Nonuniform autonomous one-dimensional exclusion nearest-neighbor reaction-diffusion models

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Abstract
In a recent article, the most general non-uniform reaction–diffusion models on a one-dimensional lattice with boundaries were considered, for which the time evolution equations of correlation functions are closed and the stationary profile can be obtained using a transfer-matrix method. Here, models are investigated for which the equation of relaxation toward the stationary profile could also be solved through a similar transfer-matrix method. A classification is given, and dynamical phase transitions are studied.

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1. Introduction

Most studies on reaction–diffusion models are focused on uniform lattices where reaction rates are site independent. Among the simplest generalizations beyond a uniform system is a lattice with alternating rates. In [1–3], the steady state configurational probabilities of an Ising spin chain driven out of equilibrium by a coupling to two heat baths has been investigated. An example is a one-dimensional Ising model on a ring, in which the evolution is according to a generalization of Glauber rates, such that spins at even (odd) lattice sites experience a temperature $T_e$ ($T_o$). In [4], the dynamics of a reaction–diffusion model with spatially alternating rates has been discussed. The response function to an infinitesimal magnetic field for the Ising–Glauber model with arbitrary exchange couplings has been studied in [5]. In [6], relaxation in the kinetic Ising model on an alternating isotopic chain has been discussed. The critical dynamics of a disordered Ising ferromagnetic chain with two coupling constants for Glauber dynamics was studied in [7].

In [8], a non-uniform extension of the Glauber model on a one-dimensional lattice with boundaries was investigated. In this model the reaction rates are based on detailed balance. There the expectation values of spins in an Ising model with nonuniform coupling constants were studied. A transfer matrix method was used to study the steady state behavior of the system in the thermodynamic limit. Different (static) phases of this system were studied, and a closed
form was obtained for this transfer matrix. In [9], the dynamics and possible dynamical phase transitions of the non-uniform extension of the Glauber model were investigated. It was shown that there are two phases: the fast phase and the slow phase. In the fast phase, the relaxation time is independent of the reaction rates at the boundaries. Changing continuously the reaction rates, the system may experience a phase transition. In the slow phase, the relaxation time changes with reaction rates at boundaries.

In [10], the most general non-uniform exclusion nearest-neighbor reaction–diffusion models on a one-dimensional lattice with boundaries were considered, for which the evolution equations of the one-point functions are closed and the stationary profile of the system can be obtained using a transfer-matrix method. It was shown that the criterion that the evolution equations be closed is the same as the case of uniform models. Such models are called autonomous. To be able to obtain a closed form for the transfer matrix, however, additional conditions are to be satisfied. Models satisfying these additional conditions as well are called superautonomous.

In this paper, we want to study the dynamics and possible dynamical phase transitions of superautonomous models using a transfer-matrix method. The scheme of the paper is as follows. In section 2, a brief review of the formalism is given. In section 3, the evolution equations governing the expectation values of the number operators are studied. It is seen that solvability through the transfer-matrix method requires additional criteria, apart from those of the superautonomy conditions obtained in [10]. In section 4, a classification is introduced for superautonomous models for which the relaxation toward the stationary profile of the expectation values of the number operators is solvable through the transfer-matrix method. In section 5, some examples are explicitly discussed. Section 6 is devoted to the concluding remarks.

2. Exclusion nearest-neighbor reaction–diffusion models with nonuniform reaction rates

Consider a one-dimensional lattice with \((L + 1)\) sites, numbered from 0 to \(L\). Each site is either empty or occupied with one particle. The evolution of the system is governed by a Hamiltonian \(\mathcal{H}\) of the form

\[
\mathcal{H} = \mathcal{H}'_0 + \left( \sum_{\alpha} \mathcal{H}_\alpha \right) + \mathcal{H}'_L,
\]

(1)

where \(\mathcal{H}_\alpha\) corresponds to the link \(\alpha\):

\[
\mathcal{H}_\alpha = \prod_{\alpha - \mu}^{\alpha} \prod_{L - \alpha - \mu}^{1} H_\alpha \prod_{\alpha}^{1} H_\alpha \prod_{\alpha}^{1} H_\alpha
\]

(2)

and

\[
\mu := \frac{1}{2}.
\]

(3)

The link \(\alpha\) links the sites \((\alpha - \mu)\) and \((\alpha + \mu)\), so that \(\alpha \pm \mu\) are integers. \(\mathcal{H}_\alpha\) is a linear operator acting on a four-dimensional space (the configuration space corresponding to the sites \((\alpha - \mu)\) and \((\alpha + \mu)\)) with a basis \(\{e_{00}, e_{01}, e_{10}, e_{11}\}\), where 0 denotes vacancy and 1 denotes occupied site. Also,

\[
\mathcal{H}'_0 = \mathcal{H}'_0 \prod_{L}^{1} H_0 \\
\mathcal{H}'_L = \prod_{L}^{1} H_L
\]

(4)
where $H'_0$ and $H'_L$ are linear operators acting on two-dimensional spaces (the configuration spaces corresponding to the sites 0 and $L$, respectively) with bases $\{e_0, e_1\}$.

The number operator in the site $i$ is denoted by $n_i$:

$$n_i = \underbrace{1 \otimes \cdots \otimes 1}_{L-i} \otimes n \otimes \underbrace{1 \otimes \cdots \otimes 1}_{i}.$$  \hspace{1cm} (5)

$n$ is an operator acting on a two-dimensional space with the basis $\{e_0, e_1\}$. The matrix form of $n$ in this basis is

$$n^a_b = \delta^a_1 \delta^b_1.$$  \hspace{1cm} (6)

The evolution equation for the expectation value of an observable $Q$ is

$$\frac{d}{dt} \langle Q \rangle = \langle Q \mathcal{H} \rangle,$$  \hspace{1cm} (7)

where

$$\langle Q \rangle = S Q \Psi,$$  \hspace{1cm} (8)

$\Psi$ is the $((2^L+1)$-dimensional) probability vector describing the system and $S$ is the covector

$$S := s \otimes \cdots \otimes s,$$  \hspace{1cm} (9)

and

$$s_a = 1.$$  \hspace{1cm} (10)

The system is called autonomous iff the Hamiltonian is so that the evolution of the expectation values of $n_i$ is closed in terms of the expectation values of $n_j$’s. In the evolution equation for the expectation value of $n_i$, the expectation values of $n_{i-1}, n_i, n_{i+1}, (n_{i-1} n_i)$ and $(n_i n_{i+1})$ occur. It is seen that the criterion that the coefficients of the last two vanish is

$$s_a [H_{-\mu} r \otimes r]^{a\dagger} = 0,$$

$$s_a [H_{+\mu} r \otimes r]^{a\dagger} = 0,$$  \hspace{1cm} (11)

respectively, where

$$r^a = -\delta^a_0 + \delta^a_1.$$  \hspace{1cm} (12)

Equation (11) should hold for all $i$’s, in order that the system be autonomous. So one can rewrite it as

$$s_a [H_{a} r \otimes r]^{a\dagger} = 0,$$

$$s_a [H_{a} r \otimes r]^{1a} = 0.$$  \hspace{1cm} (13)

It is seen that this condition is the same as the corresponding condition for uniform lattices, written for each link separately. Provided that this condition holds, one arrives at

$$\frac{d}{dt} \langle n_i \rangle = n_{i-\mu} \langle n_{i-1} \rangle + \theta_{i+\mu} \langle n_{i+1} \rangle + (\kappa_{i-\mu} + \nu_{i+\mu}) \langle n_i \rangle + (\xi_{i-\mu} + \sigma_{i+\mu}) \langle n_i \rangle, \quad 0 < i < L,$$  \hspace{1cm} (14)

where

$$\eta_a := s_a (H_a)^{a\dagger b_0 b},$$

$$\theta_a := s_a (H_a)^{1a b_0 b},$$

$$\kappa_a := s_a (H_a)^{1a 0 b},$$

$$\nu_a := s_a (H_a)^{a 1 b},$$

$$\xi_a := s_a (H_a)^{a 1 0},$$

$$\sigma_a := s_a (H_a)^{1 0 0}.$$  \hspace{1cm} (15)
For the boundary sites (the sites 0 and \( L \)), one has
\[
\frac{d}{dt}\langle n_0 \rangle = \theta_\mu \langle n_1 \rangle - (H_0^0 \, 0 + H_0^1 \, 0 - v_\mu)\langle n_0 \rangle + (H_0^1 \, 0 + \sigma_\mu),
\]
(16)
\[
\frac{d}{dt}\langle n_L \rangle = \eta_L - \mu \langle n_{L-1} \rangle - (-\kappa_{L-\mu} + H_{L-1}^0 + H_{L-1}^1)\langle n_L \rangle + (\xi_{L-\mu} + H_{L-1}^0). \tag{17}
\]

3. Relaxation toward the static solution

The evolution equations governing \( \langle n_i \rangle \) (the difference of \( \langle n_i \rangle \) with its static value) are the homogeneous parts of (14), (16) and (17), which can be written as
\[
\frac{d}{dt}\langle n_i \rangle = h_i \langle n_i \rangle, \tag{18}
\]

The eigenvalues and eigenvectors of the operator \( h \) satisfy
\[
E_x j = \eta_{j-\mu} x_{j-1} + \theta_{j+\mu} x_{j+1} + (\kappa_{j-\mu} + v_{j+\mu}) x_j, \quad \text{for } 0 < j < L, \tag{19}
\]
\[
E_x 0 = \theta_\mu x_1 - (H_0^0 \, 0 + H_0^1 \, 0 - v_\mu) x_0, \tag{20}
\]
\[
E_x L = \eta_{L-\mu} x_{L-1} - (-\kappa_{L-\mu} + H_{L-1}^0 + H_{L-1}^1) x_L, \tag{21}
\]
where \( E \) is the eigenvalue and \( x \) is the eigenvector. The solution to (19) is
\[
X_{j+\mu} = \tilde{D}_j X_j, \tag{22}
\]
where
\[
X_\alpha := \begin{bmatrix} x_{\alpha-\mu} \\ x_{\alpha+\mu} \end{bmatrix}, \tag{23}
\]
and
\[
\tilde{D}_j := \begin{bmatrix} 0 & 1 \\ -\eta_{j-\mu} \theta_{j+\mu}^{-1} - \kappa_{j-\mu} + v_{j+\mu} - E \theta_{j+\mu}^{-1} \end{bmatrix}. \tag{24}
\]
Using these, one arrives at
\[
X_\alpha = \tilde{D}_\alpha \tilde{D}_\beta X_\beta, \tag{25}
\]
where
\[
\tilde{D}_\alpha \tilde{D}_\beta := \tilde{D}_{\alpha-\mu} \cdots \tilde{D}_{\beta+\mu}. \tag{26}
\]
Equations (20) and (21) lead to
\[
X_{\mu} \propto \begin{bmatrix} \theta_\mu \\ E - \kappa_\mu - v_\mu + \xi \end{bmatrix}, \tag{27}
\]
and
\[
X_{L-\mu} \propto \begin{bmatrix} \theta_\mu \\ E - \kappa_{L-\mu} - v_{L-\mu} + \lambda \end{bmatrix}, \tag{28}
\]
respectively, where
\[
\xi := H_0^1 \, 0 + H_0^0 \, 1 + \kappa_\mu, \tag{29}
\]
\[
\lambda := H_L^1 \, 0 + H_L^0 \, 1 + v_{L-\mu}. \tag{29}
\]
Defining
\[ V := \theta_\mu \begin{bmatrix} E - \kappa_\mu - \nu_\mu + \zeta \end{bmatrix}, \] (30)
and
\[ W := [- \eta_\mu \ E - \kappa_\mu - \nu_\mu + \lambda], \] (31)
it is seen that the eigenvalues can be obtained through
\[ W \tilde{D}_{L-\mu} V = 0. \] (32)

Things become simpler if one can write \( \tilde{D}_j \) as
\[ \tilde{D}_j := \tilde{\Sigma}_{j+\mu} \tilde{\Delta}_j \tilde{\Sigma}_{j-\mu}^{-1}, \] (33)
where \( \tilde{\Delta}_j \) is diagonal and \( \tilde{\Sigma}_\alpha \) depends only on the parameters corresponding to the link \( \alpha \).

Putting
\[ \tilde{\Sigma}_\alpha = \begin{bmatrix} \tilde{a}_\alpha & \tilde{b}_\alpha \\ \tilde{c}_\alpha & \tilde{d}_\alpha \end{bmatrix}, \] (34)
and
\[ \tilde{\Delta}_j = \begin{bmatrix} \tilde{A}_j & 0 \\ 0 & \tilde{B}_j \end{bmatrix}, \] (35)
in (33), exactly similar to [9] one arrives at
\[ \tilde{A}_j = \tilde{c}_{j-\mu} \tilde{b}_{j+\mu}, \] (36)
\[ \tilde{B}_j = \tilde{d}_{j-\mu} \tilde{b}_{j+\mu}, \] (37)
and
\[ \tilde{a}_\alpha \tilde{b}_\alpha = \tilde{\phi}_{\alpha} \tilde{S}_\alpha, \] (38)
\[ \tilde{c}_\alpha \tilde{d}_\alpha = \tilde{\eta}_{\alpha} \tilde{S}_\alpha, \] (39)
\[ \tilde{a}_\alpha \tilde{d}_\alpha = (- \tilde{\phi} \tilde{k}_\alpha + \tilde{\psi}) \tilde{S}_\alpha, \] (40)
\[ \tilde{b}_\alpha \tilde{c}_\alpha = (- \tilde{\phi} \tilde{v}_\alpha - \tilde{\psi}) \tilde{S}_\alpha, \] (41)
where
\[ \tilde{S}_\alpha := \tilde{a}_\alpha \tilde{d}_\alpha - \tilde{b}_\alpha \tilde{c}_\alpha, \] (42)
and \( \tilde{\phi} \) and \( \tilde{\psi} \) are link independent. The parameters \( \tilde{k} \) and \( \tilde{v} \) are related to \( \kappa \) and \( \nu \) through
\[ \tilde{k}_{j-\mu} + \tilde{v}_{j+\mu} := \kappa_{j-\mu} + \nu_{j+\mu} - E. \] (43)
The consistency of (38) to (41) requires that \((\tilde{k} - \tilde{v})\) and \((\tilde{k} + \tilde{v} - 4 \eta \theta)\) be constant (link independent). Using
\[ (\tilde{k}_\alpha + \tilde{v}_\alpha)^2 - 4 \eta_\alpha \theta_\alpha = (\kappa_\alpha + \nu_\alpha)^2 - 4 \eta_\alpha \theta_\alpha - 2 E (\kappa_\alpha + \nu_\alpha) + E^2, \] (44)
it turns out that in order that the left-hand side be link independent for all of the values of \( E \), both \((\kappa + \nu)\) and \((\eta \theta)\) should be link independent. Combining these with the link independence of \( (\kappa - \nu) \), it is seen that the necessary and sufficient condition for the possibility of the decomposition (33) is that \( \kappa \), \( \nu \) and \( (\eta \theta) \) be link independent. This is one condition more than
the conditions for superautonomy ([9]) where it was necessary and sufficient that \((\kappa - \nu)\) and \((\kappa \nu - \eta \theta)\) be link independent.

Assuming that (33) holds, one arrives at

\[
\tilde{D}_{L-\mu} = \begin{bmatrix}
\tilde{c}_\mu \frac{\tilde{a}_\mu}{\tilde{s}_\mu} (\Xi_{L-3\mu 3\mu} - \Upsilon_{L-3\mu 3\mu}) & \tilde{b}_\mu \frac{\tilde{b}_\mu}{\tilde{s}_\mu} (\Upsilon_{L-3\mu 3\mu} - \Xi_{L-3\mu 3\mu}) \\
\tilde{c}_\mu \frac{\tilde{a}_\mu}{\tilde{s}_\mu} (\Xi_{L-3\mu 3\mu} - \Upsilon_{L-3\mu 3\mu}) & \tilde{b}_\mu \frac{\tilde{b}_\mu}{\tilde{s}_\mu} (\Upsilon_{L-3\mu 3\mu} - \Xi_{L-3\mu 3\mu}) \\
\end{bmatrix},
\]

(45)

where

\[
\Xi_{a\beta} := \frac{\tilde{c}_a}{a_a} \cdots \frac{\tilde{c}_\beta}{a_\beta},
\]

(46)

\[
\Upsilon_{a\beta} := \frac{\tilde{d}_a}{a_a} \cdots \frac{\tilde{d}_\beta}{b_\beta}.
\]

Defining \(Z_1\) through

\[
\frac{\tilde{c}_a}{a_a} = Z_1 \frac{\eta_a}{\sqrt{|\eta_a \theta_a|}},
\]

(47)

it is seen that \(Z_1\) is link independent and

\[
\frac{\tilde{d}_a}{b_a} = Z_2 \frac{\eta_a}{\sqrt{|\eta_a \theta_a|}},
\]

(48)

where

\[
Z_1 Z_2 = \text{sgn}(\eta_a \theta_a).
\]

(49)

One then arrives at

\[
E = \kappa + \nu + \sqrt{\frac{\eta \theta}{|\eta \theta|}} (Z_1^{-1} + Z_2^{-1}),
\]

(50)

where use has been made of the fact that \(\kappa, \nu\) and \((\eta \theta)\) are link independent. It is seen that (32) is equivalent to

\[
W \begin{bmatrix}
\frac{1}{\eta_{L-\mu} \theta_\mu} (Z_1^{l-2} - Z_2^{l-2}) - \frac{1}{\eta_{L-\mu} \sqrt{|\eta \theta|}} (Z_1^{l-1} - Z_2^{l-1}) \\
\frac{1}{\theta_\mu \sqrt{|\eta \theta|}} (Z_1^{l-1} - Z_2^{l-1}) - \frac{1}{\eta \theta} (Z_1^{l} - Z_2^{l}) \\
\end{bmatrix} V = 0,
\]

(51)

or

\[
0 = Z_2^l \left( \frac{1}{Z_1} + \frac{\lambda}{\sqrt{|\eta \theta|}} \right) \left( \frac{1}{Z_1} + \frac{\zeta}{\sqrt{|\eta \theta|}} \right) = Z_1^l \left( \frac{1}{Z_2^2} + \frac{\lambda}{\sqrt{|\eta \theta|}} \right) \left( \frac{1}{Z_2^2} + \frac{\zeta}{\sqrt{|\eta \theta|}} \right).
\]

(52)

In the thermodynamic limit \((L \to \infty)\), all of the unimodular values of \(Z_2^2\) are solutions to (52). If there are no further solutions, the relaxation time would be \(\tau_{\text{fast}}:\)

\[
\tau_{\text{fast}} = [-\kappa - \nu - 2 \text{Re}(\sqrt{\eta \theta})]^{-1}.
\]

(53)

Things change if there are nonunimodular solutions for \(Z_2^2\) as well. In that case, one can assume (without loss of generality) that \(|Z_2^2|\) is larger than 1. Then, (52) in the thermodynamic limit leads to

\[
\left( \frac{1}{Z_1} + \frac{\lambda}{\sqrt{|\eta \theta|}} \right) \left( \frac{1}{Z_1} + \frac{\zeta}{\sqrt{|\eta \theta|}} \right) = 0,
\]

(54)

or

\[
Z_1^{-1} = -\frac{\zeta}{\sqrt{|\eta \theta|}}.
\]

(55)
Each of (55) or (56) is acceptable, of course, provided the absolute value of the right-hand side is larger than 1. Putting these in (50), one arrives at the following time scales:

\[
\tau_\zeta = \left[-\kappa - \nu + \zeta + \zeta^{-1} \eta \theta \right]^{-1},
\]

\[
\tau_\lambda = \left[-\kappa - \nu + \lambda + \lambda^{-1} \eta \theta \right]^{-1}.
\]

One notes that each of \(\tau_\zeta\) and \(\tau_\lambda\) is larger than \(\tau_{\text{fast}}\) if the right-hand sides of (55) and (56) are larger than 1, respectively. So one arrives at the following expression for the relaxation time (\(\tau\)):

\[
\tau = \begin{cases} 
\tau_{\text{fast}}, & (-\lambda < \sqrt{\left|\eta \theta\right|}) \land (-\zeta < \sqrt{\left|\eta \theta\right|}) \\
\tau_\zeta, & (-\lambda < \sqrt{\left|\eta \theta\right|}) \land (-\zeta > \sqrt{\left|\eta \theta\right|}) \\
\tau_\lambda, & (-\lambda > \sqrt{\left|\eta \theta\right|}) \land (-\zeta < \sqrt{\left|\eta \theta\right|}) \\
\max(\tau_\zeta, \tau_\lambda), & (-\lambda > \sqrt{\left|\eta \theta\right|}) \land (-\zeta > \sqrt{\left|\eta \theta\right|}) 
\end{cases}
\]

The first case corresponds to the fast phase, where the relaxation time does not depend on the boundary rates. The other cases correspond to the slow phase, where the relaxation time does depend on the boundary rates.

4. Classification of the solvable models

The local Hamiltonian has 12 independent parameters. Of these 12 parameters, only 8 enter the evolution equations of the expectation values of the number operators. The criteria for the autonomy of the system are the two equations (13). The system is called superautonomous, if there is a basis corresponding to each link, which diagonalizes all one-point transfer matrices corresponding to the equation for the time-independent configuration of one-point functions (hence diagonalizing the whole transfer matrix). That condition could be written as the existence of the matrices \(\Delta\) and \(\Sigma\) such that

\[
D_\alpha = \Sigma_{\alpha+i} \Delta_i \Sigma_{i-\alpha},
\]

as discussed in [10]. The additional criteria for the system to be superautonomous and its relaxation be solvable through the transfer-matrix method are that \(\kappa, \nu\), and \((\eta \theta)\) be link independent. All of these five conditions can be written in terms of the following eight nonnegative independent combinations of the rates:

\[
A_\alpha := s_\alpha(H_\alpha)^{a_0} a_1^{00},
\]

\[
B_\alpha := s_\alpha(H_\alpha)^{a_1} a_0^{00},
\]

\[
D_\alpha := s_\alpha(H_\alpha)^{a_0} a_0^{01},
\]

\[
F_\alpha := s_\alpha(H_\alpha)^{a_1} a_1^{01},
\]

\[
G_\alpha := s_\alpha(H_\alpha)^{a_0} a_1^{10},
\]

\[
I_\alpha := s_\alpha(H_\alpha)^{a_1} a_1^{10},
\]

\[
K_\alpha := s_\alpha(H_\alpha)^{a_0} a_0^{11},
\]

\[
L_\alpha := s_\alpha(H_\alpha)^{a_1} a_1^{11}.
\]

The model is autonomous iff

\[
A_\alpha + D_\alpha = I_\alpha + L_\alpha,
\]

\[
B_\alpha + G_\alpha = F_\alpha + K_\alpha.
\]
One has
\[ \eta_\alpha = I_\alpha - A_\alpha, \]
\[ \kappa = -A_\alpha - D_\alpha, \]
\[ \nu = -B_\alpha - G_\alpha, \]
\[ \theta_\alpha = F_\alpha - B_\alpha. \]
\[ (62) \]

Fixing \( \kappa, \eta_\alpha, \nu \) and \( \theta_\alpha \), using (61) and (62), one is still left with two degrees of freedom. It should be noted, however, that \((\eta_\alpha \theta_\alpha)\) should be link independent, and \((\eta_\alpha - \kappa)\) and \((\theta_\alpha - \nu)\) are nonnegative. Then, the parameters entering evolution equations are \( \kappa, \nu, \eta_\alpha, \theta_\alpha, A_\alpha \) and \( B_\alpha \).

The local Hamiltonian, however, still contains two more free parameters which do not enter the time evolution equations of the expectation values of the number operators.

5. Examples

Consider a lattice each site of which is either empty (◦) or full (●). Two examples are studied here.

5.1. The diffused voting model

The reactions on a link are

- \( \circ \bullet \rightarrow \circ \circ \) with the rate \( v = e_\alpha \),
- \( \circ \bullet \rightarrow \bullet \circ \) with the rate \( e_\alpha \),
- \( \circ \bullet \rightarrow \bullet \bullet \) with the rate \( u = e_\alpha \),
- \( \bullet \circ \rightarrow \circ \circ \) with the rate \( u = h_\alpha \),
- \( \bullet \circ \rightarrow \circ \bullet \) with the rate \( h_\alpha \),
- \( \bullet \circ \rightarrow \bullet \bullet \) with the rate \( v = h_\alpha \).

The rates of injection and extraction of particles in the first (final) sites are \( a \) and \( b \) (\( a' \) and \( b' \)), respectively. The rates \( u \) and \( v \) are link independent, but the diffusion rates, \( h_\alpha \) and \( e_\alpha \), may be link dependent. Using (15) and (29) one arrives at

\[ \eta = v, \]
\[ \theta = u, \]
\[ \kappa = -v, \]
\[ \nu = -u, \]
\[ \xi = 0, \]
\[ \sigma = 0, \]
\[ \zeta = a + b - v, \]
\[ \lambda = a' + b' - u. \]
\[ (64) \]

Although the diffusion rates are link dependent, all the parameters entering the time evolution equation for \( \langle n_i \rangle_{\circ \bullet} \) are link independent. The time evolution equations are exactly the same as those obtained for the corresponding uniform model [11]. There using a different method, it was shown that the dynamical phase transition is controlled by the reaction rates at the boundaries.
Using (53), (57) and (64), one has

\[ \tau_{\text{fast}} = \frac{1}{(\sqrt{u} + \sqrt{v})^2}, \]
\[ \tau_\zeta = \left( a + b + u + \frac{uv}{a + b - v} \right)^{-1}, \]
\[ \tau_\lambda = \left( a' + b' + v + \frac{uv}{a' + b' - u} \right)^{-1}. \]  

(65)

It is seen from (58) that

\[ \tau = \begin{cases} 
\tau_\zeta, & (a + b) < (v - \sqrt{uv}) \\
\tau_\lambda, & (a' + b') < (u - \sqrt{uv}) \\
\tau_{\text{fast}}, & \text{otherwise.} 
\end{cases} \]  

(66)

The system is in the fast phase (the last case) when the boundary rates are high enough, so that it is the bulk reaction rates that determine the relaxation, and goes to the slow phase when the boundary reaction rates are less than some critical value.

5.2. Diffusion and annihilation

The reactions on a link are

\[ \circ \bullet \rightarrow \circ \circ \] with the rate \( v - e_a, \)
\[ \circ \bullet \rightarrow \bullet \circ \] with the rate \( e_a, \)
\[ \bullet \circ \rightarrow \circ \circ \] with the rate \( u - h_a, \)
\[ \bullet \circ \rightarrow \circ \bullet \] with the rate \( h_a, \)
\[ \bullet \bullet \rightarrow \circ \bullet \] with the rate \( u - e_a, \)
\[ \bullet \bullet \rightarrow \bullet \circ \] with the rate \( v - h_a. \]  

(67)

The rates of injection and extraction of particles in the first (final) sites are \( a \) and \( b \) (\( a' \) and \( b' \)), respectively. The rates \( u \) and \( v \) are link independent, but \( h_a \) and \( e_a \) could be link dependent, with their product link independent:

\[ e_a h_a = m. \]  

(68)

Also each of \( e_a \) and \( h_a \) should be less than or equal to each of \( u \) and \( v \), resulting in

\[ m \leq u v. \]  

(69)

Using (15) and (29) one arrives at

\[ \eta_a = h_a, \]
\[ \theta_a = e_a, \]
\[ \kappa = -v, \]
\[ v = -u, \]
\[ \xi = 0, \]
\[ \sigma = 0, \]
\[ \zeta = a + b - v, \]
\[ \lambda = a' + b' - u. \]  

(70)
Using (53), (57) and (70), one has

\[ \tau_{\text{fast}} = (u + v - 2\sqrt{m})^{-1}, \]
\[ \tau_\xi = \left( a + b + u + \frac{m}{a + b - v} \right)^{-1}, \]
\[ \tau_\lambda = \left( a' + b' + v + \frac{m}{a' + b' - u} \right)^{-1}, \]

and from (58)

\[ \tau = \begin{cases} 
\tau_{\text{fast}}, & (a' + b' > u - \sqrt{m}) \land (a + b > v - \sqrt{m}) \\
\tau_\xi, & (a' + b' > u - \sqrt{m}) \land (a + b < v - \sqrt{m}) \\
\tau_\lambda, & (a' + b' < u - \sqrt{m}) \land (a + b > v - \sqrt{m}) \\
\max(\tau_\xi, \tau_\lambda), & (a' + b' > u - \sqrt{m}) \land (a + b < v - \sqrt{m}) \land (a + b > v - \sqrt{m})
\end{cases} \]

It may be that both \((u - \sqrt{m})\) and \((v - \sqrt{m})\) are positive, so unlike the previous example it is possible that two phase transitions occur.

6. Concluding remarks

Autonomous models are those for which the evolution equations for the expectation values of the number operators are closed. Recently, the most general autonomous exclusion models with nearest-neighbor interactions and non-uniform reaction rates on a one-dimensional lattice were studied. In [10], using a transfer-matrix method, possible static phase transitions of such systems had been investigated. Here, a similar transfer-matrix method was used to study the dynamics and possible dynamical phase transitions of superautonomous models. It was seen that superautonomy does not guarantee that the relaxation of the system toward its stationary state be solvable using the transfer-matrix method. One further constraint is needed, which was found. It was also seen that there are two possible phases regarding the relaxation. In the fast phase, the relaxation does not depend on boundary rates. In this phase the relaxation time is controlled by only the bulk reaction rates. There may be slow phases as well. In these cases, the relaxation does receive contributions from the boundary rates. A classification was presented for the models solvable through the transfer-matrix method, and two examples were explicitly studied.

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