REGULAR BERNSTEIN BLOCKS

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Abstract. For a connected reductive group $G$ defined over a non-archimedean local field $F$, we consider the Bernstein blocks in the category of smooth representations of $G(F)$. Bernstein blocks whose cuspidal support involves a regular supercuspidal representation are called regular Bernstein blocks. Most Bernstein blocks are regular when the residual characteristic of $F$ is not too small.

Under mild hypotheses on the residual characteristic, we show that the Bernstein center of a regular Bernstein block of $G(F)$ is isomorphic to the Bernstein center of a regular depth-zero Bernstein block of $G^0(F)$, where $G^0$ is a certain twisted Levi subgroup of $G$. In some cases, we show that the blocks themselves are equivalent, and as a consequence we prove the ABPS Conjecture in some new cases.

1. Introduction

1.1. The problem. Let $F$ be a nonarchimedean local field of residual characteristic $p$, and $G$ a connected reductive $F$-group. One wants to understand the category $\mathcal{R}(G(F))$ of smooth, complex representations of $G(F)$. The Bernstein decomposition \cite{4} allows us to decompose this category into a product $\prod\{(M,\sigma)_G\} \mathcal{R}^{(M,\sigma)}_G(G(F))$ of full subcategories $\mathcal{R}^{(M,\sigma)}_G(G(F))$, called Bernstein blocks, so it is in some sense enough to understand each block. (Here, $(M,\sigma)_G$ ranges over the set of inertial classes of $G(F)$, such a class is represented by an $F$-Levi subgroup $M$ of $G$ and an irreducible, supercuspidal representation $\sigma$ of $M(F)$, and consists of the set of all pairs $(M'(F),\sigma')$ that are equivalent to $(M(F),\sigma)$ under twisting by unramified characters of $M(F)$ and conjugation by elements of $G(F)$.) Each block $\mathcal{R}^{(M,\sigma)}_G(G(F))$ is a module category over an algebra, and one can construct an appropriate Hecke algebra in a more-or-less explicit way if $(M,\sigma)_G$ has an associated Bushnell-Kutzko type \cite{6}. In particular, if $\sigma$ has depth zero, then the structure of the Hecke algebra is given by Morris \cite{12}. Thus, in principle, one way to understand the category $\mathcal{R}(G(F))$ would be to show that for each block $\mathcal{R}^{(M,\sigma)}_G(G(F))$, there is another connected reductive $F$-group $G^0$ and a depth-zero block $[M^0,\sigma^0]_{G^0}$ for $G^0(F)$ such that the blocks are equivalent as categories.

Our aim in this paper is to study the structure of Bernstein blocks using the above approach in many cases.

Let us remark that Kim and Yu \cite{16} give a construction of types (see \cite{16} §7.4 for more details), and Fintzen \cite{9} has recently shown that if $G$ splits over a tame extension of $F$ and $p$ does not divide the order of the Weyl group of $G$, then every Bernstein block for $G(F)$ admits a type arising from this construction. The Kim-Yu construction starts with a certain datum $\Sigma$, out of which it produces a sequence of types $(K^i,\rho_i)$ for groups $G^i(F)$, $0 \leq i \leq d$, where $\tilde{G} = (G^0 \subset \ldots \subset G^d = G)$ is a tower of twisted Levi subgroups of $G$ and which is part of the datum $\Sigma$. Write $(K,\rho)$ for $(K^d,\rho_d)$ and write $\mathcal{H}(G,\rho^\vee)$ (resp. $\mathcal{H}(G^0,\rho^\vee_0)$) for the Hecke algebra associated
to the type \((K, \rho)\) (resp. \((K^0, \rho_0)\)). Thus, one should expect the following precise statement, which is essentially suggested in Yu’s Conjecture ([20, Conjecture 0.2]).

**Conjecture 1.1.** There exists an algebra isomorphism

\[ \mathcal{H}(G, \rho^\vee) \sim \mathcal{H}(G^0, \rho_0^\vee). \]

While the Hecke algebra \(\mathcal{H}(G^0, \rho_0^\vee)\) is determined up to isomorphism, it (and thus the isomorphism of the Conjecture) depends on \(\Sigma\), which is not uniquely determined.

1.2. Bernstein center. In [8], Kaletha studies a large class of supercuspidal representations which he calls regular. Most supercuspidal representations are of this kind when \(p\) is not too small (see the paragraph following [8, Definition 3.7.3] to understand what is meant by “most”). We call a Bernstein block \(\mathcal{R}^{[M, \sigma]}(G(F))\) regular if \(\sigma\) is a regular supercuspidal representation. Under certain restrictions on \(p\), we show in Theorem 7.3 that

**Theorem 1.2.** The Bernstein center of a regular Bernstein block \(\mathcal{R}^{[M, \sigma]}(G(F))\) is isomorphic to the Bernstein center of a depth-zero regular Bernstein block \(\mathcal{R}^{[M^0, \sigma]}(G^0(F))\) of a twisted Levi subgroup \(G^0\) of \(G\).

When \(M = G\), this result is covered by [11, Theorem 6.1] which proves such an isomorphism for tame supercuspidal blocks. Thus, Theorem 1.2 is a generalization for regular Bernstein blocks of the main result of [11].

1.3. Hecke algebra isomorphism. Suppose \(M = G\). Then Conjecture 1.1 coincides with [20, Conjecture 0.2], which appears here as Conjecture 6.3. We show (Corollary 6.4) that the conjecture is true if the \(F\)-split rank of the center of \(G\) is at most 1; or if \(\sigma\) is generic; or if \(p\) satisfies some mild hypotheses and the restriction of \(\sigma\) to the derived group is regular in the sense of Kaletha [8].

Now let us consider the case where \(M\) is not necessarily equal to \(G\). Let \((K, \rho)\) be the \([M, \sigma]_G\)-type obtained out of the datum \(\Sigma\) by the Kim-Yu construction. Assume that the tower \(\bar{G}\) of twisted Levi subgroups contained in the datum \(\Sigma\) consists of \(F\)-Levi subgroups. Then we show (in Corollary 8.2) that Conjecture 1.1 holds. This follows as an easy consequence of the results in [16]. A special case where this condition on \(\Sigma\) holds is when \(G\) is quasi-split and \(M\) is a minimal \(F\)-Levi subgroup of \(G\). Thus, as a special case, one obtains Conjecture 1.1 for principal series blocks, generalizing the result of Roche [17].

We prove weaker versions of Conjecture 1.1 for more general situations in §10.

1.4. Consequences for the ABPS Conjecture. Aubert, Baum, Plymen, and Solleveld [3] conjecture that the set of irreducible objects in a Bernstein block has the structure of a particular twisted extended quotient. This “ABPS Conjecture” is known to hold for depth-zero Bernstein blocks due to the work of Solleveld [19]. We show, using this result of Solleveld, that under certain restrictions on \(p\), Conjecture 1.1 for a regular Bernstein block implies ABPS Conjecture for that block. Consequently, we obtain the ABPS Conjecture for many new cases (Theorem 9.6), namely those mentioned in the previous section. In particular, as the simplest case, we obtain ABPS Conjecture for principal series blocks. This generalizes a part of the main result in [3] which proves it only for principal series blocks of split groups.
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2. **Notation**

Throughout this article, \( F \) denotes a non-archimedean local field of residue characteristic \( p \). If \( G \) is a reductive \( F \)-group and \( K \) is a subgroup of \( G(F) \), we will denote \( gKg^{-1} \) by \( ^gK \) for \( g \in G(F) \). If \( \rho \) is a representation of \( K \), then \( ^g\rho \) will denote the representation \( x \mapsto \rho(g^{-1}xg) \) of \( ^gK \). If \( H \) is a subgroup of a group \( G \), then \( N_G(H) \) denotes the normalizer of \( H \) in \( G \).

Write \( B(G, F) \) for the Bruhat-Tits building of \( G(F) \). For any point \( x \in B(G, F) \), we write \( G(F)_{x,0} \) for the parahoric subgroup of \( G(F) \) associated to \( x \), and we let \( G(F)_{x:0}^{0+} \) denote the quotient of \( G(F)_{x,0} \) by its maximal pro-\( p \) subgroup. If \( G \) is a torus, then we omit \( x \) from the notation and just write \( G(F)_0 \) for the unique parahoric subgroup of \( G(F) \), and we also let \( G(F)_b \) denote the unique maximal bounded subgroup.

3. **Review of the Bernstein center**

Throughout, \( G \) denotes a connected reductive \( F \)-group. We quickly review general theory and fix some notation.

3.1. **Bernstein decomposition.** Let \( ^G G(F) \) denote the subgroup

\[ \{ g \in G(F) : |\nu(g)| = 1 \text{ for every character } \nu : G(F) \to \mathbb{C}^\times \}, \]

and let \( X_{nr}(G(F)) = \text{Hom}(G(F)/^0G(F), \mathbb{C}^\times) \). The group \( X_{nr}(G(F)) \) is called the group of **unramified characters** of \( G(F) \). Consider the pairs \((L(F), \sigma)\) where \( L \) is an \( F \)-Levi subgroup of \( G \) and \( \sigma \) is an irreducible supercuspidal representation of \( L(F) \). Denote by \([L, \sigma]_G\) its **inertial equivalence class**: the set of pairs \((L', \sigma')\), such that \( L' \) is an \( F \)-Levi subgroup of \( G \), \( \sigma' \) is an irreducible, supercuspidal representation of \( L(F) \), and \((L', \sigma') = (^gL, ^g\sigma \otimes \nu) \) for some \( g \in G(F) \) and \( \nu \in X_{nr}(L(F)) \). The set of inertial equivalence classes is called the **Bernstein spectrum** of \( G(F) \), which we denote by \( \mathcal{B}(G, F) \). Let \( \mathcal{R}(G(F)) \) denote the category of smooth representations of \( G(F) \). We say that a smooth irreducible representation \( \pi \) has **inertial support** \( \mathfrak{s} = [L, \sigma]_G \) if \( \pi \) appears as a subquotient of a representation parabolically induced from some element of the class \( \mathfrak{s} \). Define a full subcategory \( \mathcal{R}^\mathfrak{s}(G(F)) \) of \( \mathcal{R}(G(F)) \) as follows: \( \pi \in \mathcal{R}(G(F)) \) belongs to \( \mathcal{R}^\mathfrak{s}(G(F)) \) if each irreducible subquotient of \( \pi \) has inertial support \( \mathfrak{s} \). We will denote the class of irreducible objects in \( \mathcal{R}^\mathfrak{s}(G(F)) \) by \( \text{Irr}^\mathfrak{s}(G(F)) \).

**Theorem 3.1** (Bernstein). *We have a decomposition*

\[ \mathcal{R}(G(F)) = \prod_{\mathfrak{s} \in \mathcal{B}(G, F)} \mathcal{R}^\mathfrak{s}(G(F)). \]

3.2. **Hecke algebra.** Let \((\tau, W)\) be an irreducible representation of a compact open subgroup \( J \) of \( G(F) \). The Hecke algebra \( \mathcal{H}(G, \tau) \) is the space of compactly supported functions \( f : G(F) \to \text{End}_\mathbb{C}(\tau) \) satisfying

\[ f(j_1gj_2) = \tau(j_1)f(g)\tau(j_2). \]

The standard convolution algebra gives \( \mathcal{H}(G, \tau) \) the structure of an associative \( \mathbb{C} \)-algebra with identity.
We say that \((J, \tau)\) is an \(s\)-type if \(\mathcal{R}^s(G(F))\) is precisely the subcategory of \(\mathcal{R}(G(F))\) consisting of smooth representations which are generated by their \(\tau\)-isotypic component. In that case, \(\mathcal{R}^s(G(F))\) is equivalent to the category of non-degenerate modules of \(\mathcal{H}(G, \tau^\vee)\), where \(\tau^\vee\) is the dual of \(\tau\).

3.3. Bernstein center. Let \(s = [L, \sigma|_G] \in \mathfrak{B}(G, F)\) and set \(t = [L, \sigma|_L] \in \mathfrak{B}(L, F)\). The center \(\mathfrak{Z}\) (resp. \(\mathfrak{Z}^s\), resp. \(\mathfrak{Z}^t\)) of the category \(\mathcal{R}(G(F))\) (resp. \(\mathcal{R}^s(G(F))\), resp. \(\mathcal{R}^t(L(F))\)) is called its Bernstein center. Recall here that the center of an abelian category is the endomorphism ring of the identity functor. Let

\[ N^s = \{ \pi \in N(G)_c(L)(F) \mid \pi \sigma \cong \pi \nu \text{ for some } \nu \in X_{nr}(L(F)) \}, \]

and write \(W^s = N^s/L(F)\). The set \(\text{Irr}^t(L(F))\) has the structure of a complex affine variety, on which the group \(W^s\) acts algebraically. (The variety \(\text{Irr}^t(L(F))\) is (non-canonically) isomorphic to the quotient \(X_{nr}(L(F))/\{ \chi \in X_{nr}(L(F)) \mid \pi \chi = \sigma \} \).

**Theorem 3.2** (Bernstein). The Bernstein center can be viewed as the ring of regular functions on the quotient variety \(\text{Irr}^t(L(F))/W^s\).

4. Review of tame and regular supercuspidal representations

4.1. Tame supercuspidal representations. Assume that \(G\) splits over the maximal tamely ramified extension \(F^s\) of \(F\). A tamely twisted Levi subgroup of \(G'\) of \(G\) is a reductive \(F\)-subgroup such that \(G' \otimes F^s\) is an \(F^s\)-Levi subgroup of \(G \otimes F^s\).

Let \(\Sigma\) denote the datum \((\vec{G}, \pi_0, \vec{\varphi})\), where \(\vec{G} = (G^0, \ldots, G^d)\) is a tower of algebraic subgroups of \(G\),

\[ G^0 \subset \cdots \subset G^d = G, \]

such that \(Z(G^0)/Z(G)\) is anisotropic and each \(G^i\) is a tamely twisted Levi subgroup of \(G\). \(\pi_0\) is a depth-zero supercuspidal representation of \(G^0(F)\) and \(\vec{\varphi} = (\phi_0, \ldots, \phi_d)\) is such that \(\phi_i : G^i(F) \to \mathbb{C}^\times\) is a smooth character of depth \(r_i > 0\). We require the datum \(\Sigma\) to satisfy several technical conditions. (See [13, §3.1] for a precise list. Note that this source and others, e.g., [8], use the notation \(\pi_{-1}\) for the representation that we are calling \(\pi_0\).)

Yu’s construction [20] produces a supercuspidal representation \(\pi\) out of the datum \(\Sigma\). Recently, Fintzen [10] has shown that if the residue characteristic \(p\) does not divide the order of the Weyl group, this construction yields all supercuspidal representations of \(G(F)\).

4.2. Regular supercuspidal representations. Let \((S, \theta)\) be a pair consisting of tame maximal \(F\)-torus \(S\) in \(G\) and a character \(\theta : S(F) \to \mathbb{C}^\times\). A Kaletha-Howe factorization of \((S, \theta)\) is a pair \((\vec{G}, \vec{\phi})\), where \(\vec{G} = (S = G^{-1} \subset G^0 \subset \cdots \subset G^d = G)\) is a tower of tamely twisted Levi \(F\)-subgroups and \(\vec{\phi} = (\phi_{-1}, \ldots, \phi_d)\) is a sequence of characters \(\phi_i : G^i(F) \to \mathbb{C}^\times\) and satisfying

\[ \theta = \prod_{i=-1}^{d} \phi_i|_{S(F)} \]

and some additional technical conditions (see [8, Definition 3.6.2]). If the datum \((S, \theta)\) is a tame regular elliptic pair, i.e., it satisfies the conditions in [8, Definition 3.7.5], then the pair \((S, \phi_{-1})\) determines a depth-zero supercuspidal representation \(\pi_0\) of \(G^0(F)\) and the datum

\[ \Sigma = (G^0 \subset \cdots \subset G^d = G), \pi_0, (\phi_0, \ldots, \phi_d) \]
is a Yu datum and therefore produces a supercuspidal representation \(\pi(S, \theta)\) of \(G(F)\). These representations are called regular supercuspidal representations.

If \((S, \theta)\) and \((S', \theta')\) are tame regular elliptic pairs in \(G\), then \(\pi(S, \theta)\) and \(\pi(S', \theta')\) are equivalent if and only if \((S, \theta)\) and \((S', \theta')\) are conjugate in \(G(F)\).

5. A RESULT ON MULTICIPLICITY ONE UPON RESTRICTION

In this section, we will assume the following.

**Hypothesis 5.1.** The residue characteristic \(p\) of \(F\) is not a bad prime for \(G\) (see \([8] \S 2.1\)) and does not divide the order of the fundamental group of the derived group \(G^{\text{der}}\) of \(G\).

Let \((S, \theta)\) be a tame regular elliptic pair in \(G\). Let \(G'\) be an \(F\)-subgroup of \(G\) such that \(G^{\text{der}} \subset G' \subset G\). Write \(S' = S \cap G'\) and \(\theta' = \theta|_{S'(F)}\). Let \(N_{G(F)}(S', \theta'|_{S'(F)})\) denote the stabilizer of \(\theta'|_{S'(F)}\) in \(N_{G(S)}(S')(F)\).

**Definition 5.2.** The pair \((S, \theta)\) is regular in \(G'\) if \(N_{G(F)}(S', \theta'|_{S'(F)}) = S'(F)\).

**Theorem 5.3.** If \((S, \theta)\) is a tame regular elliptic pair in \(G\) which is regular in \(G'\), then \(\pi(S, \theta)|_{G(F)}\) is multiplicity free.

We observe that \((S, \theta)\) is regular in \(G'\) if and only if the components of \(\pi|_{G(F)}\) are regular supercuspidals. If \((S, \theta)\) is not regular in \(G'\), then higher multiplicities can occur, even though \((S, \theta)\) is regular in \(G\). See [1] \S 7 for an example.

Looking at general irreducible representations of \(G(F)\), there are many situations where one can prove that the restriction to \(G'(F)\) is multiplicity free. See [7] for a conjecture, together with a proof for tempered representation under some assumptions about the local Langlands correspondence. See [1] for a conjecture, reduction to the tempered case, examples, and an announcement of Theorem 5.3.

**Proof of Theorem 5.3.** Since \((S, \theta)\) is regular in \(G'\), it follows from [8] Lemma 3.6.1 that \((S', \theta')\) is a tame regular elliptic pair in \(G'\). Let \(V'\) (resp. \(V\)) be the space realizing the representation \(\pi(S', \theta')\) (resp. \(\pi(S, \theta)\)). One can describe these spaces explicitly, since each regular supercuspidal representation is induced from a representation of a compact open subgroup. We now describe an embedding of \(V'\) in \(V\).

From Yu’s construction [20], we see that

- \(\pi(S, \theta)\) is induced from a smooth irreducible representation \(\rho\) of an open subgroup \(K\) of \(G(F)\) that is compact modulo the center of \(G(F)\);
- \(\pi(S', \theta')\) is similarly induced from \((K', \rho')\); and
- we can realize \(\rho'\) as an irreducible subrepresentation of \(\rho|_{K'}\).

Let \(W\) and \(W'\) be the spaces realizing \(\rho\) and \(\rho'\). Thus, \(W'\) is a subspace of \(W\). We have

\[
W = \bigoplus_{k \in K/\text{Stab}_K(W')} k W'.
\]

Pick coset representatives \(k_1, \ldots, k_n\) of \(K/\text{Stab}_K W'\). Let

\[
\iota: \text{End}(W') \to \text{End}(W)
\]

be given by

\[
\iota: \chi' \mapsto \bigoplus_{i=1}^n k_i \chi'.
\]
Theorem 6.1. \[18, \text{Prop. 1.6.1.2}\]

Proposition 6.2. \(\pi\) and only if \(H\) canonically equivalent to the module category of \(\rho\). The result then follows from \[18, \text{Remark 1.6.1.3}\].

By Clifford theory to \((S, \theta)\) We claim that for \(g / \rho \in \text{Stab}_{G(F)}^G(V)\) is compactly induced from a representation \(\rho / \rho \in G(F)\). Then \(g / g_k^G = g_{g_k^G}^G k, k \in K\).

To see that the function \(f = \iota(f')\) is well defined, let \(g_1, g_2 \in G(F)\) and \(k_1, k_2 \in K\). Then \(g_1^G g_2^{-1} = G(F) \cap K = K\). We have

\[
\iota(f'(g_2')) = \iota(f'(g_1^G g_2^{-1} g_2')) = \iota(f'(g_1')) \rho(k_1 k_2^{-1}) \rho(k_2) = \iota(f'(g_1')) \rho(k_1).
\]

This proves that \(f\) is well defined. Now since \(S(F) \subseteq K\), it follows that \(S(F)\) preserves \(\iota(V')\). Thus

\[\text{Stab}_{G(F)}^G(V') = S(F) G(F)\]

By Clifford theory

\[
\pi(S, \theta) = \bigoplus_{g \in G(F)/\text{Stab}_{G(F)}^G(V')} \pi(S', \theta')
\]

We claim that for \(g / \rho \in \text{Stab}_{G(F)}^G(V')\), \(\pi(S', \theta') \not\equiv \pi(S', \theta')\). For suppose that \(\pi(S', \theta') \equiv \pi(S', \theta')\) for some \(g \in G(F)\). Equivalently, \(\rho(S', \theta')\) is \(G(F)\)-conjugate to \(S', \theta'\). So there exists a \(g' \in G(F)\) such that \(g' g\) stabilizes \(S', \theta'\). So \(g \in G(F) N_{G(F)}(S', \theta')\). But by hypothesis, \(N_{G(F)}(S', \theta') S(F) \subseteq S(F)\). It follows that \(g \in \text{Stab}_{G(F)}^G(V')\). This proves the claim.

Thus the summands \(\pi(S', \theta')\) for \(g \in G(F)/\text{Stab}_{G(F)}^G(V')\) are non-isomorphic and consequently, \(\pi(S, \theta) / G(F)\) is multiplicity free.

6. Hecke algebra for supercuspidal blocks

Let \((\pi, V)\) denote a supercuspidal representation of \(G(F)\). Then \(\pi / G(F)\) is a finite direct sum of finite representations of \(\circ G(F)\), all of which are \(G(F)\)-conjugate up to isomorphism. Fix an irreducible component \((\circ \pi, \circ V)\) of \(\pi / G(F)\). Then \(\text{End}_{G(F)}(\text{ind}_{G(F)}^G \circ \pi)\) is canonically isomorphic to the convolution algebra \(\mathcal{H}(G, \circ \pi)\) of \(\circ \pi\)-spherical functions \([18, \S 1.1.2]\) and the Bernstein component \(\mathcal{R}^s(G(F))\) is canonically equivalent to the module category of \(\mathcal{H}(G, \circ \pi^s)\), where \(s = [G, \pi]_G\).

**Theorem 6.1.** \([18, \text{Prop. 1.6.1.2}]\) The Hecke algebra \(\mathcal{H}(G, \circ \pi)\) is commutative if and only if \(\pi / G(F)\) is multiplicity free.

**Proposition 6.2.** The restriction \(\pi / G(F)\) is multiplicity free if the split rank of the identity component of the center of \(G\) is 0 or 1.

**Proof.** Let \(Z\) denote the center of \(G\). Then in this situation, \(G(F) / (Z : \circ G(F))\) is cyclic. The result then follows from \([18, \text{Remark 1.6.1.3}]\). \(\square\)

Now let \(\pi\) be a tame supercuspidal representation of \(G(F)\). Then by Yu’s construction, \(\pi\) is compactly induced from a representation \(\rho\) of an open, compact mod center subgroup \(K\) of \(G(F)\). Let \(\circ K\) denote the maximal compact subgroup \(K \cap \circ G(F)\) of \(K\), and let \(\circ \rho\) be any irreducible component of \(\rho / K\). Then \((\circ K, \circ \rho)\) is an \(s\)-type where \(s = [G, \pi]_G\) \([6, \text{Prop. 5.4}]\). The representation \(\pi\) is constructed out of a depth-zero representation \(\pi_0\) of a twisted Levi subgroup \(G^0\) of \(G\) together
with additional data. The representation $\pi_0$ is compactly induced from a representation $\rho_0$ of a compact mod center subgroup $K^0$ of $G^0(F)$. Define an $s_0$-type $(^GK^0, ^G\rho_0)$ analogously, where $s_0 = [G^0(F), \pi_0|_{C^0(G^0(F))}]$.

Let $\mathcal{H}(G, ^G\rho)$ (resp. $\mathcal{H}(G^0, ^G\rho_0)$) denote the Hecke algebra associated to $(^GK, ^G\rho)$ (resp. $(^GK^0, ^G\rho_0)$). Jiu-Kang Yu makes the following conjecture.

**Conjecture 6.3** (Yu’s conjecture). There is an algebra isomorphism

$$\mathcal{H}(G, ^G\rho) \cong \mathcal{H}(G^0, ^G\rho_0).$$

**Corollary 6.4.** Yu’s conjecture holds if any of the following conditions hold:

(a) The split rank of the identity component of the center of $G$ is 0 or 1.
(b) Hypothesis 5.1 holds, and the irreducible components of $\pi|_{G^0(F)}$ are regular.
(c) $\pi$ is generic.

**Proof.** By [11, Cor. 6.3], the centers of $(\mathcal{H}(G, ^G\rho))$ and $(\mathcal{H}(G^0, ^G\rho_0))$ are isomorphic. Thus, the conjecture is true if both of these Hecke algebras are commutative. From Theorem 6.1 it will be enough to show that $\pi$ and $\pi_0$ restrict without multiplicity to $^G\rho(G(F))$ and $^G\rho_0(G^0(F))$, respectively. Suppose condition (a) holds. Then our restrictions are multiplicity free from Proposition 6.2, together with the fact that the centers of $G$ and $G^0$ have the same split rank. Suppose condition (b) holds. By [8, Lemma 3.6.5], the components of $\pi_0|_{(G^0(F))}$ are regular. Thus our restrictions are multiplicity free from Theorem 5.3. Suppose condition (c) holds. From a result of Stephen DeBacker and Cheng-Chiang Tsai, $\pi_0$ is also generic, and so our restrictions are multiplicity free from [18, Remark 1.6.1.3].

Note that in proving that Yu’s conjecture holds under condition (c), we have used a result that is not yet in the literature. Readers who are unhappy about this can strengthen condition (c) to assert that both $\pi$ and $\pi_0$ are generic.

7. Bernstein center of regular blocks

Assume Hypothesis 5.1

Let $L$ be an $F$-Levi subgroup of $G$ and let $(S, \theta)$ be a tame regular elliptic pair in $L$. Write $\pi = \pi(S, \theta)$ for the associated regular supercuspidal representation of $L(F)$. The pair $(S, \theta)$ produces a chain $(S = G^{-1} \subseteq G^0 \subseteq \ldots \subseteq G^d = G)$ of twisted Levi subgroups of $G$. Write $L^0 = L \cap G^0$. Let $\phi_i : G^i(F) \to \mathbb{C}^\times$, $-1 \leq i \leq d$, be the sequence of characters obtained from a Howe factorization of $(S, \theta)$. Then $\theta = \theta_- \theta_+$, where $\theta_- = \phi_{-1}|_{S(F)}$ and $\theta_+ = \prod_{i=0}^d \phi_i|_{S(F)}$. It follows from [8, Lemma 3.6.5] that $(S, \theta_-)$ is a depth-zero tame regular elliptic pair for $L^0$. Let $\pi_0(S, \theta_-)$ denote the associated depth-zero regular supercuspidal representation of $L^0(F)$.

For $s = [L, \pi]|_G$, recall that we denote by $W^s$ the stabilizer of $s$ in $(N_G(L)(F)/L(F))$, i.e., $W^s = N^s/L(F)$, where

$$N^s = \{ n \in N_G(L)(F) \mid n\pi \cong \pi \ \text{for some} \ \nu \in X_{nu}(L(F))\}.$$ 

Similarly define $s_0, N^{s_0}$ and $W^{s_0}$ by replacing $L, \pi$ and $G$ in the above definition by $L^0, \pi_0$ and $G^0$.

**Lemma 7.1.** The stabilizer of $\theta|_{S(F)_b}$ in $N_G(S)(F)$ lies in $N_{G^0}(S)(F)$ and equals the stabilizer of $\theta_-|_{S(F)_b}$ there.

**Proof.** The proof is identical to the proof of [8, Lemma 3.6.5] when we replace $S(F)_r$ there with $S(F)_b$. 

\[\square\]
Theorem 7.2. There is a group isomorphism
\[ \iota_\pi : W^g \cong W^{s_{\pi}}. \]

Proof. Take \( n \in \mathbb{N}^g \). So \( n \in N_0(L)(F) \) and \( n(S, \theta) \) is \( L(F) \)-conjugate to \( (S, \theta\lambda) \) for some \( \lambda \in X_{\text{nr}}(S) \). This implies that there exists an \( l \in L(F) \) such that \( l^{-1}n \in \text{Stab}_{N_0(L)(F)}^{\theta \mid S(F)_{n}} \). By Lemma 7.1, \( l^{-1}n \in \text{Stab}_{N_0(L)(F)}^{\theta \mid S(F)_{n}} \). Thus we get a map \( W^g \to W^{s_{\pi}} \) induced by the map \( n \in \mathbb{N}^g \to l^{-1}n \in \mathbb{N}^{s_{\pi}} \). We claim that this map is an isomorphism. Suppose \( n_0 \in \mathbb{N}^{s_{\pi}} \). Then \( n_0 \in N_0(L^0)(F) \) such that \( l_0^{-1}n_0(S, \theta_-) = (S, \theta_-\lambda_-) \) for some \( l_0 \in L^0(F) \) and some \( \lambda_- \in X_{\text{nr}}(S) \). Thus \( l_0^{-1}n_0 \in \text{Stab}_{N_0(L^0)(F)}^{\theta_- \mid S(F)_{n_0}} \). By Lemma 7.1, \( l_0^{-1}n_0 \in \text{Stab}_{N_0(L^0)(F)}^{\theta_- \mid S(F)_{n_0}} \). To complete the proof, it suffices to show that \( l_0^{-1}n_0 \in N_0(L)(F) \).

For a torus \( T \), denote its split part by \( T_s \). Now recall that \( Z(L^0)/Z(L) \) is \( F \)-anisotropic. That is,
\[ Z(L^0)^g_s = Z(L^0)^{s_{\pi}}. \]

Let \( n = l_0^{-1}n_0 \). Since \( n \in N_0(L^0)(F) \), \( n \) preserves \( Z(L^0) \) and since \( n \) is rational, \( n \) preserves \( Z(L^0)^g_s \). Thus \( n \) preserves \( Z(L^0)^{s_{\pi}} \) and therefore also preserves \( Z_G/Z(L^0)^{s_{\pi}} = L \). This completes the proof. \( \square \)

Theorem 7.3. There is an algebra isomorphism
\[ \mathfrak{z}^g \cong \mathfrak{z}^{s_{\pi}}. \]

Proof. Write \( t = [L, \pi]_{L/F} \) and \( t_0 = [L^0(F), \pi_0]_{L^0(F)} \). By [11, Theorem 6.1], the map
\[ f : \pi \otimes \nu \in \text{Irr}^t(L(F)) \to \pi_0 \otimes (\nu|_{L^0(F)}) \in \text{Irr}^{t_0}(L^0(F)), \nu \in X_{\text{nr}}(L(F)) \]
is an isomorphism of varieties. It is clear that this map does not depend on the choice of \( \pi \), and it is clear from the construction that \( \iota_\pi \) is compatible with \( f \). Consequently, there is an isomorphism of quotient varieties,
\[ \text{Irr}^t(L(F))/W^g \cong \text{Irr}^{t_0}(L^0(F))/W^{s_{\pi}}. \]

Since \( \mathfrak{z}^g \) (resp. \( \mathfrak{z}^{s_{\pi}} \)) is the ring of regular functions on the Bernstein variety \( \text{Irr}^t(L(F))/W^g \) (resp. \( \text{Irr}^{t_0}(L^0(F))/W^{s_{\pi}} \)), the result follows. \( \square \)

8. HECKE ALGEBRA OF TAME TYPES

Let \( G \) be tamely ramified. We denote by \( B(G,F) \) the Bruhat-Tits building of \( G(F) \) and likewise for other algebraic groups. Consider the datum
\[ \Sigma = ((\tilde{G}, M^0), (y, \{t\}), \bar{r}, (K_{M^0}, \rho_{M^0}, \tilde{\phi})) \]
as in [16, §7.2]. So \( \tilde{G} = (G^0, \ldots, G^d) \) is a tamely ramified twisted Levi sequence in \( G, M^0 \) is a Levi subgroup of \( G^0, \bar{r} = (r_0, \ldots, r_d) \) is a sequence of real numbers, \( y \) is a point in \( B(M^0,F), \{t\} \) is a commutative diagram
\[ \begin{array}{c}
B(G^0,F) \downarrow \downarrow \\
\downarrow \\
B(M^0,F) \downarrow \downarrow \\
\text{B}(G^1,F) \downarrow \downarrow \\
\downarrow \\
\text{B}(M^1,F) \downarrow \downarrow \\
\text{B}(G^d,F) \downarrow \downarrow \\
\text{B}(M^d,F)
\end{array} \]
where \( M^i \) is the centralizer in \( G^i \) of the split part of the connected center of \( M^0 \). The diagram \( \{t\} \) is required to be \( \bar{s}' \)-generic relative to \( y \) (see [16, §3.5, Definition]), where \( \bar{s}' = (0, r_0/2, \ldots, r_{d-1}/2) \). The group \( K_{M^0} \) is a compact open subgroup of
$M^0(F)$ containing the parahoric $M^0(F)_{y,0}$ as a normal subgroup, and $\rho_{M^0}$ is an irreducible smooth representation of $K_{M^0}$ such that $\rho_{M^0} \mid M^0(F)_{y,0}$ contains a cuspidal representation of the finite quotient $M^0(F)_{y,0;0,+}$. Finally, $\vec{\phi} = (\phi_0, \ldots, \phi_d)$, where each $\phi_i$ is a quasi-character of $G^i(F)$. The datum $\Sigma$ is constrained by several conditions (loc. cit.).

From the datum $\Sigma$, Kim and Yu’s construction produces a sequence of Bushnell-Kutzko types $(K^i, \rho_i)$ for $G^i(F)$. Write $K = K^d$ and $\rho = \rho_d$.

**Theorem 8.1.** Let $\Sigma$ be the datum as above. Assume that for $0 \leq i \leq d$, $G^i$ is the Levi factor of a rational parabolic subgroup of $G$. Moreover, in the commutative diagram $\{ t \}$, choose $B(G^i, F) \hookrightarrow B(G^{i+1}, F)$ to be $r_i/2$-generic. Then $\Sigma$ determines an isomorphism

$$\mathcal{H}(G, \rho^\vee) \cong \mathcal{H}(G^0, \rho_0^\vee).$$

**Proof.** Let $P = G^{d-1}U$ be a parabolic subgroup of $G$ with Levi factor $G^{d-1}$ and $\mathring{P} = G^{d-1}U$ the opposite parabolic. Then by [10] Lemma 6.2(a), $K$ is decomposed with respect to $(U, G^{d-1}, U)$. Let $K_+ = K^d_+$ be as defined in [10] §7.4. Then since $K \cap U = K_+ \cap U$ and $K \cap \mathring{U} = K_+ \cap \mathring{U}$ by the genericity of $\{ t \}$, it follows that $K \cap U, K \cap \mathring{U} \subset \ker(\rho_d)$. By [10] Theorem 8.1, the support of the Hecke algebra $\mathcal{H}(G, \rho^\vee)$ is contained in $KG^{d-1}(F)K$. Thus, by [8] Theorem 7.2(ii),

$$\mathcal{H}(G, \rho^\vee) \cong \mathcal{H}(G^{d-1}, \rho_{d-1}^\vee).$$

The result follows by induction.

Assume $G$ is quasi-split and let $T$ be a maximal torus contained in an $F$-Borel subgroup. Assume Hypothesis [6.4. Let $\chi$ be a character of $T(F)$. Let $(\vec{G}, \vec{\phi})$ denote the Kaletha-Howe factorization of $(T, \chi)$ and let $r_i$ denote the depth of $\phi_i$. Put $M^0 = T$, $K_{M^0} = T(F)_0$ and $\rho_{M^0} = \chi \mid T(F)_0$. Since $T$ is contained in a Borel, $M^i = T$ for $0 \leq i \leq d$. Choose any point $y \in B(M^0, F)$ and choose $\{ t \}$ such that $B(G^i, F) \hookrightarrow B(G^{i+1}, F)$ is $r_i/2$-generic. Let $\Sigma$ be the datum consisting of these choices and let $(K^i, \rho_i)$ be the Bushnell-Kutzko types for $G^i$ constructed out of $\Sigma$ by Kim and Yu’s construction. Then $(K, \rho)$ is a $[T, \chi]_G$-type and $(K^0, \rho_0)$ is a $[T, \chi_0]_{G^0}$-type for a depth-zero character $\chi_0$ of $T(F)$. Since $T \subset G^i$ for all $i$, each $G^i$ has the same $F$-split rank as $G$, and thus each $G^i$ is an $F$-Levi subgroup of $G$. Therefore, Theorem 8.1 implies the following result.

**Corollary 8.2.** If $G$ is quasi-split and $p$ satisfies Hypothesis [5.7.1 then the Hecke algebra of a principal series block of $G(F)$ is isomorphic to the Hecke algebra of a depth-zero principal series block of a Levi subgroup $G^0(F)$ of $G(F)$. More precisely, the choice of $\Sigma$ determines an isomorphism

$$\mathcal{H}(G, \rho^\vee) \cong \mathcal{H}(G^0, \rho_0^\vee).$$

**Remark.** The Hecke algebras of depth-zero Bernstein blocks are known due to the work of Morris [12]. Therefore Corollary 8.2 can be used to produce generators and relations for the Hecke algebras of principal series blocks. When $G$ is split, this was done by Roche [17].

9. Consequences for ABPS

9.1. **Twisted extended quotient.** We recall here the notion of twisted extended quotients as given in [2 §2.1]. Let $\Gamma$ be a group acting on a topological space $X$
and let $\Gamma_x$ denote the stabilizer in $\Gamma$ of $x \in X$. Let $\xi$ denote a family of 2-cocycles $\xi_x : \Gamma_x \times \Gamma_x \rightarrow \mathbb{C}^\times$.

Define $\tilde{X}_\xi = \{ (x, \rho) \mid x \in X, \rho \in \text{Irr} \mathbb{C}[\Gamma_x, \xi_x] \}$, where $\mathbb{C}[\Gamma_x, \xi_x]$ denotes the group algebra of $\Gamma_x$ twisted by $\xi_x$. Topologize $\tilde{X}_\xi$ by requiring that a subset of $\tilde{X}_\xi$ is open if and only if its projection to the first coordinate is open in $X$. Let $\{ \phi_{\gamma,x} \mid (\gamma, x) \in \Gamma \times X \}$, denote a family of algebra isomorphisms $\phi_{\gamma,x} : \mathbb{C}[\Gamma_x, \xi_x] \rightarrow \mathbb{C}[\Gamma_{\gamma x}, \xi_{\gamma x}]$ satisfying the conditions:

(a) if $\gamma x = x$, then $\phi_{\gamma,x}$ is conjugation by an element of $\mathbb{C}[\Gamma_x, \xi_x]^\times$.

(b) $\phi_{\gamma',x}\phi_{\gamma,x} = \phi_{\gamma',\gamma x}$ for all $\gamma', \gamma \in \Gamma$ and $x \in X$.

Define a $\Gamma$-action on $\tilde{X}_\xi$ by

$\gamma \cdot (x, \rho) = (\gamma x, \rho \circ \phi_{\gamma,x}^{-1})$.

Then the twisted extended quotient of $X$ by $\Gamma$ is defined to be:

$$(X/\Gamma)_\xi := \tilde{X}_\xi/\Gamma.$$ 

**Remark 9.1.** The twisted extended quotient depends on the choices of the algebra isomorphisms $\phi_{\gamma,x}$. If $\xi_x$ is trivial for all $x \in X$, then there is a canonical choice. Namely, $\phi_{\gamma,x}$ is conjugation by $\gamma$ (see [2 §2.1]).

**9.2. ABPS Conjecture.** Write $s = [L, \pi]_L$, $t = [L, \pi]_L$ and let $W^{s,t}$ denote the stabilizer in $W^s$ of a point $t$ in $\text{Irr}^t(L(F))$.

The ABPS conjecture [2 §2.3] asserts that there exists a family of 2-cocycles $\xi_t : W^{s,t} \times W^{s,t} \rightarrow \mathbb{C}^\times$, $t \in \text{Irr}^t(L(F))$

such that there is a natural bijection

$$\text{Irr}^s(G(F)) \leftrightarrow (\text{Irr}^t(L(F))/W^s)_\xi.$$ 

This bijection is expected to be compatible with the local Langlands correspondence (loc. cit.). If $G$ is quasi-split, the cocycles in the family $\xi$ are expected to be trivial.

**9.3. An isomorphism for extended quotients.** Let $s_0$ and $t_0$ be as in [7]. Let $t_\pi$ be as in Theorem [8,2] and $f$ as in [11 Theorem 6.1]. Then for $t \in \text{Irr}^t(L(F))$,

$$t_\pi|_{W^{s,t}} : W^{s,t} \rightarrow W^{s_0,f(t)}.$$ 

Each 2-cocycle $\xi_t$, $t \in \text{Irr}^t(L(F))$, therefore defines a 2-cocycle

$$\xi^0_t|_{f(t)} : W^{s_0,f(t)} \times W^{s_0,f(t)} \rightarrow \mathbb{C}^\times.$$ 

The following theorem is then immediate from Theorem [7,2] and [11 Theorem 6.1].

**Theorem 9.3.** Suppose Hypothesis [5,4]. Then there is an isomorphism

$$t_\pi : (\text{Irr}^t(L(F))/W^s)_\xi \rightarrow (\text{Irr}^{s_0}(L^0(F))/W^{s_0})_{s_0}$$

determined by the choice of $\pi \in \text{Irr}^t(L(F))$.

Let $(K, \rho)$ (resp. $(K^0, \rho_0)$) be the type constructed by Kim and Yu for $R^s(G(F))$ (resp. $R^{s_0}(G^0(F))$). We have equivalences of categories:

$$R^s(G(F)) \rightarrow \mathcal{H}(G, \rho^\vee)\text{-Mod}, \quad R^{s_0}(G^0(F)) \rightarrow \mathcal{H}(G^0, \rho_0^\vee)\text{-Mod}.$$
Hypothesis 9.4. \( \mathcal{H}(G, \rho^\vee) \) is Morita equivalent to \( \mathcal{H}(G^0, \rho_{0}^\vee) \).

Corollary 9.5. Under the Hypothesis, the ABPS conjecture holds for the Bernstein block \( \mathcal{R}^a(G(F)) \).

Proof. For depth-zero Bernstein blocks, the ABPS conjecture holds by [19] and Morris’s presentation of Hecke algebras of depth-zero blocks [12]. The result then follows from Theorem 9.3. □

Theorem 9.6. Assume Hypothesis [27]. Assume that the twisted Levi subgroups \( G^0 \subseteq \ldots \subseteq G^d = G \) obtained in the Howe factorization of \( (S, \theta) \) in \( G \) are \( F \)-Levi subgroups of \( G \). Then ABPS conjecture holds for the Bernstein block \( \mathcal{R}^a(G(F)) \). In particular, the ABPS Conjecture holds for principal series blocks for quasi-split groups.

Proof. The first claim follows from Theorem 8.4 and Corollary 9.5. The second claim follows from Corollary 8.5 and Corollary 9.5. □

Remark 9.7. When \( G \) is a classical group, Hypothesis 9.4 for certain Bernstein blocks can be observed from results of Kim [15]. More generally, for classical groups, one should be able to prove that Hypothesis 9.4 holds for all regular Bernstein blocks using the results of Heiermann [14].

10. Partial results toward further Hecke algebra isomorphisms

We present some partial results toward Conjecture [11]. Let the notation be as in §7 and §9.3. The equivalence

\[
\mathcal{R}^a(G(F)) \cong \mathcal{H}(G, \rho^\vee) - \text{Mod},
\]

induces an isomorphism of \( \mathbb{Z}_s \) with the center of \( \mathcal{H}(G, \rho^\vee) \). View \( \mathcal{H}(G, \rho^\vee) \) as a \( \mathbb{Z}_s \)-algebra. It is then finitely generated as a \( \mathbb{Z}_s \)-module. Similarly, \( \mathcal{H}(G^0, \rho_{0}^\vee) \) is a finitely generated module over its center \( \mathbb{Z}_{s0} \).

Write \( S = \mathbb{Z}^s \setminus 0 \) (resp. \( S_0 = \mathbb{Z}^{s0} \setminus 0 \)) and let \( k_s := S^{-1}\mathbb{Z}^s \) (resp. \( k_{s0} := S_0^{-1}\mathbb{Z}^{s0} \)) denote the field of fractions of \( \mathbb{Z}^s \) (resp. \( \mathbb{Z}^{s0} \)). Write

\[
\mathcal{H}(G, \rho^\vee) := S^{-1}\mathcal{H}(G, \rho^\vee).
\]

Similarly,

\[
\mathcal{H}(G^0, \rho_{0}^\vee) := S_0^{-1}\mathcal{H}(G^0, \rho_{0}^\vee).
\]

In [5, §4.3], Bushnell and Heimiart define the notions of generic and simply generic, which are generalizations of the generic representations in quasi-split groups.

Theorem 10.1. Assume Hypothesis [27]. Assume that \( \pi_{|_{L(F)}} \) and \( \pi_{0|_{L^0(F)}} \) are multiplicity free. Then there is a vector-space isomorphism

\[
\mathcal{H}(G, \rho^\vee) \cong \mathcal{H}(G^0, \rho_{0}^\vee).
\]

If \( s \) and \( s_0 \) are simply generic, then there is a \( \mathbb{C} \)-algebra isomorphism

\[
\mathcal{H}(G, \rho^\vee) \cong \mathcal{H}(G^0, \rho_{0}^\vee).
\]

Proof. By [15, Prop. 1.8.4.1], \( \mathcal{H}(G, \rho^\vee) \) (resp. \( \mathcal{H}(G^0, \rho_{0}^\vee) \)) is a free right-module over \( \mathbb{Z} \) (resp. \( \mathbb{Z}_{s0} \)) of rank \( |W| \) (resp. \( |W_{s0}| \)). By [11] Theorem 6.1, \( \mathbb{Z} \cong \mathbb{Z}_{s0} \). The first claim is then immediate.

By [5, Theorem 5.2],

\[
\mathcal{H}(G, \rho^\vee) \cong M_n(k_s),
\]
for some integer $n$. Similarly

$$H(G^0, \rho^0) \cong M_{n_0}(k_{s_0}),$$

for some integer $n_0$. Since $k_s \cong k_{s_0}$ by Theorem 7.3 to prove $H(G, \rho^\vee) \sim H(G^0, \rho^0)$, it suffices to prove

$$\dim_{k_s} H(G, \rho^\vee) = \dim_{k_{s_0}} H(G^0, \rho^0).$$

Now $S^{-1}Z \cong S_0^{-1}Z_0$ by Theorem 7.3 and [11, Theorem 6.1]. The claim then follows from [18, Prop. 1.8.4.1].

□

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