VARIATIONS ON THE RESURGENCE OF
THE GAMMA FUNCTION

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Abstract. We review Écalle’s formalism of minors, natural-majors and real-majors, and provide explicit formulas in the Borel plane that show the resurgence of the exponential of the Stirling series. We also discuss its Stokes phenomena in the framework of alien calculus.

1. Introduction

Let Ĉ denote the Riemann surface of the logarithm. The functions

\[ \lambda(z) := \frac{\Gamma(z)}{\sqrt{2\pi} z^{e^{-z} - \frac{1}{2}}}, \quad \lambda_c(z) := z^{-c} \lambda(z) \quad (c \in \mathbb{C}) \]  

are meromorphic in Ĉ with their principal branches holomorphic in \( \mathbb{C} - \mathbb{R}_{\leq 0} \). We have \( \lambda(z) \xrightarrow{z \to \infty} 1 \) along \( \mathbb{R}_{>0} \), with \( \log(\lambda(z)) \) asymptotic to the Stirling series

\[ \tilde{\mu}(z) := \sum_{n=0}^{\infty} \frac{B_{2n+2}}{(2n+2)(2n+1)} z^{-2n-1}. \]  

We are interested in the asymptotic expansion of \( \lambda(z) \) itself, as well as that of \( \lambda_c(z) \) or \( 1/\lambda_c(z) \), in the framework of Écalle’s Resurgence Theory. Our study was prompted by M. Kontsevich, who communicated us an explicit formula for the Laplace transform of a function related to \( \lambda \), which entailed resurgence \[8\]. Our purpose here is to give various formulas for “minors”, “natural-majors” and “real-majors” of \( \lambda_c \), explaining along the way the meaning and interest of this Resurgence Theory jargon, how real-majors relate to Kontsevich’s formula and why these explicit formulas prove the resurgent character of \( \lambda \) or \( \lambda_c \) independently of that of the Stirling series. We will also discuss the Stokes phenomenon for \( \Gamma \) in the language of Écalle’s alien calculus.

Resurgence Theory originated with dynamical systems but has recently pervaded many areas of mathematics and matematical physics.

Key words and phrases. Resurgence, Stirling series, Lambert \( W \) function.

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2. Beginning of a synopsis of Resurgence Theory

2.1 A **resurgent function** is a function \( \varphi(z) \) that is holomorphic in an unbounded domain, in which it can be obtained by Borel-Laplace summation from a **resurgent series** \( \tilde{\varphi}(z) \).

2.2 A **resurgent series** is a power series \( \tilde{\varphi}(z) \) in \( z^{-1} \) whose Borel transform \( \hat{\varphi}(\xi) \) (the formal series obtained by termwise application of \( B: z^{-c} \mapsto \xi^{c-1}/\Gamma(c) \)) is convergent near \( \xi = 0 \) and is an **endlessly continuable** germ, which we call “the minor”.

2.3 Being **endlessly continuable**, roughly speaking, means that \( \hat{\varphi}(\xi) \) analytically continues to a (possibly multivalued) holomorphic function without any natural boundary: the analytic continuation is possible along any path except for a discrete set of singularities along the path, that can be circumvented, at will, to the left or to the right (with some stipulations—see [4] for the most general definition, or [2, 7]).

Borel-Laplace summation \( \hat{\varphi}(z) \) i.e. Laplace transform (or some variant) composed with Borel transform, thus gives the function \( \varphi(z) \); the formal series \( \tilde{\varphi}(z) \) (which usually is divergent everywhere) then appears as the asymptotic expansion at infinity \( (\xi) \) of \( \varphi(z) \):

\[
\begin{align*}
\text{Resurgent function } & \varphi(z) \\
\text{Laplace } & \mathcal{L}^\theta \\
\text{Borel } & \mathcal{B} \\
\text{Resurgent series } & \tilde{\varphi}(z) \\
\hat{\varphi}(\xi) \text{ endlessly continuable}
\end{align*}
\]

Here, we assume \( \hat{\varphi}(\xi) \) to have analytic continuation with at most exponential growth along the ray \( e^{i\theta}\mathbb{R}_{>0} \) for some \( \theta \in \mathbb{R} \) and

\[
\varphi(z) = \mathcal{L}^\theta \hat{\varphi}(z) = \int_0^{e^{i\theta}\infty} e^{-z\xi} \hat{\varphi}(\xi) \, d\xi
\]

in \( \Pi^\theta_\tau := \{ z \in \mathbb{C} | -\theta - \frac{\pi}{2} < \arg z < -\theta + \frac{\pi}{2}, \ \Re(ze^{i\theta}) > \tau \} \) (2.1)

with \( \tau > 0 \) large enough so that the exponential decay of the Laplace kernel ensures the integrability of \( e^{-z\xi} \hat{\varphi}(\xi) \) at infinity\(^1\).

2.4 The most elementary examples of endlessly continuable functions are the meromorphic functions—the analytic continuation has only one branch in this case—and algebraic functions. Another example is \( \hat{U}(a, b, \xi) = \frac{\xi^{a-1}}{\Gamma(a)} (1 + \xi)^{b-a-1} \), whose Laplace transform is the confluent hypergeometric function \( U(a, b, z) \) ([10, Ex. 6.105]).

\(^1\)If \( \hat{\varphi}(\xi) \) has analytic continuation with uniform exponential bound in a sector \( \arg \xi \in I \), then the Laplace transforms \( \mathcal{L}^\theta \hat{\varphi}(z) \), \( \theta \in I \), can be glued together (the Cauchy theorem entail that they match) and give rise to a function \( \mathcal{L}^I \hat{\varphi}(z) \) analytic in \( \mathcal{D}^\pi := \bigcup_{\theta \in I} \Pi^\theta_\tau \) (sectorial neighbourhood of \( \infty \) of opening \( |I| + \pi \)).
The Stirling series $\tilde{\mu}(z)$ is a resurgent series, since its Borel transform
\begin{equation}
\hat{\mu}(\xi) = \sum_{n \geq 0} \frac{B_{2n+1}}{(2n+1)!} \xi^{2n} = \xi^{-2} \left( \frac{1}{2} \coth \frac{\xi}{2} - 1 \right) \in \mathbb{C}\{\xi\}
\end{equation}
analytically continues as a meromorphic function on $\mathbb{C}$ (with simple poles on $2\pi i \mathbb{Z}^*$). The Laplace transform $L(\xi)\hat{\mu}$ is none other than $\log(\lambda(z))$, which is thus a resurgent function (see [4, pp. 244–246] or [10, Th. 5.41]).

2.5 We said power series in \S 2.2 but we may accept non-integer powers (or even more general monomials, e.g. involving powers of $\log z$). However, we need all monomials $z^{-c}$ of $\hat{\phi}(z)$ to have $\Re c > 0$ for the minor $\mathcal{B}\hat{\phi}(\xi)$ to be integrable at 0 and thus have a meaningful Laplace transform. The integrability constraint is bypassed in Écalle’s “singularity theory” ([4], [6], [12, §3.1–3.2] or [10, Sec. 6.8]), which not only widens the scope of Borel-Laplace summation but also provides the appropriate framework for alien calculus—see Section 3.

A natural-major is any endlessly continuable holomorphic function $\hat{\phi}(\xi)$ initially defined in a finite sector $\Sigma_{R,J} := \{ \xi = re^{i\theta} \in \mathbb{C} \mid 0 < r < R, \theta \in J \}$, with some $R > 0$ and some interval $J$ of length $> 2\pi$. Singularities are defined by quotienting the space of natural-majors by the subspace $\mathbb{C}\{\xi\}$ of regular germs. The equivalence class of $\hat{\phi}(\xi)$ is denoted by $\hat{\phi} = \text{sing}_0(\hat{\phi}(\xi))$. The monodromy variation induces an operator
\begin{equation}
\text{var}: \hat{\phi} \mapsto \text{var} \hat{\phi} = \hat{\phi} \quad \text{defined by} \quad \hat{\phi}(\xi) := \hat{\phi}(\xi) - \hat{\phi}(e^{-2\pi i} \xi).
\end{equation}
The image $\hat{\phi}$ is the minor of the singularity $\hat{\varphi}$; it is initially defined in a sector $\Sigma_{R,I}$ with $I := J \cap (2\pi + J)$ but is endlessly continuable too.

A singularity $\hat{\varphi}$ is called an integrable singularity if its minor $\hat{\phi}$ is uniformly integrable at the origin in every sector $\Sigma_{R,I}$ and if it has a representative $\hat{\phi}(\xi)$ that is $o(1/|\xi|)$ uniformly in every sector $\Sigma_{R,I}$. The latter condition implies that $\hat{\phi}$ is uniquely determined by $\hat{\phi}$; we use the notation $\hat{\phi} = \hat{\phi}(\xi)$. Conversely, any integrable minor $\hat{\phi}(\xi)$ is the minor of an integrable singularity; e.g. $\hat{\phi}(\xi) \in \mathbb{C}\{\xi\}$ resp. $\xi^{1/2}\mathbb{C}\{\xi\}$ $\Rightarrow$ a representative of $\hat{\phi}$ is $\hat{\phi}(\xi) := \hat{\phi}(\xi)(2\pi i)^{-\frac{2}{-\varphi}}$ resp. $\frac{1}{2\pi i} \hat{\phi}(\xi)$.

2.6 Suppose $\hat{\phi}$ is an arbitrary singularity whose minor extends analytically to a sector $\Sigma_{\infty,I}$, with $|\hat{\phi}(\xi)| \leq C e^{r|\xi|}$ on $\Sigma_{\infty,I} \cap \{ |\xi| \geq 1 \}$. Then, given $\theta \in I$, for any natural-major $\hat{\phi}(\xi)$ and any $\delta > 0$ small enough, we can consider the contour $C_{\delta,\theta} := \{ \theta' \in [\theta - 2\pi, \theta] \mapsto \delta e^{i\theta'} \}$ and the function
\begin{equation}
\mathcal{L}^{\theta,\delta}\hat{\phi}(z) := \int_{C_{\delta,\theta}} e^{-z\xi} \hat{\phi}(\xi) \, d\xi + \int_{\delta e^{i\theta}} e^{-z\xi} \hat{\phi}(\xi) \, d\xi \quad \text{holomorphic for} \quad z \in \Pi_{\theta},
\end{equation}
which does not depend on $\delta$ nor on the chosen natural-major $\hat{\phi}(\xi)$ and can be rewritten in terms of the sole natural-major as an integral $L^{\theta,\delta}\hat{\phi}(z) := \int_{\mathcal{H}_{\theta}} e^{-z\xi} \hat{\phi}(\xi) \, d\xi$ over a $\theta$-rotated Hankel contour $\mathcal{H}_{\theta}$ running from $e^{i(\theta - 2\pi)} \xi$ to $e^{i\theta} \xi$ and circling anticlockwise around 0 at distance $\delta$ (at least if $\hat{\phi}(\xi)$ itself has at most exponential growth at $\infty$ in directions $\theta$ and $\theta - 2\pi$). Moreover, as in Footnote [1] the various $L^{\theta,\delta}\hat{\phi}$, $\theta \in I$, can be glued together,
giving rise to $\mathcal{L}^I\tilde{\phi}$ holomorphic in $\mathcal{D}'_I$. This is an extension of the usual Laplace transform in the sense that $\mathcal{L}^I\tilde{\phi} = \mathcal{L}^I\phi$ for any integrable minor $\phi$.

For example, the Laplace transform of the singularity defined by

$$\tilde{I}_c(\xi) := \frac{\xi^{c-1}}{(1 - e^{-2\pi i} \xi)^c} \quad \text{for } c \in \mathbb{C} - \mathbb{Z}_{>0}, \quad \tilde{I}_c(\xi) := \frac{\xi^{c-1}}{(c-1)! \frac{\log \xi}{2\pi i}} \quad \text{for } c \in \mathbb{Z}_{>0} \quad (2.5)$$

is $\mathcal{L}^I\tilde{I}_c = z^{-c}$ for any $I$. In general, the asymptotic behaviour at $\infty$ of $\mathcal{L}^I\tilde{\phi}(z)$ only depends on the asymptotic behaviour at 0 of a natural-major: if $\tilde{\phi}(\xi)$ can be represented as a convergent series of monomials proportional to $\tilde{I}_{cn}(\xi)$, then $\mathcal{L}^I\tilde{\phi}(z)$ has an asymptotic expansion at $\infty$ given by the corresponding series of monomials proportional to $z^{-cn}$.

2.7 Resurgence and Borel summation are compatible with multiplication, the corresponding operation on singularities being the “convolution of singularities” $\tilde{\phi} \ast \tilde{\phi}$.

3. Main results

We now state our main results as two theorems, which will be proved (in Sections 4, 5) independently one of the other and independently of the Stirling series.

3.1 The Lambert function $x \mapsto W(x)$ is implicitly defined by $W(x) e^{W(x)} = x$; the canonical reference for the description of its analytic continuation is [3], the only singularities are located at $x = -e^{-1}$ and 0. One denotes by $W_0$ and $W_{-1}$ its real branches so that $-e^{-1} < x < 0 \Rightarrow W_0(x) > W_{-1}(x)$, both have square root branch points at $-e^{-1}$.

**Theorem 1.** (i) $\lambda_{3/2}(z)$ is a resurgent function obtained by $\mathcal{L}^{(-\frac{\pi}{3}, \frac{\pi}{3})}$ from the integrable singularity $\tilde{\lambda}_{3/2}$ that has natural-major and minor explicitly given, for $\arg \xi = 0$, by

$$\tilde{\lambda}_{3/2}(\xi) = \frac{1}{\sqrt{2\pi}} W_0(-e^{-1-\xi}), \quad \hat{\lambda}_{3/2}(\xi) = \frac{1}{\sqrt{2\pi}} (W_0(-e^{-1-\xi}) - W_{-1}(-e^{-1-\xi})). \quad (3.1)$$

(ii) $\chi(z) := \frac{1}{\tilde{\lambda}_{3/2}(z)}$ is a resurgent function obtained by $\mathcal{L}^{(-\frac{\pi}{3}, \frac{\pi}{3})}$ from the integrable singularity $\tilde{\chi}$ that has natural-major and minor explicitly given, for $\arg \xi = -\pi$, by

$$\tilde{\chi}(\xi) = \frac{1}{\sqrt{2\pi}} W_0(-e^{-1+\xi}), \quad \hat{\chi}(\xi) = \frac{1}{\sqrt{2\pi}} (W_0(-e^{-1+\xi}) - W_{-1}(-e^{-1+\xi})). \quad (3.2)$$

(iii) Define inductively $a_1 = 1$ and $a_k = \frac{1}{k+1} (a_{k-1} - \sum_{\ell=2}^{k-1} a_{\ell} a_{k+1-\ell})$ for $k \geq 2$, so that

$$a_2 = \frac{1}{3}, \quad a_3 = \frac{1}{36}, \quad a_4 = -\frac{1}{270}, \quad a_5 = \frac{1}{4320}, \quad a_6 = \frac{1}{17010}, \quad a_7 = -\frac{139}{5443200}, \quad \ldots \quad (3.3)$$

Then

$$\tilde{\lambda}_{3/2}(\xi) = -\frac{1}{\sqrt{2\pi}} (1 + \sum_{k \geq 1} (-1)^k a_k (2\xi)^{k/2}), \quad \hat{\lambda}_{3/2}(\xi) = \frac{1}{\sqrt{\pi}} \sum_{n \geq 0} a_{2n+1} 2^{n+1} \xi^{n+\frac{1}{2}} \quad (3.4)$$

and

$$\tilde{\chi}(\xi) = -\frac{1}{\sqrt{2\pi}} (1 + \sum_{k \geq 1} i^k a_k (2\xi)^{k/2}), \quad \hat{\chi}(\xi) = \frac{1}{\sqrt{\pi}} \sum_{n \geq 0} (-1)^n a_{2n+1} 2^{n+1} \xi^{n+\frac{1}{2}}. \quad (3.5)$$
Notice that, as a consequence of the above statements,
\[
\dot{\chi}(\xi) = i\dot{\lambda}_{3/2}(e^{-i\pi}\xi), \quad \ddot{\chi}(\xi) = i\ddot{\lambda}_{3/2}(e^{-i\pi}\xi) = -i\dot{\lambda}_{3/2}(e^{i\pi}\xi).
\] (3.6)
Since the analytic continuation of the Lambert $W$ function has no other singularities than those located at $-e^{-1}$ and 0, it follows that the minors $\dot{\lambda}_{3/2}(\xi), \ddot{\lambda}(\xi) \in \mathbb{C}\{\xi\}$ as well as the natural-majors $\ddot{\lambda}_{3/2}(\xi), \dddot{\lambda}(\xi) \in \mathbb{C}\{\xi\}$ are endlessly continuable; all their singularities are located above $2\pi i\mathbb{Z}$ and are square root branch points. On the Riemann surface of $\xi^{3/2}$, we thus have four singularity-free sectors, corresponding to arguments in
\[
J_1 := (-\frac{\pi}{2}, \frac{\pi}{2}), \quad J_2 := (\frac{\pi}{2}, \frac{3\pi}{2}), \quad J_3 := (\frac{3\pi}{2}, \frac{5\pi}{2}), \quad J_4 := (-\frac{3\pi}{2}, -\frac{\pi}{2}).
\]
The functions $\ddot{\lambda}_{3/2} = z^{-3/4}\lambda$ and $\dddot{\lambda} = z^{-3/4}$ are the Laplace transforms of their minors as in Formula (2.11) with any $\theta \in J_1$ and $\tau > 0$, hence, in the notation of Footnote [4]
\[
z^{-\frac{3}{4}}\dddot{\lambda}(z) = \mathcal{L}^{J_1}\dddot{\lambda}_{3/2}(z), \quad \frac{z^{-\frac{3}{4}}}{\dddot{\lambda}(z)} = \mathcal{L}^{J_1}\dddot{\lambda}(z) \quad \text{in} \quad \bigcup_{\theta \in J_1, \tau > 0} \Pi^\theta_\tau = \{-\pi < \arg z < \pi\} (3.7)
\]
(\text{using } J_3 \text{ instead of } J_1 \text{ amounts to a simple change of branch of the square roots). The functions } \mathcal{L}^{J_1}\dddot{\lambda}_{3/2} \text{ and } \mathcal{L}^{J_1}\dddot{\lambda} \text{ are defined in } \{-2\pi < \arg z < 0\} \text{ and differ from the previous ones by the Stokes phenomenon through } \theta = \frac{\pi}{2}, \text{ which in this case is equivalent to Euler’s reflection formula—see Section 4.}

It follows from § 2.7 that, for every $c \in \mathbb{C}$, $\lambda_c = z^{-c+\frac{3}{2}}\lambda_{3/2}$ is resurgent with $\ddot{\lambda}_c = \mathcal{I}_{c-\frac{3}{2}} \dddot{\lambda}_{3/2}$. Note that $\ddot{\lambda}_c$ is an integrable singularity if and only if $\Re c > 0$.

The sequence $(a_n)$ was defined in [9] in relation with the asymptotics of $\Gamma$. Since $\Gamma(n + \frac{3}{2}) = 2^{-n-1}(2n+1)!! \sqrt{\pi}$, the expansions (3.4)–(3.5) imply asymptotic expansions in the same domain as in (3.7):
\[
\lambda(z) \sim \ddot{\lambda}(z) := \sum_{n \geq 0} (2n+1)!! a_{2n+1} z^{-n}, \quad \frac{1}{\lambda(z)} \sim \dddot{\lambda}(-z) \quad \text{in} \quad \mathbb{C} - \mathbb{R}_{\leq 0}. \quad (3.8)
\]

3.2 Real-major[4] constitute a variant of the notion of natural-majors (see [6, pp. 81–82]): a real-major of a singularity $\check{\varphi}$ is a function $\check{\varphi}(\xi) = -2\pi i \check{\varphi}(e^{-\pi i}\xi)$ where $\check{\varphi}(\xi)$ is any natural-major. The minor and the Laplace transform of $\check{\varphi}$ can be retrieved as
\[
\check{\varphi}(\xi) = -\frac{1}{2\pi i}(\check{\varphi}(e^{\pi i}\xi) - \check{\varphi}(e^{-\pi i}\xi)), \quad \mathcal{L}^\theta \check{\varphi}(z) = \check{\mathcal{L}}^\theta \check{\varphi}(z) := \frac{1}{2\pi i} \int_{\mathcal{H}_{\theta+\pi}} e^{-z\xi} \check{\varphi}(\xi) \, d\xi. \quad (3.9)
\]
By Fourier-Laplace inversion, there are also integral formulas for a natural-major $\check{\varphi}(\xi)$ or a real-major $\check{\varphi}(\xi)$ in terms of $\varphi = \mathcal{L}^\theta \check{\varphi}$, as well as for the minor $\check{\varphi}(\xi)$ when $\check{\varphi}$ is an integrable singularity. For instance with $\theta = 0$, in which case $\varphi$ is defined at least in the half-plane $\Pi^\theta_0$, a real-major is given by
\[
\check{\varphi}_{[a]}(\xi) = \int_{u}^{+\infty} e^{-z\xi} \varphi(z) \, dz \quad \text{for } \arg \xi = 0 \quad (3.10)
\]
They owe their name to their usefulness in the summation of divergent series with real coefficients.
for any \( u \in \Pi^0 \). If \( \varphi(z) \) happens to be analytic along \( \mathbb{R}_{>0} \) and integrable at \( z = 0 \), then one can take \( u = 0 \), i.e. \( \varphi_0 = \mathcal{L}^0 \varphi \).

**Theorem 2.** If \( \Re c < \frac{1}{2} \), then \( \lambda_c = \tilde{\lambda}_c(\xi) \) in \( \mathbb{C} - \mathbb{R}_{\leq 0} \), with a real-major \( \tilde{\lambda}_c := \mathcal{L}^0 \lambda_c \) explicitly given by

\[
\tilde{\lambda}_c(\xi) = \frac{r_0(c \xi)}{(2\pi)^2} \int_{\mathbb{R}} (\xi + e^Q - Q - 1)e^{-\frac{Q}{2}} dQ \quad \text{for} \; \xi \in \mathbb{C} - \mathbb{R}_{\leq 0}.
\] (3.11)

The function \( \tilde{\lambda}_c(\xi) \) extends analytically to the Riemann surface of the logarithm except at the points that project onto \( 2\pi i \mathbb{Z} \). Is principal branch extends without any singularity to the sector \( \{ -\frac{3\pi}{2} < \arg \xi < \frac{3\pi}{2} \} \).

Note that for \( \xi \in \mathbb{C} - \mathbb{R}_{\leq 0} \) the integrand in (3.11) is well-defined, because

\[ e^Q - Q - 1 \geq 0 \quad \text{for all} \; Q \in \mathbb{R}, \] (3.12)

and has asymptotic equivalent \( e^{(c - \frac{k}{2})Q} \) as \( Q \to +\infty \) and \( |Q|^{-\frac{3}{2}} \) as \( Q \to -\infty \), whence integrability follows under our condition \( \Re c < \frac{1}{2} \).

The analytic continuation of \( \tilde{\lambda}_c(\xi) \) along any path of \( \tilde{\mathbb{C}} \) which avoids \( 2\pi i \mathbb{Z} \) can be explicitly obtained by continuously deforming the integration path in the right-hand side of (3.11). Theorem 2 thus entails a constructive proof of the resurgent character of \( \lambda_c(z) \), independently of Theorem 1.

**Remark 3.1.** \( \xi \notin 2\pi i \mathbb{Z} \) means that \( \Sigma_\xi := \{ (Q, P) \in \mathbb{C}^2 \mid P^2 = \xi + e^Q - Q - 1 \} \) is a regular complex curve (of infinite genus). When \( c \in \mathbb{Z}_{<0} \), the consideration of \( \tilde{\lambda}_c(\xi) \) or of its monodromy around a point of \( 2\pi i \mathbb{Z} \) leads to the periods \( \int_\gamma P^{-3+2c} dQ, \gamma \subset \Sigma_\xi \).

**Remark 3.2.** One can do everything with \( \nu(z) := \frac{r_0(z \xi)}{(2\pi)^2 z^2 e^{-z}} \) and \( \nu_c(z) := z^{-c} \nu(z) \) instead of \( \lambda \) and \( \lambda_c \). One obtains slightly different formulas, e.g.

\[
\Re c < 1 \Rightarrow \mathcal{L}^0 \nu_c(\xi) = 2^{-\frac{c}{2}} \int_{\mathbb{R}} (\xi + e^Q - Q - 1)e^{\frac{Q}{2}} e^{Q/2} dQ \quad \text{for} \; \xi \in \mathbb{C} - \mathbb{R}_{\leq 0}.
\] (3.13)

This formula with \( c = 0 \) is the one obtained by M. Kontsevich [3].

**Remark 3.3.** One can compute the minor \( \hat{\lambda}_c(\xi) \) from (3.9a) and (3.11) using analytic continuation in \( c \). One gets nice integral formulas for \( c \) integer, for instance

\[
\hat{\lambda}_1(\xi) = \frac{1}{2\pi i \sqrt{2}} \int_{\gamma} (-\xi + e^Q - Q - 1)^{-1/2} e^Q dQ, \quad \gamma \text{ encircling } 0.
\] (3.14)

When \( c \) is half-integer, the link with Theorem 1 and the Lambert \( W \) function is that, for \( 0 < |\xi| \ll 1 \), the two local solutions to \(-\xi + e^Q - Q - 1 = 0 \) are given by

\[
Q_{\pm}(\xi) = -1 - \xi - W_{0,-1}(\xi) = \pm (2\xi)^{1/2} + O(\xi).
\] (3.15)

### 3.3 The space of resurgent series is stable under multiplication (§2.7) and nonlinear operations like composition or substitution into a convergent series [4], [11], [7]. On the other hand, nonlinear analysis is also compatible with Borel-Laplace summation, but that is much
more elementary [11, § 3.5]. For instance, since log \( \lambda = \mathcal{L}^{-1}(\mathcal{B}\tilde{\mu}) \) as mentioned earlier, it follows that
\[
\lambda = 1 + \mathcal{L}^{1}\mathcal{B}(-1 + \exp \tilde{\mu}) \sim \exp \tilde{\mu}.
\]
Comparing with (3.8), by uniqueness of the asymptotic expansion, we get \( \tilde{\lambda} = \exp \tilde{\mu} \).

The resurgent character of \( \tilde{\lambda} \) could thus be deduced from that of the Stirling series \( \tilde{\mu} \): it is always the case that the exponential of a series whose Borel transform is meromorphic, thus resurgent, has an endlessly continuable Borel transform. However, the resulting germ usually admits no closed form formula, even if one started with an explicit meromorphic function: one obtains a function that is guaranteed to be endlessly continuable, the singularities of which can be located and analyzed by means of \( \acute{E} \)calle’s alien calculus, but there is no reason why this function should be susceptible of an expression in terms of elementary ones. As testified by Theorems 1 and 2, the Borel transform of the exponential of the Stirling series is a noticeable exception.

4. Proof of Theorem 1. Alien Calculus for \( \tilde{\lambda} \)

**Lemma 4.1.** For any \( z \in \mathbb{C} \) with \( \Re z > 0 \),
\[
\sqrt{2\pi} z^{-\frac{3}{2}} \lambda(z) = \int_{0}^{+\infty} e^{-z\xi}(q_+(\xi) - q_-(\xi)) \, d\xi,
\]  
(4.1)

where \( 0 < q_-(\xi) < q_+(\xi) \) are the roots of the equation \( q - \ln q - 1 = \xi \) for \( \xi > 0 \).

**Remark 4.1.** We noticed later that this result and its proof are essentially in [9]. That paper goes on making use of the expansion of \( q_\pm \) at \( \xi = 0 \) but does not refer to the Lambert \( W \) function which, however, allows one to write
\[
q_+(\xi) = -W_1(-e^{-1-\xi}), \quad q_-(\xi) = -W_0(-e^{-1-\xi}).
\]  
(4.2)

The connection between the expansion of \( q_\pm \) at 0 and the Lambert function was noticed in [1] and used as an illustration of “experimental mathematics”, as we later discovered.

**Proof.** We start with the formula
\[
z^{-c-1} = \int_{0}^{+\infty} e^{-z\xi} \frac{\xi^c}{\Gamma(c+1)} \, d\xi \quad \text{for } \Re c > -1 \text{ and } \Re z > 0
\]  
(4.3)

(the very basis of Borel-Laplace summation method!). Choosing \( c = z \), we get
\[
z^{-z} \Gamma(z) = z^{-z-1} \Gamma(z+1) = \int_{0}^{+\infty} e^{-zq} q^z \, dq = \int_{0}^{+\infty} e^{-z(q-\ln q)} \, dq.
\]

The map \( q \mapsto \zeta = q - \ln q \) induces a decreasing diffeomorphism \( (0,1) \to (1, +\infty) \) and an increasing diffeomorphism \( (1, +\infty) \to (1, +\infty) \). The inverse of the decreasing diffeomorphism is \( \zeta \mapsto q_-(\zeta - 1) \) and the inverse of the increasing one is \( \zeta \mapsto q_+(\zeta - 1) \), both have a square root singularity at \( \zeta = 1 \). Thus, splitting the integral in two parts and changing variable, we
get

\[ z^{-z}\Gamma(z) = -\int_{1}^{+\infty} e^{-z\zeta} q_{-}'(\zeta - 1) \, d\zeta + \int_{1}^{+\infty} e^{-z\zeta} q_{+}'(\zeta - 1) \, d\zeta \]

\[ = e^{-z} \int_{0}^{+\infty} e^{-z\zeta} (q_{+}'(\zeta) - q_{-}'(\zeta)) \, d\zeta. \]

We obtain (4.1) by multiplying by \( z e^{z} \) and integrating by parts. One gets (4.2) from \( q - \ln q = \zeta \Leftrightarrow q e^{-q} = e^{-\zeta} \Leftrightarrow -q = W_{-1}(-e^{-\zeta}) \text{ or } W_{0}(-e^{-\zeta}). \]

We thus have obtained \( \lambda_{3/2} = \mathcal{L}^{0}\hat{\lambda}_{3/2} \) with \( \hat{\lambda}_{3/2} \) as in (3.1). As already mentioned, the endless continuability and the location of the singularities of \( \hat{\lambda}_{3/2} \) are simple consequences of the complex analytic structure of \( W \). The existence of constants \( A, B > 0 \) such that \( |\hat{\lambda}_{3/2}(\zeta)| \leq A|\zeta| + B \) for \( \arg \zeta \in J_{1} \cup J_{2} \cup J_{3} \cup J_{4} \) allows us to vary the integration direction in the Laplace representation of \( \lambda_{3/2}(z) \) and get \( \lambda_{3/2} = \mathcal{L}^{(\frac{\pi}{2}, \frac{\pi}{2})}\hat{\lambda}_{3/2}. \)

Since \( q = 1 \) is a critical point of multiplicity 1 for the holomorphic function \( q \mapsto q - \log q - 1 = \xi \), the critical value \( \xi = 0 \) is a square root branch point for the two local inverse branches, \( q_{-}(\xi) \) and \( q_{+}(\xi) = q_{-}(e^{-2\pi i}\xi) \); these are two holomorphic functions of \( \xi^{1/2} \), one for each choice of the square root, which admit convergent Puiseux expansions in \( \xi^{1/2}/\mathbb{C}\{\xi^{1/2}\} \) (equivalently: \( W_{0} \) and \( W_{-1} \) have a square root branch point at \(-1/e\)). Following [9] and [1], we write the Puiseux expansion in terms of \( 2\zeta \) so as to get rational coefficients:

\[ q_{-}(\zeta) = -W_{0}(-\exp(-1 - \zeta)) = 1 + \sum_{k \geq 1} (-1)^{k} a_{k}(2\zeta)^{k/2}, \quad (4.4) \]

\[ q_{+}(\zeta) = -W_{-1}(-\exp(-1 - \zeta)) = -W_{0}(-\exp(-1 - e^{-2\pi i}\zeta)) = 1 + \sum_{k \geq 1} a_{k}(2\zeta)^{k/2} \quad (4.5) \]

with the sequence \( (a_{k})_{k \geq 1} \) of Part (iii) of Theorem 4 (the induction formula is obtained in [9] from the differential equation satisfied by \( q_{\pm} \) with respect to \((2\zeta)^{1/2}\)). The Puiseux expansion \((3.3b)\) for \( \hat{\lambda}_{3/2}(\xi) \) follows.

The relation \( q_{+}(\zeta) = q_{-}(e^{-2\pi i}\zeta) \) shows that \( \hat{\lambda}_{3/2} \) is the monodromy variation of

\[ \hat{\lambda}_{3/2}(\zeta) := -\frac{1}{\sqrt{2\pi}} q_{-}(\zeta) = \frac{1}{\sqrt{2\pi}} W_{0}(-\exp(-1 - \zeta)) \quad (4.6) \]

which certainly is \( o(1/|\xi|) \) in view of (4.4) and is thus a natural-major for \( \hat{\lambda}_{3/2} \). This concludes the proof of Part (i) of Theorem 1 as well as (3.4).

We will find a natural-major for \( \frac{1}{\lambda_{-3/2}(z)} \) starting with the formula

\[ \Re c > 0 \text{ and } \zeta > 0 \quad \Rightarrow \quad \frac{\zeta^{c}}{\Gamma(c + 1)} = \frac{1}{2\pi i} \int_{u-i\infty}^{u+i\infty} e^{\theta c} q^{-c-1} \, dq \quad \text{for any } u > 0 \quad (4.7) \]
(which can be viewed as a particular case of integral Borel transform, obtained by Fourier-Laplace inversion from (1.3)). Fix \( z > 1 \). Choosing \( \zeta = z \) and \( c = z - 1 \):

\[
\frac{z^{z-1}}{\Gamma(z)} = \frac{1}{2\pi i} \int_{u-i\infty}^{u+i\infty} e^{zq} q^{-z} \, dq = \frac{1}{2\pi i} \int_{u-i\infty}^{u+i\infty} e^{-z(-q+\log q)} \, dq \quad \text{for any} \quad u > 0. \tag{4.8}
\]

Let us fix \( u > 1 \) and perform the change of variable \( \zeta = -q + \log q \) along the integration contour \( \{q = u + it \mid t \in \mathbb{R}\} \). The new contour is \( \gamma_u := \{\zeta(t) = \zeta(t) + iv(t) \mid t \in \mathbb{R}\} \) with

\[
\zeta(t) := -u + \frac{1}{2} \ln(u^2 + t^2), \quad \zeta(t) := -t + \arctan \frac{u}{t}, \tag{4.9}
\]

We have \(-q e^{-q} = -e^\zeta\), hence \(-q = W(-e^\zeta)\) is a branch of the Lambert \( W \) function.

The even function \( t \mapsto \zeta_1(t) \) is decreasing on \((-\infty, 0]\), while the odd function \( t \mapsto \zeta_2(t) \) is decreasing; the contour \( \gamma_u \) comes from infinity in the first quadrant, reaches the real point \( \zeta(0) = -u + \ln u < -1 \), and goes to infinity in the fourth quadrant. Since \(-e^{\zeta(0)} > -e^{-1}\) and \(-u < -1\), we see that \(-u = W_{-1}(-e^{\zeta(0)})\), hence the inverse change of variable is given by \(-q = -(u + it) = W_{-1}(-e^{\zeta(t)})\), at least for \( t \) close to 0.

One sees that, by taking \( u \) close enough to 1, we can ensure that the contour \( \gamma_u \) crosses the line \( \{\Im \zeta = -1\} \) the first time between \(-1\) and \(-1 + 2\pi i\) and the second time between \(-1 - 2\pi i\) and \(-1\), hence the inverse change of variable \( q(\zeta) := W_{-1}(-e^\zeta) \) extends analytically to a neighbourhood of \( \gamma_u \), and (4.8) yields

\[
\frac{z^{z-1}}{\Gamma(z)} = \frac{1}{2\pi i} \int_{\gamma_u} e^{-z\zeta} q'(\zeta) \, d\zeta = \frac{z}{2\pi i} \int_{\gamma_u} e^{-z\zeta} q(\zeta) \, d\zeta = \frac{z}{2\pi i} \int_{\mathcal{H}_0} e^{-z\zeta} q(-1 + \xi) \, d\xi. \tag{4.10}
\]

In the last step, we have used the change of variable \( \xi = 1 + \zeta \) with the convention \( \arg(1 + \zeta(0)) = -\pi \), as well as the fact that the contour \( 1 + \gamma_u \) is homotopic to the Hankel contour \( \mathcal{H}_0 \) in \( \mathbb{C} - 2\pi i \mathbb{Z} \) relatively to \( \Re \xi \rightarrow +\infty \).

Viewed as a natural-major, \( \frac{1}{\sqrt{2\pi i}} q(-1 + \xi) \), which is none other that \( \tilde{\chi} \), is thus mapped by \( \mathcal{L}^0 \) to \( \frac{\sqrt{2\pi e^{-z-2} e^{-z}}}{\Gamma(z)} \), which is

\[
\frac{1}{\lambda_{-3/2}}.
\]

Finally, since \( \arg \xi = -\pi \Rightarrow \tilde{\chi}(\xi) = \frac{i}{\sqrt{2\pi}} W_{-1}(-e^{-1+\xi}) = -\frac{i}{\sqrt{2\pi}} q_{+}(e^{i\xi}), \) we can use the Puiseux expansion (1.5), \( q_{+}(e^{i\xi}) = 1 + \sum_{k \geq 1} i^k a_k(2\xi)^k/2 \in \mathbb{C}\{\xi^{1/2}\} \), which shows that \( \tilde{\chi} := \text{sing}_0(\tilde{\chi}(\xi)) \) is an integrable singularity with a minor

\[
\tilde{\chi}(\xi) = -\frac{i}{\sqrt{2\pi}} (q_{+}(e^{i\xi}) - q_{+}(e^{-i\xi})) = \frac{i}{\sqrt{2\pi}} (W_{-1}(-e^{-1+\xi}) - W_{0}(-e^{-1+\xi}))
\]

whose Puiseux expansion is (3.5b). The proof of Theorem 1 is complete.

Alien calculus and Stokes phenomenon

A point on the line-segment \((0, \omega)\) close to \( \omega \in \tilde{\mathbb{C}} \) can be written \( \omega + \xi \) with the convention \( \arg \xi = \arg \omega - \pi \). Given \( \tilde{\varphi} \), there are finitely many points \( \omega_1 < \omega_2 < \cdots < \omega_{r-1} \) on \((0, \omega)\) such that the analytic continuation of the minor \( \hat{\varphi} \) exists along every path \( \gamma_{\varepsilon}, \varepsilon \in \{+, -\}^{r-1} \),
obtained by following \((0, \omega)\) and circumventing each \(\omega_j\) to the right (resp. left) if \(\varepsilon_j = +\) (resp. 
\(-\)) without going backwards. Then Écalle defines two operators:

\[
\Delta^+_\omega \tilde{\varphi} := \text{sing}_0 \left( \text{cont}_{\gamma_+ \cdots} \hat{\varphi}(\omega + \xi) \right), \quad \Delta^-_\omega \tilde{\varphi} := \sum_{\varepsilon} \frac{p(\varepsilon) q(\varepsilon)}{r^j} \text{sing}_0 \left( \text{cont}_{\gamma_-} \hat{\varphi}(\omega + \xi) \right) \quad (4.11)
\]

with \(p(\varepsilon) := \text{number of} '+' \text{ in} \varepsilon\) and \(q(\varepsilon) := r - 1 - p(\varepsilon)\). The operators \(\Delta^\pm_\omega\) are derivations for \(\tilde{\varphi}\) and generate an \(\infty\)-dimensional free Lie algebra of derivations acting on the algebra of singularities, giving rise to interesting algebraic combinatorics questions \([4]\). The operators \(\Delta^\pm_\omega\) (which can be expressed in terms of \(\Delta_\omega'\), \(\omega' \in (0, \omega)\)) satisfy a modified Leibniz rule and have a natural interpretation in terms of Stokes phenomena \([5, 4, 6, 10]\), which we now illustrate on \(\hat{\lambda}_c\). The analytic continuation of \(W_0\) and \(W_{-1}\) is known well enough so that (3.11) allows one to compute directly, with \(\omega^\pm_m := 2\pi m e^{\pm \frac{\pi}{2}}\) for \(m \geq 1\),

\[
\Delta^\pm_{\omega^m} \hat{\lambda}_c = \pm \frac{1}{m} \hat{\lambda}_c, \quad \Delta^+_{\omega^m} \hat{\lambda}_c = \hat{\lambda}_c, \quad \Delta^+_{\omega^m} \hat{\lambda}_c = \left\{ \begin{array}{ll}
- \hat{\lambda}_c & \text{for } m = 1 \\
0 & \text{for } m \geq 2
\end{array} \right. \quad (4.12)
\]

with \(c = 3/2\) and thus also with arbitrary \(c\). It follows that

\[
-\pi < \arg z < 0 \Rightarrow \mathcal{L}^J_1 \hat{\lambda}_c(z) = \mathcal{L}^J_2 \hat{\lambda}_c(z) = \sum_{m \geq 1} e^{-\omega^+_m z} \Delta^+_m \hat{\lambda}_c = \frac{1}{1 - e^{-2\pi z}} \mathcal{L}^J_2 \hat{\lambda}_c(z)
\]

\[
0 < \arg z < \pi \Rightarrow \mathcal{L}^J_4 \hat{\lambda}_c(z) = \mathcal{L}^J_1 \hat{\lambda}_c(z) = \sum_{m \geq 1} e^{-\omega^-_m z} \Delta^+_m \hat{\lambda}_c = (1 - e^{2\pi z}) \mathcal{L}^J_1 \hat{\lambda}_c(z)
\]

(which can be viewed as the solution to a Riemann-Hilbert problem—the one induced by the wall-crossing formula in Donaldson-Thomas “Doubled \(A_1\)” theory if \(c = 0\)). Recall that \(\mathcal{L}^J_1 \hat{\lambda}_{3/2} = \lambda_{3/2}\). Since \(J_1 = -\pi + J_2 = \pi + J_4\), (3.6) implies \(\mathcal{L}^J_2 \hat{\lambda}_{3/2}(z) = -i \mathcal{L}^J_1 \hat{\lambda}(e^i \pi z) = -i/\lambda_{-3/2}(e^i \pi z)\) and \(\mathcal{L}^J_4 \hat{\lambda}_{3/2}(z) = i \mathcal{L}^J_1 \hat{\lambda}(e^{-i \pi z}) = i/\lambda_{-3/2}(e^{-i \pi z})\), hence both Stokes phenomena amount to the reflection formula \(\Gamma(z) \Gamma(1 - z) = \frac{\pi}{2 \sin(\pi z)}\).

5. PROOF OF THEOREM \([2]\)

Consider \(\lambda(z)\) for \(z \in \mathbb{R}_{>0}\). We know that \(\lambda(z)\) is bounded on \([u, +\infty)\) for any \(u > 0\) hence, for any \(c \in \mathbb{C}\), the real-major for \(\lambda_c(z) = z^{-c} \lambda(z)\) that is defined by (3.10) is holomorphic for \(\Re \xi > 0\) and we get its analytic continuation to a sector of opening \(3\pi\) in the Riemann surface of the logarithm \(\hat{\mathbb{C}}\) by varying the integration half-line.

Let us do it under the assumption \(\Re c < \frac{1}{2}\). Since \(\lambda_c(z) \sim z^{-c + \frac{1}{2}}\) is then integrable at \(z = 0\), \(u = 0\) yields a real-major \(\hat{\lambda}_c(\xi) = \mathcal{L}^0 \lambda_c(\xi)\). Moreover,

\[
\lambda_c(z) \sim z^{-c} \quad \text{and} \quad \lambda_c(z) \sim \frac{1}{\sqrt{2\pi}} z^{-c - \frac{1}{2}} \quad \text{uniformly for} \ |\arg z| < \pi - \varepsilon \quad (5.1)
\]

\[\text{A proof can also be derived from the relation} \hat{\lambda} = \exp \hat{\mu} \text{ as in \([10]\)—beware of the typo in \([10], (6.99)\).} \]
for any $0 < \varepsilon < \pi$, hence we can vary the integration ray in the above Laplace transform and obtain the analytic continuation of $\tilde{\lambda}_c(\xi)$ to a sector of opening $3\pi$ in $\tilde{\mathbb{C}}$ in the form

$$\tilde{\lambda}_c(\xi) = \mathcal{L}^{J_1} \lambda_c(\xi) \quad \text{for} \quad -\frac{3\pi}{2} < \arg \xi < \frac{3\pi}{2}, \quad \text{with} \quad J_1 := (-\frac{\pi}{2}, \frac{\pi}{2}). \quad (5.2)$$

Moreover, in view of (5.1), this representation of $\tilde{\lambda}_c(\xi)$ shows that it is bounded in the domain $\{|\xi| > \tau\}$ and $|\arg \xi| < \frac{3\pi}{2} - \varepsilon$ for every $\tau, \varepsilon > 0$. Therefore Fourier-Laplace inversion yields $\lambda_c(z) = \tilde{\mathcal{L}}^{J_1} \lambda_c(z)$ for all $z \in \mathbb{C} - \mathbb{R}_{\leq 0}$ and $\theta \in J_1$ such that $\theta + \arg z \in J_1$.

We now prove Formula (3.11). For each $z > 0$,

$$\Gamma(z) = \int_0^{+\infty} e^{-x^2} \frac{dx}{x} = z^z \int_\mathbb{R} e^{-2e^Q} e^{zQ} dQ$$

(change of variable $x = z e^Q$), hence $\lambda_c(z) = \frac{1}{\sqrt{2\pi}} z^{-c+\frac{1}{2}} \int_\mathbb{R} e^{-z(e^Q - Q - 1)} dQ$ and

$$\tilde{\lambda}_c(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} dz \int_\mathbb{R} dz z^{-c+\frac{1}{2}} e^{-z(\xi + e^Q - Q - 1)}.$$

Now $\Re(\xi + e^Q - Q - 1) > 0$ by (3.12) and $\Re(\frac{3}{2} - c) > 0$, thus

$$\int_0^{+\infty} z^{-c+\frac{1}{2}} e^{-z(\xi + e^Q - Q - 1)} dz = \Gamma(\frac{3}{2} - c)(\xi + e^Q - Q - 1)^{-\frac{3}{2}}$$

by (4.3), and the conclusion follows.

To conclude the proof of Theorem 2, we observe that, to follow the analytic continuation of $\tilde{\lambda}_c(\xi)$ along a path $\Gamma$ of $\tilde{\mathbb{C}}$ that starts on $\{|\arg \xi| = 0\}$, it is sufficient to deform continuously the integration path in (3.11) so that the integration variable $Q$ avoids the zeroes of the expression $\xi + e^Q - Q - 1$. Such a continuous deformation exists as long as $0$ is not a critical value of $Q \mapsto \xi + e^Q - Q - 1$, i.e. as long as the path $\Gamma$ avoids any critical value of $Q \mapsto -e^Q + Q + 1$. These critical values are precisely the points of $2\pi i \mathbb{Z}$.

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References

[1] Jonathan M. Borwein and Robert M. Corless. “Emerging tools for experimental mathematics”. In: Amer. Math. Monthly 106.10 (1999), pp. 889–909.
[2] B. Candelpergher, J.-C. Nosmas, and F. Pham. Approche de la résurgence. Actualités Mathématiques. Hermann, Paris, 1993, pp. ii+290.
[3] R. M. Corless et al. “On the Lambert W function”. In: Adv. Comput. Math. 5.4 (1996), pp. 329–359.
[4] Jean Écalle. Les fonctions résurgentes. Tome III. Vol. 85-5. Publications Mathématiques d’Orsay. Université de Paris-Sud, Département de Mathématiques, Orsay, 1985, p. 587.
[5] Jean Écalle. Les fonctions résurgentes. Tomes I, II. Vol. 81-5 & 81-6. Publications Mathématiques d’Orsay. Université de Paris-Sud, Orsay, 1981.
[6] Jean Écalle. “Six lectures on transseries, analysable functions and the constructive proof of Dulac’s conjecture”. In: Bifurcations and periodic orbits of vector fields (Montreal, PQ, 1992). Vol. 408. NATO Adv. Sci. Inst. Ser. C. Kluwer Acad. Publ., Dordrecht, 1993, pp. 75–184.
[7] Shingo Kamimoto and David Sauzin. “Iterated convolutions and endless Riemann surfaces”. In: Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 20.1 (2020), pp. 177–215.
[8] Maxim Kontsevich. Private communication. May 2017.
[9] George Marsaglia and John C. W. Marsaglia. “A new derivation of Stirling’s approximation to n!”. In: Amer. Math. Monthly 97.9 (1990), pp. 826–829.
[10] Claude Mitschi and David Sauzin. Divergent series, summability and resurgence. I. Vol. 2153. Lecture Notes in Mathematics. Springer, [Cham], 2016, pp. xxi+298.
[11] David Sauzin. “Nonlinear analysis with resurgent functions”. In: Ann. Sci. Éc. Norm. Supér. (4) 48.3 (2015), pp. 667–702.
[12] David Sauzin. “Resurgent functions and splitting problems”. In: RIMS Kokyuroku 1493 (2006), pp. 48–117.