Differential calculus on $q$-Minkowski space

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We wish to report here on a recent approach [1] to the non-commutative calculus on $q$-Minkowski space which is based on the reflection equations (RE) with no spectral parameter. These are considered as the expression of the invariance (under the coaction of the $q$-Lorentz group) of the commutation properties which define the different $q$-Minkowski algebras. This approach also allows us to discuss the possible ambiguities in the definition of $q$-Minkowski space $M_q$ and its differential calculus. The commutation relations among the generators of $M_q$ (coordinates), $D_q$ (derivatives), $\Lambda_q$ (one-forms) and a few invariant (scalar) operators are established. The correspondence of the obtained expressions with those of [2] is given in a table at the end.

1 Covariance, the $q$-Lorentz group and the RE

As is well known, the classical $2\to1$ homomorphism $SL(2,C)/Z_2 \simeq L_+^\uparrow$ is based on the observation that the transformation of the hermitian matrix $K$, $K \mapsto AK\bar{A}^{-1} \equiv AKA^\dagger = K'$, necessarily produces another hermitian matrix $K'$ and that $detK = detK'$ if $detA = 1$. If $\sigma^\mu = (\sigma^0, \sigma^i)$ (where $\sigma^0 = I$ and $\sigma^i$ are the Pauli matrices), $K$ may be written as $K = \sigma^\mu x_\mu$ and $detK = (x^0)^2 - \vec{x}^2$. Thus, $K' = \sigma^\mu x'_\mu$ with $x'^\mu = \Lambda^\mu_\nu x^\nu$, and the correspondence $\pm A \mapsto \Lambda \in L_+^\uparrow$ realizes the covering homomorphism of the restricted Lorentz group. The antisymmetric matrix $\epsilon \equiv i\sigma^2$ realizes the equivalence of the representations $A$ and $(A^{-1})^t$ of $SL(2,C)$, $\epsilon A \epsilon^{-1} = (A^{-1})^t$; hence

$$K^\epsilon = \bar{A} K^\epsilon A^{-1} , \quad K'^\epsilon = \epsilon K^\epsilon \epsilon^{-1} .$$

Clearly, since $\epsilon(\sigma^\mu)^* \epsilon^{-1} = (\sigma^0, -\sigma^i) \equiv \rho^\mu$, another set of hermitian matrices (not related by a similarity transformation to $\sigma^\mu$) may be introduced. All this reflects that for $SL(2,C)$ the representations $A$ and $\bar{A} = \epsilon A^* \epsilon^{-1}$ are inequivalent since the complex conjugation $*$ is an external automorphism; $A$ and $A^*$ define the two fundamental

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representations under which undotted and dotted spinors transform. In contrast, the vector representation $D^{\frac{1}{2}, \frac{1}{2}}$ is real and the $*$-conjugation, which in general produces inequivalent representations ($*: D^{j,j'} \mapsto D^{j',j}$), acts trivially on it ($K$ is hermitian).

The crucial idea [3, 4, 5, 6, 2, 7] to deform the Lorentz group was to replace the $SL(2, C)$ matrices $A$ by the generator matrix $M$ of the quantum group $SL_q(2, C)$. Due to the fact that the hermitian conjugation ($M^\dagger_{ij} = M_{ji}^*$) includes the $*$-operation, an extra copy $\tilde{M}$ of $SL_q(2, C)$ was introduced, with entries not commuting with those of $M$. The $R$-matrix form of the commutation relations among the quantum group generators $(a, b, c, d)$ of $M$ and $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ of $\tilde{M}$ may be expressed by

$$R_{12}M_1M_2 = M_2M_1R_{12},$$
$$R_{12}\tilde{M}_1\tilde{M}_2 = \tilde{M}_2\tilde{M}_1R_{12},$$
$$R_{12}M_1\tilde{M}_2 = \tilde{M}_2M_1R_{12}.$$

As in the classical case (eq. (2)), here $M$ and $\tilde{M}$ are related by requiring $\tilde{M}^{-1} = M^\dagger$. This condition is consistent with all relations in (3) provided that the deformation parameter $q$ is real, $q \in \mathbb{R}$. The set of generators $(a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ satisfying $\text{det}_q M = 1 = \text{det}_q \tilde{M}$, the commutation relations (3) and the conditions $\tilde{M}^{-1} = M^\dagger$ define the quantum Lorentz group $L_q$. There are, however, other possibilities if we allow for different $R$ matrices in (3) reflecting different commutation properties.

To introduce the $q$-Minkowski algebra $\mathcal{M}_q$ it is natural to extend (1) to the quantum case by stating that the algebra generated by the entries of $K$ is a comodule for the coaction $\phi$ defined by

$$\phi: K \mapsto K' = M K \tilde{M}^{-1}, \quad K'_{is} = M_{ij} \tilde{M}^{-1}_{ks} K_{jl} \equiv \Lambda_{il,js} K_{jl},$$

where it is assumed that the matrix elements of $K$ commute with those of $M$ and $\tilde{M}$ but not among themselves. Much in the same way as the commuting properties of $q$-two-vectors (or, better said here, $q$-spinors) are preserved by the coaction of $M$ and $\tilde{M}$, we now demand that the commuting properties of the entries of $K$ are preserved by (4). More specifically, in order to identify the elements of $K$

$$K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with the generators of $\mathcal{M}_q$ we require, as in the classical case,

a) a reality property preserved by (4),
b) a set of commutation relations for the elements of $K$ (a ‘presentation’ of the algebra $\mathcal{M}_q$) preserved by (4),
c) a (real) $q$-Minkowski length $l_q$, defined through the $q$-determinant $\text{det}_q K$ of $K$, invariant under the $q$-Lorentz transformation (4).

The reality condition $K = K^\dagger$ (a) is consistent with (4) since $\tilde{M}^{-1} = M^\dagger$ as in the classical case. Let us now explore (b) to express the six basic relations for the elements of $K$ which define a $q$-Minkowski algebra.

We may describe them (4) by means of a general RE (see [8, 9] and references therein; see also [10, 11] in the context of braided algebras)

$$R^{(1)} K_1 R^{(2)} K_2 = K_2 R^{(3)} K_1 R^{(4)}.$$ (6)
The four $4 \times 4$ matrices have now to be found by demanding the invariance of (3) under (4). From $R^{(1)}K_{1}^{4}R(2)K_{2}^{4} = K_{2}^{4}R^{(1)}K_{1}^{4}$ and (4) we obtain

$$R^{(1)}M_{1}K_{1}\tilde{M}_{1}^{-1}R(2)M_{2}K_{2}\tilde{M}_{2}^{-1} = M_{2}K_{2}\tilde{M}_{2}^{-1}R^{(3)}M_{1}K_{1}\tilde{M}_{1}^{-1}R^{(4)}$$

(7)

an expression which may be written as

$$R^{(1)}M_{1}M_{2}K_{1}R(2)K_{2}\tilde{M}_{1}^{-1}\tilde{M}_{2}^{-1} = M_{2}M_{1}K_{2}R^{(3)}K_{1}\tilde{M}_{2}^{-1}\tilde{M}_{1}^{-1}R^{(4)}$$

(8)

if

$$R^{(2)}M_{2}\tilde{M}_{1} = \tilde{M}_{1}M_{2}R^{(2)} , \quad R^{(3)}M_{1}\tilde{M}_{2} = \tilde{M}_{2}M_{1}R^{(3)}$$

(9)

and which is equal again to (3) if

$$R^{(1)}M_{1}M_{2} = M_{2}M_{1}R^{(1)} , \quad R^{(4)}M_{2}\tilde{M}_{1} = \tilde{M}_{1}M_{2}R^{(4)}$$

(10)

Comparing these equations (3) and (10) with (3) we find that the $q$-Lorentz group (3) leads to

$$R^{(1)} = R_{12} \text{ or } R_{21}^{-1} , \quad R^{(2)} = R_{21} , \quad R^{(3)} = R_{12} , \quad R^{(4)} = R_{21} \text{ or } R_{12}^{-1}$$

(11)

The solution $R^{(1)} = R_{12}$, $R^{(4)} = R_{21}$ gives

$$R_{12}K_{1}R_{21}K_{2} = K_{2}R_{12}K_{1}R_{21}$$

(12)

For $K$ expressed as in (5), eq. (12) is equivalent to the six basic relations (3) (10)

$$\alpha\beta = q^{-2}\beta\alpha \quad , \quad \alpha\gamma = q^{2}\gamma\alpha \quad , \quad [\alpha, \delta] = 0 \quad , \quad [\beta, \gamma] = q^{-1}\lambda(\delta - \alpha)\alpha \quad , \quad [\delta, \beta] = q^{-1}\lambda\alpha\beta \quad , \quad [\gamma, \delta] = q^{-1}\lambda\gamma\alpha$$

(13)

which characterize $\mathcal{M}_{q}$. We shall adopt the point of view that this associative, non-commutative algebra or ‘quantum space’ is the primary object on which the non-commutative differential calculus will be constructed. We may then give the following Definition

The quantum Minkowski space-time algebra is the non-commutative associative algebra $\mathcal{M}_{q}$ generated by the four elements of $K$, subjected to the reality conditions $\alpha = \alpha^{*}$, $\delta = \delta^{*}$, $\beta^{*} = \gamma$, $\gamma^{*} = \beta$, and satisfying the commutation relations (13) defined by (12).

The central (commuting) elements of $\mathcal{M}_{q}$ may be obtained by using the $q$-trace $tr_{q}$ (12), (13). The linear one is the $q$-trace of $K$

$$c_{1} \equiv tr_{q}K \equiv tr(DK) = q^{-1}\alpha + q\delta \quad , \quad D = diag(q^{-1}, q) = e^{q}e^{qt}$$

(14)

($e^{q}$ is the $q$-antisymmetric tensor) and satisfies $tr_{q}\mathcal{O} = tr_{q}(MQM^{-1})$ since $M^{t}D(M^{-1})^{t} = D$. The higher order central elements are given by

$$c_{n} \equiv tr_{q}K^{n} \quad , \quad Kc_{n} = c_{n}K \quad .$$

(15)

The $q$-trace $tr_{q}K = c_{1}$ is central but not invariant. As in the classical case, where $1/2 tr(\sigma_{\mu}x^{\mu}) = x^{0}$, we may identify $1/[2] tr_{q}K$ with the time coordinate. The first
two central elements \( c_1 \) and \( c_2 \) are algebraically independent; for \( n > 2 \) the \( c_n \) are polynomial functions of them due to the characteristic equation for \( K \),

\[
qK^2 - c_1K + \frac{q}{2}(q^{-1}c_1^2 - c_2)I = 0 .
\]

(16)

The \( q \)-determinant (condition c) \( \det_q K \) of \( K \) is obtained by means of the \( q \)-antisymmetrizer \( P_- \), which is a rank one \( 4 \times 4 \) projector. It is defined through

\[
(\det_q K)P_- = -qP_- K_1 \hat{R} K_1 P_- = (\alpha \delta - q^2 \gamma \beta) P_- ,
\]

(17)

where \( \hat{R} = \mathcal{P} R \) with \( \mathcal{P} \) the permutation operator. We now identify this real, central (since \( \det_q K = q(q^{-1}c_1^2 - c_2)/[2] \)) and invariant element with the square \( l_q \) of the \( q \)-Minkowski invariant length \([4, 6, 7, 2]\)

\[
l_q \equiv \det_q K = \alpha \delta - q^2 \gamma \beta , \quad l_q \in \mathcal{M}_q .
\]

(18)

If \( K \) transforms by \([3]\) say, contravariantly, then

\[
K^{ij}_q = \hat{R}^{ij,kl}_q K_{kl} \quad (K^\epsilon = \hat{R}^\epsilon K) ,
\]

(19)

where \( \hat{R}^\epsilon = (1 \otimes \epsilon^q) \hat{R}(1 \otimes (\epsilon^q)^{-1}) \), transforms covariantly i.e. \( K^\epsilon = \hat{M} K^\epsilon \hat{M}^{-1} \) (with our definition of \( \hat{R}^\epsilon, K^\epsilon \) reduces for \( q=1 \) to \( K^\epsilon \) in \([2]\) but for a sign). When \( \det_q K \neq 0 \), \( K^\epsilon = -q^{-1} (\det_q K)^{-1} \). The ‘length’ of a \( q \)-Minkowski vector is then given by the central \( L_q \)-invariant \( q \)-trace of \( K^\epsilon K \)

\[
l_q \equiv \det_q K = -q[2] tr_q(KK^\epsilon) = -q[2] tr_q(K^\epsilon K) , \quad [l_q, K] = 0 .
\]

(20)

Let us now introduce a \( q \)-Minkowski tensor \( g_{ij,kl} \) by means of the expression

\[
g_{ij,kl} = -q^{-1}[2] D_{si} \hat{R}^{es}_j \hat{R}^{es}_k \hat{l}_{im} \hat{R}^{im}_{jm,kl} \epsilon^{-1}_{it} , \quad (g = -q^{-1}[2] D_{1} \mathcal{P} \hat{R}^{12}) ,
\]

(21)

so that

\[
l_q = -q[2] D_{si} \hat{R}^{es}_j \hat{R}^{es}_k K_{ij} K_{kl} = q^2 g_{ij,kl} K_{ij} K_{kl} .
\]

(22)

Since

\[
\Lambda^\epsilon g \Lambda = g , \quad \Lambda_{rs,ij} \Lambda_{rs,kl} = g_{ij,kl} ,
\]

(23)

the tensor \( g_{ij,kl} \) may be identified with the \( q \)-Minkowski metric.

Let us come back to the other solutions of \([8]\) in \([1]\). The possibility \( R^{(1)} = R_{12} \), \( R^{(4)} = R_{12}^{-1} \) implies replacing \([12]\) by

\[
R_{12} K_1 R_{21} K_2 = q^2 K_2 R_{12} K_1 R_{12}^{-1} ,
\]

(24)

which defines a new \( K \). The commutation properties of its entries, however, are again given by \([3]\) although now \( \det_q K = 0 \). Indeed, multiplying \([2]\) by \( P_- \mathcal{P} \) from the left and by \( \mathcal{P} P_- \) from the right, we obtain using \( \hat{R}^\pm = q^\mp P_+ - q^\mp P_- \) that
(1 - q^4)P_+K_1\hat{R}K_1P_- = 0 \text{ and } (q^4 \neq 1) \ det_q K = 0. \text{ Also, the insertion of the } R\text{-matrix identity } q^2R_{12}^{-1} = R_{21} - q\lambda[2]P_+\mathcal{P} \text{ in eq. (24) reproduces (12),}

\[ R_{12}K_1R_{21}K_2 = K_2R_{12}K_1R_{21} - q\lambda[2]K_2R_{12}K_1P_+\mathcal{P} = K_2R_{12}K_1R_{21}, \tag{25} \]

since \( \det_q K = 0 \). This means that nothing is gained by considering (24) as a separate case, and it may be discarded. The other two solutions \( R^{(1)} = R_{21}^{-1}, R^{(4)} = R_{12}^{-1} \) and \( R^{(1)} = R_{21}^{-1}, R^{(4)} = R_{21} \) may be easily seen, respectively, the same as (12) and (24); thus, if the \( q \)-Lorentz group is defined by eqs. (3) we are led uniquely to (12) or (13) as the relations defining the \( q \)-Minkowski algebra \( \mathcal{M}_q \).

## 2 Deformed derivatives and \( q \)-De Rham complex

The development of a non-commutative differential calculus (see, e.g., [14, 13, 12, 11, 10] and references therein) requires including derivatives and differentials. We shall now do this for the \( q \)-Minkowski space by extending the RE to accommodate them appropriately; in this way, the \( q \)-derivatives (\( D_q \)) and the \( q \)-forms (\( \Lambda_q \)) algebras will be defined by RE. Consider first an object \( Y \) transforming covariantly \( i.e., \)

\[ Y \mapsto Y' = \tilde{M}YM^{-1} , \quad Y = \begin{bmatrix} u & v \\ w & z \end{bmatrix}. \tag{26} \]

Proceeding now as for (3) with \( Y \) replacing \( K \), the invariance of the commutation properties of the matrix elements of \( Y \) gives

\[ R^{(1)} = R_{12} \text{ or } R_{21}^{-1}, \quad R^{(2)} = R_{12}^{-1}, \quad R^{(3)} = R_{21}^{-1}, \quad R^{(4)} = R_{21} \text{ or } R_{12}^{-1}. \tag{27} \]

These four possibilities again reduce to two,

\[ R_{12}Y_1R_{12}^{-1}Y_2 = Y_2R_{21}^{-1}Y_1R_{21} , \tag{28} \]
\[ q^2R_{21}^{-1}Y_1R_{21}^{-1}Y_2 = Y_2R_{21}^{-1}Y_1R_{21} , \tag{29} \]

of which we shall retain only (28) since (29) leads (in similarity with (24)) to the same algebra plus the condition \( \det_q Y = 0 \) (see (30) below).

The (central and \( q \)-Lorentz invariant) \( q \)-determinant is defined through

\[ (\det_q Y)P_- = (-q^{-1})P_+Y_1\hat{R}^{-1}Y_1P_- = (uz - q^{-2}vw)P_- . \tag{30} \]

Since \( Y \) is covariant, we may define (cf. (19)) a contravariant \( Y^c \) by \( Y^c = (\hat{R}^c)^{-1}Y \) (when \( \det_q Y \neq 0 \), \( Y^c = -q(\det_q Y)^{-1}Y \)); then, (cf. (20))

\[ \Box_q \equiv \det_q Y = -q^{-1}[2]tr_q(YY^c) = -q^{-1}[2]tr_q(Y^cY) , \quad [\Box_q, Y] = 0 , \tag{31} \]

where \( \Box_q \) becomes the central and \( L_q \)-invariant \( q \)-D’Alembertian once the components of \( Y \) are associated with the \( q \)-derivatives. As the \( K \) matrix entries were associated with the generators of \( \mathcal{M}_q \), we shall consider the elements of \( Y \) as those of the algebra \( \mathcal{D}_q \) of the \( q \)-Minkowski derivatives.
The next step in constructing the non-commutative $q$-Minkowski differential calculus is to establish the commutation properties among coordinates and derivatives. We need extending the classical relation $\partial_{\eta} x^\nu = x^\nu \partial_{\mu} + \delta_{\mu}^\nu$, $\partial^\dagger = -\partial$, to the non-commutative case in a $q$-Lorentz invariant way. This may be done by means of an inhomogeneous RE of the form

$$Y_2 R^{(1)} K_1 R^{(2)} = R^{(3)} K_1 R^{(4)} Y_2 + \eta J,$$

(32)

where $\eta$ is a constant, $\eta J \rightarrow I_4$ in the $q \rightarrow 1$ limit, and $J$ is invariant,

$$J \mapsto \tilde{M}_2 M_1 J \tilde{M}_1^{-1} M_2^{-1} = J, \quad \tilde{M}_2 M_1 J = J M_2 \tilde{M}_1.$$

(33)

This equation exhibits the need of having $K$ and $Y$ transforming e.g. contravariantly and covariantly; indeed, the assumption that they transform in the same manner (both as $K$, say) leads to $M_1 M_2 J = J M_1 \tilde{M}_2$ which cannot be fulfilled already in the $q = 1$ case since it would imply the equivalence of unequivalent representations.

Again, an analysis similar to those of Sec.1 leads to

$$R^{(1)} = R_{12} \text{ or } R_{21}^{-1}, \quad R^{(2)} = R_{21}, \quad R^{(3)} = R_{12}, \quad R^{(4)} = R_{12}^{-1} \text{ or } R_{21}.$$

(34)

As for $J$, setting $J \equiv J' \mathcal{P}$ in eq. (33) gives $\tilde{M}_2 M_1 J' = J M_1 \tilde{M}_2$, hence $J = R_{12} \mathcal{P}$ (the same result follows if we set $J = \mathcal{P} J'$). This means that there are, in principle, four basic possibilities consistent with covariance expressing the commutation properties of coordinates (elements of $K$) and derivatives (entries of $Y$). These read

$$Y_2 R_{12} K_1 R_{21} = R_{12} K_1 R_{12}^{-1} Y_2 + \eta_1 R_{12} \mathcal{P};$$

(35)

$$Y_2 R_{21}^{-1} K_1 R_{21} = R_{21} K_1 R_{21}^{-1} Y_2 + \eta_2 R_{21} \mathcal{P};$$

(36)

$$Y_2 R_{12} K_1 R_{21} = R_{12} K_1 R_{12}^{-1} Y_2 + \eta_3 R_{12} \mathcal{P};$$

(37)

$$Y_2 R_{21}^{-1} K_1 R_{21} = R_{12} K_1 R_{12}^{-1} Y_2 + \eta_4 R_{12} \mathcal{P}.$$

(38)

Due to the fact that these expressions now involve $K$ and $Y$, they are all unequivalent. In fact, we do not need assuming that the four $Y'$s appearing in each of the equations (35–38) are the same; all that it is demanded is that they all transform as in (26).

Let us now look at the hermiticity properties of $K$ and $Y$. It is clear that, since $R^t = R^\dagger$ ($q$ is real), eqs. (12) and (28) are independently consistent with the hermiticity of $K$ and the antihermiticity of $Y$. However, this is no longer the case if the inhomogeneous equations are included. Keeping the physically reasonable assumption that $K$ is hermitian, eq. (33) gives

$$Y_2^\dagger R_{21}^{-1} K_1 R_{21} = R_{12} K_1 R_{21} Y_2^\dagger - \eta_1 R_{12} \mathcal{P};$$

(39)

i.e., $Y^\dagger$ satisfies the commutation relations given by the second inhomogeneous equation (36) for $\eta_2 = -\eta_1$ (of course, $Y'^\dagger = \tilde{M} Y'^\dagger M^{-1}$ again since $\tilde{M} = (M^{-1})^\dagger$). Thus, we need accommodating $Y^\dagger$ by means of another reflection equation, eq. (36) for $Y^\dagger$. Having selected (35) for $Y$ and (36) for $Y^\dagger$, we may now consider the possibilities (37) or (38). It turns out that they are inconsistent with the previous relation (12), what may be seen with a little effort by acting on (12) with an additional $Y$ and using (37) or (38). For instance, multiplying (12) from the left by $Y_3 R_{23} R_{13}$ and using twice (37)
Then, the relation (37) (and, analogously, (38)) is inconsistent with (12). In contrast, properties of similar calculations show that (35) and (36) are consistent with the commutation $\delta$ the $R$.

Now, since $R_{12}R_{21} = I + \lambda R_{12}\mathcal{P}$, we conclude that the two terms in (12) are different. Then, the relation (37) (and, analogously, (38)) is inconsistent with (12). In contrast, similar calculations show that (35) and (36) are consistent with the commutation properties of $\mathcal{M}_q$ and $\mathcal{D}_q$.

In order to have the inhomogeneous term in the simplest form (the analogue of the $\delta^\mu_\nu$ of the $q = 1$ case) it is convenient to take $\eta_1 = q^2$ and to redefine $Y^\dagger$ as $\tilde{Y} = -q^{-4}Y^\dagger$. In this way, the equations describing the commutation relations of the generators of the algebras of coordinates ($K$), derivatives ($Y$) and their hermitian conjugates ($Y^\dagger \propto \tilde{Y}$) are given by

$$Y_2 R_{12} K_1 R_{21} = R_{12} K_1 R_{12}^{-1} Y_2 + q^2 R_{12} \mathcal{P}$$

$$\tilde{Y}_2 R_{21}^{-1} K_1 R_{21} = R_{12} K_1 R_{21} \tilde{Y}_2 + q^{-2} R_{12} \mathcal{P} .$$

Notice that, although we identified $Y$ with the derivatives and $\tilde{Y}$ with their hermitians, the reciprocal assignment is also possible. We may also introduce a RE for $Y$ and $\tilde{Y}$; consistency selects from (27) the solution

$$\tilde{Y}_2 R_{21}^{-1} Y_1 R_{21} = R_{21}^{-1} Y_1 R_{12}^{-1} \tilde{Y}_2 .$$

The determination of the commutation relations for the $q$-De Rham complex requires introducing the exterior derivative $d$ (2); we shall assume that $d^2 = 0$ and that it satisfies the Leibniz rule. To the four generators of the $\mathcal{M}_q$ (entries of $K$) and of $\mathcal{D}_q$ ($Y$) Minkowski algebras we now add the four elements of $dK$ ($q$-one-forms), which generate the De Rham complex algebra $\Lambda_q$ (the degree of a form is defined as in the classical case). Clearly, $d$ commutes with the $q$-Lorentz coaction (1), so that

$$dK^\dagger = M dK \tilde{M}^{-1} .$$

Applying $d$ to (12) we obtain

$$R_{12} dK_1 R_{21} K_2 + R_{12} K_1 R_{21} dK_2 = dK_2 R_{12} K_1 R_{21} + K_2 R_{12} dK_1 R_{21} .$$

We now use that $R_{12} = R_{21}^{-1} + \lambda \mathcal{P}$ (and the same for 1$\leftrightarrow$2) to replace one $R$ in each term in such a way that the terms in $\mathcal{P} K_1 R_{21} dK_2$ and in $\mathcal{P} dK_1 R_{21} K_2$ may be cancelled. In this way we obtain two solutions of (46); since the relations obtained

and the YBE for different ordering of the $R$-matrices, we obtain, respectively, for the left and right hand sides

$$\text{l.h.s.} : \quad R_{23} R_{13} K_2 R_{12} K_1 R_{21} R_{32} R_{31} Y_3 R_{32}^{-1} R_{31}^{-1} \quad + \eta_3 R_{12} R_{13} K_1 R_{31} \mathcal{P}_{32} + \eta_3 R_{12} \mathcal{P}_{13} R_{23} R_{21} K_2$$

$$\text{r.h.s.} : \quad R_{23} R_{13} K_2 R_{12} K_1 R_{21} R_{32} R_{31} Y_3 R_{32}^{-1} R_{31}^{-1} \quad + \eta_3 \mathcal{P}_{23} R_{13} R_{12} K_1 R_{21} + \eta_3 R_{23} K_2 R_{32} \mathcal{P}_{13} R_{21} .$$

The cubic and the first linear terms coincide, and the last ones can be rewritten, respectively, as

$$\eta_3 R_{12} R_{21} (\mathcal{P}_{13} R_{21} K_2) \quad , \quad \eta_3 (\mathcal{P}_{13} R_{21} K_2) R_{12} R_{21} .$$

Now, since $R_{12} R_{21} = I + \lambda R_{12} \mathcal{P}$, we conclude that the two terms in (12) are different. Then, the relation (37) (and, analogously, (38)) is inconsistent with (12). In contrast, similar calculations show that (35) and (36) are consistent with the commutation properties of $\mathcal{M}_q$ and $\mathcal{D}_q$. 
are not invariant under hermitian conjugation, we may use one of them for \(dK\) and the other for the hermitian conjugate \(dK^\dagger\)

\[
R_{12}K_1R_{21}dK_2 = dK_2R_{12}K_1R_{12}^{-1} , \quad R_{12}dK_1^\dagger R_{21}K_2 = K_2R_{12}dK_1^\dagger R_{12}^{-1} ,
\]  

(47)

from which follows that

\[
R_{12}dK_1R_{21}dK_2 = -dK_2R_{12}dK_1R_{12}^{-1} , \quad R_{12}dK_1^\dagger R_{21}dK_2^\dagger = -dK_2^\dagger R_{12}dK_1^\dagger R_{12}^{-1} .
\]

(48)

We expect the \(q\)-determinant of \(dK\) to vanish; using the first eq. in (48) we check that

\[
tr_q(dK \ dK^\epsilon) = 0 ,
\]

(49)

where \(dK^\epsilon = \hat{R}^\epsilon dK\) (cf. (19)) and, in fact, \(P_+dK_1\hat{R}dK_1P_+ = 0\).

Finally, to complete the full set of commutation relations, we need those of \(dK\) and \(Y\) (and their hermitians). They are given in general by

\[
Y_2R_{21}dK_1R_{21} = R_{12}dK_1R_{21}Y_2 , \quad \tilde{Y}_2R_{12}dK_1^\dagger R_{21} = R_{12}dK_1^\dagger R_{12}^{-1}\tilde{Y}_1 ,
\]

(51)

where the second expression corresponds to the only possible consistent solution for \(50\) now written for \(\tilde{Y}\) and \(dK^\dagger\). Notice that eq. \(50\) is, as any RE, characterized by the transformation properties of its entries, and that \(Y\) and \(\tilde{Y}\) as well as \(dK\) and \(dK^\dagger\) transform in the same manner due to the condition \(\hat{M} = (M^{-1})^\dagger\). With the same structure of \(50\) (with \(Y\) (\(dK\)) replaced by \(\tilde{Y}\) (\(dK^\dagger\))) and hence with the same solutions \(34\), it is possible to introduce the commutation properties of \(\tilde{Y}, dK\) and \(Y, dK^\dagger\). Consistency gives

\[
\tilde{Y}_2R_{21}dK_1R_{21} = R_{12}dK_1R_{21}\tilde{Y}_2 , \quad Y_2R_{12}dK_1^\dagger R_{21} = R_{12}dK_1^\dagger R_{12}^{-1}Y_1 .
\]

(52)

Eqs. (12), (28), (13), (17-18) and (51-52) define the full differential calculus on \(\mathcal{M}_q\). The identification of the RE algebras generators (entries of \(K, Y\) and \(dK\) matrices) with the ones of \(\mathcal{M}_q, \mathcal{D}_q\) and \(\Lambda_q\) of \(\mathbb{P}\) is provided by

\[
K \equiv \begin{bmatrix} qD & B \\ A & C/q \end{bmatrix} , \quad Y \equiv \begin{bmatrix} \partial_D & \partial_A/q \\ q\partial_B & \partial_C \end{bmatrix} , \quad dK \equiv \begin{bmatrix} qdD & dB \\ dA & dC/q \end{bmatrix} .
\]

(53)

3 Non-commutative differential calculus and invariant operators

Using the matrices \(K, Y\) and \(dK\), we have introduced invariant differential operators as the \(q\)-Minkowski length \(l_q\) \((20)\), the \(q\)-D’Alembertian operator \(\Box_q\) \((31)\) and the exterior derivative \(d\) which, as its invariance suggests, has the form

\[
d = tr_q(dKY) .
\]

(54)
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| Reflection equations | $SL_q(2, C)$-matrix formulation | $q$-Lorentz group $R$-matrices formulation |
|----------------------|----------------------------------|------------------------------------------|
| $R_{12}K_1R_{21}K_2 = K_2R_{12}K_1R_{21}$ | $x^i x^j = R_{ij}^{kl} x^k x^l$ | |
| $R_{12}Y_1R_{21}^{-1}Y_2 = Y_2R_{21}^{-1}Y_1R_{21}$ | | $\partial_i x^j = \delta_i^j + R_{ij}^{kl} \partial_k \partial_l$ |
| $Y_2R_{12}K_1R_{21} = R_{12}K_1R_{12}^{-1}Y_2 + q^2 R_{12} P$ | | $(\xi^i \equiv dx^i) \quad \xi^i \xi^j = -R_{ij}^{kl} \xi^k \xi^l$ |
| $R_{12} dK_1 R_{21} dK_2 = -dK_2 R_{12} dK_1 R_{12}^{-1}$ | | $x^i \xi^j = R_{ij}^{kl} \xi^k x^l$ |
| $R_{12}Y_1R_{21}^{-1}Y_2 = Y_2R_{21}^{-1}Y_1R_{21}$ | | $\partial_i \xi^j = R_{ij}^{kl} \xi^k \partial_l$ |
| $Y_2R_{21}^{-1} dK_1 R_{21} = R_{12} dK_1 R_{21} Y_2$ | | $\partial_i x^j = \delta_i^j + R_{ij}^{kl} x^k \partial_l$ |
| $R_{21}Y_1Y_2 = Y_2R_{12} Y_1 R_{21}$ | | $x^i \xi^j = R_{ij}^{kl} \xi^k x^l$ |
| $Y_2R_{21}^{-1} dK_1 R_{21} = R_{12} dK_1 R_{21} Y_2 + q^{-2} R_{12} P$ | | |
| $0 = tr_q (K z K^t)$ | | |
| $L_q = \partial_q (dK z dK^t)$ | | |
| $0 = tr_q (K z dK^t)$ | | |
| $Y_{12} = q^{-2} Y_{12} - q^{2} K^t$ | | |
| $l_q dK = q^{-2} dK z l_q$ | | |
| $L_q = q^{-2} L q + g_{ij} x^j$ | | |
| $L_q = q^{-2} L q + q^{-2} s + (q^2 + 1)$ | $\Delta L = q^{-2} L \Delta + q^{-2} E + (q^2 + 1)$ | |
| $d q K = q^{-2} d K q$ | $\Delta x^i = q^{-2} x^i \Delta + q^2 g_{ij} \partial_j x^i$ | |
| $d q d K = q^{-2} d K d q$ | $\Delta \xi^i = q^{-2} \xi^i \Delta$ | |
| $d = tr_q (dK Y)$ | $d = \xi^i \partial_i$ | |
| $d \cdot K = (dK) + K zd$ | $dz x^i = \xi^i + x^i d$ | |
| $Y z d = q^2 d Y + (q^2 - 1) d K^t q$ | $\partial_i d = q^2 d \partial_i - (1 - q^{-2}) g_{ij} \xi^j \Delta$ | |
| $d (dK) = -(dK) d$ | $dz \xi^i = -\xi^i d$ | |
| $dl_q = l_q d - q^2 W$ | $dz L = L d + W$ | |
| $s q K = q^{-2} K s + K - q^{-1} L q Y^t$ | $E x^i = q^{-2} x^i E + x^i + q \lambda L \xi g_{ij} \partial_j$ | |
| $s q s Y = q^{-2} s Y + Y - q \lambda K^t q$ | $\partial_i E = q^{-2} E \partial_i + \partial_i + q^{-1} g_{ij} x^j \Delta$ | |
| $s d K = d K s$ | $E \xi^i = \xi^i E$ | |
| $s l_q = q^{-2} l_q s + (q^2 + 1) l_q$ | $E L = q^{-2} LE + (q^2 + 1) L$ | |
| $d s = d + q^{-2} s d - q \lambda W q$ | $d E = d + q^{-2} E d + q^{-1} \lambda W \Delta$ | |
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