A Q-ANALOGUE OF THE MULTIPLICATIVE CALCULUS:
Q-MULTIPLICATIVE CALCULUS

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Abstract. In this paper, we propose q-analog of some basic concepts of multiplicative calculus and we called it as q-multiplicative calculus. We successfully introduced q-multiplicative calculus and some basic theorems about derivatives, integrals and infinite products are proved within this calculus.

1. Introduction. A q-analog, also called a q-generalization or q-extension is a mathematical expression parameterized by a quantity q that generalized a known expression and reduces to the known expression in the limit q → 1. q-Calculus was also expressed by q-analogues of the traditional lines of ordinary calculus. The terms in q calculus were first defined and called by Euler in the eighteenth century. After that, many applications were found in the 19th century; for example, Jacobi’s triple product identity and the theory of q-hypergeometric functions. However, q-calculus gained a systematical organization with Jackson [6]. Within the last quarter of the 20th century, applications of q-calculus became much more significant in some areas of mathematics and physics [5, 9, 10, 11].

As a different calculus, multiplicative calculus, M.Grossman and Robert Katz [5] produced a new definition of calculus in 1972. The fundamental point of this calculus is the use of division and multiplication instead of subtraction and addition; and this newly acquired calculus was named after non-Newtonian calculus. Later on, in the last quarter of the 20th century, this new calculus was a topic for new study areas and used more often. Particularly Stanley and Bashirov [1, 14] have directed the attention of the researchers to this topic by conducting plenty of research and analyzed it as multiplicative calculus. As much as it has become a topic for mathematical studies, it has also become a topic for economics and finance [4].

In this study, we introduced q-analogues of the multiplicative calculus and called it after q-multiplicative calculus. Thanks to q-analogue of the multiplicative derivative and integral, we have acquired new definitions and theories and presented them with their substantiation.

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2. Notations and preliminaries for q-calculus. At this point, we are going to show necessary q-calculus definitions in order to understand this paper. For the detailed information about q-calculus, please look at [3, 6, 7, 8, 12, 13].

Let \( q \in (0, 1) \). A q-natural number \([n]_q\) is defined by
\[
[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + ... + q^{n-1}
\]
the factorial of a q-number \([n]_q\) is defined by
\[
[n]_q! = \begin{cases} 1 & n = 0 \\ [n]_q[n-1]_q[n-2]_q \ldots [1]_q & n = 1, 2, 3, \ldots \end{cases}
\]
the q-analogue of \((x-a)^n_q\) is the polynomial
\[
(x-a)^n_q = \begin{cases} 1 & n = 0 \\ [x-a]_q[x-qa]_q[x-q^2a]_q \ldots [x-q^{n-1}a]_q & n = 1, 2, 3, \ldots \end{cases}
\]
q-binomial coefficient is given by
\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{[n]_q!}{k![n-k]!}
\]
where \( n, k \in \mathbb{N} \).

2.1. q-derivative.

Let \( q \in (0, 1) \) and \( f(x) \) be an arbitrary function. The q-differential is defined by
\[
d_q f(x) = f(qx) - f(x) \\
d_q x = (q - 1)x
\]
and the q-derivative of a function \( f \) given on a subset of \( \mathbb{R} \) is defined by
\[
D_q[f(x)] = \frac{d_q f(x)}{d_q x} = \frac{f(qx) - f(x)}{(q - 1)x}, \quad x \neq 0, \quad (1)
\]
\[
D_q[f(0)] = \lim_{x \to 0} D_q[f(x)]
\]
where \( qx \) and \( x \) should be in the domain of \( f \) and \( D_q \) is q-difference operator.

It is clear that if \( f(x) \) is differentiable, the well-known limit
\[
\lim_{q \to 1} D_q[f(x)] = \frac{df(x)}{dx} = f'(x).
\]

High q-derivatives is denoted as \( D_q^n[f(x)] = D_q^{n-1}[D_q[f(x)]] \). The nth q-derivative is defined by its values at the points \( \{q^k x, \ k = 0, 1, 2, \ldots, n \} \) through the identity
\[
D_q^n[f(x)] = (1 - q)^{-n} x^{-n} \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q (-1)^k q^{-k(n-k)} q^{-k(k-1)/2} f(xq^k).
\]
In the standard approach to two q-analogues of classical exponential function \( e^x \) are used:

\[
e^x_q = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!}, \\
E^x_q = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{x^k}{[k]_q!}.
\]

These two q-exponential functions under q-derivative are given by

\[
D_q[e^{nx}_q] = n - q e^{nx}_q \quad \text{and} \quad D_q[E^{nx}_q] = n - q E^{nx}_q.
\]

Generally, we used \( e^x_q \) in our processes. The q-analogues of the sine and cosine functions can be defined in analogy with their well-known Euler expressions in terms of the exponential function.

\[
\sin_q x = e^{ix}_q - e^{-ix}_q \quad \text{and} \quad \cos_q x = \frac{e^{ix}_q + e^{-ix}_q}{2i}
\]

\[
D_q[\sin \alpha x] = \frac{\alpha}{1 - q} \cos \alpha x \quad \text{and} \quad D_q[\cos \alpha x] = - \frac{\alpha}{1 - q} \sin \alpha x.
\]
Notice, it is clear that if \( f(x) \) is integrable, then
\[
\lim_{q \to 1} \int_0^b f(x) d_q x = \int_0^b f(x) dx.
\]

**Theorem 1.** (see [8]) If \( f'(x) \) exists in a neighborhood of \( x = 0 \) and \( f(x) \) is continuous at \( x = 0 \) where \( f'(x) \) denotes the ordinary derivative of \( f(x) \), we have
\[
\int_a^b D_q[f(x)]d_q x = f(b) - f(a). \tag{7}
\]

3. **q-multiplicative calculus.**

3.1. **q-multiplicative derivative.**

In this chapter, we will explain new definitions and theorems being the main objective of our article. First of all, we will present \( q \)-analogue of multiplicative derivative, where it is called \( q \)-multiplicative derivative. Afterwards, we will see some features and theorems on these definitions with proves. It can be looked at books and articles published recently for more details on multiplicative calculus [1, 2, 5, 14]. Notations of multiplicative calculus are as follows in the literature: The multiplicative derivative of a function \( f(x) \) is denoted as \( \frac{d^*}{dx^*} f(x) \) or sometimes as \( f^*(x) \). Its explicit definition is given by [14]
\[
f^*(x) = \lim_{\Delta x \to 0} \left( \frac{f(x + \Delta x)}{f(x)} \right)^{\frac{1}{\Delta x}}, \quad f(x) > 0 \tag{8}
\]
\( f^{*(k)} \) is high multiplicative derivatives and \( f^{*(0)} = f \).

Now, by using the \( q \)-analog, we give a new definition and introduce \( q \)-multiplicative derivative.

**Definition 1.** Let \( q \in (0, 1) \) and \( f(x) \) be a positive function. The \( q \)-analogue of the multiplicative derivative, the \( q \)-multiplicative derivative, is
\[
D^*_q[f(x)] := \left( \frac{f(qx)}{f(x)} \right)^{\frac{1}{(q-1)x}}. \tag{9}
\]

Here, we changed the places of \( x + \Delta x \) and \( qx \) in the (8) within the expression of (9), however, we did not limit and from \( x + \Delta x = qx \), \( \Delta x = (q-1)x \), we found the expression (9). It can be easily seen that, the limit of (9) when it approaches to \( q \to 1 \), is obtained as follows:
\[
\lim_{q \to 1} D^*_q[f(x)] = \frac{d^*}{dx^*} f(x).
\]

The \( n \)th \( q \)-multiplicative derivative can be denoted by \( D^{*(n)}_q[f(x)] \) for \( n=0,1,2,\ldots \), or sometimes as \( f^{*(n)}_q(x) \).

In addition, shortly, \( q^* \)-derivative can be used instead of \( q \)-multiplicative derivative. Definition (1), helped us to take steps towards \( q \)-multiplicative calculus which is a more extended version of multiplicative calculus. Now, we can discover classical theorem, definition and results of multiplicative calculus by using (9).
Theorem 2. Let \( q \in (0, 1) \) and \( f(x) \) be a \( q \)-differentiable positive function. Then we can obtain \( q \)-multiplicative derivative of \( f(x) \) as

\[
D_q^*[f(x)] = e_q^{D_q[\ln f(x)]}.
\]  

(10)

Proof.

\[
D_q^*[f(x)] = e_q^{\ln\left(\frac{f(qx)}{f(x)}\right)^{\frac{1}{q-1}}} = e_q^{\frac{1}{q-1}\ln\left(\frac{f(qx)}{f(x)}\right)} = e_q^{\frac{\ln f(qx) - \ln f(x)}{(q-1)x}} = e_q^{D_q[\ln f(x)]}
\]

for \( \ln f(x) = (\ln f)(x) \).

Note that, the \( q \)-multiplicative derivation by using expression (9) in calculations will cause difficulty for us. It can be clearly seen that in this theorem, we can easily make calculations by using classical \( q^{*} \) derivative transactions. Therefore, generally expression (10) will be used for \( q^{*} \) derivative in further theorem and applications.

In general, we have

\[
D_q^*[f(x)] = f_q D_q[\ln f(x)] = \frac{f_q}{f(x)}.
\]

consider function \( f(x) = x \)

\[
e_q^{D_q[\ln x]} = e_q^{\frac{\ln x}{x-1}} \quad \text{and} \quad e_q^{\frac{D_q[x]}{x}} = e_q^{\frac{1}{x}}.
\]

Corollary 1. Let \( q \in (0, 1) \) and \( f(x) \) positive function. The \( n \)th \( q \)-multiplicative derivative of \( f(x) \) is given by

\[
D_q^{(n)}[f(x)] = e_q^{D_q^{(n)}[\ln f(x)]} \quad n = 0, 1, 2, ...
\]  

(11)

Proof. By using theorem 2 , we have second \( q^{*} \) derivative as

\[
D_q^{**}[f(x)] = e_q^{D_q[\ln (e_q^{D_q[\ln f(x)]})]} = e_q^{D_q[\ln f(x)]} = e_q^{D_q^{(2)}[\ln f(x)]}.
\]

Let’s continue the same transaction in order to find other high-level derivations:

\[
D_q^{*(3)}[f(x)] = e_q^{D_q[\ln (e_q^{D_q^{(2)}[\ln f(x)]})]} = D_q[\ln f(x)] = e_q^{D_q[\ln f(x)]}
\]

when we repeat this procedure for \( n \) times, if the \( n \)th \( q^{*} \) derivative of \( f(x) \) exists at \( x \), then it is obtained by

\[
D_q^{(n)}[f(x)] = e_q^{D_q^{(n)}[\ln f(x)]}.
\]
Corollary 2. Let $q \in (0, 1)$ and $f(x)$ positive function. Then

$$D_q^{*n}[f(x)] = e_q^{(1-q)^{-n} x^{-n} \sum_{k=0}^{n-1} \binom{n}{k} (-1)^k q^{-k(n-k)} q^{-k(k-1)/2} \ln f(xq^k)^{q^k}.} \quad (12)$$

Proof. It is easily seen that we can denote to proof of $n$th $q^*$ derivative from (2). \hfill \Box

3.1.1. The operation properties of $q$-multiplicative derivative.

We have given general definition about $q^*$ derivative. Now, some rules with their proof obtained from these definitions are as follows. Suppose that $f$ and $g$ are $q^*$ differentiable and $\alpha, \beta$ is a positive constant. Then, the list below can be easily presented:

i) Constant rule: $D_q^* \left[ \alpha f(x) \right] = D_q^* \left[ f(x) \right]$, 

ii) Product Rule: $D_q^* \left[ f(x)g(x) \right] = D_q^* \left[ f(x) \right] D_q^* \left[ g(x) \right]$, 

iii) Quotient Rule: $D_q^* \left[ \frac{f(x)}{g(x)} \right] = \frac{D_q^* \left[ f(x) \right]}{D_q^* \left[ g(x) \right]}$, 

iv) Chain Rule: $D_q^* \left[ f(g(x)) \right] = D_{q,g}^* \left[ f(g(x)) \right] g_q(x)$, 

v) Power Rule: $D_q^* \left[ f(x)^{g(x)} \right] = (D_q^* \left[ f(x) \right])^{g(x)} f(x) D_q^* \left[ g(x) \right]$. 

We proved some of the rules as follows. The rule (i) can be proved as:

$$D_q^* \left[ \alpha f(x) \right] = \left[ \frac{\alpha f(qx)}{\alpha f(x)} \right] \left[ \frac{(q-1)x}{(q-1)x} \right] = \frac{f(qx)}{f(x)} = D_q^* \left[ f(x) \right].$$

The rule (ii) can be proved as:

$$D_q^* \left[ f(x)g(x) \right] = e_q^{D_q^*[\ln(f(x)g(x))]} = e_q^{D_q^*[\ln f(x)+\ln g(x)]} = e_q^{D_q^*[\ln f(x)]} e_q^{D_q^*[\ln g(x)]} = D_q^* \left[ f(x) \right] D_q^* \left[ g(x) \right].$$

The rule (iv) can be proved as:

$$D_q^* \left[ f(g(x)) \right] = \left[ \frac{f(g(qx))}{f(g(x))} \right] \left[ \frac{1}{(q-1)x} \right] = \left[ \frac{f(g(qx))}{f(g(x))} \right] \left[ \frac{q^k(qx)}{q^k} \right] = D_{q,g}^* \left[ f(g(x)) \right] g_q(x).$$
The rule (v) can be proved as:

\[ D_q^n[f(x)^g(x)] = e_qD_q[\ln(f(x))^{g(x)}] \]

\[ = e_qD_q[g(x) \ln f(x)] \]

\[ = e_q^g(qx)D_q[\ln f(x)] + \ln f(x)D_q[g(x)] \]

\[ = e_q^g(qx)\ln f(x)D_q[g(x)] \]

\[ = (D_q^n[f(x)])^g(qx) f(x)^D_q[g(x)]. \]

**Theorem 3.** Let \( f(x) \) and \( g(x) \) be two functions, Then, for any nonnegative integer \( n \)

\[ D_q^n[f(x)^g(x)] = e_q \sum_{k=0}^{n} \binom{n}{k} q^k \ln(f(q^n-kx)) D_q^{n-k}[g(x)]. \]  

(13)

**Proof.** Firstly, let’s begin to find 1st level derivations of the left-hand side of expression 13 step by step

\[ D_q^1[f(x)^g(x)] = e_q D_q[\ln f(x)] \]

\[ = e_q^1 \ln f(x)D_q[g(x)] + g(x)D_q[\ln f(x)]. \]

For 2nd level derivation, we can obtain

\[ D_q^2[f(x)^g(x)] = e_q D_q[q^2 \ln f(x)] \]

\[ = e_q^2 \ln f(q^2x)D_q^2[g(x)] + [2]_q D_q[g(x)]D_q[\ln f(x)] + g(x)D_q[\ln f(x)]D_q^2[\ln f(x)]. \]

For 3nd level derivation, we can obtain

\[ D_q^3[f(x)^g(x)] = e_q D_q[q^3 \ln f(x)] \]

\[ = e_q^3 \ln f(q^3x)D_q^3[g(x)] + [3]_q D_q[g(x)]D_q[\ln f(x)] + [3]_q D_q[g(x)]D_q[\ln f(q^2x)] + g(x)D_q^2[\ln f(x)]. \]

Following the same procedure, for calculate of \( D_q^n[f(x)^g(x)] \) can be found as follows

\[ D_q^n[f(x)^g(x)] = e_q \sum_{k=0}^{n} \binom{n}{k} q^k \ln(f(q^n-kx)) D_q^{n-k}[g(x)]. \]

Example 1. Consider exponential function \( f(x) = [\alpha]_q x^\lambda \) with \( \lambda > 0 \), let’s find its \( q \)-derivative

\[ D_q^n[f(x)] = e_q D_q[\ln[\alpha]_q x^\lambda] = e_q^\lambda = \lambda. \]

Notice that, the constant obtained from \( q \)-derivative of exponential function in this example, is also multiplicative rate. In other words, \( \lambda \) is factor multiplied in unit of time.

In addition, we formed the table below. Analyzing the table, you will see that some results are same with multiplicative derivative especially first three order.
Also, the functions analyzed in this table include multiplication and exponentiation which are the most convenient for q-multilcative differentiation.

| \( f(x) \) | \( f^*(x) \) | \( f_q^*(x) \) |
|-----------|-----------|-----------|
| \( k \)   |          1 |          1 |
| \( k e^{nx} \) | \( e^k \) | \( e_q^{|k|} \) |
| \( k b^x \)  | \( b \)  | \( b \)   |
| \( k x^n \)  | \( e^{\frac{n}{q}} \) | \( e_q^{rac{n}{q-1}} \) |
| \( k e^{g(x)} \) | \( g^*(x) \) | \( g_q^*(x) \) |
| \( g(x)^k \)  | \( g^*(x)^k \) | \( g_q^*(x)^k \) |
| \( (g(x))^{h(x)} \) | \( g^*(x)^{h(x)} \) | \( g_q^*(x)^{h(x)} \) |
| \( e^{\sin \alpha x} \) | \( e^{\alpha \cos \alpha x} \) | \( e_q^{\cos \alpha x} \) |
| \( e^{\cos \alpha x} \) | \( e^{\sin \alpha x} \) | \( e_q^{\cos \sin \alpha x} \) |

Table 1 – \( q^* \) derivative rules

Now, we will mention some properties of continuous functions in q-multiplicative calculus. The behavior of \( q^* \) derivative in a neighborhood of a local extreme point is described.

**Theorem 4.** Suppose that \( f(x) \) be a continuous function on a segment \([a, b]\) and \( c \in (a, b) \) be a point of its local maximum.

i) if \( 0 < a < b \), then there exists \( \mu \in (0, 1) \) such that

\[
D_q^*[f(c)] \left\{ \begin{array}{l}
\geq 1 \quad \forall q \in (\mu, 1) \\
\leq 1 \quad \forall q \in (1, \mu^{-1})
\end{array} \right\}
\] (14)

ii) if \( a < b < 0 \), then there exists \( \mu \in (0, 1) \) such that

\[
D_q^*[f(c)] \left\{ \begin{array}{l}
\leq 1 \quad \forall q \in (\mu, 1) \\
\geq 1 \quad \forall q \in (1, \mu^{-1})
\end{array} \right\}
\] (15)

Furthermore, \((\forall q \in (\mu, \mu^{-1}))(\exists \xi \in (a, b)) : D_q^*[f(\xi)] = 1.\)

**Proof.** As (i) and (ii) are proven in a similar pattern, we only express one of them. As the maximum point of \( f(x) \) function is \( c \), there exists \( \varepsilon > 0 \). \( f(x) \leq f(c) \), for all \( x \in (c - \varepsilon, c + \varepsilon) \subset (a, b) \). Suppose that \( 0 < a < b \), \( q_0 \in (0, 1) \), such that \( c - \varepsilon < q_0 c < c \). Now, for all \( q \in (q_0, 1) \), it is valid \( q e < c \) and \( \frac{1}{q e - c} < 0 \)

\[
\frac{f(qc)}{f(c)} \leq 1 \quad \Rightarrow \quad \left( \frac{f(qc)}{f(c)} \right)^{\frac{1}{q e - c}} \geq 1^{\frac{1}{q e - c}} = 1
\]

\[
D_q^*[f(c)] \geq 1.
\]

Similarly, there exists \( q_1 \in (0, 1) \) such that \( c < c/q_1 < c + \varepsilon \) and for all \( q \in (1, q_1^{-1}) \) it is valid \( q e > c \) and \( \frac{1}{q e - c} > 0 \)

\[
D_q^*[f(c)] \geq 1.
\]
Proof. In a similar way, it can be proved like the proof of theorem 4. Suppose that \( c \in (a, b) \) is maximum value for \( f(x), a < b < 0, q_2 \in (0, 1) \), such that \( cq_2 \in (c, c + \varepsilon) \). Now for all \( q \in (q_2, 1) \), it is valid \( qc > c \) and \( \frac{1}{qc - c} > 0 \)

\[
\frac{f(qc)}{f(c)} \leq 1 \Rightarrow \left( \frac{f(qc)}{f(c)} \right)^{\frac{1}{q-1}} \leq 1^{\frac{1}{q-1}} = 1 \tag{17}
\]

\( D_q^*[f(c)] \leq 1. \)

So we can denote by \( \mu = \max\{q_0, q_1\} \).

Similarly, there exists \( q_3 \in (0, 1) \) such that \( c - \varepsilon < c/q_3 < c \) and for all \( q \in (1, q_3^{-1}) \) it is valid \( qc < c \) and \( \frac{1}{qc - c} < 0 \)

\[
\frac{f(qc)}{f(c)} \leq 1 \Rightarrow \left( \frac{f(qc)}{f(c)} \right)^{\frac{1}{q-1}} \geq 1^{\frac{1}{q-1}} = 1 \tag{18}
\]

\( D_q^*[f(c)] \geq 1. \)

So we can denote by \( \mu = \max\{q_2, q_3\} \). As the function \( f \) has a derivation at \( c \) point, there are \( q^* \) derivatives from the left and the right and they are equal to each other. Therefore, we can obtain from (17-20) \( c \in (a, b), D_q^*[f(c)] \geq 1 \Rightarrow D_q^*[f(c)] \leq 1 \Rightarrow D_q^*[f(c)] = 1. \)

**Theorem 5.** Suppose that \( f(x) \) be a continuous function on a segment \([a, b]\) and \( c \in (a, b) \) be a point of its local minimum.

i) If \( 0 < a < b \), then there exists \( \mu \in (0, 1) \) such that

\[
D_q^*[f(c)] \begin{cases} \leq 1 & \forall q \in (\mu, 1) \\ \geq 1 & \forall q \in (1, \mu^{-1}) \end{cases} \tag{20}
\]

ii) if \( a < b < 0 \), then there exists \( \mu \in (0, 1) \) such that

\[
D_q^*[f(c)] \begin{cases} \geq 1 & \forall q \in (\mu, 1) \\ \leq 1 & \forall q \in (1, \mu^{-1}) \end{cases} \tag{21}
\]

Furthermore, \( (\forall q \in (\mu, \mu^{-1}) (\exists c \in (a, b)) : D_q^*[f(c)] = 1. \)

Proof. In a similar way, it can be proved like the proof of theorem 4. \( \square \)

Note that, if is differentiable for all \( x \in (a, b) \), then \( \lim_{q \to 1} D_q^*[f(x)] = f^*(x). \) So, if \( c \in (a, b) \) is a point of local extreme of \( f(x) \), we obtain \( f^*(c) = D^*_q[f(c)] = 1. \)

**Theorem 6.** (q-Multiplicative Rolle) Let \( f(x) \) be a continuous function on \([a, b]\) satisfying \( f(a) = f(b) \). Then there exists \( \mu \in (0, 1) \), such that

\[
(\forall q \in (\mu, \mu^{-1}))(\exists \xi \in (a, b)) : D_q^*[f(\xi)] = 1. \tag{22}
\]
Proof. If \( f(x) \) is a constant function on \((a, b)\), we have \( D^q[f(x)] = 1 \). However \( f(x) \) has its extreme value in some point in \((a, b)\). So according to theorem (4,5), it is easily found \( c \in (a, b) \), \( D^q[f(c)] = 1 \).

Theorem 7 (q-Multiplicative Mean Value Theorem). Let \( f(x) \) be continuous function on the closed interval \([a, b]\) and which is \( q^* \)-differentiable at every point of \((a, b)\). Then there exists \( \mu \in (0, 1) \) such that

\[
(\forall q \in (\mu, \mu^{-1}) \exists \xi \in (a, b)) : D^q[f(\xi)] = \left(\frac{f(b)}{f(a)}\right)^{\frac{1}{q-\mu}}. \tag{23}
\]

Proof. Let

\[
G(x) = \frac{f(x)}{f(a)} \left(\frac{f(a)}{f(b)}\right)^{\frac{x-a}{b-a}}, \quad \forall x \in [a, b]
\]

\( G(x) \) be a function continuous on \([a, b]\) and it is \( q^* \)-differentiable on \((a, b)\). Additionally, it is easily found that \( G(a) = G(b) = 1 \). The hypotheses of Rolle’s Theorem for \( q^* \)-derivatives are convenient and there is some \( c \) in \((a, b)\), such that \( D^q[G(c)] = 1 \).

Thus we can apply \( D^q[G(c)] = 1 \) to (23)

\[
1 = D^q[f(c)] \left(\frac{f(a)}{f(b)}\right)^{\frac{x-a}{b-a}}.
\]

which then implies

\[
D^q[f(c)] = \left(\frac{f(b)}{f(a)}\right)^{\frac{1}{q-\mu}}. \tag{24}
\]

Theorem 8. (q-Multiplicative Taylor’s Theorem) Let \( f(x) \) be a continuous function on some interval \((a, b)\) and \( c \in [a, b] \). Then q-multiplicative Taylor formula is defined by

\[
f(x) = \prod_{k=0}^{n} (D^q[k][f(a)])^{\frac{(x-a)^k}{(k+1)q^k}} R_{n+1}(x) \tag{25}
\]

where

\[
(x - a)^0_q = 1, \quad (x - a)^k_q = \prod_{i=0}^{k-1} (x - aq^i), \quad (k \in \mathbb{N}).
\]

\( R_{n+1}(x) \) is a remainder term in q-multiplicative Taylor formula and there exists \( \mu \in (0, 1) \) such that for all \( q \in (\mu, 1) \), and some \( c \) between \( a \) and \( b \), then

\[
R_{n+1}(x) = (D^q[n+1][f(c)])^{\frac{(x-a)^{n+1}}{(n+1)q^{n+1}}}. \tag{26}
\]

Example 2. Consider function \( f(x) = e^{x^n} \) and \( a = 1 \). Let us expand \( f(x) \) using q*-Taylor’s formula. For \( k \leq n \), we have
\[ D_q^*[f(x)] = e^{D_q[\ln e^x]} = e^{D_q[x^n]} \\
= e^{[n]_q x^{n-1}} \]

\[ D_q^{[2]}[f(x)] = e^{D_q^{[2]}[x^n]} = e^{[n]_q [n-1]_q x^{n-2}} \]

\[ D_q^{[k]}[f(x)] = e^{[n]_q [n-1]_q ... [n-k+1]_q x^{n-k}} \]

\[ D_q^{[k]}[f(1)] = e^{[n]_q [n-1]_q ... [n-k+1]_q} \quad \text{for} \quad x = 1 \]

Then, it can be obtained

\[ e^x = \prod_{k=0}^{n} \left( e^{[n]_q [n-1]_q ... [n-k+1]_q} \right)^{\frac{(x-1)^k}{[k]_q}} = \prod_{k=0}^{n} e^{[n]_q [n-1]_q ... [n-k+1]_q (x-1)^k} \]

3.2. \textit{q}-multiplicative integral.

We introduced the \textit{q}-integral and will establish a multiplicative integral relation to the respective \textit{q}-multiplicative integral. Riemann multiplicative integral of \( f \) on \([a, b]\) is as follows: Let \( f \) be a positive bounded function on \([a, b]\). \( W = \{ x_0, x_1, ..., x_m \} \) on \([a, b]\) is a partition, \( \xi_k, \quad k = 1, 2, ..., m \) is any point between \( x_{k-1} \) and \( x_k \). \( \Delta s_k \) is \( x_k - x_{k-1} \). According to the definition of integral product from Bashirov et al. [1] is given by

\[ P(f, W) = \prod_{k=1}^{m} f(\xi_k)^{\Delta s_k} \]

and Riemann integral of \( \ln f \) on \([a, b]\) is given by

\[ \prod_{k=1}^{m} f(\xi_k)^{\Delta s_k} = e^{\sum_{k=1}^{m} (\ln f(\xi_k)) \Delta s_k}. \]

The symbol of multiplicative integral is denoted as

\[ \int_{a}^{b} f(x)^{dx} \]

and if \( f \) is a positive function and the integrable of \( \ln f \) on \([a, b]\) exists, then the multiplicative integral of \( f \) on \([a, b]\) exists and is defined by

\[ \int_{a}^{b} f(x)^{dx} = e\int_{a}^{b} \ln f(x)^{dx}. \quad (26) \]

\textbf{Definition 2.} Let \( f \) be a positive bounded function on \( 0 < a < b \). The \textit{q}-analogue of the multiplicative integral, the \textit{q}-multiplicative integral, can be defined by

\[ \int_{a}^{b} f(x)^{dqx} = e_q^{\int_{a}^{b} \ln f(x)^{dqx}} \quad q \in (0, 1) \quad (27) \]

and the definite \textit{q}-multiplicative integral is defined as
Now, we denote Riemann sum for \( q \)-multiplicative integral (shortly \( q^* \) integral).

Let \( \Omega_q \) be the linear operators, \( \Omega_q (F(x)) = F(qx) \) (see [8]). By definition of \( D_q \) and by the fact that \( D_q[F(x)] = f(x) \). Suppose that \( F(x) = \int \ln f(x) d_q x \) for power of (28), then, we can get \( D_q[F(x)] = \ln f(x) \). Then we have

\[
\frac{1}{(q-1)x}(\Omega_q - 1)F(x) = \frac{F(qx) - F(x)}{(q-1)x} = \ln f(x).
\]

We can formally compute that

\[
F(x) = \frac{1}{1-\Omega_q}((1-q)x \ln f(x))
= (1-q) \sum_{i=0}^{\infty} \Omega_q^i(x \ln f(x))
= (1-q)(x \ln f(x) + qx \ln f(qx) + q^2x \ln f(q^2x) + ...)
= (1-q)x \sum_{i=0}^{\infty} q^i \ln f(q^i x)
\]

Substituting (29) into (27), we have

\[
e^\int_q \ln f(x) d_q x = e^{(1-q)x \sum_{i=0}^{\infty} q^i \ln f(q^i x)}.
\]

Notice that

\[
\lim_{q \to 1} \int_a^b f(x) d_q x = \int_a^b f(x) dx.
\]

### 3.2.1. The operation properties of \( q \)-multiplicative integral.

Let \( f \) and \( g \) be \( q^* \) integrable on \([a, b]\). Then, we can easily show the following rules of \( q^* \) integral.

(i) Power Rule: \( \int_a^b (f(x)^k) d_q x = \int_a^b (f(x)^{d_q x})^k \quad k \in \mathbb{R} \)

(ii) Product Rule: \( \int_a^b (f(x)g(x)) d_q x = \int_a^b f(x) d_q x \int_a^b g(x) d_q x \)

(iii) Quotient Rule: \( \int_a^b (f(x)/g(x)) d_q x = \int_a^b f(x) d_q x / \int_a^b g(x) d_q x \)

(iv) \( \int_a^b f(x) d_q x = \int_a^c f(x) d_q x + \int_c^b f(x) d_q x \quad a \leq c \leq b. \)

The rule (iii) can be proved:
\[
\int_a^b \left( \frac{f(x)}{g(x)} \right) dq = e_q \int_a^b (\ln f(x) - \ln g(x)) dx = e_q^b (\ln f(x)) - e_q^a (\ln g(x)) = e_q^b f(x) - e_q^a g(x).
\]

**Example 3.** Let \( f(x) = e_q^{\sin_q(kx)} \) where \( k \) is a constant. Then \( q^* \) integral of \( f(x) \) is obtained by

\[
\int e_q^{\sin_q(kx)} dq = e_q \int \ln e_q^{\sin_q(kx)} dx = e_q \int \sin_q(kx) dx = e_q \frac{1}{k} \cos_q(kx).
\]

**Example 4.** Let \( f(x) = e_q^{kx} \) where \( k \in \mathbb{Z}^+ \). Then \( q^* \) integral of \( f(x) \) is obtained by

\[
\int e_q^{kx} dq = e_q \int \ln e_q^{kx} dx = e_q \int \frac{k(1-q)x}{1-q^2} dq = e_q \frac{kx^2}{1-q^2}.
\]

Notice that, \( \lim_{q \to 1} \int e_q^{kx} dq = \frac{kx^2}{1}. \) This result is also the same as option (c) of table 2 in [8].

We present a framework of the basic rules for \( q^* \) integral under table 2, with rules for the classic multiplicative integral added for comparison. The rules are easy to check.

| Rule                        | Framework |
|-----------------------------|-----------|
| \( I^*(f) \)                | \( I_q^*(f) \) |
| \( \int (f(1)) dx = c \)   | \( \int (f(1)) dq = c \) |
| \( \int (k) dx = c k^x \)  | \( \int (k) dq = c k^x \) |
| \( \int (e^{kx}) dx = c e^{kx} \) | \( \int (e^{kx}) dq = c e_q^{kx} \) |
| \( \int (e^{kx^n}) dx = c e_q^{kx^{n+1}} \) | \( \int (e^{kx^n}) dq = c e_q^{kx^{n+1}} \) |
| \( \int (\cos x) dx = c \sin x \) | \( \int (\cos x) dq = c e_q^{\sin x} \) |
| \( \int (\sin x) dx = c \cos x \) | \( \int (\sin x) dq = c e_q^{\cos x} \) |

*Table 2 - \( q^* \) integral rules*
Theorem 9. (Fundamental theorem of calculus for q-multiplicative integral) Let \( f(x) \) be a differentiable positive and continuous on a segment \( 0 \leq a < b \), \( D_q^*[f(x)] = f(x) \), then

\[
\int_a^b f(x)^{d_q}x = \frac{F(b)}{F(a)}. \tag{31}
\]

Proof. Let \( D_q^*[F(x)] = f(x) \) and \( F(x) \) is continuous on \( [a, b] \). Then we can denote by

\[
F(b) = e_q^\int_0^b \ln f(x)dx = e_q^{\frac{(1-q)b}{q} \sum_{i=0}^{\infty} q^i \ln (q^ib)}.
\]

Similarly, we can show \( F(a) \) for finite \( a \). Now, using (28)

\[
\int_a^b f(x)^{d_q}x = e_q^\int_0^b \ln f(x)dx - \int_0^a \ln f(x)dx = e_q^\int_0^b \ln f(x)dx - e_q^\int_0^a \ln f(x)dx = \frac{F(b)}{F(a)}.
\]

This proves the theorem. \( \square \)

Corollary 3. Let \( f(x) \) be \( q^* \) differentiable and \( D_q^*[f(x)] \) be \( q^* \) integrable on \( [a, b] \), then

\[
\int_a^b (D_q^*[f(x)])^{d_q}x = \frac{f(b)}{f(a)}. \tag{32}
\]

Proof.

\[
\int_a^b (D_q^*[f(x)])^{d_q}x = \int_a^b (e_q^\int_0^b [\ln f(x)]^{d_q}x = e_q^\int_a^b \ln f(x)^{d_q}x
\]

from theorem 1

\[
\int_a^b (D_q^*[f(x)])^{d_q}x = e_q^\int_0^b \ln f(x)^{d_q}x = e_q^\int_0^b \ln f(b) - \ln f(a)
\]

\[
= e_q^\int_0^b \frac{f(b)}{f(a)}
\]

\( \square \)

Theorem 10. (q-Multiplicative Integration by Parts) Let \( f(x) \) be a \( q^* \) differentiable and \( g(x) \) be \( q \)-differentiable, they are continuous on a segment \( 0 \leq a < b \), then

\[
\int_a^b ((D_q^*[f(x)])^{d_q}x = \frac{f(b)g(b)}{f(a)g(a)} \int_a^b \frac{1}{f(x)^{q\alpha(x)}}. \tag{33}
\]
Proof. Using the definition of $q^*$integral, we have

$$
\int_{a}^{b} ((D_q^*[f(x)](x))^{q^*_v})^{d_q} = \int_{q}^{1} \left( e^{q^*_v} D_q^{\ln f(x)} \right)^{d_q} = e^{q^*_v} \int_{q}^{1} g(x) D_q^{\ln f(x)} dx.
$$

Using the product rule for $q$-derivative, we have

$$
e^{q^*_v} D_q^{\ln f(x)} = e^{\ln f(qx) D_q^{\ln g(x)} + g(x) D_q^{\ln f(x)}}$$

we can apply (32) to obtain

$$
e^{q^*_v} D_q^{\ln f(x)} = e^{\ln f(qx) D_q^{\ln g(x)} + g(x) D_q^{\ln f(x)}}$$

$$
e^{q^*_v} D_q^{\ln f(x)} = e^{\ln f(qx) D_q^{\ln g(x)} + g(x) D_q^{\ln f(x)}}$$

4. Conclusion. The purpose of this article is to apply $q$-analogues on multiplicative calculus by analyzing multiplicative calculus which has been a topic for most recent researches. According to our research, we strongly believe that $q$-multiplicative calculus, which is the extended version of multiplicative calculus, can show the way for further research fields with new definitions theories and applications.

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