Haupt–Kapovich theorem revisited

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On the 100th anniversary of O. Haupt’s paper
“Ein Satz über die Abelschen Integrale”
and 20th anniversary of M. Kapovich’s paper
“Periods of abelian differentials and dynamics”

Abstract
A theorem of O. Haupt, rediscovered by M. Kapovich and celebrated by his proof invoking Ratner theory, describes the set of de Rham cohomology classes on a topological orientable surface, which can be realized by an abelian differential in some respective complex structure, in purely topological terms. We make an attempt to describe similarly pairs and triples of cohomology classes, which can be realized by abelian differentials in some complex structure. This leads us to some interesting problems in algebraic geometry of curves, and gives an unexpected local description of the Teichmüller space.

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1 Introduction

The problem which the Haupt–Kapovich theorem concerns is the following. Let $S$ be a topological genus $g > 1$ orientable surface. Its first cohomology $H^1(S, \mathbb{C})$ is a complexification of a lattice endowed with an integral skew-symmetric form, the intersection pairing. If the complex structure $I$ on the surface is fixed, making it into a complex curve, one can consider the space $H^{1,0}(S, I)$ of cohomology classes of holomorphic 1-forms. Whenever another complex structure is chosen, another $H^{1,0}$ subspace in cohomology arises, so it of course cannot be reconstructed from the linear-algebraic data of topological origin on $H^1(S, \mathbb{C})$. But can the union of all the $H^{1,0}$ subspaces for all possible complex structures be captured by topology? The answer is positive, and is given by the following

**Theorem** (O. Haupt, 1920 [H], M. Kapovich, 2000 [K]). The union

$$\bigcup_{I \in T_S} H^{1,0}(S, I) \subset H^1(S, \mathbb{C})$$

of all the $H^{1,0}$ subspace over all possible complex structures is precisely the subset of the classes $[\alpha]$ which satisfy the following conditions:

1. $\sqrt{-1}[\alpha] \wedge [\bar{\alpha}] \geq 0$,
2. Whenever the set of periods $\left\{ \int_\gamma \alpha \right\}_{\gamma \in H_1(S, \mathbb{Z})}$ is a lattice in $\mathbb{C}$, one has $\sqrt{-1}[\alpha] \wedge [\bar{\alpha}]$ greater than the covolume of this lattice.

Here $T_S$ stands for the Teichmüller space of complex structures on $S$ modulo isotopy. The necessity of these conditions is obvious: the first follows from the fact that $\sqrt{-1}dz \wedge d\bar{z}$ is a positive volume form on a Riemannian surface; as for the second, if $\alpha$ is a holomorphic representative, the multivalued integral $\int_{\mathbb{C}}^{z_0} \alpha$ descends to a holomorphic mapping onto an elliptic curve, the quotient of $\mathbb{C}$ by the periods of $\alpha$, and the degree of a map from a genus $g > 1$ curve onto an elliptic curve is at least two, so its area w.r.t. the class $[\alpha]$ is at least twice as the area of the elliptic curve, i.e. the covolume of the period lattice. However, the sufficiency of these conditions is nontrivial. Although it can be proved by elementary means (and this way has been proven by Haupt), a more elegant proof by Kapovich is based on the ergodic theory, especially the Ratner theorem.

Since the subset of realizable classes is invariant under scaling, one may consider it as a question about characterization of linear subspaces, which fall into $H^{1,0}(S, I)$ for some complex structure $I$. Bogomolov noticed that a variation of this question turns out to be interesting:

**Problem.** Describe the union

$$\bigcup_{I \in T_S} \text{Gr}(k, H^{1,0}(S, I)) \subset \text{Gr}(k, H^1(S, \mathbb{C}))$$

in terms of topology of $S$. 

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In this paper, we slightly modify the proof of Kapovich in order to fit better for dealing with this problem.

The paper is organized as follows. In the Section 2 one discusses the openness of the set of representable classes (or pairs, or triples). In the subsection 2a the notation is fixed and familiar facts about the Teichmüller space, Gauss–Manin connection, Kodaira–Spencer class for algebraic curves are recollected. In the subsection 2b one introduces the Kapovich–Torelli mappings, and restates a openness theorem due to Hejhal and Thurston, which is of great importance in the Kapovich’s paper. In the subsection 2c a weaker version of Hejhal–Thurston theorem for pairs and triples of abelian differentials is proved, and the algebraic-geometric condition on the pairs in which this theorem fails, is outlined. In the subsection 2d a geometric construction of certain examples of failure of Hejhal–Thurston theorem for pairs and triples is provided.

The Section 3 invokes ergodic-theoretic considerations. Moore’s theorem implies that the sets of representable classes are dense, which in particular implies the Proposition 3.2 which is one of the main results of the paper.

Theorem (Proposition 3.2). Let \((S, I)\) be a complex curve of genus at least four, and \(\tau \subset H^{1,0}(S, I)\) be a generic three-dimensional subspace. There exist a neighborhood \(U \subset \text{Gris}(3, 2g)\) in the isotropic Graßmannian w. r. t. the intersection pairing containing \(\tau\) s. t. for any \(\tau' \in U\) there exists a unique complex structure \(I' = I(\tau')\) s. t. one has \(\tau' \subset H^{1,0}(S, I')\). In other words, deformation of a generic triple of abelian differentials determines a unique local deformation of a complex structure, so that the deformed cohomology classes would be represented by abelian differentials in the deformed complex structure.

In the Section 4 one describes some geometry of the isoperiodic locus with prospect on so-called lesser isoperiodic foliation defined in the subsection 2c (Definition 8). There we also prove a similar statement, but restricted to hyperelliptic curves:

Theorem (Proposition 4.6). Let \((S, I)\) be a hyperelliptic complex curve of genus at least three, and \(\tau \subset H^{1,0}(S, I)\) be a generic two-dimensional subspace. There exist a neighborhood \(U \subset \text{Gris}(2, 2g)\) containing \(\tau\) s. t. for any \(\tau' \in U\) there exists a unique hyperelliptic complex structure \(I' = I(\tau')\) s. t. one has \(\tau' \subset H^{1,0}(S, I')\). In other words, deformation of a generic pair of abelian differentials determines a unique local deformation of a hyperelliptic complex structure, so that the deformed cohomology classes would be represented by abelian differentials in the deformed complex structure.

In what follows, we use italics when reminding the well-known definitions, and bold when introducing our own.
2 Hejhal–Thurston ‘holonomy’ theorem

2a Three incarnations of Kodaira–Spencer tensor

This subsection is a reminder of what is widely known. For an introduction, see an excellent book [TrAG].

Shape operator of the Hodge bundle. Consider the space $\mathcal{C}_x(T^*S \otimes TS)$ of tensors $I \in C^\infty(T^*S \otimes TS)$ satisfying $I^2 = -\text{Id}_{TS}$ with the usual topology on smooth sections of a vector bundle. Any such tensor gives rise to a complex structure since $\dim S = 2$. The diffeomorphism group $\text{Diff}(S)$ acts on this space via pullbacks.

Definition 1. The quotient $\mathcal{T}(S) = C_x(S)/\text{Diff}^0(S)$ by the connected component of the diffeomorphism group is called the Teichmüller space.

Fact. The Teichmüller space of a genus $g$ surface is isomorphic, as a quotient of a Fréchet manifold by a Fréchet–Lie group, to a finite-dimensional smooth manifold of dimension $6g - 6$ (actually, an open ball). It carries a natural complex structure and a holomorphic fibration called the universal curve $\mathcal{U}_S \to \mathcal{T}_S$ over a point $I$ biholomorphic to $(S, I)$.

Let $H^1_\omega \mathcal{T}_S \to \mathcal{T}_S$ be the sheaf $R^1 \pi_*(\mathcal{Z}_{\mathcal{U}_S})$. It is actually a sheaf of sections of the local system $H^1_\omega \mathcal{T}_S$ with stalk over the point $I$ being the integral first cohomology of the fiber $\pi^{-1}(I)$. It can be complexified to obtain the vector bundle $H^1 \mathcal{T}_S = H^1_\omega \mathcal{T}_S \otimes \mathcal{O}_\mathcal{T}_S$ with fiber over the point $I$ being the vector space $H^1(\pi^{-1}(I), \mathbb{C}) \approx H^1(S, \mathbb{C})$.

Definition 2. The only connection $\nabla^{GM}$ on the bundle $H^1 \mathcal{T}_S$ making the local sections of $H^1_\omega \mathcal{T}_S \subset H^1 \mathcal{T}_S$ parallel is called the Gauss–Manin connection.

Since in each neighborhood it possesses a basis of parallel sections, this connection is flat, and since the Teichmüller space is simply connected, it gives a trivialization of the bundle, i.e. the projection $H^1 \mathcal{T}_S \to V := H^1(S, \mathbb{C})$ which we shall denote by $\pi$.

Definition 3. The cohomology bundle $H^1 \mathcal{T}_S$ has a subbundle $\Omega \mathcal{T}_S \subset H^1 \mathcal{T}_S$ s.t. $\Omega \mathcal{T}_S |_I = H^{1,0}(S, I) \subset H^1(S, I) = H^1 \mathcal{T}_S(I)$, i.e. the bundle of abelian differentials. It is called the Hodge (sub)bundle and is not parallel w. r. t. the Gauss–Manin connection. It can be also defined abstractly as $\pi_*(\Omega_{\mathcal{U}_S}/\mathcal{T}_S)$.

Since no abelian differential has real coefficients, the Hodge subbundle does not intersect its complex conjugate. Hence one can consider the projection $\omega: H^1 \mathcal{T}_S \to \Omega \mathcal{T}_S$ with kernel $\Omega \mathcal{T}_S$.

Definition 4. The Kodaira–Spencer tensor $KS: T\mathcal{T} \mathcal{O}_{\mathcal{T}_S} \to \Omega \mathcal{T}_S \cong (\Omega \mathcal{T}_S)^*$ is the shape operator of the Gauss–Manin connection w. r. t. this splitting:

$$KS_\omega(\alpha) = \omega(\nabla^{GM}_\omega \alpha).$$

It is a standard fact that this operator is linear over $C^\infty(\mathcal{T}_S)$ in both indices.
First cohomology space of the tangent bundle. The tangent space \( T_I Cx(S) \) is the space of all tensors \( A: TS \to TS \) s. t. the operator \( I + \varepsilon A \) for \( \varepsilon^2 = 0 \) squares to \( -\text{Id}_{TS} \). Expanding this, one gets \( I^2 + \varepsilon(IA + IA) + \varepsilon^2 A^2 = -\text{Id} = I^2 \), or just \( IA + IA = 0 \). The space of such tensors is acted upon by the Lie algebra of the diffeomorphism group, i. e. the Lie algebra of vector fields.

**Fact.** The quotient
\[
\{ A: TS \to TS \mid AI + IA = 0 \} / \{ \text{Lie}_v I \mid v \in C^\infty(TS) \}
\]
is indeed isomorphic to the tangent space of the Teichmüller space at \( I \).

This quotient resembles the first de Rham cohomology: indeed, it is the quotient of 1-forms (with vector fields coefficients) subject to some closedness condition by 1-forms coming from vector-valued 0-forms (i. e. vector fields) via derivation.

Let \( E \to X \) be a complex vector bundle over a complex manifold. To give a complex structure on its total space in which the fibers are complex submanifolds and the projection map is holomorphic is the same as to give the Dolbeault operator
\[
\bar{\partial}: \Gamma(E) \to \Omega^{0,1}(X, E).
\]
Its extension to all forms with coefficients in sections of \( E \) via the usual Leibniz rule satisfies \( \bar{\partial}^2 = 0 \), hence one can speak of cohomology with coefficients in a holomorphic vector bundle. In particular, \( H^0_\bar{\partial}(E) \) is just the space of holomorphic sections of \( E \).

**Fact.** The tangent space to the Teichmüller space at point \( I \) can be canonically identified with the cohomology space \( H^1(S, TS) \) for the standard holomorphic structure on the tangent bundle of the complex curve \( (S, I) \).

More geometric way to see the correspondence between the first-order deformations of the complex structure and the first cohomology of the holomorphic tangent bundle is as follows. Consider a first-order deformation \( \mathfrak{X} \to \Delta = \text{Spec } \mathbb{C}[h]/(h^2) \) with central fiber \( \mathfrak{X}_0 \approx X \), and the short exact sequence of vector bundles \( TX \to T\mathfrak{X}|_X \to \mathcal{O}_X \). The isomorphism classes of extensions \( E \to E' \to 0 \) are precisely the first cohomology classes from \( H^1(X, E) \).

**Fact.** The Kodaira–Spencer tensor defined above is the same as the mapping
\[
\text{KS}: H^{1,0}(S, I) \times H^1(S, TS) \to H^{0,1}(S, I)
\]
which can be written down on the representatives by the rule
\[
\text{KS}(\alpha \times v) = \alpha(v(x)),
\]
where \( \alpha \in \Omega^{1,0}(S, I) \) is a closed \((1, 0)\)-form, \( v \in \Omega^1(S, TS) \) is a vector-valued \( 1 \)-form vanishing on \((1, 0)\)-vector, and the result lies in the space \( \Omega^{0,1}_{\text{cl}}(S, I) \) of closed \((0, 1)\)-forms.
Multiplication of holomorphic 1-forms. Let $X$ be an $n$-dimensional complex projective manifold, $K_X$ be its canonical bundle, i.e., the top exterior power of the holomorphic cotangent bundle, and $E$ a holomorphic vector bundle.

**Fact** (Serre duality). $H^i(E) \approx H^{n-i}(E^* \otimes K_X)^*$.  

Let $\Omega^p = \Omega^{p,0}$ be the complex exterior $p$-th power of the holomorphic cotangent bundle. The cohomology of these bundle has the following interpretation:

**Fact** (Dolbeault theorem). There is a canonical isomorphism $H^q(\Omega^p) \approx H^{p,q}(X, I)$.  

Applying these isomorphisms, the Kodaira–Spencer map for complex curves can be rewritten as a map  

$$KS: H^0(K_S) \times H^0(T_S^* \otimes K_S)^* \rightarrow H^1(0_S),$$

or applying the Serre duality again on the right and by adjunction  

$$KS: H^0(K_S)^2 \rightarrow H^0(K_S)^* \otimes H^0(K_S)^*.$$  

**Fact.** The Kodaira–Spencer map $KS: H^0(K_S)^2 \rightarrow H^0(K_S)^* \otimes H^0(K_S)^*$ is dual to the symmetric multiplication of holomorphic 1-forms, i.e., the natural map $H^0(K_S)^2 \rightarrow H^0(K_S^2)$.  

2b Hejhal–Thurston theorem and Kapovich–Torelli mappings  

Let us go back to the differential-geometric setting. So we have the Teichmüller space $T_S$, the vector space $V = H^1(S, \mathbb{C})$, the vector bundle $H^1 T_S \rightarrow T_S$ with global trivialization $H^1 T_S \rightarrow V$, and the subbundle $\Omega T_S \subset H^1 T_S$. Fix an integer $k$ and consider the bundle of Grassmannian varieties $\text{Gr}(k, \Omega T_S) \rightarrow T_S$. The composition of inclusion and projection $\Omega T_S \rightarrow H^1 T_S \rightarrow V$ induces the map $\text{Gr}(k, \Omega T_S) \rightarrow \text{Gr}(k, V)$ which we shall denote by $\kappa_k$.  

**The extreme case: $k = g$.** In this case each fiber of the fibration $\text{Gr}(g, \Omega T_S)$ is a single point corresponding to the whole $g$-dimensional space $\Omega_1 = H^{1,0}(S, I)$, hence $\text{Gr}(g, \Omega T_S)$ is canonically biholomorphic to $T_S$. In this case the mapping $\kappa_g: T_S \rightarrow \text{Gr}(g, V)$ is known as the Torelli mapping.  

**Fact** (local Torelli theorem for curves). The Torelli mapping $\kappa_g: T_S \rightarrow \text{Gr}(g, 2g)$ is locally a holomorphic embedding.  

The image of this embedding is known as the Schottky locus, and is rather difficult to describe.
The opposite case: $k = 1$. In this case each fiber of the fibration $Gr(1, \Omega T S) = P(\Omega T S) \to T S$ is a projective space. The mapping $\kappa_1: P(\Omega T S) \to P(V)$ has been studied by Kapovich (indeed, its image is the projectivization of the cone of representable classes from Haupt–Kapovich theorem). Hence

**Definition 5.** We shall call the mapping $\kappa_1$ the Kapovich mapping, and its image the Kapovich locus. In the intermediate cases $1 < k < g$ we shall refer to the maps $\kappa_k$ as to the Kapovich–Torelli mappings, and images thereof as the Kapovich–Schottky loci.

The following is what Kapovich called the ‘holonomy theorem’, and ascribed it to Hejhal and Thurston.

**Theorem.** The Kapovich map $\kappa_1: P(\Omega T S) \to P(V)$ is open.

Kapovich established it by analytical means considering the uniform convergence of developing mappings. We shall prove a stronger statement by the means of differential geometry.

**Proposition 2.1.** The differential of the Kapovich mapping $\kappa_1: P(\Omega T S) \to P(V)$ is everywhere surjective.

**Proof.** The part of Gauss–Manin connection preserving the Hodge subbundle splits the tangent bundle $TP(\Omega T S)$ into the vertical subbundle, which is isomorphic to $T_{(a)}P(\Omega I) = \text{Hom}(\langle a \rangle, \Omega(S, I)/\langle a \rangle)$ for any line $\langle a \rangle \in \Omega(S, I)$ (and maps by the derivative of the Kapovich mapping isomorphically onto the tangent space $T_{(a)}P(\Omega I) \subset T_{(a)}P(V)$), and the horizontal subbundle $\text{Hor}_{(a), I} \approx T I T S$. By definition, the differential $d\kappa_1|_{\text{Hor}_{(a), I}}: T I T S \to \text{Hom}(\langle a \rangle, H^{0,1}(S, I))$ is the restriction of the Kodaira–Spencer tensor $KS: T \Omega T S \times \Omega T S \to \Omega T S$ onto the line spanned by $a$.

Hence the Proposition 2.1 is equivalent to the following ‘nondegeneracy’ assertion about the Kodaira–Spencer tensor:

**Proposition 2.2.** For any complex structure $I$ on $S$ and any nonzero class $a \in H^{1,0}(S, I)$ the map given $T I T S \to H^{0,1}(S, I)$ by

$$v \mapsto KS_v(\alpha)$$

is surjective. Equivalently, for any nonzero section $\alpha \in H^0(K_S)$ any linear functional from $H^0(K_S)^*$ can be represented by

$$\beta \mapsto v(\alpha \otimes \beta)$$

for some $v \in H^0(K_S^2)^*$.

**Proof.** The restatement of the proposition is equivalent to the statement that the map $m_\alpha: H^0(K_S) \to H^0(K_S^2)$ given by $m_\alpha(\beta) = \alpha \otimes \beta$ is injective. This is obvious though, since whenever both $\alpha$ and $\beta$ are nonzero, their product is nonzero away from its $4g - 4$ zeroes (counted with multiplicity).
Definition 6. We shall call the range of the map $m_\alpha$ the dividend subspace and denote it by $L_\alpha \subset H^0(K^2_S)$. It is precisely the space of holomorphic quadratic differentials divisible by $\alpha$.

2c Weak Hejhal–Thurston theorem for pairs and triples

In order to generalize the Kapovich’s proof to the subspaces in cohomology other than lines, one should determine where the differential of the Kapovich–Torelli map is surjective. The first thing we need to notice is that, in a sense, it never is: the space $V$ carries the skew-symmetric intersection pairing (which we shall denote by $\omega$) given by the wedge product of forms, and two holomorphic 1-forms on a curve wedge multiply to zero. Therefore the Kapovich–Schottky locus lies within the isotropic Graßmannian $\text{Gr}(k, V)$, which is a closed subset of $\text{Gr}(k, V)$. In its turn, in order to conclude when the Kapovich–Torelli mapping $\kappa_k: \text{Gr}(k, \Omega^1 T_S) \rightarrow \text{Gris}(k, V)$, one needs to understand first what is the tangent space to the isotropic Graßmannian. Let us start from the standard observation that the symplectic form induces a well-defined pairing $\tau \times V/\tau \rightarrow \mathbb{C}$ for any isotropic subspace $\tau \subset V$.

**Lemma.** Let $\tau \in \text{Gris}(k, V)$ be an isotropic subspace. A vector $v \in T_\tau \text{Gr}(k, V) =$ $\text{Hom}(\tau, V/\tau)$ is tangent to the isotropic Graßmannian $\text{Gris}(k, V) \subset \text{Gr}(k, V)$ iff the corresponding map satisfies

$$\omega(x, v(y)) = \omega(y, v(x))$$

for any $x, y \in \tau$.

We shall call such maps balanced.

**Proof.** The infinitesimal displacement of the plane $\tau$ by the mapping $\theta$ consists of the vectors $\{x + \varepsilon \theta(x): x \in \tau\}$ for $\varepsilon^2 = 0$. This plane is isotropic whenever for any $x, y \in \tau$ one has $0 = \omega(x + \varepsilon \theta(x), y + \varepsilon \theta(y)) = \omega(x, y) + \varepsilon(\omega(\theta(x), y) + \omega(y, \theta(x)))$, which is equivalent to the balancedness condition since $\omega(x, y) = 0$ for any $x, y \in \tau$.

Therefore, the Kapovich–Torelli mapping has surjective differential at point $\tau \subset H^{1,0}(S, I)$ iff any balanced map $\theta: \tau \rightarrow V/\tau$ can be represented as the Kodaira–Spencer map for some deformation $v$, i. e.

$$\eta \mapsto \text{KS}_\eta(\eta)|_{\tau} \in \text{Hom}(\tau, H^{0,1}(S, I)) \subset \text{Hom}(\tau, V/\tau).$$

**Definition 7.** If $\tau \subset H^0(K^1_S)$ is a subspace, we call the kernel of the multiplication map $\tau \otimes H^0(K^1_S) \rightarrow H^0(K^2_S)$ the obscurant subspace. We call a subspace $\tau$ (or a tuple of forms spanning it) coprime if $\dim \tau = 2$ and the obscurant subspace is one-dimensional, or $\dim \tau = 3$ and the obscurant subspace is three-dimensional (note that in this case the multiplication map is surjective). Otherwise we call $\tau$ linked.
Proposition 2.3. Let $\tau \subset H^0(K_S)$, and $\dim \tau = 2$ or 3. Then the derivative of the Kapovich–Torelli mapping at $\tau \in \text{Gr}(\dim \tau, \Omega T_S)$ is surjective iff $\tau$ is coprime. Moreover, in the case $\dim \tau = 3$ and $\tau$ being coprime the derivative is bijective.

Proof. Let us again consider the space $H^0,1(S, I)$, the target of a mapping $\theta \in \text{Hom}(\tau, H^0,1(S, I)) \subset \text{Hom}(\tau, V/\tau)$, as the dual of the space of abelian differentials $\Omega(S, I)$. In the case $k = 2$, our goal is to show that if $\theta$ is balanced, then it can be realized as the value of the Kodaira–Spencer tensor on some deformation $v \in H^0(K^2_S)$. Let $\alpha, \beta \in \tau$ be a basis. We know what the values of $\theta$ on $\alpha$ and $\beta$ are: they should be realized as restrictions of $v$ onto the dividend subspaces $L_\alpha$ and $L_\beta$ (which are identified with the space of abelian differentials by the mappings $m_\alpha$ and $m_\beta$). The balancedness condition means that these functionals on $L_\alpha$ and $L_\beta$ coincide on the line spanned by $\alpha \otimes \beta$. Provided this line exhausts the intersection $L_\alpha \cap L_\beta$, one can uniquely extend this pair of functionals to the subspace spanned by $L_\alpha \cup L_\beta$ and then to whole space $H^0(K^2_S)$. Otherwise any $\xi \in L_\alpha \cap L_\beta$ noncollinear to $\alpha \otimes \beta$ gives a hyperplane in $T\text{Gr}(2, V)$, in which the range of the derivative of the Kapovich–Torelli map is contained. Note however that the intersection $L_\alpha \cap L_\beta$ has the same dimension as the kernel of the map $\tau \otimes H^0(K_S) = L_\alpha \oplus L_\beta \to H^0(K^2_S)$, i.e. the obscurant subspace. Moreover, the obscurant subspace is canonically identified with the subspace of $H^0(K^2_S)$ consisting of quadratic differentials divisible by all the 1-forms in $\tau$ (one can project the kernel onto $L_\alpha$ along $L_\beta$, consider the range of the projection as a subspace in $L_\alpha \subset H^0(K^2_S)$, and note that the composite map does not depend on the choice of $\alpha$ and $\beta$).

In the case $k = 3$ and $\tau = \langle \alpha, \beta, \gamma \rangle$ one knows the values of the desired functional $v$ on the subspaces $L_\alpha$, $L_\beta$ and $L_\gamma$. The balancedness condition implies again that these values agree on the lines $\alpha \otimes \beta$, $\beta \otimes \gamma$ and $\gamma \otimes \alpha$. Then it can be extended to a functional on the linear hull of $L_\alpha \cup L_\beta \cup L_\gamma$ no matter what its values were, iff no other relation on monomials would show up after taking symmetric products (i.e., the obscurant subspace is three-dimensional). The space of relations is precisely the kernel of the natural mapping $L_\alpha \oplus L_\beta \oplus L_\gamma \to H^0(K^2_S)$, and it is three-dimensional iff this map is surjective, i.e. the union $L_\alpha \cup L_\beta \cup L_\gamma$ spans the whole $H^0(K^2_S)$. In this case the deformation is determined by its values on three forms $\alpha$, $\beta$, $\gamma$ uniquely, so the differential of the Kapovich–Torelli map is bijective at this point.

Note that in the case $k = 3$ it is not enough to merely ask for the pairwise intersections $L_\alpha \cap L_\beta$, $L_\beta \cap L_\gamma$ and $L_\gamma \cap L_\alpha$ to be one-dimensional, or even to be one-dimensional and not to lie within one plane (much like it happens with the Borromean rings). A counterexample is given by any hyperelliptic curve of genus greater than two, cf. Proposition 2.3.

Corollary 2.4. 1. Let $(S, I)$ be a curve of genus at least two and $\tau \subset \Omega(S, I)$ a plane spanned by a pair of coprime abelian differentials. Then it has a local $(g-2)$-dimensional family of deformations which preserve the periods.
of the differentials from $\tau$, and when $\tau$ is coprime, the dimension of the deformation space equals exactly $g - 2$.

2. Let $(S, I)$ be a curve of genus at least three and $\tau \subset \Omega(S, I)$ a three-dimensional coprime subspace. Then any local deformation preserving the periods of the differentials from $\tau$ is trivial.

**Definition 8.** Analogously to the case $k = 1$, in which the fibers of the projection $P(\Omega T S) \rightarrow P(V)$ are known as isoperiodic foliation, we shall refer to the fibers of the projection $\text{Gr}(2, \Omega T S) \rightarrow \text{Gr}(2, V)$ as to the lesser isoperiodic foliation, since its leaves, after being projected to the Teichmüller space, lie within the projections of the leaves of the usual isoperiodic foliation.

A similar foliation in a slightly more Teichmüller-theoretic context, for a pair of meromorphic differentials with real periods, appeared in an early version of a paper of Grushevsky and Krichever [GK] under the name small foliation. Unlike the lesser isoperiodic foliation, which we know to have singularities, their foliation is conjectured to be smooth.

**Proposition 2.5.** Let $\alpha$ and $\beta$ be two linked 1-forms on a curve $S$ (i.e. they span a linked plane). Then they have at least two zeroes in common. If they have exactly two common zeroes, $S$ must be hyperelliptic. In general, if they have exactly $n$ common zeroes, then the gonality of $S$ is no greater than $n$.

**Proof.** Let $\xi \in L_\alpha \cap L_\beta$ be a holomorphic quadratic differential. Since it is divisible by both $\alpha$ and $\beta$, it vanishes at each point $z$ at least up to order $\max\{\text{ord}_z \alpha, \text{ord}_z \beta\}$. The meromorphic function $\frac{\xi}{(\alpha \otimes \beta)}$ has poles exactly at points $z$ in which one has $\text{ord}_z \xi < \text{ord}_z \alpha + \text{ord}_z \beta$. This means that in order for this function to be nonconstant (i.e. to have at least two poles or one double pole), so that $\xi$ could be not proportional to $\alpha \otimes \beta$ and hence $\alpha$ and $\beta$ be linked, they must have at least two common zeroes. \qed

**Corollary 2.6.** Two holomorphic 1-forms on a genus two curve are always coprime. If two forms on a genus three curve are linked, this curve is hyperelliptic.

**Proof.** Indeed, if they are linked, they must have at least two common zeroes, and if they have three, then the meromorphic function $\frac{\alpha}{\beta}$ would have one zero and one pole. \qed

**Corollary 2.7.** The locus of pairs $(I, \tau) \in \text{Gr}(2, \Omega T S)$, where $\tau \subset H^{1,0}(S, I)$ is linked, lies inside a subvariety of codimension two.

**Proof.** The coincidence of at least two zeroes is an analytical codimension two condition. \qed

**Definition 9.** We shall call the locus of $k$-planes $(k = 2, 3)$ in cohomology which can be represented by the pairs or triples of coprime holomorphic forms the coprime Kapovich–Schottky locus.
The following Proposition can be deduced from the above considerations, but is actually an elementary computation of dimensions.

**Proposition 2.8.** For \( k \geq 4 \), the derivative of the Kapovich–Torelli mapping \( \kappa_k : \text{Gr}(k, \Omega \mathcal{T} S) \to \text{Gris}(k, V) \) is never surjective.

**Proof.** The isotropic Graßmannian \( \text{Gris}(k, 2g) \) is of codimension \( 1 + 2 + \cdots + (k - 1) = k(k - 1)/2 \) inside the usual Graßmannian \( \text{Gr}(k, 2g) \), which has dimension \( k(2g - k) \), hence its dimension equals \( 2gk - (3k^2 - k)/2 \). The \( k \)-th Graßmannian bundle of the Hodge bundle has dimension \( 3g - 3 + k(g - k) = gk + 3g - k^2 - 3 \). In order for the Kapovich–Torelli map to have surjective derivative, one must have \( gk + 3g - k^2 - 3 \geq 2gk - (3k^2 - k)/2 \), or equivalently \( g(k - 3) - (k^2 - k - 6)/2 \leq 0 \).

For \( k \leq 3 \), this implies \( k - (k + 2)/2 \leq 0 \), or \( k/2 \leq 1 \), which is impossible whenever \( k > 3 \).

The problem of determination of the Kapovich–Schottky locus for \( k > 3 \) may be interesting, but cannot be covered by the generalization of Kapovich’s method.

### 2d Sheaf-theoretic appearance of the obscurant subspaces

**Definition 10.** Let \( \tau \subset H^0(K_S) \) be an \( m \)-dimensional subspace spanned by holomorphic 1-forms, which we view as an injective homomorphism of sheaves \( T \to \mathcal{O} \otimes \tau^* \simeq \mathcal{O}^m \). Its cokernel is called the **normal sheaf** of \( \tau \), and is denoted by \( \nu_\tau \).

The reason for the name is as follows. Suppose \( A \) is an abelian surface, and \( \iota : S \to A \) is a holomorphic mapping with at worst normal crossings from a smooth curve, the image of which is not contained in any elliptic curve. Let \( \tau = \iota^* H^0(\Omega^1_A) \) be the space of restrictions of holomorphic 1-forms on \( A \) to \( S \). Then \( \nu_\tau \) is precisely the normal bundle of \( S \) inside \( A \). Note that in this case the adjunction formula implies that the normal bundle is isomorphic to the canonical bundle. This can be generalized as follows:

**Proposition 2.9 ([BSY]).** Provided the forms \( \alpha_i \) spanning \( \tau \) have no zero in common, the sheaf \( \nu_\tau \) is a rank \( m - 1 \) vector bundle with determinant isomorphic to the canonical bundle of \( S \).

**Proof.** One can give a basis of \( g - 1 \) holomorphic section of this sheaf at every point. The short exact sequence \( T \to \mathcal{O}^m \to \nu_\tau \) gives an isomorphism \( T \otimes \Lambda^{m-1} \nu_\tau \cong \Lambda^m \mathcal{O}^m = \mathcal{O} \), hence \( \Lambda^{m-1} \nu_\tau \cong K \).

The short exact sequence of sheaves \( T \to \mathcal{O} \otimes \tau^* \to \nu_\tau \) gives rise to the long exact sequence. Since we are interested in the case \( m > 1 \) (and hence necessarily \( g > 1 \)), \( H^0(T) = 0 \), and the sequence reads

\[
H^0(\mathcal{O}) \otimes \tau^* \to H^0(\nu_\tau) \to H^1(T) \to H^1(\mathcal{O}) \otimes \tau^* \to H^1(\nu_\tau).
\]

It gives the following characterization of the obscurant subspace:
Proposition 2.10. The obscurant subspace of $\tau$ is the Serre dual of the first cohomology space $H^1(\nu_\tau)$ of the normal bundle of $\tau$.

Proof. The Serre dual of $H^1(\nu_\tau)$ is the kernel of the map $H^1(O)^* \otimes \tau \to H^1(T)^*$ dual to the substitution of tangent vectors, which by Serre duality is the map $\tau \otimes H^0(K) \to H^0(K^2)$ given by the symmetric multiplication of forms. \hfill \Box

Note also that the Euler characteristic $\chi(\nu_\tau) = (3 - m)(g - 1)$ depends only on the dimension of $\tau$. In most cases the connecting homomorphism is zero. However, in certain interesting cases it is not.

Proposition 2.11. The range of the connecting homomorphism $H^0(\nu_\tau) \to H^1(T)$ is precisely the space of first-order deformations of the curve which preserve periods of all 1-forms in $\tau$.

Proof. It equals the kernel of the map $H^1(T) \to H^1(O) \otimes \tau^*$, Serre dual to the multiplication map $\tau \otimes H^0(K) \to H^0(K^2)$, hence is precisely the annihilator of quadratic differentials of the form $\alpha \otimes -\alpha$, $\alpha \in \tau$.

Example. Let $X$ be a curve on an abelian surface $A$, and $\tau$ be the plane of restrictions to $X$ of holomorphic forms on $A$. The canonical bundle is isomorphic to normal, so any 1-form gives a variation of the curve inside a surface. When this 1-form is a restriction of a 1-form on the surface, the variation of the curve is just a parallel shift inside a torus, and hence changes not the complex structure. Otherwise it gives a nonzero class in $H^1(T)$.

Now we restrict our attention to the case of two holomorphic 1-forms.

Definition 11. Let $\alpha, \beta$ be two forms spanning a plane $\tau$. By its overlap we shall mean the divisor

$$Y = Y_\tau = \sum p_i z_i, \quad p_i = \min\{\text{ord}_{z_i} \alpha, \text{ord}_{z_i} \beta\}.$$ 

It is clear that $Y_\tau$ only depends on $\tau$ and not on a choice of a basis of it.

If overlap is empty, the sheaf $\nu = \nu_\tau$ is invertible and isomorphic to the canonical bundle by Proposition 2.9. In general, we have a mapping $O \otimes \tau \to K$ given by $\beta \oplus -\alpha$, which annihilates the range of the map $T \xrightarrow{\alpha \oplus \beta} O \otimes \tau^*$ and hence factorizes through its cokernel $O \otimes \tau^* \to \nu$. If $Y_\tau \neq 0$, the natural mapping $\nu_\tau \to K$ has both kernel and cokernel. What is true is the following

Proposition 2.12. Let $\tau \subset H^0(K)$ be a 2-plane, and $Y$ its overlap. Then its normal sheaf includes into the short exact sequence $O_Y \to \nu_\tau \to K(-Y)$, where $O_Y$ is a torsion sheaf.

Proof. Any coherent sheaf on a curve has the torsion subsheaf with locally free quotient (i.e. vector bundle). In this case, it is supported on the overlap. \hfill \Box
Fix an isomorphism $\nu \simeq \mathcal{O}_Y \oplus K(-Y)$. Then one has an exact sequence
$$0 \to \mathcal{C}^2 \to H^0(\mathcal{O}_Y) \oplus H^0(K(-Y)) \to H^1(T) \to H^1(\mathcal{O} \otimes \tau^*) \to H^1(K(-Y)) \to 0,$$
so the obscurant subspace is the dual of $H^1(K(-Y))$, which by Serre duality is $H^0(K \otimes K^*(Y)) = H^0(\mathcal{O}(Y))$. If $A$ and $B$ are divisors of zeroes of the forms $\alpha, \beta \in \tau$, then the latter space can be identified with $H^0(K^2(-A - B + Y))$. This is obviously the space of quadratic differentials with enough zeroes to be divisible by $\alpha$ and $\beta$, which agrees with the description of the obscurant subspace for two forms from the proof of the Proposition 2.3.

**Proposition 2.13.** The set of pairs $(I, \tau) \in \text{Gr}(3, \Omega T_S)$ s. t. the triple $\tau \subset H^{1,0}(S, I)$ is linked, lies within a subvariety of codimension at least one.

**Proof.** We know that linked triples are precisely the triples where the rank of $H^1(\nu_\tau)$ jumps. By semicontinuity, it is enough to find a pair $(I, \tau)$ in which this rank equals three. But this follows from a well-known refinement of the Max Noether’s theorem:

**Fact** (Noether’s theorem on quadratic differentials). *Let $S$ be a non-hyperelliptic curve of degree greater than two. Then there exist three holomorphic 1-forms $\alpha, \beta, \gamma \in \Omega(S)$ s. t. the space $H^0(K^2)$ is spanned by quadratic differentials of the form $\alpha \otimes -$, $\beta \otimes -$, $\gamma \otimes -$.*

**Proof.** For the proof, see [FK, III.11.20, p. 149].}

**2e Counterexamples to the Hejhal–Thurston theorem for pairs and triples**

In this subsection, we present a number of examples of linked pairs and triples, in which the differential of the Kapovich–Torelli mapping fails to be surjective, in order to give the geometric flavour of these exceptional cases.

**Ramified covers of low-genera curves.** It follows from the Corollary 2.4 that in order to present a pair $(S, \tau)$ of a curve and a linked triple of abelian differentials it is enough to present a curve with a triple of abelian differentials which possess nontrivial deformations, isoperiodic for each of these differentials. The simplest case is a ramified cover $\pi: S \to C$ of a genus three curve $C$ together with a triple $\tau = \pi^* \Omega(C) \subset \Omega(S)$. Any variation of the ramification locus gives a deformation of the curve $S$, and periods of the pullbacks of the 1-forms on the curve $C$ stay the same (since the curve $C$ itself is unchanged). Similar construction gives an example of a linked pair. Let $C'$ be a genus two curve, and $\pi': S \to C'$ is a double cover ramified at $2n > 2$ points. Then the pullback of any holomorphic 1-form on $C'$ has $2n + 2 \times 2$ zeroes, hence the genus of $S$ equals $g = n + 3$. Variations of the ramification locus produce a $2n$-dimensional family of deformations isoperiodic for all the differentials from $\pi'^* \Omega(C')$, which is greater than $g - 2 = n + 1$ predicted by the Corollary 2.4.
Hyperelliptic genus three curves. We have noticed in the Corollary that if a genus three curve carries two linked 1-forms, it is necessarily hyperelliptic. Now we show the converse: any hyperelliptic genus three curve has a pair of linked 1-forms, and such pairs (or rather planes spanned by such pairs) are parametrized by a rational curve.

The canonical map for a hyperelliptic curve of genus three sends it two-to-one onto the Veronese curve in $\mathbb{P}^2$, i.e. a quadric. Let $a, b, c$ and $d$ be four points on this quadric. The canonical map has the characteristic property that each hyperplane (here, projective line) cuts out a canonical divisor, i.e. corresponds to an abelian differential up to scaling. Let us denote such a differential corresponding to the line $ab$ by $\omega_{ab}$, etc. Then the differentials $\omega_{ab}$ and $\omega_{ad}$ are linked: they both divide the quadratic differential $\omega_{ab} \otimes \omega_{ad}$ and the quadratic differential $\omega_{ab} \otimes \omega_{cd}$ as well. The plane spanned by such a pair depends only on the point $a$ where the corresponding lines intersect, hence such planes are parametrized by the Veronese curve, which is rational.

Generic genus four curves. The canonical image $C$ of a generic genus four curve is an intersection of a quadric $Q$ and a cubic in $\mathbb{P}^3$. The lines on the quadric are the same as the trisecants of $C \subset \mathbb{P}^3$. Let $a$ and $c$ be two lines on $Q$ from one family and $b$ and $d$ two lines from the other, so that $abcd$ is a spatial quadrilateral. Again, if the lines $a$ and $b$ intersect, we shall denote by $\omega_{ab}$ an abelian differential (unique up to scaling) with zero locus cut out by the plane $ab$. Then the differentials $\omega_{ab}$ and $\omega_{ad}$ are linked: they both divide the quadratic differential $\omega_{ab} \otimes \omega_{ad}$ and the quadratic differential $\omega_{ab} \otimes \omega_{cd}$ as well. The plane spanned by such a pair again depends on the line $a$ only, hence the planes of linked pairs for a generic genus four curves are parametrized by the variety of lines on a quadric surface, i.e. two rational curves.

Sections of abelian threefolds. Any threefold can be embedded into $\mathbb{P}^7$, and hence any abelian threefold $A$ possesses lots of curves cut out by sections by different $\mathbb{P}^3$’s (precisely, parametrized by Gr(6,8), which has dimension 16). If $S \subset A$ is such a section, then $\tau = H^{1,0}(A)|_S \subset H^{1,0}(S)$ is linked, since any other section has holomorphic forms with the same periods, and if $\tau$ were coprime, the Corollary would imply that all of these sections would be trivial deformations of $S$, i.e. parallel shifts inside the torus. They only have three-dimensional space, however.

3 Input from the ergodic theory

The current picture is the following. We have Kapovich–Torelli mappings $\kappa_k : \text{Gr}(k, \Omega T_S) \to \text{Gr}(k, V)$ ($k = 1, 2, 3$), and their images are the Kapovich–Schottky loci. Over some subsets of these loci (which are open not just within these loci but even within the isotropic Graßmannian), their fibers are of dimension $2g - 3, g - 2$ and 0, respectively. For $k = 1$, all the fibers are smooth of
First, it is clear that the Kapovich–Schottky loci are contained in certain open subset of the isotropic Graßmannian: the form \( q(x) = \sqrt{-1} \omega(x, \bar{x}) \) is positive definite on any \( H^{1,0} \) subspace.

**Definition 12.** The locus of isotropic \( k \)-planes in a symplectic space \((V, \omega)\) on which the form \( q(x) = \sqrt{-1} \omega(x, \bar{x}) \) is positive definite, is called the **Hodge–Riemann Graßmannian** and denoted by \( \text{Gris}^+(k, V) \).

Much more can be said by the means of the ergodic theory. The space of all complex structures \( \text{Cx}(S) \) is acted on by not merely the connected component of the diffeomorphism group, but the whole diffeomorphism group. Therefore the quotient \( \text{MCG}(S) = \text{Diff}(S)/\text{Diff}^0(S) \), called the **mapping class group**, acts on the Teichmüller space, and, via pullbacks, on the cohomology. On the space \( V \) it acts through its quotient \( \text{Sp}(2g, \mathbb{Z}) \), preserving the intersection pairing, and hence the isotropic Graßmannians \( \text{Gris}(k, V) \subset \text{Gr}(k, V) \) and the Hodge–Riemann Graßmannians \( \text{Gris}^+(k, V) \). If \( f \in \text{Diff}(S) \) and \( \alpha \) is a holomorphic 1-form w. r. t. complex structure \( I \), then \( f^\ast \alpha \) is holomorphic w. r. t. \( f^\ast I \), so the Hodge bundle is also preserved under the MCG(S)-action, and the Kapovich–Torelli mappings are MCG(S)-equivariant. Therefore the Kapovich–Schottky loci and the exceptional loci within those are \( \text{Sp}(2g, \mathbb{Z}) \)-invariant.

**Lemma.** The Hodge–Riemann Graßmannian \( \text{Gris}^+(k, 2g) \) is acted upon transitively by the group \( \text{Sp}(2g, \mathbb{R}) \), and is isomorphic as a homogeneous space to \( \text{Sp}(2g, \mathbb{R})/(U(k) \times \text{Sp}(2g - 2k, \mathbb{R})) \).

**Proof.** Let \( \tau \in \text{Gris}^+(k, 2g) \) be some isotropic plane. It has no real vectors, since the form \( q(x) = \sqrt{-1} \omega(x, \bar{x}) \) would vanish on it. Therefore the span \( \langle \tau, \bar{\tau} \rangle \) has dimension \( 2k \), and is the complexification of a real subspace \( \tau_\mathbb{R} \subset \mathbb{R}^{2g} \). Since the subspace \( \tau \) was positive, the subspace \( \tau_\mathbb{R} \) is symplectic. The subspace \( \tau \subset \tau_\mathbb{R} \otimes \mathbb{C} \) gives rise to a complex structure operator \( J \) on \( \tau_\mathbb{R} \), which preserves the symplectic form (i. e. \( \omega(Jx, Jy) = \omega(x, y) \)) and has the symmetric form \( g(x, x) = \omega(x, Jx) \) positive definite. Therefore the Hodge–Riemann Graßmannian \( \text{Gris}^+(k, 2g) \) is the same thing as the Graßmannian of \( 2k \)-dimensional real symplectic subspaces in \( \mathbb{R}^{2g} \) endowed with a suitable complex structure operator. Such pairs are acted upon by the real symplectic group \( \text{Sp}(2g, \mathbb{R}) \) transitively, and the stabilizer of any pair is isomorphic to the subgroup \( U(k) \times \text{Sp}(2g - 2k, \mathbb{R}) \).

**Fact** (C. Moore). Let \( G \) be a semisimple Lie group, \( \Gamma \subset G \) a lattice, and \( H \) a noncompact Lie subgroup in \( G \). Then \( \Gamma \) acts ergodically on \( G/H \). In particular, the group \( \text{Sp}(2g, \mathbb{Z}) \) acts ergodically on the Hodge–Riemann Graßmannian \( \text{Gris}^+(k, 2g) \approx \text{Sp}(2g, \mathbb{R})/(U(k) \times \text{Sp}(2g - 2k, \mathbb{R})) \) for \( k < g \).

From this, one can conclude:

**Proposition 3.1.** The Kapovich–Schottky loci for \( k = 1, 2, 3 \) and \( g > k \) are dense in the Euclidean topology on the Hodge–Riemann Graßmannian.
Proof. Indeed, they contain the coprime Kapovich–Schottky loci, which are open by Proposition 2.3 and Corollaries 2.7 and 2.13 and are invariant under an ergodic action by Moore’s theorem.

Proposition 3.2. Let \((S, I)\) be a complex curve of genus at least four, and \(\tau \subset H^{1,0}(S, I)\) be a generic three-dimensional subspace. There exist a neighborhood \(U \subset \text{Gris}(3, 2g)\) containing \(\tau\) s. t. for any \(\tau' \in U\) there exists a unique complex structure \(I' = I(\tau')\) s. t. one has \(\tau' \subset H^{1,0}(S, I')\). In other words, deformation of a generic triple of abelian differentials determines a unique local deformation of a complex structure, so that the deformed cohomology classes would be represented by abelian differentials in the deformed complex structure.

Proof. Immediate from the density of the Kapovich–Schottky locus (Proposition 3.1) and local bijectivity of the Kapovich–Torelli mapping \(\kappa_3: \text{Gr}(3, \Omega T_S) \to \text{Gris}(3, V)\) in generic point (Propositions 2.3 and 2.13).

We hope to qualify somehow the word ‘generic’ by applying Ratner theory to classify the exceptional orbits in spirit of the original Kapovich’s proof in a subsequent revision of the paper.

4 Intrinsic geometry of the isoperiodic locus

4a Flat structure on the isoperiodic foliation

The following ‘relative periods’ coordinates on the isoperiodic leaf are well-known in Teichmüller theory [GK] [HM].

Fact. Fix a cohomology class \([\alpha] \in H^1(S, \mathbb{C})\), an isoperiodic family of complex structures w. r. t. \([\alpha]\), and let \(\alpha_I \in \Omega(S, I)\) be a holomorphic 1-form representing it in the complex structure \(I\). Let \(z_1^I, z_1^I, \ldots, z_{2g-3}^I\) be its zeroes (so that the point \(z_i^I\) varies smoothly as we vary the complex structure \(I\)). Then the tuple of functions

\[ \Xi_i(I) = \int_{z_0^I}^{z_i^I} \alpha_I \]

for \(i = 1, 2, \ldots, 2g - 3\) gives a local coordinate system on the isoperiodic leaf of the class \([\alpha]\).

Note that each function \(\Xi_i\) is defined up to an additive constant, i. e. a period of \(\alpha\).

Let us give a more algebraic handling of the above construction. Much like we considered the normal sheaf of a plane in \(H^0(K)\), one can consider a normal sheaf for just one form \(\alpha\). It is isomorphic to the structure sheaf \(\mathcal{O}_Z\) of the subscheme \(Z = Z_\alpha\) of zeroes of \(\alpha\), and the long exact sequence reads:

\[ 0 \to H^0(\mathcal{O}) \to H^0(\mathcal{O}_Z) \to H^1(T) \to H^1(\mathcal{O}) \to 0. \]
Under the Serre duality the last arrow can be viewed as the arrow $H^0(K_S^2)^* \to H^0(K_S)^*$ dual to the arrow $H^0(K_S) \to H^0(K_S^2)$ given by $\xi \mapsto \xi \otimes \alpha$. Its kernel is hence precisely the tangent space to the universal isoperiodic deformation w. r. t. $\alpha$.

**Example.** Let $E$ be an elliptic curve with a holomorphic form $dz$, $p: S \to E$ a ramified cover, and $\alpha = p^*dz$. Then $Z$ is the ramification divisor of $p$, and the isoperiodic deformation w. r. t. $\alpha$ is given by the variation of the ramification points on $E$. Variation of each point is prescribed by a vector, which can be paired with $dz$ to obtain a number. Hence the variation is described by a tuple of numbers at each ramification point. The trivial deformations are precisely the ones coming from a variation of the ramification locus, which arises from a shift on the elliptic curve, i.e. for which the numbers associated to the ramification points are all the same (so that the section of $H^0(\mathcal{O}_Z)$ is constant).

**Proposition 4.1.** Let $v \in H^1(T)$ be an isoperiodic deformation w. r. t. $\alpha$, which we view both as a variation of complex structure $I \mapsto I + \varepsilon v$ and a section $s \in H^0(\mathcal{O}_Z)$. Then one has $\text{Lie}_v \Xi(I) = s(z_i) - s(z_0)$.

Whenever we have a section $s \in H^0(\mathcal{O}_Z)$, we shall denote the corresponding deformation by $v_s \in H^1(T)$.

**Proposition 4.2.** Let $s \in H^0(\mathcal{O}_Z)$ be a section. The value of its image under the connecting homomorphism in $H^1(T) \cong H^0(K^2)^*$ on a quadratic differential $\omega$ is given by

$$v_s(\omega) = \sum_{i=0}^{2g-3} s(z_i) \text{Res}_{z_i} \left( \frac{\omega}{\alpha} \right). \quad (4.1)$$

**Proof.** Computation for Čech–Dolbeault double resolution [Z].

It is clear that $v_s$ vanishes on each quadratic differential of the form $\alpha \otimes \omega$, since the quotient $\omega/\alpha$ is holomorphic and has zero residues in this case, so that the corresponding deformations indeed preserve the periods of $\alpha$. It is also clear that the right-hand side vanishes for $v$ a constant vector, since in this case it is the sum of residues of a meromorphic 1-form, which is zero on a compact curve.

**4b Lesser isoperiodic leaf in a greater isoperiodic leaf**

The lesser isoperiodic leaf of $(\alpha, \beta)$ is obviously contained in the tangent space to the isoperiodic leaf of $\beta$, and one can describe the map on the tangent spaces $H^0(K)/\langle \alpha, \beta \rangle \to H^0(\mathcal{O}_{Z_{\alpha}})/\text{const}.$

Again, we examine thoroughly the case of a curve on an abelian surface. The forms $\alpha$ and $\beta$ give two parallel foliations on the surface, and the zeroes of $\alpha$ and $\beta$ correspond to the points where the curve is tangent to the foliation. Hence, interpreting 1-form as a displacement of the curve, we should look at its values in the zeroes of $\alpha$. Note that $\beta$ is never zero in a zero of $\alpha$, hence the
mapping \( H^0(K) \rightarrow H^0(\mathcal{O}_{Z_{\alpha}}) \) given by
\[
\gamma \mapsto s_{\gamma}^{\beta} = \left( \frac{\gamma(z_0)}{\beta(z_0)}, \frac{\gamma(z_1)}{\beta(z_1)}, \frac{\gamma(z_2)}{\beta(z_2)}, \ldots, \frac{\gamma(z_{2g-3})}{\beta(z_{2g-3})} \right)
\] (4.2)
is well-defined. Moreover, after quotienting out the constants, the mapping vanishes on \( \alpha \) and \( \beta \), giving the desired map.

It is clear that the same formula works for any pair of 1-forms \( \alpha \) and \( \beta \) with disjoint zeroes, not necessarily with discrete period lattice. Moreover, if zeroes are not disjoint, we can write down the map \( H^0(K(\mathcal{O}_{Z_{\alpha}})) \rightarrow H^0(\mathcal{O}_{Z_{\alpha}}) \) by the same formula since any section of \( K(\mathcal{O}_{Z_{\alpha}}) \) can be represented by a 1-form vanishing at each zero of \( \alpha \) where \( \beta \) also vanishes.

The formula (4.1) for \( v_\gamma(\omega) \) yields
\[
v_\gamma(\omega) = \sum_{i=0}^{2g-3} \frac{\gamma(z_i)}{\beta(z_i)} \text{Res}_{z_i} \left( \frac{\omega}{\alpha} \right),
\]
and for \( \omega = \beta \otimes - \) by Cauchy’s index theorem one has
\[
v_\gamma(\omega) = 2\pi i \sum_{i=0}^{2g-3} \frac{\gamma(z_i)}{\beta(z_i)} \int_{\partial \Delta_i} \frac{\beta \otimes -}{\alpha} = 2\pi i \sum_{i=0}^{2g-3} \int_{\partial \Delta_i} \frac{\gamma \otimes -}{\alpha} = 0
\]
since the functions \( \gamma/\beta \) are holomorphic in the small neighborhoods of \( z_i \) (which we here denote by \( \Delta_i \)). Hence these deformations indeed preserve the periods of \( \beta \).

Let \( \iota \) be a hyperelliptic involution on a curve \( S \) again. We know that the zero locus of a holomorphic 1-form is preserved by \( \iota \), so let us order the zeroes \( z_i \in Z = Z_{\alpha} \) in such a way that \( \iota(z_{2i}) = z_{2i+1} \). In what follows, we assume that no differential we consider vanishes at a fixed point of the hyperelliptic involution.

**Proposition 4.3.** Under such an ordering, the tangent space of the lesser isoperiodic leaf for \( \alpha \) and \( \beta \), provided they have no zeroes in common, is the space of sections \( s \in H^0(\mathcal{O}_{Z}) \) with \( s(z_{2i}) = s(z_{2i+1}) \) modulo constants. In particular, it does not depend on \( \beta \).

**Proof.** Any isoperiodic deformation w. r. t. \( \alpha \) preserving also periods of \( \beta \) is given by a section of the form (4.2). Since the involution \( \iota \) is hyperelliptic, for any two forms \( \beta, \gamma \in \Omega(S) \) one has
\[
s_{\gamma}^{\beta}(z_{2i}) = \frac{\gamma(z_{2i})}{\beta(z_{2i})} = \frac{-\gamma(z_{2i})}{-\beta(z_{2i})} = \frac{(\iota^* \gamma)(z_{2i})}{(\iota^* \beta)(z_{2i})} = \frac{\gamma(\iota(z_{2i}))}{\beta(\iota(z_{2i}))} = \frac{\gamma(z_{2i+1})}{\beta(z_{2i+1})} = s_{\gamma}^{\beta}(z_{2i+1}).
\]
The space of sections with \( s(z_{2i}) = s(z_{2i+1}) \) has dimension \( g - 1 \), and its image in \( H^1(T) \) has dimension \( g - 2 \). By Corollary 2.4, it is the tangent space to the lesser isoperiodic leaf. \( \square \)
This implies that if a first-order deformations of a hyperelliptic curve preserves periods of two holomorphic 1-forms without common zeroes, it preserves periods of any holomorphic 1-form. Of course there cannot exist such an analytic deformation, since it would contradict the Torelli theorem. This really means that the lesser isoperiodic leaves in the Teichmüller space have maximal possible tangency along the locus of hyperelliptic curves.

This might look a little weird: it implies e.g. that the span of $L_\alpha \cap L_\beta$ for any coprime $\alpha$ and $\beta$ contains all the monomials of the form $\omega \otimes \omega'$, i.e. the multiplication map $H^0(K_S)^2 \to H^0(K_S^2)$ is not surjective. One ought not be afraid of this: note that $\iota^*\omega = -\omega$, so $\iota$ acts on the range of the multiplication map $H^0(K_S)^2 \to H^0(K_S^2)$ as identity. However, there are always quadratic differentials which are acted upon by $\iota$ as $-\text{Id}$. The hyperelliptic curves of genus greater than two are the only counterexamples to the infinitesimal Torelli theorem for curves, this is due to Max Noether [TAG, Ch. VIII].

**Proposition 4.4.** Let $v \in H^1(T)$ be a first-order deformation. It preserves the hyperelliptic involution iff $\iota^*v = v$.

Proof. Clear on the level of cocycles.

Note that if $\iota^*v = -v$, then one has $v(\omega \otimes \omega') = -(\iota^*v)(\omega \otimes \omega') = -v(\iota^*\omega \otimes \iota^*\omega') = -v((-\omega) \otimes (-\omega')) = -v(\omega \otimes \omega')$. In other words, the $(-1)$-eigenspace of $\iota^*$ on $H^1(T)$ is the space of first-order deformations which are isoperiodic for any class. This can be seen e.g. from the fact that the connecting homomorphism $H^0(K) \to H^1(T)$ is equivariant w.r.t. the hyperelliptic involution.

**Proposition 4.5.** Let $s \in H^0(O_Z)$ be a section which gives a deformation $v_s \in H^1(T)$. Then it satisfies $s(z_i) = -s(z_{i+1})$, maybe after adding a constant section. In particular, a hyperelliptic deformation of a hyperelliptic curve never preserves periods of more than one holomorphic 1-form.

**Example.** Since this is not really important for us, we shall consider an example instead of giving a proof. Let $E$ be an elliptic curve, and $D$ is an effective divisor symmetric w.r.t. $z \mapsto -z$. Let $S$ be a ramified cover of $E$ with ramification divisor $D$. Then $S$ admits a hyperelliptic involution s.t. the diagram

\[
\begin{array}{c}
S \\
\downarrow p
\end{array} \longrightarrow \begin{array}{c}
\mathbb{CP}^1 \\
\downarrow \text{2:1}
\end{array}
\]

is commutative. Let $\alpha = p^*dz$, the isoperiodic deformations for $\alpha$ are given by the displacements of points in the ramification divisor $D$. If we want the hyperelliptic involution to survive, we should move the points $\zeta \in D$ and $-\zeta \in D$ in opposite directions, so that their images would be interchanged by $z \mapsto -z$ as well.

**Example.** This last assertion can also be demonstrated in a more specific situation. Let $S$ be a hyperelliptic curve on an abelian surface $A$, and $\alpha, \beta$ be
restrictions of the forms on $A$ to $S$. The hyperelliptic involution on $S$ extends to $A$, and $S$ projects to a rational curve on the K3 surface $A/\iota$. An isoperiodic deformation would produce a variation of this hyperelliptic curve inside $A$ and hence give rise to a family of rational curves on a K3 surface. However, rational curves on a K3 surface in a given homology class are discrete.

The Proposition 4.5 can be furnished in a manner similar to our main Proposition 3.2:

**Proposition 4.6.** Let $(S, I)$ be a hyperelliptic complex curve of genus at least three, and $\tau \subset H^{1,0}(S, I)$ be a generic two-dimensional subspace. There exist a neighborhood $U \subset \text{Gris}(2, 2g)$ containing $\tau$ such that for any $\tau' \in U$ there exists a unique hyperelliptic complex structure $I' = I(\tau')$ such that one has $\tau' \subset H^{1,0}(S, I')$. In other words, deformation of a generic pair of abelian differentials determines a unique local deformation of a hyperelliptic complex structure, so that the deformed cohomology classes would be represented by abelian differentials in the deformed complex structure.

**Proof.** The mapping class group preserves the hyperelliptic locus. Otherwise the proof is parallel to the proof of the Proposition 3.2.

**Example.** Note that the assumption on nonvanishing of 1-forms in the fixed points of $\iota$ is necessary. The counterexample can be given by a double cover of genus two curve $C$ ramified in a divisor, which is preserved by the hyperelliptic involution of $C$ (it is clear from a commutative diagram like above that such a cover is also hyperelliptic). The pullbacks of two forms on $C$ have their periods preserved while we vary the ramification divisor (provided it stays symmetric w. r. t. the involution). Note that in this case the pullbacks vanish in the preimages of the ramification divisor, which are preserved by the hyperelliptic involution of the double cover. Also note that such double covers have genus at least four, so their existence does not contradict the following

**Proposition 4.7.** Any pair of holomorphic linked classes on a genus three curve admits an isoperiodic deformation in which they are coprime.

**Proof.** By Corollary 2.6 genus three curve with linked classes is hyperelliptic. By the above Proposition, any first-order deformation of a hyperelliptic curve preserving periods of two differentials preserves period of the third. On the other hand, any pair of abelian differentials on genus three curve possess a one-parameter analytic family of isoperiodic deformation. If it lies within the hyperelliptic locus, it preserves the periods of all the three differentials, and hence is trivial by Torelli theorem. Therefore it intersects the complement of the hyperelliptic locus, which gives the desired deformation.

Note that this Proposition cannot be pushed any further since there exist ramified covers of genus two curves with deformations of dimension larger than predicted. The hyperelliptic curves are also the instance of triples of classes in which the derivative of the Kapovich–Torelli mapping is degenerate, but its fibers are still of predicted dimension.
4c Sections of determinantal varieties and reciprocity law

It is not as simple to describe the range of the map $\gamma \mapsto s_{\gamma}$ for non-hyperelliptic curves. Something can be said, though.

Let $S$ be a genus $g$ curve, and $\tau \subset \Omega(S)$ be a subspace. Let us denote by $H^1(T)$ the range of the connecting homomorphism $H^{0}(\nu_{\tau}) \to H^{1}(T_{S})$, which is the space of first-order deformations preserving the periods of forms from $\tau$.

**Definition 13.** Let $\alpha \in \Omega(S)$ be a holomorphic differential. The set

$$Q(S, \alpha) = \bigcup_{\alpha \in \tau} H^1_{\tau}(T) \subset H^1_{\alpha}(T)$$

is called the doubly isoperiodic cone.

**Proposition 4.8.** The doubly isoperiodic cone for a generic pair $(S, \alpha)$ has dimension $2g - 4$ (and hence codimension one in $H^1_{\alpha}(T_S)$).

**Proof.** Note that if $H^1_{\tau}$ and $H^1_{\tau'}$ have nonzero intersection, it means that $H^1_{\tau, \tau'}$ is nonzero. That is, the self-intersection of this cone correspond to triples of forms possessing nontrivial infinitesimal deformations preserving periods. Since generic triple has no such deformations, this means that an open subset of the cone is fibered with fibers $H^1_{\tau}$ (of dimension $g - 2$) over the base parametrizing different choices for $\tau$ (i.e. $P(\Omega/\langle \alpha \rangle)$, which has dimension $g - 2$ as well).

**Definition 14.** Let $V$ be a vector space, and $v \in V$ be a nonzero vector. The cone of elements in $\text{Hom}(\Omega/\langle v \rangle, \Omega^*)$ having nonzero kernel is called the linear-algebraic doubly isoperiodic cone and denoted by $\Omega(V, v)$.

**Proposition 4.9.** Obviously, the doubly isoperiodic cone $Q(S, \alpha)$ is the intersection of the cone $\Omega(\Omega(S), \alpha)$ by the subspace $H^1_{\alpha}(T_S)$ embedded via the Kodaira–Spencer maps. This intersection is unlikely.

**Proof.** The ambient space $\text{Hom}(\Omega/\langle \alpha \rangle, \Omega^*)$ has dimension $(g-1)g = g^2 - g$. In order to give an element in the cone $\Omega(\Omega, \alpha)$, one has to specify the kernel (there is $(g - 2)$-dimensional space of possibilities for it), and the map to $\Omega^*$ (which can be chosen out of $(g - 2)g$-dimensional variety). Hence the dimension of the linear-algebraic doubly isoperiodic cone is $(g - 2)(g + 1) = g^2 - g - 2$, and its codimension equals two. Hence its intersection with the subspace $H^1_{\alpha}(T) \subset \text{Hom}(\Omega/\langle \alpha \rangle, \Omega^*)$ consisting of the Kodaira–Spencer operators of the deformations preserving the periods of $\alpha$, is unlikely.

This Proposition gives a condition on the possible position of $H^1_{\alpha}$ subspace within the tangent space to the moduli of abelian varieties. It can be probably easily described by the means of representation theory, since the cone $\Omega(V, v)$ is a union of orbits of a parabolic subgroup in $\text{GL}(V)$ stabilizing the line spanned by $v$.

It is tempting to claim that since the cone $Q(S, \alpha)$, or rather its projectivization, is a hypersurface in $P(H^1_{\alpha}) = \mathbb{P}^{2g - 4}$ swept by a family of $(g - 3)$-dimensional subspaces $P(H^1_{\tau})$ parametrized by $\mathbb{P}^{g - 2} = P(\Omega/\langle \alpha \rangle)$, it is actually a smooth quadric. This is actually the case for $g = 3$. 

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Proposition 4.10. Let $X$ be a genus three curve, and a form $\alpha \in \Omega(X)$ has zeroes $z_0, z_1, z_2, z_3 \in X$. There exists a number $B$ s. t. no matter which 1-forms $\beta, \gamma \in \Omega(X)$ we take (provided they have no zeroes in common and, together with $\alpha$, span $\Omega(X)$), one has

$$\left[ \frac{\gamma(z_0)}{\beta(z_0)} : \frac{\gamma(z_1)}{\beta(z_1)} : \frac{\gamma(z_2)}{\beta(z_2)} : \frac{\gamma(z_3)}{\beta(z_3)} \right] = B,$$

where the braces denote the cross-ratio. Moreover, any quadruple of numbers with cross-ratio $B$ can be realized as the section $s_B^3$ for some $\beta, \gamma \in \Omega(X)$.

Proof. Any pair $\beta', \gamma' \in \Omega(X)$ can be obtained from the pair $\beta, \gamma$ by the operations $(\beta, \gamma) \mapsto (\beta + c\alpha, \gamma + c\alpha)$, $(\beta, \gamma) \mapsto (c\beta, c\gamma)$, $(\beta, \gamma) \mapsto (\beta + c\beta)$, $(\beta, \gamma) \mapsto (\gamma, \beta)$. The first operation does not change the vector $s_B^3$ at all, since $\alpha$ vanishes at $z_i$'s. The second operation scales the quadruple and hence does not change its cross-ratio. The third operation adds a constant quadruple $(c, c, c, c)$, so does not affect the cross-ratio as well. The last operation acts on the quadruple as $(a, b, c, d) \mapsto (a^{-1}, b^{-1}, c^{-1}, d^{-1})$. It is elementary to check that

$$\frac{a^{-1} - c^{-1}}{b^{-1} - d^{-1}} : \frac{a^{-1} - d^{-1}}{b^{-1} - c^{-1}} = \frac{b - c}{ac} : \frac{c - b}{a - d} = \frac{cd - ad}{bd - d} : \frac{bc - ac}{bd - d} = \frac{a - c}{b - c} : \frac{a - d}{b - d}.$$

On the other hand, any two quadruples with the same cross-ratio can be related by the chain of operations $(a, b, c, d) \mapsto (ka, kb, kc, kd)$, $(a, b, c, d) \mapsto (a + k, b + k, c + k, d + k)$, $(a, b, c, d) \mapsto (a^{-1}, b^{-1}, c^{-1}, d^{-1})$. □

This can be thought of as a Weil-type reciprocity law. However, whereas the usual reciprocity law involves two functions and values of each of them at zeroes and poles of the other, this reciprocity law involves values of two holomorphic forms at zeroes of the third, so in order to deduce from the Weil reciprocity law itself, some terms must be canceled out. It is worth mentioning that there is another way to assign a cross-ratio to a canonical divisor on a genus three curve: namely, consider its canonical embedding into $\mathbb{CP}^2$ as a quartic curve, a canonical divisor is a linear section of its image, i.e. four points on a projective line. It would be nice to check if their cross-ratio equals the cross-ratio defined above.

Take $B \neq \pm 1$. Then the set of quadruples $(a, b, c, d) \in \mathbb{C}^4$ with $[a : b : c : d] = B$, after quotienting out the line spanned by $(1, 1, 1, 1)$, is a quadratic cone. This endows the greater isoperiodic leaf for $(X, \alpha)$, which is three-dimensional, with a meromorphic field of quadratic cones, which is degenerate e.g. along the hyperelliptic locus. By the above Proposition, any lesser isoperiodic leaf, which in this case has dimension one, is tangent to this field of cones.

However, the linear-algebraic cone $\Omega(\Omega, \alpha)$ has degree greater than two for $g > 3$ (since the characterization of $(g - 1) \times g$-matrices with nontrivial kernel involves the determinantal expression in its minors). Hence it sections, even unlikely, cannot be quadratic cones. This can also be viewed as a reciprocity law, i.e. certain degree $g - 1$ algebraic relation on the values of the $2g - 2$-tuple of numbers $\left\{ \frac{\alpha(z_i)}{\gamma(z_i)} \right\}_{i=0}^{2g-3}$. It would be interesting to write it down explicitly.
Example. Let $S$ be a genus $g = 4$ curve. In this case, the projectivization $\text{P}Q(S,\alpha)$ is a hypersurface swept by lines parametrized by $\mathbb{P}^2$. It is cut out, on the other hand, as a plane section of a determinantal variety $\text{P}Q(\Omega(S),\alpha)$, which has degree three. Hence $\text{P}Q(S,\alpha)$ is a cubic threefold. It is known that the so-called Fano surface of lines on a smooth cubic threefold is of general type, and hence cannot be $\mathbb{P}^2$. Therefore the doubly isoperiodic cone for a genus four curve is never smooth. This implies the following

**Proposition 4.11.** For any homolorphic 1-form $\alpha$ on a genus four curve $S$ there exist a linked three-dimensional plane $\tau \subset \Omega(S)$ containing $\alpha$.

**Proof.** Indeed, the singularities of the doubly isoperiodic cone correspond to linked triples (cf. the proof of Proposition 4.8). \hfill $\square$

Note that there is another way of associating a singular cubic threefold to a non-hyperelliptic genus four curve, via the linear system of cubics passing through its canonical image $\text{vdGK}$. It does not require a choice of a holomorphic 1-form, so it’s probably not the same, though it would be interesting to find a direct relation between these threefolds.

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