Decomposition of optical force into conservative and nonconservative components

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We present a multipole expansion theory for optical force exerting on a particle immersed in generic monochromatic free-space optical field. Based on the theory, we have, for the first time, successfully decomposed the optical force on a spherical particle of arbitrary size into a conservative and a nonconservative parts, which are, respectively, written as a gradient of a scalar function and curl of a vector function in an explicit and analytical form. As a result, a scalar potential and a vector potential can be defined, up to gauge freedoms, for the optical force. The decomposition shed light on the understanding of the optical force and pave a new way to engineer optical force for various purposes such as equilibrium statistical mechanics as well as optical micromanipulation.

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I. MULTIPOLE EXPANSION OF OPTICAL FORCE

Physically, the period-averaged optical force acting on a particle in any monochromatic optical field is written in two terms, the interception (extinction) force \( \langle \vec{F}_{\text{int}} \rangle \) and recoil force \( \langle \vec{F}_{\text{rec}} \rangle \). Based on the T-matrix method [1, 2], the multipole field theory [3, 4], and the irreducible tensor theory [5], the optical force can be written in terms of the electric and magnetic multipoles of various orders induced on the particle. To be specific, the interception force \( \langle \vec{F}_{\text{int}} \rangle \) is given by

\[
\langle \vec{F}_{\text{int}} \rangle = \sum_{l=1}^{\infty} \langle \vec{F}_{\text{int}}^{(l)} \rangle,
\]

\[
\langle \vec{F}_{\text{int}}^{(l)} \rangle = \langle \vec{F}_{\text{int}}^{e(l)} \rangle + \langle \vec{F}_{\text{int}}^{m(l)} \rangle
\]

\[
\langle \vec{F}_{\text{int}}^{e(l)} \rangle = \frac{1}{2l!} \text{Re} \left( \nabla^{(l)} \vec{E}^{*} \right) \cdot \vec{\Omega}_{\text{elec}}^{(l)}
\]

\[
\langle \vec{F}_{\text{int}}^{m(l)} \rangle = \frac{1}{2l!} \text{Re} \left( \nabla^{(l)} \vec{B}^{*} \right) \cdot \vec{\Omega}_{\text{mag}}^{(l)}
\]

and the recoil force \( \langle \vec{F}_{\text{rec}} \rangle \) reads

\[
\langle \vec{F}_{\text{rec}} \rangle = \sum_{l=1}^{\infty} \langle \vec{F}_{\text{rec}}^{(l)} \rangle,
\]

\[
\langle \vec{F}_{\text{rec}}^{(l)} \rangle = \langle \vec{F}_{\text{rec}}^{x(l)} \rangle + \langle \vec{F}_{\text{rec}}^{m(l)} \rangle + \langle \vec{F}_{\text{rec}}^{x(l)} \rangle
\]
where $k$ is the wave number in the transparent (lossless) background medium where the particle is located, the quantities $\vec{E}$ and $\vec{B}$ denote, respectively, the incident electric and magnetic fields where the particle is immersed, $\epsilon$ is the Levi-Civita tensor, whose components $\epsilon_{ijk}$ are antisymmetric with respect to the permutation of any pair of indices, and the superscripts “e”, “m”, and “x” denote, respectively, the contribution due to the electric multipoles, magnetic multipoles and hybrid term. The electric and magnetic multipoles of order $l$ are described by totally symmetric and traceless rank-$l$ tensors $\hat{\mathbb{O}}^{(l)}_{\text{elec}}$ and $\hat{\mathbb{O}}^{(l)}_{\text{mag}}$, respectively. The lower order cases with $l = 1$, 2, 3, 4, and 5 correspond to the dipole, quadrupole, octupole, hexadecapole, and dotriacontapole, respectively. The multiple contribution between two tensors of ranks $l$ and $l'$, denoted by $\hat{\mathbb{A}}^{(l)} \otimes \hat{\mathbb{B}}^{(l')}$, yields a tensor of rank $l + l' - 2m$ given by

$$
\hat{\mathbb{A}}^{(l)} \otimes \hat{\mathbb{B}}^{(l')} = \hat{\mathbb{A}}^{(l)}_{i_1 i_2 \cdots i_m i_{m+1} i_{m+2} \cdots i_{m+l'}} \hat{\mathbb{B}}^{(l')}_{i_{m+1} i_{m+2} \cdots i_{m+l'}} \delta_{i_1 i_2 \cdots i_m i_{m+1} i_{m+2} \cdots i_{m+l'}}, \quad 0 \leq m \leq \min \{l, l'\},
$$

with the summation over repeated indices assumed. Here the tensor contraction is made consecutively over two nearest indices in two index sequences. Equations (1) and (2) imply that the interception force is simply the coupling of the electric multipoles with the external electric field, governed by Eq. (1a), plus the coupling between the magnetic multipoles and the external magnetic field, determined by Eq. (1d). The recoil force originates from the coupling between the electric multipoles of adjacent orders, denoted by Eq. (2a), the magnetic multipoles of adjacent orders, delineated by Eq. (2b), and the electric and magnetic multipoles of the same order, depicted by Eq. (2c). Equations (1) and (2) constitute the complete multipole expansion of optical force on any particle, up to arbitrary orders of multipoles.

In the following, we give, for a spherical particle, the electric (magnetic) 2$^l$-pole moments $\hat{\mathbb{O}}^{(l)}_{\text{elec}}$ $\hat{\mathbb{O}}^{(l)}_{\text{mag}}$, in terms of the Mie coefficients $a_l$ ($b_l$) and the multiple gradient of the incident electric (magnetic) field. For a general (non-spherical) particle, the multipole moments are determined by its T-matrix $\hat{\mathbb{O}}$, instead of the simple Mie coefficients. The T-matrix for a general non-spherical scatterer can only be evaluated numerically.
where \( l \) denotes the unit dyad of dimension 3, and the symbol \( \otimes \) represents the tensor product so that the term in the square brackets means \( m \) consecutive times of tensor product, resulting in a rank-\( l \) tensor independent of \( m \). In component form, it reads

\[
\mathbf{\hat{\gamma}}_{1l12\cdots l} = \mathbf{S}_l \left[ \delta_{i_1 i_2} \delta_{i_3 i_4} \cdots \delta_{i_{l-2} i_{l-1}} \mathbf{M}_{1l12\cdots l} \right].
\]

It can be proved that

\[
\mathbf{n} \times \left[ \mathbf{n}^{(l-2m-1)} \mathbf{M}_{1l12\cdots l} \right] = \frac{l}{l-2m} \mathbf{n} \times \left[ \mathbf{n}^{(l-1)} \mathbf{M}_{1l12\cdots l} \right],
\]

where \( \mathbf{n} = \mathbf{r}/r \) is the unit vector in radial direction, and \( \mathbf{n}^{(m)} \) denotes a rank-\( m \) tensor resulting from the tensor product of \( m \) vectors \( \mathbf{n} \). For instance, \( \mathbf{n}^{(0)} = 1 \), \( \mathbf{n}^{(1)} = \mathbf{n} \), \( \mathbf{n}^{(2)} = \mathbf{n} \otimes \mathbf{n} \), and \( \mathbf{n}^{(3)} = \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{n} \). The totally symmetric and traceless multipole moments, \( \mathbf{O}_{\text{elec}}^{(l)} \) and \( \mathbf{O}_{\text{mag}}^{(l)} \), which are therefore usually referred to as 2\( l \)-pole moments, are derived based on theory of multipole fields [4].

\[
\mathbf{\hat{\gamma}}_{\text{elec}}^{(l)} = \mathbf{\hat{\gamma}}_{\text{mag}}^{(l)} = \gamma_{\text{elec}}^{(l)} \mathbf{M}_{\text{elec}}^{(l)},
\]

\[
d_{l,m} = \frac{1}{4m} \frac{l!}{m!} \frac{\Gamma(l-m+\frac{1}{2})}{\Gamma(l+\frac{1}{2})} \frac{1}{l!} \quad \text{with} \quad d_{l,0} = 1,
\]

where \( \lfloor x \rfloor \) gives the greatest integer less than or equal to \( x \) and \( \Gamma(x) \) denotes the Gamma function. The lower order cases with \( l = 1, 2, 3, 4, \) and \( 5 \) correspond to the dipole, quadrupole, octupole, hexadecapole, and dotriacontapole moments, respectively. For instance, \( \mathbf{O}_{\text{elec}}^{(1)} \) reduces to the electric dipole moment \( \mathbf{p}_e \), \( \mathbf{O}_{\text{mag}}^{(1)} \) is the magnetic dipole moment \( \mathbf{m}_m \), \( \mathbf{O}_{\text{elec}}^{(2)} \) represents the electric quadrupole moment \( \mathbf{Q}_e \), and \( \mathbf{O}_{\text{mag}}^{(2)} \) delineates the magnetic quadrupole moment \( \mathbf{Q}_m \).

The electric and magnetic polarizabilities, \( \gamma_{\text{elec}}^{(l)} \) and \( \gamma_{\text{mag}}^{(l)} \), depend on the Mie coefficients [3] \( a_l \) and \( b_l \) of a spherical particle through

\[
\gamma_{\text{elec}}^{(l)} = \frac{4l(2l+1)!!}{(l+1)} \frac{i \pi \varepsilon_0 a_l}{\varepsilon^{2l+1}}, \quad \gamma_{\text{mag}}^{(l)} = \frac{4l(2l+1)!!}{(l+1)} \frac{i \pi b_l}{\mu_0 \mu^{2l+1}}.
\]

In the Système International d’Unités (SI), the dimensions of \( \gamma_{\text{mag}}^{(l)} \) and \( \gamma_{\text{elec}}^{(l)} \) differ by a factor of \( c^2 \), viz, \( \gamma_{\text{mag}}^{(l)} / \gamma_{\text{elec}}^{(l)} = c^2 (b_l/a_l) \).

**II. DECOMPOSITION INTO GRADIENT AND CURL FORCES**

To decompose the optical force acting on a spherical particle into a sum of an irrotational term of zero curl and a solenoidal (divergenceless) term of zero divergence, or, equivalently, the conservative and non-conservative parts, it is convenient to write the optical force in terms of the multiple gradients \( \mathbf{\hat{\gamma}}_{\text{elec}}^{(l)} \) and \( \mathbf{\hat{\gamma}}_{\text{mag}}^{(l)} \) of the electric and magnetic fields defined in [4], instead of multipole moments \( \mathbf{O}_{\text{elec}}^{(l)} \) and \( \mathbf{O}_{\text{mag}}^{(l)} \). After lengthy algebra, the results read

\[
\langle F_{\text{int}}^{(l)} \rangle = \frac{1}{2l} \Re \sum_{m=0}^{l-1} c_{l,m} k^{4m} \mathbf{\hat{\gamma}}_{\text{elec}}^{(l)} \left[ \nabla^{(l-2m)} \mathbf{E}^* \right]_{\text{elec}},
\]

\[
\langle F_{\text{int}}^{(l)} \rangle = \frac{1}{2l} \Re \sum_{m=0}^{l-1} c_{l,m} k^{4m} \mathbf{\hat{\gamma}}_{\text{mag}}^{(l)} \left[ \nabla^{(l-2m)} \mathbf{B}^* \right]_{\text{mag}},
\]
The decomposition of the optical force into the irrotational and the solenoidal terms starts with Eqs. (9) and (10) as

\[
\langle \mathbf{F}_{\text{rec}}(l) \rangle = \frac{-1}{4\pi \varepsilon_0} \frac{(l + 2) k^{2l+3}}{(l + 1)! (2l + 3)!} \text{Im} \sum_{m=0}^{\lfloor l/2 \rfloor} g_{l,m} k^{4m+2} \eta_{\text{elec}}^{(l)} \frac{\gamma_{\text{elec}}^{(l-2m)} \gamma_{\text{elec}}^{(l-2m-1)}}{\gamma_{\text{elec}}^{(l-2m-1)}} + \text{Im} \sum_{m=0}^{\lfloor l/2 \rfloor} f_{l,m} k^{4m} \eta_{\text{mag}}^{(l)} \frac{\gamma_{\text{mag}}^{(l-2m)} \gamma_{\text{mag}}^{(l-2m-1)}}{\gamma_{\text{mag}}^{(l-2m-1)}} \quad (10a)
\]

\[
\langle \mathbf{F}_{\text{rec}}(l) \rangle = \frac{-1}{4\pi} \frac{(l + 2) k^{2l+3}}{(l + 1)! (2l + 3)!} \text{Im} \sum_{m=0}^{\lfloor l/2 \rfloor} g_{l,m} k^{4m+2} \eta_{\text{elec}}^{(l)} \frac{\gamma_{\text{elec}}^{(l-2m)} \gamma_{\text{elec}}^{(l-2m-1)}}{\gamma_{\text{elec}}^{(l-2m-1)}} + \text{Im} \sum_{m=0}^{\lfloor l/2 \rfloor} f_{l,m} k^{4m} \eta_{\text{mag}}^{(l)} \frac{\gamma_{\text{mag}}^{(l-2m)} \gamma_{\text{mag}}^{(l-2m-1)}}{\gamma_{\text{mag}}^{(l-2m-1)}} \quad (10b)
\]

\[
\langle \mathbf{F}_{\text{rec}}(l) \rangle = \frac{Z_0}{4\pi} \frac{k^{2l+2}}{l ! (2l + 1)!} \text{Re} \sum_{m=0}^{\lfloor (l-1)/2 \rfloor} h_{l,m} k^{4m} \eta_{\text{hyb}}^{(l)} \left[ \gamma_{\text{elec}}^{(l-2m)} \gamma_{\text{elec}}^{(l-2m-1)} \gamma_{\text{elec}}^{(l-2m-1)} \right] \frac{\gamma_{\text{elec}}^{(l-2m)} \gamma_{\text{elec}}^{(l-2m-1)}}{\gamma_{\text{elec}}^{(l-2m-1)}} \quad (10c)
\]

where \( \eta_{\text{elec}}^{(l)} = \gamma_{\text{elec}}^{(l)} \gamma_{\text{elec}}^{(l)} \), \( \eta_{\text{mag}} = \gamma_{\text{mag}}^{(l)} \gamma_{\text{mag}}^{(l)} \), and \( \eta_{\text{hyb}}^{(l)} = \gamma_{\text{elec}}^{(l)} \gamma_{\text{mag}}^{(l)} \), are products of polarizabilities. In deriving Eqs. (9) and (10), we have used the following mathematical identities

\[
\sum_{m=0}^{\lfloor (l-1)/2 \rfloor} f_{l,m} k^{4m} \eta_{\text{elec}}^{(l)} \frac{\gamma_{\text{elec}}^{(l-2m)} \gamma_{\text{elec}}^{(l-2m-1)}}{\gamma_{\text{elec}}^{(l-2m-1)}} + \sum_{m=0}^{\lfloor (l-1)/2 \rfloor} g_{l,m} k^{4m+2} \eta_{\text{elec}}^{(l)} \frac{\gamma_{\text{elec}}^{(l-2m)} \gamma_{\text{elec}}^{(l-2m-1)}}{\gamma_{\text{elec}}^{(l-2m-1)}}
\]

and

\[
\left[ \sum_{m=0}^{\lfloor (l-1)/2 \rfloor} f_{l,m} k^{4m} \eta_{\text{elec}}^{(l)} \frac{\gamma_{\text{elec}}^{(l-2m)} \gamma_{\text{elec}}^{(l-2m-1)}}{\gamma_{\text{elec}}^{(l-2m-1)}} \right] \frac{\gamma_{\text{elec}}^{(l-2m)} \gamma_{\text{elec}}^{(l-2m-1)}}{\gamma_{\text{elec}}^{(l-2m-1)}} = \sum_{m=0}^{\lfloor (l-1)/2 \rfloor} h_{l,m} k^{4m} \left[ \gamma_{\text{hyb}}^{(l)} \right] \frac{\gamma_{\text{elec}}^{(l-2m)} \gamma_{\text{elec}}^{(l-2m-1)}}{\gamma_{\text{elec}}^{(l-2m-1)}} \quad (11b)
\]

with

\[
c_{l,m} = \frac{(-1)^m}{m!} \frac{l!}{\Gamma(l - m + \frac{1}{2})} \frac{\Gamma(l - 2m)}{l^2} \frac{\Gamma(l + 1/2) \Gamma(l - 2m)}{l(l+1)(2l-2m+1)} = \frac{(-1)^m}{m!} \frac{\Gamma(l - m + \frac{1}{2})}{l^2} \frac{\Gamma(l + 1/2) \Gamma(l - 2m)}{l(l+1)(2l-2m+1)} \quad (11c)
\]

\[
f_{l,m} = \frac{(-1)^m}{m!} \frac{l!}{\Gamma(l - m + \frac{1}{2})} \frac{\Gamma(l - 2m)}{l^2} \frac{\Gamma(l + 1/2) \Gamma(l - 2m)}{l(l+1)(2l-2m+1)} \quad (11d)
\]

\[
g_{l,m} = \frac{(-1)^m}{m!} \frac{l!}{\Gamma(l - m + \frac{1}{2})} \frac{\Gamma(l - 2m)}{l^2} \frac{\Gamma(l + 1/2) \Gamma(l - 2m)}{l(l+1)(2l-2m+1)} \quad (11e)
\]

\[
h_{l,m} = \frac{(-1)^m}{m!} \frac{l!}{\Gamma(l - m + \frac{1}{2})} \frac{\Gamma(l - 2m)}{l^2} \frac{\Gamma(l + 1/2) \Gamma(l - 2m)}{l(l+1)(2l-2m+1)} \quad (11f)
\]

The decomposition of the optical force into the irrotational and the solenoidal terms starts with Eqs. (9) and (10), as well as the definitions Eq. (10), To achieve high readability, we assume the particle is located in the regime with \( z > 0 \) and write the incident wave, in which the particle is immersed, in terms of the plane wave spectrum representation \( \langle 12 \rangle \), given below,

\[
\mathbf{E} = \mathbf{E}_{\text{inc}} = \int_{-\infty}^{\infty} \mathbf{e}^{k_{\parallel} \mathbf{r}} d\mathbf{u}_x d\mathbf{u}_y \quad \text{and} \quad \mathbf{H} = \mathbf{H}_{\text{inc}} = \frac{1}{Z_0} \int_{-\infty}^{\infty} \mathbf{h}^{k_{\parallel} \mathbf{r}} d\mathbf{u}_x d\mathbf{u}_y, \quad (12)
\]
where $k$ is the wave number in the background medium,
\[
\vec{u} = u_x \hat{e}_x + u_y \hat{e}_y + u_z \hat{e}_z, \quad \text{with} \quad u_z = \begin{cases} \sqrt{1 - u_x^2 - u_y^2}, & \text{if } u_x^2 + u_y^2 \leq 1 \\ i\sqrt{u_x^2 + u_y^2 - 1}, & \text{if } u_x^2 + u_y^2 > 1 \end{cases}, \quad \text{and} \quad \vec{u} \cdot \vec{u} = 1,
\]

whereas the electric and magnetic plane wave spectra $\vec{e}_\vec{u}$ and $\vec{h}_\vec{u}$ satisfy
\[
\vec{u} \cdot \vec{e}_\vec{u} = \vec{u} \cdot \vec{h}_\vec{u} = 0, \quad \vec{h}_\vec{u} = \vec{u} \times \vec{e}_\vec{u}, \quad \vec{e}_\vec{u} = -\vec{u} \times \vec{h}_\vec{u}. \tag{13}
\]

With the plane wave spectrum representation Eqs. \[12\], the multiple gradients in \[14\] can be written as
\[
\begin{align*}
\leftrightarrow^{\leftrightarrow^{(n)}}_{\text{elec}} &= \frac{(ik)^{n-1}}{n} \sum_{j=0}^{n-1} \int_{-\infty}^{\infty} \vec{u}^{(n-1-j)} \vec{e}_\vec{u}^{(j)} e^{ik\vec{u} \cdot \vec{r}} du_x du_y, \tag{14a} \\
\leftrightarrow^{\leftrightarrow^{(n)}}_{\text{mag}} &= \frac{(ik)^{n-1}}{n c} \sum_{j=0}^{n-1} \int_{-\infty}^{\infty} \vec{u}^{(n-1-j)} \vec{h}_\vec{u}^{(j)} e^{ik\vec{u} \cdot \vec{r}} du_x du_y, \tag{14b}
\end{align*}
\]

where $\vec{u}^{(n)}$ denotes the tensor product of $n$ vectors $\vec{u}$, viz, $\vec{u}^{(0)} = 1$, $\vec{u}^{(1)} = \vec{u}$, $\vec{u}^{(2)} = \vec{u} \otimes \vec{u}$, and $\vec{u}^{(3)} = \vec{u} \otimes \vec{u} \otimes \vec{u}$, with the symbol $\otimes$ representing the tensor product.

After lengthy algebra, the interception parts $\langle \vec{F}_{\text{int}}^{(e)(l)} \rangle$ and $\langle \vec{F}_{\text{int}}^{(m)(l)} \rangle$ of the optical force involving order $l$ multipoles can be rewritten as
\[
\begin{align*}
\langle \vec{F}_{\text{int}}^{(e)(l)} \rangle &= \frac{1}{2l!} \sum_{m=0}^{l} \frac{|l|_{\text{elec}}^{(l)}}{m_{\text{elec}}^{(l)}} c_{lm} k^{4m} \text{Re} \left[ j_{\text{elec}}^{(l)} t_{\text{elec}}^{(l-2m)} \right], \tag{15a} \\
\langle \vec{F}_{\text{int}}^{(m)(l)} \rangle &= \frac{1}{2l!} \sum_{m=0}^{l} \frac{|l|_{\text{mag}}^{(l)}}{m_{\text{mag}}^{(l)}} c_{lm} k^{4m} \text{Re} \left[ j_{\text{mag}}^{(l)} t_{\text{mag}}^{(l-2m)} \right], \tag{15b}
\end{align*}
\]

where
\[
\begin{align*}
t_{\text{elec}}^{(n)} &= \left[ \nabla^{(n)} \vec{E}^* \right] \leftrightarrow^{\leftrightarrow^{(n)}}_{\text{elec}} \\
&= \frac{1}{2} \left[ \nabla \vec{D}_{\text{ee}}^{(n)} - \nabla \times \vec{S}_{\text{ee}}^{(n)} - 2i\omega \text{Re} \vec{S}_{\text{em}}^{(n)} \right] \\
&\quad - \frac{(n-1)\omega^2}{2n} \left[ \nabla \vec{D}_{\text{mm}}^{(n-1)} - \nabla \times \vec{S}_{\text{mm}}^{(n-1)} - 2i\omega \text{Re} \vec{S}_{\text{em}}^{(n-1)} \right], \tag{15b} \\
t_{\text{mag}}^{(n)} &= \left[ \nabla^{(n)} \vec{B}^* \right] \leftrightarrow^{\leftrightarrow^{(n)}}_{\text{mag}} \\
&= \frac{1}{2} \left[ \nabla \vec{D}_{\text{mm}}^{(n)} - \nabla \times \vec{S}_{\text{mm}}^{(n)} - 2i\omega \text{Re} \vec{S}_{\text{em}}^{(n)} \right] \\
&\quad - \frac{(n-1)\omega^2}{2n c^2} \left[ \nabla \vec{D}_{\text{ee}}^{(n-1)} - \nabla \times \vec{S}_{\text{ee}}^{(n-1)} - 2i\omega \text{Re} \vec{S}_{\text{em}}^{(n-1)} \right]. \tag{15c}
\end{align*}
\]

The field moments in the reciprocal space are defined as follows.
\[
\begin{align*}
D_{\text{ee}}^{(n)} &= \left[ (\nabla^{(n-1)} \vec{E})^{(n)} \times (\nabla^{(n-1)} \vec{E}^*) \right], \\
D_{\text{mm}}^{(n)} &= \left[ (\nabla^{(n-1)} \vec{B}) \times (\nabla^{(n-1)} \vec{B}^*) \right], \\
\vec{S}_{\text{ee}}^{(n)} &= \left[ (\nabla^{(n-1)} \vec{E})^{(n-1)} \times (\nabla^{(n-1)} \vec{E}^*) \right]^{(2)} \hat{\epsilon}, \\
\vec{S}_{\text{mm}}^{(n)} &= \left[ (\nabla^{(n-1)} \vec{B})^{(n-1)} \times (\nabla^{(n-1)} \vec{B}^*) \right]^{(2)} \hat{\epsilon}, \\
\vec{S}_{\text{em}}^{(n)} &= \left[ (\nabla^{(n-1)} \vec{E})^{(n-1)} \times (\nabla^{(n-1)} \vec{B}^*) \right]^{(2)} \hat{\epsilon}. \tag{16}
\end{align*}
\]
It is noted that $D_{ee}^{(n)}$ and $D_{mm}^{(n)}$ are real while $S_{em}^{(n)}$ and $\tilde{S}_{em}^{(n)}$ are purely imaginary. In Eqs. (10) we have defined the second kind of multiple tensor contraction, denoted by $(m) \cdot (n)$,

$$\tilde{\mathbf{A}}^{(m)} \cdot \tilde{\mathbf{B}}^{(n)} = \tilde{A}_{k_1 k_2 \ldots k_m i_{m+1} i_{m+2} \ldots i_n} \tilde{B}_{k_1 k_2 \ldots k_m j_{m+1} j_{m+2} \ldots j_n}, \quad 0 \leq m \leq \min[n, n'] \quad (17)$$

which differs from Eq. (9) in that the tensor contraction is made over the corresponding left most indices, instead of over the nearest indices, in the two index sequences. Some simple examples are

$$\langle \tilde{\mathbf{F}}_{\text{rec}}^{(l)} \rangle = \frac{-c_l k^{2l+3}}{4\pi} \left\{ \sum_{m=0}^{[(l-1)/2]} f_{l,m} k^{4m} \text{Im} \left[ \eta_{\text{elec}}^{(l)} \tilde{r}_{\text{ee}}^{(l-2m)} \right] + \sum_{m=0}^{[(l-2)/2]} g_{l,m} k^{4m+2} \text{Im} \left[ \eta_{\text{elec}}^{(l)} \tilde{r}_{\text{ee}}^{(l-2m-1)} \right] \right\} \quad (19)$$

$$\langle \tilde{\mathbf{F}}_{\text{rec}}^{(l)} \rangle = -\frac{c_l k^{2l+3}}{4\pi} \left\{ \sum_{m=0}^{[(l-1)/2]} f_{l,m} k^{4m} \text{Im} \left[ \eta_{\text{mag}}^{(l)} \tilde{r}_{\text{mm}}^{(l-2m)} \right] + \sum_{m=0}^{[(l-2)/2]} g_{l,m} k^{4m+2} \text{Im} \left[ \eta_{\text{mag}}^{(l)} \tilde{r}_{\text{mm}}^{(l-2m-1)} \right] \right\} \quad (19)$$

where

$$\tilde{r}_{\text{ee}}^{(n)} = \left[ \tilde{M}_{\text{elec}}^{(n)} \ldots \tilde{M}_{\text{elec}}^{(n)} \right] = \left[ \tilde{M}_{\text{elec}}^{(n)} \right] \left[ \tilde{M}_{\text{elec}}^{(n)} \right] \ldots \left[ \tilde{M}_{\text{elec}}^{(n)} \right]$$

$$= \left[ \frac{\nabla D_{ee}^{(n)} - \nabla \times \tilde{S}_{ee}^{(n)} - 2i\omega \text{Re} \tilde{S}_{em}^{(n)}}{2(n+1)} + \frac{i\omega}{(n+1)} \tilde{S}_{em}^{(n)} \right]$$

$$- \frac{(n-1)\omega^2}{2(n+1)} \left[ \nabla D_{mm}^{(n-1)} - \nabla \times \tilde{S}_{mm}^{(n-1)} - 2i\omega \text{Re} \tilde{S}_{em}^{(n-1)} \right] \quad (20a)$$

$$\tilde{r}_{\text{mm}}^{(n)} = \left[ \tilde{M}_{\text{mag}}^{(n)} \ldots \tilde{M}_{\text{mag}}^{(n)} \right] = \left[ \tilde{M}_{\text{mag}}^{(n)} \right] \left[ \tilde{M}_{\text{mag}}^{(n)} \right] \ldots \left[ \tilde{M}_{\text{mag}}^{(n)} \right]$$

$$= \left[ \frac{\nabla D_{mm}^{(n)} - \nabla \times \tilde{S}_{mm}^{(n)} - 2i\omega \text{Re} \tilde{S}_{em}^{(n)}}{2(n+1)} + \frac{i\omega}{(n+1)c^2} \tilde{S}_{em}^{(n)} \right]$$

$$- \frac{(n-1)\omega^2}{2(n+1)c^2} \left[ \nabla D_{mm}^{(n-1)} - \nabla \times \tilde{S}_{mm}^{(n-1)} - 2i\omega \text{Re} \tilde{S}_{em}^{(n-1)} \right] \quad (20b)$$

with $c_l = (l+2)/[(l+1)(2l+3)]$.

Finally, the hybrid term $\langle \tilde{F}_{\text{rec}}^{(l)} \rangle$ of the recoil force, given by (10c), can be cast, for our purpose, into

$$\langle \tilde{\mathbf{F}}_{\text{rec}}^{(l)} \rangle = \frac{Z_0}{4\pi} \left\{ \sum_{m=0}^{[(l-1)/2]} \frac{k^{2l+2}}{l!(2l+1)!} h_{l,m} k^{4m} \text{Re} \left[ \eta_{\text{hyb}}^{(l)} \tilde{r}_{\text{ee}}^{(l-2m)} \right] \right\} \quad (21)$$

where, written in a symmetric form with respect to the electric and magnetic fields (possessing so called “electric-magnetic democracy” (13)),

$$\tilde{r}_{\text{em}}^{(n)} = \left[ \mathbb{M}_{\text{elec}}^{(n)} \ldots \mathbb{M}_{\text{elec}}^{(n)} \right] (2) \cdot \tilde{\epsilon}$$

$$= \left[ \frac{i(n-1)\omega}{2n c^2} \nabla \times \left( \tilde{S}_{ee}^{(n-1)} + c^2 \tilde{S}_{mm}^{(n-1)} \right) + \frac{i(n-1)(n-2)\omega k^2}{2n^2 c^2} \nabla \times \left[ \tilde{S}_{ee}^{(n-2)} + c^2 \tilde{S}_{mm}^{(n-2)} \right] \right] - \frac{1}{n} \tilde{S}_{em}^{(n)}$$

$$- \frac{(n-1)}{2n} \nabla \times \left( \tilde{S}_{em}^{(n-1)} - \tilde{S}_{me}^{(n-1)} \right) + \frac{(n-1)(n-2)k^2}{2n^2} \nabla \times \left[ \tilde{S}_{em}^{(n-2)} - \tilde{S}_{me}^{(n-2)} \right] + \frac{(n-1)k^2}{n^2} \tilde{S}_{em}^{(n-1)} \times (22)$$
The two new field moments, \( \mathbf{G}^{(n)}_{em} \) and \( \mathbf{G}^{(n)}_{me} \), in the reciprocal space are given by

\[
\mathbf{G}^{(n)}_{em} = [\nabla^{(n-1)} \mathbf{E}] \cdot [\nabla^{(n)} \mathbf{B}]^*, \quad \mathbf{G}^{(n)}_{me} = [\nabla^{(n-1)} \mathbf{B}] \cdot [\nabla^{(n)} \mathbf{E}]^*.
\]

(23)

In the dipolar limit, with \( l = 1 \), the hybrid term depends only on the complex Poynting vector \( \mathbf{S}_{em}^{(1)} = \mathbf{E} \times \mathbf{B}^* \), which is interpreted as a lateral optical force resulting from the Belinfante spin momentum \( \mathbf{S} \) in two-wave interference.

The recoil force, given by \( \mathbf{S}_{em}^{(n)} \), depends on \( \mathbf{S}_{em}^{(n)} \), which is neither irrotational nor solenoidal. So we need to decompose \( \mathbf{S}_{em}^{(n)} \) into a gradient and curl parts. To this end, we write

\[
\mathbf{S}_{em}^{(n)} = -\nabla \varphi_{s}^{(n)} + \nabla \times \psi_{s}^{(n)}.
\]

(24)

It is worked out that

\[
\varphi_{s}^{(n)} = -\frac{i\kappa^{2n}}{2\omega} \sum_{m=n}^{\infty} \frac{1}{k^{2m}} [D_{ee}^{(m)} - c^{2}D_{mm}^{(m)}],
\]

(25a)

\[
\psi_{s}^{(n)} = -\frac{4k^{2n-1}}{2} \sum_{m=n}^{\infty} \frac{1}{k^{2m}} [\mathbf{G}_{em}^{(m)} - \mathbf{G}_{me}^{(m)*}],
\]

(25b)

which result in

\[
\mathbf{S}_{em}^{(n)} = -\frac{k^{2n-2}}{2} \nabla \times \sum_{m=n}^{\infty} \frac{1}{k^{2m}} [\mathbf{G}_{em}^{(m)} - \mathbf{G}_{me}^{(m)*}] + \frac{i\kappa^{2n}}{2\omega} \nabla \sum_{m=n}^{\infty} \frac{1}{k^{2m}} [D_{ee}^{(m)} - c^{2}D_{mm}^{(m)}].
\]

(26)

Since \( D_{ee}^{(n)} \) and \( D_{mm}^{(n)} \) are real, the real part \( \text{Re} \mathbf{S}_{em}^{(n)} \) is divergenceless, as stated in the text after Eq. (18). Equations (19 - 22), together with (20), (25a), and (25b), constitute the decomposition of the recoil force into a conservative and a nonconservative parts.

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