The Bounds for Eigenvalues of Normalized and Signless Laplacian Matrices

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In this paper, we obtain the bounds of the extreme eigenvalues of a normalized and signless Laplacian matrices using by their traces. In addition, we determine the bounds for k-th eigenvalues of normalized and signless Laplacian matrices.

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1 Introduction

Let $G(V, E)$ be a simple graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set of $E$. For $v_i \in V$, the degree of $v_i$, the set of neighbours of $v_i$ denoted by $d_i$ and $N_i$, respectively. The cardinality of $N_i$ is denoted by $c_{ij}$; i.e, $|N_i| = c_{ij}$. If $v_i$ and $v_j$ adjacency, we denote $v_i \sim v_j$ or shortly use $i \sim j$.

The adjacency matrix, Laplacian matrix and diagonal matrix of vertex degree of a $G$ graph denoted by $A(G)$, $L(G)$, $D(G)$, respectively. Clearly

$L(G) = D(G) - A(G)$.

The normalized Laplacian matrix of $G$ is defined as $\mathcal{L}(G) = D^{-1/2}(G)L(G)D^{-1/2}(G)$ i.e, $\mathcal{L}(G) = [\ell_{ij}]_{n \times n}$, where

\[
\ell_{ij} = \begin{cases}
1 & \text{if } i = j \\
\frac{1}{\sqrt{d_i d_j}} & \text{if } i \sim j \\
0 & \text{otherwise}.
\end{cases}
\]

The signless Laplacian matrix of $G$ is defined as $Q(G) = D(G) + A(G)$ i.e, $Q(G) = [q_{ij}]_{n \times n}$, where

\[
q_{ij} = \begin{cases}
d_i & \text{if } i = j \\
1 & \text{if } i \sim j \\
0 & \text{otherwise}.
\end{cases}
\]

Since $\mathcal{L}(G)$ normalized Laplacian matrix and $Q(G)$ signless Laplacian matrix are real symetric matrices, their eigenvalues are real. We denote the eigenvalues of $\mathcal{L}(G)$ and $Q(G)$ are by

$\lambda_1(\mathcal{L}(G)) \geq \cdots \geq \lambda_n(\mathcal{L}(G))$
\[ \lambda_1(Q(G)) \geq \cdots \geq \lambda_n(Q(G)) \]

, respectively.

Now we give some bounds for normalized Laplacian matrix and signless Laplacian matrix.

**Oliveira and de Lima’s bound** [1]. For a simple connected graph \( G \) with \( n \) vertices and \( m \) edges, \( \Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta \)

\[ \lambda_1(Q(G)) \leq \max_i \left\{ d_i + \sqrt{d_i^2 + 8d_i m_i} \right\} \]  

where \( m_i = \frac{1}{d_i} \sum_{j \sim i} d_j \).

**Another Oliveira and de Lima’s bound** [1].

\[ \lambda_1(Q(G)) \leq \max_i \left\{ d_i + \sqrt{d_i m_i} \right\} \]  

where \( m_i = \frac{1}{d_i} \sum_{j \sim i} d_j \).

**Li, Liu et al. bound’s** [2,3].

\[ \lambda_1(Q(G)) \leq \frac{\Delta + \delta - 1}{2} \sqrt{(\Delta + \delta - 1)^2 + 8(2m - (n - 1)\delta)} \]  

**Rojo and Soto’s bound** [4]. If \( \lambda_1 \) is the largest eigenvalue of \( \mathcal{L} \), then

\[ |\lambda_1(\mathcal{L}(G))| \leq 2 - \min_{i < j} \left( \frac{|N_i \cap N_j|}{\max\{d_i, d_j\}} \right) \]  

where the minimum is taken over all pairs \( (i, j) \), \( (1 \leq i < j \leq n) \).

In this paper, we find an extreme eigenvalues of normalized Laplacian matrix and signless Laplacian matrix of a \( G \) graph with using their traces.

To obtain bounds for eigenvalues of \( \mathcal{L}(G) \) and \( Q(G) \) we need the following lemmas and theorems.

**Lemma 1.** Let \( W \) and \( \lambda = (\lambda_j) \) be nonzero column vectors, \( e = (1, 1, \ldots, 1)^T \), \( C = I_n - \frac{e^T e}{n} \) \( m = \frac{\lambda_1 + \lambda_n}{n} \), \( s^2 = \frac{\lambda_1 \lambda_n}{n} \) and \( I_n \) is an Identity matrix. Let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). Then

\[ -s \sqrt{nW^T CW} \leq W^T \lambda - mW^T e = W^T C \lambda \leq s \sqrt{nW^T CW} \]

\[ \sum_j (\lambda_j - \lambda_n)^2 = n[s^2 + (m - \lambda_n)^2] \]

\[ \sum_j (\lambda_1 - \lambda_j)^2 = n[s^2 + (\lambda_1 - m)^2] \]

\[ \lambda_n \leq m - \frac{s}{\sqrt{n} - 1} \leq m + \frac{s}{\sqrt{n} - 1} \leq \lambda_1. \]
**Theorem 1.** Let $A$ be a $n \times n$ complex matrix. Conjugate transpose of $A$ denoted by $A^*$. Let $B = AA^*$ whose eigenvalues are $\lambda_1(B) \geq \lambda_2(B) \geq \cdots \geq \lambda_n(B)$. Then

\[
m - s\sqrt{n - 1} \leq \lambda_2^2(B) \leq m - \frac{s}{\sqrt{n - 1}}
\]

and

\[
m + \frac{s}{\sqrt{n - 1}} \leq \lambda_1^2(B) \leq m + s\sqrt{n - 1}
\]

where $m = \frac{tr B}{n}$ and $s^2 = \frac{tr B^2}{n} - m$.

## 2 Main Results for Normalized Laplacian Matrix

**Theorem 2.** Let $G$ be a simple graph and $\mathcal{L}(G)$ be a normalized Laplacian matrix of $G$. If the eigenvalues of $\mathcal{L}(G)$ are $\lambda_1(\mathcal{L}(G)) \geq \lambda_2(\mathcal{L}(G)) \geq \cdots \geq \lambda_n(\mathcal{L}(G))$, then

\[
\lambda_n(\mathcal{L}(G)) \leq \sqrt{\left(1 + \frac{2}{n} \sum_{i\sim j, i<j} \frac{1}{d_id_j} \right)^2 + \frac{tr[L(G)]^4 - nm^2}{n(n-1)}} \tag{5}
\]

\[
\lambda_1(\mathcal{L}(G)) \geq \sqrt{\left(1 + \frac{2}{n} \sum_{i\sim j, i<j} \frac{1}{d_id_j} \right)^2 + \frac{tr[L(G)]^4 - nm^2}{n(n-1)}} \tag{6}
\]

\[
\lambda_1(\mathcal{L}(G)) \leq \sqrt{1 + \frac{2}{n} \sum_{i\sim j, i<j} \frac{1}{d_id_j} + \left(\frac{tr[L(G)]^4}{n} - m^2\right)(n-1)} \tag{7}
\]

**Proof.** Obviously,

\[
tr[L(G)]^2 = n + 2 \sum_{i\sim j, i<j} \frac{1}{d_id_j}
\]

and

\[
tr[L(G)]^4 = \sum_{i=1}^n \left(1 + \sum_{i\sim j} \frac{1}{d_id_j}\right)^2 + 2 \sum_{i<j} \left(\sum_{k \in N_i \cap N_j} \frac{1}{d(kd_id_j)} - \sum_{i\sim j} \frac{2}{d_id_j}\right)^2
\]

Since $\mathcal{L}(G)$ real symmetric matrix, we find the result from Theorem 1.

**Example 1.** Let $G = (V, E)$ with $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{(1, 2), (1, 5), (2, 3), (2, 4), (2, 6), (3, 4), (3, 5), (4, 5), (5, 6)\}$.

| $\lambda(\mathcal{L}(G))$ | (4) | (6)(lower bound) | (7)(upper bound) |
|--------------------------|-----|-----------------|-----------------|
| 1.86                     | 2   | 1.34            | 1.93            |
3 Main Results for Signless Laplacian Matrix

Theorem 3. Let $G$ be a simple graph and $Q(G)$ be a signless Laplacian matrix of $G$. If the eigenvalues of $Q(G)$ are $\lambda_1(Q(G)) \geq \lambda_2(Q(G)) \geq \cdots \geq \lambda_n(Q(G))$, then

$$\lambda_n(Q(G)) \leq \sqrt{\left(1 + \frac{2}{n} \sum_{i \sim j, i < j} \frac{1}{d_i d_j}\right) + \sqrt{\frac{\text{tr}[Q(G)]^4 - nm^2}{n(n-1)}}}$$

(8)

$$\lambda_1(Q(G)) \geq \sqrt{\left(1 + \frac{2}{n} \sum_{i \sim j, i < j} \frac{1}{d_i d_j}\right) + \sqrt{\frac{\text{tr}[Q(G)]^4 - nm^2}{n(n-1)}}}$$

(9)

$$\lambda_1(Q(G)) \leq \sqrt{1 + \frac{2}{n} \sum_{i \sim j, i < j} \frac{1}{d_i d_j} + \sqrt{\left(\frac{\text{tr}[Q(G)]^4}{n} - m^2\right)(n-1)}}$$

(10)

Proof. Clearly

$$\text{tr}[Q(G)]^2 = n + 2 \sum_{i \sim j, i < j} \frac{1}{d_i d_j}$$

and

$$\text{tr}[Q(G)]^4 = \sum_{i=1}^{n} \left(1 + \sum_{i \sim j} \frac{1}{d_i d_j}\right)^2 + 2 \sum_{i < j} \left(\sum_{k \in N_i \cap N_j} \frac{1}{d_k \sqrt{d_i d_j}} - \sum_{i \sim j} \frac{2}{d_i d_j}\right)^2$$

Since $Q(G)$ real symmetric matrix, we found the result from Theorem 1.

Example 2. Let $G = (V, E)$ with $V = \{1, 2, 3, 4, 5, 6, 7\}$ and $E = \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (2, 3), (3, 5), (4, 5), (4, 6)\}$. 

| $\lambda(Q(G))$ | (1) | (2) | (3) | (9)(lower bound) | (10)(upper bound) |
|------------------|-----|-----|-----|------------------|-------------------|
| 7.67             | 9.08| 9.74| 9.34| 4.58             | 7.76              |

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