Integer quantum Hall effect of interacting electrons: dynamical scaling and critical conductivity

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We report on a study of interaction effects on the polarization of a disordered two-dimensional electron system in a strong magnetic field. Treating the Coulomb interaction within the time-dependent Hartree-Fock approximation we find numerical evidence for dynamical scaling with a dynamical critical exponent \( z = 1 \) at the integer quantum Hall plateau transition in the lowest Landau level. Within the numerical accuracy of our data the conductivity at the transition and the anomalous diffusion exponent are given by the values for non-interacting electrons, independent of the strength of the interaction.

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While the occurrence of quantized plateaus in the Hall conductivity of two-dimensional systems in strong magnetic fields is the most striking aspect of the quantum Hall effect (QHE), it is the transitions between these plateaus that have recently attracted a lot of attention, both experimentally and theoretically. Our understanding of the QHE is not complete without an understanding of the plateau transitions. Furthermore, the plateau transitions are among the most extensively studied examples of quantum phase transitions. The origin of the plateaus is the localization of the electrons due to disorder, with the plateau transitions corresponding to the localization-delocalization transitions in the disorder potential. Theoretical studies have focused on the effect of disorder on non-interacting electrons in a strong magnetic field. From numerical and analytical calculations the following picture emerged for weak disorder: in every Landau level there exists a critical energy \( E_c \) at which the localization length \( \xi \) diverges as a power law \( |E - E_c|^{-\nu} \) with a localization length exponent \( \nu = 2.35 \pm 0.03 \) independent of Landau level index and correlation length of the disorder potential. At all other energies electrons are exponentially localized. In the region of localized states the longitudinal conductivity vanishes and the Hall conductivity is quantized in multiples of \( e^2/h \). At the critical energies the Hall conductivity jumps by \( e^2/h \) and the longitudinal conductivity takes on a finite value \( \sigma_c \) with \( \sigma_c \approx 0.5e^2/h \) in the lowest Landau level. The eigenstates at the critical energies show multifractal fluctuations leading to correlation functions being characterized by an anomalous diffusion exponent \( \eta \approx 0.4 \).

Dynamical correlation functions show scaling behavior as a function of the variable \( qL_\omega \), where \( q \) is the wave-vector and \( L_\omega = 1/\sqrt{\rho_0\hbar\omega} \) is the relevant frequency-dependent length scale, the effective system size with mean level spacing \( \hbar\omega \). Comparing this result to the definition of the dynamical critical exponent \( L_\omega \propto \omega^{-1/z} \) and observing that the density of states (DOS) \( \rho_0 \) is finite at the transition, the dynamical critical exponent is found to be \( z = 2 \).

Experimentally, strong evidence for scaling behavior near the plateau transitions was observed. In particular, the observed values for the localization length exponent \( \nu = 2.3 \pm 0.1 \) and the critical conductivity \( \sigma_c \approx 0.5e^2/h \) are in remarkable agreement with the theory for non-interacting electrons. However, in high-frequency measurements scaling with a dynamical critical exponent \( z = 1 \) was observed in contrast to the value \( z = 2 \) for non-interacting electrons. This discrepancy has been attributed to the influence of the Coulomb interactions between the electrons that are always present in the experiments. In fact, numerical simulations in which the interactions are treated in self-consistent Hartree-Fock (HF) approximation show a linear suppression of the DOS at the Fermi energy irrespective of the position of the Fermi energy. According to the argument presented above at the critical energy this non-critical effect leads to a reduction of the dynamical critical exponent \( z \) from 2 to 1. The influence of the Coulomb interactions on other critical properties of the QH transitions was also studied within the self-consistent HF approximation. From the scaling of the participation ratio of the HF eigenstates it was found that the localization length exponent \( \nu \) and the fractal dimension \( D_2 = 2 - \eta \) were not changed by the interactions. In lieu of the critical conductivity the Thouless numbers of the HF eigenenergies were studied and also found to be unchanged by the interactions. However, not much is known about the relation of the conductivity of an interacting electron system and the Thouless numbers of the HF eigenvalues.

In this paper we overcome some of the deficiencies of the previous studies by directly calculating the irreducible dynamical polarization \( \Pi^{\text{irr}}(q,\omega) \). This allows us...
to obtain the critical quantities $z$, $\sigma_c$, and $\eta$ from the scaling limit of the polarization. Let us first sketch our approach. As the starting point for our numerical calculations we use the self-consistent Hartree-Fock (TDHF) approximation [11,12]. The corresponding conserving approximation for the two-particle Green’s function is the time-dependent Hartree-Fock (TDHF) approximation [13]. Within this approximation we evaluated the dynamical polarization $\Pi^{\text{irr}}(q, \omega)$ including vertex corrections. The disorder-averaged polarization is extrapolated to infinite system size and vanishing $q$ and $\omega$. In this limit, the polarization obeys the scaling form

$$\Pi^{\text{irr}}(q, \omega) = \frac{\sigma^* q^2/e^2}{\sigma^* q^2/e^2 \chi_q^{\text{irr}}} - i\omega.$$ (1)

The Onsager conductivity $\sigma^*$ is related by the Einstein relation

$$\sigma^* = e^2 \chi_q^{\text{irr}} D$$ (2)

to the diffusion coefficient $D(q, \omega)$ and the static susceptibility $\chi_q^{\text{irr}}$. For vanishing wave-vector it coincides with the Kubo conductivity $\sigma = e^2 \omega \Pi^{\text{irr}}/i q^2$. Due to the occurrence of the Coulomb gap even at criticality the susceptibility $\chi_q^{\text{irr}} \propto q$. The conductivity $\sigma^*(q, \omega)$ is found to be a function of the scaling variable $x = q^2/\chi_q^{\text{irr}} \hbar \omega$ with the limiting behavior

$$\sigma^* (x) \rightarrow \left\{ \begin{array}{ll}
\sigma^*_{\text{ex}}, & x \rightarrow 0, \\
\sigma^*_{\alpha}, & x \rightarrow \infty.
\end{array} \right.$$ (3)

Since $x \propto q/\omega$ the Coulomb gap changes the dynamical critical exponent $z$ from its non-interacting value 2 to 1. On the other hand, we find that the critical DC conductivity $\sigma^*_{\alpha} = 0.5 \pm 0.1 e^2/h$ and the anomalous diffusion exponent $\eta = 0.4 \pm 0.1$ are independent of the strength of the interactions and in agreement with previous calculations for non-interacting electrons [11,12]. These are the central results of our study.

In the remainder of the paper we present our model, our calculations and our results, relegating a more detailed discussion to a separate publication [15]. We consider the situation where the cyclotron energy $\hbar \omega_c$ is much larger than both the disorder and the Coulomb energy and the Fermi energy is in the lowest Landau level onto which we project the Hamiltonian. As a model for the disorder we use the random Landau matrix [16]. The strength of the disorder potential is characterized by the level broadening in self-consistent Born approximation $\Gamma$ and its range is taken to be zero. The electrons interact via the bare Coulomb interaction $V(q) = 2\pi e^2/\kappa q$. The strength of the interactions is given by the ratio $\gamma = (e^2/\kappa l)/\Gamma$, where $\kappa$ is the dielectric constant and $l$ is the magnetic length. The square systems of linear size $L$ with periodic boundary conditions contained $N$ electrons and from $N_{\Phi} = L^2/2\pi l^2 = 169$ to 900 flux quanta. In the presence of interactions physical quantities depend on the filling factor $\nu_F = N/N_{\Phi}$. We choose $\nu_F = 1/2$ in all our calculations so that the Fermi energy coincides with the critical energy of the plateau transition. Including the HF interaction the eigenvalues $\epsilon_\alpha$ and eigenfunctions $|\alpha\rangle$ of the Hamiltonian are calculated self-consistently [11,12]. A major simplification arises in this calculation since in the lowest Landau level the exchange interaction

$$V_{\text{ex}}(q, q') = -\delta_{q,-q'} \sqrt{2\pi^3 e^2 \kappa \hbar} \frac{e^2 q^2}{4 I_0(q^2 l^2/4)}$$ (4)

and hence the HF interaction

$$V_{\text{HF}}(q, q') = \delta_{q,-q'} (V(q) + V_{\text{ex}}(q))$$ (5)

become local ($I_0$ is the Bessel function) [17,18].

Perhaps the most characteristic feature of the HF spectrum is the formation of a Coulomb gap in the single-particle DOS [19]. In particular, the DOS at the Fermi energy $\rho(E_F)$ decreases linearly with $l/\gamma L$. This is shown in Fig. 1 [11,12]. The suppression of the DOS is not related to the phase transition but occurs for all values of the Fermi energy. Only when the Fermi energy coincides with the critical energy does it change the critical dynamics of the system.

The bare polarization $\Pi^0(q, q', \omega)$ is given by

$$\Pi^0(q, q', \omega) = \sum_{\alpha, \beta} (\alpha|\beta \rangle q \langle \beta|\alpha\rangle q') \times \left[ \frac{f_\alpha (1 - f_\beta)}{\hbar \omega + \epsilon_\beta - \epsilon_\alpha - i\delta} - \frac{f_\beta (1 - f_\alpha)}{\hbar \omega + \epsilon_\beta - \epsilon_\alpha + i\delta} \right],$$ (6)

where $(\beta|\alpha\rangle_q = (\beta|e^{-iqr}|\alpha\rangle, f_\alpha$ is the Fermi function (we take $T = 0$ in our calculations) and $\delta$ is an infinitesimal parameter of the order of the mean level spacing. After averaging over disorder (we use between

![FIG. 1. Single-particle density of states at the critical energy as a function of $l/\gamma L$ for $N_{\Phi}$ between 100 and 900 and $\gamma$ between 0.15 and 1.](image-url)
13 and 100 realizations of the disorder) $\Pi^0$ becomes translationally and rotationally invariant $\Pi^0(q, q', \omega) = \delta(q + q')\Pi^0(q, \omega)$. The real part of the static bare polarization coincides in the limit $q \to 0$ with the single particle DOS, $\lim_{q \to 0} \text{Re} \Pi^0(q, \omega = 0) = \chi_0^0 = \rho(E_F)$ [53].

The TDHF approximation for the irreducible and the screened polarizations are shown in Fig. 2. In the lowest Landau level the ladder sum for the vertex corrections can be transformed into a RPA-like bubble sum with the Coulomb potential replaced by $A \Pi^0(q, \omega)$ [20, 15]. Similarly, the screened polarization $\Pi^{\text{scr}}$ is related to the bare polarization through a bubble sum containing the Hartree-Fock potential $V_{\text{HF}}(q)$ [15].

$$
\Pi^{\text{irr}}(q, \omega) = \Pi^0(q, \omega) (1 - V_{\text{ex}}(q) \Pi^{\text{irr}}(q, \omega)) \quad (7a)
$$

$$
\Pi^{\text{scr}}(q, \omega) = \Pi^0(q, \omega) (1 - V_{\text{HF}}(q) \Pi^{\text{scr}}(q, \omega)) \quad (7b)
$$

In principle, the vertex corrections should be applied to $\Pi^0(q, q')$ since it becomes diagonal only after disorder averaging. We checked for small system sizes that using Eq. (7b) only introduces errors comparable to the statistical uncertainties for the relevant parameter range.

Since the approximations leading to $\Pi^{\text{irr}}$ and $\Pi^{\text{scr}}$ are conserving and due to Eq. (7), the polarizations can be parameterized in the limit of infinite system size and vanishing $q$ and $\omega$ through a susceptibility $\chi_q$ and a diffusion coefficient $D(q, \omega)$,

$$
\Pi^A(q, \omega) = \frac{\chi_q^A D^A q^2}{D^A q^2 - i\omega}, \quad (8)
$$

with $A = \{0, \text{irr}, \text{scr}\}$. The Einstein relation $\sigma^A(q, \omega) = e^2 \chi_q^A D^A(q, \omega)$ defines the Onsager conductivity $\sigma^*$. From Eq. (6) it is apparent that the dressing of $\chi_q$ and $D$ leaves the Onsager conductivity unchanged

$$
\sigma^* = e^2 \chi_q^0 D^0 = e^2 \chi_q^{\text{irr}} D^{\text{irr}} = e^2 \chi_q^{\text{scr}} D^{\text{scr}}. \quad (9)
$$

For vanishing wave-vector the Onsager conductivity coincides with the Kubo conductivity $\sigma^*(q = 0, \omega) = \sigma(\omega) = \lim_{q \to 0} e^2 \omega \Pi^{\text{irr}}(q, \omega)/q^2$.

The polarization can be rewritten in terms of $\sigma^*$,

$$
\Pi^{\text{irr}}(q, \omega) = \chi_q^{\text{irr}} \frac{\sigma^* x}{\sigma^* x - i e^2/\hbar}. \quad (10)
$$

with $x = q^2/\chi_q^{\text{irr}} \hbar \omega$. If dynamical scaling holds then $\sigma^* x$ is a function of $q^2/\omega$. In order to extract the scaling behavior from our numerical data for finite system sizes, we first calculate the static susceptibility $\chi_q^{\text{irr}}$ from the real part of the irreducible polarization. Due to the occurrence of the linear Coulomb gap (cf. Fig. 1), $\chi_q^{\text{irr}} \propto q$ for small $q$ in the thermodynamic limit, as is shown in Fig. 4.

Next, the imaginary part of $\Pi^{\text{irr}}$ is divided by the numerically obtained $\chi_q^{\text{irr}}$. The resulting function is then simultaneously extrapolated to infinite system size and $q$ and $\omega$ to zero for fixed value of $x$ as is shown in Fig. 4. This choice of scaling variable allows for reasonable extrapolations, while other choices like $q^2/\rho_0 \hbar \omega$ do not give satisfactory results. Since $\chi^{\text{irr}}$ is asymptotically linear in $q$, $x \propto q/\omega$ in the thermodynamic limit and the dynamical critical exponent $\gamma = 1$.

Figure 4 shows the resulting conductivity $\sigma^*$ as a function of $x$. In the limit $x \to 0$, $\sigma^*$ coincides with the DC Kubo conductivity $\sigma_c$. We find that $\sigma_c$ does not depend on the strength of the interaction up to $\gamma = 0.6$ and is
given by \( \sigma_c = 0.5 \pm e^2/h \). In the opposite limit \( x \to \infty \), the conductivity reflects multifractal density fluctuations and scales like \( x^{1-\eta} \). A numerically more accurate determination of \( \eta \) is possible from the scaling of the static polarization \( F \). We find \( \eta = 0.4 \pm 0.1 \) independent of the interactions and in agreement with published data for non-interacting electrons \( F \). Comparing our result to \( F \) we see that the irreducible polarization reflects the multifractal exponent \( D_2 \) of the HF eigenstates similar to the case of non-interacting electrons.

Let us make a few remarks on our results. Although the conductivity \( \sigma_c \) is finite at the transition, the vanishing susceptibility \( \chi^\text{irr} \) implies non-diffusive behavior of the HF-particles, \( D(q, \omega) \propto q^{-1} \) for \( q \to 0 \). A central issue is the validity of the HF approach that has recently been questioned \( F \). The change in the dynamical critical exponent is due to the use of the unscreened Coulomb interaction. The dielectric function \( \kappa \) is given by \( \kappa(q, \omega) = 1 + V(q) \Pi^\text{irr}(q, \omega) \). From eq. \( F \) it follows that screening behaves differently in the limits of small and large \( x \). In the limit of small \( x \), that is relevant to transport, screening is inefficient and \( \kappa \to 1 \). The use of the unscreened HF approximation thus might be justified for the calculation of the frequency-dependent conductivity. On the other hand, in the limit of large \( x \) we observe static screening, \( \kappa(q, \omega) = 1 + \chi^\text{irr}_q V(q) \). For slow processes, during which the ground state has time to relax, the statically screened HF approximation is more appropriate. It leads to a finite susceptibility and is described by \( z = 2 \). A final judgment on the validity of the HF approach to the critical quantum Hall system might only by possible if one succeeds in a renormalization group treatment of the neglected correlations \( F \).

In summary, we have presented results on the influence of Coulomb interactions on the critical properties of the integer quantum Hall system. Within the time-dependent Hartree-Fock approximation we find the critical conductivity \( \sigma_c \) and the anomalous diffusion exponent \( \eta \) are unchanged by interactions while the dynamical critical exponent \( z \) changes from 2 to 1. The origin of this behavior is the non-critical reduction of the single-particle density of states.

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