Quantum repeated games revisited

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Abstract

We present a scheme for playing quantum repeated $2 \times 2$ games based on Marinatto and Weber’s approach to quantum games. As a potential application, we study the twice repeated Prisoner’s Dilemma game. We show that results not available in the classical game can be obtained when the game is played in the quantum way. Before we present our idea, we comment on the previous scheme of playing quantum repeated games proposed by Iqbal and Toor. We point out the drawbacks that make their results unacceptable.

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1. Introduction

The Marinatto–Weber (MW) idea of quantum $2 \times 2$ games introduced in [1] has found applications in many branches of game theory. The MW approach to evolutionary games [2] and Stackelberg equilibrium [3] are merely two of many applications. In [4] and [5], we have shown that the MW idea is applicable to finite games in the extensive form as well. Consequently, this scheme of playing quantum games can be applied to many other game-theoretical problems. In this paper, we deal with the problem of quantization of twice repeated $2 \times 2$ games. Since a finitely repeated game is just a special case of a finite extensive game, we apply the method based on [4] and [5] to play the repeated game in the quantum way. The idea of quantum-repeated games was first introduced in [6], where the authors adapt the MW scheme for the twice repeated Prisoner’s Dilemma (PD). Then, they investigate if results that are unavailable when the game is played classically can occur in the quantum area. The point of this paper [6] is to provide sufficient conditions for players’ cooperation in the (PD) game.

We examine the idea of Iqbal and Toor before we define our scheme. Firstly, we study the problem of cooperation and prove that player’s cooperation in the game defined by the protocol proposed in [6] is not possible. Secondly, we check whether Iqbal and Toor’s scheme is actually in accordance with the concept of repeated game. It turns out that the discussed scheme does not include the classical twice repeated PD; hence, it cannot be the quantum realization of this game in the spirit of the MW approach. To support our arguments, we propose the new
protocol for a twice repeated $2 \times 2$ game and show that our idea generalizes the classical twice repeated game. Our paper also includes the proof of the advantage of the quantum scheme over the classical one. Namely, we prove that both players can benefit from playing the game via our protocol. Moreover, we show that in contrast to the situation encountered in the classical game, the cooperation of the players is possible for some sort of PD games played repeatedly if our quantum approach is used.

It is worth noting that the idea of quantum games by Marinatto and Weber that we are going to use is one of the many different ways to put repeated games, or more generally, extensive games in the quantum domain. The general (at the highest mathematical level) idea of quantum strategies that takes into consideration players’ knowledge about previous course of a sequential game was presented in [7]. In this paper, in some ways, we deal with a special kind of these strategies constructed by means of the identity and the Pauli operator $X$. Note also that Iqbal and Toor’s proposal [6] is not the only paper in which quantum repeated PD was studied. In [8], the authors proposed the quantum scheme for iterated PD. The simplified framework about strategies that the authors employed is based on the assumption that the total number of rounds in a repeated game must be random, infinite or at least unknown to the players. Since we focus on finitely repeated games where the players know the number of rounds, our scheme considers strategies that are contingent on all stages in the game.

Studying our paper requires a little background in game theory. All notions such as extensive game, information set, strategy, equilibrium, subgame-perfect equilibrium, etc, used in the paper are explained in an accessible way, for example, in [9] and [10]. The adequate preliminaries can also be found in [4], where quantum games in an extensive form are examined.

2. Twice-repeated games and the Prisoner’s Dilemma

The PD is one of the most fundamental problems in game theory (the general form of the PD according to [11] is given in figure 1(a)). It demonstrates why the rationality of players can lead them to an inefficient outcome. Although the payoff vector $(R, R)$ is better to both players than $(P, P)$, they cannot obtain this outcome since each player’s strategy $C$ (cooperation) is strictly dominated by $D$ (defection). As a result, the players end up with payoff $P$ corresponding to the unique Nash equilibrium $(D, D)$.

A similar scenario occurs in the case of finitely repeated PD game. The concept of a finitely repeated game assumes playing a static game (a stage of the repeated game) for a fixed number of times. Additionally, the players are informed about the results of consecutive
stages. In the twice repeated case, it means that each player’s strategy specifies an action at the first stage and four actions at the second stage where a particular action at the second stage is chosen depending on what of the four outcomes of the first stage has occurred. It is clearly visible if we write twice the repeated game in the extensive form (see figure 2). The first stage of the twice repeated PD in the extensive form is simply the game depicted in figure 1(b), where the players specify an action C or D at the information set 1.1 or 2.1, respectively (the information sets of player 2 are marked by a dotted line connecting the nodes to show the lack of knowledge of the second player about the previous move of the first player). When the players choose their actions, the result of the first stage is announced. Since they have knowledge about the results of the first stage, they can choose different actions at the second stage depending on the previous result; hence, the next four game trees in figure 1 are required to describe the repeated game. The game tree exhibits ten information sets (five for each player labeled 1 and 2, respectively) at which each of the players has two moves. Thus, each of them has \(2^5 = 32\) strategies as they specify C or D at their own five information sets.

To find the Nash equilibrium in a finitely repeated game, it is convenient to use the property that the equilibrium profile always implies the Nash equilibrium at the last stage of the game. Therefore, to find the Nash equilibrium in the twice repeated PD, it is sufficient to consider strategy profiles that generate the profile \((D, D)\) at the second stage. Then, it follows that \(D\) is the best response for each player at the first stage as well. By induction, it can be shown that playing the action \(D\) at each stage of a finitely repeated PD constitutes the unique Nash equilibrium. It is worth noting that if a single stage of a repeated game has more than one equilibrium, different Nash equilibria may be played at the last stage depending on the results of previous stages. For example, let us consider the Battle of the Sexes (BoS) game given by the following bimatrix:

\[
\Gamma : \begin{bmatrix}
O & F \\
\alpha, \beta & (\gamma, \gamma) \\
\beta, \gamma & (\alpha, \beta)
\end{bmatrix},
\]

where \(\alpha > \beta > \gamma\). (1)

It has two pure Nash equilibria, namely, \((O, O)\) and \((F, F)\). Let us now examine the twice repeated BoS. Obviously, its game tree is like that of the one in figure 2. Let us assign appropriate sums of two-stage payoff outcomes to each possible profile (like it has been done in the case of the twice repeated PD). Then, we find many different Nash equilibria. One of these is to play the Nash equilibrium \((O, O)\) at the first stage, keep playing \((O, O)\) at the second stage if the outcome of the first one is \((O, O)\) or \((O, F)\), otherwise to play the stage-game Nash equilibrium \((F, F)\). Although such a situation is impossible in the classically played...
twice repeated PD game, it can appear in the quantum counterpart as we will see in the rest of this paper.

3. Comment on ‘Quantum-repeated games’ by Iqbal and Toor [6]

Let us recall the MW approach to playing PD repeatedly introduced in [6]. According to this concept, the two-stage PD is placed in $\mathcal{H} = (\mathbb{C}^2)^{\otimes 4}$ complex Hilbert space with the computational basis. The game starts with the preparation of a 4-qubit pure state represented by a unit vector in $\mathcal{H}$. The general form of this state is described as follows:

$$|\psi_{\text{in}}\rangle = \sum_{x_1,x_2,x_3,x_4=0,1} \lambda_{x_1,x_2,x_3,x_4} |x_1,x_2,x_3,x_4\rangle,$$

where $\lambda_{x_1,x_2,x_3,x_4} \in \mathbb{C}$ and $\sum_{x_1,x_2,x_3,x_4=0,1} |\lambda_{x_1,x_2,x_3,x_4}|^2 = 1$. (2)

Players’ moves are identified with the identity operator $\sigma_0$ and the bit flip Pauli operator $\sigma_1$. Player 1 is allowed to act on the first and third qubits, and player 2 acts on the second and fourth ones. In the first stage of the game, the two first qubits are manipulated. Let $\rho_{\text{in}}$ be the density operator for the initial state (2). Then, state $\rho$ after the player’s actions takes the form

$$\rho = \sum_{x_1,x_2=0,1} p_{x_1} q_{x_2} (\sigma_{x_1}^1 \otimes \sigma_{x_2}^1) \rho_{\text{in}} (\sigma_{x_1}^3 \otimes \sigma_{x_2}^3),$$

and

$$\sum_{x_1=0,1} p_{x_1} = \sum_{x_2=0,1} q_{x_2} = 1,$$

where $p_{x_1}$ ($q_{x_2}$) is the probability of applying $\sigma_{x_1}^1$ ($\sigma_{x_2}^3$) to the first (second) qubit. Next, the other two qubits are manipulated. It corresponds to the second stage of the classical game. The operation $\sigma_{x_4}^3$ on the third qubit with probability $p_{x_4}$ and operation $\sigma_{x_3}^4$ on the fourth qubit with probability $q_{x_3}$ change the state $\rho$ to

$$\rho_{\text{fin}} = \sum_{x_1,x_2=0,1} p_{x_1} q_{x_2} (\sigma_{x_1}^1 \otimes \sigma_{x_2}^1) \rho (\sigma_{x_3}^3 \otimes \sigma_{x_4}^4),$$

and

$$\sum_{x_1=0,1} p_{x_1} = \sum_{x_3=0,1} q_{x_3} = 1.$$ (4)

The next step is to measure the final state $\rho_{\text{fin}}$ in order to determine final payoffs. The measurement is defined by the four payoff operators $X_{i,j}$, $i, j = 1, 2$ associated with particular player $i$ and stage $j$. That is,

$$X_{1,1} = (R(00)\langle 00| + S(01)\langle 01| + T|10\rangle \langle 10| + P|11\rangle \langle 11|) \otimes \mathbb{I}^{\otimes 2};$$

$$X_{1,2} = \mathbb{I}^{\otimes 2} \otimes (R(00)\langle 00| + S(01)\langle 01| + T|10\rangle \langle 10| + P|11\rangle \langle 11|);$$

$$X_{2,1} = (R(00)\langle 00| + T|01\rangle \langle 01| + S|10\rangle \langle 10| + P|11\rangle \langle 11|) \otimes \mathbb{I}^{\otimes 2};$$

$$X_{2,2} = \mathbb{I}^{\otimes 2} \otimes (R(00)\langle 00| + T|01\rangle \langle 01| + S|10\rangle \langle 10| + P|11\rangle \langle 11|).$$ (5)

Then, the expected payoff $E_{i,j}$ for player $i$ at stage $j$ when player 1 chooses strategy $(\sigma_{x_1}^1, \sigma_{x_2}^3)$ and player 2 chooses $(\sigma_{x_3}^4, \sigma_{x_4}^1)$ is obtained by the following formula:

$$E_{i,j}(\sigma_{x_1}^1, \sigma_{x_2}^3, \sigma_{x_3}^4, \sigma_{x_4}^1) = \text{tr}(X_{i,j}\rho_{\text{fin}}).$$ (6)

The authors took up the issue of cooperation in the two-stage PD game. They put the initial state

$$|\psi_{\text{in}}\rangle = \lambda_{0000}\langle 0000| + \lambda_{0011}\langle 0011| + \lambda_{1100}\langle 1100| + \lambda_{1111}\langle 1111|$$ (7)
and fix the PD payoffs
\[ T = 5, \quad R = 3, \quad P = 1, \quad S = 0. \] (8)

Next, they identify \( \sigma_0 \) and \( \sigma_1 \) as actions of cooperation and defection, respectively, and claim that conditions
\[ |\lambda_{0000}|^2 + |\lambda_{0011}|^2 \leq \frac{1}{2}, \quad |\lambda_{0011}|^2 + |\lambda_{1111}|^2 \leq \frac{1}{2} \] (9)
are sufficient to choose \( \sigma_0 \) by both players (thereby cooperating) at the first stage given that the players have chosen \( \sigma_1 \) at the second one. We raise below two objections concerning the results of [6].

3.1. The incompatibility of the protocol (2)–(6) and theory of repeated games

The main fault of the protocol (2)–(6) is that the twice repeated game cannot be described in this way. In fact, this protocol quantizes the game PD played twice when the players are not informed about a result of the first stage. It is notable, for example, when we re-examine the way of finding the optimal solution provided in [6]. The authors analyze the game backward, first by focusing on the Nash equilibria at the second stage. They set condition for the profile \((\sigma_i^3, \sigma_i^4)\) to be the Nash equilibrium at the second stage. Next, given that \((\sigma_i^3, \sigma_i^4)\) is fixed, they determine the set of amplitudes for which the profile \( (E_0, E_1, (\sigma_0^3, \sigma_0^4), (\sigma_1^3, \sigma_1^4)) \) is the Nash equilibrium of the game implied by (2)–(6). This method to find the Nash equilibria is not correct since it does not include the possibility that players make their actions depending on a result of the first stage. Although the problem seems to be insignificant where a single stage of a repeated game has unique Nash equilibrium, it becomes important in remaining cases. Let us consider the initial state (7) satisfying the requirement
\[ 2(|\lambda_{0000}|^2 + |\lambda_{1110}|^2) = |\lambda_{0011}|^2 + |\lambda_{1111}|^2 \] (10)
and let us take (8) to be the PD’s payoffs. Then, the expected payoffs for the players at the second stage of the game defined by the scheme (2)–(6) are as follows:
\[
E_{1,2}(\cdot, \sigma_0^1), (\cdot, \sigma_0^3) = E_{1,2}(\cdot, \sigma_1^1), (\cdot, \sigma_1^3) = 5|\lambda|; \\
E_{1,2}(\cdot, \sigma_0^1), (\cdot, \sigma_1^3) = E_{1,2}(\cdot, \sigma_1^3), (\cdot, \sigma_0^3) = 10|\lambda|;
\] (11)
where \(|\lambda| = |\lambda_{0000}|^2 + |\lambda_{1110}|^2\) and \(i = 1, 2\). Results of (11) imply the continuity of Nash equilibria in the second stage (it is easy to note, for example, when we draw a 2 × 2 bimatrix with entries defined by (11)), among them \( (\cdot, \sigma_0^3), (\cdot, \sigma_1^3) \) and \( (\cdot, \sigma_1^3), (\cdot, \sigma_0^3) \). Bearing in mind the remark in section 2 about possible profiles in the BoS game, the correct protocol for quantum repeated games should allow one to assign a payoff outcome (by the measurement (5)) to a strategy profile, where different Nash equilibria are played at the second stage depending on actions chosen at the first one. However, an example of a profile where the players play \((\cdot, \sigma_0^3), (\cdot, \sigma_1^3)\) at the second stage if a result of the first stage is \((\sigma_0^3, \cdot), (\sigma_0^3, \cdot)\), and they play \((\cdot, \sigma_1^3), (\cdot, \sigma_0^3)\) in other cases cannot be achieved by the scheme (2)–(6). Since there is a 2-qubit register allotted to the second stage, it allows one to write only one pair of actions \((\sigma_i^3, \sigma_i^4)\) before the measurement is made.

An argument against the scheme proposed in [6] can be expressed in another way. Namely, all the results included in [6] can be obtained by considering the simplified protocol (2)–(6) where the sequential procedure (3) and (4) for determining the final state \( \rho_{\text{fin}} \) is simply replaced with
\[
\rho_{\text{fin}} = \bigotimes_{j=1}^{4} \sigma_0^j \rho_{\text{in}} \bigotimes_{j=1}^{4} \sigma_0^j.
\] (12)
In this case, the first and second players simultaneously pick \((\sigma_1^1, \sigma_1^2)\) and \((\sigma_2^1, \sigma_2^2)\), respectively, having essentially only four strategies each. However, as we mentioned in the previous section, each player has 32 strategies in the classical twice repeated game. As a result, protocol (2)–(6) cannot coincide with the classical case if \(|\psi_{in}\rangle = |0000\rangle\). Despite the fact that the authors assume that a player knows her opponent’s action taken previously, schemes (2)–(6) do not take it into consideration. In consequence, a game being quantized by (2)–(6) differs from the game in figure 2 in that the nodes 1.2, 1.3, 1.4 and 1.5 (2.2, 2.3, 2.4 and 2.5) lie at the same information set (i.e. should be connected with a dotted line).

3.2. The misconception about the cooperative strategy in the PD game played according to the MW approach

Another fault in the approach presented in [6] that we are going to discuss is based on misinterpreting the operators \(\sigma_0\) and \(\sigma_1\) as cooperation and defection in the protocol given by (2)–(6). Although the authors correctly assumed that the set \(\{\sigma_0, \sigma_1\}\) in the quantum game (2)–(6) is the counterpart of the classical moves in the PD game (see comment in [12] on the terminology in the MW scheme), the specificity of the MW scheme does not allow to associate particular classical actions with particular operators \(\sigma_0\) and \(\sigma_1\) if additional assumptions are not made.

Let us consider the initial state \(|\psi_{in}\rangle\) in which the two first qubits associated with the first stage are prepared in the state \(|x_1, x_2\rangle\) for some fixed values \(x_1, x_2 \in \{0, 1\}\). Then, the first stage of the game given by (2)–(6) is isomorphic to the classical PD game. When the initial state is \(|00\rangle\), then \(\sigma_0\) corresponds to the action \(C\) and \(\sigma_1\) corresponds to \(D\). However, when the initial state is \(|11\rangle\), the action ‘cooperate’ ought to be identified with \(\sigma_1\) and the action ‘defect’ with \(\sigma_0\) since by putting \(\rho_{fin} = (\sigma_1^1 \otimes \sigma_1^2)|11\rangle \langle 11| (\sigma_1^2 \otimes \sigma_1^2)\) into the formula (6), we have

\[
(tr(X_{1,1}\rho_{fin}), \, tr(X_{2,1}\rho_{fin})) = \begin{cases} 
(R, R), & \text{if } (k_1, k_2) = (1, 1); \\
(S, T), & \text{if } (k_1, k_2) = (1, 0); \\
(T, S), & \text{if } (k_1, k_2) = (0, 1); \\
(P, P), & \text{if } (k_1, k_2) = (0, 0).
\end{cases}
\tag{13}
\]

That is, the outcome of the game does not only depend exclusively on the operators but also depends on the initial state and on what the final state \(\rho_{fin}\) can be obtained through the available operators. Thus, the identification of operators with actions taken in the classical game without taking into consideration the form of the initial state is not correct. The misidentification assumed in [6] causes that condition (9) does not solve the problem formulated in this paper. It is clearly visible when we take, for example, the initial state \(|\psi_{in}\rangle = |1100\rangle\). It satisfies the inequalities (9); thus, \(\sigma_0\) is optimal at the first stage for each player. In fact, \(\sigma_0\) represents the action ‘defect’ as is shown in (13). If we take \(|\psi_{in}\rangle = |1100\rangle\) in the game (2)–(6), the payoff corresponding to the strategy profile \((\sigma_0, \sigma_0)\) at the first stage and \((\sigma_1, \sigma_1)\) at the second one equals 2P for each player—total payoff for the defection. Thus, condition (9) does not imply the cooperation at the first stage.

Quite opposite, it turns out that the players never cooperate when they play the game defined by (2)–(6). Let us consider any initial state (2) in which the first and second qubits are prepared in a way that for \((s_1, s_2) = (\sigma_1^1, \sigma_1^2)\), we have

\[
(E_{1,1}(s_1, s_2), \, E_{2,1}(s_1, s_2)) = \begin{cases} 
(R', R'), & \text{if } (k_1, k_2) = (1, 1); \\
(S', T'), & \text{if } (k_1, k_2) = (1, 0); \\
(T', S'), & \text{if } (k_1, k_2) = (0, 1); \\
(P', P'), & \text{if } (k_1, k_2) = (0, 0).
\end{cases}
\tag{14}
\]
where the values $T'$, $R'$, $P'$, $S'$ meet the requirements of the PD given in figure 1(a), so the operators $\sigma_0$ and $\sigma_1$ can be regarded as cooperation and defection, respectively. Next, let us estimate the difference

$$E_1((\sigma_1^1, \sigma_0^1), s_2) - E_1((\sigma_1^0, \sigma_0^3), s_2)$$

for any $s_2 = (\sigma_2^3, \sigma_4^3)$. (15)

where $E_1 = E_{1.1} + E_{1.2}$. Since the same actions are taken on the third and fourth qubits, we have $E_{1.2}((\sigma_1^3, \sigma_0^3), s_2) = E_{1.2}((\sigma_1^4, \sigma_0^4), s_2)$; therefore, the value $E_1$ depends only on $E_{1.1}$. Thus, for $s_2 = (\sigma_2^3, \sigma_4^3)$, we obtain from (14) that

$$0 < E_1((\sigma_1^1, \sigma_0^3), s_2) - E_1((\sigma_1^0, \sigma_0^3), s_2) = \begin{cases} T' - R', & \text{if } \kappa_2 = 0; \\ P' - S', & \text{if } \kappa_2 = 1. \end{cases}$$

(16)

In a similar way, we can prove that the strategy $(\sigma_1^1, \sigma_0^3)$ of player 1 is strictly dominated by $(\sigma_1^0, \sigma_0^4)$. As a result, we conclude that $(\sigma_1^1)$ is the best response of player 1 at the first stage. Symmetry of payoffs in PD implies that strategy $(\sigma_2^3, \sigma_4^3)$ of player 2 is strictly dominated by $(\sigma_2^0, \sigma_4^0)$ as well as $(\sigma_3^1, \sigma_4^1)$ is strictly dominated by $(\sigma_3^2, \sigma_4^2)$. Thus, there is no Nash equilibrium such that the players choose $\sigma_0$ (cooperation) at the first stage.

### 4. The MW approach to twice repeated quantum games

In this section, we propose a scheme of playing the twice repeated $2 \times 2$ quantum game that is free from the faults we have pointed in the previous section. Our construction is based on the protocol that we proposed in [4], where general finite extensive quantum games were considered. Since a repeated game is a special case of an extensive game, we can adapt this concept. Next, we examine whether better results, when compared to the classical case, can be obtained from our protocol. In particular, we re-examine the problem of cooperation studied in [6].

#### 4.1. Construction of a twice repeated $2 \times 2$ quantum game via the MW protocol

Let us consider a $2 \times 2$ game defined by the outcomes $O_{1,2}, t_1, t_2 = 0, 1$. The twice repeated $2 \times 2$ quantum game played according to the MW approach is as follows:

Let $\mathcal{H} = (\mathbb{C}^2)^{\otimes 10}$ be a complex Hilbert space with the computational basis {$|x_1, x_2, \ldots, x_{10}\rangle$}, $x_j = 0, 1$. Then, the initial state of the game is a 10-qubit pure state represented by a unit vector in the space $\mathcal{H}$

$$|\psi_{\text{in}}\rangle = \sum_{x=0}^{2^{10}-1} \lambda_x |x\rangle,$$

where $\lambda_x \in \mathbb{C}$ and $\sum_x |\lambda_x|^2 = 1$. (17)

where the sum is over all possible decimal values of $x = (x_{10} = (x_1 x_2 \ldots x_{10})_2$. The players are allowed to apply operators $\sigma_0$ and $\sigma_1$. The qubits with odd indices are manipulated by player 1 and the qubits labeled by even indices are manipulated by player 2. Such an assignment implies 32 possible strategies for each player as they specify five operations $\sigma_{i,j}^p$ (where $j$ and $k$ indicate qubit number and operation number, respectively) on their own qubits. We denote a player $i$’s strategy by $\tau_i = (\sigma_{i,1}^1, \sigma_{i,2}^1, \sigma_{i,3}^1, \sigma_{i,4}^1, \sigma_{i,5}^1, \sigma_{i,6}^1, \sigma_{i,7}^1, \sigma_{i,8}^1)$, where $i = 1, 2$. The profile $\tau = (\tau_1, \tau_2)$ gives rise to the final state

$$|\psi_{\text{fin}}\rangle = \bigotimes_{j=1}^{10} \sigma_{j,i}^p |\psi_{\text{in}}\rangle.$$
If each of the players applies $r'_1$ and $r'_2$ with probabilities $p_t$ and $q_t$, respectively, it implies the state $|\psi_{fin}^{t,r'}\rangle$ (defined by (18)) with probability $p_t q_t$, and then, the final state is the density operator associated with the ensemble $\{p_t q_t, |\psi_{fin}^{t,r'}\rangle\}$,

$$\rho_{\text{fin}} = \sum_{t,t'} p_t q_t |\psi_{\text{fin}}^{t,r'}\rangle \langle \psi_{\text{fin}}^{t,r'}|.$$  \hfill (19)

Until now, a difference between the concept in [6] and our protocol lies in the dimension of the space $\mathcal{H}$. The next difference is clearly visible in a description of measurement operators. The measurement on $\rho_{\text{fin}}$ that determines an outcome of the game is described by a collection $\{X_1, X_{2,00}, X_{2,01}, X_{2,10}, X_{2,11}\}$ where its components are defined as follows:

$$X_1 = \sum_{x_1,x_2 \in \{0,1\}} O_{x_1,x_2} |x_1,x_2\rangle \langle x_1,x_2| \otimes \mathbb{1}^{\otimes 8};$$

$$X_{2,00} = \sum_{x_5,x_6 \in \{0,1\}} O_{x_5,x_6} |00\rangle \langle 00| \otimes |x_3,x_4\rangle \langle x_3,x_4| \otimes \mathbb{1}^{\otimes 6};$$

$$X_{2,01} = \sum_{x_5,x_6 \in \{0,1\}} O_{x_5,x_6} |01\rangle \langle 01| \otimes |x_5,x_6\rangle \langle x_5,x_6| \otimes \mathbb{1}^{\otimes 4};$$

$$X_{2,10} = \sum_{x_7,x_8 \in \{0,1\}} O_{x_7,x_8} |10\rangle \langle 10| \otimes \mathbb{1}^{\otimes 4} \otimes |x_7,x_8\rangle \langle x_7,x_8| \otimes \mathbb{1}^{\otimes 2};$$

$$X_{2,11} = \sum_{x_9,x_{10} \in \{0,1\}} O_{x_9,x_{10}} |11\rangle \langle 11| \otimes \mathbb{1}^{\otimes 6} \otimes |x_9,x_{10}\rangle \langle x_9,x_{10}|.$$  \hfill (21)

Then, the expected outcomes for player $i$: $E_{i,1}$ at the first stage and $E_{i,2}$ at the second stage are calculated using the following formulas:

$$E_{i,1} = \text{tr}(X_i \rho_{\text{fin}}), \quad E_{i,2} = \text{tr}\left(\sum_{t_1,t_2} X_{2,t_1,t_2} \rho_{\text{fin}}\right).$$  \hfill (22)

Let us give justification of our construction. Note that $2^{10}$ is the minimal dimension of the space $\mathcal{H}$ required to play the twice repeated $2 \times 2$ game. Since a player’s strategy in a twice repeated $2 \times 2$ game specifies an action at the first stage and at each of the four subgames fixed by the outcome of the first stage, the quantum protocol needs a 5-qubit register to write player’s strategy. The first two qubits are used to perform operations at the first stage of the repeated game. Then, given the form of $X_1$ and strategies of players restricted to manipulate the first and second qubits, the protocol (17)–(22) coincides, in fact, with the MW scheme of playing a $2 \times 2$ quantum game [1]. The remaining eight qubits are used to define players’ moves at the second stage. That is, by pairing consecutive qubits from the third qubit onward, actions at the second stage are defined on the appropriate pair of qubits depending on the outcome at the previous stage. For example, given that the outcome $O_{10}$ has occurred at the first stage (that is the outcome 10 on the first two qubits has been measured), the expected outcome $E_{i,2}$ depends only on operation on $x_7$ and $x_8$, i.e. $E_{i,2} = \text{tr}(X_{2,10} \rho_{\text{fin}})$. Then, the players play the second stage in the same way as in the protocol (2)–(6). However, in contrast to the previous idea, each player specifies her move for each possible outcome $O_{x_7,x_8}$.

A game generated by our scheme naturally coincides with the classical case when appropriate initial state is prepared. We prove this fact by means of a convenient sequential approach to (17)–(22) provided in the next section.
4.2. The extensive form of a quantum twice repeated $2 \times 2$ game

The protocol (17)–(22) enables us to put a game into the extensive form by using a similar method to what was described in [4]. The extensive form is obtained through sequential calculating the final state $\rho_{\text{fin}}$, according to the following procedure. To begin with, the players manipulate the first pair of qubits. Then the measurement in the computational basis is made on these qubits (as a result, an outcome $O_{1,1}$ of the first stage is returned). The measured outcome is sent to the players. Depending on the measurement outcome $t_1, t_2$ that occurs with probability $p(t_1, t_2)$, the players act on the next pair of qubits: if $t_1, t_2$ is observed, then players 1 and 2 manipulate qubits $2t + 3$ and $2t + 4$, respectively, where $t = (t_1, t_2)$ is a decimal representation of a binary number $t_{1,2}$. The procedure can be formally described as follows.

**Sequential procedure**

1. \( (\sigma_z \otimes \sigma_z) |\psi_{\text{fin}}\rangle = |\psi\rangle \)

   The players perform their operations $\sigma_z^1$ and $\sigma_z^2$ on the initial state $|\psi_{\text{fin}}\rangle$.

2. \( \frac{M_{1,1,2}}{\sqrt{\langle \psi | M_{1,1,2} | \psi \rangle}} = |\psi_{1,1,2}\rangle \)

   The first two qubits in the state $\rho$ are measured.

   The measurement is described by a collection $\{M_{1,1,2} : M_{1,1,2} = |t_1, t_2\rangle \langle t_1, t_2| \otimes \mathbb{1}^{\otimes 8}, t_1, t_2 = 0, 1\}$.

3. \( p(t_1, t_2) \left\{ (\sigma_z^2 \otimes \sigma_z^4) |\psi_{1,1,2}\rangle \right\} \)

   Given that the outcome $t_1, t_2$ has been observed, players 1 and 2 perform operations $\sigma_z^2$ and $\sigma_z^4$ on the post-measurement state.

It turns out that we can prove

**Proposition 4.1.** The density operator for the ensemble \( \left\{ p(t_1, t_2), (\sigma_z^2 \otimes \sigma_z^4) |\psi_{1,1,2}\rangle \right\} \) in the sequential procedure and the density operator $|\psi_{\text{fin}}\rangle \langle \psi_{\text{fin}}|$ associated with state (18) determine the same outcomes $E_{1,1}$ and $E_{1,2}$ given the measurements (20) and (21).

**Proof.** Let us put $\rho = |\psi\rangle \langle \psi|$. Given that $|\psi_{1,1,2}\rangle \langle \psi_{1,1,2}| = M_{1,1,2} \rho M_{1,1,2} / p(t_1, t_2)$, the state $\rho'_{\text{fin}}$ can be written as

\[
\rho'_{\text{fin}} = \sum_{i_1, i_2 = 0}^1 \sigma_z^2 \otimes \sigma_z^4 M_{i_1, i_2} \rho M_{i_1, i_2} \sigma_z^{2+3} \otimes \sigma_z^{2+4}.
\]

Since the first and second qubits are measured, any operation $\sigma_i$ for which $j \neq 1, 2$ does not influence the measurement. Therefore, we obtain

\[
\rho'_{\text{fin}} = \sum_{i_1, i_2 = 0}^1 M_{i_1, i_2} \sigma_z^2 \otimes \sigma_z^4 \rho \sigma_z^{2+3} \otimes \sigma_z^{2+4} M_{i_1, i_2}.
\]

Note that $X_{i_1, i_2} M_{i_1, i_2} = \delta_{i_1, i_1'} \delta_{i_2, i_2'}$, where $\delta_{i, i'}$ is the Kronecker’s delta, and $\epsilon = (i_1, i_2)$, and $\epsilon' = (i_1', i_2')$. Using the form (24) of $\rho'_{\text{fin}}$, we obtain

\[
\text{tr} \left( \sum_{\epsilon'} X_{\epsilon_1, \epsilon_2} \rho'_{\text{fin}} \right) = \text{tr} \left( \sum_{\epsilon} X_{\epsilon_1, \epsilon_2} \sigma_z^{2+3} \otimes \sigma_z^{2+4} \rho \sigma_z^{2+3} \otimes \sigma_z^{2+4} \right).
\]

Observe that for each $\epsilon$, the trace of each term of the sum on the right-hand side of equation (25) depends only on operation $\sigma'_j$ on a qubit $j$, where $j \in \{1, \ldots, 10\}$ and $10$. Thus, equation (25) holds when also the rest of operations $\sigma'_j$ are added,

\[
\text{tr} \left( \sum_{\epsilon} X_{\epsilon_1, \epsilon_2} \rho'_{\text{fin}} \right) = \text{tr} \left( \sum_{\epsilon} X_{\epsilon_1, \epsilon_2} \sigma'_j \rho_{\text{fin}} \otimes \sigma'_j \right).
\]
Consequently, the left-hand side of (26) is equal to the expected outcome $E_{i,2}$ associated with the final state $|\psi_{\text{fin}}\rangle$. To prove that $\rho_{\text{fin}}$ also determines the expected outcome $E_{i,1}$, let us see that $X_i$ and $\{M_{i,1,2}\}$ are the same projective measurements up to the eigenvalues. Hence,

$$\text{tr}(X_i \rho_{\text{fin}}) = \text{tr}(X_i \sigma_{\kappa_1}^{2j+3} \otimes \sigma_{\kappa_2}^{2j+4} \rho \sigma_{\kappa_1}^{2j+3} \otimes \sigma_{\kappa_2}^{2j+4}).$$  

(27)

Since $\rho = \sigma_{\kappa_1}^1 \otimes \sigma_{\kappa_2}^2 \rho \sigma_{\kappa_1}^2 \otimes \sigma_{\kappa_2}^1$, we obtain

$$\text{tr} \left( X_i \rho_{\text{fin}} \right) = \text{tr} \left( X_i \otimes \sigma_{\kappa_1}^j \rho \sigma_{\kappa_1}^j \right).$$  

(28)

Equations (26) and (28) show that the state determined by the sequential procedure and state (18) set the same outcomes $E_{i,1}$ and $E_{i,2}$ for $i = 1, 2$. Using the same way as above and the linearity of the trace, it can be proved that the equivalence is true if the players pick nondegenerate mixed strategies as well.

Having a sequential approach that conforms with the protocol (17)–(22), we are able to analyze a quantum repeated game through its extensive form. It can facilitate the work significantly, bearing in mind a $32 \times 32$ bimatrix associated with the normal representation of a twice repeated $2 \times 2$ game. Let us study the game tree drawn from the sequential procedure if the initial state (17) takes the form

$$|\psi_{\text{fin}}\rangle = \lambda_0 |0\rangle^{\otimes 10} + \lambda_1 |1\rangle^{\otimes 10}.$$  

(29)

Let us use the sequential procedure step-by-step. At first, the players manipulate $\sigma_{\kappa_1}^1$ and $\sigma_{\kappa_2}^2$. Hence, we obtain the following state:

$$\sigma_{\kappa_1}^1 \otimes \sigma_{\kappa_2}^2 |\psi_{\text{fin}}\rangle = \lambda_0 |\kappa_1, \kappa_2\rangle |0\rangle^{\otimes 8} + \lambda_1 |\kappa_1, \kappa_2\rangle |1\rangle^{\otimes 8},$$  

(30)

where $\kappa_j$ is the negation of $\kappa_j$. A game tree at this phase is just the game tree corresponding to a $2 \times 2$ game (see figure 1(b)), where operation $\sigma_{\kappa_j}^j$ for $j = 1, 2, \kappa_j = 0, 1$ is associated with respective branches of that game tree. After a sequence of actions $\left(\sigma_{\kappa_1}^1, \sigma_{\kappa_2}^2\right)$, the measurement $\{M_{i,1,2}\}$ is made. Let us focus on the cases when the measurement outcome 00 or 11 has been observed. The form of (30) tells us that the measurement outcomes 00 and 11 are possible only if the profile at the first stage takes the form of $\left(\sigma_{\kappa_1}^1, \sigma_{\kappa_2}^2\right)$, where $\kappa = 0, 1$. Then, the probability $p(00)$ ($p(11)$) that the measurement outcome 00 (11) will occur is equal to $\mid \lambda_\kappa \mid^2$ ($\mid \lambda_\bar{\kappa} \mid^2$). Thus, the game tree is extended to include random actions 00 and 11 with respective probabilities after the both histories $\left(\sigma_{\kappa_1}^1, \sigma_{\kappa_2}^2\right)$. Since further moves of the players depend only on the measurement, the pair of histories $\left(\sigma_{\kappa_1}^1, \sigma_{\kappa_2}^2, 00\right), \left(\sigma_{\kappa_1}^1, \sigma_{\kappa_2}^2, 01\right)$ and the pair $\left(\sigma_{\kappa_1}^1, \sigma_{\kappa_2}^2, 10\right), \left(\sigma_{\kappa_1}^1, \sigma_{\kappa_2}^2, 11\right)$ constitute two separate information sets. Next, given that 00 (11) has occurred, the players, according to the sequential procedure, manipulate the third and fourth (the ninth and the tenth) qubits at the second stage. Therefore, another extensive form of a $2 \times 2$ game is added to each sequence $\left(\sigma_{\kappa_1}^1, \sigma_{\kappa_2}^2, \kappa\right)$, where $\kappa, \ell = 0, 1$. In consequence, we obtain a game tree shown in figure 3 (a part of the game tree after histories of $\left(\sigma_{\kappa_1}^1, \sigma_{\kappa_2}^2\right), \kappa = 0, 1$ can be constructed in the similar way). Each outcome associated with a terminal history of the so-defined extensive game is determined by a pure state from the ensemble given by the sequential procedure. For example, after the sequence $\left(\sigma_{\kappa_1}^1, \sigma_{\kappa_2}^2\right)$, the post-measurement state takes the form of $|0\rangle^{\otimes 2} |1\rangle^{\otimes 8}$ (up to a global phase factor) with probability $\mid \lambda_\kappa \mid^2$, and the players choose the sequence $\left(\sigma_{\kappa_1}^3, \sigma_{\kappa_2}^4\right)$. Then, the total outcome $E_i := E_{i,1} + E_{i,2}$ associated with sequence $\left(\sigma_{\kappa_1}^1, \sigma_{\kappa_2}^2, 00\right), \left(\sigma_{\kappa_1}^1, \sigma_{\kappa_2}^2, 11\right)$ is calculated according to formulas (22),

$$E_i = \text{tr} \left( \left( X_i + \sum_{\ell_1,\ell_2} X_{2,\ell_1,\ell_2} \right) |0\rangle \langle 0|^{\otimes 2} \otimes |\kappa_3, \kappa_4\rangle |\kappa_3, \kappa_4\rangle \otimes |1\rangle \langle 1|^{\otimes 6} \right).$$  

(31)
Figure 3. The extensive form for a twice repeated PD played through the protocol (17)–(22) when the initial state is on the form of (29).

The extensive approach allows us to see directly that our scheme coincides with the classical twice repeated $2 \times 2$ game when $|\psi_{in}\rangle = |0\rangle^\otimes 10$. Without loss of generality, let the outcomes $O_{\iota_1,\iota_2}$ be the payoff outcomes corresponding to the PD game. Then, putting $|\lambda_0\rangle^\otimes 2$ in (29) and assuming $\sigma_0^j := C$, $\sigma_1^j := D$, the game in figure 3 depicts exactly the classical twice repeated PD game (compare figures 2 and 3).

4.3. Twice-repeated PD game played by means of the protocol (17)–(22)

Let us study the twice repeated PD game played with the use of our scheme. Analysis of our protocol with the initial state of the general form (17) is a laborious task and it deserves a separate paper to report about. Nevertheless, we can derive many interesting features with less effort considering the initial state of the form

$$|\psi_{in}\rangle = \bigotimes_{j=1}^5 |\psi_j\rangle,$$

where $|\psi_j\rangle$ is the state of the $2j - 1$ and $2j$ qubit. (32)

Let us consider first the problem of optimization of the equilibrium payoffs, given a space of initial states as a domain.

**Proposition 4.2.** There are infinitely many settings of the initial state (17) for which the twice repeated PD game played with the use of the protocol (17)–(22) has a unique subgame perfect equilibrium with the equilibrium payoff $(2Q, 2Q)$ such that $Q > P$.

**Proof.** Let us put the initial state (32) into the protocol (17)–(22) assuming that $|\psi_j\rangle = |\psi\rangle$ for any $j$. Then, the measurement $\{M_{\iota_1,\iota_2}\}$ on the first pair of qubits does affect others' qubits. Moreover, given that the outcome $O_{\iota_1,\iota_2}$ has occurred, the expected outcome $E_{i,2}$ depends only on manipulating on one pair of qubits $|\psi\rangle$ due to the form of (21). Therefore, regardless of the first stage outcome $O_{\iota_1,\iota_2}$, the players are faced with a $2 \times 2$ quantum game at the second stage (played via the MW approach). That is, the players are faced with the problem

$$(|\psi\rangle\langle\psi|, [\sigma_0, \sigma_1], X'_i),$$

(33)
where players 1 and 2 apply operators from the set \{σ₀, σ₁\} on the first and second qubits of \(|φ⟩\), respectively. The outcome operator \(X'_i\) for player \(i\) takes the form
\[
X'_i = \sum_{y₁,y₂=0,1} O_{y₁,y₂} |y₁,y₂⟩⟨y₁,y₂|.
\]
and the expected outcome is equal to
\[
E_i(σ^{k₁}_i, σ^{k₂}_i) = \text{tr}(σ^{k₁}_i ⊗ σ^{k₂}_i |φ⟩⟨φ| σ^{k₁}_i ⊗ σ^{k₂}_i X'_i).
\]
Obviously, the first stage game is also described exactly as the triple \((33)\). Since a quantum game according to the MW approach is a game expressed by a bimatrix, it leads us to the conclusion that the protocol \((17)–(22)\) with the initial state \(|φ⟩⊗^{⊗n}\) can, in fact, be treated as a twice repeated bimatrix game generated by \((33)\).

Let us substitute \(O_{y₁,y₂}\) for the payoffs of the PD game in the game \((33)\) and examine it toward uniqueness of the Nash equilibria. We put a state \(|φ⟩ = λ₁|00⟩ + λ₂|11⟩\), for which the amplitudes of \(|φ⟩\) satisfy the condition
\[
0 < |λ₀|^2 < \frac{\min[|T - R|, |P - S|]}{|T - R| + |P - S|}.
\]
Then, the inequalities
\[
E₁(σ₁₀, σ₁₁) > E₁(σ₁₁, σ₂₁) \quad \text{and} \quad E₂(σ₁₀, σ₂₀) > E₂(σ₁₁, σ₁₁)
\]
are true for any \(κ₁, κ₂ = 0, 1\). Inequalities (37) imply the unique Nash equilibrium \((σ₁₀, σ₁₁)\). Moreover, the first inequality of condition (36) ensures that
\[
E₁(σ₁₀, σ₂₀) = |λ₀|^2 R + |λ₁|^2 P > P.
\]
Since the game constructed in the proof can be regarded as a classical twice repeated game, we are allowed to use all facts of classical repeated game theory. One of these facts tells us that a unique stage-game Nash equilibrium implies, for any finite number of repetitions, a unique subgame-perfect equilibrium in which the stage-game Nash equilibrium is played in every stage \([9, \text{proposition 157.2}]\). This completes the proof.

Of course, the protocol \((17)–(22)\) can be re-formulated for any finitely repeated \(2 × 2\) game, and then, the statement analogical to proposition 4.2 can be formulated. Unfortunately, the number of qubits required in our protocol grows exponentially with the number of stages. For example, in the case of a game repeated three times, the protocol \((17)–(22)\) needs 32 qubits to describe the third stage. In general, \(\sum_{j=1}^{n} 2^{2j-1}\) qubits are required for a \(2 × 2\) game repeated \(n\) times.

We shall now re-examine the problem of cooperation considered in \([6]\). We demonstrated in the comment on Iqbal and Toor’s results that the cooperation at the first stage cannot lead to an equilibrium in the game defined by their scheme. However, we also showed that this protocol does not take into consideration a player’s move at the second stage as a function of the result of the first stage. Therefore, the cooperation problem in \([6]\) was not studied in a proper way. The following example proves that the cooperation of players can be a part of a reasonable solution if the twice repeated PD game is played via our scheme.

**Example 4.1.** Let us consider the PD game with payoff vectors
\[
O_{00} = (4, 4), \quad O_{01} = (0, 5), \quad O_{10} = (5, 0), \quad O_{11} = (1, 1)
\]
and insert them into \((20)\) and \((21)\). Let us take, for example, the initial state \((17)\) of the form
\[
|ψ_{in}⟩ = |0⟩⊗² ⊗ (\sqrt{0.6}|0⟩⊗² + \sqrt{0.4}|1⟩⊗²) ⊗ |0⟩⊗₆³.
\]
Figure 4. The extensive form for the twice repeated PD (39) played through the protocol (17)–(22) with an update on the initial state (40).

A game specified in this way differs from the classical one only in the subgame following the outcome $O_{00}$ of the first stage because then $E_{i2}$ depends on operations on the entangled third and fourth qubits. Since two first qubits in the state $|00\rangle$ imply the classical PD game at the first stage, we are permitted to identify the action 'cooperate' and the action 'defect' with $\sigma_0$ and $\sigma_1$, respectively, assuming $C := \sigma_0$ and $D := \sigma_1$. Moreover, the quantum measurement after the first stage is trivialized in this case and it coincides with the classical observation of actions in an extensive game. It follows that both the game defined by (17)–(22), (39) and (40), and the classical game can be represented by the same game tree as well as the same payoffs except payoffs when 00 has been measured on the first pair of qubits after the first stage. Let us determine now the payoff outcomes at the second stage given that the post-measurement state of the first pair of qubits is $|00\rangle$ (in other words, when player 1’s strategy $\tau_1 = (\sigma_{1\kappa_1}, \sigma_{3\kappa_3}, \sigma_{5\kappa_5}, \sigma_{7\kappa_7}, \sigma_{9\kappa_9})$ and player 2’s strategy $\tau_2 = (\sigma_{2\kappa_2}, \sigma_{4\kappa_4}, \sigma_{6\kappa_6}, \sigma_{8\kappa_8}, \sigma_{10\kappa_{10}})$ make the strategy profile $(\tau_1, \tau_2) = (\sigma_{1\kappa_1}, \sigma_{3\kappa_3}, \sigma_{5\kappa_5}, \sigma_{7\kappa_7}, \sigma_{9\kappa_9}, \sigma_{2\kappa_2}, \sigma_{4\kappa_4}, \sigma_{6\kappa_6}, \sigma_{8\kappa_8}, \sigma_{10\kappa_{10}}))$. Given the initial state (21), the payoff outcome $E_{i2}$ for each $\kappa_3, \kappa_4 \in \{0, 1\}$ and $i = 1, 2$ is as follows:

$$E_{i2}(\sigma_{1\kappa_1}, \sigma_{3\kappa_3}, \sigma_{5\kappa_5}, \sigma_{7\kappa_7}, \sigma_{9\kappa_9}, \sigma_{2\kappa_2}, \sigma_{4\kappa_4}, \sigma_{6\kappa_6}, \sigma_{8\kappa_8}, \sigma_{10\kappa_{10}}) = 0.6O_{\kappa_3, \kappa_4} + 0.4O_{\kappa_3, \kappa_4}.$$ (41)

The extensive form of the game with expected payoffs $E_{i1} + E_{i2}$ are shown in figure 4.

Let us examine this game for subgame perfect equilibria. Such a profile has to induce the Nash equilibrium in any subgame fixed by an outcome at the first stage. In our case, it is a profile in which both players apply $\sigma_1$ to qubits from the third qubit onward. Consequently, in quest of subgame perfect equilibria, we take only the following profiles into consideration:

$$(\tau_1, \tau_2) \in \left\{ \sigma_{1\kappa_1} \times \sigma_{3\kappa_3} \times \prod_{j=3}^{10} \sigma_{j\kappa_j} \right\}. \quad (42)$$

Then, it turns out that the noncooperative subgame-perfect equilibrium is still preserved. If one of the players picks $\sigma_1$ at the first stage, the best response of the other one is to pick $\sigma_1$ too. As a result, the profile $(\tau_1', \tau_2') = \prod_{j=1}^{10} \sigma_{j\kappa_j}$ constitutes a subgame-perfect equilibrium. However, in contrast to the classical twice repeated PD, there is another subgame-perfect equilibrium $(\tau_1'', \tau_2'')$ in which each player chooses $\sigma_0$ (cooperates) at the first stage, i.e. $\tau_1'' = (\sigma_{0\kappa_1}, \sigma_{3\kappa_3}, \sigma_{5\kappa_5}, \sigma_{7\kappa_7}, \sigma_{9\kappa_9})$ and $\tau_2'' = (\sigma_{0\kappa_2}, \sigma_{4\kappa_4}, \sigma_{6\kappa_6}, \sigma_{8\kappa_8}, \sigma_{10\kappa_{10}})$. Moreover, only the latter equilibrium is reasonable since it yields the payoff 6.2 instead of 2 for each player.
Example 4.1 shows that the cooperation of players is possible when the twice repeated PD game is played according to our scheme. Unfortunately, the example does not solve this problem for any PD game. The condition $2R > T + S$ imposed on the payoffs admits to select an arbitrary large number $T$ (if a sufficiently small number $S$ is selected). We suppose that an appropriately large $T$ may convince the players to defect even if the game is played in quantum domain.

5. Conclusion

Our paper proves that repeated games can be quantized. That is, we have shown that by appropriately modifying the MW scheme for $2 \times 2$ quantum games can indeed generalize a twice repeated game. In addition, such a quantized game can be further analyzed by strategic as well as extensive form games. Our results also indicate (with the use of the twice repeated PD) that playing repeated games in the quantum domain can give superior results in comparison with the classical ones. At the same time, we have answered why the previous approach [6] cannot be treated as a correct protocol for quantum repeated games. The main objection is that the protocol [6] is unable to consider a full set of strategies available to players. In contrast to Iqbal and Toor’s scheme, the protocol defined in this paper is free from the mentioned fault.

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