Three-dimensional physics and the pressure of hot QCD

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Abstract

We update Monte Carlo simulations of the three-dimensional SU(3) + adjoint Higgs theory, by extrapolating carefully to the infinite volume and continuum limits, in order to estimate the contribution of the infrared modes to the pressure of hot QCD. The sum of infrared contributions beyond the known 4-loop order turns out to be a smooth function, of a reasonable magnitude and specific sign. Unfortunately, adding this function to the known 4-loop terms does not improve the match to four-dimensional lattice data, in spite of the fact that other quantities, such as correlation lengths, spatial string tension, or quark number susceptibilities, work well within the same setup. We outline possible ways to reduce the mismatch.

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1. Introduction

Motivated primarily by experimental heavy ion collision programs at RHIC, LHC and FAIR, there are on-going large-scale efforts to determine the pressure of hot Quantum Chromodynamics (QCD) through numerical Monte Carlo simulations of the full four-dimensional (4d) three-flavour [1]–[3] or even four-flavour [4] theory. Due to the significant cost of simulating light dynamical fermions near the chiral limit, these efforts typically make use of so-called staggered quarks. At any finite lattice spacing, the staggered action does not possess the flavour symmetries of the continuum theory, and is also problematic in the flavour-singlet sector. Although these problems will eventually be overcome by the use of less compromised fermion discretizations, it would appear welcome in the meanwhile to explore complementary avenues as well, in order to offer timely crosschecks on the results that are being produced.

One potentially useful avenue in this respect is provided by effective field theory methods. In the limit of a high temperature, $T$, and a small gauge coupling, $g$, QCD develops a hierarchy of three momentum scales, $\pi T \gg gT \gg g^2T/\pi$. The largest scale, $\pi T$, can be systematically integrated out, yielding a so-called Dimensionally Reduced effective field theory [5, 6], or “Electrostatic QCD” (EQCD) [7]. EQCD has previously been used to determine non-perturbatively quantities such as spatial correlation lengths [8], the spatial string tension [9]–[11], and quark number susceptibilities [12]. In all cases, a surprisingly good match to the results of 4d lattice simulations was found, even down to temperatures very close to that of the deconfining crossover, $T_c$ (see, e.g., refs. [13, 14]). We would therefore like to explore the extent to which a similar success could be achieved for the pressure (within pure Yang-Mills theory), and subsequently perhaps apply the same methods to physical QCD.

A way to apply EQCD to the study of the pressure on the non-perturbative level, as well as first numerical results, were put forward a number of years ago [15]. The practical results of ref. [15] suffered from two problems, however: certain 4-loop logarithmic terms, whose form was not understood at the time, were effectively missed [16]; and the systematic error from the continuum extrapolation carried out was probably underestimated, given that the approach to the continuum limit has turned out to be more delicate than anticipated [17]. The purpose of this paper is, therefore, to update the analysis of ref. [15], both by making use of the new theoretical ingredients in refs. [16, 17], and by increasing the numerical effort manifold. Unfortunately, the match to 4d lattice data does not improve despite all these efforts. On the other hand, the fact that systematic errors are now under control, implies that the discrepancy needs to be taken seriously, and this offers us the possibility to speculate on the kind of physics that might be missing in our approach. In particular, we wish to discuss the role of higher dimensional operators within the EQCD framework, as well as the new qualitative features that should be expected from more radically improved effective theories.

We start by specifying the general setup of our approach (Sec. 2); go on to discuss the details of the lattice formulation within EQCD (Sec. 3); present the main results (Sec. 4); and conclude with a discussion and outlook (Sec. 5).
2. General setup

At high temperatures, the pressure of QCD can be written as \[ p_{\text{QCD}}(T) = p_{\text{hard}}(T) + p_{\text{soft}}(T) . \] (2.1)

Here \( p_{\text{hard}} \) is a matching coefficient (defined in the \( \overline{\text{MS}} \) scheme) which gets contributions only from the hard scale, \( k \sim \pi T \), and is computable in perturbation theory, while

\[
p_{\text{soft}}(T) \equiv \lim_{V \to \infty} \frac{T}{V} \ln \int D A_0^a D A_0^a \exp(-S_E) \bigg|_{\overline{\text{MS}}},
\] (2.2)

where \( V = \int d^d x \) is the \( d \)-dimensional volume \( (d \equiv 3 - 2\epsilon) \), represents the contributions of the soft scales. The effective action can be written as

\[
S_E = \int d^d x \left\{ \frac{1}{2} \text{Tr} [F_{ij}^2] + \text{Tr} [D_i, A_0]^2 + m_3^2 \text{Tr} [A_0^2] + \lambda_3 (\text{Tr} [A_0^2])^2 + \ldots \right\}.
\] (2.3)

Here \( F_{ij} = (i/g_3^2) [D_i, D_j] \), \( D_i = \partial_i - i g_3 A_i \), \( A_i = A_i^a T^a \), \( A_0 = A_0^a T^a \), and \( T^a \) are hermitean generators of SU(3). In the following we set \( \epsilon \to 0 \) in all finite quantities, whereafter the dimensionalities of \( g_3^2 \) and \( \lambda_3 \) are GeV.

Now, \( p_{\text{soft}} \) is scale-dependent, just like \( p_{\text{hard}} \). Within the truncated form of Eq. (2.3), the scale dependence can be worked out explicitly [18]. Defining the dimensionless ratios

\[
x \equiv \frac{\lambda_3}{g_3^2},
\]

\[
y \equiv \frac{m_3^2 (\mu = g_3^2)}{g_3^4},
\]

and ignoring terms of \( O(g^8) \) in terms of 4d power counting, we can write

\[
p_{\text{soft}}(T) = -T g_3^6 \left\{ \mathcal{F}_{\overline{\text{MS}}}(x, y) + \frac{y d_A C_A}{(4\pi)^2} \ln \frac{\tilde{\mu}}{g_3^2} - \frac{d_A C_A^3}{(4\pi)^4} \left( \frac{43}{3} - \frac{27}{32} \pi^2 \right) \ln \frac{\tilde{\mu}}{g_3^2} \right\},
\] (2.6)

where \( \tilde{\mu} \) is the \( \overline{\text{MS}} \) scale parameter; \( \mathcal{F}_{\overline{\text{MS}}} \) is by definition the vacuum energy density of the (truncated) EQCD, computed with \( \tilde{\mu} = g_3^2 \) and scaled dimensionless by dividing with \( g_3^6 \); and \( d_A \equiv N_c^2 - 1, C_A \equiv N_c, N_c \equiv 3 \). The function \( \mathcal{F}_{\overline{\text{MS}}} \) can, in turn, be written as

\[
\mathcal{F}_{\overline{\text{MS}}}(x, y) = -\frac{d_A C_A^3}{(4\pi)^4} \left[ \left( \frac{43}{12} - \frac{157}{768} \pi^2 \right) \ln \frac{1}{2 C_A} + \beta_3 \right] + \mathcal{F}_{\overline{\text{MS}}}^{4\text{-loop}}(x, y) + \mathcal{F}_{\overline{\text{MS}}}^{R}(x, y).
\] (2.7)

Here \( \beta_3 = -0.2 \pm 0.8 \) is the non-perturbative “Linde term” from the pure three-dimensional Yang-Mills theory \[ \text{[19, 20]}, \] estimated numerically in refs. \[ \text{[21–23]}, \] while \( \mathcal{F}_{\overline{\text{MS}}}^{4\text{-loop}} \) is the 4-loop perturbative contribution sensitive to the adjoint Higgs field \( A_0 \) \[ \text{[18]}, \] specified for completeness in appendix A. The remainder, \( \mathcal{F}_{\overline{\text{MS}}}^{R} \), is what we address in the following.

For dimensional reasons, the function \( \mathcal{F}_{\overline{\text{MS}}}^{R} \) necessarily depends on the parameter \( y \) (it gets contributions from five loops and beyond and, before the rescaling with \( g_3^6 \), therefore contains
terms of the form $g_3^{8+2n}/m_3^{1+n}$, with $n \geq 0$). Thus no information is lost (i.e. no integration constant is needed) if we take a partial derivative with respect to $y$, leading to a condensate:

$$\partial_y F^{R}_{MS}(x, y) = \langle \text{Tr} [\hat{A}_0^2] \rangle_{MS} = \langle \text{Tr} [\hat{A}_0^2] \rangle_{MS, \mu=g_3^2} - \langle \text{Tr} [\hat{A}_0^2] \rangle^{4\text{-loop}}_{MS, \mu=g_3^2},$$

where $\hat{A}_0 \equiv A_0/g_3$. For a non-perturbative study, we need to change the scheme from $\overline{\text{MS}}$ to the lattice, and then the relation in Eq. (2.8) goes over into

$$\partial_y F^{R}_{MS}(x, y) = \lim_{a \to 0} \left\{ \langle \text{Tr} [\hat{A}_0^2] \rangle_a - y^{1/2} f_0(\hat{m}) - y^{-1/2} f_1(\hat{m}) - y f_2(\hat{m}) - y^{-1} f_3(\hat{m}) \right\},$$

where

$$\hat{m} \equiv a g_3^2 y^{1/2},$$

and $f_i$ are functions that have recently been determined numerically in ref. [17]. The task, therefore, is to measure $\langle \text{Tr} [\hat{A}_0^2] \rangle_a$ on the lattice, for values of $x$ and $y$ corresponding to the physical finite-temperature QCD (cf. Fig. 1), and insert then the result in Eq. (2.9).
3. Lattice simulations

The lattice study is carried out with the standard Wilson discretized action,
\[ S_a \equiv \beta \sum_{x} \sum_{i<j} \left( 1 - \frac{1}{3} \text{Re} \text{Tr} \left[ P_{ij}(x) \right] \right) - \frac{12}{\beta} \sum_{x,i} \text{Tr} \left[ \hat{A}_0(x) U_i(x) \hat{A}_0(x+i) U_i^\dagger(x) \right] \]
\[ + \sum_x \left\{ \alpha \text{Tr} \left[ \hat{A}_0^2(x) \right] + \frac{216 x_{\text{latt}}}{\beta} \left( \text{Tr} \left[ \hat{A}_0^2(x) \right] \right)^2 \right\} , \]  
(3.1)
where \( U_i(x) \) is a link matrix; \( x+i \equiv x + a \hat{e}_i \), with \( \hat{e}_i \) a unit vector; \( P_{ij}(x) \) is the plaquette; and
\[ \beta \equiv \frac{6}{g_3^8 a} . \] (3.2)

The bare mass parameter is given by [25]
\[ \alpha = \frac{36}{\beta} \left\{ 1 + \frac{6}{\beta^2} y_{\text{latt}} - (6 + 10 x_{\text{latt}}) \frac{3.175911535625}{4\pi \beta} \right. 
- \left. \frac{3}{8\pi^2 \beta^2} \left[ (60 x_{\text{latt}} - 20 x_{\text{latt}}^2)(\ln \beta + 0.08849) + 34.768 x_{\text{latt}} + 36.130 \right] \right\} . \] (3.3)

For historical reasons, we have implemented in most of the runs a partial \( O(a) \) improvement [26], by taking
\[ x_{\text{latt}} \equiv x + \frac{0.328432 - 0.835282 x + 1.167759 x^2}{\beta} , \] (3.4)
but for the mass parameter no improvement was carried out, since the additive \( O(a) \) terms have not been determined: \( y_{\text{latt}} \equiv y \). The improvement of \( x_{\text{latt}} \) turns out to play little practical role if properly taken into account in the subtraction of Eq. (2.9); in fact we did not implement it in the last sets of runs, corresponding to \( N_f = 2 \) as well as to \( \beta = 240 \) with \( N_f = 0 \). In any case our setup conforms with ref. [17], where the functions \( f_i \) in Eq. (2.9) were determined numerically for general \( x_{\text{latt}} \).

The input parameters in any run are \( x, y, \beta \). The phase diagram of the system in this space has a “disordered”, or symmetric phase, as well as a symmetry broken phase [27]. The simulations we carry out represent the physical QCD only in the symmetric phase [24]. It turns out that on the physical stripe (Fig. 1) the symmetric phase is actually metastable, but strongly enough so for any practical effects to be miniscule in large enough volumes (see, however, the discussion in Sec. 5).

The three-dimensional SU(3) + adjoint Higgs theory is a confining gauge theory, and possesses a mass gap (for a practical demonstration see, e.g., ref. [8]). Thereby all finite volume effects must be exponentially suppressed. For completeness we have checked this explicitly: the condensate \( \langle \text{Tr} [\hat{A}_0^2] \rangle_a \) as a function of \( \beta/N = 6/g_3^8 L \), where \( L \) is the extent of the box, is shown in Fig. 2. No finite-volume effects are visible in the range \( \beta/N < 1 \), or \( L > 6/g_3^8 \), to which we restrict in the following. The parameters used for the production runs are listed in Table 1. 
Figure 2: Finite-volume values for \( \langle \text{Tr}[\hat{A}_0^2] \rangle_a \), as a function of the physical extent \( \beta/N = 6/g_3^2 L \) of the box \( (L = aN) \), for a small \( y \) (left) and large \( y \) (right). No volume dependence is visible for \( \beta/N < 1.0 \). At small \( y \) and small volumes, the metastability of the physical phase is too weak to hold the system there for any length of Monte Carlo time.

4. Numerical results

In Fig. 3 we show \( \langle \text{Tr}[\hat{A}_0^2] \rangle_a \) at two values of \( y \), as a function of the lattice spacing \( 1/\beta = a g_3^2 / 6 \), and after the subtraction of the various terms in Eq. (2.9). The figure indicates that after the subtractions, a continuum limit \( (1/\beta \to 0) \) can be taken; and that perturbation theory does converge even at the smallest \( y \) (corresponding to the lowest temperature), in the sense that each subtraction is smaller than the previous one, and that the remainder is at most of the same order as the last subtraction. Increasing \( y \), perturbation theory converges faster, and the significance loss due to the subtractions becomes substantial.

In Fig. 4 we show a magnification of the fully subtracted results, for selected values of \( y \). This highlights the regular-looking approach to the continuum limit.

In order to carry out the continuum extrapolation, we have tested four different fit functions: a constant fit, a fit linear in \( 1/\beta \), a fit including a linear term and the logarithm \( (\ln(\beta))/\beta \), as well as a quadratic fit. The results are shown in Fig. 5. We observe that all dependences apart from the first one lead to consistent results; the situation is thus quite different from ref. [12], where fits allowing for a logarithmic term were the only ones leading to a sensible outcome. In other words, the 3-loop and 4-loop subtractions in Eq. (2.9), which were not available at the time of ref. [12], have effectively removed any possible logarithmic terms. In the following, intercepts obtained from linear extrapolations (squares) will be used.
Table 1: The continuum parameters $x, y$ (cf. Eqs. (2.16), (2.56)); the lattice couplings $\beta$ (cf. Eq. (4.22)); and the box sizes $N (V = a^3 N^3)$ used for the production runs. Values marked with a star correspond to $N_t = 2$, others to $N_t = 0$. In the few cases where the box size has the superscript 2, two independent runs were launched. In total, our sample consists of 186 lattices.

| $x$   | $y$   | $\beta_N$ |
|-------|-------|-----------|
| 0.0060 | 6.389567 | 24_{48}  |
|       | 32_{48} | 32_{40, 64, 96} | 40_{64, 80} | 54_{64, 96} | 67_{120} | 80_{96, 144} | 120_{144}^2, 200 | 240_{512} |
| 0.0075 | *6.621393 | 32_{144} | 40_{144} | 54_{144} | 67_{144} | 80_{144} | 120_{144} | 240_{512} |
| 0.0100 | *5.307970 | 32_{48} | 40_{80} | 54_{96} | 67_{120} | 80_{144} | 120_{144} | 240_{512} |
| 0.0120 | 3.856558 | 32_{48} | 40_{64, 80} | 54_{96} | 67_{120} | 80_{144} | 120_{144} | 240_{512} |
| 0.0130 | *3.994547 | 32_{144} | 40_{144} | 54_{144} | 67_{144} | 80_{144} | 120_{144} | 240_{512} |
| 0.0200 | 2.979720 | 32_{48} | 40_{80} | 54_{96} | 67_{120} | 80_{144^2} | 120_{144} | 240_{512} |
| 0.0260 | *3.085255 | 32_{144} | 40_{144} | 54_{144} | 67_{144} | 80_{144} | 120_{144} | 240_{512} |
| 0.0300 | 1.518356 | 32_{48} | 40_{80} | 54_{108} | 67_{120} | 80_{120} | 120_{200} | 240_{512} |
| 0.0310 | *1.32508 | 32_{48} | 40_{80} | 54_{108} | 67_{120} | 80_{120} | 120_{200} | 240_{512} |
| 0.0350 | 1.142577 | 32_{48} | 40_{80} | 54_{84, 96} | 67_{120} | 80_{84, 144} | 120_{144} | 240_{512} |
| 0.0450 | 0.901336 | 32_{48} | 40_{80} | 54_{108} | 67_{120} | 80_{120} | 120_{200} | 240_{512} |
| 0.0600 | 0.690251 | 32_{48} | 40_{64, 80} | 54_{96} | 67_{120} | 80_{144} | 120_{200} | 240_{512} |
| 0.0860 | *0.710991 | 32_{144} | 40_{144} | 54_{144} | 67_{144} | 80_{144} | 120_{144} | 240_{512} |
| 0.1000 | 0.498801 | 32_{48} | 40_{80} | 54_{108} | 67_{120} | 80_{120} | 120_{200} | 240_{512} |
| 0.1300 | *0.483006 | 32_{144} | 40_{144} | 54_{144} | 67_{144} | 80_{144} | 120_{144} | 240_{512} |
| 0.1370 | 0.357377 | 32_{176} | 40_{176} | 54_{176} | 67_{176} | 80_{176, 320} | 120_{176, 320} | 240_{512} | 240_{512} |

The dashed line in Fig. 5 shows the curve

$$
\langle \text{Tr} [\hat{A}_0^2]^{RMS}_{\bar{\text{MS}}} \rangle \approx \frac{1}{y^{3/2}} \left( c_1 + \frac{c_2}{y^{1/2}} + \frac{c_3}{y} \right),
$$

(4.1)

with $c_1 = 0.0200(8), c_2 = -0.0191(14), c_3 = 0.0149(6)$. This representation describes our data well ($\chi^2 / \text{d.o.f.} = 2.6$) in the whole $y$-range considered.

The next task is to integrate Eq. (2.9) in order to determine the function $\mathcal{F}^{RMS}_{\bar{\text{MS}}}$. Since Eq. (2.9) contains a partial rather than a total derivative, and on the physical stripe the parameter $x$ changes (cf. Fig. 1), an integration is strictly speaking not possible\footnote{In ref. [15] this problem was tackled by also measuring the condensate $\partial_x \mathcal{F}^{\bar{\text{MS}}}_{\bar{\text{MS}}} = \langle (\text{Tr} [\hat{A}_0^2])^2 \rangle_x - \ldots$. However, some of the renormalization constants needed for the $\overline{\text{MS}}$ conversion remain unknown, and in any case the practical effects from this condensate appeared to be too small to change the qualitative behaviour.}; however, the leading $x$-dependent term of $\mathcal{F}^{RMS}_{\bar{\text{MS}}}$ corresponds to a contribution of the type $p_{\text{soft}} \sim g_3^0 \mu_3 / m_3 \sim$...
Figure 3: Shown is $\langle \text{Tr} [\hat{A}^2] \rangle_a$ at a small and large $y$, after the consecutive subtraction of the functions $f_0, f_1, f_2, f_3$ of Eq. (2.9), corresponding to 1-loop, 2-loop, 3-loop and 4-loop effects, respectively.

g^3 T^4$, which is of higher order than other effects we have ignored. More importantly, all effects containing the parameter $x$ are numerically subdominant for $N_c = 3$ (cf. Eq. (4.1)). Thereby, close to the physical stripe, the remainder in Eq. (4.1) can simply be integrated to

$$
\mathcal{F}_{\text{MS}}(x, y) \simeq -\frac{2}{y^{1/2}} \left( c_1 + \frac{c_2}{2y^{1/2}} + \frac{c_3}{3y} \right).
$$

(4.2)

We note that the numerical value of the integral $\mathcal{F}_{\text{MS}}$ is only weakly dependent on the ansatz used for the continuum extrapolation.

We can now compare our result with 4d lattice data. In the following we consider pure glue [28], for which systematic errors are best under control. To get the EQCD result, we need to evaluate both $p_{\text{hard}}$ and $p_{\text{soft}}$ (Eq. (2.1)). In practice, we take part of the terms in $p_{\text{soft}}$, namely those specified explicitly in Eqs. (2.6), (2.7), and combine them with the hard contribution $p_{\text{hard}}$, specified in Eqs. (5), (6) of ref. [29]. Given that the non-perturbative constant $\beta_G$ contains numerical errors and that $p_{\text{hard}}$ contains a perturbative constant, denoted by $\Delta_{\text{hard}}$ in ref. [29], which remains unknown, we treat the combination $\Delta_{\text{hard}} + d_A C_A^3 \beta_G$ as a free parameter. The remaining soft contribution, $\delta p_{\text{soft}} = -T g_3^6 \mathcal{F}_{\text{MS}}$ (cf. Eq. (2.6)), is given by the sum of Eqs. (4.2), (A.1). The free parameter $\Delta_{\text{hard}} + d_A C_A^3 \beta_G$ is fixed by minimizing the $\chi^2$-difference of 4d lattice data [28] and our full result in the range $T/T_c \geq 3.0$. The outcome of the fit is shown in Fig. 6 (It corresponds to $\Delta_{\text{hard}} + d_A C_A^3 \beta_G = -7.24$.)

It can be observed from Fig. 6 that our high-temperature result and the low-temperature 4d lattice result depart already at $T \approx 3.2 T_c$, and that the fit is in general not particularly good. In fact, compared with the fit in ref. [29] which assumed $\mathcal{F}_{\text{MS}} = 0$, $\chi^2$ increases about
Figure 4: A magnification of the fully subtracted results from data of the type in Fig. 3 for selected values of $y$, indicated on the right. Multiple data points at the same $\beta$ correspond to independent runs, or different volumes (cf. Table 1).

30-fold. (The fit of ref. [29] corresponds to $\Delta_{\text{hard}} + d_A C_A^3 \beta_G = -1.84$.) The reason for the increase is that $\langle \text{Tr} [\hat{A}_0^2] \rangle_{\text{MS}}$ in Eq. (4.1) is positive, whereby $\mathcal{F}_{\text{MS}}$ in Eq. (4.2) is negative, whereby $p_{\text{soft}}/T^4$ gets a contribution increasing rapidly at small temperatures (cf. Eq. (2.6)). This is completely unlike the behaviour of the 4d lattice data.

Now, one possible reason for the mismatch could be that, starting at $O(g^7)$, higher order operators should be added to the EQCD action [16]. It seems, however, that in practice such operators cannot change the result in a substantial way. Indeed, for $N_f = 0$, the higher order operators were determined in ref. [30], and the only one contributing at $O(g^7 T^4)$ reads

$$\delta S_E \sim \int d^d x \frac{g^2 C_A}{(4\pi)^3 T^2} D_i^2 A_0^a D_j^2 A_0^a.$$ (4.3)

Treating this perturbatively (as a “vertex”), and working in dimensional regularization, yields a contribution

$$\delta p_{\text{soft}} \sim \frac{d_A C_A}{(4\pi)^2 T} \frac{g^2 m_3^5}{4\pi} \sim \frac{1}{30} \frac{d_A C_A^4}{(4\pi)^3} g^7 T^4,$$ (4.4)

where we inserted $C_A = 3$ and $m_3 \sim g T$. In contrast, the effect that we have determined numerically in this paper has the magnitude (cf. Eqs. (2.6), (4.2))

$$\delta p_{\text{soft}} \sim 2c_1 \frac{g^3 T}{m_3} \sim 20 \frac{d_A C_A^4}{(4\pi)^3} g^7 T^4.$$ (4.5)
This is larger than Eq. (4.4) by more than two orders of magnitude. Another effect of $O(g^7)$ is that the parameter $y$ should be computed at the 3-loop order; however, this only modifies the relation of $y$ and $T/T_c$, and cannot revert the upward trend of our data at small $T/T_c$. Therefore, it seems unlikely that infrared sensitive effects (from momenta $k \sim g T$) describable with simple improvements of EQCD would be the cause for the mismatch; the problems are perhaps more likely related to the treatment of the ultraviolet modes ($k \sim \pi T$).

5. Conclusions and outlook

The purpose of this paper has been to approximate the non-perturbative contribution of the dynamics represented by a (truncated) effective theory called EQCD, Eq. (2.3), to the pressure of hot QCD. The result is constituted by the sum of the non-perturbative Linde term (Eq. (2.7)), perturbatively known terms up to 4-loop level (Eqs. (2.6), (A.1)), as well as an ultraviolet finite all-orders remainder that we have estimated numerically (Eq. (4.2)).

On the technical side, the main content of our study was to carefully carry out the continuum extrapolation needed for estimating the remainder, Eq. (4.2). As has been illustrated in Fig. 5, the continuum extrapolation appears now to be under control, thanks partly to the recent determination of the functions $f_i$ in Eq. (2.9) [17]. Therefore, as we have argued, the representation in Eq. (4.2) should be free of substantial systematic errors close to the
Figure 6: Left: the best fit of our result to 4d lattice data for \( p/T^4 \) at \( N_f = 0 \) \[28\]. Our result comes out as a function of \( T/\Lambda_{\text{MS}} \); for converting to \( T/T_c \) we scanned the interval \( T_c/\Lambda_{\text{MS}} = 1.0 \ldots 1.35 \) (dark band). For comparison, with the light band we show the outcome for \( F_{\text{MS}} = 0 \) \[29\]. Right: the corresponding trace anomaly, \( (e - 3p)/T^4 = Td(p/T^4)/dT \).

physical stripe of Fig. 1. Comparing with the lower order terms in Eq. (A.1), Eq. (4.2) also has a reasonable magnitude, and indicates that perturbation theory within EQCD is in fact a useful tool at all parameter values corresponding to the 4d theory. In principle, the numerical result could also be compared with improved resummation methods defined within EQCD (see, e.g., refs. \[31\]–\[33\]).

At the same time, from the phenomenological point of view, our result is somewhat of a disappointment: as shown in Fig. 6, the match to 4d lattice data in the range \( T/T_c \geq 3.0 \) is not particularly smooth. In fact, as shown in the figure, including the newly determined \( \gtrsim 5\)-loop remainder decreases the quality of the fit significantly. This implies that, unfortunately, we are not in a position to realize our original goal of offering consolidated crosschecks for the \( N_f \neq 0 \) QCD pressure in the interesting temperature range \( T/T_c \sim 1.5 \ldots 3.0 \).

On the other hand, our study raises the theoretical question of what kind of effects could be responsible for the mismatch between our results and that of 4d lattice simulations. On the mundane side, one possibility would be a substantial contribution from the condensate denoted by \( \partial_x F_{\text{MS}} = \langle (\text{Tr}[\hat{A}_0^2])^2 \rangle_a - \ldots \). Unfortunately its systematic inclusion would require a significant amount of new analytic and numerical work; moreover, as order-of-magnitude estimates and previous preliminary simulations suggest, it appears unlikely that this condensate could significantly change the qualitative behaviour that we have observed.

On a more adventurous note, let us point out that, qualitatively, the reason for the mismatch is that the condensate in Eq. (4.1) is too large, and grows rapidly as \( y \) (or \( T \)) decreases (note that for \( N_f = 0 \), the range \( T/T_c = 10^0 \ldots 10^3 \) corresponds to \( y \approx 0.3 \ldots 1.2 \), or
$1/y \simeq 3.3 \ldots 0.8$, cf. Fig. [1]. Since the condensate measures the mean squared fluctuation of the field $A_0$, this means that the colour-electric gauge field fluctuates too much. Indeed, if the remainder in Fig. [5] continued to decrease also for $1/y \gtrsim 0.5$ (in principle the remainder can even become negative), then the match in Fig. [6] could conceivably improve dramatically. Perhaps one reason for the large fluctuations at small $y$ could be that the physical phase of EQCD is merely metastable [24] (see also Fig. 3 of ref. [27])? If so, improved effective theories of the type in refs. [34]–[39], which do not make any explicit distinction between the scales $k \sim gT$ and $k \sim \pi T$ and which are by construction free of the metastability problem, might yield smaller fluctuations and a better outcome.

Of course, if no scale separation is made between the scales $k \sim gT$ and $k \sim \pi T$, then one should in principle also account for corrections of $O(g^8)$ to the hard part of the pressure ($p_{\text{hard}}$, Eq. (2.1)), rendering the analysis very hard. Nevertheless, it might be interesting to explore if a phenomenologically successful recipe could be found even without this last step.

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**Appendix A. Four-loop three-dimensional vacuum energy density**

The function $F_4^{\text{loop}}$, defined through Eq. (2.7) and representing the contribution of the adjoint scalar $A_0$ to the vacuum energy density of EQCD, with $\bar{\mu} = \frac{g_2^2}{3}$ and with $N_c = 3$ so that only a single scalar self-coupling appears, can be written as [18]:

$$F_4^{\text{loop}}(x, y) = -\frac{d_A}{(4\pi)^2} \frac{y^2}{3} + \frac{d_A}{(4\pi)^3} \frac{y}{2} \left\{ C_A \left( 3 - 2 \ln(4y) \right) + x(d_A + 2) \right\} + \frac{d_A}{(4\pi)^3} \frac{y^2}{2} \left\{ C_A^2 \left[ \frac{89}{12} + \frac{\pi^2}{3} - \frac{11}{3} \ln 2 \right] + \frac{x}{2} C_A(d_A + 2) \left[ 2 \ln(4y) - 1 \right] \right\} + x^2(d_A + 2) \left[ 3 - \ln(16y) \right] - \frac{x^2}{4} (d_A + 2)^2 \right\} + \frac{d_A}{(4\pi)^4} \left\{ C_A^3 \left[ \frac{43}{8} - \frac{491}{1536} \pi^2 \right] \ln(4y) + \frac{311}{256} + \frac{43}{32} \ln 2 \right\}.

^2$We have inserted the value $\gamma_{10} = \frac{\pi^2}{24} - \ln^2 2/2$ [40], not known analytically in ref. [18].
+ \frac{11}{3} \ln^2 2 - \frac{461}{9216} \pi^2 + \frac{491}{1536} \pi^2 \ln 2 - \frac{1793}{512} \zeta(3)
+ \frac{5x}{2} C_A^2 \left[ \left( 1 - \frac{\pi^2}{8} \right) \ln(4y) + 1 + 3 \ln 2 - \frac{17}{96} \pi^2 + \frac{\pi^2}{8} \ln 2 - \frac{35}{16} \zeta(3) \right]
+ \frac{x}{4} C_A^2 (d_A + 2) \left[ \ln^2 (4y) - \ln(4y) - \frac{43}{6} + \frac{11}{3} \ln 2 - \frac{\pi^2}{3} \right]
+ \frac{x^2}{4} C_A (d_A + 2) \left[ - \ln^2 (4y) + \left( 9 - \frac{\pi^2}{4} \right) \ln(4y) \right.
\left. - 8 + 6 \ln 2 + \frac{31}{24} \pi^2 + \frac{\pi^2}{4} \ln 2 + 8 \ln^2 2 - \frac{21}{4} \zeta(3) \right]
\frac{-x^2}{8} C_A (d_A + 2)^2 \left[ 2 \ln(4y) + 1 \right]
+ \frac{x^3}{4} (d_A + 2)^2 \left[ \ln(16y) - 1 \right] + \frac{x^3}{24} (d_A + 2)^3
+ \frac{x^3}{48} (d_A + 2)(d_A + 8) \left[ - \pi^2 \ln(2y) + \frac{\pi^2}{2} - 21 \zeta(3) \right].
(A.1)

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