ON THE TWO-POINT CORRELATION FUNCTION IN
DYNAMICAL SCALING AND SCHRÖDINGER INVARIANCE

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ABSTRACT

The extension of dynamical scaling to local, space-time dependent rescaling factors
is investigated. For a dynamical exponent $z = 2$, the corresponding invariance group
is the Schrödinger group. Schrödinger invariance is shown to determine completely the
two-point correlation function. The result is checked in two exactly solvable models.

Keywords: Dynamical scaling; Glauber dynamics; Schrödinger invariance; conformal
invariance

1. Introduction

The concept of scaling has proved to be a very fruitful one in describing phase
transitions of statistical systems. For static critical phenomena, the renormalization
group (see e.g. Ref. 1) has elucidated the universal quantities characterizing a uni-
versality class and has provided approximation schemes for their calculation. More
recently, conformal invariance, at least in two dimensions, has yielded exact results
for critical exponents and amplitude ratios and also for the multipoint correlation
functions (see e.g. Ref. 2).

Much less is known for time-dependent problems\(^3\). However, it was recognised
that dynamical scaling may arise in various situations, that is, the two-point corre-
lation function $C(\vec{r}, t)$ satisfies

$$C(\lambda \vec{r}, \lambda^z t) = \lambda^{-2z} C(\vec{r}, t)$$

(1.1)

where $z$ is the dynamic critical exponent and $x$ is the scaling dimension. Examples
include the time-dependent behaviour at a static critical point\(^1,3\) or the ordering
process following the quench of a system from an initial state at high temperatures to
a final state below the critical temperature\(^4\). In this case, the corresponding static
system is not critical. For a recent experimental example (with conserved order
parameter) for the development of dynamical scaling at late times in $\text{Mn}_{0.67} \text{Cu}_{0.33}$,
see Ref. 5.
Eq. (1.1) can be recast into the form
\[ C(r, t) = t^{2x/z} \Phi \left( \frac{r^z}{t} \right) \] (1.2)
defining the scaling function \( \Phi(u) \) where \( u = r^z/t \) is the scaling variable. What can be said about \( \Phi(u) \) ?

We propose to generalize the dynamic scaling (1.1) from the global form with \( \lambda = \text{const.} \) to a local one, where the rescaling factor becomes space-time dependent
\[ \lambda = \lambda(\vec{r}, t). \] (1.3)
This is analogous to the introduction of conformal invariance in statics\(^6\). In fact, the direct generalization of conformal invariance to dynamics, with \( \lambda = \lambda(\vec{r}) \), was attempted\(^7\) some time ago, being restricted to two space dimensions and to the case where the static system is at a second order critical point. The approach to be presented here is not subject to any of these restrictions.

Here, we shall concentrate on the special case \( z = 2 \) with a non-conserved order parameter. The group of local scale transformations is then the Schrödinger group, to be defined in the next section. We shall show that
\[ \Phi(u) = \Phi_0 \exp \left( -\frac{1}{2} \mathcal{M} u \right) \] (1.4)
where \( \mathcal{M} \) and \( \Phi_0 \) are constants. This result will be confirmed in section 3 for the \( d \)-dimensional spherical model with a non-conserved order parameter\(^8\) and for the one-dimensional Glauber-Ising model\(^9\). In section 4, we conclude with a brief outlook for the case \( z \neq 2 \).

2. The Schrödinger group

The Schrödinger group \( \text{Sch}(d) \) in \( d \) space dimensions is defined\(^10,11\) by the following space-time transformations
\[ \vec{r} \rightarrow \vec{r}' = \frac{\mathcal{R} \vec{r} + \vec{v} t + \vec{a}}{\gamma t + \delta} \] (2.1)
\[ t \rightarrow t' = \frac{\alpha t + \beta}{\gamma t + \delta}; \quad \alpha \delta - \beta \gamma = 1 \]
where \( \mathcal{R} \) is a rotation matrix and \( \alpha, \beta, \gamma, \delta, \vec{v}, \vec{a} \) are parameters. The Schrödinger group is the maximal group which transforms\(^10\) solutions of the Schrödinger equation
\[ i \frac{\partial}{\partial t} + \frac{1}{2m} \sum_{j=1}^{d} \frac{\partial^2}{\partial r_j \partial r_j} \psi = 0 \] (2.2)
into other solutions of (2.2), viz. \( (\vec{r}, t) \rightarrow g(\vec{r}, t), \psi \rightarrow T_g \psi \)
\[ (T_g \psi)(\vec{r}, t) = f_g \left( g^{-1}(\vec{r}, t) \right) \psi \left( g^{-1}(\vec{r}, t) \right) \] (2.3)
where \( f_g \) is the companion function which has been worked out explicitly. For diffusive processes, (2.2) may be replaced by the Helmholtz equation, with \( 2im = D^{-1} \), where \( D \) is the diffusion constant. It is also known that Euclidean free field theory is invariant under (2.3). In particular, choosing \( \vec{v} = \vec{a} = 0, \beta = \gamma = 0 \) and \( \alpha = 1/\delta \), we recover the global scale transformation \( \vec{r} \to \alpha \vec{r}, t \to \alpha^2 t \), which corresponds to \( z = 2 \). For simplicity, we take \( d = 1 \) in the sequel, but all the results to be described here generalize immediately to arbitrary \( d \).

The infinitesimal generators are

\[
X_n = -t^{n+1} \partial_t - \frac{n+1}{2} t^n \partial_r - \frac{n(n+1)}{4} \mathcal{M} t^{n-1} r^2 \quad ; \quad n = -1, 0, 1 \quad (2.4a)
\]

\[
Y_n = -t^{n+1/2} \partial_r - \left( n + \frac{1}{2} \right) \mathcal{M} t^{n-1/2} r \quad ; \quad n = -\frac{1}{2}, \frac{1}{2} \quad (2.4b)
\]

\[
M_n = -t^n \mathcal{M} \quad ; \quad n = 0 \quad (2.4c)
\]

where the terms \( \sim \mathcal{M} \) in \( X_n \) and \( Y_n \) come from the companion function. The commutation relations are

\[
[X_n, X_m] = (n-m)X_{n+m}
\]

\[
[X_n, Y_m] = \left( \frac{n}{2} - m \right) Y_{n+m}
\]

\[
[X_n, M_m] = -mM_{n+m}
\]

\[
[Y_n, Y_m] = (n-m)M_{n+m}
\]

\[
[Y_n, M_m] = [M_n, M_m] = 0 \quad (2.5)
\]

and it follows that the set \( \{ X_{-1}, X_0, X_1, Y_{-1/2}, Y_{1/2}, M_0 \} \) spans a six-dimensional subalgebra. In order to implement Schrödinger invariance on the correlation functions, we have to replace in (1.1) the factor \( \lambda^{-2x} \) by the corresponding Jacobian

\[
\phi_1(r'_1, t'_1) \ldots \phi_n(r'_n, t'_n) > = \prod_{i=1}^{n} \left| \frac{\partial(r'_i, t'_i)}{\partial(r_i, t_i)} \right|^{-x_i/(2+d)} < \phi_1(r_1, t_1) \ldots \phi_n(r_n, t_n) > . \quad (2.6)
\]

This is the analogue of the definition of the quasiprimary fields of conformal invariance. In particular, derivative fields are excluded by (2.6).

We now examine the consequences for the two-point function

\[
F(r_1, r_2; t_1, t_2) = < \phi_1(r_1, t_1), \phi_2(r_2, t_2) > . \quad (2.7)
\]

Translations are generated by \( X_{-1} \) and \( Y_{-1/2} \) and imply that \( F = F(r_1 - r_2, t_1 - t_2) \). Global scale transformations are generated by \( X_0 \). Writing \( r = r_1 - r_2 \) and \( t = t_1 - t_2 \), we have

\[
\left( t \partial_t + \frac{1}{2} r \partial_r + \frac{1}{2} (x_1 + x_2) \right) F = 0 \quad (2.8)
\]

yielding, with \( x = x_1 + x_2 \)

\[
F(r, t) = t^{-x/2} G(u) \quad ; \quad u = r^2/t \quad (2.9)
\]
and reproducing (1.2). The new information comes from the Galilei transformation, generated by $Y_{1/2}$. By translation invariance, we can put $r_2 = t_2 = 0$. Then

$$(t\partial_t + Mr) t^{-x/2}G(u) = 0$$

which gives an equation for $G(u)$

$$\frac{dG}{du} + \frac{M}{2}G = 0$$

and thus, with $\Phi_0$ being a normalization constant

$$F(r, t) = \Phi_0 t^{-x/2} \exp\left(-\frac{M r^2}{2 t}\right).$$

We still have to see whether this is consistent with the special Schrödinger transformation generated by $X_1$. Using translational invariance, we put $r_2 = t_2 = 0$ and have

$$(t^2\partial_t + tr\partial_r + \frac{M}{2} r^2 + 2r_1^2) t^{-x/2}G(u) = 0$$

and we can see that this implies two conditions. The first one is just (2.11), while the second one is $x = 2x_1$ or, since $x = x_1 + x_2$

$$x_1 = x_2$$

which means that the two scaling operators $\phi_1, \phi_2$ have to be the same in order to have a non-vanishing two-point correlation function. We summarize our result

$$<\phi_1(\vec{r}, t)\phi_2(\vec{0}, 0)> = \delta_{1,2}\Phi_0 t^{-x_1} \exp\left(-\frac{M |\vec{r}|^2}{2 t}\right)$$

where we have restored the $d$-dimensional case. For the special case $d = 2$, this is in agreement with the conformal invariance approach of Ref. 7. Comparing with the corresponding result of conformal invariance$^6$ of a static critical point, we note the importance of the contribution arising from the companion term in the Schrödinger group which is parametrized by the non-universal constant $M$.

One may go on and consider higher correlation functions. Furthermore, one can show that invariance under translations, dilatations, space rotations and Galilei transformations imply the full Schrödinger invariance if the interactions are short-ranged. This will be presented in detail elsewhere$^{12}$.

3. Comparison with exactly solvable models

We now compare the result for the two-point function (1.4,2.15) with two exactly solvable time-dependent models which have $z = 2$.

The first model we consider is the $O(N)$-symmetric time-dependent Ginzburg-Landau model$^8$. Initially, the system is at very high temperatures, but at time
$t = 0$, it is quenched to zero temperature. In the $N \to \infty$ limit, the structure function was calculated exactly\(^8\) for late times in $d$ spatial dimensions

$$C(\vec{k}, t) = M_0^2 L^d(t) \exp \left( -k^2 L^2(t) \right)$$  \hspace{0.5cm} (3.1)

where $M_0$ is the equilibrium magnetization, $L(t) = (2Dt)^{1/2}$ is the typical domain size and $D$ is the diffusion constant. This can be rewritten in direct space

$$C(\vec{r}, t) = 2^{-d/2} M_0^2 \exp \left( -\frac{d}{8D} \frac{\vec{r}^2}{t} \right)$$  \hspace{0.5cm} (3.2)

in agreement with (1.4,2.15) and we read off $x = 0$. Since the renormalization group eigenvalue $y = d - x$ and $y = d$ at a first-order transition, this last result is probably not too surprising.

As a second example, we take the one-dimensional Ising model with Glauber dynamics\(^9\). If the system is in thermal equilibrium at temperature $T$, the spin-spin correlation function is known exactly\(^9\) \((t > 0)\)

$$C(\vec{a} - \vec{b}, t) = <\sigma(0)\sigma(t)> = e^{-\alpha t} \sum_{\ell = -\infty}^{\infty} \eta|a-b+\ell| I_\ell(\gamma \alpha t)$$  \hspace{0.5cm} (3.3)

where $\alpha$ is the transition rate, $\eta = \tanh J/k_B T$, $\gamma = \tanh 2J/k_B T$, $J$ is the exchange integral of the Ising model and $I_\ell$ is a modified Bessel function. To analyse this, we recall the asymptotic expansion\(^{13}\), as $x \to \infty$

$$I_\ell(x) \simeq (2\pi x)^{-1/2} \exp \left( x - \frac{\ell^2}{2x} \right) \left( 1 + O(x^{-1}) \right)$$  \hspace{0.5cm} (3.4)

and, writing $r = a - b$, we have

$$C(r, t) \simeq e^{-\alpha(1-\gamma)t} (2\pi \gamma \alpha t)^{-1/2} \left\{ \exp \left( -\frac{r^2}{2\gamma \alpha t} \right) + \sum_{\ell \neq 0} \eta|\ell| \exp \left( -\frac{(r + \ell)^2}{2\gamma \alpha t} \right) \right\}.$$  \hspace{0.5cm} (3.5)

Now take the simultaneous scaling limit $r \to \infty$, $t \to \infty$ such that $u = r^2/t$ stays fixed. Then the leading term becomes

$$C(r, t) \sim e^{-\alpha(1-\gamma)t} (2\pi \gamma \alpha t)^{-1/2} \exp \left( -\frac{1}{2\gamma \alpha} \frac{r^2}{t} \right)$$  \hspace{0.5cm} (3.6)

where each of the terms neglected is an exponentially small correction-to-scaling term.

The first factor in (3.6) describes the off-critical relaxation towards equilibrium and we can identify the well-known relaxation time $\tau^{-1} = \alpha(1 - \gamma)$. At the critical point $T = 0$, we have $\gamma = 1$ and this factor becomes unity. Alternatively, we can
define another temperature-time scaling limit, where \( t \to \infty, T \to 0 \) such that \( t \exp^{-4J/k_BT} \) is kept fixed. We then recover the anticipated scaling form and find agreement with (1.4,2.15). We read off \( 2x = 1/2 \).

4. Conclusions and outlook

The hypothesis of Schrödinger invariance in critical dynamics with \( z = 2 \) was shown to predict the scaling function of the two-point correlation function in the case of a non-conserved order parameter. This finding is supported by results from exactly solvable models. The theory will be developed more systematically elsewhere\(^{12}\).

We comment briefly on the possibility to generalize beyond the case \( z = 2 \). It can be argued\(^{12}\) that for \( u \) large

\[
\Phi(u) \sim \exp \left( - \text{const.} \ u^{1/(z-1)} \right). \tag{4.1}
\]

We are not aware of any calculation in critical dynamics which either supports or excludes (4.1). However, (4.1) is supported in static, but strongly anisotropic systems, where now \( \theta = \nu_\parallel/\nu_\perp \) measures the anisotropy and enters in the scaling form (1.1), (1.2) instead of \( z \). Examples are provided by Lifshitz points in the spherical model\(^{14}\) (\( \theta = 1/2, 2, 3 \)) and two-dimensional directed percolation\(^{15}\) (\( \theta \simeq 1.58 \)). Conformal invariance, valid only for \( d = 2 \) and at an isotropic (or static) critical point, suggests that\(^7\) \( \Phi(u) \sim e^u \) independent of \( z \) or \( \theta \). Although this result is in agreement with ours for \( \theta = 2 \) (or \( z = 2 \)), it is at variance with (4.1) if \( \theta \neq 2 \). Unfortunately, in none of the models with \( \theta \neq 2 \) studied so far the correlation function was calculated in \( d = 2 \) so that a direct comparison has not yet been possible. A detailed account will be given in a separate paper\(^{12}\).

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