Dilatonic Black Holes, Naked Singularities and Strings

P. H. Cox, B. Harms and Y. Leblanc

Department of Physics and Astronomy, The University of Alabama
Box 870324, Tuscaloosa, AL 35487-0324

We extend a previous calculation which treated Schwarzschild black hole horizons as quantum mechanical objects to the case of a charged, dilaton black hole. We show that for a unique value of the dilaton parameter $\alpha$, which is determined by the condition of unitarity of the S matrix, black holes transform at the extremal limit into strings.

PACS numbers: 4.60.+n, 11.17.+y, 97.60.lf
Several authors [1–4] have discussed the relationship between strings and the singularities encountered in general relativity. In Refs. [1,2] string-like actions were derived from two different, although presumably equivalent, analyses of the scattering of massless, point-like particles from a Schwarzschild black hole. Holzhey and Wilczek [3] have suggested that extremal charged, dilatonic black holes may evolve from membranes to strings when the dilaton parameter increases to one. Witten [4] has pointed out that the “cosmic censorship” hypothesis may be wrong and that string theory may provide useful insights into the study of naked singularities.

In this letter we analyze the scattering of a massless, pointlike particle from a charged, dilatonic black hole. We obtain the dependence of the phase shift and the string tension on the dilaton parameter $a$. We show that for the extremal, dilatonic black hole there is a value of $a$ for which a black hole transforms into a string.

In Ref. [5] the phase shift of a massless particle scattering from a Schwarzschild black hole was calculated. ’t Hooft [1] showed that under rather general assumptions about quantum gravitodynamics the S matrix which arises from this phase shift can be written in terms of a “string-like” action with imaginary string tension equivalent. We first extend this calculation by deriving the Green function which determines the phase shift for a massless particle scattering from a charged, dilatonic black hole. As in Ref. [5], the metric of space-time surrounding a black hole with a massless particle falling in from spherical angle $\Omega' = (\theta', \phi')$ can under certain conditions be represented by gluing together two solutions of Einstein’s field equations. The solutions are cut at the null surface defined by setting one of the Kruskal coordinates to zero ($u = 0$) and glued after a nonconstant shift of the other Kruskal coordinate,

$$\delta v(\Omega) = p_{in} f(\Omega, \Omega')$$

where $p_{in}$ is the ingoing particle’s momentum with respect to the Kruskal coordinates and the Green function $f$ satisfies the equation, at $u = 0$,

$$\frac{A}{g} \Delta f - \frac{g_{uv}}{g} f = 32\pi A^2 \delta^2(\Omega, \Omega')|_{u=0},$$
where $\Delta$ is the angular Laplacian. The functions $A, g$ and $g_{uv}$ are identified by writing the metric in the form

$$ds^2 = 2 A(u, v) \, dudv + g(u, v) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \tag{3}$$

The necessary conditions for the matching are

$$A_{,u} \big|_{u=0} = g_{,u} \big|_{u=0} = 0. \tag{4}$$

To find $A(u, v)$ and $g(u, v)$ we start from the metric for a charged, dilaton black hole written in terms of the usual space-time coordinates

$$ds^2 = \lambda^2 \, dt^2 - \lambda^{-2} \, dr^2 - \bar{R}^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right), \tag{5}$$

where ($a$ is the dilaton parameter)

$$\lambda^2 = \left(1 - \frac{r_+}{r} \right) \left(1 - \frac{r_-}{r} \right)^{(1-a^2)/(1+a^2)}, \tag{6}$$

and

$$\bar{R}^2 = r^2 \left(1 - \frac{r_-}{r} \right)^{2a^2/(1+a^2)} \tag{7}$$

The parameters $r_+$ and $r_-$ are related to the mass, $M$, and the charge, $Q$, by

$$2 \, M = r_+ + \frac{(1-a^2)}{(1+a^2)} \, r_-$$

$$Q^2 = \frac{r_+ \, r_-}{1+a^2}. \tag{8}$$

The Kruskal coordinates are obtained as usual (see, e.g., Ref. [3]). $u$ and $v$ are related to the time, $t$, and the “tortoise” coordinate, $\xi$, by

$$u = e^{\alpha \xi} \, e^{\alpha t}$$

$$v = -e^{\alpha \xi} \, e^{-\alpha t}, \tag{9}$$

where $\alpha$ is a constant whose value will be determined by the conditions stated in Eq.(4) and $\xi$ is given by
\[ \xi = \int \frac{dr}{\lambda^2} . \]  

(10)

Using the transformations in Eq.(9), we find

\[ A(u, v) = -\frac{\lambda^2 e^{2\alpha \xi}}{2\alpha^2} \]
\[ g(u, v) = R^2 . \]  

(11)

In order for \( A, v \) and \( g, v \) to vanish at \( u = 0 \) (\( r = r_+ \)) we must have

\[ \left. \frac{d\lambda^2}{dr} \right|_{r=r_+} - 2 \alpha = 0 , \]  

(12)

which gives for \( \alpha \)

\[ \alpha = \frac{1}{2} r_+ \left( \frac{r_+ - r_-}{r_+} \right)^b , \]  

(13)

where \( b = (1 - a^2)/(1 + a^2) \).

We are now ready to determine the Green function \( f \). We multiply through Eq.(2) by \( g \) and use

\[ \Gamma = g_{,uv} \left. \frac{A}{A} \right|_{u=0} = \frac{r_+ - r_-}{r_+} + \frac{a^2 r_-}{r_+(1 + a^2)} \]  

(14)

If we take the north pole as the direction of incidence, Eq.(2) becomes,

\[ \Delta f - \Gamma f = -2 \pi \kappa \delta(\theta) , \]  

(15)

where we have defined \( \kappa \) to be

\[ \kappa = -16 A g |_{u=0} . \]  

(16)

Evaluating \( \kappa \) we find

\[ \kappa = 32 e^{-1} r_+^2 \left( 1 - \frac{r_-}{r_+} \right)^{2a^2/1+a^2} , \]  

(17)

where the constant of integration has been chosen such that the “tortoise” coordinate reduces in the limit \( a \to 0 \) to the form obtained for the Schwarzschild solution.

4
The solution of Eq.(15) proceeds the same as in Ref. [5]. The Green function is expanded in spherical harmonics and upon determining the expansion coefficients it can be expressed as a sum over Legendre polynomials \( P_l(\cos \theta) \),

\[
f = \kappa \sum_l \frac{l + \frac{1}{2}}{l(l+1) + \Gamma} P_l(\cos \theta) ,
\]

where \( \Gamma \) is the constant defined in Eq.(14).

The calculation of the S matrix \([1,2]\) gives

\[
\langle u(\Omega) | v(\Omega) \rangle = N \exp \left( i \int f^{-1}(\Omega, \Omega') v(\Omega) u(\Omega) \right) ,
\]

where \( f^{-1} \) is \((\Gamma - \Delta)/2\pi \kappa\) and \( N \) is a normalization constant. In the momentum representation this becomes

\[
\langle p_{\text{out}}(\Omega) | p_{\text{in}}(\Omega) \rangle = N' \int Du(\Omega) \int Dv(\Omega) \times \exp \left[ \int d^2\sigma \left( \frac{i}{2\pi \kappa} (\Gamma u v + \partial_\Omega v \partial_\Omega u) + i u p_{\text{out}} - i v p_{\text{in}} \right) \right] .
\]

neglecting the term due to the curvature (\( \Gamma \) term), switching to the ‘membrane coordinates’ \( x^0, x_3 \) defined by (in the metric where \( x^2 = x^2 - x_0^2 \))

\[
x^0 = (v + u)/2 ; \quad x_3 = (v - u)/2 ,
\]

and writing the result as a covariant expression gives

\[
\langle p_{\text{out}}(\Omega) | p_{\text{in}}(\Omega) \rangle = C \int Dx^\mu(\sigma) Dg^{ab}(\sigma) \times \exp \left[ \int d^2\sigma \left( -\frac{T}{2} \sqrt{g} g^{ab} \partial_a x^\mu \partial_b x^\mu + i x^\mu p^\mu(\sigma) \right) \right] ,
\]

where \( T \), the string tension equivalent, is

\[
T = \frac{i}{\pi \kappa} ,
\]

and the integration \( d^2\sigma, (\sigma_1 = x_1, \sigma_2 = x_2) \) is over the variables orthogonal to the membrane coordinates generated by \( \theta \) and \( \phi \). The argument of the exponential in Eq.(22) is the Polyakov action except for the factor of \( i \). To remove the factor of \( i \) and thus restore
unitarity we can consider the case where \( r_- > r_+ \) and set the dilaton parameter \( a = 1/\sqrt{3} \). In this region \( \kappa \) is pure imaginary, and the action is truly of the Polyakov form.

Of course, the case \( r_- > r_+ \) represents a naked singularity classically, so the picture indicated by this calculation is that charged, dilaton black holes undergo a transition as their parameters evolve to the extreme case (here \( r_+ = r_- \)) where they would become naked singularities, instead become strings, for this specific value of the dilaton parameter \( a \). Presumably the event horizon then becomes the world sheet of the string.

In this paper we have shown that 't Hooft's idea that the horizon of a black hole may be regarded as a quantum mechanical object can be extended to the case of charged, dilaton black holes. Using the constraint of unitarity, we have calculated a unique value, \( a = 1/\sqrt{3} \), for the dilaton parameter. We have found that for this value black holes undergo a transition to true strings at the extremal limit for a particular value of the dilaton parameter. We are exploring the possibility that similar transitions may occur between black p-branes and strings [7]. In this case the unitarity constraint is expected to give a dilaton parameter dependent upon the space-time dimensionality. Our considerations have been for quantum mechanical objects which undergo a transition into strings rather than producing a naked singularity. It is tempting to speculate that this represents a new way of implementing the "cosmic censorship" hypothesis in such processes as aspherical gravitational collapse, which has been suggested as a possible explanation of the observed extragalactic gamma ray bursts. This paper shows that instead of a naked singularity fundamental strings may be produced whose massless modes are detected on earth as gamma rays. If appropriate detectors can be designed and constructed, the massive string modes might also be detected for such an event.

**ACKNOWLEDGMENTS**

This research was supported in part by the U.S. Department of Energy under Grant No. DE-FG05-84ER40141 and in part by the Texas National Research Laboratory Commission
under Grant No. RCFY92-117.
† Permanent address: Department of Physics, Texas A&I University, Kingsville, TX 78363.

[1] G. ’t Hooft, Nuc. Phys. B335, 138(1990).

[2] H. Verlinde and E. Verlinde, PUPT-1279 (1991).

[3] C. Holzhey and F. Wilczek, IASSNS-HEP-91/71 (1991).

[4] E. Witten, IASSNS-HEP-92/24 (1992).

[5] T. Dray and G. ’t Hooft, Nuc. Phys. B253, 173(1985).

[6] R. Adler, M. Bazin, and M. Schiffer, Introduction to General Relativity, McGraw-Hill, New York (1965).

[7] G.T. Horowitz and A. Strominger, Nuc. Phys. B360, 197(1991).