Uniqueness and reconstruction theorems for pseudodifferential operators with a bandlimited Kohn-Nirenberg symbol

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Abstract Motivated by the problem of channel estimation in wireless communications, we derive a reconstruction formula for pseudodifferential operators with a bandlimited symbol. This reconstruction formula uses the diagonal entries of the matrix of the pseudodifferential operator with respect to a Gabor system. In addition, we prove several other uniqueness theorems that shed light on the relation between a pseudodifferential operator and its matrix with respect to a Gabor system.

Keywords Gabor analysis · Pseudodifferential operators · Shannon sampling · Bandlimited Kohn-Nirenberg symbol · Reconstruction · Uniqueness · Wireless channels

1 Introduction

The mathematical formulation of orthogonal frequency division multiplexing (OFDM) in wireless communications uses several fundamental notions from time-frequency analysis. On the one hand, Gabor expansions are used to transform digital information into an analog signal. On the other hand, pseudodifferential operators are used to model the distortion of a signal by the physical channel. Inevitably the rigorous analysis of the communication system leads to new and interesting questions in time-frequency analysis that are quite relevant for communication engineering.
In this paper we study a problem arising in channel estimation. Which information is required to determine the symbol of a pseudodifferential operator? How can an operator be reconstructed from such information?

To put the discussion on a firm basis, let us describe an extremely simplified model of signals and the transmission in wireless communications. Our discussion is based on the engineering monograph [17], see also [15, 40] for the mathematical models. Let \( \pi(z)f(t) = e^{2\pi iz^2t}f(t - z_1) \) denote the time-frequency shift by \( z = (z_1, z_2) \in \mathbb{R}^2 \), and let \( \Lambda = a\mathbb{Z} \times b\mathbb{Z} \) denote a lattice in the time-frequency plane with lattice parameters \( a, b > 0 \). In orthogonal frequency division multiplexing (OFDM) a string of numbers \( c_\lambda \), the “digital” information, is used as the coefficient sequence for a Gabor series of the form

\[
\sum_{k \in \mathbb{Z}} \sum_{|l| \leq B} c_{kl} e^{2\pi ilbt} g(t - ak) = \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda)g(t),
\]

for \( t \in \mathbb{R} \).

The pulse \( g \) is usually taken to be a characteristic function, but in nonstationary environments pulses with better frequency concentration are preferable [2, 16, 18, 21, 34]. The analog signal thus built is then transmitted from a sender to a receiver and distorted or transformed by physical processes.

The second link between wireless communications and time-frequency analysis is the description of the distortion of the signal \( f \) during the physical transmission. As a result of multipath propagation and of the Doppler effect, the received signal is a superposition of time-frequency shifts. Specifically, the received signal can be written as

\[
\hat{f}(t) = \int \hat{\sigma}(\eta, u) \pi(-u, \eta) f(t) du d\eta,
\]

for \( t \in \mathbb{R} \).

Here \( \hat{\sigma} \) is the Fourier transform of a symbol \( \sigma \) on \( \mathbb{R}^2 \) and is called the spreading function that indicates the amplitude of each occurring time-frequency shift. In the standard mathematical language the distortion \( f \to \hat{f} \) is just the pseudodifferential operator (in the Kohn-Nirenberg calculus) with symbol \( \sigma \), and is usually written as

\[
\hat{f}(t) = \sigma^{KN} f(t) = \int \sigma(t, \xi) e^{2\pi it\xi} \hat{f}(\xi) d\xi,
\]

for \( t \in \mathbb{R} \).

For physical reasons the time delay and the Doppler shift must be bounded, and therefore the spreading function \( \hat{\sigma} \) has a compact support. Equivalently, the symbol \( \sigma \) is bandlimited, i.e., analytic and of exponential type. From the perspective of analysis, such pseudodifferential operators are extremely special and are only the elementary building blocks for the study of difficult operators [19]. For wireless communications, pseudodifferential operators with bandlimited symbols are precisely the appropriate model.
At the receiver, the distorted analog signal \( \tilde{f} \) is analyzed by taking correlations with time-frequency shifts of the given pulse (or some other pulse). Thus the data to be analyzed are therefore the numbers

\[
y_{\lambda} = \langle \tilde{f}, \pi(\lambda)g \rangle = \sum_{\mu \in \Lambda} c_{\mu} \langle \sigma^{KN} \pi(\mu)g, \pi(\lambda)g \rangle,
\]

(1)

for \( \lambda \in \Lambda \).

The task of the engineer is now to recover and estimate the original data \( c_{\lambda} \) from the received information \( y_{\lambda} \). The central object here is the matrix \( H \) with entries

\[
H_{\lambda,\mu} = \left( \langle \sigma^{KN} \pi(\mu)g, \pi(\lambda)g \rangle \right),
\]

(2)

for all \( \lambda, \mu \in \Lambda \), of the pseudodifferential operator with respect to the set of time-frequency shifts \( \{ \pi(\lambda)g : \lambda \in \Lambda \} \). In wireless communications this matrix is called the channel matrix. Its estimation and inversion are among the principal engineering tasks.

A fundamental mathematical problem concerns the relation between the channel matrix and the symbol. This range of questions has been studied in time-frequency analysis, e.g., in [13, 15], yet the models and assumptions of wireless communications pose new and intriguing problems. An important objective is to recover or approximate the symbol \( \sigma \) from partial information about the channel matrix; this is the problem of channel estimation. Usually, in real wireless communication systems, pilot tones are used to estimate some entries of the channel matrix on the diagonal [6, 17, 41]. The problem then is to recover the entire matrix (2) and subsequently to solve the system \( y = Hc \).

The engineering models lead to the mathematical question when and how the channel matrix is completely determined by its diagonal. For arbitrary operators, this question does not even make sense, but for operators with a bandlimited symbol, as we will see, one can recover the symbol completely from the diagonal of the channel matrix. Our first contribution is a precise reconstruction formula for the symbol from the diagonal of the channel matrix. In other words, the matrix is completely determined by its diagonal! The hypothesis that \( \sigma \) is band-limited suggests a connection to the sampling theory of band-limited functions. Indeed, once this connection is established (which we do in Lemma 1 below), one may apply results from the Shannon sampling theory and obtains the following reconstruction formula.

**Theorem 1** Let \( \Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d \), \( g \in S(\mathbb{R}^d) \). If \( \sigma \in S'(\mathbb{R}^{2d}) \), supp \( \hat{\sigma} \subseteq Q_{\epsilon} = \left[ -\frac{1}{2a} + \epsilon, \frac{1}{2a} - \epsilon \right]^d \times \left[ -\frac{1}{2b} + \epsilon, \frac{1}{2b} - \epsilon \right]^d \) for some \( \epsilon > 0 \), then there exists a kernel \( K \in S(\mathbb{R}^{2d}) \) such that

\[
\sigma = \sum_{\lambda \in \Lambda} \langle \sigma^{KN} \pi(\lambda)g, \pi(\lambda)g \rangle T_{\lambda}K
\]

with convergence in \( S' \).

**Theorem 1** provides a theoretical answer to a crucial point of channel estimation: how can the channel be estimated from (partial) knowledge of the diagonal of the channel matrix?
We formulate several versions of this theorem that reflect various models and assumptions in wireless communications. In engineering it is usually assumed that the channel is a Hilbert-Schmidt operator and \( \sigma \in L^2(\mathbb{R}^d) \). In this case, the series (3) converges in \( L^2 \), and (3) is valid under weaker assumptions on the pulse \( g \). In our opinion, however, the distributional version offers a better model for signal propagation since “point scatterers” correspond precisely to point measures in the Kohn-Nirenberg symbol. Moreover, for the stable recovery of the coefficients \( c_\lambda \) from (1), the channel matrix must be invertible, which is certainly not the case for a Hilbert-Schmidt operator.

Theorem 1 can be interpreted as a result about operator identification and shares several aspects with the work of Pfander et al. [22, 29, 31]. In a series of papers, they investigate the question under what conditions a symbol \( \sigma \) can be recovered from a single measurement \( \sigma^{KN} f \) on a suitable distribution \( f \). Similar to Theorem 1, their answer is expressed as a sampling theorem and is valid for bandlimited symbols.

Our second contribution is the analysis of some common assumptions in the engineering community. A common assumption in the wireless communications literature is that the channel matrix is diagonal so that its inversion becomes trivial. Equivalently, this means that the channel matrix is diagonalized by the time-frequency shifts of a suitable function. There are numerous papers building on this assumption, e.g. [8, 9, 24, 35]. We show that this assumption cannot withstand mathematical scrutiny.

We prove that if the underlying Gabor system is a frame for \( L^2(\mathbb{R}^d) \), but not a basis, then the corresponding channel matrix \( H \) cannot be diagonal, unless the operator \( \sigma^{KN} \) is identically zero. We further prove that for a non-zero pseudodifferential operator with a bandlimited symbol and a Gaussian window, the channel matrix cannot vanish identically, quite independently of the spanning properties of the Gabor system.

The paper is organized as follows: In Section 2, we summarize the mathematical preliminaries. In Section 3 we present the reconstruction formula for the symbol of the pseudodifferential operator, and in Section 4 we present the proposed uniqueness results.

## 2 Preliminaries

We collect some concepts and definitions from time-frequency analysis. The precise details and proofs can be found in [11] or in [12].

The Fourier transform of a function \( f \in L^2(\mathbb{R}^d) \) is defined as

\[
\hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} dx,
\]

for \( \xi \in \mathbb{R}^d \).

The two fundamental operators in time-frequency analysis are the translation operators \( T_x \) and the modulation operators \( M_\xi \) defined by

\[
T_x f(t) = f(t-x) \quad \text{and} \quad M_\xi f(t) = e^{2\pi i \xi \cdot t} f(t),
\]

for \( t, x, \xi \in \mathbb{R}^d \).
Their compositions are the time-frequency shift operators \( \pi \) defined as

\[
\pi(z)f(t) = M_{\xi} T_x f(t) = e^{2\pi i \xi \cdot t} f(t-x),
\]

for \( z = (x, \xi) \in \mathbb{R}^{2d} \).

A set of time-frequency shifts of a non-zero window function \( g \in L^2(\mathbb{R}^d) \) with respect to a lattice \( \Lambda = a \mathbb{Z}^d \times b \mathbb{Z}^d \subseteq \mathbb{R}^{2d}, a, b > 0, \)

\[
\mathcal{G}(g, \Lambda) = \{ \pi(\lambda)g : \lambda \in \Lambda \}
\]
is called a Gabor system. If there exist constants \( A, B > 0 \) such that for all \( f \in L^2(\mathbb{R}^d) \)

\[
A \| f \|_2^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g \rangle|^2 \leq B \| f \|_2^2,
\]

then the set \( \mathcal{G}(g, \Lambda) \) is called a Gabor frame with frame bounds \( A \) and \( B \).

The short-time Fourier transform (STFT) of a function or distribution \( f \) with respect to a non-zero window \( g \) is defined as

\[
V_g f(x, \xi) = \int_{\mathbb{R}^d} f(t)g(t-x)e^{-2\pi i \xi \cdot t} dt = \langle f, M_{\xi} T_x g \rangle = \langle f, \pi(z)g \rangle,
\]

for \( z = (x, \xi) \in \mathbb{R}^{2d} \).

The Rihaczek distribution of two functions \( f, g \in L^2(\mathbb{R}^d) \) is defined as

\[
R(f, g)(x, \xi) = f(x)\overline{g(\xi)}e^{-2\pi ix \cdot \xi},
\]

for \( x, \xi \in \mathbb{R}^d \). The Rihaczek distribution and the short-time Fourier transform are related in the following way:

\[
\hat{R}(f, g) = UF_g f,
\]

where \( UF(\xi, x) = F(-x, \xi) \) and \( x, \xi \in \mathbb{R}^d \).

Let \( \sigma \) be a (measurable) function or a tempered distribution on \( \mathbb{R}^{2d} \). The bilinear form

\[
\langle \sigma^{KN} f, g \rangle = \langle \sigma, R(g, f) \rangle,
\]

where \( \sigma^{KN} \) is a pseudodifferential operator in the Kohn-Nirenberg calculus with Kohn-Nirenberg symbol \( \sigma \). If \( \hat{\sigma} \) is a locally integrable function, then the Kohn-Nirenberg transform can also be written as

\[
\sigma^{KN} f(x) = \int_{\mathbb{R}^{2d}} \hat{\sigma}(\eta, u)M_{\eta} T_{-u} f(x) du d\eta.
\]
To describe general classes of pseudodifferential operators with bandlimited symbols we use the terminology of operator Paley-Wiener spaces as introduced by Pfander et al. [29, 31] and define

\[ \text{OPW}^p(Q) = \{ H : H : S(\mathbb{R}^d) \rightarrow S'(\mathbb{R}^d), \text{supp} \hat{\sigma} \subseteq Q, \sigma \in L^p(\mathbb{R}^{2d}) \}, \]  

for \( 1 \leq p \leq \infty \) and a compact set \( Q \subseteq \mathbb{R}^d \).

### 3 Reconstruction formula

We first deal with the question if and how the symbol of a pseudodifferential operator \( \sigma^{KN} \) can be reconstructed from the diagonal of the channel matrix.

Let \( \Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d \) be a lattice in \( \mathbb{R}^{2d} \), \( g \in L^2(\mathbb{R}^d) \) a non-zero window function, and \( \mathcal{G}(g, \Lambda) \) be the corresponding Gabor system. The matrix \( H \) of a pseudodifferential operator \( \sigma^{KN} \) with respect to the Gabor system \( \mathcal{G}(g, \Lambda) \) is defined as follows:

\[ H_{\lambda,\mu} = \langle \sigma^{KN} \pi(\mu)g, \pi(\lambda)g \rangle, \]

for \( \lambda, \mu \in \Lambda \).

When \( \sigma^{KN} \) describes a wireless channel, the matrix \( H \) describes the action of a wireless channel on certain transmit pulses and is therefore called the channel matrix.

We remark that for the solution of the linear equation (1) \( H \) must be invertible. In wireless communications it is customary to assume that \( \mathcal{G}(g, \Lambda) \) is a Riesz basis for the generated subspace [21, 24, 25, 39].

We first derive an alternative expression for the diagonal entries of \( H \) in terms of the Rihaczek distribution of \( g \). Let \( \mathcal{F}L^1(\mathbb{R}^{2d}) \) denote the Fourier algebra on \( \mathbb{R}^{2d} \) consisting of all functions on \( \mathbb{R}^{2d} \) with integrable Fourier transform.

We have the following well-known lemma.

**Lemma 1** Let \( \sigma^{KN} \in \text{OPW}^p(Q) \), for \( 1 \leq p < \infty \), \( Q \subseteq \mathbb{R}^d \) a compact set and \( g \in L^1(\mathbb{R}^d) \cap \mathcal{F}L^1(\mathbb{R}^{2d}) \). The diagonal entries of \( H \) can be written as follows

\[ H_{\lambda,\lambda} = \langle \sigma^{KN} \pi(\lambda)g, \pi(\lambda)g \rangle = \sigma \ast \mathcal{R}(g, g)^*(\lambda), \]

where \( \lambda \in \Lambda \) and \( f^*(x) = \bar{f}(-x) \).

**Proof** Under the given assumptions, \( \sigma \) is infinitely differentiable and \( D^\alpha \sigma \) is bounded for all multi-indices \( \alpha \). The standard theory of pseudodifferential operators implies that \( \sigma^{KN} \) is bounded on \( L^2(\mathbb{R}^d) \) [11, 19]. Consequently the mapping \( \lambda \in \mathbb{R}^{2d} \rightarrow \langle \sigma^{KN} \pi(\lambda)g, \pi(\lambda)g \rangle \) is continuous, and the channel matrix is well-defined.

Using the definition of the Rihaczek distribution (4), we get

\[ \| \mathcal{R}(g, g)^* \|_1 = \| g \ast \hat{g} \|_1 = \| g \|_1 \cdot \| \hat{g} \|_1 < \infty. \]

Since \( \sigma \in L^p(\mathbb{R}^{2d}) \), the convolution \( \sigma \ast \mathcal{R}(g, g)^* \) is well-defined in the \( L^p \)-sense.

From the intertwining property of the Rihaczek distribution [14, Lemma 4.2], we have

\[ R(\pi(\lambda)g, \pi(\lambda)g)(z) = \mathcal{R}(g, g)(z - \lambda). \]
Combining the definition of the Kohn-Nirenberg transform (6) and (9), we obtain

\[
\langle \sigma^{KN} \pi(\lambda) g, \pi(\lambda) g \rangle = \langle \sigma, R(\pi(\lambda) g, \pi(\lambda) g) \rangle
\]

\[
= \int_{\mathbb{R}^{2d}} \sigma(z) R(\pi(\lambda) g, \pi(\lambda) g)(z) dz
\]

\[
= \int_{\mathbb{R}^{2d}} \sigma(z) R(g, g)(z - \lambda) dz
\]

\[
= \int_{\mathbb{R}^{2d}} \sigma(z) R(g, g)^*(\lambda - z) dz
\]

\[
= \sigma * R(g, g)^*(\lambda).
\]  

(10)

In general, (10) is valid for almost every \( \lambda \in \mathbb{R}^{2d} \). Since \( \text{supp} F(\sigma * R(g, g)^*) = \text{supp} \hat{\sigma} \cdot \text{supp} \hat{R(g, g)^*} \subseteq \text{supp} \hat{\sigma} \), is compact, \( \sigma * R(g, g)^* \) is an analytic function, and therefore (10) is valid for every \( \lambda \in \mathbb{R}^{2d} \).

By combining the observation of Lemma 1 with the classical sampling theorem for band-limited functions, we obtain a reconstruction formula for the symbol of a pseudodifferential operator from the diagonal of the channel matrix. In the formulation of the multivariate version of the Shannon-sampling theorem, we need the “sinc”-function adapted to a lattice \( \Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d \subseteq \mathbb{R}^{2d} \), namely

\[
sinc_{a,b}(x) = \prod_{j=1}^{d} \frac{\sin \pi a x_j}{\pi a x_j} \prod_{j=d+1}^{2d} \frac{\sin \pi b x_j}{\pi b x_j}
\]

Then every function \( \sigma \in L^2(\mathbb{R}^{2d}) \) with \( \text{supp} \hat{\sigma} \subseteq Q = [-\frac{1}{2a}, \frac{1}{2a}]^d \times [-\frac{1}{2b}, \frac{1}{2b}]^d \) possesses the cardinal series expansion

\[
\sigma = \sum_{\lambda \in \Lambda} \sigma(\lambda) T_\lambda \text{sinc}_{a,b}
\]

with convergence in \( L^2 \) and uniformly. For the general theory of Shannon sampling we refer to [3, 26, 27].

The first reconstruction formula is stated for symbols \( \sigma \) in \( L^p(\mathbb{R}^{2d}), 1 \leq p < \infty \).

**Theorem 2** Let \( \sigma^{KN} \in \text{OPW}^p(Q) \), for \( 1 \leq p < \infty \), \( Q = [-\frac{1}{2a}, \frac{1}{2a}]^d \times [-\frac{1}{2b}, \frac{1}{2b}]^d \) and \( g \in L^1(\mathbb{R}^d) \cap \text{FL}^1(\mathbb{R}^d) \). Choose \( \varphi \in C^\infty_c(\mathbb{R}^{2d}) \) such that \( \varphi = 1 \) on \( Q \), define \( K = \mathcal{F}^{-1} \left( \frac{\varphi}{\mathcal{U}V g} \right) \) and assume that \( \mathcal{U}V g \) does not vanish on \( \text{supp} \varphi \). Then the symbol \( \sigma \) can be reconstructed from the diagonal entries \( H_{\lambda,\lambda} = \langle \sigma^{KN} \pi(\lambda) g, \pi(\lambda) g \rangle \) of the channel matrix via the modified cardinal series

\[
\sigma = \frac{1}{(ab)^d} \sum_{\lambda \in \Lambda} H_{\lambda,\lambda} T_\lambda (\text{sinc}_{a,b} * K).
\]  

(11)
The sum converges absolutely, uniformly and in $L^p(\mathbb{R}^d)$. If $p = 1$, the reconstruction formula is still valid, but the series converges only in $L^q(\mathbb{R}^d)$ for $q > 1$, but not in $L^1(\mathbb{R}^d)$.

**Proof** We apply the multivariate version of the classical Shannon-Whittaker-Kotelnikov sampling theorem with the lattice $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$ to the bandlimited function $\sigma \ast R(g, g) \ast$.

We recall from Lemma 1 that $H_{\lambda,\lambda} = \sigma \ast R(g, g) \ast(\lambda)$ and write

$$\sigma \ast R(g, g) \ast = \frac{1}{(ab)^d} \sum_{\lambda \in \Lambda} H_{\lambda,\lambda} T_\lambda \text{sinc}_{a,b}. \quad (12)$$

According to the $L^p$-theory of the cardinal series [1, 38] the sum converges absolutely, uniformly, and in $L^p(\mathbb{R}^d)$ for $1 < p < \infty$. This sampling expansion holds pointwise also for $p = 1$, but the convergence is then only in $L^q(\mathbb{R}^d)$ for $q > 1$.

Since $R(g, g) \ast \in L^1(\mathbb{R}^d)$, formula (5) implies that $\hat{U}V_{gg} \in \mathcal{F}L^1(\mathbb{R}^d)$. Since $\hat{U}V_{gg} \neq 0$ on supp $\varphi$, we apply the Wiener-Lévy theorem, see [32, Theorem 1.3.1], to conclude that there exists a function $\psi \in \mathcal{F}L^1(\mathbb{R}^d)$ such that $\psi = \frac{1}{\hat{U}V_{gg}}$ on supp $\varphi$. Since $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, $\varphi$ is also in $\mathcal{F}L^1(\mathbb{R}^d)$ and we have $\varphi \psi = \frac{\varphi}{\hat{U}V_{gg}} \in \mathcal{F}L^1(\mathbb{R}^d)$. Thus we conclude that $K = \mathcal{F}^{-1}\left(\frac{\varphi}{\hat{U}V_{gg}}\right) \in L^1(\mathbb{R}^d)$.

Now we claim that

$$\sigma \ast R(g, g) \ast \ast K = \sigma. \quad (13)$$

To prove this, we take the Fourier transform of (13) and obtain

$$\hat{\sigma} \cdot R(g, g) \ast \cdot \hat{K} = \hat{\sigma} \cdot \hat{U}V_{gg} \cdot \hat{U}V_{gg}^{-1} \cdot \varphi = \hat{\sigma},$$

because $\varphi = 1$ on supp $\hat{\sigma}$.

Finally we combine (12) and (13) to compute

$$\sigma = \sigma \ast R(g, g) \ast \ast K = \frac{1}{(ab)^d} \sum_{\lambda \in \Lambda} H_{\lambda,\lambda} T_\lambda \text{sinc}_{a,b} \ast K. \quad (14)$$

Since we convolve $\sigma \ast R(g, g) \ast \in L^p(\mathbb{R}^d)$ with $K \in L^1(\mathbb{R}^d)$, the series in (14) inherits the convergence properties from (12).

Similar reconstruction formulas are valid when one or several side-diagonals of the channel matrix $H$ are known [28, Chap.4].

In the case $p = 2$ we can weaken the assumptions on $g$ considerably to $g \in L^2(\mathbb{R}^d)$. This case is important because it treats the (unjustified) assumption of the engineering community that wireless channels are Hilbert-Schmidt operators [22, 30]. In addition, it covers the rectangular window $g = \chi_{[a, b]^d}$ corresponding to OFDM without pulse-shaping [6, 10, 41].
Proposition 1  With the notation of Theorem 2 assume that $\sigma \in \text{OPW}^2(Q)$ and $g \in L^2(\mathbb{R}^d)$. Then $\sigma$ can be reconstructed from $(H_{\lambda,\lambda})_{\lambda \in \Lambda}$ by

$$\sigma = \frac{1}{(ab)^d} \sum_{\lambda \in \Lambda} H_{\lambda,\lambda} T_{\lambda}(\text{sinc}_{a,b} \ast K).$$  \hspace{1cm} (15)

with convergence in $L^2(\mathbb{R}^{2d})$ and uniform convergence.

Proof  The proof requires only a minor modification. Since $\hat{R}(g,g)^\ast = U \hat{V} g g$ is bounded, the product $\hat{\sigma} \cdot \hat{R}(g,g)^\ast$ is in $L^2(\mathbb{R}^{2d})$ with support in $Q$. Thus $\sigma \ast R(g,g)^\ast \in L^2(\mathbb{R}^{2d})$ is bandlimited and the sampling reconstruction (12) holds with uniform convergence and convergence in $L^2(\mathbb{R}^{2d})$.

Finally, since $U \hat{V} g g$ does not vanish on $\text{supp} \varphi$ by assumption, the multiplier $\hat{K} = \varphi \cdot \frac{U \hat{V} g g}{\text{supp} \varphi}$ is bounded, and therefore the operator $F \mapsto F \ast K$ is bounded on $L^2(\mathbb{R}^{2d})$. Consequently, the deconvolution formulas (13) and (14) are well-defined on $L^2(\mathbb{R}^{2d})$ and the reconstruction (15) follows.

Finally we formulate a distributional version of the reconstruction theorem. This version is not just for the sake of mathematical generalization, but is necessary for the accurate modelling of physical channels. For example, a single point scatterer with time delay $\tau$ and Doppler shift $\nu$ has the point measure $\delta(\tau,\nu)$ as its spreading function. A typical spreading function is usually written as a distributional part plus a random component [23]. By adapting the hypothesis of Theorem 2 we obtain the following statement.

Proposition 2  Let $\sigma \in S'((\mathbb{R}^d)^2)$, $\text{supp} \hat{\sigma} \subseteq Q = [-\frac{1}{2a} + \varepsilon, \frac{1}{2a} - \varepsilon]^d \times [-\frac{1}{2b} + \varepsilon, \frac{1}{2b} - \varepsilon]^d$ for some $\varepsilon > 0$ and $g \in S(\mathbb{R}^d)$. Choose $\varphi \in C^\infty_c((\mathbb{R}^d)^2)$ such that $\varphi = 1$ on $Q$, $\text{supp} \varphi \subseteq [-\frac{1}{2a}, \frac{1}{2a} ]^d \times [-\frac{1}{2b}, \frac{1}{2b} ]^d$, and assume that $U \hat{V} g g$ does not vanish on $\text{supp} \varphi$. Then the symbol $\sigma$ can be reconstructed from the diagonal entries $H_{\lambda,\lambda} = \langle \sigma^{KN} \pi(\lambda) g, \pi(\lambda) g \rangle$ of the channel matrix via the modified cardinal series

$$\sigma = \frac{1}{(ab)^d} \sum_{\lambda \in \Lambda} H_{\lambda,\lambda} T_{\lambda} F^{-1} \left( \frac{\varphi}{U \hat{V} g g} \right),$$  \hspace{1cm} (16)

with distributional convergence.

Proof  If $g \in S(\mathbb{R}^d)$, then $R(g,g) \in S((\mathbb{R}^d)^2)$ and thus $\sigma \ast R(g,g)^\ast \in S'((\mathbb{R}^d)^2)$ with supp $F(\sigma \ast R(g,g)^\ast) \subseteq Q$. The distributional version of the sampling theorem [4] now yields that

$$\sigma \ast R(g,g)^\ast = \frac{1}{(ab)^d} \sum_{\lambda \in \Lambda} (\sigma \ast R(g,g)^\ast)(\lambda) T_{\lambda} F^{-1} \varphi,$$

with distributional convergence. Since $\sigma \ast R(g,g)^\ast$ is an entire function of at most polynomial growth on $\mathbb{R}^{2d}$ (by the theorem of Paley-Wiener [33]) the pointwise
evaluations are well-defined. Likewise, since $\sigma^{KN}$ is continuous from $S(\mathbb{R}^{d})$ to $S'(\mathbb{R}^{d})$, the mapping $\lambda \mapsto \langle \sigma^{KN} \pi(\lambda)g, \pi(\lambda)g \rangle$ is continuous, therefore, as in Lemma 1, $H_{\lambda,\lambda} = (\sigma * R(g,g)^*)(\lambda)$ for $\lambda \in \Lambda$.

To conclude, we observe that $\varphi \cdot \mathcal{U}V_{g}^{-1}$ is in $S(\mathbb{R}^{2d})$ and thus also $K = \mathcal{F}^{-1} \left( \varphi \cdot \mathcal{U}V_{g}^{-1} \right) \in S(\mathbb{R}^{2d})$. Consequently, the deconvolution formulas (13) and (14) make sense in $S'(\mathbb{R}^{2d})$, and the reconstruction formula is proved.

We conclude this section with a few remarks on the applicability of the reconstruction theorems.

(i) Proposition 2 may be relevant for the numerical implementation of the reconstruction formula. By assuming a slightly smaller spectrum $Q_{\epsilon}$, the expanding kernel $\mathcal{F}^{-1} \left( \varphi \cdot \mathcal{U}V_{g}^{-1} \right)$ is in $S(\mathbb{R}^{2d})$ and decays rapidly. If $\sigma \in L^{p}(\mathbb{R}^{2d})$ instead of $S'(\mathbb{R}^{d})$, then the expansion is localized and converges rapidly. This implies that the symbol $\sigma$ can be approximated well on a rectangle $K = [L_{1}, L_{2}]^{d} \times [M_{1}, M_{2}]^{d}$ from the diagonal entries $H_{\lambda,\lambda}$ taken in a small neighborhood of $K$.

(ii) For concrete wireless channels the support of the spreading function is an extremely small rectangle of area much smaller than one. Such channels are called underspread in the engineering literature. As explained in [17, Chap. 1], realistic wireless communications scenarios yield a size for $\text{supp} \hat{\sigma}$ of order $10^{-4} - 10^{-3}$. Therefore the reconstruction of the symbol can be carried out with a rather coarse lattice.

(iii) In the engineering practice the diagonal entries are estimated only on a sublattice $\Lambda_{p} \subseteq \Lambda$ by means of pilot symbols, see [17, Chap. 5] and [6, 41]. In this case one applies Theorem 2 to a sublattice of the full lattice.

4 Uniqueness results

The reconstruction results of the previous section imply that a pseudodifferential operator with a bandlimited symbol is uniquely determined by the diagonal of the channel matrix. In this section, we prove further uniqueness results that illustrate the relation between a pseudodifferential operator and the corresponding channel matrix under various assumptions on the Gabor system and the symbol. For notational simplicity, we now work in dimension $d = 1$.

Let $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ be a lattice, let $g \in L^{2}(\mathbb{R})$ and $\mathcal{G}(g, \Lambda)$ the corresponding Gabor system in $L^{2}(\mathbb{R})$. We assume that

$$H_{\lambda,\mu} = \langle \sigma^{KN} \pi(\mu)g, \pi(\lambda)g \rangle = 0,$$

for all $\lambda, \mu \in \Lambda$.

Of course, if $\mathcal{G}(g, \Lambda)$ spans $L^{2}(\mathbb{R})$, then obviously $\sigma^{KN} = 0$. Under the basic assumptions of wireless communications (bandlimited symbol and Gabor Riesz sequence) the conclusion is not so obvious.
Before we state the next theorem, we rewrite the general entries of the channel matrix. We write $\lambda = (\lambda_1, \lambda_2), \mu = (\mu_1, \mu_2) \in \mathbb{R}^2$. From the definition of $\sigma^{KN}$, see (6), we have

$$\langle \sigma^{KN} \pi(\mu)g, \pi(\lambda)g \rangle = \langle \hat{\sigma}, UV_{\pi(\mu)g} \pi(\lambda)g \rangle.$$ 

Using the covariance property of the STFT (e.g., [12, Ch. 3]), we compute

$$V_{\pi(\mu)g} \pi(\lambda)g(-x, \xi) = e^{-2\pi i (\xi - \lambda_2)\lambda_1} e^{2\pi i \mu_2 (-x - \lambda_1)} T_{\lambda - \mu} V_g g(-x, \xi)$$

Writing $G = UV_g g$, we conclude that (17) is equivalent to

$$\langle \hat{\sigma}, M_{(\mu_2, -\lambda_1)} T_{\lambda - \mu} G \rangle = 0,$$

for all $\lambda, \mu \in \Lambda$. Here $\mu_2 \in b\mathbb{Z}$ and $\lambda_1 \in a\mathbb{Z}$, so the modulations are taken with respect to the lattice $\Lambda' = b\mathbb{Z} \times a\mathbb{Z}$. Combining (17) and (18), we obtain

$$\langle \hat{\sigma}, M_{\mu} T_{\lambda} G \rangle = 0,$$

for all $\lambda \in \Lambda, \mu \in \Lambda' = b\mathbb{Z} \times a\mathbb{Z}$.

The first uniqueness theorem treats the case of Gaussian Gabor systems.

**Theorem 3** Let $\varphi(x) = e^{-\pi x^2}$ be the Gaussian function and $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ an arbitrary lattice. Assume that the symbol $\sigma \in S'(\mathbb{R}^2)$ possesses a compactly supported Fourier transform in some $L^p(\mathbb{R}^{2d}), 1 \leq p \leq \infty$. If

$$H_{\lambda, \mu} = \langle \sigma^{KN} \pi(\mu)\varphi, \pi(\lambda)\varphi \rangle = 0,$$

for all $\lambda, \mu \in \Lambda$, then $\sigma^{KN}$ is identically zero.

**Proof** The proof is based on the fact that the Gaussian $g$ is a (strictly) totally positive function. This means that for two arbitrary sequences of real numbers $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$ the matrix

$$\left( g(x_j - y_k), j, k = 1, \ldots, n \right)$$

has a strictly positive determinant and is thus invertible. See [36, 37] for the fundamental properties of totally positive functions.

It is profitable to use the Weyl calculus of pseudodifferential operators, which is formulated by means of the (cross-) Wigner distribution of $f, g \in L^2(\mathbb{R})$

$$W(f, g)(x, \xi) = \int_{\mathbb{R}} f(x + \frac{t}{2}) g(x - \frac{t}{2}) e^{-2\pi i \xi \cdot t} dt.$$ 

The Wigner distribution satisfies the following covariance property ([11] or [12, Prop. 4.3.2c]):

$$W(\pi(\mu)\pi(\lambda)f, \pi(\lambda)g) = e^{-\pi i \mu_1 \mu_2} M_{\mu} T_{\lambda} W(f, g).$$
where \( \tilde{\mu} = (\mu_2, -\mu_1) \) for \( \mu = (\mu_1, \mu_2) \in \mathbb{R}^2 \). The Wigner distribution of the Gaussian \( \varphi(t) = e^{-\pi t^2} \) is the Gaussian

\[
W(\varphi, \varphi)(x, \xi) = \sqrt{2} e^{-\pi (x^2 + \xi^2)/2}
\]

with Fourier transform \( \Phi(\xi, x) = \hat{W}(\varphi, \varphi)(\xi, x) = 2^{-\frac{3}{2}} e^{-2\pi (\xi^2 + x^2)}. \) Now assume that \( \tilde{\sigma} \in L^1(\mathbb{R}^2) \) and set \( \hat{\tau}(\xi, x) = e^{\pi i \xi \xi} \hat{\sigma}(\xi, x) \), then we have

\[
\langle \sigma \rho N, \pi(\lambda)g, \pi(\mu)\pi(\lambda)g \rangle = \langle \tau, W(\pi(\mu)\pi(\lambda)g, \pi(\lambda)g) \rangle \tag{21}
\]

(See [11] or [12, Chap. 14.3] for the transition between the Kohn-Nirenberg calculus and the Weyl calculus.) Consequently, \( \langle \sigma \rho N, \pi(\lambda)\varphi, \pi(\nu)\varphi \rangle = 0 \) for all \( \lambda, \nu \in \Lambda \) holds, if and only if

\[
\langle \tau, W(\pi(\mu)\pi(\lambda)\varphi, \pi(\lambda)\varphi) \rangle = e^{\pi i \mu \lambda} \langle \tau, M_{-\lambda} T_{\mu} \varphi \rangle = e^{\pi i \mu \lambda} e^{-2\pi i \mu \lambda} \langle \hat{\tau} \cdot T_{\mu} \tilde{\Phi} \rangle = 0 \tag{22}
\]

for all \( \lambda, \mu \in \Lambda \), or equivalently for all \( \lambda \in \Lambda \) and \( \tilde{\mu} \in \Lambda' \). By the Poisson summation formula with the dual lattice \( \Lambda' = a^{-1} \mathbb{Z}^d \times b^{-1} \mathbb{Z}^d \) identity (22) is equivalent to

\[
\sum_{\nu \in \Lambda' \cap a^{-1} \mathbb{Z}^d} T_{\nu}(\hat{\tau} T_{\mu} \tilde{\Phi})(\xi, x) = \sum_{j, k = -\infty}^{\infty} \hat{\tau} \left( \xi + \frac{j}{a}, x + \frac{k}{b} \right) e^{-\pi (\xi + \frac{j}{a} + b l_2)^2/2 + (x + \frac{k}{b} + a l_1)^2/2} = 0
\]

for almost all \( x, \xi \in \mathbb{R}^d \) and for all \( \tilde{\mu} = (b l_2, -a l_1) \in \Lambda' \). Since the last expression is periodic and \( \hat{\tau} \) has compact support, the last sum is actually finite. Thus for sufficiently large \( L \in \mathbb{N} \) we obtain the condition

\[
\sum_{j, k = -L}^{L} \hat{\tau} \left( \xi + \frac{j}{a}, x + \frac{k}{b} \right) e^{-\pi (\xi + \frac{j}{a} + b l_2)^2/2 + (x + \frac{k}{b} + a l_1)^2/2} = 0
\]

for almost all \( x, \xi \in [0, 1/a] \times [0, 1/b] \) and all \( \tilde{\mu} = (b l_2, -a l_1) \in \Lambda' \).

Since Gaussian functions are totally positive, the (square) matrix with entries

\[
e^{-\pi (\xi + \frac{j}{a} + b l_2)^2/2}, |j| \leq L, l_1 = 0, \ldots, 2L \text{ is invertible, and likewise the matrix }\]

\[
e^{-\pi (x + \frac{k}{b} + a l_1)^2/2}, |k| \leq L, l_2 = 0, \ldots, 2L \text{ is invertible. From this we conclude that }\]

\[
\tau(\xi + \frac{j}{a}, x + \frac{k}{b}) = 0 \text{ for all } j, k \text{ and almost all } x, \xi. \text{ Consequently } \hat{\tau} = 0 \text{ almost everywhere and thus } \hat{\sigma} = 0. \]

The above proof requires that \( \hat{\sigma} \) is defined almost everywhere. With a different and more explicit proof [28] Theorem 3 remains valid under the weaker assumption that \( \sigma \in S'(\mathbb{R}^{2d}) \) with \( \text{supp } \hat{\sigma} \) compact.

Finally, we investigate an important assumption made in wireless communications. To facilitate the inversion of the channel matrix in (1), it is commonly assumed
that the channel matrix is a diagonal matrix \([8, 9, 24, 35]\). Clearly, if the Gabor system \(G(g, \Lambda)\) is a (Riesz) basis for \(L^2(\mathbb{R}^d)\), then there is a bijection between operators and channel matrices, and thus there exist diagonal channel matrices.

However, if the Gabor system is a frame, then this assumption is never satisfied.

**Theorem 4** Let \(\sigma^{KN}\) be a bounded operator on \(L^2(\mathbb{R}^d)\), \(\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d\) be a lattice with \(ab < 1\) and \(g \in L^2(\mathbb{R}^d)\) such that \(G(g, \Lambda)\) is a frame for \(L^2(\mathbb{R}^d)\). If
\[
\langle \sigma^{KN}\pi(\mu)g, \pi(\lambda)g \rangle = d_\mu \delta_{\lambda\mu},
\]
for all \(\lambda, \mu \in \Lambda\), then \(\sigma^{KN}\) is identically zero.

**Proof** It follows from (23) that \(\sigma^{KN}\pi(\mu)g\) is orthogonal to the linear span of the set \(\{\pi(\lambda)g : \lambda \in \Lambda, \lambda \neq \mu\}\) for all \(\forall \mu \in \Lambda\).

Since \(\{\pi(\lambda)g : \lambda \in \Lambda\}\) is a frame for \(L^2(\mathbb{R}^d)\), according to \([7, \text{Lemma IX}]\), there are two possibilities for the set \(\{\pi(\lambda)g : \lambda \in \Lambda, \lambda \neq \mu\}\). Either

- \(\{\pi(\lambda)g : \lambda \in \Lambda, \lambda \neq \mu\}\) is incomplete in \(L^2(\mathbb{R}^d)\), or
- \(\{\pi(\lambda)g : \lambda \in \Lambda, \lambda \neq \mu\}\) is again a frame for \(L^2(\mathbb{R}^d)\).

If the set \(\{\pi(\lambda)g : \lambda \in \Lambda, \lambda \neq \mu\}\) is incomplete in \(L^2(\mathbb{R}^d)\), then the set \(\{\pi(\lambda)g : \lambda \in \Lambda\}\) has to be a Riesz basis for \(L^2(\mathbb{R}^d)\). Otherwise it would not be possible to have an incomplete set after removing one single element. It follows from the density and duality theory of frames \([5, 12, 20]\) that in this case \(ab = 1\). This contradicts the assumption \(ab < 1\).

Therefore the set \(\{\pi(\lambda)g : \lambda \in \Lambda, \lambda \neq \mu\}\) is again a frame for \(L^2(\mathbb{R}^d)\). This implies that the orthogonal complement of \(\{\pi(\lambda)g : \lambda \in \Lambda, \lambda \neq \mu\}\) has to be equal to the set \(\{0\}\). That is,
\[
\sigma^{KN}\pi(\mu)g \perp \text{span}\{\pi(\lambda)g : \lambda \in \Lambda, \lambda \neq \mu\},
\]
for all \(\mu \in \Lambda\), is only possible if \(\sigma^{KN}\pi(\mu)g = 0, \forall \mu \in \Lambda\) and therefore \(\sigma^{KN} \equiv 0\).

It seems that some fundamental algorithms of wireless communications in nonstationary environments are based on the incorrect assumption that the channel matrix is diagonal. Nevertheless the intuition of the communication engineers is perfectly correct and can be supported by rigorous mathematical results. Indeed, if the symbol \(\sigma\) is smooth and the pulse \(g\) of the Gabor system possesses a minimal amount of time-frequency concentration, then the channel matrix decays rapidly off its diagonal \([13, 15]\). Hence, from a numerical point of view, the channel matrix can be approximated well by a diagonal matrix. This idea was used for improved equalization methods in \([16, 42]\).

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