Null Surfaces: Counter-term for the Action Principle and the Characterisation of the Gravitational Degrees of Freedom

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January 7, 2015

Abstract

Constructing a well-posed variational principle and characterizing the appropriate degrees of freedom that need to be fixed at the boundary are non-trivial issues in general relativity. For spacelike and timelike boundaries, one knows that (i) the addition of a counter-term [like the Gibbons-Hawking-York (GHY) counter-term] will make the variational principle well-defined and (ii) the degrees of freedom to be fixed on the boundary are contained in the induced 3-metric. These results, however, do not directly generalize to null boundaries on which the 3-metric becomes degenerate. In this work, we address the following questions: (i) What is the counter-term that needs to be added on a null boundary to make the variational principle well-defined? (ii) How do we characterize the degrees of freedom which need to be fixed at the boundary? We show that the counter-term to be added is $2\sqrt{q}(\Theta + \kappa)$ and that the degrees of freedom to be fixed on the surface are in the induced 2-metric on a null surface, $q^{ab}$, and the tangent vector $\ell^a$ to the null congruence on the surface. We also demonstrate that the degrees of freedom in $\ell^a$ can be eliminated by choosing suitable coordinates. This allows one to identify the physical degrees of freedom of the gravitational field with components $q^{ab}$ of the 2-metric in a suitable $(1+1+2)$ double null parametrization of the spacetime. The implications are discussed.

1 Introduction and Summary

Just like any other field theory, the dynamics of gravity can be obtained from an action, the Einstein-Hilbert action. On varying the action with respect to the metric, we obtain an equations-of-motion term and a boundary term. The equations of motion turn out to be second order in the derivatives of the metric. But the boundary term is unusual, as it contains variations both of the metric and its first derivatives normal to the boundary [1]. Thus, setting the variation of the action to zero will lead to the equations of motion on the bulk only if we fix the metric and its normal derivatives on the boundary. The problem with such a structure is that it makes the variational principle ill-defined [2]. In general, the equations of motion and the boundary conditions will turn out to be inconsistent [2, 3].

There is a widely accepted prescription for resolving this issue. One adds an extra term to the action, called a counter-term, such that the surface term in the variation of the new action contains only variations of the metric and does not involve the variations of the normal derivatives. Thus, we need to fix only the metric on the boundary and the variational principle becomes well-posed. The most

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commonly used counter-term is the Gibbons-Hawking-York (GHY) counter-term [4, 5], although there are other counter-terms available in the literature [6].

The GHY prescription has been around for a long time and is now textbook material (see, e.g., Chapter 6 of [1]). The counter-term is an integral, over the boundary surface \( \partial V \) of the spacetime region \( V \) under consideration, of essentially the product of the square root of the determinant of the 3-metric on the surface and the divergence of the unit normal to the surface. The procedure of its construction uses (a) the unit normal to \( \partial V \), (b) the induced metric \( h_{ab} \) on \( \partial V \) and (c) the covariant derivative on \( \partial V \) compatible with the induced metric \( h_{ab} \). These structures are well-defined for spacelike or timelike regions of the boundary surface. The trouble is, we can have a third kind of region on the boundary surface, viz. a null surface, which is ubiquitous in general relativity. The horizons of black holes, for example, are null surfaces. It is generally accepted that entropy and temperature can be associated with black hole horizons [7, 8, 9, 10, 11] and — more generally — with any local Rindler horizon [12, 13], which is also a null surface. These properties are not shared by any spacelike or timelike surface. The trouble with the standard prescription is that it is not directly applicable to a null surface as the normal to a null surface has a zero norm and the natural 3-metric on a null surface is degenerate. Also, there is no natural notion of a covariant derivative on a null surface making the usual procedure inapplicable.

A possible work-around is to consider a timelike surface infinitesimally separated from the null surface ("stretched horizon"), perform the calculations and then take the null limit. As we shall demonstrate, this approach does give a proper, finite expression for the boundary term and the counter-term when the null limit is taken. But, given the importance of this problem, one would like to have a first-principle approach, leading to a prescription for the counter-term to be added on the null surface purely based only on properties of the surface, without recourse to any limiting procedure. Surprisingly, this issue does not seem to have attracted sufficient attention in the literature. As far as we know, there is no approach available in the literature that allows one to start from first principles and find the counter-term for a null surface. (The only related reference we could find was [14] which treats a very specialized metric and is not applicable to a general null surface and does not address several important technical and conceptual issues.)

There is another important reason for attempting this study. For the non-null surfaces, we also obtain, from the variational principle itself, the result that only the components of the induced metric (modulo 3D diffeomorphisms) need to be fixed at the boundary [15]. So, in a timelike/spacelike surface one can identify the gravitational degrees of freedom as contained in the 6 components of the induced 3-metric \( h_{ab} \). Given the freedom in transforming the 3 coordinates on \( \partial V \), this leaves 3 independent degrees of freedom. (It is possible to reduce it further to 2 using the freedom in foliation but that reduction is irrelevant to the aspects we want to stress.) When we try to obtain the results for a null surface as a limit of the results on a non-null surface, the question arises as how to characterize these degrees of freedom in the null limit. The properly defined induced metric for null surfaces is essentially a 2-metric \( q_{ab} \). It is not clear how the six degrees of freedom in the induced metric \( h_{ab} \) on the non-null surface could relate to the three degrees of freedom in the 2-metric \( q_{ab} \) on taking the null limit.

We stress that it should be possible to tackle the above issues from first principles, by just studying the form of the variation of the action \( \delta A_{\rho \nu} \), on the null boundary. (We shall use the notation \( \delta A_{\rho \nu} \) for the boundary term in the variation of the Einstein-Hilbert action for a boundary \( \partial V \) of the spacetime region \( V \) under consideration.). The algebraic structure of \( \delta A_{\rho \nu} \) should by itself reveal, without we having to do anything, (a) the counter-term that needs to be added to make the variation well-defined and (b) the variables which need to be fixed on the boundary. (Such a ‘constructive’ and simple procedure was not available in literature until recently [15] even for non-null boundary.) It is important to perform this computation for a null boundary.

We will provide comprehensive discussion of these and related issues from several different perspectives in this paper. Because of the rather extensive nature of the discussion, we first summarize the key results obtained in this work.

- We briefly review the procedure of obtaining the counter-term and identifying the variables which need to be fixed on the boundary just from examining the algebraic structure of \( \delta A_{\rho \nu} \) on non-null boundary. Then, we perform a very similar analysis and obtain \( \delta A_{\rho \nu} \), on a general null surface. The final result is given by

\[
[\delta A_{\rho \nu}]_{\text{null}} = \int_{\partial V} d\lambda d^2 z_\perp \left\{-2\delta \sqrt{\det(q + \kappa)} + \sqrt{q} \left[\Theta_{ab} - (\Theta + \kappa) q_{ab}\right] \delta q^{ab} + P_\rho \delta \ell^c\right\}, \quad (1)
\]

(There is also an ignorable total derivative term on the 3-surface, which has been omitted here for simplicity but can be found in the main discussion later on.) The general result as given
in Eq. (1) can be interpreted by the following identifications: \( \ell_a = \partial_a \phi \) is the normal to the \( \phi = \text{constant boundary surface} \) which is null for a specific value of \( \phi \), \( k^a \) is an auxiliary null vector and \( q_{ab} = g_{ab} + \ell_a \ell_b + k_a \kappa_b, \kappa_a \ell_b \) is the induced metric. The integration is over a parameter \( \lambda \), varying along the null geodesic congruence such that \( \ell^a = dx^a/d\lambda \) and two coordinates \((z^1, z^2)\) that are constant along the null geodesics, with \( d^2 z^\pm = dz^1 dz^2 \). The symmetric tensor \( \Theta_{ab} = g^m_{a} q^b_{m} \nabla_m \ell_n \) is the second fundamental form, \( \Theta = \Theta^a_a \) is the expansion scalar and \( \kappa \) is the non-affinity coefficient on the null surface, i.e. \( \ell^a \nabla_a \ell^b = \kappa \ell^b \). (Using \( \ell_a = \partial_a \phi \), as opposed to \( \ell_a = A \partial_a \phi \) for some scalar \( A \), will render \( \kappa = 0 \) if all the \( \phi = \text{constant surfaces} \) are null but not when \( \phi = \text{constant is null only for a specific value of} \phi \); this is why \( \kappa \neq 0 \) even though \( \ell_a = \partial_a \phi \); see discussion later.). The symbol \( q \) is used for the determinant of the 2-metric \( q_{AB} \). The conjugate momentum to \( q^{ab} \) is \( \sqrt{q} \Theta_{ab} - (\Theta + \kappa) q_{ab} \) and the conjugate momenta to \( \ell^a \) turn out to be

\[
P_c = -\sqrt{q} b_c [\nabla_c \ell^b + \nabla^b \ell_c - 2 \delta^b_c (\Theta + \kappa)]
\]

We have also carried out the analysis for \( \ell_a \) having the form \( A \partial_a \phi \), with \( A = 1 \) on the null surface, but with \( \ell^2 = 0 \) everywhere. In that case also, we obtain a decomposition of the same structure as in Eq. (1).

The structure of \([\delta A_{av}]_{null} \) is similar to the familiar decomposition that we obtain for the surface term on a non-null surface:

\[
[\delta A_{av}]_{non-null} = \int \frac{d^3 x}{\delta v} \left[ \delta (2 \sqrt{|N|} K) - \sqrt{|N|} (K_{ab} - K h_{ab}) \delta h_{ab} \right]
\]

(Again, there is a total 3-derivative on the surface that we have omitted here but can be found in the main text.) Thus, Eq. (1) suggests that the counter-term to be added on a null surface is \( 2 \sqrt{|N|} (\Theta + \kappa) \) and that the intrinsic quantities to be fixed on the null surface are \( q^{ab} \) and \( \ell^a \).

- To get a better understanding of the result in Eq. (1), we derive it in a different way. We start with the result in Eq. (3) and then take a careful null limit to arrive back at Eq. (1). The total variation term in Eq. (1) is generated by the corresponding term in Eq. (3), while the \( \delta q^{ab} \) (3 degrees of freedom) term and the \( \delta \ell^a \) (3 degrees of freedom since \( \ell^2 = 0 \)) term are generated by the \( \delta h_{ab} \) (6 degrees of freedom) term, with the various extra terms that are generated cancelling out. Thus, we obtain a direct proof that the 3 degrees of freedom that are to be fixed on the null surface are in one-to-one correspondence with the 3 degrees of freedom that are to be fixed on a non-null surface, albeit in a different form. That is, the 6 degrees of freedom in \( h_{ab} \) translates into (i) three degrees of freedom contained in the induced 2-metric \( q_{ab} \) on the null surface and (ii) three degrees of freedom contained in the components of the tangent vector \( \ell^a \) to the null congruence on the null surface. (As far as we know, this simple and elegant characterization of the degrees of freedom is new and has not been noted before in literature.)

- We undertake a more careful counting of the number of degrees of freedom taking into account the freedom of choice of coordinates. We first demonstrate that the \( \delta \ell^a \) term in Eq. (1) can be eliminated by infinitesimal coordinate transformations on the null surface, allowing us to provide a very natural characterization of the gravitational degrees of freedom as described by the 2-metric \( q_{ab} \) in a \( (1+1+2) \) null surface foliation. We also show that we cannot eliminate any of the three degrees of freedom that need to be fixed in \( q^{ab} \) using the residual freedom in coordinate transformations.

- We illustrate these general results in two natural coordinate systems adapted to describe the metric near an arbitrary null surface. We first review the explicit construction of both these coordinate systems, which we call the Gaussian Null Coordinates (GNC) and Null Surface Foliation (NSF). The GNC metric has the feature that the null surface arises naturally as the limit of a sequence of non-null surfaces. Thus, we can easily apply the GHY prescription on a nearby non-null surface and take the null limit. In the NSF, the fiducial null surface appears as one member of a set of null surfaces. Hence, the limiting procedure cannot be applied directly (and would need the explicit construction of an infinitesimally separated non-null surface). But the NSF has some other advantages. In both cases, we find that we obtain explicit results in agreement with Eq. (1).

The conventions used in this paper are as follows: We use the metric signature \((- , + , + , +)\). The fundamental constants \( G, h \) and \( c \) have been set to unity. The Latin indices, \( a, b, \ldots \), run over all space-time indices, and are hence summed over four values. Greek indices, \( \alpha, \beta, \ldots \), are used when we specialize
to indices corresponding to a codimension-1 surface, i.e. a $3-$ surface, and are summed over three values. Upper case Latin symbols, $A, B, \ldots$, are used for indices corresponding to two-dimensional hypersurfaces, leading to sums going over two values. We shall use $\equiv$ to indicate equalities that are only valid on the null surface. In order to allow the reader to grasp the main spirit of the paper without getting bogged down by details, we have delegated most of the explicit calculation to appendices. The reader may thus refer to the calculations only as and when required.

2 The information contained in the boundary term of the variation of the action

The variation of an action, generically, contains two terms: (a) A bulk term, the vanishing of which will give us the equations of motion and (b) a boundary term, which is usually ignored. But the boundary term contains significant amount of information! By a careful analysis of this term, we can determine the true degrees of freedom which needs to be fixed at the boundary and even the structure of the counter-term that needs to be added to the action principle to make it well-defined. Let us illustrate this idea in electromagnetism and gravity.

2.1 Warm-up: Electromagnetic field

As a warm up, let us consider the simple case of electromagnetism. The action for the free electromagnetic field is given by

$$A = -\frac{1}{16\pi} \int_V F_{ik} F^{ik} d^4x; \quad F_{ik} = \partial_i A_k - \partial_k A_i$$

(4)

Let us begin by assuming that $A_i$ are the dynamical variables and vary them in the action. This leads to

$$\delta A = -\frac{1}{4\pi} \int_V [\partial_k F^{ik}] \delta A_k d^4x + \frac{1}{4\pi} \int_{\partial V} F^{ik} \delta A_i d\sigma_k = -\frac{1}{4\pi} \int_V [\partial_k F^{ik}] \delta A_k d^4x + \frac{1}{4\pi} \int E \cdot \delta A d^3x$$

(5)

where we have assumed that the boundary contributions arise from $t = \text{constant}$ surfaces. Since bulk and boundary variations are independent, $\delta A = 0$ will require vanishing of these two terms individually. Let us first assume (rather naively) that $\delta A_i$ is completely arbitrary in the bulk $V$. Then the vanishing of the bulk term will lead to the equations of motion $\partial_k F^{ik} = 0$. Once this is granted, we next want the on-shell boundary term to vanish. This is, of course, possible if we specify $A_i$ on the boundary; but that is not required. The algebraic structure of the boundary term already tells us that we need not fix $A_0$ at all. Further, even as regards $A$ we only need to fix it modulo the addition of a gradient: $A \rightarrow A + \nabla F$. This is because, $E \cdot \nabla F = \nabla \cdot (EF)$ on-shell (since $\nabla \cdot E = 0$) which allows this term to be converted into a surface integral at spatial infinity and ignored. Thus, we only need to fix the part of $A$ which is unaffected by the addition of a gradient; viz., the $B = \nabla \times A$. In other words:

We do not tell the action principle what needs to be fixed at the boundary; instead, the action principle tells us what should be fixed.

In the case of the electromagnetism it is the magnetic field that is to be fixed at the boundary. We naively thought it is $A_i$ but the algebraic structure of the boundary term in the variation of the action corrects our mistake and tells us that it is the magnetic field.

2.2 Gravity: Non-null boundary

One can do exactly the same in the case of gravity. We will first assume (again naively) that $g_{ab}$ are the dynamical variables and vary them in the Einstein-Hilbert action

$$16\pi A_{EH} = \int_V d^4x \sqrt{-g} R,$$

(6)
where $\mathcal{V}$ indicating the spacetime region under consideration, over which the integration is performed. On varying the metric, we get

$$16\pi \delta A_{\text{EH}} = - \int_{\mathcal{V}} d^4x \sqrt{-g} G^{ab} \delta g_{ab} + \int_{\partial \mathcal{V}} d^3x \sqrt{-g} \nabla_c \delta R_{ab}$$

$$\equiv - \int_{\mathcal{V}} d^4x \sqrt{-g} G^{ab} \delta g_{ab} + \int_{\partial \mathcal{V}} d^3x \sqrt{-g} \nabla_c \delta w^c$$

$$= - \int_{\mathcal{V}} d^4x \sqrt{-g} G^{ab} \delta g_{ab} + \int_{\partial \mathcal{V}} d^3S \ n_c \delta w^c$$

$$\equiv - \int_{\mathcal{V}} d^4x \sqrt{-g} G^{ab} \delta g_{ab} + \int_{\partial \mathcal{V}} d^3S Q[n_c]$$

(7)

where $d^3S$ is the appropriate integration measure on the boundary, $n_c$ is the normal with constant norm (0 or $\pm 1$) on the surface, $w^c \equiv (g^{ab} \delta \Gamma_{ab}^c - g^{ck} \delta \Gamma_{ak}^c)$ and we have defined the function $Q[A_c]$ for any one-form $A_c$ as

$$Q[A_c] = A_c (g^{ab} \delta \Gamma_{ab}^c - g^{ck} \delta \Gamma_{ak}^c) .$$

(8)

which will prove to be useful throughout the paper. Straightforward algebra now leads to the following alternate expression for $Q[n_c]$:

$$Q[n_c] = \nabla_a (\delta n_a^c) - \delta (2 \nabla_a n^a) + \nabla_a n_b \delta g^{ab}; \quad \delta n_a^c \equiv \delta n_a + g^{ab} \delta n_b$$

(9)

Note that $\delta n^a_n a = 0$, which implies that the vector $\delta n^a_a$ lives on the boundary. The description so far is completely general and holds for both null and non-null boundaries with suitable measure $n_c d^3S$ for integrating vector fields.

For the non-null case [15], we have the relation $\nabla_a V^a = D_a V^a - c a b V^b$ valid for any $V^a$ which satisfies $V^a n_a = 0$, where $c_i \equiv n^i \nabla_j n_i$. Further, the integration measure is now $d^3S = \sqrt{h} d^3x$ with $n_a$ normalised to $\pm 1$ so that the surface term becomes the integral of $d^3x \sqrt{\left| h \right|} Q[n_c]$. This will lead to the expression:

$$Q[n_c] = D_a (\delta n_a^c) - \delta (2 \nabla_a n^a) + (\nabla_a n_b - c n_a b) \delta h^{ab}$$

(10)

Note that this algebraic manipulation has naturally led us to the combination $(\nabla_a n_b - c n_a b)$ without us having to introduce any geometrical considerations. This combination, of course, is the negative of the extrinsic curvature [1] defined by $K_{ab} = - h_{ab} \nabla_c n_b$ and has the following properties: (a) $K_{ij} = K_{ji}$; (b) $K_{i} n^i = 0$; (c) $K = g^{ij} K_{ij} = - \nabla_a n^a$. The second property tells us that $K_{ij} \delta g^{ij} = K_{ij} \delta h^{ij}$ which has been used to arrive at Eq. (10). The surface term is an integral of $Q[n_c]$ over $d^3x \sqrt{\left| h \right|}$ which allows the first term $D_a (\delta n_a^c)$ in Eq. (10) to be converted to a 2-dimensional surface integral and ignored. The rest of the terms will reduce to the following expression for the boundary term of EH action:

$$[\delta A_{ab}]_{\text{non-null}} = \int_{\partial \mathcal{V}} d^3x \left[ \delta (2 \sqrt{\left| h \right|} K) - \sqrt{\left| h \right|} (K_{ab} - K h_{ab}) \right] \delta h^{ab}.$$

(11)

We need to set $[\delta A_{ab}]_{\text{non-null}}$ to zero with appropriate boundary conditions to obtain the equations of motion in the bulk from the action principle $\delta A = 0$. Unlike in the case of, e.g., electromagnetism, this is not straightforward because we will need to set both $\delta [K \sqrt{h}] = 0$ and $\delta h^{ab} = 0$ on the boundary. This is not acceptable, since $\delta K$ contains variations of normal derivatives of $h_{ab}$ as well. Setting both $\delta [K \sqrt{h}] = 0$ and $\delta h^{ab} = 0$, in general, will lead to inconsistency with the Einstein’s equations. (For a discussion of this and other issues with fixing the metric and its derivatives at the boundaries, see, e.g., [2].)

Incredibly enough, the algebraic structure of Eq. (11) itself suggests a way out. Since the term involving $\delta [K \sqrt{h}]$ is the variation of a boundary integral, we can add this as a counter-term to the action such that the surface term arising from the new action does not depend on the variations of the normal derivatives of the metric. Once this is done, we only have to set $\delta h^{ab} = 0$ on the boundary. This leads to the GHY counter-term [4, 5], viz. integral of $-2 K \sqrt{h}$ over the boundary, which is the most commonly used one for this purpose.

Thus, again, we do not have to tell the action what variables to fix on the boundary or even what counter-terms to add! Instead, the algebraic structure of the boundary term itself tells us not only (i) what degrees of freedom need to be fixed on the boundary [viz. $h_{ab}$] but also (ii) what counter-term to add to the action to make the variation well-defined.
3 Boundary Term of the Einstein-Hilbert Action: Null Surfaces

The crucial difference between non-null and null boundaries is the following: If the surface is non-null, we can define an induced metric $h_{ab}$ and a covariant derivative operator $D_a$ compatible with it (i.e., $D_a h_{bc} = 0$). We can then re-express $\nabla_a (\delta n^a)$ in terms of the covariant derivative $D_a (\delta n^a)$ on the boundary which leads to significant simplification. We cannot do this for a null surface, which makes a vital algebraic difference. We, therefore, need to use some other procedure to analyse the algebraic structure of the boundary variation on a null-surface.

One possible approach to handle a null surface would be to treat the null surface as a limit of a sequence of non-null surfaces, apply the prescription given above on the non-null surfaces and then take the limit in which this sequence goes over to a null surface. This can be done with the help of a parameter which labels the surfaces such that a particular value of the parameter corresponds to the null surface; we will do this in Section 3.4. However, we would also like to formulate a first-principle prescription that can be applied to the null surface as a surface in its own right and not as a fringe member of some family. This will be done in Section 3.3. As a preamble to these two derivations, we shall first recall several features of null surfaces relevant to our task.

3.1 Null surfaces

A null surface in a four-dimensional spacetime $M$ is a three-dimensional submanifold $N$ with the criterion that the metric $\gamma_{\mu\nu}$ obtained by the restriction of the full metric $g_{ab}$ to the hypersurface $N$ is degenerate, which means that it is possible to find non-zero vectors $v^\mu$ on $N$ such that $\gamma_{\mu\nu} v^\mu = 0$. We shall assume that the full $g_{ab}$ is non-degenerate, i.e., every non-zero four-dimensional vector is mapped by the metric to a non-zero one-form. If $v^a$ is the four-dimensional push-forward of $v^\mu$ and $w^a$ is the four-dimensional push-forward of $w^\mu$, another vector at the same point on $N$, then $g_{ab} v^a w^b = \gamma_{\mu\nu} v^\mu w^\nu = 0$. Thus, $w^a$ is orthogonal to every vector at that point on the surface including itself. A set of such null vectors on the surface, one vector per point, gives a null vector field $\ell^a$ normal to $N$. (There will be only one null curve on the surface passing through each point on $N$. So the only freedom we have in the choice of the null vector is the freedom of scaling by a scalar factor.) Note that $\ell^a$ has the peculiar property that it is both tangent and normal to the surface: tangent since we had started with the vector being on the null surface and hence we will be moving along the surface if we move along the orbits of $\ell^a$; and normal since we have $g_{ab} w^a \ell^b = 0$ for any vector $w^a$ on the surface.

If $N$ corresponds to some $=\text{constant}$ surface, the corresponding normal one-form can be expressed as $\ell_a = A \partial_\phi$, where $A$ is a scalar. Since we have assumed the four-dimensional metric $g_{ab}$ to be non-degenerate, $A \neq 0$. As proved in Appendix A.2.1, we will then have

$$\ell^a \nabla_a \ell_b := \kappa \ell_b; \quad \kappa \equiv \ell^a \partial_a (\ln A) - \frac{k}{2} \partial_a (\ell^a \ell_a)$$

(12)

Note that while we assume $\ell^2 = 0$ on the null surface, it can in general be non-zero off the surface. So $\kappa$ has two contributions: one arises because $A \neq 1$ and the other because $\ell^2 \neq 0$ off the surface. Very often, we will set $A := 1$ on the null surface making $\ell_a := \nabla_a \phi$ on it. The above result shows that $\kappa$ can be still non-zero on the null surface due to the second term in Eq. (222).

Since $\ell^a \nabla_a \ell_b := \kappa \ell_b$ is a geodesic equation we see that the orbits of $\ell^a$ are geodesics. We may call $\kappa$ as the non-affinity coefficient because $\kappa = 0$ if we choose an affine parametrization. Since $\ell^a$ lies on the null surface, the null surface can be thought of as being filled by a congruence of null geodesics. (If e.g., $\ell^a$ is the Killing field normal to the horizon of a stationary black hole in an asymptotically flat spacetime, we can identify $\kappa$ with the surface gravity of the horizon [16]). Two specific parametrizations of the metric on a general null surface are provided in Section 4.

3.2 Structure of the boundary term

To study the boundary term on a null surface, it is more convenient to use the notion of a surface gradient rather than normal to the surface and rewrite the variation of the Einstein-Hilbert action in the following form:

$$16\pi \delta A_{\text{EH}} = - \int_M d^4 x \sqrt{-g} \bar{G}^{ab} \delta g_{ab} + \int_{\partial Y} d^3 x v_c \left[ \sqrt{-g} \left( g^{ab} \delta \Gamma^c_{ab} - g^{cd} \delta \Gamma^a_{cd} \right) \right]$$

(13)

We have again converted the total divergence $\nabla_a w^a$ in Eq. (7) to a term on the boundary and defined $v_c$ to be the surface gradient (not the unit normal) to the surface that bounds the volume. If the surface
is given by the equation $\phi = \text{constant}$, then \( v_c \equiv \pm \partial_c \phi \) with the appropriate sign chosen as per the conventions of Gauss’ theorem (see Appendix A.1). When the volume integral is done as usual with the higher value of \( \phi \) at the upper limit of the integration, \( \partial_c \phi \) or \( -\partial_c \phi \) is used depending on whether \( \phi \) or \( -\phi \) increases on going from inside the volume to outside through the surface. (Note that $\sqrt{-g}$ appears in Eq. (13) since we have used the surface gradient instead of the unit normal. This is appropriate at this stage since \( \mathcal{V} \) could be null in which case there is no natural notion of “unit” normal \( n_a \) or $\sqrt{|h|}$. If we choose to use the unit normal \( n_a \), then \( n_a \sqrt{|h|} \) will replace \( v_a \sqrt{-g} \); see Appendix B.)

We have earlier defined the useful function \( Q[A_c] \) for any one-form \( A_c \) as

$$Q[A_c] \equiv A_c(g^{ab}\delta \Gamma_{ab} - g^{ck}\delta \Gamma_{ak}) = \nabla_a(\delta A^a) - \delta(2\nabla_a A^a) + (\nabla_a A_b) \delta g^{ab}$$

(14)

where \( \delta A^a \equiv \delta A^a + g^{ab}\delta A_b \). Adding a $\sqrt{-g}$ and manipulating, we can obtain

$$\sqrt{-g}Q[v_c] = \sqrt{-g}\nabla_c[\delta v^c] - 2\delta(\sqrt{-g}\nabla_a v^a) + \sqrt{-g}(\nabla_a v_b - g_{ab}\nabla_c v^c)\delta g^{ab},$$

(15)

When \( v_c = \partial_c \phi \) is the surface gradient, the integral of this expression gives the surface term of the Einstein-Hilbert action on a \( \phi = \text{constant} \) surface, irrespective of whether the surface is timelike, spacelike or null. The surface term $\sqrt{-g}Q$ depends only on the surface gradient, \( \partial_c \phi \), at the \( \phi = \phi_0 \) surface. It does not depend on the normalization we may choose for \( v_c \), i.e., it does not depend on \( B \) if we choose to write \( v_c = B\partial_c \phi \). It also does not depend on the behaviour of \( v_c \) away from the surface. But the decomposition in Eq. (15) needs to be rewritten if we choose to work with \( v_c = B\partial_c \phi \). Also, the decomposition cares about the behaviour of \( v_c \) away from the surface due to the presence of derivatives of \( v_c \).

Let us now concentrate on the null-surface and denote the surface gradient on it by \( v_c \equiv \ell_c = \nabla_c \phi \). The expression \( \delta \ell^a \equiv \ell_a \delta \ell_a + \ell^a\delta \ell_a = \ell(\ell^2) \) is zero whenever \( \ell_a \) a constant norm (even if the norm is zero) and the variation preserves this norm, which we shall assume is the case. So we can have \( \ell_a \delta \ell^a = 0 \) even on a null surface when the surface gradient has zero norm. This would imply that \( \delta \ell^a \) again lives on the boundary, i.e., on the null surface. This would suggest — following the route we took for non-null surfaces — converting the first term in Eq. (15) into a surface term on the 3–surface \( \phi \equiv \phi_0 \). For this, we used the notion of surface covariant derivative on the null surface, which, in turn, requires a projector on to the surface. Based on the analogy with non-null surfaces (where \( h_{ab} = g_{ab} - \epsilon n_an_b \) is the projector), if we try \( h_{ab} = g_{ab} - \ell_a\ell_b \), it does not work because now: \( h_{ab}\ell^a = \ell_b - \ell^a\ell_a \ell_b = \ell_b 
eq 0 \), since \( \ell_a\ell^a = 0 \) on the null surface. Thus, there is no straightforward extension of the projector in the non-null case to the null case, essentially because the metric is degenerate.

The solution to this problem was suggested by Carter in [17]. Since we are not able to define a projector by using \( \ell_a \) alone, we choose an auxiliary vector \( k^a \) such that \( \ell_a k^a = -1 \) at \( \phi = \phi_0 \). Then, as we prove in Appendix A.2.2, \( q_{ab} = g_{ab} + \ell_a k_b + \ell_b k_a \) is the object on the null surface analogous to \( h_{ab} \). Since \( k^a \) is under our control, we shall assume that \( \ell_a k^a = -1 \) and \( k_a k^a = 0 \) to be valid everywhere. Thus, we have

$$\ell_a \ell^a := 0; \quad \ell_a k^a = -1; \quad k_a k^a = 0.$$

(16)

The first relation is enforced only on the null surface. Further, we shall assume that these relations are respected by the variations.

### 3.3 Boundary contribution in terms of 2-surface variables

In order to find an expression for the boundary term on a null surface in terms of the 2-surface variables, similar to the expression for non-null surfaces in Eq. (11), we shall start with the expression for the surface term in Eq. (15), with \( v_c = \partial_c \phi \) being the surface gradient. We may take our null normal to be \( \ell_a = A\partial_a \phi = A\nu_a \), where \( A \) is some scalar. Then, recalling the definition of \( Q \) in Eq. (8), we can write the boundary term on the null surface in the following form:

$$\sqrt{-g}Q[v_a] = \sqrt{-g}Q[\ell_a] = \frac{1}{A} \left\{ \sqrt{-g}\nabla_c[\delta \ell^c] - 2\delta(\sqrt{-g}\nabla_a \ell^a) + \sqrt{-g}(\nabla_a \ell_b - g_{ab}\nabla_c \ell^c)\delta g^{ab} \right\}.$$

(17)

But since we do not seem to have any natural way of fixing the factor \( A \) (unlike the case in the non-null case where the condition \( n_a n^a = \epsilon \) was a natural choice) we shall make the choice \( A = 1 \). This offers two advantages in our manipulations. First, it eliminates the \( 1/A \) factor in Eq. (17). Secondly, we will have \( \delta \ell_a = \partial_a \delta \phi = 0 \) as the scalar \( \phi \) labelling the null surface is not being varied. With the choice \( A = 1 \), the boundary term on the null surface becomes

$$\sqrt{-g}Q[\ell_a] = \sqrt{-g}\nabla_c[\delta \ell^c] - 2\delta(\sqrt{-g}\nabla_a \ell^a) + \sqrt{-g}(\nabla_a \ell_b - g_{ab}\nabla_c \ell^c)\delta g^{ab}.$$

(18)
Our first task will be to separate out a surface term from the first term in Eq. (18). We shall label this term as \( \sqrt{-g}Q_1 \). We have

\[
\sqrt{-g}Q_1 = \sqrt{-g} \nabla_a \delta \ell^a_1 \mid = \partial_a [\sqrt{-g} \delta \ell^a_1] 
\]

(19)

Let us consider this expression in the coordinates \((\phi, y_1, y_2, y_3)\), where \((y_1, y_2, y_3)\) are some coordinates introduced on the \(\phi\)-constant null surface under consideration. Then, the partial derivatives with respect to \((y_1, y_2, y_3)\) are on the null surface and the \(\phi\)-derivative is off the surface. Thus, we have

\[
\sqrt{-g}Q_1 = \partial_\phi [\sqrt{-g} \delta \ell^\phi_1] + \partial_a [\sqrt{-g} \delta \ell^a_1], 
\]

(20)

where \(a\) runs over \((y_1, y_2, y_3)\). Now, \(\delta \ell^\phi_1 = \delta \ell^a_1 \ell_a = \delta (\ell_a \ell^a)\). We have assumed this to be zero on the null surface but not off it. Hence, the first term in Eq. (20) is not zero and \(\sqrt{-g}Q_1\) is not a total surface derivative on the null surface.

In order to separate out a total surface derivative on the null surface from \(\sqrt{-g}Q_1\), we can use one of the two projectors, \(\Pi^a_b\) and \(q^a_b\), that we have introduced in Appendix A.2.2. On \(\delta \ell^a_1\). They are defined by

\[
\Pi^a_b = \delta^a_b + k^a \ell_b, 
\]

(21)

\[
q^a_b = \delta^a_b + k^a \ell_b + \ell^c k_b. 
\]

(22)

Now, \(\Pi^a_b\) projects orthogonal to \(\ell_a\) but \(q^a_b\) projects orthogonal to both \(\ell_a\) and \(k_a\). Since our aim is to separate out derivatives along the surface (i.e., orthogonal to \(\ell_a\)) even if they are not orthogonal to \(k_a\), we shall use the projector \(\Pi^a_b\). Thus,

\[
\sqrt{-g}Q_1 = \partial_a [\sqrt{-g} \Pi^a_b \delta \ell^b_1] - \partial_a [\sqrt{-g} \Pi_b^c \delta \ell^c_1] - \partial_a [\sqrt{-g} k^c \ell_b \delta \ell^c_1] 
\]

\[
= \partial_a [\sqrt{-g} \Pi^a_b \delta \ell^b_1] - \sqrt{-g} (\ell_a \ell^c) \nabla_k b - \sqrt{-g} k^b \nabla b [\delta (\ell_a \ell^c)] 
\]

\[
= \partial_a [\sqrt{-g} \Pi^a_b \delta \ell^b_1] - \sqrt{-g} k^b \partial_b [\delta (\ell_a \ell^c)], 
\]

(23)

where the last step was obtained by using our assumption \(\delta (\ell_a \ell^c) = 0\) on the null surface. The first term in Eq. (23) is a surface derivative on the null surface as \(\Pi^a_b \ell_a = 0\). The second term in Eq. (23) is a bit of a bother. With \(\ell_a = \partial_\phi \phi, \delta (\ell_a \ell^c) = \delta \phi^\alpha\). Hence, this term has variations of the derivatives of the metric. We shall take out the \(\delta\) to obtain

\[
- k^b \partial_b [\delta (\ell_a \ell^c)] = - \delta [k^b \partial_b (\ell_a \ell^c)] + \delta k^b \partial_b (\ell_a \ell^c). 
\]

(24)

The vector \(\delta k^a\) lies on the null surface as \(\ell_a \delta k^a = \delta (\ell_a k^a) = 0\) as per our assumption \(\delta \ell_a = 0\). Since \(\ell_a \ell^c = 0\) on the null surface, the last term in Eq. (24) is zero and we are left with

\[
- k^b \partial_b [\delta (\ell_a \ell^c)] = - \delta [k^b \partial_b (\ell_a \ell^c)]. 
\]

(25)

Substituting in Eq. (23), we obtain

\[
\sqrt{-g}Q_1 = \partial_a [\sqrt{-g} \Pi^a_b \delta \ell^b_1] - \sqrt{-g} [k^b \partial_b (\ell_a \ell^c)] 
\]

\[
= \partial_a [\sqrt{-g} \Pi^a_b \delta \ell^b_1] - \delta [\sqrt{-g} k^b \partial_b (\ell_a \ell^c)] - \frac{\sqrt{-g}}{2} [k^b \partial_b (\ell_a \ell^c)] g_{ij} \delta g^{ij}. 
\]

(26)

Thus, we have written \(\sqrt{-g}Q_1\) in a form where all the derivatives of the metric have been smuggled into surface derivatives and total variations.

Substituting Eq. (26) back in Eq. (18), we find the following expression occurring in two places:

\[
\nabla_a \ell^a + \frac{k^a}{2} \partial_a (\ell_b \ell^b) = \delta_\phi \nabla_a \ell^b + k^a \ell_b \nabla_a \ell^b = \Pi^a_b \nabla_a \ell^b. 
\]

(27)

Thus, the boundary term on the null surface reduces to

\[
\sqrt{-g}Q(\ell_a) = \partial_a [\sqrt{-g} \Pi^a_b \delta \ell^b_1] - 2 \delta (\sqrt{-g} \Pi^a_b \nabla_a \ell^b) + \sqrt{-g} (\nabla_a \ell_b - g_{ab} \Pi^c_d \nabla_c \ell^d) \delta g^{ab}. 
\]

(28)

We have succeeded in separating out a total derivative on the surface and a total variation to remove all derivatives of the metric. In order to understand what degrees of freedom need to be fixed on the null surface, we shall now decompose \(g^{ab}\) into \(q^{ab}\), \(\ell^a\) and \(k^a\).
First, note that the relations \( q^{ab}\ell_a = 0 \) and \( q^{ab}k_a = 0 \) are respected by the variations. This is because we have assumed Eq. (16) to be valid even on variation. Thus, terms of the form \( \ell_a\ell_b\delta q^{ab} \), \( \ell_a k_b\delta q^{ab} \) and \( k_a k_b\delta q^{ab} \) would reduce to zero. For example,
\[
\ell_a k_b\delta q^{ab} = \delta (q^{ab}\ell_a) k_b - q^{ab} k_b \delta \ell_a = 0 .
\]
(29)

Using this result, we can simplify \( g_{ab}\delta q^{ab} \) as follows:
\[
g_{ab}\delta q^{ab} = g_{ab} \left[ \delta (q^{ab}\ell_a) - \delta (\ell^a q^b) - \delta (\ell^b q^a) \right] = g_{ab}\delta q^{ab} + 2 (\ell_a k_b + \ell_b k_a) \delta (\ell^a q^b)
\]
(30)
where we have also used \( \ell_a \delta k^a = 0 \), arising from our assumptions of \( \delta \ell_a = 0 \) and \( \delta (\ell_a k^a) = 0 \).

Next, we shall simplify \( (\nabla_a \ell_b) \delta q^{ab} \). First, note that this is a symmetric object since we have in the following form:
\[
(\nabla_a \ell_b) \delta q^{ab} = \Theta_{ab} \delta q^{ab} - 2 (k^b \nabla_a \ell_b) \delta \ell^a .
\]
(33)

We shall also write \( \Pi^b_a \nabla_a \ell_b \) in the following form:
\[
\Pi^b_a \nabla_a \ell_b = (q^m_a - \ell^m k_a) \nabla_a \ell_b = \Theta + \kappa ,
\]
(34)
making use of Appendix A.2.2, Appendix A.2.5 and Appendix A.2.1.

We have thus derived all the results that we need. Substituting Eq. (30), Eq. (33) and Eq. (34) in Eq. (28), we get
\[
\sqrt{-g} \mathcal{Q}'[\ell_a] = \partial_a \left( \sqrt{-g} \Pi^b_a \delta \ell^b \right) - 2\delta \left[ \sqrt{-g} (\Theta + \kappa) + \sqrt{-g} [\Theta_{ab} - (\Theta + \kappa) q_{ab}] \delta q^{ab} \right]
\]
\[+ 2\sqrt{-g} \left( [\Theta + \kappa] k_c + \ell^b \nabla_c k_b \right) \delta \ell^c .
\]
(35)

Thus, the surface term in the action for a null surface takes the following form:
\[
\delta A_{null} = \int_{\partial V} d^3 x \sqrt{-g} \mathcal{Q}'[\ell_a]
\]
\[= \int_{\partial V} d^3 x \left\{ \sqrt{-g} \nabla_a \left( \Pi^a \delta \ell^b \right) - 2\delta \left[ \sqrt{-g} (\Theta + \kappa) \right] + \sqrt{-g} [\Theta_{ab} - (\Theta + \kappa) q_{ab}] \delta q^{ab} \right.
\]
\[+ 2\sqrt{-g} \left( [\Theta + \kappa] k_c + \ell^b \nabla_c k_b \right) \delta \ell^c \right\} .
\]
(36)

Apart from \( \delta \ell^a = \delta \Theta + g^{ab} \delta \ell_b \), the definition of the rest of the symbols can be found in Appendix A.2.1, Appendix A.2.2 and Appendix A.2.5. In the first line, we have a structure like we have in the non-null case in Eq. (260) (in Appendix B) reproduced below for convenience:
\[
[\delta A_{null}]_{non-null} = \int_{\partial V} d^3 x \left[ \sqrt{|h|} D_a (\delta n^a) + \delta (2\sqrt{|h|} K) - \sqrt{|h|} (K_{ab} - K h_{ab}) \delta h^{ab} \right] .
\]
The first term in Eq. (36) is the total 3-derivative term (since $\Pi^a_b = \Pi^a_{b\ell_a} = 0$), the second term can be cancelled by a counter-term and the third term is the term that may be killed by fixing $q^{ab}$ on the surface. We have a crucial extra term with $\delta \ell^a$ which we shall deal with when we discuss degrees of freedom. Specializing to the coordinate system $(\lambda, z^1, z^2)$ introduced in Appendix A.2.3 and using Eq. (231) in Appendix A.2.7, Eq. (36) reduces to

$$\delta A_{null} = \int_{\partial V} d\lambda d^2z_{\perp} \sqrt{q} Q[\ell_a]$$

$$= \int_{\partial V} d\lambda d^2z_{\perp} \left\{ \sqrt{q} \nabla_a \left( \Pi^a_{b\delta \ell_b} \right) - 2\delta \left[ \sqrt{q}(\Theta + \kappa) \right] + \sqrt{q} \Theta_{ab} - (\Theta + \kappa)q_{ab} \right\} \delta q^{ab}$$

$$+ 2\sqrt{q} \delta \ell^c \left[ (\Theta + \kappa) \kappa_c + \ell^b \nabla_c k_b \right] \right\}. \tag{38}$$

This is our final result. We see that the action principle has again told us what to do! On a null surface we need to add the integral of $(\Theta + \kappa)$ as the counter-term. Further we need to set $\delta q^{ab} = 0$ and $\delta \ell^a = 0$ on the boundary which requires fixing the set $(q^{ab}, \ell^a)$ (six independent components, since $\ell^2 = 0$) on the boundary. This is analogous to fixing $h_{ab}$ (six independent components) on the non-null surface. Note that in the latter case, we can use the freedom in the choice of coordinates to set 3 of the 6 components in $h_{ab}$ to a preassigned form. But in the usual approach, there is no natural foliation to divide the six degrees of freedom in $h_{ab}$ into $(3+3)$. But the situation is better as regards null-surface. We shall show in Section 5 that one can choose the coordinates to set $\delta \ell^a = 0$ automatically. Thus, the genuine degrees of freedom can be identified with the 3 independent components of $q_{ab}$ in a $(1+1+1)$ foliation with no further coordinate transformations being allowed.

The expression Eq. (38) also tells us that the momentum conjugate to $q^{ab}$ is $\sqrt{q} \Theta_{ab} - (\Theta + \kappa)q_{ab}$ and the momentum conjugate to $\ell^a$ is $2\sqrt{q} \left[ (\Theta + \kappa) k_c + \ell^b \nabla_c k_b \right]$. We can write the momentum conjugate to $\ell^a$ in another form as

$$P_c = 2\sqrt{q} \left[ \nabla_a \ell^a k_c + \ell^b \nabla_c k_b - \kappa k_c \right] = 2\sqrt{q} \left[ \nabla_a \ell^a k_c - k_b \nabla_c \ell^b - \kappa k_c \right]$$

$$= -2\sqrt{q} \kappa_b \left[ \nabla_a \ell^a - \delta^a_c \left( \nabla_a \ell^a - \kappa \right) \right] = -2\sqrt{q} \kappa_b \left[ \nabla_c \ell^b - \delta^b_c \left( \Theta + \kappa \right) \right]. \tag{39}$$

where, $2\kappa = -k^a \nabla_a \ell^2$ and we have used Eq. (226).

In the cases where $\kappa = 0$, as would happen if $\ell^2 = 0$ everywhere, the counter-term would be the integral of $\sqrt{q} \Theta$. In this case, we can make use of Eq. (219) in Appendix A.2.5 to write the counter-term as

$$-2 \int_{\partial V} d\lambda d^2z_{\perp} \sqrt{q} \Theta = -2 \int_{\partial V} d\lambda d^2z_{\perp} \frac{\partial \sqrt{q}}{\partial \lambda} = -2 \int_{\partial V} d^2z_{\perp} \sqrt{q} \left|_{\lambda_2}^{\lambda_1} \right. = -2 \left. \left[ S(\lambda_2) - S(\lambda_1) \right] \right), \tag{40}$$

where $\lambda_1$ and $\lambda_2$ are the limits of integration of $\lambda$. Thus, the counter-term in this case, apart from an overall factor, is the difference in the area of the 2-surface orthogonal to $\ell$ and $k$ on the null surface, denoted by $S(\lambda)$, at the limits of $\lambda$ integration. A similar interpretation can be given to the counter-term for the non-null case, the integral of $-2\sqrt{|h|} K$ over the boundary surface, if we assume that the Lie bracket of the normal vector $n^a$ with any of the coordinate basis vectors on the surface is zero (see Appendix E). This, in turn, requires $n^a$ to be a tangent vector to a geodesic congruence, which may not be true in general. With this assumption, $-\sqrt{|h|} K$ is the rate of change along $n^a$ of the volume of a 3-surface element orthogonal to $n^a$.

In the next subsection, we shall show that a similar expression for the surface term can also be obtained if we start with $\ell_a = A \partial_a \phi$ but assume $A = 1$ on the null surface and $\ell^a \ell_a = 0$ everywhere.

### 3.3.1 Working With the Choice $\ell_a = A \partial_a \phi$

Since $\ell_a = A \partial_a \phi$, we do not have the results $\nabla_a \ell_b = \nabla_b \ell_a$ everywhere and $\delta \ell_a = 0$ everywhere that we had used profusely in the last section. But since we are fixing $A = 1$ on the surface, we do have the result $\delta \ell_a := 0$ valid only on the null surface. Let us start with the divergence term, which can be written as

$$\sqrt{-g} \nabla_e \left[ \delta \ell^e \right] = \sqrt{-g} \nabla_e \left[ \left( \delta^e_a + k^e \ell_d \right) \delta \ell^d \right] = \delta \ell_e = 0 \tag{41}$$

where, as in the previous section, we have introduced the projector $(\delta^e_a + k^e \ell_d) = \Pi^e_d$ with $\ell_a \Pi^d_{a\ell_d} = 0$. Then, the first term in the above expression can be converted to a three divergence part, leading to

$$\sqrt{-g} \nabla_e \left[ \delta \ell^e \right] = \partial_a \left[ \sqrt{-g} \Pi^d_{a\ell_d} \delta \ell^d \right] = \partial_d \left[ \sqrt{-g} k^e \ell_d \delta \ell^d \right]. \tag{42}$$

10
Now, the vector $\ell^a$, being null everywhere, satisfies the relation $\ell^2 = 0$ everywhere, making $\ell_a \delta \ell_a^c = \delta (\ell^2) = 0$. Thus, the extra term in Eq. (42) vanishes leading to

$$\sqrt{-g} \nabla_a [\delta \ell_a^c] = \partial_a \left[ \sqrt{-g} \Pi^a_{\ell} \delta \ell_a^c \right]$$

(43)

The second term corresponds to the counter-term with the following expression:

$$\delta \left( 2 \sqrt{-g} \nabla_a \ell^a \right) = \delta \left[ 2 \sqrt{-g} (\Theta + \kappa) \right]$$

(44)

Finally, we also have the result

$$(\nabla_a \ell_b) \delta g^{ab} = \Theta_{ab} \delta q^{ab} - \kappa \ell_b \delta k^b - \delta \ell^a \left(k^b \nabla_a \ell_b + k^b \nabla_b \ell_a \right),$$

(45)

along with the following identity:

$$g_{ab} \delta g^{ab} = q_{ab} \delta q^{ab} - 2 k_a \delta \ell^a - 2 \ell_b \delta k^b.$$  

(46)

Hence, the boundary term takes the following form,

$$\sqrt{-g} Q [\ell_a] = \partial_a \left[ \sqrt{-g} \Pi^{a}_{\ell} \delta \ell_a^c \right] - \delta \left[ 2 \sqrt{-g} (\Theta + \kappa) \right] + \sqrt{-g} [\Theta_{ab} - (\Theta + \kappa) q_{ab}] \delta q^{ab} + \sqrt{-g} [2 (\Theta + \kappa) k_a - k^b \nabla_a \ell_b - k^b \nabla_b \ell_a] \delta \ell^a - k^a \delta \ell_a \left(2 \sqrt{-g} \Theta + \sqrt{-g} \kappa \right)$$

$$= \partial_a \left[ \sqrt{-g} \Pi^{a}_{\ell} \delta \ell_a^c \right] - \delta \left[ 2 \sqrt{-g} (\Theta + \kappa) \right] + \sqrt{-g} [\Theta_{ab} - (\Theta + \kappa) q_{ab}] \delta q^{ab} + \sqrt{-g} \delta \ell^a \left[ 2 (\Theta + \kappa) k_a + \ell_b \nabla_a \ell_b - k^b \nabla_b \ell_a \right]$$

(47)

Since $\delta \ell_a := 0$, vanishing on the null surface, we get

$$\sqrt{-g} Q [\ell_a] = \partial_a \left[ \sqrt{-g} \Pi^{a}_{\ell} \delta \ell_a^c \right] - \delta \left[ 2 \sqrt{-g} (\Theta + \kappa) \right] + \sqrt{-g} [\Theta_{ab} - (\Theta + \kappa) q_{ab}] \delta q^{ab} + \sqrt{-g} \delta \ell^a \left[ 2 (\Theta + \kappa) k_a + \ell_b \nabla_a \ell_b - k^b \nabla_b \ell_a \right]$$

(48)

Specializing to the coordinate system $(\lambda, z^1, z^2)$ introduced in Appendix A.2.3 and using Eq. (231) in Appendix A.2.7, the surface term in the action based on Eq. (48) reduces to

$$\delta A_{null} = \int_{\partial \mathcal{N}} d\lambda d^2 z \sqrt{-q} Q [\ell_a]$$

$$= \int_{\partial \mathcal{N}} d\lambda d^2 z \left\{ \sqrt{-q} \nabla_a (\Pi^a_{\ell} \delta \ell_a^c) - 2 \delta \left[ \sqrt{-q} (\Theta + \kappa) \right] + \sqrt{-q} [\Theta_{ab} - (\Theta + \kappa) q_{ab}] \delta q^{ab} + \sqrt{-q} \delta \ell^a \left[ 2 (\Theta + \kappa) k_a + \ell_b \nabla_a \ell_b - k^b \nabla_b \ell_a \right] \right\}.$$  

(49)

Again, the expression Eq. (48) tells us that the momentum conjugate to $q^{ab}$ is $\sqrt{-q} [\Theta_{ab} - (\Theta + \kappa) q_{ab}]$ and the momentum conjugate to $\ell^a$ is $2 \sqrt{-q} \left[ (\Theta + \kappa) k_a + \ell_b \nabla_a \ell_b \right]$. We can write the momentum conjugate to $\ell^a$ in another form as

$$P_c = \sqrt{-q} \left[ 2 (\nabla_a \ell^a) k_c - k^b \nabla_a \ell_b + k^b \nabla_b \ell_a \right] = - \sqrt{-q} k_b \left[ \nabla_c \ell^b + \nabla^b \ell_c - 2 \delta^b_c (\nabla_a \ell^a) \right].$$

(50)

### 3.4 Null surface as a limit of sequence of non-null surfaces

The description given above provides a somewhat abstract identification of the results between non-null surfaces and null surfaces. We would next like to make this concrete and obtain the result for the null surface when it is treated as a limit of a sequence of non-null surfaces.

Let the relevant null surface be part of a family of surfaces, not all of which are necessarily null, characterized by the constant values of a scalar function $\phi(x)$. The null surface is specified by $\phi(x) = \phi_0$. (That is, the surface $\phi = \phi_0$ is null but the other $\phi =$constant surfaces may be spacelike, timelike or even null.) We shall label the null surface as $\mathcal{N}$ and also introduce the scalar function $r = \phi - \phi_0$. Thus, we can represent the null surface by the symbol $\mathcal{N}$ or by one of the two equations, $\phi = \phi_0$ or $r = 0$, as per convenience. The normal to the surfaces has the form $\ell_a = A \nabla_a \phi$, for some scalar function $A(x)$. If the surfaces were spacelike or timelike, we could have fixed $A(x)$ by demanding $\ell_a \ell_a = \pm 1$. But now, we have $\ell_\phi \ell^\phi = A^2 g^{\phi \phi}$ with $g^{\phi \phi} = 0$ at $\phi = \phi_0$. Thus, imposing this constraint would make $A$ diverge at $\phi = \phi_0$. Hence, we cannot impose any such normalization condition without introducing infinities.
We shall now examine the decomposition of the surface term of the Einstein-Hilbert action for a non-null surface, provided in Eq. (260) in Appendix B, and obtain the null surface limit of each of the terms. For convenience, we reproduce Eq. (260) below:

\[
[\delta A_{\text{ov}}]_{\text{non-null}} = \int_{\partial V} d^3x \left[ \sqrt{|h|} D_a(\delta n^a) + \delta(2\sqrt{|h|}K) - \sqrt{|h|}(K_{ab} - K_{ab}) \delta h^{ab} \right].
\]

In obtaining the limit, we shall assume that the metric components are finite and that the metric determinant is finite (and non-zero) when we take the null limit. (Later on, we will demonstrate these conditions and the final result in specific coordinate systems.)

The expression for the normal is given in Eq. (237) in Appendix B: \(n_b = N\nabla_b \phi\). On the other hand, we have used the expression \(\ell_a = \partial_a \phi\) in obtaining Eq. (36). So, we have the identification \(n_a = N\ell_a\). Now, \(N\) is defined in terms of \(g^{\phi \phi}\) in Eq. (238) in Appendix B: \(g^{\phi \phi} = \epsilon/N^2\). So, the null limit can be imposed by demanding that \(N \to \infty\). Near the null surface \(\phi = \phi_0\), defining \(r = \phi - \phi_0\), we shall take \(g^{\phi \phi}\) to have the behaviour \(g^{\phi \phi} = Br^n + O(r^{n+1})\) for a finite non-zero \(B\) and some positive integer \(n\). Thus, the null limit can also be represented by \(r \to 0\).

Let us start with the surface boundary term in Eq. (51). This term has been analyzed in Appendix F.1 to show that it can be written in the form:

\[
\sqrt{|h|} D_a(\delta n^a) = \partial_a \left[ \sqrt{-g} \ell^a \ell^b \delta h^{ab} \right] = 2\partial_a \left[ \sqrt{-g} \ell^a \delta \ln N \right].
\]

The first term in Eq. (54) is finite in the \(N \to \infty\) limit. In the second term in Eq. (54), we use the expansion for \(g^{\phi \phi}\) near the null surface and see that the \(r\) factor cancels (as we have taken the variation to not affect the coordinates) and the term is finite in the null limit of \(N \to \infty\) or \(r \to 0\). Thus, we have decomposed \(\sqrt{|h|} D_a(\delta n^a)\) in such a way that each term in the decomposition is clearly finite in the null limit.

On the other hand, the surface boundary term for a null surface in Eq. (36) has been shown in Appendix F.1 to reduce to

\[
\partial_a \left[ \sqrt{-g} \Pi^a_b \delta \ell^b \right] = \partial_a \left[ \sqrt{-g} \ell^a \delta \ell^b \right] = \partial_a \left[ \sqrt{-g} \ell^a \delta \ln N \right].
\]

Thus, from Eq. (52) and Eq. (53), we obtain the result that the null surface limit of the surface boundary term in Eq. (51) is

\[
\sqrt{|h|} D_a(\delta n^a) \, r \to 0 = \partial_a \left[ \sqrt{-g} \ell^a \delta \ln N \right].
\]

Next, let us look at the counter-term in Eq. (51). From Appendix F.2, a similar analysis shows that the null limit of the variation of the counter-term turns out to be

\[
\delta \left( 2\sqrt{|h|}K \right) \, r \to 0 = -2\delta \left[ \sqrt{-g} \ell^a \ell^b \delta h^{ab} \right] + 2\partial_a \left[ \sqrt{-g} \ell^a \delta \ln N \right].
\]

Consider now the last set of terms in Eq. (51). Among them, the \(h_{ij} \delta h^{ij}\) term is manipulated in Appendix F.3 leading to

\[
\sqrt{|h|} K h_{ij} \delta h^{ij} \, r \to 0 = -\sqrt{-g} \left[ (\Theta + \kappa) q_{ab} \delta q^{ab} + 2\sqrt{-g} (\Theta + \kappa) k_i \delta q^b - 2\sqrt{-g} (\Theta + \kappa) \delta \ln N \right]
\]

Again, one can see that each term is finite in the null limit under our assumptions.

The last term to be considered in Eq. (51) is the \(K_{ij} \delta h^{ij}\) term. The pertinent expression is derived in Appendix F.4 as

\[
-\sqrt{|h|} K_{ij} \delta h^{ij} \, r \to 0 = \sqrt{-g} \left[ \Theta_{ij} \delta q^{ij} + 2\delta \ell^a \ell^b \nabla_i k_j + 2 (\partial_i \ln N) \delta \ell^a + 2\ell^a (\partial_i \ln N) \delta \ln N \right].
\]

Adding Eq. (54), Eq. (55), Eq. (56) and Eq. (57), we obtain the following result when we take the null limit of the integrand on the right hand side of Eq. (51):

\[
\sqrt{|h|} D_a(\delta n^a) + \delta(2\sqrt{|h|}K) - \sqrt{|h|}(K_{ab} - K_{ab}) \delta h^{ab}
\]

\[
\left[ \sqrt{-g} \ell^a \delta \ell^b \right] - 2\delta \left[ \sqrt{-g} (\Theta + \kappa) \right] + \sqrt{-g} (\Theta_{ab} - (\Theta + \kappa) q_{ab}) \delta q^{ab}
\]

\[
+ 2\sqrt{-g} \ell^a \left[ (\Theta + \kappa) k_i \right] + \ell^a \nabla_i k_j \right]
\]

\[
+ 2\partial_a \left[ \sqrt{-g} \ell^a \delta \ln N \right] - 2\delta \left[ \sqrt{-g} \ell^a \partial_a \ln N \right] - 2\sqrt{-g} (\Theta + \kappa) \delta \ln N
\]

\[
- \sqrt{-g} \left[ \ell^a \partial_a \ln N \right] g_{ij} \delta q^{ij} - 2\sqrt{-g} \left[ \ell^a \partial_a \ln N \right] \delta \ln N
\]

\[
+ \sqrt{-g} \left[ 2 (\partial_i \ln N) \delta \ell^i + 2\ell^a (\partial_i \ln N) \delta \ln N \right]
\]
We find that the first two lines reproduce the result in Eq. (36). Thus, we should be able to show that the rest of the terms cancel out. This is proved in Appendix F.5 and we obtain the following expression for the surface term:

$$[\delta A_{\text{surf}}]_{\text{non-null}}^{r=0} = \int_{\partial V} d^3x \left\{ \sqrt{-g} \nabla_a \left[ \Pi^a_b \delta \ell^b_k \right] - 2\delta \left[ \sqrt{-g} (\Theta + \kappa) \right] + \sqrt{-g} (\Theta_{ab} - (\Theta + \kappa) q_{ab}) \delta q^{ab} + 2\sqrt{-g} \delta \ell^a \left[ (\Theta + \kappa) k_i + \ell^i \nabla_j k_j \right] \right\} \ .$$

where we have replaced the surface index $a$ in the first term in Eq. (58) with the four-dimensional index $a$ as $\Pi^a_b = 0$. Thus, we have re-derived the result in Eq. (36), with proper canonical momenta conjugate to $\ell^a$, reaffirming our faith in the correctness of this result.

This analysis gives us a different perspective. Since we are familiar with the decomposition of the surface term in the non-null case, we can now trace back where each term in the null limit comes from and see where the degrees of freedom that we fix on the boundary for the non-null case are hidden. From Eq. (54) and Eq. (55), we see that the surface term and the total variation term (which is to be cancelled by a counter-term) generate the surface term and the total variation term, respectively, in the null case, in addition to one extra term each. These extra terms appeared in our group of extra terms that we have shown to reduce to zero. Hence, the $\delta q^{ab}$ and the $\delta \ell^a$ terms originate from the $\delta h^{ab}$ terms in the non-null case. There were six degrees of freedom in $h^{ab}$ while there are three degrees of freedom in the symmetric matrix $q^{ab}$ constrained by the seven constraints $q^{ab} b_k = 0$ and $q^{ab} k_b = 0$, which will be respected by the variation by virtue of our demand that $\ell, k$ are left unchanged. (Although this appears at first sight to be eight constraints, one constraint, namely $q^{ab} b_k = 0$, is common.) The vector $\delta \ell^a$ appears to have four components. But we have already imposed the constraint $\ell^a = k_a w^a = 0$ even on variation, reducing one degree of freedom. Thus, the six components in variations of $h^{ab}$ go in to three components of $\delta q^{ab}$ and three components of $\delta \ell^a$. We shall show in Section 5 that the three degrees of freedom in $\delta \ell^a$ can be removed by making use of surface diffeomorphisms.

In the next section, we shall illustrate the result Eq. (38) in the case of two specific parametrizations for a general null surface.

### 4 Boundary Term on The Null Surface for Specific Parameterizations

In this section, we shall verify the validity of the result in Eq. (38) by working out two specific cases. We shall introduce two parametrizations for a general null surface and calculate the boundary term of the Einstein-Hilbert action on the null surface. In both cases, we find results in agreement with Eq. (38).

#### 4.1 Gaussian Null Coordinates (GNC)

The first parametrization that we shall discuss is what is commonly known as Gaussian null coordinates (GNC), in analogy with the Gaussian normal coordinates [18]. Gaussian normal coordinates are constructed by extending the coordinates on a non-null hypersurface to a spacetime neighbourhood using geodesics normal to the surface. This prescription does not work for a null surface as the “normal” geodesics lie on the null surface itself. Hence, the construction of Gaussian null coordinates is carried out by making use of certain, uniquely defined, auxiliary null geodesics. As far as we know, this coordinate chart in the neighborhood of a null surface was introduced by Moncrief and Isenberg in [19] and hence may also be termed as Moncrief-Isenberg coordinates. The construction of GNC is also described in [20], [21] and [22].

#### 4.1.1 Construction of the Coordinates

We shall now describe the construction of GNC. Our description will be less technical and more intuitive. We shall mostly follow the notation of [22]. Note that [22] has the GNC metric in Eq. 3.2.7 but then claims that it can be further constrained and writes it as in Eq. 3.2.8, i.e makes the replacement $a \rightarrow r a$. We do not subscribe to this claim and will use Eq. 3.2.7.

Consider a smooth null surface $N$ in a spacetime manifold $M$. Let $g_{ab}$ represent the metric on $M$. We shall take the spacetime to be four-dimensional but the following construction can be easily extended to an $n$-dimensional Lorentzian spacetime as long as the extra dimensions are spacelike. Take a spacelike 2-surface, $\zeta$, on $N$ and introduce coordinates $(x^1, x^2)$ on that surface. We have shown that the null surface
$N$ is generated by null geodesics. The null geodesics cannot be along the spacelike surface at any point, since any vector that lies on the spacelike surface has to be spacelike. Thus, one necessarily moves away from the spacelike surface by travelling along any of these null geodesic. Therefore, the null geodesics can be used to define coordinates on the null surface as follows. Let $u$ be an affine parameter along the null geodesics with $u=0$ on $\zeta$ and increasing towards future. Let each null geodesic be labelled by the coordinates $(x^1, x^2)$ of the point at which it intersects $\zeta$. Then, any point on $N$ in a neighbourhood of $\zeta$, sufficiently small so that the geodesics do not cross, can be assigned the coordinates $(u, x^1, x^2)$, where $(x^1, x^2)$ corresponds to the label given to the null geodesic passing through that point and $u$ corresponds to the affine parameter at that point. Henceforth, when we talk about, say, coordinates on the null surface, it is to be understood that we are talking about such a neighbourhood. We shall call the null surface, there is a unique vector surrounding spacetime. We shall do this with the help of a new set of null geodesics. At each point on the null surface (so that the null geodesics do not cross) can be assigned the coordinates $(u, x^1, x^2)$ of that point. The fourth coordinate may be chosen as an affine parameter along the null geodesic. Let $r$ be that affine parameter such that $r=0$ represents the null surface and $k=-\partial/\partial r$. (The sole purpose of the minus sign is to reproduce the exact form of the metric in [22] by off-setting the fact that we demand $\ell_a k^a$ to be $-1$ while [22] equates the same to $+1$.) These constraints uniquely determine the affine parameter. Any $r'=A(u, x^1, x^2)r + B(u, x^1, x^2)$ is also an affine parameter, but the condition $r'=0$ on the null surface will lead to $B=0$ and the condition $\ell_a k^a=-1$ for $k=-\partial/\partial r'$ fixes the value of $A$ to be unity. Having chosen this affine parameter, any point in a sufficiently small neighbourhood of the null surface (so that the null geodesics do not cross) can be assigned the coordinates $(u, r, x^1, x^2)$, where the set $(u, x^1, x^2)$ corresponds to the label of the null geodesic that passes through that point while $r$ represents the affine parameter value at that point.

Let us now turn our attention to the form of the metric in the neighbourhood of the null surface. We shall start by writing down the form of the metric on the null surface. Since $\ell$ is null everywhere on the null surface, we can write down our first metric component: $g_{uu} = g_{ab} \ell^a \ell^b = 0$, everywhere on the null surface. Next, since the null surface is represented by $r=\text{constant}$, $X_A = \partial/\partial x^A$ at the null surface has to lie on the null surface. This means that $X_A^a \ell_a = 0$, which implies $g_{ab} X_A^a X_B^b = q_{AB} = 0$ everywhere on the null surface. Finally, denoting the rest of the components relevant for vectors on the null surface, $g_{AB} = g_{ab} X_A^a X_B^b$, by $q_{AB}$, we have the following list of components of the metric on the null surface:

$$g_{uu} = 0; \ g_{uA} = 0; \ g_{AB} = q_{AB};$$

Let us next determine the metric components in the neighbourhood of the null surface that we are considering. First, since $k = -\partial/\partial r$ is null everywhere, we have $g_{ab} k^a k^b = g_{rr} = 0$ everywhere. To find the rest of the components, let us write down the geodesic equation governing the null geodesics that are integral curves of $k$. Since $r$ is an affine parameter, we have

$$k^a \nabla_a k^b = 0 \implies \Gamma^r_{rr} = 0 \implies \Gamma^a_{arr} = 0,$$

where we contracted with $g_{ab}$ in the last step. Putting $a = r$ tells us that $g_{rr}$ remains zero along the geodesics. But if we put $a = \mu$ where $x_\mu$ is any one of the three surface coordinates $(u, x^1, x^2)$, we have

$$\Gamma^{\mu}_{rr} = \frac{1}{2} \left(-\partial_r g_{rr} + 2 \partial_\mu g_{r\mu} \right) = \partial_\mu g_{r\mu} = 0.$$

Thus, $g_{r\mu}$ is also constant along the geodesics. (This is a common feature for affinely parametrized geodesics. For null geodesics, if $\partial/\partial \phi$ is the tangent vector with $\phi$ being the affine parameter, then we shall have $\partial_\phi g_{a\mu} = 0$. For non-null geodesics, we can get the same constraint provided $\phi$ is taken as the proper time or proper length or a constant multiple thereof, so that we can put $\partial_\phi g_{a\mu} = 0$ as in the last step above.) To fix the values of $g_{r\mu}$, note that we have the constraints $k^a \ell_a = -1$ and $\ell_a X_A^a = 0$ at...
$r = 0$, which translate into $g_{ru} = 1$ and $g_{rA} = 0$. From Eq. (62), we conclude that $g_{ru} = 1$ and $g_{rA} = 0$ everywhere in the region under consideration. Finally, defining $q_{AB} = g_{ab}X_{A}X_{B}$, with the definition $X_{A} = \partial / \partial x^{A}$ extended everywhere, and respecting Eq. (60), we have all the components of the metric as follows:

$$g_{rr} = g_{r\alpha} = 0; \quad g_{ru} = 1; \quad g_{uu} = -2r\alpha; \quad g_{uA} = -r\beta_{A}; \quad g_{AB} = q_{AB},$$

(63)

where $\alpha$ and $\beta^{A}$ are finite at $r = 0$ so that $g_{uu}$ and $g_{ru}$ vanish at $r = 0$. The $2 \times 2$ matrix $q_{AB}$ is positive definite (since $X_{A}$ are spacelike vectors). The choice of the sign for $g_{uA}$ is not of importance as the sign can be flipped by changing the coordinates on the spatial surface, $x^{A} \rightarrow -x^{A}$. On the other hand, the sign of $g_{uu}$ does have a physical significance. With this particular choice, the vector $\ell^{a}$ becomes timelike for $r > 0$ and spacelike for $r < 0$ when $\alpha > 0$. Also,

$$g^{rr} = 2\alpha + r^{2}\beta^{2},$$

where $\beta^{2} = q_{AB}\beta^{A}\beta^{B}$, which is a positive quantity since $q_{AB}$ is positive definite. Thus, the normal to $r$-constant surfaces, $\partial_{a}r \equiv r_{a}$, satisfies the condition

$$r_{a}r_{a} = 2\alpha + r^{2}\beta^{2}.$$  

The second term is strictly positive. But the first term dominates as $r \rightarrow 0$. With $\alpha > 0$, $r$-constant surfaces near $r = 0$ are time-like in the $r > 0$ region and space-like in the $r < 0$ region. In fact, all $r$-constant surfaces in the $r > 0$ region become time-like by this choice.

So, finally, here is the line element in GNC:

$$ds^{2} = -2r\alpha du^{2} + 2drdu - 2r\beta_{A}dudx^{A} + q_{AB}dx^{A}dx^{B}$$

(64)

Note that there are six independent functions ($\alpha, \beta_{A}, q_{AB}$) in this metric. We have used up the freedom of choosing the four coordinates to eliminate four out of the ten degrees of freedom in the metric. [The inverse metric and the Christoffel symbols corresponding to this metric are given in Appendix D.1. The Ricci tensor components are provided in [19].] Introducing a time coordinate by $t = u + r$ in place of $u$, we can rewrite this metric in the standard ADM form as,

$$ds^{2} = -N^{2}dt^{2} - 2(1 + r\alpha)(dr - N^{r}dt)(dr - N^{r}dt)$$

$$+ 2r\beta_{A}(dr - N^{r}dt)(dx^{A} - N^{A}dt) + q_{AB}(dx^{A} - N^{A}dt)(dx^{B} - N^{B}dt)$$

(65)

$$N^{r} = \frac{1 + 2r\alpha + r^{2}\beta^{2}}{-2 - 2r\alpha + r^{2}\beta^{2}}$$

(66)

$$N^{A} = -r\beta^{A}\left[\frac{3 + 4r\alpha}{-2 - 2r\alpha + r^{2}\beta^{2}}\right]$$

(67)

$$N^{2} = 2r\alpha + 2r\beta_{A}N^{r}N^{A} - 2(1 + r\alpha)(N^{r})^{2} + q_{AB}N^{A}N^{B}$$

(68)

This expression shows that the relationship between the GNC metric components and the standard ADM variables ($N, N_{0}, h_{\alpha\beta}$) is not simple. In particular the degrees of freedom in $h_{\alpha\beta}$ comes in entangled in terms of the other degrees of freedom in GNC variables. As we shall see, this has some implications for the variational principle.

### 4.2 Surface Term on a Null Surface as a limit in GNC

Gaussian null coordinates (GNC) provide a situation where $r = \text{constant}$ surfaces are time-like for $r > 0$ and null for $r = 0$. In order to show the validity of our previous derived result for null surfaces, Eq. (38), we will obtain the boundary term on the null surface as the limit of the boundary term on a time-like surface. Consider the following expression for the term to be integrated on the boundary of a time-like or space-like surface as given in Eq. (51):

$$Q\sqrt{|h|} = \sqrt{|h|}D_{a}(\delta n_{a}^{i}) + \delta(2K\sqrt{|h|}) - \sqrt{|h|}(K_{ij} - Kh_{ij})\delta h^{ij},$$

(69)

where $Q$ as given in Eq. (234) in Appendix B has to be evaluated for the unit normal. We shall use the object $P_{ab} = -\sqrt{|h|}(K_{ab} - Kh_{ab})$, which, apart from a constant factor, represents the canonical momentum conjugate to $h_{\alpha\beta}$ in ADM formalism [1]. Then, the above equation will take the form

$$Q\sqrt{|h|} = \sqrt{|h|}D_{a}(\delta n_{a}^{i}) + \delta(2K\sqrt{|h|}) + P_{ij}\delta h^{ij}$$

(70)
Further, we have the following relations

\[ P^{ab} h_{ab} = 2\sqrt{|h|} K; \quad P_{ab} \delta h^{ab} = -P^{ab} \delta h_{ab} \]  

(71)

Thus, we can also write the boundary term \( Q\sqrt{|h|} \) using the above expressions for \( P^{ab} h_{ab} \) and \( P_{ab} \delta h^{ab} \) leading to

\[ Q\sqrt{|h|} = \sqrt{|h|} D_u (\delta n^a_+) + h_{ij} \delta P^{ij} \]  

(72)

The idea is to evaluate this term for an \( r = \) constant surface with \( r > 0 \), which corresponds to a timelike surface, and then take the \( r \to 0 \) limit.

We shall first assume that the variations preserve the GNC form, i.e., we will only vary the functions present in the GNC metric, viz. \( (\alpha, \beta_A, q^{AB}) \). Further since the metric only has the combination \( \alpha \) and \( r \beta_A \) occurring in it, the variations \( r \delta \alpha, r \delta \beta_A \) will vanish in the \( r \to 0 \) limit. So we are essentially restricting ourselves to variations with \( \delta \ell^a = 0, \delta q^{AB} \neq 0 \) at this stage. (We will describe a more general situation later on.)

When the evaluation of \( h_{ab} \delta P^{ab} \) is carried out and the limit \( r \to 0 \) is taken carefully, we get the result

\[ -h_{ab} \delta (\sqrt{|h|}(K^{ab} - K h^{ab})) = \partial_a (\sqrt{|q|} \frac{\delta \alpha}{\alpha}) + \sqrt{|q|} [-2\delta \alpha - \frac{1}{2} \delta q^{AB} \partial_a q_{AB} - q^{AB} \partial_a (\delta q_{AB})], \]  

(73)

where \( q \) is the determinant of the 2-metric \( q_{AB} \). The first term in Eq. (73) precisely cancels the \( \sqrt{|h|} D_u (\delta n^a_+) \) term so that we readily obtain

\[ Q\sqrt{|h|} = \sqrt{|q|} [-2\delta \alpha - \frac{1}{2} \delta q^{AB} \partial_a q_{AB} - q^{AB} \partial_a (\delta q_{AB})] \]  

(74)

This result may be rewritten in the following form:

\[ Q\sqrt{|h|} = -2\sqrt{|q|} \delta \alpha - \frac{1}{2} \sqrt{|q|} q^{AB} \partial_a \delta q_{AB} - \sqrt{|q|} \partial_a [\delta (ln \sqrt{|q|})], \]  

(75)

We can also derive the same result starting from Eq. (69). Under the variations that we are considering, each of the terms in Eq. (69) have the following limits as \( r \to 0 \):

\[ \sqrt{|h|} D_u (\delta n^a_+) = -\partial_a \left( \sqrt{|q|} \frac{\delta \alpha}{\alpha} \right) \]  

(76)

\[ \delta (2K \sqrt{|h|}) = \delta \left( 2\partial_a \sqrt{q} + 2\alpha \sqrt{q} - \sqrt{q} \frac{\partial_a \alpha}{\alpha} \right) \]  

(77)

\[ \sqrt{|h|}(K_{ij} - K h_{ij}) \delta h^{ij} = \frac{\delta \alpha}{\alpha} \partial_a \sqrt{q} + \frac{1}{2} \sqrt{q} q^{CD} \partial_a q_{CD} \]  

\[ + \frac{2}{\sqrt{q}} \delta \sqrt{q} \partial_a \sqrt{q} + 2 \delta \sqrt{q} \left( \alpha - \frac{1}{2} \frac{\partial_a \alpha}{\alpha} \right) \]  

(78)

The difference of the last two terms can be simplified and written as

\[ \delta (2K \sqrt{|h|}) - \sqrt{|h|}(K_{ij} - K h_{ij}) \delta h^{ij} = -2\delta \alpha \sqrt{q} + \partial_a \left( \sqrt{q} \frac{\delta \alpha}{\alpha} \right) \]  

\[ - \frac{1}{2} \sqrt{q} q^{AB} \partial_a \delta q_{AB} - \sqrt{q} \partial_a (\delta \ln \sqrt{q}) \]  

(79)

Combining with the total derivative term, we again arrive at Eq. (75) as the null limit for Eq. (69):

\[ Q\sqrt{|h|} = \sqrt{|h|} D_u (\delta n^a_+) + \delta (2K \sqrt{|h|}) - \sqrt{|h|}(K_{ij} - K h_{ij}) \delta h^{ij}, \]  

\[ = -2\sqrt{|q|} \alpha - \frac{1}{2} \sqrt{|q|} q^{AB} \partial_a \delta q_{AB} - \sqrt{|q|} \partial_a (\delta \ln \sqrt{|q|}) \]  

(80)

For the Gaussian null coordinates, we choose the normal \( v_a \) to be \( \ell_a = \partial_a r \), the surface gradient to the \( r = 0 \) surface. Then, the normal vector has the following elements in \((u, r, x^1, x^2)\) coordinates:

\[ \ell_a = (0, 1, 0, 0); \quad \ell^a = g^{ar} \]  

(81)

(Note that this is not the \( \ell \) we used in Section 4.1 in constructing the GNC metric.) The auxiliary vector (see Appendix A.2.2), defined by the two conditions \( \ell_a k^a = \ell^a k_a = -1 \) and \( k_a k^a = 0 \) on the null surface,
can be chosen to be $k_a = -g_{ar}$ and $k^a = (0, -1, 0, 0)$. Then, we shall have the induced metric on the null surface defined as, $q_{ab} = g_{ab} + \ell_a k_b + \ell_b k_a$. The explicit form of the induced metric on the $r = 0$ surface is
\begin{equation}
q_{ab} = q_{AB} \delta_a^A \delta_b^B, \quad q^{ab} = q^{AB} \delta^a_A \delta^b_B,
\end{equation}
while $q_a^a = \text{diag}(0, 0, 1, 1)$. Next, let us look at the second fundamental form for the null surface (see Appendix A.2.5):
\begin{equation}
\Theta_{ab} = \theta^{\alpha a}_a \gamma_b \nabla_m \epsilon_n
\end{equation}
Calculating $\Theta_{ab}$ for our case, we find that the non-zero components are
\begin{equation}
\Theta_{AB} = \frac{1}{2} \partial_a q_{AB},
\end{equation}
so that the contraction of the above tensor $\Theta_{ab}$ leads to the the trace
\begin{equation}
\Theta = \frac{1}{2} q^{AB} \partial_a q_{AB}.
\end{equation}
Thus, we get the following expression for $(\Theta_{ab} - \Theta q_{ab}) \delta q^{ab}$ in GNC:
\begin{equation}
(\Theta_{ab} - \Theta q_{ab}) \delta q^{ab} = \frac{1}{2} \partial_a q_{AB} \delta q^{AB} + \frac{2}{q}(\nabla_\epsilon q)(\nabla_\epsilon q)(86)
\end{equation}
Comparing this result with Eq. (75), we find that, for GNC, the surface term is expressible in terms of the induced quantities $\Theta_{ab}$ and $\Theta$ in the following fashion:
\begin{equation}
\delta A_{\text{null}} = \int du d^2 x \left[ -2\sqrt{-q} \delta \alpha - \delta (2\sqrt{-q} \Theta) + (\Theta_{ab} - \Theta q_{ab}) \delta q^{ab} \sqrt{-q} \right]
\end{equation}
\begin{equation}
= \int du d^2 x \left[ -\delta (2\sqrt{-q} (\Theta + \alpha)) + [\Theta_{AB} - (\Theta + \alpha) q_{AB}] \delta q^{AB} \sqrt{-q} \right](87)
\end{equation}
To analyze this result, let us see what each of the terms in the general expression Eq. (15) give when we substitute the metric for GNC and $v_a = \ell_a = (0, 1, 0, 0)$. Then, we find the following on the $r = 0$ surface:
\begin{equation}
\sqrt{-g} \nabla_a (\delta n^a_\alpha) = 2\sqrt{-q} \delta \alpha;(88)
\end{equation}
\begin{equation}
-2\delta (\sqrt{-g} \nabla_\epsilon \ell^\epsilon) = -2\delta (\sqrt{-q} (\Theta + 2\alpha));(89)
\end{equation}
\begin{equation}
\sqrt{-g} (\nabla_a \ell_b - g_{ab} \nabla_c \ell^c) \delta g^{ab} = \sqrt{-q} [\Theta_{AB} - q_{AB} (\Theta + 2\alpha)] \delta q^{AB}.(90)
\end{equation}
The second equation uses the fact that, $\nabla_a \ell^\epsilon$ is equal to $\Theta + 2\alpha$, which stems from the fact that the norm $\ell^2$ does not vanish everywhere in GNC and thus has additional contribution to $\nabla_a \ell^\epsilon$. This can be seen explicitly from Eq. (226). We have mentioned just after Eq. (15) that the decomposition depends on the behaviour of $\ell_a$ away from the $r = 0$ surface, although the full surface term does not. To drive this point home, let us consider $v_a = \ell_a = (-2\alpha, 1, -r \beta_1, -r \beta_2)$, corresponding to a different vector $\ell^\epsilon = (1, 0, 0, 0)$. Note that $\ell_a \to \partial_a r$ as $r = 0$. Then, the decomposition in Eq. (15) on the $r = 0$ surface becomes:
\begin{equation}
\sqrt{-g} \nabla_a (\delta n^a_\alpha) = -2\sqrt{-q} \delta \alpha;(91)
\end{equation}
\begin{equation}
-2\delta (\sqrt{-g} \nabla_\epsilon \ell^\epsilon) = -2\delta (\sqrt{-q} \Theta);(92)
\end{equation}
\begin{equation}
\sqrt{-g} (\nabla_a \ell_b - g_{ab} \nabla_c \ell^c) \delta g^{ab} = \sqrt{-q} [\Theta_{AB} - q_{AB} \Theta] \delta q^{AB}.(93)
\end{equation}
One can check that both these decompositions add up to the result given in Eq. (87) but the individual contributions of the three terms are different.

Therefore, for the variations we have considered, we only need to fix $q_{AB}$ on the surface, by construction. Even though we varied the six components in the GNC metric, $\alpha$ and $\beta^A$ appear in the combination $r\alpha$ and $r\beta^A$. As the null surface remains $r = 0$ even after variation, when we take the limit $r \to 0$ contributions from $\delta \alpha$ and $\delta \beta^A$ vanish. Since the components of $\ell^a$ is given by Eq. (81), this corresponds to variations with $\delta \ell^a = 0$ and our general result in Eq. (38) shows that we only need to fix $q_{AB}$ on the surface. So everything is consistent. In order to get the full structure of Eq. (38), we need to consider unconstrained variations of the GNC line element, which we shall take up next.
4.3 Surface Term in GNC for Unconstrained Variations

We shall now consider the null surface described in the GNC coordinates but allow the variations to be arbitrary but finite. That is, the variations are not restricted to be of GNC form. We shall start from the expression for the surface term in terms of the variation of the Christoffel symbols given in Eq. (232), Appendix B:

\[ \sqrt{-g(Q[v_c]} = \sqrt{-g(v_c)}(g^{ab}\delta \Gamma^c_{ab} - g^{ck}\delta \Gamma^a_{ak}). \]  

(94)

Substituting the base metric as the GNC metric and \( v_c = \partial_r r \), we obtain

\[ \sqrt{-gQ} = \sqrt{g} (\delta \Gamma^r_{ru} + q^{AB}\delta \Gamma^r_{AB} - \delta \Gamma^u_{wu} - \delta \Gamma^a_{Aa}). \]  

(95)

Evaluating the \( \Gamma \)s for arbitrary variations from the GNC metric, we obtain the following expansion:

\[ \begin{aligned}
\sqrt{-gQ} &= \sqrt{g} \left[ \partial_r (\delta g_{uu}) - \frac{1}{2} \partial_u q_{AB} \delta q^{AB} - q^{AB} \partial_u (\delta q_{AB}) \\
&\quad - \frac{\delta g^{rr}}{2} q^{AB} \partial_r q_{AB} - \partial_u (\delta g_{ur}) \\
&\quad + \delta g^{ru} \left( -2 \alpha - \frac{q^{AB}}{2} \partial_u q_{AB} \right) \\
&\quad + \delta g^{rC} \left( -\beta_C - \frac{1}{2} q^{AB} \partial_C q_{AB} + q^{AB} \partial_A q_{BC} \right) \\
&\quad + q^{AB} \partial_A (\delta g_{uA}) \right]. \\
\end{aligned} \]  

(96)

It will be fruitful to convert the variations of the inverse metric into variations of the metric. For example,

\[ \delta g^{rr} = -g^{ra} g^{rb} \delta g_{ab} := -g^{ru} g^{ru} \delta g_{uu} = -\delta g_{uu}. \]  

(97)

Similarly, we can obtain

\[ \begin{aligned}
\delta g^{ru} &= -\delta g_{ru} \quad (98) \\
\delta g^{rC} &= -q^{AC} \delta g_{uA}. \quad (99)
\end{aligned} \]

Using these expressions in Eq. (96) and manipulating the terms, we can obtain

\[ \begin{aligned}
\sqrt{-gQ} &= \partial_A \left( \sqrt{g} q^{AB} \delta g_{uA} \right) - \partial_u \left( \sqrt{g} \delta g_{ru} \right) \\
&\quad + \delta (\sqrt{g} \partial_u q_{uu}) + 2\delta (g_{ur} \partial_u \sqrt{g}) + \delta [2 \sqrt{g} \alpha (g_{ru} - 1)] \\
&\quad + \delta g_{uu} \partial_r (\sqrt{g}) + \sqrt{g} \beta_B \delta g_{uB} + \alpha \sqrt{g} q^{AB} \delta q_{AB} - \frac{\sqrt{g}}{2} \partial_u q_{AB} \delta q^{AB} - \sqrt{g} q^{AB} \partial_u (\delta q_{AB}) .
\end{aligned} \]  

(100)

We will now show that this result matches with the one in Eq. (38). To do this, let us start with the following expression:

\[ \begin{aligned}
\Theta + \kappa &= \nabla_u \ell^a + \frac{1}{2} k^a \partial_u \ell^2 \\
&= \nabla_u \ell^a + \frac{1}{2} k^a \partial_u g^{rr}.
\end{aligned} \]  

(101)

Varying it, we arrive at:

\[ \begin{aligned}
2\delta (\Theta + \kappa) &= 2\delta (\nabla_u \ell^a) + k^a \partial_u g^{rr} + k^a \partial_u \delta g^{rr} \\
&= 2\partial_u \delta g^{ur} + \partial_u \delta g^{rC} + 2\partial_A \delta g^{rA} + 2 \delta g^{rr} \partial_u \ln \sqrt{-g} + 2 \delta g^{rC} \partial_u \ln \sqrt{-g} \\
&\quad + 2 \delta g^{rA} \partial_A \ln \sqrt{-g} + 2 \delta_u \delta \ln \sqrt{-g} - 2 \alpha \delta g^{ur} \\
&= -2 \partial_u \delta g_{ur} - \partial_u \delta g_{uu} - 2 \alpha \delta g_{ur} - 2 \beta_B \delta g_{uB} - 2 \partial_A (q^{AB} \delta g_{uA}) - \delta g_{ur} (q^{AB} \partial_u q_{AB}) \\
&\quad - \delta g_{uu} (q^{AB} \partial_u q_{AB}) - q^{AB} \delta g_{uB} (q^{CD} \partial_A q_{CD}) + 2 \partial_u \delta \ln \sqrt{-g}.
\end{aligned} \]  

(102)

Using this result, the variation of the counter-term becomes

\[ \begin{aligned}
2\delta \left[ \sqrt{-g} (\Theta + \kappa) \right] &= 2 \sqrt{-g} \delta (\Theta + \kappa) + 2 \sqrt{-g} (\Theta + \kappa) \delta g_{ur} - \sqrt{-g} (\Theta + \kappa) q_{AB} \delta q^{AB} \\
&= \sqrt{-g} \left[ -2 \partial_u \delta g_{ur} - \partial_u \delta g_{uu} - 2 \partial_A (q^{AB} \delta g_{uA}) - \delta g_{ur} (2\Theta + 2\kappa + q^{AB} \partial_u q_{AB}) \\
&\quad - \delta g_{uu} q^{AB} \partial_u q_{AB} - \delta g_{uB} (2 \beta_B + q^{AB} q^{CD} \partial_A q_{CD}) + 2 \partial_u \delta \ln \sqrt{-g} - (\Theta + \kappa) q_{AB} \delta q^{AB} \right].
\end{aligned} \]  

(103)
Adding to Eq. (100), we get:

\[
\sqrt{-g}Q + 2\delta \left[ \sqrt{-g} (\Theta + \kappa) \right] = -3\sqrt{-g} \partial_\mu \delta g_{\nu \mu} - \partial_A (q^{AB} \delta g_{B}) - \delta g_{\nu \mu} \left( \frac{1}{2} q^{AB} \partial_A q_{AB} + 2 \Theta + 2\kappa \right) 
\]

- \delta g_{\nu \mu} \left( \frac{1}{2} q^{AB} \partial_A q_{AB} \right) - \beta^B \delta g_{B} - \frac{1}{2} q^{AB} \delta g_{B} q^{CD} \partial_A q_{CD}

+ 2\partial_\mu \ln \sqrt{-g} - (\Theta + \kappa) q_{AB} \delta q^{AB} - \Theta_{AB} \delta q^{AB} - q^{AB} \partial_\mu \delta q_{AB}

= [\Theta_{AB} - (\Theta + \kappa) q_{AB}] \delta q^{AB} - 2\sqrt{-g} \partial_\mu \delta g_{\nu \mu} - \sqrt{-g} \partial_\Lambda (q^{AB} \delta q_{AB})

- \sqrt{-g} \left[ \delta g_{\nu \mu} (3\Theta + 2\kappa) + \delta g_{\nu \mu} \left( \frac{1}{2} q^{AB} \partial_A q_{AB} \right) + \delta g_{\nu \mu} \left( \beta^B + \frac{1}{2} q^{AB} q^{CD} \partial_A q_{CD} \right) \right]

- 2\partial_\mu \ln \sqrt{-g} - \partial_\Lambda (q^{AB} \delta q_{AB})

= [\Theta_{AB} - (\Theta + \kappa) q_{AB}] \delta q^{AB} - \partial_\Lambda \left( \sqrt{-g} \delta g_{\nu \mu} \right) - \partial_\Lambda \left( \sqrt{-g} q^{AB} \delta g_{AB} \right)

- 2\sqrt{-g} (\Theta + \kappa) \delta g_{\nu \mu} - \sqrt{-g} \delta g_{\nu \mu} \left( \frac{1}{2} q^{AB} \partial_A q_{AB} \right) - \sqrt{-g} \beta^B \delta g_{B} 
\]

In this expression the total divergence term, the counter-term involving \( \Theta + \kappa \) and one involving the variation with \( [\Theta^{ab} - (\Theta + \kappa) q^{ab}] \) are easy to interpret and we have seen it earlier. The extra term which appears, in addition to these, has the following form:

\[
Q_{\text{extra}} = +2\sqrt{-g} (\Theta + \kappa) \delta g^{ar} + \sqrt{-g} \delta g^{rr} \left( \frac{1}{2} q^{AB} \partial_A q_{AB} \right) + \sqrt{-g} \beta^B \delta g^{B} 
\]

These are precisely the terms which arises from the variation of \( \delta \ell^a = \delta g^{ar} \). One can also show that the coefficient of \( \delta \ell^a \) matches with the one in our general expression derived earlier in Eq. (38). Hence the above result can be taken as a verification of our general result. If we had restricted our variations to GNC parameters only, then we would have \( \delta q^{AB} = r \delta \beta^B = 0 \), \( \delta g^{rr} = 2r \delta \alpha = 0 \) and \( \delta g^{ar} = 0 \). Hence we would retrieve our earlier result that only \( q^{AB} \) needs to be fixed.

### 4.4 Null Surface Foliation (NSF)

In this section, we introduce another parametrization for the spacetime metric near a null surface. The fiducial null surface is now taken to be a member of a set of null surfaces, unlike the case for GNC where only the \( r = 0 \) surface in the family of \( r = \text{constant} \) surfaces was taken to be null. We shall call the resulting form of the metric the null surface foliation (NSF) metric. Here we shall give the main outline of its construction. The details can be found in [23, 24].

#### 4.4.1 Construction

Consider a four-dimensional spacetime manifold specified by \( (M, g_{ab}) \). Let our fiducial null surface be one member of a family of null hypersurfaces in this spacetime. Using the fact that null surfaces are spanned by null geodesics, it is possible to introduce a natural coordinate system adapted to our family of null hypersurfaces in the following manner. We first select one of the co-ordinates, say \( x^3 \), such that \( x^3 = \text{constant} \) represents the set of null surfaces with a choice of \( x^3 = 0 \) on \( S \), our fiducial null surface. Then, we choose a spacelike Cauchy surface, \( \Sigma_t \), and denote the intersection of \( S \) with \( \Sigma_t \), a two dimensional surface, by \( S_t \). Let us define the coordinates on this surface as \( x^a \equiv (x^1, x^2) \), with the two corresponding basis vectors lying on \( S_t \) denoted by \( e_A = \partial/\partial x_A \) (with the foreknowledge that the fourth coordinate will be chosen to be constant on \( S_t \)). At every point \( P \) on \( S_t \), there are exactly two future-pointing null directions orthogonal to it, among which one direction will lie on \( S \). Let \( \ell \) denote a null vector field tangent to this direction at every point on \( S_t \). Thus, \( \ell \) will satisfy the relation \( \ell, e_A = 0 \). The same exercise can be repeated replacing \( S \) with every other \( x_3 = \text{constant} \) surface so that we will have the coordinates, \( x_1 \) and \( x_2 \) (as well as \( x_3 \), which has already been defined all over spacetime), and the vector field \( \ell \) defined all over \( \Sigma_t \). Within this arrangement, we can construct a coordinate system near \( \Sigma_t \) in the following way: (a) Take the coordinates \( x^1, x^2, x^3 \) as constant all along the null geodesics passing through every point \( P(x^1, x^2, x^3) \) on \( \Sigma_t \) and moving in the direction of \( \ell^a \) and (b) take \( t \) to be an affine parameter distance along these geodesics with, say, \( t = 0 \) on \( \Sigma_t \). Let us order our four coordinates as \( (t, x^1, x^2, x^3) \). In this coordinate system, the null vector is given by \( \ell^a = (1, 0, 0, 0) \). Thus, the null condition \( \ell^2 = 0 \) implies \( g_{tt} = 0 \). On the other hand, the geodesic condition with affine parametrization,
Since it is possible to eliminate four metric coefficients out of the ten in dependent coefficients to six. This coordinate transformation has been carried out in the line element near the null surface may be written as

\[ ds^2 = -N^2 dt^2 + \left( \frac{M dx^3}{N} + \epsilon N dt \right)^2 + q_{AB} \left( dx^A + m^A dx^3 \right) \left( dx^B + m^B dx^3 \right), \]  

(106)

with \( \epsilon = \pm 1 \). The \( t = \) constant surfaces are taken as a spacelike surfaces. The \( x^3 = \) constant surfaces are null with the metric given by \( ds^2 = q_{AB} dx^A dx^B \). The inverse metric and Christoffel symbols of this metric are given in Appendix D.3.

This metric may be rewritten in the following condensed form:

\[ ds^2 = \bar{M}^2(dx^3)^2 + 2M \epsilon dt dx^3 + 2q_{AB} m^A dx^B dx^3 + q_{AB} dx^A dx^B, \]  

(107)

with \( \bar{M}^2 = M^2/N^2 + q_{AB} m^A m^B \). This metric has seven parameters, one more than the GNC metric in Eq. (64). Since it is possible to eliminate four metric coefficients out of the ten independent coefficients required to parametrize a symmetric four-dimensional metric by using the four coordinate transformations, there must be a coordinate transformation that we could make to reduce the number of parameters to six. This coordinate transformation has been carried out in Appendix D.2 leading to

\[ ds^2 = (\bar{M}^2 - 2\ell \frac{\partial \ln M}{\partial x^3}(dx^3)^2 + 2\epsilon M dt dx^3 + 2q_{AB} m^A dx^B dx^3 + q_{AB} dx^A dx^B \]  

\[ = \bar{M}^2(dx^3)^2 + 2\epsilon M dt dx^3 + 2q_{AB} m^A dx^B dx^3 + q_{AB} dx^A dx^B, \]  

(108)

Another way to get rid of \( M \) is to go over the construction of the coordinate system once again and note that the condition \( \partial_t g_{33} = 0 \) out of the four conditions \( \partial_t g_{ta} = 0 \) has not been used. Now, once the coordinate system \( (x_1, x_2, x_3) \) has been defined on \( \Sigma_t \) and we have decided that \( \Sigma_t \) is a \( t = \) constant surface for the fourth coordinate \( t \), we have the basis vector \( e_3 = \partial/\partial x_3 \) corresponding to the \( x_3 \) coordinate at every point on \( \Sigma_t \). We can then demand \( \ell, e_3 = 1 \) everywhere on \( \Sigma_t \). The four conditions, \( \ell, \ell = 0, \ell, e_3 = 1 \) and \( \ell, e_A = 0 \), fixes \( \ell \) uniquely. Then, we shall have \( M = g_{33} = \ell, e_3 = 1 \) on \( \Sigma_t \) and, by virtue of \( \partial_t g_{33} = 0 \), everywhere else. Thus, from now on, we will assume that the parameter \( M = 1 \) and do the rest of the calculations. The final form of the metric we shall use is

\[ ds^2 = -N^2 dt^2 + \left( \frac{dx^3}{N} + \epsilon N dt \right)^2 + q_{AB} \left( dx^A + m^A dx^3 \right) \left( dx^B + m^B dx^3 \right), \]  

(109)

which on expansion becomes

\[ ds^2 = \bar{M}^2(dx^3)^2 + 2M dt dx^3 + 2q_{AB} m^A dx^B dx^3 + q_{AB} dx^A dx^B, \]  

(110)

with \( \bar{M}^2 = 1/N^2 + q_{AB} m^A m^B \).

4.4.2 Surface Term in the NSF Metric under constrained variation

In this case, with the coordinates \( (t, x^1, x^2, x^3) \), we shall take the normal to the null surface as

\[ \ell_a = \partial_a x_3 = (0, 0, 0, 1); \quad \ell^a = (\epsilon, 0, 0, 0). \]  

(111)

We shall choose the auxiliary vector

\[ k^a = (\bar{M}^2/2\epsilon, 0, 0, -1); \quad k_a = (-\epsilon, -m_1, -m_2, -\bar{M}^2/2). \]  

(112)

where we have defined \( \bar{M}^2 = (1/N^2 + m^2) \). One can verify that \( k^a k_a = 0 \) and \( \ell^a k_a = -1 \). Then, we have

\[ q_{ab} dx^a dx^b = q_{AB} dx^A dx^B \]  

(113)

and

\[ q^{ab} dx_a dx_b = m^2 dx_0^2 - 2m^4 dx_0 dx_A + q^{AB} dx_A dx_B \]  

(114)

where \( q^{AB} \) is the inverse of \( q_{AB} \). Finally, we have the mixed form we have

\[ q^a_b dx_a dx^b = -\epsilon m_B dx_0^B + \delta^A_B dx_A dx^B \]  

(115)
The non-zero components of $\Theta_{ab}$ are
\[ \Theta_{AB} = \frac{\epsilon}{2M} \partial_t q_{AB}, \] 
and
\[ \Theta = \left( \frac{\epsilon}{M} \right) \frac{1}{\sqrt{\xi}} \partial_t \sqrt{\xi}, \]
where $q$ is the determinant of the 2-metric $q_{AB}$. We shall now write down the surface term. The surface term, $\sqrt{-g}Q[\xi]$, can be decomposed using Eq. (15). The different terms in this expansion give:
\[ \sqrt{-g} \nabla_a [\delta \ell^a_\perp] = -\partial_t \left( \epsilon \sqrt{q} \frac{\delta M}{M} \right) \]
\[ -\delta \left( 2\sqrt{-g} \nabla_a \ell^a \right) = -\delta \left( 2M \sqrt{\xi} \Theta \right) = -\delta [2\partial_t (\epsilon \sqrt{\xi})] \]
\[ -\sqrt{-g} \nabla_a \xi^b \delta g_{ab} = M \sqrt{\xi} \Theta_{AB} \delta q^{AB}, \]
\[ -\sqrt{-g} g_{ab} \nabla_c \delta g_{ab} = -\sqrt{-g} \Theta \left[ 2\epsilon M \delta \left( \frac{\epsilon}{M} \right) + q_{AB} \delta q^{AB} \right]. \]

Our aim was to see if we can fix just $q^{AB}$ on the surface, just as we only need to fix $h_{\alpha\beta}$ on the surface for a spacelike or timelike surface. We have variations of $M$ in Eq. (121). But we have already seen in Section 4.4.1 that $M$ can be set to unity with a suitable choice of coordinates. Then, the surface term takes the form
\[ \delta A_{null} = \int_{\partial \mathcal{V}} d^2 x_{\perp} \left\{ -\delta [2\sqrt{\xi}(\Theta + \kappa)] + \sqrt{\xi}[\Theta_{AB} - (\Theta + \kappa)q_{AB}]\delta q^{AB} \right\}. \]

We see that we get back the exact expression we had obtained for GNC in Eq. (87), with $\alpha$ and $q_{AB}$ replaced by their corresponding quantities $\kappa$ and $q_{AB}$. Note that the variations we have considered sets $\delta \ell^a = 0$ and hence we pick up only the remaining terms obtained earlier. To reproduce the terms involving $\delta \ell^a$ we need to consider a more general form of the variation, which we shall now take up.

### 4.4.3 Surface Term in NSF for unconstrained variation

We shall now take the original metric to be the NSF metric but allow the variations to be arbitrary, i.e. the variations are not restricted to those which preserve the original NSF form. We shall start from the expression for the surface term in terms of the variation of the Christoffel symbols given in Eq. (232) in Appendix B:
\[ \sqrt{-g}Q[\xi] = \sqrt{-g_{\alpha\beta}} (g_{\alpha\beta} \delta \Gamma_{\alpha\beta}^c - g_{\alpha}^c \delta \Gamma_{\alpha}^a). \]

Substituting the base metric as the GNC metric and $v_c = \partial_c x^3$, we obtain:
\[ Q[\xi] = g^{tt} \delta \Gamma_{tt}^3 + 2g^{tA} \delta \Gamma_{tA}^3 + 2g^{3A} \delta \Gamma_{3A}^3 - g^{3t} \delta \Gamma_{tt}^3 - g^{tt} \delta \Gamma_{tt}^A - g^{3t} \delta \Gamma_{3t}^3 + q^{AB} \delta \Gamma_{AB}^3. \]

Let us now evaluate each term individually and then put them together. For that, we have the following individual expressions:
\[ g^{tt} \delta \Gamma_{tt}^3 = \frac{1}{2} g^{tt} \partial_t \delta g_{tt} \]
\[ 2g^{tA} \delta \Gamma_{tA}^3 = -\epsilon m^A (\delta g^{3a} \partial_t m_A + \delta g_{tA} \partial_t q_{AB} + \epsilon \partial_A \delta g_{tt}) \]
\[ g^{3A} \delta \Gamma_{3A}^3 = \frac{\epsilon}{2} \left[ \delta g^{3a} \partial_t \left( m^2 + \frac{1}{N^2} \right) + \delta g^{3A} \partial_t m_A + \epsilon \partial_A \delta g_{tt} \right] \]
\[ q^{AB} \delta \Gamma_{AB}^3 = \frac{1}{2} q^{AB} \left[ -\epsilon \partial_t \delta q_{AB} + \delta g^{3C} (-\partial_{c} q_{AB} + \partial_{CA} q_{CB} + \partial_{DB} q_{CA}) + \delta g^{3t} (-\delta q_{AB} + \partial_{A} m_B + \partial_{B} m_A) \right. \]
\[ + \delta g^{AC} (-\partial_{t} q_{CB} + \partial_{AC} q_{BD} + \partial_{DB} q_{CA}) + \delta g^{3t} (-\delta q_{AB} + \epsilon (\partial_{A} \delta g_{tB} + \partial_{B} \delta g_{tA}) \right) \]
\[ \delta \Gamma_{tt}^t = \frac{1}{2} \left[ -\frac{1}{N^2} \partial_t \delta g_{tt} - \epsilon \partial_t \delta g_{tt} - \epsilon m^A (-\partial_A \delta g_{tt} + 2\partial_t \delta g_{tA}) \right] \]
\[ \delta \Gamma_{tt}^A = \frac{1}{2} \left[ \delta g^{A3} \partial_t m_A + \delta q^{AB} \partial_t q_{AB} - \epsilon m^A \partial_A \delta g_{tt} + q^{AB} \partial_t \delta q_{AB} \right] \]
Then substitution of these results in (Eq. (124)) leads to the following expression:

\[
Q[v_\varphi] = \frac{\epsilon}{2} g^{tt} \partial_t g_{tt} - m^A (\delta g^{33} \partial_t m_A + \delta g^{AB} \partial_t q_{AB} + \epsilon \partial_A \delta g_{tt}) \\
+ \frac{\epsilon}{2} \left[ \delta g^{33} \partial_t \left( m^2 + \frac{1}{N^2} \right) + \delta g^{3A} \partial_t m_A + \epsilon \partial_A \delta g_{tt} \right] \\
+ \frac{1}{2} \epsilon g^{AB} \left[ -\partial_t \delta q_{AB} + \delta g^{33} (\partial_t q_{AB} + \partial_A m_B + \partial_B m_A) + \delta g^{3C} (\partial_t q_{AB} + \partial_A q_{CB} + \partial_B q_{CA}) \right] \\
+ \delta g^{3t} (-q^{AB} \partial_t q_{AB}) + \epsilon (\partial_A \delta g_{tt} + \partial_t \delta g_A) \\
- \frac{1}{2} \left[ \frac{1}{N^2} \partial_t \delta g_{tt} - \epsilon \partial_3 \delta g_{tt} - m^A (\partial_A \delta g_{tt} + 2 \partial_t \delta g_A) \right] \\
+ \frac{\epsilon}{2} \left[ \delta q^{3A} \partial_t m_A + \delta q^{AB} \partial_t q_{AB} - m^A \partial_A \delta g_{tt} + q^{AB} \partial_t \delta q_{AB} \right].
\]

\[
Q[v_\varphi] = \frac{\epsilon}{2} g^{tt} \partial_t g_{tt} - m^A (\delta g^{33} \partial_t m_A + \delta g^{AB} \partial_t q_{AB} + \epsilon \partial_A \delta g_{tt}) \\
+ \frac{\epsilon}{2} \left[ \delta g^{33} \partial_t \left( m^2 + \frac{1}{N^2} \right) + \delta g^{3A} \partial_t m_A + \epsilon \partial_A \delta g_{tt} \right] \\
+ \frac{1}{2} \epsilon g^{AB} \left[ -\partial_t \delta q_{AB} + \delta g^{33} (\partial_t q_{AB} + \partial_A m_B + \partial_B m_A) + \delta g^{3C} (\partial_t q_{AB} + \partial_A q_{CB} + \partial_B q_{CA}) \right] \\
+ \delta g^{3t} (-q^{AB} \partial_t q_{AB}) + \epsilon (\partial_A \delta g_{tt} + \partial_t \delta g_A) \\
- \frac{1}{2} \left[ \frac{1}{N^2} \partial_t \delta g_{tt} - \epsilon \partial_3 \delta g_{tt} - m^A (\partial_A \delta g_{tt} + 2 \partial_t \delta g_A) \right] \\
+ \frac{\epsilon}{2} \left[ \delta q^{3A} \partial_t m_A + \delta q^{AB} \partial_t q_{AB} - m^A \partial_A \delta g_{tt} + q^{AB} \partial_t \delta q_{AB} \right].
\]

Now the variation of the expansion scalar can be obtained as

\[
2 \delta \Theta = 2 \delta (\sqrt{-g} \Theta) \\
= 2 \partial_t \delta \sqrt{-g} + 2 \partial_3 \delta \sqrt{-g} + 2 \partial_A \delta \sqrt{-g} + 2 \delta g^{33} \partial_t \ln \sqrt{-g} + 2 \partial_t (\delta \ln \sqrt{-g})
\]

which can be used subsequently to obtain the following result:

\[
2 \delta (\sqrt{-g} \Theta) = 2 \sqrt{-g} \partial_t \delta \sqrt{-g} + 2 \sqrt{-g} \partial_3 \delta \sqrt{-g} + 2 \sqrt{-g} \partial_A \delta \sqrt{-g} + 2 \delta g^{33} \partial_t \ln \sqrt{-g} \\
+ 2 \sqrt{-g} \partial_t (\delta \ln \sqrt{-g}) - 2 \sqrt{-g} \partial_t (\delta \Theta) \sqrt{-g} - \sqrt{-g} \partial_t q_{AB} \delta q^{AB}
\]

Also we have the results

\[
\delta g^{tt} = -\delta g^{33} \quad \delta g_{tA} = -\epsilon g_{tA} \delta g^{33} - \epsilon q_{AB} \delta g^{AB} - \epsilon g_{AB} \delta g^{33}.
\]

As usual, we have the standard counter-term and variation proportional to \(\delta q^{AB}\) which are by now familiar. Thus, we arrive at the following result:

\[
\sqrt{-g} Q + 2 \delta (\sqrt{-g} \Theta) = \frac{\epsilon}{2} \partial_t (\sqrt{-g} \Theta) - \partial_A (\sqrt{-g} m^A \delta g_{tt}) \\
+ \epsilon \partial_A (\sqrt{-g} q^{AB} \delta g_{tt}) + (\Theta_{AB} - \partial q_{AB}) \delta q^{AB} + \sqrt{-g} Q_{extra}
\]

In addition to boundary terms and counter-terms, we have the following extra terms:

\[
\sqrt{-g} Q_{extra} = -\sqrt{-g} \delta g^{3t} q^{AB} \partial_t q_{AB} - \epsilon \sqrt{-g} \delta g^{3A} [m_A q^{CD} \partial_t q_{CD} + \partial_t m_A] \\
+ \sqrt{-g} g^{33} \left[ -\partial_t \left( m^2 + \frac{1}{N^2} \right) - \frac{\epsilon}{2} \left( m^2 + \frac{1}{N^2} \right) q^{AB} \partial_t q_{AB} \right]
\]

These terms are all proportional to \(\delta g^{3A}\) i.e. to \(\delta \ell^a\), as to be expected, with the proper coefficients dictated by Eq. (48). This matches with our earlier conclusion. If we restrict to variations of NSF parameters, then it is clear that \(\delta \ell^a = 0\). Then, in accordance with the general result presented in Eq. (38), the only variation consists of \(q^{AB}\), as we have shown explicitly in the previous section.

5 Discussion of the Number of Degrees of Freedom

We shall now try to enumerate and identify the metric degrees of freedom that we have to fix on the boundary in our approach. Consider first the surface term in the spacelike surface. Initially, the metric has ten degrees of freedom but we have the freedom of making four coordinate choices. We slice the spacetime into \(t = \) constant surfaces, thereby fixing the \(t\)-coordinate of each spacetime point. Then, we introduce the normal to the \(t\)-constant surfaces. There are four components to the normal \(n_a\). But writing \(n_a = -N \partial_t t\) reduces the degrees of freedom available in \(n_a\) to be varied to one (since the coordinates are kept constant during the variation). This degree of freedom, which is in \(N\), is used to satisfy the constraint \(\delta(n_a n^a) = 0\), the condition that the normalization is preserved by the variation, leaving the ten metric degrees of freedom free to flap around. Out of these ten, only six, the degrees of freedom in \(g_{a\beta} = h_{a\beta}\), appear in the boundary term in the variation of the Einstein-Hilbert action after we remove the counter-term part and the surface term on the \(3\)-surface, and hence only these need to be
fixed on the $t = \text{constant}$ surface taken as the initial surface. But we have three more coordinate choices to make. The choice of three coordinates on the initial $t = \text{constant}$ surface can be used to remove three of the six degrees of freedom in $h_{\alpha\beta}$ on the surface. This leaves 3 degrees of freedom. (It can be argued that one of these three degrees of freedom just gives the location of the 3-surface in the 4-dimensional spacetime and only the other two degrees are real physical degrees of freedom [18]; this is irrelevant to our purpose and hence we will ignore this additional reduction of one degree of freedom.)

Let us now consider the surface term on a null surface. The surface term was initially specified in terms of the metric. But then we introduced two vectors $\ell^a$ and $k^a$ into the mix. (Note that we cannot take, say, $\ell_a$ and $k^a$ as independent under the variation. Once we are given the metric and $\ell_a$, the $k^a$ is fixed.) The degrees of freedom that we then have, ostensibly, in the system are the ten metric components, $g_{ab}$, and the four components each of the two null vectors $\ell^a$ and $k^a$, giving us eighteen degrees of freedom in total (with the gauge freedom in choosing the four coordinates unutilized). Obviously, there should exist several constraints amongst these variables, which we will now examine.

We thus notice that, in the spacelike case, the normal $n_a$ ended up completely constrained and hence there were only the metric degrees of freedom left free. If we are attempting a counting of the degrees of freedom of the metric that needs to be fixed on the null surface, we should similarly take care that the extra degrees of freedom that we have introduced in $\ell^a$ and $k^a$ are constrained appropriately. In our derivation of Eq. (36), we have assumed the conditions $\ell_a = \partial_a \phi$ (which may also be written as $\ell_a = \partial_a (c - \phi = \partial_a r)$ and $\delta \ell_a = 0$. Thus, $\ell_a$, and equivalently $\ell^a$, is completely constrained in terms of the metric and the coordinates. We also have introduced the four degrees of freedom in $k^a$. We have carried out our calculations imposing the constraints $\ell.\ell = 0$, $\ell.\k = -1$ and $k.k = 0$. The two last equations can be satisfied by constraining two degrees of freedom in $k^a$. (We shall demonstrate this shortly.) Meanwhile, consider the first constraint $\ell.\ell = 0$. Since we are working with $\delta \ell_a = 0$, $\delta (\ell.\ell) = 0$ is the same as $\delta g^{ab} \ell_a \ell_b = \delta g^{rr} = 0$, i.e., this constraint cannot be satisfied unless we constrain a particular degree of freedom in the metric. We shall come back to this point at the end.

Let us now get to the task of constraining $k^a$. In Appendix C, the form of the components of $\delta \ell_a, \delta k_a, \delta \ell^a$ and $\delta k^a$ in terms of the original canonical null basis, once the constraints are imposed, is provided in Eq. (271). Right now, we shall try to make use of the degrees of freedom in the variations of $k$ to satisfy $\delta (k.k) = 0$ and $\delta (\ell.k) = 0$. In our case, we have $\delta \ln A = 0$. The variations of $k$ on the null surface, $r = 0$, have to take the form:

$$\delta k^a = -f_2 \ell^a + R^A_2 e^a_A$$
$$\delta k_a = f_2 \ell_a + f_1 k_a + R^A_2(e_A)_a$$

(131)
(132)

We can constrain this further. Note that

$$\delta (k.k) = \delta g_{ab} k^a k^b + 2 k_a \delta k^a = k_a \left(2 \delta k^a - \delta g^{ab} k_b\right).$$

(133)

Thus, $\delta (k.k) = 0$ implies

$$2 \delta k^a - \delta g^{ab} k_b = 2 (A k^a + B^A e^a_A) \implies \delta k^a = \frac{\delta g^{ab} k_b}{2} + A k^a + B^A e^a_A.$$  

(134)

Using the other constraint, $\delta (\ell.k) = 0 \Rightarrow \ell_a \delta k^a = 0$, we can fix $A$ in terms of the metric variations as

$$A = \frac{\ell_a k_b \delta g^{ab}}{2}.$$  

(135)

Thus, in Eq. (134), the only degrees of freedom remaining in $\delta k^a$ are in $B^A$, which means that the $f_2$ in Eq. (131) is fixed by the constraints. Thus, we have used up two degrees of freedom in $k$ to satisfy the constraints $\delta (k.k) = 0$ and $\delta (\ell.k) = 0$. The two remaining degrees of freedom, in terms of $\delta k^a$, are in $R^A_2$ in Eq. (131). This was expected as we had only two constraints but four components of $\delta k^a$ to be fixed. The corresponding ambiguity in the specification of the original $k^a$ is well-known [25]. In order to remove the ambiguity for $\delta k^a$, let us fix a gauge by choosing $R^A_2$ to be zero. In terms of $e_A$, this may be written as $\delta k^a (e_A)_a = 0$. Another way to state the same constraint is

$$g^{ab}_a \delta k_b = 0,$$  

(136)

which is a form better suited for our formalism as we have not explicitly introduced $e_A$ when dealing with the boundary term.
Having, thus fixed the degrees of freedom in $\ell^a$ and $k^a$, let us examine the boundary term. Eq. (36) tells us that one set of constraints that may be imposed on the null surface to kill the boundary term is given by
\[ \delta q^{ab} := 0; \quad \delta \ell^a := 0. \] (137)

Let us first take the case of $\delta \ell^a$. Note that $\delta \ell^a = \delta q^{ar}$ in terms of the metric degrees of freedom. Out of these degrees, $\delta \ell^r = 0$ due to our constraint $\delta (\partial_r \ell) = 0$. Only the surface components, $\delta \ell^a = \delta q^{ar}$, remain non-vanishing. Now, we have worked with variations that do not affect the coordinates. But since coordinate choices can be thought of as gauge degrees, we can try to eliminate $\delta \ell^a$ by a suitable choice of the gauge. If we eliminate $\delta \ell^a$ using a coordinate choice, we are essentially fixing a gauge in which $\delta q^{ar} = 0$. We shall next try to add an infinitesimal coordinate transformation on top of our variation to reduce $\delta \ell^r$ to zero.

Let the coordinate transformation be given by $x^a \rightarrow x^a = x^a + \xi^a$. We should make sure that this transformation does not affect any of our other constraints. The constraints on $\ell, \ell\cdot k$ and $k\cdot k$ are constraints on scalars and hence will not be affected by our coordinate transformation. (Note that we are considering the variation of the object at the same physical spacetime point, since we are imagining a physical surface which bounds the spacetime volume we are interested in and we are attempting to eliminate certain degrees of freedom on that surface. Hence, it is the variation under the tensor transformation law that has to be considered here, and not the Lie variation.)

The constraint $\delta \ell_a = 0$ will be affected by the coordinate transformation and hence we need to impose appropriate conditions on $\xi^a$ for this constraint. Under our original variation, $\ell_a$ did not change due to the constraint, i.e., $\ell'_a = \ell_a$. Now, the coordinate transformation will change this to
\[ \ell'^a_a = \ell_a = \partial_a \xi^r. \] (138)
Requiring no change will enforce $\partial_a \xi^r = 0$, i.e., one can only have a constant shift in the $r$–direction. This has to be the form everywhere in the spacetime as we have taken $\delta \ell_a = 0$ everywhere. A constant shift leaves all tensor quantities invariant and hence is of no use to us. We may therefore set $\xi^r = 0$. Thus, the form of $\xi^a$ should be $(0, \xi^a)$, i.e., an infinitesimal transformation on the $r = \text{constant}$ surface, everywhere in spacetime. Further, the position of the null surface, $r = 0$, will now be at $r' = 0$ in the new coordinate system.

Next, we shall consider our constraint, Eq. (136), that $q^c_a \delta k^b = 0$. On coordinate transformation, the vector $k^a = k^a + \delta k^a$ goes to
\[ k'^a = k^a + k^b \partial_b \xi^a \] (139)
\[ = k^a + \delta k^a + (k^b + \delta k^b) \partial_b \xi^a. \] (140)
Thus, the variation from the original $k^a$ becomes
\[ \delta' k^a = \delta k^a + (k^b + \delta k^b) \partial_b \xi^a. \] (141)
We need to demand that this be zero when acted on by the original $q^c_a$. (Our constraint on $\delta k^a$ is that it should be a linear combination of the original $\ell^a$ and $k^a$.) This demand gives us the following condition:
\[ q^c_a (k^b + \delta k^b) \partial_b \xi^a = 0. \] (142)
Although this appears to be a set of four conditions, the conditions $q^c_a \ell_c = 0$ and $q^c_a k_c = 0$ mean that there are only two independent constraints here.

Having thus listed the constraints that $\xi^a$ has to satisfy, we now get to the task of eliminating $\delta \ell^a$. Under the coordinate transformation, the vector $\ell'^a = \ell^a + \delta \ell^a$ goes to
\[ \ell'^{ma} = \ell^m + \ell^b \partial_b \xi^a. \] (143)
We want the coordinate transformation to reduce $\ell'^a$ back to $\ell^a$. Putting $\ell'^a = \ell^a$, we get the constraint
\[ \delta \ell^a + \ell^b \partial_b \xi^a = 0. \] (144)
The $r$–component of this equation is just
\[ \delta \ell^r = 0, \] (145)
since we have already imposed $\partial_a \xi^r = 0$. This means that, if $\delta \ell^r$ is not zero initially, then it cannot be set to zero by the coordinate transformation respecting $\delta \ell_a = 0$. But in our case $\delta \ell^r$ is indeed constrained to be zero by $\delta \ell^2 = 0$. 

24
If we set $\ell^a = \partial / \partial \lambda$, then the condition for putting $\delta \ell^a$ to zero on the surface is

$$\partial_\lambda \xi^a = -\delta \ell^a,$$  

(146)
on the surface. This fixes how $\xi^a$ varies in the $\lambda$–direction. This direction is on the null surface as $\ell^a \ell_a = 0$ at the null surface. Thus, this condition will not conflict with Eq. (142) as it constrains derivatives along $k^a + \delta k^a$, which does not lie on the null surface ($\ell^a \ell_a = k^a k_a = -1$).

Now, suppose we perform an infinitesimal coordinate transformation according to Eq. (146) to set $\delta \ell^a$ to zero. This means that we only have $\delta q^{ab}$ left to be set to zero on the null surface. This is a set of three conditions. To see this, note that only the combination $[\Theta_{ab} - (\Theta + \kappa) q_{ab}] \delta q^{ab}$ occurs in Eq. (38). The symmetric object $\Theta_{ab} = (\Theta + \kappa) q_{ab}$ has 10 components and satisfies 7 constraints in the form of the set of four constraints $[\Theta_{ab} - (\Theta + \kappa) q_{ab}] \ell^a = 0$ plus the set of another four constraints $[\Theta_{ab} - (\Theta + \kappa) q_{ab}] k^a = 0$ from which one constraint $[\Theta_{ab} - (\Theta + \kappa) q_{ab}] \ell^a k^b = 0$ is to be subtracted as it occurs in both sets. Thus, $\Theta_{ab} - (\Theta + \kappa) q_{ab}$ has only three components and only the corresponding components of $q^{ab}$ need to be fixed. But the question that should be asked is if we have any coordinate degrees of freedom left that we can use to reduce this number further.

As we have shown, the constraint $\delta \ell_a = 0$ allows only surface coordinate transformations. But these surface transformations can be used to set $\delta \ell^a$ to zero as per Eq. (146). Suppose we have fixed such a gauge that $\ell^a$ does not vary on the null surface and is along $\partial / \partial \lambda$. Still, coordinate transformations that respect

$$\partial_\lambda \xi^a = 0$$  

(147)
are allowed. We also have to respect Eq. (142). Since $\xi^r$ is set to zero, only $\xi^a$ components occur in these constraints. Now, the derivative is along $k^a + \delta k^a$. From our original constraint $q_{\alpha}^a \delta k^b = 0$, we know that $\delta k^b$ can be expressed in terms of $\ell^a$ and $k^a$ alone. The derivative along $\ell^a$ of $\xi^a$ is fixed to be zero by virtue of the above constraint, $\partial_\lambda \xi^a = 0$. Also, $\delta k^b$ does not have any component along $k^b$ on the null surface as per Eq. (131). Thus, Eq. (142) is a statement about the derivative of $\xi^a$ along $k^a$.

We can rewrite this constraint as

$$q_{\alpha}^a k^b \partial_\lambda \xi^a = 0. \quad (148)$$

As we have already stated, this is a set of only two independent equations. Let us now choose $\lambda$ as a coordinate, the second coordinate specified after $r$. Let the other two coordinates be denoted by $x^A$ and be represented by capital Latin letters in indices. Then, we have $q_{\lambda}^a = q_{\alpha}^a \ell^a = 0$. Thus, the above equation does not constrain $\xi^\lambda$. The constraint may be written as

$$q_{\alpha}^a k^b \partial_\lambda \xi^A = 0, \quad (149)$$

which constrains only two components of $\xi$. This constraint does not restrict our freedom to choose coordinates on the surface $r = 0$, since we may tune $\partial_\lambda \xi^A$ (which has to occur in the constraint since $k^r = -1$ is non-zero) in order to satisfy it, whatever behaviour we may choose for the coordinates on the $r = 0$ surface.

Now let us see what components of $\delta q^{ab}$, if any, may be set to zero by coordinate transformations respecting our constraints. Since we have $q^{ab} \ell_b = 0$ and $\delta q^{ab} \ell_b = \delta (q^{ab} \ell_b) = 0$ (see Eq. (29)), we need to consider only the surface components $q^{a\beta}$ and $\delta q^{a\beta}$. Let $q^{a\beta} = q^{a\beta} + \delta q^{a\beta}$. Now we perform a coordinate transformation using $\xi^a = (0, \xi^\alpha)$ to obtain

$$q'^{\alpha\beta} = q^{\alpha\beta} + q^{\alpha \eta} \partial_\mu \xi^\beta + q^{\mu \beta} \partial_\mu \xi^\alpha. \quad (150)$$

Thus, the variation from the original $q^{a\beta}$ becomes

$$\delta' q^{a\beta} = \delta q^{a\beta} + q^{\alpha \eta} \partial_\mu \xi^\beta + q^{\mu \beta} \partial_\mu \xi^\alpha. \quad (151)$$

Can we tune the $\xi^\alpha$ to eliminate any of the degrees of freedom in $\delta' q^{a\beta}$? $\delta' q^{a\beta}$ can be any function of $\lambda$ and the other two $x^A$ coordinates on the $r = 0$ surface. But $\xi^\alpha$ are functions only of the $x^A$ due to Eq. (147). Thus, we may be able to eliminate some degree of freedom of $\delta' q^{a\beta}$ on some $\lambda = \lambda_1$ 2-surface by tuning $\xi^\lambda$ and follow it up with the elimination of the same degree of freedom on two other 2-surfaces, $\lambda = \lambda_2$ with $A = 1, 2$, by tuning one each of the two $\xi^A$. But once this is done, $\xi^\alpha$ will be fixed everywhere on the 3-surface. The degree of freedom that we have fixed to some known values on three 2-surfaces may take some arbitrary values elsewhere on the 3-surface and we would not have any coordinate choices left to rein it in. Hence, we do not have the freedom to eliminate any of the three degrees of freedom in $q_{\alpha\beta}$ using coordinate transformations.
Before we conclude this section, let us come back to the point we had promised we shall discuss. We had pointed out that the constraint \( \ell, \ell := 0 \) will lead to \( g^{\phi\phi} = g^{\phi r} \) being fixed on the null surface. The initial variational problem involved arbitrary variations of all components of the metric. After introducing our formalism and fixing certain gauges to eliminate the extra degrees of freedom that we have put in, we seem to have also frozen one degree of freedom in the metric, leaving us with one degree of freedom less than at our starting point. If we unfreeze this degree, will its variation appear in the surface term and give us one more degree of freedom to fix on the surface? Here we appeal to the success of the limiting procedure. The degree of freedom \( g^{\phi\phi} \) in the ADM case corresponds to \( N \). Since \( N \) does not have to be fixed on the boundary in the ADM case, we should expect that the same should hold for the null case.

In the case of a spacelike surface, it is easy to argue that \( h_{a b} \) needs to be fixed at the boundary. Further since we have the freedom of three coordinate transformations on the surface, it is obvious that we can control 3 components out of the 6 components of \( h_{a b} \), reducing the number of degrees of freedom to 3. We have provided here the corresponding arguments in the case of null surface. The variational principle tells us that we now need to fix \((q^{AB}, \ell^a)\) having 6 degrees of freedom of which 3 degrees of freedom in \( \ell^a \) can be taken care of by coordinate choice, leaving again 3 degrees of freedom. While there is no nice characterisation of these 3 degrees of freedom contained in \( h_{a b} \), we now have a simple way of identifying them with the degrees of freedom in \( q^{AB} \). It is, however, known that one can further reduce the number of degrees of freedom by 1 by using the foliation freedom. Since this is common to description with non-null surface and null surface, we do not expect any new features to emerge in this context. Nevertheless it will be useful to provide an explicit demonstration of the final reduction of the degrees of freedom from 3 to 2 both in the spacelike and null cases. We hope to return to these issues in a future work.

6 Conclusions

Our aim in this work was to find out the structure of the boundary term in the variation of the Einstein-Hilbert action when the boundary is a null surface. We wanted to do this from first principles, adhering to the philosophy: You don’t tell action what variations to consider; the action will tell you that, in the case of Einstein-Hilbert action we expect the action to also tell us what counter-term needs to be added to make the action well-defined.

We first undertook a general analysis of the surface term for a null surface \( \phi = \phi_0 \), or \( r = \phi - \phi_0 = 0 \), with \( \ell_a = \partial_a \phi \) and \( k^a \) being the auxiliary null vector satisfying \( k^a \ell_a = -1 \). We took \( \phi \) as one of the coordinates and considered variations that vary the metric but do not affect the coordinates. (This assumption also means \( \delta \ell_a = 0 \).) The variations were taken to respect the following constraints: i) \( \ell, \ell := 0 \), ii) \( k, k = 0 \) and iii) \( \ell, k = -1 \), with the first constraint being imposed only on the null surface. In this case, we found that the surface term on the null surface can be expressed in the following form (see Eq. (38)):

\[
\delta A_{null} = \int_{\partial V} d\lambda d^2 z \{ \sqrt{\gamma} \nabla_a (\Pi^a b \delta b^b) - 2\delta \sqrt{\gamma} (\Theta + \kappa) + \sqrt{\gamma} (\Theta_{ab} - (\Theta + \kappa) q_{ab}) \delta q^{ab} + \delta \ell^c P_c \},
\]

where, taking the normal \( \ell_a = \partial_a \phi \) for the null boundary being a \( \phi \) constant surface and \( k^a \) to be the auxiliary null vector, we have \( \Pi^a b = \delta^a b + k^a \ell_b, \delta b^b = \delta q^{ab} + g^{ab} \delta b_b \) and \( q_{ab} = g_{ab} + \ell a \ell b + k a k b \) is the induced metric on the null surface (see Appendix A.2.2). The integration is over a parameter \( \lambda \), varying along the null geodesics such that \( \ell^a = d x^a / d \lambda \), and two coordinates \((z^1, z^2)\) that are constant along the null geodesics (see Appendix A.2.3). The capital Latin letters in the indices run over \((z^1, z^2)\). \( \Theta_{ab} = g^{a'}_{a} q^{b'}_{b} \nabla_{m} \ell_{a} \) is the second fundamental form, \( \Theta = \Theta^a_{a} \) is the expansion scalar and \( \kappa \) is the non-affinity coefficient on the null surface. (The definitions of \( \Theta_{ab}, \Theta \) and \( \kappa \) can be found in Appendix A.2.5 and Appendix A.2.1.) \( q \) is the determinant of the 2-metric \( q_{AB} \). The conjugate momentum to \( q^{ab} \) turns out to be \( \sqrt{\gamma} (\Theta_{ab} - (\Theta + \kappa) q_{ab}) \) while the conjugate momentum to \( \ell^a \) is \( P_a = -\sqrt{\gamma} k_b [\nabla_a \ell^b + \nabla^b \ell_a - 2 \delta^b (\Theta + \kappa)] \). We have also repeated the whole analysis for a null vector of the form \( \ell_a = A \ell_a \phi \), with \( A = 1 \) on the null surface, and satisfying the condition \( \ell^2 = 0 \) everywhere to obtain the same decomposition as in Eq. (152).

We note that the structure is very similar to the structure in the non-null boundary (see Eq. (51)). The first term in Eq. (152) is the total 3-derivative term (due to the properties of \( \Pi^a b \)), the second term can be cancelled by a counter-term \( 2 \sqrt{\gamma} (\Theta + \kappa) \) added to the action and the third term can be
killed by fixing $q^{ab}$ on the surface. There is an extra term that has the variation of $\ell^c$ but we have shown in Section 5 that $\delta \ell^c$, the components on the surface, can be put to zero on the null surface using appropriate infinitesimal coordinate transformations on the surface. ($\delta \ell^c = \delta \ell^c$ is already zero due to the imposed constraints.) It is also shown in Section 5 that only three degrees of freedom in $q^{ab}$ need to be fixed on the surface. Further, it is demonstrated that there is no freedom left in the choice of coordinates to reduce this number to make contact with the non-null case [18]. It is then argued that the constraint $\delta \ell_a = 0$ is the culprit and that this must be relaxed to eliminate one more degree of freedom. But the detailed investigation in this direction has been kept for a future work.

We have also verified the result in Eq. (152) using two parametrizations of the metric near an arbitrary null surface: the GNC and NSF metrics. For GNC, the null surface is one of the set of surfaces of constant coordinate $r$, with the other surfaces being timelike or spacelike. Thus, there is a natural way to obtain the results on the null surface as a limit of results on spacelike or timelike surfaces. On the other hand, NSF metric has the fiducial null surface as one member of a set of null surfaces, $x_3 = 1$ is constant, and hence does not provide a natural framework for obtaining the results on the null surface as a limit. In both cases, we consider variations that preserve the structure of the underlying metric and obtain the same structure for the surface term (see Eq. (87) and Eq. (122)) as in Eq. (152).

We summarize in Tab. 1 the similarities and differences between non-null surfaces and null surfaces with respect to the intrinsic geometry and treatment of the boundary term.

| Properties | Non-null | Null |
|------------|---------|------|
| Normal “Dimension” | $n_a; n_a n^a = \epsilon = \pm 1$ | $\ell_a; \ell_a \ell^a = 0$ |
| Induced Metric | $h_{\alpha\beta} = g_{ab} \epsilon_{\alpha a} \epsilon_{\beta b}$ | $q_{AB} = g_{ab} \epsilon_{\alpha a} \epsilon_{\beta b}$ |
| Auxiliary Vector | $-\epsilon n^a$ | $k^a; \ell_a k^a = -1$ |
| Integration Measure | $d^3x \sqrt{|h|}$ | $d^2x d\lambda \sqrt{q}$ |
| Second Fundamental Form | $K_{ab} = -h^{mn} \nabla_m n_b$ | $\Theta_{ab} = q_{ab} q^{cd} \nabla_m n_n$ |
| Counter-term | $-\int d^3x \sqrt{|h|} 2K$ | $\int d\lambda d^2z_\perp \sqrt{q} 2 (\Theta + \kappa)$ |
| Boundary Term | $\sqrt{|h|} D_\alpha \delta n^a$ | $\sqrt{|q|} D_\alpha \left( \Pi a b \delta \ell^b \right)$ |
| Degrees of Freedom | $h_{\alpha\beta}$ | $q_{AB}, \ell^c$ |
| Conjugate Momentum to Induced Metric | $-\sqrt{|h|} \left[ K^{ab} - h^{ab} K \right]$ | $\sqrt{|q|} h_b \left[ \nabla_c \ell^b + \nabla^b \ell_c - 2 \delta^b_a (\nabla_a \ell^a) \right]$ |

The defining characteristic that distinguished a non-null surface and a null surface is the surface gradient, whose norm is non-zero for a non-null surface and zero for a null surface. The norm of the surface gradient for a non-null surface can be normalized to ±1 by multiplying with a suitable scalar. The induced metric on a non-null surface is a 3-metric $h_{\alpha\beta}$ while the induced metric on a null surface is a 2-metric $q_{ab}$. Hence, the “dimension” of the null surface has been listed as 2. For a null surface, we define an auxiliary vector $k^a$ to be such that $\ell_a k^a = -1$ and $k^a k_a = 0$. For a non-null surface, the term auxiliary vector is not generally used but we have put $-\epsilon n^a$ in the respective place in the table as this vector satisfies $(\pm \epsilon n^a) n_a = -1$. The rest of the entries in the table are quantities which we have already discussed in the analysis of our results on the decomposition of the boundary term on a null surface.

**Acknowledgments**

The research of TP is partially supported by J.C. Bose research grant of DST, India. KP and SC are supported by the Shyama Prasad Mukherjee Fellowship from the Council of Scientific and Industrial Research (CSIR), India. KP would like to thank Kinjal Lochan for discussions.
Appendices

A Some Requisite Pedagogical and Background Material

A.1 Gauss’ Theorem

In order to explain the conventions regarding the use of the Gauss’ theorem, we refer to the proof of the theorem given in Chapter 3 of [25]. The Gauss’ theorem is stated in the following form:

$$\int_V d^4x \sqrt{-g} \nabla_a A^a = \int_{\partial V} d\Sigma_a A^a ,$$  \hspace{1cm} (153)

with \( d\Sigma_a \) being the directed surface element on the boundary of the integration volume. Note that [25] uses Greek indices to run over all spacetime indices and Latin indices to run over indices on a surface, opposite of the convention we have adopted. Then, the proof is illustrated using a set of \( x^0 = \text{constant} \) surfaces. The following expression is obtained:

$$\int_V d^4x \sqrt{-g} \nabla_a A^a = \int d^3x \sqrt{-g} A^0 \bigg|_{x^0=1} ,$$  \hspace{1cm} (154)

This may be rewritten as

$$\int_V d^4x \sqrt{-g} \nabla_a A^a = \int d^3x \sqrt{-g} A^0 \bigg|_{x^0=1} - \int d^3x \sqrt{-g} A^0 \bigg|_{x^0=0} = \int d^3x \sqrt{-g} A^0 n_a ,$$  \hspace{1cm} (155)

with \( n_a = \partial_a x^0 \) at \( x^0 = 1 \) and \( n_a = -\partial_a x^0 \) at \( x^0 = 0 \). This depends on the fact that the integration is carried out from \( x^0 = 0 \) to \( x^0 = 1 \). If the integration was done the other way, the signs would have flipped.

So we may formulate the following rule. Let us assume that \( x^0 \) is going to be integrated from lower value to higher value. In that case, \( \partial_a x^0 \) or \( -\partial_a x^0 \) is to be used at an \( x^0 = \text{constant} \) surface according to whether \( x^0 \) or \( -x^0 \) is increasing as we move from inside the integration volume to outside through the surface.

A.1.1 Decomposition of \( \sqrt{-g} \) in Terms of Determinant of Metric on a 2-surface

One relevant question is whether there is a decomposition of \( \sqrt{-g} \) in terms of \( \sqrt{q} \), \( q \) being the determinant of the 2-metric \( q_{AB} \) on the null surface, akin to the decomposition \( \sqrt{-g} = N \sqrt{|h|} \) in the timelike and spacelike case. We shall prove in this appendix a preliminary result on the decomposition of the determinant of a 4 \( \times \) 4 metric in terms of the determinant of a 2 \( \times \) 2 submatrix, which will be later applied to a null surface.

We start by writing down a general result relating the determinant of a 2 \( \times \) 2 submatrix to the determinant of the whole matrix. We shall prove the result working with the metric written in the coordinates \((\phi, x^1, x^2, x^3)\) with the components on the \( \phi = \text{constant} \) surface being denoted by \( h_{\alpha\beta} \) (refer Appendix B):

\[
g_{ab} = \begin{pmatrix} g_{\phi\phi} & g_{\phi1} & g_{\phi2} & g_{\phi3} \\ g_{1\phi} & h_{11} & h_{12} & h_{13} \\ g_{2\phi} & h_{21} & h_{22} & h_{23} \\ g_{3\phi} & h_{31} & h_{32} & h_{33} \end{pmatrix} .
\]  \hspace{1cm} (156)

In this case, we can use the definition of an inverse matrix element applied to \( g^{\phi\phi} \) to write

\[
g = \frac{h}{h^{\phi\phi}} ,
\]  \hspace{1cm} (157)

where \( h \) is the determinant of \( h_{\alpha\beta} \), the 3 \( \times \) 3- matrix obtained by deleting the \( \phi \)-column and \( \phi \)-row from \( g_{ab} \). Now, we can play the same game again with \( h_{\alpha\beta} \). The determinant of the 2 \( \times \) 2 matrix \( q_{AB} \), \( A, B = 2, 3 \), defined by \( q_{AB} = h_{AB} \) satisfies an analogue of Eq. (157):

\[
h = \frac{q}{h^{11}} ,
\]  \hspace{1cm} (158)
where $h^{11}$ is the 11-th component of the matrix $h^{\alpha\beta}$, the inverse of the matrix $h_{\alpha\beta}$. Substituting for $h$ in Eq. (157), we obtain

$$g = \frac{q}{g^{\phi\phi}h^{11}}. \quad (159)$$

Now, the denominator above can be expanded as follows:

$$g^{\phi\phi}h^{11} = g^{\phi\phi}g^{11} - (g^{1\phi})^2, \quad (160)$$

which is easiest to obtain by using the formula $h^{ab} = g^{ab} - \epsilon^{a\nu}n^{\nu}.b$. Thus, we obtain a relation relating the determinant of a $2 \times 2$ submatrix with the determinant of the full $4 \times 4$ matrix:

$$g = \frac{q}{g^{\phi\phi}g^{11} - (g^{1\phi})^2}. \quad (161)$$

### A.2 Null Surfaces

In this section, we shall discuss various pedagogical and otherwise useful material on null surfaces that will be needing in the main text.

#### A.2.1 The Non-affinity Coefficient $\kappa$

In this appendix, we shall prove that $\ell^a \nabla_a \ell_b \propto \ell_b$, for $\ell_a = A \partial_a \phi$, to a null surface $\phi = \phi_0$. We have

$$\ell^a \nabla_a \ell_b = \ell^a \nabla_a (A \partial_b \phi) = \frac{\ell^a}{A} \partial_a A + \ell^a A \nabla_a \ell_b \phi$$

$$= \frac{\ell^a}{A} \partial_a A + \ell^a A \nabla_b \ell_a \phi = \frac{\ell^a}{A} \partial_a A + \ell^a A \nabla_b \frac{\ell_a}{A}$$

$$= \ell^a \frac{\ell_a}{A} \partial_a A + \ell^a \nabla_b \ell_a = \ell^a \partial_a (\ln A) \ell_b + \frac{1}{2} \partial_b (\ell^a \ell_a) \quad (162)$$

Consider some coordinate system with $\phi$ as one of the coordinates, say ($\phi, x_1, x_2, x_3$). In such a coordinate system,

$$\ell_a = A \partial_a \phi = (A, 0, 0, 0) \quad (163)$$

Now, $\partial_b (\ell^a \ell_a)$ will only have the $\phi-$component at the null surface. This is because $\ell^a \ell_a = 0$ all along the null surface and hence only $\partial_b (\ell^a \ell_a) \neq 0$ at the null surface. Thus, we obtain $\partial_b (\ell^a \ell_a) \propto \ell_b$ and

$$\ell^a \nabla_a \ell_b = \kappa \ell_b, \quad (164)$$

where $\kappa$ is a scalar. It may be termed the non-affinity coefficient as it will be zero for an affine parameterization of the null geodesics [26]. An explicit expression for $\kappa$ is derived in Appendix A.2.6.

#### A.2.2 Induced Metric on a Null Surface

In this appendix, we shall discuss how to find the induced metric on a null surface. If we choose an auxiliary vector $k^a$ such that $k^a \ell_a = -1$ on the null surface $\phi = \phi_0$ (note that we have not specified $k^a k_a$ yet), then we have the following results on the null surface. First, consider the object

$$\Pi^a_b = \delta^a_b + k^a \ell_b \quad (165)$$

It is easy to verify that $\Pi^a_b \ell_a = 0$, so any vector $L^a$ can be acted on by $\Pi^a_b$ to give $M^b = \Pi^b_a L^a$ such that $M^a \ell_a = 0$. To verify that an operator $P$ is a projector, we need to verify $P^2 = P$, which is satisfied in this case as $\Pi^a_b \Pi^b_a = \Pi^a_a$.

While $\Pi^a_b$ may be good as a projector, $h_{ab}$ had the additional status of being the induced metric on the surface. Now, $\Pi_{ab}$ does have the property that if we look at the components on the surface (represented by Greek letters, meant to run over $\{x^1, x^2, x^3\}$ in a coordinate system $\{\phi, x^1, x^2, x^3\}$), they do satisfy $\Pi_{ab} = g_{\alpha\beta}$ as $\ell_a = 0$. But if we want a symmetric object, we may turn to

$$q_{ab} = g_{ab} + \ell_a k_b + k_a \ell_b = \Pi_{ab} + \ell_a k_b \quad (166)$$

This also has the property $q_{ab} \ell_a = 0$ and, in addition, $q_{ab}^{\mu\nu} = 0$ at $\phi = \phi_0$ (while $\ell_0^a \ell^a = \ell^a$). To see if this is a projector, consider

$$q_{ab}^{\mu} q_{\mu}^{b} = q_{c}^{c} + \ell^a k_b q_{ab}^{c} \quad (167)$$
Thus, in order for $q_0^a$ to be a projector, we need the additional condition $\ell^a k_b q_c^b = 0$, or $k_b q_c^b = 0$. This means
\begin{equation}
k_b q_c^b = k_c - k_c + k_b k_c = 0,
\end{equation}
which requires $k^a k_a = 0$, i.e., we need to choose $k^a$ to be a null vector.

Thus, for a null surface with normal $\ell_a$, we choose an auxiliary vector $k^a$ such that (i) $\ell_a k^a = -1$ and (ii) $k_b k^a = 0$. Then, the symmetric object $q_{ab}$ is such that $q_0^a$ acts as a projector to the space orthogonal to $\ell^a$. In fact, it also acts a projector to the space orthogonal to $k^a$. Thus, it projects to a 2-dimensional subspace of the tangent space on the 3-surface. We may say that it is a projector to the $v$ with, e.g., $k^2 = 0$.

Thus, $q_0^a$ projects to the subspace spanned by $e_A^a$. The metric induced on the surface formed by the $e_A^a$ is
\begin{equation}
g_{ab} e_A^a e_B^b = (\ell_a k_b - \ell_b k_a) e_A^a e_B^b = q_{ab} e_A^a e_B^b = q_{AB}.
\end{equation}
Now, $q_{AB}$ contains the whole information about $q_{ab}$ since $q_{ab} \ell^a$ and $q_{ab} k^b$ are already constrained to be zero. Thus, $q_{AB}$, and hence $q_{ab}$, represents the induced metric on the two-dimensional surface spanned by $e_A$ and orthogonal to $\ell$ and $k$.

To get the induced metric on the 3-surface, note that the vector space on the 3-surface is spanned by $(\ell, e_A)$ as $\ell^a \ell_a = 0$, $e_A^a \ell_a = 0$ but $k^a \ell_a = -1$. Thus, the induced metric on the 3-surface consists of the components $g_{ab} e_A^a e_B^b = q_{AB}$, $g_{ab} \ell^a \ell^b = 0$ and $g_{ab} k^a \ell^b = 0$. Hence, the 3-metric is also effectively $q_{AB}$.

Let us now introduce the dual basis [16] to the canonical null basis. We need a set of four linearly independent one-forms, $v_{*}^{(a)}$, such that $v_{*}^{(i)} v_{*}^{(j)} = \delta_{ij}$. Denoting the inverse of the 2-metric $q_{AB}$ by $q^{AB}$, we introduce a set of two one-forms, $e_A^a$, such that
\begin{equation}
e_A^a = q^{AB} g_{ab} e_B^b = q^{AB} (e_B)_{a}.
\end{equation}
Multiplying both sides by $q_{AC}$ and $g^{ac}$, we get the inverse relation
\begin{equation}
e_C^a = q_{AC} g^{ac} e_A^a.
\end{equation}
Then, it can be easily checked that
\begin{equation}
v_{*}^{(a)} = (\kappa - \mathbf{k}, e_A^a),
\end{equation}
with $\kappa$ representing the one-form with components $k_a$ etc., provides the required dual basis. In particular, we have
\begin{equation}
e_A^a e_B^b = e_A^a q^{BC} (e_C)_{a} = q^{BC} g_{ab} e_C^b e_A^a = q^{BC} q_{CA} = \delta_{AB}.
\end{equation}
The canonical null basis and the dual basis allows us to write down the following decomposition of the Kronecker tensor:
\begin{equation}
\delta_{ab} = -\ell^a k_b - k^a \ell_b + e_A^a e_B^b,
\end{equation}
which is just the relation we demanded between the canonical null basis and its dual, but now with the explicit basis vectors and dual one-forms put in. Raising the lower index, we obtain
\begin{align}
g^{ab} &= -\ell^a k^b - k^a \ell^b + e_A^a (e^A)^b \\
&= -\ell^a k^b - k^a \ell^b + q^{AB} e_A^a e_B^b,
\end{align}
which implies
\begin{equation}
q^{ab} = q^{AB} e_A^a e_B^b.
\end{equation}
A.2.3 Erecting a Coordinate System on the Null Surface

Suppose we have a set of coordinates \( x^a = (x^1, x^2, x^3, x^4) \) charting the four-dimensional spacetime with the null surface under consideration. Any set of three continuous, infinitely differentiable functions, say \( y^a = (y^1, y^2, y^3) \), of the spacetime coordinates \( x^a \) constitutes a system of coordinates on the null surface provided the set of values of these functions at each and every point on the null surface is unique. Then, the coordinate basis is the set of three vectors

\[
e^a_\alpha = \frac{\partial x^a}{\partial y^\alpha}.
\]

If \( g_{ab} \) be the components of the metric of the ambient spacetime in the coordinates \( x^a \), the induced metric on the null surface is given by [25]

\[
h_{\alpha\beta} = g_{ab} e^a_\alpha e^b_\beta.
\]

This is a 3-metric and the determinant of this 3-metric, \( h \), will be zero as the surface is a null surface. The easiest way to see this and to work on the null surface is to erect a coordinate system naturally suited to the null nature of the surface [25]. The null surface is filled by a congruence of null geodesics, the integral curves of the normal vector \( \ell^a \). We choose a parameter \( \lambda \) varying smoothly on the null generators such that the displacements along the generators are of the form \( dx^\alpha = \ell^\alpha d\lambda \). (In this paper, the null surface is taken as a \( \phi = \) constant surface and the normal is taken as \( \ell^a = A\partial_\alpha\phi \) for some \( A \). \( \ell^\alpha \) is then fixed and \( \lambda \) has to be chosen appropriately. If our aim is just to erect a coordinate system on a given null surface, we may first choose \( \lambda \) to be some parameter which varies along the null geodesics and then choose \( \ell^a = dx^\alpha/d\lambda \).) If we further ensure that \( \lambda \) varies smoothly for displacements across geodesics, we may choose it as one of the coordinates on the null surface. The other two coordinates are to be chosen as two smooth functions \( z^A = (z^1, z^2) \) that are constant on each null geodesic. They act as a unique label for each null geodesic. In this coordinate system, varying \( \lambda \) would correspond to a displacement along a particular null geodesic while varying the set \( z^A \) would correspond to displacements across the generators along points of equal \( \lambda \). The basis vectors in the coordinate system \( (\lambda, z^A) \) are

\[
e^\lambda = \frac{\partial x^\lambda}{\partial \lambda}; \quad e^a_A = \frac{\partial x^a}{\partial z^A}, \quad A = 1, 2.
\]

Note that the identification \( \ell^\alpha = dx^\alpha/d\lambda = \partial x^\alpha/\partial \lambda \) is possible because the coordinates \( z^A \) have been chosen to be constant along the null geodesics. Note that previously we had chosen an auxiliary vector \( k^a \) and then demanded that \( e^a_\lambda k^a = 0 \). But while we are working purely on the null surface, there is no notion of an auxiliary null vector, as no vector on the null surface will satisfy \( k^a \ell_a = -1 \). In fact, having defined the coordinate basis vectors on the null surface, if we now we want to move into the ambient 4-dimensional spacetime and define \( q_{ab} \), etc., we can specify \( k^a \) uniquely by the four conditions \( k^a k_a = 0, k^a \ell_a = 1 \) and \( g_{ab} k^b e^a_A = 0 \). (This is how we defined \( k^a \) in the case of GNC metric in Section 4.1.)

The components of the induced metric are \( h_{\lambda\lambda} = g_{ab} \ell^a \ell^b = 0 \) (since \( \ell^a \) is null), \( h_{\lambda A} = g_{ab} \ell^a e^b_A = 0 \) (since \( e^b_A \) lies on the surface and \( \ell^a \) is the normal) and \( q_{AB} \equiv g_{ab} e^a_A e^b_B \). In the coordinate order \( (\lambda, z^1, z^2) \), the matrix form is

\[
h_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q_{11} & q_{12} \\ 0 & q_{12} & q_{22} \end{pmatrix}.
\]
The determinant of $h_{ab}$, $h$, is obviously zero and will remain zero even if we transform to a different coordinate system on the null surface as the determinant changes only by a Jacobian factor. The line element is 2-dimensional:

\[ ds^2 = q_{AB}dz^Adz^B. \]  

(186)

Keeping this coordinate system as a reference, we can explore other coordinate systems on the null surface. The metric in any coordinate system $y^\alpha = (y^1, y^2, y^3)$ on the surface is given by

\[ h'_{\alpha\beta} = q_{AB} \frac{\partial x^A}{\partial y^\alpha} \frac{\partial x^B}{\partial y^\beta} \]  

(187)

In general, none of the components need to be zero, although the determinant will vanish. But consider the special case $(y^1(\lambda, z^1, z^2), y^2(z^1, z^2), y^3(z^1, z^2))$. This coordinate system has coordinates $y^2$ and $y^3$ constant on the null geodesics. $y^1$ may now be considered as the parameter varying along the null geodesics and we will have $\ell^a = A(y^1, y^2, y^3)\partial x^a/\partial y^1$ (with $A$ becoming unity if we choose $x^1 = \lambda$). In this case, the metric will again take the form

\[ h'_{ab} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & q'_{11} & q'_{12} \\ 0 & q'_{12} & q'_{22} \end{pmatrix}. \]  

(188)

### A.2.4 Directed Surface Element for the Null Surface

For any set of coordinates $y^\alpha = (y^1, y^2, y^3)$ on the null surface with associated basis vectors $e^a_\alpha = \partial x^a/\partial y^\alpha$, the invariant directed surface element for the null surface is given by [25]

\[ d\Sigma_a = \epsilon_{abcd}e^b_1e^c_2e^d_3dy^1dy^2dy^3. \]  

(189)

Now, $\epsilon_{abcd}e^b_1e^c_2e^d_3$ is proportional to the normal $\ell_a$ (for a null surface, there is no unique normalized normal, but our treatment here will work for any choice of the normal) as its contraction with any vector on the null surface, expressible as a linear combination of $e^a_\alpha$, is zero. Thus, we may write

\[ \epsilon_{abcd}e^b_1e^c_2e^d_3 = f\ell_a \]  

(190)

for some scalar function $f$. Contracting with $k^a$, we obtain

\[ f = -\epsilon_{abcd}k^b e^c_1e^d_3. \]  

(191)

Since $f$ is a scalar, it can be evaluated in any coordinate system. So we choose the 4-dimensional coordinates $x^a$ to be $(\phi, y^1, y^2, y^3)$ with $\phi$ constant on the null surface. The form of the normal will then be $\ell_a = A\partial_\phi$ for some scalar function $A$. In this coordinate system, borrowing the notation $\ddagger$ from [25] to indicate equalities valid in a specified coordinate system,

\[ f \equiv \epsilon_{\phi 123}k^\phi \ddagger = \frac{\epsilon_{\phi 123}}{A}, \]  

(192)

where we have used the condition $k^a\ell_a \ddagger = k^\phi\ell_\phi = -1$ to find $k^\phi$. If we choose the $\epsilon_{abcd}$ tensor so that $\epsilon_{\phi 123}$ is positive, then we shall have

\[ f \ddagger = \frac{\sqrt{-g}}{A}, \]  

(193)

with the choice $\ell_a = \partial_\phi$ used in our treatment of the boundary term leading to

\[ f \ddagger = \sqrt{-g}. \]  

(194)

Thus, for the null surface represented as $\phi = $ constant and the normal being specified as $\ell_a = A\partial_\phi$, the directed null surface element, in a coordinate system where $\phi$ is one of the coordinates, becomes

\[ d\Sigma_a \equiv \sqrt{-g} \ell_a dy^1dy^2dy^3 = \sqrt{-g}\partial_\phi dy^1dy^2dy^3. \]  

(195)

Note that, while this result is not generally covariant, it is valid in any coordinate system with $\phi$ as one of the coordinates. In the case of $\ell_a = \partial_\phi$, it reduces to simply

\[ d\Sigma_a \equiv \sqrt{-g}\ell_a dy^1dy^2dy^3. \]  

(196)
In fact, nowhere in the derivation of Eq. (195) and Eq. (196) did we have to make use of the assumption of the surface being null. In the case of non-null surfaces, we have the extra luxury of specifying the normalization of \( \ell_a \) uniquely to give \( n_a = \ell_a = N \partial_\phi \) (see Eq. (237)). The auxiliary vector \( k^a \) should then be taken as \( k^a = -\epsilon_{ab}^a \) as the only condition we demand of \( k^a \) in the above derivation is \( k^a \ell_a = -1 \). Then, \( A = N \) and \( \sqrt{-g}/A = \sqrt{|h|} \) and we arrive back at the familiar result for surface element for a non-null case.

We have mentioned that Eq. (195) and Eq. (196) are valid only when \( \phi \) is taken as a coordinate system. One way to formulate a fully covariant expression for the surface element is as follows. Note that all the treatment till Eq. (191) is fully covariant. The non-covariance sneaked in at the evaluation of \( f \). Now, given \( \ell_a \) and the triad of vectors \( e^a_b \), we can formulate a tetrad of vectors \( (-k^a, e^a_\alpha) \), with \( k^a \) being some vector which satisfies \( k^a \ell_a = -1 \). Using this tetrad and the metric, we can construct independent scalars \( g_{\alpha \beta} = g_{ab}(-k^a)(-k^b) \), \( g_{\alpha 1} = g_{ab}(-k^a)e^1_b \), etc. In the coordinate system \(( \phi, y^1, y^2, y^3 \)\), these scalars are just the components of \( g_{ab} \). But now imagine taking the matrix corresponding to the metric \( g_{ab} \) and replacing each component with the corresponding scalar to get a matrix of scalars. The determinant of this matrix is also a scalar. Let us denote it by \( g \). Hence, from Eq. (191), we have

\[
\left[ \sqrt{-g} \right] = \left[ \sqrt{-g} \right] \Rightarrow f = \sqrt{-g},
\]

which is an equality between scalars and hence valid in all coordinate systems if valid in one. Note that this also means

\[
\sqrt{-g} = -\epsilon_{abcd} e^b_a e^c_\alpha e^d_b,
\]

from Eq. (191).

In the special coordinate system \(( \lambda, z^1, z^2 \)\) introduced in Appendix A.2.3 with \( \ell_a = \partial_\phi \), the directed surface element reduces to [25]

\[
d\Sigma_a = \ell_a \sqrt{g} d\lambda dz^1 dz^2,
\]

where \( g \) is the determinant of the 2-metric \( g_{AB} \). Note the difference in minus sign from the analogous expression in [25]. This minus sign difference arose because [25] defines the normal as \( \ell_a = -\partial_\phi \) so that \( \ell^a \) is future-directed.

### A.2.5 Second Fundamental Form \( \Theta_{ab} \) and Expansion Scalar \( \Theta \)

In this section, we shall introduce the second fundamental form \( \Theta_{ab} \) and the expansion scalar \( \Theta \) for a null surface. The terminology follows [26]. But [26] works with a foliation of null surfaces while we have only specified the \( \phi = \phi_0 \) surface to be a null surface among the \( \phi = \text{constant} \) surfaces. Hence, we cannot blindly carry forward the results in [26]. But we shall see that the results that we require are in fact unaltered.

Following [26], we use the projector in Eq. (165) to introduce the extension of the second fundamental form for the null surface at any point, with respect to the chosen normal \( \ell \) (since, unlike the non-null case, the normalization of the normal is not fixed and hence the choice of \( \ell \) is not unique), to the tangent space of the four-dimensional spacetime manifold at that point:

\[
\Theta_{ab} \equiv \Pi^c_a \Pi^d_b \nabla_c \ell_d.
\]

Now, consider the following object on the null surface:

\[
g^d_a g^b_c \nabla_c \ell_d = (\Pi^c_a + \ell \ell^a_b) (\Pi^d_b + \ell \ell^d_b) \nabla_c \ell_d = \Theta_{ab} + \Pi^c_a \ell \ell^d_b \nabla_c \ell_d + \ell \ell^a_b \Pi^d_b \nabla_c \ell_d + \ell \ell^a_b \ell \ell^d_b \nabla_c \ell_d = \Theta_{ab} + \Pi^c_a \ell \ell^d_b \nabla_c \ell_d + \kappa \kappa_a \Pi^d_b \ell_d + \kappa \kappa_a \ell \ell^d_b \ell_d
\]

\[
= \Theta_{ab} + \frac{1}{2} \Pi^c_a \ell \ell^d_b \nabla_c (\ell \ell^d_b),
\]

where we have made use of the results \( \Pi^a_b \ell_a = 0 \) and \( \ell \ell^a_a = 0 \) on the null surface. The second term in Eq. (201) contains an expression of the form \( \nabla_b (\ell \ell^a_b) \). Let us simplify this expression to a form that will be more useful to us. We write

\[
\nabla_b (\ell \ell^a_b) = g_{bc} \nabla^c (\ell \ell^a_b) = (q_{bc} - \ell \ell^c_b - k_b k_c) \nabla^c (\ell \ell^a_b),
\]

33
where we have used Eq. (166) in the second line. In Eq. (202), the third term has the combination

$$\ell^c \nabla_c (\ell^a \ell_a) = \ell^c \partial_c (\ell^a \ell_a) = 0, \quad (203)$$

since \( \ell^c \partial_c \) is a derivative along the null surface and the value of \( \ell^a \ell_a \) is zero throughout the null surface. Next, consider the first term in Eq. (202), which has

$$q_{bc} \nabla^c (\ell^a \ell_a) = q_{bc}^d \partial_d (\ell^a \ell_a) = q_{bc}^d \partial_d (\ell^a \ell_a), \quad (204)$$

in the coordinate system in which \( \phi \) is one of the coordinates, since the only direction along which \( \ell^a \ell_a \) varies on the null surface is off the surface i.e the direction along which \( \phi \) varies.

Now, using Eq. (163),

$$q_{ba}^c \propto q_{ba}^c \ell_b = 0. \quad (205)$$

Thus, \( q_{ba}^c \) is identically zero on the null surface and the first term in Eq. (202) also vanishes. (To prove that a tensor quantity is identically zero, it is sufficient to prove that it is zero in one coordinate system.) So, we arrive at the result that

$$\nabla_b (\ell^a \ell_a) = -k^c \nabla_c (\ell^a \ell_a) \ell_b. \quad (206)$$

Using this result in Eq. (201), we obtain

$$\Theta_{ab} = q_{ab}^d \nabla_d \ell_d. \quad (207)$$

Instead of labouring forth with "the four-dimensional extension of the second fundamental form", we shall henceforth take the liberty of referring to \( \Theta_{ab} \) as just the second fundamental form of the null surface. To be precise, we should also add that it is the second fundamental form with respect to the normal \( \ell_a \) under consideration but we shall consider this to be understood henceforth.

The \( \Theta_{ab} \) is a symmetric object. To see this, we use Eq. (163) and decompose \( \nabla_a \ell_b \) as

$$\nabla_a \ell_b = A \nabla_a \nabla_b \phi + \ell_b \partial_a \ln A. \quad (208)$$

The first term in Eq. (208) is symmetric while the second term does not contribute to \( \Theta_{ab} \). Hence, proved.

We shall next prove a relation between \( \Theta_{ab} \) and the Lie derivative of \( q_{ab} \) along \( \ell \). The Lie derivative formula is

$$\mathcal{L}_{\ell} q_{ab} = \ell^c \nabla_c q_{ab} + q_{cd} \nabla_b \ell^d + + q_{db} \nabla_a \ell^d. \quad (209)$$

Substituting for \( q_{ab} \) from Eq. (166), we have

$$\mathcal{L}_{\ell} q_{ab} = \ell^c (\ell_a \nabla_d k_b + k_b \nabla_d \ell_a + \ell_b \nabla_d k_a + k_a \nabla_d \ell_b) + \nabla_b \ell_a + (\ell_b k_a + k_b \ell_a) \nabla_a \ell^d + \nabla_a \ell_b + (\ell_b k_a + k_b \ell_a) \nabla_a \ell^d. \quad (210)$$

Contracting with \( q_{am}^a q_{bn}^b \), and using \( q_{ab}^a \ell_a = 0 \) and \( q_{ab}^a k_a = 0 \), we obtain

$$q_{am}^a q_{bn}^b \mathcal{L}_{\ell} q_{ab} = q_{am}^a q_{bn}^b (\nabla_a \ell_b + \nabla_b \ell_a) = \Theta_{mn} + \Theta_{nm} = 2 \Theta_{mn} \quad (211)$$

which implies

$$\Theta_{mn} = \frac{1}{2} q_{mn}^a q_{ab} \ell q_{ab}. \quad (212)$$

Finally, let us look at the trace of \( \Theta_{ab} \):

$$\Theta = q^{ab} \Theta_{ab} = q^{ab} \Theta_{ab}. \quad (213)$$

We shall refer to \( \Theta \) as the expansion scalar on the null surface. More specifically, it is the expansion along \( \ell^a \) of the 2-surface on the null surface orthogonal to \( k^a \). The reason for this terminology becomes clear if we take the trace of Eq. (212). We obtain

$$\Theta = \frac{1}{2} q^{ab} \ell q_{ab}. \quad (214)$$

This equation may be further manipulated to a form easier to interpret. We can make use of the canonical null basis introduced in Appendix A.2.2. We have

$$q_{ab} = q_{AB} e^A_a e^B_b, \quad q^{ab} = q^{AB} e_A^a e_B^b. \quad (215)$$
where summation over $A, B = 1, 2$ is implied. Then,

$$
\Theta = \frac{1}{2} q^{AB} e^a_A e^b_B \mathcal{L}_\ell (q_{CD} e^c_C e^D_B) = \frac{1}{2} q^{AB} \mathcal{L}_\ell q_{AB} + q^{AB} q_{BC} e^a_A \mathcal{L}_\ell e^a_C
$$

$$= \frac{1}{2} q^{AB} \mathcal{L}_\ell q_{AB} + e^a_A \mathcal{L}_\ell e^a_A = \frac{1}{2} q^{AB} \mathcal{L}_\ell q_{AB} - e^a_A \mathcal{L}_\ell e^a_A .
$$

(216)

where we have used $e^a_A e^B_a = \delta^B_A$, $q^{AB} q_{BC} = \delta^A_C$ and $\mathcal{L}_\ell (e^a_A e^B_a) = \mathcal{L}_\ell 2 = 0$ along the way. Now, let $e^a_A$ be the coordinate vectors $e^a_i$ and $e^a_i$ in the special coordinate system $(\lambda, z^1, z^2)$ introduced in Appendix A.2.3. Since $e^a_A$ and $\ell^a$ are then members of a coordinate basis, their Lie bracket should be zero, i.e,

$$[\ell, e_A] = 0 \Rightarrow \mathcal{L}_\ell e^a_A = 0 .
$$

(217)

Enforcing this condition, we obtain

$$\Theta = \frac{1}{2} q^{AB} \mathcal{L}_\ell q_{AB} = \frac{\mathcal{L}_\ell \sqrt{q}}{\sqrt{q}} ,
$$

(218)

where $q$ is the determinant of the 2-metric $q_{AB}$. Thus, we see that $\Theta$ represents the fractional change in an area element of the $\lambda = 2$-surface on the null surface as it is Lie dragged along $\ell^a$. We can replace the Lie derivative in the above expression with an ordinary derivative since $q_{AB} = q_{0\ell} e^a_A e^B_B$ is a scalar under 4-dimensional coordinate transformations with the same physical vectors. Thus, $\mathcal{L}_\ell q_{AB} = \ell^a \partial_a q_{AB}$ which becomes $\mathcal{L}_\ell q_{AB} = dq_{AB} / d\lambda$ for the parameter $\lambda$ introduced in Appendix A.2.3. Hence,

$$\Theta = \frac{1}{\sqrt{\sqrt{q}}} \frac{d\sqrt{q}}{d\lambda} .
$$

(219)

### A.2.6 $\kappa$ and $\Theta$ in terms of ($\ell, k$)

In this appendix, we shall derive expressions for the non-affinity coefficient $\kappa$ and $\Theta$ in terms of $\ell_a$ and auxiliary vector $k_a$, for $\ell_a$ being the null normal, $\ell_a = A \partial_a \phi$, to a surface $\phi = \phi_0$.

Let us first define the auxiliary vector $k^a$ (need not be null for manipulations in this section) such that $k^a \ell_a = -1$. Then,

$$\ell^a \nabla_a \ell_b = \ell^a \nabla_a (A \partial_b \phi) = \ell^a \frac{\ell_b}{A} \partial_a A + \ell^a A \nabla_a \nabla_b \phi
$$

$$= \ell^a \frac{\ell_b}{A} \partial_a A + \ell^a A \nabla_b \phi = \ell^a \frac{\ell_b}{A} \partial_a A + \ell^a A \nabla_b \ell_a (\frac{\ell_a}{A})
$$

$$= \ell^a \frac{\ell_b}{A} \partial_a A + \ell^a \nabla_b \ell_a = \ell^a \partial_a (\ln A) \ell_b + \frac{1}{2} \partial_b (\ell^a \ell_a)
$$

(220)

Substituting in Eq. (220), we obtain

$$\ell^a \nabla_a \ell_b = [\ell^a \partial_a (\ln A)] \ell_b - \frac{k^a}{2} \partial_a (\ell^a \ell_a) \ell_b
$$

$$= [\ell^a \partial_a (\ln A) - \frac{k^a}{2} \partial_a (\ell^a \ell_a)] \ell_b
$$

(221)

Hence, we see that $\ell^a \nabla_a \ell_b = \kappa \ell_b$ with the non-affinity coefficient $\kappa$ given by the formula

$$\kappa = \ell^a \partial_a (\ln A) - \frac{k^a}{2} \partial_a (\ell^a \ell_a)
$$

(222)

Using this expression, we can also evaluate

$$\nabla_a \ell^a = q^{ab} \nabla_a \ell_b - \ell^a k^b \nabla_a \ell_b - \ell^b k^a \nabla_a \ell_b = \Theta + \kappa - \frac{k^a}{2} \partial_a (\ell^b \ell_b)
$$

(223)

$$= \Theta + \kappa - \frac{k^a}{2} \partial_a (\ell^b \ell_b) .
$$

(224)

Let us define

$$\tilde{\kappa} \equiv - \frac{k^a}{2} \partial_a (\ell^b \ell_b) = -k^a \ell^b \nabla_a \ell_b
$$

(225)

a measure of how much $\ell_a \ell^a$ varies as we move away from the null surface. If $\nabla_a \ell_b$ is symmetric, like in the case when we have $\ell_b = \nabla_b \phi$, we shall get $\tilde{\kappa} = \kappa$. If $\nabla_a \ell_b$ is antisymmetric, i.e if $\ell_b$ is a Killing vector, then we get $\tilde{\kappa} = -\kappa$. Eq. (224) now takes the form

$$\nabla_a \ell^a = \Theta + \kappa + \tilde{\kappa} .$$

(226)
A.2.7 Decomposition of $\sqrt{-g}$ in Terms of $\sqrt{q}$

We shall use the result Eq. (161) which is applicable even when we take the null limit. In the limit the $\phi = \text{constant surface under consideration is null, } g^{\phi \phi} = 0$ and $h = 0$. Taking the limit $g^{\phi \phi} \rightarrow 0$ on Eq. (161), we get

$$g = \frac{-q}{(g^{1 \phi})^2}.$$  \hspace{1cm} (227)

For the null surface, we need $q$ to be the determinant of the 2-metric $q_{AB} = g_{ab} e^a_A e^b_B$ (see Eq. (171)). To apply result Eq. (227), we shall specialize to a coordinate system such that $e^a_A$ are coordinate basis vectors and $\ell_a = \partial_a \phi$. On the null surface, we shall introduce the two coordinates $z^A = (z^1, z^2)$, constant on the null geodesics and a third coordinate $\mu$ which is a parameter varying along the null geodesics such that the vectors $e^a_A$ lie on a $\mu = \text{constant surface}$ and $e^a_A = \partial x^a/\partial z^A$. We shall also have $e^a = (1/M) \partial x^a/\partial \mu$ for some scalar $M$. In the case where we choose the special coordinate system $(\lambda, z^1, z^2)$ introduced in Appendix A.2.3, we will have $M = 1$ and $e^a = \partial x^a/\partial \lambda$. In the coordinates $(\phi, \mu, z^1, z^2)$, $g_{AB} = q_{AB}$, $g_{A\mu} = \delta_{AB} (\ell^a/M) e^a_A = 0$ and $g_{\mu \mu} \propto g_{ab} \ell^a \ell^b = 0$. Also, consider $g_{ab} \ell^a k^b = (g_{ab}/M) k^b$. Now, $g_{ab} \ell^a k^b = \ell_a k^a = k^\phi$. Hence, only $k^\phi$ contributes to $g_{ab} \ell^a k^b$. Since $g_{ab} \ell^a k^b = -1$, we have $k^\phi = -1$ and we obtain $g_{ab} = M$. Thus, in the coordinates $(\phi, \mu, z^1, z^2)$, the metric takes the form

$$g_{ab} = \begin{pmatrix}
q_{\phi \phi} & q_{\phi 1} & q_{\phi 2} \\
M & 0 & 0 \\
q_{\phi 1} & 0 & q_{11} & q_{12} \\
q_{\phi 2} & 0 & q_{12} & q_{22}
\end{pmatrix}. \hspace{1cm} (228)$$

Using Eq. (227) (or by direct calculation of the determinant from the above matrix), we can write down

$$g = -M^2 q,$$  \hspace{1cm} (229)

and

$$\sqrt{-g} = |M| \sqrt{q}. \hspace{1cm} (230)$$

If we specialize to $\mu = \lambda$, we would have $g = q$ and

$$\sqrt{-g} = \sqrt{q}.$$  \hspace{1cm} (231)

B Boundary Conditions for Spacelike and Timelike Surfaces: An Alternate Approach

In this appendix, we shall detail an alternate approach to arrive at the standard prescription for fixing the boundary conditions on spacelike and timelike surfaces [15]. As we have seen in Eq. (7), the surface term of the Einstein-Hilbert action is the integral of the quantity

$$\sqrt{-g} Q[v_c] = \sqrt{-g} v_c (g^{ab} \delta \Gamma^c_{ab} - g^{ck} \delta \Gamma^c_{ak})$$  \hspace{1cm} (232)

over the boundary of the spacetime region under consideration. Here, $v_c$ is the surface gradient. To be concrete, let us take $\phi = \phi_0$ to be the surface, with $\phi_0$ a constant. Then $v_c = \pm \partial_c \phi$, with the sign decided according to the conventions of the Gauss’ theorem (see Appendix A.1). Let us assume that our surface is such that $\phi$ increases on going from inside the integration volume to outside through the surface. Hence, we shall use $v_c = \partial_c \phi$ from now on. It is clear that Eq. (232) contains the variations of the metric and its derivatives. We can separate out the terms with the variations of the metric and its derivatives in Eq. (232) and write

$$\sqrt{-g} Q[v_c] = v_c \left[ 2\sqrt{-g} P^{dcea} \delta g_{ab} - 4\sqrt{-g} P^{cda} b \delta (\partial_d g_{ab}) \right],$$  \hspace{1cm} (233)

where $P^a_{bcd} = 1/2 (g^{ab} \delta^c_d - g^{bc} \delta^d_a)$. Since $v_c = \partial_c \phi$, the coefficient of $\delta (\partial_d g_{ab})$ is $-4\sqrt{-g} P^{dab}$. We can easily check that $P^{dab} \neq 0$ in general. Hence, Eq. (232) contains variations of the normal derivatives of the metric.

Now, let us consider the case of timelike and spacelike surfaces as the boundary and see how we can fix boundary conditions without having to fix both the metric and its normal derivatives at the boundary. Generalizing Eq. (232), we shall define $Q[A_c]$ for any vector $A_c$ to be

$$Q[A_c] \equiv A_c (g^{ab} \delta \Gamma^c_{ab} - g^{ck} \delta \Gamma^c_{ak})$$  \hspace{1cm} (234)
For timelike or spacelike boundaries, we can normalize the surface gradient to obtain the unit normal \( n^a \) such that
\[
n = \frac{v^a}{\sqrt{|g_{\phi\phi}|}}; \quad n_n^a = \epsilon,
\]
with \( \epsilon = 1 \) for a timelike surface and \( \epsilon = -1 \) for a spacelike surface. We shall also demand that the normalization be preserved under the variation of the metric, i.e.,
\[
\delta(n_n^a n_n^a) = 0.
\]

We choose a coordinate system with \( \phi \) taken as one of the coordinates. Let the other coordinates, the coordinates charting the \( \phi = \text{constant} \) surface, be labelled \( (x_1, x_2, x_3) \). As mentioned in the introduction, the Greek indices (other than \( \phi \)) run over \( (x_1, x_2, x_3) \) while the Latin indices run over \( (\phi, x_1, x_2, x_3) \). Let us use \( N \) for the normalization factor in \( n^a \), i.e.
\[
n^a = N \partial^a \phi = v^a N,
\]
where \( N \) is to be assumed positive so that \( n^a \) and \( \partial^a \phi \) point in the same direction. The normalization \( n^a n^a = \epsilon \) then relates \( N \) to the metric as
\[
g_{\phi\phi} = \epsilon / N^2.
\]

Another convention that is used often is to demand that \( \phi \) increases in the direction of the normal vector \( n^a \), i.e., \( n^a \partial_a \phi > 0 \). In this case, we would write
\[
\tilde{n}^a = \epsilon N v^a,
\]
with \( N \) positive, where we have put the tilde just to distinguish the normal in this convention from the \( n^a \) in Eq. (237). Now, from the standard procedure of calculating the inverse of a metric we know that
\[
g = \epsilon N^2 h = -N^2 |h|.
\]
Thus, we arrive at the following decomposition for \( \sqrt{-g} \):
\[
\sqrt{-g} = N \sqrt{|h|}.
\]

Using this expression and Eq. (237), we can rewrite the expression for the surface term in Eq. (232), in the case of timelike or spacelike surfaces, as
\[
\sqrt{-g} Q[v_c] = \sqrt{|h|} n_c (g^{ab} \delta \Gamma^c_{ab} - g^{ck} \delta \Gamma^a_{ak})
\]
\[
= \sqrt{|h|} Q[v_c].
\]
As we can see from Eq. (233), this expression contains the variations of the metric as well as its normal derivatives. Our aim is to discover a counter-term such that, when added to the Einstein-Hilbert action, the surface term obtained in the variation will contain only the variations of the metric. That is, we would like to express the surface term in the form
\[
\delta A_{\alpha\nu} = \delta[X] + Y_{ab} \delta g^{ab}.
\]
Then, it is \(-X\) that we have to add to the action as the counter-term.

Proceeding towards this goal, let us first manipulate \( Q[A_c] \) for an arbitrary vector \( A^c \). We have the following relations:
\[
\delta(\nabla A_b) = \nabla \delta A_b - \Gamma^c_{ab} A_c; \quad \delta(\nabla A^c) = \nabla \delta A^c + A^e \delta \Gamma^a_{ec}.
\]
Using these relations, we can rewrite the expression for $Q[A_c]$ in Eq. (234) as follows:

\[
Q[A_c] = \{[\nabla_a, \delta], g^{ab}\} A_b
= \nabla_a (\delta A^a_\perp) - \delta (2 \nabla_a A^a) + \nabla_a A_b \delta g^{ab},
\]

where

\[
\delta A^a_\perp = \delta A^a + g^{ab} \delta A_b.
\]

In the first line of Eq. (246), we have used the notation $[A, B]$ for the commutator, $AB - BA$, and $\{A, B\}$ for the anticommutator, $AB + BA$.

Let us now specialize to the case of $n_a$. Then, we have

\[
Q[n_a] = \nabla_a (\delta n^a_\perp) - \delta (2 \nabla_a n^a) + \nabla_a n_b \delta g^{ab},
\]

where $\delta n^a_\perp = \delta n^a + g^{ab} \delta n_b$. Using Eq. (236), we can see that

\[
\delta n^a_\perp n_a = 0,
\]

which means that the vector $\delta n^a_\perp$ lies on the surface $\phi = \phi_0$. This property can be used to decompose the first term of Eq. (248) in the following manner:

\[
\nabla_a (\delta n^a_\perp) = \delta^b_a \nabla_a (\delta n^b_\perp)
= (h^b_a + e^n b n_b) \nabla_a (\delta n^b_\perp)
= h^b_a \nabla_a (\delta n^b_\perp) + e^n b n_b \nabla_a (\delta n^b_\perp)
= D_a (\delta n^a_\perp) - \epsilon (n^a \nabla_a n_b) \delta n^b_\perp
= D_a (\delta n^a_\perp) - e a b \delta n^b_\perp.
\]

On the way to obtaining the above expression, we have introduced the induced metric

\[
h_{ab} = g_{ab} - e^n b n_b,
\]

and a derivative operator $D_a$ such that $D_a V^b = h^a_b h^d_b \nabla_a V^d$. For vectors $V^a$ on the surface (i.e., satisfying $V^a n_a = 0$), $D_a$ is the natural covariant derivative on the $3-$surface (see [25] and Chapter 12 in [1]) compatible with the induced surface metric.

To proceed further, let us look at the nature of the variations we are considering. The $\delta$'s are field variations on the metric and leave the coordinates unchanged, which is why they could be taken inside integrals over spacetime and derivatives with respect to the coordinates. The scalar $\phi$ is taken to be fixed during the variation, which means that the foliating surfaces $\phi = \text{constant}$ are kept fixed during the variation. In our case, this is not an extra assumption as we have already taken $\phi$ to be one of our coordinates. Now, from Eq. (237),

\[
n_a = N \partial_a \phi \Rightarrow \delta n_a = \delta N \partial_a \phi = \delta (\ln N) n_a.
\]

This shows that $\delta n_a$ is in the direction of $n_a$. We can write this in terms of variations of the metric using the constraint $n^a n_a = \epsilon$, equivalent to Eq. (238). We see that

\[
\delta \ln N = \frac{1}{2} \delta \ln N^2 = \frac{1}{2} \delta \ln g^{\phi \phi} = - \frac{\epsilon N^2}{2} \delta g^{\phi \phi} = - \frac{\epsilon}{2} \delta g^{ab} n_a n_b.
\]

Thus, we obtain the following explicit expression for the variation of $n_a$ in terms of the metric:

\[
\delta n_a = - \frac{\epsilon}{2} \delta g^{ij} n_i n_j n_a.
\]

Now, the second term in Eq. (250) contains

\[
a_b \delta n^a_\perp = a_b \delta n^b + a^b \delta n_b = a_b n_a \delta g^{ab} + a_b \delta n_a g^{ab} = a_b n_a \delta g^{ab},
\]

where we have first used the definition of $\delta n^a_\perp$ from Eq. (247), and then applied the result $a^b \delta n_b = 0$, which we know from Eq. (252), twice. Thus, Eq. (250) can be rewritten as

\[
\nabla_a (\delta n^a_\perp) = D_a (\delta n^a_\perp) - e a b \delta n^b.
\]
So, Eq. (248) takes on the form
\[ Q[n_c] = D_a(\delta n^a_a) - \delta(2\nabla_a n^a) + (\nabla_a n_b - \epsilon n_a a_b) \delta g^{ab} \]  
(257)

Our manipulations have naturally produced the quantity \( \nabla_a n_b - \epsilon n_a n_b \). This quantity is nothing but the negative of the extrinsic curvature [1] defined by
\[ K_{ab} = -h^c_\Sigma \nabla_c n_b = -\nabla_a n_b + \epsilon n_a a_b , \]
where we have used the definition of the induced metric given in Eq. (251). \( K_{ab} \) has the following properties:
\[ K_{ij} = K_{ji}; \quad K_{ij} n^j = 0; \quad K = g^{ij} K_{ij} = -\nabla_a n^a \]
(258)

The second equation tells us that \( K_{ij} \delta g^{ij} = K_{ij} \delta h^{ij} \). Using this result and the results in the last three equations, we can rewrite the surface term of the Einstein-Hilbert action, in the following useful form:
\[ [\delta A_{av}]_{\text{non-null}} = \int_{\partial V} d^3x \sqrt{\vert h \vert} Q[n_c] = \int_{\partial V} d^3x \left[ \sqrt{\vert h \vert} D_a(\delta n^a_a) + \delta(2\sqrt{\vert h \vert} K) - 2K\delta(\sqrt{\vert h \vert}) - \sqrt{\vert h \vert}(K_{ab} - K h_{ab}) \delta h^{ab} \right] \]
(260)

On integrating over the 3–surface, the first term in Eq. (260) is a 3–divergence which can be converted to a boundary term. If the surface bounding the integration volume is a closed surface, then this term goes to zero. The second term in Eq. (260) can be killed off by adding the counter-term \(-2\sqrt{\vert h \vert} K\) to the Einstein-Hilbert Lagrangian. The last term in Eq. (260) can be put to zero by fixing the induced metric \( h_{ab} \) on the boundary.

If we choose to work in the convention where the normal is chosen to be the one in Eq. (239), then we would have the surface term in the form
\[ [\delta A_{av}]_{\text{non-null}} = \int_{\partial V} d^3x \epsilon \sqrt{\vert h \vert} Q[\tilde{n}_c] = \int_{\partial V} d^3x \left[ \epsilon \sqrt{\vert h \vert} D_a(\delta \tilde{n}^a_a) + \delta(2\epsilon \sqrt{\vert h \vert} K) - \epsilon \sqrt{\vert h \vert}(K_{ab} - K h_{ab}) \delta h^{ab} \right] \]
(261)

C  Decompositions of Variations of \((\ell, k)\) in Canonical Null Basis

The null normal vector is taken as
\[ \ell_a = A \partial_a \phi , \]
(262)
where \( \phi = \phi_0 \) represents the null surface. Let us define \( r = \phi - \phi_0 \), so that the null surface limit may be represented by \( r \to 0 \). In the following, we shall assume \( \ell_a \) to be null only on the null surface. Thus, we may write \( \ell^2 = r A_1 \) for some suitable function \( A_1 \). We shall take an auxiliary vector \( k^a \) such that \( k^a \ell_a = -1 \) and \( k^a k_a = 0 \) everywhere. The two relations ensure that \( \ell^a \) and \( k^a \) are linearly independent. Further, we shall choose two linearly independent vectors \( e^i_1 \) and \( e^i_2 \), collectively represented by \( e^i_A \), such that \( \ell. e^A_A = 0 \) and \( k. e^A_A = 0 \) everywhere. The four vectors \( (\ell^a, k^a, e^A_A) \), \( A = 1, 2 \), are linearly independent and form a basis at any point in spacetime. To construct the dual basis for one-forms (see Appendix A.2.2), we need the help of the tensor \( \Pi^a_b = \delta^a_b + k^a \ell_b \) and the scalar \( q_{AB} = g_{ab} e_A^a e_B^b \) introduced in Appendix A.2.2. Let \( \ell_a = \Pi_{ab} \ell^b = \ell_a + \ell^2 k_a \) and \( e^a_A = q^{AB} g_{ab} e^b_B \), where \( q^{AB} \) is the inverse of \( q_{AB} \), i.e., \( q_{AB} q^{BC} = \delta^A_C \). Then, it is easy to check that \( (-k_a, \ell_a, e^A_A) \), with \( A = 1, 2 \), is the required dual basis. While this is the rightful claimant to the post of the dual basis, we prefer to use the basis \( (k_a, \ell_a, e^A_a) \), which, albeit not possessing the elite status of the canonical dual, is the prettier partner to the canonical null basis for vectors.

Then, under variations of the background metric with the surfaces of constant \( \phi \) left undisturbed, we get the following expressions for variations of \( \ell^a \) and \( k^a \) as an expansion in terms of our chosen bases:
\[
\begin{align*}
\delta \ell_a &= (\delta \ln A) \ell_a , \\
\delta \ell^a &= P_1 \ell^a + Q_1 k^a + R_1^A e^a_A , \\
\delta k^a &= P_2 \ell^a + Q_2 k^a + R_2^A e^a_A , \\
\delta k_a &= P_3 \ell_a + Q_3 k_a + R_3^A e_A^a .
\end{align*}
\]
(263-266)
Any of the coefficients on the RHS can be picked out by appropriately contracting with the members of the vector and one-form bases. Few of these coefficients \( P, Q \) and \( R \) can be determined by imposing the following conditions: \( \delta(\ell.\ell) = 0, \delta(\ell.k) = 0 \) and \( \delta(k.k) = 0 \), where the first condition is imposed only on the null surface. Near the null surface, we may write \( \delta(\ell.\ell) = rA_2 \), for a suitable \( A_2 \).

The first condition leads to

\[
ra_2 = \delta(\ell \ell^a) = \delta \ell^a \ell^a + \ell_a \delta \ell^a = (\delta \ln A) \ell_a + \ell_a (P_1 \ell^a + Q_1 k^a + R_1^A e_A^a) = rA_1 (\delta \ln A + P_1) - Q_1 \\
\implies Q_1 = r \left[ A_1 (\delta \ln A + P_1) - A_2 \right].
\]

(267)

In other words, \( Q_1 \) goes to zero at the null surface.

The second condition \( \delta(\ell.k) = 0 \) can be expanded in two ways leading to

\[
0 = \delta(\ell^a k_a) = \delta \ell^a k_a + \ell_a \delta k_a = k_a (P_1 \ell^a + Q_1 k^a + R_1^A e_A^a) + \ell_a \left( P_2 \ell^a + Q_3 k^a + R_2^A e_A^a \right) = -Q_3 - P_1 + r A_1 P_3 \\
\implies Q_3 + P_1 = rA_1 P_3 ,
\]

(268)

and

\[
0 = \delta(k \ell^a) = k^a \delta \ell_a + \ell_a \delta k^a = k^a (\delta \ln A) \ell_a + \ell_a \left( P_2 \ell^a + Q_2 k^a + R_2^A e_A^a \right) = -\delta \ln A - Q_2 + r A_1 P_2 \\
\implies \delta \ln A + Q_2 = rA_1 P_2 ,
\]

(269)

Finally, the last constraint leads to another relation.

\[
0 = \delta(k k^a) = k_a \delta k^a + k^a \delta k_a = k_a \left( P_2 \ell^a + Q_2 k^a + R_2^A e_A^a \right) + k^a \left( P_3 \ell_a + Q_3 k_a + R_3^A e_A^a \right) = -P_2 - P_3 .
\]

(270)

Incorporating all these results, we arrive at the following variations:

\[
\delta \ell_a = (\delta \ln A) \ell_a, \\
\delta \ell^a = -f_1 \ell^a + r \left[ A_1 (\delta \ln A - f_1) - A_2 \right] k^a + R_1^A e_A^a , \\
\delta k^a = -f_2 \ell^a - (\delta \ln A + r A_1 f_2) k^a + R_2^A e_A^a , \\
\delta k_a = f_2 \ell_a + (f_1 + r A_1 f_2) k_a + R_3^A e_A^a .
\]

(271)

D More on GNC and NSF

D.1 Inverse Metric and Christoffel Symbols in GNC

With coordinates ordered as \((u, r, x^A)\), the metric as presented in Eq. (64) and its inverse may be written in matrix notation as

\[
g_{ab} = \begin{pmatrix}
-2r \alpha & -r \beta_A \\
1 & 0 & 0 \\
-r \beta_A & 0 & q_{AB}
\end{pmatrix}, \quad g^{ab} = \begin{pmatrix}
0 & 1 & 0 \\
1 & 2r \alpha + r^2 \beta^2 & r \beta_A \\
0 & r \beta_A & q^{AB}
\end{pmatrix}
\]

(272)

where \( q^{AB} = (q^{-1})^{AB} \) is the inverse matrix of \( q_{AB} \), \( \beta^A = q^{AB} \beta_B \) and \( \beta^2 = \beta_A \beta^A \). The determinant of the metric is \( g = q \), where \( q \) is the determinant of the 2-metric \( q_{AB} \). Let \( \hat{D}_A \) be the covariant derivative operator associated with the two-dimensional metric \( q_{AB} \). For example,

\[
\hat{D}_A \beta_B = \partial_A \beta_B - \hat{\Gamma}_{AB}^C \beta_C ,
\]

(273)

where

\[
\hat{\Gamma}_{AB}^C = \frac{q^{CD}}{2} ( -\partial_D q_{AB} + \partial_A q_{DB} + \partial_B q_{DA} ) ,
\]

(274)
and $\bar{D}_A \beta = \partial_A \beta$ for the two-dimensional scalar $\beta$. Introducing $\bar{\alpha} = -2r\alpha$ and $\bar{\beta}_A = -r\beta_A$, the Christoffel symbols have the following expressions \cite{22}:

\begin{align}
\Gamma^u_{uu} &= -\frac{1}{2} \partial_u \bar{\alpha} \\
\Gamma^u_{uA} &= -\frac{1}{2} \partial_u \bar{\beta}_A \\
\Gamma^u_{AB} &= -\frac{1}{2} \partial_u q_{AB} \\
\Gamma^u_{rr} &= \Gamma^u_{rA} = 0 \\
\Gamma^r_{uA} &= \frac{1}{2} (\partial_r \bar{\alpha} - \bar{\beta}_A \partial_r \bar{\beta}_C) \\
\Gamma^r_{AB} &= \frac{1}{2} (\partial_u q_{AB} + (\bar{\beta}_C - \bar{\alpha}) \partial_r q_{AB}) + \frac{1}{2} \left( \bar{D}_A \beta_B + \bar{D}_B \beta_A \right) \\
\Gamma^r_{AB} &= \frac{1}{2} (\bar{\beta}_B - \partial_r \beta_A) + \frac{1}{2} \bar{\beta}_B (\partial_u q_{AB} + \partial_A \bar{\beta}_B - \partial_B \bar{\beta}_A) \\
\Gamma^r_{rr} &= 0 \\
\Gamma^A_{BC} &= \frac{1}{2} \bar{\beta}^A \partial_r q_{BC} + \Gamma^A_{BC} \\
\Gamma^A_{Bu} &= \frac{1}{2} \bar{\beta}^A \partial_r \bar{\beta}_B + \frac{1}{2} q^{CA} \partial_u q_{BC} + \frac{1}{2} q^{CA} (\partial_B \bar{\beta}_C - \partial_C \bar{\beta}_B) \\
\Gamma^A_{Br} &= \frac{1}{2} q^{CA} \partial_r q_{BC} \\
\Gamma^A_{uu} &= \frac{1}{2} \bar{\beta}^A \partial_u \bar{\alpha} - \frac{1}{2} q^{CA} \partial_u \bar{\beta}_C + q^{AC} \partial_u \bar{\beta}_C \\
\Gamma^A_{ur} &= \frac{1}{2} q^{CA} \partial_r \bar{\beta}_C \\
\Gamma^A_{rr} &= 0
\end{align}

\subsection*{D.2 Eliminating the Extra Parameter M in the NSF Metric}

The NSF metric was given in Eq. (107) and is of the form

\[ ds^2 = \bar{M}^2 (dx^3)^2 + 2M t dx^1 + 2q_{AB} m^A dx^B dx^3 + q_{AB} dx^A dx^B. \] (291)

The condition that the null vector $t^a = \partial / \partial t$ is affinely parametrized leads to $\partial_t M = 0$. So, we have $M = M(x^1, x^2, x^3)$.

Let us make the coordinate transformation to a new coordinate $\bar{t} = t M(x^1, x^2, x^3)$ so that

\[ M \partial_t = \partial_{\bar{t}} - \bar{t} \frac{\partial M}{\partial x^1} dx^1 - \bar{t} \frac{\partial M}{\partial x^2} dx^2 - \bar{t} \frac{\partial M}{\partial x^3} dx^3. \] (292)

The metric in the new coordinates would take the form

\[ ds^2 = (\bar{M}^2 - 2\bar{t} \frac{\partial \ln M}{\partial x^3}) (dx^3)^2 + 2\bar{t} dx^1 + 2q_{AB} m^A - \bar{t} \frac{\partial M}{\partial x^1} dx^B dx^3 + q_{AB} dx^A dx^B. \] (293)

We can see that this metric is of the same form as we would have obtained by putting $M = 1$ in Eq. (291), $\partial / \partial \bar{t}$ is a null geodesic with affine parameterization, as we would have expected since $\bar{t}$ is a linear function of $t$ and hence also an affine parameter. Thus, we have managed to encode the same information as in Eq. (291) with one less parameter. But one difference with Eq. (291) is that while $\partial / \partial x_3$ was taken to be a spacelike vector in Eq. (291), the vector $\partial / \partial x_3$ in Eq. (293) (which is a different vector) is not necessarily spacelike.
D.3 Inverse Metric and Christoffel Symbols for the NSF Metric

The NSF metric corresponding to the line element in Eq. (107) and its inverse in matrix form are given below in the coordinates \((t, x^1, x^2, x^3)\):

\[
g_{ab} = \begin{pmatrix} 0 & 0 & \frac{\epsilon M}{m_A} & 0 \\ 0 & q_{AB} & \frac{m_A}{m^2 + \frac{\epsilon^2}{M^2}} & 0 \\ \frac{\epsilon^2}{M^2} & \frac{m_A}{m^2 + \frac{\epsilon^2}{M^2}} & m_A^2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g^{ab} = \begin{pmatrix} \frac{1}{\frac{\epsilon^2}{M^2}} & -\frac{m_A \epsilon}{q^{AB}} & 0 \\ -\frac{m_A \epsilon}{q^{AB}} & q^{AB} & 0 \\ 0 & 0 & \frac{1}{m_A^2} \end{pmatrix},
\]

where \(q^{AB} = (q^{-1})^{AB}\) is the inverse matrix of \(q_{AB}\), \(m^A = q^{AB} m_B\) and \(m^2 = m_A m^A\). The determinant of the metric is \(g = M^2 q\), where \(q\) is the determinant of the 2-metric \(q_{AB}\).

Introducing \(\tilde{D}_A\) as the covariant derivative operator corresponding to the two-dimensional metric \(q_{AB}\), as in Appendix D.1, the Christoffel symbols for the above metric are

\[
\Gamma^t_{tt} = \partial_t \ln M \\
\Gamma^t_{At} = -\frac{1}{2} \frac{\partial A}{\partial t} \ln M + \frac{\epsilon}{2M} q_{AB} \partial_t m^B \\
\Gamma^t_{AB} = \frac{1}{2N^2} \partial_t q_{AB} + \frac{\epsilon}{2M} \left( \tilde{D}_A m_B + \tilde{D}_B m_A \right) - \frac{\epsilon}{2M} \partial_t q_{AB} \\
\Gamma^t_{xt} = \frac{1}{2} m^A \partial_A \ln M + \frac{\epsilon}{2M} \partial_t \left( \frac{M^2}{N^2} \right) + \frac{\epsilon}{2M} m_A \partial_t m^A \\
\Gamma^t_{x^A x^3} = \frac{1}{2} \frac{\partial A}{\partial t} \left( m^2 + \frac{M^2}{N^2} \right) + \frac{\epsilon m_A}{2M} \partial_A \left( m^2 + \frac{M^2}{N^2} \right) - \frac{\epsilon M}{N^2} \partial_A m - \frac{\epsilon m_A m^B}{2M} \partial_A q_{AB} \\
\Gamma^t_{Bt} = \frac{1}{2} \frac{\partial B}{\partial t} q_{BC} + \tilde{\Gamma}^t_{BC} \\
\Gamma^A_{BC} = \frac{\epsilon m_A}{2M} \partial_t q_{BC} + \tilde{\Gamma}^A_{BC} \\
\Gamma^A_{tx^3} = -\frac{\epsilon}{2} q^{AB} \partial_B M + \frac{q^{AB}}{2} \partial_t m^B \\
\Gamma^A_{B x^3} = -\frac{m_A}{2M} \partial_B M + \frac{\epsilon m_A}{2M} \partial_t m_B + \frac{1}{2} q^{AC} \left( \partial_B m_C - \partial_C m_B \right) + \frac{1}{2} q^{AC} \partial_A q_{BC} \\
\Gamma^A_{t} = 0 \\
\Gamma^A_{x^3 x^3} = \frac{\epsilon m_A}{2M} \partial_t \left( m^2 + \frac{M^2}{N^2} \right) - \frac{m^A}{m_A} \partial_A \ln M - \frac{1}{2} q^{AB} \partial_B \left( m^2 + \frac{M^2}{N^2} \right) + q^{AB} \partial_B m^B \\
\Gamma^x_{x^3 x^3} = -\frac{\epsilon}{2M} \partial_t \left( m^2 + \frac{M^2}{N^2} \right) + \epsilon m_A \partial_t m^A \\
\Gamma^x_{A x^3} = -\frac{1}{\epsilon} \partial_A m - \partial_t q_{AB} \\
\Gamma^x_{A x^3} = \Gamma^x_{A t} = 0.
\]

E Interpretation of the Counter-term in the Non-null Case

We have seen that the counter-term in the null case can be interpreted as the difference in the 2-surface areas orthogonal to \(\ell\) and \(k\) at the boundaries of the null congruence on the null surface (see Eq. (40)) when \(\ell^2 = 0\) everywhere. The counter-term to be added to the boundary in the non-null case is the integral over the boundary of the expression \(2\sqrt{|h|} \nabla_a n^a\) (see Eq. (260)). We shall work with the case where the boundary is a spacelike surface, a \(t\)-constant surface in coordinates \((t, y_1, y_2, y_3)\), and provide an interpretation for the counter-term. Working along the same lines, a similar interpretation can be given for the counter-term on a general non-null surface. The metric has the ADM form [1]

\[
ds^2 = -N^2 dt^2 + h_{\alpha \beta} (dx^\alpha + N^\alpha dt) (dx^\beta + N^\beta dt),
\]

(311)
with the inverse metric components given by
\[
g^{tt} = -\frac{1}{N^2}; \quad g^{ta} = \frac{N^\alpha}{N^2}; \quad g^{ab} = h^{\alpha \beta} - \frac{N^\alpha N^\beta}{N^2}. \tag{312}
\]

The normal according to our conventions (see Appendix B) is \( n_a = \partial_a t \). The counter-term integrand \( \sqrt{h} \nabla_a n^a \) can be manipulated as follows:
\[
\sqrt{h} \nabla_a n^a = \frac{\sqrt{h}}{\sqrt{g}} \partial_a (\sqrt{-g} n^a) = \frac{1}{N} \partial_a \left( \frac{N \sqrt{h} n^a}{\sqrt{h}} \right) = n^a \partial_a \left( \sqrt{h} \right) + \frac{\sqrt{h}}{N} \partial_a (N n^a). \tag{313}
\]

We have \( n^a = g^{at} = (1/N, N^\alpha/N) \), the second term in Eq. (313) can be written in the following form:
\[
\frac{\sqrt{h}}{N} \partial_a (N n^a) = \frac{\sqrt{h}}{N} \partial_a N^\alpha. \tag{314}
\]

Let \( e^a_\alpha, \alpha = 1, 2, 3, \) represent the coordinate basis vectors on the 3-surfaces normal to \( n^a \). They satisfy \( e^a_\alpha n_a = 0 \). We shall consider the case where these basis vectors are Lie transported along \( n^a \). So we have
\[
[n^a, e^a_\alpha] = 0 \implies n^b \partial_b e^a_\alpha - e^b_\alpha \partial_b n^a = 0. \tag{315}
\]

In the coordinates where \( e^a_\alpha \) are basis vectors, we would have \( e^a_\alpha = \delta^a_\alpha \). Hence, the above condition reduces to
\[
\partial_a n^a = 0 \implies \partial_a N = 0 \text{ and } \partial_a N^\alpha = 0. \tag{316}
\]

These conditions also imply that \( n^a \) are tangent vectors to affinely parametrized geodesics. Using \( n_a = N \nabla_a t \) and \( \nabla_a \nabla_b t = \nabla_b \nabla_a t \), we can reduce the geodesic equation to the condition
\[
n^a \nabla_a n_b = 0 \implies h^{ab} \partial_a \ln N = 0. \tag{317}
\]

Since \( h^{ab} \) has only the spatial \( h^{\alpha \beta} \) components, Eq. (316) implies the RHS is zero. Hence, \( n^a \) should satisfy the geodesic condition. Using Eq. (314) and Eq. (316), Eq. (313) reduces to
\[
\sqrt{h} \nabla_a n^a = n^a \partial_a \left( \sqrt{h} \right), \tag{318}
\]

which is the change in the volume of a 3-surface element along \( \ell^a \). Taking \( \tau \) to be the parameter along the integral curves of \( n^a \), we may also write the above equation as
\[
\sqrt{h} \nabla_a n^a = \frac{\partial}{\partial \tau}. \tag{319}
\]

Unlike the null case in Eq. (40), where the derivative was on the surface, this derivative is off the boundary surface and hence will not get integrated in the boundary integration to be interpreted as the difference in the volume of the 3-surface element at two different points.

## F Details of Various Calculations

### F.1 Derivation of Eq. (54)

The boundary term in the non-null case has the following expression:
\[
\sqrt{|h|} D_a (\delta n^a_\perp) = \partial_a \left( \sqrt{|h|} \delta n^a_\perp \right)
= \partial_a \left( \sqrt{|h|} (N \delta \ell^a + 2 \ell^a \delta N) \right)
= \partial_a \left( \sqrt{|h|} \delta \ell^a + 2 \ell^a \delta N \right), \tag{320}
\]

where we have used \( \delta \ell_a = 0 \) to get to the second line and Eq. (241) in the last line. On the other hand, the surface term on the null surface in Eq. (36) can be manipulated as follows:
\[
\partial_a \left[ \sqrt{-g} 1^a_\beta \delta \ell^\beta_\perp \right] = \partial_a \left[ \sqrt{-g} 1^a_\beta \delta \ell^\beta_\perp \right] = \partial_a \left[ \sqrt{-g} \delta \ell^a_\perp \right] + \partial_a \left[ \sqrt{-g} k^a \ell_b \delta \ell^b_\perp \right]
= \partial_a \left[ \sqrt{-g} \delta \ell^a_\perp \right] + \partial_a \left[ \sqrt{-g} k^a \ell_b \delta \ell^b_\perp \right]
= \partial_a \left[ \sqrt{-g} \delta \ell^a_\perp \right], \tag{321}
\]

where we have used \( \delta \ell_a = 0 \) to write \( \delta \ell^a_\perp = \delta \ell^a + g^{ab} \delta \ell_a = \delta \ell^a \) and used \( \delta (\ell^2) = \ell_a \delta \ell^a = 0 \) on the null surface.
F.2 Derivation of Eq. (55)

The variation of the counter-term can be written as:
\[
\delta \left( 2K \sqrt{|h|} \right) = -\delta \left( 2\sqrt{|h|} \nabla_a n^a \right)
\]
\[
= -2\delta \left[ \frac{1}{N} \partial_a \left( \sqrt{-g} g^{\phi \phi} n_b \right) \right]
\]
\[
= -2\delta \left[ \frac{1}{N} \partial_a \left( \sqrt{-g} g^{\phi \phi} N \right) \right]
\]
\[
= -2\delta \left[ \partial_a \left( \sqrt{-g} g^{\phi \phi} \right) \right] - 2\delta \left[ \partial_a \left( \ln N \right) \sqrt{-g} g^{\phi \phi} \right]
\]
\[
= -2\delta \left[ \partial_a \left( \sqrt{-g} g^{\phi \phi} \right) \right] - 2\delta \left[ \sqrt{-g} \ell^a \partial_a \left( \ln N \right) \right]
\]
\[
= -2\delta \left[ \partial_a \left( \sqrt{-g} g^{\phi \phi} \right) \right] + \delta \left[ \sqrt{-g} \left( \partial_\phi g^{\phi \phi} + \frac{g^{\alpha \phi}}{g^{\phi \phi}} \partial_\alpha g^{\phi \phi} \right) \right] ,
\]
(322)
where the last line has been obtained using the relation \( g^{\phi \phi} = \epsilon / N^2 \). In the limit \( N \to \infty \), it is clear that all terms in Eq. (322) are finite under our assumptions. For a null surface, we have the following result using Eq. (224):
\[
\partial_a \left( \sqrt{-g} \ell^a \right) = \sqrt{-g} \nabla_a \ell^a = \sqrt{-g} \left\{ (\Theta + \kappa) - \frac{1}{2} k^a \nabla_a \ell^2 \right\} = \sqrt{-g} \left\{ (\Theta + \kappa) - \frac{1}{2} k^a \partial_a g^{\phi \phi} \right\}
\]
(323)
As we are interested in considering variations such that \( g^{\phi \phi} \) is kept zero on the surface (since we demanded \( \delta (\ell_a \ell^a) = 0 \) in obtaining Eq. (36)), the last term in Eq. (323) can be manipulated as follows in the null limit:
\[
-\frac{1}{2} k^a \partial_a g^{\phi \phi} = -\frac{1}{2} k^\phi \partial_\phi g^{\phi \phi} = \frac{1}{2} \partial_\phi g^{\phi \phi} ,
\]
(324)
where we have used \( k^a \ell_a = k^\phi = -1 \) in the last line. Thus, we can write the null limit of Eq. (322) as
\[
\delta \left( 2\sqrt{|h|} K \right) \rceil \to 0 - 2\delta \left[ \sqrt{-g} (\Theta + \kappa) \right] - \delta \left[ \sqrt{-g} \partial_\phi g^{\phi \phi} \right] + \delta \left[ \sqrt{-g} \left( \partial_\phi g^{\phi \phi} + \frac{g^{\alpha \phi}}{g^{\phi \phi}} \partial_\alpha g^{\phi \phi} \right) \right]
\]
\[
= -2\delta \left[ \sqrt{-g} (\Theta + \kappa) \right] + \delta \left[ \sqrt{-g} \partial_\phi g^{\phi \phi} \right]
\]
\[
= -2\delta \left[ \sqrt{-g} (\Theta + \kappa) \right] - 2\delta \left[ \sqrt{-g} \ell^a \partial_a \ln N \right]
\]
(325)

F.3 Derivation of Eq. (56)

We will first calculate the quantity \( h_{ij} \delta h^{ij} \):
\[
h_{ij} \delta h^{ij} = (h_{ij} \delta g^{ij} - 2\epsilon h_{ij} n^i \delta n^j)
\]
\[
= h_{ij} \delta g^{ij}
\]
\[
= (g_{ij} \delta g^{ij} - \epsilon n_i n_j \delta g^{ij})
\]
\[
= (g_{ij} \delta g^{ij} - \epsilon N^2 \ell_i \ell_j \delta g^{ij})
\]
\[
= (g_{ij} \delta g^{ij} - \epsilon N^2 \delta g^{\phi \phi})
\]
\[
= (g_{ij} \delta g^{ij} + 2\ell \ln N)
\]
(326)
On the null side, we have the following result:
\[
q_{ab} \delta g^{ab} = q_{ab} \delta g^{ab} = g_{ab} \delta g^{ab} + 2\ell_k \delta g^{ab}
\]
\[
= g_{ab} \delta g^{ab} + 2k_b \delta g^{ab} = g_{ab} \delta g^{ab} + 2k_b \delta g^{ab}
\]
(327)
Now, if we look at the derivation of Eq. (55), it is easy to see that the expression is valid without the \( \delta \) too. Thus, we have
\[
\sqrt{|h|} K \rceil \to 0 - \sqrt{-g} \left[ (\Theta + \kappa) + \ell^a \partial_a \ln N \right]
\]
(328)
We are ready now to attack the last term in Eq. (51):
\[
\sqrt{|h|} K h_{ij} \delta h^{ij} \rceil \to 0 - \sqrt{-g} \left[ (\Theta + \kappa) + \ell^a \partial_a \ln N \right] \times \left[ g_{ij} \delta g^{ij} + 2\ell \ln N \right]
\]
\[
= - \sqrt{-g} (\Theta + \kappa) q_{ab} \delta q^{ab} + 2\sqrt{-g} (\Theta + \kappa) k_b \delta \ell^b - 2\sqrt{-g} (\Theta + \kappa) \delta \ln N
\]
\[
- \sqrt{-g} \ell^a \partial_a \ln N g_{ij} \delta g^{ij} - 2\sqrt{-g} (\ell^a \partial_a \ln N) \delta \ln N
\]
(329)
F.4 Derivation of Eq. (57)

We have:

\[- \sqrt{|h|} K_{ij} \delta h^{ij} = - \sqrt{|h|} K_{ij} \delta g^{ij} + 2 \varepsilon \sqrt{|h|} K_{ij} n^i \delta n^j\]

\[- \sqrt{|h|} K_{ij} \delta g^{ij}\]

\[- \sqrt{|h|} \delta g^{ij} \left[- \nabla_i n_j + \epsilon_n m^m \nabla_m n_j \right] = \sqrt{-g} \left[ \nabla_i \ell_j + \ell_j \partial_i \ln N - \epsilon N^2 \ell_i \ell^m \nabla_m \ell_j - \epsilon N^2 \ell_i \ell^m \nabla_m \ln N \right] \delta g^{ij} \quad (330)\]

The third term in Eq. (330) is

\[- \epsilon N^2 \ell_i \ell^m \nabla_m \ell_j \delta g^{ij} = - \frac{1}{2} \epsilon N^2 \ell_i \nabla_j (\ell^2) \delta g^{ij}\]

\[- \frac{1}{2 g_{\phi \phi}} \left( \partial_j g^{\phi \phi} \right) \delta \ell^i = \delta \ell^i \partial_j \ln N \quad (331)\]

where we have used the symmetry of \( \nabla_a \ell_b \) in the second step. This term adds with the second term in Eq. (330), while the last term in Eq. (330) gives

\[- \epsilon N^2 \ell_i \ell^m \nabla_m \ln N \delta \ell^i = - \epsilon N^2 \ell_i \ell^m \left( \nabla_m \ln N \right) \delta \ell^i = - N^2 \ell^m \left( \partial_m \ln N \right) \delta \ell^i \quad (332)\]

where we have used \( \ell_a \delta \ell^a = \delta \phi = \delta g^{\phi \phi} \) in the first step. Adding everything up, we obtain

\[- \sqrt{|h|} K_{ij} \delta h^{ij} = \sqrt{-g} \left[ \nabla_i \ell_j \delta g^{ij} + 2 \left( \partial_i \ln N \right) \delta \ell^j + 2 \ell^i \left( \partial_i \ln N \right) \delta \ln N \right] \quad (333)\]

All terms here are finite in the null limit. In the last term, \( \delta \ln N \) is finite under our assumptions and \( \ell^i \left( \partial_i \ln N \right) \) has been shown to be finite in Eq. (322). \( \left( \partial_i \ln N \right) \delta \ell^i \) can also be easily shown to be finite. The first term in Eq. (333) needs to be decomposed in the null limit. This is done as follows:

\[\nabla_i \ell_j \delta g^{ij} \overset{r=0}{=} \nabla_i \ell_j \left[ \delta g^{ij} - 2 \delta \left( \ell^i \ell^j \right) \right] = \nabla_i \ell_j \delta g^{ij} - \delta k^i \partial_i \ell^j - 2 \delta \ell^i \ell^j \nabla_i \ell_j - \Theta_{ij} \delta g^{ij} + 2 \delta \ell^i \ell^j \nabla_i \ell_j \quad (334)\]

In the last step, we have used Eq. (32) for the first term and \( \delta \alpha = 0 \) was used to put \( \delta \kappa^i \partial_i \ell^2 = 0 \). Putting it all together, we arrive at

\[- \sqrt{|h|} K_{ij} \delta h^{ij} \overset{r=0}{=} - \sqrt{-g} \left[ \Theta_{ij} \delta g^{ij} + 2 \delta \ell^i \ell^j \nabla_i \ell_j + 2 \left( \partial_i \ln N \right) \delta \ell^i + 2 \ell^i \left( \partial_i \ln N \right) \delta \ln N \right] \quad (335)\]

F.5 Derivation of Eq. (59)

The extra terms in Eq. (58) are

\[\text{Extra Terms} = 2 \partial_\alpha \left[ \sqrt{-g} \ell^\alpha \delta \ln N \right] - 2 \delta \left[ \sqrt{-g} \ell^\alpha \partial_\alpha \ln N \right] - 2 \sqrt{-g} \left( \Theta + \kappa \right) \delta \ln N \]

\[+ \sqrt{-g} \left[ 2 \left( \partial_\ell \ln N \right) \delta \ell^i + 2 \ell^i \left( \partial_\ell \ln N \right) \delta \ln N \right] \quad (336)\]

We shall start by working on the first, second and the fourth of these extra terms. These are

\[2 \partial_\alpha \left[ \sqrt{-g} \ell^\alpha \delta \ln N \right] - 2 \delta \left[ \sqrt{-g} \ell^\alpha \partial_\alpha \ln N \right] + \sqrt{-g} \left( \ell^\alpha \partial_\alpha \ln N \right) \delta \ln N \]

Substituting back, we have the extra terms as

\[\text{Extra Terms} = 2 \left\{ \left[ \partial_\alpha \left( \sqrt{-g} \ell^\alpha \right) \right] \delta \ln N + \sqrt{-g} \delta \ell^\alpha \partial_\ell \ln N + \sqrt{-g} \left( \ell^\alpha \partial_\ell \ln N \right) \delta \ln N \right\} \quad (338)\]
Now, from Eq. (323) and Eq. (324), we know that the first term in Eq. (338) can be manipulated as

\[
\partial_\alpha \left( \sqrt{-g} \phi \right) - \sqrt{-g} (\Theta + \kappa) \delta \ln N = \left[ \sqrt{-g} \left( \nabla_a \left( \sqrt{-g} \phi \right) - \Theta - \kappa - \partial_\phi \left( \sqrt{-g} \phi \right) \right) \right] \delta \ln N
\]

\[
= \left( \frac{-\sqrt{-g}}{2} \partial_\phi \phi^\phi - \sqrt{-g} \partial_\phi \phi \right) \delta \ln N
\]

\[
= \left( \frac{-\sqrt{-g}}{2} \partial_\phi \phi^\phi \right) \delta \ln N
\]

\[
= \left( \sqrt{-g} \frac{\ell}{\kappa} \partial_\phi \phi^\phi \right) \delta \ln N \nabla_a \ell_a - \Theta - \kappa - \partial_\phi \left( \sqrt{-g} \phi \right) \right) \delta \ln N
\]

\[
\delta \ln N = \left( \sqrt{-g} \frac{\ell}{\kappa} \partial_\phi \phi^\phi \right) \delta \ln N
\]

\[
\delta \ln N = \left( \sqrt{-g} \frac{\ell}{\kappa} \partial_\phi \phi^\phi \right) \delta \ln N
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\delta \ln N = \left( \sqrt{-g} \frac{\ell}{\kappa} \partial_\phi \phi^\phi \right) \delta \ln N
\]

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\delta \ln N = \left( \sqrt{-g} \frac{\ell}{\kappa} \partial_\phi \phi^\phi \right) \delta \ln N
\]

where we have made use of Eq. (238) and \( \ell_a = g^{a\phi} \). In the second step, we have put the term

\[- \left( \sqrt{-g} \ell^a \partial_\phi \sqrt{-g} \right) \delta \ln N\]

to zero as we know that \( \ell^a = \ell^a \ell_a = 0 \) on the null surface while \( \sqrt{-g} \) and its derivatives are assumed to be finite everywhere and \( \delta \ln N \) is finite under our assumption about the behaviour of \( g^{\phi\phi} \) near the surface.

Thus, the first and the last terms in Eq. (338) are identical. The middle term should also be then of the same form with an extra factor of \(-2\). Indeed, we have

\[
\delta \ell^\phi \partial_\phi \phi \ln N = \frac{-2\ell^\phi}{N^2} \partial_\phi \phi \ln N \delta \ln N = -2\ell^\phi \left( \partial_\phi \phi \ln N \right) \delta \ln N
\]

so that Eq. (338) reduces to

**Extra Terms = 0!**

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