Invariant vector means and complementability of Banach spaces in their second duals

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Abstract. Let $X$ be a Banach space. Fix a torsion-free commutative and cancellative semigroup $S$ whose torsion-free rank is the same as the density of $X^{**}$. We then show that $X$ is complemented in $X^{**}$ if and only if there exists an invariant mean $M : \ell_\infty(S, X) \to X$. This improves upon previous results due to Bustos Domecq (J Math Anal Appl 275(2):512–520, 2002), Kania (J Math Anal Appl 445:797–802, 2017), Goucher and Kania (Studia Math 260:91–101, 2021).

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1. Introduction

Invariant means on amenable groups are an important tool in many parts of Mathematics, especially in Harmonic analysis (see [8,9]). For the basic properties of invariant means, we refer the reader to [8]. Invariant means and their generalizations for vector-valued functions also play an important role in the stability of functional equations and selections of set-valued functions (see [1,5,6,16]).

The space of all bounded functions from a set $S$ into a Banach space $X$ is denoted by $\ell_\infty(S, X)$. Let us recall the definition of an amenable semigroup (see [3]).
**Definition 1.1.** A semigroup \((S, +)\) is called left [resp. right] amenable if and only if there exists a linear map \(L: \ell_\infty(S, \mathbb{R}) \to \mathbb{R}\) such that
\[
\inf f(S) \leq L(f) \leq \sup f(S), \quad f \in \ell_\infty(S, \mathbb{R})
\]
\[
L(a f) = L(f), \quad a \in S, \quad f \in \ell_\infty(S, \mathbb{R}),
\]
\[
[L(f_a) = L(f), \quad a \in S, \quad f \in \ell_\infty(S, \mathbb{R})],
\]
where
\[
a f(x) = f(a + x), \quad a, x \in S, \quad f \in \ell_\infty(S, \mathbb{R}),
\]
\[
[f_a(x) = f(x + a), \quad a, x \in S, \quad f \in \ell_\infty(S, \mathbb{R})].
\]
If both left and right invariant means exist, then \(S\) is called amenable.

**Remark 1.2.** In the above definition the first condition
\[
\inf f(S) \leq L(f) \leq \sup f(S)
\]
is equivalent to conditions \(L(\mathbb{1}_S) = 1\) and \(|L(f)| \leq \|f\| := \sup |f(S)|\).

It is known that every commutative semigroup is amenable (an easy consequence of the Markov–Kakutani fixed point theorem, see [15, Theorem 5.23]).

Certain generalizations of invariant means were investigated for vector-valued functions in [5] and the existence thereof appears to be related to properties such as reflexivity.

Some generalized definition of an invariant mean has been used by many mathematicians as a folklore (e.g. by Pełczyński [14]). The explicit form of this definition can be found e.g. in the work of Ger [6].

**Definition 1.3.** Let \((S, +)\) be a left [right] amenable semigroup and \(X\) be a Banach space. A linear map \(M: \ell_\infty(S, X) \to X\) is called a left [right] \(X\)-valued invariant mean if
\[
\|M\| \leq 1,
\]
\[
M(c \mathbb{1}_S) = c, \quad c \in X,
\]
\[
M(a f) = M(f), \quad a \in S, f \in \ell_\infty(S, X),
\]
\[
[M(f_a) = M(f), \quad a \in S, f \in \ell_\infty(S, X)],
\]
where
\[
a f(x) = f(a + x), \quad a, x \in S, \quad f \in \ell_\infty(S, X),
\]
\[
[f_a(x) = f(x + a), \quad a, x \in S, \quad f \in \ell_\infty(S, X)].
\]
If \(M\) is a left and right invariant mean, then \(M\) is called an \(X\)-valued invariant mean.

If in the above definition the norm of map \(M\) is equal to at most \(\lambda \geq 1\), then \(M\) is called an \(X\)-valued invariant \(\lambda\)-mean.
The existence of such invariant means for a fixed Banach space and for all amenable semigroups has been studied by Domecq [4, Theorem 1 and 2] and by the author in [12]. However, as observed by Lipecki in his Mathematical Review (MR1943762) of Bustos Domecq’s paper, the proof of Theorem 2 contains a gap (a flawed choice of the semigroup, so we cannot use the Principle of Local Reflexivity). This gap was corrected by Kania [10].

Goucher and Kania [7] consider the following question (communicated privately to T. Kania by J.M.F. Castillo).

Suppose that a Banach space $X$ admits an invariant mean with respect to every/some commutative group. Must $X$ be complemented in $X^{**}$?

They proved the following (see [7, Theorem A], [10, Theorem 1.2])

**Theorem 1.4.** Let $X$ be a Banach space and $\lambda \geq 1$. Then the following assertions are equivalent.

1. $X$ is complemented in $X^{**}$ by a projection of norm at most $\lambda$;
2. for every amenable semigroup $S$ there exists an $X$-valued invariant $\lambda$-mean on $S$;
3. for every commutative semigroup $S$ there exists an $X$-valued invariant $\lambda$-mean on $S$;
4. for every free commutative group $G$ of rank $|X^{**}|$ there exists an $X$-valued invariant $\lambda$-mean on $G$;
5. there exists an $X$-valued invariant $\lambda$-mean on the additive group of $X^{**}$.

It is also demonstrated ([7, Remark 1.1]) that there exists a commutative noncancellative semigroup $S$ (that could be chosen as large as one wishes) such that there exists an $X$-valued invariant mean on $S$.

In this paper we will prove that if $X$ is a Banach space and there exists an invariant $X$-valued mean on any arbitrary commutative cancellative semigroup $S$ of torsion-free rank $\text{dens} \; X^{**}$, then $X$ is $\lambda$-complemented in $X^{**}$.

2. Preliminaries

First we recall the definition of torsion-free rank (see [2]).

**Definition 2.1.** Let $S$ be a commutative cancellative semigroup. A set $A \subset S$ is independent if $\sum_{i=1}^{n} k_i a_i = \sum_{i=1}^{n} m_i a_i$ for any $n \in \mathbb{N}$ and $a_i \in A$, $k_i, m_i \in \mathbb{N}_0$, $i \in \{1, \ldots, n\}$ implies $k_i = m_i$ for $i \in \{1, \ldots, n\}$.

Let further $\mathcal{A}_0$ be the family of all independent sets $L$ in $S$ consisting only of elements whose order is infinite and such that $L$ is maximal with respect to these properties. The cardinal number of any set in $\mathcal{A}_0$ is called a torsion-free rank of $S$ and is denoted by $r_0(S)$ (all the sets in $\mathcal{A}_0$ have the same cardinal number).
The density character of a Banach space $X$, denoted $\text{dens } X$, is the smallest cardinal $\kappa$ for which $X$ has a dense subset of cardinality $\kappa$.

**Lemma 2.2.** Let $X$ be an infinite-dimensional Banach space. Then

1. if $\mathcal{B}$ is a linearly independent subset of $X$ and $\mathbb{F}$ is a countable dense subfield of a scalar field of $X$, then $|\text{span}_\mathbb{F}\mathcal{B}| = |\mathcal{B}|$;
2. for every closed subspace $Y$ of $X$ there exists a linearly independent subset $\mathcal{B}$ of $X$ such that $|\mathcal{B}| = \text{dens } X$, $\text{span } \mathcal{B} = X$, $\text{span } (\mathcal{B} \cap Y) = Y$. Moreover, we can assume that the norm of each $x \in \mathcal{B}$ is equal to 1.

**Proof.** 1. We observe that

$$|\mathcal{B}| \leq |\text{span}_\mathbb{F}\mathcal{B}| = |\bigcup_{n \in \mathbb{N}} (\mathbb{F} \cdot \mathcal{B})^n| \leq |\mathbb{N}| \cdot \sup_{n \in \mathbb{N}} |(\mathbb{F} \cdot \mathcal{B})^n| = |\mathbb{N}| \cdot |\mathbb{F} \cdot \mathcal{B}| = |\mathbb{N}| \cdot |\mathbb{F}| \cdot |\mathcal{B}| = |\mathcal{B}|.$$

2. Let $Y$ be a closed subspace of $X$, $D$ be a dense subset of $X$ such that $|D| = \text{dens } X$ and $K$ be a dense subset of $Y$ such that $|K| \leq \text{dens } X$. Let further

$$D_1 := \left\{ \frac{x}{\|x\|} : x \in K \setminus \{0\} \right\},$$
$$D_2 := \left\{ \frac{x}{\|x\|} : x \in D \setminus \{0\} \right\}.$$

Let further $\mathcal{B}_1$ be a maximal linearly independent subset of $D_1$ and $\mathcal{B}$ be a maximal linearly independent subset of $D_1 \cup D_2$ such that $\mathcal{B}_1 \subset \mathcal{B}$. We have $\text{span } \mathcal{B}_1 = Y$, $\text{span } \mathcal{B} = X$ and $|\mathcal{B}| \leq |D_1| + |D_2| = \text{dens } X$. Let $\mathbb{F} = \mathbb{Q}$ when $X$ is a real space or $\mathbb{F} = \mathbb{Q}(i)$ when $X$ is a complex space.

Since $\text{span}_\mathbb{F}\mathcal{B}$ is dense in $X$, $|\mathcal{B}| = |\text{span}_\mathbb{F}\mathcal{B}| \geq \text{dens } X$. We also note that the norm of each $x \in \mathcal{B}$ is equal to 1.

□

We will also require the version of the principle of local reflexivity due to Lindenstrauss and Rosenthal [11]. We denote by $\kappa : X \to X^{**}$ the canonical embedding from a Banach space $X$ into the second dual.

**Theorem 2.3.** Let $X$ be a Banach space. Then for every finite-dimensional subspace $F \subset X^{**}$ and each $\varepsilon \in (0,1]$ there exists a linear map $P_F^\varepsilon : F \to \kappa(X)$ such that

1. $(1-\varepsilon)\|x\| \leq \|P_F^\varepsilon(x)\| \leq (1+\varepsilon)\|x\|$, $x \in F$;
2. $P_F^\varepsilon(x) = x$ for $x \in F \cap \kappa(X)$.

It is a standard fact that subgroups and quotients of amenable groups are amenable. Using exactly the same ideas one can prove that if a Banach space admits an invariant mean with respect to a group, then it also does so with respect to subgroups and quotients of the group (see [12, Theorem 3.12] and [7, Lemma 2.3]). We would like to get a similar result for quotients of semigroups.
define the quotient semigroup $S/G$. Throughout this section we fix an infinite-dimensional Banach space $X$. If there exists an $X$-valued invariant $\lambda$-mean $M: \ell_\infty(S, X) \to X$, then there exists an $X$-valued invariant $\lambda$-mean $M: \ell_\infty(S/G, X) \to X$.

Proof. We define a map $M_1: \ell_\infty(S/G, X) \to X$ by the formula

$$M_1(f) := M(\psi(f)), \quad f \in \ell_\infty(S/G, X),$$

where $\psi(f)(s) = f([s]_G)$ for $s \in S$ and $f \in \ell_\infty(S/G, X)$. Since $\psi$ is linear, $\|\psi(f)\| = \|f\|$ and

$$\psi([t]_G f)(s) = [t]_G f([s]_G) = f([t + s]_G) = \psi(f)(t + s) = (t \psi(f))(s),$$

$$\psi(f[t]_G)(s) = f[t]_G([s]_G) = f([s + t]_G) = \psi(f)(s + t) = (\psi(f)t)(s),$$

for all $s, t \in S$, $f \in \ell_\infty(S/G, X)$, then $M_1$ is an $X$-valued invariant $\lambda$-mean on $S/G$. \hfill $\square$

3. Main results

Throughout this section we fix an infinite-dimensional Banach space $X$, $\lambda \geq 1$. Let $\gamma$ be a cardinal number. We denote by $S_\gamma$ the commutative semigroup comprising all finite subsets of $\gamma$ endowed with the operation of taking the union of sets. It is easy to observe that $|S_\gamma| = \gamma$.

Theorem 3.1. Let $\gamma$ be an infinite cardinal number. If there exists an $X$-valued invariant $\lambda$-mean $M: \ell_\infty(S_\gamma, X) \to X$, then for every subspace $E$ of $X^{**}$ such that $\text{dens} E = \gamma$ there exists a linear map $P: E \to X$ such that $\|P\| \leq \lambda$ and $P(x) = x$ for $x \in \kappa(X) \cap E$.

Proof. Let $\mathbb{K}$ be a scalar field of $X$. In view of Lemma 2.2 there exists a linearly independent subset $\mathcal{B}$ of $E$ such that $\text{span} \mathcal{B} = E$, $\text{span} (\mathcal{B} \cap \kappa(X)) = \kappa(X) \cap E$, $|\mathcal{B}| = \text{dens} E = \gamma$. Let $T: \gamma \to \mathcal{B}$ be a bijection and $M: \ell_\infty(S_\gamma, X) \to X$ be an $X$-valued invariant $\lambda$-mean.

For $A \in S_\gamma$ we define $\varepsilon_A := \frac{1}{|A| + 1}$ and $P^A_{\text{span} T(A)}$ is a fixed linear operator satisfying the conditions of Theorem 2.3.
We define the map $P: E \to X$ in the following way (on the dense subspace $\text{span}\mathcal{B}$, the map is simply continuously extended to the closure): for $x \in \text{span}\mathcal{B}$ we put $P(x) := M(\phi_x)$, where

$$
\phi_x(A) := \begin{cases}
P^A_{\text{span}T(A)}(x), & x \in \text{span}T(A) \\
0, & x \notin \text{span}T(A),
\end{cases}, \quad A \in S_\gamma
$$

when $x \in \mathcal{B}$ and

$$
\phi_x(A) := \sum_{i=1}^n \lambda_i \phi_{x_i}(A), \quad A \in S_\gamma,
$$

when $x = \sum_{i=1}^n \lambda_i x_i$, $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$, $x_1, \ldots, x_n \in \mathcal{B}$.

For $x, y \in \text{span}\mathcal{B}$ and $\alpha \in \mathbb{K}$ we notice that $\phi_{\alpha x + y} = \alpha \phi_x + \phi_y$. Thus

$$
P(\alpha x + y) = M(\phi_{\alpha x + y}) = \alpha M(\phi_x) + M(\phi_y) = \alpha P(x) + P(y),
$$

so $P$ is linear on $\text{span}\mathcal{B}$.

Let $x = \sum_{i=1}^n \lambda_i x_i$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$, $x_1, \ldots, x_n \in \mathcal{B}$. Let further $A_0 \in S_\gamma$ be such that $x_1, \ldots, x_n \in T(A_0)$.

We observe that

$$
\|P(x)\| = \|M(\phi_x)\| = \|M(\phi_x(\cdot \cup A_0))\| \leq \lambda \sup_{A \in S_\gamma} \|\phi_x(A \cup A_0)\| = \lambda \sup_{A \in S_\gamma} \left\| \sum_{i=1}^n \lambda_i P^\epsilon_{\text{span}T(A \cup A_0)}(x_i) \right\|
$$

$$
= \lambda \sup_{A \in S_\gamma} \left\| P^\epsilon_{\text{span}T(A \cup A_0)} \left( \sum_{i=1}^n \lambda_i x_i \right) \right\| = \lambda \sup_{A \in S_\gamma} \left\| P^\epsilon_{\text{span}T(A \cup A_0)}(x) \right\|
$$

$$
\leq \lambda \sup_{A \in S_\gamma} \left( 1 + \epsilon_{A \cup A_0} \right) \|x\| \leq \lambda \left( 1 + \frac{1}{1 + |A_0|} \right) \|x\|.
$$

Since $A_0$ is arbitrary, we get $\|P(x)\| \leq \lambda\|x\|$.

Moreover, if $x \in \kappa(X)$, then from the properties of $\mathcal{B}$ we get $x_1, \ldots, x_n \in \kappa(X)$ and

$$
\phi_x(A \cup A_0) = \sum_{i=1}^n \lambda_i \phi_{x_i}(A \cup A_0) = \sum_{i=1}^n \lambda_i P^\epsilon_{\text{span}T(A \cup A_0)}(x_i)
$$

$$
= \sum_{i=1}^n \lambda_i x_i = x, \quad A \in S_\gamma.
$$

Hence

$$
P(x) = M(\phi_x) = M(\phi_x(\cdot \cup A_0)) = x.
$$

\qed
Theorem 3.2. Let $S$ be a commutative cancellative semigroup of torsion-free rank $\delta$, $\gamma = \max(\delta, \omega)$. If there exists an $X$-valued invariant $\lambda$-mean $M_S: \ell_\infty(S, X) \to X$, then there exists an $X$-valued invariant $\lambda$-mean $M: \ell_\infty(S_\gamma, X) \to X$.

Proof. First we observe that we can assume that $S$ contains only elements of infinite order. Indeed the set $G$ of all elements of finite order is a group and a torsion-free rank of $S/G$ is equal to $\gamma$. In view of Lemma 2.4 there exists an $X$-valued invariant $\lambda$-mean on $S/G$.

Let $A \subset S$ be a maximal linearly independent set. Hence $|A| = \delta$.

- First assume that $|A| = \gamma$ and let $A = \{x_\alpha : \alpha < \gamma\}$. For each $x \in S$ we define a set

$$D_x := \{x_1, \ldots, x_n \in A : \exists k, k_1, \ldots, k_n \in \mathbb{N} \exists I \subset \{1, \ldots, n\} kx + \sum_{i \in I} k_i x_i = \sum_{i \notin I} k_i x_i\}.$$

First, we show that the above set is well-defined. If there exist $k, m \in \mathbb{N}$, $k_1, \ldots, k_n, m_1, \ldots, m_n \in \mathbb{N} \cup \{0\}$, $x_1, \ldots, x_n \in A$, and $I, J \subset \{1, \ldots, n\}$ such that $k_i \neq 0$ for $i \in I$, $m_i \neq 0$ for $i \in J$ and

$$kx + \sum_{i \in I} k_i x_i = \sum_{i \notin I} k_i x_i,$$

then

$$mkx + \sum_{i \in I} mk_i x_i + \sum_{i \notin I} km_i x_i = kmx + \sum_{i \in J} km_i x_i + \sum_{i \notin J} mk_i x_i,$$

whence

$$\sum_{i \in I} mk_i x_i + \sum_{i \notin J} km_i x_i = \sum_{i \in J} km_i x_i + \sum_{i \notin I} mk_i x_i,$$

so

$$\sum_{i \in I \cap J} mk_i x_i + \sum_{i \in I \setminus J} (mk_i + km_i) x_i + \sum_{i \notin I \cup J} km_i x_i = \sum_{i \in J \setminus I} km_i x_i + \sum_{i \notin J \cup I} (km_i + mk_i) x_i + \sum_{i \notin I \cup J} mk_i x_i.$$

As $A$ is linearly independent, we have $I \setminus J = J \setminus I = \emptyset$, which means that $I = J$. Thus we get that $km_i = mk_i$ for $i \in \{1, \ldots, n\}$, so $D_x$ is well-defined.

We define a map $\varphi: \ell_\infty(S_\gamma, X) \to \ell_\infty(S, X)$ by the formula

$$\varphi(f)(x) := f(\{\alpha < \gamma : x_\alpha \in D_x\}), \ x \in S, \ f \in \ell_\infty(S_\gamma, X).$$

It is easy to observe that $\varphi$ is linear, $\|\varphi(f)\| = \|f\|$ for $f \in \ell_\infty(S_\gamma, X)$ and $\varphi(c1_{S_\gamma}) = c1_S$ for $c \in X$. 
Let $M_{S}: \ell_{\infty}(S, X) \to X$ be an $X$-valued invariant $\lambda$-mean. We define $M: \ell_{\infty}(S_{\gamma}, X) \to X$ by the formula

$$M(f) := M_{S}(\varphi(f)), \ f \in \ell_{\infty}(S_{\gamma}, X).$$

From the properties of $\varphi$ we obtain that $M$ is linear, $M(c \mathbb{1}_{S_{\gamma}}) = c$ for $c \in X$, and $\|M\| \leq \|M_{S}\| \leq \lambda$.

Now we show that $M$ is invariant. Let $f \in \ell_{\infty}(S_{\gamma}, X)$ and $A \in S_{\gamma}$. Since $A = \{\alpha_{1}, \ldots, \alpha_{n}\}$, from the invariance on each singleton $\{\alpha_{i}\}$ we obtain

$$M(Af) = M(\{\alpha_{1}\}(\{\alpha_{2}, \ldots, \alpha_{n}\} f)) = M(\{\alpha_{2}, \ldots, \alpha_{n}\} f) = \ldots = M(\{\alpha_{n}\} f) = M(f), \ f \in \ell_{\infty}(S_{\gamma}, X).$$

Hence we need to prove the invariance on each singleton, so we can assume that $A = \{\beta\}$ for some $\beta < \gamma$. Let $Z := \{x \in S : x_{\beta} \notin D_{x+x_{\beta}}\}$. We show that

$$Z \cap (m_{x_{\beta}} + Z) = \emptyset, \ m \in \mathbb{N}. \quad (3.1)$$

Suppose that $x \in Z \cap (m_{x_{\beta}} + Z)$ for some $m \in \mathbb{N}$. Then there exists $y \in Z$ such that $x = m_{x_{\beta}} + y$. Hence $x_{\beta} \notin D_{y+x_{\beta}} \cup D_{y+(m+1)x_{\beta}}$ but on the other hand, if $x_{\beta} \notin D_{y+x_{\beta}}$, then $x_{\beta} \in D_{y+(m+1)x_{\beta}}$, so we have a contradiction.

Since $S$ is cancellative, from (3.1) we obtain that

$$(nx_{\beta} + Z) \cap (mx_{\beta} + Z) = \emptyset, \ m, n \in \mathbb{N}_{0}, \ m > n. \quad (3.2)$$

Let $g \in \ell_{\infty}(S, X)$ be such that $g(x) = 0$ for $x \in S \setminus Z$. From (3.2) we get

$$n\|M_{S}(g)\| = \|\sum_{i=1}^{n} M_{S}(ix_{\beta}g)\| = \|M_{S}(\sum_{i=1}^{n} ix_{\beta}g)\| \leq \lambda\|\sum_{i=1}^{n} ix_{\beta}g\| \leq \lambda\|g\|, \ n \in \mathbb{N},$$

so $M_{S}(g) = 0$.

For each $y \in S$ we have

- if $x_{\beta} \notin D_{y}$, then $D_{y+x_{\beta}} = D_{y} \cup \{x_{\beta}\}$, so

$$\varphi(\{\beta\}f)(y) = f(\{\alpha < \gamma : x_{\alpha} \in D_{y}\} \cup \{\beta\}) = f(\{\alpha < \gamma : x_{\alpha} \in D_{y+x_{\beta}}\}) = (x_{\beta}\varphi(f))(y);$$

- if $x_{\beta} \in D_{y}$ and $x_{\beta} \in D_{y+x_{\beta}}$, then $D_{y+x_{\beta}} = D_{y}$, so

$$\varphi(\{\beta\}f)(y) = f(\{\alpha < \gamma : x_{\alpha} \in D_{y}\} \cup \{\beta\}) = f(\{\alpha < \gamma : x_{\alpha} \in D_{y}\}) = f(\{\alpha < \gamma : x_{\alpha} \in D_{y+x_{\beta}}\}) = (x_{\beta}\varphi(f))(y);$$

- if $x_{\beta} \notin D_{y+x_{\beta}}$, then $y \in Z$. 


Hence

$$(\varphi(\{\beta\}f) - x_\beta \varphi(f))(y) = 0, \ y \in S \setminus Z,$$

so

$$M(\{\beta\}f) = M_S(\varphi(\{\beta\}f) = M_S(x_\beta \varphi(f)) = M_S(\varphi(f)) = M(f).$$

- Now assume that $|A| < \gamma$. Hence $\gamma = \omega$. Let $N = |A|$, $A = \{x_1, \ldots, x_N\}$. Since $S$ can be embedded in a group, for each $x \in S$ there exist $k(x) \in \mathbb{N}$, $k_1(x), \ldots, k_N(x) \in \mathbb{Z}$ such that $k(x)x = \sum_{i=1}^{N} k_i(x)x_i$. We define a map

$$\varphi: \ell_\infty(S_\omega, X) \to \ell_\infty(S, X)$$

by the formula

$$\varphi(f)(x) := f(\{\alpha \in \omega : \alpha k(x) \leq |k_1(x)|\})$$

$x \in S$, $f \in \ell_\infty(S_\omega, X)$.

It is easy to observe that $\varphi$ is linear, $\|\varphi(f)\| \leq \|f\|$ for $f \in \ell_\infty(S_\omega, X)$ and $\varphi(c1_{S_\omega}) = c1_{S}$ for $c \in X$.

Let $M_S: \ell_\infty(S, X) \to X$ be an $X$-valued invariant $\lambda$-mean. We define $M: \ell_\infty(S_\omega, X) \to X$ by the formula

$$M(f) := M_S(\varphi(f)), \ f \in \ell_\infty(S_\omega, X).$$

From the properties of $\varphi$ we obtain that $M$ is linear, $M(c1_{S_\omega}) = c$ for $c \in X$, and $\|M\| \leq \lambda$.

Now we show that $M$ is invariant. Let $f \in \ell_\infty(S_\omega, X)$ and $A \in S_\omega$. Similarly as in the previous case we need only to prove the invariance on each singleton, so we can assume that $A = \{\beta\}$ for some $\beta \in \omega$. Let

$$Z := \left\{ x \in S : |k_1(x)| < \beta k(x) \right\}.$$ 

We show that

$$Z \cap (2m\beta x_1 + Z) = \emptyset, \ m \in \mathbb{N}. \hspace{1cm} (3.3)$$

Suppose that $x \in Z \cap (mx_\beta + Z)$ for some $m \in \mathbb{N}$. Then there exists $y \in Z$ such that $x = 2m\beta x_1 + y$. Hence

$$k(y)[y + 2m\beta x_1] = [k_1(y) + 2mk(y)\beta]x_1 + \sum_{i=2}^{N} k_i(y)x_i,$$

which gives us

$$\beta k(y) > k_1(y) + 2m\beta k(y) > -\beta k(y) + 2\beta k(y) = \beta k(y),$$

so we have a contradiction.

Since $S$ is cancellative, from (3.3) we obtain that

$$(2n\beta x_1 + Z) \cap (2m\beta x_1 + Z) = \emptyset, \ m, n \in \mathbb{N}_0, \ m > n. \hspace{1cm} (3.4)$$
Now observe that for $x \in S \setminus Z$ we have
\[
\varphi(f_{\{\beta\}}(x)) = f_{\{\beta\}}\left(\{\alpha \in \omega : \alpha k(x) \leq |k_1(x)|\}\right)
\]
\[
= f\left(\{\alpha \in \omega : \alpha k(x) \leq |k_1(x)|\} \cup \{\beta\}\right)
\]
\[
= f\left(\{\alpha \in \omega : \alpha k(x) \leq |k_1(x)|\}\right) = \varphi(f(x)),
\]
so from (3.4) we obtain that
\[
n\|M(f - f_{\{\beta\}})\| = n\|MS(\varphi(f) - \varphi(f_{\{\beta\}}))\|
\]
\[
= \|\sum_{i=1}^{n} MS((\varphi(f) - \varphi(f_{\{\beta\}}))_{2i}\beta x_1)\|
\]
\[
= \|MS\left(\sum_{i=1}^{n} (\varphi(f) - \varphi(f_{\{\beta\}}))_{2i}\beta x_1\right)\| \leq \lambda \|\varphi(f) - \varphi(f_{\{\beta\}})\|
\]
for every $n \in \mathbb{N}$, which means that $M(f_{\{\beta\}}) = M(f)$.

\[\square\]

Using Theorems 1.4, 3.1 and 3.2 we obtain the following

**Corollary 3.3.** The following assertions are equivalent:

1. $X$ is complemented in $X^{**}$ by a projection of norm at most $\lambda$;
2. for every amenable semigroup $S$ there exists an $X$-valued invariant $\lambda$-mean on $S$;
3. for any cancellative semigroup $S$ of torsion-free rank $\delta$, $\text{dens} X^{**} = \max(\delta, \omega)$, there exists an $X$-valued invariant $\lambda$-mean on $S$.

The following example shows that in general in the third assertion of the previous corollary the torsion-free rank of semigroup $S$ cannot be less than the density of $X$.

**Example 3.4.** Let $\Gamma$ be an uncountable set such that $|\Gamma|$ is a regular cardinal number. We define the set
\[
X := \{f \in \ell_\infty(\Gamma) : |\{\alpha \in \Gamma : f(\alpha) \neq 0\}| < |\Gamma|\}.
\]
It is easy too see that $X$ is a Banach space. Since $1_{\{\alpha\}} \in X$ for $\alpha \in \Gamma$, $\text{dens} X = |\Gamma|$.

Let $S$ be an amenable semigroup, $|S| < \text{dens} X$ and $L : \ell_\infty(S, \mathbb{R}) \to \mathbb{R}$ be an invariant mean. We define $M : \ell_\infty(S, X) \to X$ by the formula
\[
M(g)(\alpha) := L(g(\cdot)(\alpha)), \ g \in \ell_\infty(S, X), \ \alpha \in \Gamma.
\]
First, we observe that
\[
\{\alpha \in \Gamma : M(g)(\alpha) \neq 0\} = \{\alpha \in \Gamma : L(g(\cdot)(\alpha)) \neq 0\}
\]
\[
\subset \bigcup_{s \in S} \{\alpha \in \Gamma : g(s)(\alpha) \neq 0\}
\]
and since $|\Gamma|$ is regular, we have

$$|\{\alpha \in \Gamma : M(g)(\alpha) \neq 0\}| \leq |S| \cdot \sup_{s \in S} |\{\alpha \in \Gamma : g(s)(\alpha) \neq 0\}| < |\Gamma|,$$

so $M$ is well-defined.

It is easy to see that $M$ is linear. We have also

$$\|M(g)\| = \sup_{\alpha \in \Gamma} |M(g)(\alpha)| = \sup_{\alpha \in \Gamma} |L(g(\cdot))(\alpha)|$$

$$\leq \sup_{\alpha \in \Gamma} \sup_{s \in S} |g(s)(\alpha)| = \sup_{s \in S} \|g(s)\| = \|g\|, \ g \in \ell_\infty(S, X),$$

and

$$M(c \mathbb{1}_S)(\alpha) = L(c(\alpha) \mathbb{1}_S) = c(\alpha), \ c \in X, \ \alpha \in \Gamma.$$ 

Finally, we observe that

$$M(\alpha g)(\alpha) = L(g(\cdot + \alpha)(\alpha)) = L(g(\cdot))(\alpha)$$

$$= M(g)(\alpha), \ g \in \ell_\infty(S, X), \ \alpha \in S, \ \alpha \in \Gamma,$$

so $M$ is an $X$-valued invariant mean.

In the paper of Pelczyński and Sudakov [13, Theorem 1] it is shown that $X$ isn’t complemented in its bidual.

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