CLIQUE POLYNOMIALS AND CHORDAL GRAPHS

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Abstract. The ordinary generating function of the number of complete subgraphs of $G$ is called a clique polynomial of $G$ and is denoted by $C(G, x)$. A real root of $C(G, x)$ is called a clique root of the graph $G$. Hajiabolhasan and Mehrabadi showed that the clique polynomial has always a real root in the interval $[-1, 0)$. Moreover, they showed that the class of triangle-free graphs has only clique roots. Here, we generalize their result by showing that the class of $K_4$-free chordal graphs has also only clique roots. Moreover, we show that this class has always a clique root $-1$. We finally conclude the paper with several important questions and conjectures.

1. Introduction and Motivation

For a finite and simple graph $G = (V, E)$, a complete subgraph of $G$ on $k$ vertices is called a $k$-clique. For a subset $U \subseteq V(G)$, the subgraph induced on $U$ will be denoted by $G[U]$. We recall that an edge which joins two vertices of a cycle but is not itself an edge of the cycle is called a chord of the cycle. A graph is chordal if each cycle of length at least four has a chord. We also recall that the Clique Polynomial [1] of the graph $G$, denote it by $C(G, x)$ is defined, as follows

$$C(G, x) := 1 + \sum_{\emptyset \neq U \subseteq V(G) : G[U] \text{ is a clique}} x^{|U|},$$

or equivalently

$$C(G, x) := 1 + \sum_{k=1}^{\omega(G)} c_k(G),$$

in which $c_k(G)$ is the number of $k$-cliques in $G$ and $\omega(G)$ is the size of the largest clique of $G$.

Hajiabolhasan and Mehrabadi [1] showed that the clique polynomial of any simple graph has always a real root in the interval $[-1, 0)$. We will call it the clique root of $G$. They also showed that class of triangle-free graphs has only clique roots.

In the recent years, proving that a particular class of graphs polynomial has only real roots has attracted the attention of many researchers working in the area of algebraic graph theory. This is partly because of the interesting properties of polynomials with only real roots like being unimodal.

Our main goal here is to contribute in this line of research by extending the
above result for the class of $K_4$-free chordal graphs. More precisely, we will prove the following.

**Theorem 1.1.** The class of $K_4$-free chordal graphs has only clique roots. In particular, they have always a clique root $-1$.

We will also give the following immediate corollary of the above theorem, which is indeed a new algebraic proof of Turan’s graph theorem for planar $K_4$-free graphs.

**Corollary 1.2.** If $G$ is a $K_4$-free connected planar graph with $n$ vertices and $m$ edges, then we have

\[
m \leq \frac{n^2}{3}
\]

2. **Chordal Graphs and Clique Polynomials**

In this section, we investigate the important class of chordal graphs. This class of graphs is very important in computer science, specially from the computational complexity viewpoint. Many hard problems in general graphs have easy solutions in the class of chordal graphs. As we will see, the clique polynomial of chordal graphs can give us important insights into the structure of these graphs.

**Definition 2.1.** A graph is chordal if every cycle of length greater than three has a chord. A vertex of a graph is simplicial if its neighbors induces a clique in the graph.

One of the important properties of a chordal graph is that it has always a clique decomposition. For the sake completeness, here we quickly review the idea of decomposing a chordal graph into cliques. For detailed information, one can refer to [1].

**Definition 2.2.** For given graphs $G_1$ and $G_2$, we say that a graph $G$ arises from $G_1$ and $G_2$ by pasting along $S$ if we have $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = S$. In this case, the graphs $G_i$ are called the simplicial summands of $G$.

**Remark 2.1.** From the above definition, it is clear that a graph is chordal if it can be constructed recursively by pasting complete graphs along cliques. It is not hard to see that this process is independent of the order in which complete graphs paste to each other. Indeed, this recursive construction gives us a clique decomposition of chordal graphs which is essential to obtain their clique polynomials.

For simplicity of arguments, we use the notation $G_1 \cup_S G_2$ whenever $G_1$ and $G_2$ are pasted along $S$. The following lemma is key to obtain an explicit formula for the clique polynomial of chordal graphs. The proof is straightforward and left to the reader as a simple exercise.
Proposition 2.1. Let $G_1$ and $G_2$ be two simple graphs and $G = G_1 \cup Q G_2$ be their pasting along the $i$-clique $Q$. Then, we have

$$C(G, x) = C(G_1, x) + C(G_2, x) - (x + 1)^i, \quad (i \geq 1).$$

By the successive application of the formula (2.1) and the recursive construction of chordal graphs, we can obtain the following explicit formula for the clique polynomial of chordal graphs.

Theorem 2.2. Let $G$ be a chordal graph defined as a pasting of the complete graphs $\{G_i\}_{i=1}^r$ of sizes $n_i$’s, respectively. That is, $G = G_1 \cup Q_1 G_2 \cup Q_2 \cdots \cup Q_{r-1} G_r$, where $\{Q_j\}_{j=1}^{r-1}$ are cliques of sizes $l_j$’s, respectively. Then, we have

$$C(G, x) = \sum_{i=1}^r (x + 1)^{n_i} - \sum_{j=1}^{r-1} (x + 1)^{l_j}$$

As an immediate consequence of the above theorem, we have the following interesting result.

Corollary 2.3. Any chordal graphs $G$ without isolated vertices has always a clique root $-1$. The multiplicity of this root is equal to the size of the smallest clique in the pasting process of the recursive construction of $G$.

3. Proofs of the Main Results

Here, we first give a proof of the following proposition which is a weaker version of Theorem 1.1. From now on, for simplicity of arguments, we will assume that our graphs are connected.

Proposition 3.1. The class of $K_4$-free planar chordal graphs has only clique roots. In particular, $-1$ is always a clique root.

Proof. For a given $K_4$-free planar graph $G$, by Euler Formula, we have

$$n - m + f = 2,$$

where $n$, $m$ and $f$ are the number of vertices, edges and faces of a planar embedding of $G$, respectively. Moreover, if $G$ is a chordal graphs, then we observe that

$$f = t + 1,$$

where $t$ is the number of triangles (triangular faces) of $G$. Hence, form (3.1) and (3.2), we conclude that

$$n - m + t = 1,$$

or equivalently

$$1 - n + m - t = 0.$$ 

By last identity and considering the fact that the clique polynomial of a $K_4$-free graph $G$ is $C(G, x) = 1 + nx + mx^2 + tx^3$, we get

$$C(G, -1) = 1 - n + m - t = 0.$$
That is, the clique polynomial of any $K_4$-free planar chordal graph $G$ has always $-1$ as a clique root. Therefore, we obtain the following multiplicative decomposition of $C(G, x)$

\[(3.4) \quad C(G, x) = (1 + x)(1 + (n - 1)x + (m - n + 1)x^2).\]

The final step of the proof is to show that the quadratic polynomial:

\[(3.5) \quad Q(G, x) = 1 + (n - 1)x + (m - n + 1)x^2,\]

has always a real root. To this end, we actually prove that $Q(G, x)$ is a clique polynomial of a triangle-free graph $\tilde{G}$ which is obtained from the original graph $G$ based on the idea of a spanning tree of $G$.

For a given $K_4$-free chordal graph $G$, pick up an arbitrary vertex $v \in V(G)$. Now, we construct a spanning tree of $G$ rooted at the vertex $v$. We will denote it by $T_G$. Clearly, this tree has $n$ vertices and $n - 1$ edges. Now in the graph $G$, delete all $n - 1$ edges of the tree $T_G$ and call the resulting graph $\hat{G}$. This graph has clearly $n$ vertices and $m - (n - 1)$ edges. Finally, by construction $\hat{G}$ is triangle-free graph and $v$ is an isolated vertex. Thus, the graph $\tilde{G}$ obtained from $\hat{G}$ by deleting the vertex $v$ has $n - 1$ vertices and $m - n + 1$ edges, as required.

\[\square\]

**Remark 3.1.** The following figure shows the process of obtaining the graph $\tilde{G}$ from the original graph $G$ for a sample graph $G$.

\[
G: \quad \Rightarrow T_G: \quad \Rightarrow \hat{G}: \quad \Rightarrow \tilde{G}:
\]

Fig 1. The Construction of Triangle-Free graph $\tilde{G}$

Now, we are ready to give a proof of our main theorem.

**Proof of Theorem 1.1.** We first note that by Corollary 1.2, any connected chordal graph has always a clique root $-1$. Now, the rest of the proof is exactly the same as the proof of Proposition 3.1.

\[\square\]

Next we give an algebraic proof of Turan’s Graph Theorem [3] for $K_4$-free graphs which is indeed Corollary 1.2.

**Proof of Corollary 1.2.** We first note that since we want to prove an upper bound for the maximum possible number of edges, without loss of generality we assume that the graph $G$ is chordal.

As we saw in the proof of Theorem 1.1 if $G$ is a given $K_4$-free chordal graph with $n$ vertices and $m$ edges, then the following quadratic equation

\[Q(G, x) = 1 + (n - 1)x + (m - n + 1)x^2,\]
has only real zeros. Hence, its discriminant is nonnegative and therefore we have the following inequality:

\[(n - 1)^2 - 4(m - n + 1) \geq 0,\]

which is equivalent to

\[(n - 1)^2 - 4(m - n + 1) \geq 0,\]

which is equivalent to

\[(n + 1)^2 - 1 \leq \frac{n^2}{3},\]

On the other hand, we have the inequality

\[(n + 1)^2 - 1 \leq \frac{n^2}{3},\]

which is equivalent to the obvious inequality \((n - 3)^2 \geq 0\). Thus, the inequalities \((3.6)\) and \((3.7)\) immediately implies the Turan’s inequality for \(K_4\)-free graphs.

\[\square\]

4. Open problems and questions

We already showed that the class of connected \(K_4\)-free chordal graphs has only real roots. Now, one might ask whether the class of connected \(K_5\)-free chordal graphs has the same property or not.

Unfortunately, this is not true in general. For example, the graph \(K_4 + 1\) (a complete graph \(K_4\) plus one edge) has only two clique roots. Indeed, we have

\[C(K_4 + 1) = 1 + 5x + 7x^2 + 4x^3 + x^4 = (1 + x)(1 + 4x + 3x^2 + x^3).\]

Since the cubic polynomial \(\phi(x) = 1 + 4x + 3x^2 + x^3\) has the first derivative \(\phi'(x) = 3(x + 1)^2 + 1\) which is always positive, by the first derivative criteria, \(\phi(x) = 1 + 4x + 3x^2 + x^3\) has exactly one real root. Thus, we come up with the following first open question.

**Open Question 1.** Which subclasses of \(K_5\)-free chordal graphs have only clique roots?

Recall that the class of 3-trees are those graphs which can be constructed recursively by starting with a complete graph \(K_4\), and then repeatedly adding vertices in such a way that each added vertex has exactly three neighbors that form a clique (triangle).

By the above definition, it is not hard to see that the class of 3-trees is a subclass of \(K_5\)-free chordal graphs. Next, we come up with the following conjecture.

**Conjecture 1.** The class of 3-trees has only clique roots.

Considering the fact that any connected chordal graph has a clique root \(-1\) and the recursive definition of of chordal graphs, we made the following stronger conjecture.
Conjecture 2. The class of connected $K_5$-free chordal graphs with the clique root $-1$ of multiplicity 2 has only clique roots.

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