MINIMAL NON-NILPOTENT GROUPS WHICH ARE SUPERSOLVABLE

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Abstract. The structure of a group which is not nilpotent but all of whose proper subgroups are nilpotent has interested the researches of several authors both in the finite case and in the infinite case. The present paper generalizes some classic descriptions of M. Newman, H. Smith and J. Wiegold in the context of supersolvable groups.

1. Introduction

Let $\mathfrak{N}$ be the class of all nilpotent groups. A group $G$ is said to be a minimal non-nilpotent group, or $\mathfrak{N}$-critical group, or Schmidt group, or $MNN$-group, if it doesn’t belong to $\mathfrak{N}$ but all of whose proper subgroups belong to $\mathfrak{N}$. We will use the last terminology in the present paper. It is evident already from these 4 ways to call the same mathematical object that there is a wide literature on the topic. Many authors are still interested in studying $MNN$-groups, because they play an important role from the point of view of the general theory. The first example of finite $MNN$-group is probably the symmetric group $S_3$ of order 6. We know that $S_3$ can be written as the semidirect product of a cyclic group $C_3$ of order 3 by a cyclic group $C_2$ of order 2, which acts by inversion on $C_3$. Already for $S_3$ the condition of being an $MNN$-group determines its structure, in fact, we have a semidirect product and this allows us to have a deep knowledge of the whole group. At this point the following question becomes natural.

What is the influence of being an $MNN$-group on the group structure?

In the finite case a first answer is due to a famous contribution of O. Yu. Schmidt and more details can be found in [8]. His methods and techniques showed that the question can be seen from a different prospective, involving the theory of classes of groups and conditions which are weaker of being nilpotent. A recent contribution in this direction has been given by J.C. Beidleman and H. Heineken in [1] Theorem 2], where they generalize the description of O. Yu. Schmidt to the context of saturated formation of finite groups.

On another hand, classic descriptions of $MNN$-groups in the infinite case were given by M. Newman, H. Smith and J. Wiegold in [4, 9, 10]. Among these groups, there are Tarski groups [5] so it is a common use the imposition of suitable finiteness conditions in order to treat separately the Tarski groups.

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Now we illustrate the new idea of the present paper. Consider the following subset of the subgroup lattice \( \mathcal{L}(G) \) of \( G \)
\[
(1.1) \quad \mathcal{M}(G) = \{ H \leq G : H \notin \mathcal{N} \}.
\]
\( \mathcal{M}(G) = \{ G \} \) if and only if \( |\mathcal{M}(G)| = 1 \), that is, \( G \) is the unique non-nilpotent subgroup, that is, \( G \) is an \( MNN \)-group. It turns out that we may extend significantly the classifications in \([4, 8, 9]\), dealing with \( (1.1) \) when \( |\mathcal{M}(G)| = m \geq 1 \). For the case \( m = 2 \) we can be more precise and details are illustrated in Section 2, preparing the main results which are in Section 3. For higher values of \( m \) we have not found deep restrictions on the group structure and, to the best of our knowledge, it is an open problem.

2. The Case \( m = 2 \)

The motivation of studying \( (1.1) \) is clear once we note that \( |\mathcal{M}(G)| \) gives a measure of how many \( MNN \)-subgroups are contained in \( G \), and so, of how \( G \) is far from the usual classifications in \([4, 8, 9]\). Of course, \( |\mathcal{M}(G)| = 2 \) if and only if \( G \notin \mathcal{N} \) and we have just 1 \( MNN \)-subgroup \( K \) of \( G \). Going on, the situation is a little bit more complicated. Already the case \( |\mathcal{M}(G)| = 3 \) needs of more attention.

Lemma 2.1. \( |\mathcal{M}(G)| = 2 \) if and only if \( G \notin \mathcal{N} \) and \( G \) contains a maximal normal subgroup \( K \) which is an \( MNN \)-group.

Proof. Since \( |\mathcal{M}(G)| = 2 \), we have \( \mathcal{M}(G) = \{ G, K \} \), where \( K < G \). So \( K \) is an \( MNN \)-group. If there is a subgroup \( H \) of \( G \) such that \( K < H < G \), then \( H \in \mathcal{N} \) and so \( K \in \mathcal{N} \). This contradiction implies that \( K \) is a maximal subgroup of \( G \). Now for each \( x \in G \), \( K^x \leq G \). But \( K^x \not\in \mathcal{N} \), so \( K^x = K \). Then \( K \) is normal in \( G \). \( \square \)

Lemma 2.2. Assume \( |\mathcal{M}(G)| = 2 \) and \( K \) as in Lemma 2.1. Then \( G/K \) is of prime order and \( G' \leq K \).

Proof. Since \( G/K \) has only two subgroups, \( G/K \) is of prime order. Since \( G/K \) is abelian, \( G' \leq K \). \( \square \)

Remark 2.3. Assume \( |\mathcal{M}(G)| = 2 \). Then \( K \) in Lemma 2.1 is a characteristic subgroup of \( G \).

Proof. Let \( \alpha \in \text{Aut}(G) \). Then \( \alpha(K) \simeq K \not\in \mathcal{N} \), so \( \alpha(K) = K \). \( \square \)

In order to proceed we recall the Hall’s Criterion of nilpotence in \([6, 5.2.10]\).

Theorem 2.4 (P. Hall, see \([6]\)). Let \( N \) be a normal subgroup of a group \( G \). If \( N \in \mathcal{N} \) and \( G/N' \in \mathcal{N} \), then \( G \in \mathcal{N} \).

Remark 2.5. Assume \( |\mathcal{M}(G)| = 2 \) and \( G' < K \) with \( K \) as in Lemma 2.1. Then \( G \) is solvable with a non-trivial non-nilpotent homomorphic image.

Proof. Since \( G' < K \) and \( K \) is an \( MNN \)-group, \( G' \in \mathcal{N} \) and so \( G \) is solvable. Theorem 2.4 implies \( G/G'' \not\in \mathcal{N} \), which is the requested homomorphic image. \( \square \)

Remark 2.6. Let \( K \) be as in Remark 2.5. If \( \mathcal{M}(G) = \{ G, K \} \), then \( \mathcal{M}(G/G'') = \{ G/G'', K/G'' \} \).
Proof. Remark 2.5 shows that $G/G'' \not\in \mathfrak{N}$. Each subgroup of $G/G''$ is of the form $H/G''$, where $G'' \leq H \leq G$. Then $H/G'' \in \mathfrak{N}$, whenever $H \neq K$ and $H \neq G$. Therefore, $K/G'' \not\in \mathfrak{N}$ by Theorem 2.4. 

A group $G$ is locally graded if every nontrivial finitely generated subgroup of $G$ have a finite image. The next result recalls [10, Theorem 2].

**Theorem 2.7** (H. Smith, see [10]). Let $G$ be a locally graded group and suppose that, for some positive integer $b(G)$, every non-nilpotent subgroup of $G$ is subnormal of defect $\leq b(G)$ in $G$. Then $G$ is solvable.

Now Remark 2.5 can be reformulated in the following way.

**Proposition 2.8.** Assume $|\mathcal{M}(G)| = 2$. If $G$ is locally graded, then $G$ is solvable.

**Proof.** Let $K$ be as in Lemma 2.1. All non-nilpotent subgroups of $G$ are subnormal. Then $G$ is solvable by Theorem 2.7. 

**Lemma 2.9.** Assume $|\mathcal{M}(G)| = 2$ and $K$ as in Remark 2.5. If $M \neq K$ is a maximal normal subgroup of $G$, then $[K,M] \neq 1$.

**Proof.** Assume $[K,M] = 1$. Then $M \leq C_G(K)$. If $M = C_G(K)$, then $M \cap K = Z(K)$ and so $MK/M \simeq K/(M \cap K) \simeq K/Z(K)$ is cyclic. This gives $K$ abelian. If $G = C_G(K)$, then $K \leq Z(G)$ and again $K$ is abelian. Both cases contradict $K \not\in \mathfrak{N}$. The result follows.

**Proposition 2.10.** Assume $|\mathcal{M}(G)| = 2$ and $K$ as in Remark 2.5. If $M$ is a maximal subgroup of $G$ whose elements have coprime order with those of $K$, then $K$ is the unique maximal subgroup of $G$.

**Proof.** $K$ is periodic by the classification of M.Newman and J.Wiegold in [3]. Then $G$ is periodic and so $M$. $M \cap K = \{1\}$ from the relation $(|m|, |k|) = 1$ for each $m \in M$ and $k \in K$. Then, $[M,K] \leq M \cap K = \{1\}$. Lemma 2.9 implies that $M = K$ and the result follows.

Recall that $\pi(G)$ denotes the set of prime divisors of the order of the elements of $G$.

**Corollary 2.11.** Assume $|\mathcal{M}(G)| = 2$ and $K$ non-finitely generated. If $K$ has maximal subgroups, then $G$ is a Chernikov group of derived length at most 3 with $|\pi(G)| \leq 2$.

**Proof.** By the classification of M.Newman and J.Wiegold in [3], $K$ is a metabelian Chernikov $p$-group for some prime $p$ (see [3], p.242, lines +5 and +6). From Lemma 2.2 the result follows.

**Lemma 2.12.** Assume $|\mathcal{M}(G)| = 2$ and $K$ non-finitely generated. Then $G$ is locally nilpotent. In particular, each maximal subgroup of $G$ is normal and of prime index.

**Proof.** Every finitely generated subgroup $H$ of $G$, such that $H \neq G$ and $H \neq K$, is nilpotent. Then $G$ is locally nilpotent. The remaining part of the result follows easily.

**Corollary 2.13.** Assume $|\mathcal{M}(G)| = 2$ and $K$ non-finitely generated. Then $G$ is solvable.

**Proof.** This follows from Lemma 2.12 and Proposition 2.8.
A concrete situation is described as follows.

Example 2.14. Write \( A = C_{2^n} \) for the quasicyclic 2-group, \( B = \langle x \rangle \) and \( C = \langle y \rangle \), where \( x \) and \( y \) have order 2. Consider \( K = A \times B \), which is the well-known locally dihedral 2-group [6, p.344], and \( G = K \times C \). By construction, \( \mathcal{L}(K) = \mathcal{L}(A) = \{K, B, (H, B)\} \), where \( \{1\} \neq H < A \). Of course, \( B \in \mathcal{R} \). On the other hand, \( \langle H, B \rangle \leq Z_i(K) \) for some \( i \geq 1 \), since \( K \) is \( \omega \)-hypercentral. Then \( (H, B) \in \mathcal{R} \). We conclude that \( K \) is an \( MNN \)-group. Now, the presence of \( K \) implies that \( G \) is not an \( MNN \)-group. By construction, \( \mathcal{L}(G) = \mathcal{L}(K) = \{G, C, \langle L, C \rangle\} \), where \( \{1\} \neq L < K \). Noting that \( \langle L, C \rangle = L \times C \), we have \( L \times C \in \mathcal{R} \). Then \( \mathcal{M}(G) = \{G, K\} \). Note that \( K \) is the unique maximal subgroup of \( G \). Now, \( A \) is the unique maximal subgroup of \( K \) that is is not an \( MNN \)-group. Note also that \( A \) is the unique maximal subgroup of \( K \). We have all this is needed in order to state that \( G \) satisfies Proposition 2.8 and Corollaries 2.11, 2.13, adding a finite cyclic group to a given \( MNN \)-group. Then, choosing a suitable order for the cyclic group, we may give examples for Proposition 2.10.

3. Main Theorems

In order to proceed with the proof of the main theorem of the present section, we recall [4, Lemma 3.2] and [4, Theorem 2.12], respectively.

Lemma 3.1 (M.Newman–J.Wiegold, see [4]). Let \( G \) be a finitely generated non-nilpotent group all of whose proper subgroups are locally nilpotent and \( \gamma_\infty(G) \) be the last term of the lower central series of \( G \). If \( G/\gamma_\infty(G) \) is nontrivial, then \( G \) is finite.

Theorem 3.2 (M.Newman–J.Wiegold, see [4]). If \( G \) is a group in which every pair of proper normal subgroups generates a proper subgroup, then \( G/G' \) is a locally cyclic \( p \)-group for some prime \( p \) and \( G' = \gamma_\infty(G) \).

We should recall also some notations from [11]. Let \( n \geq 1 \), \( i \) and \( j \) be two distinct integers in \( \{1, 2, \ldots, n\} \), \( p_i, p_j \) primes, \( d_i, d_j \geq 1 \), \( \pi(d_i) \) be the set of prime divisors of \( d_i \) and \( q_i \in \pi(d_i) \). An \( F_{(p_i, d_i)} \)-group is a Frobenius group whose kernel is an elementary abelian group of order \( p_i^{m_i} \) with cyclic complement of order \( d_i \), where \( m_i \) is the exponent of \( p_i \) modulo \( q_i \). The next result quotes [11, Theorem 1].

Theorem 3.3. In a non-nilpotent finite group \( G \), all \( MNN \)-subgroups are subnormal if and only if

\[(3.1) \quad G/Z_\infty(G) = G_1 \times G_2 \times \ldots \times G_n,\]

where \( G_i \) is an \( F_{(p_i, d_i)} \)-group, and \( (d_i, d_j) = 1 \) for any \( i \neq j \) with \( i, j \in \{1, 2, \ldots, n\} \).

Our main result is the following and describes [11, Theorem 1] in a special case.

Theorem 3.4. Assume \( K \) as in Remark 2.7. If \( K \) is finitely generated, then \( G \) is a finite supersolvable group. Furthermore,

\[(3.2) \quad G/Z_\infty(G) = G_1 \times G_2 \times \ldots \times G_n,\]

where \( G_i \) is an \( F_{(p_i, d_i)} \)-group and \( (d_i, d_j) = 1 \) for any \( i \neq j \) with \( i, j \in \{1, 2, \ldots, n\} \).
Proof: An application of Lemma 3.1 to \( K \) implies that \( K \) is finite. Then \( G \) is finite by Lemma 2.2. More precisely, \( G = K \langle x \rangle \), where \( |\langle x \rangle| = |G/K| = q \) for some prime \( q \). By Theorem 3.2 we may deduce that \( |K'/K'| \) is a cyclic group of order \( p^r \) for some prime \( p \) and some \( r \geq 1 \). Then \( K = K'(y) \), where \( |\langle y \rangle| = p^r \), and so \( G = \langle K', x, y \rangle = K'(x, y) \), where \( K' \) is nilpotent finitely generated of class \( c \). We know from [6, 5.2.18] that a finitely generated nilpotent group has a central series whose factors are cyclic with prime or infinite orders and so \( K' = S \) is supersolvable and we have the following series \( \{1\} = Z_0(S) \triangleleft Z_1(S) \triangleleft \ldots \triangleleft Z_c(S) = S \triangleleft K \triangleleft G \), where \( Z_1(S)/Z_0(S), \ldots, Z_c(S)/Z_{c-1}(S) \) are cyclic groups of prime order. We have just seen that \( K/S \) is a cyclic group. \( G/K \) is cyclic by Lemma 2.2. Note that each term of this series is normal in \( G \). Therefore \( G \) is supersolvable.

Independently, the only fact that \( G \) is finite allows us to apply [11] Theorem 1 and so \( G/Z_\infty(G) \) is the direct product of Frobenius groups as claimed. \( \square \)

It is interesting the following consequence of Theorem 3.4.

Corollary 3.5. If \( G \) is a finite solvable group with \( |\mathcal{M}(G)| = 2 \), then it is supersolvable.

Remark 3.6. Theorem 3.3 relates \( G/Z_\infty(G) \) with \( |\mathcal{M}(G)| \). Recall that nilpotent finitely generated groups are supersolvable (see [6]). Then we are saying in Theorem 3.4 that small values of \( |\mathcal{M}(G)| \) imply that \( G \) is a (finite) supersolvable group which is not nilpotent. Furthermore we are describing, thanks to \( G/Z_\infty(G) \), how much is big the difference from being supersolvable and not being nilpotent.

The remainder of this section illustrates another aspect of (1.1).

We recall from [6, §13.3] that

\[
\omega(G) = \bigcap_{s \notin G} N_G(S)
\]

is the Wielandt subgroup of a group \( G \). \( \omega(G) \) is always a \( T \)-group, that is, a group in which the normality is a transitive relation. Solvable \( T \)-groups were classified by D.J. Robinson in 1964 (see [2]) and more generally the groups in which all the subgroups are subnormal were classified by W. Möhres in [3] (see also [2, §12.2]). These are related to \( MN \)-groups by [9, Theorem 3.1], which is quoted below.

Theorem 3.7 (H. Smith, see [9]). Let \( G \) be a solvable \( MNN \)-group and suppose that \( G \) has no maximal subgroups. Then:

(i) \( G \) is a countable \( p \)-group for some prime \( p \) and \( G/G' \simeq C_{p^\infty} \);

(ii) every subgroup of \( G \) is subnormal;

(iii) every hypercentral image of \( G \) is abelian and \( G' = \gamma_\infty(G) \);

(iv) every radicable subgroup of \( G \) is central;

(v) \( HG' = G \) implies \( H = G \) for every subgroup \( H \) of \( G \) and \( C_G(G') \) is abelian.

In particular, \( G \) has no proper subgroups of finite index;

(vi) \( G' \) is not the normal closure in \( G \) of a finite subgroup;

(vii) \( Z(G) = Z_\infty(G) \).

We have all it is necessary in order to prove the second main result of this section.

Theorem 3.8. Assume \( |\mathcal{M}(G)| = 2 \) and \( K \) non-finitely generated as in Lemma 2.1. If \( K \) has no maximal subgroups, then \( K/\omega(K) \) is non-trivial, non-abelian, of infinite exponent and at least of countable abelian rank.
Proof. From Corollary 2.13, *G* is solvable. Assume ω(*K*) = *K*. *K* is a *T*-group and Theorem 3.7 (ii) every subgroup of *K* is subnormal. Both these conditions imply that *K* is a Dedekind group, then *K* ∈ *ℜ*. This contradiction shows that *K*/ω(*K*) is non-trivial.

Assume [*K*/ω(*K*), *K*/ω(*K*)) = 1. Then

\[(3.4) \quad [K, K] ≤ ω(K) = \bigcap_{S \in S_n} N_K(S) = \bigcap_{S ≤ K} N_K(S) = norm(K) ≤ Z_2(K),\]

where the last inequality is due to a famous result of E. Schenkman [7]. Therefore *K* ∈ *ℜ*, which is a contradiction. This implies that *K*/ω(*K*) cannot be abelian.

The fact that *K*/ω(*K*) is of infinite exponent follows by the classification of W. M"ohres and precisely by [3, Theorem].

Note that *K*/ω(*K*) has no maximal subgroups. Then *K*/ω(*K*) has no proper subgroups of finite index. On another hand, we know from [2, 5.3.6] that a solvable group with finite abelian rank and no proper subgroups of finite index must be nilpotent. This implies that *K*/ω(*K*) cannot be of finite abelian rank, and so, at least of countable abelian rank. □

Unfortunately, we cannot think to Example 2.14 in case of Theorem 3.8, since in Example 2.14 there are maximal subgroups. However, a satisfactory description is offered by the following result.

**Corollary 3.9.** Assume |M(*G*)| = 2 and *K* non-finitely generated as in Lemma 2.1. If *K* has no maximal subgroups, then *K* has the series

\[(3.5) \quad \{1\} \triangleleft ω(*G*) = K^{(d)} \triangleleft K^{(d-1)} \triangleleft \ldots \triangleleft K' \triangleleft K \triangleleft *G*,\]

where ω(*K*) = γ_3(ω(*K*)) ⊗ *L*, *L* is the subgroup generated by the involutions of ω(*K*), *K*/K' ≃ C_{p^∞} for some prime *p*, there exists some *i* ∈ {1, ..., *d*} such that *K*^{(i+1)}/K' is the direct product of infinitely many copies of C_{p^∞}, *G*/K is of prime order.

Proof. *G* is solvable by Corollary 2.13, *K* is a solvable *MN*N-group with no maximal subgroups and it must be a periodic *p*-group, by Theorem 3.7. The fact that ω(*K*) is a semidirect product of *L* and γ_3(ω(*K*)) follows from the classification of periodic solvable *T*-groups and can be found for instance in [6, Exercises 13.4, n.10, p.394]. Now the rest of the result follows from the combination of Lemma 2.1, Theorem 3.8 [9, Exercises 13.4, n.10, p.394] and Theorem 3.7. □

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