Equivariant prequantization and the moment map

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Abstract

If $\omega$ is a closed $G$-invariant 2-form and $\mu$ a moment map, then we obtain necessary and sufficient conditions for equivariant pre-quantizability that can be computed in terms of the moment map $\mu$. We also compute the obstructions to lift the action of $G$ to a pre-quantization bundle of $\omega$. Our results are valid for any compact and connected Lie group $G$.

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1 Introduction

In Geometric Quantization, a closed 2-form $\omega \in \Omega^2(M)$ on a manifold $M$ is said to be pre-quantizable if there exists a principal $U(1)$-bundle $P \to M$ with connection $\Xi$ such that $\text{curv}(\Xi) = \omega$. A classical result of Weil and Kostant (e.g. see [6]) states that $\omega$ is pre-quantizable if and only if $\omega$ is integral, i.e., its cohomology class lies in the image of the map $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$.

If a group of symmetries $G$ acts on $M$ and $\omega$ is $G$-invariant, it is natural to ask if it is possible to pre-quantize $\omega$ maintaining the symmetry. In more detail, we say that a $G$-invariant 2-form $\omega$ is $G$-equivariant pre-quantizable if there exists a lift of the action to a principal $U(1)$-bundle $P \to M$ with connection $\Xi$ such that $\Xi$ is $G$-invariant and $\text{curv}(\Xi) = \omega$. In this case, the moment $\text{Mom}^\Xi$ of $\Xi$ gives a moment map for $\omega$. In particular, a necessary condition for $\omega$ to be $G$-equivariant pre-quantizable is that the action of $G$ should be Hamiltonian. If $\mu$ is a moment map for $\omega$, we say that $(\omega, \mu)$ is $G$-equivariant pre-quantizable if there exists a $G$-equivariant $U(1)$-bundle with connection $(P, \Xi)$ such that $\text{curv}(\Xi) = \omega$ and $\text{Mom}^\Xi = \mu$. For a compact group $G$, a necessary and sufficient condition for $G$-equivariant pre-quantizability can be obtained in terms of $G$-equivariant cohomology of $M$ (e.g. see [3, 7]). The problem is that
this condition cannot be computed directly in terms of $\omega$ and $\mu$. In the case in which $G$ is a torus and $\omega$ is symplectic, there is another characterization of $G$-equivariant pre-quantizability that can be expressed directly in terms of $\omega$ and $\mu$ (e.g. see [3, Example 6.10]). Precisely, $(\omega, \mu)$ is $G$-equivariant pre-quantizable if and only if $\omega$ is integral and for a fixed point $x \in M^G$ we have $\mu_X(x) \in \mathbb{Z}$ for any $X \in \ker(\exp_G)$.

In this paper we generalize the preceding result to the case of non-abelian groups by using the concept of equivariant holonomy. We show that, given a pre-quantization bundle $(P, \Xi)$ for $\omega$, the obstruction to lift the action to $P$ is given by a group homomorphism $\Lambda^{E, \mu}: H_1(G, \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$. Under mild assumptions, this obstruction is equivalent to a map $\Delta^{E, \mu}: \ker(\exp_G) \to \mathbb{R}/\mathbb{Z}$ that can be computed in terms of the moment map $\mu$. Moreover, if $G$ is compact and we do not fix $\mu$, then the obstruction is given by the restriction of $\Lambda^{E, \mu}$ to the torsion subgroup of $H_1(G, \mathbb{Z})$, and we have a similar results for the restriction of $\Delta^{E, \mu}$ to a subset $\Xi_G \subset \ker(\exp_G)$. In the important case of a compact symplectic manifold we obtain that $(\omega, \mu)$ is $G$-equivariant pre-quantizable if and only if $\omega$ is integral and $\max_M(\mu_X) \in \mathbb{Z}$ for any $X \in \ker(\exp_G)$. Furthermore, $\omega$ is $G$-equivariant pre-quantizable if an only if $\omega$ is integral and $\max_M(\mu_X) \in \mathbb{Z}$ for any $X \in \Xi_G$.

2 Equivariant holonomy

In this section we recall the definition and properties of equivariant holonomy introduced in [1] and [2]. Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$ and let $M$ be a connected and oriented manifold. A $G$-equivariant $U(1)$-bundle is a principal $U(1)$-bundle $P \to M$ in which $G$ acts (on the left) by principal bundle automorphisms. If $\phi \in G$ and $y \in P$, we denote by $\phi_P(y)$ the action of $\phi$ on $y$. In a similar way, for $X \in \mathfrak{g}$ we denote by $X_P \in \mathfrak{X}(P)$ the corresponding vector field on $P$ defined by $X_P(x) = \frac{d}{dt}|_{t=0} \exp(-tX)P(x)$.

We denote by $I$ the interval $[0, 1]$. If $\gamma: I \to M$ is a curve, we define the inverse curve $\overline{\gamma}: I \to M$ by $\overline{\gamma}(t) = \gamma(1-t)$. Moreover, if $\gamma_1$ and $\gamma_2$ are curves with $\gamma_1(1) = \gamma_2(0)$ we define $\gamma_1 * \gamma_2: I \to \mathbb{R}$ by $\gamma_1 * \gamma_2(t) = \gamma_1(2t)$ for $t \in [0, 1/2]$ and $\gamma_1 * \gamma_2(t) = \gamma_2(2t-1)$ for $t \in [1/2, 1]$. For any $\phi \in G$ we define $C^0(M) = \{ \gamma: I \to M \mid \gamma \text{ is piecewise smooth and } \gamma(1) = \phi_M(\gamma(0)) \}$, and $C^\infty_x(M) = \{ \gamma \in C^0(M) \mid \gamma(0) = x \}$. Note that if $e \in G$ is the identity element, then $C^\infty_x(M) = C_e(M)$ is the space of loops based at $x$. If $\phi \in G$ and $\gamma$ is a curve on $M$ then we define $\phi \cdot \gamma$ by $(\phi \cdot \gamma)(t) = \phi_M(\gamma(t))$.

Let $\Xi$ be a $G$-invariant connection on a $G$-equivariant $U(1)$-bundle $P \to M$. If $\phi \in G$ and $\gamma \in C^\infty(M)$, the $\phi$-equivariant holonomy $\text{hol}_\Xi^\phi(\gamma) \in U(1)$ of $\gamma$ is characterized by the property $\overline{\gamma}(1) = \phi_P(\gamma(0)) \cdot \exp(2\pi i \text{hol}_\Xi^\phi(\gamma))$ for any $\Xi$-horizontal lift $\overline{\gamma}: I \to P$ of $\gamma$. Note that if $\gamma \in C^\infty_x(M)$ is a loop on $M$, then $\text{hol}_\Xi^e(\gamma)$ is the ordinary holonomy of $\gamma$. The following result is proved in [2].

**Proposition 1** If $P \to M$ is a $G$-equivariant principal $U(1)$-bundle, and $\Xi$ is a $G$-invariant connection on $P$, then for any $\phi, \phi' \in G$ we have
then the Lie algebra action can be exponentiated to an action of $G$ associated to the action of $G$ where $X^\ast G$ is a submanifold of $G$. Precisely, the lift is defined by setting $L^\phi_{(\gamma \cdot \zeta)}(\gamma \ast \zeta) = \hat{\phi}^\ast_\gamma (\gamma \cdot \zeta) = \hat{\phi}^\ast_\gamma (\gamma \cdot \zeta) = \hat{\phi}^\ast_\gamma (\gamma \cdot \zeta) = \hat{\phi}^\ast_\gamma (\gamma \cdot \zeta) = \hat{\phi}^\ast_\gamma (\gamma \cdot \zeta)

If $\Xi$ is a $G$-invariant connection on a principal $U(1)$ bundle $P \to M$ then $\frac{1}{2\pi} d\Xi$ projects onto a closed $G$-equivariant 2-form $\text{curv}(\Xi) \in \Omega^2(M)$ called the curvature of $\Xi$. We have the following generalization of the classical Gauss-Bonnet Theorem for surfaces

**Proposition 2** If $\Sigma \subset M$ is a 2-dimensional submanifold with boundary $\partial \Sigma = \bigcup_{i=1}^k \gamma_i$, with $\gamma_i \in \mathcal{C}(M)$ then we have $\sum_{i=1}^k \text{hol}^\Xi(\gamma_i) = \int_{\Sigma} \text{curv}(\Xi) \mod \mathbb{Z}$.

If $\omega \in \Omega^2(M)$ is a closed $G$-invariant 2-form, we say that the action is Hamiltonian if there exists a moment map for $\omega$, i.e., if there exists a $G$-equivariant map $\mu : g \to \Omega^0(M)$ such that $i_{X^\ast \omega} = d(\mu(X))$ for any $X \in g$. In the case of a $G$-invariant connection $\Xi$, its curvature $\text{curv}(\Xi)$ has a moment map $\text{Mom}^\Xi : g \to \Omega^0(M)$ defined by $\text{Mom}^\Xi(x) = -\frac{1}{2\pi} \Xi(X_P(y))$ for any $y \in P$ such that $\pi(y) = x$. Furthermore, we have the following result (see [2])

**Proposition 3** For any $X \in g$ and $x \in M$ we define $\tau_{x,X}(s) = \exp(sX)_M(x)$. Then $\tau_{x,X} \in \mathcal{C}^\exp_x(X)(M)$ and we have $\text{hol}^\Xi_{\exp}(\tau_{x,X}) = \text{Mom}^\Xi_x(x) \mod \mathbb{Z}$.

### 3 Lifting the action to a pre-quantization bundle

Let $\omega$ be a closed $G$-invariant 2-form and let $(P, \Xi)$ be a (non-equivariant) pre-quantization of $\omega$. A classical problem in Symplectic Geometry is if the action of $G$ can be lifted to $P$ leaving $\Xi$ invariant (e.g., see [1], [4], [6], [7] and references therein). If $\mu$ is a moment map for $\omega$, we can also impose that $\text{Mom}^\Xi = \mu$. We study these problems in detail in the next Sections.

#### 3.1 Lifting the action with the moment map fixed

Let $P \to M$ be a principal $U(1)$-bundle, $\Xi$ a connection on $P$ and let $\mu : g \to \Omega^0(M)$ be a moment map for $\omega = \text{curv}(\Xi)$. We say that the action of $G$ on $M$ admits a $(\Xi, \mu)$-lift to $P$ if there exists a lift of the action of $G$ to $P \to M$ by automorphisms such that $\Xi$ is $G$-invariant and $\text{Mom}^\Xi = \mu$.

At the infinitesimal level it is possible to lift the action of $g$ to $P$ in such a way that $L_{X^\ast \mu} \Xi = 0$ and $\iota_{X^\ast \mu} \Xi = -\mu_X$ for any $X \in g$ (e.g., see [3], [6], [7]). Precisely, the lift is defined by setting $X_P(y) = H^\Xi_\gamma(x) = \mu_X(x) \xi_P(y)$, where $\gamma = \pi(y)$. $H^\Xi_\gamma(X_N)$ is the $\Xi$-horizontal lift of $X_N$ and $\xi_P$ the vector field associated to the action of $U(1)$ on $P$. If $G$ is connected and simply connected, then the Lie algebra action can be exponentiated to an action of $G$ on $P$ (e.g., see [6]) and hence the action of $G$ admits a $(\Xi, \mu)$-lift to $P$. However, if $G$
is connected but not simply connected there could be obstructions to lift the action of $G$ to $P$ (e.g. see [2, 7]).

Let $\rho_G: G \to G$ be the universal covering group of $G$. The action of $G$ induces an action of $\tilde{G}$ on $M$ and, as $\tilde{G}$ is simply connected, this action admits a $(\Xi, \mu)$-lift to $P$, i.e., we have a group homomorphisms $\beta: \tilde{G} \to \text{Aut}P$. The homomorphism $\beta$ defines a group homomorphism $G \to \text{Aut}P$ if and only if $\beta(\ker \rho_G) = \text{id}_P$, i.e., if and only if $\phi = \text{id}_P$ for any $\phi \in \ker \rho_G$. The group $\ker \rho_G$ can be identified with the fundamental group $\pi_1(G)$ that it is well known to be an abelian group and hence we have $\ker \rho_G \simeq \pi_1(G) \simeq H_1(G, \mathbb{Z})$. The explicit isomorphism is given by the map that assigns to $\phi \in \ker \rho_G$ the class of $\rho_G(\gamma)$, where $\gamma$ is any curve on $\tilde{G}$ joining $1_G$ and $\phi$.

If $\phi \in \ker \rho_G$ and $y \in P$ then we have $\phi(y) = y \cdot \exp(-2\pi i \cdot \Lambda^\Xi(\phi))$ for certain $\Lambda^\Xi(\phi) \in \mathbb{R}/\mathbb{Z}$. The action admits a $(\Xi, \mu)$-lift to $P$ if and only if $\Lambda^\Xi(\phi) = 0 \mod \mathbb{Z}$ for any $\phi \in \ker \rho_G \simeq H_1(G, \mathbb{Z})$ and $y \in P$. We study this condition in terms of the $\tilde{G}$-equivariant holonomy of $\Xi$. From the definition of equivariant holonomy we obtain the following

**Proposition 4** If $\phi \in \ker \rho_G$ then for any $\gamma \in C^\phi(\mu_x, \mu_Y)(M) = C^\phi(\mu_x)(M)$ we have $\Lambda^\Xi(\phi) = \text{hol}^\Xi(\gamma) - \text{hol}_x(\gamma)$.

In particular, by applying the preceding proposition to the constant curve $c_x$, with $x = \pi(y)$ we obtain $\Lambda^\Xi(\phi) = \text{hol}^\Xi(c_x)$.

**Proposition 5** $\Lambda^\Xi(\phi)$ is a constant function on $P$ (i.e., it does not depend on $y \in P$).

**Proof.** If $y' \in P$ is another point and $x' = \pi(x)$, we choose a curve $\zeta$ with $\zeta(0) = x$ and $\zeta(1) = x'$. As $\phi \in \ker \rho$ we have $(\phi \cdot \zeta) = \zeta$ and hence

$0 = \text{hol}^\Xi(\zeta * c_x * \zeta * \overline{c}_{x'} * \overline{\overline{c}}_{x'}) = \text{hol}^\Xi(\zeta * c_x * \zeta) + \text{hol}_x^\Xi(\overline{c}_{x'})$

$= \text{hol}^\Xi(c_x) - \text{hol}^\Xi(c_{x'})$.

We denote $\Lambda^\Xi(\phi)$ simply by $\Lambda^\Xi(\phi)$, and we have a well defined group homomorphism $\Lambda^\Xi(\phi): \ker \rho_G \to \mathbb{R}/\mathbb{Z}$ and the following

**Proposition 6** The action of $G$ on $M$ admits a $(\Xi, \mu)$-lift if and only if the homomorphism $\Lambda^\Xi(\phi): \ker \rho_G \simeq H_1(G, \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ vanishes.

We can obtain an equivalent conditions in terms of Lie algebra $\mathfrak{g}$ of $G$. Let $\exp_G: \mathfrak{g} \to G$ and $\exp_{\tilde{G}}: \mathfrak{g} \to \tilde{G}$ be the exponential maps and define $\ker \exp_G = \{ X \in \mathfrak{g} : \exp(X) = 1_G \}$. We have $\ker(\ker \exp_G) \subset \ker \rho_G$, and we define $\Delta^\Xi(\phi): \ker \exp_G \to \mathbb{R}/\mathbb{Z}$ by $\Delta^\Xi(\phi)(X) = \Lambda^\Xi(\phi)(\exp_G X)$. By applying Propositions 3 and 4 to the curve $\tau_{x, X}$ we obtain the following

**Proposition 7** If $X \in \ker \exp_G$ then
a) for any $x \in M$ we have $\Delta^\Xi(\phi)(X) = \mu_X(x) - \text{hol}^\Xi(\tau_{x, X}) \mod \mathbb{Z}$.

e) If $x \in M$ and $X_M(x) = 0$ then $\Delta^\Xi(\phi)(X) = \mu_X(x) \mod \mathbb{Z}$.
In particular, if \( x \in M^G \) is a fixed point for the action of \( G \), then \( \Delta^{\Xi,\mu}(X) = \mu_X(x) \mod \mathbb{Z} \) for any \( X \in \ker \exp_G \).

**Corollary 8** If \( \omega \) is integral and \( \mu \) is a moment map for \( \omega \), then for any \( X \in \ker \exp_G \) and any \( x, x' \in M \) such that \( X_M(x) = X_M(x') = 0 \) we have \( \mu_X(x) = \mu_X(x') \in \mathbb{Z} \). In particular, for any two fixed points \( x, x' \in M^G \) we have \( \mu_X(x) = \mu_X(x') \in \mathbb{Z} \).

**Proof.** We have \( 0 = d\mu_X(x) = i_{X_M}\omega(x) \), and as \( \omega \) is non-degenerate, we conclude that \( X_M(x) = 0 \). By Proposition 7 we have \( \Delta^{\Xi,\mu}(X) = \mu_X(x) \mod \mathbb{Z} \).

**Corollary 10** If \( M \) is compact and \( \omega \) is symplectic then for any \( X \in \ker \exp_G \) we have \( \Delta^{\Xi,\mu}(X) = \max_M(\mu_X) \mod \mathbb{Z} = \min_M(\mu_X) \mod \mathbb{Z} \).

In particular we obtain the following

**Corollary 11** If \( \omega \) is symplectic and integral, and \( M \) is compact, then for any \( X \in \ker \exp_G \) we have \( \max_M(\mu_X) - \min_M(\mu_X) \in \mathbb{Z} \).

We recall that a Lie group is called exponential if the exponential map \( \exp_G : \mathfrak{g} \to G \) is surjective. For example, if \( G \) is compact and connected, then \( G \) and \( \hat{G} \) are exponential (e.g. see [5, Proposition 6.10]). We say that \( G \) is w-exponential if \( \ker \rho_G \subset \exp_G(\mathfrak{g}) \). For example, any compact group is w-exponential. For a w-exponential group \( G \) we have \( \exp_G(\ker \exp_G) = \ker \rho_G \) and hence it is equivalent to work with \( \Delta^{\Xi,\mu} \) or with \( \Lambda^{\Xi,\mu} \).

**Theorem 12** If \( G \) is w-exponential, then the action of \( G \) on \( M \) admits a \((\Xi, \mu)\)-lift to \( P \) if and only if \( \Delta^{\Xi,\mu}(X) = 0 \mod \mathbb{Z} \) for any \( X \in \ker \exp_G \).

**Example 13** Let \( M \) be a connected manifold in which \( G = S^1 \) acts. We set \( \omega = 0 \) and \( \mu^{\ast} : \mathbb{R} \to \Omega^0(M) \) given by \( \mu^{\ast}_X(x) = cX \) for any \( x \in M \) and \( X \in \mathbb{R} \). Clearly \( \omega \) is \( G \)-invariant, \( \mu^{\ast} \) is a moment map for \( \omega \) and \( \ker \exp_{S^1} = 2\pi \mathbb{Z} \).

Let \( (P, \Xi) \) be the trivial bundle and connection. If \( X = 2\pi c \) then we have \( \Delta^{\Xi,\mu^{\ast}}(X) = \mu^{\ast}_X(x) - \text{hol}(\tau_{\Xi}, X) \mod \mathbb{Z} = 2\pi c \mod \mathbb{Z} \). We conclude that the action admits a \((\Xi, \mu)\)-lift to \( P \) if and only if \( 2\pi c \in \mathbb{Z} \).

**Corollary 14** If \( G \) is w-exponential and \( M^G \neq \emptyset \), then the action admits a \((\Xi, \mu)\)-lift to \( P \) if and only if for a fixed point \( x \in M^G \) we have \( \mu_X(x) \in \mathbb{Z} \) for any \( X \in \ker \exp_G \).

If \( G \) is a torus and \( \omega \) is symplectic, the Atiyah-Guillemin-Sternberg convexity Theorem implies the existence of fixed points, and hence the preceding theorem can be applied. For non abelian groups we cannot assert that \( M^G \neq \emptyset \), but we have the following

**Corollary 15** If \( G \) is w-exponential, \( M \) is compact and \( \omega \) is symplectic, then the action admits a \((\Xi, \mu)\)-lift to \( P \) if and only if for any \( X \in \ker \exp_G \) we have \( \max_M(\mu_X) \in \mathbb{Z} \).
4 Lifting the action with an arbitrary moment map

In this Section we assume that $G$ is a connected and compact Lie group. Let $P \to M$ be a principal $U(1)$-bundle, $\Xi$ a connection on $P$ and $\omega = \text{curv}(\Xi)$. We say that the action of $G$ on $M$ by automorphisms such that $\Xi$ is $G$-invariant. In this case $\text{Mom}^G$ is a moment map for $\omega$ and hence, there exists a $\Xi$-lift if and only if there exists a $(\Xi, \mu)$-lift for a moment map $\mu$.

We assume that the action is Hamiltonian and that $\mu$ is a moment map for $\omega$. As we have seen, the obstruction to the existence of a $(\Xi, \mu)$-lift is given by $\Delta^{\Xi, \mu} \in \text{Hom}(H_1(G, \mathbb{Z}), \mathbb{R} / \mathbb{Z})$. Any other moment map is given by $\mu' = \mu + b$ where $b \in H^1(\mathfrak{g})$, i.e., $b \in \text{Hom}(\mathfrak{g}, \mathbb{R})$ satisfies the condition $b([X, Y]) = 0$ for any $X, Y \in \mathfrak{g}$. If $\Lambda^{\Xi, \mu} \neq 0$, we can try to use $b$ to cancel $\Delta^{\Xi, \mu}$, but only for the elements that are not torsion. Precisely, let $T_G$ be the torsion subgroup of $\ker \rho_G \simeq H^1(G, \mathbb{Z})$. At the Lie algebra level, we have $\Delta^{\Xi, \mu} = \Delta^{\Xi, \mu} + b \mod \mathbb{Z}$ and we define $\mathfrak{t}_G$ as the space of $X \in \ker \exp_G$ such that there exists $n \in \mathbb{N}$ with $nX \in \ker \exp_G$. Then we have $\exp_G(\mathfrak{t}_G) = T_G$ and we have the following result

**Lemma 16** If $X \in \mathfrak{t}_G$ and $b \in H^1(\mathfrak{g})$ then $b(X) = 0$.

**Proof.** If $X \in \mathfrak{t}_G$ and $n \in \ker \exp_G$ then for any moment map $\mu$ we have $n\Delta^{\Xi, \mu}(X) = \Delta^{\Xi, \mu}(nX) = 0 \mod \mathbb{Z}$. If $\mu' = \mu + \lambda b$ then $0 = n\Delta^{\Xi, \mu}(X) - n\Delta^{\Xi, \mu}(X) = n\lambda b(X) \mod \mathbb{Z}$ for any $\lambda \in \mathbb{R}$, and hence $b(X) = 0$. ■

Then we have the following

**Theorem 17** Let $G$ be a compact and connected Lie group. Then

a) The restriction of $\Lambda^{\Xi, \mu}$ to $T_G$ is independent of the moment map $\mu$.

b) There exists a $\Xi$-lift to $P$ if and only if $\Lambda^{\Xi, \mu}(\phi) = 0$ for any $\phi \in T_G$.

c) There exists a $\Xi$-lift to $P$ if and only if $\Delta^{\Xi, \mu}(X) = 0$ for any $X \in \mathfrak{t}_G$.

**Proof.** Clearly c) and b) are equivalent and a) follows from Lemma 16. We prove b). We only need to prove that if $\Lambda^{\Xi, \mu}(\phi) = 0$ for any $\phi \in T_G$ then there exists a $\Xi$-lift to $P$ (the converse follows from a)). We know that $H_1(G, \mathbb{Z}) \simeq \ker \rho_G \simeq T_G \oplus \mathbb{Z}^k$. We choose a system of generators $\phi_1, \ldots, \phi_k$ for the free part of $\ker \rho_G$. If $G$ is compact, $H^1(\mathfrak{g})$ it can be identified with $H^1(G, \mathbb{R}) \simeq \mathbb{R}^k$ by the map that assigns to $b \in H^1(\mathfrak{g})$ the $G$-invariant 1-form $\alpha_b$ such that $\alpha_b|_\mathfrak{g} = b$. Hence, if $\Lambda^{\Xi, \mu}(\phi_i) = c_i$, and $\phi_i = \exp_G(X_i)$, then there exists $b \in H^1(\mathfrak{g}) \simeq H^1(G, \mathbb{R})$ such that $-c_i = \int_{\rho_\phi} \alpha_b \mod \mathbb{Z} = b(X_i) \mod \mathbb{Z}$, where $\theta_i(t) = \exp(tX_i)$. We set $\mu' = \mu + b$ and we have $\Lambda^{\Xi, \mu'}(\phi_i) = \Delta^{\Xi, \mu'}(X_i) = \Delta^{\Xi, \mu}(X_i) + b(X_i) \mod \mathbb{Z} = c_i - c_i = 0$. Furthermore, if the restriction of $\Lambda^{\Xi, \mu}$ to $T_G$ vanish, then we have $\Lambda^{\Xi, \mu'}(\phi) = 0$ for any $\phi \in \ker \rho_G$. ■

If $P_i \to M_i$, are $U(1)$-bundles with connections $\Xi_i$ and moment maps $\mu_i$ for $\text{curv}(\Xi_i)$, $i = 1, 2$ then $\mu_1 + \mu_2$ is a moment map for $\text{curv}(\Xi_1 \otimes \Xi_2) = \text{curv}(\Xi_1) + \text{curv}(\Xi_2)$ and we have $\Delta^{\Xi_1 \otimes \Xi_2, \mu_1 + \mu_2} = \Delta^{\Xi_1, \mu_1} + \Delta^{\Xi_2, \mu_2}$. As a consequence of Theorem 17 we obtain the following classical result (e.g. see [4, 7]).
Corollary 18 If \( G \) is a compact and connected Lie group then there exists \( r_G \in \mathbb{N} \) such that for any Hamiltonian action of \( G \) on \((M, \omega)\) and any pre-quantization bundle \((P, \Xi)\) for \( \omega \) there exists a \( \Xi^{\otimes r_G} \)-lift of the action to \( P^{\otimes r_G} \).

Example 19 If \( G = SO(n) \), \( n > 1 \) then \( H^1(G, \mathbb{Z}) \simeq \mathbb{Z}_2 \), and hence \( r_{SO(n)} = 2 \).

Example 20 If \( H^1(G, \mathbb{Z}) \) does not have torsion, then \( r_G = 1 \), and any Hamiltonian action with integral \( \omega \) can be lifted to \( P \).

5 Equivariant pre-quantization

In the cases in which \( \Lambda^{\Xi, \mu}(X) \) only depends on \( \mu \) it is easy to obtain conditions for equivariant pre-quantization from the results of Section 3. In particular we have the following results:

Theorem 21 a) If \( G \) is \( w \)-exponential, \( M \) is compact and \( \omega \) is symplectic, then \((\omega, \mu)\) is \( G \)-equivariant pre-quantizable if and only if \( \omega \) is integral and for any \( X \in \ker \exp G \) we have \( \max_M(\mu_X) \in \mathbb{Z} \).

b) If \( G \) and \( M \) are compact and \( \omega \) is symplectic, then \( \omega \) is \( G \)-equivariant pre-quantizable if and only if \( \omega \) is integral and for any \( X \in \mathfrak{g} \) we have \( \max_M(\mu_X) \in \mathbb{Z} \).

Example 22 We consider the usual symplectic structure on \( S^2 \) given by the volume form \( \text{vol}_g \in \Omega^2(S^2) \) and the usual moment map \( h: \mathfrak{so}(3) \to \Omega^0(S^2) \) given by \( h_X(x) = \langle \vec{v}_X, x \rangle \), where \( \vec{v}: \mathfrak{so}(3) \to \mathbb{R}^3 \) is the map that assigns to \( X = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \in \mathfrak{so}(3) \) the vector \( \vec{v}_X = (c, -b, a) \).

As \( \exp X \) is a rotation of angle \( ||\vec{v}_X|| \) around the axis determined by \( \vec{v}_X \), \( \ker(\exp G) \) corresponds under \( \vec{v} \) to the union of the spheres of radius \( 2\pi k \) for \( k \in \mathbb{Z} \). If \( X \in \ker(\exp G) \) and \( ||\vec{v}_X|| = 2\pi k \) we have \( \max_M(h_X) = 2\pi k \).

The form \( \omega = \lambda \text{vol}_g \) is integral if and only if \( \lambda = \frac{n}{4\pi} \) for \( n \in \mathbb{Z} \). But in this case we have \( \max_M(\lambda h_X) = \frac{nk}{2} \) that is integer for any \( k \) if and only if \( n \) is even. Hence we have

Proposition 23 a) The form \( \lambda \text{vol}_g \) is pre-quantizable if and only if \( \lambda = \frac{n}{4\pi} \) for \( n \in \mathbb{Z} \).

b) The form \( \lambda \text{vol}_g \) is \( SO(3) \)-equivariant pre-quantizable if and only if \( \lambda = \frac{n}{4\pi} \) for \( n \in \mathbb{Z} \).

Remark 24 Note that if \( \lambda = \frac{n}{4\pi} \), then \( \lambda \text{vol}_g \) is \( SU(2) \)-equivariant pre-quantizable because \( SU(2) \) is the universal cover of \( SO(3) \).

Corollary 25 If \( G \) is a compact and connected Lie group and we have a Hamiltonian action of \( G \) on a compact manifold \((M, \omega)\) such that \( \omega \) is integral and symplectic then we have \( \max \mu_X(x) \in \mathbb{Q} \) for any \( X \in \mathfrak{g} \).
We also obtain the following generalization of Example 6.10 in [3]:

**Theorem 26** a) If $G$ is $w$-exponential and $M^G \neq \emptyset$, then $(\omega, \mu)$ is $G$-equivariant pre-quantizable if and only if $\omega$ is integral and for a fixed point $x \in M^G$ we have $\mu_X(x) \in \mathbb{Z}$ for any $X \in \ker \exp_G$.

b) If $G$ is compact and $M^G \neq \emptyset$, then $\omega$ is $G$-equivariant pre-quantizable if and only if $\omega$ is integral and for a fixed point $x \in M^G$ we have $\mu_X(x) \in \mathbb{Z}$ for any $X \in \mathfrak{g}_G$.

As a consequence of Corollary 18 we obtain the following

**Corollary 27** If $G$ is a compact and connected Lie group then there exists $r_G \in \mathbb{N}$ such that for any Hamiltonian action of $G$ on a manifold $(M, \omega)$ with $\omega$ is integral, the form $r_G \cdot \omega$ is $G$-equivariant pre-quantizable.

**Remark 28** As commented above, if $H_1(G, \mathbb{Z})$ does not have torsion, then $r_G = 1$ and any Hamiltonian action of $G$ on a manifold $(M, \omega)$ with $\omega$ is integral is $G$-equivariant pre-quantizable.

**Example 29** We consider the action of $S^1$ on $S^2$ by rotations around the $z$ axis and the form $\omega = \frac{1}{4\pi} \text{vol}_g \in \Omega^2(S^2)$ with the moment map $\mu_X = \frac{1}{4\pi} h_{X,M}$ for $X \in \mathbb{R}$, and where $M = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{array} \right)$. For $X = 2\pi k \in \ker \exp_G = 2\pi \mathbb{Z}$ we have $\max_M(\mu_X) = 0$, and hence $(\omega, \mu)$ is not $S^1$-equivariant pre-quantizable. However, as $H_1(S^1, \mathbb{Z})$ does not have torsion, we know that $\omega$ is $S^1$-equivariant pre-quantizable with another moment map $\mu' = x + b$ with $b_X = cX$ and $c$ constant. For example, it is enough to take $c = -\frac{1}{4\pi}$ as then we have $\max_M(\mu'_X) = k - \frac{1}{2\pi} 2\pi k = 0 \mod \mathbb{Z}$.

The case in which $\Lambda^{\Xi,\mu}(X)$ depends on $\Xi$ is more complicated because it can happen that $(\omega, \mu)$ is $G$-equivariant pre-quantizable but $\Lambda^{\Xi,\mu} \neq 0$ (see Example 31 below). If $(P, \Xi)$ and $(P', \Xi')$ are two pre-quantizations of $\omega$, then there exists a group homomorphism $\beta: H_1(M, \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ such that $\text{hol}^\Xi(\gamma) = \exp^\Xi(\gamma) + \beta(\gamma)$ for any $\gamma \in C(M)$. Then we have $\Lambda^{\Xi,\mu}(X) = \Lambda^{\Xi,\mu}(X) + \beta(\tau_{X,\mathbb{R}})$. Note that if the homology class $[\tau_{X,\mathbb{R}}] \in H_1(M, \mathbb{Z})$ vanishes for any $X \in \ker \exp_G$, then $\Lambda^{\Xi,\mu}(X)$ is independent of $\Xi$. This happens for example if $H_1(M, \mathbb{Z}) = 0$.

**Example 30** If in Example 73 we consider a manifold $M$ such that $H_1(M, \mathbb{Z}) = 0$, then we obtain that $(0, \mu^\circ)$ is $G$-equivariant pre-quantizable if and only if $2\pi c \in \mathbb{Z}$.

However, if $H_1(M, \mathbb{Z}) \neq 0$ the situation could be different, and it could be possible to cancel $\Lambda^{\Xi,\mu}$ with $\beta$:

\footnote{The connection $\Xi' \otimes \Xi^{-1}$ on the bundle $P' \otimes P^{-1}$ is a flat connection and $\beta$ is the holonomy of $\Xi' \otimes \Xi^{-1}$.}
Example 31 We apply Example 13 to the case in which $M = T^2$ is the 2-torus and $G = S^1$ acts on $T^2$ by rotations. We have $\text{Hom}(H_1(T^2, \mathbb{Z}), \mathbb{R}/\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z}$, and if $X = 2\pi z$ then for any $p \in \mathbb{R}/\mathbb{Z}$ there exists $\beta \in \text{Hom}(H_1(T^2, \mathbb{Z}), \mathbb{R}/\mathbb{Z})$ such that $\beta([r_x, X]) = p$. If we chose $p = -2\pi z c \text{mod} \mathbb{Z}$ then we have $\Lambda^\Xi, \mu(X) = \Lambda^\Xi, \mu(X) + \beta([r_x, X]) = 0 \text{ mod } \mathbb{Z}$. Hence $(0, \mu^c)$ is $G$-equivariant pre-quantizable for any $c \in \mathbb{R}$.

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