I. INTRODUCTION

The lack of a formalism useful to describe, on the same footing, many different out of equilibrium systems, is one of the most important unsolved problems in thermostatistics. While the equilibrium thermodynamics for ergodic systems can be derived from the statistical theory of Gibbs, out of equilibrium systems exhibit a great diversity, thus making difficult the unification of their description within a single theory.

Fluctuation theorems (FT) [1–11] are exact relations for the probability distributions for the values $W$ of observables $W$ which are functionals of the stochastic state-space system trajectories (e.g. work, heat or more generally, different forms of trajectory-dependent entropy production) in processes driven by arbitrary protocols. These relations, valid even beyond the linear regime, contain as particular cases the Green-Kubo formula, the Onsager reciprocity relations and the second law of thermodynamics for systems with equilibrium steady states in the limit of small entropy production [3, 12]. For systems with non-equilibrium steady states (NESS), they provide a generalization of the second law of thermodynamics [8] which gives a formal framework for the phenomenological theory of Oono and Paniconi [13] for the thermodynamics of NESS. In this case, also a generalized fluctuation-dissipation relation (FDR) can be derived [14–22, 42], as well as some ‘symmetry’ relations for the response functions involved in this generalized FDR [23, 24]. For a recent review on fluctuation theorems and their applications, see [25].

For all these reasons, FTs seem to be a good starting point in order to derive a more general theory unifying the physics of many different out-of-equilibrium systems. In fact, many features of Markovian systems are well described within the same formalism.

In spite of this, non-Markovian systems demand much more work and comprehension.

In the previous recent years, some important progress has been made in this line. For example, the work fluctuations theorems (Jarzinsky identity and Crooks fluctuation theorem) as well as the fluctuation theorem for the total entropy production have been derived for non-Markovian systems evolving with ergodic dynamics [26–29]. On the other hand, an asymptotic FT relevant for glassy dynamics has been derived in [30], while the validity of the work relations has been studied in the context of anomalous dynamics for the first time in [31]. A discussion on the role of substitute Markov processes has been presented in [36].

However, a lot of work remains to be done. For systems with NESS the entropy production associated to the total heat exchanged with the bath does not represent a relevant lower bound for the entropy change in the system [8]. In fact, the entropy production which gives the most accurate lower bound should be related to the transitions of the system between different steady states. When detailed balance holds, all the heat exchanged with the reservoir is used for the system in order to jump from one steady state to the other, however, without detailed balance, part of this heat is used in order to maintain an steady-state with non-vanishing currents. In this case, the correct lower bound is given by the excess entropy, associated to the excess heat, which is the energy delivered or absorbed for the system when it jumps between different steady states [8]. Then, a description in terms of the Hatano-Sasa functional, or more generally, in terms of the non-adiabatic entropy production [32, 33] is needed. As far as we know, these quantities have not been studied yet for non-Markov dynamics.

The aim of this paper is to fill this gap. In fact, we show that the non-adiabatic entropy production, as defined for Markovian systems, can satisfy an integral fluctuation theorem (IFT) if the system fulfills certain “stability”
condition which we discuss below. In this context it is possible to extend the second law for transitions between NESS and the validity of the generalized FDR to non-Markov dynamics.

The paper is organized as follows. In section II we present the basic notations and introduce the evolution operators for some “substitute Markov processes” relevant for the definition of generalized fluxes. In section III we introduce our concept of stability and discuss an important property of stable systems. In section IV we introduce the non-adiabatic entropy production and we discuss some conclusions remarks are given in VII.

II. PRELIMINARIES

A. Basic definitions and notations

To start, we will consider continuous time processes for a system that can be described either by discrete or continuous configurations. A configuration will be denoted by \( x \), which may generically represent a single variable, a vector, or a field. The system can be additionally driven by a set of time-dependent external parameters which will be denoted by \( \lambda \).

Trajectories from the initial time \( t_i = 0 \) to the final time \( t_f = t \) will be denoted by vectors, so that \( \hat{e} = (e(\tau))_{\tau=0}^t \), with \( e = (x, \lambda) \).

Otherwise stated, stochastic functionals (like entropy productions) will be denoted by calligraphic capital letters, while specific values for these functionals will be denoted by the same, capital but latin character. For example, the non-adiabatic entropy production functional will be denoted by \( S_{\text{na}}[x; \lambda] \), while for the specific values of this quantity we will write \( S_{\text{na}} \).

We assume, as a fundamental requirement in our theory, that the system is ergodic.

Let us denote by \( p(x, t) \) the probability density function (PDF) for the system to be in the vicinity of configuration \( x \) at time \( t \) and by \( p^\lambda(x; \lambda) \) the steady state PDF for a constant protocol \( \lambda(t) = \lambda \). Given the non-Markovian nature of the process, one should be careful in order to introduce this distribution, as the system at any time (except for the initial time \( t_i \)) is correlated with its previous history. Ergodicity ensures that at constant protocols the system forgets its initial preparation for long enough times, so \( p^\lambda(x; \lambda) \) should be thought as the limiting PDF reached by the system at time \( t \to \infty \), if the protocol is stopped at time \( t' \ll t \) with value \( \lambda(t') = \lambda \).

The initial PDF will be assumed arbitrary and denoted by \( p_0(x) \). On the other hand, conditional probabilities will be denoted by \( K \).

Finally, the evolution operators for the substitute Markov processes (which will be introduced in the next paragraph), will be generically denoted by \( \hat{\mathcal{L}} \).

B. Substitute Markov processes

In general terms, the evolution equation for the one-time probability density of non-Markovian processes involves convolutions in time, making explicit the fact that the system is correlated with its past. Fokker-Planck equations with fractional time derivatives constitute very known examples of this fact. However, the evolution equation can, under very general conditions, be written in a convolutionless (not memoryless) form in terms of an evolution operator related to some ‘substitute’ Markov process having the same one-time probability as the original non-Markovian process [34, 35]. The evolution equation reads

\[
\partial_t p(x, t) = \hat{\mathcal{L}}(t; t_i, [\lambda]) p(x, t),
\]

where we have made explicit the fact that the evolution operator depends on the initial time, and also on the whole protocol up to time \( t \). This reflects the effect of the memory of the process. The form of the operator also depends on the initial PDF of the system at time \( t_i \), where it is prepared without any correlation with previous times. This means that to each particular preparation of the system, corresponds a different Markovian substitute. When the process is Markovian, the correlation with the previous history is no longer present and the given operator generates a semigroup, independently of the initial PDF. In this case one recovers the usual Master (Fokker-Planck or Kramers) equation for discrete (continuous) variables.

In what follows, we will drop for simplicity the initial time and the protocol dependency on the arguments of the evolution operator, so we will just write \( \hat{\mathcal{L}}(t) \).

The evolution equation (1) can be easily derived from the general properties of conditional probabilities[34]. In fact, let \( K(x, s|x', t) \) be the conditional probability for the system to be in the vicinity of \( x \) at time \( s \) given that it was around \( x' \) at time \( t \). One then can write for any process, being or not Markovian

\[
p(x, s) = \int dx' K(x, s|x', t) p(x', t).
\]

It is worth noting that if the process is non-Markovian, the conditional probability will depend on the whole history up to time \( s \). From the previous equation it is easy to see that

\[
\partial_t p(x, t) = \int dx' \dot{M}(x, x', t) p(x', t),
\]
where \( \hat{M}(x, x', t) = \lim_{t \to t+} \partial_s K(x, s|x', t) \), from which it is immediate to identify the operator \( \hat{\mathcal{L}} \).

### III. STABILITY

We say that a non-Markovian system is stable if it, being prepared at the initial time (where there is no correlation with the previous history) with the steady-state PDF corresponding to \( \lambda \), evolves in such way that if \( \lambda \) does not change in time, then for all times the PDF remains unchanged. With a Markovian process in mind, it may seem to be counter-intuitive that the stability criterion could be violated, but for non-Markovian dynamics the violation may indeed occur in many realistic models. In order to understand why it is possible, we first make the reader to note that for Markovian systems there is no difference between preparing the system in an arbitrary state in a very remote moment in the past and to start to observe it at \( t = 0 \), when it has evolved to its steady-state, and to directly prepare the system in the steady-state at \( t = 0 \). This is so, because previous history does not matter at all. However, this is not the case for non-Markovian dynamics. Consider the conditional probability for the system to be around \( x \) at time \( t \), given that it was for sure at \( x_0 \) at time \( t_0 = 0 \), in two different contexts, when the system is prepared initially at \( t = 0 \) with the steady-state PDF without any correlation with its previous history and when the system were prepared at \( t = -\infty \) in an arbitrary state, letting the value of \( \lambda \) unchanged. Let us denote these propagators by \( K(x, t|x_0) \) and \( K_{\infty}(x, t|x_0) \) respectively. It is clear that, in general, \( K(x, t|x_0) \neq K_{\infty}(x, t|x_0) \). Also note that \( \int dx_0 K_{\infty}(x, t|x_0)p^x(x_0; \lambda) = p^x(x; \lambda) \). Then, it may be possible that

\[
\int dx_0 K(x, t|x_0)p^x(x_0; \lambda) \neq p^x(x; \lambda),
\]

leading to the instability that we are describing. In words, in both cases the position at \( t = 0 \) is sampled from the steady-state PDF, but in the first case, the system is not really in its steady-state which, for non-Markovian systems, is not entirely determined by the one-time PDF. The stability criterion introduced above is in fact abstract since in practice the typical, and to be best of our knowledge the only, achievable way to prepare a system so that it is described by its steady PDF, is allowing it to evolve for a long enough time keeping the values of the external parameters fixed, as considered in [26]. However, we will see below that one does not actually need to be able to prepare the system sampling its initial values from the steady PDF, and that considering an arbitrary initial distribution, the problem of the validity of the fluctuation theorem for the non-adiabatic entropy production is mapped to the problem of its validity in the hypothetical case we are discussing (see equations (13) and (14) below). Then, the main lesson extracted from this abstract concept is that the lack of stability gives information about the dynamics of the system in realistic situations: the fluctuation theorem for the non-adiabatic entropy production does not hold for any initial preparation, even in experimentally accessible conditions if one considers the evolution of the system at finite times and not in the asymptotic regime considered, for example, in [30]. A practical way to identify unstable systems in realistic situations will be discussed in terms of the fluctuation-dissipation theorem in a future work [46].

To continue with our discussion, let us introduce the Hatano-Sasa functional \( \mathcal{Y} \) defined as

\[
\mathcal{Y}[\bar{x}; \bar{\lambda}] = \int_0^\infty d\tau \bar{\lambda}(\tau) \partial_\lambda \phi(x(\tau); \lambda(\tau)),
\]

with \( \phi(x; \lambda) = -\ln p^x(x; \lambda) \). Let us also introduce \( \hat{P}_x(x, Y, t) \), the joint PDF for the system to be in configuration \( x \) at time \( s = t \), having observed a value of the Hatano-Sasa functional \( Y \), when the initial condition is sampled from the steady state PDF, and its Laplace transform \( \hat{P}_x(x, Y, t) = \int dY e^{-Y Y} P_x(x, Y, t) \). Let us denote \( \hat{P}_x(x, t) = P_x(x, Y = 1, t) \). It turns out that for stable systems, we have the identity

\[
\hat{P}_x(x, t) = p^x(x, \lambda(t)).
\]

The proof of this statement is based on a general property of correctly defined substitute Markov evolution operators which has been shown in [26], say

\[
\hat{\mathcal{L}}(t)p^x(x; \lambda(t)) = 0 \quad \forall t \geq 0,
\]

for arbitrary protocols if the system is prepared in its steady-state. It is worth to say that, for the evolution operator to be correctly defined, a full phase-space description (inclusion of all the degrees of freedom) is needed, as suggested by Ref.[36]. In that reference, it has been shown that for correlated noises the velocity ceases to be an slow variable and should be included in the determination of the substitute evolution operator, even if one is studying the probability of a quantity which does not depend on velocity. Equation (6) can be shown by simple inspection by noting the evolution equation for \( \hat{P}_x(x, t) \)

\[
\partial_t \hat{P}_x(x, t) = \hat{\mathcal{L}}(t)\hat{P}_x(x, t) - \hat{\lambda}(t) \partial_\lambda \phi(x; \lambda(t))\hat{P}_x(x, t),
\]

with initial condition \( \hat{P}_x(x, 0) = p^x(x; \lambda_0) \) and using the identity given by equation (7). Equation (8) can be obtained by Laplace transforming the evolution equation for \( P_x(x, Y, t) \), which corresponds to the same equation for the PDF of the \( x \) variable plus an extra term associated to the current in the \( Y \) direction \( \dot{J}_Y(t) = \dot{\lambda}(t) \partial_\lambda \phi(x; \lambda(t))P_x(x, Y, t) \)

\[
\partial_t P_x(x, Y, t) = \hat{\mathcal{L}}(t)P_x(x, Y, t) - \partial_Y J_Y(t).
\]

The above reasoning seems to be correct, but there is some important condition that has to be satisfied: stability. In fact, if stability does not hold, then (7) and
correspondingly (6) do not hold either. In order to see this, let us assume that (7) holds for arbitrary protocols, but the system is unstable. Then, for constant protocols one should have, by virtue of instability, that \( p(x, dt) \neq p^\ast(x; \lambda_0) \). On the other hand, by virtue of (7) we have \( p(x, dt) = p^\ast(x; \lambda_0) + dt \hat{L}(0)p^\ast(x; \lambda_0) = p^\ast(x; \lambda_0) \). This contradiction is solved only if (7) does not hold for unstable systems.

As previously announced, in section V we study a simple unstable system. The skeptic reader could argue that this instability is artificial, since the violation of (7) may be associated to a by-product of the substitute Markov process provided that one does not have a method to define it correctly in the general case. For this reason, we discuss the instability of the model system presented in V not by means of substitute Markov processes, but directly using generalized Langevin equations (GLE) methods, which are always correct [36]. On the other hand, in section VI we study an stable system, where the generalized FDR holds.

Before finishing with this section, we would like to briefly discuss about some systems where stability could be violated. If one considers out-of equilibrium degrees of freedom performing non-Markovian dynamics, some “effective” non-local time-dependency on the protocol can be self-generated by performing local transformations of variables, a feature exclusively associated to non-Markovianity (see section V). On the other hand, if the non-Markovian noise acting on the system does not satisfy an equilibrium FDT of the second kind (at least in terms of an effective temperature), the system is unstable.

### IV. NON-ADIABATIC ENTROPY PRODUCTION

#### A. Definition

This paragraph is devoted to present the main definition of non-adiabatic entropy production and its physical meaning. This entropy production have been introduced for the first time in Refs. [32, 33] for Markovian systems. We will define it here in the same way, but it will be written in a different (although equivalent) form. We define

\[
\mathcal{S}_{na}[\vec{x}; \vec{\lambda}] = -\ln \frac{p(x(t_f), t_f)p^\ast(x(t_i); \lambda(t_i))}{p_0(x(t_i))p^\ast(x(\lambda(t_i)))} + \mathcal{Y}[\vec{x}; \vec{\lambda}],
\]

(10)

Note that, as in [33], a splitting of the non-adiabatic entropy production into a boundary contribution and a driving contribution becomes apparent. The first term, accounts for the relaxation of the system to the steady-state, while the second term is only non-zero in the presence an external protocol.

It is also worth saying that in [32] this quantity have been defined as the logratio of the path probabilities of two different systems, obtaining (10) as a result. We use here (10) as formal definition, avoiding any reference to path probabilities and hence, making this definition extensible to non-Markov dynamics without identifying any ‘dual’ dynamics, since, even if for Markovian systems the dual dynamics can be straightforwardly associated to a system with different interactions [23, 24, 37], for non-Markov dynamics this identification could be more intricate.

#### B. Integral fluctuation theorem

In order to start with the derivation of the IFT for the non-adiabatic entropy production, we point out first that, according to equation (10) we can write

\[
e^{-\mathcal{S}_{na}} = \frac{p(x, t)p^\ast(x_0; \lambda_0)}{p_0(x_0)p^\ast(x(t))}e^{-\mathcal{Y}},
\]

(11)

where \(x_0\) corresponds to the initial position while \(\lambda_0 = \lambda(0)\). Let us now introduce the conditional probability for the system to be in configuration \(x\) at time \(t\), and having observed a value of the Hatano-Sasa functional \(Y\), given that it was in configuration \(x_0\) at \(t_i = 0\). The initial value of the Hatano-Sasa functional has not to be specified since it is always zero. Let us denote this conditional probability by \(K(x, Y, t|x_0)\). For any observable of the form \(\Lambda(x, Y, t, x_0)\) one can write

\[
\langle \Lambda(x, Y, t, x_0) \rangle = \int dx dx_0 dY \Lambda(x, Y, t, x_0) K(x, Y, t|x_0)p_0(x_0),
\]

(12)

which implies

\[
\langle e^{-\mathcal{S}_{na}} \rangle = \int dx_0 p_0(x_0) \int dx \int dY \frac{p(x, t)p^\ast(x_0; \lambda_0)}{p_0(x_0)p^\ast(x(t))} K(x, Y, t|x_0)e^{-\mathcal{Y}} = \int dx \frac{p(x, t)}{p^\ast(x; \lambda(t))} \int dY P_\ast(x, Y, t)e^{-\mathcal{Y}},
\]

(13)

which directly leads to

\[
\langle e^{-\mathcal{S}_{na}} \rangle = \int dx P_\ast(x, t) \frac{p(x, t)}{p^\ast(x; \lambda(t))},
\]

(14)

If the system is stable in the sense discussed in the previous section, we have

\[
\langle e^{-\mathcal{S}_{na}} \rangle = 1.
\]

(15)
Equation (15) is the first main result of this paper. It encodes most of the fundamental aspects for the non-adiabatic entropy production to be a meaningful thermodynamic quantity for ergodic systems. It is worth noting that if the invariant measure of the system corresponds to the Boltzmann-Gibbs distribution and the system is initially prepared in this state, the previous result reduces to the Jarzynski relation, already derived in [26] for general non-Markovian ergodic systems following an approach based on Markov substitute processes, and in [27, 29] for non-linear generalized Langevin systems by means of an approach based on functional probabilities of trajectories.

C. Second law

Let us introduce the excess entropy functional $S_{\text{ex}}$ [8] as

$$S_{\text{ex}}[\tilde{x}; \lambda] = - \int_0^t d\tau \hat{J}_i \phi(x; \lambda).$$

(16)

For discrete spaces, the integral in (16) should be replaced by a sum as $\sum_{k=1}^N \left[ \hat{J}_i \phi(x_{k-1}; \lambda_i) - \hat{J}_i \phi(x_k; \lambda_i) \right]$, where $k = 1, 2, \ldots, N$ labels the set of time instants when the system jumps between different configurations. From (10) and (16) we can write

$$S_{\text{ex}} = \Delta S_{\text{a}} + S_{\text{ex}},$$

(17)

with $\Delta S_{\text{a}} = -\ln p(x(t), t)/p_0(x_0)$ the entropy change of the system. From (15), (17), and the Jensen inequality, it follows

$$\langle \Delta S_{\text{a}} \rangle \geq -\langle S_{\text{ex}} \rangle.$$

(18)

Equation (18), which is a direct consequence of (15), is the second main result of our paper. It represents the second law of thermodynamics for transitions between NESS exactly as expressed in [8].

D. Generalized fluxes and forces

Another important property (not directly derived from (15) but also crucial in order to build a coherent thermostatistics) is that the time derivative of the average non-adiabatic entropy production can be expressed as a sum of products of generalized fluxes and forces, as for Markov dynamics [33]. From the definition (10), it follows that

$$\frac{d}{dt} \langle S_{\text{ex}} \rangle = -\int dx \partial_i p(x, t) \ln \frac{p(x, t)}{p^*(x; \lambda(t))}.$$

(19)

Recalling now (3), we can write

$$\frac{d}{dt} \langle S_{\text{ex}} \rangle = -\int dx dx' \dot{M}(x', t)p(x', t) \ln \frac{p(x, t)}{p^*(x; \lambda(t))}.$$  \tag{20}

From (3) and the normalization condition for PDFs, one can see that $\int dx \dot{M}(x, x', t) = 0$. Then, we can safely add a zero to equation (20) as

$$\int dx dx' \dot{M}(x', x, t)p(x, t) \ln \frac{p(x, t)}{p^*(x; \lambda(t))} = 0,$$

(21)

obtaining

$$\frac{d}{dt} \langle S_{\text{ex}} \rangle = -\int dx dx' \mathcal{J}(x, x', t) \ln \frac{p(x, t)}{p^*(x; \lambda(t))},$$

(22)

with the fluxes $\mathcal{J}(x, x', t) = \dot{M}(x', x, t)p(x', t) - \dot{M}(x', x, t)p(x, t)$. Note that $\mathcal{J}(x, x', t) = -\dot{M}(x', x, t)$, as it should be. Changing now $x$ by $x'$ in (22), summing the resulting equation term by term with (22) and dividing by two, we finally obtain

$$\frac{d}{dt} \langle S_{\text{ex}} \rangle = \frac{1}{2} \int dx dx' \mathcal{J}(x, x', t) \mathcal{F}(x, x', t),$$

(23)

with the forces $\mathcal{F}(x, x', t) = \frac{\partial(x; \lambda(t))p(x, t)}{\partial(x; \lambda(t))p(x', t)}$. Equation (23) constitutes the third main result of our work.

For general non-Markovian dynamics it is a hard task to build the evolution operator for the Markovian substitute process (except for Gaussian and two-level systems [34, 35]), so, the generalized currents can be hard to compute, however, equation (23) may be very important from the conceptual (and hopefully also from the experimental) point of view. We also point out that this property is valid even if the system is unstable.

E. Generalized fluctuation-dissipation relation

As expressed previously, if the stable system is initially prepared in the steady-state compatible with some values of the external protocols, then equation (15) reduces to the Hatano-Sasa identity

$$\left\langle \exp \left[ -\sum_i \int_0^t d\tau \partial_i \phi(x; \lambda_i) \right] \right\rangle = 1,$$

(24)

where we have explicitly introduced the index $i$ to label all the external parameters. Suppose that at the initial time we have $\lambda_i(t = 0) = \lambda_{i0}$ and that for $t > 0$ we have $\lambda_i(t) = \lambda_{i0} + \delta \lambda_i(t)$, with $|\delta \lambda_i|/\lambda_{i0} \ll 1$. Then, introducing the observables $b_i(t) = \partial_i \phi(x(t); \lambda_0)$ and repeating the same steps as in Ref. [18], we have the generalized FDR

$$\langle b_i(t) \rangle = \sum_j \int_0^t dt' \frac{d}{dt} \langle b_i(t) b_j(t') \rangle \delta \lambda_j(t'),$$

(25)

where $\langle \ldots \rangle$ denotes averages in the perturbed system, while $\langle \ldots \rangle_{\text{ss}}$ denotes averages in the unperturbed system, where the parameters are kept fixed at their initial values. Note that, if stable, the system prepared in the
steady-state remains there always as long as one does not perturb it. This justifies the double-s subscript.

Equation (25) constitutes the fourth main result of our paper. It is, in our opinion, a rather important result since it is commonly claimed to hold exclusively for Markovian dynamics. This is, to the best of our knowledge, the first time that this relation is extended, in this particular way, to non-Markovian systems, clarifying that the crucial conditions for (25) to hold are ergodicity and stability and not Markovianity. We however point out that FDRs for non-Markov dynamics have been studied before (see for example the pioneering work [34], and the more recent work [47]).

V. A MODEL SYSTEM EXHIBITING INSTABILITY

In this section we will study a model system exhibiting instability. In this case, the Hatano-Sasa relation is violated and correspondingly, the modified FDR does not hold. In this model, we consider the steady-state distribution of a genuine non-equilibrium degree of freedom for which the equation of motion is obtained from the original equation of motion by means of a local transformation, generating a non-local dependency on the external protocol. The system is ergodic, however if it is prepared at $t = 0$ without any correlation with the past, in such a way that the initial position is sampled from the corresponding steady-state PDF, and the external protocol is kept constant, it abandons its initial state and returns to it after a transient time.

Consider the following GLE:

$$\int_0^t d\tau \gamma(t - \tau) \dot{x}(\tau) = -k(x(t) - x_c(t)) + \xi(t),$$

where the Gaussian noise $\xi(t)$ have zero mean and a second cumulant $(\langle \xi(t) \xi(t') \rangle = T \gamma(|t - t'|))$, with $T$ the temperature of the bath. This relation is not sufficient in order to ensure the ergodicity of the dynamics (see for example [38]). We here assume that the system is ergodic. We also point out that equation (26) can serve as a model for a particle dragged through a viscoelastic liquid by an optical trap, which is experimentally accessible [40].

If we consider as the external protocol the position of the trap center, we see that the system described by (26) reaches an equilibrium state for constant $x_c$, which means that the degree of freedom $x$ is able to equilibrate. Imagine now that we consider as external protocol not the position of the trap center, but its velocity $v_c(t)$ as in the experiment in [39], such that a constant protocol means constant velocity. In this case, the variable $x$ is a genuine out-of-equilibrium degree of freedom which is not even allowed to reach an steady value. We can however recover a variable capable to reach an steady value if we consider the quantity $y(t) = x(t) - x_c(t)$, which is also a genuine out of equilibrium degree of freedom. In terms of the variable $y$, the equation of motion reads

$$\int_0^t d\tau \gamma(t - \tau) \dot{y}(\tau) = -ky(t) + F(t) + \xi(t),$$

where the force $F$ is given by

$$F(t) = -\int_0^t d\tau \gamma(t - \tau)v_c(\tau).$$

Note that even for constant velocity, this force is time dependent. This non local in time dependency on the external protocol will have drastic consequences. Also note that for Markov dynamics with $\gamma(t) \sim \delta(t)$, the dependency on the external protocol becomes local.

The solution of (27) can be found by means of the Laplace transform and reads

$$y(t) = G_1(t)y_0 - \int_0^t d\tau G_1(t - \tau)v_c(\tau) + \eta(t),$$

where $\eta(t) = \int_0^t d\tau G_2(t - \tau)\xi(\tau)$. The quantities $G_1(t)$ and $G_2(t)$ are the inverse Laplace transforms of $G_1(u)$ and $G_2(u)$ given by

$$\hat{G}_2(u) = |u^2(\gamma(u) + k)^{-1}; \hat{G}_1(u) = \hat{\gamma}(u)\hat{G}_2(u).$$

The noise $\eta$ has zero mean and correlator

$$\Delta(t, t') = \langle \eta(t)\eta(t') \rangle = \frac{T}{k} \left[ G_1(|t - t'|) - G_1(t)G_1(t') \right],$$

as can be easily shown by direct calculation in the Laplace space. Note that ergodicity requires that $\lim_{t \to \infty} G_1(t) = 0$, while compatibility with the initial conditions demands $G_1(t = 0) = 1$. We also note that from (30) we can obtain the following identity

$$G_1(t) = 1 - kH(t),$$

with $H(t) = \int_0^t d\tau G_2(\tau)$. One can see, by using (29), (31), and (32), and using the properties of $G_1$ that for constant $v_c$ the system reaches the following distribution

$$p(y; v_c) = \sqrt{\frac{k}{2\pi T}} \exp \left[ - \frac{k(y + \gamma_{\text{eff}} v_c)^2}{2T} \right],$$

where $\gamma_{\text{eff}} = \gamma_{\text{eff}}(t \to \infty)$, and $\gamma_{\text{eff}}(t) = \int_0^t d\tau G_1(\tau)$. Now let us assume that the initial position of the particle is sampled from this distribution, which means that

$$\langle y_0 \rangle = -\gamma_{\text{eff}} v_c,$n

$$\langle \delta y^2 \rangle = \frac{T}{k},$$

where $\delta y_0 = y_0 - \langle y_0 \rangle$. Using this in (29) for constant $v_c$, we obtain that still $\langle \delta y^2(t) \rangle = \langle \delta y_0^2 \rangle = \frac{T}{k}$, with $\delta y(t) = y(t) - \langle y(t) \rangle$, however, the mean value of the process is

$$\langle y(t) \rangle = -[\gamma_{\text{eff}} G_1(t) + \gamma_{\text{eff}}(t)]v_c \neq \langle y_0 \rangle.$$
As the process is Gaussian, this is enough to ensure that 
\( p(y, t) \neq p(y; v_c) \) for finite times, thus, the system is
unstable. Note however that
\[
\lim_{t \to \infty} \langle y(t) \rangle = \langle y_0 \rangle, \tag{37}
\]
which means that the system decays again to the steady-state PDF.

We remark once again that this instability is the result of sampling the initial position from the steady-state PDF at the very beginning of the evolution, where the system is uncorrelated with the bath and its previous history. Once the system reaches the steady state after a long time in interaction with the thermal bath, this instability disappears. In other words, in this model the instability is associated to the fact that initially, even if the system is prepared with the steady-state PDF, the non-local force defined by (28) depends on time even for constant protocols. This does not happen for a system prepared far away in the past. When this time-dependent force relaxes, the instability disappears.

Finally, it is worth noting that in the Markovian case the instability is no longer present. In fact, if \( \gamma(t) = \delta(t) \), then \( G_1(t) = e^{-kt} \) and \( \gamma_{\text{eff}}(t) = \gamma_{\text{eff}}[1 - G_1(t)] \), with \( \gamma_{\text{eff}} = 1/k \). Using this in (36), we immediately see that \( \langle y(t) \rangle = \langle y_0 \rangle \) for all times.

VI. OVERDAMPED HARMONIC OSCILLATOR COUPLED TO TWO NON-MARKOVIAN BATHS

We will now test the validity of (25) for non-Markovian ergodic systems considering a very simplistic (and more unrealistic than the previous one) model, but useful in order to illustrate our findings. Consider an overdamped harmonic oscillator, coupled to two non-Markovian baths
\[
\int_0^t d\tau \gamma_1(t - \tau) + \gamma_2(t - \tau) = -kx(t) + f(t) + \xi_1(t) + \xi_2(t), \tag{38}
\]
with Gaussian noises with variances \( \langle \xi_\nu(t)\xi_\nu(t') \rangle = T_{\nu\nu}\gamma_\nu(|t - t'|) \), \( \nu = 1, 2 \). The force \( f \) will be considered as an external protocol. We can identify the heat exchanged with each reservoir
\[
dQ_\nu(t) = \left[ \int_0^t d\tau \gamma_\nu(t - \tau) \dot{x}(\tau) - \xi_\nu(t) \right] \dot{x}(t) dt, \tag{39}
\]
the energy change
\[
dE(t) = kx(t)\dot{x}(t) dt, \tag{40}
\]
and the work
\[
dW(t) = f(t)\dot{x}(t) dt, \tag{41}
\]
from where the first law of thermodynamics follows
\[
dE(t) = dW(t) - dQ(t), \tag{42}
\]
with the total heat \( dQ = dQ_1 + dQ_2 \). In order to ensure the ergodicity and the stability of the dynamics, we take the two baths to be identical, so \( \gamma_1(t) = \gamma_2(t) = \frac{1}{2}\gamma(t) \). Introducing then the effective temperature \( T_{\text{eff}} = \frac{1}{2}(T_1 + T_2) \), we can write for the dynamics of the system
\[
\int_0^t d\tau \gamma(t - \tau) \dot{x}(\tau) = -kx(t) + f(t) + \xi(t), \tag{43}
\]
with \( \langle \xi(t)\xi(t') \rangle = T_{\text{eff}}\gamma(|t - t'|) \). The solution of this equation is given by
\[
x(t) = G_1(t)x_0 + \int_0^t d\tau G_2(t - \tau)f(\tau) + \eta(t), \tag{44}
\]
with \( G_1, G_2, \) and \( \eta \) as given in V. Then, the steady-state PDF for this system is given by the Boltzmann-Gibbs distribution
\[
p^G(x; f) = \frac{k}{2\pi T_{\text{eff}}} \exp \left[ -\frac{(kx - f)^2}{2kT_{\text{eff}}} \right]. \tag{45}
\]
As (45) is similar to an equilibrium PDF, one should be tempted to believe that the average total entropy production rate in the steady state is zero, however, this is incorrect. One can, for example, read an illuminating discussion about this point in Ref. [43], where the entropy production of a spin model in one and two dimensions have been studied. As the authors correctly pointed out, an steady-state PDF of the Boltzmann-like type is not sufficient to ensure that the system is in equilibrium. What really defines equilibrium is detailed balance, or in a more macroscopic language, the vanishing of the average entropy production. In this sense, one should also note that, even if the coarse-grained description given by (43) correctly estimates the non-adiabatic entropy production, it severely underestimates the total entropy production, as pointed out in the second reference in [33] for a Markovian system. Then, (45) represents a genuine NESS.

Let us assume that the system is initially prepared in the steady state associated to the force \( f_0 \) and that \( f(t) = f_0 + \delta f(t) \). In this case we identify the observable \( b(t) \) as
\[
b(t) = \frac{f_0 - kx(t)}{kT_{\text{eff}}}, \tag{46}
\]
Note now that we can rewrite (44) as follows
\[
x(t) = G_1(t)x_0 + \frac{f_0}{k}(1-G_1(t)) \int_0^t d\tau G_2(t-\tau)\delta f(\tau) + \eta(t), \tag{47}
\]
where we have made use of (32). From (45) we see that \( \langle x_0 \rangle = f_0/k \), which implies
\[
\langle b(t) \rangle = -(T_{\text{eff}})^{-1} \int_0^t d\tau G_2(t-\tau)\delta f(\tau). \tag{48}
\]
Now, for the unperturbed system we can write

$$b(t) = -(T_{\text{eff}})^{-1} \left[ G_1(t)(x_0 - \frac{f_0}{k}) + \eta(t) \right],$$  \hspace{1cm} (49)

from where it follows recalling (31)

$$\langle b(t) b(t') \rangle_{\text{ss}} = (kT_{\text{eff}})^{-1} G_1(t - t'); \hspace{1cm} t > t'.$$  \hspace{1cm} (50)

Now, using (32), we can write

$$\frac{d}{dt} \langle b(t) b(t') \rangle_{\text{ss}} = -(T_{\text{eff}})^{-1} G_2(t - t').$$  \hspace{1cm} (51)

Then, comparing (51) with (48), we conclude that

$$\langle b(t) \rangle = \int_0^t dt' \frac{d}{dt} \langle b(t) b(t') \rangle_{\text{ss}} \delta f(t'),$$  \hspace{1cm} (52)

which completes the proof. Then, we have checked the general result (25) for a genuine non-Markovian system with NESS.

VII. CONCLUDING REMARKS AND PERSPECTIVES

We have shown that, in strong contrast with Markovian systems, non-Markov dynamics may be unstable, in the sense that a system prepared in such a way that the initial positions are sampled from the steady-state PDF, may depart from this state at finite times even if the external protocols are kept constant.

For stable systems the non-adiabatic entropy production satisfies an integral fluctuation theorem and the second law of thermodynamics for transitions between NESS holds, exactly as for Markov dynamics. On the other hand, the generalized FDR is also verified, clarifying that, contrary to what is often asserted that this relation only holds for Markovian systems, the conditions which need to be fulfilled are ergodicity and stability. However, if the stability condition fails, it turns out that the generalized FDR does not hold anymore. We believe that this is the reason why the generalized FDR has remained elusive up to now for non-Markov dynamics. It is common to see this issue discussed in the literature by the study of models violating the stability condition (see for example the model discussed in Ref. [18] related to a molecular motor with an internal relaxation time, which is a reliable model for the experimental situation presented in Ref. [45]).

The time derivative of the average non-adiabatic entropy production can be written as a sum of products of generalized fluxes and forces, even without stability. This could be relevant for the experimental determination of this quantity if one is able to determine the steady-state distribution of the system, since the particle current can also be determined in the experiment. On the other hand, the determination of the currents may give direct information about the properties of the evolution operator.

Some interesting open questions remain to be answered. First, the integral fluctuation theorem (15) suggests that a detailed fluctuation theorem may also hold if one introduces a dual system so that the non-adiabatic entropy production can be expressed as the logratio of the forward path probability of the original system and the time-reversed path probability of the dual system. This also could make easy to identify an adiabatic entropy production, such that the total entropy production can be split as in [32]. Second, it would be interesting to generalize the FDR to the case when the system is unstable. In this case, we speculate that some ‘violation’ terms should appear and it would be interesting to investigate their precise form and physical meaning. It would be also interesting to relate that case to the recent results presented in [44]. Third, some immediate improvements of the present theory can be developed in order to describe a wider variety of systems. For example, an extension of this theory to describe also non-ergodic systems, can be attempted in the spirit of Refs. [41, 42]. In those references, generalized Hatano-Sasa identities have been obtained in terms of functions which are not related to the steady-state PDF. This may be relevant if the steady state is not univocally determined, as it is the case for non-ergodic systems. With this improvement, the same formalism could be used in order to describe such complex systems as glasses far from asymptotic states.

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