Abstract. The multiple-input multiple-output (MIMO) detection problem, a fundamental problem in modern digital communications, is to detect a vector of transmitted symbols from the noisy outputs of a fading MIMO channel. The maximum likelihood detector can be formulated as a complex least-squares problem with discrete variables, which is NP-hard in general. Various semidefinite relaxation (SDR) methods have been proposed in the literature to solve the problem due to their polynomial-time worst-case complexity and good detection error rate performance. In this paper, we consider two popular classes of SDR-based detectors and study the conditions under which the SDRs are tight and the relationship between different SDR models. For the enhanced complex and real SDRs proposed recently by Lu et al., we refine their analysis and derive the necessary and sufficient condition for the complex SDR to be tight, as well as a necessary condition for the real SDR to be tight. In contrast, we also show that another SDR proposed by Mobasher et al. is not tight with high probability under mild conditions. Moreover, we establish a general theorem that shows the equivalence between two subsets of positive semidefinite matrices in different dimensions by exploiting a special “separable” structure in the constraints. Our theorem recovers two existing equivalence results of SDRs defined in different settings and has the potential to find other applications due to its generality.

Key words. MIMO detection, semidefinite relaxation, tight relaxation, equivalent relaxation

AMS subject classifications. 90C22, 90C20, 90C46, 90C27

1. Introduction. Multiple-input multiple-output (MIMO) detection is a fundamental problem in modern digital communications [33, 36]. The MIMO channel can be modeled as

\[ r = Hx^* + v, \]

where \( r \in \mathbb{C}^m \) is the vector of received signals, \( H \in \mathbb{C}^{m \times n} \) is a complex channel matrix, \( x^* \) is the vector of transmitted symbols, and \( v \) is the vector of additive Gaussian noises. Moreover, each entry of \( x^* \) is drawn from a discrete symbol set \( S \) determined by the modulation scheme.

The MIMO detection problem is to recover the transmitted symbol vector \( x^* \) from the noisy channel output \( r \), with the information of the symbol set \( S \) and the channel matrix \( H \). Under the assumption that each entry of \( x^* \) is drawn uniformly and independently from the symbol set \( S \), it is known that the maximum likelihood detector can achieve the optimal detection error rate performance. Mathematically, it can be formulated as a discrete least-squares problem:

\[
\begin{align*}
\min_{x \in \mathbb{C}^n} & \quad \|Hx - r\|_2^2 \\
\text{s.t.} & \quad x_i \in S, \ i = 1, 2, \ldots, n,
\end{align*}
\]
where $x_i$ denotes the $i$-th entry of the vector $x$ and $\| \cdot \|_2$ denotes the Euclidean norm. In this paper, unless otherwise specified, we will focus on the $M$-ary phase shift keying ($M$-PSK) modulation, whose symbol set is given by

$$ S_M := \{ z \in \mathbb{C} : |z| = 1, \, \arg(z) \in \{2j\pi/M, \, j = 0, 1, \ldots, M-1\} \}, $$

where $|z|$ and $\arg(z)$ denote the modulus and argument of a complex number, respectively. As in most practical digital communication systems, throughout the paper we require $M = 2^b$ where $b \geq 1$ is an integer$^1$.

Many detection algorithms have been proposed to solve problem (1.2) either exactly or approximately. However, for general $H$ and $r$, problem (1.2) has been proved to be NP-hard [32]. Hence, no polynomial-time algorithms can find the exact solution (unless $P = NP$). Sphere decoding [4], a classical combinatorial algorithm based on the branch-and-bound paradigm, offers an efficient way to solve problem (1.2) exactly when the problem size is small, but its expected complexity is still exponential [11]. On the other hand, some suboptimal algorithms such as linear detectors [23, 6] and decision-feedback detectors [35, 5] enjoy low complexity but at the expense of substantial performance loss: see [36] for an excellent review.

Over the past two decades, semidefinite relaxation (SDR) has gained increasing attention in non-convex optimization [7, 18, 34]. It is a celebrated technique to tackle quadratic optimization problems arising from various signal processing and wireless communication applications, such as beamforming design [24, 15], sensor network localization [3, 2, 27], and angular synchronization [25, 1, 37]. Such SDR-based approaches can usually offer superior performance in both theory and practice while maintaining polynomial-time worst-case complexity.

For MIMO detection problem (1.2), the first SDR detector [30, 20] was designed for the real MIMO channel and the binary symbol set $S = \{+1, -1\}$. Notably, it is proved that this detector can achieve the maximal possible diversity order [12], meaning that it achieves an asymptotically optimal detection error rate when the signal-to-noise ratio (SNR) is high. It was later extended to the more general setting with a complex channel and an $M$-PSK symbol set in [28, 19], which we refer to as the conventional SDR or (CSDR). However, this conventional approach fails to fully utilize the structure in the symbol set $S$. To overcome this issue, researchers have developed various improved SDRs and we consider the two most popular classes below. The first class proposed in [22] is based on an equivalent zero-one integer programming formulation of problem (1.2). Four SDR models were introduced and two of them will be discussed in details later (see (ESDR1-T) and (ESDR2-T) further ahead). The second class proposed in [17] further enhances (CSDR) by adding valid cuts, resulting in a complex SDR and a real SDR (see (ESDR-X) and (ESDR-Y) later on).

In this paper, we focus on two key problems in SDR-based MIMO detection: the tightness of SDRs and the relationship between different SDR models. Firstly, note that SDR detectors are suboptimal algorithms as they replace the original discrete optimization problem (1.2) with tractable semidefinite programs (SDPs). Hence, after solving an SDP, we need some rounding procedure to make final symbol decisions. However, under some favorable conditions on $H$ and $v$, an SDR can be tight, i.e., it has an optimal rank-one solution corresponding to the true vector of transmitted symbols. Such tightness conditions are of great interest since they give theoretical guarantees on the optimality of SDR detectors. While it has been well studied for the

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$^1$Our results in section 3 also hold for the more general case where $M$ is a multiple of four.
simple case \([9, 10, 14, 26]\) where \(H \in \mathbb{R}^{m \times n}, \ v \in \mathbb{R}^m, \) and \(S = \{+1, -1\}, \) for the more general case where \(H \in \mathbb{C}^{m \times n}, \ v \in \mathbb{C}^m, \) and \(S = \mathcal{S}_M (M \geq 4), \) tightness conditions for SDR detectors have remained unknown until very recently. The authors in [17] showed that (CSDR) is not tight with probability one under some mild conditions. On the other hand, their proposed enhanced SDRs are tight\(^2\) if the following condition is satisfied:

\[
\lambda_{\text{min}}(H^\dagger H) \sin \left(\frac{\pi}{M}\right) > \|H^\dagger v\|_\infty,
\]

where \(\lambda_{\text{min}}(\cdot)\) denotes the smallest eigenvalue of a matrix, \((\cdot)^\dagger\) denotes the conjugate transpose, and \(\|\cdot\|_\infty\) denotes the \(L_\infty\)-norm. To the best of our knowledge, this is the best condition that guarantees a certain SDR to be tight for problem (1.2) in the \(M\)-PSK settings.

Secondly, researchers have noticed some rather unexpected equivalence between different SDR models independently developed in the literature. The earliest one of such results is reported in [21], where three different SDRs for the high-order quadrature amplitude modulation (QAM) symbol sets are proved to be equivalent. Very recently, the authors in [16] showed that the enhanced real SDR proposed in [17] is equivalent to one SDR model in [22]. It is worth noting that while these two papers are of the same nature, the proof techniques are quite different and it is unclear how to generalize their results at present.

In this paper, we make contributions to both problems. For the tightness of SDRs, we sharpen the analysis in [17] to give the necessary and sufficient condition for the complex enhanced SDR to be tight, and a necessary condition for the real enhanced SDR to be tight. Specifically, for the case where \(M \geq 4\), we show that the enhanced complex SDR (ESDR-X) is tight if and only if

\[
H^\dagger H + \text{Diag}(\text{Re}(\text{Diag}((x^*)^{-1}H^\dagger v))) - \cot \left(\frac{\pi}{M}\right) \text{Diag}(\text{Im}(\text{Diag}((x^*)^{-1}H^\dagger v))) \succeq 0,
\]

while the enhanced real SDR (ESDR-Y) is tight only if

\[
H^\dagger H + \text{Diag}(\text{Re}(\text{Diag}((x^*)^{-1}H^\dagger v))) - \cot \left(\frac{2\pi}{M}\right) \text{Diag}(\|\text{Im}(\text{Diag}((x^*)^{-1}H^\dagger v)))\| \succeq 0,
\]

where \(A \succeq 0\) means that the matrix \(A\) is positive semidefinite (PSD), \(\text{Diag}(x)\) denotes a diagonal matrix whose diagonals are the vector \(x, \) and \(\text{Re}(\cdot), \) \(\text{Im}(\cdot),\) and \(\|\cdot\|\) denote the entrywise real part, imaginary part, and absolute value of a number/vector/matrix, respectively. Moreover, we prove that one of the SDR models proposed in [22] is generally not tight: under some mild assumptions, its probability of being tight decays exponentially with respect to the number of transmitted symbols \(n.\)

For the relationship between different SDR models, we propose a general theorem showing the equivalence between two subsets of PSD cones. Specifically, we prove the correspondence between a subset of a high-dimensional PSD cone with a special “separable” structure and the one in a lower dimension. Our theorem covers both equivalence results in [21] and [16] as special cases, and has the potential to find other applications due to its generality.

The paper is organized as follows. We introduce the existing SDRs for (1.2) in section 2 and analyze their tightness in section 3. In section 4, we propose a general
theorem that establishes the equivalence between two subsets of PSD cones in different dimensions, and discuss how our theorem implies previous results. Section 5 provides some numerical results to validate our analysis. Finally, section 6 concludes the paper.

We summarize some standard notations used in this paper. We use \(x_i\) to denote the \(i\)-th entry of a vector \(x\) and \(X_{i,j}\) to denote the \((i,j)\)-th entry of a matrix \(X\). We use \(|\cdot|\), \(\|\cdot\|_2\), and \(\|\cdot\|_\infty\) to denote the entrywise absolute value, the Euclidean norm, and the \(L_\infty\) norm of a vector, respectively. For a given number/vector/matrix, we use \(\cdot^\dagger\) to denote the conjugate transpose, \(\cdot^\top\) to denote the transpose, and \(\text{Re}(\cdot)/\text{Im}(\cdot)\) to denote the entrywise real/imaginary part. We use \(\text{Diag}(x)\) to denote the diagonal matrix whose diagonals are the vector \(x\), and \(\text{diag}(X)\) to denote the vector whose entries are the diagonals of the matrix \(X\). Given an \(m \times n\) matrix \(A\) and the index sets \(\alpha \subset \{1,2,\ldots,m\}\) and \(\beta \subset \{1,2,\ldots,n\}\), we use \(A[\alpha,\beta]\) to denote the submatrix with entries in the rows of \(A\) indexed by \(\alpha\) and the columns indexed by \(\beta\). Moreover, we denote the principal submatrix \(A[\alpha,\alpha]\) by \(A[\alpha]\) in short. For two matrices \(A\) and \(B\) of appropriate size, \(\langle A, B \rangle := \text{Re}(\text{Tr}(A^\dagger B))\) denotes the inner product, \(A \otimes B\) denotes the Kronecker product, and \(A \succeq B\) means \(A - B\) is PSD. For a set \(A\) in a vector space, we use \(\text{conv}(A)\) to denote its convex hull. For a random variable \(X\) and measurable sets \(B\) and \(C\), \(\text{Prob}(X \in B|C)\) denotes the conditional probability given \(C\), and \(\mathbb{E}[X]\) denotes the expectation of \(X\). Finally, the symbols \(i\), \(1_n\), \(I_n\), and \(S^n_+\) represent the imaginary unit, the \(n\times1\) all-one vector, the \(n\times n\) identity matrix, and the \(n\)-dimensional PSD cone, respectively.

2. Review of semidefinite relaxations. In this paper, we focus on the \(M\)-PSK setting with the symbol set \(S_M\) given in (1.3). To simplify the notations, we let \(s \in \mathbb{C}^M\) be the vector of all symbols, where
\[
s_j = e^{i\theta_j} \quad \text{and} \quad \theta_j = \frac{(j - 1)2\pi}{M}, \quad j = 1, 2, \ldots, M,
\]
and further we let \(s_R = \text{Re}(s)\) and \(s_I = \text{Im}(s)\).

The objective in (1.2) can be written as
\[
\|Hx - r\|_2^2 = x^\dagger Qx + 2\text{Re}(c^\dagger x) + r^\dagger r = \langle Q, xx^\dagger \rangle + 2\text{Re}(c^\dagger x) + r^\dagger r,
\]
where we define
\[
Q = H^\dagger H \quad \text{and} \quad c = -H^\dagger r.
\]
By introducing \(X = xx^\dagger\) and discarding the constant \(r^\dagger r\), we can reformulate (1.2) as
\[
\min_{x,X} \langle Q, X \rangle + 2\text{Re}(c^\dagger x)
\]
\[
s.t. \quad X_{i,i} = 1, \quad i = 1, 2, \ldots, n,
\]
\[
x_i \in S_M, \quad i = 1, 2, \ldots, n,
\]
\[
X = xx^\dagger,
\]
where the constraint \(X_{i,i} = 1\) comes from \(X_{i,i} = |x_i|^2 = 1\). The conventional SDR (CSDR) in [28, 19] simply drops the discrete symbol constraints \(x \in S_M\) and relaxes
the rank-one constraint to $X \succeq xx^\dagger$, resulting in the following relaxation:

$$\min_{x, X} \langle Q, X \rangle + 2\text{Re}(c^\dagger x)$$

subject to $X_{i,i} = 1, \ i = 1, 2, \ldots, n,$

$$X \succeq xx^\dagger,$$

where $x \in \mathbb{C}^n$ and $X \in \mathbb{C}^{n \times n}$. Since $X \succeq xx^\dagger$ is equivalent to $\begin{bmatrix} 1 & x^\dagger \\ x & X \end{bmatrix} \succeq 0,$

the above (CSDR) is an SDP on the complex domain. Moreover, for the simple case where $H \in \mathbb{R}^{m \times n}, v \in \mathbb{R}^m,$ and $M = 2$, a real SDR similar to (CSDR) has the form:

$$\min_{x, X} \langle Q, X \rangle + 2c^T x$$

subject to $X_{i,i} = 1, \ i = 1, 2, \ldots, n,$

$$X \succeq xx^\top,$$

where $x \in \mathbb{R}^n, X \in \mathbb{R}^{n \times n},$ and we redefine $Q = H^T H$ and $c = -H^T r$ (cf. (2.1)). The problem (2.3) has also been extensively studied in the literature [30, 20, 9, 10, 14, 26]. It is proved in [9, 10] that (2.3) is tight if and only if

$$H^T H + [\text{Diag}(x^* )]^{-1} \text{Diag}(H^T v) \succeq 0,$$

while (CSDR) is not tight for $M \geq 4$ with probability one under some mild conditions [17].

Recently, a class of enhanced SDRs was proposed in [17]. Instead of simply dropping the constraints $x_i \in S_M$ as in (CSDR), the authors replaced the discrete symbol set $S_M$ by its convex hull to get a continuous relaxation:

$$\min_{t, x, X} \langle Q, X \rangle + 2\text{Re}(c^\dagger x)$$

subject to $X_{i,i} = 1, \ i = 1, 2, \ldots, n,$

$$x_i = \sum_{j=1}^M t_{i,j} s_j, \ \sum_{j=1}^M t_{i,j} = 1, \ i = 1, 2, \ldots, n,$$

$$t_{i,j} \geq 0, \ j = 1, 2, \ldots, M, \ i = 1, 2, \ldots, n,$$

$$X \succeq xx^\dagger,$$

where $x \in \mathbb{C}^n, X \in \mathbb{C}^{n \times n},$ and $t \in \mathbb{R}^{Mn}$ is the concatenation of $M$-dimensional vectors $t_1, t_2, \ldots, t_n$ with $t_i = [t_{i,1}, t_{i,2}, \ldots, t_{i,M}]^T$. The authors in [17] further proved that (ESDR-X) is tight if condition (1.4) holds. We term the above SDP as “ESDR-X”, where “E” stands for “enhanced” and “X” refers to the matrix variable. The same naming convention is adopted for all the SDRs below.

We can also formulate (2.2) in the real domain and then use the same technique to get a real counterpart of (ESDR-X). Let

$$y = \begin{bmatrix} \text{Re}(x) \\ \text{Im}(x) \end{bmatrix}, \ \hat{Q} = \begin{bmatrix} \text{Re}(Q) & -\text{Im}(Q) \\ \text{Im}(Q) & \text{Re}(Q) \end{bmatrix}, \ \text{and} \ \hat{c} = \begin{bmatrix} \text{Re}(c) \\ \text{Im}(c) \end{bmatrix},$$

...
then the real enhanced SDR (ESDR-Y) is given by

\[
\min_{t_i, y_i} \langle \bar{Q}, Y \rangle + 2 \bar{c}^T y
\]

(ESDR-Y)

\[
\text{s.t.} \quad \mathcal{Y}(i) = \sum_{j=1}^{M} t_{i,j} K_j, \quad \sum_{j=1}^{M} t_{i,j} = 1, \quad i = 1, 2, \ldots, n,
\]

\[
t_{i,j} \geq 0, \quad j = 1, 2, \ldots, M, \quad i = 1, 2, \ldots, n,
\]

\[
Y \succeq y y^T,
\]

where \( t \in \mathbb{R}^{Mn} \), \( y \in \mathbb{R}^{2n} \), \( Y \in \mathbb{R}^{2n \times 2n} \), and we define

\[
\mathcal{Y}(i) := \begin{bmatrix} 1 & y_i & y_{n+i} \\ y_i & Y_{i,i} & Y_{i,n+i} \\ y_{n+i} & Y_{n+i,i} & Y_{n+i,n+i} \end{bmatrix}, \quad i = 1, 2, \ldots, n.
\]

In (ESDR-Y), these 3 × 3 matrices are constrained in a convex hull whose extreme points are

\[
K_j = \begin{bmatrix} 1 & s_{R,j} & s_{I,j} \\ s_{R,j} & 1 & s_{R,j} \\ s_{I,j} & s_{R,j} & 1 \end{bmatrix}^T, \quad j = 1, 2, \ldots, M,
\]

where \( s_{R,j} = \text{Re}(s_j) \) and \( s_{I,j} = \text{Im}(s_j) \). It has been shown that (ESDR-Y) is tighter than (ESDR-X) [17, Theorem 4.1], and hence (ESDR-Y) is tight whenever (ESDR-X) is tight.

Now we turn to another class of SDRs developed from a different perspective in [22], which is applicable to a general symbol set. The idea is to introduce binary variables to express \( x_i \in S_M \) by

\[
x_i = t_i^T s, \quad i = 1, 2, \ldots, n,
\]

where \( t_i = [t_{i,1}, t_{i,2}, \ldots, t_{i,M}]^T \), \( \sum_{j=1}^{M} t_{i,j} = 1 \), and \( t_{i,j} \in \{0, 1\} \). The above constraints (2.7) can be rewritten in a compact form as \( x = St \), where \( S = \mathbf{I}_n \otimes s^T \) and we concatenate all vectors \( t \) to get \( t = [t_1^T, \ldots, t_n^T]^T \in \mathbb{R}^{Mn} \). Similarly, we can also formulate (2.7) in the real domain as \( y = S \hat{t} \), where

\[
y = \begin{bmatrix} \text{Re}(x) \\ \text{Im}(x) \end{bmatrix} \quad \text{and} \quad \hat{S} = \begin{bmatrix} \text{Re}(S) \\ \text{Im}(S) \end{bmatrix} = \begin{bmatrix} \mathbf{I}_n \otimes s_{R} \\ \mathbf{I}_n \otimes s_{I} \end{bmatrix}.
\]

By introducing \( T = \hat{t} \hat{t}^T \in \mathbb{R}^{Mn \times Mn} \), the problem (1.2) is equivalent to

\[
\min_{T,i} \langle \bar{Q}, T \rangle + 2 \bar{c}^T t
\]

(2.9)

\[
\text{s.t.} \quad \sum_{j=1}^{M} t_{i,j} = 1, \quad i = 1, 2, \ldots, n,
\]

\[
t_{i,j} \in \{0, 1\}, \quad j = 1, 2, \ldots, M, \quad i = 1, 2, \ldots, n,
\]

\[
T = \hat{t} \hat{t}^T,
\]

where

\[
\bar{Q} = \hat{S}^T \bar{Q} \hat{S} \quad \text{and} \quad \bar{c} = \hat{S}^T \hat{c}.
\]
To derive an SDR for (2.9), we first allow \( t_{i,j} \) to take any value between 0 and 1. For the rank-one constraint \( T = tt^T \), the authors in [22] proposed four ways of relaxation and we will introduce two of them in the following\(^3\). We first partition \( T \) as an \( n \times n \) block matrix

\[
T = \begin{bmatrix}
T_{1,1} & T_{1,2} & \cdots & T_{1,n} \\
T_{2,1} & T_{2,2} & \cdots & T_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
T_{n,1} & T_{n,2} & \cdots & T_{n,n}
\end{bmatrix},
\]

where \( T_{i,j} \in \mathbb{R}^{M \times M} \) for \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, n \). In the first model, we relax \( T = tt^T \) to \( T \succeq tt^T \) and impose constraints on the diagonal elements:

\[
\begin{aligned}
& \min_{t,T} \langle Q, T \rangle + 2c^T t \\
& \text{s.t. } t_{i,j} \geq 0, \sum_{j=1}^M t_{i,j} = 1, j = 1, 2, \ldots, M, \ i = 1, 2, \ldots, n, \\
& \text{diag}(T_{i,i}) = t_i, \ i = 1, 2, \ldots, n, \\
& T \succeq tt^T,
\end{aligned}
\]

(ESDR1-T)

where \( t \in \mathbb{R}^{Mn} \) and \( T \in \mathbb{R}^{Mn \times Mn} \). The second model further requires \( T_{i,i} \) to be a diagonal matrix, leading to the following SDR:

\[
\begin{aligned}
& \min_{t,T} \langle Q, T \rangle + 2c^T t \\
& \text{s.t. } t_{i,j} \geq 0, \sum_{j=1}^M t_{i,j} = 1, j = 1, 2, \ldots, M, \ i = 1, 2, \ldots, n, \\
& T_{i,i} = \text{Diag}(t_i), \ i = 1, 2, \ldots, n, \\
& T \succeq tt^T.
\end{aligned}
\]

(ESDR2-T)

Since (ESDR2-T) puts more constraints on the variables \( t \) and \( T \), (ESDR2-T) is tighter than (ESDR1-T). Notably, it is shown in [16] that (ESDR2-T) is equivalent to (ESDR-Y), and hence (1.4) is also a sufficient condition for (ESDR2-T) to be tight.

Table 1 summarizes all SDR models discussed in this paper, where we highlight our contributions on the tightness of different SDRs in bold.

3. Tightness of semidefinite relaxations.

3.1. Tightness of (ESDR-X). Let \( X^* = x^*(x^*)^\dagger \), and the key idea of showing the tightness of (ESDR-X) is to certify \((x^*, X^*)\) as the optimal solution by considering the Karush-Kuhn-Tucker (KKT) conditions of (ESDR-X). Our derivation is based on [17, Theorem 4.2] and we provide a simplified version for completeness.

**Theorem 3.1** ([17, Theorem 4.2]). Suppose that \( M \geq 4 \). Then \((x^*, X^*)\) is the optimal solution of (ESDR-X) if and only if there exist

\[
\lambda_i \in \mathbb{R}, \ \mu_{i,-1} \geq 0, \text{ and } \mu_{i,1} \geq 0, \ i = 1, 2, \ldots, n,
\]

\(^3\)Our formulations are slightly different from the original ones in [22] since they used the equality constraints to eliminate one variable for each \( t_i \) before relaxing the PSD constraint. However, in numerical tests we found that this variation only causes a negligible difference in the optimal solutions of the SDRs.
Table 1
Summary of SDR models in this paper.

| SDR model | Origin | Domain | Dimension of PSD cone | Comments |
|-----------|--------|--------|-----------------------|----------|
| CSDR      | Ma et al. [19] | $\mathbb{C}$ | $n + 1$ | tight with probability 0 [17] |
| ESDR-X    | CSDP2 in Lu et al. [17] | $\mathbb{C}$ | $n + 1$ | tight if and only if (1.5) holds |
| ESDR-Y    | ERSDP in Lu et al. [17] | $\mathbb{R}$ | $2n + 1$ | tight only if (1.6) holds |
| ESDR1-T   | Model II in Mobasher et al. [22] | $\mathbb{R}$ | $Mn + 1$ | tight with probability no greater than $(2/M)^n$ |
| ESDR2-T   | Model III in Mobasher et al. [22] | $\mathbb{R}$ | $Mn + 1$ | equivalent to ESDR-Y [16] |

such that $\mathbf{H}$ and $\mathbf{v}$ in (1.1) satisfy

$$(x_i^*)^{-1}(\mathbf{H}^\dagger \mathbf{v})_i = \lambda_i + \frac{\mu_{i-1}}{2} e^{-i\frac{\pi}{M}} + \frac{\mu_i}{2} e^{i\frac{\pi}{M}}, \quad i = 1, 2, \ldots, n,$$

and $\mathbf{Q} + \text{Diag}(\lambda) \succeq 0$.

The authors in [17] further derived the sufficient condition (1.4), under which the conditions in Theorem 3.1 are met by choosing $\lambda_i = -\lambda_{\min}(\mathbf{Q})$ for $i = 1, 2, \ldots, n$. To strengthen their analysis, we view the conditions in Theorem 3.1 as a semidefinite feasibility problem. To be specific, if we define

$$(3.1) \quad z_i = (x_i^*)^{-1}(\mathbf{H}^\dagger \mathbf{v})_i, \quad i = 1, 2, \ldots, n,$$

and

$$\mathcal{C}(\lambda) = \left\{ z \in \mathbb{C} : \exists \mu_{-1}, \mu_1 \geq 0 \text{ s.t. } z = \lambda + \frac{\mu_{i-1}}{2} e^{-i\frac{\pi}{M}} + \frac{\mu_i}{2} e^{i\frac{\pi}{M}} \right\},$$

then Theorem 3.1 states that (ESDR-X) is tight if and only if the following problem is feasible:

$$\text{find } \lambda \in \mathbb{R}^n$$
$$\text{s.t. } \mathbf{Q} + \text{Diag}(\lambda) \succeq 0,$$
$$z_i \in \mathcal{C}(\lambda_i), \quad i = 1, 2, \ldots, n.$$

Each constraint $z_i \in \mathcal{C}(\lambda_i)$ turns out to be a simple inequality on $\lambda_i$. To see this, we plot $\mathcal{C}(\lambda_i)$ as the shaded area in Figure 1. It is clear from the figure that

$$z_i \in \mathcal{C}(\lambda_i) \iff |\text{Im}(z_i)| \leq \left( -\lambda_i + \text{Re}(z_i) \right) \tan \left( \frac{\pi}{M} \right),$$

which leads to

$$z_i \in \mathcal{C}(\lambda_i) \iff \lambda_i \leq \text{Re}(z_i) - |\text{Im}(z_i)| \cot \left( \frac{\pi}{M} \right).$$

This, together with (3.2), gives the necessary and sufficient condition for (ESDR-X) to be tight and we formally state it in Theorem 3.2.
Suppose that $M \geq 4$. Then (ESDR-X) is tight if and only if

$$Q + \text{Diag}(\text{Re}(z)) - \cot\left(\frac{\pi}{M}\right) \text{Diag}(|\text{Im}(z)|) \succeq 0,$$

where $z = [z_1, z_2, \ldots, z_n]^T \in \mathbb{C}^n$.

Note that (3.3) is exactly the same as (1.5) if we recall the definitions of $Q$ in (2.1) and $z_i$ in (3.1). Furthermore, if we set $M = 2$ and $H, v$ to be real in (1.5), it becomes the same as the previous result (2.4). Hence, our result extends (2.4) to the more general case where $M \geq 4$ and $H, v$ are complex. Finally, the sufficient condition (1.4) in [17] can be derived from our result. Since

$$\text{Re}(z_i) - |\text{Im}(z_i)| \cot\left(\frac{\pi}{M}\right) = -\frac{1}{\sin\left(\frac{\pi}{M}\right)} |z_i| \geq -\frac{1}{\sin\left(\frac{\pi}{M}\right)} \|H^\dagger v\|_\infty,$$

we have

$$\text{Diag}(\text{Re}(z)) - \cot\left(\frac{\pi}{M}\right) \text{Diag}(|\text{Im}(z)|) \succeq -\frac{1}{\sin\left(\frac{\pi}{M}\right)} \|H^\dagger v\|_\infty I_n.$$  

Combining this with $Q \succeq \lambda_{\min}(Q)I_n$, we can see that (1.4) is a stronger condition on $H$ and $v$ than (1.5).

3.2. Tightness of (ESDR-Y). Similar to Theorem 3.1, we have the following characterization for (ESDR-Y) to be tight. Since the proof technique is essentially the same as that in [17], we put the proof in a separate technical report [13].

**Theorem 3.3.** Suppose that $M \geq 4$. Let the transmitted symbol vector $x^*$ be

$$x^*_i = s_{u_i}, \quad u_i \in \{1, 2, \ldots, M\}, \quad i = 1, 2, \ldots, n,$$

and define

$$\hat{e} = \begin{bmatrix} \text{Re}(v) \\ \text{Im}(v) \end{bmatrix} \in \mathbb{R}^{2n}, \quad \hat{H} = \begin{bmatrix} \text{Re}(H) & -\text{Im}(H) \\ \text{Im}(H) & \text{Re}(H) \end{bmatrix} \in \mathbb{R}^{2n \times 2n},$$

$$y^* = \begin{bmatrix} \text{Re}(x^*) \\ \text{Im}(x^*) \end{bmatrix} \in \mathbb{R}^{2n}, \quad Y^* = y^*(y^*)^T \in \mathbb{R}^{2n \times 2n}.$$
Then \((y^*, Y^*)\) is the optimal solution of (ESDR-Y) if and only if there exist \(\lambda \in \mathbb{R}^{2n}\), \(\mu \in \mathbb{R}^n\), and \(g \in \mathbb{R}^{2n}\) that satisfy

\[
\tilde{H}^T \tilde{v} = g + (\Lambda + M) y^*,
\]

(3.5)

\[
(\Gamma_i, K_{u_i}) \geq (\Gamma_j, K_j), \quad j = 1, 2, \ldots, M, \quad i = 1, 2, \ldots, n,
\]

and

\[
\hat{Q} + \Lambda + M \succeq 0,
\]

where \(K_j\) is defined in (2.6), \(\hat{Q}\) is defined in (2.5), and

\[
\Lambda = \text{Diag}(\lambda), \quad M = \begin{bmatrix} 0 & \text{Diag}(\mu) & 0 \\ \text{Diag}(\mu) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Gamma_i = \begin{bmatrix} 0 & g_i & g_{n+i} \\ g_i & \lambda_i & \mu_i \\ g_{n+i} & \mu_i & \lambda_{n+i} \end{bmatrix}.
\]

Furthermore, (3.5) and (3.6) in Theorem 3.3 can be simplified to the following inequalities on \((\lambda, \mu)\) (see Appendix A):

\[
\sin^2 \left( \frac{\theta_{u_i} + \Delta \theta_j}{2} \right) \lambda_i + \cos^2 \left( \frac{\theta_{u_i} + \Delta \theta_j}{2} \right) \lambda_{n+i} - \sin (2\theta_{u_i} + \Delta \theta_j) \mu_j \leq \text{Re}(z_i) - \cot \left( \frac{\Delta \theta_j}{2} \right) \text{Im}(z_j), \quad j \in \{1, 2, \ldots, M\} \setminus \{u_i\}, \quad i = 1, 2, \ldots, n.
\]

(3.8)

Here \(\Delta \theta_j = \theta_j - \theta_{u_i}\), \(\theta_{u_i}\) is the phase of the \(i\)-th transmitted symbol \(x^*_i\), and \(z_i\) is defined in (3.1). Similar to (3.2), we formulate the conditions in Theorem 3.3 as a semidefinite feasibility problem as follows:

\[
\begin{align*}
\text{find} & \quad \lambda \in \mathbb{R}^{2n} \text{ and } \mu \in \mathbb{R}^n \\
\text{s.t.} & \quad \hat{Q} + \Lambda + M \succeq 0, \\
& \quad (3.8) \text{ is satisfied},
\end{align*}
\]

(3.9)

where \(\Lambda\) and \(M\) are defined in (3.7). However, unlike problem (3.2) where every inequality only involves one dual variable, problem (3.9) has inequalities with three variables coupled together and it is unclear how to choose the “optimal” \(\lambda\) and \(\mu\). In the following, we give a simple necessary condition for (ESDR-Y) being tight based on (3.9).

**Theorem 3.4.** Suppose that \(M \geq 4\). If (ESDR-Y) is tight, then

\[
\text{find } \lambda \in \mathbb{R}^{2n} \text{ and } \mu \in \mathbb{R}^n \\
\text{s.t. } \hat{Q} + \Lambda + M \succeq 0, \\
\text{where } (3.8) \text{ is satisfied},
\]

(3.9)

where \(\Lambda = H^*H\) and \(z\) is defined in (3.1).

Before proving Theorem 3.4, we first introduce the following lemma.

**Lemma 3.5.** Suppose that \(V\) is a PSD matrix in \(\mathbb{R}^{2n}\) and is partitioned as

\[
V = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix},
\]

where \(A = A^T\), \(C = C^T\), and \(A, B, C \in \mathbb{R}^{n \times n}\). Then

\[
U = \frac{1}{2}(A + C) + \frac{i}{2}(B^T - B)
\]

is a PSD matrix in \(\mathbb{C}^n\).
Proof. We observe that
\[ U = \frac{1}{2} \left[ I_n - iI_n \right] \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \left[ I_n - iI_n \right]. \]
The result follows immediately.

Proof of Theorem 3.4. If (ESDR-Y) is tight, we can find \( \lambda \in \mathbb{R}^{2n} \) and \( \mu \in \mathbb{R}^n \) that satisfy the constraints in (3.9). By Lemma 3.5, the constraint \( \hat{Q} + \Lambda + M \succeq 0 \) implies
\begin{equation}
Q + \text{Diag}(\bar{\lambda}) \succeq 0,
\end{equation}
where \( \bar{\lambda} \in \mathbb{R}^n \) is given by
\[ \bar{\lambda}_i = \frac{1}{2}(\lambda_i + \lambda_{n+i}), \ i = 1, 2, \ldots, n. \]
Fix \( i \in \{1, 2, \ldots, n\} \) and let \( \hat{z}_i \) and \( \hat{z}_{n+i} \) denote Re(\( z_i \)) and Im(\( z_i \)), respectively. If \( \hat{z}_{n+i} \geq 0 \), we set \( \Delta \theta_j = \frac{2\pi}{M} \) in (3.8) to get
\[ \sin^2 \left( \theta_{u_i} + \frac{\pi}{M} \right) \lambda_i + \cos^2 \left( \theta_{u_i} + \frac{\pi}{M} \right) \lambda_{n+i} - \sin \left( 2\theta_{u_i} + \frac{2\pi}{M} \right) \mu_j \leq \hat{z}_i - \cot \left( \frac{\pi}{M} \right) |\hat{z}_{n+i}|. \]
Since \( M \geq 4 \), we can also set \( \Delta \theta_j = \frac{2\pi}{M} + \pi \) to get
\[ \cos^2 \left( \theta_{u_i} + \frac{\pi}{M} \right) \lambda_i + \sin^2 \left( \theta_{u_i} + \frac{\pi}{M} \right) \lambda_{n+i} + \sin \left( 2\theta_{u_i} + \frac{2\pi}{M} \right) \mu_j \leq \hat{z}_i + \tan \left( \frac{\pi}{M} \right) |\hat{z}_{n+i}|. \]
Adding the above two inequalities and dividing both sides by two, we have
\begin{equation}
\bar{\lambda}_i = \frac{1}{2}(\lambda_i + \lambda_{n+i}) \leq \hat{z}_i - \cot \left( \frac{2\pi}{M} \right) |\hat{z}_{n+i}|. \tag{3.12}
\end{equation}
If \( \hat{z}_{n+i} < 0 \), we can also arrive at (3.12) by setting \( \Delta \theta_j \) to be \( -\frac{2\pi}{M} \) and \( -\frac{2\pi}{M} + \pi \), respectively. Finally, Theorem 3.4 follows from (3.11) and (3.12).

Note that (3.10) is the same as (1.6) if we recall the definitions of \( Q \) in (2.1) and \( z_i \) in (3.1). Moreover, since (ESDR-Y) is tighter than (ESDR-X), (ESDR-Y) will also be tight if (1.5) holds. Therefore, we have both a necessary condition (1.6) and a sufficient condition (1.5) for (ESDR-Y) to be tight.

3.3. Tightness of (ESDR1-T). In the same spirit, we first give a necessary and sufficient condition for (ESDR1-T) to be tight. Since the technique is essentially the same as that used in Theorem 3.3, we omit the proof details due to the space limitation.

Theorem 3.6. Suppose that \( M \geq 4 \). Let the transmitted symbol vector \( x^* \) be
\[ x^*_i = s_{u_i}, \ u_i \in \{1, 2, \ldots, M\}, \ i = 1, 2, \ldots, n, \]
and define
\[ t^*_{i,u_i} = 1, \ t^*_{i,j} = 0, \ j \neq u_i, \ i = 1, 2, \ldots, n, \]
\[ T^* = t^* (t^*)^T. \]
\( (t^*, T^*) \) is the optimal solution of (ESDR1-T) if and only if there exist \( \alpha \in \mathbb{R}^n \) and \( \gamma \in \mathbb{R}^{Mn} \) such that
\[
\text{Diag}(1 - 2t^*)\gamma = -2\hat{S}^T\hat{H}^T\hat{v} + \alpha \otimes 1_M,
\]
and
\[
\hat{Q} + \text{Diag}(\gamma) \succeq 0.
\]
where \( \hat{S} \) is defined in (2.8), \( \hat{H}, \hat{v} \) are defined in (3.4), and \( \hat{Q} \) is defined in (2.10).

Now we provide a corollary that will serve as our basis for further derivation.

**Corollary 3.7.** If (ESDR1-T) is tight, then there exist
\[
\alpha \in \mathbb{R}^n \text{ and } \gamma_1, \gamma_2, \ldots, \gamma_n \in \mathbb{R}^M
\]
that satisfy
\[
\begin{align*}
\gamma_{i,j} &= \begin{cases}
-2\text{Re}[s_j(\hat{H}^T\hat{v})_i] + \alpha_i, & \text{if } j \neq u_i, \\
2\text{Re}[s_j(\hat{H}^T\hat{v})_i] - \alpha_i, & \text{if } j = u_i,
\end{cases} \\
& \quad j = 1, 2, \ldots, M, \ i = 1, 2, \ldots, n,
\end{align*}
\]
and
\[
\begin{align*}
\mathbf{w}^T \text{Diag}(\gamma_i) \mathbf{w} &\geq 0, \ i = 1, 2, \ldots, n,
\end{align*}
\]
for any \( \mathbf{w} \in \mathbb{R}^M \) such that \( \mathbf{w}^T \mathbf{s}_R = \mathbf{w}^T \mathbf{s}_I = 0 \).

**Proof.** By Theorem 3.6, if (ESDR1-T) is tight, we can find \( \alpha \in \mathbb{R}^n \) and \( \gamma \in \mathbb{R}^{Mn} \) that satisfy (3.13) and (3.14). Let \( \gamma \) be partitioned as \( \gamma = [\gamma_1^T, \gamma_2^T, \ldots, \gamma_n^T]^T \) where \( \gamma_j \in \mathbb{R}^M \) is the \( j \)-th block of \( \gamma \). It is straightforward to verify that (3.13) is equivalent to (3.15).

Moreover, for any \( i \in \{1, 2, \ldots, n\} \) and any \( \mathbf{w} \in \mathbb{R}^M \) that satisfies \( \mathbf{w}^T \mathbf{s}_R = \mathbf{w}^T \mathbf{s}_I = 0 \), we set \( \mathbf{w} = [\mathbf{w}_1^T, \mathbf{w}_2^T, \ldots, \mathbf{w}_n^T]^T \in \mathbb{R}^{Mn} \) to be
\[
\mathbf{w}_j = \begin{cases}
0, & \text{if } j \neq i, \\
\mathbf{w}, & \text{if } j = i.
\end{cases}
\]
It is simple to check that \( \hat{S}\mathbf{w} = 0 \). Therefore, recalling that \( \hat{Q} = \hat{S}^T\hat{Q}\hat{S} \), by (3.14) we have
\[
\mathbf{w}^T(\hat{Q} + \text{Diag}(\gamma)) \mathbf{w} = \mathbf{w}^T \text{Diag}(\gamma) \mathbf{w} = \mathbf{w}^T \text{Diag}(\gamma_i) \mathbf{w} \geq 0.
\]
The proof is complete.

In practice, the symbol set \( \mathcal{S} \), such as the one in (1.3) considered in this paper, is symmetric with respect to the origin. Therefore, we can find \( u'_i \in \{1, 2, \ldots, M\} \) that satisfies \( s_{u'_i} = -s_{u_i} \). Now let \( \mathbf{w} \in \mathbb{R}^M \) be
\[
\mathbf{w}_j = \begin{cases}
0, & \text{if } j \notin \{u_i, u'_i\}, \\
1, & \text{if } j \in \{u_i, u'_i\}, \ j = 1, 2, \ldots, M.
\end{cases}
\]
We have \( \mathbf{w}^T \mathbf{s}_R = s_{R,u_i} + s_{R,u'_i} = 0 \) and \( \mathbf{w}^T \mathbf{s}_I = s_{I,u_i} + s_{I,u'_i} = 0 \). Hence, when (ESDR1-T) is tight, Corollary 3.7 implies that
\[
\mathbf{w}^T \text{Diag}(\gamma_i) \mathbf{w} = \gamma_{u_i} + \gamma_{u'_i} = 4\text{Re}[s_{u_i}(\hat{H}^T\hat{v})_i] = 4\text{Re}[(x^*_i)^T(\hat{H}^T\hat{v})_i] \geq 0.
\]
This immediately leads to the following upper bound on the tightness probability of 
(ESDR1-T).

**Corollary 3.8.** Suppose that the symbol set $S$ is symmetric with respect to the 
origin and $0 \notin S$. We further assume that
(a) The entries of $x^*$ are drawn from $S$ uniformly and independently;
(b) $x^*, H$, and $v$ are mutually independent; and
(c) the distribution of $H$ and $v$ are continuous.

Then we have

$$\text{Prob}(\text{(ESDR1-T) is tight}) \leq \left(\frac{1}{2}\right)^n.$$ 

**Proof.** Let $z_i = (x_i^*)^\dagger (H^\dagger v)_i$, $i = 1, 2, \ldots, n$. Since $H$ and $v$ are independent 
continuous random variables, the event

$$\bigcup_{i=1}^{n} \bigcup_{s \in S} \{\text{Re}(s^\dagger (H^\dagger v)_i) = 0\}$$

happens with probability zero. Hence, because of the symmetry of $S$, with prob-
bility one exactly half of the symbols $s \in S$ satisfy $\text{Re}(s^\dagger (H^\dagger v)_i) > 0$ for each 
$i \in \{1, 2, \ldots, n\}$ when $H$ and $v$ are given. By the assumption that $x_i^*$ is uniformly 
distributed over $S$, we obtain

$$\text{Prob}(\text{Re}(z_i) \geq 0 \mid H, v) = \frac{1}{2} \text{ almost surely.}$$

Moreover, $\{z_i\}_{i=1}^n$ are mutually independent conditioned on $H$ and $v$. This leads to

$$\text{Prob}(\text{Re}(z_i) \geq 0, i = 1, 2, \ldots, n) = \mathbb{E}_{H, v}[\text{Prob}(\text{Re}(z_i) \geq 0, i = 1, 2, \ldots, n \mid H, v)]$$

$$= \mathbb{E}_{H, v}\left[\prod_{i=1}^{n} \text{Prob}(\text{Re}(z_i) \geq 0 \mid H, v)\right]$$

$$= \left(\frac{1}{2}\right)^n.$$

Finally, Corollary 3.8 follows from the fact that the tightness of (ESDR1-T) implies
$\text{Re}(z_i) \geq 0, i = 1, 2, \ldots, n.$

It is worth noting that all the assumptions in Corollary 3.8 are mild: they are 
satisfied if we use the $M$-PSK or QAM modulation scheme and the entries of $H$ and $v$ follow the complex Gaussian distribution.

Intuitively, we will expect that (ESDR1-T) is less likely to recover the transmitted 
symbols with an increasing symbol set size $M$. In the following, we present a more 
refined upper bound on the tightness probability specific to the $M$-PSK setting and 
the proof can be found in [13].

**Theorem 3.9.** Suppose that $M$-PSK is used with $M \geq 4$ and the same assump-
tions in Corollary 3.8 hold. Then we have

$$\text{Prob}(\text{(ESDR1-T) is tight}) \leq \left(\frac{2}{M}\right)^n.$$
From Corollary 3.8 and Theorem 3.9, we can see that the tightness probability of \((ESDR1-T)\) is bounded away from one regardless of the noise level, and it tends to zero exponentially fast when the number of transmitted symbols \(n\) increases. This is in sharp contrast to \((ESDR-X)\) and \((ESDR-Y)\), whose tightness probabilities will approach one if the noise level is sufficiently small and the number of received signals \(m\) is sufficiently large compared to \(n\) [17, Theorem 4.5].

4. Equivalence between different SDRs. In this section, we focus on the relationship between different SDR models of (1.2). Related to the SDRs discussed so far, a recent paper [16] proved that \((ESDR2-T)\) is equivalent to \((ESDR-Y)\) for the MIMO detection problem with a general symbol set. An earlier paper [21] compared three different SDRs in the QAM setting and showed their equivalence. Compared with those in section 2, the SDRs considered in [21] differ greatly in their motivations and structures, and the two equivalence results are proved using different techniques. In this section, we provide a more general equivalence theorem from which both results follow as special cases. This not only reveals the underlying connection between these two works, but also may potentially lead to new equivalence between SDRs.

4.1. Review of previous results. In [16], the authors established the following correspondence between a pair of feasible points of \((ESDR2-T)\) and \((ESDR-Y)\):

\[
Y = \hat{S} \hat{T} \hat{S}^T \quad \text{and} \quad y = \hat{S} \hat{t},
\]

where \(\hat{S} \in \mathbb{R}^{2n \times M_n}\) is defined in (2.8). In [21], the authors considered the feasible set of a virtually-antipodal SDR (VA-SDR):

\[
\text{(VA-SDR)} \quad \begin{bmatrix} 1 & b^T \\ b & B \end{bmatrix} \in S^{2n+1}_+ \\
\text{s.t.} \quad B_{i,i} = 1, \ i = 1, 2, \ldots, qn,
\]

and that of a bounded-constrained SDR (BC-SDR):

\[
\text{(BC-SDR)} \quad \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \in S^{2n+1}_+ \\
\text{s.t.} \quad 1 \leq X_{i,j} \leq (2^q - 1)^2, \ i = 1, 2, \ldots, n,
\]

where \(q \geq 1\) is an integer. We refer interested readers to [21] and references therein for their derivations. The authors proved the equivalence between (VA-SDR) and (BC-SDR) by showing the following correspondence:

\[
X = WBW^T \quad \text{and} \quad x = Wb,
\]

where

\[
W = [I_n \ 2I_n \ 4I_n \ \ldots \ 2^{q-1}I_n] \in \mathbb{R}^{n \times qn}.
\]

Note that both equivalence results in (4.1) and (4.2) fall into the following form:

\[
\left\{ \begin{bmatrix} 1 & y^T \\ y & Y \end{bmatrix} \in \mathcal{F}_1 \right\} = \left\{ \begin{bmatrix} 1 & y^T \\ y & Y \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} 1 & t^T \\ t & T \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & P^T \end{bmatrix} : \begin{bmatrix} 1 & t^T \\ t & T \end{bmatrix} \in \mathcal{F}_2 \right\},
\]

where \(\mathcal{F}_1\) is a subset of \(S^{k+1}_+\), \(\mathcal{F}_2\) is a subset of \(S^{d+1}_+\), and we call \(P \in \mathbb{R}^{k \times d}\) as the transformation matrix. Moreover, both the transformation matrices \(S\) in (4.1) and \(W\) in (4.2) have a special “separable” property that we now define for ease of presentation.
Definition 4.1. A matrix \( P \in \mathbb{R}^{k \times d} \) is called separable if there exist a partition of rows \( \alpha_1, \alpha_2, \ldots, \alpha_l \) and a partition of columns \( \beta_1, \beta_2, \ldots, \beta_l \) such that

\[
P[\alpha_i; \beta_j] = 0, \quad \forall i \neq j.
\]

In other words, a matrix is separable if, after possibly rearranging rows and columns, it has a block diagonal structure. In particular, for the transformation matrix \( S \) in (4.1), the corresponding row and column partitions are given by

\[
\alpha_i = \{i, n + i\}, \quad \beta_i = \{(i - 1)M + 1, (i - 1)M + 2, \ldots, iM\}, \quad i = 1, 2, \ldots, n;
\]

for the transformation matrix \( W \) in (4.2), they are given by

\[
\alpha_i = \{i\}, \quad \beta_i = \{i, i + n, i + 2n, \ldots, i + (q - 1)n\}, \quad i = 1, 2, \ldots, n.
\]

4.2. A general equivalence theorem. Now we are ready to present our main equivalence result.

Theorem 4.2. Suppose that the matrix \( P \in \mathbb{R}^{k \times d} \) is separable with row partition \( \alpha_1, \alpha_2, \ldots, \alpha_l \) and column partition \( \beta_1, \beta_2, \ldots, \beta_l \). Moreover, define

\[
k_i := |\alpha_i|, \quad d_i := |eta_i|, \quad \text{and} \quad P_i := P[\alpha_i, \beta_i] \in \mathbb{R}^{k_i \times d_i}, \quad i = 1, 2, \ldots, l,
\]

where we use \(| \cdot |\) to denote the cardinality of a set. Then given arbitrary constraint sets \( A_i \subset \mathbb{R}^{d_i \times d_i} \) for \( i = 1, 2, \ldots, l \), the following set

\[
\begin{bmatrix}
1 & y^T \\
y & Y
\end{bmatrix} \in S_{+}^{k+1}
\]

s.t. \( Y = P T P^T, \quad y = P t, \) \quad (4.5)

\[
\begin{bmatrix}
1 & t^T \\
t & T
\end{bmatrix} \in S_{+}^{d+1},
\]

\[
\begin{bmatrix}
1 & t[\beta_i] \\
t[\beta_i] & T[\beta_i]
\end{bmatrix} \in A_i, \quad i = 1, 2, \ldots, l,
\]

where the variables are \( y \in \mathbb{R}^k, \quad Y \in \mathbb{R}^{k \times k}, \quad t \in \mathbb{R}^d, \quad \text{and} \quad T \in \mathbb{R}^{d \times d}, \) is the same as

\[
\begin{bmatrix}
1 & y^T \\
y & Y
\end{bmatrix} \in S_{+}^{k+1}
\]

s.t. \( Y[\alpha_i] = P_i T(i)^T P_i^T, \quad y[\alpha_i] = P_i t(i), \)

\[
\begin{bmatrix}
1 & (t(i))^T \\
t(i) & T(i)
\end{bmatrix} \in S_{+}^{d+1},
\]

\[
\begin{bmatrix}
1 & (t(i))^T \\
t(i) & T(i)
\end{bmatrix} \in A_i, \quad i = 1, 2, \ldots, l,
\]

where the variables are \( y \in \mathbb{R}^k, \quad Y \in \mathbb{R}^{k \times k}, \quad t(i) \in \mathbb{R}^{d_i}, \quad \text{and} \quad T(i) \in \mathbb{R}^{d_i \times d_i} \) with \( i = 1, 2, \ldots, l \).

The following lemma will be useful in our proof.

Lemma 4.3 ([8, Theorem 7.3.11]). Let \( A \in \mathbb{R}^{p \times n} \) and \( B \in \mathbb{R}^{q \times n} \) where \( p \leq q \). Then \( A^T A = B^T B \) if and only if there exists a matrix \( U \in \mathbb{R}^{q \times p} \) with \( U^T U = I_p \) such that \( B = UA \).
Proof of Theorem 4.2. Without loss of generality, we assume the transformation matrix \( P \in \mathbb{R}^{k \times d} \) is in the form

\[
P = \begin{bmatrix}
P_1 & P_2 & \cdots & P_l
\end{bmatrix},
\]

where \( P_i \in \mathbb{R}^{k_i \times d_i} \), \( \sum_{i=1}^l k_i = k \), and \( \sum_{i=1}^l d_i = d \).

For one direction, suppose that \((y, Y, t, T)\) satisfies the constraints in (4.5). Then it is straightforward to see that \((y, Y)\) also satisfies the constraints in (4.6) together with

\[
t^{(i)} = t[i], \quad T^{(i)} = T[i], \quad i = 1, 2, \ldots, l.
\]

The other direction of the proof is more involved. Given \((y, Y)\) and the variables \(\{t^{(i)}, T^{(i)}\}_{i=1}^l\) in (4.6), our goal is to construct \((t, T)\) satisfying the conditions in (4.5). To simplify the notations, we define

\[
\hat{Y} := \begin{bmatrix} 1 & y \end{bmatrix}^T, \quad \hat{T}^{(i)} := \begin{bmatrix} 1 & (t^{(i)})^T \\ T^{(i)} \end{bmatrix}, \quad \text{and} \quad \hat{P}_i := \begin{bmatrix} 1 & 0 \\ 0 & P_i \end{bmatrix}.
\]

Let \( r = \max\{k, d\} \). Since \( \hat{Y} \in \mathbb{S}_\times^{k+1} \), it can be factorized as

\[
\hat{Y} = \tilde{V}^\top \tilde{V},
\]

where \( \tilde{V} \in \mathbb{R}^{(r+1) \times (k+1)} \). The above factorization can be done because \( r \geq k \). Further, we partition \( \tilde{V} \) as

\[
\tilde{V} = \begin{bmatrix} v & V_1 & V_2 & \cdots & V_l \end{bmatrix},
\]

where \( v \in \mathbb{R}^{r+1} \) and \( V_i \in \mathbb{R}^{(r+1) \times k_i} \) contains the columns of \( \tilde{V} \) indexed by \( \alpha_i \) for \( i = 1, 2, \ldots, l \). Moreover, we have \( v^\top v = \hat{Y}_{1,1} = 1 \). Similarly, \( T^{(i)} \) can be factorized as

\[
\hat{T}^{(i)} = (\hat{Z}^{(i)})^\top \hat{Z}^{(i)}, \quad i = 1, 2, \ldots, l,
\]

where \( \hat{Z}^{(i)} \in \mathbb{R}^{(d_i+1) \times (d_i+1)} \) and is partitioned as

\[
\hat{Z}^{(i)} = \begin{bmatrix} \hat{z}^{(i)} & Z^{(i)} \end{bmatrix}.
\]

Combining (4.9) with the equality constraints in (4.6), we get

\[
\begin{bmatrix} \hat{1} \\ y[\alpha_i] \end{bmatrix} = \hat{P}_i \hat{T}^{(i)} \hat{P}_i^\top = (\hat{Z}^{(i)} \hat{P}_i^\top) (\hat{Z}^{(i)} \hat{P}_i^\top), \quad i = 1, 2, \ldots, l,
\]

where \( \hat{Z}^{(i)} \hat{P}_i^\top \in \mathbb{R}^{(d_i+1) \times (k_i+1)} \). On the other hand, the factorization in (4.8) implies

\[
\begin{bmatrix} \hat{1} \\ y[\alpha_i] \end{bmatrix} = [v \ V_i]^\top [v \ V_i],
\]

where \( [v \ V_i] \in \mathbb{R}^{(r+1) \times (k_i+1)} \). By Lemma 4.3, we can find \( U_i \in \mathbb{R}^{(r+1) \times (d_i+1)} \) with \( U_i^\top U_i = I_{d_i+1} \) such that

\[
[v \ V_i] = U_i \hat{Z}^{(i)} \hat{P}_i^\top.
\]
Substituting (4.7) and (4.10) into (4.11), we get
\begin{equation}
(4.12) \quad v = U_i z^{(i)} \quad \text{and} \quad V_i = U_i Z^{(i)} P^T.
\end{equation}

Finally, we define
\begin{equation}
(4.13) \quad R = \begin{bmatrix} v & U_1 Z^{(1)} & U_2 Z^{(2)} & \ldots & U_l Z^{(l)} \end{bmatrix} \in \mathbb{R}^{(r+1) \times (d+1)},
\end{equation}
whose columns indexed by $\beta_i$ are given by $U_i Z^{(i)}$, and construct $(t, T)$ by
\begin{equation}
(4.14) \quad \begin{bmatrix} 1 & t^T \\ t & T \end{bmatrix} = R^T R.
\end{equation}

Next we verify that $(t, T)$ in (4.13) indeed satisfies all the constraints in (4.5). The positive semidefiniteness is evident by our construction. For the equality constraints, by using (4.12) we have
\begin{equation}
R \begin{bmatrix} 1 & 0 \\ 0 & P^T \end{bmatrix} = \begin{bmatrix} v & U_1 Z^{(1)} P^T & U_2 Z^{(2)} P^T & \ldots & U_l Z^{(l)} P^T \end{bmatrix} \\
= \begin{bmatrix} v & V_1 & V_2 & \ldots & V_l \end{bmatrix} \\
= \tilde{V}.
\end{equation}

Hence, we get
\begin{equation}
\begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix} \begin{bmatrix} 1 & t^T \\ t & T \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & P^T \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & P^T \end{bmatrix} R^T R \begin{bmatrix} 1 & 0 \\ 0 & P^T \end{bmatrix} \\
= \tilde{V}^T \tilde{V} \\
= \begin{bmatrix} 1 & y^T \\ y & Y \end{bmatrix},
\end{equation}

which is equivalent to $Y = PTP^T$ and $y = Pt$. Lastly, note that
\begin{equation}
(4.15) \quad \begin{bmatrix} 1 & t[i] \beta_i \\ t[i] \beta_i & T[i] \beta_i \end{bmatrix} = \begin{bmatrix} v & U_i Z^{(i)} \end{bmatrix}^T \begin{bmatrix} v & U_i Z^{(i)} \end{bmatrix} \\
= \begin{bmatrix} U_i z^{(i)} & U_i Z^{(i)} \end{bmatrix}^T \begin{bmatrix} U_i z^{(i)} & U_i Z^{(i)} \end{bmatrix} \\
= (\tilde{Z}^{(i)})^T U_i^T U_i \tilde{Z}^{(i)} \\
= (\tilde{Z}^{(i)})^T \tilde{Z}^{(i)} \\
= T[i] = \begin{bmatrix} 1 & t[i] T[i] \end{bmatrix},
\end{equation}

where we used (4.13) in (4.14), $v = U_i z^{(i)}$ (cf. (4.12)) in (4.15), and $U_i^T U_i = I_{d_i+1}$ in (4.16). Hence, the remaining constraints in (4.5) are also satisfied because of the conditions on $T[i]$ in (4.6).

The proof of Theorem 4.2 is complete. \(\square\)

Two remarks are in order. Firstly, the variables in (4.5) are in a high-dimensional PSD cone $S^{d+1}_+$, while those in (4.6) are in the Cartesian product of smaller PSD cones $S^{k+1}_+ \times S^{d_1+1}_+ \times S^{d_2+1}_+ \times \cdots \times S^{d_l+1}_+$. When $k$, $d_1$, $d_2$, $\ldots$, $d_l$ are much smaller than $d$, using (4.6) instead of (4.5) can achieve dimension reduction without any additional cost. This can bring substantially higher computational efficiency for solving
the corresponding SDP in practice (see section 5). Secondly, Theorem 4.2 is very
general and thus could be applicable to a potentially wide range of problems. It is
worth highlighting that we require no assumptions on the sets \( \mathcal{A}_i \) that constrain the
submatrices, as well as the row and column partitions of the separable matrix \( \mathbf{P} \). This
enables us to accommodate both the equivalence results (4.1) and (4.2), as we will
show next.

4.2.1. Equivalence between (ESDR2-T) and (ESDR-Y). As we noted be-
fore, the transformation matrix \( \hat{\mathbf{S}} \) in (4.1) is separable with the row and column par-
titions given in (4.3) and we have

\[
\hat{\mathbf{S}}[\alpha_i, \beta_i] = \begin{bmatrix} s^R_i \\ s^L_i \end{bmatrix} \in \mathbb{R}^{2 \times M}, \ i = 1, 2, \ldots, n.
\]

Moreover, we can see that the feasible set of (ESDR2-T) is in the form of (4.5) with
the set \( \mathcal{A}_i \) given by

\[
\mathcal{A}_i = \left\{ t \in \mathbb{R}^M, \ \sum_{j=1}^M t_j = 1, \ t_j \geq 0, \ j = 1, 2, \ldots, M \right\}
= \left\{ \sum_{j=1}^M t_j \mathbf{E}_j : \sum_{j=1}^M t_j = 1, \ t_j \geq 0, \ j = 1, 2, \ldots, M \right\}
= \text{conv} \{ \mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_M \},
\]

where

\[
\mathbf{E}_j = \begin{bmatrix} 1 \\ e_j \end{bmatrix} \begin{bmatrix} 1 \\ e_j \end{bmatrix}^T, \ j = 1, 2, \ldots, M,
\]

and \( e_j \in \mathbb{R}^M \) is the \( j \)-th unit vector. Applying Theorem 4.2 to (ESDR2-T) gives the
following equivalent formulation:

\[
\begin{bmatrix} 1 \\ y^T \\ Y \end{bmatrix} \in \mathbb{S}^{2n+1}_{+},
\]

s.t.

\[
\begin{bmatrix} 1 \\ y_i \\ Y_{i,i} \\ y_{n+i} \\ Y_{n+i,n+i} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ s^T_i \\ s_i \end{bmatrix} \begin{bmatrix} 1 \\ (t(i))^T \\ T(i) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ s_R \ s_I \end{bmatrix},
\]

\[
\begin{bmatrix} 1 \\ t(i) \\ (t(i))^T \\ T(i) \end{bmatrix} \in \mathbb{S}^{M+1}_{+},
\]

\[
\begin{bmatrix} 1 \\ t(i) \\ (t(i))^T \\ T(i) \end{bmatrix} \in \text{conv} \{ \mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_M \}, \ i = 1, 2, \ldots, n.
\]

Since each matrix \( \mathbf{E}_j \) is PSD, their convex hull is a subset of \( \mathbb{S}^{M+1}_{+} \) and hence the
PSD constraints in (4.17) are redundant. Furthermore, note that

\[
\begin{bmatrix} 1 \\ s^R_i \\ 0 \\ s^L_i \end{bmatrix} \mathbf{E}_j \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ s^R_i \\ 0 \\ s^L_i \end{bmatrix} \begin{bmatrix} e_j \\ e_j^T \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ s_R \ s_I \end{bmatrix} = \begin{bmatrix} 1 \\ s_{R,j} \\ s_{L,j} \end{bmatrix},
\]
which is exactly the matrix $K_j$ defined in (2.6). Therefore, we can conclude that (4.17) is the same as (ESDR-Y), and hence (ESDR2-T) and (ESDR-Y) are equivalent.

### 4.2.2. Equivalence between (VA-SDR) and (BC-SDR)

Similarly, we observe that the transformation matrix $W$ in (4.2) is separable with row and column partitions given in (4.4), and let

$\mathbf{w}^T := W[\alpha_i, \beta_i] = [1, 2, 4, \ldots, 2^{q-1}]$.

The feasible set in (VA-SDR) conforms to (4.5) with the set $\mathcal{A}_i$ given by

$\mathcal{A}_i = \left\{ \begin{bmatrix} 1 & b^T \\ b & B \end{bmatrix} : b \in \mathbb{R}^q, B \in \mathbb{R}^{q \times q}, \text{diag}(B) = 1 \right\}$.

Hence, by applying Theorem 4.2 to (VA-SDR), we get the following equivalent formulation:

$$\begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \in \mathbb{S}^{n+1}_+ \quad \text{s.t.} \quad X_{i,i} = \mathbf{w}^T \mathbf{B}^{(i)} \mathbf{w}, \ x_i = \mathbf{w}^T \mathbf{b}^{(i)},$$

$$\begin{bmatrix} 1 \\ \mathbf{b}^{(i)} \end{bmatrix} \in \mathbb{S}^{q+1}_+, \quad \text{diag}(\mathbf{B}^{(i)}) = 1, \quad i = 1, 2, \ldots, n.$$

Next we argue that all the constraints $x_i = \mathbf{w}^T \mathbf{b}^{(i)}$ are redundant, i.e., the set in (4.19) is equivalent to

$$\begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \in \mathbb{S}^{n+1}_+ : \ X_{i,i} = \mathbf{w}^T \mathbf{B}^{(i)} \mathbf{w}, \ \mathbf{B}^{(i)} \in \mathbb{S}^q_+, \ \text{diag}(\mathbf{B}^{(i)}) = 1, \quad i = 1, 2, \ldots, n.$$  

To show this, we need to prove that, for any $x, X, B^{(1)}, \ldots, B^{(n)}$ satisfying the constraints in (4.20), there must exist $\mathbf{b}^{(i)} \in \mathbb{R}^q$ such that

$$x_i = \mathbf{w}^T \mathbf{b}^{(i)} \text{ and } \begin{bmatrix} 1 \\ \mathbf{b}^{(i)} \end{bmatrix} \begin{bmatrix} \mathbf{b}^{(i)^T} \\ \mathbf{B}^{(i)} \end{bmatrix} \succeq 0, \quad i = 1, 2, \ldots, n.$$

Fix $i \in \{1, 2, \ldots, n\}$. Note that the PSD constraints in (4.20) implies

$$\begin{bmatrix} 1 & x_i^2 \\ x_i & X_{i,i} \end{bmatrix} \succeq 0 \iff x_i^2 \leq X_{i,i}.$$

When $X_{i,i} = 0$, we must have $x_i = 0$ and we can achieve (4.21) by simply letting $\mathbf{b}^{(i)} = 0$. Otherwise, we have $X_{i,i} > 0$ and hence we can let

$$\mathbf{b}^{(i)} = \frac{x_i}{X_{i,i}} \mathbf{B}^{(i)} \mathbf{w}.$$

Since $X_{i,i} = \mathbf{w}^T \mathbf{B}^{(i)} \mathbf{w}$, we can see that $\mathbf{w}^T \mathbf{b}^{(i)} = (\mathbf{w}^T \mathbf{B}^{(i)} \mathbf{w}) x_i / X_{i,i} = x_i$.

To verify the PSD constraint in (4.21), it suffices to show that $\mathbf{B}^{(i)} \succeq \mathbf{b}^{(i)} (\mathbf{b}^{(i)^T})$. Note that

$$\begin{bmatrix} X_{i,i} & \mathbf{w}^T \mathbf{B}^{(i)} \\ \mathbf{B}^{(i)} \mathbf{w} & \mathbf{B}^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{w}^T \mathbf{B}^{(i)} \mathbf{w} & \mathbf{w}^T \mathbf{B}^{(i)} \\ \mathbf{B}^{(i)} \mathbf{w} & \mathbf{B}^{(i)} \end{bmatrix} = \begin{bmatrix} \mathbf{w}^T \\ \mathbf{I}_q \end{bmatrix} \mathbf{B}^{(i)} \begin{bmatrix} \mathbf{w} & \mathbf{I}_q \end{bmatrix} \succeq 0,$$
which implies the Schur complement is also PSD, i.e.,
\[ B^{(i)} - \frac{1}{X_{i,i}} B^{(i)} w w^T B^{(i)} \succeq 0. \]

This, together with (4.22) and (4.23), shows
\[ b^{(i)} (b^{(i)})^T = \frac{x_i^2}{X_{i,i}} B^{(i)} ww^T B^{(i)} \preceq \frac{X_{i,i}}{x_i^2} B^{(i)} ww^T B^{(i)} \preceq B^{(i)}. \]

Hence both conditions in (4.21) are satisfied.

Finally, to show that (4.20) is the same as (BC-SDR), we need the following lemma.

**Lemma 4.4.** Let \( w \in \mathbb{R}^q \) be the vector defined in (4.18). It holds that
\[ \{ x : \exists B \in \mathbb{S}^q_+ \text{ s.t. } x = w^T B w, \text{ diag}(B) = 1_q \} = \{ 1 \leq x \leq (2^q - 1)^2 \}. \]

**Proof.** See Appendix B.

Putting all pieces together, we have proved that (VA-SDR) is equivalent to (BC-SDR) by showing the correspondence (4.2).

**5. Numerical results.** In this section, we present some numerical results. Following standard assumptions in the wireless communication literature (see, e.g., [31, Chapter 7]), we assume that all entries of the channel matrix \( H \) are independent and identically distributed (i.i.d.) following a complex circular Gaussian distribution with zero mean and unit variance, and all entries of the additive noise \( v \) are i.i.d. following a complex circular Gaussian distribution with zero mean and variance \( \sigma^2 \). Further, we choose the transmitted symbols \( x_1^*, x_2^*, \ldots, x_n^* \) from the symbol set \( S_M \) in (1.3) independently and uniformly. We define the SNR as the received SNR per symbol:
\[ \text{SNR} := \frac{\mathbb{E}[\|Hx^*\|^2]}{n \mathbb{E}[\|v\|^2]} = \frac{mn}{n \cdot m \sigma^2} = \frac{1}{\sigma^2}. \]

We first consider a MIMO system where \( (m,n) = (16,10) \) and \( M = 8 \). To evaluate the empirical probabilities of SDRs not being tight, we compute the optimal solutions of (ESDR-X), (ESDR-Y), and (ESDR1-T) by the general-purpose SDP solver SeDuMi [29] with the desired accuracy set to \( 10^{-6} \). The SDR is decided to be tight if the output \( \hat{x} \) returned by the SDP solver\(^4\) satisfies \( \| \hat{x} - x^* \|_\infty \leq 10^{-4} \). We also evaluate the empirical probabilities of conditions (1.4)–(1.6) not being satisfied. We run the simulations at 8 SNR values in total ranging from 3 dB to 24 dB. For each SNR value, 10,000 random instances are generated and the averaged results are plotted in Figure 2.

We can see from Figure 2 that our results (1.5) and (1.6) provide better characterizations than the previous tightness condition (1.4) in [16]. The empirical probability of (ESDR-X) not being tight matches perfectly with our analysis given by the necessary and sufficient condition (1.5). The probability of (1.6) not being satisfied is also a good approximation to the probability of (ESDR-Y) not being tight, underestimating the latter roughly by a factor of 9. Moreover, the numerical results also validate

\(^4\)The output \( \hat{x} \) is directly given by the optimal solution in (ESDR-X), while it is obtained from the relation (2.5) between \( x \) and \( y \) in (ESDR-Y) and the relation (2.7) between \( x \) and \( t \) in (ESDR1-T).
our analysis that \((\text{ESDR}_1\cdot T)\) is not tight with high probability. In fact, \((\text{ESDR}_1\cdot T)\) fails to recover the vector of transmitted symbols in all 80,000 instances.

Next, we compare the optimal values as well as the CPU time of solving \((\text{ESDR}_X)\) and \((\text{ESDR}_2\cdot T)\). Table 2 shows the relative difference between the optimal values of \((\text{ESDR}_Y)\) (denoted as \(\text{opt}_{\text{ESDR}_Y}\)) and \((\text{ESDR}_2\cdot T)\) (denoted as \(\text{opt}_{\text{ESDR}_2\cdot T}\)) averaged over 300 simulations, which is defined as \(\frac{|\text{opt}_{\text{ESDR}_Y} - \text{opt}_{\text{ESDR}_2\cdot T}|}{|\text{opt}_{\text{ESDR}_2\cdot T}|}\).

We can see from Table 2 that the difference is consistently in the order \(10^{-7}\) in various settings, which verifies the equivalence between \((\text{ESDR}_Y)\) and \((\text{ESDR}_2\cdot T)\). In Figure 3, we plot the average CPU time consumed by solving \((\text{ESDR}_Y)\) and \((\text{ESDR}_2\cdot T)\) in an 8-PSK system with increasing problem size \(n\). For fair comparison, both SDRs are implemented and solved by ScDuMi and we repeat the simulations for 300 times. With the same error performance, we can see that \((\text{ESDR}_Y)\) indeed solves the MIMO detection problem (1.2) more efficiently and saves roughly 90% of the computational time in our experiment.

### Table 2

| SNR | Relative diff. in optimal objective values. |
|-----|--------------------------------------------|
|     | \((m, n) = (4, 4)\) | \((m, n) = (6, 4)\) | \((m, n) = (10, 10)\) | \((m, n) = (15, 10)\) |
| 5dB | 4.62e\(-7\)       | 5.10e\(-7\)       | 6.26e\(-7\)       | 7.73e\(-7\)       |
| 10dB| 5.67e\(-7\)       | 4.06e\(-7\)       | 6.50e\(-7\)       | 7.16e\(-7\)       |
| 15dB| 5.94e\(-7\)       | 3.83e\(-7\)       | 7.80e\(-7\)       | 5.58e\(-7\)       |

6. Conclusions. In this paper, we studied the tightness and equivalence of various existing SDR models for the MIMO detection problem (1.2). For the two SDRs \((\text{ESDR}_X)\) and \((\text{ESDR}_Y)\) proposed in [17], we improved their sufficient tightness condition and showed that the former is tight if and only if (1.5) holds while the latter is tight only if (1.6) holds. On the other hand, for the SDR \((\text{ESDR}_1\cdot T)\) proposed in [22], we proved that its tightness probability decays to zero exponentially fast with an increasing problem size under some mild assumptions. Together with known results, our analysis provides a more complete understanding of the tightness conditions for existing SDRs. Moreover, we proposed a general theorem that unifies...
Previous results on the equivalence of SDRs [21, 16]. For a subset of PSD matrices with a special “separable” structure, we showed its equivalence to another subset of PSD matrices in a potentially much smaller dimension. Our numerical results demonstrated that we could significantly improve the computational efficiency by using such equivalence.

Due to its generality, we believe that our equivalence theorem can be applied to SDPs in other domains beyond MIMO detection and we would like to put this as a future work. Additionally, we noticed that the SDRs for problem (1.2) combined with some simple rounding procedure can detect the transmitted symbols successfully even when the optimal solution has rank more than one. Similar observations have also been made in [12]. It would be interesting to extend our analysis to take the postprocessing procedure into account.

Appendix A. Simplification of (3.5) and (3.6). Fix $i \in \{1, 2, \ldots, n\}$. From (3.5), we have

\[
(\hat{H}^T \hat{v})_i = g_i + \lambda_i y_i^* + \mu_i y_{n+i}^* \quad \text{and} \quad (\hat{H}^T \hat{v})_{n+i} = g_{n+i} + \mu_i y_i^* + \lambda_{n+i} y_{n+i}^*,
\]

which can be written in a matrix form:

\[
(A.1) \quad \begin{bmatrix}
(\hat{H}^T \hat{v})_i \\
(\hat{H}^T \hat{v})_{n+i}
\end{bmatrix} = \begin{bmatrix}
g_i \\
g_{n+i}
\end{bmatrix} + \begin{bmatrix}
\lambda_i & \mu_i \\
\mu_i & \lambda_{n+i}
\end{bmatrix} \begin{bmatrix}
y_i^* \\
y_{n+i}^*
\end{bmatrix}.
\]

Recall the definitions of $\Gamma_i$ in (3.7) and $K_j$ in (2.6). Then

\[
(A.2) \quad \langle \Gamma_i, K_j \rangle = 2 \begin{bmatrix}
\cos(\theta_j) \\
\sin(\theta_j)
\end{bmatrix}^T \begin{bmatrix}
g_i \\
g_{n+i}
\end{bmatrix} + \begin{bmatrix}
\cos(\theta_j) \\
\sin(\theta_j)
\end{bmatrix}^T \begin{bmatrix}
\lambda_i & \mu_i \\
\mu_i & \lambda_{n+i}
\end{bmatrix} \begin{bmatrix}
y_i^* \\
y_{n+i}^*
\end{bmatrix}.
\]

Using (A.1), we have

\[
(A.3) \quad \begin{bmatrix}
\cos(\theta_j) \\
\sin(\theta_j)
\end{bmatrix}^T \begin{bmatrix}
g_i \\
g_{n+i}
\end{bmatrix} = \begin{bmatrix}
\cos(\theta_j) \\
\sin(\theta_j)
\end{bmatrix}^T (\hat{H}^T \hat{v})_i - \begin{bmatrix}
\cos(\theta_j) \\
\sin(\theta_j)
\end{bmatrix}^T (\hat{H}^T \hat{v})_{n+i} + \begin{bmatrix}
\lambda_i & \mu_i \\
\mu_i & \lambda_{n+i}
\end{bmatrix} \begin{bmatrix}
y_i^* \\
y_{n+i}^*
\end{bmatrix},
\]

FIG. 3. Average CPU time of solving (ESDR-Y) and (ESDR2-T) when $M = 8$.  
\[\text{Average CPU Time (in sec.)}\]

\[\text{Problem Size, } n\]

\[10^{-2}\]

\[10^{-1}\]

\[10^{0}\]

\[10^{1}\]

\[10^{2}\]

\[10^{3}\]
Hence, when \( j, B \subset A \), we get

\[
\langle \mathbf{T}_i, \mathbf{K}_j \rangle = 2 \left[ \begin{array}{c} \cos(\Delta \theta_j) \\ \sin(\Delta \theta_j) \end{array} \right]^T \left[ \begin{array}{c} \hat{z}_i \\ \hat{z}_{n+i} \end{array} \right] - 2 \left[ \begin{array}{c} \cos(\theta_j) \\ \sin(\theta_j) \end{array} \right]^T \left[ \begin{array}{c} \lambda_i \\ \mu_i \end{array} \right] \left[ \begin{array}{c} \mu_i \\ \lambda_{n+i} \end{array} \right] \left[ \begin{array}{c} y_i^* \\ y_{n+i}^* \end{array} \right]
\]

In particular, when \( j = u_i \), the above becomes

\[
\langle \mathbf{T}_i, \mathbf{K}_{u_i} \rangle = 2 \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]^T \left[ \begin{array}{c} \hat{z}_i \\ \hat{z}_{n+i} \end{array} \right] - \left[ \begin{array}{c} y_i^* \\ y_{n+i}^* \end{array} \right]^T \left[ \begin{array}{c} \lambda_i \\ \mu_i \end{array} \right] \left[ \begin{array}{c} \mu_i \\ \lambda_{n+i} \end{array} \right] \left[ \begin{array}{c} y_i^* \\ y_{n+i}^* \end{array} \right].
\]

Hence, when \( j \neq u_i \), (3.6) is equivalent to

\[
2 \left[ \begin{array}{c} 1 - \cos(\Delta \theta_j) \\ -\sin(\Delta \theta_j) \end{array} \right]^T \left[ \begin{array}{c} \hat{z}_i \\ \hat{z}_{n+i} \end{array} \right] \geq \left[ \begin{array}{c} y_i^* - \cos(\theta_j) \\ y_{n+i}^* - \sin(\theta_j) \end{array} \right]^T \left[ \begin{array}{c} \lambda_i \\ \mu_i \end{array} \right] \left[ \begin{array}{c} \mu_i \\ \lambda_{n+i} \end{array} \right] \left[ \begin{array}{c} y_i^* - \cos(\theta_j) \\ y_{n+i}^* - \sin(\theta_j) \end{array} \right]
\]

or

\[
\left[ \begin{array}{c} -\cos \left( \frac{\Delta \theta_j}{2} \right) \\ -\cos \left( \frac{\Delta \theta_j}{2} \right) \end{array} \right]^T \left[ \begin{array}{c} \hat{z}_i \\ \hat{z}_{n+i} \end{array} \right] \geq \left[ \begin{array}{c} \sin(\theta_{u_i} + \frac{\Delta \theta_j}{2}) \\ \cos(\theta_{u_i} + \frac{\Delta \theta_j}{2}) \end{array} \right]^T \left[ \begin{array}{c} \lambda_i \\ \mu_i \end{array} \right] \left[ \begin{array}{c} \mu_i \\ \lambda_{n+i} \end{array} \right] \left[ \begin{array}{c} \sin(\theta_{u_i} + \frac{\Delta \theta_j}{2}) \\ \cos(\theta_{u_i} + \frac{\Delta \theta_j}{2}) \end{array} \right],
\]

which is exactly the same as (3.8).

Appendix B. Proof of Lemma 4.4. To simplify the notations, we use \( A \) and \( B \) to denote the left-hand side and the right-hand side in (4.24), respectively.

We first prove that \( A \supset B \). Let \( C = \{ B \in S^q_+ : \text{diag}(B) = 1_q \} \), and we can view \( A \) as the image of the convex set \( C \) under the affine mapping \( B \mapsto \langle B, w w^T \rangle \). Therefore, the set \( A \) is also convex. Moreover, note that both the rank-one matrices \( 1_q 1_q^T \) and \( [-1_{q-1}] [-1_{q-1}]^T \) belong to \( C \). Direct computations show that

\[
w^T 1_q 1_q^T w = \left( \sum_{i=1}^q 2^{i-1} \right)^2 = (2^q - 1)^2,
\]

\[
w^T \left[ -1_{q-1} \right] \left[ -1_{q-1} \right]^T = \left( 2^q - 1 - \sum_{i=1}^{q-1} 2^{i-1} \right)^2 = 1,
\]

and hence both 1 and \( (2^q - 1)^2 \) belong to \( A \). Finally, the convexity of \( A \) implies \( B \subset A \).

Now we prove the other direction, i.e., \( A \subset B \). This is equivalent to showing

\[
1 \leq w^T B w \leq (2^q - 1)^2, \quad \forall \ B \in C.
\]

For the upper bound, we first note that \( B \in S^q_+ \) implies

\[
(B.1) \quad |B_{i,j}| \leq \sqrt{B_{i,i} B_{j,j}} = 1, \quad i \neq j, \quad 1 \leq i, j \leq q.
\]

Since every entry of the matrix \( w w^T \) is positive, we have

\[
w^T B w = \langle B, w w^T \rangle \leq \langle 11^T, w w^T \rangle = (2^q - 1)^2
\]

for any \( B \in C \), and hence the upper bound holds.
For the lower bound, it clearly holds when \( q = 1 \). When \( q > 1 \), for any matrix \( B \in \mathcal{C} \) we partition it as

\[
B = \begin{bmatrix}
B' & b' \\
(b')^T & 1
\end{bmatrix},
\]

where \( B' \in \mathbb{R}^{(q-1) \times (q-1)} \) and \( b' \in \mathbb{R}^{q-1} \). Note that we have \( |b'_i| \leq 1 \) for \( 1 \leq i \leq q-1 \) (cf. (B.1)), and \( B \in \mathbb{S}_+^q \) implies \( B' \preceq b'(b')^T \). Further, we let \( w' = [1 \ 2 \ 4 \ \ldots \ 2^{q-2}]^T \in \mathbb{R}^{q-1} \) such that \( w = [(w')^T \ 2^{q-1}]^T \) (cf. (4.14)). We have

\[
\begin{align*}
(w')^TBw &= (w')^TBw' + 2^q(w')^Tb' + (2^q-1)^2 \\
&\geq (w')^Tb'(b')^Tw' + 2^q(w')^Tb' + (2^q-1)^2 \\
&= ((w')^Tb' + 2^{q-1})^2.
\end{align*}
\]

Since

\[
(w')^Tb' = \sum_{i=1}^{q-1} 2^{i-1}b'_i \geq -\sum_{i=1}^{q-1} 2^{i-1} = -2^{q-1} + 1,
\]

we immediately get \( w^TBw \geq (-2^{q-1} + 1 + 2^{q-1})^2 = 1 \), and hence the lower bound also holds.

The proof is now complete.

**Acknowledgments.** The authors would like to thank Professors Zi Xu and Cheng Lu for their useful discussions on an earlier version of this paper.

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