Electromagnetic back-reaction from currents on a straight string

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Abstract

Charge carriers moving at the speed of light along a straight, superconducting cosmic string carry with them a logarithmically divergent slab of electromagnetic field energy. Thus no finite local input can induce a current that travels unimpeded to infinity. Rather, electromagnetic back-reaction must damp this current asymptotically to nothing. We compute this back-reaction and find that the electromagnetic fields and currents decline exactly as rapidly as necessary to prevent a divergence. We briefly discuss the corresponding gravitational situation.

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I. INTRODUCTION

Cosmic strings are effectively-one-dimensional objects that may have formed as topological defects during spontaneous symmetry breaking in the early universe, or as string theory objects at the end of “brane inflation” [1]. For a review, see Ref. [2]. Strings may be of cosmological and astronomical significance in a number of ways, such as producing CMB anisotropies [3, 4], sourcing gravitational waves [5], or gravitationally lensing astrophysical objects [6]. The strongest constraints on cosmic strings come from non-observation of gravitational waves in pulsar timing experiments [7, 8].

Superconductivity is a common property of cosmic strings. In many models, strings may carry a conserved current comprised of massless charge carriers [9]. In general the charge carriers might carry some charge other than the usual electromagnetic one, and so might be coupled to some other field or no field at all, but here we will consider only the case where the charge carriers on the string are electromagnetically charged. Superconductivity gives additional possibilities for detection of cosmic strings, such as emission of charge carriers that decay into cosmic rays [10] and emission of bursts of electromagnetic radiation [11].

In this paper we will consider the electromagnetic effects of charges and currents on a straight, static string. We will initially treat the string as a line of zero width, but later we will introduce, in a heuristic way, a length δ characteristic of the width of the string core.

In the absence of external fields, there is an exact solution giving the fields of charges and currents on a string [12]. For a string lying on the x axis, in cylindrical coordinates,

\[ E(x, \rho, \theta) = 2J^t(x)\rho^{-1}\hat{\rho}, \]

\[ B(x, \rho, \theta) = 2J^x(x)\rho^{-1}\hat{\theta}, \]

where \( J^x \) is the electric current flowing in the positive x direction and \( J^t \) the electric charge. Since there is no electric field pointing in the x direction, there is no back-reaction and the currents do not dissipate [12]. Note, however, that the fields in Eq. (1) have divergent total energy. Since the fields go as \( 1/\rho \), the energy density \( \varepsilon \) goes as \( 1/\rho^2 \). The energy per unit length \( \mathcal{E} \) is then

\[ \mathcal{E} = \int \varepsilon \rho d\rho d\theta \propto \int \frac{1}{\rho} d\rho, \]

which diverges logarithmically as \( \rho \to \infty \), even for a finite-width string.

Now suppose we attempt to induce a current on a string, by applying some external electric field for some finite time in a localized region. If the field points toward the right, we will produce some positive charge carriers moving to the right and some negative charge carriers moving to the left. In the absence of back-reaction, these charge carriers would move unimpeded to infinity, as shown in Fig. [1] and the solution would approach the exact solution of Eq. (1). But that solution is impossible to obtain, because our procedure can only supply a finite energy to the string, leaving us with a paradox.

Clearly back-reaction must decrease the currents. In fact, it must decrease the current and charge density asymptotically to zero, because the final solution to this problem cannot allow for any stationary currents in the final state. If any part of the initial train of charge carriers continues to propagate unchanged along the string (at some constant, nonzero population density), then the paradox still exists.

Indeed, we will find that the initial current, or more accurately, the induction of the initial current, leads at later times to an electric field that points oppositely to the applied field, which reduces the current. But how can the current be reduced? As the charge carriers are
FIG. 1. A train of charge carriers of length $L$, moving at the speed of light, carries with it a disk-shaped slab of electromagnetic energy.

massless, they move at the speed of light and cannot be slowed, and by charge conservation they cannot be destroyed. Instead, the effect of the field is to remove charge carriers from the initial current and create charge carriers with the same charge but the opposite direction of motion. Alternatively, we may say that the charge carriers scatter off the field and reverse their direction of motion. We will find that the electric fields and currents produced by back-reaction drop off exactly as quickly as is necessary to prevent this paradox, and that they eventually go to zero at infinity.

The rest of the paper is organized as follows: In Sec. II, we find an integral equation for the electric fields and currents resulting from a $\delta$-function source. In Sec. III, we solve this equation for the electric fields and currents on the future light cone of the point where the source is applied. In Sec. IV, we find the electric field and current in the interior of the future light-cone. In Sec. V, we find the current contained in some finite strip of the spacetime diagram. In Sec. VI, we examine the effects of a general source. We conclude in Sec. VII with a discussion of our results and how they might be relevant to gravitational back-reaction on a string subject to some spatial displacement.

We use metric signature $(+,-,-,-)$ and work in units where $c = 1$ and $\hbar = 1$. We express electromagnetic quantities in Gaussian units, so the squared electric charge is $q^2 = \alpha \approx 1/137$.

II. THE INDUCED ELECTRIC FIELD AND THE RESULTING CURRENT

Charges and currents on a string give rise to electric fields, which can induce further currents. Consider a string lying along the $x$ axis with some linear charge density $J^t(x, t)$ and current $J^x(x, t)$. The 3-dimensional charge-current density 4-vector is then $j^\mu(X) = J^\mu(x)\delta(y)\delta(z)$, and the induced electromagnetic potential in Lorenz gauge is

$$A^\mu(X) = \int G(X, X')j^\mu(X')d^4X' = \int G(x, t, x', t')J^\mu(x', t')dx' dt', \quad (3)$$

where $G$ is the retarded Green’s function.

We will henceforth be interested only in the electric field at locations on the string, and only in the component in the $x$ direction, because this is the field that leads to changes in the string currents. It is given in terms of the potential by

$$E_x(x, t) = F^{10} = -\partial_x A^t - \partial_t A^x. \quad (4)$$

After this, we will drop the subscript $x$ on $E$. 3
Applying the derivatives of Eq. (4) to Eq. (3) gives

\[ E(x,t) = - \int \left( J^t(x',t') \partial_x G(x,t,x',t') + J^x(x',t') \partial_t G(x,t,x',t') \right) dx'dt' \].

(5)

Since the Green’s function depends only on \( x - x' \) and \( t - t' \), we can write \( \partial_x G = -\partial_x' G \), \( \partial_t G = -\partial_t' G \). Then we integrate by parts to find

\[ E(x,t) = - \int G(x,t,x',t') \left( \partial_x' J^t(x',t') + \partial_t' J^x(x',t') \right) dx'dt' \].

(6)

The effect of an electric field on a superconducting string is to induce a current \[2, 9\],

\[ \partial_x J^t + \partial_t J^x = q^2 E_x \].

(7)

Putting Eq. (7) in Eq. (6), we see that an electric field applied to the string at time \( t' \) leads to an additional electric field at any later time \( t \),

\[ E(x,t) = -q^2 \int G(x,t,x',t') E(x',t') dx'dt' \].

(8)

Equation (8) is incomplete, because on the right-hand side \( E \) means the total electric field, including external sources, whereas on the left it means only the field induced by the string current. If we include some external field \( E_{ext} \), we have

\[ E(x,t) = E_{ext} - q^2 \int G(x,t,x',t') E(x',t') dx'dt' \].

(9)

This integral equation allows us, in principle, to find the electric field in the presence of a string responding to any given applied external field.

In the absence of any electric field, charge carriers move freely at the speed of light in the positive and negative \( x \) directions on the string. It is therefore convenient to change to a null coordinate system with

\[ u = \frac{t + x}{\sqrt{2}} \],

(10a)

\[ v = \frac{t - x}{\sqrt{2}} \].

(10b)

The retarded Green’s function in two dimensions in these coordinates is

\[ G = \frac{\delta(v - v')}{u - u'} + \frac{\delta(u - u')}{v - v'} \].

(11)

and Eq. (9) becomes

\[ E(u,v) = E_{ext} - Q \left( \int_0^u \frac{E(u',v)}{u - u'} du' + \int_0^v \frac{E(u,v')}{v - v'} dv' \right) \]

(12)

where \( Q = q^2 \). We interpret this integral equation as follows: The electric field at each point is given by an integral over all electric fields on that point’s past light cone. The integral
over \( u' \) gives the contribution from all fields earlier in time and to the left, while the integral over \( v' \) gives the contribution from all fields earlier in time and to the right.

Unfortunately, the integrals in Eq. (12) have a divergence at the upper limit of integration, which results from our treatment of the string as a one-dimensional object, giving it infinite self-inductance. To fix this problem, we add in a constant \( \delta \), characteristic of the string’s width, to both of the denominators, which gives

\[
E(u, v) = E_{\text{ext}} - Q \left( \int_{0}^{u} \frac{E(u', v)}{u - u' + \delta} \, du' + \int_{0}^{v} \frac{E(u, v')}{v - v' + \delta} \, dv' \right).
\] (13)

We now split up the current into parts that propagate to the right and to the left,

\[
J^u = \frac{J^t + J^x}{\sqrt{2}}, \quad \text{(14a)}
\]
\[
J^v = \frac{J^t - J^x}{\sqrt{2}}. \quad \text{(14b)}
\]

Using the continuity equation, \( \partial_t J^t + \partial_x J^x = 0 \), and Eq. (7), we find

\[
\partial_u J^u = -\partial_v J^v = \frac{Q}{2} E. \quad \text{(15)}
\]

A positive value of \( J^u \) represents positive charge carriers moving to the right. In the absence of any electric field, these carriers move freely at the speed of light, so \( J^u \) does not depend on \( u \). A positive value of \( J^v \) represents positive charge carriers moving to the left. In the absence of any electric field, these carriers move freely to the left, so \( J^v \) does not depend on \( v \).

Once we have determined the electric field, we can determine the current via

\[
J^u = \frac{Q}{2} \int_{0}^{u} E(u', v) \, du', \quad \text{(16a)}
\]
\[
J^v = -\frac{Q}{2} \int_{0}^{v} E(u, v') \, dv'. \quad \text{(16b)}
\]

### III. THE ELECTRIC FIELD AND CURRENT ON THE NULL LINES

We will start by considering the case where the applied field is a \( \delta \)-function at the origin, \( E_{\text{ext}} = \delta(x)\delta(t) = \delta(u)\delta(v) \). In this case, there will be singular currents and fields on the \( u \) and \( v \) axes and nonsingular currents in the chronological future of the origin, \( u, v > 0 \). We will find the singular currents and fields first.

We write the total field,

\[
E = E_{\text{ext}} + E_{\text{int}} + E_R + E_L, \quad \text{(17)}
\]

where \( E_{\text{int}} \) is the “interior” electric field in the chronological future of the origin and \( E_R, E_L \) are the singular fields on the positive \( u \) and \( v \) axes, respectively. A positive value of \( E_R \) or \( E_L \) represents an electric field pointing rightward and \( E_L(u, v) = E_R(v, u) \) by symmetry.

To find \( E_R \), we use Eq. (17) in Eq. (13) and only consider points \( (u, v) \) on the positive \( u \)-axis. Since there is nothing in the past to the right of the the right null line, the second
term of Eq. (13) does not contribute. Furthermore, we see that $E_{\text{int}}$ and $E_L$ cannot be in the past of the $u$-axis. Thus we use $E = E_{\text{ext}} + E_R$ on both sides of Eq. (13), and integrate the resulting $\delta(u)$, to get

$$E_R(u, v) = -Q \left( \frac{\delta(v)}{u + \delta} + \int_0^u \frac{E_R(u', v)}{u - u' + \delta} \, du' \right). \quad (18)$$

We can separate out the $\delta$-function in $E_R$ by writing

$$E_R(u, v) = -f(u) \delta(v). \quad (19)$$

Then $f$ satisfies

$$f(u) = h(u) - (f * h)(u). \quad (20)$$

where

$$h(z) = \frac{Q}{z + \delta}, \quad (21)$$

and $*$ indicates a convolution, defined by

$$(f * h)(u) = \int_0^u f(u') h(u - u') \, du'. \quad (22)$$

Then, with the Laplace transforms

$$\mathcal{L}[f](w) = F(w), \quad (23a)$$
$$\mathcal{L}[h](w) = H(w) = Qe^{\delta w}E_1(\delta w), \quad (23b)$$

where $E_1$ is the exponential integral function, we perform the transform of Eq. (20) to get

$$F(w) = H(w) - F(w)H(w), \quad (24)$$

and so

$$F(w) = \frac{H(w)}{1 + H(w)} = \frac{Qe^{\delta w}E_1(\delta w)}{1 + Qe^{\delta w}E_1(\delta w)}. \quad (25)$$

We now take the inverse Laplace transform of $F(w)$. As discussed in the appendix, $F(w)$ has a branch cut on the negative real axis and no singularities in the right half-plane, and does not diverge at $w = 0$. Thus the inverse Laplace transform is

$$f(u) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} F(w)e^{wu} \, dw. \quad (26)$$

Furthermore, the integrand falls quickly enough at infinity that we can deform the contour of integration to enclose only the negative real axis. Thus,

$$f(u) = \frac{1}{2\pi i} \int_0^{+\infty} (F(-a - i0) - F(-a + i0))e^{-au} \, da. \quad (27)$$

Now using the property of $E_1$ that

$$E_1(-x \pm i0) = -\text{Ei}(x) \mp i\pi, \quad (28)$$
and changing variables to $\alpha = \delta a$, we get

$$f(u) = \frac{Q}{\delta} \int_0^\infty \frac{e^{-\alpha(u/\delta+1)}}{(1 - Q e^{-\alpha} \text{Ei}(\alpha))^2 + (Q \pi e^{-\alpha})^2} d\alpha. \quad (29)$$

Equations (19,29) give an exact solution to Eq. (18), but the integral cannot be done in closed form, so we must make approximations. Since we inserted $\delta$ by hand we do not expect this expression to be good for $u$ comparable to $\delta$, so we take $u \gg \delta$. Then the exponential term in the numerator suppresses the integrand unless $\alpha \ll 1$, which means we can neglect factors of $e^{-\alpha}$ everywhere, and use the small-argument approximation $\text{Ei}(\alpha) = \ln \alpha + \gamma$, where $\gamma$ is the Euler-Mascheroni constant. This gives us a simplified integral equation,

$$f(u) = -\frac{Q}{\delta} \int_0^\infty \frac{e^{-\alpha u/\delta}}{(1 - Q (\ln \alpha + \gamma))^2 + (Q \pi)^2} d\alpha. \quad (30)$$

Now we make the much stronger approximation that $\ln(u/\delta) \gg 1$, meaning that $u$ is many orders of magnitude larger than $\delta$. This is justified because we are concerned with a paradox that occurs only when logarithms become large. Then we observe that for the great majority of the range of integration before the integral is cut off by the exponential at $\alpha \sim \delta/u$, $\ln \alpha$ has a value near $\ln(\delta/u)$. Thus we approximate the denominator in Eq. (30) by its value at $\alpha = \delta/u$ and perform the integral. We can ignore $\gamma$ by comparison with $\ln(\delta/u)$, and the second term in the denominator by comparison with the first, to get

$$f(u) = -\frac{Q}{u \left(1 + Q \ln(u/\delta)\right)^2}, \quad (31)$$

which gives, by Eq. (19),

$$E_R(u, v) = \frac{Q}{u \left(1 + Q \ln(u/\delta)\right)^2}. \quad (32)$$

Note that we do not consider $Q \ln(\alpha)$ to be much greater than 1, because $Q$ may be small.

We are now prepared to find the right-moving singular current. From Eqs. (16,19), we get

$$J_R^u(u, v) = \frac{Q}{2} \int_0^u \left(E_{\text{ext}}(u', v) + E_R(u', v)\right) du' = -\frac{Q}{2} \delta(v) \int_0^u \left(\delta(u') - f(u')\right) du'. \quad (33)$$

Because our approximations are not accurate for small $u$, it is better to change the range of integration, as follows. By the definition of the Laplace transform, $\int_0^\infty f(u) du = F(0) = 1$, so $\int_0^\infty (\delta(u) - f(u)) du = 0$, and thus

$$J_R^u(u, v) = \frac{Q}{2} \delta(v) \int_u^\infty f(u') du'. \quad (34)$$

This means that the current falls asymptotically to zero at late times, which is what we expect from our general arguments above.

This integral is easily done, and tells us that the right-going singular current is

$$J_R^u(u, v) = \frac{Q}{2} \frac{\delta(v)}{1 + Q \ln(u/\delta)}. \quad (35)$$
The expressions for the left-going singular electric field and current are found from $E_L(u,v) = E_R(v,u)$ and $J^e_L(u,v) = -J^e_R(v,u)$ and Eqs. (32,35). We see that the singular currents are always right-going and go to zero as $u$ or $v$ goes to infinity. The singular electric fields, after the initial kick, are negative and going to zero faster than the singular currents. Examining the singular current more closely along either the $u$ or $v$ axis, we see that half of the initial current has been scattered away when the varying coordinate reaches $\delta e^{1/Q}$. We will discuss the scale of this decline in Sec. [VII]

This solution for the singular currents solves the problem of the divergence of the energy contained in the electric field. The currents decline exactly as quickly as is required to cancel the logarithmic divergence along the null lines.

IV. THE ELECTRIC FIELD AND CURRENT ON THE INTERIOR

We are now interested in finding the electric field and currents in the chronological future of the source, $E_{\text{int}}$, $J^e_{\text{int}}$, and $J^e_{\text{ext}}$. We begin to solve for $E_{\text{int}}$ via the same process as for the singular electric field. We note that $E_{\text{ext}}$ is not on the past light-cone of any point in the interior and that we need both integrals from Eq. (13), which we now write as

$$E_{\text{int}}(u,v) = f(u)h(v) + f(v)h(u) - (E_{\text{int}}(\cdot,v)*h)(u) - (E_{\text{int}}(u,\cdot)*h)(v).$$

(36)

Where we solved for the singular field using one Laplace transform from the domain $u \to w$, here we will solve by taking a double Laplace transform from $u \to w$ and $v \to y$. We let $\mathcal{H} = \mathcal{L}[E_{\text{int}}]$, so

$$\mathcal{H}(w,y) = F(w)H(y) + F(y)H(w) - \mathcal{H}(w,y)(H(w) + H(y))$$

(37)

and

$$\mathcal{H}(w,y) = \frac{F(w)H(y) + F(y)H(w)}{1 + H(w) + H(y)} = \frac{1}{1 + H(w) + H(y)} - \frac{1}{1 + H(w)} - \frac{1}{1 + H(y)} + 1.$$  

(38)

The properties of $\mathcal{H}$ are discussed in the appendix. There are no singularities in the right half-plane, so the double inverse Laplace transform is

$$E_{\text{int}}(u,v) = -\frac{1}{4\pi^2} \int_{-i\infty}^{i\infty} e^{uw} dw \int_{-i\infty}^{i\infty} e^{vy} dy \mathcal{H}(w,y)$$

(39)

and we can deform the contours as before to find

$$E_{\text{int}}(u,v) = -\frac{1}{2\pi^2} \int_0^\infty \int_0^\infty \left[ \mathcal{H}(-a - i0, -b - i0) - \mathcal{H}(-a + i0, -b - i0) 
- \mathcal{H}(-a - i0, -b + i0) + \mathcal{H}(-a + i0, -b + i0) \right] e^{-\alpha u - \beta v} da db.$$ 

(40)

Note that only the first term of $\mathcal{H}$ depends on both $w$ and $y$; as a result, the subtractions in Eq. (40) cancel the other terms in pairs. We change variables to $\alpha = \delta a$ and $\beta = \delta b$, then simplify to obtain

$$E_{\text{int}}(u,v) = \frac{2}{\delta^2} \int_0^\infty \int_0^\infty \frac{CQ^2 e^{-\alpha(1+u/\delta)} - \beta(1+v/\delta)}}{C^4 + 2C^2 Q^2 \pi^2 (e^{-2\alpha} + e^{-2\beta}) + Q^4 \pi^4 (e^{-2\alpha} - e^{-2\beta})^2} da db.$$ 

(41)
where \( C = 1 - Q \left( e^{-\alpha \text{Ei}(\alpha)} + e^{-\beta \text{Ei}(\beta)} \right) \). We now make the approximations \( u, v \gg \delta \), which give

\[
\frac{2Q^2}{\delta^2} \int_0^\infty \int_0^\infty \frac{e^{-(\alpha u + \beta v)/\delta}}{(1 - Q(\ln(\alpha \beta) + 2\gamma))^3 + 4Q^2\pi^2(1 - Q(\ln(\alpha \beta) + 2\gamma))} \, d\alpha \, d\beta. \tag{42}
\]

Now, we apply the strong approximation \( \ln(u/\delta), \ln(v/\delta) \gg 1 \), as in Sec. III. Furthermore, the cubic term is much greater than the linear term, and so we ignore the linear term in its entirety. Thus, the interior electric field is

\[
E_{\text{int}}(u, v) = \frac{2Q^2}{uv (1 + Q \ln(uv/\delta^2))^3}. \tag{43}
\]

We see that the interior electric field always points to the right, going to zero when \( u \) or \( v \) goes to infinity. It declines faster than the singular electric field.

We may find the interior currents using Eq. (16). In the solution for \( J_{\text{int}}^u \), we see immediately that \( E_R \) and \( E_{\text{ext}} \) will not contribute, because we are interested in points for which \( v > 0 \). The integral of \( E_L(u', v) \) is trivial and yields \(-f(v)\). From the definition of the Laplace transform,

\[
\int_0^\infty E_{\text{int}}(u, v) \, du = \frac{1}{2\pi i} \int_{i\infty}^{i\infty} e^{yv} \frac{F(0)H(y) + F(y)H(0)}{1 + H(0) + H(y)} \, dy. \tag{44}
\]

Then, because \( H(0) = \infty \), this simplifies to

\[
\int_0^\infty E_{\text{int}}(u, v) \, du = \frac{1}{2\pi i} \int_{i\infty}^{i\infty} F(y)e^{yv} \, dy = \mathcal{L}^{-1}[F](v) = f(v). \tag{45}
\]

We see that

\[
\int_0^\infty E(u', v) \, du' = 0, \tag{46}
\]

and therefore

\[
\int_0^u E(u', v) \, du' = -\int_u^\infty E(u', v) \, du'. \tag{47}
\]

Using this in Eq. (16), we find

\[
J_{\text{int}}^u(u, v) = \frac{Q}{2} \int_0^u E_{\text{int}}(u', v) \, du' = -\frac{Q^2}{2v} \frac{1}{(1 + Q \ln(uv/\delta^2))^2}, \tag{48}
\]

and by symmetry

\[
J_{\text{int}}^v(u, v) = \frac{Q^2}{2u} \frac{1}{(1 + Q \ln(uv/\delta^2))^2}. \tag{49}
\]

We see that the interior currents are always left-going and go to zero as either \( u \) or \( v \) goes to infinity. They decline faster than the singular currents, and so the paradox is also solved for the interior.

The behavior of the interior and singular currents is summarized in Fig. 2.
FIG. 2. The quantities $J^u$ and $J^v$ represent currents flowing in the $u$ and $v$ directions, respectively. Since $J^R_R$ is positive and $J^L_L$ is negative, both represent right-going currents. Since $J^R_R$ is negative and $J^L_L$ is positive, both represent left-going currents.

V. THE CURRENT CONTAINED IN A STRIP

Since the singular current is right-going while the interior current is left-going, it is interesting to know how much these currents cancel. For example, suppose we measure the current in $u$ flowing over a range of $v'$ from 0 to $v$. Then we find

$$ I^u_{\text{strip}}(u, v) = \int_0^v (J^u_R(u, v') + J^u_{\text{int}}(u, v')) dv'. \quad (50) $$

Using Eqs. (19,34), we may write

$$ I^u_{\text{strip}}(u, \infty) = \frac{Q}{2} \int_0^\infty \int_0^\infty (\delta(v') f(u') - E_{\text{int}}(u', v') + \delta(u') f(v')) du' dv'. \quad (51) $$

The third term does not contribute for nonzero $u$. We then integrate over $v'$ first. In the first term we get $f(u')$. For the second term, Eq. (45) gives

$$ \int_0^\infty E_{\text{int}}(u', v') dv' = f(u'), \quad (52) $$

and thus $I^u_{\text{strip}}(u, \infty) = 0$. Then

$$ \int_0^v (J^u_R(u, v') + J^u_{\text{int}}(u, v')) dv' = - \int_v^\infty (J^u_R(u, v') + J^u_{\text{int}}(u, v')) dv'. \quad (53) $$

We may now solve for $I^u_{\text{strip}}$ in the strip for finite $v$. The result is

$$ I^u_{\text{strip}}(u, v) = \frac{Q}{2} \frac{1}{1 + Q \ln(uv/\delta^2)}. \quad (54) $$

Because this current is positive for any $(u, v)$, we conclude that the right-going singular current is larger in magnitude than the total right-going interior current for any strip of finite $v$. As the size of the strip becomes very large ($v \to \infty$), the scattered current will cancel out the singular current, so the total current in the strip approaches zero.

While we have only examined the right-going current, the left-going case is analogous.
VI. THE TOTAL CURRENT

We have found the current everywhere. It is now useful to define the total currents \( J_{\text{tot}}^u \) and \( J_{\text{tot}}^v \) by combining our solutions. Because these solutions are for a \( \delta \)-function applied field \( E_{\text{ext}} \), the total currents are the Green’s function current solutions. They are given by

\[
J_{\text{tot}}^u(u, v) = J_R^u(u, v)\theta(u) + J_{\text{int}}^u(u, v)\theta(u)\theta(v),
\]

\[
J_{\text{tot}}^v(u, v) = J_L^v(u, v)\theta(v) + J_{\text{int}}^v(u, v)\theta(u)\theta(v),
\]

where the Heaviside \( \theta \) functions ensure that we only look at points in the future of the origin.

We use this result to consider the effects of a non-singular \( E_{\text{ext}} \). If we apply an initial electric field over a finite space-time region, the resulting charges, currents, and fields can be found integrating the solution for the \( \delta \)-function source. For a general externally applied field \( E_{\text{ext}} \), the current in \( u \) is given by

\[
J^u(u, v; E_{\text{ext}}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_{\text{ext}}(u', v')J_{\text{tot}}^u(u - u', v - v')du'dv',
\]

and the current in \( v \) is analogous.

Now, we will consider the case where the applied electric field is a top-hat distribution with width \( d \),

\[
E_{\text{ext}}(x, t) = \delta(t)(\theta(x + d) - \theta(x)),
\]

or

\[
E_{\text{ext}}(u, v) = \sqrt{2}\delta(u + v)(\theta(u - v + \sqrt{2}d) - \theta(u - v)).
\]

If we first integrate over \( u' \), then Eq. (56) becomes

\[
J^u(u, v; E_{\text{ext}}) = \sqrt{2} \int_0^{d/\sqrt{2}} J_{\text{tot}}^u(u + v', v - v')dv'.
\]

Consider the current in a region that is \( u \)-like connected to the source distribution, \( v \leq d/\sqrt{2} \), as shown in Fig. 3. If we take \( u \gg d \), then we may neglect \( v' \) in comparison to \( u \), and Eq. (59)
becomes
\[ J^u(u, v; E_{\text{ext}}) = \sqrt{2} \int_0^{d/\sqrt{2}} (J^u_R(u, v - v') + J^u_{\text{int}}(u, v - v')\theta(v - v')) \, dv'. \] (60)

If we then consider where \( J^u_R(u, v - v') \) and \( \theta(v - v') \) are supported, we may change the upper limit of integration and rewrite Eq. (60) as
\[ J^u(u, v; E_{\text{ext}}) = \sqrt{2} \int_0^v (J^u_R(u, v - v') + J^u_{\text{int}}(u, v - v')) \, dv'. \] (61)

Using Eqs. (50,54), we find
\[ J^u(u, v; E_{\text{ext}}) = \frac{Q}{\sqrt{2}} \frac{1}{1 + Q \ln(uv/\delta^2)}. \] (62)

To what degree does the current retain the top-hat shape of the source? The current at the leading edge is only the singular current; the current at the trailing edge is lower by factor
\[ \frac{1 + Q \ln(u/\delta)}{1 + Q \ln(u/\delta) + Q \ln(v/\delta)}. \] (63)

This is never less than 1/2 and goes to 1 as \( u \) becomes exponentially larger than \( v \). So there is some distortion, but it is never large and decreases as the current diminishes over time.

VII. DISCUSSION

When an electric field is applied to a superconducting string, it induces a current, but the current is reduced because of the effect of self-inductance [2], which one can interpret as the need to create not only the currents on the string but the associated electromagnetic fields. The self-inductance is
\[ L = 2 \ln(R/\delta) \]
where \( R \) is some radius cutoff. However, in the case of a straight, static string exposed to a momentary field, there is no cutoff, so the self-inductance is formally infinite, and no persistent current can be produced. This is another way of describing the paradox that any persistent current would carry with it an infinite amount of field energy.

We found above that currents are induced initially, but they then drop inversely with \( \ln(u/\delta) \) or \( \ln(v/\delta) \). The inverse-logarithm drop-off cancels the logarithmic divergence that would otherwise occur in the electromagnetic field energy, preventing the paradox.

Since \( \ln(u/\delta) \) increases rapidly when \( u \sim \delta \), but then more and more slowly for increasing \( u \), the initial current drops off quickly at the start but the decrease becomes less and less when \( u \gg \delta \). Half of the singular current has vanished when \( u = \delta e^{1/Q} \). The time at which a fraction \( f \) of the right-moving singular current remains is given by
\[ u = \delta e^{(1-f)/(Qf)}. \]

Now let \( \mu \) be the cosmic string tension. Multiplying by Newton’s constant \( G \) gives a dimensionless measure \( G\mu \), which must be less than \( 10^{-8} \) to avoid conflict with pulsar timing observations [7, 8]. Strings of this scale would have a thickness \( \delta \approx 10^{-29} \) cm. For this \( \delta \), half of the current has vanished when \( u \) is about the size of the observable universe, so the back-reaction never decreases the current more than this. On the other hand, smaller decreases start quite rapidly; one quarter of the current has vanished when \( u \) is about one angstrom. Figure 4 shows the rapid initial drop-off of the current, followed by the long tail.
This study of electromagnetic back-reaction may be of theoretical interest in examining the gravitational back-reaction of a string. In this scenario, a static straight string is suddenly displaced in space in a local region. We can then predict how the left- and right-going kinks generated by this displacement evolve over time. We expect the effect of gravitational back-reaction to be similar to what we found here for electromagnetism, except that in place of the squared charge $Q$ the gravitational self-coupling will be proportional to $G\mu$. For a string with $G\mu \approx 10^{-8}$, the kinks would be damped by about one part in one million at the length scale of the observable universe. Thus, we would not expect any observable damping due to this effect in the gravitational case.

These results are applicable to straight strings and concern the divergent effects associated with a string of zero width or an infinite transverse space. In the case of loops, or infinite strings that are not straight, there will be other effects, which are not divergent [13], but nevertheless are likely to be larger in realistic situations than the effects we discuss here.

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Appendix A: Analytic structure of $F$ and $H$

In order to take inverse Laplace transforms, we need to know the analytic structure of the functions $F$ and $H$. First consider $F$, given by Eq. (25). The function $E_1(z)$ has a logarithmic singularity at $z = 0$ and a branch cut on the negative real axis, and is otherwise analytic. In Eq. (25) the logarithmic singularity cancels between the numerator and the denominator, so there is no divergence at the origin, but the branch cut remains.
The only possibility for additional singularities in $F$ not present in $H$ is for the denominator of Eq. (25) to vanish. We can show that this cannot happen as follows. From the definition of $E_1$, we can write $e^z E_1(z) = \int_0^\infty dt e^{-z/t}$ and take the contour to run first in the positive real direction and then in the imaginary direction at positive real infinity, where it does not contribute. Thus $E_1(z) e^z = \int_0^\infty dx e^{-x/(z + x)}$. For positive real $z$, the result is positive. For $z$ with positive imaginary part, $1/(z + x)$ always has negative imaginary part, and vice versa, so nowhere in the domain of definition of $E_1$ can $e^z E_1(z)$ be a negative real number. Thus $1 + H(z)$ can never vanish.

Deformation of the integration contour from $-i\infty \ldots i\infty$ to enclose only the negative real axis yields circular contours at radius $r \to \infty$. As $w \to \infty$, $E_1(w) \approx e^{-w}/w$. Thus, $F(w) \sim 1/w$ for large $|w|$. Thus a contour of radius $r$ yields $\int_0^{\pi/2} rd\theta \exp(-uw)/w$. The magnitude of the integrand is less than $\exp(-w)$ for large $w$. Nevertheless, it is still possible to deform the contours in Eq. (38). For each $w$, let us deform the contour in $y$. We may discover some isolated points $y_i$ where $H(y_i) = -1 - H(w)$. At such a point, there will be contributions of the form $2\pi i/H'(y_i)$. Now we will deform the contour in $w$. If there is any point where $H'(y_i) = 0$, there will be a contribution from deforming the contour across a pole. But in fact there is no such point. We have

$$H'(y) \propto e^y E_1(y) - \frac{1}{y} = \int_0^\infty dx e^{-x} \left[ \frac{1}{y + x} - \frac{1}{y} \right] = -\frac{1}{y} \int_0^\infty dx e^{-x} \frac{x}{y + x}$$

(A1)

If $y$ is positive and real, the integrand is always positive. If $y$ has positive imaginary part, then the integrand has negative imaginary part, and vice versa. So the integral can never vanish. Thus there are no poles in $1/H'(y_i)$.

As $y \to \infty$ with $w$ fixed, $H \sim 1/y$, so there is no contribution from contours at infinity, as above. Once we have deformed the $y$ contour, we can deform the $w$ contour with again a contribution $\sim 1/w$ in the $w \to \infty$ limit. Thus there is no obstacle to reaching Eq. (40).

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