Covariant action for bouncing cosmologies in modified Gauss-Bonnet gravity

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Abstract

Cyclic universes with bouncing solutions are candidates for solving the big bang initial singularity problem. Here we seek bouncing solutions in a modified Gauss-Bonnet gravity theory, of the type \( R + f(G) \), where \( R \) is the Ricci scalar, \( G \) is the Gauss-Bonnet term, and \( f \) some function of it. In finding such a bouncing solution we resort to a technique that reduces the order of the differential equations of the \( R + f(G) \) theory to second order equations. As general relativity is a theory whose equations are of second order, this order reduction technique enables one to find solutions which are perturbatively close to general relativity. We also build the covariant action of the order reduced theory.

1. Introduction

The physics of the big bang initial singularity is still an open problem in cosmology and fundamental physics. A complete description of the universe must avoid at all costs spacetime singularities as their existence makes the future physically unpredictable. It is supposed that in a quantum gravity regime new physics sets in and spacetimes singularities get a proper description. In such a regime there are several possibilities, but an intriguing one is to suppose that at some tiny scale, of the order of few Planck units, the universe undergoes a bounce \([1, 2]\), such that a previously collapsing universe expands back again originating our own visible universe, yet to possibly collapse again in a cyclic way, giving rise to a cyclic universe \([3]\).

A modified Friedmann equation of the type, \( H^2 = \frac{1}{3\kappa} \rho \left( 1 - \frac{\rho}{\rho_c} \right) \), has been predicted in loop quantum cosmology \([4–7]\). Here, \( H = H(t) \) is the Hubble function defined in terms of the scale factor \( a(t) \) by \( H \equiv \frac{\dot{a}}{a} \), with \( t \) being the time cosmic parameter and a dot denotes differentiation with respect to \( t \), \( \kappa = 8\pi \), we put Newton’s constant \( G_N \) to one and the speed of light \( c \) to one, \( \rho = \rho(t) \) is the matter energy density, and \( \rho_c \) is a critical energy density given by \( \rho_c = \frac{\gamma}{2\sqrt{3} \pi} \rho_p \), where \( \rho_p \) is the Planck density \( \rho_p \equiv \frac{1}{G_N^2 \hbar} \), with \( \hbar \) being the Planck constant that we likewise put equal to one, \( \gamma = \frac{\ln 3}{\sqrt{8\pi}} \) is the Barbero-Immirzi parameter, and so \( \frac{\gamma}{\sqrt{8\pi}} \) is a number of order 1. Thus, in this loop quantum cosmology scheme the bounce indeed occurs at around the Planck length \( l_p \).

It is of interest to reproduce the modified Friedmann equation of the kind shown through a theory with an appropriate effective covariant action without having to go into the details of loop quantum cosmology. But general relativity gives the pure Friedmann equation, and the corresponding action, the Einstein-Hilbert action, is unique in the sense that in four spacetime dimensions it is the only action that gives a theory with second order field equations for the metric. Now, loop quantum gravity, and thus loop quantum cosmology, has no additional degrees of freedom in relation to general relativity. Moreover, in loop quantum cosmology, the modified Friedmann equation \( H^2 = \frac{1}{3\kappa} \rho \left( 1 - \frac{\rho}{\rho_c} \right) \), is just the Friedmann equation with an
altered source and so clearly there are no new degrees of freedom. So, it seems that without invoking an action with extra degrees of freedom one cannot get the appropriate modified Friedmann equation obtained through a quantum version of general relativity, i.e., through loop quantum cosmology itself. On the other hand, invoking an action, and so a theory, with extra degrees of freedom would generically yield a different modified Friedmann equation.

Sotiriou [8] has considered this conundrum in some detail using $f(R)$ gravity theory [9], where $R$ is the Ricci scalar and $f$ some function of it. Now, higher-order theories, such as $f(R)$, go beyond the Einstein-Hilbert action, and so involve, under metric variation, higher-order differential equations. These introduce new degrees of freedom such as new extra gravitational fields. The upshot is that a modified Friedmann equation would not look like the one of loop quantum cosmology. However, by envisaging such a theory as an effective field theory [10] and through an order reduction technique [11, 12], Sotiriou was able to find a modified Friedmann equation of loop quantum cosmology type without new degrees of freedom [8]. Thus, even a theory with new degrees of freedom, as $f(R)$ gravity has in general, can, upon specific considerations, be treated for some purposes as a theory with the same degrees of freedom as general relativity [8]. Such a treatment has also been used in [13], but in adopting this procedure one must be careful, since if order reduction is to be used in conjunction with effective field theory then one must be aware of the problems it can raise [10].

It is certainly of significance to extend Sotiriou’s analysis and see whether different higher-order gravities can as well yield bouncing universes. Here we consider a modified Gauss-Bonnet gravity theory with action $R + f(G)$ [14–19], where $G$ is the Gauss-Bonnet term, given by $G = R^2 - 4 R^{ab} R_{ab} + R^{abcd} R_{abcd}$, with $R_{abcd}$, $R_{ab}$, and $R$ being the Riemann tensor, the Ricci tensor, and the Ricci scalar, respectively, and $f(G)$ means a function of $G$. Although in four spacetime dimensions $G$ is a topological invariant and does not contribute to the dynamics, $f(G)$ is non trivial even in four dimensions. In this connection, it is impressive that modified Gauss-Bonnet gravity has been proven to be perfectly viable, since it passes solar system tests [20] (see also [14]), naturally leads to late-time cosmic acceleration [20], and is stable under linear cosmological perturbations [21].

Our idea is to have a bouncing cosmology coming out of a still relatively simple theory such as $R + f(G)$, while taking into account terms that involve altogether, the Riemann and the Ricci curvatures and the Ricci scalar. It would be interesting to extend our results for more general theories including functions of all possible curvature invariants consistent with the procedure of effective field theories.

Now, $R + f(G)$ gravity is likewise an higher-order theory and, as in $f(R)$, involves under metric variation higher-order differential equations which in turn introduce new degrees of freedom, and ultimately the modified Friedmann equation would not be the one we seek, of the type provided by loop quantum cosmology. As in [8] one can bypass this problem, sticking to $R + f(G)$ gravity and using it as an effective field theory of the higher-order theory together with an order reduction technique procedure. This approach avoids the unwanted ghostly modes that arise in higher-order theories and it further has the advantage that the physical solutions one is interested in can be connected perturbatively to the general relativity solutions. Some other solutions of the full theory, like solutions of the equivalent scalar-tensor representation of $R + f(G)$ gravity, may be of interest in other contexts, but of no use to get the modified Friedmann equation of loop quantum cosmology. We note in addition that second-order theories can still be pathological as, for instance, in the case of a system for which the equations develop an imaginary propagation speed. We assume that the order reduction procedure we use does not yield second-order equations with this unwanted behavior. One can wonder why the scheme proposed is a valid scheme. The answer can lie on the essential features of the fundamental theory, yet to be formulated, which may impose to be itself expressed in this way, when viewed as an effective theory.

In brief, our aim is to find from $R + f(G)$ gravity, an effective action invariant under diffeomorphisms, that yields a modified field equation with a modified source that yields a bounce of loop quantum cosmology type. We extend this into $R + f(G)$ gravity the approach and the results for $f(R)$ theories given in [8].

The paper is organized as follows. In Sec. 2 we present the $f(G)$ theory and the order reduced field equations, i.e., the second order field equations that are perturbatively close to general relativity. In Sec. 3 we choose the ansatz of a Friedmann-Lemaître-Robertson-Walker (FLRW) line element and deduce a modified first Friedmann equation for the evolution of the universe. In Sec. 4 we assume the simplest possible model,
i.e., a universe with zero cosmological constant, zero spatial curvature, and composed of a stiff fluid and derive the Lagrangian and the conditions obeyed by the \( f(G) \) Gauss-Bonnet modified gravity to have a universe with a bounce. In Sec. 5 we conclude.

2. Modified Gauss-Bonnet \( f(G) \) gravity theory and order reduced field equations

The action for the modified Gauss-Bonnet \( f(G) \) gravity in four dimensions reads as

\[
S = S_{\text{grav}}(g_{ab}) + S_{\text{matter}}(g_{ab}, \psi),
\]

where \( S_{\text{grav}}(g_{ab}) \) is the appropriate gravitational action for the modified Gauss-Bonnet theory that depends on the metric \( g_{ab} \), and \( S_{\text{matter}}(g_{ab}, \psi) \) is the matter action that depends on the metric \( g_{ab} \) and on matter fields \( \psi \). The gravitational action is

\[
S_{\text{grav}} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} L_{\text{grav}},
\]

where \( \kappa = 8\pi \), \( g \) is the determinant of the metric \( g_{ab} \), \( a, b \) are spacetime indices, and \( L_{\text{grav}} \) is the Lagrangian density for the gravity sector given by

\[
L_{\text{grav}} = R + f(G),
\]

where \( R \) is the Ricci scalar and \( f(G) \) is a function of the Gauss-Bonnet term \( G \) defined as

\[
G = R^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd},
\]

with \( R_{abcd} \) being the Riemann tensor and \( R_{ab} \) the Ricci tensor. In Eq. (3) the term \( R \) is the Einstein-Hilbert Lagrangian, and the term \( f(G) \) is the term that yields the modified gravity we want to consider.

The principle of least action, \( \delta S = 0 \), applied to Eqs. (1)-(3) yields

\[
R_{ab} - \frac{1}{2}g_{ab}R - \frac{1}{2}g_{ab}f(G) + f'(G)[2RR_{ab} + 2R_{acde}R_b^{cde} - 4R_{cb}R_a^c - 4R^{abcd}R_{abcd}]
- 4g_{ab}R^{cd}\nabla_c \nabla_d f'(G) + 4R_{cb}^c \nabla_b f'(G) + 4R_{ac}^c \nabla_a f'(G) + 2g_{ab}R\Box f'(G)
- 2R\nabla_a \nabla_b f'(G) + 4R_{abcd}^c \nabla^d f'(G) - 4R_{ab} \nabla f'(G) = \kappa T_{ab},
\]

where \( T_{ab} \), defined by \( T_{ab} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{ab}} \), is the stress-energy tensor. In Eq. (5) a prime designates derivation with respect to \( G \), \( \nabla_a \) is the metric covariant derivative and we define the d’Alembertian as \( \Box = g^{ab}\nabla_a \nabla_b \). In \( f(G) \) gravity, in contrast to \( f(R) \) gravity, the equations of motion (5) not only depend on \( R \) and \( R_{ab} \), but also on \( R_{abcd} \) and on \( G \) itself. Now, the Riemann tensor can be defined in terms of the Weyl tensor \( C_{abcd} \), the Ricci tensor, the Ricci scalar, and the metric \( g_{ab} \) as \( R_{abcd} = C_{abcd} + \frac{1}{6} (g_{ac}R_{db} + g_{bd}R_{ca} - g_{ad}R_{cb} - g_{bc}R_{da}) + \frac{1}{2} (g_{ad}g_{cb} - g_{ac}g_{db})R \). We proceed by making the simplifying assumption that the spacetime has zero Weyl tensor, \( C_{abcd} = 0 \). This is in line with what we will do next, when working with a FLRW line element for which the Weyl tensor \( C_{abcd} \) vanishes. For \( C_{abcd} = 0 \), we have that \( R_{abcd} \) can be written in terms of \( R_{ab} \) and \( R \) as

\[
R_{abcd} = \frac{1}{2} (g_{ac}R_{db} + g_{bd}R_{ca} - g_{ad}R_{cb} - g_{bc}R_{da}) + \frac{1}{6} (g_{ad}g_{cb} - g_{ac}g_{db})R.
\]

With Eq. (6) one can simplify Eq. (4) to

\[
G = \frac{2}{3} R^2 - 2R_{ab}R^{ab}.
\]

Equations (6) and (7) can then be put directly into Eq. (5) defining our equation of motion.
We may further parametrize the modified Gauss-Bonnet function \( f(G) \) as
\[
f(G) = 2\Lambda + \epsilon \varphi(G),
\] where \( \Lambda \) is a cosmological constant term, \( \varphi(G) \) is a function of the invariant \( G \) and \( \epsilon \) is a dimensionless parameter.

Two comments are in order here in regard to Eqs. (3) and (8). First, one can think of \( \varphi \) as a function incorporating all possible corrections to the Einstein-Hilbert action. For instance, if \( \varphi \) is thought of as an expansion, not necessarily analytic, then one possibility is that it could look like \( \varphi = \cdots + a_i l_p^{6i} G + a_2 l_p^2 G + a_3 l_p^2 G \ln l_p G + a_4 l_p^2 G^2 + \ldots \), where the characteristic scale was set to be \( l_p \) and the \( a_i \) are coefficients without units. The expansion could have other different terms, see [14–19] for some other terms that can appear. Second, the \( \epsilon \) parameter is not a priori a small parameter. It could, if we wished, be absorbed in \( \varphi \). For instance, in the expansion just given it could be absorbed by the \( a_i \)s. Moreover, clearly a parameter such as \( \epsilon \) is desired such that when it goes to zero general relativity is brought back. Indeed, when \( \epsilon \) is set to zero in Eq. (8), the action (3) gives simple general relativity, so that \( \epsilon \) serves essentially to indicate the deviation from general relativity.

We can now start our plan. Rather than considering it as an exact theory, we want to consider the \( R + f(G) \) gravity, Eqs. (3) and (8), as an effective field theory perturbatively close to general relativity in such a way as to not change the original degrees of freedom. To do so we, work at lowest order in \( \epsilon \). For implementing this plan, from Eqs. (3) and (8) we see we must have that \( \epsilon \varphi \ll \mathcal{R} \) at the range of curvatures considered. For instance, suppose that \( \epsilon \varphi(G) \sim R^2/\rho_c \), for some critical density \( \rho_c \). If then the universe’s critical density at a bounce is of the order of the Planck density, then at around that scale essentially \( R \ll \rho_c \sim l_p^{-2} \).

To maintain the original degrees of freedom of general relativity, and thus get rid of the unwanted extra degrees of freedom of the \( R + f(G) \) gravity, we apply an order reduction technique to the field equations. The parameter \( \epsilon \) allows to develop in a transparent way this technique, working out to lowest order Eq. (5). To proceed we substitute \( f(G) \) and \( f'(G) \) with \( \epsilon = 0 \) in Eq. (5), and express \( R \) and \( R_{ab} \) at the lowest order. This process yields,
\[
R^T = -4\Lambda - \kappa T,
\]
\[
R^T_{ab} = -\frac{\kappa}{2}g_{ab}T - \Lambda g_{ab} + \kappa T_{ab},
\]
where we denote lowest order values by using the superscript \( T \). At lowest order, \( \epsilon = 0 \), and using Eqs. (9) and (10), we obtain for Eqs. (6) and (7) the following expressions
\[
R^T_{abcd} = -\frac{\kappa}{2} (g_{ad} T_{cb} + g_{bc} T_{da} - g_{ac} T_{db} - g_{bd} T_{ca}) - \frac{1}{3} (g_{ac} g_{bd} - g_{ad} g_{bc}) (\Lambda + \kappa T),
\]
\[
G^T = \frac{2}{3} \kappa^2 T^2 - 2\kappa^2 T_{ab} T^{ab} + \frac{8}{3} \Lambda^2 + \frac{4}{3} \Lambda \kappa T.
\]

The application of order reduction is equivalent to replacing \( R, R_{ab}, R_{abcd} \), and \( G \) by, respectively, \( R^T, R^T_{ab}, R^T_{abcd} \), and \( G^T \) in Eq. (5). This procedure brings us to the expression
\[
R_{ab} - \frac{1}{2} g_{ab} R + \epsilon \left[ -\frac{1}{2} g_{ab} \varphi_T + \varphi_T \left( 2R^T R_{ab} - 4R^T_{cd} g^{ef} R^T_{ef} + 2R^T_{acde} g^{ef} g^{gh} R^T_{efgh} - 4g^{cd} g^{ef} R^T_{acde} R^T_{df} \right) - 4R^T_{ab} \varphi_T - 4g_{ab} g^{ef} R^T_{ef} \nabla_c \nabla_d \varphi_T + 4g^{cd} R^T_{da} \nabla_c \nabla_b \varphi_T + 4g^{cd} R^T_{db} \nabla_c \nabla_a \varphi_T + 2g_{ab} R^T \varphi_T \right] - 2R^T \nabla_a \nabla_b \varphi_T + 4g^{ef} g^{cd} R^T_{ef} \nabla_d \nabla_c \varphi_T = \kappa T_{ab},
\]
where \( \varphi_T = \varphi(G^T) \) and \( \varphi_T = \varphi'(G^T) \), with the prime denoting differentiation with respect to the Gauss-Bonnet invariant \( G^T \). Eq. (13) is the order reduced field equation that we wanted.
3. Modified first Friedmann equation

Our next step is to derive a modified first Friedmann equation from the order reduced field equation, Eq. (13). To do so we choose a FLRW line element of the type

\[ ds^2 = -dt^2 + a^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \]

(14)

where \( t \) is the time coordinate, \((r, \theta, \phi)\) are the spatial coordinates, \( a = a(t) \) is the cosmological scale factor, and \( k = -1, 0, 1 \) yields hyperbolic, flat, and spherical spaces, respectively. We assume a perfect fluid description for the fields involved, so the stress-energy tensor is

\[ T_{ab} = \left( \rho + p \right) u_a u_b + pg_{ab}, \]

(15)

where \( u_a \) is the fluid’s 4-velocity, \( \rho = \rho(t) \) and \( p = p(t) \) are the energy density and pressure of the fluid, respectively, with \( p \) being given by

\[ p = w \rho, \]

(16)

and \( w \) being a number. Defining the Hubble function \( H = H(t) \) by

\[ H = \frac{\dot{a}}{a}, \]

(17)

where a dot means derivative with respect to time \( t \), the zero-zero component of Eq. (13) is

\[ \frac{6k}{a^2} + 6H^2 + 2\Lambda - 2k\rho + k \varphi^T - \frac{4}{3} \epsilon (\Lambda - k\rho) \left[ 6H \dot{\varphi}^T + \varphi^T (2\Lambda + k\rho(1 + 3w)) \right] = 0. \]

(18)

An independent equation can be taken by noting that, assuming a FLRW metric, i.e., Eq. (14), one has \( \nabla_a T^{ab} = 0 \) in Eq. (13). This is equivalent to energy conservation which reads

\[ \dot{\rho} = -3H(1 + w)\rho. \]

(19)

Now, using the chain rule, one has \( \dot{\varphi}^T = \frac{\partial \varphi^T}{\partial \varphi} \frac{\partial \varphi^T}{\partial \rho} \dot{\rho} \). Substituting \( \dot{\rho} \) from Eq. (19) one finds

\[ \dot{\varphi}^T = 4\varphi^T H k\rho(1 + w)[\Lambda + 2k\rho(1 + 3w) - 3\Lambda w]. \]

(20)

After placing Eq. (20) into Eq. (18), we get

\[
H^2 = \frac{k}{a^2} - \frac{\Lambda}{3} + \frac{1}{3} k\rho + \frac{\epsilon}{18} \left[ -3 \varphi^T + \frac{96k^2 \rho \varphi^{''T}(1 + w)(k\rho - \Lambda)(\Lambda + 2k\rho(1 + 3w) - 3\Lambda w)}{a^2} \right. \\
+ 4(\Lambda - k\rho) \left. \left( 8k\rho \varphi^{''T}(1 + w)(k\rho - \Lambda)(\Lambda + 2k\rho(1 + 3w) - 3\Lambda w) + \varphi^{''T}(2\Lambda + k\rho(1 + 3w)) \right) \right]. \]

(21)

When substituting Eq. (20) into Eq. (18), we have used the lowest order value for \( H^2 \), namely, \( H^2 = -\frac{k}{a^2} - \frac{\Lambda}{3} + \frac{1}{3} k\rho \). Eq. (21) is the modified first Friedmann equation for the \( R + f(G) \) gravity theory after a process of order reduction.

4. Bouncing cosmology in Gauss-Bonnet modified gravity

Let us now assume the simplest possible model, i.e., zero cosmological constant, zero spatial curvature, and a stiff fluid so that

\[ \Lambda = 0, \quad k = 0, \quad w = 1, \]

(22)
respectively. First, with these assumptions we have that Eqs. (12), (15), (16), and (22) yield
\[ \rho^2 = -\frac{3}{16\kappa^2} G^T. \] (23)
Thus, from Eq. (23) we see that \( G^T \) must be negative. Then, using Eq. (23) and the assumptions of Eq. (22), one finds that Eq. (21) turns into
\[ H^2 = \frac{1}{3} \kappa \rho - \epsilon \left( (G^T)^2 \phi'' + \frac{1}{6} G^T \phi' + \frac{1}{6} \phi^2 \right). \] (24)
Notice that, by taking \( \phi \) one finds that Eq. (24) returns the first Friedmann equation of general relativity as expected, i.e., \( H^2 = \frac{1}{3} \kappa \rho \).

We want to retrieve a cyclic universe dynamical equation with a bounce. Following e.g. [8], the appropriate Friedmann equation with a bounce can be written as
\[ H^2 = \frac{1}{3} \kappa \rho \left( 1 - \frac{\rho}{\rho_c} \right), \] (25)
where \( \rho_c \) is the critical energy density at which the bounce occurs.

Comparing Eqs. (24) with (25) we get \( \epsilon \left( (G^T)^2 \phi'' + \frac{1}{6} \phi^2 T + \frac{1}{6} \phi^2 T \right) = \frac{\kappa G^2}{\rho_c}. \) This equation upon using Eq. (23) turns into \( \epsilon \left( -(G^T)^2 \phi'' + \frac{1}{6} G^T \phi' + \frac{1}{6} \phi^2 \right) = \frac{G^2}{16 \kappa \rho_c}. \) In the solution for \( \phi(G) \) we can drop the superscript \( T \) since to \( \epsilon \) order \( f(G) \) and \( f(G^T) \) are the same, see Eq. (8). So, the equation is
\[ \epsilon \left( -G^2 \phi'' + \frac{1}{6} G \phi' - \frac{1}{6} \phi \right) = \frac{G}{16 \kappa \rho_c}. \] (26)
This is a differential equation for \( \phi \). Now we are studying the dynamics of the universe for times that are close to, but still not quite at the Planck scale, say 10 times the Planck scale. Since the quantity \( \rho_c \) sets the scale of the problem, we rescale Eq. (26) by writing \( \bar{G} = G/\rho_c^2 \) and \( \bar{\phi} = \phi/\rho_c \). So Eq. (26) is now
\[ \epsilon \left( -\bar{G}^2 \phi'' + \frac{1}{6} \bar{G} \phi' - \frac{1}{6} \phi \right) = \frac{\bar{G}}{16 \kappa \rho_c}, \] (27)
where a dash denotes now differentiation with respect to \( \bar{G} \). The solution for Eq. (27) reads \( \bar{\phi}(\bar{G}) = \bar{c}_1 \sqrt{\bar{G}} + \bar{c}_2 \bar{G} - \frac{3 G \ln (1/G)}{20 \kappa \rho_c}, \) where \( \bar{c}_1 \) and \( \bar{c}_2 \) are dimensionless constants of integration. Now, \( \sqrt{\bar{G}} \) is a term that is not a correction term in comparison to \( R \), so we put \( \bar{c}_1 = 0 \) and the \( \bar{G} \) is a pure divergence so we can drop the corresponding terms. So finally we get the solution \( \bar{\phi}(\bar{G}) = \frac{3 G}{20 \kappa \rho_c} \ln |\bar{G}|, \) where \( |\bar{G}| \) means the absolute value of \( G \). Recovering \( \phi \) in terms of \( G \) we have
\[ \epsilon \phi(G) = \frac{3}{40 \kappa \rho_c} \ln \left| \frac{G}{\rho_c^2} \right|. \] (28)
Then, the Lagrangian for the gravity sector of the modified Gauss-Bonnet \( R + f(G) \) gravity is \( \mathcal{L}_{\text{grav}} = R + f(G) = R + \epsilon \phi(G), \) see Eq. (3), which after using Eq. (28) gives
\[ \mathcal{L}_{\text{grav}} = R + \frac{3}{40 \kappa \rho_c} \ln \left| \frac{G}{\rho_c^2} \right|. \] (29)
This is the Lagrangian we were after.

The approximation we use is valid for \( R \ll \rho_c \sim l_p^{-2} \). So, at \( \rho_c \) the approximation breaks down, since there \( R \sim \rho_c \sim l_p^{-2} \) and one cannot deduce that Eq. (29) yields Eq. (25) at the bounce. Our claim is that at stages close to the bounce Eq. (25) holds still as a good approximation to our Lagrangian, describing correctly the collapsing universe just before and just after the bounce. Typically, the equation is valid for
\( \rho \sim 0.1 \rho_c \), say, in which case a typical length \( l \) would have \( l \sim 3 \rho_c \). On the other hand Eq. (25) is an equation that comes from loop quantum gravity and so supposedly correct at the bounce itself. Thus, presumably, the Lagrangian Eq. (29) holds good for the bounce.

We now note that the Lagrangian for the \( f(R) \) gravity theory yields that the first order correction to \( R \) is an \( R^2 \) term, indeed \( \mathcal{L}_{\text{grav},f(R)} = R + \frac{1}{16 \pi} \rho_c \frac{R^2}{\rho_c} \). This Lagrangian is to be compared to the Lagrangian we have found in Eq. (29) for the \( R + f(G) \) theory. Eq. (29) has the zeroth order Einstein-Hilbert term \( R \) plus the term \( \ln |G| \) as the first order correction to general relativity.

Interesting to note that Lagrangians of the type of our Eq. (29) were found too using different approaches where some reconstruction methods were used by demanding that the scale factor undergoes a bounce at some cosmological time \([15–19]\). In this way specific \( R + f(G) \) gravity bouncing models in the early universe were selected.

5. Conclusions

We have derived an effective Lagrangian, and so a covariant action, for the modified Gauss-Bonnet \( R + f(G) \) gravity theory which yields a cosmological solution for a bouncing universe. Since the theory under study generically involves higher-order differential equations which might lead to physical instabilities, we have used an order reduction technique procedure. By means of this approach, one can indeed avoid the unwanted ghostly modes arising in these theories to construct viable physical solutions which can be connected perturbatively to the solutions of general relativity. Notice that higher-order derivatives are not the only source of possible instabilities since also second order theories can lead to pathologies. However, in order to give a definite answer about this issue, one would need to make a thorough perturbative analysis such as to understand if the proposed bounce solution is smooth and dynamically viable.

Let us in addition stress that there has been a growing interest in the understanding of the big bang initial singularity, and universes with a bounce of the sort we have presented here are viable candidates to finding a solution to this problem. It would be interesting to investigate, by means of the same approach used here, more general theories including functions of all possible curvature invariants.

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