bar{\Lambda}^{h-1} X, \quad \bar{X} \rightarrow \Lambda^{h-1} \bar{\Lambda}^h \bar{X}.
\]

(6)

Thus \( X \) resp. \( \bar{X} \) must be a homogeneous polynomial of order \((h, \bar{h} - 1)\) resp. \((h - 1, \bar{h})\), counted according to the rules that each \( u_j = \partial^j \phi \) carries (conformal) weight \((j, 0)\), each \( \bar{u}_j \) weight \((0, j)\), and each \( v \) weight \((1, 1)\).

Consider now a conservation law for which \( h \leq \bar{h} \) (the case \( \bar{h} < h \) follows by interchanging \( \partial \) and \( \bar{\partial} \) throughout). We shall show that by appropriate (and possibly repeated) modifications (4) all dependency on the variable \( u \) may be removed from the expressions for the currents, and all dependency on \( v \) may be removed from the expression for \( \bar{X} \).

Focussing attention on the \( u \) dependence we may write

\[
\bar{X} = \sum_{\alpha} \bar{C}_\alpha(\bar{u}, v) u_1^{\alpha_1} \cdots u_2^{\alpha_2} u_h^{\alpha_h-1} \equiv \sum_{\alpha} \bar{C}_\alpha \bar{u}^\alpha,
\]

(7)

where the sum runs over vectors \( \alpha = (\alpha_{h-1}, \ldots, 0) \) of integers \( \alpha_j \geq 0 \) such that \( \sum_{j=1}^{h-1} j \alpha_j \leq h - 1 \). There is a similar expression for \( X \). We can arrange the vectors in lexical order (which is a total ordering), and identify the term in \( \bar{X} \) with the largest \( \alpha \)-vector,

\[
\hat{\alpha}(\bar{X}) \equiv \bar{a}.
\]

(8)

We generally let \( \hat{\alpha}(\cdot) \) denote the functional which extracts the largest \( \alpha \)-vector from a given expression. The term referred to above is \( \bar{C}_\bar{a} \bar{u}^\bar{a} \). Likewise, we can identify the term in \( X \) with the largest \( \alpha \)-vector,

\[
\hat{\alpha}(X) \equiv a,
\]

(9)
i.e. the term $C_{a}u^{a}$.

Note that if $\alpha = (0, \ldots, 0, \alpha_{j}, \ldots, \alpha_{1}) > (0, \ldots, 0, 0) \equiv 0$, with $j \leq h$ the largest integer for which $\alpha_{j} > 0$ ($j \leq h - 1$ in the case of $X$), then

$$\partial \alpha \equiv \hat{\alpha}(\partial u^{\alpha}) = (0, \ldots, 1, \alpha_{j} - 1, \ldots, \alpha_{1}) > \alpha, \quad (10)$$

$$\bar{\partial} \alpha \equiv \hat{\alpha}(\bar{\partial} u^{\alpha}) = (0, \ldots, 0, \alpha_{j}, \ldots, \alpha_{1} - 1) < \alpha, \quad (11)$$

(with rather obvious modifications of the explicit expressions if $j = 1$ or $\alpha_{1} = 0$).

For now, assume that $\bar{a} > 0$. Then, the term in $\partial X$ with the largest $\alpha$-vector is contained in the expression $\bar{C}_{a} \partial u^{a}$, and the term in $\bar{\partial} X$ with the largest $\alpha$-vector is contained in the expression $(\bar{\partial} C_{a}) u^{a}$. Note that we are assured to have $\bar{\partial} C_{a} \neq 0$, since by assumption $\bar{h} \geq h > 0$, the last inequality being a consequence of $\bar{a} > 0$. Since $\partial \bar{X} = \bar{\partial} X$, we must have that

$$\partial \bar{a} = a. \quad (12)$$

The statements above can be made more concise by introducing a notation for calculations which are exact only to leading order in the $\alpha$-sequence: Let

$$A \equiv \sum_{\alpha} C_{a}(A) u^{\alpha} \simeq B \equiv \sum_{\alpha} C_{a}(B) u^{\alpha} \quad (13)$$

denote that $\hat{\alpha}(A) = \hat{\alpha}(B) \equiv c$, and that $C_{c}(A) = C_{c}(B)$. Thus, we have that

$$X \simeq C_{a}u^{a}, \quad \bar{X} \simeq \bar{C}_{a}\bar{u}^{a}, \quad (14)$$

and, for $\bar{a} > 0$,

$$\partial \bar{X} \simeq \bar{C}_{a}\partial u^{a}, \quad \bar{\partial} \bar{X} \simeq (\bar{\partial} C_{a}) u^{a} \simeq (\bar{\partial} C_{a}) \partial u^{a} \simeq \bar{\partial} \left( C_{a} \partial u^{a} \right) \simeq \bar{\partial} \partial \left( C_{a} u^{a} \right). \quad (15)$$

The last two equivalences are true because of the inequalities $\hat{\alpha}(\bar{\partial} u^{a}) < \hat{\alpha}(\partial u^{a})$, and $\hat{\alpha}(u^{a}) < \hat{\alpha}(\bar{\partial} u^{a})$. Equation (15) means that the leading order part of the conservation law is trivial, and may be removed by a redefinition (1). This process may be continued until we end up with $\bar{a} = 0$ after a finite number of steps.

Thus, we have shown that $X$ can be reduced to a polynomial in $\bar{u}$ and $v$. It remains to show that also the dependency on $v$ can be eliminated by the addition of trivial conservation laws. It turns out that we may equally well assume the more general form $\bar{X} = \bar{X}(\bar{u}, \phi)$, and show that an arbitrary dependency on $\phi$ can be eliminated. Assuming this form we find

$$\partial \bar{X} \simeq (\partial \bar{X}/\partial \phi) \partial \phi, \quad (16)$$

provided $\partial \bar{X}/\partial \phi \neq 0$. This should be identified with

$$\bar{\partial} X \simeq (\bar{\partial} C_{a}) \partial \phi \simeq \bar{\partial} \partial D, \quad (17)$$

This should refer to the rest of the text.
where
\[ D(\bar{u}, \phi) = \int d\phi' C_a(\bar{u}, \phi') \]
is a well-defined function provided \( \phi \) takes values in a simple-connected space. Thus, the order \( \partial \phi \)-part of the conservation law above can be eliminated by the addition of a trivial conservation law, \( X \rightarrow X - \partial D, \bar{X} \rightarrow \bar{X} - \bar{\partial} D \), leaving us in the promised situation where \( \bar{X} = \mathcal{P}(\bar{u}) \). From this form it follows by the use of the equation of motion (1) that
\[ \partial \bar{X} = \sum_i \left( \partial \bar{X} / \partial \bar{u}_i \right) \partial \bar{u}_i = v \bar{Q}(\bar{u}), \]
where \( \bar{Q} \) is a homogeneous polynomial of rank \( \bar{h} - 1 \). This in turn requires \( X \) to be of the form
\[ X = v \bar{Q}(\bar{u}), \] (18)
so that
\[ \bar{\partial} X = v \left( \bar{u}_1 + \sum_i \bar{u}_{i+1} \frac{\partial}{\partial \bar{u}_i} \right) \bar{Q}(\bar{u}). \]
Thus, the conservation law (3) is satisfied if and only if
\[ \bar{Q}(\bar{u}) = \left( \bar{u}_1 + \sum_i \bar{u}_{i+1} \frac{\partial}{\partial \bar{u}_i} \right) Q(\bar{u}). \] (19)

Before closing this section we stress that the purpose of our presentation has been to introduce and illustrate the methods to be used on the general case below, not to find the already well known conservation laws for the Liouville model. Since the (conformally improved) energy momentum tensor for the Liouville model
\[ \bar{T} = \frac{1}{2} \bar{u}_1^2 - \bar{u}_2 = \frac{1}{2} (\bar{\partial} \phi)^2 - \bar{\partial}^2 \phi, \]
satisfies the conservation law \( \partial \bar{T} = 0 \), it follows that any polynomial \( \bar{Q}(\bar{T}) \) in the variables \( \bar{T} \equiv (\bar{T}, \bar{\partial} \bar{T}, \bar{\partial}^2 \bar{T}, \ldots) \) also satisfies \( \partial \bar{Q}(\bar{T}) = 0 \).

The reader should further note that almost no use have been made of the explicit expression for the force term \( v = v(\phi) \). We mainly needed the fact that \( \partial v / \partial \phi \propto v \) to derive the explicit form (18) for the current \( X \).

Therefore, the results of this section are easily generalized to more general forces \( v(\phi) \), i.e. those which are the gradients, \( v = -\partial w / \partial \phi \), of a potential \( w \) which satisfies an homogeneous, finite order, constant coefficient differential equation,
\[ \left( \frac{\partial^n}{\partial \phi^{n+1}} + c_n \frac{\partial^n}{\partial \phi^n} + \cdots + c_0 \right) w(\phi) = 0. \]
This includes all polynomial potentials. Then all translation invariant conservation laws (3) can be transformed to a form where

\[ \bar{X} = \mathcal{P}(\bar{u}), \quad X = \sum_{j=0}^{n} w_j Q_j(\bar{u}), \]  

(20)

with an \((n+1)\)-component vector

\[ \mathbf{w} \equiv (w, \partial w/\partial \phi, \ldots, \partial^n w/\partial \phi^n). \]

This, of course, agrees with the general form of the conserved energy momentum tensor,

\[ \bar{X} = \frac{1}{2} \bar{u}_2^2 - \alpha \bar{u}_2, \quad X = w_0 - 2\alpha w_1, \]

where \(\alpha\) can be adjusted arbitrarily by the addition of a trivial conservation law. It is also in agreement with the infinitely many conservation laws for a free field theory, \(v(\phi) = \phi\), where e.g. the next order conservation law reads

\[ \partial \bar{u}_2^2 = -\bar{\partial} \bar{u}_1^2, \]  

(21)

and sinh-Gordon model, \(v(\phi) = \sinh \phi\), where e.g. the next order conservation law reads

\[ \partial \left( \frac{1}{2} \bar{u}_1^2 - \bar{u}_2 \right)^2 = -\bar{\partial} \left( e^{-\phi} \bar{u}_1^2 \right). \]  

(22)

Note that these expressions exemplify the facts that

(i) some of the conservation laws for the conformal theory can be obtained as a limiting case of the affine conservation laws, by shifting the \(\phi\)-field by an infinite constant,

(ii) some of the conservation laws for the free field theory can be obtained as a limiting case of the affine conservation laws, by extracting the pieces which are of leading order as \(\phi \to 0\),

(iii) the conservation laws for the sine-Gordon model can be obtained from the sinh-Gordon conservation laws by making the replacement \(\phi \to i\phi\). This will lead to expressions which involve complex currents, but where the real and imaginary parts must be separately conserved. However, in each case only one of the parts constitutes a nontrivial conservation law.

Since our procedure is of a restrictive rather than constructive nature, the question whether there exist models beyond the examples above with additional conservation laws is left open. But by investigating limiting behaviours like those above, the possible form of such laws can be further constrained in each specific model.
3 The general case

In this section we show that the results of the previous section can be generalized to the models of a \(m\)-component field \(\phi = (\phi_1, \ldots, \phi_m)\) satisfying the equation

\[
\partial \bar{\partial} \phi = -v = -C w
\]

where \(C\) is a \(m \times n\) constant matrix, and each component of \(w = (w_1, \ldots, w_n)\) is an exponential expression

\[
w_i = \exp \left( \sum_{j=1}^{m} k_{ij} \phi_j \right).
\]

The most interesting special cases of (23) are the Toda resp. the affine Toda field equations (one equation for each Lie resp. affine Lie algebra). These are integrable systems with an infinite number of conservation laws. However, equation (23) is more than sufficient to restrict the form of the conserved currents to the level we intend in this paper. The analysis turns out to be a modest extension of our example in the previous section—most of the effort goes into establishing and explaining notation.

In a local translation invariant conservation law \(\bar{\partial} X = \partial \bar{X}\) the currents \(X, \bar{X}\) will a priori depend on \((x, t)\) through the matrices \(u\) \((u_{ij} \equiv \partial^j \phi_i)\), \(\bar{u}\) \((\bar{u}_{ij} \equiv \bar{\partial}^j \phi_i)\), and the vector \(w\). Equation (23) is formally invariant under rescalings

\[
\partial \rightarrow \Lambda \partial, \quad \bar{\partial} \rightarrow \bar{\Lambda} \bar{\partial}, \quad w \rightarrow \Lambda \bar{\Lambda} w,
\]

although the latter is not necessarily implementable through a transformation on the fields \(\phi\). However, a true symmetry is not necessary for our purposes. We only need a power counting assignment which is invariant under replacements like

\[
\partial \bar{\partial} \phi \rightarrow -v = -C w, \quad \partial w_i \rightarrow w_i \sum_j k_{ij} u_{j1}, \quad \bar{\partial} w_i \rightarrow w_i \sum_j k_{ij} \bar{u}_{j1}.
\]

Thus we may assign \(u_{ij}\) a weight \((j, 0)\), \(\bar{u}_{ij}\) a weight \((0, j)\), \(w_i\) a weight \((1, 1)\), and consider each conserved current to be a homogenous polynomial in these variables, \(X\) of weight \((\bar{\hbar}, \hbar - 1)\) and \(\bar{X}\) of weight \((\hbar - 1, \bar{\hbar})\). It is sufficient to consider the case \(\hbar \leq \bar{\hbar}\), since the opposite case follows by interchanging \(\partial\) and \(\bar{\partial}\) in all expressions. We begin by demonstrating that all dependence on the variable \(u\) may be eliminated from \(\bar{X}\) by the addition of trivial conservation laws. Focussing on the \(u\)-dependence we write

\[
\bar{X} = \sum_{\beta} \tilde{C}_\beta(\bar{u}, w) u^\beta \equiv \sum_{\beta} \tilde{C}_\beta(\bar{u}, w) \prod_{ij} u_{ij}^{\beta_{ij}},
\]

where \(\beta\) is a matrix of non-negative integers. For each \(\beta\) we define the vector \(\alpha(\beta)\) of integers by \(\alpha_j = \sum_i \beta_{ij}\), and conversely for each vector \(\alpha\) we define \(B(\alpha)\)
to be the inverse image of this mapping: \( \mathcal{B}(a) \equiv \{ \beta \mid \alpha(\beta) = a \} \). We again order the \( \alpha \)'s lexically, and let \( \hat{\alpha} (\cdot) \) denote the functional which extracts the largest \( \alpha \)-vector from a given expression. Thus, with

\[
a = \hat{\alpha}(X), \quad \mathcal{B} = \mathcal{B}(a), \quad \bar{a} = \hat{\alpha}(\bar{X}), \quad \bar{\mathcal{B}} = \mathcal{B}(\bar{a}),
\]

we have that

\[
X \simeq \sum_{\beta \in \mathcal{B}} C_{\beta} u^{\beta}, \quad \bar{X} \simeq \sum_{\beta \in \mathcal{B}} \bar{C}_{\beta} u^{\beta}
\]

to leading order in the \( \alpha \)'s. By analysing the action of \( \partial \) and \( \bar{\partial} \) one realizes that

\[
\hat{\alpha}(\bar{\partial} u^{\beta}) < \hat{\alpha}(u^{\beta}) < \hat{\alpha}(\partial u^{\beta}),
\]

provided \( \hat{\alpha}(u^{\beta}) > 0 \).

For now, assume that \( \bar{a} > 0 \). For each \( \beta \in \bar{\mathcal{B}} \) let \( \partial \beta \) be the set of \( \beta' \)-matrices such that \( \partial u^{\beta} \) contain a term proportional to \( u^{\beta'} \), i.e. so that

\[
\partial u^{\beta} = \sum_{\beta' \in \partial \beta} c_{\beta\beta'} u^{\beta'},
\]

with all the (integer) coefficients \( c_{\beta\beta'} > 0 \). Further let

\[
\partial b(\beta) = \{ \beta' \in \partial \beta \mid \alpha(\beta') = \hat{\alpha}(\partial \beta) \},
\]

so that

\[
\partial u^{\beta} \simeq \sum_{\beta' \in \partial b(\beta)} c_{\beta\beta'} u^{\beta'},
\]

and let

\[
\partial \bar{\mathcal{B}} \equiv \bigcup_{\beta \in \bar{\mathcal{B}}} \partial b(\beta).
\]

It turns out that the sets \( \partial b(\beta) \) are disjoint, \( \partial b(\beta_1) \cap \partial b(\beta_2) = \emptyset \) if \( \beta_1 \neq \beta_2 \), so that for any element in \( \beta' \in \partial \bar{\mathcal{B}} \) one may reconstruct the \( \beta \) from which it arose. This can be seen as follows: Assume that \( \beta \) is a \( r \times n \) matrix, with at least one non-zero entry in the \( n \)'th column. For each entry \( \beta_{in} \neq 0 \) of \( \beta \), there exists a \( r \times (n + 1) \) matrix \( \beta' \) in \( \partial b(\beta) \) with the properties that

(i) \( \beta'_{i,n+1} = 1 \),

(ii) \( \beta'_{in} = \beta_{in} - 1 \),

(iii) \( \beta'_{j,n+1} = 0 \) for all \( j \neq i \),

(iv) \( \beta'_{jm} = \beta_{jm} \) for all \( j \neq i, m \neq n \).
Thus, for any $\beta'$ in $\partial\bar{B}$ the corresponding $\beta$ is found by adding the last (non-vanishing) column to the next but last column.

We then have that

$$\partial\bar{X} \simeq \sum_{\beta \in B} \bar{C}_\beta \partial u^\beta \simeq \sum_{\beta \in B} \bar{C}_\beta c_{\beta\beta'} u^{\beta'} = \sum_{\beta' \in \partial\bar{B}} \bar{C}_{\beta'} u^{\beta'},$$

(32)

where $\bar{C}_{\beta'} \equiv \bar{C}_\beta c_{\beta\beta'}$ (no summation convention). Likewise

$$\bar{\partial}X \simeq \sum_{\beta \in B} (\bar{\partial}C_\beta) u^\beta.$$

(33)

Thus, by comparing (32) and (33) one finds that $\partial\bar{B} = B$, and

$$\bar{\partial}C_{\beta'} = \bar{C}_{\beta'} = \bar{C}_\beta c_{\beta\beta'} \quad \text{(no summation convention),}$$

(34)

since $\partial X = \partial\bar{X}$ by assumption.

For each $\beta' \in \partial b(\beta)$ let $\hat{C}_{\beta\beta'} = C_{\beta'}/c_{\beta\beta'}$ (by construction we are assured that $c_{\beta\beta'} \neq 0$). It follows from (34) that $\bar{\partial} \left( \bar{C}_{\beta\beta'_1} - \bar{C}_{\beta\beta'_2} \right) = 0$ for any two $\beta'_1, \beta'_2 \in \partial b(\beta)$. Thus, $\hat{C}_{\beta\beta'}$ is ‘almost’ independent of $\beta'$. To define a truly $\beta'$-independent quantity, let $\hat{C}_\beta$ be the average over $\beta'$ of all the $\hat{C}_{\beta\beta'}$’s. This leads to the representation

$$C_{\beta'} = \hat{C}_\beta c_{\beta\beta'} + C_{\beta'}^0,$$

(35)

where $\bar{\partial}C_{\beta'}^0 = 0$. Applying the various relations above to (32) we find

$$\partial\bar{X} \overset{(32)}{=} \sum_{\beta' \in \partial\bar{B}} \left( \bar{\partial}C_{\beta'} \right) u^{\beta'} \overset{(33)}{=} \sum_{\beta \in B} \left( \bar{\partial}\hat{C}_\beta c_{\beta\beta'} \right) u^{\beta'} = \sum_{\beta \in B} \left( \bar{\partial}\hat{C}_\beta \right) \sum_{\beta' \in \partial b(\beta)} c_{\beta\beta'} u^{\beta'}$$

$$\overset{(34)}{=} \sum_{\beta \in B} \left( \bar{\partial}\hat{C}_\beta \right) \partial u^\beta \overset{(37)}{=} \bar{\partial} \sum_{\beta \in B} \hat{C}_\beta \partial u^\beta \overset{(38)}{=} \bar{\partial} \bar{\partial} \sum_{\beta \in B} \hat{C}_\beta u^\beta.$$

Equation (36) demonstrates that the conservation law is trivial to leading order as long as $\bar{a} > 0$.

Having shown that $\bar{X}$ can be reduced to a polynomial in $\bar{u}$ and $w$ through a modification (4), it remains to prove that also the $w$-dependency can be eliminated in the same manner. We may equally well assume the more general form $\bar{X} = \bar{X}(\bar{u}, \phi)$, and show that an arbitrary dependency on $\phi$ can be eliminated. Assuming this form we find that

$$\partial\bar{X} \simeq \sum_{i} \left( \partial\bar{X}/\partial \phi_i \right) \partial \phi_i,$$

(37)
which requires that
\[ X \simeq \sum_i C_i(\bar{u}, \phi) \partial \phi_i, \quad \bar{D}X \simeq \sum_i (\partial C_i) \partial \phi_i. \] (38)

By comparing (37) and (38) one finds \( \partial \bar{X} / \partial \phi_i = \partial C_i \). Since \( \partial^2 \bar{X} / \partial \phi_i \partial \phi_j = \partial^2 \bar{X} / \partial \phi_j \partial \phi_i \) this in turn implies
\[ \bar{D} \left[ \frac{\partial C_i}{\partial \phi_j} - \frac{\partial C_j}{\partial \phi_i} \right] \equiv \bar{D}F_{ij} = 0. \] (39)

Thus, \( F_{ij} = -F_{ji} \) is independent of \( \bar{u} \) and \( \phi \). We may view \( C_i \) as the electro-

magnetic gauge potential corresponding a constant field strength \( F \). It is easy to verify that \( \frac{1}{2} \sum_j F_{ij} \phi_j \) is one possible solution for \( C_i \) (provided the space is simple-

connected, which we assume). The general form must be a gauge transformation of this potential, i.e.
\[ C_i = \frac{1}{2} \sum_j F_{ij} \phi_j + \frac{\partial D(\bar{u}, \phi)}{\partial \phi_i}. \] (40)

Thus
\[ X \simeq \sum_i \frac{\partial D}{\partial \phi_i} \partial \phi_i - \frac{1}{2} \sum_{ij} F_{ij} \phi_i \partial \phi_j \simeq \partial D - \frac{1}{2} \sum_{ij} F_{ij} \phi_i \partial \phi_j. \] (41)

By a modification \( \partial \bar{X} - \partial X = \sum_{ij} F_{ij} \phi_i v_j + \text{other terms} \), (43)

where the
\[ \text{other terms} = \sum_i \frac{\partial X^0(\bar{u}, \phi)}{\partial \phi_i} \bar{u}_{i+1} + \sum_{ij} \frac{\partial X^0(\bar{u}, \phi)}{\partial u_{i,j}} \bar{u}_{i+1,j} - \sum_{ij} \frac{\partial X^0(\bar{u}, \phi)}{\partial \bar{u}_{ij}} \partial \bar{u}_{ij} \] (44)

are free from terms with the combination \( \phi_i v_j \). Thus, \( \bar{D}X \) cannot possibly be a conservation law unless \( F_{ij} = 0 \). By these arguments we have restricted \( \bar{X} \) to the form
\[ \bar{X} = \bar{P}(\bar{u}) \] (45)
where \( \bar{P} \) is a homogeneous polynomial of rank \((0, \bar{h})\). From this form it follows by the use of the equation of motion (23) that

\[
\partial \bar{X} = \sum_{ij} \left( \partial \bar{X} / \partial \bar{u}_{ij} \right) \partial \bar{u}_{ij} = \sum_i w_i \bar{Q}_i(\bar{u}),
\]

where the \( \bar{Q}_i \)'s are homogeneous polynomials of rank \((0, \bar{h} - 1)\). This in turn requires \( X \) to be of the form

\[
X = \sum_i w_i \bar{Q}_i(\bar{u}), \tag{46}
\]

so that

\[
\bar{\partial} X = \sum_i w_i \left( \sum_j k_{ij} \bar{u}_{j1} + \sum_{j\ell} \bar{u}_{j,\ell+1} \frac{\partial}{\partial \bar{u}_{j\ell}} \right) \bar{Q}_i.
\]

Thus, the conservation law (3) is satisfied if and only if

\[
\bar{Q}_i(\bar{u}) = \left( \sum_j k_{ij} \bar{u}_{j1} + \sum_{j\ell} \bar{u}_{j,\ell+1} \frac{\partial}{\partial \bar{u}_{j\ell}} \right) \bar{Q}_i(\bar{u}) \tag{47}
\]

for all \( i \).

As in the previous section we finally note that the results of this section can be extended to more general equations

\[
\partial \bar{\partial} \phi = -v,
\]

where the forces \( v_i \) may e.g. include polynomials in \( \phi \). The conserved currents may still be transformed to the form (45), (46), with a more general set of \( w_i \)'s. A sufficient requirement for the restriction to this form is that

(i) all forces \( v_i \) are expressible as linear combinations of derivatives \( \partial w_j / \partial \phi_k \).

(ii) the set of \( w_i \)'s is closed under differentiations, i.e. that each \( \partial w_j / \partial \phi_k \) is expressible as a linear combination of the \( w_i \)'s.

## 4 Conclusions

In conclusion, we have investigate the possible form of local translation invariant conservation laws associated with the relativistic field equations

\[
\partial \bar{\partial} \phi_i = -v_i(\phi)
\]

for a multicomponent field \( \phi \). Under the assumptions that

(i) the \( v_i \)'s can be expressed as linear combinations of partial derivatives \( \partial w_j / \partial \phi_k \) of a set of functions \( w_j(\phi) \),
(ii) the space of functions spanned by the $w_j$’s is closed under partial derivations, and

(iii) the fields $\phi$ take values in a simple-connected space,

the local conservation laws can be transformed to the either form

$$\partial \bar{P} = \bar{\partial} \sum_j w_j Q_j,$$  \hspace{1cm} (49)

where $\bar{P}$ and $Q_j$ are homogeneous polynomials in the variables $\bar{\partial}\phi_i$, $\bar{\partial}^2\phi_i$, . . . ),

or to the parity transformed form of (49), obtained by making the replacements $\partial \equiv \partial_t + \partial_x \Rightarrow \bar{\partial} \equiv \partial_t - \partial_x$ in all expressions.

Our results are in agreement with the conclusions of Freedman et. al. [15], who (among other things) sought for “covariant” forms,

$$\partial_{\mu} J^{\mu \alpha ... \omega} = 0,$$

of the conservation laws for the 2-dimensional sine-Gordon equation. They indeed found that the 4th order conservation law could be written in such a covariant form, but that the additional conservation laws were (in our sense) trivial. We have shown that this results extends to the conservation laws of all orders, and to all field theories in the class specified above.

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