Operator-valued Kernels and Control of Infinite dimensional Dynamic Systems

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Motivation: Green kernels to solve PDEs

To solve the heat equation $\partial_s u(s, y) = \Delta u(s, y)$ on $\mathbb{R}^d$ with $u(t, x) = f_t(\cdot)$ for a given $t$, one just has to find the Green kernel $k(s, t, y, x)$ s.t.

$$\partial_s k(s, t, y, x) = \Delta_y k(s, t, y, x), \forall s, y \text{ and } k(t, t, y, x) = \delta_y(x), \forall y$$

then the solution is obtained through a kernel integral operator $u = Kf$, i.e.

$$u(s, y) = \int_x k(s, t, y, x)f_t(x)dx,$$

and we know that actually this is the heat kernel

$$k(s - t, x, y) = \frac{1}{(4\pi(s - t))^d} e^{-\frac{\|x-y\|^2}{4(s-t)}} \text{ for } s \geq t.$$ 

What about $\partial_s u = \Delta u + v$ where $v(s, y)$ is a control? Is it possible to find a notion of Green kernel for Linear-Quadratic optimal control problems?
Motivation: Green kernels to solve PDEs

To solve the heat equation \( \partial_s u(s, y) = \Delta u(s, y) \) on \( \mathbb{R}^d \) with \( u(t, x) = f_t(\cdot) \) for a given \( t \), one just has to find the Green kernel \( k(s, t, y, x) \) s.t.

\[
\partial_s k(s, t, y, x) = \Delta_y k(s, t, y, x), \quad \forall s, y \text{ and } k(t, t, y, x) = \delta_y(x), \quad \forall y
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k(s - t, x, y) = \frac{1}{(4\pi(s - t))^d} e^{-\frac{\|x - y\|^2_d}{4(s-t)}} \quad \text{for } s \geq t.
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What about \( \partial_s u = \Delta u + v \) where \( v(s, y) \) is a control? Is it possible to find a notion of Green kernel for Linear-Quadratic optimal control problems?

Yes! This is what we are going to see in this talk by focusing on the Hilbert space of controllable trajectories.
Time-varying infinite-dimensional LQ optimal control

Let \( (V, \| \cdot \|_V) \) and \( (H, \| \cdot \|_H) \) be two separable Hilbert spaces, and \( U \) a Hilbert space. We assume that \( V \subset H \), with continuous injection. Identifying \( H \) to its dual, we have also the inclusion \( H \subset V' \) with continuous injection, where \( V' \) is the dual of \( V \).

\[
\begin{align*}
\min_{y(\cdot), u(\cdot)} & \chi_{y_0}(y(t_0)) + g(y(T)) \\
& + (y(t_0), J_0 y(t_0))_H + \int_{t_0}^T [(M(t)y(t), y(t))_H + (N(t)u(t), u(t))_U] dt \\
\text{s.t.} & \quad \frac{dy}{dt} + A(t)y(t) = B(t)u(t), \text{ a.e. in } [t_0, T]
\end{align*}
\]

- state \( y(t) \in V \), control \( u(t) \in U \), \( \exists \alpha > 0, \beta \in \mathbb{R}, \forall z \in V, \langle A(t)z, z \rangle_{V' \times V} + \beta \|z\|_H^2 \geq \alpha \|z\|_V^2 \)
- \( A(t) \in \mathcal{L}(V, V') \), \( B(\cdot) \in L^\infty(t_0, T; \mathcal{L}(U, H)) \), \( M(\cdot) \in L^\infty(t_0, T; \mathcal{L}(H, H)) \), \( N(\cdot) \in L^\infty(t_0, T; \mathcal{L}(U, U)) \), \( M(t) \geq 0 \) and \( N(t) \geq \nu \text{Id}_U \) (\( \nu > 0 \)), \( J_0 \succ 0 \),
- differentiable terminal cost \( g : V \to \mathbb{R} \), indicator function \( \chi_{y_0} \),
- \( y(\cdot) : [t_0, T] \to V \) absolutely continuous, \( N(\cdot)^{1/2}u(\cdot) \in L^2([t_0, T]) \)
Time-varying infinite-dimensional LQ optimal control

Let \((V, \|\cdot\|_V)\) and \((H, \|\cdot\|_H)\) be two separable Hilbert spaces, and \(U\) a Hilbert space. We assume that \(V \subset H\), with continuous injection. Identifying \(H\) to its dual, we have also the inclusion \(H \subset V'\) with continuous injection, where \(V'\) is the dual of \(V\).

\[
\min_{y(\cdot), u(\cdot)} \chi_{y_0}(y(t_0)) + g(y(T)) \\
+ (y(t_0), J_0 y(t_0))_H + \int_{t_0}^T [(M(t)y(t), y(t))_H + (N(t)u(t), u(t))_U] dt \\
\text{s.t.} \quad \frac{dy}{dt} + A(t)y(t) = B(t)u(t), \ \text{a.e. in} \ [t_0, T]
\]

\[
\rightarrow L(y(t_j)_{j \in [J]}) \\
\rightarrow \|y(\cdot)\|_{\mathcal{H}_K}^2 \\
\rightarrow y(\cdot) \in \mathcal{H}_K
\]

- state \(y(t) \in V\), control \(u(t) \in U\), \(\exists \alpha > 0, \beta \in \mathbb{R}, \forall z \in V, \langle A(t)z, z \rangle_{V' \times V} + \beta \|z\|^2_H \geq \alpha \|z\|^2_V\)
- \(A(t) \in \mathcal{L}(V, V')\), \(B(\cdot) \in L^\infty(t_0, T; \mathcal{L}(U, H))\), \(M(\cdot) \in L^\infty(t_0, T; \mathcal{L}(H, H))\), \(N(\cdot) \in L^\infty(t_0, T; \mathcal{L}(U, U))\), \(M(t) \geq 0\) and \(N(t) \geq \nu \text{Id}_U\) \((\nu > 0)\), \(J_0 \succ 0\),
- differentiable terminal cost \(g : V \rightarrow \mathbb{R}\), indicator function \(\chi_{y_0}\), “loss function“
- \(L : (\mathbb{R}^Q)^J \rightarrow \mathbb{R} \cup \{\infty\}\),
- \(y(\cdot) : [t_0, T] \rightarrow V\) absolutely continuous, \(N(\cdot)^{1/2}u(\cdot) \in L^2([t_0, T])\)
LQ optimal control is a kernel regression!

By rewriting the LQ problem, we can turn it into a loss+regularizer problem in a “machine learning” (regression) fashion.

\[
\begin{align*}
\min_{y(\cdot), u(\cdot)} & \quad \chi y_0(y(t_0)) + g(y(T)) \\
+ (y(t_0), J_0 y(t_0))_H + \int_{t_0}^T \left[(M(t)y(t), y(t))_H + (N(t)u(t), u(t))_U\right]dt \\
\text{s.t.} & \quad \frac{dy}{dt} + A(t)y(t) = B(t)u(t), \text{ a.e. in } [t_0, T]
\end{align*}
\]

\[
\begin{align*}
\min_{y(\cdot), u(\cdot)} & \quad L(y(t_j)_{j \in [J]}) \\
+ |y(\cdot)|^2_{\mathcal{H}_K} \\
\text{s.t.} & \quad y(\cdot) \in \mathcal{H}_K
\end{align*}
\]

We will see that the regression is over a reproducing kernel Hilbert space (RKHS) $\mathcal{H}_K$ with a kernel $K$ depending on $[t_0, T], A, B, M, N$. The space $\mathcal{H}_K$ plays the role of a Sobolev space for LQ optimal control (similarly to Poisson’s equation).
The classical way of solving LQ optimal control: the Riccati equation

The functional \( u(\cdot) \mapsto J(u(\cdot)) = \int_{t_0}^{T} [(M(t)y(t), y(t))_H + (N(t)u(t), u(t))_U]dt \) is quadratic and strictly convex. It has a unique minimum \( u(\cdot) \), which is computed as follows: the forward-backward system of equations

\[
\begin{align*}
\frac{dy}{dt} + A(t)y(t) + B(t)N^{-1}(t)B^*(t)p(t) &= 0, & y(t_0) &= y_0 \\
-\frac{dp}{dt} + A^*(t)p(t) - M(t)y(t) &= 0, & p(T) &= 0,
\end{align*}
\]

has a unique solution. Moreover, we have the decoupling property

\[
p(t) = P(t)y(t)
\]

in which \( P(t) \in \mathcal{L}(H; H) \) is symmetric and positive semidefinite. The operator \( P(t) \) is defined by solving a system similar to (1) for each \( t \in [t_0, T] \) and \( h \in H \)

\[
\begin{align*}
\frac{dc}{ds} + A(s)\xi(s) + B(s)N^{-1}(s)B^*(s)\eta(s) &= 0, & \xi(t) &= h, \\
-\frac{d\eta}{ds} + A^*(s)\eta(s) - M(s)\xi(s) &= 0, & \eta(T) &= 0 \forall s \in (t, T),
\end{align*}
\]

and then setting \( \eta(t) = P(t)h \).
The classical way of solving LQ optimal control: the Riccati equation
(cont.)

If \( \varphi(\cdot) \in L^2(t_0, T; H) \) satisfies \( \frac{d\varphi}{dt} + A(t)\varphi(t) \in L^2(t_0, T; H) \), then \( \Psi(t) = P(t)\varphi(t) \) satisfies

\[
-\frac{d\Psi}{dt} + A^*(t)\Psi(t) \in L^2(t_0, T; H),
\]

and

\[
-\frac{d\Psi}{dt} + A^*(t)\Psi(t) + P(t)\left[ \frac{d\varphi}{dt} + A(t)\varphi(t) + B(t)N^{-1}(t)B^*(t)\Psi(t) \right] = M(t)\varphi(t).
\]

This formally can be written as

\[
-\frac{dP}{dt} + P(t)A(t) + A^*(t)P(t) + P(t)B(t)N^{-1}(t)B^*(t)P(t) = M(t), \quad P(T) = 0.
\]  

(4)

The optimal state \( y(\cdot) \) for the LQR control problem is solution of the equation

\[
\frac{dy}{dt} + (A(t) + B(t)N^{-1}(t)B^*(t)P(t))y(t) = 0, \quad y(t_0) = y_0.
\]  

(5)

and the optimal control \( u(\cdot) \) is given by \( u(t) = -N^{-1}(t)B^*(t)P(t)y(t) \).

We will use in the sequel the semi-group (a.k.a. evolution family)

\[
\partial_t \Phi_{A,P}(t, s) + (A(t) + B(t)N^{-1}(t)B^*(t)P(t))\Phi_{A,P}(t, s) = 0, \quad \Phi_{A,P}(s, s) = \text{Id}_H.
\]  

(6)
Reproducing kernel Hilbert spaces (RKHS)

A RKHS \((\mathcal{H}_k, \langle \cdot, \cdot \rangle_{\mathcal{H}_k})\) is a Hilbert space of real-valued functions over a set \(\mathcal{T}\) if one of the following equivalent conditions is satisfied

\[
\exists k : \mathcal{T} \times \mathcal{T} \rightarrow \mathbb{R} \text{ s.t. } k_t(\cdot) = k(\cdot, t) \in \mathcal{H}_k \text{ and } f(t) = \langle f(\cdot), k_t(\cdot) \rangle_{\mathcal{H}_k} \text{ for all } t \in \mathcal{T} \text{ and } f \in \mathcal{H}_k
\]
(reproducing property)

the topology of \((\mathcal{H}_k, \langle \cdot, \cdot \rangle_{\mathcal{H}_k})\) is stronger than pointwise convergence
i.e. \(\delta_t : f \in \mathcal{H}_k \mapsto f(t)\) is continuous for all \(t \in \mathcal{T}\).

\[
|f(t) - f_n(t)| = |\langle f - f_n, k_t \rangle_{\mathcal{H}_k}| \leq \|f - f_n\|_{\mathcal{H}_k} \|k_t\|_{\mathcal{H}_k} = \|f - f_n\|_{\mathcal{H}_k} \sqrt{k(t, t)}
\]

For \(\mathcal{T} \subset \mathbb{R}^d\), Sobolev spaces \(\mathcal{H}^s(\mathcal{T}, \mathbb{R})\) satisfying \(s > d/2\) are RKHSs.

\[
\begin{align*}
H^1_0 &= \{ f \mid f(0) = 0, \exists f' \in L^2(0, \infty) \} \\
\langle f, g \rangle_{H^1_0} &= \int_0^\infty f' g' \, dt \\
k(t, s) &= \min(t, s).
\end{align*}
\]

Other classical kernels

\[
k_{\text{Gauss}}(t, s) = \exp \left(-\|t - s\|_{\mathbb{R}^d}^2/(2\sigma^2)\right) \quad k_{\text{poly}}(t, s) = (1 + \langle t, s \rangle_{\mathbb{R}^d})^2.
\]
Two essential tools for computations

**Representer Theorem (e.g. [Schölkopf et al., 2001])**

Let $L : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{\infty\}$, strictly increasing $\Omega : \mathbb{R}^+ \rightarrow \mathbb{R}$, and

$$\bar{f} \in \operatorname{argmin}_{f \in \mathcal{H}_k} L \left( (f(t_n))_{n \in [N]} \right) + \Omega \left( \|f\|_k \right)$$

Then $\exists \,(a_n)_{n \in [N]} \in \mathbb{R}^N$ s.t. $\bar{f}(\cdot) = \sum_{n \in [N]} a_n k(\cdot, t_n)$

$\hookrightarrow$ Optimal solutions lie in a finite dimensional subspace of $\mathcal{H}_k$.

**Finite number of evaluations $\implies$ finite number of coefficients**

**Kernel trick**

$$\langle \sum_{n \in [N]} a_n k(\cdot, t_n), \sum_{m \in [M]} b_m k(\cdot, s_m) \rangle_{\mathcal{H}_k} = \sum_{n \in [N]} \sum_{m \in [M]} a_n b_m k(t_n, s_m)$$

$\hookrightarrow$ On this finite dimensional subspace, no need to know $(\mathcal{H}_k, \langle \cdot, \cdot \rangle_{\mathcal{H}_k})$. 
Vector-valued reproducing kernel Hilbert space (vRKHS)

Let $\mathcal{T}$ be a non-empty set. A Hilbert space $(\mathcal{H}_K, \langle \cdot, \cdot \rangle_K)$ of $V$-vector-valued functions defined on $\mathcal{T}$ is a vRKHS if there exists a matrix-valued kernel $K : \mathcal{T} \times \mathcal{T} \rightarrow \mathcal{L}(V', V)$ such that the reproducing property holds:

$$K(\cdot, t)p \in \mathcal{H}_K, \quad p^\top f(t) = \langle f, K(\cdot, t)p \rangle_K, \quad \text{for } t \in \mathcal{T}, \ p \in V', \ f \in \mathcal{H}_K$$

There is a one-to-one correspondence between $K$ and $(\mathcal{H}_K, \langle \cdot, \cdot \rangle_K)$, so changing $\mathcal{T}$ or $\langle \cdot, \cdot \rangle_K$ changes $K$. We also have a representer theorem for

$$\mathcal{J}(y(\cdot)) = L((y(t_n))_{n=1}^N, \|y(\cdot)\|_{\mathcal{H}_K}^2) = 0$$

for a given extended-valued function $L : H^N \times [0, +\infty] \rightarrow \mathbb{R} \cup \{+\infty\}$. (7)

[Micchelli and Pontil, 2005, Theorem 4.2]

If for every $z \in H^N$ the function $h : \xi \in \mathbb{R}_+ \mapsto L(z, \xi) \in \mathbb{R}_+ \cup \{+\infty\}$ is strictly increasing and $\hat{y}(\cdot) \in \mathcal{H}_K$ minimizes the functional (20), then $\hat{y}(\cdot) = \sum_{n=1}^N K(\cdot, t_n)z_n$ for some $\{z_n\}_{n=1}^N \subseteq H$. In addition, if $L$ is strictly convex, the minimizer is unique.
Hilbert space of trajectories

We consider the subset $\mathcal{H}$ of $L^2(t_0, T; H)$ defined as follows

$$\mathcal{H} = \{ y(\cdot) \in L^2(t_0, T; H) \mid \frac{dy}{dt} + A(t)y(t) = B(t)u(t), \text{ with } u(\cdot) \in L^2(t_0, T; U) \}.$$ 

There is not necessarily a unique choice of $u(\cdot)$ for a given $y(\cdot) \in \mathcal{H}$ (for instance if $B(t)$ is not injective for some $t$). Therefore, with each $y(\cdot) \in \mathcal{H}$, we associate the control $u(\cdot)$ having minimal norm based on the pseudoinverse of $B(t)^\oplus$ of $B(t)$ for the $U$-norm

$$\| \cdot \|_{N(t)} := \| N(t)^{1/2} \cdot \|_U,$$

$$u(t) = B(t)^\oplus \left[ \frac{dy}{dt} + A(t)y(t) \right] \text{ a.e. in } [t_0, T], \rightarrow \text{ we get rid of the control!}$$

whence $u(\cdot)$ minimizes $\int_{t_0}^{T} (N(t)u(t), u(t))_U \, dt$ among the controls admissible for $y(\cdot) \in \mathcal{H}$. We consequently equip $\mathcal{H}$ with the norm

$$\| y(\cdot) \|_{\mathcal{H}}^2 = (y(t_0), J_0y(t_0))_H + \int_{t_0}^{T} [(M(t)y(t), y(t))_H + (N(t)u(t), u(t))_U] \, dt,$$

with $J_0$ s.t. $(J_0 + P(t_0))$ invertible. Then $\mathcal{H}$ has the structure of a Hilbert space.
Hilbert space of trajectories is a RKHS with explicit kernel!

\[
\mathcal{H} = \{ y(\cdot) \in L^2(t_0, T; H) \mid \frac{dy}{dt} + A(t)y(t) = B(t)u(t), \text{ with } u(\cdot) \in L^2(t_0, T; U) \}. \tag{9}
\]

\[
\|y(\cdot)\|_{\mathcal{H}}^2 = (y(t_0), J_0 y(t_0))_H + \int_{t_0}^{T} [(M(t)y(t), y(t))_H + (N(t)u(t), u(t))_U]dt, \tag{10}
\]

**Theorem (Main result)**

We assume the coercivity of the drift, the strong convexity of the objective, and the invertibility of \((J_0 + P(t_0))\) conditions. Set \(K(s, t) \in \mathcal{L}(H, H)\) as

\[
K(s, t) = \Phi_{A, P}(s, 0)(J_0 + P(t_0))^{-1}\Phi_{A, P}^*(t, 0) + \int_{t_0}^{\min(s, t)} \Phi_{A, P}(s, \tau)B(\tau)N^{-1}(\tau)B^*(\tau)\Phi_{A, P}^*(t, \tau)d\tau.
\]

Then the space \((\mathcal{H}, \| \cdot \|_{\mathcal{H}})\) defined by (9),(10) is a RKHS associated with the kernel \(K\).

where \(\partial_t\Phi_{A, P}(t, s) + (A(t) + B(t)N^{-1}(t)B^*(t)P(t))\Phi_{A, P}(t, s) = 0, \quad \Phi_{A, P}(s, s) = \text{Id}_H.\)

Proof is mostly integration by parts (if we guess the form of the kernel).
Decomposition of the kernel into null-control and null-initial condition

From now on, we denote $\mathcal{H}$ by $\mathcal{H}_K$. We split the kernel $K$ into

$$K(s, t) = K^0(s, t) + K^1(s, t)$$  \hspace{1cm} (11)

$$K^0(s, t) := \Phi_{A, P}(s, 0)(J_0 + P(t_0))^{-1}\Phi_{A, P}^*(t, 0),$$  \hspace{1cm} (12)

$$K^1(s, t) := \int_{t_0}^{\min(s, t)} \Phi_{A, P}(s, \tau)B(\tau)N^{-1}(\tau)B^*(\tau)\Phi_{A, P}^*(t, \tau)d\tau. $$

The kernel $K^1$ is instrumental for the LQR. Consider the Hilbert subspace of $\mathcal{H}^1_K$ of functions with initial value equal to 0, equipped with $\|\cdot\|_{\mathcal{H}_K}$,

$$\mathcal{H}^1_K = \{y(\cdot) \mid \frac{dy}{dt} + A(t)y(t) = B(t)u(t), y(t_0) = 0, \text{ with } u(\cdot) \in L^2(t_0, T; U)\}. $$  \hspace{1cm} (13)

**Proposition**

The Hilbert space $\mathcal{H}^1_K$ is a RKHS associated with the operator-valued kernel $K^1(s, t)$.
Example of heat equation with distributed control

We here focus on bounded $B(\cdot) \in L^\infty$ and parabolic equations (unbounded/hyperbolic would require a few changes). Take $V = H^1(\mathbb{R}^d, \mathbb{R})$, $H = L^2(\mathbb{R}^d, \mathbb{R})$, $A(\cdot) \equiv -\Delta$ and $B(\cdot) \equiv \text{Id}_H$, then the heat equation with distributed control writes as

$$\frac{dy}{dt} = \Delta y(t) + u(t), \quad y(t_0) = y_0 \in H. \quad (14)$$

As objective, take $J_0 = \lambda \text{Id}_H$ with $\lambda > 0$, $M(\cdot) \equiv 0$ and $N(\cdot) \equiv \text{Id}_H$, thus $P(\cdot) \equiv 0$, and $\Phi_{A,P}(t,s) = \Phi_A(t,s)$. In this well-known context, the (integral) operator $\Phi_A(t,s) = e^{-A(t-s)}$ is merely the heat semi-group associated to the heat kernel, for $t > s$,

$$k(t-s,x,y) = \frac{1}{(4\pi(t-s))^{d/2}} e^{-\|x-y\|^2_{d/4(t-s)}}.$$

Using that $A$ is self-adjoint and the known expression of the Fourier transform of a normalized Gaussian, one can show that $\int_0^{2s} k(\tau,x,y) d\tau = k(s^2,x,y)$ and consequently that, for $t > s$,

$$K_1(s,t) = \frac{1}{2}[\int_0^{2s} e^{-A\tau} d\tau] \circ e^{-A(t-s)}$$

is a kernel integral operator with kernel $k_1 = k(t-s+s^2,x,y)/2$. On the other hand $K_0(s,t) = e^{-A(t+s)}/\lambda$ has for kernel $k_0 = k(t+s,x,y)/\lambda$. This allows for explicit handling of the kernel $K$ in applied cases with various objective functions.
Solving control problems: Final nonlinear term - Mayer problem

We consider the dynamic system

$$\frac{dy}{dt} + A(t)y(t) = B(t)u(t), \quad y(t_0) = y_0. \quad (15)$$

We want to find the pair $y_0, u(\cdot)$ in order to minimize

$$J(u(\cdot), y_0) := g(y(T)) + \frac{1}{2} (y(t_0), J_0 y(t_0))_H + \frac{1}{2} \int_{t_0}^T [(M(t)y(t), y(t))_H + (N(t)u(t), u(t))_U] dt,$$

where $h \mapsto g(h)$ is a Gâteaux differentiable function on $H$. Using the norm $\|\cdot\|_{H_K}$ defined in (10), this problem can be formulated as minimizing a functional on $H_K$, namely

$$\mathcal{J}(y(\cdot)) := g(y(T)) + \frac{1}{2} \|y(\cdot)\|_{H_K}^2. \quad (16)$$

If $\hat{y}(\cdot)$ is a minimizer, it satisfies the Euler equation

$$\left(Dg(\hat{y}(T)), \zeta(T)\right)_H + (\hat{y}(\cdot), \zeta(\cdot))_{H_K} = 0, \quad \forall \zeta(\cdot) \in H_K. \quad (17)$$

By the reproducing property $\left(Dg(\hat{y}(T)), \zeta(T)\right)_H = (K(\cdot, T)Dg(\hat{y}(T), \zeta(\cdot))_{H_K}$ and (17) yields immediately the equation for $\hat{y}(\cdot)$

$$K(\cdot, T)Dg(\hat{y}(T)) + \hat{y}(\cdot) = 0. \quad (18)$$
Solving control problems: recovering the standard solution of the LQR

We can now go back to the standard LQR problem, where the initial state \( y_0 \) is known. The state \( y(\cdot) \) can be written as follows \( y(s) = \Phi_A(s, 0)y_0 + \zeta(s) \) where \( \zeta(\cdot) \) satisfies

\[
\frac{d\zeta}{ds} + A(s)\zeta(s) = B(s)u(s), \quad \zeta(t_0) = 0.
\]

Therefore \( \zeta(\cdot) \in \mathcal{H}_{K^1} \). We write \( y_0(s) = \Phi_A(s, 0)y_0 \) and

\[
J(u(\cdot)) = \int_{t_0}^{T} (M(t)y_0(t), y_0(t))_H dt + \int_{t_0}^{T} (M(t)\zeta(t), \zeta(t))_H dt + 2\int_{t_0}^{T} (M(t)y_0(t), \zeta(t))_H dt + \int_{t_0}^{T} (N(t)u(t), u(t))_U dt.
\]

The problem amounts to minimizing \( J(\zeta(\cdot)) = \|\zeta(\cdot)\|_{\mathcal{H}_K}^2 + 2\int_{t_0}^{T} (M(t)y_0(t), \zeta(t))_H dt \) on the Hilbert space \( \mathcal{H}_{K^1} \). Since

\[
J(\zeta(\cdot)) = \|\zeta(\cdot)\|_{\mathcal{H}_K}^2 + 2\left( \zeta(\cdot), \int_{t_0}^{T} K^1(\cdot, t)M(t)y_0(t)dt \right)_H,
\]

the minimizer is obtained immediately by the formula \( \hat{\zeta}(s) = -\int_{t_0}^{T} K^1(s, t)M(t)y_0(t)dt \).
More general objectives: state constraints and intermediary points

More generally one may consider several constrained time points:

\[ \mathcal{J}(y(\cdot)) = L((y(t_n))_{n=1}^N, \|y(\cdot)\|^2_{\mathcal{H}_K}) \]  

(20)

for a given extended-valued function \( L : H^N \times [0, +\infty] \to \mathbb{R} \cup \{+\infty\} \).

[ Micchelli and Pontil, 2005, Theorem 4.2 ]

If for every \( z \in H^N \) the function \( h : \xi \in \mathbb{R}_+ \mapsto L(z, \xi) \in \mathbb{R}_+ \cup \{+\infty\} \) is strictly increasing and \( \hat{y}(\cdot) \in \mathcal{H}_K \) minimizes the functional (20), then

\[ \hat{y}(\cdot) = \sum_{n=1}^N K(\cdot, t_n)z_n \]

for some \( \{z_n\}_{n=1}^N \subseteq H \).

In addition, if \( L \) is strictly convex, the minimizer is unique.
Conclusion

In a nutshell

• finding an RKHS somewhere allows for simpler computations

• in LQ optimal control, RKHSs come from vector spaces of trajectories

• in linear estimation, kernels come from covariances of optimal errors (explains the duality between estimation & control), *The RKHSs underlying linear SDE Estimation, Kalman filtering and their relation to optimal control*, Aubin-Frankowski & Bensoussan, 2022, Pure and Applied Functional analysis (to appear, available on arXiv)

Objective:

• re-read known optimal control/estimation problems through kernel lens

• use nonlinear embeddings on the state, apply it to stochastic optimal control, and optimization over measures

• Koopman operator and Model Predictive Control as possible applications

Thank you for your attention!
Micchelli, C. A. and Pontil, M. (2005). On learning vector-valued functions. *Neural Computation, 17*(1):177–204.

Schölkopf, B., Herbrich, R., and Smola, A. J. (2001). A generalized representer theorem. In *Computational Learning Theory (CoLT)*, pages 416–426.