Hierarchical clustering and formation of power-law correlation in 1-dimensional self-gravitating system

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August 31, 2000

Abstract
The process of formation of fractal structure in one-dimensional self-gravitating system is examined numerically. It is clarified that structures created in small spatial scale grow up to larger scale through clustering of clusters, and form power-law correlation.

1 Introduction
1-dimensional self-gravitating system has long been used to study systems interacting through self-gravity [Oor32, HF67, RMS8, YGS92, TGK94]. The model helped us to understand various properties, such as motion of stars normal to the galactic plane, relaxation properties of elliptical galaxies, and so on.

Recently we have found that fractal structure dynamically emerges in the system [KK00]. It is surprising that the fractal structure is created from non-fractal initial conditions.

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The emergence of fractal structure has several important meanings. First, there are many fractal (or power-law type) structure in nature. In astrophysics the distribution of young stars, galaxies and cluster of galaxies are known to obey power-law \cite{Lar95, NTHN98, DCGE94}. However there is no universal explanation about the origin of power-law behavior in general. Since the dynamical emergence of fractal structure arises in self-gravitational model, there can be a relation between the origin of these fractal structures.

Second. There are many studies about the relaxation process of 1-dimensional self-gravitating system and it is reported that relaxation is quite slow \cite{RM88, TGK94, and references therein}. It is interesting if there is any relation between the slow relaxation in the model and the dynamical emergence of fractal structure.

Since the system is Hamiltonian system with many particles, one may expect that most initial condition lead the system to thermal equilibrium. It is interesting that some initial conditions, e.g., uniformly random in spatial distribution, evolves to form fractal spatial structure, instead of just become thermalized. Hence to study the origin of the dynamical emergence of fractal structure may give unique insight to non-thermal behavior and structure formation in many-body systems in general \cite{KK92, KK94}.

Therefore it is quite interesting to investigate the reason why the fractal structure emerges from non-fractal initial conditions in 1-dimensional self-gravitating system. In order to do that, it is essentially important to know in detail the process how the fractal structure is created, not just to observe the grown-up fractal structure.

In this letter we continue the study to investigate the process how the fractal structure is developed through time evolution.

The model is presented in section 2. In section 3 we will examine in detail how the power-law correlation is formed, and the final section is for summary and discussions.

## 2 Model

The model we use in this letter is the same as the one used in our previous letter \cite{KK00}, that is, the one-dimensional self-gravitating system. The Hamiltonian is

\[
H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + 2\pi Gm^2 \sum_{i>j} |x_i - x_j| , \quad -\infty < x_i < \infty. \tag{1}
\]

This model represents one-dimensional motion of parallel sheets of infinite extent interacting through Newtonian gravity.
The model is simple and tractable, and yet it also contains essential feature of gravitational interaction.

Throughout this paper we set
\[ m \equiv \frac{1}{N} \quad \text{and} \quad 4\pi G \equiv 1. \quad (2) \]

3 Formation of power-law correlation

In the previous letter \[KK00\] we showed that fractal structure emerges from non-fractal initial conditions. Typical initial conditions we used are those of virial ratio \(2E_{\text{kin}}/E_{\text{pot}} = 0\). (Spatial distribution is set to be random.) These state of zero velocity dispersion corresponds to the limiting case of zero thermal fluctuation.

To analyze the spatial structure we use two-point correlation function \(\xi(r)\), which is defined as
\[
dP = ndV(1 + \xi(r)) \quad (3)
\]
where \(dP\) stands for probability to find another particle in volume \(dV\) at distance \(r\) from a particle. \(n\) is the average number density. If particles are distributed independently and no spatial structure is formed, then \(dP\) does not depend on \(r\) and \(\xi(r) = 0\). If, on the other hand, \(\xi(r)\) increases as \(r\) decreases, the probability that we find another particle near one particle gets large, and we can understand cluster-like structures are formed. If the \(\xi(r)\) does not have characteristic length, then we can understand that the cluster system does not have characteristic length scale.

Fig.1 shows temporal evolution of \(\xi(r)\) for early stage of time. We can clearly see how the power-law behavior of correlation function grows up. That is, power-law structure is first created in small scale, then gradually grows up to larger scale.

The graphs are aligned from bottom to top as time proceeds because we are looking at small-sized scale here \((r < 0.002)\) and particles initially separated are coming near, which increase \(\xi(r)\) for small \(r\).

The time dependence of two point correlation function \(\xi(r)\) tells us when and where the power-law structure is created and how it grows up. However \(\xi(r)\) does not tell what kind of structure is actually formed. So we need another quantity to understand the detail of the structure formation.

Since we are dealing with a gravitational system, interaction is attractive. Each particle, initially distributed in the 1-dimensional space, attracts each other by gravitational force, and small clusters are formed here and there. These clusters attracts and moves around each other, as well as their internal
oscillation (rotation, in 3D). While these processes go on, the order of particles given at initial condition is exchanged, and initial distribution, aligned on the $x$-axis as seen in the $(x,v)$ phase space ($\mu$-space), get folded many times. Hence the degree of folding is a good quantity to understand how the cluster structure is formed.

We can measure the number of folding as follows. Since we are considering the case where initial conditions have zero velocity dispersion, distribution of particles in $(x,v)$ space ($\mu$-space) in each initial condition is on a curve, and we can label particles along the curve. For example, if we take the initial condition as

$$v_i(t=0) = 0, \quad i = 1, 2, \cdots, N,$$

we can label the particles as

$$x_1(0) < x_2(0) < \cdots < x_N(0). \quad (4)$$

Then we can count $f(t)$, the number of folding at time $t$, as

$$f(t) \equiv \text{number of } i \quad (1 < i < N) \text{ which satisfies}$$

$$(x_{i+1}(t) - x_i(t)) \cdot (x_i(t) - x_{i-1}(t)) < 0 \quad (5)$$

Fig. 2 shows how the initial distribution becomes folded. Time dependence of number of folding $f(t)$ shown in this figure and time dependence of the two point correlation function $\xi(r)$ shown in Fig. 1 corresponds well. That is, power-law behavior of $\xi(r)$ develops from small scale as the number
of folding increases, and become most clear when $f(t)$ is at a maximum at $t = t_{16}$.

The process of structure formation, as seen in two quantities $\xi(r)$ and $f(t)$, can also be understood in the real space of particle distribution. Fig. 3 represents time evolution of particle distribution of particles $(x_i(t), v_i(t))$, $8200 < i < 8300$ for $t = t_8, t_{13}, t_{15}$, and $t_{16}$, where $t_\ell$ are time steps defined as

$$t_\ell \equiv \frac{5}{64} \ell .$$

(These time steps are defined just for practical purpose.)

In these figures we connect each particles on $(x, v)$ space according to the order defined at the initial condition (4) so as to enhance the structure of folding. Vertical steps on the graphs represent the location where particles are clustered.

At $t = t_8$ we can see several small clusters are formed. These clusters gradually approach each other to form larger clusters, as seen in $t = t_{13}$, $t_{15}$ and $t_{16}$ in the figure. Through this process power-law correlation is developed.

In later time the clusters get together and larger clusters are developed. In the left of Fig. 4 we show a snapshot of the distribution of particles shown in the previous figure Fig. [3]. This cluster itself is a part of a larger cluster system shown in the right of the figure.
Figure 3: Distribution in \((x,v)\) space at \(t = t_8\) (up:left), \(t_{13}\) (up:right), \(t_{15}\) (down:left) and \(t_{16}\) (down:right) for particles No. 8201 \sim 8299. Particles are linked according to the order assigned at \(t = 0\).

Figure 4: continued from Fig.3 (left) and \(\mu\)-space (wide ranged) (right) at \(t = t_{55}\)
4 Summary and discussions

In this letter we clarified the process of formation of power-law correlation in one-dimensional self-gravitating system, found in our previous letter [KK00]. Power-law behavior of two-point correlation function begins in small spatial scale, then grows up to larger scale. Number of foldings in the order of particles gets large when the power-law behavior is clearly seen. These results and phase space portraits represents the process of formation of fractal structure as follows: tiny clusters are formed first, then the clusters get together to form larger clusters, and so on.

The process we showed is quite similar to the “hierarchical clustering scenario” [Lee80, Man77], which is used for describe the large-scale structure of universe. It is known that two-point correlation function of galaxies and cluster of galaxies obey power-law [DCGE94]. An explanation for the origin of the power-law structure is that density fluctuation of small scale is created first to become some structure of small scale (bottom-up), and, according to the “hierarchical clustering scenario” the structure made in small scale attract each other through gravitational interaction and merge into larger structure, and so on. The initial state is now considered to be of small temperature fluctuation.

In this letter we have shown one example which obeys this “hierarchical clustering scenario” by purely gravitational interaction. An important fact is that our result shows that power-law correlation emerges through hierarchical clustering. Also the initial condition we found for fractal structure to be created is the zero velocity dispersion, which coincides with zero temperature fluctuation considered in the cosmology.

The clustering process is developed by gravitational interaction. Hence similar process can be actually occurring in nature where the dynamics of the system is governed by gravity. One candidate is the power-law distribution of young stars in star-forming regions [Lar95, NTH98].

The emergence of power-law correlation can be considered as a subject of structure formation in general. Our result shows a process of generating fractal structure by purely dynamical interaction.

In this letter we have clarified how the fractal structure is formed from non-fractal initial conditions in one-dimensional self-gravitating system. We hope that this result helps us to understand why the structure is created.
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