Asymptotic stress fields near the crack tip in perfect plastic materials under mixed mode loading (plane strain conditions)

L V Stepanova

1Samara National Research University, Moskovskoe Shosse 34A, Samara, Russia, 443086

Abstract. In the paper the asymptotic stress fields in the vicinity of the crack tip in perfectly plastic Mises materials under mixed mode loading for the full range of the mode mixities are presented. This objective is engendered by the necessity of considering all the values of the mixity parameter for the full range of the mode mixities for plane strain conditions to grasp stress tensor components behaviour in the vicinity of the crack tip as the mixity parameter is changing from 0 to 1. To gain a better understanding of the stress distributions all values of the mixity parameter to within 0.1 were considered and analysed. The asymptotic solution to the statically determinate problem is obtained by the eigenfunction expansion method. Steady-state stress distributions for the full range of the mode mixities are found. The type of the mixed mode loading is controlled by the mixity parameter changing from zero for pure mode II loading to 1 for pure mode I loading. It is shown that the analytical solution is described by different relations in different sectors, the value of which is changing from 7 sectors to 5 sectors. At loadings close to pure mode II, seven sectors determine the solution whereas six and five sectors define the solution for the mixity parameter higher 0.33 and less than 0.89 respectively for plane strain conditions. The number of sectors depends on the mixity parameter. The angular stress distributions are not fully continuous and radial stresses are discontinuous for some values of the mixity parameter. It is interesting to note that the characteristic feature of the asymptotic solution obtained is the presence of a segment of values of the mixity parameter for which the solution does not depend on the mixity parameter (the solution does not depend on the mixity parameter for the mixity parameter from 0.89 to 1 and the solution coincides with the solution for mode I crack in perfect plastic materials for plane strain conditions). Thus, the salient point of the study is that the asymptotic solution is described by the same formulae for all values of the mixity parameter from 0.89 to 1 for plane strain.

1. Introduction
Despite the fact that the study of stress fields in the vicinity of the crack tip in a perfectly plastic material goes back to the work of Prandtl [1] and the characterization of cracks in perfectly plastic materials is the classical problem of conventional fracture mechanics this subject continues to arouse interest in the scientific community, especially in connection with the need for a mathematical description of stress fields at the crack tip under mixed mode loading conditions in nonlinear materials [2-9]. This problem remains an open problem [2]. The asymptotic plane strain and plane crack tip fields for both power law hardening and perfectly plastic materials have been presented by Hutchinson [10,11], Rice [12] and Rice and Rosengren [13] under pure mode I and pure mode II conditions and...
Shih [14,15] under mixed mode conditions. For power law hardening materials the asymptotic crack tip stress and strain fields possess the well-known HRR singularity. The crack-tip solutions for perfectly plastic materials do not have any singularity.

In [16] elastic perfectly-plastic asymptotic plane stress crack tip fields have been constructed by assembling elastic, constant stress and fan sectors under a complete range of mixed mode I/II states of loading. The angular stress distributions are fully continuous, and do not contain the stress discontinuities which have been a feature of many previously proposed solutions. The analytic solutions are verified by finite element solutions under contained yielding conditions. The structure of the elastic perfectly-plastic fields is compared to the structure of the asymptotic strain hardening fields.

In [2] a classic problem in nonlinear fracture mechanics has been reconsidered. This solution is limited to the following key assumptions: plane stress (so three-dimensional effects are neglected), fully yielded material around the crack tip (so linear elastic sectors are not considered in the asymptotic analysis), and small-scale yielding. While it is clear that more realistic assumptions are important, the correct solution of this classic (and easier!) problem is also important. On the important matter of three-dimensional effects, the pure mode I summary provided in [17,18] on plane stress, plane strain and three-dimensional effects is interesting. A key point is that the theory of plane stress for mode I fracture is useful, but it is understood that plane stress is for very thin specimens, i.e., an important limiting case.

In the present study the classic nonlinear fracture problem of a fully yielded, mixed mode stationary crack in a perfectly plastic material for conditions of plane strain is reconsidered. It is shown that the stress distribution coincides with the solution for Mode I problem for all values of the mixity parameter starting from $M^* = 0.8896724$.

2. Mathematical statement of the problem. Basic equations

Polar coordinates $r, \theta$ are introduced and centered at the crack tip. With reference to the polar coordinates, the equilibrium equations can be written as

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \sigma_{rr} - \sigma_{\theta\theta} = 0, \quad \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + 2 \sigma_{r\theta} = 0. \quad (1)$$

For elastic-perfectly plastic materials, the analytical solutions [1,7] numerical calculations [18,19] indicated that the stresses near a crack tip are non-singular within both the plastic and elastic sectors around the crack tip. Accordingly, all stress components near the crack tip are only functions of the polar angle $\theta$, but not the distance from the crack tip $r$. Thus, the asymptotic expansion for the stress components can be sought in the following form

$$\sigma_{rr}(r, \theta) = \sigma_{rr}^{(0)}(\theta) + r^\alpha \sigma_{rr}^{(1)}(\theta), \quad r \to 0. \quad (2)$$

Thus, the partial differential equations in (1) reduce to ordinary differential equations

$$\sigma_{rr}^{(0)} + \sigma_{rr}^{(1)} - \sigma_{\theta\theta}^{(0)} = 0, \quad \sigma_{r\theta}^{(0)} + 2 \sigma_{r\theta}^{(1)} = 0 \quad (3)$$

Under the plane strain conditions the von Mises yield criterion can be expressed as

$$\left(\sigma_{rr}^{(1)} - \sigma_{\theta\theta}^{(1)}\right)^2 + 4 \left(\sigma_{r\theta}^{(1)}\right)^2 = 4k^2 \quad (4)$$

where $k$ is the yield strength in shear and $k = \sigma_0 / \sqrt{3}$, $\sigma_0$ is the tensile yield stress. The yield criterion (4) is satisfied automatically if the stress components can be expressed as

$$\sigma_{rr}^{(1)}(\theta) = \sigma^{(1)} - k \cos 2\theta (\theta), \quad \sigma_{r\theta}^{(1)}(\theta) = \sigma^{(1)} + k \cos 2\theta (\theta), \quad \sigma_{\theta\theta}^{(1)}(\theta) = k \sin 2\theta (\theta). \quad (5)$$

The equilibrium considerations alone require that the traction and along the border between any two sectors must be continuous, but radial stress is not. Therefore, one can pose the following conditions on the lines $\theta = \theta$, separating the sectors:

$$\sigma_{rr}(\theta^+) = \sigma_{rr}(\theta^-), \quad \sigma_{r\theta}(\theta^+) = \sigma_{r\theta}(\theta^-). \quad (6)$$

The traction free conditions on the crack flanks are
The near-tip plasticity mixity parameter is defined by Shih [14,15] in terms of opening and shear stresses ahead of the crack tip as

$$ M^r = (2 / \pi) \arctan \left[ \lim_{r \to 0^+} \frac{\sigma_{yy}(r, \theta = 0)}{\sigma_{rr}(r, \theta = 0)} \right] \quad (8) $$

with $M^r = 1$ for the pure mode I and $M^r = 0$ for the pure mode II.

3. Crack-tip stress fields under mode I and mode II loading

The asymptotic solutions in perfectly plastic materials for Mode I and Mode II loadings are well known. For further analysis one can adduce the solutions for Mode I loading

$$ \{ -\pi \leq \theta \leq -3\pi/4 \quad \sigma_{rr}^{(0)} = (1/2)(1 + \cos 2\theta), \quad \sigma_{\theta\theta}^{(0)} = (1/2)(1 - \cos 2\theta), \quad \sigma_{r\theta}^{(0)} = -(1/2)\sin 2\theta, \}
\{ -3\pi/4 \leq \theta \leq -\pi/4 \quad \sigma_{rr}^{(0)} = \sigma_{\theta\theta}^{(0)} = \theta + 1/2 + 3\pi/4, \sigma_{r\theta}^{(0)} = -1/2, \}
\{ -\pi/4 \leq \theta \leq \pi/4 \quad \sigma_{rr}^{(0)} = \frac{\pi}{2} + \frac{1}{2}\cos 2\theta, \quad \sigma_{\theta\theta}^{(0)} = \frac{\pi}{2} + \frac{1}{2}\cos 2\theta, \quad \sigma_{r\theta}^{(0)} = \frac{1}{2}\sin 2\theta, \}
\{ \pi/4 \leq \theta \leq 3\pi/4 \quad \sigma_{rr}^{(0)} = \sigma_{\theta\theta}^{(0)} = -\theta + 1/2 + 3\pi/4, \sigma_{r\theta}^{(0)} = 1/2, \}
\{ -3\pi/4 \leq \theta \leq \pi \quad \sigma_{rr}^{(0)} = (1/2)(1 + \cos 2\theta), \quad \sigma_{\theta\theta}^{(0)} = (1/2)(1 - \cos 2\theta), \quad \sigma_{r\theta}^{(0)} = -(1/2)\sin 2\theta, \}

and for Mode II loading

$$ \{ -\pi = \theta_1 \leq \theta \leq \theta_2 = -3\pi/4 \quad \sigma_{rr}^{(0)} = \frac{1}{2} + \frac{1}{2}\cos 2\theta, \quad \sigma_{\theta\theta}^{(0)} = \frac{1}{2} - \frac{1}{2}\cos 2\theta, \quad \sigma_{r\theta}^{(0)} = \frac{1}{2}\sin 2\theta, \}
\{ \theta_2 \leq \theta \leq \theta_4 = -5\pi/8 + 1/4 \quad \sigma_{rr}^{(0)} = \sigma_{\theta\theta}^{(0)} = \theta + 1/2 + 3\pi/4, \quad \sigma_{r\theta}^{(0)} = -1/2, \}
\{ \theta_4 \leq \theta \leq \theta_5 = 5\pi/8 + 1/4 \quad \sigma_{rr}^{(0)} = \sigma_{\theta\theta}^{(0)} = -\theta, \quad \sigma_{r\theta}^{(0)} = 1/2, \}
\{ \theta_5 \leq \theta \leq \theta_6 = 3\pi/4 \quad \sigma_{rr}^{(0)} = -\frac{1}{4} - \frac{1}{4}\cos 2(\theta - \theta_5 - \pi/4), \quad \sigma_{\theta\theta}^{(0)} = -\frac{1}{4} - \frac{1}{4}\cos 2(\theta - \theta_5 - \pi/4), \}
\{ \theta_6 \leq \theta \leq \theta_7 = 3\pi/4 \quad \sigma_{rr}^{(0)} = \sigma_{\theta\theta}^{(0)} = \theta - 1/2 - 3\pi/4, \quad \sigma_{r\theta}^{(0)} = -1/2, \}
\{ \theta_7 \leq \theta \leq \pi \quad \sigma_{rr}^{(0)} = -\frac{1}{2} - \frac{1}{2}\cos 2\theta, \quad \sigma_{\theta\theta}^{(0)} = -\frac{1}{2} + \frac{1}{2}\cos 2\theta, \quad \sigma_{r\theta}^{(0)} = \frac{1}{2}\sin 2\theta. \}

4. Crack tip fields for Mode Mixed Loading

Having obtained the asymptotic solutions (8) and (9) and having perused a number of the previously reported results [7] one can generalize the asymptotic solutions for mixed mode loading with the mixity parameter $M^r$:
\[
\begin{align*}
\begin{cases}
-\pi = \theta_1 \leq \theta \leq \theta_2 = -3\pi / 4 & \sigma_{rr}^{(0)} = \frac{1}{2} + \frac{1}{2} \cos 2\theta, \quad \sigma_{\theta\phi}^{(0)} = \frac{1}{2} - \frac{1}{2} \cos 2\theta, \quad \sigma_{r\phi}^{(0)} = -\frac{1}{2} \sin 2\theta, \\
\theta_3 \leq \theta \leq \theta_5 & \sigma_{rr}^{(0)} = \frac{1}{2} + \frac{3\pi}{4}, \quad \sigma_{\theta\phi}^{(0)} = \theta + 1/2 + 3\pi/4, \quad \sigma_{r\phi}^{(0)} = -1/2,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\theta_5 \leq \theta \leq \theta_6 & \sigma_{rr}^{(0)} = \theta + 1/2 + 3\pi/4, \quad \sigma_{\theta\phi}^{(0)} = -1/2, \\
\theta_6 \leq \theta \leq \pi & \sigma_{rr}^{(0)} = -\frac{1}{2} \sin 2(\theta - \theta_3 + 3\pi/4), \quad \sigma_{\theta\phi}^{(0)} = \theta + 1/2, \quad \sigma_{r\phi}^{(0)} = 1/2.
\end{cases}
\end{align*}
\]

Unknows \(\theta_1, \theta_4, \theta_5, \theta_6\) can be found from the continuity conditions (6) leading to equations

\[
\begin{align*}
\theta_3 &= \frac{1}{4} \lg \left(\frac{\pi}{2} \right) - \frac{1}{2} \frac{\pi}{4}, \quad \theta_4 = \frac{1}{4} \left(\frac{\pi}{2} \right) - \frac{1}{2} \frac{\pi}{8}, \\
\theta_5 &= \frac{1}{4} \left(\frac{\pi}{2} \right) + \frac{1}{2} \frac{\pi}{4}, \quad \theta_6 = \frac{1}{4} \left(\frac{\pi}{2} \right) + \frac{1}{2} \frac{5\pi}{8}.
\end{align*}
\]

One can see from Eqs. (11) that the stress distribution (10) containing seven sectors is valid for values of the mixity parameter from the interval \(0 \leq M^r \leq 0.3301952\). For this value of the mixity parameter the angle \(\theta_5\) is equal to \(3\pi/4\) and the sector \(\theta_5 \leq \theta \leq \theta_6\) vanishes from the solution (10).

Thus, the stress distribution (10) is valid for all the values from the interval \(0 \leq M^r \leq 0.3301952\). For the values of the mixity parameter larger than \(M^r = 0.3301952\) the asymptotic solution is determined by the following formulae

\[
\begin{align*}
\begin{cases}
-\pi = \theta_1 \leq \theta \leq \theta_2 = -3\pi / 4 & \sigma_{rr}^{(0)} = \frac{1}{2} + \frac{1}{2} \cos 2\theta, \quad \sigma_{\theta\phi}^{(0)} = \frac{1}{2} - \frac{1}{2} \cos 2\theta, \quad \sigma_{r\phi}^{(0)} = -\frac{1}{2} \sin 2\theta, \\
\theta_3 \leq \theta \leq \theta_5 & \sigma_{rr}^{(0)} = \frac{1}{2} + \frac{3\pi}{4}, \quad \sigma_{\theta\phi}^{(0)} = \theta + 1/2 + 3\pi/4, \quad \sigma_{r\phi}^{(0)} = -1/2,
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\theta_5 \leq \theta \leq \theta_6 & \sigma_{rr}^{(0)} = \theta + 1/2 + 3\pi/4, \quad \sigma_{\theta\phi}^{(0)} = -1/2, \\
\theta_6 \leq \theta \leq \pi & \sigma_{rr}^{(0)} = -\frac{1}{2} \sin 2(\theta - \theta_3 + 3\pi/4), \quad \sigma_{\theta\phi}^{(0)} = \theta + 1/2, \quad \sigma_{r\phi}^{(0)} = 1/2.
\end{align*}
\]
\[
\begin{align*}
\theta_4 & \leq \theta \leq \theta_s, \quad \sigma_{rr}^{(0)} = \sigma_{\theta\theta}^{(0)} = -\theta + (1 / 2) \tan(\pi M^p / 2), \quad \sigma_{r\theta}^{(0)} = 1 / 2, \\
(12) \\
\theta_s & \leq \theta \leq \theta_b, \\
\sigma_{rr}^{(0)} &= -\theta_s + \frac{1}{2} \tan(\pi M^p / 2) + \frac{1}{2} \cos 2(\theta - \theta_s - \pi / 4), \\
\sigma_{\theta\theta}^{(0)} &= -\theta_s + \frac{1}{2} \tan(\pi M^p / 2) - \frac{1}{2} \cos 2(\theta - \theta_s - \pi / 4), \\
\sigma_{r\theta}^{(0)} &= -\frac{1}{2} \sin 2(\theta - \theta_s - \pi / 4).
\end{align*}
\]

Since for all the values \( \theta \) from the interval \(-\pi \leq \theta \leq 0\) the asymptotic solution has the same structure (10) the angles \( \theta_s, \theta_b \) can be found by the use of Eqs. (11a) whereas the angles \( \theta_s, \theta_b \) are determined from the following equations

\[
-\theta_s + \frac{1}{2} \tan \left( \frac{\pi}{2} M^p \right) - \frac{1}{2} \cos 2 \left( \theta_s - \theta - \frac{\pi}{4} \right) = -\frac{1}{2} + \frac{1}{2} \cos 2 \theta_s, \quad \sin 2 \left( \theta_s - \theta - \frac{\pi}{4} \right) = \sin 2 \theta_s.
\]

The angular distributions of the stress components are shown in figures 1-3. One can see the stress components defined by Eqs. (10) in (12). It should be noted that for at the values of the mixity parameter \( M^p \) close to 1 one would expect that the asymptotic solution should tend to the asymptotic presentation (12) continuously. However, the solution has one special feature. It turns out that according to (12) there is no any finite limit of the functions \( \sigma_{\theta\theta}^{(0)}(\theta) \) and \( \sigma_{rr}^{(0)}(\theta) \) for \( \theta_s \leq \theta \leq \theta_b \) \( \theta_s \leq \theta \leq \theta_b \) at \( M^p \to 1 \). Analysis of the solution (12) has shown that the angular distributions of the stress components coincide with the angular distributions of the stress components for the tensile load as the mixity parameter is approaching to \( M^p = 0.8896724 \). For all the values of the mixity parameter from the interval \( 0.8896724 \leq M^p \leq 1 \) the asymptotic solution is described by the formulae (8) which are the solution for the pure mode I loading (figures 1-3). As one can be seen from Eqs. (8) the solution contains five sectors. Figures 1-3 show the variations of the stress components from the pure Mode II loading to the pure Mode I loading. Red curves of figures 1-3 show the angular distributions for the pure Mode II (the first limiting case \( M^p = 0 \)). The stress components \( \sigma_{rr}^{(0)}, \sigma_{\theta\theta}^{(0)} \) are antisymmetric functions whereas the shear stress \( \sigma_{r\theta}^{(0)} \) is symmetric. The curves shown by blue points correspond to the second limiting case of the Mode I loading (\( M^p = 1 \)). Here, on the contrary, the stress components \( \sigma_{rr}^{(0)}, \sigma_{\theta\theta}^{(0)} \) are symmetrical functions with respect to the vertical axis, whereas the shear stress \( \sigma_{r\theta}^{(0)} \) is antisymmetrical.

It should be underscored that the asymptotic analysis of the mixed mode crack tip fields performed here was stipulated by the need to find eigenvalue spectra of nonlinear eigenvalue problems resulting from the mixed mode crack problems in power law creeping materials [20-22]. For the eigenspectrum to be found it is necessary to study multi-parametric two-point boundary value problems which can be solved by the shooting techniques. However, the choice of the shooting parameters is shown to be one of difficult problems for mixed mode loadings [20-22]. The angular distributions and the characteristic value of the mixity parameter obtained here will allow us to find accurate crack-tip fields corresponding new eigenvalues. It underlines the importance of a detailed and accurate asymptotic analysis such as this.
5. Conclusions

In the paper the asymptotic stress fields in the vicinity of the crack tip in perfectly plastic Mises materials under mixed mode loading for the full range of the mode mixities are presented. This objective is engendered by the necessity of considering all the values of the mixity parameter for the full range of the mode mixities for plane strain conditions to grasp stress tensor components behaviour in the vicinity of the crack tip as the mixity parameter is changing from 0 to 1. To gain a better understanding of the stress distributions all values of the mixity parameter to within 0.1 were considered and analyzed. The asymptotic solution to the statically determinate problem is obtained by the eigenfunction expansion method. Steady - state stress distributions for the full range of the mode mixities are found. The type of the mixed mode loading is controlled by the mixity parameter changing from zero for pure mode II loading to 1 for pure mode I loading. It is shown that the
analytical solution is described by different relations in different sectors, the value of which is changing from 7 sectors to 5 sectors. At loadings close to pure mode II, seven sectors determine the solution whereas six and five sectors define the solution for the mixity parameter higher 0.33 and less than 0.89 and higher 0.89 respectively for plane strain conditions. The number of sectors depends on the mixity parameter. The angular stress distributions are not fully continuous and radial stresses are discontinuous for some values of the mixity parameter. It is interesting to note that the characteristic feature of the asymptotic solution obtained is the presence of a segment of values of the mixity parameter for which the solution does not depend on the mixity parameter (the solution does not depend on the mixity parameter for the mixity parameter from 0.89 to 1 and the solution coincides with the solution for mode I crack in perfect plastic materials for plane strain conditions). Thus, the salient point of the study is that the asymptotic solution is described by the same formulae for all values of the mixity parameter from 0.89 to 1 for plane strain.

![Figure 3](image-url)

**Figure 3.** Distributions of the shear stress $\sigma_{\theta}^{(n)}$ in the vicinity of the crack tip for different values of the mixity parameter (plane strain condition).

### 6. Acknowledgments

Financial support from the Russian Foundation of Basic Research (project No. 19-01-00631) is gratefully acknowledged.

### 7. References

[1] Prandtl L 1920 *Goettinger Nachr, Math.-Phys. Kl* 74.
[2] Loghin A and Joseph P 2020 *Journal of the Mechanics and Physics of Solids* 139 103890.
[3] Turis M and Ivankova O 2020 *MATEC Web of Conferences* 310 00028.
[4] Hozdez J, Langlois M, Witz J F, Limodin N, Najjar D, Charkaluk E, Osmond P, Forre A and Szmytka F 2019 *Journal of Solids and Structures* 171 92.
[5] Lesiu G, Smolnicki M, Mech R, Ziety A and Fragassa C 2020 *Engineering Failure Analysis* 109 104354.
[6] Basnet R, Timisina S, Le K H and Kim J S 2018 *International Journal of Engineering Science* 123 127.
[7] Stepanova L V and Yakovleva E M 2015 *Journal of Mechanics of Materials and Structures* 10(3) 367.
[8] Stepanova L V 2018 *Procedia Structural Integrity* 13 255.
[9] Larisa S and Ekaterina Y 2016 *Procedia Structural Integrity* 2 793.
[10] Hutchinson J W 1968 *J Mech Phys Solids* 16 13.
[11] Hutchinson J W 1968 *J Mech Phys Solids* **16** 337.
[12] Rice J R 1968 *Journal of Applied Mechanics* **35** 379.
[13] Rice J R and Rosengren G F 1968 *J Mech Phys Solids* **16** 1-12.
[14] Shih C F 1973 Elastic-Plastic Analysis of Combine Mode Crack Problems *Ph.D. thesis, Harvard University, Cambridge, Massachusetts*.
[15] Shih C F 1974 *Fracture Analysis, ASTM STP* **560** 187.
[16] Rahman M and Hancock J W 2006 *International Journal of Solids and Structures* **43** 3692.
[17] Zhu X K and Chao Y J 2001 *Journal of the Mechanics and Physics of Solids* **49(8)** 363.
[18] Zhu X K and Chao Y J 2000 *International Journal of Solids and Structures* **37** 577.
[19] Dong P and Pan J 1990 *International Journal of Fracture* **45** 243.
[20] Stepanova L V and Yakovleva E M 2014 *PNRPU Mechanics Bulletin* **3** 129.
[21] Stepanova L and Yakovleva E 2016 *AIP Conference Proceedings* **1785** 030030.
[22] Stepanova L 2009 *Journal of Applied Mechanics and Technical Physics* **50(1)** 137.