Approximation bounds for norm constrained neural networks with applications to regression and GANs

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Abstract

This paper studies the approximation capacity of ReLU neural networks with norm constraint on the weights. We prove upper and lower bounds on the approximation error of these networks for smooth function classes. The lower bound is derived through the Rademacher complexity of neural networks, which may be of independent interest. We apply these approximation bounds to analyze the convergences of regression using norm constrained neural networks and distribution estimation by GANs. In particular, we obtain convergence rates for over-parameterized neural networks. It is also shown that GANs can achieve optimal rate of learning probability distributions, when the discriminator is a properly chosen norm constrained neural network.

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1 Introduction

The expressiveness and approximation capacity of neural networks has been an active research area in the past few decades. The universal approximation property of shallow neural networks with one hidden layer and various activation functions was widely discussed in the 1990s [Cybenko, 1989; Hornik, 1991; Pinkus, 1999]. It was also shown that shallow neural networks can achieve attractive approximation rates for certain functions [Barron, 1993]. The recent breakthrough of deep learning has attracted much research on the approximation theory of deep neural networks. The approximation rates of ReLU deep neural networks have been well studied for many function classes, such as continuous functions [Yarotsky, 2017, 2018; Shen et al., 2020], smooth functions [Yarotsky and Zhevnerchuk, 2020; Lu et al., 2021], piecewise smooth functions [Petersen and Voigtlaender, 2018], shift-invariant spaces [Yang et al., 2022a] and band-limited functions [Montanelli et al., 2021].

In practice, neural network models are trained by minimizing certain loss functions on observed data. The approximation theory provides estimates on the bias of the model, while the sample complexity of the model controls how well it can generalize to unseen data by learning from finite observed samples [Anthony and Bartlett, 2009; Shalev-Shwartz and Ben-David, 2014; Mohri et al., 2018]. In modern applications, the number of training samples is often

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smaller than the number of weights in neural networks. For the generalization performance in this case, as pointed out by Bartlett [1998], the size of the weights is more important than the size of networks. The recent works [Neyshabur et al., 2015b; Bartlett et al., 2017; Golowich et al., 2020; Barron and Klusowski, 2019] also show that the sample complexity of deep neural networks can be controlled by certain norms of the weights. However, in the approximation theory literature, the approximation rates of deep neural networks are characterized by the number of weights [Yarotsky, 2017, 2018; Yarotsky and Zhevnerchuk, 2020] or the number of neurons [Shen et al., 2020; Lu et al., 2021], rather than the size of weights.

Besides, many regularization methods have been introduced to enforce Lipschitz constraint on neural networks (for example, spectral normalization [Miyato et al., 2018] and weight penalty [Brock et al., 2019]). It has been demonstrated that the Lipschitz constraint on neural networks can improve robustness to adversarial examples [Cisse et al., 2017], and stabilize the training of Generative Adversarial Networks (GAN, [Goodfellow et al., 2014; Arjovsky and Bottou, 2017; Arjovsky et al., 2017]). However, these regularization methods often make explicit or implicit restrictions on some norms of the weights, which largely reduce the expressive power of the models. For instance, Huster et al. [2019] showed that ReLU neural networks with certain constraints on the weights cannot represent some simple functions, such as the absolute value function. Hence, it is desirable to study how norm constrains on the weights affect the approximation capacity of neural networks.

In this paper, we give upper and lower bounds on the approximation error of ReLU neural networks with certain norm constrain on the weights for smooth function classes. To be concrete, let $\phi_\theta : \mathbb{R}^d \to \mathbb{R}$ be a function computed by a multi-layer ReLU neural network with width $W$ and depth $L$, where $\theta$ represents the collection of weights. In the $\ell$-th layer, the neural network computes an affine transformation $T_\ell(x) = A_\ell x + b_\ell$ and then applies the ReLU activation function element-wise (no activation in the output layer). When all the biases $b_\ell = 0$, it is natural to consider the constraint on the product of matrix norm $\prod_{\ell=0}^L \|A_\ell\| \leq K$, which controls the generalization ability [Bartlett et al., 2017; Golowich et al., 2020]. We generalize this idea to general bias $b_\ell$ and define the norm constraint $\kappa(\theta) \leq K$ as \eqref{eq:constraint}, which suitably constrains the bias. Our main results estimate the approximation error for Hölder continuous function $f \in \mathcal{H}^\alpha(\mathbb{R}^d)$ with smoothness index $\alpha > 0$. We show that if the width $W$ and depth $L$ are sufficiently large ($W$ needs to grow with $\sqrt{K}$), then it holds that

$$\sup_{f \in \mathcal{H}^\alpha} \inf_{\kappa(\theta) \leq K} \|f - \phi_\theta\|_{C([0,1]^d)} \lesssim K^{-\alpha/(d+1)}.$$  

In addition, if $d > 2\alpha$, then for any neural networks with width $W \geq 2$ and depth $L$,

$$\sup_{f \in \mathcal{H}^\alpha} \inf_{\kappa(\theta) \leq K} \|f - \phi_\theta\|_{C([0,1]^d)} \gtrsim (K\sqrt{L})^{-2\alpha/(d-2\alpha)}.$$  

The advantage of our approximation upper bound is that it only depends on the norm constraint so that it can be combined with the generalization bounds in [Bartlett et al., 2017; Golowich et al., 2020] and applied to over-parameterized neural networks. For comparison, in [Shen et al., 2020; Lu et al., 2021], the approximation error is bounded by the width and depth, but there is no restriction on the weights. Yarotsky [2017, 2018]; Yarotsky and Zhevnerchuk [2020] obtained approximation bounds in terms of the number of non-zero weights. Although this can be regarded as the sum of zero-norm of the weights, it is more like a constraint on the network architecture, rather than a constraint on the size of the weights. In [Petersen and Voigtlaender, 2018; Bölcskei et al., 2019; Schmidt-Hieber, 2021], the authors also provide
approximation results of deep neural networks with bound on the maximum value of the weights. But these bounds can not directly control the generalization. On the contrary, our norm constraint provides a bound on the Rademacher complexity of the network (see Lemma 2.3 and Golowich et al. [2020]).

To illustrate the application of the approximation bounds, we study the regression problem of estimating an unknown function \( f_0 \in \mathcal{H}^\alpha \) from its noisy samples. Combining the empirical process theory with our approximation bounds, we can estimate the convergence rate of the empirical risk minimization using norm constrained neural networks. In particular, we obtain convergence rates for over-parameterized neural networks, which give statistical guarantee for neural networks used in practice. We also apply our results to generative adversarial networks. It is shown that, if a properly chosen norm constrained neural network is used as the discriminator, GAN is able to achieve the optimal convergence rate of learning probability distributions.

The rest of the paper is organized as follows. In Section 2, we define the norm constraint on neural networks and give some preliminary results. Section 3 presents and proves our main results on the approximation bounds for norm constrained neural networks. In Section 4, we apply our results to study the convergence rates of two machine learning algorithms. Finally, Section 5 concludes this paper with a discussion on possible future directions of research.

1.1 Notation

The set of positive integers is denoted by \( \mathbb{N} := \{1, 2, \ldots \} \). For convenience, we also use the notation \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). The cardinality of a set \( S \) is denoted by \( |S| \). We use \( \|x\|_p \) to denote the \( p \)-norm of a vector \( x \in \mathbb{R}^d \). For a multi-index \( s = (s_1, \ldots, s_d) \in \mathbb{N}_0^d \), the symbol \( \partial^s \) denotes the partial differential operator \( \partial^s := (\frac{\partial}{\partial x_1})^{s_1} \cdots (\frac{\partial}{\partial x_d})^{s_d} \) and we use the convention that \( \partial^0 \) is the identity operator when \( s = 0 \). If \( X \) and \( Y \) are two quantities, we denote \( X \land Y := \min\{X, Y\} \) and \( X \lor Y := \max\{X, Y\} \). We use \( X \lesssim Y \) or \( Y \gtrsim X \) to denote the statement that \( X \leq CY \) for some constant \( C > 0 \). We denote \( X \asymp Y \) when \( X \lesssim Y \lesssim X \). Finally, we introduce the covering number and packing number to measure the complexity of a set in a metric space.

**Definition 1.1** (Covering and Packing numbers). Let \( \rho \) be a metric on \( \mathcal{M} \) and \( S \subseteq \mathcal{M} \). For \( \epsilon > 0 \), a set \( T \subseteq \mathcal{M} \) is called an \( \epsilon \)-covering (or \( \epsilon \)-net) of \( S \) if for any \( x \in S \) there exists \( y \in T \) such that \( \rho(x, y) \leq \epsilon \). A subset \( U \subseteq S \) is called an \( \epsilon \)-packing of \( S \) (or \( \epsilon \)-separated) if any two elements \( x \neq y \) in \( U \) satisfy \( \rho(x, y) > \epsilon \). The \( \epsilon \)-covering and \( \epsilon \)-packing numbers of \( S \) are denoted respectively by

\[
\mathcal{N}_c(S, \rho, \epsilon) := \min\{|T| : T \text{ is an } \epsilon\text{-covering of } S\},
\]
\[
\mathcal{N}_p(S, \rho, \epsilon) := \max\{|U| : U \text{ is an } \epsilon\text{-packing of } S\}.
\]

It is not hard to check that \( \mathcal{N}_p(S, \rho, 2\epsilon) \leq \mathcal{N}_c(S, \rho, \epsilon) \leq \mathcal{N}_p(S, \rho, \epsilon) \).
2 Neural networks with norm constraints

Let \( L, N_1, \ldots, N_L \in \mathbb{N} \). We consider the function \( \phi : \mathbb{R}^d \to \mathbb{R}^k \) that can be parameterized by a ReLU neural network of the form

\[
\begin{align*}
\phi_0(x) &= x, \\
\phi_{\ell+1}(x) &= \sigma(A_{\ell}\phi_{\ell}(x) + b_{\ell}), \quad \ell = 0, \ldots, L - 1, \quad (2.1) \\
\phi(x) &= A_L\phi_L(x),
\end{align*}
\]

where \( A_{\ell} \in \mathbb{R}^{N_{\ell+1} \times N_{\ell}} \), \( b_{\ell} \in \mathbb{R}^{N_{\ell+1}} \) with \( N_0 = d \) and \( N_{L+1} = k \). The activation function \( \sigma(x) := x \lor 0 \) is the Rectified Linear Unit function (ReLU, [Nair and Hinton, 2010]) and it is applied element-wise. The numbers \( W := \max\{N_1, \ldots, N_L\} \) and \( L \) are called the width and depth of the neural network, respectively. We denote by \( \mathcal{N}(W, L) \) the set of functions that can be parameterized by ReLU neural networks with width \( W \) and depth \( L \). When the input dimension \( d \) and output dimension \( k \) are clear from contexts, we simply denote it by \( \mathcal{N}(W, L) \). Sometimes, we will use the notation \( \phi_{\theta} \in \mathcal{N}(W, L) \) to emphasize that the neural network function \( \phi_{\theta} \) is parameterized by \( \theta := ((A_0, b_0), \ldots, (A_{L-1}, b_{L-1}), A_L) \).

Next, we introduce a special class of neural network functions \( S\mathcal{N}(W, L) \) which contains functions of the form

\[
\tilde{\phi}(x) = \tilde{A}_L\sigma(\tilde{A}_{L-1}\sigma(\cdots\sigma(\tilde{A}_0\tilde{x})))) , \quad \tilde{x} := \begin{pmatrix} x \\ 1 \end{pmatrix}, \quad (2.2)
\]

where \( \tilde{A}_{\ell} \in \mathbb{R}^{N_{\ell+1} \times N_{\ell}} \) with \( N_0 = d + 1 \) and \( \max\{N_1, \ldots, N_L\} = W \). Since these functions can also be written in the form (2.1) with \( b_{\ell} = 0 \) for all \( 1 \leq \ell \leq L - 1 \), we know that \( S\mathcal{N}(W, L) \subseteq \mathcal{N}(W, L) \). There is a natural way to introduce norm constraint on the weights: for any \( K \geq 0 \), we denote by \( S\mathcal{N}(W, L, K) \) the set of functions in the form (2.2) that satisfies

\[
\prod_{\ell=0}^L \| \tilde{A}_{\ell} \| \leq K,
\]

where \( \|A\| \) is some norm of a matrix \( A = (a_{i,j}) \in \mathbb{R}^{m \times n} \) and, for simplicity, we only consider the operator norm defined by \( \|A\| := \sup_{\|x\|_\infty \leq 1} \|Ax\|_\infty \) in this paper. It is well-known that \( \|A\| \) is the maximum 1-norm of the rows of \( A \):

\[
\|A\| = \max_{1 \leq i \leq m} \sum_{j=1}^n |a_{i,j}|.
\]

Hence, we make a constraint on the 1-norm of the incoming weights of each neuron.

To introduce norm constraint for the class \( \mathcal{N}(W, L) \), we observe that any \( \phi \in \mathcal{N}(W, L) \) parameterized as (2.1) can be written in the form (2.2) with

\[
\tilde{A}_L = (A_L, 0), \quad \tilde{A}_{\ell} = \begin{pmatrix} A_{\ell} & b_{\ell} \\ 0 & 1 \end{pmatrix} , \quad \ell = 0, \ldots, L - 1,
\]

and

\[
\prod_{\ell=0}^L \| \tilde{A}_{\ell} \| = \| A_L \| \prod_{\ell=0}^{L-1} \max\{ \| (A_{\ell}, b_{\ell}) \|, 1 \} . \quad (2.3)
\]
Hence, we define the norm constrained neural network $\mathcal{NN}(W, L, K)$ as the set of functions $\phi_\theta \in \mathcal{NN}(W, L)$ of the form (2.1) that satisfies the following norm constraint on the weights

$$\kappa(\theta) := \|A_L\| \prod_{\ell=0}^{L-1} \max\{\|(A_\ell, b_\ell)\|, 1\} \leq K. \quad (2.4)$$

The following proposition summarizes the relation between the two neural network classes $\mathcal{NN}(W, L, K)$ and $\mathcal{SNN}(W, L, K)$. It shows that we can essentially regard these two classes as the same when studying their expressiveness.

**Proposition 2.1.** $\mathcal{SNN}(W, L, K) \subseteq \mathcal{NN}(W, L, K) \subseteq \mathcal{SNN}(W + 1, L, K)$.

**Proof.** By the definition (2.4) and the relation (2.3), it is easy to see that $\mathcal{NN}(W, L, K) \subseteq \mathcal{SNN}(W + 1, L, K)$. Conversely, for any $\hat{\phi} \in \mathcal{SNN}(W, L, K)$ of the form (2.2), by the absolute homogeneity of the ReLU function, we can always rescale $\hat{A}_\ell$ such that $\|\hat{A}_\ell\| \leq K$ and $\|\hat{A}_\ell\| = 1$ for $\ell \neq L$. Since the function $\hat{\phi}$ can also be parameterized in the form (2.1) with $\theta = (\hat{A}_0, (\hat{A}_1, 0), \ldots, (\hat{A}_{L-1}, 0), \hat{A}_L)$ and $\kappa(\theta) = \prod_{\ell=0}^{L} \|\hat{A}_\ell\| \leq K$, we have $\hat{\phi} \in \mathcal{NN}(W, L, K)$.

The sample complexity of $\mathcal{SNN}(W, L, K)$ has been studied in the recent works [Neyshabur et al., 2015b, 2018; Bartlett et al., 2017; Golowich et al., 2020]. By Proposition 2.1, these sample complexity bounds can also be applied to $\mathcal{NN}(W, L, K)$. We will use the Rademacher complexity to derive lower bounds for the approximation capacity of norm constrained neural networks.

**Definition 2.2** (Rademacher complexity). Given a set $S \subseteq \mathbb{R}^n$, the Rademacher complexity of $S$ is denoted by

$$\mathcal{R}_n(S) := \mathbb{E}_{\xi_{1:n}} \left[ \sup_{(s_1, \ldots, s_n) \in S} \frac{1}{n} \sum_{i=1}^{n} \xi_i s_i \right],$$

where $\xi_{1:n} = \{\xi_i\}_{i=1}^{n}$ is a sequence of i.i.d. Rademacher random variables which take the values 1 and $-1$ with equal probability $1/2$.

**Lemma 2.3.** For any $x_1, \ldots, x_n \in [-B, B]^d$ with $B \geq 1$, let $S := \{(\phi(x_1), \ldots, \phi(x_n)) : \phi \in \mathcal{SNN}_{d,1}(W, L, K)\} \subseteq \mathbb{R}^n$, then

$$\mathcal{R}_n(S) \leq \frac{1}{n} K \sqrt{2(L + 2 + \log(d + 1))} \max_{1 \leq j \leq d + 1} \sqrt{\sum_{i=1}^{n} x_{i,j}^2} \leq B K \sqrt{2(L + 2 + \log(d + 1))},$$

where $x_{i,j}$ is the $j$-th coordinate of the vector $\tilde{x}_i = (x_{i,1}, 1) \in \mathbb{R}^{d+1}$. When $W \geq 2$,

$$\mathcal{R}_n(S) \geq \frac{K}{2} \sqrt{2/n} \max_{1 \leq j \leq d + 1} \sqrt{\sum_{i=1}^{n} x_{i,j}^2} \geq \frac{K}{2} \sqrt{2/n}.$$

**Proof.** The upper bound is from Golowich et al. [2020, Theorem 3.2].

For the lower bound, we consider the linear function class $\mathcal{F} := \{x \mapsto a^T \hat{x} : a \in \mathbb{R}^{d+1}, \|a\|_1 \leq K/2\}$. Observing that $a^T \tilde{x} = \sigma(a^T \tilde{x}) - \sigma(-a^T \tilde{x})$, we conclude that $\mathcal{F} \subseteq$
Proposition 2.4 (Rescaling). Every \( \phi \in \mathcal{N}(W, L, K) \) can be written in the form (2.1) such that \( \|A_L\| \leq K \) and \( \|(A_\ell, b_\ell)\| \leq 1 \) for \( 0 \leq \ell \leq L - 1 \).

Proof. We first parameterize \( \phi \) in the form (2.1) and denote \( k_\ell := \max\{\|(A_\ell, b_\ell)\|, 1\} \) for all \( 0 \leq \ell \leq L - 1 \). We let \( \tilde{A}_\ell = A_\ell / k_\ell \), \( \tilde{b}_\ell = b_\ell / (\prod_{i=0}^{\ell-1} k_i) \), \( \tilde{A}_L = A_L / \prod_{i=0}^{L-1} k_i \) and consider the new parameterization of \( \phi \):

\[
\tilde{\phi}_{\ell+1}(x) = \sigma(\tilde{A}_\ell \tilde{\phi}_\ell(x) + \tilde{b}_\ell), \quad \tilde{\phi}_{0}(x) = x.
\]

It is easy to check that \( \|\tilde{A}_L\| \leq K \) and

\[
\|(A_\ell, b_\ell)\| = \frac{1}{k_\ell} \left\| \left( A_\ell \frac{b_\ell}{\prod_{i=0}^{\ell-1} k_i} \right) \right\| \leq \frac{1}{k_\ell} \|(A_\ell, b_\ell)\| \leq 1,
\]

where the second inequality is due to \( k_\ell \geq 1 \).

Next, we show that \( \phi_\ell(x) = \left( \prod_{i=0}^{\ell-1} k_i \right) \tilde{\phi}_\ell(x) \) by induction. For \( \ell = 1 \), by the absolute homogeneity of the ReLU function,

\[
\phi_1(x) = \sigma(A_0 x + b_0) = k_0 \sigma(\tilde{A}_0 x + \tilde{b}_0) = k_0 \tilde{\phi}_1(x).
\]

Inductively, one can conclude that

\[
\phi_{\ell+1}(x) = \sigma(A_\ell \phi_\ell(x) + b_\ell) = \left( \prod_{i=0}^{\ell} k_i \right) \sigma \left( \tilde{A}_\ell \tilde{\phi}_\ell(x) + \tilde{b}_\ell \right) = \left( \prod_{i=0}^{\ell} k_i \right) \tilde{\phi}_{\ell+1}(x),
\]

where the last inequality is due to Khintchine inequality, see Ledoux and Talagrand [1991, Lemma 4.1] and Haagerup [1981].
where the third equality is due to induction. Therefore,
\[ \phi(x) = A_L \phi_L(x) = A_L \left( \prod_{i=0}^{L-1} k_i \right) \tilde{\phi}_L(x) = \tilde{A}_L \phi_L(x), \]
which means \( \phi \) can be parameterized by \((\tilde{A}_0, \tilde{b}_0), \ldots, (\tilde{A}_{L-1}, \tilde{b}_{L-1}), \tilde{A}_L)\) and we finish the proof. \( \square \)

In the following proposition, we summarize some basic operations on neural networks. These operations will be useful for the construction of neural networks, when we study the approximation capacity.

**Proposition 2.5.** Let \( \phi_1 \in \mathcal{NN}_{d_1,k_1}(W_1, L_1, K_1) \) and \( \phi_2 \in \mathcal{NN}_{d_2,k_2}(W_2, L_2, K_2) \).

(i) If \( d_1 = d_2, k_1 = k_2, W_1 \leq W_2, L_1 \leq L_2 \) and \( K_1 \leq K_2 \), then \( \mathcal{NN}_{d_1,k_1}(W_1, L_1, K_1) \subseteq \mathcal{NN}_{d_2,k_2}(W_2, L_2, K_2) \).

(ii) (Composition) If \( k_1 = d_2 \), then \( \phi_2 \circ \phi_1 \in \mathcal{NN}_{d_1,k_2}(\max\{W_1, W_2\}, L_1 + L_2, K_2 \max\{K_1, 1\}) \).

Let \( A \in \mathbb{R}^{d_2 \times d_1} \) and \( b \in \mathbb{R}^{d_2} \). Define the function \( \phi(x) := \phi_2(Ax + b) \) for \( x \in \mathbb{R}^{d_1} \), then \( \phi \in \mathcal{NN}_{d_1,k_2}(W_2, L_2, K_2 \max\{\|A, b\|, 1\}) \).

(iii) (Concatenation) If \( d_1 = d_2 \), define \( \phi(x) := (\phi_1(x), \phi_2(x)) \), then \( \phi \in \mathcal{NN}_{d_1,k_1+k_2}(W_1 + W_2, L_1 + L_2, \max\{K_1, K_2\}) \).

(iv) (Linear Combination) If \( d_1 = d_2 \) and \( k_1 = k_2 \), then, for any \( c_1, c_2 \in \mathbb{R}, c_1 \phi_1 + c_2 \phi_2 \in \mathcal{NN}_{d_1,k_2}(W_1 + W_2, \max\{L_1, L_2\}, |c_1|K_1 + |c_2|K_2) \).

**Proof.** By Proposition 2.4, we can parameterize \( \phi_i, i = 1, 2, \) in the form (2.1) with parameters \(((A_0^{(i)}, b_0^{(i)}), \ldots, (A_{L_i-1}^{(i)}, b_{L_i-1}^{(i)}), A_{L_i}^{(i)})\) such that \( \|A_{L_i}^{(i)}\| \leq K_i \) and \( \|A_{L_i}^{(i)}, b_{L_i}^{(i)}\| \leq 1 \) for \( \ell \neq L_i \).

(i) We can assume that \( A_1^{(1)} \in \mathbb{R}^{W_2 \times W_1} \) and \( b_1^{(1)} \in \mathbb{R}^{W_1} \), \( 0 \leq \ell \leq L_1 - 1 \), by adding suitable zero rows and columns to \( A_1^{(1)} \) and \( b_1^{(1)} \) if necessary (this operation does not change the norm). Then, \( \phi_1 \) can also be parameterized by the parameters
\[
\begin{pmatrix}
A_0^{(1)}, b_0^{(1)}
A_1^{(1)}, b_1^{(1)}
\vdots
A_{L_1-1}^{(1)}, b_{L_1-1}^{(1)}
\end{pmatrix},
\]
where \( \text{Id} \) is the identity matrix. Hence, \( \phi_1 \in \mathcal{NN}_{d_2,k_2}(W_2, L_2, K_2) \).

(ii) By (i), we can assume \( W_1 = W_2 \) without loss of generality. Then, \( \phi_2 \circ \phi_1 \) can be parameterized by
\[
\begin{pmatrix}
A_0^{(1)}, b_0^{(1)}
A_1^{(1)}, b_1^{(1)}
A_{L_1-1}^{(1)}, b_{L_1-1}^{(1)}
A_0^{(2)}, b_0^{(2)}
A_1^{(2)}, b_1^{(2)}
\vdots
A_{L_2-1}^{(2)}, b_{L_2-1}^{(2)}
A_{L_1}^{(1)}, b_{L_1}^{(1)}
A_{L_2}^{(2)}
\end{pmatrix}.
\]

We observe that
\[
\|A_0^{(2)} A_1^{(1)}, b_0^{(2)}\| = \|A_0^{(2)}, b_0^{(2)}\| A_1^{(1)} 0 = \|A_2^{(2)}, b_0^{(2)}\| A_1^{(1)} 0 \leq \max\{K_1, 1\}.
\]

Hence, \( \phi_2 \circ \phi_1 \in \mathcal{NN}_{d_1,k_2}(W_1, L_1 + L_2, K_2 \max\{K_1, 1\}) \).
For the function $\phi(x) := \phi_2(Ax + b)$, we can similarly parameterize it by
\[
\left(\left( A_0^{(2)} A_0^{(2)} b + b_0^{(2)} \right), \left( A_1^{(2)} b_1^{(2)} \right), \ldots, \left( A_{L_2-1}^{(2)} b_{L_2-1}^{(2)} \right), A_{L_2}^{(2)} \right).
\]
Using
\[
\left\| \left( A_0^{(2)} A_0^{(2)} b + b_0^{(2)} \right) \right\| = \left\| \left( A_0^{(2)} b_0^{(2)} \right) \left( \begin{array}{c} A_0^1 \\ 0 \\ \end{array} \right) \right\| \leq \max\{\|(A, b)\|, 1\},
\]
we conclude that $\phi \in \mathcal{N}(W_2, L_2, K_2 \max\{\|(A, b)\|, 1\})$.

(iii) By (i), we can assume that $L_1 = L_2$. Then, $\phi$ can be parameterized by the parameters $(A_0, b_0), \ldots, (A_{L_1-1}, b_{L_1-1}, A_{L_1})$ where
\[
A_0 := \left( A_0^{(1)} \right), b_0 := \left( b_0^{(1)} \right), \quad A_\ell := \left( A_\ell^{(1)} \right), b_\ell := \left( b_\ell^{(1)} \right), \ell \neq 0.
\]
Notice that $\|A_{L_1}\| = \max\{\|A_{L_1}^{(1)}\|, \|A_{L_1}^{(2)}\|\} \leq \max\{K_1, K_2\}$ and
\[
\left\| (A_0, b_0) \right\| = \left\| \left( A_0^{(1)} b_0^{(1)} \right) \right\| \leq 1,
\]
\[
\left\| (A_\ell, b_\ell) \right\| = \left\| \left( A_\ell^{(1)} b_\ell^{(1)} \right) \right\| \leq 1, \quad 0 < \ell < L_1.
\]

(iv) Replacing the matrix $A_{L_1}$ in (iii) by $(c_1 A_{L_1}^{(1)} , c_2 A_{L_1}^{(2)})$, the conclusion follows from
\[
\left\| (c_1 A_{L_1}^{(1)} , c_2 A_{L_1}^{(2)}) \right\| \leq |c_1| \left\| A_{L_1}^{(1)} \right\| + |c_2| \left\| A_{L_1}^{(2)} \right\| \leq |c_1| K_1 + |c_2| K_2.
\]

\section{Approximation of smooth functions}

In this section, we study how well norm constrained neural networks approximate smooth functions. To begin with, let us introduce the notion of regularity of functions.

\textbf{Definition 3.1} (Hölder classes). Let $d \in \mathbb{N}$ and $\alpha = r + \beta > 0$, where $r \in \mathbb{N}_0$ and $\beta \in (0, 1]$. We denote the Hölder class $\mathcal{H}^\alpha(\mathbb{R}^d)$ as
\[
\mathcal{H}^\alpha(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \to \mathbb{R}, \max_{\|s\| \leq r} \sup_{x \in \mathbb{R}^d} |\partial^s f(x)| \leq 1, \max_{\|s\|=r} \sup_{x \neq y} \frac{|\partial^s f(x) - \partial^s f(y)|}{\|x - y\|^{\beta}} \leq 1 \right\},
\]
where the multi-index $s \in \mathbb{N}_0^d$. Denote $\mathcal{H}^\alpha := \{ f : [0, 1]^d \to \mathbb{R}, f \in \mathcal{H}^\alpha(\mathbb{R}^d) \}$ as the restriction of $\mathcal{H}^\alpha(\mathbb{R}^d)$ to $[0, 1]^d$.

It should be noticed that for $\alpha = r + 1$, we do not assume that $f \in C^r$. Instead, we only require that $f \in C^r$ and its derivatives of order $r$ are Lipschitz continuous. In particular, when $\alpha = 1$, $\mathcal{H}^1$ is the set of bounded 1-Lipschitz continuous functions:
\[
\|f\|_{L^\infty} \leq 1 \text{ and Lip } (f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|} \leq 1.
\]
We will also denote Lip 1 := $\{ f : \text{ Lip } (f) \leq 1 \}$ for convenience. Thus, $\mathcal{H}^1 \subseteq \text{ Lip } 1$. 


Since the ReLU function is 1-Lipschitz, it is easy to see that, for any $\phi_\theta \in \mathcal{NN}(W, L, K)$,

$$\text{Lip} (\phi_\theta) \leq \kappa(\theta) \leq K.$$

However, it was shown by Huster et al. [2019] that some simple 1-Lipschitz functions, such as $f(x) = |x|$, cannot be represented by $\mathcal{NN}(W, L, K)$ for any $K < 2$. Their result implies that norm constrained neural networks have a restrictive expressive power. Nevertheless, since two-layer neural networks are universal, $\mathcal{NN}(W, L, K)$ can approximate any continuous functions when $W$ and $K$ are sufficiently large. In the following, we will try to quantify the approximation error

$$E(\mathcal{H}^\alpha, \mathcal{NN}(W, L, K)) := \sup_{f \in \mathcal{H}^\alpha} \inf_{\phi \in \mathcal{NN}(W, L, K)} \|f - \phi\|_{C([0, 1]^d)},$$

where $C([0, 1]^d)$ is the space of continuous functions on $[0, 1]^d$ equipped with the sup-norm. Our main results can be summarized in the following theorem.

**Theorem 3.2.** Let $d \in \mathbb{N}$ and $\alpha = r + \beta > 0$, where $r \in \mathbb{N}_0$ and $\beta \in (0, 1]$.

1. There exists $c > 0$ such that for any $K \geq 1$, any $W \geq cK(2d + \alpha)/(2d + 2)$ and $L \geq 2[\log_2(d + r)] + 2$,

$$E(\mathcal{H}^\alpha, \mathcal{NN}(W, L, K)) \lesssim K^{-\alpha/(d + 1)}.$$

2. If $d > 2\alpha$, then for any $W, L \in \mathbb{N}$, $W \geq 2$ and $K \geq 1$,

$$E(\mathcal{H}^\alpha, \mathcal{NN}(W, L, K)) \gtrsim (K\sqrt{L})^{-2\alpha/(d - 2\alpha)}.$$

We note that the (implied) constants in the theorem only depend on $d$ and $\alpha$. We also note that the lower bound is derived from the upper bound of Rademacher complexity in Lemma 2.3, which is independent of the width $W$. Notice that the lower bound of Rademacher complexity in Lemma 2.3 is also independent of the depth $L$. When assuming more control over Schatten norm of the parameter matrices, Golowich et al. [2020] obtained sample complexity upper bounds that are independent of the size of neural networks. Consequently, one can obtain size-independent lower bound of approximation error for such neural networks.

### 3.1 Upper bounds

The upper bound in Theorem 3.2 is proved by an explicit construction of norm constrained neural networks that approximate the local Taylor polynomials. Following the constructions in [Yarotsky, 2017, 2018; Yarotsky and Zhevnerchuk, 2020; Lu et al., 2021], we first consider the approximation of the quadratic function $f(x) = x^2$ and then extend the approximation to monomials.

**Lemma 3.3.** For any $k \in \mathbb{N}$, there exists $\phi_k \in \mathcal{NN}(k, 1, 3)$ such that $\phi_k(x) = 0$ for $x \leq 0$, $\phi_k(x) \in [0, 1]$ for $x \in [0, 1]$ and

$$|x^2 - \phi_k(x)| \leq \frac{1}{2k^2}, \quad x \in [0, 1].$$

**Proof.** The construction is based on the integral representation of $x^2$:

$$x^2 = \int_0^x 2x - 2bdx = \int_0^x 2\sigma(x - b)db = \int_0^1 2\sigma(x - b)db, \quad x \in [0, 1]. \quad (3.1)$$
We can approximate the integral by Riemann sum. For any $k \in \mathbb{N}$, define

$$\phi_k(x) = \frac{1}{k} \sum_{i=1}^{k} 2\sigma \left( x - \frac{2i - 1}{2k} \right).$$

Then, by Proposition 2.5, $\phi_k \in \mathcal{NN}(k, 1, K)$ with

$$K = \frac{2}{k} \left( 1 + \frac{2i - 1}{2k} \right) = 3.$$

It is easy to see that $\phi_k(x) = 0$ for $x \leq 0$. Since $\phi_k$ is an increasing function, we have $0 = \phi_k(0) \leq \phi_k(x) \leq \phi_k(1) = 1$ for $x \in [0, 1]$.

For any $x \in (0, 1]$, let us denote $i_x = \left\lfloor kx \right\rfloor \in \{1, \ldots, k\}$, then $x \in ((i_x - 1)/k, i_x/k]$. If $i < i_x$, then

$$\int_{(i-1)/k}^{i/k} 2\sigma(x - b)db = \int_{(i-1)/k}^{i/k} 2x - 2b db = \frac{2x}{k} - \frac{2i - 1}{k^2} = \frac{2}{k} \sigma \left( x - \frac{2i - 1}{2k} \right).$$

If $i > i_x$, then

$$\int_{(i-1)/k}^{i/k} 2\sigma(x - b)db = 0 = \frac{2}{k} \sigma \left( x - \frac{2i - 1}{2k} \right).$$

Therefore,

$$\left| x^2 - \phi_k(x) \right| = \sum_{i=1}^{k} \int_{(i-1)/k}^{i/k} 2\sigma(x - b)db - \sum_{i=1}^{k} \frac{2}{k} \sigma \left( x - \frac{2i - 1}{2k} \right) \left| \int_{(i-1)/k}^{i/k} 2\sigma(x - b) - 2\sigma \left( x - \frac{2i_x - 1}{2k} \right) db \right| \leq \int_{(i-1)/k}^{i/k} \left| b - \frac{2i_x - 1}{2k} \right| db = \frac{1}{2k^2},$$

where we use the Lipschitz continuity of ReLU in the inequality.

**Remark 3.4.** Our construction is based on the integral representation (3.1), which can be regarded as an infinite width neural network. This construction is different from the construction in Yarotsky [2017], which use the teeth function $T_i = T_1 \circ T_{i-1} = T_1 \circ \cdots \circ T_1$ to construct the approximator

$$f_k(x) = x - \sum_{i=1}^{k} 4^{-i} T_i(x),$$

where $T_i(x) = 2x$ for $x \in [0, 1/2]$ and $T_i(x) = 2(1 - x)$ for $x \in [1/2, 1]$. It can be shown that $f_k$ achieves the approximation error $|x^2 - f_k(x)| \leq 2^{-(k+1)}$. Since $T_1 \in \mathcal{NN}(2, 2, 7)$, by Proposition 2.5, this compositional property implies $T_i \in \mathcal{NN}(2, 2i, 7^i)$ and consequently one can show that $f_k \in \mathcal{NN}(2k + 1, 2k, \frac{1}{4} (\frac{7}{4})^{k+1} - \frac{1}{4})$. Hence, in the construction of Yarotsky [2017], the approximation error decays exponentially with the depth but only polynomially with the norm constraint $K$. On the contrary, in our construction, the network has a finite norm constraint but the approximation error decays only quadratically on the width.
Using the relation $xy = 2\left(\left(\frac{x+y}{2}\right)^2 - \left(\frac{x}{2}\right)^2 - \left(\frac{y}{2}\right)^2\right)$, we can approximate the product function by neural networks and then further approximate any monomials $x_1 \cdots x_d$.

**Lemma 3.5.** For any $k \in \mathbb{N}$, there exists $\psi_k \in \mathcal{NN}(6k, 2, 216)$ such that $\psi_k : [-1, 1]^2 \to [-1, 1]$ and

$$|xy - \psi_k(x, y)| \leq \frac{3}{k^2}, \quad x, y \in [-1, 1].$$

Furthermore, $\psi_k(x, y) = 0$ if $xy = 0$.

**Proof.** Let $\phi_k \in \mathcal{NN}(k, 1, 3)$ be the network in Lemma 3.3 and define $\tilde{\phi}_k(x) = \phi_k(x) + \phi_k(-x)$.

By Proposition 2.5, $\tilde{\phi}_k \in \mathcal{NN}(2k, 1, 6)$. Since $\phi_k(x) = 0$ for $x \leq 0$, we have $\tilde{\phi}_k(x) = \phi_k(|x|)$ and the approximation error is

$$|x^2 - \tilde{\phi}_k(x)| = |x^2 - \phi_k(|x|)| \leq \frac{1}{2k^2}, \quad x \in [-1, 1].$$

Using the fact that $xy = 2\left(\left(\frac{x+y}{2}\right)^2 - \left(\frac{x}{2}\right)^2 - \left(\frac{y}{2}\right)^2\right)$, we consider the function

$$\tilde{\psi}_k(x, y) := 2\tilde{\phi}_k\left(\frac{x+y}{2}\right) - 2\tilde{\phi}_k\left(\frac{x}{2}\right) - 2\tilde{\phi}_k\left(\frac{y}{2}\right).$$

Then, $\tilde{\psi}_k(x, y) = 0$ if $xy = 0$, and, for any $x, y \in [-1, 1]$,

$$|xy - \tilde{\psi}_k(x, y)| \leq 2\left|\left(\frac{x+y}{2}\right)^2 - \tilde{\phi}_k\left(\frac{x+y}{2}\right)\right| + 2\left|\left(\frac{x}{2}\right)^2 - \tilde{\phi}_k\left(\frac{x}{2}\right)\right| + 2\left|\left(\frac{y}{2}\right)^2 - \tilde{\phi}_k\left(\frac{y}{2}\right)\right| \leq \frac{3}{k^2}.$$

By Proposition 2.5, $\tilde{\psi}_k \in \mathcal{NN}(6k, 1, 36)$.

Finally, let $\chi(x) = \sigma(x) - \sigma(-x) - 2\sigma(\frac{1}{2}x - \frac{1}{2}) + 2\sigma(-\frac{1}{2}x - \frac{1}{2}) = (x \lor -1) \land 1$, then $\chi \in \mathcal{NN}(4, 1, 6)$. We construct the target function as

$$\psi_k(x, y) = \chi(\tilde{\psi}_k(x, y)) = (\tilde{\psi}_k(x, y) \lor -1) \land 1.$$

Then, for any $x, y \in [-1, 1]$,

$$|xy - \psi_k(x, y)| \leq |xy - \tilde{\psi}_k(x, y)| \leq \frac{3}{k^2}.$$

By Proposition 2.5, $\psi_k \in \mathcal{NN}(6k, 2, 216)$.

**Lemma 3.6.** For any $d \geq 2$ and $k \in \mathbb{N}$, there exists $\phi \in \mathcal{NN}(6dk, 2\lceil \log_2 d \rceil, 6d^{3\lceil \log_2 d \rceil})$ such that $\phi : [-1, 1]^d \to [-1, 1]$ and

$$|x_1 \cdots x_d - \phi(x)| \leq \frac{6d}{k^2}, \quad x = (x_1, \ldots, x_d)^T \in [-1, 1]^d.$$

Furthermore, $\phi(x) = 0$ if $x_1 \cdots x_d = 0$.

**Proof.** We firstly consider the case $d = 2^m$ for some $m \in \mathbb{N}$. For $m = 1$, by Lemma 3.5, there exists $\phi_1 \in \mathcal{NN}(6k, 2, 216)$ such that $\phi_1 : [-1, 1]^2 \to [-1, 1]$ and $|x_1 x_2 - \phi_1(x_1, x_2)| \leq 3k^{-2}$ for any $x_1, x_2 \in [-1, 1]$. We define $\phi_m : [-1, 1]^{2m} \to [-1, 1]$ inductively by

$$\phi_{m+1}(x_1, \ldots, x_{2m+1}) = \phi_1(\phi_m(x_1, \ldots, x_{2m}), \phi_m(x_{2m+1}, \ldots, x_{2m+1})).$$
Then, \( \phi_m(x_1, \ldots, x_{2m}) = 0 \) if \( x_1 \cdots x_{2m} = 0 \) because this equation is true for \( m = 1 \). Next, we inductively show that \( \phi_m \in \mathcal{N}(3k2^{2m}, 2m, 216^m) \) and

\[
|x_1 \cdots x_{2m} - \phi_m(x_1, \ldots, x_{2m})| \leq (2^m - 1)\epsilon.
\]

where we denote \( \epsilon := 3k^{-2} \), i.e. the approximation error of \( \phi_1 \).

It is obvious that the assertion is true for \( m = 1 \) by construction. Assume that the assertion is true for some \( m \in \mathbb{N} \), we will prove that it is true for \( m + 1 \). By Proposition 2.5 and the construction of \( \phi_{m+1} \), we have \( \phi_{m+1} \in \mathcal{N}(3k2^{2m+1}, 2m+2, 216^{m+1}) \). For any \( x_1, \ldots, x_{2m+1} \in [-1, 1] \), we denote \( s_1 := x_1 \cdots x_{2m}, t_1 := x_{2m+1} \cdots x_{2m+1}, s_2 := \phi_m(x_1, \ldots, x_{2m}) \) and \( t_2 := \phi_m(x_{2m+1}, \ldots, x_{2m+1}) \), then \( s_1, t_1, s_2, t_2 \in [-1, 1] \). By the hypothesis of induction,

\[
|s_1 - s_2|, |t_1 - t_2| \leq (2^m - 1)\epsilon.
\]

Therefore,

\[
|s_1 t_1 - s_2 t_2| = |s_1 t_1 - \phi_1(s_2, t_2)|
\]

\[
\leq |s_1 t_1 - s_1 t_2| + |s_1 t_2 - s_2 t_2| + |s_2 t_2 - \phi_1(s_2, t_2)|
\]

\[
\leq |t_1 - t_2| + |s_1 - s_2| + \epsilon \leq (2^{m+1} - 1)\epsilon.
\]

Hence, the assertion is true for \( m + 1 \).

For general \( d \geq 2 \), we choose \( m = \lfloor \log_2 d \rfloor \), then \( 2^{m-1} < d \leq 2^m \). We define the target function \( \phi : [-1, 1]^d \rightarrow [-1, 1] \) by

\[
\phi(x) := \phi_m \left( \begin{pmatrix} \text{Id}_d & 0_{(2^m-d) \times d} \end{pmatrix} x + \begin{pmatrix} 0_{d \times 1} \\ 1_{(2^m-d) \times 1} \end{pmatrix} \right),
\]

where \( \text{Id}_d \) is the \( d \times d \) identity matrix, \( 0_{p \times q} \) is the \( p \times q \) zero matrix and \( 1_{(2^m-d) \times 1} \) is an all ones vector. By Proposition 2.5, \( \phi \in \mathcal{N}(3k2^{2m}, 2m, 216^m) \subseteq \mathcal{N}(6dk, 2 \lfloor \log_2 d \rfloor, 6^3 \lfloor \log_3 d \rfloor) \) and the approximation error is

\[
|x_1 \cdots x_d - \phi(x)| \leq (2^m - 1)\epsilon \leq 2de = 6dk^{-2}.
\]

Furthermore, \( \phi(x) = 0 \) if \( x_1 \cdots x_d = 0 \) because \( \phi_m \) has such property. \( \square \)

In Lemma 3.6, we constructed neural networks to approximate monomials. We can then approximate any \( f \in \mathcal{H}^\alpha \) by approximating its local Taylor expansion

\[
p(x) = \sum_{n \in \{0,1,\ldots,N\}^d} \psi_n(x) \sum_{|s| \leq r} \frac{\partial^s f(x)}{s!} \left( x - \frac{n}{N} \right)^s,
\]

where we use the usual conventions \( s! = \prod_{i=1}^d s_i! \) and \( (x - \frac{n}{N})^s = \prod_{i=1}^d (x_i - \frac{n_i}{N})^{s_i} \). The functions \( \{ \psi_n \}_n \) form a partition of unity of \( [0, 1]^d \) and each \( \psi_n \) is supported on a sufficiently small neighborhood of \( n/N \).

**Theorem 3.7.** For any \( N, k \in \mathbb{N} \) and \( h \in \mathcal{H}^\alpha \) with \( \alpha = r + \beta \), where \( r \in \mathbb{N}_0 \) and \( \beta \in (0, 1] \), there exists \( \phi \in \mathcal{N}(W, L, K) \) where

\[
W = 6(r + 1)(d + r)d^r(N + 1)^d k,
\]

\[
L = 2 \lfloor \log_2 (d + r) \rfloor + 2,
\]

\[
K = 6^3 \lfloor \log_2 (d + r) \rfloor + 1(r + 1)d^r N(N + 1)^d,
\]

such that

\[
\|h - \phi\|_{L^\infty([0,1]^d)} \leq 2d^r (N^{-\alpha} + 6(r + 1)(d + r)k^{-2}).
\]
Proof. Let
\[ \psi(t) = \sigma(1 - |t|) = \sigma(1 - \sigma(t) - \sigma(-t)) \in [0, 1], \quad t \in \mathbb{R}, \]
then \( \psi \in \mathcal{N}(2, 2, 3) \) and the support of \( \psi \) is \([-1, 1]\). For any \( n = (n_1, \ldots, n_d) \in \{0, 1, \ldots, N\}^d \), define
\[ \psi_n(x) := \prod_{i=1}^{d} \psi(Nx_i - n_i), \quad x = (x_1, \ldots, x_d)^T \in \mathbb{R}^d, \]
then \( \psi_n \) is supported on \( \{x \in \mathbb{R}^d : \|x - n/N\|_\infty \leq 1/N\} \). The functions \( \{\psi_n\}_n \) form a partition of unity of the domain \([0, 1]^d\):
\[
\sum_{n \in \{0, 1, \ldots, N\}^d} \psi_n(x) = \prod_{i=1}^{d} \sum_{n_i=0}^{N} \psi(Nx_i - n_i) \equiv 1, \quad x \in [0, 1]^d.
\]

Let \( p(x) \) be the local Taylor expansion (3.2). For convenience, we denote \( p_{n,s}(x) := \psi_n(x)(x - n/N)^s \) and \( c_{n,s} := \partial^s h(n/N)/s! \). Then, \( p_{n,s} \) is supported on \( \{x \in \mathbb{R}^d : \|x - n/N\|_\infty \leq 1/N\} \) and
\[
p(x) = \sum_{n \in \{0, 1, \ldots, N\}^d} \sum_{\|s\|_1 \leq r} c_{n,s} p_{n,s}(x).
\]
Using Taylor’s Theorem with integral remainder (see Petersen and Voigtländer [2018, Lemma A.8] for example), it can be shown that the approximation error is
\[
|f(x) - p(x)| = \left| \sum_n \psi_n(x)f(x) - \sum_n \psi_n(x) \sum_{\|s\|_1 \leq r} c_{n,s} \left(x - \frac{n}{N}\right)^s \right|
\leq \sum_n \psi_n(x) \left| f(x) - \sum_{\|s\|_1 \leq r} c_{n,s} \left(x - \frac{n}{N}\right)^s \right|
\leq \sum_n \left| f(x) - \sum_{\|s\|_1 \leq r} c_{n,s} \left(x - \frac{n}{N}\right)^s \right|
\leq \sum_{\|x - n/N\|_\infty < 1/N} \sum_{\|s\|_1 \leq r} c_{n,s} \left(x - \frac{n}{N}\right)^s
\leq 2^d d^r \|x - n/N\|_\infty^\alpha
\leq 2^d d^r N^{-\alpha}.
\]

Let \( \Phi_D \in \mathcal{N}(6Dk, 2[\log_2 D], 6^3[\log_2 D]) \) be the \( D \)-product function constructed in Lemma 3.6. Then, we can approximate \( p_{n,s} \) by
\[
\phi_{n,s}(x) := \Phi_{d+\|s\|_1}(\psi(Nx_1 - n_1), \ldots, \psi(Nx_d - n_d), \ldots, x_i - n_i/N, \ldots),
\]
where the term \( x_i - n_i/N \) appears in the input only when \( s_i \neq 0 \) and it repeats \( s_i \) times. (When \( d = 1 \) and \( s = 0 \), we simply let \( \phi_{n,0}(x) = \psi(Nx-n) \).) Since \( x_i - n_i/N = \sigma(x_i - n_i/N) - \sigma(-x_i + n_i/N) \) and \( \|s\|_1 \leq r \), by Proposition 2.5, we have \( \phi_{n,s} \in \mathcal{N}(6(d + r)k, 2[\log_2(d + r)] + 2, 6^3[\log_2(d + r)] + 1N) \). By Lemma 3.6, the approximation error is
\[
|p_{n,s}(x) - \phi_{n,s}(x)| \leq 6(d + r)k^{-2}.
\]
Finally, by Proposition 2.5, \( \phi \) shows the existence of any such that \( W \). Therefore, we can approximate \( p(x) \) by

\[
\phi(x) = \sum_{n \in \{0, \ldots, N\}^d} \sum_{\|s\|_1 \leq r} c_{n,s} \phi_{n,s}(x).
\]

Observe that \( |c_{n,s}| = |\partial^s f(n/N)/s| \leq 1 \) and the number of terms in the inner summation is

\[
\sum_{\|s\|_1 \leq r} 1 = \sum_{j=0}^{r} \sum_{\|s\|_1 = j} 1 \leq \sum_{j=0}^{r} d^j \leq (r + 1)d^r.
\]

The approximation error is, for any \( x \in [0,1]^d \),

\[
|p(x) - \phi(x)| = \left| \sum_{n} \sum_{\|s\|_1 \leq r} c_{n,s} p_{n,s}(x) - \sum_{n} \sum_{\|s\|_1 \leq r} c_{n,s} \phi_{n,s}(x) \right|
\leq \sum_{n} \sum_{\|s\|_1 \leq r} |c_{n,s}| |p_{n,s}(x) - \phi_{n,s}(x)|
\leq \sum_{n: \|x - n/N\|_{\infty} < 1/k} \sum_{\|s\|_1 \leq r} |p_{n,s}(x) - \phi_{n,s}(x)|
\leq 6 \cdot d^j (r + 1)(d + r)d^k k^{-2}.
\]

Hence, the total approximation error is

\[
|h(x) - \phi(x)| \leq |h(x) - p(x)| + |p(x) - \phi(x)| \leq 2^d d^{r}(N^{-\alpha} + 6(r + 1)(d + r)k^{-2}).
\]

Finally, by Proposition 2.5, \( \phi \in \mathcal{N}(6(r+1)(d+r)d^r(N+1)^d k, 2[d \log_2(d+r)] + 2, 6^3[d \log_2(d+r)] + 1)(r + 1)d^r N(N + 1)^d) \).

Using the construction in Theorem 3.7, we can give a proof of the approximation upper bound in Theorem 3.2.

**Proof of Theorem 3.2 (Upper bound).** We choose \( N = \lceil k^{2/\alpha} \rceil \) in the Theorem 3.7, then it shows the existence of \( \phi \in \mathcal{N}(W, L, K) \) with

\[
W = 6(r + 1)(d + r)d^r(N + 1)^d k \asymp k^{2d/\alpha + 1},
L = 2[d \log_2(d + r)] + 2,
K = 6^3[d \log_2(d+r)] + 1)(r + 1)d^r N(N + 1)^d \asymp k^{2(d+1)/\alpha},
\]

such that \( \|h - \phi\|_{L^\infty([0,1]^d)} \leq 2^d d^{r}(N^{-\alpha} + 6(r + 1)(d + r)k^{-2}) \lesssim k^{-2} \). Therefore, \( k \asymp K^{\alpha/(2d+2)} \), \( W \asymp k^{2d/\alpha + 1} \asymp K^{(2d+1)/(2d+2)} \) and we have the approximation bound

\[
|h - \phi\|_{L^\infty([0,1]^d)} \lesssim k^{-2} \lesssim K^{-\alpha/(d+1)}.
\]

Since increasing \( W \) and \( L \) can only decrease the approximation error, the bound holds for any \( W \gtrsim K^{(2d+\alpha)/(2d+2)} \) and \( L \geq 2[d \log_2(d + r)] + 2 \). \( \square \)
3.2 Lower bounds

In this section, we present two methods that give lower bounds for the approximation error using norm constrained neural networks. Both methods use the Rademacher complexity (Lemma 2.3) to lower bound the approximation capacity. The first method is inspired by Maiorov and Ratsaby [1999], which characterized the approximation order by pseudo-dimension (or VC dimension [Vapnik and Chervonenkis, 1971]). This method compares the packing numbers of neural networks $\mathcal{N}(W, L, K)$ and the target function class $\mathcal{H}^\alpha$ on a suitably chosen data set. The second method establishes the lower bound by finding a linear functional that distinguishes the approximator and target classes. Using the second method, we give explicit constant on the approximation lower bound in Theorem 3.10, but it only holds for $H^1$.

Let us begin with the estimation of the packing number of $\mathcal{H}^\alpha$. We first construct a series of subsets $\mathcal{H}^\alpha_N \subseteq \mathcal{H}^\alpha$ with high complexity and simple structure. To this end, we choose a $C^\infty$ function $\psi : \mathbb{R}^d \to [0, \infty)$ which satisfies $\psi(0) = 1$ and $\psi(x) = 0$ for $\|x\|_\infty \geq 1/4$, and let $C_{\psi, \alpha} > 0$ be a constant such that $C_{\psi, \alpha} \psi \in \mathcal{H}^\alpha(\mathbb{R}^d)$. For any $N \in \mathbb{N}$, we consider the function class

$$\mathcal{H}^\alpha_N := \left\{ h_\alpha(x) = \frac{C_{\psi, \alpha}}{N^\alpha} \sum_{n \in \{0, \ldots, N-1\}^d} a_n \psi(Nx - n) : a \in A_N \right\}, \quad (3.3)$$

where we denote $A_N := \{ a = (a_n)_{n \in \{0, \ldots, N-1\}^d} : a_n \in \{1, -1\} \}$ as the set of all sign vectors indexed by $n$. Observe that, for the function $\psi_n(x) := \frac{C_{\psi, \alpha}}{N^\alpha} \psi(Nx - n)$,

$$\sup_{x \in \mathbb{R}^d} |\partial^s \psi_n(x)| = N^\|s\|_1 - \alpha C_{\psi, \alpha} \sup_{x \in \mathbb{R}^d} |\partial^s \psi(x)| \leq 1, \quad \|s\|_1 \leq r,$n

$$\sup_{x \neq y} \frac{|\partial^s \psi_n(x) - \partial^s \psi_n(y)|}{\|x - y\|_\infty^\beta} = N^{r-\alpha} C_{\psi, \alpha} \sup_{x \neq y} \frac{|\partial^s \psi(x) - \partial^s \psi(y)|}{N^{\beta} \|x - y\|_\infty^\beta} \leq 1, \quad \|s\|_1 = r,$$

where $\alpha = r + \beta > 0$, with $r \in \mathbb{N}_0$, $\beta \in (0, 1]$ and we use the fact $C_{\psi, \alpha} \psi \in \mathcal{H}^\alpha(\mathbb{R}^d)$. Therefore, $\psi_n$ is also in $\mathcal{H}^\alpha(\mathbb{R}^d)$. Since the functions $\psi_n$ have disjoint supports and $a_n \in \{1, -1\}$, one can check that each $h_\alpha$ is in $\mathcal{H}^\alpha(\mathbb{R}^d)$ and hence $\mathcal{H}^\alpha_N \subseteq \mathcal{H}^\alpha$.

Next, we consider the packing number of $\mathcal{H}^\alpha_N$ on the set $\Lambda_N := \{ n/N : n \in \{0, \ldots, N-1\}^d \}$. For convenience, we will denote the function values of a function class $\mathcal{F}$ on $\Lambda_N$ by

$$\mathcal{F}(\Lambda_N) := \{(f(n/N))_{n \in \{0, \ldots, N-1\}^d} : f \in \mathcal{F}\} \subseteq \mathbb{R}^m,$$

where $m = |\Lambda_N| = N^d$ is the cardinality of $\Lambda_N$. Observe that, for $h_\alpha \in \mathcal{H}^\alpha_N$,

$$h_\alpha(n/N) = \frac{C_{\psi, \alpha}}{N^\alpha} \sum_{i \in \{0, \ldots, N-1\}^d} a_i \psi(n - i) = \frac{C_{\psi, \alpha}}{N^\alpha} a_n, \quad (3.4)$$

where the last equality is because $\psi(n - i) = 1$ if $n = i$ and $\psi(n - i) = 0$ if $n \neq i$. We conclude that

$$\mathcal{H}^\alpha_N(\Lambda_N) = \{ C_{\psi, \alpha} N^{-\alpha} a : a \in A_N \} = C_{\psi, \alpha} N^{-\alpha} A_N.$$

We will estimate the packing number of $\mathcal{H}^\alpha_N(\Lambda_N)$ under the metric

$$\rho_2(x, y) := \left( \frac{1}{m} \sum_{i=1}^m (x_i - y_i)^2 \right)^{1/2} = m^{-1/2} \|x - y\|_2, \quad x, y \in \mathbb{R}^m. \quad (3.5)$$

The following combinatorial lemma is sufficient for our purpose.
Lemma 3.8. Let $A := \{a = (a_1, \ldots, a_m) : a_i \in \{1, -1\}\}$ be the set of all sign vectors on $\mathbb{R}^m$. For any $m \geq 8$, there exists a subset $B \subseteq A$ whose cardinality $|B| \geq 2^{m/4}$, such that any two sign vectors $a \neq a'$ in $B$ are different in more than $\lceil m/8 \rceil$ places.

Proof. For any $a \in A$, let $U(a)$ be the set of all $a'$ which are different from $a$ in at most $k = \lceil m/8 \rceil$ places. Then,

$$|U(a)| \leq \sum_{i=0}^{k} \binom{m}{i} \leq \left(\frac{me}{k}\right)^k \leq (16e)^{m/8} \leq 64^{m/8} = 2^{5m/4},$$

where the second inequality is from Vershynin [2018, Exercise 0.0.5]. We can construct the set $B = \{a_1, \ldots, a_n\}$ as follows. We take $a_1 \in A$ arbitrarily. Suppose the elements $a_1, \ldots, a_j \in A$ have been chosen, then $a_{j+1}$ is taken arbitrarily from $A \setminus \bigcup_{i=1}^{j} U(a_i)$. Then, by construction, $a_{j+1}$ and $a_i$ ($1 \leq i \leq j$) are different in more than $\lceil m/8 \rceil$ places. We do this process until the set $A \setminus \bigcup_{i=1}^{n} U(a_i)$ is empty. Since

$$2^m = |A| \leq \sum_{i=1}^{n} |U(a_i)| \leq n2^{3m/4},$$

we must have $|B| = n \geq 2^{m/4}$.

By Lemma 3.8, when $m = N^d \geq 8$, there exists a subset $B_N \subseteq A_N$ whose cardinality $|B_N| \geq 2^{m/4}$, such that any two vectors $a \neq a'$ in $B_N$ are different in more than $\lceil m/8 \rceil$ places. Thus,

$$\rho_2(a, a') = m^{-1/2}||a - a'||_2 \geq 2m^{-1/2}\lceil m/8 \rceil^{1/2} > 1/2.$$

By equation (3.4), this implies that

$$\rho_2(h_a(\Lambda_N), h_{a'}(\Lambda_N)) > C_{\psi, a} 2^{N^d/4}.$$

In other words, $\{h_a(\Lambda_N) : a \in B_N\}$ is a $\frac{1}{2}C_{\psi, a} N^{-\alpha}$-packing of $\mathcal{H}_N^\alpha(\Lambda_N)$ and hence we can lower bound the packing number

$$N_p(\mathcal{H}_N^\alpha(\Lambda_N), \rho_2, \frac{1}{2}C_{\psi, a} N^{-\alpha}) \geq N_p(\mathcal{H}_N^\alpha(\Lambda_N), \rho_2, \frac{1}{2}C_{\psi, a} N^{-\alpha}) \geq 2^{m/4} = 2^N.$$

(3.6)

On the other hand, one can upper bound the packing number of a set in $\mathbb{R}^m$ by its Rademacher complexity due to Sudakov minoration for Rademacher processes, see Ledoux and Talagrand [1991, Corollary 4.14] for example.

Lemma 3.9 (Sudakov minoration). There exists a constant $C > 0$ such that for any set $S \subseteq \mathbb{R}^m$ and any $\epsilon > 0$,

$$\log N_p(S, \rho_2, \epsilon) \leq C \frac{mR_m(S)^2 \log \left(2 + \frac{1}{\sqrt{mR_m(S)}}\right)}{\epsilon^2}.$$
Together with Lemma 3.9, we can upper bound the packing number

$$\log \mathcal{N}_p(\Phi(\Lambda_N), \rho_2, \epsilon) \leq C \frac{K^2 L}{\epsilon^2}, \quad (3.7)$$

for some constant $C > 0$.

Now, we are ready to prove our main lower bound for approximation error in Theorem 3.2. The idea is that, if the approximation error $E$ for some constant $C > 0$ contradicts the conclusion of Lemma 3.9, then the packing numbers of $\mathcal{H}^\alpha(\Lambda_N)$ and $\Phi(\Lambda_N)$ are close, and hence we can compare the lower bound (3.6) and upper bound (3.7). We will show that this leads to a contradiction when the approximation error is too small.

**Proof of Theorem 3.2 (Lower bound).** Denote $\Lambda_N := \{n/N : n \in \{0, \ldots, N - 1\}^d\}$ and $\Phi = \mathcal{NN}(W, L, K)$ as above. We have shown (by (3.6) and (3.7)) that, when $N^d \geq 8$, there exists $C_1, C_2 > 0$ such that the packing number

$$\log_2 \mathcal{N}_p(\mathcal{H}^\alpha(\Lambda_N), \rho_2, 3C_1 N^{-\alpha}) \geq N^d/4, \quad (3.8)$$

and for any $\epsilon > 0$,

$$\log_2 \mathcal{N}_p(\Phi(\Lambda_N), \rho_2, \epsilon) \leq C_2 \frac{K^2 L}{\epsilon^2}. \quad (3.9)$$

Assume the approximation error $\mathcal{E}(\mathcal{H}^\alpha, \Phi) < C_1 N^{-\alpha}$, where $N \geq 8^{1/d}$ will be chosen later. Using (3.8), let $\mathcal{F}$ be a subset of $\mathcal{H}^\alpha$ such that $\mathcal{F}(\Lambda_N)$ is a $3C_1 N^{-\alpha}$-packing of $\mathcal{H}^\alpha(\Lambda_N)$ with $\log_2 |\mathcal{F}(\Lambda_N)| \geq N^d/4$. By assumption, for any $f_i \in \mathcal{F}$, there exists $g_i \in \Phi$ such that $\|f_i - g_i\|_\infty \leq C_1 N^{-\alpha}$. Let $\mathcal{G}$ be the collection of all $g_i$. Then, $\log_2 |\mathcal{G}(\Lambda_N)| \geq N^d/4$ and, for any $i \neq j$,

$$\rho_2(g_i(\Lambda_N), g_j(\Lambda_N))$$

$$\geq \rho_2(f_i(\Lambda_N), f_j(\Lambda_N)) - \rho_2(f_i(\Lambda_N), g_i(\Lambda_N)) - \rho_2(g_j(\Lambda_N), f_j(\Lambda_N))$$

$$\geq \rho_2(f_i(\Lambda_N), f_j(\Lambda_N)) - \|f_i - g_i\|_\infty - \|g_j - f_j\|_\infty$$

$$> 3C_1 N^{-\alpha} - C_1 N^{-\alpha} - C_1 N^{-\alpha}$$

$$= C_1 N^{-\alpha}.$$

In other words, $\mathcal{G}(\Lambda_N)$ is a $C_1 N^{-\alpha}$-packing of $\Phi(\Lambda_N)$. Combining with (3.9), we have

$$\frac{N^d}{4} \leq \log_2 \mathcal{N}_p(\Phi(\Lambda_N), \rho_2, C_1 N^{-\alpha}) \leq C_2 \frac{K^2 L}{C_1 N^{-2\alpha}},$$

which is equivalent to

$$N^{d-2\alpha} \leq 4C_1^{-2} C_2 K^2 L. \quad (3.10)$$

Now, we choose $N = \max\{\lceil(5C_1^{-2} C_2 K^2 L)^{1/(d-2\alpha)}\rceil, \lceil 8^{1/d} \rceil\}$, then (3.10) is always false. This contradiction implies $\mathcal{E}(\mathcal{H}^\alpha, \Phi) \geq C_1 N^{-\alpha} \gtrsim (K^2 L)^{-\alpha/(d-2\alpha)}$. \qed

Finally, we provide an alternative method to prove the lower bound in Theorem 3.2 when $\alpha = 1$. We observe that, for any $f \in \mathcal{H}^1$ and $\phi \in \mathcal{NN}(W, L, K)$, by Hahn-Banach theorem,

$$\|f - \phi\|_{C([0, 1]^d)} = \sup_{\|T\| \neq 0} \frac{|Tf - T\phi|}{\|T\|} \geq \sup_{\|T\| \neq 0} \frac{|Tf| - |T\phi|}{\|T\|},$$
where $T$ is any bounded linear functional on $C([0,1]^d)$ with operator norm $\|T\| \neq 0$. Thus, for any nonzero linear functional $T$,

$$ E(\mathcal{H}^1, \mathcal{NN}(W,L,K)) \geq \sup_{f \in \mathcal{H}^1} \inf_{\phi \in \mathcal{NN}(W,L,K)} \frac{|Tf| - |T\phi|}{\|T\|} $$

$$ \geq \frac{1}{\|T\|} \left( \sup_{f \in \mathcal{H}^1} |Tf| - \sup_{\phi \in \mathcal{NN}(W,L,K)} |T\phi| \right) = \frac{1}{\|T\|} \left( \sup_{f \in \mathcal{H}^1} Tf - \sup_{\phi \in \mathcal{NN}(W,L,K)} T\phi \right). $$

Hence, to provide a lower bound of $E(\mathcal{H}^1, \mathcal{NN}(W,L,K))$, we only need to find a linear functional $T$ that distinguishes $\mathcal{H}^1$ and $\mathcal{NN}(W,L,K)$. In order to use the Rademacher complexity bounds for neural networks (Lemma 2.3), we will consider the functional

$$ T_n h := \frac{1}{n} \sum_{i=1}^n h(x_i) - \int_{[0,1]^d} h(x) dx, \quad h \in C([0,1]^d), $$

where the points $x_1, \ldots, x_n \in [0,1]^d$ will be chosen appropriately. Notice that, when $\{x_i\}_{i=1}^n$ are randomly chosen from the uniform distribution on $[0,1]^d$, $T_n h$ is the difference of empirical average and expectation. The optimal transport theory [Villani, 2008] provides a lower bound for $\sup_{f \in \mathcal{H}^1} T_n f$, while the Rademacher complexity upper bounds $\sup_{\phi \in \mathcal{NN}(W,L,K)} T_n \phi$ in expectation by symmetrization argument.

**Theorem 3.10.** For any $W, L \in \mathbb{N}$, $K \geq 1$ and $d \geq 3$,

$$ E(\mathcal{H}^1, \mathcal{NN}(W,L,K)) \geq c_d \left( K \sqrt{L + 2} + \log(d+1) \right)^{-2/(d-2)}, $$

where $c_d = (d-2)^{-d/(d-2)}(d+1)^{-(d+1)/(d-2)}$.

**Proof.** Define the functional $T_n$ on $C([0,1]^d)$ by (3.11). It is easy to check that $\|T_n\| \leq 2$. We have shown that

$$ E(\mathcal{H}^1, \mathcal{NN}(W,L,K)) \geq \frac{1}{2} \left( \sup_{f \in \mathcal{H}^1} T_n f - \sup_{\phi \in \Phi} T_n \phi \right) $$

where we denote $\Phi = \mathcal{NN}(W,L,K)$ to simplify the notation. Our analysis is divided into three steps.

**Step 1:** Lower bounding $\sup_{f \in \mathcal{H}^1} T_n f$. Observe that $\mathcal{H}^1 \subseteq \text{Lip} 1$ and, for any $g \in \text{Lip} 1$, the function $f = g - \min_{x \in [0,1]^d} g(x) \in \mathcal{H}^1$ satisfies $T_n f = T_n g$. We conclude that

$$ \sup_{f \in \mathcal{H}^1} T_n f = \sup_{g \in \text{Lip} 1} T_n g. $$

By the Kantorovich-Rubinstein duality [Villani, 2008, Remark 6.5],

$$ \sup_{g \in \text{Lip} 1} T_n g = \mathcal{W}_1 \left( \frac{1}{n} \sum_{i=1}^n \delta_{x_i}, \mathcal{U} \right) := \inf_{\mu} \int_{[0,1]^d \times [0,1]^d} \|x - y\|_\infty d\mu(x,y) $$

is the 1-Wasserstein distance between the discrete distribution $\frac{1}{n} \sum_{i=1}^n \delta_{x_i}$ and the uniform distribution $\mathcal{U}$ on $[0,1]^d$, where the infimum is taken over all joint probability distribution (also called coupling) $\mu$ on $[0,1]^d \times [0,1]^d$, whose marginal distributions are $\frac{1}{n} \sum_{i=1}^n \delta_{x_i}$ and $\mathcal{U}$ respectively. It is enough to estimate the 1-Wasserstein distance.
We notice that, for any $r \in [0, 1/2]$,

$$\mathcal{U}\left(\left\{ y \in [0, 1]^d : \min_{1 \leq i \leq n} \|x_i - y\|_\infty \geq rn^{-1/d} \right\}\right)$$

$$= 1 - \mathcal{U}\left(\left\{ y \in [0, 1]^d : \min_{1 \leq i \leq n} \|x_i - y\|_\infty < rn^{-1/d} \right\}\right)$$

$$\geq 1 - \sum_{i=1}^n \mathcal{U}\left(\left\{ y \in [0, 1]^d : \|x_i - y\|_\infty < rn^{-1/d} \right\}\right)$$

$$\geq 1 - n(2rn^{-1/d})^d = 1 - 2^d r^d.$$ 

Hence, for any coupling $\mu$ and $r \in [0, 1/2]$,

$$\int_{[0,1]^d \times [0,1]^d} \|x - y\|_\infty d\mu(x, y) = \int_{\bigcup_{i=1}^n \{x_i\} \times [0,1]^d} \|x - y\|_\infty d\mu(x, y)$$

$$\geq \int_{\bigcup_{i=1}^n \{x_i\} \times [0,1]^d} \min_{1 \leq i \leq n} \|x_i - y\|_\infty d\mu(x, y)$$

$$= \int_{[0,1]^d \times [0,1]^d} \min_{1 \leq i \leq n} \|x_i - y\|_\infty d\mu(y)$$

$$\geq (1 - 2^d r^d)rn^{-1/d}.$$ 

As a consequence, for any $n$ points $x_1, \ldots, x_n \in [0, 1]^d$,

$$\sup_{f \in \mathcal{H}} T_n f \geq \sup_{r \in [0, 1/2]} (1 - 2^d r^d)rn^{-1/d} = 2^{-1}(d+1)^{-1/4}n^{-1/4},$$

where the supremum is attained when $r = 2^{-1}(d+1)^{-1/4}$.

**Step 2:** Upper bounding $\sup_{f \in \mathcal{H}} T_n f$. Let $X_{1:n} = \{X_i\}_{i=1}^n$ be $n$ i.i.d. samples from the uniform distribution $\mathcal{U}$ on $[0,1]^d$. We are going to upper bound

$$\mathcal{I}_n := \mathbb{E}_{X_{1:n}} \left[ \sup_{\phi \in \Phi} \left( \frac{1}{n} \sum_{i=1}^n \phi(X_i) - \int_{[0,1]^d} \phi(x) dx \right) \right] = \mathbb{E}_{X_{1:n}} \left[ \sup_{\phi \in \Phi} \left( \frac{1}{n} \sum_{i=1}^n \phi(X_i) - \mathbb{E}_{X \sim \mathcal{U}}[\phi(X)] \right) \right].$$

We introduce a ghost sample dataset $X'_{1:n} = \{X'_i\}_{i=1}^n$ drawn i.i.d. from $\mathcal{U}$, independent of $X_{1:n}$. Then,

$$\mathcal{I}_n = \mathbb{E}_{X_{1:n}} \left[ \sup_{\phi \in \Phi} \left( \frac{1}{n} \sum_{i=1}^n \phi(X_i) - \mathbb{E}_{X'_{1:n}} \frac{1}{n} \sum_{i=1}^n \phi(X'_i) \right) \right]$$

$$\leq \mathbb{E}_{X_{1:n}, X'_{1:n}} \left[ \sup_{\phi \in \Phi} \frac{1}{n} \sum_{i=1}^n (\phi(X_i) - \phi(X'_i)) \right].$$

Let $\xi_{1:n} = \{\xi_i\}_{i=1}^n$ be a sequence of i.i.d. Rademacher variables independent of $X_{1:n}$ and $X'_{1:n}$.
Then, by symmetry, we can bound $\mathcal{I}_n$ by Rademacher complexity:

$$\mathcal{I}_n \leq \mathbb{E}_{X_1^n, X'_1^n, \xi_1^n} \left[ \sup_{\phi \in \Phi} \frac{1}{n} \sum_{i=1}^{n} \xi_i (\phi(X_i) - \phi(X'_i)) \right]$$

$$= \mathbb{E}_{X_1^n, X'_1^n, \xi_1^n} \left[ \sup_{\phi \in \Phi} \frac{1}{n} \sum_{i=1}^{n} \xi_i \phi(X_i) + \sup_{\phi \in \Phi} \frac{1}{n} \sum_{i=1}^{n} -\xi_i \phi(X'_i) \right]$$

$$= 2 \mathbb{E}_{X_1^n, \xi_1^n} \left[ \sup_{\phi \in \Phi} \frac{1}{n} \sum_{i=1}^{n} \xi_i \phi(X_i) \right]$$

$$= 2 \mathbb{E}_{X_1^n} \left[ \mathcal{R}_n(\Phi(X_1^n)) \right],$$

where we denote $\Phi(X_1^n) := \{(\phi(X_1), \ldots, \phi(X_n)) \in \mathbb{R}^n : \phi \in \Phi\}$ and the second last equality is due to the fact that $X_i$ and $X'_i$ have the same distribution and the fact that $\xi_i$ and $-\xi_i$ have the same distribution.

By Lemma 2.3, for any $X_1^n \subseteq [0, 1]^d$,

$$\mathcal{R}_n(\Phi(X_1^n)) \leq \sqrt{2K} \sqrt{L + 2 + \log(d+1)n^{-1/2}}.$$

Hence, there exists $x_1, \ldots, x_n \in [0, 1]^d$ such that

$$\sup_{\phi \in \mathcal{N}(W, L, K)} T_n \phi \leq \mathcal{I}_n \leq 2 \mathbb{E}_{X_1^n} \left[ \mathcal{R}_n(\Phi(X_1^n)) \right].$$

**Step 3:** Optimizing $n$. We have shown that there exists $T_n$ such that

$$\mathcal{E}(\mathcal{H}_1, \mathcal{N}(W, L, K)) \geq \frac{1}{2} \left( \sup_{f \in \mathcal{H}_1} T_n f - \sup_{\phi \in \Phi} T_n \phi \right)$$

$$\geq ds^{-1}n^{-1/d} - \sqrt{2tn^{-1/2}},$$

where $s = 4(d+1)^{1+1/d}$ and $t = K \sqrt{L + 2 + \log(d+1)}$. In order to optimize over $n$, we can choose

$$n = \left\lfloor \left( \frac{s}{d-2} \right) \right\rfloor.$$

Then, since $st \geq 2$, we have $n \geq \left( \frac{s}{d-2} \right) - 1 \geq \frac{1}{2} \left( \frac{s}{d-2} \right)$ and

$$\mathcal{E}(\mathcal{H}_1, \mathcal{N}(W, L, K)) \geq ds^{-1} \left( \frac{s}{d-2} - 2t \right) - d/(d-2)$$

$$= (d-2)^{-d/(d-2)} t^{-2/(d-2)}$$

$$= c_d \left( K \sqrt{L + 2 + \log(d+1)} \right)^{-2/(d-2)},$$

where $c_d = (d-2)^{-d/(d-2)} (d+1)^{-(d+1)/(d-2)}$. 

**4 Applications to machine learning**

In this section, we apply Theorem 3.2 to two typical machine learning algorithms: regression by neural networks and distribution estimation by GANs. For regression, the goal is to estimate an unknown function $f_0$ from its noisy samples. One of the useful and effective
methods is the empirical risk minimization, which estimates $f_0$ by minimizing some risk on the observed samples over some chosen hypothesis class. When $f_0$ is in some continuous function class and the hypothesis class is a ReLU neural network, the convergence rates of this estimator have been derived by [Schmidt-Hieber, 2020; Nakada and Imaizumi, 2020]. Here, we make a norm constraint on the weights and study the convergence rate of the corresponding estimator. As a consequence, our results provide statistical guarantee for overparameterized networks, see Theorem 4.1 and Corollary 4.3. For distribution estimation, a GAN implicitly estimates the data distribution by training a generator that transports an easy-to-sample distribution to the data distribution, and a discriminator that distinguishes samples produced by the generator from true samples. It has been shown that GANs perform extremely well in practice [Gulrajani et al., 2017; Miyato et al., 2018; Brock et al., 2019]. We can combine the error analysis in Huang et al. [2022] with Theorem 3.2 to derive convergence rate for GANs with norm constrained neural networks as discriminator, which gives statistical guarantee on the performance of GANs, see Theorem 4.6 and Corollary 4.11.

In the statistical analysis of learning algorithms, we often require that the hypothesis class is uniformly bounded. For any $B > 0$, we will use the notations

$$
\mathcal{NN}^B_{d,k}(W, L) := \{ \phi \in \mathcal{NN}_{d,k}(W, L) : \phi(x) \in [-B, B]^k, \forall x \in \mathbb{R}^d \},
$$

$$
\mathcal{NN}^B_{d,k}(W, L, K) := \{ \phi \in \mathcal{NN}_{d,k}(W, L, K) : \phi(x) \in [-B, B]^k, \forall x \in \mathbb{R}^d \},
$$

which represent the neural network classes uniformly bounded by $B$. Note that we can truncate the output of $\phi \in \mathcal{NN}_{d,k}(W, L, K)$ by applying $\chi_B(x) = (x \lor -B) \land B$ element-wise. Since

$$
\chi_B(x) = \sigma(x) - \sigma(-x) - (B + 1)\sigma\left(\frac{x}{B+1}\right) + (B + 1)\sigma\left(-\frac{x}{B+1}\right),
$$

it is not hard to see that $\chi_B \circ \phi \in \mathcal{NN}^B_{d,k}(\max\{W, 4k\}, L + 1, (2B + 4)\max\{K, 1\})$ by Proposition 2.5. Therefore, the approximation upper bound in Theorem 3.2 also holds true for $\mathcal{NN}^1_{d,k}(W, L, K)$ when $L \geq 2\lceil\log_2(d + r)\rceil + 3$.

### 4.1 Regression

Suppose we have a set of $n$ samples $S_n = \{(X_i, Y_i)\}_{i=1}^n \subseteq [0, 1]^d \times \mathbb{R}$ which are independently and identically generated from the regression model

$$
Y_i = f_0(X_i) + \eta_i, \quad X_i \sim \mu, \quad i = 1, \ldots, n,
$$

where $\mu$ is the marginal distribution of the covariates $X_i$ supported on $[0, 1]^d$, and $\eta_i$ is an i.i.d. Gaussian noise independent of $X_i$ with $\mathbb{E}[\eta_i] = 0$ and $\mathbb{E}[\eta_i^2] = V^2$, where $V \geq 0$. We aim to estimate the unknown target function $f_0 \in \mathcal{H}^a$ by the empirical risk minimizer (ERM)

$$
\arg\min_{\phi_0 \in \mathcal{NN}^1_{d,k}(W, L, K)} \mathcal{L}_n(\phi_0) := \arg\min_{\phi_0 \in \mathcal{NN}^1_{d,k}(W, L, K)} \frac{1}{n} \sum_{i=1}^n (\phi_0(X_i) - Y_i)^2. \quad (4.1)
$$

The performance of the estimation is measured by the expected risk

$$
\mathcal{L}(\phi_0) := \mathbb{E}_{(X,Y)}[(\phi_0(X) - Y)^2] = \mathbb{E}_{X \sim \mu}[(\phi_0(X) - f_0(X))^2] + V^2.
$$

It is equivalent to evaluate the estimator by the excess risk

$$
\|\phi_0 - f_0\|_{\mathcal{L}^2(\mu)}^2 = \mathcal{L}(\phi_0) - \mathcal{L}(f_0).
$$
In deep learning, the optimization problem (4.1) is generally solved by first order methods such as gradient descent or stochastic gradient descent on the parameters $\theta$. Assume that $\hat{\theta}_n$ is the output of a solver, say stochastic gradient descent, with optimization error $\epsilon_{opt} \geq 0$, i.e.,

$$\mathcal{L}_n(\hat{\theta}_n) \leq \inf_{\phi_\theta \in \mathcal{N}\mathcal{N}^1(W, L, K)} \mathcal{L}_n(\phi_\theta) + \epsilon_{opt}. \quad (4.2)$$

Then, for any $\phi_\theta \in \mathcal{N}\mathcal{N}^1(W, L, K)$,

$$\|\hat{\theta}_n - f_0\|_{L^2(\mu)}^2 = \mathcal{L}(\hat{\theta}_n) - \mathcal{L}(f_0)$$

$$= \left[ \mathcal{L}(\hat{\phi}_n) - \mathcal{L}_n(\hat{\phi}_n) \right] + \left[ \mathcal{L}_n(\hat{\phi}_n) - \mathcal{L}_n(\phi_\theta) \right] + \left[ \mathcal{L}_n(\phi_\theta) - \mathcal{L}(\phi_\theta) \right] + \left[ \mathcal{L}(\phi_\theta) - \mathcal{L}(f_0) \right]$$

$$\leq \left[ \mathcal{L}(\hat{\phi}_n) - \mathcal{L}_n(\hat{\phi}_n) \right] + \epsilon_{opt} + \left[ \mathcal{L}_n(\phi_\theta) - \mathcal{L}(\phi_\theta) \right] + \|\phi_\theta - f_0\|_{L^2(\mu)}^2.$$

Observing that $E_{S_n} \mathcal{L}_n(\phi_\theta) = \mathcal{L}(\phi_\theta)$ and taking the infimum over $\phi_\theta$, we get

$$E_{S_n} \left[ \|\hat{\theta}_n - f_0\|_{L^2(\mu)}^2 \right] \leq \inf_{\phi_\theta \in \mathcal{N}\mathcal{N}^1(W, L, K)} \|\phi - f_0\|_{L^2(\mu)}^2 + E_{S_n} \left[ \mathcal{L}(\hat{\phi}_n) - \mathcal{L}_n(\hat{\phi}_n) \right] + \epsilon_{opt}, \quad (4.3)$$

where we decompose the excess risk into three terms: approximation error $\inf_\phi \|\phi - f_0\|_{L^2(\mu)}^2$, statistical (generalization) error $E_{S_n} [\mathcal{L}(\hat{\phi}_n) - \mathcal{L}_n(\hat{\phi}_n)]$ and optimization error $\epsilon_{opt}$.

**Theorem 4.1.** Assume $f_0 \in \mathcal{H}^\alpha$ with $\alpha = r + \beta > 0$, where $r \in N_0$ and $\beta \in (0, 1]$. There exists $c > 0$ such that for any $W \geq cK(2d+\alpha)/(2d+2)$ and any $L \geq 2[\log_2(d+r)]+3$ independent of $n$, if we choose

$$K \sim n^{(d+1)/(2d+4\alpha+2)},$$

then, for any estimator $\hat{\theta}_n \in \mathcal{N}\mathcal{N}^1(W, L, K)$ satisfying (4.2),

$$E_{S_n} \left[ \|\hat{\theta}_n - f_0\|_{L^2(\mu)}^2 \right] - \epsilon_{opt} \lesssim n^{-\alpha/(d+2\alpha+1)} \log n.$$

**Proof.** Using the error decomposition (4.3), we only need to estimate the approximation error and stochastic error. For the approximation error, by Theorem 3.2 and the choice of $W$ and $L$,

$$\inf_{\phi \in \mathcal{N}\mathcal{N}^1(W, L, K)} \|\phi - f_0\|_{L^2(\mu)}^2 \lesssim K^{-2\alpha/(d+1)}.$$

For the statistical error,

$$E_{S_n} \left[ \mathcal{L}(\hat{\phi}_n) - \mathcal{L}_n(\hat{\phi}_n) \right]$$

$$= E_{S_n} \left[ \|\hat{\phi}_n - f_0\|_{L^2(\mu)}^2 + V^2 - \left( \frac{1}{n} \sum_{i=1}^n (\hat{\phi}_n(X_i) - f_0(X_i))^2 - 2\eta_i (\hat{\phi}_n(X_i) - f_0(X_i)) + \eta_i^2 \right) \right]$$

$$= E_{X_1:n} \left[ \|\hat{\phi}_n - f_0\|_{L^2(\mu)}^2 - \frac{1}{n} \sum_{i=1}^n (\hat{\phi}_n(X_i) - f_0(X_i))^2 \right] + 2E_{S_n} \left[ \frac{1}{n} \sum_{i=1}^n \eta_i (\hat{\phi}_n(X_i) - f_0(X_i)) \right]$$

$$\leq E_{X_1:n} \left[ \sup_{f \in \mathcal{F}} \mathbb{E}_X[f^2(X)] - \frac{1}{n} \sum_{i=1}^n f^2(X_i) \right] + 2E_{X_1:n} \mathbb{E}_{\eta_1:n} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n \eta_i f(X_i) \right]$$

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where $\mathcal{F} := \{ \phi - f_0 : \phi \in \mathcal{N}\mathcal{L}^1(W, L, K) \}$ and $X_{1:n} = \{X_i\}_{i=1}^n$ is the sequence of samples. By a standard symmetrization argument (similar to step 2 in the proof of Theorem 3.10), one can obtain
\[
\mathbb{E}_{X_{1:n}} \left[ \sup_{f \in \mathcal{F}} \mathbb{E}_X [f^2(X)] - \frac{1}{n} \sum_{i=1}^n f^2(X_i) \right] \leq 2 \mathbb{E}_{X_{1:n}} \left[ \mathcal{R}_n(\mathcal{F}^2(X_{1:n})) \right],
\]
where we denote $\mathcal{F}^2(X_{1:n}) := \{(f^2(X_1), \ldots, f^2(X_n)) : f \in \mathcal{F} \} \subseteq \mathbb{R}^n$. Since $\|f_0\|\infty \leq 1$ and $\|f\|\infty = \|\phi - f_0\|\infty \leq 2$ for any $f \in \mathcal{F}$, by the structural properties of Rademacher complexity (see Bartlett and Mendelson [2002, Theorem 12]), we have
\[
\mathbb{E}_{X_{1:n}} \left[ \mathcal{R}_n(\mathcal{F}^2(X_{1:n})) \right] \leq 8 \mathbb{E}_{X_{1:n}} \mathcal{R}_n(\mathcal{F}(X_{1:n})) \leq 8 \left( \mathbb{E}_{X_{1:n}} \mathcal{R}_n(\Phi(X_{1:n})) + \frac{\|f_0\|\infty}{\sqrt{n}} \right) \approx \frac{K}{\sqrt{n}},
\]
where $\Phi(X_{1:n}) := \{(\phi(X_1), \ldots, \phi(X_n)) : \phi \in \mathcal{N}\mathcal{L}^1(W, L, K)\}$ and we use Lemma 2.3 in the last inequality. On the other hand, the Gaussian complexity can be bounded by Rademacher complexity [Bartlett and Mendelson, 2002, Lemma 4]:
\[
\mathbb{E}_{X_{1:n}} \mathbb{E}_{\eta_{1:n}} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^n \eta_i f(X_i) \right] \lesssim \mathbb{E}_{X_{1:n}} \left[ \mathcal{R}_n(\mathcal{F}(X_{1:n})) \right] \log n \lesssim \frac{K \log n}{\sqrt{n}}.
\]
Hence,
\[
\mathbb{E}_{S_n} \left[ \mathcal{L}(\hat{\phi}_n) - \mathcal{L}_n(\hat{\phi}_n) \right] \lesssim \frac{K \log n}{\sqrt{n}}.
\]

In summary, the error decomposition (4.3) implies
\[
\mathbb{E}_{S_n} \left[ \|\hat{\phi}_n - f_0\|^2_{L^2(\mu)} - \epsilon_{opt} \right] \lesssim K^{-2\alpha/(d+1)} + \frac{K \log n}{\sqrt{n}}.
\]
If we choose $K \approx n^{(d+1)/(2d+4\alpha+2)}$, then
\[
\mathbb{E}_{S_n} \left[ \|\hat{\phi}_n - f_0\|^2_{L^2(\mu)} - \epsilon_{opt} \right] \lesssim n^{-\alpha/(d+2\alpha+1)} \log n.
\]

**Remark 4.2.** We have estimated the learning rate of the ERM in expectation (with respect to the observed samples). High probability bounds on the error $\|\hat{\phi}_n - f_0\|^2_{L^2(\mu)}$ can be similarly derived by using concentration inequalities for random processes, see [Boucheron et al., 2013; Anthony and Bartlett, 2009; Shalev-Shwartz and Ben-David, 2014; Mohri et al., 2018] for more details.

The constrained optimization problem (4.1) may be difficult to optimize in practice. As an alternative, one can use the regularized empirical risk minimization
\[
\arg\min_{\phi_\theta \in \mathcal{N}\mathcal{L}^1(W, L)} \mathcal{L}_{n,\lambda}(\phi_\theta) := \arg\min_{\phi_\theta \in \mathcal{N}\mathcal{L}^1(W, L)} \frac{1}{n} \sum_{i=1}^n (\phi_\theta(X_i) - Y_i)^2 + \lambda \kappa(\theta), \quad \lambda > 0. \tag{4.4}
\]
Assume that $\hat{\phi}_{n,\lambda} \in \mathcal{N}\mathcal{L}^1(W, L)$ parameterized by $\hat{\theta}_{n,\lambda}$ is the output of an optimization solver, say stochastic gradient descent, with optimization error $\epsilon_{opt} \geq 0$, i.e., $\hat{\theta}_{n,\lambda}$ is an $\epsilon_{opt}$-optimal solution of (4.4) satisfying
\[
\mathcal{L}_{n,\lambda}(\hat{\phi}_{n,\lambda}) \leq \inf_{\phi_\theta \in \mathcal{N}\mathcal{L}^1(W, L)} \mathcal{L}_{n,\lambda}(\phi_\theta) + \epsilon_{opt}. \tag{4.5}
\]
Then, for any $K \geq 0$ and $\phi_\theta \in \mathcal{N}\mathcal{N}^1(W, L, K)$, we have
\[
\mathcal{L}_n(\hat{\phi}_{n, \lambda}) + \lambda \kappa(\hat{\theta}_{n, \lambda}) = \mathcal{L}_{n, \lambda}(\hat{\phi}_{n, \lambda}) \leq \mathcal{L}_n(\phi_\theta) + \lambda \kappa(\theta) + \epsilon_{\text{opt}}.
\]
Taking the infimum over all $\phi_\theta \in \mathcal{N}\mathcal{N}^1(W, L, K)$, we get
\[
\mathcal{L}_n(\hat{\phi}_{n, \lambda}) + \lambda \kappa(\hat{\theta}_{n, \lambda}) \leq \inf_{\phi_\theta \in \mathcal{N}\mathcal{N}^1(W, L, K)} \mathcal{L}_n(\phi_\theta) + \lambda K + \epsilon_{\text{opt}}.
\] (4.6)

Hence, $\hat{\phi}_{n, \lambda}$ can be regarded as a solution of the constrained optimization problem (4.1) with optimization error bounded by $\lambda K + \epsilon_{\text{opt}}$ for certain $K$. As a corollary, we show that the regularized ERM can achieve the same convergence rate of ERM in Theorem 4.1, when there is no noise and $\lambda$ is chosen appropriately.

**Corollary 4.3.** Under the assumption of Theorem 4.1 with zero noise $\eta_i = Y_i - f_0(X_i) = 0$, there exists $c > 0$ such that for any
\[
W \geq c n^{(2d+\alpha)/(4d+8\alpha+4)}, \quad L \geq 2[\log_2(d + r)] + 3, \quad \lambda \asymp n^{-1/2},
\]
and any estimator $\hat{\phi}_{n, \lambda} \in \mathcal{N}\mathcal{N}^1(W, L)$ satisfying (4.5) with optimization error\
\[
\epsilon_{\text{opt}} \lesssim n^{-\alpha/(d+2\alpha+1)},
\]
we have
\[
\mathbb{E} S_n \left[ \| \hat{\phi}_{n, \lambda} - f_0 \|^2_{L^2(\mu)} \right] \lesssim n^{-\alpha/(d+2\alpha+1)} \log n.
\]

**Proof.** By Theorem 3.2, there exists $c_0 > 0$ such that for any $W \geq c_0 K^{(2d+\alpha)/(2d+2)}$ and $L \geq 2[\log_2(d + r)] + 3$,\
\[
\mathcal{E}(\mathcal{H}^\alpha, \mathcal{N}\mathcal{N}(W, L, K)) \lesssim K^{-\alpha/(d+1)}.
\]
Since the noise $\eta_i = 0$, inequality (4.6) implies
\[
\kappa(\hat{\theta}_{n, \lambda}) \leq \frac{1}{\lambda} \inf_{\phi_\theta \in \mathcal{N}\mathcal{N}^1(W, L, K)} \frac{1}{n} \sum_{i=1}^{n} (\phi_\theta(X_i) - f_0(X_i))^2 + K + \frac{\epsilon_{\text{opt}}}{\lambda} \lesssim \lambda^{-1} K^{-2\alpha/(d+1)} + K + \lambda^{-1} \epsilon_{\text{opt}}.
\]
If $\lambda \asymp K^{-1} K^{-2\alpha/(d+1)}$ and $\epsilon_{\text{opt}}$ for $\hat{\phi}_{n, \lambda} \in \mathcal{N}\mathcal{N}^1(W, L, K)$ with $K \leq \tilde{K} \lesssim K$. Using inequality (4.6) again, we have
\[
\mathcal{L}_n(\hat{\phi}_{n, \lambda}) \leq \inf_{\phi_\theta \in \mathcal{N}\mathcal{N}^1(W, L, \tilde{K})} \mathcal{L}_n(\phi_\theta) + \lambda \tilde{K} + \epsilon_{\text{opt}},
\]
which implies $\hat{\phi}_{n, \lambda}$ is a solution of the constrained optimization problem with optimization error $\lambda \tilde{K} + \epsilon_{\text{opt}}$. Now, we choose $\tilde{K} \asymp K \asymp n^{(d+1)/(2d+4\alpha+2)}$ and
\[
W \geq c_0 K^{(2d+\alpha)/(2d+2)} \asymp n^{(2d+\alpha)/(4d+8\alpha+4)}.
\]
Then, $\lambda \asymp n^{-1/2}$ and $\epsilon_{\text{opt}} \lesssim \lambda K \lesssim n^{-\alpha/(d+2\alpha+1)}$. Therefore, Theorem 4.1 implies
\[
\mathbb{E} S_n \left[ \| \hat{\phi}_{n, \lambda} - f_0 \|^2_{L^2(\mu)} \right] \lesssim n^{-\alpha/(d+2\alpha+1)} \log n + \epsilon_{\text{opt}} + \lambda \tilde{K} \lesssim n^{-\alpha/(d+2\alpha+1)} \log n,
\]
which completes the proof. □

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Remark 4.4. Thanks to the norm constraint, both Theorem 4.1 and Corollary 4.3 hold with no requirement on the upper bound of the size of network. As a consequence, we can allow the width $W$ and depth $L$ large enough such that the number of weights is greater than the number of samples, i.e., over-parameterization is allowed. Although the regularized optimization problem of the form (4.4) is highly nonconvex, for over-parameterized models, the optimization error $\epsilon_{opt}$ of stochastic gradient descent decays linearly to zero as the number of iterations increase under certain conditions [Allen-Zhu et al., 2019; Du et al., 2019; Nguyen, 2021; Liu et al., 2022]. Hence, with the help of the approximation results with norm constraint in this paper, it may be possible to close the gap between the current theory of approximation, generalization and optimization and further demystify why over-parameterized neural networks work well in practice.

4.2 Generative adversarial networks

Suppose we have $n$ i.i.d. samples $S_n = \{X_i\}_{i=1}^n$ from an unknown probability distribution $\mu$ supported on $[0,1]^d$. Generative adversarial networks implicitly estimate the data distribution $\mu$ by training a generator $g : \mathbb{R}^k \rightarrow [0,1]^d$ and a discriminator $f : [0,1]^d \rightarrow \mathbb{R}$ against each other. To be concrete, we choose an easy-to-sample source distribution $\nu$ (for example, uniform or Gaussian distribution) and compute the generator $g$ by minimizing the distance between the empirical distribution $\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}$ and the push-forward distribution $g\#\nu$:

$$\argmin_{g \in G} d_F(\hat{\mu}_n, g\#\nu) := \argmin_{g \in G} \sup_{f \in F} \mathbb{E}_{\mu}[f] - \mathbb{E}_{g\#\nu}[f],$$

(4.7)

where $d_F$ is the Integral Probability Metric (IPM, Müller [1997]) with respect to the discriminator class $F$, and the push-forward measure $g\#\nu$ of a measurable set $S \subseteq [0,1]^d$ is defined by $g\#\nu(S) = \nu(g^{-1}(S))$. In practice, the generator and discriminator classes are often parameterized by neural networks. If the training is successful, $g\#\nu$ should be close to the target distribution $\mu$ in some sense. In general, we can evaluate the performance by another IPM with respect to the evaluation class $H$

$$d_H(\mu, g\#\nu) := \sup_{h \in H} \mathbb{E}_{\mu}[h] - \mathbb{E}_{g\#\nu}[h].$$

For instance, in the Wasserstein GAN [Arjovsky et al., 2017], $H = \text{Lip} 1$ is the 1-Lipschitz class and $d_H = \mathcal{W}_1$ is the Wasserstein distance by Kantorovich-Rubinstein duality [Villani, 2008]. In Sobolev GAN [Mroueh et al., 2018], $H$ is a Sobolev class.

Assume that $\hat{\gamma}_n \in G$ is a solution of the problem (4.7) with optimization error $\epsilon_{opt} \geq 0$:

$$d_F(\hat{\mu}_n, (\hat{\gamma}_n)\#\nu) \leq \argmin_{g \in G} d_F(\hat{\mu}_n, g\#\nu) + \epsilon_{opt}.$$  

(4.8)

Similar to the analysis for regression, we have the following error decomposition for GANs.

Lemma 4.5 (Huang et al. [2022], Lemma 9). Assume that $F$ is symmetric ($f \in F$ implies $-f \in F$), $\mu$ and $g\#\nu$ are supported on $[0,1]^d$ for all $g \in G$. Then, for any $\hat{\gamma}_n \in G$ satisfying (4.8),

$$d_H(\mu, (\hat{\gamma}_n)\#\nu) \leq 2\mathcal{E}(H^n, F) + \inf_{g \in G} d_F(\hat{\mu}_n, g\#\nu) + [d_F(\mu, \hat{\mu}_n) \land d_H(\mu, \hat{\mu}_n)] + \epsilon_{opt}.$$
Note that the error \( d_{\mathcal{H}^0}(\mu, (\hat{g}_n)\#\nu) \) is decomposed into four error terms: (1) discriminator approximation error \( \mathcal{E}(\mathcal{H}^\alpha, \mathcal{F}) \) measuring how well the discriminator \( \mathcal{F} \) approximates the evaluation class \( \mathcal{H}^\alpha \); (2) generator approximation error \( \inf_{g \in \mathcal{G}} d_F(\hat{\mu}_n, g\#\nu) \) measuring the approximation capacity of the generator; (3) statistical error \( d_F(\mu, \hat{\mu}_n) \wedge d_{\mathcal{H}^0}(\mu, \hat{\mu}_n) \) due to the fact that we only have finite samples; and (4) the optimization error \( \epsilon_{\text{opt}} \). When \( \mathcal{F} = \mathcal{N}(W, L, K) \) is a class of norm constrained neural networks, Theorem 3.2 provides an upper bound on the discriminator approximation error. Since any function \( f \in \mathcal{N}(W, L, K) \) is \( K \)-Lipschitz, the generator approximation error can be bounded by

\[
\inf_{g \in \mathcal{G}} d_F(\hat{\mu}_n, g\#\nu) \leq K \inf_{g \in \mathcal{G}} W_1(\hat{\mu}_n, g\#\nu),
\]

where \( W_1 = d_{\text{Lip}_1} \) is the Wasserstein distance. The approximation capacity of generative networks in Wasserstein distance have been studied recently by [Perekrestenko et al., 2020, 2021; Yang et al., 2022b]. Finally, the statistical error can be bounded using empirical process theory.

**Theorem 4.6.** Let \( \mu \) be a probability distribution supported on \([0, 1]^d\) and \( \alpha = r + \beta > 0 \), where \( r \in \mathbb{N}_0 \) and \( \beta \in (0, 1] \). Assume that the generator \( \mathcal{G} \) and source distribution \( \nu \) satisfy \( \inf_{g \in \mathcal{G}} W_1(\hat{\mu}_n, g\#\nu) = 0 \) for any samples \( S_n = \{X_i\}_{i=1}^n \). There exists \( c > 0 \) such that, if the discriminator is chosen as \( \mathcal{F} = \mathcal{N}(W, L, K) \) with

\[
W \geq cK^{(2d+\alpha)/(2d+2)}, \quad L \geq 2[\log_2(d + r)] + 2, \quad K \asymp n^{(d+1)/d},
\]

then, for any GAN estimator \( \hat{g}_n \in \mathcal{G} \) satisfying (4.8),

\[
\mathbb{E}_{S_n}[d_{\mathcal{H}^\alpha}(\mu, (\hat{g}_n)\#\nu)] - \epsilon_{\text{opt}} \lesssim n^{-\alpha/d} \vee n^{-1/2}(\log n)^\tau,
\]

where \( \tau = 1 \) if \( 2\alpha = d \), and \( \tau = 0 \) otherwise.

**Proof.** By Theorem 3.2 and our choice of \( W \) and \( L \), the discriminator approximation error satisfies

\[
\mathcal{E}(\mathcal{H}^\alpha, \mathcal{F}) \lesssim K^{-\alpha/(d+1)}.
\]

If we choose \( K \asymp n^{(d+1)/d} \), then \( \mathcal{E}(\mathcal{H}^\alpha, \mathcal{F}) \lesssim n^{-\alpha/d} \). Since any \( f \in \mathcal{F} \) is \( K \)-Lipschitz,

\[
\inf_{g \in \mathcal{G}} d_F(\hat{\mu}_n, g\#\nu) = \inf_{g \in \mathcal{G}} \sup_{f \in \mathcal{F}} \mathbb{E}_{\hat{\mu}_n}[f] - \mathbb{E}_{g\#\nu}[f] \leq K \inf_{g \in \mathcal{G}} W_1(\hat{\mu}_n, g\#\nu) = 0,
\]

by assumption. Using a standard symmetrization argument (similar to step 2 in the proof of Theorem 3.10), the statistical error \( \mathbb{E}_{S_n}[d_{\mathcal{H}^\alpha}(\mu, \hat{\mu}_n)] \) can be bounded by Rademacher complexity, which can be further bounded by Dudley’s entropy integral (see Huang et al. [2022, Lemma 12] for more details):

\[
\mathbb{E}_{S_n}[d_{\mathcal{H}^\alpha}(\mu, \hat{\mu}_n)] \lesssim \inf_{0 < c < 1/2} \left( \delta + \frac{1}{\sqrt{n}} \int_{\delta}^{1/2} \sqrt{\log \mathcal{N}(\mathcal{H}^\alpha, \| \cdot \|_\infty, \epsilon)} \, de \right).
\]

By Kolmogorov and Tikhomirov [1961], we have the following bound for the covering number

\[
\log \mathcal{N}(\mathcal{H}^\alpha, \| \cdot \|_\infty, \epsilon) \lesssim \epsilon^{-d/\alpha}.
\]

Then, a simple calculation shows (see Huang et al. [2022, Page 9])

\[
\mathbb{E}_{S_n}[d_{\mathcal{H}^\alpha}(\mu, \hat{\mu}_n)] \lesssim n^{-\alpha/d} \vee n^{-1/2}(\log n)^\tau.
\]

The conclusion then follows from Lemma 4.5. \( \square \)
Remark 4.7. The assumption that the generator approximation error is zero can be fulfilled by sufficiently large neural network class \( \mathcal{G} = \mathcal{NN}(W_1, L_1) \). More precisely, it was shown in [Yang et al., 2022b; Huang et al., 2022] that if \( \nu \) is absolutely continuous and \( n \leq W_1^2 L_1 \) then \( \inf_{g \in \mathcal{G}} W_1(\hat{\nu}_n, g_{\#}\nu) = 0 \) for any samples \( S_n = \{X_i\}_{i=1}^n \).

Remark 4.8. For nonparametric density estimation, Liang [2021]; Singh et al. [2018] established the minimax optimal rate \( O(n^{-(\alpha+\beta)/(2\beta+d)} \vee n^{-1/2}) \) for learning distributions in a Sobolev class with smoothness \( \beta \), when the evaluation class is another Sobolev class with smoothness \( \alpha \). The learning rate in Theorem 4.6 matches this optimal rate with \( \beta = 0 \) up to a logarithmic factor, without making any assumptions on the regularity of the target distribution.

Remark 4.9. The optimization problem (4.7) implicitly assume that we can compute the expectation \( E_{g_{\#}\nu}[f] = E_{\nu}[f \circ g] \). This expectation can be estimated by the empirical average \( E_{\nu_m}[f \circ g] \), where \( \nu_m = \frac{1}{m} \sum_{i=1}^m \delta_{Z_i} \) is the empirical distribution of \( m \) random samples \( \{Z_i\}_{i=1}^m \) from \( \nu \). Since \( \nu \) is easy to sample, we can take \( m \) as large as we want. Hence, in stead of (4.7), one can use

\[
\arg\min_{g \in \mathcal{G}} d_{\mathcal{F}}(\hat{\mu}_n, g_{\#}\nu_m) := \arg\min_{g \in \mathcal{G}} \sup_{f \in \mathcal{F}} E_{\nu_m}[f] - E_{g_{\#}\nu_m}[f].
\]

Suppose \( \hat{g}_{n,m} \in \mathcal{G} \) is a solution with optimization error \( \epsilon_{opt} \). Using the argument in Huang et al. [2022], one can show that \( \hat{g}_{n,m} \) achieves the same rate as \( \hat{g}_n \) in Theorem 4.6, if \( m \) is sufficiently large.

It has been demonstrated that Lipschitz continuity of the discriminator is a key condition for a stable training of GANs [Arjovsky and Bottou, 2017; Arjovsky et al., 2017]. In the original Wasserstein GAN [Arjovsky et al., 2017], the Lipschitz constraint on the discriminator is implemented by weight clipping. In the follow-up works, several regularization methods have been proposed to enforce Lipschitz condition, such as gradient penalty [Gulrajani et al., 2017; Petzka et al., 2018], weight normalization [Miyato et al., 2018] and weight penalty [Brock et al., 2019]. In Theorem 4.6, the Lipschitz constant is controlled by the norm constraint \( \kappa(\theta) \leq K \). We can also establish the minimax optimal rate for the corresponding GAN estimator regularized by weight penalty:

\[
\arg\min_{g \in \mathcal{G}} d_{\mathcal{F},\lambda}(\hat{\mu}_n, g_{\#}\nu) := \arg\min_{g \in \mathcal{G}} \sup_{\phi_\theta \in \mathcal{F}} E_{\bar{\nu}_n}[\phi_\theta] - E_{g_{\#}\nu}[\phi_\theta] - \lambda \kappa(\theta)^2, \quad \lambda > 0, \tag{4.9}
\]

where \( \mathcal{F} = \mathcal{NN}(W, L) \) is a neural network class. The following proposition explains the relation between the regularized problem (4.9) and the constrained optimization problem (4.7).

Proposition 4.10. For any probability distributions \( \mu \) and \( \nu \) defined on \( \mathbb{R}^d \), any \( \lambda, K > 0 \),

\[
d_{\mathcal{F},\lambda}(\mu, \nu) = \frac{d_{\mathcal{F}_K}(\mu, \nu)^2}{4\lambda K^2},
\]

where \( \mathcal{F} = \mathcal{NN}(W, L) \) and \( \mathcal{F}_K := \mathcal{NN}(W, L, K) \).

Proof. Observe that, for any \( a \geq 0 \),

\[
\sup_{\phi_\theta \in \mathcal{F}, \kappa(\theta) = a} E_{\mu}[\phi_\theta] - E_{\nu}[\phi_\theta] = a \sup_{\phi_\theta \in \mathcal{F}, \kappa(\theta) = 1} E_{\mu}[\phi_\theta] - E_{\nu}[\phi_\theta],
\]

\[
\sup_{\phi_\theta \in \mathcal{F}, \kappa(\theta) = a} E_{\mu}[\phi_\theta] - E_{\nu}[\phi_\theta] = d_{\mathcal{F}_K}(\mu, \nu)^2 = \frac{d_{\mathcal{F}_K}(\mu, \nu)^2}{4\lambda K^2},
\]

where \( \mathcal{F} = \mathcal{NN}(W, L) \) and \( \mathcal{F}_K := \mathcal{NN}(W, L, K) \).
because if $\phi_\theta$ is parameterized by $\theta = ((A_0, b_0), \ldots, (A_{L-1}, b_{L-1}), A_L)$, then $a_\phi$ can be parameterized by $\theta' = ((A_0, b_0), \ldots, (A_{L-1}, b_{L-1}), aA_L)$ and $\kappa(\theta') = a\kappa(\theta)$. Thus,

$$
d_{F_K}(\mu, \nu) = \sup_{0 \leq a \leq K} \sup_{\phi_\theta \in \mathcal{F}, a(\theta) = a} E_{\mu}[\phi_\theta] - E_{\nu}[\phi_\theta] = K \sup_{\phi_\theta \in \mathcal{F}, a(\theta) = 1} E_{\mu}[\phi_\theta] - E_{\nu}[\phi_\theta].
$$

Therefore,

$$
d_{F,\lambda}(\mu, \nu) = \sup_{\phi_\theta \in \mathcal{F}} E_{\mu}[\phi_\theta] - E_{\nu}[\phi_\theta] - \lambda \kappa(\theta)^2
= \sup_{a \geq 0} \sup_{\phi_\theta \in \mathcal{F}, a(\theta) = a} E_{\mu}[\phi_\theta] - E_{\nu}[\phi_\theta] - \lambda a^2
= \sup_{a \geq 0} a \frac{d_{F_K}(\mu, \nu) - \lambda a^2}{K}
= \frac{d_{F_K}(\mu, \nu)^2}{4\lambda K^2},
$$

where the supremum is achieved at $a = \frac{1}{2\lambda K} d_{F_K}(\mu, \nu)$ in the last equality. \hfill \square

Combining Proposition 4.10 with Theorem 4.6, we can obtain the learning rate of the solution of the regularized optimization problem (4.9).

**Corollary 4.11.** Under the assumption of Theorem 4.6, let $W, L, K$ be the parameters in Theorem 4.6 and $\lambda = \frac{1}{4K^2} \asymp n^{-2(d+1)/d}$, then for any GAN estimator $\hat{g}_{n,\lambda} \in \mathcal{G}$ satisfying

$$
d_{F,\lambda}(\hat{\mu}_n, (\hat{g}_{n,\lambda})^\# \nu) \leq \arg\min_{g \in \mathcal{G}} d_{F,\lambda}(\hat{\mu}_n, g^\# \nu) + \epsilon_{\text{opt}},
$$

where $\mathcal{F} = \mathcal{N}(W, L)$, we have

$$
E_{\nu}[d_{H_0}(\mu, (\hat{g}_{n,\lambda})^\# \nu)] - \sqrt{\epsilon_{\text{opt}}} \lesssim n^{-\alpha/d} \lor n^{-1/2} (\log n)^\tau,
$$

where $\tau = 1$ if $2\alpha = d$, and $\tau = 0$ otherwise.

**Proof.** Since $\lambda = \frac{1}{4K^2}$, by Proposition 4.10,

$$
d_{F_K}(\hat{\mu}_n, (\hat{g}_{n,\lambda})^\# \nu)^2 = d_{F,\lambda}(\hat{\mu}_n, (\hat{g}_{n,\lambda})^\# \nu) \leq \arg\min_{g \in \mathcal{G}} d_{F,\lambda}(\hat{\mu}_n, g^\# \nu) + \epsilon_{\text{opt}}
= \arg\min_{g \in \mathcal{G}} d_{F_K}(\hat{\mu}_n, g^\# \nu)^2 + \epsilon_{\text{opt}},
$$

where we denote $F_K = \mathcal{N}(W, L, K)$. As a consequence,

$$
d_{F_K}(\hat{\mu}_n, (\hat{g}_{n,\lambda})^\# \nu) \leq \sqrt{\arg\min_{g \in \mathcal{G}} d_{F_K}(\hat{\mu}_n, g^\# \nu)^2 + \epsilon_{\text{opt}}} \leq \arg\min_{g \in \mathcal{G}} d_{F_K}(\hat{\mu}_n, g^\# \nu) + \sqrt{\epsilon_{\text{opt}}},
$$

which means $\hat{g}_{n,\lambda} \in \mathcal{G}$ is a solution of (4.7) with discriminator $F_K$ and optimization error $\sqrt{\epsilon_{\text{opt}}}$. Hence, we can apply Theorem 4.6. \hfill \square
5 Conclusions and future work

This paper has established upper and lower approximation bounds for ReLU neural networks with norm constraint on the weights. We used these bounds to analyze the convergence rate of estimating Hölder continuous functions by norm constrained neural networks. In particular, our results can be applied to over-parameterized neural networks, which are widely used in practice. We also showed that GAN can achieve optimal rate of learning probability distributions, when the discriminator is a properly chosen norm constrained neural network. Our results provide statistical guarantees on the performance of norm constrained neural networks.

Norm constrained or regularized neural networks have been widely used in practical applications [Neyshabur et al., 2015a; Miyato et al., 2018; Brock et al., 2019]. But the theory of their approximation and generalization capacity is still very limited. We hope that this work can motivate more study on this field. In the following, we list some possible directions for future research.

- There is a gap between the upper and lower bounds in Theorem 3.2. In [Yarotsky, 2018; Shen et al., 2020], the optimal approximation rates, in terms of the numbers of weights and neurons, are derived through the so-called bit extraction technique [Bartlett et al., 2019]. By using this technique, one can approximately discretize the input and reduce the approximation problem to an interpolation problem [Shen et al., 2020; Lu et al., 2021]. This helps us avoid computing the outer summation in the local Taylor approximation (3.2). Hence, we think it is worth to explore whether one can apply bit extraction technique to construct norm constrained neural networks that have better approximation rates.

- The lower bound in Theorem 3.2 is derived through the upper bound for Rademacher complexity in Lemma 2.3. This upper bound is independent of the width, but depends on the depth. It is still unclear whether it is possible to obtain size-independent bounds without further assumption on the weights of neural networks.

- In the definition of norm constraint (2.4), we restrict ourselves to the operator norm induced by $\| \cdot \|_\infty$ for the weight matrices. It will be interesting to extend the results to other norms. A more fundamental question is how different norms affect the approximation and generalization capacity?

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