PICONE’S IDENTITY FOR \( p \)-BIHARMONIC OPERATOR AND ITS APPLICATIONS

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Abstract. In this article we prove the nonlinear analogue of Picone’s identity for \( p \)-biharmonic operator. As an application of our result we show that the Morse index of the zero solution to a \( p \)-biharmonic boundary value problem is 0. We also prove a Hardy type inequality and Sturmian comparison principle. We also show the strict monotonicity of the principle eigenvalue and linear relationship between the solutions of a system of singular \( p \)-biharmonic system.

1. Introduction

The classical Picone’s identity says that for differentiable functions \( v > 0 \) and \( u \geq 0 \),
\[
|\nabla u|^2 + \frac{u^2}{v^2} |\nabla v|^2 - 2 \frac{u}{v} \nabla u \nabla v = |\nabla u|^2 - \nabla \left( \frac{u^2}{v} \right) \nabla v \geq 0.
\]
(1.1) has an enormous applications to second-order elliptic equations and systems, see for instance, [1, 2, 3, 12] and the references therein. Nonlinear analogue of (1.1) is established by J. Tyagi [13]. In order to apply (1.1) to \( p \)-Laplace equations, (1.1) is extended by W. Allegretto and Y.X.Huang [4]. Nonlinear analogue of Picone’s type identity for \( p \)-Laplace equations is established by K. Bal [5].

In [6], D.R.Dunninger established a Picone identity for a class of fourth order elliptic differential inequalities. This identity says that if \( u, v, a \Delta u, A \Delta v \) are twice continuously differentiable functions with \( v(x) \neq 0 \) and \( a \) and \( A \) are positive weights, then
\[
\text{div} \left[ u \nabla (a \Delta u) - a \Delta u \nabla u - \frac{u^2}{v} \nabla (A \Delta v) + A \Delta v \nabla \left( \frac{u^2}{v} \right) \right] = -\frac{u^2}{v} \Delta (A \Delta v) + u \Delta (a \Delta u) + (A - a)(\Delta u)^2
\]
(1.2)
\[
- A \left( \Delta u - \frac{u}{v} \Delta v \right)^2 + A \frac{2 \Delta v}{v} \left( \nabla u - \frac{u}{v} \nabla v \right)^2.
\]

With some simplifications in (1.2), we obtain the following identity:

Let \( u \) and \( v \) be twice continuously differentiable functions in \( \Omega \) such that \( v > 0, -\Delta v > 0 \) in \( \Omega \). Denote
\[
L(u, v) = \left( \Delta u - \frac{u}{v} \Delta v \right)^2 - \frac{2 \Delta v}{v} \left( \nabla u - \frac{u}{v} \nabla v \right)^2.
\]
(1.3)

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\[ R(u, v) = |\Delta u|^2 - \Delta \left( \frac{u^2}{v} \right) \Delta v. \]

Then (i) \( L(u, v) = R(u, v) \) (ii) \( L(u, v) \geq 0 \) and (iii) \( L(u, v) = 0 \) in \( \Omega \) if and only if \( u = \alpha v \) for some \( \alpha \in \mathbb{R} \).

Nonlinear analogue of (1.3) is established by G. Dwivedi and J. Tyagi [7]. Picone’s identity for \( p \)-biharmonic operator is established by J. Jaróš [10]. Picone’s identity for elliptic differential operators are discussed by N. Yoshida [14] and J. Jaróš [11].

In this article we establish the nonlinear analogue of Picone’s identity for \( p \)-biharmonic operator. We also discuss some qualitative results in the spirit of W. Allegretto and Y.X. Huang [4] and J. Tyagi [13].

The plan of the paper is as follows: Section 2 deals with nonlinear analogue of Picone’s identity. In section 3, we give several application of Picone’s identity to \( p \)-biharmonic equations.

2. Main Results

Throughout this article, we assume the following hypotheses, unless otherwise stated.

(i) \( \Delta^2_p := \Delta(|\Delta u|^{p-2} \Delta u) \), denotes \( p \)-biharmonic operator.
(ii) \( \Omega \) denotes any domain in \( \mathbb{R}^n \).
(iii) \( 1 < p < \infty \).
(iv) \( f : \mathbb{R} \rightarrow (0, \infty) \) be a \( C^2 \) function.

First we state Young’s inequality, which will be used later.

**Lemma 2.1.** If \( a \) and \( b \) are two nonnegative real numbers and \( p \) and \( q \) are such that \( \frac{1}{p} + \frac{1}{q} = 1 \), then

\[ ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \]

equality holds if and only if \( a^p = b^q \).

**Proof.** For proof we refer to [5]. \( \square \)

Next we give Picone’s identity for \( p \)-biharmonic operator.

**Lemma 2.2.** Let \( u \geq 0, v > 0 \) be twice continuously differentiable functions in \( \Omega \) and \( -\Delta v > 0 \) in \( \Omega \). Denote

\[ R(u, v) = |\Delta u|^p - \Delta \left( \frac{u^p}{v^{p-1}} \right) |\Delta v|^{p-2} \Delta v \]
\[ L(u, v) = |\Delta u|^p + \frac{(p-1)}{v^p} |\Delta v|^p - \frac{p}{v^{p-1}} |\Delta v|^{p-2} \Delta u \Delta v \]
\[ - \frac{p(p-1)}{v^{p-1}} |\Delta v|^{p-2} \left( \nabla u - \frac{u}{v} \nabla v \right)^2. \]

Then (i) \( L(u, v) = R(u, v) \), (ii) \( L(u, v) \geq 0 \), (iii) \( L(u, v) = 0 \) in \( \Omega \) if and only if \( u = \alpha v \) for some \( \alpha \in \mathbb{R} \).

**Proof.** Let us expand \( R(u, v) \):

\[ R(u, v) = |\Delta u|^p - \Delta \left( \frac{u^p}{v^{p-1}} \right) |\Delta v|^{p-2} \Delta v \]
By Lemma 2.1, with $a \geq 0$. This proves that (I) holds.

Now consider (II): Since $|\Delta u| |\Delta v| \geq \Delta u \Delta v$, therefore, (II) holds.

Now consider (I):

\[
(I) = |\Delta u|^p + \frac{(p - 1)u^p}{v^p} |\Delta v|^p - \frac{pu^{p-1}}{v^{p-1}} |\Delta v|^{p-2} \Delta u \Delta v
\]

By Lemma 2.1 with $a = |\Delta u|$ and $b = \frac{|\Delta v|^{p-1}}{v^{p-1}}$, we get

\[
|\Delta u|^p + \frac{(p - 1)u^p}{v^p} |\Delta v|^p - \frac{pu^{p-1}}{v^{p-1}} |\Delta v|^{p-2} \Delta u \Delta v \geq 0.
\]

This proves that (I) holds. This proves the (ii), that is, $L(u, v) \geq 0$.

Now $L(u, v) = 0$ in $\Omega$ implies that $\nabla u - \frac{u}{v} \nabla v = 0$, provided $u(x_0) \neq 0$ for some $x_0 \in \Omega$. This gives $\nabla \left( \frac{u}{v} \right) = 0$, that is, $u = \alpha v$ for some $\alpha \in \mathbb{R}$. This completes the proof.

Now we establish the nonlinear analogue of Picone’s identity for $p$-biharmonic operator.
**Lemma 2.3.** Let \( u \) and \( v \) be twice continuously differentiable functions in \( \Omega \) such that \( u \geq 0 \) and \(-\Delta v > 0\) in \( \Omega \). Let \( f : \mathbb{R} \to (0, \infty) \) be a \( C^2 \) function such that \( f'(y) \geq (p - 1)|f(y)| \) for all \( y \in \mathbb{R} \) and \( f''(y) \leq 0 \), \( \forall y \in \mathbb{R} \). Denote

\[
L(u, v) = |\Delta u|^p - \frac{pu^{p-1}|\Delta v|^p \Delta u}{f} + \frac{u p f'(v)|\Delta v|^p}{f^2} \left( \frac{1}{2} \frac{|\Delta v|^{p-2}u^{p-2}}{f} \right) \left[ \left( \frac{2u f'}{f} \nabla v - p \nabla u \right)^2 + p(p-1)|\nabla u|^2 \right].
\]

\[
R(u, v) = |\Delta u|^p - \Delta \left( \frac{u^p}{f(v)} \right) |\Delta v|^{p-2} \Delta v.
\]

Then (i) \( L(u, v) = R(u, v) \), (ii) \( L(u, v) \geq 0 \), (iii) \( L(u, v) = 0 \) in \( \Omega \) if and only if \( u = \alpha v \) for some \( \alpha \in \mathbb{R} \).

**Proof.** Let us expand the \( R(u, v) \):

\[
R(u, v) = |\Delta u|^p - \Delta \left( \frac{u^p}{f(v)} \right) |\Delta v|^{p-2} \Delta v
\]

\[
= |\Delta u|^p + \frac{u p f'(v)|\Delta v|^p}{f^2} - \frac{pu^{p-1}|\Delta v|^p \Delta u}{f^2} \left( \frac{1}{2} \frac{|\Delta v|^{p-2}u^{p-2}}{f} \right) \left[ \left( \frac{2u f'}{f} \nabla v - p \nabla u \right)^2 + p(p-1)|\nabla u|^2 \right],
\]

\[
= \left( \frac{\Delta v|\Delta v|^{p-2}p^{p-2}}{f} \right) \left( p(p-1)|\nabla u|^2 + \frac{2u^2 f^2}{f^2} |\nabla v|^2 - \frac{2pf'u(\nabla u, \nabla v)}{f} \right),
\]

\[
R(u, v) = \left( \frac{\Delta v|\Delta v|^{p-2}p^{p-2}}{f} \right) \left( p(p-1)|\nabla u|^2 + \frac{2u^2 f^2}{f^2} |\nabla v|^2 - \frac{2pf'u(\nabla u, \nabla v)}{f} \right).
\]

First we consider (ii).

\[
(\text{ii}) = -\frac{\Delta v|\Delta v|^{p-2}p^{p-2}}{f} \left[ p(p-1)|\nabla u|^2 + \frac{2u^2 f^2}{f^2} |\nabla v|^2 - \frac{2pf'u(\nabla u, \nabla v)}{f} \right]
\]

\[
+ \left( \frac{p\nabla u}{\sqrt{2}} \right)^2 - \left( \frac{p\nabla u}{\sqrt{2}} \right)^2
\]

\[
= -\frac{1}{2} \frac{\Delta v|\Delta v|^{p-2}p^{p-2}}{f} \left[ \left( \frac{2uf'}{f} \nabla v - p \nabla u \right)^2 + p(p-1)|\nabla u|^2 \right].
\]
This completes (i), that is, \( L(u, v) = R(u, v) \). Also (II) \( \geq 0 \), since \(-\Delta v > 0\).

Next consider (I).

\[
\text{(I)} = \left( |\Delta u|^p + \frac{u^p f'(v) |\Delta v|^p}{f^2} - \frac{p u^{p-1} |\Delta v|^{p-2} |\Delta u| |\Delta v|}{f(v)} \right) \\
+ \frac{p u^{p-1} |\Delta v|^{p-2}}{f} (|\Delta u| |\Delta v| - \Delta u \Delta v),
\]

clearly second term of above equation is nonnegative. So on using Young’s inequality (Lemma 2.1) with \( a = |\Delta u|, b = \frac{(u|\Delta v|)^{p-1}}{f} \), we get

\[
\frac{|\Delta u|^{p-1}|\Delta v|^{p-1}}{f} \leq \frac{|\Delta u|^p}{p} + \frac{(u|\Delta v|)^{(p-1)q}}{q f^q}
\]

\[
p \frac{|\Delta u|^{p-1}|\Delta v|^{p-1}}{f} \leq |\Delta u|^p + (p-1) \frac{(u|\Delta v|)^{(p-1)q}}{f^q},
\]

equality holds when

\[
(2.2) \quad |\Delta u| = \frac{u|\Delta v|}{(f(v))^\frac{p}{q}}.
\]

Now on using \( f'(y) \geq (p-1)[f(y)]^{\frac{p-2}{p}} \), we get

\[
|\Delta u|^p + \frac{(u^p f'(v) |\Delta v|^p)}{f^2} - \frac{p u^{p-1} |\Delta v|^{p-1}}{f} \geq 0
\]

and equality holds when

\[
(2.3) \quad f'(y) = (p-1)[f(y)]^{\frac{p-2}{p}}
\]

This gives (I) \( \geq 0 \).

Now (III) \( \geq 0 \), since \(-\Delta v > 0\) and \( f''(v) \leq 0 \). This proves (ii). \( \square \)

### 3. Applications

In this section we will give some applications of nonlinear Picone’s identity following the spirit of [4].

**Hardy type result.** We start with establishing a Hardy type inequality for p-biharmonic operator.

**Theorem 3.1.** Let there be a \( v \in C_\infty^c(\Omega) \) such that

\[
\Delta(|\Delta u|^{p-2} \Delta v) \geq \lambda g f(v), \quad v > 0 \text{ in } \Omega, \quad -\Delta v > 0 \text{ in } \Omega,
\]

for some \( \lambda > 0 \) and a nonnegative continuous function \( g \) then for any \( u \in C_\infty^c(\Omega) \); \( u \geq 0 \) it holds that

\[
(3.1) \quad \int_\Omega |\Delta u|^p dx \geq \lambda \int_\Omega g |u|^p dx,
\]

where \( f \) satisfies \( f'(y) \geq (p-1)[f(y)]^{\frac{p-2}{p}} \).

**Proof.** Take \( \phi \in C_\infty^c(\Omega), \phi > 0 \). By Lemma 2.3 we have

\[
0 \leq \int_\Omega L(\phi, v) dx
\]

\[
= \int_\Omega R(\phi, v) dx = \int_\Omega \left( |\Delta \phi|^p - \Delta \left( \frac{\phi^p}{f(v)} \right) |\Delta v|^{p-2} \Delta v \right) dx
\]
\[ \int_\Omega |\Delta \phi|^p dx - \int_\Omega \frac{\phi^p}{f(v)} |\Delta^2 \phi|^{p-2} \Delta \phi dx \]

\[ \leq \int_\Omega |\Delta \phi|^p dx - \lambda \int_\Omega \phi^p g dx \]

letting \( \phi \to u \), we get

\[ \int_\Omega |\Delta u|^p dx \geq \lambda \int_\Omega g |u|^p dx. \]

\[ \square \]

**Strumium comparison principle.** Comparison principles play vital role in study of partial differential equations. Here, we establish nonlinear version of Sturmium comparison principle for p-biharmonic operator.

**Theorem 3.2.** Let \( f_1 \) and \( f_2 \) are two weight functions such that \( f_1 < f_2 \) and \( f \) satisfies \( f'(y) \geq (p-1)[f(y)]^{\frac{p-2}{p}} \). If there is a positive solution \( u \) satisfying

\[ \Delta^2_p v = f_1(x)|u|^{p-2} u \text{ in } \Omega \]

\[ u = 0 = \Delta u \text{ on } \partial \Omega. \]

\[ (3.2) \]

Then any nontrivial solution \( v \) of

\[ \Delta^2_p v = f_2(x)f(v) \text{ in } \Omega \]

\[ u = 0 = \Delta u \text{ on } \partial \Omega, \]

\[ (3.3) \]

must change sign.

**Proof.** Let us assume that there exists a solution \( v > 0 \) of \( (3.3) \) in \( \Omega \). Then by Picone’s identity we have

\[ 0 \leq \int_\Omega L(u,v) dx = \int_\Omega R(u,v) dx \]

\[ = \int_\Omega |\Delta u|^p - \Delta \left(\frac{u^p}{f(v)}\right) |\Delta^2 u|^{p-2} \Delta v dx \]

\[ = \int_\Omega \left( f_1(x)u^p - f_2(x)u^p \right) dx \]

\[ = \int_\Omega (f_1 - f_2) u^p dx < 0, \]

which is a contradiction. Hence, \( v \) changes sign. \[ \square \]

**Strict Monotonicity of principle eigenvalue in domain.** Consider the indefinite eigenvalue problem

\[ \Delta^2_p u = \lambda g, \text{ in } \Omega, \]

\[ u = 0 = \Delta u \text{ on } \partial \Omega, \]

\[ (3.4) \]

where \( g(x) \) is indefinite weight function.

**Theorem 3.3.** Let \( \lambda_1^+ (\Omega) > 0 \) be the principle eigenvalue of \( (3.4) \), then suppose \( \Omega_1 \subset \Omega_2 \) and \( \Omega_1 \neq \Omega_2 \). Then \( \lambda_1^+ (\Omega_1) > \lambda_1^+ (\Omega_2) \), if both exist.

**Proof.** Let \( u_i \) be a positive eigenfunction associated with \( \lambda_i^+ (\Omega_i), i = 1, 2 \). Evidently for \( \phi \in C_0^\infty (\Omega_i) \), we have

\[ 0 \leq \int_{\Omega_i} L(\phi_1, u_2) dx \]
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\[\int_{\Omega_1} |\Delta \phi|^p dx - \int_{\Omega_1} \phi p \int_{\Omega_1} \Delta (|\Delta u_2|^{p-2} \Delta u_2) dx \]

\[= \int_{\Omega_1} |\Delta \phi|^p dx - \lambda^+_1(\Omega_2) \int_{\Omega_1} g(x) \phi^p dx.\]

Letting \(\phi \to u_1\), we get

\[0 \leq \int_{\Omega} L(u_1, u_2) dx = (\lambda^+_1(\Omega_1) - \lambda^+_1(\Omega_2)) \int_{\Omega_1} g \phi^p dx,\]

this gives \(\lambda^+_1(\Omega_1) - \lambda^+_1(\Omega_2) > 0\), as if \(\lambda^+_1(\Omega_1) = \lambda^+_1(\Omega_2)\), we conclude that \(u_1 = ku_2\), which is not possible as \(\Omega_1 \subset \Omega_2\) and \(\Omega_1 \nsubseteq \Omega_2\). This completes the proof. \(\square\)

**Quasilinear System with Singular nonlinearity.** We will use Picone’s identity to establish a linear relationship between solutions of a quasilinear system with singular nonlinearity. Consider the singular system of elliptic equations

\[\Delta^2 p u = f(v), \text{ in } \Omega,\]

\[\Delta^2 p v = (f(v))^2, \text{ in } \Omega,\]

\[u > 0, v > 0 \text{ in } \Omega,\]

\[u = 0 = v \text{ on } \partial \Omega,\]

\[\Delta u = 0 = \Delta v \text{ on } \partial \Omega,\]

where \(f\) satisfies \(f'(y) \geq (p - 1)|f(y)|^{\frac{p-2}{2}}\).

**Theorem 3.4.** Let \((u, v)\) be a weak solution of \((3.5)\) and \(f\) satisfy \(f'(y) \geq (p - 1)|f(y)|^{\frac{p-2}{2}}\), then \(u = c_1 v\), where \(c_1\) is a constant.

**Proof.** Let \((u, v)\) be weak solution of \((3.5)\). Then for any \(\phi_1, \phi_2 \in H^2(\Omega) \cap H_0^1(\Omega)\), we have

\[\int_{\Omega} |\Delta u|^p \Delta \phi_1 dx = \int_{\Omega} f(v) \phi_1 dx \]

\[\int_{\Omega} |\Delta v|^p \Delta \phi_2 dx = \int_{\Omega} \frac{f^2(v)}{u^{p-2}} \phi_2 dx.\]

Choosing \(\phi_1 = u\) and \(\phi_2 = \frac{u^2}{f(v)}\) in \((3.6)\) and \((3.7)\) respectively, we get

\[\int_{\Omega} |\Delta u|^p dx = \int_{\Omega} u f(v) dx\]

\[= \int_{\Omega} |\Delta v|^p \Delta u \left(\frac{u^2}{f(v)}\right) dx,\]

which gives

\[\int_{\Omega} \left( |\Delta u|^p - |\Delta v|^p |\Delta u|^{p-2} \right) \left(\frac{u^2}{f(v)}\right) dx = 0,\]

or

\[\int_{\Omega} R(u, v) dx = 0,\]

this gives \(R(u, v) = 0\), which in turn implies that \(u = c_1 v\). \(\square\)
Morse Index. Let us consider the problem
\[(3.8)\]
\[
\Delta^2 u = a(x)f(u), \quad \text{in } \Omega, \\
u = 0 = \Delta u, \quad \text{on } \partial \Omega.
\]
The morse index of the solution of the \((3.8)\) is the number of negative eigenvalues of the linearized operator
\[
\Delta^2 - a(x)f'(u)
\]
acting on \(H^2(\Omega) \cap H_0^1(\Omega)\), that is, the number of eigenvalue \(\lambda\) such that \(\lambda < 0\) and the boundary value problem
\[(3.9)\]
\[
\Delta^2 w - a(x)f'(u)w = \lambda w, \quad \text{in } \Omega, \\
w = 0 = \Delta w, \quad \text{on } \partial \Omega
\]
has a nontrivial solution \(w\) in \(H^2(\Omega) \cap H_0^1(\Omega)\).

Theorem 3.5. Let us consider \((3.8)\). Suppose \(f'(0) \leq 1 \leq f'(s), \forall s \in (0, \infty)\) and \(f(0) = 0\). Let \(a(x)\) be a positive continuous function in \(\bar{\Omega}\). Then the trivial solution of \((3.8)\) has morse index 0.

Proof. Let \(v \in H^2(\Omega) \cap H_0^1(\Omega)\) be a positive solution of \((3.8)\). Then
\[(3.10)\]
\[
\int_\Omega |\Delta v|^p - 2 \Delta v \Delta \psi dx = \int_\Omega a(x)f(v) \psi dx, \quad \forall \psi \in H^2(\Omega) \cap H_0^1(\Omega).
\]
For any \(0 \neq w \in H^2(\Omega) \cap H_0^1(\Omega)\), let us take \(\psi = \frac{w^2}{f(v)}\) as a test function in \((3.10)\), we get
\[(3.11)\]
\[
\int_\Omega |\Delta v|^p - 2 \Delta v \Delta \left( \frac{w^2}{f(v)} \right) dx = \int_\Omega a(x)f(v) \frac{w^2}{f(v)} dx,
\]
on using \(R(u, v) \geq 0\), we get
\[(3.12)\]
\[
\int_\Omega |\Delta w|^p dx \geq \int_\Omega a(x)w^2 dx \geq \int_\Omega a(x)f'(0)w^2 dx.
\]
Consider the eigenvalue problem associated with the linearization of \((3.8)\) at 0, which is nothing but
\[(3.13)\]
\[
\Delta^2 w - a(x)f'(0)w = \lambda w, \quad \text{in } \Omega, \\
w = 0 = \Delta w, \quad \text{on } \partial \Omega
\]
By variational characterization of the eigenvalue in \((3.13)\), from \((3.12)\), we get that \(\lambda \geq 0\) and corresponding eigenfunction is positive. Which proves the claim. \(\square\)

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