SIGNATURE CHANGE AND CLIFFORD ALGEBRAS

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Abstract

Given the real Clifford algebra of a quadratic space with a given signature, we define a new product in this structure such that it simulates the Clifford product of a quadratic space with another signature different from the original one. Among the possible applications of this new product, we use it in order to write the minkowskian Dirac equation over the euclidean spacetime and to define a new duality operation in terms of which one can find self-dual and anti-self-dual solutions of gauge fields over Minkowski spacetime analogous to the ones over Euclidean spacetime and without needing to complexify the original real algebra.

1 Introduction

Clifford algebras (CA) are very important in theoretical and mathematical physics. It is almost impossible to list all its applications but the interested reader can found some of them in references [1]. Now, in order to define the CA of a given vector space $V$ one need to endow $V$ with a (in general symmetric) bilinear form $g_{p,q}$. One interesting fact is that the structure of the CA depends not only on the dimension of $V$ but also on the signature of $g$. In another words, real Clifford algebras $Cl_{p,q}$ and $Cl_{p',q'}$ ($p + q = p' + q' = n$) are in general not isomorphic, that is, when we change the signature we get in general different Clifford algebras. For example, the Clifford algebras $Cl_{1,3}$ and $Cl_{3,1}$ are not isomorphic – indeed, in terms of matrix algebras, the former is isomorphic to the algebra of $2 \times 2$ quaternionic matrices while the latter is isomorphic to the algebra of $4 \times 4$ real matrices.

Changing the signature of a given space may sound in principle an artificial process but undoubtly it is a very important thing in modern physics. For example, the euclidean formulation of field theories is a fundamental tool in modern physics. Indeed, sometimes it seems to be even crucial, as in the theory of instantons, in finite temperature field theory and in lattice gauge theory. Going from an euclidean to a minkowskian theory or vice-versa involves changing the signature of the metric over the spacetime and in general a minkowskian theory is transformed in an euclidean theory by analytical continuation, that is, by making $t \rightarrow it$. The interpretation of making $t \rightarrow it$ is not trivial – in relation to this, see [3] (and references therein) where it was interpreted as a rotation in a five dimensional spacetime.

Despite the ingenuity of an approach like [3] in interpreting $t \rightarrow it$ as a rotation in a five dimensional space, we believe it is unsatisfactory since the use of an additional time coordinate in spacetime appears to us to be meaningless. Indeed it would be much more satisfactory if one could find an approach to describe the signature change where there is no need to introduce extra dimensions. In the case where the problem involves CA the situation is even more problematic since the signature changed CA and the original one may not be isomorphic. A possible way to overcome this situation is to complexify the original real CA since the structure of complex CA depends only on the dimension of $V$, but this approach can also be seen as a result of introducing an extra dimension (see below).

The objective of this paper is to introduce an algebraic approach to the change of signature in CA where there is no need to introduce any extra dimension to describe it. The signature change appears as a
transformation on the algebraic structure underlying the theory. The idea is to propose an operation that “simulates” the product properties of the signature changed space in terms of the original space or vice-versa. The particular case we have in mind is the one involving the four dimensional spacetime, where we introduce an operation “simulating” the properties of minkowski spacetime in terms of an euclidean spacetime and vice-versa. This operation will be called “vee product” (since it will be denoted by a \( \lor \) in order to distinguish it from the usual Clifford product that will be denoted by juxtaposition). The advantage of this approach is obvious since we can retain the “physics” (the minkowskian properties) in a suitable mathematical world (the euclidean spacetime). The fact that we can define the vee product in terms of the Clifford product means that we can describe the minkowskian properties in terms of euclidean spacetime and vice-versa.

Our approach has been inspired by the work of Lounesto \cite{4}. But there are some differences in the method and interpretation which the reader can compare. Moreover, Lounesto only discussed some problems involving the case of signature change corresponding to opposite signatures, that is, \( (p, q) \) and \( (q, p) \). Cases like \( (p, q) \) and \( (p + q, 0) \) have not been considered by Lounesto and this is the case we have when considering minkowskian and euclidean spacetimes. Some applications of this are discussed here – and some others can be found in \cite{5}.

As an application, we are first interested in this paper in studying the Dirac equation. First of all, the version of Dirac equation we obtain by making \( t \rightarrow it \) – we shall call it the euclidean Dirac equation – has physical properties that are obviously different from the original Dirac equation – which we shall call the minkowskian Dirac equation. The question we want to address in this context is if it is possible to obtain a new equation that exhibit the same physical properties of the original equation. More specifically, our idea is to write the minkowskian Dirac equation in the euclidean spacetime, which should obviously be different from an euclidean Dirac equation in an euclidean spacetime. Minkowskian and euclidean spacetimes are different worlds, both mathematically and physically speaking, and we want to simulate the minkowskian scenario in an euclidean world. The idea is to write an equation in euclidean spacetime in terms of the vee product such that it is equivalent to the original equation in terms of the Clifford product in Minkowski spacetime. Of course the Dirac equation in terms of the vee product is expected to be different from the euclidean Dirac equation.

In order to introduce our approach we need first of all to take some care. Dirac equation is in general formulated in terms of \( 4 \times 4 \) complex matrices - the gamma matrices - obeying the relation \( \Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2g_{\mu\nu} \), where \( g_{\mu\nu} = \text{diag}(1,1,1,1) \) in the euclidean case and \( g_{\mu\nu} = \text{diag}(1,-1,-1,-1) \) in the minkowskian case. This algebra is the matrix representation of the complex Clifford algebra \( C\ell_c(4) \). On the other hand, this algebra is the complexification of the real algebras \( C\ell_{1,3} \) and \( C\ell_{4,0} \) associated with the minkowskian and euclidean spacetimes, respectively, that is, \( C\ell_c(4) \simeq \mathbb{C} \otimes C\ell_{1,3} \simeq \mathbb{C} \otimes C\ell_{4,0} \). Moreover, \( C\ell_c(4) \) is also isomorphic to the real algebra \( C\ell_{4,1} \), while the real algebras \( C\ell_{1,3} \) and \( C\ell_{4,0} \) - which are also isomorphic\footnote{However \( C\ell_{3,1} \) and \( C\ell_{4,0} \) are not isomorphic, and we shall see how to consider this case also. Anyway, it is not the isomorphism between \( C\ell_{1,3} \) and \( C\ell_{4,0} \) that matters in this discussion.} - are isomorphic to the even subalgebra of \( C\ell_{4,1} \). All these facts show that when we complexify the real algebra \( C\ell_{1,3} \) getting \( \mathbb{C} \otimes C\ell_{1,3} \simeq C\ell_c(4) \simeq C\ell_{4,1} \) we are introducing an extra dimension to the minkowskian spacetime such that the extra dimension is of the type of euclidean time. In the same way, when we complexify the real algebra \( C\ell_{4,0} \) getting \( \mathbb{C} \otimes C\ell_{4,0} \simeq C\ell_c(4) \simeq C\ell_{4,1} \) we are introducing an extra dimension to the euclidean spacetime such that the extra dimension is of the type of minkowskian time.

Now, our idea is to not use any extra dimension, and \textit{one certain way to do this} is to avoid using any of these complexified structures. This can be achieved using the real formulation of Dirac theory due to Hestenes \cite{6}. One can easily formulate the Dirac theory in terms of the Clifford algebra \( C\ell_{1,3} \), which is isomorphic to the algebra of \( 2 \times 2 \) quaternionic matrices. A Dirac spinor in this way is represented by a pair of quaternions, but rather we prefer to use the form of Dirac equation called Dirac-Hestenes equation in terms of the so called Dirac-Hestenes spinor \footnote{The reason for our choice is simple. The...}.
Dirac-Hestenes spinor is represented by an element of the even subalgebra $Cl^{++}_{1,3}$ while a Dirac spinor is an element of an ideal of $Cl_{1,3}$. Both formulations are equivalent [9] but if we use the former one we avoid the problem of considering the transformation between different ideals (in fact left and right ideals), which will happen in the latter case, and in order to not do unnecessary work – and even hidden some fundamental facts – we prefer to use the Dirac-Hestenes equation.

As another application, we discuss the problem of finding self-dual and anti-self-dual solutions of gauge fields. Since the group being abelian or not is irrelevant for this matter we shall restrict our attention to the abelian case. As is well-known, in the Minkowski spacetime there does not exist (real) solutions to the problem $* F = \pm F$, where $F$ is the 2-form representing the electromagnetic field and $*$ is the Hodge star operator. However, in an euclidean spacetime this problem has solutions and they are given by $E = \pm B$, where $E$ and $B$ are the electric and magnetic components of $F$. Now, using the operation we discussed above, we can define a new Hodge-like operator on Minkowski spacetime such that in relation to this new operator we have self-dual and anti-self-dual solutions for that problem on Minkowski spacetime. Moreover, using this Hodge-like operator we can completely simulate an euclidean metric while still working on Minkowski spacetime. We also show that the relation between those Hodge-like operators is given by the parity operation.

We organized this paper as follows. In section 2 we briefly discuss the Clifford algebras. In section 3 we discuss the transformation from euclidean to minkowskian spacetimes and vice-versa from an algebraic point of view. Our idea is to define a new Clifford product in terms of the original Clifford product that defines the algebra we are working. This new product simulates the algebra of Minkowski spacetime inside the algebra of the euclidean spacetime. Then in section 4 we discuss the minkowskian Dirac equation over the euclidean spacetime using our approach. Of course that one could be interested in the other way, that is, to simulate an euclidean world inside a minkowskian spacetime. Indeed we can consider any other possibility, as we discuss in section 5. Finally in section 6 we discuss how to obtain solutions corresponding to self-dual and anti-self-dual gauge fields and the definition of a Hodge-like operation with many interesting properties.

## 2 Mathematical Preliminaries

There are many different ways to define Clifford algebras [1], each of them emphasizing different aspects. Our approach has been chosen due to the direct introduction to Clifford product [11] which will be fundamental in the next sections.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for $\mathbb{R}^{p,q}$, where $\mathbb{R}^{p,q}$ is a real vector space of dimension $n = p+q$ endowed with an interior product $g : \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \to \mathbb{R}$. Writing the quadratic form $\sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} x_i x_j$ as the square of the linear expression $\sum_{i=1}^{n} x_i e_i$ and assuming the distributive property we obtain the well-known expression for the Clifford algebra $Cl(\mathbb{R}^{p,q}, g) \equiv Cl_{p,q}$,

$$e_i e_j + e_j e_i = 2 g_{ij} \quad (1)$$

where $g_{ij}$ are the metric components. This defines the Clifford product, which has been denoted by juxtaposition.

There is a product $\wedge$, called the exterior product, underlying the Clifford algebra. It is an associative, bilinear and skew-symmetric product of vectors. Furthermore, by applying it to our orthogonal basis we can construct a new vector space $\Lambda^2(\mathbb{R}^{p,q})$ whose elements are called bivectors, i.e., $\wedge : \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \to \Lambda^2(\mathbb{R}^{p,q})$. The skew-symmetric property allows us to extend the definition to $\Lambda^n(\mathbb{R}^{p,q})$. In general $\wedge : \Lambda^k(\mathbb{R}^{p,q}) \times \Lambda^l(\mathbb{R}^{p,q}) \to \Lambda^{k+l}(\mathbb{R}^{p,q})$.

If $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^{p,q}$ then 1 and the Clifford products $e_i, \ldots, e_i$, $(1 \leq i_1 < i_2 < \ldots < i_k \leq n)$ will establish a basis for $Cl_{p,q}$ which has dimension $2^n$. If $\{e_1, \ldots, e_n\}$ is an orthogonal basis then $e_1 \cdots e_n = e_1 \wedge \cdots \wedge e_n$, which is usually called volume element. It follows that $Cl_{p,q}$ and $\Lambda(\mathbb{R}^{p,q}) = \Lambda(n) = \bigoplus_{k=0}^{n} \Lambda^k(\mathbb{R}^{p,q})$.
are \( A \in Cl_{p,q} \) takes the form
\[
A = A_0 + A_1 + \ldots + A_n
\]  
(2)
where \( A_r \), called an \( r \)-vector, belongs to \( \Lambda^r(\mathbb{R}^{p,q}) \subset Cl_{p,q} \), \( (r = 0, 1, \ldots, n) \). More explicitly,
\[
A = a_0 + a_1^i e_i + a_2^{ij} e_{ij} + \ldots + a_1^{\ldots n} e_{1\ldots n}
\]  
(3)
It is convenient to define a projector \( \langle \rangle_r : \Lambda(V) \rightarrow \Lambda^r(V) \), i.e., \( \langle A \rangle_r = A_r \).

An important property is that Clifford algebra is a \( \mathbb{Z}_2 \)-graded algebra, i.e., we can divide it into even \( (Cl_{p,q}^+ \) and odd \( (Cl_{p,q}^- \) grades. \( Cl_{p,q}^- \) is a sub-algebra of \( Cl_{p,q} \) called even sub-algebra. In our example, \( Cl_{4,0}^+ = \{1,e_{12},e_{13},e_{23}\} \) and \( Cl_{4,0}^- = \{e_1,e_2,e_3,e_{123}\} \). Some important identities that we will use later are \( (a \in \mathbb{R}^{p,q}, B \in \Lambda^r(\mathbb{R}^{p,q}) \subset Cl_{p,q}) \):
\[
aB = a \cdot B + a \wedge B
\]  
(4)
\[
a \wedge B \equiv \langle aB \rangle_{r+1} = \frac{1}{2}(aB + (-1)^r Ba)
\]  
(5)
\[
a \cdot B \equiv \langle aB \rangle_{r-1} = \frac{1}{2}(aB - (-1)^r Ba)
\]  
(6)

3 The Vee Product

Let \( V \) be a vector space of dimension \( n = 4 \). We have five different Clifford algebras depending on the signature: \( Cl_{4,0}, Cl_{4,1}, Cl_{2,2}, Cl_{1,3} \) and \( Cl_{0,4} \). With the (possible) exception of \( Cl_{2,2} \) the importance of the others in modern physics is more than obvious.

First we shall consider the case involving the algebras \( Cl_{1,3} \) and \( Cl_{4,0} \). Let \( A, B \in Cl_{4,0} \) and \( AB \) be its Clifford product. Now we define a new product, which we call a vee product, \( A \vee B \) simulating the \( Cl_{1,3} \) Clifford product in \( Cl_{4,0} \). After the selection in \( \mathbb{R}^{4,0} \) of an arbitrary unit vector \( e_0 \) to represent the fourth dimension and the completion of the basis with three other orthonormal vectors \( e_i \), we define for \( u, v \in \mathbb{R}^{4,0} \)
\[
\mathbf{u} \vee \mathbf{v} := (-1)[\mathbf{vu} - 2(\mathbf{v} \cdot \mathbf{e}_0)(\mathbf{e}_0 \cdot \mathbf{u})]
\]  
(7)
Using this product and the \( Cl_{4,0} \) standard basis it is easy to prove that \( e_0 \vee e_0 = 1 \) and \( e_i \vee e_i = -1 \) \( (i = 1, 2, 3) \), while \( e_0^2 = e_0 e_0 = 1 \) \( (\mu = 0, 1, 2, 3) \). Moreover, for \( u, w \in \mathbb{R}^{4,0}, \)
\[
uw + wu = 2u \cdot w = 2(u_0w_0 + u_1w_1 + u_2w_2 + u_3w_3)
\]  
(8)
but if we use the vee product then
\[
u \vee w + w \vee u = -wu + 2(w \cdot e_0)(e_0 \cdot u) - uw + 2(u \cdot e_0)(e_0 \cdot w)
\]  
\[
= 2(u_0w_0 - u_1w_1 - u_2w_2 - u_3w_3)
\]
A little bit more general case is when one has a vector and a \( k \)-graded element, i.e., \( v \) and \( B_k \)
\[
B_k \vee v = (-1)^k[vB_k - 2(v \cdot e_0)(e_0 \cdot B_k)]
\]  
(9)
\[
v \vee B_k = (-1)^k[B_k v - 2(B_k \cdot e_0)(e_0 \cdot v)]
\]  
(10)
Now with the help of (6) one can see that $\lor$ is associative, that is, $v \lor (u \lor w) = (v \lor u) \lor w$. Moreover, the vee product preserves the multivectorial structure since

$$\frac{1}{2}[u, v]_{\lor} = \frac{1}{2}(u \lor v - v \lor u) = \frac{1}{2}(-vu + 2u0v0 + uv - 2u0v0) = \frac{1}{2}(uv - vu) = \frac{1}{2}[u, v] = u \land v$$

Finally, we can generalize those expression as

$$A_l \lor B_k = (-1)^{kl}[B_k A_l - 2(B_k \cdot e_0)(e_0 \cdot A_l)]$$

### 4 Dirac equation and vee product

In this section we come to consider the results recently introduced and their applications to Dirac equation. We shall use Dirac-Hestenes equation \[7, 8, 9, 12, 14\] as discussed in the introduction. As starting point for our analysis we take the Dirac-Hestenes equation in $Cl_{1, 3}$

$$\nabla \psi \gamma_{21} - m \psi \gamma_{0} = 0$$

(12)

where $\nabla$ denotes the Dirac operator, that is,

$$\nabla = \gamma_0 \partial_0 - \gamma_1 \partial_1 - \gamma_2 \partial_2 - \gamma_3 \partial_3$$

and $\gamma_\mu \ (\mu = 0, 1, 2, 3)$ are interpreted as vectors in $Cl_{1, 3}$ and $\psi = \psi(x) \in Cl_{1, 3}^+, \forall x \in M$, where $M$ is the minkowskian manifold. Now let us to ask a question: How can one simulate the minkowskian Dirac equation in an euclidean formulation? The answer is given for the vee product.

Firstly, multiplying on the right for $\gamma_{12}$ we can write (12) as

$$\nabla \psi - m \psi \gamma_{12} = 0$$

(13)

Considering $\psi \in Cl_{4, 0}^+, \forall x \in M$ and using e-notation for the $Cl_{4, 0}$ elements, the Dirac equation (13) can be written in the euclidean spacetime using the $\lor$ product as

$$\nabla \lor \psi - m \psi \lor e_{012} = 0$$

(14)

where $\nabla = e^\mu \partial_\mu$ with $e^\mu = e_\mu$. Note that $e_{012} = e_0 e_1 e_2 = e_0 \lor e_1 \lor e_2$.

Let us see how this equation appears in terms of the original Clifford product in euclidean spacetime. First we split the Dirac operator in temporal and space parts,

$$\nabla \lor \psi = e_0 \lor \partial_0 \psi + e_i \lor \partial_i \psi$$

Working on the temporal part we have

$$e_0 \lor \partial_0 \psi = \partial_0 \psi e_0 - 2(\partial_0 \psi \cdot e_0)(e_0 \cdot e_0) = \partial_0 \psi e_0 - 2\left[\frac{1}{2}(\partial_0 \psi e_0 - e_0 \partial_0 \psi)\right] = e_0 \partial_0 \psi$$

where we have used $\partial_0 \psi \cdot e_0 = \frac{1}{2}(\partial_0 \psi e_0 - e_0 \partial_0 \psi)$ For the space part we have

$$e_i \lor \partial_i \psi = \partial_i \psi e_i - 2[(\partial_i \psi \cdot e_0) e_i] = \partial_i \psi e_i$$

Therefore

$$\nabla \lor \psi = e_0 \partial_0 \psi + \partial_i \psi e_i$$

(15)
It is easy to see that
\[ \nabla \vee \nabla \vee \psi = \Box_M \psi \tag{16} \]
where \( \Box_M = \partial_0^2 - \sum_{i=1}^3 \partial_i^2 \), while \( \nabla^2 \psi = \Box_E \psi \) with \( \Box_E = \partial_0^2 + \sum_{i=1}^3 \partial_i^2 \).

The massive term in (13) will be
\[ m \psi \vee e_{012} = m e_{12} \psi e_0 \tag{17} \]
Now we can write an equation simulating the minkowskian Dirac equation in \( Cl_{4,0} \):
\[ e_0 \partial_0 \psi + \partial_i \psi e_i - m e_{12} \psi e_0 = 0 \tag{18} \]
If one considers a charged fermion field \( \psi \) in interaction with the electromagnetic field \( A \) we will add to the Dirac equation (13) the term \( e_A \psi \gamma_{12} \) where \( A \in \mathbb{R}^{1,3} \), applying the vee product we get
\[ A \vee \psi \vee e_{12} = e_{12} \psi A - 2(\psi \cdot e_0) A_0 \tag{19} \]
It is important to remark that we have never changed the algebra – we have working with \( Cl_{4,0} \). It can be proved that solutions of (13) are solutions of (18) too. For a general \( \psi \in Cl_{1,3}^+, \forall x \in M \), we get in (13) eight coupled differential equations; transforming this original \( \psi \) solution to its \( Cl_{4,0} \) version and checking it with (18) we recover exactly the same coupled system.

5 The General Case

Lounesto has studied the case involving the opposite signatures (+−−−) and (−+++). For \( a, b \in Cl_{1,3} \) the tilt transformation, based in the even-odd decomposition \( Cl_{1,3} = Cl_{1,3}^+ \oplus Cl_{1,3}^- \) first emphasized by Clifford, was defined as
\[ ab \quad \text{Cl}_{1,3} \text{product} \rightarrow b_+ a_+ + b_- a_+ + b_- a_- + b_+ a_- \quad \text{Cl}_{1,3} \text{product} \]
where the subscripts minus and plus are the odd and even parts. We reinterpret this transformation as a new product \( \vee_t \) given by
\[ A_l \vee_t (3,1) B_k = (-1)^{kl} B_k A_l \tag{20} \]
The meaning of this expression is that we are defining a mapping from an algebra \( Cl \) to its opposite algebra \( Cl_{op} \), and it is not difficult to show that \( Cl_{op}^{op} = Cl_{q,p} \).
In order to consider the general case, we consider the vee and tilt products. There are indeed some similarities
Vee product: \[ A_l \vee B_k = (-1)^{kl}[B_k A_l - 2(B_k \cdot e_0)(e_0 \cdot A_l)] \tag{21} \]
Tilt product: \[ A_l \vee B_k = (-1)^{kl} B_k A_l \tag{22} \]
It is interesting to see which is the purpose of the term with the temporal component \( e_0 \). The vee product simulates the change of signature as
where these signs correspond to the square of the canonical basis elements. All the squares change except for the temporal component. If we set up the inverse problem \((+-++) \rightarrow (++++)\), the general expression for the corresponding vee product will be the same. This is very welcome since, for example, we can get back the Euclidean Dirac equation from the Minkowskian one as in (18) just by using again the \(\lor\) product. The same holds in the opposite direction, of course.

For the tilt product,

\[
\begin{array}{c|c}
(+---) & (-+++) \\
\hline
\text{Cl}_{1,3} & \text{Cl}_{3,1} \\
\end{array}
\]

we change all the squares and now we do not need to subtract any temporal term. Again the opposite problem \((-+++) \rightarrow (+---)\) keeps up the same form. For the most interesting cases in physics we have

\[
(++++) \leftrightarrow (+---) \quad \downarrow \quad \downarrow \\
(----) \leftrightarrow (-+++)
\]

As we know the change of signature in the same row is done using the vee product and between rows of the same column using the tilt product. Now if we want to do the change in a diagonal way, i.e. \((++++) \leftrightarrow (-++++)\) then we will need to compose vee and tilt product. All this products can be extended to another signatures. For example

\[
\begin{array}{cccc}
(++++) & (----) & (-----) & (++++) \\
\downarrow & \downarrow & \downarrow & \downarrow \\
(----) & (++++) & (++++) & (+++++)
\end{array}
\]

If we want to change the signature along the same row we only need to know the canonical basis element \(e_\mu\) which doesn’t change its square, then

\[
A_l \lor B_k = (-1)^{kl}[B_k A_l - 2(B_k \cdot e_\mu)[e^\mu \cdot A_l]]
\]  

where \(\mu\)'s are not summed. And for the change between rows in the same column

\[
A_l \lor_t B_k = (-1)^{kl}B_k A_l
\]

As one can easily see there is no condition limiting this work to four dimensions, so our scheme holds in any dimension.

### 6 Other Applications

While in our discussion of the Dirac equation we were interested in simulating “Minkowskian properties” in an Euclidean spacetime, in this section we are interested in simulating “Euclidean properties” in Minkowski spacetime. Therefore in relation to the notation we are interested in simulating the product of \(\text{Cl}_{4,0}\) in terms of the generators \(\{\gamma_\mu\} (\mu = 0, 1, 2, 3)\) of \(\text{Cl}_{1,3}\).

#### 6.1 The Hodge Star Operator

The Hodge star operator \(*\) in Minkowski spacetime can be written using \(\text{Cl}_{1,3}\) as

\[
*\Phi = \Phi \gamma_5,
\]

where \(\Phi \in \text{Cl}_{1,3}\) is a multivector field and \(\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3\) and by the tilde we denoted the reversion operation such that

\[
\tilde{A}_k = (-1)^{k(k-1)/2}A_k
\]
for $A_k \in \Lambda^k$.

Using the vee product we can write the Hodge star operator $\star$ corresponding to euclidean spacetime in terms of the algebra $Cl_{1,3}$ of Minkowski spacetime as\footnote{Note that the Hodge operator in euclidean spacetime is denoted by an asterisk while the one in Minkowski spacetime is denoted by a star.}

$$\star \Phi = \tilde{\Phi} \vee \gamma_5.$$  \hfill (27)

It is easy to see that $\gamma_0 \vee \gamma_1 \vee \gamma_2 \vee \gamma_3 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$ and that $\gamma_5 \vee \gamma_5 = 1$ while $\gamma_5 \gamma_5 = -1$. In order to rewrite the above expression using the definition of the vee product it is convenient to split $\Phi$ into even and odd parts, that is, $\Phi = \Phi_+ + \Phi_-$ where $\Phi_{\pm} = \pm \hat{\Phi}_{\pm}$, where by the hat we denoted the graded involution, that is,

$$\hat{A}_k = (-1)^k A_k,$$  \hfill (28)

where $A_k \in \Lambda^k$. Now we can write using the definition of the vee product that

$$\Phi \vee \gamma_5 = \tilde{\Phi}_+ \vee \gamma_5 + \tilde{\Phi}_- \vee \gamma_5$$

$$= \gamma_5 \tilde{\Phi}_+ + 2 \gamma_123(\gamma_0 \cdot \tilde{\Phi}_+) + \gamma_5 \tilde{\Phi}_- + 2 \gamma_123(\gamma_0 \cdot \tilde{\Phi}_-),$$  \hfill (29)

and

$$\Phi_+ \vee \gamma_5 = \gamma_5 \gamma_0 \tilde{\Phi}_+ \gamma_0, \quad \Phi_- \vee \gamma_5 = - \gamma_5 \gamma_0 \tilde{\Phi}_- \gamma_0.$$  \hfill (30)

We have therefore

$$\Phi \vee \gamma_5 = \gamma_5 \gamma_0 \tilde{\Phi} \gamma_0,$$  \hfill (31)

and the euclidean Hodge star operator $\star$ can be written as

$$\star \Phi = \gamma_5 \gamma_0 \tilde{\Phi} \gamma_0,$$  \hfill (32)

where $\tilde{\Phi} = \tilde{\Phi} = \tilde{\Phi}$. Moreover, since the parity operation $\mathcal{P}$ in $Cl_{1,3}$ is given by \footnote{Note that the Hodge operator in euclidean spacetime is denoted by an asterisk while the one in Minkowski spacetime is denoted by a star.}

$$\mathcal{P}(\Phi) = \gamma_0 \Phi \gamma_0,$$  \hfill (33)

and if we rewrite (32) as

$$\star \Phi = - \gamma_0 \tilde{\Phi} \gamma_5 \gamma_0,$$  \hfill (34)

we have that

$$\star \Phi = -\mathcal{P}(\star \Phi).$$  \hfill (35)

This expression clearly shows that $\star$ is written only in terms of operations on Minkowski spacetime.

### 6.2 Differential and Codifferential Operators

In Minkowski spacetime the differential $d$ and codifferential $\delta$ operators can be written in terms of Dirac operator $\nabla$ as \footnote{Note that the Hodge operator in euclidean spacetime is denoted by an asterisk while the one in Minkowski spacetime is denoted by a star.}

$$d \Phi = \frac{1}{2} \left( \nabla \Phi + \tilde{\Phi} \tilde{\nabla} \right), \quad \delta \Phi = \frac{1}{2} \left( \nabla \Phi - \tilde{\Phi} \tilde{\nabla} \right),$$  \hfill (36)
and such that $\nabla = d + \delta$, where $\nabla = \gamma^\mu \partial_\mu$ and the right action of the Dirac operator denoted by $\bar{\nabla}$ is defined as $\bar{\Phi} \bar{\nabla} = (\partial_\mu \Phi) \gamma^\mu$. With the above definition we have that $\delta = *d*$, where $*$ is the Hodge star operator in Minkowski spacetime.\footnote{One can find different definitions for the codifferential operator $\delta$ but this is completely irrelevant for our purpose that is to give examples of applications of our method.}

Now we want to write the differential and codifferential operators corresponding to the case of euclidean spacetime using the algebra of Minkowski spacetime. Let us denote these operators by $\bar{\delta}d$ and $\delta\bar{d}$, respectively, the check being used to distinguish them from their counterparts in Minkowski spacetime. Since the differential operator is defined independently of the existence of a metric structure on a manifold, we have that $\bar{\delta}d = d$; on the other hand, the codifferential operator requires a metric for its definition and therefore we have $\delta\bar{d} \neq \delta$. In order to write the euclidean version of these operators in Minkowski spacetime the recipe is to replace the usual Clifford product by the vee product in formulas (36), that is,

$$\bar{\delta}d = \frac{1}{2} \left( \nabla \vee \Phi + \hat{\Phi} \vee \bar{\nabla} \right), \quad \delta\bar{d} = \frac{1}{2} \left( \nabla \vee \Phi - \hat{\Phi} \vee \bar{\nabla} \right),$$

(37)

The expressions $\nabla \vee \Phi = \gamma^\mu \partial_\mu \Phi$ and $\Phi \vee \bar{\nabla} = \partial_\mu \Phi \vee \gamma^\mu$ can be calculated as before. We have for $\Phi_k \in \Lambda^k$ that ($i = 1, 2, 3$)

$$\nabla \vee \Phi_k = \gamma^0 \partial_0 \Phi + (-1)^k (\partial_i \Phi_k) \gamma^i, \quad \Phi_k \vee \bar{\nabla} = (-1)^k \gamma^i \partial_i \Phi_k + \partial_0 \Phi_k \gamma^0,$$

and therefore

$$\nabla \vee \Phi = \gamma^0 \partial_0 \Phi + \partial_i \Phi \gamma^i, \quad \Phi \vee \bar{\nabla} = \gamma^i \partial_i \Phi + \partial_0 \Phi \gamma^0.$$

(38)

(39)

Now, using the expression for $\bar{\delta}d$ given in (37) we see that

$$\bar{\delta}d = \frac{1}{2} \left( \gamma^0 \partial_0 \Phi + \partial_i \Phi \gamma^i + \gamma^i \partial_i \Phi + \partial_0 \Phi \gamma^0 \right) = \frac{1}{2} \left( \gamma^\mu \partial_\mu \Phi + \partial_\mu \hat{\Phi} \gamma^\mu \right) = d\Phi,$$

(40)

that is, $\bar{\delta}d = d\delta$ as expected.

On the other hand, using the expression for $\delta\bar{d}$ given in (38) we have that

$$\delta\bar{d} = \frac{1}{2} \left( \gamma^0 \partial_0 \Phi + \partial_i \Phi \gamma^i - \gamma^i \partial_i \Phi - \partial_0 \hat{\Phi} \gamma^0 \right) = \frac{1}{2} \left( \left[ (\gamma^0 \partial_0 \Phi - \gamma^i \partial_i \Phi) - \left( \partial_0 \hat{\Phi} \gamma^0 - \partial_i \hat{\Phi} \gamma^i \right) \right] \right),$$

(41)

and clearly $\delta\bar{d} \neq \delta\bar{d}$, as expected. Using (32) for the euclidean Hodge star operator $*$ written in Minkowski spacetime we have that

$$d* \Phi = \frac{1}{2} \left( -\gamma_5 \gamma^\mu \gamma_0 \partial_\mu \bar{\Phi} \gamma_0 + \gamma_5 \gamma_0 \partial_\mu \bar{\Phi} \gamma_0 \gamma^\mu \right)$$

(42)

and therefore

$$d* \Phi = \frac{1}{2} \left( \gamma_5 \partial_\mu \Phi \gamma_0 \gamma^\mu \gamma_0 \gamma_0 + \gamma_5 \gamma_0 \partial_\mu \bar{\Phi} \gamma_5 \right) = \frac{1}{2} \left( \gamma_0 \gamma^\mu \gamma_0 \partial_\mu \Phi - \partial_\mu \bar{\Phi} \gamma^\mu \gamma_0 \right)$$

(43)

and comparing this with (37) we have

$$\delta = *d* \neq *d*,$$

(44)

as expected.
6.3 Self-Dual and Anti-Self-Dual Solutions of Gauge Field Equations

Since for what follows it is irrelevant whether or not we are considering abelian or non-abelian gauge fields, we shall consider the electromagnetic field – U(1) gauge fields – as an example for our discussion.

Let us consider the free Maxwell equations in Minkowski spacetime,
\[ \text{d}F = 0, \quad \delta F = 0. \] (45)

These equations don’t have self-dual and/or anti-self-dual solutions, that is, solutions satisfying the conditions \( F = \pm \ast F \) in the real case. Such solutions are possible in Minkowski spacetime only if we complexify the underlying algebra considering a complex \( F \). On the other hand, self-dual and/or anti-self-dual solutions of Maxwell equations exist in euclidean spacetime without the need of any complexification. This can be described in our scheme by writing the equivalent Maxwell equations in Minkowski spacetime using the vee product.

The equations
\[ \check{\text{d}}F = 0, \quad \check{\delta}F = 0, \] (46)
are written in terms of the real Clifford algebra of Minkowski spacetime and admit self-dual and anti-self-dual solutions satisfying \( F = \pm \ast F \). This condition reads
\[ F = \pm \gamma_5 \gamma_0 \bar{F} \gamma_0 = \pm \gamma_5 \gamma_0 \tilde{F} \gamma_0. \] (47)

Now writing
\[ F = E + \gamma_5 B, \] (48)
where
\[ E = \frac{1}{2} (F - \gamma_0 F \gamma_0), \quad \gamma_5 B = \frac{1}{2} (F + \gamma_0 F \gamma_0), \] (49)
and such that \( E \gamma_0 = -\gamma_0 E \) and \( \gamma_5 B \gamma_0 = \gamma_0 \gamma_5 B \), we can see that (47) is satisfied for
\[ E = \pm B, \] (50)
which are the usual self-dual and anti-self-dual solutions of the euclidean case but now written in terms of Minkowski spacetime.

7 Conclusions

Given the Clifford algebra of a quadratic space with a given signature, we have defined a new product in this structure such that it simulates the Clifford product of a quadratic space with another arbitrary signature different from the original one. We have used this in order to give an algebraic approach to the so called Wick rotation. We have used this new product in order to simulate the product associated with the Minkowski spacetime in terms of the Clifford algebra of the euclidean spacetime. We have also shown how to write the minkowskian Dirac equation in euclidean spacetime and in the other way how to write the Hodge star operator and the differential and codifferential operators corresponding to the euclidean case in terms of Minkowski spacetime, discussing self-dual and anti-self-dual solutions for the gauge field equations in this case.

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References

[1] See, for example, *Clifford Algebras and their applications in Mathematical Physics*, J.S.R. Chisholm and A.K. Common (eds.), D. Reidel Publ. Co. (1986); A. Micali, R. Boudet and J. Helmstetter (eds.), Kluwer (1992); R. Delanghe, F. Brackx and H. Serras (eds.), Kluwer (1993); V. Dietrich, K. Habetha and G. Jank (eds.), Kluwer (1998). See also *Clifford (geometric) algebras – with applications in Physics, Mathematics and Engineering*, W. Baylis (ed.), Biekhauser (1996).

[2] J. Glimm and A. Jaffe, *Quantum Physics*, 2nd. ed., Springer Verlag (1987).

[3] P. van Nieuwenhuizen and A. Waldron, Phys. Lett. B 389, 29 (1996); see also hep-th 9611043, 9702047.

[4] P. Lounesto, *Clifford algebras and Spinors*, Cambridge University Press (1997).

[5] W.M. Pezzaglia and J.J. Adams, “Should metric signature matter in Clifford algebra formulations of physical theories?”, preprint gr-qc 9704048.

[6] I. Porteous, *Clifford Algebras and the Classical Groups*, Cambridge University Press (1995).

[7] V.L. Figueiredo, E.C. Oliveira and W.A. Rodrigues, Int. J. Theor. Phys. 29, 371 (1990).

[8] D. Hestenes, *Space-Time Algebra*, Gordon and Breach, NY, (1966)

[9] W.A. Rodrigues, Q.A.G. de Souza, J.Vaz and P. Lounesto, Int. J. Theor. Phys. 35, 1849 (1996).

[10] J.E. Gilbert and M.A.M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*, Cambridge University Press (1991).

[11] M. Riesz, *Clifford Numbers and Spinors*, Fundamental Theories of Physics, Kluwer Academic Publishers, (1993)

[12] P. Lounesto, *Clifford Algebra and Hestenes Spinors*, Found. Phys. 23, 1203 (1993)

[13] W.K. Clifford, *Applications of Grassmann’s Extensive Algebra*, American Journal of Mathematics 1, 350-358 (1878).

[14] J.M. Parra, *On Dirac and Dirac-Darwin-Hestenes equations*, Clifford Algebras and Their Applications in Mathematical Physics, Proceedings of 2nd. International Conference, R. Boudet, J. Helmstetter and A. Micali (eds), Kluwer Academic Publishers, (1989)