A Unified Approach to Computation of Integrable Structures

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Abstract We expose a unified computational approach to integrable structures (including recursion, Hamiltonian, and symplectic operators) based on geometrical theory of partial differential equations.

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1 Introduction

Although there seems to be no commonly accepted definition of integrability for general systems of partial differential equations (PDEs), [21, 30], some features of these systems are beyond any doubt. These are:

– infinite hierarchies of (possibly, nonlocal) symmetries and/or conservation laws,
– (bi-)Hamiltonian structures,
– recursion operators.

Of course, these properties are closely interrelated, even if this is not so straightforward as it may seem at first glance (see, e.g., [2]).
Methods to establish the integrability property are numerous and often reduce to finding a Lax pair for a particular PDE. In spite of their efficiency, all these methods possess two drawbacks that are quite serious, to our opinion:

- they are unable to deal with general PDEs directly and to be applied require reduction of the initial system to evolution form (which is not always done in a correct way, see, e.g., the discussion in [7]),
- they lack a well-defined language for working with nonlocal objects that are essential in most applications.

In a series of papers [7, 8, 10–15] we presented our approach to integrable structures which is, at least partially, free from these disadvantages. In [16] a review of the geometrical theory illustrated with many examples from the above papers can be found.

Essentially, the approach is based on two constructions naturally associated to the equation at hand: those of tangent and cotangent coverings (see Sect. 6) that serve as exact conceptual counterpart to tangent and cotangent bundles in classical differential geometry. These coverings are also differential equations, and we prove that all operators responsible for the integrability properties (i.e., Hamiltonian, symplectic, and recursion ones) can be identified with higher or generalized symmetries and cosymmetries of these equations (Sect. 5). Consequently, all computations boil down to solving two linear equations:

\[ \ell_E^\ast (\phi) = 0 \]

and

\[ \ell^\ast E (\psi) = 0, \]

where \( \ell_E \) is the linearization operator of the initial equation and \( \ell^\ast E \) is its formally adjoint (see Sects. 3 and 4) lifted to tangent or cotangent coverings. The solutions that possess certain additional properties (expressed in terms of the Schouten and Nijenhuis brackets and the Euler operator) deliver the needed operators.

In this paper, we tried to expose our method in such a way that the interested reader could use it as an operational manual for computations. As a tutorial example we chose the Korteweg-de Vries equation because all results for this equation are well known and can be easily checked, even by hand. More complicated equations are also briefly discussed. In particular, we will consider equations in more than two independent variables which are naturally presented in non-evolutionary form (Kadomtsev-Petviashvily and Plebanski equations). The related examples appear here for the first time. In Sect. 11, we describe a freely available software package that was used in our computations [27].

## 2 Differential Equations and Solutions

Our general framework is based on [1], but we will give a coordinate-based exposition here.

For an equation in \( n \) independent variables \( x^i \) and \( m \) unknown functions \( u^j \) we consider the jet space \( J^\infty(n,m) \) with the coordinates \( x^i, u^j_\sigma, \) where \( i = 1, \ldots, n, \ j = 1, \ldots, m \) and \( \sigma = i_1 i_2 \ldots i_{|\sigma|}, \) where \( 0 \leq i_k \leq n \) and \( i_1 \leq i_2 \leq \cdots \leq i_{|\sigma|}, \) is an ordered multi-index of finite, but unlimited length \( |\sigma| \). Denote by \( \pi : J^\infty(n,m) \to \mathbb{R}^n \) the projection to the space of independent variables \( x^1, \ldots, x^n. \)

The vector fields

\[ D_i = \frac{\partial}{\partial x^i} + \sum_{j,\sigma} u^j_\sigma \frac{\partial}{\partial u^j_\sigma}, \quad i = 1, \ldots, n, \]  

(1)
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where $\sigma i$ is the ordered multiindex obtained by $\sigma$ and $i$ after reordering, are called total derivatives. They define an $n$-dimensional distribution on $J^\infty(n, m)$ that is called the Cartan (or higher contact) distribution and is denoted by $\mathcal{C}$. Dually, the Cartan distribution is the annihilator of the system of differential 1-forms

$$\omega^j_\sigma = du^j_\sigma - \sum_i u^j_{\sigma i} dx^i, \quad j = 1, \ldots, m, \quad |\sigma| \geq 0,$$

the so-called Cartan, or higher contact forms.

Let a differential equation be given by

$$F^l\left(x^i, \ldots, \partial^{|\sigma|} u^j_{\sigma}, \ldots\right) = 0, \quad l = 1, \ldots, r.$$  

Then we consider all its differential consequences, or prolongations,

$$D_{\sigma} F^l = 0, \quad l = 1, \ldots, r, \quad |\sigma| \geq 0,$$

where $D_{\sigma} = D_{\sigma 1} \circ \cdots \circ D_{\sigma k}$ for $\sigma = i_1 \ldots i_k$. We take the (usually, infinite-dimensional) hypersurface $\mathcal{E} \subset J^\infty(n, m)$ defined by zeros of (4) as a geometrical image of (3).

We assume that the following two conditions hold: first, if $G$ is a function on $J^\infty(n, m)$ and $G|_{\mathcal{E}} = 0$ then $G$ is a differential consequence of $(F^1, \ldots, F^r)$; second, for any differential operator in total derivatives $\Delta$ such that $\Delta(F^l) = 0, l = 1, \ldots, r$, we have $\Delta|_{\mathcal{E}} = 0$.

The second condition excludes gauge equations from our consideration.

Differential operators in total derivatives are said to be $\mathcal{C}$-differential operators.

By (4), all total derivatives are tangent to $\mathcal{E}$ and thus the Cartan distribution induces an $n$-dimensional distribution on $\mathcal{E}$, which we call by the same name. Its $n$-dimensional integral manifolds are solutions of $\mathcal{E}$.

**Example 1** Consider the KdV equation

$$u_t = uu_x + u_{xxx}. $$

The corresponding jet space is $J^\infty(2, 1)$ with the coordinates

$$x, \quad t, \quad u_{i,k} = u_{x \underbrace{x \ldots x}_{i \text{ times}} t \underbrace{t \ldots t}_{k \text{ times}}}$$

and the total derivatives

$$D_x = \frac{\partial}{\partial x} + \sum_{i,k \geq 0} u_{i+1,k} \frac{\partial}{\partial u_{i,k}}, \quad D_t = \frac{\partial}{\partial t} + \sum_{i,k \geq 0} u_{i,k+1} \frac{\partial}{\partial u_{i,k}}.$$ 

Thus, the space $\mathcal{E}$ is defined by the relations

$$u_{i,k+1} = D^l_x D^k_t (u_{0,0} u_{1,0} + u_{3,0}),$$

from where it follows that all partial derivatives containing $t$ may be expressed via the ones containing $x$ only. The elements of the first group are called principal derivatives, and the remaining ones are called parametric derivatives [20].
The functions \( x, t \), and the parametric derivatives \( u_i = u_{i,0} \) may be taken for coordinates on \( \mathcal{E} \) (internal coordinates). In terms of these coordinates, the total derivatives are

\[
D_x = \frac{\partial}{\partial x} + \sum_{i \geq 0} u_{i+1} \frac{\partial}{\partial u_i}, \quad D_t = \frac{\partial}{\partial t} + \sum_{i \geq 0} D^i_x (uu_1 + u_3) \frac{\partial}{\partial u_i},
\]

while the Cartan forms on \( \mathcal{E} \) are given by

\[
\omega_i = du_i - u_{i+1} \, dx - D^i_x (uu_1 + u_3) \, dt, \quad i \geq 0.
\]

**Example 2** The KP equation

\[
u_{yy} = u_{tx} - u^2_x - uu_{xx} - \frac{1}{12} uu_{xxxx}
\]

is defined in the jet space \( J^\infty(3, 1) \). Its total derivatives can be written as

\[
D_t = \frac{\partial}{\partial t} + \sum_{\sigma \in P} u_{\sigma t} \frac{\partial}{\partial u_{\sigma}}, \quad D_x = \frac{\partial}{\partial x} + \sum_{\sigma \in P} u_{\sigma x} \frac{\partial}{\partial u_{\sigma}}, \quad D_y = \frac{\partial}{\partial y} + \sum_{\sigma \in P} u_{\sigma y} \frac{\partial}{\partial u_{\sigma}},
\]

where \( P \) is the set of multi-indexes that do not contain more than one instance of \( y \). This is the set of multi-indexes of all parametric derivatives. Note that in \( D_y \) the coordinates \( u_{\sigma y} \), where \( \sigma \in P \) is of the form \( \sigma = \tau y \), are principal, and must be replaced with

\[
D_{\tau} \left( u_{tx} - u^2_x - uu_{xx} - \frac{1}{12} uu_{xxxx} \right).
\]

### 3 Linearization and Symmetries

A (non-trivial) symmetry of the distribution \( \mathcal{E} \) (on \( J^\infty(n, m) \) or on \( \mathcal{E} \)) is a \( \pi \)-vertical vector field that preserves this distribution. In other words,

\[
X = \sum_{\sigma, j} a^j_{\sigma} \frac{\partial}{\partial u_{\sigma}^j}
\]

is a symmetry if and only if

\[
[D_i, X] = 0, \quad i = 1, \ldots, n.
\]

Equation (8) implies that

\[
a^j_{\sigma i} = D_i (a^j_{\sigma}), \quad i = 1, \ldots, n, \quad j = 1, \ldots, m, \quad |\sigma| \geq 0,
\]

from which it follows that any symmetry is of the form

\[
\mathcal{E}_\phi = \sum_{\sigma, j} D_\sigma (\phi^j) \frac{\partial}{\partial u_{\sigma}^j}
\]

on \( J^\infty(n, m) \). Here \( \phi = (\phi^1, \ldots, \phi^m) \) is an arbitrary smooth vector function on \( J^\infty(n, m) \). Vector fields of the form (9) are called evolutionary and \( \phi \) is called the generating function.
(or characteristic). Generating functions are often identified with the corresponding symmetries and form a Lie $\mathbb{R}$-algebra with respect to the Jacobi (or higher contact) bracket

$$\{\phi_1, \phi_2\} = \mathcal{E}_{\phi_1}(\phi_2) - \mathcal{E}_{\phi_2}(\phi_1).$$

(10)

Given an equation $\mathcal{E}$, its symmetry algebra $\text{sym } \mathcal{E}$ is formed by the generating functions $\phi_i$ satisfying the system of linear equations (defining equations for symmetries)

$$\ell_{\mathcal{E}}(\phi) = 0,$$

(11)

where the operator $\ell_{\mathcal{E}}$ is defined as follows. For any smooth vector function $F = (F^1, \ldots, F^r)$ on $J^\infty(n, m)$ set

$$\ell_F = \sum_{\sigma} \frac{\partial F^\sigma}{\partial u^\beta} D_{\sigma}.$$

This operator is called the linearization of $F$. It is a matrix differential operator in total derivatives acting on generating functions and producing vector functions of the same dimension as $F$. Then, if $\mathcal{E}$ is defined by (3) one has

$$\ell_{\mathcal{E}} = \ell_F|_E.$$

The restriction of $\ell_F$ to $\mathcal{E}$ is well defined since $\ell_F$ is an operator in total derivatives.

**Example 3** For the KdV equation (5), the defining equation for symmetries is

$$D_t(\phi) = u_1 \phi + u D_x(\phi) + D^3_x(\phi),$$

(12)

where $\phi = \phi(x, t, u, u_1, \ldots, u_k)$ and $D_x, D_t$ are given by (6).

### 4 Conservation Laws and Cosymmetries

Consider an equation $\mathcal{E} \subset J^\infty(n, m)$ that, for simplicity, we will assume to be topologically trivial. This means that we assume that (3) and its prolongations (4) be topologically trivial. This holds at least locally for most equations, and it is a reasonable hypothesis in computations. Define the spaces $\Lambda^q_{\mathcal{E}}$ of horizontal differential forms that are generated by the elements

$$\omega = a dx^i_1 \wedge \cdots \wedge dx^i_q, \quad a \in C^\infty(\mathcal{E}).$$

The horizontal de Rham differential $d_h : \Lambda^q_{\mathcal{E}} \rightarrow \Lambda^{q+1}_{\mathcal{E}}$ is defined by

$$d_h \omega = \sum_i D_i(a) dx^i_1 \wedge dx^i_2 \wedge \cdots \wedge dx^i_q,$$

and $d_h \circ d_h = 0$.

An $(n-1)$-form $\omega \in \Lambda^{n-1}_{\mathcal{E}}$ is a conservation law of $\mathcal{E}$ if $d_h \omega = 0$. A conservation law is trivial if it is of the form $d_h \theta$ for some $\theta \in \Lambda^{n-2}_{\mathcal{E}}$. Thus we define the group of equivalence classes of conservation laws to be the cohomology group $\text{CL}(\mathcal{E}) = H^{n-1}_{\mathcal{E}}(\mathcal{E}) = \text{ker } d_h/ \text{im } d_h$.
To facilitate computation of nontrivial conservation laws, their generating functions (or characteristics) are very useful [1, p. 29] (see also [24–26]). To present the definition, recall that for any \( \mathcal{C} \)-differential operator \( \Delta : V \to W \) its formal adjoint \( \Delta^* : \hat{W} \to \hat{V} \) is uniquely defined, where \( \hat{A} = \text{hom}(A, \Lambda^n_0) \). Namely, for a scalar operator \( \Delta = \sum_\sigma a_\sigma D_\sigma \) one has

\[
\Delta^* = \sum_\sigma (-1)^{|\sigma|} D_\sigma \circ a_\sigma
\]

and for a matrix one \( \Delta = \|\Delta_{ij}\| \) we set

\[
\Delta^* = \|\Delta^*_{ji}\|.
\]

Now, let \( V \) be the space to which \( F = (F^1, \ldots, F^r) \) belongs, \( F^j \) being the functions from equation (3). Consider a conservation law \( \omega \in \Lambda^{n-1}_h(\mathcal{C}) \) and extend it to a horizontal \( (n - 1) \)-form \( \bar{\omega} \) on \( J^\infty(n, m) \). Then, due to the regularity condition, since \( d_h(\bar{\omega}) \) must vanish on \( \mathcal{C} \), we have

\[
d_h(\bar{\omega}) = \Delta(F)
\]

for some \( \mathcal{C} \)-differential operator \( \Delta : V \to \Lambda^n_0(J^\infty) \). The characteristic \( \psi_\omega \) of \( \omega \) is defined by

\[
\psi_\omega = \Delta^*(1)_{|\mathcal{C}}.
\]

Then \( \omega \) is trivial if and only if \( \psi_\omega = 0 \).

Example 4 Let \( \mathcal{C} \) be a system of evolution equations

\[
u_j^i = f^j(\nu, t, \ldots, \nu_k^\alpha, \ldots), \quad j, \alpha = 1, \ldots, m,
\]

where subscript \( k \) indicates the number of derivatives over \( x \). Let \( \omega = X dx + T dt \) be a conservation law. Then its characteristic is

\[
\psi = \left( \delta X \over \delta u_1, \ldots, \delta X \over \delta u_m \right).
\]

where

\[
\frac{\delta}{\delta u^a} = \sum_\sigma (-1)^{|\sigma|} D_\sigma \circ \frac{\partial}{\partial u^a}
\]

are the variational derivatives.

Generating functions satisfy the equation

\[
\ell^*_\mathcal{C}(\psi) = 0.
\]

Arbitrary solutions of this equation are called cosymmetries of \( \mathcal{C} \). The space of cosymmetries is denoted by \( \text{cosym}(\mathcal{C}) \). Thus, construction (13) determines the inclusion

\[
\delta : \text{CL}(\mathcal{C}) \to \text{cosym}(\mathcal{C}).
\]
Example 5 For the KdV equation (5) its cosymmetries are the solutions of
\[ D_t(\psi) = u D_x(\psi) + D_x^3(\psi), \] (16)
as it follows from the expression of the adjoint operator of \( \ell_{\mathcal{E}} \) (see (12)).

5 Nonlocal Extensions

An invariant, geometric way to introduce nonlocal objects (e.g., variables, operators) to the
initial setting is based on the concept of differential coverings [1, Chap. 6], see also the
collection of papers [17]. To introduce the concept, consider two well known examples.

Example 6 Let
\[ u_t = uu_x + u_{xx} \]
be the Burgers equation. Then the Cole-Hopf substitution
\[ u = -\frac{2v_x}{v} \]
introduces a new variable \( v \) which is nonlocally expressed in terms of the old one and satisfies the heat equation
\[ v_t = v_{xx}. \]

Example 7 The Miura transformation
\[ w_x = u + \frac{1}{6} w^2 \]
introduces a nonlocal variable \( w \) and transforms the KdV equation (5) to
\[ w_t = w_{xxx} - \frac{1}{6} w^2 w_x, \]
i.e., to the modified KdV equation.

The geometrical construction that generalizes these and similar examples is as follows. Consider two equations \( \mathcal{E} \subset J^\infty(n, m) \) and \( \tilde{\mathcal{E}} \subset J^\infty(n, \tilde{m}) \) with the same collection of independent variables \( x^1, \ldots, x^n \). A smooth surjection \( \tau: \tilde{\mathcal{E}} \rightarrow \mathcal{E} \) is called a differential covering (or simply a covering) of \( \mathcal{E} \) by \( \tilde{\mathcal{E}} \) if its differential takes the Cartan distribution on \( \tilde{\mathcal{E}} \) to that on \( \mathcal{E} \). Coordinates in the fiber of \( \tau \) are called nonlocal variables.

In coordinates, this definition means that the total derivatives on \( \tilde{\mathcal{E}} \) are of the form
\[ \tilde{D}_i = D_i + X_i, \quad i = 1, \ldots, n, \]
where \( X_i \) are \( \tau \)-vertical vector fields that satisfy the relations
\[ D_i(X_j) - D_j(X_i) + [X_i, X_j] = 0, \quad 1 \leq i < j \leq n. \] (17)
Any \( \mathcal{E} \)-differential operator \( \Delta \) on \( \mathcal{E} \) can be lifted to a \( \mathcal{E} \)-differential operator \( \tilde{\Delta} \) on \( \tilde{\mathcal{E}} \) substituting all total derivatives \( D_i \) by \( \tilde{D}_i \).
Example 8 Consider the case of three independent variables \( x, y, z \). Let \( \omega = a \, dx \wedge dy + b \, dx \wedge dz \) be a conservation law of \( E \) which has only two nontrivial components. Set \( \tilde{E} = E \times \mathbb{R}^\infty \) and

\[
\begin{align*}
\tilde{D}_x &= Dx + w_x \frac{\partial}{\partial w} + w_{xx} \frac{\partial}{\partial w_x} + \cdots, \\
\tilde{D}_y &= Dy + a \frac{\partial}{\partial w} + D_x(a) \frac{\partial}{\partial w_x} + \cdots, \\
\tilde{D}_z &= Dz + b \frac{\partial}{\partial w} + D_x(b) \frac{\partial}{\partial w_x} + \cdots,
\end{align*}
\]

where \( w, w_x, w_{xx}, \ldots \) are coordinates in \( \mathbb{R}^\infty \). Then \( \tau : E \times \mathbb{R}^\infty \rightarrow E \) is a covering.

The same construction works for \( n \neq 3 \). If \( n = 2 \) we get a 1-dimensional covering. For \( n > 2 \) it is important to remember that the initial conservation law must have only two nonzero components.

An interesting and a well known class of coverings comes from the Wahlquist-Estabrook construction [28, 29].

Example 9 For the KdV equation, consider the coverings

\[
\begin{align*}
\tilde{D}_x &= Dx + X, \\
\tilde{D}_t &= Dt + T
\end{align*}
\]

such that the coefficients of the fields \( X \) and \( T \) depend on \( u_0, u_1, u_2 \) and nonlocal variables only. Then one can show that for any such a covering

\[
\begin{align*}
X &= u_0^2 A + u_0 B + C, \\
T &= \left( 2u_0 u_2 - u_1^2 + \frac{2}{3} u_0^3 \right) A + \left( u_2 + \frac{1}{2} u_0^2 \right) B + u_1 [B, C] \\
&\quad + \frac{1}{2} [B, [C, B]] + u_0 [C, [C, B]] + D,
\end{align*}
\]

where the fields \( A, B, C, \) and \( D \) are independent of \( u_i \) and fulfill the commutator relations

\[
\begin{align*}
[A, B] &= [A, C] = [C, D] = 0, \\
[B, [B, [B, C]]] &= 0, \\
[B, D] + [C, [C, [C, B]]] &= 0, \\
[A, D] + \frac{1}{2} [C, B] + \frac{3}{2} [B, [C, [C, B]]] &= 0.
\end{align*}
\]

Denote by \( g \) the free Lie algebra with four generators and defining relations (18). Then any covering satisfying the above formulated ansatz is defined by a representation of \( g \) in the Lie algebra of vector fields on the covering.

The next example is crucial for the subsequent constructions.
Example 10 The class of coverings that will be introduced in this example delivers the solution to the following factorization problem. Assume that two $C$-differential operators

$$ \Delta : V \to W, \quad \Delta' : V' \to W' $$

are given, where $V$, $W$, $V'$, $W'$ are spaces of smooth vector functions on $E$ of dimensions $l$, $s$, $l'$, and $s'$, respectively. The problem is: find all operators $A : V \to V'$ such that there exists a commutative diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\Delta} & W \\
A \downarrow & & \downarrow B \\
V' & \xrightarrow{\Delta'} & W'
\end{array}
$$

with some operator $B : W \to W'$. Since the problem has obvious trivial solutions of the form $A = A' \circ \Delta$, we are interested in the space

$$\mathcal{A}(\Delta, \Delta') = \left\{ A \mid \Delta' \circ A = B \circ \Delta \right\}$$

consisting of nontrivial solutions. Obviously, elements of $\mathcal{A}(\Delta, \Delta')$ define maps from $\ker \Delta$ to $\ker \Delta'$. Assume that $E \subset J^{\infty}(n, m)$ with the coordinates $x^i, u^\sigma_{\alpha}$. Consider the space $J^{\infty}(n, m + l)$ with additional coordinates $v^\beta_{\sigma}$.

There is a natural surjection $\xi : J^{\infty}(n, m + l) \to J^{\infty}(n, m)$, and let $E_{\Delta}$ be defined by

$$\Delta(v) = 0$$

in $\xi^{-1}(E)$. Then $\xi : E_{\Delta} \to E$ is evidently a covering. We call it the $\Delta$-covering.

To every operator

$$A = \left\| \sum_{\sigma} a^\sigma_{\alpha} D_{\sigma} \right\|,$$

there corresponds a vector function

$$\Phi_A = \left( \sum_{\sigma, \alpha} a^l_{\alpha} v^\sigma_{\alpha}, \ldots, \sum_{\sigma, \alpha} a^{l'}_{\alpha} v^\sigma_{\alpha} \right)$$

on $E_{\Delta}$. The function $\Phi_A$ vanishes if and only if $A = A' \circ \Delta$ for some operator $A'$, so that the function $\Phi_A$ corresponds to the equivalence class of $A$.

As was indicated above, the operator $\Delta'$ can be lifted to $\tilde{\Delta}'$ in the covering under consideration. Consider a function $\Phi_A$ that satisfies the equation

$$\tilde{\Delta}'(\Phi_A) = 0.$$ 

It can be shown [10] that such functions are in one-to-one correspondence with the elements of $\mathcal{A}(\Delta, \Delta')$ given by (19).
6 Tangent and Cotangent Coverings

These two coverings serve, in the geometry of differential equations, as counterparts to tangent and cotangent bundles in differential geometry of finite-dimensional manifolds [16]. Technically, both can be obtained as particular cases of Example 10 above.

Consider an equation $\mathcal{E} \subset J^\infty(n, m)$ and take the operator $\ell_\mathcal{E}$ for $\Delta$ in the construction of the $\Delta$-covering. In more detail, this means that we consider a new dependent variable $q = (q^1, \ldots, q^m)$ and extend $\mathcal{E}$ by the equation

$$\ell_\mathcal{E}(q) = 0.$$  

The resulting equation is denoted by $\mathcal{T}(\mathcal{E}) \subset J^\infty(n, 2m)$ and the projection $\tau: \mathcal{T}(\mathcal{E}) \to \mathcal{E}$ is called the tangent covering to $\mathcal{E}$. A important property of $\tau$ is that its holonomic sections, i.e., sections $\phi: \mathcal{E} \to \mathcal{T}(\mathcal{E})$ that preserve the Cartan distributions, are symmetries of $\mathcal{E}$.

Example 11 The tangent covering for the KdV equation is the system

$$
\begin{align*}
    u_t &= uu_x + u_{xxx}, \\
    q_t &= u_x q + u_{qq} + q_{xxx}
\end{align*}
$$

that projects to $\mathcal{E}$ by $(x, t, \ldots, u_k, \ldots, q_i, \ldots) \mapsto (x, t, \ldots, u_k, \ldots)$.

Dually, consider the jet space $J^\infty(n, m + r)$, where $r$ is from (3), with the new dependent variable $p = (p^1, \ldots, p^r)$ and extend $\mathcal{E}$ by

$$\ell^*_\mathcal{E}(p) = 0.$$  

The equation $\mathcal{T}^*(\mathcal{E}) \subset J^\infty(n, m + r)$ together with the natural projection $\tau^*: \mathcal{T}^*(\mathcal{E}) \to \mathcal{E}$ is the cotangent covering to $\mathcal{E}$. Its holonomic sections are cosymmetries of $\mathcal{E}$. Note also that $\mathcal{T}^*(\mathcal{E})$ is always an Euler-Lagrange equation with the Lagrangian $L = F^1 p^1 + \cdots + F^r p^r$.

Example 12 If $\mathcal{E}$ is the KdV equation then $\mathcal{T}^*(\mathcal{E})$ is of the form

$$
\begin{align*}
    u_t &= uu_x + u_{xxx}, \\
    p_t &= up_x + p_{xxx}
\end{align*}
$$

cf. (16).

There exist two canonical inclusions,

$$\nu: \text{sym}(\mathcal{E}) \to \text{CL}(\mathcal{T}^*\mathcal{E})$$

and

$$\nu^*: \text{cosym}(\mathcal{E}) \to \text{CL}(\mathcal{T}\mathcal{E})$$

(defined below), that play a very important role in practical computations described in Sects. 7–10 below.
Moreover, one can prove (see [16]) that the image of $\nu$ consists of the elements that are fiber-wise linear with respect to the projection $\tau: T^* E \to E$. Dually, the image of $\nu^*$ coincides with conservation laws that are linear along the fibers of the tangent covering.\footnote{To be more precise, the images of $\nu$ and $\nu^*$ consist of equivalence classes that contain at least one fiber-wise linear element.}

In what follows, we assume that $r$ (the number of equations) equals $m$ (the number of unknown functions).

Consider a symmetry $\phi$ of the equation $\mathcal{E}$ and extend it to some $\tilde{\phi}$ defined on the ambient space $J^\infty(n, m + r)$. Then due to the Green formula [1, p. 191] (see [24, p. 41] for more details) one has

$$\langle \ell_F (\tilde{\phi}), p \rangle - \langle \tilde{\phi}, \ell_F^*(p) \rangle = d_h \omega_{\tilde{\phi}},$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between the spaces $A$ and $\hat{A}$, while the correspondence $\phi \mapsto \omega_{\tilde{\phi}}$ is a $\mathcal{E}$-differential operator. We set

$$\nu(\phi) = \omega_{\tilde{\phi}} | T^* E,$$

and this is a well defined operation. Note that in the right-hand side of the above formula and in similar situations, to simplify notation, when writing a differential form we actually mean its de Rham cohomology class.

**Example 13** Let $\phi$ be a symmetry of the KdV equation. Then, due to (12) and (21), one has

$$\left( D_t(\phi) - D_x^3(\phi) - u D_x(\phi) - u_x \phi \right) p - \phi (u p_x + p_{xxx} - p_t)$$

$$= D_t(\phi p) - D_x (u p \phi - 3 p_x D_x(\phi) + D_x^2(p \phi)),$$

i.e.,

$$\nu(\phi) = \phi p dx + \left( u p \phi - 3 p_x D_x(\phi) + D_x^2(p \phi) \right) dt.$$  \hspace{1cm} (22)

In a dual way, if $\psi$ is a cosymmetry of $\mathcal{E}$ then extend it to some $\tilde{\psi}$ on $J^\infty(n, 2m)$, use the Green formula

$$\langle \ell_F^*(\tilde{\psi}), q \rangle - \langle \tilde{\psi}, \ell_F(q) \rangle = d_h \omega_{\tilde{\psi}},$$

and set

$$\nu^*(\psi) = \omega_{\tilde{\psi}} | T^* \mathcal{E},$$

with similar considerations as $\nu$.

**Example 14** For a cosymmetry $\psi$ of the KdV equation, using (16) and (20), we arrive at the correspondence

$$\nu^*(\psi) = \psi q dx + \left( u q \psi - 3 q_x D_x(\psi) + D_x^2(q \psi) \right) dt,$$

which in this particular case coincides with (22).
Example 15 Consider the Plebanski (or second heavenly) equation
\[ u_{xz} + u_{ty} + u_{tt}u_{xx} - u_{tx}^2 = 0. \] (23)

Its linearization is self-adjoint (so that it is Lagrangian), hence the tangent covering coincides with the cotangent one; they are defined through the equation
\[ p_{xz} + p_{ty} + u_{tt}p_{xx} + u_{xx}p_{tt} - 2u_{tx}p_{tx} = 0. \]

The \( u \)-translation symmetry \( \partial/\partial u \) has the generating function \( \phi = 1 \), so we have
\[ p_{xz} + p_{ty} + u_{tt}p_{xx} + u_{xx}p_{tt} - 2u_{tx}p_{tx} = D_x(p_z + u_{tt}p_x - u_{tx}p_t) + D_t(p_y + u_{xx}p_t - u_{tx}p_x) = 0, \]
thus the corresponding conservation law is
\[ \nu (1) = (p_y + u_{xx}p_t - u_{tx}p_x) dx \wedge dy \wedge dz + (u_{tx}p_t - p_z - u_{tt}p_x) dt \wedge dy \wedge dz. \] (24)

7 Recursion Operators for Symmetries

Consider an equation \( \mathcal{E} \) and its tangent covering \( \tau : \mathcal{T}\mathcal{E} \rightarrow \mathcal{E} \). Let \( \tilde{\ell}_{\mathcal{E}} \) be the lifting of the linearization operator to this covering and \( \Phi = (\Phi^1, \ldots, \Phi^m) \), where
\[ \Phi^\alpha = \sum_{\alpha' \sigma} a_{\alpha' \sigma} q_{\sigma}, \]
is a solution of the equation
\[ \tilde{\ell}_{\mathcal{E}}(\Phi) = 0. \]

Then, by Example 10, the operator
\[ \mathcal{R} = \left\| \sum_{\sigma} a_{\alpha' \sigma} D_\sigma \right\| \]
takes elements of \( \text{ker} \ell_{\mathcal{E}} \) to themselves. In other words, these operators act on symmetries of \( \mathcal{E} \) and thus are said to be recursion operators.

Example 16 Let
\[ u_t = u_{xx} \]
be the heat equation. Then the space of the tangent covering is given by
\[ u_t = u_{xx}, \quad q_t = q_{xx} \]
and the defining equation for recursion operators is
\[ \tilde{D}_t(\Phi) = \tilde{D}_x^2(\Phi), \] (25)
where
\[ \tilde{D}_x = D_x + \sum_{i \geq 0} q_{i+1} \frac{\partial}{\partial q_i}, \quad \tilde{D}_t = D_t + \sum_{i \geq 0} q_{i+2} \frac{\partial}{\partial q_i}, \]
(recall that \( q_i = q_x \ldots x, i \) times \( x \)) and
\[ \Phi = a_0 q + a_1 q_1 + \cdots + a_k q_k, \quad a_i \in C^\infty(\mathcal{E}). \]
The first three nontrivial solutions of (25) are
\[ \Phi_{00} = q, \quad \Phi_{10} = q_1, \quad \Phi_{11} = tq_1 + \frac{x}{2}, \]
to which there correspond the recursion operators
\[ \mathcal{R}_{00} = \text{id}, \quad \mathcal{R}_{10} = D_x, \quad \mathcal{R}_{11} = tD_x + \frac{x}{2}. \]
It can be shown that the entire algebra of recursion operators for the heat equation is generated by \( \mathcal{R}_{10} \) and \( \mathcal{R}_{11} \) with the only relation
\[ [\mathcal{R}_{10}, \mathcal{R}_{11}] = \frac{1}{2}. \]
This algebra is isomorphic to the universal enveloping algebra for the 3-dimensional Heisenberg algebra.

Example 17 Taking the tangent covering (20) to the KdV equation and solving \( \tilde{\ell}_\mathcal{E}(\Phi) = 0 \) in this covering leads to the only solution \( \Phi = q \), i.e., it yields the identity operator. This is not a surprise, because it is well known that the KdV equation does not admit local recursion operators.

Let us now take a cosymmetry \( \psi = 1 \) of the KdV equation and consider the corresponding conservation law on the tangent covering
\[ \nu^*(1) = q \, dx + (uq + q_2) \, dt. \]
Then the corresponding nonlocal variable (see Example 8) \( Q^1 \) satisfies the equations
\[ Q^1_x = q, \quad Q^1_t = qu + q_{xx}. \]
In the extended setting the linearized equation \( \tilde{\ell}_\mathcal{E}(\Phi) = 0 \), where
\[ \Phi = A_1 Q^1 + a_0 q + a_1 q_1 + \cdots + a_k q_k, \]
acquires a new nontrivial solution
\[ \Phi_1 = q_2 + \frac{2}{3} uq + \frac{1}{3} u_1 Q^1 \]
to which the Lenard recursion operator
$R = D_x^2 + \frac{2}{3}u + \frac{1}{3}u_1 D_x^{-1}$ (26)

corresponds.

If we consider the next cosymmetry $\psi = u$ then the corresponding nonlocal variable will be defined by

$Q^3 = qu$, $Q^4 = qu^2 + q_{xx}u - q_xu_x + qu_{xx},$

which will lead to another recursion operator

$$
D_x^4 + \frac{4}{3}u D_x^2 + 2u_1 D_x + \frac{4}{9} (u^2 + 3u_2) + \frac{1}{3} (uu_1 + u_3) D_x^{-1} + \frac{1}{9}u_1 D_x^{-1} \circ u,
$$

the square of (26).

Given an equation $E$ and two recursion operators $R_1, R_2$: $\text{sym } E \rightarrow \text{sym } E$, the Nijenhuis bracket

$$[[R_1, R_2]] : \text{sym } E \times \text{sym } E \rightarrow \text{sym } E$$

is defined by

$$[[R_1, R_2]](\phi_1, \phi_2) = \{R_1(\phi_1), R_2(\phi_2)\} + \{R_2(\phi_1), R_1(\phi_2)\}$$

$$- R_1([\{R_2(\phi_1), \phi_2\} + \{\phi_1, R_2(\phi_2)\}]) - R_2([\{R_1(\phi_1), \phi_2\} + \{\phi_1, R_1(\phi_2)\}])$$

$+ (R_1 \circ R_2 + R_2 \circ R_1)(\phi_1, \phi_2),$

where $\{\cdot, \cdot\}$ is the Jacobi bracket given by (10). An operator $R$ is called hereditary if $[[R, R]] = 0$. This property was introduced in [3–6]. It can be shown that if this property holds and $R$ is invariant with respect to a symmetry $\phi$ then the symmetries $\phi_i = R^i(\phi)$ form a commuting hierarchy (see, for example, [9, Chap. 4, Sect. 3.1]).

8 Symplectic Structures

Let us now solve the equation

$$\tilde{\ell}^*_\mathcal{E} (\Psi) = 0$$

on the tangent covering for $\Psi$ of the form

$$\Psi = (\Psi^1, \ldots, \Psi^r), \quad \Psi^\alpha = \sum_{\alpha'} a_{\alpha \alpha'} \Psi^{\alpha'}.$$

The corresponding $\mathcal{E}$-differential operator

$$J = \left\{ \sum_{\sigma} a_{\alpha \sigma} D_\alpha \right\},$$

due to the results of Example 10, takes symmetries of $\mathcal{E}$ to its cosymmetries.

Let us call a conservation law $\omega$ admissible (with respect to $J$) if

$$\delta(\omega) \in \text{im} J,$$
where the operator $\delta: \text{CL}(\mathcal{E}) \to \text{cosym}(\mathcal{E})$ is from (15). Then $\mathcal{S}$ determines a bracket on the set of admissible conservation laws by

$$\{\omega, \omega'\}_{\mathcal{S}} = L_{\partial_{\phi}}(c'),$$

where $L_{\partial_{\phi}}(c') = \ell_{c'}(\phi)$, $\phi$ is such that $\delta(\omega) = \mathcal{S}(\phi)$ and $c'$ is a representative of the cohomology class $\omega'$. This bracket is well defined on equivalence classes of nontrivial admissible conservation laws. It is skew-symmetric if

$$\mathcal{S}^* \circ \ell_{\mathcal{S}} = \ell_{\mathcal{S}}^* \circ \mathcal{S},$$

i.e., if $\ell_{\mathcal{S}}^* \circ \mathcal{S}$ is a self-adjoint operator. In the case of evolution equations, (28) means the $\mathcal{S}$ is a skew-adjoint operator.

Let now an operator $\mathcal{S}$ satisfy (28). Let us deduce the conditions under which the bracket (27) fulfills the Jacobi identity. To this end, consider a vector function $\phi = (\phi^1, \ldots, \phi^m)$ on $J^\infty(n, m)$. Then, since $\mathcal{E}$ satisfies the regularity condition from Sect. 2, (28) implies

$$\ell_{\mathcal{S}}^* \mathcal{S} \ell_F(\phi) - \mathcal{S}^* \ell_F(\phi) = \bar{\Delta}_\phi(F),$$

where $\bar{\Delta}_\phi$ is a $C^\infty$-differential operator. Put $\Delta_\phi = \bar{\Delta}_\phi|_{\mathcal{E}}$ and define the map

$$\delta \mathcal{S}: \text{sym} \mathcal{E} \times \text{sym} \mathcal{E} \to \text{cosym} \mathcal{E}$$

by

$$(\delta \mathcal{S})(\phi_1, \phi_2) = (\Theta_{\phi_1} \mathcal{S})(\phi_2) - (\Theta_{\phi_2} \mathcal{S})(\phi_1) + \Delta_{\phi_2}^*(\phi_1).$$

Then the equality

$$\delta \mathcal{S} = 0$$

guarantees the Jacobi identity for the bracket $\{\cdot, \cdot\}_{\mathcal{S}}$. Operators satisfying (28) and (29) are called symplectic structures.

**Example 18** The KdV equation does not possess local symplectic structures, but if we extend the space $\mathfrak{T} \mathcal{E}$ with nonlocal variables $Q^1$ and $Q^3$ (see Sect. 7) and solve the equation

$$\tilde{\ell}_{\mathcal{E}}(\Psi) = 0$$

for $\Psi$ of the form

$$\Psi = A^3 Q^3 + A^1 Q^1 + a^0 q + a^1 q_1 + \cdots + a^k q_k,$$

there will be two nontrivial solutions

$$\Psi_1 = Q_1, \quad \Psi_3 = q_1 + \frac{1}{3} u Q_1 + \frac{1}{3} Q_3$$

with the corresponding nonlocal symplectic operators

$$\mathcal{S}_1 = D_x^{-1}$$

and

$$\mathcal{S}_3 = D_x + \frac{1}{3} u D_x^{-1} + \frac{1}{3} D_x^{-1} \circ u.$$
9 Hamiltonian Structures

Consider now the cotangent covering to \( E \) and the lifting \( \tilde{\ell}_E \) of the linearization operator to this covering. If \( \Phi = (\Phi^1, \ldots, \Phi^m) \) is of the form

\[
\Phi^\alpha = \sum_{\alpha'} a_{\alpha' \sigma} q_{\alpha'}^\sigma
\]

and satisfies

\[
\tilde{\ell}_E (\Phi) = 0
\]

then, due to Example 10, the operator

\[
\mathcal{H} = \left\| \sum_{\sigma} a_{\alpha' \sigma} D_\sigma \right\|
\]

takes cosymmetries of the equation \( E \) to its symmetries.

For any such an operator, consider the bracket

\[
\{ \omega, \omega' \} \mathcal{H} = L_H (\delta \omega) (\omega')
\]

(30)
defined on the space \( \text{CL}(E) \) of conservation laws. Here \( \delta : \text{CL}(E) \to \text{cosym}(E) \) is the map (15). This bracket is skew-symmetric if

\[
(\ell_E \circ \mathcal{H})^* = \ell_E \circ \mathcal{H}.
\]

(31)

For evolution equations one usually requires an additional condition of existence of a conservation law \( \chi \) such that \( u_t = H (\delta \chi) \). For such equations (31) implies \( \mathcal{H}^* = -\mathcal{H} \).

For any two \( C \)-differential operators \( \mathcal{H}_1, \mathcal{H}_2 : \text{cosym}(E) \to \text{sym}(E) \) satisfying (31) define their Schouten bracket [16, p. 13] (see [24, p. 226] for a more general definition)

\[
[[\mathcal{H}_1, \mathcal{H}_2]] : \text{cosym}(E) \times \text{cosym}(E) \to \text{sym}(E)
\]

by

\[
[[\mathcal{H}_1, \mathcal{H}_2]](\psi_1, \psi_2) = \mathcal{H}_1 \left( L_{\mathcal{H}_2 \psi_1} (\psi_2) \right) - \mathcal{H}_2 \left( L_{\mathcal{H}_1 \psi_1} (\psi_2) \right) + \left\{ \mathcal{H}_1(\psi_2), \mathcal{H}_2(\psi_1) \right\} - \left\{ \mathcal{H}_1(\psi_1), \mathcal{H}_2(\psi_2) \right\},
\]

(32)

where \{\cdot, \cdot\} is the Jacobi bracket [1, p. 48] and

\[
L_\phi (\psi) = \mathcal{E}_\phi (\psi) + \ell^*_\phi (\psi)
\]

for any \( \phi \in \text{sym}(E) \) and \( \psi \in \text{cosym}(E) \).

An operator \( \mathcal{H} \) satisfying (31) and such that

\[
[[\mathcal{H}, \mathcal{H}]] = 0
\]

is called a Hamiltonian structure on \( E \). For such a structure, the bracket (30) satisfies the Jacobi identity. Two Hamiltonian structures \( \mathcal{H}_1, \mathcal{H}_2 \), are compatible if

\[
[[\mathcal{H}_1, \mathcal{H}_2]] = 0.
\]
Equations admitting compatible structures are called bi-Hamiltonian. One can apply the Magri scheme [19] to generate infinite series of commuting symmetries and conservation laws.

**Example 19** In the case of the KdV equation, we solve the equation

\[ \tilde{D}(\Phi) = u_1 \Phi + u \tilde{D}_x(\Phi) + \tilde{D}_x^3(\Phi), \quad \Phi = a^0 p + a^1 p_1 + \cdots + a^k p_k, \]

in the cotangent covering (Example 12) and obtain two nontrivial solutions

\[ \Phi_1 = p_1, \quad \Phi_3 = p_3 + \frac{2}{3} u p_1 + \frac{1}{3} u_1 p_0 \]

with the corresponding (and well known!) Hamiltonian operators

\[ \mathcal{H}_1 = D_x, \quad \mathcal{H}_3 = D_x^3 + \frac{2}{3} u D_x + \frac{1}{3} u_1. \]

Consider the \( x \)-translation \( u_x \) which is a symmetry of the KdV equation and the corresponding nonlocal variable \( P^1 \) in the cotangent covering. We have

\[ P^1_x = p u_x, \quad P^1_t = p(uu_1 + u_3) + 2u_1 - p_1 u_2. \]

One obtains a new solution

\[ \Phi_5 = p_5 + \frac{4}{3} u p_3 + 2u_1 p_2 + \frac{4}{9} (u^2 + 3u_2) p_1 + \left( \frac{4}{9} uu_1 + \frac{1}{3} u_3 \right) p - \frac{1}{9} u_1 P_1 \]

in the extended setting to which the nonlocal Hamiltonian structure

\[ \mathcal{H}_5 = D_x^5 + \frac{4}{3} u D_x^3 + 2u_1 D_x^2 + \frac{4}{9} (u^2 + 3u_2) D_x + \left( \frac{4}{9} uu_1 + \frac{1}{3} u_3 \right) - \frac{1}{9} u_1 D_x^{-1} \circ u_1 \]

corresponds.

**Example 20** Consider the KP equation (Example 2). The cotangent covering is defined through the further equation

\[ \tilde{\ell}_\varepsilon(\Phi) = \tilde{D}_{yy}(\Phi) - \tilde{D}_{tx}(\Phi) + 2u_x \tilde{D}_x(\Phi) + u_{xx} \Phi + u \tilde{D}_x(\Phi) + \frac{1}{12} \tilde{D}_{xxxx}(\Phi) = 0, \]

with \( \Phi = \sum_\sigma a^\sigma p_\sigma \) (here the sum extends to a finite number of multi-indexes of parametric coordinates \( \sigma \in P \), and \( a^\sigma \) are smooth functions on \( \varepsilon \)). The nontrivial solution

\[ \Phi = p_{xx} \]

can be found. A simple computation shows that the corresponding operator \( \mathcal{H} = D_x^2 \) satisfies (31) and fulfills the condition \( \mathcal{H} = D_x^2 \) satisfies (31) and fulfills the condition \( [[\mathcal{H}, \mathcal{H}]] = 0 \), hence it is Hamiltonian.
In [18] the following evolution form of the KP equation is introduced:

\[ \psi_x = u_y, \quad u_x = v, \quad v_x = w, \quad w_x = 12(u_t - uv - \psi_y). \] (33)

Then, one local Hamiltonian operator (Eq. (51) in [18]) is obtained. After applying the change of coordinate formulae for Hamiltonian operators (see [13]) it is readily proved that it corresponds to the above \( \Phi \).

10 Recursion Operators for Cosymmetries

Finally, let us consider the equation \( \ell^*_E(\psi) = 0 \) lifted to the cotangent covering. An operator \( \psi \) that fulfills the equation takes an element \( \psi \in \text{cosym}(\mathcal{E}) \) into an element \( \Psi(\psi) \in \text{cosym}(\mathcal{E}) \), i.e., it is a recursion operators for cosymmetries.

**Example 21** If \( \mathcal{E} \) is the KdV equation then the equation

\[ \tilde{D}_t(\psi) = u \tilde{D}_x(\psi) + \tilde{D}_3(\psi) \]

possesses no nontrivial solution. But when we extend \( \mathcal{E}^* \) by the nonlocal variable \( P^1 \) (see the previous section) a solution

\[ \psi = p_2 + \frac{2}{3} u p - \frac{1}{3} P^1 \]

arises to which the recursion operator

\[ \tilde{R} = D^2_x + \frac{2}{3} u - D^{-1}_x \circ u_1 \]

corresponds.

**Example 22** Consider the Plebanski equation (Example 15). The linearization of this equation is self-adjoint, hence the operators that we defined in this paper (recursion for symmetries, symplectic, Hamiltonian, recursion for cosymmetries) act in the same spaces. Now, extend the (co)tangent covering with the nonlocal variable \( R \) defined by

\[ R_t = u_{tx} p_t - p_x - u_{tx} p_x, \quad R_x = p_y + u_{xx} p_t - u_{tx} p_x. \]

The extension of the equation \( \tilde{\ell}_E(\Phi) = 0 \) admits the solution linear with respect to \( p_\sigma \) and \( R_\sigma \)

\[ \Phi_2 = R, \] (34)

besides the trivial solution \( \Phi_1 = p \).

In [22] the following evolution form of Plebanski equation is introduced:

\[ u_t = q, \quad q_t = \frac{1}{u_{xx}} (q_x^2 - q_y - u_{xz}). \] (35)

Then, a local Hamiltonian operator (Eq. (11) in [22]) and a nonlocal Hamiltonian operator (Eq. (27) in [22]) are obtained. After applying the change of coordinate formulae for Hamiltonian operators (see [13]) it is readily proved that they correspond to \( \Phi_1 \) and \( \Phi_2 \). Different
changes of coordinates from (23) to (35) transform $\Phi_1$ and $\Phi_2$ to recursion operators for symmetries or cosymmetries and symplectic operators.

It is interesting to remark that, while in the evolutionary form all such operators are mutually different because they act in different spaces, in the original formulation of the equation they coincide with $\Phi_1$ and $\Phi_2$, whose expressions are considerably simpler than that of their ‘evolutionary’ counterparts.

11 Computer Support

The computations in this paper were done by the help of CDIFF, a symbolic computation package devoted to computations in the geometry of DEs and developed by P.K.H. Gragert, P.H.M. Kersten, G.F. Post and G.H.M. Roelofs at the University of Twente, The Netherlands, with latest additions by one of us (R.V.). CDIFF runs in the computer algebra system REDUCE, which recently has become free software and can be downloaded here [23].

The ‘Twente’ part of CDIFF package is included in the official REDUCE distribution, and most of it is documented. Additional software, especially geared towards computations for differential equations with more than two independent variables, can be downloaded at the Geometry of Differential Equations website http://gdeq.org. The software includes a user guide [27] and many example programs which cover most examples presented in this paper.

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