New approach to $K^0 - \bar{K}^0$ mixing

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Abstract

A new QCD calculation of the $B$ parameter of $K^0 - \bar{K}^0$ mixing is presented. It makes use of polynomial kernels in dispersion integrals in order to practically eliminate the contributions of the unknown pseudoscalar strange continuum. This approach avoids the arbitrariness and instability inherent to the Borel exponential kernels used in previous sum rules calculations. A simultaneous calculation of the mixed quark gluon condensate $$\langle g\bar{q}\sigma_{\mu\nu}\frac{\sigma^a}{2}\sigma_{\mu\nu}q \rangle$$ which enters in the expression for $B$ is presented. Finally the K-meson decay constant $f_K$ is calculated to five loops.

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1 Introduction

In the Standard Model (SM) the mixing of the two eigenstates of strangeness [1] is predicted as a higher order process [2] which contributes to the $K_L - K_S$ mass difference through the so-called $\Delta S = 2$ box diagram.

The $K_L - K_S$ mass difference $\Delta m$ is a sum of a long distance dispersive contribution $\Delta m_L$ and a short distance one $\Delta m_S$ proportioned to the matrix element

$$\langle \bar{K}^0(p')|\theta_{\Delta S=2}|K^0(p)\rangle$$ (1)

with

$$\theta_{\Delta S=2} = (\bar{s}\gamma_\mu(1-\gamma_5)d)(\bar{s}\gamma_\mu(1-\gamma_5)d)$$ (2)

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Neglecting anomalous dimension factors the parameter $B$ is defined as

$$\langle \bar{K}^0(p')|\theta_{\Delta S=2}|\bar{K}^0(p)\rangle = \frac{16}{3} B f_K^2 (p.p)$$

(3)

And $B = 1$ in the vacuum saturation approximation.

More sophisticated calculations of $B$ followed using quark and bag models, lattice calculations and QCD sum rule techniques [3–10]. Unfortunately no single value for $B$ has emerged.

I present here a new approach. Starting with a 3-pt function involving two pseudoscalar currents in addition to the $\Delta S = 2$ four quark operator

$$A(p,p')(p.p') = i^2 \int dx \, dy \, e^{ipx-ip'y} \langle 0|T j_5(x) \, \theta_{\Delta S=2}(0) \, j_5(y)|0 \rangle$$

(4)

Dispersion relations for this quantity will be written and intermediate states inserted. The $K$-meson poles carry the sought for information in addition there is the contribution of the strange pseudoscalar continuum of which not much is known except that it is dominated by two radial excitations of the $K$, $K (1460)$ and $K (1830)$. In order to damp the unknown contribution of the continuum Borel (Laplace) transforms of correlators have been used in which case the damping is provided by an exponential kernel $e^{-t/M^2}$. If the parameter $M^2$, the squared of the Borel mass, is small the damping is good but the contribution of the unknown higher order condensates increase rapidly. If $M^2$ increases the contribution of the unknown condensates decreases but the damping in the resonance region worsens. An intermediate value of $M^2$ has to be chosen. Because $M^2$ is an unphysical parameter the results should be independent of it in a relatively broad window which is not always the case. The choice of the parameter which signals the onset of perturbative QCD is another source of uncertainty.

In this work I shall use polynomial kernels in order to eliminate the contributions of the unknown continuum. The coefficients of these polynomials are chosen to make the roots coincide with the masses of the radial excitation of the $K$ and involve none of the arbitrariness and instability inherent to the use of exponential kernels.

$A(p,p')^{QCD}$ is made of a factorisable ($f$) and a non-factorisable ($nf$) part. The $nf$ part is dominated by the non-perturbative quark gluon mixed condensate.

$$g \langle \bar{q} \, \sigma_{\mu\nu} \frac{\lambda^2}{2} G_{\mu\nu}(a) \, q \rangle \equiv m_0^2 \langle \bar{q}q \rangle$$

(5)

Estimates of $m_0^2$ in the literature range from 0.3 $GeV^2$ to 1.2 $GeV^2$. The method used here allows for an independent determination of $m_0^2$ which will be used. As a final application the value the $K$ decay constant is calculated to 5-loops.
2 Calculation of the B parameter

Consider the 3-point function

\[ A(p, p') \langle p, p' \rangle = i^2 \int \ dx \ dy \ e^{ipx - ip'y} \ 0 | T j_5(x) \ 0 | j_5(y) \ 0 \]  

where \( j_5(x) = \bar{d}(x) i \gamma_5 s(x) \) is the pseudoscalar current. The amplitude \( A(t = p^2, t' = p'^2, p,p') \) will be studied at fixed \( p.p' \) and will be denoted by \( A(t, t') \).

\( A(t, t') \) possesses a double pole, two single poles and cuts on the real \( t, t' \) axes which extend from \( th = (m_K + 2m_\pi)^2 \) to infinity stemming from the strange pseudoscalar intermediate states.

\[ A(t, t')(p.p') = \frac{2f_K^2 m_K^4}{(m_s + m_d)^2(t - m_K^2)(t' - m_K^2)} \langle K^0 | \theta_{\Delta S=2} | \bar{K}^0 \rangle + \frac{\Phi(t)}{(t - m_K^2)} + \frac{\Phi(t')}{(t' - m_K^2)} + ... \]  

Consider now the double integral in the complex \( t \) and \( t' \) planes

\[ \frac{1}{(2\pi i)^2} \int_c \int_{c'} dt \ dt' P(t) P(t') A(t, t')(p.p') \]

where \( c \) and \( c' \) are the contours shown on fig.1, \( f_K \) is the \( K \) decay constant \( (f_K = .114 \ GeV) \) and \( P(t) \) is a so far arbitrary entire function.

Because \( \Phi(t), \Phi(t') \) have no singularities inside the contours of integration the single poles do not contribute to the double integral and we are left with

\[ \frac{2f_K^2 m_K^4}{(m_s + m_d)^2}(K^0 | \theta_{\Delta S=2} | \bar{K}^0) P^2(m_K^2) = \]

\[ \frac{1}{(2\pi i)^2} \int_{th}^R dt \ P(t) \int dt' P(t') Disc_{t,t'} A(t, t')(p.p') \]  

\[ + \frac{1}{(2\pi i)^2} \oint dt \ P(t) \oint dt' P(t') A(t, t')(p,p') \]  

The first integral on the r.h.s of eq. \( 8 \) represents the contribution of the pseudoscalar strange continuum. \( P(t) \) is now chosen to be a second order polynomial whose roots coincide with the masses squared of the radial excitations of the \( K, K(1460) \) and \( K(1870) \)

\[ P(t) = 1 - .768 \ GeV^{-2} t + .14 \ GeV^{-4} t^2 \]  

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$P(t)$ and $P(t')$ essentially eliminate the contributions of the continuum. On the circle of large radius $R$, $A(t, t')$ can safely be replaced by $A^{QCD}(t, t')$.

So that, using $\langle K^0 | \theta_{\Delta s=2} | \bar{K}^0 \rangle = \frac{16}{3} f_K^2 B(p,p')$ gives

$$32 \frac{f_K^4 m_K^4}{3 (m_s + m_d)^2} P^2(m_K^2) B = \frac{1}{(2\pi i)^2} \oint dt dt' P(t) P(t') A^{QCD}(t, t')$$  \hspace{1cm} (10)

$A^{QCD}(t, t')$ is the sum of a factorisable and a non-factorisable part \[4,5\]

$$A^{QCD} = A_{f}^{QCD} + A_{nf}^{QCD}$$  \hspace{1cm} (11)

where

$$A_{f}^{QCD} = \frac{8}{3} \Pi_5(t) \Pi_5(t')$$  \hspace{1cm} (12)

$$\Pi_5(t) = -\frac{3}{8\pi^2} m_s \ln(-t) + \frac{\langle dd + \bar{s}s \rangle}{t} - \frac{m_s \langle a_s GG \rangle}{8} \frac{1}{t^2} + \frac{0}{t^3} + ...$$  \hspace{1cm} (13)

and

$$A_{nf}^{QCD} = \frac{2}{3} m_0^2 \langle \bar{q}q \rangle^2 \left( \frac{1}{t^2} + \frac{1}{t^2 t} \right)$$

$$+ \frac{1}{4\pi^2} m_s^2 \langle \bar{q}q \rangle \frac{1}{tt'} \left( \ln(-\frac{t}{\mu^2}) + \ln(-\frac{t'}{\mu^2}) \right)$$

$$- \left[ \frac{4\pi^2}{9} \langle \bar{q}q \rangle^2 \langle a_s GG \rangle + \frac{13}{288} m_0^4 \langle \bar{q}q \rangle^2 \right] \frac{1}{t^2 t'} + ...$$  \hspace{1cm} (14)

The expressions for $B_f$ and $B_{nf}$ follow:

$$\frac{4 f_K^4 m_K^4}{(m_s + m_d)^2} P^2(m_K^2) B_f = I^2$$  \hspace{1cm} (15)

where

$$I = \frac{1}{2\pi i} \oint dt P(t) \Pi_5(t)$$

$$= -\frac{3m_s}{8\pi^2} \frac{1}{2\pi i} \oint dt P(t) \ln(-t) + \langle \bar{d}d + \bar{s}s \rangle + \frac{a_1 m_s}{8} \langle a_s GG \rangle$$  \hspace{1cm} (16)

Because $\ln(-t)$ posses a cut on the positive $t -$ axis which starts at the origin, the integral over the circle in the equation above can be transformed in an integral over the real axis so that
\[ I = -\frac{3m_s}{8\pi^2} \int_0^R dt \, P(t) + \langle \bar{d}d + \bar{s}s \rangle + \frac{a_1m_s}{8} \langle a_sGG \rangle \] (17)

The choice of \( R \) is determined by stability considerations. It should not be too small as this would invalidate the operator product expansion on the circle, nor should it be too large because \( P(t) \) would start enhancing the contribution of the continuum instead of suppressing it. We seek an intermediate range of \( R \) for which the integral (17) is stable.

As discussed above, our choice for \( P(t) \) is
\[ P(t) = 1 - 0.768t + 0.140t^2 \]
which vanishes at the radial excitations of the \( K \) and is very small in a broad interval around them. The integral \( i(R) = \int_0^R dt \, P(t) \) is seen to be stable for \( 2 \text{ GeV}^2 \lesssim R \lesssim 4 \text{ GeV}^2 \), \( i(R) \approx 0.83 \text{ GeV}^2 \) as shown in fig. 2. Then
\[ -(m_s + m_d)I = - (m_s + m_d) \langle \bar{d}d + \bar{s}s \rangle \]
\[ + \frac{3}{8\pi^2} m_s(m_s + m_d) i(R) - \frac{a_1m_s}{8} (m_s + m_d) \langle a_sGG \rangle \] (18)

Turn now to the contribution of the \( nf \) part. Similar manipulations lead to
\[ \frac{4f_k^4m_K^4}{(m_s + m_d)^2} P^2(m_K^2) B_{nf} = \]
\[ -\frac{1}{2} a_1 m_0^2 \langle \bar{q}q \rangle^2 + \frac{3m_0^2}{16\pi^2} \langle m_s\bar{q}q \rangle \ln \frac{R}{\mu^2} \]
\[ - \frac{3}{8} \frac{4\pi^2}{9} \langle \bar{q}q \rangle^2 \langle a_sGG \rangle + \frac{13}{288} m_0^4 \langle \bar{q}q \rangle^2 \] \( a_1^2 \) (19)

Values of \( m_0^2 \), which parametrizes the quark-gluon mixed condensate vary over a large range in the literature. The method presented here allows an independent evaluation of this quantity.

The integral
\[ \frac{1}{(2\pi i)^2} \int_c P(t)(t - m_K^2) \int_c dt' P(t')(t' - m_K^2) A^{QCD}(t, t') = I_f + I_{nf}' = 0 \] (20)
Vanishes because the singularities inside \( C, C' \) have been removed.
\[ I_f' = \frac{8}{3} I_f'^2 \] (21)
\[ I' = \frac{1}{2\pi i} \int dt \, P(t)(t - m_K^2) \left\{ -\frac{3m_s}{8\pi^2} \ln t - t + \frac{\langle \bar{d}d + \bar{s}s \rangle}{t} - \frac{m_s}{8} \langle a_sGG \rangle \frac{1}{t^2} \right\} \]
\[ = -m_K^2 \langle \bar{d}d + \bar{s}s \rangle - \frac{3m_s}{8\pi^2} i(R) - \frac{m_s}{8} \langle a_sGG \rangle (1 + a_1 m_K^2) \] (22)
where

\[ i'(R) = \int_0^R dt \, P(t)(t - m_K^2) \quad (23) \]

The n.f contribution is

\[ I'_{nf} = -\frac{4}{3} m_0^2 \langle \bar{q}q \rangle^2 \left( m_K^2(1 + a_1 m_K^2) - \frac{m_s m_0^2 \langle \bar{q}q \rangle}{2\pi^2 m_0^2} m_K^2 [i''(R) - m_K^2 \ln \frac{R}{\mu^2}] \right) \]

\[ - \left[ \frac{4\pi^2}{9} \langle \bar{q}q \rangle^2 \langle a_s G G \rangle + \frac{13}{288} m_0^4 \langle \bar{q}q \rangle^2 \right] (1 + a_1 m_a^2)^2 \]

\[ - \frac{1}{\mu^2} \int_0^R dt \, [1 + a_1 m_K^2 - (a_1 - a_2 m_K^2)t - a_2 t^2] \quad (24) \]

where

\[ i''(R) = \int_0^R dt \, [1 + a_1 m_K^2 - (a_1 - a_2 m_K^2)t - a_2 t^2] \quad (25) \]

Eqs (20, 25) determine \( m_0^2 \) which in turn yields \( B_{nf} \) when inserted in eq. (19). The condensate \( \langle \bar{d}d + \bar{s}s \rangle \) dominates our equations. This quantity was determined in [14] from a study of the 2-point correlator

\[ \Psi_5(t) = i \int dx \, e^{iqx} \langle 0 | T \partial_\mu A_\mu^d(x) \partial_\nu A_\nu^s(0) | 0 \rangle \]

with the result

\[ \Psi_5(0) = - (m_s + m_d) \langle \bar{d}d + \bar{s}s \rangle = 2f_K^2 m_K^2 P(m_K^2) + \delta_5 \]

\[ (26) \]

and it was obtained

\[ \delta_5 \approx \frac{1}{2\pi i} \int \frac{dt}{t} \frac{dt}{t} P(t) \, \Psi^{QCD}_5(t) = .0014 \, GeV^4 \]

or

\[ \Psi_5(0) = (.39 \pm .03) \times 10^{-2} \, GeV^4 \]

\[ (27) \]

This, with \( (m_s + m_d) = (108 \pm 8) \, MeV \), yield finally

\[ m_0^2 = .90 \, GeV^2 \]

\[ B = .55 \pm .08 \]

\[ (28) \]

(29)
3 $f_K$ to five loops

The method is further used in the calculation of the kaon decay constant $f_K$. Start with the correlator

$$\Pi_{\mu\nu}(t = q^2) = i \int dx \ e^{iqx} \langle 0 | T A_\mu(x) A_\nu(0) | 0 \rangle = (q_\mu q_\nu - q_{\mu\nu} q^2) \Pi^{(1)}(t) + q_\mu q_\nu \Pi^{(0)}(t)$$

Then consider

$$\Pi(t) = \Pi^{(0+1)}(t)$$

And consider

$$\int_0^R dt \ P(t) \Pi(t) = 2f_K^2 \Pi(m_K^2)$$

$$= \frac{1}{\pi} \int_0^R dt \ P(t) \ Im \ \Pi(t) + \frac{1}{2\pi i} \int dt \ P(t) \ \Pi^{QCD}(t)$$

(30)

As before the polynomial $P(t)$ is chosen in order to eliminate the contribution of the integral on the cut. We have now to take into account the axial-vector resonances in addition to the pseudoscalar ones. That is $K_1(1273)$ and $K_1(1402)$ in addition to $K(1460)$ and $K(1830)$. The choice

$$P(t) = 1 - 1.42t + .648t^2 - .093t^3$$

(31)

with the coefficients in appropriate powers of $GeV$ achieves the purpose of eliminating the contribution of the continuum. Here

$$\Pi^{QCD}(t) = \Pi_{pert}(t) + \frac{c_1}{t} + \frac{c_2}{t^2} + \frac{c_3}{t^3} + ...$$

(32)

$$4\pi \ Im \ \Pi_{pert}(t) = 1 + a_s(r) + a_s^2(r) l_2(t, r) + a_s^3(r) l_3(t, r) + a_s^4(r) l_4(t, r)$$

(33)

The $l_i(t, r)$ and the strong coupling constant $a_s(r)$ are known to 5-loop order [8,15] and the non-perturbative condensates are given by [9].

$$c_1 = + \frac{3m_s^2}{4\pi^2} (1 + \frac{7}{3} a_s)$$

$$c_2 = \frac{1}{12} (1 - \frac{11}{18} a_s) \langle a_s GG \rangle + (1 - \frac{a_s}{3}) \langle m_s \bar{s}s \rangle$$

$$c_3 = - a_s \frac{32\pi^2}{9} [ \langle \bar{q}q \rangle \langle \bar{s}s \rangle - \frac{1}{9} \langle qq \rangle^2 - \frac{1}{9} \langle \bar{s}s \rangle ]$$

(34)
The integral of $\Pi_{pert}(t)$ over the circle is transformed into an integral over the cut once again and finally

$$2f_K^2 P(m_K^2) = \frac{1}{\pi} \int_0^R dt \, P(t) \, Im \, \Pi_{pert}(t) + c_1 - a_1 c_2 - a_2 c^3$$  \hspace{1cm} (35)$$

The integral is stable for $1 \lesssim r \lesssim 3.5 \text{GeV}^2$ and the non perturbative contribution are small. The final result is

$$f_K = (0.110 \pm 0.003) \text{GeV}$$  \hspace{1cm} (36)$$

The error is obtained by varying the renormalization scale $\mu$ which enters in the QCD expressions $\Pi_{pert}$. 
Figure 2: The variation of $i(R) = \int_0^R dt \, P(t)$ as a function of $R$ in Gev.

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