Comment on ‘Simulating thick atmospheric turbulence in the lab with application to orbital angular momentum communication’

Jeffrey H Shapiro
Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
E-mail: jhs@mit.edu

Abstract. Recently, Rodenburg et al (2014 New J. Phys. 16 033020) presented an approach for simulating propagation over a long path of uniformly distributed Kolmogorov-spectrum turbulence by means of a compact laboratory arrangement that used two carefully placed and controlled spatial light modulators. We show that their simulation approach mimics the behavior of plane-wave propagation, rather than general beam-wave propagation. Thus, the regime in which their orbital angular momentum (OAM) cross-talk results accurately represent the behavior to be expected in horizontal-path propagation through turbulence may be limited to collimated-beam OAM modes whose diameters are sufficient that turbulence-induced beam spread is negligible.
1. Introduction

The use of orbital angular momentum (OAM) beams in free-space optical (FSO) communication has attracted considerable interest of late, both for increasing the data rate of classical communications [1, 2] and the secret-key rate of quantum communications [3, 4]. Of course, such OAM FSO systems are subject to performance degradation arising from atmospheric turbulence, thus prompting work to understand the nature of that degradation [5, 6, 7], and how adaptive optics might mitigate it [8]. Almost exclusively, however, studies of turbulence effects on OAM propagation, such as [5, 6, 7, 8], have presumed thin, phase-screen turbulence in the vicinity of the receiver pupil. Consequently, these works do not properly characterize the effects that would be encountered in propagation over a long path of uniformly-distributed turbulence, i.e., thick turbulence. Recently, Rodenburg et al. [9] presented an approach for simulating propagation over long path of uniformly distributed Kolmogorov-spectrum turbulence by means of a compact laboratory arrangement that used two carefully placed and controlled spatial light modulators. We show that their simulation approach mimics the behavior of plane-wave propagation, rather than general beam-wave propagation. Thus, the regime in which their OAM cross-talk results accurately represent the behavior to be expected in horizontal-path propagation through turbulence may be limited to collimated-beam OAM modes whose diameters are sufficient that turbulence-induced beam spread is negligible.

We begin, in section 2, by showing how the extended Huygens-Fresnel principle [10] can be used to characterize the average cross-talk between OAM modes that have propagated through thick turbulence. Then, in section 3, we demonstrate that [9] may only capture that behavior when turbulence-induced beam spread can be neglected for their collimated-beam OAM modes [11]. There, we also show that this no beam-spread condition is barely satisfied by the simulation parameters they establish for 785-nm-wavelength light to propagate over a 1-km-long path through uniformly distributed $C_n^2 = 1.8 \times 10^{-14} \text{m}^{-2/3}$ Kolmogorov-spectrum turbulence from an 18.2-cm-diameter transmit pupil to an 18.2-cm-diameter receive pupil.

2. Cross-Talk Characterization via the Extended Huygens-Fresnel Principle

Let $\{\Psi_\ell(\rho)\}$, for $\rho = (x, y)$, be the complex field envelopes for a set of wavelength-$\lambda$ orthonormal OAM modes on the circular transmitter pupil $A_0 = \{\rho : |\rho| \leq D/2\}$ in the $z = 0$ plane, where $\ell$ indexes their orbital angular momenta. Likewise, let $\{\psi_\ell(\rho')\}$, for $\rho' = (x', y')$, be a set of orthonormal OAM modes on the circular receiver pupil $A_L = \{\rho' : |\rho'| \leq D/2\}$ in the $z = L$ plane that are extracted by a mode converter in that pupil. From the extended Huygens-Fresnel principle, we have that the complex field envelope, $\zeta_\ell(\rho')$, of the field produced in the $z = L$ plane by transmission of $\Psi_\ell(\rho)$...
from $\mathcal{A}_0$ is

$$\zeta_\ell (\rho') = \int_{\mathcal{A}_0} d\rho \, \Psi_\ell (\rho) h_L (\rho', \rho), \quad (1)$$

where $h_L (\rho', \rho)$ is the atmospheric Green’s function. The unnormalized average cross-talk between received OAM modes $\ell$ and $\ell'$ in the $\mathcal{A}_L$ pupil is therefore

$$C_{\ell, \ell'} \equiv \left< \left| \int_{\mathcal{A}_L} d\rho' \, \psi_{\ell'}^* (\rho') \zeta_\ell (\rho') \right|^2 \right>, \quad (2)$$

where angle brackets denote averaging over the turbulence ensemble. It follows that $C_{\ell, \ell'}$ is completely characterized by the mutual coherence function of the atmospheric Green’s function, viz.,

$$C_{\ell, \ell'} = \int_{\mathcal{A}_L} d\rho' \int_{\mathcal{A}_L} d\rho' \int_{\mathcal{A}_0} d\rho \int_{\mathcal{A}_0} d\rho \, \psi_{\ell'}^* (\rho') \psi_{\ell'} (\rho') \Psi_{\ell}^* (\rho_1) \Psi_{\ell} (\rho_2)$$

$$\times \langle h^*_L (\rho_1, \rho_1) h_L (\rho_2', \rho_2) \rangle. \quad (3)$$

For Kolmogorov-spectrum turbulence, we have that \cite{10, 12, 13}

$$\langle h^*_L (\rho_1, \rho_1) h_L (\rho_2', \rho_2) \rangle = \frac{e^{-ik(|\rho_1' - \rho_1|^2 - |\rho_2' - \rho_2|^2)/2L}}{(\lambda L)^2} e^{-D(\rho_1' - \rho_2', \rho_1 - \rho_2)/2}, \quad (4)$$

where $k = 2\pi/\lambda$ is the wave number, the fraction on the right is due to vacuum propagation, and

$$D(\rho_1' - \rho_2', \rho_1 - \rho_2) \equiv 2.91k^2 \int_0^L dz \, C_n^2 (z) (|\rho_1' - \rho_2'| z/L + (\rho_1 - \rho_2) (1 - z/L))^{5/3}, \quad (5)$$

is due to turbulence, whose strength profile along the path is $C_n^2 (z)$. The initial derivation of this mutual coherence function employed the Rytov approximation \cite{10, 12}, hence $D(\rho_1' - \rho_2', \rho_1 - \rho_2)$ was termed the two-source, spherical-wave, wave structure function, and the validity of (4) and (5) was limited to the weak-perturbation regime before the onset of saturated scintillation. Later \cite{13}, it was shown that (4) and (5) could be obtained from the small-angle approximation to the linear transport equation, making them valid well into saturated scintillation.

Several key points are worth noting here. First, non-uniform turbulence distributions lead to there being several coherence lengths of potential interest, including: (a) the transmitter-pupil coherence length

$$\rho_0 \equiv \left( 2.91k^2 \int_0^L dz \, C_n^2 (z) (1 - z/L)^{5/3} \right)^{-3/5}, \quad (6)$$

which quantifies the beam spread incurred in propagation from $z = 0$ to $z = L$, because transmission of a complex field envelope $E_0 (\rho)$ from $\mathcal{A}_0$ yields a complex field envelope $E_L (\rho')$ in $\mathcal{A}_L$ whose average irradiance is

$$\langle |E_L (\rho')|^2 \rangle =$$

$$\int_{\mathcal{A}_0} d\rho_1 \int_{\mathcal{A}_0} d\rho_2 \, E_0^* (\rho_1) E_0 (\rho_2) \frac{e^{-ik(|\rho' - \rho_1|^2 - |\rho' - \rho_2|^2)/2L}}{(\lambda L)^2} e^{-((\rho_1 - \rho_2) / \rho_0)^{5/3}/2}, \quad (7)$$
and (b), the receiver-pupil coherence length
\[ \rho_0' \equiv \left( 2.91k^2 \int_0^L dz C_n^2(z)(z/L)^{5/3} \right)^{-3/5}, \tag{8} \]
which quantifies the angle-of-arrival spread for a point-source transmission \( E_0(\rho) = \delta(\rho) \) from \( A_0 \), because a diffraction-limited, focal-length \( f > 0 \) lens in \( A_L \) yields an average image-plane irradiance
\[ \langle |E_{L'}(\rho)|^2 \rangle = \int_{A_L} d\rho_1' \int_{A_L} d\rho_2' e^{ik\rho' \cdot (\rho_1' - \rho_2')/L'} \langle |\rho_1' - \rho_2'|/\rho_0' \rangle^{5/3}, \tag{9} \]
where \( 1/L' = 1/L - 1/f \). For a uniform distribution of turbulence—\( C_n^2(z) \) constant from \( z = 0 \) to \( z = L \)—these two coherence lengths coincide,
\[ \rho_0 = \rho_0' = (1.09k^2C_n^2L)^{-3/5}. \tag{10} \]

In addition to \( \rho_0 \) and \( \rho_0' \), there is one more coherence length we need to introduce. Suppose that a plane wave is transmitted from the \( z = 0 \) plane, i.e., \( E_0(\rho) = E_0 \) for all \( \rho \) in that plane. The mutual coherence function of the resulting \( z = L \) field, found from (4) and (5) with \( A_0 \) extended to cover the entire \( z = 0 \) plane, obeys
\[ \langle E_{L'}^*(\rho_1)E_L(\rho_2) \rangle = \int_{A_L} d\rho_1 \int_{A_L} d\rho_2 |E_0|^2 e^{-ik(\rho_1 - \rho_2)^2}/2 \langle |\rho_1' - \rho_2'|/\rho_0 \rangle^{5/3} e^{-D(\rho_1' - \rho_2, \rho_1,\rho_2)/2}, \tag{11} \]
for Kolmogorov-spectrum turbulence, where the second equality follows from integrating in sum and difference coordinates, \( \rho_+ = (\rho_1 + \rho_2)/2 \) and \( \rho_- = \rho_1 - \rho_2 \). Note that
\[ D(\rho_1' - \rho_2', \rho_1 - \rho_2) = \left( 2.91k^2 \int_0^L dz C_n^2(z) \right) |\rho_1' - \rho_2'|^{5/3}, \tag{12} \]
is the plane-wave structure function of the Rytov theory, whose coherence length is
\[ \rho_p = \left( 2.91k^2 \int_0^L dz C_n^2(z) \right)^{-3/5} \tag{13} \]
in general, and
\[ \rho_p = (2.91k^2C_n^2L)^{-3/5} \tag{14} \]
for uniformly-distributed turbulence.

3. The Rodenburg et al Simulator

Rodenburg et al [9] performed a laboratory experiment scaled to simulate propagation of 785-nm-wavelength OAM modes over a 1-km-long path of uniformly distributed, Kolmogorov-spectrum turbulence with \( C_n^2 = 1.8 \times 10^{-14} \text{ m}^{-2/3} \). In what follows, however, we shall stick with the unscaled path geometry, rather than the scaled version Rodenburg et al used in their experiments.
In [9], the position and strength of the two phase screens were chosen to match the following propagation parameters for the propagation path specified above—the Fried parameter version of the plane-wave coherence length, \(r_0 = (6.88)^{3/5} \rho_P\); the plane-wave log-amplitude variance, \(\sigma_x^2 = 0.31k^{7/6}C_n^2 L^{11/6}\); the normalized variance, \(\sigma_p^2\), of the power collected by the 18.2-cm-diameter receiver pupil from a plane-wave transmission; and the density of branch points in that receiver pupil, \(\rho_{BP}\)—see [9] for the details. Rodenburg et al give \(r_0\) values and locations for the two phase screens they claim will simulate propagation through the thick turbulent path described above: 

\[r_0^1 = 3.926 \text{ cm}, \quad r_0^2 = 3.503 \text{ cm}, \quad z_1 = 171.7 \text{ m}, \quad \text{and} \quad z_2 = 1.538 \text{ m}.
\]

To connect with the extended Huygens-Fresnel principle theory laid out in section 2, we note that Rodenburg et al’s phase screens correspond to the impulsive \(C_n^2\) distribution:

\[C_n^2(z) = N_{n1}^2 \delta(z - z_1) + N_{n2}^2 \delta(z - z_2), \quad (15)\]

where

\[N_{nm}^2 = 6.88/2.91k^{2}3^{5/3} \quad \begin{cases} 8.14 \times 10^{-12} \text{ m}^{-1/3} & \text{for } m = 1 \\ 9.84 \times 10^{-12} \text{ m}^{-1/3} & \text{for } m = 2. \end{cases} \quad (16)\]

At this point it is easy to see the limitation of the Rodenburg et al simulation. Their two-screen \(C_n^2(z)\) distribution yields 

\[\rho_0 = \left[2.91k^2 \left(N_{n1}^2(1-z_1/L)^{5/3} + N_{n2}^2(1-z_2L)^{5/3}\right)\right]^{-3/5} = 8.3 \text{ mm}, \quad (17)\]

and 

\[\rho_0' = \left[2.91k^2 \left(N_{n1}^2(z_1/L)^{5/3} + N_{n2}^2(z_2L)^{5/3}\right)\right]^{-3/5} = 7.19 \text{ cm}, \quad (18)\]

for the transmit and receive pupil coherence lengths, whereas the uniformly-distributed turbulence they are trying to simulate would have 

\[\rho_0 = \rho_0' = (1.09k^2C_n^2 L)^{-3/5} = 1.38 \text{ cm}. \quad (19)\]

Given the above coherence-length discrepancies, one cannot expect the Rodenburg et al simulator to yield accurate results for \(C_{\ell,\ell'}\) for all choices of the OAM modes \(\Psi_\ell(\rho)\). The question then becomes when could it provide an accurate cross-talk assessment? Because the Rodenburg et al simulator matched propagation parameters for a plane-wave source, it is reasonable to suggest collimated-beam OAM modes as the natural candidates for accurate \(C_{\ell,\ell'}\) determination via that simulator. Indeed, although [9] does not say so, the cross-talk results reported therein were obtained with precisely such modes [11], i.e.,

\[\Psi_\ell(\rho) = \sqrt{\frac{4}{\pi D^2}} e^{i \ell \phi}, \quad \text{for } |\rho| \leq D/2, \quad (20)\]

where \(\phi\) is the azimuthal angle of \(\rho\).

Because the \(D \rightarrow \infty\) limit of (20) is an OAM-modulated plane wave, we can expect that the \(C_{\ell,\ell'}\) results from [9] should be accurate when \(D\) is large enough that turbulence-induced beam spread can be ignored. For a simple and optimisitic initial
assessment of whether beam spread is insignificant in the [9] scenario, we will replace its 18.2-cm-diameter circular pupils with square pupils having 18.2 cm sides and calculate

\[ F_0 \equiv \int_{-D/2}^{D/2} dx' \int_{-D/2}^{D/2} dy' \langle |\zeta_0(\rho')|^2 \rangle \]  

for

\[ \Psi_0(\rho) = \frac{1}{D}, \quad \text{for } |x|, |y| \leq D/2 \]  

when \( \rho_0 = 3.8 \text{ mm} \) (the \( z = 0 \) plane coherence length for the Rodenburg et al simulator), and compare that result with the corresponding vacuum-propagation (\( \rho_0 = \infty \)) result. After some algebra we get

\[ F_0 = \int_{-1}^{1} dv_x \int_{-1}^{1} dv_y \frac{\sin[\pi \sqrt{D_f v_x} (1 - |v_x|)]}{\pi v_x} \frac{\sin[\pi \sqrt{D_f v_y} (1 - |v_y|)]}{\pi v_y} \times \frac{\sin[\pi \sqrt{D_f v_x}]}{\pi \sqrt{D_f v_x}} \frac{\sin[\pi \sqrt{D_f v_y}] e^{-D/\rho_0} D^3/2}}{\pi \sqrt{D_f v_y}} \]  

where \( D_f = D^4/(\lambda L)^2 \) is the Fresnel-number product of the transmitter-receiver geometry. Equation (23) yields \( F_0 = 0.929 \) for vacuum propagation and \( F_0 = 0.719 \) for the turbulent case. Inasmuch as Rodenburg et al’s circular pupils inscribe our square pupils, and

\[ \Psi_\ell(\rho) = e^{i\ell \phi} \frac{1}{D}, \quad \text{for } |x|, |y| \leq D/2 \]  

with \( \ell \neq 0 \) has higher spatial-frequency content than does \( \Psi_0(\rho) \), we believe that the results from [9] are on the edge of providing an accurate cross-talk assessment. A more definitive statement about the validity of [9] would require full comparison between its experimental results and numerical evaluation of (2), using (4) and (5) with the parameter values for the horizontal-path scenario [9] chose to simulate.

Conclusions

We have shown that the two-screen turbulence simulator from [9] does not properly represent the Green’s-function mutual coherence for a uniform distribution of Kolmogorov-spectrum turbulence. As a result, the average cross-talk predictions from [9] for reception without adaptive optics—predictions that can be directly compared with those obtained from the extended Huygens-Fresnel principle—may only be valid for that paper’s collimated-beam OAM modes when turbulence-induced beam spread is insignificant. Moreover, if the results for cross-talk without adaptive optics are suspect, then those obtained for cross-talk mitigation with adaptive optics must also be regarded with skepticism.

References

[1] Gibson G, Courtial J, Padgett M J, Vasnetsov M, Pas’ko V, Barnett S M, Franke-Arnold S 2004 Opt. Express 12 5448
Comment on ‘Simulating thick atmospheric turbulence in the lab...’

[2] Wang J, Yang J-Y, Fazal I M, Ahmed N, Yan Y, Huang H, Ren Y, Yue Y, Dolinar S, Tur M, Willner A E 2012 Nature Photon. 6 488
[3] Malik M, O’Sullivan M, Rodenburg B, Mirhosseini M, Leach J, Lavery M P J, Padgett M J, Boyd R W 2012 Opt. Express 20 13195
[4] Leach J, Bolduc E, Gauthier D J, Boyd R W 2012 Phys. Rev. A 85 060304(R)
[5] Paterson C 2005 Phys. Rev. Lett. 94 153901
[6] Rodenburg B, Lavery M P J, Malik M, O’Sullivan M N, Mirhosseini M, Robertson D J, Padgett M, Boyd R W 2012 Opt. Lett. 37 3735
[7] Ren Y, Huang H, Xie G, Ahmed N, Yan Y, Erkmen B I, Chandrasekaran N, Lavery M P J, Steinhoff N K, Tur M, Dolinar S, Neifeld M, Padgett M J, Boyd R W, Shapiro J H, Willner A E 2013 Opt. Lett. 38 4062
[8] Ren Y, Xie G, Huang H, Bao C, Yan Y, Ahmed N, Lavery M P J, Erkmen B I, Dolinar S, Tur M, Neifeld M A, Padgett M J, Boyd R W, Shapiro J H, Willner A E 2014 Opt. Lett. 39 2845
[9] Rodenburg B, Mirhosseini M, Malik M, Magaña-Loaiza O S, Yanakas M, Maher L, Steinhoff N K, Tyler G A, Boyd R W 2014 New J. Phys. 16 033020
[10] Shapiro J H in Strohbehn J W, ed 1978 Laser Beam Propagation in the Atmosphere (Berlin: Springer-Verlag) Ch. 6
[11] Rodenburg B, 2014 private communication
[12] Lutomirski R F, Yura H T 1971 Appl. Opt. 10 1652
[13] Ishimaru A Wave Propagation and Scattering in Random Media, vol. 2 1978 (New York, NY: Academic) Ch. 20