Pricing principle via Tsallis relative entropy in incomplete market

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Abstract

A pricing principle is introduced for non-attainable \(q\)-exponential bounded contingent claims in an incomplete Brownian motion market setting. The buyer evaluates the contingent claim under the “distorted Radon-Nikodym derivative” and adjustment by Tsallis relative entropy over a family of equivalent martingale measures. The pricing principle is proved to be a time consistent and arbitrage-free pricing rule. More importantly, this pricing principle is found to be closely related to backward stochastic differential equations with generators \(f(y)|z|^2\) type. The pricing functional is compatible with prices for attainable claims. Except translation invariance, the pricing principle processes lots of elegant properties such as monotonicity and concavity etc. The pricing functional is showed between minimal martingale measure pricing and conditional certainty equivalent pricing under \(q\)-exponential utility. The asymptotic behavior of the pricing principle for ambiguity aversion coefficient is also investigated.

Keywords: Tsallis relative entropy, quadratic BSDE, pricing principle

1. Introduction

1.1. Model setup and motivation

Given a time \(T < \infty\). Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\) be a filtered probability space. \((\mathcal{F}_t)_{0 \leq t \leq T}\) is the augmented filtration generated by a \((m + n)\)-dimensional standard Brownian motion \(\mathbf{W} = (W, W^\perp)\), where \(W\) and \(W^\perp\) are the first \(m\) and the last \(n\) components. We also assume \((\mathcal{F}_t)_{0 \leq t \leq T}\) satisfies the usual hypotheses, completeness and right-continuity.

Consider a market model with \(m\) traded assets whose price processes evolve in the following simple model,

\[
S_t = S_0 + \int_0^t \lambda_s ds + W_t, \ 0 \leq t \leq T,
\]

where \(S_0 \in \mathbb{R}^m\) and \(\lambda\) is a uniformly bounded \(\mathbb{R}^m\)-valued predictable process with respect to \(\mathcal{F}_t^W = \sigma(W_s, 0 \leq s \leq t),\ t \in [0, T]\). For a general price process, this can be achieved under a suitable assumption on its drift and volatility matrix.

We denote by \(\mathcal{Q}\) the set of all probability measures on \((\Omega, \mathcal{F}_T)\), equivalent with respect to \(\mathbb{P}\). Define \(\mathcal{M}\) as the set of probability measures \(\mathcal{Q} \subset \mathcal{Q}\) such that \((S_t)_{t \in [0, T]}\) is a local martingale with respect to \(\mathcal{Q}\).

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By the predictable martingale representation theorem of Brownian motion, for any $Q \in \mathcal{M}$ with density process $D_{Q,P} = \mathbb{E}_P[D_t^Q | \mathcal{F}]$, there exists a unique $\mathbb{R}^n$-valued predictable process $(\alpha_t^Q)_{t \in [0,T]}$ such that $\int_0^t \alpha_s^Q \cdot dW_s^\perp$ is a local integral martingale on $[0,T]$ and

$$D_t^Q = \mathcal{E}(-\lambda \cdot W + \alpha_t^Q \cdot W^\perp), \quad 0 \leq t \leq T.$$  

(1.1)

We also denote $D_{s,t}^Q = D_{t}^Q / D_{s}^Q$, $0 \leq s \leq t \leq T$.

One of the important problems in mathematical finance is to price the non-attainable contingent claims in the above incomplete market. In the incomplete case, a general contingent claim is not feasible to create perfectly replicating portfolios. From an economic point of view, it means that such a claim will have an intrinsic risk. Therefore, the market must develop the arbitrage-free pricing in order to specify the appropriate price for the given contingent claim.

Föllmer and Schweizer (1990) propose the minimal martingale measure pricing to solve the above problem. The minimal martingale measure (MEMM) $Q_{\text{min}}$ is defined by

$$\frac{dQ_{\text{min}}}{dP} = \mathcal{E}(-\lambda \cdot W) = \exp \left( -\int_0^T \lambda_s \cdot dW_s - \frac{1}{2} \int_0^T |\lambda_s|^2 ds \right),$$

see Theorem 3.5 in Föllmer and Schweizer (1990). In particular, since $Q_{\text{min}} \sim P$, for each $Q \in \mathcal{M}$, then we have $D_{Q,Q_{\text{min}}} = \mathcal{E}(\alpha^Q \cdot W^\perp)$. Föllmer and Schweizer (1990) also show that minimal martingale measure $Q_{\text{min}}$ is related to the relative entropy among all martingale measures. The relative entropy (also called Kullback-Leibler divergence) between any two probability measures $Q$ and $P$ is defined by

$$H_1(Q|P) := \mathbb{E}_P \left[ \frac{dQ}{dP} \ln \frac{dQ}{dP} \right] = \mathbb{E}_P[D_T^{Q,P} \ln D_T^{Q,P}], \quad \text{when } Q \ll P.$$

Relative entropy theory has been proved to have important applications in mathematical finance. For example, the reader can refer to Frittelli (2000a), Delbaen et al. (2002), Klöppel and Schweizer (2007) and Föllmer and Schied (2016).

The purpose of this paper is to provide new ideas and insights for pricing the non-attainable contingent claim in incomplete market. We first give the economic motivations for our problem. Consider a decision maker (DM) who is concerned about model misspecification. The DM has a baseline model $Q_{\text{min}}$, but also considers alternative models in $\mathcal{M}_f$, which is the set of equivalent local martingale measures with finite discrepancy regarding to the baseline model and will be given specifically in Section 3. For some contingent claim $\xi$, the DM evaluates it by the following pricing principle

$$F_t(\xi) := \text{ess inf}_{Q \in \mathcal{M}_f} \left( \mathbb{E}_{Q_{\text{min}}} \left[ (D_{t,T}^{Q,Q_{\text{min}}})^q \xi \mid \mathcal{F}_t \right] + \frac{1}{\gamma} H_{q,t}(Q|Q_{\text{min}}) \right), \quad t \in [0,T],$$  

(1.2)

where $\gamma > 0$ is the ambiguity aversion coefficient, and

$$H_{q,t}(Q|Q_{\text{min}}) = \mathbb{E}_{Q_{\text{min}}} \left[ (D_{t,T}^{Q,Q_{\text{min}}})^q \ln_q D_{t,T}^{Q,Q_{\text{min}} \mid \mathcal{F}_t} \right].$$

1Many thanks to one referee for his/her suggestions on the expression of the motivation.
is a conditional Tsallis relative entropy characterizing the discrepancy. Tsallis relative entropy is put forward by Tsallis (1988) from the viewpoints of statistical physics (see more details in Tsallis (2009)). When $Q \ll P$, it is defined by

$$H_q(Q|P) := \int \left( \frac{dQ}{dP} \right)^q \ln_q \left( \frac{dQ}{dP} \right) dP = \mathbb{E}_P \left[ (D_{T}^{Q,P})^q \ln_q D_{T}^{Q,P} \right],$$

where $q > 0$, $q \neq 1$, and $\ln_q(\cdot)$ denotes the generalized $q$-logarithm function (see Section 2 for a detailed expression). The parameter $q$ can be viewed as a bias or distortion of the original probability measure.

As a consequence, for a given alternative model $Q \in \mathcal{M}_f$, the DM evaluates the expected payoff via $\mathbb{E}_{Q^{\text{min}}}[ (D_{t,T}^{Q,Q^{\text{min}}})^q \xi | \mathcal{F}_t ]$ with a weight $(D_{t,T}^{Q,Q^{\text{min}}})^q$. When $q \in (0, 1)$, the DM put less weight on events with large $D_{t,T}^{Q,Q^{\text{min}}}$; when $q > 1$, the DM put more weight on events with large $D_{t,T}^{Q,Q^{\text{min}}}$. The DM also penalizes models which are too far away from the baseline model $Q^{\text{min}}$. The discrepancy is measured by conditional Tsallis relative entropy. In this setting, the pricing principle (1.2) is a robustness evaluation. In particular, when $q = 1$, this is exactly the formulation of Hansen and Sargent (2001), where $H_1(Q|Q^{\text{min}})$ is the relative entropy. In this way, our pricing principle can be presented as an extension of Hansen and Sargent (2001) with the relative entropy replaced by Tsallis relative entropy.

### 1.2. The main contributions of this work

The pricing principle (1.2) induced by Tsallis relative entropy seems not to be necessarily time consistent or arbitrage-free, because it is a robustness valuation problem and has the nonlinearity brought by the distortion parameter $q$. The main contribution of this work is that we identify a connection between this pricing principle and backward stochastic differential equations (BSDEs) with generators $f(y)|z|^2$ type. This kind of BSDEs are recently investigated by Bahlali and his coauthors (see Bahlali, Eddahbi and Ouknine (2017); Bahlali and Tangpi (2018); Bahlali (2020)) and by Zheng, Zhang and Feng (2021) for more refined results.

We summarize our main theoretical contributions as follows. First, Theorem 2.1 provides the integral representation for the conditional Tsallis relative entropy between equivalent martingale measure and original reference measure. In addition, we also obtain the representation of the conditional Tsallis relative entropy between equivalent martingale measure and minimal martingale measure. When $q = 1$, these representation results are exactly the corresponding formulations of relative entropy (see Choulli and Stricker (2005) or Skidas (2003)).

Second, and most important, we find a close connection between the pricing principle (1.2) and a specific quadratic BSDE. In order to investigate the pricing principle, we introduce two related problems of BSDEs (Problem 2 and Problem 3) in Section 3. Problem 2 is a dual formulation, and it is an optimization problem of BSDEs. Problem 3 is a specific quadratic BSDE whose generator involves $y$ and is in form of $g(t,y,z,z^\perp) = -\lambda_z \cdot z - f(y)|z|^2$. After solving the uniqueness solution of this BSDE, we show that both Problem 2 and Problem 3 are equal to the pricing principle induced by Tsallis relative entropy (see Theorem 3.1). To the best of our knowledge, these results are new, which is a complement and enrichment to the relationship between classical relative entropy and quadratic BSDE. Theorem 3.1 indicates
that the pricing principle is a time consistent and arbitrage-free pricing rule, and it is also compatible with risk neutral pricing principle for the attainable claims.

Finally, we study the properties of the pricing principle. The pricing principle processes lots of elegant properties, such as monotonicity, concavity, time consistency and so on. However, different from the relative entropy or exponential preference’s situation, the existence of the distortion parameter \( q \neq 1 \) renders this pricing principle does not satisfy the translation invariance or cash-additivity. Translation invariance is sometimes criticized by some scholars (for example, El Karoui and Ravanelli (2009), Han et al. (2021)), and our pricing principle sheds some lights on this direction.

We further investigate conditional certainty equivalent for the contingent claim under \( q \)-exponential utility, which is proved to be the robust representation by a family of equivalent measures (maybe not martingale measures), see Proposition 4.3. Proposition 4.3 states that our pricing principle is between minimal martingale measure pricing and conditional certainty equivalent pricing under \( q \)-exponential utility, which indicates it is a good choice among the arbitrage-free prices. Moreover, we consider the asymptotic behavior of the pricing principle when ambiguity aversion coefficient goes to zero or infinity.

1.3. Related literature

We end the introduction with some related literature. Another widely used method for valuation in incomplete markets is indifference valuation. Especially in exponential form, translation invariance can lead to analytical tractability as well as attractive properties. It is known to all that expected exponential utility indifference pricing, especially in a dynamic context, involves a quadratic BSDE and relative entropy theory. See Rouge and El Karoui (2000), Mania and Schweizer (2005), Barrieu and El Karoui (2003), Frei, Malamud and (2011), Henderson and Liang (2014) or Hu, Imkeller and Müller (2005). Quadratic BSDEs have been considered by Kobylanski (2000), Briand and Hu (2006, 2008), Xing and Žitković (2018) and so on.

This paper contributes to the literature on pricing the contingent claims in an incomplete market without using classical relative entropy. Recently, more and more non-relative entropy models have entered the vision of scholars. The papers closest to ours are Meyer-Gohde (2019), Ma and Tian (2021) and Maenhout, Vedolin and Xing (2021).

Meyer-Gohde (2019) derives a generalization of model uncertainty framework of Hansen and Sargent (2001), using Tsallis relative entropy, and finds calibrations that match detection error probabilities yield comparable asset pricing implications across models. Ma and Tian (2021) consider the generalized entropic risk measures, which is related to conditional certainty equivalent of \( q \)-exponential utility, see subsection 4.2. Maenhout, Vedolin and Xing (2021) apply Cressie-Read divergence (a general relative entropy which closes to Tsallis relative entropy) to understand portfolio choice and general equilibrium asset pricing.

Frittelli (2000b) constructs a theory of value based on agent’s preference and coherent with the no arbitrage principle. Leitner (2008) offers a convex monotonic pricing functional for non-attainable bounded contingent claims, defined as the convex conjugate of a generalized entropy penalty functional. Horst, Pirvu and Dos Reis (2010), Cherdito et al. (2016) and Kardaras, Xing and Žitković (2017) study the equilibrium pricing using the translation invariant preferences. Faidi, Matoussi and Mnif (2011) and Faidi, Matoussi and Mnif (2017) investigate the related stochastic control problem by relative entropy and \( \phi \)-divergences.
Laeven and Stadje (2014) and Calvia and Rosazza Gianin (2020) examine the dynamic risk measures and related BSDEs in the jump situation, and the readers can refer to Rosazza Gianin (2006), Jiang (2008) and Delbaen, Peng and Rosazza Gianin (2010) for the Brownian motion case.

The remainder of this paper is organized as follows. Section 2 introduces the definition of Tsallis relative entropy, and provides its integral representations. Section 3 presents our main results. The properties of the pricing principle are investigated in Section 4. Section 5 concludes the paper. All the proofs are relegated to Appendix.

2. Tsallis relative entropy and its integral representation

Similar to relative entropy, Tsallis relative entropy can also be interpreted as how far away \( Q \) is from the reference \( P \); see more details in Tsallis (2009). We will use Tsallis relative entropy to explore its potential implications in pricing principle. This section introduces the definition of Tsallis relative entropy and investigates its integral representation.

Let’s first introduce the \( q \)-exponential function and its inverse, the \( q \)-logarithm function, which are defined respectively as follows

\[
\exp_q(x) := \begin{cases} 
[1 + (1 - q)x]^{1/q}, & x \geq -\frac{1}{1-q} \text{ and } 0 < q < 1, \\
[1 + (1 - q)x]^{-1/q}, & x < -\frac{1}{1-q} \text{ and } q > 1,
\end{cases}
\]

and

\[
\ln_q(x) := \begin{cases} 
x^{1-q-1}, & x \geq 0 \text{ and } 0 < q < 1, \\
x^{1/q-1} & x > 0 \text{ and } q > 1.
\end{cases}
\]

We always assume that \( q > 0 \) and \( q \neq 1 \), and \( \exp_q(\cdot) \) and \( \ln_q(\cdot) \) are well defined. \( \text{Dom}(\ln_q) \) and \( \text{Dom}(\exp_q) \) represent the domains for \( \ln_q(\cdot) \) and \( \exp_q(\cdot) \) respectively. One can easily get that if \( q \rightarrow 1 \), then \( \ln_q(x) \rightarrow \ln(x), x > 0, \) and \( \exp_q(x) \rightarrow \exp(x), x \in \mathbb{R} \).

For any two probability measures \( Q \) and \( P \) on \( (\Omega, \mathcal{F}_T) \), Tsallis (1988, 2009) introduces the \( q \)-generalization of the relative entropy (also called Tsallis relative entropy) as follows:

\[
H_q(Q|P) := \left\{ \begin{array}{ll}
\int (\frac{dQ}{dP})^q \ln_q(\frac{dQ}{dP})dP, & Q \ll P, \\
+\infty, & \text{others},
\end{array} \right.
\]

where \( q > 0 \) and \( q \neq 1 \).

Obviously, \( H_q(Q|P) \) is well-defined when \( 0 < q < 1 \). For \( q > 1 \), if \( Q \ll P \), then \( Q(\{\frac{dQ}{dP} = 0\}) = \int_{\{\frac{dQ}{dP} = 0\}} \frac{dQ}{dP}dP = 0 \). Hence, \( H_q(Q|P) = \int (\frac{dQ}{dP})^{q-1} \ln_q(\frac{dQ}{dP})dQ \) is also well-defined.

Similarly, if \( Q \ll P \) on \( \mathcal{F}_T \), for each \( t \in [0, T] \),

\[
H_{q,t}(Q|P) := \mathbb{E}_P \left[ (D_{t,T}^{Q,P})^q \ln_q D_{t,T}^{Q,P} \mid \mathcal{F}_t \right],
\]

denotes the conditional Tsallis relative entropy of \( Q \) and \( P \) under \( \mathcal{F}_t \).

In the following, we are going to establish the integral representations for the (conditional) Tsallis relative entropy.

**Theorem 2.1.** Let \( q > 0 \) and \( q \neq 1 \). For any given \( Q \in \mathcal{M} \), suppose \( H_q(Q|P) < +\infty \), then the following results hold.
(i) For each $t \in [0, T]$, we have

$$H_{q,t}(Q|P) = \frac{1}{(D_t^q)^q} \left( \mathbb{E}_P \left[ (D_t^q)^q \ln_q D_t^q \mid \mathcal{F}_t \right] - (D_t^q)^q \ln_q D_t^q \right)$$

and

$$H_{q,t}(Q|P) = \frac{q}{2} \mathbb{E}_P \left[ \int_t^T (D_s^q)^q (|\lambda_s|^2 + |\alpha_s^Q|^2) ds \mid \mathcal{F}_t \right],$$

where $\alpha^Q$ is determined by (1.1).

(ii) In particular, we have that

$$H_q(Q|P) = \frac{q}{2} \mathbb{E}_P \left[ \int_0^T (D_s^q)^q (|\lambda_s|^2 + |\alpha_s^Q|^2) ds \right].$$

Besides, we get

$$H_q(Q_{\text{min}}|P) = \frac{q}{2} \mathbb{E}_{Q_{\text{min}}} \left[ \int_0^T (D_s^q)^q |\lambda_s|^2 ds \right].$$

**Corollary 2.1.** Let $q > 0$ and $q \neq 1$. For any $Q \in \mathcal{M}$, if $H_q(Q|Q_{\text{min}}) < +\infty$, then, for each $t \in [0, T]$, we have

$$H_{q,t}(Q|Q_{\text{min}}) = \frac{q}{2} \mathbb{E}_{Q_{\text{min}}} \left[ \int_t^T (D_s^q)^q \cdot |\alpha_s^Q|^2 ds \mid \mathcal{F}_t \right],$$

and in particular,

$$H_q(Q|Q_{\text{min}}) = \frac{q}{2} \mathbb{E}_{Q_{\text{min}}} \left[ \int_0^T (D_s^q)^q \cdot |\alpha_s^Q|^2 ds \right],$$

where $\alpha^Q$ is determined by (1.1).

**Remark 2.1.** The readers can refer to [Choulli and Stricker (2003)] and [Skidas (2003)] for the integral representation results of the classical relative entropy. When $q = 1$, our representation results are exactly the corresponding formulations of classical relative entropy.

### 3. Main results

Before introducing our main results, let’s introduce some notations. Suppose $q > 0$, $q \neq 1$, and the aversion coefficient $\gamma > 0$. We denote by $\mathcal{L}_q^\gamma(\mathcal{F}_T)$ for the set of all random variables on $(\Omega, \mathcal{F}_T)$ such that $\exp_q(-\gamma \xi) \in L^1(P)$ and $\mathbb{E}_P[\exp_q(-\gamma \xi)] \in \text{Dom}(\ln_q)$. Define

$$\mathcal{L}_q^\gamma(\mathcal{F}_T, b) := \{ \xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T) \mid \exists m_1, m_2 \in \mathbb{R}, \text{ s.t. } 0 < m_1 \leq \exp_q(-\gamma \xi) \leq m_2, \ P-a.s. \}.$$  

Let $(\theta_t)_{t \in [0, T]}$ be a $\mathbb{R}^n$-valued locally square integrable predictable process on $(\Omega, \mathcal{F}_T, P)$. For each $t \in [0, T]$, define $M_t := \int_0^t \theta_s \cdot dW_s$. If the process $M$ is a BMO$(P)$ martingale, then
the stochastic exponential $\mathcal{E}(M)$ is a uniformly integrable $\mathbb{P}$-martingale by Theorem 2.3 in Kazamaki (1994). Then we can define an equivalent martingale measure $\mathbb{Q}^\theta$ in $\mathcal{M}$ as follows:

$$
\frac{d\mathbb{Q}^\theta}{d\mathbb{P}} : = \mathcal{E}(-\lambda \cdot W + \theta \cdot W^\perp)_{T}
$$

$$
= \exp \left( - \int_0^T \lambda_s \cdot dW_s + \int_0^T \theta_s \cdot dW^\perp_s - \frac{1}{2} \int_0^T (|\lambda_s|^2 + |\theta_s|^2)ds \right).
$$

Denote $\Theta$ by the set of all $\mathbb{R}^n$-valued predictable processes $(\theta_t)_{t\in[0,T]}$ such that $\int_0^T \theta_s \cdot dW^\perp_s$ is a BMO($\mathbb{P}$) martingale and the following backward equation

$$
Y^\theta_t = \mathbb{E}_{\mathbb{Q}^\theta} \left[ \xi + \int_t^T \mu(Y^\theta_s) |\theta_s|^2 ds \bigg| \mathcal{F}_t \right], \quad t \in [0,T], \quad \xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b),
$$

(3.1)

has a unique bounded solution $Y^\theta$ with $\mathbb{E}_{\mathbb{Q}^\theta} \left[ \int_0^T |\mu(Y_s^\theta) \cdot |\theta_s|^2 ds \right] < +\infty$, where

$$
\mu(x) := \frac{1 - (1 - q) \gamma x}{q} = \frac{1}{q} \left( \exp_q(-\gamma x) \right)^{1-q}.
$$

The main result of the paper is the connection among the following three problems.

- **Problem 1**: For any contingent claim $\xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b)$, pricing principle is defined by

$$
F_t(\xi) := \text{ess inf}_{Q \in \mathcal{M}_f} \left( \mathbb{E}_{Q^\text{min}} \left[ (D_{i,x}^Q)^q \xi | \mathcal{F}_t \right] + \frac{1}{\gamma} H_{q,t}(Q | Q^\text{min}) \right), \quad t \in [0,T],
$$

(3.2)

where

$$
\mathcal{M}_f := \{ Q \in \mathcal{M} | H_{q}(Q | Q^\text{min}) < +\infty \}.
$$

- **Problem 2**: For any contingent claim $\xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b)$,

$$
\tilde{F}_t(\xi) := \text{ess inf}_{\theta \in \Theta} Y^\theta_t, \quad t \in [0,T],
$$

(3.3)

where $Y^\theta$ is the solution of (3.1).

- **Problem 3**: Consider the following BSDE: for each $\xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b),$

$$
\begin{cases}
    dY_s = \left( \lambda_s \cdot Z_s + \frac{3}{2} \cdot \frac{|Z^\perp_s|^2}{\mu(Y_s)} \right) ds + Z_s \cdot dW_s + Z^\perp_s \cdot dW^\perp_s, & s \in [0,T], \\
    Y_T = \xi.
\end{cases}
$$

(3.4)

**Theorem 3.1.** Suppose $\gamma > 0, q > 0$ and $q \neq 1$. For each $\xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b)$, we have

$$
F_t(\xi) = \tilde{F}_t(\xi) = Y_t(\xi), \quad \forall t \in [0,T],
$$

(3.5)

where $Y_t(\xi)$ is the solution of BSDE (3.4). Moreover, the essential infima in (3.2) and (3.3) can be achieved.
Our objective is to provide a pricing principle for the non-attainable contingent claim in incomplete market. Problem 1 is the starting point and motivation for our study in the introduction. This pricing rule is fascinating and attractive. The buyer evaluates the contingent claim $\xi$ under the “distorted Radon-Nikodym derivative” and adjustment by conditional Tsallis relative entropy over a family of equivalent martingale measures.

Specifically, the influence of distortion parameter can be understood in the following ways. When $0 < q < 1$, it enhances the small values of Radon-Nikodym derivative and reduces the larger values. However, when $q$ is greater than one, the opposite is true. Hence, we can interpret $q$ as a measure of agent’s pessimism.

To solve the Problem 1, we introduce the Problem 2 and Problem 3. Problem 2 is a dual formulation, where the value $Y^\theta$ has a recursive representation. The buyer assesses the contingent claim $\xi$ under a martingale measure $Q^\theta$, corrects it by the backward equation (3.1), and finally takes the worst case under a family of equivalent martingale measures $\mathcal{M}^\theta$, which is defined by

$$\mathcal{M}^\theta = \{Q^\theta \in \mathcal{M} \mid \theta \in \Theta\}.$$  

Then the pricing problem (3.3) becomes an optimization problem of BSD Es. The dynamic formulation in BSDE (3.4) of Problem 3 is important, because it is not clear that $F_t(\xi)$ defined in (3.2) is time consistent, due to the time subscript $t$ in $D^0_{t,T}$ and the distortion parameter $q$. Similarly, we cannot easily obtain that $\tilde{F}(\xi)$ is dynamically consistent because it is also an optimization problem for a series of recursions. The optimal value process is described as the solution of the BSDE (3.4).

The kind of BSDEs with generators $f(y)|z|^2$ are firstly investigated by Bahlali, Eddahbi and Ouksnine (2017), and subsequently studied by Bahlali and Tangpi (2018), Bahlali (2020) and Zheng, Zhang and Feng (2021). The existence of solution to BSDE (3.4) can be obtained by the existing literature, while the uniqueness part of BSDE (3.4) is not available in the current literature. Since the generator of BSDE (3.4) is concave in $(y, z^\perp)$, motivated by the $\theta$-method in Briand and Hu (2008) dealing with the convex generator, we can derive the uniqueness result by some subtle transformations. We give it in the following proposition.

**Proposition 3.1.** Suppose $\gamma > 0$, $q > 0$ and $q \neq 1$. For any $\xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b)$, then BSDE (3.4) admits a unique solution $(Y, Z) = (Y, Z, Z^\perp)$ in which $\int_0^T Z_s \cdot d\tilde{W}_s$ is a BMO($\mathbb{P}$) martingale, and $Y$ is continuous and bounded, specifically, for each $t \in [0, T]$, $Y_t \in \mathcal{L}_q^\gamma(\mathcal{F}_t, b)$.

On the one hand, by Proposition 3.1 and Theorem 3.1, for each $t \in [0, T]$, the pricing principle is a mapping that $F_t : \mathcal{L}_q^\gamma(\mathcal{F}_T, b) \to \mathcal{L}_q^\gamma(\mathcal{F}_t, b)$ by

$$F_t(\xi) = Y_t(\xi) = Y_t, \quad \xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b),$$

where $Y_t$ is the solution for BSDE (3.4). In particular, $F(\xi) := F_0(\xi) = Y_0(\xi) = Y_0$. We often omit $\xi$ when there is no ambiguity. BSDE formulation (3.4) and the result in (3.5) indicate that $F(\xi)$ is a time consistent pricing rule.

On the other hand, for any contingent claim $\xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b)$, the solution of BSDE (3.4) satisfies the fact that $\int_0^T Z_s \cdot d\tilde{W}_s$ is a BMO($\mathbb{P}$) martingale and $Y_t \in \mathcal{L}_q^\gamma(\mathcal{F}_t, b)$, $\forall t \in [0, T]$. These results imply that $\mu(Y)$ is bounded (strict greater than a positive constant) and $\int_0^T Z_s^\perp \cdot$.  

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$dW_s$ is a BMO($\mathbb{P}$) martingale, then $Y_s$ is a martingale under the equivalent martingale measure $Q^\xi$ in $\mathcal{M}$, i.e.,

$$Y_t(\xi) = Y_t = \mathbb{E}_{Q^\xi}[\xi | \mathcal{F}_t], \quad 0 \leq t \leq T,$$

where

$$\frac{dQ^\xi}{d\mathbb{P}} = D_T^{Q^\xi, \mathbb{P}} = \mathcal{E}(-\lambda \cdot W - \frac{\gamma Z}{2\mu(Y)} \cdot W),$$

Hence, the pricing principle $F$ for the contingent claim $\xi$ is an arbitrage-free pricing rule.

This pricing principle, connected with the above BSDEs, is very interesting and exciting. The pricing mechanism is nonlinear yielding the price in terms of a conditional nonlinear expectation. Since BSDE's generator involves $y$, the pricing principle will not have cash additivity or translation invariance property generally. Translation invariance has always been controversial (for example, El Karoui and Ravanelli (2009), Han et al. (2021)), and our pricing principle will shed some lights on this direction.

We end this section with some useful remarks.

**Remark 3.1.** The pricing principle (3.2) is compatible with risk neutral pricing principle for the attainable contingent claim. It is also related to the conditional certainty equivalent of some utility function when the contingent claim is completely unhedged.

- If the contingent claim is attainable, i.e., $\xi \in \mathcal{L}_q(\mathcal{F}_T^W, b)$, then $Z = 0$, and

$$F_t(\xi) = \mathbb{E}_{Q^\min} [\xi | \mathcal{F}_t^W], \quad t \in [0, T].$$

Then this pricing functional is compatible with risk neutral pricing principle.

- If the contingent claim is completely unhedged, i.e., $\xi \in \mathcal{L}_q(\mathcal{F}_T^{W^\perp}, b)$, then $Z = 0$, and

$$F_t(\xi) = -\frac{1}{\gamma} \ln_q \mathbb{E}_{Q^\min} [\exp_q(-\gamma \xi) | \mathcal{F}_t^{W^\perp}]$$

$$= U^{-1}(\mathbb{E}_{Q^\min} [U(\xi) | \mathcal{F}_t^{W^\perp}])), \quad t \in [0, T],$$

where $U$ is the $q$-exponential utility, i.e. $U(x) := 1 - \exp_q(-\gamma x)$. In this situation, $F_t(\xi)$ equals to the conditional certainty equivalent under $q$-exponential utility. This result will be expanded further in subsection 4.2.

- For the general contingent claim $\xi \in \mathcal{L}_q(\mathcal{F}_T, b)$, we will prove that

$$U^{-1}(\mathbb{E}_{Q^\min} [U(\xi) | \mathcal{F}_t]) \leq F_t(\xi) \leq \mathbb{E}_{Q^\min}[\xi | \mathcal{F}_t], \quad t \in [0, T],$$

where $U$ is the $q$-exponential utility, see Proposition 4.4.

**Remark 3.2.** When $q = 1$ and the contingent claim is completely unhedged, i.e., $\xi \in L^\infty(\mathcal{F}_T^{W^\perp})$, by Proposition 6.4 in Barrieu and El Karoui (2009), we have known that this pricing functional is the conditional certainty equivalent of exponential utility under the minimal martingale measure, i.e.,

$$F_t(\xi) = -\frac{1}{\gamma} \ln \mathbb{E}_{Q^\min} [\exp(-\gamma \xi) | \mathcal{F}_t^{W^\perp}], \quad t \in [0, T].$$
Remark 3.3. For each contingent claim $\xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b)$, the essential infimum in (3.2) can be achieved at the equivalent martingale measure $Q^* \in \mathcal{M}_f$, i.e.,

$$dQ^* = \mathcal{E}(-\lambda \cdot W + \alpha Q^* \cdot W^\perp)_T,$$

where $\alpha Q^* = -\frac{\gamma Z^\perp}{\mu(Y)}$. While the essential infimum in (3.3) can be obtained by

$$\theta^* = -\frac{\gamma Z^\perp}{\mu(Y)}.$$

The readers can find these results in the proofs process of Theorem 3.1, see Lemma 6.2 and Lemma 6.3 in Appendix 6.2.

Remark 3.4. For each $t \in [0, T]$, the pricing principle $F_t(\xi)$ is the price to buy the contingent claim $\xi$ for DM. Similarly, we can also define the price to sell the claim $\xi \in \mathcal{L}_q^{-\gamma}(\mathcal{F}_T, b)$ by

$$-F_t(-\xi) := \text{ess sup}_{Q \in \mathcal{M}_f}\left(\mathbb{E}_{Q^{\text{min}}}[\left(D_t^{Q_{\text{min}}}\right)^\gamma \xi | \mathcal{F}_t] - \frac{1}{\gamma}H_{q,t}(Q|Q^{\text{min}})\right).$$

(3.8)

In this paper, we focus on the buyer’s price $F$.

4. Some properties of the pricing principle

The goal of this section is to study the properties for the pricing principle $F$. In subsection 4.1, we introduce the basic properties of the pricing rule. We discuss its relationship with conditional certainty equivalent under $q$-exponential utility in subsection 4.2. Subsection 4.3 considers the asymptotic behavior of the pricing principle when the aversion coefficient goes to zero or infinity.

4.1. Basic properties

We claim that the pricing principle satisfies normalization, monotonicity, concavity, time consistency, and cash subadditivity or superadditivity and so on.

Proposition 4.1. Let $\gamma > 0$, $q > 0$ and $q \neq 1$. For each $t \in [0, T]$, the pricing principle $F_t : \mathcal{L}_q^\gamma(\mathcal{F}_T, b) \to \mathcal{L}_q^\gamma(\mathcal{F}_T, b)$, defined in (3.2), satisfies the following properties:

(i) Normalization: $F_t(0) = 0$.

(ii) Monotonicity: For $\xi, \eta \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b)$ with $\xi \geq \eta$, then $F_t(\xi) \geq F_t(\eta)$.

(iii) Concavity: If $\kappa \xi + (1 - \kappa)\eta \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b)$ for all $\kappa \in [0, 1]$, then

$$F_t(\kappa \xi + (1 - \kappa)\eta) \geq \kappa F_t(\xi) + (1 - \kappa)F_t(\eta).$$

(iv) Quasi-concavity: If $\kappa \xi + (1 - \kappa)\eta \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b)$ for all $\kappa \in [0, 1]$, then

$$F_t(\kappa \xi + (1 - \kappa)\eta) \geq \min\{F_t(\xi), F_t(\eta)\}.$$
(v) Scaling property: Let $\kappa \xi, \xi \in L_q^r(\mathcal{F}_T, b)$. Then $F_t(\kappa \xi) \geq \kappa F_t(\xi)$ if $\kappa \in [0, 1]$; $F_t(\kappa \xi) \leq \kappa F_t(\xi)$ if $\kappa \geq 1$.

(vi) Time consistency: For $\xi \in L_q^r(\mathcal{F}_T, b)$, then

$$F_s(\xi) = F_s(F_t(\xi)), \quad 0 \leq s \leq t \leq T.$$

**Proof.** (i)-(iii) can be easily verified by the definition in (3.2). (iv) and (v) follow directly from (iii) and (i). (vi) uses the results in Theorem 3.1. □

**Proposition 4.2.** Let $\gamma > 0$, $q > 0$ and $q \neq 1$. For each $t \in [0, T]$, given the pricing principle $F_t : L_q^r(\mathcal{F}_T, b) \rightarrow L_q^r(\mathcal{F}_T, b)$, defined in (3.2). Let $c$ be a constant, and $\xi, \xi + c \in L_q^r(\mathcal{F}_T, b)$. Then we have that

(i) Cash superadditivity: If $0 < q < 1$, $c \leq 0$ or $q > 1$, $c \geq 0$, then

$$F_t(\xi + c) \geq F_t(\xi) + c.$$

(ii) Cash subadditivity: If $0 < q < 1$, $c \geq 0$ or $q > 1$, $c \leq 0$, then

$$F_t(\xi + c) \leq F_t(\xi) + c.$$

**Proof.** Case (i): If $0 < q < 1$, $c \leq 0$, by the definition of the pricing principle, it implies

$$F_t(\xi + c) = \text{ess inf}_{Q \in \mathcal{M}_f} \left( \mathbb{E}_{Q^\text{min}} \left[ (D_{t,T}^{Q^\text{min}})^q (\xi + c) \mid \mathcal{F}_t \right] + \frac{1}{\gamma} H_{q,t}(Q \mid Q^\text{min}) \right)$$

$$\geq F_t(\xi) + \text{ess inf}_{Q \in \mathcal{M}_f} \left( \mathbb{E}_{Q^\text{min}} \left[ c \cdot (D_{t,T}^{Q^\text{min}})^q \mid \mathcal{F}_t \right] \right)$$

$$\geq F_t(\xi) + \text{ess inf}_{Q \in \mathcal{M}_f} \left( \mathbb{E}_{Q^\text{min}} \left[ D_{t,T}^{Q^\text{min}} \mid \mathcal{F}_t \right] \right)^q$$

$$= F_t(\xi) + c,$$

where we used the Jensen’s inequality. When $q > 1$ and $c \geq 0$, the proof is similar.

Case (ii): If $0 < q < 1$, $c \geq 0$, we get

$$F_t(\xi + c) \leq F_t(\xi) + \text{ess sup}_{Q \in \mathcal{M}_f} \left( \mathbb{E}_{Q^\text{min}} \left[ c \cdot (D_{t,T}^{Q^\text{min}})^q \mid \mathcal{F}_t \right] \right)$$

$$\leq F_t(\xi) + \text{ess sup}_{Q \in \mathcal{M}_f} \left( \mathbb{E}_{Q^\text{min}} \left[ D_{t,T}^{Q^\text{min}} \mid \mathcal{F}_t \right] \right)^q$$

$$= F_t(\xi) + c.$$

When $q > 1$ and $c \leq 0$, the proof is similar. □

**Remark 4.1.** Delbaen, Penq and Rosazza Gianin (2010) study dynamic convex risk measure or utility process, satisfying cash additivity, and they find that it is related to a kind of BSDEs with generators are independent from $y$. Cash or constant additivity has always been a controversial property for contingent claim pricing in academia. In order to model stochastic or ambiguous interest rates, El Karoui and Ravanelli (2009) present a kind of the theory, and it turns out to be related to a specific BSDE involving $y$. From the results of Propositions 4.1 and 4.2, pricing principle $F$ has very good properties and is a suitable choice for contingent claim pricing.
4.2. Conditional certainty equivalent under q-exponential utility

Let $\gamma > 0$, $q > 0$ and $q \neq 1$. Consider the following function:

$$U(x) := 1 - \exp_q(-\gamma x), \ x \in \text{Dom}(U),$$

which is called $q$-exponential utility. Obviously, $U$ is a strictly increasing and concave continuous function defined on $\text{Dom}(U)$.

For each $t \in [0, T]$, we consider the conditional certainty equivalent of the law of $\xi$ under minimal martingale measure $Q^{\min}$ by setting

$$CE_q(\xi|F_t) := U^{-1}(E_{Q^{\min}}[U(\xi)|F_t]), \ \forall \xi \in L^\gamma_q(F_T, b).$$

Directly calculation results in the following

$$CE_q(\xi|F_t) = -\frac{1}{\gamma} \ln_q E_{Q^{\min}}[\exp_q(-\gamma \xi)|F_t], \ \forall \xi \in L^\gamma_q(F_T, b). \quad (4.1)$$

When $t = 0$, we denote $CE_q(\xi|F_0)$ by $CE_q(\xi)$.

By Theorem 4.2 in Ma and Tian (2021) or Theorem 3.2 in Zheng, Zhang and Feng (2021), we can easily obtain that for each contingent claim $\xi \in L^\gamma_q(F_T, b)$, then

$$CE_q(\xi|F_t) = y_t, \ \forall t \in [0, T], \quad (4.2)$$

where $y$ is the unique solution to the following BSDE

$$\begin{cases}
  dy_s = \left(\lambda_s \cdot z_s + \frac{\gamma}{2} \cdot \frac{|z_s|^2 + |z^\perp|^2}{\mu(y_s)}\right) ds + z_s \cdot dW_s + z^\perp_s \cdot dW^\perp_s, & s \in [0, T], \\
y_T = \xi.
\end{cases} \quad (4.3)$$

Comparing with BSDE (3.4), BSDE (4.3) is easier to solve and it has an explicit solution characterized by conditional certainty equivalent of $q$-exponential utility. BSDE (4.3) admits a unique solution $(y, z) = (y, z, z^\perp)$ in which $y$ is continuous and bounded, and $\int_0^T \pi_s \cdot d\tilde{W}^\min_s$ is a BMO($Q^{\min}$) martingale, where $\tilde{W}^\min = (W^{-\lambda}, W^\perp)$ is a Brownian motion under $Q^{\min}$, and $W^{-\lambda} = W + \int_0^\cdot \lambda_s ds$.

**Remark 4.2.** Ma and Tian (2021) prove that the solution of BSDE (4.3) is conditional certainty equivalent of $q$-exponential utility, however, they do not investigate its representation and not explore its potential applications in pricing theory.

In general, conditional certainty equivalent of $q$-exponential utility is not compatible with the risk neutral pricing for attainable claims. In fact,

$$CE_q(\xi|F_t) = E_{\tilde{Q}^\xi} [\xi | F_t], \ \forall t \in [0, T],$$

where

$$\frac{d\tilde{Q}^\xi}{d\tilde{P}} := D_{T}^{\tilde{Q}^\xi, \tilde{P}} = \mathcal{E}(-\lambda \cdot W - \frac{\gamma z}{2\mu(y)} \cdot W - \frac{\gamma z^\perp}{2\mu(y)} \cdot W^\perp)_T.$$

The equivalent measure $\tilde{Q}^\xi$ is not an equivalent martingale measure if $z \neq 0$. For example, for some $\xi$ in $L^\gamma_q(F^W_T, b)$. 

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When the contingent claim is completely unhedged, i.e., \( \xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b) \), then \( Z = z = 0 \), where \( Z \) and \( z \) are part of solutions to BSDEs \( (3.6) \) and \( (4.3) \) respectively. In this case, we get \( (3.6) \) holds, i.e.,

\[
F_t(\xi) = CE_q(\xi|\mathcal{F}_t), \quad \forall t \in [0, T].
\]

However, they are not equal to each other generally.

Define

\[
Q_f := \{ Q \in \mathcal{Q} \mid H_q(Q|Q^{\min}) < +\infty \}.
\]

The following Proposition 4.3 further reveals the differences and connections between pricing principles \( F_t(\xi) \) and \( CE_q(\xi|\mathcal{F}_t) \).

**Proposition 4.3.** Suppose \( \gamma > 0 \), \( q > 0 \) and \( q \neq 1 \). For any \( \xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b) \) and \( t \in [0, T] \), then we have

\[
CE_q(\xi|\mathcal{F}_t) = \operatorname{ess inf}_{Q \in Q_f} \mathbb{E}_{Q^{\min}} \left[ (D_{t,T}^{Q,Q^{\min}})^q \xi + \frac{q}{2\gamma} \int_t^T (D_{t,s}^{Q,Q^{\min}})^q (|\beta_s^Q|^2 + |\alpha_s^Q|^2) ds \mid \mathcal{F}_t \right],
\]

where \( \beta^Q \) and \( \alpha^Q \) are determined by \( D^{Q,Q^{\min}} = \mathcal{E}(\beta^Q.W^{-\lambda} + \alpha^Q.W^\lambda) \). Moreover, the essential infimum can be attained. In particular,

\[
CE_q(\xi) = \inf_{Q \in Q_f} \left( \mathbb{E}_{Q^{\min}}[\langle (D_{T,T}^{Q,Q^{\min}})^q \xi \rangle + \frac{1}{\gamma} H_q(Q|Q^{\min})] \right). \tag{4.7}
\]

By comparing Proposition 4.3 with \( (3.2) \), we find that conditional certainty equivalent pricing is a robust representation over a family of equivalent measures in \( Q_f \), while \( F \) is a robust representation over a family of equivalent martingale measures in \( \mathcal{M}_f \). Combining \( (3.2) \) and \( (4.6) \) together, we can finally obtain the following results.

**Proposition 4.4.** Let \( \gamma > 0 \), \( q > 0 \) and \( q \neq 1 \). For each \( t \in [0, T] \) and \( \xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b) \), then it implies

\[
CE_q(\xi|\mathcal{F}_t) \leq F_t(\xi) \leq \mathbb{E}_{Q^{\min}}[\xi|\mathcal{F}_t]. \tag{4.8}
\]

Proposition 4.4 indicates that our pricing rule \( F \) is between minimal martingale measure pricing and conditional certainty equivalent pricing under \( q \)-exponential utility function.

**Remark 4.3.** The sell price of the contingent claim \( \xi \) can be studied similarly. In fact, for each \( t \in [0, T] \) and \( \xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b) \cap \mathcal{L}_q^{-\gamma}(\mathcal{F}_T, b) \), then

\[
CE_q(\xi|\mathcal{F}_t) \leq F_t(\xi) \leq \mathbb{E}_{Q^{\min}}[\xi|\mathcal{F}_t] \leq -F_t(-\xi) \leq -CE_q(-\xi|\mathcal{F}_t). \tag{4.9}
\]

Specifically, \( [F_t(\xi), -F_t(-\xi)] \) is the arbitrage-free pricing interval for contingent claim \( \xi \).
4.3. The effects of ambiguity aversion coefficient

In order to highlight the influence of the aversion parameter \( \gamma \), we denote \( F_t(\xi) \) by \( F_t(\xi, \gamma) \) for any \( \xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b) \) in this subsection.

**Proposition 4.5.** Let \( \gamma > 0, \ q > 0 \) and \( q \neq 1 \). For each \( t \in [0, T] \), the pricing principle \( F_t(\xi, \gamma) \) has the following properties.

(i) It is decreasing in \( \gamma \), i.e., if \( \gamma \leq \gamma' \) and \( \xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b) \cap \mathcal{L}_q^{\gamma'}(\mathcal{F}_T, b) \), then we have

\[
F_t(\xi, \gamma) \leq F_t(\xi, \gamma').
\]

(ii) For any \( \kappa > 0 \), if \( \kappa \xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b) \), then

\[
F_t(\kappa \xi, \gamma) = \kappa F_t(\xi, \kappa \gamma).
\]

**Proof.** For any \( \kappa > 0 \), if \( \kappa \xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b) \), then it implies that \( \xi \in \mathcal{L}_q^{\kappa \gamma}(\mathcal{F}_T, b) \). Thus, the results are immediately consequences of the definition of the pricing principle \( F_t(\xi, \gamma) \).

The readers can refer to Rouge and El Karoui (2000) and Mania and Schweizer (2005), and these authors also give the asymptotic results for large and small aversion coefficients of indifference pricing theory in exponential utility and relative entropy framework.

The following proposition is the asymptotic results of our pricing principle.

**Proposition 4.6.** Let \( q > 0 \) and \( q \neq 1 \). For each \( t \in [0, T] \) and \( \xi \in \bigcap_{\gamma > 0} \mathcal{L}_q^\gamma(\mathcal{F}_T, b) \), then we have

\[
\lim_{\gamma \to +\infty} F_t(\xi, \gamma) = \operatorname{ess inf}_{Q \in \mathcal{M}_f} \mathbb{E}_{Q^{\text{min}}} \left[ (D_{t,T}^{Q^{\text{min}}})^q \xi \mid \mathcal{F}_t \right], \quad \mathbb{P} - \text{a.s.}
\]

and

\[
\lim_{\gamma \to 0} F_t(\xi, \gamma) = \mathbb{E}_{Q^{\text{min}}} [\xi \mid \mathcal{F}_t], \quad \mathbb{P} - \text{a.s.}
\]

Loosely speaking, the interpretation of Proposition 4.6 is that, in the small risk aversion limit, our pricing principle converges to minimal martingale measure pricing; in the large ambiguity aversion limit, our pricing principle converges to the “distorted Radon-Nikodym derivative” pricing over a family of equivalent martingale measures \( \mathcal{M}_f \). By Proposition 4.6, we can immediately draw the following corollary.

**Corollary 4.1.** Let \( \gamma > 0, \ q > 0 \) and \( q \neq 1 \). For each \( t \in [0, T] \) and \( \xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b) \), then

\[
\operatorname{ess inf}_{Q \in \mathcal{M}_f} \mathbb{E}_{Q^{\text{min}}} \left[ (D_{t,T}^{Q^{\text{min}}})^q \xi \mid \mathcal{F}_t \right] \leq F_t(\xi, \gamma) \leq \mathbb{E}_{Q^{\text{min}}} [\xi \mid \mathcal{F}_t].
\]
5. Conclusions

The paper introduces a pricing principle for contingent claims in incomplete markets via Tsallis relative entropy theory. The agent evaluates the contingent claim under the “distorted Radon-Nikodym derivative” and adjustment by Tsallis relative entropy over a family of equivalent martingale measures. In order to investigate the pricing principle, we introduce two equivalent problems of BSDEs. Fortunately, similar to connections of quadratic BSDEs and equivalent martingale measures. In order to investigate the pricing principle, we introduce two Radon-Nikodym derivative and adjustment by Tsallis relative entropy over a family of Tsallis relative entropy theory. The agent evaluates the contingent claim under the “distorted Lemma 6.1.

5. Conclusions

The paper introduces a pricing principle for contingent claims in incomplete markets via Tsallis relative entropy theory. The agent evaluates the contingent claim under the “distorted Radon-Nikodym derivative” and adjustment by Tsallis relative entropy over a family of equivalent martingale measures. In order to investigate the pricing principle, we introduce two equivalent problems of BSDEs. Fortunately, similar to connections of quadratic BSDEs and equivalent martingale measures. In order to investigate the pricing principle, we introduce two Radon-Nikodym derivative and adjustment by Tsallis relative entropy over a family of Tsallis relative entropy theory. The agent evaluates the contingent claim under the “distorted

The pricing principle processes lots of elegant properties, such as monotonicity, time consistency, concavity and so on. It is also compatible with risk neutral pricing principle when contingent claim is attainable. Our pricing principle sheds some lights on arbitrage-consistency, concavity and so on. It is also compatible with risk neutral pricing principle when ambiguity aversion coefficient goes to zero or infinity.

6. Appendix: The Proofs

In this appendix, we will give the proofs of the previous theorems and propositions.

6.1. Proof of Theorem 2.1

Lemma 6.1. Let \( q > 0 \) and \( q \neq 1 \). For \( Q \ll P \), \( D_{t,T}^{Q,P} = E_F[D_{t,T}^{Q,P} | F_t] \), suppose that \( H_q(Q|P) < +\infty \). Then \( (D_{t,T}^{Q,P})^q \ln q D_{t,T}^{Q,P} \) is a \( P \)-submartingale on \([0, T]\). Furthermore, \( (D_{t,T}^{Q,P})^q \ln q D_{t,T}^{Q,P} \) is a uniformly integrable \( P \)-submartingale.

Proof. Since the process \( D_{t,T}^{Q,P} \) is a martingale and \( f(x) = x^q \ln q (x), x \in \text{Dom}(\ln q) \), is convex and bounded from below, then Jensen’s inequality yields, for \( 0 \leq t \leq T \),

\[
E_P[f(D_{t,T}^{Q,P}) | F_t] \geq f(E_P[D_{t,T}^{Q,P} | F_t]) = f(D_{t,T}^{Q,P}).
\]

Since \( H_q(Q|P) = E_F[(D_{t,T}^{Q,P})^q \ln q D_{t,T}^{Q,P}] < +\infty \), then \( (D_{t,T}^{Q,P})^q \ln q D_{t,T}^{Q,P} \) is a uniformly integrable \( P \)-submartingale on \([0, T]\).

Proof of Theorem 2.1. Due to the fact that

\[
H_q(Q|P) = E_{F}[(D_{t,T}^{Q,P})^q \ln q D_{t,T}^{Q,P}] = \frac{1}{1-q} = \frac{1}{1-q} < +\infty,
\]

we have \( D_{t,T}^{Q,P} \in L^{1\ln q}(P) \). Let’s prove (2.3). Indeed,

\[
H_{q,t}(Q|P) = E_{F}\left[ (D_{t,T}^{Q,P})^q \ln q D_{t,T}^{Q,P} | F_t \right] = E_{F}\left[ \frac{D_{t,T}^{Q,P} - (D_{t,T}^{Q,P})^q}{1-q} | F_t \right]
\]

\[
= \frac{1}{(D_{t,T}^{Q,P})^q} E_{F}\left[ \frac{(D_{t,T}^{Q,P})^q - (D_{t,T}^{Q,P})^q}{1-q} | F_t \right]
\]

\[
= \frac{1}{(D_{t,T}^{Q,P})^q} E_{F}\left[ \left( (D_{t,T}^{Q,P})^q \ln q D_{t,T}^{Q,P} - (D_{t,T}^{Q,P})^q \ln q D_{t,T}^{Q,P} \right) | F_t \right].
\]
where the third equality is derived from \( \mathbb{E}_P[D_{\xi,T}^{Q,P} | \mathcal{F}_t] = 1 \).

Since
\[
dD_t^{Q,P} = D_t^{Q,P}(-\lambda_t \cdot dW_t + \alpha_t^Q \cdot dW_t^\perp), \quad D_0^{Q,P} = 1,
\]
applying Itô’s formula to \( (D_t^{Q,P})^q \ln q D_t^{Q,P} \), it yields that
\[
d(D_t^{Q,P})^q \ln q D_t^{Q,P} = \frac{q}{2} (D_t^{Q,P})^q (|\lambda_t|^2 + |\alpha_t^Q|^2) dt + \frac{D_t^{Q,P} - q(D_t^{Q,P})^q}{1-q} (-\lambda_t \cdot dW_t + \alpha_t^Q \cdot dW_t^\perp)
\]
\[
= \frac{q}{2} (D_t^{Q,P})^q (|\lambda_t|^2 + |\alpha_t^Q|^2) dt + L_t,
\]
where \( L \) is the local martingale in the Doob-Meyer decomposition of the \( P \)-submartingale process \( (D_t^{Q,P})^q \ln q D_t^{Q,P} \). By Lemma 6.1, \( (D_t^{Q,P})^q \ln q D_t^{Q,P} \) is of class \( D \), and this further implies \( L \) is in fact a uniformly integrable martingale by Theorem 1.4.1 of Karatzas and Shreve (1998).

The result (2.4) is an immediate consequence of integrating from \( t \) to \( T \), taking the conditional expectations on both sides, and the equality (2.3). Setting \( t = 0 \), then (2.5) holds obviously.

To prove equality (2.6), it is enough to show \( H_q(Q_{\text{min}}|\mathbb{P}) < +\infty \). In fact,
\[
H_q(Q_{\text{min}}|\mathbb{P}) = \frac{1}{1-q} \left( 1 - \mathbb{E}_P[(\mathcal{E}(-\lambda \cdot W)_T)^q] \right)
\]
and
\[
\mathbb{E}_P[(\mathcal{E}(-\lambda \cdot W)_T)^q] = \mathbb{E}_P \left[ (\mathcal{E}(-q\lambda \cdot W)_T) \cdot \exp \left( \frac{q^2 - q}{2} \int_0^T |\lambda_s|^2 ds \right) \right]
\]
\[
\leq C_0 \mathbb{E}_P[(\mathcal{E}(-q\lambda \cdot W)_T)] = C_0,
\]
where the inequality is from the boundedness of \( \lambda \) and \( C_0 \) is a suitable positive constant. We then can get the finiteness of Tsallis relative entropy between \( Q_{\text{min}} \) and \( \mathbb{P} \). \( \square \)

6.2. Proof of Theorem 3.1

This subsection gives the proof of Theorem 3.1.

**Lemma 6.2.** Suppose \( \gamma > 0 \), \( q > 0 \) and \( q \neq 1 \). For each \( \xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b) \), we have
\[
Y_t(\xi) = \tilde{F}_t(\xi), \quad \forall t \in [0,T], \quad (6.1)
\]
where \( Y_t(\xi) \) is the solution of BSDE (3.4). Moreover, the essential infimum in (3.3) can be achieved.

**Proof.** By the definition of \( \tilde{F} \), for each \( \xi \in \mathcal{L}_q^\gamma(\mathcal{F}_T, b) \), we only need to show that
\[
Y_t = \text{ess inf}_{\theta \in \Theta} Y_t^\theta, \quad t \in [0,T], \quad \mathbb{P} - a.s.,
\]
where \( Y \) is the uniqueness solution of BSDE (3.4).

For any \( \theta \in \Theta \), due to the fact that \( \xi \) is bounded and backward equation (3.1) has a unique solution \( Y^\theta \) with \( \mathbb{E}^\theta \left[ |\xi + \int_0^T \frac{[\mu(Y_s^\theta)]^2}{2\gamma} |\theta_s|^2 ds \right] < \infty \), by martingale representation theorem, then there exists a \( \mathbb{R}^{m+n} \)-valued predictable process \( \bar{\varphi} = (\varphi, \varphi^\perp) \) with \( \int_0^T |\bar{\varphi}_s|^2 ds < +\infty \), \( \mathbb{Q}^\theta \)-a.s., such that

\[
dY^\theta = -\frac{1}{2\gamma} \mu(Y^\theta_s) \cdot |\theta_s|^2 ds + \varphi_s \cdot dW^\theta_s + \varphi^\perp_s \cdot dW^\perp_{s}, \quad Y^\theta_T = \xi, \tag{6.2}
\]

where the process

\[
W^\theta = \left( \frac{W^\theta_{t-\lambda}}{W^\perp_{t-\theta}} \right) = \left( \frac{W_t + \int_0^t \lambda_s ds}{W_t^\perp - \int_0^t \theta_s ds} \right), \quad 0 \leq t \leq T,
\]

is a \( \mathbb{Q}^\theta \) Brownian motion on \( (\Omega, \mathcal{F}_T) \) by Girsanov theorem.

On the other hand, BSDE (3.4) implies that

\[
dY_s = \left( \lambda_s \cdot Z_s + \frac{\gamma}{2} \cdot \frac{|Z^\perp_s|^2}{\mu(Y_s)} \right) ds + Z_s \cdot dW_s + Z^\perp_s \cdot dW^\perp_s
\]

\[
= \left( \frac{\gamma}{2} \cdot \frac{|Z^\perp_s|^2}{\mu(Y_s)} + \theta_s \cdot Z^\perp_s \right) ds + Z_s \cdot dW^\theta_s + Z^\perp_s \cdot dW^\perp_s, \quad Y_T = \xi. \tag{6.3}
\]

Let’s subtract these two equations (6.2) and (6.3), then it derives

\[
d(Y_s - Y^\theta_s) = \left( \frac{\gamma}{2} \cdot \frac{|Z^\perp_s|^2}{\mu(Y_s)} + \theta_s \cdot Z^\perp_s + \frac{1}{2\gamma} \mu(Y^\theta_s) \cdot |\theta_s|^2 \right) ds
\]

\[
\quad + (Z_s - \varphi_s) \cdot dW^\theta_s + (Z^\perp_s - \varphi^\perp_s) \cdot dW^\perp_s
\]

\[
= - \frac{\mu(Y_s) - \mu(Y^\theta_s)}{2\gamma} |\theta_s|^2 ds + \frac{\mu(Y_s)}{2\gamma} |\theta_s + \frac{\gamma Z^\perp_s}{\mu(Y_s)}|^2 ds
\]

\[
\quad + (Z_s - \varphi_s) \cdot dW^\theta_s + (Z^\perp_s - \varphi^\perp_s) \cdot dW^\perp_s.
\]

From Proposition 3.1 we know that \( \int_0^T \bar{Z}_s \cdot d\bar{W}_s \) is a BMO martingale under \( \mathbb{P} \). Since \( \theta \in \Theta \) and boundedness of \( \lambda \), by Theorem 3.3 in [Kazamaki (1994)]\(^1\), we also can get that \( \int_0^T \bar{Z}_s \cdot d\bar{W}^\theta_s \) is a BMO martingale under \( \mathbb{Q}^\theta \).

For \( n \geq 1 \), define a stopping time sequence,

\[
\tau_n := \inf\{r \in [0, T] \mid \int_0^r |\varphi_s|^2 ds \geq n \} \wedge T.
\]

For any \( t \in [0, \tau_n] \), it implies that

\[
Y_t - Y^\theta_t = \mathbb{E}^\theta[Y_{\tau_n} - Y^\theta_{\tau_n} | \mathcal{F}_t] + \mathbb{E}^\theta \left[ \int_t^{\tau_n} \frac{\mu(Y_s) - \mu(Y^\theta_s)}{2\gamma} |\theta_s|^2 ds | \mathcal{F}_t \right]
\]

\[
\quad - \mathbb{E}^\theta \left[ \int_t^{\tau_n} \frac{\mu(Y_s)}{2\gamma} |\theta_s + \frac{\gamma Z^\perp_s}{\mu(Y_s)}|^2 ds | \mathcal{F}_t \right]. \tag{6.4}
\]
Due to the fact that \( Y \) is bounded and \( \mu(Y_s) > 0, s \in [0, T] \), then
\[
Y_t - Y^\theta_t \leq \mathbb{E}_Q \left[ Y_n - Y^\theta_n \mid F_t \right] + \mathbb{E}_Q \left[ \int_t^T \frac{\mu(Y_s) - \mu(Y^\theta_s)}{2\gamma} |\theta_s|^2 ds \mid F_t \right].
\]

Since the boundedness of \( Y \) and \( Y^\theta \) and \( \theta \in \Theta \), letting \( n \) approach infinity, it gives
\[
Y_t - Y^\theta_t \leq \mathbb{E}_Q \left[ \int_t^T \frac{\mu(Y_s) - \mu(Y^\theta_s)}{2\gamma} |\theta_s|^2 ds \mid F_t \right] = \mathbb{E}_Q \left[ \int_t^T q - \frac{1}{2q} (Y_s - Y^\theta_s) |\theta_s|^2 ds \mid F_t \right].
\]

By Lemma C3 in [Schroder and Skiadas (1999)], it implies that
\[
Y_t \leq Y^\theta_t, \quad 0 \leq t \leq T, \quad \mathbb{P} \text{ - a.s.}
\]

Thus, we have \( Y_t \leq \text{ess inf}_{\theta \in \Theta} Y^\theta_t, \quad t \in [0, T], \quad \mathbb{P} \text{ - a.s.} \)

Finally, from the procedure of above proof, if we choose
\[
\theta^* := -\frac{\gamma Z^\perp}{\mu(Y)}
\]
in (6.4) and verify \( \theta^* \in \Theta \), then it concludes \( Y = Y^\theta^* \).

In fact, since \( \mu(Y) \) is bounded, then \( \int_0^t \theta_s^* \cdot dW^\perp_s \) is obviously a BMO(\( \mathbb{P} \)) martingale by Proposition 3.1. Backward equation (3.1) has at most one solution by Lemma C3 in [Schroder and Skiadas (1999)]. For the given \( \theta^* \), we can show that \( Y \) is the solution of (3.1).

Besides,
\[
\mathbb{E}_{Q^\theta^*} \left[ \int_0^T |\mu(Y_s)| \cdot |\theta^*_s|^2 ds \right] = \mathbb{E}_{Q^\theta^*} \left[ \int_0^T \frac{\gamma^2 |Z^\perp_s|^2}{2\mu(Y_s)} ds \right] \leq C_1 \mathbb{E}_{Q^\theta^*} \left[ \int_0^T |Z^\perp_s|^2 ds \right] < +\infty,
\]

where \( C_1 \) is a suitable positive constant.

Hence, it yields the assertion, i.e., (6.1) holds. \( \square \)

**Remark 6.1.** Lemma 6.2 shows that the solution of BSDE (3.1) equals to the essential infimum of backward equations (3.1) under a family of equivalent martingale measures \( \mathcal{M}^\Theta \).

It is a robust dual representation result for BSDE (3.1). Essentially, the key tool to obtain this robust dual representation depends on the Legendre transform of the concave generator of BSDE (3.1).

For the robust representation results of BSDEs with concave or convex generators, the reader can refer to [El Karoui, Peng and Quenez (1997)] with Lipschitz generators, [Barrieu and El Karoui (2009), Lazrak and Quenez (2003)] or [El Karoui and Ravanelli (2009)] with quadratic growth generators.
Lemma 6.3. Suppose $\gamma > 0$, $q > 0$ and $q \neq 1$. For any $\xi \in \mathcal{L}_q^r(\mathcal{F}_T, b)$, for each $t \in [0, T]$, we have

$$Y_t(\xi) = F_t(\xi) = \text{ess inf}_{Q \in \mathcal{M}_f} \mathbb{E}_{Q^{\min}} \left[ (D^{Q, \min}_{t,T})^q \xi + \frac{q}{2\gamma} \int_t^T (D^{Q, \min}_{s,T})^q |\alpha_s^Q|^2 ds \mid \mathcal{F}_t \right]. \quad (6.5)$$

Moreover, the essential infimum can be attained. In particular,

$$Y_0(\xi) = F(\xi) = \inf_{Q \in \mathcal{M}_f} \mathbb{E}_{Q^{\min}} \left[ (D^{Q, \min}_{0,T})^q \xi + \frac{q}{2\gamma} \int_0^T (D^{Q, \min}_{s,T})^q |\alpha_s^Q|^2 ds \right] \quad (6.6)$$

$$= \inf_{Q \in \mathcal{M}_f} \left( \mathbb{E}_{Q^{\min}} \left[ (D^{Q, \min}_{0,T})^q \xi \right] + \frac{1}{\gamma} H_q(Q|Q^{\min}) \right). \quad (6.7)$$

**Proof.** For any $Q \in \mathcal{M}_f$, since

$$dD^{Q, \min}_s = D^{Q, \min}_s \alpha_s^Q \cdot dW^\perp_s, \quad D^{Q, \min}_0 = 1,$$

applying Itô’s formula to $(D^{Q, \min}_T)^q$, it derives that

$$d(D^{Q, \min}_s)^q = q(D^{Q, \min}_s)^q \alpha_s^Q \cdot dW^\perp_s + \frac{q(q-1)}{2}(D^{Q, \min}_s)^q |\alpha_s^Q|^2 ds.$$ 

From the finiteness of Tsallis relative entropy between $Q$ and $Q^{\min}$, we get $(D^{Q, \min}_T)^q \in L^1(Q^{\min})$. For $q > 1$, by Jensen’s inequality we know that the process $(D^{Q, \min}_s)^q$ is a uniformly integrable submartingale under $Q^{\min}$. Similarly, for $0 < q < 1$, $(D^{Q, \min}_s)^q$ is a uniformly integrable supermartingale under $Q^{\min}$. Thus, for $q > 0$ and $q \neq 1$ we know that $(D^{Q, \min}_s)^q$ is of class $D$.

On the other hand, BSDE (3.4) can be rewritten as follows,

$$dY_s = \frac{\gamma}{2} \cdot \frac{|Z_s^\perp|^2}{\mu(Y_s)} ds + Z_s \cdot dW_s^{-\lambda} + Z_s^\perp \cdot dW^\perp_s, \quad Y_T = \xi, \quad (6.8)$$

where $dW_s^{-\lambda} = dW_s + \lambda_s ds$. By Girsanov theorem, we know that $W^{\min} = (W^{-\lambda}, W^\perp)$ is a Brownian motion under minimal martingale measure $Q^{\min}$, and $\langle W^{-\lambda}, W^\perp \rangle = 0$. Since $Y$ is boundedness and $\int_0^t Z_s \cdot dW_s$ is a BMO martingale under $\mathbb{P}$, then, by Theorem 3.3 in Kazamaki [1994], $\int_0^t Z_s \cdot dW^{\min}_s$ is a BMO martingale under $Q^{\min}$.

Using Itô’s formula to $Y(D^{Q, \min}_T)^q$, we get that

$$dY_s(D^{Q, \min}_s)^q = Y_s d(D^{Q, \min}_s)^q + (D^{Q, \min}_s)^q dY_s + dY_s d(D^{Q, \min}_s)^q$$

$$= (D^{Q, \min}_s)^q \left( \frac{q(q-1)}{2} Y_s \cdot |\alpha_s^Q|^2 + \frac{\gamma}{2} \cdot \frac{|Z_s^\perp|^2}{\mu(Y_s)} + q \alpha_s^Q \cdot Z_s^\perp \right) ds$$

$$+ (D^{Q, \min}_s)^q \left( Z_s \cdot dW_s^{-\lambda} + (Z_s^\perp + q Y_s^Q \alpha_s^Q) \cdot dW^\perp_s \right).$$

The stochastic integral term $L = \int_0^t (D^{Q, \min}_s)^q \left( Z_s \cdot dW_s^{-\lambda} + (Z_s^\perp + q Y_s^Q \alpha_s^Q) \cdot dW^\perp_s \right)$ is a local martingale under $Q^{\min}$ by Itô’s formula. Let $\tau_n$ be a sequence of stopping times converging to $T$ such that $\tau_n \geq t$ and $L^{\tau_n}$ is a true martingale. Then we get
\[
Y_t = \mathbb{E}_{Q_{\min}} \left[ Y_{\tau_n} (D_{t,\tau_n}^{Q,\min})^q \mid \mathcal{F}_t \right] - \mathbb{E}_{Q_{\min}} \left[ \int_t^\tau (D_{s,s}^{Q,\min})^q \Gamma_s ds \mid \mathcal{F}_t \right], \tag{6.9}
\]

where
\[
\Gamma_s := \frac{q(q - 1)}{2} Y_s \cdot |\alpha_s^Q|^2 + \frac{\gamma}{2} \cdot \frac{|Z_s^\perp|^2}{\mu(Y_s)} + q \alpha_s^Q \cdot Z_s^\perp.
\]

By the definition of \( \mu(\cdot) \), it implies that
\[
\Gamma_s = \frac{q}{2\gamma} (q - 1) \gamma Y_s \cdot |\alpha_s^Q|^2 + \frac{\gamma}{2} \cdot \frac{|Z_s^\perp|^2}{\mu(Y_s)} + q \alpha_s^Q \cdot Z_s^\perp
\]
\[
= \frac{q}{2\gamma} (\mu(Y_s)q - 1) \cdot |\alpha_s^Q|^2 + \frac{\gamma}{2} \cdot \frac{|Z_s^\perp|^2}{\mu(Y_s)} + q \alpha_s^Q \cdot Z_s^\perp
\]
\[
= \frac{\mu(Y_s)}{2\gamma} \cdot q |\alpha_s^Q|^2 + \frac{\gamma Z_s^\perp}{\mu(Y_s)} - \frac{q}{2\gamma} \cdot |\alpha_s^Q|^2.
\]

Putting the \( \Gamma \) into (6.9), we get
\[
Y_t = \mathbb{E}_{Q_{\min}} \left[ Y_{\tau_n} (D_{t,\tau_n}^{Q,\min})^q \mid \mathcal{F}_t \right] + \frac{q}{2\gamma} \mathbb{E}_{Q_{\min}} \left[ \int_t^\tau (D_{s,s}^{Q,\min})^q \cdot |\alpha_s^Q|^2 ds \mid \mathcal{F}_t \right]
\]
\[
- \frac{1}{2\gamma} \mathbb{E}_{Q_{\min}} \left[ \int_t^\tau (D_{s,s}^{Q,\min})^q \mu(Y_s) |q\alpha_s^Q + \frac{\gamma Z_s^\perp}{\mu(Y_s)}|^2 ds \mid \mathcal{F}_t \right]. \tag{6.10}
\]

Since \( (D^{Q,\min})^q \) is of class \( D \), \( Y \) and \( \mu(Y) \) are bounded, sending \( n \) to infinity, by monotone convergence theorem, which yields that
\[
Y_t = \mathbb{E}_{Q_{\min}} \left[ (D_{t,T}^{Q,\min})^q \xi \mid \mathcal{F}_t \right] + \frac{q}{2\gamma} \mathbb{E}_{Q_{\min}} \left[ \int_t^T (D_{s,s}^{Q,\min})^q \cdot |\alpha_s^Q|^2 ds \mid \mathcal{F}_t \right]
\]
\[
- \frac{1}{2\gamma} \mathbb{E}_{Q_{\min}} \left[ \int_t^T (D_{s,s}^{Q,\min})^q \mu(Y_s) \cdot |q\alpha_s^Q + \frac{\gamma Z_s^\perp}{\mu(Y_s)}|^2 ds \mid \mathcal{F}_t \right].
\]

Hence,
\[
Y_t \leq \text{ess inf}_{Q \in \mathcal{M}_f} \mathbb{E}_{Q_{\min}} \left[ (D_{t,T}^{Q,\min})^q \xi + \frac{q}{2\gamma} \int_t^T (D_{s,s}^{Q,\min})^q \cdot |\alpha_s^Q|^2 ds \mid \mathcal{F}_t \right].
\]

Moreover, the essential infimum is actually achieved if we can show \( Q^* \in \mathcal{M}_f \), where
\[
\alpha^{Q^*} := -\frac{\gamma Z^\perp}{q\mu(Y)}. \tag{6.11}
\]

Since \( \int_0^\tau Z_s \cdot dW^{\min}_s \) is a BMO martingale under \( Q^{\min} \) and \( \mu(Y) \) is bounded from below, then \( \int_0^\tau \alpha^{Q^*} \cdot dW^\perp_s \) is also a BMO martingale under \( Q^{\min} \). By Theorem 2.3 in Kazamaki (1994), we get \( \mathcal{E}(\alpha^{Q^*} \cdot W^\perp) \) is a uniformly integrable martingale under \( Q^{\min} \). Thus, we can define
\[
\frac{dQ^*}{dQ^{\min}} := \mathcal{E}(\alpha^{Q^*} \cdot W^\perp)_T = D_T^{Q^*,Q^{\min}},
\]

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and $Q^* \in \mathcal{M}$.

By the following Lemma 6.4 we know that $H_{q}(Q^* \mid Q^{\min}) < +\infty$. To sum up, (6.3) is true. Setting $t = 0$, then (6.6) and (6.7) can be easily obtained by (2.8) in Corollary 2.1.

Lemma 6.4. Suppose $\gamma > 0$, $q > 0$ and $q \neq 1$. Suppose $\alpha_{Q^*} = -\frac{z}{q - 1}$. We have

$$E_{Q^{\min}} \left[ (D_{T}^{Q^*, Q^{\min}})^{q} \right] = E_{Q^{\min}} \left[ E(\alpha_{Q^*} \cdot W^{-1})_{T}^{q} \right] < +\infty,$$

which means $H_{q}(Q^* \mid Q^{\min}) < +\infty$.

**Proof.** When $0 < q < 1$, it holds obviously by Hölder inequality.

Next, we are going to prove the situation of $q > 1$. This case is not trivial. Using Itô’s formula to $\ln \mu(Y)$, it implies that

$$d \ln \mu(Y) = \frac{(q - 1)\gamma}{q \mu(Y)} d\mu(Y) - \frac{1}{2} \frac{(q - 1)^{2}\gamma^{2}}{q^{2}} \frac{1}{\mu^{2}(Y)} d\mu(Y)^{2}$$

$$= \frac{(q - 1)\gamma}{q \mu(Y)} Z_s \cdot dW_{s}^{-\lambda} - \frac{1}{2} \frac{(q - 1)^{2}\gamma^{2}}{q^{2}} \frac{|Z_s|^{2}}{\mu^{2}(Y)} ds - (q - 1)\alpha_{s}^{Q^*} \cdot dW_{s}^{1} + \frac{q - 1}{2} |\alpha_{s}^{Q^*}|^{2} ds,$$

where $W^{-\lambda}$ is defined the same as in (6.8).

Integrating both sides from zero to $T$, it derives that

$$(q - 1) \left( \int_{0}^{T} \alpha_{s}^{Q^*} \cdot dW_{s}^{1} - \frac{1}{2} \int_{0}^{T} |\alpha_{s}^{Q^*}|^{2} ds \right)$$

$$= \ln \frac{\mu(Y)}{\mu(\xi)} + \int_{0}^{T} \frac{(q - 1)\gamma}{q \mu(Y)} Z_s \cdot dW_{s}^{-\lambda} - \int_{0}^{T} \frac{1}{2} \frac{(q - 1)^{2}\gamma^{2}}{q^{2}} \frac{|Z_s|^{2}}{\mu^{2}(Y)} ds.$$

Taking the exponent on both sides of the above equality, it implies

$$E(\alpha_{Q^*} \cdot W^{-1})_{T}^{q - 1} = \frac{\mu(Y)}{\mu(\xi)} E \left( \frac{(q - 1)\gamma}{q} \frac{Z}{\mu(Y)} \cdot W^{-\lambda} \right)_{T}.$$

Therefore, we get that

$$E(\alpha_{Q^*} \cdot W^{-1})_{T}^{q} = \frac{\mu(Y)}{\mu(\xi)} E \left( \frac{(q - 1)\gamma}{q} \frac{Z}{\mu(Y)} \cdot W^{-\lambda} \right)_{T} E(\alpha_{Q^*} \cdot W^{-1})_{T}.$$

Since $\int_{0}^{(q - 1)\gamma} \frac{Z}{\mu(Y)} \cdot dW_{s}^{-\lambda}$ and $\int_{0}^{(q - 1)\gamma} \alpha_{s}^{Q^*} \cdot dW_{s}^{1}$ are both the BMO martingales under $Q^{\min}$.

By Theorem 2.3 in Kazamaki [1994], we get $E \left( \frac{(q - 1)\gamma}{q} \frac{Z}{\mu(Y)} \cdot W^{-\lambda} + \alpha_{Q^*} \cdot W^{-1} \right)$ is a uniformly integrable martingale under $Q^{\min}$. Due to the fact that $\mu(Y)$ is bounded, we can get that

$$E_{Q^{\min}} \left[ E(\alpha_{Q^*} \cdot W^{-1})_{T}^{q} \right] \leq C_{2} E_{Q^{\min}} \left[ E \left( \frac{(q - 1)\gamma}{q} \frac{Z}{\mu(Y)} \cdot W^{-\lambda} + \alpha_{Q^*} \cdot W^{-1} \right)_{T} \right]$$

$$= C_{2} < +\infty,$$

where $C_{2}$ is a suitable positive constant.
Remark 6.2. In the situation of \( q > 1 \), by Theorem 3.4 in \([\text{Kazamaki} (1994)](1994)\) and BMO martingale property of \( \int_0^t \alpha^Q_s \cdot dW^\perp_s \) under \( \mathbb{Q}^\text{min} \), then we get \( \mathcal{E}(\alpha^Q \cdot W^\perp) \) satisfies the reverse Hölder inequality (\( R_p \)) for some \( p > 1 \), i.e.,

\[
\mathbb{E}_{\mathbb{Q}^\text{min}} \left[ \mathcal{E}(\alpha^Q \cdot W^\perp)^p_T \right] \leq \tilde{C}_p \mathbb{E}(\alpha^Q \cdot W^\perp)^p_0 = \tilde{C}_p < +\infty,
\]

where \( \tilde{C}_p \) is a positive constant. But, we can't simply get the magnitude relationship between \( q \) and \( p \). If \( p \geq q \), then (6.12) is true by Hölder inequality again. However, if \( p < q \), we can not get the result directly. Lemma 6.4 essentially depends on the structure of BSDE (3.4).

Proof of Theorem 3.1. Theorem 3.1 is immediately obtained by combining Lemma 6.2 and Lemma 6.3.

6.3. Proof of Proposition 4.3

Proof. For any \( \mathbb{Q} \in \mathcal{Q}_f \), noticing that

\[
dD^\mathbb{Q} \mathbb{Q}^\text{min} = D^\mathbb{Q} \mathbb{Q}^\text{min}_0 (\beta^\mathbb{Q} \cdot dW^{\perp -\lambda} + \alpha^\mathbb{Q} \cdot dW^\perp), \quad D^\mathbb{Q} \mathbb{Q}^\text{min}_0 = 1
\]

and

\[
dy_s = \frac{\gamma}{2} \frac{|z_s|^2 + |z^\perp_s|^2}{\mu(y_s)} ds + z_s \cdot dW^{\perp -\lambda} + z^\perp_s \cdot dW^\perp, \quad y_T = \xi,
\]

where \( \beta^\mathbb{Q} \) and \( \alpha^\mathbb{Q} \) are determined by the martingale representation theorem of Brownian motion.

By the similar arguments in Lemma 6.3, we can get

\[
y_t = \mathbb{E}_{\mathbb{Q}^\text{min}} \left[ (D_{t,T}^\mathbb{Q} \mathbb{Q}^\text{min})^q \xi | \mathcal{F}_t \right] + \frac{q}{2\gamma} \mathbb{E}_{\mathbb{Q}^\text{min}} \left[ \int_t^T (D_{t,s}^\mathbb{Q} \mathbb{Q}^\text{min})^q \cdot (|\beta^\mathbb{Q}_s|^2 + |\alpha^\mathbb{Q}_s|^2) ds | \mathcal{F}_t \right] - \frac{1}{2\gamma} \mathbb{E}_{\mathbb{Q}^\text{min}} \left[ \int_t^T (D_{t,s}^\mathbb{Q} \mathbb{Q}^\text{min})^q \mu(y_s) \cdot \left( |q \beta^\mathbb{Q}_s + \frac{\gamma z_s}{\mu(y_s)}|^2 + |q \alpha^\mathbb{Q}_s + \frac{\gamma z^\perp_s}{\mu(y_s)}|^2 \right) ds | \mathcal{F}_t \right].
\]

Therefore,

\[
y_t \leq \text{ess inf} \mathbb{E}_{\mathbb{Q}^\text{min}} \left[ (D_{t,T}^\mathbb{Q} \mathbb{Q}^\text{min})^q \xi + \frac{q}{2\gamma} \int_t^T (D_{t,s}^\mathbb{Q} \mathbb{Q}^\text{min})^q \cdot (|\beta^\mathbb{Q}_s|^2 + |\alpha^\mathbb{Q}_s|^2) ds | \mathcal{F}_t \right].
\]

Besides, the essential infimum is actually achieved if we can show \( \mathbb{Q}^* \in \mathcal{Q}_f \), where

\[
\alpha^\mathbb{Q}^* := - \frac{\gamma z^\perp}{q \mu(y)} \quad \text{and} \quad \beta^\mathbb{Q}^* := - \frac{\gamma z}{q \mu(y)}.
\]

(6.13)

In fact, since \( \int_0^t z_s \cdot dW^\perp_s \) is a BMO martingale under \( \mathbb{Q}^\text{min} \) and \( \mu(y) \) is bounded from below, by Theorem 2.3 in \([\text{Kazamaki} (1994)](1994)\), we can get \( \mathcal{E}(\beta^\mathbb{Q}^* \cdot W^{\perp -\lambda} + \alpha^\mathbb{Q}^* \cdot W^\perp) \) is a uniformly integrable martingale under \( \mathbb{Q}^\text{min} \), which implies \( \mathbb{Q}^* \in \mathcal{Q} \).

Next, we claim \( \mathbb{Q}^* \in \mathcal{Q}_f \), i.e., \( H_q(\mathbb{Q}^* || \mathbb{Q}^\text{min}) < +\infty \). When \( q < 1 \), it is obvious. For \( q > 1 \), applying Itô’s formula to \( \frac{1}{1-q} \ln \mu(y) \), it derives that

\[
\frac{q}{1-q} d \ln \mu(y_s) = q \left( \beta^\mathbb{Q}^* \cdot dW^{\perp -\lambda} + \alpha^\mathbb{Q}^* \cdot dW^\perp - \frac{1}{2} (|\beta^\mathbb{Q}^*|^2 + |\alpha^\mathbb{Q}^*|^2) ds \right),
\]

\[
\cdots
\]

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which leads to
\[
E_{Q_{\min}}[(D_T^{(Q,\min)})^q] = E_{Q_{\min}}[\mathcal{E}(\beta^Q \cdot W^{-\lambda} + \alpha^Q \cdot W^q)_T] \\
= E_{Q_{\min}}\left[\left(\frac{\mu(\xi)}{\mu(y_0)}\right)^{q/(1-q)}\right] < +\infty.
\]

For any \( t \in [0, T] \), applying Theorem 2.1 to \( H_{q,t}(Q|Q_{\min}) \), it completes the proof. \( \square \)

6.4. Proof of Proposition 4.6

Proof. From Proposition 4.5 (i), we know that for each \( t \in [0, T] \) and each bounded \( \xi \in \bigcap_{\gamma > 0} \mathcal{L}_q^\gamma(F_T, b) \), \( F_t(\xi, \gamma) \) is decreasing with respect to \( \gamma \).

By the definition of \( F_t \), for any \( \gamma > 0 \), we obviously get
\[
F_t(\xi, \gamma) \geq \text{ess inf}_{Q \in M_f} E_{Q_{\min}}\left[(D_{t,T}^{(Q,\min)})^q \xi | F_t\right].
\]

Thus, it implies
\[
\lim_{\gamma \to +\infty} F_t(\xi, \gamma) \geq \text{ess inf}_{Q \in M_f} E_{Q_{\min}}\left[(D_{t,T}^{(Q,\min)})^q \xi | F_t\right].
\]

On the other hand, for any \( Q \in M_f \), for any \( \gamma > 0 \), we have
\[
F_t(\xi, \gamma) \leq E_{Q_{\min}}\left[(D_{t,T}^{(Q,\min)})^q \xi | F_t\right] + \frac{1}{\gamma} H_{q,t}(Q|Q_{\min})
\]
\[
= \frac{1}{\gamma (D_{t,T}^{(Q,\min)})^q} \left( E_{Q_{\min}}\left[(D_{t,T}^{(Q,\min)})^q \ln_q D_T^{(Q,\min)} | F_t\right] - (D_{t,T}^{(Q,\min)})^q \ln_q D_T^{(Q,\min)} \right)
\]
\[
+ E_{Q_{\min}}\left[(D_{t,T}^{(Q,\min)})^q \xi | F_t\right],
\]
where the last equality follows from (2.3) in Theorem 2.1. Sending \( \gamma \) to infinity, and taking the infimum with \( Q \) on \( M_f \), (6.10) holds.

By Proposition 4.3, for each \( \gamma > 0 \), we have
\[
C E_q(\xi|F_t) \leq F_t(\xi, \gamma) \leq E_{Q_{\min}}[\xi | F_t].
\]

In order to prove (4.11), we only need to show that for each \( \xi \in \bigcap_{\gamma > 0} \mathcal{L}_q^\gamma(F_T, b) \),
\[
\lim_{\gamma \to 0} -\frac{1}{\gamma} \ln_q E_{Q_{\min}}[\exp_q(-\gamma \xi) | F_t] = E_{Q_{\min}}[\xi | F_t], \quad \forall t \in [0, T]. \quad (6.14)
\]

In fact, for each \( t \in [0, T] \), setting
\[
h(\gamma) = E_{Q_{\min}}[\exp_q(-\gamma \xi) | F_t].
\]

Then \( h(\gamma) \overset{a.s.}{\to} 1 \) when \( \gamma \) goes to zero. Since \( \xi \) is bounded, we have that
\[
h'(\gamma) = -E_{Q_{\min}}[\exp_q(-\gamma \xi) \xi | F_t], \quad \mathbb{P} - a.s.
\]
Therefore, it implies that
\[
\lim_{\gamma \to 0} \frac{1}{\gamma} \ln_q \mathbb{E}_{Q^{\min}} [\exp_q (-\gamma \xi) | \mathcal{F}_t] \\
= \lim_{\gamma \to 0} \frac{1 - h(\gamma)^{1-q}}{\gamma (1 - q)} \\
= \lim_{\gamma \to 0} \frac{h'(\gamma)}{(h(\gamma))^{q}} \\
= \mathbb{E}_{Q^{\min}} [\xi | \mathcal{F}_t].
\]

\[\Box\]

6.5. Proof of Proposition 3.1

Suppose \( \gamma > 0, q > 0 \) and \( q \neq 1 \). Define
\[\Lambda := \left\{ x < \frac{1}{(1-q)\gamma}, \ 0 < q < 1, \ x > \frac{1}{(1-q)\gamma}, \ q > 1 \right\}.\]

We denote \( f(y) := \frac{\gamma}{2\mu(y)} = \frac{\gamma}{2} \exp^{q-1}(-\gamma y), y \in \Lambda. \) Then \( f : \Lambda \to \mathbb{R} \) is locally integrable.

Define \( W^{-\lambda} := W + \int_0^\lambda \lambda ds \), we know that \( W^{\min} = (W^{-\lambda}, W^\perp) \) is a Brownian motion under minimal martingale measure \( Q^{\min} \) and \( \langle W^{-\lambda}, W^\perp \rangle = 0 \). Obviously, Proposition 3.1 is equivalent to the following proposition by Girsanov transformation, see Theorem 3.3 in Kazamaki (1994).

Proposition 6.1. Suppose \( \gamma > 0, q > 0 \) and \( q \neq 1 \). For any \( \xi \in \mathcal{L}^\gamma_q(\mathcal{F}_T, b) \), then the following BSDE
\[
Y_t = \xi - \int_t^T f(Y_s) \cdot |Z_s^\perp|^2 ds - \int_t^T Z_s \cdot dW_s^{-\lambda} - \int_t^T Z_s^\perp \cdot dW_s^\perp, \ t \in [0,T],
\]

admits a unique solution \( (Y, Z) = (Y, Z, Z^\perp) \) in which \( \int_0^T Z_s \cdot dW^{\min}_s \) is a BMO(\( Q^{\min} \)) martingale, and \( Y \) is continuous and bounded, specifically, for each \( t \in [0,T], Y_t \in \mathcal{L}^\gamma_q(\mathcal{F}_t, b). \)

Proof. For any \( \xi \in \mathcal{L}^\gamma_q(\mathcal{F}_T, b) \), then there exist two positive constants \( m_1 \) and \( m_2 \) such that \( 0 < m_1 \leq \exp_{q}(-\gamma \xi) \leq m_2 \). Therefore,
\[
\tilde{m}_2 := -\frac{1}{\gamma} \ln_q m_2 \leq \xi \leq -\frac{1}{\gamma} \ln_q m_1 =: \tilde{m}_1.
\]

Since \( \ln_q m_1 > -\frac{1}{1-q} \) if \( 0 < q < 1 \), and \( \ln_q m_2 < -\frac{1}{1-q} \) if \( q > 1 \), it implies that
\[\xi \in [\tilde{m}_2, \tilde{m}_1] \subseteq \Lambda.\]

Moreover, for any \( y \in \Lambda, \tilde{z} = (z, z^\perp) \in \mathbb{R}^{m+n}, \)
\[|F_\Lambda(y, z^\perp)| = f(y)|z^\perp|^2 \leq f(y)|\tilde{z}|^2,\]

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then, by Proposition 4.4 in Zheng, Zhang and Feng (2021), there exists a solution $(Y, Z) = (Y, \bar{Z})$ in which $\int_0^T \mathbb{I} \cdot dW_s^2$ is a BMO($\mathcal{Q}^{\text{min}}$) martingale, and $Y$ is continuous and bounded, specifically, for each $t \in [0, T]$, $Y_t \in \mathbb{M}_2$, or $Y_t \in L^2(F_t, b)$.

Besides, using Theorem A1 in Bahlali, Eddahbi and Ouahnine (2017), in fact, there exist a maximal solution and a minimal solution, in $[\bar{m}_2, \bar{m}_1]$, for BSDE (6.15). Suppose $(Y^L, Z^L)$ and $(Y^U, Z^U)$ are minimal solution and maximal solution for BSDE (6.15) respectively.

Due to the fact that the generator $F_\Lambda(y, z^\perp) = -f(y)|z^\perp|^2$ is concave in $(y, z^\perp)$, motivated by the $\theta$-method in Briand and Hu (2008) dealing with the convex generator, we can derive the uniqueness of the solution to BSDE (6.15) by some subtle transformations. The main difficulty is that we have to be careful about the domain of the function $F_\Lambda$.

Case (i): $0 < q < 1$. For any $\theta \in (0, 1)$, setting

\[
\delta_\theta Y = Y^L - \theta Y^U, \delta_\theta Z = Z^L - \theta Z^U, \delta_\theta Z^\perp = Z^\perp|^2 - \theta Z^\perp.
\]

We have that

\[
\delta_\theta F_\Lambda = F_\Lambda(Y^L, Z^\perp) - \theta F_\Lambda(Y^U, Z^\perp).
\]

We have

\[
\frac{\delta_\theta Y}{1 - \theta} = \frac{\theta}{1 - \theta} (Y^L - Y^U) + Y^L \leq Y^L \leq \bar{m}_1 < \frac{1}{(1 - q)\gamma}.
\]

On the other hand, by the concavity of $F_\Lambda$ in $(y, z^\perp)$, it implies

\[
F_\Lambda(Y^L, Z^\perp) = F_\Lambda(\theta Y^U + (1 - \theta) \frac{\delta_\theta Y}{1 - \theta} \theta Z^\perp + (1 - \theta) \frac{\delta_\theta Z^\perp}{1 - \theta})
\]

\[
\geq \theta F_\Lambda(Y^U, Z^\perp) + (1 - \theta) F_\Lambda(\frac{\delta_\theta Y}{1 - \theta} \frac{\delta_\theta Z^\perp}{1 - \theta}).
\] (6.16)

Using Itô’s formula to $\exp_q(-\frac{\gamma}{1 - \theta} \delta_\theta Y)$, it obtains that

\[
d \exp_q(-\frac{\gamma}{1 - \theta} \delta_\theta Y) = - \exp_q(-\frac{\gamma}{1 - \theta} \delta_\theta Y)\frac{\gamma}{1 - \theta} (\delta_\theta Z^\perp \cdot dW^\perp_s + \delta_\theta Z_s \cdot dW^\perp_s)
\]

\[
+ \exp_q(-\frac{\gamma}{1 - \theta} \delta_\theta Y)\frac{\gamma}{1 - \theta} \delta_\theta F_\Lambda(s)ds
\]

\[
+ \frac{q}{2(1 - \theta)^2} \exp_q^{2q}(\frac{\gamma}{1 - \theta} \delta_\theta Y) (|\delta_\theta Z_s^\perp|^2 + |\delta_\theta Z_s|^2)ds
\]

\[
\geq - \exp_q(-\frac{\gamma}{1 - \theta} \delta_\theta Y)\frac{\gamma}{1 - \theta} (\delta_\theta Z^\perp \cdot dW^\perp_s + \delta_\theta Z_s \cdot dW^\perp_s)
\]

\[
+ \frac{q}{2(1 - \theta)^2} \exp_q^{2q}(\frac{\gamma}{1 - \theta} \delta_\theta Y) |\delta_\theta Z_s|^2 ds,
\]

where the last inequality is derived from (6.16) and the definition of $F_\Lambda$.

For each $\theta \in (0, 1)$, since $\exp_q(-\frac{\gamma}{1 - \theta} \delta_\theta Y)$ is bounded, and $\delta_\theta Z^\perp$ and $\delta_\theta Z$ is square integrable, then the stochastic integral term is a martingale under $\mathcal{Q}^{\text{min}}$. Taking the conditional expectation under $\mathcal{Q}^{\text{min}}$, we get that

\[
\exp_q(-\frac{\gamma}{1 - \theta} \delta_\theta Y_t) \leq \mathbb{E}_{\mathcal{Q}^{\text{min}}} \left[ \exp_q(-\gamma \xi) \left| \mathcal{F}_t \right. \right], \quad t \in [0, T].
\]
Therefore, for each $t \in [0, T],$
\[
Y^L_t - \theta Y^U_t = \delta_0 Y_t \\
\geq - (1 - \theta) \frac{1}{\gamma} \ln \mathbb{E}_{Q^\text{min}} [\exp(-\gamma \xi) | \mathcal{F}_t] \\
\geq (1 - \theta) \tilde{m}_2.
\]
Sending $\theta$ to 1, we get $Y^L - Y^U \geq 0$, which gives the uniqueness result.

Case (ii): $q > 1$. For any $\theta \in (0, 1)$, setting
\[
\tilde{\delta}_0 Y = Y^U - \theta Y^L, \tilde{\delta}_0 Z = Z^U - \theta Z^L, \tilde{\delta}_0 Z^\perp = Z^{1, U} - \theta Z^{1, L}.
\]
Similarly, we get
\[
\frac{\tilde{\delta}_0 Y}{1 - \theta} = \frac{\theta}{1 - \theta} (Y^U - Y^L) + Y^U \geq Y^U \geq \tilde{m}_2 > \frac{1}{(1 - q)\gamma}
\]
and
\[
\tilde{\delta}_0 F_\Lambda(s) = F_\Lambda(Y^U, Z^{1, U}) - \theta F_\Lambda(Y^L, Z^{1, L}) \\
\geq (1 - \theta) F_\Lambda\left(\frac{\tilde{\delta}_0 Y}{1 - \theta}, \frac{\tilde{\delta}_0 Z^\perp}{1 - \theta}\right).
\]

Applying Itô’s formula to $\ln \mu(\frac{\tilde{\delta}_0 Y}{1 - \theta})$, we get
\[
d \ln \mu(\frac{\tilde{\delta}_0 Y_s}{1 - \theta}) \\
= \frac{(q - 1)\gamma}{q(1 - \theta)} \mu^{-1}(\frac{\tilde{\delta}_0 Y_s}{1 - \theta}) d \tilde{\delta}_0 Y_s - \frac{1}{2} \frac{(q - 1)^2 \gamma^2}{q^2(1 - \theta)^2} \mu^{-2} (\frac{\tilde{\delta}_0 Y_s}{1 - \theta}) \langle d \tilde{\delta}_0 Y_s \rangle^2 \\
= \frac{(q - 1)\gamma}{q(1 - \theta)} \mu^{-1}(\frac{\tilde{\delta}_0 Y_s}{1 - \theta}) (\tilde{\delta}_0 Z_s \cdot dW^{-\lambda}_s + \tilde{\delta}_0 Z^\perp_s \cdot dW^\perp_s) \\
- \frac{(q - 1)\gamma}{q(1 - \theta)} \mu^{-1}(\frac{\tilde{\delta}_0 Y_s}{1 - \theta}) \tilde{\delta}_0 F_\Lambda(s) ds \\
- \frac{1}{2} \frac{(q - 1)^2 \gamma^2}{q^2(1 - \theta)^2} \mu^{-2} (\frac{\tilde{\delta}_0 Y_s}{1 - \theta}) (|\tilde{\delta}_0 Z^\perp_s|^2 + |\tilde{\delta}_0 Z_s|^2) ds \\
\leq b_s \cdot dW^{-\lambda}_s - \frac{1}{2} |b_s|^2 ds + (q - 1)a_s \cdot dW^\perp_s - \frac{1}{2} (q - 1)^2 |a_s|^2 ds \\
- \frac{(q - 1)\gamma}{q} \mu^{-1}(\frac{\tilde{\delta}_0 Y_s}{1 - \theta}) F_\Lambda\left(\frac{\tilde{\delta}_0 Y_s}{1 - \theta}, \frac{\tilde{\delta}_0 Z^\perp_s}{1 - \theta}\right) ds \\
= b_s \cdot dW^{-\lambda}_s - \frac{1}{2} |b_s|^2 ds + (q - 1)a_s \cdot dW^\perp_s + \frac{1}{2} (q - 1)|a_s|^2 ds,
\]
where
\[
b_s = \frac{(q - 1)\gamma}{q(1 - \theta)} \mu^{-1}(\frac{\tilde{\delta}_0 Y_s}{1 - \theta}) \tilde{\delta}_0 Z_s
\]
and

\[ a_s = \frac{\gamma}{q(1-\theta)} \mu^{-1}(\delta_\theta Y_s \frac{\delta_\theta Y_s}{1-\theta}) \tilde{\delta}_\theta Z_s. \]

Integrating both sides from \( t \) to \( T \) and taking the exponent on both sides of the above equality, we get

\[ \mu(\xi) \mathcal{E}(-a \cdot W^\perp)_t^T \leq \mu(\frac{\delta_\theta Y_t}{1-\theta}) \mathcal{E} \left( b \cdot W^{-\lambda} - a \cdot W^\perp \right)_t^T, \]

which means

\[ \exp_q^{1-q}(-\gamma \xi) \mathcal{E}(-a \cdot W^\perp)_t^T \leq \exp_q^{1-q}(-\gamma \frac{\delta_\theta Y_t}{1-\theta}) \mathcal{E} \left( b \cdot W^{-\lambda} - a \cdot W^\perp \right)_t^T. \]

For each \( \theta \in (0, 1) \), due to the boundedness of \( \mu^{-1}(\delta_\theta Y_t) \), and the definitions of \( a \) and \( b \), \( \mathcal{E} \left( b \cdot W^{-\lambda} - a \cdot W^\perp \right) \) then is a martingale under \( Q^{\min} \). Taking the conditional expectation under \( Q^{\min} \), and using the reverse Hölder inequality, we have, for each \( t \in [0, T] \),

\[ \exp_q^{1-q}(\frac{\delta_\theta Y_t}{1-\theta}) \geq \mathbb{E}_{Q^{\min}} \left[ \exp_q^{1-q}(\frac{\delta_\theta Y_t}{1-\theta}) \mathcal{E}(-a \cdot W^\perp)_t^T | \mathcal{F}_t \right] \]

\[ \geq \mathbb{E}_{Q^{\min}} \left[ \mathcal{E}(-a \cdot W^\perp)_t^T | \mathcal{F}_t \right]^{q} \cdot \mathbb{E}_{Q^{\min}} \left[ \exp_q(-\gamma \xi) | \mathcal{F}_t \right]^{1-q}. \]

Finally, we get \( \exp_q(-\gamma \frac{\delta_\theta Y_t}{1-\theta}) \geq \mathbb{E}_{Q^{\min}} \left[ \exp_q(-\gamma \xi) | \mathcal{F}_t \right] \). Therefore, for each \( t \in [0, T] \),

\[ Y^U_t - \theta Y^L_t = \frac{\tilde{\delta}_\theta Y_t}{1-\theta} \quad \leq -(1-\theta) \frac{1}{\gamma} \ln_q \mathbb{E}_{Q^{\min}} \left[ \exp_q(-\gamma \xi) | \mathcal{F}_t \right] \]

\[ \leq -(1-\theta)m_1. \]

Sending \( \theta \) to 1, we get \( Y^U - Y^L \leq 0 \), which gives the uniqueness result. \( \square \)

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