Net bundles over posets and K-theory

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Abstract

We continue studying net bundles over partially ordered sets (posets), defined as the analogues of ordinary fibre bundles. To this end, we analyze the connection between homotopy, net homology and net cohomology of a poset, giving versions of classical Hurewicz theorems. Focusing our attention on Hilbert net bundles, we define the K-theory of a poset and introduce functions over the homotopy groupoid satisfying the same formal properties as Chern classes. As when the given poset is a subbase for the topology of a space, our results apply to the category of locally constant bundles.

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1 Introduction

This paper continues the discussion of invariants of a partially ordered set begun in [10, 11], comparing them with the analogous topological invariants when the partially ordered set is a suitable base for the topology of a topological space ordered under inclusion. Whereas previously, the emphasis has been on the fundamental groupoid, connections, curvature, simplicial cohomology and Čech cohomology, it is now on homology, locally constant cohomology and K-theory with the fundamental groupoid continuing its ubiquitous role.

Thus we prove an analogue of the Hurewicz isomorphism showing that the first homology group of a poset with values in $\mathbb{Z}$ is isomorphic to its Abelianized homotopy group. We further consider an arcwise locally contractible space and a poset made up of a subbase of contractible open sets ordered under inclusion and show that the first homology groups with values in $\mathbb{Z}$ of the topological space coincides with that of the poset as do the first cohomology groups with values in an Abelian group.

The pivotal notions are those of net bundle and quasinet bundle. These are bundles over a poset where the fibres are objects in a category together with a functor from the poset mapping an element of the poset to the corresponding fibre. In a quasinet bundle, the functor takes values in the monomorphisms, in a net bundle in the isomorphisms.

We show that a poset net bundle with pathwise connected fibres gives rise to a short exact sequence of homotopy groups as does a net bundle of topological spaces with connected and locally contractible fibres.

We define a functor from the category of net bundles of topological spaces to the category of fibre bundles over the poset with the Alexandroff topology. The category of 1–cocycles with values in a group of the global space of the original net bundle regarded as a net bundle over the poset with the opposite ordering is equivalent to the category of homomorphisms of the homotopy group of the fibre bundle with values in the group. The fibre bundle associated with a principal $G$–net bundle has locally constant transition functions.

The remainder of the paper is largely devoted to studying Hilbert net bundles and their K-theory. Here there is a functor from Hilbert net bundles to vector bundles.
over the poset with the Alexandroff topology. The category of Hilbert net bundles is equivalent to the category of unitary finite-dimensional representations of the homotopy group of the poset. Consequently, its K-ring is isomorphic to the representation ring of its homotopy group. But it is also equivalent to the category of locally constant vector bundles over the poset with the Alexandroff topology. This leads to an isomorphism of the corresponding K-rings.

We prove an analogue of the Thom isomorphism asserting that the first cohomology of the poset with values in a group is isomorphic to the first cohomology of a projective net bundle with values in the same group.

The first Chern class of a Hilbert net bundle is defined as an element of the net cohomology group $H^1(K, T)$. After this, the Chern K-classes are defined as elements of the reduced K-theory and Chern functions as complex-valued functions on the homotopy classes of paths, allowing one to recover the first Chern class by evaluating a suitable polynomial. When $K$ is a subbase for the topology of a space $M$, our results describe locally constant vector bundles over $M$, the first Chern class is an element of the singular cohomology $H^1(M, T)$ and the Chern functions are defined on the homotopy groupoid of $M$.

The ideas in the present paper can be applied to the theory of secondary classes. Our results imply that when the poset $K$ is a subbase for the topology of a manifold, each finite-dimensional Banach net bundle over $K$ has well-defined secondary Chern classes (for example, see [6 §4.19],[11 §1(g)]). It would be of interest to find a description of such classes in terms of the properties of $K$ such as its net cohomology.

An appendix is devoted to describing the simplicial sets associated with a poset, explaining the related notion of homotopy and proving that the homotopy group of a product of symmetric simplicial sets is equal to the product of the homotopy groups.

2 Background and Notations.

For the basic notions on the simplicial set of a poset $K$ and its homotopy theory, including the definition of path, we refer the reader to [12,11]. However the basic definitions and terminology can also be found in the Appendix.

The fundamental covering of $K$ is given by the family of sets

$$V_a := \{ o \in K : a \leq o \}, \quad a \in \Sigma_0(K),$$

which provide a base for the Alexandroff topology of $K$. We denote the associated space by $\tau K$ (see [11 §2.3]).

In order to simplify the exposition, in the present paper we shall always assume that the poset $K$ is pathwise connected. This implies that the isomorphism class of the homotopy group $\pi_1(K, a)$ does not depend on the choice of $a \in \Sigma_0(K)$.

In the same way, each space $M$ is assumed to be arcwise connected, and its homotopy group is denoted by $\pi_1(M)$. A pivotal result that will be used in the present paper is the
following: when $M$ is Hausdorff, and $K$ is a subbase of arcwise and simply connected open subsets of $M$ ordered under inclusion, there is an isomorphism $\pi_1(M) \simeq \pi_1(K, a)$, $a \in \Sigma_0(K)$ (see [12, Thm.2.18]).

We denote the identity map of a set $S$ by $id_S$.

A multiplicative semigroup of polynomials with coefficients in a ring $R$ is defined by

$$1 + hR[[h]] := \{1 + \sum_{k=1}^{r} x_k h^k \mid r \in \mathbb{N}, x_k \in R\}.$$ 

3 Abelian (co)homology.

3.1 Net cohomology.

The net cohomology of the poset $K$ with values in an Abelian group $A$, written additively, is the cohomology of the simplicial set $\tilde{\Sigma}_*(K)$ with values in $A$. To be precise, one can define the set $C^n(K, A)$ of $n$–cochains of $K$ with values in $A$ as the set of functions $v : \tilde{\Sigma}_n(K) \to A$. $C^n(K, A)$ inherits from $A$ the structure of an Abelian group:

$$(v + w)(x) := v(x) + w(x), \quad (−v)(x) := −v(x), \quad x \in \tilde{\Sigma}_n(K),$$

for any $v, w \in C^n(K, A)$. The coboundary operator $d$ defined by

$$dv(x) = \sum_{k=0}^{n} (-1)^k v(\partial_k x), \quad x \in \tilde{\Sigma}_n(K),$$

is a mapping $d : C^n(K, A) \to C^{n+1}(K, A)$ satisfying the equation $ddv = \iota$, $v \in C^n(K, A)$, where $\iota$ is the trivial cochain. Clearly $d$ is a group morphism. An $n$–cochain $z$ is said to be an $n$–coycle if belongs to the kernel of $d$. We denote the group of $n$–cocycles by $Z^n(K, A)$. Since $dC^{n-1}(K, A)$ is a subgroup $B^n(K, A)$ of $C^n(K, A)$, we define the net cohomology with coefficients in $A$

$$H^n(K, A) := Z^n(K, A)/B^n(K, A).$$

The functors $H^n(−, −)$, $n \in \mathbb{N}$, are contravariant w.r.t the poset and covariant w.r.t. to the group; moreover, net cohomology has long exact sequences (see [9, Lemma 1.1]).

Remark 3.1. Let $K$ be connected, $a \in \Sigma_0(K)$ and $\text{Hom}(\pi_1(K, a), A)$ denote the set of morphisms from $\pi_1(K, a)$ to $A$. Using [10, Prop.3.8] and the fact that $A$ is Abelian, there is an isomorphism $H^1(K, A) \simeq \text{Hom}(\pi_1(K, a), A)$. This allows one to compute several net cohomology groups.

3.2 Net homology.

In the present section, we introduce the notion of the net homology of a poset.
Let $A$ be an Abelian group with identity $0 \in A$. For every $n \in \mathbb{N}$, we let $C_n(K, A)$ denote the free Abelian group generated by formal linear combinations of elements of $\Sigma_n(K)$ with coefficients in $A$, and define the boundary morphism as the $A$-linear map

$$b_n : C_n(K, A) \to C_{n-1}(K, A) , \quad b_n x := \sum_{k=0}^{n} (-1)^k \partial_k x . \quad (3.1)$$

It is clear that $b_{n-1} b_n = 0$; we define $Z_n(K, A) := \ker b_n$, $B_n(K, A) := b_{n+1}(C_{n+1}(K, A))$, and the net homology group $H_n(K, A) := Z_n(K, A)/B_n(K, A)$ (for $n = 0$, we define $C_0(K, A) := A$, $B_0(K, A) := 0$ and $b_0 := 0$, so that $H_0(K, A) = A$). In the sequel, we will use the notation $b \equiv b_n$, so that $b_{n-1} b_n \equiv b^2 = 0$. Moreover, we will use the same notation for elements of $Z_n(K, A)$ and $H_n(K, A)$, identifying cycles with the corresponding net homology classes.

When $A = \mathbb{Z}$, $C_n(K, \mathbb{Z})$ reduces to the Abelian group generated by the elements of $\Sigma_n(K)$. Elements of $Z_1(K, \mathbb{Z})$ satisfy the relation

$$\sum_i k_i (\partial_0 b_i - \partial_1 b_i) = 0 , \quad \sum_i k_i b_i \in Z_1(K, \mathbb{Z}) , \quad (3.2)$$

so that, $b \in \Sigma_1(K) \cap Z_1(K, \mathbb{Z})$ if, and only if, $\partial_0 b = \partial_1 b$. We establish other useful relations. Given $a \in \Sigma_0(K)$ consider the degenerate 1-simplex $\sigma_0 a$ and note that $\sigma_0 a = b(\sigma_0 \sigma_0 a)$. So that,

$$\sigma_0 a = 0 \in H_1(K, \mathbb{Z}) . \quad (3.3)$$

If $b \in \Sigma_1(K)$ and $\tilde{b}$ is defined by $\partial_0 \tilde{b} := \partial_1 b$, $\partial_1 \tilde{b} := \partial_0 b$, $\tilde{b} := |b|$, then defining $c := ([b]; b, \sigma_0 \partial_0 b, \tilde{b})$ we obtain

$$bc = b - \sigma_0 \partial_0 b + \tilde{b} ,$$

implying (by (3.3))

$$\tilde{b} + b = 0 \in H_1(K, \mathbb{Z}) . \quad (3.4)$$

Now, for every $n \in \mathbb{N}$ we consider the map

$$C^n(K, A) \times C_n(K, A) \to A \quad (3.5)$$

obtained extending the evaluation $(v, x) \mapsto v(x)$, $v \in C^n(K, A)$, $x \in \Sigma_n(K)$, by linearity. Some elementary computations show that (3.5) induces a bilinear map

$$H^n(K, A) \times H_n(K, A) \to A . \quad (3.6)$$

We now analyze the connection between homotopy and net homology, providing a version of a classical result ([4, Thm.2.A.1]). To this end, we make some preliminary remarks on $\pi_1(K, a)$, $a \in \Sigma_0(K)$, and the associated Abelianized group $\pi_1(K, a)_{ab}$.

We recall that $\pi_1(K, a)_{ab}$ is defined as the quotient of $\pi_1(K, a)$ by the commutator subgroup, that is the normal subgroup generated by elements of the form $p_1 p_2 p_1^{-1} p_2^{-1}$, $p_1, p_2 \in \pi_1(K, a)$; by construction, $\pi_1(K, a)_{ab}$ is the universal Abelian group such that
each group morphism $\pi_1(K, a) \to A$, with $A$ Abelian, factorizes through a morphism $\pi_1(K, a)_{ab} \to A$. For each $p \in \pi_1(K, a)$, we denote the corresponding class in $\pi_1(K, a)_{ab}$ by $[p]$, so that $[p * p'] = [p] * [p'] = [p'] * [p] = [p' * p]$, $p, p' \in K(a)$.

**Remark 3.2.** Let $c \in \bar{\Sigma}_2(K)$. Then the path $p_c := \partial_1 c * \partial_0 c * \partial_2 c$ is homotopic to the constant path.

For each $a \in \Sigma_0(K)$, we denote the set of paths starting and ending in $a$ by $K(a)$.

**Remark 3.3.** Let $p' \in K(a')$, $a' \in \Sigma_0(K)$, $a' \neq a$; then there is a path $\gamma_{a,a'}$, starting in $a'$ and ending in $a$, so that we can define $p := \gamma_{a,a'} * p * \gamma_{a,a'}^{-1} \in K(a)$.

**Remark 3.4.** Let $p := b_n * \cdots * b_1 \in K(a)$, $a \in \Sigma_0(K)$. Then for every index $i = 0, \ldots, n$, we define the path

$$p_i := b_i * \cdots * b_1 * b_n * \cdots * b_{i+1} \in K(\partial_0 b_i).$$

We say that $p_i$ is obtained from $p$ by a shuffle. If $\gamma_{a,\partial_i b_i}$ is a path as in the previous remark, then we define

$$\hat{p}_i := \gamma_{a,\partial_i b_i} * p_i * \gamma_{a,\partial_i b_i}^{-1} \in K(a).$$

In this way, we find (independently of the choice of $\gamma_{a,\partial_i b_i}$)

$$[\hat{p}_i] = [\gamma_{a,\partial_i b_i} * b_i * \cdots * b_1 * b_n * \cdots * b_{i+1} * \gamma_{a,\partial_i b_i}] = [\gamma_{a,\partial_i b_i} * b_i * \cdots * b_1] * [b_n * \cdots * b_{i+1} * \gamma_{a,\partial_i b_i}] = [b_n * \cdots * b_{i+1} * \gamma_{a,\partial_i b_i}] * [\gamma_{a,\partial_i b_i} * b_i * \cdots * b_1] = [p].$$

Of course, if $\partial_0 b_i = a$, then we can pick $\gamma_{a,\partial_i b_i} = \sigma_0 a$ and $[p] = [p_i]$. Note that if, for some $i$, $p_i$ is homotopic to a constant path, then $[\hat{p}_i] = [\gamma_{a,\partial_i b_i} * \gamma_{a,\partial_i b_i}]$ and $[p]$ is the identity of $\pi_1(K, a)_{ab}$.

**Lemma 3.5.** Let $a \in K$ and $p \in K(a)$ be a path of the form

$$p = \ldots * b_m * \ldots * b_1 * b_m * \ldots * b_1, \quad m = 2, \ldots \tag{3.7}$$

Then $[p] = [\sigma_0 a] \in \pi_1(K, a)_{ab}$.

**Proof.** $p$ is of the form $p = p_2 * (\bar{b}_1 * p_1 * \bar{b}_1)$, where $(\bar{b}_1 * p_1 * \bar{b}_1), p_2 \in K(a)$. Suppose that $p_1$ contains the 1-simplex $b_m$ and $p_2 \bar{b}_m$. Shuffle $\bar{b}_1 * p_1 * \bar{b}_1$ to give a path $p_1'$ ending with $b_m$; then, by Remark 3.4, $[\bar{b}_1 * p_1 * \bar{b}_1] = [\gamma_{a,\partial_i b_m} * p_1 * \gamma_{a,\partial_i b_m}]$. Shuffle $p_2$ to get a path $p_2'$ beginning with $\bar{b}_m$. Then $[p_2] = [\gamma_{a,\partial_i b_m} * p_2' * \gamma_{a,\partial_i b_m}]$. It follows that

$$[p] = [\gamma_{a,\partial_i b_m} * p_2' * p_1' * \gamma_{a,\partial_i b_m}].$$

Now, $p_2' * p_1'$ contains $\bar{b}_m * b_m$ and $b_1 * \bar{b}_1$, and these can be removed without changing the homotopy class; we then have a path of the same type with fewer 1-simplices. Suppose on the other hand that there is no $b_m$ contained in $p_1$ such that $p_2$ contains $\bar{b}_m$. Then both $p_1$ and $p_2$ are of the form (3.7) and the result follows by induction.  \[\Box\]
Theorem 3.6. Let \((K, \leq)\) be a pathwise connected poset, and \(a \in \Sigma_0(K)\). Then there is an isomorphism \(H_1(K, \mathbb{Z}) \simeq \pi_1(K, a)_{ab}\).

Proof. Step 1. Let \(a \in \Sigma_0(K)\), and \(p := b_n * \cdots * b_1\) a generic path in \(K(a)\). With this notation, we define the group morphism

\[
T : K(a) \rightarrow C_1(K, \mathbb{Z}) , \quad Tp := \sum_{i=0}^{n} b_i .
\]

(3.8)

Since

\[
bTp = \sum_{i=0}^{n} (\partial_0 b_i - \partial_1 b_i) = a + \sum_{i=1}^{n-1} (\partial_0 b_i - \partial_1 b_{i+1}) - a = 0 ,
\]

we conclude that \(T\) actually takes values in \(Z_1(K, \mathbb{Z})\). Step 2. We verify that \(T\) factorizes through a map

\[
T : \pi_1(K, a) \rightarrow H_1(K, \mathbb{Z})
\]

(3.9)

(note the slight abuse of the notation \(T\)). To this end, we consider \(c \in \Sigma_2(K)\), \(p := b_n * \cdots * b_1 \in K(a)\), and an elementary deformation of \(p\) performed replacing a pair \(b_i = \partial_0 c, b_{i+1} = \partial_2 c, i \in \{0, \ldots, n\}\), by \(\partial_1 c\). If we denote the deformed path by \(p_c\), then

\[
Tp_c = \sum_{i=0}^{n} b_i - \sum_{k=0}^{2} (-1)^k \partial_k c = Tp - b_2 c .
\]

In the same way, an elementary deformation performed replacing \(b_i = \partial_1 c\) by \(\partial_0 c * \partial_2 c\) gives rise to the operation

\[
Tp_c = \sum_{i=0}^{n} b_i + \sum_{k=0}^{2} (-1)^k \partial_k c = Tp + b_2 c .
\]

This implies that \((3.9)\) is well-defined. Step 3. We verify that \((9)\) is surjective. To this end, note that a generic element of \(Z_1(K, \mathbb{Z})\) can be written up to homology in the form

\[
x = \sum_{j=1}^{m} b_j ,
\]

no signs being needed since a \(b_j\) can be replaced by \(\tilde{b}_j\) if necessary. We show that there is a bijection \(f\) on \(1, 2, \ldots, m\) and \(0 = m_0 < m_1 < \cdots < m_{\ell} = m\) and paths \(p_k = q_k * f(m_k) * f(m_{k-1}) * \cdots * f(m_{k-1}+1) * \partial k \in K(a)\) for \(k = 1, 2, \ldots, \ell\). This suffices since setting \(p := p_\ell * \cdots * p_1, Tp = x\) up to homology. We set \(f(1) = 1\) and since \(x \in Z_1(K, \mathbb{Z})\) we may pick a \(b_j\) with \(\partial_0 b_j = \partial_1 b_1\) and set \(f(2) = j\). We continue in this way, i.e. pick a \(b_k\) with \(\partial_0 b_k = \partial_1 b_j\) and set \(f(3) = k\). After say \(r\) steps this process terminates when two conditions are fulfilled, \(\partial_1 f(r) = \partial_0 b_1\) and there is no \(b_s\) with \(\partial_0 b_s = \partial_1 f(r)\). We set \(m_1 := r\) and pick a path \(q_1\) with \(\partial_0 q = \partial_0 b_1\) and \(\partial_1 q = a\). The sum over the remaining 1–simplices is still in \(Z_1(K, \mathbb{Z})\) and the argument may be
repeated to reach the desired conclusion. Note that by Lemma 3.5 $T_p = T_{p'}$ implies $[p] = [p']$. Step 4. Let $p := b_n \cdots b_1 \in \ker T$, i.e. $x := \sum b_i \in B_1(K, \mathbb{Z})$. Then there are 2–cycles $c_1, \ldots, c_m$ such that

$$x = \sum_{j \in J} (\partial_0 c_j - \partial_1 c_j + \partial_2 c_j) - \sum_{k \in I} (\partial_0 c_k - \partial_1 c_k + \partial_2 c_k),$$

where $I \cup J = \{1, \ldots, m\}$. Thus,

$$\sum_i b_i + \sum_{j} (\partial_1 c_j + \overline{\partial_1 c_j}) + \sum_k (\partial_0 c_k + \overline{\partial_0 c_k} + \partial_2 c_k + \overline{\partial_2 c_k}) =$$

$$\sum_j (\partial_0 c_j + \overline{\partial_1 c_j} + \partial_2 c_j) + \sum_k (\partial_0 c'_k + \overline{\partial c'_k} + \partial_2 c'_k),$$

where $c'_k$ is the two simplex got by exchanging the vertices 0 and 1 of $c_k$. Adding on $\sum_j T(\gamma_j \ast \overline{\gamma_j}) + \sum_k T(\gamma_k \ast \overline{\gamma_k})$, where $\gamma_j$ is a path from $\partial_0 c_j$ to $a$ and $\gamma_k$ a path from $\partial_0 c_k$ to $a$, to each side, we get an element $y$ of $B_1(K, \mathbb{Z})$. From the form of the left hand side, we see that there is a path $p_1$ homotopic to $p$ with $T_{p_1} = y$ and, from the form of the right hand side, there is a path $p_2$ homotopic to $\sigma_0 a$ with $T_{p_2} = y$. Hence $[p] = 0$. \hfill \Box

**Corollary 3.7.** Let $(K, \leq)$ be a pathwise connected poset. For every Abelian group $A$, there is an isomorphism $H^1(K, A) \simeq \text{Hom}(H_1(K, \mathbb{Z}), A)$.

**Proof.** By Rem 3.1 we have an isomorphism $H^1(K, A) \simeq \text{Hom}(\pi_1(K, a), A)$. Since $A$ is Abelian, each morphism $\phi \in \text{Hom}(\pi_1(K, a), A)$ factorizes through an element of $\text{Hom}(H_1(K, \mathbb{Z}), A)$. \hfill \Box

Now, let $M$ be a Hausdorff space and let $H_1(M, A)$, $H^1(M, A)$ denote respectively the singular homology and the singular cohomology of $M$ with coefficients in $A$. We fix a poset $M_{\prec}$ providing a subbase of arcwise and simply connected open subsets of $M$.

**Theorem 3.8.** Let $M$ be a Hausdorff, locally arcwise and simply connected space. Then there is an isomorphism $H_1(M, \mathbb{Z}) \simeq H_1(M_{\prec}, \mathbb{Z})$. For every Abelian group $A$ there is an isomorphism $H^1(M_{\prec}, A) \simeq \text{Hom}(H_1(M, \mathbb{Z}), A)$, and this implies the isomorphism

$$H^1(M_{\prec}, A) \simeq H^1(M, A).$$

**Proof.** Let $a \in \Sigma_0(M_{\prec})$. By [12] Thm.2.18, we have isomorphisms $H_1(M, \mathbb{Z}) \simeq \pi_1(M)_{ab} \simeq \pi_1(M_{\prec}, a)_{ab}$ and, by the classical Hurewicz theorem there is an isomorphism $H_1(M, \mathbb{Z}) \simeq \pi_1(M)_{ab}$. The isomorphism (3.10) follows since, by the universal coefficient theorem, $H^1(M, A) \simeq \text{Hom}(H_1(M, \mathbb{Z}), A)$ (see [4] §3.1)). \hfill \Box

When $M$ is a manifold it makes sense to consider the de Rham cohomology $H^1_{dR}(M, \mathbb{R})$, and we have

**Corollary 3.9.** Let $M$ be a connected manifold. Then there is an isomorphism $H^1(M_{\prec}, \mathbb{R}) \simeq H^1_{dR}(M, \mathbb{R})$. 

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4 Homotopy and net cohomology of net bundles.

4.1 Net structures.

A quasinet bundle with base $K$ is given by a 4-ple $\mathcal{X} := (X, \pi, J, K)$, where $X$ is a set called the total space, $\pi : X \to K$ is a surjective map with fibres $X_a := \pi^{-1}(a)$, $a \in \Sigma_0(K)$, and $J$ is a family of injective maps $J_b : X_{\partial b} \to X_{\partial b}$, $b \in \Sigma_1(K)$, called the net structure of $\mathcal{X}$, satisfying the cocycle relations $J_{\partial c}J_{\partial c} = J_{\partial c}$, $c \in \Sigma_2(K)$, and $J_{\sigma a} = id_{X_a}, a \in \Sigma_0(K)$. When each $J_b$ is a bijective map, we say that $\mathcal{X}$ is a net bundle.

The fibres of a net bundle $\mathcal{X}$ are all isomorphic (see [11]); a distinguished fibre $X_a$ of $\mathcal{X}$ will be called the standard fibre of $\mathcal{X}$ and emphasized with the notation $F$.

Analogous definitions apply when considering categories with more structure than the category of sets; in that case, the net structure shall be a family of monomorphisms (or isomorphisms) in the appropriate category. Of particular interest for our purposes will be Hilbert (quasi)net bundles, group (quasi)net bundles, C*-algebra (quasi)net bundles and (quasi)net bundles of topological spaces.

The first class of quasinet bundles will be studied in [5]. Quasinet bundles of C*-algebras motivated our work. They arise as follows.

Example 4.1. In the algebraic approach to quantum field theory (see [3], for example), the set of quantum observables is presented as a net of C*-algebras: by this we mean a map $\mathcal{A}$, assigning a C*-algebra $\mathcal{A}(o)$ to elements $o$ of a distinguished poset $K$ of contractible open subsets of spacetime, and interpreted as the algebra of quantum observables localized in the open set $o \in K$. The map $\mathcal{A}$ preserves the order, in the sense that $\mathcal{A}(o_1) \subseteq \mathcal{A}(o_2)$ whenever $o_1 \subseteq o_2$. This structure may be replaced by a quasinet bundle of C*-algebras defined as follows: we consider the set

$$\hat{\mathcal{A}} := \{(o, \mathcal{A}(o)) , \ o \in K\}$$

endowed with the natural projection $\pi : \hat{\mathcal{A}} \to K$ onto the first component and net structure $J_b$, $b \in \Sigma_1(K)$, defined by the inclusion $\mathcal{A}(\partial_1 b) \subseteq \mathcal{A}(\partial_0 b)$.

Now let $\hat{\mathcal{X}} := (\hat{X}, \hat{\pi}, \hat{J}, K)$ be a quasinet bundle. A map $T : X \to X$ is said to be a morphism if $\hat{\pi}T = \pi$ (this implies that $T$ restricts to maps $T_a : X_a \to X_a, a \in \Sigma_0(K)$) and

$$\hat{J}_bT_{\partial b} = T_{\partial b}J_b , \ b \in \Sigma_1(K).$$

In this case, we use the notation $T \in (\mathcal{X}, \hat{\mathcal{X}})$; if each $T_a, a \in \Sigma_0(K)$, is a bijective map, then we say that $T$ is an isomorphism. In this way, the set of quasinet bundles becomes a category. The full subcategory of net bundles with fibre $F$ will be denoted by $\mathfrak{B}(K, F)$.

Every net bundle $\mathcal{X} := (X, \pi, J, K)$ defines a net bundle

$$\mathcal{X}_o := (X_o, \pi_o, J^o, K^o),$$

where $K^o$ is the dual poset (see Appendix [A.2]), $X_o := X$, $\pi_o := \pi$, and the net structure $J^o$ is defined by $J^o := J_{(\partial_1 b, \partial_0 b)}^{-1}$, $b \in \Sigma_1(K^o)$. This provides an isomorphism $\mathfrak{B}(K, F) \simeq \mathfrak{B}(K^o, F)$, which has been described in cohomological terms in [11 §7].
Now, let $G$ denote the group of automorphisms of $F$. By [10, Prop.3.8], [11, Thm.6.7], for each $a \in \Sigma_0(K)$ there are isomorphisms

$$\mathfrak{B}(K, F) \simeq H^1(K, G) \simeq \hat{\text{Hom}}(\pi_1(K, a), G),$$

where $\mathfrak{B}(K, F)$ is the set of isomorphism classes of net bundles with fibre $F$ and $\hat{\text{Hom}}(\pi_1(K, a), G)$ denotes the set of equivalence classes of morphisms from the homotopy group $\pi_1(K, a)$ to $G$, two morphisms being equivalent if they differ by an inner automorphism of $G$.

The notion of morphism can be generalized to allow changes of the base $K$; to this end, we introduce the notion of pullback.

Let $\eta : K' \to K$ be a morphism of posets. Given a quasinet bundle $X$ over $K$, we define a quasinet bundle $X^{\eta}$, by considering the set

$$X^{\eta} := \{(x, a') \in X \times \Sigma_0(K') : \pi(x) = \eta(a')\}$$

endowed with the obvious projection $\pi^{\eta} : X^{\eta} \to K'$ and the net structure $J^{\eta}_b(x, \partial_1 b') := (J^{\eta}_b(x', \partial_1 b'), x \in X_{\partial_1 \eta(y')}$.

By restricting the notion of pullback to the subcategory of net bundles with fibre $F$, we obtain a functor $\eta^* : \mathfrak{B}(K, F) \to \mathfrak{B}(K', F)$, $\eta^*X := X^{\eta}$.

We are now able to define the notion of a generalized morphism from a quasinet bundle $\hat{X} := (\hat{X}, \hat{\pi}, \hat{J}, \hat{K})$ to a quasinet bundle $\hat{X} := (X, \pi, J, K)$ as a pair $(\eta, T)$, where $\eta : \hat{K} \to K$ is a morphism of posets, and $T \in (\hat{X}^{\eta}, \hat{X})$ is a morphism in the usual sense.

We also recall that a local section of a quasinet bundle $(X, \pi, J, K)$ is given by a map $\sigma$ from a subposet $V \subseteq K$ into $X$, such that $\pi \sigma = id_V$ and $J_b \sigma(\partial_1 b) = \sigma(\partial_0 b)$, $b \in \Sigma_1(V)$. If $V = K$, then $\sigma$ is said to be global. We denote the set of local sections of $\hat{X}$ over $V$ by $S(V; \hat{X})$ (for $V = K$, we use the analogous notation for the set of global sections).

### 4.2 Poset net bundles.

In the present section, we consider the notion of poset net bundle. The interesting applications are when the fibre is the poset arising from the topology of a space. In particular, in the following we relate the homotopy of a poset net bundle to that of the underlying poset.

A poset net bundle is a net bundle $(X, \eta, J, K)$ where each fibre $X_a$, $a \in \Sigma_0(K)$, is endowed with an ordering $\leq_a$ and each map $J_b$, $b \in \Sigma_1(K)$, is an isomorphism of posets.

A standard argument for net bundles allows one to conclude that each poset net bundle admits a fixed poset $(F, \leq)$ as standard fibre, endowed with poset isomorphisms $V_a : F \to X_a$, $a \in \Sigma_0(K)$.

**Lemma 4.2.** Let $(X, \eta, J, K)$ be a poset net bundle. Then there is a canonical ordering $\prec$ on the total space $X$ making $\eta$ a poset epimorphism. In this way, morphisms of poset net bundles give rise to poset morphisms of the underlying total spaces.
Proof. Define \( x_1 < x_0 \iff a_1 := \eta(x_1) \leq \eta(x_0) =: a_0 \) and \( J_{(a_0,a_1)}(x_1) \leq_{a_0} x_0 \).

The previous lemma illustrates the main advantage of considering poset net bundles: in fact, the usual net bundles \((X, p, J, K)\) correspond to partial orderings where each fibre \(X_a\) has the “discrete” order relation \( x \leq x' \iff x = x', \ x, x' \in X_a\) (see \[\text{III \& IV} \], whereas a poset net bundle may be endowed with a more interesting partial ordering.

By the previous lemma it makes sense to consider the natural maps

\[
\eta^* : H^1(K, G) \to H^1(X, G), \tag{4.3}
\]

\[
\eta_* : \pi_1(X, x) \to \pi_1(K, a), \quad x \in X_a, \tag{4.4}
\]

for every poset net bundle \((X, \eta, J, K)\) and group \(G\).

**Theorem 4.3.** Let \((X, \eta, J, K)\) be a poset net bundle with pathwise connected fibre \(F\). Then for each \(a \in \Sigma_0(K)\) there is a morphism \(j_a : \pi_1(X_a, x) \to \pi_1(X, x)\) with \(\pi_1(X_a, x) \simeq \pi_1(F, a')\), \(x \in X_a, a' \in \Sigma_0(F)\), and an exact sequence

\[
\pi_1(X_a, x) \xrightarrow{j_a} \pi_1(X, x) \xrightarrow{\eta_*} \pi_1(K, a) \to 0. \tag{4.5}
\]

In particular, if \((F, \leq)\) is simply connected, then \(\eta_*\) is an isomorphism.

**Proof.** The morphism \(j_a\) is induced by the order-preserving inclusion \(j_a : (X_a, \leq_a) \to (X, \leq)\). Clearly,

\[
\eta_* j_a[p] = [\sigma_0 a], \quad [p] \in \pi_1(X_a, x),
\]

where \(\sigma_0 a, a \in \Sigma_0(K)\), denotes, as usual, the degenerate 1-simplex. In other words, \(j_a(\pi_1(X_a, x)) \subseteq \ker \eta_*\). Let \(p := b_n \ast \cdots \ast b_1\) be a closed path in \(\Sigma_1(K)\). We fix \(|v_1| \in X_{|b_1|}\) and define recursively

\[
\begin{align*}
|v_{k+1}| &= J_{|b_{k+1}|; \partial_0 b_k}^{-1} J_{|b_k|; \partial_0 b_k}^{-1} |v_k| \\
\partial_0 v_{k+1} &= J_{|b_{k+1}|; \partial_0 b_k}^{-1} |v_{k+1}|, \quad i = 0, 1 \\
v_k &= (|v_k|; \partial_0 v_k, \partial_1 v_k)
\end{align*}
\]

(recalling that \(\partial_0 b_k = \partial_1 b_{k+1}, k = 1, \ldots, n - 1\)). In this way, we obtain a path \(\tilde{p}_0 := \tilde{b}_n \ast \cdots \ast \tilde{b}_1\) in \(\Sigma_1(X)\) such that \(\eta_* \tilde{p}_0 = p\). Since \(\tilde{p}_0\) may be not closed, we consider a path \(\tilde{p}_1\) in \(\Sigma_1(X_{\partial_0 b_1} \simeq F)\) starting in \(\partial_0 v_0\) and ending in \(\partial_1 v_1\), and define \(\tilde{p} := \tilde{p}_0 \ast \tilde{p}_1\). By construction, \(\eta_*(\tilde{p})\) coincides with the closed path

\[
b_n \ast \cdots \ast b_1 \ast \sigma_0 \partial_1 b_1 \ast \cdots \ast \sigma_0 \partial_1 b_1
\]

where \(\sigma_0 \partial_1 b_1\) is the degenerate 1-simplex in \(\Sigma_1(K)\). The closed path constructed above is clearly homotopic to \(p\). For the converse, let \(\tilde{p} := v_n \ast \cdots \ast v_1\) be a closed path in \(X\) with \(\tilde{p} \in \ker \eta_*\). We prove that \(\tilde{p}\) is homotopic to a path in \(X_a\) for some \(a \in \Sigma_0(K)\). To this end, we consider the path in \(K\)

\[
\eta_*(\tilde{p}) := b_n \ast b_{n-1} \ast \cdots \ast b_1, \quad b_k := \eta_* v_k, \quad k = n, \ldots, 1.
\]
Since \( \tilde{p} \in \ker \eta_* \), we may find 2-simplices \( c_1, \ldots, c_m \in \Sigma_2(K) \) providing elementary deformations of \( \eta_*(\tilde{p}) \) to the constant path

\[
\sigma_0 a \quad a := \partial_1 b_n \quad \sigma_0 a := (a; a, a).
\]

In particular we may have, for example, an elementary deformation contracting \( b_2 * b_1 \) to \( \partial_1 c_1 \). Now, \( v_2 * v_1 \) is a path in \( \eta^{-1}(V_{c_1}) \); using the definition of local chart \((\mathbf{11} \; \S 4.3)\) and Lemma \((\mathbf{L} 2)\) we conclude that there is an isomorphism of posets

\[
\eta^{-1}(V_{c_1}) \simeq V_{c_1} \times F.
\]

The image of \( v_2 * v_1 \) under the above isomorphism can be written, according to \((\mathbf{A} 1)\) as a pair \((p^\alpha, p^\mu)\), where \( p^\alpha = \partial_1 c_1 \) is a path in \( V_{c_1} \) and \( p^\mu \) is a path in \( F \). Applying \((\mathbf{A} 1)\), we conclude that \( v_2 * v_1 \) is homotopic to a path \( v_m' * \ldots * v_1' \) such that

\[
\eta_*(v_m' * \ldots * v_1') = \partial_1 c_1 * \sigma_0 a * \ldots * \sigma_0 a.
\]

Thus, \( \tilde{p} \) is homotopic to a path of the type

\[
\tilde{p}_1 := v_n * v_{n-1} * \ldots * v_3 * v_m' * \ldots * v_1',
\]

Note that if there is a \( c_2 \in \Sigma_2(K) \) contracting \( b_3 * \partial_1 c_1 \), then \( v_3 * v_m' * \ldots * v_1' \) is a path in \( \eta^{-1}(V_{c_1}) \simeq V_{c_2} \times F \), so that we can again apply the above argument. Moreover, the same procedure applies to ampliations of \( \eta_*(\tilde{p}) \). Iterating the above operations for each deformation of \( \eta_*(\tilde{p}) \), we conclude that \( \tilde{p} \) is homotopic to a path \( \tilde{p}_m \) such that

\[
\eta_*(\tilde{p}_m) = \sigma_0 a * \ldots * \sigma_0 a, \text{ i.e. } \tilde{p}_m \text{ is a path in } X_0. \text{ This proves } \ker \eta_* = j_a (\pi_1 (X_0, x)).
\]

**Corollary 4.4.** Let \((X, \eta, J, K)\) be a poset net bundle with simply connected fibre \((F, \leq)\). Then the maps \((\mathbf{4.3}), (\mathbf{4.3})\) define group isomorphisms.

**Proof.** It suffices to note that \( \pi_1 (F, a') = 0 \) and apply \((\mathbf{4.2}), (\mathbf{4.5})\).

The previous theorem has a well-known counterpart, involving topologies of spaces; in this case, \((\mathbf{4.5})\) is a long exact sequence involving higher homotopy groups. One might hope to generalize Thm\((\mathbf{4.3})\) and get a long exact sequence by introducing higher homotopy groups for posets.

Now let \( \mathcal{X} := (X, p, J, K) \) be a net bundle of topological spaces, so that each \( J_b, b \in \Sigma_1(K) \), is a homeomorphism. We fix a poset \( \mathcal{X}_{<,a} \) of open subsets \( U \subset X_a, U \neq \emptyset \), ordered under inclusion and forming a subbase for \( X_a \). We require that each \( J_b \) is a poset isomorphism from \( \mathcal{X}_{<,\partial_1 b} \) to \( \mathcal{X}_{<,\partial_0 b} \). Then defining \( \mathcal{X}_{<} := \cup_a \mathcal{X}_{<,a} \) and letting \( p_{<} : \mathcal{X}_{<} \to K \) be the obvious projection, we conclude that

\[
\mathcal{X}_{<} := (X_{<}, p_{<}, J, K) \quad (\mathbf{4.6})
\]

is a poset net bundle, called the **poset net bundle associated with \( \mathcal{X} \). By \([\mathbf{12}, \text{Thm.2.8}]\), the topological homotopy group \( \pi_1 (X_a) \) is isomorphic to \( \pi_1 (X_{<,a}, U), a \in \Sigma_0(K), U \in \Sigma_0(X_{<,a}) \); thus, by Thm\((\mathbf{4.3})\) we conclude the following:
Corollary 4.5. Let $X$ be a net bundle of topological spaces with fibre a Hausdorff, locally arcwise and simply connected space $M$. For each $a \in \Sigma_0(K)$, there is an exact sequence

$$\pi_1(M) \overset{i_a}{\rightarrow} \pi_1(X_o, U) \overset{j_a}{\rightarrow} \pi_1(K, a) \rightarrow 0 \ .$$  \hfill (4.7)

Example 4.6. For the notion of principal net bundle, we refer the reader to [11]. Let $\mathcal{P} := (P, \pi, J, K, R)$ be a principal net bundle with a connected Lie group $G$ as standard fibre. Then (provided the net structure is defined by means of isomorphisms of Lie groups), $\mathcal{P}$ is also endowed with the structure of a topological net bundle and the previous corollary applies.

The global space of a net bundle of topological spaces $X := (X, p, J, K)$ can be topologized in the following way: pick for each $a \in \Sigma_0(K)$ a subbase $\mathcal{U}_a$ for the topology of $X_a$. If $U \in \mathcal{U}_a$, we define the cylinder with base $U$ to be

$$T_{a,U} := \{ x \in J_{(o,a)}(U), o \in V_a \} .$$ \hfill (4.8)

Clearly $p(T_{a,U}) = V_a$. The family $\{T_{a,U}\}$ is a subbase for a topology on $X$ and we denote the associated topological space by $\tau X$. This topology is independent of the choice of subbases. Note that there is a continuous bijection

$$T_{a,U} \rightarrow V_a \times U \ , \ x \mapsto (p(x), J_{(a,p(x))}x) .$$

The projection $p$ is therefore continuous as a map from $\tau X$ to $\tau K$.

Lemma 4.7. Let $X := (X, p, J, K)$, $\hat{X} := (\hat{X}, \hat{p}, \hat{J}, \hat{K})$ be net bundles of topological spaces and $f \in (X, \hat{X})$ a morphism. Then $f : \tau X \rightarrow \tau \hat{X}$ is continuous.

Proof. It suffices to show that if $\hat{V}$ is open in $\hat{X}_a$ then $f^{-1}(T_{a,\hat{V}}) = T_{a,f^{-1}(\hat{V})}$. If $x \in J_{(o,a)}(f^{-1}(\hat{V}))$ then $f(x) \in fJ_{(o,a)}(f^{-1}(\hat{V})) \subset J_{(o,a)}(\hat{V})$. Hence $T_{a,f^{-1}(\hat{V})} \subset f^{-1}(T_{a,\hat{V}})$. If $f(x) \in J_{(o,a)}(\hat{V})$ then $x \in f^{-1}J_{(o,a)}(\hat{V}) = J_{(o,a)}f^{-1}(\hat{V})$ so $f^{-1}(T_{a,\hat{V}}) \subset T_{a,f^{-1}(\hat{V})}$. \hfill $\square$

When $X$ is trivial, the previous Lemma implies that there is a homeomorphism

$$\tau X \cong \tau K \times M ,$$ \hfill (4.9)

where $\tau K$ is the space $K$ with the Alexandroff topology (see [11, §2.3]). The previous elementary remark gives an idea of the local behaviour of $\tau X$ for a generic $X$, and implies that the bundle $p : \tau X \rightarrow \tau K$ is locally trivial when $X$ is locally trivial.

If $s \in S(V_a; X)$ and $(a) \in U$, then $s(o) \in T_{a,U}$ for each $o \in V_a$ and $s^{-1}(T_{a,U}) = V_a$; we conclude that each $s \in S(V_a; X)$ defines a local section $s : V_a \rightarrow \tau X$.

We now consider the associated net bundle of topological spaces $X_o$, see (4.6), and the associated poset $X_{o,<}$, see (4.6) and Lemma 4.2.
Theorem 4.8. Let $\mathcal{X} := (X, p, J, K)$ be a net bundle of topological spaces with standard fibre a space $M$. Then the map $p : \tau X \to \tau K$ defines a fibre bundle with fibre $M$. Moreover, there is an isomorphism

$$\pi_1(X_{o,<}, U) \simeq \pi_1(\tau X), \quad U \in \Sigma_0(X_{o,<}), \quad (4.10)$$

and for every group $G$ there are isomorphisms

$$H^1(X_{o,<}, G) \simeq \hat{\text{Hom}}(\pi_1(\tau X), G). \quad (4.11)$$

Proof. In order to prove the theorem it just remains to check the details in (4.10). To this end, let us consider the poset $X_{cyl}$ with elements the sets (4.8), ordered under inclusion. Since $X_{cyl}$ is a subbase for $\tau X$, we have an isomorphism $\pi_1(X_{cyl}, T) \simeq \pi_1(\tau X), T \in \Sigma_0(X_{cyl})$. Thus, in order to prove (4.10), it suffices to construct a poset isomorphism from $X_{o,<}$ to $X_{cyl}$. By the definition of $X_{o,<}$, $X_{o,<}$ coincides with $X_{<}$ as a set, thus elements of $X_{o,<}$ are given by open sets $U \in X_{<,a}, a \in \Sigma_0(K^o)$. The order relation for $X_{o,<}$ is given by $U < U' \iff a \geq a'$ and $J^{o}(a, a')(U) \subseteq U'$, i.e. $U \subseteq J_{(a, a')}(U')$ (see Lemma 4.2).

We consider the bijective map

$$X_{<} \to X_{cyl}, \quad U \in X_{<,a} \mapsto T_{a,U}. \quad (4.12)$$

By construction, $U < U'$ implies $V_a \subseteq V_{a'}$ and $U \subseteq J_{(a, a')}(U')$, thus $T_{a,U} \subseteq T_{a',U'}$, and the theorem is proved.

Corollary 4.9. Let $G$ be a group. For every principal net $G$-bundle $\mathcal{P} := (P, p, J, R, K)$, the fibre bundle $p : \tau P \to \tau K$ has locally constant transition maps.

Proof. For every $a \in \Sigma_0(K), V_a$ is an open neighbourhood of $a$ trivializing $\tau P$, so there are local charts $\theta_a : V_a \times G \rightarrow p^{-1}(V_a), a \in \Sigma_0(K)$, giving rise to locally constant transition maps (see [11, Lemma 5.6]).

5 K-theory of a poset.

5.1 Net bundles of Banach spaces.

We now specialize our discussion to the case of quasinet bundles of Banach spaces. These objects have already been studied in a combinatorial setting ([5]). In the present paper, we emphasize the topological viewpoint.

Banach quasinet bundles are quasinet bundles having Banach spaces as fibres, and net structure given by bounded, injective, linear operators. If $\mathcal{E} := (E, \pi, J, K), \hat{\mathcal{E}} := (\hat{E}, \hat{\pi}, \hat{J}, \hat{K})$ are Banach quasinet bundles, then we denote the set of bounded morphisms from $\mathcal{E}$ into $\hat{\mathcal{E}}$ by $(\mathcal{E}, \hat{\mathcal{E}})$. If $T \in (\mathcal{E}, \hat{\mathcal{E}})$, then each $T_a : E_a \to \hat{E}_a, a \in \Sigma_0(K)$, is a bounded linear operator satisfying the relations

$$T_{a}J_{b} = \hat{J}_{b}T_{a}, \quad b \in \Sigma_1(K). \quad (5.1)$$
If $\mathcal{E}, \hat{\mathcal{E}}$ are Banach net bundles, then ($K$ being pathwise connected) (5.1) implies that $\|\mathcal{T}_a\|$ does not depend on the choice of $a \in \Sigma_0(K)$.

The Banach quasinet bundle $\mathcal{E}$ is said to be finite dimensional if the dimension of the fibres $E_a$, $a \in \Sigma_0(K)$, has an upper bound $d \in \mathbb{N}$. Since each $J_b$, $b \in \Sigma_1(K)$, between vector spaces with the same finite dimension is also surjective, we have

**Lemma 5.1.** Let $\mathcal{E}$ be a finite-dimensional Banach quasinet bundle. Then $\mathcal{E}$ is a Banach net bundle if and only if the rank function $d(a) := \dim(E_a)$, $a \in \Sigma_0(K)$, is constant.

A Banach quasinet bundle $\mathcal{E}$ is said to be a Hilbert quasinet bundle if each fibre $E_a$, $a \in \Sigma_0(K)$, is a Hilbert space, and each $J_b$, $b \in \Sigma_1(K)$, preserves the scalar product.

Some remarks on the classification of Banach net bundles follow. If we want to apply (4.2) to Banach net bundles, then we have to pick the group $G$ to take account of the structure of interest. For example, we may consider the group of invertible operators of a Banach space, or the unitary group $U$ of a Hilbert space $\mathcal{H}$ if we are interested in isomorphisms preserving the Hermitian structure. In particular, in the finite-dimensional case, $G$ is the complex linear group $\mathbb{G}L(d)$, $d \in \mathbb{N}$, or the unitary group $U(d)$.

Let us denote the class of isomorphism of Hilbert net bundles with fibre $\mathcal{H}$ by $\hat{H}_{\text{net}}(K, \mathcal{H})$, and the Čech cohomology of the dual poset $K^\circ$ by $\hat{H}_{\text{net}}^1(K^\circ, \mathcal{U})$. Applying [11] Thm.8.1, we find

**Proposition 5.2.** For each poset $K$ and $a \in \Sigma_0(K)$, there are isomorphisms

$$\hat{H}_{\text{net}}(K, \mathcal{H}) \simeq H^1(K, \mathcal{U}) \simeq H^1_\mathbb{C}(K^\circ, \mathcal{U}) \simeq \hat{\text{Hom}}(\pi_1(K, a), \mathcal{U}).$$

Unlike ordinary vector bundles, a finite-dimensional Banach net bundle cannot in general be endowed with an Hermitian structure, in fact the equivalence relation induced by inner automorphisms leaves the determinant map $\det(\chi)$, $\chi \in \text{Hom}(\pi_1(K, a), \mathbb{G}L(d))$, $p \in \pi_1(K, a)$, invariant. Thus, if $\chi$ does not take values in $U(d)$, the same is true of every $\chi' \in \text{Hom}(\pi_1(K, a), \mathbb{G}L(d))$ equivalent to $\chi$.

We denote the category of Hilbert quasinet bundles over $K$ by $\mathcal{H}_{\text{net}}(K)$, and the full subcategory of Hilbert net bundles by $H_{\text{net}}(K)$. For finite dimensional Hilbert quasinet bundles, we use the analogous notations $\mathcal{V}_{\text{net}}(K)$ and $V_{\text{net}}(K)$.

Some algebraic structures are naturally defined on $\mathcal{H}_{\text{net}}(K)$: (1) The direct sum $\mathcal{E} \oplus \hat{\mathcal{E}}$; (2) The adjoint $\ast : (\mathcal{E}, \hat{\mathcal{E}}) \to (\hat{\mathcal{E}}, \mathcal{E})$; (3) The tensor product $\mathcal{E} \otimes \hat{\mathcal{E}}$; (4) The symmetry $\theta \in (\mathcal{E} \otimes \hat{\mathcal{E}}, \hat{\mathcal{E}} \otimes \mathcal{E})$, $\theta_a(v \otimes \hat{v}) := \hat{v} \otimes v$, $v \in E_a$, $\hat{v} \in \hat{E}_a$; (5) The conjugate net bundle $\overline{\mathcal{E}} := (\mathcal{E}, \pi, J, K)$; (6) The antisymmetric tensor powers $\wedge^r \mathcal{E}$, $r \in \mathbb{N}$.

There is a Banach quasinet bundle of morphisms $B(\mathcal{E}, \hat{\mathcal{E}})$ associated with Hilbert quasinet bundles $\mathcal{E}$, $\hat{\mathcal{E}}$. It is defined as follows: for each $a \in K$, consider the vector space $(E_a, \hat{E}_a)$ of linear operators from $E_a$ into $\hat{E}_a$ and the disjoint union $B(E_a, E'_a) := \bigcup_p (E_a, E'_a)$. Then define the net structure $I_b(t) := J_b t J_b^{-1}$, $t \in (E_d b, E_d b)$, $b \in \Sigma_1(K)$. In the finite-dimensional case, we clearly have an isomorphism of Banach quasinet bundles

$$B(\mathcal{E}, \hat{\mathcal{E}}) \to \mathcal{E} \otimes \hat{\mathcal{E}}.$$  

(5.2)
The above isomorphism induces an Hermitian structure on $B(\mathcal{E}, \hat{\mathcal{E}})$, making it into a Hilbert quasinet bundle.

**Lemma 5.3.** Every morphism $T \in (\mathcal{E}, \hat{\mathcal{E}})$ defines a section of $B(\mathcal{E}, \hat{\mathcal{E}})$.

*Proof.* If $T \in (\mathcal{E}, \hat{\mathcal{E}})$, then we may regard $T$ as a map $T : \Sigma_0(K) \rightarrow B(\mathcal{E}, \hat{\mathcal{E}})$, $T(a) := T_a$.

By the definition of morphism, we find that $T$ satisfies the compatibility property w.r.t. the net structure of $B(\mathcal{E}, \hat{\mathcal{E}})$, i.e. $I_b \circ T(\partial_1 b) = T(\partial_0 b)$.

The existence of the adjoint and norm on $qH_{\text{net}}(K)$ imply that the space $(\mathcal{E}, \mathcal{E})$ is a unital $\mathcal{C}^*$-algebra. We can now prove the following:

**Lemma 5.4.** The category $qH_{\text{net}}(K)$ has subobjects.

*Proof.* Consider $\mathcal{E} \in qH_{\text{net}}(K)$, $\mathcal{E} := (E, \pi, J, K)$ and a projection $P \in (\mathcal{E}, \mathcal{E})$. We define $P\mathcal{E} := (PE, \pi', J', K)$, where $PE := \cup_0 P_a E_a$, $\pi'$ is the natural map from $PE$ onto $K$, and $J'_{b,v} := J_{b,v}$, $v \in P_{\partial_b} E_{\partial_b}$, $b \in \Sigma_1(K)$, $a \in \Sigma_0(K)$. Since $J_b P_{\partial_b} = P_{\partial_b} J_b$, we conclude that $J'_{b,v} P_{\partial_b} E_{\partial_b} \subseteq P_{\partial_b} E_{\partial_b}$, so that $J'$ is a well-defined net structure, and $P\mathcal{E}$ is a Hilbert quasinet bundle.

Hilbert quasinet bundles associated with projections as in the previous lemma are called direct summands. A Hilbert net bundle $\mathcal{E}$ is said to be irreducible if it does not admit direct summands different from 0 and $\mathcal{E}$. We summarize the results of the present section in the following proposition.

**Proposition 5.5.** The category $qH_{\text{net}}(K)$ with arrows bounded morphisms is a symmetric tensor $\mathcal{C}^*$-category with subobjects, direct sums, and identity object given by the trivial Hilbert net bundle $\iota := (K \times \mathbb{C}, \pi, j, K)$. (When $K$ is pathwise connected) $\iota$ is simple, i.e. $(\iota, \iota) = \mathbb{C}$.

In the next lemma, we establish an equivalence between the presence of nowhere zero global sections and trivial direct summands.

**Lemma 5.6.** Let $\mathcal{E} := (E, \pi, J, K)$ be a Hilbert quasinet bundle. There is a nowhere zero section $\sigma \in S(K; \mathcal{E})$ if and only if $\mathcal{E}$ has a trivial direct summand of rank one. Thus, an irreducible Hilbert net bundle is nontrivial if and only if it lacks nowhere zero sections.

*Proof.* If $\mathcal{E}$ has a trivial direct summand, then it is clear that there is a nowhere zero section $\sigma : K \rightarrow E$. Conversely, since the net structure $J$ involves isometric maps, if there is such a section $\sigma$ then up to normalization we may assume that $\|\sigma(o)\| = 1$, $o \in K$. Given the trivial net bundle $\iota := (K \times \mathbb{C}, p, j, K)$, we define the map $I_\sigma(o, z) := z\sigma(o)$, $o \in K$, $z \in \mathbb{C}$. It is clear that $I_\sigma$ is injective. Since $\sigma(o) \in E_o$, $o \in K$, $I_\sigma$ preserves the fibres. We now verify that $I_\sigma$ preserves the net structure, i.e. $I_\sigma \circ j = J \circ I_\sigma$ for every $b \in \Sigma_1(K)$, we compute

$$I_\sigma \circ j_b(\partial_1 b, z) = I_\sigma(\partial_0 b, z) = z\sigma(\partial_0 b) = z J_b \circ \sigma(\partial_1 b) = J_b \circ I_\sigma(\partial_1 b, z),$$

where we used the fact that $J_b \circ \sigma(\partial_1 b) = \sigma(\partial_0 b)$. This proves that $I_\sigma$ is a net bundle morphism. □
Corollary 5.7. Let \( d \in \mathbb{N} \) and \( \mathcal{E} := (E, \pi, J, K) \) be a rank \( d \) Hilbert net bundle. If there are \( d \) linearly independent sections of \( \mathcal{E} \), then \( \mathcal{E} \) is trivial.

The above corollary shows that the existence of sufficient global sections means that the Hilbert net bundle is trivial, in contrast to the usual topological setting. The reason is that, a section of a Hilbert net bundle not zero over some \( o \in K \) is nowhere zero. Of course, this is not true for ordinary vector bundles. The Chern functions introduced in Sec. 5.3 measure the obstruction to the existence of global sections.

5.2 Basic Properties of K-theory.

In the sequel, we always assume that our Hilbert net bundles are finite-dimensional.

Let \( \mathcal{E} \) be a Hilbert net bundle over \( K \). We denote the isomorphism class of \( \mathcal{E} \) by \( [\mathcal{E}] \). Direct sum and tensor product induce a natural semiring structure on the set \( \mathcal{V}_{\text{net}}(K) \) of isomorphism classes of finite-dimensional Hilbert net bundles. We define \( K^0(K) \) to be the Grothendieck ring associated with \( \mathcal{V}_{\text{net}}(K) \), and denote the semiring morphism assigning to each isomorphism class \( [\mathcal{E}] \) the associated element of \( K^0(K) \) by

\[
\mathcal{V}_{\text{net}}(K) \to K^0(K) \quad [\mathcal{E}] \mapsto [\mathcal{E}].
\]

By definition, every element of \( K^0(K) \) can be written as \( [\mathcal{E}] - [\mathcal{F}] \), with \( [\mathcal{E}] \), \( [\mathcal{F}] \in \mathcal{V}_{\text{net}}(K) \). In analogy with the usual K-theory, we characterize Hilbert net bundles \( \mathcal{E}, \mathcal{E}' \to K \) with \( [\mathcal{E}] = [\mathcal{E}'] \): as this result depends only on the definition of Grothendieck ring, we omit the proof.

Lemma 5.8. Let \( \mathcal{E}, \mathcal{E}' \) be Hilbert net bundles over \( K \). Then \( [\mathcal{E}] = [\mathcal{E}'] \in K^0(K) \) if and only if there exists a Hilbert net bundle \( \mathcal{F} \) such that \( \mathcal{E} \oplus \mathcal{F} \) is isomorphic to \( \mathcal{E}' \oplus \mathcal{F} \).

The next two results indicate a drastic difference between ordinary K-theory and net K-theory. In fact, we are going to prove that every Hilbert net bundle with a complement is trivial (in ordinary K-theory, every vector bundle has a complement, see [5, I.6.5]).

Proposition 5.9. Let \( \mathcal{T}_d \) be the trivial rank \( d \) Hilbert net bundle. Then every direct summand of \( \mathcal{T}_d \) is trivial.

Proof. Let \( \mathcal{E}' \) be a Hilbert net subbundle of \( \mathcal{T}_d \), and \( P \in (\mathcal{T}_d, \mathcal{T}_d) \) the projection associated with \( \mathcal{E}' \). Then \( P \) is a section of the bundle \( B(\mathcal{T}_d, \mathcal{T}_d) \) (see Lemma 5.3). Now, \( B(\mathcal{T}_d, \mathcal{T}_d) \) is isomorphic to the trivial net bundle \( K \times \mathcal{M}(d) \) (where \( \mathcal{M}(d) \) denotes the matrix algebra of order \( d \)). Thus, every section of \( B(\mathcal{T}_d, \mathcal{T}_d) \) is constant. In particular, \( P \) is a constant section whose range is a fixed vector subspace \( V \) of \( \mathbb{C}^d \), and \( \mathcal{E}' \) is isomorphic to the trivial bundle \( K \times V \). \(\square\)

Corollary 5.10. Let \( \mathcal{E}, \hat{\mathcal{E}} \) be Hilbert net bundles such that \( \mathcal{E} \oplus \hat{\mathcal{E}} \simeq \mathcal{T}_n \) for some \( n \in \mathbb{N} \). Then \( \mathcal{E}, \hat{\mathcal{E}} \) are trivial.
Hilbert net bundles $E$, $F$ (not necessarily of the same rank) are said to be stably equivalent if $E \oplus T_j$ is isomorphic to $F \oplus T_k$ for some $j, k \in \mathbb{N}$. It follows from the previous corollary that if a Hilbert net bundle is stably equivalent to a trivial net bundle, then it is trivial. The set $V^s_{\text{net}}(K)$ of stable equivalence classes is an Abelian semiring w.r.t. the operation of direct sum and tensor product. By definition, there is a natural epimorphism

$$\hat{V}_{\text{net}}(K) \to V^s_{\text{net}}(K) \quad (5.4)$$

mapping each $\{E\} \in \hat{V}_{\text{net}}(K)$ into its stable equivalence class. The kernel of (5.4) is given by the set of isomorphism classes of trivial Hilbert net bundles, hence labelled by the non-negative integers. For the notion of representation ring, we recommend [13] to the reader; the following result is a direct consequence of [12, Thm.2.8].

**Theorem 5.11.** Let $a \in \Sigma_0(K)$. Then the category $V_{\text{net}}(K)$ is equivalent to the category of unitary, finite-dimensional representations of $\pi_1(K, a)$, so that $K^0(K)$ is isomorphic to the representation ring of $\pi_1(K, a)$.

**Corollary 5.12.** Let $E := (E, \pi, J, K)$ be a Hilbert net bundle. Then there is a unique decomposition $E = \bigoplus_{i=1}^r n_i V_i$, where each $V_i$ is irreducible, $n_i \in \mathbb{N}$, and $n_i V_i$ denotes the direct sum of $n_i$ copies of $V_i$.

**Proof.** In fact, the above decomposition corresponds to the decomposition of the representation $\chi : \pi_1(K, a) \to U(d)$ associated with $E$ into irreducibles. Note that a Hilbert net bundle is irreducible if and only if the associated representation of $\pi_1(K, a)$ is irreducible. 

**Corollary 5.13.** Let $(K, \leq)$ be a poset such that $\pi_1(K, a)$ is Abelian. Then every Hilbert net bundle over $K$ is a direct sum of line net bundles. The ring $K^0(K)$ is generated by $H^1(K, \mathbb{T})$ as a $\mathbb{Z}$-module.

Let us now consider the rank function $\rho : \hat{V}_{\text{net}}(K) \to \mathbb{N}$, assigning to the (isomorphism class of a) Hilbert net bundle the corresponding rank. It is clear that $\rho$ is a semiring epimorphism, so that it makes sense to consider the associated extension $\rho : K^0(K) \to \mathbb{Z}$. The kernel of $\rho$ is called the reduced net K-theory of $K$ and denoted by $\tilde{K}^0(K)$. In this way we have a direct sum decomposition

$$K^0(K) = \mathbb{Z} \oplus \tilde{K}^0(K) \quad (5.5)$$

so that $\tilde{K}^0(K)$ embeds into $K^0(K)$. The reduced group $\tilde{K}^0(K)$ encodes the nontrivial part of the K-theory of $K$: if $K$ is simply connected, then $\tilde{K}^0(K) = 0$. By the definition of stable equivalence class and recalling (5.4, 5.5), we conclude that the canonical map $\hat{V}_{\text{net}}(K) \to K^0(K)$ factorizes through $V^s_{\text{net}}(K)$, in such a way that the following diagram commutes:

$$\xymatrix{ \hat{V}_{\text{net}}(K) \ar[r] \ar[d] & V^s_{\text{net}}(K) \ar[d] \ar[r] & \tilde{K}^0(K) \ar[d] & \ar[l] \ar[r] & \tilde{K}^0(K) } \quad (5.6)$$
Net K-theory satisfies natural functorial properties. If \( \eta : K' \to K \) is a poset morphism, then the pullback induces a natural ring morphism \( \eta^* : K^0(K) \to K^0(K') \). If \( \mathcal{X} := (X, \eta, J, K) \) is a poset net bundle with fibre \( F \), then by (4.3) we find an exact sequence of rings

\[
Z \to K^0(K) \xrightarrow{\eta^*} K^0(X) \to K^0(F) .
\] (5.7)

Note that the images of \( Z \) w.r.t. the above maps correspond to trivial representations of the homotopy groups \( \pi_1(K, a), \pi_1(X, x), \pi_1(F, a') \); so that, (5.7) restricts to an exact sequence of reduced K-groups

\[
0 \to \tilde{K}^0(K) \xrightarrow{\eta^*} \tilde{K}^0(X) \to \tilde{K}^0(F) .
\]

We conclude that if \( K^0(F) = Z \) (i.e. \( \tilde{K}^0(F) = 0 \)), then \( \tilde{K}^0(X) \) and \( \tilde{K}^0(K) \) are isomorphic.

Let us now consider a rank \( d \) Hilbert net bundle \( E := (E, \pi, J, K) \). For every fibre \( E_a, a \in \Sigma_0(K) \), we consider the associated projective space \( PE_a \). Each unitary \( J_b, b \in \Sigma_1(K) \), defines a homeomorphism \( [J]_b : PE_{a,b} \to PE_{a,b} \). Defining \( PE := \cup_a PE_a \) and the obvious projection \( [\pi] : PE \to K \), we obtain a net bundle of topological spaces

\[
\mathcal{PE} := (PE, [\pi], [J], K) ,
\] (5.8)
called the projective net bundle associated with \( E \). We denote by

\[
\mathcal{PE}_\preceq := (PE_\preceq, [\pi]_\preceq, [J], K)
\]
the poset net bundle associated with \( \mathcal{PE} \) in the sense of Sec.4.2.

**Theorem 5.14** (The Thom isomorphism). Let \( (E, \pi, J, K) \) be a rank \( d \) Hilbert net bundle. For every group \( G \), the pullback over \( PE_\preceq \) induces isomorphisms

\[
[\pi]_\preceq^* : H^1(K, G) \to H^1(PE_\preceq, G) , \quad [\pi]^* : K^0(K) \to K^0(PE_\preceq) .
\] (5.9)

**Proof.** Since the projective space is pathwise and simply connected, we apply Cor.4.4, Cor.4.5 and conclude that the map \( [\pi]_\preceq : PE_\preceq \to K \) induces an isomorphism

\[
[\pi]_{\preceq,*} : \pi_1(PE_\preceq, v) \to \pi_1(K, a) , \quad v \in PE_{\preceq,a} .
\] (5.10)

Since we find

\[
Z^1(K, G) \simeq \text{Hom}(\pi_1(K, a), G) , \quad Z^1(PE_\preceq, G) \simeq \text{Hom}(\pi_1(PE_\preceq, v), G) ,
\]
for every group \( G \), the theorem is proved at the level of cohomology. The isomorphism at the level of K-theory follows by Thm.5.11 and (5.10). \( \square \)
In contrast to ordinary geometry, the Thom isomorphism really is an isomorphism and not a monomorphism. But this result is not as useful as its topological counterpart, since there is not a well-defined notion of canonical line net bundle with base $PE_a$. We shall return to this point in the sequel (Rem. 5.16).

For every topological space $Y$, we denote the category of vector bundles over $Y$ by $V_{\text{top}}(Y)$, and the subcategory (not full) of locally constant vector bundles with arrows locally constant morphisms by $V_{\text{lc}}(Y)$ (see for example [7, Ch.I.2]). Now, a Hilbert net bundle $\mathcal{E} := (E, \pi, J, K)$ is a net bundle of topological spaces in a natural way, so there is an associated vector bundle $\pi : \tau E \to \tau K$ (see Thm. 4.8).

**Theorem 5.15.** For every Hilbert net bundle $\mathcal{E} := (E, \pi, J, K)$, the projection $\pi$ defines a continuous map $\pi : \tau E \to \tau K$, and $\tau E$ becomes a locally constant vector bundle over $\tau K$. Thus, there is a natural functor

$$\tau_* : V_{\text{net}}(K) \to V_{\text{top}}(\tau K), \quad \mathcal{E} \mapsto \tau E,$$

providing an isomorphism $V_{\text{net}}(K) \simeq V_{\text{lc}}(\tau K)$. For every net bundle of topological spaces $\mathcal{X} := (X, p, \Phi, K)$, the topological pullback $p^*(\tau E)$ defines a functor

$$p^* : V_{\text{net}}(K) \to V_{\text{lc}}(\tau X).$$

**Proof.** $\pi : \tau E \to \tau K$ is a vector bundle as a direct consequence of Thm. 4.8. The functoriality of the map (5.11) follows by Lemma 4.7. Applying Cor. 4.9 to the $U(d)$-cocycle associated with $\mathcal{E}$ we see that $\tau E$ is locally constant. Finally, $p^*$ takes values in $V_{\text{lc}}(\tau X)$ since the pullback of a locally constant bundle is locally constant.  

We emphasize the fact that $\tau K$ is just a $T_0$-space, thus we cannot use the usual machinery of differential geometry, as in [7, Ch.I] for example, to study properties of the locally constant bundle $\tau E$.

**Remark 5.16.** The fibre bundle $[\pi] : \tau PE \to \tau K$ has simply connected fibres, thus the topological version of Thm. 4.3 and the fact that $\tau PE$ is locally constant, imply that we have isomorphisms

$$[\pi]_* : \pi_1(\tau PE) \to \pi_1(\tau K), \quad [\pi]^* : V_{\text{lc}}(\tau K) \to V_{\text{lc}}(\tau PE).$$

Now, let us consider the topological pullback $[\pi]^*(\tau E) \to \tau PE$. By definition, $[\pi]^*(\tau E)$ has as elements pairs $(v, \xi) \in E_a \times PE_a$, $a \in \tau K$, and we can define the canonical line bundle $\tau L \to \tau PE$ by

$$\tau L := \{(v, \xi) \in [\pi]^*(\tau E) : v \in \xi\}.$$

As in [5, Ch.IV.2], the restriction of $\tau L$ over $PE_a$, $a \in \tau K$, is just the usual canonical line bundle $L_a \to PE_a$. Since $L_a$ is not locally constant $\tau L$ is not locally constant, so it does not belong to the image of $[\pi]^*$, and $[\pi]^*(\tau E) = \tau L \oplus E'$ is direct sum of not locally constant bundles. We conclude that the splitting principle cannot be applied in the category of Hilbert net bundles.
5.3 Chern classes for Hilbert net bundles.

As we saw in the previous section, the algebraic properties of the category of Hilbert net bundles over a poset $K$ are drastically different from those of vector bundles over a topological space. This fact is also reflected in the construction of Chern classes. As we saw in Theorem 5.15 and will see in §6, Hilbert net bundles are strictly related to locally constant vector bundles, which have trivial Chern classes when the base space is a manifold.

A different approach to Chern classes may be based on the picture of Hilbert net bundles as unitary representations of the homotopy group. From this point of view, there are some results ([15, 14]) on Chern classes associated with representations of a discrete group $G$ (in our case, the homotopy group). But unfortunately, such Chern classes do not fit in well with our aims, as they are expressed in terms of the group cohomology of $G$, or, equivalently, in terms of the singular cohomology of a suitable Eilenberg-McLane space associated with $G$. In both the cases, an immediate interpretation in terms of properties of the initial poset is lost; moreover, such classes vanish when the above-mentioned space is homotopic to a manifold.

In the following, we propose a different construction of Chern classes, in terms of homotopy-invariant complex functions on the path groupoid of the poset.

5.3.1 The first Chern class.

Let $d \in \mathbb{N}$ and $\mathcal{E}$ be a Hilbert net bundle of rank $d$. According to (4.2), $\mathcal{E}$ is characterized by a cohomology class $z \in H^1(K, \mathbb{U}(d))$. By the functoriality of $H^1(K, \cdot)$, the determinant map $\det : \mathbb{U}(d) \to \mathbb{T}$ induces a map $\det_* : H^1(K, \mathbb{U}(d)) \to H^1(K, \mathbb{T})$. We define the first Chern class of $\mathcal{E}$ to be

$$c_1(\mathcal{E}) := \det_* z \in H^1(K, \mathbb{T}) .$$

By the elementary properties of determinants, we find $c_1(\mathcal{E} \oplus \mathcal{E}') = c_1(\mathcal{E})c_1(\mathcal{E}')$, and obtain an epimorphism of Abelian groups

$$c_1 : K_0(K) \to H^1(K, \mathbb{T}) . \quad (5.14)$$

The first Chern class is natural in the sense that, if $\eta : K' \to K$ is a morphism, then

$$\eta^*c_1(\mathcal{E}) = c_1(\eta^*\mathcal{E}) . \quad (5.15)$$

From a categorical point of view, the first Chern class encodes the cohomological obstruction for $\mathcal{E}$ to be a special object: if $c_1(\mathcal{E}) = 0$, then the totally antisymmetric line net bundle $\lambda^d\mathcal{E}$ is trivial, and every normalized section $R \in S(K; \lambda^d\mathcal{E})$ is a solution of the equation ([2, (3.19)]).

Now, $c_1(\mathcal{E})$ can be regarded as a morphism from $\pi_1(K, a)$ into $\mathbb{T}$. By Thm 3.6 we conclude that $c_1(\mathcal{E})$ factorizes through a morphism

$$\hat{c}_1(\mathcal{E}) \in \text{Hom}(H_1(K, \mathbb{Z}), \mathbb{T}) , \quad (5.16)$$

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that we call the *Abelianized first Chern class*. By Cor.3.7 \( \text{H}^1(K,\mathbb{T}) \) is isomorphic to \( \text{Hom}(\text{H}_1(K,\mathbb{Z}),\mathbb{T}) \), thus it is essentially equivalent to consider \( \hat{c}_1(\mathcal{E}) \) instead of \( c_1(\mathcal{E}) \); but since \( \text{H}_1(K,\mathbb{Z}) \) is generally easier to compute, it is usually convenient to use (5.16).

### 5.3.2 Chern K-classes

We now adapt a classical construction to net K-theory ([5, IV.2.17]). Let \( \mathcal{E} := (E, \pi, J, K) \) be a rank \( d \) Hilbert net bundle. We consider the antisymmetric tensor powers \( \lambda^k \mathcal{E} \), \( k = 1, \ldots, d \), and define

\[
k_i(\mathcal{E}) := \sum_{k=0}^{i} (-1)^k \binom{d-k}{i-k} [\lambda^k \mathcal{E}] , \quad i = 1, \ldots, d .
\]  

(5.17)

Some elementary computations involving the dimensions of antisymmetric tensor powers imply that \( \rho(k_i(\mathcal{E})) = 0 \), thus \( k_i(\mathcal{E}) \in \tilde{K}_0^0(K) \). We call the classes \( k_i(\mathcal{E}) \) the *Chern K-classes*. Keeping (5.14) in mind, we find

\[
c_1(k_1(\mathcal{E})) = -c_1[\mathcal{E}] .
\]

Applying the well-known identity

\[
\lambda^i(\mathcal{E} \oplus \mathcal{E}') = \oplus_{l+m=i} \lambda^l \mathcal{E} \otimes \lambda^m \mathcal{E}'
\]  

(5.18)

to \( \mathcal{E}' = \mathcal{T}_k \), we see that if \( \mathcal{E} \) has rank \( d \) and admits a trivial, rank \( k \) direct summand, then \( k_i(\mathcal{E}) = 0 \), \( k \leq i \leq d \). We define the *total Chern K-class* to be

\[
k(\mathcal{E}) := 1 + \sum_{i=1}^{d} k_i(\mathcal{E}) h^i \ , \quad \mathcal{E} \in V_{net}(K) .
\]

By definition, \( k(\mathcal{E}) = k(\mathcal{E} \oplus \mathcal{T}_k) \) for every \( k \in \mathbb{N} \). This fits in well with the idea that \( k(\mathcal{E}) \) should encode the nontrivial properties of \( \mathcal{E} \), and implies that the classes \( k_i \) are well-defined for elements of \( V_{net}(K) \). The above considerations and the naturality of the pullback yield

**Proposition 5.17.** We have

\[
k_i(\mathcal{E} \oplus \mathcal{E}') = \sum_{l+m=i} k_l(\mathcal{E}) k_m(\mathcal{E}') \ , \quad \mathcal{E}, \mathcal{E}' \in V_{net}(K) .
\]  

(5.19)

Thus, the total Chern K-class factorizes through a morphism

\[
k : V_{net}^s(K) \to 1 + h\tilde{K}^0(K)[[h]] , \quad k(\mathcal{E} \oplus \mathcal{E}') = k(\mathcal{E}) k(\mathcal{E}') ,
\]

such that \( \eta^*k(\mathcal{E}) = k(\eta^*\mathcal{E}) \) for every poset morphism \( \eta : K' \to K \).

### 5.3.3 Chern functions.

Let \( \Pi_1(K) \) denote the set of paths of \( K \). Since each 1–simplex is a path of length 1, \( \Sigma_1(K) \) is contained in \( \Pi_1(K) \). Given a set \( S \), a map \( f : \Pi_1(K) \to S \) is said to be *homotopy-invariant* if \( f(p) = f(p') \) whenever \( p \) is homotopic to \( p' \).
Let $\mathcal{E} := (E, \pi, J, K)$ be a rank $d$ Hilbert net bundle with associated $\mathbb{U}(d)$-cocycle $z \in Z^1(K, \mathbb{U}(d))$. By [10, Eq.32], $z$ can be extended to a $\mathbb{U}(d)$-valued, homotopy-invariant map on $\Pi_1(K)$; in particular, if $p$ is homotopic to a constant path, then $z(p)$ is the identity $1 \in \mathbb{U}(d)$. Using the trace map $\text{Tr}$ and the exterior powers $\wedge^k$, $k = 1, \ldots, d$, we define the maps

$$
\chi_z^k(p) := \text{Tr} \wedge^k z(p) , \quad p \in \Pi_1(K).
$$

Clearly, the restriction of $\chi_z^k$ to $\tilde{\Sigma}_1(K)$ yields an element of $C^1(K, \mathbb{C})$. On the other hand, if we restrict $\chi_z^k$ to $K(a)$, $a \in \Sigma_0(K)$, then by homotopy invariance we find that $\chi_z^k$ can be regarded as the character of the representation of $\pi_1(K, a)$ associated with $\lambda^k \mathcal{E}$, $k = 1, \ldots, d$. In particular, since the trace is the identity map for rank one representations, we find

$$
\chi_z^d = c_1(\mathcal{E}) , \quad d = \rho(\mathcal{E}) .
$$

Let us denote the ring of bounded, homotopy-invariant maps from $\Pi_1(K)$ to $\mathbb{C}$, by $R^1(K, \mathbb{C})$. By analogy with the previous section, we introduce the Chern functions

$$
\zeta_i \mathcal{E} \in R^1(K, \mathbb{C}) , \quad \zeta_i \mathcal{E} := \sum_{k=0}^i (-1)^k \binom{d-k}{i-k} \chi_z^k , \quad i = 1, \ldots, d .
$$

For $i > d$, we set $\zeta_i \mathcal{E} := 0$. If $\mathcal{T}_d$ is the trivial Hilbert net bundle of rank $d$, then (5.18) implies $\zeta_i \mathcal{T}_d = 0$, $i = 1, \ldots, d$. If $\mathcal{L} \in V_{\text{net}}(K)$ is a line net bundle with $\mathbb{T}$-cocycle $\zeta$, then from (5.20) we find

$$
\chi_1^1 = c_1(\mathcal{L}) \quad \Rightarrow \quad c_1 \mathcal{L} = 1 - c_1(\mathcal{L}) .
$$

Since the trace is additive on direct sums and multiplicative on tensor products, the argument of Prop.5.17 shows that

$$
c_i(\mathcal{E} \oplus \mathcal{E}') = \sum_{l+m=i} c_l \mathcal{E} \cdot c_m \mathcal{E}' ,
$$

so that

$$
c_{d+k}(\mathcal{E} \oplus \mathcal{T}_k) = 0 , \quad c_i(\mathcal{E} \oplus \mathcal{T}_k) = c_i \mathcal{E} , \quad i, k = 1, \ldots .
$$

The previous equalities yield the usual interpretation of the functions $\zeta_i \mathcal{E}$ as obstructions to the triviality of $\mathcal{E}$. Moreover, (5.24) implies that each $\zeta_i \mathcal{E}$ depends only on the class of $\mathcal{E}$ in $V_{\text{net}}(K)$. For the first Chern class, we have the following result.

**Lemma 5.18.** For a Hilbert net bundle $\mathcal{E}$ of rank $d$, we have

$$
c_1(\mathcal{E}) = 1 + \sum_{i=1}^d (-1)^i \zeta_i \mathcal{E} .
$$

**Proof.** Let us recall the obvious identity

$$
\sum_{j=0}^N (-1)^j \binom{N}{j} = 0 , \quad N \in \mathbb{N} .
$$

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We put $\chi^0_z := 1$, so that, by the definition of $c_i \mathcal{E}$, the r.h.s. of (5.28) coincides with
\[
\sum_{i=0}^d (-1)^i \sum_{k=0}^i (-1)^k \binom{d-k}{i-k} \chi_z^k.
\] (5.27)

Let us put together the coefficients of each $\chi^k_z$ and set $j := i - k$, so that in particular $(-1)^{i+k} = (-1)^{j+2k} = (-1)^j$. Then we find that the quantity (5.27) coincides with
\[
\sum_{k=0}^d \left( \sum_{j=0}^{d-k} \binom{d-k}{j} (-1)^j \right) \chi_z^k.
\]

Using (5.26), we conclude that the terms between the brackets vanish, except in the case $d = k$, which provides the coefficient $\chi^d_z$. Thus, the r.h.s. of (5.25) coincides with $\chi^d_z$, which is equal to $c_1(\mathcal{E})$ by (5.20).

**Theorem 5.19.** Defining the polynomial
\[
\mathfrak{c} \mathcal{E}(h) := 1 + \sum_{i=1}^d c_i \mathcal{E} h^i
\]
for each $\mathcal{E} \in V_{\text{net}}(K)$ gives a natural morphism
\[
\mathfrak{c} : V_{\text{net}}^d(K) \to 1 + h \mathcal{R}^1(K, \mathbb{C})[[h]] , \quad \mathfrak{c}(\mathcal{E} \oplus \mathcal{E}') = \mathfrak{c} \mathcal{E} \cdot \mathfrak{c} \mathcal{E}' ,
\] (5.28)
such that $c_1(\mathcal{E}) = \mathfrak{c} \mathcal{E}(-1)$.

**Proof.** After Prop.5.17, (5.23) and (5.25), the only nontrivial assertion that we have to verify is the naturality of (5.28). Let $\eta : K' \to K$ be a morphism and $z \in Z^1(K, \mathcal{U}(d))$ the cocycle associated with $\mathcal{E}$. Then $\eta^* z \in Z^1(K', \mathcal{U}(d))$ is the cocycle associated with the pullback $\eta^* \mathcal{E}$, and clearly $\chi^k_{\eta^* z} = \chi^k_z \circ \eta_1$, where $\eta_1 : \Pi_1(K') \to \Pi_1(K)$ is the map induced by $\eta$. This shows that the functions $c_i \mathcal{E}$ are natural, i.e. $c_i(\eta^* \mathcal{E}) = \eta^* c_i \mathcal{E} := c_i \mathcal{E} \circ \eta_1$.

It is instructive to give the details when $\mathcal{E} = \bigoplus_i^d \mathcal{L}_i$ is the direct sum of line net bundles. Since $\mathfrak{c}$ is a homomorphism, we get
\[
\mathfrak{c} \mathcal{E}(h) = \prod_{i=1}^d \mathfrak{c} \mathcal{L}_i(h) = \prod_{i=1}^d [1 + \hat{c}_i \mathcal{L}_i h] = \prod_{i=1}^d [1 + (1 - c_1(\mathcal{L}_i)) h] ,
\] (5.29)
so that we obtain
\[
c_i \mathcal{E} = \sum_{1 \leq k_1 < \ldots < k_i \leq d} [1 - c_1(\mathcal{L}_{k_1})] \cdots [1 - c_1(\mathcal{L}_{k_i})] , \quad i = 1, \ldots, d .
\] (5.30)
Remark 5.20. Let us regard the Chern functions as maps from $\pi_1(K,a)$ to $\mathbb{C}$. Since the trace is invariant under the adjoint action of $\pi_1(K,a)$, each $c_i E$ factorizes through a map on the orbit space $[\pi_1(K,a)]$ with elements the classes $(p') := \{p*p*p^{-1}, p \in \pi_1(K,a)\}$. Let $\eta : \pi_1(K,a) \to [\pi_1(K,a)]$ denote the natural projection and $T : \pi_1(K,a) \to H_1(K,\mathbb{Z})$ the Abelianization map defined according to Thm 3.6. Since $Tp' = T(p*p*p^{-1}) \in H_1(K,\mathbb{Z})$, $p,p' \in \pi_1(K,a)$, there is a map $\hat{T} : [\pi_1(K,a)] \to H_1(K,\mathbb{Z})$ such that $T$ can be obtained as the composition

$$\pi_1(K,a) \xrightarrow{\eta} [\pi_1(K,a)] \xrightarrow{T} H_1(K,\mathbb{Z}).$$

Of course, when $\pi_1(K,a)$ is Abelian the above maps are isomorphisms and each $c_i E$ factorizes through a map from $H_1(K,\mathbb{Z})$ to $\mathbb{C}$.

The following simple result shows how the Chern functions can be used to compute $V^*_{net}(K)$ explicitly when $K$ has homotopy group $\mathbb{Z}$.

Proposition 5.21. Let $\pi_1(K,a) = \mathbb{Z}$, $a \in \Sigma_0(K)$, and

$$S := \{ (\chi - 1)^{-1} \in \mathbb{C} : \chi \in \mathbb{T} - \{1\} \}.$$ 

Then $V^*_{net}(K)$ is isomorphic to the multiplicative semigroup of polynomials

$$P(h) := 1 + \sum_{i=1}^{d} a_i h^i \in 1 + h\mathbb{C}[[h]], \quad a_i \in \mathbb{C},$$

whose zeroes belong to $S$.

Proof. Since $\pi_1(K,a) = H_1(K,\mathbb{Z}) = \mathbb{Z}$ is Abelian, every Hilbert net bundle $E$ is the direct sum of line net bundles $\mathcal{L}_1, \ldots, \mathcal{L}_d$ with associated cocycles $z_i : \pi_1(K,a) \to \mathbb{T}$, $i = 1, \ldots, d$, thus (5.29) applies. Now, each $z_i : \mathbb{Z} \to \mathbb{T}$ is uniquely determined by $\chi_i := z_i(1) \in \mathbb{T}$; so that, the $d$-ple $\{\chi_i\} \in \mathbb{T}^d$ is a complete invariant for $E$. Eliminating any term $\chi_i = 1$, we obtain a $r$-ple $\{\chi_i\} \in \mathbb{T}^r$, $r \leq d$, which is a complete invariant for the stable equivalence class of $E$. It is now clear that the polynomial

$$P(h) := \prod_{i=1}^{r} [1 + (1 - \chi_i)h] \in 1 + h\mathbb{C}[[h]]$$

has the allowed zeroes. Since the set $\{\chi_i\}$ can be reconstructed from the set of zeroes of $P$, the map $\{[E] \mapsto P\}$ is one-to-one and obviously surjective. \qed

5.4 A sketch of equivariant $K$-theory.

Let $K$ be a poset and $G$ a group inducing a left action on $K$ by poset automorphisms. Then for each $n \in \mathbb{N}$ the set of $n$-simplices $\Sigma_n(K)$ becomes a $G$-set in the natural way, and we denote the images of the action of $g \in G$ on $a \in \Sigma_0(K)$, $b \in \Sigma_1(K)$, $c \in \Sigma_2(K)$, by $ga \in \Sigma_0(K)$, $gb \in \Sigma_1(K)$, $gc \in \Sigma_2(K)$.
A Hilbert net $G$-bundle is given by a Hilbert net bundle $\mathcal{E} := (E, p, J, K)$ such that:

1. $E$ is endowed with a left $G$-action;
2. The projection $p$ is a $G$-map, so that each $g \in G$ induces a bijective map $g_a : E_a \to E_{ga}$, $a \in \Sigma_0(K)$;
3. every $g_a$ is a unitary operator satisfying the condition

$$g_{ab} J_b = J_g g_{ab}, \quad b \in \Sigma_1(K) \quad (5.31)$$

Let $\mathcal{E}, \hat{\mathcal{E}}$ be Hilbert net $G$-bundles over $K$. A morphism $T \in (\mathcal{E}, \hat{\mathcal{E}})$ is said to be $G$-equivariant if $T(gv) = gTv$, $g \in G$, $v \in E$. We denote the set of $G$-equivariant morphisms by $(\mathcal{E}, \hat{\mathcal{E}})_G$ and the associated net bundle by $B(\mathcal{E}, \hat{\mathcal{E}})_G$. By construction, $B(\mathcal{E}, \hat{\mathcal{E}})_G$ is a Hilbert net bundle endowed with a trivial $G$-action. In this way, we obtain a category $V_{net}(K; G)$ naturally endowed with direct sums and tensor products, and, by using the usual construction, we can define the equivariant $K$-theory $K^0_G(K)$.

We now give some elementary properties when $K$ has the trivial $G$-action. Firstly, note that $(5.31)$ implies that each $g \in G$ defines a section of $B(\mathcal{E}, \hat{\mathcal{E}})_G$. Moreover, when $G$ is compact averaging w.r.t. Haar measure provides a projection

$$P_G \in (\mathcal{E}, \mathcal{E}) \quad (5.32)$$

and we denote the associated Hilbert net bundle by $\mathcal{E}_G := P_G \mathcal{E} \in V_{net}(K)$. Applying $P_G$ to $\hat{\mathcal{E}} \otimes \overline{\mathcal{E}}$, we find $P_G B(\mathcal{E}, \hat{\mathcal{E}}) = B(\mathcal{E}, \hat{\mathcal{E}})_G$. Let $v \in \hat{\mathcal{E}}$, $w \in \mathcal{E}$, and $Tvw \in B(\mathcal{E}, \hat{\mathcal{E}})_G$, $Tvw' := v \cdot (w, w')$; if $(\mathcal{E}, \mathcal{E})_G \simeq \mathbb{C}$, then we have a canonical map

$$\mathcal{E} \otimes B(\mathcal{E}, \hat{\mathcal{E}})_G \to \hat{\mathcal{E}}, \quad w' \otimes Tvw \mapsto P_G(Tvw') \cdot v \quad (5.33)$$

**Theorem 5.22.** Let $G$ be a compact group acting trivially on $K$. Then there is a splitting $K^0_G(K) \simeq K^0(K) \otimes R(G)$.

**Proof.** We use a classical argument (see [13, Prop.2.2] for details): for each irreducible representation $\alpha$ of $G$, we consider the associated $G$-module $H_\alpha$ and the corresponding trivial $G$-bundle $\mathcal{T}_\alpha := (H_\alpha \times K, \pi, j, K)$ such that $(\mathcal{T}_\alpha, \mathcal{T}_\alpha)_G \simeq \mathbb{C}$. Then we define the map

$$R(G) \otimes K^0_0(K) \to K^0_G(K), \quad [\alpha] \otimes [\mathcal{E}] \to [\mathcal{T}_\alpha \otimes \mathcal{E}] \quad (5.34)$$

The fact that $(5.34)$ is an isomorphism follows by observing that the isotypical decomposition

$$\mathcal{E} \simeq \oplus_\alpha \mathcal{T}_\alpha \otimes B(\mathcal{T}_\alpha, \mathcal{E})_G$$

(defined as in $(5.33)$) provides the desired inverse. \hfill \Box

### 6 Locally constant bundles.

Let $M$ be a Hausdorff space; we fix a poset $M_\prec$ of arcwise and simply connected open subsets of $M$ providing a subbase for the topology of $M$. Before proceeding, we recall some well-known facts:
The universal cover \( \tilde{M} \) is a right homogeneous \( \pi_1(M) \)-space. For each finite-dimensional, unitary representation \( u \) of \( \pi_1(M) \) over a Hilbert space \( H_u \), we define

\[
E_u := \tilde{M} \times_{\pi_1(M)} H_u
\]

(6.1)
as the quotient of \( \tilde{M} \times H_u \) by the equivalence relation

\[
(x, v) \sim (x\gamma, u(\gamma)v), \quad \gamma \in \pi_1(M).
\]
The natural projection \( p : \tilde{M} \to M \) induces a projection \( \pi : E_u \to M \) and it can be verified that \( E \) is indeed a locally constant vector bundle. If \( T \in (u, \hat{u}) \) is an intertwiner, then the map

\[
T^lc(x, v) := (x, T v), \quad x \in M, \ v \in H_u,
\]
induces a morphism from \( E_u \) to \( E_u \).

(2) Let \( M \) be a manifold. A vector bundle \( E \to M \) is locally constant if and only if it admits a flat connection ([7, Ch.I]). (3) By ([7, Prop.II.3.1]), this implies that the ordinary Chern classes of locally constant bundles over manifolds vanish.

We denote the category of locally constant vector bundles over \( M \) by \( V_{lc}(M) \). The interpretation of elements of \( V_{lc}(M) \) in terms of representation of \( \pi_1(M) \) provides an equivalence with the category of finite-dimensional, unitary representations of \( \pi_1(M) \). Moreover, \( V_{lc}(M) \) is naturally equipped with tensor product and direct sums, so that the set \( \hat{V}_{lc}(M) \) of isomorphism classes of locally constant vector bundles becomes a semiring.

**Theorem 6.1.** Let \( M \) be a Hausdorff, locally arcwise and simply connected space. Then the categories \( V_{net}(M) \) and \( V_{lc}(M) \) are equivalent. Moreover, \( K^0(M) \) is isomorphic to the representation ring of \( \pi_1(M) \), and describes the Grothendieck ring associated with \( \hat{V}_{lc}(M) \).

**Proof.** It is well-known that the map \( \{u \mapsto E_u\} \) defined by (6.1) provides an equivalence between the category \( \text{Rep}(\pi_1(M)) \) of finite-dimensional, unitary representations of \( \pi_1(M) \) and \( V_{lc}(M) \). Moreover, \( \text{Rep}(\pi_1(M)) \) is equivalent to \( V_{net}(M) \) by [10, Prop.3.8]. Thus, \( V_{net}(M) \) is equivalent to \( V_{lc}(M) \), and the proof follows by applying Theorem 5.11

The functor \( F : V_{net}(M) \to V_{lc}(M) \) giving the above equivalence can be explicitly described as follows. If \( \mathcal{E} \in V_{net}(M) \), then by Prop 5.2 there is \( d \in \mathbb{N} \) and a unitary representation \( z : \pi_1(M), a \to U(d), a \in \Sigma_0(M) \), defining a representation \( u \) of \( \pi_1(M) \) according to [12 Thm.2.18] (in concrete terms, if \( \gamma : [0, 1] \to M \) is a closed curve, define \( u(\gamma) := z(p) \), where \( p \in M_a(a) \) is an approximation of \( \gamma \) in the sense of [12 Def.2.13]). We define \( F(\mathcal{E}) := E_u \). If \( T \in (\mathcal{E}, \mathcal{E}) \) is a morphism, then according the proof of [10, Prop.3.8] we have an intertwiner \( T(a) \in (z, \hat{z}) = (u, \hat{u}) \), providing a morphism from \( E_u \) to \( E_u \).

**Corollary 6.2.** Let \( \pi_1(M) \) be Abelian. Then every locally constant vector bundle over \( M \) decomposes into a direct sum of locally constant line bundles. The ring \( K^0(M) \) is generated as a \( \mathbb{Z} \)-module by the singular cohomology \( H^1(M, \mathbb{T}) \).
Apply Thm.3.8.

Example 6.3. When $M$ is the circle $S^1$, we find $H_1(S^1, \mathbb{T}) = \pi_1(S^1) = \mathbb{Z}$, so that each locally constant vector bundle is direct sum of locally constant line bundles. Moreover, $H^1(S^\infty, \mathbb{T}) = H^1(S^1, \mathbb{T}) = \mathbb{T}$, and this implies $K^0(S^\infty) \simeq \mathbb{Z}[\mathbb{T}]$ (i.e., the ring of formal linear combinations of elements of $\mathbb{T}$ with coefficients in $\mathbb{Z}$).

The homotopy groupoid of $M$ is given by the category with objects the points of $M$ and set of arrows the set $\tilde{\pi}_1(M)$ of homotopy classes of continuous curves in $M$. We denote the ring of bounded, complex functions on $\tilde{\pi}_1(M)$ by $R^1(M, \mathbb{C})$ and the Abelian semigroup of stable equivalence classes of locally constant vector bundles, defined as in (5.4), by $V_{lc}(M)$.

Theorem 6.4. For each locally constant vector bundle $E \to M$ of rank $d$, there are functions $c_iE \in R^1(M, \mathbb{C})$, $i = 1, \ldots, d$, such that $c_i(E \oplus E') = \sum_{l+m=i} c_lE \cdot c_mE'$. The polynomial

$$cE(h) := 1 + \sum c_i E h^i$$

defines a natural morphism

$$c : V_{lc}(M) \to 1 + hR^1(M, \mathbb{C})[[h]] , \quad c(E \oplus E') = cE \cdot cE' .$$

When $\pi_1(M)$ is Abelian, $R^1(M, \mathbb{C})$ can be replaced by the ring of bounded complex functions on the singular homology $H_1(M, \mathbb{Z})$.

Proof. By [12, Thm.2.18], the homotopy groupoid of $M$ is isomorphic to the homotopy groupoid of $M_{\prec}$, so that there is a ring isomorphism $R^1(M_{\prec}, \mathbb{C}) \simeq R^1(M, \mathbb{C})$. Thus, we apply Thm.3.8, Thm.5.19 and Rem.5.20.

A Simplicial sets.

A simplicial set is a contravariant functor from the simplicial category $\Delta^+$ to the category of sets. $\Delta^+$ is a subcategory of the category of sets having as objects $n := \{0, 1, \ldots, n-1\}$, $n \in \mathbb{N}$ and as mappings the order preserving mappings. A simplicial set has a well known description in terms of generators, the face and degeneracy maps, and relations. We use the standard notation $\partial_i$ and $\sigma_j$ for the face and degeneracy maps, and denote the compositions $\partial_i\partial_j$, $\sigma_i\sigma_j$, respectively, by $\partial_{ij}$, $\sigma_{ij}$. A path in a simplicial set is an expression of the form

$$p := b_n * b_{n-1} * \cdots * b_1 ,$$

where the $b_i$ are 1–simplices and $\partial_0 b_i = \partial_1 b_{i+1}$ for $i = 1, 2, \ldots, n-1$. We set $\partial_1 p := \partial_1 b_1$, $\partial_0 p := \partial_0 b_n$ and

$$\ell(p) := n$$

the length of $p$. Concatenation gives us an obvious associatiive composition law for paths and in this way we get a category without units.
Homotopy provides us with an equivalence relation \( \sim \) on this structure. This is the equivalence relation generated by a finite sequence \( s(i) \), with \( i = 1, \ldots, k \) say, of elementary deformations. An elementary deformation of a path consists in replacing a 1–simplex \( \partial_1c \) of the path by a pair \( \partial_0c, \partial_2c \), where \( c \in \Sigma_2 \), or, conversely in replacing a consecutive pair \( \partial_0c, \partial_2c \) of 1–simplices of \( p \) by a single 1–simplex \( \partial_1c \). The former type of deformation is called an ampliation of the path, the latter a contraction. Quotienting by this equivalence relation yields the homotopy category of the simplicial set.

We shall mainly use symmetric simplicial sets. These are contravariant functors from \( \Delta^s \) to the category of sets, where \( \Delta^s \) is the full subcategory of the category of sets with the same objects as \( \Delta^+ \). A symmetric simplicial set also has a description in terms of generators and relations, where the generators now include the permutations of adjacent vertices, denoted \( \tau_i \). In a symmetric simplicial set we define the reverse of a 1–simplex \( b \) to be the 1–simplex \( \tilde{b} := \tau_0b \) and the reverse of a path \( p = b_n \ast b_{n-1} \ast \cdots \ast b_1 \) is the path \( \tilde{p} := \tau_0b_1 \ast b_0b_2 \ast \cdots \ast \tau_0b_n \). The reverse acts as an inverse after taking equivalence classes so the homotopy category becomes a homotopy groupoid. Given a symmetric simplicial set \( \tilde{\Sigma}_* \) we denote its homotopy groupoid by \( \pi_1(\tilde{\Sigma}_*) \).

### A.1 Homotopy of products

Consider a pair \( \tilde{\Sigma}_*^\alpha \) and \( \tilde{\Sigma}_*^\mu \) of symmetric simplicial sets. Let \( \Pi_1^\alpha \) and \( \Pi_1^\mu \) denote the corresponding set of paths, and let \( \sim_\alpha \) and \( \sim_\mu \) denote the corresponding homotopy equivalence relations. Now, consider the product simplicial set \( \tilde{\Sigma}_*^\alpha \times \tilde{\Sigma}_*^\mu \). Let \( \Pi_1^\alpha \times \Pi_1^\mu \) be the set of paths of the product simplicial set and denote the homotopy equivalence relation by \( \sim \). A path \( p \) in this set is a pair \( (p^\alpha, p^\mu) \), where \( p^\alpha, p^\mu \) are paths in \( \Pi_1^\alpha \) and \( \Pi_1^\mu \) respectively with \( \ell(p^\alpha) = \ell(p^\mu) \). Note that a homotopy in this set is a finite sequence \( s(i) = (s^\alpha(i), s^\mu(i)) \), with \( i = 1, \ldots, m \) to say, where \( s^\alpha(i) \) and \( s^\mu(i) \) are either both ampliations or both contractions of the paths \( p^\alpha \) and \( p^\mu \) for any \( i = 1, \ldots, m \). In particular note that, if \( p \sim q \), then \( p^\alpha \sim_\alpha q^\alpha \) and \( p^\mu \sim_\mu q^\mu \).

Our aim is to show that the fundamental groupoid of the product simplicial set is equal to the product of the fundamental groupoids. To this end, consider the set \( \Pi_1^\alpha \times \Pi_1^\mu \) product of paths. Note that the elements \( p \) of this set are pairs \( (p^\alpha, p^\mu) \) of paths where, in general, \( \ell(p^\alpha) \neq \ell(p^\mu) \). Finally, observe that the proof of our claim follows once we have shown that the identity map

\[ \Pi_1^\alpha \times \Pi_1^\mu \ni p \to p \in \Pi_1^\alpha \times \Pi_1^\mu, \]

is injective and surjective up to equivalence. It is easily seen to be surjective. In fact let \( p \in \Pi_1^\alpha \times \Pi_1^\mu \). Assume that \( \ell(p^\alpha) = \ell(p^\mu) + k \). If \( a \) is the starting point of the path \( p^\mu \) in \( \Sigma_*^\mu \), then we consider the path \( p^\mu \ast (\sigma_0a)^*k \), where \( (\sigma_0a)^*k \) is the \( k \)-fold composition of the degenerate 1–simplex \( \sigma_0a \). Clearly \( p^\mu \ast (\sigma_0a)^*k \sim_\mu p^\mu \) and \( (p^\alpha, p^\mu \ast (\sigma_0a)^*k) \in \Pi_1^\alpha \times \Pi_1^\mu \).

For injectivity we must show that

\[ p^\alpha \sim_\alpha q^\alpha, \ p^\mu \sim_\mu q^\mu \Rightarrow p \sim q. \quad (A.1) \]
If \( p' \sim p \) and \( q' \sim q \) then it obviously suffices to pose the question for \( p' \) and \( q' \) instead. \( p = \sigma_0 \partial_1 p \) is homotopic to \( p \) so it suffices to suppose that \( \ell(p) = \ell(q) \). Pick sequences \( s^\alpha_0 \) and \( s^\mu_0 \) of elementary deformations leading from \( p^\alpha \) to \( q^\alpha \) and from \( p^\mu \) to \( q^\mu \). Since \( \ell(p^\alpha) = \ell(q^\alpha) \) and \( \ell(p^\mu) = \ell(q^\mu) \), each sequence contains as many ampliations as contractions but they may not have the same length. If \( \ell(s^\alpha_0) < \ell(s^\mu_0) \), say, we may adjoin pairs consisting of an ampliation adding on \( \sigma_0 \partial_1 p^\alpha \) and a contraction removing it again. We may therefore replace \( s^\alpha_0 \) and \( s^\mu_0 \) by sequences \( s^\alpha_1 \) and \( s^\mu_1 \) of the same length. We now proceed inductively: if after \( n-1 \) steps we have sequences \( s^\alpha_n \) and \( s^\mu_n \) and if the \( n \)-th elements of \( s^\alpha_n \) and \( s^\mu_n \) are both ampliations or both contractions, set \( s^\alpha_{n+1} := s^\alpha_n \) and \( s^\mu_{n+1} := s^\mu_n \). If this is not the case and the \( n \)-th element of \( s^\alpha_n \), say, is a contraction, \( s^\alpha_{n+1} \) is obtained by inserting \( \sigma_0 \partial_1 p^\alpha \) as \( n \)-th member and \( s^\mu_{n+1} \) by adding \( \sigma_0 \partial_1 p^\mu \) as a final element. The process terminates with sequences \( s^\alpha_k \) and \( s^\mu_k \), say, having ampliations and contractions in the same relative positions. Setting \( s(i) := (s^\alpha_k(i), s^\mu_k(i)) \). \( s \) is then a sequence of deformations from \( p \) to \( q \ast \sigma_0 a \ast \cdots \ast \sigma_0 a \sim q \), where \( a = \sigma_0 \partial_1 (p^\alpha, p^\mu) \), completing the proof.

### A.2 Simplicial sets of a poset

We shall be concerned here with two different simplicial sets that can be associated with a poset and we give their definitions not just for a poset but for an arbitrary category \( \mathcal{C} \). The first denoted \( \Sigma_*(\mathcal{C}) \) is just the usual nerve of the category. Thus the 0–simplices are just the objects of \( \mathcal{C} \), the 1–simplices are the arrows of \( \mathcal{C} \) and a 2–simplex \( c \) is made up of its three faces which are arrows satisfying \( \partial_0 c \partial_2 c = \partial_1 c \). The explicit form of higher simplices will not be needed in this paper. The homotopy category of \( \Sigma_*(\mathcal{C}) \) is canonically isomorphic to \( \mathcal{C} \) itself. The second simplicial set \( \tilde{\Sigma}_*(\mathcal{C}) \) is a symmetric simplicial set. It is constructed as follows. Consider the poset \( P_n \) of non-void subsets of \( \{0, 1, \ldots, n-1\} \) ordered under inclusion. Any mapping \( f \) from \( \{0, 1, \ldots, m-1\} \) to \( \{0, 1, \ldots, n-1\} \) induces an order preserving mapping from \( P_m \) to \( P_n \). Regarding the \( P_n \) as categories, we have realized \( \Delta^n \) as a subcategory of the category of categories. We then get a symmetric simplicial set where an \( n \)-simplex of \( \tilde{\Sigma}_*(\mathcal{C}) \) is a functor from \( P_n \) to \( \mathcal{C} \).

Given a poset \( K \) with order relation \( \leq \). We recall that \( K \) is upward directed whenever for any pair \( o, \hat{o} \in K \) there is \( \hat{o} \) such that \( o, \hat{o} \leq \hat{o} \). It is downward directed if the dual poset \( K^\circ \) is upward directed. The dual \( K^\circ \) of \( K \) is the poset having the same elements as \( K \) and opposite order relation \( \preceq \), i.e., \( a \preceq \hat{a} \) if, and only if, \( a \geq \hat{a} \).

When we specialize to a poset \( K \) the nerve \( \Sigma_*(K) \) and \( \tilde{\Sigma}_*(K) \) admit the following representation (see [11] for details). A 0–simplex of \( \Sigma_*(K) \) is just an element of the poset. For \( n \geq 1 \), an \( n \)-simplex \( x \) is formed by \( n + 1 \) \((n-1)\)-simplices \( \partial_0 x, \ldots, \partial_n x \), and by a 0–simplex \( |x| \) called the support of \( x \) such that \( |\partial_0 x|, \ldots, |\partial_n x| \leq |x| \). The nerve \( \Sigma_*(K) \) turns out to be a subsimplicial set of \( \Sigma_*(K) \). To see this, it is enough to define \( f_0(a) := a \) on 0–simplices and, inductively, \( |f_n(x)| := \partial_0 \cdots 11x \) and \( \partial_i f_n(x) := f_{n-1}(\partial_i x) \). So we obtain a simplicial map \( f_* : \Sigma_*(K) \to \Sigma_*(K) \). We sometimes adopt the following notation: \( (o; a, \hat{a}) \) is the 1–simplex of \( \Sigma_1(K) \) whose support is \( o \) and whose 0– and 1–face
are, respectively, $a$ and $\tilde{a}$; $(a, \tilde{a})$ is the 1-simplex of the nerve $\Sigma_1(K)$ whose 0- and 1-face are, respectively, $a$ and $\tilde{a}$.

The poset $K$ is said to be pathwise connected whenever the simplicial set $\tilde{\Sigma}^*(K)$ is pathwise connected. The homotopy groupoid $\pi_1(K)$ of $K$ is defined as $\pi_1(\tilde{\Sigma}^*(K))$; in particular, choosing a reference point $a \in \Sigma_0(K)$ we get the homotopy group $\pi_1(K, a)$. We observe that $K$ is pathwise connected if, and only if, its dual $K^\circ$ is pathwise connected and that $\pi_1(K)$ is isomorphic to $\pi_1(K^\circ)$ (see [11]). In the present paper we will consider only pathwise connected posets $K$. Thus we shall say that $K$ is simply connected whenever $\pi_1(K)$ is trivial. Note that, when $K$ is upward directed, $\tilde{\Sigma}^*(K)$ admits a contracting homotopy. So in this case $K$ is simply connected. The same happens when $K$ downward directed since $K$ and $K^\circ$ have isomorphic homotopy groupoids.

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