Cluster mathematical model for conjugate thermal processes of the isothermal bodies system in the fluid flow

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Abstract. There has been developed a cluster mathematical modelling method for non-stationary thermal processes, occurring in complex technical system structures, which thermal models represent a system of clusters. Each cluster has a core, combining the heat-emitting elements that fall into the said cluster, a shell of the cluster, and a fluid, flowing through the cluster. The state of the thermal process in each cluster at any specific time is characterized by three state variables, namely, by the temperatures of the core, the temperatures of the shell and the ambient temperature. At that, the elements of each cluster (core, shell, fluid flow) exchange heat with each other and the elements of the adjacent clusters. As opposed to existing methods, the cluster method allows modelling thermal processes that occur in complex technical systems, taking into account non-uniform temperature distribution in the fluid flow, conjugate nature of heat transfer between the fluid flow, the cores and the shells of the clusters. The mathematical model, describing state variables of thermal processes in the cluster thermal model, represents a system of block matrix differential equations with matrix and vector blocks, corresponding to the respective clusters of the thermal model. The solution of equations of the mathematical model is the values of the state variables of the thermal processes in the clusters. The methodology of the application of the cluster method is shown on the example of a real technical system.

1. Introduction
When designing technical systems (TS) designed for various purposes (electronic, chemical engineering, etc.) it is necessary to have tools for mathematical and computer modelling of thermal processes that occur in the structures, representing a system of solid heat-emitting (active) and dissipative (passive) elements, encapsulated in a shell (case), through which a cooling fluid is pumped. The fluid flow (liquid, gas, air) inside the shells of the TS accumulates heat, passing through a network of channels that are formed by active and passive elements of the structure, and, thus, increases its enthalpy (heat content), which is then transferred and conveyed by the flow to the upstream TS elements, creating both their extra heating and excess enthalpy. Thermal processes that occur in active and passive elements and the fluid flow, passing over them, interact with each other and the ambient environment, and the mathematical model that describes them shall be formulated as conjugate.

The mathematical model of interacting with each other thermal processes, occurring in the solid elements and the fluid flow, represents a system of non-stationary, non-linear, and, in some cases, stochastic differential equations of heat transfer, Navier-Stokes motion and energy in the fluid flow both inside and outside the TS, and is solved together with the respective boundary and matching conditions at the solid-liquid interface. Taking into account the fact that the number of equations in the
mathematical model is by an order of magnitude more than that of elements in the TS (running into tens and hundreds of thousands), it is quite difficult to solve the problem both mathematically and computationally, even with state-of-the-art supercomputers. This paper describes the cluster method for mathematical modeling of thermal processes, in which the complex TS structure is represented by a thermal model in the form of a system, consisting of clusters, each of which includes a core, combining heat-emitting elements that fall into the cluster, a shell of the cluster, and a fluid, flowing through it. This being said, the state of the thermal process in each $k$-th cluster $(k = 1,2,...,K)$ at any specific time $t$ is characterized by three state variables, namely, the temperature of the core $T_{c,k}(t)$, the temperature of the shell $T_{s,k}(t)$ and the ambient temperature $T_{a,k}(t)$ in the cluster. The cluster-based approach in modelling of thermal processes in complex TS structures allows to find temperature distributions in the fluid pumped through the TS, temperatures of active and passive elements, and temperature distributions inside the TS case. The application of the method is considered on the example of the thermal process, occurring in the real TS.

2. Cluster thermal and mathematical models of thermal processes in the TS

In the cluster thermal model, the complex TS structure is divided into clusters, $k = 1,2,...,K$ (Fig. 1), each of which consists of the core, combining active elements that fall into the cluster, its shell, and the fluid, flowing around the core and the shell. In real TS structures, all active and passive elements are, firstly, in conductive thermal interaction with each other carried out through multiple solid elements of connecting joints, fixtures, mountings; secondly, convective heat exchange with the fluid flow, pumped through the cluster; thirdly, in radiant heat exchange with each other and the fluid flow in the cluster; and, fourthly, convective-radiant heat exchange between the shell and the ambient environment. Thus, the temperature of the environment and the elements rapidly mixes and equalizes within the volume of the local TS structure. Therefore, the volumes and the form of clusters can be chosen in a way that it can be taken, with an accuracy sufficient for technical practice, that the temperatures of the core, the shell and the ambient environment within a single cluster are isothermal ones. The state of the thermal process in each cluster $k$, $k = 1,2,...,K$ at any specific time is fully determined by three state variables, namely, the temperature of the core, the temperature of the shell and the ambient temperature, i.e. by the vector $T_k(t) = (T_{c,k}(t), T_{s,k}(t), T_{a,k}(t))^T$. The fluid inflows to the $k$-th cluster and outflows from it with the temperatures $T_{a,in,k}(t)$ and $T_{a,out,k}(t)$ respectively. The total heat dissipation power in the $k$-th cluster, $P_k(t)$, is given by

$$P_k(t) = \sum_{n=1}^{N_k} Q_{n,k}(t),$$

where $Q_{n,k}(t)$ is the heat dissipation power of the $n$-th element in the $k$-th cluster.
pation power in the core of the k-th cluster is equal to \( \Phi_k(t) \), and the temperature of the ambient environment outside the shell equals \( T_e \).

The mathematical model of thermal processes in the cluster thermal model (Fig. 1) is based on the following prerequisites: a) the fluid inside the cluster shell is incompressible, the rate of the fluid flow changes along the flow direction and is constant in terms of cross-section, the convective heat flow in the fluid far exceeds the heat flow of the thermal conductivity, the internal heat sources caused by the fluid viscosity are small to negligible compared to the heat emission energy of the active elements; b) the radiation between the clusters, the cores, the shells and the environment inside and outside of the clusters, as well as the dependence of thermophysical properties of materials of the TS’s solid elements and the environment inside and outside the clusters on the temperature is disregarded.

In this condition, the mathematical model of thermal processes in the k-th cluster consists of three equations that describe three state variables of the thermal process in the cluster, namely, the temperature of the core, the temperature of the shell and the ambient temperature \( T_k(t) = (T_{c,k}(t), T_{s,k}(t), T_{a,k}(t))^T \) [1, 2]:

- for the core of the k-th cluster
  
  \[
  h_{c,k} \frac{dT_{c,k}(t)}{dt} + J_{c,s,k}(T_{c,k}, T_{s,k}, t) + J_{c,a,k}(T_{c,k}, T_{a,k}, t) - \]
  
  \[-J_{c,k-1-c,k}(T_{c,k-1}, T_{c,k}, t) + J_{c,k-c,k+1}(T_{c,k}, T_{k+1}, t) = \Phi_k(t),
  \]
  
  \( T_{c,k}(t = 0) = T_e \),

  where
  
  \( J_{c-s,k}(T_{c,k}, T_{s,k}, t) = g_{c-s,k}^\text{cond} \cdot (T_{c,k}(t) - T_{s,k}(t)) \)

  is the conductive heat flow between the core of the k-th cluster and its shell transferred by the conductive heat transfer \( g_{c-s,k}^\text{cond} \) in the absence of the conductive contact between the core and the shell \( g_{c-s,k}^\text{cond} = 0 \);

  \( J_{c-a,k}(T_{c,k}, T_{a,k}, t) = g_{c-a,k}^\text{conv} \cdot (T_{c,k}(t) - T_{a,k}(t)) \)

  is the convective heat transfer between the core of the k-th cluster and the fluid flow pumped through the k-th cluster, \( g_{c-a,k}^\text{conv} = \alpha_c \cdot S_{c-a,k} \) is the convective heat transfer, \( \alpha_c \) is the heat-exchange coefficient, \( S_{c-a,k} \) is the heat-release surface of the core of the k-th cluster;

  \( J_{c,k-1-c,k}(T_{c,k-1}, T_{c,k}, t) = g_{c-k-1-c,k}^\text{cond} \cdot (T_{c,k-1}(t) - T_{c,k}(t)), \)

  \( J_{c,k-c,k+1}(T_{c,k}, T_{k+1}, t) = g_{c-k-c,k+1}^\text{cond} \cdot (T_{c,k}(t) - T_{k+1}(t)) \)

  are the conductive heat flows between the cores of the \( k-1 \) and \( k+1 \) clusters and the \( k \)-th and \( k+1 \) clusters, \( g_{c-k-1-c,k}^\text{cond} \) and \( g_{c-k-c,k+1}^\text{cond} \) are the conductive heat transfer, in the absence of conductive thermal contact between the adjacent cores of two clusters the heat transfer \( g_{c-k-1-c,k}^\text{cond} \) and \( g_{c-k-c,k+1}^\text{cond} \) equals zero; on the contrary, in the case of the ideal thermal contact between the adjacent cores, the thermal contact resistance \( R_{c-k-1-c,k}^\text{cond} = 1/g_{c-k-1-c,k}^\text{cond} \) and \( R_{c-k-c,k+1}^\text{cond} = 1/g_{c-k-c,k+1}^\text{cond} \) equals zero;

  \( h_{c,k} = \rho_{c,k} c_{c,k} V_{c,k} \) is the specific heat capacity of the core of the k-th cluster with the density \( \rho_{c,k} \), the specific heat capacity \( c_{c,k} \), the volume \( V_{c,k} \); \( \Phi_k(t) \) is the aggregate energy of the internal heat sources (energy of consumption of the active elements) in the core of the k-th cluster;

- for the shell of the k-th cluster
  
  \[
  h_{s,k} \frac{dT_{s,k}(t)}{dt} - J_{s-s,k}(T_{s,k}, T_{s,k}, t) + J_{s-e,k}(T_{s,k}, T_e, t) - \]
  
  \[-J_{s-e,k}(T_{s,k}, T_{a,k}, t) - J_{s,k-1-s,k}(T_{s,k-1}, T_{s,k}, t) + J_{s,k-s,k+1}(T_{s,k}, T_{s,k+1}, t) = 0,
  \]
  
  \( T_{s,k}(t = 0) = T_e \),

  where
  
  \( J_{s-s,k}(T_{s,k}, T_{s,k}, t) = g_{s-s,k}^\text{conv} \cdot (T_{s,k}(t) - T_{s,k}(t)) \)

  are the convective heat flows from the external shell surface to the ambient environment and from the internal shell surface to the fluid flow inside the k-th cluster, respectively, \( g_{s-s,k} = \alpha_{s-s,k} S_{s-s,k} \) and
are the convective heat transfer with heat-exchange coefficients $\alpha_{s-e,k}$ and $\alpha_{s-a,k}$, and outside $S_{s-e,k}$ and inside $S_{s-a,k}$ heat-release surfaces of the shell:

$$J_{s,k-1-s,k}(T_{s,k-1}, T_{s,k}, t) = g_{s(k-1)-s,k}^{\text{cond}} \cdot (T_{s,k-1}(t) - T_{s,k}(t)),$$

$$J_{s,k-s,k+1}(T_{s,k}, T_{s,k+1}, t) = g_{s(s-k+1)-k+1}^{\text{cond}} \cdot (T_{s,k}(t) - T_{s,k+1}(t))$$

are the conductive heat flows between the shells of the $k-1$, $k$-th and $k$-th, $k+1$ clusters with the heat transfer $g_{s,k-1-s,k}^{\text{cond}}$ and $g_{s,k-s,k+1}^{\text{cond}}$, in the absence of the conductive thermal contact between the shells of the adjacent clusters the heat transfer $g_{s,k-1-s,k}^{\text{cond}}$ and $g_{s,k-s,k+1}^{\text{cond}}$ equals zero, otherwise, in the case of ideal thermal contact between the shells of the adjacent clusters the thermal contact resistance $R_{s,k-1-s,k}^{\text{cond}} = 1/g_{s,k-1-s,k}^{\text{cond}}$ and $R_{s,k-s,k+1}^{\text{cond}} = 1/g_{s,k-s,k+1}^{\text{cond}}$ equals zero; $h_s = \rho_s c_s V_s$ is the total heat capacity of the shell of the $k$-th cluster with the density $\rho_s$, the specific heat capacity $c_s$, and the volume $V_s$;

- for the fluid, flowing in the $k$-th cluster

$$h_{a,k} \frac{dT_{a,k}(t)}{dt} - J_{c-a,k}(T_{c,k}, T_{a,k}, t) + J_{s-a,k}(T_{s,k}, T_{a,k}, t) + J_{a,k}(T_{a,k,\text{out}}, T_{a,k,\text{in}}, t) = 0,$$

$$T_{a,k}(t = 0) = T_e, \quad k = 1, \ldots, m$$

where

$$J_{a,k}(T_{a,k,\text{out}}, T_{a,k,\text{in}}, t) = c_{a,k} G_k (T_{a,k,\text{out}}(t) - T_{a,k,\text{in}}(t))$$

is the enthalpy flow of the fluid, accumulating heat from the heat-emitting elements of the TS in the said cluster; $G_k = \rho_{a,k,\text{in}} v_{a,k,\text{in}} S_{a,k,\text{in}} = \rho_{a,k,\text{out}} v_{a,k,\text{out}} S_{a,k,\text{out}}$ is the mass flow of the fluid, inflowing to the $k$-th cluster through the opening with the section $S_{a,k,\text{in}}$ at the rate $v_{a,k,\text{in}}$ and the temperature $T_{a,k,\text{in}}(t)$ and outflowing from the $k$-th cluster through the opening with the section $S_{a,k,\text{out}}$ at the rate $v_{a,k,\text{out}}$ and the temperature $T_{a,k,\text{out}}(t)$; $h_{a,k} = \rho_{a,k} c_{a,k} V_{a,k}$ is the total heat capacity of the fluid, flowing through the $k$-th cluster with the density $\rho_{a,k}$, the specific heat capacity $c_{a,k}$, and the volume $V_{a,k}$.

With an accuracy sufficient for engineering practice, it can be taken that the average temperature of the fluid flow within a single cluster is tied to the temperatures of the flow at the input and the output of the cluster via the correlation $T_{a,k}(t) = T_{a,k,\text{out}}(t) - T_{a,k,\text{in}}(t)$ [3, 4, 5], therefore, the flow $J_{a,k}(T_{a,k,\text{out}}, T_{a,k,\text{in}}, t)$ may be expressed as $J_{a,k}(T_{a,k,\text{out}}, T_{a,k,\text{in}}, t) = 2c_{a,k} G_k (T_{a,k}(t) - T_{a,k,\text{in}}(t))$.

By applying the expressions for the thermal flows to equations (1), (2), (3), we will get the following mathematical model for the thermal processes for the state variables in the $k$-th cluster in matrix view:

$$H_k \frac{dT_k(t)}{dt} + G_k \cdot T_k(t) = P_k(t),$$

$$T_k(t = 0) = T_e I,$$

where $I = (1, 1, 1)^T$ is the unit vector; $H_k = \text{diag}[h_{c,k}, h_{s,k}, h_{a,k}]$ is the diagonal matrix of the total heat capacity of the core, the shell and the fluid flow of the $k$-th cluster; $G_k$ is the matrix of the thermal conduction of the $k$-th cluster equals

$$G_k = \begin{pmatrix}
G_k^{(1)} & -g_{c,s,k}^{\text{cond}} & -g_{c-a,k}^{\text{cond}} \\
-g_{c-s,k}^{\text{cond}} & G_k^{(2)} & g_{s-a,k}^{\text{cond}} \\
-g_{a-s,k}^{\text{cond}} & g_{s-a,k}^{\text{cond}} & G_k^{(3)}
\end{pmatrix},$$

$P_k(t)$ is the vector of the right-hand side of matrix equation (7)

$$P_k(t, \omega) = \begin{pmatrix}
\Phi_k(\omega) + g_{c,k-1-c,k}^{\text{cond}} T_{c,k-1}(t) + g_{c,k-c,k+1}^{\text{cond}} T_{c,k+1}(t) \\
g_{s-c,k-1-s,k}^{\text{cond}} T_{s,k-1}(t) + g_{s,s-k-1+s,k}^{\text{cond}} T_{s,k+1}(t)
\end{pmatrix}.$$

Taking into account the evident congruence $T_{a,k,\text{in}}(t) = T_{a,k-1,\text{out}}(t)$ and the recurrence relation $T_{a,k,\text{in}}(t) = 2T_{a,k}(t) - T_{a,k-1,\text{in}}(t)$, $k = 1, 2, \ldots, K$, we will get the block matrix equation expressed in terms of the temperature $T_{a,k,\text{in}}$ of the fluid flow at the input of the TS that is known a priori
where \( \mathbf{T}(t) = (T_1(t), T_2(t), ..., T_K(t))^T \) is the block vector of temperatures of the clusters, the \( k \)-th vector block of which equals \( T_k(t) = (T_{c,k}(t), T_{s,k}(t), T_{a,k}(t))^T, \ k = 1,2, ..., K; \)
\( H = \text{diag}(H_1, H_2, ..., H_K) \) is the diagonal block matrix of the total heat capacity of the solid elements of the clusters, consisting of the diagonal block matrices \( H_k = \text{diag}(h_{c,k}, h_{s,k}, h_{a,k}), \ k = 1,2, ..., K; \)
\( Q(t) = (Q_1(t), Q_2(t), ..., Q_K(t))^T \) is the block vector of the energies \( \Phi_k(t) \) in each cluster, the temperature of the environment \( T_e \) and the temperature at the input of the TS \( T_{a,in} \) that are known a priori, the \( k \)-th vector block of which equals \( Q_k(t) = (\Phi_k(t), g_{s,a,k}^\text{con}, T_e, 2(-1)^{k-1}c_{a,k}G_kT_{a,in})^T; \)
\( \mathbf{G} \) is \( 3K \times 3K \) determinate block matrix of the thermal conduction of the clusters with the following structure

\[
\mathbf{G} = \begin{pmatrix}
G_{11} & G_{12} & 0 & \cdots & 0 \\
G_{21} & G_{22} & G_{23} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
G_{K,1} & G_{K,2} & G_{K,3} & \cdots & G_{KK}
\end{pmatrix}
\] (8)

with the diagonal block matrices \( G_{ii} \) equal \( G_i \) (5) and diagonal block matrices

\[
G_{i,i-1} = \text{diag}(g^\text{cond}_c[i-1,i], g^\text{cond}_s[i-1,i], 4c_{a,i}G_i), i = 2,3, ..., K,
G_{i,i+1} = \text{diag}(g^\text{cond}_c[i,i+1], g^\text{cond}_s[i,i+1], 4c_{a,i}G_i), i = 1,2, ..., K - 1,
G_{ij} = (-1)^{i-j}\text{diag}(0, 0, 4c_{a,j}G_j), i = 3,4, ..., K, \ j = 1,2, ..., K - 2.
\]

The equation system (7) is the first-order block matrix linear ordinary differential equations, which can be solved with the known numerical techniques.

3. Application of the cluster method

Let’s consider the application of the cluster method on the example of the stationary thermal process in the TS, which is a device in a flat case, containing multiple electronic elements cooled by the air flow that is pumped from the ambient environment to the TS input. The cluster thermal model of the TS consists of five clusters (Fig. 2) with heat generation capacities of 22, 10, 15, 8, and 14 W in each cluster. The ambient temperature and the equal temperature of the fluid pumped to the input of the TS equal \( T_e = T_{a,in} = 23^\circ C \).

The calculated isotherms of the temperature distribution inside the TS, including the temperatures of the cores and the shells with the partitions in the structure along the air flow from the inlet to the outlet, which are derived from the solution of the steady-state equations (7) of the mathematical model (where \( d/dt = 0 \) are shown in fig. 2. The obtained data also show that the temperature distribution of

![Fig. 2. Calculated isotherms for the distribution of the temperature of the solid phase of TS by the clusters along the fluid flow](image-url)
the cooling fluid pumped through the case of the TS is non-linear, in contrast to the existing methods for modelling of thermal processes in the complex TS, in which it is often assumed that the temperature distribution in the pumped fluid flow is uniform or linear. Comparison of the results of computational simulation of the considered TS with experimental data shows that the simulation error does not exceed 11%, which is acceptable for the practice of engineering design.

4. Conclusion
The cluster modeling method demonstrated in this paper allows to determine the temperature distributions in the heat-emitting elements, the TS case, and the fluid flow, taking into account the unevenness of temperature distributions and the conjugate nature of heat transfer. The thermal model of the TS structure represents a system of clusters, in each of which the state of the thermal process is characterized by three isothermal temperatures, namely, the temperature of the core of the cluster, which combines all the heat-emitting elements that fall into the cluster, the temperature of the cluster shell and the temperature of the fluid flow within its volume. Moreover, all elements of a single cluster (core, shell, fluid), as well as the elements of the adjacent clusters, interacting with them, are in a state of conjugated heat exchange between themselves. The mathematical model is based on the cluster thermal model and represents a block matrix system of equations, with matrix and vector blocks corresponding to different clusters of the thermal model. The application of the developed method is shown on the example of the real TS, which is a computing system, consisting of several (five) electronic modules that exchange heat with the flow of the cooling fluid inside the TS case and the environment. Publication is made as part of national assignment for SRISA RAS (fundamental scientific research 47 GP) on the topic No.0065-2019-0001 (AAAA-A19-119011790077-1).

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