THE ALMOST-ININVARIANT SUBSPACE PROBLEM FOR BANACH SPACES

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Abstract. We show that for any bounded operator $T$ acting on an infinite dimensional Banach space there exists a rank one operator $F$ such that $T + F$ has invariant subspace of infinite dimension and codimension. This extends to arbitrary Banach spaces a previous result that was proved only in the reflexive case. We also show that, for any fixed $\varepsilon > 0$, there exists $F$ as above such that $\|F\| < \varepsilon$.

1. Introduction

The Invariant Subspace Problem is one of the most famous problem in Operator Theory, and is concerned with the search of non-trivial, closed, invariant subspaces for bounded operators acting on a separable Banach space. Considerable success has been achieved over the years both for the existence of such subspaces for many classes of operators, as well as for non-existence of invariant subspaces for particular examples of operators. However, the question whether every bounded operator on a reflexive Banach space has an invariant subspace is still open. In particular, the question is open for the important case of a separable Hilbert space. We refer the reader to the monograph by Radjavi and Rosenthal [RR03] for an overview and to the book by Chalendar and Partington [CP11] for more recent approaches to the Invariant Subspace Problem.

In this paper we give a positive answer to a question closely related to the Invariant Subspace Problem, but not equivalent to it. For the remaining of the paper it is assumed that all Banach spaces that appear are complex, and by "subspace" we mean a norm closed subspace. We say that a subspace $Y$ of a Banach space $X$ is almost-invariant for $T$ if there exists a finite dimensional subspace $E$ such that $TY \subseteq Y + E$. The smallest dimension of such an $E$ is called the defect of $Y$ for $T$. It is not hard to see that if $Y$ is finite dimensional or finite codimensional then $Y$ is almost-invariant for any for any bounded operator $T$. Therefore the question whether a bounded operator
$T$ has an almost-invariant subspace $Y$ (the almost-invariant subspace problem) is non-trivial only if we require $Y$ to be both infinite dimensional and infinite codimensional. This motivates the notions of a half-space, a subspace that is infinite dimensional and codimensional. These concepts were introduced in a paper by Androulakis, Popov, Tcaciuc, and Troitsky \cite{APTT09}, and one of the first results in that paper was that a bounded operator $T$ has an almost-invariant half-space $Y$ if and only if there exists a finite rank operator $F$ such that $T + F$ has an invariant half-space. In the Hilbert space setting, invariant subspaces for perturbations of operators have been studied for a long time. For example, Brown and Pearcy \cite{BP71} proved that for any $T \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is an infinite-dimensional separable Hilbert space, and for any $\varepsilon > 0$, there exists a compact operator $K$ with norm at most $\varepsilon$ such that $T + K$ has an invariant half-space. As an immediate consequence of Voiculescu’s \cite{V76} famous non-commutative Weyl-von Neumann Theorem it follows that there exists a compact operator $K$ such that $T + K$ has a reducing half-space, that is, a half-space that is invariant for both $T - K$ and $(T - K)^*$.

The main result of \cite{APTT09} was the following theorem, which was used in that paper to prove the existence of almost-invariant half-spaces for certain classes of weighted shifts, and it will also be important here. Recall that a sequence $\{x_n\}_n$ in a Banach space is called minimal if, for every $k \in \mathbb{N}$, $x_k$ does not belong to the closed linear span of the set $\{x_n : n \neq k\}$ (see also \cite[Section 1.f]{LT77}).

**Theorem 1.1.** \cite[Theorem 3.2]{APTT09}, \cite[Remark 1.3]{MPR12} Let $X$ be a Banach space and $T \in \mathcal{B}(X)$ satisfying the following conditions:

1. The unbounded component of the resolvent set $\rho(T)$ contains $\{z \in \mathbb{C} : 0 < |z| < \varepsilon\}$ for some $\varepsilon > 0$.
2. There is a vector $e \in X$ whose orbit $\{T^n e\}_{n=0}^\infty$ is a minimal sequence.

Then $T$ has an almost-invariant half-space with defect at most one.

The almost-invariant half-space problem was solved for reflexive Banach spaces by Popov and Tcaciuc in \cite{PT13}. An important step was the following theorem which proves the existence of almost-invariant half-spaces provided a certain spectral condition holds, and this result will also feature in our proof of the general case:

**Theorem 1.2.** \cite[Theorem 2.3]{PT13} Let $X$ be an infinite dimensional Banach space and let $T \in \mathcal{B}(X)$ such that there exists $\mu \in \partial\sigma(T)$ that is not an eigenvalue. Then $T$ admits an almost-invariant half-space with defect one.
In the same paper, for the Hilbert space case, the authors prove the existence of "good" perturbations that are also of small norm. This gave a substantial improvement over the result of Brown and Pearcy mentioned above, by showing that for any bounded operator \( T \), and for any \( \varepsilon > 0 \) there exists a finite rank operator \( F \) with norm at most \( \varepsilon \) such that \( T + F \) has an invariant half-space. Moreover, when either the boundary of \( T \) or \( T^* \) does not consist entirely of eigenvalues, \( F \) can be taken to be rank one. In [TW17] this result was extended to the reflexive case.

Partial solutions for general Banach spaces were given by Sirotkin and Wallis, first for weakly-compact operators and for quasinilpotent operators in [SW14], then for strictly singular operators in [SW16]. In the latter paper they also showed that any bounded operator acting on a Banach space admits a compact perturbation that has an invariant half-space. Common almost-invariant half-spaces for algebras of operators have been studied in [P10], [MPR13], and [SW16]. In [MPR13] the authors show that whenever a norm closed algebra of operators on a Hilbert space admits a common almost-invariant half-space, then it actually admits a common invariant half-space. This result was extended in [SW16] to norm closed algebras of operators on Banach spaces.

In Section 2 of this paper we solve the almost-invariant subspace problem in full generality. In Section 3 we refine the method to obtain perturbation small in norm.

2. The Main result

For a Banach space \( X \), we denote by \( \mathcal{B}(X) \) the algebra of all bounded operators on \( X \). When \( T \in \mathcal{B}(X) \), we write \( \sigma(T), \sigma_p(T), \sigma_{ess}(T), \rho(T) \) and \( \partial \sigma(T) \) for the spectrum of \( T \), point spectrum of \( T \), the essential point spectrum of \( T \), the resolvent set of \( T \) and the topological boundary of the spectrum, respectively. The closed span of a set \( \{x_n\}_n \) of vectors in \( X \) is denoted by \([x_n]\). A sequence \((x_n)_{n=1}^\infty \) in \( X \) is called a basic sequence if any \( x \in [x_n] \) can be written uniquely as \( x = \sum_{n=1}^\infty a_n x_n \), where the convergence is in norm (see [LT77] section 1.a] for background on Schader bases and basic sequences).

An important ingredient in our proof is the following \( w^* \)-analogue of the Bessaga-Pelczynski selection principle. An outline of the proof first appeared in a paper by Johnson and Rosenthal (see Theorem III.1 and Remark III.1 in [JR72]). For a shorter proof see the recent paper of González and Martínez-Abejón [GM12].

**Theorem 2.1.** [JR72][GM12] If \((x_n^*)\) is a semi-normalized, \( w^* \)-null, sequence in a dual Banach space \( X^* \), then there exists a basic subsequence \((y_n^*)\) of \((x_n^*)\), and a bounded sequence \((y_n)\) in \( X \) such that \( y_i^*(y_j) = \delta_{ij} \) for all \( 1 \leq i, j < \infty \).
We begin by proving an essential step for the general case, step that deals with the situation when $T^*$ satisfies a spectral condition similar to the one in the hypothesis of Theorem 1.2.

**Theorem 2.2.** Let $X$ be a separable Banach space and $T \in \mathcal{B}(X)$ a bounded operator such that $\partial \sigma(T^*) \setminus \sigma_p(T^*) \neq \emptyset$. Then $T$ has an almost-invariant half-space with defect at most one.

**Proof.** Let $\lambda \in \partial \sigma(T^*) \setminus \sigma_p(T^*)$ and without loss of generality assume $\lambda = 0$, otherwise work with $T - \lambda I$. Let $(\lambda_n)$ be a sequence in the resolvent $\rho(T^*)$ such that $\lambda_n \to 0$. Then we have that $\|(\lambda_n I - T^*)^{-1}\| \to \infty$ and, from Uniform boundedness principle, it follows that there exists $e^* \in X^*$ such that $\|(\lambda_n I - T^*)^{-1}e^*\| \to \infty$. Put $h_n^* := (\lambda_n I - T^*)^{-1}e^*$ and $x_n^* := h_n^*/\|h_n^*\|$. Easy calculations show that

\begin{equation}
T^*x_n^* = \lambda_n x_n^* - \frac{e^*}{\|h_n^*\|}
\end{equation}

Claim 1: $(x_n^*)$ has a subsequence that is $w^*$-null.

From Banach-Alaoglu we have that $B_{X^*}$, the unit ball of $X^*$, is $w^*$-compact, and since $X$ is separable, $B_{X^*}$ is also $w^*$-metrizable. Therefore, by passing to a subsequence, we can assume that $x_n^* \xrightarrow{w^*} y^*$ for some $y^* \in X^*$. Remains to show that $y^* = 0$. Since $\lambda_n \to 0$, $x_n^* \xrightarrow{w^*} y^*$, and $\|h_n^*\| \to \infty$ we have that

\begin{equation}
T^*x_n^* \xrightarrow{w^*} T^*y^* \text{ and } \lambda_n x_n^* - \frac{e^*}{\|h_n^*\|} \xrightarrow{w^*} 0
\end{equation}

From (1) and (2) it follows that $T^*y^* = 0$. However $0$ is not an eigenvalue, so we must have that $y^* = 0$ and the Claim 1 is proved.

Claim 2: $(x_n^*)$ has a subsequence $(x_{n_k}^*)$ such that $[x_{n_k}^*]^\top$ is a half-space of $X$.

By eventually passing to a subsequence we can assume that $x_n^* \xrightarrow{w^*} 0$. From Theorem 2.1 by passing to a further subsequence, we can assume that $(x_n^*)$ is a basic sequence and there exists $(x_n) \subseteq X$ such that $x_n^*(x_k) = \delta_{nk}$ for any $n, k \in \mathbb{N}$. It is routine to check that both $(x_n)$ and $(x_n^*)$ are linearly independent, and that $[x_{2n+1}] \subseteq [x_{2n}^*]^\top$, therefore $[x_{2n}^*]^\top$ is infinite dimensional. We also have that for any $n \in \mathbb{N}$ $x_{2n}^*([x_{2n}^*]^\top) = 0$, therefore $[x_{2n}^*]^\top$ is infinite codimensional as well and Claim 2 is proved.

In view of the previous Claims, by passing to a subsequence we may assume that $(x_n^*)$ is a basic sequence and that $Z := [x_n^*]^\top = [h_n^*]^\top$ is a half-space of $X$.

Note that for any $z \in Z$, and for any $n \in \mathbb{N}$, we have that

\[h_n^*(Tz) = T^*h_n^*(z) = (\lambda_n h_n^* - e^*)z = \lambda_n h_n^*(z) - e^*(z) = -e^*(z)\]
If $Z \subseteq \ker e^*$ then we have that for all $n \in \mathbb{N}$ and for all $z \in Z$, $h_n^*(Tz) = 0$. Hence $TZ \subseteq Z$ and we are done.

Otherwise, we can find $z_0 \in Z$ such that $z_0 \notin \ker e^*$. Put $f := Tz_0$ and for any $z \in Z$ define a scalar $\alpha_z$ by $\alpha_z := \frac{e^*_\ast(z)}{e^*_\ast(z_0)}$. Then, for any $n \in \mathbb{N}$ and $z \in Z$ we have:

\[
h^*_n(Tz - \alpha_z f) = h^*_n(Tz - \alpha_z f) = h^*_n(Tz - \frac{e^\ast(z)}{e^\ast(z_0)} Tz_0) = h^*_n(Tz) - \frac{e^\ast(z)}{e^\ast(z_0)} h^*_n(Tz_0) = -e^\ast(z) + \frac{e^\ast(z)}{e^\ast(z_0)} e^\ast_n(z_0) = 0
\]

Therefore, for any $z \in Z$ we have that $Tz - \alpha_z f \in Z$, so

\[
Tz = Tz - \alpha_z f + \alpha_z f \in Z + [f], \text{ for all } z \in Z
\]

It follows that $TZ \subseteq Z + [f]$, hence $Z$ is an almost-invariant half-space for $T$ with defect $[f]$. \hfill \Box

We are now ready to prove the general case.

**Theorem 2.3.** Let $X$ be a separable Banach space. Then any bounded operator $T \in \mathcal{B}(X)$ has an almost-invariant half-space with defect at most one.

**Proof.** If $\partial \sigma(T) \setminus \sigma_p(T) \neq \emptyset$ or $\partial \sigma(T^\ast) \setminus \sigma_p(T^\ast) \neq \emptyset$ then by applying Theorem 1.2 or Theorem 2.2 respectively, we obtain that $T$ has an almost-invariant half-space with defect at most one. Therefore, remains to consider the situation when any value in $\partial \sigma(T) = \partial \sigma(T^\ast)$ is an eigenvalue for both $T$ and $T^\ast$.

Easy calculations show that an eigenvector for $T^\ast$ cancels any eigenvector of $T$ corresponding to different eigenvalues. It follows that when $\partial \sigma(T) = \partial \sigma(T^\ast)$ is infinite, we can actually build an invariant half-space as span of countably many eigenvectors corresponding to a countably infinite subset of $\partial \sigma(T)$ such that the complement in $\partial \sigma(T)$ is also infinite (see proof of Theorem 2.7 in [PT13] for details).

Remains to consider the case when $\partial \sigma(T)$ is finite. In this situation we have $\partial \sigma(T) = \sigma(T)$. We can assume without loss of generality that $\sigma(T)$ is a singleton. Indeed if $\sigma(T) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, for each $1 \leq i \leq n$ consider the Riesz projection $P_i$ associated to $\lambda_i$. That is, $P_i^2 = P_i$, $P_i T = T P_i$ (so each $X_i := P_i X$ is a $T$-invariant subspace of $X$), $\sigma(T|_{P_i X}) = \{\lambda_i\}$, and $P_1 + P_2 + \cdots + P_n = I$. It follows that one of the subspaces $X_i$ is infinite dimensional and, for that particular $i$, consider the operator $S := T|_{X_i} : X_i \to X_i$. If $S$ has an almost-invariant half-space $Y \subseteq X_i$, then the same $Y$...
is also an almost-invariant half-space for $T$, with the same defect. Therefore, we may assume $\sigma(T) = \{\lambda\}$ and, by eventually replacing $T$ with $T - \lambda I$, we may also assume $\lambda = 0$.

Next we show that either we can find a vector $z$ such that the orbit $\{T^n z\}$ is a minimal sequence, or there exists an infinite dimensional $T$-invariant subspace $Y$ such that restriction of $T$ to $Y$ has dense range. The proof is similar to the second half of the proof of Theorem 2.7 in [PT13], we include the argument here for the sake of completeness. For any $n \in \mathbb{N}$, denote by $Y_n = \overline{T^n X}$, with $Y_0 := X$. We have that each $Y_n$ is invariant under $T$, $Y_{n+1} = \overline{TY_n}$ and $X \supseteq Y_1 \supseteq Y_2 \supseteq \ldots$. Also note that for any $j, n \in \mathbb{N}$ and any $y \in Y_j$ we have that $T^n(y) \in Y_{j+n}$. Note that we can assume each $Y_j$ is infinite dimensional; indeed, otherwise, if $j$ is the smallest index for which $Y_j$ is finite dimensional, then any half-space of $Y_{j-1}$ containing $Y_j$ is an invariant half-space for $T$. If $Y_1$ is of infinite codimension in $X$, then $Y_1$ is an invariant half-space for $X$ and we are done. Therefore we can assume that $Y_1$ is of finite codimension in $X$, hence complemented in $X$, and we can write $X = Y_1 \oplus Z$, where $Z$ is finite dimensional. If $Z = \{0\}$ then $T$ has dense range. Otherwise, let $\{z_1, z_2, \ldots, z_k\}$ be a basis for $Z$ and assume the orbit $\{T^n z_j\}_n$ is not minimal for any $1 \leq j \leq k$. For any $1 \leq j \leq k$ denote by $p_j$ the smallest index such that $T^{p_j} z_j \in [T^n z_j]_{n \not\equiv p_j}$. It is easy to see that for this choice of $p_j$ we actually have that $T^{p_j} z_j \in [T^n z_j]_{n \equiv p_j}$ (see, e.g. Lemma 2.6 in [PT13]), thus $T^{p_j} z_j \in Y_{p_j+1}$, for any $1 \leq j \leq k$. If we let $p_0 := \max\{p_1, p_2, \ldots, p_k\}$, it follows that $T^{p_0} z_j = T^{p_0-p_j} (T^{p_j} z_j) \in Y_{p_0+1}$ for any $1 \leq j \leq k$. Therefore, since $\{z_1, z_2, \ldots, z_k\}$ is a basis for $Z$, we have that $T^{p_0} z \in Y_{p_0+1}$ for any $z \in Z$. We also have that $T^{p_0} y \in Y_{p_0+1}$ for any $y \in Y_1$, and since $X = Y_1 \oplus Z$ it follows that $T^{p_0} x \in Y_{p_0+1}$ for any $x \in X$. This means that $\overline{T^{p_0} X} \subseteq Y_{p_0+1}$, so $Y_{p_0} \subseteq Y_{p_0+1}$. On the other hand, $Y_{p_0+1} \subseteq Y_{p_0}$, therefore $Y_{p_0+1} = Y_{p_0}$ and the last equality means that $T|_{Y_{p_0}}$ has dense range.

If we find a vector $z$ such that the orbit $\{T^n z\}$ is a minimal sequence, we can apply Theorem 1.1 and obtain that $T$ has an almost-invariant half-space with defect at most one. Otherwise, there exists $Y$ an infinite dimensional subspace of $X$ such that $T|_Y$ has dense range. Consider $S := T|_Y : Y \to Y$. Since $S$ has dense range it follows that $S^*$ is injective. Note that $\sigma(S) = \sigma(T) = \{0\}$, therefore $0 \in \sigma(S^*) = \sigma(S)$ is not an eigenvalue. We can now apply Theorem 2.2 to conclude that $S$, hence also $T$, has an almost-invariant half-space with defect at most one. 

$\square$
3. Perturbations of small norm

We proved in the previous section that for any \( T \in (X) \) we can find a rank one perturbation \( F \) such that \( T + F \) has an invariant half-space. In this section we show that, under the same spectral assumptions as in Theorem 2.2, such \( F \) may be chosen to be small in norm. When the spectral conditions are not satisfied we still can find a finite rank perturbation \( F \) of small norm, but not necessarily rank one, such that \( T + F \) has an invariant half-space. In [TW17] the authors proved the following theorem, which we will also use here.

**Theorem 3.1.** [TW17, Proposition 2.2] Let \( X \) be an infinite dimensional Banach space and let \( T \in \mathcal{B}(X) \) such that there exists \( \mu \in \partial \sigma(T) \) that is not an eigenvalue. Then for any \( \varepsilon > 0 \) there exists a rank one operator \( F \) with \( \|F\| < \varepsilon \) such that \( T + F \) has an invariant half-space.

Thus, this theorem gives the existence of perturbations of small norm when the boundary of the spectrum of \( T \) has non-eigenvalues. We begin by proving a companion theorem to the one above, in the situation when \( T^* \) satisfies a similar type of spectral condition.

**Theorem 3.2.** Let \( X \) be a Banach space and \( T \in \mathcal{B}(X) \) a bounded operator such that \( \partial \sigma(T^*) \setminus \sigma_p(T^*) \neq \emptyset \). Then for any \( \varepsilon > 0 \) there exists a rank one operator \( F \) with \( \|F\| < \varepsilon \) such that \( T + F \) has an invariant half-space.

**Proof.** Fix \( \varepsilon > 0 \). Let \( \lambda \in \partial \sigma(T^*) \setminus \sigma_p(T^*) \) and, as before, without loss of generality assume \( \lambda = 0 \). Given \( \lambda_n \in \rho(T^*), \lambda_n \to 0 \), consider vectors \( e^* \in X^*, \|e^*\| = 1, h_n^* := (\lambda_n I - T^*)^{-1}e^* \) and \( x_n^* := h_n^*/\|h_n^*\| \), and \( (x_n) \) in \( X \) as in the proof of Theorem 2.2.

By passing to a subsequence, consider also \( (x_n) \) a bounded sequence in \( X \) biorthogonal to \( (x_n^*) \), as given by Theorem 2.1. Let \( M \) be such that \( \|x_n\| \leq M \) for all \( n \in \mathbb{N} \) and by passing to a further subsequence assume that \( \sum_{n=1}^{\infty} \|h_n\|^{-1} < \varepsilon/M \) (recall that \( \|h_n\| \to \infty \)), and \( Z := [x_n^*]^{\top} \) is a half-space.

Define \( f \in X \) by

\[
 f := \sum_{n=1}^{\infty} \frac{1}{\|h_n\|} x_n
\]

We have

\[
 \left\| \sum_{n=1}^{\infty} \frac{1}{\|h_n\|} x_n \right\| \leq \sum_{n=1}^{\infty} \left\| \frac{1}{\|h_n\|} x_n \right\| \leq \sum_{n=1}^{\infty} \frac{M}{\|h_n\|} \leq M \frac{\varepsilon}{M} = \varepsilon
\]
Therefore $f$ is well defined and $\|f\| \leq \varepsilon$. Note that for all $n$, the bounded functional $h^*_n$ satisfies
\[
h^*_n(f) = \|h^*_n\| x^*_n(f) = \|h^*_n\| x^*_n\left(\sum_{i=1}^{\infty} \frac{1}{\|h_i\|} x_i\right) = \|h^*_n\| \sum_{i=1}^{\infty} \frac{1}{\|h_i\|} x^*_n(x_i) = 1
\]

Consider now the rank one operator $F := e^* \otimes f$, that is, for any $x \in X$, $F(x) = e^*(x)f$. We have that $\|F\| = \|e^*\|\|f\| < \varepsilon$ and will show that $Z$ is an invariant half-space for $T + F$. To this end, it is enough to show that for any $z \in Z$, and any $n \in \mathbb{N}$, we have that $h^*_n(Tz + Fz) = 0$. Indeed:
\[
h^*_n(Tz + Fz) = h^*_n(Tz) + h^*_n(Fz) = T^*h^*_n(z) + e^*(z)h^*_n(f)
\]
\[
= \lambda_n h^*_n(z) - e^*(z) + e^*(z)
\]
\[
= \lambda_n h^*_n(z) = 0.
\]

Therefore $(T + F)(Z) \subseteq Z$ and this concludes the proof.

Next we will prove the result in its full generality, when no assumptions on the spectrum are made.

**Theorem 3.3.** Let $X$ be a Banach space and $T \in \mathcal{B}(X)$ a bounded operator. Then for any $\varepsilon > 0$ there exists a finite rank operator $F$ with $\|F\| < \varepsilon$ such that $T + F$ has an invariant half-space. Moreover, if $\partial \sigma(T) \setminus \sigma_p(T) \neq \emptyset$ or $\partial \sigma(T^*) \setminus \sigma_p(T^*) \neq \emptyset$, $F$ can be taken to be rank one.

**Proof.** Fix $\varepsilon > 0$. Note that the ”moreover” part is simply Theorem 3.1 when $\partial \sigma(T) \setminus \sigma_p(T) \neq \emptyset$ and Theorem 3.2 when $\partial \sigma(T^*) \setminus \sigma_p(T^*) \neq \emptyset$. Remains to consider the case when $\partial \sigma(T) = \partial \sigma(T^*)$ consist only of eigenvalues. If these sets are infinite, the same argument as in the proof of Theorem 2.7 in [PT13] (also used in Theorem 2.3 in the previous section ) shows that $T$ actually has an invariant half-space. When $\partial \sigma(T) = \partial \sigma(T^*)$ is finite, we can assume as we did in the proof of Theorem 2.3 that $T$ is quasinilpotent, and 0 is an eigenvalue for both $T$ and $T^*$.

Denote by $N$ the kernel of $T$ and by $R$ the closure of the range of $T$. If $N$ is infinite dimensional, than any subspace of $N$ that is a half-space will be an invariant half-space for $T$. Since $T^*$ is injective it follows that the range of $T$ is not dense in $X$. Clearly $R$ is infinite dimensional, and if it is infinite codimensional as well, then $R$ is an invariant half-space for $T$. Therefore we may assume that $N$ is finite dimensional and $R$ is finite codimensional. Denote by $n := \dim(N)$ and by $m := \text{codim}(R)$, and write $X = N \oplus Y$. 
and $X = R \oplus Z$. Fix bases $\{f_1, f_2, \ldots, f_n\}$ of $N$ and $\{g_1, g_2, \ldots, g_m\}$ of $Z$. We will consider separately the cases $n \leq m$ and $n > m$.

If $n \leq m$, consider the rank $n$ operator $G : N \to Z$ defined by $G(f_i) = g_i$, for any $1 \leq i \leq n$. Extend $G$ to $X$ by letting $G|_Y = 0$. It is easy to verify that for any scalar $\alpha \neq 0$, $T + \alpha G$ is injective. Recall that the essential spectrum is stable under compact perturbations, and that the spectrum of a compact perturbation of a quasinilpotent operator is at most countable, with 0 the only possible accumulation point (see e.g. [AA02], Corollary 7.50)). It follows that $0 \in \partial \sigma(T + \alpha G)$ and since $T + \alpha G$ is injective, $0$ is not an eigenvalue for $T + \alpha G$. Choose $\alpha > 0$ such that $\|\alpha G\| < \varepsilon/2$. We can apply Theorem 3.1 for $T + \alpha G$ and find $F_0 \in \mathcal{B}(X)$ a rank one operator such that $\|F_0\| < \varepsilon/2$ and $T + \alpha G + F_0$ has an invariant half-space. Then $F := \alpha G + F_0$ is an operator of rank $n + 1$ that satisfies the conclusion.

If $m < n$, consider the rank $m$ operator $G : N \to Z$ defined by $G(f_i) = g_i$, for any $1 \leq i \leq m$, and extend $G$ to $X$ by letting $G|_Y = 0$. It follows easily that for any scalar $\alpha \neq 0$, $T + \alpha G$ has dense range. The same argument as in the previous paragraph gives that $0 \in \partial \sigma(T + \alpha G)$ for any $\alpha \neq 0$. Since $T + \alpha G$ has dense range, it follows that $(T + \alpha G)^*$ is injective and $0 \in \partial \sigma(T + \alpha G)^*$ is not an eigenvalue for $(T + \alpha G)^*$. Pick $\alpha > 0$ such that $\|\alpha G\| < \varepsilon/2$, and apply Theorem 3.2 for $T + \alpha G$. As before, we can find $F_0 \in \mathcal{B}(X)$ a rank one operator such that $\|F_0\| < \varepsilon/2$ and $T + \alpha G + F_0$ has an invariant half-space. Setting $F := \alpha G + F_0$ we obtain the conclusion, and this ends the proof.

\[ \square \]

REFERENCES

[AA02] Y.A. Abramovich and C.D. Aliprantis. An Invitation to Operator Theory (2000), ISBN 0-8219-2146-6.
[AK06] F. Albiac, N. Kalton, Topics in Banach space theory. Graduate Texts in Mathematics, 233. Springer, New York, 2006.
[APTT09] George Androulakis, Alexey I Popov, Adi Tcaciuc and Vladimir G. Troitsky. Almost Invariant Half-spaces of Operators on Banach Spaces Integr.Equ.Oper.Theory 65 (2009), 473–484.
[BP71] A. Brown, C. Pearcy, Compact restrictions of operators Acta Sci. Math. (Szeged) 32 (1971), 271–282.
[CP11] I. Chalendar, J.R. Partington, Modern approaches to the invariant-subspace problem Cambridge Tracts in Mathematics, 188. Cambridge University Press, Cambridge, 2011.
[GM12] Manuel González, Antonio Martinez-Abeyón On basic sequences in dual Banach spaces, JMAA 395(2) (2012), 813-814.
[JR72] W.B. Johnson, H.P. Rosenthal On $w^*$-basic sequences and their applications to the study of Banach spaces, Studia Math. 43 (1972), 77-92.
[LT77] J. Lindenstrauss, L. Tzafriri, Classical Banach spaces. I. Sequence spaces. Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92. Springer-Verlag, Berlin-New York, 1977.
[MPR12] Laurent W. Marcoux, Alexey I. Popov and Heydar Radjavi, *On Almost-invariant Subspaces and Approximate Commutation* (preprint), arXiv:1204.4621 [math.FA] (2012).

[MPR13] Laurent W. Marcoux, Alexey I. Popov and Heydar Radjavi, *On Almost-invariant Subspaces and Approximate Commutation*, J. Funct. Anal. **264**(4) (2013), 10881111.

[P10] Alexey I. Popov *Almost invariant half-spaces of algebras of operators* Integr.Equ.Oper.Theory **67**(2) (2010), 247-256

[PT13] Alexey I. Popov and Adi Tcaciuc. *Every Operator has Almost-invariant Subspaces*, J. Funct. Anal. **265**(2) (2013), 257265.

[RR03] H. Radjavi, P. Rosenthal, *Invariant subspaces, Second edition*. Dover Publications, Inc., Mineola, NY, 2003.

[SW14] Gleb Sirotkin and Ben Wallis. *The structure of almost-invariant half-spaces for some operators* J. Funct. Anal. **267**(2014), 2298-2312.

[SW16] Gleb Sirotkin and Ben Wallis. *Almost-invariant and essentially-invariant halfspaces* Linear Algebra Appl. **507** (2016), 399-413

[TW17] Adi Tcaciuc, Ben Wallis. *Controlling almost-invariant halfspaces in both real and complex setting* Integr.Equ.Oper.Theory **87**(1) (2017), 117-137.

[V76] D. V. Voiculescu, *A non-commutative Weyl-von Neumann theorem* Rev. Roumaine Math. Pures Appl. 21 (1976), 97-113.

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