EULER-POINCARÉ REDUCTION FOR SYSTEMS WITH
CONFIGURATION SPACE ISOTROPY

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Abstract. This paper concerns Lagrangian systems with symmetries, near
points with configuration space isotropy. Using twisted parametrisations cor-
responding to phase space slices based at zero points of tangent fibres, we
deduce reduced equations of motion, which are a hybrid of the Euler-Poincaré
and Euler-Lagrange equations. Further, we state a corresponding variational
principle.

1. Introduction. It is well known that the presence of symmetry in a physical
system is a key ingredient in reducing the dimension of the phase space of a given
problem. In particular, mechanics is one of the fields where symmetry and the
associated conservation laws have received particular attention starting with the
works of Euler and Lagrange. In recent decades, some key achievements are: the
theory of symplectic reduction and reduction by stages (see, for example, [1], [4],
[14], [18], or [24]), the Marle-Guillemin-Sternberg normal form (the Hamiltonian
slice theorem) [6, 11], and the energy-momentum method [9, 12, 17, 19].

The present paper considers Lagrangian systems on tangent bundles, with lifted
symmetries and configuration-space isotropy. The geometrical framework for study-
ing such systems is given by degenerate parametrisations (“slice coordinates”) for
neighbourhoods of phase space points with configuration-space isotropy. More pre-
cisely, the configuration space in the neighbourhood of the given symmetric point

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is modeled as a twisted product, which is the base space of a principal bundle with fibre the isotropy group of the given point. Using an appropriate version of the Euler-Poincaré reduction together with a connection on the principal bundle just mentioned, we deduce the Euler-Poincaré-Lagrange bundle equations and interpret them in terms of a non-standard variational principle. We note that these equations, in the case of trivial isotropy, coincide with the Lagrange-Poincaré equations for the special case when the configuration space is a trivial bundle (see [4]). The present paper is the Lagrangian counterpart to [23].

We apply our approach to simple mechanical systems. From a practical point of view, the change to slice coordinates on $TQ$ reorganises the dynamics as a coupled system with two parts: a rigid-body-like system (corresponding to the invertible part of the locked inertia tensor) and a non-symmetric simple mechanical system. The metric on $Q$ is split into a “reduced locked inertia” block and a “reduced mass” block, the two being coupled by a “Coriolis”-type term. The reduced Lagrangian appears naturally and the Euler-Poincaré-Lagrange bundle equations are consequently determined. As a concrete example, we calculate the reduced Lagrangian and the equations of motion in the case of three collinear point masses interacting via a bonding potential $V$ depending only on the pairwise distances.

Slice parametrisations have been successfully applied to stability and bifurcation problems. In the case of compact and free group actions, the Hamiltonian slice theorem was used to describing generic bifurcations near a relative equilibrium in the case of symmetric Hamiltonian systems (see [16] and [22]). Slice coordinates were also essential in determining stability of relative equilibria in the case of non-free group actions with singular momentum values in simple mechanical systems (see [21]). The Lagrangian framework developed in the present paper may see future applications in dynamics and control. For instance, this framework may allow the extension of the method of controlled Lagrangians (i.e., kinetic and potential shaping) [2, 3] to systems with configuration space isotropy.

2. Geometry. Basic definitions and notation. In this section we briefly review the geometry of tangent bundles near points with configuration space isotropy. For more details, see [5], [7], [13], [18], and [23].

Let $G$ be a Lie group, with Lie algebra $g$, and consider a smooth left action of $G$ on a finite-dimensional manifold $M$, written $(g, q) \mapsto g \cdot q$. For every $\xi \in g$ and $z \in M$, the infinitesimal action of $\xi$ on $z$ is $\xi \cdot z = \frac{d}{dt} \exp (t\xi) \cdot z|_{t=0}$. The isotropy subgroup of $z \in M$ is $G_z := \{g \in G \mid g \cdot z = z\}$. An action is free if all of the isotropy subgroups $G_z$ are trivial.

An action is proper if the map $(g, z) \mapsto (z, g \cdot z)$ is proper (i.e., the preimage of every compact set is compact). Note that this is always the case if $G$ is compact. It is well-known that if the action is proper, all isotropy subgroups are compact.

Given a $G$ action $\Phi : G \times Q \to Q$, the group $G$ has a tangent lift action on $TQ$, given by $g \cdot v = T\Phi_g (v)$. In this context, the space $Q$ called the configuration space or base space. For any $q \in Q$, the isotropy group $G_q$ is called the configuration space isotropy of any point $v \in T_q Q$.

Let $K$ be a Lie subgroup of $G$, and $S$ a manifold on which $K$ acts. Consider the following two left actions on $G \times S$:

- $K$ acts by twisting: $k \cdot (g, s) = (gk^{-1}, k \cdot s)$
- $G$ acts by left multiplication: $\gamma \cdot (g, s) = (\gamma g, s)$.
It is easy to show that these actions are free and proper and commute. The twisted product $G \times_K S$ is the quotient of $G \times S$ by the twist action of $K$. It is a smooth manifold, and the projection

$$\pi_K : G \times S \to G \times_K S$$

$$(g,s) \mapsto [g,s]_K$$

is a principal bundle with fibre $K$. The left multiplication action of $G$ commutes with the twist action and drops to a smooth $G$ action on $G \times_K S$, given by $\gamma \cdot [g,s]_K = [\gamma g,s]_K$.

Now consider a $G$ action on a manifold $M$, and a point $z \in M$, and let $K = G_z$ be the isotropy subgroup of $z$. A tube for the $G$ action at $z$ is a $G$-equivariant diffeomorphism from some twisted product $G \times_K S$ to an open neighbourhood of $G \cdot z$ in $M$, that maps $[e,0]_K$ to $z$. The space $S$ may be embedded in $G \times_K S$ as $\{[e,s]_K : s \in S\}$; the image of the latter by the tube is called a slice.

The slice theorem of Palais [20] (see also [18, 2.3.28]) states that tubes always exist for smooth proper actions of a Lie group $G$ on manifold $M$. One version of the theorem is the following: given $z \in M$, with isotropy group $K = G_z$, there always exists a $K$-invariant Riemannian metric on a neighbourhood of $z$. (This is due to the compactness of $K$, see [18] for details.) Let $N$ be the orthogonal complement to $g \cdot z$. Then there exists a $K$-invariant neighbourhood $S$ of 0 in $N$ such that the map

$$\tau : G \times_K S \to M$$

$$[g,s]_K \mapsto g \cdot \exp_z s$$

(where $\exp_z$ is the Riemannian exponential) is a tube at $z$ for the $G$ action. The $K$-invariant complement $N$ to $g \cdot z$ is sometimes called a linear slice to the $G$ action at $z$. The twisted product $G \times_K N$ may be identified with the normal bundle to the orbit $G \cdot z$. If $M$ is an open subset of a vector space, with $G$ acting linearly, as in the example in Section 5.1, then $S$ can be identified with a neighbourhood of the origin in a linear subspace of $M$ itself, and the tube defined by $\tau([g,s]_K) = g \cdot (q_0 + s)$.

### Configuration space slices.

Consider a Lagrangian $L : TQ \to \mathbb{R}$, invariant under a proper tangent-lifted action of a Lie group $G$. Let $q_0 \in Q$ and $K = G_{q_0}$. By Palais’ slice theorem, the following map

$$\tau : G \times_K S \to Q$$

$$[g,s]_K \mapsto g \cdot \exp_{q_0} s$$

is a tube at $q_0$ for the $G$ action. We consider the following composition,

$$G \times S \xrightarrow{\pi_K} G \times_K S \xrightarrow{\tau} Q,$$

and we regard the composition $\tau \circ \pi_K : G \times S \to Q$ as a degenerate “parametrisation” of $Q$ in a neighbourhood of $q_0$, defining the “slice coordinates” $(g,s)$. This parametrisation is semi-global in the sense that it is global in the group direction and local in the slice direction. The tangent lift of $\tau \circ \pi_K$ gives the parametrisation $T(G \times S) \to TQ$. In this paper we will describe the dynamics on $TQ$, with configurations in the neighbourhood of the group orbit $G \cdot q_0$, by pulling them back to $T(G \times S)$.
We denote by \( \mathfrak{g} \) and \( \mathfrak{k} \) the Lie algebras of \( G \) and \( K \). Throughout the paper we identify \( TG \) with \( G \times \mathfrak{g} \) using the left trivialisation given by:
\[
TG \xrightarrow{\cong} G \times \mathfrak{g}
\]
\[
T_L g \zeta \mapsto (g, \zeta)
\]
where \( L_g \) is left multiplication by \( g \). Similarly, we make the identification
\[
T(G \times S) \cong TG \times TS \cong G \times \mathfrak{g} \times TS \quad (2.3)
\]
Note that \( TS \) is trivial, as \( S \) is a subset of a vector space. We write elements of \( TS \) as \((s, \dot{s})\). The left multiplication action of \( G \) and the twist action of \( K \) on \( G \times S \) both lift to free, proper, commuting actions on \( T(G \times S) \).

**The tangent bundle.** Next we describe the tangent bundle \( T(G \times_K S) \) using a degenerate parametrisation by \( G \times \mathfrak{k} \perp TS \subset T(G \times S) \).

Fix a \( K \)-invariant complement of \( \mathfrak{k} \) in \( \mathfrak{g} \), which we denote \( \mathfrak{k} \perp \) (such a complement can always be found by averaging over \( K \), since \( K \) is compact). Consider the projection \( \pi_K : G \times S \to G \times_K S \). Its tangent map is a \( K \)-invariant \( G \)-equivariant surjection. If we describe points in \( T(G \times_K S) \) as \( T\pi_K(g, \zeta, s, \dot{s}) \), then we have two kinds of degeneracy in our coordinates: first, \((g, s)\) is not uniquely determined by \( T\pi_K\); second, given a choice of \((g, s)\), the tangent vector \((\zeta, \dot{s}) \in T_{(g,s)}(G \times S)\) is not uniquely determined because \( T_{(g,s)}\pi_K \) has a kernel,
\[
\ker T_{(g,s)}\pi_K = \mathfrak{k} \cdot (g, s) = \{(-\xi, \xi \cdot s) \mid \xi \in \mathfrak{k}\}.
\]
We therefore have the splitting:
\[
T_{(g,s)}(G \times S) \xrightarrow{\text{left-triv.}} \mathfrak{g} \oplus T_s S = (\mathfrak{k} \perp \oplus T_s S) \oplus \mathfrak{k} \cdot (g, s)
\]
(2.4)
\[
= (\mathfrak{k} \perp \oplus T_s S) \oplus \{-\xi^t, \xi^t \cdot s \mid \xi^t \in \mathfrak{k}\}
\]
for all \((g, s) \in (G, S)\). This defines a connection on the principal bundle \( \pi_K : G \times S \to G \times_K S \), with horizontal subspaces \((\mathfrak{k} \perp \oplus T_s S)\) (with left-trivialisation) at each \((g, s) \in G \times S\).

We eliminate the second kind of degeneracy by restricting \( T\pi_K \) to \( G \times \mathfrak{k} \perp TS \subset T(G \times S) \). Our new parametrisation of \( T(G \times_K S) \) is
\[
T\pi_K : G \times \mathfrak{k} \perp TS \longrightarrow T(G \times_K S). \quad (2.5)
\]
Note that for any \((g, s) \in G \times S\), the map \( T_{(g,s)}\pi_K \) is an isomorphism from \( \mathfrak{k} \perp \times T_s S \) to \( T_{(g,s)\cup}(G \times_K S) \). Composing \( \pi_K \) with the tube \( \tau \) from Equation 2.2 gives a map \((g, s) \mapsto g \cdot \exp_{q_0} s\). Differentiating this gives
\[
T(\tau \circ \pi_K) : G \times \mathfrak{k} \perp TS \longrightarrow TQ
\]
(2.6)
\[
(g, \zeta, s, \dot{s}) \mapsto g \cdot (\zeta \cdot \exp_{q_0} s + T_s \exp_{q_0}(\dot{s}))
\]
This formalises the observation that \( T_q Q \cong \mathfrak{g} \cdot q \oplus S \) near the point at which the slice \( S \) is defined.

Since \( G \times \mathfrak{k} \perp TS \) and \( T\pi_K \) are \( K \)-invariant, the map \( T\pi_K \) descends to the quotient by \( K \),
\[
\overline{T\pi_K} : (G \times \mathfrak{k} \perp TS) / K \longrightarrow T(G \times_K S).
\]
It is easily checked that this map is a \( G \)-equivariant diffeomorphism. The \( K \) action on \( G \times \mathfrak{k} \perp TS \) is exactly the twist action on \( G \times (\mathfrak{k} \perp TS) \) given by the adjoint
action on \( K \) on \( \mathfrak{k}^\perp \) and the tangent-lifted action of \( K \) on \( TS \). Thus \( T\pi_K \) may be written
\[
T\pi_K : G \times_K (\mathfrak{k}^\perp \times TS) \longrightarrow T(G \times_K S)
\]
and we see that \( T\pi_K \) is actually a tube for \( T(G \times_K S) \) around \( 0 \in [e,0]_K \).

From now on, for simplicity of notation, we will identify \( \tilde{\sigma} \) with \( [e,0]_K \).

Let \( \tilde{\sigma} : [e,0] \times T(S) \longrightarrow T(G \times_K S) \);

\[
\tilde{\sigma} t \in G \times_K \mathfrak{s} \quad \forall t \in [e,0].
\]

3. Equations of motion. Let \( L : TQ \rightarrow \mathbb{R} \) be a \( G \)-invariant Lagrangian and let \( q_0 \in Q \), with isotropy group \( K = G_{q_0} \). From the previous section, there exists a local \( G \)-equivariant diffeomorphism \( G \times_K S \rightarrow Q \) taking \( [e,0]_K \) to \( q_0 \), where \( S \) is a linear slice at \( q_0 \). For simplicity of notation, we identify \( Q \) with \( G \times_K S \) and \( q_0 \) with \( [e,0]_K \).

We define \( \hat{L} : T(G \times S) \rightarrow \mathbb{R} \) by \( \hat{L} = L \circ T\pi_K \). Any curve \( \hat{c}(t) \) in \( G \times S \) defines a curve \( (\hat{c}(t), \hat{\dot{c}}(t)) \) in \( T(G \times S) \). Using left-trivialised coordinates, this can be written as \( (\hat{c}(t), \hat{\dot{c}}(t)) = (g(t), \xi(t), s(t), \dot{s}(t)) \), which is a path in \( G \times \mathfrak{g} \times TS \), with \( \xi(t) := g(t)^{-1} \dot{g}(t) \) := \( Tg(t)^{-1} \dot{g}(t) \in \mathfrak{g} \), where \( \dot{g}(t)^{-1} \) is left multiplication by \( g(t)^{-1} \). For any path \( \hat{c}(t) \), define \( c(t) := (\pi_K \circ \hat{c})(t) \). Then by the chain rule, \( T\pi_K \hat{c}(t), \hat{\dot{c}}(t) = c(t), \dot{c}(t) \), so the actions \( \int_a^b \hat{L}(\hat{c}(t), \hat{\dot{c}}(t)) \, dt \) and \( \int_a^b \hat{L}(c(t), \dot{c}(t)) \, dt \) are identical. Thus, a curve \( \hat{c}(t) = (g(t), s(t)) \) is a solution of the Euler-Lagrange equations for \( \hat{L} \) if and only if the curve \( c(t) := (\pi_K \circ \hat{c})(t) = [g(t), s(t)]_K \) is a solution of the Euler-Lagrange equations for \( L \).

Since our configuration space is \( G \times S \) and \( G \) acts only on the first factor, one can combine Euler-Lagrange reduction and Euler-Poincaré reduction to arrive at the following extended Euler-Poincaré reduction theorem (for details, see [8, Section 7.3]):

**Theorem 3.1** (Euler-Poincaré Reduction). Let \( G \) be a Lie group, \( S \) a manifold, \( \tilde{L} : T(G \times S) \rightarrow \mathbb{R} \) a left-invariant Lagrangian, and \( \hat{L} : \mathfrak{g} \times TS \rightarrow \mathbb{R} \) be defined by \( \hat{L}(\xi,s) := \tilde{L}(e,\xi,s) \). For any curve \( g(t) \in G \), let \( \xi(t) = g(t)^{-1} \dot{g}(t) \). Then a curve \( (g(t), s(t)) \) satisfies the Euler-Lagrange equations for Lagrangian \( \tilde{L} \) if and only if the following reduced variational principle holds:
\[
\delta \int_a^b \hat{L}(\xi(t), s(t), \dot{s}(t)) \, dt = 0
\]
for variations \( (\delta \xi, \delta s) \) with \( \delta \xi \) of the form \( \delta \xi = \dot{\eta} + ad_\xi \eta \), where \( \eta(t) \) is an arbitrary path in \( \mathfrak{g} \) which vanishes at the endpoints, i.e., \( \eta(a) = \eta(b) = 0 \). This principle is equivalent to:
\[
\frac{d}{dt} \frac{\delta \hat{L}}{\delta \xi} = ad_\xi \frac{\delta \hat{L}}{\delta \xi} \quad \text{(Euler-Poincaré equation)} \quad \text{and} \quad \frac{d}{dt} \frac{\delta \hat{L}}{\delta \dot{s}} = \frac{\delta \hat{L}}{\delta \dot{s}} \quad \text{(Euler-Lagrange equation)}.
\]

We now apply this theorem in the particular case outlined before the theorem, in which \( \hat{L} = L \circ T\pi_K \). For every path \( \hat{c}(t) \) in \( G \times S \), let \( S(\hat{c}) = \int_a^b \hat{L}(\hat{c}(t), \hat{\dot{c}}(t)) \, dt \). Let
if  \( \tilde{\exp} \) is simple, then it follows that
\[
\tilde{L}(\tilde{\gamma}_s(t), \tilde{\gamma}_s(t)) = L \circ T\pi_K(\tilde{\gamma}_s(t), \dot{\tilde{\gamma}}_s(t)) = L \circ T\pi_K(\tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)).
\]
Therefore, defining
\[
\delta \tilde{\gamma}(t) := \left. \frac{d}{ds} \tilde{\gamma}_s(t) \right|_{s=0},
\]

it follows that
\[
DS(\tilde{\gamma}) \cdot \delta \tilde{\gamma} = \left. \frac{d}{ds} S(\tilde{\gamma}_s) \right|_{s=0} = \left. \frac{d}{ds} \right|_{s=0} \int_a^b \tilde{L}(\tilde{\gamma}_s(t), \dot{\tilde{\gamma}}_s(t)) \, dt = 0. \quad (3.7)
\]
In particular, for any path \( \eta^\perp(t) \) in \( \mathfrak{k} \), consider the family of paths \( k(t,s) := \exp(s \eta^\perp(t)) \). Then
\[
\delta \tilde{\gamma}(t) = \left. \frac{d}{ds} k(t,s) \dot{\gamma}_s(t) \right|_{s=0} = \eta^\perp(t) \cdot \dot{\gamma}(t).
\]
If \( \tilde{\gamma}(t) = (g(t), s(t)) \) for all \( t \), then \( k(t,s)c(t) = (g(t)k(t,s)^{-1}, k(t,s)c(t)) \), so
\[
\delta \tilde{\gamma}(t) = \eta^\perp(t) \cdot (g(t), s(t)) = (-g(t)\eta^\perp(t), \eta^\perp \cdot s(t)) \in T_{(g(t), s(t))}(G \times S).
\]
It follows from Equation 3.7 that, for any path \( \eta^\perp(t) \) in \( \mathfrak{k} \), variations of \( \tilde{\gamma}(t) \) of the form
\[
(\delta g(t), \delta s(t)) = (-g(t)\eta^\perp(t), \eta^\perp \cdot s(t))
\]
lead to \( \delta S = 0 \). Thus it suffices to consider only variations in some complement to these variations, for example variations of the form \( (\delta g, \delta s) \) such that \( \delta g(t) = g(t)\eta^\perp(t) \) for some path \( \eta^\perp(t) \) in \( \mathfrak{k}^\perp \). Note that these are variations in the horizontal direction of the connection defined in Equation 2.4, and we have shown that only these variations are important since \( \tilde{L} \) is degenerate in the vertical direction.

In summary, we have shown:

**Lemma 3.2.** To find critical points of the action functional for \( \tilde{L} \), it suffices to consider only variations of the form \( (\delta g, \delta s) \) such that \( \delta g(t) = g(t)\eta^\perp(t) \) for some path \( \eta^\perp(t) \) in \( \mathfrak{k}^\perp \).

In general, if \( \delta g(t) = g(t)\eta(t) \) and \( \xi(t) = g^{-1}(t)\dot{g}(t) \), for all \( t \), then the proof of Euler-Poincaré reduction shows that \( \delta \xi = \eta + \text{ad}_\xi \eta \). By the previous lemma, it suffices to consider only those \( \delta \xi \) of the form
\[
\delta \xi = \eta^\perp + \text{ad}_\xi \eta^\perp.
\]
Thus we obtain the following consequence of Euler-Poincaré reduction:

**Theorem 3.3.** Consider a \( G \)-invariant Lagrangian \( L : T(G \times_K S) \to \mathbb{R} \) and define \( \tilde{L} : T(G \times S) \to \mathbb{R} \) by \( \tilde{L} = L \circ T\pi_K \). Let \( \tilde{l}(\xi, s, \dot{s}) = \tilde{L}(g, \xi, s, \dot{s}) \). For every curve \( \tilde{c}(t) \) in \( G \times S \), let \( c(t) \) be the curve in \( G \times_K S \) defined by \( c = \pi_K \circ \tilde{c} \). Let \( \tilde{c}(t) = (g(t), s(t)) \) and \( \xi(t) = g^{-1}(t)\dot{g}(t) \) for all \( t \). Then \( c(t) \) satisfies the Euler-Lagrange equations for \( L \) if and only \( \tilde{c}(t) \) satisfies
\[
\delta \int_a^b \tilde{l}(\xi(t), s(t), \dot{s}(t)) \, dt = 0, \quad (3.8)
\]
using variations in \( \xi \) of the form
\[
\delta \xi = \eta^\perp + \text{ad}_\xi \eta^\perp, \quad (3.9)
\]
where $\eta^\perp(t)$ is an arbitrary path in $g^\perp$ which vanishes at the endpoints, i.e., $\eta^\perp(a) = \eta^\perp(b) = 0$; and variations of $s(t)$ that vanish at the endpoints.

Using the standard “integration by parts” argument, the variational principle in Theorem 3.3 is equivalent to:

$$0 = \int_a^b \left( \frac{\delta I}{\delta \xi}, (\dot{\eta}^\perp + \text{ad}_\xi \eta^\perp) \right) + \left( \frac{\delta I}{\delta s}, \delta s \right) + \left( \frac{\delta I}{\delta \dot{s}}, \delta \dot{s} \right) dt \quad (3.10)$$

$$= \int_a^b \left( \frac{\delta I}{\delta \xi}, \eta^\perp \right) + \left( \text{ad}_\xi \left( \frac{\delta I}{\delta \xi} \right), \eta^\perp \right) + \left( \frac{\delta I}{\delta s}, \delta s \right) + \left( \frac{\delta I}{\delta \dot{s}}, \delta \dot{s} \right) dt$$

for all paths $\eta^\perp(t)$ in $t^\perp$ and all variations $\delta s(t)$ of $s(t)$. We now break $\frac{\delta I}{\delta \xi}$ into $t^*$ and $t^\perp$ components. Just as we write the $t$ and $t^\perp$ components of $\xi$ as $\xi^t$ and $\xi^\perp$, respectively, we will write the $t^*$ and $t^\perp$ components of $\frac{\delta I}{\delta \xi}$ as $\frac{\delta I}{\delta \xi^t}$ and $\frac{\delta I}{\delta \xi^\perp}$. So

$$\frac{\delta I}{\delta \xi} = \text{Pr}_{t^*} \left( \frac{\delta I}{\delta \xi^t} \right) + \text{Pr}_{t^\perp} \left( \frac{\delta I}{\delta \xi^\perp} \right) = \frac{\delta I}{\delta \xi^t} + \frac{\delta I}{\delta \xi^\perp}.$$

With this notation, we have

$$\left< -\frac{d}{dt} \frac{\delta I}{\delta \xi^\perp}, \eta^\perp \right> = \left< \text{Pr}_{t^\perp} \left( -\frac{d}{dt} \frac{\delta I}{\delta \xi^\perp} \right), \eta^\perp \right> = \left< -\frac{d}{dt} \frac{\delta I}{\delta \xi^\perp}, \eta^\perp \right>.$$

We will use a line over $\text{ad}_\xi$ to indicate projection onto $t^\perp$, so, in particular,

$$\overline{\text{ad}_\xi} \frac{\delta I}{\delta \xi} := \text{Pr}_{t^\perp} \left( \text{ad}_\xi \frac{\delta I}{\delta \xi} \right).$$

Since $\eta^\perp$ and $\delta s$ are arbitrary, Equation 3.10 is equivalent to the following “Euler-Poincaré-Lagrange” equations:

$$\frac{d}{dt} \frac{\delta I}{\delta \xi^\perp} = \overline{\text{ad}_\xi} \frac{\delta I}{\delta \xi^\perp} \quad \text{and} \quad \frac{d}{dt} \frac{\delta I}{\delta \xi^\perp} = \frac{\delta I}{\delta \xi^\perp.} \quad (3.11)$$

The first of the Equations 3.11 can be rewritten, breaking both the $\xi$ and the $\frac{\delta I}{\delta \xi}$ on the right hand side into $t^\perp$ and $t$ components:

$$\frac{d}{dt} \frac{\delta I}{\delta \xi^\perp} = \overline{\text{ad}_\xi^t} \frac{\delta I}{\delta \xi^\perp} = \overline{\text{ad}_\xi^t} \frac{\delta I}{\delta \xi^\perp} + \overline{\text{ad}_\xi^\perp} \frac{\delta I}{\delta \xi^t} + \overline{\text{ad}_\xi} \frac{\delta I}{\delta \xi^t}.$$

The second term on the last line is zero, because $\text{ad}_\xi^t \frac{\delta I}{\delta \xi^t} \in t^*$ and the overline represents projection onto $t^\perp$. For the third term note that, since $t^\perp$ is $K$-invariant, $\text{ad}_{\eta^t} \xi^\perp \in t^\perp$ for all $\eta^t \in t$. It follows that for all $\nu^t \in t^*$

$$\left< \overline{\text{ad}_\xi} \nu^t, \eta^\perp \right> = \left< \nu^t, \text{ad}_\xi^\perp \eta^\perp \right> = -\left< \nu^t, \text{ad}_{\eta^t} \xi^\perp \right> = 0.$$
Hence the overline can be omitted from the third term. Thus the first of Equations 3.11 is equivalent to:

\[
\frac{d}{dt} \frac{\delta \tilde{l}}{\delta \xi} = \text{ad}_{\xi}^{\perp} \frac{\delta \tilde{l}}{\delta \xi} + \text{ad}_{\perp}^{\star} \frac{\delta \tilde{l}}{\delta \xi}. \tag{3.12}
\]

We now use the fact that \( \tilde{L} = L \circ T_{\pi_K} \) in a second way: since \( T_{(g,s)} \pi_K \) has kernel \( \mathfrak{k} \cdot (g, s) \), which in left-trivialised coordinates equals

\[
\{(-\xi^t, \xi^t \cdot s) \in T_{(g,s)} (G \times S) : \xi^t \in \mathfrak{k}\},
\]
we obtain \( \tilde{L}(g, \xi, s) = \hat{L}(g, \xi - \xi^t, s, s + \xi^t \cdot s) \) for all \( \xi \in \mathfrak{k} \), and thus:

\[
0 = \left\langle \frac{\delta \hat{L}}{\delta \xi^t}, -\xi^t \cdot s \right\rangle + \left\langle \frac{\delta \hat{L}}{\delta s}, \xi^t \cdot s \right\rangle = \left\langle \frac{\delta \tilde{L}}{\delta \xi^t}, -\xi^t \cdot s \right\rangle + \left\langle s \circ \frac{\delta \tilde{L}}{\delta s}, \xi^t \cdot s \right\rangle,
\]
for all \( \xi^t \in T_{\xi} \mathfrak{k} \simeq \mathfrak{k} \). Since \( \xi^t \) is arbitrary, this is equivalent to

\[
\frac{\delta \tilde{L}}{\delta \xi^t} = s \circ \frac{\delta \tilde{L}}{\delta s}.
\]

Since \( \tilde{l}(\xi, s, \delta) = \tilde{L}(e, \xi, s, \delta) = \tilde{L}(g, \xi, s, \delta) \), the previous two equations are also valid with \( \tilde{L} \) replaced with \( l \). In summary, Equations 3.11 are equivalent to

\[
\frac{d}{dt} \frac{\delta \tilde{l}}{\delta \xi} = \text{ad}_{\xi}^{\perp} \frac{\delta \tilde{l}}{\delta \xi} + \text{ad}_{\perp}^{\star} \left( s \circ \frac{\delta \tilde{l}}{\delta s} \right) \quad \text{and} \tag{3.14}
\]

\[
\frac{d}{dt} \frac{\delta \tilde{l}}{\delta s} = \frac{\delta \tilde{l}}{\delta s}.
\]

Up to this point we have considered a general base curve \( (\xi(t), s(t)) \) in \( g \times S \) for the variations. However, we can restrict attention to paths in \( \mathfrak{k}^{\perp} \times S \), since we are interested in paths \( c(t) \) in \( G \times_K S \). Indeed, as we have seen in the previous section, the projection \( \pi_K : G \times S \to G \times_K S \) is a principal bundle, with a connection given by the splittings in Equation 2.4. Given any path \( c(t) \) in \( G \times_K S \), and an element \((g_0, s_0) \in \pi_{K}^{-1}(c(0))\) (the fibre over \( c(0) \)), there exists a unique horizontal lift of \( c(t) \) to a path \( \hat{c}(t) \) such that \( \hat{c}(0) = (g_0, s_0) \). Note that \( \hat{c}(t) \) is horizontal if and only if \( \hat{c}(t) = (g(t), s(t)) \) and \( g^{-1}(t)\hat{g}(t) \in \mathfrak{k}^{\perp} \) for all \( t \). In left-trivialised coordinates, if \( (\hat{c}(t), \hat{c}(t)) = (g(t), \xi(t), s(t), \delta(t)) \) then \( \hat{c}(t) \) is a horizontal curve if and only if \( \xi^t(t) = 0 \) for all \( t \). Note that \( c(t) \) satisfies the Euler-Lagrange equations for \( L \) if and only if its horizontal lifts \( \hat{c}(t) \) satisfy the Euler-Lagrange equations for \( \tilde{L} \). Thus a curve \( c(t) \) satisfies the Euler-Lagrange equations for \( L \) if and only if any (and hence all) of its horizontal lifts to \( G \times S \) satisfies Equations 3.14 with \( \xi \) replaced by \( \xi^t \).

We have now eliminated all \( \mathfrak{k} \) components from Equations 3.14. Thus it makes sense to restate these equations in terms of the restriction of \( \tilde{l} \) to \( \mathfrak{k}^{\perp} \times TS \), which we denote

\[
\tilde{l} : \mathfrak{k}^{\perp} \times TS \to \mathbb{R}. \tag{3.15}
\]

Since

\[
\frac{\delta \tilde{l}}{\delta \xi^t} = \frac{\delta \tilde{l}}{\delta \xi^t}, \quad \frac{\delta \tilde{l}}{\delta s} = \frac{\delta \tilde{l}}{\delta s}, \quad \text{and} \quad \frac{\delta \tilde{l}}{\delta s} = \frac{\delta \tilde{l}}{\delta s},
\]
the same equations apply with \( \tilde{l} \) replaced by \( \tilde{l} \) throughout.
Observation 3.4. The equations of motion and all calculations are naturally performed on the reduced model space $\mathfrak{k}^\perp \times TS$. This will be illustrated in the example presented in Section 5.

We have proven the following:

**Theorem 3.5 (Euler-Poincaré Bundle Equations).** Let $L$ and $\tilde{L}$ be as above. For every curve $c(t)$ in $G \times K S$, let $(g(t), s(t))$ be a horizontal lift of $c(t)$. Define $\xi^\perp(t) = g^{-1}(t)\dot{g}(t)$, which is always in $\mathfrak{k}^\perp$ by the definition of a horizontal lift. Then $c(t)$ satisfies the Euler-Lagrange equations for $L$ if and only $(\xi^\perp(t), s(t))$ satisfies

$$
\frac{d}{dt} \left( \frac{\delta L}{\delta \xi^\perp} \right) = \text{ad}_{\xi^\perp}^* \left( \frac{\delta L}{\delta \xi^\perp} \right) + \text{ad}_{\xi^\perp}^* \left( s \circ \frac{\delta L}{\delta s} \right) \quad \text{and} \quad (3.16)
$$

$$
\frac{d}{dt} \left( \frac{\delta L}{\delta \dot{s}} \right) = \frac{\delta L}{\delta \dot{s}} .
$$

Note that the first equation above is a modified version of the Euler-Poincaré equation, while the second equation is an Euler-Lagrange equation.

**Reconstruction.** The reconstruction of the curve $g(t)$ along the group is given by the non-autonomous differential equation

$$
\dot{g}(t) = g(t)\xi^\perp(t) \quad \text{with} \quad g(0) = g_0 .
$$

**Remark 3.6.** If the isotropy subgroup $K$ is trivial, then the theorem above reduces to a simple consequence of the Lagrange-Poincaré equations given in [4].

4. **Variational principle.** In the previous section, we derived the Euler-Poincaré bundle equations, stated in Theorem 3.5, from the variational principle stated in Theorem 3.3. Note however that the variational principle in Theorem 3.3 is stated in terms of the function $\tilde{l} : \mathfrak{g} \times TS \to \mathbb{R}$, while the Euler-Poincaré bundle equations involve only the function $\tilde{l} : \mathfrak{k}^\perp \times TS \to \mathbb{R}$, which is the restriction of $\tilde{l}$ to $\mathfrak{k}^\perp \times TS$.

Can we state a corresponding variational principle on $\mathfrak{k}^\perp \times TS$?

One approach is to replace variations $\delta \xi^\perp$ (an arbitrary variation in the $\mathfrak{k}$ direction) with equivalent variations in the $T_s S$ direction, using the fact, established in the previous section, that

$$
\left\langle \frac{\delta \tilde{l}}{\delta \xi^\perp}, -\delta \xi^\perp \right\rangle + \left\langle \frac{\delta \tilde{l}}{\delta \dot{s}}, \delta \xi^\perp \cdot s \right\rangle = 0
$$

for all $\delta \xi^\perp \in T_{\xi^\perp} \simeq \mathfrak{k}$ (see Equation 3.13 and following remarks). It follows from the previous equation that

$$
\delta \tilde{l}(\xi(t), s(t), s(t)) = \left\langle \frac{\delta \tilde{l}}{\delta \xi^\perp}, \delta \xi^\perp \right\rangle + \left\langle \frac{\delta \tilde{l}}{\delta \xi^\perp}, \delta \xi^\perp \right\rangle + \left\langle \frac{\delta \tilde{l}}{\delta \dot{s}}, \delta \dot{s} \right\rangle + \left\langle \frac{\delta \tilde{l}}{\delta \dot{s}}, \delta \dot{s} \right\rangle
$$

$$
= \left\langle \frac{\delta \tilde{l}}{\delta \xi^\perp}, \delta \xi^\perp \right\rangle + \left\langle \frac{\delta \tilde{l}}{\delta \dot{s}}, \delta \dot{s} \right\rangle + \left\langle \frac{\delta \tilde{l}}{\delta \dot{s}}, \frac{d}{dt} \delta \dot{s} + \delta \xi^\perp \cdot s \right\rangle ,
$$

since $\delta \dot{s} = \frac{d}{dt} \delta s$. In Theorem 3.3, the variations in $\xi$ are of the form

$$
\delta \xi = \eta^\perp + \text{ad}_\xi \eta^\perp , \quad (4.17)
$$
and so $\delta \xi^t = \Pr_{\xi} (\text{ad}_\xi \eta^\perp)$ and $\delta \xi^\perp = \dot{\eta}^\perp + \Pr_{\eta^\perp} (\text{ad}_\eta \eta^\perp)$. We introduce the notation $(\text{ad}_\xi \eta^\perp)^t := \Pr_{\xi} (\text{ad}_\xi \eta^\perp)$ and $(\text{ad}_\eta \eta^\perp)^\perp := \Pr_{\eta^\perp} (\text{ad}_\eta \eta^\perp)$. Thus

$$
\delta \xi^t = (\text{ad}_\xi \eta^\perp)^t \quad \text{and} \quad \delta \xi^\perp = \dot{\eta}^\perp + (\text{ad}_\eta \eta^\perp)^\perp.
$$

It follows that in this case,

$$
\delta \tilde{l} (\xi(t), s(t), \dot{s}(t)) = \left\langle \frac{\delta \tilde{l}}{\delta \xi^t}, \dot{\eta}^\perp + (\text{ad}_\eta \eta^\perp)^\perp \right\rangle \\
+ \left\langle \frac{\delta \tilde{l}}{\delta s}, \frac{d}{dt} \delta s + (\text{ad}_\eta \eta^\perp)^\perp \cdot s \right\rangle.
$$

(4.18)

Note that the $\xi$ component of $\delta \xi$ no longer appears in the formula. If $\xi(t) = \xi^\perp(t) \in \xi^\perp$, then $\tilde{l}$ may be replaced by $\tilde{l}$ in the above equation. Thus we have shown that, at a base curve $(\xi^\perp(t), s(t), \dot{s}(t))$, the condition

$$
\delta \int_a^b \tilde{l} (\xi(t), s(t), \dot{s}(t)) \ dt = \int_a^b \delta \tilde{l} (\xi(t), s(t), \dot{s}(t)) \ dt = 0,
$$

(4.19)

for variations as described in Theorem 3.3, is equivalent to the condition

$$
\delta \int_a^b \tilde{l} (\xi^\perp(t), s(t), \dot{s}(t)) \ dt = \int_a^b \tilde{l} (\xi(t), s(t), \dot{s}(t)) \ dt = 0,
$$

(4.20)

for variations of the form

$$
\left( \dot{\eta}^\perp + (\text{ad}_\eta \eta^\perp)^\perp, \delta s, \frac{d}{dt} \delta s + (\text{ad}_\eta \eta^\perp)^\perp \cdot s \right),
$$

where $\eta^\perp(t)$ is an arbitrary path in $g^\perp$ that vanishes at the endpoints, and $\delta s$ is an arbitrary variation of $s$ that vanishes at the endpoints. Thus Theorem 3.3 implies the following variational principle:

**Proposition 4.1.** Consider a $G$–invariant Lagrangian $L: T(G \times K) S \to \mathbb{R}$ and define $\tilde{L}: T(G \times S) \to \mathbb{R}$ by $\tilde{L} = L \circ T\pi_K$. Let $\tilde{l}(\xi, s, \dot{s}) = \tilde{L}(g, \xi, s, \dot{s})$, and let $\tilde{l}$ be the restriction of $\tilde{l}$ to $\xi^\perp \times TS$. For every curve $c(t)$ in $G \times S$, let $\tilde{c}(t)$ be the horizontal lift of $c(t)$ to $G \times K$, in the sense described above. Let $c(t) = (g(t), s(t))$ and $\xi(t) = \tilde{g}^{-1}(t) \tilde{g}(t)$ for all $t$. Then $c(t)$ satisfies the Euler-Lagrange equations for $L$ if and only $\tilde{c}(t)$ satisfies

$$
\delta \int_a^b \tilde{l} (\xi^\perp(t), s(t), \dot{s}(t)) \ dt = 0,
$$

(4.21)

for variations of the form

$$
\left( \dot{\eta}^\perp + (\text{ad}_\eta \eta^\perp)^\perp, \delta s, \frac{d}{dt} \delta s + (\text{ad}_\eta \eta^\perp)^\perp \cdot s \right),
$$

where $\eta^\perp(t)$ is an arbitrary path in $g^\perp$ that vanishes at the endpoints, and $\delta s$ is an arbitrary variation of $s$ that vanishes at the endpoints.

**Remark 4.2.** By Theorem 3.5, the variational principle in the previous proposition is equivalent to the Euler-Poincaré bundle equations (Equations 3.16).
5. Example: Simple mechanical systems. In this section we consider the theory above in the case of simple mechanical systems. We also provide an example by considering a three-point-mass system at a collinear configuration.

Let $L : TQ \to \mathbb{R}$ be a Lagrangian characterised by a non-degenerate left $G$-invariant Riemannian metric $K(\cdot, \cdot) : TQ \times TQ \to \mathbb{R}$ and a left $G$-invariant potential $V : Q \to \mathbb{R}$. The Lagrangian reads:

$$L(q, v_q) = \frac{1}{2} K(v_q, v_q) - V(q). \quad (5.22)$$

Let $q_0$ be a symmetric point with isotropy group $K$. We take $q_0$ to be the base of the slice coordinates. Let $S$ be the orthogonal complement with respect to the given metric to the tangent orbit through $q_0$, i.e.,

$$S = (gq_0)^{\perp} = \{ (q, v_q) \mid K(v_q, \xi_Q(q_0)) = 0 \}. \quad (5.23)$$

For simplicity we confine ourselves to a linear $G$ action, so that points in the configuration space $Q$ can be represented as $q = g(q_0 + s) \equiv [g, s]_K$ (see Section 2). Since our analysis is performed for finite-dimensional configuration spaces $Q$, we can take advantage of the matrix representation of the metric $K$.

We represent each velocity vector $v_q \in T_qQ$ as $v_q = (g, \xi^\perp, s, \dot{s}) \in G \times \mathfrak{t}^\perp \times TS$, using the parametrisation in Equation 2.5. A direct derivation of $q = g(q_0 + s)$ gives $v_q = g(\xi^\perp(q_0 + s) + \dot{s})$ and the dynamics of the system takes place in the model space $G \times \mathfrak{t}^\perp \times TS$.

Under the change of variable $(q, v_q) = (g, \xi^\perp, s, \dot{s})$ the kinetic term becomes:

\[
\frac{1}{2} K(v_q, v_q) = \frac{1}{2} K((\xi^\perp, \dot{s}), (\xi^\perp, \dot{s})) = \frac{1}{2} K((\xi^\perp, 0), (\xi^\perp, 0)) + \frac{1}{2} K((\xi^\perp, 0), (0, \dot{s})) + \frac{1}{2} K((0, \dot{s}), (\xi^\perp, 0)) + \frac{1}{2} K((0, \dot{s}), (0, \dot{s})).
\]

We define the reduced locked inertia tensor $I : S \to \mathcal{L}(\mathfrak{t}^\perp^*, \mathfrak{t}^\perp)$, the Coriolis term $C : S \to \mathcal{L}(T_sS, (\mathfrak{t}^\perp^*))$, and the reduced mass $m : S \to \mathcal{L}(T_sS, T_sS)$ as following:

\[
\langle \mathbb{I}(s)\xi^\perp, \eta^\perp \rangle := K((\xi^\perp, 0), (\eta^\perp, 0)), \quad (5.24)
\]

\[
\langle C(s)s, \xi^\perp \rangle := K((0, \dot{s}), (\xi^\perp, 0)), \quad \text{and} \quad (5.25)
\]

\[
\langle m(s)s, \dot{u} \rangle := K((0, \dot{s}), (0, \dot{u})). \quad (5.26)
\]

Note that $\mathbb{I}(s)$ is a non-degenerate inner product on $\mathfrak{t}^\perp$ and it represents the restriction of the usual locked inertia tensor to the Lie algebra $\mathfrak{t}^\perp$. Similarly, $m(s)$ is a non-degenerate inner product on $T_sS$ and it represents the restriction of the kinetic metric to the subspace $T_sS$ (the terminology comes from the fact that in classical mechanics the kinetic metric is given by the mass tensor of the system). The mixed map $C(s) s$ couples the system and the terminology is inherited from the usual Coriolis forces in mechanics. Our choice of coordinates with the slice based at $q_0 \equiv [e, 0]_K$ enforces $C(0) = 0$, i.e., the system is decoupled at the base.

In coordinates $(\xi^\perp, s, \dot{s}) \in \mathfrak{t}^\perp \times TS$ the reduced Lagrangian $\bar{l}$ (introduced by the mapping 3.15) reads as

$$\bar{l}(\xi^\perp, s, \dot{s}) = \frac{1}{2} \langle \mathbb{I}(s)\xi^\perp, \xi^\perp \rangle + \langle C(s)s, \xi^\perp \rangle + \langle \dot{s}, C^* (s)\xi^\perp \rangle + \frac{1}{2} \langle m(s)s, \dot{s} \rangle - V(s), \quad (5.27)$$

where $C^T$ stands for the adjoint operator of $C$ (note that in matrix representation $C^*(s)$ is simply the transpose of $C$, i.e., $C^*(s) = C^T(s)$). The Euler-Poincaré bundle equations may now be written as given by Theorem 3.5.

### 5.1. Collinear three-point-mass systems

An example of a simple mechanical system at an isotropic configuration point is provided by an $SO(3)$-invariant three-point-mass system in a collinear configuration.

Let $m_1, m_2, m_3$ be three point masses aligned along the $Oz$ axes in a Cartesian system of reference of the Euclidian space $E$. The point masses $m_1, m_2, m_3$ are interacting though means of a given bonding potential $V$ depending only on the pairwise distances. Example of such systems are given by Eulerian configurations in the classical Newtonian three-body problem (although the collinear configuration need not be a relative equilibrium), or collinear molecules formed by three atoms.

The configuration space we use the reduced Jacobi-Bertrand-Haretu coordinates (see [12]), that is, by the relative vector $r_1$ determined by $m_1, m_2$, and the vector $r_2$ describing the position of $m_3$ relative to the center of mass of the $m_1 m_2$, subsystem. The symmetry group is $SO(3)$ and acts diagonally on the configurations space $Q = \mathbb{R}^3 \times \mathbb{R}^3$ (where we ignore possible collisions) and by cotangent lift on $TQ = T\mathbb{R}^6$. We denote by $v_1$ and $v_2$ the velocity vectors with based at $r_1$ and $r_2$, respectively. The Lagrangian of the system is

$$L(r_1, r_2, v_1, v_2) = \frac{1}{2} \begin{pmatrix} v_1 & v_2 \end{pmatrix} \begin{pmatrix} \text{diag}(M_1) & 0 \\ 0 & \text{diag}(M_2) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + V(r_1, r_2) \quad (5.28)$$

where

$$M_1 := \frac{m_1 m_2}{m_1 + m_2} \quad \text{and} \quad M_2 := \frac{(m_1 + m_2) m_3}{m_1 + m_2 + 3}$$

are the reduced masses of the system, and $V$ is $SO(3)$-invariant.

Let $r_1^0 = (0, 0, \lambda_1)$, $r_2^0 = (0, 0, \lambda_2)$ be the (initial) collinear configuration of the system. The isotropy subgroup of $(r_1^0, r_2^0)$ is the $SO(2)$ group of rotations around the $Oz$ axis. The isotropy subalgebra $\mathfrak{k}$ consists in instantaneous rotations about the $Oz$ axis which we denote by $so(2)_z$. Its complement $\mathfrak{k}^\perp$ is formed by instantaneous rotations about $Ox$ and $Oy$ denoted by $so(3)_{xy}$. The slice $S$ is given by all vectors $(w_1, w_2) \in \mathbb{R}^3 \times \mathbb{R}^3$ such that

$$\begin{pmatrix} w_1 & w_2 \end{pmatrix} \begin{pmatrix} \text{diag}(M_1) & 0 \\ 0 & \text{diag}(M_2) \end{pmatrix} \begin{pmatrix} \xi_{\mathbb{R}^3}(0, 0, \lambda_1) \\ \xi_{\mathbb{R}^3}(0, 0, \lambda_2) \end{pmatrix} = 0 \quad \text{for all} \quad \xi_{\mathbb{R}^3} \in so(3).$$

In this case $S$ is a four-dimensional subspace of $\mathbb{R}^6$. A choice to describe $S$ is

$$S := \{(s_1, s_2, s_3, -\rho s_1, -\rho s_2, s_4) \mid s_i \in \mathbb{R}, \; i = 1, 2, 3, 4\} \quad (5.29)$$

where $\rho = \frac{M_1 \lambda_1}{M_2 \lambda_2}$. Using the definitions given in Equations 5.24, 5.25, and 5.26 and denoting $M := M_1 + \rho^2 M_2$ and $\Delta := M_1 (\lambda_1 + s_3) - \rho M_2 (\lambda_2 + s_4)$, we obtain:
\[\mathbb{I}(s) = \begin{pmatrix} M_1(\lambda_1 + s_3)^2 + M_2(\lambda_2 + s_4)^2 + M s_3^2 & -M s_2^2 \\ -M s_1 s_2 & M_1(\lambda_1 + s_3)^2 + M_2(\lambda_2 + s_4)^2 + M s_1^2 \end{pmatrix},\]

\[\mathbb{C}(s) = \begin{pmatrix} 0 & -\Delta M_1 s_2 \\ \Delta & 0 \end{pmatrix},\]

\[m(s) = \begin{pmatrix} M & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & M_1 & 0 \\ 0 & 0 & 0 & M_2 \end{pmatrix} =: m.\]

Using Theorem 3.5, the dynamics near the collinear configuration \((r_0^1, r_0^2)\) is described in the reduced model space \(so(3)_{xy} \times \mathbb{R}^4\) and it is given by curves \((\xi^\perp(t), s(t))\) that satisfy

\[\frac{d}{dt} \left( \delta \ell \frac{\delta \ell}{\delta \xi^\perp} \right) = \text{ad}_{\xi^\perp}^* \left( \delta \ell \frac{\delta \ell}{\delta \xi^\perp} \right) + \text{ad}_{s^*}^* \left( s \circ \delta \ell \right)\]

and

\[\frac{d}{dt} \left( \delta s \frac{\delta s}{\delta s} \right) = \delta \ell \frac{\delta s}{\delta s}.\]

We calculate:

\[\text{ad}_{\xi^\perp}^* \left( \delta \ell \frac{\delta \ell}{\delta \xi^\perp} \right) = \text{Pr}_{xy} \left( \text{ad}_{s^*}^+ \left( \delta \ell \frac{\delta \ell}{\delta \xi^\perp} \right) \right) + \text{Pr}_{xy} \left( \left( \frac{\delta \ell}{\delta \xi^1}, \frac{\delta \ell}{\delta \xi^2}, 0 \right) \times (\xi_x, \xi_y, 0) \right) = 0.\]

The diamond operator definition as specialised in the present context is: for all \(s \in S = \mathbb{R}^4\) and \(\sigma \in T^* S = T^* \mathbb{R}^4\), \(\langle s \circ \sigma, \xi \rangle = \langle \sigma, \xi^* \rangle\) for any \(\xi \in \text{so}(2)_z\). In fact, \(s \circ \sigma\) is the momentum map corresponding to the co-tangent lifted action of the isotropy group \(SO(2)_z\) on the slice \(T^* S = T^* \mathbb{R}^4\). Writing the infinitesimal action of \(\text{so}(2)_z\) on \(S\) as embedded in \(\mathbb{R}^6\) (Equation 5.29) we have that for all \(\xi = (0, 0, \xi_z) \in \text{so}(2)_z\)

\[\xi \cdot (s_1, s_2, s_3, -\rho s_1, -\rho s_2, s_4) = ((0, 0, \xi_z) \times (s_1, s_2, s_3), (0, 0, 0) \times (-\rho s_1, -\rho s_2, s_4)) = (-\xi_z s_2, s_z s_1, 0, \xi_z \rho s_2, -\xi_z \rho s_1, 0).\]

Thus the infinitesimal action of \(\text{so}(2)_z\) on \(S\) is given by

\[\xi \cdot (s_1, s_2, s_3, s_4) = (-\xi_z s_2, s_z s_1, 0, 0).\]

The corresponding momentum map \(s \circ \sigma : T^* S \to (\text{so}(2)_z)^*\) is

\[s \circ \sigma = -\sigma_1 s_2 + \sigma_2 s_1.\]

We now calculate the second term of Equation 5.30:
\[ \text{ad}_{\xi_x}^s \left( s \cdot \frac{\delta I}{\delta s} \right) = \left( 0, 0, s \cdot \frac{\delta I}{\delta s} \right) \times (\xi_x, \xi_y, 0) \]
\[ = \left( 0, 0, -\frac{\delta I}{\delta \xi_x} s_2 + \frac{\delta I}{\delta \xi_y} s_1 \right) \times (\xi_x, \xi_y, 0) \]
\[ = \left( -\xi_y \left( -\frac{\delta I}{\delta \xi_x} s_2 + \frac{\delta I}{\delta \xi_y} s_1 \right), -\xi_x \left( -\frac{\delta I}{\delta \xi_x} s_2 + \frac{\delta I}{\delta \xi_y} s_1 \right), 0 \right). \]

Using the Lagrangian \( \bar{l} \) expression as given by Equation 5.27 and the calculations above, Equations 5.30 become
\[ \frac{d}{dt} \left( I(s) \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} + C(s) \dot{s} \right) = \left( -\frac{\partial I}{\partial \xi_x} s_2 + \frac{\partial I}{\partial \xi_y} s_1 \right) \left( -\xi_y \right) \]
whereas Equations 5.31 take the form
\[ \frac{d}{dt} \left( (\xi^\perp)^T C(s) + m \dot{s} \right) = \frac{\partial}{\partial s} \left( \frac{1}{2} (\xi^\perp)^T C(s) \xi^\perp + (\xi^\perp)^T C(s) \dot{s} + V(s) \right) + \frac{1}{2} \dot{s}^T m \dot{s}. \]

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