Limit evolution in some Markovian models of opinion dynamics with large number of participants

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Abstract. We consider a discrete time Markov process describing dynamics in a society of $K$ agents with special opinion exchange rules. In the limit as $K$ goes to infinity, we prove that the process converges to some limiting dynamical system and afterwards we discuss some properties of that system.

1. Introduction
It is well known that Markov stochastic processes are widely used for the construction of a variety models in physics, engineering, social sciences and other applied domains. In this paper we provide rigorous results related to a special Markovian model of opinion dynamics. The idea of this model was inspired by papers [1–3]. Lorenz [2] proposed alternative dynamical systems that mimic WD and HK, two famous agent-based models of opinion dynamics [4–7]. The approach of [2] was not based on individual agent opinions but rather on iterations of some nonlinear mapping of probability distributions on a finite opinion set. This approach used a notion of Interactive Markov Chains that was previously proposed in [1]. This concept is close to so called nonlinear Markov processes [8–12]. The paper [2] contained very interesting numerical results and their discussion from the physical point of view (a bifurcation diagram etc). It was shown in [2] that the alternative models have a limit behavior similar to the classical models, in particular, for uniform initial distribution they can demonstrate properties of stabilization, clustering, polarization etc. The paper [3] was devoded to a slightly different model that additionally used concepts of internal opinions and external actions (see also [13] for similar approach to different models).

First we define a multi-dimensional Markovian model in terms of a particle system with conditionally independent transitions. Then we pass to an aggregated Markovian model and afterwards, by letting the number of particles go to infinity, we rigorously prove convergence to a nonlinear dynamical system similar to that in [2, 3]. Since we assume that particles are identical our system takes intermediate place between models with local interaction and mean-field models (see, for example, [14–17]).
2. Model
Consider a community of $K$ participants (agents). Suppose that agent’s opinions evolve in the discrete time $t \in \mathbb{Z}_+$ and take their values in a finite set $\Theta = \{\theta_1, \ldots, \theta_N\}$. To be concrete, let $N$ be even and $\theta_i = \frac{i}{N+1}$. At each time moment $t$, the agents independently perform public actions based on their internal opinions:

$$a_i(t) = \begin{cases} 0, & X_i(t) < 0.5 \\ 1, & X_i(t) > 0.5 \end{cases}$$

Let $Q^-(t)$ be a population share corresponding to the action 0 and $Q^+(t)$ be a population share corresponding to the action 1, namely,

$$Q^+(t) = \frac{1}{K} \sum_{i=1}^{K} a_i(t), \hspace{1cm} Q^-(t) = \frac{1}{K} \sum_{i=1}^{K} (1 - a_i(t))$$

Note that $Q^+(t) + Q^-(t) = 1$. Let $f : \Theta \to [0, 1]$ be the probability for an agent with the opinion $\theta_i$ to change it under the influence of the environment. This function can be interpreted as inertia of opinion. The probability that the agent $i$ keeps his opinion value $\theta_i$ unchanged is equal to $1 - f(\theta_i)$.

Any agent $i$ is influenced by the actions of the whole population. At each time instant $t$ independently of other agents the agent $i$ shifts his opinion from $\theta_i$ to $\theta_{i+1}$ or $\theta_{i-1}$ with probabilities $f(\theta_i)Q^+(t)$ and $f(\theta_i)Q^-(t)$ correspondingly.

3. Markov process with discrete state space
Denote by $\vec{X}(t) = (X_1(t), \ldots, X_K(t))$ opinions of all agents. To describe the evolutions of the system we introduce a Markov process on the state space is $\Theta^K$ as follows.

1) Particles $X_i$ are conditionally independent for fixed $\vec{X}(t)$

$$P\left( X_1(t+1) = \theta_1^{t+1}, \ldots, X_K(t+1) = \theta_K^{t+1} \mid X_1(t) = \theta_1^t, \ldots, X_K(t) = \theta_K^t \right) =$$

$$= \prod_{i=1}^{K} P\left( X_i(t+1) = \theta_i^{t+1} \mid X_1(t) = \theta_1^t, \ldots, X_K(t) = \theta_K^t \right)$$

Thus, for a given current state $\vec{X}(t)$, the components $X_1$, ..., $X_K$ perform their transitions on the time step $t \to t+1$ independently.

Let $\theta_i^t = x$, $\theta_i^{t+1} = y$. Then we introduce notation $b_{xy} = P(X_i(t+1) = y \mid \vec{X}(t) = \vec{\theta}^t)$. It is convenient to think $b_{xy}$ as entries of the following matrix

$$\begin{pmatrix}
1 - f(\theta_1)Q^+(t) & f(\theta_1)Q^+(t) & 0 & \ldots & 0 & 0 \\
0 & 1 - f(\theta_2)Q^+(t) & f(\theta_2)Q^+(t) & \ldots & 0 & 0 \\
0 & 0 & 1 - f(\theta_3)Q^+(t) & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 - f(\theta_{N-1})Q^+(t) & f(\theta_{N-1})Q^+(t) \\
0 & 0 & 0 & \ldots & f(\theta_N)Q^-(t) & 1 - f(\theta_N)Q^-(t)
\end{pmatrix}$$

This matrix contains conditional probabilities for transition of a single particle. However $(X_i(t), t \in \mathbb{Z}_+)$ is not a Markov process due to dependence of $p_{xy} = p_{xy}(\vec{X}(t))$ on the total vector $\vec{X}(t)$.\"
4. New state space. Occupation numbers
For each \( \theta_j \in \Theta \) define the "occupation" numbers: \( \nu_j(t) = \# \{ k : X_k(t) = \theta_j \} \). Vectors \( \vec{\nu} = (\nu_1, \ldots, \nu_N) \) take values in the set

\[
\mathcal{G}_K = \left\{ (n_1, \ldots, n_N) : n_k \geq 0, \sum_{j=1}^N n_j = K \right\}.
\]

Lemma. "Agglomerated" process \((\vec{\nu}(t), t = 0, 1, \ldots)\) is a Markov chain on \( \mathcal{G}_K \).

Proof. Suppose that the random value \( \eta_{i-1}^+(t) \) denotes the number of particles transferred from \( \theta_{i-1} \to \theta_i \) during the time step \( t \to t + 1 \). In the same way \( \eta_{i+1}^-(t) \) is the number of particles transferred from \( \theta_{i+1} \to \theta_i \) and \( \eta_0^0(t) \) is the number of particles in \( \theta_i \) that did not change their opinion.

In this case

\[
\begin{align*}
\eta_{i-1}^+(t) &\sim_{\text{cond}} \text{Bin}(\nu_{i-1}(t), f(\theta_{i-1})Q^+(t)) \\
\eta_{i+1}^-(t) &\sim_{\text{cond}} \text{Bin}(\nu_{i+1}(t), f(\theta_{i+1})Q^-(t)) \\
\eta_0^0(t) &\sim_{\text{cond}} \text{Bin}(\nu_i(t), 1 - f(\theta_i))
\end{align*}
\]

Here the notation \( \sim_{\text{cond}} \) stands for the conditional distribution given \( \vec{\nu}(t) \) and \( \text{Bin}(n, p) \) is binomial distribution. It means that, for example, the probabilities \( P(\eta_{i-1}^+(t) = m \mid \vec{\nu}(t)) \) are binomial.

Similar conclusion holds for the triple

\[
(\eta_0^0(t), \eta_i^+(t), \eta_i^-(t)) \sim_{\text{cond}} \text{Poli}(\nu_i(t); 1 - f(\theta_1), f(\theta_1)Q^+, f(\theta_1)Q^-)
\]

where \( \text{Poli}(n; r_1, r_2, r_3) \) is the multinomial distribution.

Notice that \( \nu_i(t + 1) = \eta_0^0(t) + \eta_{i-1}^+(t) + \eta_{i+1}^-(t) \) and that \( Q^+(t) \) and \( Q^-(t) \) depend only on \( \vec{\nu}(t) \)

\[
Q^+(t) = \frac{1}{K} \sum_{i=N/2+1}^N \nu_i(t), \quad Q^-(t) = \frac{1}{K} \sum_{i=1}^{N/2} \nu_i(t)
\]

If we fix the history of the process \( \vec{\nu}(t), \ldots, \vec{\nu}(1), \vec{\nu}(0) \) then distribution of the vector \( \vec{\nu}(t+1) \) does not depend on the vector \( \vec{\nu}(s) \) at earlier moments in time \( s < t \). Hence, the Markov property is satisfied

\[
P(\vec{\nu}(t+1) = |\vec{\nu}(t), \ldots, \vec{\nu}(0)) = P(\vec{\nu}(t+1) = \mid\vec{\nu}(t))
\]

Corollary 1. The Markov chain \( \vec{X}(t) \) has two absorbing states: \( (\theta_1, \ldots, \theta_1) \) and \( (\theta_N, \ldots, \theta_N) \). The Markov chain \( \vec{\nu}(t) \) also have two absorbing states: \( (K, 0, \ldots, 0) \) and \( (0, \ldots, 0, K) \). All other states are inessential and communicate with one another.

Corollary 2. Let parameters \( N, f \) and \( K \) be fixed and \( t \to \infty \). Then there exist limits

\[
\lim_{t \to} P(\vec{X}(t) = \vec{\theta}) \quad \text{and} \quad \lim_{t \to} P(\vec{\nu}(t) = \vec{n}).
\]

These limits are zero for inessential states and nonzero for the absorbing states. The values of these limits depend on the distributions \( \vec{X}(0) \) and \( \vec{\nu}(0) \).
5. Large particle number limit

Systems with a large number of agents $K$ are of particular interest. Define process of ratios, dividing the $\vec{v}(t)$ by $K : \rho_j^{(K)}(t) = \frac{\nu_j(t)}{K}$. Let $N$ and the opinion set $\Theta = \{\theta_1, \ldots, \theta_N\}$ be fixed.

We have the sequence of Markov processes $\vec{\rho}(t) = \left(\rho_1^{(K)}(t), \ldots, \rho_N^{(K)}(t)\right)$ defined on state spaces $\mathcal{G}_K = \{(r_1, \ldots, r_N) : r_k \geq 0, \sum_{j=1}^N r_j = 1\}$ which depend on $K$.

Then we prove the following theorem.

**Theorem.**

1) Fix $t$. Suppose that there exists a limit $\vec{\rho}^{(K)}(t) \xrightarrow{P} \vec{\rho}^{(\infty)}(t)$ as $K \to \infty$ and $\vec{\rho}^{(\infty)}(t)$ is a deterministic vector. Then there exist a deterministic limit

$$\vec{\rho}^{(K)}(t + 1) \xrightarrow{P} \vec{\rho}^{(\infty)}(t + 1)$$

2) Let sequence of initial distributions $\vec{X}(0) \in \Theta^K$ be such that there is a limit $\vec{\rho}^{(K)}(0) \xrightarrow{P} \vec{\rho}^{(\infty)}(0)$ as $K \to \infty$. Then the sequence of random vectors $\{\vec{\rho}^{(\infty)}(t), \ t = 0, 1, 2, \ldots\}$ is a solution to the following system

$$\begin{align*}
\rho_1(t + 1) &= \rho_1(t) - \rho_1(t)f(\theta_1)\rho^+(t) + \rho_2(t)f(\theta_2)\rho^-(t) \\
\vdots \\
\rho_i(t + 1) &= (1 - f(\theta_i))\rho_i(t) + \rho_{i-1}(t)f(\theta_{i-1})\rho^+(t) + \rho_{i+1}(t)f(\theta_{i+1})\rho^-(t) \\
\vdots \\
\rho_N(t + 1) &= \rho_N(t) - \rho_N(t)f(\theta_N)\rho^-(t) + \rho_{N-1}(t)f(\theta_{N-1})\rho^+(t)
\end{align*}$$

Here we denote $\vec{\rho}(t) \equiv \vec{\rho}^{(\infty)}(t)$ and $\rho^+(t) = \sum_{i=1}^N \rho_i(t)$, $\rho^-(t) = \sum_{i=i}^{N/2+1} \rho_i(t)$.

Note that transformation of the right side is nonlinear in $\vec{\rho}(t)$. So $(\vec{\rho}(t), t = 0, 1, \ldots)$ is a trajectory of some nonlinear dynamical system on the set of probability distributions

$$\mathcal{G}_\infty = \left\{(r_1, \ldots, r_N) : r_k \geq 0, \sum_{j=1}^N r_j = 1\right\}.$$ 

Notice that $\mathcal{G}_\infty$ does not depend on $K$.

We can also represent the resulting dynamical system in the matrix form

$$\vec{\rho}(t + 1) = \vec{\rho}(t)A(\vec{\rho}(t))$$

where

$$A(\vec{\rho}) = I_N + \rho^-D(f)J_1 + \rho^+D(f)J_2$$

and $I_N$ is identity matrix of size $N \times N$, $D(f) = \text{diag}(f(\theta_1), \ldots, f(\theta_N))$,

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -1 \end{pmatrix}, \quad J_2 = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}.$$
Proof. The idea is to calculate the Laplace transform for the joint distribution of the vector components $\left( \rho_i^{(K)}(t + 1), \ i = 1, \ldots, N \right)$ and pass to the limit.

1. Consider the generating function $\mathcal{V}(t + 1)$.

$$
\pi_{\mathcal{V}(t+1)}(s) = \mathbb{E}\prod_i s_i^{\nu_i(t+1)} = \mathbb{E}\mathbb{E}\prod_i s_i^{\eta_i^0(t)} s_i^{-1} s_{i+1}^{\eta_i^+(t)} |\mathcal{V}(t)).
$$

Since the process is conditionally independent and $(\eta_i^0(t), \eta_i^+(t), \eta_i^-(t)) \sim_{\text{cond Poli}}(\cdot)$, then

$$
\mathbb{E}\prod_i \mathbb{E}(s_i^{\eta_i^0(t)} s_i^{-1} s_{i+1}^{\eta_i^+(t)} |\mathcal{V}(t)) = \mathbb{E}\prod_i (p_{i,i} s_i + p_{i,i+1} s_{i+1} + p_{i,i+1} s_{i+1})^{\nu_i(t)}
$$

Proceed to the Laplace transform for the normalized occupation numbers $\rho_j^{(K)}(t) = \frac{\nu_j(t)}{K}$

$$
\phi_{\mathcal{V}^{(K)}(t+1)}(\lambda) = \mathbb{E}\prod_i e^{-\lambda_i \rho_i(t+1)} = \pi_{\mathcal{V}(t+1)}(e^{-\lambda/K}) = \mathbb{E}\prod_i (p_{i,i} e^{-\lambda_i/K} + p_{i,i+1} e^{-\lambda_{i+1}/K} + p_{i,i-1} e^{-\lambda_{i-1}/K})^{\nu_i(t)}
$$

Next, we make technical calculations (details are omitted): we pass to the logarithm, expand $e^{-\lambda}$ according to Taylor, regroup according to powers $K^{-1}$ and make the reverse transition to $\exp$.

As a result of all transformations, we substitute the vector $\bar{\lambda} = (0, \ldots, 0, \lambda_i, 0, \ldots, 0)$

$$
\phi_{\mathcal{V}^{(K)}(t+1)}(\lambda_i) = \mathbb{E}\exp(-\lambda_i (p_{i,i} \rho_i^{(K)}(t) + p_{i-1,i} \rho_i^{(K)}(t) + p_{i+1,i} \rho_i^{(K)}(t))) \
\cdot \exp(\lambda_i^2 (\frac{1}{2K} C + \bar{o}(1/K)))
$$

where

$$
C = (p_{i,i} - p_{i,i}^2) \rho_i^{(K)}(t) + (p_{i-1,i} - p_{i-1,i}^2) \rho_i^{(K)}(t) + (p_{i+1,i} - p_{i+1,i}^2) \rho_i^{(K)}(t)
$$

and $p_{i,j}^{(K)}$ depend on $K$.

By assumption

$$
\mathcal{V}^{(K)}(t) \xrightarrow{P} \tilde{\rho}(t), \quad K \to \infty,
$$

where $\mathcal{V}^{(K)}(t)$ is deterministic limit.

By Slutsky’s theorem it follows that if a continuous function $g(\cdot)$ is given and sequence of random vectors converges in probability to the limit, then $g(\mathcal{V}^{(K)}(t)) \xrightarrow{P} g(\tilde{\rho}(t))$.

Applying Slutsky’s theorem, we get:

$$
\mathbb{E}\exp(-\lambda_i \rho_i(t + 1)) \xrightarrow{K \to \infty} \exp(-\lambda_i (p_{i,i} \rho_i(t) + p_{i-1,i} \rho_{i-1}(t) + p_{i+1,i} \rho_{i+1}(t))
$$

Notice that the right hand side is the Laplace transform of the deterministic variable. The pointwise convergence of the Laplace transform implies the convergence of distribution functions, thus

$$
\rho_i^{(K)}(t + 1) \xrightarrow{d} (p_{i,i}^{(\infty)} \rho_i(t) + p_{i-1,i}^{(\infty)} \rho_{i-1}(t) + p_{i+1,i}^{(\infty)} \rho_{i+1}(t))
$$
In this case $\rho_i^{(K)}(t+1)$ converge in probability to a deterministic random variable in distribution, and hence.

$$
\rho_i^{(K)}(t+1) \xrightarrow{P} (1 - f(\theta_i))\rho_i(t) + f(\theta_{i-1})\left( \sum_{j=\lfloor N/2 \rfloor}^N \rho_j(t) \rho_{i-1}(t) \right)
+ f(\theta_{i+1})\left( \sum_{j=1}^{\lfloor N/2 \rfloor} \rho_j(t) \rho_{i+1}(t) \right)
\square
$$

It can be shown that nonlinear dynamical systems $\overline{\rho}(t)$ can have multiple fixed points (stationary solutions of the evolution equation). It is very interesting to describe sets of stable and nonstable fixed points and to find basins of attraction.

6. Conclusions and future work
We provided a mathematically rigorous proof of the statement about convergence of opinion distribution to the solution of some system of nonlinear evolution equations in the limit when the number of agents grows. In future we plan to study stability properties of fixed points of the limit dynamical system. We hope that in futures these results will open doors to computer simulations of more realistic opinion dynamics models.

7. References
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