PAC Learnability of nuclear masses

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Abstract

After more than 80 years from the seminal work of Weizsäcker and the liquid drop model of the atomic nucleus, theoretical errors over nuclear masses (\(\sim\) MeV) are order of magnitudes larger than experimental ones (\(\lesssim\) keV). Predicting the mass of atomic nuclei is with precision is extremely challenging due to the non–trivial many–body interplay of protons and neutrons in nuclei, and the complex nature of the nuclear strong force. This paper argues that the arduous development of nuclear physics in the passed century is due to the exploration of a system on the limit of the knowledgeable, defined within the statistical theory of learning.
The atomic nucleus is the system in nature which embodies three of the four fundamental forces. The nuclear strong force determines the binding of protons and neutrons (nucleons) forming the atomic nuclei. The electromagnetic force drives the proton repulsion within the nucleus and the attraction of electrons to the nucleus, forming atoms and molecules. The nuclear weak force regulates nuclear beta decays and the ratio between protons and neutrons in nature. The interplay of these forces generates the binding energy by which nuclei are held together, and therefore the nuclear masses. The binding energy in turn establish properties of every nuclear process, such as stellar burning.

A model providing an accurate reproduction of nuclear masses would represent a stepping stone testifying to the understanding of the atomic nucleus. It would foster applications of this knowledge to other fields of science and technology. Many relevant properties of the nucleus are extremely sensitive to this quantity: decay lifetimes and reaction rates having 5th order polynomial or even exponential sensitivity to the difference of binding energy between two neighbouring nuclei; this implies the need of extreme accuracy in the modeling of the system that guarantees prediction of binding energies with the highest possible precision. The difficulty of improving precision of nuclear mass models was hinted in the context of chaotic quantum systems \cite{1, 2}: the statistical distribution of masses shows a chaotic behaviour that is formidable to deterministically reproduce.

In this paper, the problem of devising a model capable of reproducing and predicting nuclear binding energies will be considered as a statistical learning problem \cite{3}. The physicists use experimental data regarding atomic nuclei to draw conclusion about the system. Therefore, physicists can be considered as devising a specific model belonging to a class of models Λ by fitting the related free parameters and selecting functional forms to minimize the deviation between model and experimental data.

In nuclear structure (that concern the state of an isolated nucleus), the data available is related to the finite number of nuclear isotopes available in the lab and overall that can possibly exist. In particular in the case of theories that concern exclusively with ground state properties (as for mass models and density functional theory), only the few ground state properties (i.e. binding energy and radii) of these nuclei so far discovered are available in the dataset. As of the last atomic mass evaluation \cite{4}, 3435 nuclei have been measured in the laboratories around the world. Due to the lack of a comprehensive model of nuclear binding energies, is not known how many nuclei could exist in total, but is safe to assume
that the current number will not change by more than an order of magnitude (the current consensus doubles the number of nuclei discovered for 7000 existing [5]) and in practice will not be measured in the foreseeable future. This limited number of data available poses a limit on the possible precision and predictive power of the model selected with learning theory from the class of models \( \Lambda \).

Statistical learning introduces concepts useful to frame this aspect of research in physics within quantifiable boundaries [6]. The objective of a general learning problem is the minimization of the total risk functional \( R(\alpha) \). That is, finding the set of parameters \( \alpha \) by which a given model generalizes to a theory by best reproducing all possible data that could be taken under consideration. However, physics works with a finite set of experimental data. These data constitutes a training set by which physicists want to extrapolate a general behaviour, and promote a model over available data to a generalized theory that can describe a range of possible data not yet measured. Therefore what a physicist minimizes is the empirical risk, usually defined as residual over a number \( l \) of data, of a model function or functional \( f_\alpha \) which takes a set \( \alpha \) of parameters belonging to a space \( \Lambda \) which defines the class of models under consideration. That is, for the empirical risk,

\[
R_{\text{emp}}(\alpha) = \frac{1}{l} \sum_{i=1}^{l} (y_i - f_\alpha(x))^2, \tag{1}
\]

the value of \( \alpha \) which minimizes \( R_{\text{emp}}(\alpha) \) have to be found.

Under specific conditions defined by the empirical risk minimization principle (ERM), the minimum of \( R_{\text{emp}}(\alpha) \) converges (in probability) to the minimum of the total risk functional \( R(\alpha) \) when the number of data is large \( (l \to \infty) \) [3]. Therefore enabling the generalization over situations not yet encountered.

One could argue that physics sits in an agnostic learning paradigm, being that is not possible to suppose that the best solution available belongs to the model class \( \Lambda \) under consideration. However, the general class of models under consideration have the guarantee to possibly reproduce data: perceptrons are universal approximators [7], and similarly in DFT is guaranteed the existance of a universal functional [8] that will reproduce density of states (therefore, ground state properties) of a given physical system. In the following we will therefore use non–agnostic PAC learning.

When the number of data is “small” a good empirical model defined by \( R_{\text{emp}} \) does not guarantee a corresponding predictivity (i.e., good total risk). In the case of limited number
data, the “complexity” of the model plays a role regarding the best possible result and cannot be automatically pruned by fitting techniques. This is well known by the physicist community as the Occam’s razor principle [9] (cf. also [10]). Statistical learning theory can further quantify the impact of the tradeoff between complexity and data availability through the structural risk minimization (SRM) induction principle [6, 11].

The complexity of a function is defined through the concept of Vapnik and Chervonekis (VC)–dimension [12]. The VC–dimension is, for a set of boolean functions $\Theta_\alpha(x)$ with $\alpha \in \Lambda$, the maximum number $h$ of input vectors $x_1, \ldots, x_h$ that can be shattered, i.e. separated in the $2^h$ possible ways by the function in set $\Lambda$. The definition can be generalized for a bounded, real model $a \leq f_\alpha(x) \leq b$, with $a, b \in \mathbb{R}$ (in the mass model case, e.g. $a = 0$ MeV and $b = 8.7945$ MeV is the maximum binding energy per nucleon, that is of $^{62}$Ni isotope), defining a corresponding set of boolean functions

$$\Theta_\alpha(x, \beta) = \theta(f_\alpha(x) - \beta),$$

with $\theta$ the Heaviside unit step function ({$\theta(z) = 0$ for $z < 0$, and $\theta(z) = 1$ for $z \geq 0$}). The VC–dimension of the set of real valued $f_\alpha(x)$ corresponds to the VC–dimension of the set of the indicator functions $\Theta_\alpha(x, \beta)$ in Eq. (2).

If we consider a polynomial in $f : \mathbb{N}^2 \to \mathbb{R}$ the VC–dimension is given by the number of points in $\mathbb{N}^2$ the related indicator function (2) can shatter. A lower bound on its dimension is given by lifting the polynomial to the space of its monomials, and generating a set of points associated with each of the terms of the basis of polynomials. A useful polynomial to compare with mass-models is,

$$E(N, Z) = \sum_{i,j=0}^{N} a_{ij} A^i Z^j,$$

where $A$ is the total number of nucleons, $Z$ the atomic charge or number of protons, and the binding energy $E$ is parametrized as a polynomial of these variables. The VC–dimension of such polynomial is therefore $h = (N + 1)^2$.

For a real valued feed–forward neural network, the VC–dimension $h$ was demonstrated to be $O(N) \leq h \leq O(N^2)$ [13, 14], with $N$ the number of weights. The lower limit of the given band remains essentially valid as lower bound if the activation function is a sigmoid.

The lower bound of examples needed for learning was demonstrated in the case of binary classifier [14], and the understanding of the limit case study has been improved reaching the
exact bound [16]. Later on, the bound of number of examples needed was demonstrated for bounded functions [17] in \( \mathbb{Z} \), and has been extended based on fat–shattering dimension to include noisy data [18] for agnostic learning. This work will consider to the general lower bound of examples needed to possibly learn in a PAC learning setting, derived from the Hoeffding’s inequality [19],

\[
m \geq \frac{1}{\epsilon} \left[ \ln(h) + \ln(1/\delta) \right],
\]

where \( \epsilon \) is the generalization error, that is the difference between the average of the distribution of the loss (e.g. RMSD) over the training set, respect to the average value of an ideal distribution. At variance with the empirical risk minimization, there is no guarantee that the function \( f_\alpha \) with \( \alpha \in \Lambda \) with error \( \epsilon \) exists, but only a \( 1 - \delta \) probability.

Therefore, there is probability \( 1 - \delta \) that exist within our hypothesis space \( \Lambda \) a function where the training error of the distribution is \( \epsilon \) away from the total error over all possible inputs.

![Graph](image)

**FIG. 1.** Comparison of error lower bound and results of neural network training for a given number of data \( m \). \( \sigma \) root mean square deviation of a feedforward neural network with 50 nodes in the input layer, one hidden layer comprised of 1000 nodes and 1 output layer, all with rectified linear unit activation function, on the \( k \)-fold cross validation of the AME16 dataset (points). Related \( \epsilon \) lower bound from (4) with VC–dimension defined as directly proportional to number of weights (line).

Eq. (4) can be now used to estimate the number of data needed for a given precision, and to estimate the minimum expected error \( \epsilon \) using \( m \) data points to train a model with VC-
FIG. 2. Comparison of error lower bound and results of neural network training with a given number of nodes \( n \) and hidden layers, figure (top) and close-up at small error/deviation (bottom). 

\( \sigma \) root mean square deviation of a feedforward neural network on 10-fold cross-validation of the AME16 dataset (points). The neural network consist of 50 input layer, 1 output layer, a number of hidden layers specified by the following color coding: blue (1), green (2), red (4), cyan (8), magenta (16), yellow (32), black (64). Each of the hidden layers has \( n \) nodes. \( \epsilon \) corresponding lower bounds from [4] with VC–dimension defined as directly proportional to number of weights (line).

dimension \( h \). The dataset used consists of the 2016 Atomic Mass Evaluation [4], considering all nuclei with \( A > 16 \), including the phenomenological estimates, for a total of 3336 nuclei and associated masses.

At first, the PAC–learning bounds have been validated considering a training of neural network based on a feed–forward neural network. The network must take \( A, Z \) integer doublets and give back a floating point number that represent \( E \). Therefore, the network is a model \( f_\alpha : \mathbb{N}^2 \rightarrow \mathbb{R} \), with \( \alpha \in \Lambda \) are the parameters of the network. The network is composed of an input layer with 50 sigmoid nodes, and a single output node to give \( E \). In between a number of hidden nodes and layers with rectified linear unit (reLU) activation function, that is varied to test different VC–dimensional networks. The number of weights for such a network is \( 51n + n^L \) with \( n \) the number of nodes in a layer and \( L \) the number of layers, therefore its VC–dimension is at least \( h \geq 51n + n^L \). The structure has been chosen after a hyperparameter optimization for good performance. Rectified linear unit has been
chosen for its piecewise linear structure, that guarantees the neural network dimensionality bounds of [13, 14].

The training loss is evaluated and optimized as root mean square deviation (RMSD) between calculated and experimental binding energies in a training set. The RMSD in Figs. 1, 2 are evaluated on the validation set, using k–fold cross validation technique, which guarantees an assessment of the fitting procedure which reduces the bias in the test/validation selection [20, 21]. The uncertainty related to the RMSD is calculated as standard deviation for different folds. The results presented in Figs. 1, 2 testify to the reliability of PAC learning bounds for the treatment of this system, the RMSD $\sigma$ of models for different neuron number approach the PAC limit of $\epsilon$ when the number of neurons can be trained to reliably describe the system.

The Weizsäcker semi–empirical mass formula [22, 23] is one of the first attempts to describe the binding energy $BE$ of an atomic nucleus in function of powers of the number of nucleons $A$ and protons $Z$,

$$E(A, Z) = a_vA - a_sA^{2/3} - a_C\frac{Z(Z-1)}{A^{1/3}} - a_a\frac{(A - 2Z)^2}{A} + a_p\delta_p\frac{\sqrt{A}}{A},$$  \hspace{1cm} (5)

where $\delta_p$ is 1 for $A$ and $Z$ even, 0 for $A$ or $Z$ odd, -1 for $A$ and $Z$ odd. The coefficients of the semi–empirical model mass formula, consist in the well known volume $a_v$, surface $a_s$, Coulomb $a_C$, asymmetry $a_a$ and pairing $a_p$ terms. It is of notice that its fractional contributions are not directly related to Eq. (3). However, it is straightforward to relate this function to a VC-dimension of at least $(N = 6) + 1)^2$, if we consider with no bias all the possible combinations of radius, surface, volume and symmetry terms, or 6 if we consider the parametrization fixed by the physical derivation.

Density functional theory is an extremely successful model for atomic nuclei [24, 25] and it is used to describe systems composed of many quantum particles in general [26]. It is based on the Hohenberg-Kohn theorems [8], that is: i) ground state properties of a many–fermion system are uniquely determined by the fermion density, and ii) there exists one and only one density that minimize the functional, that is the ground state density and its energy density functional returns the ground state energy. This latter preposition is known as variational principle. In other words, the model functional of density $\rho$, $E[\rho(x)]$ is given by the expectation value of some Hamiltonian $H$, will be minimized varying the densities with some Lagrangian constrains (e.g. that the densities contain the correct number of particles).
and its minimum will correspond to the exact ground state density and corresponding energy.

The functional $E_\alpha[\rho(x)]$ is usually a complicated combination of densities derived from a pseudo-potential \cite{27,28}. The parameters $\alpha$ of the pseudo-potential are tuned to reproduce physical ground state properties and then the densities and solutions are derived thanks to the variational principle. Therefore, the same principle of risk minimization and consequent bounds apply. However, to calculate the exact VC-dimension of a complicated functional is not easy considering the non-linearity of the operation the functional applies on the density functions. But a conservative lower bound can be provided considering that,

$$h(E[\rho(x)]) \leq h(E(x)).$$  \hspace{1cm} (6)

The very popular Skyrme density functional is composed of a contact interaction, with a momentum-dependent term (which translates in derivatives of the densities in the functional) and a density dependent term. The functional can be related to a polynomial expansion (plus the density dependent term) of the density using \cite{6}, reducing to the VC-complexity of a second order polynomial \cite{29} over two dimensions (neutrons and protons) and 8 constrains on the parameters, therefore with dimension at least $h_{\text{Skyrme}} \geq 2((N = 2) + 1)^2 - 8 = 10$. As a title of example, in the case of the Gogny functional \cite{30} the pseudo-potential is composed by two Gaussians with different widths and a density-dependent term. The Gaussian itself has VC-dimension 3, and there are 8 terms for each, making the VC-dimension at least $h_{\text{Gogny}} \geq 24$.

Interestingly, theorem 6.8 and following of \cite{6} state that a good rate of convergence can be reached only for smooth functions. Despite being derived from pseudo-potentials with difficult discontinuities (e.g. Skyrme pseudo-potential is a combination of Dirac $\delta$) the resulting densities are smooth and therefore can be converged.

A long standing problem in the creation of nuclear density functionals and mass models is the number and type of data that has to be included in their creation. This work moves towards the quantification of the answer to this problem. The constrains provided in Table I represent the bounds for different nuclear physics model considering a training only on mass information. The considered cases of Skyrme and Gogny functional include information concerning properties nuclear matter (an hypothetical system composed of an infinite number of neutrons or an homogeneous combination of protons and neutron), spectra, deformation and other information. The objective of developing a functional is more general than an
TABLE I. Properties of different models to describe nuclear physics masses. “Neural network” refers to a neural network with 50 nodes in the input layer, one node in the output layer, with sigmoid activation function, and one hidden layer comprised of 1000 nodes with reLU activation function, for a total of 52000 weights. The columns represent i) the lower bound on VC–dimension for the given model, as explained in the text; ii) root mean square deviation $\sigma$ of experimental and calculated masses with state-of-the-art models in reference; iii) lower bound on the $\epsilon$ error provided by the Hoeffing inequality in PAC–learning; iv) number of data points needed to reach a generalization error $\epsilon$ of 100 keV with 99% probability, a considerable improvement to current bounds. To be noted that the result for Gogny D1M in [32], contains beyond DFT corrections.
in the Weiszacker semi–empirical mass formula surface and Coulomb parameters are highly correlated [33]. This decreases the VC dimensionality but also the post– and pre–dictive power of the model.

The close values of the bound $\epsilon$ and the deviation $\sigma$ of state of the art models, testifies to the approach of the limits of learnable as defined by PAC learning. The difficulty in further improving mass models to reach a predictive and precise estimate of nuclear masses might be then not be a shortcoming on some specific model, but explained instead as a necessary richness in information content unlearnable on the basis of masses (or few ground state properties) alone.

Concluding, this work for the first time has investigated the VC–dimension related to a many-body method and its implication regarding the performance that a given training can reach. This suggests that many–body methods might not only be judged by their computational complexity, as in the field of Hamiltonian complexity studies [34, 35], but also in terms of their information complexity represented by VC–dimension and PAC learning bounds.

Acknowledgment

The Quadro P6000 GPU used for this research was donated by the NVIDIA Corporation. This work benefited from discussions with the participants to the workshop “Novel approaches for the description of heavy nuclei” organized at Lund University 19–21 March 2019, with the contribution of Newton Alumni Fellowship of the Royal Society. The source code used will be included in the supplementary material of the publication and publicly released on a git server at a later date.

Methods

The adopted geometry of the neural network and training: 50 input nodes, 1 output nodes and training making use of 0.5 dropout of the hidden layers have been obtained as the most consistent values after a tree Parzen estimate hyperparameter optimization (based on Expected Improvement method). Other than sigmoid activation function, also rectified linear unit and softmax have been tried in all the possible combination between input, hidden
layers and output. The resulting RMSD was not significantly impacted, testifying to the robustness of the PAC–learnable boundary. Rectified linear unit in the hidden layer has been chosen for simplicity in calculating the VC dimensionality.

The root mean square deviation (RMSD) has been calculated in cross-validation [20]. It consist in dividing the dataset into training and validation exclusive subsets. The procedure is repeated several times with different separation of training and validation dataset guaranteeing a bias-free assessment of the fitting procedure which improves on Bootstrap method [21]. This method consist in:

- Divide the training set in a number of equivalent subsets $k$ (usually, but not necessarily, randomly picked). This makes up the $k$–fold. A popular option, empirically verified to perform well in a variety of situation, is $k = 10$.

- Train the set on a set composed of $k − 1$ subsets, and validate it on the remaining one.

- Repeat the training $k$ times, so that training and validation are considered over all the possible validation sets.

- From the RMSD resulting from the combination of training–validation, consider the average and the standard deviation of RMS deviations.

The average RMSD and its deviation will inform on the performance of the model and cost function chosen, and its resilience to modification of the dataset and therefore predictive power. Where otherwise not specified, $k = 10$ has been used. Other $k$–fold choices were also tested, including a “complete cross-validation”, that is a number of folds equal to the number of data.

The Weiszacker model represented in Table I, has been obtained with a RMSD optimization over AME12 database [36] and validated using the AME16 database [4]. The cost function adopted is the modified $\tilde{\chi}^2$,

$$\tilde{\chi}^2 = \sum_i (f_\alpha(x_i) - y_i)^2/(\log(\Delta y_i))^2,$$

(7)

that has been adjusted in the case measured data $y_i$ have errors $\Delta y_i$ that span different orders of magnitude, as for the case of errors in mass measurements, from [37].
Mass Model

Neural network approaches to fit to nuclear masses have been tried with specific configurations for neural network structure and propagation strategy [38, 39], obtaining results between 0.7 and 5 MeV of RMSD on masses on specific tests set. The result of this work, obtained with a randomly initialized model that does not introduce bias through the geometry, is 2.54 MeV on the larger AME16 dataset in Figs. 3 and 4. To be noted that the best results in [38] are obtained with the networks with the least number of weights, i.e. lower VC–dimension.

Also models to reproduce HFB calculations [40] and other nuclear properties, such has radii [41], have been recently introduced. Moreover, neural networks have recently emerged as possible method for providing additional corrections and correlations (arguably representing missing shell–effects) to a previously devised mass–model with excellent results [42, 43].

![Graph](image.png)

FIG. 3. Difference between the neural network postdiction and experimental binding energy in MeV in function of the atomic mass number. The neural network is composed of 1000 rectified linear unit nodes in the hidden layer, is the best resulting out of the 10–fold cross validation over AME12 dataset. Similarly to most mass models, the familiar arches in correspondence of the magic numbers are present.

The parameters of the obtained Weiszacker formula used in Table [4] are reported in Table II, with a striking resemblance in quantity and uncertainties with [33] (results obtained independently and with different fitting procedures). The related uncertainties have been computed with the covariance matrix calculated as inverse of the Hessian, which is composed by derivatives of the cost function respect to the free parameters. By linear approximation
FIG. 4. Segre chart of the isotopes with the difference between the neural network postdiction and experimental result in MeV. The neural network is composed of 1000 rectified linear unit nodes in the hidden layer, is the best resulting out of the 10–fold cross validation over AME12 dataset. That results in a root mean square deviation on the AME16 dataset of 2.54 MeV.

It is possible to estimate errors related to a parameter or observable \( [28, 37, 44] \)

\[
\begin{align*}
    a_v &= 15.40 \pm 0.014 \\
    a_s &= 16.71 \pm 0.042 \\
    a_C &= 0.701 \pm 0.001 \\
    a_o &= 22.56 \pm 0.037 \\
    a_p &= 11.88 \pm 0.823 
\end{align*}
\]

TABLE II. Coefficients in MeV of the semi–empirical mass formula \( [5] \) obtained fitting AME12. Notice the uncertainties related to the different quantities, especially concerning pairing that testifies the softness of the regression of Weizsacker mass formula respect to pairing, that is also the reason why different formulations are used for this term.

Support vector machine \( [45] \) obtained interesting results of accuracy and would be also an interesting object of study in light of PAC learning bounds which are well established \( [6] \).

[1] Åberg S 2002 Nature 417 499–501 ISSN 1476-4687 URL https://www.nature.com/articles/417499a
[2] Bohigas O and Leboeuf P 2002 *Phys. Rev. Lett.* **88**(9) 092502 URL https://link.aps.org/doi/10.1103/PhysRevLett.88.092502

[3] Vapnik V N 1999 *IEEE Transactions on Neural Networks* **10** 988–999 ISSN 1045-9227

[4] Wang M, Audi G, Kondev F, Huang W, Naimi S and Xu X 2017 *Chinese Physics C* **41** 030003 URL http://stacks.iop.org/1674-1137/41/i=3/a=030003

[5] Nazarewicz W 2018 *Nature Physics* **14** 537–541 URL https://doi.org/10.1038/s41567-018-0163-3

[6] Vapnik V N 1995 *The Nature of Statistical Learning Theory* (Berlin, Heidelberg: Springer-Verlag) ISBN 0-387-94559-8

[7] Hornik K, Stinchcombe M and White H 1989 *Neural Networks* **2** 359 – 366 ISSN 0893-6080 URL http://www.sciencedirect.com/science/article/pii/0893608089900208

[8] Hohenberg P and Kohn W 1964 *Phys. Rev.* **136**(3B) B864–B871 URL https://link.aps.org/doi/10.1103/PhysRev.136.B864

[9] Blumer A, Ehrenfeucht A, Haussler D and Warmuth M K 1987 *Information Processing Letters* **24** 377 – 380 ISSN 0020-0190 URL http://www.sciencedirect.com/science/article/pii/0020019087901141

[10] Domingos P 1999 *Data Mining and Knowledge Discovery* **3** 409–425 ISSN 1573-756X URL https://doi.org/10.1023/A:1009868929893

[11] Shawe-Taylor J, Bartlett P L, Williamson R C and Anthony M 1998 *IEEE Transactions on Information Theory* **44** 1926–1940 ISSN 0018-9448

[12] Vapnik V and Chervonenkis A 1971 *Theory of Probability & Its Applications* **16** 264–280 (Preprint https://doi.org/10.1137/1116025) URL https://doi.org/10.1137/1116025

[13] Maass W 1995 *Handbook of Brain Theory and Neural Networks* ed Arbib M A (MIT Press) pp 1000–1003

[14] Koiran P and Sontag E D 1996 *Advances in neural information processing systems* pp 197–203

[15] Ehrenfeucht A, Haussler D, Kearns M and Valiant L 1989 *Information and Computation* **82** 247 – 261 ISSN 0890-5401 URL http://www.sciencedirect.com/science/article/pii/0890540189900023

[16] Kontorovich A and Pinelis I 2016 *CoRR* abs/1606.08920 to be published in annals of statistics (Preprint 1606.08920) URL http://arxiv.org/abs/1606.08920

[17] Haussler D 1992 *Information and Computation* **100** 78 – 150 ISSN 0890-5401 URL http:
[18] Bartlett P L, Long P M and Williamson R C 1996 Journal of Computer and System Sciences 52 434 – 452 ISSN 0022-0000 URL \text{http://www.sciencedirect.com/science/article/pii/089054019290010D}

[19] Hoeffding W 1963 Journal of the American Statistical Association 58 13–30 (Preprint https://www.tandfonline.com/doi/pdf/10.1080/01621459.1963.10500830) URL https://www.tandfonline.com/doi/abs/10.1080/01621459.1963.10500830

[20] Lachenbruch P A and Mickey M R 1968 Technometrics 10 1–11 ISSN 0040-1706 URL \text{http://www.jstor.org/stable/1266219}

[21] Efron B 1983 Journal of the American Statistical Association 78 316–331 ISSN 0162-1459 URL https://www.tandfonline.com/doi/abs/10.1080/01621459.1983.10477973

[22] Weizsäcker C F v 1935 Zeitschrift für Physik 96 431 ISSN 0044-3328 URL http://dx.doi.org/10.1007/BF01337700

[23] Bethe H A and Bacher R F 1936 Rev. Mod. Phys. 8(2) 82–229 URL https://link.aps.org/doi/10.1103/RevModPhys.8.82

[24] Bender M, Heenen P H and Reinhard P G 2003 Rev. Mod. Phys. 75(1) 121–180 URL \text{https://link.aps.org/doi/10.1103/RevModPhys.75.121}

[25] Ring P and Schuck P 1980 The Nuclear Many–Body Problem (Berlin: Springer)

[26] Sholl D S and Steckel J A 2009 Density Functional Theory: A Practical Introduction (John Wiley and sons) ISBN 9780470373170

[27] Perlińska E, Rohoziński S G, Dobaczewski J and Nazarewicz W 2004 Phys. Rev. C 69(1) 014316 URL \text{https://link.aps.org/doi/10.1103/PhysRevC.69.014316}

[28] Bennaceur K, Idini A, Dobaczewski J, Dobaczewski P, Kortelainen M and Raimondi F 2017 Journal of Physics G: Nuclear and Particle Physics 44 045106 URL \text{http://stacks.iop.org/0954-3899/44/i=4/a=045106}

[29] First order derivatives go to 1, and other terms cancel

[30] Dechargé J and Gogny D 1980 Phys. Rev. C 21(4) 1568–1593 URL \text{https://link.aps.org/doi/10.1103/PhysRevC.21.1568}

[31] Kortelainen M, Lesinski T, Moré J, Nazarewicz W, Sarich J, Schunck N, Stoitsov M V and Wild S 2010 Phys. Rev. C 82(2) 024313 URL \text{https://link.aps.org/doi/10.1103/PhysRevC.82.024313}
