Stationary properties of a Brownian gyrator with non-Markovian baths

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(Dated: February 20, 2020)

Abstract

We investigate the stochastic behavior of a two-temperature Langevin system with non-Markovian thermal reservoirs. The model describes an overdamped Brownian particle in a quadratic potential and coupled to heat baths at different temperatures. The reservoirs are characterized by Gaussian white and colored noises and a dissipation memory kernel. The stationary states present non-trivial average rotational motion influenced by stochastic torques due to harmonic, friction and fluctuating thermal forces. However, the Markovian limit leads to a vanishing average torque produced by fluctuating thermal forces. We also study the effects of memory on the stochastic heat and the entropy production in the steady-state regime.
I. INTRODUCTION

Many-body systems in thermodynamic equilibrium are characterized by distinct fundamental relationships between few macroscopic quantities, despite the huge number of microscopic degrees of freedom \[1, 2\]. Nevertheless, as length-scales and time-scales are reduced, non-trivial fluctuations play a relevant role in the microscopic and mesoscopic domains. In fact, the investigation of these effects is a very important research topic for the understanding and technological developments of microscopic engines, in noisy environments \[3, 12\].

Thermal fluctuations on small-scale objects can be studied through a stochastic device proposed by Filliger and Reimann \[13\], which is called Brownian gyrator. This system consists of a Brownian particle trapped in a two-dimensional harmonic well and coupled to thermal baths at distinct temperatures. The reservoirs act along orthogonal spatial directions. The quadratic potential is supposed to be asymmetric, with principal axes different from the perpendicular directions associated with baths (see Fig. 1).

Brownian gyrators present interesting non-equilibrium properties which are described theoretically \[13–16\] and also in the context of experiments \[17, 18\]. The interplay between asymmetric potentials and orthogonal spatial directions associated with thermal baths may give rise to emergent rotational motion, associated with the probability current \[14\]. As the Brownian particle is coupled to heat reservoirs at different temperatures, the system behaves as a Brownian engine that exhibits systematic gyration (about the potential minimum) induced by thermal fluctuations.

From a theoretical point of view, Brownian gyrators are modeled as two-temperature Langevin dynamics in a quadratic potential \[13–16\]. In fact, many studies analyze the problem in terms of a Brownian particle with Markovian thermal reservoirs \[13–16\]. However, the presence of time-correlated stochastic forces can also lead to out-of-equilibrium properties \[19, 20\]. Since the interesting fluctuating-induced behavior of Brownian gyrators are related to non-equilibrium effects, it could be interesting to consider the influence of memory on these stochastic systems.

In this work, we consider a Brownian gyrator coupled to non-Markovian thermal reservoirs. The system is treated in the overdamped approximation, along the lines of discussions presented in \[19, 22\]. The long-time limit leads to out-of-equilibrium stationary states as bath temperatures are distinct. The steady-state rotational motion is affected by torques
produced by thermal fluctuating and friction forces, in addition to the torque exerted by harmonic forces. This is in contrast to the Markovian limit, which gives a vanishing mean torque due to thermal fluctuating forces \[13\]. Memory also affects the stochastic heat exchanged with baths as well as the entropy production in the steady-state regime.

The paper is organized as follows. In Sec. II we introduce the Brownian gyrator through a two-temperature Langevin system. The probability density for stationary states is presented in Sec. III. The average gyration properties are calculated in Sec. IV. We discuss, in Sec. V, the stochastic heat exchanges. The entropy variation of steady-states is determined in Sec. VI. Finally, we present the conclusions in Sec. VII.

II. TWO-TEMPERATURE LANGEVIN DYNAMICS

We consider an overdamped Brownian particle moving in a two-dimensional space. The system is described by two degrees of freedom, \(x_1\) and \(x_2\), and a potential field given by

\[
V(x) = \frac{k}{2} x^T \begin{pmatrix} 1 & u \\ u & 1 \end{pmatrix} x,
\]

where \(x = (x_1, x_2)\), \(x^T\) is the transpose of \(x\), \(k > 0\) and \(u\) are parameters. In order to characterize a harmonic trap, we assume that \(-1 < u < 1\), which leads to a potential landscape with global minimum at the origin. For non-trivial values of the parameter \(u\), the principal axes \(Y_i\) of the harmonic potential do not coincide with the Cartesian frame \(x_i\), as represented in Fig. I. This possibility of different eigenframes is an important ingredient that affects the stationary behavior of the model \[13\].

The dynamical evolution of the position variables is given by the Langevin equations

\[
\gamma \dot{x}_i + \int_0^t dt' K(t - t') \dot{x}_i(t') = -\partial_i V + f_i,
\]

where \(\gamma\) is a friction coefficient, \(K(t)\) is the dissipative memory kernel, and \(f_i\) is an stochastic force related to the thermal environment, with temperature \(T_i\), acting on \(x_i\). The initial conditions are \(x_i(0) = 0\) and \(\dot{x}_i(0) = 0\). The stochastic system described by (2) is a prototype of Brownian gyrators \[13\] \[16\] \[18\] \[23\] including non-Markovian properties associated with baths. We assume the \(f_i\) to be Gaussian noises, with temporal correlations \[19\] \[21\] \[24\], \(\text{3}\)
FIG. 1. Brownian motion with distinct thermal reservoirs. Position variable $x_i$ is affected by a bath at temperature $T_i$. The harmonic potential presents principal axes $Y_i$.

with cumulants given by (with $k_B = 1$)

$$
\langle f_i(t) \rangle_c = 0,
$$

$$
\langle f_i(t) f_j(t') \rangle_c = T_i \delta_{ij} [2\gamma \delta(t - t') + K(t - t')],
$$

$$
K(t) = \frac{\Gamma}{\tau} \exp \left(-\frac{|t|}{\tau}\right),
$$

where $T_i$ is the effective temperature of bath $i$, with $i = 1, 2$. $\Gamma$ is a friction coefficient, $\tau$ is the correlation time-scale, and $\delta_{ij}$ is the Kronecker delta. We assume that the noises are independent, for simplicity sake. Langevin forces described by (3) present distinct time-scales effects associated with thermal baths, which have been used to explore physical systems with fast and slow stochastic dynamics [19–21, 24].

For our Brownian particle in a harmonic trap, each non-Markovian bath is coupled to a specific position variable, but thermal fluctuations affect both degrees of freedom due to the interaction $V(x_1, x_2)$. The limiting case of vanishing coupling parameter, $u \to 0$, leads to the $x_i$ evolving independently of each other, under the influence of a single reservoir. As a result, if bath temperatures are equal, we find equilibrium states characterized by a canonical distribution [19, 20, 22]. Otherwise, for non-zero values of $u$ and distinct effective temper-
atures, we expect the system to achieve a long-time behavior with rich out-of-equilibrium properties.

We investigate the two-temperature Brownian system [2] through time-averaging procedures [22, 25–27]. These techniques are very appropriate to deal with linear Langevin dynamics with general types of noises [28–30], and even systems with non-linear force fields [31–33]. The main idea is to study the temporal evolution of average quantities such as moments and/or cumulants by means of integral transformations. Here, we focus on the cumulants of the degrees of freedom $x_i$. The main point is that taking averages related to $x_i$ correspond to taking averages of quantities associated with thermal noises. Technical details are shown in Appendix A. This approach allows us to study many distinct stationary properties of the system, such as the distribution function and the average heat exchanged with reservoirs.

In the next section, we investigate the long-time behavior of the joint probability density related to position variables.

III. PROBABILITY DENSITY

We are interested in determining the distribution associated with $x_i$, in order to characterize the stochastic behavior of the system. Along the lines of the formalism we are adopting, it is interesting to write the instantaneous distribution function as [22, 27]

$$P(x, t) = \langle \delta (x - x(t)) \rangle,$$

(4)

where the averages $\langle \cdot \rangle$ are taken over the noise realizations. Then, we express the delta functions as Fourier integrals,

$$\delta (x - x(t)) = \int \frac{d^2z}{4\pi^2} \exp \left[ i z^T (x - x(t)) \right].$$

(5)

This gives us the probability density in terms of a Fourier transform of the characteristic function. However, since we defined the stochastic forces in terms of noise cumulants, it is more appropriate to deal with the cumulant-generating function, which is obtaining taking
the logarithm of characteristic function. As a result, we find

\[ P(x, t) = \int \frac{d^2z}{4\pi^2} \exp \left[ iz^T x + \mathcal{K}(z, t) \right], \quad (6) \]

where

\[ \mathcal{K}(z, t) = -\frac{1}{2} z^T \begin{pmatrix} \langle x_1^2(t) \rangle_c & \langle x_1(t) x_2(t) \rangle_c \\ \langle x_1(t) x_2(t) \rangle_c & \langle x_2^2(t) \rangle_c \end{pmatrix} z, \quad (7) \]

is the instantaneous join cumulant-generating function. However, we may write the cumulants of each \( x_i \) as a linear superposition of the cumulants of stochastic forces. Then, we can use these cumulants to obtain (7) and, consequently, an analytic expression for the join probability density.

### A. Stationary cumulants

The harmonic potential as well as the Gaussian nature of the noises lead to a full investigation about the instantaneous cumulants. These quantities present many transient effects, which are irrelevant as the system achieves stationary states, which is the regime we are interested in. Then, for the long-time limit \( t \to \infty \), we can write steady-states cumulants as

\[ b_{ij} = \lim_{t \to \infty} \langle x_i(t) x_j(t) \rangle_c. \quad (8) \]

Technically, these stationary cumulants are obtained by considering contour integration methods, as discussed in Appendix B. Then, alter all calculations, we have

\[ b_{11} = \frac{\zeta (2 - u^2)}{2k (1 - u^2)} \left[ \zeta + k\tau \Gamma \right] T_1 + \frac{(\zeta + k\tau \Gamma) u^2 T_2}{2k (1 - u^2) \left[ \zeta + k\tau \Gamma (2 - u^2) \right]}, \quad (9) \]

\[ b_{22} = \frac{(\zeta + k\tau \Gamma) u^2 T_1 + \left[ \zeta (2 - u^2) + k\tau \Gamma (4 - 3u^2) \right] T_2}{2k (1 - u^2) \left[ \zeta + k\tau \Gamma (2 - u^2) \right]}, \quad (10) \]

\[ b_{12} = -\frac{u (T_1 + T_2)}{2k (1 - u^2)}, \quad (11) \]
where
\[ \zeta = (\gamma + \Gamma)^2 + k\tau \left[ 2\gamma + k\tau \left( 1 - u^2 \right) \right], \]  
(12)
is a positive quantity. It is clear that these cumulants contain all information about the
non-Markovian effects on the fluctuations associated with position variables. Notice that \( b_{11} \)
and \( b_{22} \) are positive, and \( b_{12} \) is negative.

**B. Stationary distribution**

Since we calculated all cumulants for stationary states, we can express the steady-state
join distribution as
\[ P_s(x) = \frac{1}{2\pi \sqrt{\det B}} \exp \left( -\frac{1}{2} x^T B^{-1} x \right), \]  
(13)
where
\[ B = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix}, \]  
(14)
and \( B^{-1} \) and \( \det B \) are the inverse and the determinant of \( B \), respectively. The general
stationary state is given by a non-equilibrium Gaussian distribution. Nevertheless, it is
straightforward to perceive that equilibrium states are achieved whenever the heat reservoirs
present the same temperatures, \( T_i \to T \),
\[ P_s(x) \to \frac{k \sqrt{1 - u^2}}{2\pi T} \exp \left[ -\frac{V(x)}{T} \right]. \]  
(15)
which is the canonical distribution for the two-dimensional linear system defined by the
potential \( [1] \).

Another important special case is obtained by taking the Markovian limit \( \tau \to 0 \). For
this case, we have
\[ b_{11} \to \frac{2T_1 - (T_1 - T_2) u^2}{2k (1 - u^2)}, \]  
(16)
\[ b_{22} \rightarrow \frac{2 T_2 + (T_1 - T_2) u^2}{2k (1 - u^2)}, \tag{17} \]
\[ b_{12} \rightarrow -\frac{u (T_1 + T_2)}{2k (1 - u^2)}, \tag{18} \]

These expressions allow us to write a stationary distribution which is in agreement with earlier works on Brownian gyrators with thermal white noises at different temperatures \[14, 15, 27\]. That memoryless case may present some interesting properties due to the mismatch between the principal axes associated with the distribution function and the underlying harmonic potential \[14, 15\]. For stationary states, the system can exhibit a complex probability flux and non-zero mean angular velocity. One can say that non-equilibrium fluctuations induce stochastic rotation as the bath temperatures are different \[13, 15, 18, 23, 34\].

Although we have a Brownian system with time-correlated thermal noises, it is possible to check that the potential energy \([1]\) and the probability function \([13]\) present different principal axes. This mismatch condition can also lead to effective rotational behavior. Then, it is worth investigating how non-Markovian baths can influence the induced stochastic rotation of the system.

In the next section, we analyze the effects of memory on the average rotational quantities.

**IV. STOCHASTIC GYRATION PROPERTIES**

We want to study the effects of non-Markovian baths on the average rotational properties of the system. This is possible by considering the torque due to all forces acting on the Brownian particle. It is important to mention that earlier theoretical studies describe Brownian gyrators with solely thermal white noises \[13, 15, 18, 23, 34\]. In this work with non-Markovian reservoirs, one can shows that, by evaluating the torques \(\mathcal{T}\) about the origin of all forces, the overdamped Langevin system \([2]\) leads to a rotational equation of motion of the type

\[ \mathcal{T}_{\text{diss}} + \mathcal{T}_V + \mathcal{T}_f = 0, \tag{19} \]

where the torques are given by

\[ \mathcal{T}_{\text{diss}} = -\gamma \varepsilon_{ij} x_i \dot{x}_j - \int_0^t dt' K(t - t') \varepsilon_{ij} x_i (t) \dot{x}_j (t'), \tag{20} \]
\[ T_V = -\varepsilon_{ij} x_i \partial_j V, \] (21)

\[ T_f = \varepsilon_{ij} x_i f_j, \] (22)

and \( \varepsilon_{ij} \) is a two-index Levi-Civita symbol, proper for a two dimensional geometry. We also assume Einstein summation convention. The presence of a memory kernel for dissipation implies that \( T_{\text{diss}} \) should be interpreted as the torque due to friction forces. Otherwise, \( T_V \) and \( T_f \) are the torques exerted by harmonic and thermal fluctuating forces, respectively.

It is possible to identify, through relationships between angular momentum and angular velocity, a quantity called weighted angular velocity \[23\], which is the angular velocity (about the origin) multiplied by the squared distance form the origin. The weighted angular velocity is closely related to the dissipative torque \( T_{\text{diss}} \). The average of \( T_{\text{diss}} \) may be evaluated by means of the noise averages and integral transformations, as we did for the stationary cumulants. If we apply similar techniques to find the average angular velocity \[14, 15, 18\], probably we would obtain a complicated mathematical structure. Also, for our model system with memory kernel, a direct study of \( T_{\text{diss}} \) would lead to cumbersome expressions. Nevertheless, the balance equation \[19\] indicates that the torque due to friction can be estimated by using the torques produced by harmonic and thermal forces. These quantities are feasible to calculate and analyze as follows.

**A. Average torque due to harmonic force**

We obtain analytic expressions for all average torques by using time-averaging treatments. In particular, harmonic forces contribute to the average rotation through the stationary state relation

\[ \langle \varepsilon_{ij} x_i \partial_j V \rangle_s = ku \left( \langle x_1^2 \rangle_c - \langle x_2^2 \rangle_c \right), \] (23)

which is related to the variances of position variables. The subscript \( s \) stands for the long-time limit. Since we already calculated those quantities for stationary states in IIIA, we may write

\[ \langle T_V \rangle_s = \frac{u (T_2 - T_1) (\zeta + 2k\tau \Gamma)}{\zeta + k\tau \Gamma (2 - u^2)}, \] (24)
where $\zeta$ is given by (B5). The average torque (24) depends on the difference between the temperatures of thermal baths. The same is true for the others average torques. This dependence indicates the importance of non-equilibrium thermal fluctuations on the stochastic rotation. We also notice a non-linear dependence on the coupling parameter $u$, in addition to contributions coming from noise temporal correlations.

It is worth mentioning that the simplest case of Brownian gyrator with thermal white noises exhibits a linear dependence on the coupling parameter $u$. In fact, by taking the Markovian limit of the average harmonic torque, we find

$$\langle T_V \rangle_s \to u (T_2 - T_1) \quad \text{as} \quad \Gamma, \tau \to 0,$$

This is consistent with the memoryless case of an overdamped Brownian system, at different effective temperatures, in a two-dimensional confining potential \[13, 23\].

**B. Average torque due to thermal noises**

It is reasonable that non-Markovian baths affect the instantaneous behavior of all stochastic torques. Nevertheless, we find that thermal fluctuating forces also contribute non-trivially to the steady-state rotational properties. As shown in Appendix C, thermal noises produce an average torque of the type

$$\langle T_f \rangle_s = \frac{4\Gamma \tau k u (T_1 - T_2)}{[2 (\gamma + k \tau) + \Gamma]^2 - 4 (k \tau u)^2}.$$

Notice the dependence on friction constant $\Gamma$ and persistence $\tau$. The average thermal torque (26) is nontrivial as $\Gamma$ and $\tau$ are non-zero. This is very different from the Markovian limiting case, which leads to a null average thermal torque \[13, 23\],

$$\langle T_f \rangle_s \to 0 \quad \text{as} \quad \Gamma, \tau \to 0.$$

The average torque exerted by friction forces is directly related to the average torque produced by harmonic forces if memory effects are vanishing. However, for the general non-Markovian case, both dissipative and fluctuating forces may lead to relevant contributions to the average rotation.
Therefore, it seems that Brownian gyrators with friction memory kernel may exhibit systematic rotational dynamics distinct from the Markovian case. Noise temporal correlations can also influence the energetic balance and heat exchanges with thermal environments. Then, it is important to develop a detailed analysis about the energetic considerations of the model as well as its irreversibility properties.

In the next section, we study how memory may affect the behavior of the stochastic energetics of the system.

V. STEADY-STATE HEAT EXCHANGE

We can also investigate the stochastic behavior of the heat flow and energetic variations for the our two-temperature Langevin dynamics. In order to do that, we follow the stochastic thermodynamics formalism discussed by Sekimoto [35, 36]. For this kind of formalism, it is more appropriate to adopt calculus manipulation in Stratonovich sense [37, 38].

Based on the overdamped Langevin equations (2), which present trivial external work contributions
\[ d'W = dV - d'Q_1 - d'Q_2 = 0, \]
we can write the stochastic heat exchanged as
\[ d'Q_i = \partial_i V dx_i = J_i dt, \]
where
\[ J_i = f_i \dot{x}_i - \gamma \dot{x}_i^2 - \int_0^t dt' K(t - t') \dot{x}_i(t) \dot{x}_i(t'). \]

is the heat flux related to bath \( i \), at temperature \( T_i \). The stochastic equation (28) represents the energetic balance for the coordinate \( x_i \), and (30) is interpreted as the power contributions of thermal baths. Noise temporal correlations contribute to the stochastic heat by means of the friction memory kernel. The heat flow in (30) is consistent with the kind of non-Markovian system defined by (2). For vanishing contributions of the memory kernel, which
is corresponds to $\Gamma \to 0$ in (3), we obtain the memoryless case

$$J_i \to f_i \dot{x}_i - \gamma \dot{x}_i^2 \quad \text{as} \quad \Gamma \to 0,$$

which is in agreement with the stochastic heat flux obtained from a Brownian system with Markovian thermal baths [35, 36].

The balance equation (28) relates the dissipative and injected power of reservoirs to the internal energy variation. The stochastic heat exchanges during a time interval between $t$ and $t + t_o$ is given by

$$Q_i = \int_t^{t + t_o} dt' \, \dot{x}_i \, \partial_i V.$$

With these expressions, it is straightforward to write a stochastic version of the first law of the thermodynamics, which reads $Q_1 + Q_2 = V (t + t_o) - V (t)$. This relation depends on the trajectory along which the system evolves from $t$ to $t + t_o$. However, our main interest is to analyze the average properties in the steady-state regime. As a result, for the long-time limit, we have

$$\langle Q_1 \rangle_s + \langle Q_2 \rangle_s = 0, \quad \text{as} \quad t \to \infty,$$

which is a constraint on the average heat flowing from one bath to the other as the stationary state is achieved. From a technical perspective, the constraint (33) allows us to investigate the heat exchanges through the system in terms of a specific position variable or reservoir. Then, for simplicity, we choose $x_1$ (or reservoir 1) to work with.

It is possible to develop many mathematical calculations for the linear model we are studying, specially for the steady-state stochastic energetics (technical details shown in Appendix D). Then, by considering the long-time limit $t \to \infty$, we find the stationary average heat

$$\langle Q_1 \rangle_s = \frac{(T_1 - T_2) \, t_o k u^2 \Psi}{2 \gamma}.$$

with

$$\Psi = \frac{\gamma (\gamma + \Gamma) + k \tau [2 \gamma + \Gamma + k \tau (1 - u^2)]}{(\gamma + \Gamma)^2 + k \tau [2 \gamma + \Gamma + (\Gamma + k \tau) (1 - u^2)]}.$$
FIG. 2. Stationary average heat flow through the system. For $T_1 > T_2$ in long-time limit, the system absorbs, on average, an amount of heat from reservoir 1 and releases the same amount into reservoir 2.

This is valid for any values of $u^2 < 1$. Notice the positive quantity $\Psi$ is less than unit. It is important to bear in mind that the system is simultaneously in contact with thermal baths at different temperatures. As a result, for $T_1 > T_2$, average heat (related to a time interval $t_o$) is absorbed from reservoir 1 and delivered into reservoir 2, as depicted in Fig. 2. In fact, the expression (34) is consistent with the interpretation that, for heat flowing into the system through $x_1$, the average heat exchange $\langle Q_1 \rangle_s$ should be positive whenever $T_1 > T_2$. Another point is that (34) exhibits a very complicated dependence on the noise temporal correlations as well as the interaction parameter $u$.

Clearly, the average heat (34) may be null by taking equal bath temperatures, provided that $u$ is non-zero, for any finite values of friction constants and persistence time-scale. For the particular case of Markovian thermal baths, it is possible to recover results consistent with the memoryless Brownian gyrator. In fact, this can be done by taking the limit $\Gamma \to 0$ in (35), which leads to $\Psi \to 1$. Then, we find

$$\langle Q_1 \rangle_s \to \frac{(T_1 - T_2) t_o k u^2}{2\gamma},$$

which is in agreement with the average heat obtained from a two-temperature Langevin system with Gaussian white noises [35, 36]. Notice that by taking the limit $\tau \to 0$ in (35), we find an average heat with friction coefficient $\gamma + \Gamma$.

As each degree of freedom is affected by a single bath with a specific temperature $T_i$, the whole system experiences an irreversible heat exchange with reservoirs, provided that
$T_1 \neq T_2$. In the next section, we consider the entropic changes of the model.

VI. STATIONARY ENTROPY VARIATION

The irreversibility aspects of our Langevin system is characterized by determining the total entropy generation for the long-term run. In order to do that, we follow the approach developed previously by the authors [27], which studied a similar two-temperature Brownian system with friction memory kernel. Accordingly, we assume that, during the time interval between $t$ and $t + t_o$, the total entropy change of the the system and the medium is given by

$$\Delta S = \Delta S_{sys} + \Delta S_m,$$

(37)

where $\Delta S_{sys}$ is the change of entropy associated with the system, which is the Brownian particle subjected to the harmonic potential, and $\Delta S_m$ is the entropy variation related to the thermal medium, which are the non-Markovian heat reservoirs.

We define, as usual, the entropy of the system in term of the well-known Gibbs-Shannon entropy formula [39, 40] (we assume Boltzmann constant $k_B = 1$)

$$S_{sys}(t) = -\int d^2x P(x, t) \ln P(x, t),$$

(38)

where $P(x_1, x_2, t)$ is the time-dependence join distribution [6]. Clearly, the entropy variation of the system is of the type $\Delta S_{sys} = S_{sys}(t + t_o) - S_{sys}(t)$. The entropy change of the thermal medium is associated with the stochastic heat exchanges relate to thermal reservoirs [6]. Since we are interested in the average properties, we can assume

$$\Delta S_m = -\frac{\langle Q_1 \rangle}{T_1} - \frac{\langle Q_2 \rangle}{T_2},$$

(39)

which is the average version of entropy variation associated with the stochastic heat flowing through the thermal environment. It is worth mentioning that the definition of stochastic heat in (30) indicates that a positive heat quantity accounts to increase the internal energy of the system.

The main interest here is to determine the stationary behavior of the total entropy variation. However, as the system achieves the steady-state regime, the probability distribution
becomes time-independent. As a result, for the long-term run, it straightforward to perceive that the entropy change of the system is typically null, since $\Delta S_{sys} \to 0$ as $t \to \infty$. This means that, for stationary states, there exists an entropic balance between internal system and surroundings. On the other hand, the entropy change of the thermal medium is related to the steady-state heat exchanges, which gives

$$\Delta S_m^s = \frac{T_1 - T_2}{T_1 T_2} \langle Q_1 \rangle_s. \quad (40)$$

Here, we use the first law form in (33), which shows that thermal energy injected by one reservoir is, on average, dissipated into the other.

Therefore, combining all relevant entropic contributions for the stationary states, we obtain

$$\Delta S_{tot} \to \Delta S_m^s = \sigma t_o \quad as \quad t \to \infty, \quad (41)$$

where

$$\sigma = \frac{(T_1 - T_2)^2 k u^2 \Psi}{2 \gamma T_1 T_2}, \quad (42)$$

is the total steady-state entropy production rate, which is clearly a non-negative quantity. The parameter $\Psi$ given by (35) is a positive quantity. Notice that the persistence parameter $\tau$ and the coupling parameter $u$ affect the entropy generation. In fact, the non-Markovian properties of the baths play a role in the heat flow and, consequently, the steady-state entropy behavior of the system. We can obtain results in agreement with equilibrium statistical thermodynamics for non-zero values of $u$ and equal bath temperatures. In this case, microscopic reversibility is recovered and stationary states are characterized by a Boltzmann-Gibbs distribution, as shown in [IIIB]

The rate of entropy production (42) is consistent with the particular case of a Brownian dynamics with white noises and distinct temperatures. This limiting case is obtained straightforwardly by considering $\Gamma \to 0$ in (42). Consequently, we find

$$\sigma \to \frac{(T_1 - T_2)^2 k u^2}{2 \gamma T_1 T_2} \quad as \quad \Gamma \to 0. \quad (43)$$
This entropy generation rate is due to a non-zero average heat flowing through the system, which leads to out-of-equilibrium stationary states, stressing the role heat reservoirs at distinct temperatures play.

Earlier works also discuss the effects of memory on the irreversibility associated with Langevin systems with many thermal baths [19, 24]. Our model presents qualitative results analogous to those investigations. For example, we find that the memory effect contributes to the stationary entropy generation, but these contributions are trivial as the bath temperatures become equal, even for finite values of noise temporal correlations $\tau$. This is also observed in a one-dimensional Brownian system coupled to many heat reservoirs [19, 20].

Langevin dynamics with multiple reservoirs present intriguing physical properties. For simple linear models analogous to the one studied in this work, we believe it is important to develop further investigations with the inclusion of inertia, distinct kinds of non-Markovian noises, and athermal reservoirs. Also, it would be very interesting to consider model systems with non-linear force fields, where it is possible to identify some noise-induced effects [41–43].

VII. CONCLUSIONS

We study an elementary Brownian gyrator with non-Markovian thermal baths. The joint probability density associated with position variables is calculated analytically by means of a time-averaging formalism. We find that memory affects the stochastic rotational properties in the long-time limit. For a finite memory time-scale, we show there exists non-trivial average torques due to friction forces, harmonic and fluctuating thermal forces. Nevertheless, for the memoryless limit, we observe a null average torque due to thermal noises. The stochastic energetic properties of the model is investigated for steady-states. The average heat and the entropy production show a memory-dependent behavior. As the baths present different temperatures, memory contributes to the long-time behavior of the entropy generation. Otherwise, for reservoirs with equal temperatures, the entropy production is null, even for finite memory time-scale, and equilibrium is recovered.
ACKNOWLEDGMENT

This work is supported by the Brazilian funding agencies CNPq and CAPES (Finance Code 001).

Appendix A: Frequency-domain representation of Langevin equations

Following [22, 27], we introduce the integral form

\[ x_j(t) = \lim_{\epsilon \to 0} \int dq \frac{dq}{2\pi} e^{(iq+\epsilon)t} \hat{x}_j(iq+\epsilon), \quad (A1) \]

where \( \hat{x}_j(s) \) is is the Laplace transform of \( x_j(t) \). Now, we take the Laplace transform of the Langevin equations, which gives

\[ \hat{x}_1 = \frac{1+\tau s}{p} \left\{ \left[ \gamma \tau s^2 + (\gamma + \Gamma + k\tau) s + k \right] \hat{f}_1 - ku (1+\tau s) \hat{f}_2 \right\}, \quad (A2) \]

\[ \hat{x}_2 = \frac{1+\tau s}{p} \left\{ -ku (1+\tau s) \hat{f}_1 + \left[ \gamma \tau s^2 + (\gamma + \Gamma + k\tau) s + k \right] \hat{f}_2 \right\}, \quad (A3) \]

where

\[ p = p_+ p_-, \]

\[ p_\pm = \gamma \tau s^2 + [\gamma + \Gamma + k\tau (1 \pm u)] s + k (1 \pm u). \quad (A4) \]

We also need the Laplace transforms of noise cumulants

\[ \left\langle \hat{f}_i(s_1) \hat{f}_j(s_2) \right\rangle_c = \frac{T_i \delta_{ij}}{s_1 + s_2} \left\{ 2\gamma + \frac{[2 + \tau (s_1 + s_2)]\Gamma}{(1+\tau s_1)(1+\tau s_2)} \right\}. \quad (A5) \]

Therefore, in frequency-domain representation, the cumulants related to position variables are written as linear combination of noises cumulants.
Appendix B: Variances for steady-states

We consider the change of variables

\[ y_1 = x_1 + x_2, \]
\[ y_2 = x_1 - x_2. \]  \hfill (B1)

Then, we use the integral relation (A1) and the Laplace forms of noise cumulants (A5), which give \( \langle y_i y_j \rangle_c \) as integral expressions that can be calculated applying the methods of residues. Then, for the long-time limit, one may find

\[ \lim_{t \to \infty} \langle y_1^2(t) \rangle_c = \frac{T_1 + T_2}{k (1 + u)}, \]  \hfill (B2)

\[ \lim_{t \to \infty} \langle y_1(t) y_2(t) \rangle_c = \frac{(T_1 - T_2) (\zeta + 2k\tau\Gamma)}{k (\zeta + k\tau\Gamma (2 - u^2))}, \] \hfill (B3)

\[ \lim_{t \to \infty} \langle y_2^2(t) \rangle_c = \frac{T_1 + T_2}{k (1 - u)}, \] \hfill (B4)

where

\[ \zeta = (\gamma + \Gamma)^2 + k\tau [2\gamma + k\tau (1 - u^2)]. \] \hfill (B5)

These expressions allow us to find the stationary cumulants of \( x_i \) by taking the inverse transform of \( y_i \).

Appendix C: Stochastic torque due to thermal noises

The average torque exerted by fluctuating thermal forces is given by

\[ \langle T_f \rangle = \langle x_1 f_2 - x_2 f_1 \rangle, \] \hfill (C1)

where can use the integral form (A1) in order to determine the average torque for stationary states. Then, we have

\[ \langle T_f \rangle_s = \frac{k u (T_1 - T_2)}{\pi} \lim_{\epsilon \to 0} \int dq_1 \, \frac{\gamma [1 - \tau^2 (iq_1 + \epsilon)^2]}{p_+ (iq_1 + \epsilon) p_- (iq_1 + \epsilon)} \left\{ \frac{1 + \tau (iq_1 + \epsilon)}{1 - \tau (iq_1 + \epsilon)} \right\}. \] \hfill (C2)
This integral over $q_1$ does not present any convergence problem. As a result, by using the methods of residues, we obtain
\[
\langle T_f \rangle_s = \frac{4 \Gamma \tau k u (T_1 - T_2)}{2 (\gamma + k \tau + \Gamma)^2 - 4 (k \tau)^2}.
\] (C3)

**Appendix D: Average stochastic heat**

The average heat associated with the thermal bath 1 is of the form
\[
\langle Q_1 \rangle = C_1 + C_2 + \frac{k u}{4} [\Upsilon_1 (t + t_o) - \Upsilon_1 (t)],
\] (D1)

where
\[
C_1 = \frac{k}{2} \left[ \langle x_1^2 (t + t_o) \rangle - \langle x_1^2 (t) \rangle \right],
\] (D2)
\[
C_2 = \frac{k u}{8} \left[ \langle y_1^2 (t + t_o) \rangle - \langle y_1^2 (t) \rangle + \langle y_2^2 (t + t_o) \rangle - \langle y_2^2 (t) \rangle \right],
\] (D3)

and
\[
\Upsilon_1 (t) = \int_0^t dt' (\langle y_1 \dot{y}_2 \rangle - \langle y_2 \dot{y}_1 \rangle).
\] (D4)

Then, in terms of the integral representation (A1), we can evaluate (D4) for the long-term run ($t \gg 1$), which gives
\[
\Upsilon^s_1 = \frac{(T_1 - T_2)}{\pi} \lim_{\epsilon \rightarrow 0} \int dq_1 (iq_1 + \epsilon) \left\{ \frac{\gamma [1 - \tau^2 (i q_1 + \epsilon)^2] + \Gamma}{p_+ (i q_1 + \epsilon)} - \frac{\gamma [1 - \tau^2 (-i q_1 - \epsilon)^2] + \Gamma}{p_- (-i q_1 - \epsilon)} \right\},
\] (D5)

This integral over $q_1$ does not satisfy the Jordan’s lemma, since the integrand is a rational function of the type $1/q_1$. Then, we should take into account integration along the semicircular contour, which asymptotic form gives
\[
\frac{4 \pi t (T_1 - T_2)}{\gamma} \text{ as } |q_1| \rightarrow \infty.
\] (D6)

After evaluating all residue contributions, it is possible to show that
\[
\Upsilon^s_1 = \frac{2 t u (T_1 - T_2) \hat{\phi}_1}{\gamma} \hat{\phi}_2,
\] (D7)
where

\[
\hat{\gamma}_1 = \gamma (\gamma + \Gamma) + k\tau \left[2\gamma + \Gamma + k\tau \left(1 - u^2\right)\right],
\]

and

\[
\hat{\gamma}_2 = (\gamma + \Gamma)^2 + k\tau \left[2\gamma + \Gamma + (\Gamma + k\tau) \left(1 - u^2\right)\right].
\]

The cumulants of \(x_i\) and \(y_i\) are time-independent for stationary states, which results

\[
C_1, C_2 \to 0 \quad \text{as} \quad t \to \infty.
\]

Therefore, the steady-state form of (D1) is given by

\[
\langle Q_1 \rangle_s = \frac{t_0 k u^2 (T_1 - T_2) \hat{\gamma}_1}{2\gamma} \hat{\gamma}_2.
\]

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