ON *k*-JET AMPLENESS OF LINE BUNDLES ON HYPERELLIPTIC SURFACES

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Abstract. We study *k*-jet ampleness of line bundles on hyperelliptic surfaces using vanishing theorems. Our main result states that on a hyperelliptic surface of an arbitrary type a line bundle of type \((m, m)\) with \(m \geq k + 2\) is *k*-jet ample.

1. Introduction

The concepts of higher order embeddings: *k*-spandness, *k*-very ampleness and *k*-jet ampleness were introduced and studied in a series of papers by M.C. Beltrametti, P. Francia and A.J. Sommese, see [BeFS1989], [BeS1988], [BeS1993]. The last notion is of our main interest in the present work.

The problem of *k*-jet ampleness has been studied on certain types of algebraic surfaces. In [BaSz2-1997] Th. Bauer and T. Szemberg characterise *k*-jet ample line bundles on abelian surfaces with Picard number 1. For an ample line bundle \(L\) on a K3 surface Th. Bauer, S. Di Rocco and T. Szemberg in [BaDRSz2000], and S. Rams and T. Szemberg in [RSz2004] explore for which \(n\) the line bundle \(nL\) is *k*-jet ample.

There are also several papers concerning *k*-jet ampleness in higher dimensions, e.g. [BaSz1997] study *k*-jet ampleness on abelian varieties, [DR1999] on toric varieties, [BeSz2000] on Calabi-Yau threefolds, [ChI2014] on hyperelliptic varieties.

In the present paper we prove that on a hyperelliptic surface of an arbitrary type a line bundle of type \((m, m)\) with \(m \geq k + 2\) is *k*-jet ample. Note that a line bundle of type \((k+2, k+2)\) is numerically equivalent to \((k+2)L_1\) where \(L_1 = (1, 1)\). By theory of hyperelliptic surfaces we know that \(L_1\) is ample, so our result is consistent with results obtained on other algebraic surfaces with Kodaira dimension 0. Our approach uses vanishing theorems of the higher order cohomology groups — Kawamata-Viehweg Theorem and Norimatsu Lemma.

Proof of the fact that a line bundle of type \((k+2, k+2)\) is *k*-jet ample on any hyperelliptic surface \(S\) can be found also in [Ch2014]. The authors use the fact that \(S\) is covered by an abelian surface divided by the group action, and the results of [PP2004]. We provide a self-contained and more elementary proof of this fact.

2. Preliminaries

Let us set up the notation and basic definitions. We work over the field of complex numbers \(\mathbb{C}\). We consider only smooth reduced and irreducible projective varieties. By \(D_1 \equiv D_2\) we denote the numerical equivalence of divisors \(D_1\) and \(D_2\). By a curve we understand an irreducible subvariety of dimension 1. In the notation we follow [PAG2004].

Date: July 21, 2015.

2010 Mathematics Subject Classification. 14C20, 14F17, 14E25.

Key words and phrases. *k*-jet ampleness, generation of *k*-jets, hyperelliptic surfaces, vanishing theorems.
Let $X$ be a smooth projective variety of dimension $n$. Let $L$ be a line bundle on $X$, and let $x \in X$.

**Definition 2.1.**

(1) We say that $L$ generates $k$-jets at $x$, if the restriction map

$$H^0(X, L) \longrightarrow H^0(X, L \otimes O_X/m^k_{x})$$

is surjective.

(2) We say that $L$ is $k$-jet ample, if for every points $x_1, \ldots, x_r$ the restriction map

$$H^0(X, L) \longrightarrow H^0(X, L \otimes O_{X}/(m^k_{x_1} \otimes \ldots \otimes m^k_{x_r}))$$

is surjective, where $\sum_{i=1}^{r} k_i = k + 1$.

Note that 0-jet ampleness is equivalent to being spanned by the global sections, and 1-jet ampleness is equivalent to very ampleness.

The notion of $k$-jet ampleness generalises the notion of very ampleness and $k$-very ampleness (see [BeS1993], Proposition 2.2). We recall the definition of $k$-very ampleness as we mention this notion in the proof of the main theorem:

**Definition 2.2.** We say that a line bundle $L$ is $k$-very ample if for every 0-dimensional subscheme $Z \subset X$ of length $k + 1$ the restriction map

$$H^0(X, L) \longrightarrow H^0(X, L \otimes O_{Z})$$

is surjective.

In other words $k$-very ampleness means that the subschemes of length at most $k + 1$ impose independent conditions on global sections of $L$.

We also recall the definition of the Seshadri constant:

**Definition 2.3.** The Seshadri constant of $L$ at a given point $x \in X$ is the real number

$$\varepsilon(L, x) = \inf \left\{ \frac{LC}{\text{mult}_x C} : C \supset x \right\},$$

where the infimum is taken over all irreducible curves $C \subset X$ passing through $x$.

If $\pi: \tilde{X} \longrightarrow X$ is the blow-up of $X$ at $x$, and $E$ is an exceptional divisor of the blow-up, then equivalently the Seshadri constant may be defined as (see e.g. [PAG2004] vol. I, Proposition 5.1.5):

$$\varepsilon(L, x) = \sup \{ \varepsilon : \pi^*L - \varepsilon E \text{ is nef} \}.$$ 

We will use two vanishing theorems for the higher order cohomology groups — Kawamata-Viehweg Vanishing Theorem and Norimatsu Lemma.

**Theorem 2.4** (Kawamata-Viehweg Vanishing Theorem; [Laz1997], Vanishing Theorem 5.2). Let $D$ be an nef and big divisor on $X$. Then

$$H^i(X, K_X + D) = 0 \quad \text{for } i > 0.$$ 

**Definition 2.5** ([PAG2004], vol. II, Definition 9.1.7). We say that $D = \sum D_i$ is a simple normal crossing divisor (or an SNC divisor for short) if $D_i$ is smooth for each $i$, and $D$ is
defined in a neighbourhood of any point in local coordinates \((z_1, \ldots, z_n)\) as \(z_1 \cdot \ldots \cdot z_k = 0\) for some \(k \leq n\).

**Theorem 2.6** (Norimatsu Lemma; [PAG2004], vol. I, Vanishing Theorem 4.3.5). Let \(D\) be an ample divisor on \(X\) and let \(F\) be an SNC divisor on \(X\). Then

\[
H^i(X, K_X + D + F) = 0 \quad \text{for} \ i > 0.
\]

3. Hyperelliptic surfaces

First let us recall the definition of a hyperelliptic surface.

**Definition 3.1.** A hyperelliptic surface \(S\) (sometimes called bielliptic) is a surface with Kodaira dimension equal to 0 and irregularity \(q(S) = 1\).

Alternatively (see [Bea1996], Definition VI.19), a surface \(S\) is hyperelliptic if \(S \cong (A \times B)/G\), where \(A\) and \(B\) are elliptic curves, and \(G\) is an abelian group acting on \(A\) by translation and acting on \(B\), such that \(A/G\) is an elliptic curve and \(B/G \cong \mathbb{P}^1\). \(G\) acts on \(A \times B\) coordinatewise. Hence we have the following situation:

\[
\begin{align*}
S & \cong (A \times B)/G & \Phi & \rightarrow & A/G \\
\downarrow & & \Psi & &  \\
B/G & \cong \mathbb{P}^1 & & &
\end{align*}
\]

where \(\Phi\) and \(\Psi\) are the natural projections.

Hyperelliptic surfaces were classified at the beginning of 20th century by G. Bagnera and M. de Franchis in [BF1907], and independently by F. Enriques i F. Severi in [ES1909-10]. They showed that there are seven non-isomorphic types of hyperelliptic surfaces, characterised by the action of \(G\) on \(B \cong \mathbb{C}/(\mathbb{Z} \omega \oplus \mathbb{Z})\) (for details see e.g. [Bea1996], VI.20). For each hyperelliptic surface we have that the canonical divisor \(K_S\) is numerically trivial.

In 1990 F. Serrano in [Se1990], Theorem 1.4, characterised the group of classes of numerically equivalent divisors \(\text{Num}(S)\) for each of the surface’s type:

**Theorem 3.2** (Serrano). A basis of the group \(\text{Num}(S)\) for each of the hyperelliptic surface’s type and the multiplicities of the singular fibres in each case are the following:

| Type of a hyperelliptic surface | \(G\) | \(m_1, \ldots, m_s\) | Basis of \(\text{Num}(S)\) |
|---------------------------------|-------|-----------------|-----------------|
| 1                              | \(\mathbb{Z}_2\) | 2, 2, 2, 2 | \(A/2, B\) |
| 2                              | \(\mathbb{Z}_2 \times \mathbb{Z}_2\) | 2, 2, 2, 2 | \(A/2, B/2\) |
| 3                              | \(\mathbb{Z}_4\) | 2, 4, 4 | \(A/4, B\) |
| 4                              | \(\mathbb{Z}_4 \times \mathbb{Z}_2\) | 2, 4, 4 | \(A/4, B/2\) |
| 5                              | \(\mathbb{Z}_3\) | 3, 3, 3 | \(A/3, B\) |
| 6                              | \(\mathbb{Z}_3 \times \mathbb{Z}_3\) | 3, 3, 3 | \(A/3, B/3\) |
| 7                              | \(\mathbb{Z}_6\) | 2, 3, 6 | \(A/6, B\) |

Let \(\mu = \text{lcm}\{m_1, \ldots, m_s\}\) and let \(\gamma = |G|\). Given a hyperelliptic surface, its basis of \(\text{Num}(S)\) consists of divisors \(A/\mu\) and \((\mu/\gamma)B\).
We say that \( L \) is a line bundle of type \((a, b)\) on a hyperelliptic surface if \( L \equiv a \cdot A/\mu + b \cdot (\mu/\gamma)B \). In \( \text{Num}(S) \) we have that \( A^2 = 0, B^2 = 0, AB = \gamma \). Note that a divisor \( b \cdot (\mu/\gamma)B \equiv (0, b), b \in \mathbb{Z}, \) is effective if and only if \( b \cdot (\mu/\gamma) \in \mathbb{N} \) (see \([\text{Ap1998}]\), Proposition 5.2).

The following proposition holds:

**Proposition 3.3** (see \([\text{Se1990}]\), Lemma 1.3). Let \( D \) be a divisor of type \((a, b)\) on a hyperelliptic surface \( S \). Then

1. \( \chi(D) = ab; \)
2. \( D \) is ample if and only if \( a > 0 \) and \( b > 0; \)
3. If \( D \) is ample then \( h^0(D) = \chi(D) = ab.\)

Now we recall a bound for the self-intersection of a curve. The adjunction formula, applied to the normalisation of a curve \( C \), implies the following formula:

**Proposition 3.4** (Genus formula, \([\text{GH1978}]\), Lemma, p. 505). Let \( C \) be a curve on a surface \( S \), passing through \( x_1, \ldots, x_r \) with multiplicities respectively \( m_1, \ldots, m_r \). Let \( g(C) \) denote the genus of the normalisation of \( C \). Then

\[
g(C) \leq \frac{C^2 + C.K_S}{2} + 1 - \sum_{i=1}^{r} \frac{m_i(m_i - 1)}{2}.
\]

Note that:

**Observation 3.5.** A curve \( C \) on a hyperelliptic surface has genus at least 1. Indeed, otherwise the normalisation of \( C \), of genus zero, would be a covering (via \( \Phi \)) of an elliptic curve \( A/G \). This contradicts the Riemann-Hurwitz formula.

4. Main result

Our main result is the following

**Theorem 4.1.** Let \( S \) be a hyperelliptic surface. Let \( L \) be a line bundle of type \((m, m)\) with \( m \geq k + 2 \) on \( S \). Then \( L \) is \( k \)-jet ample.

By the results of M. Mella and M. Palleschi, see \([\text{MP1993}]\), Theorems 3.2-3.4, we know that \( L \equiv (a, b) \) with at least one of the coefficients strictly smaller than \( k + 2 \) is not \( k \)-very ample on an arbitrary hyperelliptic surface, in particular it is not \( k \)-very ample on a hyperelliptic surface of type 1. A line bundle which is not \( k \)-very ample is not \( k \)-jet ample. Therefore the line bundle \( L \equiv (k + 2, k + 2) \) is the first natural object of study.

**Proof.** We will prove that \( L \equiv (k + 2, k + 2) \) is \( k \)-jet ample and as a consequence we will get that a line bundle of type \((m, m)\) with \( m \geq k + 2 \) is \( k \)-jet ample.

Let \( r \geq 1 \). We have to check that for each choice of distinct points \( x_1, \ldots, x_r \in S \) the map

\[
H^0(X, L) \longrightarrow H^0 \left( X, L \otimes \mathcal{O}_X/(m_{x_1}^{k_1} \otimes \ldots \otimes m_{x_r}^{k_r}) \right)
\]

is surjective, where \( \sum_{i=1}^{r} k_i = k + 1 \).

We consider the standard exact sequence:

\[
0 \longrightarrow (K_S + L) \otimes m_{x_1}^{k_1} \otimes \ldots \otimes m_{x_r}^{k_r} \longrightarrow K_S + L \longrightarrow (K_S + L) \otimes \mathcal{O}_X/(m_{x_1}^{k_1} \otimes \ldots \otimes m_{x_r}^{k_r}) \longrightarrow 0.
\]
By the long sequence of cohomology, surjectivity of the map
\[ H^0(K_S + L) \longrightarrow H^0\left((K_S + L) \otimes \mathcal{O}_X/(m_{x_1}^{k_1} \otimes \ldots \otimes m_{x_r}^{k_r})\right) \]
is implied by vanishing of \( H^1\left((K_S + L) \otimes m_{x_1}^{k_1} \otimes \ldots \otimes m_{x_r}^{k_r}\right) \).

By the projection formula, we have that
\[ H^1\left((K_S + L) \otimes m_{x_1}^{k_1} \otimes \ldots \otimes m_{x_r}^{k_r}\right) \cong H^1\left(\pi^*(K_S + L) - \sum_{i=1}^{r} k_i E_i\right) \cong H^1\left(K_{\tilde{S}} - \sum_{i=1}^{r} E_i + \pi^*L - \sum_{i=1}^{r} k_i E_i\right) \cong H^1\left(K_{\tilde{S}} + \pi^*L - \sum_{i=1}^{r} (k_i + 1) E_i\right). \]

We will show that \( H^1\left(K_{\tilde{S}} + \pi^*L - \sum_{i=1}^{r} (k_i + 1) E_i\right) = 0 \), using vanishing theorems. We consider separately case \( r = 1 \), and separately case \( r \geq 2 \).

First let \( r = 1 \). We show that \( \pi^*L - (k+2)E \) is nef and big, hence by Kawamata-Viehweg vanishing theorem we get that \( H^1(K_{\tilde{S}} + \pi^*L - (k+2)E) = 0 \).

We have that
\[ \pi^*L - (k+2)E = \pi^*((k+2,k+2)) - (k+2)E = (k+2)(\pi^*(1,1) - E). \]

By [Fa2015], Theorem 3.1, we know that on a hyperelliptic surface the Seshadri constant of a line bundle of type \((1,1)\) at an arbitrary point \( x \) is at least 1. Therefore
\[ \sup \{ \varepsilon : \pi^*L - \varepsilon E \text{ is nef} \} = \varepsilon(L,x) = (k+2) \cdot \varepsilon((1,1),x) \geq k+2, \]
hence the line bundle \( \pi^*L - (k+2)E \) is nef. Thus to prove that \( \pi^*L - (k+2)E \) is also big, it is enough to show that \( (\pi^*L - (k+2)E)^2 > 0 \), which is equivalent to prove that \( L^2 > (k+2)^2 \).

The last inequality holds, as
\[ L^2 = (k+2,k+2)^2 = 2(k+2) \cdot (k+2) = 2(k+2)^2. \]

The case \( r = 1 \) is proved.

Now let \( r \geq 2 \). We will prove that \( H^1(K_{\tilde{S}} + \pi^*L - \sum_{i=1}^{r} (k_i + 1) E_i) = 0 \). The proof will be divided in several cases, depending on the position of points \( x_1, \ldots, x_r \).

Let \( k = 1 \). Generation of 1-jets is by definition equivalent to 1-very ampleness. The line bundle \( L \equiv (3,3) \) is 1-very ample on any hyperelliptic surface by [MP1993], Theorems 3.2.3.4.

Let \( k \geq 2 \).

If \( k_i = 0 \) for some \( i \), then we can consider points \( x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r \). In this case without lose of generality we may take a smaller \( r \). From now on we assume that \( k_i \geq 1 \) for every \( i \). Obviously, \( r \in [2, k+1] \) as \( \sum_{i=1}^{r} k_i = k + 1 \).

For simplicity, we present a proof for hyperelliptic surfaces of type 1. For surfaces of other types the proof is analogous. The small differences are listed in Remark 4.15.

The proof consists of a few steps which we describe briefly before we turn to the details.

First, in Case I, we consider a situation where on each singular fibre \( A/2 \), on each fibre \( A, \) and on each fibre \( B \) there are points \( x_i \) with the sum of multiplicities \( k_i \) equal to at most \( \frac{k+1}{2} \).

Then, in Cases II and III, we consider a situation where there exists a fibre \( A/2, \) respectively \( A, \) on which there are some points \( x_i \) with the sum of multiplicities \( k_i \) greater than \( \frac{k+1}{2} \).
In both cases we have two possibilities: (a) the sum of multiplicities of points lying on any fibre $B$ is smaller than $\frac{k+1}{2}$; (b) there exists a fibre $B$ for which the sum of multiplicities of points on this fibre is at least $\frac{k+1}{2}$. Therefore we divide Cases II and III into two subcases: respectively IIa, IIb and IIIa, IIIb.

Finally, in Case IV we consider the situation where some points $x_i$ lie on a fixed fibre $B$ and their sum of multiplicities does not exceed $\frac{k+1}{2}$, moreover for each fibre $A/2$ and for each fibre $A$ the sum of multiplicities of points lying on this fibre is at most $\frac{k+1}{2}$. This covers all possibilities.

In all the cases described above we prove that $H^1\left(K_S + \pi^*L - \sum_{i=1}^r (k_i + 1)E_i\right) = 0$, using Kawamata-Viehweg vanishing theorem in Cases I and IIIa, and Norimatsu lemma in Cases IIa, IIb, IIIb and IV. In Cases I and IIIa we show that a divisor $M = \pi^*L - \sum_{i=1}^r (k_i + 1)E_i$ is big and nef, while in Cases IIa, IIb, IIIb and IV we define an appropriate SNC divisor $F$ and prove that a divisor $N = M - F$ is ample, using Nakai-Moishezon criterion.

In all the cases by $C \equiv (\alpha, \beta)$ we denote a reduced irreducible curve on $S$ passing through $x_1, \ldots, x_r$ with multiplicities respectively $m_1, \ldots, m_r$, where $m_i \geq 0$ for all $i$, and there exists $j$ with $m_j > 0$. We have $\tilde{C} = \pi^*C - \sum_{i=1}^r m_i E_i$.

Define $k_i^W = \begin{cases} k_i & \text{if } x_i \in W \text{ for a reduced fibre } W \text{ of } \Phi \text{ or } \Psi, \\ 0 & \text{otherwise.} \end{cases}$

Let $r_W$ be the number of points from the set $\{x_1, \ldots, x_r\}$ which are contained in $W$.

Let us now move on to considering the cases in more detail.

**Case I.** For an arbitrary fibre $W$ (where $W = A/2$, or $W = A$, or $W = B$) the sum of multiplicities of the points $x_i$ lying on this fibre is at most $\frac{k+1}{2}$, i.e.

$$\sum_{i=1}^{r_W} k_i^W \leq \frac{k+1}{2}.$$ 

Since $\sum_{i=1}^{r_W} k_i^W \leq \frac{k+1}{2}$, in particular we have that $r_W \leq \frac{k+1}{2}$.

We will show that the line bundle $M = \pi^*L - \sum_{i=1}^r (k_i + 1)E_i$ is big and nef.

**Lemma 4.2.** $M$ is a nef line bundle.

**Proof.** We ask whether $M\tilde{C} \geq 0$. We have to check that

$$(\star) = M\tilde{C} = \left(\pi^*L - \sum_{i=1}^r (k_i + 1)E_i\right) \cdot \left(\pi^*C - \sum_{i=1}^r m_i E_i\right) \geq 0$$

Let us consider the following cases:

(1) $C = A/2$, or $C = B$, or $C = A$. Then for $i = 1, \ldots, r$ we have $m_i = 1$, hence

$$(\star) \geq LC - \sum_{i=1}^{r_W} (k_i^W + 1) \geq (k+2) - \sum_{i=1}^{r_W} (k_i^W + 1) \geq (k+2) - \left(\sum_{i=1}^{r_W} k_i^W + r_W\right) \geq (k+2) - \left(\frac{k+1}{2} + \frac{k+1}{2}\right) > 0.$$ 

(2) $C$ is not a fibre. Hence $\alpha > 0$ and $\beta > 0$. We prove the inequality $M\tilde{C} \geq 0$ in Proposition [19] at the end of the proof of the main theorem.
Lemma 4.3. $M$ is big.

Proof. Since $M$ is nef, it is enough to prove that $M^2 > 0$. As $M^2 = 2(k+2)^2 - \sum_{i=1}^{r} (k_i + 1)^2$, we ask whether $2(k+2)^2 > \sum_{i=1}^{r} (k_i + 1)^2$. It suffices to show that

$$2(k+2)^2 > \left( \sum_{i=1}^{r} k_i^2 \right) + 2(k+1) + r,$$

where $r$ at the end of the formula is of the greatest possible value $k+1$. Since $\sum_{i=1}^{r} k_i^2 \leq (\sum_{i=1}^{r} k_i)^2 = (k+1)^2$, our goal is to prove the inequality

$$2(k+2)^2 > (k+1)^2 + 3k + 3,$$

which is elementary. \qed

Case IIa. There exists a fibre $A/2$ on which there are points $x_1, \ldots, x_s$ with the sum of multiplicities $\sum_{i=1}^{s} k_i > \frac{k+1}{2}$ and on each fibre $B$ we have $\sum_{i=1}^{r_{i}} k_i^B < \frac{k+1}{2}$.

Obviously, for any other fibre $A/2$ and for any fibre $A$ we have $\sum_{i=1}^{r_{i}} k_i^W < \frac{k+1}{2}$.

We write

$$M = \pi^*L - \sum_{i=1}^{r} (k_i + 1) = \pi^*(A/2) - \sum_{i=1}^{s} E_i + \pi^*((k+1, k+2)) - \sum_{i=1}^{s} k_i E_i - \sum_{i=s+1}^{r} (k_i + 1) E_i,$$

so $M = N + F$, where $N = \pi^*((k+1, k+2)) - \sum_{i=1}^{s} k_i E_i - \sum_{i=s+1}^{r} (k_i + 1) E_i$ and $F = \pi^*(A/2) - \sum_{i=1}^{s} E_i = A/2$. Clearly $F$ is a smooth and reduced divisor, hence $F$ is an SNC divisor. It remains to show that $N$ is ample.

Lemma 4.4. $N$ is ample.

Proof. We check that $N^2 > 0$, and $N\tilde{C} > 0$.

Let us estimate $N^2$

$$N^2 = 2(k+1)(k+2) - \sum_{i=1}^{r} k_i^2 - 2 \sum_{i=s+1}^{r} k_i - (r - s) \geq k^2 + 2k + 1 + s > 0,$$

as $\sum_{i=s+1}^{r} k_i < \frac{k+1}{2}$, $\sum_{i=1}^{r} k_i^2 \leq (\sum_{i=1}^{r} k_i)^2 = (k+1)^2$ and $r \leq k+1$.

Now we check whether

$$(\ast) = N\tilde{C} = \left( \pi^*((k+1, k+2)) - \sum_{i=1}^{s} k_i E_i - \sum_{i=s+1}^{r} (k_i + 1) E_i \right) \cdot \left( \pi^*C - \sum_{i=1}^{r} m_i E_i \right) > 0$$

We consider the following cases:

(1) $C$ is the fibre $A/2$ for which $\sum_{i=1}^{s} k_i > \frac{k+1}{2}$. Then $m_i = 1$ for all $i$, and

$$(\ast) = k + 2 - \sum_{i=1}^{s} k_i \geq k + 2 - (k + 1) = 1 > 0,$$

as $\sum_{i=1}^{s} k_i \leq \sum_{i=1}^{r} k_i = k + 1$. 7
(2) \( C \) is a different fibre \( A/2 \), or \( C = A \), or \( C = B \). All \( m_i = 1 \), hence
\[
(*) \geq k + 1 - \sum_{i=1}^{r_W} (k_i^W + 1) > k + 1 - \left( \frac{k + 1}{2} + \frac{k + 1}{2} \right) = 0,
\]
as \( \sum_{i=1}^{r_W} k_i^W < \frac{k + 1}{2} \) and \( r_W < \frac{k + 1}{2} \).
(3) \( C \) is not a fibre — see Proposition 4.9.

\[\square\]

**Case IIb.** Points \( x_1, \ldots, x_s \) lie on a fixed fibre \( A/2 \) with \( \sum_{i=1}^{s} k_i > \frac{k + 1}{2} \), moreover there exists a fibre \( B \) such that \( x_s, \ldots, x_l \) lie on \( B \) with \( \sum_{i=s}^{t} k_i \geq \frac{k + 1}{2} \).

We define \( F := \tilde{A}/2 + \tilde{B} = \tilde{A}/2 + B \). Clearly \( F \) is an SNC divisor. We define \( N := M - F = \pi^*(((k + 1, k + 1)) + E_s - \sum_{i=1}^{t} E_i - \sum_{i=t+1}^{r} (k_i + 1)E_i \). It remains to check that \( N \) is ample.

**Lemma 4.5.** \( N \) is ample.

**Proof.** Analogously to Case IIa, we show that \( N^2 > 0 \), and that \( N \tilde{C} > 0 \).

We estimate \( N^2 \) from below, using inequalities \( \sum_{i=t+1}^{r} k_i \leq \frac{k + 1}{2} \), \( \sum_{i=1}^{r} k_i^2 \leq (k + 1)^2 \) and \( r \leq k + 1 \).

\[
N^2 = 2k^2 + 4k + 1 - \sum_{i=1}^{r} k_i^2 - 2 \sum_{i=t+1}^{r} k_i - (r - t) \geq
2k^2 + 4k + 1 - (k + 1)^2 - 2 \cdot \frac{k + 1}{2} - (k + 1 - t) = k^2 + t - 2.
\]

Hence \( N^2 > 0 \), because \( k \geq 2 \).

Now we check that
\[
(*) = N \tilde{C} > 0
\]

We consider the following cases:
(1) \( C \) is the fibre \( A/2 \) for which \( \sum_{i=1}^{s} k_i \geq \frac{k + 1}{2} \). Then
\[
(*) = k + 1 + 1 - \sum_{i=1}^{s} k_i \geq k + 2 - (k + 1) = 1 > 0,
\]
as \( \sum_{i=1}^{s} k_i \leq k + 1 \).
(2) \( C \) is a different fibre \( A/2 \), or \( C = A \). Then
\[
(*) \geq k + 1 - \sum_{i=1}^{r_W} (k_i^W + 1) > k + 1 - \left( \frac{k + 1}{2} + \frac{k + 1}{2} \right) = 0,
\]
as \( \sum_{i=1}^{r_W} k_i^W < \frac{k + 1}{2} \) and \( r_W < \frac{k + 1}{2} \).
(3) \( C \) is the fibre \( B \) for which \( \sum_{i=1}^{t} k_i \geq \frac{k + 1}{2} \). Then
\[
(*) = (k + 1) + 1 - \sum_{i=1}^{t} k_i \geq k + 2 - (k + 1) = 1 > 0.
\]
(4) \( C \) is a different fibre \( B \). Then
\[
(*) \geq (k + 1) - \sum_{i=1}^{r_W} (k_i^W + 1) > (k + 1) - \left( \frac{k + 1}{2} + \frac{k + 1}{2} \right) = 0,
\]
as \( \sum_{i=1}^{rW} k_i^W < \frac{k+1}{2} \) and \( rW < \frac{k+1}{2} \).

(5) \( C \) is not a fibre — see Proposition 4.9. \( \square \)

**Case IIIa.** Points \( x_1, \ldots, x_s \) lie on a fixed fibre \( A \) with \( \sum_{i=1}^{s-1} k_i > \frac{k+1}{2} \), and for each fibre \( B \) we have \( \sum_{i=1}^{rW} k_i^W < \frac{k+1}{2} \).

Obviously in this case, for any other fibre \( A \) and for any fibre \( A/2 \) we have \( \sum_{i=1}^{rW} k_i^W < \frac{k+1}{2} \).

Let \( M = \pi^*L - \sum_{i=1}^{r} (k_i + 1)E_i \). We have already showed in Case I that \( M^2 > 0 \). It remains to prove that \( \widetilde{M} \) jest nef.

**Lemma 4.6.** \( M \) is nef.

**Proof.** We ask whether
\[
(*) = M\widetilde{C} \geq 0
\]

We consider the following cases:

(1) \( C \) is the fibre \( A \) for which \( \sum_{i=1}^{s} k_i > \frac{k+1}{2} \). Then
\[
(*) = 2(k + 2) - \sum_{i=1}^{s} (k_i + 1) \geq 2(k + 2) - \sum_{i=1}^{r} k_i - s \geq 2(k + 2) - (k + 1) - (k + 1) = 2 > 0.
\]

(2) \( C \) is a different fibre \( A \), or \( C = A/2 \), or \( C = B \). Then
\[
(*) \geq k + 2 - \sum_{i=1}^{rW} (k_i^W + 1) \geq k + 2 - \left( \frac{k + 1}{2} + \frac{k + 1}{2} \right) = 1 > 0.
\]

(3) \( C \) is not a fibre — see Proposition 4.9. \( \square \)

**Case IIIb.** Points \( x_1, \ldots, x_s \) lie on a fixed fibre \( A \) with \( \sum_{i=1}^{s-1} k_i > \frac{k+1}{2} \), and there exists a fibre \( B \), such that \( x_s, \ldots, x_t \) lie on \( B \) and \( \sum_{i=s}^{t} k_i \geq \frac{k+1}{2} \).

We define \( F := \widetilde{B} = \pi^*B - \sum_{i=s}^{t} E_i \). Clearly \( F \) is an SNC divisor. Let \( N := M - F = \pi^*((k + 2, k + 1)) - \sum_{i=1}^{s-1} (k_i + 1)E_i - \sum_{i=s}^{t} k_i E_i - \sum_{i=t+1}^{r} (k_i + 1)E_i \). It remains to check that \( N \) is ample.

**Lemma 4.7.** \( N \) is ample.

**Proof.** Let us estimate \( N^2 \):
\[
N^2 = 2(k + 2)(k + 1) - \sum_{i=1}^{s-1} (k_i + 1)^2 - \sum_{i=s}^{t} k_i^2 - \sum_{i=t+1}^{r} (k_i + 1)^2 =
2(k + 2)(k + 1) - \sum_{i=1}^{r} k_i^2 - 2 \left( \sum_{i=1}^{s-1} k_i - \sum_{i=t+1}^{r} k_i \right) - (s - 1) - (r - t) \geq
2k^2 + 6k + 4 - (k + 1)^2 - 2(k + 1) - s + 1 - r + t \geq
k^2 + 2k + 2 - (k + 1) - (k + 1) + t = k^2 + t > 0.
\]

Now we check that
\[
(*) = N\widetilde{C} > 0
\]

Let us consider the following cases:
(1) $C$ is the fibre $A$ for which $\sum_{i=1}^{s} k_i > \frac{k+1}{2}$. Then

\[
(*) = 2(k + 1) - \sum_{i=1}^{s-1} (k_i + 1) - k_s \geq 2(k + 1) - \sum_{i=1}^{s} k_i - (s - 1) \geq 2(k + 1) - (k + 1) - (k + 1) + 1 = 1 > 0.
\]

(2) $C$ is the fibre $B$ for which $\sum_{i=s}^{t} k_i \geq \frac{k+1}{2}$. Then

\[
(*) = k + 2 - \sum_{i=s}^{t} k_i \geq k + 2 - (k + 1) = 1 > 0.
\]

(3) $C$ is a different fibre $A$ or a different fibre $B$, or $C = A/2$. Then

\[
(*) \geq k + 1 - \sum_{i=s}^{r_W} (k_i + 1) \geq k + 1 - \left(\frac{k + 1}{2} + \frac{k + 1}{2}\right) = 1 > 0.
\]

(4) $C$ is not a fibre — see Proposition 4.9. \hfill \Box

**Case IV.** Points $x_1, \ldots, x_s$ lie on a fixed fibre $B$ with $\sum_{i=1}^{s} k_i > \frac{k+1}{2}$, and for each fibre $A/2$, and for each fibre $A$ the sum of multiplicities of the points lying on this fibre does not exceed $\frac{k+1}{2}$.

We define $F := \tilde{B} = \pi^*B - \sum_{i=1}^{s} E_i$. Of course $F$ is an SNC divisor. We define $N := M - F = \pi^*((k + 2, k + 1)) - \sum_{i=1}^{s} k_i E_i - \sum_{i=s+1}^{r} (k_i + 1) E_i$. It remains to prove that $N$ is ample.

**Lemma 4.8.** $N$ is ample.

**Proof.** Analogously to Case IIa,

\[N^2 = k^2 + 2k + 1 + s > 0.\]

We have to check that

\[
(*) = N\tilde{C} > 0
\]

Let us consider the following cases:

(1) $C$ is the fibre $B$ for which $\sum_{i=1}^{s} k_i > \frac{k+1}{2}$. Then

\[
(*) = k + 2 - \sum_{i=1}^{s} k_i \geq k + 2 - (k + 1) = 1 > 0.
\]

(2) $C$ is a different fibre $B$ or $C = A$. Then

\[
(*) \geq k + 2 - \sum_{i=1}^{r_W} (k_i^{W} + 1) \geq k + 2 - \left(\frac{k + 1}{2} + \frac{k + 1}{2}\right) = 1 > 0.
\]

(3) $C = A/2$.

If $r_W < \frac{k+1}{2}$, then

\[
(*) = k + 1 - \sum_{i=1}^{r_W} (k_i^{W} + 1) > k + 1 - \left(\frac{k + 1}{2} + \frac{k + 1}{2}\right) = 0.
\]
Otherwise \( r_W = \frac{k+1}{2} \). Hence the situation is as follows: \( s \) points whose sum of multiplicities is greater than \( \frac{k+1}{2} \) lie on the fixed fibre \( B \), and there exists a fibre \( A/2 \) with \( \frac{k+1}{2} \) points whose sum of multiplicities does not exceed \( \frac{k+1}{2} \), which implies that all the multiplicities of points lying on \( A/2 \) equal 1. Hence \( x_1, \ldots, x_s \) lie on \( B \) and \( x_s, \ldots, x_r \) lie on \( A/2 \). Therefore

\[
(*) = k + 1 - k_s - \sum_{i=s+1}^{r} (k_i + 1) = k + 1 - 2(r - s) = k + 2 - 2 \cdot \frac{k + 1}{2} = 1 > 0
\]

(4) \( C \) is not a fibre — see Proposition 4.9  \( \square \)

We have considered all the possible positions of \( x_1, \ldots, x_r \). Hence the proof will be completed if we show that for a curve \( C \) not being a fibre the inequality respectively \( MC > 0 \) or \( NC > 0 \) holds. We prove this fact in the proposition below.

**Proposition 4.9.** We have the following inequalities for a curve \( C \) which is not a fibre:

- \( MC \geq 0 \) in Cases I and IIIa,
- \( NC > 0 \) in Cases IIa, IIb, IIIb and IV.

**Proof.** By assumption \( C \equiv (\alpha, \beta) \) with \( \alpha > 0 \) and \( \beta > 0 \). We have to prove that

- \( \left( \sum_{i=1}^{r} k_i \right) + 1 \)(\( \alpha + \beta \)\) - \( \sum_{i=1}^{r} (k_i + 1)m_i \geq 0 \) in Cases I and IIIa;
- \( \sum_{i=1}^{r} k_i \)(\( \alpha + \beta \)\) + \( \alpha - \sum_{i=1}^{s} k_i m_i - \sum_{i=s+1}^{r} (k_i + 1)m_i > 0 \) in Case IIa;
- \( \sum_{i=1}^{r} k_i \)(\( \alpha + \beta \)\) + \( m_s - \sum_{i=1}^{t} k_i m_i - \sum_{i=t+1}^{r} (k_i + 1)m_i > 0 \) in Case IIb;
- \( \sum_{i=1}^{r} k_i \)(\( \alpha + \beta \)\) + \( \beta - \sum_{i=1}^{s-1} (k_i + 1)m_i - \sum_{i=s}^{t} k_i m_i - \sum_{i=t+1}^{r} (k_i + 1)m_i > 0 \) in Case IIIb;
- \( \sum_{i=1}^{r} k_i \)(\( \alpha + \beta \)\) + \( \beta - \sum_{i=1}^{s-1} k_i m_i - \sum_{i=s+1}^{r} (k_i + 1)m_i > 0 \) in Case IV.

Observe that in all the situations above, Proposition 4.9 will be proved if we show that

\[
(*) \quad \left( \sum_{i=1}^{r} k_i \right)(\alpha + \beta) \geq \sum_{i=1}^{r} (k_i + 1)m_i.
\]

Let \( D \equiv (4, 4) \). Since \( h^0(D) = 4 \cdot 4 = 16 \), for an arbitrary point \( x \) there exists a divisor \( D_x \in |D| \) such that \( \text{mult}_x D_x = 5 \) (vanishing up to order 5 imposes 15 conditions). Hence there are two possibilities: either \( \alpha \leq 4 \) and \( \beta \leq 4 \), and then \( C \) and \( D_x \) may have a common component \( C' \) (the curve \( C \) is irreducible); or \( \alpha > 4 \) or \( \beta > 4 \), and then by Bézout’s Theorem

\[
4(\alpha + \beta) = (\alpha, \beta).(4, 4) = CD = CD_x \geq \text{mult}_x C \cdot \text{mult}_x D_x \geq 5m_i.
\]

Proof of the proposition in each case will be completed in Lemma 4.10 and Lemma 4.12 \( \square \)

**Lemma 4.10.** Proposition 4.9 holds if \( \alpha > 4 \) or \( \beta > 4 \).
Proof. We begin with a useful observation:

**Observation 4.11.** Let $S$ be a hyperelliptic surface of any type, let $C \equiv (\alpha, \beta)$, where $\alpha > 4$ or $\beta > 4$. To prove $(\star)$, it suffices to prove the inequality

$$(\star^\star) \quad r(\alpha + \beta) \geq 2 \sum_{i=1}^{r} m_i.$$ 

Proof. We have already observed that by Bézout’s Theorem, for every $i \in \{1, \ldots, r\}$ we have that $\alpha + \beta \geq \frac{5}{2} m_i \geq m_i$, hence

$$(k_i - 1)(\alpha + \beta) \geq (k_i - 1)m_i.$$ 

Summing up these $r$ inequalities with inequality $(\star^\star)$ we obtain the inequality $(\star)$. \qed

Now we will show that if $\alpha > 4$ or $\beta > 4$ then $(\star^\star)$ is satisfied for $r \geq 3$. Let us denote $i$-th inequality in the inequality $(\star^\star)$ by $(\star^\star_i)$, i.e.

$$(\star^\star_i) \quad (\alpha + \beta) \geq 2m_i.$$ 

The inequality $(\star^\star_i)$ is satisfied for $m_i < 4$. Indeed, $C$ is not a fibre and by assumption we have $\alpha \geq 5$ or $\beta \geq 5$, hence $\alpha + \beta \geq 5 + 1 = 6 \geq 2m_i$ for $m_i \leq 3$. Therefore we may assume that $m_i \geq 4$.

We delete from $(\star)$ all the inequalities $(\star^\star_i)$ with $m_i < 4$ and consider a modified inequality $(\star)$, possibly with a smaller number of points $r$. It may even happen that $r < 2$ in the modified $(\star)$.

Now we will prove $(\star)$ assuming that $m_i \geq 4$ for all $i$, and $r \geq 3$. Equivalently, we want to show that

$$r^2(\alpha + \beta)^2 \geq 4 \left( \sum_{i=1}^{r} m_i \right)^2.$$ 

By inequality between means, it is enough to check that

$$r^2(\alpha + \beta)^2 \geq 4r \sum_{i=1}^{r} m_i^2$$ 

It suffices to check that

$$r(2\alpha \beta) \geq 2 \sum_{i=1}^{r} m_i^2.$$ 

By the genus formula $2\alpha \beta \geq \sum_{i=1}^{r} m_i^2 - \sum_{i=1}^{r} m_i$ (see Proposition 3.4 and Observation 3.5), hence it is enough to prove that

$$r \left( \sum_{i=1}^{r} m_i^2 - \sum_{i=1}^{r} m_i \right) \geq 2 \sum_{i=1}^{r} m_i^2$$ 

$$(r - 2) \left( \sum_{i=1}^{r} m_i^2 \right) - r \sum_{i=1}^{r} m_i \geq 0$$
We assume that $m_i \geq 4$, so $\sum_{i=1}^{r} m_i^2 \geq 4 \sum_{i=1}^{r} m_i$. Hence it is enough to show that
\[
4(r - 2) \left( \sum_{i=1}^{r} m_i \right) - r \sum_{i=1}^{r} m_i \geq 0
\]
\[
(3r - 8) \cdot (4r) \geq 0
\]
which is obviously true for $r \geq 3$.

To finish the proof of Lemma, it remains to check that the assertion holds for $r < 3$. If $r = 0$, then every inequality (\text{*ii}) is satisfied, which together with Observation 4.11 completes the proof of the lemma. Hence we have to consider cases $r = 1$ (when in the inequality (\text{*}) all but one inequalities (\text{**}i) hold), and $r = 2$.

Let $r = 2$. We prove Lemma in Cases I and IIIa, and separately in all the remaining cases.

Let us consider Cases I and IIIa. We have to prove the inequality
\[
(k_1 + k_2 + 1)(\alpha + \beta) - (k_1 + 1)m_1 - (k_2 + 1)m_2 \geq 0
\]
Analogously to Observation 4.11 it suffices to prove the inequality $(1+1)(\alpha + \beta) + (\alpha + \beta) \geq 2m_1 + 2m_2$. Indeed, adding two fulfilled inequalities of the form $(k_i - 1)(\alpha + \beta) \geq (k_i - 1)m_i$, we obtain the assertion.

Therefore we have to check that
\[
3(\alpha + \beta) \geq 2m_1 + 2m_2.
\]
In the linear system of divisor $D \equiv (4, 4)$, for each points $x_1, x_2$, there exists such a divisor $D_{x_1,x_2} \equiv D$ that its multiplicity at $x_1$ equals 4, and multiplicity at $x_2$ equals 2.
Since $\alpha > 4$ or $\beta > 4$ and the curve $C$ is irreducible, $C$ is not a component of $D$. Therefore by Bézout’s Theorem we have
\[
4(\alpha + \beta) = CD = CD_{x_1,x_2} \geq 4m_1 + 2m_2.
\]
Analogously
\[
4(\alpha + \beta) \geq 2m_1 + 4m_2.
\]
Summing up two inequalities above, we obtain
\[
8(\alpha + \beta) \geq 6m_1 + 6m_2.
\]
Hence
\[
3(\alpha + \beta) \geq \frac{9}{4} m_1 + \frac{9}{4} m_2 \geq 2m_1 + 2m_2,
\]
and the assertion is proved.

Now let us consider Cases IIa, IIb, IIIb and IV. There are two possibilities:

(a) One of the points $x_1, x_2$ (without loss of generality $x_2$) lies respectively: on the fixed fibre $A/2$ in Case IIa, on the intersection of the fixed fibres $A/2$ and $B$ in Case IIb, on the intersection of the fixed fibres $A$ and $B$ in Case IIIb, on the fixed fibre $B$ in Case IV. Then the desired inequalities $N\bar{C} > 0$ are implied by the inequality
\[
(k_1 + k_2)(\alpha + \beta) \geq (k_1 + 1)m_1 + k_2m_2.
\]

By observation analogous to Observation 4.11 it suffices to show that
\[
2(\alpha + \beta) \geq 2m_1 + m_2.
\]
The assertion holds by inequality (4.1).
(b) None of the points \(x_1, x_2\) lies on the fixed fibre respectively \(A/2, A/2\) intersected with \(B, A\) intersected with \(B, A\), respectively \(A/2, A/2\) intersected with \(B, A\) intersected with \(B, A\), and \(B, A\) intersected with \(B, A\). By assumption of Case respectively IIa, IIb, IIIb and IV, at the beginning there was a point \(x_3\) on those fixed fibres or on the intersection of the fixed fibres, and the inequality \((\ast\ast_3)\) is satisfied. We restore this inequality, and we want to prove that
\[
(k_1 + k_2 + k_3)(\alpha + \beta) \geq (k_1 + 1)m_1 + (k_2 + 1)m_2 + k_3m_3
\]
By observation analogous to Observation 4.11 it is enough to prove that
\[
3(\alpha + \beta) \geq 2m_1 + 2m_2 + m_3
\]
In the linear system of divisor \(D \equiv (4, 4)\) for each points \(x_1, x_2, x_3\) there exists such a divisor \(D_{x_1,x_2,x_3} \equiv D\) that its multiplicity at \(x_1\) equals 3, its multiplicity at \(x_2\) equals 3, and its multiplicity at \(x_3\) equals 2.
By Bézout’s Theorem we obtain
\[
(4.2) \quad 4(\alpha + \beta) = CD = CD_{x_1,x_2,x_3} \geq \sum_{i=1}^{3} \text{mult}_{x_i} C \cdot \text{mult}_{x_i} D_{x_1,x_2,x_3} \geq 3m_1 + 3m_2 + 3m_3.
\]
Hence
\[
3(\alpha + \beta) \geq \frac{9}{4} m_1 + \frac{9}{4} m_2 + \frac{6}{4} m_3 \geq 2m_1 + 2m_2 + m_3,
\]
and we are done.

Now let \(r = 1\). At the beginning the number of points \(r\) was at least 2, hence while deleting from the inequality \((\ast\ast)\) the inequalities \((\ast\ast_i)\) with \(m_i \leq 3\), we deleted all by one inequalities.
We restore one of the deleted inequalities \((\ast\ast_i)\): an arbitrary one in Cases I and IIIa; in Cases IIa, IIb, IIIb and IV — the inequality corresponding to \(x_i\) lying respectively on the fixed fibre \(A/2\), on the intersection of the fixed fibres \(A/2\) and \(B\), on the intersection of the fixed fibres \(A\) and \(B\), on the fixed fibre \(B\) (if \(x_1\) is not in such a position; an arbitrary inequality \((\ast\ast_i)\) otherwise). We obtained the inequality with \(r = 2\) which was already proved in subcase (a). \(\square\)

**Lemma 4.12.** Proposition 4.9 holds if \(\alpha \leq 4\) and \(\beta \leq 4\).

*Proof.* We denote
\[
(\ast_i) \quad (\alpha + \beta)k_i \geq (k_i + 1)m_i.
\]
Observe the following property:

**Remark 4.13.** For an arbitrary \(m_i\) the inequality \((\ast_i)\) is satisfied if \(m_i \leq \frac{1}{2}(\alpha + \beta)\). Indeed, \(2m_i \geq \left(1 + \frac{1}{k_i}\right) m_i\) for all \(k_i\).

In particular, if \(m_i = 0\) or \(m_i = 1\), then the inequality \((\ast_i)\) is satisfied, since \(\alpha \geq 1\) and \(\beta \geq 1\).

The multiplicities of \(C\) at \(x_1, \ldots, x_r\) satisfy genus formula, i.e. \(2\alpha\beta \geq \sum_{i=1}^{r} m_i^2 - \sum_{i=1}^{r} m_i\) (see Proposition 3.4 and Observation 3.5). In particular, for any \(x_i\) we have an upper bound \(2\alpha\beta \geq m_i^2 - m_i\).
If the inequality \((\star_i)\) holds for some \(m_i\), then it holds also for all multiplicities \(n_i < m_i\). After renumbering the points we assume that the multiplicities are decreasing: \(m_1 \geq m_2 \geq \ldots \geq m_r\).

In the table below, for every curve \(C \equiv (\alpha, \beta)\) with \(0 < \alpha \leq 4\) and \(0 < \beta \leq 4\), we present the following quantities:

- an upper bound for the maximal possible multiplicity \(m_i\) at any point \(x_i\), obtained from genus formula,
- an upper bound for the multiplicity \(m_i\), for which by Remark 4.13 the inequality \((\star_i)\) holds,
- all possible values of \(m_1\) greater than the number from the previous column, and for each such \(m_1\) the greatest possible value of \(m_2\) and \(m_i\) for \(i > 2\) (or an upper bound in the latter case), all obtained from genus formula.

| Case no. | \(C \equiv (\alpha, \beta)\) | \(\max m_i\) | \(\max m_i\) such that \((\star_i)\) holds | \(m_1\) | \(m_2\) | \(m_i\) for \(i > 2\) |
|----------|-------------------------------|--------------|---------------------------------|------|------|----------------|
| i        | \((4, 4)\)                    | 6            | 4                               | 6    | 2    | 1              |
| ii       | \((4, 3)\) or \((3, 4)\)      | 5            | 3                               | 5    | 2    | 3 \leq 2       |
| iii      | \((4, 2)\) or \((2, 4)\)      | 4            | 3                               | 4    | 2    | \leq 2         |
| iv       | \((4, 1)\) or \((1, 4)\)      | 3            | 2                               | 3    | 2    | 1              |
| v        | \((3, 3)\)                    | 4            | 3                               | 4    | 3    | 1              |
| vi       | \((3, 2)\) or \((2, 3)\)      | 4            | 2                               | 4    | 1    | 3 \leq 1       |
| vii      | \((3, 1)\) or \((1, 3)\)      | 3            | 2                               | 3    | 1    | 1              |
| viii     | \((2, 2)\)                    | 3            | 2                               | 3    | 2    | 1              |
| ix       | \((2, 1)\) or \((1, 2)\)      | 2            | 1                               | 2    | 2    | 1              |
| x        | \((1, 1)\)                    | 2            | 1                               | 2    | 1    | 1              |

Note that in each case (i)-(x), the inequality \((\star_i)\) holds for each multiplicity \(m_i\) where \(i > 2\). Therefore we would be done if we could prove \((\star)\) for \(r = 2\) and points \(x_1, x_2\) with multiplicities \(m_1, m_2\) listed in the table.

Simple computations give us that

**Remark 4.14.**

1. For \(r = 2\), if \(m_1 + m_2 \leq \alpha + \beta\), then \((\star)\) holds.
2. Moreover for \(r = 2\), if \(m_1 + m_2 \leq \alpha + \beta + 1\), then the inequality from Cases I and IIIa holds, i.e. \((\alpha + \beta) \left(\sum_{i=1}^{r} k_i + 1\right) \geq \sum_{i=1}^{r} (k_i + 1)m_i\).

Let us consider the cases listed in the table. In cases (ii), (iii), (iv), (vii) the inequality \((\star)\) holds by Remark 4.14(2). In cases (i), (v), (vi), (viii), (ix) we repeat the reasoning with restoring a certain inequality \((\star_3)\), if it is needed. Case (x) is the most subtle — in some subcases we have to restore two inequalities \((\star_3)\) and \((\star_4)\).

We explain the cases (i) and (x) in detail.

Case (i): \(C \equiv (4, 4)\).

It is enough to prove \((\star)\) in two subcases: \(r = 2, m_1 = 6, m_2 = 2\), and \(r = 2, m_1 = 5, m_2 = 4\). In the first subcase the we get the assertion by Remark 4.14(1). In the second subcase, in some situations we have to restore the fulfilled inequality \((\star_3)\) to prove \((\star)\). Our reasoning differs with respect to cases of the main theorem:
In Cases I and IIIa the inequality (∗) holds by Remark 4.14(2).

In Cases IIa, IIIb, IV there are two possibilities:

• One of the points \(x_1, x_2\) (without loss of generality \(x_2\)) lies respectively: on the fixed fibre \(A/2\) in Case IIa, on the intersection of the fixed fibres \(A\) and \(B\) in Case IIIb, on the fixed fibre \(B\) in Case IV. Then it is enough to prove that
\[
(\alpha + \beta)(k_1 + k_2) + \min\{\alpha, \beta\} > (k_1 + 1)m_1 + k_2m_2
\]
\[
8(k_1 + k_2) + 4 > 5(k_1 + 1) + 4k_2
\]
\[
3k_1 + 4k_2 > 1
\]
The last inequality holds as \(k_i \geq 1\) for all \(i\).

• None of the points \(x_1, x_2\) lies on the fixed fibre respectively \(A/2\), intersected with \(B\), and \(B\). By assumption of Cases respectively IIa, IIIb and IV, at the beginning there was a point \(x_3\) on the fixed fibre or on the intersection of the fixed fibres, and the maximal possible by genus formula multiplicity \(m_3\) is equal to 1. We restore the inequality (∗\(_3\)) and we want to prove that
\[
(\alpha + \beta)(k_1 + k_2 + k_3) + \min\{\alpha, \beta\} > (k_1 + 1)m_1 + (k_2 + 1)m_2 + k_3m_3
\]
\[
3k_1 + 4k_2 + 7k_3 > 5
\]
The inequality holds.

In Case IIb there are also two possibilities:

• One of the points \(x_1, x_2\) is the intersection point of the fixed fibres \(A/2\) and \(B\). Then the inequality
\[
(\alpha + \beta)(k_1 + k_2) + m_2 > (k_1 + 1)m_1 + k_2m_2
\]
\[
3k_1 + 4k_2 > 1
\]

obviously holds.

• None of the points \(x_1, x_2\) lies on the intersection of the fixed fibres \(A/2\) and \(B\). Then by assumption of Case IIb there was a point \(x_3\) with \(m_3 = 1\) on this intersection. We restore the inequality (∗\(_3\)) and have to check that the following inequality is satisfied
\[
(\alpha + \beta)(k_1 + k_2 + k_3) + m_3 > (k_1 + 1)m_1 + (k_2 + 1)m_2 + k_3m_3
\]
\[
3k_1 + 4k_2 + 7k_3 > 8
\]
which is true.

Case (x): \(C \equiv (1, 1)\).

We consider a situation \(r = 2, m_1 = 2, m_2 = 1\). If needed, we restore one or two inequalities (∗\(_i\)) and prove (∗) for \(r = 3, m_1 = 2, m_2 = 1, m_3 = 1\) or for \(r = 4, m_1 = 2, m_2 = 1, m_3 = 1, m_4 = 1\). Here come the details:

- In Cases I and IIIa the inequality (∗) holds by Remark 4.14(2).
- In Cases IIa, IIIb, IV there are two possibilities:

• One of the points \(x_1, x_2\) (say \(x_2\)) lies respectively: on the fixed fibre \(A/2\) in Case IIa, on the intersection of the fixed fibres \(A\) and \(B\) in Case IIIb, on the fixed fibre \(B\) in Case IV. Then it suffices to prove that
\[
(\alpha + \beta)(k_1 + k_2) + \min\{\alpha, \beta\} > (k_1 + 1)m_1 + k_2m_2
\]
which gives
\[
k_2 > 1
\]
If $k_2 = 1$, then respectively: on the fixed fibre $A/2$, on the fixed fibre $B$ but outside the intersection of the fixed fibres $A$ and $B$ (otherwise $k = 1$, but we have excluded such a situation at the beginning of main theorem’s proof), on the fixed fibre $B$, there was originally at least one more point, as for points on the fibre $\sum k_i^W > \frac{k+1}{2}$. If $x_1$ is the described point, it suffices to prove that

$$(\alpha + \beta)(k_1 + k_2) + \min\{\alpha, \beta\} > k_1m_1 + k_2m_2$$

$$k_2 + 1 > 0$$

The inequality holds. If the described point is different from $x_1$, then it has multiplicity $m_3 = 1$. We have to check that

$$(\alpha + \beta)(k_1 + k_2 + k_3) + \min\{\alpha, \beta\} > (k_1 + 1)m_1 + k_2m_2 + k_3m_3$$

$$k_2 + k_3 > 1$$

The inequality holds.

- None of the points $x_1, x_2$ lies on the fixed fibre respectively $A/2$, $A$ intersected with $B$, and $B$. By assumption of Case respectively IIa, IIIb and IV, at the beginning there was a point $x_3$ on the fixed fibre or on the intersection of the fixed fibres, and the maximal possible by genus formula multiplicity $m_3$ is equal to 1. We restore the inequality $(\star_3)$, and we have to check that

$$(\alpha + \beta)(k_1 + k_2 + k_3) + \min\{\alpha, \beta\} > (k_1 + 1)m_1 + (k_2 + 1)m_2 + k_3m_3$$

$$k_2 + k_3 > 2$$

If $k_3 = 1$, then by assumption of Case IIa, IIIb, IV respectively on the fixed fibre $A/2$, on the fixed fibre $B$ but outside the intersection of $A$ and $B$, on the fixed fibre $B$, there is a point $x_4$. In Case IIIb $x_4$ may be equal to one of the points $x_1, x_2$ — we obtain an inequality which was already proved. Otherwise in Case IIIb, and in Cases IIa and IV $x_4$ is different from $x_1, x_2, x_3$ and $m_4 = 1$. We have to prove that

$$(\alpha + \beta)(k_1 + k_2 + k_3 + k_4) + m_3 > (k_1 + 1)m_1 + (k_2 + 1)m_2 + k_3m_3 + k_4m_4$$

$$k_2 + k_3 + k_4 > 2$$

The inequality holds.

- In Case IIb there are two possibilities as well:

  - One of the points $x_1, x_2$ (say $x_2$) is the intersection point of the fixed fibres $A/2$ and $B$. Then we have to prove that

    $$(\alpha + \beta)(k_1 + k_2) + m_2 > (k_1 + 1)m_1 + k_2m_2$$

    $$k_2 > 1$$

  If $k_2 = 1$, then by assumption of Case IIb there is at least one more point $x_3$ on the fixed fibre $A/2$. If $x_3 = x_1$ then it is enough to check that

    $$(\alpha + \beta)(k_1 + k_2) + m_2 > k_1m_1 + k_2m_2$$

    $$k_2 + 1 > 0$$

    The inequality holds. Otherwise $m_3 = 1$, and it suffices to show that

    $$(\alpha + \beta)(k_1 + k_2 + k_3) + m_2 > (k_1 + 1)m_1 + k_2m_2 + k_3m_3$$

    $$k_2 + k_3 > 1$$
The inequality holds.

• None of the points \( x_1, x_2 \) in the intersection point of the fixed fibres \( A/2 \) and \( B \). Then on this intersection there is a point \( x_3 \) with \( m_3 = 1 \). Hence it is enough to prove that

\[
(\alpha + \beta)(k_1 + k_2 + k_3) + m_3 > (k_1 + 1)m_1 + (k_2 + 1)m_2 + k_3m_3
\]

\[
k_2 + k_3 > 2
\]

If \( k_3 = 1 \) then there is one more point \( x_4 \) on the fixed fibre \( A/2 \). If \( x_4 \) is equal to \( x_1 \) or \( x_2 \) then we get the already proved inequality

\[
(\alpha + \beta)(k_1 + k_2 + k_3) + m_3 > (k_1 + 1)m_1 + k_2m_2 + k_3m_3
\]

If \( x_4 \) is different from \( x_1, x_2, x_3 \) then \( m_4 = 1 \), and we obtain the inequality

\[
(\alpha + \beta)(k_1 + k_2 + k_3 + k_4) + m_3 > (k_1 + 1)m_1 + (k_2 + 1)m_2 + k_3m_3 + k_4m_4
\]

which has been proved as well. \( \square \)

The proof of the Lemma 4.10 and Lemma 4.12 is completed, hence the main theorem has been proved. \( \square \)

We have presented the detailed proof for hyperelliptic surfaces of type 1. Below we list the small differences which occur in the proof for hyperelliptic surfaces of types 2-7.

**Remark 4.15.**

• For a hyperelliptic surface of even type there is no Case IIb nor IIIb, as the divisor \((\mu/\gamma)B \equiv (0, 1)\) is not effective on such a surfaces. Hence while proving that respectively \( M \tilde{C} \geq 0 \) and \( N \tilde{C} > 0 \), we do not consider the curve \( C \equiv (\mu/\gamma)B \).

• For a hyperelliptic surface of type 3, 4, 7 there are more types of singular fibres \( mA/\mu \). Hence in Cases I, IIa, IIIa, IV, and for a surfaces of type 3 and 7 also in Cases IIb and IIIb, we have to consider the intersection of \( C \) with singular fibres of all admissible types \( mA/\mu \), but they are estimated from below by the intersection with a fibre \( A/\mu \).

• For a hyperelliptic surface of type 3, 4 and 7 we consider additional cases — points \( x_1, \ldots, x_s \) lie on a fixed singular fibre \( mA/\mu \), where \( 1 < m \leq \frac{4}{2} \). These cases are analogous to Cases IIIa, IIIb.

Proof of Theorem 4.11 for \( r = 1 \) gives us the following:

**Corollary 4.16.** Let \( S \) be a hyperelliptic surface. Let \( L \) be a line bundle of type \((k+2, k+2)\) on \( S \). Then \( L \) generates \( k \)-jets at any point \( x \).

**Acknowledgments.** The author would like to thank Tomasz Szemberg and Halszka Tutaj-Gasińska for many helpful discussions and improving the readability of the paper.

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