ON HOMOLOGICAL STABILITY FOR ORTHOGONAL GROUPS
AND SPECIAL ORTHOGONAL GROUPS

MASAYUKI NAKADA

Abstract. The problem of homological stability helps us to catch the structure of group homology. We calculate homological stability of special orthogonal groups, and we also calculate the stability of orthogonal groups with determinant-twisted coefficients under a certain good situation. We also get some results about the structure of these homology.

Contents

1. Introduction
1.1. Notations and Preliminaries
1.2. Results
2. Involution \( \sigma \)
3. Induction algorithm
4. On structures of \( \sigma \)-coinvariant part
5. Proof of theorem 1.3
6. Applications
6.1. Proof of theorem 1.5
6.2. Some applications
References

1. Introduction

1.1. Notations and Preliminaries. In this paper, \( F \) is an infinite Pythagorean field of \( \text{char}(F) \neq 2 \). A field \( F \) is Pythagorean if the sum of any two squares is a square. Other than algebraically closed fields, the real numbers \( \mathbb{R} \) is a typical example. Let \( q(x) = \sum_{i=1}^{n} x_i^2 \) be the Euclidean quadratic form on \( F^n \). We denote by \( O_n = O_n(F, q) \) the corresponding orthogonal group and by \( SO_n = SO_n(F, q) \) the corresponding special orthogonal group of degree \( n \). We denote by \( S = S(F^n) \) the unit sphere \( \{ x \in F^n ; q(x) = 1 \} \). We write by \( \mathbb{Z}^t \) the determinant-twisted \( O_n \)-module which admits the twisted action by the determinant.

We consider, for any integer \( n \geq 0 \), that \( F^n \) is isometrically embedded in \( F^{n+1} \) as \( x \mapsto (0, x) \). This defines an inclusion map

\[
\text{inc}: O_n \to O_{n+1}, \quad A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}
\]

for \( n \geq 1 \) and, in case \( n = 0 \), we consider \( O_0 \) as the trivial group. The inclusion \( (1.1) \) induces the map of homology groups

\[
H_i(\text{inc}): H_i(O_n) \to H_i(O_{n+1})
\]

Date: June 16, 2011.
2010 Mathematics Subject Classification. 20J05.
Key words and phrases. Group homology, Homological stability, Scissors congruence.
in the $i$-th degree. We understand the coefficient of homology is $\mathbb{Z}$ if it is omitted.

It is well known that every isomorphic embedding induces an inclusion $\text{inc}: O_n \hookrightarrow O_{n+1}$ which are conjugate each other by the theorem of Witt and induces the same map in homology. In the same way, from the inclusion $\text{inc}: SO_n \hookrightarrow SO_{n+1}$ we have $H_i(\text{inc}): H_i(SO_n) \rightarrow H_i(SO_{n+1})$.

$H_i(SO_n)$ admits an involution induced from the short exact sequence
\[(1.3) \quad 1 \rightarrow SO_n \xrightarrow{i} O_n \xrightarrow{\det} \mathbb{Z}/2 \rightarrow 1.\]

Here $\mathbb{Z}/2$ means the multiplicative group $\{\pm 1\}$, $i$ is the natural inclusion and $\det$ is the determinant homomorphism. We denote by $\sigma$ this involution.

1.2. Results. The problem of homological stability of $O_n$ was first studied by Sah in [5] in case $F = \mathbb{R}$. Cathelineau generalised the result of Sah for any infinite Pythagorean fields in [2]. In [2], Cathelineau proved the following; let $O_n = O_n(F, q)$ be the orthogonal group over an infinite Pythagorean field $F$ with Euclidean quadratic form $q$.

**Proposition 1.1** ([2], [5]). The map $H_i(\text{inc}): H_i(O_n) \rightarrow H_i(O_{n+1})$ is bijective for $i < n$, and surjective for $i \leq n$.

In Sah’s paper [5] only $H_2$ of the stability for $SO_n(\mathbb{R})$ was studied. In [2] Cathelineau proved the following result for $SO_n(\mathbb{F}, q)$:

**Proposition 1.2** ([2]). The map $H_i(\text{inc}): H_i(SO_n, \mathbb{Z}[1/2]) \rightarrow H_i(SO_{n+1}, \mathbb{Z}[1/2])$ is bijective for $2i < n$, and surjective for $2i \leq n$.

In case $F$ is quadratically closed, it is known that the obstruction to stability for $SO_n$ with coefficient $\mathbb{Z}[1/2]$ is the Milnor $K$ group $K_n^M(F)$ [2].

We make precise the result of Cathelineau’s result (proposition 1.2). Our result for special orthogonal groups is the following:

**Theorem 1.3.** The map $H_i(\text{inc}): H_i(SO_n) \rightarrow H_i(SO_{n+1})$ is bijective for $2i < n$, and surjective for $2i \leq n$.

In the proof of theorem 1.3 we also get the following corollary (see section 6).

**Corollary 1.4.** For $2i < n$, the group $H_i(SO_n)$ is isomorphic to its own $\sigma$-invariant part.

Another implication is:

**Theorem 1.5.** The map $H_i(\text{inc}, \mathbb{Z}^l): H_i(O_n, \mathbb{Z}^l) \rightarrow H_i(O_{n+1}, \mathbb{Z}^l)$ is bijective for $2i < n$, and surjective for $2i \leq n$.

The group $H_n(O_n, \mathbb{Z}^l)$ plays an important role in the problem of spherical scissors congruence (see [3]).

Acknowledgements. The author would like to thank Masana Harada for his helpful supports.

2. Involution $\sigma$

In this section we study the involution on $H_i(SO_n)$ induced from the extension (1.3).

Let $R$ be a commutative ring, $G$ be a group and $M$ be a left $RG$-module. Let $\gamma$ be an element in $G$. We can define an endomorphism $\gamma_*$ on $H_i(G, M)$ by

$$
\gamma_*([g_1|\ldots|g_i] \otimes m) = [\gamma g_1 \gamma^{-1}|\ldots|\gamma g_i \gamma^{-1}] \otimes \gamma m
$$

using the standard bar resolution. Here \(g_1, \ldots, g_i \in G\) and \(m \in M\). Notice that

\[(2.1)\]
the endomorphism \(\gamma\) is chain homotopic to the identity.

For the proof, see [3, Lemma 5.4] for instance.

The sequence (1.3) splits by

\[\iota: \mathbb{Z}/2 \to O_n, \quad \pm 1 \mapsto \begin{pmatrix} \pm 1 & 0 \\ 0 & 1_{n-1} \end{pmatrix},\]

where \(1_{n-1}\) means the \((n-1)\)-unit matrix.

The sequence (1.3) induces the action of \(\mathbb{Z}/2 = \{\pm 1\}\) on \(SO_n\). The \((-1)\)-action

\[-1 \cdot g := \begin{pmatrix} 1 & 0 \\ 0 & 1_{n-1} \end{pmatrix} g \begin{pmatrix} 1 & 0 \\ 0 & 1_{n-1} \end{pmatrix}^{-1}\]

defines an involution \(\sigma\) on \(H_i(SO_n)\).

Observe that the following diagram

\[(2.2)\]

\[
\begin{array}{ccc}
H_i(SO_n) & \xrightarrow{H_i(\ker)} & H_i(SO_{n+1}) \\
\sigma \downarrow & & \sigma \downarrow \\
H_i(SO_n) & \xrightarrow{H_i(\ker)} & H_i(SO_{n+1})
\end{array}
\]

is commutative, for

\[
\text{inc}(\sigma \cdot g) = \begin{pmatrix} 1 & 0 \\ 0 & 1_{n-1} \end{pmatrix} g \begin{pmatrix} -1 & 0 \\ 0 & 1_{n-1} \end{pmatrix}^{-1}
\]

\[
= \begin{pmatrix} (-1)^{-1} & 0 \\ 0 & 1_{n-1} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1_{n-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1_{n-1} \end{pmatrix} \begin{pmatrix} (-1)^{-1} & 0 \\ 0 & 1_{n-1} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1_{n-1} \end{pmatrix} = \begin{pmatrix} (-1)^{-1} & 0 \\ 0 & 1_{n-1} \end{pmatrix}
\]

where \(\begin{pmatrix} (-1)^{-1} & 0 \\ 0 & 1_{n-1} \end{pmatrix} \in SO_{n+1}\), so that this action on homology \(H_i(SO_{n+1})\)

is trivial. Moreover we have

\[(2.3)\]

\[\text{Im } H_i(\ker) \subseteq H_i(SO_{n+1})^\sigma,\]

for

\[-1 \cdot (\text{inc}(g)) = -1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1_n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1_n \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} = \text{inc}(g),\]

where \(H_i(SO_{n+1})^\sigma\) is \(\sigma\)-invariant part of \(H_i(SO_n)\).

From the above argument, there exists a map

\[(2.4)\]

\[H_i(\ker)_{\sigma}': H_i(SO_n)_\sigma \to H_i(SO_{n+1}),\]
where we denote by $H_i(\text{SO}_n)_\sigma$ $\sigma$-coinvariant part of $H_i(\text{SO}_n)$, so that the diagram

\[
\begin{array}{ccc}
H_i(\text{SO}_n) & \xrightarrow{H_i(\text{inc})} & H_i(\text{SO}_{n+1}) \\
\downarrow \pi & & \downarrow \pi \\
H_i(\text{SO}_n)_\sigma & \xrightarrow{H_i(\text{inc})}_\sigma & H_i(\text{SO}_{n+1})_\sigma
\end{array}
\]

is commutative, where the map $\pi$ is the natural projection, and the map $H_i(\text{inc})_\sigma$ factors as $\pi \circ H_i(\text{inc})'_\sigma$.

3. Induction algorithm

In this section we shall prove the following inductive statement on $q$ for each fixed $n$ and $i$:

**Lemma 3.1.**

(3.1) $H_q(\text{SO}_n)_\sigma \rightarrow H_q(\text{SO}_{n+1})_\sigma$ is bijective for $q \leq i$ if the following two conditions are satisfied;

(3.2) $H_q(\text{O}_n) \rightarrow H_q(\text{O}_{n+1})$ is bijective for $q \leq i$

and

(3.3) $H_q(\text{SO}_n) \rightarrow H_q(\text{SO}_{n+1})$ is bijective for $q < i$.

First we compare the Lyndon-Hochschild-Serre spectral sequences (see for instance [4]) on $\text{O}_n$. Let $E^r$ be the $r$-th term of the Lyndon-Hochschild-Serre homology spectral sequence associated to (1.3), and $\tilde{E}^r$ be that on $\text{O}_{n+1}$. Since we use the bar resolutions, $E^1_{0,q}$ can be seen to be generated by the classes of the form $[a] \otimes c$, where $a$ is an element of $\mathbb{Z}/2$ and $c \in H_q(\text{SO}_n)$. We denote by $\delta$ the horizontal map of the double bar complex which induces $E^2_0$. Then from the definition of $\delta$, it holds that

$$\delta([a] \otimes c) = a \cdot c - c.$$ 

Hence its class is zero in $\mathbb{Z}/2$-coinvariant part $H_0(\mathbb{Z}/2, H_q(\text{SO}_n))$.

Thus we get that for any $q$, $E^\infty_{0,q} = H_0(\mathbb{Z}/2, H_q(\text{SO}_n))$ injects into $E^\infty_{0,q}$, so that

(3.4) $E^2_{0,q} = E^3_{0,q} = \cdots = E^\infty_{0,q}$.

The same is true for $\tilde{E}^r_{0,q}$. Hence we have from (3.4) that there exists a natural injection $H_i(\text{SO}_n)_\sigma \rightarrow H_i(\text{O}_n)$.

We denote by $\{F_p\}$ the filtration on the $i$-th homology $H_i(\text{O}_n)$ induced by the Lyndon-Hochschild-Serre spectral sequence obtained from (1.3), and by $\{\tilde{F}_p\}$ that on $H_i(\text{O}_{n+1})$. We write the filtration on $H_i(\text{O}_n)$ by

$$0 = F_{-1} \subseteq F_0 \subseteq F_1 \subseteq \cdots \subseteq F_p \subseteq \cdots \subseteq H_i(\text{O}_n).$$

Here we treat the filtration only on the $i$-th homology, so that this filtration is of length $i$ on $H_i(\text{O}_n)$.

Recall that if $p + q = i$ then

$$F^\infty_{p,q} = F_p / F_{p-1}.$$ 

First we compare the ‘left’ parts.

**3.5.** Under the condition that (3.2) and (3.3) are satisfied, the map

$$H_i(\text{O}_n)/F_0 \rightarrow H_i(\text{O}_{n+1})/\tilde{F}_0$$

is bijective.
To prove 3.6 first we check the following 3.7.

3.6. If both the condition 3.2 and 3.3 are satisfied, the following map of spectral sequences

\[ E_{p,q}^\infty(\text{inc}): E_{p,q}^\infty \to \tilde{E}_{p,q}^\infty \]

is bijective for \( p + q \leq i \) and \( q < i \).

Proof. First we calculate \( E^2 \)-terms. The condition says that for \( q < i \), the map

\[ E_{p,q}^2(\text{inc}): E_{p,q}^2 \to \tilde{E}_{p,q}^2 \]

is bijective. Moreover, let \( d_{p,q}^2: E_{p,q}^2 \to E_{p-2,q+1}^2 \) and \( \tilde{d}_{p,q}^2: \tilde{E}_{p,q}^2 \to \tilde{E}_{p-2,q+1}^2 \) be the corresponding differentials, then we get

\[ d_{p,q}^2 \circ E_{p,q}^2(\text{inc}) = E_{p-2,q+1}^2(\text{inc}) \circ d_{p,q}^2 \]

for \( q < i - 1 \), and from (3.4) we obtain

\[ d_{2,i-1}^2 \circ E_{2,i-1}^2(\text{inc}) = 0 = E_{0,i}^2(\text{inc}) \circ d_{2,i-1}^2. \]

Hence, at \( E^3 \)-terms

\[ E_{p,q}^3(\text{inc}): E_{p,q}^3 \to \tilde{E}_{p,q}^3 \]

is bijective for \( p + q \leq i \) and \( q < i \), or for \( q < i - 1 \) and any \( p \). Moreover, we also obtain that

\[ d_{p,q}^3 \circ E_{p,q}^3(\text{inc}) = E_{p-3,q+2}^3(\text{inc}) \circ d_{p,q}^3 \]

for \( q < i - 2 \) and that

\[ d_{3,i-2}^3 \circ E_{3,i-2}^3(\text{inc}) = 0 = E_{0,i}^3(\text{inc}) \circ d_{3,i-2}^3. \]

Repeating this process until \( r = i + 1 \), both \( E_{p,q}^r \) and \( \tilde{E}_{p,q}^r \) are degenerate at \( p + q \leq i \), and therefore for \( p + q \leq i \) and \( q < i \) the map

\[ E_{p,q}^\infty(\text{inc}): E_{p,q}^\infty \to \tilde{E}_{p,q}^\infty \]

is bijective.

\[ \square \]

From 3.6 using five lemmas repeatedly, we have the following:

3.7. If the condition 3.2 and 3.3 are satisfied, the natural map

\[ F_p: F_p/F_0 \to \tilde{F}_p/\tilde{F}_0 \]

is bijective.

Now we begin the proof of lemma 3.7.

Suppose that

\[ H_q(\text{inc}): H_q(O_n) \to H_q(O_{n+1}) \]

is bijective for \( q \leq i \) and

\[ H_q(\text{inc}): H_q(SO_n) \to H_q(SO_{n+1}) \]

is bijective for \( q < i \). Then by 3.7 the map

\[ H_q(\text{inc})/F_0: H_q(O_n)/F_0 \to H_q(O_{n+1})/\tilde{F}_0 \]

becomes bijective. Notice that \( H_i(SO_n)_\sigma \cong F_0 \), and \( H_i(SO_{n+1})_\sigma \cong \tilde{F}_0 \) respectively.

Hence the middle and right vertical maps in the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & H_i(SO_n)_\sigma & \longrightarrow & H_i(O_n) & \longrightarrow & H_i(O_n)/F_0 & \longrightarrow & 0 \\
\downarrow H_i(\text{inc}) & & & & \downarrow H_i(\text{inc}) & & \downarrow H_i(\text{inc})/F_0 \\
0 & \longrightarrow & H_i(SO_{n+1})_\sigma & \longrightarrow & H_i(O_{n+1}) & \longrightarrow & H_i(O_{n+1})/\tilde{F}_0 & \longrightarrow & 0
\end{array}
\]
(where the injectivities of left arrows are from (3.4)) are bijective and the horizontal columns in the above are exact. That is, using five lemma, the left vertical map is also bijective. This concludes the claim of lemma 3.1. □

We define the map \( \rho_i \) as the composition of natural inclusion and projection:

\[
(3.8) \quad \rho_i : H_i(\mathbb{SO}_n)^\sigma \to H_i(\mathbb{SO}_n) \to H_i(\mathbb{SO}_n)_\sigma.
\]

Let us see the following commutative diagram

\[
(3.9)
\begin{array}{ccccccccc}
H_i(\mathbb{SO}_n)^\sigma & \xrightarrow{\rho_i} & H_i(\mathbb{SO}_n) & \to & H_i(\mathbb{SO}_n)_\sigma & / & \text{Im} \rho_{n,i} & \to & 0 \\
H_i(\mathbb{SO}_{n+1})^\sigma & \xrightarrow{\rho_i} & H_i(\mathbb{SO}_{n+1}) & \to & H_i(\mathbb{SO}_{n+1})_\sigma & / & \text{Im} \rho_i & \to & 0
\end{array}
\]

where the left vertical map \( H_i(\text{inc})^\sigma \) means the restriction. Here if \( H_i(\text{inc})_\sigma \) is surjective then, using (3.9), \( \rho_i : H_i(\mathbb{SO}_{n+1})^\sigma \to H_i(\mathbb{SO}_{n+1})_\sigma \) in (3.9) is surjective, and hence

\[
\text{Coker} \rho_i = H_i(\mathbb{SO}_{n+1})_\sigma / \text{Im} \rho_i
\]
is zero.

We have proved the following proposition.

**Lemma 3.2.** If \( H_i(\text{inc})_\sigma : H_i(\mathbb{SO}_n) \to H_i(\mathbb{SO}_{n+1})_\sigma \) is surjective, then

\[
\text{Coker} \{ \rho_i : H_i(\mathbb{SO}_{n+1})^\sigma \to H_i(\mathbb{SO}_{n+1})_\sigma \} = 0.
\]

4. **On structures of \( \sigma \)-coinvariant part**

In this section we prove the following lemma:

**Lemma 4.1.** For all \( n \) and all \( i \), the following isomorphism exists:

\[
(4.1) \quad (H_i(\mathbb{SO}_n)/H_i(\mathbb{SO}_n)^\sigma) \otimes_{\mathbb{Z}/2} \mathbb{Z} \cong \text{Coker} \rho_i.
\]

Let us first notice that the involution \( \sigma : H_i(\mathbb{SO}_n) \to H_i(\mathbb{SO}_n) \) induces an involution endomorphism on \( H_i(\mathbb{SO}_n)/H_i(\mathbb{SO}_n)^\sigma \).

**4.2.** There exists an isomorphism

\[
\text{Coker} \rho_i \cong (H_i(\mathbb{SO}_n)/H_i(\mathbb{SO}_n)^\sigma)_\sigma,
\]

where \( \rho_i \) is the map defined in (3.8).

For the left exactness of the functor \( \bullet \otimes_{\mathbb{Z}/2} \mathbb{Z} \) of taking \( \sigma \)-coinvariant part, we get the following commutative diagram

\[
(4.3)
\begin{array}{ccccccccc}
0 & \to & H_i(\mathbb{SO}_n)^\sigma & \to & H_i(\mathbb{SO}_n) & \xrightarrow{\text{proj}} & H_i(\mathbb{SO}_n)^\sigma & \to & 0 \\
\downarrow \rho_i & & \downarrow & & \downarrow & & \downarrow & & \\
H_i(\mathbb{SO}_n)^\sigma \otimes_{\mathbb{Z}/2} \mathbb{Z} & \to & H_i(\mathbb{SO}_n) \otimes_{\mathbb{Z}/2} \mathbb{Z} & \xrightarrow{\text{proj}_\sigma} & (H_i(\mathbb{SO}_n)/H_i(\mathbb{SO}_n)^\sigma) \otimes_{\mathbb{Z}/2} \mathbb{Z} & \to & 0
\end{array}
\]

where \( \text{proj} \) is the natural projection and \( \text{proj}_\sigma = \text{proj} \otimes_{\mathbb{Z}/2} \mathbb{Z} \), horizontal sequences are exact, and each vertical map sends \( x \) to \( x \otimes 1 \).

Next we see that each element in the module \( (H_i(\mathbb{SO}_n)/H_i(\mathbb{SO}_n)^\sigma)_\sigma \) is annihilated by 2.

**4.4.** There exists the isomorphism

\[
\left( \frac{H_i(\mathbb{SO}_n)}{H_i(\mathbb{SO}_n)^\sigma} \right)_\sigma \cong \frac{H_i(\mathbb{SO}_n)}{H_i(\mathbb{SO}_n)^\sigma} / 2 \frac{H_i(\mathbb{SO}_n)}{H_i(\mathbb{SO}_n)^\sigma}.
\]
Hence if we denote the class of a sequence and hence we get lemma 4.1.

(5.1)

Therefore using the exact sequence 4.6 we have

\[ H_1(SO_n \sigma) \cong \frac{H_1(SO_n)}{H_1(SO_n)\sigma} / \text{Im } \tau \]

Now if we use the claims of 4.2 and 4.4 we have the following isomorphisms:

\[ H_1(SO_n \sigma) \cong \frac{H_1(SO_n)}{H_1(SO_n)\sigma} \cong \frac{H_1(SO_n)}{2H_1(SO_n)\sigma} \]

and hence we get lemma 4.1.

5. Proof of theorem 1.3

If \( H_1(\text{inc}) : H_1(SO_n \sigma) \rightarrow H_1(SO_{n+1} \sigma) \) is surjective, then

\[ \text{Coker } \rho_i : H_1(SO_{n+1})^\sigma \rightarrow H_1(SO_{n+1})_\sigma \] is 0.

Adding lemma 4.1 we have

\[ \frac{H_1(SO_{n+1})}{H_1(SO_{n+1})^\sigma} / 2H_1(SO_{n+1})^\sigma \cong \text{Coker } \rho_i : H_1(SO_{n+1})^\sigma \rightarrow H_1(SO_n \sigma) \] is 0.

Therefore \( H_1(SO_{n+1})/H_1(SO_{n+1})^\sigma \) has neither torsion elements whose orders are divided by 2 nor torsion-free elements.

Now we get the following:

Lemma 5.1. If the map \( H_1(SO_n \sigma) \rightarrow H_1(SO_{n+1} \sigma) \) is bijective, then

\[ \frac{H_1(SO_n)}{H_1(SO_{n+1})^\sigma} \otimes \mathbb{Z}/2 = 0 \]

From lemma 3.1 the sufficient condition to satisfy the assumption of lemma 5.1 is (3.2) and (3.3).

We already have a result concerning to (3.2). Recall that Cathelineau proved in [2] that

(5.1) \( H_1(O_n) \rightarrow H_1(O_{n+1}) \) is bijective for \( i < n \) and surjective for \( i \leq n \).
Cathelineau proved in [2] that

\[ H_i(SO_n, \mathbb{Z}[1/2]) \cong H_i(SO_n, \mathbb{Z}[1/2]) \quad \text{for } 2i < n. \tag{5.2} \]

Notice that with coefficient \( \mathbb{Z}[1/2] \) we have \( H_i(SO_n, \mathbb{Z}[1/2])^{\sigma} \cong H_i(SO_n, \mathbb{Z}[1/2]) \).

Because \( \mathbb{Z}[1/2] \) is \( \mathbb{Z} \)-flat module, we have that the sequence

\[ 0 \to H_i(SO_n)^{\sigma} \otimes_\mathbb{Z} \mathbb{Z}[1/2] \to H_i(SO_n) \otimes_\mathbb{Z} \mathbb{Z}[1/2] \to \left( \frac{H_i(SO_n)}{H_i(SO_n)^{\sigma}} \right) \otimes_\mathbb{Z} \mathbb{Z}[1/2] \to 0 \]

is still exact. Now we have that \( H_i(SO_n)^{\sigma} \otimes_\mathbb{Z} \mathbb{Z}[1/2] \cong H_i(SO_n, \mathbb{Z}[1/2])^{\sigma} \) and \( H_i(SO_n) \otimes_\mathbb{Z} \mathbb{Z}[1/2] \cong H_i(SO_n, \mathbb{Z}[1/2]) \), and therefore from the above exact sequence we get that

\[ \left( \frac{H_i(SO_n)}{H_i(SO_n)^{\sigma}} \right) \otimes_\mathbb{Z} \mathbb{Z}[1/2] \cong H_i(SO_n, \mathbb{Z}[1/2])^{\sigma}. \]

Hence we obtain the following reformulated Cathelineau’s formula:

\[ \frac{H_i(SO_n)}{H_i(SO_n)^{\sigma}} \otimes_\mathbb{Z} \mathbb{Z}[1/2] = 0 \quad \text{for } 2i < n. \tag{5.3} \]

We start the proof of theorem 1.3 inductively.

First, the map

\[ H_0(SO_3) \xrightarrow{\cong} H_0(SO_4) \xrightarrow{\cong} H_0(SO_5) \xrightarrow{\cong} \cdots \]

are all bijective since we have \( H_0(SO_n) = \mathbb{Z} \) for any \( n \) and maps are natural. Here we set \( SO_0 = \) trivial group.

From (5.1), the homological stability of \( O_n \) at \( H_1 \) is the following:

\[ 0 \cong H_1(O_0) \to H_1(O_1) \to H_1(O_2) \xrightarrow{\cong} H_1(O_3) \xrightarrow{\cong} \cdots. \]

We have, from lemma 3.1 the homological stability of \( H_1(SO_n) \) below:

\[ H_1(SO_1)^{\sigma} \to H_1(SO_2)^{\sigma} \xrightarrow{\cong} H_1(SO_3)^{\sigma} \to \cdots. \]

Thus from lemma 5.1 we get that

\[ \frac{H_1(SO_m)}{H_1(SO_m)^{\sigma}} \left/ 2 \frac{H_1(SO_m)}{H_1(SO_m)^{\sigma}} \right. = 0 \quad \text{for } m > 2. \tag{5.4} \]

We also have from Cathelineau’s theorem that

\[ \frac{H_1(SO_m)}{H_1(SO_m)^{\sigma}} \otimes_\mathbb{Z} \mathbb{Z}[1/2] = 0 \quad \text{for } m > 2. \tag{5.5} \]

Combining (5.4) and (5.5), we have that

\[ \frac{H_1(SO_m)}{H_1(SO_m)^{\sigma}} = 0 \quad \text{for } m > 2. \tag{5.6} \]

The condition (5.3) means the existence of following isomorphisms:

\[ H_1(SO_m)^{\sigma} \cong H_1(SO_m) \cong H_1(SO_m) \sigma. \]

Hence we get the following diagram.

\[
\begin{array}{c}
H_1(SO_2) \cong H_1(SO_3) \cong H_1(SO_4) \cong \cdots \\
\uparrow \cong \uparrow \cong \uparrow \\
H_1(SO_2)^{\sigma} \cong H_1(SO_3)^{\sigma} \cong H_1(SO_4)^{\sigma}
\end{array}
\]

\[
\begin{array}{c}
H_1(SO_2) \cong H_1(SO_3) \cong H_1(SO_4) \cong \cdots \\
\uparrow \cong \uparrow \cong \uparrow \\
H_1(SO_2)^{\sigma} \cong H_1(SO_3)^{\sigma} \cong H_1(SO_4)^{\sigma}
\end{array}
\]
From the above diagram we have the stability of \( H_1(\text{SO}_n) \) is
\[
H_1(\text{SO}_2) \to H_1(\text{SO}_3) \xrightarrow{\sigma} H_1(\text{SO}_4) \xrightarrow{\sigma} \cdots \tag{5.7}
\]
and therefore we get that the map
\[
H_1(\text{SO}_n) \to H_1(\text{SO}_{n+1})
\]
is surjective for \( n \geq 2 \) and bijective for \( n > 2 \).

Next we see the homological stability of \( H_2(\text{SO}_n) \). From (5.1), the homological stability of \( \text{O}_n \) at \( H_2 \) is the following:
\[
H_2(\text{O}_2) \to H_2(\text{O}_3) \xrightarrow{\sigma} H_2(\text{O}_4) \xrightarrow{\sigma} \cdots \ .
\]
Using this with the stability (5.7) of \( H_1(\text{SO}_n) \), we have
\[
H_2(\text{SO}_3) \sigma \xrightarrow{\sigma} H_2(\text{SO}_4) \sigma \xrightarrow{\sigma} H_2(\text{SO}_5) \sigma \xrightarrow{\sigma} \cdots \ .
\]

Using this with the stability (5.7) of \( H_1(\text{SO}_n) \), we have
\[
H_2(\text{SO}_3) \sigma \to H_2(\text{SO}_4) \sigma \to H_2(\text{SO}_5) \sigma \to \cdots \ .
\]

Since we have the condition
\[
\frac{H_2(\text{SO}_m)}{H_2(\text{SO}_m)^{\sigma}} = 0
\]
for \( m > 4 \) as in the case of \( H_1 \), again we get the diagram
\[
\begin{array}{ccc}
H_2(\text{SO}_4)^{\sigma} & \xrightarrow{\sim} & H_2(\text{SO}_4)^{\sigma} \\
\uparrow & & \uparrow \\
H_2(\text{SO}_4) & \xrightarrow{\sim} & H_2(\text{SO}_4) \\
\uparrow & & \uparrow \\
H_2(\text{SO}_2) & & \end{array}
\]
Therefore we get that the map
\[
H_2(\text{SO}_n) \to H_2(\text{SO}_{n+1})
\]
is surjective for \( n \geq 4 \) and bijective for \( n > 4 \).

Now we finish the proof of main theorem (1.3) by repeating this argument inductively.

\[\square\]

Remark 5.2. Cathelineau showed in [2, Theorem 1.3 and Theorem 1.5] that if \( F \) is a quadratically closed field, the kernel of \( H_n(\text{SO}_{2n}, \mathbb{Z}[1/2]) \to H_n(\text{SO}_{2n+1}, \mathbb{Z}[1/2]) \) is the \( \mathbb{Z}[1/2] \)-tensored Milnor \( K \)-group \( K_n^M(F) \otimes \mathbb{Z}[1/2] \) of the field \( F \).

We refer to corollary (1.4). Notice that in the proof of theorem (1.3) we also get a result about the structure of the homology group of special orthogonal groups.

Let us recall that when the degree of homology is equal to \( n \), we get the following commutative diagram.
\[
\begin{array}{ccc}
H_n(\text{SO}_{2n}) & \xrightarrow{\sim} & H_n(\text{SO}_{2n+1}) \\
\uparrow & & \uparrow \\
H_n(\text{SO}_{2n}) & \xrightarrow{\sim} & H_n(\text{SO}_{2n+2}) \\
\uparrow & & \uparrow \\
H_n(\text{SO}_{2n}) & & \end{array}
\]

\[\square\]
To prove that the bijectivity of the arrows $H_n(\text{SO}_m)^\sigma \to H_n(\text{SO}_m)$ for $m \geq 2n+1$, we inductively proved and used the fact that $H_n(\text{SO}_m)/H_n(\text{SO}_m)^\sigma = 0$ for $m \geq 2n+1$. This implies that for $m \geq 2n+1$ the homology $H_n(\text{SO}_m)$ entirely consists of its $\sigma$-invariant part.

6. Applications

In this section we will prove theorem 1.5 and get some applications.

6.1. Proof of theorem 1.5. As before we consider $\mathbb{Z}$ as $\mathbb{Z}\mathcal{O}_n$-module with trivial $\mathcal{O}_n$-action, and $\mathbb{Z}^t$ as determinant-twisted $\mathbb{Z}\mathcal{O}_n$-module. We consider the group ring $\mathbb{Z}[\mathbb{Z}/2]$ as an $\mathcal{O}_n$-module. $\mathbb{Z}[\mathbb{Z}/2]$ has generators $\epsilon$ and $\sigma$ as $\mathbb{Z}$-module and we define an $\mathcal{O}_n$-action on $\mathbb{Z}[\mathbb{Z}/2]$ defined by

$$g \cdot \epsilon = \begin{cases} 
\epsilon, & \text{if } \det g = 1, \\
\sigma, & \text{if } \det g = -1,
\end{cases}$$

$$g \cdot \sigma = \begin{cases} 
\sigma, & \text{if } \det g = 1, \\
\epsilon, & \text{if } \det g = -1.
\end{cases}$$

Then we have the following two short exact sequences of $\mathcal{O}_n$-modules:

$$1 \to \mathbb{Z}^t \to \mathbb{Z}[\mathbb{Z}/2] \to \mathbb{Z} \to 1,$$

where the left map sends 1 to $\epsilon - \sigma$, and

$$1 \to \mathbb{Z} \to \mathbb{Z}[\mathbb{Z}/2] \to \mathbb{Z}^t \to 1,$$

where the left map sends 1 to $\epsilon + \sigma$.

Notice that from Shapiro’s lemma (see [3, Lemma 5.5]) we have

$$H_i(\mathcal{O}_n, \mathbb{Z}[\mathbb{Z}/2]) \cong H_i(\text{SO}_n)$$

for any $i$ and $n$.

From the sequence (6.2), we get the Bockstein long exact sequence

$$\cdots \to H_{i+1}(\mathcal{O}_n) \to H_i(\mathcal{O}_n, \mathbb{Z}^t) \to H_i(\text{SO}_n) \to H_i(\mathcal{O}_n) \to H_{i-1}(\mathcal{O}_n, \mathbb{Z}^t) \to \cdots,$$

and from (6.3) we get

$$\cdots \to H_{i+1}(\mathcal{O}_n, \mathbb{Z}^t) \to H_i(\mathcal{O}_n) \to H_i(\text{SO}_n) \to H_i(\mathcal{O}_n, \mathbb{Z}^t) \to H_{i-1}(\mathcal{O}_n) \to \cdots,$$

respectively.

Remark 6.1. We can check easily that the map $H_i(\mathcal{O}_n) \to H_i(\text{SO}_n)$ induced from $\mathbb{Z} \to \mathbb{Z}[\mathbb{Z}/2]$ in (6.3) is the transfer map [11, 9 of Chapter III].

Let us see the following commutative diagram:

$$\cdots \to H_{n+1}(\mathcal{O}_{2n}, \mathbb{Z}^t) \to H_{n+1}(\text{SO}_{2n}) \to H_{n+1}(\mathcal{O}_{2n}) \to H_n(\mathcal{O}_{2n}, \mathbb{Z}^t) \to \cdots$$

$$\cdots \to H_{n+1}(\mathcal{O}_{2n+1}, \mathbb{Z}^t) \to H_{n+1}(\text{SO}_{2n+1}) \to H_{n+1}(\mathcal{O}_{2n+1}) \to H_n(\mathcal{O}_{2n+1}, \mathbb{Z}^t) \to \cdots$$

$$\cdots \to H_n(\mathcal{O}_{2n}) \to H_n(\mathcal{O}_{2n}) \to \cdots$$

$$\cdots \to H_n(\mathcal{O}_{2n+1}) \to H_n(\mathcal{O}_{2n+1}) \to \cdots$$

where the horizontal sequences are (6.5) and the vertical maps are stability maps.

We already have stability results (proposition 1.1 and theorem 1.3). Therefore, using five lemma, we obtain theorem 1.5.
Remark 6.2. If the range can be expanded, from the Bockstein exact sequence \((6.6)\), we can expand the stability range of \(H_\ast(\text{SO}_n)\). But this may be false because the kernel of \(H_n(\text{SO}_{2n}) \to H_n(\text{SO}_{2n+1})\) may not be trivial (see remark \((5.2)\)).

6.2. Some applications. First we see the stability map \(H_i(\text{inc}, \mathbb{Z}^l) : H_i(\text{O}_n, \mathbb{Z}^l) \to H_i(\text{O}_{n+1}, \mathbb{Z}^l)\).

Let \(C_l\) be the free abelian group generated by the set of all ordered \((l + 1)\)-tuples (denoted by \((v_0, \ldots, v_l)\)) of points of \(n\)-dimensional unit sphere \(S = S(F^{n+1})\) with the understanding that such an \(l\)-cell is zero if \(v_0 = v_1\), and let \(\partial\) be

\[
\partial(v_0, \ldots, v_l) = \sum_{j=0}^{l} (-1)^j(v_0, \ldots, \hat{v}_j, \ldots, v_l).
\]

\(\text{O}_{n+1}\) acts diagonally on \(C_l\). \((\text{C}_\ast, \partial)\) is a chain complex of \(\text{O}_{n+1}\)-modules, which is acyclic with augmentation \(\mathbb{Z}\).

We define a spectral sequence \('E^1_{p,q}\) as the hyperhomology \(H_p(\text{O}_{n+1}, \mathbb{C}^l_q)\) which converges to the homology \(H_p+q(\text{O}_n, \mathbb{Z}^l)\). Observe that

\[
'E^1_{i,0} \cong H_i(\text{O}_n, \mathbb{Z}^l) \quad \text{for } i \geq 0.\]

To see this, we use Shapiro’s lemma; we have \(H_i(\text{O}_n, \mathbb{C}^l_0) \cong H_i(\text{Stab}(v_0), (\mathbb{Z}(v_0))^l)\), where \(\text{Stab}(v_0)\) is the stabilizer at \(v_0 \in S\), since \(\text{O}_{n+1}\) acts on \(S\) transitively. From Witt’s theorem \(\text{Stab}(v_0) \cong \text{O}_n\) for any \(v_0\), and we get \((6.7)\).

Next we calculate \('E^2_{i,0}\). From Shapiro’s lemma we get

\[
'E^2_{i,0} \cong \bigoplus_{(v_0, v_1)} H_i(\text{Stab}((v_0, v_1)), \mathbb{Z}^l) \otimes \mathbb{Z}(v_0, v_1)
\]

where \((v_0, v_1)\) runs all representatives of the set of \(1\)-cells decomposed to \(\text{O}_{n+1}\)-orbits. The differential \(d^1: 'E^1_{i,1} \to 'E^1_{i,0}\) sends \(c \otimes (v_0, v_1)\) to \(c \otimes (v_1) - c \otimes (v_0)\), where \(c\) is an \(i\)-cycle of \(\text{Stab}((v_0, v_1))\). From Shapiro’s lemma in reverse, \(c \otimes (v_i)\) represents an element of \(H_i(\text{O}_{n+1}, \mathbb{C}^l_0)\) for each \(i \geq 0, 1\). If \(v_0\) and \(v_1\) are linearly independent, then, from Witt’s theorem, we can find an element \(g \in \text{O}_{n+1}\) which sends \(v_0\) to \(v_1\) and centralizes \(\text{Stab}((v_0, v_1))\). Moreover we can find such element \(g\) in \(\text{SO}_{n+1}\) inverting its determinant if necessary, so that \(c \otimes (v_1) - c \otimes (v_0)\) is homologous to zero. If \(v_0 = v_1\), then we get \(c \otimes (v_1) - c \otimes (v_0)\) is homologous to \(2c \otimes (v_1)\) because the determinant of the reflection of \(v_0\) is \(-1\) and this reflection centralizes \(\text{Stab}((v_0, v_1))\). Therefore we obtain the following:

\[
'E^2_{i,0} \cong H_i(\text{O}_n, \mathbb{Z}^l) \otimes \mathbb{Z}/2 \quad \text{for } i \geq 0.
\]

Hence we obtain from \((6.8)\) that

\[
H_i(\text{O}_n, \mathbb{Z}^l) \to H_i(\text{O}_{n+1}, \mathbb{Z}^l) \quad \text{factors through the quotient } H_i(\text{O}_n, \mathbb{Z}^l) \otimes \mathbb{Z}/2.
\]

Applying this result \((6.9)\) to the stability theorem \((6.5)\) we get the following result.

**Corollary 6.3.** \(H_i(\text{O}_n, \mathbb{Z}^l) \cong H_i(\text{O}_n, \mathbb{Z}^l) \otimes \mathbb{Z}/2\) for \(2i < n\).

Finally, we introduce a pair of useful results. These results can be considered as the generalization of \((5.2)\).

The map \(H_i(\text{SO}_n) \to H_i(\text{O}_n)\) in Bockstein exact sequence \((6.5)\) coincides the composite map \(H_i(\text{SO}_n) \to H_i(\text{SO}_n)_\sigma \to H_i(\text{O}_n)\) induced from Lyndon-Hochschild-Serre exact sequence associated to \((1.3)\). Recall that the map \(H_i(\text{SO}_n)_\sigma \to H_i(\text{O}_n)\) is injective (proved at \((3.2)\)). Then we get the following result from the sequence \((6.5)\):

**Corollary 6.4.** \(\text{Coker}\{H_i(\text{O}_n, \mathbb{Z}^l) \to H_i(\text{SO}_n)\} \cong H_i(\text{SO}_n)_\sigma\) for \(i \geq 0\), where the map \(H_i(\text{O}_n, \mathbb{Z}^l) \to H_i(\text{SO}_n)\) is the map in \((6.3)\).
In the same way, from (6.6), we get the following:

**Corollary 6.5.** Coker$\{H_i(O_n) \to H_i(SO_n)\} \cong (H_i(SO_n)\sigma)_{\sigma}$ for $i \geq 0$, where $(H_i(SO_n)\sigma)_{\sigma}$ is determinant-twisted $\sigma$-invariant part.

**References**

[1] K. S. Brown. *Cohomology of Groups*, volume 87 of *Grad. Texts in Math*. Springer-Verlag, 1982.
[2] J.-L. Cathelineau. Homology stability for orthogonal groups over algebraically closed fields. *Ann. Scient. Éc. Norm. Sup.*. **40**:487–517, 2007.
[3] J. L. Dupont. *Scissors Congruence, Group Homology and Characteristic Classes*. Nankai Tracts in Mathematics. World Scientific, 2001.
[4] J. McCleary. *A User’s Guide to Spectral Sequences*, volume 38 of *Stud. Adv. Math*. Cambridge, 2001.
[5] C. H. Sah. Homology of classical Lie groups made discrete, I. Stability theorems and Schur multipliers. *Comment. Math. Helv.*, **61**:308–347, 1986.

**Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto 606-8502, Japan**

*E-mail address: masayuki@math.kyoto-u.ac.jp*