Strong solutions for jump-type stochastic differential equations with non-Lipschitz coefficients

Zhun Gou, Ming-hui Wang and Nan-jing Huang

Department of Mathematics, Sichuan University, Chengdu, People's Republic of China

ABSTRACT
In this paper, the existence and pathwise uniqueness of strong solutions for jump-type stochastic differential equations are investigated under non-Lipschitz conditions. A sufficient condition is obtained for ensuring the non-confluent property of strong solutions of jump-type stochastic differential equations. Moreover, some examples are given to illustrate our results.

1. Introduction
Jump-type stochastic differential equations (JSDEs), as natural extensions of stochastic differential equations (SDEs), have been widely applied to many fields of science and engineering such as physics, astronomy, finance, ecology, biology and so on. As for the applications in physics, Chudley and Elliott [5] applied JSDEs to describe atomic diffusion typically consists of jumps between vacant lattice sites. Bergquist et al. [1] illustrated the quantum jumps in a single atom by JSDEs. Gleyzes et al. [13] employed JSDEs to analyze the observation that the microscopic quantum system exhibits at random times sudden jumps between its states. Pellegrini [23] proved the existence and uniqueness of a solution for the jump-type stochastic Schrödinger equations. As for the applications in finance, Shreve [28] and Tankov [29] have enumerated many financial models which can be described by JSDEs. Some related work, we refer to [24–26]. Thus, it would be necessary to study some properties of solutions to JSDEs. In this paper, we mainly investigate some qualitative properties of solutions to JSDEs under non-Lipschitz conditions.

The linear growth condition guarantees that the solutions for JSDEs has no finite explosion time with probability one. However, the linear growth condition may not be satisfied in some practical situations. For instance, in the mathematical ecological models of [17,21], the coefficients do not satisfy the linear growth condition while non-explosion is still guaranteed. Some non-explosive results for general SDEs without jumps under the linear growth condition can be found in [8,9,18]. Thus, one natural question is: can we relax the linear growth condition for JSDEs? The first task of this paper is to provide a new sufficient super linear growth condition for ensuring the non-explosion of strong solutions for JSDEs.
In general, the usual method for studying the pathwise uniqueness of strong solutions for SDEs with Lipschitz conditions is to employ Gronwall’s inequality to demonstrate that the distance \(E^{1/2}[|\tilde{X}(t) - X(t)|^2]\) between two solutions \(\tilde{X}(t)\) and \(X(t)\) vanishes [15]. Unfortunately, as pointed out by Fang and Zhang [10], the usual method employed in the previous literature is not applicable without the usual Lipschitz condition. In 1971, Yamada and Watanabe [32] showed that the Lipschitz condition can be relaxed to the Hölder condition for one-dimensional SDEs. Recently, the pathwise uniqueness of strong solutions for SDEs with non-Lipschitz conditions has been studied by many authors (see, for example [18,27]). However, to the best of our knowledge, there are only a few papers dealing with the pathwise uniqueness of strong solutions for JSDEs with non-Lipschitz conditions (see [11,20]). The second task of this paper is to give a new non-lipschitz condition to guarantee the pathwise uniqueness of strong solutions to JSDEs.

On the other hand, the closely related non-confluent property (also known as the non-contact property) of strong solutions for SDEs with the Lipschitz condition has been studied by several authors (see, for example, [7,30] and the references therein). Moreover, some sufficient conditions are derived for ensuring the non-confluent property of strong solutions for SDEs without jumps with non-lipschitz coefficients in [10,18]. However, the non-confluent property of strong solutions for SDEs with jumps had not been studied until a sufficient condition was established by Xi and Zhu [31]. The third task of this paper is to give a new sufficient condition for ensuring the non-confluent property of strong solutions for SDEs with jumps.

The rest of this paper is structured as follows. Section 2 presents some necessary preliminaries including assumptions and lemmas. In Section 3, we obtain main results concerned with the non-explosion and pathwise uniqueness of strong solutions for JSDEs with super linear growth and non-Lipschitz conditions. Before concluding this paper, the non-confluent property of strong solutions for JSDEs is investigated in Section 4.

2. Preliminaries

Let \(\{p_1(t)\}\) and \(\{p_2(t)\}\) be two \(\mathcal{F}_t\)-Poisson point processes on \(U_1\) and \(U_2\) with characteristic measures \(v_1(du)\) and \(v_2(du)\), respectively, such that \(\{B_t\}, \{p_1(t)\}, \{p_2(t)\}\) are independent of each other. Let \(N_1(ds, du)\) and \(N_2(ds, du)\) be Poisson random measures associated with \(\{p_1(t)\}\) and \(\{p_2(t)\}\), respectively. Moreover, suppose that \(b : \mathbb{R} \to \mathbb{R}\) and \(\sigma : \mathbb{R} \to \mathbb{R}\) are two continuous functions, \(c_1 : \mathbb{R} \times U_1 \to \mathbb{R}\) and \(c_2 : \mathbb{R} \times U_2 \to \mathbb{R}\) are two Borel functions.

In this paper, we consider the following JSDE:

\[
X(t) = X_0 + \int_0^t \sigma(X(s)) \, dB_s + \int_0^t \int_{U_1} c_1(X(s-), u)N_1(ds, du) \\
+ \int_0^t b(X(s)) \, ds + \int_0^t \int_{U_2} c_2(X(s-), u)N_2(ds, du)
\]  

(1)

with \(\mathbb{E}[|X_0|^2] < \infty\), where

\[
\tilde{N}_1(dt, du) = N_1(dt, du) - v(dz) \, dt
\]
Lemma 2.1 ([11]): Following JSDE

\[ \text{naturalfiltration generated by } \{B_t\} \]

\[ \{ \text{countably many } N_t \} \text{ is the compensated Poisson random measure of } N_t \]

Assumption 2.2: Assume that there exists a non-decreasing and continuously differentiable function \( \rho: [0, \infty) \to [0, \infty) \) such that \( \rho(x) > 0 \) for \( x > 0 \) satisfying

\[ \int_{0+} \frac{ds}{\rho(s)} = \infty. \]  

(3)

Clearly, the following functions satisfy (3):

\[ \rho(x) = x(x > 0); \quad \rho(x) = -x \ln x \left( 0 < x \leq \frac{1}{e} \right); \]

\[ \rho(x) = x \ln(- \ln x) \left( 0 < x \leq \frac{1}{e} \right); \quad \rho(x) = 1 - x^2 \left( 0 < x \leq \frac{1}{e} \right). \]

Assumption 2.3: Suppose that there exists a constant \( \delta > 0 \) such that, for any \( x, y \in \mathbb{R} \) with \( 0 < |x - y| \leq \delta_0 \),
(i) \(\max\{(x - y)(b(x) - b(y)), |\sigma(x) - \sigma(y)|^2\} \leq |x - y|^{2 - \alpha} \rho(|x - y|^{\alpha})\);

(ii) \(\int_{U_1} \max\{|c_1(x, u) - c_1(y, u)|^{\alpha}, |x - y|^{\alpha - 1} \cdot |c_1(x, u) - c_1(y, u)|\} v_1(du) \leq \rho(|x - y|^{\alpha})\);

(iii) \(\int_{U_2} \max\{|c_2(x, u) - c_2(y, u)|^{\alpha}, |x - y|^{\alpha - 1} \cdot |c_2(x, u) - c_2(y, u)|\} v_2(du) \leq \rho(|x - y|^{\alpha})\).

Here \(0 \leq \alpha < +\infty\) is a fixed constant and \(\rho\) is defined in Assumption 2.1.

**Remark 2.1:** In particular, Assumption 2.3 reduces to the Lipschitz case when \(\alpha = 0\). Thus it is sufficient to consider the situation for \(0 < \alpha < +\infty\).

**Assumption 2.4:** Assume that there exist two non-decreasing, continuous functions \(\rho_1, \rho_2 : [0, \infty) \rightarrow [0, \infty)\) satisfying:

\[\int_{0^+} \frac{ds}{\rho_i(s)} = \infty, \quad i = 1, 2,\]

where \(\rho_1\) is concave. In addition, suppose that there exists a constant \(\delta_0 > 0\) such that, for any \(x, y \in \mathbb{R}\) with \(0 < |x - y| \leq \delta_0\),

(i) \((x - y)(b(x) - b(y)) + \int_{U_1} |c_2(x, u) - c_2(y, u)| v_2(du) \leq |x - y|\rho_1(|x - y|),\)

(ii) \(|\sigma(x) - \sigma(y)|^2 + \int_{U_1} |c_1(x, u) - c_1(y, u)|^2 v_1(du) \leq \rho_2(|x - y|)\).

Here \(c_1(x, u)\) is non-decreasing for each fixed \(u\).

**Assumption 2.5:** Suppose that

\[v_i(u \in U : \text{there exist } x, y \in \mathbb{R} \text{ with } x \neq y \text{ such that } |x - y + c_i(x, u) - c_i(y, u)| \leq \delta|x - y|) = 0, \quad i = 1, 2,\]

where \(\delta > 0\) is a fixed constant. In addition, assume that, for any \(x, y \in \mathbb{R}\),

(i) \((x - y)(b(x) - b(y)) \leq |x - y|^{2 + \alpha} \rho(|x - y|^{-\alpha});\)

(ii) \(|\sigma(x) - \sigma(y)|^2 \leq |x - y|^{2 + \alpha} \rho(|x - y|^{-\alpha});\)

(iii) \(\int_{U_1} |c_i(x, u) - c_i(y, u)| v_i(du) \leq |x - y|^{1 + \alpha} \rho(|x - y|^{-\alpha}), \quad i = 1, 2,\)

Here \(0 \leq \alpha < +\infty\) is a fixed constant and \(\rho\) is defined in Assumption 2.1.

**Remark 2.2:** If \(\alpha = 0\), then Assumption 2.5 reduces to the assumption in Corollary 3.3 in [31].

In order to obtain our main results, we also need the following lemmas.
Lemma 2.2 ([22]): Suppose that $X(t) \in \mathbb{R}$ is an Itô-Lévy process of the following form:

$$dX(t) = b(t, \omega) \, dt + \sigma(t, \omega) \, dB_t + \int_{\mathbb{R}} \gamma(t, u, \omega) \overline{N}(dt, du),$$

where

$$\overline{N}(x) = \begin{cases} N(dt, du) - \nu(du) \, dt, & \text{if } |u| < R; \\ N(dt, du), & \text{if } |u| \geq R \end{cases}$$

for some $R \in [0, +\infty)$. Let $f \in C^2(\mathbb{R}^2)$ and define $Y(t) = f(t, X(t))$. Then $Y(t)$ is again an Itô-Lévy process and

$$dY(t) = \frac{\partial f}{\partial t}(t, X(t)) \, dt + \frac{\partial f}{\partial x}(t, X(t)) \left[ b(t, \omega) \, dt + \sigma(t, \omega) \, dB_t \right]$$

$$+ \frac{1}{2} \sigma^2(t, X(t)) \frac{\partial^2 f}{\partial x^2}(t, X(t))$$

$$+ \int_{|z| < R} \left\{ f(t, X(t-)) + \gamma(t, u) - f(t, X(t-)) - \frac{\partial f}{\partial x}(t, X(t-)) \gamma(t, u) \right\} \nu(du) \, dt$$

$$+ \int_{\mathbb{R}} \left\{ f(t, X(t-)) + \gamma(t, u) - f(t, X(t-)) \right\} \overline{N}(dt, du).$$

In the sequel, for any $f \in C^n(\mathbb{R})$, we will replace $\left( \partial^n / \partial^n x \right) f(x)$ by $D^n f(x)$ for convenience.

Lemma 2.3: Let $u(t)$ and $g(t)$ be non-negative continuous functions, and $f(t)$ a non-negative continuously differentiable and non-decreasing function for all $t \geq 0$. Furthermore, suppose that $\rho : [0, +\infty) \to [0, +\infty)$ is a non-negative and non-decreasing continuous function with

$$\rho(t) = 0 \iff t = 0 \quad \text{and} \quad \int_{0+} \frac{ds}{\rho(s)} = \infty.$$

Then the inequality

$$u(t) \leq f(t) + \int_0^t g(s) \rho(u(s)) \, ds$$

implies the inequality

$$u(t) \leq \Omega^{-1} \left[ \Omega(f(t)) + \int_0^t g(s) \, ds \right],$$

where

$$\Omega(t) = \int_0^t \frac{ds}{\rho(s)}, \quad \forall \ t > 0.$$
Proof: Let
\[ v(t) = f(t) + \int_0^t g(s) \rho(u(s)) \, ds \]
and
\[ \phi(t) = \Omega(v(t)) - \Omega(f(t)) - \int_0^t g(s) \, ds. \]
Then \( \max\{u(t), f(t)\} \leq v(t) \).
By direct computations, we have
\[
\phi'(t) = \frac{f'(t) + g(t) \rho(u(t))}{\rho(v(t))} - \frac{f'(t)}{\rho(f(t))} - g(t)
\]
\[
= \frac{\rho(f(t)) - \rho(v(t))}{\rho(f(t)) \cdot \rho(v(t))} f'(t) + g(t) \frac{\rho(u(t)) - \rho(v(t))}{\rho(v(t))}
\]
\[
\leq 0.
\]
This shows that \( \phi(t) \) is non-increasing and
\[
\phi(t) = \Omega \left( f(t) + \int_0^t g(s) \rho(u(s)) \, ds \right) - \Omega(f(t)) - \int_0^t g(s) \, ds \leq \phi(0) = 0.
\]
Moreover, since \( \Omega(t) \) is increasing, one has
\[
\Omega(u(t)) \leq \Omega \left( f(t) + \int_0^t g(s) \rho(u(s)) \, ds \right) \leq \Omega(f(t)) + \int_0^t g(s) \, ds
\]
and so
\[
u(t) \leq \Omega^{-1} \left( \Omega(f(t)) + \int_0^t g(s) \, ds \right).
\]
On the other hand, define \( \Psi(t) = \int_1^t (ds/\rho(s)) \). Then \( \Psi(t) \) is increasing and satisfies \( \Psi(0) = -\infty \) since \( \int_0^t (ds/\rho(s)) = \infty \). Letting \( f(t) = 0 \), it follows that
\[
\Psi(u(t)) \leq \Psi(v(t)) = G(0) + \int_0^t \Psi'(v(s)) v'(s) \, ds = \Psi(0)
\]
\[
+ \int_0^t \frac{\Psi(u(s))}{\Psi(v(s))^2} g(s) \, ds \leq \Psi(0) + \int_0^t g(s) \, ds.
\]
Since \( \Psi(0) = -\infty \) and \( \int_0^t g(s) \, ds < +\infty \), we have \( \Psi(u(t)) = -\infty \) and \( u(t) = 0 \) consequently.

Remark 2.3: If \( f(t) = k > 0 \) is a constant function, then Lemma 2.3 reduces to the corresponding result in [3].
3. Non-explosion and Pathwise Uniqueness

Theorem 3.1: Under Assumption 2.2, the solutions for JSDE (1) do not explode in finite time.

Proof: Define
\[ \phi(x) = \exp \left\{ \int_0^x \frac{ds}{sY(s) + 1} \right\}, \quad x \geq 0. \]

By simple computations, we have
\[ \phi'(x) = \phi(x) \frac{1}{xY(x) + 1} \geq 0, \quad \phi''(x) = \phi(x) \frac{1 - Y(x) - xY'(x)}{(xY(x) + 1)^2} \leq 0. \]

Clearly, \( \phi(x) \) is a concave function with \( \phi(x) \to \infty \) as \( x \to \infty \). Moreover,
\[ D^{(1)} \phi(x^2) = \phi'(x^2) \cdot 2x, \quad D^{(2)} \phi(x^2) = 2\phi'(x^2) + \phi''(x^2) \cdot 4x^2. \]

Since \( \phi''(x) \leq 0 \), we know that
\[ \phi(y) \leq \phi(x) + (y - x)\phi'(x), \quad \forall x, y \in [0, \infty). \]

It follows that
\[ \phi \left( (X(s-) + c_1 (X(s-), u))^2 \right) - \phi(X^2(s-)) - D^{(1)} \phi \left( (X^2(s-)) \cdot c_1(X(s-), u) \right) \leq \phi'(X^2(s-)) \left[ 2X(s-)c_1(X(s-)) + c_1^2(X(s-), u) \right] - \phi'(X^2(s-)).2X(s-)c_1 (X(s-), u) \]
\[ = \phi'(X^2(s-))c_1^2(X(s-), u). \]

Let
\[ \tau_R := \inf \{ t \geq 0 : \text{max}(|\bar{X}(t)|, |X(t)|) \geq R \}. \]

Applying Lemma 2.2, one has
\[ \mathbb{E} \left[ \phi(X^2(t \wedge \tau_R)) \right] \]
\[ = \mathbb{E} [\phi(|X_0|^2)] + \mathbb{E} \left[ \int_0^{t \wedge \tau_R} D^{(1)} \phi(X^2(s))b(X(s)) + \frac{1}{2} \sigma^2(X(s))D^{(2)} \phi(X^2(s)) \, ds \right] \]
\[ + \mathbb{E} \left[ \int_0^{t \wedge \tau_R} \int_{U_1} \phi(|X(s-) + c_1(X(s-), u)|^2) - \phi(X^2(s-)) \right. \]
\[ - D^{(1)} \phi(|X^2(s-)|) \cdot c_1(X(s-), u) v_1(du) \, ds \]
\[ + \mathbb{E} \left[ \int_0^{t \wedge \tau_R} \int_{U_2} \phi \left( (X(s-) + c_2(X(s-), u))^2 \right) - \phi((X^2(s-))v_2(du) \, ds \right]. \]
\[ \begin{align*}
&\leq \mathbb{E}[\phi(|X_0|^2)] + \mathbb{E}\left[ \int_0^{t\wedge \tau_R} \phi'(X^2(s)) \cdot [2X(s)b(X(s)) + \sigma^2(X(s))] \right. \\
&+ 2(X^2(s))\phi''(X^2(s))\sigma^2(X(s)) \, ds \\
&+ \int_{U_1} \phi'(X^2(s-)) \cdot |c_1(X(s-), u)|^2 v_1(du) \, ds \\
&\left. + \mathbb{E}\left[ \int_0^{t\wedge \tau_R} \int_{U_3} \phi'(X^2(s-)) \cdot (2|c_2(X(s-), u)|^2 + X^2(s-))v_2(du) \, ds \right] \right].
\end{align*} \]

Since \( \phi''(x) \leq 0 \) and \( \int_{U_3} 1 \nu_2(du) \leq \int_{U_2} 1 \nu_2(du) \leq M \), by Assumption 2.2, we have
\[ \mathbb{E}\left[ \phi(X^2(t \wedge \tau_R)) \right] \leq \phi \left( \mathbb{E}[|X_0|^2] \right) + \mathbb{E}\left[ \int_0^{t\wedge \tau_R} \phi'(X^2(s)) \cdot (M + 1)\mu [X^2(s)\Upsilon(X^2(s)) + 1] \, ds \right] \]
\[ = \phi \left( \mathbb{E}[|X_0|^2] \right) + (M + 1)\mu \int_0^t \mathbb{E}[\phi(X^2(s \wedge \tau_R))] \, ds. \]

Thus, it follows from Gronwall’s inequality that
\[ \mathbb{E}\left[ \phi(X^2(t \wedge \tau_R)) \right] \leq \phi \left( \mathbb{E}[|X_0|^2] \right) \cdot e^{\mu(M+1)t}. \]

Letting \( |X(t \wedge \tau_R)| \to \infty \), we have \( t \to \infty \) as \( \mathbb{E}[|X_0|^2] < \infty \). Therefore, the solution has no finite explosion time.

\textbf{Theorem 3.2:} Under Assumptions 2.2 and 2.3, the pathwise uniqueness of strong solutions for JSDE (2) holds.

\textbf{Proof:} By the assumptions imposed on \( \rho \), we can find a strictly decreasing sequence \( \{a_n\} \subset (0, 1] \) such that

(i) \( a_0 = 1 \);
(ii) \( \lim_{n \to \infty} a_n = 0 \);
(iii) \( \int_{a_n}^{a_{n-1}} (1/\rho(r)) \, dr = n \) for every \( n \geq 1 \).

Clearly, for each \( n \geq 1 \), there exists a continuous function \( \rho_n \) on \( \mathbb{R} \) such that

(i) \( \rho_n(r) \) has a supported set \( (a_n, a_{n-1}) \);
(ii) \( 0 \leq \rho_n(r) \leq 2/n \rho(r) \) for every \( r > 0 \);
(iii) \( \int_{a_n}^{a_{n-1}} \rho_n(r) \, dr = 1 \).

Now we consider the following sequence of functions:
\[ \psi_n(r) = \int_0^{|r|} \int_0^r \rho_n(u) \, du \, dv, \quad r \in \mathbb{R}, \ n \geq 1. \]
Clearly, $\psi_n$ is even and twice continuously differentiable (except at $r = 0$) with the following properties:

(i) $|\psi'_n(r)| \leq 1$, $r \neq 0$;
(ii) $\lim_{n \to \infty} \psi_n(r) = |r|$, $r \neq 0$;
(iii) $\psi''_n(r) \leq (2/n\rho(r))I(a_n,a_{n-1})(r)$, $r \neq 0$.

Furthermore, for each $r > 0$, the sequence $\{\psi_n(r)\}_{n \geq 1}$ is non-decreasing. Note that for each $n \in \mathbb{N}$, $\psi_n$, $\psi'_n$ and $\psi''_n$ all vanish on the interval $(-a_n,a_n)$. By direct computations, we have, for $0 \neq x \in \mathbb{R}$,

$$D\psi_n(|x|^\alpha) = \frac{d}{dx} \psi_n(|x|^\alpha) = \psi'_n(|x|^\alpha) \cdot \alpha x \cdot |x|^\alpha - 2$$

and

$$D^2\psi_n(|x|^\alpha) = \psi''_n(|x|^\alpha) \cdot \alpha^2 |x|^{2\alpha - 2} + \psi'_n(|x|^\alpha) \cdot \alpha(\alpha - 1)|x|^\alpha - 2.$$ 

Next we suppose that $X$ and $\tilde{X}$ are two solutions for (2) of the following forms:

$$X(t) = X_0 + \int_0^t \sigma(X(s)) \, dB_s + \int_0^t \int_{U_1} c_1(X(s-), u) \tilde{N}_1(ds, du)$$

$$+ \int_0^t b(X(s)) \, ds + \int_0^t \int_{U_2} c_2(X(s-), u) N_2(ds, du)$$

and

$$\tilde{X}(t) = X_0 + \int_0^t \sigma(\tilde{X}(s)) \, dB_s + \int_0^t \int_{U_1} c_1(\tilde{X}(s-), u) \tilde{N}_1(ds, du)$$

$$+ \int_0^t b(\tilde{X}(s)) \, ds + \int_0^t \int_{U_2} c_2(\tilde{X}(s-), u) N_2(ds, du)$$

for all $t \geq 0$, where $x, \tilde{x} \in \mathbb{R}$.

Denote $\Delta_t := \tilde{X}(t) - X(t)$ for all $t \geq 0$ and define

$$S_{\delta_0} = \inf \{t \geq 0 : |\Delta_t| \geq \delta_0\} = \inf \{t \geq 0 : |\tilde{X}(t) - X(t)| \geq \delta_0\}.$$ 

For $R > 0$, let

$$\tau_R := \inf \{t \geq 0 : \max(|\tilde{X}(t)|, |X(t)|) \geq R\}.$$ 

Then, by Theorem 3.1, we have $\tau_R \to \infty$ a.s. as $R \to \infty$. Denote $t' = t \wedge \tau_R \wedge S_{\delta_0}$ and

$$\Delta_{c_i} = c_i(\tilde{X}(s-), u) - c_i(X(s-), u), \quad i = 1, 2.$$
Applying Lemma 2.2, we have

\[
\mathbb{E}\left[\psi_n(|\Delta_t|^{\alpha})\right] = \mathbb{E}\left[\int_0^{t^*} I_{(\Delta_s, \neq 0)} \left\{ D\psi_n(|\Delta_s|^{\alpha})(b(\bar{X}(s)) - b(X(s))) + \frac{1}{2} D^2\psi_n(|\Delta_s|^{\alpha})|\sigma(\bar{X}(s)) - \sigma(X(s))|^2 \, ds \right\} \right] \\
+ \mathbb{E}\left[\int_0^{t^*} \int_U \{ \psi_n(|\Delta_s + \Delta_{c_1}|^{\alpha}) - \psi_n(|\Delta_s|^{\alpha}) - I_{(\Delta_s, \neq 0)} D\psi_n(|\Delta_s|^{\alpha}) \cdot \Delta_{c_1} \} v_1(du) \, ds \right] \\
+ \int_0^{t^*} \int_U \{ \psi_n(|\Delta_s + \Delta_{c_2}|^{\alpha}) - \psi_n(|\Delta_s|^{\alpha}) \} v_2(du) \, ds \\
= J_1 + J_2.
\]

Since

\[|\psi'_n(r)| \leq 1, \quad \psi''_n(r) \leq \frac{2}{n\rho(r)} I_{(a_n, a_{n-1})}(r),\]

it follows from Assumption 2.3 that

\[J_1 \leq \mathbb{E}\left[\int_0^{t^*} I_{(\Delta_s, \neq 0)} \left\{ |\psi'_n(|\Delta_s|^{\alpha})| \cdot |\alpha| \cdot |\Delta_s|^{\alpha-2} (\bar{X}(s) - X(s))(b(\bar{X}(s)) - b(X(s))) + \frac{1}{2} \left\{ |\psi''_n(|\Delta_s|^{\alpha})| \cdot \alpha^2 |\Delta_s|^{2\alpha-2} + |\alpha(\alpha - 1)| \cdot \psi'_n(|\Delta_s|^{\alpha}) \cdot |\Delta_s|^{\alpha-2} \right\} |\sigma(\bar{X}(s)) - \sigma(X(s))|^2 \, ds \right\} \right] \]

\[\leq \mathbb{E}\left[\int_0^{t^*} I_{(\Delta_s, \neq 0)} \left[ |\alpha| \cdot |\Delta_s|^{\alpha-2} |\Delta_s|^{2-\alpha} \rho(|\Delta_s|^{\alpha}) + \frac{1}{2} \left\{ \frac{2}{n\rho(|\Delta_s|^{\alpha})} I_{(a_n, a_{n-1})}(|\Delta_s|^{\alpha}) \alpha^2 |\Delta_s|^{2\alpha-2} + |\alpha(\alpha - 1)| \cdot |\Delta_s|^{\alpha-2} \right\} |\Delta_s|^{2-\alpha} \rho(|\Delta_s|^{\alpha}) \right] \, ds \right] \]

\[\leq \mathbb{E}\left[\int_0^{t^*} \left( \frac{1}{2} |\alpha(\alpha - 1)| + |\alpha| \right) \cdot \rho(|\Delta_s|^{\alpha}) + \frac{\alpha^2 |\Delta_s|^{\alpha}}{n} I_{(a_n, a_{n-1})}(|\Delta_s|^{\alpha}) \, ds \right] \]

\[\leq \left( \frac{1}{2} |\alpha(\alpha - 1)| + |\alpha| \right) \mathbb{E}\left[\int_0^{t^*} \rho(|\Delta_s|^{\alpha}) \, ds \right] + \frac{\alpha^2 a_{n-1}^{\alpha}}{n} t'. \]

Regarding \(J_2\), by Lagrange's mean value theorem and the fact that \(|\psi'_n(r)| \leq 1\), we have the following cases:
Case I. For 0 < \alpha \leq 1, since \((A + B)^\alpha \leq A^\alpha + B^\alpha\) for all \(A, B \geq 0\), we know that there exists some \(\xi_1 \in [\|\Delta_s\|^\alpha, (\|\Delta_s\| + |\Delta_{c_i}|)^\alpha]\) such that
\[
\psi_n(|\Delta_s + \Delta_{c_i}|^\alpha) - \psi_n(|\Delta_s|^\alpha) \leq \psi_n(|\Delta_s|^\alpha + |\Delta_{c_i}|^\alpha) - \psi_n(|\Delta_s|^\alpha) \\
\leq |\psi'_n(\xi_1)| \cdot |\Delta_s|^\alpha + |\Delta_{c_i}|^\alpha - |\Delta_s|^\alpha | \\
\leq |\Delta_{c_i}|^\alpha.
\]

Case II. For 1 < \alpha < +\infty, since \((A + B)^{\alpha-1} \leq (2^{\alpha-2} + 1)(A^{\alpha-1} + B^{\alpha-1})\) for all \(A, B \geq 0\), there exists some \(\xi_2 \in [\min\{\Delta_s, \Delta_s + \Delta_{c_i}\}, \max\{\Delta_s, \Delta_s + \Delta_{c_i}\}]\) such that
\[
\psi_n(|\Delta_s + \Delta_{c_i}|^\alpha) - \psi_n(|\Delta_s|^\alpha) \leq \alpha|\psi'_n(\xi_2|^\alpha)| \cdot |\xi_2|^\alpha - 1 \cdot |\Delta_s + \Delta_{c_i} - \Delta_s| \\
\leq \alpha(|\Delta_s| + |\Delta_{c_i}|)^{\alpha-1}|\Delta_{c_i}| \\
\leq \alpha(2^{\alpha-2} + 1)(|\Delta_s|^{\alpha-1}|\Delta_{c_i}| + |\Delta_{c_i}|^\alpha),
\]
where the second inequality follows from
\[
0 \leq |\xi_2| \leq \max\{|\Delta_s|, |\Delta_s + \Delta_{c_i}|\} \leq |\Delta_s| + |\Delta_{c_i}|.
\]

Thus,
\[
J_2 \leq \mathbb{E}\left[\int_0^t \alpha(2^{\alpha-2} + 1) + 1 + 1)\rho(|\Delta_s|^\alpha) + \alpha\rho(|\Delta_s|^\alpha) + [\alpha(2^{\alpha-2} + 1) + 1)\rho(|\Delta_s|^\alpha) \right] ds \\
\leq \alpha(2^{\alpha} + 3) + 2 \cdot \mathbb{E}\left[\int_0^t \rho(|\Delta_s|^\alpha) ds \right] \\
\]
and so
\[
\mathbb{E}\left[\psi_n(|\Delta_r|^\alpha)\right] \leq p(\alpha)\mathbb{E}\left[\int_0^t \rho(|\Delta_s|^\alpha) ds \right] + \frac{\alpha^2 a_{n-1}}{n} t,
\]
where
\[
p(\alpha) = \frac{1}{2}|\alpha - 1| + |\alpha| + \alpha(2^{\alpha} + 3) + 2.
\]
Since \(\lim_{n \to \infty} \psi_n(r) = |r|\), letting \(n \to \infty\) yields
\[
\mathbb{E}\left[|\Delta_r|^\alpha\right] \leq p(\alpha)\mathbb{E}\left[\int_0^t \rho(|\Delta_s|^\alpha) ds \right] \\
\leq p(\alpha)\mathbb{E}\left[\int_0^{t \wedge T_R} \rho\left(|\Delta_{s\wedge S_0}|^\alpha\right) ds \right] \\
\leq p(\alpha)\int_0^{t} \rho \left(\mathbb{E}\left[|\Delta_{s\wedge S_0}|^\alpha\right]\right) ds,
\]
where the last inequality follows from Jensen’s inequality. It follows from Theorem 3.1, Fatou’s lemma and the monotone convergence theorem that
\[
\mathbb{E}[|\Delta_t \wedge S_0|^\alpha] \leq \lim_{R \to \infty} \mathbb{E}[|\Delta_r|^\alpha] \leq p(\alpha)\int_0^t \rho \left(\mathbb{E}[|\Delta_{s\wedge S_0}|^\alpha]\right) ds.
\]
Applying Lemma 2.3 yields that \(\mathbb{E}[|\Delta_t \wedge S_0|^\alpha] \to 0\) and so \(\Delta_t \wedge S_0 = 0\) a.s..
On the set \( \{ S_{\delta_0} \leq t \} \), we have \( |\Delta_t'| \geq \delta_0 \). Observing that \( 0 = \mathbb{E}[|\Delta_t \wedge S_{\delta_0}|^{\alpha}] \geq \delta_0^{\alpha} \mathbb{P}\{ S_{\delta_0} \leq t \} \), we have \( \mathbb{P}\{ S_{\delta_0} \leq t \} = 0 \) and hence \( \Delta_t = 0 \) a.s., which is the desired result. \( \blacksquare \)

**Remark 3.1:** We would like to point out that the proof method of Theorem 3.2 is similar to the one of Theorem 2.4 in [31].

**Theorem 3.3:** Under Assumptions 2.2 and 2.3, JSDE (1) has a unique non-explosive strong solution.

**Proof:** Similar to the proof of Theorem 2.2 in [20], applying Theorems 3.1 and 3.2, we know that there exists a unique non-explosive strong solution for (2). Thus, by Lemma 2.1, there also exists a unique strong non-explosive solution for (1). \( \blacksquare \)

**Corollary 3.1:** Under Assumptions 2.2 and 2.4, JSDE (1) has a unique non-explosive strong solution.

**Proof:** Similar to the proof of Theorem 3.2, replacing \( \rho(r) \) by \( \rho_2(r) \), we have

\[
D^2 \psi_n(|\Delta_s|) \sigma(\tilde{X}(s)) - \sigma(X(s)))^2 \leq \frac{2}{n \rho_2(|\Delta_s|)} I_{(an, an-1)}(|\Delta_s|) \cdot \rho_2(|\Delta_s|) = \frac{2}{n} I_{(an, an-1)}.
\]

For convenience, we denote \( \Delta_{c_1} = c_1(\tilde{X}(s - u), u) - c_1(X(s - u), u) \). By Taylor’s expansion, there exists some \( \eta = \Delta_s + \theta \Delta_{c_1} \) with a constant \( \theta \in (0, 1) \) such that

\[
\int_{U_1} \left[ \psi_n(|\Delta_s + \Delta_{c_1}|) - \psi_n(|\Delta_s|) - I_{[\Delta_s \neq 0]} D\psi_n(|\Delta_s|) \cdot (\Delta_{c_1}) \right] v_1(du) = \int_{U_1} \frac{1}{2} D^2 \psi_n(|\eta|) \cdot |\Delta_{c_1}|^2 v_1(du). \tag{6}
\]

Since \( c_1(x, u) \) is non-decreasing for fixed \( u \), we have \( \Delta_s \cdot \Delta_{c_1} \geq 0 \) and \( |\eta| \geq |\Delta_s| \). Thus, it follows from (6) that

\[
\int_{U_1} \left[ \psi_n(|\Delta_s + \Delta_{c_1}|) - \psi_n(|\Delta_s|) - I_{[\Delta_s \neq 0]} \cdot D\psi_n(|\Delta_s|) \cdot (\Delta_{c_1}) \right] v_1(du) \leq \frac{1}{2} \cdot \frac{2}{n \rho_2(|\Delta_s|)} \int_{U_1} |\Delta_{c_1}|^2 I_{(an, an-1)}(|\eta|) v_1(du) \leq \frac{I_{(an, an-1)}}{n}.
\]

The rest proof can be completed by the similar arguments to Theorems 3.2 and 3.3, and so we omit it here. \( \blacksquare \)

**Remark 3.2:** We would like to mention that Corollary 3.1 can be easily extended to the multi-dimensional case (see Theorem 3.3 of Fu and Li [11]).
Example 3.1: Consider the following SDE:

\[
X(t) = X_0 - \int_0^t |X(s)| \ln |X(s)| \, ds + \int_0^t \sqrt{|X(s)|} \, dB_s + \int_0^t \int_{|u| \leq 1} \sqrt{|X(s)|} \tilde{N}_1(ds, du)
+ \int_0^t \int_{|u| > 1} \gamma |u|X(s) N_2(ds, du).
\]  

(7)

Here \( \gamma \) is a positive constant such that \( \int_{|u| \leq 1} |\gamma u|^2 \nu(du) = 1 \). It is easy to show that, for any \( x > 0 \), the coefficient \( b(x) = -x \cdot \ln x \) satisfies Assumptions 2.1 and 2.2, and for any \( x \geq 0 \), the coefficient \( \sigma(x) = \sqrt{x} \) satisfies Assumption 2.2. Thus, \( b(x) \) and \( \sigma(x) \) are both non-Lipschitzian due to

\[
\lim_{x \to 0^+} b'(x) = \lim_{x \to 0^+} \sigma'(x) = +\infty.
\]

Furthermore,

\[
|b(x) - b(y)| = | \int_x^y 1 + \ln t \, dt | \leq \int_0^{|x-y|} |1 + \ln t| \, dt = |1 + \ln |x-y||
\]

for all \( 0 < x, y < 1/e \), and

\[
(\sigma(x) - \sigma(y))^2 \leq |x - y|
\]

for all \( x, y \geq 0 \). Thus, the coefficients of (7) satisfy Assumptions 2.2 and 2.4. By Corollary 3.1, we know that (7) has a unique non-explosive strong solution.

4. Non-confluent property

In this section, we present the non-confluent property of strong solutions to (1).

Definition 4.1: Suppose that (1) has a unique non-explosive strong solution \( X(t) \) for any initial value \( X_0 \). Then \( X(t) \) is said to have the non-confluent property if, for any \( x, y \in \mathbb{R} \) with \( x \neq y \),

\[
P\{X^x(t) \neq X^y(t), \text{ for all } t \geq 0\} = 1,
\]

where \( X^x \) and \( X^y \) denote the strong solutions of (1) with initial conditions \( x \) and \( y \), respectively.

Theorem 4.1: Suppose that Assumption 2.5 holds and (1) has a unique non-explosive strong solution \( X(t) \) for any initial value \( X(0) = x \in \mathbb{R} \). Then \( X(t) \) has the non-confluent property.
Proof: Consider the function \( R(x) = |x|^{-\alpha} \). For any \( x, y \in \mathbb{R} \) with \( x \neq 0 \) and \( |x + y| \geq \delta|x| \), where \( \delta \) is a fixed constant. We now claim that

\[
R(x + y) - R(x) - DR(x) \cdot y = \frac{|x|^\alpha - |x + y|^\alpha}{|x + y|^\alpha \cdot |x|^\alpha} - DR(x) \cdot y \leq K \frac{|x|^\alpha + |x|^{\alpha-1}|y|}{|x|^{2\alpha}}, \tag{8}
\]

where \( K > 0 \) is a constant. Indeed, it is enough to see that

\[-DR(x) \cdot y = \alpha x|x|^{-\alpha-2}y \leq \alpha \frac{|x|^{\alpha-1}|y|}{|x|^{2\alpha}}.\]

For \( |x| \leq |x + y| \), (8) is automatically satisfied. Thus, it is sufficient to consider the case for \( |x| \geq |x + y| \). For \( 0 \leq \alpha \leq 1 \),

\[|x|^\alpha - |x + y|^\alpha \leq |x|^\alpha.\]

For \( 1 < \alpha < +\infty \), there exists some \( \xi \in (|x + y|, |x|) \) such that

\[|x|^\alpha - |x + y|^\alpha = \alpha|\xi|^{\alpha-1}(|x| - |x + y|) \leq \alpha|x|^{\alpha-1}|y|.\]

Since \( |x + y| \geq \delta|x| \), (8) holds for \( K = (1/\delta^\alpha)(1 + 2\alpha) \).

For any \( x, \tilde{x} \in \mathbb{R} \) with \( x \neq \tilde{x} \), let \( X(t) \) and \( \tilde{X}(t) \) be two strong solutions for (1) of the following forms:

\[X(t) = x + \int_0^t \sigma(X(s)) \, dB_s + \int_0^t \int_{U_1} c_1(X(s-), u) \tilde{N}_1(ds, du) + \int_0^t b(X(s)) \, ds + \int_0^t \int_{U_2} c_2(X(s-), u) N_2(ds, du),\]

\[\tilde{X}(t) = \tilde{x} + \int_0^t \sigma(\tilde{X}(s)) \, dB_s + \int_0^t \int_{U_1} c_1(\tilde{X}(s-), u) \tilde{N}_1(ds, du) + \int_0^t b(\tilde{X}(s)) \, ds + \int_0^t \int_{U_2} c_2(\tilde{X}(s-), u) N_2(ds, du).\]

Denote \( \Delta_t = \tilde{X}(t) - X(t) \). Then \( |\Delta_0| = |x - \tilde{x}| > 0 \). For any \( 1/|\Delta_0| < n \in \mathbb{N} \) and \( R > \max\{|\tilde{x}|, |x|\} \), define

\[T_{1/n} := \inf \left\{ t \geq 0 : |\Delta_t| \leq \frac{1}{n} \right\},\]

\[T_0 := \inf \{ t \geq 0 : |\Delta_t| = 0 \},\]

\[\tau_R := \inf \{ t \geq 0 : \max\{|\tilde{X}(t)|, |X(t)|\} \geq R \}.\]

Obviously, \( T_0 = \lim_{n \to \infty} T_{1/n} \) and \( \lim_{R \to \infty} \tau_R = \infty \) a.s..
Let \( t' = t \land \tau_R \land T_{1/n} \) and

\[
\Delta_{c_i} = c_i(\bar{X}(s), u) - c_i(X(s), u), \quad i = 1, 2.
\]

It follows from (4), (8) and Lemma 2.2 that

\[
\mathbb{E} \left[ |\Delta_{t'}|^{-\alpha} \right] = |\Delta_0|^{-\alpha} + \mathbb{E} \left[ \int_0^{t'} -\alpha \Delta s \cdot |\Delta s|^{-\alpha-2} \cdot (b(\bar{X}(s)) - b(X(s))) + \frac{1}{2} |\sigma(\bar{X}(s)) - \sigma(X(s))|^2 \cdot \alpha + 1 \cdot |\Delta s|^{-\alpha-2} \, ds \right] + \mathbb{E} \left[ \int_0^{t'} \int_{U_1} \left( R(\Delta s + \Delta_{c_1}) - R(\Delta s) - D_R(\Delta s) \cdot \Delta_{c_1} \right) \nu_1(du) \, ds \right] + \mathbb{E} \left[ \int_0^{t'} \int_{U_2} \left( R(\Delta s + \Delta_{c_2}) - R(\Delta s) \right) \nu_2(du) \, ds \right] \leq |\Delta_0|^{-\alpha} + \mathbb{E} \left[ \int_0^{t'} \left( \alpha + \frac{1}{2} \alpha(\alpha + 1) \right) \cdot \rho(|\Delta s|^{-\alpha}) \, ds \right] + \mathbb{E} \left[ \int_0^{t'} \int_{U_1} K \left( |\Delta s|^{-\alpha} + |\Delta s|^{-\alpha-1} |\Delta_{c_1}| \right) \nu_1(du) \, ds \right] + \mathbb{E} \left[ \int_0^{t'} \int_{U_2} K' \left( |\Delta s|^{-\alpha} + |\Delta s|^{-\alpha-1} |\Delta_{c_2}| \right) \nu_2(du) \, ds \right] \leq |\Delta_0|^{-\alpha} + \mathbb{E} \left[ \int_0^{t'} K_1 |\Delta s|^{-\alpha} + K_2 \rho(|\Delta s|^{-\alpha}) \, ds \right],
\]

where

\[
K' = \frac{1}{\delta^\alpha}(1 + \alpha), \quad K_1 = M(K + K'), \quad K_2 = \alpha + \frac{1}{2} \alpha(\alpha + 1) + K + K'.
\]

Let \( \rho_0(x) = K_1 x + K_2 \rho(x) \). Then \( \rho_0 \) is also concave and non-decreasing with \( \rho_0(0) = 0 \). Moreover, there exists some \( x_0 \geq 0 \) such that, for any \( x \in [0, x_0] \), either \( \rho(x) \geq x \) or \( \rho(x) \leq x \) holds. Thus,

\[
\int_0^{x_0} \frac{1}{\rho_0(s)} \, ds \geq \int_0^{x_0} \frac{1}{(K_1 + K_2) \max\{\rho(s), s\}} \, ds = +\infty
\]

and consequently,

\[
\int_{x_0+}^{1+} \frac{1}{\rho_0(s)} \, ds = +\infty.
\]

Let \( u(t') = E[|\Delta_{t'}|^{-\alpha}] \). Then, by Jensen's inequality,

\[
0 \leq u(t') \leq |\Delta_0|^{-\alpha} + \int_0^{t'} \rho_0(u(s \land \tau_R \land T_{1/n})) \, ds.
\]
Applying Lemma 2.3,
\[ 0 \leq u(t') \leq \Omega^{-1} \left( \Omega \left( |\Delta_0|^{-\alpha} \right) + t' \right) < +\infty. \]

Letting \( R \to \infty \) and by Fatou's lemma, one has
\[
0 \leq u(t \wedge T_{1/n}) = \mathbb{E} \left[ |\Delta_{t \wedge T_{1/n}}|^{-\alpha} \right] \leq \Omega^{-1} \left( \Omega \left( |\Delta_0|^{-\alpha} \right) + t \wedge T_{1/n} \right) \leq \Omega^{-1} \left( \Omega \left( |\Delta_0|^{-\alpha} \right) + t \right) < +\infty.
\]

Since \( |\Delta_{T_{1/n}}| \leq 1/n \) holds on the set \( \{ T_{1/n} < t \} \) and \( R(x) = |x|^{-\alpha} \) is non-increasing for \( x > 0 \), it follows that
\[
\left( \frac{1}{n} \right)^{-\alpha} \mathbb{P}\{ T_{1/n} < t \} \leq \mathbb{E}[|\Delta_{t \wedge T_{1/n}}|^{-\alpha} \cdot I_{\{ T_{1/n} < t \}}] \leq \mathbb{E}[|\Delta_{t \wedge T_{1/n}}|^{-\alpha}] \leq \Omega^{-1} \left( \Omega \left( |\Delta_0|^{-\alpha} \right) + t \right) < +\infty
\]
and so
\[
\mathbb{P}\{ T_{1/n} < t \} \leq n^{-\alpha} \cdot \Omega^{-1} \left( \Omega \left( |\Delta_0|^{-\alpha} \right) + t \right).
\]
This shows that \( \mathbb{P}\{ T_0 < t \} = 0 \) holds for all \( t \geq 0 \). Thus, letting \( t \to \infty \), we have \( \mathbb{P}\{ T_0 < \infty \} = 0 \). In other words, \( |\Delta_0| > 0 \) a.s. on the interval \( [0, \infty) \), which completes the proof.

\[\Box\]

**Remark 4.1:** If \( \alpha = 0 \), our results reduces to Corollary 3.3 in [31].

**Example 4.1:** Consider the following SDE:
\[
X(t) = X_0 - \int_0^t \left[ X^3(s) + X^{1/3}(s) \right] ds + \int_0^t 2X(s) dB_s + \int_0^t \int_{|u| \leq 1} \gamma |u|X(s) \tilde{N}_1(ds, du) + \int_0^t \int_{|u| > 1} \gamma |u|X(s) N_2(ds, du). \tag{9}
\]
Here \( \gamma \) is a positive constant such that \( \int_{|u| \leq 1} |\gamma u|^2 \nu(du) = 1 \). Note that
\[ xb(x) = -x^4 - x^{4/3} \]
and
\[
(x - y)(b(x) - b(y)) = -(x - y)(x^{1/3} - y^{1/3} + x^3 - y^3) \leq -(x^{1/3} - y^{1/3})^2 \left[ (x^{1/3} + \frac{1}{2} y^{1/3})^2 + \frac{3}{4} y^{1/3} \right] - (x - y)^2 \left( (x + \frac{1}{2} y)^2 + \frac{3}{4} y^2 \right).
\]
We can check that the coefficients of (9) satisfy Assumptions 2.2, 2.3 and 2.5 for \( \alpha = 0 \). Thus, employing Theorem 3.3, we see that (9) has a unique non-explosive strong solution \( X(t) \) for any initial value \( X(0) = x \in \mathbb{R} \). According to Theorem 4.1, we know that \( X(t) \) has the non-confluent property.
5. Conclusions

This paper is devoted to study some qualitative properties of strong solutions for a class of JSDEs with the super linear growth and non-lipschitz conditions. We have obtained the non-explosive property of strong solutions for JSDEs with the super linear growth condition by applying similar arguments in [9]. By employing the Bihari-Lasalle inequality, we have also established the pathwise uniqueness of strong solutions to JSDEs with the non-Lipschitz condition, in which \( \mathbb{E}[|\tilde{X}(t) - X(t)|^\alpha] \) vanishes up to an appropriately defined stopping time by constructing a sequence of smooth functions. Moreover, we have showed the non-confluent property of strong solutions for JSDEs under some mild conditions.

These findings of the research have led the authors to the following main contributions: (i) it was relaxed for the usual linear growth condition which guarantees the non-explosive property of solutions; (ii) a generalized non-Lipschitz condition was given to guarantee the existence and uniqueness of the solution to the JSDEs; (iii) the method developed by [11,31] also works for uniqueness problem with respect to the JSDEs under the non-Lipschitz condition constructed in this paper, i.e. the non-Lipschitz condition in our paper has universality; (iv) non-confluent property of strong solutions to the JSDEs has been obtained under the nonlinear condition.

We would like to mention that JSDEs considered in this paper are all driven by Brownian motions and Poisson processes. It is well known that JSDEs driven by Lévy processes have attracted much attention recently (see, for example, [12,14,19,33]). Therefore, it would be crucial and interesting to extend the results of this paper to JSDEs driven by general Lévy processes. We also note that various theoretical results with applications for SDEs driven by fractional Brownian motions (fBms) have been studied extensively in literature; for instance, we refer the reader to [2,4,6,16,34] and the references therein. Thus, it would be important to extend our results to JSDEs driven by fBms. We plan to address these problems as we continue our research.

Acknowledgments

The authors are grateful to the editors and reviewers whose helpful comments and suggestions have led to much improvement of the paper.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was supported by the National Natural Science Foundation of China (11471230, 11671282).

References

[1] J.C. Bergquist, R.G. Hulet, W.M. Itano, and D.J. Wineland, Observation of quantum jumps in a single atom, Phys. Rev. Lett. 57(14) (1986), pp. 1699–1702.
[2] F. Biagini, Stochastic Calculus for Fractional Brownian Motion and Applications, 2nd ed., Springer, London, 2008.
[3] I. Bihari, A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations, Acta Math. Acad. Sci. Hungar. 7(1) (1956), pp. 81–94.
[4] G. Binotto, I. Nourdin, and D. Nualart, *Weak symmetric integrals with respect to the fractional Brownian motion*, Ann. Probab. 46(4) (2018), pp. 2243–2267.

[5] C.T. Chudley and R.J. Elliott, *Neutron scattering from a liquid on a jump diffusion model*, Proc. Phys. Soc. 77(2) (1961), pp. 353–361.

[6] C. Czichowsky, R. Peyre, W. Schachermayer, and J. Yang, *Shadow prices, fractional Brownian motion, and portfolio optimisation under transaction costs*, Finance Stoch. 22(1) (2018), pp. 161–180.

[7] M. Émery, *Non confluence des solutions d'une equation stochastique lipschitzienne*, in Séminaire de Probabilités XV 1979/80, Springer, Berlin, 1981, pp. 587–589.

[8] S. Fang and T. Zhang, *A class of stochastic differential equations with non-lipshitzian coefficients: Pathwise uniqueness and no explosion*, C. R. Math. 337(11) (2003), pp. 737–740.

[9] S. Fang and T. Zhang, *Stochastic differential equations with non-lipshitz coefficients: I. Pathwise uniqueness and large deviation*, preprint (2003). Available at arXiv, math/0311032.

[10] S. Fang and T. Zhang, *A study of a class of stochastic differential equations with non-lipshitzian coefficients*, Probab. Theory Relat. Fields 132(3) (2005), pp. 356–390.

[11] Z. Fu and Z. Li, *Stochastic equations of non-negative processes with jumps*, Stoch. Process. Appl. 120(3) (2010), pp. 306–330.

[12] H. Geman, P. Carr, D.B. Madan, and M. Yar, *Stochastic volatility for Levy processes*, Math. Finance 13(3) (2003), pp. 345–382.

[13] S. Gleyzes, S. Kuhr, C. Guerlin, J. Bernu, S. Deleglise, U.B. Hoff, M. Brune, J.M. Raimond, and S. Haroche, *Quantum jumps of light recording the birth and death of a photon in a cavity*, Nature 446(7133) (2007), pp. 297–300.

[14] S.W. He, J.G. Wang, and J.A Yan, *Semimartingale Theory and Stochastic Calculus*, 2nd ed., Routledge, New York, 2018.

[15] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd ed., North-Holland, Amsterdam, 1989.

[16] J.H. Kang, B.Z. Yang, and N.J. Huang, *Pricing of FX options in the MPT/CIR jump-diffusion model with approximative fractional stochastic volatility*, Physica A (2019), p. 121871. Available at https://doi.org/10.1016/j.physa.2019.121871.

[17] R.Z. Khasminskii and F.C. Klebaner, *Long term behavior of solutions of the lotka-volterra system under small random perturbations*, Ann. Appl. Probab. 11(3) (2001), pp. 952–963.

[18] G. Lan and J.C. Wu, *New sufficient conditions of existence, moment estimations and non confluence for sdes with non-lipshitzian coefficients*, Stoch. Process. Appl. 124(12) (2014), pp. 4030–4049.

[19] M.J. Landis, J.G. Schraiber, and M. Liang, *Phylogenetic analysis using Lévy processes: Finding jumps in the evolution of continuous traits*, Syst. Biol. 62(2) (2013), pp. 193–204.

[20] Z. Li and L. Mytnik, *Strong solutions for stochastic differential equations with jumps*, Ann. Inst. H. Poincaré Probab. Statist. 47(4) (2010), pp. 1055–1067.

[21] X. Mao, G. Marion, and E. Renshaw, *Environmental brownian noise suppresses explosions in population dynamics*, Stoch. Process. Appl. 97(1) (2002), pp. 95–110.

[22] B.K. Øksendal and A. Sulem, *Applied Stochastic Control of Jump Diffusions*, 2nd ed., Springer, Berlin, 2007.

[23] C. Pellegrini, *Existence, uniqueness and approximation of the jump-type stochastic Schörderinger equation for two-level systems*, Stoch. Process. Appl. 120(9) (2010), pp. 1722–1747.

[24] Y. Ren, W. Yin, and R. Sakthivel, *Stabilization of stochastic differential equations driven by G-Brownian motion with feedback control based on discrete-time state observation*, Automatica 95 (2018), pp. 146–151.

[25] P. Revathi, R. Sakthivel, and Y. Ren, *Stochastic functional differential equations of Sobolev-type with infinite delay*, Stat. Probab. Lett. 109 (2016), pp. 68–77.

[26] R. Sakthivel, P. Revathi, Y. Ren, and G. Shen, *Retarded stochastic differential equations with infinite delay driven by Rosenblatt process*, Stoch. Anal. Appl. 36(2) (2018), pp. 304–323.

[27] J. Shao, F.Y. Wang, and C. Yuan, *Harnack inequalities for stochastic (functional) differential equations with non-lipshitzian coefficients*, Elect. J. Probab. 17(27) (2012), pp. 1–18.
[28] S.E. Shreve, *Stochastic Calculus for Finance II: Continuous-Time Models*, Springer-Verlag, New York, 2004.

[29] P. Tankov, *Financial Modelling with Jump Processes*, Chapman and Hall/CRC, London, 2003.

[30] A. Uppman, *Sur le flot d'une equation differentielle stochastique*, in *Séminaire de Probabilités XVI 1980/81*, Springer, Berlin, 1982, pp. 268–284.

[31] F. Xi and C. Zhu, *Jump type stochastic differential equations with non-lipschitz coefficients and feller and strong feller properties*, J. Differ. Equ. 266(8) (2019), pp. 4668–4711.

[32] T. Yamada and S. Watanabe, *On the uniqueness of solutions of stochastic differential equations*, J. Math. Kyoto Univ. 11(1) (1971), pp. 155–167.

[33] B.Z. Yang, J. Yue, and N.J. Huang, *Equilibrium price of variance swaps under stochastic volatility with Lévy jumps and stochastic interest rate*, Int. J. Theor. Appl. Finance 22(4) (2019), p. 1950016, (33 pages).

[34] J. Yue and N.J. Huang, *Fractional Wishart processes and ε-fractional Wishart processes with applications*, Comput. Math. Appl. 75 (2018), pp. 2955–2977.