Tucker tensor factor models: matricization and mode-wise PCA estimation

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Abstract

High-dimensional, higher-order tensor data are gaining prominence in a variety of fields, including but not limited to computer vision and network analysis. Tensor factor models, induced from noisy versions of tensor decompositions or factorizations, are natural potent instruments to study a collection of tensor-variate objects that may be dependent or independent. However, it is still in the early stage of developing statistical inferential theories for the estimation of various low-rank structures, which are customary to play the role of signals of tensor factor models. In this paper, we attempt to “decode” the estimation of a higher-order tensor factor model by leveraging tensor matricization. Specifically, we recast it into mode-wise traditional high-dimensional vector/fiber factor models, enabling the deployment of conventional principal components analysis (PCA) for estimation. Demonstrated by the Tucker tensor factor model (TuTFaM), which is induced from the noisy version of the widely-used Tucker decomposition, we summarize that estimations on signal components are essentially mode-wise PCA techniques, and the involvement of projection and iteration will enhance the signal-to-noise ratio to various extent. We establish the inferential theory of the proposed estimators, conduct rich simulation experiments, and illustrate how the proposed estimations can work in tensor reconstruction, and clustering for independent video and dependent economic datasets, respectively.

Keywords: Iterative projected estimation; matricization; principal components; tensor reconstruction; tensor subspace; Tucker decomposition; unsupervised learning
1 Introduction

With the advancements in data acquisition techniques, high-dimensional higher-order tensor data objects are becoming increasingly prevalent in various fields. Here higher-order is in the sense that the number of modes or ways of the tensor is no less than three (Kolda and Bader, 2009). For example, in the computer vision and medical imaging fields, 3D images and video sequences are naturally described as 3rd-order tensors, where each mode corresponds to different domains, such as spatial dimensions, color channels, and frames; in the network data analysis fields, transport networks and social networks can be formulated as tensor objects, where each mode indicates participating entities and types of relations (Bi et al., 2021; Chen et al., 2022; Lu et al., 2008). The use of higher-order tensor data objects enables a more comprehensive and expressive representation of complex data, allowing for capturing interactions and dependencies among multiple variables or dimensions. However, it comes into common view that our ability to acquire and analyze tensor data has surpassed our capability to develop fundamental mathematical and statistical theories that can justify and rationalize the efficacy of models and methodologies.

In this paper, we attempt to “decode” the estimation of higher-order tensor factor models and develop the corresponding inferential theories. Specifically, we aim to develop PCA-type estimations, which are the most widely-used techniques for tensor factor analysis with various variants for different low-rank structures (Zare et al., 2018). We start from a Tucker tensor factor model (TuTFaM), which is a noise version of the widely-used Tucker decomposition (Tucker, 1966), i.e., the low-rank signal is in the form of a product of a low-dimensional core tensor (representing latent tensor factors) and factor loading matrices across all modes. In detail, let $X_t \in \mathbb{R}^{p_1 \times \cdots \times p_D}$, where $t \in \{1, \cdots, T\}$, denote the $D$th-order tensor observations that may be weakly correlated or uncorrelated. Then TuTFaM is expressed as

$$X_t = S_t + E_t,$$

where $S_t$ is the unknown low-rank signal part, $E_t \in \mathbb{R}^{p_1 \times \cdots \times p_D}$ is the noise tensor, $F_t \in \mathbb{R}^{k_1 \times \cdots \times k_D}$ is the random low-dimensional ($k_d \ll p_d$) latent tensor factor, $A_d \in \mathbb{R}^{p_d \times k_d}$ is the unknown and deterministic mode-$d$ loading matrix for $d \in [D]$, and $\times_d$ is the tensor $d$-mode (matrix) product. In this context, high dimensionality implies that the sample or time points size $T$ and the size of each mode $p_d$ can be significantly large. Moreover, it is
possible for the mode-wise size $p_d$ to surpass the value of $T$. In addition, model (1) reduces to a bilinear matrix factor model when $D = 2$ \cite{Wang2019, ChenFan2023, Yu2022}, and a vector factor model when $D = 1$ \cite{Forni2000, BaiNg2002, Bai2003, Lametal2011, LamYao2012}.

\section{1.1 Motivation: matricization and mode-wise PCA estimation}

The proposed estimation procedures are motivated by tensor matricization (also called unfolding or flattening), that is, the elements of an observed tensor object $X_t \in \mathbb{R}^{p_1 \times \cdots \times p_D}$ is reordered into a matrix along each mode $d \in [D]$, where the mode-$d$ fibers are arranged as the columns of the resulting matrix $X_t^{(d)} \in \mathbb{R}^{p_d \times p_{-d}}$ with $p_{-d} = \prod_{i \neq d}^{D} p_i$, and hence the $(i_d, j_d)$th element of $X_t^{(d)}$ is the $(i_1, \cdots, i_D)$th element of $X_t$ satisfying $j_d = 1 + \sum_{d' \neq d}^{D} (i_{d'} - 1) \prod_{m \neq d}^{D} p_m$. The corresponding latent tensor factor is flattened as $F_t^{(d)} \in \mathbb{R}^{k_d \times k_{-d}}$ with $k_{-d} = \prod_{i \neq d}^{D} k_i$. Therefore, one has the equivalent matricized representation of model (1) as

$$X_t^{(d)} = A_d F_t^{(d)} A_{[D]/\{d\}}^\top + \varepsilon_t^{(d)}, \quad d \in [D], \quad (2)$$

where $A_{[D]/\{d\}} := A_D \otimes \cdots \otimes A_{d+1} \otimes A_{d-1} \otimes \cdots \otimes A_1$ with $\otimes$ being the Kronecker product and $\varepsilon_t^{(d)} \in \mathbb{R}^{p_d \times p_{-d}}$ is the mode-$d$ matricization of the noise tensor $\varepsilon_t$.

Let $X^{(d)} = (X_1^{(d)}, \cdots, X_T^{(d)}) \in \mathbb{R}^{p_d \times (Tp_{-d})}$ be the mode-$d$ design matrix with dimension $p_d$ and sample size $Tp_{-d}$, and $E^{(d)} = (\varepsilon_1^{(d)}, \cdots, \varepsilon_T^{(d)}) \in \mathbb{R}^{p_d \times (Tp_{-d})}$ be the noise matrix. Then $Z^{(d)} = (Z_1^{(d)}, \cdots, Z_T^{(d)}) \in \mathbb{R}^{k_d \times (Tp_{-d})}$ can be regarded as the latent factor matrix of $X_t^{(d)}$, where $Z_t^{(d)} = F_t^{(d)} A_{[D]/\{d\}}^\top$. Hence, based on tensor observations $\{X_t\}_{t=1}^{T}$ of interest, model (2) can be reformulated in matrix form as

$$X^{(d)} = A_d Z^{(d)} + E^{(d)}, \quad d \in [D]. \quad (3)$$

This model is essential the matrix form of a $p_d$-dim \textit{vector factor model} with sample size $Tp_{-d}$ \cite{Bai2003} in that each column of $X^{(d)}$ is the mode-$d$ fiber $X_{t,i_1 \cdots i_{d-1} : i_d+1 \cdots i_D}$ with the $j$th entry being $x_{t,i_1 \cdots i_{d-1} j i_{d+1} \cdots i_D}$ for $j \in [p_d]$. Then it is natural to perform the principal components estimation stated in \cite{Bai2003} (also called the spectral method in \cite{ChenFan2023, Fan2023}) for the mode-wise fiber factor model (1.3). That is, if the sample variance-covariance matrix (2nd sample moment of $X^{(d)}$) is denoted by

$$\hat{M}_d = \frac{1}{Tp} X^{(d)}(X^{(d)})^\top = \frac{1}{Tp} \sum_{t=1}^{T} X_t^{(d)}(X_t^{(d)})^\top, \quad (4)$$
then one obtains the estimator of the deterministic loadings \( A_d \) as follows

\[
\hat{A}_d = \sqrt{p_d} \ \text{eig}(\hat{M}_d, k_d), \quad d \in [D],
\]

(5)

where \( p = \prod_{d=1}^D p_d \) and \( \text{eig}(U, r) \in \mathbb{R}^{n \times r} \) is the matrix with the top \( r \) eigenvectors of any non-negative definite placeholder matrix \( U \in \mathbb{R}^{n \times n} \) as columns. Next, one can obtain the estimated latent tensor factor and signal part, respectively

\[
\hat{F}_t = \frac{1}{p} X_t \times_1 \hat{A}_1^\top \times_2 \cdots \times_D \hat{A}_D^\top \quad \text{and} \quad \hat{S}_t = \hat{F}_t \times_1 \hat{A}_1 \times_2 \cdots \times_D \hat{A}_D.
\]

(6)

Such an estimation procedure that is derived from equations (2) to (6) is named mode-wise PCA (moPCA) since one derives estimators of loading matrices \( A_d \) mode by mode through flattening the observed tensor objects. Chen et al. (2022)’s TOPUP and TIPUP estimation approaches for tensor time series are consistent with the mode-wise spirit of moPCA; when \( D \) reduces to 2, for the existing works on the bilinear form matrix factor models, their estimation procedures are along row (mode-1) and column (mode-2) respectively, say, the \( \alpha \)-PCA in Chen and Fan (2023) and the initial estimation in Yu et al. (2022) for dependent or independent observations, and the estimation in Wang et al. (2019) for matrix time series.

1.2 Related works

Two branches of PCA estimations PCA estimation is the most commonly used method for high-dimensional factor analysis (Chen et al., 2020). It is implemented by the truncated eigen-decomposition or singular value decomposition on some second-moment-type statistics such as the matrix \( \hat{M}_d \) (Chen and Fan, 2023, Remark 2). Existing works apply PCA either on the constructed auto-covariance matrix for time series data, such as Chen et al. (2022), Han et al. (2020), Han et al. (2022) (tensor time series), Wang et al. (2019), Chen et al. (2020) (matrix time series), and Lam et al. (2011), Lam and Yao (2012) among others (vector time series); or on the constructed contemporaneous variance-covariance (CVC) matrix for weakly correlated or even independent \( D \)th-order tensor observations, such as Chen and Fan (2023), Yu et al. (2022) (matrix factor models) and Bai and Ng (2002), Bai (2003) among others (vector factor models); there are also several works that combine both auto-covariance and CVC, such as dynamic (Forni et al. 2015, 2017) and static (Zhang et al. 2023) vector factor models for vector time series. Note that, CVC matrix not only incorporates contemporaneous information but also includes time-lagged correlation.
information when stationary dynamic factors are accommodated, though it may not explicitly capture the auto-correlation information of observations. It inspires us to construct CVC-based PCA estimations to fill out the gap for estimation of low-rank structure for uncorrelated or weakly correlated higher-order tensor-variate collection.

**Projection and Iteration** By adhering to the principle of mode-wise thought, moPCA estimation can be further enhanced based on variants of the matricized model (2). On the one hand, the projection technique (Lu et al., 2008; Wang et al., 2019) can be implemented on the raw tensors driven by the practices in modern machine learning and data science to enhance the signal-to-noise ratio and convergence rate. This is evidenced by the projected estimation in Yu et al. (2022) that extends the mode-wise estimation in Chen and Fan (2023) for 2nd-order tensor (matrix) observations. On the other hand, the iteration technique (Zhang and Xia, 2018; Yuan and Zhang, 2016) can be applied to further enhance the signal-to-noise ratio and convergence rate. This is evidenced by the iterative projected estimations in Han et al. (2020) that extend the mode-wise estimation in Chen et al. (2022) for tensor time series. These results motivate us to develop the projected mode-wise PCA estimation and the iterative projected mode-wise PCA estimation for weakly correlated or uncorrelated tensor observations in Sections 2 and 3, respectively.

**Other works** CP decomposition is another widely used tensor decomposition method, the induced CP tensor factor model has been studied by Han et al. (2021) along the line of auto-covariance type estimation. Besides tensor factor models, there are limited statistical inference for higher-order tensors on topics such as tensor completion (Yuan and Zhang, 2016, 2017; Zhang, 2019; Xia and Yuan, 2019; Xia et al., 2021 among others), tensor decomposition (Sun et al., 2017; Tang et al., 2020; Zhang and Xia, 2018; Zhang and Han, 2019; Zhou et al., 2022 among others), supervised learning (Zhou et al., 2013; Johndrow et al., 2017; Chen et al., 2019; Zhou et al., 2023; Li and Zhang, 2017 among others), and etc.

**1.3 Organization**

The remainder of this article is organized as follows. Section 2 develops a set of tensor PCA that involves the projection technique, called PmoPCA, and elucidates the rationale of matricization and mode-wise thought from the perspective of tensor subspace learning and optimization of tensor decomposition. Section 3 develops the other set of tensor PCA techniques that involve both iteration and projection for the purpose of better signal-to-noise
ratio and convergence rate. Section 4 establishes asymptotic properties of the proposed estimators including convergence rates and asymptotic distributions. Section 5 evaluates the finite sample performance of the proposed estimations. Section 6 analyzes a moving MNIST video dataset and an international import-export transport network dataset, which contains temporally uncorrelated and correlated tensor observations, respectively. Section 7 concludes the paper with a short summary. All the technique proofs are left in the Supplementary.

2 Projected mode-wise PCA estimation and tensor subspace learning

The tensor factor model TuTFaM, like vector or matrix factor models (Lam et al., 2011; Wang et al., 2019; Chen and Fan, 2023), suffers the identification problem. That is, two sets of \((D+1)\)-tuples \(\{A_1, \ldots, A_D, F_t\}\) and \(\{A_1 H_1, \ldots, A_D H_D, F_t \times_1 H_1^{-1} \times_2 \cdots \times_D H_D^{-1}\}\) lead to the identical model, for any nonsingular matrices \(\{H_d \in \mathbb{R}^{k_d \times k_d} : d \in [D]\}\). However, the mode-wise factor loading space \(\mathcal{M}(A_d)\), which is spanned by the columns of \(A_d\), is uniquely defined in the sense that \(\mathcal{M}(A_d)\) and \(\mathcal{M}(A_d H_d)\) are identical for \(d \in [D]\). Note that, the lack of uniqueness of \(\{A_d\}_{d=1}^D\) brings the advantage of flexible rotation of an estimated factor loading matrix. To facilitate our estimation, we assume \(p_d^{-1} A_d^T A_d\) approximates to the identity matrix \(I_{k_d}\) of size \(k_d\) when \(p_d\) is quite large (Assumption 3). This restriction confines the invertible transformation matrix to the orthogonal one, which aids in simplifying the structure of the estimated loading matrix. Once such \(\{A_d\}_{d=1}^D\) is specified, the induced latent tensor factor \(F_t\) is determined. Therefore, the \(D+1\) tuple \(\{A_1, \ldots, A_D, F_t\}\) in TuTFaM is estimable. Hence, for our proposed estimations and inferential theories, we will focus on the estimation of mode-wise factor loading spaces \(\mathcal{M}(A_d)\) under the orthogonal rotation restriction.

2.1 Projected mode-wise PCA estimation

The framework of moPCA estimation in the Subsection 1.1 is based on purely marginal mode information. Nevertheless, it is more informative to estimate each loading by integrating data information from all modes so that the induced loading estimators can incorporate joint influence among all modes and the signal of the low-rank approximation
can be strengthened. This insight inspires us to give a more delicate consideration to the matricized factor model (2).

In the estimation procedure of moPCA, we treat $F_t^{(d)}$ and $A_{[D]/\{d\}}$ together and notate their product as $X_t^{(d)}$, which acts as the latent factor in the fiber factor model (3). In essence, this is equivalent to replace each unknown $A_{d'}$ ($d' \in [D]/\{d\}$) with an identity matrix in the slice matrix factor model (2). Such treatment is intuitive and expedient but at the cost of losing all interrelation information among modes except mode-$d$. Instead, one may want to incorporate the interaction influence of non-$d$ modes. We address the problem by projecting $X_t^{(d)}$ into non-$d$ modes hyperspace, that is, projecting the rows of $X_t^{(d)}$ onto the column space of $A_{[D]/\{d\}}$. This is implemented by multiplying $A_{[D]/\{d\}}$ from right on both sides of equation (2), yielding the following matricized model

$$Y_{t,d}^{(d)} := \frac{1}{p-d}X_t^{(d)} A_{[D]/\{d\}} \approx A_d F_t^{(d)} + \tilde{E}_t^{(d)},$$

where $\tilde{E}_t^{(d)} = \mathcal{E}_t^{(d)} A_{[D]/\{d\}}/p-d$ is the noise matrix, and the approximation is induced from the identification conditions. This new model for a fixed tensor observation incorporates the parameters $A_D, \cdots A_{d+1}, A_{d-1} \cdots A_1$ explicitly, and plays a role in the induced sample variance-covariance matrix later, impacting the estimation of $A_d$.

Let $Y^{(d)} = (Y_{1,d}^{(d)}, \cdots, Y_{T,d}^{(d)}) \in \mathbb{R}^{p_d \times (Tk_{-d})}$ be the observation of sample size $Tk_{-d}$ and dimension $p_d$, and $F^{(d)} = (F_1^{(d)}, \cdots, F_T^{(d)}) \in \mathbb{R}^{k_d \times (Tk_{-d})}$ be the latent factor matrix. By concatenating $Y_{t,d}^{(d)}$ for $t \in [T]$, we have the following vector factor model

$$Y^{(d)} \approx A_d F^{(d)} + \tilde{E}^{(d)},$$

where $\tilde{E}^{(d)} = (\mathcal{E}_1^{(d)}, \cdots, \mathcal{E}_T^{(d)}) \in \mathbb{R}^{p_d \times (Tk_{-d})}$ is the noise matrix. This new projected model is again a $p_d$-dim vector factor model while with sample size $Tk_{-d}$ and each column of $Y^{(d)}$ is a projected fiber,

$$\sum_{i_1, \cdots, i_{d-1}, i_{d+1}, \cdots, i_D} X_{t,i_1,\cdots,i_{d-1},i_{d+1} \cdots, i_D} a_1 a_{i_1,i_1} \cdots a_{d-1,i_{d-1},i_{d-1}} a_{d+1,i_{d+1},i_{d+1}} \cdots a_{D,i_D,i_D} ;$$

which is more complicated than the column fiber vector in the vector factor model (3) and apparently contains the information of non-$d$ modes.

Again we perform the method of principal components for the mode-wise projected-fiber vector factor model (8). However, we can not work on $(T^{p_d})^{-1} \sum_{t=1}^{T} Y_{t,d}^{(d)} Y_{t,d}^{(d)\top}$ directly like on equation (4) because $Y_{t,d}^{(d)}$ in representation (7) contains unknown matrix parameters $A_D, \cdots A_{d+1}, A_{d-1} \cdots A_1$. Therefore we consider substituting the unknown $Y_{t,d}^{(d)}$ by a rea-
sonable estimator below,

\[ \hat{Y}_{t,d}^{(d)} := \frac{1}{p-d} \mathcal{A}_t^{(d)} \breve{\mathbf{A}}_{(D)/(d)}, \]  

(9)

where \( \breve{\mathbf{A}}_{(D)/(d)} = \breve{\mathbf{A}}_D \otimes \cdots \breve{\mathbf{A}}_{d+1} \otimes \breve{\mathbf{A}}_{d-1} \otimes \cdots \breve{\mathbf{A}}_1 \) and \{\breve{\mathbf{A}}_d\}'s are some estimators that behave as the projected matrices. The moPCA estimators \{\breve{\mathbf{A}}_d\}'s in (5) have the natural priority in playing the role of \{\breve{\mathbf{A}}_d\}'s in equation (9). Consequently, based on the following sample variance-covariance matrix

\[ \breve{\mathbf{M}}_d = \frac{1}{TP_d} \sum_{t=1}^{T} \breve{Y}_{t,d}^{(d)} \breve{Y}_{t,d}^{(d)\top}, \quad d \in [D], \]  

(10)

by the method of principal components (spectral method), we obtain estimators of loading matrices, and the resulting estimated latent factors and signal parts as

\[ \breve{\mathbf{A}}_d = \sqrt{p_d} \text{eig}(\breve{\mathbf{M}}_d, k_d), \quad d \in [D], \]

\[ \breve{\mathbf{F}}_t = \frac{1}{p} \mathcal{A}_t \times_1 \breve{\mathbf{A}}_1 \times_2 \cdots \times_D \breve{\mathbf{A}}_D^\top, \quad t \in [T], \]  

(11)

\[ \breve{\mathbf{S}}_t = \breve{\mathbf{F}}_t \times_1 \breve{\mathbf{A}}_1 \times_2 \cdots \times_D \breve{\mathbf{A}}_D, \quad t \in [T]. \]

Such an estimation procedure is named projected mode-wise PCA (PmoPCA) since it implements projection on the mode-wise matricized factor model. Such projection technique is widely used in tensor/matrix factor model estimations, such as Han et al. (2020)’s iTOPUP and iTIPUP estimation approaches for tensor time series, and Yu et al. (2022)’s projected estimation for matrix observations (\( D = 2 \)).

PmoPCA is superior to moPCA in terms of signal-to-noise ratio under TuTFaM in the sense that the order of the smallest nonzero eigenvalues of \( \breve{\mathbf{M}}_d \) in equation (10) becomes higher than the zero ones compared to \( \mathbf{M}_d \) in equation (4); refer to Proposition 1 and Corollary 1 in the Supplementary.

2.2 Tensor subspace learning

In this subsection, we demonstrate the philosophy of matricization and mode-wise thought from the perspective of tensor subspace learning and optimization on decomposition models.

Tensor subspace learning is one popular way for feature extraction to conduct tasks such as recognition, classification, and reconstruction in SIAM and data science (He et al., 2005; Lu et al., 2008). For the trivial low-rank approximation and specific Tucker decomposition, the latter of which is used to represent the signal part in model (1), one may denote their
tensor subspaces

\[ S_{\text{Trivial}} = \left\{ \sum_{j=1}^{r} f_j A_j | f = (f_1, \cdots, f_r)^T \in \mathbb{R}^r \right\}, \]

\[ S_{\text{Tucker}} = \left\{ \sum_{i_1=1}^{k_1} \cdots \sum_{i_D=1}^{k_D} f_{i_1 \cdots i_D} (A_{1,i_1} \cdots A_{D,i_D}) | F \in \mathbb{R}^{k_1 \times \cdots \times k_D} \right\}, \]

respectively, where the former is spanned by the tensor basis or eigenfaces \( \{ A_j \in \mathbb{R}^{p_1 \times \cdots \times p_D} : j \in [r] \} \) with entries of \( f \) being factor coefficients, and the latter is spanned by the rank-one tensor basis or eigenfaces \( \{ A_{1,i_1} \cdots A_{D,i_D} \in \mathbb{R}^{p_1 \times \cdots \times p_D} : i_d \in [k_d], d \in [D] \} \) with entries of \( F = (f_{i_1 \cdots i_D})_{k_1 \times \cdots \times k_D} \) being factor coefficients, \( A_{d,i_d} \in \mathbb{R}^{p_d} \) being the \( i_d \)th column of \( A_d \), and \( \circ \) being the outer product. Compared to eigenfaces \( A_j \) of \( S_{\text{Trivial}} \), eigenfaces of \( S_{\text{Tucker}} \) have the specific mode-wise outer product forms, which extract more structural information from the tensor data. Tensor subspace is relatively low-dimensional and yet preserves sufficient variability in an input tensor object. In a geometrical perspective, one may look on the tensor subspace analysis as projecting the raw high-dimensional tensor \( X_t \) onto the low-dimensional tensor subspace; and coordinates of the tensor basis once estimated, act as extracted latent features. Estimation of the tensor subspace is achieved by minimizing the following squared error with respect to the tensor basis and coefficients

\[
\min_{\{A_j\}_{j=1}^{r}} \sum_{t=1}^{T} \left\| X_t - \sum_{j=1}^{r} f_{t,j} A_j \right\|_F^2 = \min_{A,F} \| X - AF^\top \|_F^2, \tag{12}
\]

where \( X = (\text{vec}(X_1), \cdots, \text{vec}(X_T)) \in \mathbb{R}^{p \times T} \), \( \hat{A} = (\text{vec}(A_1), \cdots, \text{vec}(A_r)) \in \mathbb{R}^{p \times r} \), \( \hat{F} = (f_1, \cdots, f_T)^\top \in \mathbb{R}^{T \times r} \), \( A = A_D \otimes \cdots \otimes A_1 \in \mathbb{R}^{p \times k} \), \( F = (\text{vec}(F_1), \cdots, \text{vec}(F_T))^\top \in \mathbb{R}^{T \times k} \) with \( k = \Pi_{d=1}^{D} k_d \), and \( \| \cdot \|_F \) and \( \text{vec}(\cdot) \) are the Frobenius norm and the vectorization operator of a matrix or tensor, respectively.

For the upper problem of expressions (12) (trivial case with simple eigenfaces \( A_j \)), solving \( \min_{A,F} \| X - AF^\top \|_F^2 \) with the corresponding vectorized tensor factor model \( \text{vec}(X_t) = (\text{vec}(A_1), \cdots, \text{vec}(A_r)) f_t + \text{vec}(\epsilon_t) \) for \( t \in [T] \) is the same to perform the principal components method for the vector factor model (3) in the seminal work of Bai (2003) up to a normalized constant. Therefore the trivial (vectorized) PCA estimation for the subspace learning analysis of the left side is to conduct the truncated eigen-decomposition on the sample variance-covariance matrix \( XX^\top = \sum_{t=1}^{T} \text{vec}(X_t)\text{vec}(X_t)^\top \).
For the lower problem of expressions (12) (Tucker case with complicated rank-one eigenfaces), however, solving \( \min_{A, F} \| X - A F^\top \|_F^2 \) with the corresponding vectorized model \( \text{vec}(X_t) = (A_D \otimes \cdots \otimes A_1) \text{vec}(F_t) + \text{vec}(\xi_t) \) by conducting trivial PCA can NOT recover the matrices \( \{ A_d \}_{d=1}^D \) from the estimator of \( A = A_D \otimes \cdots \otimes A_1 \), and its direct minimization is NP-hard (Hillar and Lim, 2013). Hence the PCA-type methods specifically for Tucker subspace learning should be re-investigated. Note that the left side minimization can be transferred to the maximization of \( \sum_{t=1}^T \| \hat{X}_t \times_1 A_1^\top \cdots \times_D A_D^\top F \|_F^2 \) (this sum of squared norm is named total tensor scatter in Lu et al. (2008)) or \( \text{tr}\{ A_D^\top \sum_{t=1}^T \hat{X}_t^{(d)} \hat{X}_t^{(d)\top} A_d \} \) (Petersen and Pedersen, 2012 equation (541)) with respect to \( \{ A_d \}_{d=1}^D \), where \( \hat{X}_t^{(d)} = X_t^{(d)} A_{[D]/\{d\}} \) is the projection of \( X_t^{(d)} \) onto the column space of \( A_{[D]/\{d\}} \) (Kolda and Bader, 2009 Subsection 4.2). Then, a tensor PCA is doable by performing truncated eigen-decomposition on the mode-wise sample variance-covariance matrix \( \sum_{t=1}^T \hat{X}_t^{(d)} \hat{X}_t^{(d)\top} \) over \( [D] \) iteratively given the estimation of other modes, which is a tractable approximate solution for the original optimization problem. Based on different choices of the estimation of \( A_{[D]/\{d\}} \), the mode-wise PCA has many variants, say, moPCA, PmoPCA, the estimation in Barigozzia et al. (2022), Chen and Lam (2024), and the estimation in the next section.

\section{Iterative projected mode-wise PCA estimation}

Note that, in PmoPCA, the loading matrices for data projection are neither updated in each projection (step) over \( [D] \) nor iterated in equation (9). Involving updating and iteration will augment the signal-to-noise ratio and accelerate convergence rates of estimators; refer to Proposition 6 and Corollary 1 in the Supplementary. This drives us to do updating in each projection step across modes iteratively, yielding the \((s + 1)\)st iteration in the projection-updated estimator of \( \gamma_{t,d}^{(d,s+1)} \)

\begin{equation}
\gamma_{t,d}^{(d,s+1)} := \frac{1}{p_{-d}} \hat{X}_t^{(d)} \hat{A}_{[D]/\{d\}}^{(s+1)},
\end{equation}

where \( \hat{A}_{[D]/\{d\}}^{(s+1)} = (\hat{A}_D^{(s)} \otimes \cdots \otimes \hat{A}_{d+1}^{(s)} \otimes \hat{A}_{d-1}^{(s+1)} \otimes \cdots \otimes \hat{A}_1^{(s+1)}) \) and \( \hat{A}_{d'}^{(s)} \) is the \( s \)th iteration estimator for \( d' \in [D]/\{d\} \). Again the moPCA estimators \( \{ \hat{A}_{d'}^{(0)} \} \) are good candidates for the initial estimators \( \{ \hat{A}_{d'}^{(0)} \} \). The induced mode-wise sample variance-covariance matrix is

\begin{equation}
\hat{M}_d^{(s+1)} = \frac{1}{Tp_d} \sum_{t=1}^T \gamma_{t,d}^{(d,s+1)} \gamma_{t,d}^{(d,s+1)\top}, \ d \in [D].
\end{equation}
By the method of principal components, the estimators of loading matrices, and the resulting latent factors and signal parts are obtained as

\[
\begin{align*}
\widehat{A}_d^{(s+1)} &= \sqrt{p_d} \text{eig}(\widehat{M}_d^{(s+1)}, k_d), \quad d \in [D], \\
\widehat{F}_t^{(s+1)} &= \frac{1}{p} \mathcal{X}_t \times_1 \widehat{A}_1^{(s+1)^\top} \times_2 \cdots \times_D \widehat{A}_D^{(s+1)^\top}, \quad t \in [T], \\
\widehat{S}_t^{(s+1)} &= \widehat{F}_t^{(s+1)} \times_1 \widehat{A}_1^{(s+1)} \times_2 \cdots \times_D \widehat{A}_D^{(s+1)}, \quad t \in [T].
\end{align*}
\] (15)

Such an estimation procedure is named iterative projected mode-wise PCA (IPmoPCA) since it implements iteration and projection simultaneously. Note that we still use the notations for moPCA in representing IPmoPCA estimators hereafter since the proof is under the moPCA-initial setting. The IPmoPCA estimation is summarized in Algorithm 1.

**Algorithm 1** IPmoPCA

1: **Input**: Tensor observations \( \{\mathcal{X}_t\}_{t=1}^T \), factor numbers \( \{k_d\}_{d=1}^D \), the error tolerance \( \epsilon \), the maximum number of iterations \( m \), the initial estimators \( \{\widehat{A}_d^{(0)}\}_{d=1}^D \)

2: **Output**: Estimation \( \{\widehat{A}_d^{(s+1)}\}_{d=1}^D \), \( \{\widehat{F}_t^{(s+1)}\}_{t=1}^T \), \( \{\widehat{S}_t^{(s+1)}\}_{t=1}^T \)

3: for \( s \) in 0 : \( m \) do

4: for \( d \) in 1 : \( D \) do

5: compute \( \widehat{\gamma}_{t,d}^{(d+1)} \) in equation (13) for \( t \in [T] \), \( \widehat{M}_d^{(s+1)} \) in equation (14), \( \widehat{A}_d^{(s+1)} \) in equation (15)

6: end for

7: Break the for loop, if \( \max_{d\in[D]} \left\{ p_d^{-1} \| \widehat{A}_d^{(s+1)^\top} \widehat{A}_d^{(s+1)^\top} - \widehat{A}_d^{(s)^\top} \widehat{A}_d^{(s)^\top} \| \right\} \leq \epsilon \)

8: end for

9: for \( t \) in 1 : \( T \) do

10: compute \( \widehat{F}_t^{(s+1)} \) and \( \widehat{S}_t^{(s+1)} \) in equation (15)

11: end for

12: return \( \{\widehat{A}_d^{(s+1)}\}_{d=1}^D \), \( \{\widehat{F}_t^{(s+1)}\}_{t=1}^T \), \( \{\widehat{S}_t^{(s+1)}\}_{t=1}^T \)

**Remark 1.** (Comparison with HOOI) IPmoPCA estimation can be regarded as a special higher-order orthogonal iteration (HOOI) \([\text{De Lathauwer et al., 2000}]\) by taking the observations as a \((D + 1)\)-th order tensor with mode-(\(d + 1\)) being time points. Model (1) can be rewritten as follows

\[
\mathcal{Z} = \mathcal{G} \times_1 \mathcal{A}_1 \times_2 \cdots \times_D \mathcal{A}_D + \mathcal{R},
\]

where \( \mathcal{Z} \in \mathbb{R}^{p_1 \times \cdots \times p_D \times T} \) is the \((D + 1)\)th-order tensor, the signal part of which has been
expressed as the Tucker decomposition formation \cite{Tucker1966, Kolda2009} with the \((D+1)\)th loading matrix as the \(T\)-dim identity matrix. Then \(Z_{t} = X_{t}, G_{t} = F_{t}, \) and \(R_{t} = E_{t}, \) for \(t \in [T]\). If one adopts the classical Tucker decomposition algorithm HOOI, then only the first \(D\) loading matrices need to be calculated. More specifically, for \(d \in [D]\), when estimating \(A_{d}\), eigen-decomposition is conducted on \(Q_{d}^{(s+1)} \otimes A_{d}^{(s+1)}\), which has the same column space as \(\hat{M}_{d}^{(s+1)}\) in equation (14), where \(Q_{d}^{(s+1)} = Z^{(d)} \left( I_{T} \otimes \hat{A}_{d}^{(s+1)} \right)\).

**Remark 2.** (Comparison with MPCA and MOP-UP) IPmoPCA has the same algorithmic form as multilinear principal component analysis (MPCA) \cite{Lu2008} except the difference in the stopping rule and up to some normalization constants. MPCA is essentially a tensor object dimension reduction framework with the Tucker low-rank structure, where their mode-wise loading matrices \(\{A_{d}\}\) are directly obtained by maximizing the so-called total tensor scatter \(\sum_{t=1}^{T} \|X_{t} \times A_{1}^{T} \times \cdots \times A_{D}^{T}\|_{F}^{2}\) \cite{Lu2008, equation (4)} (written in our notations). This is exactly the transferred maximization problem of the lower minimization problem in our expressions (12). Then MPCA is equivalent to performing the mode-wise PCA on the iterated-projected estimator \(\hat{Y}_{t,d}^{(d,s+1)}\) in equation (13) up to the constant \(p_{-d}\), and the stopping rule of the surrounding for loop is based on the distance of total tensor scatters. Similar to IPmoPCA and MPCA, a recent mode-wise principal subspace pursuit (MOP-UP) framework \cite{Tang2023}, though with a different additive mode-wise linear forms low-rank structure, is actually the mode-wise PCA on the iterated-projected estimator like \(\hat{Y}_{t,d}^{(d,s+1)}\) up to the constant \(p_{-d}\), except that the projection matrices are replaced by \(\hat{A}_{d}^{(s+1)}_{\{D\}/\{d\},\perp}\), i.e. the orthogonal complement of \(\hat{A}_{d}^{(s+1)}_{\{D\}/\{d\}}\).

**Remark 3.** (Factor numbers) When we develop the mode-wise PCA estimations, we assume the factor numbers \(\{k_{d}\}_{d=1}^{D}\) are fixed. But when they are unknown, we need to determine the number of factors before the estimation. Here, take the moPCA as an example, we use the general ratio-based estimators as follows

\[
\hat{k}_{d} = \arg \max_{1 \leq j \leq k_{\text{max}}} \frac{\hat{\lambda}_{j}(\hat{M}_{d})}{\hat{\lambda}_{j+1}(\hat{M}_{d})}, \quad d \in [D],
\]

where \(\hat{\lambda}_{1}(\hat{M}_{d}) \geq \hat{\lambda}_{2}(\hat{M}_{d}) \geq \cdots \geq \hat{\lambda}_{p_{d}}(\hat{M}_{d}) \geq 0\) are the ordered eigenvalues of \(\hat{M}_{d}\) and \(k_{\text{max}}\) is a preassigned upper bound. Under some assumptions, the order of the top \(k_{d}\) eigenvalues of \(\hat{M}_{d}\) is higher than the last \(p_{d} - k_{d}\) ones. Thus the above estimator can provide a consistent estimator for \(k_{d}\), which is proved in the Supplementary. Note that the similar ratio-based estimators obtained based on \(\hat{M}_{d}\) and \(\hat{M}_{d}^{(s+1)}\) also have the consistency property.
4 Theoretical results

In this section, we develop the theoretical properties of the proposed methods. In Subsection 4.1, we develop the necessary assumptions. In Subsection 4.2, we first establish the convergence rates and the asymptotic distributions of estimated loading matrices, and the convergence rates of the estimated latent factors and signal parts of PmoPCA in Theorems 4.1-4.3; next, we establish the convergence rates of estimated loading matrices of IPmoPCA in Theorem 4.4.

4.1 Assumptions

Our assumptions for inference of TuTFaM can be regarded as higher-order extensions of those used in the large-dimensional vector factor model (Bai, 2003) and in the high-dimensional matrix factor model (Chen and Fan, 2023; Yu et al., 2022), respectively.

**Assumption 1.** \(\alpha\)-mixing. The vectorized factor process \(\{\text{vec}(F_t)\}\) and noise process \(\{\text{vec}(E_t)\}\) are \(\alpha\)-mixing. A vector process \(\{u_t\}\) is said to be \(\alpha\)-mixing, if
\[
P \left( \bigcap_{h=1}^{\infty} \alpha(h) \frac{1}{\gamma} < \infty \right)
\]
for some \(\gamma > 2\), where the mixing coefficient \(\alpha(h)\) represents \(\sup_A \sup_{B} \left| P(A \cap B) - P(A)P(B) \right|\) with \(C_i^j\) as the \(\sigma\)-field generated by \(\{u_t : i \leq t \leq j\}\). \(\alpha\)-mixing is more general than stationarity as a probabilistic tool characterizing variable dependence. It states that the degree of dependence decreases as the distance between two given sets of random variables goes to infinity. Assumption 1 is a regular condition for deriving asymptotic distribution.

**Assumption 2.** Tensor factor. For any \(t \in [T]\), assume tensor factor \(F_t\) is of fixed dimension \(k_1 \times \cdots \times k_D\) with \(\mathbb{E}(F_t) = 0\) and \(\mathbb{E}\|F_t\|_F^2 \leq c < \infty\) for some constant \(c\). The mode-wise matricized latent tensor factor \(F_t^{(d)}\) satisfies
\[
\frac{1}{T} \sum_{t=1}^{T} F_t^{(d)} F_t^{(d)\top} \xrightarrow{p} \Sigma_d, \quad d \in [D],
\]
where \(\Sigma_d \in \mathbb{R}^{k_d \times k_d}\) is a positive definite matrix and has distinct eigenvalues.

That the sample second-order moment of mode-\(d\) matricization of \(F_t\) converges to some positive definite matrix \(\Sigma_d\) can be derived under the \(\alpha\)-mixing condition in Assumption 1. Refer to Athreya and Lahiri (2006, Chapter 16) and Francq and Zakoian (2019, Appendix A.3) for more details. Without loss of generality, we arrange the diagonal elements of \(\Lambda_d\) in descending order to guarantee unique eigen-decomposition and identifiable eigenvectors, where \(\Lambda_d\) is from the spectral decomposition, \(\Sigma_d = \Gamma_d \Lambda_d \Gamma_d^\top\). Assumption 2 is an extension
of Assumption A in Bai (2003) and Assumption B in Yu et al. (2022) for higher-order tensor observations respectively.

**Assumption 3.** Mode-wise loading matrix.

1. For \( d \in [D] \), there exists some positive constant \( c_d \) such that \( \| A_d \|_{\text{max}} \leq c_d \), where \( \| \cdot \|_{\text{max}} \) is the maximum absolute value of the entries of a matrix;
2. As \( \min_d \{ p_d \} \to \infty \), \( \| p_d^{-1} A_d^T A_d - I_{k_d} \| \to 0 \) for \( d \in [D] \), where \( \| \cdot \| \) is the matrix \( L_2 \)-norm.

Assumption 3.1 assumes the boundedness of entries of mode-wise loading matrices; Assumption 3.2 assumes the asymptotic normalization through orthogonalization and makes our model work within the strong factor regime. They guarantee that the factors have nontrivial contribution to most of the entries of tensor observations. Assumption 3 is standard in confining loading matrices, acting as the mode-wise counterpart of Assumption F1 (a) plus (c) in Stock and Watson (2002), and is consistent with similar assumptions in the literature (Bai, 2003; Lam et al., 2011; Wang et al., 2019; Chen and Fan, 2023; Yu et al., 2022).

**Assumption 4.** Cross modes interrelation of noise. For any \( i_d, j_d \in [p_d], d \in [D] \), and \( t \in [T] \), there exists some positive constant \( c \) such that,

1. \( \mathbb{E}(\xi_t) = 0, \mathbb{E}(e_{t,i_1 \cdots i_D})^8 \leq c \);
2. \( \sum_{i_1}^{p_i} \cdots \sum_{i_D}^{p_{i_D}} | \mathbb{E}(e_{t,i_1 \cdots i_D} e_{t,i_1' \cdots i_D'}) | \leq c \);
3. \( \sum_{s=1}^{T} \sum_{i_1,j_1=1}^{p_i} \cdots \sum_{i_D,j_D=1}^{p_D} | \text{cov}(e_{t,i_1 \cdots i_D} e_{t,j_1 \cdots j_D}, e_{s,i_1' \cdots i_D'} e_{s,j_1' \cdots j_D'}) | \leq c \).

Assumption 4.1 assumes zero-mean and bounded eighth-order moment of each entry of the noise tensor. It is consistent with Assumption C.1 in Bai (2003), Assumption 2.2 in Chen and Fan (2023), and Assumption D.1 in Yu et al. (2022). Recall that the entry of the \((2D)\)th-order covariance tensor of noise is \( \mathbb{E}(e_{t,i_1 \cdots i_D} e_{t,i_1' \cdots i_D'}) \). It describes the across-modes dependency, which is assumed to be weak in Assumption 4.2. This reduces to what was stated in Assumption 4.2 in Chen and Fan (2023), and Assumption D.2(2) in Yu et al. (2022). Assumption 4.3 gives bounds to restrict weak correlation across different modes of the \((4D)\)th-order covariance tensor \( \text{cov}(\xi_t \circ \xi_t, \xi_s \circ \xi_s) \). This reduces to the mode-2 tensor counterpart expressed in Assumption 4.3 in Chen and Fan (2023), and Assumption D.3 in Yu et al. (2022). Overall, we allow weak dependence across modes and observations of the noise in Assumption 4, which is quite general incorporating independent tensor observations as special cases.

**Assumption 5.** Weak dependence between factor and noise. There exists some positive
Suppose that \( e \in \mathbb{R}^{p_d} \), where \( \|e\| \leq 1 \).

Assumptions 1-5 hold, then there exists an asymptotic orthogonal matrix, which is a positive definite matrix with eigenvalues bounded away from 0 and infinity.

Let \( \{F_t^{(d)} u^\top E_t^{(d)} v\} \). Assumptions 5.1 assumes the weak temporal correlation of the matricized process \( \{F_t^{(d)} u^\top E_t^{(d)} v\} \). Assumptions 5.2 and 5.3 assume the weak correlation across modes for the higher-order moments between the factor tensor and noise tensor. It is consistent with Assumption D in Bai (2003) and Assumption E in Yu et al. (2022). Note that Chen and Fan (2023) and Chen et al. (2022) assume uncorrelation between the noise and factor terms directly.

**Assumption 6.** Central limit theorems. Let \( E_{t,i,i}^{(d)} \in \mathbb{R}^{p_d} \) be the \( i \)th row of \( E^{(d)} \) for \( i \in [p_d] \),

\[
\frac{1}{\sqrt{T} p_{-d}} \sum_{t=1}^{T} F_t^{(d)} A_{[D]/(d)}^\top E_{t,i,i}^{(d)} \xrightarrow{L} N_{k_d}(0, V_d), \quad d \in [D],
\]

where \( V_d \) denotes \( \lim_{T \to \infty} (T p_{-d})^{-1} \mathbb{E}(\sum_{t=1}^{T} \sum_{s=1}^{T} F_t^{(d)} A_{[D]/(d)}^\top E_{t,i,i}^{(d)} E_{s,i,i}^{(d)} A_{[D]/(d)} F_s^{(d)} ^\top) \), which is a positive definite matrix with eigenvalues bounded away from 0 and infinity.

The process \( \{F_t^{(d)} A_{[D]/(d)}^\top E_{t,i,i}^{(d)}\} \) converges to Gaussian distribution by the central limit theorem for the \( \alpha \)-mixing process (Athreya and Lahiri 2006, Francq and Zakoian 2019) under Assumptions 1-5. Assumption 6 is the mode-wise extension of Assumption F.4 in Bai (2003).

### 4.2 Theoretical properties

Let \( \tilde{A}_d \) be the diagonal matrix with diagonal elements being the top \( k_d \) eigenvalues of \( \tilde{M}_d \), and denote the below asymptotic orthogonal transformation matrix,

\[
\tilde{H}_d = \frac{1}{T p_{-d}} \sum_{t=1}^{T} F_t^{(d)} A_{[D]/(d)}^\top \tilde{A}_{[D]/(d)} A_{[D]/(d)}^\top F_t^{(d)} ^\top \tilde{A}_d \tilde{A}_d^{-1}.
\]

We present the following convergence rate of the PmoPCA estimators \( \tilde{A}_d \).

**Theorem 4.1** Suppose that \( T \) and \( \{p_d\}_{d=1}^{D} \) tend to infinity, and \( \{k_d\}_{d=1}^{D} \) are fixed. If Assumptions 1-5 hold, then there exists an asymptotic orthogonal matrix \( \tilde{H}_d \), such that

\[
\frac{1}{p_d} \|\tilde{A}_d - A_d \tilde{H}_d\|_F^2 = O_p \left\{ \frac{1}{T p_{-d}} + \frac{1}{p_d} + \sum_{d' \neq d} \left( \frac{1}{T p_{dd'}} + \frac{1}{T^2 p_{dd'}^2} + \frac{1}{p_{dd'}^3} \right) \right\}.
\]
This result is the same as that in Theorem 3.1 for projected estimators in [Yu et al. (2022)] for 2nd-order tensor observations. Let \( \tilde{A}_{d,i} \in \mathbb{R}^{k_d} \) be the \( i \)th row of \( \tilde{A}_d \) and we present the results on the row-wise asymptotic normality of PmoPCA estimators.

**Theorem 4.2** Suppose that \( T \) and \( \{p_d\}_{d=1}^D \) tend to infinity, \( \{k_d\}_{d=1}^D \) are fixed, and Assumptions 1-6 hold. For \( d \in [D] \) and \( i \in [p_d] \),

(i) if \( T_{p-d} = o\left\{ p^2 + \sum_{d' \neq d} \left( T_{p_{d'}}^2 + T_{p_{d'}} \sum_{d'' \neq d'} p_{d'} p_{d''}^4 \right) \right\} \), then

\[
\sqrt{T_{p-d}}(\tilde{A}_{d,i} - \tilde{H}_d^\top A_{d,i}) \xrightarrow{d} N(0, \Lambda_d^{-1} \Gamma_d^\top \Gamma_d \Lambda_d^{-1});
\]

(ii) if \( p^2 + \sum_{d' \neq d} \left( T_{p_{d'}}^2 + T_{p_{d'}} \sum_{d'' \neq d'} p_{d'} p_{d''}^4 \right) = O(T_{p-d}) \), then

\[
\tilde{A}_{d,i} - \tilde{H}_d^\top A_{d,i} = O_p \left\{ \frac{1}{p} + \sum_{d' \neq d} \left( \frac{1}{\sqrt{T_{p_{d'}}}} + \frac{1}{T_{p_{d'}}} + \frac{1}{p_{d'} p_{d''}^2} \right) \right\}.
\]

Theorem 4.2 shows that \( \tilde{A}_d \) is a good estimator of \( A_d \tilde{H}_d \) in the sense that, \( \tilde{A}_{d,i} \) is asymptotic consistent without limit restriction on \( T \) and \( \{p_d\}_{d=1}^D \). However, the asymptotic normality can not be achieved unless \( D \leq 2 \). When \( D = 3 \), for establishing asymptotic normality, the condition \( p_2/p_3 = o(1) \) and \( p_3/p_2 = o(1) \) need to hold simultaneously.

In the following theorem, we show the consistency of estimated tensor factors \( \tilde{F}_t \in \mathbb{R}^{k_1 \times \cdots \times k_D} \) in terms of the Frobenius norm and signal parts \( \tilde{S}_t \in \mathbb{R}^{p_1 \times \cdots \times p_D} \) element-wise.

**Theorem 4.3** Suppose that \( T \) and \( \{p_d\}_{d=1}^D \) tend to infinity, and \( \{k_d\}_{d=1}^D \) are fixed. If Assumptions 1-5 hold, then for \( i_d \in [p_d], \ d \in [D] \) and \( t \in [T] \), we have

\[
\begin{aligned}
(a) \quad & \left\| \tilde{F}_t - F_t \times_1 \tilde{H}_1^{-1} \times_2 \cdots \times_D \tilde{H}_D^{-1} \right\|_F \\
& = O_p \left\{ \frac{1}{\sqrt{p}} + \sum_{d=1}^D \frac{1}{\sqrt{T_{p_{d}}}} + \sum_{d_1=1}^D \sum_{d_2 \neq d_1}^D \left( \frac{1}{\sqrt{T_{p_{d_1}p_{d_2}}}} + \frac{1}{T_{p_{d_1}}} + \frac{1}{p_{d_1} p_{d_2}^2} \right) \right\}.

(b) \quad & |\tilde{s}_{t,i_1\cdots i_D} - s_{t,i_1\cdots i_D}| = O_p \left\{ \frac{1}{\sqrt{p}} + \sum_{d=1}^D \left( \frac{1}{\sqrt{\sum_{d'=1}^D T_{p_{d'}}}} + \frac{1}{\sqrt{T_{p_{d}}}} \right) + \sum_{d_1=1}^D \sum_{d_2 \neq d_1}^D \frac{1}{p_{d_1} p_{d_2}^2} \right\}.
\end{aligned}
\]

Note that, compared with the results of moPCA estimators (Proposition 4 in the Supplementary), the convergence rates of \( \tilde{F}_t \) and \( \tilde{s}_{t,i_1\cdots i_D} \) have been enhanced.

Recall that, the \((s+1)\)st iteration loading matrix estimator \( \tilde{A}_d^{(s+1)} \) represented in equation (15) is \( \sqrt{p_d} \) times the top \( k_d \) eigenvectors of \( \tilde{M}_d^{(s+1)} \) represented in equation (14). Let \( \tilde{A}_d^{(s+1)} \) be the diagonal matrix whose diagonal elements are the top \( k_d \) eigenvalues of \( \tilde{M}_d^{(s+1)} \)
in descending order. The asymptotic orthogonal matrix constructed from the representation of \( \hat{\mathbf{A}}^{(s+1)}_d \) in the proof of Theorem 4.4 is

\[
\hat{\mathbf{H}}^{(s+1)}_d = \frac{1}{T_{pp-d}} \sum_{t=1}^{T} \mathcal{F}^{(d)}_t \mathbf{A}^\top_{[D]/(d)} \hat{\mathbf{A}}^{(s+1)}_{[D]/(d)} \hat{\mathbf{A}}^{(s+1)}_{[D]/(d)} \mathbf{A}_{[D]/(d)} \mathcal{F}^{(d)}_t \mathbf{A}^\top_d \hat{\mathbf{A}}^{(s+1)}_d \hat{\mathbf{A}}^{(s+1)}_d^{-1}.
\]

Let \( w_d^{(s)} = p_d^{-1}\| \hat{\mathbf{A}}_d^{(s)} - \mathbf{A}_d \hat{\mathbf{H}}_d^{(s+1)} \|_F^2 \) for non-negative integers \( s \). Theorem 4.4 below establishes the consistency of \( \hat{\mathbf{A}}^{(s+1)}_d \) in the Frobenius norm.

**Theorem 4.4** Suppose that \( T \) and \( \{p_d\}_{d=1}^D \) tend to infinity, and \( \{k_d\}_{d=1}^D \) are fixed. If Assumptions 1-5 hold, there exists an asymptotic orthogonal matrix \( \hat{\mathbf{H}}^{(s+1)}_d \), such that for integers \( s \geq 0 \), we have

\[
w_d^{(s+1)} = \frac{1}{p_d} \| \hat{\mathbf{A}}^{(s+1)}_d - \mathbf{A}_d \hat{\mathbf{H}}_d^{(s+1)} \|_F^2 = O_p \left\{ \frac{1}{T_{p-d}} + \gamma_d^{(s+1)} \right\},
\]

where

\[
\gamma_d^{(1)} = \frac{1}{p_d^2} + \sum_{d' \neq d} \frac{1}{T_{pp-d'}} + \left( \frac{1}{T_{p-d}} + \frac{1}{p_d^2} \right) \left( \sum_{d'=d+1}^D w_{d'}^{(0)^2} + \sum_{d'=1}^{d-1} w_{d'}^{(1)^2} \right)
\]

\[
+ \sum_{d'=d+1}^D \left\{ \frac{1}{T_{p-d}} + \frac{1}{T_{p-d'}} + \frac{w_{d'}^{(0)}}{T} \right\} w_{d'}^{(1)}
\]

\[
+ \sum_{d'=1}^{d-1} \left\{ \frac{1}{T_{p-d}} + \frac{1}{T_{p-d'}} + \frac{w_{d'}^{(1)}}{T} \right\} \left( \sum_{d''=d'+1}^D w_{d''}^{(0)^2} + \sum_{d''=1}^{d'-1} w_{d''}^{(1)^2} \right) + \frac{w_d^{(1)}}{T_{p-d'}}
\]

and

\[
\gamma_d^{(s+1)} = \frac{1}{p_d^2} + \sum_{d' \neq d} \frac{1}{T_{pp-d'}} + \left( \frac{1}{T_{p-d}} + \frac{1}{p_d^2} \right) \left( \sum_{d'=d+1}^D w_{d'}^{(s)^2} + \sum_{d'=1}^{d-1} w_{d'}^{(s+1)^2} \right)
\]

\[
+ \sum_{d'=d+1}^D \left\{ \frac{1}{T_{p-d}} + \frac{1}{T_{p-d'}} + \frac{w_{d'}^{(s)}}{T} \right\} \left( \sum_{d''=d'+1}^D w_{d''}^{(s-1)^2} + \sum_{d''=1}^{d'-1} w_{d''}^{(s)^2} + \frac{w_{d'}^{(s)}}{T_{p-d'}} \right)
\]

\[
+ \sum_{d'=1}^{d-1} \left\{ \frac{1}{T_{p-d}} + \frac{1}{T_{p-d'}} + \frac{w_{d'}^{(s+1)}}{T} \right\} \left( \sum_{d''=d'+1}^D w_{d''}^{(s)^2} + \sum_{d''=1}^{d'-1} w_{d''}^{(s+1)^2} + \frac{w_{d'}^{(s+1)}}{T_{p-d'}} \right) + \frac{w_d^{(s+1)}}{T_{p-d'}}
\]

In Theorem 4.4, one can see that convergence rates of IPmoPCA estimators for loading matrices of the \( (s+1) \)st iteration depend on that of former iterations, projection steps, and moPCA estimators. Thus the convergence rate of IPmoPCA estimators may be enhanced compared to that of moPCA estimators (Proposition 2 in the Supplementary) since \( T_{p-d} \) may run faster than \( p_d^2 \) but slower than \( 1/\gamma_d^{(s+1)} \). For tensor time series under the TuTFaM setting, Han et al. (2020) presented iTOPUP and iTIPUP methods by adding iteration and projection on TOPUP and TIPUP methods in Chen et al. (2022) respectively. Under assumptions of strong factor and fixed factor numbers, convergence rates of estimated loading matrices are all \( O_p(1/T_{p-d}) \) for estimators of TIPUP, iTIPUP, and iTOPUP.

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It is understandable that the above three estimators have faster convergence rates. Under two stringent assumptions on noise $\mathcal{E}_t$, itself uncorrelated (white) across time and uncorrelated with factors $\mathcal{F}_t$ for all time lags, the expectation of their mode-wise sample auto-covariance matrix remains only the time-lag factor terms. In contrast, under mild assumptions on noise $\mathcal{E}_t$, itself weakly correlated and correlated with factors, our proposed IPmoPCA estimators integrate contemporaneous information from mode-wise sample variance-covariance matrices, whose expectation incorporates not only the factor terms but also the error terms and the interrelation terms between factor and noise. The rich information of data beyond the factor terms, despite complicating various orders of summands of $\gamma_d^{(1)}$ in the convergence rate expression, makes our estimation outperform auto-covariance based methods, and more effective in a wider application scope. Refer to the numerical studies in assessing errors of estimated loading matrices, signal parts, and factor numbers. Our strategy sacrifices the convergence rate but maintains more structural information so as to guarantee good estimation. In addition, the IPmoPCA estimators may achieve the same convergence rate as the above three auto-covariance based estimators provided that the order of $1/Tp_{-d}$ is higher than $\gamma_d^{(s+1)}$ in equation (17).

**Remark 4. (Comparison of convergence rate)** Let IPmoPCA-$i$, $i \in \mathbb{N}$ be the IPmoPCA estimation with $i$ times iterations. We compare convergence rates of estimators for $A_1$ by methods mentioned by far including IPmoPCA estimators with iteration up to thrice for tensors with equal mode size $q$, referred to Table 1. The results indicate that the convergence rate of IPmoPCA-3 is comparable with that by auto-covariance based methods and is faster than those of moPCA and PmoPCA as the order of tensors increases. Thus increasing the number of iterations is a way to acquire ideal theoretical rates.

| Methods      | $D = 2$      | $D = 3$      | $D = 4$      | $D = 10$     |
|--------------|--------------|--------------|--------------|--------------|
| iTOPUP/iTIPUP | $\frac{1}{Tq}$ | $\frac{1}{Tq^2}$ | $\frac{1}{Tq^3}$ | $\frac{1}{Tq^4}$ |
| moPCA        | $\frac{1}{Tq} + \frac{1}{q^2}$ | $\frac{1}{Tq} + \frac{1}{q^2}$ | $\frac{1}{Tq} + \frac{1}{q^2}$ | $\frac{1}{Tq} + \frac{1}{q^2}$ |
| PmoPCA       | $\frac{1}{Tq} + \frac{1}{q^2}$ | $\frac{1}{Tq^2} + \frac{1}{q^3}$ | $\frac{1}{Tq^3} + \frac{1}{q^4}$ | $\frac{1}{Tq^4} + \frac{1}{q^5}$ |
| IPmoPCA-3    | $\frac{1}{Tq} + \frac{1}{q^2}$ | $\frac{1}{Tq^2} + \frac{1}{q^3}$ | $\frac{1}{Tq^3} + \frac{1}{q^4}$ | $\frac{1}{Tq^4} + \frac{1}{q^5}$ |
5 Simulation

In this section, we compare the performance of five methods including moPCA, PmoPCA, IPmoPCA, iTIPUP and PROJ, where PROJ is proposed by Chen and Lam (2024), which is an iterative projection estimation based on a pre-averaging initial estimator, and the projection is on the direction aligning to the strongest estimated factor. The estimation errors of loading matrices, signal parts, and factor numbers estimators, as well as the computation time, are reported. Note that, the moPCA estimators are used as projected matrices and initial estimators for PmoPCA and IPmoPCA, respectively.

5.1 Simulation settings

The simulation data is generated by model (1) in the case of $D = 3$ with $k_1 = 2$, $k_2 = 3$, and $k_3 = 4$. The specific data generating procedure is as follows. First, the loading matrix $A_d$ is taken as $\sqrt{p_d}$ times the first $k_d$ left singular vectors of the SVD decomposition on the matrix with standard normal elements for $d \in [3]$. Second, the tensor factor $F_t \in \mathbb{R}^{2 \times 3 \times 4}$, $t \in [T]$ is generated from the following model

$$F_t = \phi F_{t-1} + \sqrt{1 - \phi^2} V_t,$$

where $\text{vec}(V_t) \sim N_k(0, I_k)$, $k = 24$, and $\phi$ controls the temporal correlation of tensor factors. Note that each element of $F_t$ follows the AR(1) model with unit variance. At last, the tensor noise $E_t \in \mathbb{R}^{p_1 \times p_2 \times p_3}$, $t \in [T]$ is generated from the following model

$$E_t = \psi E_{t-1} + \sqrt{1 - \psi^2} U_t,$$

where $\text{vec}(U_t) \sim N(0, \Delta_D \otimes \cdots \otimes \Delta_1)$, and the diagonal elements of $\Delta_d$ are unit and the off diagonal elements are $1/p_d$. Besides the parameter $\psi$ controls the temporal correlation of tensor noises, the adoption of the array-normal distribution of $U_t$ (Hoff, 2011; Fosdick and Hoff, 2014) makes the tensor noise possessing intra-mode and inter-modes correlation as well. We consider four different scenarios of $\phi$ and $\psi$ resulting in different type tensor observations.

Scenario I: (Uncorrelated) Let $\phi = 0$ and $\psi = 0$, which means both $F_t$ and $E_t$, hence the tensor observations $X_t$, have no temporal correlation.

Scenario II: (Correlated factors) Let $\phi = 0.6$ and $\psi = 0$, which means the noise $E_t$ has no temporal dependence and the tensor observations $X_t$ has temporal correlation due to the factor $F_t$’s temporal correlation.
Scenario III: (Correlated noises) Let $\phi = 0$ and $\psi = 0.8$, which means the factor $F_t$ has no temporal dependence and the tensor observations $X_t$ has temporal correlation due to the noise $E_t$’s temporal correlation.

Scenario IV: (Correlated factors and noises) Let $\phi = 0.6$ and $\psi = 0.8$, which means both $F_t$ and $E_t$, hence the tensor observations $X_t$, are temporal correlated.

For each scenario, five sizes $(20, 20, 20)$, $(50, 20, 20)$, $(50, 50, 50, 50)$, $(100, 50, 50, 50)$ and $(100, 100, 100, 100)$ are considered for $(T, p_1, p_2, p_3)$. The number of replications is 100. Five methods, i.e., iTIPUP(1), moPCA, PmoPCA, IPmoPCA and PORJ, are compared on the simulated data. Note that, iTOPUP is excluded for its memory intractability and time consumption under high dimensionality, and iTIPUP(1) means the auto-covariance lag-1 case.

5.2 Comparison of estimated loading matrices

In this subsection, we display the results of the estimation error of estimated loading matrices. Recall that the loading matrices are not identifiable, only the column spaces are uniquely determined. Hence, the measurement focusing on the column space distance is used to evaluate the estimation errors, which are defined as follows

$$D(\hat{A}_d, A_d) = p_d^{-1} \left\| \hat{A}_d \hat{A}_d^\top - A_d A_d^\top \right\|,$$

where $A_d$ is moPCA estimator for mode-$d$. The closer the column spaces of $\hat{A}_d$ and $A_d$ get to each other, the closer $D(\hat{A}_d, A_d)$ goes to 0. For PmoPCA estimator $\tilde{A}_d$ and IPmoPCA estimator $\hat{A}_d^{(s)}$, the measurement can be similarly defined.

We report the simulation results as the averaged column space distance over 100 replications. Because the results for different modes are similar, only the ones for mode-1 are reported in Figure [1] for simplicity, and the ones for other modes are left in S4 of the Supplementary. From Figure [1], we can conclude four points. First, all five methods, i.e., iTIPUP(1), moPCA, PmoPCA, IPmoPCA, and PORJ, show consistency as the size increases in not only the sample size $T$ but also the mode sizes $p_d$ for $d \in [3]$. Second, in Scenarios I and III, there exists some outliers for iTIPUP(1) because the temporal uncorrelatedness of the tensor factors ($\phi = 0$) violates the basic assumptions of auto-covariance type methods. Third, in Scenarios I and III, PROJ performs well; while in Scenarios II and IV, its performance is not that satisfactory, it may be due to the information insufficiency of their single-direction projection. At last, our methods (moPCA, PmoPCA, IPmoPCA)
outperform iTIPUP(1) under all scenarios, and ourperform PROJ under Scenarios II and IV, which illustrates the robustness of our approach for a wider application scope. In addition, moPCA, PmoPCA, and IPmoPCA have similar performance although IPmoPCA is slightly better sometimes.

Figure 1: The results for mode-1 loading matrix

Figure 2: The results for signal parts
5.3 Comparison of estimated signal parts

In this subsection, we display the results of the estimation error of estimated signal parts, which is measured by the root mean square error (RMSE) defined below,

\[
\text{RMSE}_S = \sqrt{\frac{1}{TP} \sum_{t=1}^{T} \| \tilde{S}_t - S_t \|_F^2},
\]

where \( \tilde{S}_t \) is moPCA estimator for the signal part. For PmoPCA estimator \( \tilde{S}_t \) and IPmoPCA estimator \( \tilde{S}_t^{(s)} \), the measurement can be similarly defined.

We report the simulation results as the averaged RMSE over 100 replications in Figure 2. From Figure 2 we can see a similar trend to the estimated loading matrix in Figure 1. All five methods show consistency as the size increases; there still exists some outliers for iTIPUP(1) in Scenarios I and III; the performance of PROJ is still not that satisfactory in Scenarios II and IV; our methods outperform iTIPUP(1) under all scenarios and outperform PROJ under Scenarios II and IV.

5.4 Comparison of estimated factor numbers

In this subsection, we display the results of the estimation error of estimated factor numbers, and compare the computation time. Note that the estimation of factor numbers with PROJ is not included considering its tedious computation time.

First, the estimation error of estimated factor numbers is measured by the accuracy defined as follows

\[
\text{acc} = \frac{1}{D} \sum_{d=1}^{D} I(\hat{k}_d = k_d) * 100\%,
\]

where \( \hat{k}_d \) is defined in equation (16). Then the average values of acc among 100 replications are reported as the final results in Table 2 from which we can see that our approach can estimate the factor numbers correctly under all scenarios, but iTIPUP(1) can not identify the right number of factors when the latent tensor factor is temporal uncorrelated (Scenarios I and III). The result is similar to that of factor loadings and signal parts, which shows that our approach is robust in a wider application scope again.

Second, the averaged computation time over 100 replications of the five methods are reported in Figure 3 from which we can see that our methods have faster computational speed, especially for large sizes. The computation time of PmoPCA estimation starts from using moPCA as projected matrices in representation (9), and that of IPmoPCA estimation
Table 2: The result for estimated factor numbers

| Methods (size) | Scenario I | | Scenario II |
|----------------|------------|------------------|--------------|
| acc\textsubscript{ITIPUP} | size 1 | size 2 | size 3 | size 4 | size 5 |
| acc\textsubscript{moPCA} | 82.0 | 78.0 | 88.0 | 88.3 | 91.3 |
| size 1 | 100 | 100 | 100 | 100 | 100 |
| size 2 | 100 | 100 | 100 | 100 | 100 |
| size 3 | 100 | 100 | 100 | 100 | 100 |
| size 4 | 100 | 100 | 100 | 100 | 100 |
| size 5 | 100 | 100 | 100 | 100 | 100 |

starts from using moPCA as initial estimators in representation [13].

Figure 3: The results for running time

6 Real data analysis

We demonstrate the proposed approaches on two real datasets. The first one is the Moving MNIST data [Srivastava et al., 2015] (http://www.cs.toronto.edu/~nitish/unsupervised_video/), which has serially uncorrelated tensor observations. The second one is the import-export transport networks data, which is analyzed in Chen et al. (2022), and has temporally correlated tensor observations. The goal of the analysis is from two aspects, one is about the explanation of the estimated loading matrix as well as the corresponding mode-wise clustering result, and the other is about the performance of tensor
reconstruction. Five methods, i.e., iTIPUP, moPCA, PmoPCA, IPmoPCA, and PROJ, are adopted in accordance with simulation studies. Note that, the moPCA estimators are used as projected matrices and initial estimators for PmoPCA and IPmoPCA respectively.

6.1 Moving MNIST data: serial independent

The Moving MNIST data contains 10,000 serial independent video sequences. Each video contains two numbers rotating in a counterclockwise direction to switch relative positions across 20 frames; refer to the frames of the first video sample in Figure 4. Each frame has a spatial resolution of 64 × 64 pixels, then a video can be formulated as a 3rd-order tensor $\mathcal{X}_t \in \mathbb{R}^{20 \times 64 \times 64}$ for $t \in [10000]$, where the 1st mode is the frame mode, and the 2nd and the 3rd modes are the pixel row and column coordinate modes, respectively. The data is centralized to satisfy our assumptions, and $(k_1, k_2, k_3) = (2, 3, 4)$ is used as the size of the latent factors.

![Figure 4: The 20 frames of video 1 are represented from left to right, top to bottom sequentially](image)

**Clustering of frames** We first show the mode-wise clustering results based on the estimated loading matrices. Note that, for the sake of interpretability, we only consider the clustering of frames by conducting hierarchical clustering on the estimated mode-1 loading matrix. The hierarchical clustering plots of the five methods are displayed in Figure 5. From Figure 5 we can see that, our methods (moPCA, PmoPCA, IPmoPCA) can clearly separate the two groups of frames before and after relative position interchange, while iTIPUP(1) and PROJ can not.

**Tensor reconstruction** We evaluate the performance of the tensor reconstruction numerically by the following reconstruction error (RE)

$$RE = \frac{\|\hat{\mathcal{X}} - \mathcal{X}\|_F}{\|\mathcal{X}\|_F},$$
where $\mathcal{X} \in \mathbb{R}^{20 \times 64 \times 64 \times 10000}$ is the 4th-order tensor, mode-4 slices of which are the tensor observations. The results of intra-sample RE are reported in the first row of Table 3. We can see that, the proposed three approaches are competitive and outperform iTIPUP(1) and PROJ methods in the RE merit. Furthermore, we also report the computation time of the above methods in the second row of Table 3. The computation time of PmoPCA estimation starts from using moPCA as projected matrices, and that of IPmoPCA estimation starts from using moPCA as initial estimators. We can see that the proposed three approaches are much faster than iTIPUP(1) and PROJ.

### Table 3: Comparison of reconstruction errors for Moving MNIST data

|       | iTIPUP | moPCA  | PmoPCA | IPmoPCA | PROJ  |
|-------|--------|--------|--------|---------|-------|
| RE    | 0.9516 | 0.9166 | 0.9146 | 0.9146  | 0.9247|
| TIME  | 587.83s| 97.28s | 27.35s | 73.53s  | 1804.03s|

Figure 5: The results for estimated mode-1 loading matrix

#### 6.2 Import-export transport network data: serial dependent

The import-export transport networks data is shared by authors of Chen et al. (2022). It contains the monthly export volume among 22 countries on 15 product categories from...
January 2010 to December 2016 (84 months). The detailed list of countries and products is left in S5 of the Supplementary. It is preprocessed by conducting the three-month moving average to reduce the effect of incidental transactions of large trades or unusual shipping delays. The resulting 3rd-order tensor time series are denoted as $X_t \in \mathbb{R}^{22 \times 22 \times 15}$, $t \in [82]$, where the three modes are export countries, import countries, and product categories, respectively. We use factor numbers $(k_1, k_2, k_3) = (4, 4, 6)$ following Chen et al. (2022), then six methods, i.e., iTIPUP(1), iTIPUP(2), moPCA, PmoPCA, IPmoPCA and PROJ are used to fit the whole dataset. To improve interpretability, after applying the varimax rotation, the rotated loadings are multiplied by 30 and truncated to integers.

| method | factor | BE  | BU  | CA  | DK  | FI  | FR  | DE  | GR  | IS  | IR  | MX  | NO  | PO  | PT  | ES  | SE  | CH  | TR  | US  | UK  |
|--------|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| iTIPUP(1) | 1    | -1  | 1   | -29 | 0   | 0   | 0   | 1   | 0   | 0   | -1  | 1   | 1   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | -1  |
|         | 2    | 1   | 1   | 0   | 1   | 1   | 1   | -1  | 1   | 1   | 1   | 0   | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 1   | 0   | 1   |
|         | 3    | 8   | 1   | -1  | 2   | 1   | 10  | 24  | 1   | 3   | 1   | 6   | 8   | -1  | 1   | 4   | 1   | 6   | 3   | 5   | 2   | 1   |
|         | 4    | -2  | 0   | 1   | 0   | 0   | 1   | 3   | 0   | 1   | 0   | -2  | 1   | 30  | 0   | 0   | 0   | -1  | 1   | 0   | 0   | 1   |
| iTIPUP(2) | 1    | -1  | 0   | -29 | 0   | 0   | 0   | 1   | 0   | 1   | 0   | -1  | 1   | 1   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   |
|         | 2    | 1   | 1   | 0   | 1   | 1   | 1   | -1  | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 30  |
|         | 3    | -7  | 0   | 2   | -10 | -9  | -23 | 0   | -2  | 0   | -5  | -7  | 2   | 0   | -3  | 0   | -5  | -2  | -4  | -1  | 0   | 0   |
|         | 4    | -2  | 0   | 1   | 0   | 0   | 1   | 3   | 0   | 1   | 0   | -2  | 1   | 30  | 0   | 0   | 0   | -1  | 1   | 0   | 0   | 1   |
| moPCA  | 1    | 3   | 1   | -1  | 1   | 1   | 1   | -4  | 1   | 1   | 1   | 1   | 3   | 0   | -29 | 1   | 1   | 1   | 1   | 2   | 0   | 1   | 1   |
|         | 2    | 0   | 1   | -1  | 1   | 1   | 1   | -1  | 1   | 1   | 1   | 0   | 0   | 0   | 1   | 1   | 1   | 1   | 1   | 1   | 0   | 1   |
|         | 3    | 8   | 1   | 0   | 2   | 1   | 10  | 24  | 1   | 2   | 1   | 6   | 8   | -2  | 1   | 4   | 1   | 6   | 3   | 5   | 2   | 2   |
|         | 4    | -7  | 0   | 2   | 1   | 10  | 24  | 1   | 2   | 1   | 6   | 8   | -2  | 1   | 4   | 1   | 6   | 3   | 5   | 2   | 1   | 7   |
| IPmoPCA | 1    | -1  | 1   | -29 | 0   | 0   | 0   | 2   | 0   | 1   | 0   | -1  | 1   | 1   | 0   | 1   | 0   | 0   | 0   | 0   | 0   | 0   |
|         | 2    | 1   | 1   | 0   | 1   | 1   | -1  | 1   | 1   | 1   | 1   | 0   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 30  |
|         | 3    | -7  | 0   | 2   | -1  | 0   | -9  | -23 | 0   | -2  | 0   | -5  | -7  | 2   | 0   | -3  | 0   | -5  | -2  | -4  | -1  | 0   |
|         | 4    | 3   | 1   | 0   | 1   | 1   | 0   | -2  | 1   | 0   | 1   | 3   | 0   | -29 | 1   | 1   | 1   | 2   | 0   | 1   | 1   | 0   |
| PROJ   | 1    | -1  | 1   | -29 | 0   | 0   | 0   | 2   | 1   | 1   | 1   | 0   | -1  | 1   | 1   | 0   | 1   | 0   | -1  | 0   |
|         | 2    | 0   | 1   | -1  | 1   | 1   | 0   | 2   | 4   | 0   | 1   | 0   | 0   | 3   | 30  | 3   | 1   | 0   | 1   | 1   | 1   |
|         | 3    | -2  | 1   | -1  | 1   | 1   | -1  | -25 | 0   | 0   | 0   | -1  | 4   | 3   | 0   | 1   | 1   | 0   | -2  | 0   | 5   | -14 |
|         | 4    | 0   | 0   | 0   | 0   | -1  | 3   | 8   | 0   | 0   | 0   | -1  | 0   | -2  | 1   | -22 | 1   | -4  | 0   | 3   | 0   | 4   |

**Export countries** For the export countries, results for the six methods are shown in Table 4. We can see that the results of iTIPUP(1), iTIPUP(2), PmoPCA, and IPmoPCA are very close, where Canada, the United States, and Mexico load heavily on the export factors 1, 2, and 4 respectively, and most of the European countries mainly load on the factor 3, especially Germany. For the result of the moPCA, the order of factors is changed, where Canada and Mexico load heavily on export factors 4 and 1 respectively. For the
result of PROJ, Canada and Mexico load heavily on the export factors 1 and 2 respectively, Germany and Norway mainly load on factors 3 and 4, respectively, and the United Kingdom mainly loads on both factors 3 and 4. These results are coincident with the heatmap and hierarchical clustering results based on the loading matrices of the six methods in S5 of the Supplementary.

Table 5: Estimation of loading matrix $A_2$

| method   | factor | BE | BU | CA | DK | FI | FR | DE | GR | IS | IT | MX | NO | PO | PT | ES | SE | CH | TR | US | UK |
|----------|--------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| iTIPUP(1)| 1      | 1  | 0  | 0  | 0  | 0  | 1  | 1  | 0  | 0  | 0  | 1  | 0  | 0  | 0  | 1  | 0  | -29| 0  |
|          | 2      | -2 | 1  | -29| 0  | 1  | -2 | -5 | 0  | 1  | 0  | -2 | 0  | 2  | 0  | 1  | 0  | 1  | 0  | -2 | 0  | 1  |
|          | 3      | -8 | 0  | 6  | -2 | -1 | -16| -10| 0  | -3 | 0  | 0  | -9 | 0  | -1 | -4 | -1 | -7 | -3 | -6 | -3 | 0  | -12|
|          | 4      | 2  | 1  | -1 | 1  | 0  | 0  | 1  | 0  | 0  | 0  | 2  | 1  | -29| 1  | 1  | 1  | 1  | 0  | 3  | 1  | 0  |
| iTIPUP(2)| 1      | 1  | 0  | 0  | 0  | 0  | 1  | 1  | 0  | 1  | 0  | 1  | 0  | 0  | 1  | 0  | 0  | 0  | 1  | 0  | 0  |
|          | 2      | -2 | 1  | -29| 0  | 1  | -2 | -5 | 0  | 1  | 0  | -2 | 0  | 2  | 0  | 1  | 0  | 1  | 0  | 1  | 0  |
|          | 3      | -8 | 0  | 6  | -2 | -1 | -16| -10| 0  | -3 | 0  | 0  | -9 | 0  | -1 | -5 | -1 | -7 | -3 | -6 | -3 | 0  |
|          | 4      | 2  | 0  | -1 | 1  | 0  | 1  | 0  | 0  | 0  | 2  | 1  | -29| 1  | 1  | 1  | 1  | 0  | 3  | 1  | 0  |
| moPCA    | 1      | 1  | 0  | 0  | 0  | 0  | 1  | 1  | 0  | 0  | 1  | 1  | 0  | 0  | 1  | 0  | 0  | 0  | 1  | 0  |
|          | 2      | 1  | 0  | 24 | 0  | 0  | 1  | -1 | 0  | 0  | 1  | 1  | 0  | 19 | 1  | 0  | 0  | 0  | 0  | 1  |
|          | 3      | -8 | 0  | 2  | -2 | -1 | -18| 3  | 0  | -3 | 0  | 0  | -9 | 0  | -1 | -5 | -1 | -7 | -3 | -6 | -3 | 0  |
|          | 4      | 3  | 1  | 2  | 1  | 0  | 30 | 1  | -1 | 1  | 2  | 1  | 0  | 1  | 4  | 1  | 1  | 1  | 1  | 1  |
| PmoPCA   | 1      | 1  | 0  | 0  | 0  | 0  | 1  | 1  | 0  | 0  | 1  | 1  | 0  | 1  | 0  | 1  | 0  | 0  | 1  | 0  |
|          | 2      | 3  | 0  | 30 | 1  | 0  | 3  | 5  | 1  | 0  | 1  | 3  | 1  | -1 | 1  | 0  | 0  | 0  | 1  | 0  |
|          | 3      | -8 | 0  | 6  | -2 | -1 | -16| -9 | 0  | -3 | 0  | 0  | -9 | 0  | -1 | -5 | -1 | -7 | -3 | -6 | -3 | 0  |
|          | 4      | 2  | 1  | -1 | 1  | 0  | 2  | -1 | 1  | 0  | 0  | 2  | 2  | -29| 1  | 1  | 1  | 0  | 4  | 1  | 0  |
| IPmoPCA  | 1      | 1  | 0  | 0  | 0  | 0  | 1  | 1  | 1  | 0  | 0  | 1  | 1  | 1  | 0  | 1  | 1  | 0  | 0  | 1  |
|          | 2      | -2 | 1  | -29| 0  | 1  | -2 | -5 | 0  | 1  | 0  | -2 | 0  | 2  | 0  | 1  | 1  | 0  | 1  | -1 | 0  |
|          | 3      | 0  | 1  | -5 | 3  | 2  | 17 | 11 | 1  | 4  | 1  | 2  | 10| 1  | 2  | 5  | 2  | 8  | 4  | 7  | 4  |
|          | 4      | 2  | 1  | -1 | 1  | 0  | 1  | -1 | 1  | 0  | 0  | 2  | 1  | -29| 1  | 1  | 1  | 1  | 0  | 3  | 1  |
| PROJ     | 1      | 1  | 0  | 1  | 0  | 1  | 1  | 1  | 0  | 0  | 0  | 0  | 0  | 1  | 1  | 1  | 1  | 1  | 0  | 0  | -29|
|          | 2      | 0  | 0  | -28| 1  | 0  | -4 | 1  | 0  | 1  | 0  | -1 | -4 | 2  | 0  | 0  | 0  | 1  | 0  | 2  |
|          | 3      | -1 | 0  | 8  | 1  | 0  | 7  | -3 | 1  | -2 | 1  | -2 | 3  | 1  | -1 | -1 | 1  | 0  | -1 | -5 | 0  |
|          | 4      | -6 | 0  | 3  | 0  | 0  | -22| -7 | -1 | 1  | 1  | 0  | -2 | -4 | 0  | 0  | -2 | -14| -2 | -5 | -2 |

**Import countries** For the import countries, results for the six methods are shown in Table 5. The results of the moPCA show that the United States, Mexico and Canada, and Germany load heavily on export factors 1, 2, and 4 respectively, and most of the European countries mainly load on factor 3, especially for France and the United Kingdom. For the results of iTIPUP(1), iTIPUP(2), PmoPCA, and IPmoPCA, the United States, Canada, and Mexico load heavily on export factors 1, 2, and 4 respectively, and most of the European countries mainly load on factor 3, especially for France, the United Kingdom, and Germany. For the results of PROJ, the United States, Canada, and the United Kingdom load heavily on export factors 1, 2, and 3 respectively, and most of the European countries mainly load on factor 4, especially France and Spain. These results are coincident with the heatmap.
and hierarchical clustering results based on the loading matrices of the six methods in S5 of the Supplementary.

Table 6: Estimation of loading matrix $A_3$

| method      | factor | $C_1$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $C_6$ | $C_7$ | $C_8$ | $C_9$ | $C_{10}$ | $C_{11}$ | $C_{12}$ | $C_{13}$ | $C_{14}$ | $C_{15}$ |
|-------------|--------|-------|-------|-------|-------|-------|-------|-------|-------|-------|---------|---------|---------|---------|---------|---------|
| iTIPUP(1)   | 1      | 1     | -2    | 1     | 1     | 1     | -1    | 0     | 3     | -1    | 0       | -1      | 0       | -29     | 0       | -4      |
|             | 2      | 0     | 0     | 1     | -29   | 1     | 2     | -1    | 1     | 1     | -1      | 1       | 1       | -2      | 0       | 1       |
|             | 3      | 0     | -2    | 0     | 0     | 29    | 1     | 1     | 0     | 1     | 1       | 4       | 3       | 0       | 0       | 7       |
|             | 4      | 0     | 1     | -4    | 1     | 1     | 1     | 3     | 0     | -2    | 0       | -2      | 2       | -1      | 1       | -29     |
|             | 5      | 6     | 7     | 6     | 1     | 1     | 21    | 1     | 1     | 10    | 2      | 0       | -5      | 16      | 1       | 1       |
|             | 6      | 2     | -2    | 3     | 0     | 5     | -3    | 1     | -5    | 8     | 3       | 10      | 4       | 4       | 4       | 2       |
| iTIPUP(2)   | 1      | 1     | -2    | 1     | 1     | 1     | -1    | 0     | 3     | -1    | 0       | -1      | 0       | -29     | 0       | -4      |
|             | 2      | 0     | 0     | 1     | -29   | 1     | 2     | -1    | 1     | 1     | -1      | -2      | 0       | 1       | 0       | 0       |
|             | 3      | 0     | -2    | 0     | 0     | 29    | 1     | 1     | 0     | 1     | 1       | 4       | 3       | 0       | 0       | 7       |
|             | 4      | 0     | 1     | -4    | 1     | 1     | 1     | 3     | 0     | -2    | 0       | 0       | -2      | -1      | 1       | -29     |
|             | 5      | 6     | 7     | 6     | 1     | 1     | 21    | 1     | 1     | 10    | 2      | 0       | -5      | 16      | 1       | 1       |
|             | 6      | 2     | -1    | 3     | 0     | 5     | -3    | 1     | -5    | 8     | 3       | 10      | 4       | 4       | 4       | 2       |
| moPCA       | 1      | 1     | -1    | 0     | 0     | 2     | 1     | 0     | 3     | -1    | 0       | 1       | 1       | -29     | 0       | -6      |
|             | 2      | 0     | 0     | 1     | -29   | 1     | 1     | 1     | 2     | -1    | 1       | 1       | 0       | 1       | 1       | -2      |
|             | 3      | 1     | 2     | 0     | 1     | -28   | 0     | 0     | 1     | 1     | 0       | 1       | 0       | 1       | 1       | -7      |
|             | 4      | 0     | 0     | 2     | 0     | 0     | 0     | 0     | 0     | 3     | -1      | 0       | 0       | 2       | 0       | 30      |
|             | 5      | 6     | 6     | 7     | 1     | 3     | 17    | 2     | 8     | 7     | 2       | 4       | 19      | 4       | 0       | 9       |
|             | 6      | 0     | 0     | 2     | 0     | 0     | -2    | 1     | 1     | 1     | 1       | 1       | 30      | 0       | 0       | 0       | 5       |
| PmoPCA      | 1      | 1     | -2    | 0     | 1     | 1     | -1    | 3     | -1    | 1     | 1       | -2      | 0       | -29     | 1       | -3      |
|             | 2      | 0     | 0     | 1     | -29   | 1     | 2     | -1    | 1     | 1     | 1       | -2      | 0       | 1       | 1       | 0       |
|             | 3      | 0     | -2    | 0     | 0     | 29    | 1     | 1     | 0     | 1     | 1       | 4       | 2       | 0       | 0       | 8       |
|             | 4      | 5     | 7     | 7     | 1     | 2     | 20    | 0     | 10    | 2     | 0       | -4      | 17      | 1       | 4       | -4      |
|             | 5      | 2     | 0     | 2     | 0     | 0     | -3    | 1     | 1     | 1     | 1       | 1       | 0       | 0       | 30      |
|             | 6      | -2    | 4     | -1    | 1     | -4    | 5     | -1    | 6     | -7    | -2      | -9      | -2      | -2      | -3      | 26      |
| IPmoPCA     | 1      | 1     | -2    | 1     | 1     | 1     | -1    | 0     | 3     | -1    | 0       | -1      | 0       | -29     | 0       | -4      |
|             | 2      | 0     | 0     | 1     | -29   | 1     | 2     | -1    | 1     | 1     | -1      | -2      | 0       | 1       | 0       | 0       |
|             | 3      | 1     | 3     | 1     | -28   | 0     | 0     | 1     | 0     | 0     | -3      | -2      | 1       | 1       | -6      |
|             | 4      | 1     | 0     | 5     | 0     | 0     | -2    | 1     | 3     | 1     | 1       | 3       | 2       | 0       | 30      |
|             | 5      | 6     | 7     | 6     | 1     | 1     | 21    | 1     | 10    | 2     | 0       | -5      | 16      | 1       | 1       | -3      |
|             | 6      | 2     | -2    | 3     | 1     | 5     | -3    | 1     | -5    | 7     | 3       | 10      | 4       | 4       | 4       | 2       |
| PROJ        | 1      | 0     | 1     | 1     | 30    | 0     | 1     | 1     | 1     | 1     | 1       | 1       | 1       | 0       | 1       | -1      |
|             | 2      | 1     | -1    | 0     | 0     | -1    | 1     | 0     | 2     | 1     | 1       | 2       | 1       | 2       | -3      | -29     |
|             | 3      | 6     | 8     | -2    | 1     | 21    | 8     | 1     | 4     | 1     | 0      | 1       | 16      | 8       | -1      | 4       |
|             | 4      | -5    | 5     | 1     | 1     | 0     | 0     | 1     | -6    | 0     | 0       | 28      | -1      | 7       | 0       | -1      |
|             | 5      | 1     | 14    | -1    | 0     | 5     | 1     | 2     | 1     | -1    | 0      | 5       | -2      | -24     | 3       | 5       |
|             | 6      | 3     | 4     | 9     | 1     | -5    | 4     | 1     | 6     | 1     | 1       | -6      | 6       | 0       | 26      |

**Product categories** As for the product categories, results for the six methods are shown in Table 6. The results of the moPCA show that factors 1, 2, 3, 4, and 6 are the Machinery & Electrical factor, Mineral Products factor, Chemicals & Allied Industries factor, Transportation factor, and Stone & Glass factor, respectively. Factor 5 has heavy loadings on Plastics & Rubbers and Metals. The results of iTIPUP(1), iTIPUP(2), and IPmoPCA are similar, where factors 1, 2, 3, 4, and 6 are the Machinery & Electrical factor, Mineral...
Products factor, Chemicals & Allied Industries factor, Transportation factor, and Miscellaneous factor, respectively. Factor 5 has heavy loadings on Plastics & Rubbers and Metals. For PmoPCA, factor 4 is the mixed factor of Plastics & Rubbers and Metals, and factor 5 is the Transportation factor. For PROJ, factors 1, 2, 4, and 6 are the Mineral Products factor, Transportation factor, Stone & Glass factor, and Miscellaneous factor, respectively. Factors 3 and 5 are both mixed factors of two product categories. These results can be illustrated by the heatmap and hierarchical clustering results based on the loading matrices in S5 of the Supplementary.

Table 7: RE values for the Import-Export Transport dataset

|                | iTIPUP(1) | iTIPUP(2) | moPCA  | PmoPCA  | IPmoPCA  | PROJ  |
|----------------|-----------|-----------|--------|---------|----------|-------|
| (2,2,2)        | 0.566542  | 0.566548  | 0.575356| 0.567267| 0.566519 | 0.737432|
| (4,2,2)        | 0.492059  | 0.492065  | 0.499865| 0.493276| 0.491989 | 0.717259|
| (5,5,5)        | 0.283764  | 0.283767  | 0.294388| 0.284301| 0.283761 | 0.609906|
| (10,10,5)      | 0.162273  | 0.162268  | 0.165199| 0.162877| 0.162268 | 0.363559|
| time           | 12.53s    | 14.90s    | 1.54s   | 1.73s   | 10.73    | 83.05s |

Tensor reconstruction The results of averaged RE values over 10 folds cross-validation for aforementioned six methods under various factor number settings are in Table 7. We can see that our proposed three methods are doable for highly correlated tensor time series even though we designed our methodology for weakly dependent and independent tensor observations.

7 Conclusion

In this article, we bond the estimation of low-rank structure under TuTFaM with that of the well-developed high-dimensional vector factor models through tensor matricization; refer to S6.2 of the Supplementary. Under such a framework, we develop two sets of tensor PCA variants to estimate the components in the low-rank signal under TuTFaM. Particularly, IPmoPCA which is based on both iteration and projection, is a refinement of PmoPCA that involves projection only, in terms of enhancement in convergence rate and signal-to-noise ratio; refer to Supplementary for proof. Note that though our angle of view is on the decomposition of data objects directly, there is another line of work focusing on the

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estimation of covariance tensors with low-rank and/or sparsity assumptions (Fan et al., 2021, 2013, 2016; Tsiligkaridis et al., 2013; Zhou, 2014; Greenewald et al., 2019); refer to S6.1 of the Supplementary for moPCA of TuTFaM.

Our intention of contribution is to show that, the PCA or spectral method for any order tensor observations follows the same vein of the mode-wise thought, though tensor objects have larger sizes and more complicated structures with the growing number of modes. This article is essentially the extension of the following two works. On the one hand, it follows the line of CVC-based PCA estimation methods in Chen and Fan (2023) and extends the estimation of factor models from the 2nd-order tensor (matrix) to the higher-order tensor. On the other hand, it extends the work of Chen et al. (2022) in terms of application scope, i.e., from the tensor time series to the more general uncorrelated or weakly correlated tensor observations. In addition, our work is under the assumption of homogeneous factor strength. As one reviewer pointed out, recently there has been arising research interest in factor analysis under the practical heterogeneous factor strength situation (Bai and Ng, 2023; Freyaldenhoven, 2022; Uematsu and Yamagata, 2022; Chen and Lam, 2024; Zhang et al., 2023), where there coexist strong and weak factors owing to varying column strength measures. Note that the sporadic such works evidenced their superiority in estimating factor numbers, not to mention the model flexibility and practicability. This motivates us to extend the proposed IPmoPCA approach under the heterogeneous factor case in the near future.

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