On the Closed Form Solution for the Geodesics in SdS Space

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“Dedicated to Niels Henrik Abel; The Great Pioneer into the Theory of Functions and Algebraic Equations”

The closed form solution for the geodesics of classical particles in SdS space is obtained in terms of hyperelliptic modular functions and multiple hypergeometric functions. The closed form solution for the five roots of the fifth degree polynomial is found giving the branch places on the genus two Riemann surface. ‘The Inversion Problem’, for the genus two hyperelliptic integral, is solved in a closed form. The solution is shown to reduce to elliptic functions when the cosmological constant is zero. Current observational data is in favor of a cosmological constant model\(^1\). This solution is important in astrophysical applications of measuring the cosmological constant.

I. Introduction

To calculate the geodesics in the centrally symmetric gravitational field of a Schwarzschild black hole in a universe with a cosmological constant(\(\Lambda \neq 0\)), i.e. Schwarzschild de-Sitter space(SdS space), the Hamilton-Jacobi method is applied to the metric\(^2-5\)

\[
ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\phi\phi}d\phi^2 + g_{\theta\theta}d\theta^2
\]

\[
g_{tt} = \left(1 - \frac{1}{3} \sigma \Lambda r^2 - \frac{r_g}{r}\right)c^2, g_{rr} = \left(1 - \frac{1}{3} \sigma \Lambda r^2 - \frac{r_g}{r}\right)^{-1}, g_{\phi\phi} = -r^2 \sin^2 \theta, g_{\theta\theta} = -r^2
\]

Where \(\sigma = \pm 1\) depending on whether or not the universe is de-Sitter or Anti-de-Sitter, respectively. \(\Lambda\) is the cosmological constant and was first introduced by Einstein\(^6\) and \(r_g\) was introduced by Schwarzschild. The functional form of

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the geodesics when \( r_g = 0 \) have been studied, they are given in terms of genus zero circular functions. When \( \Lambda = 0 \) the geodesics are given by genus one elliptic functions. In this paper, the functional form of the geodesics when \( \Lambda \) and \( r_g \) are both non-zero are found, in terms of genus two hyperelliptic theta functions and multiple hypergeometric functions.

II. Derivation of the Hyperelliptic \( \phi \)-Integral

Let a particle of mass \( m \) be constrained to move in the equatorial plane, \( \theta = 0 \), and let the action be given by

\[
S = -Et + L\phi + S_r
\]

where \( E \) is the total energy of the particle and \( L \) is its angular momentum. By the conservation of four momentum,

\[
g^{ij} \frac{dS}{dx^i} \frac{dS}{dx^j} = m^2 c^2
\]

we can insert \( S \) from (3) and \( g^{ij} \) from (2) into (4) and obtain,

\[
S_r = \int_{r_0}^{r} \frac{\left( \frac{E^2}{c^2} - (1 - \frac{4}{3} \Lambda \sigma r^2 - \frac{r_g}{r})\left( \frac{L^2}{r^2} + m^2 c^2 \right) \right)^{\frac{1}{2}}}{\left( 1 - \frac{4}{3} \Lambda \sigma r^2 - \frac{r_g}{r} \right)}.
\]

Differentiating \( S_r \) w.r.t. \( L \) and using

\[
\phi + \frac{\partial S_r}{\partial L} = \text{const} = 0
\]

\[
\phi = \int_{r_0}^{r} \frac{Ldr}{r^2\left( \frac{E^2}{c^2} - (1 - \frac{4}{3} \Lambda \sigma r^2 - \frac{r_g}{r})\left( \frac{L^2}{r^2} + m^2 c^2 \right) \right)^{\frac{1}{2}}}
\]

This integral is a hyperelliptic integral of genus, \( g = 2 \). The problem is to invert this integral, i.e. express the upper limit \( r \) as a function of \( \phi \), and \( \phi \) as a function of \( r \), this is known as the ‘Inversion Problem’. The inversion problem was first conceived by Euler.

III. The Solution to the Inversion Problem for the Hyperelliptic Integral of Arbitrary Genus

The following will be an account given by H.F. Baker and H. Exton on the application of theta functions and multiple hypergeometric functions to obtain the closed form solution to the arbitrary genus hyperelliptic integral, except for the branch places on the Riemann surface. This requires the solution to the \( n^{th} \) degree polynomial, for its \( n \) roots. The general solution to the inversion
problem is given by solving \( g \) of the \( 2g + 1 \) equations, for the \( g \) variable places \( x_1, \ldots, x_g \),

\[
\frac{\partial^2 (u|u^{b,a})}{\partial^2 (u|u^{b',a'})} = A(b)(b - x_1) \ldots (b - x_g) \tag{8}
\]

The notation Baker uses for the generalized theta function is,

\[
\vartheta(u|u^{b,a}) = \vartheta(u; \frac{1}{2}m, \frac{1}{2}m') = e^{\alpha u^2} \Theta(v; \frac{1}{2}m, \frac{1}{2}m') \tag{9}
\]

where \( a \) is an arbitrary \( g \times g \) symmetrical matrix, since putting (9) into (8) gives,

\[
\frac{\partial^2 (u|u^{b,a})}{\partial^2 (u|u^{b',a'})} = \frac{\Theta^2(v; \frac{1}{2}m, \frac{1}{2}m')}{\Theta^2(v; \frac{1}{2}k, \frac{1}{2}k')} \tag{10}
\]

where \( m, m', k \) and \( k' \) are integers. Riemann’s theta functions are defined as,

\[
\Theta(v; \frac{1}{2}m, \frac{1}{2}m') = \sum_{n} e^{2\pi i n^2 (m + \frac{1}{2}m')} + i\pi m(n + \frac{1}{2}m') \tag{11}
\]

with, \( \Sigma = \sum_{n_1 = -\infty}^{+\infty} \ldots \sum_{n_g = -\infty}^{+\infty} \). \( b \) is a \( g \times g \) matrix, in general asymmetrical. \( b \) is a \( g \times g \) symmetrical matrix. In (10), \( u \) denotes the \( g \) quantities,

\[
u_{i,a_1} x_{1,a_1} + \ldots + u_{i,a_g} x_{g,a_g} = u_i \tag{12}
\]

with \( i = 1, \ldots, g \). This is known as Abel’s Theorem. Riemann’s Normal Integral of the first kind is defined as,

\[
\int_a^x \frac{(x, 1)_{i,g-1} \, dx}{y} \tag{13}
\]

where

\[
(x, 1)_{i,g-1} = A_{i,g-1} x^{g-1} + \ldots + A_{i,0} \tag{14}
\]

and the Abel coefficients \( A_{i,g-1}, \ldots, A_{i,0} \) are constants determined experimentally. The denominator in (13) is given by,

\[
y^2 = 4(x - a_1) \ldots (x - a_g)(x - c_1) \ldots (x - c_g)(x - c) \tag{15}
\]

where \( a_1, \ldots, a_g, c_1, \ldots, c \) are branch places and \( x \) is a variable on the one complex dimensional Riemann surface of genus, \( g \). Now since the sum of terms is finite in (13), then by combining (13)(14)and(15) an arbitrary term in \( u_{i,a}^x \) may be transformed to Extonian form\(^{10} \), using a Möbius transformation,

\[
\int_0^z t^{a-1}(1 - t)^{c-a-1}(1 - tz_1)^{c-\frac{1}{2}} \ldots (1 - tz_g)^{c-\frac{1}{2}} \, dt
\]

\[
= \frac{z^a}{a} F^{(2g+1)}(a, \frac{1}{2}, \ldots, \frac{1}{2}, 1 + a - c; a + 1; z_1, \ldots, z_{2g}, z) \tag{16}
\]

where \( Re(a) \) and \( Re(c - a) \) are greater than zero. \( a \) and \( c \) are appropriately chosen constants. The variables \( z_1, \ldots, z_n \) depend on the roots of \( y \) and the
lower limit. While \( z \) depends on the upper limit of the hyperelliptic integral. The condition that \( z_1, \ldots, z_{2g} \) and \( z \) each have their absolute value less than one ensures that the Lauricella function \( F^{(2g+1)}_{D} \) converges, making the variables in Abel’s theorem well defined. The convergence takes place in a unit sphere in \( 2g + 1 \) hyperspace. The Lauricella function has the expansion,

\[
F^{(2g+1)}_{D}(a, \frac{1}{2}, \ldots, \frac{1}{2}; 1 + a - c; a + 1; z_1, \ldots, z_{2g+1}, z) = \sum_{(m)=-\infty}^{+\infty} \frac{(a, m_1 + \ldots + m_{2g+1})(\frac{1}{2}, m_1) \ldots (1 + a - c, m_{2g+1})z_1^{m_1} \ldots z_{2g+1}^{m_{2g+1}}}{(a + 1, m_1 + \ldots + m_{2g+1})m_1! \ldots m_{2g+1}!}
\]

(17)

where \( (m) = m_1 \ldots m_{2g} \) and \( z_i \) have the form from the Möbius Transformation,

\[
z_i = \frac{r_j - a}{r_i - a}
\]

(19)

where \( a \) is the lower limit of the hyperelliptic integral (13), and

\[
z = \frac{x - a}{r_i - a}
\]

(20)

Now let,

\[
\frac{1}{2} \Omega_{m,m'} = m_1 \omega_{r,1} + \ldots + m_g \omega_{r,g} + m'_1 \omega'_{r,1} + \ldots + m'_g \omega'_{r,g}
\]

(21)

where the \( \omega \) have the representation,

\[
\omega = C \times F^{(2g-1)}_{D}(a, \frac{1}{2}, \ldots, \frac{1}{2}; c; z_1, \ldots, z_{2g-1})
\]

(22)

and where \( m \) and \( m' \) are equal to +1 when integrating across a period loop right to left, and -1 when crossing from left to right and 0 when not crossing a period loop. \( C \) is a constant that depends on the branch places and Abel’s coefficients. \( \omega \) is convergent in a unit sphere in \( 2g - 1 \) hyperspace. \( a \) and \( c \) are the appropriate constants. Now define an integral of the first kind via,

\[
\pi iv^{r,a}_r = h_{r,1}u^{x,a}_1 + \ldots + h_{r,g}u^{x,a}_g
\]

(23)

with \( r = 1, \ldots, g \) and where,

\[
2h \omega = \pi i
\]

(24)

\[
2h \omega' = \pi i \tau
\]

(25)

where Riemann’s condition for hyperelliptic integrals is imposed [16–18].

\[
\tau = \tau^t
\]

(26)
With the above equations, Riemann’s theta function becomes,

\[ \Theta(v; \frac{1}{2}m, \frac{1}{2}m') = \sum e^{2\pi iv(n+\frac{1}{2}m') + i\pi(n+\frac{1}{2}m')^2 + m(n+\frac{1}{2}m')} \]  \tag{27} 

where the quadratic forms are given by,

\[ v(n + \frac{1}{2}m') = vn + \frac{1}{2}m'v = v_1n_1 + \ldots + v_gn_g + \frac{1}{2}v_1m'_1 + \ldots + \frac{1}{2}v_gm'_g \]  \tag{28} 

and similarly,

\[ \tau(n + \frac{1}{2}m') = \tau n^2 + \tau nm' + \frac{1}{4}\tau m'^2 \\
= (\tau_1n_1^2 + 2\tau_1n_1n_2 + \ldots + \tau_{g,g}n_g^2) \\
+ \sum_{s=1}^{g} \sum_{r=1}^{g} \tau_{r,s}n_rm'_s \\
+ \frac{1}{4}(\tau_1m'_1^2 + 2\tau_1m'_1m'_2 + \ldots + \tau_{g,g}m'_g^2) \]  \tag{29} 

It has been proven by Riemann that for the integral of the first kind, when it has the given period scheme, then the imaginary part of, 

\[ \tau_1n_1^2 + 2\tau_1n_1n_2 + \ldots + \tau_{g,g}n_g^2 \]  \tag{30} 

is positive for all integer values of \( n_1 \) and \( n_2 \), with the exception of \( n_1 = n_2 = 0 \). Therefore the modulus of \( e^{i\pi\tau n^2} \) is less than unity and the function (27) converges for all values of its argument, \( v \). The last thing to define in (8), is the constant \( A(b) \), it is defined as,

\[ A(b) = (\epsilon \frac{d}{dx}(x - a_1)\ldots(x - a_g)(x - c_1)\ldots(x - c_g)(x - c)b)_{x=b}^{-\frac{1}{2}} \]  \tag{31} 

where \( \epsilon \) is +1 when \( u^{b,a} \) is an odd half period and −1 when it is an even half period.

IV. The Complete Solution to the Inversion Problem of the Genus Two Hyperelliptic Integral

The solution to the inversion of the genus two hyperelliptic integral consists of two parts. The first part is to express the upper limits, \( x_1 \) and \( x_2 \), in terms of the angles, \( u_1 \) and \( u_2 \), in Abel’s theorem,

\[ u_1^{x_1,a_1} + u_1^{x_2,a_2} = u_1 \]  \tag{32} 

\[ u_2^{x_1,a_1} + u_2^{x_2,a_2} = u_2 \]  \tag{33} 

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and secondly, express the angles $u_1$ and $u_2$ in terms of $x_1$ and $x_2$, so that $u_1$, $u_2$, $x_1$ and $x_2$ become well defined coordinates on Kummer’s quartic surface. The solution to the first part is given by, with $g = 2$,

$$
\frac{\Theta^2(v_1, v_2; \frac{1}{2} m_1, \frac{1}{2} m_2, \frac{1}{2} m'_1, \frac{1}{2} m'_2)}{\Theta^2(v_1, v_2; \frac{1}{2} k_1, \frac{1}{2} k_2, \frac{1}{2} k'_1, \frac{1}{2} k'_2)} = A(b)(b - x_1)(b - x_2)
$$

(34)

where by (23) we have,

$$
\pi i v_1 = \pi i v_{x,a}^{x,a} = h_{1,1} u_{x,a}^{x,a} + h_{1,2} u_{x,a}^{x,a}
$$

(35)

$$
\pi i v_2 = \pi i v_{x,a}^{x,a} = h_{2,1} u_{x,a}^{x,a} + h_{2,2} u_{x,a}^{x,a}
$$

(36)

where the $h_{i,j}$ are the elements of the following matrix,

$$
h = \frac{\pi i}{\Delta} \begin{pmatrix}
\omega_{2,2} & -\omega_{1,2} \\
-\omega_{2,1} & \omega_{1,1}
\end{pmatrix}
$$

and

$$
\Delta = \det(\omega) \neq 0
$$

(37)

Now the solution to the second part of the inversion problem is facilitated by first choosing,

$$
x_2 = a_2
$$

(38)

by (13), we have

$$
\begin{align*}
u_1 &= u_{x,a}^{x,a} = \int_{a_1}^{x_1} \frac{(A_{1,0} + A_{1,1} x)}{y} dx \\
u_2 &= u_{x,a}^{x,a} = \int_{a_1}^{x_1} \frac{(A_{2,0} + A_{2,1} x)}{y} dx
\end{align*}
$$

(39) \hspace{1cm} (40)

where,

$$
y^2 = 4x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0
$$

(41)

which can be rewritten as,

$$
y^2 = 4(x - a_1)(x - a_2)(x - c_1)(x - c_2)(x - c)
$$

(42)

Now let (39) and (40) be written as,

$$
\begin{align*}
u_{x,y} &= \int_{y}^{x} \frac{(A_{k,0} + A_{k,1} t)}{\sqrt{4(t - r_1)(t - r_2)(t - r_3)(t - r_4)(t - r_5)}} dt
\end{align*}
$$

(43)

where $k = 1, 2$, $i_1, i_2, i_3, i_4$ and $i_5 \in \mathbb{N}$ mod 5 where $\mathbb{N} \in [0, 1, \ldots, \infty)$, so that,

$$
\begin{align*}
r_0, r_1, r_2, r_3, r_4 &\in (a_1, a_2, c_2, c_1, c)
\end{align*}
$$

(44)

The integral(43) may written as a product of factors, via the Möbius transformation,

$$
z = \frac{x - y}{r_{i_1} - y}
$$

(45)
we have,

\[ u_{k}^{x,y} = B(A_{k,0} + A_{k,1}y) \times \]
\[ \int_{0}^{z} (1-z)^{-(k_1)}(1-zx_{i_1})^{-k_2}(1-zx_{i_2})^{-k_3}(1-zx_{i_3})^{-k_4}(1-zx_{i_4})^{-k_5}(1-zx_{i_5})^{-k_6} dz \]
\[ + B(r_{i_1} - y)A_{k,1} \times \]
\[ \int_{0}^{z} z(1-z)^{-\frac{1}{2}}(1-zx_{i_1})^{-\frac{1}{2}}(1-zx_{i_2})^{-\frac{1}{2}}(1-zx_{i_3})^{-\frac{1}{2}}(1-zx_{i_4})^{-\frac{1}{2}}(1-zx_{i_5})^{-\frac{1}{2}} dz \]

(46)

where

\[ B \equiv (y, r_{i_1}, r_{i_2}, r_{i_3}, r_{i_4}, r_{i_5}) = \frac{1}{2\pi} \sqrt{\frac{r_{i_1} - y}{(r_{i_1} - y)(r_{i_2} - y)(r_{i_4} - y)(r_{i_5} - y)}} \] (47)

and

\[ x_{i_1} = \frac{r_{i_1} - y}{r_{i_2} - y}; x_{i_2} = \frac{r_{i_3} - y}{r_{i_3} - y}; x_{i_3} = \frac{r_{i_3} - y}{r_{i_4} - y}; x_{i_4} = \frac{r_{i_1} - y}{r_{i_4} - y} \] (48)

The two integrals (46) are of Extonian form, so that the integral (43) is given by,

\[ u_{k}^{x,y} = \text{phef}(A_{k,0}, A_{k,1}; x, y) + \text{shef}(A_{k,1}; x, y) \] (49)

where “phef” stands for “primary hyper-elliptic function” and “shef” stands for “secondary hyper-elliptic function”. They are defined as,

\[ \text{phef}(A_{k,0}, A_{k,1}; x, y) \equiv B(A_{k,0} + A_{k,1}y)zF_{D}(5)(1, 1, \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; 2; x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}, z) \] (50)

\[ \text{shef}(A_{k,1}; x, y) \equiv \frac{1}{2}BA_{k,1}(r_{i_1} - y)z^{2}F_{D}(5)(2, 1, \frac{1}{2}; \frac{1}{2}; \frac{1}{2}; 2; 3; x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}, z) \] (51)

Both (50) and (51) are defined via the twelve slotted, five variable, Lauricella D multiple hypergeometric function, \( F_{D}(2g+1) \), with \( 2g+1 = 5 \). Ref. 10. These functions converge when the moduli of the five variable parameters \( x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5} \) and \( z \) have their absolute values less than unity. The convergence takes place in a unit sphere in five dimensional hyperspace. In this case (50) and (51) may be expanded into their Lauricella series, i.e.

\[ \text{phef}(A_{k,0}, A_{k,1}; x, y) = B(A_{k,0} + A_{k,1}y) \times \]
\[ \sum_{(m) = 0}^{\infty} \frac{(1, m_1 + m_2 + m_3 + m_4 + m_5)(\frac{1}{2}, m_1)(\frac{1}{2}, m_2)(\frac{1}{2}, m_3)(\frac{1}{2}, m_4)(\frac{1}{2}, m_5)}{(2, m_1 + m_2 + m_3 + m_4 + m_5)m_1!m_2!m_3!m_4!m_5!} \times x_{i_2}^{m_1}x_{i_3}^{m_2}x_{i_4}^{m_3}x_{i_5}^{m_4}z^{m_5+1} \] (52)

\[ \text{shef}(A_{k,1}; x, y) = \frac{1}{2}B A_{k,1}(r_{i_1} - y) \times \]
\[ \sum_{(m) = 0}^{\infty} \frac{(2, m_1 + m_2 + m_3 + m_4 + m_5)(\frac{1}{2}, m_1)(\frac{1}{2}, m_2)(\frac{1}{2}, m_3)(\frac{1}{2}, m_4)(\frac{1}{2}, m_5)}{(3, m_1 + m_2 + m_3 + m_4 + m_5)m_1!m_2!m_3!m_4!m_5!} \times x_{i_2}^{m_1}x_{i_3}^{m_2}x_{i_4}^{m_3}x_{i_5}^{m_4}z^{m_5+2} \] (53)
with
\[ |x_{i2}| < 1; |x_{i3}| < 1; |x_{i4}| < 1; |x_{i5}| < 1; |z| < 1 \tag{54} \]

The quadrupoly periodic double theta function is given from (27) to (29),
\[
\Theta(v_1, v_2; \frac{1}{2} m_1, \frac{1}{2} m_2, \frac{1}{2} m'_1, \frac{1}{2} m'_2) = \sum_{n_1 = -\infty}^{\infty} \sum_{n_2 = -\infty}^{\infty} e^{\pi i (v_1(2n_1 + m'_1) + v_2(2n_2 + m'_2))} \\
\times q^{(2n_2 + m'_2)^2} q'^{(2n_1 + m'_1)^2} \rho(2n_1 + m'_1)(2n_2 + m'_2)(-1)(m_1 n_1 + m_2 n_2)(m_1 m'_1 - m_2 m'_2) \tag{55} 
\]
where
\[ q = e^{\frac{i}{4} \pi \tau_1}, q' = e^{\frac{i}{4} \pi \tau_2}, r = e^{\frac{i}{4} \pi \tau_1} \tag{56} \]
\[ m_1 = \rho, m_2 = \rho', \lambda, m'_1 = \sigma, m'_2 = \sigma' = \mu \tag{57} \]
so that \( a_r \) in Forsyth's paper is equal to
\[ a_r = (i)^{\sigma' \rho' + \rho \sigma} \rho(2n + \sigma')(2m + \sigma) \tag{58} \]
where \( n = n_1 \) and \( m = n_2 \) are the summation indices. Forsyth gives a review of Rosenhain's theory,\(^{22-23}\) of the fifteen ratios of the quadrupoly periodic theta functions. From the list given, the first ratio is selected and is equal to,
\[
\frac{\Theta^2(v_1, v_2; \frac{1}{2}, \frac{1}{2}, 0, 1)}{\Theta^2(v_1, v_2; \frac{1}{2}, \frac{1}{2}, 0, 0)} = \sqrt{\frac{a_1 - a_2}{(a_1 - c_1)(a_1 - c_2)(a_1 - c)}} (x_1 - c) \tag{59} 
\]
where (31)(38)(55)and (59) have been used. The double theta function used in (59) is,
\[
\theta_{13} = \sum_{n_1 = -\infty}^{+\infty} \sum_{n_2 = -\infty}^{+\infty} e^{\pi i (v_1(2n_1 + v_2(2n_2 + 1))} \\
\times q^{(2n_2 + 1)^2} q'^{(2n_1)^2} \rho(2n_1)(2n_2 + 1)(-1)(n_1 + n_2)(-1) \tag{60} 
\]
where Forsyth’s notation is adopted,
\[ \theta_{13} = \Theta(v_1, v_2; \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}) \tag{61} \]
and in the denominator of (59), is
\[
\theta_{12} = \sum_{n_1 = -\infty}^{+\infty} \sum_{n_2 = -\infty}^{+\infty} e^{\pi i (v_1(2n_1 + v_2(2n_2))} q^{(2n_2)^2} q'^{(2n_1)^2} \rho(2n_1)(2n_2)(-1)^{n_1 + n_2} \tag{62} 
\]
with,
\[ \theta_{12} = \Theta(v_1, v_2; \frac{1}{2}, \frac{1}{2}, 0, 0). \tag{63} \]
This completes the inversion of the genus two hyperelliptic integral. The analytic continuation formulae are given in Exton. If one wants the solution with all four
coordinates \( x_1, x_2, u_1 \) and \( u_2 \) in (32) and (33) one does not choose \( x_2 = a_2 \). Proceed as above adding on the appropriate multiple hypergeometric functions and selecting another theta function. This was not done here to keep the solution as concise as possible while keeping its closed form (although all four coordinates may play a role in the geometric interpretation of the solution when applied to the problem at hand). Now in order to have a closed form solution we must find the roots of the quintic, \( r_1, r_2, r_3, r_4 \) and \( r_5 \) in a closed form.

V. The Complete Solution to the Quintic

A. The Tschirnhausian Transformation to Bring’s Normal Form

The initial Tschirnhausian transformation used is a generalization of Bring’s\(^{24-25}\), but a simplification of Cayley’s\(^{26}\), with a quartic substitution,

\[
\text{Eq1} = x^4 + dx^3 + cx^2 + bx + a + y
\]

(64)
to the general quintic equation,

\[
\text{Eq2} = x^5 + mx^4 + nx^3 + px^2 + qx + r.
\]

(65)

Then by the process of elimination between (64) and (65), the following 25 equations are obtained,

\[
M_{15} = 1, M_{14} = d, M_{13} = c, M_{12} = b, M_{11} = a + y, M_{25} = m - d
\]

\[
M_{24} = n - c, M_{23} = p - b, M_{22} = -y + q - a, M_{21} = r, M_{35} = n + dm - m^2 - c
\]

\[
M_{34} = p - b - mn + dn, M_{33} = q - a - mp - y + dp, M_{32} = r - mq + dq, M_{31} = dr - mr
\]

\[
M_{45} = -cm - m^3 + b - p + dm^2 + 2mn - dn, M_{44} = a + dm + y - dp + m^2 n - q + n^2 + mp - cn
\]

\[
M_{43} = np - r + dmp - dq + mp - m^2 p - cp, M_{42} = np - r + dr - m^2 q - cq + dmq + nq,
\]

\[
M_{41} = -m^2 r - cr + nr + dm r
\]

\[
M_{55} = bm - 2mp + q - y + cn - 2dmn - a + 3m^2 n - cm^2 + dm^3 - n^2 - m^4 + dp
\]

\[
M_{54} = -dmp + dm^2 n + cp + dq + 2mn^2 - cmn - m^3 n - 2np - mq + m^2 p + m^2 q - m^3 p + dr
\]

\[
M_{53} = cq + 2mnp - dnp - p^2 + bp - nq + dm^2 p - mr - dmq - cmp + m^2 q - m^3 p + r
\]

\[
M_{52} = bq - cmq - nr - dm r - dmq + cq + m^2 r - m^3 q + 2mnq - pq + dm^2 q - m^3 p + dr
\]

\[
M_{51} = br - m^3 r - dm r + dm^2 r + 2mnr - pr - cmr
\]

(66)
These equations are then substituted into the equation,

$$\det(M) = \begin{vmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{31} & M_{32} & M_{33} & M_{34} & M_{35} \\ M_{41} & M_{42} & M_{43} & M_{44} & M_{45} \\ M_{51} & M_{52} & M_{53} & M_{54} & M_{55} \end{vmatrix} = 0$$

which generates a polynomial which will be reduced to Bring’s equation (80).

Let this equation be,

$$y^5 + p_4 y^4 + p_3 y^3 + p_2 y^2 + Ay + B = 0. \quad (67)$$

Now take coefficients of the fourth, third, and second order terms in (67), respectively, and set them to zero. Solving,

$$p_4 = 0 \quad (68)$$

for a, one has

$$a = \frac{1}{5} dm^3 + \frac{1}{5} bm - \frac{3}{5} dmn - \frac{2}{5} m^2 + \frac{4}{5} q - \frac{1}{5} cm^2 + \frac{2}{5} c n - \frac{4}{5} mp + \frac{4}{5} m^2 n + \frac{3}{5} dp - \frac{1}{5} m^4 \quad (69)$$

Substituting a and Bring’s substitution for b and c, i.e.

$$b = \alpha d + \xi \quad (70)$$

$$c = d + \eta \quad (71)$$

into (67), which becomes

$$y^5 + Ay + B = 0 \quad (72)$$

with the third and second order coefficients in (67), $p_3$ and $p_2$ set to zero. $p_3$ is a quadratic in d, so that in may be written as,

$$p_3 = p_{32} d^2 + p_{31} d + p_{30} = 0. \quad (73)$$

Setting the coefficients, $p_{32}$, $p_{31}$ and $p_{30}$ each to zero, gives

$$p_{32} = \left( -\frac{2}{5} m^2 + n \right) \alpha^2 + \left( -\frac{17}{5} m^2 n - \frac{17}{5} mp + \frac{4}{5} n^3 - 2n^2 - \frac{13}{5} nm - \frac{4}{5} n^4 \\ + 3p + 4q \right) \alpha + \frac{22}{5} m^2 p + \frac{21}{5} mpn - \frac{19}{5} pn + \frac{12}{5} nm^4 + 5r - \frac{2}{5} m^6 - \frac{18}{5} n^2 m^2 \\ + 2q + \frac{19}{5} n^2 m - 4m^3 n + \frac{8}{5} m^3 p + 3qm^2 - 3mr - \frac{3}{5} p^2 - 3nq - 5qn \\ - 2mp - \frac{2}{5} m^4 - \frac{12}{5} m^3 p + n^3 + \frac{4}{5} m^5 - \frac{3}{5} n^2 \\ = 0. \quad (74)$$
Since $p_{32}$ is quadratic in $\alpha$ we solve it for $\alpha$, and obtain,

$$
\alpha = \frac{1}{2}(-13mn - 10n^2 + 4m^3 + 20q + 17m^2n - 4m^4 + 15p - 17mp \\
\quad + (-40qm^4 + 80qm^2 + 40m^3p + 60np^2 - 15n^2m^4 - 100n^2q - 80mpn^2 + 200m^2r + 260m^2nq \\
\quad - 225p^2 - 120m^3r + 40m^5p + 265m^2p^2 - 40qm^3 - 80m^4p - 20qmn \\
\quad + 360m^2pn + 30n^3m^3 + 600pq - 510mp^2 - 120n^3m - 680mpq \\
\quad + 300mnp - 500nr - 80pn^2 - 170m^3pn + 60n^3))^{1/2}/(2m^2 - 5n). \\
(75)
$$

So $\alpha$ is a number calculated directly from the coefficients of the quintic. Now the coefficient multiplying the linear term in $d$, $p_{31}$, is also linear in both $\eta$ and $\xi$. Solving it for $\eta$ and substituting this into the zeroth term in $d$, $p_{30}$, gives a quadratic equation in $\xi$, i.e. let

$$
p_{30} = \xi_2\xi^2 + \xi_1\xi + \xi_0 = 0 \\
(76)
$$

and solve, giving,

$$
\xi = \frac{1}{2}((-\xi_1 + (\xi_1^2 - 4\xi_2\xi_0)^{1/2})/\xi_2. \\
(77)
$$

Where the equations for $\xi_0$, $\xi_1$ and $\xi_2$, depend only on the coefficients of the quintic, and are given in the pre-print$^{27}$. $p_2$ is cubic in $d$, i.e.

$$
p_2 = d_3d_1^3 + d_2d_1^2 + d_1d_0 \\
(78)
$$

solving using Cardano’s rule on Maple, gives

$$
d = \frac{1}{6}(36d_1d_2d_3 - 108d_0d_2^2 - 8d_2^3 \\
\quad + 12\sqrt[3]{(4d_1^3d_3 - d_1^2d_2^2 - 18d_1d_2d_3d_0} \\
\quad + 27d_0^2d_3 - 4d_0d_1d_2d_3)^{1/2}/d_3 \\
\quad - \frac{2}{3}(3d_2d_3 - d_2^2)/(d_3(36d_1d_2d_3 - 108d_0d_2^2 - 8d_2^3 \\
\quad + 12\sqrt[3]{(4d_1^3d_3 - d_1^2d_2^2 - 18d_1d_2d_3d_0} \\
\quad + 27d_0^2d_3 - 4d_0d_1d_2d_3)^{1/2}) \\
\quad - \frac{1}{3}d_2/d_3. \\
(79)
$$

Where $d_0$, $d_1$, $d_2$, and $d_3$, are also given in the pre-print$^{27}$. Now (67) has been transformed to (72), where $A$ and $B$ depend only on the coefficients of the quintic(65). So that equation(72), with a linear transformation, becomes Bring’s normal form,

$$
z^5 - z - s = 0 \\
(80)
$$
where,

\[ y = (-A)^{\frac{1}{4}} z \]  
\[ s = -B/(-A)^{\frac{1}{4}} \]

We have used the fact that \( p_4 \) is linear in \( a \) to get \( p_4 = 0 \). Then \( b, c \) and \( d \) were considered a point in space, on the curve of intersection of a quadratic surface \( p_3 \), and a cubic surface, \( p_2 \). Giving \( p_3 = p_2 = 0 \), as required. 

**B. The Solution to Bring’s Normal Form**

Bring’s normal form is solvable. Any polynomial that can be transformed to the form,

\[ z^n - az^m - b = 0 \]  

(83)

can be solved with the hypergeometric equation. The solution is given by considering \( z = z(s) \) and differentiating Bring’s equation (80) w.r.t. \( s \) four times, then after making the substitution,

\[ t = \frac{5^5}{4^4} s^4 \]

(84)

one obtains the fourth order Fuchian generalized hypergeometric differential equation,

\[ t^3(1 - t) \frac{d^4 z}{dt^4} + t^2 \left( \frac{9}{2} - 7t \right) \frac{d^3 z}{dt^3} + t \left( \frac{51}{16} - \frac{411}{40} t \right) \frac{d^2 z}{dt^2} + \left( \frac{3}{32} - \frac{183}{80} t \right) \frac{dz}{dt} + \frac{231}{160000} z = 0 \]

(85)

whose solution is

\[ z(s) = \, _4F_3 \left( \left[ -\frac{1}{20}, \frac{3}{20}, \frac{7}{20}, \frac{11}{20} \right], \left[ \frac{3}{4}, \frac{1}{2}, \frac{1}{2} \right], \frac{5^5}{4^4} s^4 \right) \]

(86)

**C. Reverse Tschirnhausian Transformation**

When undoing the Tschirnhausian transformation one must be careful to generate and keep all possible roots. All roots are calculated below. Incorrect roots are eliminated by direct substitution. Firstly, calculate \( y \), with (81). We now undo the Tschirnhausian transformation by substituting \( d, c, b, a \) and \( y \) into the quartic substitution, (64). The resulting equation is solved using the quadratic method. The quadratic method introduces all the possible roots for the quartic. Let the general quartic,

\[ x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0 \]

(87)
have coefficients,
\[
\begin{align*}
a_0 &= a + y \\
a_1 &= b \\
a_2 &= c \\
a_3 &= d.
\end{align*}
\]

There are twelve possible roots to the quartic, let them be contained in the array, \(x_{mn}\) with \(m \in (1, 2, 3)\) and \(n \in (1, 2, 3, 4)\). Only four of these roots satisfy the quartic for a given set of coefficients \(a_0, a_1, a_2\) and \(a_3\). Using nested loops to enter the candidate solutions into both the general quintic (65) and the quartic (87), let the root that satisfies them both be \(r_1\). This root is then factored out of the quintic (65), giving the following equations,
\[
\begin{align*}
a_0 &= q + r_1 p + r_1^2 m + mr_1^3 + r_1^4 \\
a_1 &= p + r_1 n + r_1^2 m + r_1^3 \\
a_2 &= n + mr_1 + r_1^2 \\
a_3 &= m + r_1
\end{align*}
\]

These equations are then substituted into the general quartic, which gives another twelve roots, \(x_{mn}\). This time four of these roots satisfy both the quartic and quintic, let these roots be \(r_2, r_3, r_4\) and \(r_5\). They were found the same way as \(r_1\), using nested loops. The final array has five non-zero elements, the five complex roots of the general quintic equation (65).

\[\text{D. Conclusion}\]

The five formulae for the roots of the general quintic equation are obtained, without root ambiguity, in agreement with Gauss’ ‘Fundamental Theorem of Algebra’ when the polynomial is of degree five.

\[\text{VI. Determination of the Genus Two Hyperelliptic Periods for Arbitrary Parameters}\]

Now from the hyperelliptic function (49), the periods can be calculated exactly for arbitrary branch places calculated in the previous section, this gives,
\[
\omega'_{1,1} = u_k^{c_1, a_1} = phef_k(c_1, a_1) + shef_k(c_1, a_1)
\]
where
\[
\begin{align*}
\text{phef}_k(c_1, a_1) &= \frac{(A_{k,0} + A_{k,1} a_1)}{i \sqrt{(c - a_1)(a_2 - a_1)(c_2 - a_1)}} \\
\times F_D^{(5)}(1, \frac{1}{2}, 1, 1, \frac{1}{2}, 2; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 1, 1, 1)
\end{align*}
\]
\[ shf_k(c_1, a_1) = \frac{A_k,1(c_1 - a_1)}{2\sqrt{(c - a_1)(a_2 - a_1)(c_2 - a_1)}} \times F_D^{(5)}(2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 3; \frac{c_1 - a_1}{c - a_1}, \frac{c_1 - a_1}{a_2 - a_1}, \frac{c_1 - a_1}{c_2 - a_1}, 1, 1) \]  

which becomes,

\[ \omega'_{k,1} = \frac{\pi (A_k,0 + A_k,1 a_1)}{2 \sqrt{(c - a_1)(a_2 - a_1)(c_2 - a_1)}} F_D^{(3)}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1; 1, x_1, x_2, x_3) \]

\[ + \frac{A_k,1(c_1 - a_1)\pi}{4 \sqrt{(c - a_1)(a_2 - a_1)(c_2 - a_1)}} F_D^{(3)}(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}; 2; x_1, x_2, x_3) \]

with,

\[ x_1 = \frac{c_1 - a_1}{a_2 - a_1}; x_2 = \frac{c_1 - a_1}{c_2 - a_1}; x_3 = \frac{c_1 - a_1}{c - a_1} \]

and similarly,

\[ \omega_{k,2} = -u_k^{c, a_2} = -shf_k(c, a_2) - shf_k(c, a_2) \]

\[ \omega_{k,2} = -\frac{(A_k,0 + A_k,1 a_2)\pi}{2 \sqrt{(a_1 - a_2)(c_2 - a_2)(c_1 - a_2)}} F_D^{(3)}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1; 1, x_1, x_2, x_3) \]

\[ - \frac{\pi A_k,1(c - a_2)}{4 \sqrt{(a_1 - a_2)(c_2 - a_2)(c_1 - a_2)}} F_D^{(3)}(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}; 2; x_1, x_2, x_3) \]

\[ x_1 = \frac{c - a_2}{a_1 - a_2}; x_2 = \frac{c - a_2}{c_2 - a_2}; x_3 = \frac{c - a_2}{c_1 - a_2} \]

and,

\[ \omega'_{k,2} = u_k^{c_2, a_2} = shf_k(c_2, a_2) + shf_k(c_2, a_2) \]

\[ \omega'_{k,2} = \frac{\pi (A_k,0 + A_k,1 a_1)}{2 \sqrt{(a_1 - a_2)(c_2 - a_2)(c_1 - a_2)}} F_D^{(3)}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1; 1, x_1, x_2, x_3) \]

\[ + \frac{\pi A_k,1(c_2 - a_2)}{4 \sqrt{(a_1 - a_2)(c_2 - a_2)(c_1 - a_2)}} F_D^{(3)}(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}; 2; x_1, x_2, x_3) \]

\[ x_1 = \frac{c_2 - a_2}{a_1 - a_2}; x_2 = \frac{c_2 - a_2}{c_2 - a_2}; x_3 = \frac{c_2 - a_2}{c_1 - a_2} \]

and finally,

\[ \omega_{k,1} = u_k^{a,c_1} = shf_k(a, c_1) + shf_k(a, c_1) \]

\[ \omega_{k,1} = u_k^{c_2,a_1} + \omega_{k,2} \]

where,

\[ u_k^{c_2,a_1} = \frac{(A_k,0 + A_k,1 a_1)\pi}{2 \sqrt{(a_2 - a_1)(c_1 - a_1)(c - a_1)}} F_D^{(3)}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1; 1, x_1, x_2, x_3) \]

\[ + \frac{A_k,1(c_2 - a_1)\pi}{4 \sqrt{(a_2 - a_1)(c_1 - a_1)(c - a_1)}} F_D^{(3)}(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}; 2; x_1, x_2, x_3) \]
\[ x_1 = \frac{c_2 - a_1}{c - a_1}; x_2 = \frac{c_2 - a_1}{c_1 - a_1}; x_3 = \frac{c_2 - a_1}{a_2 - a_1} \]  

(104)

Riemann’s moduli are given by,
\[
\tau_{1,1} = \frac{\omega_{2,2}\omega'_{1,1} - \omega_{1,2}\omega'_{2,1}}{\Delta}
\]
\[
\tau_{2,2} = \frac{-\omega_{2,1}\omega'_{1,2} + \omega_{1,1}\omega'_{2,2}}{\Delta}
\]

and by Riemann’s condition (26),
\[
\tau_{1,2} = \frac{\omega_{2,1}\omega'_{1,1} - \omega_{1,1}\omega'_{2,1}}{\Delta}
\]
\[
= \frac{-\omega_{2,2}\omega'_{1,2} + \omega_{1,2}\omega'_{2,2}}{\Delta}
\]
\[
\equiv \tau_{2,1}
\]

(105)

(106)

VII. The Solution to the Hyperelliptic $\phi$ Integral

The solution to the hyperelliptic $\phi$ integral (7) is with $x_1 = r_g/r$,
\[
r = \frac{r_g\theta_{12}^2\sqrt{a_1 - a_2}}{c\sqrt{a_1 - a_2}\theta_{12}^2 - \sqrt{(a_1 - c_1)(a_1 - c_2)(a_1 - c_2)}\theta_{13}^2}
\]

(107)

where $\theta_{13}$ and $\theta_{12}$ are given in (60) and (62). The hyperelliptic $\phi$ integral (10) may be written as
\[
-\frac{\phi}{2} = \int_{a_0}^{x_1} \frac{A_{1,1} x dx}{y} = p\text{hef}(A_{1,1}; x_1, a_0) + s\text{hef}(A_{1,1}; x_1, a_0)
\]

(108)

\[ p\text{hef}(A_{1,1}; x_1, a_0) = B(a_0)A_{1,1} \times \]
\[
\sum_{(m)=0}^{\infty} \frac{(1, m_1 + m_2 + m_3 + m_4 + m_5)(\frac{1}{2}, m_1)(\frac{1}{2}, m_2)(\frac{1}{2}, m_3)(\frac{1}{2}, m_4)(\frac{1}{2}, m_5)}{(2, m_1 + m_2 + m_3 + m_4 + m_5)(m_1 m_2 m_3 m_4 m_5)\times x_2^{m_1} x_3^{m_2} x_4^{m_3} x_5^{m_4} x_6^{m_5+1}}
\]

(109)

\[ s\text{hef}(A_{1,1}; x_1, a_0) = \frac{1}{2} B(r_1 - a_0)A_{1,1} \times \]
\[
\sum_{(m)=0}^{\infty} \frac{(2, m_1 + m_2 + m_3 + m_4 + m_5)(\frac{1}{2}, m_1)(\frac{1}{2}, m_2)(\frac{1}{2}, m_3)(\frac{1}{2}, m_4)(\frac{1}{2}, m_5)}{(3, m_1 + m_2 + m_3 + m_4 + m_5)(m_1 m_2 m_3 m_4 m_5)\times x_2^{m_1} x_3^{m_2} x_4^{m_3} x_5^{m_4} x_6^{m_5+2}}
\]

(110)

and
\[
\frac{\phi'}{2} = \int_{a_0}^{x_1} \frac{(A_{2,0} + A_{2,1}) x dx}{y} = p\text{hef}(A_{2,0}, A_{2,1}; x_1, a_0) + s\text{hef}(A_{2,1}; x_1, a_0)
\]

(111)
\[ phef(A_{2,0}, A_{2,1}; x_1, a_0) = B \times (a_0 A_{2,1} + A_{2,0}) \]
\[ \sum_{(m)=0}^{+\infty} (1, m_1 + m_2 + m_3 + m_4 + m_5) (1, m_1)(1, m_2)(1, m_3)(1, m_4)(1, m_5) \]
\[ \times x_2^{m_1} x_3^{m_2} x_4^{m_3} x_5^{m_4} z^{m_5 + 1} \]
\[ \sum_{(m)=0}^{+\infty} (2, m_1 + m_2 + m_3 + m_4 + m_5) (1, m_1)(1, m_2)(1, m_3)(1, m_4)(1, m_5) \]
\[ \times x_2^{m_1} x_3^{m_2} x_4^{m_3} x_5^{m_4} z^{m_5 + 2} \]
\[ (112) \]
\[ (113) \]

Abel’s coefficients \( A_{2,0} \) and \( A_{2,1} \) depend on the cosmological constant and are determined by experimentation and observation. So with
\[ x_2 = \frac{a_1 - a_0}{a_2 - a_0}; x_3 = \frac{a_1 - a_0}{c - a_0}; x_4 = \frac{a_1 - a_0}{c_1 - a_0}; x_5 = \frac{a_1 - a_0}{c_2 - a_0} \]
\[ (114) \]
where the branch places are related to the apsidal points by,
\[ a_1 = \frac{r_g}{r_1}; a_2 = \frac{r_g}{r_2}; c = \frac{r_g}{r_3}; c_1 = \frac{r_g}{r_4}; c_2 = \frac{r_g}{r_5} \]
\[ (115) \]
and the variable place is given by,
\[ z = \frac{x_1 - a_0}{a_1 - a_0} \]
\[ (116) \]
with
\[ |x_2| < 1; |x_3| < 1; |x_4| < 1; |x_5| < 1; |z| < 1 \]
\[ (117) \]
so that the variables in the argument of the theta functions are,
\[ v_1 = \frac{-\omega_{1,2} \phi' + \omega_{2,3} \phi}{\Delta} \]
\[ v_2 = \frac{\omega_{1,1} \phi' - \omega_{2,1} \phi}{\Delta} \]
\[ (118) \]
where the periods \( \omega_{i,j} \) are given in section VI. With this choice the equations of Forsyth and Baker are identical to these, in the case of \( q = 2 \). The roots of the quintic \( a_1, a_2, c_1, c_2 \) and \( c \) are given by setting the coefficients in(41) equal to
\[ \lambda_0 = 4r = \frac{4Am^2c^2r_g^4}{3L^2} \]
\[ \lambda_1 = 4q = 0 \]
\[ \lambda_2 = 4p = \frac{4}{3} Ar_g^2 - \frac{4m^2c^2r_g^2}{L^2} + \frac{4r_g^2E^2}{L^2c^2} \]
\[ \lambda_3 = 4n = \frac{4m^2c^2r_g^2}{L^2} \]
\[ \lambda_4 = 4m = -4 \]
\[ (119) \]
then substituting into the solution to the quintic in section V.

16
A. Discussion

The geodesics of a particle in SdS space are determined by three coordinates, the radial coordinate, \( r \), and two angular coordinates \( \phi \) and \( \phi' \). \( \phi \) is the actual physical coordinate, while \( \phi' \) is an angular coordinate that comes from Abel’s theorem. \( r \) is a function of both \( \phi \) and \( \phi' \), by the quadrupoly periodic theta functions of two variables. Now \( \phi \) and \( \phi' \) depend on \( r \), i.e. \( \phi = \phi(r) \) and \( \phi' = \phi'(r) \) and they can be expressed as a linear combination of phef and shef. Thus \( \phi = \phi(r) = -2 \text{phef}(r) - 2 \text{shef}(r) \) is the closed form solution to the calculation of the geodesics in SdS space.

VIII. The Reduction of the Hyperelliptic Functions to Elliptic Functions when \( \Lambda = 0 \)

When the cosmological constant is set to zero and the particles are subject to gravitation alone, two roots of the quintic(42) go to zero. Let these two roots be \( c_2 \) and \( a_2 \). Then \( \omega'_{1,2} = \omega'_{2,2} = 0 \). Then the following equations are obtained for the Riemann moduli,

\[
\tau_{1,1} = (\omega_{2,2}\omega'_{1,1} - \omega_{1,2}\omega'_{2,1})/\Delta \neq 0 \tag{120}
\]
\[
\tau_{2,2} = (\omega_{2,1}\omega'_{1,2} + \omega_{1,1}\omega'_{2,2})/\Delta = 0 \tag{121}
\]
\[
\tau_{1,2} = (\omega_{2,1}\omega'_{1,1} - \omega_{1,1}\omega'_{2,1})/\Delta = (\omega_{2,2}\omega'_{1,2} + \omega_{1,2}\omega'_{2,2})/\Delta = 0 \tag{122}
\]

Let Abel’s coefficients be functions of the cosmological constant, i.e.

\[
A_{1,1} \equiv A_{1,1}(\Lambda), A_{2,0} \equiv A_{2,0}(\Lambda), A_{2,1} \equiv A_{2,1}(\Lambda) \tag{123}
\]

Taking the limit as \( \Lambda \to 0 \) with (108), (111),(118), one has

\[
v_1 = \lim_{\Lambda \to 0} -\frac{\omega_{1,2}\phi' + \omega_{2,2}\phi}{\Delta} = 0 \tag{124}
\]
gives,

\[
A_{2,0}(0) = 0 \tag{125}
\]
\[
A_{2,1}(0) = \frac{-\omega_{2,2}}{\omega_{1,2}} \tag{126}
\]

and

\[
v_2 = \lim_{\Lambda \to 0} \frac{\omega_{1,1}\phi' - \omega_{2,1}\phi}{\Delta} = \lim_{\Lambda \to 0} \frac{\phi}{\omega_{1,2}} \tag{127}
\]
or,

\[
v_2 = \lim_{\Lambda \to 0} \frac{2}{\omega_{1,2}} (\text{phef}(A_{1,1}(\Lambda); x, y) + \text{shef}(A_{1,1}(\Lambda); x, y)) \tag{128}
\]
Letting $\lim_{\Lambda \to 0} A_{1,1}(\Lambda) = A_{1,1}(0) = \omega_{1,2}$ and using the identity,
\[
zF_D^{(5)}\left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 2; x_2, x, x_3, z\right) - \frac{1}{2} x^2 zF_D^{(5)}\left(2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{1}{2}; 3; x_2, x, x_3, z\right) = zF_D^{(3)}\left(1, \frac{1}{2}, \frac{1}{2}; 2; x_2, x_3, z\right)
\]
(129)
to obtain the triple hypergeometric function,
\[
v_2 = \frac{1}{2} \frac{(x-y)}{(a_1-y)(c_1-y)(c-y)} F_D^{(3)}\left(1, \frac{1}{2}, \frac{1}{2}; 2; \frac{a_1-y}{c_1-y}, \frac{a_1-y}{c-y}, \frac{x-y}{a_1-y}\right)
\]
(130)
that reduces to the incomplete elliptical integral of the first kind\textsuperscript{35–36}, when $y \to a_1$. So that $\phi$ becomes,
\[
v_2 = \phi = \frac{-2}{\sqrt{c-a_1}} F(\sin^{-1} \sqrt{\frac{x-a_1}{c_1-a_1}}, \sqrt{\frac{c_1-a_1}{c-a_1}}) \left(131\right)
\]
With these definitions,
\[
\lim_{\Lambda \to 0} \left(\theta_{1,1}^2, \theta_{2,1}^2\right) = \frac{\theta_{1,1}^2}{\theta_{2,1}^2} = \frac{1}{k} sn^2(v)
\]
(132)
where $sn(v)$ is Jacobi's elliptic function\textsuperscript{37}. Now with the choice
\[
c = \frac{r_2}{r_1}, c_1 = \frac{r_2}{r_2}, a_1 = \frac{r_2}{r_3}, k = \sqrt{\frac{r_2 r_1 - r_3 r_1}{r_2 r_1 - r_3 r_2}}
\]
(133)
which gives the Schwarzschild black hole solution without the cosmological constant, as a genus one elliptic function, which is the same as Ref. 7, it is
\[
\left(\frac{1}{r} - \frac{1}{r_1}\right) = \left(\frac{1}{r_3} - \frac{1}{r_1}\right) sn^2(v_1)
\]
(134)
Lastly, the moduli become 1 and
\[
\tau_{1,1} = \frac{\omega'_{2,1}}{\omega_{2,1}} = \lim_{\Lambda \to 0} \frac{\text{phec}_2(c_1, a_1) + \text{shef}_2(c_1, a_1))}{\text{lim}_{\Lambda \to 0}(w_{2,1}^{a_1} + \omega_{2,1})}
\]
(135)
Using the identities,
\[
F_D^{(3)}\left(1, \frac{1}{2}, \frac{1}{2}; 1; x, x, z\right) - \frac{1}{2} x F_D^{(3)}\left(3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 2; x, x, z\right) = 2 F_1\left(\frac{1}{2}, \frac{1}{2}; 1; z\right)
\]
(136)
\[
\pi F_D^{(3)}\left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, x, y\right) - \frac{1}{2} F_D^{(3)}\left(3, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 2; 1, x, y\right) = \frac{2}{\sqrt{x}} F(\sin^{-1} \sqrt{\frac{y}{x}}, \sqrt{\frac{y}{x}})
\]
(137)
where \( \mathbb{F}_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; z) \) is the Gauss function for the complete elliptical integral of the first kind and \( F \) is the elliptic function for the incomplete elliptical integral of the first kind. Then the periods of section VI, (90) to (104) reduce to elliptic functions, i.e.,

\[
\omega'_{k,1} = A_{k,1} \frac{1}{\sqrt{c - a_1}} F_D(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 1; \frac{c_1 - a_1}{c - a_1})
= A_{k,1} \frac{1}{\sqrt{c - a_1}} \pi \mathbb{F}_1(\frac{1}{2}, \frac{1}{2}; 1; z)
= A_{k,1} \frac{1}{\sqrt{c - a_1}} F\left(\frac{\pi}{2}, \sqrt{\frac{c_1 - a_1}{c - a_1}}\right),
\]

(138)

similarly,

\[
u^{c_2,a_1}_k \rightarrow u^{0,a_1}_k = A_{k,1} \frac{1}{\sqrt{c_1 - a_1}} F\left(\sin^{-1}\left(\frac{a_1}{a_1 - c}\right), \sqrt{\frac{a_1 - c}{a_1 - c_1}}\right)
\]

(139)

\[
\omega_{k,2} = -A_{k,1} \frac{1}{\sqrt{c - c_1}} F\left(\sin^{-1}\left(\frac{c}{c - a_1}\right), \sqrt{\frac{c - a_1}{c - c_1}}\right)
\]

(140)

\[
\omega_{k,1} = u^{0,a_1}_k + \omega_{k,2}
\]

(141)

and lastly,

\[
\omega'_{k,2} = 0
\]

(142)

It has been shown that the solution for the geodesics in SdS space in terms of multiple hypergeometric and hyperelliptic modular functions reduce to elliptic functions in the case of pure Schwarzschild space.

**IX. Conclusion**

The geodesics of particles in the combined Schwarzschild black hole and a non-zero cosmological constant, are given by ratios of quadrupoly periodic theta functions of two variables. The inverse functions are obtained as multiple hypergeometric functions of five variables, thus solving the Inversion Problem for hyperelliptic integrals of the first kind, i.e. genus two. It is a closed form solution since the branch places are found by solving the quintic equation for it’s five roots. The five formulae for the five roots of the quintic were obtained, in agreement with Gauss’ Fundamental Theorem of Algebra when the polynomial is of degree five. The closed form solution for the geodesics in SdS space was found to be \( \phi = \phi(r) \), which is a linear combination of \( phef(r) \) and \( shef(r) \) which are convergent twelve slotted five variable hypergeometric functions.

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