Lie algebra of homogeneous operators
of a vector bundle

Lecomte P.B.A. and Zihindula Mushengezi E.

July 30, 2020

Abstract

We prove that for a vector bundle $E \to M$, the Lie algebra $D_E(E)$ generated by all differential operators on $E$ which are eigenvectors of $L_E$ and the Lie algebra $D_G(E)$ obtained by Grothendieck construction over the $\mathbb{R}$–algebra $A(E) := Pol(E)$, coincide.

This allows us to compute all the derivations of $A(E)$ and to obtain a Lie algebraic characterization of the vector bundle $E \to M$ with the Lie algebra of zero-weight derivations of the $\mathbb{R}$–algebra $A(E)$.

1 Definitions

Let $\pi : E \to M$ be a vector bundle, $L_E$ be the Lie derivative in the direction of the Euler vector field and $D(E, M)$ be the Lie algebra of all differential operators $D : C^\infty(E) \to C^\infty(E)$, where $C^\infty(E)$ is the algebra of smooth functions of $E$.

A such differential operator is called "homogeneous" if it is sum of differential operators which are eigenvectors with constant eigenvalues of $L_E$.

Let now consider the Lie sub-algebra $D_E(E)$ of $D(E, M)$ of homogeneous differential operators.

Denoting by $D^k(E, M)$ the space of all differential operators of $k$–th order, we have

$$D_E(E) = \bigcup_{k \geq 0} D^k_E(E)$$

where $D^k_E(E)$ is the space generated by

$$\{T \in D^k(E, M) | \exists \lambda \in \mathbb{Z} : L_E T = \lambda T\}$$

1Where $L_E$ is the Lie derivative in the direction of the Euler vector field of $E$. 

1
The associative algebra $\mathcal{A}(E)$ of functions which are fiberwise polynomial coincides with $D_0^E(E)$.

One has

$$\mathcal{A}(E) = \bigoplus_{\lambda \in \mathbb{N}} \mathcal{A}^\lambda(E);$$

with, for all $u \in \mathcal{A}^\lambda(E)$, $L_E(u) = \lambda u$.

We denote by $D_G(E)$ the Lie algebra obtained by Grothendieck construction on the associative algebra as follows.

$$D_G(E) = \bigcup_{k \geq 0} D_G^k(E),$$

where $D_G^0(E) = \mathcal{A}(E)$ and

$$D_G^k(E) = \{ T \in \text{End}(\mathcal{A}(E)) : \forall f \in \mathcal{A}(E), [T, f] \in D_G^k(E) \}$$

Our aims is to show that

$$D_E(E) = D_G(E)$$

2 The topological space $\mathcal{A}(E)$

All open subset $V \subset \mathbb{R}^p$ admits a fundamental sequence of compact subsets, ie, an increasing sequence $(K_m)$ of compacts in $V$ such that $\bigcup_m K_m = V$ and $K_m \subset \text{int}(K_m)$; where $\text{int}(X)$ means the interior of $X$.

Let $E$ be a manifold and consider the associative algebra $C^\infty(E)$ of all smooth functions. Consider an at most countable atlas $(V_\alpha, \varphi_\alpha)$ of $E$.

Let $(K_{m,\alpha})$ be a fundamental sequence of compact subsets of $\varphi_\alpha(V_\alpha)$.

For all $(s, m) \in \mathbb{N} \times \mathbb{N}_0$ and $f \in C^\infty(E)$, we put

$$p_{s,m,\alpha}(f) = \sup_{x \in K_{m,\alpha}, |\lambda| \leq s} |\partial^\lambda(f \circ \varphi^{-1}_\alpha)|,$$

with $\lambda$ a multi-index. These $p_{s,m,\alpha}$ as defined above are semi-norms on $C^\infty(E)$. They provide with the space $C^\infty(E)$ an Hausdorff, locally convex and complete topological space structure, ie, a FRECHET space structure. This topology has the following property.

A sequence $(f_k)$ of functions in $C^\infty(E)$ converges to zero if and only if for all chart $(V, \varphi)$ of $E$, for all compact subset $K$ of $\varphi(V)$ and all multi-index $\lambda$, the sequence of all restrictions of $\partial^\lambda(f \circ \varphi^{-1}_\alpha)$ to $K$ converge uniformly to zero.
Since topology defined on $C^\infty(E)$ is associated with a countable family of semi-norms, then $C^\infty(E)$ is metrizable.

In the following lines, the space $\mathcal{A}(E)$ is endowed with the topology induced by that of $C^\infty(E)$.

As a result, a function $\Phi : \mathcal{A}(E) \to \mathcal{A}(E)$ is continuous if for every sequence $(P_n)$ in $\mathcal{A}(E)$, $\Phi(P_n) \to \Phi(P)$ whenever $P_n \to P$.

Generally, if a topological vector space $V$ is a direct sum of two vector subspaces $V_1$ and $V_2$, this is not enough to say that $V$ is also their topological sum.

Let $\pi : E \to M$ be a vector bundle.

**Proposition 2.1** For all $r \in \mathbb{N}$, the linear application

$$ p_r : \mathcal{A}(E) \to \mathcal{A}^r(E) : u = u_0 + u_1 + \ldots + u_s \mapsto u_r $$

is continuous.

**Proof.** Let $(P_n)$ be a sequence that converges to zero in $\mathcal{A}(E)$. Let $K$ be a compact in $\varphi(V)$, where $(V, \varphi)$ is an adapted chart of the vector bundle $E$. Consider a fundamental sequence $(K_m)$ of compacts in $\varphi(V)$ such as $0 \in K_m$, for all $m$. There exists $m \in \mathbb{N}$ such that $K \subset K_m$.

Locally, one can write $P_n = d_n \sum_{i=0}^s P_{ni}$, where $(P_{ni} \circ \varphi^{-1})(x, \xi) = \sum_{|\alpha|=i} A_n^\alpha(x) \xi^\alpha$ and $d_n \in \mathbb{N}$. We will show that the sequence $(P^n_r)$ also converges to zero for all $r \in \mathbb{N}$.

Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that

$$ \forall n \geq N, \quad \sup_{(x,\xi) \in K_m, |\lambda|+|\mu| \leq s} |\partial^\lambda \overline{\partial^\mu}(P_n \circ \varphi^{-1})(x, \xi)| < \varepsilon \quad (*) $$

As the above inequality is particularly true for $\xi = 0$, we have

$$ \forall n \geq N, \quad \sup_{(x,\xi) \in K_m, |\lambda|+|\mu| \leq s} |\partial^\lambda \overline{\partial^\mu}(P^0_n \circ \varphi^{-1})(x, \xi)| < \varepsilon $$

It means the sequel $(P^0_n)$ converges to zero in $\mathcal{A}(E)$.

The relation $(*)$ is still true for $|\mu| = r, \ (r \in \mathbb{N})$ and so the sequel $(A^n_r)$ converges to zero.

Now, if we put $P^n_r = \sum_{|\mu|=r} \xi^\mu A^n_\mu$, one has

$$ \sup_{(x,\xi) \in K_m, |\mu| = r} |\partial^\lambda \overline{\partial^\mu}(P^n_r \circ \varphi^{-1})(x, \xi)| = \sup_{(x,\xi) \in K_m, |\mu| \leq s} \sum_{\nu \leq r} |C_{\mu,\nu}||\xi^\nu||\partial^\lambda (A^n_\mu \circ \varphi^{-1})(x)| $$

and then $P^n_r \to 0$. ■
3 Algebras identification: $\mathcal{D}_E(E) \cong \mathcal{D}_G(E)$

We first start by showing that given a vector bundle $\pi: E \to M$, a differential operator of $\mathcal{D}(E)$ is entirely determined by its values on the fiberwise polynomial functions.

Let us state the following preliminary result which can be justified as in [11] p.7; the main ingredient of the proof being the Taylor development.

**Lemma 3.1** If $u \in \mathcal{A}(E)$ is such as $j^l_a u = 0$, then we can write, in the neighborhood of $a$,

$$u = \sum_{i=1}^N u_{i_1} \cdots u_{i_l} \quad (i)$$

with $u_{i_j} \in \mathcal{A}(E)$ and $u_{i_j}(a) = 0, \forall (i, j) \in [1, N] \times [0, l]$.

The following proposition will allow us to extend each element of $\mathcal{D}_G(E)$ into a differential operator of $\mathcal{D}(E)$.

**Proposition 3.2** For all endomorphism $D: \mathcal{A}(E) \to \mathcal{A}(E)$ of $\mathbb{R}$–vector spaces such that

$$j^l_a(u) = 0 \Rightarrow D(u)(a) = 0, \forall u \in \mathcal{A}(E),$$

there exists $\hat{D} \in \mathcal{D}(E)$ such that

$$\hat{D}(v) = D(v), \forall v \in \mathcal{A}(E).$$

**Proof.** This statement results from the fact that for all $f \in C^\infty(E), a \in E$ and for all integer $k \in \mathbb{N}$, there exist $u \in \mathcal{A}(E)$ such that

$$j^k_a(f) = j^k_a(u).$$

Let then $D \in \mathcal{D}_G^k(E)$. One can therefore set, for all $f \in C^\infty(E)$,

$$\hat{D}(f)_a = D(u)_a,$$

the polynomial function $u$ having same $k$-order jet as $f$ in $a \in E$. ■

**Corollary 3.3** For all $D \in \mathcal{D}_G^l(E), (l \geq 0)$, there is a unique differential operator $\hat{D} \in \mathcal{D}^l(E, M)$ such that

$$\hat{D}(u) = D(u), \forall u \in \mathcal{A}(E).$$
**Proof.** Let $D \in \mathcal{D}_G^l(E)$. Consider a function $u \in \mathcal{A}(E)$ such that $j^l_0(u) = 0$. Let us show that we have $D(u)_a = 0$. We proceed by induction on $l$. Indeed, this statement being true for $l = 0$, suppose, by induction hypothesis that it is for $k < l$. So, when $k = l$, with the notations of the previous Lemma 3.1 we have that

$$D(u) = \sum_{i=1}^N u_{i_0} D(u_{i_1} \ldots u_{i_l}) + \sum_{i=1}^N [D, u_{i_0}] (u_{i_1} \ldots u_{i_l})$$

vanishes in $a$ and then, the desired result follows.  

At this point we have already established that the elements of $\mathcal{D}_G(E)$, like those of $\mathcal{D}_E(E)$, can be seen as restrictions of the differential operators of $\mathcal{D}(E)$ on the $\mathbb{R}$–algebra $\mathcal{A}(E)$. Thus, the elements of $\mathcal{D}_G(E)$ locally decompose into expressions comprising polynomial functions along the fibers of $E$. But nothing tells us that these polynomials are of bounded degree independently of charts; as it turned out to be the case with the elements of $\mathcal{D}_E(E)$. The following lemma will allow us to prove a result which states that this is also true for the elements of $\mathcal{D}_G(E)$.

**Lemma 3.4** Let $D \in \mathcal{D}_G^l(E)$. There is $N_s \in \mathbb{N}$ such that

$$\mathcal{A}^s(E) = \mathcal{F}_{N_s}(E),$$

where, for $N \in \mathbb{N}$, by definition, one has

$$\mathcal{F}_N(E) = \{ u \in \mathcal{A}^s(E) : r \geq N \Rightarrow pr_r \hat{D}(u) = 0 \}. $$

**Proof.**

Note that

$$\mathcal{A}^s(E) = \{ u \in \mathcal{C}^\infty(E) : L_E u = su \}$$

is closed in $\mathcal{C}^\infty(E)$; it is therefore a Baire space. For $N \in \mathbb{N}$, it is the same for

$$\mathcal{A}^s_N(E) = \{ u \in \mathcal{A}^s(E) : r \geq N \Rightarrow pr_r \hat{D}(u) = 0 \}. $$

This last set is an intersection of the closed due to the continuity of the differential operators and that of $pr_r$, by virtue of the previous Proposition 3.3 (Relative to the topology for which $\mathcal{C}^\infty(E)$ is a Fréchet space.) Observe that we have

$$\mathcal{A}^s(E) = \bigcup_{N \in \mathbb{N}} \mathcal{A}^s_N(E).$$

\footnote{It shown in \cite{12}}
Thus, there is $N_s \in \mathbb{N}$ such that $\mathcal{A}_{N_s}^s(E)$ is interior non-empty. However, any open set of a topological vector space containing the origin is absorbing. So we can write

$$\mathcal{A}^s(E) = \mathcal{A}_{N_s}^s(E).$$

Indeed, let $v \in \text{Int}(\mathcal{A}_{N_s}^s(E))$. There is then an open set $\mathcal{U}$ containing the origin such that

$$v + \mathcal{U} \subset \text{Int}(\mathcal{A}_{N_s}^s(E)) \subset \mathcal{A}^s(E).$$

Now consider any element $u \in \mathcal{A}^s(E)$. As $\mathcal{U}$ is absorbent, there is $\kappa > 0$ such that $\kappa u \in \mathcal{U}$. Therefore,

$$\kappa u + v \in \mathcal{A}^s(E)$$

and thus $u \in \mathcal{A}^s(E)$. □

**Proposition 3.5** The unique differential operator $\hat{D} \in \mathcal{D}(E)$ associated with $D \in \mathcal{D}_G^l(E)$ is written locally

$$\hat{D} = \sum_{|\alpha| + |\beta| \leq l} u^{\alpha,\beta} \partial_\alpha \overline{\partial}_\beta \quad (*)$$

where the $u^{\alpha,\beta}$ are polynomials in $(y^1, \ldots, y^n)$ of a maximum degree bounded independently of charts.

**Proof.** Recall first that

$$\partial_\alpha \overline{\partial}_\beta (x^\gamma y^\delta) = \begin{cases} \frac{\gamma!}{(\gamma-\alpha)! (\delta-\beta)!} x^{\gamma-\alpha} y^{\delta-\beta} & \text{if } \alpha \leq \gamma \text{ and } \beta \leq \delta \\ 0 & \text{if not} \end{cases}$$

For $|\alpha| + |\beta| = 0$,

$$u^{00} = \hat{D}(1) \in \mathcal{A}^0 \oplus \cdots \oplus \mathcal{A}^{N_0}(E),$$

for some $N_0 \in \mathbb{N}$, by virtue of the previous Lemma 3.4.

Assume by induction hypothesis that the proposition is true for $|\alpha| + |\beta| < t$.

Let $U$ be a trivialization domain over which $\hat{D}$ is of the form $(*)$.

Consider a function $\rho \in \mathcal{A}^0(E)$ such that $\rho = \pi^*_E(f)$, $f$ being a function with compact support in $U$, zero outside $U$ and equal to 1 in an open $V$ of $U$.

For $|\gamma| + |\delta| = t$, we have

$$\hat{D}(\rho x^\gamma y^\delta) = \sum_{\alpha < \gamma \text{ or } \beta < \delta} u^{\alpha,\beta} \frac{\gamma!}{(\gamma-\alpha)! (\delta-\beta)!} x^{\gamma-\alpha} y^{\delta-\beta} + \gamma! \delta! u^\gamma \delta$$

6
Observe that there is $N_{|\delta|}$ such that
\[ \hat{D}(\rho x^\gamma y^\delta) \in A^0(E) \oplus \cdots \oplus A^{N_{|\delta|}}(E). \]

This comes from the previous Lemma 3.4, and we have that for $\alpha < \gamma$ or $\beta < \delta$, the recurrence hypothesis can be applied to $u^{\alpha,\beta}$.

We deduce that $u^{\gamma,\delta}$ is polynomial of bounded degree, independently of chart; and this completes the proof of the proposition. ■

From the previous proposition and from the local characterization of the algebra $D^E(E)$ we deduce the following result.

**Theorem 3.6** Let $E \to M$ be a vector bundle. Quantum Poisson algebras $D_G^E(E)$ and $D^E(E)$ defined by
\[ D^E(E) = \bigcup_{k \geq 0} D^k(E), \]
and
\[ D_G^E(E) = \bigcup_{k \geq 0} D_G^k(E), \]
where we have set
\[ D_G^0(E) = A(E), D_G^{k+1}(E) = \{ T \in \text{End}(A(E)) : [T, A(E)] \subset D_G^k(E) \} \]
and
\[ D_G^k(E) = A(E), D_G^k(E) = \{ T \in D^k(E, M) | \exists r \in \mathbb{Z} : L(T) = rT \}, \]
coincide up to isomorphism.

## 4 Derivations of the associative algebra $A(E)$

Let $E \to M$ be a vector bundle. In the following lines, $A(E)$ still designates the associative algebra of polynomial functions along the fibers of the bundle $E \to M$. We use the equality $D^E(E) = D_G^E(E)$ to determine all the derivations of the $\mathbb{R}$-algebra $A(E)$. We denote by $\text{Vect}(E)$ the space of vector fields of $E$.

**Proposition 4.1** A linear map $D : A(E) \to A(E)$ is a derivation of $A(E)$ if and only if $D$ is the restriction to $A(E)$ of an element of $D^1(E) \cap \text{Vect}(E)$.

In other words,
\[ \text{Der}(A(E)) = D^1(E) \cap \text{Vect}(E)|_{A(E)}. \]
Proof. The inclusion $\text{Der}(\mathcal{A}(E)) \supset \mathcal{D}_E^1(E) \cap \text{Vect}(E)$ is obvious. Now let $D \in \text{Der}(\mathcal{A}(E))$. We have, for all $u \in \mathcal{A}(E)$, the following equality

$$[D, \gamma_u] = \gamma_{D(u)}.$$

Indeed, for all $u, v \in \mathcal{A}(E)$, we can write

$$[D, \gamma_u](v) = D(uv) - uD(v) = D(u)v = \gamma_{D(u)}(v).$$

Therefore, $D \in \mathcal{D}_E^1(E) = \mathcal{D}_E^1(E) \cap \text{Vect}(E)$. Observe that as $\mathbb{R}$–vector spaces, we have

$$\mathcal{D}_E^1(E) = (\mathcal{D}_E^1(E) \cap \text{Vect}(E)) \oplus \mathcal{A}(E).$$

Let us then set $D = D_c + w$ with $D_c \in \mathcal{D}_E^1(E) \cap \text{Vect}(E)$ and $w \in \mathcal{A}(E)$. As $D(1) = D_c(1) = 0$, we conclude that $w = 0$. Therefore, we obtain the following inclusion

$$\text{Der}(\mathcal{A}(E)) \subset \mathcal{D}_E^1(E) \cap \text{Vect}(E);$$

which completes the demonstration.  

We propose in the following lines a result which relates the Lie algebra of the infinitesimal automorphisms $\text{Aut}(E)$ of the vector bundle $E$ and that of zero-weight derivations of the $\mathbb{R}$–algebra $\mathcal{A}(E)$.

**Proposition 4.2** The algebra of zero-weight derivations of $\mathcal{A}(E)$ is given by

$$\text{Der}^0(\mathcal{A}(E)) = \text{Aut}(E)\mid_{\mathcal{A}(E)}.$$  

**Proof.** The inclusion

$$\text{Der}^0(\mathcal{A}(E)) \supset \text{Aut}(E)\mid_{\mathcal{A}(E)}$$

is immediate. For $D \in \text{Der}^0(\mathcal{A}(E))$, by adopting the notations of the Corollary 3.3, we obtain an element $\hat{D} \in \text{Vect}(E)$ such that $D = \hat{D}\mid_{\mathcal{A}(E)}$. Since $[\mathcal{E}_E, D]\mid_{\mathcal{A}(E)} = 0$, then we have

$$[\mathcal{E}_E, \hat{D}] = 0,$$

and the result is established.  

We will deduce from the previous proposition a Pursell-Shanks type result, by virtue of the following proposition taken from [8].

**Proposition 4.3** Let $E \to M$ and $F \to N$ be two vector bundles of respective ranks $n, n' > 1$ with $H^1(M, \mathbb{Z}/2) = 0$. Lie algebras $\text{Aut}(E)$ and $\text{Aut}(F)$ are isomorphic if and only if the vector bundles $E \to M$ and $F \to N$ are.

8
Here is now the result of Lie-algebraic characterization announced.

**Corollary 4.4** Let $E \to M$ and $F \to N$ be two vector bundles with ranks greater than 1. If $H^1(M, \mathbb{Z}/2) = 0$, then, the vector bundles $E \to M$ and $F \to N$ are isomorphic in and only if the Lie algebras of zero-weight derivations $\text{Der}^0(\mathcal{A}(E))$ and $\text{Der}^0(\mathcal{A}(F))$ are.

We also know that given two vector bundles $E \to M$ and $F \to N$, any isomorphism $\Psi : \mathcal{A}(E) \to \mathcal{A}(F)$ of associative algebras induces a Lie algebras isomorphism by

$$\hat{\Psi} : \text{Der}(\mathcal{A}(E)) \to \text{Der}(\mathcal{A}(F)) : D \mapsto \Psi \circ D \circ \Psi^{-1}.$$ 

Moreover, if $\Psi$ is graduated, the induced isomorphism respects the Lie subalgebras $\text{Der}^0(\mathcal{A}(E))$ and $\text{Der}^0(\mathcal{A}(F))$ of zero-weight derivations of $\text{Der}(\mathcal{A}(E))$ and $\text{Der}(\mathcal{A}(F))$ respectively.

We propose in this section another way to determine the zero-weight derivations of $\mathcal{A}(E)$ and in doing so, we show that such an isomorphism $\hat{\Psi} : \text{Der}(\mathcal{A}(E)) \to \text{Der}(\mathcal{A}(F))$ preserves the Euler vector field.

**Theorem 4.5** Let $\pi : E \to M$ be a vector bundle of rank $n$. 
(a) The Lie algebra of homogeneous zero-weight derivations of the associative algebra $\mathcal{A}(E)$ is given by

$$\text{Der}^0(\mathcal{A}(E)) \cong \text{Vect}(M) \oplus gl(E^*).$$

where we have set $gl(E^*) = \Gamma(\text{Hom}(E^*, E^*))$.
(b) The center $Z(\text{Der}^0(\mathcal{A}(E)))$ of this Lie algebra is formed by the real multiples of the Lie derivative in the direction of the Euler vector field.

**Proof.** Let $D \in \text{Der}^0(\mathcal{A}(E))$. Since $D$ respects the graduation of $\mathcal{A}(E)$, its restriction to $\mathcal{A}^0(E) = \{ f \circ \pi : f \in C^\infty(M) \}$ comes down to the action of a vector field $X \in \text{Vect}(M)$ by

$$\pi^* f \mapsto \pi^* L_X f.$$ 

Consider the restriction of $D$ to $\mathcal{A}^1(E)$, this last space being identified with $\Gamma(E^*)$. In fact, with $u \in \Gamma(E^*)$ we associate $\tilde{u} \in \mathcal{A}^1(E)$ defined by

$$\tilde{u}(a) = u_x(a),$$

for $a \in E_x$. 

9
Thus, the linear application $D : \Gamma(E^*) \rightarrow \Gamma(E^*)$ is a differential operator, and it is of order one. Indeed, for any $u \in \Gamma(E^*)$, such that $j^x_1u = 0$ in an open $U$ of $M$ containing $x$, we can consider a decomposition

$$u = \sum_i f_i u_i, \quad f_i \in A^0(E), u_i \in \Gamma(E^*)$$

where the $f_i$ and the $u_i$ vanish in $x$. Therefore, the differential operator $D$ acting on the sections of the bundle $E^*$ is written locally

$$D(u) = A(u) + \sum_i A^i(\partial_i u)$$

with $A^i, A \in C^\infty(U, gl(n, \mathbb{R}))$. Observe that $\Gamma(E^*)$ is a $A^0(E)$–module; we define then for any $f \in A^0(E)$, a zero-order differential operator acting on the sections of the bundle $E^*$ by

$$\gamma_f : \Gamma(E^*) \rightarrow \Gamma(E^*) : u \mapsto fu.$$ 

We then have on the one hand,

$$D(fu) = fD(u) + D(f)u = fA(u) + f\sum_i A^i(\partial_i u) + D(f)u,$$

for any $f \in A^0(E)$ and $u \in \Gamma(E^*) \cong A^1(E)$.

And on the other hand,

$$D(fu) = A(fu) + \sum_i A^i(\partial_i(fu)) = A(fu) + \sum_i A^i(f\partial_i u) + \sum_i A^i(u\partial_i f).$$

In addition, we have

$$D(f)u = (X.f)u,$$

for a vector field $X \in Vect(M)$.

And more, for all $f \in A^0(E)$ and $A \in gl(E^*)$, we have that $[A, \gamma_f] = 0$, this last bracket being that of the commutators in the algebra of endomorphisms of the $\mathbb{R}$–vector space $\Gamma(E^*)$.

We deduce that $A^i = X^i id$ and we can therefore write

$$D(u) = \nabla_X u + B(u),$$

with $X \in Vect(M)$ and $B \in gl(E^*)$. 

10
We have assumed given, in what precedes, a connection on the vector bundle $E \to M$, and this is still the case in the following lines.

Observe that the derivation $L_E \in \text{Der}^0(A(E))$ is zero in $A^0(E)$ and that it coincides with the identity on $\Gamma(E^*)$; which therefore corresponds to the case $X = 0$, $A = id$.

Since $A^0(E)$ and $A^1(E)$ generate the all $\mathbb{R}$–algebra $A(E)$, the part (a) of the theorem is thus established.

Let $D_{X,A} \in Z(\text{Der}^0(A(E)))$, $D_{Y,B} \in \text{Der}^0(A(E))$ and $u \in \Gamma(E^*)$.

We must have

$$0 = [D_{X,A}, D_{Y,B}](u) = D_{X,A}(D_{Y,B}(u)) - D_{Y,B}(D_{X,A}(u)) = \nabla_X(\nabla_Y u + B(u)) + A(\nabla_Y u + B(u)) - \nabla_Y(\nabla_X u + A(u)) - B(\nabla_X u + A(u))$$

$$= (R(X,Y) + \nabla_{[X,Y]}u + (\nabla_X B)(u) - (\nabla_Y A)(u) + [A,B](u)$$

This relation being true for all $Y,B$; by setting $Y = 0$, we obtain

$$[A,B] = 0 \text{ and } \nabla_X B = 0,$$

whatever $B$. The first equality gives $A = \kappa id$. The second gives, by setting $B = f id, f \in A^0(E)$, $X.f id = 0$ and thus $X = 0$. ($M$ is assumed to be connected.)

**Corollary 4.6** Let $E \to M$ and $F \to N$ be two vector bundles.

Any isomorphism of associative algebras $\Phi : A(E) \to A(F)$ induces an isomorphism of Lie algebras $\hat{\Psi} : \text{Der}^r(A(E)) \to \text{Der}^r(A(F))$ such that

$$\hat{\Psi}(\mathcal{E}_E) = \mathcal{E}_F \quad \text{and} \quad \hat{\Psi}(\text{Der}^r(A(E))) = \text{Der}^r(A(F))$$

with $\text{Der}^r(A(E))$ (resp.$\text{Der}^r(A(F))$) designating the $\mathbb{R}$–vector space of $r$-weight derivations of $A(E)$ (resp. $A(F)$).

**Proof.** Since $\Psi$ induces an graded isomorphism between $A(E)$ and $A(F)$, the proof of this can be founded in [12], we will denote the two by $\Psi$. Thus, by definition, for all $D \in \text{Der}(A(E))$, we have $\hat{\Psi}(D) = \Psi \circ D \circ \Psi^{-1}$; and thus, $\Psi$ respecting the gradations, it comes, for all $D \in \text{Der}^r(A(E))$ and all $u \in A^s(F)$,

$$(\hat{\Psi}(D))(u) = \Psi(D(\Psi^{-1}(u))) \in A^{r+s}(F),$$

because $\Psi^{-1}(u) \in A^s(E)$. Hence the inclusion

$$\hat{\Psi}(\text{Der}^r(A(E))) \subset \text{Der}^r(A(F)).$$
In addition, from
\[ \hat{\Psi}(Z(Der^0(A(E)))) = Z(Der^0(A(F))), \]
given the previous theorem 4.5 we conclude that there exists \( \kappa \in \mathbb{R} \setminus \{0\} \) such that
\[ \hat{\Psi}(E) = \kappa F. \]
Therefore, for any \( u \in A^1(F) \), on the one hand we have
\[ (\hat{\Psi}(E)(u)) = \Psi \circ E \circ \Psi^{-1}(u) = \Psi(\Psi^{-1}(u)) \quad \text{since} \quad \Psi^{-1}(u) \in A^1(E) \]
And on the other,
\[ (\hat{\Psi}(E)(u)) = (\kappa F)(u) = \kappa u \]
Therefore, \( \kappa = 1; \) and we have that \( \hat{\Psi} \) preserves the Euler vector field. \( \blacksquare \)
References

[1] De Wilde M, Lecomte P, Some Characterizations of Differential operators on Vector Bundles, In: E.B. Christoffel, Eds: Butzer P, Feher F, Brihäuser Verlag, Basel (1981),pp. 543-549

[2] Dieudonné J, Éléments d’analyse T.3, Cahiers scientifiques, Fascicules XXXIII, Gauthier-Villars, Paris, (1970).

[3] Grabowski J, Isomorphisms of algebras of smooth functions revisited, Archiv Math. (to appear) (electronic version at http://arXiv.org/abs/math.DG/0310295)

[4] Grabowski J, Poncin N, Automorphisms of quantum and classical Poisson algebras, Comp. Math., 140 (2004), pp. 511-527

[5] Grabowski J, Poncin N, Lie-algebraic characterizations of manifolds, Central Europ. J. of Math., 2(5) (2005), pp. 811-825

[6] Grabowski J, Poncin N, On quantum and classical Poisson algebras, Banach Center Publ. 76, Warszawa (2007), pp. 313-324.

[7] Koriyama A, Maeda Y, Omori H, On Lie algebras of vector fields Trans. Amer. Math. Soc., 226 (1977), pp. 89-117

[8] Lecomte P, On the infinitesimal automorphisms of a vector bundle, J. Math. pure et appl. Go (1981), pp. 229-239

[9] Lecomte P, On some sequence of graded Lie algebras associated to manifolds, Ann. Glob. Anal. Geom., 12 (1994), pp. 183-192

[10] Lecomte P.B.A, Note on the Linear Endomorphisms of a Vector Bundle., Manuscripta mathematica 32 (1980): 231-238

[11] Lecomte P, Sur l’algèbre de Lie des sections d’un fibré en algèbres de Lie, Ann. Inst. Fourier, XXX, Fasc. 4, (1980), pp. 35-50.

[12] Lecomte P.B.A, Leuther T, Zihindula Mushengezi E, On a Lie algebraic characterization of vector bundles, SIGMA 8 (2012), 004:10 pages, 2012.

[13] Pursell L E, Shanks M E, The Lie algebra of a smooth manifold, Proc. Amer. Math. Soc. 5 (1954), pp. 468.