SHADOWING, EXPANSIVENESS AND STABILITY OF DIVERGENCE-FREE VECTOR FIELDS

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Abstract. Let $X$ be a divergence-free vector field defined on a closed, connected Riemannian manifold. In this paper, we show the equivalence between the following conditions:

- $X$ is in the $C^1$-interior of the set of expansive divergence-free vector fields.
- $X$ is in the $C^1$-interior of the set of divergence-free vector fields which satisfy the shadowing property.
- $X$ is in the $C^1$-interior of the set of divergence-free vector fields which satisfy the Lipschitz shadowing property.
- $X$ has no singularities and $X$ is Anosov.

1. Introduction and statement of the results

Let $M$ be an $n$-dimensional, $n \geq 3$, closed, connected and smooth Riemannian manifold, endowed with a volume form, which has associated a measure $\mu$, called the Lebesgue measure, and let $d$ denote the Riemannian distance. Let $\mathfrak{X}(M)$ be the set of vector fields and let $\mathfrak{X}_s(M)$ be the set of divergence-free vector fields, both defined on $M$ and endowed with the $C^s$ Whitney topology, $s \geq 1$. From now on, we consider $s = 1$. A vector field $X$ has associated a flow, denoted by $X^t$, $t \in \mathbb{R}$. Denote by $\text{Per}(X)$ the union of the closed orbits of $X$ and by $\text{Sing}(X)$ the union of the singularities of $X$. A subset of $M$ is said to be regular if it has no singularities. Denote by $\text{Crit}(X)$ the set of the closed orbits and the singularities associated to $X$. A singularity $p$ is linear if there exist smooth local coordinates around $p$ such that $X$ is linear and equal to $DX(p)$ in these coordinates (see [17, Definition 4.1]).

Take a $C^1$-vector field and a regular point $x$ in $M$ and let $N_x := X(x) \perp \subset T_xM$ denote the $(\dim(M) - 1)$-dimensional normal bundle of $X$ at $x$ and $N_{x,r} = N_x \cap \{u \in T_xM : \|u\| < r\}$, for $r > 0$. Since, in general, $N_x$ is not $DX_x^t$-invariant, we define the linear Poincaré flow

$$P_X^t(x) := \Pi_{X^t(x)} \circ DX_x^t,$$
where $\Pi_{X^t(x)} : T_{X^t(x)}M \to N_{X^t(x)}$ is the canonical orthogonal projection.

Let $\Lambda$ be a compact, $X^t$-invariant and regular set. If $N_\Lambda$ admits a $P_X^t$-invariant splitting $N_\Lambda = N_\Lambda^1 \oplus N_\Lambda^u$, such that there is $\ell > 0$ satisfying
\[
\|P_X^t(x)\|_{N_\Lambda^1} \leq \frac{1}{2} \text{ and } \|P_X^{-t}(X^t(x))\|_{N_\Lambda^u} \leq \frac{1}{2},
\]
for any $x \in \Lambda$, we say that $\Lambda$ is hyperbolic. A vector field $X$ is said to be Anosov if the whole manifold $M$ is hyperbolic. Let $\mathcal{A}_\mu(M)$ denote the set of Anosov $C^1$-divergence-free vector fields.

Take $T > 0$ and $\delta > 0$. A map $\psi : \mathbb{R} \to M$ is a $(\delta, T)$-pseudo-orbit of a flow $X^t$ if, for any $\tau \in \mathbb{R}$, $d(X^t(\psi(\tau)), \psi(\tau + t)) < \delta$, for any $|t| \leq T$.

Take $\epsilon > 0$. A pseudo-orbit $\psi$ of a flow $X^t$ is $\epsilon$-shadowed by some orbit of $X^t$ if there is $x \in M$ and an increasing homeomorphism $\alpha : \mathbb{R} \to \mathbb{R}$, called reparametrization, which satisfies $\alpha(0) = 0$ and such that $d(X^{\alpha(t)}(x), \psi(t)) < \epsilon$, for every $t \in \mathbb{R}$. 

**Definition 1.1.** A $C^1$-vector field $X$ satisfies the shadowing property if for any $\epsilon > 0$ there is $\delta > 0$ such that any $(\delta, T)$-pseudo-orbit $\psi$, for $T > 0$, is $\epsilon$-shadowed by some orbit of $X$. Let $\mathcal{S}^1(M)$ and $\mathcal{S}^1_\mu(M)$ denote the sets of vector fields in $\mathcal{X}^1(M)$ and $\mathcal{X}^1_\mu(M)$, respectively, satisfying the shadowing property.

In the mid 1990’s (see [13]) it was shown that a dissipative diffeomorphism in the $C^1$-interior of the set of diffeomorphisms with the shadowing property is structurally stable. More recently, Lee and Sakai (see [7]) proved that if $X \in \text{int}\mathcal{S}^1(M)$ and has no singularities then $X$ satisfies the Axiom A and the strong transversality conditions, where $\text{int}S$ stands for the $C^1$-interior of a set $S \subset \mathcal{X}^1(M)$.

Now, we introduce a weaker definition.

**Definition 1.2.** A $C^1$-vector field $X$ satisfies the Lipschitz shadowing property if there are positive constants $\ell$ and $\delta_0$ such that any $(\delta, T)$-pseudo-orbit $\psi$, with $T > 0$ and $\delta \leq \delta_0$ is $\ell \delta$-shadowed by an orbit of $X$. Let $\mathcal{LS}^1(M)$ and $\mathcal{LS}^1_\mu(M)$ denote the sets of vector fields in $\mathcal{X}^1(M)$ and $\mathcal{X}^1_\mu(M)$, respectively, satisfying the Lipschitz shadowing property.

It is immediate, from the previous definitions, that $\mathcal{LS}^1(M) \subset \mathcal{S}^1(M)$ and $\mathcal{LS}^1_\mu(M) \subset \mathcal{S}^1_\mu(M)$. In [16], Tikhomirov proved that, for dissipative vector fields, Lipschitz shadowing is equivalent to structural stability. Recently, Pilyugin and Tikhomirov proved the same result for dissipative diffeomorphisms (see [12]). We can find in [11] the proof of that Anosov vector fields satisfy the Lipschitz shadowing property.

Let us now present the notion of expansive vector field.
Definition 1.3. A $C^1$-vector field $X$ is expansive if for any $\epsilon > 0$ there is $\delta > 0$ such that if $d(X^t(x), X^{\alpha(t)}(y)) \leq \delta$, for all $t \in \mathbb{R}$, for $x, y \in M$ and a continuous map $\alpha : \mathbb{R} \to \mathbb{R}$ with $\alpha(0) = 0$, then $y = X^s(x)$, where $|s| \leq \epsilon$. Denote by $E^1(M) \subset X^1(M)$ the set of expansive vector fields and by $E^1_\mu(M) \subset X^1_\mu(M)$ the set of divergence-free expansive vector fields, both endowed with the $C^1$ Whitney topology.

In 1970’s, Mañé proved that if a dissipative diffeomorphism $f$ is in the $C^1$-interior of the set of expansive diffeomorphisms then $f$ is Axiom A and satisfies the quasi-transversality condition (see [8]). Later, in [9], Moriyasu, Sakai and Sun proved the same result for dissipative vector fields. Moreover, they proved that if $X \in \text{int} E^1(M)$ and has the shadowing property then $X$ is Anosov. Recently, Pilyugin and Tikhomirov proved that an expansive dissipative diffeomorphism having the Lipschitz shadowing property is Anosov (see [12]).

In this article, we intend to characterize divergence-free vector fields, with a topological property of Anosov systems, such as topological stability under $C^1$-open conditions: shadowing and expansiveness. We prove the following:

**Theorem 1.** For the divergence-free setting, one has that

$$\text{int } E^1_\mu(M) = \text{int } S^1_\mu(M) = \text{int } LS^1_\mu(M) = A^1_\mu(M).$$

2. Definitions and auxiliary results

In this section, we state some definitions and present some results that will be used in the proofs.

Let $\Lambda$ be a compact, $X^t$-invariant and regular set. Consider a splitting $N = N^1 \oplus \cdots \oplus N^k$ over $\Lambda$, for $1 \leq k \leq n - 1$, such that all the subbundles have constant dimension. This splitting is dominated if it is $P^t_X$-invariant and there exists $\ell > 0$ such that, for every $0 \leq i < j \leq k$ and every $x \in \Lambda$, one has

$$\|P^\ell_X(x)\big|_{N^i}\| \cdot \|P^{-\ell}_X(X^\ell(x))\big|_{N^j_{X^\ell(x)}}\| \leq \frac{1}{2}, \ \forall \ x \in \Lambda.$$

The following result can be obtained following the ideas presented in [10, Proposition 2.4].

**Theorem 2.1.** Let $X \in X^1_\mu(M)$ and let $U$ be a small $C^1$-neighbourhood of $X$. Then, for any $\epsilon > 0$, there exist $l, \tau > 0$ such that, for any $Y \in U$ and any closed orbit $x$ of $Y^t$ of period $\pi(x) > \tau$,

- either $P^\ell_Y$ admits an $l$-dominated splitting over the $Y^t$-orbit of $x$
or else for any neighbourhood $U$ of $x$, there exists an $\epsilon$-$C^1$-perturbation $\tilde{Y}$ of $Y$, coinciding with $Y$ outside $U$ and along the orbit of $x$, such that $P_\tilde{Y}^{\pi(x)}(x) = \text{id}$, where id denotes the identity on $N_x$.

To prove Theorem 1, we also need to state the definition of star vector field.

**Definition 2.1.** A $C^1$-vector field $X$ is a star vector field if there exists a $C^1$-neighborhood $U$ of $X$ in $\mathfrak{X}^1(M)$ such that if $Y \in U$ then every point in $\text{Crit}(Y)$ is hyperbolic. Moreover, a $C^1$-divergence-free vector field $X$ is a divergence-free star vector field if there exists a $C^1$-neighborhood $U$ of $X$ in $\mathfrak{X}^1_\mu(M)$ such that if $Y \in U$ then every point in $\text{Crit}(Y)$ is hyperbolic. The set of star vector fields is denoted by $\mathcal{G}^1(M)$ and the set of divergence-free star vector fields is denoted by $\mathcal{G}^1_\mu(M)$.

Accordingly with this definition, in [6, Theorem 1] it is proved the following result.

**Theorem 2.2.** If $X \in \mathcal{G}^1_\mu(M)$ then $\text{Sing}(X) = \emptyset$ and $X$ is Anosov.

A 3-dimensional proof of this result is presented in [5] and a version for 4-dimensional symplectic Hamiltonian vector fields can be found in [3].

The following result says that the linear Poincaré flow cannot admit a dominated splitting over the set of regular points of $M$ if the vector field has a linear hyperbolic singularity of saddle-type.

**Proposition 2.3.** [17, Proposition 4.1] If $X \in \mathfrak{X}^1(M)$ admits a linear hyperbolic singularity of saddle-type then $P^t_X$ does not admit any dominated splitting over $M \setminus \text{Sing}(X)$.

A vector field $X$ is topologically mixing if, given any nonempty open sets $U, V \subset M$, there is $T > 0$ such that, for any $t \geq T$, we have $X^t(U) \cap V \neq \emptyset$. We end this section with a result stating that, $C^1$-generically, the divergence-free vector fields are topologically mixing.

**Theorem 2.4.** [2, Theorem 1.1] There exists a $C^1$-residual subset $\mathcal{R} \subset \mathfrak{X}^1_\mu(M)$ such that if $X \in \mathcal{R}$ then $X$ is a topologically mixing vector field.

### 3. Proof of the theorem

**Lemma 3.1.** If $X \in \text{int} \mathcal{E}^1_\mu(M)$ then any closed orbit of $X$ is hyperbolic.

**Proof.** Take $X \in \text{int} \mathcal{E}^1_\mu(M)$ and $U$ a $C^1$-neighbourhood of $X$ in $\mathcal{E}^1_\mu(M)$. Let $p$ be a point in a closed orbit of $X$ with period $\pi > 0$ and $U_p$ a
small neighbourhood of $p$ in $M$. By contradiction, assume that there is an eigenvalue $\lambda$ of $P_0^\pi(p)$ such that $|\lambda| = 1$.

Applying Zuppa’s Theorem (see [18]), we can find $Y \in \mathcal{U}$ such that $Y \in \mathcal{X}^\pi_\mu(M)$, $Y^\pi(p) = p$ and $P_0^\pi(p)$ has an eigenvalue $\lambda$ with $|\lambda| = 1$.

**Remark 3.1.** Notice that if $P_0^\pi(p)$ has not an eigenvalue $\lambda$ with $|\lambda| = 1$, it has an eigenvalue $\bar{\lambda}$ such that $|\bar{\lambda}| \approx 1$. So, we just have to perform a $C^1$-conservative perturbation $Z$ of $Y$, by [4, Lemma 3.2], such that $P_Z^\pi(p)$ has an eigenvalue $\bar{\lambda}$ with $|\bar{\lambda}| = 1$.

Accordingly with Moser’s Theorem (see [10]), there is a smooth conservative change of coordinates $\varphi_p : U_p \to T_pM$ such that $\varphi_p(p) = \tilde{0}$. Let $f_Y : \varphi_p^{-1}(N_p) \to \Sigma$ be the Poincaré map associated to $Y^t$, where $\Sigma$ denotes the Poincaré section through $p$, and take $V$ a $C^1$-neighbourhood of $f_Y$.

By [4, Lemma 3.2], taking $\mathcal{T}$ a small flowbox of $Y^{[0,t_0]}(p)$, $0 < t_0 < \pi$, we have that there are $Z \in \mathcal{U}$, $f_Z \in \mathcal{V}$ and $\epsilon > 0$ such that: $Z^t(p) = Y^t(p)$, $t \in \mathbb{R}$; $P_0^\epsilon(p) = P_0^\epsilon(p)$; $Z|_{T^1} = Y\big|_{T^1}$ and

$$f_Z(x) = \begin{cases} \varphi_p^{-1} \circ P_Y^\pi(p) \circ \varphi_p(x), & x \in B_{\epsilon/4}(p) \cap \varphi_p^{-1}(N_p) \\ f_Y(x), & x \notin B_{\epsilon}(p) \cap \varphi_p^{-1}(N_p). \end{cases}$$

Notice that $P_Z^\pi(p)$ still has an eigenvalue $\lambda$ with $|\lambda| = 1$.

Since $Z \in \mathcal{E}^\mu_\mu(M)$, for a sufficiently small $\epsilon > 0$, there is $0 < \delta' < \epsilon$ such that if $d(Z^t(x), Z^\alpha(t)(y)) \leq \delta$, for any $t \in \mathbb{R}$, $x, y \in M$ and $\alpha : \mathbb{R} \to \mathbb{R}$ continuous such that $\alpha(0) = 0$, then $y = Z^s(x)$, where $|s| \leq \epsilon$.

Take $0 < \delta' < \delta$ such that if $x, y \in M$ satisfy $d(x, y) < \delta'$ then $d(Z^t(x), Z^t(y)) < \delta$, for $0 \leq t \leq \pi$.

Firstly, assume that $\lambda = 1$ and fix the associated non-zero eigenvector $v$ such that $\|v\| < \delta'$. Take $\varphi_p^{-1}(v) \in \varphi_p^{-1}(N_p) \setminus \{p\}$ and note that

$$f_Z(\varphi_p^{-1}(v)) = \varphi_p^{-1} \circ P_Y^\pi(p) \circ \varphi_p(\varphi_p^{-1}(v)) = \varphi_p^{-1} \circ P_Y^\pi(p)(v) = \varphi_p^{-1}(v).$$

So, $d(p, \varphi_p^{-1}(v)) = d(p, f_Z(\varphi_p^{-1}(v))) = \|v\| < \delta'$. Then, as was mentioned before, $d(Z^t(p), Z^\alpha(t)(\varphi_p^{-1}(v)) < \delta$, for $0 \leq t \leq \pi$. Therefore, we can find a continuous function $\alpha : \mathbb{R} \to \mathbb{R}$, with $\alpha(0) = 0$, such that $d(Z^t(p), Z^\alpha(t)(\varphi_p^{-1}(v))) < \delta$, for every $t \in \mathbb{R}$. Now, since $Z \in \mathcal{E}^\mu_\mu(M)$, $\varphi_p^{-1}(v) = Z^s(p)$, for $|s| \leq \epsilon$. This is a contradiction, because $\varphi_p^{-1}(v) \in \varphi_p^{-1}(N_p) \setminus \{p\}$.

Now, if $|\lambda| = 1$ but $\lambda \neq 1$, we point out that, by [4, Lemma 3.2], we can find $W \in \mathcal{U}$ such that $P_W^\pi(p)$ is a rational rotation. Then, there is $T \neq 0$ such that $P_W^{T+\pi}(p) = id$. So, we can go on with the previous
Lemma 3.2. If $X \in \text{int } S^1_\mu(M)$ then any closed orbit of $X$ is hyperbolic.

Proof. Take $X \in \text{int } S^1_\mu(M)$, $\mathcal{U}$ a $C^1$-neighbourhood of $X$ in $S^1_\mu(M)$ and $p$ be a closed orbit of $X$ with period $\pi > 0$. By contradiction, assume that there is an eigenvalue $\lambda$ of $P^x_\mu(p)$ such that $|\lambda| = 1$.

By Zuppa’s Theorem (see [18]), we can find $Y \in \mathcal{U}$ such that $Y \in X^\infty_\mu(M)$, $Y^\pi(p) = p$ and $P^x_\mu(p)$ has an eigenvalue $\lambda$ with $|\lambda| = 1$, as we remarked in the proof of Lemma 3.1.

Consider $\varphi$ and $Z \in \mathcal{U}$ as described in the proof of Lemma 3.1 and

$$f_Z(x) = \begin{cases} \varphi_p^{-1} \circ P^x_\mu(p) \circ \varphi_p(x) & , x \in B_{\epsilon_0}(p) \cap \varphi_p^{-1}(N_p) \\ f_Y(x) & , x \notin B_{\epsilon_0}(p) \cap \varphi_p^{-1}(N_p) \end{cases},$$

where $\epsilon_0 > 0$ is small.

As it was explained in the proof of Lemma 3.1 we can assume $\lambda = 1$ and fix the associated non-zero eigenvector $v$ such that $\|v\| = \epsilon_0/2$.

Define $T_i = \{sv : 0 \leq s \leq 1\}$.

Since $Z \in S^1_\mu(M)$, for any $\epsilon > 0$ there is $\delta > 0$ such that every $(\delta, T)$-pseudo-orbit is $\epsilon$-shadowed by some orbit $y$ of $Z^t$, for $T > 0$. Fix $0 < \epsilon \leq \frac{\epsilon_0}{4}$. The idea now is to construct a $(\delta, T)$-pseudo-orbit of $Z^t$, adapting the strategy described on [7] Proposition A]. Let us present the highlights of that proof.

Let $x_0 = p$ and $t_0 = 0$. Since $p$ is a parabolic closed orbit, we construct a finite sequence $\{(x_i, t_i)\}_{i=0}^n$, where $I \in \mathbb{N}$, $t_i > 0$, $x_i \in \varphi_p^{-1}(T_i)$, for $1 \leq i \leq I$, such that: $x_I = \varphi_p^{-1}(v)$; $d(Z^t(f_Z(x_i)), Z^t(x_{i+1})) < \delta$, for $|t| \leq T$ and $0 \leq i \leq I - 1$; $Z^t(x_i) = f_Z(x_i)$. So, letting $S_n = \sum_{i=0}^n t_i$, for $0 \leq n \leq I$, the map $\psi : \mathbb{R} \to M$ defined by

$$\psi(t) = \begin{cases} Z^t(x_0) & , t < 0 \\ Z^{t-S_n}(x_{n+1}) & , S_n \leq t < S_{n+1}, 0 \leq n \leq I - 2 \\ Z^{t-S_{I-1}}(x_I) & , t \geq S_{I-1} \end{cases},$$

is a $(\delta, T)$-pseudo-orbit of $Z^t$. Now, since $Z \in \mathcal{U}$, there is a reparametrization $\alpha$ and a point $y \in B_{\epsilon}(p) \cap \varphi_p^{-1}(N_{p,\epsilon})$ which $\epsilon$-shadows $\psi$; that is, $d(Z^\alpha(y), \psi(t)) < \epsilon$, for any $t \in \mathbb{R}$. Note that, since $\lambda = 1$,

$$d(x_0, x_I) = d(p, \varphi_p^{-1}(v)) = d(p, f_Z(\varphi_p^{-1}(v))) = \|v\| = \frac{\epsilon_0}{2} > 2\epsilon.$$

But, since $Z$ has the shadowing property,

$$d(x_0, x_I) \leq d(x_0, Z^{\alpha(S_{I-1})}(y)) + d(Z^{\alpha(S_{I-1})}(y), \psi(S_{I-1})) < 2\epsilon,$$
Lemma 3.3. If $X \in \mathcal{X}^1_\mu(M)$ has a singularity then, for any neighbourhood $V$ of $X$, there is an open and nonempty set $U \subset V$ such that any $Y \in U$ has a linear hyperbolic singularity.

Proof. Let $p$ be a singularity of $X \in \mathcal{X}^1_\mu(M)$ and $\epsilon > 0$. By a small $C^1$-conservative perturbation of $X$ (see [4]), we can find $X_1, \epsilon$-C$^1$-close to $X$, with a hyperbolic singularity $p$. Denote by $V$ a $C^1$-neighbourhood of $X_1$ in $\mathcal{X}^1_\mu(M)$ where the analytic continuation of $p$ is well-defined. Now, by Zuppa’s Theorem (see [18]), there is a smooth vector field $X_2 \in V$ with a hyperbolic singularity $p_2$. If the eigenvalues of $DX_2(p_2)$ satisfy the nonresonance conditions of the Sternberg linearization theorem (see [15]) then there is a smooth diffeomorphism conjugating $X_2$ and its linear part around $p_2$. If the nonresonance conditions are not satisfied then we can perform a $C^1$-conservative perturbation of $X_2$, so that the eigenvalues satisfy the nonresonance conditions. So, since the set of divergence-free vector fields satisfying the nonresonance conditions is an open and dense set in $\mathcal{X}^1_\mu(M)$, there is a $C^1$-neighbourhood $U$ of $X_2$ in $V$ such that any vector field $X_3 \in U$ is conjugated to its linear part, meaning that $X_3$ has a linear hyperbolic singularity. □

Proof of Theorem 1. Take $X \in \text{int}\mathcal{E}^1_\mu(M)$ and let $U$ be a $C^1$-neighbourhood of $X$ in $\mathcal{E}^1_\mu(M)$, small enough such that Theorem 2.1 holds.

Recall that a conservative version of Pugh and Robinson’s General Density Theorem (see [13]) asserts that, $C^1$-generically, the closed orbits are dense in $M$. Denote by $\mathcal{P}R_\mu(M)$ the Pugh and Robinson’s residual set in $\mathcal{X}^1_\mu(M)$ and by $\mathcal{R}$ the residual set given by Theorem 2.4.

By contradiction, assume that there is $p \in \text{Sing}(X)$. By Lemma 3.3, there is $Y \in U \cap \mathcal{R} \cap \mathcal{P}R_\mu(M)$ such that $p \in \text{Sing}(Y)$ is linear hyperbolic, and so of saddle-type. So, by Proposition 2.3, $P^T_Y$ does not admit any dominated splitting over $M\backslash \text{Sing}(Y)$.

We point out that, by Lemma 3.1, any closed orbit of $Y$ is hyperbolic. Now, as in the proof of [6, Lemma 3.1], take a closed orbit $x$ of $Y$ with arbitrarily large period. So, by Theorem 2.1 there are constants $\ell, \tau > 0$ such that $P^\ell_Y$ admits an $\ell$-dominated splitting over the $Y'$-orbit of $x$ with period $\pi(x) > \tau$. Since $Y \in \mathcal{R}$, by the volume preserving Arnaud Closing Lemma (see [11, p.13]), there is a sequence of vector fields $Y_n \in U \cap \mathcal{R}$, $C^1$-converging to $Y$, and, for every $n \in \mathbb{N}$, $Y_n$ has a closed orbit $\gamma_n = \gamma_n(t)$ of period $\pi_n$ such that $\lim_{n \to \infty} \gamma_n(0) = x$ and $\lim_{n \to \infty} \pi_n = +\infty$. Therefore, by Theorem 2.1, $P^\ell_{Y_n}$ admits an $\ell$-dominated splitting over the orbit $\gamma_n$, for large $n$. Choosing $i \in J \subseteq \mathbb{N}$, there is
a sequence of $Y_i$ with $P_{Y_i}$ having an $\ell$-dominated splitting on a closed orbit $p_i$ and such that the dimensions of the invariant bundles do not depend on $i$. Then, given that $M = \limsup_{n} \gamma_n = \bigcap_{N \in \mathbb{N}} \left( \bigcup_{n \geq N} \gamma_n \right)$, we prove that $P_{Y_i}$ admits a dominated splitting over $M \setminus \text{Sing}(Y)$. But this is a contradiction. So, $\text{Sing}(X) = \emptyset$ and, by Lemma 3.1, one has that if $X \in \text{int} \mathcal{S}_\mu^1(M)$ then $X \in \mathcal{G}_\mu^1(M)$. Then, by Theorem 2.2, $X$ is Anosov.

Now, take $X \in \text{int} \mathcal{S}_\mu^1(M)$. Applying Lemma 3.2 we can follow an analogous strategy to that one described above and prove that if $X \in \text{int} \mathcal{S}_\mu^1(M)$ then $\text{Sing}(X) = \emptyset$ and $X$ is Anosov.

In order to conclude the proof of Theorem 1 it is enough to see that $\mathcal{L}\mathcal{S}_\mu^1(M) \subset \mathcal{S}_\mu^1(M)$ and that $\mathcal{A}_\mu^1(M) \subset \mathcal{L}\mathcal{S}_\mu^1(M)$, by [11, Theorem 1.5.1].

Acknowledgements

I would like to thank my supervisors, Mário Bessa and Jorge Rocha, whose suggestions and guidance enabled me to develop this work.

The author was supported by Fundação para a Ciência e a Tecnologia, SFRH/BD/33100/2007.

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