Newtonian analogue of corresponding space–time dynamics of rotating black holes: implication for black hole accretion

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ABSTRACT
Based on the conserved Hamiltonian for a test particle, we have formulated a Newtonian analogue of Kerr space–time in the ‘low energy limit of the test particle motion’. In principle, this can be used comprehensively to describe general relativistic (GR) features of Kerr space–time, but with less accuracy for high spin. The derived potential, which has an explicit velocity dependence, contains the entire relativistic features of corresponding space–time, including the frame dragging effect, unlike other prevailing pseudo-Newtonian potentials for the Kerr metric where such an effect is either totally missing or introduced in an ad hoc manner. The particle dynamics with this potential precisely reproduce the GR results within a maximum ~10 per cent deviation in energy for a particle orbiting circularly in the vicinity of a rapidly corotating black hole. GR epicyclic frequencies are also well reproduced with the potential, although with a relatively higher percentage of deviation. For counter-rotating cases, the obtained potential replicates the GR results with precise accuracy. The Kerr–Newtonian potential also approximates the radius of marginally stable and marginally bound circular orbits with reasonable accuracy for a < 0.7. Importantly, the derived potential can imitate the experimentally tested GR effects, such as perihelion advancement and bending of light with reasonable accuracy. Thus, the formulated Kerr–Newtonian potential can be useful to study complex accreting plasma dynamics and its implications around rotating black holes in the Newtonian framework, avoiding GR gas dynamical equations.

Key words: accretion, accretion discs – black hole physics – gravitation.

1 INTRODUCTION
Spinning black holes (BHs) have wide physical implications ranging from exotic frame dragging to controlling some of the highest energetic phenomena in the observed Universe. Astrophysical BHs that mostly exhibit in two extreme mass limits – the stellar mass BHs of ~ (5–10) M⊙ in BH X-ray binaries (BHXRBs) and supermassive BHs (SMBHs) of ≳ 10⁶ M⊙ residing in the centre of all galaxies (active galactic nuclei and quasars) – are realized in the physical Universe through the accretion of gaseous plasma around them and its related phenomena (e.g. Bisnovatyi-Kogan & Lovelace 2001; Ho 2008, and references therein). BH spin powers the accretion flow and governs the accretion dynamics, especially in the inner regions in its vicinity (Meier 1999; Bhattacharya, Ghosh & Mukhopadhyay 2010), consequently describing diverse accretion-related phenomena from quasi-periodic oscillations (QPOs; Stella & Vietri 1999; Mukhopadhyay 2009) to powering astrophysical jets (De Villers et al. 2005; Bhattacharya et al. 2010). The spin of the BH is plausibly responsible for accretion disc precession in its inner regions through the Bardeen–Peterson effect (Schawinski et al. 2007), which, in turn, regulates the precession of relativistic jets. Apart from accretion-powered jets, astrophysical jets might also be powered by the direct extraction of the rotational energy of SMBHs in active galactic nuclei (Blandford & Znajek 1977). Recent studies indicate that SMBH spin could enhance the observed luminosity in BH accreting systems by several orders of magnitude and might play a predominant role in defining observed active galactic nucleus classes (see Rajesh & Mukhopadhyay 2010; Mukhopadhyay, Bhattacharya & Sreekumar 2012; Ghosh & Konar, submitted to MNRAS, and references therein). Galactic mergers drive SMBH binaries to coalescence, determining the final state of BH spin (Rezzolla et al. 2008; Martínez-Sansigre & Rawlings 2011). The co-evolution of SMBHs (both spin and mass) and their host galaxies remains one of the outstanding problems in cosmic structure formation (Cattaneo et al. 2009). Because of the universal and indispensable nature of BH spin, its effect on astrophysical processes, such as accretion-related phenomena, cannot possibly be ignored.

BHs are exact classical solutions of field equations in Einstein’s theory of general relativity. The formulation of a precise accretion flow model around a central BH requires a combination of...
numerous physical processes, such as the following: advective two-temperature relativistic plasma dynamics; magnetohydrodynamic (MHD) turbulent diffusive terms including viscosity, resistivity and thermal conductivity; detailed radiative processes; several local physics and non-linear effects of collisionless plasma (Sharma et al. 2007; Cremaschini, Miller & Tessarotto 2012). This is a complex and tricky subject, especially accretion flow in the vicinity of BHs, where general relativistic (GR) effects are important. It becomes yet more difficult when outflows and jets are included and perturbative effects are incorporated in the flow. Such a complex physical system with GR equations often becomes inconceivable in practice. To avoid the GR fluid equations, most authors study accretion and its related processes around BHs using fluid equations in the Newtonian framework. Notwithstanding, authors often use a simple Newtonian potential without considering the essential GR effects in investigating Keplerian accretion dynamics around non-rotating BHs (Shakura & Sunayev 1973; Pringle 1981). The only impression it accommodates from general relativity is that the innermost edge of the disc truncates at the marginally stable circular orbit of Schwarzschild geometry. The use of a spherically symmetric Newtonian potential often gives satisfactory results for accretion phenomena around non-rotating BHs within a limit of accuracy, excluding the very inner regions of the disc where GR effects are important. However, the exterior solution of a rotating BH, whose spin is purely a GR effect, is described by Kerr geometry, which, without having any spherical symmetry, does not have any Newtonian analogue, unlike the Schwarzschild metric. The only recourse authors employ is the pseudo-Newtonian approach by taking into account a few relativistic features of Kerr geometry and accommodating it into gas dynamical equations in the Newtonian framework, in order to avoid cumbersome GR equations.

Pseudo-Newtonian potentials (PNPs) have been extensively used in the astrophysical literature, especially with regard to accretion flow around BHs after the leading work of Paczyński & Witt (1980), hereafter PW80. The corresponding potential, although introduced ad hoc, quite precisely reproduces the last stable circular orbit in Schwarzschild geometry and has been widely featured in the literature to study accretion dynamics around non-rotating BHs. Several other PNPs have been proposed for accretion flows, either to describe episodic frequency or the fluid dynamical aspect of rotating as well as non-rotating BHs in the equatorial plane (Nowak & Wagoner 1991; Artemova, Bjórnsson & Novikov 1996; Mukhopadhyay & Misra 2003, hereafter MM03). Mukhopadhyay (2002), hereafter M02, prescribed a PNP to describe the fluid dynamics of accretion flow around a rotating BH in the equatorial plane, deriving directly from the Kerr metric. Based on this method, Ghosh (2004) developed a PNP corresponding to the Hartle–Thorne metric, which describes an exterior solution of a rotating hard surface. Ghosh & Mukhopadhyay (2007), hereafter GM07, formulated a generalized pseudo-Newtonian vector potential, useful for studying accretion gas dynamics around a rotating BH in the equatorial plane. Both these PNPs (M02, GM07), which have been methodologically derived from the metric itself, are found to be valid for the entire regime of the Kerr parameter; however, perturbative effects and epicyclic frequencies are not best described by them. None the less, both the PNPs of M02 and GM07 have been used in several hydrodynamical/hydromagnetic accretion studies with admirable success (Chan, Psaltis & Özel 2005; Lipunov & Gorbovskoy 2007; Shafee, Narayan & McClintock 2008; Bhattacharya et al. 2010). A few other PNPs have also been proposed ad hoc to describe the generalized Kerr geometry (Semerák & Karas 1999; Chakrabarti & Mondal 2006).

Although PNPs mimic a few GR features of corresponding space-times to a certain extent, a single PNP corresponding to a particular metric still lacks the uniqueness to describe all GR effects simultaneously, with reasonable accuracy. Unlike the PNPs of M02 and GM07, most PNPs are arbitrarily proposed in an ad hoc way without direct correspondence to the metric. PNPs in a generic way are formulated or prescribed to reproduce circular orbits, best suited to study Keplerian accretion flow. Nevertheless, a more fundamental issue regarding PNPs is that a PNP is not a physical analogue of local gravity, is not based on any robust physical theory and does not satisfy the Poisson equation. A PNP is simply a mathematical mimicking of certain GR features of the corresponding metric, which is used instead of the Newtonian potential in the Newtonian framework fluid equations. Also, certain unique GR features, such as perihelion precession, are not well reproduced with most PNPs. Recently, Wegg (2012) proposed, ad hoc, a couple of PNPs by modifying the PNP of PW80 to reproduce precessional effects in general relativity for orbits with large apoapsis. However, these are not very effective in the vicinity of the Schwarzschild BH. Things become yet more intriguing when formulating a PNP corresponding to the Kerr geometry, because unique features of Kerr space–time, such as frame dragging and gravitomagnetic effects, necessitate the explicit information of these effects in the corresponding PNP. Although the PNPs of M02 and GM07 plausibly contain the information of these effects, because they have been derived from the GR metric, they do not exhibit them explicitly. Other PNPs simply accommodate these terms in an ad hoc fashion.

Recently, Tejeda & Rosswog (2013), hereafter TR13, formulated a generalized effective potential in the same vein as that of a Newtonian for a Schwarzschild BH, describing particle motion around it, based on a proper axiomatic procedure. The potential that is developed directly from the corresponding metric can be viewed as some kind of Newtonian analogue to the GR metric, which has an explicit dependence on the radial velocity and angular velocity of a test particle. This generalized potential reproduces exactly several relativistic features of the corresponding Schwarzschild geometry. As articulated earlier regarding the importance of rotating BHs in astrophysical scenarios, following TR13, we aim to develop a generalized effective potential of a Kerr BH in the equatorial plane for the motion of a test particle. The potential would then be an appropriate Newtonian or potential analogue of Kerr space–time, which we refer to as the Kerr–Newtonian potential. This kind of potential would then be useful to study accretion-related phenomena around rotating BHs in a more effective way.

In the next section, we derive the Kerr–Newtonian potential starting from the Kerr metric. Subsequently, in Section 3, we compare various relativistic features with our potential. In Section 4, we compare the effectiveness of our potential with other existing PNPs in the literature, in reproducing the GR features of Kerr geometry. Finally, we finish in Section 5 with a discussion and summary.

2 FORMULATION OF THE GENERALIZED POTENTIAL

The Kerr space–time in the Boyer–Lindquist coordinate system is given by

\[
ds^2 = - \left( 1 - \frac{2r_s}{r} \right) c^2 dt^2 + \frac{4a r_s \sin^2 \theta}{\Sigma} c dr d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left( r^2 + a^2 + \frac{2a r_s \sin^2 \theta}{\Sigma} \right) \sin^2 \theta d\phi^2, \tag{1}
\]
where $\Delta = r^2 + a^2 - 2mr$, $\Sigma = r^2 + a^2\cos^2 \theta$, $r_c = GM/c^2$ and $a = I/(Mc)$, which is called the Kerr parameter. The Lagrangian density of the particle of mass $m$ in the Kerr space–time is then given by

$$2\mathcal{L} = -(1 - \frac{2r_c}{\Sigma})c^2 \left( \frac{dt}{d\tau} \right)^2 - \frac{4ar_s \sin^2 \theta c \frac{dr}{d\tau}}{\Sigma} \frac{d\phi}{d\tau} \frac{dr}{d\tau}$$

$$+ \frac{\Delta}{\Lambda} \left( \frac{d\tau}{d\tau} \right)^2 + \frac{\Delta}{\Sigma} \left( \frac{d\phi}{d\tau} \right)^2$$

$$+ \left( r^2 + a^2 + \frac{2ar_s a^2 \sin^2 \theta}{\Sigma} \right) \sin^2 \theta \left( \frac{d\phi}{d\tau} \right)^2.$$  \(2\)

From the symmetries, we obtain two constants of motion corresponding to two ignorable coordinates $t$ and $\phi$ given by

$$\mathcal{P}_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = -(1 - \frac{2r_c}{\Sigma})c^2 \frac{dr}{d\tau} - \frac{2ar_s \sin^2 \theta}{\Sigma} \frac{d\phi}{d\tau} = \text{constant} = -\epsilon$$  \(3\)

and

$$\mathcal{P}_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -\frac{2ar_s \sin^2 \theta}{\Sigma} \frac{dr}{d\tau} + \left( r^2 + a^2 + \frac{2ar_s a^2 \sin^2 \theta}{\Sigma} \right) \sin^2 \theta \frac{d\phi}{d\tau} = \text{constant} = \lambda,$$  \(4\)

where $\epsilon$ and $\lambda$ are the specific energy and specific angular momentum of the orbiting particle, respectively. Here, $\dot{t}$ and $\dot{\phi}$ represent the derivatives of $t$ and $\phi$ with respect to proper time $\tau$. For particle motion in the equatorial plane ($\theta = \pi/2$), by solving the above two equations, we obtain

$$\frac{dr}{d\tau} = \frac{(\epsilon/c^2)(r^3 + ar + 2ra^2) - (2ar_s \lambda)/c}{r \Delta},$$ \(5\)

$$\frac{d\phi}{d\tau} = \frac{(\epsilon/c)2ar_s + (r - 2r_c)\lambda}{r \Delta}.$$ \(6\)

Using $\mathcal{L} = -(1/2)m^2c^2$ and substituting equations (5) and (6) in equation (2), we obtain

$$\epsilon^2 - c^4 \left( 1 + \frac{a^2}{r^2} \right)^2 - \frac{r^2 a^2 \epsilon^2}{r^5 c^2}$$

$$+ \frac{2ar_s \epsilon}{r \Delta} = -\frac{Gm}{r} + \frac{\Delta}{2r^2} \left( 1 - \frac{2r_c}{r} \right).$$ \(7\)

Using equations (5) and (6), we find

$$\frac{dr}{d\tau} = \frac{\epsilon}{c^2} \left( \frac{(r - 2r_c) + (2ar_s/c)\phi}{\Delta} \right).$$ \(8\)

The basis of our potential formulation is the low energy limit of the test particle motion (TR13), which is $\epsilon/c^2 \sim 1$. We write $E = (\epsilon^2 - c^4)/(2c^2)$ considering a locally inertial frame for test particle motion, which will reduce to the total mechanical energy ($\equiv$ Hamiltonian) in Newtonian mechanics in the non-relativistic limit with $a = 0$. The second term in the above definition of $E$ is the rest mass energy of the particle, which is subtracted from relativistic energy due to the low energy limit, in analogy to Newtonian Hamiltonian. Computing $\lambda$ from equation (6) and substituting in equation (7) and using equation (8), we finally obtain the generalized Hamiltonian ($E_{\text{GN}}$) of a test particle around Kerr space–time in the low energy limit as

$$E_{\text{GN}} = -\frac{Gm}{r} + \left( \frac{1}{2} \frac{r^2}{\Delta} \right)$$

$$\left( \frac{r - 2r_c}{2r^2} \right)^2,$$ \(9\)

where overdots represent the derivative with respect to coordinate time $t$. With $a = 0$, $E_{\text{GN}}$ reduces to that of Schwarzschild geometry. The generalized Hamiltonian $E_{\text{GN}}$ in the low energy limit should be equivalent to the Hamiltonian in the Newtonian framework. The effective Hamiltonian in the Newtonian regime with the generalized potential in the equatorial plane will then be equivalent to $E_{\text{GN}}$ in equation (9). Thus,

$$E_{\text{GN}} = \frac{1}{2}(r^2 + r^2\dot{\phi}^2) + V_{\text{GN}} - \dot{\phi} \frac{\partial V_{\text{GN}}}{\partial \phi} - \dot{\phi} \frac{\partial V_{\text{GN}}}{\partial \phi},$$ \(10\)

where $T = (1/2)(r^2 + r^2\dot{\phi}^2)$ is the non-relativistic specific kinetic energy of the test particle and $V_{\text{GN}}$ is the most generalized form of the potential in Newtonian analogue of Kerr space–time in the spherical geometry with test particle motion in the equatorial plane, which contains the entire information of the source. The potential $V_{\text{GN}}$ is then given by

$$V_{\text{GN}} = -\frac{Gm}{r} \left( 1 - \omega \phi \right) - (G_1 r^2 + G_2 r^2\dot{\phi}^2) + \frac{r^2 + r^2\dot{\phi}^2}{2},$$ \(11\)

where $G_1 = \frac{r^3}{(r - 2r_c)\Delta}$, $G_2 = \frac{\Delta}{(r - 2r_c)^2}$.

Note that all the dynamical quantities expressed are specific quantities. In the Newtonian limit, $G_1 = G_2 = 1$ and $\omega = 2ar_s/c(r - 2r_c)$. In equation (11), $\omega \phi$ in the potential arises due to the effect of frame dragging. The potential $V_{\text{GN}}(=V_{\text{GN}})$ is a modified potential deviating from the exact Newtonian (spherical symmetric part). The subscript KN denotes Kerr–Newtonian. The potential is an explicit velocity-dependent potential containing all gravitational effects of Kerr space–time for a stationary observer. Thus, the potential in equation (11) contains the explicit information of gravitomagnetic and frame dragging effects, which has been obtained directly from the Kerr metric by solving geodesic equations of motion. Putting $a = 0$, the potential reduces to that in Schwarzschild geometry. Unlike most other PNPs, which are either derived or prescribed for particle motion in a circular orbit, the potential in equation (11) is applicable for generalized orbital dynamics. It is to be noted that we have restricted ourselves to deriving a Kerr–Newtonian potential corresponding to particle motion in the equatorial plane. The formulation of a more generalized Kerr–Newtonian potential for off-equatorial particle orbits is immensely complicated within our present approach, where the necessary use of the Carter constant seems to be a prerequisite (see GM07). Such a study will be pursued in the near future.

Although the Kerr–Newtonian potential, in principle, should precisely reproduce all orbits in the exact Kerr geometry, the form of the potential in equation (11) diverges at $r = 2r_c = 2GM/c^2$. This is happening precisely because of the presence of $[(r - 2r_c) + (2ar_s/c)\phi]^2$ in the denominator of the Hamiltonian $E_{\text{GN}}$ in equation (9), which has been obtained while replacing the conserved specific angular momentum $\lambda$ by $\phi$. Thus, the potential in the form given in equation (11) would not be useful within the range $r \lesssim 2r_c$. Note that for the Kerr BH, the horizon $r_H = r_s$ for maximal spin. However, such a radial zone of range $r \lesssim 2r_c$ is in the extreme vicinity...
Variation of potential with radial distance

\[ \phi \propto \omega r + \dot{\omega} \frac{\dot{r}}{a} \]  

\[ \Lambda_{\text{KN}} = \frac{\partial L_{\text{KN}}}{\partial \dot{\phi}} = -\frac{GM\omega}{r} + \frac{G\dot{r}^2\phi(2 + \omega \dot{\phi})}{2(1 + \omega \dot{\phi})^2} - \frac{G\dot{r}^2\omega}{2(1 + \omega \dot{\phi})^2}. \]  

Obtaining the specific Hamiltonian from equation (13), the radial motion of the particle in the presence of this potential is then given by

\[ r^2 = \frac{2}{G_1} \left( E_{\text{KN}} + \frac{GM}{r} \right) (1 + \omega \dot{\phi})^2 - \frac{G_1 \dot{r}^2 \phi^2}{G_1^2}. \]  

Here, \( E_{\text{KN}} \) is the conserved specific Hamiltonian of the particle motion in the Kerr–Newtonian, which is equivalent to \( E_{\text{GR}} \), and \( \dot{r} \) is identical to the expression in the exact Kerr geometry in the low energy limit. Next, we compute the equations of motion of a test particle using the Kerr–Newtonian potential. For the \( r \) coordinate, we obtain

\[ \left( 1 - \frac{A_1 B_1}{B_1} \right) \ddot{r} + \left[ A_1 + \frac{A_1 B_1}{B_1} \left( B_1 + B_4 + A_4 B_1 \right) \right] \dot{r}^2 - A_1 \dot{\phi}^2 + A_4 + \frac{A_1}{B_1} B_5 \dot{r}^4 = 0. \]  

Similarly, for the \( \phi \) coordinate, we have

\[ \left( 1 - \frac{A_1}{B_1} \right) \ddot{\phi} + \left[ \frac{1}{B_1} \left( B_1 + B_3 + B_4 + A_4 B_1 \right) \right] \dot{\phi}^2 - \frac{A_1 B_5}{B_1} \phi^2 + \frac{1}{B_1} (B_6 + A_1 B_5) \dot{r}^3 = 0. \]
Here,
\[
A_1 = \frac{1}{2(r - 2r_s)} \left[ 2a^2(r - 3r_s) - 4rr_r(r - 2r_s) \right] \frac{\omega \phi}{r \Delta} + \frac{\omega \phi}{1 + \omega \phi},
\]
\[
A_2 = \frac{G_s}{2r} \left[ 2(r - 3r_s)(r - 2r_s) - 4r_s r^3 + \Delta \frac{\omega \phi}{1 + \omega \phi} \right],
\]
\[
A_3 = \frac{\omega}{1 + \omega \phi}, \quad A_4 = \frac{GM}{r^2 G_{\delta}} \left[ 1 - \frac{4ar_s}{c} \right] \frac{r - r_s}{(r - 2r_s)^2},
\]
\[
B_1 = (G_{\delta} r^2 + G_{\omega \phi} r^2), \quad B_2 = \frac{r^2 \phi}{2(r - 2r_s)} G_{\delta} \omega \phi (3 + \omega \phi),
\]
\[
B_3 = \frac{r^2 \phi (1 + \omega \phi)}{2(r - 2r_s)} \left[ (2r - 3r_s)(r - 2r_s) - 4(r_s/r) a^2 \right] (2 + \omega \phi),
\]
\[
B_4 = \frac{4GM}{r^2 \phi} \omega (1 + \omega \phi),
\]
\[
B_5 = \frac{\omega}{2} \left\{ G_{\delta} (1 - \omega \phi) \frac{2a^2 r^2(r - 3r_s) - 4r_s r(r - 2r_s)}{(r - 2r_s)^2[D^2(1 + \omega \phi)]} \right\}.
\]

Equations (14)–(17) provide the complete particle dynamics around Kerr BHs in the Kerr–Newtonian framework. They reduce to the expressions corresponding to Schwarzschild case, with \(a = 0\).

### 3 COMPARISON OF GR FEATURES WITH THE KERR–NEWTONIAN POTENTIAL

In this section, we compare various GR features with the Kerr–Newtonian potential for different values of Kerr parameter \(a\). As argued earlier, we use the form of potential given in equation (11) in which the potential will be generically valid beyond \(r > 2r_s\).

#### 3.1 Dynamics of circular orbit

The circular orbit of the test particle is determined by conditions
\[
\dot{r} = 0, \quad \dot{\phi} \neq 0.
\]

We use the above conditions for circular orbits using equations (14)–(16) from which we obtain the specific angular momentum \(\lambda_{KN}|c\), the specific Hamiltonian \(E_{KN}|c\) and the specific angular velocity \(\phi_{KN}|c\) numerically. The symbol \(|c\) corresponds to the dynamical quantities in the circular orbit. Alternatively, \(\lambda_{KN}|c\) and \(E_{KN}|c\) can be directly obtained from equation (15) by replacing \(\phi\) with \(\lambda_{KN}\) and its corresponding derivative, and subsequently using prerequisite circular orbit conditions. In that case, it is then possible for us to obtain analytical expressions for \(\lambda_{KN}|c\) and \(E_{KN}|c\). Then, \(\lambda_{KN}|c\) is given by
\[
\lambda_{KN}|c = \frac{-Q_1 \pm \sqrt{Q_1^2 - 4R_1}}{2},
\]
where
\[
Q_1 = \left[ 6a^3 r_s r_c - 6r_s a c r (r^2 + a^2) \right] \left[ a^2 r (r - 2r_s) - r (r - 3r_s)(r^2 + a^2) \right],
\]
\[
R_1 = \left[ \frac{GM}{a^2} (r^2 + a^2) \right] \left[ (r^2 + 3a^2) - 2a^2 r \right] \left[ a^2 r (r - 2r_s) - r (r - 3r_s)(r^2 + a^2) \right].
\]

Similarly, \(E_{KN}|c\) is given by
\[
E_{KN}|c = \frac{\left( \frac{\lambda_{KN}|c}{c} \right)^2 (r - 2r_s)}{r [1 + (a/r^3)^2]} + \frac{r_s (2ac \lambda_{KN} - a^2 c^2)}{r [1 + (a/r^3)^2]}. \tag{20}
\]

Then, \(\phi_{KN}|c\) is computed from a quadratic relation \(P_2 \ddot{\phi}_{KN}|c + Q_2 \phi_{KN}|c + R_2 = 0\) obtained using equation (14), where
\[
P_2 = \frac{\omega^2 \left( \lambda_{KN}|c + \frac{GM}{r} \right) - \frac{GM}{r} \lambda_{KN}|c}{2},
\]
\[
Q_2 = \frac{2\omega (\lambda_{KN}|c + \frac{GM}{r}) - \frac{GM}{r} \lambda_{KN}|c}{2},
\]
\[
R_2 = \frac{\lambda_{KN}|c + \frac{GM}{r}}{2}.
\]

For \(a = 0\), this quadratic relation becomes linear and reduces to that in TR13. For \(a \neq 0\), we obtain the physically correct solution of \(\phi_{KN}|c\), given by
\[
\Omega_{KN}|c = -\frac{Q_2 - \sqrt{Q_1^2 - 4P_2 R_2}}{2P_2}. \tag{21}
\]

The corresponding relativistic results in the Kerr geometry are given by (Bardeen 1973)
\[
\lambda_{K}|c = \frac{\sqrt{GM} r^2 - 2a \sqrt{r^2 + a^2} - a^2}{r(r^2 - 3r_s r + 2a \sqrt{r^2 + a^2})}, \tag{22}
\]
\[
\epsilon_{K}|c = \frac{(r^2 - 2a \sqrt{r^2 + a^2})}{r(r^2 - 3r_s r + 2a \sqrt{r^2 + a^2})}, \tag{23}
\]
\[
\Omega_{K}|c = \frac{\phi_{K}|c}{c} \frac{\sqrt{GM}}{r^2 + 1/4 a^2}. \tag{24}
\]

Note that the actual specific Hamiltonian in the Kerr geometry is \(E_{K}|c = (\epsilon_{K}|c - c^2)/(2c^2)\), which is actually plotted.

Fig. 2 shows the variation of specific angular momentum with \(r\) for both corotating and counter-rotating circular orbits with the Kerr–Newtonian potential and has been compared with the corresponding relativistic geometry. The angular momentum profiles corresponding to the Kerr–Newtonian potential reproduce the GR results quite accurately.

In Fig. 3, we exhibit similar profiles for the corresponding specific Hamiltonian with the Kerr–Newtonian potential, which lies within an error of \(\sim 10\) per cent, in the vicinity of the rapidly corotating BH. However, the counter-rotating GR results are reproduced with the Kerr–Newtonian potential with precise accuracy. The angular frequency profiles are displayed in Fig. 4, which also reproduce the GR results, but with less accuracy in the inner regions of the flow for high BH spin. A maximum error of \(\sim 36\) per cent is obtained in the vicinity of an extremely corotating BH. Conversely, for counter-rotating particle orbits, the Kerr–Newtonian potential quite accurately reproduces the corresponding GR values.

The two salient GR features corresponding to the particle motion in the circular orbit in Kerr geometry are the marginally stable and marginally bound orbits (\(r_{ms}\) and \(r_{mb}\), respectively). As usual, we use the conditions \(dE_{KN}|c/dr = 0\) and \(E_{KN}|c = 0\) using equations (19) and (20) to obtain numerical values of \(r_{ms}\) and \(r_{mb}\), respectively, corresponding to the Kerr–Newtonian potential. It is found that for the counter-rotating case, \(r_{mb}\) is almost exactly replicated with the Kerr–Newtonian potential and \(r_{ms}\) is precisely reproduced within a maximum error of \(\sim 1.2\) per cent, as depicted in Fig. 5. However, for the corotating case, we obtain the real solution for \(r_{ms}\) and \(r_{mb}\) up to the Kerr parameter \(a \sim 0.7\). Here, \(r_{mb}\) is reproduced exactly.
Figure 2. Variation of specific angular momentum with radial distance \( r \) for both corotating and counter-rotating circular orbits. Solid and dashed curves are for the exact Kerr geometry and the Kerr–Newtonian framework, respectively, and \( r \) and \( a \) are expressed in units of \( r_s \). The specific angular momentum is in units of \( GMc^{-1} \).

whereas \( r_{ms} \) is reproduced within a reasonable accuracy with a maximum error margin of \( \sim 10 \) per cent, up to the specified value of \( a \sim 0.7 \) (see Fig. 5). This limiting description of \( r_{ms} \) and \( r_{mb} \) up to \( a \sim 0.7 \) is because of the expression under the square root in equation (19), which for \( a > 0.7 \) becomes negative at a radial distance larger than \( r_{ms} \) and \( r_{mb} \). Note that circular geodesics can still be described for \( a \gtrsim 0.7 \), but at radii \( r \gtrsim 3r_s \).

3.2 Orbital perturbation

Perturbation in accretion flow is mostly studied in order to understand the instabilities in the accreting system. Small perturbation in the flow, which is linked to the epicyclic frequency, and its coupling to BH spin, can be related to QPOs in BHXRBs. TR13 computed the epicyclic frequency for a test particle motion with their Schwarzschild–Newtonian potential, and compared it with the exact GR value, which they found to be highly accurate. Thus, using equations (16) and (17), we estimate the radial epicyclic frequency for a test particle motion in a circular orbit in the equatorial plane, which will be influenced by the spin of BH. Thus, \( r \) and \( \phi \) and their derivatives will be perturbed according to

\[
r \rightarrow r + \delta r, \quad \dot{r} \rightarrow \delta \dot{r}, \quad \ddot{r} \rightarrow \ddot{r},
\]

in the flow.
Figure 3. Same as Fig. 2, but with the variation of the specific Hamiltonian of the particle in a circular orbit with radial distance $r$, in units of $c^2$. Other parameters are the same as in Fig. 2.
Figure 4. Same as Figs 2 and 3, but with the variation of the angular frequency of the particle in a circular orbit with $r$, in units of $c^3 GM^{-1}$. 
\[ \phi \rightarrow \phi + \delta \phi, \quad \dot{\phi} \rightarrow \dot{\phi} + \delta \dot{\phi}, \quad \ddot{\phi} = \delta \ddot{\phi}. \]  

Inserting equations (25) and (26) into equations (16) and (17), and using equation (14), we obtain the linearized perturbed equations. By solving these, the radial epicyclic frequency \( \kappa \) is computed, given by

\[ \kappa^2 = -\dot{\phi}_|c|^2 \left[ 1 - a^2 \left( r^2 - 10r_s + 10r^2 \right) + \frac{8r(r - rs)a^4}{r^3(r - 2r_s)} \right] - \dot{\phi}_|c|^2 \left[ \frac{\Delta}{2r(r - r_s)} - \frac{\Delta(3r - 2r_s)a^2}{2r^2(r - 2r_s)^3} \right] \frac{\omega \dot{\phi}_|c|}{1 + \omega \dot{\phi}_|c|} 
+ \dot{\phi}_|c|^2 \left[ \frac{D_1}{r - 2r_s} \frac{\omega \dot{\phi}_|c|}{1 + \omega \dot{\phi}_|c|} + D_2 \frac{\omega \dot{\phi}_|c|}{1 + \omega \dot{\phi}_|c|} \right] 
- \frac{2GM}{r^3} \left[ \frac{r - r_s}{c} \frac{a}{r - 2r_s} + a^2 \left( 2 - \frac{5r_s}{r} \right) \left( 1 + \omega \dot{\phi}_|c| \right) \right] 
+ \frac{4GMrs a}{r^2} \frac{r - r_s}{c(r - 2r_s)^2} + \mathcal{F}_1 \mathcal{F}_2 \Delta \frac{5r^2 - 14rs + 10r_s^2}{r(r - r_s)} 
- 2(r - r_s)(r - 2r_s) \dot{\phi}_|c| \left( 1 + \omega \dot{\phi}_|c| \right) \]  

From equation (27), \( D_1, D_2, \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are given in Appendix B. The expression for \( \kappa \) in equation (27) exactly reduces to that in the Schwarzschild case with \( a = 0 \). Because of the cumbersome and long nature of the equations, the derivation of \( \kappa \) is given in Appendix B. We compare the value of \( \kappa \) corresponding to the Kerr–Newtonian potential with the exact relativistic result in Kerr geometry, which is given by (Semeráčk & Zácek 2000)

\[ \kappa^2 = \left( \frac{\Omega_k|c|}{r} \right)^2 \left[ \Delta - 4[\sqrt{(r,r) - a}] \right]. \]  

It must be mentioned that we have only derived the radial epicyclic frequency with no expression for the vertical epicyclic frequency. This is because in order to have an expression for the vertical epicyclic frequency or to study perturbations perpendicular to the equatorial plane, it is necessary to derive a \( \theta \)-dependent Kerr–Newtonian potential valid for the off-equatorial potential, and subsequently a \( \theta \)-dependent equation of motion. However, this is beyond the scope of our present study.

Fig. 6 shows the comparison of radial dependence of \( \kappa \) obtained from the Kerr–Newtonian potential with that in general relativity, which exhibits a maximum error of \(~35 \text{ per cent}\) in the vicinity of a rapidly corotating BH. Here, also, the Kerr–Newtonian potential quite precisely reproduces the corresponding GR results for a counter-rotating BH.

### 3.3 Orbital precession

Using equations (14) and (15), we compute \( d\phi/dr \) in the Kerr–Newtonian, which we compare with the corresponding GR expression obtained using equations (6)–(8). In the Schwarzschild case, the expressions of \( d\phi/dr \) in both the Schwarzschild–Newtonian framework as well as in exact general relativity are similar, giving identical orbital trajectory and perihelion precession (TR13). This exactness guarantees that the bending of light or gravitational lensing (Bhadra, Biswas & Sarkar 2010) in the Schwarzschild–Newtonian framework can also reproduce an identical GR result. Nevertheless, in Fig. 7, we compare \( d\phi/dr \) as a function of \( r \) corresponding to both the Kerr–Newtonian potential and its GR counterpart, in the low energy limit of the test particle motion. It shows that \( d\phi/dr \) corresponding to the Kerr–Newtonian potential is almost identical to the corresponding GR result. In Fig. 8, we display elliptic-like trajectories for a particle orbit corresponding to the Kerr–Newtonian potential in the \( x-y \) plane, obtained from the equations of motion, and we compare the nature of trajectories with the GR results for a corotating BH. We obtain the plots of an elliptical trajectory using Cartesian transformation adopting the method of the Euler–Cromer algorithm, which preserves energy conservation. For all cases, the test particle starts from an apoapsis \( r_s = 60 r_s \) with a fixed eccentricity \( e = 0.714 \). Fig. 8 shows that the orbital trajectory corresponding to the Kerr–Newtonian potential resembles the GR result well, but with less accuracy for a rapidly spinning BH. The value of the apsidal precession can be estimated using the relation of the orbital trajectory. The apsidal precession or the perihelion advancement \( \Psi \) is given by the relation

\[ \Psi = \Pi - \pi = \int_{r_p}^{r_s} \frac{d\phi}{dr} dr - \pi, \]  

where \( \Pi \) is the usual half orbital period of the test particle and \( r_p \) is the periapsis of the orbit. Alternatively, we can easily compute the apsidal precession directly from the trajectory profiles. In Table 1, we display the values of apsidal precession corresponding to the Kerr–Newtonian potential and compare them with the GR results, for different values of the Kerr parameter \( a \). We use a similar set of orbital parameters as used in Fig. 8. We find that the maximum deviation of \( \Psi \) for the Kerr–Newtonian potential from that of the
Newtonian analogue of Kerr black holes

Figure 6. Same as Figs 2–4, but with the variation of the epicyclic frequency of the particle in a circular orbit with $r$, in units of $c^3 GM^{-1}$.

exact GR result is no more than $\sim 12$ per cent (fourth column of Table 1), corresponding to an extremely rotating Kerr BH.

Moreover, because of the similar nature of the orbital trajectory corresponding to the Kerr–Newtonian potential and its GR counterpart, we can conservatively predict that the Kerr–Newtonian potential would also reproduce the corresponding GR bending of light with reasonable accuracy.

4 A COMPARATIVE ANALYSIS OF THE KERR–NEWTONIAN POTENTIAL

The essential philosophy of the procedure adopted in the present work to derive the Newtonian-like analogous potential of the corresponding Kerr geometry is to reproduce the geodesic equations of motion of test particles with reasonable accuracy, if not exactly. Therefore, not only does the adopted method demand that the dynamical profiles (such as conserved angular momentum and conserved energy) and the temporal features (such as angular and epicyclic frequencies) are reproduced with precise/good accuracy

$^1$Error (per cent) = $\left(\frac{|GR_{values} - KN_{values}|}{GR_{values}}\right) \times 100$. 

Error (per cent) = $\left(\frac{|GR_{values} - KN_{values}|}{GR_{values}}\right) \times 100$. 

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Figure 7. Variation of orbital trajectory $d\phi/dr$ in the radial direction $r$ for both corotating and counter-rotating particle motion with conserved specific Hamiltonian $E = 0.02$ and with specific angular momentum $\lambda = 3.5$. Solid and dashed curves are for the exact Kerr geometry and the Kerr–Newtonian framework, respectively. Here, energy, angular momentum and radius are in units of $c^2$, $GMc^{-1}$ and $GMc^{-2}$, respectively.

but also, most importantly, it guarantees the replication of the orbital trajectory of test particle motion with reasonable accuracy. In Fig. 9, we show the percentage deviation of various dynamical quantities in circular geodesics obtained with the Kerr–Newtonian potential from those of pure GR as a function of the Kerr parameter $a$, at two different radii. It should be noted that because of the non-appearance of stable circular orbits corresponding to counter-rotating BHs at $r \leq 6r_s$, we do not obtain any physically correct value of radial epicyclic frequencies for counter-rotating circular orbits at those radii, as reflected in Fig. 9(d).

The form of the Kerr–Newtonian potential in equation (11) is generically valid beyond $r \sim 2r_s$. However, no real solutions for $r_{m1}$ and $r_{n1}$ exist beyond $a \sim 0.7$ for circular geodesics, which seems mainly due to the approximation of the low energy limit of the test particle motion used to derive the Kerr–Newtonian potential. Only when restricted to the radial range $r \gtrsim 3r_s$ can the Kerr–Newtonian potential describe circular geodesics with reasonable accuracy for any $a$, including $a > 0.7$. However, for counter-rotating particle orbits, the entire spectrum of GR features can be described by the Kerr–Newtonian potential with precise accuracy for all values of the Kerr parameter $a$. In contrast, most other prevailing PNPs corresponding to Kerr space–time (see the introduction) mostly put emphasis on reproducing the last stable circular and/or marginally bound orbits. None of these can reproduce the entire spectrum of GR features, even with marginal accuracy and within acceptable error limits. Moreover, because the Kerr–Newtonian potential has been derived from the conserved Hamiltonian of the motion, it contains the explicit information of the velocity of the test particle,
Figure 8. Comparison of the elliptic-like trajectory of the particle orbit in the equatorial plane in Kerr space–time corresponding to the Kerr–Newtonian potential with that in exact general relativity, projected in the $x$–$y$ plane. For all cases, the particle starts from an apogee $r_a = 60 r_s$ with eccentricity $e = 0.714$. Figs 8(a)–(c) correspond to $a = 0.5$, whereas Figs 8(d)–(f) correspond to $a = 1.0$. Solid, long-dashed, short-dashed and dot-dashed curves in all the figures correspond to Newtonian, Schwarzschild, Kerr–Newtonian and exact Kerr geometry, respectively. The velocities are expressed in units of $c$.

Table 1. Comparison of the values of apsidal precession between Kerr and Kerr–Newtonian for different $a$, where $r_a = 60 r_s$ and $e = 0.714$.

| $a$ | Kerr       | Kerr–Newtonian | Error (per cent) |
|-----|------------|----------------|------------------|
| 0.0 | 1.5095     | 1.5095         | 0.0000           |
| 0.3 | 1.4314     | 1.4831         | 3.6118           |
| 0.5 | 1.3844     | 1.4604         | 5.4897           |
| 0.8 | 1.3276     | 1.4317         | 7.8412           |
| 0.95| 1.2791     | 1.4145         | 10.5855          |
| 1.0 | 1.2401     | 1.3915         | 12.2087          |

as should be the case for any relativistic analogue, as well as explicit information of the frame dragging, unlike most of the PNPs corresponding to Kerr geometry.

The PNPs corresponding to Kerr geometry are found to be less accurate than their Schwarzschild counterparts. Most of the PNPs corresponding to Kerr geometry are free-fall type potentials (e.g. Artemova et al. 1996; M02; MM03), without explicit information of frame dragging. Free-fall type PNPs might have some merit in mimicking spherically symmetric space–times, but can be questionable when describing axially symmetric rotating BHs. Artemova et al. (1996) proposed two types of PNPs, which can only reproduce the location of $r_{\text{ms}}$. Moreover, it has been pointed out by M02 that the PNPs of Artemova et al. (1996) are only valid for corotating BHs. For counter-rotating BHs, they provide the
incorrect value of $r_{\text{ms}}$. Also, the PNPs give huge errors when reproducing $r_{\text{mb}}$ ($\sim$500 per cent) and the specific energy in the innermost region ($\sim$50 per cent), corresponding to counter-rotating BHs. Nevertheless, a few features of Keplerian accretion discs, such as optical depth and temperature in the GR paradigm, can be reproduced by their PNPs, within an acceptable error limit, $\sim$(10–20) per cent, for corotating BHs. They cannot describe the orbital trajectories.
Figure 10. Elliptic orbital trajectory in the equatorial plane up to \( r \sim 2r_s \) corresponding to the Kerr–Newtonian potential. The solid line is for the Kerr–Newtonian potential and the dashed line is for Kerr geometry. The particle orbit lies between \( r_a = 30r_s \) and \( r_h = 2.01r_s \) for the Kerr parameter \( a = 1 \).

Semerák & Karas (1999) prescribed a PNP, ad hoc, taking into account the effect of frame dragging through a correction term. It is a three-dimensional potential and has been prescribed to be useful for off-equatorial orbits. Although the significance of geodesic equations of motion was apparently considered while prescribing their potential, the PNP cannot reproduce \( r_{\text{mis}} \) and \( r_{\text{lab}} \), or other features of circular geodesics, with reasonable accuracy. Semerák & Karas pointed out that the PNP is unable to approximately reproduce GR profiles of angular momentum and energy as well as the orbital trajectories, even within acceptable error limits. Corresponding to the Kerr–Newtonian potential, the specific energy marginally deviates from the GR results, but with a maximum error margin of \( \sim 10 \) per cent in the vicinity of an extremely spinning BH, for the corotating case. For counter-rotating BHs, the Kerr–Newtonian potential reproduces nearly exact GR results for circular geodesics.

The PNP prescribed by M02 gives comparatively far better results than the above stated PNPs, in mimicking the key GR features of Kerr geometry. The PNP of M02 has been derived from the corresponding metric and it exactly reproduces \( r_{\text{mis}} \) for all values of \( a \). Moreover, the marginally bound orbit \( r_{\text{lab}} \) can also be reproduced by this PNP for all values of \( a \) within an error margin of \( \sim 5 \) per cent. Also, the profile of the conserved specific energy for circular geodesics can be approximately reproduced, at least, within the acceptable error limits (see fig. 5 in GM07). However, the potential of M02 cannot reproduce well the corresponding GR angular and epicyclic frequencies; the error margins for these parameters are as high as \( \sim 180 \) and \( \sim 800 \) per cent, respectively, for an extremely spinning BH (MM03). Also, this potential is unable to reproduce orbital trajectories properly. In MM03, two PNPs were prescribed for describing temporal effects such as angular and epicyclic frequencies as well as specific energies around the Kerr geometry. However, none of these can reproduce well both specific energy and angular frequencies simultaneously; for the logarithmically modified potential, the deviation in specific energy is more than 30 per cent whereas the deviations in epicyclic frequencies for the second-order expansion potential range from 25 to 170 per cent when \( a \lesssim 0.9 \). Moreover, these potentials cannot reproduce \( r_{\text{ph}} \).

A few more PNPs corresponding to the generalized Kerr geometry (three-dimensional) also exist in the literature, one of which is GM07, which is an extension of M02 and thus exhibits similar behaviour with a similar type of limitation. Another is prescribed by Chakrabarti & Mondal (2006) where the information of frame dragging has been introduced ad hoc. In the equatorial plane, this potential is valid approximately up to the Kerr parameter \( a \sim 0.8 \). At this value of \( a = 0.8 \), the value of \( r_{\text{lab}} \) corresponding to this PNP deviates by more than 20 per cent from the exact GR result. Also, the dynamical profiles and the orbital trajectories are less accurately reproduced by this potential compared to the Kerr–Newtonian potential.

Thus, we can conclude that none of the prevailing PNPs corresponding to Kerr geometry can reproduce well all the essential GR features simultaneously, within a reasonable margin of error. In contrast, within the criteria of the low energy limit, the Kerr–Newtonian potential can approximate most of the GR features of Kerr geometry with precise/reasonable accuracy for \( -1 \lesssim a \lesssim 0.7 \). For \( a > 0.7 \), the circular geodesics can still be treated by the Kerr–Newtonian potential accurately if we restrict to the radial range \( r \gtrsim 3r_s \). For general orbital trajectories (without confining to circular orbits only), however, the Kerr–Newtonian potential can be effectively used for \( r \gtrsim 2r_s \) without any restriction on \( a \). For instance, we have obtained the elliptic orbital trajectory down to \( r \sim 2r_s \) for \( a = 1 \), using the Kerr–Newtonian potential, as shown in Fig. 10. Because the Kerr–Newtonian potential describes the corresponding GR orbital trajectories with reasonable accuracy, the potential can well reproduce the experimentally tested GR effects, such as perihelion advancement or gravitational bending of light, within an acceptable margin of error.

5 DISCUSSION

A PNP corresponding to Kerr geometry is more inconspicuous because it is necessary to mimic several explicit Kerr features such as frame dragging and gravitomagnetic effects and therefore it is more complex. The Kerr–Newtonian potential formulated in this work, invoking a physically correct methodology, is found to approximate all the Kerr features with reasonable accuracy for \( a < 0.7 \) unlike the prevailing PNPs for the Kerr space–time. The formulated Kerr–Newtonian potential has been derived under the low energy limit \( (\epsilon/c^2 \sim 1) \) approximation and also restricting the particle orbits in the equatorial plane only, but even in such restricted circumstances, it is more complicated in comparison to that in the pure Schwarzschild case. The analytical form of the Kerr–Newtonian potential, which has been evaluated from the conserved Hamiltonian (9), restricts its applicability to \( r \gtrsim 2r_s \). This would not cause any major difficulties for astrophysical scenarios because accretion studies are mainly focused on regions with \( r > 2r_s \).

Although the robustness of the Kerr–Newtonian potential in equation (11), and its ability to mimic most of the GR features within an acceptable error margin, is quite appreciable, we need to remember that any analogous modified Newtonian description of relativistic geometry is inherently approximate in nature. Moreover, we have also assumed the criteria of the low energy limit to derive the said potential. The low energy limit criterion might be suitable to
describe static geometries, for which the results would be precisely exact and the GR features in its entirety would be reproduced with remarkable accuracy (Sarkar, Ghosh & Bhadra 2014). For axially symmetric Kerr geometry, it provides more limitations. Because of this, although the counter-rotating particle orbits have been accurately described, the Kerr–Newtonian potential cannot be used for $a > 0.7$ when describing the innermost circular geodesics that essentially determine the gravitational energy to be extracted from matter accreting on to the BH; the potential can be employed for $a > 0.7$ only when restricted to the radial range $r \geq 3r_s$. For general orbital trajectories, however, no such restriction on $a$ needs to be imposed; the geodesics are permissible over the entire radial range of $r \geq 2r_s$, for any value of $a$. Therefore, the derived potential could be used comfortably in studying realistic astrophysical processes around rapidly spinning BHs, at least for $a < 0.7$.

The most appropriate physical system to use this kind of potential is an accreting BH, because accretion of gaseous plasma around BHs is one of the few plausible ways to realize the presence of astrophysical BHs in the observed Universe. Realistic accreting plasma dynamics is extremely complex, comprising several microphysical processes. GR plasma equations with all the underlaid physical processes become extremely tedious, which then inevitably necessitates the study of these systems in the Newtonian hydrodynamical/MHD framework, but with a correct Newtonian analogue of GR effects. The Newtonian framework gives us the freedom to construct more robust accretion flow models with the detailed inclusion of two-temperature non-equilibrium plasma dynamics, the effect of collisionless plasma, precise radiative transfer equations and other necessary turbulent diffusive terms, especially around spinning BHs.

It is worth mentioning that BH accretion and related processes are also studied through GR MHD (numerical) simulations (e.g. Abramowicz & Fragile 2013, and references therein; McKinney & Gammie 2004; Hawley & Krolik 2006; Komissarov et al. 2007; McKinney, Tchekhovskoy & Blandford 2012). Recently, a few full three-dimensional GR radiative MHD codes have been developed to study BH accretion, which include COSMOS++ (Dibi et al. 2012; Fragile et al. 2012), KORAL (Sadowski et al. 2014) and GRHYDRO (Mösta et al. 2014). The latter code has been built within the framework of the Einstein tool kit. Such a simulation study, however, requires an expensive fast computing system, and even with such facility, the dynamical study can be performed at present only for a very limited time duration, considering only a subset of physics. Simulating accretion discs for a very large range of scales that can be present in a real system is also very difficult, if not impossible, with present-day computational facilities. So, PNPs are still useful to understand the underlying physics of accretion discs/jets.

Perhaps rotating BHs are universally present both in the local Universe in BHXRBs and in the centre of galaxies. The BH spin is directly responsible for plausibly powering astrophysical jets, generating QPOs in BHXRBs, increasing the radiative efficiency of accretion flow and several other accretion-related processes. It is extremely difficult to incorporate these effects with the appropriate physics in the accreting plasma dynamics in the exact GR framework. Then, the Kerr–Newtonian potential, in principle, becomes effective for the study of accretion flow and its implications around rotating BHs, avoiding GR fluid equations. The real test is to use them in real accretion scenarios in numerical and MHD simulation studies.

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Here, we provide the Cartesian transformation of the acceleration terms given by equations (16) and (17) for the test particle motion in the equatorial plane, corresponding to the Kerr–Newtonian potential. Using the following identities (see TR13),

\[ r = \sqrt{x^2 + y^2}, \]  
\[ \dot{r} = \frac{x \dot{x} + y \dot{y}}{\sqrt{x^2 + y^2}}, \]  
\[ \phi = \frac{(x \dot{y} - y \dot{x})}{x^2 + y^2}, \]

the accelerations of the particle motion in the \( x \) and \( y \) directions are then given by

\[ \ddot{x} = \frac{\dot{x} (\ddot{r} - r \dot{\phi}) - y (\dot{r} \dot{\phi} + 2 \dot{r} \phi)}{r}, \]
\[ \ddot{y} = \frac{\dot{y} (\ddot{r} - r \dot{\phi}) + x (\dot{r} \dot{\phi} + 2 \dot{r} \phi)}{r}, \]

where \( \dot{r} \) and \( \dot{\phi} \) are given by equations (16) and (17), respectively. With \( a = 0 \), the corresponding acceleration terms in equations (A4) and (A5) exactly reduce to those given in TR13. The corresponding \( r \) and \( \phi \) equations in Kerr geometry in the equatorial plane are given by

\[ \ddot{r} - 2 \frac{r - 2r_s}{r^2} \left[r^2 + 2a^2 r(r - 2r_s) - (3r^2 + a^2)(r - 2r_s) \frac{\omega \phi}{2} \right] \dot{r}^2 - \left[ \frac{\Delta^2}{r^2} - \frac{\alpha^2}{r^2} \right] \dot{\phi}^2 + \frac{GM}{r^3} (1 + \omega \phi)^2 \]
\[ \times \left[ \frac{c}{\epsilon} (r - 2r_s) \left( 1 - \frac{2 \alpha^2}{r^2} \right) + \frac{a^2}{r} (3r - 4r_s) - \frac{2 \Delta a}{r - 2r_s} \frac{\phi}{c (1 + \omega \phi)} \right] = 0 \]

and

\[ \ddot{\phi} + \frac{1}{\Delta} \left[ 2(r - 3r_s) - \frac{4a^2 r_s}{r^2} \frac{r - r_s}{r - 2r_s} + \frac{\alpha^2 (3r^2 + a^2)(r - 2r_s)}{r^2} \right] \dot{\phi} + \frac{ar c}{r^2 \Delta} (1 + \omega \phi)^2 \dot{r} = 0, \]

respectively. Equations (A6) and (A7) exactly reduce to that in the Schwarzschild case with \( a = 0 \) (see TR13).

Here, we show the derivation of radial epicyclic frequency \( \kappa \) corresponding to the Kerr–Newtonian potential. Following Section 3.2, the linearized perturbed equations are given by

\[ \delta \ddot{r} = \delta r \dot{\phi} \left[ 1 - a^2 (r^2 - 10r_s + 10r_s^2) + 8 \frac{r_s (r - r_s) a^2}{r^3 (r - 2r_s)^2} \right] + \delta r \left[ \frac{\Delta}{2r(r - r_s)} - \frac{\Delta (3r - 2r_s) a^2}{2r^2 (r - 2r_s)^2} \right] \frac{\omega \phi}{1 + \omega \phi} \]

\[ - \delta \dot{\phi} \left[ \frac{1}{r - 2r_s} \left[ D_1 \frac{\omega \phi}{1 + \omega \phi} + D_2 \frac{\omega \phi (1 - \omega \phi)}{(1 + \omega \phi)^2} \right] + \frac{2GM}{r^3} \left[ (r - 2r_s)(r - 4r_s) + a^2 \left( 1 - \frac{5r_s}{r} \right) \right] (1 + \omega \phi) \right] \]

\[ - \delta \dot{\phi} \left[ \frac{4GM \alpha}{c} + \frac{r - r_s}{r - 2r_s} \right] \left[ \frac{\Delta}{r(r - 2r_s)} \left[ \frac{5r^2 - 14r_s + 10r_s^2}{r(r - 2r_s)} - 2(r - r_s)(r - 2r_s) \right] \phi (1 + \omega \phi) \right] \]

\[ + \delta \phi \left[ \frac{4GM \alpha}{c} + \frac{r - r_s}{r(r - 2r_s)} \right] (1 + \omega \phi) \left[ 1 + \omega \phi \right] \]

\[ \left[ \frac{1}{r - 2r_s} \left[ D_1 + \frac{2GM \alpha}{c} + \frac{\omega \phi (1 - \omega \phi)}{(1 + \omega \phi)^2} \right] \right] \right\} \]  

and

\[ \delta \ddot{\phi} = - \frac{\delta r \dot{\phi}}{2(r - 2r_s)} \left[ \omega \phi \left[ (3 + \omega \phi) \right] \right] - \frac{2GM \alpha}{c} \left[ \frac{r - r_s}{r(r - 2r_s)} \right] (1 + \omega \phi)^3 \]

\[ - \frac{1}{\Delta} \left[ 2(r - 3r_s)(r - 2r_s) - 4 \frac{\alpha^2}{r} \right] \left[ 2 + \omega \phi \right] (1 + \omega \phi), \]

respectively. Here, \( D_1 = -(G_\Sigma/2r)[2r - 2r_s(r - 3r_s) - 4a^2 r_s^2/(r - 2r_s)] \) and \( D_2 = (G_\Sigma/2r) \). Equations (B1) and (B2) reduce to that in the Schwarzschild case with \( a = 0 \). We assume perturbed quantities \( \delta r = \delta r_0 e^{\kappa t} \) and \( \delta \phi = \delta \phi_0 e^{\kappa t} \) for harmonic oscillations, where \( \kappa \) is the
radial epicyclic frequency. Here, \( \delta r_0 \) and \( \delta \phi_0 \) are amplitudes and \( \kappa = \sqrt{(-1)} \) (Semerák & Zácek 2000; TR13). With the substitution of \( \delta r \) and \( \delta \phi \), equations (B1) and (B2) reduce to

\[
-k^2 \delta r = \dot{\phi}_c^2 \left[ 1 - a^2 (r^2 - 10a + 10r_i^2) + 8 \frac{r_i(r - r_i)a^4}{r^3(r - 2r_i)^3} \right] \delta r + \dot{\phi}_c^2 \left[ \frac{\Delta}{2r(r - r_i)} - \frac{\Delta(3r - 2r_i)a^2}{2r^2(r - 2r_i)^3} \right] \frac{\omega \dot{\phi}_c}{1 + \omega \phi_c} \delta r
\]

\[
- \dot{\phi}_c^2 \left[ \frac{\Delta}{2r(r - r_i)} + \Delta \frac{\omega \phi_c}{1 + \omega \phi_c} \right] \delta r + \frac{2GM \alpha}{r^5} \left[ (r - 2r_i)(r - 4r_i) + a^2 \left( 2 - \frac{5r_i}{r} \right) \right] \frac{\omega \dot{\phi}_c}{1 + \omega \phi_c} \delta r
\]

\[
- \dot{\phi}_c^2 \left[ \frac{\Delta}{2r(r - r_i)} + \Delta \frac{\omega \phi_c}{1 + \omega \phi_c} \right] \delta r + \frac{2GM \alpha}{r^5} \left[ (r - 2r_i)(r - 4r_i) + a^2 \left( 2 - \frac{5r_i}{r} \right) \right] \frac{\omega \dot{\phi}_c}{1 + \omega \phi_c} \delta r
\]

\[
+ \frac{4GM \alpha}{r^5} \left[ (r - 2r_i)(r - 4r_i) + a^2 \left( 2 - \frac{5r_i}{r} \right) \right] \frac{\omega \dot{\phi}_c}{1 + \omega \phi_c} \delta r
\]

and

\[
k \delta \phi = \frac{\dot{\phi}_c}{2(r - 2r_i)} \left[ 3 + \omega \phi_c \right] t \delta r - \frac{\dot{\phi}_c}{2(r - 2r_i)} \frac{2GM \alpha}{r^5} \left[ (r - 2r_i)(r - 4r_i) + a^2 \left( 2 - \frac{5r_i}{r} \right) \right] \frac{\omega \dot{\phi}_c}{1 + \omega \phi_c} \delta r
\]

\[
+ \frac{\dot{\phi}_c}{2(r - 2r_i)} \left[ 3 + \omega \phi_c \right] t \delta r.
\]

respectively. Solving equations (B3) and (B4), we eventually solve for radial epicyclic frequency \( \kappa \) given by equation (28) in Section 3.2. \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) in equation (28) are given by

\[
\mathcal{F}_1 = \frac{4GM \alpha}{r^5} \left[ (r - 2r_i)(r - 4r_i) + a^2 \left( 2 - \frac{5r_i}{r} \right) \right] \frac{\omega \dot{\phi}_c}{1 + \omega \phi_c} \delta r
\]

\[
+ \frac{\dot{\phi}_c}{2(r - 2r_i)} \left[ 3 + \omega \phi_c \right] t \delta r.
\]

respectively.

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