ESTIMATES FOR NORMS OF RANDOM POLYNOMIALS

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Abstract. This paper contains some estimates for the integral-uniform norm and the uniform norm of a wide class of random polynomials. The family of integral-uniform norms introduced in [6] is a natural generalization of the maximum norm taken over a net. We prove some properties of the integral-uniform norms. The given application of the established estimates demonstrates that the integral-uniform norms may be useful whenever one is interested in the properties of a function distribution.

Key words: integral-uniform norm; random polynomials with respect to a general function system; trigonometric polynomials with random coefficients.

1. Introduction

In this paper some estimates for mathematical expectation of norms of random polynomials of the type

\[ \sum_{j=1}^{n} a_j \xi_j(\omega)f_j(x) \]

are presented. Here \( \{\xi_i\}_{i=1}^{n} \) is a set of independent random values defined on \( (\Omega, P) \) and \( \{f_i\}_{i=1}^{n} \) is a set of functions on another probability space \( (X, \mu) \). The norms here are taken in a space of functions, which depend only on the space variable \( x \) with fixed \( \omega \).

Similar estimates for various systems of functions \( \{f_i\}_{i=1}^{n} \) and random variables \( \{\xi_i\}_{i=1}^{n} \) have been widely applied in analysis since 1930s. In 1954 Salem and Zygmund [14] established a number of estimates for the uniform norm of random trigonometric polynomials. In particular in [14], it was shown that

\[ \mathbb{E}\| \sum_{k=-n}^{n} r_k(\omega)e^{ikt} \|_{\infty} \asymp (n \log n)^{1/2}, \]

where \( r_k(\omega) \) are the Rademacher functions, here and further the expression \( A_n \asymp B_n \) stands for \( cA_n \leq B_n \leq CA_n \) with some constants \( c, C \). This estimate along with Khinchin’s inequality reflects subtle differences between a finite dimensional subspace of \( L_\infty \) and its natural embeddings in \( L_p \) spaces with \( 1 \leq p < \infty \).

By now various methods for estimating the uniform norm of random polynomials (1) have been developed (e.g. see [5], [8], [9]). A first lower estimate for the uniform norm...
of a random polynomial (1) with respect to a general function system was established by Kashin and Tzafriri in [5]–[7], this result will be formulated in Section 2.

In [6] Kashin and Tzafriri introduced the following norm

\[ \|f\|_{m,\infty} := \int_X \cdots \int_X \max \{ |f(x_1)|, \ldots, |f(x_m)| \} \, d\mu(x_1) \cdots d\mu(x_m), \]

where \( f \) is a function defined on a measure space \((X, \mu)\), \( \mu(X) = 1 \). This norm is a natural generalization of \( \| \cdot \|_\infty \)-norm taken over a net, we call it the integral-uniform norm. One can easily see that for every integrable function \( f \in L_1(X) \) we get

\[ \|f\|_1 = \|f\|_{1,\infty}, \]

and

\[ \|f\|_{m,\infty} = \int_0^{1-\lambda_f(t)} (1 - (1 - \lambda_f(t))^m) \, dt, \]

where

\[ \lambda_f(t) := \mu\{ \tau : |f(\tau)| > t \}. \]

It is also easy to notice, that for \( f \in L_\infty(X) \) the following inequalities take place

\[ \|f\|_1 \leq \|f\|_{m,\infty} \leq \|f\|_\infty \]

and \( \|f\|_{m,\infty} \to \|f\|_\infty \) as \( m \to \infty \). Using a trivial inequality \( \max(|a|, |b|) \leq |a| + |b| \) and the definition of the integral-uniform norm (2), we get

\[ \|f\|_{n,\infty} \leq \left( \frac{n}{m} + 1 \right) \|f\|_{m,\infty}, \quad m < n \]

for all \( f \in L_1(X) \). For the integral-uniform norm of an indicator \( \chi_\Delta \) of a set \( \Delta \subset X \) the identity (3) implies

\[ \|\chi_\Delta\|_{m,\infty} = 1 - (1 - \mu\Delta)^m, \]

Thus, if we take \( m \) of order \( 1/\mu\Delta \) then \( \|\chi_\Delta\|_{m,\infty} \) is of order one.

The technique used in [5], [7] for estimating the uniform norm of random polynomials turned out to be applicable for estimating the integral-uniform norm (2). In fact, an estimate for the integral-uniform norm of random polynomials (1) for a special case of parameter \( m \) was implicitly obtained in [5]–[7].

In Section 2 we present some generalizations of the results from [5]–[7] for both the case of the \( \| \cdot \|_{m,\infty} \)-norm with an arbitrary parameter \( m \) and a wider class of function systems \( \{f_i\}_1^n \). The generalizations are obtained by the same method as in [5]–[7], which relies on a multidimensional version of the central limit theorem with precise estimate of the error term. In Section 3 we shall show that under some additional constraints on \( \{\xi_i\}_1^n \) and \( m \) the established estimate is precise in sense of order. In Section 4 we mention some properties of the integral-uniform norm, in particular, its properties are illustrated on some inequalities for the integral-uniform norms of trigonometric polynomials. In addition, in Section 4 we present an application of the established estimates. This application uses a simple geometrical lemma which could be of independent interest. Most of the results have presented here been announced by the author in [3].

I would like to express my special thanks to B.S. Kashin for his numerous useful comments and advices, also I am very grateful to E.M. Semenov for interesting discussions.
2. The Lower Estimates for the Integral-Uniform Norms of Random Polynomials.

In [6], [7] Kashin and Tzafriri proved that whenever systems of functions \( \{f_i\}_i \) and \( \{\xi_i\}_i \), defined on probability spaces \((X, \mu)\) and \((\Omega, P)\) respectively, satisfy the following conditions

(a) \( \|f_i\|_2 = 1 \) and \( \|f_i\|_3 \leq M \) for every \( i \);
(b) \( \| \sum_{i=1}^n c_i f_i \|_2 \leq M \left( \sum_{i=1}^n |c_i|^2 \right)^{1/2} \) for all sets of coefficients \( \{c_i\}_i \);
(c) \( \{\xi_i\}_i \) are independent variables, such that \( E \xi_i = 0, \) \( E |\xi_i|^2 = 1 \) and \( E |\xi_i|^3 \leq M^3 \).

Then there exist positive constants \( q = q(M), \) \( C_j = C_j(M), j = 1, 2, 3 \) such that

\[
P \left\{ \omega \in \Omega : \| \sum_{i=1}^n a_i \xi_i(f_i) f_i \|_{L^\infty(X)} \leq C_1 \left( \sum_{i=1}^n |a_i|^2 \right)^{1/2} (1 + \log R)^{1/2} \right\} \leq \frac{C_2}{R^q},
\]

where

\[
R := \frac{\left( \sum_{i=1}^n |a_i|^2 \right)^2}{\sum_{i=1}^n |a_i|^4},
\]

and hence

\[
E \| \sum_{i=1}^n a_i \xi_i f_i \|_{L^\infty(X)} \geq C_3 \left( \sum_{i=1}^n |a_i|^2 \right)^{1/2} (1 + \log R)^{1/2}.
\]

The proof of these estimates practically involved the estimate of \( \| \sum_{i=1}^n a_i \xi_i f_i \|_{m, \infty} \) for a special value of parameter \( m \), precisely \( m \asymp (1 + \log R)^2 R^{1/2 + \epsilon} \).

In this paper a generalization of the inequalities (6), (8) for both the case of integral-uniform norm and a wider class of random polynomials is established. In particular, it is shown that if \( R(\{a_i\}) \asymp n \) then the estimates (6), (8) stay true whenever functions \( \{f_i\}_i \) satisfy instead of (b) the following condition

(b') \( \| \sum_{i=1}^n c_i f_i \|_2 \leq M n^p \left( \sum_{i=1}^n |c_i|^2 \right)^{1/2} \) for all sets of coefficients \( \{c_i\}_i \) with some constants \( M > 0 \) and \( p \in [0, 1/2) \).

**Theorem 1.** Let \( \{f_i\}_i \) and \( \{\xi_i\}_i \) be sets of functions defined on probability spaces \((X, \mu)\) and \((\Omega, P)\) respectively, which satisfy (a) and (c). Let also \( \{a_i\}_i \) be a fixed set of coefficients and for all sets of coefficients \( \{c_i\}_i \) the following inequality hold

\[
\| \sum_{i=1}^n c_i f_i \|_2 \leq M \left( R(\{a_i\}) \right)^p \left( \sum_{i=1}^n |c_i|^2 \right)^{1/2},
\]

where \( R = R(\{a_i\}) \) defined by (7); \( M > 0 \) and \( p \in [0, 1/2) \) are some constants.

Then there exist positive constants \( q' = q'(p), \) \( C'_j = C'_j(p, M), j = 1, 2, 3 \) such that the following estimates take place

\[
P \left\{ \omega \in \Omega : \| \sum_{i=1}^n a_i \xi_i(\omega) f_i \|_{m, \infty} \leq C'_1 \left( \sum_{i=1}^n |a_i|^2 \log P \right)^{1/2} \right\} \leq \frac{C'_2}{P^{q'}};
\]

\[
E \| \sum_{i=1}^n a_i \xi_i f_i \|_{m, \infty} \geq C'_3 \left( \sum_{i=1}^n |a_i|^2 \log P \right)^{1/2},
\]
where \( P := \min(m, R) + 1 \).

**Corollary 1.** Let the coefficients \( \{a_i\}_{i=1}^n \) satisfy \( R(\{a_i\}_{i=1}^n) \ll n \) and functions \( \{f_i\}_{i=1}^n \) and \( \{\xi_i\}_{i=1}^n \) satisfy (a) and (c) respectively. Let also \( \{f_i\}_{i=1}^n \) satisfy

\[
\| \sum_{i=1}^{n} \varepsilon_if_i \|_2 \leq M^{1/2+p}
\]

for all signs \( \varepsilon_i = \pm 1 \) with some constants \( p \in [0, 1/2), M > 0 \). Then the estimates (9), (10) hold for the random polynomial (1).

To prove the Corollary it suffices to notice that Lemma from Sec. 4 implies the condition (b′) (with another \( p \in [0, 1/2] \)) for the functions \( \{f_i\}_{i=1}^n \).

The proof of Theorem 1 essentially follows the pattern of the proof of (6), (8) from [7]. In Section 3 it is shown that under some additional constraints on \( \xi_k \) and \( m \) the estimate (10) is precise in sense of order.

**Proof of Theorem 1.**

**Step 1.** We can re-scale the coefficients \( \{a_i\}_{i=1}^n \) so that \( \sum_{i=1}^{n} |a_i|^2 = 1 \). Let \( \varepsilon = \varepsilon(M) = \frac{1}{4}(\frac{3}{4M^2})^3 \). Consider the set

\[
E_1 := \left\{ x : \sum_{i=1}^{n} |a_i|^3|f_i(x)|^3 < \frac{M^3}{\varepsilon} \sum_{i=1}^{n} |a_i|^3 \right\}.
\]

Then assumption (a) and Chebyshev’s inequality imply

\[
\mu(E_1) \frac{M^3}{\varepsilon} \sum_{i=1}^{n} |a_i|^3 \leq \int \sum_{i=1}^{n} |a_i|^3|f_i(x)|^3 d\mu(x) \leq M^3 \sum_{i=1}^{n} |a_i|^3,
\]

so it follows that \( \mu E_1 \geq 1 - \varepsilon \).

Next consider the function

\[
f(x) := \sum_{i=1}^{n} |a_i|^2|f_i(x)|^2,
\]

which satisfies \( \|f\|_1 = 1 \) and

\[
\|f\|_{3/2} \leq \sum_{i=1}^{n} |a_i|^2\|f_i\|_{3/2} = \sum_{i=1}^{n} |a_i|^2\|f_i\|_3^2 \leq M^2.
\]

Consider also the set \( E_2 := \{ x : f(x) > \frac{1}{4} \} \). Since \( \int_{E_2} f \leq \frac{1}{4} \) it follows

\[
\frac{3}{4} \leq \int_{E_2} f \leq \|f\|_{3/2} \mu(E_2)^{1/3} \leq M^2 \mu(E_2)^{1/3}
\]

and therefore \( \mu E_2 \geq (\frac{3}{4M^2})^3 \). Now consider the set

\[
E_3 := \left\{ x \in E_2 : f(x) < 2(\frac{4M^2}{3})^3 \right\}.
\]

For measure \( \mu(E_2 \setminus E_3) \) we have the estimate

\[
2(\frac{4M^2}{3})^3 (\mu E_2 - \mu E_3) \leq \int_{E_2 \setminus E_3} f(x) d\mu(x) \leq \|f\|_1 = 1
\]
so that
\[ \mu E_3 \geq \frac{1}{2}\left(\frac{3}{4M^2}\right)^3 = 2\varepsilon. \]

Finally note that the set \( E := E_1 \cap E_3 \) has the following properties

(i) \( \mu E \geq \varepsilon(M) = \frac{3}{4} \left(\frac{3}{4M^2}\right)^3; \)
(ii) \( \sum_{i=1}^{n} |a_i|^3 |f_i(x)|^3 < \frac{256}{27} M^6 \sum_{i=1}^{n} |a_i|^3 \) for all \( x \in E; \)
(iii) For \( x \in E \) the function \( f(x) = \sum_{i=1}^{n} |a_i|^2 |f_i(x)|^2 \) satisfies
\[ \frac{1}{4} < f(x) < 2 \left(\frac{4M^2}{3}\right)^2 =: \gamma(M). \]

**Step 2.** Define a new measure \( \nu \) on \( X \) by
\[ d\nu(x) = \begin{cases} \frac{d\mu(x)}{f(y)d\mu(x)}, & x \in E, \\ d\mu(x), & x \in E^c. \end{cases} \]

One can easily see that \( \nu \) is a probability measure on \( X \). Define also functions \( g_i(x) \), \( 1 \leq i \leq n \) by
\[ g_i(x) := \begin{cases} f_i(x), & x \in E, \\ \frac{1}{2}f_i(x) \left(\frac{f(y)d\mu(x)}{f(x)d\mu(E)}\right)^{1/2}, & x \in E^c. \end{cases} \]

The functions \( g_i \) have the following properties:

(i) \( \|g_i\|_{L_2(\nu)} = 1 \) for \( 1 \leq i \leq n; \)
(ii) \( \|\sum_{i=1}^{n} c_i g_i\|_{L_2(\nu)} = \|\sum_{i=1}^{n} c_i f_i\|_{L_2(\mu)} \leq MR^p \left(\sum_{i=1}^{n} |c_i|^2\right)^{1/2} \) for all sets of coefficients \( \{c_i\}_{i=1}^{n}; \)
(iii) For all \( x \in E \) the following identity takes place
\[ g(x) := \sum_{i=1}^{n} |a_i|^2 |g_i(x)|^2 = \frac{1}{\mu E} \int_E f(y)d\mu(y) =: K^2, \]
and \( \frac{1}{4} \leq K^2 < \gamma(M); \)
(iv)
\[ \sum_{i=1}^{n} |a_i|^3 |g_i(x)|^3 \leq \beta(M) \sum_{i=1}^{n} |a_i|^3 \]
for all \( x \in E \), where \( \beta(M) = 10^5 M^{18}; \)
(v) Finally note that for \( x \in E \) and \( \omega \in \Omega \) a.s.
\[ |\sum_{i=1}^{n} a_i \xi_i(\omega) g_i(x)| \leq 5M^3 |\sum_{i=1}^{n} a_i \xi_i(\omega) f_i(x)|. \]

**Step 3.** Note that if there exists a set \( F \subset E^m \) such that \( \nu^m(F) \geq (\nu E)^m/2 \) and for some \( \omega_0 \in \Omega \)
\[ \max_{1 \leq j \leq m} \left( |\sum_{i=1}^{n} a_i \xi_i(\omega_0) g_i(x_j)| \right) \geq \rho \quad \text{for } (x_1, \ldots, x_m) \in F, \]
then for the set
\[ F_0 = \left\{ x \in E : |\sum_{i=1}^{n} a_i \xi_i(\omega_0) g_i(x) \mid \geq \rho \right\}. \]
we have \((\nu E - \nu F_0)^m \leq (\nu E)^m / 2\) so that
\[
\mu F_0 \geq C(M)\nu F_0 \geq C(M)\left[1 - \left(\frac{1}{2}\right)^{1/m}\right]\nu E.
\]
Taking into account (v) and (iii) from step 2 (see also (5)), we get
\[
\| \sum_{i=1}^{n} a_i \xi_i(\omega_0) f_i(x) \|_{\mu, m, \infty} \geq \| \chi_{F_0}(x) \cdot \sum_{i=1}^{n} a_i \xi_i(\omega_0) f_i(x) \|_{\mu, m, \infty}
\geq \frac{\rho}{5M^3} \| \chi_{F_0} \|_{\mu, m, \infty} = \frac{\rho}{5M^3} (1 - (1 - \mu F_0)^m)
\geq \frac{\rho}{5M^3} (1 - \left(1 - C(M)\left[1 - \left(\frac{1}{2}\right)^{1/m}\right]\nu E\right)^m).
\]
Using the inequality \(1 - (\frac{1}{2})^{\frac{m}{2}} \leq \frac{1}{2m}\) and (i) from step 1 we get
\[
\| \sum_{i=1}^{n} a_i \xi_i(\omega_0) f_i(x) \|_{\mu, m, \infty} \geq \frac{\rho}{5M^3} \left(1 - \left(1 - \frac{C(M)\nu E}{2m}\right)^m\right) \geq C'(M)\rho.
\]
Thus, to prove (9) for \(\{f_i\}_1^n\) on \((X, \mu)\) it suffices to prove it for \(\{g_i\}_1^n\) on \((X, \nu)\). Define \(F \subset E^m\) by
\[
F := \left\{ (x_j)_{j=1}^m \in E^m : \frac{1}{m^2} \sum_{j,k=1 \atop j \neq k}^{m} \left| \sum_{i=1}^{n} |a_i|^2 |g_i(x_j) g_i(x_k)| \right|^2 \leq 2 \left( \frac{MR^p}{\varepsilon(M)} \right)^2 \sum_{i=1}^{n} |a_i|^4 \right\}.
\]
To estimate \(\nu F\) notice that
\[
\frac{1}{\nu(E)^m} \int_E \cdots \int_E \frac{1}{m^2} \sum_{j,k=1 \atop j \neq k}^{m} \left| \sum_{i=1}^{n} |a_i|^2 |g_i(x_j) g_i(x_k)| \right|^2 \, d\nu(x_1) \cdots d\nu(x_m)
\leq \frac{1}{(\nu(E)m)^2} \sum_{j,k=1 \atop j \neq k}^{m} \int_E \int_E \left| \sum_{i=1}^{n} |a_i|^2 |g_i(x_j) g_i(x_k)| \right|^2 \, d\nu(x_j) \, d\nu(x_k)
\leq \left( \frac{MR^p}{\nu(E)m} \right)^2 \sum_{j,k=1 \atop j \neq k}^{m} \int_E \int_E \left| \sum_{i=1}^{n} |a_i|^4 |g_i(x_j)|^2 \right|^2 \, d\nu(x_j) \, d\nu(x_k)
\leq \left[ \frac{MR^p}{\varepsilon(M)} \right]^2 \sum_{i=1}^{n} |a_i|^4.
\]
From Chebyshev’s inequality we have \(\nu^m(F) \geq \nu(E)^m / 2\).

**Step 4.** For \(x \in E\) and \(\rho > 0\) define
\[
E_\rho(x) := \left\{ \omega \in \Omega : \sum_{i=1}^{n} a_i \xi_i(\omega) g_i(x) > \rho \right\}.
\]
As we have seen in step 3 in order to prove the theorem it suffices to show that there exist some constants \(\alpha(M, p) \in (0, 1), q' = q'(p) > 0\) and \(K_0(M)\) such that for every \((x_j)_{j=1}^m \in F\) and \(\rho := \alpha K(2 \log P)^{1/2}\) the following estimate takes place
\[(*) \quad P \left\{ \Omega \setminus \bigcup_{j=1}^{m} E_\rho(x_j) \right\} \leq K_0 P^{-q'}.
\]
Then

Proposition 1. see Corollary 17.2 in [1].

with an estimate for the error term. We use the following result due to Rotar' [13] (or Step 5. In order to prove (**) we shall use a sharper version of the central limit theorem implies (*) and therefore (9). The aim of the remaining steps is to prove (**).

\[ E|\eta| \leq (E|\eta|^2)^{1/2} \left( \frac{\sum_{j=1}^{m} E_{\rho}(x_j)}{\kappa} \right)^{1/2} \]

Thus, the inequality

\begin{align*}
(\star) \quad E|\eta| & \geq (1 - K_0 P^{-q'}) (E|\eta|^2)^{1/2} \\
(\star \star) \quad E|\eta| & \geq (1 - K_0 P^{-q'}) (E|\eta|^2)^{1/2}
\end{align*}

implies (*) and therefore (9). The aim of the remaining steps is to prove (**).

**Step 5.** In order to prove (**) we shall use a sharper version of the central limit theorem with an estimate for the error term. We use the following result due to Rotar' [13] (or see Corollary 17.2 in [1]).

**Proposition 1.** Let \( \{X_i\}_{i=1}^{h} \) be a set of random vectors in \( \mathbb{R}^d \) such that \( E X_i = 0, 1 \leq i \leq h \)

\[ \sup_{A \in C} |Q_h(A) - \Phi_{0,V}(A)| \leq K_1(d) h^{-1/2} \rho_3 \lambda^{-3/2}, \]

where \( K_1(d) < \infty \) is a constant, \( C \) denotes the class of all Borel convex subsets of \( \mathbb{R}^d \),

\[ \rho_3 := h^{-1} \sum_{i=1}^{h} E|X_i|^3, \]

\( \lambda \) is the smallest eigenvalue of the matrix \( V = h^{-1} \sum_{i=1}^{h} \text{cov}(X_i) \) (recall that the covariance matrix of a random vector \( Y = (y_1, \ldots, y_d) \) such that \( EY = 0 \) defined by \( \text{cov}(Y) := \{E(y_j y_k)\}_{j,k=1}^{d} \), \( Q(A) \) is the probability that \( h^{-1/2} \sum_{i=1}^{h} X_i \) belongs to a convex set \( A \) and, finally, \( \Phi_{0,V} \) denotes the normal distribution with the density

\[ \phi_{0,V}(Y) := (2\pi)^{-d/2}(\det V)^{-1/2} \exp \left\{ -\frac{1}{2}(Y, V^{-1}Y) \right\}, \quad Y \in \mathbb{R}^d. \]

We shall apply Proposition 1 twice: for one- and two-dimensional cases.

For fixed \( x \in E \) let

\[ X_i(\omega) := a_i \xi_i(\omega) g_i(x) \quad 1 \leq i \leq n. \]

Then

\[ \rho_3 = \frac{1}{n} \sum_{i=1}^{n} E|X_i|^3 = \frac{1}{n} \sum_{i=1}^{n} |a_i|^3 E|\xi_i|^3 |g_i(x)|^3 \leq \]

\[ \leq \frac{M^3}{n} \sum_{i=1}^{n} |a_i|^3 |g_i(x)|^3 \leq \frac{M^3 \beta(M)}{n} \sum_{i=1}^{n} |a_i|^3, \]

\[ \lambda = V = \frac{1}{n} \sum_{i=1}^{n} \text{cov}(X_i) = \frac{1}{n} \sum_{i=1}^{n} |a_i|^2 E|\xi_i|^2 |g_i(x)|^2 = \frac{K^2}{n}. \]
So Preposition 1 implies
\[ |P(E_\rho(x)) - \frac{1}{K} \left( \frac{n}{2\pi} \right)^{1/2} \int_{x^2/2}^{\infty} e^{-\frac{x^2}{2K^2}} dy \leq K_1 \frac{M^3 \beta(M)}{K^3} \sum_{i=1}^{n} |a_i|^3. \]

By a change of variable in the integral we get
\[ |P(E_\rho(x)) - \frac{1}{(2\pi)^{1/2} K} \int_{\rho}^{\infty} e^{-\frac{x^2}{2K^2}} dy \leq K_2(M) \sum_{i=1}^{n} |a_i|^3 \leq K_2 \left( \sum_{i=1}^{n} |a_i|^4 \right)^{1/2} = \frac{K_2}{R^{1/2}}. \]

It is well-known that
\[ \int_{\frac{z}{2}}^{\infty} e^{-t^2/2} dt \approx \frac{1}{z} e^{-z^2/2}, \quad z > 1. \]

Therefore, when \( \alpha(M, p) \) satisfies \( 0 < \alpha^2 < 1/2 \) and \( R > R_0(M) \) we can neglect the error term in the application of the central limit theorem, so we have
\[ P(E_\rho(x)) \approx K_0^{-1} e^{-\frac{x^2}{2K^2}} \]
which implies
\[ E|\eta| := \sum_{j=1}^{m} P(E_\rho(x_j)) \approx mK^{-1} e^{-\frac{x^2}{2K^2}}. \]

Note here, that by taking if necessary \( K_0(M) \) large enough we can neglect the case \( R < R_0(M) \).

Note that
\[ E|\eta|^2 = \sum_{j=1}^{m} P(E_\rho(x_j)) + \sum_{j,k=1}^{m} P(E_\rho(x_j) \cap E_\rho(x_k)) = \]
\[ = E|\eta| + \sum_{j,k=1}^{m} P(E_\rho(x_j) \cap E_\rho(x_k)) \]
and \( E|\eta| \leq K_3 m^{-1/2}(E|\eta|)^2 \leq K_3 P^{-1/2}(E|\eta|)^2 \), where \( K_3 = K_3(M) > 0 \). So to prove \((**)\) it is enough to show that
\[ (***) \sum_{j,k=1}^{m} P(E_\rho(x_j) \cap E_\rho(x_k)) \leq (1 + K_4 P^{-q})(E|\eta|)^2 \]
for some constants \( K_4 = K_4(M, p) > 0, q = q(M) > 0, \alpha(M, p) > 0 \).

**Step 6.** Let us split the index set \( \{(i, j) : 1 \leq i \neq j \leq n\} \) into two sets. Let
\[ \sigma_1 = \{(j, k) : 1 \leq j \neq k \leq m, \sum_{i=1}^{n} |a_i|^2 g_i(x_j)g_i(x_k) < \frac{1}{8}\}. \]

Since \((x_j)_{j=1}^{n} \in F\) (see Step 3) it follows that
\[ |\sigma_1^{\ominus}| \leq 8^2 \sum_{j,k=1}^{m} \sum_{\substack{i=1 \atop j \neq k}}^{n} |a_i|^2 g_i(x_j)g_i(x_k)^2 \leq 128 \left( \frac{mMR^p}{\varepsilon(M)} \right)^2 \sum_{i=1}^{n} |a_i|^4 = \frac{128}{R} \left( \frac{mMR^p}{\varepsilon(M)} \right)^2. \]
Thus, whenever \( \alpha^2(M, p) < 1/2 - p \) we have
\[
\sum_{(j, k) \in \sigma_1} \mathbb{P}(E_{\rho}(x_j) \cap E_{\rho}(x_k)) \leq \frac{128}{R} \left( \frac{mMR^p}{\varepsilon(M)} \right)^2 \rho^{-1} e^{-\frac{\rho^2}{2K^2}} < \frac{K_4(M)}{R^{1/2-p}} (E[\eta])^2.
\]

**Step 7.** For fixed pair \( s = (j, k) \in \sigma_1 \) consider a set of 2-dimensional random vectors defined by
\[
X_i^s(\omega) := (a_i \xi_i(\omega) g_i(x_j), a_i \xi_i(\omega) g_i(x_k)); \quad 1 \leq i \leq n.
\]
To estimate the error term in the central limit theorem for these random vectors, notice that
\[
\rho_3^s := \frac{1}{n} \sum_{i=1}^{n} |a_i|^3 E[|\xi_i|^3 (|g_i(x_j)|^2 + |g_i(x_k)|^2)]^{3/2} \leq \frac{8M^3 \beta(M)}{n} \sum_{i=1}^{n} |a_i|^3.
\]
\[
V^s = \frac{1}{n} \begin{pmatrix}
\sum_{i=1}^{n} |a_i|^2 |g_i(x_j)|^2 & \sum_{i=1}^{n} |a_i|^2 g_i(x_j)g_i(x_k) \\
\sum_{i=1}^{n} |a_i|^2 g_i(x_j)g_i(x_k) & \sum_{i=1}^{n} |a_i|^2 |g_i(x_k)|^2
\end{pmatrix} = \frac{1}{n} \begin{pmatrix}
K^2 & \sum_{i=1}^{n} |a_i|^2 g_i(x_j)g_i(x_k) \\
\sum_{i=1}^{n} |a_i|^2 g_i(x_j)g_i(x_k) & K^2
\end{pmatrix}.
\]
Hence,
\[
\det V^s = \frac{1}{n^2} \left( K^4 - \left| \sum_{i=1}^{n} |a_i|^2 |g_i(x_j)g_i(x_k)|^2 \right| > \frac{1}{n^2} \left( \frac{1}{16} - \frac{1}{64} \right) = \frac{3}{64n^2},
\]
and
\[
\text{trace } V^s = \frac{2K^2}{n}.
\]
Note that the matrix \( V^s \) is positive so both its eigenvalues are positive. Let \( \lambda_2 \geq \lambda_1 > 0 \) be the eigenvalues, taking into account that \( \lambda_1 + \lambda_2 = \text{trace } V^s = 2K^2/n \), we get
\[
\frac{3}{64n^2} < \det V^s = \lambda_2 \lambda_1 < \frac{2K^2}{n} \lambda_1,
\]
hence,
\[
\lambda_1 > \frac{3}{128nK^2}.
\]
So the central limit theorem (Proposition 1) for \( X_i^s \) gives
\[
\left| \mathbb{P}(E_{\rho}(x_j) \cap E_{\rho}(x_k)) - \frac{1}{2\pi (\det V^s)^{1/2}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(y, (V^s)^{-1}y)} dy_1 dy_2 \right| \leq
\]
\[
\leq K_1(2)n^{-1/2}\rho_3^s \lambda_1^{-3/2} < K_5 \sum_{i=1}^{n} |a_i|^3,
\]
for a constant $K_5 = K_5(M) < \infty$. Taking into account $\sum_{i=1}^{n} |a_i|^3 \leq R^{-1/2}$, we get

\[
\sum_{s=(j,k) \in \sigma_1} P(E_\rho(x_j) \cap E_\rho(x_k)) < 
K_5^2 \int_{\rho}^{\infty} \int_{\rho}^{\infty} \sum_{s \in \sigma_1} \frac{1}{2\pi (\det nV^s)^{1/2}} e^{-\frac{1}{2}(Y, (nV^s)^{-1} Y)} dy_1 dy_2.
\]

If we choose $\alpha(M) < 1/5$ the error term $K_5 m^2 R^{-1/2} \leq K_6(M) R^{-1/4} |E| |\eta||^2$, thus, to prove (***), it remains to estimate the integral term

\[
\int_{\rho}^{\infty} \int_{\rho}^{\infty} \sum_{s \in \sigma_1} \frac{1}{2\pi (\det nV^s)^{1/2}} e^{-\frac{1}{2}(Y, (nV^s)^{-1} Y)} dy_1 dy_2.
\]

We shall compare it with the expression

\[
\sum_{(j,k) \in \sigma_1} P(E_\rho(x_j)) \cdot P(E_\rho(x_k)) = \int_{\rho}^{\infty} \int_{\rho}^{\infty} \sum_{s \in \sigma_1} \frac{1}{2\pi K^2} e^{-\frac{1}{2}(y_1^2 + y_2^2)} dy_1 dy_2 + w
\]

\[
= \frac{|\sigma_1|}{2\pi K^2} \int_{\rho}^{\infty} \int_{\rho}^{\infty} e^{-\frac{1}{2}(y_1^2 + y_2^2)} dy_1 dy_2 + w,
\]

where $w \leq 2K_2 m^2 R^{-1/2}$ (see Step 5), as before we can ensure that $w = o(R^{-1/4} |E| |\eta||^2)$. In order to compare the two integral expressions, notice that

\[
(nV^s)^{-1} = \frac{1}{\det(nV^s)} \begin{pmatrix} K^2 & -\sum_{i=1}^{n} |a_i|^2 g_i(x_j)g_i(x_k) \\ -\sum_{i=1}^{n} |a_i|^2 g_i(x_j)g_i(x_k) & K^2 \end{pmatrix}.
\]

Let

\[
c_s := \sum_{i=1}^{n} |a_i|^2 g_i(x_j)g_i(x_k) \quad \text{for } s = (i, j) \in \sigma_1,
\]

then

\[
(nV^s)^{-1} = \begin{pmatrix} \frac{K^2}{K^4 - |c_s|^2} & -\frac{c_s}{K^4 - |c_s|^2} \\ -\frac{c_s}{K^4 - |c_s|^2} & \frac{K^2}{K^4 - |c_s|^2} \end{pmatrix}.
\]

Now we have

\[
\int_{\rho}^{\infty} \int_{\rho}^{\infty} \sum_{s \in \sigma_1} \frac{1}{2\pi (\det nV^s)^{1/2}} e^{-\frac{1}{2}(Y, (nV^s)^{-1} Y)} dy_1 dy_2 =
\]

\[
\int_{\rho}^{\infty} \int_{\rho}^{\infty} \sum_{s \in \sigma_1} \frac{1}{2\pi (K^4 - |c_s|^2)^{1/2}} \exp \left( -\frac{K^2 (y_1^2 + y_2^2)}{2(K^4 - |c_s|^2)} + \frac{c_s y_1 y_2}{K^4 - |c_s|^2} \right) dy_1 dy_2.
\]

Let also

\[
a_s := \frac{K^2}{K^4 - |c_s|^2}; \quad b_s := \frac{c_s}{K^4 - |c_s|^2}, \quad s \in \sigma_1.
\]
Notice, that for any $L > 1$ the following inequality holds
\[
\int_{L \rho}^{\infty} \int_{\rho}^{\infty} \sum_{s \in \sigma_1} \frac{1}{2\pi(K^4 - |c_s|^2)^{1/2}} e^{(-\frac{a_s}{2}(y_1^2 + y_2^2) + b_s y_1 y_2)} dy_1 dy_2 \leq 4 \int_{L \rho}^{\infty} \int_{\rho}^{\infty} \sum_{s \in \sigma_1} e^{-\frac{a_s-b_s}{2}(y_1^2 + y_2^2)} dy_1 dy_2 \times \frac{1}{L \rho^2} \sum_{s \in \sigma_1} e^{-\frac{a_s-b_s}{2}(L^2+1)\rho^2}.
\]
Since for all $s \in \sigma_1$ we have
\[
a_s - b_s = \frac{K^2 - c_s}{K^2 - |c_s|^2} = \frac{1}{K^2 + c_s} \geq \frac{1}{\gamma(M) + \frac{1}{8}} =: \gamma'(M) > 0,
\]
it follows that
\[
\int_{L \rho}^{\infty} \int_{\rho}^{\infty} \sum_{s \in \sigma_1} \frac{1}{2\pi(\det(nV^s))^{1/2}} e^{(-\frac{a_s}{2}(y_1^2 + y_2^2) + b_s y_1 y_2)} dy_1 dy_2 \leq \frac{K \gamma m^2}{L \rho^2} e^{-\frac{1}{2} \gamma'(M)(L^2+1)\rho^2}.
\]
Set $L^2 + 1 := \frac{4}{\gamma(M) K^2}$ and get
\[
\frac{K \gamma m^2}{L \rho^2} e^{-\frac{1}{2} \gamma'(M)(L^2+1)\rho^2} \leq K_8(M) \frac{(E[\eta])^2}{\rho^{a^2(M,p)}}
\]
with a constant $K_8(M) < \infty$. Therefore, there exists a constant $K_9(M) < \infty$ such that whenever $\alpha^2(M,p) < 1/5$ the following inequality holds
\[
\sum_{(j,k) \in \sigma_1} P(E_{\rho}(x_j)) \cap P(E_{\rho}(x_k)) \leq K_9(M) \frac{(E[\eta])^2}{\rho^{a^2(M,p)}} + \int_{L \rho}^{L \rho} \int_{\rho}^{L \rho} \sum_{s \in \sigma_1} \exp \left\{ -\frac{a_s}{2}(y_1^2 + y_2^2) + b_s y_1 y_2 \right\} dy_1 dy_2.
\]

**Step 8.** To finish the proof of the Theorem it remains to compare the expression
\[
A := \sum_{s \in \sigma_1} \frac{1}{2\pi(\det(nV^s))^{1/2}} e^{(-\frac{a_s}{2}(y_1^2 + y_2^2) + b_s y_1 y_2)}
\]
with the expression
\[
B := \frac{1}{2\pi K^2} |\sigma_1| e^{-\frac{1}{2K^2}(y_1^2 + y_2^2)}
\]
in the range $\rho \leq y_1, y_2 \leq L \rho$. We are going to show that $A \leq B(1 + K_10 R^{-q'})$ pointwise in that range with some constants $K_10(M) < \infty$, $q'(p) > 0$. In fact, assume for a moment we have shown it, then integrate this inequality over the domain $\rho \leq y_1, y_2 \leq L \rho$ and get
\[
\int_{\rho}^{L \rho} \int_{\rho}^{L \rho} \sum_{s \in \sigma_1} \frac{1}{2\pi(\det(nV^s))^{1/2}} e^{(-\frac{a_s}{2}(y_1^2 + y_2^2) + b_s y_1 y_2)} dy_1 dy_2 < (1 + \frac{K_10}{R^{q'}}) \int_{\rho}^{\infty} \int_{\rho}^{\infty} \frac{|\sigma_1|}{2\pi K^2} e^{-\frac{y_1^2+y_2^2}{2K^2}} dy_1 dy_2 \leq (1 + \frac{K_10}{R^{q'}})((E[\eta])^2 + w),
\]
where the error term \( w = o(R^{-1/4})(E[\eta])^2 \) (see step 7). This finally implies (***) and, thus, the theorem statement.

To prove this inequality split the index set \( \sigma_1 \) into subsets
\[
\sigma_r := \{ s \in \sigma_1 : 2^{-r} \leq |c_s| < 2^{-r+1} \}, \quad r = 4, 5, \ldots
\]
Clearly, \( \sigma_1 = \bigcup_{r \geq 4} \sigma_r \). We can estimate \(|\sigma_r|\) as follows (see Step 3)
\[
\frac{|\sigma_r|}{2^{2r}} \leq \sum_{s \in \sigma_1} |c_s|^2 \leq 2m^2 \left( \frac{MR^p}{\varepsilon(M)} \right)^2 \sum_{i=1}^n |a_i|^4
\]
so that
\[
|\sigma_r| \leq \min \left( m^2, 2^{2r+1} \left( \frac{MR^p}{\varepsilon(M)} \right)^2 \frac{m^2}{R} \right).
\]
For \( s \in \sigma_r \) and \( \rho \leq y_1, y_2 \leq L\rho \) we have
\[
\left| (a_s - \frac{1}{K^2})(y_1^2 + y_2^2) - 2b_s y_1 y_2 \right| = \left| \frac{(c_s/K)^2(y_1^2 + y_2^2) - 2c_s y_1 y_2}{K^4 - c_s^2} \right| \leq 25 \left( 2^{-2r+2}(y_1^2 + y_2^2) + 2^{-r+2}y_1 y_2 \right) \leq 200 \cdot 2^{-r} L^2 \rho^2.
\]
Moreover, if \( s \in \sigma_r \) with \( r \geq 4 \), then
\[
\det(nV^*) = K^4 - c_s^2 \geq K^4 - 2^{-2r+2} \geq \frac{K^4}{(1 + K_{11}2^{-r})^2},
\]
where \( K_{11} \) is an absolute constant. Now we can say that
\[
A \leq \frac{B \cdot S}{|\sigma_1|},
\]
where
\[
S := \sum_{r=4}^{\infty} |\sigma_r| (1 + K_{11}2^{-r}) e^{\frac{200}{8}L^2\rho^2} = \sum_{r=4}^{\left\lfloor \frac{1-2p}{4 \log_2 R} \right\rfloor + 1} + \sum_{r=\left\lceil \frac{1-2p}{4 \log_2 R} \right\rceil}^{\infty} = S_1 + S_2.
\]
We can estimate \( S_1 \) as follows
\[
S_1 \leq \frac{1-2p}{4} \log_2 R \cdot 2R^{1-2p} \left( \frac{MR^p}{\varepsilon(M)} \right)^2 m^2 \frac{R}{R} \left( 1 + \frac{K_{11}}{16} \right) e^{\frac{200}{8}L^2\rho^2}.
\]
Notice, that \( e^{\frac{200}{8}L^2\rho^2} = P^{\frac{200}{8}K^2L^2} \). Choose \( \alpha(M, p) \) such that
\[
\frac{200}{8} \alpha^2 K^2 L^2 < \frac{1-2p}{20}.
\]
This condition on \( \alpha(M, p) \) is compatible with the previously imposed ones \( (\alpha^2 < \min\{\frac{1}{2} - p, \frac{1}{25}\}) \). So we get
\[
S_1 \leq K_{12} |\sigma_1| R^{\frac{1-2p}{4}}
\]
with a constant \( K_{12} = K_{12}(M, p) < \infty \). Further,
\[
S_2 \leq \sum_{r=\left\lceil \frac{1-2p}{4 \log_2 R} \right\rceil}^{\infty} |\sigma_r| (1 + K_{11}R^{\frac{(1-2p)}{4}})(1 + K_{13}R^{\frac{(1-2p)}{4}}) \leq |\sigma_1|(1 + K_{14}R^{\frac{(1-2p)}{4}})
\]
with some constants $K_{13} = K_{13}(M, p)$ and $K_{14} = K_{14}(M, p) < \infty$. Thus, we have
\[
|S| \leq (1 + (K_{12} + K_{14})R^{-(1-2p)/5})|\sigma_1|
\]
which implies $A \leq B(1 + K_{10}R^{-(1-2p)/5})$ and completes the proof of Theorem 1.

3. The upper estimate.

Let us show now that with some restrictions on $\{\xi_k\}_{1}^{n}$ and $m \leq n$ the estimate (10) is precise in sense of order. The following theorem states this explicitly.

**Theorem 2.** Let $\xi_k$ be independent variables for which the following exponential estimate takes place

\[
P\left\{ \left| \sum_{k=1}^{n} c_k \xi_k \right| > t \left( \sum_{k=1}^{n} c_k^2 \right)^{1/2} \right\} \leq C_4 e^{-t^2C_5}
\]

for all sets of coefficients $\{c_k\}_{1}^{n}$ with some absolute positive constants $C_4, C_5$. Then

\[
E\left\| \sum_{k=1}^{n} \xi_k f_k \right\|_{m, \infty} \leq C_6 \left( \sum_{k=1}^{n} ||f_k||_{\infty}^2 \right)^{1/2} \cdot \sqrt{1 + \log m}
\]

for all function systems $\{f_k\}_{1}^{n} \subset L_1(X)$ and all $m \geq 1$ with an absolute constant $C_6 > 0$. And since for all bounded functions $f \in L_\infty$ the integral-uniform norm $\|f\|_{m, \infty} \leq \|f\|_\infty$ for bounded functions (12) implies

\[
E\left\| \sum_{k=1}^{n} \xi_k f_k \right\|_{m, \infty} \leq C_6 \left( \sum_{k=1}^{n} ||f_k||_{\infty}^2 \right)^{1/2} \cdot \sqrt{1 + \log m}.
\]

**Proof.** For $x \in X$ define
\[
\eta_x(\omega) := \sum_{k=1}^{n} f_k(x)\xi_k(\omega);
\]
\[
\mu_t(x_1, \ldots, x_m) := P\left\{ \max_{1 \leq h \leq m} (||\eta_{x_h}||) > t \max_{1 \leq h \leq m} \left( \sum_{k=1}^{n} |f_k(x_h)|^2 \right)^{1/2} \right\}.
\]

Using the exponential estimate (11), we get that
\[
\mu_t(x_1, \ldots, x_m) \leq \sum_{h=1}^{m} P\left\{ ||\eta_{x_h}|| > t \max_{1 \leq h \leq m} \left( \sum_{k=1}^{n} |f_k(x_h)|^2 \right)^{1/2} \right\} \leq mC_4 e^{-C_5t^2}.
\]

Note that for $t_0 = (\frac{3}{C_5} \log m)^{1/2}$ we have $\mu_{t_0}(x_1, \ldots, x_m) \leq C_4 m^{-2}$. Now we can estimate
\[
E \max_{1 \leq h \leq m} ||\eta_{x_h}|| \leq \left( \max_{1 \leq h \leq m} \sum_{k=1}^{n} |f_k(x_h)|^2 \right)^{1/2} \left( \sqrt{\frac{3}{C_5}} \log m + \sum_{t=t_0}^{\infty} (t+1)\mu_t(x_1, \ldots, x_m) \right)
\]
\[
\leq \left( \max_{1 \leq h \leq m} \sum_{k=1}^{n} |f_k(x_h)|^2 \right)^{1/2} \left( \sqrt{\frac{3}{C_5}} \log m + C_4 m \sum_{t=t_0}^{\infty} (t+1)e^{-C_5t^2} \right)
\]
\[
\leq C \max_{1 \leq h \leq m} \left( \sum_{k=1}^{n} |f_k(x_h)|^2 \right)^{1/2} \sqrt{1 + \log m}.
\]
Integrating the last inequality with respect to \( x_1, \ldots, x_m \), we get (see (2))

\[
E \left\| \sum_{k=1}^{n} \xi_k f_k \right\|_{m,\infty} \leq \int_X \cdots \int_X E \max_{1 \leq h \leq m} |\eta_{x_h}| d\mu(x_1) \cdots d\mu(x_m)
\]

\[
\leq C \left\| \left( \sum_{k=1}^{n} |f_k|^2 (1 + \log m) \right)^{1/2} \right\|_{m,\infty}.
\]

This completes the proof.

**Corollary 2.** For uniformly bounded functions \( \{f_k\}^n_1 \subset L_\infty(X) \) with \( \|f_k\|_\infty \leq M \) and independent random variables \( \{\xi_k\}^n_1 \), satisfying the exponential estimate (11), Theorem 2 implies

\[
E \left\| \sum_{k=1}^{n} a_k \xi_k f_k \right\|_{m,\infty} \leq MC_6 \left( \sum_{k=1}^{n} |a_k|^2 \right)^{1/2} \cdot \sqrt{1 + \log m}.
\]

Hence, whenever \( m \leq n \) and \( m = O(R(\{a_k\}^n_1)) \) (see (7)), then the inequality (10) from Theorem 1 is precise in sense of order for all uniformly bounded function systems \( \{f_k\}^n_1 \) and independent random variables \( \{\xi_k\}^n_1 \), satisfying the exponential estimate (11). In particular, it is true for trigonometric polynomials with random coefficients.

**Remark.** If we take a sequence of (multivariate) trigonometric polynomials of order at most \( n \) as the functions \( \{f_k\}^n_1 \) and apply Theorem 2 with the parameter \( m = n \) then, taking into account (14) (see below), we get the well-known upper estimate for the expectation of the uniform norm of a random trigonometric polynomial, e.g. exposed in J.-P. Kahane’s book (see Th. 3 Ch. 6 [4]).

4. **Some properties of the integral-uniform norms and application.**

The following Theorem compares the integral-uniform norm of an integrable function \( f \in L_1(X) \) with its average over an arbitrary subset of \( X \).

**Theorem 3.** For each \( f \in L_1(X) \) \( ((X, \mu) \text{ is a probability space}) \) and arbitrary measurable \( \Delta \subset X \) \( (\mu \Delta \equiv |\Delta| > 0) \) the following inequality holds

\[
\|f\|_{m,\infty} \geq (1 - (1 - |\Delta|)^m) \cdot \frac{1}{|\Delta|} \int_{\Delta} |f|.
\]

**Proof.** Clearly, it suffices to prove (13) for the case when \( \text{supp} f \subset \Delta \). Using the formula (3) we get

\[
\|f\|_{k+1,\infty} - \|f\|_{k,\infty} = \int_0^\infty (1 - (1 - \lambda_f(t))^{k+1}) \, dt - \int_0^\infty (1 - (1 - \lambda_f(t))^k) \, dt
\]

\[
= \int_0^\infty \lambda_f(t)(1 - \lambda_f(t))^k \, dt
\]

\[
\geq (1 - |\Delta|)^k \int_0^\infty \lambda_f(t) \, dt = (1 - |\Delta|)^k \|f\|_1.
\]

Sum this inequality up from \( k = 1 \) to \( k = m - 1 \) and get

\[
\|f\|_{m,\infty} - \|f\|_1 \geq \|f\|_1 \sum_{k=1}^{m-1} (1 - |\Delta|)^k,
\]
which implies

$$||f||_{m,\infty} \geq ||f||_1 \frac{1 - (1 - |\Delta|)^m}{|\Delta|}.$$  

To complete the proof notice that $||f||_1 = \int_\Delta |f|$ since $\text{supp} f \subset \Delta$.

For trigonometric polynomials of order at most $n$ the identity (5) implies

(14) \quad $||P_n||_n,\infty \asymp ||P_n||_\infty$.  

In fact, for a set $E := \{x \in [0, 2\pi] : |P_n(x)| \geq ||P_n||_\infty/2\}$ the Bernstein inequality implies $\mu E \geq 1/n$, evaluating $||\chi_E||_{n,\infty}$ from (5), we get (14). If $n \geq m$ then (4) and (14) for trigonometric polynomials of order at most $n$ imply

$$||P_n||_\infty \leq C \frac{n}{m} ||P_n||_{m,\infty},$$

where $C > 0$ is an absolute constant. For the Fejér kernels this inequality is precise in sense of order, in fact, when $n \geq m$ one can prove that

$$||K_n||_{m,\infty} \asymp m; \quad ||D_n||_{m,\infty} \asymp m(1 + \log \frac{n}{m}),$$

where $K_n$ is the Fejér kernel and $D_n$ is the Dirichlet kernel.

For the integral-uniform norm as for any shift invariant norm (e.g. see [2]) the following analog of the Bernstein inequality takes place.

**Proposition 2.** \footnote{See [2] for more general cases of Bernstein-type inequalities.} For the integral-uniform norm of the derivative of trigonometric polynomial $P_n$ of order at most $n$ the following inequality holds

(15) \quad $||P_n^{(r)}||_{m,\infty} \leq n^r ||P_n||_{m,\infty}, \quad r = 1, 2, \ldots.$  

**The idea of the proof.** Use the M. Riesz Interpolation Formula [12] (or see Ch. 2.4 [11]) for derivative of a trigonometric polynomial of order at most $n$:

$$P_n'(x) = \sum_{k=1}^{2n} (-1)^{k+1}\lambda_k P_n(x + x_k),$$

where $\lambda_k := \frac{1}{4n \sin^2(x_n/2)}$; \quad $x_n := \frac{2k - 1}{2n}\pi$.

And notice that $\sum_{k=1}^{2n} \lambda_k = n$. \square

It is well-known that $L_\infty$-norm of trigonometric polynomials of order at most $n$ is equivalent to its $L_\infty$-norm taken over the uniform net $\{\frac{s}{4n}2\pi\}_{s=1}^{4n}$, precisely

(16) \quad $||P_n||_\infty \asymp \max_{1 \leq s \leq 4n} (|P_n(\frac{s}{4n}2\pi)|).$

This fact easily follows from the classical Bernstein inequality for the uniform norm. Using Proposition 2 one can prove an analog of (16) for the integral-uniform norm. For
a vector $x = (x_k)_{k=1}^N \in \mathbb{R}^N$ define $\|x\|_{m,\infty}$ by
\[
\|x\|_{m,\infty} := \frac{1}{N^m} \sum_{k_1=1}^N \cdots \sum_{k_m=1}^N \max_{1 \leq j \leq m} |x_{k_j}|.
\]
For trigonometric polynomials we have the following

**Theorem 4.** There exist positive constants $C_9, C_{10}$ such that for all trigonometric polynomials $P_n$ of order at most $n$ the following inequalities hold
\[
P_n \leq \|P_n\|_{m,\infty} \leq C_9 \|P_n\|_{m,\infty},
\]
where $P_n := (P_n(t_k))_{k=1}^{8n}$, $t_k := \frac{k}{8n} 2\pi$.

**Proof.** Let $\Delta_k := [t_k, t_{k+1})$, and $\delta := 2\pi/(8n)$. The family of semi-intervals $\{\Delta_k\}_{k=1}^{8n}$ splits the circle $[0, 2\pi]$, so for each $x \in [0, 2\pi]$ there exists a unique $k(x)$ such that $x \in \Delta_{k(x)}$. Thus, for any net $x_1, \ldots, x_m$ we have
\[
\left| \max_{1 \leq j \leq m} \left( |P_n(x_j)| \right) - \max_{1 \leq j \leq m} \left( |P_n(t_{k(x_j)})| \right) \right| \leq \max_{1 \leq j \leq m} \left( |P_n(x_j) - P_n(t_{k(x_j)})| \right)
\]
\[
\leq \max_{1 \leq j \leq m} \left( \int_{\Delta_{k(x_j)}} |P'_n| \right).
\]
Integrating this inequality over $x_1, \ldots, x_m$ we get
\[
(2\pi)^m \|P_n\|_{m,\infty} - \|P_n\|_{m,\infty} \leq \delta^m \sum_{j_1, \ldots, j_m} \max_{1 \leq s \leq m} \int_{\Delta_{s}} |P'_n| \leq 
\]
\[
\leq \delta \sum_{j_1, \ldots, j_m} \int_{\Delta_1} \cdots \int_{\Delta_m} \max_{1 \leq s \leq m} |P'_n(x_s)| dx_1 \cdots dx_m = \delta (2\pi)^m \|P'_n\|_{m,\infty}.
\]
Applying (15) to estimate the righthand-side we get
\[
\|P_n\|_{m,\infty} - \|P_n\|_{m,\infty} \leq \delta n \|P_n\|_{m,\infty} \leq \frac{2\pi}{8} \|P_n\|_{m,\infty}.
\]
Since $\pi/4 < 1$, it implies (17) and completes the proof.

Now we give an application of Theorem 1 demonstrating the potential utility of the family of integral-uniform norms. In [10] S. Montgomery-Smith and E.M. Semenov reduced a certain problem from functional analysis to the following

**Problem.** Let $\{f_i\}_{i=1}^n$ be a system of functions defined on a measure space $(X, \mu)$, $\mu X = 1$ and $\|f_i\|_1 = 1$. The question is if there exists a sequence of signs $\{\theta_i\}_{i=1}^n$, $\theta_i = \pm 1$ such that for every $k = 1, \ldots, n$ the following estimate takes place
\[
\sup_{\theta \in \mathbb{R}^n} \sum_{\mu e = 2^{-k}} 2^k \int_{e} \left| \sum_{i=1}^n \theta_i f_i(x) \right| d\mu(x) \geq c_0 \sqrt{n}k,
\]
where $c_0$ is an absolute positive constant.

Using Theorem 1 we prove the following theorem, which partially solves the Problem.

**Theorem 5.** For a set of functions $\{f_i\}_{i=1}^n$ on $(X, \mu)$, $\mu X = 1$ such that $\|f_i\|_1 = 1$ and $\|f_i\|_3 \leq M$, there exists a sequence of signs $\{\theta_i\}_{i=1}^n$, $\theta_i = \pm 1$, such that for every $k = 1, \ldots, \log n$ the inequality (18) takes place with an absolute constant $c_0 > 0$. 
Proof. Fix some $\delta \in (0, 1/2)$. Assume first that for the function system $\{f_i\}_{i=1}^n$ there exists a set of signs $\{\theta_i\}_{i=1}^n$, $\theta_i = \pm 1$ such that

$$\| \sum_{i=1}^n \theta_i f_i \|_1 \geq n^{1+\delta/2}.$$  

Then it obviously implies the assertion of the Theorem. In fact, let $k < n^\delta$ then

$$\sup_{x \in X} \frac{1}{\varepsilon} \left( \sum_{i=1}^n \theta_i f_i(x) \right) \geq \| \sum_{i=1}^n \theta_i f_i \|_1 \geq n^{1/2+\delta/4}.$$  

Now assume the opposite, i.e. that

$$\| \sum_{i=1}^n \theta_i f_i \|_1 \leq n^{3/2+\delta}.$$  

for all sets of signs $\{\theta_i\}_{i=1}^n$, $\theta_i = \pm 1$. We are going to show that (19) and the boundness of functions $f_i$ in $L_3$ imply (b') from Section 2 with some $p < 1/2$. We need the following geometrical

Lemma. Let $\{w_i\}_{i=1}^n$ be a set of vectors in a linear space with a norm $\| \cdot \|$ (or a seminorm) such that $\| w_i \| = 1$ and

$$\| \sum_{i=1}^n \theta_i w_i \| \leq C_1 n^{1/2+\delta}$$  

for all sets of signs $\{\theta_i\}_{i=1}^n$, $\theta_i = \pm 1$ with some constants $\beta \in [0, 1/2)$, $C_1 > 0$. Then

$$\| \sum_{i=1}^n a_i w_i \| \leq C_2 (\beta, C_1) n^{1/4+\delta} \left( \sum_{i=1}^n a_i^2 \right)^{1/4}$$  

for all sets of coefficients $\{a_i\}_{i=1}^n$. One cannot improve this estimate in the sense that there exist a norm $\| \cdot \|$, vectors $\{w_i\}_{i=1}^n$ and coefficients $\{a_i\}_{i=1}^n$ such that (20) holds and (20') is precise in sense of order.²

Let us first finish the proof of the theorem. Notice that (19), Hölder’s inequality and the triangle inequality for $\| \cdot \|$-norm imply for the function $F^\theta := \sum_{i=1}^n \theta_i f_i$,

$$\| F^\theta \|_2 \leq \| F^\theta \|_1^{1/2} \| F^\theta \|_3^{1/4} \leq n^{1/4+\delta} (M n)^{1/4} = M^{1/4} n^{1/4+\delta}.$$  

Thus, we can apply the Lemma for a set of functions $\{f_i/\|f_i\|_2\}_{i=1}^n$ in $L_2$ with $\beta = 3/8 + \delta/4$ and get

$$\| \sum_{i=1}^n a_i f_i \|_2 \leq C(M) n^{7+2\delta/16} \left( \sum_{i=1}^n a_i^2 \right)^{1/2}.$$  

Therefore, the function system $\{f_i/\|f_i\|_2\}_{i=1}^n$ satisfies the conditions (a), (b) (with another $M$) and $p = (7 + 2\delta)/16 < 1/2$. Let $\{g_i\}_{i=1}^n$ be the Rademacher functions and

²Geometrically the Lemma implies that the convex hull of the set $B_\infty^d \cup (n^{1/2+\delta} B_1^d)$ has an inscribed sphere with radius of order $n^{1/4+\delta/2}$, here $B_\infty^d$ denotes the $d$-dimensional cube whose vertices have coordinates $\pm 1$, and $B_1^d = \{v \in \mathbb{R}^d : \sum_{i=1}^d |v_i| \leq 1\}$ (generalized octahedron).
\[ m = 2^k \leq n, \text{ now we can apply Theorem 1 (see (9) and the Remark to Theorem 1)} \]

\[ \text{to the random polynomial } F^\xi = \sum_1^n \xi_i f_i \text{ and get} \]

\[ P\{ \omega \in \Omega : \| F^\xi \|_{2^k, \infty} \leq C'_1(nk)^{\frac{1}{2}} \} \leq C'_2 2^{-qk}. \]

Summing these inequalities up from \( k = k_0 \) to \( \log n \) we get

\[ P \bigcup_{k=k_0}^{\log n} \left\{ \omega \in \Omega : \| F^\xi \|_{2^k, \infty} \leq C'_1(nk)^{\frac{1}{2}} \right\} \leq C 2^{-qk_0}. \]

Thus, there exists a set of signs \( \{ \theta_i \}_{i=1}^n \) such that

\[ \| F^\theta \|_{2^k, \infty} = \| \sum_{i=1}^n \theta_i f_i \|_{2^k, \infty} \geq C(M, \delta) \sqrt{nk} \]

for each \( k = k_0, \ldots, \log n \). Moreover, we can take the constant \( C(M, \delta) \) large enough in order to make the inequality (21) hold for all \( k = 1, \ldots, \log n \).

Let us show now that (21) implies (18). Consider the set \( e^*_k \subset X, \mu e^*_k = 2^{-k} \) such that

\[ \int_{e^*_k} |F^\theta(x)|d\mu = \sup_{\mu E = 2^{-k}} \int_{e^*_k} \left| \sum_{i=1}^n \theta_i f_i(x) \right|d\mu, \]

and let \( \chi e^*_k \) be its characteristic function. Using the triangle inequality for the integral-uniform norm and the fact that \( \| g \|_{m, \infty} \leq m \| g \|_1 \) for each \( g \) (see (4)), obtain

\[ \| F^\theta \|_{2^k, \infty} \leq \| F^\theta \cdot \chi e^*_k \|_{2^k, \infty} + \| F^\theta \cdot (1 - \chi e^*_k) \|_{2^k, \infty} \leq \]

\[ \leq 2^k \int_{e^*_k} |F^\theta(x)|d\mu + \sup_{x \notin e^*_k} |F^\theta(x)| \leq 2 \cdot 2^k \int_{e^*_k} |F^\theta(x)|d\mu. \]

This inequality along with (21) and (22) implies (18) for all \( k = 1, \ldots, \log n \). To complete the proof of the Theorem it remains to prove the Lemma.

**Proof of the Lemma.** Consider a set of coefficients \( \{a_i\}_{1}^n \) such that \( \sum_1^n a_i^2 = n \) and \( |a_i| < \sqrt{n} \). Define the following index sets

\[ \sigma_k = \{ j : 2^{-k} \sqrt{n} \leq |a_i| < 2^{-k+1} \sqrt{n} \} \quad k \geq 1. \]

From Chebyshev’s inequality it follows that \( |\sigma_k| \leq 2^{2k} \). Define

\[ w^{(k)} := \sum_{j \in \sigma_k} w_j. \]

From the triangle inequality it follows that \( \| w^{(k)} \| \leq |\sigma_k| 2^{-k+1} \sqrt{n} \leq 2^{k+1} \sqrt{n} \). In order to estimate

\[ W := \sum_{i=1}^n a_i w_i = \tilde{w}^{(K)} + \sum_{k=1}^K w^{(k)} \]

notice that the residual term \( \tilde{w}^{(K)} \) belongs to the convex hull of the vectors \( 2^{-K} \sqrt{n} \sum_1^n \pm w_i \). So we get

\[ \| \tilde{w}^{(K)} \| \leq 2^{-K} \sqrt{n} \cdot C_{11} n^{\frac{1}{2} + \beta}. \]
Assembling all the facts we get
\[ \|W\| \leq \sum_{k=1}^{K} \|w^{(k)}\| + \|\tilde{w}^{(K)}\| \leq \sqrt{n} \sum_{k=1}^{K} 2^{k+1} + C_{11} 2^{-K} n^{1+\beta} < \sqrt{n}(2^{K+2} + C_{11} 2^{-K} n^{1+\beta}). \]

Choose \( K \approx (1/4 + \beta/2) \log n \) to obtain
\[ \|W\| \leq C_{12} (C_{11}) n^{\frac{1}{4} + \frac{\beta}{2}}, \]
which completes the proof of (20') and Theorem 5.

To show that (20') is precise consider a linear space with a basis \( \{w_{k}\}_{1}^{n} \) and the norm
\[ \| \sum_{k=1}^{n} a_k w_k \| := \max \left( \sum_{k=1}^{[n^{1/2+\beta}]} |a_k|, \max_{k > n^{1/2+\beta}} |a_k| \right). \]

Obviously, \( \|w_k\| = 1 \) and \( \| \sum_{i=1}^{n} \pm w_k \| \leq n^{1/2+\beta} \). Consider a vector \( W := \sum_{k=1}^{[n^{1/2+\beta}]} w_k \), for which \( \|W\| = [n^{1/2+\beta}] \). Clearly, the inequality (20') is precise for \( W \).

**References**

[1] R.N. Bhattacharya and R. Ranga Rao, *Normal Approximation and Asymptotic Expantions*, J. Wiley (1976)
[2] R. DeVore and G. Lorentz, *Constructive Approximation*, Springer Verlag, 1993
[3] P.G. Grigoriev, *Estimates for norms of random polynomials and their application*, Math. Notes \textbf{v. 69}, \textbf{N 6} (2001)
[4] J.-P. Kahane, *Some Random Series of Functions*. Heath mathematical monographs. Lexington, Mass., 1968.
[5] B. Kashin and L. Tzafriri, *Lower estimates for the supremum of some random processes*, East J. on Approx. \textbf{v. 1}, \textbf{N 1} (1995), 125–139.
[6] B. Kashin and L. Tzafriri, *Lower estimates for the supremum of some random processes, II*, East J. on Approx. \textbf{v. 1}, \textbf{N 3} (1995), 373–377.
[7] B. Kashin and L. Tzafriri, *Lower estimates for the supremum of some random processes, II*, Preprint, Max–Plank Institut für Mathematik, Bonn/95-85
[8] M. Ledoux and M. Talagrand, *Probability in Banach Spaces*, Springer Verlag, 1991.
[9] M. Marcus and G. Pisier, *Random Fourier Series with Applications to Harmonic Analysis*, Princeton Univ. Press, 1981.
[10] S. Montgomery-Smith and E.M. Semenov, *Embeddings of rearrangement invariant spaces that are not strictly singular*, Positivity \textbf{v. 4}, (2000), 397–402.
[11] S.M. Nikolskii, *Approximation of the Multivariate Functions and Embedding Theorems*, Nauka, Moscow, 1969 (in Russian)
[12] M. Riesz, *Formule d’interpolation pour la dérivée d’un polynome trigonométrique*, C. R. Acad. Sci. Paris 158 (1914), 1152–1154.
[13] V.I. Rotar', *Nonuniform estimate of the rate of convergence in multidimensional central limit theorem*, Theory of Probab. and Appl. \textbf{v. 15} (1970), 647–665
[14] R. Salem, A. Zygmund, *Some properties of trigonometric series whose terms have random signs*, Acta Math. \textbf{91}(1954) 245–301.