MATHEMATICAL AND NUMERICAL ANALYSIS OF TIME-DEPENDENT GINZBURG–LANDAU EQUATIONS IN NONCONVEX POLYGONS BASED ON HODGE DECOMPOSITION

BUYANG LI† AND ZHIMIN ZHANG‡

Abstract. We prove well-posedness of time-dependent Ginzburg–Landau system in a nonconvex polygonal domain, and decompose the solution as a regular part plus a singular part. We see that the magnetic potential is not in $H^1$ in general, and the finite element method (FEM) may give incorrect solutions. To remedy this situation, we reformulate the equations into an equivalent system of elliptic and parabolic equations based on the Hodge decomposition, which avoids direct calculation of the magnetic potential. The essential unknowns of the reformulated system admit $H^1$ solutions and can be solved correctly by the FEMs. We then propose a decoupled and linearized FEM to solve the reformulated equations and present error estimates based on proved regularity of the solution. Numerical examples are provided to support our theoretical analysis and show the efficiency of the method.

Key words. superconductivity, reentrant corner, singularity, well-posedness, finite element method, convergence, Hodge decomposition

AMS subject classifications. 35Q56, 35K61, 65M12, 65M60

1. Introduction. The Ginzburg–Landau theory, initially introduced by Ginzburg and Landau [16] and subsequently extended to the time-dependent case by Gor’kov and Eliashberg [18], are widely used to describe the phenomena of superconductivity in both low and high temperatures [11, 22]. In a two-dimensional domain $\Omega \subset \mathbb{R}^2$, the time-dependent Ginzburg–Landau model (TDGL) is governed by two equations (with the Lorentz gauge),

\[
\begin{align*}
\eta \frac{\partial \psi}{\partial t} + \left( \frac{i}{\kappa} \nabla + A \right)^2 \psi + (|\psi|^2 - 1) \psi - i \eta \kappa \psi \nabla \cdot A &= 0, \\
\frac{\partial A}{\partial t} + \nabla \times (\nabla \times A) - \nabla (\nabla \cdot A) + \text{Re} \left[ \psi^* \left( \frac{i}{\kappa} \nabla + A \right) \psi \right] &= \nabla \times f,
\end{align*}
\]

where $\eta$ and $\kappa$ are given positive constants, the order parameter $\psi$ is an unknown complex scalar function and $\psi^*$ denotes the complex conjugate of $\psi$, the real-vector valued function $A = (A_1, A_2)$ denotes the unknown magnetic potential, and the scalar function $f$ denotes the external magnetic field, and we have used the notations

\[
\nabla \times A = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}, \quad \nabla \cdot A = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2},
\]

\[
\nabla \times f = \left( \frac{\partial f}{\partial x_2}, -\frac{\partial f}{\partial x_1} \right), \quad \nabla \psi = \left( \frac{\partial \psi}{\partial x_1}, \frac{\partial \psi}{\partial x_2} \right).
\]

The natural boundary and initial conditions for this problem are

\[
\begin{align*}
\nabla \psi \cdot n &= 0, \quad A \cdot n = 0, \quad \nabla \times A &= f, \quad \text{on } \partial \Omega \times (0, T), \\
\psi(x, 0) &= \psi_0(x), \quad A(x, 0) = A_0(x), \quad \text{in } \Omega,
\end{align*}
\]

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†Department of Mathematics, Nanjing University, Nanjing, 210093, Jiangsu, P.R. China. (buyangli@nju.edu.cn)
‡Beijing Computational Science Research Center, Beijing, 100084, P.R. China.
‡Department of Mathematics, Wayne State University, Detroit, MI 48202, USA. (zzhang@math.wayne.edu)
where \( \mathbf{n} \) denotes the unit outward normal vector on the boundary \( \partial \Omega \).

The TDGL has been widely studied both theoretically and numerically. Existence and uniqueness of the solution for (1.1)-(1.2) in a smooth domain were proved by Chen et al. [8], where equivalence of (1.1)-(1.2) to the Ginzburg–Landau equations under the temporal gauge was proved. Various numerical methods for solving the TDGL were reviewed in [12,14]. In contrast with the many numerical approximation schemes, numerical analysis of the model seems very limited so far. Error analysis of a Galerkin finite element method (FEM) with an implicit backward Euler time-stepping scheme was presented in [7,13], where optimal-order convergence rate of the numerical solution was proved for sufficiently regular solution. A linearized Crank–Nicolson scheme was proposed in [24] for a regularized TDGL under the temporal gauge without error analysis. An alternating Crank–Nicolson scheme was proposed in [25] and error estimates were presented for a regularized TDGL under the grid-ratio restriction \( \tau = O(h^{11/12}) \), where \( \tau \) and \( h \) are the time-step size and spatial mesh size. Although convergence of the numerical solutions has been proved in [7,13,25] in smooth domains, these error estimates may not hold in a domain with corners, where the regularity of the solution may not satisfy the conditions required in the analysis. It has been reported in [15,24] that the numerical solution of the magnetic potential by the FEM often exhibits undesired singularities around a corner. To resolve this problem, a mixed FEM was proposed in [4] to approximate the triple \((\nabla \times \mathbf{A}, \nabla \cdot \mathbf{A}, \mathbf{A})\) in a finite element subspace of \(H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)\), which requires less regularity of \( \mathbf{A} \) intuitively, and error estimates of the finite element solution were presented under the assumption that \( \mathbf{A} \) is in \( H^1_n(\Omega) := \{ \mathbf{a} \in H^1(\Omega)^3 : \mathbf{a} \cdot \mathbf{n} = 0 \text{ on } \partial \Omega \} \). Recently, an optimal-order error estimate of the FEM with a linearized Crank–Nicolson scheme was presented in [15] without restriction on the grid ratio, but the analysis requires stronger regularity of the solution and the domain. On one hand, existing theoretical and numerical analysis of the model all require the magnetic potential to be in \( H^1_n(\Omega) \). In a domain with reentrant corners, however, the magnetic potential may not be in \( H^1_n(\Omega) \) and well-posedness of the TDGL remains open. On the other hand, numerical approximations of the TDGL in domains with reentrant corners are important for physicists to study the effects of surface defects in superconductivity [2,26], which are often accomplished by solving (1.1)-(1.2) directly with the finite element or finite difference methods, without being aware of the danger of these numerical methods.

In this paper, we study the TDGL in a nonconvex polygon, possibly with reentrant corners. We shall prove that the system (1.1)-(1.4) is well-posed, with \( \mathbf{A} \in L^\infty((0,T);H^s(\Omega)^2) \) for some \( s \in (0,1) \) which depends on the interior angles of the reentrant corners. As shown in the numerical examples, with such low-regularity, the FEM may give an incorrect solution for the magnetic potential \( \mathbf{A} \), which further pollutes the numerical solution of \( \psi \) due to the coupling of equations. We are interested in reformulating (1.1)-(1.4) into an equivalent form which can be solved correctly by the FEMs, as they are preferred when using software packages and when other equations are coupled with the Ginzburg–Landau equations. Our idea is to apply the Hodge decomposition \( \mathbf{A} = \nabla \times \mathbf{u} + \nabla \psi \), and consider the projection of (1.2) onto the divergence-free and curl-free subspaces, respectively. Then (1.1)-(1.4) is reformulated as

\[
\eta \frac{\partial \psi}{\partial t} + \left( \frac{i}{\kappa} \nabla + \mathbf{A} \right)^2 \psi + (|\psi|^2 - 1) \psi - i \eta \kappa \psi \nabla \cdot \mathbf{A} = 0, \tag{1.5}
\]

\[
\Delta p = -\nabla \times \left( \text{Re} \left[ \psi^* \left( \frac{i}{\kappa} \nabla + \mathbf{A} \right) \psi \right] \right) \tag{1.6}
\]

\[
\Delta q = \nabla \cdot \left( \text{Re} \left[ \psi^* \left( \frac{i}{\kappa} \nabla + \mathbf{A} \right) \psi \right] \right) \tag{1.7}
\]

\[
\frac{\partial u}{\partial t} - \Delta u = f - p, \tag{1.8}
\]
(1.9) \[ \frac{\partial v}{\partial t} - \Delta v = -g, \]

with the boundary and initial conditions

(1.10) \[ \nabla \psi \cdot n = 0, \quad p = 0, \quad \nabla q \cdot n = 0, \quad u = 0, \quad \nabla v \cdot n = 0, \quad \text{on } \partial \Omega \times (0, T], \]
(1.11) \[ \psi(x, 0) = \psi_0(x), \quad u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad \text{in } \Omega, \]

where \( \nabla \times p \) and \( \nabla q \) are just the divergence-free and curl-free parts of \( \text{Re}[\psi^* (\frac{i}{\kappa} \nabla + A) \psi] \), respectively, \( u_0 \) and \( v_0 \) are defined by

\[
\begin{cases}
-\Delta u_0 = \nabla \times A_0 & \text{in } \Omega, \\
u_0 = 0 & \text{on } \partial \Omega,
\end{cases}
\]

\[
\begin{cases}
\Delta v_0 = \nabla \cdot A_0 & \text{in } \Omega, \\
\partial_n v_0 = 0 & \text{on } \partial \Omega,
\end{cases}
\]

with \( \int_\Omega v_0(x) \, dx = 0 \). We shall prove that the solution of the projected TDGL \((1.5)-(1.11)\) coincides with the solution of \((1.1)-(1.4)\). Then we propose a decoupled and linearized FEM to solve \((1.5)-(1.11)\), and establish error estimates based on proved regularity of the solution. Our main results are presented in Section 2, and we prove these results in Section 3-5. In Section 6, we present numerical examples to support our theoretical analysis. Due to limitations on pages, derivations of the system \((1.5)-(1.11)\) are presented in a separate paper \[21\], where the efficiency of the method is shown via numerical simulations in comparison with the traditional approaches of solving the TDGL directly under the temporal gauge and the Lorentz gauge.

2. Main results. For any nonnegative integer \( k \), we let \( W^{k,p}(\Omega) \), \( \text{and } W^{k,p}(\Omega) \) denote the conventional Sobolev spaces of real-valued and complex-valued functions defined in \( \Omega \), respectively, with \( H^k(\Omega) = W^{k,2}(\Omega) \), \( \mathcal{H}^k(\Omega) = W^{k,2}(\Omega) \), \( L^2(\Omega) = H^0(\Omega) \) and \( L^2(\Omega) = \mathcal{H}^0(\Omega) \); see \[1\]. For a positive real number \( s_0 = k + s \), with \( s \in (0, 1) \), we define \( H^{s_0}(\Omega) = (H^k(\Omega), H^{k+1}(\Omega))[s] \) via the complex interpolation; see \[3\]. We denote \( H^{s_0} = H^{s_0}(\Omega) \), \( \mathcal{H}^{s_0} = \mathcal{H}^{s_0}(\Omega) \), \( L^p = L^p(\Omega) \), \( L^p = L^p(\Omega) \), and let \( H^1 \) denote the subspace of \( H^1 \) consisting of functions whose traces are zero on \( \partial \Omega \). For any two functions \( f, g \in L^2 \) we define

\[
(f, g) = \int_\Omega f(x) g(x)^* \, dx,
\]

where \( g(x)^* \) denotes the complex conjugate of \( g(x) \), and define

\[
L^p = L^p \times L^p, \quad H^s = H^s \times H^s, \quad H^s_0(\Omega) := \{ a \in H^1 \times H^1 : a \cdot n = 0 \text{ on } \partial \Omega \},
\]

\[
\mathcal{H}_n(\text{curl, div}) = \{ a \in L^2 : \nabla \times a \in L^2, \quad \nabla \cdot a \in L^2 \text{ and } a \cdot n = 0 \text{ on } \partial \Omega \},
\]

\[
H(\text{curl}) = \{ g \in L^2 : \nabla \times g \in L^2 \}.
\]

**Definition 2.1. (Weak solutions of \((1.1)-(1.4)\)).** Let \( \omega \) denote the maximal interior angle of the nonconvex polygon \( \Omega \). The pair \((\psi, A)\) is called a weak solution of \((1.1)-(1.4)\) if

\[
\psi \in C([0, T]; L^2) \cap L^\infty((0, T); \mathcal{H}^1) \cap L^2((0, T); H^1),
\]

\[
\partial_t \psi, \Delta \psi \in L^2((0, T); L^2), \quad \| \psi \| \leq 1 \quad \text{a.e. in } \Omega \times (0, T),
\]

\[
A \in C([0, T]; L^2) \cap L^\infty((0, T); \mathcal{H}_n(\text{curl, div})),
\]

\[
\partial_t A \in L^2((0, T); L^2), \quad \nabla \times A, \nabla \cdot A \in L^2((0, T); H^1),
\]

for any \( s \in (1/2, \pi/\omega) \), with \( \psi(\cdot, 0) = \psi_0, \quad A(\cdot, 0) = A_0 \), and the variational equations

\[
\int_0^T \left[ \left( \frac{\partial \psi}{\partial t}, \varphi \right) + \left( \left( \frac{i}{\kappa} \nabla + A \right) \psi, \left( \frac{i}{\kappa} \nabla + A \right) \varphi \right) \right] \, dt
\]
\begin{align}
(2.1) \quad & + \int_0^T \left[ (|\psi|^2 - 1)\psi - i\eta\kappa\psi\nabla \cdot A, \phi \right] dt = 0, \\
& \int_0^T \left[ \left( \frac{\partial A}{\partial t}, a \right) + (\nabla \times A, \nabla \times a) + (\nabla \cdot A, \nabla \cdot a) \right] dt \\
& = \int_0^T \left[ (f, \nabla \times a) - \left( \text{Re} \left[ \psi^* \left( \frac{i}{\kappa} \nabla + A \right) \psi \right], a \right) \right] dt,
\end{align}

hold for all \( \varphi \in L^2((0,T); \mathcal{H}^1) \) and \( a \in L^2((0,T); \mathcal{H}_n(\text{curl}, \text{div})) \).

**Definition 2.2. (Weak solutions of (1.5)-(1.11))** Let \( \omega \) denote the maximal interior angle of the nonconvex polygon \( \Omega \). The quintuple \((\psi, p, q, u, v)\) is called a weak solution of (1.5)-(1.11) if

\[
\psi \in C([0,T]; \mathcal{L}^2) \cap L^\infty((0,T); \mathcal{H}^1) \cap L^2((0,T); \mathcal{H}^{1+}),
\]

\[
\partial_t \psi, \Delta \psi \in L^2((0,T); \mathcal{L}^2), \quad |\psi| \leq 1 \quad \text{a.e. in} \quad \Omega \times (0,T),
\]

\[
p \in L^\infty((0,T); \mathcal{H}^1), \quad q \in L^\infty((0,T); \mathcal{H}^1), \quad u \in C([0,T]; \mathcal{H}^1), \quad v \in C([0,T]; \mathcal{H}^1),
\]

\[
\partial_t u, \partial_t v, \Delta u, \Delta v \in L^\infty((0,T); \mathcal{L}^2) \cap L^2((0,T); \mathcal{H}^1),
\]

for any \( s \in (1/2, \pi/\omega) \), with \( \psi(\cdot,0) = \psi_0, u(\cdot,0) = u_0, v(\cdot,0) = v_0 \), and the variational equations

\[
\int_0^T \left[ \left( \eta \frac{\partial \psi}{\partial t}, \varphi \right) + \left( \left( \frac{i}{\kappa} \nabla + A \right) \psi, \left( \frac{i}{\kappa} \nabla + A \right) \varphi \right) \right] dt = 0,
\]

(2.3)

\[
\int_0^T (\nabla p, \nabla \xi) dt = \int_0^T \left( \text{Re} \left[ \psi^* \left( \frac{i}{\kappa} \nabla + A \right) \psi \right], \nabla \times \xi \right) dt
\]

(2.4)

\[
\int_0^T (\nabla q, \nabla \zeta) dt = \int_0^T \left( \text{Re} \left[ \psi^* \left( \frac{i}{\kappa} \nabla + A \right) \psi \right], \nabla \zeta \right) dt
\]

(2.5)

\[
\int_0^T \left[ \left( \frac{\partial u}{\partial t}, \theta \right) + (\nabla u, \nabla \theta) \right] dt = \int_0^T (f - p, \theta) dt,
\]

(2.6)

\[
\int_0^T \left[ \left( \frac{\partial v}{\partial t}, \vartheta \right) + (\nabla v, \nabla \vartheta) \right] dt = -\int_0^T (q, \vartheta) dt,
\]

(2.7)

hold for all \( \varphi \in L^2((0,T); \mathcal{H}^1) \), \( \xi, \theta \in L^2((0,T); \mathcal{H}^1) \) and \( \zeta, \vartheta \in L^2((0,T); \mathcal{H}^1) \).

Our first result is the well-posedness and equivalence of the systems (1.1)-(1.4) and (1.5)-(1.11), which are presented in the following theorem.

**Theorem 2.1. (Well-posedness and equivalence of the two systems)** If \( f \in L^\infty((0,T); \mathcal{L}^2) \cap L^2((0,T); H(\text{curl})) \), \( \psi_0 \in \mathcal{H}^1 \), \( A_0 \in \mathcal{H}_n(\text{curl}, \text{div}) \) and \( |\psi_0| \leq 1 \quad \text{a.e. in} \quad \Omega \), then the system (1.1)-(1.4) admits a unique weak solution in the sense of Definition 2.1 and the system (1.5)-(1.11) admits a unique solution which coincides with the solution of (1.1)-(1.4).

Moreover, if we let \( x_j, j = 1, \cdots, m \), be the reentrant corners of the domain \( \Omega \), then the solution has the decomposition

\[
\psi(x,t) = \Psi(x,t) + \sum_{j=1}^m \alpha_j(t) \Phi(|x - x_j|)|x - x_j|^{\pi/\omega} \cos(\pi \Theta_j(x)/\omega),
\]

\[
A = \nabla \times u + \nabla v
\]
with
\[
\begin{align*}
    u(x, t) &= \bar{u}(x, t) + \sum_{j=1}^{m} \beta_j(t) \Phi(|x-x_j|)|x-x_j|^\pi/\omega_j \sin(\pi \Theta_j(x)/\omega_j), \\
    v(x, t) &= \bar{v}(x, t) + \sum_{j=1}^{m} \gamma_j(t) \Phi(|x-x_j|)|x-x_j|^\pi/\omega_j \cos(\pi \Theta_j(x)/\omega_j),
\end{align*}
\]
where $\Phi \in L^2((0, T); H^2)$, $\bar{u}, \bar{v} \in L^\infty((0, T); H^2)$, $\Phi(x)$ is a smooth cut-off function which equals 1 in a neighborhood of 0, $\Theta_j(x)$ is the angle shown in Figure 1, and $\alpha_j, \beta_j, \gamma_j \in L^2(0, T)$.

Further regularity of the solution is presented below, which is needed in the analysis of the convergence of the numerical solution.

**Theorem 2.2. (Further regularity)**

If $f \in C([0, T]; H(\text{curl}))$, $\nabla \times f \in L^2((0, T); H(\text{curl}))$, $\partial_t f \in L^2((0, T); L^2)$, $\psi_0 \in H^1$, $\Delta \psi_0 \in L^2$, $A_0 \in H_n(\text{curl, div})$, $\nabla \cdot A_0, \nabla \times A_0 \in H^1$, $|\psi_0| \leq 1$ a.e. in $\Omega$, and the compatibility conditions
\[
\partial_n \psi_0 = 0 \text{ and } \nabla \times A_0 = f(\cdot, 0) \text{ on } \partial \Omega
\]
are satisfied, then the solution of (1.5)-(1.11) possesses the regularity
\[
\psi \in C([0, T]; H^{1+s}), \quad \partial_t \psi \in L^2((0, T); H^{1+s}), \quad \partial_{tt} \psi \in L^2((0, T); L^2),
\]
\[
p, q \in L^\infty((0, T); H^1), \quad u, v \in C([0, T]; H^{1+s}),
\]
\[
\partial_t u, \partial_t v \in L^2((0, T); H^{1+s}), \quad \partial_{tt} u, \partial_{tt} v \in L^2((0, T); L^2)
\]
for any $s \in (1/2, \pi/\omega)$.

To solve the reformulated system (1.5)-(1.11), we propose a decoupled and linearized Galerkin FEM. For this purpose, we let $\pi_h$ be a quasi-uniform triangulation of the domain $\Omega$ and denote the mesh size by $h$. Let $V_h^1$ denote the space of complex-valued $C^0$ piecewise linear functions subject to the triangulation, let $V_h^0$ denote the space of real-valued $C^0$ piecewise linear functions, and set $\tilde{V}_h^1 = \{ \varphi \in V_h^1 : \varphi = 0 \text{ on } \partial \Omega \}$.

Clearly, $V_h^1, \tilde{V}_h^1$ and $\tilde{V}_h^0$ are finite dimensional subspaces of $H^1$, $H^1$ and $H^1$, respectively. Let $I_h$ be the commonly used Lagrange interpolation operator onto the finite element spaces. For any positive integer $N$, we let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of the time interval $[0, T]$ and set $\tau = T/N$. For any sequence of functions $\varphi^n$, we define $D_\tau \varphi^{n+1} := (\varphi^{n+1} - \varphi^n)/\tau$, and we define a cut-off function $\chi : \mathbb{C} \to \mathbb{C}$ by
\[
\chi(z) = z/\max(|z|, 1), \quad \forall z \in \mathbb{C},
\]
which is Lipschitz continuous and satisfies that $|\chi(z)| \leq 1$, $\forall z \in \mathbb{C}$.

We look for $\psi_h^{n+1} \in V_h^1$, $p_h^{n+1}, q_h^{n+1} \in \tilde{V}_h$ and $v_h^{n+1}, u_h^{n+1} \in V_h^1$ satisfying the equations

\[
(D_\tau \psi_h^{n+1}, \varphi) + ((ik^{-1} \nabla + A_h^\tau) \psi_h^{n+1}, (ik^{-1} \nabla + A_h^\tau) \varphi)
\]
\[+(\psi_h^{n+1}, \varphi) + (i\eta \nabla ((\psi_h^{n+1})^*) \varphi) = 0,
\]

\[
(\nabla p_h^{n+1}, \nabla \xi) = (\text{Re}[\chi(\psi_h^*) (ik^{-1} \nabla \psi_h^{n+1} + A_h^\tau \psi_h^{n+1})], \nabla \times \xi)
\]

\[\text{Re}[\chi(\psi_h^*) (ik^{-1} \nabla \psi_h^{n+1} + A_h^\tau \psi_h^{n+1})] = 0,
\]

\[\tau < \eta/4
\]

\[\forall \varphi \in V_h^1, \xi, \theta \in \tilde{V}_h, \psi_h \in V_h^1, \text{with } A_h^\tau = \nabla \times u_h^n + \nabla v_h^n, \text{where } u_h^n \in \tilde{V}_h \text{ and } v_h^n \in V_h^1
\]

are solved from

\[\nabla u_h^n, \nabla \xi) = (A, \nabla \times \xi), \quad \forall \xi \in \tilde{V}_h,
\]

\[\nabla v_h^n, \nabla \zeta) = (A, \nabla \cdot \zeta), \quad \forall \zeta \in V_h^1,
\]

and $\psi_h^n$ is the Lagrange interpolation of $\psi^0$.

For the proposed scheme, we have the following theorem concerning the convergence of the numerical solution.

**Theorem 2.3.** (Convergence of the finite element solution)

The finite element system (2.8)-(2.12) admits a unique solution $(\psi_h^n, p_h^n, q_h^n, u_h^n, v_h^n)$ when $\tau < \eta/4$ and, under the assumptions of Theorem 2.2,

\[
\max_{1 \leq n \leq N} \left(\|u^n - u_h^n\|_{H^1} + \|v^n - v_h^n\|_{H^1} + \|A^n - A_h^n\|_{L^2} + \|\psi^n - \psi_h^n\|_{L^2} \right) \leq C(\tau + h^s),
\]

where $C$ is a positive constant independent of $\tau$ and $h$.

In the rest part of this paper, we prove Theorem 2.1, 2.3. To simplify the notations, we denote by $C$ a generic positive constant which may be different at each occurrence but is independent of $n$, $\tau$ and $h$.

3. **Proof of Theorem 2.1** In this section, we prove well-posedness of the Ginzburg–Landau equations in a nonconvex polygon and equivalence of the two formulations 1.11 and 1.12. Compared with smooth domains, in a nonconvex polygon, the space $H_0^1(\Omega)$ is not equivalent to $H_0^1(\Omega)$ and is not embedded into $L^p$ for large $p$. Convergence of the nonlinear terms of the approximating solutions needs to be proved based on the weaker embedding $H_0^1(\text{curl, div}) \hookrightarrow\hookrightarrow L^4$ in the compactness argument, and uniqueness of solution needs to be proved based on weaker regularity of the solution.

3.1. **Preliminaries.** Firstly, we cite a lemma concerning the regularity of Poisson’s equations in a nonconvex polygon 10 17.

**Lemma 3.1.** The solution of the Poisson equations

\[
\begin{cases}
\Delta w = g \quad \text{in } \Omega, \\
w = 0 \quad \text{on } \partial \Omega,
\end{cases}
\]

satisfies that (the Neumann problem requires $\int_\Omega g(x) \, dx = \int_\Omega w(x) \, dx = 0$)

\[
\|w\|_{W^{1,p_s}} + \|w\|_{H^{1+s}} \leq C_s \|g\|_{L^2}, \quad \forall \ s \in (1/2, \pi/\omega),
\]

where $p_s = 2/(1-s) > 4$ when $s \in (1/2, \pi/\omega)$. 


where \( \psi \) to control the order parameter pointwisely. We first need the following lemma.

**Lemma 3.2.** \( H_0(\text{curl}, \text{div}) \hookrightarrow H^s \hookrightarrow L^{p_s} \) for any \( s \in (1/2, \pi/\omega) \).

**Proof.** From \( [6] \) we know that \( \mathbf{A} \) has the decomposition \( \mathbf{A} = \nabla \times \mathbf{u} + \nabla v \), where \( \mathbf{u} \) and \( v \) are the solutions of

\[
\begin{align*}
-\Delta u &= \nabla \times \mathbf{A} & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

and

\[
\begin{align*}
\Delta v &= \nabla \cdot \mathbf{A} & \text{in } \Omega, \\
\partial_v v &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

respectively, with \( \int_{\Omega} v(x) \, dx = 0 \). For the two Poisson’s equations, Lemma 3.1 implies that

\[
\|u\|_{H^{1+s}} + \|v\|_{H^{1+s}} \leq C_s(\|\nabla \times \mathbf{A}\|_{L^2} + \|\nabla \cdot \mathbf{A}\|_{L^2}), \quad \forall s \in (1/2, \pi/\omega).
\]

Thirdly, we introduce a lemma concerning the embedding of discrete Sobolev spaces.

**Lemma 3.3.** Let \( \theta_h \in \hat{V}_h^1 \), \( \varphi_h \in V_h^1 \) with \( \int_{\Omega} \varphi_h(x) \, dx = 0 \). If we define \( \Delta_h \vartheta_h \in \hat{V}_h^1 \) and \( \Delta_h \vartheta_h \in V_h^1 \) by

\[
\begin{align*}
(\Delta_h \vartheta_h, \varphi) &= - (\nabla \theta_h, \nabla \varphi), & \forall \varphi & \in \hat{V}_h^1, \\
(\Delta_h \vartheta_h, \varphi) &= - (\nabla \theta_h, \nabla \varphi), & \forall \varphi & \in V_h^1,
\end{align*}
\]

then

\[
\|\nabla \vartheta_h\|_{L^2} \leq C \|\Delta_h \vartheta_h\|_{L^2}, \quad \text{and} \quad \|\nabla \vartheta_h\|_{L^2} \leq C \|\Delta_h \vartheta_h\|_{L^2}.
\]

**Proof.** Let \( \vartheta \) be the solution of the Poisson’s equation

\[
\Delta \vartheta = \Delta_h \vartheta_h
\]

with the Dirichlet boundary condition \( \vartheta = 0 \) on \( \partial \Omega \). Then \( (\nabla (\vartheta - \vartheta_h), \nabla \xi_h) = 0 \) for any \( \xi_h \in \hat{V}_h^1 \), which implies that, via the standard \( H^1 \)-norm error estimate and Lemma 3.1,

\[
\|\nabla (\vartheta - \vartheta_h)\|_{L^2} \leq C \|\vartheta - \vartheta_h\|_{H^{1+s}} \leq C \|\Delta_h \vartheta_h\|_{L^2+s}.
\]

Since \( s > 1/2 \), by applying the inverse inequality we obtain that

\[
\|\nabla (I_h \vartheta - \vartheta_h)\|_{L^2} \leq C h^{-1/2} \|\nabla (I_h \vartheta - \vartheta_h)\|_{L^2} \leq C \|\Delta_h \vartheta_h\|_{L^2} + \|\Delta_h \vartheta_h\|_{L^2}.
\]

Thus \( \|\nabla \vartheta_h\|_{L^2} \leq \|\nabla (I_h \vartheta - \vartheta_h)\|_{L^2} + \|\Delta_h \vartheta_h\|_{L^2} \leq C \|\Delta_h \vartheta_h\|_{L^2} \). The proof for \( \vartheta_h \) is similar.

### 3.2. Existence of weak solutions for (1.1)-(1.4)

In this subsection, we prove existence of weak solutions for the system (1.1)-(1.4) by constructing approximating solutions in finite dimensional spaces and then applying a compactness argument. Firstly, we need the following lemma to control the order parameter pointwisely.

**Lemma 3.4.** For any given \( \mathbf{A} \in L^\infty((0,T); H_0(\text{curl}, \text{div})) \), the equation (1.1) has at most one weak solution \( \psi \in L^\infty((0,T); H^1) \cap H^1((0,T); L^2) \) in the sense of (1.1). If the solution exists then it satisfies that \( |\psi| \leq 1 \) a.e. in \( \Omega \times (0,T) \).

**Proof.** From Lemma 3.2 we see that \( H_0(\text{curl}, \text{div}) \hookrightarrow H^s \hookrightarrow L^4 \) and so \( \mathbf{A} \in L^\infty((0,T); L^4) \). Uniqueness of the solution can be proved easily based on the regularity assumption of \( \psi \). To prove \( |\psi| \leq 1 \) a.e. in \( \Omega \times (0,T) \), we integrate (1.1) against \( \psi^2 |(\psi^2 - 1)_+| \) and consider the real part, where \( (|\psi|^2 - 1)_+ \) denotes the positive part of \( |\psi|^2 - 1 \). For any \( t' \in (0,T) \) we have

\[
\int_\Omega \left[ \frac{2}{4} (\psi(x,t')^2 - 1)^2_+ \right] \, dx + \int_0^{t'} \int_\Omega |\psi)|^2 |(\psi^2 - 1)_+| \, dx \, dt
\]
which implies that \( \int_{\Omega} \langle |\psi(x,t')|^2 \rangle^2 \, dx = 0 \). Thus \( |\psi| \leq 1 \) a.e. in \( \Omega \times (0,T) \). \( \square \)

Secondly, we construct approximating solutions in finite dimensional spaces. For this purpose, we let \( \phi_1, \phi_2, \cdots \) be the eigenfunctions of the Neumann Laplacian, which form a basis of \( \mathcal{H}^1 \). Let \( M : H_n(\text{curl}, \text{div}) \to (H_n(\text{curl}, \text{div}))^\prime \) be defined by

\[
(Mu, v) = (\nabla \times u, \nabla \times v) + (\nabla \cdot u, \nabla \cdot v), \quad \text{for } u, v \in H_n(\text{curl}, \text{div}).
\]

Since the bilinear form on the right-hand side is coercive on the space \( H_n(\text{curl}, \text{div}) \), which is compactly embedded into \( L^2 \), the spectrum of \( M \) consists of a sequence of eigenvalues which tend to infinity, and the corresponding eigenvectors \( a_1, a_2, a_3, \cdots \) form a basis of \( H_n(\text{curl}, \text{div}) \) [9, 23].

We define \( \mathcal{V}_N = \text{span}\{\phi_1, \phi_2, \cdots, \phi_N\} \) and \( X_N = \text{span}\{a_1, a_2, \cdots, a_N\} \), which are finite dimensional subspaces of \( \mathcal{H}^1 \) and \( H_n(\text{curl}, \text{div}) \), respectively, and we look for \( \Psi_N(t) \in \mathcal{V}_N, \Lambda_N(t) \in X_N \) such that

\[
\begin{align*}
&\left( \eta \frac{\partial \Psi_N}{\partial t}, \varphi \right) + \left( \left( \frac{i}{\kappa} \nabla + \Lambda_N \right) \Psi_N, \left( \frac{i}{\kappa} \nabla + \Lambda_N \right) \varphi \right) \\
&\quad + \left( (|\Psi_N|^2 - 1)\Psi_N - i\eta \kappa \chi(\Psi_N) \nabla \cdot \Lambda_N, \varphi \right) = 0, \\
&\left( \frac{\partial \Lambda_N}{\partial t}, a \right) + (\nabla \times \Lambda_N, \nabla \times a) + (\nabla \cdot \Lambda_N, \nabla \cdot a) \\
&\quad + \left( \text{Re} \left[ \Psi_N \left( \frac{i}{\kappa} \nabla + \Lambda_N \right) \Psi_N \right], a \right) = (f, \nabla \times a),
\end{align*}
\]

for any \( \varphi \in \mathcal{V}_N \) and \( a \in X_N \) at any \( t \in (0,T) \), with the initial conditions \( \Psi(0) = \Pi_N \psi_0 \) and \( \Lambda(0) = \Pi_N A_0 \), where \( \Pi_N \) and \( \Pi_N \) are the projections of \( \mathcal{H}^1 \) and \( H_n(\text{curl}, \text{div}) \) onto the subspaces \( \mathcal{V}_N \) and \( X_N \), respectively.

Existence and uniqueness of solutions for the ODE problem [3.1]-[3.2] are obvious. To present estimates of the semi-discrete solution \( (\Psi_N, \Lambda_N) \), we substitute \( \varphi = \partial_t \Psi \) and \( a = \partial_t \Lambda \) into the equations, and sum up the two results. Then we obtain that

\[
\begin{align*}
\frac{d}{dt} \int_\Omega &\left( \frac{1}{2} \left( \left| \frac{i}{\kappa} \nabla \Psi_N + \Lambda_N \Psi_N \right|^2 \right) + \frac{1}{2} (|\Psi_N|^2 - 1)^2 + |\nabla \times \Lambda_N - f|^2 + |\nabla \cdot \Lambda_N|^2 \right) \, dx \\
&+ \int_\Omega \left( \left| \frac{\partial \Lambda_N}{\partial t} \right|^2 + \eta \left| \frac{\partial \Psi_N}{\partial t} \right|^2 \right) \, dx
\end{align*}
\]
which further imply that
\[ \partial_t \Psi_N \rightarrow \psi \quad \text{weakly in} \quad L^\infty((0,T);H^1), \]
\[ \Psi_N \rightarrow \psi \quad \text{weakly in} \quad L^p((0,T);H^1) \quad \text{for any} \quad 1 < p < \infty, \]
\[ \partial_t \Psi_N \rightarrow \partial_t \psi \quad \text{weakly in} \quad L^2((0,T);L^2), \]
\[ \Lambda_N \rightarrow \Lambda \quad \text{weakly* in} \quad L^\infty((0,T);H_n(\text{curl, div})), \]
\[ \Lambda_N \rightarrow \Lambda \quad \text{weakly in} \quad L^p((0,T);H_n(\text{curl, div})) \quad \text{for any} \quad 1 < p < \infty, \]
\[ \partial_t \Lambda_N \rightarrow \partial_t \Lambda \quad \text{weakly in} \quad L^2((0,T);L^2), \]
\[ \Lambda_N \rightarrow \Lambda \quad \text{strongly in} \quad L^\infty((0,T);L^{4+\varepsilon}) \quad \text{for any} \quad 1 < p < \infty, \]
which further imply that
\[ \Psi_N \Lambda_N \rightarrow \psi \Lambda \quad \text{strongly in} \quad L^2((0,T);L^2 \times L^2), \]
\[ \nabla \Psi_N \Lambda_N \rightarrow \nabla \psi \Lambda \quad \text{weakly in} \quad L^2((0,T);L^{4/3}), \]
\[ \Psi_N|A|^2 \rightarrow \psi|\Lambda|^2 \quad \text{strongly in} \quad L^2((0,T);L^{4/3}), \]
\[ \left( \frac{i}{\kappa} \nabla + \Lambda_N \right) \Psi_N \rightarrow \left( \frac{i}{\kappa} \nabla + \Lambda \right) \psi \quad \text{weakly in} \quad L^2((0,T);L^2), \]
\[ \Psi_N^* \left( \frac{i}{\kappa} \nabla + \Lambda_N \right) \Psi_N \rightarrow \psi^* \left( \frac{i}{\kappa} \nabla + \Lambda \right) \psi \quad \text{weakly in} \quad L^2((0,T);L^{4/3} \times L^{4/3}), \]
\[ \left( \frac{i}{\kappa} \nabla + \Lambda_N \right) \Psi_N \Lambda_N \rightarrow \left( \frac{i}{\kappa} \nabla + \Lambda \right) \psi \Lambda \quad \text{weakly in} \quad L^2((0,T);L^{4/3}). \]

By applying Gronwall’s inequality, we obtain that
\[ \| \Psi_N \|_{L^\infty((0,T);H^1)} + \| \partial_t \Psi_N \|_{L^2((0,T);L^2)} + \| \Lambda_N \|_{L^\infty((0,T);H_n(\text{curl, div}))} + \| \partial_t \Lambda_N \|_{L^2((0,T);L^2)} \leq C, \]
where the constant \( C \) does not depend on \( N \).

Thirdly, since \( H^1 \leftrightarrow L^p \) for any \( 1 < p < \infty \) and \( H_n(\text{curl, div}) \leftrightarrow L^{4+\varepsilon} \) for some \( \varepsilon > 0 \), by the Aubin–Lions compactness argument [20], there exist
\[ (3.3) \quad \psi \in L^\infty((0,T);H^1) \cap H^1((0,T);L^2), \]
\[ (3.4) \quad \Lambda \in L^\infty((0,T);H_n(\text{curl, div})) \cap H^1((0,T);L^2), \]
and a subsequence of \( (\Psi_N, \Lambda_N)_{N=1}^\infty \), denoted by \( (\Psi_{N_m}, \Lambda_{N_m})_{m=1}^\infty \), such that

For any given \( \varphi \in L^2((0,T);V_N) \) and \( a \in L^2((0,T);X_N) \), integrating \([4.1] - [4.2]\) with respect to time and letting \( N = N_m \rightarrow \infty \), we derive \([2.1] - [2.2]\). In other words, \( \psi \in L^\infty((0,T);H^1) \cap H^1((0,T);L^2) \) is a weak solution of \([1.1]\) in the sense of \([2.1]\), and \( \Lambda \in L^\infty((0,T);H_n(\text{curl, div})) \cap H^1((0,T);L^2) \) is a weak solution of \([1.2]\) in the sense of \([2.2]\).

The conditions of Lemma [3.3] are satisfied, which implies that \( |\psi| \leq 1 \) a.e. in \( \Omega \times (0,T) \).

Finally, we prove the additional regularity of the solution specified in Definition [2.1]. From Lemma [3.2] we see that
\[ \Lambda \in L^\infty((0,T);H_n(\text{curl, div})) \rightarrow L^\infty((0,T);H^4) \rightarrow L^\infty((0,T);L^4) \]
for any $s \in (1/2, \pi/\omega)$. From (1.1) we see that
\[
\frac{1}{\kappa^2} \Delta \psi = \eta \partial_t \psi + \frac{i}{\kappa} \nabla \cdot (\mathbf{A} \psi) + \frac{i}{\kappa} \mathbf{A} \cdot \nabla \psi + |\mathbf{A}|^2 \psi + (|\psi|^2 - 1) \psi - i \eta \kappa \psi \nabla \cdot \mathbf{A} \quad \text{a.e. in } \Omega,
\]
which imply that
\[
\|\Delta \psi\|_{L^2} \leq C \|\partial_t \psi\|_{L^2} + C \| \nabla \cdot \mathbf{A} \|_{L^2} \|\psi\|_{L^\infty} + C \|\mathbf{A}\|_{L^4} \|\nabla \psi\|_{L^4} + C \|\mathbf{A}\|_{L^2}^2 + C \|(|\psi|^2 - 1) \psi\|_{L^2}
\]
\[
\leq C + C \|\partial_t \psi\|_{L^2} + C \| \nabla \psi\|_{L^4}
\]
\[
\leq C + C \|\partial_t \psi\|_{L^2} + C \| \nabla \psi\|_{L^4}^{1-(4/p_s)/(2-4/p_s)} \| \nabla \psi\|_{L^{p_s}}^{1/(2-4/p_s)}
\]
\[
\leq C + C \|\partial_t \psi\|_{L^2} + C \| \nabla \psi\|_{L^4}^{1/(2-4/p_s)}
\]
where we have used (5.3) and Lemma 3.1 in the last inequality. Since $1/(2-4/p_s) < 1$, the last inequality implies $\|\Delta \psi\|_{L^2} \leq C + C \|\partial_t \psi\|_{L^2}$, and so
\[
\|\Delta \psi\|_{L^2((0,T);L^2)} \leq C + C \|\partial_t \psi\|_{L^2((0,T);L^2)} \leq C,
\]
which further implies $\psi \in L^2((0,T);H^{1+})$ by Lemma 3.1. From (1.2) we see that
\[
\|\nabla \times (\nabla \times \mathbf{A}) - \nabla (\nabla \cdot \mathbf{A})\|_{L^2((0,T);L^2)}
\]
\[
\leq C \|\partial_t \mathbf{A}\|_{L^2((0,T);L^2)} + C \|\psi^4 \left(i \kappa^{-1} \nabla \psi + \mathbf{A} \psi\right)\|_{L^2((0,T);L^2)} + C \|\nabla \times \mathbf{f}\|_{L^2((0,T);L^2)}
\]
\[
\leq C \|\partial_t \mathbf{A}\|_{L^2((0,T);L^2)} + C \|\nabla \psi\|_{L^2(\Omega)} + C \|\mathbf{A}\|_{L^2((0,T);L^2)} + C \|\nabla \times \mathbf{f}\|_{L^2((0,T);L^2)}
\]
\[
\leq C.
\]
Note that $w = \nabla \times \mathbf{A} - \mathbf{f}$ satisfies the equation
\[
- \Delta w = \nabla \times (\nabla \times w) = \nabla \times \mathbf{f},
\]
with $w = 0$ on $\partial \Omega$ and $\mathbf{f} = \nabla \times (\nabla \times \mathbf{A}) - \nabla (\nabla \cdot \mathbf{A}) - \nabla \times \mathbf{f} \in L^2((0,T);L^2)$. The energy estimate of $w$ gives
\[
\|w\|_{L^2((0,T);H^1)} \leq C \|\mathbf{f}\|_{L^2((0,T);L^2)} \leq C.
\]
Thus $\nabla (\nabla \cdot \mathbf{A}) = \nabla \times \mathbf{w} - \mathbf{f} \in L^2((0,T);L^2)$, which indicates that $\nabla \cdot \mathbf{A} \in L^2((0,T);H^1)$.

Existence of a weak solution of (1.1)-(1.4) in the sense of Definition 2.1 has been proved.

### 3.3. Uniqueness of the weak solution.

Suppose that there are two solutions $(\psi, \mathbf{A})$ and $(\Psi, \mathbf{A})$ for the system (1.1)-(1.4) in the sense of Definition 2.1. Let $e = \psi - \Psi$ and $\mathbf{E} = \mathbf{A} - \mathbf{A}$. Then we have
\[
\int_0^T \left[ (\eta \partial_t e, \varphi) + \frac{1}{\kappa^2} (\nabla e, \nabla \varphi) + (|\mathbf{A}|^2 e, \varphi) \right] dt
\]
\[
= \int_0^T \left[ - \frac{i}{\kappa} (\mathbf{A} \cdot \nabla e, \varphi) - \frac{i}{\kappa} (\mathbf{E} \cdot \nabla \Psi, \varphi) + \frac{i}{\kappa} (e \mathbf{A}, \nabla \varphi) + \frac{i}{\kappa} (\Psi \mathbf{E}, \nabla \varphi)
\]
\[
- (|\mathbf{A}|^2 - |\mathbf{A}|^2) \Psi, \varphi - (|\psi|^2 - 1) \psi - (|\Psi|^2 - 1) \Psi, \varphi \right] \right] dt
\]
\[
- \int_0^T (i \eta \kappa \psi \nabla \cdot \mathbf{E} + i \eta \kappa \epsilon \nabla \cdot \mathbf{A}, \varphi) \right] dt,
\]
\[
\int_0^T \left[ (\partial_t \mathbf{E}, \mathbf{a}) + (\nabla \times \mathbf{E}, \nabla \times \mathbf{a}) + (\nabla \cdot \mathbf{E}, \nabla \cdot \mathbf{a}) \right] dt
\]
we obtain that

\[
\frac{\eta}{2} \|e(t, t')\|^2_{L^2} + \int_0^{t'} \left( \frac{1}{\kappa^2} \|\nabla e\|^2_{L^2} + \|A e\|^2_{L^2} \right) \, dt
\]

for any \( \varphi \in L^2((0, T); H^1) \) and \( a \in L^2((0, T); H_n(\text{curl}, \text{div})) \). By choosing \( \varphi(x, t) = e(x, t)1_{(0, t')}(t) \) and \( a(x, t) = E(x, t)1_{(0, t')}(t) \), and using the regularity estimate

\[
\text{ess sup}_{t \in (0, T)} (\|\nabla \varphi\|_{L^2} + \|\nabla \psi\|_{L^2} + \|A\|_{L^4} + \|\nabla A\|_{L^4}) \leq C,
\]

we obtain that

\[
\frac{\eta}{2} \|e(t, t')\|^2_{L^2} + \int_0^{t'} \left( \frac{1}{\kappa^2} \|\nabla e\|^2_{L^2} + \|A e\|^2_{L^2} \right) \, dt
\]

and

\[
\frac{1}{2} \|e(t, t')\|^2_{L^2} + \int_0^{t'} \left( \|\nabla \times E\|^2_{L^2} + \|\nabla \cdot E\|^2_{L^2} \right) \, dt
\]

for any \( \varphi \in L^2((0, T); H^1) \) and \( a \in L^2((0, T); H_n(\text{curl}, \text{div})) \). By choosing \( \varphi(x, t) = e(x, t)1_{(0, t')}(t) \) and \( a(x, t) = E(x, t)1_{(0, t')}(t) \), and using the regularity estimate

\[
\text{ess sup}_{t \in (0, T)} (\|\nabla \varphi\|_{L^2} + \|\nabla \psi\|_{L^2} + \|A\|_{L^4} + \|\nabla A\|_{L^4}) \leq C,
\]

we obtain that

\[
\frac{\eta}{2} \|e(t, t')\|^2_{L^2} + \int_0^{t'} \left( \frac{1}{\kappa^2} \|\nabla e\|^2_{L^2} + \|A e\|^2_{L^2} \right) \, dt
\]

and

\[
\frac{1}{2} \|e(t, t')\|^2_{L^2} + \int_0^{t'} \left( \|\nabla \times E\|^2_{L^2} + \|\nabla \cdot E\|^2_{L^2} \right) \, dt
\]

where \( \epsilon \) is an arbitrary positive number. By choosing \( \epsilon < \frac{1}{4} \min(1, \kappa^{-2}) \) and summing up the last two inequalities, we obtain that

\[
\frac{\eta}{2} \|e(t, t')\|^2_{L^2} + \frac{1}{2} \|E(t, t')\|^2_{L^2} \leq \int_0^{t'} \left( \frac{1}{\kappa^2} \|e\|^2_{L^2} + C \|E\|^2_{L^2} \right) \, dt,
\]

which implies

\[
\max_{t \in (0, T)} \left( \frac{\eta}{2} \|e\|^2_{L^2} + \frac{1}{2} \|E\|^2_{L^2} \right) = 0
\]

via Gronwall’s inequality. Uniqueness of the weak solution is proved.
3.4. Equivalence of (1.1)-(1.4) and (1.5)-(1.11). Let \((\psi, A)\) be the unique solution of (1.1)-(1.4) and, for the given \(\psi\) and \(A\), we let \((p, q, u, v)\) be the solution of (1.6)-(1.9). Since \(\text{Re}[\psi^*(\frac{i}{\kappa} \nabla + A)\psi] \in L^\infty((0,T);L^2)\), the standard regularity estimates of Poisson’s equations yield that

\[
p, q \in L^\infty((0,T); H^1),
\]

\[
u, v \in L^\infty((0,T); H^1) \cap L^2((0,T); H^{1+s}), \quad \partial_t u, \partial_t v, \Delta u, \Delta v \in L^2((0,T); L^2).
\]

By setting \(\tilde{A} = \nabla \times u + \nabla v\), we have \(\tilde{A} \in L^\infty((0,T); L^2) \cap L^2((0,T); H_0^1)\) and \(\partial_t \tilde{A} \in L^2((0,T); (H_0^1(\text{curl, div}))')\). Since \(\text{Re}[\psi^*(\frac{i}{\kappa} \nabla + A)\psi] = \nabla \times p + \nabla q\), the integration of (1.8) against \(\nabla \cdot a\) minus the integration of (1.9) against \(\nabla \times a\) gives

\[
\int_0^T \left[ \left( \frac{\partial \tilde{A}}{\partial t}, a \right) + (\nabla \times \tilde{A}, \nabla \times a) + (\nabla \cdot \tilde{A}, \nabla \cdot a) \right] dt
\]

\[
= \int_0^T (f, \nabla \times a) dt - \int_0^T \left( \text{Re}[\psi^*(\frac{i}{\kappa} \nabla + A)\psi], a \right) dt
\]

for any \(a \in L^2((0,T); H_0^1(\text{curl, div}))\), with \(\tilde{A}_0 = A_0\). Comparing the above equation with (2.2), we derive that \(\tilde{A} = A\). Thus \(\Delta u = \nabla \times A \in L^\infty((0,T); L^2) \cap L^2((0,T); H^1)\) and \(\Delta v = \nabla \cdot A \in L^\infty((0,T); L^2) \cap L^2((0,T); H^1)\), and from (1.8)-(1.9) we further derive that \(\partial_t u, \partial_t v \in L^\infty((0,T); L^2), H^1\).

Overall, (1.5)-(1.11) has a solution \((\psi, p, q, u, v)\) which possesses the regularity specified in Definition 2.2 satisfying (2.3)-(2.7) with \(A = \nabla \times u + \nabla v\), where \((\psi, A)\) coincides with the unique solution of (1.1)-(1.4). Based on the regularity of \(\psi, p, q, u\) and \(v\), uniqueness of the solution for (1.5)-(1.11) can be proved in a similar way as Section 3.3. We omit the proof due to the limitation on pages.

3.5. Singularity of the solution. From the analysis in the last two subsections we see that

\[
\begin{aligned}
-\Delta u &= \nabla \times A \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega, \\
\Delta v &= \nabla \cdot A \quad \text{in } \Omega, \\
\partial_n v &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \(\nabla \times A, \nabla \cdot A \in L^\infty((0,T); L^2)\). For each fixed \(t\), the solutions of the two Poisson’s equations have the decomposition [19]

\[
u(x, t) = \sum_{j=1}^m \beta_j(t) \Phi(|x - x_j|)|x - x_j|^\pi/\omega_j \sin(\pi \Theta_j(x)/\omega_j) + \tilde{u}(x, t),
\]

\[
v(x, t) = \sum_{j=1}^m \gamma_j(t) \Phi(|x - x_j|)|x - x_j|^\pi/\omega_j \cos(\pi \Theta_j(x)/\omega_j) + \tilde{v}(x, t),
\]

where

\[
\sum_{j=1}^m |\beta_j(t)| + \sum_{j=1}^m |\gamma_j(t)| + \|\tilde{u}(\cdot, t)\|_{H^2} + \|\tilde{v}(\cdot, t)\|_{H^2} \leq C \|\nabla \times A\|_{L^2} + C \|\nabla \cdot A\|_{L^2}.
\]

Thus

\[
\|\beta_j\|_{L^\infty(0,T)} + \|\gamma_j\|_{L^\infty(0,T)} + \|\tilde{u}\|_{L^\infty((0,T); H^2)} + \|\tilde{v}\|_{L^\infty((0,T); H^2)}
\]

\[
\leq C(\|\nabla \times A\|_{L^\infty((0,T); L^2)} + \|\nabla \cdot A\|_{L^\infty((0,T); L^2)}) \leq C.
\]

The singular part of \(\psi\) can be derived in a similar way.

The proof of Theorem 2.1 is completed.
4. Proof of Theorem 2.2 In this section, we prove further regularity of the solution under some compatibility conditions. We need the following lemma concerning the maximal $L^p$ regularity of parabolic equations in a Lipschitz domain [27].

**Lemma 4.1.** The solution of the equation

\[
\begin{cases}
\partial_t u - \Delta u = f & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega, \\
u(x,0) = 0 & \text{for } x \in \Omega \quad \text{and} \quad \begin{cases}
\partial_t v - \Delta v = g & \text{in } \Omega, \\
\partial_n v = 0 & \text{on } \partial \Omega, \\
v(x,0) = 0 & \text{for } x \in \Omega
\end{cases}
\end{cases}
\]

satisfy that

\[
\|\partial_t u\|_{L^p((0,T);L^2)} + \|\Delta u\|_{L^p((0,T);L^2)} \leq C_p\|f\|_{L^p((0,T);L^2)},
\]

\[
\|\partial_t v\|_{L^p((0,T);L^2)} + \|\Delta v\|_{L^p((0,T);L^2)} \leq C_p\|g\|_{L^p((0,T);L^2)},
\]

for any $1 < p < \infty$.

Rewriting (1.1) as

\[
\eta \frac{\partial(\psi - \psi_0)}{\partial t} - \frac{1}{\kappa^2} \Delta(\psi - \psi_0) = -g,
\]

with

\[
g = \frac{i}{\kappa} \nabla \cdot (A\psi) + \frac{i}{\kappa} A \cdot \nabla \psi + |A|^2 \psi + (|\psi|^2 - 1) \psi - i\eta \kappa \psi \nabla \cdot A - \frac{i}{\kappa} \Delta \psi_0,
\]

and applying Lemma 4.1 (here we need the compatibility condition $\partial_n \psi_0 = 0$ on $\partial \Omega$), we derive that, for any given $1 < p < \infty$,

\[
\|\partial_t(\psi - \psi_0)\|_{L^p((0,T);L^2)} + \|\Delta(\psi - \psi_0)\|_{L^p((0,T);L^2)} \leq C\|g\|_{L^p((0,T);L^2)}
\]

\[
\leq C\|\nabla \cdot A\|_{L^p((0,T);L^2)} + C\|A\|_{L^\infty((0,T);L^4)} \|\nabla \psi\|_{L^p((0,T);L^4)} + C\|A\|_{L^{2p'}((0,T);L^4)}^2 + C
\]

\[
\leq C\|\nabla \psi\|_{L^p((0,T);L^{p\ast})} \|\Delta \psi\|_{L^p((0,T);L^{p\ast})} + C
\]

\[
\leq C\|\nabla \psi\|_{L^p((0,T);L^{p\ast})} \|\Delta \psi\|_{L^p((0,T);L^{p\ast})} + C
\]

which implies that $\|\partial_t \psi\|_{L^p((0,T);L^2)} + \|\Delta \psi\|_{L^p((0,T);L^2)} \leq C$. In other words, we have

\[
\psi \in \bigcap_{p>1} W^{1,p}((0,T); L^2) \cap L^p((0,T); H^{1+p}) \hookrightarrow L^\infty((0,T); W^{1,4}).
\]

Let $\overline{w} = \nabla \cdot A$ and consider the divergence of (1.2), i.e.

\[
\frac{\partial \overline{w}}{\partial t} - \Delta \overline{w} = -\text{Re} \left[ \nabla^* \cdot \left( \frac{i}{\kappa} \nabla \psi + A\psi \right) + \psi^* \left( \frac{i}{\kappa} \Delta \psi + \psi \nabla \cdot A + A \cdot \nabla \psi \right) \right],
\]

with the boundary condition $\partial_n w = 0$ on $\partial \Omega$. The standard energy estimates of the above equation give

\[
\|\partial_t \overline{w}\|_{L^2((0,T);L^2)} + \|\Delta \overline{w}\|_{L^2((0,T);L^2)}
\]

\[
\leq C\|\overline{w}_0\|_{H^1} + C\|\nabla \psi^* \cdot (i\kappa^{-1} \nabla \psi + A\psi) + \psi^* (i\kappa^{-1} \Delta \psi + \psi \nabla \cdot A + A \cdot \nabla \psi)\|_{L^2((0,T);L^2)}
\]

\[
\leq C\|\nabla \cdot A\|_{H^1} + C\|\nabla \psi^*\|_{L^1((0,T);L^4)} \|\nabla \psi\|_{L^4((0,T);L^4)} + \|A\|_{L^4((0,T);L^4)}
\]
In other words, we have
\[ C(\|\Delta \psi\|_{L^2((0,T);L^2)} + \|\nabla \cdot A\|_{L^2((0,T);L^2)} + \|A\|_{L^4((0,T);L^4)}\|\nabla \psi\|_{L^4((0,T);L^4)}) \]
\[ \leq C \]
If we let \( w = \nabla \times A - f \) and consider the curl of (1.2), in a similar way one can prove
\[ \|\partial_t w\|_{L^2((0,T);L^2)} + \|\Delta w\|_{L^2((0,T);L^2)} \leq C. \]

The last two inequalities imply that

\[ \partial_t A \in L^2((0, T); H_n (\text{curl}, \text{div})) \to L^2((0, T); L^4). \]

Consider the time derivative of (1.1) and denote \( 0 = \partial \psi \). We have
\[ \eta \frac{\partial \psi}{\partial t} - \frac{1}{\kappa^2} \Delta \psi = -\tilde{g}, \]
with the boundary condition \( \partial_n \psi = 0 \) on \( \partial \Omega \), where
\[ \tilde{g} = (i\kappa - i\eta \kappa)\psi \nabla \cdot A + (i\kappa - i\eta \kappa)\psi \nabla \cdot \tilde{\nabla} + 2i\kappa^{-1} \nabla \psi + 2i\kappa^{-1} A \cdot \nabla \psi + 2A \cdot \nabla \psi + |A|^2 \psi + (\psi \psi^* + \psi^* \psi) \psi + (|\psi|^2 - 1) \psi. \]

The energy estimates of the equation give that
\[ \|\partial_t \psi\|_{L^2((0,T);L^2)} + \|\Delta \psi\|_{L^2((0,T);L^2)} + \|\nabla \psi\|_{L^\infty((0,T);L^2)} \]
\[ \leq C \|g\|_{L^2((0,T);L^2)} \]
\[ \leq C \|\psi\|_{L^\infty((0,T);L^2)} \|\nabla \cdot A\|_{L^2((0,T);L^\infty)} + C \|\nabla \cdot \tilde{\nabla}\|_{L^2((0,T);L^2)} \]
\[ + C \|A\|_{L^2((0,T);L^4)} \|\nabla \psi\|_{L^\infty((0,T);L^2)} + C \|A\|_{L^4((0,T);L^4)} \|\nabla \psi\|_{L^2((0,T);L^4)} \]
\[ + C \|A\|_{L^\infty((0,T);L^4)} \|\tilde{\nabla}\|_{L^2((0,T);L^4)} + C \|\psi\|_{L^2((0,T);L^2)} \]
\[ \leq C \|\psi\|_{L^2((0,T);L^2)} \frac{1}{2} \|\partial_t \psi\|_{L^2((0,T);L^2)} + C \]
\[ \leq C \|\nabla \psi\|_{L^2((0,T);L^2)} \|\nabla \psi\|_{L^2((0,T);L^2)} + C \]
\[ \leq C \|\nabla \psi\|_{L^2((0,T);L^2)} + C \|\psi\|_{L^2((0,T);L^2)} \]
\[ \leq C \|\partial_t \psi\|_{L^2((0,T);L^2)} + \|\Delta \psi\|_{L^2((0,T);L^2)} + \|\nabla \psi\|_{L^\infty((0,T);L^2)} \]
\[ \leq C. \]

In other words, we have
\[ \|\partial_t \psi\|_{L^2((0,T);L^2)} + \|\partial_t \psi\|_{L^2((0,T);H^{1+\nu})} + \|\partial_t \psi\|_{L^\infty((0,T);H^1)} \leq C. \]

Now we consider the time derivative of (1.5), i.e.
\[ \Delta \tilde{p} = -\nabla \times \text{Re} [\psi^* (i\kappa^{-1} \nabla + A) \psi]^* \]
\[ \Delta \tilde{q} = \nabla \cdot \text{Re} [\psi^* (i\kappa^{-1} \nabla + A) \psi]^* \]
\[ \frac{\partial \tilde{u}}{\partial t} - \Delta \tilde{u} = \tilde{f} - \tilde{p}, \]
\[ \frac{\partial \dot{v}}{\partial t} - \Delta \dot{v} = -\dot{q}, \]

with the boundary conditions \( \dot{p} = 0, \partial_n \dot{q} = 0, \dot{u} = 0 \) and \( \partial_n \dot{v} = 0 \) on \( \partial \Omega \). In particular, the boundary condition \( \dot{u} = 0 \) on \( \partial \Omega \) at the time \( t = 0 \) requires the compatibility condition \( \nabla \times A_0 = f_0 \) on \( \partial \Omega \). Since

\[
\| [\psi^* (i\kappa^{-1} \nabla + A) \psi] \|_{L^2(0,T);L^2)} \leq \| \psi^* (i\kappa^{-1} \nabla + A) \psi + |\psi|^2 \mathbf{A} \|_{L^2(0,T);L^2)}
\]

\[
\leq \| \psi^* \|_{L^2((0,T);L^\infty)} \| i\kappa^{-1} \nabla \psi + A \psi \|_{L^\infty((0,T);L^2)} + \| i\kappa^{-1} \nabla \psi \|_{L^2((0,T);L^2)} + \| A \|_{L^\infty((0,T);L^2)} \| \dot{\psi} \|_{L^2((0,T),L^2)} + \| \dot{\mathbf{A}} \|_{L^2((0,T);L^2)} \leq C,
\]

the energy estimates of (4.1)-(4.5) give

\[
\| \nabla \dot{\varrho} \|_{L^2((0,T);L^2)} + \| \dot{\nabla} q \|_{L^2((0,T);L^2)} \leq \| [\psi^* (i\kappa^{-1} \nabla + A) \psi] \|_{L^2((0,T);L^2)} \leq C,
\]

and then the energy estimates of (4.6)-(4.7) give

\[
\| \partial_t \dot{u} \|_{L^2((0,T);L^2)} + \| \Delta \dot{u} \|_{L^2((0,T);L^2)} + \| \nabla \dot{u} \|_{L^\infty((0,T);L^2)} \leq C \| f - \dot{p} \|_{L^2((0,T);L^2)} \leq C,
\]

\[
\| \partial_t \dot{v} \|_{L^2((0,T);L^2)} + \| \Delta \dot{v} \|_{L^2((0,T);L^2)} + \| \nabla \dot{v} \|_{L^\infty((0,T);L^2)} \leq C \| \dot{q} \|_{L^2((0,T);L^2)} \leq C,
\]

which further imply that \( \partial_t \dot{u}, \partial_t \dot{v} \in L^2((0,T); H^{1+s}) \).

The proof of Theorem 2.3 is completed.

5. Proof of Theorem 2.3 The proof consists of two parts. In the first part, we prove the boundedness of the finite element solution and the invertibility of the linear systems, which are independent of the regularity of the exact solution. In the second part, we present error estimates of the finite element solution based on a mathematical induction on the \( L^4 \) norm of \( A_h^n = \nabla \times u_h^n + \nabla v_h^n \), which is needed to control the nonlinear terms in the equations.

5.1. Stability of the finite element solution. Substituting \( \varphi = \psi_h^{n+1} \) into (2.8) and considering the real part, we derive that

\[
D_r \left( \frac{\eta}{2} \| \psi_h^{n+1} \|_{L^2}^2 \right) + \| (i\kappa^{-1} \nabla + A_h^n) \psi_h^{n+1} \|_{L^2}^2 + \int_\Omega \| \psi_h^n \|_{H^1}^2 \| \psi_h^{n+1} \|_{L^2}^2 \, dx = \| \psi_h^{n+1} \|_{L^2}^2,
\]

which together with the discrete Gronwall’s inequality implies that, when \( \tau < \eta/4 \),

\[
\max_{0 \leq n \leq N-1} \| \psi_h^{n+1} \|_{L^2}^2 + \sum_{n=0}^{N-1} \tau \| (i\kappa^{-1} \nabla + A_h^n) \psi_h^{n+1} \|_{L^2}^2 \leq C.
\]

Since \( |\chi(\psi_h^n)| \leq 1 \), by substituting \( \xi = p_h^{n+1} \) into (2.9) and substituting \( \zeta = q_h^{n+1} \) into (2.10), we obtain

\[
\| \nabla p_h^{n+1} \|_{L^2} + \| \nabla q_h^{n+1} \|_{L^2} \leq C \| (i\kappa^{-1} \nabla + A_h^n) \psi_h^{n+1} \|_{L^2},
\]

which together with (5.1) gives

\[
\sum_{n=0}^{N-1} \tau \| \nabla p_h^{n+1} \|_{L^2} + \sum_{n=0}^{N-1} \tau \| \nabla q_h^{n+1} \|_{L^2} \leq C.
\]
Then, substituting $\theta = D_r u_h^{n+1}$ into (2.11) and $\vartheta = D_r v_h^{n+1}$ into (2.12), we derive that

$$
\sum_{n=0}^{N-1} \tau \left( \| D_r u_h^{n+1} \|_{L^2}^2 + \| D_r v_h^{n+1} \|_{L^2}^2 \right) + \max_{0 \leq n \leq N-1} \left( \| \nabla u_h^{n+1} \|_{L^2}^2 + \| \nabla v_h^{n+1} \|_{L^2}^2 \right)
\leq C \sum_{n=0}^{N-1} \tau \| f^{n+1} \|_{L^2} + C \sum_{n=0}^{N-1} \tau \| p_h^{n+1} \|_{L^2} + C \sum_{n=0}^{N-1} \tau \| q_h^{n+1} \|_{L^2} \leq C.
$$

(5.3)

From the above derivations it is not difficult to see that the linear systems defined by (2.8)-(2.12) are invertible when $\tau < \eta/4$, and the discrete solution $(\psi_h^n, p_h^n, q_h^n, u_h^n, v_h^n)$ solved from (2.8)-(2.12) is uniformly bounded in $L^\infty_\tau (L^2) \times L^2_\tau (H^1) \times L^2_\tau (H^1)$ with respect to the time-step size $\tau$ and spatial mesh size $h$.

### 5.2. Error estimates

Note that the exact solution $(\psi, p, q, u, v)$ satisfies the equations

$$
(D_r \psi^{n+1}, \varphi) + ((i \kappa - \nabla + A^n) \psi^{n+1}, (i \kappa - \nabla + A^n) \varphi)
\quad + ((|\psi|^2 - 1) \psi^{n+1}, \varphi) + (i \kappa A^n, \nabla (|\psi^{n+1}|^2) \varphi) = (E^n_{\psi^{n+1}}, \varphi),
$$

(5.4)

$$
(\nabla p^{n+1}, \nabla \xi) = (\Re \chi (\psi^n)^*(i \kappa - \nabla \psi^{n+1} + A^n \psi^{n+1}), \nabla \xi) + (E^n_p, \nabla \xi)
$$

(5.5)

$$
(\nabla q^{n+1}, \nabla \zeta) = (\Re \chi (\psi^n)^*(i \kappa - \nabla \psi^{n+1} + A^n \psi^{n+1}), \nabla \zeta) + (E^n_q, \nabla \zeta)
$$

(5.6)

$$
(D_r u^{n+1}, \theta) + (\nabla u^{n+1}, \nabla \theta) = (f^{n+1} - p^{n+1}, \theta) + (E^n_u, \theta)
$$

(5.7)

$$
(D_r v^{n+1}, \vartheta) + (\nabla v^{n+1}, \nabla \vartheta) = (-q^{n+1}, \vartheta) + (E^n_v, \vartheta),
$$

(5.8)

for all $\varphi \in \mathcal{V}_h^1$, $\xi, \zeta \in \mathcal{V}_h^1$ and $\vartheta, \vartheta \in \mathcal{V}_h^1$, with

$$
(\nabla u^0, \nabla \xi) = (A_0, \nabla \xi), \quad \forall \xi \in \mathcal{V}_h^1,
$$

(5.9)

$$
(\nabla v^0, \nabla \zeta) = (A_0, \nabla \zeta), \quad \forall \zeta \in \mathcal{V}_h^1,
$$

(5.10)

where

$$
E^n_{\psi^{n+1}} = \eta (D_r \psi^{n+1} - \partial_t \psi^{n+1}) + \frac{i}{\kappa} \nabla \cdot ((A^n - A^{n+1}) \psi^{n+1}) + \frac{i}{\kappa} (A^n - A^{n+1}) \cdot \nabla \psi^{n+1}
\quad + (|A^n|^2 - |A^{n+1}|^2) \psi^{n+1} + (|\psi|^2 - |\psi^{n+1}|^2) \psi^{n+1} + i \kappa \psi^{n+1} \nabla \cdot (A^{n+1} - A^n)
$$

$$
E^n_p = E^n_{\psi^{n+1}} = \Re \chi (\psi^n)^*(\psi^{n+1} - \psi^n) \nabla \psi^{n+1} + ((\psi^{n+1})^* A^{n+1} - (\psi^n)^* A^n) \psi^{n+1}
$$

$$
E^n_u = D_r u^{n+1} - \partial_t u^{n+1}
$$

$$
E^n_v = D_r v^{n+1} - \partial_t v^{n+1},
$$

are truncation errors due to the time discretization, which satisfy that

$$
\sum_{n=0}^{N-1} \tau \left( \| E^n_{\psi^{n+1}} \|_{L^2}^2 + \| E^n_p \|_{L^2}^2 + \| E^n_q \|_{L^2}^2 + \| E^n_u \|_{L^2}^2 + \| E^n_v \|_{L^2}^2 \right) \leq C \tau^2.
$$

Let $R_h : H^1 \rightarrow \mathcal{V}_h^1$ and $\hat{R}_h : H^1 \rightarrow \hat{V}_h^1$ denote the Ritz projection operator onto the finite element spaces, i.e.

$$
(\nabla (\phi - R_h \phi), \nabla \varphi) = 0 \quad \forall \phi \in \mathcal{H}_h^1 \quad \text{and} \quad \varphi \in \mathcal{V}_h^1,
$$

$$
(\nabla (\phi - \hat{R}_h \phi), \nabla \varphi) = 0 \quad \forall \phi \in H^1 \quad \text{and} \quad \varphi \in \hat{V}_h^1.
$$
Then $R_h$, restricted to $H^1$, is just the Ritz projection from $H^1$ onto $V_h^1$, and we have

\[
\| \phi - R_h \phi \|_{L^2} + h^s \| \nabla (\phi - R_h \phi) \|_{L^2} \leq C h^{2s} \| \phi \|_{H^{1+s}}, \quad \forall \phi \in H^{1+s},
\]

\[
\| \phi - \tilde{R}_h \phi \|_{L^2} + h^s \| \nabla (\phi - \tilde{R}_h \phi) \|_{L^2} \leq C h^{2s} \| \phi \|_{H^{1+s}}, \quad \forall \phi \in \tilde{H}^1 \cap H^{1+s}.
\]

Let $e_{\psi,h}^{n+1} = \psi_h^{n+1} - R_h \psi^{n+1}$, $e_{p,h}^{n+1} = p_h^{n+1} - \tilde{R}_h p^{n+1}$, $e_{q,h}^{n+1} = q_h^{n+1} - R_h q^{n+1}$, $e_{u,h}^{n+1} = u_h^{n+1} - \tilde{R}_h u^{n+1}$, $e_{v,h}^{n+1} = v_h^{n+1} - R_h v^{n+1}$. The difference between (2.8)-(2.14) and (5.4)-(5.10) gives that $u_0 = \tilde{R}_h u^0$, $v_0 = R_h v^0$ and

\[
(\eta D_r e_{\psi,h}^{n+1}, \varphi) + \kappa^{-2} (\nabla e_{\psi,h}^{n+1}, \nabla \varphi)
\]

\[
= (\eta D_r (\psi^{n+1} - R_h \psi^{n+1}), \varphi) - (E_{\psi}^{n+1}, \varphi) - \frac{i}{\kappa} (A^n_h \cdot \nabla e_{\psi,h}^{n+1}, \varphi)
\]

\[
+ \frac{i}{\kappa} (A^n_h \cdot \nabla (\psi^{n+1} - R_h \psi^{n+1}), \varphi) - \frac{i}{\kappa} ((A^n_h - A^n) \cdot \nabla \psi^{n+1}, \varphi)
\]

\[
+ \frac{i}{\kappa} (\nabla \psi^{n+1}, \nabla \varphi) - \frac{i}{\kappa} \left( (\psi^{n+1} - R_h \psi^{n+1}) A^n_h, \nabla \varphi \right) + \frac{i}{\kappa} \left( \psi^{n+1} (A^n_h - A^n), \nabla \varphi \right)
\]

\[
- \left( (|A^n_h|^2 - |A^n|^2) \psi^{n+1}, \varphi \right) - \left( |A^n_h|^2 e_{\psi,h}^{n+1}, \varphi \right) - \left( |A^n_h|^2 (\psi^{n+1} - R_h \psi^{n+1}), \varphi \right)
\]

\[
- \left( (|A^n_h|^2 - 1) \psi^{n+1} - (|\psi|^2 - 1) \psi^{n+1}, \varphi \right) + (i \kappa A^n_h, \nabla ((e_{\psi,h}^{n+1})^*) \varphi)
\]

(5.11)

\[
(\nabla e_{p,h}^{n+1}, \nabla \xi)
\]

\[
= -(E_{p}^{n+1}, \nabla \times \xi) + \left( \text{Re} \left[ (\chi(\psi_h^n)^* - \chi(R_h \psi)^*) (i \kappa^{-1} \nabla \psi^{n+1} + A^n \psi^{n+1}) \right], \nabla \times \xi \right)
\]

\[
- \left( \text{Re} \left[ (\chi(\psi_h^n)^* - \chi(R_h \psi)^*) (i \kappa^{-1} \nabla \psi^{n+1} + A^n \psi^{n+1}) \right], \nabla \times \xi \right)
\]

\[
+ \left( \text{Re} \left[ \chi(\psi_h^n)^* (i \kappa^{-1} \nabla e_{\psi,h}^{n+1} + (A^n_h - A^n) \psi^{n+1} + A^n \psi^{n+1}) \right], \nabla \times \xi \right)
\]

(5.12)

\[
(\nabla e_{q,h}^{n+1}, \nabla \zeta)
\]

\[
= -(E_{q}^{n+1}, \nabla \zeta) + \left( \text{Re} \left[ (\chi(\psi_h^n)^* - \chi(R_h \psi)^*) (i \kappa^{-1} \nabla \psi^{n+1} + A^n \psi^{n+1}) \right], \nabla \zeta \right)
\]

\[
- \left( \text{Re} \left[ (\chi(\psi_h^n)^* - \chi(R_h \psi)^*) (i \kappa^{-1} \nabla \psi^{n+1} + A^n \psi^{n+1}) \right], \nabla \zeta \right)
\]

\[
+ \left( \text{Re} \left[ \chi(\psi_h^n)^* (i \kappa^{-1} \nabla e_{\psi,h}^{n+1} + (A^n_h - A^n) \psi^{n+1} + A^n \psi^{n+1}) \right], \nabla \zeta \right)
\]

(5.13)

\[
(D_r e_{\psi,h}^{n+1}, \theta) + (\nabla e_{u,h}^{n+1}, \nabla \theta)
\]

(5.14)

\[
= (D_r (u^{n+1} - \tilde{R}_h u^{n+1}), \theta) + (p^{n+1} - p_h^{n+1}, \theta) - (E_{u}^{n+1}, \theta),
\]

(5.15)

\[
(D_r e_{v,h}^{n+1}, \varphi) + (\nabla e_{v,h}^{n+1}, \nabla \varphi)
\]

\[
= (D_r (v^{n+1} - R_h v^{n+1}), \varphi) + (q^{n+1} - q_h^{n+1}, \varphi) - (E_{v}^{n+1}, \varphi),
\]

for all $\varphi \in V_h^1$, $\xi, \varphi \in V_h^1$ and $\zeta, \varphi \in V_h^1$, with $\| e_{\psi,h}^0 \|_{L^2} \leq C h^{2s}$, $\| e_{p,h}^0 \|_{H^1} \leq C h^s$ and $e_{u,h}^0 = e_{v,h}^0 = 0$. Substituting $\theta = D_r e_{\psi,h}^{n+1}$ and $\varphi = D_r e_{v,h}^{n+1}$ into (5.14)-(5.15), we get

\[
\| D_r e_{\psi,h}^{n+1} \|_{L^2}^2 + \| \Delta_h e_{u,h}^{n+1} \|_{L^2}^2 + D_r \| \nabla e_{u,h}^{n+1} \|_{L^2}^2
\]

\[
\leq C \| D_r (u^{n+1} - \tilde{R}_h u^{n+1}) \|_{L^2}^2 + C \| p^{n+1} - p_h^{n+1} \|_{L^2}^2 + C \| E_{u}^{n+1} \|_{L^2}^2,
\]
\[ \| D_\tau e^{n+1}_{v,h} \|_{L^2}^2 + \| \Delta_h e^{n+1}_{v,h} \|_{L^2}^2 + D_\tau \| \nabla e^{n+1}_{v,h} \|_{L^2}^2 \leq C \| D_\tau (v^{n+1} - R_h v^{n+1}) \|_{L^2}^2 + C \| q^{n+1} - q_h^{n+1} \|_{L^2}^2 + C \| E^{n+1}_v \|_{L^2}^2, \]

where \( \Delta_h e^{n+1}_{v,h} \) and \( \Delta_h e^{n+1}_{v,h} \) are defined in Lemma 3.3. By Lemma 3.3, the last two inequalities imply that
\[
C^{-1} \| e^{n+1}_{u,h} \|_{W^{1,4}}^2 + D_\tau \| \nabla e^{n+1}_{v,h} \|_{L^2}^2 \\
\leq C \| p^{n+1} - p_h^{n+1} \|_{L^2}^2 + C \| D_\tau (u^{n+1} - \tilde{R}_h u^{n+1}) \|_{L^2}^2 + C \| E^{n+1}_u \|_{L^2}^2, \\
C^{-1} \| e^{n+1}_{v,h} \|_{W^{1,4}}^2 + D_\tau \| \nabla e^{n+1}_{v,h} \|_{L^2}^2 \\
\leq C \| q^{n+1} - q_h^{n+1} \|_{L^2}^2 + C \| D_\tau (v^{n+1} - R_h v^{n+1}) \|_{L^2}^2 + C \| E^{n+1}_v \|_{L^2}^2.
\]

The sum of the last two inequalities gives
\[
C^{-1} \| e^{n+1}_{A,h} \|_{L^2}^2 + D_\tau (\| \nabla e^{n+1}_{u,h} \|_{L^2}^2 + \| \nabla e^{n+1}_{v,h} \|_{L^2}^2) \\
\leq C \| p^{n+1} - p_h^{n+1} \|_{L^2}^2 + C \| q^{n+1} - q_h^{n+1} \|_{L^2}^2 + C \| E^{n+1}_u \|_{L^2}^2 + C \| E^{n+1}_v \|_{L^2}^2 \\
+ C \| D_\tau (u^{n+1} - \tilde{R}_h u^{n+1}) \|_{L^2}^2 + C \| D_\tau (v^{n+1} - R_h v^{n+1}) \|_{L^2}^2.
\]

(5.16)

At this moment, we invoke a mathematical induction on
\[
\| A^n_h \|_{L^2} \leq \max_{0 \leq n \leq N} \| A^n \|_{L^2} + 1.
\]

Since
\[
\| A^n_h - A^n \|_{L^2} \leq \| \nabla \times (\tilde{R}_h u^0 - u^0) \|_{L^2} + \| \nabla (R_h v^0 - v^0) \|_{L^2} \leq Ch^{-1/2}(\| u^0 \|_{H^{1+s}} + \| v^0 \|_{H^{1+s}}),
\]

there exists a positive constant \( h_1 \) such that (5.17) holds for \( n = 0 \) when \( h < h_1 \). In the following, we present estimates of the finite element solution by assuming that (5.17) holds for \( 0 \leq n \leq m \), for some nonnegative integer \( m \). We shall see that if (5.17) holds for \( 0 \leq n \leq m \), then it also holds for \( n = m + 1 \).

Substituting \( \xi = e^{n+1}_{p,h} \) in (5.12), it is not difficult to derive that
\[
\| \nabla e^{n+1}_{p,h} \|_{L^2} \leq C \| E^{n+1}_p \|_{L^2} + C \| e^n_{\psi,h} \|_{L^2} \| ik^{-1} \nabla \psi^{n+1} + A^n \psi^{n+1} \|_{L^4} \\
+ C \| \psi^n - R_h \psi^n \|_{L^4} \| ik^{-1} \nabla \psi^{n+1} + A^n \psi^{n+1} \|_{L^4} \\
+ C \| \nabla e^{n+1}_{\psi,h} \|_{L^2} + \| A^n_h - A^n \|_{L^2} + \| A^n_h \|_{L^2} \| e^{n+1}_{\psi,h} \|_{L^2} \\
+ C \| (\nabla (\psi^{n+1} - R_h \psi^{n+1}) \|_{L^2} + \| A^n_h \|_{L^2} \| \psi^n - R_h \psi^n \|_{L^4} \|_{L^4} \\
\leq C \| E^{n+1}_p \|_{L^2} + C \| e^n_{\psi,h} \|_{H^1} + C \| \psi^{n+1} - R_h \psi^n \|_{H^1} + C \| A^n_h - A^n \|_{L^2}.
\]

(5.18)

Similarly, by substituting \( \xi = e^{n+1}_{q,h} \) in (5.12), one can derive that
\[
\| \nabla e^{n+1}_{q,h} \|_{L^2} \leq C \| E^{n+1}_q \|_{L^2} + C \| e^n_{\psi,h} \|_{H^1} + C \| \psi^n - R_h \psi^n \|_{H^1} + C \| A^n_h - A^n \|_{L^2}.
\]

(5.19)

Substituting \( \phi = e^{n+1}_{\psi,h} \) in (5.11), we obtain that
\[
\frac{1}{2} \| e^{n+1}_{\psi,h} \|_{L^2}^2 + \kappa^{-2} \| \nabla e^{n+1}_{\psi,h} \|_{L^2}^2 \leq C \| D_\tau (\psi^{n+1} - R_h \psi^{n+1}) \|_{L^2}^2 + C \| e^{n+1}_{\psi,h} \|_{L^2}^2 + C \| E^{n+1}_v \|_{L^2}.
\]
+ C\|A_h^n\|L^4 \|\nabla e_{\psi,h}^{n+1}\|L^2 \|e_{\psi,h}^{n+1}\|L^4 + C\|A_h^n\|L^4 \|\nabla (\psi_{\psi,h}^{n+1} - R_h \psi_{\psi,h}^{n+1})\|L^2 \|e_{\psi,h}^{n+1}\|L^4 \\
+ C\|A_h^n - A^n\|L^4 \|\nabla \psi_{\psi,h}^{n+1}\|L^4 \|e_{\psi,h}^{n+1}\|L^4 + C\|e_{\psi,h}^{n+1}\|L^4 \|\nabla \psi_{\psi,h}^{n+1}\|L^2 \\
+ C\|\psi_{\psi,h}^{n+1} - R_h \psi_{\psi,h}^{n+1}\|L^4 \|e_{\psi,h}^{n+1}\|L^4 + C\|A_h^n\|L^4 \|\nabla \psi_{\psi,h}^{n+1}\|L^2 \\
+ C\|\psi_{\psi,h}^{n+1}\|L^4 \|\nabla e_{\psi,h}^{n+1}\|L^2 \|e_{\psi,h}^{n+1}\|L^4 + C\|A_h^n\|L^4 \|\nabla \psi_{\psi,h}^{n+1}\|L^2 \\
+ C\|A_h^n - A^n\|L^4 \|\nabla e_{\psi,h}^{n+1}\|L^2 \|e_{\psi,h}^{n+1}\|L^4 \\
\leq \epsilon_1 \|\nabla \psi_{\psi,h}^{n+1}\|L^2 + C\|\psi_{\psi,h}^{n+1}\|L^2 + C\|A_h^n - A^n\|L^2 \\
(5.20)
+ C\|E_{\psi,h}^{n+1}\|L^2 + C\|D_r (\psi_{\psi,h}^{n+1} - R_h \psi_{\psi,h}^{n+1})\|L^2 \\
(5.21)

By choosing \( \epsilon_1 \) and \( \epsilon \) small enough, the term \((C\epsilon_1 + \epsilon)\|\nabla e_{\psi,h}^{n+1}\|L^2\) on the right-hand side of the last inequality can be eliminated by the left-hand side. Since \\
\|A_h^n - A^n\|L^4 \leq C\|\nabla e_{\psi,h}^{n+1}\|L^2 + C\|\nabla e_{\psi,h}^{n+1}\|L^2 + C\|\nabla e_{\psi,h}^{n+1}\|L^2 + C\|\nabla (u^n - R_h v^n)\|L^4 \\
\leq C\|\nabla e_{\psi,h}^{n+1}\|L^2 + C\|\nabla e_{\psi,h}^{n+1}\|L^2 + C\|u^{n+1}\|H^{1+s} + ||v^{n+1}\|H^{1+s}\|H^s\), \\
the inequality (5.21) reduces to \\
\frac{\epsilon_1}{C} \|e_{\psi,h}^{n+1}\|L^2 + \frac{1}{2\kappa_2} \|\nabla e_{\psi,h}^{n+1}\|L^2 + D_r (\epsilon_1 \|\nabla e_{\psi,h}^{n+1}\|L^2 + \epsilon_1 \|\nabla e_{\psi,h}^{n+1}\|L^2 + \frac{\eta}{2} \|e_{\psi,h}^{n+1}\|L^2) \\
\leq C\|\nabla e_{\psi,h}^{n+1}\|L^2 + C\|\nabla e_{\psi,h}^{n+1}\|L^2 + C\|\nabla e_{\psi,h}^{n+1}\|L^2 \\
+ C\|E_{\psi,h}^{n+1}\|L^2 + C\|E_{\psi,h}^{n+1}\|L^2 + C\|E_{\psi,h}^{n+1}\|L^2 + C\|E_{\psi,h}^{n+1}\|L^2 \\
+ C\|D_r (\psi_{\psi,h}^{n+1} - R_h \psi_{\psi,h}^{n+1})\|L^2 + C\|D_r (\psi_{\psi,h}^{n+1} - R_h \psi_{\psi,h}^{n+1})\|L^2 \\
+ C\|D_r (\psi_{\psi,h}^{n+1})\|L^2 + C\|D_r (\psi_{\psi,h}^{n+1})\|L^2 \\
+ C\|D_r (\psi_{\psi,h}^{n+1})\|L^2 + C\|D_r (\psi_{\psi,h}^{n+1})\|L^2 \\n(5.22)
\max_{0 \leq n \leq m} \left( \|\nabla e_{\psi,h}^{n+1}\|L^2 + \|\nabla e_{\psi,h}^{n+1}\|L^2 \right) + \sum_{n=0}^m \tau \|e_{\psi,h}^{n+1}\|L^2 \leq C_1 (\tau^2 + h^{2s})

for some positive constant \( C_1 \). In particular, the last inequality implies that \\
\max_{0 \leq n \leq m} \|e_{\psi,h}^{n+1}\|L^2 \leq C (\tau^2 + h^{2s}).
If $\tau \geq h$, then we have
\[
\|e_{A,h}^{m+1}\|_{L^4}^2 \leq \frac{1}{\tau} \sum_{n=0}^{m} \tau \|e_{A,h}^{n+1}\|_{L^4}^2 \leq C(\tau + h^{2s}/\tau) \leq C(\tau + h^{2s-1});
\]
if $\tau \leq h$, then we have
\[
\|e_{A,h}^{m+1}\|_{L^4}^2 \leq Ch^{-1}\|e_{A,h}^{m+1}\|_{L^4}^2 \leq C(\tau^2/h + h^{2s-1}) \leq C(h + h^{2s-1}).
\]
Overall, we have $\|e_{A,h}^{m+1}\|_{L^4}^2 \leq C(\tau + h + h^{2s-1})$ and so
\[
\|A_h^{m+1} - A^{m+1}\|_{L^4} \leq \|e_{A,h}^{m+1}\|_{L^4} + \|\nabla \times (u^{m+1} - \tilde{R}_h u^{m+1})\|_{L^4} + \|\nabla (v^{m+1} - \tilde{R}_h v^{m+1})\|_{L^4} \leq C(\tau^{1/2} + h^{1/2} + h^{s-1/2}).
\]
There exist positive constants $\tau_2$ and $h_2$ such that when $\tau < \tau_2$ and $h < h_2$ we have
\[
\|A_h^{m+1} - A^{m+1}\|_{L^4} \leq 1,
\]
and this completes the mathematical induction on (6.1) in the case that $\tau < \tau_2$ and $h < h_2$. Thus (5.22) holds for $m = N - 1$ with the same constant $C_1$, provided $\tau < \tau_2$ and $h < h_2$.

If $\tau \geq \tau_2$ or $h \geq h_2$, from (5.1)-(5.3) we see that
\[
\max_{0 \leq n \leq N-1} \left( \|\nabla e_{u,h}^{n+1}\|_{L^2}^2 + \|\nabla e_{v,h}^{n+1}\|_{L^2}^2 + \|e_{\psi,h}^{n+1}\|_{L^2}^2 \right) \leq C_2 \leq C_2 \left( \tau_2^{-2} + h_2^{-2s} \right)(\tau^2 + h^{2s}) \]
for some positive constant $C_2$. From (5.22) and (5.23) we see that for any $\tau$ and $h$ we have
\[
\max_{0 \leq n \leq N-1} \left( \|\nabla e_{u,h}^{n+1}\|_{L^2}^2 + \|\nabla e_{v,h}^{n+1}\|_{L^2}^2 + \|e_{\psi,h}^{n+1}\|_{L^2}^2 \right) \leq \left[ C_1 + C_2 \left( \tau_2^{-2} + h_2^{-2s} \right) \right](\tau^2 + h^{2s}).
\]

The proof of Theorem 2.3 is completed.

6. Numerical example. We consider an artificial example, the equations
\[
(6.1) \quad \eta \frac{\partial \psi}{\partial t} + \left( \frac{i}{\kappa} \nabla + A \right)^2 \psi + (|\psi|^2 - 1)\psi - i\eta \kappa \psi \cdot A = g,
\]
\[
(6.2) \quad \frac{\partial A}{\partial t} + \nabla \times (\nabla \times A) - \nabla(\nabla \cdot A) + \text{Re} \left[ \psi^* \left( \frac{i}{\kappa} \nabla + A \right) \psi \right] = g + \nabla \times f,
\]
in an L-shape domain $\Omega$ whose longest side has unit length, centered at the origin, with $\eta = 1$ and $k = 10$. The functions $f = \nabla \times A \in C^1([0,T];H^2)$, $g \in C([0,T];L^2)$ and $g \in C([0,T];L^2)$ are chosen corresponding to the exact solution
\[
\psi = \tau^2 \Phi(r)r^{2/3}\cos(2\theta/3),
\]
\[
A = \left( (4t^2\Phi(r)r^{-1/3}/3 + t^2\Phi'(r)r^{2/3}) \cos(\theta/3), \quad (4t^2\Phi(r)r^{-1/3}/3 + t^2\Phi'(r)r^{2/3}) \sin(\theta/3) \right),
\]
where $(r, \theta)$ denotes the polar coordinates, the cut-off function $\Phi(r)$ is defined by
\[
\Phi(r) = \begin{cases} 
0.1 & \text{if } r < 0.1, \\
\Upsilon(r) & \text{if } 0.1 \leq r \leq 0.4, \\
0 & \text{if } r > 0.4,
\end{cases}
\]
Fig. 2. Quasi-uniform triangulations with $M = 16, 32, 64$. 

and $\Upsilon'(r)$ is the unique 7th order polynomial satisfying the conditions $\Upsilon'(0.1) = \Upsilon'(0.4) = \Upsilon''(0.1) = \Upsilon''(0.4) = \Upsilon'''(0.1) = \Upsilon'''(0.4) = 0$ and $\Upsilon(0.1) = \Upsilon(0.4) = 0$. It is easy to check that the exact solution $(\psi, A)$ satisfies the boundary and initial conditions (1.3)-(1.4) with $\psi_0 = 0$ and $A_0 = (0, 0)$.

The L-shape domain is triangulated quasi-uniformly, as shown in Figure 2, with $M$ nodes per unit length on each side, and we denote by $h = 1/M$ for simplicity.

Firstly, we solve (6.1)-(6.2) directly by the FEM with piecewise linear finite elements and a linearized backward Euler scheme, and we denote the numerical solution by $(\tilde{\psi}^N_h, \tilde{A}^N_h)$. In a convex or smooth domain, convergence of the numerical solution $(\tilde{\psi}^N_h, \tilde{A}^N_h)$ can be proved based on the method of [7, 13]. Here we are interested in the question: whether the numerical solution converges to the correct solution in a nonconvex polygonal domain? To answer this question, we present the errors of the numerical solution in Table 1 with $\tau = h$ for several different $h$. One can see that the errors do not decrease as the mesh is refined. In other words, the numerical solution $(\tilde{\psi}^N_h, \tilde{A}^N_h)$ does not converge to the correct solution, nor does the physical quantity $|\tilde{\psi}^N_h|$ converge to $|\psi^N|$.

Secondly, we solve the projected TDGL corresponding to (6.1)-(6.2) by the proposed method and denote the numerical solution by $(\psi^N_h, A^N_h)$. We present the errors of the numerical solution in Table 2 where the convergence rate of $\psi^N_h$ is calculated by the formula

$$\text{convergence rate of } \psi^N_h = \log\left(\frac{\|\psi^N_h - \psi^N\|_{L^2}/\|\psi_{h/2} - \psi^N\|_{L^2}}{\log 2}\right)/\log 2$$

based on the finest mesh size $h$ (the same formula is used for $|\psi^N_h|$ and $A^N_h$). We see that the convergence rates of $\psi^N_h$, $|\psi^N_h|$ and $A^N_h$ are better than $O(h^{2/3})$, which is the worst convergence rate proved in Theorem 2.3. The numerical results are consistent with our theoretical analysis and indicate that our method is efficient for solving the Ginzburg–Landau equations in a domain with reentrant corners.

7. Conclusions. We have proved the well-posedness of the time-dependent Ginzburg–Landau superconductivity model in a nonconvex polygonal domain. Due to the singularity of the magnetic potential, direct application of the finite element method to the original Ginzburg–Landau equations may yield an incorrect solution. Based on the Hodge decomposition, we reformulated the equations into an equivalent system, which avoids direct calculation of the magnetic potential, and therefore can be solved correctly by finite element methods. Then a decoupled and linearized FEM was proposed and convergence rate of the numerical solution was established based on proved regularity of the essential unknowns of the reformulated system. Numerical examples show the effectiveness of the proposed method in comparison with the traditional approach. For simplicity, we have focused on nonconvex polygons in this paper. Nevertheless, the results can be extended to nonconvex curved polygons without essential change of the argument.

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Table 1
Errors of the finite element solution \((\psi_N^h, A_N^h)\) with \(\tau = h\).

| \(h\)  | \(\|\psi_N^h - \psi_N^h\|_{L^2}\) | \(\|\psi_N^h| - |\psi_N^h|\|_{L^2}\) | \(\|A_N^h - A_N^h\|_{L^2}\) |
|-------|---------------------------------|---------------------------------|---------------------------------|
| 1/16  | 4.2113E-03                      | 3.7007E-03                      | 8.3961E-02                      |
| 1/32  | 3.1847E-03                      | 2.0651E-03                      | 8.1396E-02                      |
| 1/64  | 2.9884E-03                      | 1.6286E-03                      | 7.9709E-02                      |
| 1/128 | 2.9170E-03                      | 1.4624E-03                      | 7.8779E-02                      |
| 1/256 | 2.8734E-03                      | 1.3875E-03                      | 7.8210E-02                      |

convergence rate \(O(h^{0.02})\) \(O(h^{0.07})\) \(O(h^{0.01})\)

Table 2
Errors of the finite element solution \((\psi_N^h, A_N^h)\) with \(\tau = h\).

| \(h\)  | \(\|\psi_N^h - \psi_N^h\|_{L^2}\) | \(\|\psi_N^h| - |\psi_N^h|\|_{L^2}\) | \(\|A_N^h - A_N^h\|_{L^2}\) |
|-------|---------------------------------|---------------------------------|---------------------------------|
| 1/16  | 2.7608E-03                      | 2.4889E-03                      | 2.9448E-02                      |
| 1/32  | 8.0517E-04                      | 7.0163E-04                      | 1.4861E-02                      |
| 1/64  | 3.1147E-04                      | 2.8685E-04                      | 8.0870E-03                      |
| 1/128 | 1.3066E-04                      | 1.2664E-04                      | 4.3397E-03                      |
| 1/256 | 6.1047E-05                      | 6.0252E-05                      | 2.3748E-03                      |

convergence rate \(O(h^{1.09})\) \(O(h^{1.07})\) \(O(h^{0.87})\)

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