Direct Data-Driven State-Feedback Control of Linear Parameter-Varying Systems*

Chris Verhoek¹, Roland Tóth¹,², and Hossam S. Abbas³

¹Control Systems group, Dept. of Electrical Engineering, Eindhoven University of Technology, Eindhoven, The Netherlands.
²Systems and Control Lab, Institute for Computer Science and Control, Budapest, Hungary.
³Institute for Electrical Engineering in Medicine, Universität zu Lübeck, Lübeck, Germany.

Monday 27th May, 2024

Abstract

The framework of linear parameter-varying (LPV) system has shown to be a powerful tool for the design of controllers for complex nonlinear systems using linear tools. In this work, we derive novel methods that allow to synthesize LPV state-feedback controllers directly from a single sequence of data and guarantee stability and performance of the closed-loop system, without knowing the model of the plant. We show that if the measured open-loop data from the system satisfies a persistency of excitation condition, then the full open-loop and closed-loop input-scheduling-state behavior can be represented using only the data. With this representation, we formulate synthesis problems that yield controllers that guarantee stability and performance in terms of infinite horizon quadratic cost, generalized $H_\infty$-norm and $\ell_2$-gain of the closed-loop system. The controllers are synthesized by solving an SDP with a finite set of LMI constraints. Additionally, we provide a synthesis method to handle noisy measurement data. Competitive performance of the proposed data-driven synthesis methods is demonstrated w.r.t. model-based synthesis that have complete knowledge of the true system model in multiple simulation studies, including a nonlinear unbalanced disc system.

Keywords: Data-Driven Control, Linear Parameter-Varying Systems, State-Feedback Control, Behavioral systems, $H_\infty$ control.

1 Introduction

Due to increasing performance, environmental, etc., requirements, control of new generations of engineering systems is increasingly more challenging, as the dynamic behaviors of these systems is becoming dominated by nonlinear (NL) effects. A particularly interesting framework to deal with such challenges is the class of linear parameter-varying (LPV) systems [1]. LPV systems have a linear input-(state)-output relationship, but this relationship itself is dependent on a measurable, time-varying signal, referred to as the scheduling signal. This scheduling signal is used to express the NL/time-varying/exogenous components that are affecting the system, allowing to describe a wide range of NL systems in terms of LPV surrogate models [1]. In practice, LPV modeling and model-based control methods have been successfully deployed in many engineering problems, proving versatility of the framework to meet with the increasing complexity challenges and performance expectations [2].

However, despite of the powerful LPV model-based control solutions, their deployment is becoming more challenging, as modeling of the next generation of engineering systems via first-principles is increasingly complex, time-consuming and often lacks sufficient accuracy. Hence, engineers frequently need to obtain models based on data to analyze and control such systems. Despite the tremendous progress in LPV and NL model identification, e.g., [1, 3, 4, 5, 6, 7, 8, 9], several aspects of the model identification toolchain such as model structure selection, are either under-developed or are rather complex and demand several iterations. Moreover, research on identification for control in the linear time-invariant (LTI) case has shown that, to synthesize a controller to achieve a given

*Corresponding author Chris Verhoek. c.verhoek@tue.nl
performance objective in model-based control, only some dynamical aspects of the system are important [10]. This means that focusing the identification process on obtaining only such information accurately for synthesis can achieve higher model-based control performance for a given experimentation budget. This lead to the idea that fusing the control objective into system identification and even accomplishing synthesis of a controller directly from data, i.e., direct data-driven control, carries many benefits [11], while it also simplifies the overall modeling and control toolchain. Based on this motivation, data-driven LPV control methods [12, 13, 14] have been introduced, which have provided promising solutions to the direct control design problem, however, often without guarantees on the stability and performance of the resulting closed-loop system.

In the LTI case, a cornerstone result, the so-called Fundamental Lemma [15], allows the design of direct data-driven controllers with closed-loop stability and performance guarantees. Using this lemma, numerous powerful results have been developed for LTI systems on data-driven simulation [16], stability and performance analysis [17, 18, 19], and (predictive) control [20, 21, 22, 23], many of which also provide robustness against noise. The extension of Willems’ Fundamental Lemma for the class of LPV systems has been derived in [24] and it has already been used for the development of direct LPV data-driven predictive control design [25] and direct data-driven dissipativity analysis of LPV systems [26]. In this work, we further extend this promising approach with the development of data-driven LPV state-feedback controller synthesis methods that can provide closed-loop stability and performance guarantees based on just a single (short) sequence of measurement data from the system. More specifically, our contributions in this paper are:

C1: We derive data-driven open- and closed-loop LPV representations from a single sequence of input-scheduling-state measurement data of an unknown LPV system that has a state-space representation with affine dependence;
C2: We develop fully data-based analysis and synthesis methods of LPV state-feedback controllers to guarantee closed-loop stability;
C3: We develop direct data-driven analysis and synthesis methods of LPV state-feedback controllers with guaranteed quadratic, $H^\infty$-norm and $\ell_2$-gain-based performance of the closed-loop;
C4: We extend the methods from Contribution C2 to work with noisy state measurements.
C5: We present an extensive set of simulation studies to demonstrate the capabilities of our methods.

The data-driven analysis and synthesis techniques that we present in this paper can be efficiently solved as a semi-definite program (SDP), subject to a finite set of linear matrix inequalities (LMIs), which are constructed using only the measurement data from the unknown LPV system.

The paper is structured as follows. The problem setting is introduced in Section 2, followed by the open- and closed-loop data-driven LPV representations in Section 3, providing C1. Based on the derived representations, the direct LPV data-driven controller synthesis methods are formulated in Section 4 and 5 giving C2 and C3. We extend the results of Section 4 to work with noise-infected data in Section 6, providing C4. We finish the paper with comparative simulation studies in Section 7, demonstrating competitive performance of the proposed methods as part of C5. Finally, the conclusions are drawn in Section 8. To support the clarity and readability of our results, we have added a review of the existing model-based state-feedback analysis and synthesis methods in the appendix.

Notation

$\mathbb{R}$ denotes the set of real numbers, while the set of all integers is denoted by $\mathbb{Z}$. The set of real symmetric $n \times n$ matrices is denoted as $\mathbb{S}^n$. For sets $A$ and $B$, $B^A$ indicates the collection of all maps from $A$ to $B$. The $n \times m$ zero matrix is denoted by $0_{n \times m}$, where in case $m = n$ we write $0_n$. The identity matrix is denoted by $I_n \in \mathbb{S}^n$, while $1_n$ denotes the vector $[1 \cdots 1]^\top \in \mathbb{R}^n$. $X > 0$ and $X < 0$ ($X \geq 0$ and $X \leq 0$) denote positive/negative (semi) definiteness of a symmetric matrix $X \in \mathbb{S}^n$, respectively. The Kronecker product of $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{p \times q}$ is denoted by $A \otimes B \in \mathbb{R}^{np \times mq}$, and for $C \in \mathbb{R}^{m \times k}$, $D \in \mathbb{R}^{q \times l}$, the following identity holds [27]:

$$ (A \otimes B)(C \otimes D) = AC \otimes BD. \tag{1} $$

The Redheffer star product is $\Delta \star L$, which for $\Delta \in \mathbb{R}^{n \times m}$ and $L \in \mathbb{R}^{p \times q}$ with $n < p$ and $m < q$ gives the upper linear fractional transformation (LFT). Furthermore, the Moore-Penrose (right) pseudo-inverse is denoted by $\dagger$. We use $(\cdot)$ to denote a symmetric term in a quadratic expression, e.g.,
\((s)^\top Q(a - b) = (a - b)^\top Q(a - b)\) for \(Q \in \mathbb{R}^{n \times n}\) and \(a, b \in \mathbb{R}^n\). Block diagonal concatenation of matrices is given by \(\text{blkdiag}\).

## 2 Problem setting

Consider a discrete-time LPV system that can be represented by the following LPV-SS representation:

\[
\begin{align*}
    x_{k+1} &= A(p_k)x_k + B(p_k)u_k, \\
    y_k &= x_k,
\end{align*}
\]

where \(k \in \mathbb{Z}\) is the discrete time, \(x_k \in \mathbb{R}^{n_x}\), \(u_k \in \mathbb{R}^{n_u}\), \(y_k \in \mathbb{R}^{n_y}\) and \(p_k \in \mathbb{P}\) are the state, input, output and scheduling signals, respectively, and \(\mathbb{P} \subset \mathbb{R}^{n_p}\) is a compact, convex set which defines the range of the scheduling signal. The matrix functions \(A : \mathbb{P} \rightarrow \mathbb{R}^{n_x \times n_x}\) and \(B : \mathbb{P} \rightarrow \mathbb{R}^{n_x \times n_u}\) are considered to have affine dependency on \(p_k\), which is a common assumption in practice, cf. \([28, 29]\).

\begin{align*}
    A(p_k) &= A_0 + \sum_{i=1}^{n_p} p_{k,i}A_i, \\
    B(p_k) &= B_0 + \sum_{i=1}^{n_p} p_{k,i}B_i,
\end{align*}

where \(\{A_i\}_{i=1}^{n_p}\) and \(\{B_i\}_{i=1}^{n_p}\) are real matrices with appropriate dimensions. The solutions of (2) are collected in the set

\[
\mathcal{B} = \{(x, p, u) \in (\mathbb{R}^{n_x} \times \mathbb{P} \times \mathbb{R}^{n_u})^\mathbb{Z} \mid (2) \text{ holds } \forall k \in \mathbb{Z}\},
\]

which we refer to as the behavior. To stabilize (2), we can design an LPV state-feedback controller \(K(p_k)\) corresponding to the control-law

\[
    u_k = K(p_k)x_k,
\]

where we choose the LPV state-feedback controller \(K : \mathbb{P} \rightarrow \mathbb{R}^{n_x \times n_u}\) to have affine dependence on \(p_k\)

\[
K(p_k) = K_0 + \sum_{i=1}^{n_p} p_{k,i}K_i,
\]

similar to (2). The well-known state-feedback problem is to design \(K(p_k)\) such that it ensures asymptotic (input-to-state) stability and minimizes a given performance measure (e.g., \(\ell_2\)-gain) of the closed-loop system

\[
x_{k+1} = A_{CL}(p_k)x_k = (A(p_k) + B(p_k)K(p_k))x_k,
\]

under all scheduling trajectories \(p \in \mathbb{P}^\mathbb{Z}\), where the solution set of (6) is \(\mathcal{B}_{CL} = \mathcal{B}_{|\mathcal{X}=K(p)x} \subseteq \mathcal{B}_{\mathbb{X}, p}\). For such a design of \(K(p_k)\), one will need the exact mathematical description of (2), which can be unrealistic in practical situations. In this paper, we consider the design of \(K(p_k)\) for an unknown LPV system (2), based only on the measured data-set

\[
\mathcal{D}_{N_d} = \{u^d_k, x^d_k, p^d_k\}_{k=1}^{N_d},
\]

which is a single trajectory from (2). This problem is formalized in the following problem statement.

### Problem statement

Consider a data-generating system that can be represented with (2). Given the data-dictionary \(\mathcal{D}_{N_d}\) sampled from the data-generating system. How to synthesize a state-feedback controller \(K(p_k)\), based on only \(\mathcal{D}_{N_d}\), that ensures stability and performance of the closed-loop (6)?

In order to solve this problem, we first need to construct data-driven representations of \(\mathcal{B}\) and \(\mathcal{B}_{CL}\), i.e., data-based realizations of (2) and (6), corresponding to Contribution C1.

## 3 Data-Driven LPV Representations

In this section, we derive novel data-driven LPV-SS representations that we will use for state-feedback controller design. These results can be seen as the state-space counterpart of data-driven LPV-IO representations used in \([24, 25]\). Based on the data-dictionary \(\mathcal{D}_{N_d}\), we construct data-driven representations of both the open-loop (2) and the closed-loop (6) systems, allowing later to derive controller synthesis algorithms using only the information in \(\mathcal{D}_{N_d}\).
3.1 Open-loop data-driven LPV representation

First note that by separating the coefficient matrices in (2) from the signals in (3), we can rewrite the state equation (2a) by introducing auxiliary signals \( p_k \otimes x_k \) and \( p_k \otimes u_k \):

\[
x_{k+1} = A \begin{bmatrix} x_k \\ p_k \otimes x_k \end{bmatrix} + B \begin{bmatrix} u_k \\ p_k \otimes u_k \end{bmatrix},
\]

where \( A = [A_0 \ A_1 \ \cdots \ A_{n_p}] \), \( B = [B_0 \ B_1 \ \cdots \ B_{n_p}] \). Then, given the measured data-dictionary \( D_{N_d} \) from (2), the construction of the following matrices

\[
U = [u_1^d \ \cdots \ u_{N_d-1}^d], \quad U^p = [p_1^d \otimes u_1^d \ \cdots \ p_{N_d-1}^d \otimes u_{N_d-1}^d],
\]

\[
X = [x_1^d \ \cdots \ x_{N_d-1}^d], \quad X^p = [p_1^d \otimes x_1^d \ \cdots \ p_{N_d-1}^d \otimes x_{N_d-1}^d], \quad X_+ = [x_1^d \ \cdots \ x_{N_d}^d],
\]

where \( U^p \in \mathbb{R}^{n_p \times n_d}, U \in \mathbb{R}^{n_x \times n_d}, X^p \in \mathbb{R}^{n_p \times n_d \times n_d} \) and \( X, X_+ \in \mathbb{R}^{n_x \times n_d} \), allows to write the relationship between the system parameters \( A, B \) and the data in \( D_{N_d} \) as

\[
X_+ = A \begin{bmatrix} X \\ X^p \end{bmatrix} + B \begin{bmatrix} U \\ U^p \end{bmatrix}.
\]

Note that (10) holds true due to the linearity property of LPV systems along a given \( p \) trajectory. This is a well-known fact and often used in LPV subspace identification, cf. [30]. Instead of estimating the state equation (2a) by introducing auxiliary signals \( p_k \otimes x_k \) and \( p_k \otimes u_k \), we use the data in (11) as a representation, avoiding any loss of information.

Next, we derive a data-driven representation of the closed-loop system (6), where (2) with (3) is directly as a representation, avoiding any loss of information.

The data-based representation (11) is well-posed under the condition that the data-set \( D_{N_d} \) is persistently exciting (PE). Note that the PE condition is always defined with respect to a certain system class/representation, which is in this case a static-affine LPV-SS representation with full state observation. Contrary to the PE condition for shifted-affine LPV-IO representations [24], the PE condition for the system class considered in this paper is simpler, and corresponds to the existence of the right pseudo-inverse \( G^\dagger \) of \( G \), giving the following PE condition for \( D_{N_d} \) to imply existence of (11).

**Condition 1** (Persistency of Excitation). If \( G \) has full row rank, i.e., \( \text{rank}(G) = (1 + n_p)(n_x + n_u) \), then \( D_{N_d} \) is persistently exciting w.r.t. (2) and (3).

Based on this condition, we need at least \( N_d \geq (1 + n_p)(n_x + n_u) \) data points for (11). Moreover, as we assume full state observation, the parameters \( n_x, n_p, n_u \) are also known and thus no additional conditions are required to ensure existence of (11).

3.2 Closed-loop data-driven LPV representation

Next, we derive a data-driven representation of the closed-loop system (6), where (2) with (3) is controlled via the feedback law (5), where the controller has static-affine dependence, as defined in [50]. The following theorem gives conditions such that we can represent (6) using only the data-set \( D_{N_d} \), i.e., this result provides a data-based parameterization of the closed-loop system with the state-feedback law (5).

**Theorem 1** (Data-based closed-loop representation). Given the data-dictionary \( D_{N_d} = \{u_k, p_k, x_k \}^N_{k=1} \), measured from (2) and satisfying Condition 1, let \( X_+ \) and \( G \) be defined as in (9) and (11), under \( D_{N_d} \). For an LPV controller \( K \) given by (5), the closed-loop system (6) is represented equivalently as

\[
x_{k+1} = X_+ V \begin{bmatrix} x_k \\ p_k \otimes x_k \end{bmatrix},
\]

where \( V = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \).
where $V \in \mathbb{R}^{N_n \times (1+n_p+n_p^2)}$ is any matrix that satisfies

$$
\mathcal{M}_{CL} := \begin{bmatrix}
I_{n_x} & 0 & 0 \\
0 & I_{n_p} \otimes I_{n_x} & 0 \\
K_0 & I_{n_p} \otimes K_0 & I_{n_p} \otimes K
\end{bmatrix} = \begin{bmatrix}
X_P \\
U_P \\
U
\end{bmatrix} V,
$$

(13)

**Proof.** The satisfaction of Condition 1 provides via the Rouché-Capelli theorem that there always exists a $V$ such that (13) holds. Now, let us write the controller (5) as

$$
u_k = K \begin{bmatrix} x_k \\
p_k \otimes x_k \end{bmatrix}, \quad K = \begin{bmatrix} K_0 & K_1 & \cdots & K_{n_p} \end{bmatrix},
$$

(14)

such that, when substituted in (3), we obtain the closed-loop

$$x_{k+1} = \begin{bmatrix} A_0 & B_0 K_0 \end{bmatrix} x_k + \begin{bmatrix} B_0 \end{bmatrix} \begin{bmatrix} p_k \otimes x_k \end{bmatrix} + \begin{bmatrix} \bar{B} \end{bmatrix} \cdot \begin{bmatrix} p_k \otimes (K_0 x_k + \bar{K} p_k \otimes x_k) \end{bmatrix},
$$

(15)

where $\bar{A} = \begin{bmatrix} A_1 & \cdots & A_{n_p} \end{bmatrix}$, similarly for $\bar{B}, \bar{K}$. Using the Kronecker property (1), we can rewrite (15) as

$$x_{k+1} = \begin{bmatrix} A_0 + B_0 K_0 & A + B_0 \bar{K} + B(I_{n_p} \otimes K_0) & B(I_{n_p} \otimes \bar{K}) \end{bmatrix} \begin{bmatrix} x_k \\
p_k \otimes x_k \end{bmatrix},
$$

(16)

where $\mathcal{M}$ now fully defines the relationship between the signals in the closed-loop LPV system. $\mathcal{M}$ can be rewritten as

$$\mathcal{M} = [\mathcal{A} \quad \mathcal{B}] \mathcal{M}_{CL} \begin{bmatrix} \mathcal{X} \\
\mathcal{U} \\
\mathcal{U}_P \end{bmatrix} \mathcal{V}.
$$

(17)

Substituting the relation $[\mathcal{A} \quad \mathcal{B}] = \mathcal{X}_+ \mathcal{G}^\dagger$ and (13) into (17) yields $\mathcal{M} = \mathcal{X}_+ \mathcal{V}$, which, through (13), gives the data-based closed-loop representation (12), equivalent to the model-based closed-loop representation (6).

Note that, under Condition 1, $\mathcal{V} = \mathcal{G}^\dagger \mathcal{M}_{CL}$ would be only a particular (minimum 2-norm solution) of (13), while Theorem 1 allows for an affine subspace of solutions in terms of $\mathcal{V}$ in the orthogonal projection of $\mathcal{M}_{CL}$ on the range of $\mathcal{G}$. Furthermore, for a given $\mathcal{V}$, condition (13) allows to recover the controller $K(p_k)$. To this end, let us introduce a partitioning of $\mathcal{V} := [V_0 \quad \mathcal{V} \quad \bar{V}]$, with

$$\mathcal{V} = \begin{bmatrix} V_1 & \cdots & V_{n_p} \end{bmatrix}, \quad \bar{V} = \begin{bmatrix} V_{n_p+1} & \cdots & V_{n_p} \end{bmatrix},$$

such that, via (13), we have

$$K_0 = U V_0, \quad \bar{K} = U \bar{V}.$$

Hence, the control law can be recovered, and is fully defined in terms of the data $\mathcal{D}_{N_n}$ as

$$u_k = U \begin{bmatrix} V_0 & \bar{V} \end{bmatrix} \begin{bmatrix} x_k \\
p_k \otimes x_k \end{bmatrix}.$$

Note that, in order to represent the control law, the term $\bar{V}$ is redundant and only required to fulfill the data relations in (13). We will now derive synthesis algorithms that allows us to find $\mathcal{V}$, and thus synthesize LPV state-feedback controllers using only the information in $\mathcal{D}_{N_n}$.

### 4 Data-driven synthesis of stabilizing LPV controllers

Based on the data-driven representation of the closed-loop system, developed in Section 3.2, we will derive data-driven controller synthesis methods that can be solved as an SDP. We show that the controllers synthesized by these methods ensure closed-loop stability and can achieve a wide range of performance targets. We first solve the direct data-driven stability analysis problem, from which the synthesis methods for stabilizing LPV state-feedback control can be derived via an extension of Lemma 2. This is followed by the derivation of the direct data-driven methods that allow to design LPV state-feedback controllers with different performance targets, which is discussed in Section 5.
4.1 Closed-loop data-driven stability analysis

We first solve the direct data-driven stability analysis problem. Quadratic asymptotic stability of the LPV system \( \mathbb{P} \), i.e., boundedness and convergence of the state-trajectories to the origin under \( u \equiv 0 \), is implied with the existence of a Lyapunov function \( V(x) = x^TP^{-1}x > 0 \) \( \forall x \in \mathbb{R}^{n_x} \setminus \{0\} \), with \( P^{-1} \in \mathbb{S}^{n_x} \) that satisfies \( V(x_{k+1}) - V(x_k) < 0 \) under all \( (x,p,0) \in \mathcal{M} \). Working this out for a given state-feedback controller \( K(p_k) \) as in \([5]\), we obtain the well-known condition to analyze closed-loop stability in a model-based sense, see, e.g., Lemma \([2]\) or \([31]\). That is, the closed-loop LPV system \([4]\) is quadratically, asymptotically stable if

\[
\begin{bmatrix}
    P & PM^T(p) \\
    M(p)P & P
\end{bmatrix} \succ 0, \quad M(p) = M \begin{bmatrix}
     I_{n_x} \\
     p \otimes I_{n_x} \\
     p \otimes p \otimes I_{n_x}
\end{bmatrix},
\]

for all \( p \in \mathbb{P} \). This allows us to use the closed-loop representation in \([3]\) to derive the following theorem, which provides a computable method to analyze the stability of the unknown LPV system in closed-loop with \( K(p_k) \) in a fully data-driven setting.

**Theorem 2** (Data-driven feedback stability analysis). Given a data-set \( \mathcal{D}_{N_k} \), satisfying Condition \([7]\) from a system that can be represented by \([2]\). For a \( P \in \mathbb{S}^{n_x} \), let \( F \in \mathbb{R}^{N_x-1 \times n_x(1+n_p+n_x^2)} \) and \( F_Q \in \mathbb{R}^{(N_x-1)(1+n_p+n_x^2)} \) be defined such that

\[
F(p) := V \begin{bmatrix}
    I_{n_x} \\
    p \otimes I_{n_x} \\
    p \otimes p \otimes I_{n_x}
\end{bmatrix} = F \begin{bmatrix}
    I_{n_x} \\
    p \otimes I_{n_x} \\
    p \otimes p \otimes I_{n_x}
\end{bmatrix} = \begin{bmatrix}
    I_{N_x-1} \\
    p \otimes I_{N_x-1}
\end{bmatrix},
\]

Then the LPV state-feedback controller \( K(p) \) stabilizes \([2]\) under the feedback-law \([5]\), if there exists an \( F_Q \), a \( P > 0 \) and a multiplier \( \Xi \in \mathbb{S}^{4n_p,n_x} \), which satisfy

\[
\mathcal{M}_{CL} \begin{bmatrix}
    P & 0 & 0 \\
    0 & I_{n_p} \otimes P & 0 \\
    0 & 0 & I_{n_p} \otimes I_{n_p} \otimes P
\end{bmatrix} = \mathcal{G} F,
\]

and the LMI conditions in \([61]\), with \( \Delta(p) = \text{diag}(p) \otimes I_{2n_x} \), and where \( W \) and \( L_{11}, \ldots, L_{22} \) are given by:

\[
\begin{align*}
W &= \begin{bmatrix}
    P_0 \quad \mathcal{X}_+ F_Q \\
    (\mathcal{X}_+ F_Q)\quad P_0
\end{bmatrix}, \quad L_{11} = \begin{bmatrix}
    0_{2n_x,n_p} \\
    0_{n_p \times 2n_x,n_p}
\end{bmatrix}, \quad L_{12} = \begin{bmatrix}
    1_{n_p} \otimes I_{2n_x} \\
    1_{n_p} \otimes 0 \otimes I_{2n_x}
\end{bmatrix}, \\
\mathcal{X}_+ &= \text{blkdiag}(X_+, I_{n_p} \otimes X_+), \quad \mathcal{X}_+ = \text{blkdiag}(P, 0_{n_p,n_x}), \quad L_{21} = \begin{bmatrix}
    I_{n_p} \otimes [I_{n_x} \quad 0] \\
    0_{n_x \times 2n_x,n_p}
\end{bmatrix}, \quad L_{22} = \begin{bmatrix}
    1_{n_p} \otimes I_{n_x} \\
    0 \quad 1_{n_p} \otimes 0 \otimes I_{n_x}
\end{bmatrix}.
\end{align*}
\]

**Proof.** Substituting the relation \([17]\) of the data-based closed-loop representation into \([18]\) results in

\[
\begin{bmatrix}
    P \quad PV^T(p)X^T_+ \\
    X^+_V(p)P \quad P
\end{bmatrix} \succ 0, \quad \forall p \in \mathbb{P}, \quad \text{where } V(p) = V \begin{bmatrix}
    I_{n_x} \\
    p \otimes I_{n_x} \\
    p \otimes p \otimes I_{n_x}
\end{bmatrix}
\]

and \( V \) is restricted by \([13]\), corresponding to \( \mathcal{M}_{CL} = \mathcal{G} V \). To this end, let us introduce the matrix function \( F(p) := V(p)P \), resembling \([19]\). Substituting \( F(p) \) in \([22]\) results in

\[
\begin{bmatrix}
    P \quad (X_+ F(p))^T \\
    X_+ F(p) \quad P
\end{bmatrix} \succ 0, \quad \forall p \in \mathbb{P},
\]

while the substitution of \( F(p) \) in \([13]\) yields

\[
\mathcal{M}_{CL} \begin{bmatrix}
    I_{n_x} \\
    p \otimes I_{n_x} \\
    p \otimes p \otimes I_{n_x}
\end{bmatrix} = \mathcal{G} F(p),
\]

\footnote{Choosing a parameter-varying Lyapunov function can reduce conservatism of the analysis, but causes \([13]\) to have 3rd-order polynomial dependency on \( p \), making it difficult to arrive to an SDP form of the analysis and synthesis problems. See \([22]\) for a possible extension.}
which couples $P$ with condition (13). Note that (23) is not an LMI, due to the quadratic dependence of $F$ on $p$. Using Kronecker property (1) we can simplify (24) as

$$
\mathcal{M}_{\text{CL}} \begin{bmatrix}
P & 0 & 0 \\
0 & I_{n_p} \otimes P & 0 \\
0 & 0 & I_{n_p} \otimes I_{n_p} \otimes P
\end{bmatrix} \begin{bmatrix}
I_{n_x} \\
p_k \otimes I_{n_x} \\
p_k \otimes p_k \otimes I_{n_x}
\end{bmatrix} = \mathcal{G}F(p_k).
$$

(25)

This immediately reveals the required scheduling dependency of $F(p_k)$ that is inherited from the left-hand side, cf. (19), which allows us to further simplify to (20). Now we can employ Lemma 4 in the Appendix to formulate the conditions as a finite number of LMI constraints.

Consider the definition of $F_Q$ in (19). With the application of Kronecker property (1) twice on $X_+ F(p_k)$, we can write it as

$$
X_+ F(p) = \begin{bmatrix}
I_{n_x} \\
p_k \otimes I_{n_x}
\end{bmatrix}^\top X_+ F_Q \begin{bmatrix}
I_{n_x} \\
p_k \otimes I_{n_x}
\end{bmatrix},
$$

(26)

which is a quadratic form. This allows to write (23) as (50) with $W$ as in (21) and $L(p) = \text{blkdiag} \left( \begin{bmatrix} I_{n_x} \\
p_k \otimes I_{n_x} \end{bmatrix}, \begin{bmatrix} I_{n_x} \\
p_k \otimes I_{n_x} \end{bmatrix} \right)$. Thus, by representing $L(p)$ as the LFT $L(p) = \Delta(p) \ast \bar{L} = L_{22} + L_{21}(I - L_{11} \Delta(p))^{-1} L_{12}$ with $L_{11}, \ldots, L_{22}$ as in (21), we obtain the LMI conditions of Theorem 2 that are linear and multi-convex in $p$, which concludes the proof.

If we assume that $\mathbb{P}$ is a polytopic scheduling set and have the additional requirement that $\Xi_{22} < 0$, multi-convexity of (61) allows to equally represent these constraints by a finite set of LMIs, specified at the vertices of $\mathbb{P}$; see also [33, 34].

**Remark 1.** Regarding the proof of Theorem 3, the following remarks are important to mention:

i) In case of a given control law $K(p_k)$, the matrix $\mathcal{M}_{\text{CL}}$ is fixed, because $X_+$ and $\mathcal{G}$ are fully determined by $\mathcal{D}_{N_d}$. Hence, Lyapunov stability analysis of the closed-loop system results in finding a $V$ and a positive definite $P$ that satisfy (13) and (22) for all $p \in \mathbb{P}$, providing a fully data-driven method to analyze whether the controller $K(p_k)$ stabilizes an unknown LPV system using only the measurements in $\mathcal{D}_{N_d}$.

ii) Condition (24) is crucial for selecting the scheduling dependency structure of both $F$ and $P$. In particular, choosing the scheduling dependence of $F$ such that it matches with the cumulative dependence of the left-hand side, allows to drop the $p_k$ terms in (24) and formulate the condition on the level of the involved matrices only, making the relation scheduling independent. This significantly reduces the complexity of the synthesis condition that is derived based on (23) and (24).

iii) Note that when partitioning $F$ and $F_Q$ in (19) as $\begin{bmatrix} F_0 & F & \bar{F} \end{bmatrix}$ and $\begin{bmatrix} F_{11} & F_{12} \\
F_{21} & F_{22} \end{bmatrix}$, the matrix $F_Q$ is based on just the re-shuffling of the terms of $F$, i.e., $F_{11} \in \mathbb{R}^{(N_d-1) n_p \times n_x}$ results from the scheduling independent term $F_0$, matrices $F_{12} \in \mathbb{R}^{(N_d-1) n_p \times n_p}$ and $F_{21} \in \mathbb{R}^{(N_d-1) n_p \times n_x}$ result from $F$, while $F_{22} \in \mathbb{R}^{(N_d-1)n_p \times n_p}$ is based on $\bar{F}$.

### 4.2 Stabilizing controller synthesis

We have now obtained a set of fully data-based linear constraints that can be solved as an SDP, allowing to analyze closed-loop stability of the feedback interconnection of a given LPV controller with an unknown LPV system. In this section, we further extend this result by deriving linear constraints for data-driven synthesis of a stabilizing LPV controller. Note that in this case, the controller is a decision variable, which yields the linear conditions in Theorem 2 nonlinear. The following result recasts this problem, which allows to *synthesize* the LPV controller using a set of linear constraints.

**Theorem 3** (Data-driven stabilizing feedback synthesis). Given a data-set $\mathcal{D}_{N_d}$, satisfying Condition 7 from a system that can be represented by (2). For a $P \in \mathbb{S}^{n_x}$, let the matrices $F \in \mathbb{R}^{(N_d-1) n_p \times (1+n_p+n_p)}$ and $F_Q \in \mathbb{R}^{(N_d-1)(1+n_p) \times n_x(1+n_p)}$ be defined as in (19). If there exist a $P > 0$ and $F_Q$, $Y_0 \in \mathbb{R}^{n_p \times n_p}$, $\bar{Y} \in \mathbb{R}^{n_p \times n_x}$, $\Xi \in \mathbb{S}^{n_p \times n_p}$ that satisfy

$$
\begin{bmatrix}
P & 0 & 0 \\
0 & I_{n_p} \otimes P & 0 \\
Y_0 & \bar{Y} & 0 \\
0 & I_{n_p} \otimes Y_0 & I_{n_p} \otimes \bar{Y}
\end{bmatrix} = \mathcal{G}F,
$$

(27)
and the LMI conditions in (61) in the Appendix with \( \Delta(p), W, L_{11}, \ldots, L_{22} \) as given in (21), then

\[
K_0 = Y_0 P^{-1}, \quad \hat{K} = \hat{Y} (I_{n_p} \otimes P)^{-1},
\]

(28)
gives an LPV state-feedback controller \( K(p) \) in terms of (50) that guarantees stability of the closed-loop interconnection (6).

**Proof.** The proof of this result is built on the proof of Theorem 2. Note that there are multiplications of decision variables in the constraint (20). Substituting

\[
M \rightarrow \frac{1}{2} X + \frac{1}{2} X^\top = \Xi_2
\]

(22) is infeasible, and recasting (20) as (27) gives that, if there exist \( Y_0 \in \mathbb{R}^{n_u \times n_x}, \hat{Y} \in \mathbb{R}^{n_u \times n_u} \) that satisfy (27), then extracting a controller realizaton in terms of (28) gives an \( \mathcal{M}_{\text{CL}} \) that satisfies (20).

Under a convex, polytopic \( P \) and additionally having \( \Xi_2 < 0 \), the conditions of Theorem 3 are convex in \( p \) and the decision variables. Then, this result provides a set of LMI conditions for an SDP which can easily be solved using off-the-shelf methods. Hence, we obtained easily verifiable conditions for the direct design of a stabilizing LPV state-feedback controller from only a single data sequence \( \mathcal{D}_{N_a} \), collected from the unknown LPV system, provided that \( \mathcal{D}_{N_a} \) satisfies Condition 1. The next section presents extensions of this synthesis approach by incorporating performance measures.

## 5 Data-driven control synthesis with performance objectives

### 5.1 Quadratic performance-based synthesis

We first derive a data-based controller synthesis algorithm that minimizes the infinite horizon quadratic cost

\[
J(x, u) = \sum_{k=0}^\infty x_k^\top Q x_k + u_k^\top R u_k, \quad Q \succeq 0, R > 0.
\]

(30)
The known conditions for model-based LPV controller synthesis to achieve quadratic performance, see, e.g., [31] and Lemma 3, can be reformulated in a fully data-based setting. By using the data-driven closed-loop representation derived in Section 3.2, we can replace the term \( A_{\text{CL}}(p) P \) in (63a) by \( X_a F(p) \) with \( F(p) \) as in [19]. The following result provides LMI conditions for the data-driven synthesis of an LPV controller that minimizes (30).

**Theorem 4** (Data-driven quadratic performance optimal synthesis). Given a data-set \( \mathcal{D}_{N_a} \), satisfying Condition 1 from a system that can be represented by \( F \). For a \( P \in \mathbb{S}^{n_x} \), let the matrices \( F \in \mathbb{R}^{n_a \times n_u (1+n_p \times n_u)} \) and \( F_Q \in \mathbb{R}^{(N_a-1)(1+n_p \times n_u)} \) be defined as in (19). Let \( P \) be the minimizer of \( \sup_{P \in \mathbb{S}^{n_x}} \text{trace}(P) \) among all possible choices of \( P > 0, \Xi \in \mathbb{S}^{n_u \times n_u}, \quad F_Q, \text{and } Y \), such that, for all \( p \in P \), both (27) and (61) are satisfied, where \( \Delta(p) = \text{diag}(p) \otimes I_{2n_a} \), and

\[
W = \begin{bmatrix}
P_0 & (X_a F_Q)\top & [PQ^{1/2}P]^{\top} & (R^{1/2}Y)\top \\
X_a F_Q & P_0 & 0 & 0 \\
[PQ^{1/2}P] & 0 & I_{n_x} & 0 \\
(R^{1/2}Y) & 0 & 0 & I_{n_u}
\end{bmatrix},
\]

(31a)

\[
L_{11} = 0_{2n_a n_x}, \quad L_{12} = 0_{2n_a n_x}, \quad L_{21} = 0_{2n_x n_u}, \quad L_{22} = 0_{2n_x n_u}
\]

(31b)

Then, the LPV state-feedback controller \( K(p) \) as in (5) with gains (28) is a stabilizing controller for (2) and achieves the minimum of \( \sup_{(x,p) \in \mathcal{G}_{\text{CL}}} J(x, u = K(p)x) \).
Proof. Similar to the proof of Theorem 3, first, the matrix inequality (55b) of Lemma 2 is converted into a data-based counterpart using $X_{p} P(p)$, thanks to (24). Then we can rewrite the resulting data-based matrix inequality in the quadratic form (59) with (61) and $L(p) = \text{blkdiag} \left( \left[ I_{n_x} \right], \left[ I_{n_x} \right], I_{n_x}, I_{n_u} \right)$. With this formulation, the $S$-procedure in Lemma 1 can be readily applied to derive the required LMI synthesis conditions, multi-convex in $p$, and $P > 0$.

Note again that if $\Xi_{22} < 0$ and $\mathbb{P}$ is a convex polytopic set, due to multi-convexity, the LMI conditions in Theorem 4 are only required to be solved at the vertices of $\mathbb{P}$.

5.2 $H_2^2$-norm performance-based synthesis

To consider induced gains-based performance metrics, e.g., $H_2$ or $\ell_2$ performance, we will introduce a representation of general controller configurations in terms of the closed-loop generalized plant concept, see (32) in Appendix B.3 for a detailed derivation. Consider the generalized disturbance signal $w_k \in \mathbb{R}^{n_x}$, the generalized performance signal $z_k \in \mathbb{R}^{n_x+n_u}$, a state-feedback LPV controller $K(p)$ as in (5) and the tuning matrices $Q, R$ as in (30). The closed-loop generalized plant is now given by

$$x_{k+1} = (A(p_k) + B(p_k)K(p_k))x_k + w_k,$$

$$z_k = \begin{bmatrix} Q & 0 \\ R & K(p_k) \end{bmatrix} x_k.$$

We can characterize the performance of $w \rightarrow z$ in terms of the so-called $H_2^2$-norm $\| \Sigma \|_{H_2^2}$ which for an exponentially stable (32) is defined as

$$\| \Sigma \|_{H_2^2} := \left( \limsup_{N \rightarrow \infty} \mathbb{E} \left\{ \frac{1}{N} \sum_{k=0}^{N} z_k^T z_k \right\} \right)^{\frac{1}{2}},$$

when $w_k$ is a white noise signal. Here $\mathbb{E}$ denotes the expectation w.r.t. $w$. We can now formulate the data-based analog of Lemma 4 [33], Lem. 1, which synthesizes a controller that guarantees a bound $\gamma$ on the $H_2^2$-norm and closed-loop stability of (32).

**Theorem 5** (Data-driven $H_2^2$-norm performance synthesis). Given a performance objective $\gamma > 0$ and a data-set $\mathcal{D}_{N_d}$, satisfying Condition 2, from the system represented by (2). If there exist matrices $P \in \mathbb{S}^{n_x}$ and $S \in \mathbb{S}^{n_u}$, $\Xi \in \mathbb{S}^{n_x+n_u}$, $F_Q$ as in (19), and $\mathcal{Y} = [Y_0 \ Y]$, such that (33) and

$$\begin{bmatrix} S & R \mathcal{Y} I_{n_u} \\ (\mathcal{Y})^T I_{n_x} \end{bmatrix} > 0, \quad P - I_{n_x} > 0, \quad \text{trace}(QP) + \text{trace}(S) < \gamma^2,$$

are satisfied for all $p \in \mathbb{P}$, where in (61),

$$W = \begin{bmatrix} P_0 - I_{n_x} & X_+ F_Q \\ (X_+ F_Q)^T & P_0 \end{bmatrix}, \quad I_{n_x} = \text{blkdiag}(I_{n_x}, 0),$$

and $\Delta(p), L_{11}, \ldots, L_{22}$ are as in (21), then, the LPV state-feedback controller $K(p)$ in (5) with gains (28) is a stabilizing controller for (2) and achieves an $H_2^2$-norm of the closed-loop system (32) that is less than $\gamma$.

**Proof.** The proof follows the same lines as for Theorem 3 and Theorem 4. The quadratic form (59) is obtained with $W$ as in (35) and $L(p) = \text{blkdiag} \left( \left[ I_{n_x} \right], \left[ I_{n_x} \right], I_{n_x}, I_{n_u} \right)$, as in the proof of Theorem 3.


\footnotetext[2]{This is a well-known LMI condition for quadratic performance-based LPV state-feedback controller synthesis, see, e.g., [31].}

\footnotetext[3]{This is an LPV-extension of the classical $H_2$-norm for LTI systems. There are multiple formulations, but we take here the definition from [32].}
5.3 $\ell_2$-gain performance-based synthesis

Another widely used performance metric is the (induced) $\ell_2$-gain of a system, which is defined as the infimum of $\gamma > 0$ such that for all trajectories in $\mathcal{B}$, with $x_0 = 0$, we have $\|z\|_2 \leq \gamma \|w\|_2$, where $\| \cdot \|_2$ denotes the $\ell_2$-norm of a signal, cf. [36]. Following the well-known result in [37] on $\ell_2$-gain LPV state-feedback synthesis, see also Lemma 5 we now formulate a fully data-based method to synthesize a state-feedback controller $K(p)$ that guarantees stability and an $\ell_2$-gain bound $\gamma$ for the closed-loop system (52).

**Theorem 6** (Data-driven $\ell_2$-gain performance synthesis). Given a performance objective $\gamma > 0$ and a data-set $\mathcal{D}_{N_s}$, satisfying Condition 4 from the system represented by (2). If there exist matrices $P \in \mathbb{S}^{n_p}$ with $P > 0$ and $\Xi \in \mathbb{S}^{n_p \times n_{\xi}}$, $F_Q$ as in (21), and $Y = \begin{bmatrix} Y_0 & Y \end{bmatrix}$, such that (27) and (61) are satisfied for all $p \in \mathbb{P}$, where in (61) $\Delta(p) = \text{diag}(p) \otimes I_{2n_x}$, and

$$ W = \begin{bmatrix} P_0 & (X_+ F_Q) \top & (Q \frac{1}{2} P) \top & (R \frac{1}{2} Y) \top & \gamma I_{n_x} \top & 0 \end{bmatrix}, $$

$$ L_{11} = \begin{bmatrix} 0_{2n_x n_p}, & 0 \end{bmatrix}, \quad L_{12} = \begin{bmatrix} 1_{n_p} \otimes I_{2n_x} & 0_{2n_x n_p \times 2n_x + n_n} \end{bmatrix}, $$

$$ L_{21} = \begin{bmatrix} I_{n_p} \otimes 0_{n_x \times 2n_x n_p} \end{bmatrix}, \quad L_{22} = \begin{bmatrix} I_{n_x} & 0_{n_x \times 2n_x + n_n} \end{bmatrix}, $$

then the LPV state-feedback controller $K(p)$ as in (5) with gains (28) is a stabilizing controller for (2) and achieves an $\ell_2$-gain of the closed-loop system (32) that is less than $\gamma$.

**Proof.** The proof follows the same lines as for Theorem 5 where the quadratic form (59) is obtained with $W$ and $L(p) := \text{blkdiag}(\begin{bmatrix} I_{n_x} & I_{n_x} & I_{n_x} \end{bmatrix} \otimes I_{n_x} \otimes I_{n_x})$, from which $L_{11}, \ldots, L_{22}$ in (36) are derived.

With $\Xi_{22} < 0$ and $\mathbb{P}$ being a convex polytope, we can again use the argument of multi-convexity such that the LMI conditions in Theorem 5 are required to be solved only at the vertices of the scheduling set $\mathbb{P}$.

**Remark 2.** A common practice in $\mathcal{H}_2^\infty$-norm/$\ell_2$-gain-based synthesis algorithms is to generate controllers that minimize $\gamma$ subject to the constraints. The results in Theorem 5 and Theorem 6 allow for formulating the SDPs with the minimization of $\gamma$ as well, as $\gamma$ appears linearly in the conditions.

**Remark 3.** Our results consider analysis and control synthesis problems where $A, B, K$ are affinely dependent on $p$. This structural dependency in the closed-loop system is clearly revealed in our derivations for the synthesis methods by choosing the partitioning of $F_Q$ as in Remark 4 (iii). Hence, $F_{11}$ represents the terms that are independent of the scheduling, $F_{12}, F_{21}$ represent the affine dependence on $p$ in the closed-loop and $F_{22}$ represents the polynomial dependence, emerging from the multiplication of $B(p)$ and $K(p)$. This clear distinction allows us to have control over the dependency of $K$. More specifically, if we enforce $F_{22} = 0$, the synthesized controller is scheduling independent, which results in the synthesis of a robust state-feedback controller.

6 Handling noisy measurement data

In this section, we consider the situation where the measurements in $\mathcal{D}_{N_s}$ are infected by measurement noise. We want to highlight that there are direct data-driven LPV control methods available in the literature that can handle noise, e.g., [13, 38]. These methods, however, consider the problem of finding a controller that stabilize all the systems that could have generated $\mathcal{D}_{N_s}$, reminiscent to a set-membership approach. The method we present in this section is focused on measurement noise, which can be seen as the LPV extension of [13, Sec. V.A]. Hence, compared to [13, 38] the disturbance is not affecting the system dynamics. Moreover, the synthesis method we present does not consider a particular bound or statistical property on the noise itself, only a signal-to-noise ratio (SNR) like condition with respect to the data-dictionary.
6.1 Setting

Consider the system (2), but now suppose that our state observations are corrupted by a noise term \( \epsilon \), i.e.,

\[
 z_k = x_k + \epsilon_k. 
\] (37)

The goal is now to design a stabilizing controller using the measurements of \( z_k \), i.e., our data-dictionary is given as

\[
 \mathcal{D}_N^\epsilon := \{ u_k^d, p_k^d, z_k^d \}_{k=1}^{N_d}. 
\] (38)

Let us collect the noise samples associated to the measurements in \( \mathcal{D}_N^\epsilon \) in the matrices

\[
 E := [\epsilon_1^d \ldots \epsilon_{N_d-1}^d], \quad E_+ := [\epsilon_2^d \ldots \epsilon_{N_d}^d],
\] (39)

such that the noisy state-observations in \( \mathcal{D}_N^\epsilon \) are collected as

\[
 Z := X + E, \quad Z_+ := X_+ + E_+.
\] (40)

Similar to (9), we define \( Z^p \) and \( E^p \) as \([p_1^d \otimes z_1^d \ldots p_{N_d-1}^d \otimes z_{N_d-1}^d]\) and \([p_1^d \otimes \epsilon_1^d \ldots p_{N_d-1}^d \otimes \epsilon_{N_d-1}^d]\), respectively. Note that \( Z^p = X^p + E^p \). We again consider a persistence of excitation-like condition, similar to Condition 1. However, now we also require that ‘the one-step-ahead signal relationship’ is not lost due to the noise. Therefore, we consider the following assumptions

**Assumption 1.** Both the matrices \([Z^T \quad (Z^p)^T \quad U^T \quad (U^p)^T]^T \) and \( Z_+ \) have full row rank.

**Assumption 2.** For some \( \epsilon > 0 \), it holds that \( R_-R_+^T \ll \epsilon Z_+ Z_+^T \), where \( R_- := A \begin{bmatrix} E \\ E^p \end{bmatrix} - E_+ \).

In [18] Sec. V.A, the discussion that follows the LTI variants of these assumptions is directly applicable to our LPV counterparts. That is, it is easy to observe that Assumption [1] may in fact be satisfied for arbitrary \( \epsilon_k \). Hence, we need Assumption [2] which essentially says that the (spurious) behavior in the noise signal that could have been observed as system dynamics is \( \epsilon \) times smaller than the actual behavior coming from the system. Hence, \( \epsilon \) can be seen as an indication of the noise level, such as the SNR. Note that if Assumption [1] is satisfied, there always exists a large enough \( \epsilon \) such that Assumption [2] holds.

6.2 Stabilizing controller synthesis with noisy data

We are now ready to present the direct data-driven synthesis method for noisy data. We consider here only the synthesis of a stabilizing LPV controller, as the extension of the results that follow to synthesis for the earlier discussed performance metrics is trivial.

Intuitively, it is not possible to find a stabilizing LPV controller for arbitrary \( \epsilon \), i.e., arbitrary SNR. As the following result illustrates, the synthesis problem provides an LPV controller that is guaranteed to stabilize the closed-loop as long as \( \epsilon \) satisfies an upper bounded provided by the solution of the synthesis procedure, that is, for a given (upper bound on) \( \epsilon \) associated with \( \mathcal{D}_N^\epsilon \), the stabilizing synthesis procedure is as follows.

**Theorem 7** (Data-driven stabilizing synthesis from noisy data). Given \( \mathcal{D}_N^\epsilon \) such that Assumptions [1] and [2] hold. If there exist an \( \alpha > 0 \), a matrix \( P \in \mathbb{S}_{n^d} \), \( F_Q \in \mathbb{R}^{(N_d-1)(1+n_p) \times n_x(1+n_p)} \), \( Y_0 \in \mathbb{R}^{n_x \times n_x} \), and \( \bar{Y} \in \mathbb{R}^{n_u \times n_u n_p} \) that satisfy \( \frac{\alpha^2}{4\pi^2} > \epsilon \), \( P > 0 \) and

\[
 \begin{bmatrix}
 P & 0 & 0 \\
 0 & I_{n_p} \otimes P & 0 \\
 Y_0 & \bar{Y} & 0 \\
 0 & I_{n_p} \otimes Y_0 & I_{n_p} \otimes \bar{Y}
 \end{bmatrix}
 \begin{bmatrix}
 Z^p \\
 U^p \\
 Z^+ F(p)^T \\
 U^+(p)^T 
\end{bmatrix} =
 \begin{bmatrix}
 Z \\
 U \\
 Z^+ F(p) \\
 U^+(p)
 \end{bmatrix} \geq 0,
\] (41)

\[
 \begin{bmatrix}
 P - \alpha Z_+ Z_+^T \\
 (Z_+ + F(p))^T \\
 Z_+ F(p) \\
 I_{N_d-1} \\
 F(p)
 \end{bmatrix} \geq 0,
\] (42)

for all \( p \in \mathcal{P} \), where \( F(p), F \) are constructed with \( F_Q \) as in (19), then the LPV state-feedback controller \( K(p) \) as in (5) with gains (28) is a stabilizing controller for (2).

**Proof.** The concept of this proof is based on [18] Thm. 5. Note that the matrix equality condition in (41) follows directly from the proof of Theorem 3. We first derive a data-based representation of
the closed-loop LPV system, parametrized by \( \mathcal{V} \) and the noise trajectories \( [9] \). Recall \( \mathcal{M} \) and \( \mathcal{M}_{cl} \) from (the proof of) Theorem \( 7 \). We start off with \( 17 \):

\[
\mathcal{M} = \begin{bmatrix} A & B \end{bmatrix}, \quad \mathcal{M}_{cl} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} Z \\ Z^p \\ U \\ U^p \end{bmatrix} \mathcal{V} = \begin{bmatrix} A & B \end{bmatrix} \mathcal{G} \mathcal{V} + \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} E \\ E^p \\ 0 \\ 0 \end{bmatrix} \mathcal{V} = \left( X_+ + A \begin{bmatrix} E \\ E^p \end{bmatrix} \right) \mathcal{V} = (Z_+ + R_-) \mathcal{V}, \quad (43)
\]

where \( \mathcal{V} \) satisfies

\[
\begin{bmatrix}
I_{n_\mathcal{M}} & 0 & 0 \\
0 & I_{n_\mathcal{M}} \otimes I_{n_\mathcal{M}} & 0 \\
K_0 & I_{n_\mathcal{M}} \otimes K_0 & I_{n_\mathcal{M}} \otimes K_0
\end{bmatrix}
\begin{bmatrix}
Z \\
Z^p \\
U \\
U^p
\end{bmatrix}
\mathcal{V}.
\]

Now that we have a parametrization of \( \mathcal{M} \) in terms of the measured data, the noise (by means of \( R_- \)) and \( \mathcal{G} \), we can substitute the parametrization into \( 18 \). Recall that \( F(p) = V(p)P \), then

\[
\begin{bmatrix}
P \\
(Z_+ + R_-) \mathcal{V}(p)P \\
(P \mathcal{V}(p))((Z_+ + R_-)^T)
\end{bmatrix} \succ 0 \iff (Z_+ + R_-) \mathcal{V}(p)P \mathcal{V}(p)(Z_+ + R_-)^T - P = Z_+ \mathcal{V}(p)P \mathcal{V}(p)Z_+^T + R_- \mathcal{V}(p)P \mathcal{V}(p)R_-^T + Z_+ \mathcal{V}(p)P \mathcal{V}(p)R_-^T < 0.
\]

We now use a special case of Young’s relation \( 39 \), given by

Let \( P \succ 0, \delta > 0, \quad X^T SY + Y^T SX \lesssim \delta X^T SY \). Applying this to \( 44 \) with \( X = (Z_+ \mathcal{V}(p))^T, S = P \) and \( Y = (R_- \mathcal{V}(p))^T \), we obtain

\[
\begin{bmatrix}
(P \mathcal{V}(p))((Z_+ + R_-)^T)
\end{bmatrix} \succ 0 \iff (Z_+ + R_-) \mathcal{V}(p)P \mathcal{V}(p)(Z_+ + R_-)^T - P = \Theta_0(\delta).
\]

We have reached the last steps in the proof. Suppose we have found a solution to \( 42 \). Working out the matrix inequalities in \( 19 \) and applying the Schur complement gives, cf. \( 20 \).

\[
\begin{align*}
\Theta_1(\alpha) & := Z_+ F(p)P^{-1} F^T(p)Z_+^T + \alpha Z_+ Z_+^T < 0, \\
\Theta_2 & := (P \mathcal{V}(p))((Z_+ + R_-)^T) - I_{n_{\mathcal{M}}-1} \prec 0.
\end{align*}
\]

Note that \( \Theta_2 \) is equivalent to \( \mathcal{V}(p)P \mathcal{V}(p)Z_+^T + \delta Z_+ Z_+^T + (1 + \delta^{-1})R_- R_-^T - P \). Finally, we observe that the right-hand side of \( 17 \) is equivalent to \( \Theta_1(\alpha) + (\delta - \alpha)Z_+ Z_+^T + (1 + \delta^{-1})R_- R_-^T \), i.e.,

\[
\Theta_0(\delta) \prec (\delta - \alpha)Z_+ Z_+^T + (1 + \delta^{-1})R_- R_-^T.
\]

Hence, solutions to \( 42 \) provide a stabilizing LPV controller if

\[
(\delta - \alpha)Z_+ Z_+^T + (1 + \delta^{-1})R_- R_-^T < 0 \iff R_- R_-^T < \frac{\alpha - \delta}{1 + \delta^{-1}} Z_+ Z_+^T.
\]

For \( \delta = \frac{\alpha^2}{4 + \alpha^2} > \epsilon \), which concludes the proof.

The matrix (in)equalities of Theorem \( 7 \) are quadratic in \( p \). To solve the problem in an SDP subject to LMI constraints, we employ the full-block \( \mathcal{S} \)-procedure of Lemma \( 3 \), as with the earlier synthesis problems.

**Proposition 1** (Convex synthesis problem for noisy data). Given \( \mathcal{D}_{N_3} \) such that Assumptions \( 2 \) and \( 6 \) hold. The matrix (in)equalities \( 41, 42 \) hold if and only if there exists an \( \alpha > 0 \), an \( F_Q \),
a $P > 0$, matrices $Y_0$ and $Y$, and a multiplier $\Xi \in \mathbb{S}^{2n_p(3n_d+N_d-1)}$ that satisfy [41] and the LMI conditions in [61], where $\Delta(p)$, $W$ and $L_{11}, \ldots, L_{22}$ are given by:

$$\Delta(p) = \text{blkdiag}(\text{diag}(p) \otimes I_{n_x}, \text{diag}(p) \otimes I_{n_x}, \text{diag}(p) \otimes I_{N_d-1}, \text{diag}(p) \otimes I_{n_x})$$  \hspace{1cm} (48a)

$$W = \begin{bmatrix}
    P - \alpha Z_+ Z_+^T & 0 & 0 & 0 & 0 & 0 & 0 \\
    (Z_+ F_Q)^\top & P & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & I_{N_d-1} & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & F_Q & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & F_Q & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & F_Q & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & F_Q \\
\end{bmatrix} \hspace{1cm} (48b)$$

$$L_{11} = 0_{n_p(3n_x+N_d-1)},$$

$$L_{12} = \text{blkdiag}(1_{n_p} \otimes I_{n_x}, 1_{n_p} \otimes I_{n_x}, 1_{n_p} \otimes I_{N_d-1}, 1_{n_p} \otimes I_{n_x}),$$

$$L_{21} = \text{blkdiag}(0_{n_x \times n_x n_p}, 0_{n_x \times n_x n_p}, 0_{N_d-1 \times (N_d-1) n_p}, 0_{n_x n_x n_p}),$$

$$L_{22} = \text{blkdiag}(0_{n_x n_x n_x}, 0_{n_x n_x n_x}, 0_{N_d-1 \times (N_d-1) n_x}, 0_{n_x n_x n_x}),$$  \hspace{1cm} (48c)

where $Z_+ := \text{blkdiag}(Z_+, I_{n_p} \otimes Z_+)$.  

**Proof.** The proof is fairly straightforward. First, we use the relationship in [26] and [19] to write [42] as

$$\begin{bmatrix}
    I_{n_x} \\
    P - \alpha Z_+ Z_+^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix}
    P - \alpha Z_+ Z_+^T & 0 & 0 & 0 & 0 & 0 & 0 \\
    (Z_+ F_Q)^\top & P & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & I_{N_d-1} & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & F_Q & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & F_Q & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & F_Q & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & F_Q \\
\end{bmatrix} \begin{bmatrix}
    I_{n_x} & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & I_{n_x} & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & I_{N_d-1} & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & F_Q & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & F_Q & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & F_Q & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & F_Q \\
\end{bmatrix} \begin{bmatrix}
    I_{n_x} & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & I_{n_x} & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & I_{N_d-1} & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & F_Q & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & F_Q & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & F_Q & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & F_Q \\
\end{bmatrix} \succ 0,  \hspace{1cm} (49)$$

$$\begin{bmatrix}
    I_{n_x} \\
    I_{n_x} \\
    I_{N_d-1} \\
    I_{N_d-1} \\
    I_{n_x} \\
    I_{n_x} \\
    I_{n_x} \\
\end{bmatrix} \begin{bmatrix}
    I_{n_x} & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & I_{n_x} & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & I_{N_d-1} & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & F_Q & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & F_Q & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & F_Q & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & F_Q \\
\end{bmatrix} \begin{bmatrix}
    I_{n_x} & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & I_{n_x} & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & I_{N_d-1} & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & F_Q & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & F_Q & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & F_Q & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & F_Q \\
\end{bmatrix} \begin{bmatrix}
    I_{n_x} & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & I_{n_x} & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & I_{N_d-1} & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & F_Q & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & F_Q & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & F_Q & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & F_Q \\
\end{bmatrix} \succ 0.  \hspace{1cm} (50)$$

Combining the above two matrix inequalities allows to write them in the form of [59] with $W$ as in [48] and

$$L(p) := \text{blkdiag}(\begin{bmatrix}
    I_{n_x} \\
    I_{n_x} \\
    I_{N_d-1} \\
    I_{N_d-1} \\
    I_{n_x} \\
    I_{n_x} \\
    I_{n_x} \\
\end{bmatrix}, \begin{bmatrix}
    I_{n_x} \\
    I_{n_x} \\
    I_{N_d-1} \\
    I_{N_d-1} \\
    I_{n_x} \\
    I_{n_x} \\
    I_{n_x} \\
\end{bmatrix}),$$  \hspace{1cm} (51)

$L(p)$ can be written in LFR form with $L_{11}, \ldots, L_{22}$ as given in [59]. Application of Lemma [1] concludes the proof.  

If $P$ is a convex polytope and $\Xi_{22} \prec 0$, the LMI conditions of Proposition [1] are convex in $p$. Hence, they are required to be solved only at the vertices of the scheduling set $P$, again using the multi-convexity argument.

We have now presented a direct data-driven control method that allows to synthesize LPV controllers directly from noise-infected measurement data. Moreover, the extension of this result towards the earlier presented performance metrics is straight-forward. This allows for real-life application of our methods, as is already demonstrated in [40].

Before we conclude this section, we want to remark on the computational complexity and conservatism of the bound on $\epsilon$. As remarked in [18], the bound $\frac{\lambda^2}{3\lambda + 2\alpha} > \epsilon$ can be theoretically conservative. Increasing the SNR in the data-dictionary, which can be accomplished by, e.g., increasing $N_d$ or the magnitude of $u_k$ and $x_k$ (assuming $\varepsilon_k$ to be independent of $u_k$ and $x_k$), while maximizing $\alpha$ can minimize the conservativeness of the conclusions drawn from Theorem [7]. Moreover, as the SDP of Proposition [1] grows with $N_d$, one can make the trade-off between conservatism and computational complexity.
7 Simulation studies

In this section, the proposed data-driven LPV controller synthesis methods are applied in three simulation studies to demonstrate their effectiveness and compare their achieved performance w.r.t. model-based methods. All the examples have been implemented using MATLAB 2020a and the resulting SDPs have been solved using the YALMIP toolbox [41] with MOSEK solver [42]. In the first example, we compare our proposed data-driven methodology with model-based design in terms of stabilizing robust feedback control synthesis, while in the second example we showcase the method in optimal $H_2$-norm-based design. In the third example, we test our method on an unbalanced disc system with an $\ell_2$-gain objective and we show that our data-driven LPV method can also be successfully deployed on a nonlinear system by exploiting the principle of LPV embedding.

7.1 Simulation study 1: Data-driven scheduling independent state feedback

The LPV system in this example is taken from [43]. The considered LPV system can be represented as (2) with $n_x = 2$, $n_u = 1$ and $n_p = 2$, where $A$ has affine scheduling dependence and $B$ is scheduling independent and are characterized by

$$A_0 = \begin{bmatrix} 0.2485 & -1.0355 \\ 0.8910 & 0.4065 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.0063 & -0.0938 \\ 0 & 0.0188 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.0063 & -0.0938 \\ 0 & 0.0188 \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0.3190 \\ -1.3080 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$  

Furthermore, the scheduling set is defined as $\mathcal{P} := [-1, 1] \times [-1, 1]$. Based on the PE condition (Condition 1), $N_d \geq (1 + n_p)(n_x + n_u) = 9$. Therefore, only 9 samples of input-scheduling data are generated based on random samples from a uniform distribution $U(-1, 1)$. By simulating the system with this $u$ and $p$ sequence, and a random initial condition generated similarly, results in $D_{N_d}$ from which $G$ is constructed. An a posteriori check of $\text{rank}(G) = 9$ yields that Condition 1 is satisfied.

Now, a data-driven robust (scheduling independent) state-feedback controller is designed via Theorem 4, see Remark 3. Hence, we choose $F_{22} = 0$ and we set $Q = R = I$. Solving the associated LMIs at the vertices of $\mathcal{P}$ yields

$$K_0 = \begin{bmatrix} 0.4832 & 0.4839 \end{bmatrix}, \quad \hat{P} = \begin{bmatrix} 1.6436 & -0.4595 \\ -0.4595 & 3.0426 \end{bmatrix},$$

where $K_0$ is obtained using (28). Note that choosing $F_{22} = 0$ automatically renders $\hat{K} \approx 0$ up to numerical precision. To validate these results, the corresponding model-based conditions of Lemma 5 have been used to design a robust state-feedback controller for the exact model of the LPV system, which resulted in a $K_0$ and $\hat{P}$ that are equivalent to (52) up to numerical precision. This shows that in terms of the robust controller synthesis case, our methods are truly the data-based counterparts of model-based synthesis methods.

7.2 Simulation study 2: Data-driven scheduling dependent state-feedback

Most of the LPV state-feedback control synthesis methodologies require constant $B$ matrices in the LPV-SS representation of the generalized plant. In this example, we compare our methods to the model-based design approaches in [44], which is one of the few works that considers LPV state-feedback synthesis under a scheduling dependent $B$-matrix. With this example, we show that we can handle a scheduling-dependent $B$ as good as the proposed model-based approach in [44]. Furthermore, we demonstrate the controller synthesis method using noisy data, as discussed in Section 6. 

\footnote{For experimental results on the application of our methods on real world systems, see [39].}
7.2.1 LPV system and data generation

The LPV system from [44] can be represented as (2) with \( n_x = 4, n_u = 1 \) and \( n_p = 2 \), where \( A \) and \( B \) have affine scheduling dependence. The matrices are characterized by

\[
A_0 = \begin{bmatrix}
0.8 & -0.25 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0.2 & 0.03 \\
0 & 0 & 1 & 0
\end{bmatrix}, \quad A_1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0.8\alpha & -0.5\alpha & 0 & \alpha \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad A_2 = 0_{4 \times 4},
\]

\[
B_0 = \begin{bmatrix}
0.5 \\
0 \\
0.5 \\
0
\end{bmatrix}, \quad B_1 = 0_{4 \times 1}, \quad B_2 = \begin{bmatrix}
0.5 \\
0 \\
-0.5 \\
0
\end{bmatrix}.
\]

Moreover, \( \mathbb{P} = [-1, 1] \times [-1, 1] \) and the parameter \( \alpha \geq 0 \) in [44] that is used to modify the scheduling range without affecting \( \mathbb{P} \) is chosen as in [44], i.e., \( \alpha = 0.53 \). We generate \( \mathcal{D}_{N_d} \), with \( N_d \geq (1 + n_p)(n_x + n_u) = 15 \), by applying \( u \) and \( p \) sequences from \( \mathcal{U}(-1, 1) \) on the system (initialized with a randomly chosen initial condition \( x_1 \)). After constructing \( \mathcal{G} \) from \( \mathcal{D}_{N_d} \), the rank check in Condition 1 yields that \( \mathcal{D}_{N_d} \) is persistently exciting for this system.

We will now only use \( \mathcal{D}_{N_d} \) to synthesize a stabilizing LPV controller using Theorem 3 and a stabilizing LPV controller that guarantees closed-loop quadratic performance using Theorem 5 for the considered system, which we assume to be unknown. We compare these to the model-based counterparts discussed in [44], which require complete model-knowledge.

7.2.2 Stabilizing state-feedback LPV controller

Using Theorem 3 we design a stabilizing data-driven LPV state-feedback controller based on the measured data-set. Next to the LMI constraints of Theorem 3 we add \( 0.1 < \text{trace}(P) < 10 \) to improve numerical conditioning of the problem due to the inversion of \( P \) in (28). Solving the synthesis on the vertices of \( \mathbb{P} \) yields an LPV controller of the form of (5) with

\[
K_0 = \begin{bmatrix}
-0.2553 & 0.1265 & -0.3894 & -0.4131 \\
\end{bmatrix}, \quad K_1 = \begin{bmatrix}
-0.3066 & 0.1611 & -0.0445 & -0.4056 \\
\end{bmatrix},
\]

\[
K_2 = \begin{bmatrix}
-0.0844 & 0.0389 & -0.0652 & -0.1650 \\
\end{bmatrix}, \quad (53a)
\]

and (inverse) Lyapunov matrix \( \hat{P} = P^{-1} \)

\[
\hat{P} = \begin{bmatrix}
0.7096 & -0.1998 & 0.3649 & 0.5581 \\
-0.1998 & 0.2808 & -0.0171 & -0.1321 \\
0.3649 & -0.0171 & 1.7044 & 0.3473 \\
0.5581 & -0.1321 & 0.3473 & 1.1987
\end{bmatrix}, \quad (53b)
\]

where \( K_0, K_1, K_2 \) are computed by (28).

To compare our result, we design a controller of the form of (5) based on the model-based synthesis conditions in [44, Cor. 3] using exact knowledge of the system equations. We also add the additional constraint \( 0.1 < \text{trace}(P) < 10 \) to improve numerical conditioning. The model-based design yields:

\[
K_0^m = \begin{bmatrix}
-0.2447 & 0.0990 & -0.4591 & -0.4232 \\
\end{bmatrix}, \quad K_1^m = \begin{bmatrix}
-0.3120 & 0.2052 & 0.0132 & -0.3834 \\
\end{bmatrix},
\]

\[
K_2^m = \begin{bmatrix}
-0.0676 & 0.0362 & -0.0683 & -0.1451 \\
\end{bmatrix}, \quad (54a)
\]

and (inverse) Lyapunov matrix \( \hat{P}^m = (P^m)^{-1} \)

\[
\hat{P}^m = \begin{bmatrix}
0.9094 & -0.2632 & 0.4766 & 0.7356 \\
-0.2632 & 0.3495 & -0.0369 & -0.1916 \\
0.4766 & -0.0369 & 2.1779 & 0.4587 \\
0.7356 & -0.1916 & 0.4587 & 1.5495
\end{bmatrix}, \quad (54b)
\]

It is clear that, from a numerical perspective, the results (53) and (54) are approximately similar.

Comparing the two obtained controllers in simulation with the considered LPV system yields the results plotted in Figure 1 which shows similar performance up to small numerical errors. This shows that our data-driven synthesis method, which only uses the measured data-set \( \mathcal{D}_{N_d} \) for the controller synthesis, is truly a counterpart of the model-based synthesis methods.
7.2.3 Quadratic performance state-feedback LPV controller

Next, we design a data-driven state-feedback LPV controller that ensures quadratic performance using the developed tools in Theorem 4, using only the information in $\mathcal{D}_{N_t}$. We choose $Q = I_d$ and $R = 1$ for the performance objective $\mathcal{H}_2^d$. We solve the LMIs of Theorem 4 as an SDP on the vertices of $\mathcal{P}$, where we choose the objective function as $\max \text{trace}(P)$, which yields an LPV controller of the form $\tilde{K}$, with

\[
K_0 = \begin{bmatrix} -0.0989 & 0.0878 & -0.3018 & -0.2412 \end{bmatrix}, \quad K_1 = \begin{bmatrix} -0.2043 & 0.1308 & -0.0349 & -0.2645 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.0963 & 0.0593 & -0.0123 & -0.1767 \end{bmatrix},
\]

and $\text{trace}(P) = 0.9151$. Again, we compare this result with a model-based synthesis approach, see [15 Prop. 1] using a constant Lyapunov matrix. The model-based controller synthesis yields

\[
K_0^m = \begin{bmatrix} -0.0082 & 0.0332 & -0.2916 & -0.1401 \end{bmatrix}, \quad K_1^m = \begin{bmatrix} -0.1081 & 0.0676 & -0.0002 & -0.1353 \end{bmatrix}, \quad K_2^m = \begin{bmatrix} 0.0062 & -0.0026 & -0.0116 & 0.0088 \end{bmatrix},
\]

and $\text{trace}(P^m) = 0.7146$.

We compare the model-based controller to the direct data-driven LPV controller via simulation, and the computation of the $\mathcal{H}_2^d$-norm of the generalized plant in closed-loop for both controllers over an equidistant grid of 250 points over $\mathcal{P}$. The simulation results are given in Figure 1 which shows how both controllers have a similar performance. The maximum of the achieved $\mathcal{H}_2^d$-norm over the grid can be seen as an approximation of the ‘true’ $\mathcal{H}_2^d$-norm of the LPV system. The resulting values for the data-based and the model-based controller are $5.27$ and $5.79$, respectively, which indicates that both controllers are very similar in terms of achieved performance. The difference between the two values is attributed to numerical differences during the solution of the problems.

From this example, we can conclude that the proposed data-driven control synthesis machinery provides competitive controllers w.r.t. model-based approaches by using only a few data samples measured from the system, rather than a full model. This considerably simplifies the overall LPV modeling and control toolchain, avoiding the two-step process of obtaining a model and then designing a model-based controller and respecting the control performance objectives in exploitation of the data. Moreover, it can cope with $p$-dependence of the $B$ matrix of the plant dynamics.

7.2.4 Stabilizing controller synthesis with noisy measurements

Finally, we employ the methods discussed in Section 6 on the considered system. We use the exact same parametrization as before. In this case, we simulate the data-generating system for $N_d = 26$, while we add noise $\epsilon_k$ to our measurements taken from a normal distribution with variance $0.1$. The input and scheduling sequences are generated from a uniform distribution $\mathcal{U}(-1, 1)$. The state-measurements in our data-dictionary $\mathcal{D}_{N_t}$ are shown in Figure 2. Because of the nature of the simulation study, we can calculate $\epsilon$ in Assumption 2 by solving a simple minimization over $\epsilon$ subject to $R_r^T - \epsilon Z_r^T Z_r^T < 0$. This provides us with $\epsilon = 0.2172$ for the obtained $\mathcal{D}_{N_t}$.

Next, we synthesize a stabilizing LPV controller for the system using the methods provided in Theorem 7 and Proposition 1, and $\mathcal{D}_{N_t}$. Next to testing feasibility of (42), we additionally maximize over $\alpha$ to look for the solution with the largest $\epsilon$ that still guarantees stability. The problem is
successfully solved on the vertices of $\mathcal{P}$ and yields the LPV controller parametrization:

$$K_0^* = \begin{bmatrix} -0.233 & 0.2199 & -0.4428 & -0.5837 \end{bmatrix}, \quad K_1^* = \begin{bmatrix} -0.2355 & 0.1640 & 0.0834 & -0.4688 \end{bmatrix},$$

$$K_2^* = \begin{bmatrix} -0.0856 & 0.1715 & -0.174 & -0.0352 \end{bmatrix},$$

the Lyapunov matrix $\tilde{P}_e = (P_e)^{-1}$:

$$\tilde{P}_e = \begin{bmatrix} 7.6842 & -5.1997 & 0.6303 & -3.4561 \\ -5.1997 & 6.4229 & 0.1364 & 5.0478 \\ 0.6303 & 0.1364 & 9.3075 & 0.1633 \\ -3.4561 & 5.0478 & 0.1633 & 14.8499 \end{bmatrix},$$

and the value $\alpha = 0.0160$, which yields a $\epsilon_{\text{opt}} = 6.312 \cdot 10^{-5}$. Hence, based on the statement of Theorem 7, the obtained controller does not guarantee stability of the closed-loop with this noise level. However, as highlighted under Proposition 8, the bound on $\epsilon$ coming from the optimization can be fairly conservative. Indeed, when simulating the controller in closed-loop with the system under the feedback law $u_k = K(p_k)z_k$ in Simulation study 2, where $K(p)$ is synthesized with $\mathcal{D}_{\mathcal{N}_e}$, the controller is fed with the noisy measurements, the closed-loop is stabilized and regulated to the origin without any problems, as depicted in Figure 3. In fact, if we choose our inputs for the data-generation to be larger such that the SNR goes up (i.e., $\epsilon$ shrinks), approximately the same controller is obtained for a higher $\alpha$. Also increasing the number of data-points effectively improves the SNR, at the cost of computational power. This example demonstrates two important aspects. First, and most important, it shows that it is possible to achieve stabilizing LPV control synthesis directly from noisy measurements. Second, it demonstrates that the currently available methods that consider noise are rather conservative, motivating further research into the direction of noise handling and stochastic aspects in direct data-driven control.

7.3 Simulation study 3: Application on the nonlinear unbalanced disc system

To demonstrate applicability of the proposed data-driven control on a real-world unstable nonlinear system, we consider the unbalanced disc setup, depicted in Figure 4. This system consists of a DC motor that is connected to a disc containing an off-centered mass. Thus, the system behavior mimics the behavior of a rotational pendulum.

7.3.1 System dynamics and data-generation

The nonlinear dynamics can be represented by the differential equation:

$$\ddot{\theta}(t) = -\frac{mg}{J} \sin(\theta(t)) - \frac{1}{\tau} \dot{\theta} + \frac{K_m}{\tau} u(t),$$

(57)
For the angular position \( \theta \) of the disc, we need at least we need a scaled \( P \) from which we can immediately define \( \mathcal{P} \) as \([-0.22, 1]\). For a better formulation of the LFRs, we scaled \( p \) such that \( \mathcal{P} := [-1, 1] \).

For the data generation, we have implemented the nonlinear dynamical equations (57) in MATLAB, which are solved using an ODE45 solver at a fixed sampling time of \( T_s = 0.01 \). By Condition 1, we need at least \( N_d = 6 \) data points. Hence, \( D_{N_d} \) is generated by exciting the system with a uniform randomly generated input of length 6, where \( u \) is in the range \([-10, 10]\). Moreover, as it is assumed throughout the paper, the state measurements are noise-free (apart from Section 6). A posteriori, the scheduling \( p \) is determined using the aforementioned scheduling map, with which \( \mathcal{G} \) is constructed. The obtained data-dictionary \( D_{N_d} \) that we use to synthesize the controllers is shown in Figure 5. Note that we also depicted \( X_t \) in the upper plots of Figure 5 by means of an additional state sample. For the obtained data-set, Condition 1 is verified, as \( \text{rank}(\mathcal{G}) = 6 \).

### 7.3.2 Controller synthesis

We now use only the data in \( D_{N_d} \) to synthesize LPV state-feedback controllers for the unbalanced disc system. Using our developed results in Theorems 4, 5 and 6, we synthesize three LPV state-
by the considered state-feedback configuration, the controllers are designed for setpoint control of corresponding input steady-state values \( x \) forced equilibrium points (control signal \( u \)). Finally, the synthesized LPV controllers are applied on the original nonlinear model, where the synthesis is (approximately) equivalent to the resulting closed-loop performance.

Table 2: Synthesis results.

| Controller | \( \gamma \) | \( \mathcal{H}_\infty \)-norm | \( \ell_2 \)-gain |
|------------|-------------|-----------------|-----------------|
| Controller 1 | N/A | 18.6 | 92.0 |
| Controller 2 | 11.6 | 11.6 | 54.7 |
| Controller 3 | 36.0 | 13.6 | 35.6 |

feedback controllers for optimal 1) quadratic, 2) \( \mathcal{H}_\infty \)-norm and 3) \( \ell_2 \)-gain performance. Our objective is to regulate the states fast and smoothly to predefined operating points using a reasonable control input. In order to achieve this objective, we have tuned the matrices \( Q \) and \( R \) for the design problems 1, 2 and 3 as follows:

\[
Q_1 = Q_2 = Q_3 = \begin{bmatrix} 8 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad R_1 = 0.5, \quad R_2 = 0.05, \quad R_3 = 0.005.
\]

which have yielded controllers with relatively similar performance. As in the previous examples, the cost function for Controller 1 is chosen as \( \max \text{trace}(P) \), while for Controller 2 we minimize the \( \mathcal{H}_\infty \)-norm \( \gamma \) during synthesis. In order to limit the aggressiveness of Controller 3, we modified the cost-function in the synthesis problem to \( \min \gamma + \lambda \text{trace}(P) \), similar to the implementation of \texttt{hinfdm} in \texttt{MATLAB}. Note that a \( \lambda > 0 \) results in larger values of \( \gamma \), i.e., the problem is regularized at the cost of performance. We choose \( \lambda = 0.1 \). The results of the LPV controller synthesis problems are given in Table 2. Additionally, we computed the maximum \( \mathcal{H}_\infty \)-norm and \( \mathcal{H}_{\infty \gamma} \)-norm of the resulting local LTI systems of the closed-loop LPV plant over a grid of \( \mathbb{P} \). These results are given in Table 2 as well, and show that for Controllers 2 and 3 the guaranteed performance by the data-based synthesis is (approximately) equivalent to the resulting closed-loop performance.

7.3.3 Simulation results

Finally, the synthesized LPV controllers are applied on the original nonlinear model, where the control signal \( u_k \) is applied to the continuous-time system in a zero-order-hold setting. Note that, by the considered state-feedback configuration, the controllers are designed for setpoint control of the origin. To show the capabilities of the controllers, we tested them in setpoint control of different forced equilibrium points \((x_{ss}, u_{ss})\), as in [17]. In order to define the forced equilibrium points, the corresponding input steady-state values \( u_{ss} \) are calculated and added to the control law, which yields \( u_k = K(p_k)(x_k - x_{ss}) + u_{ss} \). We chose to switch between \( x_{ss} = \{0 \, 0 \}^T, \{\frac{5}{7} \, 0 \}^T, \{-\frac{5}{7} \, 0 \}^T \} \), where \( x_{ss1} = \theta_{ss} \) and \( x_{ss2} = \theta_{ss} \). The simulation results are shown in Figure 6. The aggressiveness of the \( \ell_2 \)-gain performance-based controller can be explained by the choice of \( R = 0.005 \), i.e., the cost on input is mild. In general, the designed controllers provide highly competitive performance, also when compared to the model-based results in [17] Fig. 6.

From this example, we can conclude that our proposed methods allow to synthesize LPV state-feedback controllers using only data, which can guarantee stability and performance of the closed-loop systems, even when the plant is a nonlinear system. Furthermore, in most cases, one will only need \((1 + n_p)(n_x + n_u)\) data-points measured from the nonlinear system, to be able to synthesize an LPV state-feedback controller.

8 Conclusion

In this work, we have derived novel direct data-driven methods that are capable of synthesizing LPV state-feedback controllers by only using information about the to-be-controlled system in the form of a persistently exciting data-set. Formulation of these results is made possible by the proposed data-driven representations of the open-loop and closed-loop behavior of the unknown LPV system. When the LPV state-feedback controllers, provided by the introduced synthesis algorithms, are connected to the unknown data-generating system, stability and performance of the closed-loop operation is guaranteed. By means of the presented simulation studies, we demonstrate that our methods can achieve, based on only the measurement data, the same control performance as their model-based counterparts, while the latter uses perfect model knowledge of the system. Furthermore, we have demonstrated that, in line with the LPV embedding principle, the design methods can also be used for controlling nonlinear systems. We believe that our novel results can be the foundation for building a direct data-driven control framework for nonlinear systems with stability and performance.
guarantees. As a future work, we aim at the extension of the synthesis methods and the guarantees for noisy data-sets.

**Acknowledgments**

This work has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement nr. 714663), the European Union within the framework of the National Laboratory for Autonomous Systems (RRF-2.3.1-21-2022-00002), and the Deutsche Forschungs-gemeinschaft (DFG, German Research Foundation) under Project No. 419290163.

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A Full-block S-Procedure

For completeness, we give the known lemma from [48, 34] on the full-block S-procedure below. This result is instrumental and is used extensively throughout the paper.

**Lemma 1** (Full-block S-procedure [48, 34]). Given a quadratic matrix inequality

\[ L^\top(p)WL(p) > 0, \quad \forall p \in \mathbb{P}, \tag{59} \]

with \( L(p) = \Delta_p \ast \hat{L} = L_{22} + L_{21}\Delta_p(I - L_{11}\Delta_p)^{-1}L_{12}, \) where

\[
\Delta_p = \text{blkdiag}(p_1I_{n_{a1}}, p_2I_{n_{a2}}, \ldots, p_nI_{n_{an}}), \tag{60}
\]

and \( \mathbb{P} \) is convex, then (59) holds if and only if there exists a real full-block multiplier \( \Xi = \Xi^\top \) defined as \( \Xi = \begin{bmatrix} \Xi_1 & \Xi_2 \\ \Xi_2 & \Xi_{22} \end{bmatrix} \), such that

\[
\begin{bmatrix} * \\ * \end{bmatrix}^\top \begin{bmatrix} \Xi & 0 \\ 0 & -W \end{bmatrix} \begin{bmatrix} L_{11} & L_{12} \\ I & 0 \\ L_{21} & L_{22} \end{bmatrix} < 0, \quad \begin{bmatrix} * \\ * \end{bmatrix}^\top \Xi \begin{bmatrix} I \\ \Delta_p \end{bmatrix} \succeq 0, \quad \forall p \in \mathbb{P}. \tag{61}
\]

Note that (61) is not necessarily convex. At the cost of conservatism, the additional condition of \( \Xi_{22} < 0 \) makes (61) convex in \( p \) [34].
B A review of model-based LPV state-feedback control

In this section, the preliminaries for the proposed data-driven methods are discussed in terms of model-based LPV stability and performance analysis, followed by LPV state-feedback controller synthesis.

B.1 Model-based stability analysis

Asymptotic stability of the LPV system \( \mathcal{S} \), i.e., boundedness and convergence of the state-trajectories to the origin under \( u \equiv 0 \), is guaranteed with the existence of a Lyapunov function \( V(x,p) = x^T P(p)x > 0 \) with symmetric \( P : \mathbb{P} \to \mathbb{S}_{nn}^+ \) that satisfies \( V(x_{k+1}, p_{k+1}) - V(x_k, p_k) < 0 \) under all \((x,p,0) \in \mathcal{B}\).

In the remainder, we perform the analysis on the closed-loop system \( \mathcal{S} \). Stability of the closed-loop system can be verified with the following lemma:

**Lemma 2** (Stabilizing LPV state-feedback \([49]\)). The controller \( K \) in \([5]\) asymptotically stabilizes \( \mathcal{S} \) if there exists a symmetric \( P : \mathbb{P} \to \mathbb{S}_{nn}^+ \) affine in \( p \) such that

\[
\begin{bmatrix}
P(p) & (s)^T \\
A_{CL}(p)P(p) & P(p^+) \\
\end{bmatrix} > 0,
\]

(62a)

\[P(p) > 0,
\]

(62b)

for all \( p, p^+ \in \mathbb{P} \), with \( A_{CL}(p) \) as defined in \([6]\), and \( p^+ \) is the time-shift of \( p \), i.e., for \( p_k, p^+ = p_{k+1} \).

**Proof.** We give it here for completeness, see also \([49]\). If (62b) holds, then \( V(x,p) = x^T \tilde{P}(p)x > 0 \) with \( P(p) = \tilde{P}(p)^{-1}(p) \) is true for all \((x,p) \in \mathcal{B}_{CL} \), where

\[\mathcal{B}_{CL} = \{(x,p) | (x,p,K(p)x) \in \mathcal{B}\} \] .

Furthermore, (62a) under (62b) is equivalent with

\[A_{CL}^T(p)\tilde{P}(p^+)A_{CL}(p) - \tilde{P}(p) < 0, \quad \forall p, p^+ \in \mathbb{P},\]

in terms of the Schur complement. This implies that \( V(x^+, p^+) - V(x, p) < 0 \) for all vectors \( x, p^+ \in \mathbb{P} \) and \( x \in \mathbb{R}_{nn}^+ \) and hence under all \((x,p) \in \mathcal{B}_{CL} \). Based on the fact that asymptotic stability of \([6]\) is equivalent with the existence of a quadratic Lyapunov function fulfilling the above conditions \([49]\).

As it is often done in the literature, in the sequel for conditions such as (62), we will use notation like \( p \) for both the scheduling signal \( p : \mathcal{Z} \to \mathbb{P} \) and for constant vectors \( p \in \mathbb{P} \) to describe all possible values \( p_k \), i.e., the value of the signal \( p \) at time moment \( k \in \mathcal{Z} \) can take. Where possible confusion might arise, we will clarify in the text which notion we refer to.

B.2 Quadratic performance analysis

Beyond stability, design of a controller \([5]\) can be used to ensure performance specifications on the closed-loop behavior. Performance of \((5)\) can be expressed in various forms, such as the quadratic infinite-time horizon cost in \([30]\). The following condition can be used to test whether a given \( K \) stabilizes \( \mathcal{S} \) and achieves the smallest possible bound on \((50)\) under all \( p \in \mathbb{P}^Z \), i.e., achieves a minimal \( \sup_{(x,p)\in \mathcal{B}_{CL}} J(x, K(p)x) \).

**Lemma 3** (Optimal quadratic performance LPV state-feedback \([31]\)). The controller \( K \) in \([5]\) asymptotically stabilizes \( \mathcal{S} \) and achieves the minimum of \( \sup_{(x,p)\in \mathcal{B}_{CL}} J(x, K(p)x) \), if \( \sup_{p \in \mathbb{P}} \text{trace}(P(p)) < \infty \) with \( P : \mathbb{P} \to \mathbb{S}_{nn}^+ \) symmetric is minimal among all possible choices of \( P, K \) that satisfy

\[
\begin{bmatrix}
P(p) & (s)^T & (s)^T & (s)^T \\
A_{CL}(p)P(p) & P(p^+) & 0 & 0 \\
Q_2^+P(p) & 0 & I_{n_x} & 0 \\
R_2^+K(p)P(p) & 0 & 0 & I_{n_u} \\
\end{bmatrix} > 0,
\]

(63a)

\[P(p) > 0,
\]

(63b)

for all \( p, p^+ \in \mathbb{P} \).
\textbf{Proof.} If there exists a quadratic Lyapunov function \( V(k) := x_k^\top \tilde{P}(p_k)x_k > 0 \), with \( \tilde{P}(p_k) > 0 \) for all \( p_k \in \mathbb{P} \), such that

\[
V(k + 1) - V(k) \leq -(x_k^\top Q x_k + u_k^\top R u_k), \quad \forall p_k \in \mathbb{P}
\]

with \( u_k = K(p_k)x_k \), then, as the right-hand side is negative semidefinite due to \( Q \succeq 0, R > 0 \), \( K \) is an asymptotically stabilizing controller for the LPV system \([2]\), ensuring that \( x_k \to 0 \) as \( k \to \infty \).

Rewriting \(^{(64)}\), we get

\[
x_{k+1}^\top \tilde{P}(p_{k+1})x_{k+1} - x_k^\top \tilde{P}(p_k)x_k \leq -x_k^\top (Q + K^\top (p_k)RK(p_k)) x_k.
\]

Summing all terms from 0 to \( \infty \) yields

\[
\sum_{k=0}^{\infty} x_{k+1}^\top \tilde{P}(p_{k+1})x_{k+1} - x_k^\top \tilde{P}(p_k)x_k \leq -J(x, K(x)).
\]

If \(^{(65)}\) holds, then, in terms of \(^{(64)}\), \( x_k \to 0 \) as \( k \to \infty \), and the telescopic sum on the left-hand side of \(^{(66)}\) reduces to \(-x_0^\top \tilde{P}(p_0)x_0\), i.e., \( x_0^\top \tilde{P}(p_0)x_0 \geq J(x, K(x)) \). Hence, \( x_0^\top \tilde{P}(p_0)x_0 \) is an upper bound on \( J \), given that \(^{(65)}\) holds. Inequality \(^{(65)}\) is implied by

\[
A_{\text{cl}}(p)\tilde{P}(p^+)A_{\text{cl}}(p) - \tilde{P}(p) + Q + K^\top(p)RK(p) \preceq 0,
\]

holds for all \( p, p^+ \in \mathbb{P} \). Then, minimizing \( J \) over all possible state and scheduling trajectories, i.e., \( \sup_{(x,p) \in \mathbb{P}_{\text{cl}}} J(x, K(p)x) \), can be rewritten as

\[
\begin{align*}
\min_{\tilde{P}, K} & \quad x_0^\top \tilde{P}(p_0)x_0 \quad \text{ (68a)} \\
\text{s.t.} & \quad \forall p, p^+ \in \mathbb{P} : \tilde{P}(p) > 0 \quad \text{and} \quad \text{(67) holds,} \quad \text{(68b)}
\end{align*}
\]

for all possible initial conditions \( x_0 \in \mathbb{R}^{n_x} \) and \( p_0 \in \mathbb{P} \). By using the eigendecomposition of \( \tilde{P}(p) \), \(^{(68)}\) is equivalent with minimizing \( \text{trace}(\tilde{P}(p)) \) subject to \(^{(68b)}\) over all \( p \in \mathbb{P} \). Moreover, applying the Schur complement on \(^{(67)}\) w.r.t. \( \text{blkdiag}(\tilde{P}(p^+), I_{n_x}, I_{n_u}) \), followed by a congruence transformation \( \text{blkdiag}(p, I_{n_x}, I_{n_u}, I_{n_u}) \) on the resulting matrix inequality, where \( P(p) = \tilde{P}^{-1}(p) \), yields

\[
\begin{align*}
\min_{\tilde{P}, K} & \quad \sup_{p \in \mathbb{P}} \text{trace}(P(p)) \\
\text{s.t.} & \quad \forall p, p^+ \in \mathbb{P} : \text{ (63) holds,}
\end{align*}
\]

completing the proof. \hfill \blacksquare

\textbf{B.3} \( H_2^p \) performance analysis

In order to consider induced gains based performance metrics, such as the extended \( H_2 \) performance, widely used in linear control, we will introduce a representation of general controller configurations in terms of the \textit{generalized plant} concept. Consider the following LPV formulation of the plant to be controlled with state-feedback

\[
\begin{align*}
x_{k+1} & = A(p_k)x_k + B(p_k)u_k + B_w(p_k)w_k, \quad \text{(69a)} \\
y_k & = x_k \quad \text{(69b)} \\
z_k & = C_x(p_k)x_k + D_{zu}(p_k)u_k + D_{zw}(p_k)w_k, \quad \text{(69c)}
\end{align*}
\]

where \( w_k \in \mathbb{R}^{n_w} \) is a signal being the collection of generalized disturbances, such as reference signals, load disturbances, etc., while \( z_k \in \mathbb{R}^{n_z} \) is the generalized performance signal, being the collection of control objectives, such as tracking error or input usage. \( y_k = x_k \) expresses that the observed outputs of the plant are available for the controller, which is in line with the considered state-feedback objective. To avoid complexity by using performance shaping filters and to be compatible with the considered quadratic performance concept in \(^{(30)}\), consider

\[
C_x = \begin{bmatrix} Q^{\frac{1}{2}} & 0 \end{bmatrix}, \quad D_{zu} = \begin{bmatrix} 0 & R^\frac{1}{2} \end{bmatrix}, \quad D_{zw} = 0.
\]

This implies that \( w_k \in \mathbb{R}^{n_w} \) and \( z_k \in \mathbb{R}^{n_x+n_z} \). Closing the loop with feedback law \(^{(5)}\), yields the closed-loop system \(^{(52)}\).

Using the definition of the \( H_2^p \)-norm in \(^{(33)}\), the following lemma, which is a slight modification of \(^{(35)}\) Lem. 1, allows to find a bound \( \gamma \) on the \( H_2^p \)-norm of the closed-loop system and guarantee stability of \(^{(52)}\).
Lemma 4 (\(H_\infty^2\)-norm performance LPV state-feedback). The controller \(K\) in (5) asymptotically stabilizes (2) and achieves an \(H_\infty^2\)-norm of (32) that is less than a given \(\gamma > 0\), if there exist symmetric matrices \(P: \mathbb{P} \rightarrow \mathbb{S}^{n_x}\) and \(S \in \mathbb{S}^{n_x}\) such that

\[
\begin{bmatrix}
P(p^+) - I_{n_x} & A_{CL}(p)P(p) \\
(\ast)^{\top} & P(p)
\end{bmatrix} \succ 0, \\
\begin{bmatrix}
S & R^2 K(p)P(p) \\
(\ast)^{\top} & P(p)
\end{bmatrix} \succ 0, \\
\text{trace}(QP(p)) + \text{trace}(S) < \gamma^2,
\]

(71a) (71b) (71c)

for all \(p, p^+ \in \mathbb{P}\).

Proof. Applying the result of [55, Lem. 1] to (32), denoted by \(\Sigma_{CL}\), the \(H_\infty^2\)-norm can be written as

\[
||\Sigma_{CL}||_{H_\infty^2}^2 = \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N} \text{trace}\left((C(p_k)\hat{P}(p_k)C^{\top}(p_k))\right),
\]

(72)

where \(\hat{P}(p_k)\) is the controllability Gramian that satisfies

\[
A_{CL}(p_k)\hat{P}(p_k)A_{CL}^{\top}(p_k) - \hat{P}(p_{k+1}) + I_{n_x} = 0
\]

(73)

with \(\hat{P}(p_0) = 0\). Inequality (71a) implies that \(P(p) > 0\) and

\[
A_{CL}(p)P(p)A_{CL}^{\top}(p) - P(p^+) + I_{n_x} < 0,
\]

(74)

which ensures asymptotic stability of (32). Furthermore, there exists a \(W(p) = W^{\top}(p) > 0\) such that

\[
A_{CL}(p)P(p)A_{CL}^{\top}(p) - P(p^+) + I_{n_x} + W(p) = 0,
\]

(75)

which implies (73) for all \(k \geq 0\) with \(P > \hat{P}\). Moreover, (71b) implies that

\[
S > R^2 K(p)P(p)K^{\top}(p)R^2.
\]

(76)

Substituting the latter in (71c) gives

\[
\text{trace}(QP(p)) + \text{trace}(S) > \text{trace}(QP(p)) + \text{trace}(R^2 K(p)P(p)K^{\top}(p)R^2),
\]

(77)

where the right-hand side can be rewritten using the cyclic property of the trace as

\[
\text{trace}(C(p)P(p)C^{\top}(p)),
\]

(78)

with \(C(p) = [(Q^{\top}(\ast)^{\top}) (R^2 K(p_k))^{\top}]^{\top}\). Hence, finding a \(\gamma\) for which (71) holds for all \(p \in \mathbb{P}\) ensures via (71c) that (72) is upper bounded by \(\gamma\), concluding the proof. \(\blacksquare\)

B.4 \(\ell_2\)-gain performance analysis

The (induced) \(\ell_2\)-gain, as introduced in Section 5.3, can be seen as the ‘generalization’ of the \(H_\infty\) norm to the LPV case and is a widely used performance metric in model-based LPV control. The following result allows to analyze stability and \(\ell_2\)-gain performance of the closed-loop system (32).

Lemma 5 (\(\ell_2\)-gain performance LPV state-feedback). The controller \(K\) in (5) asymptotically stabilizes (2) and achieves an \(\ell_2\)-gain of (32) that is less than a given \(\gamma > 0\), if there exists a symmetric matrix \(P: \mathbb{P} \rightarrow \mathbb{S}^{n_x}\) such that

\[
\begin{bmatrix}
P(p) & (\ast)^{\top} & (\ast)^{\top} & 0 \\
A_{CL}(p)P(p) & P(p^+) & 0 & 0 & I_{n_x} \\
Q^2 P(p) & 0 & \gamma I_{n_x} & 0 & 0 \\
R^2 K(p)P(p) & 0 & 0 & \gamma I_{n_x} & 0 \\
0 & I_{n_x} & 0 & 0 & \gamma I_{n_x}
\end{bmatrix} \succ 0, \quad \quad P(p) > 0,
\]

(79a) (79b)

for all \(p, p^+ \in \mathbb{P}\).
Proof. Following standard formulation, e.g., [37], for \( \ell_2 \)-gain performance for the closed-loop LPV system [32], the following condition should hold for all \( p, p^+ \in \mathbb{P} \):

\[
\begin{bmatrix}
(\ast)^T \hat{P}(p^+) A_{CL}(p) - \hat{P}(p) & (\ast)^T P(p^+) \hat{P}(p^+) - \gamma I_{n_x} \\
\hat{P}(p^+) A_{CL}(p) & \frac{1}{\gamma} [s]^T \begin{bmatrix} \hat{Q}^2 + R^T K(p) \\
R^T K(p) \end{bmatrix} 0 \end{bmatrix} < 0
\]

(80)

with \( \hat{P}^T(p) = \hat{P}(p) > 0 \). The above matrix inequality can be rewritten as

\[
0 \prec \begin{bmatrix}
\hat{P}(p) & 0 \\
0 & \gamma I_{n_x}
\end{bmatrix} - [s]^T \begin{bmatrix}
\hat{P}^{-1}(p^+) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\hat{P}(p^+) A_{CL}(p) & \hat{P}(p^+)
\end{bmatrix}.
\]

(81)

Now, applying the Schur complement with respect to \( \text{blkdiag}(\hat{P}^{-1}(p^+), \frac{1}{\gamma} I_{n_x}, \frac{1}{\gamma} I_{n_x}) \) and the congruence transformation \( \text{blkdiag}(P(p), I_{n_x}, P(p^+), I_{n_x}, I_{n_x}) \) on the resulting matrix inequality, where \( P(p) = \hat{P}^{-1}(p) \), yields the condition

\[
\begin{bmatrix}
P(p) & 0 & (\ast)^T & (\ast)^T & (\ast)^T \\
0 & A_{CL}(p) P(p) & \gamma I_{n_x} & I_{n_x} & 0 \\
(\ast)^T P(p) & \gamma I_{n_x} & P(p^+) & 0 & 0 \\
R^T K(p) P(p) & 0 & 0 & \gamma I_{n_x} & 0 \\
0 & 0 & 0 & 0 & \gamma I_{n_x}
\end{bmatrix} > 0.
\]

Finally, pre- and post-multiplication of the above matrix inequality by the permutation matrix

\[
P = \begin{bmatrix}
I_{n_x} & 0 & 0 & 0 & 0 \\
0 & I_{n_x} & 0 & 0 & 0 \\
0 & 0 & I_{n_x} & 0 & 0 \\
0 & 0 & 0 & I_{n_x} & 0 \\
0 & 0 & 0 & 0 & I_{n_x}
\end{bmatrix}
\]

and its transpose, respectively, yields condition (79a).

\( \blacksquare \)

B.5 Model-based controller synthesis

When \( K \) is not known, the matrix inequality conditions in Lemmas 2-5 are nonlinear in the decision variables (controller parameters and Lyapunov functions) and provide an infinite number of inequalities that need to be satisfied for every point in \( \mathbb{P} \). To resolve these problems and recast these conditions to tractable controller synthesis methods, the change of variables \( Y(p) = K(p) P(p) \) can be applied in the conditions (62a), (63a), (71a), (71b) and (79a). Furthermore, the dependence on \( p \) must be defined for \( K \) and \( P \) (i.e., for \( Y(p) \)), which has an important impact on the complexity of the synthesis problem and the achievable control performance. A natural choice for the scheduling dependence of \( K \) is to assume static-affine dependence on \( p \) as in (55). On the other hand, the choice of the scheduling dependence of \( P \) is not trivial, as discussed in [32], and it is often accomplished in terms of the choice of \( Y \). These considerations allow to convert the corresponding LPV state-feedback controller synthesis problems into the minimization of a linear cost, subject to constraints defined by an infinite set of LMIs. The last step in making the synthesis problems tractable is reducing the set of constraints to a finite number of LMIs, such that the resulting SDP problem can be solved using efficient solvers. There are multiple methods available to accomplish this. As can be seen in the main body of this paper, we use the full-block \( S \)-procedure, see Appendix A, to recast the synthesis problems to an SDP of finite size.

Author Biographies

Chris Verhoek received his B.Sc. degree in Mechatronics from the Avans University of Applied Sciences and M.Sc. degree (Cum Laude) in Systems and Control from the Eindhoven University of Technology (TU/e), in 2017 and 2020 respectively. His M.Sc. thesis was selected as best thesis of the Electrical Engineering department in the year 2020. He is currently pursuing a Ph.D. degree under the supervision of Roland Tóth and Sofie Haesaert at the Control Systems Group, Dept. of Electrical Engineering, TU/e. His main research interests include (data-driven) analysis and control of nonlinear and LPV systems and learning-for-control techniques with stability and performance guarantees.
Roland Tóth received his Ph.D. degree with Cum Laude distinction at the Delft Center for Systems and Control (DCSC), Delft University of Technology (TUDelft), Delft, The Netherlands in 2008. He was a Post-Doctoral Research Fellow at TUDelft in 2009 and at the Berkeley Center for Control and Identification, University of California, Berkeley in 2010. He held a position at DCSC, TUDelft in 2011-12. Currently, he is a Full Professor at the Control Systems Group, Eindhoven University of Technology (TU/e) and a senior researcher at Systems and Control Laboratory, Institute for Computer Science and Control in Budapest (SZTAKI), Hungary. He is an Associate Editor of the IEEE Transactions on Control Systems Technology and he was the general chair of the 3rd IFAC Workshop on Linear Parameter-Varying Systems in 2019. His research interests are in identification and control of linear parameter-varying (LPV) and nonlinear systems, developing machine learning methods with performance and stability guarantees for modeling and control, model predictive control and behavioral system theory. On the application side, his research focuses on advancing reliability and performance of precision mechatronics and autonomous robots/vehicles with LPV and learning-based motion control. Prof. Tóth received the TUDelft Young Researcher Fellowship Award in 2010, the VENI award of The Netherlands Organisation for Scientific Research in 2011 and the Starting Grant of the European Research Council in 2016. He and his research team have participated in several international (FP7, IT2, etc.) and national collaborative research grants.

Hossam S. Abbas completed his Ph.D. (Summa Cum Laude) with a focus on control systems engineering from Hamburg University of Technology, Germany, in 2010. He was a research fellow at Hamburg University of Technology in 2011 and at Eindhoven University of Technology, the Netherlands, in 2013. He was a senior researcher (Humboldt fellow) at the Institute for Electrical Engineering in Medicine, Universität zu Lübeck, Germany, in 2017, and at the Medical Laser Center in Lübeck in 2019. Currently, he is a substitute professor at the Department of Electrical and Information Engineering, Faculty of Engineering, Kiel University, Germany. Additionally, he is affiliated with the Institute for Electrical Engineering in Medicine, University of Lübeck, where he is leading the Control Systems Group, and the Electrical Engineering Department, Faculty of Engineering, Assiut University, Egypt. His research interests include system identification, control of linear parameter-varying and nonlinear systems, model predictive control, and data-driven control, with applications in autonomous robots and vehicles as well as biomedical systems.