ON SYMMETRIES OF A CONTROL PROBLEM WITH GROWTH VECTOR $(4,7)$

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Abstract. We study the action of symmetries on geodesics of a control problem on a Carnot group with growth vector $(4,7)$. We show that there is a subgroup of symmetries isomorphic to $SO(3)$ and a set of points in the Carnot group with a nontrivial stabilizer of this action. We prove that each geodesic either lies in this set or do not intersect this set. In the former case the optimality of geodesics is solved completely through the identification of the quotient with Heisenberg group.

1. Introduction

This paper is motivated by the paper [8] where the first and third authors study local control of a planar mechanism with seven-dimensional configuration space whose movement is restricted by three non-holonomic conditions. The mechanism that the authors deal with is a modification of a planar mechanism generally known as trident snake robot. Trident snake robot consists of a root block in the shape of an equilateral triangle together with three one-link branches each of which is connected to one vertex of the root block via revolute joint and has a passive wheel at its very end, [9, 6]. The generalized trident snake robot introduced in [8] is a planar mechanism consisting of a root block in the shape of an equilateral triangle together with three one-link branches that have passive wheels at their ends, too. However, each of the branches is connected to one vertex of the root block via prismatic joint and one of the joints is simultaneously revolute joint, see Figure 1.

It turns out that under the assumption that the movement of the mechanism is with no slipping, its local control is described by a control problem with growth vector $(4,7)$. Indeed, each wheel determines one non-holonomic condition and...
solution space of the corresponding system forms a four–dimensional distribution in the seven–dimensional configuration space. However, this control problem is highly non–linear. Thus it is natural to approximate the problem and swap to its nilpotent approximation, \[2\].

One can see from Figure 1 that there are three canonical generators of the distribution that correspond to prismatic joints and reflect prolongations of the branches. In particular, corresponding movements commute. The most important role then plays the additional field that describes the action of revolute and prismatic joints together. On the level of nilpotent approximation, we find four vector fields \(N_0, N_1, N_2, N_3\) generating the distribution \(\mathcal{N}\) such that the only non–trivial Lie brackets are the brackets \([N_0, N_1], [N_0, N_2]\) and \([N_0, N_3]\). In particular, these brackets are not contained in the distribution and the system is controllable according to the Chow–Rashevskii theorem, \[10\].

The aim of this paper is to study the corresponding nilpotent control problem in detail. Let us note that the nilpotent approximation constructed in \[8, Section 3\] leads to slightly different vector fields than the ones we use in Section 2. However, both define equivalent left–invariant control problems on isomorphic Carnot groups. We give explicit form of the vector fields \(N_0, N_i, N_0_i, i = 1, 2, 3\) and describe corresponding Carnot group \(\mathcal{N}\) in Section 2.1. Then we write associated control problem as

\[
\dot{q} = u_0 N_0 + u_1 N_1 + u_2 N_2 + u_3 N_3
\]

for \(t > 0\) and \(q \in \mathcal{N}\) and controls \(u = (u_1, u_2, u_3, u_4)\) with boundary condition \(q(0) = q_1, q(T) = q_2\) for fixed \(q_1, q_2 \in \mathcal{N}\), where we minimize the cost functional

\[
J = \frac{1}{2} \int_0^T (u_0^2 + u_1^2 + u_2^2 + u_3^3) dt.
\]

In terms of the sub–Riemannian geometry, the solution of this problem defines a sub–Riemannian geodesics, i.e. a local minimizers of the sub–Riemannian distance. We use the Hamiltonian concept to approach this problem, \[1\]. We describe the admissible controls in Proposition 1 and the geodesics in Proposition 2.

Sub–Riemannian geodesics are not optimal for all times in general and each geodesic carries a point where it stops to be optimal. One of natural ways to find such points is to use symmetries of the system. Indeed, if there is a symmetry preserving starting point and some other point of the geodesic, then the action of

\[\text{Figure 1. Generalized trident snake robot motion in seven–dimensional configuration space in coordinates } (x, \ell_1, \ell_2, \ell_3, y, \theta, \varphi).\]
suitable symmetry can generate one-parametric family of geodesics of the same length, so all the geodesics meet in the cut point. This happens e.g. in the case of \((3, 6)\) in \([12, 11]\) and is generalized for general free distributions in \([13]\).

Let us emphasize that there is a natural decomposition of the control distribution \(\mathcal{N}\) into one-dimensional distributions generated by \(N_0\) and three-dimensional involutive distribution generated by \(N_1, N_2, N_3\). Let us note that such geometric structure is called generalized path geometry in dimension 7, \([4]\). We use this decomposition to find symmetries of the control system in Section 3. We show in Proposition 3 that the symmetry algebra is a semidirect sum of a seven-dimensional decomposition to find symmetries of the control system in Section 3. We show in Proposition 3 that the symmetry algebra is a semidirect sum of a seven-dimensional algebra of transvections that correspond to right-invariant vector fields on \(N\) and three-dimensional stabilizer of the origin isomorphic to \(so(3)\). We use this

\begin{equation}
N_0 = \partial_x - \frac{\ell_1}{2} \partial y_1 - \frac{\ell_2}{2} \partial y_2 - \frac{\ell_3}{2} \partial y_3,
\end{equation}

\begin{equation}
N_1 = \partial_{\ell_1} + \frac{x}{2} \partial y_1, \quad N_2 = \partial_{\ell_2} + \frac{x}{2} \partial y_2, \quad N_3 = \partial_{\ell_3} + \frac{x}{2} \partial y_3,
\end{equation}

where the symbol \(\partial\) stands for partial derivative. The only non-trivial brackets are

\begin{equation}
N_{01} = [N_0, N_1] = \partial y_1, \quad N_{02} = [N_0, N_2] = \partial y_2, \quad N_{03} = [N_0, N_3] = \partial y_3.
\end{equation}

These fields then determine a 2-step nilpotent Lie algebra \(\mathfrak{n}\) with the multiplicative table given in Table I.

There is a Carnot group \(N\) such that the fields \(N_0, N_1, N_2, N_3, N_{01}, N_{02}, N_{03}\) are left-invariant for the corresponding group structure. The group structure on \(N\), when identified with \(\mathbb{R}^7 = \mathbb{R}^4 \oplus \mathbb{R}^3\), reads as follows

\begin{equation}
\begin{pmatrix}
x \\
\ell_1 \\
\ell_2 \\
\ell_3 \\
y_1 \\
y_2 \\
y_3
\end{pmatrix}
= \begin{pmatrix}
x + \tilde{x} \\
\ell_1 + \tilde{\ell}_1 \\
\ell_2 + \tilde{\ell}_2 \\
\ell_3 + \tilde{\ell}_3 \\
y_1 + \tilde{y}_1 + \frac{1}{2} (x \ell_1 - \tilde{x} \ell_1) \\
y_2 + \tilde{y}_2 + \frac{1}{2} (x \ell_2 - \tilde{x} \ell_2) \\
y_3 + \tilde{y}_3 + \frac{1}{2} (x \ell_3 - \tilde{x} \ell_3)
\end{pmatrix}.
\end{equation}
In particular, $\mathcal{N} = \langle N_0, N_1, N_2, N_3 \rangle$ forms a 4–dimensional left–invariant distribution on $N$. Moreover, there is a natural decomposition

$$\mathcal{N} = \langle N_0 \rangle \oplus \langle N_1, N_2, N_3 \rangle$$

of $\mathcal{N}$ into 1–dimensional distribution and 3–dimensional involutive distribution, both left–invariant. We define the left–invariant sub–Riemannian metric $r$ on $\mathcal{N}$ by declaring $N_0, N_1, N_2, N_3$ orthonormal.

The left–invariant sub–Riemannian structure $(N, \mathcal{N}, r)$ is related to left–invariant optimal control problem written in coordinates $(x, \ell_1, \ell_2, y_1, y_2, y_3)$ as

$$\dot{q}(t) = u_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} + u_1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} + u_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} + u_3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

for $t > 0$ and $q$ in $N$ and the control $u = (u_0, u_1, u_2, u_3) \in \mathbb{R}^4$ with the boundary condition $q(0) = q_1$, $q(T) = q_2$ for fixed points $q_1, q_2 \in N$, where we minimize $\frac{1}{2} \int_0^T (u_0^2 + u_1^2 + u_2^2 + u_3^2) dt$.

2.2. Local control and Pontryagin’s maximum principle. We follow here [1] Sections 7 and 13. Left–invariant vector fields $N_0, N_1, N_2, N_3, N_{01}, N_{02}, N_{03}$ form a basis of $TN$ and determine left–invariant coordinates on $N$. Then we define corresponding left–invariant coordinates $h_0, h_1$ and $w_i, i = 1, 2, 3$ on fibers of $T^*N$ by $h_0(\lambda) = \lambda(N_0), h_i(\lambda) = \lambda(N_i), w_i(\lambda) = \lambda(N_{0i})$ for arbitrary 1–form $\lambda$ on $N$. Thus we can use $(x, \ell_1, y_1, h_0, h_1, w_i)$ as global coordinates on $T^*N$.

A geodesic is an admissible curve parametrized by constant speed whose sufficiently small arcs are length minimizers. It turns out that the geodesics are exactly projections on $N$ of normal Pontryagin extremals, i.e. integral curves of left–invariant normal Hamiltonian

$$H = \frac{1}{2}(h_0^2 + h_1^2 + h_2^2 + h_3^2),$$

since there are no strict abnormal extremals for 2–step Carnot groups. Assume $\lambda(t) = (x(t), \ell_1(t), y_1(t), h_0(t), h_1(t), w_i(t))$ in $T^*N$ is a normal extremal. Then controls $u_j$, $j = 0, 1, 2, 3$ of the system (5) satisfy $u_j(t) = h_j(\lambda(t))$ and the base system takes form

$$\dot{q} = h_0 N_0(q) + h_1 N_1(q) + h_2 N_2(q) + h_3 N_3(q)$$

|     | $N_0$ | $N_1$ | $N_2$ | $N_3$ | $N_{01}$ | $N_{02}$ | $N_{03}$ |
|-----|-------|-------|-------|-------|----------|----------|----------|
| $N_0$ | 0     | $N_{01}$ | $N_{02}$ | $N_{03}$ | 0        | 0        | 0        |
| $N_1$ | $-N_{01}$ | 0     | 0     | 0     | 0        | 0        | 0        |
| $N_2$ | $-N_{02}$ | 0     | 0     | 0     | 0        | 0        | 0        |
| $N_3$ | $-N_{03}$ | 0     | 0     | 0     | 0        | 0        | 0        |
| $N_{01}$ | 0     | 0     | 0     | 0     | 0        | 0        | 0        |
| $N_{02}$ | 0     | 0     | 0     | 0     | 0        | 0        | 0        |
| $N_{03}$ | 0     | 0     | 0     | 0     | 0        | 0        | 0        |

Table 1. Lie algebra n

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[1]
where \( q = (x, \ell_i, y_i) \). Using \( u_j(t) = \dot{h}_j(\lambda(t)) \) and the equation \( \dot{\lambda}(t) = \ddot{H}(\lambda(t)) \) for normal extremals, we can write the fiber system as

\[
\dot{h}_j = -\sum_{l=1}^{4} \sum_{k=1}^{3} c_{jl}^k h_l w_k, \quad j = 0, 1, 2, 3,
\]

\[
\dot{w}_i = 0, \quad i = 1, 2, 3,
\]

where \( c_{jl}^k \) are structure constants of the Lie algebra \( \mathfrak{n} \) for the basis \( N_0, N_1, N_2, N_3, N_{01}, N_{02}, N_{03} \), see Table \( \mathbb{I} \).

2.3. Solutions of fiber system. We see immediately from (8) that solutions \( w_1, w_2, w_3 \) are constant, thus we have

\[
w_1 = K_1, \quad w_2 = K_2, \quad w_3 = K_3
\]

for suitable constants \( K_i \). Hence the second part of the fiber system (8) takes the explicit matrix form \( \dot{h} = -\Omega_w h \), where \( h = (h_0, h_1, h_2, h_3)^t \) and

\[
\Omega_w = \begin{pmatrix}
0 & K_1 & K_2 & K_3 \\
-K_1 & 0 & 0 & 0 \\
-K_2 & 0 & 0 & 0 \\
-K_3 & 0 & 0 & 0
\end{pmatrix}.
\]

Its solution is given by \( h(t) = e^{-t\Omega_w} h(0) \), where \( h(0) \) is the initial value of vector \( h \) in the origin. If \( K_1 = K_2 = K_3 = 0 \) then \( h(t) = h(0) \) is constant and the geodesic \( (x(t), \ell_i(t), y_i(t)) \) is a line in \( N \) such that \( y_i(t) = 0 \). In next we assume that the vector \( (K_1, K_2, K_3) \) is non-zero and we denote by \( K = \sqrt{K_1^2 + K_2^2 + K_3^2} \) its length.

**Proposition 1.** The general solutions of (8) satisfying (9) for non-zero \( K \) take form

\[
h_0 = K (-C_1 \sin(Kt) + C_2 \cos(Kt)),
\]

\[
\begin{pmatrix}
h_1 \\
h_2 \\
h_3
\end{pmatrix} = (C_1 \cos(Kt) + C_2 \sin(Kt)) \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} + \begin{pmatrix} -C_3 K_3 - C_4 K_2 \\ C_3 K_1 \\ C_4 K_2 \end{pmatrix},
\]

where \( C_1, C_2, C_3, C_4 \) are real constants.

**Proof.** The solution of the system (8) is given by exponential of the matrix \( \Omega_w \) from (10). We need to analyze its eigenvalues and eigenvectors. It follows that there are (complex conjugated) imaginary eigenvalues \( \pm iK \) both of multiplicity one and the eigenvalue 0 of multiplicity two. The corresponding eigenspace of \( iK \) is generated by complex eigenvector \( \nu \) that decomposes into real and complex component as \( \Re(\nu) = (0, K_1, K_2, K_3)^t \) and \( \Im(\nu) = (K, 0, 0, 0)^t \). In the basis formed by these two vectors together with any basis of the two-dimensional eigenspace corresponding to the eigenvalue 0, the matrix \( \Omega_w \) has zeros at all positions except positions \( (\Omega_w)_{12} = - (\Omega_w)_{21} = K \). Then we get in this eigenvector basis

\[
e^{-t\Omega_w} = \begin{pmatrix}
\cos(Kt) & \sin(Kt) & 0 & 0 \\
-\sin(Kt) & \cos(Kt) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

For the choice of the basis of the eigenspace corresponding to the eigenvalue 0 given as \( (0, -K_3, 0, K_1)^t \) and \( (0, -K_2, K_1, 0)^t \), the solution can be written as the
with coefficients $C_1, C_2, C_3, C_4$. Then the formula (11) follows. \hfill \Box

Let us emphasize that the choice $C_1 = C_2 = 0$ gives constant solutions that are not relevant as control functions. Thus we assume that at least one of the constants $C_1, C_2$ is non-zero.

2.4. Solutions of base system. The base system (7) takes the explicit form

\[
\begin{aligned}
\dot{x} &= h_0, & \dot{h}_1 &= h_1, \\
\dot{h}_2 &= h_2, & \dot{h}_3 &= h_3,
\end{aligned}
\]

(13)

\[
\begin{aligned}
\dot{y}_1 &= \frac{1}{2}(xh_1 - h_0\ell_1), \\
\dot{y}_2 &= \frac{1}{2}(xh_2 - h_0\ell_2), \\
\dot{y}_3 &= \frac{1}{2}(xh_3 - h_0\ell_3).
\end{aligned}
\]

As discussed above, we are interested in solutions going through the origin, i.e. we impose the initial condition $x(0) = 0, \ell_i(0) = 0, y_i(0) = 0, i = 1, 2, 3$.

**Proposition 2.** The sub-Riemannian geodesics on Carnot group $N$ satisfying the initial condition $x(0) = 0, \ell_i(0) = 0, y_i(0) = 0, i = 1, 2, 3$ are either lines of the form

\[
(x, \ell_1, \ell_2, \ell_3, y_1, y_2, y_3)^t = (C_1 t, C_2 t, C_3 t, C_4 t, 0, 0, 0)^t
\]

parameterized by constants $C_1, C_2, C_3, C_4$ satisfying $C_1^2 + C_2^2 + C_3^2 + C_4^2 = 1$, or they are curves given by equations

\[
x = C_1 \cos(Kt) + C_2 \sin(Kt) - C_1,
\]

(15)

\[
\begin{aligned}
\begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \end{pmatrix} &= \frac{1}{K} (C_1 \sin(Kt) - C_2 \cos(Kt) + C_2) \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix} + t \begin{pmatrix} -C_3 K_3 - C_4 K_2 \\ C_4 K_1 \\ C_3 K_3 \end{pmatrix}, \\
\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &= \frac{1}{2K} (C_1^2 + C_2^2) (t K - \sin(Kt)) \begin{pmatrix} K_3 \\ K_2 \\ K_1 \end{pmatrix} + \frac{1}{2K} ((2C_1 - C_2 Kt) \sin(Kt)) \\
&- (C_1 K t + 2C_2 \cos(Kt) + 2C_2 - t C_1 K) \begin{pmatrix} -C_3 K_3 - C_4 K_2 \\ C_4 K_1 \\ C_3 K_3 \end{pmatrix},
\end{aligned}
\]

(17)

\[
K^2(C_1^2 + C_2^2) + (C_3 K_3 + C_4 K_2)^2 + C_3^2 K_3^2 + C_4^2 K_2^2 = 1.
\]

**Proof.** The line (14) corresponds to $K = 0$ and thus $h(t) = h(0)$ is constant and defines the vector of constants $(C_1, C_2, C_3, C_4)$. The length of this vector is equal to one on the level set $\frac{1}{2}$. If $K \neq 0$ we obtain $x, \dot{x}, \ell_2, \ell_3$ by direct integration of the first part of (13) and involving the initial condition, where $h(t)$ is given by (11). Substituting the results into the second part of (13) we get $y_1, y_2, y_3$ by integration. The solutions define family of curves starting at the origin such that
the Hamiltonian $H$ is constant along them. Geodesics correspond to the level set $H = \frac{1}{2}$, i.e. $h_0^2 + h_1^2 + h_2^2 + h_3^2 = 1$. According to Proposition 1 this restriction reads as \[ ]^{[18].}

3. Symmetries of the control system

Symmetries of the control system (5) coincide with the symmetries of the corresponding left–invariant sub–Riemannian structure $(N, N, r)$. These are precisely automorphisms of $N$ preserving the distribution $\mathcal{N}$ together with its decomposition (4) and sub–Riemannian metric $r$. By infinitesimal symmetry we mean a vector field such that its flow is a symmetry at each time, (4).

3.1. Infinitesimal symmetries. Infinitesimal symmetries of the sub–Riemannian structure $(N, N, r)$ are vector fields $v$ such that $\mathcal{L}_v r = 0$ and $\mathcal{L}_v(\langle N_0 \rangle) \subset \langle N_0 \rangle$ and $\mathcal{L}_v(\langle N_1, N_2, N_3 \rangle) \subset \langle N_1, N_2, N_3 \rangle$.

Proposition 3. Infinitesimal symmetries of the control system (5) form a Lie algebra generated by

\[ \begin{align*}
\partial_x + \frac{\ell_1}{2} \partial_y_1 + \frac{\ell_2}{2} \partial_y_2 + \frac{\ell_3}{2} \partial_y_3, \\
\partial_\ell_1 - \frac{x}{2} \partial_y_1, \quad \partial_\ell_2 - \frac{x}{2} \partial_y_2, \quad \partial_\ell_3 - \frac{x}{2} \partial_y_3, \\
\partial_y_1, \quad \partial_y_2, \quad \partial_y_3,
\end{align*} \]

(19)

together with

\[ \begin{align*}
v_1 &= \ell_3 \partial_\ell_2 - \ell_2 \partial_\ell_3 + y_3 \partial_\ell_1 - y_2 \partial_y_3, \\
v_2 &= \ell_1 \partial_\ell_3 - \ell_3 \partial_\ell_1 + y_1 \partial_\ell_2 - y_3 \partial_y_1, \\
v_3 &= \ell_2 \partial_\ell_1 - \ell_1 \partial_\ell_2 + y_2 \partial_\ell_3 - y_1 \partial_y_2,
\end{align*} \]

(20)

Vector fields (19) are right–invariant and generate all tranvections that form a nilpotent algebra with growth $4(7)$. Vector fields (20) generate $\mathfrak{so}(3)$ that forms the stabilizer algebra at the origin.

Proof. The statement follows by direct computation. We do not give here all details but rather explain main ideas. We start with general vector field $v = \sum_{j=0}^{3} f_j N_j + \sum_{i=1}^{3} g_i N_0_i$ for functions $f_j, g_i$ on $N$. The condition $\mathcal{L}_v(r) = 0$ implies that functions $f_j$ do not depend on $y_i$ and are linear in $x$ and $\ell_i$. The corresponding system has solution $f_0 = c_1 \ell_1 + c_2 \ell_2 + c_3 \ell_3 + c_4, f_1 = -c_1 x + c_5 \ell_2 + c_6 \ell_3 + c_7, f_2 = -c_2 x - c_5 \ell_1 + c_8 \ell_3 + c_9$ and $f_3 = -c_3 x - c_6 \ell_1 - c_8 \ell_2 + c_{10}$ for constants $c_i$.

Annihilator of $\langle N_1, N_2, N_3 \rangle$ is generated by forms $-\frac{\ell_1}{2} \partial_\ell_1 + dy_i$ for $i = 1, 2, 3$ together with $dx$. Annihilator of $\langle N_0 \rangle$ is generated by forms $\frac{\ell_2}{2} dx + dy_i$ and $\partial_\ell_i$ for $i = 1, 2, 3$. Substituting the above functions $f_j$ into $v$, evaluating the above forms on it and putting them equal to zero gives a system whose solution space is generated by (19) (20).

One can check by direct computation that fields (19) are right–invariant with respect to the group structure (3). Moreover, their brackets satisfy $[\partial_x + \sum_{i} \frac{\ell_i}{2} \partial_y_i, \partial_\ell_i - \frac{x}{2} \partial_y_i] = -\partial_y_i$ for $i = 1, 2, 3$ and the remaining brackets vanish. The fields $v_i, i = 1, 2, 3$ have the following Lie bracket structure

\[ [v_1, v_2] = -v_3, [v_1, v_3] = v_2, [v_2, v_3] = -v_1 \]

and thus form an algebra isomorphic to $\mathfrak{so}(3)$. It clearly preserves origin and forms the stabilizer subalgebra at the origin. \[ \square \]
The result is not surprising. The Lie algebra contains transvections given by  
right–invariant vector fields due to the fact that we deal with a Lie group \( N \). Moreover, there can be stabilizer at the origin which generally is a subalgebra of \( so(4) \), because it particularly preserves the metric \( r \) and distribution \( N \). Since our  
transformations generically preserve the decomposition \( \{1\} \), they preserve \( N_0 \) and thus we find \( so(3) \) generated by \( v_1, v_2, v_2 \) acting on \( \langle N_1, N_2, N_3 \rangle \).

3.2. Fixed–point sets. One can see from \( [20] \) that the action of the Lie group  
\( SO(3) \) on \( N \) generated by infinitesimal symmetry \( a_1 v_1 + a_2 v_2 + a_3 v_3 \in so(3) \) is  
given by simultaneous rotations on \( \ell_i \) and \( y_i, i = 1, 2, 3 \) with respect to the axis \( a = (a_1, a_2, a_3) \), while the coordinate \( x \) is invariant. Since all invariants of a rotation are  
multiples of its axis, the fixed points of the symmetry generated by \( a_1 v_1 + a_2 v_2 + a_3 v_3 \)  
form the set

\[
\{ (x, ka_1, ka_2, ka_3, la_1, la_2, la_3) : x, k, l \in \mathbb{R} \}.
\]

These observations following from Proposition \( 3 \) are summarized in the following two consequences, where we denote \( \ell = (\ell_1, \ell_2, \ell_3)^t \) and \( y = (y_1, y_2, y_3)^t \).

**Corollary 1.** For each \( R \in SO(3) \) the map  
\[
(x, \ell, y) \mapsto (x, R\ell, Ry)
\]

maps geodesics starting at the origin to geodesics starting at the origin.

**Corollary 2.** Set of points that are fixed by some isotropy symmetry is the union  
of sets \( [21] \) over all axes \( (a_1, a_2, a_3) \)

\[
C_n = \{(x, \ell, y) \in \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 : \forall \ell \neq 0, \exists \lambda \in \mathbb{R} \text{ such that } y = \lambda \ell \},
\]

i.e. the points such that the vectors \( \ell \) and \( y \) are collinear.

**Remark 1.** One can see from the group structure \( [3] \) that the multiplication restricts correctly to \( C_n \), so this set in fact forms a subgroup of \( N \). Moreover, \( C_n \) is  
clearly invariant with respect to the action \( [22] \).

4. Optimality of geodesics

The set of points where geodesics intersect each other and the corresponding geodesic segments have equal length is called the Maxwell set. Conjugate points are defined as critical points of the exponential map. It is proved that the normal extremal trajectory that does not contain pieces of abnormal geodesics loses its  
optimality in the conjungate point or in the Maxwel point. \( [4] \). The points where geodesics lose their optimality are called cut points, where cut locus is the set of  
cut points. In many cases, it contains sets of fixed points of symmetries. Indeed, if a geodesic meets a fixed point of a symmetry, then the action of the symmetry can  
give such set of geodesics, \( [14] \). We discuss here geodesics starting at the origin, however, the results hold for geodesics starting at arbitrary point thanks to the left–invariancy and action of vector fields \( [19] \).

4.1. Factor space. Each choice of coefficients \( C_1, C_2, C_3, C_4 \in \mathbb{R} \) and \( K_1, K_2, K_3 \in \mathbb{R} \) that satisfy \( [18] \) gives geodesics \((x(t), \ell(t), y(t))\) as described in the Proposition \( 2 \). According to \( [13] \) and \( [17] \), \( \ell(t) \) and \( y(t) \) are linear combinations of the vectors

\[
z_1 = \begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix}, \quad z_2 = \begin{pmatrix} -C_2 K_3 - C_4 K_2 \\ C_4 K_1 \\ C_3 K_1 \end{pmatrix}
\]
for any $t > 0$. The vectors $z_1$ and $z_2$ are orthogonal with respect to the Euclidean metric on $\mathbb{R}^3$ by definition. So, there is an orthogonal matrix $R \in SO(3)$ that aligns vectors $z_1$ and $z_2$ with the suitable multiples of the first two vectors of the standard basis of $\mathbb{R}^3$. Thus we get
\[ z_1 = R \begin{pmatrix} K \\ 0 \\ 0 \end{pmatrix}, \quad z_2 = R \begin{pmatrix} 0 \\ C \\ 0 \end{pmatrix}, \]
where $K = |z_1|$ is the length of $z_1$ and we denote $C = |z_2|$ the length of $z_2$. This matrix defines a representative of the geodesic class
\[ (x(t), \ell(t), y(t)) = (x(t), R\ell(t), R'y(t)). \]
The equations for this representative geodesics simplify remarkably. Namely
\[ x(\tau) = C_1(\cos \tau - 1) + C_2 \sin \tau, \]
\[ \ell_1(\tau) = C_1 \sin \tau + C_2(1 - \cos \tau), \]
\[ \ell_2(\tau) = \bar{C}_3 \tau, \]
\[ (24) \]
\[ \ell_3(\tau) = 0, \]
\[ \bar{y}_1(\tau) = (C_1^2 + C_2^2)(\tau - \sin \tau), \]
\[ \bar{y}_2(\tau) = \bar{C}_3[C_1(2 \sin \tau - \tau \cos \tau - \tau \sin \tau) + C_2(2 - 2 \cos \tau - \tau \sin \tau)], \]
\[ \bar{y}_3(\tau) = 0, \]
where $\tau = Kt$ and $\bar{C}_3 = C/K$. The level set equation (18) reads as
\[ (25) \quad K^2(C_1^2 + C_2^2 + C_3^2) = 1 \]
and determines $K > 0$ uniquely.

The factor space $N/\text{SO}(3)$ defined by the action of $\text{SO}(3)$ on $N \cong \mathbb{R}^7$ is determined by natural invariants $x, (\ell, \ell), (\ell, y), (y, y)$, where $(, )$ stands for the Euclidean scalar product on $\mathbb{R}^3$.

**Proposition 4.** Each geodesic starting at the origin defines a curve in the factor space $N/\text{SO}(3)$ given by a curve in invariants
\[ x = C_1(\cos \tau - 1) + C_2 \sin \tau, \]
\[ (\ell, \ell) = (C_1 \sin \tau + C_2(1 - \cos \tau))^2 + (\bar{C}_3 \tau)^2, \]
\[ (\ell, y) = (C_1^2 + C_2^2)(C_1 \sin \tau + C_2(1 - \cos \tau))(\tau - \cos \tau) + C_3^2 \tau[C_1(2 \sin \tau - \tau \cos \tau - \tau \sin \tau) + C_2(2 - 2 \cos \tau - \tau \sin \tau)], \]
\[ (y, y) = (C_1^2 + C_2^2)^2(\tau - \cos \tau)^2 + C_3^2[C_1(2 \sin \tau - \tau \cos \tau - \tau \sin \tau) + C_2(2 - 2 \cos \tau - \tau \sin \tau)]^2. \]

**Proof.** Follows directly from (24). \qed

4.2. **Subgroup $C_n$.** Let us recall that the subgroup $C_n$, defined by (23), consists of points in $N$ that are stabilized by some non-trivial $R \in \text{SO}(3)$ for the action from Corollary 1. Note the similarity of $C_n$ to the set $P_3$ from nilpotent $(3,6)$ sub–Riemannian problem which is known to be the set where geodesics starting at the origin lose optimality, [12]. For any point of $P_3$ there exists a one–parameter family of geodesics of equal length intersecting at this point. However, the situation in our nilpotent $(4,7)$ problem is very different.
Theorem 1. Sub–Riemannian geodesics starting at the origin either do not intersect \( C_n \) or they lie in \( C_n \) for all times.

Proof. Suppose there is an intersection of the set \( C_n \) with a sub–Riemannian geodesic \((x(t), \ell(t), y(t))\) emanating from the origin. So there is an point of intersection \((x, \bar{\ell}, \bar{y})\) of the set \( C_n \) with a sub–Riemannian geodesic \((x(t), \ell(t), y(t))\) since \( C_n \) is invariant with respect to the action (22) of \( SO(3) \). At this intersection \((x, \bar{\ell}, \bar{y})\), the collinearity of \( \bar{\ell} \) and \( \bar{y} \) is described by vanishing of the determinant
\[
\bar{\ell}_1 \bar{y}_2 - \bar{\ell}_2 \bar{y}_1 = 0.
\]
The geodesics are given by equations (24) and the determinant can be written explicitly as
\[
\bar{C}_3(d_{11}C_1^2 + 2d_{12}C_1C_2 + d_{22}C_2^2),
\]
where
\[
d_{11} = -\tau^2 - \tau \sin \tau \cos \tau - 2 \cos^2 \tau + 2,
\]
\[
d_{12} = -2 \sin \tau (2 \cos \tau - 2 + \tau \sin \tau),
\]
\[
d_{22} = 2 \cos^2 \tau + \tau \cos \tau \sin \tau - 4 \cos \tau - \tau^2 + 2.
\]
We show that the function in the bracket of (27) is never zero (unless \( C_1 = C_2 = 0 \), which is irrelevant) by showing that its discriminant \( d \) of this quadratic equation is negative for all positive times. This implies that the collinearity condition (27) is equivalent to \( \bar{C}_3 = 0 \). Then \( \bar{\ell}_2(\tau) = \bar{y}_2(\tau) = 0 \) by (24) and thus geodesic \((x(\tau), \ell(\tau), y(\tau))\) belongs to \( C_n \) for all \( \tau > 0 \).

To show that the discriminant \( d \) is negative for all positive times we compute
\[
d = -4\tau(\tau - \sin \tau)(\tau^2 + \tau \sin \tau + 4 \cos \tau - 4),
\]
hence it is sufficient to prove
\[
f(\tau) = \tau^2 + \tau \sin \tau + 4 \cos \tau - 4 > 0
\]
for all positive times \( \tau \). This can be done by combining "local" and "global" estimations of this function. The local estimation is obtained by the estimation of goniometric functions \( \sin \tau, \cos \tau \) by Taylor series. By evaluating the Taylor series of \( f \) in zero, we see we need to use the Taylor polynomial of degree seven and six, respectively. Then we get the estimation
\[
f(\tau) > -\frac{\tau^6(\tau^2 - 14)}{5040}
\]
that guarantees positivity of the function \( f \) on the interval \((0, \sqrt{14})\). On the other hand, the inequalities \( \sin \tau, \cos \tau \geq -1 \) yield a global estimation
\[
f(\tau) > \tau^2 - \tau - 8
\]
that guarantees positivity of the function \( f \) on the interval \((1 + \frac{\sqrt{33}}{2}, \infty)\). The two intervals overlap and thus \( f \) is positive for all \( \tau > 0 \).

Remark 2. The positivity of function \( f(\tau) \) from the proof above can be shown alternatively by proving the positivity of its derivative. Indeed, the derivative is given by
\[
f'(\tau) = (2 + \cos \tau) \left( \tau - \frac{3\sin \tau}{2 + \cos \tau} \right) \quad \text{and} \quad \frac{3\sin \tau}{2 + \cos \tau} \text{ is concave}.
\]
4.3. Geodesics in $C_n$. We study properties of geodesics contained in $C_n$. According to (27), this happens if and only if $\bar{C}_3 = 0$. Then the non–zero parts of geodesics (24) are

\begin{align*}
x(t) &= C_1(\cos(Kt) - 1) + C_2 \sin(Kt), \\
\bar{\ell}_1(t) &= C_1 \sin(Kt) + C_2(1 - \cos(Kt)), \\
\bar{y}_1(t) &= (C_1^2 + C_2^2)((Kt) - \sin(Kt)),
\end{align*}

and level set condition (18) reads as

\[ K^2(C_1^2 + C_2^2) = 1. \]

We show that these geodesics are preimages of geodesics in Heisenberg geometry and their optimality is well known, [1, 7, 13]. Thus, we get the following statement.

**Theorem 2.** The vertical set $\{(0,0,y) \in C_n : y \in \mathbb{R}^3\}$ is the set where the geodesics in $C_n$ starting at the origin lose their optimality. These points are Maxwell points and for geodesics defined by parameters $C_1$ and $C_2$ the cut time is

\[ t_{\text{cut}} = 2\pi \sqrt{C_1^2 + C_2^2}. \]

**Proof.** Since $\bar{\ell}_1 = \sqrt{\langle \ell, \ell \rangle}$ and $\bar{y}_1 = \sqrt{\langle y, y \rangle}$ are invariants, see (26), the expression (28) defines a curve in the factor space $C_n/\mathrm{SO}(3)$. For the choice of polar coordinates in the plane $(C_1, C_2)$ we get the standard description of geodesics on three–dimensional Heisenberg group $\mathbb{H}_3$. Indeed, the tangent space to the subgroup $C_n$ is generated by pushout vectors

\[ \bar{N}_0 = \partial_x - \frac{\bar{\ell}_1}{2} \partial_{\bar{y}_1}, \quad \bar{N}_1 = \partial_{\bar{\ell}_1} + \frac{x}{2} \partial_{\bar{y}_1}, \]

that are standard generators of Heisenberg Lie algebra. The group law (5) on the subgroup $C_n$ defines an isomorphism $C_n/\mathrm{SO}(3) \cong \mathbb{H}_3$. The cut locus of the Heisenberg group consists of the set of points

\[ \{(0, y) \in \mathbb{R}^2 \oplus \wedge^2 \mathbb{R}^2 : y \neq 0\}. \]

Namely, any geodesic from the origin loses its optimality at the point where it meets the vertical axis $(0, 0, \bar{y})$ for the first time. These points are Maxwell points and the corresponding time equals to $t_{\text{cut}} = \frac{2\pi}{K}$. Sub–Riemannian geodesics in $C_n$ going from the origin to the point $(x, \ell, \lambda \ell)$ form a preimage of the Heisenberg geodesic in
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$C_n/\text{SO}(3)$ going from the origin to the point $(x, |\ell|, \lambda |\ell|)$, where $|\ell|^2 = (\ell, \ell)$. These geodesics have the same length and they lose their optimality at the same time. □

Since the geodesics in $C_n$ are preimages of Heisenberg geodesics under the $\text{SO}(3)$–action, we can visualize them in the same way. On the left hand side of Figure 3 there is so–called Heisenberg sub–Riemannian sphere. On the right side of the same figure, there is a half–sphere with a family of geodesics from origin to the sphere.

![Figure 3. Heisenberg sub–Riemannian sphere and geodesics from origin to the points of the sphere](image)

4.4. **Sub-Riemannian geodesics out of $C_n$.** Any geodesic from the origin to the point $q = (x, \ell, y)$ defines a curve in $N/\text{SO}(3)$, see Proposition 4. This curve can be seen as factorized exponential mapping $\exp : \mathbb{R}^4 \to \mathbb{R}^4$ which is given by (26). The optimality of the geodesic is guaranteed up to the first critical point of this map, i.e. the point with zero determinant of Jacobi matrix. If the determinant is non–zero, we find the parameters $C_1, C_2, C_3$ and $\tau$ in terms of invariants of endpoint $\bar{q}$. Substituting $C_1, C_2, C_3$ and $\tau = tK$ into (24), where $K$ is given by (25), we get a geodesic $(x(t), \bar{\ell}(t), \bar{y}(t))$ from the origin to the endpoint $\bar{q} = (x, \ell, y)$ with the same invariants (26). So endpoints $q$ and $\bar{q}$ lie in the same class of $N/\text{SO}(3)$ and there is a unique transformation $R \in \text{SO}(3)$ such that $\ell = R\bar{\ell}$ and $y = R\bar{y}$, see Corollary 1. Then the geodesic from the origin to point $q$ is given by

$$t \mapsto (x(t), R\bar{\ell}(t), R\bar{y}(t)),$$

where $t \in (0, T)$ and $T = \tau \sqrt{C_1^2 + C_2^2 + C_3^2}$ is the length of this sub–Riemannian geodesic.

The formulae for geodesics in Proposition 2 in the time $t = 1$ define the unit six–dimensional sub–Riemannian sphere parameterized by $C_1, C_2, C_3, C_4$ and $K_1, K_2, K_3$. It is a disjoint union of the part in $C_n$ and outside of $C_n$. In the Figure 4 one can see two explicit cross sections of projections of the sub–Riemannian sphere with respect to parameters $(x, \ell_1, y_2)$ and a family of geodesics going from the origin to the sphere.
Figure 4. Two different cross sections of a projection of the sub-Riemannian sphere with a family of geodesics.

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