THE FOCK SPACE IN THE SLICE HYPERHOLOMORPHIC SETTING

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Abstract. In this paper we introduce and study some basic properties of the Fock space (also known as Segal-Bargmann space) in the slice hyperholomorphic setting. We discuss both the case of slice regular functions over quaternions and also the case of slice monogenic functions with values in a Clifford algebra. In the specific setting of quaternions, we also introduce the full Fock space. This paper can be seen as the beginning of the study of infinite dimensional analysis in the quaternionic setting.

1. Introduction

Fock spaces are a very important tool in quantum mechanics, and also in its quaternionic formulation; see the book of Adler [1] and the paper [31]. Roughly speaking, they can be seen as the completion of the direct sum of the symmetric or anti-symmetric, or full tensor powers of a Hilbert space which, from the point of view of Physics, represents a single particle. There is an alternative description of the Fock spaces in the holomorphic setting which, in this framework, are also known as Segal-Bargmann spaces.

In this note we work first in the setting of slice hyperholomorphic functions, namely either we work with slice regular functions (these are functions defined on subsets of the quaternions with values in the quaternions) or with slice monogenic functions (these functions are defined on the Euclidean space $\mathbb{R}^{n+1}$ and have values in the Clifford algebra $\mathbb{R}_n$), see the book [21].

Slice hyperholomorphic functions have been introduced quite recently but they have already several applications, for example in Schur analysis and to define some functional calculi. The application to Schur analysis started with the paper [6] and it is rapidly growing, see for example [3, 4, 5, 7, 8].

The applications to the functional calculus ranges from the so-called S-functional calculus, which works for $n$-tuples non necessarily commuting.
operators, to a quaternionic version of the classical Riesz-Dunford functional calculus, see [23]. The literature on slice hyperholomorphic functions and the related functional calculi is wide, and we refer the reader to the book [21] and the references therein.

We note that Fock spaces have been treated in the more classical setting of monogenic functions, see for example the book [22]. In the treatment in [22] no tensor products of Hilbert Clifford modules are involved. In the framework of slice hyperholomorphic analysis we have already introduced and studied the Hardy spaces (see [7, 8, 9]), and Bergman spaces (see [18, 20, 19]). Here we begin the study of the main properties of the Fock spaces in the slice hyperholomorphic setting.

We start by recalling the definition of the Fock space in the classical complex analysis case (for the origins of the theory see [24]). For \( n \in \mathbb{N} \) let \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \) where \( z_j = x_j + iy_j, \ x_j, y_j \in \mathbb{R} \) \((j = 1, \ldots, n)\) and denote by
\[
d\mu(z) := \pi^{-n} \prod_{j=1}^{n} dx_j dy_j
\]
the normalized Lebesgue measure on \( \mathbb{C}^n \). The Fock space of holomorphic functions \( f \) defined on \( \mathbb{C}^n \) is
\[
F_n := \{ \ f : \mathbb{C}^n \to \mathbb{C} \ \text{such that} \ \int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2} d\mu(z) < \infty \ \}. \quad (1.1)
\]
The space \( F_n \) with the scalar product
\[
\langle f, g \rangle_{F_n} = \int_{\mathbb{C}^n} f(z) \overline{g(z)} e^{-|z|^2} d\mu(z)
\]
becomes a Hilbert space and the norm is
\[
\|f\|_{F_n}^2 = \int_{\mathbb{C}^n} |f(z)|^2 e^{-|z|^2} d\mu(z), \quad f \in F_n.
\]
The space \( F_n \) is called boson Fock space and since we will treat this case in the sequel we will refer to it simply as Fock space. One of its most important properties is that it is a reproducing kernel Hilbert space. If we denote by \( \langle \cdot, \cdot \rangle_{\mathbb{C}^n} \) the natural scalar product in \( \mathbb{C}^n \) defined by
\[
\langle u, v \rangle_{\mathbb{C}^n} := \sum_{j=1}^{n} u_j \overline{v_j},
\]
for every \( u, v \in \mathbb{C}^n \) we define the function
\[
\psi_u(z) = e^{\langle u, v \rangle_{\mathbb{C}^n}} = e^{\sum_{j=1}^{n} u_j \overline{v_j}}. \quad (1.2)
\]
We have the reproducing property
\[
\langle f, \psi_u \rangle_{F_n} = f(u), \quad \text{for all} \quad f \in F_n.
\]
So there are two equivalent characterizations of the Fock space \( F_n \); one geometric, in terms of integrals (see (1.1)), and one analytic, obtained the reproducing kernel property (or, directly from (1.1)): an entire function
$f(z) = \sum_{m \in \mathbb{N}_0^n} a_m z^m$ of $n$ complex variables $z = (z_1, \ldots, z_n)$ is in $F_n$ if and only if its Taylor coefficients satisfy
\[ \sum_{m \in \mathbb{N}_0^n} m! |a_m|^2 < \infty, \]
where we have used the multi-index notation. A third characterization is of importance, namely (with appropriate identification, and with $\circ$ denoting the symmetric tensor product)
\[ F_n = \bigoplus_{k=0}^{\infty} (\mathbb{C}^n)^{\circ k}. \]
In this paper we will address some aspects of these three characterizations in the quaternionic and Clifford algebras settings.

The paper consists of four sections besides the introduction. In Section 2 we give a brief survey of infinite dimensional analysis. In Section 3 we study the quaternionic Fock space in one quaternionic variable. We then discuss, in Section 4, the full Fock space. In order to define it, we need to study tensor products of quaternionic two-sided Hilbert spaces. Tensor product of quaternionic vector spaces have been treated in the literature at various level, see e.g. [11], [31, 30]. This section in particular opens the way to study a quaternionic infinite dimensional analysis. The last section considers the case of slice monogenic functions.

2. A BRIEF SURVEY OF INFINITE DIMENSIONAL ANALYSIS

There are various ways to introduce infinite dimensional analysis. We mention here four related approaches:

1. The white noise space and the Bochner-Minlos theorem: The formula
\[ e^{-\frac{\|s\|^2}{2}} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{u^2}{2}} e^{-itu} du \] (2.1)
is an illustration of Bochner’s theorem. It is well known that there is no such formula when $\mathbb{R}$ is replaced by an infinite dimensional Hilbert space. On the other hand, the Bochner-Minlos theorem asserts that there exists a probability measure $P$ on the space $\mathcal{S}'$ of real tempered distributions such that
\[ e^{-\frac{\|s\|^2}{2}} = \int_{\mathcal{S}'} e^{i(s',s)} dP(s'). \] (2.2)
In this expression, $s$ is belongs to the space $\mathcal{S}$ of real-valued Schwartz function, the duality between $\mathcal{S}$ and $\mathcal{S}'$ is denoted by $\langle s',s \rangle$ and $\| \cdot \|_2$ denotes the $L_2(\mathbb{R}, dx)$ norm.

The probability space $L_2(\mathcal{S}', P)$ is called the white noise space, and is denoted by $\mathcal{W}$. Denoting by $Q_s$ the map $s' \mapsto \langle s',s \rangle$ we see that (2.2) induces an isometry, which we denote $Q_f$, from the Lebesgue space $L_2(\mathbb{R}, dx)$ into the white noise space. We now give an important family of orthogonal
basis \((H_\alpha, \alpha \in \ell)\) of the white noise space, indexed by the set \(\ell\) of sequences \((\alpha_1, \alpha_2, \ldots)\), with entries in
\[
\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\},
\]
where \(\alpha_k \neq 0\) for only a finite number of indices \(k\). Let \(h_0, h_1, \ldots\) denote the Hermite polynomials, and let \(\xi_1, \xi_2, \ldots\) be an orthonormal basis of \(L^2(\mathbb{R}, dx)\) (typically, the Hermite functions, but other choices are possible). Then
\[
H_\alpha = \prod_{k=1}^{\infty} h_{\alpha_k}(Q\xi_k),
\]
and, with the multi-index notation
\[
\alpha! = \alpha_1!\alpha_2!\cdots,
\]
we have
\[
\|H_\alpha\|_W^2 = \alpha!.
\]
The decomposition of an element \(f \in \mathcal{W}\) along the basis \((H_\alpha)_{\alpha \in \ell}\) is called the chaos expansion.

2. The Bargmann space in infinitely many variables: When in (1.2), \(C^n\) is replaced by \(\ell_2(\mathbb{N})\), we have the function
\[
\psi_u(z) = e^{\langle u, v \rangle_{\ell_2(\mathbb{N})}} = e^{\sum_{j=1}^{\infty} u_jv_j}.
\]
The map \(H_\alpha \mapsto z^\alpha\) is called the Hermite transform, and is unitary from the white noise space onto the reproducing kernel Hilbert space with reproducing kernel (2.5).

3. The Fock space: We denote by \(\circ\) the symmetrized tensor product and by
\[
\Gamma^0(\mathcal{H}) = \oplus_{n=0}^{\infty} \mathcal{H}^{\otimes n},
\]
the symmetric Fock space associated to a Hilbert space \(\mathcal{H}\). Then, \(\Gamma^0(L_2(\mathbb{R}, dx))\) can be identified with the white noise space via the Wiener-Itô-Segal transform defined as follows (see [37, p. 165]):
\[
\xi_\alpha = \xi_1^{\circ \alpha_1} \circ \cdots \circ \xi_m^{\circ \alpha_m} \in \mathcal{H}^{\otimes n} \mapsto H_\alpha
\]
This is the starting point of our approach to quaternionic infinite dimensional analysis; see Section 4.

4. The free setting. The full Fock space: It is defined by
\[
\Gamma(\mathcal{H}) = \oplus_{n=0}^{\infty} \mathcal{H}^{\otimes n},
\]
and allows to develop the free analog of the white noise space theory. See [36, 35] for background for the free setting. See [12] for recent applications to the theory of non commutative stochastic distributions.
We refer in particular to the papers [33, 34, 13, 14] and the books [26, 27, 28, 29, 37, 25] for more information on these various aspects.

3. The Fock space in the slice regular case

The algebra of quaternions is indicated by the symbol $\mathbb{H}$. The imaginary units in $\mathbb{H}$ are denoted by $i$, $j$, and $k$, respectively, and an element in $\mathbb{H}$ is of the form $q = x_0 + ix_1 + jx_2 + kx_3$, for $x_\ell \in \mathbb{R}$. The real part, the imaginary part and the modulus of a quaternion are defined as $\text{Re}(q) = x_0$, $\text{Im}(q) = ix_1 + jx_2 + kx_3$, and $|q|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2$, respectively. The conjugate of the quaternion $q = x_0 + ix_1 + jx_2 + kx_3$ is defined by $\bar{q} = \text{Re}(q) - \text{Im}(q) = x_0 - ix_1 - jx_2 - kx_3$ and it satisfies $|q|^2 = \bar{q}q = q\bar{q}$.

The unit sphere of purely imaginary quaternions is $S = \{q = ix_1 + jx_2 + kx_3 \text{ such that } x_1^2 + x_2^2 + x_3^2 = 1\}$.

Notice that if $I \in S$, then $I^2 = -1$; for this reason the elements of $S$ are also called imaginary units. Note that $S$ is a 2-dimensional sphere in $\mathbb{R}^4$. Given a nonreal quaternion $q = x_0 + \text{Im}(q) = x_0 + I|\text{Im}(q)|$, $I = \text{Im}(q)/|\text{Im}(q)| \in S$, we can associate to it the 2-dimensional sphere defined by $[q] = \{x_0 + I\text{Im}(q) : I \in S\}$.

This sphere has center at the real point $x_0$ and radius $|\text{Im}(q)|$. An element in the complex plane $\mathbb{C}_I = \mathbb{R} + I\mathbb{R}$ is denoted by $x + Iy$.

**Definition 3.1** (Slice regular (or slice hyperholomorphic) functions). Let $U$ be an open set in $\mathbb{H}$ and consider a real differentiable function $f : U \to \mathbb{H}$. Denote by $f_I$ the restriction of $f$ to the complex plane $\mathbb{C}_I$.

The function $f$ is (left) slice regular (or (left) slice hyperholomorphic) if, for every $I \in S$, it satisfies:

$$\overline{\partial_I}f_I(x + Iy) := \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + Iy) = 0,$$

on $U \cap \mathbb{C}_I$. The set of (left) slice regular functions on $U$ will be denoted by $\mathcal{R}(U)$.

The function $f$ is right slice regular (or right slice hyperholomorphic) if, for every $I \in S$, it satisfies:

$$(f_I\overline{\partial_I})(x + Iy) := \frac{1}{2} \left( \frac{\partial}{\partial x} f_I(x + Iy) + \frac{\partial}{\partial y} f_I(x + Iy)I \right) = 0,$$

on $U \cap \mathbb{C}_I$.

The class of slice hyperholomorphic quaternionic valued functions is important since power series centered at real points are slice hyperholomorphic: if $B = B(y_0, R)$ is the open ball centered at the real point $y_0$ and radius
If $f : B \to \mathbb{H}$ is a left slice regular function then $f$ admits the power series expansion

$$f(q) = \sum_{m=0}^{+\infty} (q - y_0)^m \frac{1}{m!} \frac{\partial^m f}{\partial x^m}(y_0),$$

converging on $B$.

A main property of the slice hyperholomorphic functions is the so-called Representation Formula (or Structure Formula). It holds on a particular class of open sets which are described below.

**Definition 3.2** (Axially symmetric domain). Let $U \subseteq \mathbb{H}$. We say that $U$ is axially symmetric if, for all $x + Iy \in U$, the whole 2-sphere $[x + Iy]$ is contained in $U$.

**Definition 3.3** (Slice domain). Let $U \subseteq \mathbb{H}$ be a domain in $\mathbb{H}$. We say that $U$ is a slice domain (s-domain for short) if $U \cap \mathbb{R}$ is non empty and if $U \cap \mathbb{C}_I$ is a domain in $\mathbb{C}_I$ for all $I \in \mathbb{S}$.

**Theorem 3.4** (Representation Formula). Let $U$ be an axially symmetric s-domain $U \subseteq \mathbb{H}$.

Let $f$ be a (left) slice regular function on $U$. Choose any $J \in \mathbb{S}$. Then the following equality holds for all $q = x + yI \in U$:

$$f(x + Iy) = \frac{1}{2} \left[ f(x + Jy) + f(x - Jy) \right] + I \frac{1}{2} \left[ J[f(x - Jy) - f(x + Jy)] \right]. \quad (3.1)$$

**Remark 3.5.** One of the applications of the Representation Formula is the fact that any function defined on an open set $\Omega_I$ of a complex plane $\mathbb{C}_I$ which belongs to the kernel of the Cauchy-Riemann operator can be uniquely extended to a slice hyperholomorphic function defined on the axially symmetric completion of $\Omega_I$ (see [21]).

We now define the Fock space in this framework.

**Definition 3.6** (Slice hyperholomorphic quaternionic Fock space). Let $I$ be any element in $\mathbb{S}$. Consider the set

$$\mathcal{F}(\mathbb{H}) = \{ f \in \mathcal{R}(\mathbb{H}) \mid \int_{\mathbb{C}_I} e^{-|p|^2} |f_I(p)|^2 d\sigma(x, y) < \infty \}$$

where $p = x + Iy$, $d\sigma(x, y) := \frac{1}{\pi} dx dy$. We will call $\mathcal{F}(\mathbb{H})$ (slice hyperholomorphic) quaternionic Fock space.

We endow $\mathcal{F}(\mathbb{H})$ with the inner product

$$\langle f, g \rangle := \int_{\mathbb{C}_I} e^{-|p|^2} \overline{g_I(p)} f_I(p) d\sigma(x, y); \quad (3.2)$$

we will show below that this definition, as well as the definition of Fock space, do not depend on the imaginary unit $I \in \mathbb{S}$.

The norm induced by the inner product is then

$$\|f\|^2 = \int_{\mathbb{C}_I} e^{-|p|^2} |f_I(p)|^2 d\sigma(x, y).$$
We have the following result:

**Proposition 3.7.** The quaternionic Fock space $\mathcal{F}(\mathbb{H})$ contains the monomials $p^n$, $n \in \mathbb{N}$ which form an orthogonal basis.

**Proof.** Let us choose an imaginary unit $I \in S$ and, for $n, m \in \mathbb{N}$, compute

$$\langle p^n, p^m \rangle = \int_{C_I} e^{-|p|^2/p^np^n} d\sigma(x, y).$$

By using polar coordinates, we write $p = \rho e^{I\theta}$ and we have

$$\langle p^n, p^m \rangle = \frac{1}{\pi} \int_0^{2\pi} \int_0^{+\infty} e^{-\rho^2} \rho^n e^{-Im\theta} \rho^m e^{In\theta} \rho d\rho d\theta$$

$$= \frac{1}{\pi} \int_0^{2\pi} \int_0^{+\infty} e^{-\rho^2} \rho^{m+n+1} e^{I(n-m)\theta} d\rho d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{I(n-m)\theta} d\theta \int_0^{+\infty} e^{-\rho^2} \rho^{m+n} d\rho^2.$$

Since $\int_0^{2\pi} e^{I(n-m)\theta} d\theta$ vanishes for $n \neq m$ and equals $2\pi$ for $n = m$, we have

$$\langle p^n, p^m \rangle = 0 \text{ for } n \neq m.$$ For $n = m$, standard computations give

$$\langle p^n, p^n \rangle = \int_0^{+\infty} e^{-\rho^2} \rho^{2n^2} d\rho^2 = n!.$$

Thus the monomials $p^n$ belong to $\mathcal{F}(\mathbb{H})$ and any two of them are orthogonal. We now show that these monomials form a basis for $\mathcal{F}(\mathbb{H})$. A function $f \in \mathcal{F}(\mathbb{H})$ is entire so it admits series expansion of the form $f(p) = \sum_{m=0}^{+\infty} p^m a_m$ and thus the monomials $p^n$ are generators. To show that they are independent, we show that if $\langle f, p^n \rangle = 0$ for all $n \in \mathbb{N}$ then $f$ is identically zero. We have:

$$\langle f, p^n \rangle = \langle \sum_{m=0}^{+\infty} p^m a_m, p^n \rangle$$

$$= \int_{C_I} e^{-|p|^2/p^np^n} \left( \sum_{m=0}^{+\infty} p^m a_m \right) d\sigma(x, y)$$

and so

$$\langle f, p^n \rangle = \lim_{r \to +\infty} \int_{|p| < r, p \in C_I} e^{-|p|^2/p^np^n} \left( \sum_{m=0}^{+\infty} p^m a_m \right) d\sigma(x, y)$$

$$= \lim_{r \to +\infty} \sum_{m=0}^{+\infty} \left( \int_{|p| < r, p \in C_I} e^{-|p|^2/p^np^n} d\sigma(x, y) \right) a_m$$

$$= \lim_{r \to +\infty} \int_0^{r} \rho^{2n} e^{-r^2} dr^2 a_n = n! a_n,$$
where we used the fact that the series expansion converges uniformly on $|p| < r$, thus we can exchange the series with the integration where needed. Thus $\langle f, p^n \rangle = 0$ for all $n$ if and only if $a_n = 0$, i.e. $f \equiv 0$.

**Proposition 3.8.** The definition of inner product (3.2) does not depend on the imaginary unit $I \in \mathbb{S}$.

**Proof.** Let $f(p) = \sum_{m=0}^{+\infty} p^m a_m; g(p) = \sum_{m=0}^{+\infty} p^m b_m \in \mathcal{F}(\mathbb{H})$ and let $I \in \mathbb{S}$. We have

$$\langle f, g \rangle = \left\langle \sum_{m=0}^{+\infty} p^m a_m, \sum_{m=0}^{+\infty} p^m b_m \right\rangle$$

$$= \int_{\mathbb{C}_I} e^{-|p|^2} \left( \sum_{m=0}^{+\infty} p^m a_m \right) \left( \sum_{n=0}^{+\infty} p^n b_n \right) d\sigma(x, y)$$

$$= \int_{\mathbb{C}_I} e^{-|p|^2} \left( \sum_{m=0}^{+\infty} \bar{a}_m p^n \right) \left( \sum_{n=0}^{+\infty} p^n b_n \right) d\sigma(x, y)$$

so that

$$\langle f, g \rangle = \sum_{n=0}^{+\infty} \int_{\mathbb{C}_I} e^{-|p|^2} \bar{a}_n p^n b_n d\sigma(x, y)$$

$$= \sum_{n=0}^{+\infty} \bar{a}_n \left( \int_{\mathbb{C}_I} e^{-|p|^2} \bar{p}^n p^n d\sigma(x, y) \right) b_n$$

$$= \sum_{n=0}^{+\infty} n! \bar{a}_n b_n,$$

which shows that the computation does not depend on the chosen imaginary unit $I$. \qed

Let us recall that the slice regular exponential function is defined by

$$e^p := \sum_{n=0}^{+\infty} \frac{p^n}{n!}.$$ We need to generalize the definition of the function $e^{zw} = \sum_{m=0}^{+\infty} \frac{(zw)^m}{m!}$, $z, w \in \mathbb{C}$ to the slice hyperholomorphic setting.

We first observe that if we set $e^{pq} = \sum_{m=0}^{+\infty} \frac{(pq)^m}{m!}$ then the function $e^{pq}$ does not have any good property of regularity: it is not slice regular neither in $p$ nor in $q$ (while $e^{zw}$ is holomorphic in both the variables). Let us consider $p$ as a variable and $q$ as a parameter and set:

$$e^{pq} = \sum_{n=0}^{+\infty} \frac{(pq)^n}{n!} = \sum_{n=0}^{+\infty} \frac{p^n q^n}{n!} \quad (3.3)$$
where the ⋆-product (see [21]) is computed with respect to the variable \( p \). It is immediate that \( e^\ast_{pq} \) is a function left slice regular in \( p \) and right regular in \( q \).

**Remark 3.9.** The definition (3.3) is consistent with the fact that we are looking for a slice regular extension of \( e^z w \). In fact, we start from the function \( e^z w = \sum_{n=0}^{+\infty} \frac{z^n w^n}{n!} \), which is holomorphic in \( z \) seen as an element on the complex plane \( \mathbb{C}_I \); we then use the Representation Formula to get the extension to \( \mathbb{H} \):

\[
\text{ext}(e^z w) = \frac{1}{2} (1 - I_q I) \sum_{n=0}^{+\infty} \frac{z^n w^n}{n!} + \frac{1}{2} (1 + I_q I) \sum_{n=0}^{+\infty} \frac{\overline{z^n w^n}}{n!} = e^q w
\]

and since \( w \) is arbitrary, we get the statement.

We now set \( k_q(p) := e^\ast_{pq} \) and we discuss the reproducing property in the Fock space.

**Theorem 3.10.** For every \( f \in \mathcal{F}(\mathbb{H}) \) we have

\[
\langle f, k_q \rangle = f(q).
\]

Moreover, \( \langle k_q, k_s \rangle = e^\ast_{qs} \).

**Proof.** We have

\[
\langle f, k_q \rangle = \int_{\mathbb{C}_I} e^{-|p|^2} e^{\ast pq} f(p) d\sigma(x, y) \\
= \int_{\mathbb{C}_I} e^{-|p|^2} \left( \sum_{n=0}^{+\infty} \frac{q^n p^n}{n!} \right) \left( \sum_{m=0}^{+\infty} p^m a_m \right) d\sigma(x, y) \\
= \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \frac{q^n}{n!} (p^m, p^n) a_m \\
= \sum_{n=0}^{+\infty} q^n a_n \\
= f(q).
\]
Similarly, we have

$$\langle k_q, k_s \rangle = \int_{C_I} e^{-|p|^2} e^{p \bar{s}} e^{\bar{p} q} d\sigma(x, y)$$

$$= \int_{C_I} e^{-|p|^2} \left( \sum_{n=0}^{+\infty} \frac{s^n p^n}{n!} \right) \left( \sum_{m=0}^{+\infty} p^m q^m \right) d\sigma(x, y)$$

$$= \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \frac{s^n q^m}{n!} (p^m, p^n) \frac{q^m}{m!}$$

$$= \sum_{n=0}^{+\infty} \frac{s^n q^n}{n!}$$

$$= e^{s \bar{q}}.$$  

□

Proposition 3.11. A function $f(p) = \sum_{m=0}^{+\infty} p^m a_m$ belongs to $F(\mathbb{H})$ if and only if $\sum_{m=0}^{+\infty} |a_m|^2 m! < \infty$.

Proof. Let us use polar coordinates; with computations similar to those in the proof of Proposition 3.8 and using the Parseval identity, we have

$$\int_{C_I} e^{-|p|^2} |f(p)|^2 d\sigma(x, y) = \lim_{r \to +\infty} \frac{1}{\pi} \int_0^r \int_0^{2\pi} e^{-\rho^2} \left( \sum_{m=0}^{+\infty} \rho^{2m} |a_m|^2 \right) \rho d\theta d\rho$$

$$= 2 \lim_{r \to +\infty} \rho^{-1} \int_0^r e^{-\rho^2} \rho^{2m+1} |a_m|^2 d\rho$$

$$= 2 \sum_{m=0}^{+\infty} |a_m|^2 m!$$

and the statement follows. □

4. Quaternion full Fock space and symmetric Fock space

Let $V$ be a right vector space over $\mathbb{H}$. Recall that a quaternionic inner product on $V$ is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{H}$ satisfies the same properties of a complex inner product, with the exception of the homogeneity requirement which is replaced by

$$\langle u \alpha, v \beta \rangle = \overline{\beta} \langle u, v \rangle \alpha,$$

and that if $V$ is complete with respect to the norm induced by the inner product, it is called a right quaternionic Hilbert space. A similar definition can be given in the case of a quaternionic vector space on the left or two-sided.
Let $\mathcal{H}$ be a two-sided quaternionic Hilbert space. Then one may consider
the quaternionic $n$-fold Hilbert space tensor power $\mathcal{H}^\otimes n$ defined by

$$
\mathcal{H}^\otimes n = \mathcal{H} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H} \quad (n \text{ times}),
$$

where all tensor product are over $\mathbb{H}$.

**Remark 4.1.** A convenient way of constructing $\mathcal{H}^\otimes n$ is inductively. Recall
that if $M$ is a left $R$-module and $N$ is a right $R$-module, then a tensor
product of them $M \otimes_R N$ is an abelian group together with a bilinear map
$\delta : M \times N \rightarrow M \otimes_R N$ which is universal in the sense that for any abelian
group $A$ and a bilinear map $f : M \times N \rightarrow A$, there is a unique group
homomorphism $\tilde{f} : M \otimes_R N \rightarrow A$ such that $\tilde{f} \otimes \delta = f$. If furthermore, $M$
is a right $S$-module and $N$ is a left $T$-module, then $SM \otimes_R N$ is a $(S,T)$-
bimodule if one defines $sz = (s \otimes 1)z$ and $zt = z(1 \otimes t)$ for $z \in SM \otimes_R N$.
Since it holds that

$$(RMS \otimes SN_T) \otimes_T P_U \cong RMS \otimes (SN_T \otimes_T P_U),$$

one can define inductively the tensor product of $M_1, \ldots, M_n$, where $M_i$ is a
$(R_{i-1},R_i)$-bi-module, and obtain a $(R_0,R_n)$-bi-module,

$$r_0M_1R_1 \otimes r_2M_2R_2 \otimes \cdots \otimes r_n, M_1R_n.$$

For more details see [32] pp. 133-135. One can also define it non-inductively
(see [17] pp. 264).

We make the convention $\mathcal{H}^\otimes 0 = H$, and the element $1 \in H$ is called the
vacuum vector and denoted by $1$. For the case of two Hilbert spaces in the
next proposition, see also [11] equation (3).

**Proposition 4.2.** Let $\langle \cdot, \cdot \rangle$ be the inner product of $\mathcal{H}$, and assume that it
satisfies also the additional property

$$
\langle u, \lambda v \rangle = \langle \overline{\lambda} u, v \rangle.
$$

Then, it induces an inner product on $\mathcal{H}^\otimes n$,

$$
\langle u_1 \otimes \cdots \otimes u_n, v_1 \otimes \cdots \otimes v_n \rangle = \langle \cdots \langle \langle u_1, v_1 \rangle u_2, v_2 \rangle u_3, v_3 \rangle \cdots \rangle u_n, v_n \rangle,
$$

with the same additional property.

**Proof.** The statement clearly holds for $n = 1$. By induction,

$$
\langle u_1 \otimes \cdots \otimes u_n\alpha, v_1 \otimes \cdots \otimes v_n\beta \rangle = \langle \langle u_1 \otimes \cdots \otimes u_{n-1}, v_1 \otimes \cdots \otimes v_{n-1} \rangle u_n\alpha, v_n\beta \rangle
$$

$$
= \overline{\overline{\beta}} \langle \langle u_1 \otimes \cdots \otimes u_{n-1}, v_1 \otimes \cdots \otimes v_{n-1} \rangle u_n, v_n \rangle \alpha
$$

$$
= \overline{\overline{\beta}} \langle u_1 \otimes \cdots \otimes u_n, v_1 \otimes \cdots \otimes v_n \rangle \alpha.
$$
and
\[
\langle v_1 \otimes \cdots \otimes v_n, u_1 \otimes \cdots \otimes u_n \rangle = \langle \langle v_1 \otimes \cdots \otimes v_{n-1}, u_1 \otimes \cdots \otimes u_{n-1} \rangle v_n, u_n \rangle \\
= \langle v_n, \langle v_1 \otimes \cdots \otimes v_{n-1}, u_1 \otimes \cdots \otimes u_{n-1} \rangle u_n \rangle \\
= \langle \langle v_1 \otimes \cdots \otimes v_{n-1}, u_1 \otimes \cdots \otimes u_{n-1} \rangle u_n, v_n \rangle \\
= \langle \langle u_1 \otimes \cdots \otimes u_{n-1}, v_1 \otimes \cdots \otimes v_{n-1} \rangle u_n, v_n \rangle \\
= \langle u_1 \otimes \cdots \otimes u_n, v_1 \otimes \cdots \otimes v_n \rangle.
\]

For the additional property, we obtain
\[
\langle u_1 \otimes \cdots \otimes u_n, \lambda v_1 \otimes \cdots \otimes v_n \rangle = \langle \langle u_1 \otimes \cdots \otimes u_{n-1}, \lambda v_1 \otimes \cdots \otimes v_{n-1} \rangle u_n, v_n \rangle \\
= \langle \langle \lambda u_1 \otimes \cdots \otimes u_{n-1}, v_1 \otimes \cdots \otimes v_{n-1} \rangle u_n, v_n \rangle \\
= \langle \lambda u_1 \otimes \cdots \otimes u_n, v_1 \otimes \cdots \otimes v_n \rangle.
\]

Additivity and positivity are obvious.

**Definition 4.3.** The quaternionic full Fock module over an Hilbert space \( H \) is the space
\[
\mathcal{F}(H) = \bigoplus_{n=0}^{\infty} H \otimes^n,
\]
with the corresponding inner product.

**Definition 4.4.** Let \( u \in H \). The right-linear map \( T_u : \mathcal{F}(H) \to \mathcal{F}(H) \) defined by
\[
T_u(u_1 \otimes \cdots \otimes u_n) = u \otimes u_1 \otimes \cdots \otimes u_n,
\]
is called the creation map. The right-linear map \( T_u^* : \mathcal{F}(H) \to \mathcal{F}(H) \) defined by
\[
T_u^*(u_0 \otimes \cdots \otimes u_n) = \langle u, u_0 \rangle u_1 \otimes \cdots \otimes u_n,
\]
is called the annihilator map.

The following result is the quaternionic counterpart of a classical result:

**Proposition 4.5.** \( T_u^* \) is the adjoint of \( T_u \).

**Proof.** The statement follows from
\[
\langle T_u^*(u_0 \otimes \cdots \otimes u_n), v_1 \otimes \cdots \otimes v_n \rangle = \langle \langle u, u_0 \rangle u_1 \otimes \cdots \otimes u_n, v_1 \otimes \cdots \otimes v_n \rangle \\
= \langle \langle u_0, u \rangle u_1 \otimes \cdots \otimes u_n, v_1 \otimes \cdots \otimes v_n \rangle \\
= \langle u_0 \otimes \cdots \otimes u_n, u \otimes v_1 \otimes \cdots \otimes v_n \rangle \\
= \langle u_0 \otimes \cdots \otimes u_n, T_u(v_1 \otimes \cdots \otimes v_n) \rangle
\]

**Remark 4.6.** Note that the isometry \( u \mapsto T_u \) is both left-linear and right-linear.

The complex-valued version of the following proposition appears in [15, 16], where the free Brownian motion is defined and studied. The derivative of the function \( X(t) \) is studied in [10].
**Proposition 4.7** (The non-symmetric quaternionic Brownian motion). Let \( \mathcal{H} = L_2(\mathbb{R}^+, dx) \), and consider \( X(t) = T_1_{[0,t]} + T^*_1_{[0,t]} \). Then
\[
\langle X(t)1, X(s)1 \rangle = \min\{t,s\}.
\]
In particular \( X(t) \) is self-adjoint, and if one consider the expectation \( E : B(F(\mathcal{H})) \rightarrow \mathbb{H} \) defined by \( E(T) = \langle T1,1 \rangle \), then
\[
E(X(s)^*X(t)) = \min\{t,s\}.
\]

**Proof.** More generally, note that
\[
\langle (T_1 + T_u^*)1, (T_1 + T_u^*)1 \rangle = \langle T_11, T_11 \rangle = \langle u, u \rangle.
\]
Since \( \langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = \min\{t,s\} \), the result follows. \( \square \)

The symmetric product \( \circ \) is defined by
\[
u_1 \circ \cdots \circ u_n = \frac{1}{n!} \sum_{\sigma \in S_n} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)},
\]
and the closed subspace of \( \mathcal{H}^{\otimes n} \) generated by all vectors of this form is called the \( n \)-th symmetric power of \( \mathcal{H} \), and denoted by \( \mathcal{H}^{\circ n} \).

**Proposition 4.8.** Let \( \langle \cdot, \cdot \rangle \) be the inner product of \( \mathcal{H} \), and assume that it satisfies also the additional property
\[
\langle u, \lambda v \rangle = \langle \bar{\lambda}u, v \rangle.
\]
Then, it induces an inner product on \( \mathcal{H}^{\circ n} \),
\[
\langle u_1 \circ \cdots \circ u_n, v_1 \circ \cdots \circ v_n \rangle = \frac{1}{n!^2} \sum_{\sigma, \tau \in S_n} \langle \cdots \langle \langle \langle u_{\sigma(1)}, v_{\tau(1)} \rangle u_{\sigma(2)}, v_{\tau(2)} \rangle u_{\sigma(3)}, v_{\tau(3)} \rangle \cdots \rangle u_{\sigma(n)}, v_{\tau(n)} \rangle,
\]
with the same additional property.

**Proof.** The result follows as in the proof of Proposition 4.2. \( \square \)

In the classical case (where \( \mathcal{H} \) is a Hilbert space over the field \( \mathbb{R} \) or \( \mathbb{C} \)), another natural inner-product is usually being used, namely the symmetric inner product. It is defined by
\[
\langle u_1 \circ \cdots \circ u_n, v_1 \circ \cdots \circ v_n \rangle = \text{per} \left( \langle u_i, v_j \rangle \right),
\]
where \( \text{per}(A) \) is called the permanent of \( A \) and has the same definition as a determinant, with the exception that the factor \( sgn(\sigma) \) is omitted. An easy computation implies that when restricted to the \( n \)-fold symmetric tensor power \( \mathcal{H}^{\circ n} \), the second inner product (i.e. the symmetric inner product) is simply \( n! \) times the first inner product (the one which is defined in Proposition 4.8). This gives rise to the following definition.
Definition 4.9. Let \( \langle \cdot , \cdot \rangle \) be the inner product of \( \mathcal{H} \), and assume that it satisfies also the additional property
\[
\langle u, \lambda v \rangle = \langle \overline{\lambda} u, v \rangle.
\]
Then, the symmetric inner product on \( \mathcal{H}^{\oplus n} \) is defined by,
\[
\langle u_1 \circ \cdots \circ u_n, v_1 \circ \cdots \circ v_n \rangle = \frac{1}{n!} \sum_{\sigma, \tau \in S_n} \langle \cdots \langle \langle u_{\sigma(1)}, v_{\tau(1)} \rangle u_{\sigma(2)}, v_{\tau(2)} \rangle \cdots \rangle u_{\sigma(n)}, v_{\tau(n)} \rangle.
\]

We now focus on the special case of the symmetric Fock space \( \mathcal{F}^o(\mathcal{H}) \) where \( p \) is a quaternion variable and \( \mathcal{H} = p\mathbb{H} \). When no confusion can arise, we will simply denote it by \( \mathcal{F}^o(\mathbb{H}) \). The following result shows the relation with the Fock space as introduced in Definition 3.6, see Proposition 3.11.

Proposition 4.10. \( \mathcal{F}^o(\mathbb{H}) \) is the space of all entire functions
\[
\sum_{n=0}^{\infty} p^na_n
\]
satisfying \( \sum_{n=0}^{\infty} |a_n|^2n! < \infty \), under an identification of \( p^{\oplus n} \) with \( p^n \).

Proof. Clearly, any element in the \( n \)-th level \( \mathcal{H}^{\oplus n} \) can be written as \( p^{\oplus n}a \) for some \( a \in \mathbb{H} \), and
\[
\langle p^{\oplus n}, p^{\oplus n} \rangle = \frac{1}{n!} \sum_{\sigma, \tau \in S_n} \langle \cdots \langle \langle p, p \rangle p, p \rangle \cdots \rangle p, p \rangle = n!
\]

5. The slice monogenic case

In this section we recall just the definition and some properties of slice monogenic functions and we show how the results obtained in Section 2 can be reformulated in this case. We work with the real Clifford algebra \( \mathbb{R}_n \) over \( n \) imaginary units \( e_1, \ldots, e_n \) satisfying the relations \( e_ie_j + e_je_i = -2\delta_{ij} \). An element in the Clifford algebra \( \mathbb{R}_n \) is of the form \( \sum_A e_A x_A \) where \( A = i_1 \ldots i_r \), \( i_\ell \in \{1,2,\ldots,n\} \), \( i_1 < \ldots < i_r \) is a multi-index, \( e_A = e_{i_1}e_{i_2} \ldots e_{i_r} \), and \( e_\emptyset = 1 \). We set \( |A| = i_1 + \ldots + i_r \) and we call \( k \)-vectors the elements of the form \( \sum_{|A| = k} e_A x_A \), if \( k > 0 \). In the Clifford algebra \( \mathbb{R}_n \), we can identify some specific elements with the vectors in the Euclidean space \( \mathbb{R}^{n+1} \): an element \( (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \) will be identified with the element \( x = x_0 + x = x_0 + \sum_{j=1}^{n} x_j e_j \) called, in short, paravector. The norm of \( x \in \mathbb{R}^{n+1} \) is defined as \( |x|^2 = x_0^2 + x_1^2 + \ldots + x_n^2 \). The real part \( x_0 \) of \( x \) will be also denoted by \( \text{Re}(x) \). Using the above identification, a function \( f : U \subseteq \mathbb{R}^{n+1} \rightarrow \mathbb{R}_n \) is seen as a function \( f(x) \) of the paravector \( x \). We will denote by \( \mathcal{S} \) the \((n-1)\)-dimensional sphere of unit 1-vectors in \( \mathbb{R}^n \), i.e.
\[
\mathcal{S} = \{ e_1 x_1 + \ldots + e_n x_n : x_1^2 + \ldots + x_n^2 = 1 \}.
\]
Note that to any nonreal paravector $x = x_0 + e_1x_1 + \ldots + e_nx_n$ we can associate a $(n-1)$-dimensional sphere defined as the set, denoted by $[x]$, of elements of the form $x_0 + |e_1x_1 + \ldots + e_nx_n|$ when $I$ varies in $\mathbb{S}$.

As it is well known, for $n \geq 3$ the Clifford algebra $\mathbb{R}^n$ contains zero divisors. Thus, in general, the result which hold in the quaternionic setting do not necessarily hold in Clifford algebra. For this reasons, we quickly revise the definitions and results given for the quaternionic Fock space. We omit the proofs since, as the reader may easily check, the proofs given in the quaternionic case are valid also in this setting.

We begin by giving the definition of slice monogenic functions (see [21]).

**Definition 5.1.** Let $U \subseteq \mathbb{R}^{n+1}$ be an open set and let $f : U \to \mathbb{R}^n$ be a real differentiable function. Let $I \in \mathbb{S}$ and let $f_I$ be the restriction of $f$ to the complex plane $C_I$. We say that $f$ is a (left) slice monogenic function if for every $I \in \mathbb{S}$, we have
\[
\frac{1}{2} \left( \frac{\partial}{\partial u} + I \frac{\partial}{\partial v} \right) f_I(u + Iv) = 0,
\]
on $U \cap C_I$. The set of (left) slice monogenic functions on $U$ will be denoted by $\mathcal{SM}(U)$.

The slice monogenic Fock spaces and their properties are as follows.

**Definition 5.2** (Slice hyperholomorphic Clifford-Fock space). Let $I$ be any element in $\mathbb{S}$. Consider the set
\[
\mathcal{F}(\mathbb{R}^{n+1}) = \{ f \in \mathcal{SM}(\mathbb{R}^{n+1}) \mid \int_{C_I} e^{-|x|^2} |f_I(x)|^2 d\sigma(u, v) < \infty \}
\]
where $x = u + Iv$, $d\sigma(u, v) := \frac{1}{\pi} dudv$. We will call $\mathcal{F}(\mathbb{R}^{n+1})$ (slice hyperholomorphic) Clifford-Fock space.

We endow $\mathcal{F}(\mathbb{R}^{n+1})$ with the inner product (which does not depend on the choice of the imaginary unit $I \in \mathbb{S}$):
\[
\langle f, g \rangle := \int_{C_I} e^{-|x|^2} g_I(x)f_I(x)d\sigma(u, v).
\]

**Proposition 5.3.** The Clifford-Fock space $\mathcal{F}(\mathbb{R}^{n+1})$ contains the monomials $x^m$, $m \in \mathbb{N}$ which form an orthogonal basis, where $x$ is a paravector in $\mathbb{R}^{n+1}$.

Starting from the function $e^{zy} = \sum_{m=0}^{+\infty} \frac{z^m y^m}{m!}$, holomorphic in $z$ that can be interpreted as an element on a complex plane $C_I$ we can extend it to a slice monogenic function as
\[
\text{ext}(e^{zy}) = \frac{1}{2} (1 - I_x I) \sum_{m=0}^{+\infty} \frac{z^m y^m}{m!} + \frac{1}{2} (1 + I_x I) \sum_{m=0}^{+\infty} \frac{z^m y^m}{m!} = e^{xy}
\]
and since $y$ is arbitrary, we get the function we need. We now consider the function $k_y(x) := e^{xy}$ and we have the reproducing property in the Clifford-Fock space.
Theorem 5.4. For every \( f \in \mathcal{F}(\mathbb{R}^{n+1}) \) we have
\[
\langle f, k_x \rangle = f(x).
\]
Moreover, \( \langle k_x, k_y \rangle = c_4^{y|y|} \).

Proposition 5.5. A function \( f(x) = \sum_{m=0}^{+\infty} x^m a_m, \ a_m \in \mathbb{R}^n \) for \( m \in \mathbb{N} \),
belongs to \( \mathcal{F}(\mathbb{R}^{n+1}) \) if and only if \( \sum_{m=0}^{+\infty} |a_m|^2 m! < \infty \).

In the case of modules over \( \mathbb{R}_n \) the full Fock module is still under investigation.

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