We review the (algebraic-)functional method devised by Galleas and further developed by Galleas and the author. We first explain the method for the simplest example: the computation of the partition function for the six-vertex model with domain-wall boundary conditions. At the heart of the method lies a linear functional equation for the partition function. After deriving this equation we outline its analysis. The result is a closed expression in the form of a symmetrized sum—or, equivalently, repeated contour integral—which can be rewritten to recover Izergin’s determinant. We comment on the connection with the method of Korepin and Izergin, the range of applicability of the functional method, and indicate how it may be adapted to the technically more involved example of the elliptic solid-on-solid model with domain walls and a reflecting end.

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* Department of Mathematical Sciences, Chalmers University of Technology and University of Gothenburg, SE-412 96 Göteborg, Sweden. julesl@chalmers.se
1 Introduction

The (algebraic-)functional method provides a way for analysing and computing key quantities, such as partition functions, for quantum-integrable models. This approach was first put forward by Galleas in 2010 [1] and subsequently developed by Galleas and the author [2–9].

Here we review the functional method for the simplest example, namely the exact computation of the domain-wall partition function, where it can be treated rigorously and provides a constructive way to recover Izergin’s determinant. We also review further examples of the functional method, in particular for the computation of the partition function of the elliptic solid-on-solid model with a reflecting end and domain walls at the other boundaries.

This text is based on a presentation delivered at the Les Houches Summer School on Integrability in June 2016 and a poster presentation at the ESI Workshop Elliptic Hypergeometric Functions in Combinatorics, Integrable Systems and Physics in March 2017. The emphasis lies on those parts of [9] that are new with respect to [7, 8].

2 The domain-wall partition function

To set the scene we give a recap of the six-vertex model, its algebraic description, the definition of the domain-wall partition function, and the exact computation of the latter by Korepin and Izergin. More details can be found in the references, see especially Chapters I and II of [9].

2.1 Set-up: the six-vertex model with domain walls

The six-vertex model is a classical statistical-physical model defined on a square lattice. The microscopic degrees of freedom are arrows pointing in either direction along each edge, subject to the (ice) rule that at every vertex two arrows point in and two point out. This leaves for the six allowed arrow configurations around each vertex shown in Figure 1 (a). We focus on the symmetric (‘zero-field’) case that is invariant under the global reversal of all arrows. The partition function

\[ Z = \sum_{\text{arrow configs}} a^\# \# + + + + + + + + b^\# + + + + c^\# + + + + \]  

(2.1)

counts the number of allowed configurations, each with a (Boltzmann) weight that keeps track of the occurring vertices. The result is a polynomial in the vertex weights \(a, b, c\).

Consider a portion of the square lattice consisting of \(L\) horizontal lines and \(L\) vertical lines. The inhomogeneous six-vertex model depends on \(L\) spectral parameters \(\lambda_i\) (one for each horizontal line of the lattice) and inhomogeneity parameters \(\mu_j\) (one for each vertical line), together with the (‘global’) crossing parameter \(\gamma\). We view the \(\lambda_i\) as variables and the other parameter as fixed. The three pairs of nonzero vertex weights are parametrized as

\[ a(\lambda) := \sinh(\lambda + \gamma) , \quad b(\lambda) := \sinh \lambda , \quad c(\lambda) := \sinh \gamma , \]  

(2.2)

where \(\lambda = \lambda_i - \mu_j\) for the vertex at which the \(i\)th horizontal line meets the \(j\)th vertical line. The corresponding partition function depends on all \(\lambda_i, \mu_j\) as well as \(\gamma\).

Algebraic formulation. For each line of the lattice let us introduce a copy of a two-dimensional vector space \(V\) with basis vectors \(|\downarrow\rangle\) and \(|\uparrow\rangle\) labelled by the two directions the
Figure 1. (a) The allowed vertex configurations for the six-vertex model. In the ‘zero-field’ case the corresponding vertex weights, from left to right, are \( a, b, c \).

(b) The allowed height profiles for Baxter’s solid-on-solid model. The local weights are \( a_{\pm}(z), b_{\pm}(z), c_{\pm}(z) \), again read from left to right and with a ‘+’ for the top row.

arrows on that line can have. Then the weights (2.2) for the six vertices in Figure 1 (a) can be encoded in the \( R \)-matrix

\[
R(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix}, \quad \begin{pmatrix} i \\ j \\ k \end{pmatrix} := \begin{pmatrix} + & 0 & 0 & 0 \\ 0 & + & 0 & 0 \\ 0 & 0 & + & 0 \end{pmatrix} . \tag{2.3}
\]

In the diagrammatic version we imagine ‘time’ increasing from the left and bottom to the right and top. The crucial property of the \( R \)-matrix is that it satisfies the Yang–Baxter equation,

\[
R_{ij}(\lambda_i - \lambda_j) R_{ik}(\lambda_i - \lambda_k) R_{jk}(\lambda_j - \lambda_k) = R_{jk}(\lambda_j - \lambda_k) R_{ik}(\lambda_i - \lambda_k) R_{ij}(\lambda_i - \lambda_j) \quad \text{or} \quad \begin{array}{c} i \\ j \\ k \end{array} = \begin{array}{c} i \\ j \\ k \end{array} . \tag{2.4}
\]

On the left the subscripts say on which factors of the \( 2^3 \)-dimensional vector space \( V_i \otimes V_j \otimes V_k \) each \( R \)-matrix acts nontrivially: \( R_{ij} = R \otimes 1, R_{jk} = 1 \otimes R \), and \( R_{ik} = (1 \otimes P)(R \otimes 1)(1 \otimes P) = (P \otimes 1)(1 \otimes R)(P \otimes 1) \) with \( P \) the permutation matrix. The diagram is a way to encode the same equation: the ‘time’ specifies the ordering, and the labels at the ‘incoming’ end of each line play the role of the subscripts (whence specifying the arguments, cf. just below (2.2)). From the \( R \)-matrix we construct for each row \( i \) the operators

\[
A(\lambda_i) := \begin{array}{c} 1 \\ 2 \\ \cdots \\ L \end{array} , \quad B(\lambda_i) := \begin{array}{c} 1 \\ 2 \\ \cdots \\ L \end{array} , \quad C(\lambda_i) := \begin{array}{c} 1 \\ 2 \\ \cdots \\ L \end{array} , \quad D(\lambda_i) := \begin{array}{c} 1 \\ 2 \\ \cdots \\ L \end{array} . \tag{2.5}
\]

Thus \( B(\lambda_i) = \langle \rightarrow | R_{iL}(\lambda_i - \mu_L) \cdots R_{i1}(\lambda_i - \mu_1) | \leftrightarrow \rangle \) and so on. These operators act on the \( 2^L \)-dimensional vector space \( \otimes_{j=1}^L V_j \) associated to the \( L \) vertical lines. A simple example of
such an action (on a dual vector) is given in (3.1) below. Since the \( R \)-matrix (2.3) satisfies the Yang–Baxter equation (2.4) the operators (2.5) obey commutation rules that are conveniently encoded in the so-called ‘\( RTT \)-relations’,

\[
\begin{array}{ccc}
i & \cdots & i' \\
1 & \cdots & L
\end{array}
\begin{array}{ccc}
i' & \cdots & i \\
1 & \cdots & L
\end{array}
= \begin{array}{ccc}
i' & \cdots & i \\
1 & \cdots & L
\end{array}
\begin{array}{ccc}
i & \cdots & i' \\
1 & \cdots & L
\end{array}.
\]

These are the defining relations of the Yang–Baxter algebra, with the entries of the \( R \)-matrix \( R_{i,i'}(\lambda_i - \lambda_i') \) playing the role of structure constants: each choice of arrows on the four external horizontal edges gives one such relation. (The bends in the horizontal lines have no significance, only their crossings do.) We will use a few of these relations in Section 3.1.

**Domain walls.** As with any statistical-physical model the goal from the viewpoint of physics is to study the thermodynamics for macroscopically large systems and correlation functions. The usual strategy is to first compute the partition function for finite but arbitrary system size \( L \) and subsequently study its asymptotic behaviour as \( L \to \infty \). Along the way one has to choose some boundary conditions. For periodic boundaries, so that the lattice is wrapped around a torus, (2.1) can be studied using the Bethe ansatz: see e.g. [10] and the references therein. This converts the computation of the partition function to the problem of solving a system of coupled algebraic equations, the Bethe-ansatz equations. The latter give enough information as \( L \to \infty \) to obtain exact expressions for macroscopic quantities such as the bulk free energy [11].

One expects the thermodynamic properties to be insensitive to the choice of boundary conditions used at the intermediate step of finite systems. It came as a great surprise that for the six-vertex model the thermodynamics does depend on the choice of boundary conditions, as Korepin and Zinn-Justin discovered in 2000 [12] while studying the case of domain walls with a particular configuration of fixed arrows on the boundary:

\[
Z(\lambda_1, \cdots, \lambda_L) := \langle \downarrow \cdots \downarrow | B(\lambda_1) \cdots B(\lambda_L) | \uparrow \cdots \uparrow \rangle = \begin{array}{ccc}
\downarrow & \cdots & \downarrow \\
\cdots & \cdots & \cdots \\
\uparrow & \cdots & \uparrow
\end{array}.
\]

(Here and later on the colours only serve to highlight the structure of the expressions.) This object, known as the domain-wall partition function, was introduced by Korepin 1982 [13]. Unlike for periodic boundary conditions, which have only been solved exactly in the limit of macroscopically large systems, the domain-wall partition function can be computed exactly for any finite system size \( L \). This is the topic of the next sections.

### 2.2 Context: Korepin–Izergin method

Let us write \( Z_L \) when we want to emphasize the system size under consideration. Korepin [13] showed that (for generic values of the inhomogeneities) the domain-wall partition function (2.7) is characterized by analytic properties,
• $Z$ is doubly symmetric: it is a symmetric function in the $\lambda_i$, and in the parameters $\mu_j$;
• $Z_L$ is a hyperbolic polynomial of degree $L-1$ in each variable $\lambda_i$: it equals $e^{-(L-1)\lambda_i}$ times a polynomial of degree $L-1$ in $e^{2\lambda_i}$ for each $i$,

together with a \textit{recursion relation} relating partition functions for successive system sizes,

$$Z_L(\lambda_1, \cdots, \lambda_L)|_{\lambda_L=\mu_L} = \text{factor} \times Z_{L-1}(\lambda_1, \cdots, \lambda_{L-1}) ,$$

with initial condition $Z_1(\lambda) = \frac{1}{1+z}$. On the right-hand side of (2.8), $Z_{L-1}$ does not depend on $\mu_L$, and the factor is a monomial in the vertex weights depending on all parameters of the model whose explicit form is not relevant for us here. Since (2.8), together with its analogues for $\lambda_i = \mu_j$ (by double symmetry), fixes $Z_L$ at sufficiently many distinct points (for generic values of the $\mu_j$) these properties uniquely determine $Z_L$ for each $L$.

Five years later Izergin [14] came up with a beautiful answer:

$$Z_L(\lambda_1, \cdots, \lambda_L) = \sinh^L \gamma \prod_{i,j=1}^L \sinh(\lambda_i - \mu_j + \gamma)$$

$$\times \frac{\det_{L \times L} \left[ \sinh(\lambda_k - \mu_i + \gamma)^{-1} \sinh(\lambda_k - \mu_i)^{-1} \right]}{\det_{L \times L} \left[ \sinh(\lambda_k - \mu_i - 1)^{-1} \right]} .$$

(The Cauchy-like determinant in the denominator is often written in the form of a product.) It is not hard to check that the function on the right-hand side of (2.9) meets all requirements, which proves that it is indeed the domain-wall partition function. In the homogeneous limit, where the $\lambda_i - \mu_j \to x$ become independent of $i$ and $j$, the (Hankel) determinant obtained from (2.9) can be used to compute the bulk free energy as $L \to \infty$ [12, 15]. A question that remains, however, is how one could cook up (2.9) in the first place. An answer is provided by the alternative method to which we turn now.

## 3 Functional method

The functional method consists of two steps:

1) use the algebraic structure underlying the model to derive a functional equation satisfied by the partition function;

2) analyse this functional equation to find a recipe for obtaining a closed formula for the partition function.

The first step is quite straightforward, the second step more involved—rather more so than the simple analysis involved in the Korepin–Izergin Section 2.2. As we will see, however, the functional approach is systematic, constructive and in fact contains the method of Korepin–Izergin.

---

1 The preceding properties can be obtained from (2.7). For symmetry in the $\lambda_i$ take the arrows on the horizontal external edges in (2.6) to be outgoing, $\widehat{\rightarrow} \cdots \widehat{\rightarrow}$. The polynomial property follows by noting that domain walls require at least one $c_-$ in every row, so there are at most $L-1$ vertices $a(\lambda_i - \mu_j)$ and $b(\lambda_i - \mu_j)$, which by (2.2) are hyperbolic monomials of degree one in $\lambda_i$. The precise form of (2.8) is found by noting that setting $\lambda_L = \mu_L$ in (2.7) forces the upper right vertex to be a $c_-$ since $b(0) = 0$ by (2.2); due to the ice rule this freezes the arrows on the upper row and right-most column.
3.1 From algebra to functional equations

The starting point for deriving the functional equation is the algebraic expression (2.7) for the domain-wall partition function. Because of the ice rule the operator \( A(\lambda_0) \) from (2.5) preserves the numbers of up- and down-pointing arrows, so \( \langle \downdownarrows \downdowarrows \downdownarrows | \rangle \) is a (dual) eigenvector:

\[
\langle \downdownarrows \downdowarrows \downdownarrows | A(\lambda_0) = \langle \downdownarrows | \otimes \downdownarrows \downdownarrows \downdownarrows = \langle \downdownarrows | \times \langle \downdownarrows \downdownarrows \downdownarrows | . \tag{3.1}
\]

Therefore we may insert \( A(\lambda_0) \) on the left of the \( B \)s in (2.7) at the expense of the \textit{eigenvalue}:

\[
\text{eigenvalue} \times Z(\lambda_1, \ldots, \lambda_L) = \langle \downdownarrows \downdowarrows \downdownarrows | \rangle A(\lambda_0) B(\lambda_1) \cdots B(\lambda_L) \langle \downdownarrows \downdownarrows \downdownarrows | . \tag{3.2}
\]

Now we can move the \( A \) past all \( B \)s using one of the relations contained in (2.6), which is of the form

\[
A(\lambda) \times B(\lambda') = \text{some structure constant} \times B(\lambda') A(\lambda) + \text{another structure constant} \times B(\lambda) A(\lambda') . \tag{3.3}
\]

In the first term on the right-hand side the \( A \) just moves past the \( B \)—up to a structure constant coming from the \( R \)-matrix in (2.6)—while in the second term \( A \) exchanges spectral parameter with the \( B \) it passes—again up to a structure constant. Thus the result of moving the \( A \) all the way to the right of the \( B \)s in (3.2) is a linear combination of terms for which the \( A \) ends up with any of the spectral parameters,

\[
\text{eigenvalue} \times Z(\lambda_1, \ldots, \lambda_L) = \sum_{\nu=0}^{L} \text{coefficient}_\nu \times \langle \downdownarrows \downdowarrows \downdownarrows | \prod_{\rho=0}^{L} B(\lambda_\rho) A(\lambda_\nu) | \downdowarrows \downdowarrows \downdowarrows | . \tag{3.4}
\]

The \( L + 1 \) coefficients come from the structure constants in (3.3) and are not hard to find when one exploits the fact that \( Z \) is symmetric in the \( \lambda_i \): this computation is routine in the context of the algebraic Bethe ansatz.

Finally use that \( | \downdowarrows \downdowarrows \downdowarrows \downdowarrows | \rangle \) is an eigenvector of \( A \) too—the computation resembles (3.1)—so \( A(\lambda_\nu) \) may be replaced by its \textit{eigenvalue}. This allows us to recognize the domain-wall partition function (2.7), depending on \( L \) out of the \( L + 1 \) different \( \lambda \)s, in each term in (3.4) to get

\[
\text{eigenvalue} \times Z(\lambda_1, \ldots, \lambda_L) = \sum_{\nu=0}^{L} \text{another coefficient}_\nu \times Z(\lambda_0, \ldots, \hat{\lambda}_\nu, \ldots, \lambda_L) , \tag{3.5}
\]

where the hat indicates that \( \lambda_\nu \) is to be omitted.

\(^2\) The precise relation is obtained from (2.6) by fixing the arrows on the horizontal external edges as \( \downdowarrows \downdowarrows \downdowarrows \downdowarrows \).
Result. The domain-wall partition function obeys the linear functional equation [5]
\[
\sum_{\nu=0}^{L} M_{\nu}(\lambda_0; \lambda_1, \cdots, \lambda_L) Z(\lambda_0, \cdots, \lambda_1, \cdots, \lambda_L) = 0 ,
\] (3.6)
where the explicit form of the coefficients obtained from (3.5) is
\[
M_0(\lambda_0; \lambda_1, \cdots, \lambda_L) := \prod_{j=1}^{L} \sinh(\lambda_j - \mu_j) - \prod_{j=1}^{L} \sinh(\lambda_j - \mu_j + \gamma) \prod_{j=1}^{L} \frac{\sinh(\lambda_j - \lambda_0 + \gamma)}{\sinh(\lambda_j - \lambda_0)}
\] (3.7a)
and, for \(1 \leq i \leq L\),
\[
M_i(\lambda_0; \lambda_1, \cdots, \lambda_L) := \prod_{j=1}^{L} \sinh(\lambda_i - \mu_j + \gamma) \prod_{j=1}^{L} \frac{\sinh(\lambda_i - \lambda_0 + \gamma)}{\sinh(\lambda_j - \lambda_i)} \prod_{j=1, j \neq i}^{L} \sinh(\lambda_j - \lambda_i) .
\] (3.7b)
This concludes the first step from the beginning of Section 3.

3.2 Analysis and solution

We reserve the notation ‘Z’ for the domain-wall partition function (2.7) and study the functional equation
\[
\sum_{\nu=0}^{L} M_{\nu}(\lambda_0; \lambda_1, \cdots, \lambda_L) F(\lambda_0, \cdots, \lambda_1, \cdots, \lambda_L) = 0 ,
\] (3.8)
with coefficients (3.7). Let us present the highlights of the analysis of this equation.

Firstly we may forget how we got (3.8) in the sense that it can be shown that sufficiently nice solutions to the equation necessarily share analytic properties with Z (cf. Section 2.2): if \(F\) is meromorphic then it is symmetric in the \(\lambda_i\), and if it is a hyperbolic polynomial in \(\lambda_i\) then it has degree \(L - 1\).

Since the number of variables in (3.8) is larger than the number of arguments of \(F\) we can specialize any single variable to a convenient value.

The greatest simplification occurs when \(\lambda_0 = \lambda_* := \mu_j - \gamma\). For \(L = 1\) this specialization in (3.8) implies that \(F = F_1\) is a constant, independent of \(\lambda_1\). The value of this constant is not determined by the functional equation, which after all is linear in \(F\).

For general \(L\) under this specialization of (3.8) can be solved for \(F(\lambda_1, \cdots, \lambda_L)\) to give
\[
F(\lambda_1, \cdots, \lambda_L) = \frac{\sum_{i=1}^{L} M_i(\lambda_*; \lambda_1, \cdots, \lambda_L)}{M_0(\lambda_*; \lambda_1, \cdots, \lambda_L) F(\lambda_*; \lambda_1, \cdots, \lambda_1, \cdots, \lambda_L)}
\]
\[
= \sum_{i=1}^{L} \frac{M_i(\lambda_*; \lambda_1, \cdots, \lambda_L)}{M_0(\lambda_*; \lambda_1, \cdots, \lambda_L)} \prod_{k=1, k \neq i}^{L} \sinh(\lambda_k - \mu_j) \times \tilde{F}(\lambda_1, \cdots, \lambda_i, \cdots, \lambda_L) ,
\] (3.9)
where the second equality is a consequence of the so-called ‘special zeroes’ that any \(F\) satisfying (3.8) possesses, and \(\tilde{F}\) is a hyperbolic polynomial of degree \(L - 2\) in each of its \(L - 1\) variables. For any solution of (3.8), if \(F\) is determined by the functional equation, then \(\mu_j - \gamma\) are eigenvalues of \(\tilde{F}\) which specialize to \(\lambda_*\) and \(\lambda_j\) respectively. For the domain-wall partition function this is a direct consequence of Korepin’s recurrence relation: \(\mu_j\) is a zero of the factor in (2.8). It may also be proven from (3.8) using the symmetry of \(F\) [9].

\[\text{\#3}\]
Interestingly, when we set $\lambda_L = \mu_L$ in (3.9) only the $L$th term in the second line survives and we precisely recover Korepin’s recurrence relation (2.8) if we would interpret $\tilde{F}$ as the solution for the functional equation for $Z_{L-1}$.

To see whether such an interpretation makes sense we plug (3.9) into (3.8) to find that if $F$ obeys (3.8) then $\tilde{F}$ does indeed obey the functional equation for $Z_{L-1}$!

It follows that we have recovered all ingredients from the Korepin–Izergin-method within our approach. Moreover, (3.9) now gives a recipe that allows us to construct $F_L = F$ in terms of $F_{L-1} = F$; crucially, unlike (2.8), our recipe does not involve any specialization on the left-hand side.

In particular, the recipe expresses any $F_L$ in terms of some $F_{L-1}$. Recall that the solution for $L = 1$ is just a constant. Thus the recipe uniquely determines $F_2$ up to a constant normalization factor. Repeating this argument it follows that the functional equation (3.8) has, up to normalization, a unique solution (in a suitable class of functions). This is a very nontrivial property for a functional equation.

By iterating (3.9) we can construct a closed direct formula for $F_L$. Fixing the normalization to match the (nonzero) value of the domain-wall partition function at any single point we have thus derived a formula for the latter in terms of a symmetrized sum:

$$Z_L(\lambda_1, \cdots, \lambda_L) = \sinh^L \gamma \sum_{\sigma \in S_L} \prod_{i<j} \sinh(\lambda_{\sigma(i)} - \mu_j + \gamma) \sinh(\lambda_{\sigma(i)} - \mu_j) \frac{\sinh(\lambda_{\sigma(i)} - \lambda_{\sigma(j)} + \gamma)}{\sinh(\lambda_{\sigma(i)} - \lambda_{\sigma(j)})}.$$  

(3.10)

Although this formula contains $L!$ terms, this is still much better than the $2^{L^2}$ terms than the domain-wall partition function naively contains, as in (2.1). One can recognize (3.10) as Baxter’s ‘perimeter Bethe ansatz’ [16] for the special case of a square lattice with domain-wall boundaries. Being symmetric in the $\lambda_i$ it is also possible to rewrite the right-hand side of (3.10) in terms of a repeated contour integral to obtain a multiple-integral formula for $Z$.

Although (3.10) is not obviously equal to Izergin’s determinant (2.9), one can directly show that the two expressions coincide using a version of Lagrange interpolation. (The latter is also the way in which Rosengren [17] first found his sum of determinants for the partition function of the elliptic solid-on-solid model with domain-wall boundary conditions starting from a symmetrized sum found by others before [18].)

4 Further examples

4.1 Comments on applicability

Having seen how the functional method works in a simple example one may wonder what the range of applicability of the approach is.

Grocery list. In the preceding text we used the following ingredients to apply the functional method successfully:

i) An algebraic expression for the quantity of interest. In principle this is available for many models using the solution of the ‘quantum-inverse scattering problem’ [19].

4 Fixing the overall normalization can be done by the leading behaviour as the spectral parameters tend to infinity [1–7], or more simply by evaluation at particular points [9].
ii) A suitable operator to insert in this algebraic expression in such a way that the resulting quantity can be computed in two ways; in particular all terms that arise in the computation should have an ‘appropriate’ form. More concretely, in Section 3.1 this amounted to the existence of an operator for which

a) both $\langle \cdots |$ and $| \cdots \rangle$ are eigenvectors;

b) the right-hand side of the relation (3.3) does not involve new operators that we cannot get rid of.

iii) The commutativity of the $B$s, and reversely, the symmetry of reasonable solutions to the functional equation. Perhaps it is possible to relax this condition; here it was convenient and used to establish the technical but important property that

iv) Reasonable solutions have ‘special zeroes’.

v) The ‘reduced’ function $\tilde{F}$ solves the functional equation for $F_{L-1}$, providing the reduction step to lower $L$.

Note that the Korepin–Izergin method also starts from (i) together with some analytic properties like the symmetry in (iii) that can be surmised from (i). Given a Korepin-type recurrence relation it is easy to locate the values of these special zeroes. On the other hand these recurrence relation follows from the functional equation; it is not yet known whether the existence of special zeroes and a Korepin-type recurrence relation are equivalent. Observe that ingredient (ii) is closely related to the computations from the algebraic Bethe ansatz.

Other examples. Luckily, the preceding ingredients are present in several other settings too. Note that $D$ from (2.5) meets requirement (ii) too, and one can indeed redo Section 3.1 with $D$ instead of $A$ to obtain another linear functional equation for the domain-wall partition function; since (3.6) already characterizes $Z$ by Section 3.2 we did not consider this other option. (It is also possible to insert $C$; as its relations with $B$ are less simple than (3.3) the resulting functional equation for $Z$ is rather more complicated [1, 2].)

Another example of ingredient (ii) is $t = A + D$ in the context of scalar products of (off-/on-shell) Bethe vectors, cf. [20], via the functional method [6]. On the other hand, for the domain-wall partition function of the Izergin–Korepin nineteen-vertex model, which was computed exactly at a particular root of unity [21], ingredient (ii)$^b$ seems to fail because the analogue of (3.3) mixes the two creation operators $B_1$ and $B_2$ for that model.

For other variations that do work one can

- upgrade the six-vertex model to Baxter’s solid-on-solid (SOS) model [3], and if so,
- further refine the SOS model to the elliptic case [4, 22];
- include one reflecting end, with domain walls on the three other boundaries [7].

The combination of these three options was treated in [8, 9]: we refer to the resulting quantity as the reflecting-end partition function for the elliptic SOS model. This is one of the technically most involved examples of the functional method, and our next topic. See [8, 9] for details.
4.2 The elliptic reflecting-end partition function

Baxter’s solid-on-solid (or interaction-round-a-face, IRF) model is a generalization of the six-vertex model that is naturally viewed as a height model: the microscopic degrees of freedom are (discrete) height variables associated to the faces (plaquettes) of the square lattice, so a configuration describes a height profile. Neighbouring heights must differ by one, allowing for the six height profiles around a vertex shown in Figure 1 (b). There is a nice correspondence between such height profiles and arrow configurations on the edges provided we know the height at any single face of the lattice. The rule to go back and forth between the two settings is the following: going anti-clockwise around a vertex, place an arrow pointing outwards (inwards) if the height increases (decreases) by one. Completing a circle we get back to the height we started at, so the arrows around the vertex must satisfy the ice rule from the six-vertex model. The result is sometimes called a generalized (six-)vertex model: it is just a version of the six-vertex model where we also keep track of (any single, whence all) heights. The partition function is a refinement of (2.1):

\[ Z = \sum_{\text{arrow configs}} \prod_{z \in \mathbb{Z}} a_+(z)^{n_a(z)} a_-(z)^{n_a(z)} b_+(z)^{n_b(z)} b_-(z)^{n_b(z)} c_+(z)^{n_c(z)} c_-(z)^{n_c(z)}. \] (4.1)

**Algebraic formulation.** We repeat the trick of passing to the inhomogeneous setting with variables \( \lambda_i \) associated to the horizontal lines and parameters \( \mu_j \) for the vertical lines. The algebraic reformulation is again based on an \( R \)-matrix containing the weights for the profiles from Figure 1 (b), where we prefer the viewpoint of the generalized vertex model to stress the similarity with (2.3):

\[ R_{ij}(\lambda, z) = \begin{pmatrix} a_+(\lambda, z) & 0 & 0 & 0 \\ 0 & b_+(\lambda, z) & c_-(\lambda, z) & 0 \\ 0 & c_+(\lambda, z) & b_-(\lambda, z) & 0 \\ 0 & 0 & 0 & a_-(\lambda, z) \end{pmatrix}, \quad i \xrightarrow{z} j \xrightarrow{z} i \]

In the diagrammatic version we have now explicitly oriented each line as indicated by a little arrow (not to be mistaken for a microscopic degree of freedom) at the ‘outgoing’ end: this will help us keeping track of the flow of ‘time’ (operator ordering) as we will have to rotate some of the figures in the presence of a reflecting end. The interesting feature of this so-called ‘dynamical’ \( R \)-matrix is that, unlike (2.4), the dynamical Yang–Baxter equation

\[ \begin{pmatrix} \varphi_+ & 0 & 0 \\ 0 & \varphi_+ & 0 \\ 0 & 0 & \varphi_+ \end{pmatrix} \]

admits a block-diagonal solution (4.2) with an elliptic parametrization for the weights:

\[ a_{\pm}(\lambda, z) = f(\lambda + \gamma), \quad b_{\pm}(\lambda, z) = f(\lambda) \frac{f(z \mp 1)}{f(z)}, \quad c_{\pm}(\lambda, z) = f(\pm 1) \frac{f(z)}{f(z)} f(\gamma), \] (4.4)

where \( f(\lambda) := -i e^{-i \pi \tau/4} \vartheta_1(i \lambda | \tau)/2 \) is essentially the odd Jacobi theta function with elliptic nome \( e^{i \pi \tau} \in \mathbb{C} \) for \( \tau \) in the upper half plane, \( \text{Im}(\tau) > 0 \). The trigonometric case \( f(\lambda) \rightarrow \)
$\sinh \lambda$ is contained in the limit $\tau \to i\infty$; if moreover $z \to \infty$ we recover (2.2) up to some factors, reproducing the setting of [7]. The appearance of an elliptic solution reflects the close connection between the elliptic SOS model and the eight-vertex model.

One can proceed to define single-row operators similar to (2.5), now depending on the height at any single face, that will obey relations akin to (2.6). However, we still have to look at the

**Reflecting end.** A reflecting end is a special choice of boundary conditions that may be imposed at any of the four boundaries and is compatible with quantum integrability. The boundary is governed by the ‘dynamical $K$-matrix’, which we take to be diagonal:

$$K_i(\lambda_i, z) = \begin{pmatrix} k_+(\lambda_i, z) & 0 \\ 0 & k_-(\lambda_i, z) \end{pmatrix}.$$  \hspace{1cm} (4.5)

At this point the orientation of the lines starts to matter: we will think of the diagram on the left-hand side of (4.5) as a horizontal line, with associated parameter $-\lambda_i$ (note the sign!), that comes in at the bottom right and bounces off the wall to continue to the top right with parameter $+\lambda_i$.

In the diagrammatic notation, quantum integrability amounts to the possibility of sliding any line through crossings of any other two lines, see (2.4), (2.6) and (4.3). From Cherednik [23] we know that in the same spirit we should demand (4.5) to obey the dynamical reflection equation

$$ \begin{pmatrix} z \\ i' \end{pmatrix} = \begin{pmatrix} k_+(\lambda_i, z) & 0 \\ 0 & k_-(\lambda_i, z) \end{pmatrix} \begin{pmatrix} z \\ i \end{pmatrix}.$$  \hspace{1cm} (4.6)

Observe that this equation features (rotated versions of) the dynamical $R$-matrix. Given (4.2) the solution (4.5) involves a free parameter $\zeta \in \mathbb{C}$ that we can associate to the reflecting end:

$$k_+(\lambda, z) = f(\zeta + \lambda) \frac{f(z + \zeta - \lambda)}{f(z + \zeta + \lambda)}, \quad k_-(\lambda, z) = f(\zeta - \lambda).$$  \hspace{1cm} (4.7)

Because we consider ‘diagonal reflection’ we actually know that the face above the reflecting line has the same height as the one below it: $\begin{pmatrix} z \\ i \end{pmatrix}$.

Sklyanin [24] taught us that in the presence of such a reflecting end one should work with double-row operators

$$A(\lambda_i, z) := \begin{pmatrix} \vdots \\ \ddots \\ z \\ 1 \\ 2 \end{pmatrix}, \quad B(\lambda_i, z) := \begin{pmatrix} \vdots \\ \ddots \\ z \\ 1 \\ 2 \end{pmatrix},$$  \hspace{1cm} (4.8)

$$C(\lambda_i, z) := \begin{pmatrix} \vdots \\ \ddots \\ z \\ 1 \\ 2 \end{pmatrix}, \quad D(\lambda_i, z) := \begin{pmatrix} \vdots \\ \ddots \\ z \\ 1 \\ 2 \end{pmatrix}.$$
By virtue of (4.3) and (4.6) these operators also obey certain relations, contained in

\[
\begin{align*}
\iota & \cdot \cdot \cdot \iota' \\
\zeta_1^2 & L \cdots
\end{align*}
\]

(4.9)

The explicit commutation rules are again found by fixing the microscopic degrees of freedom on the four external horizontal edges, and the structure constants of the resulting dynamical reflection algebra are built from the entries of the dynamical $R$-matrix.

With all these preliminaries in place the reflecting-end partition function can be defined as

\[
Z(\lambda_1, \cdots, \lambda_L; z) := \langle \cdots | B(\lambda_1, z) \cdots B(\lambda_L, z) | \cdots \rangle = \cdots
\]

(4.10)

There are $2L$ horizontal lines, pairwise connected by the reflecting end, and $L$ vertical lines. The domain walls on the three other ends look just as in (2.7). The result is consistent with the ice rule: there are equally many arrows pointing in and out of the lattice. Due to the diagonal reflection (4.5) each face along the wall has the same height $z$, and all $B$s have equal ‘dynamical’ argument. Since (4.10) is a polynomial in the weights (4.4) and (4.7), the reflecting-end partition function is an elliptic polynomial, or more precisely a ‘higher-order theta function’, in all parameters.

**Korepin–Izergin method.** The reflecting-end partition function was computed using approach the from Section 2.2 for the six-vertex model by Tsuchiya [25] and in the sos setting by Filali and Kitanine [26], yielding expressions in the form of determinants. A surprising feature of these formulae is that both the dynamical parameter $z$ and the boundary parameter $\zeta$ appear exclusively in the prefactor, not in the determinant itself. The double-row texture is, however, still visible in the determinant through crossing symmetry: all factors depending on the spectral parameters come in pairs related by $\lambda_i \mapsto -\lambda_i - \gamma$. As a result the (elliptic) polynomial degree (or order) of $Z$ in each spectral parameter is higher, but the crossing symmetry also allows one to get Korepin-like recursion relations at sufficiently many more points to characterize the reflecting-end partition function.

**Functional method.** The present setting contains several layers of complications compared to that in Section 2. Let us sketch how the functional method can be adapted. Of course (4.10) gives us ingredient (i) from Section 4.1. For (ii) a candidate is $\mathcal{A}$; by the ice rule it obeys (ii*).
The reflecting end, however, makes the relevant commutation rule more complicated,

\[
\mathcal{A}(\lambda, z) \mathcal{B}(\lambda', z) = \text{some structure constant} \times \mathcal{B}(\lambda', z) \mathcal{A}(\lambda, z) + \text{another structure constant} \times \mathcal{B}(\lambda, z) \mathcal{A}(\lambda', z) + \text{yet another structure constant} \times \mathcal{B}(\lambda, z) \mathcal{D}(\lambda', z),
\]  

(4.11)

with an additional term compared to (3.3) that might spoil (ii\textsuperscript{b}). Sklyanin [24] found the way out: replace \( \mathcal{D} \) by a certain linear combination \( \tilde{\mathcal{D}} \) of \( \mathcal{A} \) and \( \mathcal{D} \), so that (4.11) can be rewritten in terms of only \( \mathcal{A} \) and \( \tilde{\mathcal{D}} \). This also fulfils (ii\textsuperscript{a}) so that there is a similar relation with the roles of \( \mathcal{A} \) and \( \tilde{\mathcal{D}} \) reversed to tell us how to move \( \tilde{\mathcal{D}} \) past the \( \mathcal{B} \)s. Since this \( \tilde{\mathcal{D}} \) also fulfils (ii\textsuperscript{a}) we meet requirement (ii) and can derive a functional equation for the partition function.

The commutation of \( \mathcal{A} \), and the \( \tilde{\mathcal{D}} \) generated along the way, through the \( \mathcal{B} \)s in (4.10) is facilitated by the fact that the \( \mathcal{B} \)s commute, cf. the first part of (iii). The result is a linear functional equation of the form (3.6) for (4.10). The coefficient with \( \nu = 0 \) has the same structure as (3.7a), whereas those for \( 1 \leq i \leq L \) now consist of two terms, each with a structure resembling (3.7b): one comes from \( \mathcal{A} \) reaching \( | \cdots \rangle \) and the other from the \( \tilde{\mathcal{D}} \) picked up along the way doing so. See [8, 9] for the explicit expression.

Next we outline the analysis of the functional equation. As before the equation immediately reveals several analytic properties of its solutions: any reasonable (viz. meromorphic) solution is a symmetric function of the \( \lambda_i \), is crossing symmetric, and ought to have the same (elliptic) polynomial structure as the reflecting-end partition function.\textsuperscript{5} Because of the crossing symmetry there should be more ‘special zeroes’ with values that are readily guessed; we did not prove ingredient (iv) but confirmed it numerically for \( L \leq 15 \). Using these special zeroes one can recover the recurrence relation of Tsuchiya–Filali–Kitanine for the reflecting-end partition function. For the reduction step (v) a proof is again lacking, but numerical checks have been performed for \( L \leq 12 \) [8]. Despite these gaps in the rigorous treatment of the functional method in this case one can still put everything together to obtain a recipe leading to a formula for a solution, or the solution according to the aforementioned numerical checks. The result is a symmetrized sum that is equivalent to the Tsuchiya–Filali–Kitanine determinant. Since the aim of this text is to review the functional method we refer the reader interested in the explicit expression to [8, 9].

5 Summary and outlook

In short, the functional method

- contains the approach of Korepin–Izergin, as was shown in [9];
- is constructive: it provides a recipe to get a direct formula for \( Z \) [2];
- can be made rigorous, as was done for the domain-wall partition function in [9];
- is fairly general: it can for example also be applied to the elliptic solid-on-solid model with domain walls and one reflecting end [8, 9].

To stress the second point we used the terminology constructive method in [9].

\textsuperscript{5} The coefficients of the functional equation are ratios of elliptic polynomials. A comparison of their orders and norms give strong evidence for what polynomial structure the solution will have. (To turn this into a proof one has to make sure that there cannot be any cancellations between the different terms.)
To date all cases in which the functional method has been used to obtained a closed expression were previously tackled using the Korepin–Izergin method. For the six-vertex model the domain-wall partition function [1, 9] was of course first found by Korepin and Izergin [13, 14], the reflecting-end partition function [7] by Tsuchiya [25], and the corresponding (off-/on-shell) scalar products of Bethe vectors [6] by Slavnov [20], Wang [27] and Kitanine et al. [28]. In the dynamical (sos) case the domain-wall partition function [3, 4] was found by Rosengren [17], and the reflecting-end partition function [8, 9] by Filali and Kitanine [26].

Of course the real challenge is to apply the functional method to situations that have not been tackled yet. Although the functional method is more involved than the Korepin–Izergin method, it may be useful in cases where it is hard to guess the solution. One can envision a hybrid approach, in which the functional method—possibly with the guidance of a Korepin-type recurrence relation to locate the special zeroes—is used to come up with a formula that can be proven to be correct by checking that it obeys all conditions from the Korepin–Izergin method. Unfortunately the computation of the domain-wall partition function of the nineteen-vertex model [21] seems out of reach, cf. the end of Section 4.1. Another opportunity that comes to mind is the computation of n-point correlators, which can be interpreted as partition functions of lattices where the arrows on n edges are fixed. Applications in a different direction are being developed by Galleas [22, 29–31].

Finally we cannot help noticing that the functional method provides a beautiful example of the rigid structure imposed by the underlying algebra, which is reflected in the many remarkable properties of the functional equations obtained in this way.

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