Goresntein $n$-$\mathcal{X}$-injective and $n$-$\mathcal{X}$-flat modules with respect to a special finitely presented module

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Abstract. Let $R$ be a ring, $\mathcal{X}$ a class of $R$-modules and $n \geq 1$ an integer. In this paper, via special finitely presented modules, we introduce the concepts of Goresntein $n$-$\mathcal{X}$-injective and $n$-$\mathcal{X}$-flat modules. And aside, we obtain some equivalent properties of these modules on $n$-$\mathcal{X}$-coherent rings. Then, we investigate the relations among Goresntein $n$-$\mathcal{X}$-injective, $n$-$\mathcal{X}$-flat, injective and flat modules on $\mathcal{X}$-$FC$-rings ($n$-$\mathcal{X}$-coherent and $n$-$\mathcal{X}$-injective). Several known results are generalized to this new context.

Keywords: $n$-$\mathcal{X}$-coherent ring; Goresntein $n$-$\mathcal{X}$-injective module; Goresntein $n$-$\mathcal{X}$-flat module.

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1 Introduction

In 1995, Enochs et al, introduced the concept of Gorenstein injective and Gorenstein flat modules. Then, these modules have became a vigorously active area of research. For background on Gorenstein homological modules, we refer the reader to [8, 9, 12]. In 2012, Gao and Wang introduced and studied in [10] Gorenstein \( FP \)-injective modules. They established various homological properties of Gorenstein \( FP \)-injective modules mainly over a coherent ring. For more details, see [13].

Recall, that the coherent rings were first appear in Chases paper [4] without being mentioned by name. The term coherent was first used by Bourbaki in [1]. Then, the \( n \)-coherent rings by Costa in [6] introduced. Let \( n \) be a non-negative integer and \( M \) a left \( R \)-module. Then \( M \) is said to be \( n \)-presented if there is an exact sequence \( F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0 \) of left \( R \)-modules, where each \( F_i \) is finitely generated free, and a ring \( R \) is called left \( n \)-coherent if every \( n \)-presented left \( R \)-module is \( (n+1) \)-presented, and if \( n = 1 \), then \( R \) is a coherent ring, see ([6, 7]). Chen and Ding in [3] by using \( n \)-presented modules, introduced the \( n \)-FP-injective and \( n \)-flat modules. Bennis in [2] introduced the \( n \)-\( \mathcal{X} \)-injective and \( n \)-\( \mathcal{X} \)-flat modules and \( n \)-\( \mathcal{X} \)-coherent rings for any class \( \mathcal{X} \) of \( R \)-modules. Then in particular, in 2018, Zhao et al in [20] introduced the \( n \)-FP-gr-injective graded left modules, \( n \)-gr-flat graded right modules and left \( n \)-gr-coherent graded rings on a class of graded \( R \)-modules, and also they defined the special finitely presented graded left modules via projective resolutions of \( n \)-presented graded left modules, where if \( U \) is \( n \)-presented graded left module, then in the exact sequence \( F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to U \to 0 \), \( K_{n-1} = \text{Im}(F_{n-1} \to F_{n-2}) \) is called special finitely presented. In this paper, we unify and extend various homological notions, including the one cited above, to a more general context. Namely, we define the special finitely presented modules via projective resolutions of \( n \)-presented left modules in a given \( \mathcal{X} \) of \( R \)-modules. Then, we introduce and study Gorenstein \( n \)-\( \mathcal{X} \)-injective and \( n \)-\( \mathcal{X} \)-flat modules with respect to special finitely presented modules.

The paper is organized as follows:

In Section 2, some fundamental concepts and some preliminary results are stated.

In Section 3, we give some characterizations of \( n \)-\( \mathcal{X} \)-injective and \( n \)-\( \mathcal{X} \)-flat modules.

In Section 4, we introduce the notions of Gorenstein \( n \)-\( \mathcal{X} \)-injective and \( n \)-\( \mathcal{X} \)-flat modules and generalize some results of [20] to the context of \( n \)-\( \mathcal{X} \)-injective and \( n \)-\( \mathcal{X} \)-flat modules and also of [10] to the context of Gorenstein \( n \)-\( \mathcal{X} \)-injective modules. Then we obtain some equivalent characterizations of Gorenstein \( n \)-\( \mathcal{X} \)-injective and \( \mathcal{X} \)-flat modules on \( n \)-\( \mathcal{X} \)-coherent ring.
In Section 5, we introduce and investigate $n$-$\mathcal{X}$-FC rings ($n$-$\mathcal{X}$-coherent and $n$-$\mathcal{X}$-injective) whose every left module is Gorenstein $n$-$\mathcal{X}$-injective and every Gorenstein $n$-$\mathcal{X}$-injective right module is Gorenstein $n$-$\mathcal{X}$-flat. Furthermore, examples are given which show that the Gorenstein $m$-$\mathcal{X}$-injectivity (resp., the $m$-$\mathcal{X}$-flatness) does not imply the Gorenstein $n$-$\mathcal{X}$-injectivity (resp., the $n$-$\mathcal{X}$-flatness) for any $m > n$.

## 2 Preliminaries

Throughout this paper $R$ will be an associative (non necessarily commutative) ring with identity, and all modules will be unital left $R$-modules (unless specified otherwise).

In this section, some fundamental concepts and notations are stated.

Let $n$ be a non-negative integer, $M$ an left $R$-module and $\mathcal{X}$ a class of left $R$-modules. Then, $M$ is said to be Gorenstein injective (resp., Gorenstein flat) \cite{8,9} if there is an exact sequence $\cdots \rightarrow I_1 \rightarrow I_0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$ of injective (resp., flat) left modules with $M = \ker(I^0 \rightarrow I^1)$ such that $\text{Hom}(U, -)$ (resp., $U \otimes_R -$) leaves the sequence exact whenever $U$ is an injective left (resp., right) module. The Gorenstein projective modules are defined dually.

$M$ is said to be $n$-FP-injective \cite{3} if $\text{Ext}^n_R(U, M) = 0$ for any $n$-presented left $R$-module $U$. In case $n = 1$, $n$-FP-injective modules are nothing but the well-known FP-injective modules. A right module $N$ is called $n$-flat if $\text{Tor}_n^R(U, N) = 0$ for any $n$-presented left $R$-module $U$.

$M$ is said to be Gorenstein $FP$-injective \cite{10} if there is an exact sequence $E = \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots$ with $M = \ker(E^0 \rightarrow E^1)$ such that $\text{Hom}_R(P, E)$ leaves this sequence exact whenever $P$ is finitely presented with $\text{pd}_R(P) < \infty$.

A graded left $R$-module $M$ is called $n$-FP-gr-injective \cite{20} if $\text{EXT}^n_R(N, M) = 0$ for any finitely $n$-presented graded left $R$-module $N$. A graded right $R$-module $M$ is called $n$-gr-flat \cite{20} if $\text{Tor}_n^R(N, M) = 0$ for any finitely $n$-presented graded left $R$-module $N$. For more details about graded modules see \cite{11,14,15}.

From now on, $\mathcal{X}_k$ is non empty and a class of $k$-presented left $R$-modules in a given $\mathcal{X}$ for any $k \geq 0$.

An $R$-module $M$ is said to be $n$-$\mathcal{X}$-injective \cite{2} if $\text{EXT}^n_R(U, M) = 0$ for any $U \in \mathcal{X}_n$. A right $R$-module $N$ is called $n$-$\mathcal{X}$-flat \cite{2} if $\text{Tor}_n^R(U, N) = 0$ for any $U \in \mathcal{X}_n$. We use $\mathcal{X}_n$ (resp., $\mathcal{X}_n$) to denote the class of all $n$-$\mathcal{X}$-injective left $R$-modules (resp., $n$-$\mathcal{X}$-flat right $R$-modules).
A ring $R$ is called $n$-$\mathcal{X}$-coherent if every $n$-presented left $R$-modules in $\mathcal{X}$ is $(n+1)$-presented. It is clear that when $n = 0$ (resp., $n = 1$) and $\mathcal{X}$ is a class of all cyclic $R$-modules, then $R$ is Noetherian (resp., coherent).

3 $n$-$\mathcal{X}$-injective, $n$-$\mathcal{X}$-flat and special $\mathcal{X}$-presented modules

In this section, first we give a general definition of [20, Definition 3.1] and several characterizations of $n$-$\mathcal{X}$-injective and $n$-$\mathcal{X}$-flat modules, and then we introduce the Gorenstein $n$-$\mathcal{X}$-injective and Gorenstein $n$-$\mathcal{X}$-flat modules.

Definition 3.1. Let $n \geq 0$ be an integer and $U \in \mathcal{X}_n$ for any class $\mathcal{X}$ of left $R$-modules. Then the following exact sequence

$$F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to U \to 0,$$

where each $F_i$ is finitely generated free left $R$-module, exists.

Let $K_{n-1} = \text{Im}(F_{n-1} \to F_{n-2})$ and $K_n = \text{Im}(F_n \to F_{n-1})$. Then, the short exact sequence $0 \to K_n \to F_n \to K_{n-1} \to 0$ is called special short exact sequence of $U \in \mathcal{X}_n$, where $K_n$ and $K_{n-1}$ are finitely generated and finitely presented, respectively. We call the modules $K_n$ and $K_{n-1}$ special $\mathcal{X}$-generated and special $\mathcal{X}$-presented left $R$-modules, respectively.

Also, a short exact sequence $0 \to A \to B \to C \to 0$ of left $R$-modules is called special $\mathcal{X}$-pure, if for every special $\mathcal{X}$-presented $K_{n-1}$, there exists the following exact sequence:

$$0 \to \text{Hom}_R(K_{n-1}, A) \to \text{Hom}_R(K_{n-1}, B) \to \text{Hom}_R(K_{n-1}, C) \to 0,$$

where $A$ is said to be special $\mathcal{X}$-pure in $B$. Also, the exact sequence $0 \to C^* \to B^* \to A^* \to 0$ is called split special exact sequence. If $M$ is an $n$-$\mathcal{X}$-injective (resp., flat), then $\text{Ext}_R^1(K_{n-1}, M) \cong \text{Ext}^n_R(U, M) = 0$ (resp., $\text{Tor}_R^1(K_{n-1}, M) \cong \text{Tor}_n^R(U, M) = 0$) for any $U \in \mathcal{X}_n$.

In particular, if $\mathcal{X}$ is a class of graded left $R$-modules, then every special $\mathcal{X}$-generated and every special $\mathcal{X}$-presented module are special finitely generated and special finitely presented graded left $R$-modules, respectively. Also, every $n$-$\mathcal{X}$-injective left $R$-module and every $n$-$\mathcal{X}$-flat right $R$-module are $n$-$\text{FP-gr}$-injective and $n$-$\text{gr-flat}$, respectively, see [20].
Proposition 3.2. Let $\mathcal{M}$ be a class of left $R$-modules and $M$ a left $R$-module. Then the following statements are equivalent:

1. $M$ is $n$-$\mathcal{M}$-injective;
2. Every the short exact sequence $0 \to M \to A \to C \to 0$ is special $\mathcal{M}$-pure;
3. $M$ is special $\mathcal{M}$-pure in any injective left $R$-module containing it;
4. $M$ is special $\mathcal{M}$-pure in $E(M)$.

Proof. (1) $\implies$ (2) Let $U \in \mathcal{M}_n$. Then, there exists the following exact sequence

$$F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to U \to 0,$$

where each $F_i$ is finitely generated free left $R$-module. If $K_{n-1} = \text{Im}(F_{n-1} \to F_{n-2})$ is special $\mathcal{M}$-presented, then we have $\text{Ext}^1_R(K_{n-1}, M) \cong \text{Ext}^n_R(U, M) = 0$, since $M$ is $n$-$\mathcal{M}$-injective. Hence (2) follows.

(2) $\implies$ (3) There exists the short exact sequence $0 \to M \to E \to \frac{E}{M} \to 0$, where $E$ is an injective $R$-module containing $M$. So by (2), $M$ is special $\mathcal{M}$-pure in $E$.

(3) $\implies$ (4) is trivial.

(4) $\implies$ (1) Assume that $U \in \mathcal{M}_n$ and $K_{n-1}$ is special $\mathcal{M}$-presented. By (4), the short exact sequence $0 \to M \to E(M) \to \frac{E(M)}{M} \to 0$ is special $\mathcal{M}$-pure. Therefore $\text{Ext}^1_R(K_{n-1}, M) = 0$, and so from $\text{Ext}^1_R(K_{n-1}, M) \cong \text{Ext}^n_R(U, M)$ we get that $M$ is $n$-$\mathcal{M}$-injective.

The following lemma is a generalization of [18, Exercise 40].

Lemma 3.3. Let $\mathcal{M}$ be a class of left $R$-modules. Then the following statements are equivalent:

1. The exact sequence $0 \to A \to B \to C \to 0$ of $R$-modules is special $\mathcal{M}$-pure;
2. The sequence $0 \to \text{Hom}_R(K_{n-1}, A) \to \text{Hom}_R(K_{n-1}, B) \to \text{Hom}_R(K_{n-1}, C) \to 0$, is exact for every special $\mathcal{M}$-presented $K_{n-1}$;
3. The short sequence $0 \to C^* \to B^* \to A^* \to 0$ is split special exact sequence.

Proposition 3.4. Let $\mathcal{M}$ be a class of left $R$-modules. Then:

1. Every special $\mathcal{M}$-pure submodule of an $n$-$\mathcal{M}$-flat right $R$-module is $n$-$\mathcal{M}$-flat.
Every special $\mathcal{X}$-pure submodule of a left $R$-module is $n$-$\mathcal{X}$-injective.

Proof. (1) Let $A$ be a special $\mathcal{X}$-pure submodule of an $n$-$\mathcal{X}$-flat right $R$-module $B$. Then, by Lemma 3.3, the sequence $0 \to (\frac{B}{\mathcal{X}})^* \to B^* \to A^* \to 0$ is split special exact sequence. By [2, Lemma 2.8], $B^*$ is $n$-$\mathcal{X}$-injective. Then from [2, lemma 2.7], and Lemma 3.3 we deduce that $A$ is $n$-$\mathcal{X}$-flat.

(2) Let $A$ be a special $\mathcal{X}$-pure submodule of a left $R$-module $B$. Then, the exact sequence $0 \to A \to B \to \frac{B}{\mathcal{X}} \to 0$ is special $\mathcal{X}$-pure. So, by Proposition 3.2, $A$ is $n$-$\mathcal{X}$-injective.

Remark 3.5. (1) Every flat right $R$-module is $n$-$\mathcal{X}$-flat.

(2) Every injective left (resp., right) $R$-module is $n$-$\mathcal{X}$-injective.

(3) If $U \in \mathcal{X}_m$, then $U \in \mathcal{X}_n$ for any $m \geq n$.

Theorem 3.6. Let $\mathcal{X}$ be a class of left $R$-modules and $R$ a left $n$-$\mathcal{X}$-coherent ring. Then the following statements are equivalent:

(1) $RR$ is $n$-$\mathcal{X}$-injective;

(2) For any left $R$-module, there is an epimorphism $\mathcal{X} \mathcal{X}$-cover;

(3) For any right $R$-module, there is a monomorphic $\mathcal{X} \mathcal{X}$-preenvelope;

(4) Every injective right $R$-module is $n$-$\mathcal{X}$-flat;

(5) Every 1-$\mathcal{X}$-injective right $R$-module is $n$-$\mathcal{X}$-flat.

(6) Every $n$-$\mathcal{X}$-injective right $R$-module is $n$-$\mathcal{X}$-flat.

(7) Every flat left $R$-module is $n$-$\mathcal{X}$-injective.

Proof. (1) $\implies$ (3) By [2, Theorem 2.16], every right $R$-module $N$ has an $n$-$\mathcal{X}$-flat preenvelope $f : N \to F$. By [2, Theorem 2.13], $R^*$ is $n$-$\mathcal{X}$-flat, and so $\prod R^*$ is $n$-$\mathcal{X}$-flat by [2, Theorem 2.6]. On the other hand, $R^*$ is a cogenerator. Therefore, exact sequence $0 \to N \to \prod R^*$ exists, and hence homomorphism $0 \to F \to \prod R^*$ such that $hf = g$, implies that $f$ is monic.

(3) $\implies$ (4) Let $E$ be an injective right $R$-module. Then by (2), homomorphism $f : E \to F$ is a monic $n$-$\mathcal{X}$-flat preenvelope of $E$. So, the split exact sequence $0 \to E \to F \to \frac{F}{E} \to 0$ exists and implies that $E$ is $n$-$\mathcal{X}$-flat.
The proof is similar to the one of (3) $\Rightarrow$ (4).

(4) $\Rightarrow$ (6) Let $N$ be an $n\mathcal{X}$-injective right $R$-module. Then by Proposition 3.2 the exact sequence $0 \to N \to E(N) \to \frac{E(N)}{N} \to 0$ is special $\mathcal{X}$-pure. Since by (3), $E(N)$ is $n\mathcal{X}$-flat, then from Proposition 3.4 we deduce that $N$ is $n\mathcal{X}$-flat.

(5) $\Rightarrow$ (4) is clear by Remark 3.5.

(4) $\Rightarrow$ (1) By (4), $R^*$ is $n\mathcal{X}$-flat, since $R^*$ is injective. So, $R$ is $n\mathcal{X}$-injective by [2, Theorem 2.13].

(6 $\Rightarrow$ 7) Let $F$ be a flat left $R$-module, then $F^*$ is injective, so $F^*$ is $n\mathcal{X}$-flat by (6), and hence $F$ is $n\mathcal{X}$-injective.

(7 $\Rightarrow$ 2) For any $R$-module $M$, there is an $\mathcal{X}, \mathcal{X}_R$-cover $f : C \to M$. Note that $RR$ is $n\mathcal{X}$-injective, so $f$ is an epimorphism.

(2 $\Rightarrow$ 1) By hypothesis, $R$ has an epimorphism $\mathcal{X}, \mathcal{X}_R$-cover $f : D \to R$, then we have an exact sequence $0 \to Ker f \to D \to R \to 0$ with $D$ is $n\mathcal{X}$-injective. Since $R$ is projective, the sequence is split, then $RR$ is $n\mathcal{X}$-injective as an $R$-module.

\begin{proposition}
Let $\mathcal{X}$ be a class of left $R$-modules and $R$ a left $n\mathcal{X}$-coherent ring. If $\{A_i\}_{i \in I}$ is family of $R$-modules, then $\bigoplus_{i \in I} A_i$ is $n\mathcal{X}$-injective if and only if every $A_i$ is $n\mathcal{X}$-injective.
\end{proposition}

\begin{proof}
Assume that $U \in \mathcal{X}_n$. So, there exists a special exact sequence $0 \to K_n \to F_n \to K_{n-1} \to 0$ of $\mathcal{X}_n$. Since $R$ is $n\mathcal{X}$-coherent, we conclude that $U \in \mathcal{X}_{n+1}$ and $K_n$ is special $\mathcal{X}$-presented. So, if $\{A_i\}_{i \in I}$ is a family of $n\mathcal{X}$-injective left $R$-modules, we have that

$$\operatorname{Hom}(K_n, \bigoplus_{i \in I} A_i) \cong \bigoplus_{i \in I} \operatorname{Hom}(K_n, A_i);$$

one easily gets that

$$\operatorname{Ext}^n_R(U, \bigoplus_{i \in I} A_i) \cong \bigoplus_{i \in I} \operatorname{Ext}^1_R(K_n, A_i) \cong \bigoplus_{i \in I} \operatorname{Ext}^1_R(K_n, A_i) \cong \bigoplus_{i \in I} \operatorname{Ext}^n_R(U, A_i).$$

\end{proof}

\section{Gorenstein $n\mathcal{X}$-injective and $n\mathcal{X}$-flat modules}

Here, we start with the following definition of Gorenstein $n\mathcal{X}$-injective and Gorenstein $n\mathcal{X}$-flat modules. Then by using of results above, some characterizations of them are given.
Definition 4.1. Let $R$ be a ring and $\mathcal{X}$ a class of left $R$-modules. Then

(1) $G$ is called Gorenstein $n$-$\mathcal{X}$-injective left $R$-module if there exists the following exact sequence of $n$-$\mathcal{X}$-injective left $R$-modules:

$$A = \cdots \rightarrow A_1 \rightarrow A_0 \rightarrow A^0 \rightarrow A^1 \rightarrow \cdots$$

with $G = \ker(A^0 \rightarrow A^1)$ such that $\text{Hom}_R(K_{n-1}, A)$ leaves this sequence exact whenever $K_{n-1}$ is special $\mathcal{X}_n$-presented with $\text{pd}_R(K_{n-1}) < \infty$.

(1) $G$ is called Gorenstein $n$-$\mathcal{X}$-flat right $R$-module if there exists the following exact sequence of $n$-$\mathcal{X}$-flat right $R$-modules:

$$F = \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$$

with $G = \ker(F^0 \rightarrow F^1)$ such that $K_{n-1} \otimes_R F$ leaves this sequence exact whenever $K_{n-1}$ is special $\mathcal{X}_n$-presented with $\text{fd}_R(K_{n-1}) < \infty$.

For example, if $\mathcal{X}$ is a class of all cyclic $R$-modules, then every Gorenstein $1$-$\mathcal{X}$-injective left $R$-module is Gorenstein $FP$-injective, and every Gorenstein $1$-$\mathcal{X}$-flat right $R$-module is Gorenstein flat, see [2, 10].

Remark 4.2. (1) Every $n$-$\mathcal{X}$-flat right $R$-module is Gorenstein $n$-$\mathcal{X}$-flat.

(2) Every $n$-$\mathcal{X}$-injective left $R$-module is Gorenstein $n$-$\mathcal{X}$-injective.

(3) In definition, one easily gets that each $\ker(A_i \rightarrow A_{i-1})$, $\ker(A^i \rightarrow A^{i+1})$ and $\ker(F_i \rightarrow F_{i-1})$, $K^i = \ker(F^i \rightarrow F^{i+1})$ are Gorenstein $n$-$\mathcal{X}$-injective and Gorenstein $n$-$\mathcal{X}$-flat, respectively.

Lemma 4.3. Let $\mathcal{X}$ be a class of left $R$-modules and $R$ a left $n$-$\mathcal{X}$-coherent ring. If $K_{n-1}$ is a special $\mathcal{X}$-presented with $\text{fd}_R(K_{n-1}) < \infty$, then $\text{pd}_R(K_{n-1}) < \infty$.

Proof. If $\text{fd}_R(K_{n-1}) = m < \infty$, then there exists $U \in \mathcal{X}_n$ such that $\text{fd}_R(U) \leq n + m$. We show that $\text{pd}_R(U) \leq n + m$. Since $R$ is an $n$-$\mathcal{X}$-coherent, the projective resolution $\cdots \rightarrow F_{n+1} \rightarrow F_n \rightarrow \cdots \rightarrow F_0 \rightarrow U \rightarrow 0$, where any $F_i$ is finitely generated free, exists. On the other hand, above exact sequence is a flat resolution. So By [16 Proposition 8.17], $(n + m - 1)$-syzygy is flat. Hence, the exact sequence $0 \rightarrow K_{n+m-1} \rightarrow F_{n+m-1} \rightarrow \cdots \rightarrow F_0 \rightarrow U \rightarrow 0$ is a flat resolution.
Now, a simple observation shows that if \( n \geq m \) or \( n < m \), \( K_{n+m-1} \) is finitely presented and consequently by [16 Theorem 3.56], \( K_{n+m-1} \) is projective and so, \( \text{pd}_R(U) \leq n + m \) if and only if \( \text{pd}_R(K_{n-1}) \leq m \).

In the following theorem, we show that in the case of left \( n\mathcal{X} \)-coherent rings, Gorenstein \( n\mathcal{X} \)-flat and Gorenstein \( n\mathcal{X} \)-injective are determined via only the existence of the corresponding exact complexes.

**Theorem 4.4.** Let \( \mathcal{X} \) be a class of left \( R \)-modules and \( R \) a left \( n\mathcal{X} \)-coherent ring. Then

1. \( G \) is a Gorenstein \( n\mathcal{X} \)-flat right \( R \)-module if and only if there is an exact sequence

\[
F = \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots
\]

of \( n\mathcal{X} \)-flat right \( R \)-modules such that \( G = \ker(F^0 \rightarrow F^1) \).

2. \( G \) is a Gorenstein \( n\mathcal{X} \)-injective left \( R \)-module if and only if there is an exact sequence

\[
A = \cdots \rightarrow A_1 \rightarrow A_0 \rightarrow A^0 \rightarrow A^1 \rightarrow \cdots
\]

of \( n\mathcal{X} \)-injective left \( R \)-modules such that \( G = \ker(A^0 \rightarrow A^1) \).

**Proof.** (1) \((\implies)\) follows by definition.

\((\Longleftarrow)\) By definition, it suffices to show that \( K_{n-1} \otimes_R F \) is exact for every special \( \mathcal{X} \)-presented \( K_{n-1} \) with \( \text{fd}_R(K_{n-1}) < \infty \). By Lemma 4.3, \( \text{pd}_R(K_{n-1}) < \infty \). Let \( \text{pd}_R(K_{n-1}) = m \). Then we use the induction on \( m \). The case \( m = 0 \) is clear. Assume that \( m \geq 1 \). There exists a special exact sequence \( 0 \rightarrow K_n \rightarrow P_n \rightarrow K_{n-1} \rightarrow 0 \) of \( U \in \mathcal{X}_n \), where \( P_n \) is projective. Now, from the \( n\mathcal{X} \)-coherence of \( R \), we deduce that \( K_n \) is special \( \mathcal{X} \)-presented. Also, \( \text{pd}_R(K_n) \leq m - 1 \). So, the following short exact sequence of complexes exists:

\[
\cdots \rightarrow \text{ker}(f) \rightarrow K_n \rightarrow \text{coker}(f) \rightarrow \cdots
\]
By induction, $P_n \otimes_R F$ and $K_n \otimes_R F$ are exact, hence $K_{n-1} \otimes_R F$ is exact by [16, Theorem 6.10].

(2) ($\Rightarrow$) This is a direct consequence of the definition.

($\Leftarrow$) Let $K_{n-1}$ be a special $X$-presented with $\text{pd}_R(K_{n-1}) < \infty$. Then, similar proof to that of (1), $\text{Hom}_R(K_{n-1}, A)$ is exact and hence $G$ is Gorenstein $n$-$X$-injective.

**Corollary 4.5.** Let $\mathcal{X}$ be a class of left $R$-modules and $R$ a left $n$-$\mathcal{X}$-coherent ring. Then, for any left $R$-module $G$, the following assertions are equivalent:

1. $G$ is Gorenstein $n$-$\mathcal{X}$-injective;

2. There is an exact sequence $\cdots \to A_1 \to A_0 \to G \to 0$ of left $R$-modules, where every $A_i$ is $n$-$\mathcal{X}$-injective;

3. There is a short exact sequence $0 \to L \to M \to G \to 0$ of left $R$-modules, where $M$ is $n$-$\mathcal{X}$-injective and $L$ is Gorenstein $n$-$\mathcal{X}$-injective.

**Proof.** (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) follow from the definition.

(2) $\Rightarrow$ (1) For any module $G$, there is an exact sequence

$$0 \to G \to I^0 \to I^1 \to \cdots$$
where every $I^i$ is injective for any $i \geq 0$. By Remark 3.5 each $I^i$ is $n$-$\mathcal{X}$-injective. So, the exact sequence

$$\cdots \rightarrow A_1 \rightarrow A_0 \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots$$

of $n$-$\mathcal{X}$-injective left modules exists, where $G = \ker(I^0 \rightarrow I^1)$. Therefore, $G$ is Gorenstein $n$-$\mathcal{X}$-injective, by Theorem 4.4.

(3) $\implies$ (2) Assume that the exact sequence

$$0 \rightarrow L \rightarrow M \rightarrow G \rightarrow 0 \quad (1)$$

exists, where $M$ is $n$-$\mathcal{X}$-injective and $L$ is Gorenstein $n$-$\mathcal{X}$-injective. Since $L$ is Gorenstein $n$-$\mathcal{X}$-injective, there is an exact sequence

$$\cdots \rightarrow A'_2 \rightarrow A'_1 \rightarrow A'_0 \rightarrow L \rightarrow 0 \quad (2)$$

where every $A'_i$ is $n$-$\mathcal{X}$-injective. Assembling the sequences (1) and (2), we get the exact sequence

$$\cdots \rightarrow A'_2 \rightarrow A'_1 \rightarrow A'_0 \rightarrow M \rightarrow G \rightarrow 0,$$

where $M$ and $A'_i$ are $n$-$\mathcal{X}$-injective, as desired.

\[\blacksquare\]

**Corollary 4.6.** Let $\mathcal{X}$ be a class of left $R$-modules and $R$ a left $n$-$\mathcal{X}$-coherent ring. Then for any right $R$-module $G$, following assertions are equivalent:

1. $G$ is Gorenstein $n$-$\mathcal{X}$-flat;

2. There is an exact sequence $0 \rightarrow G \rightarrow B^0 \rightarrow B^1 \rightarrow \cdots$ of right $R$-modules, where every $B^i$ is $n$-$\mathcal{X}$-flat;

3. There is a short exact sequence $0 \rightarrow G \rightarrow M \rightarrow L \rightarrow 0$ of right $R$-modules, where $M$ is $n$-$\mathcal{X}$-flat and $L$ is Gorenstein $n$-$\mathcal{X}$-flat.

**Proof.** (1) $\implies$ (2) and (1) $\implies$ (3) follow from definition.

(2) $\implies$ (1) For any right $R$-module $G$, there is an exact sequence

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow G \rightarrow 0,$$

where any $P_i$ is flat for any $i \geq 0$. By Remark 3.5, every $P_i$ is $n$-$\mathcal{X}$-flat. Thus, the exact sequence

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow B^0 \rightarrow B^1 \rightarrow \cdots$$

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of \( n-\mathcal{X} \)-flat right modules exists, where \( G = \ker(B^0 \to B^1) \). Therefore by Theorem 4.4, \( G \) is Gorenstein \( n-\mathcal{X} \)-flat.

(3) \( \implies \) (2) Assume that the exact sequence

\[
0 \to G \to M \to L \to 0 \quad (1)
\]

exists, where \( M \) is \( n-\mathcal{X} \)-flat and \( L \) is Gorenstein \( n-\mathcal{X} \)-flat. Since \( L \) is Gorenstein \( n-\mathcal{X} \)-flat, there is an exact sequence

\[
0 \to L \to (F^0)′ \to (F^1)′ \to (F^2)′ \to \cdots \quad (2)
\]

where every \((F^i)′\) is \( n-\mathcal{X} \)-flat. Assembling the sequences (1) and (2), we get the exact sequence

\[
0 \to G \to M \to (F^0)′ \to (F^1)′ \to (F^2)′ \to \cdots,
\]

where \( M \) and any \((F^i)′\) are \( n-\mathcal{X} \)-flat, as desired.

\[\square\]

Proposition 4.7. Let \( \mathcal{X} \) be a class of left \( R \)-modules. Then

(1) Every direct product of Gorenstein \( n-\mathcal{X} \)-injective left \( R \)-modules is a Gorenstein \( n-\mathcal{X} \)-injective \( R \)-module.

(2) Every direct sum of Gorenstein \( n-\mathcal{X} \)-flat right \( R \)-modules is a Gorenstein \( n-\mathcal{X} \)-flat \( R \)-module.

Proof. (1) Let \( U \in \mathcal{X}_n \) and let \( \{A_i\}_{i \in I} \) be a family of \( n-\mathcal{X} \)-injective left \( R \)-modules. Then by [2, Lemma 2.7], \( \prod A_i \) is \( n-\mathcal{X} \)-injective. So, if \( \{G_i\}_{i \in I} \) is a family of Gorenstein \( n-\mathcal{X} \)-injective left \( R \)-modules, then the following \( n-\mathcal{X} \)-injective complex

\[
A_i = \cdots \to (A_i)_1 \to (A_i)_0 \to (A_i)^0 \to (A_i)^1 \to \cdots,
\]

where \( G_i = \ker((A_i)^0 \to (A_i)^1) \), induces the following exact sequence of \( n-\mathcal{X} \)-injective \( R \)-modules:

\[
\prod_{i \in I} A_i = \cdots \to \prod_{i \in I} (A_i)_1 \to \prod_{i \in I} (A_i)_0 \to \prod_{i \in I} (A_i)^0 \to \prod_{i \in I} (A_i)^1 \to \cdots,
\]

where \( \prod_{i \in I} G_i = \ker(\prod_{i \in I} (A_i)^0 \to \prod_{i \in I} (A_i)^1) \). If \( K_{n-1} \) is special \( \mathcal{X} \)-presented, then

\[
\text{Hom}_R(K_{n-1}, \prod_{i \in I} A_i) \cong \prod_{i \in I} \text{Hom}_R(K_{n-1}, A_i).
\]
By hypothesis, \(\text{Hom}_R(K_{n-1}, A_1)\) is exact, and consequently \(\prod_{i \in I} G_i\) is Gorenstein \(n-\mathcal{X}\)-injective.

(2) Let \(U \in \mathcal{X}_n\) and let \(\{I_i\}_{i \in J}\) be a family of \(n-\mathcal{X}\)-flat right \(R\)-modules. Then by \cite[Lemma 2.7]{2}, \(\bigoplus_{i \in J} I_i\) is \(n-\mathcal{X}\)-flat. So, if \(\{G_i\}_{i \in J}\) is a family of Gorenstein \(n-\mathcal{X}\)-flat right \(R\)-modules, then the following \(n-\mathcal{X}\)-flat complex

\[
I_1 = \cdots \rightarrow (I_i)_1 \rightarrow (I_i)_0 \rightarrow (I_i)^0 \rightarrow (I_i)^1 \rightarrow \cdots,
\]

where \(G_i = \ker((I_i)^0 \rightarrow (I_i)^1)\), induces the following exact sequence of \(n-\mathcal{X}\)-flat right \(R\)-modules:

\[
\bigoplus_{i \in J} I_i = \cdots \rightarrow \bigoplus_{i \in J} (I_i)_1 \rightarrow \bigoplus_{i \in J} (I_i)_0 \rightarrow \bigoplus_{i \in J} (I_i)^0 \rightarrow \bigoplus_{i \in J} (I_i)^1 \rightarrow \cdots,
\]

where \(\bigoplus_{i \in J} G_i = \ker((\bigoplus_{i \in J} I_i)^0 \rightarrow (\bigoplus_{i \in J} I_i)^1)\). If \(K_{n-1}\) is special \(\mathcal{X}\)-presented, then

\[
(K_{n-1} \otimes_R \bigoplus_{i \in J} I_i) \cong \bigoplus_{i \in J} (K_{n-1} \otimes_R I_i).
\]

By hypothesis, \((K_{n-1} \otimes_R I_i)\) is exact, and consequently \(\bigoplus_{i \in J} G_i\) is Gorenstein \(n-\mathcal{X}\)-flat.

Next, we study the Gorenstein \(n-\mathcal{X}\)-injectivity and Gorenstein \(n-\mathcal{X}\)-flatness of modules in short exact sequences.

**Proposition 4.8.** Let \(\mathcal{X}\) be a class of left \(R\)-modules and \(R\) a left \(n-\mathcal{X}\)-coherent ring. Then

1. Let \(0 \rightarrow A \rightarrow G \rightarrow N \rightarrow 0\) be an exact sequence of left \(R\)-modules. If \(A\) and \(N\) are Gorenstein \(n-\mathcal{X}\)-injective, then \(G\) is Gorenstein \(n-\mathcal{X}\)-injective.

2. Let \(0 \rightarrow K \rightarrow G \rightarrow B \rightarrow 0\) be an exact sequence of right \(R\)-modules. If \(K\) and \(B\) are Gorenstein \(n-\mathcal{X}\)-flat, then \(G\) is Gorenstein \(n-\mathcal{X}\)-flat.

**Proof.** (1) Since \(A\) and \(N\) are Gorenstein \(n-\mathcal{X}\)-injective, by Corollary \[4.3\] there exists an exact sequences \(0 \rightarrow K \rightarrow A_0 \rightarrow A \rightarrow 0\) and \(0 \rightarrow L \rightarrow N_0 \rightarrow N \rightarrow 0\) of left \(R\)-modules, where \(A_0\) and \(N_0\) are \(n-\mathcal{X}\)-injective and also, \(K\) and \(L\) are Gorenstein \(n-\mathcal{X}\)-injective. Now, we consider the following commutative diagram:
The exactness of the middle horizontal sequence with $A_0$ and $N_0$ are $n$-$\mathcal{X}$-injective, implies that $A_0 \oplus N_0$ is $n$-$\mathcal{X}$-injective by [2, Lemma 2.7]. Also, $K \oplus L$ is Gorenstein $n$-$\mathcal{X}$-injective by Proposition 4.7(1). Hence from the middle vertical sequence and Corollary 4.6, we deduce that $G$ is Gorenstein $n$-$\mathcal{X}$-injective.

(2) Since $K$ and $B$ are Gorenstein $n$-$\mathcal{X}$-flat, by Corollary 4.6 there exists an exact sequences

\[
0 \rightarrow K \rightarrow K_0 \rightarrow L_1 \rightarrow 0 \quad \text{and} \quad 0 \rightarrow B \rightarrow B_0 \rightarrow L'_1 \rightarrow 0
\]

of $R$-modules, where $K_0$ and $B_0$ are $n$-$\mathcal{X}$-flat and also, $L_1$ and $L'_1$ are Gorenstein $n$-$\mathcal{X}$-flat. Now, we consider the following commutative diagram:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & K & G & B & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & K_0 & K_0 \oplus B_0 & B_0 & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & L_1 & L_1 \oplus L'_1 & L'_1 & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0
\end{array}
\]

The exactness of the middle horizontal sequence with $K_0$ and $B_0$ are $n$-$\mathcal{X}$-flat, implies that $K_0 \oplus B_0$ is $n$-$\mathcal{X}$-flat by [2, Lemma 2.7]. Also, $L_1 \oplus L'_1$ is Gorenstein $n$-$\mathcal{X}$-flat by Proposition 4.7(2). Hence from the middle vertical sequence and Corollary 4.6 we deduce that $G$ is Gorenstein $n$-$\mathcal{X}$-flat.
The left $n$-$\mathcal{X}$-injective dimension of a left $R$-module $M$ denoted by $\text{id}_{\mathcal{X}_n}(M)$ is defined to be the least non-negative integer $m$ such that $\text{Ext}^{n+m+1}_R(U, M) = 0$ for any $U \in \mathcal{X}_n$. The left $n$-$\mathcal{X}$-flat dimension of a right $R$-module $M$ denoted by $\text{fd}_{\mathcal{X}_n}(M)$ is defined to be the least non-negative integer $m$ such that $\text{Tor}^{R}_{n+m+1}(U, M) = 0$ for any $U \in \mathcal{X}_n$. If $G$ is Gorenstein $n$-$\mathcal{X}$-injective, then $\text{id}_{\mathcal{X}_n}(G) = m$ if there is an exact sequence

$$0 \to A_m \to \cdots \to A_1 \to A_0 \to G \to 0$$

or the exact sequence

$$0 \to G \to A^0 \to A^1 \to \cdots \to A^m \to 0$$

of $n$-$\mathcal{X}$-injective left $R$-modules. Similarly, if $G$ is Gorenstein $n$-$\mathcal{X}$-flat and $\text{fd}_{\mathcal{X}_n}(G) = m$, then the above exact sequences for $n$-$\mathcal{X}$-flat right $R$-modules exists.

The following theorems are generalization of Corollaries 4.5 and 4.6 and Proposition 4.8.

**Theorem 4.9.** Let $\mathcal{X}$ be a class of left $R$-modules that is closed under kernels of epimorphisms and $R$ a left $n$-$\mathcal{X}$-coherent ring. Then, for every left $R$-module $G$, the following statements are equivalent:

1. $G$ is Gorenstein $n$-$\mathcal{X}$-injective;
2. There exists the following $n$-$\mathcal{X}$-injective resolution of $G$:

$$\cdots \to A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} G \to 0,$$

such that $\bigoplus_{i=0}^{\infty} \text{Im}(f_i)$ is Gorenstein $n$-$\mathcal{X}$-injective;
3. There exists an exact sequence

$$\cdots \to A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} G \to 0$$

of left $R$-modules, where $A_i$ has finite $n$-$\mathcal{X}$-injective dimension for any $i \geq 0$, such that $\bigoplus_{i=0}^{\infty} \text{Im}(f_i)$ is Gorenstein $n$-$\mathcal{X}$-injective.

**Proof.** (1) $\implies$ (2) By Corollary 4.5, the exact sequence

$$\cdots \to A_2 \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} G \to 0,$$
where every $A_i$ is $n$-$\mathcal{X}$-injective, exists. Consider the following exact sequences:

\[ \cdots \to A_2 \to A_1 \to A_0 \to \text{Im}(f_0) \to 0, \]
\[ \cdots \to A_3 \to A_2 \to A_1 \to \text{Im}(f_1) \to 0, \]
\[ \vdots \quad \vdots \quad \vdots \quad \vdots \]

By Proposition 3.7, $\bigoplus_{i \in I} A_i$ is $n$-$\mathcal{X}$-injective. Thus, there exists an exact sequence

\[ \cdots \to \bigoplus_{i \geq 2} A_i \to \bigoplus_{i \geq 1} A_i \to \bigoplus_{i \geq 0} A_i \to \bigoplus_{i = 0}^\infty \text{Im}(f_i) \to 0 \]

of $n$-$\mathcal{X}$-injective left $R$-modules. Consequently, Corollary 4.5 implies that $\bigoplus_{i = 0}^\infty \text{Im}(f_i)$ is Gorenstein $n$-$\mathcal{X}$-injective.

(2) $\implies$ (3) trivial.

(3) $\implies$ (1) Let the exact sequence

\[ 0 \to \bigoplus_{i \geq 0} A_i \to \bigoplus_{i \geq 1} A_i \to \bigoplus_{i \geq 2} A_i \to \cdots \to A_0 \to f_0\cdot A \to G \to 0 \]

of left $R$-modules, where $A_i$ has finite $n$-$\mathcal{X}$-injective dimension, exists. By Corollary 4.5, it is sufficient to prove that $A_i$ is $n$-$\mathcal{X}$-injective for any $i \geq 0$. Consider, the short exact sequence

\[ 0 \to \text{Im}(f_{i+1}) \to A_i \to \text{Im}(f_i) \to 0 \]

for any $i \geq 0$. Therefore, the short exact sequence

\[ 0 \to \bigoplus_{i = 0}^\infty \text{Im}(f_{i+1}) \to \bigoplus_{i = 0}^\infty A_i \to \bigoplus_{i = 0}^\infty \text{Im}(f_i) \to 0 \]

exists. By (3) and Proposition 4.8(1), $\bigoplus_{i = 0}^\infty A_i$ is Gorenstein $n$-$\mathcal{X}$-injective. Also, $\bigoplus_{i = 0}^\infty A_i$ has finite $n$-$\mathcal{X}$-injective dimension. If $\text{id}_{\mathcal{X}_n}(\bigoplus_{i = 0}^\infty A_i) = k$, then there exists an $n$-$\mathcal{X}$-injective resolution of $\bigoplus_{i = 0}^\infty A_i$:

\[ 0 \to B_k \to B_{k-1} \to \cdots \to B_0 \to \bigoplus_{i \in I} A_i \to 0. \]

Let $L_{k-1} = \ker(B_{k-1} \to B_{k-2})$ and $U \in \mathcal{X}_n$. Then, the exact sequence $0 \to B_k \to B_{k-1} \to L_{k-1} \to 0$ induces the following exact sequence:

\[ 0 = \text{Ext}_R^n(U, B_{k-1}) \to \text{Ext}_R^n(U, L_{k-1}) \to \text{Ext}_R^{n+1}(U, B_k) \to \cdots . \]

By hypothesis, $B_k$ is $(n + 1)$-$\mathcal{X}$-injective, and also $U \in \mathcal{X}_{n+1}$, since $R$ is $n$-$\mathcal{X}$-coherent. So $\text{Ext}_R^{n+1}(U, B_k) = 0$, and hence $\text{Ext}_R^n(U, L_{k-1}) = 0$. Therefore $L_{k-1}$ is $n$-$\mathcal{X}$-injective. Then with the same process, we get that $\bigoplus_{i = 0}^\infty A_i$ is $n$-$\mathcal{X}$-injective, and so by Proposition 3.7, $A_i$ is $n$-$\mathcal{X}$-injective for any $i \geq 0$. 

\[ \square \]
For the following theorem, the proof is similar to that of (1) \(\Rightarrow\) (2), (2) \(\Rightarrow\) (3) and (3) \(\Rightarrow\) (1) in Theorem 4.9.

Theorem 4.10. Let \(\mathcal{X}\) be a class of left \(R\)-modules that is closed under kernels of epimorphisms and \(R\) a left \(n\)-\(\mathcal{X}\)-coherent ring. Then, for every right \(R\)-module \(G\), the following statements are equivalent:

1. \(G\) is \(G\)-Gorenstein \(n\)-\(\mathcal{X}\)-flat;

2. There exists the following right \(n\)-\(\mathcal{X}\)-flat resolution of \(G\):

\[
0 \to G \overset{f_0}{\to} I^0 \overset{f_1}{\to} I^1 \overset{f_2}{\to} \cdots,
\]

such that \(\bigoplus_{i=0}^{\infty} \text{Im}(f^i)\) is \(G\)-Gorenstein \(n\)-\(\mathcal{X}\)-flat;

3. There exists an exact sequence

\[
0 \to G \overset{f_0}{\to} I^0 \overset{f_1}{\to} I^1 \overset{f_2}{\to} \cdots,
\]

of right \(R\)-modules, where \(I_i\) has finite \(n\)-\(\mathcal{X}\)-flat dimension for any \(i \geq 0\), such that \(\bigoplus_{i=0}^{\infty} \text{Im}(f^i)\) is \(G\)-Gorenstein \(n\)-\(\mathcal{X}\)-flat.

5 \(\mathcal{X}\)-\(FC\)-rings

A ring \(R\) is called self left \(n\)-\(\mathcal{X}\)-injective if \(R\) is an \(n\)-\(\mathcal{X}\)-injective left \(R\)-module. A ring \(R\) is called left \(\mathcal{X}\)-\(FC\)-ring if \(R\) is self left \(n\)-\(\mathcal{X}\)-injective and left \(n\)-\(\mathcal{X}\)-coherent. In this section, we investigate properties of Gorenstein \(n\)-\(\mathcal{X}\)-injective and \(n\)-\(\mathcal{X}\)-flat modules over \(\mathcal{X}\)-\(FC\)-rings generalizing several classical results. Notice that the notion of \(\mathcal{X}\)-\(FC\)-ring generalizes the classical notions of quasi-Frobenius and \(FC\) (i.e., \(IF\)) rings.

It is well-known that quasi-Frobenius (resp., \(FC\)) rings can be seen as rings over which every modules are Gorenstein injective (resp., \(FP\)-injective). Here we extend this fact as well as other known ones.

Proposition 5.1. Let \(\mathcal{X}\) be a class of left \(R\)-modules. Then every left \(R\)-module is Gorenstein \(n\)-\(\mathcal{X}\)-injective if and only if every projective left \(R\)-module is \(n\)-\(\mathcal{X}\)-injective and for any left \(R\)-module \(N\), \(\text{Hom}_R(-, N)\) is exact with respect to all special short exact sequences of \(\mathcal{X}_n\) with modules of finite projective dimension.
Proof. \((\Rightarrow)\) Let \(M\) be a projective left \(R\)-module. Then by hypothesis, \(M\) is Gorenstein \(n\)-\(\mathcal{X}\)-injective. So, the following \(n\)-\(\mathcal{X}\)-injective resolution of \(M\) exists:
\[
\cdots \to A_1 \to A_0 \to M \to 0.
\]
Since \(M\) is projective, \(M\) is \(n\)-\(\mathcal{X}\)-injective as a direct summand of \(A_0\). Also, by hypothesis and Definition 4.1, \(\text{Hom}_R(-, N)\) is exact with respect to all special short exact sequences with modules of finite projective dimension, since every left \(R\)-module \(N\) is Gorenstein \(n\)-\(\mathcal{X}\)-injective.

\((\Leftarrow)\) Choose an injective resolution \(0 \to G \to E^0 \to E^1 \to \cdots\) of \(G\) and a projective resolution \(\cdots \to F_1 \to F_0 \to G \to 0\), where every \(F_i\) is \(n\)-\(\mathcal{X}\)-injective by hypothesis. Assembling these resolutions, we get, by Remark 3.5, the following \(n\)-\(\mathcal{X}\)-injective resolution:
\[
\mathbf{A} = \cdots \to F_1 \to F_0 \to E^0 \to E^1 \to \cdots ,
\]
where \(G = \ker(E^0 \to E^1)\), \(K^i = \ker(E^i \to E^{i+1})\) and \(K_i = \ker(F_i \to F_{i-1})\) for any \(i \geq 1\). Let \(K_{n-1}\) be a special \(\mathcal{X}\)-presented module with \(\text{pd}_R(K_{n-1}) < \infty\). Then by hypothesis, we have:
\[
\text{Ext}_R^1(K_{n-1}, G) = \text{Ext}_R^1(K_{n-1}, K_i) = \text{Ext}_R^1(K_{n-1}, K_i) = 0.
\]
So, \(\text{Hom}_R(K_{n-1}, A)\) is exact, and hence \(G\) is Gorenstein \(n\)-\(\mathcal{X}\)-injective. \(\blacksquare\)

**Proposition 5.2.** Let \(\mathcal{X}\) be a class of left \(R\)-modules. Then every right \(R\)-module is Gorenstein \(n\)-\(\mathcal{X}\)-flat if and only if every injective right \(R\)-module is \(n\)-\(\mathcal{X}\)-flat and for any \(R\)-module \(N\), \(- \otimes_R N\) is exact with respect to all special short exact sequences of \(\mathcal{X}\) with modules of finite projective dimension.

**Proof.** Similar to proof that of Proposition 5.1. \(\blacksquare\)

**Theorem 5.3.** Let \(\mathcal{X}\) be a class of left \(R\)-modules and \(R\) a left \(n\)-\(\mathcal{X}\)-coherent ring. Then, the following statements are equivalent:

1. Every left \(R\)-module is Gorenstein \(n\)-\(\mathcal{X}\)-injective;
2. Every projective left \(R\)-module is \(n\)-\(\mathcal{X}\)-injective;
3. \(R\) is self \(n\)-\(\mathcal{X}\)-injective.

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Proof. (1) \implies (2) and (2) \implies (3) hold by Proposition 5.1.

(3) \implies (1) Let \( G \) be an \( R \)-module and \( \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow G \rightarrow 0 \) be any free resolution of \( G \). Then by Proposition 3.7, each \( F_i \) is \( n \)-\( \mathcal{X} \)-injective. Hence Corollary 4.5 completes the proof. \( \blacksquare \)

Examples 5.4. Let \( R = k[x^3, x^2, x^2y, xy, y^2, y^3] \) be a ring and \( \mathcal{X} \) a class of all 1-presented left \( R \)-modules. We claim that \( R \) is not 1-\( \mathcal{X} \)-injective. Suppose to the contrary, \( R \) is 1-\( \mathcal{X} \)-injective. We have \( \frac{R}{Rx^2} \) is special \( \mathcal{X} \)-presented since \( Rx^2 \cong R \) is special \( \mathcal{X} \)-generated. Also, \( \text{pd}_R(\frac{R}{Rx^2}) < \infty \). So by Proposition 5.1 and Theorem 5.3, \( \frac{R}{Rx^2} \) is projective. Therefore, the exact sequence \( 0 \rightarrow Rx^2 \rightarrow R \rightarrow \frac{R}{Rx^2} \rightarrow 0 \) splits. Thus, \( Rx^2 \) is a direct summand of \( R \) and so, \( x^2 \) is an idempotent, a contradiction.

Let \( \mathcal{X} \) be a class of graded left (right) \( R \)-modules. Then, a graded ring \( R \) will be called \( n \)-gr-regular if and only if it is \( n \)-\( \mathcal{X} \)-regular if and only if every \( n \)-presented left \( R \)-module in \( \mathcal{X} \) is projective if and only if every left \( R \)-module in \( \mathcal{X} \) is \( n \)-\( \mathcal{X} \)-injective if and only if every right \( R \)-module in \( \mathcal{X} \) is \( n \)-\( \mathcal{X} \)-flat. This is a generalization of [19, Proposition 3.11]. Notice that, when \( n = 1 \), then \( R \) is gr-regular if and only if 1-\( \mathcal{X} \)-regular, see [18].

The following example show that, for some class \( \mathcal{X} \) of \( R \)-modules and any \( m > n \), every Gorenstein \( n \)-\( \mathcal{X} \)-injective (resp., flat) is Gorenstein \( m \)-\( \mathcal{X} \)-injective (resp., flat), since by [20, Remark 3.5], every \( n \)-\( \mathcal{X} \)-injective (resp., flat) is \( m \)-\( \mathcal{X} \)-injective (resp., flat).

Examples 5.5. (1) Let \( R \) be a graded ring and \( \mathcal{X} \) a class of graded left \( R \)-module. Then for any \( m > n \), every Gorenstein \( n \)-\( \mathcal{X} \)-injective (resp., flat) is Gorenstein \( m \)-\( \mathcal{X} \)-injective (resp., flat), since by [20, Remark 3.5], every \( n \)-\( \mathcal{X} \)-injective (resp., flat) is \( m \)-\( \mathcal{X} \)-injective (resp., flat).

(2) Let \( R = k[X] \), where \( k \) is a field, and \( \mathcal{X} \) a class of graded left \( R \)-modules. Then by Remark 4.2 every graded let (resp., right) \( R \)-module is Gorenstein 2-\( \mathcal{X} \)-injective (resp., flat), since every 2-presented graded left \( R \)-module is projective. We claim that there is a graded left (resp., right) \( R \)-module \( L \) so that \( L \) is not Gorenstein 1-\( \mathcal{X} \)-injective (resp., flat). Suppose to the contrary, every graded left (resp., right) \( R \)-module is Gorenstein 1-\( \mathcal{X} \)-injective (resp., flat). If \( U \) is finitely presented graded left module, then the special exact sequence \( 0 \rightarrow L \rightarrow F_0 \rightarrow U \rightarrow 0 \) of graded left modules exists. So by Proposition 5.1 (resp., Proposition 5.2), \( U \) is projective and it follows that \( R \) is 1-\( \mathcal{X} \)-regular or \( \mathcal{X} \)-regular, contradiction, see [20, Example 3.6].

Proposition 5.6. Let \( \mathcal{X} \) be a class of left \( R \)-modules. Then, the following statements hold:
(1) If $G$ is a Gorenstein injective left $R$-module, then $\text{Hom}_R(−, G)$ is exact with respect to all special short exact sequences with modules of finite projective dimension.

(2) If $G$ is a Gorenstein flat right $R$-module, then $− \otimes_R G$ is exact with respect to all special short exact sequences with modules of finite flat dimension.

Proof. (1) Let $0 \to K_n \to P_n \to K_{n-1} \to 0$ be a special short exact sequence of $U \in \mathcal{X}_n$. It is clear that $\text{pd}_R(U) = m < \infty$, since $\text{pd}_R(K_{n-1}) < \infty$. Also, let $G$ be Gorenstein injective. Then, the following injective resolution of $G$ exists:

$$0 \to N \to A_{m-1} \to \cdots \to A_0 \to G \to 0.$$ 

So, $\text{Ext}^{n+i}_R(U, A_j) = 0$ for every $0 \leq j \leq m-1$ and any $i \geq 0$. Thus, we deduce that $\text{Ext}^{n+i}_R(U, G) \cong \text{Ext}^{n+m+i}_R(U, N) = 0$ for any $i \geq 0$. So, $\text{Ext}^1_R(K_{n-1}, G) \cong \text{Ext}^n_R(U, G) = 0$.

(2) The proof is similar to the one above. 

Now we can state the main result of this section.

Theorem 5.7. Let $\mathcal{X}$ be a class of left $R$-modules and $R$ a left $n$-$\mathcal{X}$-coherent ring. Then the following statements are equivalent:

1. $R$ is self $n$-$\mathcal{X}$-injective;

2. Every Gorenstein $n$-$\mathcal{X}$-flat left $R$-module is Gorenstein $n$-$\mathcal{X}$-injective;

3. Every Gorenstein flat left $R$-module is Gorenstein $n$-$\mathcal{X}$-injective;

4. Every flat left $R$-module is Gorenstein $n$-$\mathcal{X}$-injective;

5. Every Gorenstein projective left $R$-module is Gorenstein $n$-$\mathcal{X}$-injective;

6. Every projective left $R$-module is Gorenstein $n$-$\mathcal{X}$-injective;

7. Every Gorenstein injective right $R$-module is Gorenstein $n$-$\mathcal{X}$-flat;

8. Every injective right $R$-module is Gorenstein $n$-$\mathcal{X}$-flat;

9. Every Gorenstein $1$-$\mathcal{X}$-injective right $R$-module is Gorenstein $n$-$\mathcal{X}$-flat;

10. Every Gorenstein $n$-$\mathcal{X}$-injective right $R$-module is Gorenstein $n$-$\mathcal{X}$-flat.
Proof. (1) $\implies$ (2), (1) $\implies$ (3), (1) $\implies$ (4), (1) $\implies$ (5) and (1) $\implies$ (6) follow immediately from Theorem 3.3.

(3) $\implies$ (4), (4) $\implies$ (6) and (5) $\implies$ (6) are trivial.

(3) $\implies$ (1) Assume that $G$ is a projective left $R$-module. Then $G$ is flat and so $G$ is Gorenstein $n$-$\mathcal{X}$-injective by (3). So, similar to the proof of (3) $\implies$ (1) of Proposition 5.1, $G$ is $n$-$\mathcal{X}$-injective. Thus, the assertion follows from Theorem 3.3.

(6) $\implies$ (1) This is similar to the proof of (3) $\implies$ (1).

(1) $\implies$ (9) By Theorem 3.6 every 1-$\mathcal{X}'$-injective right $R$-module is $n$-$\mathcal{X}'$-flat. Suppose that $G$ is Gorenstein 1-$\mathcal{X}'$-injective. So, the exact sequence

$$M = \cdots \rightarrow M_1 \rightarrow M_0 \rightarrow M^0 \rightarrow M^1 \rightarrow \cdots,$$

of $n$-$\mathcal{X}$-flat right $R$-modules exists, where $G = \ker(M^0 \rightarrow M^1)$. Let $K_{n-1}$ be special $\mathcal{X}_n$-presented with $\text{f.d.}(K_{n-1}) < \infty$. Then similar to the proof of Theorem 4.4(1), $K_{n-1} \otimes_R M$ is exact, and hence $G$ is Gorenstein $n$-$\mathcal{X}'$-flat.

(9) $\implies$ (7) By Remark 3.5 every injective right $R$-module is 1-$\mathcal{X}$-injective. So, if $G$ is Gorenstein injective, then the exact sequence

$$E = \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots$$

of 1-$\mathcal{X}'$-injective right $R$-modules exists, where $G = \ker(E^0 \rightarrow E^1)$. So, if $U \in \mathcal{X}_1$ with $\text{pd}(U) < \infty$, then $U$ is special $\mathcal{X}$-presented and, by Corollary 5.6, $\text{Hom}_R(U, E)$ is exact. Therefore, $G$ is Gorenstein 1-$\mathcal{X}'$-injective. Hence, (7) follows from (9).

(7) $\implies$ (8) is trivial, since every injective $R$-module is Gorenstein injective.

(8) $\implies$ (1) Let $M$ be an injective right $R$-module. Since $M$ is Gorenstein $n$-$\mathcal{X}'$-flat, we have a long exact sequence:

$$M = \cdots \rightarrow M_1 \rightarrow M_0 \rightarrow M^0 \rightarrow M^1 \rightarrow \cdots,$$

where any $M_i$ is $n$-$\mathcal{X}'$-flat and $M = \ker(M^0 \rightarrow M^1)$. Then, the split exact sequence $0 \rightarrow M \rightarrow M^0 \rightarrow L \rightarrow 0$ implies that $M$ is $n$-$\mathcal{X}'$-flat, and hence by Theorem 3.6 we deduce that $R$ is self $n$-$\mathcal{X}'$-injective.

(1) $\implies$ (10) Suppose that $G$ is a Gorenstein $n$-$\mathcal{X}'$-injective right $R$-module. By Theorem 3.6(6), every $n$-$\mathcal{X}'$-injective right $R$-module is $n$-$\mathcal{X}'$-flat. Thus, the exact sequence

$$N = \cdots \rightarrow N_1 \rightarrow N_0 \rightarrow N^0 \rightarrow N^1 \rightarrow \cdots$$

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of $n$-$\mathcal{X}$-flat right $R$-modules exists, where $G = \ker(N^0 \to N^1)$. Then similar to proof Theorem 4.3(1), (10) follows.

(10) $\implies$ (7) is clear.

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