GLOBAL $W^{2,\delta}$ ESTIMATES FOR SINGULAR FULLY NONLINEAR ELLIPTIC EQUATIONS WITH $L^n$ RIGHT HAND SIDE TERMS

DONGSHENG LI AND ZHISU LI

Abstract. We establish in this paper a priori global $W^{2,\delta}$ estimates for singular fully nonlinear elliptic equations with $L^n$ right hand side terms. The method is to slide paraboloids and barrier functions vertically to touch the solution of the equation, and then to estimate the measure of the contact set in terms of the measure of the vertex point set. To derive global estimates from $L^n$ data, the Hardy-Littlewood maximal functions, appropriate localizations and a new type of covering argument are adopted. These methods also provide us a more direct proof of the $W^{2,\delta}$ estimates for (nonsingular) fully nonlinear elliptic equations established by L. A. Caffarelli and X. Cabré.

1. Introduction

In the present paper, we derive a priori global $W^{2,\delta}$ estimates for solutions of the singular elliptic equations including those of the following types:

(a) the equation

\begin{equation}
\operatorname{tr}(A(x)D^2u) + b(x) \cdot Du = f(x)|Du|^\gamma,
\end{equation}

where $0 \leq \gamma < 1$, $A$ is uniformly elliptic, $b$ is bounded and $f \in L^n$;

(b) the singular fully nonlinear elliptic equation

\begin{equation}
|Du|^{-\gamma} F(D^2u, Du, u, x) = f(x),
\end{equation}

where $0 \leq \gamma < 1$, $F(0, 0, \cdot, \cdot) \equiv 0$, $F$ is uniformly elliptic (see [5]), and $f \in L^n$;

(c) the famous $p$-Laplace equation

\begin{equation}
\Delta_p u := \operatorname{div}(|Du|^{p-2} Du) = f(x),
\end{equation}

where $1 < p \leq 2$ and $f \in L^n$.

For brevity, we consider solutions of singular fully nonlinear elliptic inequalities of certain type which include solutions of all the above equations. Namely, our main result will be stated in a more generalized form as follows.

Theorem 1.1. Let $0 < \lambda \leq \Lambda < +\infty$ and $0 \leq \gamma < 1$. Suppose $u \in C^0(\overline{B_1})$ is a viscosity solution of the singular fully nonlinear elliptic inequalities

\begin{equation}
|Du|^{-\gamma} P_{\lambda}(D^2u) - |Du|^{1-\gamma} \leq f \leq |Du|^{-\gamma} P_{\Lambda}(D^2u) + |Du|^{1-\gamma} \text{ in } B_1,
\end{equation}

Date: August 6, 2018.

Key words and phrases. fully nonlinear elliptic equations, global $W^{2,\delta}$ estimates, singular elliptic equations; MSC (2010): 35B45, 35D40, 35J60, 35J75.

This research is supported by NSFC.11671316. Zhisu Li is the corresponding author.
where $B_1$ is the unit open ball of $\mathbb{R}^n$, $\mathcal{P}_{\Lambda,\Lambda}^\pm$ are the Pucci extremal operators, and $f \in C^0 \cap L^n(B_1)$. Then $u \in W^{2,\delta}(B_1)$ and

$$
\|u\|_{W^{2,\delta}(B_1)} \leq C \left( \|u\|_{L^\infty(B_1)} + \|f\|_{L^n(B_1)}^{1/\gamma} \right)
$$

for any $\delta \in (0,\sigma)$, where $\sigma = \sigma(n, \lambda, \Lambda, \gamma) > 0$ and $C = C(n, \lambda, \Lambda, \gamma, \delta) > 0$.

This theorem improves essentially our previous results in [13], where the right hand side term $f$ of the equation is required to be $L^n$, and the estimate corresponding to (1.5) is just

$$
\|u\|_{W^{2,\sigma}(B_1)} \leq C \left( \|u\|_{L^\infty(B_1)} + \|f\|_{L^n(B_1)} \right).
$$

The main contribution here is in developing a systematic way to deal with the $L^n$ data, and in using delicate localization and covering arguments to derive global estimates from it in a straightforward way. Roughly speaking, first, by sliding paraboloids and some appropriate localizing barrier functions from below and above to touch the solution, and then estimating the low bound of the measure of the set of contact points by the measure of the set of vertex points, we establish a new density estimate which is corresponding to the classical Alexandroff-Bakelman-Pucci (ABP for short) estimate; then, applying a new kind of covering technique with careful localization, we obtain the desired global $W^{2,\delta}$ estimates. These methods also provide us a more direct proof of the interior $W^{2,\delta}$ estimates for (nonsingular) fully nonlinear elliptic inequalities established by L. A. Caffarelli and X. Cabré [6] (which now is recovered by Theorem 1.1 as special cases, rather than by our previous results in [13]), although the underlying key ideas are the same.

The sliding paraboloid argument we mentioned above has originated in the work of X. Cabré [4] and continued in the work of O. Savin [15], see also [12], [7] and [8]. We now give some other historical remarks concerning the $W^{2,\delta}$ estimates and the singular elliptic equations.

In 1986, F.-H. Lin [14] first established the $W^{2,\delta}$ estimates $\|D^2 u\|_{\mathcal{L}^\delta(B_1)} \leq C \|f\|_{L^n(B_1)}$ for solutions of the linear uniformly elliptic equations $a_{ij}(x) u_{ij} = f(x)$ with $u = 0$ on $\partial B_1$ and $f \in L^n(B_1)$. His method employs the Fabes-Stroock type reverse Hölder inequality, estimates for Green’s function and the ABP estimate. Later, L. A. Caffarelli and X. Cabré [6] (see also [5]) applied ABP estimate, Calderón-Zygmund cube decomposition technique, barrier function method and touching by tangent paraboloid method to obtain interior $W^{2,\delta}$ estimates for viscosity solutions of the fully nonlinear elliptic inequalities

$$
\mathcal{P}_{\Lambda,\Lambda}^- (D^2 u) \leq f \leq \mathcal{P}_{\Lambda,\Lambda}^+ (D^2 u),
$$

which can be viewed as an original form of our inequalities (1.4). As we know, the $W^{2,\delta}$ estimates have several important applications in the study of the elliptic partial differential equations, such as deriving $W^{2,p}$ estimates ($p$ is large, see [8] or [6]), proving partial regularity (see [1] or [8]), and exploring the convergence of blow down solutions (see [17]), and so on.

The investigation of singular elliptic equations of the types (1.1) and (1.2) has made much progress in recent years. The corresponding comparison principle (see [2]), ABP estimate (see [3]), Harnack inequality (see [10]) and $C^{1,\alpha}$ estimate (see [3]) have already been established. To derive $W^{2,\delta}$ estimate for singular equations, one
may naturally think that it can be an easy consequence of the classical results \cite{6}, once we have a universal control of $\|Du\|_{L^\infty}$, for instance, some $C^{1,\alpha}$ estimate, like \cite{3}. But this is always highly restricted and sometimes fails to be valid. Our method can deal with a large class of equations as illustrated above, since it does not depend on any a priori estimate of $Du$ and it does not use maximum principles. Moreover, our estimates are global, and the proof is straightforward in the sense that we do not need to separate it into interior estimates and boundary estimates. Instead, the singular case and the nonsingular case, the interior case and the boundary case, are treated all together by using delicate localization and a new type of covering lemma.

For $W^{2,\delta}$ estimate of the singular $p$-Laplace equation \eqref{1.3}, P. Tolksdorf \cite{16} proved that each $W^{1,p} \cap C^0(B_1)$ weak solution of \eqref{1.3} in $B_1$ with $f \in L^\infty(B_1)$ is $W^{2,p}_{loc} \cap W^{1,p+2}_{loc}(B_1)$. Since the $p$-Laplacian can be written as

$$\Delta_p u = |Du|^{-(2-p)} \left( \delta_{ij} - (2-p) \frac{D_i u D_j u}{|Du|^2} \right) D_{ij} u,$$

applying our Theorem 1.1 to the singular $p$-Laplace equation \eqref{1.3} with $f \in L^\infty(B_1)$, we obtain a new global $W^{2,\delta}$ estimate.

The paper is organized as follows. In Section 2, we give some notations and collect some preliminary lemmas including a new type of covering lemma. In Section 3, we first normalize Theorem 1.1 to Lemma 3.1 by rescaling argument in Subsection 3.1, then in Subsection 3.2, we establish the key density lemma and the measure decay estimate lemma, with the help of them we finally give the proof of Lemma 3.1 in Subsection 3.3.

2. Some preliminaries

In this paper, we denote by $S(n)$ the linear space of symmetric $n \times n$ real matrices and $I$ the identity matrix.

To make the sliding and touching idea more rigorous and clear, we introduce the following notations and terminologies.

Given two functions $u$ and $v : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and a point $x_0 \in \Omega$, we say that $u$ touches $v$ by below at $x_0$ in $\Omega$ and denote it briefly by $u \lessdot v$ in $\Omega$, if $u(x_0) = v(x_0)$ and $u(x) \leq v(x)$, $\forall x \in \Omega$.

For a given continuous function $u : U \subset \mathbb{R}^n \to \mathbb{R}$, we slide the concave paraboloid (of opening $\kappa > 0$ and of vertex $y$)

$$-\frac{\kappa}{2} |x - y|^2 + C$$

vertically from below in $U$ (by increasing or decreasing $C$) till it touches the graph of $u$ for the first time. If the contact point is $x_0$, we then have

$$C = u(x_0) + \frac{\kappa}{2} |x_0 - y|^2 = \inf_{x \in U} \left( u(x) + \frac{\kappa}{2} |x - y|^2 \right).$$
Given a closed set $V \subset \mathbb{R}^n$ and a continuous function $u : B_1 \to \mathbb{R}$, we now introduce the definitions of the contact sets as follows:

$$T^-(V) := T^-_{\kappa}(u, V) := \left\{ x_0 \in B_1 \mid \exists y \in V \text{ such that } u(x_0) + \frac{\kappa}{2} |x_0 - y|^2 = \inf_{x \in B_1} \left( u(x) + \frac{\kappa}{2} |x - y|^2 \right) \right\},$$

and

$$T^+(V) := T^+_{\kappa}(u, V) := T^-_{\kappa}(-u, V).$$

For simplicity, we will write $T^\pm$ instead of $T^\pm_{\kappa}(u, \overline{B_1})$ when there is no confusion. It is obvious that $T^\pm$ are closed in $B_1$. We remark further that the contact set $T^-(u, V)$ has the twofold uses of $\{ u = \Gamma_u \}$ and $\mathcal{G}_M(u, \Omega)$ in [6]: by the former, we communicate with the equation; by the latter, we measure the second derivatives of the solution.

Given $0 < \lambda \leq \Lambda$, we define the Pucci extremal operators (see also [6]) by

$$\mathcal{P}^-_{\lambda\Lambda}(X) := \lambda \sum_{e_i(X) < 0} e_i(X) + \Lambda \sum_{e_i(X) > 0} e_i(X),$$

and

$$\mathcal{P}^+_{\lambda\Lambda}(X) := \Lambda \sum_{e_i(X) < 0} e_i(X) + \lambda \sum_{e_i(X) > 0} e_i(X),$$

where $X \in S(n)$ and $e_i(X)$ denote the eigenvalues of $X$. For brevity, we will always write $\mathcal{P}^+_{\lambda\Lambda}(X)$ as $\mathcal{P}^+(X)$. For completeness and convenience, we now collect some basic properties of the Pucci extremal operators as follows:

(i) $\mathcal{P}^\pm(rX) = r\mathcal{P}^\pm(X)$, $\mathcal{P}^\pm(\Lambda X) = -r \mathcal{P}^\pm(X)$, $\forall X \in S(n)$, $\forall r \geq 0$.

(ii) $\mathcal{P}^-(X) + \mathcal{P}^-(Y) \leq \mathcal{P}^-(X + Y) \leq \mathcal{P}^-(X) + \mathcal{P}^+(Y) \leq \mathcal{P}^+(X + Y) \leq \mathcal{P}^+(X) + \mathcal{P}^+(Y)$, $\forall X, Y \in S(n)$.

(iii) If $X, Y \in S(n)$ and $X \leq Y$, then $\mathcal{P}^\pm(X) \leq \mathcal{P}^\pm(Y)$.

(iv) $\mathcal{P}^-_{\lambda\Lambda}(X) = \lambda \text{tr} X$ and $\mathcal{P}^+_{\lambda\Lambda}(X) = \lambda \text{tr} X$, provided $X \in S(n)$ and $X \geq 0$.

Now we recall the definition of the viscosity solution (see [6]). For example, we say that $u \in C^0(B_1)$ satisfies

$$F(D^2u, Du, u, x) \leq f \text{ in } B_1$$

in the viscosity sense, if $\forall \varphi \in C^2(B_1)$, $\forall x_0 \in B_1$,

$$\varphi \geq u \text{ in } U(x_0) \Rightarrow \varphi \left( x_0 \right) \geq F \left( D^2 \varphi (x_0), D\varphi(x_0), \varphi(x_0), x_0 \right) \leq f(x_0),$$

where $U(x_0) \subset B_1$ is an open neighborhood of $x_0$.

For $g \in L^1(\Omega)$, the Hardy-Littlewood maximal function of $g$ is defined by

$$\mathcal{M}(g)(x) := \sup_{r > 0} \frac{1}{|B_r(x)|} \int_{B_r(x) \cap \Omega} |g(y)| dy, \forall x \in \Omega.$$
We will use the well known weak type (1,1) property of the Hardy-Littlewood maximal operator $\mathcal{M}$, that is
\[ |\{ x \in \Omega : \mathcal{M}(g)(x) > t \}| \leq C t^{-1} \| f \|_{L^1(\Omega)}, \forall \ t > 0, \]
where $C = C(n) > 0$ depends only on the dimension $n$.

The following equivalent description of $L^p$-integrability is also needed.

**Lemma 2.1.** (see [3] Lemma 7.3.) Let $g$ be a nonnegative and measurable function in a bounded domain $\Omega \subset \mathbb{R}^n$. Suppose that $\eta > 0$, $M > 1$ and $0 < p < \infty$. Then
\[ g \in L^p(\Omega) \iff s := \sum_{k=1}^{\infty} M^{pk} \left| \left\{ x \in \Omega \mid g(x) > \eta M^k \right\} \right| < \infty, \]
and
\[ C^{-1} s \leq \| g \|_{L^p(\Omega)} \leq C (s + |\Omega|), \]
where $C > 0$ is a constant depending only on $\eta$, $M$ and $p$.

Finally, we introduce the following Vitali-type covering lemma modified from those in [12] and [13]. This lemma plays a similar role as the Calderón-Zygmund cube decomposition lemma (see [5] and [6]) usually does, but with the help of it we can obtain global estimates directly.

**Lemma 2.2.** ($(\theta, \Theta)$-type covering lemma). Let $E \subset F \subset B_1$ be measurable sets and $0 < \theta < \Theta < 1$ such that
\begin{itemize}
  \item[(i)] $|E| > \theta |B_1|$, and
  \item[(ii)] for any ball $B \subset B_1$, if $|B \cap E| \geq \theta |B|$, then $|B \cap F| \geq \Theta |B|$.
\end{itemize}
Then
\[ |B_1 \setminus F| \leq \left( 1 - \frac{\Theta - \theta}{5^n} \right) |B_1 \setminus E|. \]

**Proof.** It suffices to prove that
\[ |F \setminus E| \geq \frac{\Theta - \theta}{5^n} |B_1 \setminus E|. \]

By the Lebesgue differentiation theorem, there exists $S \subset B_1 \setminus E$, such that $|S| = |B_1 \setminus E|$ and
\[ \lim_{r \to 0} \frac{|B_r(x) \cap E|}{|B_r(x)|} = 0, \forall x \in S. \]

Hence, for each $x \in S$, there exist balls, say $B$, satisfying $x \in B \subset B_1$ and $|B \cap E| \leq \theta |B|$; we choose one of the biggest of them and denote it by $B^x$.

We assert that $|B^x \cap F| \geq \Theta |B^x|$. Otherwise, suppose that $|B^x \cap F| < \Theta |B^x|$. Since $|B_1 \cap E| > \theta |B_1|$, $|B^x \cap E| \leq \theta |B^x|$ and hence $B^x \not\subset B_1$, we may enlarge $B^x$ a little bit, denoted by $B^x$, such that $B^x \subset \bar{B}^x \subset B_1$ and $|B^x \cap F| < \Theta |\bar{B}^x|$. By the hypothesis (ii) of the lemma, $|B^x \cap E| < \theta |B^x|$, which contradicts the definition of $B^x$.

Furthermore, since $|B^x \setminus E| \geq (1 - \theta)|B^x|$, it follows from the above assertion that $|B^x \cap F \setminus E| \geq (\Theta - \theta)|B^x|$. 


Now consider the covering \( \bigcup_{x \in S} B^{x} \supseteq S \). By the Vitali covering lemma, there exists an at most countable set of \( x_i \in S \), such that \( \{ B^{x_i} \}_i \) are disjoint and \( \bigcup_i B^{x_i} \supseteq S \). Hence we have
\[
|F \setminus E| \geq \left| \left( \bigcup_i B^{x_i} \right) \cap F \setminus E \right| = \left| \bigcup_i (B^{x_i} \cap F \setminus E) \right| = \sum_i |B^{x_i} \cap F \setminus E|
\]
\[
\geq (\Theta - \theta) \sum_i |B^{x_i}| = \frac{\Theta - \theta}{5^n} \sum_i |5B^{x_i}| \geq \frac{\Theta - \theta}{5^n} \left| \bigcup_i 5B^{x_i} \right|
\]
\[
\geq \frac{\Theta - \theta}{5^n} |S| = \frac{\Theta - \theta}{5^n} |B_1 \setminus E|.
\]
This completes the proof of the lemma. \( \square \)

3. Proof of Theorem 1.1

3.1. Theorem 1.1 can be normalized to Lemma 3.1

To prove Theorem 1.1, it suffices to prove the following lemma.

**Lemma 3.1.** Let \( 0 \leq \gamma < 1 \). Assume that \( u \in C^0(B_1) \) satisfies (1.4) with \( f \in C^0 \cap L^n(B_1) \) in the viscosity sense. Then there exist constants \( \sigma = \sigma(n, \Lambda, \gamma) > 0 \) and \( \epsilon_1 = \epsilon_1(n, \Lambda, \Lambda) > 0 \), such that for any \( \delta \in (0, \sigma) \), if \( \|u\|_{L^\infty(B_1)} \leq 1/16 \) and \( \|f\|_{L^n(B_1)} \leq \epsilon_1 \), then
\[
\|u\|_{W^{2, \gamma}(B_1)} \leq C,
\]
where \( C = C(n, \Lambda, \gamma, \delta) > 0 \).

Indeed, suppose \( u \) satisfies the hypothesis of Theorem 1.1. Let
\[
\alpha := \left( 16 \|u\|_{L^\infty(B_1)} + \left( \epsilon_1^{-1} \|f\|_{L^n(B_1)} \right)^{\frac{1}{\gamma - \gamma}} + \varepsilon \right)^{-1}
\]
for any \( \varepsilon > 0 \). Then the scaled function \( \tilde{u}(x) := \alpha u(x) \) solves
\[
|D\tilde{u}|^{-\gamma} \mathcal{P}_{\Lambda, \gamma}(D^2\tilde{u}) - |D\tilde{u}|^{1-\gamma} f =: \tilde{f} \leq |D\tilde{u}|^{-\gamma} \mathcal{P}_{\Lambda, \gamma}(D^2\tilde{u}) + |D\tilde{u}|^{1-\gamma} \text{ in } B_1,
\]
and satisfies \( \|\tilde{u}\|_{L^\infty(B_1)} \leq 1/16 \) and \( \|\tilde{f}\|_{L^n(B_1)} \leq \epsilon_1 \). By Lemma 3.1 we have
\[
\|\tilde{u}\|_{W^{2, \gamma}(B_1)} \leq C = C(n, \Lambda, \gamma, \delta).
\]
Scaling back to \( u \) and letting \( \varepsilon \to 0 \), we obtain
\[
\|u\|_{W^{2, \gamma}(B_1)} \leq C \alpha^{-1} \leq C \left( \|u\|_{L^\infty(B_1)} + \|f\|_{L^n(B_1)} \right) .
\]

Theorem 1.1 thereby is proved.

3.2. Key lemmas for the proof of Lemma 3.1

Lemma 3.1 will be established via several lemmas of this subsection. The most important one is the density estimate lemma, Lemma 3.2, which is a key lemma in this paper and can be viewed as a measure theoretic ABP estimate. The strategy for the proof of Lemma 3.2 is modified from those in [15], [7] and [13]. Since the right hand side term \( f \) belongs to \( L^n \), the Hardy-Littlewood maximal functions and certain careful localization techniques have to be employed here. Note also that, in order to obtain global regularity, it is crucial to show that the contact sets are contained in the interior of \( B_1 \), in the proof of the following lemma.
Lemma 3.2. Let $0 \leq \gamma < 1$, $0 < \theta_0 < 1$ and $K \geq 1$. Assume that $u \in C^0(B_1)$ satisfies
\begin{equation}
|D u|^{-\gamma} P^{\lambda}_{\Lambda}(D^2 u) - |D u|^{1-\gamma} \leq f \text{ in } B_1
\end{equation}
in the viscosity sense, where $f \in C^0 \cap L^n(B_1)$. Then there exist constants $\epsilon_2 = \epsilon_2(n, \lambda, \theta_0) > 0$, $M = M(n, \lambda, \Lambda) > 1$, $0 < \Theta = \Theta(n, \lambda, \Lambda) < 1$ and $0 < \theta = \theta(n, \lambda, \theta_0) < \min\{\theta_0, \Theta\} < 1$ such that if $B_r(x_0) \subset B_1$ satisfies
\begin{equation}
B_r(x_0) \cap T_K^{-} \subset \{x \in B_1 : \mathcal{M}(|f|^n(x) \leq \epsilon_2 K^{(1-\gamma)n}) \geq \theta |B_r(x_0)|\}
\end{equation}
then
\[|B_r(x_0) \cap T_{K}^{-}| \geq \Theta |B_r(x_0)|.\]
(For the definitions of $T_K^{-}$ and $T_{K}^{-}$, see (2.1) in Section 2.)

Proof. For simplicity, we may assume without loss of generality that $u$ is smooth in $B_1$. Otherwise one needs to regularize $u$ using the standard $\varepsilon$-envelope method of Jensen (see for instance [6], [13] and [15]).

Take $0 < \theta < 1$ to be chosen later. It follows from (3.2) that there exists $0 < \varepsilon = \varepsilon(n, \theta) < 1$ such that
\[B_{(1-\varepsilon)r}(x_0) \cap T_K^{-} \cap \{x \in B_1 : \mathcal{M}(|f|^n(x) \leq \epsilon_2 K^{(1-\gamma)n}) \neq \emptyset\} \neq \emptyset.\]

Let $x_1 \in B_{(1-\varepsilon)r}(x_0) \cap T_K^{-} \cap \{x \in B_1 : \mathcal{M}(|f|^n(x) \leq \epsilon_2 K^{(1-\gamma)n})\}$. Then, by the definition (2.1) of $T_K^{-}$, there exists $y_1 \in B_1$ such that
\begin{equation}
P_{K,y_1}(x) := -\frac{K}{2}|x - y_1|^2 + u(x_1) + \frac{K}{2}|x_1 - y_1|^2 \leq u \text{ in } B_1.
\end{equation}

Note that $B_{3r/4}(x_1) \cap B_{r/4}(x_0)$ contains a ball of radius $\varepsilon r/2$. The proof now will be split into three steps.

Step 1. We prove that there exist $x_2 \in B_{r/2}(x_0)$ and $C = C(n, \lambda, \Lambda) > 0$ such that
\begin{equation}
u(x_2) - P_{K,y_1}(x_2) \leq CKr^2.
\end{equation}
To do this, for each $y_\nu \in B_{3r/4}(x_1) \cap B_{r/4}(x_0) \subset B_1$, we set
\[\psi(x) := \frac{x - y_\nu}{\rho} \text{ and } \phi(t) = e^A e^{-At^2} - 1 \text{ with } A > 1 \text{ to be determined later. Let } x_\tau \in B_\rho(y_\nu) \text{ such that } (u - \psi)(x_\tau) = \min_{B_\rho(y_\nu)}(u - \psi).\]

Since (3.3) implies $(u - \psi)|_{\partial B_\rho(y_\nu)} \geq 0$ and
\[\frac{1}{\rho} \frac{d}{dt} \psi(t) = \frac{1}{\rho} \frac{d}{dt} \psi(t) = -K \rho^2 \phi \left( \frac{x_1 - y_\nu}{\rho} \right) < 0,
\]
we conclude that $x_\tau \in B_\rho(y_\nu) \subset B_r(x_0) \subset B_1$ and
\begin{equation}
\frac{1}{\rho} \frac{d}{dt} \psi(t) = \frac{1}{\rho} \frac{d}{dt} \psi(t) = -K \rho^2 \phi \left( \frac{x_\tau - y_\nu}{\rho} \right) \leq P_{K,y_1}(x_\tau) + e^A Kr^2.
\end{equation}
We assert that: \( \exists A^0 = A^0(n, \lambda, \Lambda) > 1, \exists y_0^0 \in B_{3r/4}(x_1) \cap B_{r/4}(x_0) \) and \( \exists x_T^0 \in B_{\rho/4}(y_T^0) \) such that
\[
(u - \psi)(x_T^0) = \min_{B_{\rho/4}(y_T^0)} (u - \psi). \tag{3.6}
\]

Once we have proved (3.6), it will follow that
\[
|x_T^0 - x_0| \leq |x_T^0 - y_T^0| + |y_T^0 - x_0| < 3r/16 + r/4 < r/2.
\]

In view of (3.5), this establishes (3.4) by setting \( x_2 := x_T^0 \) and \( C := e^A \).

Hence we now need only to prove (3.6). Suppose by contradiction that \( \forall A > 1, \forall y_\nu \in B_{3r/4}(x_1) \cap B_{r/4}(x_0) \) and \( \forall x_T \in B_{\rho}(y_\nu) \), if they satisfy
\[
(u - \psi)(x_T) = \min_{B_{\rho}(y_\nu)} (u - \psi), \tag{3.7}
\]
then
\[
x_T \in B_{\rho}(y_\nu) \setminus B_{\rho/4}(y_\nu).
\]

Let us set \( t := |x_T - y_\nu|/\rho \) and denote by \( T \) the set of \( x_T \) while the corresponding \( y_\nu \) runs through \( B_{3r/4}(x_1) \cap B_{r/4}(x_0) \). Then \( 1/4 \leq t < 1 \) and \( T \subset B_r(x_0) \subset B_1 \). The proof now will be divided into five minor steps.

1° From (3.7), we see that
\[
\psi + \min_{B_{\rho}(y_\nu)} (u - \psi) \leq u \text{ in } B_{\rho}(y_\nu).
\]

By the definition of the viscosity solution of (3.1), we obtain
\[
|D\psi(x_T)|^{-\gamma}P_{\lambda, \Lambda}^{-}(D^2\psi(x_T)) - |D\psi(x_T)|^{1-\gamma} \leq f(x_T). \tag{3.8}
\]

Since
\[
\left| D P_{K, y_\nu}^{-}(x_T) \right| \leq CK, \quad \left| D^2 P_{K, y_\nu}^{-}(x_T) \right| \leq CK, \quad \left| \frac{\phi_t}{t} \right| \leq CAe^Ae^{-At^2} \quad \text{and} \quad |\phi_t| \geq C^{-1} A^2 e^A e^{-At^2},
\]
we deduce that
\[
|D\psi(x_T)| \leq CKAe^Ae^{-At^2},
\]
\[
|D\psi(x_T)|^\gamma \leq C \left( KAe^Ae^{-At^2} \right)^\gamma,
\]
and
\[
P_{\lambda, \Lambda}^{-}(D^2\psi)(x_T) - |D\psi(x_T)|
\geq C^{-1} K |\phi_t| - 1 - CK \left( |\phi_t| + 1 \right) - CKAe^Ae^{-At^2}
\geq C^{-1} K A e^A e^{-At^2},
\]
and consequently
\[
|D\psi(x_T)|^{-\gamma}P_{\lambda, \Lambda}^{-}(D^2\psi(x_T)) - |D\psi(x_T)|^{1-\gamma} \geq C^{-1} \left( KAe^Ae^{-At^2} \right)^{1-\gamma} \geq K^{1-\gamma},
\]
where \( A \) is sufficiently large and all the \( C \)'s depend only on \( n, \lambda \) and \( \Lambda \). Combining it with (3.8), we obtain
\[
0 < 1 \leq K^{1-\gamma} \leq f(x_T), \quad \forall x_T \in T. \tag{3.9}
\]
2° Since the inflection point of $\phi(t)$ is $t = (2A)^{-1/2}$, we can assert that

\begin{equation}
|D\varphi(X)| \leq C(n, \lambda, \Lambda),
\end{equation}

\begin{equation}
||D^2\varphi(X)|| \leq C(n, \lambda, \Lambda)
\end{equation}

and

\begin{equation}
||(D^2\varphi(X))^{-1}|| \leq C(n, \lambda, \Lambda)
\end{equation}

for all $1/4 \leq |X| < 1$, provided $A$ is large enough.

3° For any $x_T \in T$, we have

\begin{equation}
Du(x_T) = D\psi(x_T) = -K(x_T - y_1) + K\rho D\varphi\left(\frac{x_T - y_1}{\rho}\right),
\end{equation}

and

\begin{equation}
D^2u(x_T) \geq D^2\psi(x_T) = -KI + KD^2\varphi\left(\frac{x_T - y_1}{\rho}\right).
\end{equation}

Recalling (3.10), (3.11) and (3.9), it follows that

\begin{equation}
|Du(x_T)| \leq K|x_T - y_1| + K\rho \left|D\varphi\left(\frac{x_T - y_1}{\rho}\right)\right| \leq CK,
\end{equation}

and

\begin{equation}
D^2u(x_T) \geq -KI - K \left|D^2\varphi\left(\frac{x_T - y_1}{\rho}\right)\right| I \geq -CKI \geq -CK^7f(x_T)I.
\end{equation}

Combining them with (3.8) and (3.9), we deduce that

\begin{equation}
\lambda \sum_{e_i(D^2u(x_T)) > 0} e_i (D^2u(x_T)) = P_{\lambda, \Lambda} (D^2u(x_T)) - \Lambda \sum_{e_i(D^2u(x_T)) < 0} e_i (D^2u(x_T))
\end{equation}

\begin{align*}
&\leq |Du(x_T)|^\gamma f(x_T) + |Du(x_T)| + CK^7 f(x_T) \\
&\leq (CK)^7 f(x_T) + CK + CK^7 f(x_T) \\
&\leq CK^7 f(x_T).
\end{align*}

Hence the absolute value of each eigenvalue of $D^2u(x_T)$ is not greater than $CK^7 f(x_T)$, and therefore

\begin{equation}
|D_{ij} u(x_T)| \leq CK^7 f(x_T) \quad (\forall i, j = 1, 2, \ldots, n),
\end{equation}

where $C = C(n, \lambda, \Lambda) > 0$.

4° Write

\begin{equation}
X := \frac{x_T - y_1}{\rho}
\end{equation}

and

\begin{equation}
Y := \frac{Du(x_T) + K(x_T - y_1)}{K\rho}.
\end{equation}

From (3.13), we have $D\varphi(X) = Y$, and hence $X = (D\varphi)^{-1}(Y)$. By the inverse function theorem, we compute

\begin{equation}
D_Y X = (D^2\varphi(X))^{-1} = (D^2\varphi \circ (D\varphi)^{-1}(Y))^{-1},
\end{equation}

and

\begin{equation}
D_{x_T} X = D_Y X \cdot D_{x_T} Y = (D^2\varphi(X))^{-1} \cdot \frac{D^2u(x_T) + KI}{K\rho}.
\end{equation}
Since \( y_v = x_r - \rho X \), we obtain

\[
D_{x_r}y_v = I - \rho D_{x_r}X = I - (D^2 \varphi(X))^{-1} \cdot \frac{D^2 u(x_r) + K I}{K}.
\]

Thus, in view of (3.12), (3.14) and (3.9), we have

\[
|\langle D_{x_r}y_v, (i,j) \rangle| \leq \left| \delta_{ij} - \sum_k \left( (D^2 \varphi(X))^{-1} \cdot \frac{D_{kj}u(x_r)}{K} \right) \right|
\]
\[
\leq 1 + \left\| (D^2 \varphi(\gamma))^{-1} \right\|_{L^\infty(B_{1/4})} \sum_k \left( 1 + \frac{D_{kj}u(x_r)}{K} \right).
\]

and consequently

\[
(3.15) \quad \det (D_{x_r}y_v) \leq C(n, \lambda, \Lambda)K^{-\gamma}n |f(x_r)|^n.
\]

5° Consider the mapping \( y_v : x_r \mapsto y_v, \ T \mapsto B_{3r/4}(x_1) \cap B_{r/4}(x_0) \), given precisely by \( y_v = x_r - \rho X \). By the area formula and (3.13), we have

\[
|B_{3r/4}(x_1) \cap B_{r/4}(x_0)| \leq \int_T |\det(D_{x_r}y_v)| \, dx_r \leq C K^{-\gamma}n \int_T |f(x_r)|^n \, dx_r.
\]

Then, in view of the facts that \( B_{3r/4}(x_1) \cap B_{r/4}(x_0) \) contains a ball of radius \( \varepsilon r/2 \), \( T \subset B_r(x_0) \subset B_{2r}(x_1) \) and \( \mathcal{M}(|f|^n)(x_1) \leq \epsilon_2 K^{(1-\gamma)n} \), we have

\[
2^{-n}\varepsilon^n |B_{1/r^2}| \leq C K^{-\gamma}n \int_{B_{2r}(x_1)} |f(x_r)|^n \, dx_r \leq C K^{-\gamma}n \mathcal{M}(|f|^n)(x_1) |B_{2r}(x_1)| \leq 2^n C |B_1/r^n \epsilon_2.
\]

Let \( \epsilon_2 < 2^{-n}\varepsilon^n \cdot (2^nC)^{-1} = 4^{-n}C^{-1}\varepsilon^n \), we arrive at a contradiction. This proves (3.6) and completes the proof of the assertion (3.4).

**Step 2.** We now prove that there exists \( M = M(n, \lambda, \Lambda) > 1 \) such that

\[
(3.16) \quad T_{KM}(V) \subset B_r(x_0) \cap T_{KM},
\]

where

\[
V := B_{M^{-1}x_1} \left( \frac{M - 1}{M} x_2 + \frac{1}{M} y_1 \right).
\]

For each \( \tilde{x} \in T_{KM}(V) \), there exists \( \tilde{y} \in V \) such that

\[
P_{KM,\tilde{y}}(x) := -\frac{KM}{2} |x - \tilde{y}|^2 + u(\tilde{x}) + \frac{KM}{2} |\tilde{x} - \tilde{y}|^2 \leq u \text{ in } B_1.
\]

Since

\[
P_{KM,\tilde{y}}(x) - P_{KM,y_1}(x) = -\frac{K(M - 1)}{2} |x - y|^2 + R,
\]

where

\[
y := \frac{M}{M - 1} \tilde{y} - \frac{1}{M - 1} y_1
\]

(3.17)
and \( R = R(\bar{y}, K, M, y_1, \bar{x}, u(\bar{x}), x_1, u(x_1)) \) both do not depend on \( x \), we obtain
\[
P_{K,y_1}^{-}(x) - \frac{K(M-1)}{2} |x - y|^2 + R \leq u(x), \quad \forall x \in B_1.
\]
Substituting \( x_2 \in B_{r/2}(x_0) \subset B_1 \) in above, invoking (3.18) and observing that \( y \in B_{r/8}(x_2) \), we deduce that
\[
R \leq u(x_2) - P_{K,y_1}^{-}(x_2) + \frac{K(M-1)}{2} |x_2 - y|^2 \leq \left( C + \frac{M-1}{128} \right) K r^2.
\]
On the other hand, we have
\[
0 \leq u(\bar{x}) - P_{K,y_1}^{-}(\bar{x}) = P_{K,M,y}^{-}(\bar{x}) - P_{K,y_1}^{-}(\bar{x}) = -\frac{K(M-1)}{2} |\bar{x} - y|^2 + R.
\]
Hence
\[
|\bar{x} - y|^2 \leq \frac{2}{M-1} \left( C + \frac{M-1}{128} \right) r^2 = \left( \frac{2C}{M-1} + \frac{1}{64} \right) r^2 \leq \frac{1}{16} r^2,
\]
provided \( M > 1 \) is sufficiently large. Thus we obtain \(|\bar{x} - y| < r/4\) and
\[
|\bar{x} - x_2| \leq |\bar{x} - y| + |y - x_2| < r/4 + r/8 < r/2.
\]
Therefore
\[
(3.18) \quad T_{K,M}^{-}(V) \subset B_{r/2}(x_2) \subset B_r(x_0) \subset B_1.
\]
For each \( \bar{y} \in V \), we see from (3.17) that there exists \( y \in B_{r/8}(x_2) \) such that
\[
\bar{y} = \frac{M}{M-1} y + \frac{1}{M} y_1.
\]
Since \( y_1 \in \overline{B_1} \) and \( y \in B_{r/8}(x_2) \subset B_r(x_0) \subset B_1 \), by the convexity of \( B_1 \), we see that \( \bar{y} \in B_1 \). Thus we have \( V \subset B_1 \) and hence
\[
(3.19) \quad T_{K,M}^{-}(V) \subset T_{K,M}^{-}(B_1) = T_{K,M}^{-}.
\]
Combining (3.18) with (3.19) yields
\[
T_{K,M}^{-}(V) \subset B_r(x_0) \cap T_{K,M}^{-},
\]
which is exactly the assertion (3.16).

**Step 3.** We assert that
\[
(3.20) \quad |V| \leq C |T_{K,M}^{-}(V)|,
\]
where \( C = C(n, \lambda, \Lambda) > 0 \). If we have proved this, we will conclude from (3.16) that
\[
|B_r(x_0) \cap T_{K,M}^{-}| \geq |T_{K,M}^{-}(V)| \geq \frac{1}{C} |V| = \frac{1}{C} \left( \frac{M-1}{8M} \right)^n \frac{1}{\Theta} |B_r| =: \Theta |B_r(x_0)|,
\]
which proves the lemma by taking \( 0 < \theta = \theta(n, \lambda, \Lambda, \theta_0) := \frac{1}{2} \min\{\theta_0, \Theta\} < 1 \).

Hence we now need only to prove (3.20). For each \( x \in T_{K,M}^{-}(V) \), there exists a unique \( y \in V \) satisfying
\[
Du(x) = -K M(x - y)
\]
and
\[
D^2 u(x) \geq -K M I.
\]
Consider the mapping \( y : x \mapsto y, T_{KM}(V) \rightarrow V \), given precisely by
\[
y = x + \frac{1}{KM}Du(x).
\]
Since
\[
|Du(x)| = KM|x - y| \leq 2KM,
\]
and
\[
D_x y = I + \frac{1}{KM}D^2 u(x) \geq 0,
\]
we deduce from (3.1) and (3.18) that
\[
\lambda \text{tr}(D_x y) = \mathcal{P}_{\lambda,\Lambda}(D_x y) \leq P - \frac{\Lambda}{\lambda} \text{tr}(D_x y) \leq P + \frac{\Lambda}{\lambda} |D_x y| \leq 0,
\]
and
\[
\text{det}(D_x y) \leq \left( \frac{\text{tr}(D_x y)}{n} \right)^n \leq C \left( 1 + \frac{|f(x)|}{(KM)^{1 - \gamma}} \right),
\]
where the constants \( C \) (and all the \( C \)'s in the rest of this proof) depend only on \( n, \lambda \) and \( \Lambda \). Hence we can conclude, by the area formula, that
\[
\int \text{det}(D_x y) dx \leq C \left( 1 + \frac{|f(x)|}{(KM)^{1 - \gamma}} \right) \int |f(x)|^n dx.
\]
In view of the facts that \( T_{KM}(V) \subset B_r(x_0) \subset B_{2r}(x_1) \) and \( \mathcal{M}(|f|^n)(x_1) \leq \epsilon_2 K^{(1 - \gamma)n} \), we have
\[
|V| \leq C |T_{KM}(V)| + \frac{C}{(KM)^{(1 - \gamma)n}} \int_{B_{2r}(x_1)} |f(x)|^n dx
\]
\[
\leq C |T_{KM}(V)| + \frac{Cr^n}{K^{(1 - \gamma)n}} \mathcal{M}(|f|^n)(x_1)
\]
\[
\leq C |T_{KM}(V)| + \epsilon_2 C r^n.
\]
Taking \( \epsilon_2 > 0 \) to be sufficiently small such that
\[
\epsilon_2 C r^n \leq \frac{1}{2} |V| = \frac{1}{2} \left( \frac{M - 1}{8M} \right)^n |B_1| r^n,
\]
we obtain
\[
\frac{1}{2} |V| \leq C |T_{KM}(V)|,
\]
which implies (3.20) and completes the proof of Lemma 3.2.

With the Lemma 3.2 in hand, we can now prove the following measure decay estimate which concerns the decay of \( |B_1 \setminus T_i^-| \) in \( t \).
Lemma 3.3. Let $0 \leq \gamma < 1$ and $K \geq 1$. Assume that $u \in C^0(\overline{B_1})$ satisfies
\[
|Du|^{-\gamma}P_{\lambda,\Lambda}(D^2u) - |Du|^{1-\gamma} \leq f \text{ in } B_1
\]
in the viscosity sense, where $f \in C^0 \cap L^n(B_1)$. Then there exist constants $\epsilon_1 = \epsilon_1(n, \lambda, \Lambda) > 0$, $\epsilon_2 = \epsilon_2(n, \lambda, \Lambda) > 0$, $M = M(n, \lambda, \Lambda) > 1$ and $0 < \mu_0 = \mu_0(n, \lambda, \Lambda) < 1$ such that if $\text{osc } u \leq 1/8$ and $\|f\|_{L^n(B_1)} \leq \epsilon_1$, then
\[
|B_1 \setminus T_M| \leq \mu_0 \left( |B_1 \setminus T_M| + \left\{ x \in B_1 : M(|f|^n)(x) > \epsilon_2 K^{(1-\gamma)n} \right\} \right).
\]

Proof. We first prove that there exists $0 < \theta_0 = \theta_0(n, \lambda, \Lambda) < 1$ such that
\[
|B_1 \cap T_M \cap \left\{ x \in B_1 : M(|f|^n)(x) \leq \epsilon_2 K^{(1-\gamma)n} \right\}| \geq \theta_0 |B_1|.
\]

For each $\bar{x} \in T_M \setminus \overline{B_{1/2}}$, there exists $\bar{y} \in \overline{B_{1/2}}$ such that
\[
u(\bar{x}) + K \frac{1}{2} |\bar{x} - \bar{y}|^2 = \min_{x \in B_1} \left( u(x) + K \frac{1}{2} |x - \bar{y}|^2 \right) =: m(\bar{y}).
\]

Hence we conclude that $-\frac{K}{2} |x - \bar{y}|^2 + m(\bar{y}) \leq u$ in $B_1$. In particular, we have $m(\bar{y}) \leq u(\bar{y})$ and $-\frac{K}{2} |x - \bar{y}|^2 + m(\bar{y}) = u(\bar{x})$. Subtracting one from the other, we deduce that
\[
\frac{K}{2} |x - \bar{y}|^2 \leq u(\bar{y}) - u(\bar{x}) \leq \text{osc } u \leq 1/8,
\]
which implies $|\bar{x} - \bar{y}| \leq 1/2$. Thus $\bar{x} \in B_1$ and hence
\[
T_M \setminus \overline{B_{1/2}} \subset B_1.
\]

Consider, as in the proof of Lemma 3.2, the mapping $\bar{y} : \bar{x} \mapsto \bar{y}$, $T_M \setminus \overline{B_{1/2}} \to \overline{B_{1/2}}$, given by
\[
\bar{y} = \bar{x} + \frac{1}{K} Du(\bar{x}).
\]

Since $|Du(\bar{x})| = K |\bar{x} - \bar{y}| \leq 2K$, we conclude, as in (3.21), that
\[
0 \leq \det(D\bar{y}) \leq C \left( 1 + \frac{|f(\bar{x})|^n}{K^{(1-\gamma)n}} \right) \leq C(1 + |f(\bar{x})|^n),
\]
where $C = C(n, \lambda, \Lambda) > 0$. Thus it follows from the area formula and (3.23) that
\[
2^{-n} |B_1| \leq |\overline{B_{1/2}}| \leq \int_{T_M \setminus \overline{B_{1/2}}} \det(D\bar{y}) d\bar{x} \leq C |T_M \setminus \overline{B_{1/2}}| + C \int_{B_1} |f|^n.
\]

Let $\|f\|_{L^n(B_1)} \leq 2^{-(n+1)} |B_1| \cdot C^{-1}$. We obtain
\[
2^{-(n+1)} |B_1| \leq C |T_M \setminus \overline{B_{1/2}}| = C |B_1 \cap T_M \setminus \overline{B_{1/2}}| \leq C |B_1 \cap T_M|,
\]
where we have used (3.23) again. Thus
\[
B_1 \cap T_M \geq 2^{-(n+1)} C^{-1} \cdot |B_1| =: \theta_1 |B_1|,
\]
where $0 < \theta_1 = \theta_1(n, \lambda, \Lambda) < 1$.

On the other hand, by the weak type (1,1) property, we have
\[
|\left\{ x \in B_1 : M(|f|^n)(x) > \epsilon_2 K^{(1-\gamma)n} \right\}| \leq C(n) \left( \epsilon_2 K^{(1-\gamma)n} \right)^{-1} \|f\|^n_{L^n(B_1)} \leq C(n) \epsilon_2^{-1} \|f\|^n_{L^n(B_1)}.
\]
Lemma 3.3

Applying Lemma 2.2 and Lemma 3.2 to \( B \), we conclude that

\[
|B \cap T_K^c \cap \{ x \in B_1 : \mathcal{M}(|f|^n)(x) \leq \epsilon_2 K^{(1-\gamma)n} \}| \geq \frac{\theta_1}{2} |B_1| =: \theta_0 |B_1|,
\]

which is (3.22).

Now according to (3.22) and the conclusion of Lemma 3.2 applying the covering lemma, Lemma 2.3 to

\[
E := T_K^c \cap \{ x \in B_1 : \mathcal{M}(|f|^n)(x) \leq \epsilon_2 K^{(1-\gamma)n} \}
\]

and \( F := T_{KM}^c \), we thus conclude that

\[
|B_1 \setminus T_{KM}^-| = |B_1 \setminus F| \leq \left(1 - \frac{\Theta - \theta}{5^n}\right) |B_1 \setminus E| =: \mu_0 |B_1 \setminus E|
\]

\[
\leq \mu_0 \left(|B_1 \setminus T_K^-| + \left| \{ x \in B_1 : \mathcal{M}(|f|^n)(x) > \epsilon_2 K^{(1-\gamma)n} \} \right| \right).
\]

This finishes the proof of Lemma 3.3. \( \square \)

**Corollary 3.1.** Under the assumptions of Lemma 3.3, we have

\[
|B_1 \setminus T_{MK}^-| \leq C \mu^k \quad (k = 0, 1, 2, \ldots),
\]

where \( C = C(n, \lambda, \Lambda, \gamma) > 0 \) and \( 0 < \mu = \mu(n, \lambda, \Lambda, \gamma) < 1 \).

**Proof.** Let

\[
\alpha_k := |B_1 \setminus T_{MK}^-|
\]

and

\[
\beta_k := \left| \{ x \in B_1 : \mathcal{M}(|f|^n)(x) > \epsilon_2 M^{-(1-\gamma)n} \} \right|.
\]

Applying Lemma 3.3 to \( K = M^k \), \( \forall k = 0, 1, 2, \ldots \), we obtain

\[
\alpha_{k+1} \leq \mu_0 (\alpha_k + \beta_k) \quad (\forall k = 0, 1, 2, \ldots).
\]

Using the weak type (1,1) property, recalling \( \|f\|_{L^1(B_1)} \leq \epsilon_1 \), and remembering that \( \epsilon_1 \) and \( \epsilon_2 \) depend only on \( n, \lambda \) and \( \Lambda \), we conclude that

\[
\beta_k \leq C(n) \left( \epsilon_2 M^{(1-\gamma)n} \right)^{-1} \|f\|_{L^n(B_1)} \leq C_0 \left( M^{-(1-\gamma)n} \right)^k |B_1|,
\]

where \( C_0 = C_0(n, \lambda, \Lambda) > 0 \). Thus we have

\[
\sum_{i=0}^{k-1} \mu_0^{k-i} \beta_i \leq C_0 |B_1| \sum_{i=0}^{k-1} \mu_0^{k-i} \left( M^{-(1-\gamma)n} \right)^i
\]

\[
\leq C_0 |B_1| \sum_{i=0}^{k-1} \mu_1^i = C_0 |B_1| \mu_1^k \quad (\forall k = 1, 2, 3, \ldots),
\]

where \( \mu_1 = \mu_0 \) and \( \mu_0 > 0 \).
and hence
\[ \alpha_k \leq \mu_0 k |B_1| + \sum_{i=0}^{k-1} \mu_0^{k-i} \beta_i \leq (1 + C_0(k)) |B_1| \mu_1^k \leq C \mu_1^k \quad (\forall k = 1, 2, 3, \ldots), \]
where \( \mu_1 := \max \{ \mu_0, M^{-(1-\gamma)n} \} \in (0, 1) \), \( \mu := (1 + \mu_1)/2 \in (\mu_1, 1) \) and \( C := \max_{t \in \mathbb{R}} \{ \mu^{-1} \mu_1^k (1 + C_0 t) |B_1| \} \in (0, +\infty) \) are all constants depending only on \( n, \lambda, \Lambda \) and \( \gamma \). This finishes the proof of Corollary 3.1. \( \square \)

3.3. Proof of Lemma 3.1

From the above Corollary 3.1, Lemma 3.1 follows easily.

Proof of Lemma 3.1 Since \( \mathcal{P}^+(D^2 u) = -\mathcal{P}^-(D^2 u) \), by the second inequality of (1.4), it is clear that \( -u \) satisfies all the assumptions of Lemma 3.1 and hence of Corollary 3.1. Since, by definition, \( T_1^+ (u, B_1) = T_1^- (-u, B_1) \), applying Corollary 3.1 to \( -u \), we get \( |B_1 \setminus T_{1+k}^-| \leq \mu_1^k C \mu^k, \forall k \in \mathbb{Z}^+ \). Thus
\[ |B_1 \setminus T_{1+k}| \leq |B_1 \setminus T_{1+k}^-| + |B_1 \setminus T_{1+k}^+| \leq \mu_1^k C \mu^k, \forall k \in \mathbb{Z}^+. \]
and hence
\[ (3.26) \quad |B_1 \setminus T_t| \leq C t^{-\sigma}, \forall t > 0, \]
where \( \sigma := -\log \mu \). Invoking Lemma 2.1 we deduce that \( \|D^2 u\|_{L^1(B_1)} \leq C \) (see also 6 Proposition 1.1). By the interpolation theorem (see 11 Theorem 7.28), we thus obtain \( \|u\|_{W^{2,4}(B_1)} \leq C \). This completes the proof of Lemma 3.1. \( \square \)

Remark 3.1. For heuristic purpose, we give for \( u \in C^2(B_1) \) the simple and full details of deducing \( \|D^2 u\|_{L^1(B_1)} \leq C \) from (3.26). Indeed, since
\[ B_1 \cap T_t \subset \{ x \in B_1 : -t I \leq D^2 u(x) \leq t I \} \subset \{ x \in B_1 : |D^2 u(x)| \leq \sqrt{n} t \}, \]
we have
\[ \{ x \in B_1 : |D^2 u(x)| > \sqrt{n} t \} \subset B_1 \setminus T_t. \]
Hence
\[ |\{ x \in B_1 : |D^2 u(x)| > \sqrt{n} t \}| \leq |B_1 \setminus T_t| \leq C t^{-\sigma}. \]
Using Lemma 2.1 we thus obtain
\[ \|D^2 u\|_{L^1(B_1)}^\delta \leq C(n, M, \delta) \left( |B_1| + \sum_k M^{\delta k} |\{ x \in B_1 : |D^2 u(x)| > \sqrt{n} M^k \}| \right) \]
\[ \leq C(n, M, \delta) \left( |B_1| + C \sum_k M^{(\delta-\sigma)k} \right) \leq C(n, \lambda, \Lambda, \delta). \]

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Dongsheng Li: School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an 710049, China;
E-mail address: lidsh@mail.xjtu.edu.cn

Zhisu Li: School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an 710049, China;
E-mail address: lizhisu@stu.xjtu.edu.cn