Canonical Quantization of Open String and Noncommutative Geometry

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Abstract

We perform canonical quantization of open strings in the $D$-brane background with a $B$-field. Treating the mixed boundary condition as a primary constraint, we get a set of secondary constraints. Then these constraints are shown to be equivalent to orbifold conditions to be imposed on normal string modes. These orbifold conditions are a generalization of the familiar orbifold conditions which arise when we describe open strings in terms of closed strings. Solving the constraints explicitly, we obtain a simple Hamiltonian for the open string, which reveals the nature of noncommutativity transparently.

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I. INTRODUCTION

The open string gives rise to the noncommutative geometry [1] for the D-brane with a NS-NS $B$-field. The $D$-brane dynamics is described by Yang-Mills gauge fields on noncommutative space-time. This point was implied in the work of Connes, Douglas, and Schwarz [2] on the Matrix $M$-model [3] compactified on a torus in an appropriate limit. Subsequently, more direct approaches to the noncommutative geometry in the string theories were taken in refs. [4–7], where the open string dynamics in the $D$-brane background are studied. The various aspects of the noncommutative Yang-Mills gauge theories and their implications in the string theories were discussed extensively in a recent work of Seiberg and Witten [8]. In particular the equivalence of the ordinary gauge fields and the noncommutative gauge fields has been proposed and checked by comparing the ordinary Dirac-Born-Infeld theory with its noncommutative counterpart for the $D$-brane.

In order to explore further the noncommutative geometry in the string and its nonperturbative effects, we may need to develop the string field theory based on the noncommutative algebra. Bigatti and Susskind [5] also discussed recently relevance of the noncommutative geometry in the light cone quantization of open strings attached to $D$-brane, which can be easily extended to the light cone string field theory. In this respect it is important to perform canonical quantization of the open string theory in the background of $D$-brane, which will be a stepping stone toward the second quantized theory. Quantization of the open strings in the presence of $D$-branes with a $B$-field has been already discussed in the literature. In ref. [9] it was pointed out that the nontrivial boundary condition in the presence of the $B$-field modify the canonical commutation relations and leads to the noncommutativity on $D$-brane worldvolume. This point was elaborated further subsequently by Chu and Ho [10], as they examine the simplectic form obtained in terms of the mode expansion of the classical solutions. However, there are some discrepancy between two works. To resolve the discrepancy the authors of both works perform the canonical quantization, treating the mixed boundary condition as a primary constraint and employing the Dirac’s quantization method. Never-
theless, the discrepancy still remains. The purpose of this paper is to carry out the canonical quantization of the open string in the $D$-brane background with some rigor and to clarify the related problems. In the course we will be able to confirm some of the results obtained in [8] by using the conformal field theory.

II. OPEN STRING IN THE BACKGROUND OF $D$-BRANE

The bosonic part of the classical action for an open string ending on a $Dp$-brane with a $B$-field is given by

\[ I = \frac{1}{4\pi\alpha'} \int_M d^2\xi \left[ G_{\mu\nu} \sqrt{-h} \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} + B_{ij} \epsilon^{\alpha\beta} \frac{\partial X^i}{\partial \xi^\alpha} \frac{\partial X^j}{\partial \xi^\beta} \right] \]

(1)

where $\mu = 0, 1, \ldots, 9$ and $i = 0, 1, \ldots, p$. Here we consider a simple flat background first: $G_{\mu\nu} = \eta_{\mu\nu}$, $H = dB = 0$. Extension to a more general background will be discussed later.

If the $U(1)$ gauge field on the $Dp$-brane is present, $(B_{ij} + F_{ij})$ replaces $B_{ij}$ in the action Eq.(1). For simplicity we set $2\pi\alpha' = 1$ and restore it when necessary.

Choosing the metric as $h_{\alpha\beta} = \eta_{\alpha\beta} = (-, +)$, we find the canonical momenta and the Hamiltonian as

\[ P_i = \partial_\tau X^i - B_{ij} \partial_\sigma X^j, \quad P_a = \partial_\tau X^a \]

\[ H = \frac{1}{2} \left( P_i + B_{ij} \partial_\sigma X^j \right)^2 + \frac{1}{2} (P_a)^2 + \frac{1}{2} \partial_\sigma X^\mu \partial_\sigma X_\mu \]

(2a)

(2b)

where $a = p + 1, \ldots, 9$. The boundary conditions to be imposed are as follows

\[ \partial_\sigma X^i - B^i_j \partial_\sigma X^j = 0, \quad X^a = x^a \]

(3)

for $\sigma = 0, \pi$. In terms of the canonical momenta the first boundary condition is written by

\[ B^i_j P^j - M^i_j \partial_\sigma X^j = 0 \]

(4)

where $M^i_j = \eta^i_j - B^i_k B^k_j$. Since the boundary conditions are nontrivial for $X^i$, we will be concerned only with $X^i$ hereafter.
We may incorporate the boundary condition Eq.(4) into the canonical quantization, treating it as a second class constraint. Before going into the canonical quantization of the open string in the $D$-brane background, it may be useful to recall the canonical quantization of the free open string. The open string is often described as a closed string with an orbifold condition

$$X^i(\sigma) = X^i(-\sigma), \quad P^i(\sigma) = P^i(-\sigma).$$

(5)

Let us recast this procedure into the canonical quantization. The Hamiltonian for the free open string is given as

$$H = \frac{1}{2} \int \frac{d\sigma}{2\pi} \left[ (P^i)^2 + (\partial_\sigma X^i)^2 \right] = \frac{1}{2} \sum_n \eta_{ij} \left( P_n^i P_{-n}^j + n^2 X_n^i X_{-n}^j \right)$$

where $X^i = \sum_n X_n^i e^{in\sigma}$, $P^j = \sum_n P_n^j e^{-in\sigma}$. (It is assumed that appropriate real conditions are imposed on $X_n^i$ and $P_n^i$.) The boundary conditions to be imposed on the two ends of open string are as follows, $\partial_\sigma X^i(0) = \partial_\sigma X^i(\pi) = 0$. In terms of normal modes these boundary conditions are rewritten as

$$\Phi_1^i = \sum_n n X_n^i = 0, \quad \bar{\Phi}_1^i = \sum_n n(-1)^n X_n^i = 0.$$ (6)

Viewing the boundary conditions as primary constraints, we find that they generate the secondary constraints

$$\Psi_1^i = \{ H, \Phi_1^i \}_{PB} = \sum_n n P_n^i, \quad \bar{\Psi}_1^i = \{ H, \bar{\Phi}_1^i \}_{PB} = \sum_n n(-1)^n P_n^i.$$ (7)

Here the fundamental Poisson brackets are given by

$$\{ X_n^i, P_m^j \} = \eta_{ij} \delta_{nm}, \quad \{ X_n^i, X_m^j \} = 0, \quad \{ P_n^i, P_m^j \} = 0.$$ (8)

The consistency requires that $\Psi_1^i = 0$ and $\bar{\Psi}_1^i = 0$. Again in order to impose these secondary constraints consistently we should introduce the following constraints

$$\{ H, \Psi_1^i \}_{PB} = -\sum_n n^3 X_n^i = 0,$$ (9a)

$$\{ H, \bar{\Psi}_1^i \}_{PB} = -\sum_n n^3(-1)^n X_n^i = 0.$$ (9b)
By repetition we get a complete set of constraints

\[ \Phi^i_m = \sum_n n^{2m-1} X^i_n = 0, \quad \Psi^i_m = \sum_n n^{2m-1} P^i_n = 0 \]  \hspace{1cm} (10a)

\[ \bar{\Phi}^i_m = \sum_n n^{2m-1} (-1)^n X^i_n = 0, \quad \bar{\Psi}^i_m = \sum_n n^{2m-1} (-1)^n P^i_n = 0 \]  \hspace{1cm} (10b)

where \( m = 1, 2, \ldots \). Since they are of second class, one needs to construct the Dirac bracket to incorporate them into the canonical quantization. However, each constraint involves all different modes, the Dirac bracket is expected to be complicated. As for this point the following simple observation turns out to be very useful. The set of the constraints, Eq.(10a) implies

\[ -i \sum_{m=1}^{\infty} \Phi^i_m (i\sigma)^{2m-1} \frac{(2m-1)!}{(2m-1)!} = -i \sum_{n,m=1}^{\infty} X^i_n (i\sigma)^{2m-1} \frac{(2m-1)!}{(2m-1)!} = \sum_n X^i_n \sin n\sigma = 0. \]  \hspace{1cm} (11)

It follows that

\[ \chi^i_n = X^i_n - X^i_{-n} = 0, \quad n = 1, 2, \ldots, \]  \hspace{1cm} (12)

from Eq.(11) and

\[ \int_0^{2\pi} \frac{d\sigma}{\pi} \sin n\sigma \sin m\sigma = \delta(n-m) - \delta(n+m). \]  \hspace{1cm} (13)

We also see that if Eq.(12) is imposed, the constraint equation, Eq.(10a) \( \{ \Phi^i_n = 0, \ n = 1, 2, \ldots \} \), holds. Thus, two sets of constraints are equivalent to each other. If we apply this procedure to the constraints, Eq.(10b) \( \{ \Phi^i_m = 0, \ m = 1, 2, \ldots \} \), we get the same result as Eq.(12). They do not introduce additional constraints, thus they are redundant. The same procedure yields that the set of constraints \( \{ \Psi^i_n = 0, \ n = 1, 2, \ldots \} \), is equivalent to the following set of constraints,

\[ \varphi^i_n = P^i_n - P^i_{-n} = 0, \quad n = 1, 2, \ldots \]  \hspace{1cm} (14)

and that the constraints Eq.(10b) are redundant. At a glance we find that these constraints Eqs.(12, 14) are nothing but the orbifold condition Eq.(5), which introduced to describe the open string in terms of the closed string.
Now it becomes an easy task to construct the Dirac brackets. We evaluate the commutators between the constraints

\[ \{ \chi^i_n, \chi^j_m \}_P B = 0, \quad \{ \chi^i_n, \varphi^j_m \}_P B = 2\eta^{ij}\delta_{nm}, \quad \{ \varphi^i_n, \varphi^j_m \}_P B = 0. \]  \hspace{1cm} (15)

Introducing

\[ C = 2 \begin{pmatrix} 0 & \eta^{ij} \\ -\eta^{ij} & 0 \end{pmatrix} \otimes I \]  \hspace{1cm} (16)

where \( I \) is identity matrix, \( I_{nm} = \delta_{nm} \), we construct the Dirac bracket as

\[ \{ A, B \} = \{ A, B \}_P B - \{ A, \phi_M \}_P B (C^{-1})^{MN} \{ \phi_N, B \}_P B \]  \hspace{1cm} (17)

where \( \phi_M = \{ \chi^i_n, \varphi^j_m \} \). The fundamental Dirac brackets are obtained as

\[ \{ x_i, p^j \}_P B = \eta^{ij}, \quad \{ X^i_n, P^j_m \}_B = \frac{1}{2} \eta^{ij} (\delta(n-m) + \delta(n+m)) \]  \hspace{1cm} (18)

where \( n \) and \( m \) are non-zero integers. All other fundamental Dirac brackets are vanishing.

### III. CANONICAL QUANTIZATION AND D-BRANE BACKGROUND

Now let us return to the canonical quantization of the open string in the D-brane background with a B-field. As in the case of the free open string, the boundary condition may be treated as a primary constraints. In the presence of B-field, we may expand the canonical string variables in terms of normal modes as

\[ X^i(\sigma) = a^i(\sigma + c) + \sum_n X^i_n e^{in\sigma}, \quad P^i(\sigma) = \sum_n P^i_n e^{-in\sigma} \]  \hspace{1cm} (19)

where \( c \) is a constant, which will be fixed later. In the limit of strong B-field, the dynamical degrees of freedom of the open string are mostly encoded by \( a^i \). In terms of the normal modes the Hamiltonian and the boundary conditions Eqs. read as

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\[ H = \frac{1}{2}(p^i + B^j a^i)^2 + \frac{1}{2}(a^i)^2 \]
\[ + \sum_{n=1}^{\infty} \left[ (P^i_n + i n B^i_j X^j_n) \left( P^m_m - i n B^k_i X^k_n \right) + n^2 X^i_n X_{i-n} \right] \]  \hspace{1cm} (20a)
\[ \Phi_0^i = B^j_i \sum_n P^j_n - M^i_j (a^j + i \sum_n n X^j_n) = 0, \]  \hspace{1cm} (20b)
\[ \bar{\Phi}_0^i = B^j_i \sum_n P^j_n (-1)^n - M^i_j (a^j + i \sum_n n X^j_n (-1)^n) = 0. \]  \hspace{1cm} (20c)

Here choose \( c = -\pi/2 \). Note that the boundary conditions relate string coordinate variables \( \{X^i_n\} \) to the momentum variables, \( \{P^j_n\} \).

Evaluating the commutator between the Hamiltonian and the primary constraints,

\[ \{H, \Phi_0^i\}_PB = -\sum_n i n P^i_n, \]  \hspace{1cm} (21)

we find the secondary constraints, which are conjugate to the primary constraints \( \Phi_0^i \)

\[ \Psi_0^i = \sum_n n P^i_n = 0. \]  \hspace{1cm} (22)

The Dirac procedure requires further that the commutators between the secondary constraints and the Hamiltonian are vanishing

\[ \{H, \Psi_0^i\}_PB = -i \sum_n n^2 \left( B^j_i P^j_n - i n M^i_j X^j_n \right) \]  \hspace{1cm} (23)

This procedure will be continued until it does not generate additional new constraints. By repetition we get a complete set of constraints

\[ \Phi^i_m = \sum_n n^{2m} (B^j_i P^j_n - i n M^i_j X^j_n) = 0, \]  \hspace{1cm} (24a)
\[ \Psi^i_m = \sum_n n^{2m+1} P^i_n = 0, \]  \hspace{1cm} (24b)

where \( m = 0, 1, 2, \ldots \). These constraints are of second class. We also get a set of constraints, which generated by \( \bar{\Phi}_0^i \) and its commutator with the Hamiltonian. But as in the case of the free open string, they are redundant.

Since each constraint involves all different normal modes, it is desirable to disentangle them to construct the Dirac bracket. As we observed before, the set of constraints \( \{\Psi^i_m = 0, m = 0, 1, 2, \ldots\} \) is equivalent to
\[ \{ \varphi^i_m = P^i_m - P_{-m}^i = 0, \quad m = 1, 2, \ldots \}. \] (25)

By the similar procedure we disentangle the set of constraints \( \{ \Phi^i_m = 0, m = 0, 1, \ldots \} \). We lead to

\[ \sum_{m=0}^{\infty} \Phi_m^i (i\sigma)^{2m} (2m)! = \sum_n (B^i_j P_n^j - iM^i_j X_n^j) \left( \sum_{m=0}^{\infty} \frac{(i\sigma)^{2m}}{(2m)!} \right) - M^i_j a^j 
= \sum_n (B^i_j P_n^j - iM^i_j X_n^j) \cos n\sigma - M^i_j a^j \]

(26)

It follows from

\[ \int_0^{2\pi} \frac{d\sigma}{\pi} \cos n\sigma \cos m\sigma = \delta(n - m) + \delta(n + m), \]

that this set of constraints is equivalent to the following constraints

\[ \chi_0^i = B^i_j p^j - M^i_j a^j, \] (27a)

\[ \chi_n^i = B^i_j (P_n^j + P_{-n}^j) - iM^i_j n(X_n^j - X_{-n}^j) = 0, \]

(27b)

where \( p^i = P_0^i \), and \( n = 1, 2, \ldots \). The first constraint determines \( a^i \), \( a^i = (M^{-1}B)^i_j p^j \).

Assuming that this solution is used explicitly, we will remove the constraint \( \chi_0^i = 0 \) hereafter.

Taking this into account we write

\[ X^i(\sigma) = (M^{-1}B)^i_j p^j (\sigma + c) + \sum_n X_n^i e^{in\sigma}. \]

(28)

We note that \( (M^{-1}B) \) is antisymmetric.

Evaluating the commutators between the constraints, we have

\[ \{ \chi_n^i, \chi_m^j \}_{PB} = 0, \quad \{ \chi_n^i, \varphi_m^j \}_{PB} = -2inM^{ij}\delta_{nm}, \quad \{ \varphi_n^i, \varphi_m^j \}_{PB} = 0. \]

(29)

With this commutator relations we construct

\[ C = -2i \begin{pmatrix} 0 & M^{ij} \\ -M^{ij} & 0 \end{pmatrix} \otimes N \]

(30)
where $N$ is a diagonal matrix, $(N)_{nm} = n\delta_{nm}$. The Dirac bracket is defined as Eq. (17) with \( \{\phi_M\} = \{x^i_m, \varphi^i_n\} \) and $C$, which are given by Eq. (27b) and Eq. (25) respectively. The fundamental Dirac brackets are then found to be

\[
\{x^i, p^j\}_{DB} = \eta^{ij}, \quad \{X^i_n, X^j_m\}_{DB} = \frac{i}{n} (M^{-1}B)^{ij} \delta(n - m), \tag{31}
\]

\[
\{X^i_n, P^j_m\}_{DB} = \frac{1}{2} \eta^{ij} (\delta(n - m) + \delta(n + m)),
\]

where $n$ and $m$ are non-zero integers. Other fundamental brackets are vanishing. It is noted that the commutator, \( \{X^i_n, X^j_m\}_{DB} \) is modified due to the background $B$-field, which results in noncommutative geometry.

The noncommutativity becomes manifest as we evaluate the commutator

\[
\{X^i(\sigma), X^j(\sigma')\}_{DB} = -(M^{-1}B)^{i}{}_{j} \left[(\sigma + \sigma' + 2c) + \sum_{n \neq 0} \frac{1}{n} \sin n(\sigma + \sigma')\right]. \tag{32}
\]

Again the consistent choice for $c$ is $c = -\pi/2$. The reason will be clear shortly. Making use of

\[
\sum_{n \neq 0} \frac{1}{n} \sin n\theta = \begin{cases} 
\pi - \theta : & 0 < \theta < 2\pi \\
0 : & \theta = 0, 2\pi
\end{cases}
\tag{33}
\]

we have

\[
\{X^i(\sigma), X^j(\sigma')\}_{DB} = \begin{cases} 
(M^{-1}B)^{i}{}_{j} \pi : & \sigma = \sigma' = 0 \\
-(M^{-1}B)^{i}{}_{j} \pi : & \sigma = \sigma' = \pi \\
0 : & \text{otherwise.}
\end{cases}
\tag{34}
\]

Similarly, we find

\[
\{X^i(\sigma), P^j(\sigma')\}_{DB} = \eta^{ij} \left(1 + \sum_{n \neq 0} \cos n\sigma \cos n\sigma'\right)
\]

\[
\{P^i(\sigma), P^j(\sigma')\}_{DB} = 0.
\]

These commutator relations agree with those obtained in the work of Chu and Ho [7,10].
IV. NONCOMMUTATIVE GEOMETRY

It is desirable to solve the constraints explicitly if possible. The constraints are solved explicitly, there is no need for the Dirac brackets. Defining

\[ Y^i_n = \frac{1}{\sqrt{2}}(X^i_n + X^{-i}_n), \quad K^i_n = \frac{1}{\sqrt{2}}(P^i_n + P^{-i}_n), \] (35a)

\[ \bar{Y}^i_n = \frac{1}{\sqrt{2}}(X^i_n - X^{-i}_n), \quad \bar{K}^i_n = \frac{1}{\sqrt{2}}(P^i_n - P^{-i}_n), \] (35b)

where \( n = 1, 2, \ldots \), we find that the only nontrivial commutation relations are

\[ \{ Y^i_n, \bar{Y}^j_m \}_{DB} = \frac{1}{n} (M^{-1} B)^{ij} \delta_{nm}, \quad \{ Y^i_n, K^i_m \}_{DB} = \eta^{ij} \delta_{nm}, \] (36)

and all other commutators are vanishing. The constraints Eq.(25) and Eq.(27b) are read as

\[ \bar{Y}^i_n = \frac{1}{in} (M^{-1} B)^i j K^j_n, \quad \bar{K}^i_n = 0. \] (37)

Using these constraints, we can get rid of \( \bar{Y}^i_n \) and \( \bar{K}^i_n \) in favor of \( Y^i_n \) and \( K^i_n \), which satisfy the usual commutation relations. Accordingly, the Hamiltonian can be written in terms of \( Y^i_n \) and \( K^i_n \) as

\[ H = \frac{1}{2} \bar{p}^i (M^{-1})_{ij} \bar{p}^j + \frac{1}{2} \sum_{n=1} \left[ K^i_n (M^{-1})_{ij} K^j_n + n^2 Y^i_n (M)_{ij} Y^j_n \right]. \] (38)

This is precisely the Hamiltonian for a free open string in the space-time background, of which metric is given by \( M_{ij} \). Thus, the Hamiltonian can be written in terms of the usual commutative algebra. The noncommutativity arises when we identify the space-time coordinates of open strings as

\[ X^i(\sigma) = x^i + (M^{-1} B)^i j \bar{p}^j \left( \sigma - \frac{\pi}{2} \right) + \sqrt{2} \sum_{n=1} \left( Y^i_n \cos n \sigma + \frac{1}{n} (M^{-1} B)^i j K^j_n \sin n \sigma \right). \] (39)

The obtained representation for the Hamiltonian, Eq.(38) and the string coordinate variables, Eq.(39) reveals the nature of the noncommutativity in string theory. The open string prefers \( (Y^i_n, K^i_n) \) as canonical variables while the closed string prefers \( (X^i_n, P^i_n) \). Thus, in the presence of the \( D \)-brane with a \( B \)-field, the interaction between the open and closed strings...
are expected highly nontrivial. As we see, the noncommutativity is important not only in the zero mode sector but also in all other nonzero mode sectors. This representation would be useful when we discuss various stringy noncommutative effects.

In order to compare our results with those of Seiberg and Witten [8], we restore $2\pi\alpha'$, and take $G_{ij} = g_{ij}$. We also change the signature of the metric on the string worldsheet, which takes that $h_{\alpha\beta} = (+, +)$ and $B_{ij}$ is replaced with $iB_{ij}$. Following this prescription, we get the Hamiltonian and the boundary conditions as

$$H = \int \frac{d\sigma}{2\pi} \left[ \pi\alpha' g^{ij}(P_i + iB_{ik}\partial_\sigma X^k)(P_j + iB_{jl}\partial_\sigma X^l) - \frac{1}{4\pi\alpha'} g_{ij}\partial_\sigma X^i \partial_\sigma X^j \right],$$  \hspace{1cm} (40a)

$$0 = (Bg^{-1})_{ij}P_j - \frac{i}{(2\pi\alpha')^2}(G_E)_{ij}\partial_\sigma X^j \quad \text{for} \quad \sigma = 0, \pi$$  \hspace{1cm} (40b)

where $G_E$ is the effective metric seen by the open string [8]

$$(G_E)_{ij} = \left( g - (2\pi\alpha')^2 B g^{-1} B \right)_{ij}.$$  \hspace{1cm} (41)

The string coordinate and momentum variables are written by

$$X^i(\sigma) = -i(2\pi\alpha')^2 \left( G_E^{-1} B g^{-1} \right)^{ij} p_j \left( \sigma - \frac{\pi}{2} \right) + \sum_n X^i_n e^{i\sigma},$$  \hspace{1cm} (42)

$$P_i(\sigma) = \sum_n P_i^n e^{-i\sigma}.$$  \hspace{1cm}

Here we note that

$$\left( G_E^{-1} B g^{-1} \right)^{ij} = \left( \frac{1}{g + 2\pi\alpha' B} \frac{1}{g - 2\pi\alpha' B} \right)^{ij} = -\frac{1}{(2\pi\alpha')^2} \theta^{ij}. \quad (43)$$

The fundamental Dirac brackets are given as

$$\{ X^i_n, X^j_m \}_{DB} = -\frac{1}{n} \theta^{ij} \delta(n - m),$$

$$\{ X^i_n, P_{jm} \}_{DB} = \frac{1}{2} \delta^i_j \left( \delta(n - m) + \delta(n + m) \right),$$  \hspace{1cm} (44)

$$\{ x^i, p_i \}_{DB} = \delta^i_j.$$  \hspace{1cm}

Other fundamental Dirac brackets are vanishing. As a concomitant result, we have

$$\{ X^i(\sigma), X^j(\sigma') \}_{DB} = \begin{cases} 
i\pi\theta^{ij} & : \sigma = \sigma' = 0 \\ -i\pi\theta^{ij} & : \sigma = \sigma' = \pi \\ 0 & : \text{otherwise} \end{cases} \quad (45)$$

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The Hamiltonian and the string coordinate variable are written in the phase space \((Y_i^n, K_i^n)\) by

\[
H = (2\pi\alpha') \frac{1}{2} p_i (G_E^{-1})^{ij} p_j + (2\pi\alpha') \sum_{n=1} \left\{ \frac{1}{2} K_{in} (G_E^{-1})^{ij} K_{jn} - \frac{1}{(2\pi\alpha')^2} \frac{n^2}{2} Y_i^n (G_E)_{ij} Y_j^n \right\}
\]

\[
X^i(\sigma) = x^i + i\theta^{ij} p_j \left( \sigma - \frac{\pi}{2} \right) + \sqrt{2} \sum_{n=1} \left( Y_i^n \cos n\sigma + \frac{i}{n} \theta^{ij} K_{jn} \sin n\sigma \right).
\]

Thus, the obtained Hamiltonian is precisely the Hamiltonian for a free open string in spacetime with the metric given by \((G_E)_{ij}\) as we may expect. Concomitantly the spectrum of the open string in the \(D\)-brane background is determined by the effective metric \((G_E)_{ij}\). As we mentioned before, it is convenient to employ basis \(|Y_i(\sigma)\rangle\) to describe the open string, interacting with the \(D\)-brane, while the usual basis \(|X^i(\sigma)\rangle\) is more suitable for the closed string. Note that the eigenstate of \(X^i(\sigma)\), \(|X^i(\sigma)\rangle\) can be constructed as a coherent state in \(|Y_i(\sigma)\rangle\). It is quite similar to the lowest Landau level state. More detailed discussion on this point will be given somewhere else \([11]\). In ref. \([8]\), Siberg and Witten discuss the zero slope limit, where \(\alpha' \sim \epsilon^{\frac{1}{4}} \to 0\), \(g_{ij} \sim \epsilon \to 0\) while keeping \(B\), \(G_E\) and \(\theta\) fixed. The zero slope limit does not alter the noncommutative structure, but it makes the potential term dominant in the Hamiltonian as in the lowest Landau level. As we see even in the zero slope limit the nonzero mode sectors contribute to the noncommutative \(D\)-brane dynamics as well as the zero mode sector. The nonzero mode sectors would be important to understand some stringy effects in the noncommutative geometry.

**V. CONCLUDING REMARKS**

We conclude this paper with a few remarks. We find that the dynamics of \(D\)-brane with a \(B\) field can be understood in the framework of the canonical quantization. In the presence of the \(B\) field we have a mixed boundary condition, which generates an infinite number of secondary second class constraints. The set of the second class constraints is shown to be equivalent to an orbifold condition, which is a generalization of the simple one introduced when the free open string is described in terms of the closed string. The best
way to deal with the second class constraints is to solve them explicitly. Indeed we can solve the constraints explicitly without difficulty and get a simple Hamiltonian for the open string in the $D$-brane background. The Hamiltonian is found to be a free Hamiltonian for an open string in space-time with the metric $G_E$ Eq.(41). Noncommutativity arises as the orbifold condition effectively reduces the phase space for the string by half. Eq.(46) reveals the nature of noncommutativity in string theory transparently. The present work may serve as stepping stone leading us to various directions. The canonical analysis carried out in the present paper may enable us to construct the second quantized theory for the open string in the $D$-brane background, which would be an appropriate generalization of the earlier work on the open string by Witten [12]. We may also apply the same canonical quantization procedure to the open string attached to the multi-$D$-branes or to two different types of $D$-branes. Work along this direction may improve our understanding of the $AdS/CFT$ correspondence [13] and the black hole physics in string theory. In the due course one may attempt to derive the (non-Abelian) noncommutative Dirac-Born-Infeld effective action [8,14] for the $D$-brane, which remains to be an outstanding open problem.

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