Symmetry Analysis In Inflationary Cosmology

Andronikos Paliathanasis1,a)

1Institute of Systems Science, Durban University of Technology, PO Box 1334, Durban 4000, RSA

a)Corresponding author: anpaliat@phys.uoa.gr
URL: anpaliat@phys.uoa.gr

Abstract. We approach the cosmological inflation through symmetries of differential equations. We consider the general inflaton field in a homogeneous Friedmann–Lemaître–Robertson–Walker spacetime and with the use of conformal transformations we are able to write the generic algebraic solution for the field equations. We put emphasis on the inflationary models and we show how we can construct new inflationary models from already known models by using symmetry transformations.

INTRODUCTION

In the late 19th century Sophus Lie had the idea to bring to the differential equations the algebraic success of infinitesimal transformations which had attended the question of the solution of polynomial equations. While his initial expectations were never realized, he was able to establish a new method for the determination of solutions for differential equations. His pioneering work was published in 1888 [1, 2, 3] with the title “Theory of transformation groups” and defined a new area in mathematics, the Symmetries. The concept of Lie symmetries has been one of the main materials for the study of nonlinear differential equations, the main results which establish the Lie theory a main mathematical tool in modern science can be found in the works of Ovsiannikov [4], Ibragimov [5], Olver [6], Crampin [7], Leach [8] and many others.

There is a wide range of applications of the theory of symmetries in natural sciences and specifically in physics, from analytical mechanics [10], to particle physics [11], and gravitational physics [12]. An important class of symmetries which are commonly used in General Relativity are the spacetime collineations [12]. Collineations are the generators of continuous transformations where geometric objects of the gravitational theory are transformed under a specific rule, or remain invariant. The most important class of collineations are the isometries, or Killing vectors (KV), which leaves invariant the metric tensor of the theory. Furthermore, the existence of collineations can simplify the Einstein-field equations and provide important kinematic or dynamical constraints in the physical properties of the theory [13]. A classification of exact solutions in General Relativity according to the admitted collineations of the spacetime can be found in [14].

In General Relativity, the gravitational field equations is a set of ten nonlinear partial differential equations. Collineations of the Einstein tensor pass through the field equations as Lie symmetries of differential equations and in particular reduce the number of the independent variables, or the number of the independent equations. However, these Lie symmetries are restricted only to the space of independent variables, while in general there can exist continuous transformations in the space of the dependent variables which leave the field equations invariant. Hence, a detailed study of the symmetries of the field equations is required. Indeed such an analysis was the main subject of study in [16, 17].

The cosmological principle, which states that the spatial distribution in the universe is homogeneous and isotropic in large scales, is supported by the detailed analysis of the recent cosmological data [18, 19]. Mathematically, the cosmological principle is expressed with the existence of a six isometries (KVs) in the physical spacetime, which means that the only possible line element which describes the universe in large scales is the Friedmann–Lemaître–Robertson–Walker (FLRW) metric

\[
\begin{align*}
\text{d}s^2 = -N^2 (t) \text{d}t^2 + a^2 (t) \left( \frac{\text{d}r^2}{1 - kr^2} + r^2 \left( \text{d} \theta^2 + \sin^2 \theta \text{d} \phi^2 \right) \right)
\end{align*}
\]

(1)
in which $a (t)$ is the scale factor of the universe, $N (t)$ is the lapse function where without loss of generality can be set a nonzero constant, and $k$ denotes the spatial curvature of the three dimensional space, while the cosmological data indicates that $k \approx 0$.

A theoretical mechanics which explains the large-scale structure of the universe is the inflation [20]. Inflation is attributed to the existence of the “inflaton” which drives a period of acceleration in the early universe [21]. More specifically the inflaton, plays the role of a matter source in Einstein’s GR with negative pressure which displays an antigravity behavior [22]. The inflaton it is assumed that is described by a scalar field. The introduction of the scalar field in Einstein’s GR introduces new degrees of freedom in the theory which can drive the dynamics such that the inflationary era to be described. The physical particle representation of the inflaton scalar field is still unknown it can describe an “exotic” nonvisible matter source or the degrees of freedom which follow from modifications of Einstein-Hilbert’s Action [21].

The field equations in Einstein’ GR are of second-order, however in the presence of the scalar field, the field equations show unexpected complexity, as a result, exact and analytic solutions exist only for few scalar field potentials, see [23, 24, 25, 26, 27, 28] and references therein. Moreover, the main mathematical tools of our analysis are discussed. In Section b) we present a family of inflationary models which follow from a set of maximally symmetric master equations. We apply Lie theory to construct new differential equations in inflationary models and more specifically on the theoretical description of the inflaton field. The plan of the paper is as follows.

In Section b) we present the cosmological model of our consideration as also the main set of differential equations. Moreover, the main mathematical tools of our analysis are discussed. In Section b) we present a family of inflationary models which follow from a set of maximally symmetric master equations. We apply Lie theory to construct new inflationary models. Our conclusions are presented in Section a).

**SCALAR FIELD COSMOLOGY**

In the context of GR and in the presence of a scalar field $\phi (x')$ the Action Integral of the gravitational field equations is [36]

$$S = \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \left( \frac{1}{2} g_{ab} \phi' a \phi' b - V (\phi) \right),$$  \hspace{1cm} (2)

where $R$ is the Ricciscalar of the underlying manifold with metric $g_{ab}$, and $V (\phi)$ is the potential which drives the dynamics of the field $\phi (x')$.

Variation with respect to the metric tensor of (2) provides the Einstein field equations

$$R_{ab} - \frac{1}{2} R g_{ab} = \phi, a \phi, b - \frac{1}{2} g_{ab} (\phi, c \phi, c + V (\phi)),$$  \hspace{1cm} (3)

while variation with respect to the field $\phi (x')$ gives the “Klein-Gordon” equation $\phi, a g^{ab} + V, \phi = 0$.

For the spatially flat FLRW spacetime the Ricciscalar is calculated to be $R = \frac{6}{N^2} \left( \frac{N}{a} \right)^2 - \frac{\dot{a}}{a N} \dot{a}$, while by assuming that the scalar field inherits the symmetries of the spacetime it follows $\phi (x') = \phi (t)$, consequently, the latter equations are written

$$3 H^2 - \frac{1}{2} \dot{\phi}^2 - V (\phi) = 0,$$  \hspace{1cm} (4)

$$2 \dot{H} + 3 H^2 + \left( \frac{1}{2} \dot{\phi}^2 - V (\phi) \right) = 0,$$  \hspace{1cm} (5)

and

$$\ddot{\phi} + 3 H \dot{\phi} + V, \phi = 0,$$  \hspace{1cm} (6)

in which $H (t)$ is the Hubble constant defined as $H (t) = \frac{\ddot{a}}{a}$. At this point we would like to remark that we have assumed the comoving observer $u^a = \frac{1}{N} \delta^a_t$, which $u^a u_a = -1$. 
MINISUPERSPACE LAGRANGIAN

The field equations (3), (4) are of second-order in terms of the parameters \{a, \phi\}. Moreover, what it is interesting is that the field equations can be derived under the variation of the Lagrange function

\[ L(N, a, \dot{a}, \phi, \dot{\phi}) = \frac{1}{N} \left(-3a\ddot{a}^2 + \frac{1}{2}a^3\dot{\phi}^2\right) - Na^3V(\phi). \] (7)

More specifically, equations (5), (6) correspond to the Euler-Lagrange equations of (7) with respect to the variables \{a, \phi\}, while variation with respect to the lapse function \(N\) provides the constraint equation (4).

From (7) someone can define the momentum \(p_a = \frac{\dot{a}}{a}\) and \(p_\phi = \frac{\partial L}{\partial \dot{\phi}}\) and write the Hamiltonian function

\[ \mathcal{H} = N\left(-\frac{p_a^2}{3a} + \frac{p_\phi}{2a^3} + a^3V(\phi)\right). \] (8)

which from the constraint it follows that \(\mathcal{H} = 0\). At this point we would like to remark that the field equations define the singular Hamiltonian system, which is constrained by the conditions \(p_a = 0\) and (3).

It is well known that there exists a unique relation between the symmetries of dynamical systems defined by a kinetic energy and potential, equations with the collineations which define the space where the motion occurs, i.e. the kinetic metric. More specifically it is known that any generator of a symmetry vector for the dynamical system has to be a symmetry also for the geometry (37). For instance, the conservation law of momentum for the free particle follows from the translation symmetry of the Euclidean spacetime. The group of translations with the group of rotations form the group of isometries or Killing vectors of the Euclidean space.

By definition a Killing vector in a Riemannian manifold is the generator of the transformation which keeps invariant the length and the angles. On the other hand, a Homothetic vector is the generator of the transformation which keeps invariant the angles and rescales by a constant the length, whereas a Conformal vector is called the generator of the transformation which preserves the angles on the space (12). For autonomous Hamiltonian systems the “Energy” denotes the volume in the phase space. For any isometry which leaves this volume invariant in the phase space corresponds a conservation law which commutes with the Hamiltonian. As far as the Homothetic vector concerned, the solutions can be transformed under other solutions but with a rescaled “Energy” value. These two transformations relate objects which are congruent, with the identical congruent to be provided by the isometries. The situation is totally different under conformal transformations. Indeed Hamiltonian systems are not invariant under conformal transformations except if the “Energy” is zero, which means that the volume in the phase space has dimensions zero. Moreover the volume continues to be zero under conformal transformations and consequently conservation laws can be constructed.

Mathematically, that is demonstrated as follows, consider \(\mathcal{H}(p, q) = 0\) to be the energy of an autonomous Hamiltonian system and \(J(p, q)\) be a conservation law generated by a conformal vector. Then it follows that there exists a function, \(\omega\), such that \(D_t(J) = I_p + \{I, \mathcal{H}\} = \omega\mathcal{H}\); that is, \(D_t(I) = 0\), which means that \(J\) is a conservation law. These kinds of conservation laws are generated by nonlocal symmetries, which are reduced to local when \(\omega = const\) or \(\omega = 0\).

Because of the constraint equation we can say that the Energy of the Mechanical analogue is zero and construct conservation laws by using the conformal algebra of the minisuperspace. In particular, for every Conformal vector field there corresponds a conservation law for the field equations, for any function, \(V(\phi)\). Moreover, because the minisuperspace of (7) has dimension two, it admits an infinite-dimensional conformal algebra, that is, there exists an infinite number of (nonlocal) conservation laws. Of course these conservation laws are not in involution with each other, but they are with the Hamiltonian applying the constraint equation, \(\mathcal{H}(p, q) = 0\). That concept is easily generalized and in the case of conformal Killing tensors which define conservation laws polynomial in terms of the momentum \(p\) (30, 38).

As far as the dynamical system of our study is concerned, the existence of a nonlocal conservation law plus the constraint equation (7) is sufficient to prove the integrability. The generic algebraic solution is presented below.

ALGEBRAIC SOLUTION OF SCALAR FIELD COSMOLOGY

The precise meaning of the solution of a system of differential equations can be cast in several ways. Three of these are: (a) a set of explicit functions describing the variation of the dependent variables with the independent variable(s)
arbitrary function $F$ dominates the kinetic term i.e.,

From the field equations (4), (5) we can observe that in order the universe to be in an inflationary era, then the scalar specific equation of state parameter presented in [39, 40] are defined similarly [42], moreover, the recent data analysis by the Planck collaboration [19], it was found that the value of $\epsilon$ follows

for the inflationary phase to last long enough we require the second PSR parameter also to be small,

and

where now the spatially flat FLRW spacetime is written as

$$dx^2 = -e^{F(\omega)}d\omega^2 + e^{\omega/3} (dx^2 + dy^2 + dz^2).$$

It is easy to see that $F(\omega)$ is related with the Hubble functions as $F(\omega) = -2 \ln (3H(\omega))$. The latter solution is generic analytical solution for arbitrary potential. The form of the potential fixes the EoS and provides an first order-differential equation $p_\phi(\omega) = \Phi (\rho_\phi)$ which can be reduced to an algebraic equation. Otherwise, a specific function $F(\omega)$, provides always (locally) a specific potential $V(\phi (\omega))$. Closed-form solutions where that scalar field admit a specific equation of state parameter presented in [39, 40]

**INFLATIONARY SLOW-ROLL PARAMETERS**

From the field equations (4), (5) we can observe that in order the universe to be in an inflationary era, then the scalar field potential dominates the kinetic term i.e., $\frac{\dot{\phi}^2}{2} + V(\phi)$, while the field equations can be written as

$$3H^2 \approx V(\phi), \ 3\dot{\phi} \approx V_\phi$$

Consequently, the potential slow-roll parameters (PSR), $\epsilon_V = \left(\frac{V_\phi}{3V}\right)^2, \ \eta_V = \frac{V_{\phi\phi}}{V_\phi}$, have been introduced [41] in order to study the existence of the inflationary phase of the universe. Inflation occurs when $\epsilon_V << 1$, while in order for the inflationary phase to last long enough we require the second PSR parameter also to be small, $\eta_V << 1$. That is a requirement for the flatness of the scalar field potential.

On the other hand, it is possible to express the inflation in terms of the Hubble slow-roll parameters (HSR) which are defined similarly [42], $\epsilon_H = \left(\frac{H_\phi}{H}\right)^2, \ \eta_H = \frac{H_{\phi\phi}}{H^2}$. while the two different sets of slow-roll parameters are related as follows $\epsilon_V \approx \epsilon_H$ and $\eta_V \approx \eta_H + \eta_H$.

Because of the generic solution presented in the previous section we are able to write the HSR parameters in terms of function $F(\omega)$ and its derivatives as follows [43]

$$\epsilon_H = 3F', \ \eta_H = \frac{3(F')^2 - F''}{F'}.$$

The HSR can be used to define dimensionless observable parameters which are related to the inflation, the spectral index for the density perturbations $n_s \equiv 1 - 4\epsilon_H + 2\eta_H$, and the tensor to while the tensor to scalar ratio is $r = 10\epsilon_H$. Moreover, the recent data analysis by the Planck collaboration [19], it was found that the value of $n_s$, with the error is $n_{s} = 0.968 \pm 0.006$, while the range of the scalar spectral index is $n'_s = -0.003 \pm 0.007$; while parameter $r$, has been found that it has an upper boundary, that is, $r < 0.11$.

In [43], Barrow & Paliathanasis, considered that the spectral index $n_s$ and the scalar ration $r$ are related under with the relation $n_s - 1 = f (r)$ in which $f(r)$ is an analytic functions. Moreover, with the use of [43] the latter expression defines a nonlinear function differential equation. Analytic solutions of the latter equations which fit the inflationary data were determined for the case in which $f(r)$ is constant, linear, or quadratic function. This specific study is mainly focused on the symmetry of these three master equations.
MAXIMALLY SYMMETRIC EQUATIONS

The three master equations of our consideration are

\[ F'' + (F')^2 - \frac{n_0}{3} F' = 0, \]  
(14)

\[ F'' + (1 - n_1)(F')^2 - \frac{n_0}{3} F' = 0, \]  
(15)

and

\[ F'' + 3n_2 (F')^3 + (1 - n_1)(F')^2 - \frac{n_0}{3} F' = 0. \]  
(16)

We apply Lie’s theory [15] and we calculate the Lie point symmetries for the three master second-order differential equations.

For equation (14) the Lie point symmetries are

\[ X_1 = \partial_\omega , \quad X_2 = \partial_F , \quad X_3 = e^{-\frac{n_0}{3} F} \partial_\omega , \quad X_4 = e^{\frac{n_0}{3} F} \partial_F , \]  
(17)

\[ X_5 = e^{-F} \partial_F , \quad X_6 = e^{-\frac{n_0}{3} \omega} \partial_\omega , \quad X_7 = e^F (3\partial_\omega + n_0 \partial_F) , \]  
(18)

\[ X_8 = e^{\frac{n_0}{3} \omega} (3\partial_\omega + n_0 \partial_F) \]

On the other hand, the Lie point symmetries of equation (15) are calculated to be

\[ Y_1 = \partial_\omega , \quad Y_2 = \partial_F , \quad Y_3 = e^{(n_1 - 1)F - \frac{n_0}{3} \omega} \partial_\omega , \quad Y_4 = e^{\frac{n_0}{3} F - (n_1 - 1)F} \partial_F , \]  
(19)

\[ Y_5 = e^{-(n_1 - 1)F} \partial_F , \quad Y_6 = e^{-\frac{n_0}{3} \omega} \partial_\omega , \quad Y_7 = e^{(n_1 - 1)F} (3\partial_\omega + n_0 \partial_F) , \]  
(20)

\[ Y_8 = e^{\frac{n_0}{3} \omega} (3\partial_\omega + n_0 \partial_F) \]

Finally, for equation (16) the Lie point symmetries are

\[ Z_1 = \partial_\omega , \quad Z_2 = \exp \left( -\frac{n_0}{3} \right) \left( \frac{(n_1 - 1 + A(n_0, n_1, n_2))}{2} F \right) \partial_\omega , \]  
(21)

\[ Z_3 = \exp \left( -\frac{n_0}{3} \right) \left( \frac{(n_1 - 1 + A(n_0, n_1, n_2))}{2} F \right) \partial_F , \quad Z_4 = (3 - 3n_1) \partial_\omega + 2n_0 \partial_F , \]  
(22)

\[ Z_5 = \exp \left( -\frac{n_0}{3} \right) \left( \frac{(n_1 - 1 + A(n_0, n_1, n_2))}{2} F \right) \partial_F Z_6 = \exp \left( -\frac{n_0}{3} \right) \left( \frac{(n_1 - 1 + A(n_0, n_1, n_2))}{2} F \right) \partial_F \]  
(23)

\[ Z_7 = 2n_0 \cosh (A(n_0, n_1, n_2) F) \partial_F - 3(\cosh (n_1 F)) \partial_\omega , \]  
(24)

\[ Z_8 = 2n_0 \sinh (A(n_0, n_1, n_2) F) \partial_F - 3(\sinh (n_1 F)) \partial_\omega \]

where constant \( A(n_0, n_1, n_2) \) is defined as \( A(n_0, n_1, n_2) = \sqrt{(n_1 - 1)^2 + 4n_0n_2}. \)

We note with surprise that the three master equations (14)-(16) admit eight Lie point symmetries and are maximally symmetric. It is well-known that all the second-order differential equations which are invariant under the action of the \( SL(3, R) \) group, are equivalent under point transformations [1]. Hence, there exist point transformations which transform any solution of the master equations to another solution.

For instance, consider equation (14) with \( n_0 = 0 \), then, under the point transformation \( F(\omega) \to (1 - n_1) \bar{F}(\omega) \), equation (14) is written as

\[ F'' + (1 - n_1)(\bar{F}') = 0 \]  
(25)

which is nothing else than equation (15). However, at the same time the line element of the spacetime (11) transformed as follows

\[ ds^2 = -e^\omega (e^{\frac{1}{2}(1-n_1)}d\omega^2 + e^{\frac{1}{3}}(dx^2 + dy^2 + dz^2)) \]  
(26)

Consider now the classical Newtonian analogue of a free particle and an observer whose measuring instruments for time and distance are not linear. By using the measured data of the observer we reach in the conclusion that it is
not a free particle. On the other hand, in the classical system of the harmonic oscillator an observer with nonlinear measuring instruments can conclude that the system observed is that of a free particle, or that of the damped oscillator or another system. From the different observations, various models can be constructed. However, all these different models describe the same classical system and the master equations are invariant under the same group of point transformations but in a different parametrization.

In the master equations that we studied there is neither position nor time variables: the independent variable is the scale factor $\omega = 6 \ln a$, and the Hubble function is the dependent variable, $H(a)$. Therefore, we can say that at the level of the first-order approximation for the spectral indices, various representations of the variables $\{a, H(a)\}$ provide different observable values for the spectral indices. Hence we showed that the method of Lie symmetries can be used to determine new solutions from others.

CONCLUSIONS

In this article we discussed the application of symmetries for differential equations in cosmology with emphasis in the inflationary era. Because of the complexity of the nonlinear gravitational field equations Lie symmetries can play an important role in the study of the dynamical evolution of the cosmological models as also in the determination of exact and analytic solutions.

In the case of inflationary models, the novelty of the Lie’s theory is that point transformations can be applied to determine new solutions from existing ones, while to transform known solutions from one to the other. The latter is possible by applying the Lie theorem for maximally symmetric second-order differential equations.

REFERENCES

[1] S. Lie, Theorie der Transformationsgruppen I, Leipzig: B. G. Teubner (1888)
[2] S. Lie, Theorie der Transformationsgruppen II, Leipzig: B. G. Teubner (1888)
[3] S. Lie, Theorie der Transformationsgruppen III, Leipzig: B. G. Teubner (1888)
[4] L. V. Ovsiannikov, Group analysis of differential equations, Academic Press, New York, (1982).
[5] N.H. Ibragimov, CRC Handbook of Lie Group Analysis of Differential Equations, Volume I: Symmetries, Exact Solutions, and Conservation Laws, CRS Press LLC, Florida (2000)
[6] P.J. Olver, Applications of Lie Groups to Differential Equations, Springer-Verlag, New York, (1993)
[7] M. Crampin, Rep. Math. Phys. 20, 31 (1984)
[8] P.G.L. Leach, J. Math. Phys. 18, 1902 (1977)
[9] E. Noether, Nachr. d. König. Gesellsch. d. Wiss. zu Göttingen, Math-phys. Klasse, 235, (1918) (translated in English by M.A. Tavel [physics/0503066])
[10] S.F. Singer, Symmetry in Mechanics, Birkhauser Boston, Boston (2004)
[11] G. Costa and G. Fogli, Symmetry and Group Theory in Particle Physics, Lecture Notes in Physics, Springer-Verlag, Berlin (2012)
[12] G.S. Hall, Symmetries and Curvature Structure in General Relativity, World Scientific Lecture Notes in Physics 46, (2004)
[13] D.P. Mason and M. Tsamparlis, J. Math. Phys. 26, 2881 (1985)
[14] H. Stephani, D. Kramer, M. MacCallum, C. Hoensellers and E. Herlt, Exact Solutions of Einsteins’ Field Equations, Cambridge University Press, New York (2009)
[15] G.W. Bluman and S. Kumei, Symmetries of Differential Equations, Springer-Verlag, New York, (1989)
[16] S.D. Maharaj, P.G.L. Leach and R. Maartens, Gen. Relativ. Grav. 28, 35 (1996)
[17] C. Wafo Soh and F.M. Mahomed, Class. Quantum Grav. 16, 3553 (1999)
[18] P.A.R. Ade et al (Planck Collaboration), A & A 571, A22 (2014)
[19] P.A.R. Ade, et al. (Planck 2015 Collaboration), A & A 594, A20 (2016)
[20] A. Guth, Phys. Rev. D 23, 347 (1981)
[21] A.A. Starobinsky, Phys. Lett. B 91, 99 (1980)
[22] A.D. Linde, Phys. Lett. B 129, 177 (1983)
[23] J.D. Barrow, Phys. Lett. B 235, 40 (1990)
[24] G.F.R. Ellis and M.S. Madsen, Class. Quant. Grav. 8, 667 (1991)
[25] J.G. Russo, Phys. Lett. B 600, 185 (2004)
[26] S. Basilakos, M. Tsamparlis and A. Paliathanasis, Phys. Rev. D 83, 103512 (2011)
[27] A. Paliathanasis, M. Tsamplis, S. Basilakos and J.D. Barrow, Phys. Rev. D 91, 123535 (2015)
[28] A. Yu Kamenshchik, E.O. Pozdeeva, A. Tronconi, G. Venturi and S.Yu Vernov, Phys. Part. Nucl. 49, 1 (2018)
[29] R. de Ritis, G. Marmo, G. Platania, C. Rubano, P. Scudellaro and C. Stornaiolo, Phys. Rev. D 42, 1091 (1990)
[30] T. Christodoulakis, N. Dimakis and P.A. Terzis, J. Phys. A: Math. Theor. 47, 095202 (2014)
[31] S. Capozziello and K.F. Dialektopoulos, EPJC 78, 447 (2018)
[32] N. Dimakis, A. Giacomini, S. Jamal, G. Leon and A. Paliathanasis, Phys. Rev. D 95, 064031 (2017)
[33] M. Tsamparlis and A. Paliathanasis, Symmetry (MDPI) 10, 233 (2018)
[34] T. Christodoulakis, A. Karagiorgos and A. Zampeli, Symmetry 10, 70 (2018)
[35] A. Paliathanasis, A. Zampeli, T. Christodoulakis and M.T. Mustafa, Class. Quantum Grav. 35, 125005 (2018)
[36] S. Dodelson, Modern Cosmology, Academic Press, California (2003)
[37] M. Tsamparlis and A. Paliathanasis, Gen. Rel. Gravit. 43, 1861 (2011)
[38] P.A. Terzis, N. Dimakis, T. Christodoulakis, A. Paliathanasis and M. Tsamparlis, J. Geom. Phys. 101, 52 (2016)
[39] N. Dimakis, A. Karagiorgos, A. Zampeli, A. Paliathanasis, T. Christodoulakis and P.A. Terzis, Phys. Rev. D 93, 123518 (2016)
[40] J.D. Barrow and A. Paliathanasis, Phys. Rev. D 94, 083518 (2016)
[41] A.R. Liddle and D.H. Lyth, Phys. Lett. B 291, 391 (1992)
[42] E.J. Copeland, E.W. Kolb, A.R. Liddle and J.E. Lidsey, Phys. Rev. D 49, 1840 (1994)
[43] J.D. Barrow and A. Paliathanasis, Gen. Rel. Gravit. 50, 82 (2018)