Abstract

We analyze a number of natural estimators for the optimal transport map between two distributions and show that they are minimax optimal. We adopt the plugin approach: our estimators are simply optimal couplings between measures derived from our observations, appropriately extended so that they define functions on \( \mathbb{R}^d \). When the underlying map is assumed to be Lipschitz, we show that computing the optimal coupling between the empirical measures, and extending it using linear smoothers, already gives a minimax optimal estimator. When the underlying map enjoys higher regularity, we show that the optimal coupling between appropriate nonparametric density estimates yields faster rates. Our work also provides new bounds on the risk of corresponding plugin estimators for the quadratic Wasserstein distance, and we show how this problem relates to that of estimating optimal transport maps using stability arguments for smooth and strongly convex Brenier potentials. As an application of our results, we derive a central limit theorem for a density plugin estimator of the squared Wasserstein distance, which is centered at its population counterpart when the underlying distributions have sufficiently smooth densities. In contrast to known central limit theorems for empirical estimators, this result easily lends itself to statistical inference for Wasserstein distances.

1 Introduction

Optimal transport maps play a central role in the theory of optimal transport (Rachev and Rüschendorf, 1998; Villani, 2003; Santambrogio, 2015), and form an increasingly important methodology in statistics and machine learning (Kolouri et al., 2017; Panaretos and Zemel, 2019). Given two distributions \( P \) and \( Q \) supported in a set \( \Omega \subseteq \mathbb{R}^d \), an optimal transport map \( T_0 \) from \( P \) to \( Q \) is any solution to the Monge problem (Monge, 1781),

\[
\arg\min_{T \in \mathcal{T}(P,Q)} \int_{\Omega} \|x - T(x)\|^2 \, dP(x),
\]

where \( \mathcal{T}(P,Q) \) is the set of transport maps between \( P \) and \( Q \), that is, the set of Borel-measurable functions \( T : \Omega \to \Omega \) such that \( T\#P := P(T^{-1}(\cdot)) = Q \). Equivalently, we write \( T\#P = Q \) whenever \( X \sim P \) implies \( T(X) \sim Q \).

A wide range of statistical applications involve transforming random variables to ensure they follow a desired distribution. Optimal transport maps form natural choices of such transformations when
no other canonical choice is available. For instance, optimal transport maps form a useful tool for addressing label shift between train and test distributions in classification problems, and have more generally been applied to a variety of domain adaptation and transfer learning problems (Courty et al., 2016; Redko et al., 2019; Rakotomamonjy et al., 2021; Zhu et al., 2021). They have also found recent uses in distributional regression (Ghodrati and Panaretos, 2021), generative modeling (Finlay et al., 2020; Onken et al., 2021), fairness in machine learning (Gordaliza et al., 2019; Black et al., 2020; de Lara et al., 2021), and in a wide range of statistical applications to the sciences (Read, 1999; Wang et al., 2011; Schiebinger et al., 2019; Komiske et al., 2020).

An important question arising in many of these applications is that of estimating the optimal transport map between unknown distributions, on the basis of independent samples. The aim of this paper is to develop practical estimators of optimal transport maps achieving near-optimal risk. Specifically, given i.i.d. samples $X_1, \ldots, X_n \sim P$ and $Y_1, \ldots, Y_m \sim Q$, we derive estimators $\hat{T}_{nm}$ which achieve the minimax rate of convergence, typically up to polylogarithmic factors, under the loss function

$$
\|\hat{T}_{nm} - T_0\|_{L^2(P)}^2 = \int_{\Omega} \|\hat{T}_{nm}(x) - T_0(x)\|^2 dP(x).
$$

The theoretical study of such estimators was recently initiated by Hütter and Rigollet (2021), who proved that for any estimator $\hat{T}_{nm}$ with $n = m$,

$$
\sup_{P,Q} \mathbb{E}\|\hat{T}_{nm} - T_0\|_{L^2(P)}^2 \gtrsim n^{-\frac{2\alpha}{2(\alpha-1)+d}} \vee \frac{1}{n},
$$

where the supremum is taken over all pairs of distributions $(P, Q)$ for which $T_0$ lies in an $\alpha$-Hölder ball for some $\alpha \geq 1$, and satisfies several additional assumptions including a key curvature condition $A1(\lambda)$ which we define below. The lower bound (3) is reminiscent of, but generally faster than, the classical $n^{-2\alpha/(2\alpha+d)}$ minimax rate of estimating an $\alpha$-Hölder nonparametric regression function (Tsybakov, 2008), and is shown by Hütter and Rigollet (2021) to be achievable up to a polylogarithmic factor. Nevertheless, their estimator is computationally intractable in general dimension, and their work leaves open the question of developing practical optimal transport map estimators which achieve comparable risk.

**Our Contributions.** The aim of this paper is to establish the minimax optimality of several natural estimators of optimal transport maps, many of which have already been used heuristically in the statistical optimal transport literature. In particular, we analyze the following two classes of plugin estimators.

(i) **Empirical Estimators.** When no smoothness assumptions are placed on $P$ and $Q$, it is natural to study the plugin estimator based on the empirical measures

$$
P_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}, \quad Q_m = \frac{1}{m} \sum_{j=1}^{m} \delta_{Y_j}.
$$

In the special case $n = m$, there exists an optimal transport map $T_n$ from $P_n$ to $Q_n$. We show that this estimator achieves the rate $n^{-2/d}$ for all $d \geq 5$ under the squared $L^2(P_n)$ loss, assuming $T_0$ is Lipschitz and satisfies additional mild conditions. While the in-sample estimator $T_n$ is only defined over the support of $P_n$, we readily obtain estimators defined
over the entire domain by casting the extension problem as one of nonparametric regression. We show how linear smoothers and least-squares estimators can be used to interpolate \( T_n \), leading to an estimator \( \hat{T}_n \) defined over \( \Omega \) which continues to achieve the \( n^{-2/d} \) rate up to polylogarithmic factors, but now under the \( L^2(P) \) loss. This rate is minimax optimal when \( T_0 \) is Lipschitz, by equation (3). While this discussion requires the assumption \( n = m \), we show that analogous ideas extend to the general setting \( n \neq m \).

(ii) Smooth Estimators. In order to obtain faster rates of convergence when \( P \) and \( Q \) admit smooth densities \( p \) and \( q \), we next analyze the risk of the unique optimal transport map \( \hat{T}_{nm} \) between density estimators of \( p \) and \( q \). As we explain in Section 2, the Hölder smoothness of \( T_0 \) is typically expected to be of one degree greater than that of \( p \) and \( q \), and we show that our estimator nearly achieves the minimax lower bound (3) when these densities are \((\alpha - 1)\)-Hölder smooth, for any \( \alpha > 1 \). As discussed in Appendix E of Hütter and Rigollet (2021), the minimax lower bound (3) also holds under such smoothness conditions on the densities \( p \) and \( q \), as opposed to smoothness conditions on \( T_0 \).

Though our primary interest is in the two-sample problem under which both \( P \) and \( Q \) are unknown, we begin by studying analogues of the above estimators in the one-sample problem where \( P \) is assumed to be a known absolutely continuous distribution. As we shall see, in this setting there always exists an optimal transport map \( \hat{T}_n \) from \( P \) to any estimator \( \hat{Q}_n \) of \( Q \), which forms a natural plugin estimator of \( T_0 \). We bound the convergence rate of \( \hat{T}_n \) when \( \hat{Q}_n \) is either taken to be the empirical measure \( Q_n \), or a density estimator for \( Q \), mirroring the two-sample estimators (i) and (ii).

While our emphasis is on optimal transport maps, an equally important target of estimation is the optimal objective value in the Monge problem (1), which gives rise to the squared 2-Wasserstein distance \( W^2_2(P, Q) \) defined in Section 2. Our optimal transport map estimators naturally give rise to estimators for the Wasserstein distance, and in some cases we provide upper bounds on their risk which are tighter than those that could have been deduced from past literature. In fact, we show that these two estimation problems are closely related under the curvature condition \( A1(1) \) on the optimal transport problem between \( P \) and \( Q \). Specifically, we prove quantitative stability inequalities (Theorem 6), which show that for the transport map estimators \( \hat{T} \) under consideration, the following three quantities are of comparable order when \( \hat{Q} = \hat{T}_n P \),

\[
\| \hat{T} - T_0 \|_{L^2(P)}^2 \quad \left[ W^2_2(P, \hat{Q}) - W^2_2(P, Q) \right], \quad W^2_2(\hat{Q}, Q).
\]

We also establish two-sample analogues of such results (Proposition 12). As we summarize below, convergence rates of the empirical measure and certain density estimators under the Wasserstein distance are well-known, and we use the above stability bounds to deduce the risk of the corresponding plugin estimators of \( T_0 \) and \( W^2_2(P, Q) \). For example, we show there exists a density estimator \( \hat{Q}_n \) such that if \( \hat{T}_n \) is the optimal transport map from \( P \) to \( \hat{Q}_n \), then whenever \( P \) and \( Q \) admit \((\alpha - 1)\)-Hölder densities and satisfy several additional conditions, one has

\[
\mathbb{E}\| \hat{T}_n - T_0 \|_{L^2(P)}^2 \lesssim \left( \frac{\log n}{n} \right)^{2(\alpha-1)+\alpha} \sqrt{\frac{1}{n}},
\]

(4)

\[
\mathbb{E}|W^2_2(P, \hat{Q}_n) - W^2_2(P, Q)| \lesssim \left( \frac{\log n}{n} \right)^{2(\alpha-1)+\alpha} \sqrt{\frac{1}{n}}.
\]

(5)
These inequalities show that the estimator \( \hat{T}_n \) achieves the minimax rate (3), up to a polylogarithmic factor, and that the same rate also holds for the risk of \( W_2^2(P, \hat{Q}_n) \) whenever \( d > 2(\alpha + 1) \).

We build upon these estimation results to further address inference for Wasserstein distances in the high-smoothness regime \( d < 2\alpha \). We show in Section 5, under regularity conditions, that whenever \( P \neq Q \), there exists \( \sigma^2 > 0 \) such that

\[
\sqrt{n} \left( W_2^2(P, \hat{Q}_n) - W_2^2(P, Q) \right) \sim N(0, \sigma^2), \quad \text{as } n \to \infty. \tag{6}
\]

We also develop analogous results in the two-sample setting. To the best of our knowledge, this forms the first central limit theorem for a plugin estimator of the Wasserstein distance which is centered at its population counterpart, for absolutely continuous distributions \( P \) and \( Q \) in general dimension. Notice that the parametric scaling in equation (6) could have been anticipated from equation (5) when \( \alpha \) is sufficiently large. As we shall discuss, the variance \( \sigma^2 \) can easily be estimated, leading to an asymptotic confidence interval for \( W_2^2(P, Q) \).

Related Work. The two recent works of Hütter and Rigollet (2021) and Gunsilius (2021) establish \( L^2(P) \) convergence rates for transport map estimators. Gunsilius (2021) derives upper bounds on the risk of a plugin estimator for Brenier potentials, obtained via kernel density estimation of \( p \) and \( q \). This analysis results in suboptimal convergence rates for the optimal transport map \( T_0 \) itself. We show in this work that such plugin estimators do in fact achieve near-optimal convergence rates when the kernel density estimator is replaced by an orthogonal series estimator using wavelet bases.

Building upon a construction of del Barrio et al. (2020), a consistent estimator of \( T_0 \) was obtained by de Lara et al. (2021) under mild assumptions, by regularizing a piecewise constant approximation of the empirical optimal transport map \( T_n \). We do not know if quantitative convergence rates can be obtained for their estimator under stronger assumptions. Beyond these works, a wide range of heuristic estimators have been proposed in the literature (Perrot et al., 2016; Nath and Jawanpuria, 2020; Makkuva et al., 2020), but their theoretical properties remain unknown to the best of our knowledge.

Rates of convergence for the problem of estimating Wasserstein distances have arguably received more attention than that of estimating optimal transport maps. Characterizing the convergence rate of the empirical measure under the Wasserstein distance is a classical problem (Dudley, 1969; Boissard and Le Gouic, 2014; Fournier and Guillin, 2015; Weed and Bach, 2019; Lei, 2020) which immediately leads to upper bounds on the convergence rate of the empirical plugin estimator of the Wasserstein distance. While such upper bounds are generally unimprovable (Liang, 2019; Niles-Weed and Rigollet, 2019), they have recently been sharpened by Chizat et al. (2020) and Manole and Niles-Weed (2021) when \( P \neq Q \), and we employ these results to bound the convergence rates of our empirical optimal transport map estimators in Sections 3.2 and 4.2. Though the empirical plugin estimator of the Wasserstein distance is minimax optimal up to polylogarithmic factors under no assumptions on \( P \) and \( Q \), it becomes suboptimal when \( P \) and \( Q \) are assumed to have smooth densities. Weed and Berthet (2019) derive the minimax rate of estimating smooth densities under the Wasserstein distance, and we build upon their results to characterize the risk of our density plugin estimators (cf. Sections 3.3 and Sections 4.3).

The estimators for both optimal transport maps and Wasserstein distances considered throughout this work are simple to compute and nearly minimax optimal, but we make no claims that their computational efficiency is optimal. Chizat et al. (2020) analyzed distinct estimators for \( W_2^2(P, Q) \)
based on the empirical Sinkhorn divergence, which are faster to compute than \(W_2^2(P_n, Q_m)\) but are not minimax optimal. The results of Weed and Berthet (2019) imply that the Wasserstein distance between \((\alpha - 1)\)-smooth densities can be estimated at a rate \(C_d n^{-\gamma}\) when \(\alpha \gtrsim d\), for some \(\gamma\) independent of the dimension, but even in this case, their estimator requires computation time depending exponentially on \(d\). Vacher et al. (2021) analyzed an alternative estimator with more favorable computational properties; though their estimator is not minimax optimal, it can be computed in polynomial time if \(\alpha \geq C d\) for some \(C > 1\).

Central limit theorems for the empirical quadratic cost \(W_2^2(P_n, Q_m)\) around its expectation have been derived by del Barrio and Loubes (2019) under mild conditions on the underlying distributions. As we discuss in Section 5, however, the centering constant \(\mathbb{E} W_2^2(P_n, Q_m)\) in these results cannot generally be replaced by its population counterpart \(W_2^2(P, Q)\), which is a barrier to their use for statistical inference. Key exceptions are obtained when \(P\) and \(Q\) are one-dimensional (Munk and Czado, 1998; Freitag and Munk, 2005) or countable (Sommerfeld and Munk, 2018; Tameling et al., 2019), in which case non-degenerate limiting distributions for the process \(W_2^2(P_n, Q_m) - W_2^2(P, Q)\) are known up to suitable scaling. In contrast, our work derives central limit theorems with desirable centering for any absolutely continuous distributions \(P\) and \(Q\) admitting sufficiently smooth densities.

**Concurrent Work.** During the final stages of preparation of our manuscript, we became aware of the recent independent work of Deb et al. (2021) which also studies convergence rates for related plugin estimators of optimal transport maps and Wasserstein distances. A future version of this manuscript will include a more thorough comparison with their work.

**Paper Outline.** The remainder of this manuscript is organized as follows. In Section 2, we provide background on the quadratic optimal transport problem over \(\mathbb{R}^d\) and over the \(d\)-dimensional torus \(\mathbb{T}^d\). In Section 3, we state our main quantitative stability results and upper bounds for our one-sample estimators. We study two-sample estimators in Section 4. We then apply these results to obtain central limit theorems for smooth Wasserstein distances in Section 5.

**Notation.** For any \(a, b \in \mathbb{R}\), we write \(a \lor b = \max\{a, b\}\) and \(a \land b = \min\{a, b\}\), \(a_+ = a \lor 0\). The Euclidean norm on \(\mathbb{R}^d\) is denoted \(\|\cdot\|\), and the \(\ell_p\) norm of a sequence \((a_n)_{n \geq 1} \subseteq \mathbb{R}\) is written \(\|(a_n)_{n \geq 1}\|_{\ell_p} = (\sum_{n \geq 1} |a_n|^p)^{1/p}\) for all \(1 \leq p \leq \infty\). Given a closed set \(\Omega \subseteq \mathbb{R}^d\), and real numbers \(\alpha > 0\), \(1 \leq p, q \leq \infty\), the Hölder space \(C^\alpha(\Omega)\), the Besov space \(B^\alpha_{p,q}(\Omega)\), and their respective norms \(\|\cdot\|_{C^\alpha(\Omega)}\), \(\|\cdot\|_{B^\alpha_{p,q}(\Omega)}\), are defined in Appendix A. We drop the suffix \(\Omega\) from these quantities when the underlying space can be understood from context. Given a measure space \((\Omega, \mathcal{F}, \nu)\), \(L^p(\nu)\) denotes the Lebesgue space of order \(1 \leq p \leq \infty\), endowed with the norm \(\|f\|_{L^p(\nu)} = (\int_{\Omega} |f(x)|^p d\nu(x))^{1/p}\), for any measurable function \(f : \Omega \to \mathbb{R}\). When \(\nu\) is the Lebesgue measure \(\mathcal{L}\) on \(\Omega \subseteq \mathbb{R}^d\), we write \(L^p(\Omega)\) instead of \(L^p(\mathcal{L})\). For any integer \(B \geq 1\), the permutation group on \([B] = \{1, \ldots, B\}\) is denoted \(S_B\). The diameter of a set \(\Omega \subseteq \mathbb{R}^d\) is denoted \(\text{diam}(\Omega) = \sup\{\|x - y\| : x, y \in \Omega\}\), and its interior and closure are respectively denoted \(\Omega^\circ\) and \(\overline{\Omega}\). For all \(x \in \mathbb{R}^d\) and \(\epsilon > 0\), \(B(x, \epsilon) = \{y \in \mathbb{R}^d : \|x - y\| \leq \epsilon\}\). Given sequences of nonnegative real numbers \((a_n)_{n=1}^\infty\) and \((b_n)_{n=1}^\infty\), we write \(a_n \preceq b_n\) if there exist constants \(C > 0\) such that \(a_n \leq C b_n\) for all \(n \geq N\), and we also write \(a_n \asymp b_n\) if \(b_n \preceq a_n \preceq b_n\). Throughout the paper, the constants \(C\) and \(N\) are permitted to depend on the dimension \(d\), the domain \(\Omega\), and additional constants \(c_1, c_2, \ldots\) when clear from context, which we sometimes emphasize by writing \(\preceq_{c_1, c_2, \ldots}\) or \(\asymp_{c_1, c_2, \ldots}\).
2 Background on Optimal Transport

2.1 The Quadratic Optimal Transport Problem over $\mathbb{R}^d$

We provide a brief background on the optimal transport problem over $\mathbb{R}^d$ with respect to the squared Euclidean cost function, and direct the reader to Villani (2003); Santambrogio (2015) for further details. To simplify our exposition, we assume here and throughout the rest of the paper, except where otherwise specified, that all measures have support contained in a set $\Omega \subseteq \mathbb{R}^d$ satisfying the following condition.

(S1) $\Omega$ is a compact set such that $\Omega \subseteq [0, 1]^d$.

Notice that once $\Omega$ is assumed compact, the final assumption in condition (S1) can always be guaranteed by rescaling. Let $\mathcal{P}(\Omega)$ denote the set of Borel probability measures with support contained in $\Omega$, and $\mathcal{P}_{ac}(\Omega)$ the subset of such measures which are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. As we shall recall in Theorem 1 below, for any $P \in \mathcal{P}_{ac}(\Omega)$ and $Q \in \mathcal{P}(\Omega)$ the Monge problem (1) admits a minimizer $T_0$, which is uniquely defined $P$-almost everywhere. The Monge problem may, however, be infeasible when the absolute continuity condition on $P$ is removed. This observation motivated Kantorovich (1942, 1948) to develop the following convex relaxation of the Monge problem,

$$\arg\min_{\pi \in \Pi(P,Q)} \int_{\Omega} \|x - y\|^2 d\pi(x, y),$$

known as the Kantorovich problem, where $\Pi(P, Q)$ denotes the set of joint distributions on $\Omega^2$ with marginal distributions $P$ and $Q$, known as couplings of $P$ and $Q$. That is,

$$\Pi(P, Q) = \{ \pi \in \mathcal{P}(\Omega^2) : \pi(\cdot \times \Omega) = P, \pi(\Omega \times \cdot) = Q \}.$$

Notice that the Kantorovich problem is always feasible since $P \otimes Q \in \Pi(P, Q)$. It can be shown under the present setting that a minimizer $\pi$ in equation (7) always exists (Theorem 4.1, Villani (2008)), and is called an optimal coupling. In the special case where $\pi$ is supported in the graph of a map $T_0 : \Omega \to \Omega$, it must be the case that $T_0 \in T(P, Q)$ due to the marginal constraints in the definition of $\Pi(P, Q)$, and it must then follow that $T_0$ is precisely an optimal transport map from $P$ to $Q$. As we shall elaborate below, this situation turns out to characterize all optimal couplings when $P \in \mathcal{P}_{ac}(\Omega)$, and for such measures the Monge and Kantorovich problems yield equivalent solutions.

While an optimal coupling represents a transference plan for reconfiguring $P$ into $Q$, the corresponding optimal value of the objective function (7) represents the optimal cost of such a reconfiguration, which provides an easily interpretable measure of divergence between $P$ and $Q$. Specifically, it gives rise to the 2-Wasserstein distance,

$$W_2(P, Q) = \left( \inf_{\pi \in \Pi(P,Q)} \int_{\Omega} \|x - y\|^2 d\pi(x, y) \right)^{\frac{1}{2}}.$$  

which will play a recurring role throughout our development. We also refer to $W_2^2(P, Q)$ as the (quadratic) optimal transport cost between $P$ and $Q$. Notice that the above Kantorovich problem is
an (infinite-dimensional) convex program with linear constraints, and it admits a dual maximization problem given by
\[
W^2_2(P, Q) = \sup_{(\phi, \psi) \in K} \int \phi dP + \int \psi dQ,
\]
where
\[
K = \left\{ (\phi, \psi) \in L^1(\Omega) \times L^1(\Omega) : \phi(x) + \psi(y) \leq \|x - y\|^2 \text{ for all } x, y \in \Omega \right\}.
\]
In the present setting of the quadratic optimal transport problem over the compact set \(\Omega\), it can be shown that strong duality indeed holds in equation (9), and that the supremum is always achieved by some \((\phi_0, \psi_0)\) \(\in K\). Any such pair of functions is called a pair of Kantorovich potentials. In this case, notice that \((\phi_0, \phi^*_0)\), with \(\phi^*_0(y) = \inf_{x \in \Omega} \{ \|x - y\|^2 - \phi_0(x) \}\), is itself a pair of Kantorovich potentials, since replacing \(\psi_0\) by \(\phi^*_0\) can only increase the objective value (9), while retaining the constraint \((\phi_0, \phi^*_0) \in K\). If we define \(\varphi_0 = \|\cdot\|^2 - 2\phi_0\), then \(\phi^*_0\) equivalently takes the form \(\phi^*_0 = \|\cdot\|^2 - 2\varphi^*_0\), where \(\varphi^*_0(y) = \sup_{x \in \Omega} \{ (x, y) - \varphi_0(x) \}\), \(y \in \Omega\),
denotes the Legendre-Fenchel conjugate of \(\varphi_0\). Under this reparametrization, the Kantorovich dual problem is equivalent to the so-called semi-dual problem
\[
\inf_{\varphi \in L^1(P)} \int \varphi dP + \int \varphi^* dQ,
\]
in the sense that \(\varphi_0\) is a solution to the semi-dual problem if and only if \((\|\cdot\|^2 - 2\varphi_0, \|\cdot\|^2 - 2\varphi^*_0)\) is a solution to the Kantorovich dual problem (9). The significance of the semi-dual problem is in part due to its connection to the Monge problem, as described by the following result due to Knott and Smith (1984); Brenier (1991).

**Theorem 1** (Brenier’s Theorem). Let \(P \in \mathcal{P}_{ac}(\Omega)\) and \(Q \in \mathcal{P}(\Omega)\).

(i) There exists an optimal transport map \(T_0\) between \(P\) and \(Q\) which takes the form \(T_0 = \nabla \varphi_0\) for a convex function \(\varphi_0 : \mathbb{R}^d \to \mathbb{R}\) which solves the semi-dual problem (10), and which is uniquely determined \(P\)-almost everywhere up to addition of a constant.

(ii) If we further have \(Q \in \mathcal{P}_{ac}(\Omega)\), then \(\nabla \varphi^*_0\) is the (\(Q\)-almost everywhere uniquely determined) gradient of a convex function such that \(\nabla \varphi^*_0\#Q = P\), and also the solution of the Monge problem for transporting \(Q\) onto \(P\). Furthermore, for Lebesgue-almost every \(x, y \in \Omega\)
\[
\nabla \varphi^* \circ \nabla \varphi(x) = x, \quad \nabla \varphi \circ \nabla \varphi^*(y) = y.
\]

Brenier’s Theorem implies the aforementioned fact that a unique optimal transport map exists between any absolutely continuous distribution \(P\) and any distribution \(Q\), where uniqueness is always understood in the Lebesgue-almost everywhere sense. It further characterizes this map as the gradient of an optimal semi-dual potential \(\varphi_0\), which we also refer to as a Brenier potential in the sequel.

The convexity of \(\varphi_0\) already implies that it will be almost-everywhere twice differentiable. Further smoothness properties of Brenier potentials, and therefore of optimal transport maps, have been
studied via regularity theory of partial differential equations of the Monge-Ampère type, and we refer to De Philippis and Figalli (2014); Figalli (2017) for surveys. In short, denote by $p,q$ the respective Lebesgue densities of $P,Q \in \mathcal{P}_{ac}(\Omega)$, and assume $\varphi_0$ is in fact everywhere twice continuously differentiable. Then, the constraint $\nabla \varphi_0 \# P = Q$ implies by the change of variable formula that $\varphi_0$ solves the equation

$$\det (\nabla^2 \varphi_0(x)) = \frac{p(x)}{q(\nabla \varphi_0(x))}, \quad x \in \Omega. \quad (11)$$

As a direct consequence of equation (11), notice that the Hessian $\nabla^2 \varphi_0$ admits a uniformly bounded determinant whenever $p$ and $q$ are bounded, and bounded away from zero. This observation leads to the following result noted by Gigli (2011).

**Lemma 2.** Assume $\varphi_0 \in C^2(\Omega)$ and $\gamma^{-1} \leq p,q \leq \gamma$ for some $\gamma > 0$. Then, there exists a constant $\lambda > 0$, depending only on $\gamma$ and $\|\varphi_0\|_{C^2(\Omega)}$, such that $\varphi_0$ is $\lambda$-strongly convex.

Lemma 2 shows that, whenever equation (11) has positive and bounded right-hand side, smooth Brenier potentials are also strongly convex. We shall require this property in Section 3.1 to derive stability bounds for the $L^2(P)$ loss. To further obtain sufficient conditions for the Hölder smoothness of $\varphi_0$, notice that the Monge-Ampère equation (11) suggests that $\varphi_0$ admits two degrees of smoothness more than the densities $p$ and $q$. This intuition indeed turns out to hold true under suitable regularity conditions on $\Omega$, as was established in a series of publications by Caffarelli (1991, 1992a,b, 1996). The following is a summary of these results, as stated by Villani (2008) (Chapter 12).

**Theorem 3** (Caffarelli’s Regularity Theory). Assume $\Omega$ is convex, and that there exists $\gamma > 0$ such that $\gamma^{-1} \leq p,q \leq \gamma$ over $\Omega$.

(i) (Interior Regularity) Suppose there exists $\alpha > 1$, $\alpha \notin \mathbb{N}$, such that $p,q \in C^{\alpha-1}(\Omega^\circ)$. Then $\varphi_0 \in C^{\alpha+1}(\Omega^\circ)$. Moreover, for any open subdomain $\Omega'$ such that $\overline{\Omega'} \subseteq \Omega^\circ$, there exists a constant $C > 0$ depending only on $\gamma,\alpha,\Omega,\Omega',\|p\|_{C^{\alpha-1}(\Omega^\circ)},\|q\|_{C^{\alpha-1}(\Omega^\circ)}$ such that

$$\|\varphi_0\|_{C^{\alpha+1}(\Omega')} \leq C.$$

(ii) (Global Regularity) Assume $\Omega$ admits a $C^2$ boundary and is uniformly convex. Assume further that there exists $\alpha > 1$, $\alpha \notin \mathbb{N}$, such that $p,q \in C^{\alpha-1}(\Omega)$. Then, $\varphi_0 \in C^{\alpha+1}(\Omega)$.

Theorem 3(ii) implies that, under suitable conditions, the optimal transport map $T_0$ inherits one degree of smoothness more than the densities $p$ and $q$ over $\Omega$. Unlike the interior regularity result of Theorem 3(i), however, Theorem 3(ii) does not imply a uniform bound on $\|\varphi_0\|_{C^{\alpha+1}(\Omega)}$, and therefore does not preclude the possibility that the latter quantity diverges when $p,q$ vary in a $C^{\alpha-1}(\Omega)$ ball. Closely related global regularity results have also been established by Urbas (1997) under slightly stronger conditions, but we do not know if either of these results can be made uniform up to the boundary in an analogous way to the interior result of Theorem 3(i). Whenever global uniform regularity results are needed in our development, we shall sidestep this issue by working with the optimal transport problem over the torus, for which boundary considerations do not arise.
2.2 The Quadratic Optimal Transport Problem over the Flat Torus

Denote by $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ the flat $d$-dimensional torus. Specifically, $\mathbb{T}^d$ is the set of equivalence classes $[x] = \{x + k : k \in \mathbb{Z}^d\}$, for all $x \in [0, 1)^d$. Abusing notation, we typically write $x$ instead of $[x]$. $\mathbb{T}^d$ is endowed with the standard metric $d_{\mathbb{T}^d}(x, y) = \min\{\|x - y + k\| : k \in \mathbb{Z}^d\}$, $x, y \in \mathbb{T}^d$.

We identify $\mathcal{P}(\mathbb{T}^d)$ with the set of Borel measures $P$ on $\mathbb{R}^d$ such that $P([0, 1)^d) = 1$ and which are $\mathbb{Z}^d$-periodic, in the sense that $P(B) = P(k + B)$ for all $k \in \mathbb{Z}^d$ and all Borel sets $B \subset \mathbb{R}^d$. Furthermore, $\mathcal{P}_{ac}(\mathbb{T}^d)$ denotes the subset of measures in $\mathcal{P}(\mathbb{T}^d)$ which are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$. A function $f : \mathbb{T}^d \to \mathbb{R}$ is understood to be a function on $\mathbb{R}^d$ which is $\mathbb{Z}^d$-periodic, and we write $T : \mathbb{T}^d \to \mathbb{T}^d$ when $T$ is a map from $\mathbb{R}^d$ to $\mathbb{R}^d$ such that $[T(x)] = [T(y)]$ whenever $[x] = [y]$.

The optimal transport problem over $\mathbb{T}^d$ with the quadratic cost $d_{\mathbb{T}^d}^2$ largely mirrors that of the squared Euclidean cost over $\mathbb{R}^d$. In detail, define for all $P, Q \in \mathcal{P}_{ac}(\mathbb{T}^d)$ the Monge problem

$$
\min_{T \in \mathcal{T}(P,Q)} \int_{\mathbb{T}^d} d_{\mathbb{T}^d}^2(x, T(x))dP(x),
$$

where the integral is understood as being taken over $[0, 1)^d$. The Kantorovich problem and its dual give rise to the squared Wasserstein distance over $\mathcal{P}(\mathbb{T}^d)$,

$$
\mathcal{W}_2^2(P,Q) = \inf_{\pi \in \Pi(P,Q)} \int_{\mathbb{T}^d} d_{\mathbb{T}^d}^2(x,y)d\pi(x,y) = \sup_{(\varphi,\psi) \in K_T} \int \varphi dP + \int \psi dQ,
$$

where $K_T$ denotes the set of pairs of potentials $(\varphi, \psi) \in L^1(P) \times L^1(Q)$ satisfying the dual constraint $\varphi(x) + \psi(y) \leq d_{\mathbb{T}^d}^2(x,y)$ for all $x, y \in \mathbb{T}^d$. As in the Euclidean setting, the Kantorovich duality in the above display is equivalent to a semi-dual problem, whose solution characterizes the Monge problem. Indeed, the following result due to Cordero-Erausquin (1999) is an analogue of Brenier’s Theorem, together with additional properties about the optimal transport problem over $\mathbb{T}^d$.

**Proposition 4.** Let $P \in \mathcal{P}_{ac}(\mathbb{T}^d)$ and $Q \in \mathcal{P}(\mathbb{T}^d)$. Then, there exists a ($P$-almost everywhere uniquely determined) optimal transport map $T_0 = \nabla \varphi_0$ from $P$ to $Q$, where $\varphi_0 : \mathbb{R}^d \to \mathbb{R}$ is a convex function satisfying the following properties.

(i) $\|\cdot\|^2/2 - \varphi_0$ is $\mathbb{Z}^d$-periodic.

(ii) $T_0(x + k) = T_0(x) + k$ for almost every $x \in \mathbb{R}^d$ and $k \in \mathbb{Z}^d$.

(iii) For $P$-almost all $x \in \mathbb{R}^d$, $\|T_0(x) - x\| \leq \text{diam}(\mathbb{T}^d) = \sqrt{d}/2$ and $\|T_0(x) - x\| = d_{\mathbb{T}^d}(x, T_0(x))$.

Assume further that $Q \in \mathcal{P}_{ac}(\mathbb{T}^d)$, and denote the respective densities of $P, Q$ by $p, q$. Then,

(v) $\nabla \varphi_0$ is the ($Q$-almost everywhere uniquely determined) optimal transport map from $Q$ to $P$.

(vi) $(\|\cdot\|^2 - 2\varphi_0, \|\cdot\|^2 - 2\varphi_0) \in K_T$ is a pair of optimal Kantorovich potentials in equation (13).
(vii) If $\varphi_0 \in C^2(\mathbb{R}^d)$, then it solves the Monge-Ampère equation

$$\det(\nabla^2 \varphi_0(x)) q(\nabla \varphi_0(x)) = p(x), \quad x \in \mathbb{R}^d.$$  

In particular, if $\gamma^{-1} \leq p, q \leq \gamma$ for some $\gamma > 0$, then $\varphi_0$ is $\lambda$-strongly convex, for some constant $\lambda > 0$ depending only on $\gamma$ and $\|\varphi_0\|_{C^2(\mathbb{R}^d)}$.

With Proposition 4 in place, regularity properties of Brenier potentials $\varphi_0$ may be deduced from smoothness conditions on $p, q$. The following result was stated by Cordero-Erausquin (1999) without explicit mention of the uniformity of the Hölder norms appearing therein, but can readily be made uniform using Caffarelli’s interior regularity theory (Theorem 3(i); Figalli (2017), Chapter 4). We also note that this result was stated by Ambrosio et al. (2012) in the special case $d = 2$.

**Theorem 5.** Let $P, Q \in P(\mathbb{T}^d)$ be absolutely continuous with respect to the Lebesgue measure, with respective densities $p, q$ satisfying $\gamma^{-1} \leq p, q \leq \gamma$ for some $\gamma > 0$. Assume further that $p, q \in C^{\alpha - 1}(\mathbb{T}^d)$ for some $\alpha > 1$. Then, there exists a constant $C > 0$ depending only on $\alpha, \gamma, \|p\|_{C^{\alpha - 1}(\mathbb{T}^d)}$ and $\|q\|_{C^{\alpha - 1}(\mathbb{T}^d)}$ such that,

$$\|\varphi_0\|_{C^{\alpha + 1}(\mathbb{R}^d)} \leq C.$$  

### 3 Stability Bounds and the One-Sample Problem

Throughout this section, we let $P \in P_{ac}(\Omega)$ denote a known distribution, and $Q \in P_{ac}(\mathcal{Y})$ an unknown distribution from which an i.i.d. sample $Y_1, \ldots, Y_n \sim Q$ is observed. Let $T_0 = \nabla \varphi_0$ denote the unique optimal transport map from $P$ to $Q$, with respect to a convex Brenier potential $\varphi_0$. We also denote by $\phi_0 = \|\cdot\|^2 - 2\varphi_0$ and $\psi_0 = \|\cdot\|^2 - 2\varphi_0^*$ the Kantorovich potentials induced by $\varphi_0$. We assume condition (S1) holds throughout this section, and we may therefore assume without loss of generality that $|\phi_0|, |\psi_0| \leq \text{diam}(\Omega)^2 \leq d$ (Villani (2003), Remark 1.13).

Unlike the two-sample case which we discuss in Section 4, there exist canonical estimators of $T_0$ when the source distribution $P$ is known. Indeed, since $P$ is absolutely continuous, Brenier’s Theorem implies that there exists a unique optimal transport map $\hat{T}$ between $P$ and any estimator $\hat{Q}$ of $Q$. Any such transport map forms a natural estimator of $T_0$, and we analyze two such examples in this section. We first take $\hat{Q}$ to be the empirical measure of $Q$ in Section 3.2, and show that the resulting estimator $\hat{T}$ achieves the minimax risk of estimating Lipschitz optimal transport maps, under essentially no smoothness conditions on the underlying measures. In Section 3.3, we then take $\hat{Q}$ to be an orthogonal series density estimator, leading to a plugin estimator $\hat{T}$ admitting faster rates of convergence when $Q$ admits a sufficiently smooth density. In both cases, our analysis will hinge upon known upper bounds on the risk of $\hat{Q}$ under the Wasserstein distance, by invoking a key stability bound which we turn to first.

#### 3.1 A General Stability Bound

The main technical result of this section will be stated under the following curvature condition.

**A1(\lambda)** The Brenier potential $\varphi_0$ is a closed convex function such that $\varphi_0 \in C^2(\Omega)$ and $(1/\lambda)I_d \preceq \nabla^2 \varphi_0(x) \preceq \lambda I_d$ for all $x \in \Omega$. 


Condition $A_1(\lambda)$ implies in particular that $T_0$ is $\lambda$-Lipschitz over $\Omega$. As noted in Lemma 2, whenever $P$ and $Q$ both admit densities bounded away from zero and infinity, the second inequality of $A_1(\lambda)$ is sufficient to imply the first. Under this condition, we prove the following stability bounds in Appendix B.

**Theorem 6.** Let $P,Q \in \mathcal{P}_{ac}(\Omega)$, and assume condition $A_1(\lambda)$ holds for some $\lambda > 0$. Then, the following statements hold for any measure $\hat{Q} \in \mathcal{P}(\Omega)$.

(i) We have,

$$\frac{1}{\lambda} W_2^2(\hat{Q}, Q) \leq W_2^2(P, \hat{Q}) - W_2^2(P, Q) - \int \psi_0 d(\hat{Q} - Q) \leq \lambda W_2^2(\hat{Q}, Q).$$

(ii) Let $\hat{T} = \nabla \hat{\varphi}$ be the unique optimal transport map from $P$ to $\hat{Q}$. Then,

$$\frac{1}{\lambda} \| \hat{T} - T_0 \|^2_{L^2(P)} \leq W_2^2(P, \hat{Q}) - W_2^2(P, Q) - \int \psi_0 d(\hat{Q} - Q) \leq \lambda \| \hat{T} - T_0 \|^2_{L^2(P)}.$$

Theorem 6 implies that the squared $L^2(P)$ and $W_2$ losses are both comparable to the deviation $W_2^2(P, \hat{Q}) - W_2^2(P, Q)$, up to an additive term which is a linear functional of $\hat{Q}$, and which will turn out to be negligible in our applications. Remarkably, Theorem 6 also implies the following equivalence,

$$\frac{1}{\lambda} \| \hat{T} - T_0 \|^2_{L^2(P)} \leq W_2(\hat{Q}, Q) \leq \| \hat{T} - T_0 \|_{L^2(P)}.$$  \tag{14}

Notice that the second inequality always holds due to the fact that $(\hat{T}, T_0)_{\#} P$ is a suboptimal coupling of $\hat{Q}$ and $Q$. Equation (14) thus shows that the transport cost of this coupling is within a universal factor of being optimal, when the curvature condition $A_1(\lambda)$ is in force. This result will allow us to obtain upper bounds on the risk of one-sample plugin estimators $\hat{T}$ by appealing to the corresponding risk of $\hat{Q}$ under the Wasserstein distance.

Notice that one may equivalently write Theorem 6 as a stability bound in terms of the semi-dual problem. Indeed, one has by the Kantorovich duality,

$$W_2^2(P, \hat{Q}) - W_2^2(P, Q) - \int \psi_0 d(\hat{Q} - Q) = 2 \left( \int (\varphi_0 - \hat{\varphi}) dP + \int (\varphi_0^* - \hat{\varphi}^*) d\hat{Q} \right),$$

thus Theorem 6(ii) reads

$$\frac{1}{2\lambda} \| \nabla \hat{\varphi} - \nabla \varphi_0 \|^2_{L^2(P)} \leq \int (\varphi_0 - \hat{\varphi}) dP + \int (\varphi_0^* - \hat{\varphi}^*) d\hat{Q} \leq \frac{\lambda}{2} \| \nabla \hat{\varphi} - \nabla \varphi_0 \|^2_{L^2(P)}. \tag{15}$$

Equation (15) is a direct analogue of a stability bound proven by Hütter and Rigollet (2021) (Proposition 10), who show that similar inequalities hold when the measure $\hat{Q}$ appearing in the above display is replaced by $Q$. Their result assumes, however, that $\hat{\varphi}$ itself satisfies condition $A_1(\lambda)$. In contrast, we do not place any conditions on the estimator $\hat{T}$ beyond it being the optimal transport map from $P$ to $\hat{Q}$. This will permit our study of one- and two-sample estimators which are potentially nonsmooth but easy to compute, as we show next.
3.2 Upper Bounds for the One-Sample Empirical Estimators

Recall that \( Q_n = (1/n) \sum_{i=1}^n \delta_{Y_i} \) denotes an empirical measure from \( Q \). Since \( P \) is known and assumed to be absolutely continuous, a natural estimator for \( T_0 \) is the optimal transport map \( T_n \) from \( P \) to \( Q_n \),

\[
T_n = \arg \min_{T \in \mathcal{T}(P,Q_n)} \int \| x - T(x) \|^2 dP(x). \tag{16}
\]

Notice that the minimizer \( T_n \) in the above display exists and is uniquely defined \( P \)-almost everywhere, by Brenier’s Theorem. The optimization problem (16) is sometimes known as the semi-discrete optimal transport problem, for which efficient numerical solvers are well-studied (Mérigot, 2011; Levy and Schwindt, 2017).

In view of the stability bound in Theorem 6, the risk of \( T_n \) may be related to that of the empirical measure \( Q_n \) under the Wasserstein distance. This last was established for instance by Fournier and Guillin (2015), whose results imply the following bound, under no assumptions beyond (S1),

\[
E W_2^2(Q_n, Q) \lesssim \kappa_n := \begin{cases} n^{-1/2}, & d \leq 3 \\ n^{-1/2} \log n, & d = 4 \\ n^{-2/d}, & d \geq 5. \end{cases} \tag{17}
\]

The following bound on the risk of \( T_n \) is now an immediate consequence of Theorem 6, together with the fact that \( E \int \psi_0 d(Q_n - Q) = 0 \).

**Corollary 7.** Under condition \( A1(\lambda) \), we have

\[
E \| T_n - T_0 \|^2_{L^2(P)} \asymp E [W_2^2(P, Q_n) - W_2^2(P, Q)] \asymp EW_2^2(Q_n, Q) \lesssim \kappa_n.
\]

Corollary 7 implies that the one-sample empirical estimator \( T_n \) achieves the minimax lower bound (3) for estimating Lipschitz transport maps \( T_0 \), whenever \( d \geq 5 \). This result also provides a new bound on the bias of the one-sample empirical Wasserstein distance \( W_2^2(P_n, Q) \). Indeed, the following inequality can be deduced from Theorem 2 of Chizat et al. (2020), under no assumptions on \( P \) and \( Q \) beyond their compact support,

\[
EW_2^2(P, Q_n) - W_2^2(P, Q) \leq E |W_2^2(P, Q_n) - W_2^2(P, Q)| \lesssim \kappa_n. \tag{18}
\]

It was also shown by Manole and Niles-Weed (2021) that equation (18) is generally unimprovable for \( d \geq 5 \), when no further conditions are placed on \( P \) and \( Q \). In contrast, under the curvature condition \( A1(\lambda) \), Corollary 7 shows that the following stronger relation holds,

\[
EW_2^2(P, Q_n) - W_2^2(P, Q) \asymp EW_2^2(Q_n, Q). \tag{19}
\]

Since \( EW_2^2(Q_n, Q) \lesssim \kappa_n \), the above display recovers the bound (18) of Chizat et al. (2020), but \( EW_2^2(Q_n, Q) \) may also admit faster convergence rates. For instance, it was shown by Ledoux (2019) that when \( Q \) is the uniform distribution on \([0,1]^d\), \( Q_n \) achieves the following rate, which is faster...
than $\kappa_n$ whenever $d \leq 4$,\[
E W_2^2(Q_n, Q) \lesssim \begin{cases} 
 n^{-1}, & d = 1 \\
 n^{-1} \log n, & d = 2 \\
 n^{-2/d}, & d \geq 3.
\end{cases} \tag{20}
\]

In fact, the same rate then holds a fortiori for any probability distribution $Q$ given by the push-forward of the uniform distribution under any Lipschitz transport map, as noted by Fournier and Guillin (2015). In such settings, Corollary 7 implies that the same improvement in rate carries over to the bias of $W_2^2(P, Q_n)$. In fact, this rate then also holds for the squared $L^2(P)$ risk of $T_n$, thus matching the minimax lower bound (3) of Hüttner and Rigollet (2021) even when $d < 5$.

We also note that equation (19) can easily be extended to a bound on the risk of the empirical Wasserstein distance.

**Corollary 8.** Assume condition $A1(\lambda)$ holds for some $\lambda > 0$. Then,
\[
E \left| W_2^2(P, Q_n) - W_2^2(P, Q) \right| \lesssim n^{-\frac{1}{2}} + EW_2^2(Q_n, Q). \tag{21}
\]

### 3.3 Upper Bounds for One-Sample Smooth Estimators

While the empirical estimator in the previous section is shown to achieve the minimax rate of estimating Lipschitz optimal transport maps, at least when $d \geq 5$, we do not generally expect it to achieve faster rates of convergence if $T_0$ is assumed to enjoy further regularity. We instead show that such improvements can be achieved when $Q$ admits a smooth density $q$, and when the empirical measure $Q_n$ is replaced by the distribution $\hat{Q}_n$ of a density estimator $\hat{q}_n$. Specifically, define
\[
\hat{T}_n = \arg\min_{T \in T(P, \hat{Q}_n)} \int \|x - T(x)\|^2 dP(x). \tag{22}
\]

We shall focus on the special case where $\hat{q}_n$ is a wavelet density estimator, for which sharp risk estimates under the Wasserstein distance have been established by Weed and Berthet (2019). In order to appeal to their results, we shall assume throughout this section that the underlying domain is taken to be the unit cube $\Omega = [0, 1]^d$.

We briefly introduce notation from the theory of wavelets which will be needed in the sequel, and refer the reader to Appendix A for a detailed summary and references. To define a basis over the unit cube $\Omega$, we focus for concreteness on the boundary-corrected $N$-th Daubechies wavelet system, for some integer $N \geq 2$, as introduced by Cohen et al. (1993). In short, given an integer $j_0 \geq 1 + \log_2 N$, their construction leads to respective families of scaling and wavelet functions
\[
\Phi^{bc} = \{\phi_{j_0k} : 0 \leq k \leq 2^{j_0} - 1\}, \quad \Psi^{bc} = \{\psi_{j\ell k} : 0 \leq k \leq 2^{j_0} - 1, \ell \in \{0, 1\}^d \setminus \{0\}\}, \quad j \geq j_0,
\]
such that $\Psi^{bc} = \Phi^{bc} \cup \bigcup_{j = j_0}^{\infty} \Psi^{bc}_j$ forms an orthonormal basis of $L^2(\Omega)$, with the property that $\Phi^{bc}$ spans all polynomials of degree at most $N - 1$ over $\Omega$. We drop the superscript “bc” in the sequel whenever the choice of wavelet system is unambiguous. Given a probability distribution
Q ∈ P_{ac}(Ω) admitting density \( q ∈ L^2(Ω) \), one then has the representation

\[
q = \sum_{ξ ∈ Ψ} β_ξ ξ = \sum_{ξ ∈ Ψ} β_ξ ξ + \sum_{j=j_0}^{∞} \sum_{ξ ∈ Ψ_j} β_ξ ξ,
\]

where \( β_ξ = \int ξ dQ, \ ξ ∈ Ψ \),

with convergence at least in \( L^2(Ω) \). The standard truncated wavelet estimator of \( q \) (Kerkyacharian and Picard, 1992) with a threshold \( J_n > 0 \) is then given by

\[
\tilde{q}_n = \sum_{ξ ∈ Ψ} \hat{β}_ξ ξ = \sum_{ξ ∈ Ψ} \hat{β}_ξ ξ + \sum_{j=j_0}^{J_n} \sum_{ξ ∈ Ψ_j} \hat{β}_ξ ξ,
\]

where \( \hat{β}_ξ = \int ξ dQ_n, \ ξ ∈ Ψ \).

Notice that \( \tilde{q}_n \) is permitted to take on negative values, in which case it does not define a probability density. We instead define the final estimator \( \hat{q}_n \) in equation (22) by

\[
\hat{q}_n = \frac{\tilde{q}_n I(\tilde{q}_n ≥ 0)}{\int_{\tilde{q}_n > 0} \tilde{q}_n}, \text{ over } Ω.
\]  \( (23) \)

Weed and Berthet (2019) bounded the Wasserstein risk of a wavelet density estimator obtained from a distinct modification of \( \tilde{q}_n \). By appealing to \( L^∞ \) concentration inequalities for wavelet density estimators (Masry, 1997), we show in Appendix A.3.3 that their result carries over to the estimator \( \hat{q}_n \) at the expense of a polylogarithmic factor.

Lemma 9. Assume there exist \( α > 1 \) and \( M, γ > 0 \) such that \( \|q\|_{C_0(Ω)} ≤ M \) and \( γ^{-1} ≤ q ≤ γ \) over \( Ω \). Let \( 2J_n \asymp (log n/n)^{(1/(d+2(α−1))} \). Then,

\[
EW^2_2(\hat{Q}_n, Q) \lesssim_{M, γ, α} R_{T,n}(α) := \left( \frac{log n}{n} \right)^{−\frac{2α}{2α−2+α}} \sqrt{\frac{1}{n}}.
\]  \( (24) \)

Equipped with this result, we arrive at the following bound on the risk of the estimator \( \hat{T}_n \) defined in equation (22), and of the corresponding plugin estimator of the squared Wasserstein distance.

Theorem 10. Assume \( P, Q ∈ P_{ac}(Ω) \) admit respective densities \( p, q \) such that \( γ^{-1} ≤ p, q ≤ γ \), for some \( γ > 0 \). Assume further that there exists \( α > 1 \) and \( M > 0 \) such that \( \|q\|_{C_0(Ω)} ≤ M \), and choose \( 2J_n \asymp (log n/n)^{(1/(d+2(α−1))} \).

(i) (Optimal Transport Maps) Let \( R_{T,n}(α) \) be defined as in Lemma 9, and assume \( ϕ_0 \) satisfies condition \( A1(λ) \) for some \( λ > 0 \). Then,

\[
E\|\hat{T}_n − T_0\|^2_{L^2(P)} \lesssim_{M, γ, α} R_{T,n}(α).
\]  \( (25) \)

(ii) (Wasserstein Distances) Assume that for some \( λ > 0 \), \( \|ϕ_0\|_{C_α(Ω)} ≤ λ \). Then,

\[
|EW^2_2(P, \hat{Q}_n) − W^2_2(P, Q)| \lesssim_{M, γ, α} R_{T,n}(α),
\]

\[
E|W^2_2(P, \hat{Q}_n) − W^2_2(P, Q)| \lesssim_{M, γ, α} R_{W,n}(α) := \left( \frac{log n}{n} \right)^{−\frac{2α}{2α−2+α}} \sqrt{\frac{1}{n}}.
\]
Theorem 10 requires smoothness assumptions on both the density \( q \) and the potential \( \varphi_0 \); in particular, the assumption of Theorem 10(ii) requires both \( q \in C^{a-1}(\Omega) \) and \( \varphi_0 \in C^{a+1}(\Omega) \). Caffarelli’s regularity theory (Theorem 3) suggests that the former condition on \( q \) should be sufficient to imply the latter condition on \( \varphi_0 \), but such results cannot be invoked here due to the lack of smoothness of the boundary of the unit cube \([0,1]^d\). Even if the above analysis could be adapted to a domain \( \Omega \) with smooth boundary, the lack of uniformity in Caffarelli’s global regularity theory would prevent the bounds in Theorem 10 from holding uniformly in \( P \) and \( Q \), in the absence of a smoothness condition on \( \varphi_0 \). We refer to Appendix E of Hütter and Rigollet (2021) for related discussions. In Proposition 30 below, we will show that an analogue of Theorem 10 holds merely under smoothness conditions on \( \varphi \), in which case \( \hat{T}_n \) is shown to achieve the minimax rate (3) of estimating an \( \alpha \)-Hölder optimal transport map, up to polylogarithmic factors.

Theorem 10(ii) also proves that the bias of \( W^2_2(P, \hat{Q}_n) \) achieves the same convergence rate, as does its risk when \( d \geq 2(\alpha + 1) \). In the high-smoothness regime \( d < 2(\alpha + 1) \), the risk of \( W^2_2(P, \hat{Q}_n) \) does not improve beyond the parametric rate \( n^{-1/2} \), which can easily be seen to be unimprovable for this problem. The fact that the squared bias of this estimator is dominated by its variance when \( d < 2(\alpha + 1) \) will allow us to derive a central limit theorem for the empirical squared Wasserstein distance, centered around its population counterpart, in Section 5.

### 3.3.1 Proof of Theorem 10

Under the assumptions of part (i), we may apply Theorem 6 and Lemma 9 to obtain,

\[
\mathbb{E} | \hat{T}_n - T_0 |^2_{L^2(P)} \lesssim_{\lambda} \mathbb{E} W^2_2(\hat{Q}_n, Q) \lesssim_{M, \gamma, \alpha} R_{T, n}(\alpha),
\]

which immediately leads to the first claim. To prove the second claim, recall that we have assumed \( \alpha > 1 \), whence the assumption on \( \varphi_0 \) implies in particular that \( \| \varphi_0 \|_{C^2(\Omega)} \leq \lambda \). Since \( \gamma^{-1} \leq p, q \leq \gamma \), it follows by Lemma 2 that \( \varphi_0 \) satisfies condition \( \text{A1}(\lambda) \), after possibly modifying the value of \( \lambda \) in terms of \( \gamma \). We may therefore invoke Theorem 6 to obtain,

\[
\mathbb{E} W^2_2(P, \hat{Q}_n) - W^2_2(P, Q) \lesssim \mathbb{E} W^2_2(\hat{Q}_n, Q) + \mathbb{E} \int \psi_0 d(\hat{Q}_n - Q) ,
\]

(26)

\[
\mathbb{E} | W^2_2(P, \hat{Q}_n) - W^2_2(P, Q) | \lesssim \mathbb{E} W^2_2(\hat{Q}_n, Q) + \mathbb{E} \int \psi_0 d(\hat{Q}_n - Q) .
\]

(27)

Since we again have \( \mathbb{E} W^2_2(\hat{Q}_n, Q) \lesssim R_{T, n}(\alpha) \), it remains to bound the final terms of equations (26) and (27). By Lemma 26 in Appendix A.3, there almost surely exists \( N > 0 \) such that for all \( n \geq N \), we have \( \hat{q}_n = \bar{q}_n \). Recall that \( \beta_\xi \) is an unbiased estimator of \( \beta_\xi \) for all \( \xi \in \Psi \), so that

\[
q_{\xi_n} := \mathbb{E}[\bar{q}_n] = \sum_{\xi \in \Phi} \beta_\xi \xi + \sum_{j=0}^J \sum_{\xi \in \Psi_j} \beta_\xi \xi.
\]

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We thus have the bias-variance decomposition,
\[
\left| \int \psi_0(q - \tilde{q}_n) \right| \leq \left| \int \psi_0(q - q_J) \right| + \left| \int \psi_0(q_J - \tilde{q}_n) \right| =: \Delta_{1n} + \Delta_{2n}.
\]
By equations (26)–(27), we obtain
\[
|\mathbb{E}W^2_2(P, \hat{Q}_n) - W^2_2(P, Q)| \lesssim R_{T,n}(\alpha) + \Delta_{1n},
\]
\[
\mathbb{E}|W^2_2(P, \hat{Q}_n) - W^2_2(P, Q)| \lesssim R_{T,n}(\alpha) + \Delta_{1n} + \mathbb{E}[\Delta_{2n}].
\]
Notice that \(R_{W,n}(\alpha) = R_{T,n}(\alpha) \vee n^{-1/2}\), thus to prove the claim, it will suffice to prove that \(\Delta_{1n} \lesssim R_{T,n}(\alpha)\) and \(\mathbb{E}[\Delta_{2n}] \lesssim n^{-1/2}\).

**Step 1: Bounding \(\Delta_{1n}\).** Write the expansion of \(\psi_0\) in the basis \(\Psi\) as
\[
\psi_0 = \sum_{\xi \in \Phi} \gamma_\xi \zeta + \sum_{j=0}^{\infty} \sum_{\xi \in \Psi_j} \gamma_\xi \xi, \quad \text{where} \quad \gamma_\xi = \int \psi_0 \xi \text{ for all } \xi \in \Psi,
\]
where the convergence is easily seen to be uniform due to the smoothness condition on \(\psi_0\), so that,
\[
\int \psi_0(q - q_J) = \int \left( \sum_{\xi \in \Phi} \gamma_\xi \zeta + \sum_{j=0}^{\infty} \sum_{\xi \in \Psi_j} \gamma_\xi \xi \right) \left( \sum_{j=J_n+1}^{\infty} \sum_{\xi \in \Psi_j} \beta_\xi \xi \right) = \sum_{j=J_n+1}^{\infty} \sum_{\xi \in \Psi_j} \gamma_\xi \beta_\xi,
\]
by orthonormality of the basis \(\Psi\). By Lemma 22(i) in Appendix A.2, we have \(|\Psi_j| \lesssim 2^j\), therefore
\[
\Delta_{1n} \leq \sum_{j=J_n+1}^{\infty} \sum_{\xi \in \Psi_j} |\gamma_\xi \beta_\xi| \lesssim \sum_{j=J_n+1}^{\infty} 2^j \|\gamma_\xi\|_{\infty} \|\beta_\xi\|_{\Psi_j}. \quad (28)
\]
On the other hand, we have \(\|\cdot\|_{C^s(\Omega)} \lesssim \|\cdot\|_{B^{s,\infty}_{\infty,\infty}(\Omega)}\) for all \(s > 0\) by Lemma 23, thus,
\[
\|q\|_{B^{s-1,\infty}_{\infty,\infty}(\Omega)} \vee \|\psi_0\|_{B^{s+1,\infty}_{\infty,\infty}(\Omega)} \lesssim 1, \quad (29)
\]
by assumption on \(q\) and \(\varphi_0\). Furthermore, it follows by definition of Besov norm that for all \(j \geq j_0\),
\[
\|\gamma_\xi\|_{\Psi_j} \|\xi\| \lesssim \|q\|_{B^{s-\frac{1}{2},\infty}_{\infty,\infty}(\Omega)} 2^{-j\alpha} \quad \text{and} \quad \|\beta_\xi\|_{\Psi_j} \|\xi\| \lesssim \|\psi_0\|_{B^{s+\frac{1}{2},\infty}_{\infty,\infty}(\Omega)} 2^{-j\alpha}. \quad (30)
\]
Combine equations (28)–(30) to deduce
\[
\Delta_{1n} \lesssim \sum_{j=J_n+1}^{\infty} 2^j 2^{-j\alpha} \lesssim \sum_{j=J_n+1}^{\infty} 2^{-2j\alpha} \lesssim \left( \log \frac{n}{n} \right)^{-\frac{2\alpha}{\alpha - 1}} \lesssim R_{T,n}(\alpha).
\]

**Step 2: Bounding \(\mathbb{E}[\Delta_{2n}]\).** A bound on \(\mathbb{E}[\Delta_{2n}]\) immediately follows from the following simple result, which we isolate for further use below.
Lemma 11. Under the assumptions of Theorem 10(ii), we have
\[ \text{Var} \left[ \int \psi_0 \hat{q}_n \right] = \frac{1}{n} \text{Var}[\psi_0(Y)] + o \left( \frac{1}{n} \right). \]

The proof of Lemma 11 appears in Appendix D.1. Recall that \( \psi_0 \) is uniformly bounded by a constant depending only on \( d \), thus the same is true of \( \text{Var}[\psi_0(Y)] \). By Lemma 11, we deduce
\[ \mathbb{E}[\Delta_{2n}] \leq \sqrt{\text{Var}[\int \psi_0 \hat{q}_n]} \lesssim n^{-1/2}, \]
and the claim follows.

4 Empirical Stability Bounds and the Two-Sample Problem

In this section, we turn to analyzing two-sample estimators when both measures \( P, Q \in \mathcal{P}_{ac}(\Omega) \) are unknown, but i.i.d. samples \( X_1, \ldots, X_n \sim P \) and \( Y_1, \ldots, Y_m \sim Q \) are given. Here, \( \Omega \) again denotes a compact set satisfying condition (S1) except where otherwise specified. Unlike the previous section, where natural estimators for \( T_0 \) were simply given by the unique optimal transport map from \( P \) to an estimator of \( Q \), there is no canonical choice of transport map estimator when both \( P \) and \( Q \) are unknown. In fact, there do not exist any transport maps between the empirical measures of \( P \) and \( Q \) as soon as \( n \neq m \). Similarly as in the one-sample case, we shall study two classes of two-sample estimators. The first consists of estimators which interpolate the empirical in-sample optimal transport coupling, using methodologies inspired by nonparametric regression. Such estimators will achieve the optimal rate of estimating \( T_0 \) when it is Lipschitz. The second class will consist of plug-in estimators based on density estimates of \( P \) and \( Q \), and will achieve faster rates of convergence which depend on the smoothness of the densities of \( P \) and \( Q \).

The applicability of the stability bound of Theorem 6 will be limited in the first of these settings. Nevertheless, it turns out that an effective empirical analogue of this bound can be derived, to which we turn our attention first.

4.1 Empirical Stability Bounds

Denote the empirical measures of the two samples by \( P_n = (1/n) \sum_{i=1}^{n} \delta_{X_i} \) and \( Q_m = (1/m) \sum_{i=1}^{m} \delta_{Y_j} \). As previously noted, the Monge problem between \( P_n \) and \( Q_m \) is infeasible whenever \( n \neq m \). The Kantorovich problem is, however, always feasible, and takes the following form
\[ \hat{\pi} \in \arg\min_{\pi \in \mathcal{Q}_{nm}} \sum_{i=1}^{n} \sum_{j=1}^{m} \pi_{ij} \| X_i - Y_j \|^2, \]
where \( \mathcal{Q}_{nm} \) denotes the set of doubly stochastic matrices \( \pi = (\pi_{ij} : 1 \leq i \leq n, 1 \leq j \leq m) \), satisfying \( \sum_{i=1}^{n} \pi_{ij} = 1/m \) and \( \sum_{j=1}^{m} \pi_{ij} = 1/n \). We shall formulate the main stability bound of this section in terms of the quantity
\[ \Delta_{nm} = \sum_{i=1}^{n} \sum_{j=1}^{m} \hat{\pi}_{ij} \| T_0(X_i) - Y_j \|^2. \]

We obtain the following result.
Proposition 12. Let \( P, Q \in \mathcal{P}_{ac}(\Omega) \), and assume there exists \( \lambda > 0 \) such that assumption \( A1(\lambda) \) holds. Then,

\[
\frac{1}{\lambda} \mathbb{E}[\Delta_{nm}] \leq \mathbb{E}\left[ W^2_2(P_n, Q_m) - W^2_2(P, Q) \right] \leq \lambda \mathbb{E}[\Delta_{nm}].
\]

In particular, if \( (\kappa_n) \) denotes the sequence defined in equation (17), then \( \mathbb{E}[\Delta_{nm}] \leq \lambda \kappa_n^\land_m \).

To gain intuition about Proposition 12, it is fruitful to consider the special case \( n = m \). In this setting, it can be shown that the coupling \( \hat{\pi} \) in fact defines an optimal transport map, in the sense that there exists a permutation \( \tau \in S_n \) such that

\[
\pi_{ij} = I(j = \tau(i))/n, \quad \text{for all } 1 \leq i, j \leq n.
\]

Letting \( T_n : \text{supp}(P_n) \to \text{supp}(Q_n) \) denote the induced optimal transport map from \( P_n \) to \( Q_n \), given by \( T_n(X_i) = Y_{\sigma(i)} \), we have \( \Delta_{nn} = \| T_n - T_0 \|^2_{L^2(P_n)} \), and Proposition 12 then implies

\[
\mathbb{E}\| T_n - T_0 \|^2_{L^2(P_n)} \asymp \mathbb{E} W^2_2(P_n, Q_m) - W^2_2(P, Q).
\] (31)

Equation (31) is a two-sample analogue of Corollary 7, and shows that the \( L^2(P_n) \) risk of the in-sample transport map estimator is of same order as the bias of the two-sample empirical optimal transport cost. While the estimator \( T_n \) is only defined over the support of \( P_n \), we next show how it may easily be extended to the entire domain \( \Omega \).

4.2 Upper Bounds for Two-Sample Empirical Estimators

We bound the risk of two transport map estimators which interpolate the in-sample coupling \( \hat{\pi} \), and which achieve the \( \kappa_n^\land_m \) convergence rate up to polylogarithmic factors. Inspired by the classification K-nearest neighbor nonparametric regression estimator (Cover, 1968), we begin with perhaps the simplest such instance.

**One-Nearest Neighbor Estimator.** Let \( V_1, \ldots, V_n \) denote the Voronoi diagram generated by \( X_1, \ldots, X_n \), in the sense that

\[
V_j = \{ x \in \Omega : \| x - X_j \| \leq \| x - X_i \|, \; \forall i \neq j \}, \quad j = 1, \ldots, n.
\] (32)

Then, we define the one nearest neighbor estimator of \( T_0 \) by

\[
\hat{T}_{nm}^{1\text{NN}}(x) = \sum_{i=1}^n \sum_{j=1}^m (n\hat{\pi}_{ij}) I(x \in V_i) Y_j, \quad x \in \Omega.
\] (33)

In order to state an upper bound on the convergence rate of \( \hat{T}_{nm}^{1\text{NN}} \), we place the following mild condition on the support \( \Omega \).

\((S2)\) \( \Omega \) is standard, in the sense that there exist \( \epsilon_0, \delta_0 > 0 \) such that for all \( x \in \Omega \) and \( \epsilon \in (0, \epsilon_0) \),

\[
\mathcal{L}(B(x, \epsilon) \cap \Omega) \geq \delta_0 \mathcal{L}(B(x, \epsilon)),
\]

where recall that \( \mathcal{L} \) denotes the Lebesgue measure on \( \mathbb{R}^d \).
Condition (S2) arises frequently in the literature on statistical set estimation (Cuevas and Fraiman, 1997; Cuevas, 2009), and prevents $\Omega$ from admitting cusps. Under this condition, we arrive at the following upper bound, which we prove in Appendix C.1.

Proposition 13. Let $P \in \mathcal{P}_{ac}(\Omega)$ admit a density $p$ such that $\gamma^{-1} \leq p \leq \gamma$ for some $\gamma > 0$, and let $Q \in \mathcal{P}_{ac}(\Omega)$. Assume conditions $A1(\lambda)$ and (S1)–(S2). Then,

$$\mathbb{E}\|\hat{T}_{nm} - T_0\|_{L^2(P)}^2 \lesssim \lambda, \gamma, \epsilon_0, \delta_0 (\log n)^2 \kappa_{n,m}.$$  

Proposition 13 proves that the one-nearest neighbor estimator achieves the minimax rate in equation (3) of estimating optimal transport maps under the curvature condition $A1(\lambda)$ when $d \geq 5$, up to a polylogarithmic factor. This result is in stark contrast to standard risk bounds for $K$-nearest neighbor nonparametric regression, for which the number $K$ of nearest neighbors is typically required to diverge in order to achieve the minimax estimation rate of a Lipschitz continuous regression function (Györfi et al., 2006). Though increasing $K$ reduces the variance of such estimators, in our setting, Propositions 12–13 suggest that the variance of $\hat{T}_{nm}$ is already dominated by its large bias, stemming from that of the in-sample coupling $\hat{\pi}$, thereby making it sufficient to use $K = 1$ to obtain a near-optimal rate. While the one-nearest neighbor estimator is simplest to analyze, Proposition 13 seems to suggest that any linear smoother with sufficiently small bandwidth may be used to interpolate the in-sample coupling $\hat{\pi}_{nm}$ and lead to a similar rate. We leave it as an open problem to determine whether larger choices of bandwidth in such estimators can in fact lead to faster rates of convergence when stronger regularity conditions are placed on $T_0$.

Convex Least Squares Estimator. Though nearly minimax optimal, the estimator $\hat{T}_{nm}^{\text{LNN}}$ is typically not the gradient of a convex function, and is therefore not an admissible transport map in its own right. We next show how this property can be enforced using an estimator inspired by nonparametric least squares regression. Let $\mathcal{F}$ denote the class of functions $\varphi : \Omega \to \mathbb{R}$ which are convex and $\lambda$-Lipschitz, and define the least squares estimator

$$\hat{T}_{nm}^{\text{LS}} = \nabla \hat{\varphi}_{nm}, \quad \text{where} \quad \hat{\varphi}_{nm} \in \arg\min_{\varphi \in \mathcal{F}} \sum_{i=1}^n \sum_{j=1}^m \hat{\pi}_{ij} \|Y_j - \nabla \varphi(X_i)\|^2.$$  

The computation of the above infinite-dimensional optimization problem can be reduced to that of solving a finite-dimensional quadratic program, by a direct extension of well-known solvers for shape-constrained nonparametric regression with Lipschitz and convex constraints (cf. Seijo and Sen (2011), Mazumder et al. (2019), and references therein). We obtain the following upper bound by a simple extension of Proposition 13.

Proposition 14. Under the same conditions as Proposition 13, we have,

$$\mathbb{E}\|\hat{T}_{nm}^{\text{LS}} - T_0\|_{L^2(P)}^2 \lesssim \lambda, \gamma, \epsilon_0, \delta_0 (\log n)^2 \kappa_{n,m}.$$  

4.3 Upper Bounds for Two-Sample Smooth Estimators

We now study two-sample estimators under stronger smoothness assumptions on $P$ and $Q$, mirroring the smooth one-sample estimator in Section 3.3. Unlike Theorem 10, in which the smoothness of both $q$ and $\varphi_0$ were used to obtain sharp upper bounds for one-sample estimators, in the two-sample
case our analysis will also rely on the smoothness of estimators $\hat{\varphi}_{nm}$ of the potential $\varphi_0$. In order to quantify their regularity, we shall require a uniform analogue of Caffarelli’s global regularity theory (Theorem 3(ii)). Since we are unaware of such results for compact domains $\Omega \subseteq \mathbb{R}^d$, we instead assume throughout the remainder of this section that $\Omega$ is taken to be the $d$-dimensional torus $\mathbb{T}^d$, thus allowing us to appeal to Theorem 5. We note that such periodicity constraints are commonly imposed in nonparametric estimation problems to mitigate boundary issues (Efromovich, 1999; Krishnamurthy et al., 2014; Han et al., 2019). In many such cases, an alternative is to assume that the underlying probability measures place sufficiently small mass near the boundary. Such an assumption cannot be used in our context since, as before, we shall require all densities to be bounded away from zero throughout their support. It is well-known that optimal estimation rates under Wasserstein distances differ dramatically in the absence of such a condition (Bobkov and Gentil, 2005; Weed and Berthet, 2019), and we do not address this setting here.

We refer the reader to Section 2.2 for further background on the quadratic optimal transport problem over $\mathbb{T}^d$. Let $P, Q \in \mathcal{P}_{ac}(\mathbb{T}^d)$ be absolutely continuous measures admitting respective $\mathbb{Z}^d$-periodic densities $p$ and $q$. We continue to denote by $T_0$ the optimal transport map from $P$ to $Q$, with respect to the cost $d_{\mathbb{T}^d}$. As outlined in Proposition 4, $T_0$ is the gradient of a convex potential $\varphi_0 : \mathbb{R}^d \to \mathbb{R}^d$, which is uniquely determined $P$-almost everywhere up to addition of a constant, and we continue to denote by $\phi_0 = \|\cdot\|^2 - 2\varphi_0$ and $\psi_0 = \|\cdot\|^2 - 2\varphi_0^*$ the corresponding Kantorovich potentials. We also write for any Borel-measurable map $\hat{T}$,

$$\|\hat{T} - T_0\|_{L^2_{\text{per}}(P)}^2 = \int d_{\mathbb{T}^d}^2(\hat{T}(x), T_0(x)) dP(x).$$

Notice that $\|\cdot\|_{L^2_{\text{per}}(P)}$ is not a norm, but we retain the above suggestive definition by abuse of notation. Let $X_1, \ldots, X_n \sim P$ and $Y_1, \ldots, Y_m \sim Q$ denote i.i.d. samples, which we now assume to be independent of each other. Given density estimators $\hat{P}_n, \hat{Q}_m$ of $P, Q$ over $\mathbb{T}^d$, our aim is to bound the $L^2_{\text{per}}(P)$ risk of the corresponding plugin optimal transport map estimator, given by

$$\hat{T}_{nm} = \nabla \hat{\varphi}_{nm} = \arg\min_{T \in \mathcal{T}(\hat{P}_n, \hat{Q}_m)} \int d_{\mathbb{T}^d}^2(T(x), x) d\hat{P}_n(x). \quad (34)$$

As before, we shall also obtain upper bounds on the bias and risk of $W^2_2(\hat{P}_n, \hat{Q}_m)$ as a byproduct of our proofs. Indeed, our main results hinge upon the stability bounds derived in previous sections, which can easily be shown to hold in the present context.

**Proposition 15.** Assume $\varphi_0$ satisfies condition A1(\lambda), in the sense that $\varphi_0$ is a closed convex function over $\mathbb{R}^d$ satisfying

$$\frac{1}{\lambda} I_d \leq \nabla^2 \varphi_0(x) \leq \lambda I_d, \quad x \in \mathbb{R}^d.$$

Then, Theorem 6 continues to hold with $W_2$ replaced by $W_2$ and $\|\cdot\|_{L^2(P)}$ replaced by $\|\cdot\|_{L^2_{\text{per}}(P)}$.

As in the one-sample case, we focus on the situation where $\hat{P}_n$ and $\hat{Q}_m$ are wavelet density estimators. Unlike the boundary-corrected wavelet system used in Section 3.3, it will be convenient to introduce a simpler basis which guarantees that the density estimators are periodic. Specifically, we describe in Appendix A.2.2 how the standard Daubechies wavelet system may easily be periodized.
to obtain a set of \( \mathbb{Z}^d \)-periodic functions

\[
\Psi_{\text{per}} = \{1\} \cup \bigcup_{j=0}^{\infty} \Psi_{j,\text{per}}, \quad \text{where} \quad \Psi_{j,\text{per}} = \{\xi_{j,k,\ell}^{\text{per}} : 0 \leq k \leq 2^{j-1}, \ell \in \{0,1\}^d \setminus \{0\}, j \geq 0,
\]

which forms an orthonormal basis of \( L^2(\mathbb{T}^d) \) (Daubechies (1992), Section 9.3; Giné and Nickl (2016), Section 4.3). For the remainder of this section, we do not make use of the boundary-corrected basis \( \Psi_{\text{bc}} \), thus we shall drop the superscript “per” in the above display without risk of confusion.

Whenever the densities \( p,q \) lie in \( L^2(\mathbb{T}^d) \), they admit wavelet expansions of the form

\[
p = 1 + \sum_{j=0}^{\infty} \sum_{\xi \in \Psi_j} \alpha_{\xi} \xi, \quad q = 1 + \sum_{j=0}^{\infty} \sum_{\xi \in \Psi_j} \beta_{\xi} \xi,
\]

where \( \alpha_{\xi} = \int \xi dP \) and \( \beta_{\xi} = \int \xi dQ \). We then define the wavelet density estimators

\[
\tilde{p}_n = 1 + \sum_{j=0}^{J_n} \sum_{\xi \in \Psi_j} \hat{\alpha}_{\xi} \xi, \quad \tilde{q}_m = 1 + \sum_{j=0}^{J_m} \sum_{\xi \in \Psi_j} \hat{\beta}_{\xi} \xi,
\]

where \( \hat{\alpha}_{\xi} = \int \xi dP_n \) and \( \hat{\beta}_{\xi} = \int \xi dQ_m \). By orthonormality of \( \Psi \), it is straightforward to see that \( \tilde{p}_n, \tilde{q}_m \) integrate to unity, but may nevertheless be negative. As in Section 3.3, we therefore define the final density estimators by

\[
\hat{p}_n \propto \tilde{p}_n I(\tilde{p}_n \geq 0), \quad \hat{q}_m \propto \tilde{q}_m I(\tilde{q}_m \geq 0),
\]

where the proportionality constants are to be chosen such that \( \hat{p}_n \) and \( \hat{q}_m \) define respective probability distributions \( \hat{P}_n, \hat{Q}_m \in \mathcal{P}_{\text{ac}}(\mathbb{T}^d) \). As we state in Proposition 30, Appendix D, the one-sample results from Theorem 10 may readily be extended to the present periodic setting, by replacing the boundary-corrected wavelet estimator therein by the periodic wavelet estimator \( \hat{q}_m \) in the above display. Building upon this observation, we arrive at the following bound for the two-sample estimator \( \hat{T}_{nm} \) in equation (34), together with the associated plugin estimator of the quadratic optimal transport cost.

**Theorem 16.** Assume \( P,Q \in \mathcal{P}_{\text{ac}}(\mathbb{T}^d) \) admit densities \( p,q \) such that \( \gamma^{-1} \leq p,q \leq \gamma \), and such that

\[
\|p\|_{C^{\alpha-1}(\mathbb{T}^d)} \vee \|q\|_{C^{\alpha-1}(\mathbb{T}^d)} \leq M < \infty,
\]

for some \( \alpha > 1 \). Assume \( 2^{J_n} \asymp (\log n/n)^{1/(2\alpha-2)} \). Then, there exists a constant \( C > 0 \) depending only on \( d,M,\gamma,\alpha \) such that the following hold for all \( \epsilon > 0 \).

(i) **(Optimal Transport Maps)** We have,

\[
\mathbb{E} \left\| \hat{T}_{nm} - T_0 \right\|_{L^2_{\text{per}}(P)}^2 \leq C R_{T,n\wedge m}(\alpha;\epsilon), \quad \text{where} \quad R_{T,n}(\alpha;\epsilon) := n^\epsilon \begin{cases} n^{-2\alpha/(2(\alpha-1))}, & d \geq 3 \\ n^{-1}, & d < 3. \end{cases}
\]

(ii) **(Wasserstein Distances)** We have,
\[ |EW^2_2(\hat{P}_n, \hat{Q}_m) - W^2_2(P, Q)| \leq CR_{T,n}^{}(\alpha; \epsilon) \]

\[ \mathbb{E}|W^2_2(\hat{P}_n, \hat{Q}_m) - W^2_2(P, Q)| \leq CR_{W,n}^{}(\alpha; \epsilon), \text{ where } R_{W,n}^{}(\alpha; \epsilon) = n \left\{ \begin{array}{ll}
  n^{-\frac{2\alpha}{2(1-\alpha) + d}}, & d \geq 2(1 + \alpha) \\
  n^{-1/2}, & d < 2(1 + \alpha). \end{array} \right. \]

The proof appears in Appendix D.2. Theorem 16 shows that the plugin estimators \( \hat{T}_{nm} \) and \( \hat{W}^2_2(\hat{P}_n, \hat{Q}_m) \) achieve the same convergence rates as witnessed in the one-sample setting, up to an exponent \( \epsilon \) which may be made arbitrarily small. In fact, it can be seen from our proof that \( \epsilon \) may be taken to equal zero when \( \alpha \) is not an integer, at the expense of the same polylogarithmic factors as in the statement of Theorem 10. Unlike the latter result, we also note that Theorem 16 places no conditions on the regularity of \( T_0 \) or \( \varphi_0 \). Indeed, over \( \mathbb{T}^d \), these can be inferred from equation (36) due the uniform regularity result of Theorem 5.

5  A Central Limit Theorem for Smooth Wasserstein Distances

In this section, we derive limit theorems for the density plugin estimators appearing in Sections 3.3 and 4.3. We again treat the one- and two-sample settings separately.

5.1 One-Sample Central Limit Theorem

We adopt the same notation and assumptions as in Section 3.3. Specifically, throughout this subsection, \( \hat{Q}_n \) denotes the distribution of the density estimator \( \hat{q}_n \) defined in equation (23), constructed using the boundary-corrected wavelet basis over \( \Omega = [0, 1]^d \). Furthermore, \( \hat{T}_n = \nabla \hat{\varphi}_n \) denotes the optimal transport map from \( P \) to \( \hat{Q}_n \), as defined in equation (22), and \( \hat{\psi}_n = \| \cdot \|^2 - 2\hat{\varphi}_n^* \).

Recall that our stability bounds in Theorem 6 characterize the bias of \( W^2_2(P, \hat{Q}_n) \) in terms of two terms; heuristically, one has

\[ EW^2_2(P, \hat{Q}_n) - W^2_2(P, Q) \sim E \left| \int \psi_0 d(\hat{Q}_n - Q) \right| + EW^2_2(\hat{Q}_n, Q). \]

In contrast, we next show that the variance of \( W^2_2(P, \hat{Q}_n) \) is merely dominated by that of the linear functional appearing in the above display, whenever \( Q \) admits a sufficiently smooth density.

**Proposition 17.** Assume the same conditions as Theorem 10(i), and that \( d < 2\alpha \). Define

\[ R_n = W^2_2(P, \hat{Q}_n) - \int \psi_0 d\hat{Q}_n. \]

Then, \( n \text{Var}(R_n) \to 0 \) as \( n \to \infty \).

Proposition 17 is strongly inspired by del Barrio and Loubes (2019), who proved an analogue of this result when \( \hat{Q}_n \) is replaced by the empirical measure \( Q_n \). This observation led them to obtain a central limit theorem of the form

\[ \sqrt{n} \left( W^2_2(P, Q_n) - EW^2_2(P, Q_n) \right) \sim N(0, \sigma^2), \]
where \( \sigma^2 = \text{Var}[\psi_0(Y)] \). While such a result is very useful, and holds under significantly milder conditions on \( P \) and \( Q \) than those of Proposition 17, it has limited applicability to statistical inference since it is centered at \( \mathbb{E}W^2_2(P, Q_n) \) rather than \( W^2_2(P, Q) \). Notice that this centering constant cannot generally be improved when \( d \geq 5 \), since the bias of \( W^2_2(P, Q_n) \) is generically of order \( n^{-2/d} \) by Corollary 7. In contrast, under our setting, we shall see that the strong smoothness condition \( d < 2\alpha \) of Proposition 17 implies together with Theorem 10 and Lemma 11 that

\[
|\mathbb{E}W^2_2(P, Q_n) - W^2_2(P, Q)|^2 = o\left(\frac{1}{n}\right), \quad \text{and} \quad \text{Var} \left[ W^2_2(P, Q_n) \right] = \frac{\sigma^2 + o(1)}{n}.
\]

The above display shows that \( W^2_2(P, Q_n) \) has bias which is of lower order than its variance, whenever \( \sigma > 0 \), and leads to the following limit theorem centered at \( W^2_2(P, Q) \).

**Theorem 18.** Let \( d, \alpha > 1 \), \( \Omega = [0, 1]^d \), and assume \( d < 2\alpha \). Let \( P, Q \in \mathcal{P}_{ac}(\Omega) \) admit strictly positive densities \( p, q \in \mathcal{C}^{\alpha-1}(\Omega) \), such that the optimal Brenier potential \( \varphi_0 \) in the optimal transport problem from \( P \) to \( Q \) with respect to \( \|\cdot\|^2 \) lies in \( \mathcal{C}^{\alpha+1}(\Omega) \). Let \( \hat{Q}_n \) be the distribution of the boundary-corrected wavelet density estimator defined in equation (23) with respect to an i.i.d. sample \( Y_1, \ldots, Y_n \sim Q \). Then,

\[
\sqrt{n} \left( W^2_2(P, \hat{Q}_n) - W^2_2(P, Q) \right) \rightsquigarrow N(0, \sigma^2), \quad \text{as} \ n \to \infty,
\]

where \( \sigma^2 = \text{Var}[\psi_0(Y)] \) and \( \psi_0 = \|\cdot\|^2 - 2\varphi_0^\ast \).

The proof is deferred to Appendix E. To the best of our knowledge, Theorem 18 is the first known central limit theorem for a plugin estimator of the Wasserstein distance between absolutely continuous distributions in general dimension, centered at its population counterpart. We emphasize that the parametric \( \sqrt{n} \) scaling in the above result is made possible by the smoothness condition \( d < 2\alpha \). In view of Theorem 10, we do not generally expect that a central limit theorem for \( W^2_2(P, \hat{Q}_n) \) centered at its population counterpart can be obtained when \( d > 2(\alpha + 1) \), in which case we expect the squared bias of this estimator to dominate its variance. We conjecture that the threshold \( d < 2\alpha \) in Theorem 18 can be improved to \( d < 2(\alpha + 1) \), but we do not have a proof.

Due to the absolute continuity of \( Q \), notice that the variance \( \sigma^2 \) is positive if and only if \( \psi_0 \) is non-constant. It is easy to see that the latter condition holds whenever \( T_0 \) is not the identity map, thus the distributional limit in Theorem 18 is non-degenerate whenever \( P \neq Q \). When \( P = Q \), it could already have been deduced from Lemma 9 that the correct scaling for the process \( W^2_2(P, \hat{Q}_n) \) is of lower order than \( n^{-1/2} \), and we leave open the question of obtaining limit theorems under this null regime. Distinctions between limiting distributions at the null \( (P = Q) \) and away from the null \( (P \neq Q) \) are already well-known to arise for empirical estimators of Wasserstein distances in the special cases of one-dimensional (Munk and Czado, 1998; Freitag and Munk, 2005) or discrete measures (Sommerfeld and Munk, 2018; Tameling et al., 2019).

For applications to statistical inference, we also note that the variance \( \sigma^2 \) is straightforward to estimate.
Corollary 19. Assume the same conditions as Theorem 18, and assume $P \neq Q$. Let $\hat{T}_n = \nabla \hat{\varphi}_n$ denote the optimal transport map from $P$ to $\hat{Q}_n$ with respect to $\|\cdot\|^2$, and let $\tilde{\psi}_n = \|\cdot\|^2 - 2\hat{\varphi}_n$. Then,

$$\hat{\sigma}_{n}^2 = \int \tilde{\psi}_n^2 dQ_n - \left( \int \tilde{\psi}_n dQ_n \right)^2 \to^p \sigma^2 = \text{Var}[\phi_0(Y)].$$

In particular, it follows that as $n \to \infty$,

$$\sqrt{\frac{n}{\hat{\sigma}_{n}^2}} \left( W_2^2(P,\hat{Q}_n) - W_2^2(P,Q) \right) \Rightarrow N(0,1).$$

Under the conditions of Corollary 19, we deduce that $W_2^2(P,\hat{Q}_n) \pm \hat{\sigma}_{n} z_{\delta/2}/\sqrt{n}$ is an asymptotic $(1 - \delta)$-confidence interval for $W_2^2(P,Q)$ whenever $P \neq Q$, where $z_{\delta/2}$ denotes the $\delta/2$ quantile of the standard Gaussian distribution, for any $\delta \in (0,1)$. To the best of our knowledge, this is the first confidence interval for Wasserstein distances between absolutely continuous distributions in general dimension, albeit under the strong smoothness condition $2\alpha > d$.

5.2 Two-Sample Central Limit Theorem

To obtain a central limit theorem in the two-sample setting where both $P$ and $Q$ are unknown, we again focus on the quadratic optimal transport problem over $\mathbb{T}^d$.

Theorem 20. Let $\alpha, d \geq 1$ satisfy $d < 2\alpha$. Let $P, Q \in \mathcal{P}_{ac}(\mathbb{T}^d)$ be distributions admitting strictly positive densities $p, q \in C^{\alpha-1}(\mathbb{T}^d)$. Denote by $\hat{P}_n$ and $\hat{Q}_m$ the distributions of the periodic wavelet density estimators defined in equation (35), based on i.i.d. samples $X_1, \ldots, X_n \sim P$ and $Y_1, \ldots, Y_m \sim Q$, which are independent of each other. Then, as $n, m \to \infty$ such that $\frac{n}{n+m} \to \rho \in (0,1)$, we have

$$\sqrt{\frac{nm}{n+m}} \left( W_2^2(\hat{P}_n, \hat{Q}_m) - W_2^2(P,Q) \right) \Rightarrow N(0,\sigma_{\rho}^2),$$

where

$$\sigma_{\rho}^2 = \rho \text{Var}[\phi_0(X)] + (1 - \rho) \text{Var}[\psi_0(Y)],$$

and where $(\phi_0, \psi_0)$ is a pair of optimal Kantorovich potentials in the optimal transport problem from $P$ to $Q$ with respect to $d_{T^d}^2$.

Theorem 20 mirrors the one-sample result from the previous section, and shows that a two-sample central limit theorem can be obtained under the constraint of periodicity on the measures involved. The proof is deferred to Appendix E.4, where a central step is a two-sample analogue of Proposition 17, which is again inspired by del Barrio and Loubes (2019).

As before, a consistent estimator of $\sigma_{\rho}^2$ is also straightforward to obtain. Let $(\hat{\phi}_{nm}, \hat{\psi}_{nm})$ be a pair of optimal Kantorovich potentials in the $d_{T^d}^2$-optimal transport problem from $\hat{P}_n$ to $\hat{Q}_m$, and set

$$\hat{\sigma}_{nm}^2 := \frac{n}{n+m} \left[ \int \hat{\phi}_{nm}^2 dP_n - \left( \int \hat{\phi}_{nm} dP_n \right)^2 \right] + \frac{m}{n+m} \left[ \int \hat{\psi}_{nm}^2 dQ_m - \left( \int \hat{\psi}_{nm} dQ_m \right)^2 \right].$$

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The following result is a direct extension of Corollary 19, and is stated without proof.

**Corollary 21.** Assume the same conditions as Theorem 20, and assume $P \neq Q$. Then, as $n, m \to \infty$ such that $\frac{n}{n+m} \to \rho \in (0, 1)$, we have $\frac{\hat{\sigma}^2_{nm}}{\sigma^2_{\rho}} \to \frac{\sqrt{n m}}{\sigma^2_{\rho}(n + m)} \left( W_2^2(\hat{P}_n, \hat{Q}_m) - W_2^2(P, Q) \right) \sim N(0, 1), \quad (38)$

As before, Corollary 21 implies that $W_2^2(\hat{P}_n, \hat{Q}_m) \pm z_{\delta/2} \sqrt{\frac{\hat{\sigma}^2_{nm}(n + m)}{nm}}$ is an asymptotic two-sample $(1 - \delta)$-confidence interval for $W_2^2(P, Q)$, for any $\delta \in (0, 1)$.

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**A Smoothness Classes and Wavelet Density Estimation**

In this Appendix, we collect several definitions and properties about Hölder spaces, Besov spaces, and wavelet density estimators, which are used throughout our proofs.

**A.1 Hölder Spaces**

Given a closed set $\Omega \subseteq \mathbb{R}^d$, a function $f : \Omega \to \mathbb{R}$ which is differentiable up to order $k \geq 1$ in the interior of $\Omega$, and a multi-index $\beta \in \mathbb{N}^d$, we write $|\beta| = \sum_{i=1}^d \beta_i$, and for all $|\beta| \leq k$,

$$D^\gamma f = \frac{\partial^{|\gamma|} f}{\partial x_1^{\gamma_1} \ldots \partial x_d^{\gamma_d}}.$$

Given $\alpha > 0$, the Hölder space $C^\alpha(\Omega)$ is defined as the set of functions $f : \Omega \to \mathbb{R}$ which are differentiable to order $|\alpha|$ in the interior of $\Omega$, with derivatives extending continuously up to the boundary of $\Omega$, and such that the Hölder norm

$$\|f\|_{C^\alpha(\Omega)} = \sum_{j=0}^{[\alpha]} \sup_{|\gamma| = j} \|D^\gamma f\|_{L^\infty(\Omega)} + \sum_{|\gamma| = [\alpha]} \sup_{x, y \in \Omega, x \neq y} \frac{|D^\gamma f(x) - D^\gamma f(y)|}{\|x - y\|^{\alpha - [\alpha]}}$$

is finite. We also let $C_u(\Omega)$ denote the set of uniformly continuous real-valued functions on $\Omega$. Furthermore, $C^\alpha(\mathbb{T}^d)$ (resp. $C_u(\mathbb{T}^d)$) is defined as the set of $\mathbb{Z}^d$-periodic functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $f \in C^\alpha(\mathbb{R}^d)$ (resp. $f \in C_u(\mathbb{R}^d)$).
A.2 Wavelets and Besov Spaces

Recall that in Sections 3.3 and 4.3, we made use of the boundary-corrected wavelet system $\Psi^{bc}$ over the unit cube $[0, 1]^d$, and of the periodic wavelet system $\Psi^{Per}$ over the flat torus $\mathbb{T}^d$. In this section, we provide further descriptions and properties of these wavelet bases, before turning to definitions and characterizations of Besov spaces over $[0, 1]^d$ and $\mathbb{T}^d$. For concreteness, we describe these constructions in terms of the compactly-supported $N$-th Daubechies scaling and wavelet functions $\zeta_0, \xi_0 \in C^r(\mathbb{R}^d)$, where $r \geq 0.18(N - 1)$ for an integer $N \geq 2$ (Daubechies (1988); Giné and Nickl (2016), Theorem 4.2.10). Throughout the sequel and the rest of the paper, whenever we work with a Besov space $B^s_{p,q}$ or a Hölder space $C^\alpha$, we tacitly assume that the wavelet parameter $r$ is chosen to be strictly greater than the regularity parameters $|s|$ or $\alpha$.

Our exposition closely follows that of Giné and Nickl (2016), and we also refer the reader to Cohen et al. (1993); Cohen (2003); Härdle et al. (2012) and references therein for further details.

A.2.1 Boundary-Corrected Wavelets on $[0, 1]^d$

It is well-known that the Daubechies wavelet system

$$\zeta_{0k} = \zeta_0(\cdot - k), \quad \xi_{0jk} = 2^j\zeta_0(2^j(\cdot) - k), \quad j \geq 0, \ k \in \mathbb{Z},$$

forms a basis of $L^2(\mathbb{R})$, with the property that $\{\zeta_{0k} : k \in \mathbb{Z}\}$ spans all polynomials on $\mathbb{R}$ of degree at most $N - 1$. While this family may easily be periodized to obtain a basis for $L^2([0, 1])$, as in the following subsection, doing so may not accurately reflect the regularity of functions in $L^2([0, 1])$ via the decay of their wavelet coefficients, near the boundaries of the interval. This consideration motivated Meyer (1991) and Cohen et al. (1993) to introduce the so-called boundary-corrected wavelet system on $[0, 1]$, which preserves the standard Daubechies scaling functions lying sufficiently far from the boundaries of the interval, and adds edge scaling functions such that their union continues to span all polynomials up to degree $N - 1$ on $[0, 1]$. In short, given a fixed integer $j_0 \geq 1 + \log_2 N$, the construction of Cohen et al. (1993) leads to smooth scaling and wavelet edge basis functions

$$\begin{align*}
\zeta_{0jk}^{\text{left}} &\equiv \zeta_{0jk}^a \quad \text{supported in } [0, (2N - 1)/2^{j_0}], \\
\zeta_{0jk}^{\text{right}} &\equiv \zeta_{0jk}^a \quad \text{supported in } [1 - (2N - 1)/2^{j_0}, 1], \quad k = 0, \ldots, N - 1.
\end{align*}$$

In this case, if one defines

$$\zeta_{0jk}^a = 2^{-j_0} \zeta_{0jk}^a (2^{j_0} \cdot), \quad \xi_{0jk}^a = 2^{-j_0} \xi_{0jk}^a (2^{j_0} \cdot), \quad \text{for all } j \geq j_0, \ a \in \{\text{left, right}\},$$

then the family

$$\Phi^{bc}_0 = \{\zeta_{0jk}^{bc} : 0 \leq k \leq 2^{j_0} - 1\} = \{\zeta_{0jk}^{\text{left}}, \zeta_{0jk}^{\text{right}}, \zeta_{0m} : 0 \leq k \leq N - 1, \ N \leq m \leq 2^{j_0} - N - 1\}, \quad \Psi^{bc}_0 = \{\xi_{0jk}^{bc} : 0 \leq k \leq 2^j - 1, \ j \geq j_0\} = \{\xi_{0jk}^{\text{left}}, \xi_{0jk}^{\text{right}}, \xi_{0jm} : 0 \leq k \leq N - 1, \ N \leq m \leq 2^{j_0} - N - 1, \ j \geq j_0\},$$

form a basis of $L^2([0, 1])$, with the property that $\Phi^{bc}$ spans all polynomials on $[0, 1]$ of degree at most $N - 1$. We then define a tensor product wavelet basis of $L^2([0, 1]^d)$ by setting for all $j \geq j_0$...
and all $\ell = (\ell_1, \ldots, \ell_d) \in \{0,1\}^d \setminus \{0\}$,

$$\zeta_{j0k}^{bc}(x) = \prod_{i=1}^{d} \zeta_{0jk_i}^{bc}(x_i), \quad \text{and} \quad \zeta_{jk\ell}^{bc}(x) = \prod_{i: \ell_i = 0} \zeta_{0jk_i}^{bc}(x_i) \prod_{i: \ell_i = 1} \zeta_{0jk_i}^{bc}(x_i), \quad x \in [0,1]^d,$$

where in the definition of $\zeta_{j0k}^{bc}$, the index $k = (k_1, \ldots, k_d)$ ranges over $\mathcal{K}(j_0) := \{1, \ldots, 2^{j_0} - 1\}^d$, while in the definition of $\zeta_{j\ell}^{bc}$, $k$ ranges over $\mathcal{K}(j)$. In this case, the wavelet system

$$\Psi_{j0}^{bc} = \Phi^{bc} \cup \bigcup_{j=j_0}^{\infty} \Phi_{j}^{bc}, \quad \Phi^{bc} = \{\zeta_{j0k}^{bc} : k \in \mathcal{K}(j_0)\}, \quad \Psi_{j}^{bc} = \{\zeta_{j\ell}^{bc} : k \in \mathcal{K}(j)\}, \quad j \geq j_0,$$

announced in Section 3.3 forms a basis of $L^2([0,1]^d)$. We sometimes make use of the abbreviation $\Psi_{j_0 - 1} = \Phi$.

### A.2.2 Periodic Wavelets on $\mathbb{T}^d$

When working over $\mathbb{T}^d$, a simpler construction may be used due to the periodicity of the functions involved. Denote the periodization on $\mathbb{T}$ of dilations of the maps $\zeta_0, \xi_0$ by

$$\zeta_0^{\text{per}} = \sum_{k \in \mathbb{Z}} \zeta_0(\cdot - k) = 1, \quad \xi_0^{\text{per}} = \sum_{k \in \mathbb{Z}} 2^{j/2} \xi_0(2^j (\cdot - k)), \quad j \geq 0.$$

In this case, the collection

$$\Psi_0^{\text{per}} = \left\{1, \xi_{0j}^{\text{per}}(\cdot - 2^{-j}k) : 0 \leq k \leq 2^j - 1, j \geq 0\right\}$$

forms an orthonormal basis of $L^2(\mathbb{T})$, which may again be extended to $L^2(\mathbb{T}^d)$ using tensor product wavelets. Specifically, if $\xi_{jk\ell}^{\text{per}} = \prod_{i=1}^{d} (\xi_{j\ell_i}^{\text{per}})^{\ell_i}$ for all $\ell = (\ell_1, \ldots, \ell_d) \in \{0,1\}^d \setminus \{0\}$, then

$$\Psi^{\text{per}} = \{1\} \cup \bigcup_{j=0}^{\infty} \Psi_j^{\text{per}} \quad \text{with} \quad \Psi_j^{\text{per}} = \{\xi_{jk\ell}^{\text{per}} : k \in \mathcal{K}(j), \ell \in \{0,1\}^d \setminus \{0\}\}, \quad j \geq 0,$$

forms an orthonormal basis of $L^2(\mathbb{T}^d)$ (Daubechies (1992), Section 9.3; Giné and Nickl (2016), Section 4.3).
A.2.3 Properties of Boundary-Corrected and Periodic Wavelet Systems

In both of the preceding constructions, one obtains a family $\Phi$ of scaling functions and a sequence of families $(\Psi_j)_{j \geq j_0}$ of wavelet functions, such that

$$\Phi = \Psi_{j_0-1} = \{ \zeta_k : k \in K(j_0) \} = \begin{cases} \Phi^{bc}, & \Psi = \Psi^{bc} \\ \{1\}, & \Psi = \Psi^{per}, \end{cases}$$

$$\Psi_j = \{ \xi_{jk\ell} : k \in K(j), \ell \in \{0,1\} \} = \begin{cases} \Phi^{bc}_j, & \Psi = \Psi^{bc} \\ \Psi^{per}_j, & \Psi = \Psi^{per}, \end{cases} \quad j \geq j_0,$$

$$j_0 = \begin{cases} 1 + \lceil \log_2 N \rceil, & \Psi = \Psi^{bc} \\ 0, & \Psi = \Psi^{per}, \end{cases}$$

$$K(j) = \{0, \ldots, 2^j - 1\}^d, \quad j \geq j_0.$$

In both cases, the wavelet system is defined over a domain $\Omega$, which is to be understood as either $[0,1]^d$ in the boundary-corrected case, or as $\mathbb{T}^d$ (which itself may be identified with $(0,1]^d$) in the periodic case. In either of these settings, the wavelet system

$$\Psi = \Phi \cup \bigcup_{j = j_0}^{\infty} \Psi_j$$

forms a basis of $L^2(\Omega)$. The following simple result collects several properties and definitions which are common to both of the above bases.

**Lemma 22.** There exist constants $C_1, C_2 > 0$ depending only on $d,r$ and on the choice of basis $\Phi \in \{\Phi^{bc}, \Phi^{per}\}$ such that the following properties hold.

(i) $|\Phi| \leq C_1$, $|\Psi_j| \leq C_2 2^{dj_j}$ for all $j \geq j_0$.

(ii) For all $j \geq j_0$ and $\xi \in \Psi_j$, there exists a set $I_\xi \subseteq \Omega$ such that diam$(I_\xi) \leq C_1 2^{-j}$, supp$(\xi_j) \subseteq I_\xi$, and $\left\| \sum_{\xi \in \Psi_j} I(\cdot \in I_\xi) \right\|_{L^\infty} \leq C_2$.

(iii) $\xi \in C^r(\Omega)$ for all $\xi \in \Psi$.

(iv) Polynomials of degree at most $N - 1$ over $\Omega$ lie in $\text{Span}(\Phi)$.

(v) We have,

$$\sup_{0 \leq |\gamma| \leq [r]} \sum_{\zeta \in \Phi} \| D^\gamma \zeta \|_{L^\infty} \leq C_1,$$

$$\sup_{0 \leq |\gamma| \leq [r]} \sum_{\zeta \in \Psi_j} 2^{-j \left( \frac{d}{2} + |\gamma| \right)} \| D^\gamma \zeta \|_{L^\infty} \leq C_2.$$

A.2.4 Besov Spaces

We next define the Besov spaces $B^s_{p,q}(\Omega)$, for $s \in \mathbb{R} \setminus \{0\}$, $p,q \geq 1$. Once again, $\Omega$ is understood to be one of $[0,1]^d$ and $\mathbb{T}^d$, and $\Psi$ is understood to be the corresponding wavelet basis as in equation (39).
Let $f \in L^p(\Omega)$ admit the wavelet expansion

$$f = \sum_{\zeta \in \Phi} \beta_{\zeta} \zeta + \sum_{j=j_0}^{\infty} \sum_{\xi \in \Psi_j} \beta_{\xi} \xi, \quad \text{over } \Omega,$$

with convergence in $L^p(\Omega)$, where $\beta_{\xi} = \int \xi f$ for all $\xi \in \Psi$. Then, the Besov norm of $f$ may be defined by

$$\|f\|_{B_{p,q}^s(\Omega)} = \| (\beta_{\zeta})_{\zeta \in \Phi} \|_{\ell_p} + \left\| (2^{j\left(s + \frac{d}{2} - \frac{d}{p}\right)} \| (\beta_{\xi})_{\xi \in \Psi_j} \|_{\ell_p})_{j \geq j_0} \right\|_{\ell_q},$$

and we define

$$B_{p,q}^s(\Omega) = \begin{cases} \{ f \in L^p(\Omega) : \|f\|_{B_{p,q}^s(\Omega)} < \infty \}, & 1 \leq p < \infty \\ \{ f \in C_u(\Omega) : \|f\|_{B_{p,q}^s(\Omega)} < \infty \}, & p = \infty. \end{cases}$$

We shall primarily make use of Besov spaces in order to characterize Hölder continuous functions in terms of the decay of their wavelet coefficients, via the following classical result.

**Lemma 23.** For all $0 < s < r$, and $d \geq 1$, we have

$$C^s([0,1]^d) \subseteq B_{\infty,\infty}^s([0,1]^d), \quad C^s(T^d) \subseteq B_{\infty,\infty}^s(T^d),$$

and there exist $C_1, C_2 > 0$ such that

$$\| \cdot \|_{C^s([0,1]^d)} \leq C_1 \| \cdot \|_{B_{\infty,\infty}^s([0,1]^d)}; \quad \| \cdot \|_{C^s(T^d)} \leq C_2 \| \cdot \|_{B_{\infty,\infty}^s(T^d)}.$$

Furthermore, if $s \notin \mathbb{N}$, then equation (40) holds with equality, and with equivalent norms.

An analogue of Lemma 23 is well-known to hold for the Daubechies wavelet system over $\mathbb{R}^d$, in which case it can readily be proven using an equivalent characterization of Besov spaces in terms of moduli of smoothness (Giné and Nickl (2016), Section 4.3.1). Such characterizations are also available for the periodized and boundary-corrected wavelet systems (Giné and Nickl (2016), Theorem 4.3.26 and discussions in Sections 4.3.5–4.3.6), and at least in the periodized case can be shown to lead to Lemma 23 (Giné and Nickl (2016), equation (4.167)). For the boundary-corrected case, Lemma 23 is known to hold in the special case $d = 1$ (Cohen et al. (1993), Theorem 4; Giné and Nickl (2016), equation (4.152)), but we do not know of a reference stating this precise result when $d > 1$, in part due to the potential ambiguity of defining the Hölder space $C^s([0,1]^d)$ over the closed set $[0,1]^d$. We thus provide a self-contained proof of Lemma 23 in the boundary-corrected case for completeness, using standard arguments.

**Proof of Lemma 23 (Boundary-Corrected Case).** Let $\Omega = [0,1]^d$. Suppose first that $f \in B_{\infty,\infty}^s(\Omega)$ for some $s \notin \mathbb{N}$, with wavelet expansion

$$f = \sum_{\zeta \in \Phi^{bc}} \beta_{\zeta} \zeta + \sum_{j=j_0}^{\infty} \sum_{\xi \in \Psi_j^{bc}} \beta_{\xi} \xi.$$
By Lemma 22, $\xi \in C^r(\Omega)$ for all $\xi \in \Psi^{bc}$, where recall that $s < r$, thus we may define the map

$$f_\gamma = \sum_{\zeta \in \Phi^{bc}} \beta_\zeta D^\gamma \zeta + \sum_{j=j_0}^\infty \sum_{\xi \in \Phi_j^{bc}} \beta_\xi D^\gamma \xi, \quad \text{for all } 0 \leq |\gamma| \leq |s|.$$  

Notice that $\|D^\gamma \zeta\|_{L^\infty} \lesssim 1$ for all $\zeta \in \Phi^{bc}$, and for all $j \geq j_0, k \in K(j)$, $\ell \in \{0,1\}^d \setminus \{0\}$,

$$D^\gamma \xi^{bc}_{jk\ell} = 2^{(j-j_0)(\frac{d}{2}+|\gamma|)} D^\gamma \xi^{bc}_{j_0k\ell}(2^{j-j_0} \cdot)$$

Then, it follows from Lemma 22 that for all $x \in \Omega^c$,

$$|f_\gamma(x)| \leq \sum_{\zeta \in \Phi^{bc}} |\beta_\zeta D^\gamma \zeta(x)| + \sum_{j=j_0}^\infty \sum_{\xi \in \Phi_j^{bc}} |\beta_\xi D^\gamma \xi(x)|$$

$$\lesssim \|\beta_\zeta\|_{\Phi^{bc}} \|\phi_{\zeta}\|_{L^\infty} + \sum_{j=j_0}^\infty \|\beta_\xi\|_{\Phi_{\gamma}^{bc}} \|\phi_{\xi}\|_{L^\infty} 2^{(j-j_0)(\frac{d}{2}+|\gamma|)} \sum_{\xi \in \Phi_j^{bc}} I(|\zeta(x)| > 0)$$

$$\lesssim \|\beta_\zeta\|_{\Phi^{bc}} \|\phi_{\zeta}\|_{L^\infty} + \sum_{j=j_0}^\infty 2^{j(\frac{d}{2}+|\gamma|)} \|\beta_\xi\|_{\Phi_{\gamma}^{bc}} \|\phi_{\xi}\|_{L^\infty}$$

$$\lesssim \|\beta_\zeta\|_{\Phi^{bc}} \|\phi_{\zeta}\|_{L^\infty} + \left(2^{j(\frac{d}{2}+s)} \|\beta_\xi\|_{\Phi_{\gamma}^{bc}} \|\phi_{\xi}\|_{L^\infty}\right) \leq \sum_{j=j_0}^\infty 2^{(|\gamma|-s)j} \lesssim \|f\|_{B^{s}_{\infty,\infty}(\Omega)}, \quad (41)$$

where on the final line, we used the fact that $s$ is not an integer, thus $|\gamma| < s$. An analogous calculation reveals that the series defining $f_\gamma$ converges uniformly for any $0 \leq |\gamma| \leq |s|$, thus it must follow that $f$ is differentiable up to order $|s|$ with derivatives given by $D^\gamma f = f_\gamma$, which by equation (41) must satisfy $|D^\gamma f(x)| \leq C \|f\|_{B^{s}_{\infty,\infty}(\Omega)}$ for all $x \in \Omega^c$, for a constant $C > 0$ depending only on $d$ and $r$. We next prove that $D^\gamma f$ is uniformly $(s - |s|)$-Hölder continuous over $\Omega^c$, for all $|\gamma| = |s|$. For all $x, y \in \Omega^c$, we have,

$$|D^\gamma f(x) - D^\gamma f(y)| \leq \sum_{\zeta \in \Phi^{bc}} |\beta_\zeta||D^\gamma \zeta(x) - D^\gamma \zeta(y)| + \sum_{j=j_0}^\infty \sum_{\xi \in \Phi_j^{bc}} |\beta_\xi||D^\gamma \xi(x) - D^\gamma \xi(y)|.$$

Since $\zeta \in C^r(\Omega)$ for all $\zeta \in \Phi^{bc}$,

$$\sum_{\zeta \in \Phi^{bc}} |\beta_\zeta||D^\gamma \zeta(x) - D^\gamma \zeta(y)| \lesssim \|f\|_{B^{s}_{\infty,\infty}(\Omega)} \|\Phi^{bc}\| \|x - y\| \lesssim \|f\|_{B^{s}_{\infty,\infty}(\Omega)} \|x - y\|.$$
By definition of Besov norm, it will suffice to prove that

\[ s \]

second bound, let \( \| D^{\gamma} f \|_{L^2} \) be bounded above by \( \| f \|_{B_{\infty, \infty}^s(\Omega)} \). It readily follows that \( \| f \|_{B_{\infty, \infty}^s(\Omega)} \) is uniformly Hölder continuous over \( (0, 1)^d \).

Furthermore, since \( D^{\gamma} f \) is uniformly Hölder continuous over \( (0, 1)^d \), it is in particular uniformly continuous and hence extends to a continuous function over \( [0, 1]^d \), thus \( f \in C^s([0, 1]^d) \). This proves the first claim. To prove the second claim, it suffices to show that \( C^s([0, 1]^d) \subset B_{\infty, \infty}^s([0, 1]^d) \) for all \( s > 0 \), with the requisite Hölder norms. Assume \( \| f \|_{C^{s}(\Omega)} < \infty \), and let \( \beta_\xi = \int_{\Omega} f(x) \xi(x) dx \) for all \( \xi \in \Psi^{bc} \).

By definition of Besov norm, it will suffice to prove that

\[ \| (\beta_\xi)_{\xi \in \Phi^{bc}} \| \lesssim \| f \|_{C^{s}([0, 1]^d)} \]
where \((x - x_0)^\gamma = \prod_{i=1}^d (x_i - x_{0i})^{\gamma_i}\). In particular, for any given \(\xi \in \Psi_{bc}^j\), \(j \geq j_0\), choose \(x_0 \in I_\xi \cap (0, 1)^d\), where \(\text{diam}(I_\xi) \lesssim 2^{-j}\) and \(I_\xi\) is a set containing the support of \(\xi\), as defined in Lemma 22(ii). We then have,

\[
\begin{align*}
\left| \int \xi f \right| & \lesssim \int \left| \xi(x) \sum_{0 \leq |\gamma| \leq s} D^\gamma (x - x_0)^\gamma \right| dx + \int |\xi(x)||x - x_0|^s dx = \int |\xi(x)||x - x_0|^s dx,
\end{align*}
\]

where the final equality uses the fact that polynomials of degree at most \(\lfloor r \rfloor\) lie in \(\text{Span}(\Phi_{bc})\) by Lemma 22(iv), and are therefore orthogonal to \(\xi\). We thus have,

\[
|\beta_\xi| \lesssim \int |\xi(x)||x - x_0|^s dx = \int_{I_\xi} |\xi(x)||x - x_0|^s dx \lesssim 2^{dj/2} \text{diam}(I_\xi)^s \mathcal{L}(I_\xi) \lesssim 2^{-j(s + \frac{d}{2})}.
\]

The claim readily follows. \(\square\)

### A.3 Wavelet Density Estimation

We next state several properties of wavelet density estimators over \(\Omega \in \{\mathbb{T}^d, [0, 1]^d\}\), with the corresponding basis \(\Psi \in \{\Psi_{\text{per}}, \Psi_{bc}\}\) as in Section A.2.3. Let \(q \in L^2(\Omega)\) denote a probability density with corresponding probability distribution \(Q\), and with corresponding wavelet expansion

\[
q = \sum_{\xi \in \Phi} \beta_\xi \xi + \sum_{j=j_0}^\infty \sum_{\xi \in \Psi_j} \beta_\xi \xi.
\]

Given an i.i.d. sample \(Y_1, \ldots, Y_n \sim Q\) with corresponding empirical measure \(Q_n = (1/n) \sum_{i=1}^n \delta_{Y_i}\), define the corresponding wavelet density estimator

\[
\tilde{q}_n = \sum_{\xi \in \Phi} \hat{\beta}_\xi \xi + \sum_{j=j_0}^{J_n} \sum_{\xi \in \Psi_j} \hat{\beta}_\xi \xi,
\]

where \(J_n \geq j_0\) is a deterministic threshold, and \(\hat{\beta}_\xi = \int \xi dQ_n\) for all \(\xi \in \Psi_j\), \(j_0 \leq j \leq J_n\). The following simple result guarantees that \(\tilde{q}_n\) integrates to unity since \(q\) is a probability density.

**Lemma 24.** We have \(\int_\Omega \tilde{q}_n = 1\). In particular, it follows that \(\sum_{\xi \in \Phi} \hat{\beta}_\xi \int_\Omega \xi = 1\).

The proof of Lemma 24 appears in Appendix A.3.1. In the special case of the periodic wavelet system, for which \(\Phi_{\text{per}}\) consists only of the constant function 1, Lemma 24 implies that the corresponding estimated coefficient satisfies \(\hat{\beta}_1 = 1\) deterministically, thus the definition of \(\tilde{q}_n\) in equation (44) coincides with that given in Section 4.3.

With this result in place, we turn to \(L^\infty\) concentration results for \(\tilde{q}_n\), which we frequently use throughout our proofs.

**Lemma 25.** Let \(q \in \mathcal{B}^s_{\infty, \infty}(\Omega)\) for some \(s > 0\), and suppose there exists \(\gamma > 0\) such that \(\gamma^{-1} \leq q \leq \gamma\). Then, there exist constants \(v, b > 0\) depending only on \(\gamma, d\) and on the choice of wavelet system,
such that for any sequence $J_n \geq j_0$, and all $u > 0$,

$\mathbb{P} \left( \sup_{\zeta \in \Phi} |\widehat{\beta}_{\zeta} - \beta_{\zeta}| \geq u \right) \lesssim \exp \left\{ -\frac{nu^2}{b} \right\} \quad (45)$

$\mathbb{P} \left( \sup_{\xi \in \Psi_j} |\widehat{\beta}_{\xi} - \beta_{\xi}| \geq u \right) \lesssim 2^{\frac{d}{4}} \exp \left\{ -\frac{nu^2}{v + 2d/2bu} \right\}, \quad j_0 \leq j \leq J_n. \quad (46)$

In particular, if $J_n \asymp (n/ \log n)^{1/(2s + d)}$, then there exist constants $C, C', \gamma_0 > 0$ depending only on $d, \gamma, \|q\|_{B_{s,\infty}(\Omega)}$ and on the choice of wavelet system, and there almost surely exists $N > 0$ such that the following hold for all $n \geq N$,

(i) $\sup_{j_0 \leq j \leq J_n} \| (\widehat{\beta}_{\xi} - \beta_{\xi})_{\xi \in \Psi_j} \|_{L_\infty} \leq C 2^{-J_n(s + d/2)}$.

(ii) $\| \overline{q}_n \|_{B_{s,\infty}(\Omega)} \leq C'$.

(iii) $\frac{1}{\gamma_0} \leq \overline{q}_n \leq \gamma_0$.

Lemma 25(iii) can be deduced from $L^\infty$ concentration inequalities for wavelet estimators, proven for instance by Masry (1997); Guo and Kou (2019), as well as Giné and Nickl (2009) when $d = 1$. While these results are based on wavelet estimators over $\mathbb{R}^d$, they can readily be adapted to the wavelet systems considered here, as consequences of inequalities (45)–(46). For completeness, we prove the latter two bounds, and show how they imply parts (i)–(ii) of Lemma 25, in Appendix A.3.2.

Using Lemmas 24 and 25(iii), the following result is now immediate.

**Lemma 26.** Assume there exists $\gamma > 0$ such that $\gamma^{-1} \leq q \leq \gamma$ over $\Omega$. Then, there almost surely exists a constant $N > 0$ such that for all $n \geq N$, $\overline{q}_n$ is a valid probability density over $\Omega$, in the sense that $\overline{q}_n \geq 0$ and $\int_{\Omega} \overline{q}_n = 1$.

### A.3.1 Proof of Lemma 24

Recall that $\text{Span}(\Phi)$ contains all polynomials of degree at most $N - 1$ over $\Omega$, by Lemma 22(iv). In particular, it contains the constant function 1, thus if $\beta'_{\zeta} = \int_{\Omega} \zeta$, we obtain $1 = \sum_{\zeta \in \Phi} \beta'_{\zeta} \zeta$. It then follows by orthonormality of $\Psi$ that

$$\int_{\Omega} \overline{q}_n = \int_{\Omega} \left( \sum_{\zeta \in \Phi} \beta'_{\zeta} \zeta \right) \left( \sum_{\zeta \in \Phi} \widehat{\beta}_{\zeta} \zeta + \sum_{j = j_0}^{J_n} \sum_{\xi \in \Psi_j} \widehat{\beta}_{\xi} \xi \right) = \sum_{\zeta \in \Phi} \beta'_{\zeta} \widehat{\beta}_{\zeta} = \int_{\Omega} \left( \sum_{\zeta \in \Phi} \beta'_{\zeta} \zeta \right) dQ_n = 1.$$ 

This proves the claim.

### A.3.2 Proof of Lemma 25

Throughout the proof, $b, v, c, c', c'' > 0$ denote constants depending only on $d, \gamma$ and on the choice of wavelet system, whose value may change from line to line. Recall first from 22(v) that

$$\sup_{\zeta \in \Phi} \| \zeta \|_{L^\infty(\Omega)} \leq b, \quad \sup_{j \geq j_0} 2^{-jd/2} \sup_{\xi \in \Psi_j} \| \xi \|_{L^\infty(\Omega)} \leq b. \quad (47)$$
By Hoeffding’s inequality, equation (47) implies that for all $u > 0$,

$$
P \left( \sup_{\zeta \in \Phi} |\widehat{\beta}_\zeta - \beta_\zeta| \geq u \right) \leq \sum_{\zeta \in \Phi} P \left( \left| \int \zeta d(Q_n - Q) \right| \geq u \right) \lesssim \exp \left\{ -\frac{nu^2}{b^2} \right\},
$$

where we have used the fact that $|\Phi| \lesssim 1$ by Lemma 22(i). To prove equation (46), notice that for all $\xi \in \Phi_j$, $j \geq j_0$, given $Y \sim Q$,

$$
\text{Var}[\xi(Y)] \leq \int \xi^2(y)q(y)dy \leq \gamma \int \xi^2(y)dy = \gamma \leq v.
$$

Therefore, by Bernstein’s inequality, we have for all $u > 0$ and $j_0 \leq j \leq J_n$,

$$
P \left( \sup_{\xi \in \Psi_j} |\widehat{\beta}_\xi - \beta_\xi| \geq u \right) \leq \sum_{\xi \in \Psi_j} P \left( |\widehat{\beta}_\xi - \beta_\xi| \geq u \right) \lesssim 2^{dj} \exp \left\{ -\frac{nu^2}{v + 2d^2bu} \right\}.
$$

Here, the last inequality uses the fact that $|\Psi_j| \lesssim 2^d$ by Lemma 22(i) for all $j \geq j_0$. The first claim follows. To prove the second, notice that a union bound combined with the above display leads to

$$
P \left( \sup_{j_0 \leq j \leq J_n} \sup_{\xi \in \Psi_j} |\widehat{\beta}_\xi - \beta_\xi| \geq u \right) \lesssim J_n 2^{dJ_n} \exp \left\{ -\frac{nu^2}{v + 2d^2bu} \right\},
$$

whence,

$$
P \left( 2^{J_n(s + \frac{d}{2})} \sup_{j_0 \leq j \leq J_n} \left\| (\widehat{\beta}_\xi - \beta_\xi)_{\xi \in \Psi_j} \right\|_{\ell_\infty} \geq u \right) \lesssim J_n 2^{dJ_n} \exp \left\{ -\frac{nu^2}{v + 2d^2bu} \right\} \leq J_n 2^{dJ_n} \exp \left\{ -c'u^2 \log n \right\}.
$$

Choose $u^2 = \left( \frac{d}{d+2s} + 2 \right)c'$, so that

$$
P \left( 2^{J_n(s + \frac{d}{2})} \sup_{j_0 \leq j \leq J_n} \left\| (\widehat{\beta}_\xi - \beta_\xi)_{\xi \in \Psi_j} \right\|_{\ell_\infty} \leq \sqrt{2/c'} \right) \lesssim (\log n)n^{\frac{d}{d+2s}n^{-2} - \frac{d}{d+2s}} \lesssim \frac{1}{n^2}.
$$

Since this bound is summable, part (i) now follows from the Borel-Cantelli Lemma. To prove part (ii), set $q_{J_n} = \sum_{\zeta \in \Phi} \beta_\zeta \zeta + \sum_{j=j_0}^{J_n} \sum_{\xi \in \Psi_j} \beta_\xi \xi$, and notice that for all large enough $n$,

$$
\left\| \widehat{q} \right\|_{B_{\infty, \infty}} \leq \left\| \widehat{q} - q_{J_n} \right\|_{B_{\infty, \infty}} + \left\| q_{J_n} - q \right\|_{B_{\infty, \infty}} + \left\| q \right\|_{B_{\infty, \infty}}
\leq \left\| \widehat{q} - q_{J_n} \right\|_{B_{\infty, \infty}} + 2\left\| q \right\|_{B_{\infty, \infty}}
\leq \left\| (\widehat{\beta}_\zeta - \beta_\zeta)_{\zeta \in \Phi} \right\|_{\ell_\infty} + 2J_n \sup_{j_0 \leq j \leq J_n} \left\| (\widehat{\beta}_\xi - \beta_\xi)_{\xi \in \Psi_j} \right\|_{\ell_\infty} + 2\left\| q \right\|_{B_{\infty, \infty}}
\leq \left\| (\widehat{\beta}_\zeta - \beta_\zeta)_{\zeta \in \Phi} \right\|_{\ell_\infty} + \sqrt{2/c'} + 2\left\| q \right\|_{B_{\infty, \infty}}
\leq \left\| (\widehat{\beta}_\zeta - \beta_\zeta)_{\zeta \in \Phi} \right\|_{\ell_\infty} + C,
$$

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almost surely. We deduce that
\[ P \left( \| \hat{q} \|_{B_{2,1}^2} \geq 2C \right) \leq P \left( \| (\hat{\beta}_\xi - \beta_\xi) \|_{\ell_\infty} \geq C \right) \lesssim \exp(-nC^2/b), \]
which is again summable. Claim (ii) thus follows by the same argument as before. Part (iii) can be proven using a similar strategy, or can easily be obtained from adaptations of existing \( L^\infty \) concentration inequalities for wavelet density estimators over \( \mathbb{R}^d \) (Masry, 1997; Giné and Nickl, 2009; Guo and Kou, 2019). We omit the details for brevity.

With the above results in place, we are in a position to prove Lemma 9.

A.3.3 Proof of Lemma 9

It is straightforward to verify from Lemma 22 that the wavelet system \( \Psi^{bc} \) satisfies Assumptions E.1–E.6 of Weed and Berthet (2019). Since \( \gamma^{-1} \leq q \leq \gamma \) over \( \Omega \), we may thus apply their Theorem 4 to deduce that
\[ W_2^2(Q, \hat{Q}_n) \lesssim \| \hat{q}_n - q \|_{B_{2,1}^2}. \]

By Lemma 26, there almost surely exists \( N > 0 \) such that for all \( n \geq N \), \( \hat{q}_n = \bar{q}_n \). Let \( q_{J_n} = \mathbb{E}[\bar{q}_n] \), so that
\[ \mathbb{E}W_2^2(Q, \bar{q}_n) \lesssim \| \bar{q}_n - q \|_{B_{2,1}^2} \lesssim \| \bar{q}_n - q \|_{B_{2,1}^2} + \mathbb{E}\|q_{J_n} - q\|_{B_{2,1}^2}. \]

Further, by the same reasoning as the proof of their Theorem 1 and Proposition 4, one has
\[ \mathbb{E}\|q_{J_n} - q\|_{B_{2,1}^2} \lesssim 2^{-J_n\alpha}, \]
\[ \mathbb{E}\| (\hat{\beta}_\xi - \beta_\xi) \|_{\ell_2} \lesssim 1/n, \quad \mathbb{E}\| (\hat{\beta}_\xi - \beta_\xi) \|_{\ell_2} \lesssim 2^d/n, \quad j \geq j_0. \]

We thus obtain,
\[ \mathbb{E}W_2^2(Q, \bar{q}_n) \lesssim \| \bar{q}_n - q \|_{B_{2,1}^2} + \mathbb{E}\|q_{J_n} - q\|_{B_{2,1}^2}, \]
\[ \lesssim \mathbb{E}\| (\hat{\beta}_\xi - \beta_\xi) \|_{\ell_2}^2 + \mathbb{E}\left( \sum_{j=j_0}^{J_n} 2^{-j} \| (\hat{\beta}_\xi - \beta_\xi) \|_{\ell_2} \right)^2 + 2^{-2J_n\alpha} \]
\[ \lesssim \left( 1 + \left( \sum_{j=j_0}^{J_n} 2^{(\eta+1)d} \right) \left( \sum_{j=j_0}^{J_n} 2^{-2nj} \right) \right) + 2^{-2J_n\alpha}, \]

for any \( \eta \in \mathbb{R} \). Now, when \( d \geq 3 \), choose \( 1 - \frac{d}{2} < \eta < 0 \). In this case, the above display is of order
\[ \frac{2^{2(\eta+1)d-2\eta}J_n}{n} = 2^{2(d-2)J_n/n} + 2^{-2J_n\alpha} \lesssim \left( \frac{\log n}{n} \right)^{-\frac{2d}{2(\alpha-1)+\eta}}. \]
On the other hand, when \( d \leq 2 \), choose \( \eta = 0 \). In this case, the penultimate display is dominated by its second term, and is of order \( 1/n \) when \( d = 1 \) and of order \( (\log n)/n \) when \( d = 2 \). The claim follows.

B Proofs of Stability Bounds

B.1 Proof of Theorem 6

Recall that \( \varphi_0 \) denotes the Brenier potential from \( P \) to \( Q \), while \( \phi_0 = \| \cdot \|^2 - 2\varphi_0 \) and \( \psi_0 = \| \cdot \|^2 - 2\varphi_0^* \) denote the corresponding Kantorovich potentials. Since we have assumed that both \( P \) and \( Q \) are absolutely continuous distributions, Brenier’s Theorem implies that \( S_0 = \nabla \varphi_0^* = T_0^{-1} \) is the optimal transport map from \( Q \) to \( P \). Since \( \varphi_0 \) is closed, the assumption

\[
\frac{1}{\lambda} I_d \preceq \nabla^2 \varphi_0 \preceq \lambda I_d,
\]

from condition A1(\( \lambda \)) also implies (Hiriart-Urruty and Lemaréchal (2004), Theorem 4.2.2),

\[
\frac{1}{\lambda} I_d \preceq \nabla^2 \varphi_0^* \preceq \lambda I_d.
\]

Combining this bound with a second-order Taylor expansion of \( \varphi_0^* \) leads to the following key inequalities

\[
\frac{1}{2\lambda} \| x - y \|^2 \leq \varphi_0^*(y) - \varphi_0^*(x) - \langle S_0(x), y - x \rangle \leq \frac{\lambda}{2} \| x - y \|^2, \quad x, y \in \Omega. \tag{49}
\]

With these facts in place, we turn to proving the claim. It will suffice to show that

\[
\frac{1}{\lambda} \| \hat{T} - T_0 \|^2_{L^2(P)} \leq W_2^2(P, \hat{Q}) - W_2^2(P, Q) - \int \psi_0 d(\hat{Q} - Q) \leq \lambda W_2^2(\hat{Q}, Q). \tag{50}
\]

Indeed, recall that \( (\hat{T}, T_0) \# P \in \Pi(\hat{Q}, Q) \), thus \( W_2(\hat{Q}, Q) \leq \| \hat{T} - T_0 \|^2_{L^2(P)} \). Therefore, the above display implies the remaining two inequalities of the claim:

\[
\frac{1}{\lambda} W_2^2(\hat{Q}, Q) \leq W_2^2(P, \hat{Q}) - W_2^2(P, Q) - \int \psi_0 d(\hat{Q} - Q) \leq \lambda \| \hat{T} - T_0 \|^2_{L^2(P)}.
\]

It thus remains to prove equation (50), and we begin by proving the first inequality. Since \( \hat{T} \) is the optimal transport map from \( P \) to \( \hat{Q} \), we have,

\[
W_2^2(P, \hat{Q}) = \int \| \hat{T}(x) - x \|^2 dP(x)
= \int \| T_0(x) - x \|^2 dP(x) + \int 2\langle T_0(x) - x, \hat{T}(x) - T_0(x) \rangle dP(x) + \int \| \hat{T}(x) - T_0(x) \|^2 dP(x)
= W_2^2(P, Q) + \int 2\langle T_0(x) - x, \hat{T}(x) - T_0(x) \rangle dP(x) + \| \hat{T} - T_0 \|^2_{L^2(P)}.
\]

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To bound the cross term, notice that equation (49) implies

\[2 \int \langle T_0(x) - x, \tilde{T}(x) - T_0(x) \rangle dP(x)
= 2 \int \langle T_0(x) - S_0(T_0(x)), \tilde{T}(x) - T_0(x) \rangle dP(x)
\geq 2 \int \left[ \langle T_0(x), \tilde{T}(x) - T_0(x) \rangle + \varphi_0^*(T_0(x)) - \varphi_0^*(\tilde{T}(x)) + \frac{1}{2\lambda} \| \tilde{T}(x) - T_0(x) \|^2 \right] dP(x)
\]
\[= \int \left[ \| \tilde{T}(x) \|^2 - \| T_0(x) \|^2 - \| \tilde{T}(x) - T_0(x) \|^2 + 2\varphi_0^*(T_0(x)) - 2\varphi_0^*(\tilde{T}(x)) + \frac{1}{\lambda} \| \tilde{T}(x) - T_0(x) \|^2 \right] dP(x)
= \left( \frac{1}{\lambda} - 1 \right) \| T_0 \|^2_{L^2(P)} + \int \psi_0 d(\tilde{Q} - Q).
\]

We deduce

\[W_2^2(P, \tilde{Q}) \geq W_2^2(P, Q) + \frac{1}{\lambda} \| T_0 \|^2_{L^2(P)} + \int \psi_0 d(\tilde{Q} - Q),
\]

To prove the second inequality in equation (50), let \( \pi \) denote an optimal coupling between \( Q \) and \( \tilde{Q} \). Then, the measure \( \pi_{S_0} = (S_0, Id)_{\#}\pi \) is a (possibly suboptimal) coupling between \( P \) and \( \tilde{Q} \), thus

\[W_2^2(P, \tilde{Q}) \leq \int \| x - z \|^2 d\pi_{S_0}(x, z)
= \int \| S_0(y) - z \|^2 d\pi(y, z)
= \int \| S_0(y) - y \|^2 dQ(y) + \int \| y - z \|^2 d\pi(y, z) + 2 \int \langle S_0(y) - y, y - z \rangle d\pi(y, z)
= W_2^2(P, Q) + W_2^2(Q, \tilde{Q}) + 2 \int \langle S_0(y) - y, y - z \rangle d\pi(y, z).
\]

Now, notice that

\[2 \int \langle S_0(y), y - z \rangle d\pi(y, z) \leq 2 \int \left[ \varphi_0^*(y) - \varphi_0^*(z) + \frac{\lambda}{2} \| y - z \|^2 \right] d\pi(y, z)
= 2 \int \varphi_0^* d(\tilde{Q} - Q) + \lambda W_2^2(\tilde{Q}, Q),
\]

and,

\[2 \int \langle - y, y - z \rangle d\pi(y, z) = \int \left[ \| z \|^2 - \| z - y \|^2 - \| y \|^2 \right] d\pi(y, z) = \int \| \cdot \|^2 d(\tilde{Q} - Q) - W_2^2(\tilde{Q}, Q).
\]

Therefore,

\[W_2^2(P, \tilde{Q}) - W_2^2(P, Q) \leq \int \left( \| \cdot \|^2 - 2\varphi_0^* \right) d(\tilde{Q} - Q) + \lambda W_2^2(\tilde{Q}, Q) = \int \psi_0 d(\tilde{Q} - Q) + \lambda W_2^2(\tilde{Q}, Q),
\]

and the claim follows.
B.2 Proof of Corollary 8

By Theorem 6,

\[ \mathbb{E}|W_2^2(P_n, Q_n) - W_2^2(P, Q)| \leq \mathbb{E}W_2^2(Q_n, Q) + \mathbb{E}\left| \int \psi_0 d(Q_n - Q) \right|. \]

Since $|\psi_0| \leq d$, it follows from Chebyshev’s inequality that the final term is of order $n^{-1/2}$. \hfill \Box

B.3 Proof of Proposition 12

Recall from the proof of Theorem 6 (equation (49)) that, due to assumption A1(\(\lambda\)),

\[ \frac{1}{2\lambda} \|x - y\|^2 \leq \varphi_0^*(x) - \varphi_0^*(y) - \langle S_0(x), y - x \rangle \leq \frac{\lambda}{2} \|x - y\|^2, \]

for all $x, y \in \Omega$. Now, we have,

\[ W_2^2(P_n, Q_m) = \sum_{i=1}^{n} \sum_{j=1}^{m} \hat{\pi}_{ij} \|X_i - Y_j\|^2 \]

\[ = \sum_{i=1}^{n} \sum_{j=1}^{m} \hat{\pi}_{ij} \left[ \|T_0(X_i) - X_i\|^2 + 2\langle T_0(X_i) - X_i, Y_j - T_0(X_i) \rangle + \|Y_j - T_0(X_i)\|^2 \right]. \]

Notice that

\[ \mathbb{E}\left[ \sum_{i=1}^{n} \sum_{j=1}^{m} \hat{\pi}_{ij} \|T_0(X_i) - X_i\|^2 \right] = \mathbb{E}\left[ \sum_{i=1}^{n} \left( \sum_{j=1}^{m} \hat{\pi}_{ij} \right) \|T_0(X_i) - X_i\|^2 \right] \]

\[ = \mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \|T_0(X_i) - X_i\|^2 \right] = W_2^2(P, Q), \]

where we have used the marginal constraint of the coupling \(\hat{\pi}\) in the first equality of the above display. We thus obtain,

\[ \mathbb{E}\left[ W_2^2(P_n, Q_m) - W_2^2(P, Q) \right] = \mathbb{E}[\Delta_{nm}] + 2\mathbb{E}\left[ \sum_{i=1}^{n} \sum_{j=1}^{m} \hat{\pi}_{ij} \langle T_0(X_i) - X_i, Y_j - T_0(X_i) \rangle \right] \]

\[ = \mathbb{E}[\Delta_{nm}] + 2\mathbb{E}\left[ \sum_{i=1}^{n} \sum_{j=1}^{m} \hat{\pi}_{ij} \langle T_0(X_i) - S_0(T_0(X_i)), Y_j - T_0(X_i) \rangle \right]. \]

Now,

\[ 2\langle -S_0(T_0(X_i)), Y_j - T_0(X_i) \rangle \geq 2\varphi_0^*(T_0(X_i)) - 2\varphi_0^*(Y_j) + \frac{1}{\lambda} \|T_0(X_i) - Y_j\|^2, \]  \tag{52}
whence, we obtain,
\[
\mathbb{E}\left[W_2^2(P_n, Q_m) - W_2^2(P, Q)\right] \\
\geq \mathbb{E}[\Delta_{nm}] + \mathbb{E}\left[\sum_{i=1}^{n} \sum_{j=1}^{m} \hat{\pi}_{ij} \left(2\varphi_0^*(T_0(X_i)) - 2\varphi_0^*(Y_j) + \frac{1}{\lambda}\|T_0(X_i) - Y_j\|^2 + 2\langle T_0(X_i), Y_j - T_0(X_i) \rangle\right)\right]
\]

Now, notice that
\[
2\langle T_0(X_i), Y_j - T_0(X_i) \rangle = -\|T_0(X_i) - Y_j\|^2 + \|Y_j\|^2 - \|T_0(X_i)\|^2.
\]

Thus, continuing from before, we have
\[
\mathbb{E}\left[W_2^2(P_n, Q_m) - W_2^2(P, Q)\right] \\
\geq \frac{1}{\lambda} \mathbb{E}[\Delta_{nm}] + \mathbb{E}\left[\sum_{i=1}^{n} \sum_{j=1}^{m} \hat{\pi}_{ij} \left(2\varphi(T_0(X_i)) - 2\varphi_0^*(Y_j) + \|Y_j\|^2 - \|T_0(X_i)\|^2\right)\right] \\
= \frac{1}{\lambda} \mathbb{E}[\Delta_{nm}] + \mathbb{E}\left[\frac{1}{m} \sum_{j=1}^{m} \left(\|Y_j\|^2 - 2\varphi_0^*(Y_j)\right)\right] - \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} \left(\|T_0(X_i)\|^2 - 2\varphi_0^*(T_0(X_i))\right)\right] \\
= \frac{1}{\lambda} \mathbb{E}[\Delta_{nm}].
\]

This proves one of the inequalities of the claim. To obtain the other, return to equation (52) and notice that one also has
\[
2\langle -S_0(T_0(X_i)), Y_j - T_0(X_i) \rangle \leq 2\varphi_0^*(T_0(X_i)) - 2\varphi_0^*(Y_j) + \lambda \|T_0(X_i) - Y_j\|^2.
\]

The proof then proceeds analogously.

\[\square\]

### B.4 Proof of Proposition 15

The claim follows along the same lines as that of Theorem 6, thus we only provide a brief proof. It will again suffice to prove
\[
\frac{1}{\lambda}\|\hat{T} - T_0\|_{L_{\text{var}}^2(P)}^2 \leq W_2^2(P, \hat{Q}) - W_2^2(P, Q) - \int \psi_0 d(\hat{Q} - Q) \leq \lambda W_2^2(\hat{Q}, Q). \tag{53}
\]
Recall that \(\hat{T}\) is the optimal transport map from \(P\) to \(\hat{Q}\). By Proposition 4(iii), we therefore have
\[
d_{\text{var}}(\hat{T}(x), x) = \|\hat{T}(x) - x\|, \quad d_{\text{var}}(T_0(x), x) = \|T_0(x) - x\|,
\]
for Lebesgue-almost every \(x \in \Omega\). It follows that
\[
W_2^2(P, \hat{Q}) - W_2^2(P, Q) = \int \|\hat{T}(x) - x\|^2 dP(x) - \int \|T_0(x) - x\|^2 dP(x).
\]

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From here, it follows identically as in the proof of Theorem 6 that

$$W_2^2(P, \hat{Q}) - W_2^2(P, Q) \geq \frac{1}{\lambda} \| \hat{T} - T_0 \|_{L^2(P)}^2 + \int \psi_0 d(\hat{Q} - Q).$$

Since $$\| \cdot \|_{L^2(P)} \geq \| \cdot \|_{L^2_{per}(P)}$$, the first inequality follows. To prove the second inequality, let $$\pi$$ denote an optimal coupling between $$Q$$ and $$\hat{Q}$$ with respect to the cost $$d_{T_d}^2$$. Notice similarly as before that Proposition 4(iii) implies

$$W_2^2(P, \hat{Q}) = \int \| S_0(y) - y \|^2 dQ(y), \quad W_2^2(Q, \hat{Q}) = \int \| y - z \|^2 d\pi(y, z),$$

thus, since $$(S_0, Id)_{\#}\pi \in \Pi(P, \hat{Q})$$, we have

$$W_2^2(P, \hat{Q}) \leq \int d_{T_d}^2(S_0(y), z) d\pi(y, z) \leq \int \| S_0(y) - z \|^2 d\pi(y, z) = \int \| S_0(y) - y \|^2 dQ(y) + \int \| y - z \|^2 d\pi(y, z) + 2 \int \langle S_0(y) - y, y - z \rangle d\pi(y, z) = W_2^2(P, Q) + W_2^2(Q, \hat{Q}) + 2 \int \langle S_0(y) - y, y - z \rangle d\pi(y, z).$$

By the same argument as in Theorem 6, the cross term is bounded above by $$(\lambda - 1)W_2^2(\hat{Q}, Q)$$, thus the claim follows.

C Proofs of Upper Bounds for Empirical Estimators

In this Appendix, we prove Propositions 13 and 14. We begin with the following result.

**Lemma 27.** Let $$\Omega$$ satisfy conditions (S1)–(S2). Let $$P \in \mathcal{P}_{ac}(\Omega)$$ admit a density $$p$$ such that $$\gamma^{-1} \leq p \leq \gamma$$ for some $$\gamma > 0$$. Let $$V_1, \ldots, V_n$$ denote the Voronoi diagram in equation (32), based on an i.i.d. sample $$X_1, \ldots, X_n \sim P$$. Then, there exist constants $$c_1, C_1, C_2 > 0$$ depending only on $$d, \gamma, \epsilon_0, \delta_0$$ such that,

(i) For all $$\delta \in (0, 1)$$, we have,

$$\mathbb{P} \left( \max_{1 \leq i \leq n} P(V_i) \geq C_1 \log \frac{n}{\delta} \log \left( \frac{2}{\delta} \right) \right) \leq \delta + n \exp(-nc_1).$$

(ii) We have,

$$\mathbb{E} \left[ \max_{1 \leq i \leq n} \text{diam}(V_i)^2 \right] \leq C_2 \left( \frac{\log n}{n} \right)^{\frac{3}{2}}.$$

**Proof of Lemma 27.** We shall make use of the relative Vapnik-Chervonenkis inequality (Vapnik, 2013; Bousquet et al., 2003), in the following form stated by Chaudhuri and Dasgupta (2010).
Lemma 28. Let $\mathcal{B}$ denote the set of balls in $\mathbb{R}^d$. Then, there exists a universal constant $C > 0$ such that for every $\delta \in (0, 1)$, we have with probability at least $1 - \delta$ that for all $B \in \mathcal{B}$,

$$P(B) \geq \frac{Cd \log n}{n} \log \left(\frac{2}{\delta}\right) \implies P_n(B) > 0.$$ 

We now turn to the proof. Recall that $\Omega$ is a standard set by condition (S2), and recall the constants $\epsilon_0, \delta_0 > 0$ therein. Fix $1 \leq i \leq n$. Pick $x_i \in V_i$ for all $1 \leq i \leq n$, and let $\rho_i = (\epsilon_0/2d) \|x_i - X_i\|$. Since $\text{diam}(\Omega) \leq \sqrt{d}$ by condition (S1), we have $\rho_i \leq \epsilon_0$. We also have $\rho_i < \|x - X_i\|$, thus the balls $B(x_i, \rho_i)$ of radius $\rho_i$ centered at $x_i$ contain no sample points. Therefore, by Lemma 28, we have that for any $\delta \in (0, 1)$, with probability at least $1 - \delta$,

$$\max_{1 \leq i \leq n} P(B(x_i, \rho_i)) \leq C_d \frac{\log(2/\delta) \log n}{n}. \quad (54)$$

Now, since $\gamma^{-1} \leq p \leq \gamma$, the assumption of standardness on $\Omega$ leads to the bound

$$P(B(x_i, \rho_i)) \geq \gamma^{-1} \mathcal{L}(B(x_i, \rho_i) \cap \Omega) \geq \delta_0 \gamma^{-1} \mathcal{L}(B(x_i, \rho_i)) \asymp \rho_i^d,$$

thus equation (54) reduces to

$$\max_{1 \leq i \leq n} \rho_i^d \lesssim \frac{\log(2/\delta) \log n}{n}.$$

By definition of $\rho_i$, we deduce that for some constant $C_1 > 0$ not depending on $\delta$, we have with probability at least $1 - \delta$,

$$\max_{1 \leq i \leq n} \text{diam}(V_i) \leq C_1 \left(\frac{\log(2/\delta) \log n}{n}\right)^{\frac{d}{2}}.$$

To prove claim (i), notice that since the density of $P$ is bounded from above, we also have with probability at least $1 - \delta$,

$$\max_{1 \leq i \leq n} P(V_i) \leq \gamma \max_{1 \leq i \leq n} \mathcal{L}(V_i) \lesssim \max_{1 \leq i \leq n} \text{diam}(V_i)^d \lesssim \frac{\log(2/\delta) \log n}{n}.$$

To prove claim (ii), set $\delta = 2 \exp\left(-\frac{u^d n}{\log n C_1}\right)$ for any $u > 0$ to obtain

$$\mathbb{E}\left[\max_{1 \leq i \leq n} \text{diam}(V_i)^2\right] = \int_0^\infty \mathbb{P}\left(\max_{1 \leq i \leq n} \text{diam}(V_i)^2 \geq u\right) du \leq \int_0^\infty 2 \exp\left(-\frac{u^d n}{\log n C_1}\right) du \lesssim \left(\frac{\log n}{n}\right)^{\frac{d}{2}}.$$

The claim follows. \(\square\)
C.1 Proof of Proposition 13

Abbreviate \( \hat{T}_{imn}^{\text{1NN}} \) by \( \hat{T}_{imn} \). We have,

\[
\| \hat{T}_{imn} - T_0 \|_{L^2(P)}^2 = \sum_{i=1}^{n} \int_{V_i} \| \hat{T}_{imn}(x) - T_0(X_i) + T_0(X_i) - T_0(x) \|^2 dP(x)
\]

\[
\leq \sum_{i=1}^{n} \int_{V_i} \left[ \| \hat{T}_{imn}(x) - T_0(X_i) \|^2 + \| T_0(X_i) - T_0(x) \|^2 \right] dP(x).
\]

To bound the first term, notice that,

\[
\sum_{i=1}^{n} \int_{V_i} \| \hat{T}_{imn}(x) - T_0(X_i) \|^2 dP(x) = \sum_{i=1}^{n} \int_{V_i} \left\| \sum_{j=1}^{m} (n\hat{\pi}_{ij})Y_j - T_0(X_i) \right\|^2 dP(x)
\]

\[
= \sum_{i=1}^{n} P(V_i) \left\| \sum_{j=1}^{m} (n\hat{\pi}_{ij})Y_j - T_0(X_i) \right\|^2 
\]

\[
\leq \sum_{i=1}^{n} P(V_i) \sum_{j=1}^{m} (n\hat{\pi}_{ij}) \| Y_j - T_0(X_i) \|^2,
\]

by convexity of \( \| \cdot \|^2 \). Therefore, setting \( M_n = \max_{1 \leq i \leq n} P(V_i) \), we obtain

\[
\| \hat{T}_{imn} - T_0 \|_{L^2(P)}^2 \leq n\Delta_{nm} \left( \max_{1 \leq i \leq n} P(V_i) \right) + \sum_{i=1}^{n} \int_{V_i} \| T_0(X_i) - T_0(x) \|^2 dP(x).
\]

Since \( T_0 \) is \( \lambda \)-Lipschitz by condition \( \textbf{A1}(\lambda) \), the claim is now a consequence of the following simple Lemma, which we isolate for future reference.

**Lemma 29.** Under the conditions as Proposition 13, we have for any \( \lambda \)-Lipschitz map \( F : \Omega \to \Omega \),

\[
\mathbb{E} \left[ \sum_{i=1}^{n} \int_{V_i} \| F(X_i) - F(x) \|^2 dP(x) \right] \leq \lambda \gamma (\log n/n)^{2/d},
\]

\[
\mathbb{E} \left[ n\Delta_{nm} \left( \max_{1 \leq i \leq n} P(V_i) \right) \right] \leq \lambda \gamma \kappa n \log n.
\]

C.1.1 Proof of Lemma 29

The first quantity is easily bounded as follows,

\[
\mathbb{E} \left[ \sum_{i=1}^{n} \int_{V_i} \| F(X_i) - F(x) \|^2 dP(x) \right] \leq \lambda^2 \mathbb{E} \left[ \sum_{i=1}^{n} \int_{V_i} \| X_i - x \|^2 dP(x) \right] \leq \lambda^2 \mathbb{E} \left[ \sum_{i=1}^{n} P(V_i) \text{diam}(V_i)^2 \right]
\]

\[
\leq \lambda^2 \mathbb{E} \left[ \max_{1 \leq i \leq n} \text{diam}(V_i)^2 \right] \leq \lambda^2 \left( \log \frac{n}{n} \right)^2.
\]

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where the final inequality is due to Lemma 27(ii). To bound the second quantity, let $M_n = \max_{1 \leq i \leq n} P(V_i)$. By Lemma 27(i) with $\delta = 1/n^2$, there is a large enough constant $c > 0$ such that if $m_n = c(\log n)^2/n$, then $\mathbb{P}(M_n \geq m_n) \leq 1/n^2$. We have,

$$
\mathbb{E} [nM_n \Delta_{nm}] = \mathbb{E} [nM_n I(M_n \geq m_n) \Delta_{nm}] + \mathbb{E} [nM_n I(M_n < m_n) \Delta_{nm}].
$$

Notice that $\Delta_{nm}$ is bounded above by $\text{diam}(\Omega)^2$, and $0 \leq M_n \leq 1$, thus, by Proposition 12,

$$
\mathbb{E} [nM_n \Delta_{nm}] \lesssim n \mathbb{P}(M_n \geq m_n) + m_n \mathbb{E} [\Delta_{nm}] \lesssim \frac{1}{n} + (\log n)^2 \mathbb{E} [\Delta_{nm}] \lesssim (\log n)^2 \kappa_{n,m}.
$$

This proves the claim.

C.2 Proof of Proposition 14

Abbreviate $\hat{T}_{nm}$ by $\hat{T}_{nm}$. Notice first that we have

$$
\|\hat{T}_{nm} - T_0\|_{L^2(P_n)}^2 = \frac{1}{n} \sum_{i=1}^{n} \|\hat{T}_{nm}(X_i) - T_0(X_i)\|^2
$$

$$
= \sum_{i=1}^{n} \sum_{j=1}^{m} \hat{\pi}_{ij} \|\hat{T}_{nm}(X_i) - T_0(X_i)\|^2
$$

$$
\lesssim \sum_{i=1}^{n} \sum_{j=1}^{m} \hat{\pi}_{ij} \|\hat{T}_{nm}(X_i) - Y_j\|^2 + \sum_{i=1}^{n} \sum_{j=1}^{m} \hat{\pi}_{ij} \|Y_j - T_0(X_i)\|^2
$$

$$
\leq 2 \sum_{i=1}^{n} \sum_{j=1}^{m} \hat{\pi}_{ij} \|Y_j - T_0(X_i)\|^2 = 2 \Delta_{nm},
$$

where the final inequality follows by definition of $\hat{T}_{nm}$, since $\varphi_0 \in \mathcal{F}$ under assumption $\text{A1}(\lambda)$. Therefore,

$$
\|\hat{T}_n - T_0\|_{L^2(P)} = \sum_{i=1}^{n} \int_{V_i} \|\hat{T}_n - T_0\|^2 dP
$$

$$
\lesssim \sum_{i=1}^{n} \int_{V_i} \left[\|\hat{T}_{nm}(x) - T_0(X_i)\|^2 + \|\hat{T}_{nm}(X_i) - T_0(X_i)\|^2 + \|T_0(X_i) - T_0(x)\|^2\right] dP(x).
$$
By definition of \( F \) and by assumption \( A1(\lambda) \), \( \hat{T}_{nm} \) and \( T_0 \) are both \( \lambda \)-Lipschitz, thus by Lemma 29,

\[
\mathbb{E}\|\hat{T}_n - T_0\|^2_{L^2(P)} \lesssim \left(\frac{\log n}{n}\right)^2 + \mathbb{E}\left[\sum_{i=1}^{n} \int_{V_i} \|\hat{T}_{nm}(X_i) - T_0(X_i)\|^2 dP(x)\right]
\]

\[
\lesssim \left(\frac{\log n}{n}\right)^2 + \mathbb{E}\left[n \left(\max_{1 \leq i \leq n} P(V_i)\right) \|\hat{T}_{nm} - T_0\|^2_{L^2(P_n)}\right]
\]

\[
\lesssim \left(\frac{\log n}{n}\right)^2 + \mathbb{E}\left[n \left(\max_{1 \leq i \leq n} P(V_i)\right) \Delta_{nm}\right],
\]

where we used equation (55). Lemma 29 may now be applied to bound the right-hand term in the above display, leading to the claim.

\[\square\]

## D Proofs of Upper Bounds for Smooth Estimators

We begin by proving Lemma 11, which was used in the proof of Theorem 10, regarding upper bounds for one-sample density plugin estimators.

### D.1 Proof of Lemma 11

For simplicity, we write \( \Psi_{j_0-1} = \Phi \) throughout the proof. Let \( \psi_{0,J_n} = \sum_{j=j_0-1}^{J_n} \sum_{\xi \in \Psi_j} \xi \gamma_\xi \) denote the projection of \( \psi_0 \) onto \( \text{Span} \left( \bigcup_{j=j_0-1}^{J_n} \Psi_j \right) \), where recall that \( \gamma_\xi = \int \psi_0 \xi \) for all \( \xi \). We have,

\[
\int \psi_0 \hat{q}_n = \sum_{j=j_0-1}^{J_n} \sum_{\xi \in \Psi_j} \hat{\beta}_\xi \int \psi_0 \xi = \sum_{j=j_0-1}^{J_n} \sum_{\xi \in \Psi_j} \hat{\beta}_\xi \gamma_\xi = \frac{1}{n} \sum_{j=j_0-1}^{J_n} \sum_{\xi \in \Psi_j} \xi(Y_i) \gamma_\xi = \int \psi_{0,J_n} \, dQ_n,
\]

whence,

\[
\text{Var} \left[\int \psi_0 \hat{q}_n\right] = \frac{1}{n} \text{Var}[\psi_{0,J_n}(Y)] = \frac{1}{n} \text{Var}[\psi_0(Y)] + \frac{1}{n} (\text{Var}[\psi_{0,J_n}(Y)] - \text{Var}[\psi_0(Y)]).
\]

The final term of the above display will clearly be of order \( o(1/n) \) if the wavelet series of \( \psi_0 \) converges uniformly, i.e. \( \|\psi_{0,J_n} - \psi_0\|_{L^\infty} \to 0 \). Since \( \psi_0 \in C^{\alpha+1}(\Omega) \), this property is straightforward to verify as in the proof of Lemma 23. Specifically, applying Lemma 22 and the fact that \( \psi_0 \in B^{\alpha+1}_{\infty,\infty}(\Omega) \),

\[
\|\psi_{0,J_n} - \psi_0\|_{L^\infty(\Omega)} = \left\| \sum_{j=J_n+1}^{\infty} \sum_{\xi \in \Psi_j} \gamma_\xi \xi \right\| \leq \sum_{j=J_n+1}^{\infty} \|\gamma_\xi \xi \|_{L^\infty(\Omega)} \left(\sup_{\xi \in \Psi_j} \|\xi\|_{L^\infty(\Omega)}\right) \left\| \sum_{\xi \in \Psi_j} I(|\xi| > 0) \right\|
\]

\[
\lesssim \sum_{j=J_n+1}^{\infty} 2^{-j\left(\frac{\alpha}{2} + 1\right)} 2^{d/2} \lesssim 2^{-2 J_n (\alpha+1)}.
\]

The claim follows. \[\square\]
We next turn to providing proofs for the two-sample estimators in Section 4.3. We first note that the one-sample results from Section 3.3 may readily be extended to the optimal transport problem over $\mathbb{T}^d$.

**Proposition 30.** Assume $P, Q \in \mathcal{P}_{ac}(\mathbb{T}^d)$ admit densities $p, q$ such that $\gamma^{-1} \leq p, q \leq \gamma$, and such that $\|q\|_{C^\alpha(\mathbb{T}^d)} \leq M < \infty$ for some $\alpha > 1, \alpha \notin \mathbb{N}$. Let $\hat{q}_m$ be the periodic wavelet estimator defined in equation (35), with corresponding distribution $\hat{Q}_m$, and let

$$\tilde{T}_m = \arg\min_{T \in T(P, \hat{Q}_m)} \int d_{\mathbb{T}^d}^2(x, T(x))dP(x).$$

Then, there exists a constant $C > 0$ depending only on $d, M, \gamma, \alpha$ such that the following statements hold.

(i) We have, $E\|W_2^2(\hat{Q}_m, Q) \leq CR_{T,m}(\alpha)$.

(ii) We have,

$$E\|\tilde{T}_m - T_0\|_{L^2_m(P)}^2 \leq CR_{T,m}(\alpha),$$

$$|E\|W_2^2(P, \hat{Q}_m) - W_2^2(P, Q) \| \leq CR_{T,m}(\alpha),$$

$$E|W_2^2(P, \hat{Q}_m) - W_2^2(P, Q) \| \leq CR_{W,m}(\alpha).$$

Notice that the only properties of the boundary-correct wavelet basis used in the proofs of Lemma 9 and Theorem 10 are those contained in Lemmas 22 and Lemma 25 of Appendix A.2, which are also stated to hold for the periodic wavelet basis. The proof of Proposition 30 is therefore a direct extension of these results. Notice that, unlike Theorem 10, we no longer require any conditions on the smoothness of $\varphi_0$ itself, due to the torus regularity result of Theorem 5. Indeed, under the assumptions of Proposition 30, the latter implies that there exists a constant $C' > 0$ depending only on $\alpha, d, \gamma, M$ such that $\|\varphi_0\|_{C^{\alpha+1}(\mathbb{T}^d)} \leq C'$, assuming $\alpha \notin \mathbb{N}$.

### D.2 Proof of Theorem 16

The claim consists of three parts to be proven in parallel. Specifically, we shall prove that for all $\alpha > 1, \alpha \notin \mathbb{N}$,

$$E\|\hat{T}_{nm} - T_0\|_{L^2_m(P)}^2 \lesssim R_{T,n\wedge m}(\alpha),$$

$$|E\|W_2^2(\hat{P}_n, \hat{Q}_m) - W_2^2(P, Q) \| \lesssim R_{T,n\wedge m}(\alpha),$$

$$E|W_2^2(\hat{P}_n, \hat{Q}_m) - W_2^2(P, Q) \| \lesssim R_{W,n\wedge m}(\alpha).$$

(56)

(57)

(58)

To deduce the claim for any $\alpha \in \mathbb{N}$ and $\alpha > 1$, it suffices to note that $\tilde{\alpha} := \alpha - \epsilon > 1$ is not an integer for any $\epsilon \in (0, 1 \wedge (\alpha - 1))$, and the regularity assumptions in the statement of the Theorem evidently hold with $\alpha$ replaced by $\tilde{\alpha}$. We may therefore apply equations (56)–(58) with $\alpha$ replaced
by \( \tilde{\alpha} \) to deduce that, whenever \( \alpha \in \mathbb{N} \),
\[
E\|\hat{T}_{nm} - T_0\|^2_{L^2_{\text{div}}(P)} \lesssim R_{T,n\land m}(\tilde{\alpha}),
\]
\[
|E\mathcal{W}_2^2(\hat{P}_n, \hat{Q}_m) - \mathcal{W}_2^2(P,Q)| \lesssim R_{T,n\land m}(\tilde{\alpha}),
\]
\[
E|\mathcal{W}_2^2(\hat{P}_n, \hat{Q}_m) - \mathcal{W}_2^2(P,Q)| \lesssim R_{W,n\land m}(\tilde{\alpha}).
\]
Since \( \epsilon \) may be chosen arbitrarily small, the claimed rate can readily be seen to follow from the above two displays.

It thus suffices to prove equations (56)–(58) for \( \alpha \notin \mathbb{N} \). To do so, we shall frequently make use of the one-sample optimal transport problem from \( P \) to \( \hat{Q}_m \), for which convergence rates are summarized in Proposition 30. Denote by \( \hat{\varphi}_m \) the optimal Brenier potential for this problem, so that \( \hat{T}_m = \nabla \hat{\varphi}_m \) is the optimal transport map pushing \( P \) forward onto \( \hat{Q}_m \), with respect to \( d^{2}_{2,d} \). Furthermore, denote by
\[
\tilde{\varphi}_m = \|\cdot\|^2 - 2\varphi_m, \quad \tilde{\psi}_m = \|\cdot\|^2 - 2\varphi^*_m,
\]
a corresponding pair of optimal Kantorovich potentials. We proceed with four steps.

**Step 1: Regularity of Fitted Potentials.** By assumption on \( p \) and \( q \), Theorem 5 implies that \( \|\varphi_0\|_{C^0(\mathbb{T}^d)} \leq M_0 \), for a universal constant \( M_0 > 0 \) depending only on \( \gamma \) and \( M \). Therefore, Proposition 4(vii) implies that \( \varphi_0 \) is also strongly convex, and satisfies condition A1(\( \lambda \)) for some \( \lambda > 0 \) depending only on \( M_0 \) and \( \gamma \). To obtain analogous properties for the fitted potentials \( \hat{\varphi}_n \) and \( \tilde{\varphi}_m \), let \( N > 0 \) be defined as in Lemma 25, and assume for the remainder of the proof that \( n \land m \geq N \), so that \( \hat{p}_n = \hat{p}_n \) and \( \hat{q}_m = \hat{q}_m \) almost surely by Lemma 26. Then, by Lemma 25(ii) and Lemma 23, since \( \alpha \notin \mathbb{N} \), we have almost surely,
\[
\|\hat{q}_m\|_{C^0(\mathbb{T}^d)} \lesssim \|\hat{q}_m\|_{B_{\infty,1}^0(\mathbb{T}^d)} \lesssim \|q\|_{B_{\infty,1}^0(\mathbb{T}^d)} \lesssim \|q\|_{C^0(\mathbb{T}^d)} \leq M,
\]
and similarly for \( \hat{p}_n \), so there exists \( M_1 > 0 \) depending only on \( M, \gamma \) and \( d \) such that
\[
\|\hat{p}_n\|_{C^0(\mathbb{T}^d)}, \|\hat{q}_m\|_{C^0(\mathbb{T}^d)} \leq M_1, \quad \text{almost surely.}
\]
Furthermore, by Lemma 25(iii), there exists \( \gamma_0 > 0 \) depending only on \( \gamma, d \) such that
\[
\gamma_0^{-1} \leq \hat{p}_n, \hat{q}_m \leq \gamma_0, \quad \text{almost surely.}
\]

Under the preceding two displays, together with the smoothness assumptions on the population densities \( p, q \) themselves, we may apply the regularity Theorem 5 to deduce that there exists a constant \( M_2 > 0 \) depending only on \( M_0, M_1, \gamma \) such that
\[
\|\hat{\varphi}_nm\|_{C^0(\mathbb{R}^d)} \vee \|\hat{\varphi}_nm\|_{C^0(\mathbb{R}^d)} \vee \|\tilde{\varphi}_m\|_{C^0(\mathbb{R}^d)} \vee \|\tilde{\varphi}_m\|_{C^0(\mathbb{R}^d)} \leq M_2,
\]  
(59)
a almost surely. Apply Proposition 4(vii) to deduce that \( \hat{\varphi}_nm \) and \( \tilde{\varphi}_m \) satisfy the curvature condition A1(\( \lambda \)) almost surely, up to modifying the value of \( \lambda > 0 \) in terms of \( M_2 \) and \( \gamma \), namely:
\[
\lambda^{-1}I_d \leq \nabla^2\tilde{\varphi}_m(x), \nabla^2\hat{\varphi}_nm(x) \leq \lambda I_d, \quad \text{for all } x \in \mathbb{R}^d.
\]

(60)

**Step 2: Reduction to Optimal Transport Problems with Same Source Distribution.**
In order to appeal to the stability bounds of Theorem 6, we make use of the following reductions. Notice first that

\[ \mathbb{E}\left| W_2^2(\hat{P}_n, \hat{Q}_m) - W_2^2(P, Q) \right| \leq \mathbb{E}\left| W_2^2(\hat{P}_n, \hat{Q}_m) - W_2^2(P, \hat{Q}_m) \right| + \mathbb{E}\left| W_2^2(P, \hat{Q}_m) - W_2^2(P, Q) \right| \]

\[ \lesssim \mathbb{E}\left| W_2^2(\hat{P}_n, \hat{Q}_m) - W_2^2(P, \hat{Q}_m) \right| + R_{W,m}(\alpha), \quad (61) \]

where the final inequality is due to Proposition 30(ii). Similarly,

\[ \left| \mathbb{E}W_2^2(\hat{P}_n, \hat{Q}_m) - W_2^2(P, Q) \right| \leq \left| \mathbb{E}W_2^2(\hat{P}_n, \hat{Q}_m) - W_2^2(P, \hat{Q}_m) \right| + R_{T,m}(\alpha). \quad (62) \]

To obtain a similar decomposition in the \( L^2_{\text{per}}(P) \) loss, write

\[ \| \hat{T}_{nm} - T_0 \|_{L^2_{\text{per}}(P)}^2 \lesssim \| \hat{T}_{nm} - \tilde{T}_m \|_{L^2_{\text{per}}(P)}^2 + \| \tilde{T}_m - T_0 \|_{L^2_{\text{per}}(P)}^2. \quad (63) \]

The first term in the above display compares optimal transport maps with distinct source distributions, thus we proceed with the following reduction.

\[ \| \hat{T}_{nm} - \tilde{T}_m \|_{L^2_{\text{per}}(P)}^2 = \int d_{\mathcal{E}}\left( \hat{T}_{nm}(x), \tilde{T}_m(x) \right)^2 dP(x) \]

\[ = \int d_{\mathcal{E}}\left( \hat{T}_{nm}(\tilde{T}_m^{-1}(y)), y \right)^2 d\hat{Q}_m(y) \]

\[ = \int d_{\mathcal{E}}\left( \hat{T}_{nm}(\tilde{T}_m^{-1}(y)), \hat{T}_{nm}(\tilde{T}_m^{-1}(y)) \right)^2 d\hat{Q}_m(y), \quad (64) \]

where the second line follows from the fact that \( (\tilde{T}_m)_{\#} P = \hat{Q}_m \). By equation (59), \( \hat{T}_{nm} = \nabla \hat{\phi}_{nm} \) is almost surely Lipschitz with a universal constant. It follows that for all \( u, v \in \mathbb{R}^d \),

\[ d_{\mathcal{E}}\left( \hat{T}_{nm}(u), \hat{T}_{nm}(v) \right) = \min \left\{ \| \hat{T}_{nm}(u) - \hat{T}_{nm}(v) + k \| : k \in \mathbb{Z}^d \right\} \]

\[ = \min \left\{ \| \hat{T}_{nm}(u) - \hat{T}_{nm}(v + k) \| : k \in \mathbb{Z}^d \right\} \]

\[ \lesssim \min \left\{ \| u - v - k \| : k \in \mathbb{Z}^d \right\} = d_{\mathcal{E}}(u, v), \]

where we used Proposition 4(ii) on the second line, and the Lipschitz property on the third. Returning to equation (64), we deduce

\[ \| \hat{T}_{nm} - \tilde{T}_m \|_{L^2_{\text{per}}(P)}^2 \lesssim \int d_{\mathcal{E}}\left( \tilde{T}_m^{-1}(y), \tilde{T}_m^{-1}(y) \right)^2 d\hat{Q}_m(y) = \| \hat{T}_{nm} - \tilde{T}_m \|_{L^2_{\text{per}}(\hat{Q}_m)}^2. \]

Thus, together with equation (63),

\[ \mathbb{E}\| \hat{T}_{nm} - T_0 \|_{L^2_{\text{per}}(P)}^2 \lesssim \mathbb{E}\| \hat{T}_{nm} - \tilde{T}_m \|_{L^2_{\text{per}}(\hat{Q}_m)}^2 + \mathbb{E}\| \tilde{T}_m - T_0 \|_{L^2_{\text{per}}(P)}^2 \]

\[ \lesssim \mathbb{E}\| \hat{T}_{nm} - \tilde{T}_m \|_{L^2_{\text{per}}(\hat{Q}_m)}^2 + R_{T,m}(\alpha), \]

where we again made use of Proposition 30(ii) to bound the final term. Combining these facts we
deduce that, in order to prove the three bounds \((56)-(58)\), it will suffice to show
\[
\mathbb{E}\|\tilde{T}_m^{-1} - \tilde{T}_{nm}^{-1}\|_{L^2(Q_m)}^2 \lesssim R_{T,n}(\alpha),
\]  
\[
|\mathbb{E}W_2^2(\tilde{P}_n, \tilde{Q}_m) - W_2^2(P, \tilde{Q}_m)| \lesssim R_{T,n}(\alpha),
\]  
\[
\mathbb{E}|W_2^2(\tilde{P}_n, \tilde{Q}_m) - W_2^2(P, \tilde{Q}_m)| \lesssim R_{W,n}(\alpha).
\]

**Step 3: Stability Bounds.** Due to the inequalities \((60)\), the stability bounds of Proposition 15 (arising from Theorem 6) imply
\[
\mathbb{E}\|\tilde{T}_m^{-1} - \tilde{T}_{nm}^{-1}\|_{L^2(Q_m)}^2 \leq \lambda^2 \mathbb{E}W_2^2(\tilde{P}_n, P),
\]  
\[
|\mathbb{E}[W_2^2(\tilde{P}_n, \tilde{Q}_m) - W_2^2(P, \tilde{Q}_m)]| \leq \lambda \mathbb{E}W_2^2(\tilde{P}_n, P) + \mathbb{E}\int \tilde{\phi}_m d(\tilde{P}_n - P),
\]  
\[
\mathbb{E}|W_2^2(\tilde{P}_n, \tilde{Q}_m) - W_2^2(P, \tilde{Q}_m)| \leq \lambda \mathbb{E}W_2^2(\tilde{P}_n, P) + 2\mathbb{E}\int \tilde{\phi}_m d(\tilde{P}_n - P).
\]

By Proposition 30(i), \(\mathbb{E}W_2^2(\tilde{P}_n, P) \lesssim R_{T,n}(\alpha) \leq R_{W,n}(\alpha)\), and the first claim follows. Furthermore, recall that \(\tilde{\phi}_m\) is a Kantorovich potential from \(P\) to \(\tilde{Q}_m\), and is therefore independent of \(\tilde{P}_n\) since we have assumed the samples \(X_1, \ldots, X_n\) and \(Y_1, \ldots, Y_m\) to be independent of each other. To prove the claim, it will thus suffice to show that
\[
\mathbb{E}[\Delta_{1nm}] \lesssim R_{T,n}(\alpha), \quad \text{where } \Delta_{1nm} = \left| \int \tilde{\phi}_m(p - p_{J_n}) \right|,
\]  
\[
\mathbb{E}[\Delta_{2nm}] \lesssim R_{W,n}(\alpha), \quad \text{where } \Delta_{2nm} = \left| \int \tilde{\phi}_m(\tilde{P}_n - p_{J_n}) \right|,
\]
where \(p_{J_n} = \mathbb{E}[\tilde{p}_n] = \sum_{j=0}^{J_n} \sum_{\xi \in \Psi} \alpha \xi \).

**Step 4: Upper Bounds on \(\mathbb{E}[\Delta_{1nm}]\) and \(\mathbb{E}[\Delta_{2nm}]\).** We begin by bounding \(\Delta_{1nm}\), similarly as in the proof of Theorem 10. \(\tilde{\phi}_m\) is \(\mathbb{Z}^d\)-periodic by Proposition 4(i), whence it easily follows from equation \((59)\) that there exists a constant \(M_3 > 0\) depending only on \(M_2\) such that
\[
\|\tilde{\phi}_m\|_{L^\infty(T^d)} \lesssim \|\tilde{\phi}_m\|_{C^{\alpha+1}(T^d)} \leq M_3,
\]
almost surely. Notice further that we may assume without loss of generality that \(\int \tilde{\phi}_m = 0\), thus \(\tilde{\phi}_m\) admits a wavelet expansion in the periodic basis \(\Psi^\text{per}\) given by
\[
\tilde{\phi}_m = \sum_{j=0}^{\infty} \sum_{\xi \in \Psi} \tilde{\gamma}_\xi \xi, \quad \text{where } \tilde{\gamma}_\xi = \int \tilde{\phi}_m \xi \text{ for all } \xi \in \Psi,
\]
which, by equation \((73)\), almost surely satisfies
\[
\|\tilde{\gamma}_\xi\|_{L^\infty(\Psi)} \lesssim 2^{-j(\alpha+1)+\frac{d}{2}}, \quad j \geq 0.
\]
Likewise, the regularity condition on $p$ implies
\[
\| (\alpha \xi)_{\xi \in \Psi_j} \|_{\ell_{\infty}} \lesssim 2^{-j((\alpha - 1) + \frac{d}{2})}, \quad j \geq 0.
\]
Similarly as in the proof of Theorem 10, we therefore have, almost surely,
\[
\left| \int \tilde{\phi}_m(p - p_{J_n}) \right| = \left| \sum_{j = J_n + 1}^\infty \sum_{\xi \in \Psi_j} \tilde{\gamma}_\xi \alpha_\xi \right| \lesssim \sum_{j = J_n + 1}^\infty 2^{dj} \| (\tilde{\gamma}_\xi)_{\xi \in \Psi_j} \|_{\ell_{\infty}} \| (\alpha_\xi)_{\xi \in \Psi_j} \|_{\ell_{\infty}} \lesssim 2^{-2J_n \alpha} \lesssim R_{T,n}(\alpha).
\]
Equation (71) follows. To bound $\mathbb{E}[\Delta_{2nm}]$, notice first that
\[
\int \tilde{\phi}_m(p_n - p_{J_n}) = \int \left( \sum_{j = 0}^\infty \sum_{\xi \in \Psi_j} \tilde{\gamma}_\xi \right) \left( \sum_{j = 0}^\infty \sum_{\xi \in \Psi_j} (\tilde{\alpha}_\xi - \alpha_\xi) \right)
= \sum_{j = 0}^\infty \sum_{\xi \in \Psi_j} \tilde{\gamma}_\xi (\tilde{\alpha}_\xi - \alpha_\xi) = \frac{1}{n} \sum_{i = 1}^n \sum_{j = 0}^J \sum_{\xi \in \Psi_j} \tilde{\gamma}_\xi (\xi(X_i) - \alpha_\xi),
\]
by orthonormality of $\Psi$. Thus, since $\tilde{\gamma}_\xi$ is independent of $X_1, \ldots, X_n$ for all $\xi$, we have
\[
\mathbb{E}[\Delta_{2nm}^2]
= \frac{1}{n^2} \sum_{i,\ell = 1}^n \mathbb{E} \left( \left( \sum_{j = 0}^J \sum_{\xi \in \Psi_j} \tilde{\gamma}_\xi (\xi(X_i) - \alpha_\xi) \right) \left( \sum_{j = 0}^J \sum_{\xi \in \Psi_j} \tilde{\gamma}_\xi (\xi(X_\ell) - \alpha_\xi) \right) \right)
= \frac{1}{n^2} \sum_{i,\ell = 1}^n \mathbb{E} \left( \left( \sum_{j = 0}^J \sum_{\xi \in \Psi_j} \tilde{\gamma}_\xi (\xi(X_i) - \alpha_\xi) \right)^2 \right) + \frac{1}{n^2} \sum_{i,\ell = 1}^n \sum_{j = 0}^J \sum_{\xi \in \Psi_j} \mathbb{E} \mathbb{E}[\tilde{\gamma}_\xi (\xi(X_i) - \alpha_\xi)] \mathbb{E}[\tilde{\gamma}_\xi (\xi(X_\ell) - \alpha_\xi)]
= \frac{1}{n} \mathbb{E} \left[ \left( \sum_{j = 0}^J \sum_{\xi \in \Psi_j} \tilde{\gamma}_\xi (\xi(X) - \alpha_\xi) \right)^2 \right] + \frac{1}{n} \sum_{i,\ell = 1}^n \sum_{j = 0}^J \sum_{\xi \in \Psi_j} \mathbb{E} \mathbb{E}[\tilde{\gamma}_\xi (\xi(X_i) - \alpha_\xi)] \mathbb{E}[\tilde{\gamma}_\xi (\xi(X_\ell) - \alpha_\xi)]
= \frac{1}{n} \mathbb{E} \left[ \left( \sum_{j = 0}^J \sum_{\xi \in \Psi_j} \tilde{\gamma}_\xi (\xi(X) - \alpha_\xi) \right)^2 \right] \lesssim \frac{1}{n} \mathbb{E} \left[ \left( \sum_{j = 0}^J \sum_{\xi \in \Psi_j} \tilde{\gamma}_\xi (\xi(X_i) - \alpha_\xi) \right)^2 \right] + \frac{1}{n} \mathbb{E} \left[ \left( \sum_{j = 0}^J \sum_{\xi \in \Psi_j} \tilde{\gamma}_\xi (\xi(X_\ell) - \alpha_\xi) \right)^2 \right].
\]
(75)

Now, we almost surely have
\[
\left\| \sum_{j = 0}^J \sum_{\xi \in \Psi_j} \tilde{\gamma}_\xi \|_{L_{\infty}} \lesssim \sum_{j = 0}^\infty \| (\tilde{\gamma}_\xi)_{\xi \in \Psi_j} \|_{L_{\infty}} \left( \sup_{\xi \in \Psi_j} \| \xi \|_{L_{\infty}} \right) \sum_{\xi \in \Psi_j} I(|\xi| > 0) \right\|_{L_{\infty}} \lesssim \sum_{j = 0}^\infty 2^{-j((\frac{d}{2} + (\alpha + 1)) + \frac{d}{2})} < \infty,
\]
by Lemma 22 and equation (74). Furthermore, by the same argument as was used to bound $\mathbb{E}[\Delta_{1nm}]$, we also have $\sum_{j \geq 0} \sum_{\xi \in \Psi_j} |\tilde{\gamma}_\xi| |\alpha_\xi| < \infty$ almost surely. We deduce that both expectations appearing in equation (75) are finite, and we deduce that $\mathbb{E}[\Delta_{2nm}] \leq \sqrt{\mathbb{E}[\Delta_{2nm}^2]} \lesssim n^{-1/2}$. In
particular equation (72) holds. The claim follows.

\[ \square \]

E Proofs of Central Limit Theorems

We begin by stating several preliminary results which will be used in the sequel. The following is the classical Poincaré inequality for domains with bounded width (see for instance Leoni (2017), Theorem 12.17), stated in the special case of the unit cube.

**Lemma 31** (Poincaré Inequality over \([0, 1]^d\)). Let \(\Omega = [0, 1]^d\). Then, there exists a universal constant \(C_d > 0\) such that for all \(f \in C^1(\Omega)\) satisfying \(\int_\Omega f = 0\),

\[ \|f\|_{L^2(\Omega)} \leq C_d \|\nabla f\|_{L^2(\Omega)}. \]

We shall also make use of the following classical periodic Poincaré inequality (see for instance Steinerberger (2016) for a simple proof).

**Lemma 32** (Poincaré Inequality over \(T^d\)). Let \(f \in C^1(T^d)\) satisfy \(\int_{T^d} f = 0\). Then,

\[ \|f\|_{L^2(T^d)} \leq \|\nabla f\|_{L^2(T^d)}. \]

The above two results will allow us to translate convergence rates of a transport map estimator \(\hat{T}_n\) into convergence rates for the corresponding Brenier potential \(\tilde{\varphi}_n\). Next, we state the classical Efron-Stein inequality (see for instance Boucheron et al. (2013), Theorem 3.1) for bounding the variance of functions of independent random variables.

**Lemma 33** (Efron-Stein Inequality). Let \(Y_1, Y'_1, Y_2, Y'_2, \ldots, Y_n, Y'_n\) be independent random variables, and let \(R_n = f(Y_1, \ldots, Y_n)\) be a square-integrable function of \(Y_1, \ldots, Y_n\). Let

\[ R'_{ni} = f(Y_1, \ldots, Y_{i-1}, Y'_i, Y_{i+1}, \ldots, Y_n), \quad i = 1, \ldots, n. \]

Then,

\[ \text{Var}[R_n] \leq \sum_{i=1}^n \mathbb{E}(R_n - R'_{ni})^2. \]

With these results in place, we turn to the proofs of results stated in Section 5, beginning with the one-sample case.

E.1 Proof of Proposition 17

**Step 1: Convergence Rate of Kantorovich Potentials.** Recall the one-sample estimator \(\hat{T}_n\) stated in equation (22), given by the optimal transport map between \(P\) and \(\hat{Q}_n\). Let \(\hat{\varphi}_n\) denote a Brenier potential satisfying \(\hat{T}_n = \nabla \hat{\varphi}_n\), which we may assume without loss of generality satisfies \(\int_\Omega \hat{\varphi}_n = 0\). We likewise assume without loss of generality that \(\int_\Omega \varphi_0^* = 0\). By assumption \(A1(\lambda)\),
Finally, define the pair of Kantorovich potentials \((\hat{\phi}, \hat{\psi})\) corresponding to \((\varphi_n, \varphi_n^*)\) by
\[
\hat{\phi}_n = 2 \| \cdot \|^2 - \varphi_n, \quad \hat{\psi}_n = 2 \| \cdot \|^2 - \varphi_n^*.
\]

As a direct consequence of (76), we also have
\[
\mathbb{E} \| \hat{\psi}_n - \psi_0 \|_{L^2(\Omega)} \leq R_{T,n}(\alpha) = 2^{-2J_n \alpha} \vee \frac{1}{n}.
\]

**Step 2: Linearization of the Wasserstein distance.** We are now in a position to prove the claim. Let \(Y'_1 \sim Q\) denote a random variable independent of \(Y_1, \ldots, Y_n\), and let
\[
Q_n' = \frac{1}{n} \delta_{Y'_1} + \frac{1}{n} \sum_{i=2}^n \delta_{Y_i}
\]
denote the corresponding empirical measure. Let \(\hat{Q}'_n\) be the distribution with density
\[
\hat{q}'_n = \sum_{\zeta \in \Phi} \hat{\beta}'_\zeta + \sum_{j=0}^{J_n} \sum_{\xi \in \Psi_j} \hat{\beta}'_{\xi} = \sum_{j=0}^{J_n} \sum_{\xi \in \Psi_j} \hat{\beta}'_{\xi}, \quad \text{where} \quad \hat{\beta}'_{\xi} = \int \xi d\hat{Q}'_n, \ \xi \in \Psi,
\]
where we write $\Psi_{j_0-1} = \Phi$ for ease of notation. Set

$$R'_n = W_2^2(P, \hat{Q}'_n) - \int \psi_0 d\hat{Q}'_n.$$  

By the Efron-Stein inequality (Lemma 33), it will suffice to prove that $n^2 \mathbb{E}(R_n - R'_n)^2 = o(1)$. By definition of the Kantorovich potentials $(\hat{\phi}_n, \hat{\psi}_n)$, recall that

$$W_2^2(P, \hat{Q}_n) = \int \hat{\phi}_n dP + \int \hat{\psi}_n d\hat{Q}_n,$$

$$W_2^2(P, \hat{Q}'_n) = \sup_{(\phi, \psi) \in K} \int \phi dP + \int \psi d\hat{Q}'_n \geq \int \hat{\phi}_n dP + \int \hat{\psi}_n d\hat{Q}_n = W_2^2(P, \hat{Q}_n) + \int \hat{\psi}_n d(\hat{Q}'_n - \hat{Q}_n).$$

It follows that,

$$R_n - R'_n \leq \int (\hat{\psi}_n - \psi_0) d(\hat{Q}_n - \hat{Q}'_n) = \int (\hat{\psi}_n - \psi_0) \left( \sum_{j=j_0-1}^{J_n} \sum_{\xi \in \Psi_j} (\hat{\beta}_\xi - \beta_\xi) \xi \right) = \frac{1}{n} \sum_{j=j_0-1}^{J_n} \sum_{\xi \in \Psi_j} (\xi(Y_1) - \xi(Y'_1)) \int (\hat{\psi}_n - \psi_0) \xi,$$

We thus obtain,

$$n^2 \mathbb{E}(R_n - R'_n)^2 \lesssim J_n \sum_{j=j_0-1}^{J_n} \sum_{\xi \in \Psi_j} \mathbb{E}[\xi(Y)^2] \int ||\hat{\psi}_n - \psi_0||^2 |\xi|^2 \lesssim J_n \sum_{j=j_0-1}^{J_n} \int ||\hat{\psi}_n - \psi_0||^2 |\xi|^2,$$

where we used Lemma 22(ii) in the final step, together with the fact that $\gamma^{-1} \leq q \leq \gamma$, whence

$$\mathbb{E}[\xi^2(Y)] = \int \xi^2(y)q(y)dy \leq \gamma \int \xi^2(y)dy = \gamma.$$

Since $|\xi| \lesssim 2^{dj/2}$ for all $\xi \in \Psi_j$, $j \geq j_0$, by Lemma 22(v), we obtain from equation (77),

$$n^2 \mathbb{E}(R_n - R'_n)^2 \lesssim J_n \sum_{j=j_0}^{J_n} 2^{dj} \int ||\hat{\psi}_n - \psi_0||^2 \lesssim J_n 2^{dj} ||\hat{\psi}_n - \psi_0||_{L^2(\Omega)}^2 \lesssim J_n \left( 2^{J_n(d-2\alpha)} \sqrt{\frac{2^{dJ_n}}{n}} \right).$$

Since $d < 2\alpha$, the above display is of order $o(1)$, thus the claim follows from Lemma 33. \hfill \Box

### E.2 Proof of Theorem 18

Since $q$ is assumed to be positive and continuous, there must exist $\gamma > 0$ such that $\gamma^{-1} \leq q \leq \gamma$ over the compact set $\Omega = [0, 1]^d$. The conditions of Theorem 10(ii) are therefore satisfied with this choice of $\gamma$ and $M = \|q\|_{C^{\alpha-1}(\Omega)} \lor \|\varphi_0\|_{C^{\alpha+1}(\Omega)}$.

Notice that $\sigma = 0$ if and only if $P = Q$, in which case the claim is trivial due to Lemma 9. We thus assume $P \neq Q$ and $\sigma > 0$ throughout the sequel. We first derive a central limit theorem centered
at $EW^2_2(P, \hat{Q}_n)$. Notice that,
\[
\sqrt{n} \left( W^2_2(P, \hat{Q}_n) - EW^2_2(P, \hat{Q}_n) \right) = \sqrt{n} \int \psi_0 d(\hat{Q}_n - Q_{J_n}) + \sqrt{n} \left( R_n - E[R_n] \right),
\]
where recall that $Q_{J_n}$ is the distribution with density $q_{J_n} = E[\hat{q}_n]$. It follows from Proposition 17 that the final term of the above display converges to zero in probability. Regarding the first term, we have the following limit theorem.

**Lemma 34.** Assume $\sigma^2 = \text{Var}[\psi_0(Y)] > 0$. Then, under the same conditions as Theorem 18,
\[
\sqrt{n} \int \psi_0 d(\hat{Q}_n - Q_{J_n}) \rightsquigarrow N(0, \sigma^2), \quad \text{as } n \to \infty.
\]
It readily follows that, as $n \to \infty$,
\[
\sqrt{n} \left( W^2_2(P, \hat{Q}_n) - EW^2_2(P, \hat{Q}_n) \right) \rightsquigarrow N(0, \sigma^2).
\]
On the other hand, Theorem 10(ii) together with the assumption $d < 2\alpha$ imply $EW^2_2(P, \hat{Q}_n) = W^2_2(P, Q) + o(n^{-1/2})$, so that,
\[
\sqrt{n} \left( W^2_2(P, \hat{Q}_n) - W^2_2(P, Q) \right) \rightsquigarrow N(0, \sigma^2).
\]
The claim follows. It thus remains to prove Lemma 34.

**E.2.1 Proof of Lemma 34**

Recall that $\gamma_\xi = \int \xi \psi_0$ for all $\xi \in \Psi$. Note that
\[
\int \psi_0 d(\hat{Q}_n - Q_{J_n}) = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=j_0-1}^{J_n} (\xi(Y_i) - E\xi(Y_i)) \int \psi_0 \xi = \frac{1}{n} \sum_{i=1}^{n} (Z_{n,i} - E[Z_{n,i}]),
\]
where $Z_{n,i} = \sum_{j=j_0-1}^{J_n} \sum_{\xi \in \Psi_j} \xi(Y_i) \gamma_\xi$. By Lyapunov’s central limit theorem (Billingsley (1968), Theorem 7.3), it holds that
\[
\frac{1}{\sqrt{\sum_{i=1}^{n} \text{Var}[Z_{n,i}]}} \sum_{i=1}^{n} (Z_{n,i} - E[Z_{n,i}]) \rightsquigarrow N(0,1), \quad (78)
\]
provided that for some $p > 2$,
\[
\frac{\sum_{i=1}^{n} E[|Z_{n,i} - E[Z_{n,i}]|^p]}{(\sum_{i=1}^{n} E[|Z_{n,i} - E[Z_{n,i}]|^2])^{p/2}} \to 0. \quad (79)
\]
Similarly as in the proof of Lemma 11, notice that $Z_{n,i}$ is the projection of $\psi_0$ onto $\text{Span} \left( \bigcup_{j=j_0-1}^{J_n} \Psi_j \right)$ evaluated at $Y_i$, and is therefore bounded uniformly over all $n$ and $i$, due to the Hölder continuity
of $\psi_0$. Indeed, since $\psi_0 \in \mathcal{C}_0^1(\Omega)$ by Lemma 23, one has by Lemma 22,

$$\left\| \sum_{j=j_0-1}^{J_n} \sum_{\xi \in \Psi_j} \xi \gamma \xi \right\|_{L^\infty} \lesssim \sum_{j=j_0-1}^{J_n} 2^{-j(\frac{d}{2} + \alpha + 1)2/2^{2j}} \left\| \sum_{\xi \in \Psi_j} I(|\xi| > 0) \right\|_{L^\infty} \lesssim \sum_{j=j_0-1}^{\infty} 2^{-j(\frac{d}{2} + \alpha + 1)2/2^{2j}} < \infty.$$ 

It follows that for all $p > 2$, there exists a constant $C_p > 0$ such that

$$\sum_{n=1}^{n} \mathbb{E}[|Z_{n,i} - \mathbb{E}Z_{n,i}|^p] \leq C_p n.$$ 

On the other hand, by Lemma 11,

$$\sum_{n=1}^{n} \mathbb{E}[|Z_{n,i} - \mathbb{E}Z_{n,i}|^2] = n^2 \text{Var} \left[ \int \hat{\psi}_n^2 d\hat{Q}_n \right] = n \text{Var}[\psi_0(Y)] + o(n) = n(\sigma^2 + o(1)).$$

Since $\sigma > 0$, we deduce that Lypaunov’s condition holds for all $p > 2$:

$$\frac{\sum_{i=1}^{n} \mathbb{E}[|Z_{n,i} - \mathbb{E}Z_{n,i}|^p]}{(\sum_{i=1}^{n} \mathbb{E}[|Z_{n,i} - \mathbb{E}Z_{n,i}|^2])^{\frac{p}{2}}} \leq \frac{C_p n}{[n(\sigma^2 + o(1))]^{p/2}} \to 0.$$ 

The claim now follows from equation (78).

### E.3 Proof of Corollary 19

Since $\Omega$ is compact, recall that $\psi_0$, $\hat{\psi}_n$ may be taken to be uniformly bounded by a universal constant $B > 0$ depending only on $\text{diam}(\Omega)$. It follows that

$$|\hat{\sigma}_n^2 - \sigma^2| \lesssim \left| \int \hat{\psi}_n^2 dQ_n - \int \psi_0^2 dQ \right| + \left| \int \hat{\psi}_n dQ_n - \int \psi_0 dQ \right|.$$ 

We shall show that the first term vanishes in probability, and a similar argument can be used for the second. We have,

$$\left| \int \hat{\psi}_n^2 dQ_n - \int \psi_0^2 dQ \right| = \left| \int \hat{\psi}_n^2 d(Q_n - Q) - \int (\hat{\psi}_n - \psi_0)^2 dQ \right| \lesssim \left| \int \hat{\psi}_n^2 d(Q_n - Q) \right| + \left\| \hat{\psi}_n - \psi_0 \right\|_{L^2(Q)}.$$ 

Since $\hat{\psi}_n$ is convex up to translation by a quadratic function, it must be Lipschitz over $\Omega$ with a uniform constant depending only on $B$ (Hiriart-Urruty and Lemaréchal (2004), Lemma 3.1.1), and similarly for $\psi_0^2$. The set of Lipschitz functions with a uniform Lipschitz constant forms a Glivenko-Cantelli class (van der Vaart and Wellner (1996), Theorem 2.7.1), thus the first term on the right-hand side of the above display vanishes in probability. Furthermore, the second term vanishes in probability by the same argument as in Step 1 of the proof of Proposition 17 (equation (77)). We deduce that $\hat{\sigma}_n^2 \xrightarrow{p} \sigma^2$, and the claim then follows by Slutsky’s Theorem.

### E.4 Proof of Theorem 20

Since $p, q$ are assumed to be positive and continuous, notice that there must exist $\gamma > 0$ such that $\gamma^{-1} \leq p, q \leq \gamma$ over the compact set $[0, 1]^d$, and a fortiori over $\mathbb{R}^d$ by $\mathbb{Z}^d$-periodicity. The
assumptions of Theorem 16 are therefore satisfied with this choice of $\gamma$ and with $M = \|p\|_{C^{a-1}(\mathbb{T}^d)} \vee \|q\|_{C^{a-1}(\mathbb{T}^d)} < \infty$.

The proof follows the same strategy as in the one-sample case, and is again inspired by del Barrio and Loubes (2019). Define

$$R_{nm} = W_2^2(\tilde{P}_n, \tilde{Q}_m) - \int \phi_0 d\tilde{P}_n - \int \psi_0 d\tilde{Q}_n.$$ 

Let $X'_1 \sim P$ and $Y'_1 \sim Q$ be independent of $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$, and define

$$P'_n = \frac{1}{n} \delta_{X'_1} + \frac{1}{n} \sum_{i=2}^{n} \delta_{X_i}, \quad \tilde{P}_n = \sum_{j=J_0-1}^{J_n} \sum_{\xi \in \Psi_j} \hat{\alpha}_\xi \xi,$$

$$Q'_m = \frac{1}{m} \delta_{Y'_1} + \frac{1}{m} \sum_{j=2}^{m} \delta_{Y_j}, \quad \tilde{Q}_m = \sum_{j=J_0-1}^{J_m} \sum_{\xi \in \Psi_j} \hat{\beta}_\xi \xi,$$

where recall that $\Psi = \Psi_{\text{per}}$ is the periodic wavelet system, $\hat{\alpha}_\xi = \int \xi dP'_n$, and $\hat{\beta}_\xi = \int \xi dQ'_m$ for all $\xi \in \Psi$. Set

$$R'_{nm} = W_2^2(\tilde{P}_n, \tilde{Q}_m) - \int \phi_0 d\tilde{P}_n - \int \psi_0 d\tilde{Q}_m, \quad R''_{nm} = W_2^2(\tilde{P}_n, \tilde{Q}_m') - \int \phi_0 d\tilde{P}_n - \int \psi_0 d\tilde{Q}_m'.$$

By the Efron-Stein inequality (Lemma 33), notice that

$$\text{Var}(R_{nm}) \leq n \mathbb{E}(R_{nm} - R'_{nm})^2 + m \mathbb{E}(R_{nm} - R''_{nm})^2. \quad (80)$$

We shall show that the above display is of order $o(m/(n + m))$. Recall that $\tilde{T}_{nm} = \nabla \tilde{\varphi}_{nm}$ is the optimal transport map from $\tilde{P}_n$ to $\tilde{Q}_m$, and that $\tilde{\varphi}_{nm} = \|\cdot\|^2 - 2\tilde{\varphi}_{nm}$, and $\tilde{\psi}_{nm} = \|\cdot\|^2 - 2\tilde{\varphi}^*_nm$. Recall that we may choose $\tilde{\varphi}_{nm}$ such that $\int_{\mathbb{T}^d} \tilde{\varphi}_{nm} = 0$, and we likewise assume without loss of generality that $\int_{\mathbb{T}^d} \phi_0 = 0$. By the same argument as in the proof of Proposition 17, using the Kantorovich duality we have

$$R_{nm} - R'_{nm} \leq \int (\tilde{\varphi}_{nm} - \phi_0) d(\tilde{P}_n - \tilde{P}'_n) = \frac{1}{n} \sum_{j=J_0-1}^{J_n} \sum_{\xi \in \Psi_j} \left( \xi(X_1) - \xi(X'_1) \right) \int (\tilde{\varphi}_{nm} - \phi_0) \xi,$$

which as before leads to the upper bound

$$n^2 \mathbb{E}(R_{nm} - R'_{nm})^2 \leq J_n 2^{4J_n} \mathbb{E}\|\tilde{\varphi}_{nm} - \phi_0\|_{L^2(\mathbb{T}^d)}^2.$$

Apply the Poincaré inequality over $\mathbb{T}^d$ (Lemma 32), and the fact that $\gamma^{-1} \leq p \leq \gamma$ to deduce

$$\mathbb{E}\|\tilde{\varphi}_{nm} - \phi_0\|_{L^2(\mathbb{T}^d)}^2 \leq \mathbb{E}\|\nabla (\tilde{\varphi}_{nm} - \phi_0)\|_{L^2(\mathbb{T}^d)}^2 = \mathbb{E}\|\tilde{T}_{nm} - T_0\|_{L^2(\mathbb{T}^d)}^2 \leq \mathbb{E}\|\tilde{T}_{nm} - T_0\|_{L^2(P)}^2.$$

Combining the previous two displays, and applying Theorem 16(i) with the assumption $n \sim m$, we
deduce
\[ n^2 \mathbb{E}(R_{nm} - R'_{nm})^2 \lesssim J_n 2^{J_n d} R_{T; n \wedge m}(\alpha; \epsilon) \lesssim J_n n^{\epsilon} \left( 2^{-J_n (2\alpha - d)} \sqrt{\frac{J_n d}{n}} \right), \]
for an arbitrarily small constant \( \epsilon > 0 \). In particular, by choosing \( \epsilon \) sufficiently small, it follows that \( n^2 \mathbb{E}(R_{nm} - R'_{nm})^2 = o(1) \). To prove an analogous result for \( \mathbb{E}(R_{nm} - R''_{nm})^2 \), choose a distinct Brenier potential \( \hat{\varphi}_{nm} \) in the optimal transport problem from \( \hat{P}_n \) to \( \hat{Q}_m \), such that the resulting Kantorovich potentials \( (\hat{\varphi}_{nm}, \hat{\psi}_{nm}) \) satisfy \( \int_{\mathbb{T}^d} \hat{\psi}_{nm} = \int_{\mathbb{T}^d} \psi_0 \). The same reasoning as before then leads to
\[ m^2 \mathbb{E}(R_{nm} - R''_{nm})^2 \lesssim J_m 2^{dJ_m} \mathbb{E}\| \hat{\psi}_{nm} - \psi_0 \|^2_{L^2(\mathbb{T}^d)}. \] (81)

The convergence of \( \hat{\psi}_{nm} \) may be related to that of \( \hat{T}_{nm} \) in a similar way as in Step 2 of Appendix D.2, and Step 1 of Appendix E.1, thus we only provide a brief proof. In view of Lemma 32, Lemma 25(iii), and the fact that \( T_0^{-1} \) is Lipschitz by the same argument as in the proof of Theorem 16, the following holds almost surely,
\[
\| \hat{T}_{nm} - \psi_0 \|^2_{L^2(\mathbb{T}^d)} \leq \| \hat{T}_{nm} - T_0^{-1} \|^2_{L^2(\mathbb{T}^d)} \lesssim \| \hat{T}_{nm} - T_0^{-1} \|^2_{L^2(\hat{Q}_n)} \\
= \| T_0^{-1} \circ T_0 - T_0^{-1} \circ \hat{T}_{nm} \|^2_{L^2(\hat{P}_n)} \lesssim \| T_0 - \hat{T}_{nm} \|^2_{L^2(\hat{P}_n)} \lesssim \| T_0 - \hat{T}_{nm} \|^2_{L^2(\hat{P}_n)}.
\]

Combined with equation (81) and Theorem 16(i), we obtain \( m^2 \mathbb{E}(R_{nm} - R''_{nm})^2 = o(1) \) due to the condition \( d < 2\alpha \). Since \( n \gg m \), combining these bounds with equation (80) leads to
\[
\frac{nm}{n + m} \text{Var}(R_{nm}) = o(1).
\]

To obtain the limit theorem from here, notice as in the one-sample case that the above display, combined with the bias bound of Theorem 16(ii) imply, under the smooth regime \( 2\alpha > d \),
\[
\sqrt{\frac{nm}{n + m}} (\mathcal{W}_2^2(\hat{P}_n, \hat{Q}_m) - \mathcal{W}_2^2(P, Q)) = \sqrt{\frac{nm}{n + m}} \left( \int \phi_0 d(\hat{P}_n - P) + \int \psi_0 d(\hat{Q}_m - Q) \right) + o_p(1) \\
= \sqrt{m} \int \phi_0 d(\hat{P}_n - P) + \sqrt{m(1 - \rho)} \int \psi_0 d(\hat{Q}_m - Q) + o_p(1).
\]

Apply Lemma 34, which is readily seen to hold for the periodic wavelet basis under consideration, to deduce the result. \( \square \)

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