Disordered Regimes of the one-dimensional complex Ginzburg-Landau equation

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Abstract

I review recent work on the “phase diagram” of the one-dimensional complex Ginzburg-Landau equation for system sizes at which chaos is extensive. Particular attention is paid to a detailed description of the spatiotemporally disordered regimes encountered. The nature of the transition lines separating these phases is discussed, and preliminary results are presented which aim at evaluating the phase diagram in the infinite-size, infinite-time, thermodynamic limit.

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1. A prototype for studying spatiotemporal chaos

The complex Ginzburg-Landau equation (CGL):

$$\partial_t A = A + (1 + ib_1)\Delta A - (b_3 - i)|A|^2 A$$

where $A$ is a complex field, $x \in [0, L]$ and $b_3 > 0$, is of considerable importance to everybody interested in spatially extended non-equilibrium systems. It accounts for the slow modulations, in space and time, of the oscillatory state in a physical system which has undergone a Hopf bifurcation. As such, the CGL is closely related to numerous experimental situations. The resulting universality and genericity are accompanied by specific features which make this model interesting for its own sake, simply as a prototype of spatially extended dynamical systems.

There exist two important limits to CGL: the dissipative, relaxational "real Ginzburg-Landau" equation ($b_1 = 0, b_3 \to \infty$) and the dispersive, integrable nonlinear Schrödinger equation (NLS) ($b_3 = 0, b_1 \to \infty$). Studying CGL in a comprehensive manner, one encounters dynamical regimes where the relative importance of dissipation and dispersion can be tuned at will in a non-integrable, non-variational system. This has been done recently for the one-dimensional case, uncovering the "phase diagram" presented in figure 1 and discussed below.

The relative simplicity of figure 1 stems mostly from the existence of a "thermodynamic limit" for the spatiotemporally chaotic regimes observed. As a matter of fact, away from the intricacy of the bifurcation diagrams at small sizes ($L \lesssim 50$), there exists a large-size limit beyond which chaos becomes extensive and can be characterized by intensive quantities independent of system size, boundary conditions, and, to a large extent, initial conditions. Perhaps the most convincing illustration, for CGL, of this essential property of spatiotemporal chaos can be found in a recent paper by Egolf and Greenside where they show that the Lyapunov dimension is proportional to the system size $L$.

Here, I review the various disordered "phases" observed in the one-dimensional case in the large-size limit and the nature of the transitions leading to them. Emphasis is put on the physical-space structures and objects which compose the spatiotemporal chaos as they could well play a crucial role when trying to build a statistical analysis of the disordered phases. This should
also hopefully provide a clearer picture of the “elementary mechanics” at play in these regimes.

All the results reported and presented here were obtained using a pseudo-spectral code with periodic boundary conditions and second-order accuracy in space and time. Spatial resolution was typically 1024 modes for a domain of size $L = 1000$. Time steps varied like $1/\sqrt{b_3}$, with typical values as indicated in the figure captions.

2. Phases

Early work on CGL has dealt with the problem of the linear stability of its family of plane-wave solutions $A = a_k \exp i(kx + \omega_k t)$ with $a_k^2 = (1 - k^2)/b_3$ and $\omega_k = 1/b_3 - (b_1 + 1/b_3)k^2$. All these solutions are unstable for $b_1 > b_3$, a condition which defines the so-called “Benjamin-Feir line” ($BF$). For $b_1 < b_3$, plane-wave solutions with $k^2 < k_{Eckhaus}^2 = (b_3 - b_1)/(3b_3 - b_1 + 2/b_3)$ are linearly stable. The $BF$ line thus provides a first separation of the $(b_1, b_3)$ plane, indicating where the constant-modulus, phase-winding solutions might play a role.

I now give a brief account of what is known about the disordered phases. For all these spatiotemporal chaos regimes, chaos is extensive: the Lyapunov dimension (and the other quantities derived from the Lyapunov spectrum such as the entropy or the number of positive exponents) scales linearly with the system size $L$.

2.1 Defect turbulence

Above the $BF$ line and to the left of $L_1$, a strongly disordered phase is observed, best defined by a finite density of space-time zeros of $|A|$. This “defect turbulence”, also named “amplitude turbulence”, is characterized by a quasi-exponential decay of space and time correlation functions, and slower-than-exponential tails of the pdf of the phase gradient $\phi_x$.

One elementary process is the key ingredient of this strong spatiotemporal chaos: pulses of $|A|$ grow under the effect of the dispersion term; this “self-focusing” is stopped by the action of dissipation, breaking the pulse. Such events can be seen in the spatiotemporal plots presented in figure 2; regions
of almost zero amplitude are present, another indication that pulses are the relevant objects to consider when approaching the NLS limit. As one gets closer to the NLS (increasing $b_1$), the pulses get sharper and higher, whereas closer to the BF line, dissipation dominates and $|A|$ rarely overpasses the "saturation" value $1/\sqrt{b_3}$.

This disordered phase is reached from (almost) all initial conditions for parameter values to the left of $L_3$. In the triangular region delimited by BF, $L_1$ and $L_3$, it coexists with "phase turbulence", which is described below.

2.2 Phase turbulence

Phase turbulence is a weakly disordered regime observed in the region of parameter space above the BF line to the right of $L_1$, as well as in the triangular region delimited by BF, $L_1$ and $L_3$. It is best defined by the absence of space-time defects or, equivalently, by the bounded character of the pdf of $\phi_x$ and $|A|$ (see below for a discussion of this point). The absence of phase singularities implies that the "winding number" $\nu = \int_0^L \phi_x dx$ is a conserved quantity equal to a multiple of $2\pi$ which classifies the different attractors reached at given values of $b_1$ and $b_3$.

Chaos is very weak, as shown by the slow decay of space and time correlation functions, indicative of diffusive or sub-diffusive modes. This is corroborated by the flat shoulder of the spatial power spectrum of $|A|$ at low wavenumbers, reminiscent of the Kuramoto-Sivashinsky equation (KS). The dynamics is in fact very similar to that of KS, which is not surprising since this equation was originally derived to describe the phase dynamics of CGL near the BF line. The close relationship with KS suggests, in turn, that the Kardar-Parisi-Zhang interface equation (KPZ), which have been argued to account for the large-scale, long-time, properties of KS, might provide the correct description of the long-wavelength limit of phase turbulence. In such a setting, one expects exponential decay of spatial correlations and stretched-exponential decay of temporal correlations.

Spatiotemporal diagrams (figure 3) reveal that the objects involved are "shocks" of $|A|$ (similar graphs for $\phi_x$ show localized modulations of the wavenumber, and look essentially the same). Close to the BF line, the dynamics is indeed very much like in KS (figure 3a), while away from it trains
of propagative shocks become more and more frequent (figure 3b). The overall effect of $\nu$ is to introduce a general drift on the localized shocks of $|A|$ which increases with $\text{abs}(\nu)$ (figure 3c). The phase turbulence regimes thus look very different depending on $\nu$ and the distance, in the parameter plane, from $(b_1, b_3)$ to the BF line. Increasing this distance, the characteristic scales decrease (the difference of the time-scales between figures 3a and 3b is only partly due to the variation of the basic frequency like $\sqrt{b_3}$). The other important effect, the appearance of trains of propagative shocks, is an indication that KS is not a valid description away from the BF line and that other terms, incorporating odd derivatives of $\phi$, are present in the “effective phase equation” describing the corresponding phase turbulence regimes. Whether the connection to KPZ still holds in such circumstances and what is then the valid phase equation replacing KS, are open problems under investigation.

It must be mentioned, at this point, that it is still a controversial and unresolved issue to know whether phase turbulence “exists” in the thermodynamic limit, i.e., whether it is a transient phenomenon or a finite-size artifact. In sections 3.1 and 3.3 below, I discuss this point in relationship with the nature of the transition lines delimiting the phase turbulence region in figure 1. In any case, even if there always exists a very small (essentially undetectable) density of defects, the spacetime regions between those rare events are large enough so as to justify the study of phase turbulence for its own sake.

2.3 Spatiotemporal intermittency

Below the BF line, plane-wave solutions of wavenumber $k^2 \leq k^2_{\text{Eckhaus}}$ are linearly stable. This does not preclude the existence, to the left of $L_2$, of other, mostly chaotic, solutions which are easily reached for initial conditions outside the basin of attraction of the stable plane waves. These disordered regimes are spatiotemporal intermittency regimes: they consist of space-time regions of stable plane waves separated by localized objects evolving and interacting in a complex manner (figure 4). The plane waves constitute the passive, “absorbing” state while the localized objects carry the spatiotemporal disorder. The simplest of these objects are members of the family of exact solutions found by Nozaki and Bekki. These propagating “holes” of
\[ |A| \] are not defects, strictly speaking, even though \(|A|\) might remain very small in their core.\(^2\) Defects (zeros of \(A\)) do occur, but at scattered points in spacetime. This makes the density of defects difficult to estimate and rather worthless, since, for example, the average number (in space) of hole-like objects is probably a better characterisation of the dynamics. Likewise, space and time correlation functions are not the easiest quantities to measure when evaluating the coherence scales in these regimes, due to the intermittent character of the disorder. As usual with spatiotemporal intermittency,\(^4\) one can take advantage of the intrinsic binary structure of the spatiotemporal dynamics and evaluate, say, the distributions of sizes and lifetimes of the patches of plane wave solutions. These distributions are roughly exponential, yielding characteristic coherence scales.\(^2\)

The spatiotemporal intermittency regimes do exhibit space-time defects. To that extent, they may be considered as defect turbulence regimes. The key difference with the region of defect turbulence described above is that here the defects do not appear spontaneously, they are produced by the localized objects carrying the disorder.

### 2.4 The “bichaos” region

The domain of parameter space delimited by the BF, \(L_1\) and \(L_3\) lines deserves further comment. Depending on the initial conditions, one can reach one of two spatiotemporally chaotic regimes, phase turbulence or defect turbulence. A closer look at the defect turbulence reveals that it consists of localized propagating and branching hole-like objects which separate more quiescent space-time regions (figure 5). These objects do not appear spontaneously. The quiescent regions are nothing else than patches of phase turbulence, with their characteristic shocks of \(|A|\). Defect turbulence is thus a spatiotemporal intermittency regime, with the absorbing state being the phase turbulence regime.

### 3. Transitions

I now examine the various transition lines between the phases and discuss to what extent they can be considered as phase transitions.
3.1 The $L_1$ line

Line $L_1$ separates defect turbulence from phase turbulence for values of $b_1 \gtrsim 1.85$. As such, its best definition is given by the parameter values for which the density of defects $n_D$ goes to zero. It was shown\(^\text{12}\) that, for $b_1 = 3.5$, $n_D \sim (b_3 - b_3^*)^2$ with $b_3^* \approx 1.29$. No hysteresis has been detected. This transition thus appears continuous, even though one cannot exclude a crossover scenario where $n_D$ would remain small beyond $b_3^*$.

The situation is not as clear from the point of view of the correlation length $\xi$ deduced from the exponential decay of the two-point correlation function: approaching $L_1$, $\xi$ increases but does not diverge at the transition point.\(^\text{12,7}\) Although difficult to measure in the phase turbulence regimes, indirect arguments imply that $\xi$ remains finite. Only a change of behavior is observed, apparently accompanied by a discontinuity of $d\xi/db_3$. A similar situation holds for the variation of the Lyapunov dimension and the other quantities derived from the Lyapunov spectrum when crossing $L_1$, as shown by Egolf and Greenside.\(^\text{6}\) In other words, no critical behavior seems to be present at the transition, even though it is continuous.

This remark is valid for the experimental conditions within which the above results were obtained. In particular, the system sizes used ensure the extensivity of chaos, but no assessment of finite-size effects has been made in order to evaluate the infinite-size, infinite-time, “thermodynamic” limit. The main question in this respect is that of the position of $L_1$ as $L \to \infty$. If $L_1$ remains away from the BF line, the existence of phase turbulence in the thermodynamic limit is assured and the transition at $L_1$ does not show any critical behavior. Another hypothesis consists in $L_1$ moving toward BF as $L$ increases, with the correlation length at threshold increasing and finally diverging for an infinite-size system. In this case, the phase turbulence region disappears in the thermodynamic limit and the BF line is the asymptotic boundary of defect turbulence.

Preliminary results (figure 6) indicate a clear displacement of $L_1$ toward BF as $L$ increases, making the second hypothesis more likely. A similar trend was recently found by Egolf and Greenside.\(^\text{7}\) Extensive numerical work is under completion to obtain a quantitative extrapolation of the position of $L_1$ in the infinite-size limit.
3.2 The $L_2$ line

The line $L_2$, best defined as the limit of existence of spatiotemporal disordered states, is difficult to determine in a precise manner. This is mostly due to the appearance, in the transition region, of “frozen” states, i.e. spatially-disordered arrangements of localized objects with trivial time dependence (figure 7). The coherence scales of the spatiotemporal intermittency regimes increase when approaching $L_2$, but their divergence is not observed; the system “falls” on a frozen state. As often in spatiotemporal intermittency situations, the deterministic features of the system “mask” the directed-percolation-like phase transition. The line $L_2$ is thus determined only crudely, awaiting further progress. In particular, two directions seem worth investigating: a “local” approach based on the study of the interactions between the localized objects at the origin of the spatiotemporal disorder, and a “global” approach based on Lyapunov analysis of the disordered regimes.

3.3 The $L_3$ line

The $L_3$ line delimits the parameter space region where phase turbulence ceases to coexist with defect turbulence. Here I give a preliminary account of ongoing work on the nature of the breakdown of phase turbulence occurring when crossing $L_3$ (say, by decreasing $b_3$), for a winding number $\nu = 0$.

The transition is hysteretic: defect turbulence persists when crossing $L_3$ back, up to line $L_2$. The breakdown of phase turbulence occurs via the nucleation of a first defect, followed by the quasi-deterministic invasion of the phase turbulent state by the defect phase (figure 8). The speed of invasion is always finite and rather large; it is given by the average propagation speed $v_D$ of the localized objects composing the defect phase in this region of parameter space. The precursor of the first defect is apparently a local event: one of the shocks of $|A|$ composing the phase turbulence accelerates, and a depression of $|A|$ develops at its tail. When the velocity reaches $v_D$, the first hole-like object typical of the defect phase is effectively created, and quickly generates the first space-time defect. It is difficult to determine what triggers the initial acceleration of one of the shocks of $|A|$. As mentioned above, more and more trains of propagating shocks are observed in the phase turbulence regimes as one goes away, in parameter space, from the $BF$ line. Simul-
taneously, the average velocity $v_S$ of these propagative shocks increases. A possible interpretation of the nucleation is that, for parameter values beyond $L_3$, the fluctuations around $v_S$ bring the speed of one shock past a critical value, after which the shock is attracted to the hole-like solution characteristic of the defect phase. Such a critical value would be related to corresponding critical values of the local minimum of $|A|$ and local extremum of $\phi_x$, since as the velocity of the shock increases, the local depression of $|A|$ at its tail deepens, and the local phase gradient increases. Ongoing work aims at making those mostly qualitative statements more quantitative.

As described above, the breakdown of phase turbulence strongly resembles a first-order phase transition. So far, $L_3$ has been determined numerically by merely checking whether the breakdown occurs within a fixed (very long) integration time. However crude, this methodology produced a sharp transition line, all the more so as no size effects could be detected.

A quantitative estimate of the probability of nucleation is needed to improve this situation. Preliminary results on the study of the pdfs of $|A|$ and $\phi_x$ in the phase turbulence regimes provide the beginning of an answer (figure 9). Away from $L_3$ (figures 9a and 9c), these distributions have either Gaussian or strictly bounded tails. Close to $L_3$ (figures 9b and 9d), these tails are exponential, signalling a clear change of behavior on which the definition of $L_3$ can be firmly based. Nevertheless, the question of the Gaussian/bounded character of the tails away from $L_3$ remains a crucial one since it determines whether the probability of nucleation can be strictly zero or not. A plausible scenario would consist of exponential tails truncated by a finite bound moving to infinity (resp. zero) for $\phi_x$ (resp. $|A|$) when approaching $L_3$. Whereas this scenario would assure the existence of the phase turbulence in the infinite-size, infinite-time limit, the other scenario of a crossover from Gaussian to exponential tails would not. Current numerical efforts aim at providing data to decide this.

In the same spirit, another important question deals with the estimation of size-effects in this region of parameter space. So far, results for sizes $L = 1000$ and $L = 4000$ have not shown any significant difference, strengthening the case for a first-order transition.

Finally, the overall effect of a non-zero winding number $\nu$ is to give rise to earlier breakdown of the phase turbulence regime, moving $L_3$ to the right,
as expected from the general additional drift of the shocks in this case.

4. Open problems

The “phase diagram” presented in figure 1 was obtained for systems sizes $L$ large (for which the extensivity of chaos is ensured) but finite. From the above discussion of lines $L_1$ and $L_3$, it appears that an estimation of the finite-size effects is necessary, and even crucial, since the existence of phase turbulence depends on the position and nature of these transition lines in the infinite-size, infinite-time, “thermodynamic” limit. For line $L_3$, a detailed study of the pdfs of interest in the phase turbulence regimes should be able to provide an objective criterion which can help to determine precisely the breakdown of phase turbulence. In this respect, preliminary results (figure 9) are encouraging. For the line $L_2$ delimiting the domain of existence of spatiotemporal intermittency, the key point lies in finding a quantifier which can be used to determine its position efficiently. Quantities related to the Lyapunov spectrum might constitute such a quantifier.

Apart from these “global”, statistical quantities, approaches based on the study of the local objects involved in the spatiotemporally disordered dynamics (“shocks” for phase turbulence, “pulses” and “defects” for defect turbulence, “sources” and “sinks” for spatiotemporal intermittency) might also help to clarify some of the problems left open.

At any rate, and whatever the situation is in the thermodynamic limit, the “finite-size” phase diagram of figure 1 is of interest to all “experimentalists” (i.e., anybody working with finite-size systems observable during a finite time). In this respect, it shows how varied and complex statiotemporally chaotic regimes can emerge from such a simple equation as CGL. I hope the above review will also provide guidelines for dealing with similar cases, in particular when the question of what to measure arises.

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Figure Captions

Figure 1: Phase diagram in the \((b_1, b_3)\) parameter plane. Lines \(L_1\) and \(L_3\) were determined in \(12\), line \(L_2\) in \(2\). See \(2\) for numerical details.

Figure 2: Spatiotemporal representation of a defect turbulence regime in a system of size \(L = 250\) for \(b_1 = 3.5\) and \(b_3 = 0.5\) (periodic boundary conditions, timestep \(\delta t = 0.02\)). Time is running upward for \(\Delta T = 82.5\) (transient discarded). Part (a) shows the evolution of \(|A|\) (grey scale between \(|A| = 0/\text{black}\) and \(|A| = 2/\text{white}\)). Part (b) shows the corresponding evolution of \(\phi = \text{arg}(A)\) (grey scale between \(0/\text{white}\) and \(\pi/\text{black}\)).

Figure 3: Spatiotemporal evolution of \(|A|\) in phase turbulence regimes in a system of size \(L = 1000\) for \(b_1 = 3.5\) (periodic boundary conditions, timestep \(\delta t = 0.16\), transient discarded, time running upward, grey scale with white corresponding to maximum value of \(|A|\)). Part (a): close to the BF line \((\nu = 0, b_3 = 2, \Delta T = 3300, 0.68 \leq |A| \leq 0.75)\). Part (b): close to \(L_1\) \((\nu = 0, b_3 = 1.4, \Delta T = 1650, 0.69 \leq |A| \leq 1.01)\). Part (c): same as (a) but with \(\nu = -10\pi\).

Figure 4: Spatiotemporal intermittency regime in a system of size \(L = 250\) (periodic boundary conditions, timestep \(\delta t = 0.04\)) for \(b_1 = 0\) and \(b_3 = 0.5\). Time is running upward for \(\Delta T = 165\) (transient discarded). Same representation as in figure 2, but \(0 \leq |A| \leq 1.413\) in part (a).

Figure 5: Defect turbulence in the bichaos region \((L = 1000, b_1 = 1.5, b_3 = 1, \Delta T = 1312, \text{periodic boundary conditions, timestep } \delta t = 0.04, \text{and transient discarded})\). Spatiotemporal representation of \(|A|\) in grey scale from \(|A| = 1.12\) (white) to \(|A| = 0\) (black); time is running upward. Note that the hole-like propagating and branching objects typical of the defect phase do not appear spontaneously in the phase turbulent medium.

Figure 6: Variation of the defect density \(n_D\) \((\times 10^5\) on the graph) with \(b_3\) near \(L_1\). These results were obtained from runs of duration \(\Delta T \sim 20000\) after transients on systems of size \(L = 1000\) and \(2000\) with periodic boundary conditions and timestep \(\delta t = 0.04\). Although these results are
still not very precise (at such low densities, much longer runs would be necessary), the general trend is significant: higher densities are reached for larger sizes.

**Figure 7:** Frozen state observed near $L_2$ in a system of size $L = 500$ with $b_1 = -0.9$ and $b_3 = 0.18$ (periodic boundary conditions). This spatially disordered state is made of zero-velocity Nozaki-Bekki holes and shocks. $|A|$ is stationary, while $\phi$ winds regularly along time.

**Figure 8:** Breakdown of phase turbulence past the $L_3$ line in a system of size $L = 1000$ for $b_1 = 1$, $b_3 = 0.675$ and $\nu = 0$ (periodic boundary conditions, timestep $\delta t = 0.16$). Initial condition was a phase turbulent state for $b_3 = 0.7$. The breakdown occurred near $t = 12000$ as shown by the time series of $|A|$ (part (a)). Part (b) is the spatiotemporal evolution of $|A|$ for $t \in [9600, 12800]$ (grey scale from $|A| = 1.29$/white to $|A| = 1.1$/black, time running upward); at this resolution, the details of the defect phase are not visible. Part (c): zoom of (b) with a grey scale and a resolution adapted to the defect phase ($x \in [500, 1000]$, $t \in [12550, 12800]$, $0 \leq |A| \leq 1.29$); the nucleation of the first defect is obvious. Part (d): as in (b) but for the locus of the minimum (in space) of $|A|$.

**Figure 9:** Probability distribution functions of $|A|$ for phase turbulence regimes in the bichaos region. These distributions were obtained from the simulation of a system of size $L = 4000$ during $\Delta T = 11500$ sampled every 4 timesteps $\delta t = 0.16$. (periodic boundary conditions, transient discarded). Parts (a) and (c): $\nu = 0$, $b_1 = 1$, $b_3 = 0.9$. Parts (b) and (d): $\nu = 0$, $b_1 = 1$, $b_3 = 0.7$. 