LONG TIME EXISTENCE OF SOLUTION FOR THE BOSONIC MEMBRANE IN THE LIGHT CONE GAUGE

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Abstract. This paper mainly aims to establish the well-posedness on time interval $[0, \varepsilon^{-1/2}T]$ of the classical initial problem for the bosonic membrane in the light cone gauge. Here $\varepsilon$ is the small parameter measures the nonlinear effects. In geometric, the bosonic membrane are timelike submanifolds with vanishing mean curvature. Since the initial Riemannian metric may be degenerate or non-degenerate, the corresponding equation can be reduce to a quasi-linear degenerate or non-degenerate hyperbolic system of second order with an area preserving constraint via a Hamiltonian reduction. Our proof is based on a new Nash-Moser iteration scheme.

1. Introduction

Let $M$ be a compact 2-dimensional manifold, $\mathbb{R}^d$ be a $d$-dimensional Minkowski space and $\Sigma = \mathbb{R} \times M$ be a 3-dimensional submanifolds of Minkowski space $\mathbb{R}^d$. $u = (u^\mu)$ is an embedding from $\Sigma$ to $\mathbb{R}^d$. The critical points of the Nambu-Goto action

$$ S = -\int_\Sigma \mu_g $$

give rise to submanifolds $\Sigma \subset \mathbb{R}^d$ with vanishing mean curvature, where $\mu_g$ is a volume form is induced by a metric $g$ on $\Sigma$. The Euler-Lagrange equations for functional $S$ is

$$ \sqrt{|g|} \Box_g u^\mu = 0, \quad \mu = 0, \ldots, d - 1. $$

If $\sqrt{|g|} \neq 0$, above system is equivalent to

$$ (\delta_{\mu\nu} - g^{CD} \partial_C u^\mu \partial_D u^\nu) g^{AB} \partial_A \partial_B u^\nu = 0, $$

where $\mu, \nu, \ldots = 0, \ldots, d - 1$ and $A, B, C, D, \ldots = 0, 1, 2$ refer to coordinates on Minkowski space and $\Sigma$, respectively. In local coordinates, $g_{AB}$ is a Lorentzian metric on $\Sigma$ with $\partial_t$ a timelike direction, which is expressed by $g_{AB} = \eta_{\mu\nu} \partial_A u^\mu \partial_B u^\nu$, $\partial_A u^\mu = \partial_{\zeta^A} u^\mu$. Here $\eta_{\mu\nu}$ is the Minkowski metric, which has the form $\eta_{\mu\nu} du^\mu du^\nu = -(du^0)^2 + (du^1)^2 + \ldots + (du^{d-1})^2$ in Cartesian coordinates $(u^\mu, (\zeta^A) = (t, x^a)$ is a coordinates on $\Sigma$, $t$ is some global coordinate whose level sets $M_t$ foliate $\Sigma$, and $(x^a)$ is local coordinates on each $M$ with $a = 1, 2$.

The membrane system (or relativistic strings, see [1]) arises in the context of membrane, supermembrane theories and higher-dimensional extensions of string.
theory. In Lorentzian geometrically, since the critical point of the Nambu-Goto action gives rise to submanifolds with vanishing mean curvature, they are also called timelike minimal surface equations. They are one case of an important class of geometric evolution equations which is the Lorentzian analogue of the minimal submanifold equations. Since such equations possess plenty of geometric phenomenon and complicated structure (for example, they develop singularities in finite time, and degenerate and not strictly hyperbolic properties), much work is attracted in recent years. Lindblad [16] and Brendle [4] proved that the local and global well-posedness of timelike minimal surface equation with sufficiently small initial data in high dimension, respectively. The case of general codimension and local well-posedness in the light cone gauge were studied by Allen, Andersson and Isenberg [1] and Allen, Andersson and Restuccia [2], respectively. Kong and his collaborators [13] obtained a representation formula of solution and presented many numerical evidence where singularity formation is prominent. Bellettini, Hoppe, Novaga and Orlandi [3] showed that if the initial curve is a centrally symmetric convex curve and the initial velocity is zero, the string shrinks to a point in finite time. They noticed that it should be noted that the string does not become extinct there, but rather comes out of the singularity point, evolves back to its original shape and then periodically afterwards. Most recently, Nguyen and Tian [17] showed that timelike maximal cylinders in $\mathbb{R}^{1+2}$ always develop singularities in finite time and that, infinitesimally at a generic singularity, their time slices are evolved by a rigid motion or a self-similar motion. They also proved a mild generalization in non-flat backgrounds.

Before giving the motion equation of the canonical reduction of action (1.1) in the light cone gauge, we briefly recall the gauge fixing procedure, one can see [2] for more details. For gauge theories, one can see Dirac [8] for more details. We use the null coordinates $(u^+, u^-, u^m)$ and the volume form $\sqrt{w}$ on $M$ to specify the light cone gauge. More precisely, $u^+$ and the corresponding conjugate momenta $p_+$ are

$$u^+ = -p_0^t, \quad p_+ = \frac{1}{2} (c\sqrt{w})^{-\frac{1}{2}} (p_mp^m + \gamma),$$

$u^-$ and the corresponding conjugate momenta $p_-$ satisfy

$$\partial_a u^- = - (c\sqrt{w})^{-\frac{1}{2}} p_m \partial_a u^m, \quad p_- = p_0^\perp \sqrt{w},$$

where $p_0^\perp$ is a constant, $p_m = \frac{1}{2} \left( \partial_t u_\mu - N^a \partial_a u_\mu \right)$, $N^a = -g^{00} g^{0a}$, the metric $g^{AB}$ can be determined by performing the usual ADM decomposition of $g$. In order to eliminate the conjugate pairs $(x^-, p_-)$ and $(x^+, p_+)$, we provide the integrability condition

$$\int_C p_m \sqrt{w} dx^m = 0, \text{ for all closed curves } C \text{ in } \Sigma.$$ 

Furthermore, the light cone action can be obtained

$$S = \int_{\Sigma} \left( p_m \partial_t u^m - cp_+ + \partial_t [p_- x^-] \right),$$

the corresponding reduced Hamiltonian is

$$H = \frac{\sqrt{w}}{4} \left( \frac{2 p_mp^m}{\sqrt{w}} + \{u^m, u^n\} \{u_m, u_n\} \right) + p_m \{\Lambda, u^m\},$$
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where Λ is determined by \( N^a, \{ \} \) denotes the Poisson bracket associated to a symplectic structure on \( M \) in local coordinates.

Thus the equations of motion associated to the reduced Hamiltonian is the following second order system in light cone gauge (see also [11])

\[
\partial_{tt} u^m = \{ \{ u^m, u^n \}, u_n \},
\]

with the initial data

\[
u(0) = u_0, \ \partial_t u(0) = u_1,
\]

and the constraints

\[
\{ \partial_t u^m, u_m \} = 0.
\]

Direct computation shows that the right hand side of (1.2) can be written in local coordinates as

\[
\{ \{ u^m, u^n \}, u_n \} = w^{-1}(\epsilon^{ac} \epsilon^{bd} \gamma(u)_{cd} \delta_{mn} - \epsilon^{ac} \epsilon^{bd} \partial_e u_m \partial_d u_n \partial_\ell \partial_\eta u^n) \partial_\ell \partial_\eta u^n + LOT,
\]

where \( \gamma(u)_{cd} = \partial_t u^m \partial_d u_m \) is the metric, \( \epsilon^{ac} \) is the anti-symmetric symbol with two indices and LOT denotes the lower order terms. It is obviously that even the metric \( \gamma(u)_{cd} \) is Riemannian, the first term in the right hand side of above inequality can cause the symbol to be degenerate. So the hyperbolic property of equation (1.2) can not hold. As done in [2], we modify the equation (1.2) by differentiating it with respect to \( t \), then using the Jacobi identity and constraints (1.4) to obtain

\[
\partial_t \partial_{tt} u^m = \{ \{ \partial_t u^m, u^n \}, u_n \} + 2\{ \{ u^m, u^n \}, \partial_t u_n \}.
\]

Let

\[
\partial_t u^m = v^m,
\]

system (1.5) can be written in local coordinates as

\[
\partial_{tt} v^m - \partial_a (\epsilon^{ac} \epsilon^{bd} \gamma(u)_{cd} w^{-1} \partial_d v^m) = 2\partial_a (\epsilon^{ab} \epsilon^{cd} w^{-1} \partial_c u^m \partial_d u^n \partial_\eta v_n) - \epsilon^{ab} w^{-1} \partial_a (\epsilon^{cd} w^{-1} \partial_d \gamma u^n + 2\partial_a u^n \partial_b u^n \partial_\eta v_n) = 0,
\]

where \( \gamma(u)_{ab} = \epsilon^{ac} \epsilon^{bd} \gamma(u)_{cd} \), \( a, b, c, d, \ldots = 1, 2, m, n, \ldots = 1, \ldots, d - 2 \).

The corresponding initial data is

\[
u^m(0) = u^m_0, \ v^m(0) = v^m_0, \ \partial_t v^m(0) = v^m_1.
\]

We need the following condition of initial data which make that the solution \( (u^m, v^m) \) of the modified system (1.7) is also the solution of equation (1.2)

\[
\{ v^m_0, u^m_0 \} \delta_{mn} = 0, \quad v^m_1 - \{ \{ u^m_0, u^m_0 \}, u^m_0 \} \delta_{nt} = 0.
\]

When initial Riemannian metric \( \gamma(u_0)_{cd} \) is non-degenerate, second order system (1.6)-(1.7) is a strictly hyperbolic system. Allen, Anderson and Restuccia [2] showed that system (1.6)-(1.7) has a unique solution \( (u^m, v^m) \in C^1([0, T] ; H^k) \times C^k_T \) for \( k \geq 4 \). Moreover, they got a blow up criterion.

We denote by \( \partial \) any time derivative \( \partial_t \) or any coordinate derivative \( D \). One of goals in this paper is to prove the following result:
Theorem 1.1. Assume that the initial Riemannian metric $\gamma(u_0)_{ab}$ is non-degenerate, and satisfies

$$\gamma'_1|\xi|^2 \leq \gamma(u_0)_{ab}\xi_a\xi_b \leq \gamma'_2|\xi|^2, \forall \xi \in M,$$

$$\gamma'_3|\xi|^2 \leq \partial\gamma(u_0)_{ab}\xi_a\xi_b \leq \gamma'_4|\xi|^2, \forall \xi \in M,$$

where $\gamma'_1$, $\gamma'_2$, $\gamma'_3$ and $\gamma'_4$ denote positive constant.

Let $k \geq 2$, $T > 0$ and $(u_0, v_0, v_1) \in H^k \times H^k \times H^{k-1}$. Then for sufficient small $\varepsilon > 0$, nonlinear second order system (1.1) with initial data (1.3) has a unique smooth solution

$$u^m(t, x) = u_0^m + \int_0^t v^m, (t, x) \in \left[0, \frac{T}{\sqrt{\varepsilon}}\right] \times M,$$

which satisfies the constraint equation (1.4), where $v^m$ is a unique smooth solution of nonlinear degenerate hyperbolic system (1.7) with initial data (1.8)-(1.9).

The study of influence for degenerate metrics in nonlinear field equation is a very interesting problem, which can cause a degenerate nonlinear PDE. Two of famous degenerate metrics are Schwarzschild metric and Kerr metric, which are two special solutions of nonlinear vacuum Einstein field equation. The corresponding stability problem is still an open problem (see [5] for more detail). Another goal of this paper is to deal with that initial Riemannian metric $\gamma(u_0)_{cd}$ is degenerate in LCG field equations, then system (1.10)-(1.11) is a degenerate hyperbolic system. More precisely, we need to analyse the following quasi-linear toy model

$$v_{tt} - \partial_a(\gamma(t, x)\partial_b v) = g(t, x, \partial_a v, \partial_b v, \int_0^t \partial_a v),$$

where $(t, x) \in [0, \frac{T}{\sqrt{\varepsilon}}] \times M$, $\gamma(t, x)$ can be vanish, $f$ is a smooth bounded function and $g$ is a nonlinear term satisfies certain bounded conditions. The Cauchy problem of degenerate nonlinear wave equation is a very interesting problem in hyperbolic differential equations. Colombini and Spagnolo [5] showed that the following Cauchy problem of one dimensional degenerate wave equation is not well posedness

$$u_{tt} - a(t)u_{xx} = 0,$$

where $(t, x) \in [0, T] \times \mathbb{R}$, $a(t)$ is a nonnegative smooth function and oscillates an infinite number of times. On the other hand, Colombini, De Giorgi and Spagnolo [5] showed that any Cauchy problem as (1.11) is well-posed in the space of the periodic real analytic functionals. So it is interesting problem that what conditions of leading coefficients can make the degenerate nonlinear wave equation (even linear degenerate wave equation) being well-posed. We refer the paper of Han et.al. [9, 10] for coefficients analysis conditions. In order to study the long time existence of a kind of (1.10), we require that the lower order coefficients satisfies certain growth conditions, i.e. Levi conditions. For system (1.7), we assume that

$$\gamma(u_0)_{cd} = \gamma_0(x)\gamma(x)_{cd},$$

where $\gamma_0(x)$ is a $C^k$-function ($k \geq 2$), which satisfies

$$0 \leq \gamma_0(x) \leq 1, \quad |\partial_a \gamma_0(x)| \leq c_0 \gamma_0(x),$$

the Riemannian metric $\gamma_{cd}(x)$ is non-degenerate and satisfies

$$|\xi|^2 \leq \gamma(x)_{cd}\xi_d \leq \gamma_2|\xi|^2, \quad |\partial\gamma(x)_{cd}\xi_d| \leq \gamma_4|\xi|^2, \forall \xi \in M,$$
for positive constants $\gamma_1, \gamma_2, \gamma_3, \gamma_4, c_0$ and $c_1$.

Conditions (1.13) and (1.14) give the Levi condition

\[
\left| \partial_n (\gamma(u_{0,cd}) \xi_l \xi_d) \right| \leq c_2 \gamma_0(x) |\xi|^2, \quad \forall \xi \in M,
\]

where $c_2$ denotes a positive constant.

Even if the initial data make the Riemannian metric $\gamma_{cd}$ degenerate, we can prove that the well-posedness on time interval $[0, \varepsilon^{-\frac{1}{2}} T]$ is approached. More precisely, we have the following theorem.

**Theorem 1.2.** Assume that (1.13)-(1.14) holds. Let $k \geq 2, T > 0$ and $(u_0, v_0, v_1) \in H^k_1 \times H^k_1 \times H^{k-1}$. Then for sufficient small $\varepsilon > 0$, nonlinear second order system (1.2) with initial data (1.3) has a unique smooth solution

\[
u^m(t, x) = u^m_0 + \int_0^t v^m, \quad (t, x) \in \left[0, \frac{T}{\varepsilon}\right] \times M,
\]

which satisfies the constraint equation (1.4), where $v^m$ is a unique smooth solution of nonlinear degenerate hyperbolic system (1.7) with initial data (1.8)-(1.9).

We will construct a suitable Nash-Moser iteration scheme to prove Theorem 1.1-1.2. Since the proof process is similar, we only give the proof details on the scheme (For general Nash-Moser implicit function theorem, one can see [12, 14, 15]).

The organization of this paper is as follows. In Section 2, we establish the existence of linear degenerate wave equation that will arise in the linearization of equation (1.7). Section 3 is devoted to solving the initial value problem of the bosonic membrane equation (1.2) by means of a new Nash-Moser iteration scheme. Section 4 is devoted to solving the initial value problem (2.5), which is a degenerate, we can prove

\[\text{having this solution is a unique solution in } B^s_{R,T}, \text{ see } (2.6).\]

2. **Energy estimates of the linearized equation**

This section is to discuss the linearized equation (2.3), which is a degenerate linear hyperbolic equation. Throughout this section, we denote points in $\Sigma$ by $(t, x)$ with $t \in \mathbb{R}$ and $x \in M$. For $d$-dimensional compact Riemannian manifold $M$ and smooth metric $g$, the tangent space of $M$ is denoted by $T_x M$. $\Gamma^\infty(T^i M)$ is the set of all smooth tensor fields of type $(0, l), \forall l \in [1, \infty]$. For each $x \in M$, $T_x M$ is an inner product space defined as follows. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T_x M$. For any $\psi_1, \psi_2 \in T_x^* M$, the inner product is given by

\[
(\psi_1, \psi_2)_{T_x^* M} = \sum_{i_1, i_2, \ldots, i_l = 1}^n \psi_1(e_{i_1}, \ldots, e_{i_l}) \psi_2(e_{i_1}, \ldots, e_{i_l}).
\]

In view of above inner product, $\Gamma^\infty(T^i M)$ are inner product spaces endowed with

\[
(\psi_1, \psi_2)_{\Gamma^\infty(T^i M)} = \int_M (\psi_1, \psi_2)_{T_x^* M} \sqrt{\gamma}, \quad \forall \psi_1, \psi_2 \in \Gamma^\infty(T^i M).
\]

We denote $L^2(M, \Gamma^\infty(T^i M))$ the completions of $\Gamma^T M$ in above inner product. The Sobolev space $H^l(M)$ is the completion of $C^\infty(M)$ with respect to the norm

\[
\|u\|_{H^l}^2 = \|u\|_{L^2}^2 + \sum_{i=1}^{l} \|D^i u\|_{L^2(M, \Gamma^\infty(T^i M))}^2.
\]
In this paper case, we consider the Sobolev space and function space of functions on $M$ in a fixed coordinate. For any $l \in [1, \infty]$, $\mathbf{H}^l$ denotes the Sobolev space of functions on $M$ whose derivatives of up to order $l$ are square integrable. The corresponding norm is

$$
\|u\|_{L^2}^2 = \int_M |u|^2 \sqrt{w}, \quad \|u\|_{H^l}^2 = \|u\|_{L^2}^2 + \int_M |Du|^2 \sqrt{w},
$$

where $Du$ denote any coordinate derivative $\partial_u u$.

We denote the spatially weighted Lebesgue spaces by $\mathbf{L}^2(M)$, which is equipped with the norm

$$(2.1) \quad \|u\|_{\mathbf{L}^2} = \int_M \left(1 + |x|^2\right)^{\frac{1}{2}} |u|^2 \sqrt{w}.$$ 

Then the Fourier transform is an isomorphism between $\mathbf{L}^2$ and $\mathbf{H}^l$.

Next we introduce some other spaces which is used in this paper. $\mathbf{L}^\infty$ spaces and $\mathbf{W}^{1,\infty}$ spaces equip with the norms

$$
\|u\|_{\mathbf{L}^\infty} = \text{ess sup}_{M} |u|, \quad \|u\|_{\mathbf{W}^{1,\infty}} = \|u\|_{\mathbf{L}^\infty} + \|Du\|_{\mathbf{L}^\infty},
$$

respectively.

$\mathbf{C}^r([0,T]; \mathbf{H}^l)$ denotes the function spaces with the norm

$$
\|u\|_{\mathbf{C}^r([0,T]; \mathbf{H}^l)} = \sup_{[0,T]} \sum_{i=0}^r \|\partial^i u\|_{\mathbf{H}^l}.
$$

$\mathbf{C}^2_T := \cap_{i=0}^2 \mathbf{C}^r([0,T]; \mathbf{H}^{l-i})$ denotes function spaces with the spacetime norms

$$
\|u\|_T^2 = \sum_{i=0}^2 \|\partial^i u\|_{\mathbf{H}^{l-i}}, \quad |||u|||_{T,T} = \sup_{[0,T]} \|u(\cdot)\|_T.
$$

Generic constants are denoted by $c_0, c_1, \ldots$, their values may vary in the same formula or in the same line.

We denote

$$
\mathcal{B}(u)^a \partial_a v^m = 2 \partial_a (e^{ac} e^{cd} w^{-1} \partial_c u^m \partial_d u^n \partial_b v_n) - e^{ac} w^{-\frac{1}{2}} \partial_d (e^{cd} w^{-\frac{1}{2}}) (\gamma_{bd} \partial_d u^m + 2 \partial_a u^m \partial_b u^n \partial_d v_n).
$$

Then equation (1.7) can be rewritten as

$$(2.2) \quad \partial_t v^m - \partial_a (e^{ac} e^{bd} \gamma(u)_{cd} \partial_b v^m) + \mathcal{B}(u)^a \partial_a v^m = 0.$$ 

It follows from (1.6) that

$$(2.3) \quad \gamma(u)_{cd} = \gamma(u_0)_{cd} + \partial_c u_0^a \int_0^t \partial_d v_n + \partial_d u_{0n} \int_0^t \partial_c v^m + \int_0^t \int_0^v \gamma(v)_{cd},$$

which implies that system (2.2) is equivalent to

$$
\begin{align*}
\partial_t v^m & - \partial_a (e^{ac} e^{bd} \gamma(u_0)_{cd} \partial_b v^m) - \partial_a (e^{ac} e^{bd} \partial_d v_n) (\partial_c v^m) \\
& - \partial_a (e^{ac} e^{bd} \partial_d u_{0n} (\partial_c v^m) \partial_b v^m) - \partial_a (e^{ac} e^{bd} \gamma v^m) \\
& + \mathcal{B}(u)^a \partial_a v^m = 0.
\end{align*}
$$

(2.4)

Since we assume that $\gamma(u_0)_{cd}$ is a degenerate Riemannian metric, system (2.4) is a degenerate quasi-linear wave equation. Linearizing quasi-linear degenerate wave
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The equation around \( v \) leads to the following linearized equation with an external force

\[
\partial_t h^m - \partial_a (\epsilon^{ac} \epsilon^{bd} \gamma(u_0)_{cd} \partial_b h^m) - \varepsilon \partial_a (\epsilon^{ac} \epsilon^{bd} \partial_c u^0_n (\int_0^t \partial_d v_n) \partial_b h^m) \\
+ \varepsilon \partial_a (\epsilon^{ac} \epsilon^{bd} \partial_c u^0_n (\int_0^t \partial_d h_n) \partial_b v^m) - \varepsilon \partial_a (\epsilon^{ac} \epsilon^{bd} (\int_0^t \partial_c h^n) \partial_d v^m) - \varepsilon \partial_a (\epsilon^{ac} \epsilon^{bd} (\int_0^t \partial_c v^m) \partial_d h_n) \partial_b v^m)
\]

(2.5) \( + \varepsilon B(u)^a \partial_a h^m = g(t, x) \),

where

\[
B(u)^a \partial_a h^m = 2 \partial_a (\epsilon^{ab} \epsilon^{cd} w^{-1} (\int_0^t \partial_c h^m) (\int_0^t \partial_d v_n) \partial_b v_n) \\
+ 2 \partial_a (\epsilon^{ab} \epsilon^{cd} w^{-1} (\int_0^t \partial_c v^m) (\int_0^t \partial_d h_n) \partial_b v_n) \\
+ 2 \partial_a (\epsilon^{ab} \epsilon^{cd} w^{-1} (\int_0^t \partial_c v^m) (\int_0^t \partial_d h_n) \partial_b h_n) \\
- \epsilon^{ab} w^{-\frac{1}{2}} \partial_c (\epsilon^{cd} w^{-\frac{1}{2}}) (\gamma_{bd} \partial_a h^m + (\int_0^t \partial_b h^n) (\int_0^t \partial_d v_n) \partial_a v^m) \\
- \epsilon^{ab} w^{-\frac{1}{2}} \partial_c (\epsilon^{cd} w^{-\frac{1}{2}}) (\int_0^t \partial_b h^n) (\int_0^t \partial_d h_n) \partial_a v^m + 2 (\int_0^t \partial_b v_n) (\int_0^t \partial_a h^m) \partial_d v_n
\]

This following theorem is the main result in this section, which states the existence of linearized equation with an external force \( g(t, x) \) of the nonlinear bosonic membrane equation (2.2). Here the external force \( g(t, x) \) denotes the error term which is produced by carrying out the Nash-Moser iteration scheme in next section.

For \( T > 0, k \geq 2 \) and \( 0 < R < 1 \), we define

\[
B^k_{R,T} := \{ u \in C^k_T : |||u|||_{k,T} \leq R < 1 \}.
\]

**Theorem 2.1.** Let \( k \geq 2 \). Assume that (1.12)-(1.14) and \( v \in B^k_{R,T} \) hold. Then for any initial data \((h_0, h_1) \in H^{k-2} \times H^{k-2}\), there exists an \( H^{k-2} \) solution \( h(t, x) \) of (2.2) on \([0,T] \times M\). Moreover, for any \( 1 \leq s \leq k-2 \) and sufficient small \( \varepsilon > 0 \), there holds

\[
|||h|||_{s,T} \leq c_3 (|||h_0|||_{s,T} + |||h_1|||_{s,T} + |||g|||_{s,T}).
\]

Before proving Theorem 2.1, i.e. the local existence of solution for the linearized system of (2.5), we first consider a toy model

\[
\partial_t h - g(x) \partial_a (\rho_{ab}(t, x) \partial_b h) + B(t, x) \partial_a h
\]

(2.8) \( -\varepsilon (Du, Dv) (\int_0^t \partial_a h = g(t, x), \)
with the initial data
\begin{equation}
  h(0) = h_0, \quad \partial_t h(0) = h_1,
\end{equation}
where $D$ denotes any coordinate derivative $\partial_a v$, $\rho_{ab}(t, x)$, $B(t, x)$ and $g(t, x)$ are $C^k$ for some integer $k \geq 2$.

Assume that $f$ can be controlled by a constant in the norm $\| \cdot \|_{s, T}$ for some $s \in [1, k - 1]$ when $(Du, Dv) \in B^{k}_{s, T}$. The coefficient $\rho_{ab}(t, x)$, $B(t, x)$ satisfying
\begin{align}
  \rho_0 |\xi|^2 &\leq \rho_{ab}\xi_a \xi_b \leq \rho_1 |\xi|^2, \quad \rho_2 |\xi|^2 \leq \partial \rho_{ab}\xi_a \xi_b \leq \rho_3 |\xi|^2, \quad \forall \xi \in M, \\
  0 \leq g(x) &\leq 1, \quad x \in M,
\end{align}
and the Levi condition
\begin{equation}
  \|(B - \rho_{a}\rho_{b})\xi_a \xi_b\| \leq c_5 g |\xi|^2, \quad \xi \in M,
\end{equation}
for some positive constants $c_0$ and $c_2 \leq p_3$.

The result concerning the existence of equation (2.8) is the following theorem.

**Theorem 2.2.** Assume that $k \geq 2$ and (2.10)-(2.12) holds. Then for any initial data $(h_0, h_1) \in H^{k-1} \times H^{k-2}$, there exists an $H^{k-1}$ solution $h(t, x)$ of (2.8)-(2.9) on $[0, T] \times M$. Moreover, for any $1 \leq s \leq k - 1$ and small $\varepsilon > 0$, there holds
\begin{equation}
  \|h\|_{s, T} \leq c_7 (\|h_0\|_{s, T} + \|h_1\|_{s, T} + \|g\|_{s, T}).
\end{equation}

Before giving the proof of the above theorem, we need to carry out some priori estimates on the solution $h$ of equation (2.8).

**Lemma 2.3.** Let $h$ be a $H^{2}$-solution to (2.8) with initial data (2.7). Assume that (2.10)-(2.12) holds. Then there exists a nonnegative $C^1$ function $\varphi$ in $[0, T] \times M$ satisfying
\begin{equation}
  \frac{\partial_t \varphi}{\varphi} \leq c_8, \quad \frac{|D\varphi|}{\varphi} \rho \leq c_9.
\end{equation}

For any $\lambda > c_{10}$, there holds
\begin{equation}
  (\lambda - c_{10}) \int_{[0, T] \times M} e^{-\lambda t} \varphi (h_t^2 + g \partial_a \h \partial_b h + h^2) \sqrt{w} \leq \int_{\{0\} \times M} e^{-\lambda t} \varphi (h_0^2 + g \partial_a \h \partial_b h + h^2) \sqrt{w}
\end{equation}
\begin{equation}
  + \int_{[0, T] \times M} e^{-\lambda t} \varphi (\varepsilon f^2 (\int_0^t \partial_a h)^2 + g^2) \sqrt{w}.
\end{equation}

**Proof.** Taking the inner product of the linearized equation (2.8) with $2e^{-\lambda t} \varphi h_t$, we have
\begin{equation}
  \partial_t [e^{-\lambda t} \varphi (h_t^2 + g \partial_a \h \partial_b h)] + \lambda e^{-\lambda t} \varphi (h_t^2 + g \partial_a \h \partial_b h) - 2e^{-\lambda t} \partial_a \varphi (g \rho_{ab} \varphi h_t \partial_b h)
\end{equation}
\begin{equation}
  = e^{-\lambda t} \varphi h_t^2 + e^{-\lambda t} \varphi g \rho_{ab} \partial_a \h \partial_b h + e^{-\lambda t} \varphi g \partial_a \rho_{ab} \partial_a \h \partial_b h
\end{equation}
\begin{equation}
  -2e^{-\lambda t} \partial_a \varphi g \rho_{ab} \partial_a \h \partial_b h - 2e^{-\lambda t} \varphi (B - g \partial_a \rho_{ab} \partial_b \h) h_t
\end{equation}
\begin{equation}
  + 2e^{-\lambda t} \varphi h_t f \int_0^t \partial_a h + 2e^{-\lambda t} \varphi h_t g.
\end{equation}

Note that
\begin{equation}
  \partial_t (e^{-\lambda t} \varphi h^2) + \lambda e^{-\lambda t} \varphi h^2 = \varphi t e^{-\lambda t} h^2 + 2e^{-\lambda t} \varphi h h_t.
\end{equation}
Summing up (2.16)-(2.17), we get
\begin{equation}
  \partial_t [e^{-\lambda t} \varphi (h_t^2 + g \partial_a \h \partial_b h + h^2)] + \lambda e^{-\lambda t} \varphi (h_t^2 + g \partial_a \h \partial_b h + h^2) - 2e^{-\lambda t} \partial_a (g \rho_{ab} \varphi h_t \partial_b h)
\end{equation}
Then we have the following result by taking (2.20)
\[ \forall \Box \]
\[ \square \]
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Lemma 2.3, we need to assume that
Thus we obtain
\[ \int^{t} \partial_{a} h \]
\[ +2 \varepsilon e^{-\lambda t} \phi_{t} g. \]

Using Cauchy inequality, by (2.10), (2.12), (2.14) and (2.18), we derive
\[ \partial_{t}[e^{-\lambda t} \varphi(h_{t}^{2} + \varrho_{pa} \partial_{a} h \partial_{b} h + h^{2})] + \lambda e^{-\lambda t} \varphi(h_{t}^{2} + \varrho_{pa} \partial_{a} h \partial_{b} h + h^{2}) - 2e^{-\lambda t} \partial_{a}(\varrho_{pa} \phi_{t} \partial_{b} h) \]
\[ \leq \left(2 \frac{\phi_{t}}{\phi} \right) \varrho_{pab} + \frac{|B - \varrho_{pa} \rho_{ab}| + 2)e^{-\lambda t} \phi_{t}^{2}}{\phi} \varrho_{pab} + \frac{|B - \varrho_{pa} \rho_{ab}|}{\varrho} e^{-\lambda t} \phi_{b}^{2} \]
\[ +2 \lambda e^{-\lambda t} \phi_{b}^{2} + e^{-\lambda t} \phi_{f}^{2} \left( \int_{0}^{t} \partial_{b} h \right)^{2} + e^{-\lambda t} \phi_{g}^{2} \]
\[ \leq c_{10} e^{-\lambda t} \phi(h_{t}^{2} + \varrho_{pa} \partial_{a} h \partial_{b} h + h^{2}) \]
\[ + \varepsilon e^{-\lambda t} \phi_{f}^{2} \left( \int_{0}^{t} \partial_{a} h \right)^{2} + e^{-\lambda t} \phi_{g}^{2}. \]

Choosing \( \lambda > c_{10} \), then inequality (2.19) leads to
\[ \partial_{t}[e^{-\lambda t} \varphi(h_{t}^{2} + \varrho_{pa} \partial_{a} h \partial_{b} h + h^{2})] + \lambda e^{-\lambda t} \varphi(h_{t}^{2} + \varrho_{pa} \partial_{a} h \partial_{b} h + h^{2}) - 2e^{-\lambda t} \partial_{a}(\varrho_{pa} \phi_{t} \partial_{b} h) \]
\[ \leq \varepsilon e^{-\lambda t} \phi_{f}^{2} \left( \int_{0}^{t} \partial_{a} h \right)^{2} + e^{-\lambda t} \phi_{g}^{2}. \]

Thus we obtain
\[ \int_{(T) \times M} e^{-\lambda t} \varphi(h_{t}^{2} + \varrho_{pa} \partial_{a} h \partial_{b} h + h^{2}) \sqrt{w} \]
\[ + \lambda e^{-\lambda t} \varphi(h_{t}^{2} + \varrho_{pa} \partial_{a} h \partial_{b} h + h^{2}) \sqrt{w} \]
\[ \leq \int_{(0,T) \times M} e^{-\lambda t} \varphi(h_{t}^{2} + \varrho_{pa} \partial_{a} h \partial_{b} h + h^{2}) \sqrt{w} \]
\[ + \int_{(0,T) \times M} e^{-\lambda t} \varphi(e^{f}(\int_{0}^{t} \partial_{a} h)^{2} + g^{2}) \sqrt{w}, \]

which combining with (2.10) gives (2.15). This completes the proof.

To derive classical energy estimates of solutions by eliminating the weight \( \varphi \) in
Lemma 2.3, we need to assume that
\[ 0 < \varphi(x) \leq 1, \quad x \in M. \]

Then we have the following result by taking \( \varphi(t,x) = \varphi(x)^{-1} \) in Lemma 2.3 for
\( \forall (t,x) \in [0,T] \times M. \)
Lemma 2.4. Let $h$ be a $H^2$-solution to (2.3) with initial data (2.7). Assume that (2.11), (2.12) and (2.20) holds. Then for any $\lambda > \lambda_1$, there holds

$$
(\lambda - c_{11}) \int_{[0,T] \times M} e^{-\lambda t} \left( g^{-1} h_t^2 + \partial_a h \partial_b h + g^{-1} h^2 \right) \sqrt{w} 
$$

$$
\leq \int_{\{0\} \times M} e^{-\lambda t} \left( g^{-1} h_t^2 + \partial_a h \partial_b h + g^{-1} h^2 \right) \sqrt{w} 
\quad + \int_{[0,T] \times M} e^{-\lambda t} \left( \varepsilon f^2 \left( \int_0^t \partial_a h \right)^2 + g^2 \right) \sqrt{w}.
$$

(2.21)

Next we eliminate $g^{-1}(x)$ in energy inequality (2.21).

Lemma 2.5. Let $h$ be a $H^2$-solution to (2.3) with initial data (2.7). Assume that (2.11), (2.12) and (2.20) holds. Then for any $\lambda > \lambda_0$, there holds

$$
\lambda \int_{[0,T] \times M} e^{-\lambda t} \left( h_t^2 + (Dh)^2 + h^2 \right) \sqrt{w} 
\leq c_{12} \int_{\{0\} \times M} e^{-\lambda t} \left( h_0^2 + h^2 + (Dh_0)^2 + (Dh)^2 \right) \sqrt{w} 
\quad + c_{12} \int_{[0,T] \times M} e^{-\lambda t} \left( |g|^2 + |Dg|^2 \right) \sqrt{w} 
\quad + c_{12} \varepsilon \int_{[0,T] \times M} e^{-\lambda t} \left( \int_0^t \partial_a h \right)^2 \sqrt{w},
$$

(2.22)

where $\lambda_0$ depends on $\rho_0, \rho_1$ and the $C^1$ norm of $B$.

Proof. we introduce an auxiliary function $\hat{h}$, which satisfies

$$
\partial_t \hat{h} - \partial_b \hat{h} - \hat{h} = g, \ x \in M,
\hat{h}(0, x) = h_0, \ \partial_t \hat{h}(0, x) = h_1.
$$

(2.23)

Let

$$
\hat{h} = h - \hat{h},
$$

then it follows from (2.8) and (2.23) that

$$
\partial_t \hat{h} - g(x) \partial_a (\rho_{ab}(t, x) \partial_b \hat{h}) + B(t, x) \partial_a \hat{h} - \varepsilon f \int_0^t \partial_a h = \hat{g}, \ x \in M,
$$

(2.24)

$$
\hat{h}(0, x) = 0, \ \partial_t \hat{h}(0, x) = 0, \ x \in M,
$$

(2.25)

where

$$
\hat{g} = g(x) \partial_a (\rho(t, x) \partial_b \hat{h}) + \partial_b \hat{h} + \hat{h} - B(t, x) \partial_b \hat{h}.
$$

(2.26)

Using the similar deriving process with (2.18) and taking $\varphi = g^{-1}$, by (2.24) we get

$$
\partial_t [e^{-\lambda t} \left( g^{-1} \hat{h}_t^2 + \rho_{ab} \partial_a \hat{h} \partial_b \hat{h} \right) + \left( g^{-1} \hat{h}_t^2 \right)] + \lambda e^{-\lambda t} \left( g^{-1} \hat{h}_t^2 + \rho_{ab} \partial_a \hat{h} \partial_b \hat{h} + g^{-1} \hat{h}^2 \right) 
\quad - 2e^{-\lambda t} \partial_a (\rho_{ab} \hat{h}_t \partial_b \hat{h}) 
\quad = -2e^{-\lambda t} \partial_a \varphi^{-1} \rho_{ab} \hat{h}_t \partial_b \hat{h} - 2e^{-\lambda t} (-g^{-1} B + \partial_a \rho_{ab}) \partial_b \hat{h}_t 
\quad + e^{-\lambda t} (\partial_t \rho_{ab}) \partial_a h \partial_b h - 2e^{-\lambda t} \varphi^{-1} \hat{h}_t f \int_0^t \partial_a h + 2e^{-\lambda t} g^{-1} \hat{h}_t \hat{g}.
$$

(2.27)
To avoid an extra loss of derivatives of integrating (2.27), by (2.20), we can rewrite the last term in (2.27) as
\[ e^{-\lambda t} g^{-1} h_t g = \partial_a (e^{-\lambda t} \rho_{ab} \partial_b h \partial_t h) - \partial_t (e^{-\lambda t} \rho_{ab} \partial_a h \partial_b h) + e^{-\lambda t} \rho_{ab} \partial_a h \partial_b h - \lambda e^{-\lambda t} \rho_{ab} \partial_a h \partial_b h \]
(2.28)\[+e^{-\lambda t} (\partial_t \rho_{ab}) \partial_a h \partial_b h + e^{-\lambda t} g^{-1} (\delta_t h_t + \delta_t \delta h) - e^{-\lambda t} g^{-1} \delta_t B \partial_a h. \]
Inserting (2.28) into (2.27), using Cauchy inequality, (2.10) and (2.20), then integrating in \([0, T] \times M\), we obtain
\[ \int_{(T) \times M} e^{-\lambda t} (g^{-1} h_t^2 + (\delta h)^2) \sqrt{w} + (\lambda - c_{13}) \int_{[0, T] \times M} e^{-\lambda t} (g^{-1} h_t^2 + (\delta h)^2) \sqrt{w} \leq \int_{(T) \times M} e^{-\lambda t} (\delta h)^2 \sqrt{w} + c_{14} \int_{[0, T] \times M} e^{-\lambda t} ((\partial_t h_t)^2 + (\delta h)^2) \sqrt{w} \]
(2.29)\[+c_{14} \varepsilon \int_{[0, T] \times M} e^{-\lambda t} f^2 (\int_0^t \partial_a h)^2 \sqrt{w}. \]
It follows from (2.20) that
(2.30) \[ g^{-1} h_t^2 + (\delta h)^2 + g^{-1} \delta h^2 \geq |h_t - \bar{h}_t|^2 + |\delta h - \bar{\delta} h|^2, \]
which combining with (2.29) gives that
\[ c_{15} \int_{[0, T] \times M} e^{-\lambda t} (h_t^2 + (\partial_t h)^2) \sqrt{w} \leq \int_{(T) \times M} e^{-\lambda t} (h_t^2 + (\delta h)^2) \sqrt{w} + c_{16} \int_{[0, T] \times M} e^{-\lambda t} (h_t^2 + (\partial_t h)^2) \sqrt{w} \]
(2.31)\[+c_{16} \varepsilon \int_{[0, T] \times M} e^{-\lambda t} f^2 (\int_0^t \partial_a h)^2 \sqrt{w}. \]
Multiplying (2.28) both side by 2\(e^{-\lambda t} \bar{h}\) and integrating on \([0, T] \times M\), we have
\[ \int_{(T) \times M} e^{-\lambda t} (\bar{h}^2 + (\partial_t \bar{h})^2) \sqrt{w} + \lambda \int_{[0, T] \times M} e^{-\lambda t} (\bar{h}^2 + (\partial_t \bar{h})^2) \sqrt{w} \leq \int_{[0, T] \times M} e^{-\lambda t} (\bar{h}^2 + (\partial_t \bar{h})^2) \sqrt{w} + c_{17} \int_{[0, T] \times M} e^{-\lambda t} (|g|^2)^2 \sqrt{w} \]
(2.32)\[+c_{17} \varepsilon \int_{[0, T] \times M} e^{-\lambda t} (|g|^2)^2 \sqrt{w}. \]
Differentiating (2.28) with respect to \(x\), by the similar process of getting (2.32), we derive
\[ \int_{(T) \times M} e^{-\lambda t} ((\partial_t D h)^2 + (\partial_t \delta h)^2) \sqrt{w} + \lambda \int_{[0, T] \times M} e^{-\lambda t} ((\partial_t \bar{h})^2 + (\partial_t \bar{h})^2) \sqrt{w} \leq \int_{[0, T] \times M} e^{-\lambda t} (h_0^2 + h_1^2 + (Dh_0)^2 + (Dh_1)^2) \sqrt{w} \]
(2.33)\[+c_{18} \int_{[0, T] \times M} e^{-\lambda t} (|g|^2 + |Dg|^2)^2 \sqrt{w}. \]
Inequalities (2.32), (2.33) give the control of two terms in the right hand side of (2.31). Thus substituting (2.32), (2.33) in (2.31), (2.22) is obtained. This completes the proof. □

In what follows, we plan to obtain the estimate for \( ||h||_s \) by considering the equations of spacial derivatives of \( h \). For any multi-index \( \alpha \in \mathbb{Z}_+^2 \) with \( |\alpha| = s \), applying \( D^\alpha \) to both sides of (2.8) to get

\[
(2.34) \partial_t D^\alpha h - \varphi_\alpha (\rho_{ab} \varphi_b D^\alpha h) + B \partial_a D^\alpha h - \varepsilon f(Du, Dv) \int_0^t D^\alpha + 1h = F_\alpha,
\]

where the nonlinear terms

\[
F_\alpha = D^\alpha g + \sum_{s_1 + s_2 = \alpha, |s_1| \geq 1} (D^{s_1}(\varphi_{ab})) \partial_{s_1} \partial_b D^{s_2} h
\]

\[
+ \sum_{s_1 + s_2 = \alpha, |s_1| \geq 1} (D^{s_1}(\varphi_a \varphi_b)) \partial_b D^{s_2} h - \sum_{s_1 + s_2 = \alpha, |s_1| \geq 1} (D^{s_1} B) \partial_a D^{s_2} h
\]

\[
(2.35) + \varepsilon \sum_{s_1 + s_2 = \alpha, |s_1| \geq 1} (D^{s_1} f(Du, Dv)) \int_0^t DD^{s_2} h.
\]

For convenience, we denote all spacial derivatives of \( h \) of the order \( s \) by a column vector of \( m(s) \) components

\[
u_s^T = (\partial_1^s h, \partial_1^{s-1} \partial_2 h, \ldots, \partial_2^s h).
\]

It follows from putting together the equations corresponding to all \( \alpha \) with \( |\alpha| = s \) in (2.34) that

\[
(2.36) \partial_t u_s - \varphi_\alpha (\rho_{ab} \varphi_b u_s) + B_s \partial_a u_s - \varepsilon f_s(Du, Dv) \int_0^t Du_s = F_\alpha,
\]

where \( B_s, f_s \) are \( (m \times m) \)-matrices and \( \tilde{F}_\alpha \) is an \( m \)-vector given by

\[
\tilde{F}_\alpha = (F_s(0,\ldots,0), F_s(-1,\ldots,0), \ldots, F_s(0,\ldots,0,\ldots,0)).
\]

It is obviously that the following Levi condition holds

\[
|B_s - \varphi_a \rho_{ab} I| \leq c_{19} \varphi, \ x \in M.
\]

For \( s \geq 1 \), we set

\[
C_s = \sum_{a,b=1}^2 (||D^2(\varphi_{ab})||_s + ||D(\varphi_a \varphi_b I)||_s) + ||B_s||_s.
\]

**Lemma 2.6.** Let \( h \) be a \( H^2 \)-solution to (2.8) with initial data (2.7). Assume that (2.11), (2.12) and (2.20) holds. Then for sufficient small \( \varepsilon > 0 \), there holds

\[
(2.37) ||h||_{s,T} \leq c_{20} (||h_0||_{s,T} + ||h_1||_{s,T} + ||g||_{s,T}).
\]

**Proof.** The proof is based on the induction. For \( s = 1 \), Lemma 2.5 gives the case by choosing a suitable \( \lambda \). We assume that (2.37) holds for all \( 1 \leq j \leq s \), and we prove that (2.37) holds for \( s + 1 \). Since (2.37) has the same structure with (2.8), the result in Lemma 2.5 can be used directly. Note that the vector \( D^s h \) satisfies (2.36). By (2.22), we have

\[
\lambda \int_{[0,T] \times M} e^{-\lambda t}((D^s \partial_s h)^2 + (D^s + 1 h)^2 + (D^s h)^2) \sqrt{w}
\]
Note that (2.43). Furthermore, we can apply (2.42). This completes the proof.

Thus choosing a large enough, by (2.39), we get

$$\int_{[0,T] \times M} e^{-\lambda t} ((D^s h_0)^2 + (D^s h_1)^2 + (D^{s+1} h_0)^2 + (D^{s+1} h_1)^2) \sqrt{w}$$

where $c$ is a constant depending on $C_s$.

By (2.35), we drive

$$\int_{[0,T] \times M} e^{-\lambda t} (|F_\alpha|^2 + |DF_\alpha|^2) \sqrt{w}$$

where $D^\alpha$ denotes the nonlinear terms involving $D^\alpha$ with $|\alpha| \leq s$.

It follows from (2.38) and (2.39) that

$$\int_{[0,T] \times M} e^{-\lambda t} (\lambda(D^s \partial_t h)^2 + (\lambda - c)(D^{s+1} h)^2 + \lambda(D^s h)^2) \sqrt{w}$$

where $F'_\alpha$ denotes the nonlinear terms involving $D^\alpha$ with $|\alpha| \leq s$.

Note that

$$\|F'_\alpha\|_{L^2} \leq c_{24} (\|h\|_{H^s} + T^2 \|f\|_{L^2} |h|_{s,T}).$$

Thus choosing a large enough, by (2.39), we get

$$\|D^s \partial_t h\|_{L^2} + \|D^s h\|_{H^s} \leq c_{25} (\|h_0\|_{H^{s+1}} + \|h_1\|_{H^{s+1}} + \|g\|_{H^{s+1}} + \|h\|_{H^s})$$

Furthermore, we can apply $D^{s-1}$ to both sides of (2.8), then deriving a similar estimate with $2.42$. We conclude that

$$\sum_{j=0}^{s+1} \|\partial_t^j D^{s+1-j} h\|_{L^2} \leq c_{26} (\|h_0\|_{H^{s+1}} + \|h_1\|_{H^{s+1}} + \|g\|_{H^{s+1}} + \|h\|_{H^s})$$

Note that $\|h\|_s$ is equivalent to $\|h\|_{H^s} + \|\partial_t h\|_{H^s}$. Hence (2.37) can be derived by (2.43). This completes the proof. \hfill \Box

Proof of Theorem 2.2. The proof is based on an approximation. For any small $\delta > 0$, we consider the regularized equation with the initial data (2.9) in $[0, T] \times M$

$$\partial_t h - (\rho(x) + \delta)\partial_t (\rho_{ab}(t, x)\partial_a h) + B(t, x)\partial_a h - \varepsilon f(Du, Dv) \int_0^t \partial_a h = g(t, x).$$
Above system is a strictly hyperbolic equation, which is equivalent to
\[ (2.44) \]
\[ \partial_t \mathbf{h} + \mathbf{A} \mathbf{h} = \mathbf{g}, \]
with initial data \( \mathbf{h} = (h_0, h_1, 0)^T \), where \( \mathbf{h} = (\hat{h}, h, z)^T \), \( \mathbf{g} = (g, 0, 0)^T \) and
\[ \mathbf{A} = \begin{pmatrix} 0 & -(\gamma(x) + \delta)\partial_a (\rho_{ab}(t, x) \partial_b) + B(t, x) \partial_a \varepsilon f \\ -1 & 0 \\ 0 & -\partial_a \end{pmatrix}. \]
Note that \( f \) is bounded in the norm \( ||\cdot||_{s,T} \) with \( 1 \leq s \leq k - 1 \) and all coefficients in the operator \( \mathbf{A} \) are \( \mathbb{C}^k \) for some integer \( k \geq 2 \). Then linear equation \( (2.44) \) admits an \( \mathbb{H}^k \)-solution \( \mathbf{h}_3 \). By \( (2.37) \) in Lemma 2.6, for \( s \leq k - 1 \), we have
\[ (2.45) \]
\[ ||h_3||_{s,T} \leq c_3 (||h_0||_{s,T} + ||h_1||_{s,T} + \|g\|_{s,T}). \]
where \( c_3 \) denotes a constant depending on \( \delta \). Thus by a standard argument, there exists a function \( h \) and a sequence \( \delta_j \rightarrow 0 \) such that \( h_{\delta_j} \rightarrow h \) in \( \mathbb{H}^{k-2} \). Here \( h \) is solution of \( (2.3) \). This completes the proof.

**Proof of Theorem 2.1.** By \( (1.12) \), directly computation shows that
\[ \partial_a (\epsilon^{ac} \epsilon^{bd} \gamma(u_0)_{cd} \partial_b h^m) = \gamma_0(x) \partial_a (\epsilon^{ac} \epsilon^{bd} \gamma(x)_{cd} \partial_b h^m) \\ + (\partial_a \gamma(x)) \epsilon^{ac} \epsilon^{bd} \gamma(x)_{cd} \partial_b h^m. \]
Using the second condition in \( (1.13) \) and the first condition in \( (1.14) \), we derive
\[ \| (\partial_a \gamma(x)) \epsilon^{ac} \epsilon^{bd} \gamma(x)_{cd} \partial_b h^m \|_s \leq c_{\gamma_1} \| h^m \|_{s+1}, \]
\[ \| \epsilon^{ac} \epsilon^{bd} \partial_d u^m (\int_0^t \partial_d v^n) \|_{s,T} \leq c_{27T} \| u_0 \|_{s+1} \| v \|_{s+1,T} \leq c_{T,R}, \]
\[ \| \epsilon^{ac} \epsilon^{bd} \partial_d u^m (\int_0^t \partial_d v^n) \|_{s,T} \leq c_{T,R}, \]
\[ \| \epsilon^{ac} \epsilon^{bd} \gamma (\int_0^t v^n)_{cd} \|_{s,T} \leq c_{T,R}, \]
where constants \( c_{\gamma_1} \) and \( c_{T,R} \) depend on \( \gamma_1 \) and \( T, R \), respectively.
Thus we can set
\[ g(x)(\partial_a \rho_{ab}(t, x) \partial_b h^m) - \partial_a (O(\epsilon) \partial_b h^m) = \partial_a (\epsilon^{ac} \epsilon^{bd} \gamma(u_0)_{cd} \partial_b h^m) \\ + \varepsilon \partial_a (\epsilon^{ac} \epsilon^{bd} \partial_c u^m (\int_0^t \partial_d v^n) \partial_b h^m) + \varepsilon \partial_a (\epsilon^{ac} \epsilon^{bd} \partial_d u^m (\int_0^t \partial_d v^n) \partial_b h^m) \]
\[ = \partial_a (\epsilon^{ac} \epsilon^{bd} \gamma(u_0)_{cd} \partial_b h^m) \\ + \varepsilon \partial_a (\epsilon^{ac} \epsilon^{bd} \partial_c u^m (\int_0^t \partial_d v^n) \partial_b h^m) + \varepsilon \partial_a (\epsilon^{ac} \epsilon^{bd} \partial_d u^m (\int_0^t \partial_d v^n) \partial_b h^m) \]
\[ (2.47) \]
In a similar way with \( (2.47) \), we set
\[ (B(t, x) + O(\epsilon)) \partial_a h^m = -(\partial_a \gamma(x)) \epsilon^{ac} \epsilon^{bd} \gamma(x)_{cd} \partial_b h^m + O(\epsilon) \partial_a h^m \\ = -(\partial_a \gamma(x)) \epsilon^{ac} \epsilon^{bd} \gamma(x)_{cd} \partial_b h^m \\ + 2\varepsilon \partial_a (\epsilon^{ac} \epsilon^{bd} \gamma(x)_{cd} \partial_b h^m) \\ + \varepsilon \partial_a (\epsilon^{ac} \epsilon^{bd} \gamma(x)_{cd} \partial_b h^m) \\ + 2\varepsilon \partial_a (\epsilon^{ac} \epsilon^{bd} \gamma(x)_{cd} \partial_b h^m) \\ - \varepsilon \epsilon^{ab} w^{-\frac{1}{2}} \partial_c (\epsilon^{cd} w^{-\frac{1}{2}}) \gamma_{bd} \partial_b h^m \]
f(Du, Dv) \int_0^t \partial_n h = O(\varepsilon) \int_0^t \partial_n h = \varepsilon \partial_u (\epsilon^{ac} \epsilon^{bd} \partial_d u^m) (\int_0^t \partial_d h_n) \partial_b v^m \\
+ \varepsilon \partial_u (\epsilon^{ac} \epsilon^{bd} \partial_d u^m) (\int_0^t \partial_d h_n) \partial_b v^m \\
+ \varepsilon \partial_u (\epsilon^{ac} \epsilon^{bd} \partial_d v^m) (\int_0^t \partial_d h_n) \partial_b v^m \\
- 2 \varepsilon \partial_u (\epsilon^{ab} \epsilon^{cd} w^{-1}) (\int_0^t \partial_d h^m) (\int_0^t \partial_d v_n) \partial_b v_n \\
- 2 \varepsilon \partial_u (\epsilon^{ab} \epsilon^{cd} w^{-1}) (\int_0^t \partial_d v^m) (\int_0^t \partial_d h_n) \partial_b v_n \\
+ \varepsilon \epsilon^{ab} w^{-\frac{1}{2}} \partial_c (\epsilon^{cd} w^{-\frac{1}{2}}) ((\int_0^t \partial_d h^m) (\int_0^t \partial_d v_n) \partial_a v^m + (\int_0^t \partial_d v^m) (\int_0^t \partial_d h_n) \partial_a v^m) \\
+ 2 \varepsilon \epsilon^{ab} w^{-\frac{1}{2}} \partial_c (\epsilon^{cd} w^{-\frac{1}{2}}) ((\int_0^t \partial_d h^m) (\int_0^t \partial_d v_n) \partial_a v^m + (\int_0^t \partial_d v^m) (\int_0^t \partial_d h_n) \partial_a v^m).

Note that assumptions (1.13)–(1.15) are equivalent to (2.10)–(2.12). Therefore using the similar process of proof of Theorem 2.2, one can prove this result.

3. Long time existence for the bosonic membrane equation

In this section, we construct a solution $u \in C_T^k \cap B_{R,T}^k$ ($k \geq 2$) of system (1.2) on $[0, \frac{T}{\sqrt{\varepsilon}}] \times M$ by a suitable Nash-Moser iteration scheme, where $B_{R,T}^k$ is defined in (2.6). We know that system (1.2) is equivalent to system (2.3) by an arrangement in section 2. Thus the main goal is to solve system (2.3).

Rescaling in (2.3) amplitude and time as

$$v^m(t, x) \mapsto \varepsilon^{-2} v^m(\sqrt{\varepsilon} t, x), \; \varepsilon > 0,$$

we are to prove the existence solution on $[0, T] \times M$ of

$$\partial_t v^m - \varepsilon^{-1} \partial_a (\epsilon^{ac} \epsilon^{bd} \gamma(u)_c d \partial_b v^m) = \varepsilon^2 \mathcal{F}(v^m),$$

where

$$\mathcal{F}(v^m) = \partial_a (\epsilon^{ac} \epsilon^{bd} \partial_c u^m) (\int_0^t \partial_d v_n) \partial_b v^m + \partial_a (\epsilon^{ac} \epsilon^{bd} \partial_d u^m) (\int_0^t \partial_d v^m) \partial_b v^m \\
+ \varepsilon \partial_a (\epsilon^{ac} \epsilon^{bd} \gamma) (\int_0^t v^m \partial_b v^m) - 2 \varepsilon \partial_a (\epsilon^{ab} \epsilon^{cd} w^{-1}) (\int_0^t \partial_d v^m) (\int_0^t \partial_d v^m) \partial_b v_n \\
+ \varepsilon \epsilon^{ab} w^{-\frac{1}{2}} \partial_c (\epsilon^{cd} w^{-\frac{1}{2}}) \gamma_{bd} \partial_b v^m + 2 (\int_0^t \partial_a v^m) (\int_0^t \partial_b v^m) \partial_d v_n).$$
Introduce an auxiliary function
\[ W^m(t, x) = v^m(t, x) - v_0^m - v_1^mt, \]
then the initial value problem (3.1)-(3.2) is equivalent to
\[ (3.4) \quad W^m(t, x) = \\partial_t W^m - \varepsilon^{-1} \partial_a (\epsilon^{ab} c^{bd} \gamma(u_0)_{cd} \partial_b W^m) - \varepsilon^2 F(W^m) = 0, \]
with zero initial data
\[ W^m(0) = 0, \quad \partial_t W^m(0) = 0. \]
We treat problem (3.5) iteratively as a small perturbation of the linear degenerate hyperbolic equation (2.5). Linearizing nonlinear system (3.5), we obtain the linearized operator
\[ (3.6) \quad \mathcal{L}_m h^m = \mathcal{L} h^m - \varepsilon^2 \partial_W F(W^m) h^m, \]
where
\[ \mathcal{L} h^m = \partial_t h^m - \varepsilon^{-1} \partial_a (\epsilon^{ab} c^{bd} \gamma(u_0)_{cd} \partial_b h^m), \]
\[ \partial_W F(W^m) h^m = \partial_a (\epsilon^{ac} c^{bd} \partial_c w^m_m (\int_0^t \partial_d W_n) \partial_b h^m) + \partial_a (\epsilon^{ac} c^{bd} \partial_c u^m_0 (\int_0^t \partial_d h_n) \partial_b W^m) + \varepsilon \partial_a (\epsilon^{ac} c^{bd} \gamma(\int_0^t W)_{cd} \partial_b h^m) + \varepsilon \partial_a (\epsilon^{ac} c^{bd} (\int_0^t \partial_d h_n) (\int_0^t \partial_d W_n) \partial_b W^m) + \varepsilon \partial_a (\epsilon^{ac} c^{bd} (\int_0^t \partial_c W^n) (\int_0^t \partial_d h_n) \partial_b W^m) - 2\varepsilon \partial_a (\epsilon^{ab} c^{cd} w^{-1} (\int_0^t \partial_c h^m) (\int_0^t \partial_d W_n) \partial_b W_n) - 2\varepsilon \partial_a (\epsilon^{ab} c^{cd} w^{-1} (\int_0^t \partial_c W^m) (\int_0^t \partial_d h_n) \partial_b W_n) - 2\varepsilon \partial_a (\epsilon^{ab} c^{cd} w^{-1} (\int_0^t \partial_c W^m) (\int_0^t \partial_d W^n) \partial_b h_n) + \varepsilon \epsilon^{ab} c^{cd} \partial_c (\epsilon^{cd} w^{-1} (\gamma_{bd} \partial_a h^m + (\int_0^t \partial_b h_n) (\int_0^t \partial_d W_n) \partial_a W^m) + 2\varepsilon (\int_0^t \partial_b W^m) (\int_0^t \partial_a h^m) \partial_d W_n) + 2\varepsilon (\int_0^t \partial_b W^m) (\int_0^t \partial_a W^m) \partial_d h_n).
\]
For the nonlinear term, by (3.3) and (3.7), direct computations show that
\[ R(h^m) := F(W^m + h^m) - F(W^m) - \partial_W F(W^m) h^m. \]
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For any \( s \geq 2 \), there holds

\[
|||R(h)|||_{s,T} \leq c_{28} \left( |||h|||^{2}_{s+2,T} (1 + |||W|||_{s+1,T}) + |||h|||^{3}_{s+2,T} \right),
\]

where \( c_{28} \) depends on \( |||u_{0}|||_{s+2} \).

Let \( \Pi_{\theta} \in C^{\infty}(R) \) such that \( \Pi_{\theta} = 0 \) for \( \theta \leq 0 \) and \( \Pi_{\theta} \rightarrow I \) for \( \theta \rightarrow \infty \). we introduce a family of smooth functions \( S(\theta') \) with \( S(\theta') = 0 \) for \( \theta' \leq 0 \) and \( S(\theta') = 1 \) for \( \theta' \geq 1 \) (see [19]). For \( W \in C^{T}_{T} \), we define

\[
\Pi_{|x'| - |x|} W(t,x) = S(|x'| - |x|)W(t,x), \quad x, x' \in M.
\]

For \( l = 0, 1, 2, \ldots, \), by setting

\[
x' := N_{l} = 2^{l},
\]
Lemma 3.2. There exist a linear map
\[ \text{Lemma 3.2.} \]
and then by (2.1), it is directly to check that
\[
\| \Pi_{|x'|} W \|_{H^1} \leq c_{s_1, s_2} N_t^{s_1-s_2} \| W \|_{H^2}, \quad \forall \ s_1 \geq s_2 \geq 0,
\]
\[
\| \Pi_{|x'|} W - W \|_{H^1} \leq c_{s_1, s_2} N_t^{s_1-s_2} \| W \|_{H^2}, \quad \forall \ 0 \leq s_1 \leq s_2.
\]
For convenience, we denote \( \Pi \) where
\[
\text{For convenience, we denote } \Pi \text{ where}
\]
and
\[
\text{and}
\]
Moreover, for
\[
\text{and}
\]
Then we plan to find the “th step” approximation solution of (3.5) has been chosen, which is \( W \neq 0 \). The “th step” approximation solution is denoted by
\[
\text{Moreover, for } 0 \leq s \leq k - 2, \text{ it holds}
\]
\[
||| h^{l+1} |||_{s,T} \leq c_{29} ||| E^l |||_{s,T}.
\]
Proof. Assume that a suitable “0th step” approximation solution of (3.5) has been chosen, which is \( W^0 \neq 0 \). The “th step” approximation solution is denoted by
\[
W^l = \sum_{i=0}^l h^i.
\]
Define
\[
E^l = \partial_t W^l - \varepsilon^{-1} \partial_u (\varepsilon^a \partial^b \gamma (u_0) c_d \partial_b W^l) - \varepsilon^2 \Pi_{N_{i+1}} F(W^l).
\]
Then we plan to find the “th step” approximation solution \( W^{l+1} \). By (3.5), we have
\[
\mathcal{G} (W^l + h^{l+1}) = \partial_t (W^l + h^{l+1}) - \varepsilon^{-1} \partial_u (\varepsilon^a \partial^b \gamma (u_0) c_d \partial_b (W^l + h^{l+1})) - \varepsilon^2 \Pi_{N_{i+1}} F(W^l + h^{l+1})
\]
\[
= - \partial_t W^l - \varepsilon^{-1} \partial_u (\varepsilon^a \partial^b \gamma (u_0) c_d \partial_b W^l) - \varepsilon^2 \Pi_{N_{i+1}} F(W^l)
\]
\[
+ \partial_t h^{l+1} - \varepsilon^{-1} \partial_u (\varepsilon^a \partial^b \gamma (u_0) c_d \partial_b h^{l+1}) - \varepsilon^2 \partial W^l \Pi_{N_{i+1}} F(W^l) h^{l+1}
\]
\[
- \varepsilon^2 \Pi_{N_{i+1}} ( \mathcal{F}(W^l + h^{l+1}) + F(W^l) + \partial W^l \mathcal{F}(W^l) h^{l+1} )
\]
\[
\tag{3.15} E^l + \mathcal{L}_\varepsilon (h^{l+1}) + R(h^{l+1}),
\]
where
\[
R(h^{l+1}) = - \varepsilon^2 \Pi_{N_{i+1}} ( \mathcal{F}(W^l + h^{l+1}) + F(W^l) + \partial W^l \mathcal{F}(W^l) h^{l+1} ) .
\]
By Theorem 2.1 in section 2, there exists a solution \( h^{l+1} \) of
\[
E^l + \mathcal{L}_\varepsilon (h^{l+1}) = 0,
\]
\[
h^{l}(0, x) = 0, \quad \partial_t h^{l}(0, x) = 0.
\]
A similar estimate with \( ||| h^{l+1} |||_{s,T} \leq c_{29} ||| E^l |||_{s,T} \).
Furthermore, one can know from (3.14) and (3.15) that
\[ E_{l+1} = R(h_{l+1}) . \]

For \( 2 \leq s_0 < \tilde{s} < s < \tilde{s} \leq k - 1 \), set
\[ s_l := \tilde{s} + \frac{s - \tilde{s}}{2^l} , \]
\[ \alpha_{l+1} := s_l - s_{l+1} = \frac{s - \tilde{s}}{2^{l+1}} . \]
By (3.18)-(3.19), it follows that
\[ s_0 > s_1 > \ldots > s_l > s_{l+1} > \ldots . \]

**Theorem 3.3.** System (3.1) with initial data (3.2) has a solution
\[ v^m(t, x) = W^m_m(t, x) + v^m_0 + \epsilon_1 v^m_1 t , \]
where \( W^m \) has the form
\[ W^m_m = \sum_{i=0}^{\infty} h_i \in C^m_T . \]

**Proof.** The proof is based on the induction. For any \( l = 0, 1, 2, \ldots \), we claim that there exists a constant \( 0 < d < 1 \) such that
\[ ||h_{l+1}||_{s_l, T} < d^{2^l} < 1 , \]
\[ ||E_{l+1}||_{s_l, T} < d^{2^l+1} , \]
\[ W^{l+1} \in B^{l+1}_{R,T} . \]

We choose a fixed sufficient small \( W^0 > 0 \) such that
\[ ||W^0||_{s_0} \ll 1, ||E^0||_{s_0} \ll 1 . \]

For the case \( l = 0 \), by (3.13), we have
\[ ||h^0||_{s_1, T} \leq c_{30}||E^0||_{s_1, T} \leq c_{30}||E^0||_{s_0, T} < 1 . \]

It follows from (3.20), (3.11), (3.16) and (3.17) that
\[ ||E^i||_{s_1, T} \leq ||R(h^i)||_{s_1, T} \leq c_{28\varepsilon^2} N^2 (||h^i||_{s_1, T}^2 (1 + ||W^0||_{s_1, T}) + ||h^i||_{s_1, T}^3) \]
\[ \leq c_{31\varepsilon^2} N^2 ||h^i||_{s_1, T}^2 \leq c(2\varepsilon||E^0||_{s_0, T})^2 . \]

It is obviously to see that (3.29)-(3.30) gives (3.20) L=0 ~ (3.21) L=0 by choosing suitable small \( \varepsilon > 0 \). So we get \( W^1 \in B^1_{R,T} . \)

Assume that (3.20)-(3.22) holds for \( 1 \leq i \leq l \), i.e.
\[ ||h^i||_{s_1, T} < 1, \]
\[ ||E^i||_{s_1, T} \leq d^{2^i} , \]
\[ W^i \in B^i_{R,T} . \]

Now we prove that (3.20)-(3.22) holds for \( l + 1 \). From (3.13) and (3.27), we have
\[ ||h^{l+1}||_{s_{l+1}, T} \leq c_{31}||E^{l+1}||_{s_{l+1}, T} < c_{31}d^{2^{l+1}} < 1 . \]

It follows from (3.9), (3.11), (3.13), (3.17), (3.28) and (3.29) that
\[ ||E^{l+1}||_{s_{l+1}, T} \leq ||R(h^{l+1})||_{s_{l+1}, T} . \]
We can choose a fixed sufficient small \( \varepsilon > 0 \) such that
\[
0 < 16\varepsilon |||E_0|||s_0,T < 1.
\]
Thus we conclude that (3.20)-(3.21) holds. Note that
\[ \begin{align*}
W_l &= \sum_{i=0}^{l} h_i
\end{align*} \]
(3.22).
Therefore, we derive
\[
\lim_{l \to \infty} |||E_l|||l,T = 0,
\]
which implies that system (3.3) with zero initial data has a solution
\[
W_{\infty} = \sum_{i=0}^{\infty} h_i \in C_T.
\]
At last, by (3.4) we obtain the solution of system (3.1) with initial data (3.2) has a solution
\[
v^m(t,x) = W_{\infty}^m + v_0^m + v_1^m t.
\]
\[\square\]
In what follows, we prove that the uniqueness of solution for system (3.1) with initial data (3.2). Assume that there exists another solution
\[
\tilde{v}^m(t,x) = \tilde{W}_{\infty}^m + v_0^m + v_1^m t.
\]
We intend to prove the following result:

**Theorem 3.4.** Assume that there exists another solution (3.31) of system (3.1) with initial data (3.2) in \( B_{R,T}^0 \). Then \( v^m(t,x) \equiv \tilde{v}^m(t,x) \) holds.

**Proof.** Let
\[
\tilde{W}_{\infty}^m = W_{\infty}^m - \tilde{W}_{\infty}^m.
\]
We plan to prove that the following initial problem
\[
\begin{align*}
\partial_t \tilde{W}_{\infty}^m - \varepsilon^{-1} \partial_a (e^{\alpha c \beta d \gamma} (u_0)_{cd} \partial_b \tilde{W}_{\infty}^m) - \varepsilon^2 (F(W_{\infty}^m) - F(\tilde{W}_{\infty}^m)) &= 0, \\
\tilde{W}_{\infty}^m(0,x) &= 0, \quad \partial_t \tilde{W}_{\infty}^m(0,x) = 0
\end{align*}
\]
(3.32)
has a solution \( \tilde{W}_{\infty}^m \equiv 0 \)

Consider the approximation system of (3.32) as
\[
G'(\tilde{W}_{\infty}^m) := \partial_t \tilde{W}_{\infty}^m - \varepsilon^{-1} \partial_a (e^{\alpha c \beta d \gamma} (u_0)_{cd} \partial_b \tilde{W}_{\infty}^m) - \varepsilon^2 (F(W_{\infty}^m) - F(\tilde{W}_{\infty}^m)) = 0.
\]
(3.33)
Then using the similar computation process with (3.15), we have
\[
L_\varepsilon(W_{\infty}^m) + E'(t,x) + R(W_{\infty}^m) - E'(t,x) = 0,
\]
(3.34)
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where $E'(t, x)$ is a function which does not depend on $\bar{W}_m^\infty$.

$$L_\varepsilon(\bar{W}_m^\infty) = \partial_t \bar{W}_m^\infty - \varepsilon^{-1} \partial_\nu(\varepsilon^\alpha \varepsilon^\beta \gamma(\nu_0, c_\alpha \partial_n \bar{W}_m^\infty)) - \varepsilon^2 \mathcal{N}_l \partial_{\bar{W}_m^\infty} \mathcal{F}(\bar{W}_m^\infty) \bar{W}_m^\infty,$$

$$R(\bar{W}_m^\infty) = \varepsilon^2 (\mathcal{F}(W_m^\infty) - \mathcal{F}(\bar{W}_m^\infty) - \partial_{\bar{W}_m^\infty} \mathcal{F}(\bar{W}_m^\infty)).$$

By Theorem 2.1 in section 2, there exists a solution $\bar{W}_m^\infty$ of

$$L_\varepsilon(\bar{W}_m^\infty) + E'(t, x) = 0,$$

$$\bar{W}_m^\infty(0, x) = 0, \quad \partial_t \bar{W}_m^\infty(0, x) = 0.$$

A similar estimate with (2.13) is derived as

$$|||\bar{W}_m^\infty|||_{s, T} \leq c_{35} |||E'|||_{s, T}.$$

Then by (3.9) and (3.34), we have

$$|||\bar{W}_m^\infty|||_{s_l, T} \leq c_{35} |||E'|||_{s_l, T} \leq |||R(\bar{W}_m^\infty)|||_{s_l, T} \leq c_{36} \varepsilon^2 N_l^2 |||\bar{W}_m^\infty|||_{s_l, T}^{2} (1 + |||W_m^\infty|||_{s_l, T}) + |||\bar{W}_m^\infty|||_{s_l, T}^3 \leq c_{36} \varepsilon^2 N_l^2 |||\bar{W}_m^\infty|||_{s_l-1, T}^2 \leq c_{38} \varepsilon^2 N_l^2 N_{l-1}^2 |||\bar{W}_m^\infty|||_{s_{l-2}, T}^2 \leq \ldots \leq c_{39} (8 \varepsilon |||W_m^\infty|||_{s_0, T})^2.$$

Choosing a suitable small $\varepsilon$ such that

$$0 < 8 \varepsilon |||W_m^\infty|||_{s_0, T} < 1.$$

Thus we obtain

$$\lim_{t \to \infty} |||\bar{W}_m^\infty|||_{s_l, T} = 0.$$

This completes the proof. □

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References

[1] P. Allen, L. Andersson, J. Isenberg, Timelike minimal submanifolds of general co-dimension in Minkowski space time. J. Hyperbolic Differ. Equ. 3 (2006) 691-700.
[2] P. Allen, L. Andersson, A. Restuccia, Local well-posedness for membranes in the light cone gauge. Commun. Math. Phys. 301 (2011) 383-410.
[3] G. Bellettini, J. Hoppe, M. Novaga, G. Orlandi, Closure and convexity results for closed relativistic strings. Complex Anal. Oper. Theory 4 (2010) 473-496.
[4] S. Brendle, Hypersurfaces in Minkowski space with vanishing mean curvature. Comm. Pure. Appl. Math. 55 (2002) 1249-1279.
[5] D. Christodoulou, The Formation of Black Holes in General Relativity, EMS Monographs in Mathematics, EMS Publishing House 2009.
[6] F. Colombini, S. Spagnolo, An example of a weakly hyperbolic Cauchy problem not well posed in $C^\infty$. Acta Math. 148 (1982) 243-253.
[7] F. Colombini, E. De Giorgi, S. Spagnolo, Sur les équations hyperboliques aves des coefficients qui dépendent que du temps. Ann. Scuola Norm. Sup. Pisa. 6 (1979) 511-559.
[8] P.A.M. Dirac, Lectures on quantum mechanics. Belfer Graduate School of Science Monographs Series, Vol. 2, New York: Belfer Graduate School of Science, 1967, Second printing of the 1964 original

[9] Q. Han, Energy estimates for a class of degenerate hyperbolic equations. *Math. Ann.* **347**, (2010) 339-364.

[10] Q. Han, J.X. Hong, C.S. Lin, On the cauchy problem of degenerate hyperbolic equations. *Trans. Amer. Math. Soc.* **358**, (2006) 4021-4044.

[11] J. Hope, Some classical solutions of relativistic membrane equations in 4-space-time dimensions. *Phys. Lett. B.* **329**, (1994) 10-14.

[12] L. Hörmander, Implicit function theorems. Stanford Lecture notes, University, Stanford 1977

[13] D.X. Kong, Q. Zhang, Q. Zhou, The dynamics of relativistic strings moving in the Minkowski space $\mathbb{R}^{1+n}$. *Commun. Math. Phys.* **269**, (2007) 135-174.

[14] J. Moser, A rapidly converging iteration method and nonlinear partial differential equations I-II. *Ann. Scuola Norm. Sup. Pisa.* **20**, (1966) 265-313, 499-535.

[15] J. Nash, The embedding for Riemannian manifolds. *Amer. Math.* **63**, (1956) 20-63.

[16] H. Lindblad, A remark on global existence for small initial data of the minimal surface equation in Minkowskian space time. *Proc. Amer. Math. Soc.* **132**, 1095-1102 (2003)

[17] L. Nguyen, G. Tian, On smoothness of timelike maximal cylinders in three dimensional vacuum spacetimes. [arXiv:1201.5183v2]

[18] T. Nishitani, The Cauchy problem for $D_t^2 - D_x a(t,x) D_x$ in Gevrey class of order $s > 2$. *Comm.PDE.* **31** 1289-1319 (2006)

[19] J.T. Schwartz, Nonlinear Functional Analysis, Gordon and Breach, New York, 1969.

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