Quantum-like Representation of Extensive Form Games: Wine Testing Game

Andrei Khrennikov
International Center for Mathematical Modeling in Physics and Cognitive Sciences
University of Växjö, S-35195, Sweden

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Abstract

We consider an application of the mathematical formalism of quantum mechanics (QM) outside physics, namely, to game theory. We present a simple game between macroscopic players, say Alice and Bob (or in a more complex form - Alice, Bob and Cecilia), which can be represented in the quantum-like (QL) way – by using a complex probability amplitude (game’s “wave function”) and noncommutative operators. The crucial point is that games under consideration are so called extensive form games. Here the order of actions of players is important, such a game can be represented by the tree of actions. The QL probabilistic behavior of players is a consequence of incomplete information which is available to e.g. Bob about the previous action of Alice. In general one could not construct a classical probability space underlying a QL-game. This can happen even in a QL-game with two players. In a QL-game with three players Bell’s inequality can be violated. The most natural probabilistic description is given by so called contextual probability theory completed by the frequency definition of probability.
1 Introduction

One should sharply distinguish between QM as a physical theory and the mathematical formalism of QM. In the same way as one should distinguish between classical Newtonian mechanics and its mathematical formalism. Nobody is surprised that differential and integral calculi which are basic in Newtonian mechanics can be fruitfully applied in other domains of science. Unfortunately, the situation with the mathematical formalism of QM is essentially more complicated – some purely mathematical specialities of QM are projected on and even identified with specialities of quantum physical systems. Although already Nils Bohr pointed out by himself [1], see also [2], [3], to the possibility to apply the mathematical formalism of QM outside of physics, the prejudices based on identification of mathematics and physics still survive (but cf. e.g. Accardi, Ballentine, De Muynck, Gudder, Landé, Mackey [4]–[19] and also [20]–[22]) and one can point out just to a few applications outside of physics. Here we discuss not reductionist models in that the quantum description appears as a consequence of composing of a system (for example, the brain, see e.g. [23], [24]) of quantum particles, but really the possibility to use the mathematical formalism of QM without direct coupling with quantum physics, see e.g. [25], [26].

One of interesting possibilities to explore quantum mathematics is provided by game theory [27], [28]. One of the main distinguishing features of the mathematical formalism of QM is the calculus of quantum probabilities. It is the calculus [29], [30] of complex probability amplitudes and self-adjoint operators, in contrast to the calculus of random variables on the Kolmogorov classical probability space [31]. The impossibility to use a fixed Kolmogorov probability space induces applications of such a probabilistic theory as Gudder’s theory of probabilistic manifolds, see also Accardi [5] for non-Kolmogorovian models. Recently Karl Hess and Walter Philipp pointed out to the old paper of Soviet mathematician Vorobjev [32] who studied the problem of the possibility to realize a number of observables on a single Kolmogorov space. This problem is equivalent to the problem of violation of Bell’s inequality which was later studied by J. Bell [33]. However, nonlocality was not involved in Vorobjev’s considerations. It is especially interesting for us that Vorobjev pointed out to a possibility to apply probabilistic models without underlaying Kolmogorov structure in game theory (in fact, he promised to come with such applications in later publications, but I was able not find anything). One may consider the present paper as a step toward realization
of the Vorobjev’s program.

We present a simple game between macroscopic players, say Alice and Bob (or in a more complex form - Alice, Bob and Cecilia), which can be represented in the quantum-like (QL) way – by using a complex probability amplitude (game’s “wave function”) and noncommutative operators. The crucial point is that games under consideration are so called extensive form games, see e.g. [34]. Here the order of actions of players is important, such a game can be represented by the tree of actions. The QL probabilistic behavior of players is a consequence of incomplete information which is available to e.g. Bob about the previous action of Alice. In general one could not construct a classical probability space underlying a QL-game. This can happen even in a QL-game with two players. In a QL-game with three players Bell’s inequality can be violated. The most natural probabilistic description is given by so called contextual probability theory [36] completed by the frequency definition of probability [37], [38]. In particular this theory provides an algorithm – quantum-like representation algorithm (QLRA) [36] which gives a possibility to represent special collections of probabilistic data for a pair of observables \(a, b\) by complex probability amplitudes (or in the abstract formalism by normalized vectors of the complex Hilbert space) and observables by self-adjoint operators \(\hat{a}, \hat{b}\). We shall use QLRA to find QL-representations of extensive form games. The probabilistic structure of our game (for two players) can be considered as Gudder’s probability manifold [15] with the atlas having two charts.

The first examples of QL-games with macroscopic players were presented in papers [39]–[42] which were based essentially on quantum logic models [43], [44]. In this paper we use contextual probabilistic arguments which provide a possibility to take into a more detailed account the probabilistic sources of the QL-behaviour of players.

Our examples are totally different from what now is widely discussed in many papers in the name of quantum games [45]. All examples with quantum coins, quantum gamblers etc. in this or that way use micro objects described by quantum physics as some hardware, while in our examples everything is totally macroscopic. However, some results obtained in the cited quantum game activity can be applied to our examples (because we use the same mathematical apparatus).

Our study of QL-games can also be considered as a contribution in clarification of problems in foundations of QM. In particular, the problems of “death of reality” and nonlocality, cf. [46], [47]. Analysis of QL-games sup-
ports the Bohr’s viewpoint on observables – the result of a measurement cannot be considered as an objective property of this systems which could be assigned to the system before the measurement starts. In our models this results are preferences of Alice and Bob in choice of wine as well as their abilities to test wine. Such things could not be assigned with a bottle of wine as its objective properties. Nevertheless, game theory shows that there are no reasons for panics – death of reality. There is reality of wine and its chemical properties as well as reality of brains which induces finally results of measurements.

The QL-behavior can be produced in the purely local framework. However, a game can be completed by interactions between players (of course, laws of special relativity are not violated). Such games are even more interesting and they could have more extended domain of applications.

2 Contextual probability

A general statistical model for observables based on the contextual viewpoint to probability will be presented. It will be shown that classical as well as quantum probabilistic models can be obtained as particular cases of our general contextual model, the Växjö model, [36].

This model is not reduced to the conventional, classical and quantum models. In particular, it contains a new statistical model: a model with hyperbolic cosh-interference that induces ”hyperbolic quantum mechanics” [36].

A physical, biological, social, mental, genetic, economic, or financial context \( C \) is a complex of corresponding conditions. Contexts are fundamental elements of any contextual statistical model. Thus construction of any model \( M \) should be started with fixing the collection of contexts of this model. Denote the collection of contexts by the symbol \( \mathcal{C} \) (so the family of contexts \( \mathcal{C} \) is determined by the model \( M \) under consideration). In the mathematical formalism \( \mathcal{C} \) is an abstract set (of “labels” of contexts).

We remark that in some models it is possible to construct a set-theoretic representation of contexts – as some family of subsets of a set \( \Omega \). For example, \( \Omega \) can be the set of all possible parameters (e.g., physical, or mental, or economic) of the model. However, in general we do not assume the possibility to construct a set-theoretic representation of contexts.

Another fundamental element of any contextual statistical model \( M \) is a
set of observables $\mathcal{O}$: each observable $a \in \mathcal{O}$ can be measured under each complex of conditions $C \in \mathcal{C}$. For an observable $a \in \mathcal{O}$, we denote the set of its possible values (“spectrum”) by the symbol $X_a$.

We do not assume that all these observables can be measured simultaneously. To simplify considerations, we shall consider only discrete observables and, moreover, all concrete investigations will be performed for dichotomous observables.

**Axiom 1:** For any observable $a \in \mathcal{O}$ and its value $\alpha \in X_a$, there are defined contexts, say $C_\alpha$, corresponding to $\alpha$-selections: if we perform a measurement of the observable $a$ under the complex of physical conditions $C_\alpha$, then we obtain the value $a = \alpha$ with probability 1. We assume that the set of contexts $\mathcal{C}$ contains $C_\alpha$-selection contexts for all observables $a \in \mathcal{O}$ and $\alpha \in X_a$.

For example, let $a$ be the observable corresponding to some question: $a = +$ (the answer “yes”) and $a = -$ (the answer “no”). Then the $C_+$-selection context is the selection of those participants of the experiment who answering “yes” to this question; in the same way we define the $C_-$-selection context. By Axiom 1 these contexts are well defined. We point out that in principle a participant of this experiment might not want to reply at all to this question or she might change her mind immediately after her answer. By Axiom 1 such possibilities are excluded. By the same axiom both $C_+$ and $C_-$-contexts belong to the system of contexts under consideration.

**Axiom 2:** There are defined contextual (conditional) probabilities $p^a_{C}(\alpha) \equiv P(a = \alpha | C)$ for any context $C \in \mathcal{C}$ and any observable $a \in \mathcal{O}$.

Thus, for any context $C \in \mathcal{C}$ and any observable $a \in \mathcal{O}$, there is defined the probability to observe the fixed value $a = \alpha$ under the complex of conditions $C$.

Especially important role will be played by “transition probabilities” $p^{\alpha|\beta}(\alpha|\beta) \equiv P(a = \alpha | C_{\beta})$, $a, b \in \mathcal{O}, \alpha \in X_a, \beta \in X_b$, where $C_{\beta}$ is the $[b = \beta]$-selection context. By axiom 2 for any context $C \in \mathcal{C}$, there is defined the set of probabilities: $\{p^a_C : a \in \mathcal{O}\}$. We complete this probabilistic data for the context $C$ by transition probabilities. The corresponding collection of data $D(\mathcal{O}, \mathcal{C})$ consists of contextual probabilities: $p^{\alpha|\beta}(\alpha|\beta), p^{b}_{C}(\beta), p^{b|a}(\beta|\alpha), p^{a}_{C}(\alpha), ...$, where $a, b, ..., \in \mathcal{O}$. Finally, we denote the family of probabilistic data $D(\mathcal{O}, \mathcal{C})$ for all contexts $C \in \mathcal{C}$ by the symbol $D(\mathcal{O}, \mathcal{C})(\equiv \cup_{C \in \mathcal{C}} D(\mathcal{O}, C))$.

**Definition 1.** (Växjö Model) An observational contextual statistical
model of reality is a triple \( M = (\mathcal{C}, \mathcal{O}, \mathcal{D}(\mathcal{O}, \mathcal{C})) \), where \( \mathcal{C} \) is a set of contexts and \( \mathcal{O} \) is a set of observables which satisfy to axioms 1, 2, and \( \mathcal{D}(\mathcal{O}, \mathcal{C}) \) is probabilistic data about contexts \( \mathcal{C} \) obtained with the aid of observables belonging \( \mathcal{O} \).

We call observables belonging the set \( \mathcal{O} \equiv \mathcal{O}(M) \) reference of observables. Inside of a model \( M \) observables belonging to the set \( \mathcal{O} \) give the only possible references about a context \( C \in \mathcal{C} \). In the definition of the Växjö Model we speak about “reality.” In our approach it is reality of contexts.

In what follows we shall consider Växjö models with two dichotomous reference observables.

3 Frequency definition of probabilities

The definition of probability has not yet been specified. In this paper we shall use the frequency definition of probability as the limit of frequencies in a long series of trials, von Mises’ approach, [37], [38]. We are aware that this approach was criticized a lot in mathematical literature. However, the main critique was directed against von Mises’ definition of randomness. If one is not interested in randomness, but only in frequencies of trials, then the frequency approach is well established, see [38].

We consider a set of reference observables \( \mathcal{O} = \{a, b\} \) consisting of two observables \( a \) and \( b \). We denotes the sets of values (“spectra”) of the reference observables by symbols \( X_a \) and \( X_b \), respectively.

Let \( C \) be some context. In a series of observations of \( b \) (which can be infinite in a mathematical model) we obtain a sequence of values of \( b : x \equiv x(b|C) = (x_1, x_2, ..., x_N, ...) \), \( x_j \in X_b \). In a series of observations of \( a \) we obtain a sequence of values of \( a : y \equiv y(a|C) = (y_1, y_2, ..., y_N, ...) \), \( y_j \in X_a \). We suppose that the principle of the statistical stabilization for relative frequencies [37], [38] holds. This means that the frequency probabilities are well defined: \( p_C^b(\beta) = \lim_{N \to \infty} \nu_N(\beta; x) \), \( \beta \in X_b \); \( p_C^a(\alpha) = \lim_{N \to \infty} \nu_N(\alpha; y) \), \( \alpha \in X_a \). Here \( \nu_N(\beta; x) \) and \( \nu_N(\alpha; y) \) are frequencies of observations of values \( b = \beta \) and \( a = \alpha \), respectively (under the complex of conditions \( C \)).

Remark. (On the notions of collective and S-sequence) R. von Mises considered in his theory two principles: a) the principle of the statistical stabilization for relative frequencies; b) the principle of randomness. A sequence of observations for which both principle hold was called a collective. [37]. However, it seems that the validity of the principle of statistical stabilization...
is often enough for applications. Here we shall use just the convergence of frequencies to probabilities. An analog of von Mises’ theory for sequences of observations which satisfy the principle of statistical stabilization was developed in [38]; we call such sequences \( S \)-sequences.

Everywhere in this paper it will be assumed that sequences of observations are \( S \)-sequences, cf. [38] (so we are not interested in the validity of the principle of randomness for sequences of observations, but only in existence of the limits of relative frequencies).

Let \( C_\alpha, \alpha \in X_a \), be contexts corresponding to \( \alpha \)-filtrations, see Axiom 1. By observation of \( b \) under the context \( C_\alpha \) we obtain a sequence:

\[
x_\alpha \equiv x(b|C_\alpha) = (x_1, x_2, ..., x_N, ...), \quad x_j \in X_b.
\]

It is also assumed that for sequences of observations \( x^\alpha, \alpha \in X_a \), the principle of statistical stabilization for relative frequencies holds true and the frequency probabilities are well defined:

\[
p^{b|a}(\beta|\alpha) = \lim_{N \to \infty} \nu_N(\beta|x^\alpha), \quad \beta \in X_b.
\]

Here \( \nu_N(\beta|x^\alpha), \alpha \in X_a \), are frequencies of observations of value \( b = \beta \) under the complex of conditions \( C_\alpha \). We can repeat all previous considerations by changing \( b|a \)-conditioning to \( a|b \)-conditioning. There can be defined probabilities \( p^{a|b}(\alpha|\beta) \).

4 Quantum-like representation algorithm — QLRA

In [36] we derived the following formula for interference of probabilities:

\[
p^{b|c}_{C}(\beta) = \sum_{\alpha} p^{a|C}_{C}(\alpha)p^{b|a}(\beta|\alpha) + 2\lambda(\beta|\alpha, C) \sqrt{\prod_{\alpha} p^{a|C}_{C}(\alpha)p^{b|a}(\beta|\alpha)},
\]

where the coefficient of interference

\[
\lambda(\beta|a, C) = \frac{p^{b|C}_{C}(\beta) - \sum_{\alpha} p^{a|C}_{C}(\alpha)p^{b|a}(\beta|\alpha)}{2\sqrt{\prod_{\alpha} p^{a|C}_{C}(\alpha)p^{b|a}(\beta|\alpha)}}.
\]

A similar representation we have for the \( a \)-probabilities. Such interference formulas are valid for any collection of contextual probabilistic data satisfying the conditions:

R1). Observables \( a \) and \( b \) are symmetrically conditioned:

\[
p^{b|a}(\beta|\alpha) = p^{a|b}(\alpha|\beta).
\]
R2). \( p^{bl}(\alpha|\beta) > 0 \) and \( p^{bl}(\beta|\alpha) > 0 \) as well as \( p^b_C(\beta) > 0 \) and \( p^c_C(\alpha) > 0 \). Suppose that also the following conditions hold:

R3). Coefficients of interference \( \lambda(\beta|a, C) \) and \( \lambda(\alpha|b, C) \) are bounded by one.

A context \( C \) such that R3) holds is called trigonometric, because in this case we have the conventional formula of trigonometric interference:

\[
p^b_C(\beta) = \sum_{\alpha} p^a_C(\alpha)p^{bl}(\beta|\alpha) + 2 \cos \theta(\beta|\alpha, C) \sqrt{\prod_{\alpha} p^a_C(\alpha)p^{bl}(\beta|\alpha)},
\]

where \( \lambda(\beta|a, C) = \cos \theta(\beta|a, C) \). Parameters \( \theta(\beta|\alpha, C) \) are said to be \( b|a \)-relative phases with respect to the context \( C \). We defined these phases purely on the basis of probabilities. We have not started with any linear space; in contrast we shall define geometry from probability.

We denote the collection of all trigonometric contexts by the symbol \( C^{tr} \).

By using the elementary formula:

\[
D = A + B + 2 \sqrt{AB} \cos \theta = |\sqrt{A} + e^{i\theta} \sqrt{B}|^2,
\]

for real numbers \( A, B > 0, \theta \in [0, 2\pi] \), we can represent the probability \( p^b_C(\beta) \) as the square of the complex amplitude (Born’s rule):

\[
p^b_C(\beta) = |\psi_C(\beta)|^2.
\]

Here

\[
\psi(\beta) \equiv \psi_C(\beta) = \sqrt{p^a_C(\alpha_1)p^{bl}(\beta|\alpha_1) + e^{i\theta_C(\beta)} \sqrt{p^a_C(\alpha_2)p^{bl}(\beta|\alpha_2)}}, \beta \in X_b,
\]

where \( \theta_C(\beta) \equiv \theta(\beta|\alpha, C) \).

The formula (5) gives the quantum-like representation algorithm - QLRA. For any trigonometric context \( C \) by starting with the probabilistic data – \( p^b_C(\beta), p^c_C(\alpha), p^{bl}(\beta|\alpha) \) - QLRA produces the complex amplitude \( \psi_C \). This algorithm can be used in any domain of science to create the QL-representation of probabilistic data (for a special class of contexts).

We point out that QLRA contains the reference observables as parameters. Hence the complex amplitude given by (5) depends on \( a, b : \psi_C \equiv \psi^{bl}_{b|a} \).

We denote the space of functions: \( \varphi : X_b \to \mathbb{C} \) by the symbol \( \Phi = \Phi(X_b, C) \). Since \( X = \{\beta_1, \beta_2\} \), the \( \Phi \) is the two dimensional complex linear
space. By using QLRA we construct the map $J^{b} : H^{tr} \to \Phi(X, C)$ which maps contexts (complexes of, e.g., physical conditions) into complex amplitudes. The representation of probability is nothing other than the famous Born rule. The complex amplitude $\psi_C(x)$ can be called a wave function of the complex of physical conditions (context) $C$ or a (pure) state. We set $e^b_\beta(\cdot) = \delta(\beta - \cdot) - \text{Dirac delta-functions concentrated in points } \beta = \beta_1, \beta_2$. The Born's rule for complex amplitudes can be rewritten in the following form: $p^b_\beta = |\langle \psi_C, e^b_\beta \rangle|^2$, where the scalar product in the space $\Phi(X_b, C)$ is defined by the standard formula: $\langle \phi, \psi \rangle = \sum_{\beta \in X_b} \phi(\beta) \bar{\psi}(\beta)$. The system of functions $\{e^b_\beta \}_{\beta \in X_b}$ is an orthonormal basis in the Hilbert space $H_{ab} = (\Phi, \langle \cdot, \cdot \rangle)$.

Let $X_b \subset \mathbb{R}$. By using the Hilbert space representation of the Born's rule we obtain the Hilbert space representation of the expectation of the observable $b$: $E(b|C) = \sum_{\beta \in X_b} \beta |\psi_C(\beta)|^2 = \sum_{\beta \in X_b} \beta \langle \psi_C, e^b_\beta \rangle \overline{\langle \psi_C, e^b_\beta \rangle} = \langle b \psi_C, \psi_C \rangle$, where the (self-adjoint) operator $b : H_{ab} \to H_{ab}$ is determined by its eigenvectors: $\hat{b} e^b_\beta = \beta e^b_\beta, \beta \in X_b$. This is the multiplication operator in the space of complex functions $\Phi(X_b, C) : \hat{b} \psi(\beta) = \beta \psi(\beta)$. It is natural to represent the $b$-observable (in the Hilbert space model) by the operator $\hat{b}$.

We would like to have Born's rule not only for the $b$-variable, but also for the $a$-variable: $p^a_\alpha = |\langle \varphi, e^a_\alpha \rangle|^2, \alpha \in X_a$.

How can we define the basis $\{e^a_\alpha \}$ corresponding to the $a$-observable? Such a basis can be found starting with interference of probabilities. We set $u^a_j = \sqrt{p^a_\alpha}, p_{ij} = p(\beta_j|\alpha_i), u_{ij} = \sqrt{p_{ij}}, \theta_j = \theta_C(\beta_j)$. We have:

$$\varphi = u^a_1 e^a_{\alpha_1} + u^a_2 e^a_{\alpha_2},$$

where

$$e^a_{\alpha_1} = (u_{11}, u_{12}), \quad e^a_{\alpha_2} = (e^{i\theta_1} u_{21}, e^{i\theta_2} u_{22})$$

The condition R1 implies that the system $\{e^a_\alpha \}$ is an orthonormal basis iff the probabilistic phases satisfy the constraint:

$$\theta_2 - \theta_1 = \pi \mod 2\pi,$$

but, as we have seen, we can always choose such phases (under the condition R1).

In this case the $a$-observable is represented by the operator $\hat{a}$ which is diagonal with eigenvalues $\alpha_1, \alpha_2$ in the basis $\{e^a_\alpha \}$. The conditional average of the observable $a$ coincides with the quantum Hilbert space average: $E(a|C) = \sum_{\alpha \in X_a} \alpha p^a_\alpha = \langle \hat{a} \psi_C, \psi_C \rangle$.
5 Wine testing game

There is a restaurant with a good collection of (only) French and Italian wines of various sorts. Couples come to this restaurant for dinners and to have more fun they play the following Wine Game which consists of two wine tests.

A1). Alice selects a bottle (without telling her friend Bob its name) and proposes him to test wine. A battle of this wine is opened in the restaurant’s kitchen, Bob gets just a glass of this wine. Alice asks him the question:

“Is it French or Italian?”

A3). If Bob answers (after testing) correctly, he gets some amount of money; if not, he loses money and Alice gets some amount of money.

The choice in A1 is not totally random, Alice has her own preferences (later she wants to share the chosen bottle with Bob).

In the second part of the game Alice and Bob interchange their roles, so Bob starts by choosing a bottle of French or Italian wines and so on.

We introduce for the first and second parts of the game the elements of the payment matrices

\[
(h^b_{FF;1}, h^b_{FI;1}, ...), (h^a_{FF;k}, h^a_{FI;k}, ...), k = 1, 2.
\]

Here the indexes \(k = 1, 2\) denote the first and second part of the game and FI,..., II combinations of choices of Alice and Bob.\(^1\) The upper indexes \(a, b\) are marks for Alice’s and Bob’s payoffs. It is natural to assume that

\[
h^b_{FF;1}, h^b_{FI;1} > 0, \quad h^b_{FI;1}, h^b_{IF;1} < 0
\]

as well as

\[
h^a_{FI;1}, h^a_{FI;1} > 0, \quad h^a_{FF;1}, h^a_{II;1} < 0.
\]

In the zero sum game

\[
h^b_{FF;k} = -h^a_{FF;k}, ..., h^b_{II;k} = -h^a_{II;k}.
\]

Each part of this game can be represented as an extensive form game, hence, by a tree, see \([35],[34]\) (or just the link \(http://en.wikipedia.org\)).

\(^1\)We also remark the Alice’s choice can be considered as an “element of reality”, since her, e.g., F, is really French wine, but Bob’s F may be in reality either French or Italian wine, cf. with discussions about realism in quantum mechanics, e.g., \([46],[47]\).
tree is very simple and it has the following branches representing actions of Alice and Bob; each branch is finished by the pair of payoffs, the symbol ”v” used for vertexes and ”act” for corresponding actions. The first part of the game is represented by the tree with the branches:

\[ v = A - act = F - v = B - act = F - (h^b_{FF,1}, h^a_{FF,1}); \]
\[ v = A - act = F - v = B - act = I - h^b_{FI,1}; h^b_{FI,1}; \]
\[ v = A - act = I - v = B - act = F - (h^b_{IF,1}, h^a_{IF,1}); \]
\[ v = A - act = I - v = B - act = I - (h^b_{II,1}, h^a_{II,1}). \]

As always, we are interested in averages of wins-losses of Alice and Bob. We consider the following probabilities:

1). Probabilities of Alice’s preferences for a bottle of French wine and respectively a bottle of Italian wine from the wine-collection of the restaurant:

\[ p_a^e(C)(F), \ p_a^e(C)(I). \]

Here the index \( C \) is related to the whole context of the game, in particular, to the collection of wines. Another restaurant has another collection of wines, and Alice would have other preferences.

2). Probabilities to recognize French wine after testing (by Bob) a bottle of French wine which was chosen by Alice for the test: \( p^{bla}(F|F) \); the probability of mistake under this condition, i.e., claiming that the wine is Italian, is then \( p^{bla}(I|F) = 1 - p^{bla}(F|F) \). In the similar way we introduce probabilities \( p^{bla}(I|I) \) and \( p^{bla}(F|I) \). Thus we have the matrix which is typically called the matrix of transition probabilities:

\[ \mathbf{P}^{bla} = (p^{bla}(\beta|\alpha)), \ \beta, \alpha = I, F. \]

3). Similarly we introduce probabilities \( p_b^e(C)(F) \) and \( p_b^e(C)(I) \) for Bob’s preferences (for the same collection of wines) as well as probabilities \( p^{a|b}(F|F), ..., p^{a|b}(I|I) \) which represent Alice’s ability to recognize the origin of wine. There is the matrix of transition probabilities \( \mathbf{P}^{a|b} = (p^{a|b}(\alpha|\beta)), \ \alpha, \beta = I, F. \)

4). Finally, we introduce probabilities that Bob will announce the result \( \beta (= F, I) \) in the game that Alice starts with the result \( \alpha \) (which is hidden from Bob):

\[ p_C^{ab}(\alpha, \beta) = p_C^e(\alpha)p^{bla}(\beta|\alpha), \]
and similar probabilities for the game which is started by Bob: \( p_C^{ba}(\beta, \alpha). \)
We remark that $p^{ab}_C(\alpha, \beta)$ is really probability on the set of all pairs $(\alpha, \beta)$:

$$\sum_{\alpha, \beta} p^{ab}_C(\alpha, \beta) = \sum_{\alpha} p^{a}_C(\alpha) \sum_{\beta} p^{b|a}_{\beta}(\beta|\alpha) = 1.$$  

This probability serves well for the first part of the game – when Alice chooses a bottle:

$$p^{a}_C(\alpha) = \sum_{\beta} p^{ab}_C(\alpha, \beta).$$

However, it could not be used in the second part of the game, since in general:

$$p^{b}_C(\beta) \neq \sum_{\alpha} p^{ab}_C(\alpha, \beta).$$

The second part of the game is served by the probability $p^{ba}_C(\beta, \alpha)$. The tricky thing, see [12], is really the combination of two games.

We point out that in general the equality

$$p^{ab}_C(\alpha, \beta) = p^{ba}_C(\beta, \alpha)$$

(8)

can be violated. This is the main source of “nonclassicality” of our game.

Then the average wins-losses in the first part of the game for Bob is given by

$$E^b_1(C) = h^b_{FF,1} p^{ab}_C(F, F) + h^b_{FI,1} p^{ab}_C(F, I) + h^b_{IF,1} p^{ob}_C(I, F) + h^b_{II,1} p^{ob}_C(I, I).$$

The average for Alice in the first part of the game (in general we can consider nonzero sum game) is given by

$$E^a_1(C) = h^a_{FF,1} p^{ob}_C(F, F) + h^a_{FI,1} p^{ob}_C(F, I) + h^a_{IF,1} p^{ob}_C(I, F) + h^a_{II,1} p^{ob}_C(I, I).$$

In the same way the averages for Alice and Bob in the second part of the game are given by

$$E^a_2(C) = h^a_{FF,2} p^{bo}_C(F, F) + h^a_{FI,2} p^{bo}_C(F, I) + h^a_{IF,2} p^{bo}_C(I, F) + h^a_{II,2} p^{bo}_C(I, I)$$

and

$$E^b_2(C) = h^b_{FF,2} p^{bo}_C(F, F) + h^b_{FI,2} p^{bo}_C(F, I) + h^b_{IF,2} p^{bo}_C(I, F) + h^b_{II,2} p^{bo}_C(I, I).$$

The averages of total wins-losses are

$$E^b(C) = E^b_1(C) + E^b_2(C), \ E^a(C) = E^a_1(C) + E^a_2(C).$$
It is convenient to introduce a “wine-observable” for Alice: \( a = F, I \). This observable appears in two different contexts. The first context, \( C \), is the context of selection of a bottle from the wine collection. Alice chooses a bottle and says herself (not Bob!) or just think – it is French wine (or it is Italian wine). The second context appears in the second part of the game when Alice should test wine proposed by Bob and after that say: it is French wine (or it is Italian wine). In fact, to be completely correct one should consider two different observables corresponding to these contexts. However, to have closer analogy with quantum mechanics, we proceed with one observable. Alice is considered as simply an apparatus which says either “French wine” or “Italian wine” (cf. with Stern-Gerlach magnet, it “says” either “spin up” or “spin down”). We remark that our cognitive example shows that it might be more natural to associate with each quantum state – wave function – its own spin-observable. We introduce a similar observable for Bob, \( b = F, I \).

6 Extensive form game with imperfect information

As was mentioned, formally wine testing game is an extensive form game. However, we should point out to one rather delicate feature of the game. We recall that a complete extensive form representation specifies: 1) the players of a game; 2) for every player every opportunity they have to move; 3) what each player can do at each of their moves; 4) what each player knows for every move; 5) the payoffs received by every player for every possible combination of moves.

Our game fulfills all those conditions besides the fourth one. In fact, the action of Alice does not specify for Bob the result of her action, Bob should guess about the country origin of the wine given by Alice. To come to the conclusion, he should perform a rather complicated analysis of the wine test. One may say that this is a game with imperfect information.

We recall that an information set is a set of decision nodes such that: 1) every node in the set belongs to one player; 2) when play reaches the information set, the player with the move cannot differentiate between nodes within the information set, i.e. if the information set contains more than one node, the player to whom that set belongs does not know which node in the
set has been reached.

If a game has an information set with more than one member that game is said to have imperfect information. A game with perfect information is such that at any stage of the game, every player knows exactly what has taken place earlier in the game, i.e. every information set is a singleton set. Any game without perfect information has imperfect information.

However, there is a problem with the second condition determining the information set. Of course, Bob does not know precisely which kind of wine is presented for the test. In this sense the set of Bob’s nodes after Alice’s action (we consider the first part of the game) forms an information set. But (and this is crucial) Bob has the possibility to analyze wine (cf. with measurement process in quantum physics). Therefore he might distinguish two actions of Alice, F and I, but only partially. I have no idea whether such a problem of analysis of actions of the opposite player was discussed in game theory?

7 Quantum-like representation

The wine testing game has a natural QL-representation. Let us consider a game with restrictions R1)–R3) on strategies, see section 4. By applying QLRA to statistical data we can construct a probability amplitude $\psi_C(\beta)$, $\beta = F, I$. To simplify considerations, we assume that the coefficients of interference are bounded by one. Thus the context $C$ is trigonometric and the probability amplitude is complex valued. It can also be represented by a unit vector of the two dimensional complex Hilbert space. We remark that in principle there are no reasons for such an assumption. In opposite to QM, Wine Game might produce hyperbolic probability amplitudes [36].

In this case we can represent the wins-losses averages in the QL-way:

$$E^b(C) = h_{FF;1}^b |\langle \psi_C, e_F^a \rangle|^2 |\langle e_F^b, e_F^a \rangle|^2 + h_{IF;1}^b |\langle \psi_C, e_I^a \rangle|^2 |\langle e_F^b, e_I^a \rangle|^2$$

$$+ h_{FF;2}^b |\langle \psi_C, e_F^b \rangle|^2 |\langle e_F^b, e_F^a \rangle|^2 + h_{IF;2}^b |\langle \psi_C, e_I^b \rangle|^2 |\langle e_F^b, e_I^a \rangle|^2$$

In the same way we represent the average for Alice. Thus the wine testing game satisfying conditions R1-R3 can be represented in the complex Hilbert space.
The QL-expression for the average is essentially simpler in the case of zero sum game with symmetry between the first and second parts: $h_{FF;1}^b = h_{FF;2}^b$, ..., $h_{II;1}^b = h_{II;2}^b$. Here

$$E^b(C) = h_{FF;1}^b (|\langle \psi_C, e_F^a \rangle|^2 - |\langle \psi_C, e_F^b \rangle|^2)$$

$$+ h_{FF;1}^b (|\langle \psi_C, e_F^a \rangle|^2 |\langle e_F^b, e_F^a \rangle|^2 - |\langle \psi_C, e_F^b \rangle|^2 |\langle e_F^b, e_F^a \rangle|^2)$$

$$+ h_{I1;1}^b (|\langle \psi_C, e_I^a \rangle|^2 |\langle e_I^b, e_I^a \rangle|^2 - |\langle \psi_C, e_I^b \rangle|^2 |\langle e_I^b, e_I^a \rangle|^2)$$

$$= h_{FF;1}^b |\langle e_F^b, e_F^a \rangle|^2 (|\langle \psi_C, e_F^a \rangle|^2 - |\langle \psi_C, e_F^b \rangle|^2) + h_{I1;1}^b |\langle e_I^b, e_I^a \rangle|^2 (|\langle \psi_C, e_I^a \rangle|^2 - |\langle \psi_C, e_I^b \rangle|^2)$$

$$+ h_{I1;1}^b |\langle e_I^b, e_I^a \rangle|^2 (|\langle \psi_C, e_I^a \rangle|^2 - |\langle \psi_C, e_I^b \rangle|^2) + h_{I1;1}^b |\langle e_I^b, e_I^a \rangle|^2 (|\langle \psi_C, e_I^a \rangle|^2 - |\langle \psi_C, e_I^b \rangle|^2)$$

8 Superposition of preferences

We now point out that we can expand e.g. vectors of the $b$-basis with respect to the $a$-basis: $c_F = c_F e_F^a + c_F e_F^b$, $c_I = c_I e_I^a + c_I e_I^b$. One might say that “Bob’s preferences are superpositions of Alice preferences.” However, we cannot assign any real meaning to such a sentence in the present game framework. Thus superposition is merely a purely mathematical representation – the geometric picture of the probabilistic structure of the game. In the same way we can expand the state $\psi_C$ with respect to the $a$-basis as well as the $b$-basis. Such expansions neither have any real meaning, just geometrical representation of probabilities. Nevertheless, such a picture is convenient for geometric representation of mental states of Alice and Bob. One may use the following geometric model: there are two basic mental states of Alice (in the context of Wine Game) $e_F^a$ and $e_F^b$. In general Alice plays in the superposition of these states $\psi = c_F e_F^a + c_F e_F^b$. In the same way Bob has two basic mental states $e_I^a$ and $e_I^b$. In general Bob plays in the superposition of these states $\psi = c_I e_I^a + c_I e_I^b$. Moreover, (at least mathematically) Bob’s mental states can be represented as superpositions of Alice’s mental states.

We can represent the average of Bob’s wins-losses in the interference form:

$$E^b(C) = (|\langle \psi_C, e_F^a \rangle|^2 - |\langle \psi_C, e_F^b \rangle|^2) (|\langle e_F^b, e_F^a \rangle|^2 + h_{I1;1}^b |\langle e_I^b, e_I^a \rangle|^2)$$
\[ + (|\langle \psi_C, e_I^a \rangle|^2 - |\bar{c}_{IF} \langle \psi_C, e_F^a \rangle + \bar{c}_{II} \langle \psi_C, e_I^a \rangle|^2 ) (h_{IF,1}^b |\langle e_F^b, e_I^a \rangle|^2 + h_{II,1}^b |\langle e_I^b, e_I^a \rangle|^2 ) \\
= (|\langle \psi_C, e_F^a \rangle|^2 - (|\langle \psi_C, e_I^a \rangle|^2 |\langle e_F^b, e_I^a \rangle|^2 + |\langle \psi_C, e_I^a \rangle|^2 |\langle e_I^b, e_I^a \rangle|^2 + 2 \cos \theta |\langle \psi_C, e_I^a \rangle|^2 |\langle e_F^b, e_I^a \rangle|^2 + |\langle \psi_C, e_I^a \rangle|^2 |\langle e_I^b, e_I^a \rangle|^2 + 2 \cos \theta |\langle \psi_C, e_I^a \rangle|^2 |\langle e_F^b, e_I^a \rangle|^2 + |\langle \psi_C, e_I^a \rangle|^2 |\langle e_I^b, e_I^a \rangle|^2 ) \\
- 2 \cos \theta |\langle \psi_C, e_I^a \rangle|^2 |\langle e_F^b, e_I^a \rangle|^2 |\langle \psi_C, e_I^a \rangle|^2 |\langle e_I^b, e_I^a \rangle|^2 ) (h_{IF,1}^b |\langle e_F^b, e_I^a \rangle|^2 + h_{II,1}^b |\langle e_I^b, e_I^a \rangle|^2 ) . \]

9 Meaning of the wave function

The wave function \( \psi_C \) was constructed on the basis of probabilities: \( p_C^a(\alpha), p_C^b(\beta), p^{b(\beta|\alpha)} \). It represents the wine collection of the restaurant as well as preferences of Alice and Bob. Moreover, it also represents their abilities to find difference between French and Italian wines.

Thus one may say such a wave function (complex probability amplitude) \( \psi_C \) has no real counterpart. We could not point out to any object in reality which is represented by \( \psi_C \). It represents the context of the wine collection as well as Bob’s and Alice’s preferences and experiences with different kinds of wines. Such a context is extremely complex. It is impossible to describe its precisely. However, the \( \psi_C \) provides some approximative representation of this context in the complex Hilbert space.

We notify that Alice and Bob are coupled through the wave function. The wave function really provides a possibility to combine probabilistic features of two cognitive systems, Alice and Bob, which could not be incorporated into a single Kolmogorov probability space.

10 The role of Bayes formula

Suppose at the moment that randomness of actions of Alice and Bob can be described by the Kolmogorov probability space \( \mathcal{P} = (\Omega, \mathcal{F}, \mathbb{P}) \) in the following way:

a). The wine-collection context \( C \) is represented by an element of \( \mathcal{F} \) which will be denoted by the same symbol.

b). Probabilities

\[ p_C^a(\alpha) = \mathbb{P}_C(A_\alpha) \equiv \frac{\mathbb{P}(A_\alpha \cap C)}{\mathbb{P}(C)}, p_C^b(\beta) = \mathbb{P}_C(B_\beta) \equiv \frac{\mathbb{P}(B_\beta \cap C)}{\mathbb{P}(C)}, \]
where \( A_\alpha = \{ \omega \in \Omega : a(\omega) = \alpha \} \), \( B_\beta = \{ \omega \in \Omega : b(\omega) = \beta \} \).

Here \( P_C \) is the conditional probability measure corresponding to the subset \( C \) of \( F \) : \( P_C(A) = \frac{P(A \cap C)}{P(C)} \).

c). Transition portabilities

\[
p^{b|a}(\beta|\alpha) = P_C(B_\beta|A_\alpha) = \frac{P(B_\beta \cap A_\alpha \cap C)}{P(A_\alpha \cap C)}.
\]

In such a representation the \( C \)-conditional Bayes formula holds:

\[
P_C(A_\alpha \cap B_\beta) = P_C(A_\alpha)P_C(B_\beta|A_\alpha).
\] (9)

Hence, here the equality (8) holds! (Because the Kolmogorovian probability is symmetric: \( P_C(A_\alpha \cap B_\beta) = P_C(B_\beta \cap A_\alpha) \).) We obtain the following equality, see [36] for details:

\[
P_C(A_\alpha)P_C(B_\beta|A_\alpha) = P_C(B_\beta)P_C(A_\alpha|B_\beta).
\] (10)

Since we want to get the QL-representation, we consider symmetrically conditioned variables \( a \) and \( b \), see R1). The condition (11) implies that

\[
P_C(A_\alpha) = P_C(B_\beta) = 1/2.
\] (11)

Thus one can construct a Kolmogorov representation of Wine Game satisfying conditions a)-c) iff selection of wines from collection is uniformly distributed between French and Italian wines (both for Alice and Bob). If not, then there is no Kolmogorov model. For example, if the probability that Alice chooses a bottle of French wine \( p^a_C(F) = 1/3 \) (and consequently the probability that she chooses a bottle of Italian wine \( p^a_C(I) = 2/3 \)), then it is impossible to construct a Kolmogorov probability space for this game. Of course, one should not forget that we assumed that the game probabilities are coupled to the Kolmogorov space via conditions a)-c) and that we would like to have symmetric transition probabilities.

The origin of this nonclassicality of the probabilistic description is impossibility to combine on a single Kolmogorov space preferences of Alice and Bob in choosing wines and their abilities to test wines. In fact, what are reasons for existence of such a space? We point out that in general the space \( \Omega \) cannot be identified with just the collection of bottles. Alice chooses a
bottle and her preferences are not completely determined until she makes
the wine order. If one likes it is possible to use the terminology that is typ-
ically used in discussions on foundations of quantum mechanics: “death of
reality.” However, in this game framework this death of reality does not look
mystically. This only means that one is not able from the very beginning to
assign to any bottle choice and test preferences of Alice and Bob. Nothing
more. Nevertheless, reality of wines could not be denied and choices and
tests of Alice and Bob are based on this wine-reality.

By using Gudder’s theory of probability manifolds [15] we can say that
we have a probability manifold with the atlas having two charts, one serves
for the first part of game and another for the second; in Accardi’s terminol-
ogy this is a non-Kolmogorovian model (he always emphasized the role of
violation of Bayes’ formula, see [4]).

We emphasize that the choice c) of the transition probabilities implies
immediately that the coefficients of interference $\lambda$ are equal zero.

11 Action at the distance?

One can consider Wine Game involving facelogy: Bob can extract some
information about the origin of wine by observing the behavior of Alice after
she has done her choice. By using the terminology of QM one can say that
there is “action at the distance.” However, even if such an action is present
in the game it is not instantaneous! Everything happens in the complete
accordance with laws of special relativity: light is reflected from Alice’s face
and Bob obtains information only when the light wave will come to his eyes.

Consideration of QL-games of the facelogy-type extends essentially the
range of possible applications of our model. However, we do not couple
directly such an action at the distance with essentially nonclassical proba-
bilistic structure. The origin of nonclassicality is the impossibility to combine
all possible preferences in a single probability space. Again by using the ter-
miminology of QM one can say that there are two incompatible measurement
settings (corresponding to two parts of Wine Game); thus we proceed in the
complete accordance with Bohr’s ideology [1].
12 Wine Game with three players

We now generalize Wine Game by considering three players, Alice, Bob, Cecilia. The first part: Alice chooses a bottle, Bob tests; the second part: Bob chooses, Cecilia tests, and the third part: Cecilia chooses, Alice tests. We shall use probabilities with indexes $a, b, c$ corresponding to Alice, Bob, Cecilia. For each part of the game we fix payment matrices. We consider symmetric game. We can write averages: for the first part $-E^{b}_{1}(C) = -E^{a}_{1}(C)$, for the second part $E^{c}_{2}(C) = -E^{b}_{2}(C)$, and for the third part $E^{c}_{3}(C) = -E^{a}_{3}(C)$.

We assume that conditions R1)–R3) which guarantee the possibility to apply QLRA hold for all pairs of observables. Thus we apply QLRA to the probabilities corresponding to the pair $a, b$. We obtain the complex probability amplitude $\psi^{c}_{ab}$ which belongs to two dimensional Hilbert space which is denoted $H_{ab}$. Observables $a, b$ are represented by self-adjoint operators $\hat{a}, \hat{b}$ which have bases of eigenvectors $\{e^{a;ab}_{\alpha}\}, \{e^{b;ab}_{\beta}\}$. We also apply QLRA to the probabilities corresponding to the pair $b, c$. We obtain a new complex probability amplitude $\psi^{c}_{bc}$ which belongs to two dimensional Hilbert space which is denoted $H_{bc}$. Observables $b, c$ are represented by self-adjoint operators $\hat{b}, \hat{c}$ which have bases of eigenvectors $\{e^{b;bc}_{\beta}\}, \{e^{c;bc}_{\gamma}\}$. Finally, consider the $H_{ca}$-representation.

These representations can be identified with the aid of unitary maps:

$$U_{ab,bc} : H_{ab} \rightarrow H_{bc}, e^{b;ab}_{\beta} \rightarrow e^{b;bc}_{\beta},$$

and

$$U_{bc,ca} : H_{bc} \rightarrow H_{ca}, e^{c;cb}_{\gamma} \rightarrow e^{c;ca}_{\gamma}.$$  

The crucial point is that $U_{ab,bc}(\psi^{c}_{ab}) = \psi^{c}_{bc}$ and $U_{bc,ca}(\psi^{c}_{bc}) = \psi^{c}_{ca}$. Therefore we can identify complex probability amplitudes $\psi^{c}_{ab}, \psi^{c}_{bc}, \psi^{c}_{ca}$ and consider a unit vector $\psi^{c}_{C}$ as representing the wine collection and preferences of Alice, Bob and Cecilia. We shall come back to this game little bit later.

We remark that this game has the structure of Gudder’s probability manifold with the atlas having three charts.

13 Simulation of Wine Game

Typically quantum probabilities are imagined as rather mysterious things. Absence of the underlying Kolmogorov space may only support such a view-
point. However, by using the frequency (von Mises) approach quantum probabilities can be easily simulated. One need not use special “quantum coins”
given by sources of photons or electrons. We simulate our game by using the
following system of dichotomous random generators (taking values F and I):

\[ g_a, g_b, g^{b|a}(\alpha), g^{a|b}(\beta). \]

Here \( g_a \) and \( g_b \) simulate choices of wine from the collection \( C \) (by Alice and
Bob, respectively); the frequencies of F and I approaches the corresponding
probabilities \( p^C_a(F), p^C_a(I), p^C_b(F), p^C_b(I) \) when the number of trial goes to
infinity. The generator \( g^{b|a}(\alpha) \) describes ability of Bob to analyze wine’s
origin under the condition that Alice selects a bottle of the \( \alpha \)-origin. For
example, the generator \( g^{b|a}(F) \) takes the value F if Bob correctly recognized
French wine (which was chosen by Alice). The generator \( g^{a|b}(\beta) \) has a similar
meaning.

Now, to simulate Wine Game, we just apply these generators consequently
in the right order, e.g., first \( g_a \) and if it takes the value F, then the generator
\( g^{b|a}(F) \). That’s all! We shall simulate probabilities and payoffs given by the
two dimensional QL-model.

In all previous considerations we started with some collection of probabil-
ities and transition probabilities and under the conditions R1)–R3) we were
able to represent Wine Game in the two dimensional complex Hilbert space.
By applying QLRA we constructed the wave function and operators \( \hat{a}, \hat{b} \).

We can also proceed in the opposite way. We can take two noncommu-
tative operators in the two dimensional Hilbert space, say \( \hat{a} \) and \( \hat{b} \), and a
normalized vector \( \psi \) in this space. Then we find (by using Born’s rule) all
probabilities which we need for Wine Game. Those probabilities will auto-
matically satisfy conditions R1)–R3). Finally, we can simulate Wine Game
by using the above scheme.

This strategy is especially convenient for generalizations of Wine Game
to spaces of high dimension. QLRA becomes very complicated [36]. Recon-
struction of the wave function is not so simple task. Therefore one can start
just with probabilities which are obtained from the mathematical formalism
of quantum mechanics. Moreover, the possibility to apply QLRA is restricted
by a number of conditions, e.g., R1), R2). One can ignore these conditions
by starting directly with a normalized vector \( \psi \).
14 Bell’s inequality: the two dimensional representation

We now come back to the Wine Game with three palavers, Alice, Bob, Cecilia. We shall use the pragmatic strategy proposed at the end of the previous section. We take probabilities and operators corresponding to a known quantum system and simulate Wine Game on the basis of these probabilities. We emphasize that we take probabilities given by the mathematical apparatus of quantum mechanics and not at all a quantum physical system by itself. We introduce a game parameter \( \theta \in [0, 2\pi) \). Alice is characterized by \( \theta = \theta_1 \), Bob by \( \theta = \theta_2 \), Cecilia by \( \theta = \theta_3 \). We take the transition probabilities corresponding to “spin 1/2 system.” For the first part of the game we have:

\[
p_{b|a}(b = F|a = F) = p_{b|a}(b = I|a = I) = \cos^2 \frac{\theta_1 - \theta_2}{2};
\]

\[
p_{b|a}(b = I|a = F) = p_{b|a}(b = F|a = I) = \sin^2 \frac{\theta_1 - \theta_2}{2}.
\]

The transition probabilities for other parts of the game are defined in a similar way, e.g.:

\[
p_{c|b}(c = F|b = F) = p_{c|b}(c = I|b = I) = \cos^2 \frac{\theta_2 - \theta_3}{2}.
\]

Let us choose the following Darice of payoffs:

\[
h_{FF} = h_{II} = +1, \ h_{IF} = h_{FI} = -1.
\]

Let us now suppose that Alice, Bob and Cecilia selects wine from the collection by using uniform random generators: \( p^b_C(\alpha) = p^b_C(\beta) = p^c_C(\gamma) = 1/2 \). We now find the average for Bob’s wins-losses in the first part of the Wine Game:

\[
E^b_1 \equiv E(\theta_1, \theta_2) = \cos^2 \frac{\theta_1 - \theta_2}{2} - \sin^2 \frac{\theta_1 - \theta_2}{2} = \cos(\theta_1 - \theta_2). \quad (12)
\]

In the same way we have for Cecilia:

\[
E^c_2 \equiv E(\theta_2, \theta_3) = \cos(\theta_2 - \theta_3) \quad (13)
\]

and finally for Alice:

\[
E^a_3 \equiv E(\theta_3, \theta_1) = \cos(\theta_3 - \theta_1). \quad (14)
\]
We set now $F=+1$ and $I=-1$, we recall that with these notations we can represent

$$E_1^b = \text{cov}(a, b), E_2^c = \text{cov}(c, b), E_3^a = \text{cov}(c, a),$$

where covariations are taken with respect to probabilities $p_{CB}^b, p_{BC}^c, p_{CA}^c$. We now ask: Can one construct a probability measure $\mathbf{P}$ and realize observables $a, b, c$ by random variables on the corresponding Kolmogorov space in such a way that

$$\mathbf{P}(a = \alpha, b = \beta) = p_{CB}^b(a = \alpha, b = \beta), \quad \mathbf{P}(b = \beta, c = \gamma) = p_{BC}^c(b = \beta, c = \gamma),$$

$$\mathbf{P}(c = \gamma, a = \alpha) = p_{CA}^c(c = \gamma, a = \alpha)?$$

(15) \hspace{1cm} (16)

The answer is negative. If representations (15), (16) can be constructed, then one can prove Bell’s inequality \cite{33}, see \cite{38} for details:

$$|\text{cov}(a, b) - \text{cov}(b, c)| \leq 1 - \text{cov}(c, a).$$

(17)

But it is known that Bell’s inequality is violated for covariations given by (12)–(12) for some choices of parameters (one could also apply Vorobjev’s theorem \cite{32}).

Thus if there is a classical probabilistic model behind Wine Game (for some set of probabilities), then averages of payments satisfy the following Bell’s inequality:

$$|E_1^b - E_2^c| \leq 1 - E_3^a.$$ \hspace{1cm} (18)

Even intuitively it is clear that there are no reasons to assume that this inequality should holds for any set of probabilities.

The expression in the left-hand side of the Bell’s inequality is equal to the average of the total win-loss of Bob in the game (i.e., in the two series of games – with Alice and Cecilia, in the first Bob tests wine and the second Cecilia does this): $E^b = E_1^b + E_2^b = E_1^b - E_2^c = \cos(\theta_1 - \theta_2) - \cos(\theta_2 - \theta_3)$.

15 Multidimensional games

Wine Game can be generalized to the Hilbert space $H$ of an arbitrary dimension. The only difference is that now the collection $C$ contains wines from $n$ countries, which are labeled by $i = 1, ..., n$. 

22
Let us consider two self-adjoint operators \( \hat{a} \) and \( \hat{b} \) and corresponding orthonormal bases of eigenvectors \( \{ e_{ai} \}_{i=1}^{n} \) and \( \{ e_{bj} \}_{j=1}^{n} \). We remark that in general we do not suppose the validity of the condition R2). In principle, operators could even commute. Of course, QLRA would not work in such a case. But our task is not reconstruct probabilities from the game, but only to simulate the game.

We also take a normalized vector \( \psi \in H \). This vector \( \psi \) describes collections of wines created by Alice and Bob as well as their experiences of testing of wines. Actions of Alice and Bob are now labeled by \( i = 1, ..., n \). The tree of this extensive form game have \( n \) nodes leaving this vertex. The Bob’s average is given by

\[
E^b = \sum_{i,j=1}^{n} h_{ij:1}^b |\langle \psi_C, e_{ai}^b \rangle|^2 |\langle e_{ij}^b, e_{aj}^a \rangle|^2 + \sum_{i,j=1}^{n} h_{ij:2}^b |\langle \psi_C, e_{bi}^b \rangle|^2 |\langle e_{ij}^b, e_{aj}^a \rangle|^2.
\]

To find conditions when the game has no underlying classical probability space, one can apply Vorobjev’s theorem which was proved for multi-valued random variables.

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