A RELATION BETWEEN THE RICCI TENSOR AND THE SPECTRUM OF THE DIRAC OPERATOR

KLAUS-DIETER KIRCHBERG

Abstract. Using Weitzenb"ock techniques on any compact Riemannian spin manifold we derive a general inequality depending on a real parameter and joining the spectrum of the Dirac operator with terms depending on the Ricci tensor and its first covariant derivatives. The discussion of this inequality yields vanishing theorems for the kernel of the Dirac operator $D$ and new lower bounds for the spectrum of $D^2$ if the Ricci tensor satisfies certain conditions.

Contents

1. Introduction 1
2. The basic Weitzenb"ock formula and first applications 2
3. The endomorphisms $E$ and $T$ 7
4. Estimates for the first eigenvalue of the Dirac operator 9
References 14

1. Introduction

In 1980 Th. Friedrich proved that, on any compact Riemannian spin manifold $M^n$ of scalar curvature $R$ with $R_{\min} := \min\{R(x)|x \in M\} > 0$, every eigenvalue $\lambda$ of the Dirac operator $D$ satisfies the inequality

$$\lambda^2 \geq \frac{nR_{\min}}{4(n-1)}$$

(see [2]). In special geometric situations, there are better estimations than (1) (see [3], [4]). For example, in case of a compact K"ahler manifold $M^{2m}$ of complex dimension $m$ with positive scalar curvature $R$, we have the estimate

$$\lambda^2 \geq \begin{cases} 
\frac{m+1}{4m} R_{\min} & \text{if } m \text{ is odd} \\
\frac{m}{4(m-1)} R_{\min} & \text{if } m \text{ is even}
\end{cases}$$

(2)

Recently it was shown in [5] and [6] that the estimate (1) can also be improved for such manifolds $M^n$ whose curvature tensor or Weyl tensor, respectively, is divergence free. It is well known that the curvature tensor $K$ of $M^n$ is divergence free if and only if the covariant derivative $\nabla R$ of the Ricci tensor $R_{\min}$ has the property

$$\nabla_X R(Y) = \nabla_Y R(X),$$

where $X$ and $Y$ are arbitrary vector fields. In this paper we generalize the results of [5] in the sense that we do not make use of the condition (3) here.

Received by the editors 29th March 2022.

1991 Mathematics Subject Classification. Primary: 53 (Differential Geometry), Secondary: 53C27, 53C25.

Key words and phrases. Dirac operator, eigenvalues, Ricci tensor.

This work was supported by the SFB 288 "Differential geometry and quantum physics" of the Deutsche Forschungsgemeinschaft.
2. The basic Weitzenböck formula and first applications

Let $M^n$ be any Riemannian spin manifold of dimension $n$ with Riemannian metric $g$ and spinor bundle $S$. Then the twistor operator

$$\mathcal{D} : \Gamma(S) \to \Gamma(TM^n \otimes S)$$

is defined by $\mathcal{D}\psi := X^k \otimes D_X \psi$ and $D_X\psi := \nabla_X \psi + \frac{1}{4} X \cdot D\psi$. Here $D := X^k \cdot \nabla_{X_k}$ denotes the Dirac operator, $(X_1, \ldots, X_n)$ is any local frame of vector fields and $(X^1, \ldots, X^n)$ is the associated coframe given by $X_j := g^{jk} \cdot X_k$, where the $g^{jk}$ denote the components of the inverse of the matrix $(g_{jk})$ with $g_{jk} := g(X_j, X_k)$. The curvature tensor $K$ of $M^n$ is defined by

$$K(X, Y)(Z) := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$ 

Moreover, we have the well known relation

$$K = \nabla g.$$ 

The Bianchi identity implies

$$g\left((\nabla_X K)(X^j, X)(Y), Z\right) = g\left((\nabla_Z \Ric)(Y) - (\nabla_Y \Ric)(Z), X\right).$$

Thus we obtain

$$A_X \psi := (\nabla_X C)(X^j, X) \psi = \frac{1}{4} X^k \cdot (\nabla_{X_k} K)(X^j, X)(X_k) \cdot \psi.$$ 

The Bianchi identity implies

$$g\left((\nabla_X K)(X^j, X)(Y), Z\right) = g\left((\nabla_Z \Ric)(Y) - (\nabla_Y \Ric)(Z), X\right).$$

Thus we obtain

$$A_X \psi := \frac{1}{4} \left((\nabla_{X_k} \Ric)(X) \cdot X^k - X^k \cdot (\nabla_{X_k} \Ric)(X)\right) \cdot \psi.$$ 

Using this a simple calculation yields the identity

$$X^k \cdot A_{X_k} \psi = -\frac{1}{4} (dR) \cdot \psi,$$

where $R := \text{tr} \Ric$ is the scalar curvature. Now, we consider the family of differential operators

$$Q^t : \Gamma(S) \to \Gamma(TM^n \otimes S)$$

depending on $t \in \mathbb{R}$ and being locally defined by $Q^t \psi := X^k \otimes Q_{X_k}^t \psi$ and

$$Q_{X_k}^t \psi := D_X\psi + t \cdot \left(\Ric - \frac{R}{n}\right)(X_k) \cdot D\psi + t \cdot \left(A_X - \frac{1}{4n} X \cdot dR\right)\psi.$$ 

Our first aim is to compute the function $|Q^t \psi|^2 := (Q_{X_k}^t \psi, Q_{X_k}^t \psi)$ for any eigenspinor $\psi$ of the Dirac operator $D$. For any spinor field $\psi$, we have the general identities

$$X^k \cdot D_{X_k} \psi = 0,$$

$$X^k \cdot \left(\Ric - \frac{R}{n}\right)(X_k) \cdot \psi = 0,$$

$$X^k \cdot \left(A_{X_k} - \frac{1}{4n} X_k \cdot dR\right)\psi = 0,$$

which imply that the image of $Q^t$ is contained in the kernel of the Clifford multiplication, i.e., it holds that

$$X^k \cdot Q_{X_k}^t \psi = 0.$$ 

Moreover, we have the well known relation

$$|D\psi|^2 = |\nabla \psi|^2 - \frac{1}{n} |D\psi|^2.$$
Lemma 2.1. Let \( \lambda \) be any eigenvalue of \( D \) and let \( \psi \) be any corresponding eigenspinor \( (D\psi = \lambda \psi) \). Then, for all \( t \in \mathbb{R} \), the equation

\[
|Q^t \psi|^2 = |\nabla \psi|^2 - \frac{\lambda^2}{n} |\psi|^2 - 2t \left( \Re \langle Ric(X^k) \nabla X_k D\psi, \psi \rangle + \Re \langle A_{\chi_k} \nabla \chi_k \psi, \psi \rangle - \lambda^2 \frac{R}{n} |\psi|^2 \right) + t^2 \left( \lambda^2 |\nabla \psi|^2 - \lambda R \frac{n}{2} |\psi|^2 \right) - 2\lambda R \frac{n}{2} \Re \langle \psi, Ric(X^k) \cdot \nabla \chi_k \psi \rangle + \lambda |\psi, Ric(X^k)\rangle^2 + \lambda^2 |\nabla \psi|^2 - \lambda R \frac{n}{2} |\psi|^2 \right) = 0.
\]

(12)

is valid.

Proof. Using the identities (3) - (6) we calculate

\[
|Q^t \psi|^2 = \langle D_{\chi_k} \psi + t(Ric - \frac{R}{n})(X_k) \psi + t(A_{\chi_k} - \frac{1}{4n} X_k \cdot dR) \psi, \psi \rangle = |D\psi|^2 + 2t \Re \langle \nabla \chi_k \psi, (Ric - \frac{R}{n})(X_k) \psi \rangle + 2t \Re \langle \nabla \chi_k \psi, (A_{\chi_k} - \frac{1}{4n} X_k \cdot dR) \psi \rangle + 2t \Re \langle Ric(X_k) \cdot \psi, (A_{\chi_k} - \frac{1}{4n} X_k \cdot dR) \psi \rangle + 2t \langle A_{\chi_k} \psi, (A_{\chi_k} - \frac{1}{4n} X_k \cdot dR) \psi \rangle = |D\psi|^2 - 2t \left( \Re \langle Ric(X^k) \nabla X_k D\psi, \psi \rangle - \Re \langle \nabla \chi_k \psi, A_{\chi_k} \psi \rangle - \lambda \frac{R}{n} |\psi|^2 + \frac{1}{2n} \Re \langle \psi, dR \cdot \psi \rangle \right) + t^2 \left( \lambda^2 |\nabla \psi|^2 - \lambda R \frac{n}{2} |\psi|^2 \right) - 2\lambda R \frac{n}{2} \Re \langle \psi, Ric(X^k) \cdot \nabla \chi_k \psi \rangle - \lambda |\psi, Ric(X^k)\rangle^2 + \lambda |\nabla \psi|^2 - \lambda R \frac{n}{2} |\psi|^2 \right) = 0.
\]

Inserting (11) in the result of this calculation we obtain (12).

By Lemma 1.4 in [5] and (5), for all spinor fields \( \psi \), we have the identity

\[
\Re \langle (Ric(X^k) \cdot \nabla X_k D\psi, \psi) \rangle + \Re \langle (A_{\chi_k} \nabla \chi_k \psi, \psi) \rangle = |\nabla D\psi|^2 - \frac{1}{8} |(D^2 - \frac{R}{4}) \psi|^2 - \frac{R}{4} |\nabla \psi|^2 + \frac{1}{2} |\nabla ric|^2 |\psi|^2 + \langle \nabla ric(X^k) \psi, \nabla X_k \psi \rangle - \text{div}(X^k \psi),
\]

where \( X^k \) is the vector field defined by

\[
X^k := \Re \left( \langle (D^2 - \frac{R}{4}) \psi, \nabla X^k \psi \rangle + \langle \nabla X^k D\psi + \frac{1}{2} Ric(X_j) \cdot \psi, X^k \cdot \nabla X_j \psi \rangle \right) X_k.
\]

Inserting (11) into (12) we obtain
Theorem 2.1. Let \( M^n \) be any Riemannian spin manifold and let \( \lambda \) be any eigenvalue of the Dirac operator \( D \). Then, for all \( t \in \mathbb{R} \), any corresponding eigenspinor \( \psi \) satisfies the equation

\[
|Q^t\psi|^2 = |\nabla \psi|^2 - \frac{\lambda^2}{4} |\psi|^2 - 2t \left( |\nabla_{\text{Ric}(X_k)}\psi, \nabla X_k\psi| + \frac{1}{4} |\text{Ric}|^2 \cdot |\nabla \psi|^2 + (\lambda^2 - \frac{R}{4}) (|\nabla \psi|^2 - (\lambda^2 - \frac{R}{4}) |\psi|^2) - \text{div}(X_\psi) \right) + \frac{t^2}{2} \left( \nabla^2 |\text{Ric}|^2 - \frac{R}{4} |\psi|^2 - \lambda \langle (\text{Ric}(X_k) \cdot A_{X_k} + A_{X_k} \cdot \text{Ric}(X_k))\psi, \psi \rangle - \langle A_{X_k} \cdot A_{X_k} \psi, \psi \rangle - \frac{1}{16n} |dR|^2 |\psi|^2 \right).
\]

Equation (14) is the basic Weitzenböck formula of our paper. In case of a divergence free curvature tensor, this formula simplifies since we have \( A_{X} = 0 \) then and \( R \) is constant (compare with Theorem 1.5 in [5]).

In the following we suppose that \( M^n \) is compact. For any continuous function \( f \), we use the notation \( f_{\min}(f_{\max}) \) for the minimum (maximum) of \( f \) on \( M^n \). Furthermore, let \( \kappa \) denote the minimum of all eigenvalues of the Ricci tensor \( \text{Ric} \) on \( M^n \). Then, for any spinor field \( \psi \), we have the inequality

\[
\kappa |\nabla \psi|^2 \leq \langle \nabla_{\text{Ric}(X_k)}\psi, \nabla X_k\psi \rangle.
\]

In the following we use the Schrödinger-Lichnerowicz formula

\[
\nabla^* \nabla = D^2 - \frac{R}{4}.
\]

Lemma 2.2. Let \( \lambda \) be any eigenvalue of the Dirac operator \( D \). Then, for any corresponding eigenspinor \( \psi \), there are the inequalities

\[
\int_{M^n} (\lambda^2 - \frac{R}{4}) (|\nabla \psi|^2 - (\lambda^2 - \frac{R}{4}) |\psi|^2) \leq \frac{R_{\max} - R_{\min}}{4} (\lambda^2 - \frac{R_{\min}}{4}) \cdot \int_{M^n} |\psi|^2
\]

Proof. Using the formula (13) we find

\[
\int_{M^n} (|\nabla \psi|^2 - (\lambda^2 - \frac{R}{4}) |\psi|^2) = 0.
\]

Hence, it follows that

\[
\int_{M^n} (\lambda^2 - \frac{R}{4}) (|\nabla \psi|^2 - (\lambda^2 - \frac{R}{4}) |\psi|^2) = -\frac{1}{4} \int_{M^n} R \left( |\nabla \psi|^2 - (\lambda^2 - \frac{R}{4}) |\psi|^2 \right).
\]

Further, we have

\[
\int_{M^n} R \left( |\nabla \psi|^2 - (\lambda^2 - \frac{R}{4}) |\psi|^2 \right) \leq \int_{M^n} R \left( |\nabla \psi|^2 - (\lambda^2 - \frac{R}{4}) |\psi|^2 \right) \leq \int_{M^n} R \left( |\nabla \psi|^2 - (\lambda^2 - \frac{R}{4}) |\psi|^2 \right) \leq \int_{M^n} (R_{\max} - R) (\lambda^2 - \frac{R}{4}) |\psi|^2 \leq \int_{M^n} (R_{\max} - R) (\lambda^2 - \frac{R_{\min}}{4}) |\psi|^2 \leq \int_{M^n} (R_{\max} - R_{\min}) (\lambda^2 - \frac{R_{\min}}{4}) |\psi|^2 = (R_{\max} - R_{\min}) (\lambda^2 - \frac{R_{\min}}{4}) \cdot \int_{M^n} |\psi|^2.
\]
This yields

\[-\frac{1}{4} \int_{M^n} R(|\nabla \psi|^2 - (\lambda^2 - \frac{R}{4})|\psi|^2) \geq -\frac{R_{\text{max}} - R_{\text{min}}}{4}(\lambda^2 - \frac{R_{\text{min}}}{4}) \cdot \int_{M^n} |\psi|^2.\]

Inserting this into (*) we obtain the first one of the inequalities (17). The second one can be proved analogously. □

The endomorphisms of the spinor bundle $E := -A_X \cdot X^k$ and $T := \text{Ric}(X_k) \cdot A_X + A_X \cdot \text{Ric}(X_k)$ occur on the right-hand side of formula (14). For any vector field $X$, the endomorphism $A_X$ is antiselfadjoint

$$ (A_X)^* = -A_X. $$

Hence, $E$ and $T$ are selfadjoint

$$ E^* = E, \quad T^* = T. $$

Moreover, we see that $E$ is nonnegative

$$ E \geq 0. $$

Let $\varepsilon$ denote the maximum of all eigenvalues of $E$ on $S$ and let $\tau$ be the minimum of the eigenvalues of $T$. Then, for any $\psi \in \Gamma(S)$, there are the inequalities

$$ \langle E\psi, \psi \rangle \leq \varepsilon |\psi|^2, $$

$$ -\langle T\psi, \psi \rangle \leq -\tau |\psi|^2. $$

In the following let $\lambda \geq 0$ be any nonnegative eigenvalue of $D$. Then, integrating equation (14) using the inequalities (13), (17), (21) and (22), in case $t \geq 0$ and $\kappa \leq 0$, we obtain

$$ 0 \leq \int_{M^n} |Q^t\psi|^2 \leq \left( \frac{n-1}{n} \lambda^2 - \frac{R_{\text{min}}}{4} \right) \cdot \int_{M^n} |\psi|^2 - 2t \left( \lambda^2 - \frac{R_{\text{min}}}{4} \right) \cdot \int_{M^n} |\psi|^2 = \left[ \frac{n-1}{n} \lambda^2 - \frac{R_{\text{min}}}{4} - 2t \left( \frac{1}{4} |\text{Ric}|_{\min}^2 - \frac{R_{\text{max}}}{n} \lambda^2 \right) \right] \cdot \int_{M^n} |\psi|^2.

Thus, in case $\kappa \leq 0$, for all $t \geq 0$, it holds that

$$ \lambda^2 - \frac{nR_{\text{min}}}{4(n-1)} - 2t \left( \frac{n}{4(n-1)} |\text{Ric}|_{\min}^2 - \frac{R_{\text{max}}}{n-1} \lambda^2 + \frac{n-1}{n} \left( \kappa - \frac{R_{\text{max}} - R_{\text{min}}}{4} \right) \right) + n \frac{t^2}{n-1} \left( |\text{Ric}|_{\min}^2 \cdot \lambda^2 - \tau \lambda + \varepsilon \right) \geq 0. $$

(23)
In case \( \kappa > 0 \), for all \( t \geq 0 \), it follows analogously that

\[
\lambda^2 - \frac{n R_{\min}}{4(n-1)} - 2t \left( \frac{n}{4(n-1)} |\text{Ric}|^2_{\min} - \frac{R_{\max}}{n-1} \lambda^2 + \frac{n \kappa}{n-1} (\lambda^2 - \frac{R_{\max}}{4}) - \frac{n(R_{\max} - R_{\min})(\lambda^2 - \frac{R_{\min}}{4})}{4(n-1)} \right) + \frac{n}{n-1} t^2 \left( |\text{Ric} - \frac{R_{\max}}{n_{\max}} \cdot \lambda^2 - \tau \cdot \lambda + \varepsilon \right) \geq 0.
\]

From (1) we know that \( \ker(D) = 0 \) if the scalar curvature \( R \) is positive (\( R_{\min} > 0 \)). Now, let us consider the case of \( R_{\min} \leq 0 \). This implies \( \kappa \leq 0 \). Inserting \( \lambda = 0 \) in (23), a simple discussion yields

**Theorem 2.2.** Let \( M^n \) be a compact Riemannian spin manifold with \( R_{\min} \leq 0 \) such that the inequality

\[
|\text{Ric}|^2_{\min} > R_{\min} \left( \kappa - \frac{R_{\max} - R_{\min}}{4} \right) + 2\sqrt{|R_{\min}| \cdot \varepsilon}
\]

is satisfied. Then the kernel of the Dirac operator is trivial (\( \ker(D) = 0 \)).

**Corollary 2.1.** Let \( M^n \) be a compact Riemannian spin manifold of constant scalar curvature \( R \leq 0 \) satisfying the inequality

\[
|\text{Ric}|^2_{\min} > R_\kappa + 2\sqrt{|R|\varepsilon}.
\]

Then there are no harmonic spinors.

**Corollary 2.2.** Let \( M^n \) be a compact Riemannian spin manifold satisfying the conditions \( R_{\min} = 0 \) and \( |\text{Ric}|_{\min} > 0 \). Then there are no harmonic spinors.

We remark that Theorem 2.2 and Corollary 2.1 are generalizations of Theorem 2.1 in \[5\]. The case of \( R_{\min} > 0 \) gives rise to the question under which conditions the inequalities (23) and (24), respectively, yield a better lower bound than (1). The answer can be given without determining the optimal parameter \( t \).

**Case 1:** \( R_{\min} > 0, \kappa \leq 0 \).

Inserting \( \lambda := \sqrt{n R_{\min}/4(n-1)} \) into (23) we obtain a contradiction for some \( t > 0 \) if and only if the coefficient of \( t \) is negative. But this is just the condition

\[
|\text{Ric}|^2_{\min} > \frac{R_{\min}}{n-1} \left( R_{\max} - \kappa + \frac{R_{\max} - R_{\min}}{4} \right) .
\]

**Case 2:** \( R_{\min} > 0, \kappa > 0 \).

The corresponding condition that can be derived from (24) analogously is

\[
|\text{Ric}|^2_{\min} > \frac{R_{\min}}{n-1} \left( R_{\max} - \kappa + \frac{R_{\max} - R_{\min}}{4} \right) + \kappa (R_{\max} - R_{\min}).
\]

It is interesting to remark that in the inequalities (27), (28) the covariant derivatives of \( \text{Ric} \) do not appear. Moreover, in case of constant scalar curvature \( R > 0 \), these conditions coincide and simplify to

\[
|\text{Ric}|^2_{\min} > \frac{R}{n-1} (R - \kappa).
\]

This generalizes the corresponding assertion in Section 2 of \[5\].
3. The endomorphisms $E$ and $T$

For further applications of the formulas (23) and (24), it is convenient to write the endomorphisms

$$E = -A_{X_k} \cdot A_{X^k} \quad \text{and} \quad T = \text{Ric}(X_k) \cdot A_{X^k} + A_{X^k} \cdot \text{Ric}(X_k)$$

of the spinor bundle $S$ in a more suitable form.

**Proposition 3.1.** There are the identities

(30) $$E = \frac{1}{4} \left| \nabla \text{Ric} \right|^2 - \frac{1}{16} |dR|^2 + \frac{1}{8} \left[ \nabla_{X_i} \text{Ric}, \nabla_{X_k} \text{Ric} \right] (X_l) \cdot X^j \cdot X^k \cdot X^l,$$

(31) $$T = \frac{1}{2} \left[ \nabla_{X_i} \text{Ric}, \text{Ric} \right] (X_l) \cdot X^k \cdot X^l.$$

**Proof.** We calculate

\[
T = \frac{1}{4} \left( \text{Ric}(X^l) \cdot (\nabla_{X_k} \text{Ric})(X_i) \cdot X^k - \text{Ric}(X^l) \cdot X^k \cdot (\nabla_{X_k} \text{Ric})(X_i) \right) + \\
\frac{1}{4} \left( (\nabla_{X_k} \text{Ric})(X_i) \cdot X^k \cdot \text{Ric}(X^l) - X^k \cdot (\nabla_{X_k} \text{Ric})(X_i) \cdot \text{Ric}(X^l) \right) = \\
\frac{1}{4} \left( X^l \cdot (\nabla_{X_k} \text{Ric})(\text{Ric}(X_i)) \cdot X^k - X^l \cdot X^k \cdot (\nabla_{X_k} \text{Ric})(\text{Ric}(X_i)) \right) + \\
+ (\nabla_{X_k} \text{Ric})(\text{Ric}(X_i)) \cdot X^k \cdot X^l - X^k \cdot (\nabla_{X_k} \text{Ric})(\text{Ric}(X_i)) \cdot X^l = \\
\frac{1}{4} \left( - (\nabla_{X_k} \text{Ric})(\text{Ric}(X_i)) \cdot X^l \cdot X^k - 2g(X^l, (\nabla_{X_k} \text{Ric} \circ \text{Ric})(X_i))X^k + \\
+ X^l \cdot (\nabla_{X_k} \text{Ric})(\text{Ric}(X_i)) \cdot X^k + 2g(X^k, (\nabla_{X_k} \text{Ric} \circ \text{Ric})(X_i))X^l + \\
+ (\nabla_{X_k} \text{Ric})(\text{Ric}(X_i)) \cdot X^k \cdot X^l + (\nabla_{X_k} \text{Ric})(\text{Ric}(X_i)) \cdot X^k \cdot X^l + \\
+ 2g((\nabla_{X_k} \text{Ric} \circ \text{Ric})(X_i), X^k)X^l \right) = \\
\frac{1}{2} \left( (\nabla_{X_k} \text{Ric} \circ \text{Ric})(X_i) \cdot X^k \cdot X^l - (\nabla_{X_k} \text{Ric} \circ \text{Ric})(X_i) \cdot X^l \cdot X^k - \\
- 2\text{tr}(\nabla_{X_k} \text{Ric} \circ \text{Ric})X^k + 2(\text{Ric} \circ \nabla_{X_k} \text{Ric})(X^k) \right) = \\
\frac{1}{2} \left( (\nabla_{X_k} \text{Ric} \circ \text{Ric})(X_i) \cdot X^k \cdot X^l - (\nabla_{X_k} \text{Ric}^2)(X_i) \cdot X^l \cdot X^k + \\
+ (\text{Ric} \circ \nabla_{X_k} \text{Ric})(X_i) \cdot X^l \cdot X^k - \text{tr}(\nabla_{X_k} \text{Ric}^2)X^k + \\
+ 2(\text{Ric} \circ \nabla_{X_k} \text{Ric})(X^k) \right) = \\
\frac{1}{2} \left( (\nabla_{X_k} \text{Ric} \circ \text{Ric})(X_i) \cdot X^k \cdot X^l - (\nabla_{X_k} \text{Ric}^2)(X_i) \cdot X^l \cdot X^k - \\
- \text{tr}(\nabla_{X_k} \text{Ric}^2)X^k \right) = \\
\frac{1}{2} \left[ \nabla_{X_k} \text{Ric}, \text{Ric} \right] (X_l) \cdot X^k \cdot X^l.
\]

The latter equation is valid since $\nabla X \text{Ric}^2$ is selfadjoint (symmetric) and, hence,

$$\left(\nabla X \text{Ric}^2\right)(X_l) \cdot X^l = -\text{tr}(\nabla X \text{Ric}^2).$$
Thus, we have \([\square]\). Now, we prove \([\square]\). It holds that

\[
E = -\frac{1}{16} \left( (\nabla_X \text{Ric})(X_i) \cdot X^j - X^j (\nabla_X \text{Ric})(X_i) \right) \left( (\nabla_{X_k} \text{Ric})(X^k) \cdot X^k \cdot (\nabla_{X_i} \text{Ric})(X^i) \right) = \\
= -\frac{1}{16} \left( (\nabla_X \text{Ric})(X_i) \cdot X^j \cdot X^k \cdot (\nabla_X \text{Ric})(X_i) \cdot X^j \cdot X^k \cdot (\nabla_X \text{Ric})(X^j) - \\
\quad -X^j \cdot (\nabla_X \text{Ric})(X_i) \cdot X^j \cdot X^k \cdot (\nabla_{X_k} \text{Ric})(X^j) \cdot X^k \cdot X^j \cdot (\nabla_{X_i} \text{Ric})(X^i) \right) = \\
= -\frac{1}{16} \left( (\nabla_X \text{Ric} \circ \nabla_{X_k} \text{Ric})(X_i) \cdot X^j \cdot X^k \cdot X^j \cdot X^k \cdot (\nabla_{X_i} \text{Ric})(X^j) \right) + \\
\quad +2X^j \cdot (\nabla_X \text{Ric} \circ \nabla_{X_k} \text{Ric})(X_i) \cdot X^j \cdot X^k \cdot X^j \cdot X^k \cdot (\nabla_{X_i} \text{Ric})(X^j) \right) = \\
= \frac{1}{8} \left( (\nabla_X \text{Ric} \circ \nabla_{X_k} \text{Ric})(X_i) \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot (\nabla_{X_i} \text{Ric})(X^j) \right) + \\
\quad +2g(\nabla_{X_i} \text{Ric} \circ \nabla_{X_k} \text{Ric})(X_i) \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot (\nabla_{X_i} \text{Ric})(X^j) \right) = \\
= \frac{1}{8} \left( -\nabla_X \text{Ric} \circ \nabla_{X_i} \text{Ric})(X_i) \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot (\nabla_{X_i} \text{Ric})(X^j) \right) + \\
\quad +g(\nabla_{X_i} \text{Ric} \circ \nabla_{X_k} \text{Ric})(X_i) \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot (\nabla_{X_i} \text{Ric})(X^j) \right) = \\
= \frac{1}{4} \text{tr}(\nabla_{X_i} \text{Ric} \circ \nabla_{X_k} \text{Ric}) + \frac{1}{8} \nabla_{X_i} \text{Ric} \circ \nabla_{X_k} \text{Ric})(X_i) \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot (\nabla_{X_i} \text{Ric})(X^j) = \\
\quad -\frac{1}{4}g(\nabla_{X_i} \text{Ric} \circ \nabla_{X_k} \text{Ric})(X^j) = \\
= \frac{1}{4} |\nabla \text{Ric}|^2 + \frac{1}{8} \nabla_{X_i} \text{Ric} \circ \nabla_{X_k} \text{Ric})(X_i) \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot (\nabla_{X_i} \text{Ric})(X^j) = \\
\quad = \frac{1}{4} |\nabla \text{Ric}|^2 - \frac{1}{16} |dR|^2 + \frac{1}{8} |\nabla_{X_i} \text{Ric} \circ \nabla_{X_k} \text{Ric})(X_i) \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot X^j \cdot (\nabla_{X_i} \text{Ric})(X^j) .
\]

Let \(\Theta\) be the 3-form on \(M^n\) defined by

\[
\Theta(X, Y, Z) := g(\nabla_X \text{Ric} \circ \nabla_Y \text{Ric})(X, Z) + g(\nabla_X \text{Ric}, \nabla_Z \text{Ric})(X, Y) + \\
\quad + g(\nabla_Y \text{Ric}, \nabla_Z \text{Ric})(X, Y) .
\]

**Corollary 3.1.** The endomorphism \(T\) acts on \(S\) via Clifford multiplication by the 3-form \(\Theta\), i.e., for all spinors \(\psi\), we have the relation

\[
T\psi = \Theta \cdot \psi .
\]

**Proof.** By \([\square]\), we have

\[
T\psi = \frac{1}{2} g(\nabla_{X_i} \text{Ric} \circ \nabla_{X_j} \text{Ric})(X_i) \cdot X^j \cdot X^k \cdot X^l \cdot \psi .
\]
Thus, if \((X_1, \ldots, X_n)\) is any local orthonormal frame, we obtain
\[
T\psi = \frac{1}{2} \sum_{j=1}^{n} g(\nabla_{X_j} \text{Ric}, \text{Ric})(X_{j}) X_{j} \cdot X_{k} \cdot X_{l} \cdot \psi + \\
\frac{1}{2} \sum_{j<k}^{n} g(\nabla_{X_j} \text{Ric}, \text{Ric})(X_{j}, X_{k}) X_{j} \cdot X_{k} \cdot X_{l} \cdot \psi + \
\frac{1}{2} \sum_{\substack{j<k<l}}^{n} g(\nabla_{X_j} \text{Ric}, \text{Ric})(X_{j}, X_{k}, X_{l}) \cdot \psi \\
= -\frac{1}{2} g(\nabla_{X_j} \text{Ric}, \text{Ric})(X_{j}, X^{k}) X^{l} \cdot \psi - \frac{1}{2} g(\nabla_{X_j} \text{Ric}, \text{Ric})(X^{k}, X_{j}) X^{l} \cdot \psi + \
\sum_{\substack{j<k, l \neq j, l}}^{n} g(\nabla_{X_j} \text{Ric}, \text{Ric})(X_{j}, X_{k}, X_{l}) \cdot \psi \\
= \sum_{j<k}^{n} g(\nabla_{X_j} \text{Ric}, \text{Ric})(X_{j}, X_{k}) X_{j} \cdot X_{k} \cdot X_{l} \cdot \psi + \
\sum_{\substack{j<k<l}}^{n} g(\nabla_{X_j} \text{Ric}, \text{Ric})(X_{j}, X_{k}, X_{l}) \cdot \psi \\
= \sum_{j<k}^{n} (g(\nabla_{X_j} \text{Ric}, \text{Ric})(X_{j}) - g(\nabla_{X_j} \text{Ric}, \text{Ric})(X_{k})) X_{j} \cdot X_{k} \cdot X_{l} \cdot \psi \\
= \sum_{j<k}^{n} (g(\nabla_{X_j} \text{Ric}, \text{Ric})(X_{j}) + g(\nabla_{X_j} \text{Ric}, \text{Ric})(X_{k}) + g(\nabla_{X_j} \text{Ric}, \text{Ric})(X_{k})) X_{j} \cdot X_{k} \cdot X_{l} \cdot \psi \\
= \Theta \cdot \psi .
\]

\[
\Box
\]

4. Estimates for the first eigenvalue of the Dirac operator

Let us introduce the notation
\[
R_{*} := \begin{cases} 
R_{\min} & \text{if } \kappa \leq 0 \\
R_{\max} & \text{if } \kappa > 0 
\end{cases} .
\]

Then the inequalities (23), (24) can be written in the unified form
\[
\alpha(t) \lambda^{2} - 2\gamma(t) \lambda \geq \beta(t) \quad (\lambda \geq 0) ,
\]
where the functions \(\alpha(t), \beta(t), \gamma(t)\) depending on \(t \geq 0\) are defined by
\[
\alpha(t) := 1 + 2t \frac{n}{n-1} \left( \frac{R_{\max}}{n} - \kappa + \frac{R_{\max} - R_{\min}}{4} \right) + \frac{n}{n-1} t^{2} |Ric - \frac{R_{\max}}{n}|_{\max} , \\
\beta(t) := \frac{n}{4(n-1)} \left( R_{\min} + 2t(|Ric|_{\min}^{2} - R_{*} \kappa + R_{\min} \frac{R_{\max} - R_{\min}}{4}) - 4t^{2} \varepsilon \right) , \\
\gamma(t) := \frac{n\pi}{2(n-1)t^{2}} .
\]
By definition, we have \( \alpha(t) \geq 1 \) and the sign of \( \gamma(t) \) depends on \( \tau \). Moreover, we see that the inequality (33) is of interest only if the function \( \beta(t) \) attains positive values for some \( t \geq 0 \). Obviously, this is the case if \( R_{\min} > 0 \). In case of \( R_{\min} \leq 0 \), this holds if the condition (25) is satisfied. Thus, from (33) we immediately obtain

**Theorem 4.1.** Let \( M^n \) be any compact \( n \)-dimensional Riemannian spin manifold and let \( \lambda \geq 0 \) be any eigenvalue of the Dirac operator. Then, for all \( t \geq 0 \) with \( \beta(t) \geq 0 \), there is the inequality

\[
\lambda \geq \frac{\sqrt{\alpha(t)\beta(t) + \gamma(t)^2} + \gamma(t)}{\sqrt{\alpha(t)\beta(t) + \gamma(t)^2} - \gamma(t)}.
\]

**Theorem 4.2.** Let \( M^n \) be a compact Riemannian spin manifold with \( \Theta = 0 \) and let \( \lambda \) be any eigenvalue of the Dirac operator. Then, for all \( t \geq 0 \), we have the inequality

\[
\lambda^2 \geq \frac{\beta(t)}{\alpha(t)}.
\]

**Proof.** By Corollary 3.1, \( \Theta = 0 \) implies \( T = 0 \). Thus, integrating (14) we obtain (35). In this case, the sign of \( \lambda \) plays no role.

**Remark 4.1.** \( M^n \) satisfies the condition \( \Theta = 0 \) in the following special situations:

(i) The covariant derivative of the Ricci tensor has the symmetry property \( (\nabla_X \text{Ric})(Y) = (\nabla_Y \text{Ric})(X) \), i.e., the curvature tensor of \( M^n \) is divergence free. This situation was investigated in [5].

(ii) The Ricci tensor commutes with its covariant derivatives, i.e., for all vector fields \( X \), it holds that

\[
[\nabla_X \text{Ric}, \text{Ric}] = 0.
\]

(36) implies \( \Theta = 0 \). This is an immediate consequence of (31) and (32).

(iii) The Ricci tensor is recurrent, i.e., there is a 1-form \( \eta \) on \( M^n \) such that, for all vector fields \( X \), the equation

\[
\nabla_X \text{Ric} = \eta(X) \cdot \text{Ric}
\]

(37) is valid. Obviously, (37) implies (36). Thus, this situation is a special case of situation (ii).

**Remark 4.2.** In case the Ricci tensor has the property

\[
[\nabla_X \text{Ric}, \nabla_Y \text{Ric}] = 0,
\]

i.e., if any two covariant derivatives of Ric commute, it follows from (30) that the endomorphism \( E \) simplifies to the function

\[
E = \frac{1}{4}|\nabla \text{Ric}|^2 - \frac{1}{16}|dR|^2.
\]

Hence, in this case we have

\[
\varepsilon = \left( \frac{1}{4}|\nabla \text{Ric}|^2 - \frac{1}{16}|dR|^2 \right)_{\max}.
\]

We see that (34) implies (38). Moreover, if Ric has pairwise different eigenvalues on \( M^n \), then (34) implies (38).

Calculating the maximum of the function \( \beta(t)/\alpha(t) \) by Theorem 4.2 we obtain
Corollary 4.1. Let $M^n$ be a compact Riemannian spin manifold satisfying the conditions $R_{\min} = 0, |\text{Ric}|_{\min} > 0$ and $[\nabla X \text{Ric}, \text{Ric}] = 0$. Then, for every eigenvalue $\lambda$ of the Dirac operator, we have the estimate

\begin{equation}
\lambda^2 \geq \frac{n}{8(n-1)} \cdot \frac{|\text{Ric}|_{\min}^2}{a + b + \sqrt{a^2 + ab + c}},
\end{equation}

where the constants $a, b, c$ are defined by

\begin{align}
a &:= \varepsilon \frac{|\text{Ric}|_{\min}^2}{\min} , \\
b &:= \frac{n}{2(n-1)} \left( n + 4 \frac{R_{\max}}{4n} - \kappa \right) , \\
c &:= \frac{n}{4(n-1)} |\text{Ric} - \frac{R}{n_{\max}}|.
\end{align}

Now, we give some simple examples.

Example 4.1. Let us consider the Riemannian product $M^4(r, \rho) := S^2(r) \times T^2(\rho)$, where $S^2(r) \subset \mathbb{R}^3$ is the standard sphere of radius $r > 0$ and $T^2(\rho) \subset \mathbb{R}^3(\rho > 0)$ the standard torus defined by $\gamma : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$, $\gamma(u, v) := (\rho(2 + \cos u) \cos v, \rho(2 + \cos u) \sin v, \rho \sin u)$. Then the Ricci tensor $\text{Ric}$ and the scalar curvature $R$ of $M^4(r, \rho)$ are given by

\begin{align}
(\ast) & \quad \text{Ric} = \frac{1}{r^2} p_S + \frac{\cos u}{\rho^2(2 + \cos u)} p_T , \\
(2\ast) & \quad R = \frac{2}{r^2} + \frac{2 \cos u}{\rho^2(2 + \cos u)} ,
\end{align}

where $p_S$ and $p_T$ denote the projections on the tangent spaces of $S^2(r)$ and $T^2(\rho)$, respectively. This implies

\begin{align}
(3\ast) & \quad |\text{Ric}|^2 = \frac{2}{r^4} + \frac{2 \cos^2 u}{\rho^2(2 + \cos u)^2} , \\
(4\ast) & \quad |\text{Ric} - \frac{R}{4}|^2 = \left( \frac{1}{r^2} - \frac{\cos u}{\rho^2(2 + \cos u)} \right)^2 , \\
(5\ast) & \quad \nabla_X \text{Ric} = -\frac{2 \sin u}{\rho^2(2 + \cos u)^2} du(X)p_T , \\
(6\ast) & \quad \frac{1}{4} |\nabla \text{Ric}|^2 - \frac{1}{16} |dR|^2 = \frac{\sin^2 u}{\rho^2(2 + \cos u)^4} .
\end{align}

Hence, we obtain

\begin{align}
(7\ast) & \quad |\text{Ric}|_{\min}^2 = \frac{2}{r^4} , \\
(8\ast) & \quad R_{\min} = 2 \left( \frac{1}{r^2} - \frac{1}{\rho^2} \right) , \quad R_{\max} = 2 \left( \frac{1}{r^2} + \frac{1}{3\rho^2} \right) , \\
(9\ast) & \quad |\text{Ric} - \frac{R}{4}|_{\max}^2 = \left( \frac{1}{r^2} + \frac{1}{\rho^2} \right)^2 .
\end{align}
\[ \kappa = -\frac{1}{\rho^2}. \]

From (*) and (5*) we see that there are the relations
\[ [\nabla_X \text{Ric}, \text{Ric}] = 0, \quad [\nabla_X \text{Ric}, \nabla_Y \text{Ric}] = 0. \]

Thus, by Proposition 3.1, Remark 4.2 and (6*), we find here
\[ \tau = 0, \quad \varepsilon = \frac{3 + 2 \sqrt{3}}{36 \rho^2}. \]

The case of \( r = \rho \) (\( R_{\text{min}} = 0 \)):

In this case, the suppositions of Corollary 4.1 are satisfied. By (7*) - (11*), the constants \( a, b, c \) are given by
\[ a = \frac{3 + 2 \sqrt{3}}{72 r^2}, \quad b = \frac{14}{9 r^2}, \quad c = \frac{4}{3 r^4}. \]

Inserting this into (1) we obtain the estimate
\[ \lambda^2 \geq 0, 116 r^{-2}. \]

The case of \( r < \rho \) (\( R_{\text{min}} > 0 \)):

By (7*) - (10*), (27) is equivalent to
\[ \frac{1}{r^4} > \frac{1}{3 \rho^2} \left( \frac{1}{r^2} - \frac{7}{\rho^2} \right). \]

For \( r < \rho \), this inequality is always satisfied. Hence, for all manifolds \( M^4(r, \rho) \) with \( r < \rho \), (35) yields a better estimate than (1). For example, let us consider the special case that \( \rho = \frac{3}{5} \sqrt{10} \). Then, by (35) and (7*) - (11*), we find
\[ \lambda^2 \geq 0, 156 r^{-2}, \]

whereas (1) yields the estimate
\[ \lambda^2 \geq \frac{1}{15} r^{-2} = 0, 06 r^{-2}. \]

Since \( M^4(r, \rho) \) is Kähler, we can also apply the estimate (3). For \( \rho = \frac{3}{5} \sqrt{10}, \) (2) and (8*) yield
\[ \lambda^2 \geq 0, 1 r^{-2}. \]

Thus, the estimate (12*) obtained by Theorem 4.2 is the better one.

The case of \( r > \rho \) (\( R_{\text{min}} < 0 \)):

For example, let us consider the special case that \( \rho = \frac{3}{5} \sqrt{10} r \), i.e., \( R_{\text{min}} = -\frac{2}{9} r^{-2} \). Then one finds that the condition (23) is satisfied. Thus, Theorem 2.2 implies that there are no harmonic spinors on \( M^4(r, \frac{3}{5} \sqrt{10} r) \). Calculating the maximum of the corresponding function \( \beta(t)/\alpha(t) \), our Theorem 4.2 yields the estimate
\[ \lambda^2 \geq 0, 061 r^{-2}. \]

Example 4.2. For \( \rho > 1 \), we consider the Riemannian product \( M^6(\rho) := S^2 \times F^2 \times T^2(\rho) \), where \( S^2 \subset \mathbb{R}^3 \) is the unit standard sphere, \( F^2 \) any compact Riemannian surface of constant
Gaussian curvature $-1$ and $T^2(\rho) \subset \mathbb{R}^3$ the torus defined in Example 4.1. Then the Ricci tensor and scalar curvature of $M^6(\rho)$ are given by

$$
\text{Ric} = p_S - p_F + \frac{\cos u}{\rho^2(2 + \cos u)} p_T,
$$

$$
R = \frac{2 \cos u}{\rho^2(2 + \cos u)},
$$

where $p_S, p_F, p_T$ denote the projections on the tangent spaces of $S^2, F^2$ and $T^2(\rho)$, respectively. This implies

\begin{align*}
|\text{Ric}|_{\text{min}}^2 &= 4, \\
R_{\text{min}} &= -\frac{2}{\rho^2}, \quad R_{\text{max}} = \frac{2}{3\rho^2}, \\
|\text{Ric} - \frac{R}{6}|_{\text{max}}^2 &= 4 \left(1 + \frac{1}{3\rho^4}\right).
\end{align*}

Moreover, here we also have $[\nabla_X \text{Ric}, \text{Ric}] = 0$ and $[\nabla_X \text{Ric}, \nabla_Y \text{Ric}] = 0$. Hence, as in Example 1 we obtain

\begin{align*}
\alpha(t) &= 1 + \left(\frac{12}{5} + \frac{28}{15} \rho^{-2}\right)t + \left(\frac{24}{5} + \frac{8}{5} \rho^{-4}\right)t^2, \\
\beta(t) &= -\frac{3}{5} \rho^{-2} + \left(\frac{12}{5} - \frac{6}{5} \rho^{-2} - \frac{4}{5} \rho^{-4}\right)t - \frac{3 + 2\sqrt{3}}{30} \rho^{-6} t^2,
\end{align*}

and (25) is equivalent to

$$
\rho^4 - \frac{1}{2} \rho^2 > \frac{1}{3} + \frac{1}{12} \sqrt{6 + 4\sqrt{3}}.
$$

The last inequality is valid for $\rho > \rho_0 \approx 1.04113$. Thus, by Theorem 2.4, $M^6(\rho)$ admits no harmonic spinors for $\rho > \rho_0$. Moreover, we are interested in the maximum $f(\rho)(t_{\text{max}})$ of the function $f(\rho)(t) := \beta(t)/\alpha(t)$ for $t \geq 0$ subject to $\rho$ describing the lower bound of $\lambda^2$ on $M^6(\rho)$. We obtain the following picture:
The lower bound is a bounded monotone increasing function of $\rho$
with $\lim_{\rho \to \infty} f(\rho)(t_{\text{max}}) \approx 0, 354$.

For example, on $M^6(2)(R_{\text{min}} = -1/2)$ we have the estimate $\lambda^2 \geq 0, 2407$.

**Remark 4.3.** The manifolds $M^4(r, \rho)$ and $M^6(\rho)$ of Example 1 and Example 2, respectively, do not satisfy the condition (3). These two examples are special cases of the following more general situation where the geometric suppositions (25), (27) and (28), respectively, can be checked easily: Let us consider a Riemannian product of the form $M^n := M^1 \times \ldots \times M^p$, where $M^n_k (k = 1, \ldots, p)$ is a compact Riemannian spin manifold of dimension $n_k$ such that $M^n_k$ is Einstein if $n_k \geq 3$. Then the Ricci tensor $\text{Ric}$ of $M^n$ satisfies the conditions (36) and (38), but not the condition (3) in general. Thus, here we obtain

$$\tau = 0, \quad \varepsilon = \frac{1}{16} \cdot \sum_{k=1}^p |dR_k|^2_{\text{max}},$$

where $R_k$ denotes the scalar curvature of $M^n_k$ and, hence, $dR_k = 0$ if $n_k \neq 2$. Moreover, we have

$$|\text{Ric}|^2_{\text{min}} = \sum_{k=1}^p \left( \frac{R_k^2}{n_k} \right)_{\text{min}},$$

$$R_{\text{min}} / \text{max} = \sum_{k=1}^p \left( \frac{R_k}{n_k} \right)_{\text{min} / \text{max}},$$

$$\kappa = \min \left\{ \left( \frac{R_k}{n_k} \right)_{k = 1, \ldots, p} \right\},$$

$$|\text{Ric} - \frac{R}{n}|^2_{\text{max}} = \left( \sum_{k=1}^p n_k \left( \frac{R_k}{n_k} - \frac{R}{n} \right)^2 \right)_{\text{max}},$$

with $R = R_1 + \ldots + R_p$ and $n = n_1 + \ldots + n_p$. Thus, we see that, in this situation, all the essential numbers can be derived from the scalar curvatures $R_1, \ldots, R_p$ and the dimensions $n_1, \ldots, n_p$.

**Remark 4.4.** Up to now we have no answer to the question if there exist compact Riemannian spin manifolds that realizes the limiting case of the inequality (34) in case of an optimal parameter $t = t_0 > 0$.

**References**

[1] H. Baum, Th. Friedrich, R. Grunewald, I. Kath, *Twistor and Killing spinors on Riemannian manifolds*, Teubner-Texte zur Mathematik, Band 124, B.G. Teubner-Verlagsgesellschaft (1991).

[2] Th. Friedrich, *Der erste Eigenwert des Dirac-Operators einer kompakten Riemannschen Mannigfaltigkeit nichtnegativer Skalarkrümmung*, Math. Nachr. 97 (1980), 117-146.

[3] Th. Friedrich, *The classification of 4-dimensional Kähler manifolds with small eigenvalue of the Dirac operator*, Math. Ann. 295 (1993), 565-574.

[4] Th. Friedrich, E.C. Kim, *Some remarks on the Hijazi inequality and generalizations of the Killing equation for spinors*, Journ. Geom. Phys. 37 (2001), 1-14.

[5] Th. Friedrich, K.-D. Kirchberg, *Eigenvalue estimates of the Dirac operator depending on the Ricci tensor*, SFB 288 Preprint No. 498 (2001).

[6] Th. Friedrich, K.-D. Kirchberg, *Eigenvalue estimates for the Dirac operator depending on the Weyl tensor*, SFB 288 Preprint No. 503 (2001).

[7] O. Hijazi, *A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors*, Commun. Math. Phys. 104 (1986), 151-162.

[8] K.-D. Kirchberg, *The first eigenvalue of the Dirac operator on Kähler manifolds*, J. Geom. Phys. 7 (1990), 449-468.

[9] W. Kramer, U. Semmelmann, G. Weingart, *Eigenvalue estimates for the Dirac operator on quaternionic Kähler manifolds*, Math. Z. 230 (1999), 727-751.
A RELATION BETWEEN THE RICCI TENSOR AND THE SPECTRUM OF THE DIRAC OPERATOR

kirchber@mathematik.hu-berlin.de
Institut für Reine Mathematik
Humboldt-Universität zu Berlin
Sitz: Rudower Chaussee 25
D-10099 Berlin, Germany