The Existence of Embedded $G$-Invariant Minimal Hypersurface

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Abstract. For a compact connected Lie group $G$ acting as isometries on a compact orientable Riemannian manifold $M^{n+1}$, we prove the existence of a nontrivial embedded $G$-invariant minimal hypersurface, that is smooth outside a set of Hausdorff dimension at most $n - 7$.

CONTENTS

0. Introduction 2
1. Terminologies 3
2. Existence of $G$-invariant Stationary Varifolds 6
3. Existence of $G$-almost minimizing varifolds in $G$-annuli 6
4. The existence of $G$-invariant replacements 11
5. Regularity of $G$-Varifolds with replacements in $G$-annuli 16
6. Cohomogeneity 1 case 18
7. Remarks 18
Acknowledgements 19
Appendices 19
Appendix A. Appendix 19
References 21
Based on the continuous version of min-max construction in [6], we prove the following theorem,

**Theorem 0.1.** Let $G$ be a compact connected Lie group acting as isometries on an orientable compact connected Riemannian manifold $M^{n+1}$ of dimension $(n+1)$ without boundary. If $n \geq 2$ and the action of $G$ is not transitive, then there exists an embedded minimal hypersurface $\Sigma^n \subset M^{n+1}$ that is invariant under the action of $G$. Moreover, $\Sigma^n$ has no boundary and is smooth outside a set $\text{Sing}\Sigma$ of Hausdorff dimension at most $n - 7$.

Invariant means for any $s \in \Sigma^n$, and all $g \in G$, we have $g.s \in \Sigma^n$. In other words, $\Sigma^n$ is a union of orbits. The statement of our Theorem 0.1 differs from Theorem 0.1 in [6] only in that our minimal hypersurface $\Sigma$ is invariant under $G$-actions. Our assumptions on the actions are very mild.

Daniel Ketover has developed in [11] an equivariant min-max for finite group actions on three-dimensional manifolds. This work is inspired by his approach, especially in the part of existence theory of invariant stationary and invariant almost minimizing varifolds. However, the regularity theory regarding $G$-invariant replacements are in a very different vein.

**0.1. Structure of the proof.** In this section, we will sketch the main ideas and structure of our proof.

First, we need to convert Theorem 0.1 into a transformation group flavored one. We follow the usual convention to define cohomogeneity as the codimension of principal orbits with respect to $M$.

**Lemma 0.2.** For a compact connected Lie group $G$ acting as on $M^{n+1}$, the action is non-transitive if and only if the cohomogeneity is non-zero.

**Proof.** It’s clear that cohomogeneity non-zero implies non-transitive. For the other direction, suppose the principal orbits are of the same dimension as $M^{n+1}$. Since orbits are a priori closed ([16]), each of them must be the entire $M$, i.e, the action is transitive. \qed

Thus, to prove Theorem 0.1 we only have to prove it for actions of cohomogeneity at least 1 by Lemma 0.2.

The cohomogeneity 1 case is significantly easier, and can be settled with an easy argument by using basic classifications as in [14]. We will deal with this in Section 6. The following arguments are for cohomogeneity at least 2.

The idea of the proof is as follows. The min-max construction in [6] can be broken down into four steps

Step 1. pulling-tight, aka, the existence of stationary varifolds,
Step 2. the existence of almost minimizing varifolds,
Step 3. the existence of smooth stable replacements for almost minimizing varifolds, Step 4. regularity of varifolds with sufficiently many smooth stable replacements.

We’ll modify each step accordingly.

First of all, our constructions are based on extracting time and again better-behaving subsequences of $G$-invariant sweepouts, so we have first to show the existence of such a thing. We will discuss this and fix our notations in Section 1.

In Section 2 we will prove that stationary with respect to $G$-invariant vector fields implies stationary in general by using an averaging construction of Lie groups. Then we can adapt the pulling-tight procedure with minimal effort, which corresponds to Step 1.

In Section 3 we will run a modified combinatorial argument as in Section 3 of [6] to produce varifolds that are almost-minimizing among equivariant deformations, which corresponds to Step 2.

In Section 4 we will use a modified argument of Section 4 of [6] to construct replacements with unknown regularity. To prove that replacements have the codimension 7 regularity, which is essential to regularity theory, we will prove that these replacements are actually minimizing on a small enough scale with respect to all deformations. This is nontrivial since by construction they’re only minimizing on a small scale with respect to $G$-invariant deformations. To this end, we use an argument first given by Lawson and Fleming in [10]. This is Step 3.

Now that we’ve proved the existence of stable replacements with codimension 7 regularity, we can simply reiterate and mimic the regularity theory developed in Section 5 of [6] to deduce the regularity of the $G$-invariant minimal hypersurface we’ve constructed. However, there are technical problems since we are no longer dealing with geodesic balls but tubes instead. Nevertheless, we will show that using the splitting of tangent cones proved in the appendix, the argument proceeds through. This is done in Section 5.

In the appendix, we will give a brief overview of basic Lie transformation group theory that we have used throughout the paper. For the moment, it’s enough to know that there is a set of equivariantly diffeomorphic orbits of the highest dimension called principal orbits. The union of all such principal orbits forms an open dense set. The cohomogeneity is the codimension of any such principal orbit in $M$. There are exceptional and singular orbits which are geometrically ”smaller” in terms of dimension.

Our paper is structured almost exactly the same as in [6]. To make comparisons easier, we intentionally try to phrase every theorem and definition as close as possible to [6] and will point out the difference explicitly.

1. Terminologies

This section corresponds to Section 0 and 1 of [6]. We will fix the terminologies and give the basic existence of a nontrivial family of $G$-invariant sweepouts.
1.1. Notations. First of all, following the convention of [6], we will fix what we mean by a generalized hypersurface.

Definition 1.1. (Definition 1.5 in [6]) A generalized hypersurface $\Gamma \subset U$ is an integral varifold whose support is of Hausdorff dimension at most $n$ so that $\Gamma$ is smooth outside a set $\text{Sing} \Gamma$ of Hausdorff dimension at most $n - 7$.

We will add stable (minimal) in front of generalized hypersurface to mean it’s stable (stationary, respectively) as a varifold.

Since we are constantly dealing with $G$-invariant objects in our paper, we will sometimes add $G$- in front of objects to indicate they are $G$-invariant. The exact definition of $G$-invariance is almost always clear from context. We will list some here.

- A $G$-varifold $V$ satisfies $g_*V = V$ for any $g \in G$.
- A $G$-vector field $X$ satisfies $g_*X = X$ for all $g \in G$.
- A $G$-isotopy $\Phi(t)$ satisfies $g^{-1} \circ \Phi(t) \circ g = \Phi(t)$ for all $t$ and $g \in G$.
- A $G$-set ($G$-neighborhood) is an (open) set which is a union of orbits.

We will adopt the same notations as in the paper [6]. However, we will sometimes add a subscript or superscript $G$ to signify $G$-invariance. We will summarize those as follows.

- $\pi_G$: the projection $\pi_G : M \mapsto M/G$ defined by $x \mapsto [x]$.
- $B^G_\rho(x), \overline{B}^G_\rho(x)$: open and closed tubes with radius $\rho$ around the orbit $G.x$.
- $X^G(M)$: the space of $G$-vector fields on $M$.
- $A^G(x, t, \tau)$: the open tube $B^G_\tau(x) \setminus \overline{B}^G_\tau(x)$.
- $\mathcal{AN}^G(x)$: the set $\{A^G(x, \tau, t)| 0 < \tau < t < r\}$.
- $d_G(U, V)$: for $G$-sets $U, V$ is defined as $d(\pi_G(U), \pi_G(V))$.
- $\text{diam}_G(U)$: the diameter of the projection $\pi_G(U)$ of $G$-set $U$ in the metric space $M/G$.

Note that all of these are well defined in that $M/G$ is also a complete Hausdorff metric space.

1.2. Basic definitions. In what follows, $M$ will denote a compact $(n + 1)$ dimensional smooth Riemannian manifold without boundary. Let $G$ be a compact connected Lie group acting smoothly as isometries on $M$, with bi-invariant Haar measure $\mu$ on $G$ normalized to $\mu(G) = 1$.

The notion of generalized smooth family is just adding $G$- to objects in [6], but we will need a weaker of sweepout.

Definition 1.2. A $G$-generalized smooth family is a $k$-parameter family of generalized hypersurfaces $\{\Gamma_t\}_{t \in [0, 1]^k}$ with the following properties

- (s0) $\mathcal{H}^n(\Gamma_t) < \infty$ for all $t \in [0, 1]^k$.
- (s1) For all $t \in [0, 1]^k$, $\Gamma_t$ is a $G$-invariant smooth hypersurface in $\Gamma_t \setminus P_t$, where $P_t$ consists of finitely many disjoint orbits;
THE EXISTENCE OF EMBEDDED G-IN Variant MINIMAL HYPERSURFACE

(s2) $H^n(\Gamma_t)$ is smooth in $t$ for $t \in (0,1)$ and continuous in $t$ for $t \in [0,1]$.
Moreover, $t \mapsto \Gamma_t$ is continuous if we use Hausdorff topology on subsets in $M$.

(s3) For any $U \subset M \setminus P_{t_0}$, there exists $\delta_0$ so that for $|t - t_0| < \delta_0$, $\Gamma_t \cap U$
the image of a smooth vector field $f_t\nu_{t_0}$ under the normal exponential map $\exp_{\Gamma_{t_0}}$ of $\Gamma_{t_0} \cap U$.

Moreover, a $G$-generalized family $\{\Gamma_t\}_{t \in [0,1]}$ is a $G$-sweepout of $M$ if there exists a
one-parameter family of $G$-invariant open sets $\{\Omega_t\}_{t \in [0,1]}$ satisfying

(sw1) $\Gamma_t \setminus \partial \Omega_t \subset P_t$ for $0 < t < 1$.
(sw2) $\Omega_0 = \emptyset, \Omega_1 = M$;
(sw3) $\text{Vol}(\Omega_t \setminus \Omega_s) + \text{Vol}(\Omega_s \setminus \Omega_t) \to 0$ as $t \to s$;

The only difference between our definition and Definition 0.2 in [6] is we define $P_t$
to be finite set consisting of orbits, instead of points. We adopt this notion because
passing through a critical point of an equivariant Morse function amounts adding
a handle bundle as in [16] instead of adding cells as in the usual Morse theory.
Thus in general we have to assume sweepouts start and end at orbits. However,
this change will not hinder our proof.

**Proposition 1.1.** If $f : M \to [0,1]$ is a $G$-equivariant Morse function in the sense
of [16]. Then $\{\Gamma_t = f^{-1}(t)\}_{t \in [0,1]}$ is a $G$-sweepout.

**Proof.** The proof is the same as proving level sets of Morse function form
a sweep-out. The only part that might need attention is to prove that there are
only finitely many orbits that might form non-smooth parts of critical submanifolds.
This can be deduced by using Lemma 4.1 in [17], the equivariant Morse lemma. □

By equivariant Morse theory as in [17], equivariant Morse functions are dense in
the space of smooth $G$-invariant functions, which comes in abundance by lifting
smooth functions on the quotient space. Thus, Proposition [1.1] is not a vacuous
statement. Moreover, by taking the gradient of those functions, we deduce the
existence of nontrivial $G$-vector fields.

For any one-parameter generalized family $\{\Gamma_t\}$, we define

$$F(\{\Gamma_t\}) = \max_{t \in [0,1]} H^n(\Gamma_t).$$

Without changing one word, the same proof of Proposition 0.5 in [6] carries over
to give,

**Proposition 1.2.** $F(\{\Gamma_t\}) \geq C(M)$ for any sweepout $\{\Gamma_t\}$, where $C(M)$ is a
positive constant depending only on $M$.

For a family $\Lambda$ of sweepouts, let

$$m_0(\Lambda) = \inf_{\{\Gamma_t\} \in \Lambda} F = \inf_{\{\Gamma_t\} \in \Lambda} \max_{t \in [0,1]} H^n(\Gamma_t).$$

By Proposition 1.2 $m_0(\Lambda) \geq C(M) > 0$. We call a sequence $\{\{\Gamma_t\}\} \subset \Lambda$ minimizing if

$$\lim_{k \to \infty} F(\{\Gamma_t\}^k) = m_0(\Lambda).$$
A sequence of hypersurfaces \( \{ \Gamma^k_t \} \) is called a min-max sequence of a family \( \Lambda \) if \( \{ \{ \Gamma_t \} \} \) is minimizing and \( \lim_k \mathcal{H}^n(\Gamma^k_t) = m_0(\Lambda) \).

We will only deal with families \( \Lambda \) closed under the following notion of homotopy.

**Definition 1.3.** We call two sweepouts \( \{ \Gamma^0_t \} \) and \( \{ \Gamma^1_t \} \) homotopic if for some two parameter \( G \)-generalized family \( \{ \Gamma_t \}_{t \in [0,1]^2} \) we have \( \Gamma_{(0,s)}^0 = \Gamma^0_s \) and \( \Gamma_{(1,s)}^1 = \Gamma^1_s \). A family \( \Lambda \) of \( G \)-sweepouts is said to be \( G \)-homotopically closed if \( \{ \Gamma_t \} \in \Lambda \), then any sweepout homotopic to \( \{ \Gamma_t \} \) is contained in \( \Lambda \).

The following smaller classes of \( G \)-homotopies will also be very useful.

**Definition 1.4.** Let \( X : [0,1] \to X(M) \) be a smooth map to the space of smooth vector fields on \( M \). Suppose \( F([0,1]) \subset X^G(M) \). Let \( \Psi_t(\cdot,\cdot) : [0,1] \times M \to M \) be the diffeomorphism corresponding to \( X(t) \). If \( \{ \Gamma_t \}_{t \in [0,1]} \) is a \( G \)-sweepout, then \( \{ \Psi_t(s,\Gamma_t) \}_{(s,t) \in [0,1]^2} \) is called a \( G \)-homotopy from \( \{ \Gamma_t \} \) to \( \{ \{ \Gamma_t \} \} \) generated by ambient isotopies.

And finally, we will give the definition of \( G \)-almost minimizing varifolds, which is essential to our regularity theory.

**Definition 1.5.** Fix \( \epsilon > 0 \) and a open \( G \)-set \( U \subset M \). Suppose \( \Omega \) is another open \( G \)-set. Then the boundary \( \partial \Omega \) of \( G \)-open set in \( M \) is \( \epsilon \)-\( G \)-almost minimizing \((\epsilon \text{-G-a.m.)}\) in \( U \) if there is no 1-parameter families of boundaries of open \( G \)-sets \( \Omega_t \), \( t \in [0,1] \), so that

\[
\begin{align*}
(a.1) & \quad (s1), (s2), (s3), (sw1), \text{ and } (sw3) \text{ of Definition 1.2 hold}, \\
(a.2) & \quad \Omega_0 = \Omega, \text{ and } \Omega_t \setminus U = \Omega \setminus U \text{ for every } t, \\
(a.3) & \quad \mathcal{H}^n(\partial \Omega_t) \leq \mathcal{H}^n(\partial \Omega) + \frac{\epsilon}{2} \text{ for all } t \in [0,1], \\
(a.4) & \quad \mathcal{H}^n(\partial \Omega_t) \leq \mathcal{H}^n(\partial \Omega) - \epsilon.
\end{align*}
\]

If there exists a sequence \( \epsilon_k \to 0 \) so that a collection \( \{ \partial \Omega^k \} \) of generalized hypersurfaces is \( \epsilon_k \)-\( G \)-a.m. in \( U \), then we say \( \{ \partial \Omega^k \} \) is \( G \)-almost minimizing in \( U \). Note that by definition, \( G \)-a.m. is a property that can be passed on into \( G \)-subsets. Thus, if \( V \subset U \) are both \( G \)-open sets, then an \( \epsilon \)-\( G \)-a.m. set in \( U \) is also \( \epsilon \)-\( G \)-a.m. in \( V \).

One major difference between almost minimizing in \([6]\) and \( G \)-almost minimizing is that for \( G \)-a.m. we’re only considering deformations under \( G \)-vector fields. In fact, this difference is significant and cannot be remedied easily, unlike the distinction between \( G \)-stationary and stationary in the next section.

Finally, we need the notion of replacement. The definition is the same as \([6]\), and we impose no invariant constraints.

**Definition 1.6.** (Definition 2.5 in \([6]\)) Let \( V \in V(M) \) be a stationary varifold and \( U \subset M \) be an open set. A stationary varifold \( V' \in V(M) \) is called a replacement for \( V \) in \( U \) if \( V' = V \) on \( M \setminus \overline{U} \), \( \|V'\|(M) = \|V\|(M) \) and \( V' \cdot V \) is a stable minimal generalized hypersurface.

### 2. Existence of \( G \)-invariant Stationary Varifolds

This section will be dedicated to proving the following proposition
**Proposition 2.1.** If \( \Lambda \) is a family of \( G \)-sweepouts closed under \( G \)-homotopies induced by ambient \( G \)-isotopies, then there exists a minimizing sequence \( \{ \Gamma^k_t \} \subset \Lambda \) so that if \( \{ \Gamma^k_t \} \) is a min-max sequence, then \( \Gamma^k_t \rightarrow V \) for some stationary \( G \)-varifold \( V \).

In \([6]\), the proof is referred to as Proposition 4.1 of \([5]\). Even though we only need to modify it minimally. However, first, we have to develop some basic facts about stationary properties under \( G \)-vector fields.

### 2.1. \( G \)-stationary implies stationary.

The idea of development in this subsection is inspired by \([11]\). By abuse of notation, we use \( \| \delta V \|_G (O) \) to denote the total first variation with respect to \( G \)-vector fields compactly supported on open set \( O \), i.e.,

\[
\| \delta V \|_G (O) = \sup \{ \delta V(\chi) | g, \chi \in G, \| \chi \| \leq 1, \text{spt}\chi \subset O \}.
\]

We will use \( \| \delta V \|_G \) to denote \( \| \delta V \|_G (M) \).

**Definition 2.1.** A \( G \)-varifold \( V \) is \( G \)-stationary if \( \| \delta V \|_G (M) = 0 \).

**Lemma 2.2.** For any \( G \)-varifold \( V \), and \( G \)-neighborhood \( O \), we have

\[
\| \delta V \|_G (O) = \| \delta V \| (O).
\]

**Proof.** It suffices to prove that for any vector field \( X \), with \( |X| \leq 1 \) supported in \( O \), there exists a \( G \)-vector field \( X_G \), with \( |X_G| \leq 1 \) so that,

\[
\delta V(X)(O) = \delta V(X_G)(O).
\]

Use \( \psi(t) \) to denote the diffeomorphisms generated by \( X \). Consider the modified diffeomorphism

\[
\psi_g(t) = g^{-1} \circ \psi(t) \circ g.
\]

Let \( X_g \) be the vector fields corresponding to \( \psi_g(t) \). Now define

\[
X_G(p) = \int_G X_g(p) d\mu(g),
\]

where the integral is carried out in \( T_pM \). By construction, \( X_G \) is supported in \( O \).

We have

\[
g_*X_G(p) = g_* \int_G X_h(g^{-1}p)d\mu(h)
\]

\[
= \int_G g_*X_h(g^{-1}p)d\mu(h)
\]

\[
= \int_G \frac{d}{dt} g \circ h^{-1} \psi(t) \circ h \circ g^{-1} d\mu(h)
\]

\[
= \int_G X_{h^{-1}}(p)d\mu(h)
\]

\[
= \int_G X_h(p)d\mu(h)
\]

\[
=X_G(p).
\]
Since $g$ and $g^{-1}$ are all isometries, we have $(g^{-1} \circ \psi(t) \circ g)_* V = g_*^{-1}(\psi(t)_*(g_* V)) = \psi(t)_* V$. By linearity of first variation, we can conclude that

$$
\delta V(X_G)(O) = \int_{G_n(O)} \text{div}_S X_G dV(x, S)
= \int_{G_n(O)} \text{div}_S \int_G X_g d\mu(g) dV(x, S)
= \int_G \int_{G_n(O)} \text{div}_S X_g dV(x, S) d\mu(g)
= \int_G \delta V(X_g)(O) d\mu(g)
= \int_G \delta V(X)(O) d\mu(g)
= \delta V(X)(O).
$$

by Fubini theorem. For the control on the norm, just note that

$$
|X_G|^2 = \langle X_G, \int_G X_h d\mu \rangle
= \int_G \langle X_G, X_h \rangle d\mu
\leq \int_G |X_G||X_h| d\mu(h)
= |X_G| \int |X_h| d\mu(h),
$$

so this yields

$$
|X_G| \leq \int_G |X_h| d\mu = 1.
$$

\[\square\]

**Corollary 1.** A $G$-stationary $G$-varifold $V$ is stationary.

**Proof.** By letting $O = M$ in Lemma 2.2 we deduce immediately the desired result. \[\square\]

2.2. **Proof of Proposition 2.1** Let $\{\Sigma^+_i\}^n$ be a minimizing sequence. We will deform it into another sequence $\{\Gamma^+_i\}^n$ using ambient $G$-isotopies so that any min-max subsequence of $\{\Gamma^+_i\}^n$ converges to a stationary varifold.

First, let’s consider the varifolds with mass bounded by $4m_0$, and call the collection $X$. Metrize it with weak-* topology by Riesz representation. Now, let $X^G$ be the subspace of $G$-varifolds in $X$. $X^G$ is closed by construction and thus a compact subset of $X$. Let $V^G_\infty = X^G \cap V_\infty$ denote the space of $G$-stationary varifolds, which is closed by construction. By our lemma, it’s a subset of the set of stationary varifolds $V_\infty$. Thus, the distance to $V^G_\infty$ is a well-defined continuous function on $X$ and thus $X^G$. Now, we can consider the annuli

$$
V_k = \{ V \in X^G | 2^{-k} \leq d(V, V^G_\infty) \leq 2^{-k+1} \}.
$$

The proof presented here is almost the same as the proof of Proposition 4.1 in [5], even without the need to change notation. In essence, we just replace $V_\infty$ with $V^G_\infty$. 
$G$-invariant stationary varifolds, and let all the vector fields used in construction to be $G$-invariant. The space of $G$-invariant manifolds is a closed subspace of the space of varifolds, which is a Banach space in our case. Thus, The basic properties like completeness, etc, descends into this closed subspace. For partition of unity, we note that it still holds in this subspace by Theorem II.2 in [9].

3. Existence of $G$-almost minimizing varifolds in $G$-annuli

In this section, we will prove the following proposition.

**Proposition 3.1.** Suppose $\Lambda$ is a family of $G$-sweepouts closed under $G$-homotopies. Then there exists a $G$-invariant function $r : M \to \mathbb{R}_+$ and a min-max sequence $\Gamma^k = \Gamma^k_t$ so that

1. $\{\Gamma^k\}$ is $G$-a.m. in every $A\mathcal{N}^G_{r(x)}(x), x \in M$;
2. $\lim_{k \to \infty} \Gamma^k \to V$ for some stationary $G$-varifold $V$.

This idea of proof is the same Section 3 of [6]. However, we will need to make some technical amendments.

### 3.1. $G$-almost minimizing varifolds.

Before coming to the proof, we introduce the basic notions for Almgren-Pitts combinatorial lemma.

**Definition 3.1.** For two $G$-open sets $U^1, U^2$, a $G$-generalized hypersurface is said to be $\epsilon$-$G$-a.m. in $(U^1, U^2)$ if it is $\epsilon$-$G$-a.m. in at least one of the two open sets. We define $CO^G$ to be the collection of pair $(U^1, U^2)$ of $G$-open sets with

$$d_G(U^1, U^2) \geq 4 \min\{\text{diam}_G(U^1), \text{diam}_G(U^2)\}.$$ 

This definition differs from Definition 3.2 in [6] in that we consider both diameter and distance on the quotient $M/G$ instead of on $M$. This shift is essential because otherwise there might be too few sets in $CO^G$. We owe this idea to [11].

The following trivial lemma utilizing only the metric space property will be of great importance.

**Lemma 3.2.** If $(U^1, U^2)$ and $(V^1, V^2)$ satisfy

$$d_G(U^1, U^2) \geq 2 \min\{\text{diam}_G(U^1), \text{diam}_G(U^2)\},$$

$$d_G(V^1, V^2) \geq 2 \min\{\text{diam}_G(V^1), \text{diam}_G(V^2)\},$$

then there exist $i, j \in \{1, 2\}$ so that $d(U^i, V^j) > 0$.

The most essential ingredient for the proof of Proposition 3.1 is the following Almgren-Pitts combinatorial lemma.

**Proposition 3.3.** (Almgren-Pitts combinatorial lemma) Let $\Lambda$ be a $G$-homotopically closed family of $G$-sweepouts. There exists a min-max sequence $\{\Gamma^N\} = \{\partial \mathcal{N}^G_{t_k(N)}\}$ so that

1. $\Gamma^N$ converges to a stationary varifold;
(2) For any \((U^1, U^2) \in \mathcal{CO}^G\), \(\Gamma^N\) is 1/N-G-a.m. in \((U^1, U^2)\) for \(N > N(U^1, U^2)\) large enough, with \(N(U^1, U^2) > 0\) depending on \((U^1, U^2)\).

Proof of Proposition 5.3 is exactly the same as proof of Proposition 3.4 in \([4]\), without even changing a word except for substituting Lemma 3.4 below for Lemma 3.1 in \([4]\) and adding \(G\)-in front of objects. We will omit the proof.

**Proof of Proposition 3.1.** The proof has essentially the same idea. We show that a subsequence of the \(\{\Gamma^k\}\) in Proposition 3.3 satisfies the requirements of Proposition 3.1.

By the existence of equivariant tubular neighborhoods \([4]\), for any \(z \in M\), there exists a nonzero \(\rho_G(z)\) so that for all \(0 < \rho \leq \rho_G(z)\), \(B^G_\rho(z)\) is a well-defined \(G\)-invariant tubular neighborhood around \(x\). Note that we cannot have uniform lower bound on \(\rho(z)\) even in simple examples like \(SO(3)\) acting on \(S^5\) by the first 3 coordinates, but apparently we can have a uniform upper bound on \(\rho_G\) by injectivity radius.

For any \(x \in M\), we fix \(k \in \mathbb{N}\) and some choice of radius \(0 < \rho(x) < \frac{1}{k} \rho_G(x)\). (The exact choice of \(\rho(x)\) doesn’t matter. We only need it to be positive. Moreover, it can be made \(G\)-invariant by pushing forward along orbits.) For all \(x \in M\), we have \((B^G_\rho, M \setminus \overline{B}^G_{\rho_\rho}(x)) \in \mathcal{CO}^G\) by construction. By Proposition 3.3 for \(k\) large, \(\Gamma^k\) is 1/k-G-almost minimizing in either \(B^G_\rho(x)\) or \(M \setminus \overline{B}^G_{\rho_\rho}(x)\). Consequently, for our choice \(\rho(x) > 0\), we have

(a) either \(\{\Gamma^k\}\) is 1/k-G-a.m. in \(B^G_\rho(y)\) for every \(y \in M\).

(b) or there exists a subsequence \(\{\Gamma^k\}\) (not relabeled) and a sequence \(\{x^k\} \subset M\) such that \(\Gamma^k 1/k-G\)-a.m. in \(M \setminus \overline{B}_{\rho_\rho}(x^k)\).

If for some choice of radius \(\alpha \rho(x) > 0\) with \(\alpha \in (0, 1]\), (a) holds, then we’re fine. If this is not the case, then we can find a subsequence of \(\{\Gamma^k\}\) (not relabeled) and a collection of points \(\{x^k_j\}_{j,k \in \mathbb{N}^+} \subset M\) so that

(i) for any fixed \(j\), \(\Gamma^k\) is 1/k-G-a.m. in \(M \setminus \overline{B}_{\rho_\rho(x^k_j)/j}(x^k_j)\) for \(k\) large enough,

(ii) \(x^k_j \xrightarrow{d_G} x_j\) for \(k \to \infty\), i.e., \(G.x^k_j\) converges to \(G.x_j\) in the quotient space \(M/G\), and \(x_j \xrightarrow{d_G} x\) for \(j \to \infty\).

**Claim.** \((*)\) For any \(J > \frac{1}{\rho(x)}\), there exists \(K_J\) so that \(\Gamma^k\) is G-1/k-a.m. in \(M \setminus \overline{B}_{1/J}(x)\) for all \(k \geq K_J\).

This can be done by choosing \(j\) with \(d_G(x_j, x) < 1/(3J)\), and more importantly

\[
\sup_{z \in M} \rho^G(z)/j \leq \frac{1}{3J}.
\]

Then take \(k\) large enough with \(d_G(x^k_j, x_j) < 1/(3J)\) and \(\Gamma^k\) 1/k-G-a.m. in \(M \setminus \overline{B}_{\rho_\rho(x^k_j)/j}(x^k_j)\). Note that

\[
\frac{\rho(x^k_j)}{j} + d_G(x^k_j, x_j) + d_G(x_j, x) < \frac{1}{J}.
\]
so we have $M \setminus \overline{B}_{r(y)}^G(x) \subset M \setminus \overline{B}_{\rho(x)y/J}^G(x)$. This proves the claim. Thus, for $y \in M \setminus \{x\}$, we can simply choose $r(y) < \rho^G(y)$ so that $B_{r(y)}^G \subset M \setminus \{x\}$. By construction we have that $\text{An}^G_{r(z)} \subset M \setminus \{x\}$ for any $\text{An}^G_{r(z)} \in \mathcal{AN}^G_{r(z)}(z)$ with $z \in M \setminus \{x\}$. By (\star), this definition of $r(y)$ satisfies the requirements in the proposition. This defines $r$ for $M \setminus \{x\}$. For $x$ itself, note that as long as $r(x) < \rho(x)$, then $\Gamma^k$ would be $1/k$-$G$-a.m. for $k$ large enough in any annulus around $x$ by (\star), since the annulus will be contained in a complement of an invariant tube. \hfill \Box

For the proof of Proposition 3.3 we need an important lemma that will help us construct dynamic competitors and glue them to get contradictions in our omitted proof of Proposition 3.3.

**Lemma 3.4.** Let $U \subset \subset U' \subset M$ be two $G$-open sets and $\{\partial \Xi_t^i\}_{t \in [0,1]}$ be a $G$-sweepout. For $\epsilon > 0$, and $t_0 \in [0,1]$, assume $\{\partial \Omega_\epsilon^i\}_{t \in [0,1]}$ is a one-parameter family of $G$-generalized hypersurfaces satisfying (a.1), (a.2), (a.3), and (a.4), with $\Omega_\epsilon = \Xi_0$. Then there is $\eta > 0$ such that the following holds for every $a, b, a', b'$ with $t_0 - \eta \leq a < a' < b' < b \leq t_0 + \eta$.

We can find a competitor $G$-sweepout $\{\partial \Xi_t^i\}_{t \in [0,1]}$ so that

(a) $\Xi_t = \Xi_t^i$ for $t \in [0,a] \cup [b,1]$ and $\Xi_t \setminus U' = \Sigma_t^{r} \setminus U'$ for $t \in (a,b)$;
(b) $\mathcal{H}^n(\partial \Xi_t^i) \leq \mathcal{H}^n(\partial \Xi_t) + \frac{\epsilon}{2}$, for every $t$;
(c) $\mathcal{H}^n(\partial \Xi_t^i) \leq \mathcal{H}^n(\partial \Xi_t) - \frac{\epsilon}{2}$ for $t \in (a',b')$.
(d) $\partial \Xi_t^i$ is $G$-homotopic to $\{\partial \Xi_t\}$.

**Proof.** The proof is the same as proof of Lemma 3.3 in [6]. There are several points worth mentioning. First, by [16], we can find invariant partition of unity subordinate to any $G$-open set. Second, when we fix normal coordinates $(z, \sigma) \in \partial \Xi_0 \cap C \times (-\delta, \delta)$, we are actually identifying the trivial normal bundle as the coordinates. In other words, we identify a $G$-invariant tubular neighborhood of $\partial \Xi_0 \cap C$ with the trivial normal bundle of $\Xi_0$. This is possible for the following reasons. All boundaries we consider are two-sided and thus naturally orientable with trivial normal bundle in the orientable ambient manifold. Thus, we can choose a unit normal field $\nu$ well-defined except at finitely many orbits. Since $G$ acts by isometries, $g_\nu \nu = \pm \nu_\nu$. Those $g$ reverse the normal will automatically form an index 2 subgroup of $G$, which is contradictory to connectedness of $G$. Thus, $G$ preserves the normal of $\partial \Xi_0 \cap C$. We can deduce that $G$-vector fields on $\partial \Xi_0 \cap C$ can be identified with $G$-smooth functions on $\partial \Xi_0 \cap C$. Moreover since the exponential map is $G$-equivariant, we see that exponentiating any $G$-vector field in the normal bundle with small enough norm would yield a $G$-invariant generalized hypersurface. Using these facts above, the $G$-invariance of our constructions can be readily verified. The index-2 subgroup argument is inspired by a conversation with Professor Robert Bryant. \hfill \Box

4. The existence of $G$-invariant replacements

This section dedicated to the proof of the following proposition

**Proposition 4.1.** Let $\{\Gamma^i\}$, $V$ and $r$ be as in Proposition 3.3. Fix $x \in M$ and consider an annulus $\text{An}^G \subset \mathcal{AN}^G_{r(x)}(x)$. Then there exists a $G$-varifold $\bar{V}$, a $G$-sequence $\{\bar{\Gamma}^i\}$ and a $G$-function $r' : M \rightarrow \mathbb{R}_+$ such that
(a) $\tilde{V}$ is a replacement for $V$ in $\text{An}^G$ and $\tilde{\Gamma}^j$ converges to $\tilde{V}$ in the sense of varifolds;
(b) $\tilde{\Gamma}^j$ is $G$-a.m. in every $\text{An}^G \in \mathcal{A}\Lambda^G_{r^j(y)}(y)$ with $y \in M$;
(c) $r'(x) = r(x)$.

**Proof.** Assume Lemma 4.2 and 4.3 below. Then exactly the same proof in Section 4.4 in [6] would carry over. The only cautious point is arguing that $\tilde{V}$ is stationary. Using the same argument, we can only deduce that it’s stationary among $G$-invariant varifolds. Invoke Corollary 4.2 to deduce that $\tilde{V}$ is in fact stationary. □

This proposition is the basis on which we can bootstrap and utilize to prove the regularity of varifolds with good replacements. Our definition of replacements is exactly the same as Definition 2.5 in [6]. We don’t require any $G$-invariance in the definition of the replacements. Instead, though our replacements are $G$-invariant by construction, they will be stable with respect to all deformations.

Now, we fix some $\text{An}^G \in \mathcal{A}\Lambda^G_{r(x)}(x)$.

**4.1. Setting.** For every $j$, consider the class $\mathcal{H}(\Omega^j, \text{An}^G)$ of $G$-sets $\Xi$ such that there exists a family $\{\Omega_t\}$ satisfying $\Omega_0 = \Omega^j$, $\Omega_1 = \Xi$, (a.1), (a.2), (a.3), for $\epsilon = \frac{1}{j}$ and $U = \text{An}^G$. Now, pick a sequence $\Gamma^{j,k} = \partial \Omega^{j,k}$ which is minimizing for the perimeter in the class $\mathcal{H}(\Omega^j, \text{An}^G)$. Up to subsequences, we can assume that

\[
\begin{align*}
\Omega^{j,k} &\text{ converges to a Caccioppoli } G \text{-set } \tilde{\Omega}^j, \\
\Gamma^{j,k} &\text{ converges to a } G \text{-varifold } \tilde{\Gamma}^j; \\
V^j &\text{ (and a suitable diagonal sequence } \tilde{\Gamma}^j = \Gamma^{j,k(j)}) \text{ converges to a } G \text{-varifold } \tilde{V}.
\end{align*}
\]

All the convergence comes from basic compactness theorem for integral currents and varifolds, and the equivalence of Cacciopoli sets with codimension-1 integral currents of finite mass. The $G$-invariance comes from the fact that $G$-invariant objects will form a closed subspace in both of these two cases.

The proof of Proposition 4.1 will be broken into three steps. First, we need to prove the following lemma for the regularity of the minimizers $\tilde{\Omega}^j$.

**Lemma 4.2.** For every $j$ and $y \in \text{An}^G$, there exists a $G$-tube $B = B^G(y) \subset \text{An}^G$ and some $k_0 \in \mathbb{N}$ with the following property. Every open $G$-set $\Xi$ such that $\partial \Xi$ is smooth except for a finite union of orbits, $\Xi \setminus B = \Omega^{j,k} \setminus B$, and $\mathcal{H}^n(\partial \Xi) < \mathcal{H}^n(\partial \Omega^{j,k})$, is contained in the collection $\mathcal{H}(\Omega^j, \text{An}^G)$ for $k \geq k_0$.

Using the above lemma, we would like to show that

**Lemma 4.3.** $\partial \tilde{\Omega}^j \cap \text{An}^G$ is a stable minimal generalized hypersurface in $\text{An}^G$ and $V^j \cap \text{An}^G = \partial \Omega^j \cap \text{An}^G$.

However, for proof of Lemma 4.3, we have to work a little bit harder than the proof of Lemma 4.2 in [6]. The idea of that proof can be utilized, but since every object and deformation are $G$-invariant, we cannot use the regularity for Plateau problem. Instead, we have to work harder for a regularity result for equivariant Plateau problem in our setting.
4.2. Proof of Lemma 4.2. Step 1 in Section 4.2 of [6] can be used unchanged. Transversality in the proof can be deduced from Theorem 6.35 (Parametric Transversality) in [13], since $\Gamma^{j,k}$ is smooth except for finitely many orbits, which corresponds to finitely many values of radius in the tube. The constructions of cones is a little different. For each $z \in \overline{B^G_j(y)}$, there is a geodesic from $z$ to $G.y$ and intersecting $G.y$ orthogonally by construction of tubular neighborhood. Denote this geodesic $[G.y, z]$. As usual, $(G.y, z) = [G.y, z] \setminus (G.x \cup \{z\})$. We let $K$ be the open cone consisting

$$K = \bigcup_{z \in \partial B^G_j(y) \cap \Omega^{j,k}} (G.y, z).$$

By construction $K$ is $G$-invariant and smooth. If we use $\exp^\perp$ to denote the exponential map of the normal bundle $N(G.y)$ of $G.y$ in $M$, then $\exp^\perp$ is a diffeomorphism on the tube $B^G_j(y)$. Now, if $K'$ is the cone over $(\exp^\perp)^{-1}\Omega^{j,k} \cap \partial B^G_j(y)$ in the normal bundle, then $\exp^\perp K' = K$. (Cone over a generalized hypersurface $S$ in the normal bundle is $C(S) = \{(p, tv) | t \in [0,1], (p, v) \in S\}$) The rest of Step 1 in Section 4.2 of [6] can be adapted easily.

Step 2 of the proof in [6] is volume estimates, which mostly carry over unchanged and consists of basic calculus on manifolds combined with tubular neighborhoods. We replace every geodesic balls in those estimates with invariant tubes instead. Note that inequality (4.5) in [6] shall have $\mathcal{H}^n(\partial K)$ on the left hand side.

The only part that needs caution is monotonicity formula estimate (4.13) in [6]. By the convergence of varifolds, we still have

$$\mathcal{H}^n(\partial \Omega^{j,k} \cap B^G_j(y)) \leq 2 \|V_j\| (B^G_j(y)).$$

However, recall that $B^G_j(y)$ is a tube around $G.y$, so we can’t use monotonicity formula immediately.

Now, invoke lemma [6,1] by choosing $20\rho < \rho_0$. By $G$-invariance of $V^j$, and monotonicity formula applied to $V^j$, we can deduce that

$$\|\partial \Omega^{j,k} \cap B^G_j(y)\| \leq \|\partial \Omega^{j,k}\| (B^G_j(y))$$

$$\leq C_y(20\rho)^{-d_\rho}C_M \|V^j\| (M)(20\rho)^n.$$
Proof. By Theorem 1.2 in [6], it suffices to prove that there exists a $G$-invariant minimizer in $\mathcal{P}(B^G_\rho(x), \Omega)$, since any minimizer will have the codimension 7 regularity.

First we can pick any minimizer $X$ in the class without requirement of $G$-invariance. Such a minimizer exists by Theorem 1.2 in [6]. Let $D = \Omega \cap \partial B^G_\rho(X)$. By transversality, $D$ is a smooth $G$-submanifold of $B^G_\rho(x)$ with boundary $\partial D = \partial \Omega \cap \partial B^G_\rho(x)$.

By definition of minimizer, have $X = \{x \in \Omega : d(x, \partial D) = \inf_{y \in \partial D} d(x, y)\}$. Therefore, $X$ is a $G$-invariant minimizer in $\mathcal{P}(B^G_\rho(x), \Omega)$. Let $X$ be a $G$-invariant minimizer in $\mathcal{P}(B^G_\rho(x), \Omega)$.

Now, define $f(x) = \int_G 1_X(gx)d\mu(g)$.

By construction we have $0 \leq f \leq 1$. Since $f$ is defined by averaging lower-semicontinuous functions, we see that $f$ is also lower-semicontinuous by Fatou lemma. Thus, the sets $X_\lambda = f^{-1}(\lambda, 1]$, are open. Moreover, $X_\lambda$ is $G$-invariant because $f$ is.

In the following paragraphs, we will prove that $X_\lambda$ is a $G$-invariant minimizer in the class $\mathcal{P}(B^G_\rho(x), \Omega)$.

Let $X_{\lambda, B} = X_\lambda \cup B^G_\rho(x)$, and $f(x)d\text{Vol}_M\cup B^G_\rho(x) = E_f$.

Recall that $X \cap M \setminus \overline{B}^G_{\rho'}(x) = X \cap M \setminus \overline{B}^G_{\rho'}(x)$, for some $0 < \rho' < \rho$. This implies $f = 1_X = 1_{X_\lambda}$ on $\mathcal{A}X^G_\rho(x, \rho', \rho)$. Thus, we deduce that $\partial X_{\lambda, B} - D, \partial E_f - D, \partial \Omega_B - D$ coincide on $\mathcal{A}X^G_\rho(x, \rho', \rho)$. Let

$$
\partial X_{\lambda, B} - D = T_{\lambda}, \\
\partial E_f - D = T.
$$

By construction, we have

$$
\int_G g_* X_B d\mu(g) = E_f = \int_0^1 X_{\lambda, B} d\lambda,
$$

and thus

$$
\int_G \partial g_* X_B d\mu(g) = \partial E_f = \int_0^1 \partial X_{\lambda, B} d\lambda.
$$

Since $G$ acts by isometries, we have $M(\partial g_* X_B) = M(\partial X_B) = M(D) + M(S)$. By lower-semicontinuity of mass and $G$-invariance of $D$, we deduce that

$$
M(\partial E_f) \leq \int_G M(\partial g_* X_B) d\mu(g) = M(D) + M(S),
$$

$$
M(E_f) \leq \int_G M(g_* X_B) d\mu(g).
$$
This implies $E_f$ is a normal current, and thus By 4.5.9(12) in [7], $X_{\lambda,B}$ are integral currents and Caccioppoli sets (interpreted in the right sense). This implies immediately that $T_{\lambda}$ are integral currents.

However, we have

$$M(\partial E_f) = M(D) + M(T),$$

$$M(\partial X_{\lambda,B}) = M(D) + M(T_{\lambda}).$$

With gradient variational formula 4.5.9(13) in [7], this immediately yields

$$\int_{[0,1]} M(T_{\lambda}) d\lambda = M(T) \leq M(S).$$

Since $T_{\lambda}$ are integral currents, we have $M(T_{\lambda}) \leq M(S)$ by definition of minimizer. This implies that $M(T_{\lambda}) = M(S)$, and $T_{\lambda}$ are area minimizers. Thus, $X_{\lambda}$ is a $G$-invariant minimizer in the class $\mathcal{P}(B^G_p(x,\Omega)).$

Professor William Allard points out that we can actually prove $\partial B^G_p(x)$ is a barrier for minimal surfaces, so the the minimizers are actually minimizing among all competitors. Consider the retraction to a sphere bundle of radius $r$ in the normal bundle by $v \mapsto v + t(\|v\| - r)^+ v$. We can verify that it decreases area if $r$ is small enough. Roughly speaking, the normal Jacobian has contributions from normal directions of order $r^{n-\dim G.x}$, with $O(1)$ contributions from tangential directions, both with respect to $G.x$. Alternatively, we can also invoke the results of [18] to deduce this barrier result.

**Corollary 2.** Let $\Omega$ be a smooth $G$-invariant open Caccioppoli set, whose boundary intersects $B^G_p(x)$ transversely. If a $G$-invariant Caccioppoli set $\Xi^G$ minimizes perimeter among $G$-invariant elements in $\mathcal{P}(B^G_p(x),\Omega)$. Then $X$ minimizes the perimeter in $\mathcal{P}(B^G_p(x),\Omega)$, without the restriction to $G$-invariant elements. Moreover, $\Xi$ is, in $U$, an open set whose boundary is smooth outside of a singular set of Hausdorff dimension at most $n-7$.

**Proof.** Pick a $G$-invariant minimizer $\Xi$ in Proposition 4.4. We have

$$\text{Per}(\Xi^G, B^G_p(x)) \leq \text{Per}(\Xi, B^G_p(x))$$

This immediately implies $\Xi^G$ is a minimizer among all competitors. The regularity result follows.

**4.4. Proof of Lemma 4.3.** Exactly the same proof as Section 4.3 of [6] applies. There are two points to be cautious of. The first is approximating $\Xi^{j,k}$ by invariant smooth functions. We can do this by the following averaging construction. Let

$$G(f)(x) = \int_G f(g.x) d\mu(g).$$

$G(f)$ is smooth by construction if $f$ is smooth. Now, just take a smooth approximation $f_n$ of $1_{\Xi^{j,k}}$ in the sense that $f_n \to 1_{\Xi^{j,k}}$ in $L^1$ and $\text{Var}(f_n, M) \to \text{Per}(\Xi^{j,k})$. Then up to subsequence, $G(f_n)$ is also a smooth approximation of $1_{\Xi^{j,k}}$. Then level sets of $G(f_n)$ would provide $G$-invariant smooth approximations of $\Xi^{j,k}$.

Finally, all our construction and competitors are $G$-invariant. To get rid of $G$-invariance restriction on minimizing, we invoke Corollary 2 to show that the minimizer is minimizing and thus stable among all competitors.
5. Regularity of $G$-Varifolds with replacements in $G$-annuli

This section corresponds to Section 5 of [6].

We apply Proposition 4.1 three times as in Section 2.4 of [6] to obtain

**Proposition 5.1.** Let $V$ and $r$ be as in Proposition 4.1. Fix $x \in M$ and $\text{An}^G \subset \text{AN}^G_{r(x)}(x)$. Then

(a) $V$ has a replacement $V'$ in $\text{An}^G$ such that,

(b) $V'$ has a replacement $V''$ in any

$$\text{An}^G \in \text{AN}^G_{r(x)}(x) \cup \bigcup_{y \not\in G.x} \text{AN}^G_{r(y)}(y).$$

(c) $V''$ has a replacement $V'''$ in any $\text{An}^G_{r'(y)} \in \text{An}^G_{r''(y)}(y)$ with $y \in M$, where $r', r''$ are both positive functions.

This section will be dedicated to prove the following proposition

**Proposition 5.2.** Let $V$ be as in Proposition 5.1. Then $V$ is induced by a minimal generalized hypersurface $\Sigma$ in the sense of Definition 1.1.

### 5.1. Tangent cones.

**Lemma 5.3.** Let $V$ be a stationary $G$-varifold in an open $G$-set $U \subset M$ having a $G$-invariant replacement in any annulus $\text{An}^G_{r(x)} \in \text{AN}^G_{r(x)}(x)$ for some positive function $r$. Then

- $V$ is integer rectifiable;
- $\theta(x, V) \geq 1$ for any $x \in U$;
- any tangent Cone $C$ to $V$ at $x$ is a minimal generalized hypersurface for general $n$ and (a multiple of) a hyperplane for $n \leq 6$ or $\dim G.x \geq n - 6$.

**Proof.** First, fix $x \in \text{supp} \|V\|$ and $0 < r < \min \{r(x)/20, \text{Inj}(M)/4\}$. Let $V'$ be the replacement of $V$ in the annulus $\text{An}^G_{r(x)}(x, r, 2r)$. We claim that $\|V'\| \neq 0$ on $\text{An}^G_{r(x)}(x, r, 2r)$. If that’s note the case, then there would be $\rho \leq r$ and $\epsilon$ such that $\text{supp}(\|V'\|) \cap \partial B^G_{\rho}(x) \neq \emptyset$ and $\text{supp}(\|V'\|) \cap \text{An}^G_{(x, \rho, \rho + \epsilon)} = \emptyset$. By choice of $\rho$ this would contradict Theorem 5.1(ii) in [6]. For the assumption of convexity in that theorem, we only need to use Theorem 5.1(ii) in [6] locally, so we can choose $r$ small enough to make the tubes very close to cylinders in local charts, which would satisfy the convexity assumption required.

Thus, $V', \text{An}^G_{r(x, r, 2r)}$ is a non-empty minimal generalized hypersurface. Thus, there exists $y \in \text{An}^G_{r(x, r, 2r)}$ with $\theta(y, V') \geq 1$. Without loss of generality, we can assume $y \in B_{2r}(x) \setminus B_r(x)$. (By $G$-invariance, we can always use some $g.y$ to substitute $y$ that sits in this geodesic annulus). By applying Lemma A.1 and letting $20r < \rho_0$, and $G$-invariance of $V$, we have

$$\|V\| (B^G_{2r}(x)) \leq \|V\| \left( \bigcup_{z \in \mathcal{B}} B_{20r}(z) \right)$$

$$\leq |\mathcal{B}| \|V\| (B_{20r}(x)).$$

(5.1)
By definition of replacements, we have

\[(5.2) \quad \|V\| (B^G_{4r}(x)) = \|V'\| (B^G_{4r}(x)).\]

Note that the collection \(B\) of balls is disjoint in Lemma A.1, so by \(G\)-invariance of \(V'\) we have

\[(5.3) \quad \|V'\| (B^G_{4r}(x)) \geq |B| \|V'\| (B^G_{2r}(y)).\]

Combining (5.1), (5.2), and (5.3), and dividing by \(|B|(20r)^n\) we deduce that

\[
\|V\| (B^G_{20r}(x)) \geq 10^n \|V'\| (B^G_{2r}(y)) \geq 10^n C_M \theta(y, V')(2r)^n.
\]

By monotonicity formula and \(\theta(y, V') \geq 1\) we can deduce that there exists constant \(C_M \geq 0\) so that

\[
\|V'\| (B^G_{2r}(y)) \geq C^{-1}_M \theta(y, V')(2r)^n.
\]

Thus, we have

\[
\|V\| (B^G_{20r}(x)) \geq 10^n C^{-1}_M \theta(y, V') \geq 10^n C_M.
\]

This implies \(\theta(x, V)\) is uniformly bounded away from 0 on \(\text{supp}(\|V\|)\) and Allard’s rectifiability theorem (5.5 in [1]) implies that \(V\) is rectifiable.

Use \(C\) to denote the tangent cone to \(V\) at \(x\) and let \(\rho_k \to 0\) a sequence with \(V_{\rho_k} \to C\). By \(G\)-invariance, the pushforward by any \(g \in G\) of these cones \(dg(C)\) and blowing up sequence \(dg(V_{\rho_k})\) are also tangent cones and blowing ups at \(g.x\). The rest of the proof goes the same as proof of Lemma 5.2 in [6]. One difference is that we have to substitute annulus in \(T_x M\) with annulus around \(i_* T_y G.y\), i.e.,

\[
\text{ann}(r, s, T_y G.y) = \{ v \in T_y M | r < \text{dist}_{T_y M}(v, T_y G.y) < s \}.
\]

The other is deducing the regularity of the cone \(C\) at \(T_y G.x\). By \(C = T_y G.x \times W\) where \(W\) is supported in \(T_y G.x\). Note that by comparison with \(C'\), \(W\) is a stable cone. This implies \(W\) is a multiple of plane if \(n - \dim G.x \leq 6\), and have singularity of Hausdorff dimension at most \(n - \dim G - 7\) if \(n - \dim G.x \geq 7\).

\section{5.2. Proof of Proposition 5.2}

**Step 1 to 4.** The same with Section 5.4 in [6].

**Step 5.** The center of our \(B^G_{4r}(x)\) are the orbits \(G.x\) instead of points. If no orbits are of dimension at least \(n - 6\), then we’re fine. However, for those of dimension at least \(n - 6\), then we shall invoke the Lemma 5.2 to deduce that the tangent cones are still hyperplanes. The same reasoning in Section 5.4 of [6] dealing with dimension lower than 7 can be used in our case to deduce the proposition.
6. Cohomogeneity 1 case

By the classification in [14], we have either $M/G = S^1$ or $M/G = [-1, 1]$. Let $\Sigma_t = \pi^{-1}(t)$ where $t \in [-1, 1]$ and we identify $S^1$ with $[-1, 1]/1 \sim 1$. By Proposition in [15], $H^n(\Sigma_t)$ depends smoothly on $t$ on principal orbits and extends continuously to 0 on exceptional orbits. If $M/G = S^1$, then every orbit is principal, so we must have a critical point, which gives a smooth minimal generalized hypersurface. If $M/G = [-1, 1]$, then $H^n(\Sigma_{-1}) = H^n(\Sigma_1) = 0$, and $H^n(\Sigma_t)$ is smooth on $(-1, 1)$ and continuous on $[-1, 1]$, so there must exists a critical point.

7. Remarks

One might wonder whether we have better control of regularity. The answer is yes in some cases. Note that by invariance of $\Sigma$, we can push forward smooth tangent planes. Thus, if a point $s$ is in the singular set $\text{Sing}\Sigma$, then $g.s$ must also belong to $\text{Sing}\Sigma$. This implies the singularity sets must also be the union of orbits, so $\text{Sing}\Sigma$ consists of orbits of dimension no larger than $n - 7$. Consequently, in some practical cases like those in [10], the singular set could roughly have dimensions or order $\frac{n}{2}$ or $\frac{n}{3}$.

However, generically, we can say nothing more about the regularity. By the work of [10], the projection of any minimal generalized hypersurface $\Sigma^n \subset M^{n+1}$ to the orbit space $M/G$ with some modified metric $V^{2/k}/g$ will be a minimal generalized hypersurface in the open manifold $M/\text{principal}/G$. Thus, the principal orbit part of $\Sigma^n$ can be reduced by quotient to a generic minimal generalized hypersurface, which imposes the $n - 7$ regularity. Meanwhile, that metric $V^{2/k}/g$ vanishes on singular orbits, so it provides no information about the singular orbits parts of $\Sigma^n$. Since the union of singular orbits can have dimension up to $n - 1$, we cannot deduce Theorem 0.1 by a reduction to orbit space $M/G$. We do need a full-blown min-max argument.

On the other hand, even though we use only one parameter here, the minimal surface we produce can have high index due to symmetry. For example, let $G = SO(2) \times SO(2)$ acting on $S^3(1) \subset \mathbb{R}^4$ through $\rho_2 \oplus \rho_2$, where $\rho_2$ is the natural representation of $SO(2)$. The cohomogeneity is one, the principal orbits are two dimensional Clifford tori, with one-dimensional exceptional orbits of circles. Thus, our min-max will produce the minimal Clifford tours, which has index 5.

Finally, why don’t we use the Almgren-Pitts version of min-max in our construction? The Almgren-Pitts theory is deeply rooted in the famous Almgren isomorphism theorem in [2].

\begin{equation}
\pi_j(Z_k(M), 0) \cong H_{j+k}(M, \mathbb{Z})
\end{equation}

However, if we’re going to consider $G$-invariant cycles, then we have to first provide a suitable version of (7.1), of which we have not considered due to a lack of $G$-invariant homology theory.
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Appendix A. Appendix

A.1. Ball covering of tubes. We will prove the following useful lemma.

**Lemma A.1.** For any \( y \in M \), there exists \( \rho_0 > 0 \) so that for any \( \rho < \rho_0 \), there exists a collection \( \mathcal{B} \) of disjoint geodesic balls of radius \( \rho \) with centers in \( G.y \) so that the concentric balls with radius \( 5\rho \) covers \( B^G_\rho(y) \). Moreover, the number of balls in this collection is at most \( C_y \rho^{-d_y} \), where \( C_y \) is a constant depending only on \( G.y \), and \( d_y = \dim G.y \).

**Proof.** Here we use a basic 5-times-radius covering theorem (putting \( \tau = 2 \) and \( \delta \) as diameter in 2.8.5 in [7]), that says for a covering using metric balls in metric space, we can find a disjoint subcollection so that 5-times-radius concentric balls of this subcollection would cover all the original balls. Now consider the covering of \( B^G_\rho(y) \) by \( \{ B_\rho(z) \mid z \in G.y \} \). We deduce that there exists a set \( \mathcal{B} \) consisting of finitely many points so that \( B_\rho(z) \cap B_\rho(z') = \emptyset \) if \( z \neq z', z, z' \in \mathcal{B} \) and

\[
B^G_\rho(y) \subset \bigcup_{z \in \mathcal{B}} B_\rho(z).
\]

Note that the cardinality of \( \mathcal{B} \) satisfies the following obvious bound

\[
|\mathcal{B}| \leq \frac{\text{Vol}(B^G_\rho(y))}{B_\rho(y)},
\]

since \( G \) acts by isometries and thus pushes forward geodesic balls to geodesic balls.

Let

\[
d_y = \dim G.y.
\]

Recall the volume of tubes in [8]. There exists \( \rho_0 > 0 \) so that for all \( \rho < \rho_0 \), we would have

\[
\text{Vol}(B^G_\rho(y)) \leq C_{d_y} \mathcal{H}^{d_y}(G.y) \rho^{n+1-d_y},
\]

for some dimensional constant \( C_{d_y} > 0 \). Moreover, by the volume of geodesic balls, we could assume that \( \text{Vol}(B_\rho(y)) \geq C_\rho \rho^{n+1} \), for some dimensional constant \( C_\rho > 0 \) by shrinking \( \rho_0 \) if necessary. If \( \rho < \rho_0 \), then there exists \( C_y > 0 \) depending only on \( G.y \) and \( M \) such that

\[
|\mathcal{B}| \leq C_y \rho^{-d_y}.
\]
\[ \text{A.2. Splitting of Tangent Cone of Integral G-varifold.} \] Let \( G.x \) be an orbit of dimension \( d_x \), and \( B^G_{\rho}(x) \) be the \( \rho \)-tubular neighborhood around \( x \). Suppose \( V \) is a rectifiable G-varifold in \( V_n \) and \( x \) is in \( \text{spt} V \). We will prove the following lemma which implies that the tangent cone splits as a product into normal directions and tangential directions to \( G.x \).

**Lemma A.2.** For any point \( y \in G.x \), there exists a tangent cone \( C_y \subset T_yM \) of \( V \), so that \( C_y + w = C_y \) for any \( w \in T_yG.x \subset T_yM \).

**Proof.** Without loss of generality, we can assume \( y = x \), since we can always pushforward our constructions by any element of \( g \). We will use \( \exp \) to denote the restriction of exponential map in \( T_yM \) inside a ball of injectivity radius. Let \( r_i \to \infty \) be a sequence so that \( (r_i)_* \exp^{-1}(V) \to C \) as varifold, where \( r_i \) is multiplication by \( r_i \) in \( T_yM \). Note that we have

\[ (r_i)_* \exp^{-1} G.y = i_* T_yG.y, \]

if \( i \) is the inclusion \( G.y \hookrightarrow M \).

By Theorem 3.5.7 of [16], we can embed \( M \) into some \( \mathbb{R}^N \) so that the action of \( G \) on \( M \) comes from a linear representation of \( G \) on \( \mathbb{R}^N \). We will also denote this action as \( \rho(g)z \) for \( z \in \mathbb{R}^N \). We will identify \( M \) as a submanifold of \( \mathbb{R}^N \) in the following reasoning.

Let \( c \in C_y \) be a point in the tangent cone. We will also regard it as a vector. We can find a sequence of points \( c_j \in V \) so that \( r_j \exp^{-1} c_j \to c \). Let \( g(t) \) be a smooth path in \( G \) so that \( g(0) = 0 \),

\[ \frac{d}{dt} \bigg|_{t=0} g(t).y = w. \]

Such a path exists by lifting a corresponding path starting with \( w \in T_yG.y \) and staying in \( G.y \approx G/G_y \).

Now, note that \( g(r_i^{-1}).c_i \in V \). If we can prove that

\[ r_i \exp^{-1}(g(r_i^{-1}).c_i) = c + w, \]

then we are done. To prove this, we need to compare \( \exp^{-1}(z) \) with \( z - y \). First, note that \( d(z - y)|_{T_xM} = \text{id}_{T_xM} = d \exp^{-1}(z)|_{T_xM}. \) Thus, \( \exp^{-1}(z) - (z - y) = O(d_M(z,y)^2) \) for \( z \in M \). Thus, we have

\[ r_i \exp^{-1}(g(r_i^{-1}).c_i) = r_i(g(r_i^{-1}).c_i - y) + r_i O(d_M(g(r_i^{-1}).c_i, y)^2) \]

\[ = r_i(\rho(g(r_i^{-1})).c_i - \rho(g(r_i^{-1})).y + \rho(g(r_i^{-1})).y) + r_i O(\|\exp^{-1}(g(r_i^{-1}).c_i)\|^2) \]

\[ = r_i(\rho(g(r_i^{-1})).r_i \exp^{-1}(c_i) + r_i O(\|\exp^{-1}(g(r_i^{-1}).c_i)\|^2) \]

\[ + r_i(\rho(g(r_i^{-1})).r_i (\rho(g(0)).y) + r_i O(\|\exp^{-1}(g(r_i^{-1})).y\|) + r_i O(\|\exp^{-1}(g(r_i^{-1}).c_i)\|^2) \]

\[ = r_i(\rho(g(r_i^{-1}))).r_i \exp^{-1}(c_i) + r_i(\rho(g(r_i^{-1})).y) + O(r_i^{-1}). \]

Let \( i \to \infty \), and we immediately get \( A.1 \). \( \square \)

**Corollary 3.** Any such cone \( C_y \) as in A.2 is a product of \( T_yG.x \) and a varifold \( W \) supported in \( i_*(T_yG.x)^\perp \).
THE EXISTENCE OF EMBEDDED G-INARIANT MINIMAL HYPERSURFACE

Proof. Take $W = C_y \cap i_x(T_y G.x) \perp$ and use Lemma A.2.

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