VECTOR-VALUED SCHRÖDINGER OPERATORS IN $L^p$-SPACES

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Abstract. In this paper we consider vector-valued Schrödinger operators of the form $\text{div}(Q\nabla u) - Vu$, where $V = (v_{ij})$ is a nonnegative locally bounded matrix-valued function and $Q$ is a symmetric, strictly elliptic matrix whose entries are bounded and continuously differentiable with bounded derivatives. Concerning the potential $V$, we assume an that it is pointwise accretive and that its entries are in $L^\infty_{\text{loc}}(\mathbb{R}^d)$. Under these assumptions, we prove that a realization of the vector-valued Schrödinger operator generates a $C_0$-semigroup of contractions in $L^p(\mathbb{R}^d; C^m)$. Further properties are also investigated.

1. Introduction

Recently, there is an increased interest in systems of parabolic equations with unbounded coefficients. Such systems appear in the study of backward-forward differential games, in connection with Nash equilibria in stochastic differential games, in the analysis of the weighted $\bar{\partial}$-problem in $\mathbb{C}^d$, in time dependent Born–Openheimer theory and also in the study of Navier–Stokes equation. For more information we refer the reader to [1, Section 6], [11], [5], [4], [14], [13] and [10].

While the scalar theory of such equations is by now well understood (see [17] and the references therein), so far there are only few articles concerned with systems. We mention the article [12] where systems of parabolic equations coupled through both, a potential term and a drift term, were considered in the $L^p$-setting. In [1, 3, 6], the authors choose a different approach. Indeed, they first constructed solutions in the space of bounded and continuous functions and only afterwards the obtained semigroup is extrapolated to the $L^p$-scale. We should point out that in the presence of an unbounded drift term the differential operator does not always generate a strongly continuous semigroup on $L^p$-spaces with respect to Lebesgue measure, see [19]. Thus, in some cases appropriate growth conditions need to be imposed on the coefficients to ensure generation of a semigroup on $L^p$ with respect to Lebesgue measure.

In this article we will consider systems of parabolic equations which are coupled only through a potential term. To be more precisely, consider the differential operator

$$\mathcal{A}u = \text{div}(Q\nabla u) - Vu =: \Delta_Q u - Vu$$

acting on vector-valued functions $u = (u_1, \ldots, u_m): \mathbb{R}^d \to \mathbb{C}^m$. Here $Q$ is a bounded, symmetric and strictly elliptic matrix with continuously differentiable entries that have bounded derivatives. The expression $\text{div}(Q\nabla u)$ should be understood componentwise, i.e. $\text{div}(Q\nabla u) = (\text{div}(Q\nabla u_1), \ldots, \text{div}(Q\nabla u_m))$. The matrix-valued function $V: \mathbb{R}^d \to \mathbb{R}^{m \times m}$ is assumed to be pointwise accretive and to have locally bounded coefficients. In contrast to the situation where an unbounded drift is
present, no additional growth assumptions on the potential $V$ are needed to ensure generation of a strongly continuous semigroup on $L^2(\mathbb{R}^d; \mathbb{R}^m)$. Indeed, following Kato [15], who considered the scalar situation, we shall construct a densely defined, m-dissipative realization $A$ of the operator $A$ in $L^2(\mathbb{R}^d; \mathbb{R}^m)$. By virtue of the Lumer–Phillips theorem $A$ generates a strongly continuous semigroup. Subsequently, we prove that this semigroups extrapolates to a consistent family of strongly continuous contraction semigroups $\{T_p(t)\}_{t \geq 0}$ on $L^p(\mathbb{R}^d; \mathbb{R}^m)$ for $1 < p < \infty$. We also give a description of the generator $A_p$ of $\{T_p(t)\}_{t \geq 0}$ and prove that the test functions form a core for the operator $A_p$.

We should point out that in our recent article [16] we were interested in proving that the domain of the vector-valued Schrödinger operator is the intersection of the domain of the diffusion part and the potential part. To that end, we had to impose growth conditions on the Potential part. Here, we allow general potential without such a growth condition. The price to pay is that we can only characterize the domain of the $L^p$-realization of our operator as the maximal $L^p$-domain.

This article is organised as follows. In Section 2 we prove a version of Kato’s inequality for vector-valued functions which is crucial in all subsequent sections. In Section 3 we construct a realization of the operator $A$ in $L^2(\mathbb{R}^d; \mathbb{R}^m)$ which generates a strongly continuous contraction semigroup. In Section 4 we extrapolate the semigroup to $L^p$-spaces, where $p \in (1, \infty)$. In the concluding Section 5 we characterize the domain of the generator as maximal domain.

Notation Let $d, m \geq 1$. By $| \cdot |$ we denote the Euclidean norm on $C^j, j = d, m$ and by $\langle \cdot, \cdot \rangle$ the Euclidean inner product. By $B(r) = \{x \in \mathbb{R}^d : |x| \leq r\}$ we denote the Euclidean ball of radius $r > 0$ and center 0. For $1 \leq p \leq \infty$, $L^p(\mathbb{R}^d; \mathbb{C}^m)$ is the $\mathbb{C}^m$-valued Lebesgue space on $\mathbb{R}^d$. For $1 \leq p < \infty$, the norm is given by

$$\|f\|_p := \left( \int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{\frac{1}{p}} = \left( \int_{\mathbb{R}^d} \left( \sum_{j=1}^m |f_j|^2 \right)^{\frac{p}{2}} \, dx \right)^{\frac{1}{p}}, \quad f \in L^p(\mathbb{R}^d; \mathbb{C}^m),$$

whereas in the case $p = \infty$ we use the essential supremum norm

$$\|f\|_{\infty} := \text{ess sup}\{|f(x)| : x \in \mathbb{R}^d\}.$$ 

For $1 < p < \infty$, $p'$ refers to the conjugate index, i.e. $1/p + 1/p' = 1$. Thus $L^{p'}(\mathbb{R}^d; \mathbb{C}^m)$ is the dual space of $L^p(\mathbb{R}^d; \mathbb{C}^m)$ and the duality pairing $\langle \cdot, \cdot \rangle_{p, p'}$ is given by

$$\langle f, g \rangle_{p, p'} = \int_{\mathbb{R}^d} \langle f(x), g(x) \rangle \, dx, \quad \text{for } f \in L^p(\mathbb{R}^d; \mathbb{C}^m), g \in L^{p'}(\mathbb{R}^d; \mathbb{C}^m).$$

By $C^\infty_c(\mathbb{R}^d; \mathbb{C}^m)$, we denote the space of all test functions, i.e. functions $f : \mathbb{R}^d \to \mathbb{C}^m$ which have compact support and derivatives of any order. The space $W^{k,p}(\mathbb{R}^d; \mathbb{C}^m)$ is the classical Sobolev space of order $k$, that is the space of all functions $f \in L^p(\mathbb{R}^d; \mathbb{C}^m)$ such that the distributional derivative $\partial^\alpha f$ belongs to $L^p(\mathbb{R}^d; \mathbb{C}^m)$ for all $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ with $|\alpha| = \sum_{j=1}^d \alpha_j \leq k$. For $1 \leq p \leq \infty$ we define $L^p_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^m)$ as the space of all measurable functions $f : \mathbb{R}^d \to \mathbb{R}^m)$ such that $\chi_{B(r)} f \in L^p(\mathbb{R}^d; \mathbb{R}^m)$ for all $r > 0$. Here $\chi_{B(r)}$ is the indicator function of the ball $B(r)$. The space $W^{k,p}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^m)$ is the space of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^m)$ such that for $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$ the distributional derivative $\partial^\alpha f$ belongs to $L^p_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^m)$. We write $H^k(\mathbb{R}^d; \mathbb{C}^m) := W^{k,2}(\mathbb{R}^d; \mathbb{C}^m)$ and $H^k_{\text{loc}}(\mathbb{R}^d; \mathbb{C}^m) := W^{k,2}_{\text{loc}}(\mathbb{R}^d; \mathbb{C}^m)$.

2. Preliminaries

Throughout this article we make the following assumptions:
Hypotheses 2.1. (1) The map $Q : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is such that $q_{ij} = q_{ji}$ is bounded and continuously differentiable with bounded derivative for all $i, j \in \{1, \ldots, d\}$ and there exist positive real numbers $\eta_1$ and $\eta_2$ such that

\[ \eta_1 |\xi|^2 \leq \langle Q(x)\xi, \xi \rangle \leq \eta_2 |\xi|^2 \]

for all $x, \xi \in \mathbb{R}^d$. (2) The map $V : \mathbb{R}^d \to \mathbb{R}^{m \times m}$ has entries $v_{ij} \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ for all $i, j \in \{1, \ldots, m\}$ and

\[ \text{Re} \langle V(x)\xi, \xi \rangle \geq 0, \]

for all $x \in \mathbb{R}^d$, $\xi \in \mathbb{C}^m$.

To simplify notations, we write for $\xi, \eta \in \mathbb{R}^d$

\[ \langle \xi, \eta \rangle_Q := \sum_{i,j=1}^d q_{ij} \xi_i \eta_j \quad \text{and} \quad |\xi|_Q := \sqrt{\langle \xi, \xi \rangle_Q}. \]

We define the operator $\Delta_Q : W^{1,1}_{\text{loc}}(\mathbb{R}^d) \to \mathcal{D}(\mathbb{R}^d)$ by setting

\[ \langle \Delta_Q u, \varphi \rangle = -\int_{\mathbb{R}^d} \langle \nabla u, \nabla \varphi \rangle_Q \, dx. \]

for any test function $\varphi \in C^\infty_c(\mathbb{R}^d)$, where $\mathcal{D}(\mathbb{R}^d)$ denotes the space of distributions.

As usual, we will say that $\Delta_Q u \in L^1_{\text{loc}}(\mathbb{R}^d)$, if there is a function $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ such that

\[ \langle \Delta_Q u, \varphi \rangle = \int_{\mathbb{R}^d} f \varphi \, dx \]

for all $\varphi \in C^\infty_c(\mathbb{R}^d)$. In this case we will identify $\Delta_Q u$ and the function $f$.

We can now prove a vector-valued version of Kato’s inequality.

Proposition 2.3. Let $u = (u_1(x), \ldots, u_m(x)) \in W^{1,p}_{\text{loc}}(\mathbb{R}^d; \mathbb{C}^m)$. Then, $|u| \in W^{1,p}_{\text{loc}}(\mathbb{R}^d)$ and

\[ \nabla |u| = \frac{1}{|u|} \sum_{j=1}^m \text{Re} (\bar{u}_j \nabla u_j) \chi_{\{u \neq 0\}}. \]

Moreover,

\[ |\nabla |u||^2_Q \leq \sum_{j=1}^d |\nabla u_j|^2_Q. \]

We can now prove a vector-valued version of Kato’s inequality.

Proposition 2.3. Let $u = (u_1, \ldots, u_m) \in H^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^m)$ be such that $\Delta_Q u_j \in L^1_{\text{loc}}(\mathbb{R}^d)$ for $j = 1, \ldots, m$. Then

\[ \Delta_Q |u| = \chi_{\{u \neq 0\}} \frac{1}{|u|} \left( \sum_{j=1}^m u_j \Delta_Q u_j + \sum_{j=1}^m |\nabla u_j|^2_Q - |\nabla |u||^2_Q \right). \]

Thus, the Kato inequality

\[ \Delta_Q |u| \geq \chi_{\{u \neq 0\}} \frac{1}{|u|} \sum_{j=1}^m u_j \Delta_Q u_j \]

holds in the sense of distributions.
We define the operator $Q$ and obtain

$$Q \Rightarrow u$$

we apply the monotone convergence theorem, using that $\|\|

In view of the Lumer–Phillips theorem, cf. [7, Theorem II-3.15], it suffices to prove

$$tions of the operator $A$.

Recall that $\lim_{\varepsilon \to 0} a_\varepsilon(u) = \|u\|$ in $L^2_{\text{loc}}(\mathbb{R}^d)$. This implies that $\lim_{\varepsilon \to 0} \Delta Q a_\varepsilon(u) = \Delta Q|u|$ in $C_c(\mathbb{R}^d)$.

Since $\nabla a_\varepsilon(u) = \sum_{j=1}^m u_j \nabla u_j$, it follows that for $\varphi \in C_c(\mathbb{R}^d)$ we have

$$\langle \Delta Q a_\varepsilon(u), \varphi \rangle = -\int_{\mathbb{R}^d} (Q \nabla a_\varepsilon(u), \nabla \varphi) \, dx = -\sum_{j=1}^m \int_{\mathbb{R}^d} \frac{u_j}{a_\varepsilon(u) + \varepsilon} \langle Q \nabla u_j, \nabla \varphi \rangle \, dx$$

$$= -\sum_{j=1}^m \int_{\mathbb{R}^d} \frac{u_j}{a_\varepsilon(u) + \varepsilon} \Delta Q u_j \varphi \, dx + \sum_{j=1}^m \int_{\mathbb{R}^d} \frac{1}{a_\varepsilon(u) + \varepsilon} \langle Q \nabla u_j, \nabla u_j \rangle \varphi \, dx$$

$$= -\sum_{j=1}^m \int_{\mathbb{R}^d} \frac{u_j}{a_\varepsilon(u) + \varepsilon} \Delta Q u_j \varphi \, dx + \sum_{j=1}^m \int_{\mathbb{R}^d} \frac{1}{a_\varepsilon(u) + \varepsilon} \langle Q \nabla u_j, \nabla u_j \rangle \varphi \, dx$$

Recall that $\lim_{\varepsilon \to 0} a_\varepsilon(u) = \|u\|$ in $L^2_{\text{loc}}(\mathbb{R}^d)$ and $\lim_{\varepsilon \to 0} \nabla a_\varepsilon(u) = \nabla |u|$ in $L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d)$. Noting that $\sum_{j=1}^m \frac{u_j}{a_\varepsilon(u) + \varepsilon}$ is uniformly bounded by 1 we see that we can apply the dominated convergence theorem in the first integral above. For the other two integrals, we apply the monotone convergence theorem, using that $Q$ is strictly elliptic and observing that $(a_\varepsilon(u) + \varepsilon)^{-1}$ decreases to $|u|^{-1}$. Note that in all integrals it is sufficient to integrate over the set $\{u \neq 0\}$. For the first and third integral, this is obvious due to the presence of $u_j$ which vanish on $\{u = 0\}$. For the second one we infer from Stampacchia’s lemma that $\nabla u_j = 0$ on $\{u = 0\}$. Thus, by letting $\varepsilon \to 0$, we obtain

$$\langle \Delta Q|u|, \varphi \rangle = \int_{\{u \neq 0\}} \sum_{j=1}^m \left( \frac{u_j}{|u|} \Delta Q u_j + \frac{1}{u} |\nabla u_j|_Q^2 \right) \varphi \, dx - \int_{\{u \neq 0\}} \frac{1}{|u|} |\nabla |u||_Q^2 \varphi \, dx.$$

This proves (2.6). Using (2.5) and (2.6), also (2.7) follows. \qed

3. Generation of semigroup in $L^2(\mathbb{R}^d, \mathbb{C}^m)$

Let us consider the differential operator $Au = \Delta Qu - Vu$, where $u = (u_1, \ldots, u_m)$. Here $\Delta Q$ acts entrywise on $u$, i.e. $\Delta Q u = (\Delta Q u_1, \ldots, \Delta Q u_m)$. We define $A$ to be the realization of $\mathcal{A}$ on $L^2(\mathbb{R}^d, \mathbb{C}^m)$ with domain

$$D(A) = \{ u \in H^1(\mathbb{R}^d, \mathbb{C}^m) : Au \in L^2(\mathbb{R}^d, \mathbb{C}^m) \}.$$

In this section we prove that $A$ generates a $C_0$-semigroup of contractions in $L^2(\mathbb{R}^d, \mathbb{C}^m)$. In view of the Lumer–Phillips theorem, cf. [7, Theorem II-3.15], it suffices to prove that $-A$ is maximal accretive, i.e. for $u \in D(A)$ we have $\langle -Au, u \rangle \geq 0$ and $I - A$ is surjective.

To that end, we follow the strategy from [15] and introduce some other realizations of the operator $\mathcal{A}$ on the space $H^{-1}(\mathbb{R}^d, \mathbb{C}^m)$, the dual space of $H^1(\mathbb{R}^d; \mathbb{C}^m)$. We define the operator $L_0$ by setting

$$L_0 u = Au, \quad u \in D(L_0) := C_c(\mathbb{R}^d; \mathbb{C}^m)$$
and the operator $L$ by

$$Lu = Au, \quad u \in D(L) := \{ u \in H^1(\mathbb{R}^d; \mathbb{C}^m) : Au \in H^{-1}(\mathbb{R}^d; \mathbb{C}^m) \}.$$  

We let $\mathcal{A}u = \Delta_Q - V^*$ be the formal adjoint of $\mathcal{A}$, where $V^*$ is the conjugate matrix of $V$. We then define the operators $\bar{L}$ and $\bar{L}_0$ in analogy to the operators $L$ and $L_0$, using the potential $V^*$ instead of $V$.

We now collect some properties of the operators $L_0$ and $L$ and the adjoints $L_0^*$. We denote the duality pairing between $H^{-1}(\mathbb{R}^d; \mathbb{C}^m)$ and $H^1(\mathbb{R}^d; \mathbb{C}^m)$ by $[\cdot, \cdot]$.

**Proposition 3.1.** Assume Hypotheses [2.7] Then the following hold

1. $L = L_0^*$ and $L = \bar{L}_0$. Consequently $\bar{L}$ and $L$ are closed.
2. $L_0$ is closable and its closure is equal to $L_0^*$.

**Proof.** (1) Let $f \in D(\bar{L})$ and $g \in C_c^\infty(\mathbb{R}^d, \mathbb{C}^m)$. Using integration by parts, we see that

$$[\bar{L} f, g] = \int_{\mathbb{R}^d} \langle \nabla \nabla f(x), g(x) \rangle dx - \int_{\mathbb{R}^d} \langle V^*(x) f(x), g(x) \rangle dx$$

$$= \int_{\mathbb{R}^d} \langle f(x), \nabla (\nabla g(x)) \rangle dx - \int_{\mathbb{R}^d} \langle f(x), V(x) g(x) \rangle dx$$

$$= [\bar{L} f, g].$$

Thus $\bar{L} = L_0^*$ and hence $\bar{L}$ is closed. In a similar way one shows that $L = \bar{L}_0^*$ and thus $L$ is also closed.

(2) Since $L_0^*$ is densely defined, $L_0$ is closable with closure $L_0^{**}$ by general theory. \[Q.E.D.\]

We can now prove the main result of this section.

**Theorem 3.2.** The operator $-L$ is maximal monotone.

**Proof. Step 1:** We first show that $-L_0^{**}$ is maximal monotone. It is easy to see that $-L_0$ is monotone. Indeed, for $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m)$ we have

$$\text{Re}[-L_0\varphi, \varphi] = \int_{\mathbb{R}^d} |\nabla \varphi(x)|^2 dx + \text{Re} \int_{\mathbb{R}^d} \langle V(x) \varphi(x), \varphi(x) \rangle dx \geq 0.$$ 

It follows that also the closure of $-L_0$, i.e. the operator $-L_0^{**}$ is monotone. As $-L_0^{**}$ is monotone, $\text{rg}(1 - L_0^{**})$, the range of $(1 - L_0^{**})$, is a closed subset of $H^{-1}(\mathbb{R}^d; \mathbb{C}^m)$. Therefore, to prove that $-L_0^{**}$ is maximal, it suffices to show that $1 - L_0^{**}$ has dense range. We prove that $(1 - L_0^{**})C_c^\infty(\mathbb{R}^d; \mathbb{C}^m)$ is dense in $H^{-1}(\mathbb{R}^d; \mathbb{C}^m)$. Since the coefficients of $\mathcal{A}$ are real, it suffices to prove that $(1 - L_0^{**})C_c^\infty(\mathbb{R}^d; \mathbb{R}^m)$ is dense in $H^{-1}(\mathbb{R}^d; \mathbb{R}^m)$. To that end, let $u \in H^1(\mathbb{R}^d; \mathbb{R}^m)$ be such that $[\varphi, u] = 0$ for all $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m)$. Then

$$u - \Delta_Q u + V^* u = 0,$$

and hence,

$$\Delta_Q u_j = u_j + \sum_{l=1}^m v_{lj} u_l,$$

for every $j \in \{1, \ldots, m\}$ in the sense of distributions. Applying [2.7], we obtain

$$\Delta_Q |u| \geq \frac{1}{|u|} \sum_{j=1}^m u_j \Delta_Q u_j \chi_{\{u \neq 0\}}$$

$$\geq \frac{\chi_{\{u \neq 0\}}}{|u|} \left( \sum_{j=1}^m u_j^2 + \sum_{j,l=1}^m v_{lj} u_l u_j \right)$$

$$\geq \frac{\chi_{\{u \neq 0\}}}{|u|} |u|^2 = |u|.$$
Thus, $\Delta |u| \geq |u|$ in the sense of distributions. Now, let $(\phi_n)_n \subset C_c^\infty(\mathbb{R}^d)$ be such that $\phi_n \geq 0$ and $\phi_n \rightarrow |u|$ in $H^1(\mathbb{R}^d)$. Then

$$0 \leq [\Delta_Q |u|, \phi_n] - [||u||, \phi_n] = - \int_{\mathbb{R}^d} \langle \nabla |u|, \nabla \phi_n \rangle_Q \, dx - \int_{\mathbb{R}^d} |u| \phi_n \, dx.$$ 

Upon $n \rightarrow \infty$, we find $-||\nabla |u||Q||^2 - ||u||^2 \geq 0$ which implies that $u = 0$. This proves that the range of $I - L_0^*$ is dense.

**Step 2:** We now prove now that $L = L_0^{**}$. We know that $L$ is a closed extension of $L_0$. Hence $L_0^{**} \subset L$. In order to get the converse inclusion, it suffices to show that $\rho(L) \cap \rho(L_0^*) \neq \emptyset$. As $L_0^{**}$ is maximal monotone we have $1 \in \rho(L_0^*)$. On the other hand, $(1 - L)D(L) \supset (1 - L_0^*)D(L_0^*) = H^{-1}(\mathbb{R}^d, C^m)$. Hence, $1 - L$ is surjective. To prove that $1 - L$ is injective, note that $\ker(1 - L) = \text{rg}(1 - L^*)^\perp$. Applying Step 1 with $V^*$ instead of $V$ we see that $-L^* = -L_0^{**}$ is maximal. Repeating the above argument, it follows that $\text{rg}(1 - L^*) = H^{-1}(\mathbb{R}^d, C^m)$ and thus $\ker(1 - L) = \{0\}$. This proves that $1 \in \rho(L)$ and ends the proof.

We can now infer that $A$ generates a strongly continuous contraction semigroup.

**Corollary 3.3.** Assume Hypotheses 2.1. Then the operator $A$ generates a $C_0$-semigroup $(T(t))_{t \geq 0}$ of contractions on $L^2(\mathbb{R}^d, C^m)$.

**Proof.** Since $-L$ is monotone, so is $-A$, the part of $-L$ in $L^2(\mathbb{R}^d; C^m)$. As $-L$ is maximal monotone, so is $-A$. Indeed, given $f \in L^2(\mathbb{R}^d; C^m) \subset H^{-1}(\mathbb{R}^d; C^m)$ we find $u \in D(A)$ such that $u - A u = f$. But then $A u = u - f$ belongs to $L^2(\mathbb{R}^d; C^m)$, proving that $u \in D(A)$ and $u - A u = f$. The claim now follows from the Lumer–Phillips theorem.

4. Extension of the semigroup to $L^p(\mathbb{R}^d, C^m)$

In this section we extrapolate the semigroup $(T(t))_{t \geq 0}$ to the spaces $L^p(\mathbb{R}^d, C^m)$, $1 \leq p < \infty$. As a first step, we prove that $(T(t))_{t \geq 0}$ is given by the Trotter–Kato product formula

$$T(t)f = \lim_{n \rightarrow \infty} \left[ e^{\frac{1}{n} \Delta_Q} e^{-\frac{t}{n} V} \right]^n f,$$

for all $t > 0$ and $f \in L^2(\mathbb{R}^d, C^m)$. Here $\{e^{\frac{1}{n} \Delta_Q}\}_{t \geq 0}$ is the semigroup generated by $\Delta_Q$ in $L^2(\mathbb{R}^d, C^m)$ and $\{e^{-\frac{t}{n} V}\}_{t \geq 0}$ is the multiplication semigroup generated by the potential $-V$, i.e. $e^{-\frac{t}{n} V}$ is multiplication with the matrix given pointwise by $\sum_{k=0}^{\infty} \frac{(-t V(x))^k}{n^k}$. To prove that the semigroup $(T(t))_{t \geq 0}$ is given by the Trotter–Kato formula (4.1) we use the following result which is also of independent interest.

**Proposition 4.1.** Assume Hypotheses 2.1. Then $C_c^\infty(\mathbb{R}^d; C^m)$ is a core for $A$.

**Proof.** Since $-A$ is maximal accretive and has real coefficients, it suffices to show that $(1 - A)C_c^\infty(\mathbb{R}^d; R^m)$ is dense in $L^2(\mathbb{R}^d; R^m)$. Let $u \in L^2(\mathbb{R}^d; R^m)$ be such that $\langle (1 - A)\varphi, u \rangle = 0$, for all $\varphi \in C_c^\infty(\mathbb{R}^d; R^m)$. Thus, $u - \Delta_Q u + V^* u = 0$ in the sense of distributions. Hence,

$$\Delta_Q u_j = u_j + \sum_{i=1}^{m} v_{ij} u_i.$$ 

In particular, $\Delta_Q u_j = \text{div}(Q \nabla u_j) \in L^2_{\text{loc}}(\mathbb{R}^d)$ for each $j \in \{1, \ldots, m\}$. Then, by local elliptic regularity, see [2 Theorem 7.1], $u_j \in H^2_{\text{loc}}(\mathbb{R}^d)$.

Therefore, $|u| = \lim_{\epsilon \rightarrow 0} (|u|^2 + \epsilon^2)^{1/2}$ belongs to $H^2_{\text{loc}}(\mathbb{R}^d)$. In particular, equation (2.6) still holds true, i.e.

$$\Delta_Q |u| \geq \frac{\chi_{\{uu \neq 0\}}}{|u|} \sum_{j=1}^{m} u_j \Delta_Q u_j$$
almost everywhere. Consequently,
\[ \Delta_Q |u| \geq \frac{\chi(u \neq 0)}{|u|} (|u|^2 + \langle Vu, u \rangle) \geq |u|. \]

Now, let \( \zeta \in C_c^\infty(\mathbb{R}^d) \) be such that \( \chi_{R(1)} \leq \zeta \leq \chi_{R(2)} \) and define \( \zeta_n(x) = \zeta(x/n) \) for \( x \in \mathbb{R}^d \) and \( n \in \mathbb{N} \). We multiply both sides of the inequality \( \Delta_Q |u| \geq |u| \) by \( \zeta_n |u| \) and integrate by parts. We obtain
\[
0 \geq \int_{\mathbb{R}^d} |u(x)|^2 \zeta_n(x) dx - \int_{\mathbb{R}^d} \Delta_Q |u(x)| \zeta_n(x) |u(x)| dx \\
= \int_{\mathbb{R}^d} |u(x)|^2 \zeta_n(x) dx + \int_{\mathbb{R}^d} \langle \nabla (\zeta_n |u|) (x), Q(x) \nabla |u|(x) \rangle dx \\
= \int_{\mathbb{R}^d} |u(x)|^2 \zeta_n(x) dx + \int_{\mathbb{R}^d} |\nabla |u|(x)|^2 \zeta_n(x) dx + \int_{\mathbb{R}^d} \langle \nabla \zeta_n(x), \nabla |u|(x) \rangle Q(x) |u(x)| dx \\
\geq \int_{\mathbb{R}^d} |u(x)|^2 \zeta_n(x) dx + \frac{1}{2} \int_{\mathbb{R}^d} \langle Q(x) \nabla \zeta_n(x), |\nabla |u|(x)|^2 \rangle dx \\
= \int_{\mathbb{R}^d} |u(x)|^2 \zeta_n(x) dx - \frac{1}{2} \int_{\mathbb{R}^d} \Delta \zeta_n(x) |u(x)|^2 dx.
\]
Here we have used in the fourth line that \( \nabla |u|^2 = 2 |u| \nabla |u| \). A straightforward computation shows
\[ \Delta \zeta_n(x) = \frac{1}{n} \sum_{i,j=1}^m \partial_i q_{ij} \partial_j \zeta(x/n) + \frac{1}{n^2} \sum_{i,j=1}^m a_{ij} \partial_i \partial_j \zeta(x/n). \]
It follows that \( \|\Delta \zeta_n\|_\infty \to 0 \) as \( n \to \infty \). Hence, letting \( n \to \infty \) in the above inequality, we obtain \( \|u\|_2 \leq 0 \), and thus \( u = 0 \). This finishes the proof. \( \square \)

**Proposition 4.2.** Assume Hypotheses\(^24\) Then the semigroup \( \{T(t)\}_{t \geq 0} \) is given by the Trotter–Kato product formula (4.1).

**Proof.** Since \( C_c^\infty(\mathbb{R}^d; \mathbb{C}^m) \subset D(\Delta_Q) \cap D(V) \), where \( D(\Delta_Q) = H^2(\mathbb{R}^d; \mathbb{C}^m) \) and \( D(V) = \{ u \in L^2(\mathbb{R}^d; \mathbb{C}^m) : V u \in L^2(\mathbb{R}^d; \mathbb{C}^m) \} \), the claim follows from Corollary III.5.8. \( \square \)

We can now extend \( \{T(t)\}_{t \geq 0} \) to \( L^p(\mathbb{R}^d, \mathbb{C}^m) \).

**Theorem 4.3.** Let \( 1 < p < \infty \) and assume Hypotheses\(^24\). Then \( \{T(t)\}_{t \geq 0} \) can be extrapolated to a \( C_0 \)-semigroup \( \{T_p(t)\}_{t \geq 0} \) on \( L^p(\mathbb{R}^d, \mathbb{C}^m) \). Moreover, if we denote by \( (A_p, D(A_p)) \) its generator, then \( A_p u = A u \), for all \( u \in C_c^\infty(\mathbb{R}^d; \mathbb{C}^m) \).

**Proof.** Let \( 1 < p < \infty \) and \( f \in L^2(\mathbb{R}^d, \mathbb{C}^m) \cap L^p(\mathbb{R}^d, \mathbb{C}^m) \). Assumption (2.4) yields \( |e^{-tV(x)} f(x)| \leq |f(x)| \) for all \( x \in \mathbb{R}^d \) and \( t \geq 0 \). So \( \|e^{-tV} f\|_p \leq \|f\|_p \), for all \( t \geq 0 \).

On the other hand, it is well-known that \( \{e^{t \Delta_Q}\}_{t \geq 0} \) is a contractive \( C_0 \)-semigroup on \( L^p(\mathbb{R}^d, \mathbb{C}^m) \). Consequently, for every \( t > 0 \), both \( e^{t \Delta_Q} \) and \( e^{-tV} \) leave the set
\[ B_p := \{ f \in L^2(\mathbb{R}^d, \mathbb{C}^m) \cap L^p(\mathbb{R}^d, \mathbb{C}^m) : \|f\|_p \leq 1 \} \]
invariant. Since \( B_p \) is a closed subset of \( L^2(\mathbb{R}^d; \mathbb{C}^m) \) as a consequence of Fatou’s lemma, it follows from the Trotter–Kato formula (4.1) that \( T(t) B_p \subset B_p \). It follows that \( \|T(t) f\|_p \leq \|f\|_p \) for all \( f \in L^2(\mathbb{R}^d; \mathbb{C}^m) \cap L^p(\mathbb{R}^d, \mathbb{R}^m) \). By density, we can extend \( T(t) \) to a contraction \( T_p(t) \) on \( L^p(\mathbb{R}^d, \mathbb{C}^m) \). The semigroup law for \( \{T_p(t)\}_{t \geq 0} \) follows immediately.

Let us prove that \( \{T_p(t)\}_{t \geq 0} \) is strongly continuous. To that end, pick \( p^* \in (1, \infty) \) and \( \theta \in (0, 1) \) such that \( 1/p = (1 - \theta)/2 + \theta/p^* \). By the interpolation inequality, we find
\[ \|T(t) f - f\|_p \leq \|T(t) f - f\|^{1-\theta}_2 \|T(t) f - f\|^{\theta}_{p^*} \leq 2^\theta \|T(t) f - f\|^{1-\theta}_2 \|f\|^{\theta}_{p^*}. \]
It follows that \( \lim_{t \to 0} T(t)f = f \) in \( L^p(\mathbb{R}^d; C^m) \) for all \( f \in C_c^\infty(\mathbb{R}^d, C^m) \). By density, the strong continuity of the semigroup \( \{T_p(t)\}_{t \geq 0} \) follows.

Let us now turn to the generator of \( \{T_p(t)\}_{t \geq 0} \). Fix \( t > 0 \) and \( f \in C_c^\infty(\mathbb{R}^d; C^m) \). Then \( f \in D(A) \) and

\[
(4.2) \quad T(t)f - f = \int_0^t AT(s)f \, ds = \int_0^t T(s)Af \, ds,
\]

where the integral is computed in \( L^2(\mathbb{R}^d; C^m) \). However, \( Af \) has compact support whence \( Af \in L^p(\mathbb{R}^d; C^m) \) and the map \( t \mapsto T_p(t)Af \) is continuous from \([0, \infty)\) into \( L^p(\mathbb{R}^d; C^m) \). Hence, (4.2) holds true in \( L^p(\mathbb{R}^d; C^m) \), i.e.

\[
T_p(t)f - f = \int_0^t T_p(s)Af \, ds.
\]

This implies that \( t \mapsto T_p(t)f \) is differentiable in \([0, \infty)\). It follows that \( f \in D(A_p) \) and \( A_p f = Af \).

**Remark 4.4.** It is also possible to extend \( T \) to a consistent contraction semigroup \( \{T_1(t)\}_{t \geq 0} \) on \( L^1(\mathbb{R}^d; C^m) \). Mutatis mutandis, the proof is that of [16] Theorem 3.7.

### 5. Maximal domain of \( A_p \) and further properties

In this section we characterize the domain \( D(A_p) \) of the generator of \( \{T_p(t)\}_{t \geq 0} \). More precisely, we prove that it is the maximal domain in \( L^p \). We first show that the space of test functions is a core for \( A_p \).

**Proposition 5.1.** Let \( 1 < p < \infty \) and assume Hypotheses 2.7. Then,

(i) the set of test functions \( C_c^\infty(\mathbb{R}^d; C^m) \) is a core for \( A_p \),

(ii) the semigroup \( \{T_p(t)\}_{t \geq 0} \) is given by the Trotter–Kato product formula.

**Proof.**

(i) Fix \( 1 < p < \infty \). Since \( -A_p \) is m-accretive and the coefficients of \( A \) are real, it suffices to show that \( (1 - A_p)C_c^\infty(\mathbb{R}^d, \mathbb{R}^m) \) is dense in \( L^p(\mathbb{R}^d; \mathbb{R}^m) \). Let \( u \in L^p(\mathbb{R}^d; \mathbb{R}^m) \) be such that \( \langle (1 - A)\varphi, u \rangle_{p,p'} = 0 \) for all \( \varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R}^m) \). So,

\[
(5.1) \quad u - \Delta Q u + V^* u = 0
\]

in the sense of distributions. In particular,

\[
\Delta Q u_j = u_j + \sum_{l=1}^m v_{j,l} u_l \in L^p_{\text{loc}}(\mathbb{R}^d)
\]

for all \( j \in \{1, \ldots, m\} \). By local elliptic regularity, see [2] Theorem 7.1, \( u_j \in W^{2,p}_{\text{loc}}(\mathbb{R}^d) \) for all \( j \in \{1, \ldots, m\} \). Then, (5.1) holds almost everywhere on \( \mathbb{R}^d \).

Consider \( \zeta \in C_c^\infty(\mathbb{R}^d) \) such that \( \chi_{B(1)} \leq \zeta \leq \chi_{B(2)} \) and define \( \zeta_n(\cdot) = \zeta(\cdot/n) \) for \( n \in \mathbb{N} \). For \( p' < 2 \) we multiply equation (5.1) by \( \zeta_n(|u|^2 + \varepsilon^2)^{-2} u \in L^p(\mathbb{R}^d, \mathbb{R}^m) \) for \( \varepsilon > 0, n \in \mathbb{N} \). Integrating by parts, we obtain

\[
0 = \int_{\mathbb{R}^d} \zeta_n(|u|^2 + \varepsilon^2)^{-2} \frac{\partial}{\partial \xi} |u|^2 \, dx + \sum_{j=1}^m \int_{\mathbb{R}^d} \left( \nabla u_j, \nabla \left( \zeta_n(|u|^2 + \varepsilon^2)^{-2} u_j \right) \right) Q \, dx
\]

\[
+ \int_{\mathbb{R}^d} \zeta_n(|u|^2 + \varepsilon^2)^{-2} \langle V^* u, u \rangle \, dx
\]

\[
\geq \int_{\mathbb{R}^d} \zeta_n(|u|^2 + \varepsilon^2)^{-2} |u|^2 \, dx + \sum_{j=1}^m \int_{\mathbb{R}^d} |\nabla u_j|_{Q(\varepsilon)}^2 \zeta_n(|u|^2 + \varepsilon^2)^{-2} \, dx
\]

\[
+ \sum_{j=1}^m \int_{\mathbb{R}^d} \langle \nabla u_j, \nabla \zeta_n \rangle_Q |u|^2 + \varepsilon^2)^{-2} u_j \, dx
\]
\[ + (p' - 2) \sum_{j=1}^{m} \int_{\mathbb{R}^d} \langle \nabla u_j, \nabla |u| \rangle \zeta_n u_j |u| \zeta_n (|u|^2 + \varepsilon^2)^{\frac{p' - 2}{p'}} \, dx \]
\[ \geq \int_{\mathbb{R}^d} \zeta_n (|u|^2 + \varepsilon^2)^{\frac{p' - 2}{p'}} |u|^2 \, dx + \sum_{j=1}^{m} \int_{\mathbb{R}^d} |\nabla u_j|_{Q(x)}^2 \zeta_n (|u|^2 + \varepsilon^2)^{\frac{p' - 2}{p'}} \, dx \]
\[ + \int_{\mathbb{R}^d} \langle \nabla |u|, \nabla \zeta_n \rangle Q(|u|^2 + \varepsilon^2)^{\frac{p' - 2}{p'}} |u| \, dx \]
\[ + (p' - 2) \int_{\mathbb{R}^d} |\nabla |u| \zeta_n| u|^2 (|u|^2 + \varepsilon^2)^{\frac{p' - 2}{p'}} \, dx. \]

It follows now from (2.5) that
\[ 0 \geq \int_{\mathbb{R}^d} \zeta_n (|u|^2 + \varepsilon^2)^{\frac{p' - 2}{p'}} |u|^2 \, dx + \int_{\mathbb{R}^d} \langle \nabla |u|, \nabla \zeta_n \rangle Q(|u|^2 + \varepsilon^2)^{\frac{p' - 2}{p'}} |u| \, dx \]
\[ + (p' - 1) \int_{\mathbb{R}^d} |\nabla |u| \zeta_n|^2 |u|^2 \, dx + \frac{1}{p'} \int_{\mathbb{R}^d} \langle \nabla ((|u|^2 + \varepsilon^2)^{\frac{p'}{2}}), \nabla \zeta_n \rangle Q \, dx \]
\[ = \int_{\mathbb{R}^d} \zeta_n (|u|^2 + \varepsilon^2)^{\frac{p' - 2}{p'}} |u|^2 \, dx - \frac{1}{p'} \int_{\mathbb{R}^d} \Delta Q \zeta_n (|u|^2 + \varepsilon^2)^{\frac{p'}{2}} \, dx. \]

Upon \( \varepsilon \to 0 \), we find
\[ \int_{\mathbb{R}^d} \zeta_n |u|^{p'} \, dx - \frac{1}{p'} \int_{\mathbb{R}^d} \Delta \zeta_n |u|^{p'} \, dx \leq 0. \]

As in the proof of Proposition 4.1, upon \( n \to \infty \), we conclude that
\[ \int_{\mathbb{R}^d} |u|^{p'} \, dx \leq 0. \]

Therefore, \( u = 0 \).

In the case when \( p' > 2 \), one multiplies in (5.4) by \( \zeta_n |u|^{p'-2} u \) and argues in a similar way.

(ii) This is an immediate consequence of (i) and \cite[Corollary II-5.8]{8}. \( \square \)

In the next result we show that the domain \( D(A_p) \) is equal to the \( L^p \)-maximal domain of \( A \).

**Proposition 5.2.** Let \( 1 < p < \infty \). Assume Hypotheses 2.1. Then
\[ D(A_p) = \{ u \in L^p(\mathbb{R}^d, \mathbb{C}^m) \cap W^{2,p}_{\text{loc}}(\mathbb{R}^d, \mathbb{C}^m) : Au \in L^p(\mathbb{R}^d, \mathbb{C}^m) \} = D_{p,\text{max}}(A). \]

**Proof.** Let us show first that \( D(A_p) \subseteq D_{p,\text{max}}(A) \). Take \( u \in D(A_p) \). Since \( C_c^\infty(\mathbb{R}^d, \mathbb{C}^m) \) is a core for \( A_p \), it follows that there exists \( (u_n)_n \subseteq C_c^\infty(\mathbb{R}^d, \mathbb{C}^m) \) such that \( u_n \to u \) and \( Au_n \to A_p u \) in \( L^p(\mathbb{R}^d, \mathbb{C}^m) \), and in particular in \( L^p_{\text{loc}}(\mathbb{R}^d, \mathbb{C}^m) \). As \( V \in L^p_{\text{loc}}(\mathbb{R}^d, \mathbb{C}^m) \), we deduce that \( V u_n \to V u \) in \( L^p_{\text{loc}}(\mathbb{R}^d, \mathbb{C}^m) \). Consequently,
\[ \Delta Q u = A_p u + V u = \lim_{n \to \infty} Au_n + V u_n \in L^p_{\text{loc}}(\mathbb{R}^d, \mathbb{C}^m). \]

So, by local elliptic regularity, we obtain \( u \in W^{2,p}_{\text{loc}}(\mathbb{R}^d, \mathbb{C}^m) \). Hence, \( Au = A_p u \) belongs to \( L^p(\mathbb{R}^d, \mathbb{C}^m) \), which shows that \( u \in D_{p,\text{max}}(A) \).

In order to prove the other inclusion it suffices to show that \( \lambda - A \) is injective on \( D_{p,\text{max}}(A) \), for some \( \lambda > 0 \). To this end, let \( u \in D_{p,\text{max}}(A) \) such that \( (\lambda - A)u = 0 \). Assume that \( p \geq 2 \). Multiplying by \( \zeta_n |u|^{p-2} u \) and integrating (by part) over \( \mathbb{R}^d \) one obtains
\[ 0 = \lambda \int_{\mathbb{R}^d} \zeta_n(x) |u(x)|^p \, dx + \int_{\mathbb{R}^d} \sum_{j=1}^{m} \langle Q \nabla u_j, \nabla \zeta_n (|u|^{p-2} u_j \zeta_n) \rangle \, dx. \]
\[
\begin{align*}
+ \int_{\mathbb{R}^d} |V(x)u(x), u(x)|u(x)|^{p-2} \zeta_n(x)dx \\
\geq \lambda \int_{\mathbb{R}^d} \zeta_n(x)|u(x)|^p dx + \int_{\mathbb{R}^d} |u(x)|^{p-2} \zeta_n(x) \sum_{j=1}^m \langle Q(x) \nabla u_j(x), \nabla u_j(x) \rangle dx \\
+ \int_{\mathbb{R}^d} \sum_{j=1}^m |u(x)|^{p-2} u_j(x) \langle Q(x) \nabla u_j(x), \nabla \zeta_n(x) \rangle dx \\
+ (p-2) \int_{\mathbb{R}^d} |u(x)|^{p-2} \zeta_n(x) \langle Q(x) \nabla |u(x)|, \nabla |u(x)| \rangle dx \\
\geq \lambda \int_{\mathbb{R}^d} \zeta_n(x)|u(x)|^p dx + \int_{\mathbb{R}^d} |u(x)|^{p-1} \langle Q(x) \nabla |u(x)|, \nabla \zeta_n(x) \rangle dx \\
\geq \lambda \int_{\mathbb{R}^d} \zeta_n(x)|u(x)|^p dx + \frac{1}{p} \int_{\mathbb{R}^d} \langle Q(x) \nabla \zeta_n(x), \nabla |u(x)|^p \rangle dx \\
\geq \lambda \int_{\mathbb{R}^d} \zeta_n(x)|u(x)|^p dx - \frac{1}{p} \int_{\mathbb{R}^d} \Delta Q \zeta_n(x)|u(x)|^p dx.
\end{align*}
\]

So, as in the proof of the above proposition, we conclude that \( u = 0 \).

The case \( p < 2 \) can be obtained similarly, by multiplying the equation \( (\lambda - \mathcal{A})u = 0 \) by \( \zeta_n(|u|^2 + \varepsilon)^{\frac{p-2}{2}} u, \varepsilon > 0 \), instead of \( \zeta_n|u|^{p-2} u \).

We end this article by giving an example which shows that generation of \( C_0 \) semigroups for scalar-valued Schrödinger operators with complex potentials can be deduced from the vector-valued case developed in the previous sections.

**Example 5.3.** Let us consider the matrix potential
\[
V(x) := \begin{pmatrix} w(x) & -v(x) \\ v(x) & w(x) \end{pmatrix} = v(x) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + w(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
where \( v \in L^{\infty}_{\text{loc}}(\mathbb{R}^d) \) and \( 0 \leq w \in L^{\infty}_{\text{loc}}(\mathbb{R}^d) \). Then Hypotheses 2.1 are satisfied and we can deduce from Theorem 4.3 and Proposition 5.2 that \( \mathcal{A}_p \), the \( L^p \)-realization of the operator
\[
\mathcal{A} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} - V
\]
with domain \( \{ u \in L^p(\mathbb{R}^d; \mathbb{C}^2) \cap W^{2,p}_{\text{loc}}(\mathbb{R}^d; \mathbb{C}^2) : \mathcal{A}u \in L^p(\mathbb{R}^d; \mathbb{C}^2) \} \),
generates a \( C_0 \) semigroup on \( L^p(\mathbb{R}^d; \mathbb{C}^2) \). Moreover \( C_c^\infty(\mathbb{R}^d; \mathbb{R}^2) \) is a core for \( \mathcal{A}_p \).

Diagonalizing the matrix \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) we see that \( \mathcal{A}_p \) is similar to a diagonal operator.

More precisely, with \( P = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \) we have
\[
P^{-1} \mathcal{A}_p P = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} - \begin{pmatrix} iv + w & 0 \\ 0 & -iv + w \end{pmatrix}.
\]

It follows that the Schrödinger operators \( \Delta \pm iv - w \) with domain
\[
\{ f \in L^p(\mathbb{R}^d) \cap W^{2,p}_{\text{loc}}(\mathbb{R}^d) : \Delta f \pm ivf - wf \in L^p(\mathbb{R}^d) \}
\]
generate \( C_0 \) semigroups on \( L^p(\mathbb{R}^d) \). Moreover \( C_c^\infty(\mathbb{R}^d) \) is a core for this operator. In general, these semigroups can not be expected to be not analytic, see [16] Example 3.5. However, imposing additional assumptions on the potential \( V \), e.g. that the numerical range is contained in a sector, one can also prove analyticity of the semigroup, see [18] Proposition 4.5. More precisely, there we find the following result:

**Proposition 5.4.** Assume Hypotheses 2.1 and there is a positive constant \( C \) such that
\[
\text{Re} \langle V(x)\xi, \xi \rangle \geq C|\text{Im} \langle V(x)\xi, \xi \rangle|, \quad \forall x \in \mathbb{R}^d, \xi \in \mathbb{C}^m.
\]
Then the semigroup \( \{ T_t \}_{t \geq 0} \) can be extended to an analytic semigroup on \( L^p(\mathbb{R}^d, C^m) \).

Using this, we see that these semigroups are analytic provided that there is a constant \( C > 0 \) such that \( |v(x)| \leq Cw(x) \) for a.e. \( x \in \mathbb{R}^d \).

**References**

[1] D. Addona, L. Angiuli, L. Lorenzi, and G. Tessitore, On coupled systems of Kolmogorov equations with applications to stochastic differential games, ESAIM Control Optim. Calc. Var., 23 (2017), pp. 937–976.

[2] S. Agmon, The \( L_p \) approach to the Dirichlet problem. I. Regularity theorems, Ann. Scuola Norm. Sup. Pisa (3), 13 (1959), pp. 405–448.

[3] L. Angiuli, L. Lorenzi, and D. Pallara, \( L^p \)-estimates for parabolic systems with unbounded coefficients coupled at zero and first order, J. Math. Anal. Appl., 444 (2016), pp. 110–135.

[4] V. Betz, B. D. Goddard, and S. Teufel, Superadiabatic transitions in quantum molecular dynamics, Proc. R. Soc. A, 465 (2009), pp. 3553–3580.

[5] G. M. Dall’Ara, Discreteness of the spectrum of Schrödinger operators with non-negative matrix-valued potentials, J. Funct. Anal., 268 (2015), pp. 3649–3679.

[6] S. Delmonte and L. Lorenzi, On a class of weakly coupled systems of elliptic operators with unbounded coefficients, Milan J. Math., 79 (2011), pp. 689–727.

[7] K.-J. Engel and R. Nagel, One-parameter semigroups for linear evolution equations, vol. 194 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[8] One-parameter semigroups for linear evolution equations, vol. 194 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[9] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[10] T. Hansel and A. Rhandi, The Oseen-Navier-Stokes flow in the exterior of a rotating obstacle: The non-autonomous case, J. Reine Angew. Math., 694 (2014), pp. 1–26.

[11] F. Haslinger and B. Helffer, Compactness of the solution operator to \( \mathcal{J} \) in weighted \( L^p \)-spaces, J. Funct. Anal., 243 (2007), pp. 679–697.

[12] M. Hieber, L. Lorenzi, J. Prüss, A. Rhandi, and R. Schnaubelt, Global properties of generalized Ornstein-Uhlenbeck operators on \( L^p(\mathbb{R}^N, \mathbb{R}^N) \) with more than linearly growing coefficients, J. Math. Anal. Appl., 350 (2009), pp. 100–121.

[13] M. Hieber, A. Rhandi, and O. Sawada, The Navier-Stokes flow for globally Lipschitz continuous initial data, Res. Inst. Math. Sci. (RIMS), (2007), pp. 159–165. Kyoto Conference on the Navier-Stokes Equations and their Applications, RIMS Kyoikuho, B1. 127.

[14] M. Hieber and O. Sawada, The Navier-Stokes equations in \( \mathbb{R}^N \) with linearly growing initial data, Arch. Ration. Mech. Anal., 175 (2005), pp. 269–285.

[15] T. Kato, On some Schrödinger operators with a singular complex potential, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 5 (1978), pp. 105–114.

[16] M. Kunze, L. Lorenzi, A. Maichine, and A. Rhandi, \( L^p \)-theory for Schrödinger systems, preprint. arxiv.org/abs/1705.03333, 2017.

[17] L. Lorenzi, Analytical methods for Kolmogorov equations. Second edition, Monograph and research notes in Mathematics, Chapman & Hall/CRC, Boca Raton, FL, 2017.

[18] A. Maichine and A. Rhandi, On a polynomial scalar perturbation of a Schrödinger system in \( L^p \)-spaces, preprint. arxiv.org/abs/1802.02772, 2018.

[19] J. Prüss, A. Rhandi, and R. Schnaubelt, The domain of elliptic operators on \( L^p(\mathbb{R}^d) \) with unbounded drift coefficients, Houston J. Math., 32 (2006), pp. 563–576.
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