Reference frame dependence of local measured Hubble constant

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Observations recently suggest 9% discrepancy between local measurement of Hubble constant and that inferred from cosmic microwave background in standard cosmology. Either standard cosmology or local measurement might expect something new. We investigate possible motional status of local measurement that might affect value of Hubble constant. Using adapted coordinate of geodesic observers in FLRW space-time and constraints inferred from observation of cosmic shear, we find that the motional status of reference frame would contribute to about 1.1% discrepancy of Hubble constant.
Recent local measurement of Hubble constant is discrepant from that inferred from cosmic microwave background (CMB) up to 9%. In last decades, the discrepancy has been verified many times. With ΛCDM as standard cosmology, CMB [1], baryon acoustic oscillation [2–5] and inverse distance ladder technique [6, 7] provide consistent Hubble constants around $67.4 \, \text{km s}^{-1} \, \text{Mpc}^{-1}$, while local measurements on such as Cepheid variable [8–12], gravitational wave [13–15], quasar [16] or tip of the red giant branch [17, 18] give larger Hubble constants. The discrepancy of Hubble constants ranges between 9% and 4%.

The larger local Hubble constant indicates a higher expanding rate of the current universe. It has inspired cosmological models beyond ΛCDM. Most models focus on gravitational repulsion aspect of the universe via a modified dark energy, such as early dark energy [19–21] or interaction [22, 23], phantom-like [24], phase transition [25, 26] and dynamical aspect [27] of dark energy. On the other hand, the local measurement of Hubble constant also can be suffered from systematics error. The error might be as results of gravitational lensing [28–30], vacuum void or density fluctuation [31–35] and local gravitational potential [36]. Most of them conclude that it’s difficult to alleviate the discrepancy of Hubble constant.

In this paper, we would explore the possibility that the discrepancy of Hubble constant might come from different motional statue of the reference frame. Namely, there might be different value of observed Hubble constant, if we measure them in different reference frames. We present coordinate transformations between geodesic and static observers in Friedmann-Lemaitre-Robertson-Walker (FLRW) space-time and find that the local measurement in adapted coordinate of the geodesic observer is different from that in the isotropic CMB frame. On the one hand, space-time beyond the description of FLRW metric suggests additional observable effect at least in redshift space. On the other hand, value of Hubble constants are different in the geodesic and CMB reference frame. It suggests that we can estimate how much discrepancy of Hubble constant can be attributed from motional status of our heliocentric reference frame. By the way, we have already known that we are not in the CMB frame [37–40]. There would be an up bound of the discrepancy of Hubble constant caused by motional status of reference frame, if we assume that redshift space distortion is originated in geodesic motion of heliocentric reference frame. More specifically, inferred from observation of amplitude of cosmic shear [40], we find that local measured Hubble constant
in the heliocentric reference frame could be larger than that inferred from CMB around 1.1%. The effect of the motional reference frame seems large enough to compare with that caused by local structure \cite{35} and can not be neglected.

This paper is organized as follows. In section [II] we introduce the adapted coordinate of geodesic observers in FLRW space-time. In this reference frame, the value of Hubble constant is shown to be different from the value in CMB frame. In section [III] using constraints inferred from observation of cosmic shear, we provide up bound of discrepancy of Hubble constant that can be attributed from motional status of heliocentric reference frame. Finally, conclusions and discussions are summarised in section [IV].

II. REFERENCE FRAME DEPENDENCE OF HUBBLE CONSTANT

In CMB frame, the universe is isotropic and described by the FLRW metric in standard cosmology,

$$\text{ds}^2 = -dt^2 + a^2(t)((dx^1)^2 + (dx^2)^2 + (dx^3)^2).$$

(1)

For a geodesic observer $u^\mu = \left( \sqrt{1 + \left(\frac{\varsigma}{a(t')}\right)^2}, 0, 0, \frac{\varsigma}{a(t')} \right)$ in CMB frame, metric of adapted coordinate of geodesic observers takes the form of

$$\text{ds}^2 = -dT^2 + a^2(T, Z) \left( dX^2 + dY^2 + \left( \frac{1 + \left(\frac{\varsigma}{a(t')}\right)^2}{1 + \varsigma^2} \right) dZ^2 \right),$$

(2)

where $a(T, Z) \equiv a(t(T, Z))$. Theoretically, there are not constraints for parameter $\varsigma$, which is an integral constant of geodesic equation. From Eq. (2), the metric is still a co-moving coordinate as $g_{00} = -1$, while coordinate time $T$ is differed from $t$ of CMB frame. We obtain the metric with a coordinate transformation,

$$\begin{cases}
T = \int_0^t \sqrt{1 + \left(\frac{\varsigma}{a(t')}\right)^2} \, dt' - \varsigma x^3 , \\
X = x^1 , \\
T = x^2 , \\
Z = \sqrt{1 + \varsigma^2} \left( x^3 - \varsigma \int_0^t \frac{dt'}{a^2 \sqrt{1 + \left(\frac{\varsigma}{a(t')}\right)^2}} \right).
\end{cases}$$

(3)

For $\varsigma = 0$, the transformation is identical. For the current universe $a \to 1$, the transformation turns to be Lorentz transformation. One can find that the metric and transformation are
non-trivial and space-time of a geodesic observer is different from description of FLRW metric.

From the adapted coordinate (Eq. (2)) in FLRW space-time, there is a general covariant transformation Eq. (3) that relates Hubble constant in the geodesic reference frame $\bar{H}_0$ and the Hubble constant $H_0$ inferred from CMB,

$$\bar{H}_0 \equiv \frac{\partial r a}{a} \bigg|_{a=1} = H_0 \sqrt{1 + \varsigma^2},$$  

where $H_0 \equiv \frac{\dot{a}(t)}{a(t)} |_{a=1}$. It indicates that $\bar{H}_0$ can not be less than $H_0$. In order to clarify Hubble constant in the motional frame in more detail, we can further calculate distance-redshift relation in low-redshift expansion,

$$d_L = \frac{\bar{z}}{H_0 \sqrt{1 + \varsigma^2}} \left( 1 + \frac{1}{2} \left( 1 - q_0 - \frac{\varsigma^2}{1 + \varsigma^2} \right) \bar{z} - \frac{1}{6} \left( 1 - q_0 - 3q_0^2 + j_0 + \frac{2(1 - q_0)\varsigma^2}{1 + \varsigma^2} - 3 \left( \frac{\varsigma^2}{1 + \varsigma^2} \right)^2 \right) \bar{z}^2 + \mathcal{O}(\bar{z}^3) \right),$$  

where deceleration parameter $q(t) \equiv -\frac{\ddot{a}}{a\dot{a}}$, jerk parameter $j(t) \equiv \frac{\dot{a}^2}{a^2}$, redshift $\bar{z} \equiv \frac{1}{a(T,0)} - 1$, and $d_L$ is isotropy part of luminosity distance. The anisotropic term of $d_L$ would emerge in sub-leading order of $\bar{z}$, while it's not important, if we only focus on Hubble constant instead of Hubble parameter. From Eq. (5), it can be shown that $\bar{z} \approx H_0 \sqrt{1 + \varsigma^2} d_L$ as $\bar{z} \rightarrow 0$. It’s consistent with Eq. (4) that $H_0 \sqrt{1 + \varsigma^2}$ is the Hubble constant $\bar{H}_0$ in geodesic reference frame. In case of $\varsigma = 0$, Eq. (5) turns to be formulation used in Refs. [8, 9, 41].

Thus, there are different values of Hubble constant in geodesic and CMB reference frame. In the case of $\varsigma = 0.4$, $\frac{\bar{H}_0}{H_0} - 1 \approx 8\%$. It’s closed to the discrepancy that Riess, et al. [8, 9] suggested. However, it’s still not enough to regard $\bar{H}_0$ as local measured Hubble constant, until we can relate these formulation with observation. We would show in next section that observation of cosmic shear suggests $\frac{\bar{H}_0}{H_0} - 1 \lesssim 3\%$.

### III. ESTIMATE PARAMETER $\varsigma$ FROM COSMIC SHEAR

Parameter $\varsigma$ is an integral constant of geodesic equation in FLRW space-time and seems not require a constraint. In fact, $\varsigma$ ought to be constrained from observation. In this section, we would estimate the parameter $\varsigma$ from cosmic shear.

From redshift surveys [37, 39], redshifts of celestial objects are observed beyond description of isotropic Hubble flow. The anisotropic part of redshift is understood as Droplet effect.
arising from peculiar velocity $V_{\text{pec}}$ of celestial objects, as there could be relative motion of celestial objects with respect to CMB frame. As shown in Ref. [40], line-of-sight total velocity, namely, the observed redshift $z_{\text{obs}}$ at low-$z$, can be expanded as

$$z_{\text{obs}}(\hat{r}) = H d + V_{\text{pec}} = H d + B_i \hat{r}_i + (Q_{ij} \hat{r}_i \hat{r}_j) d + O(d^2),$$

(6)

where observed redshift $z_{\text{obs}}$ depends on location of celestial objects $\hat{r}$, $d$ is distance of celestial objects, $B_i$ and $Q_{ij}$ are so-called bulk flow and cosmic shear, respectively. From observation [40], amplitude of bulk flow is around 300 km s$^{-1}$ and components of cosmic shear are about 3 $h$ km s$^{-1}$ Mpc$^{-1}$. As shown in Eq. (6), the anisotropic information of observed redshift is not involved in Hubble constant. If one only cares about Hubble constant, an isotropic redshift is required to be introduced. It’s what we have done in section II.

There is abundant information of the universe that might be indicated by peculiar velocity. It’s basically because bulk flow and cosmic shear are not vanished in ensemble averages of peculiar velocity [39, 40] and local overdense or void region is regard as a cause that gives rise to the peculiar velocity field beyond the description of isotropic FLRW metric. In fact, there is other possibility that peculiar velocity might involve information of observers and be originated in motional statue of observer’s reference frame. We have shown that it’s non-trivial to consider the adapted coordinate of a geodesic observer. Without local structure, we would show that the deviation of redshift from Hubble flow can still exist in geodesic reference frame.

The redshift can be calculated from the metric (Eq. (2)) directly. In low redshift approximation, redshift is proportional to velocity and can be treated as a vector with three component in space, so-called redshift space [42]. Although, redshift could be regarded as a vector in conceptual level, what we are able to calculate is still the amplitude of redshift. Therefore, we consider the components of $z_{\text{obs},i}$ from calculation of redshift along coordinate axis of $X, Y, Z$, separately. In direction of $X$ or $Y$, the components of observed redshift is $z_{\text{obs},\alpha} = \frac{1}{a(T,0)} - 1$ and $z_{\text{obs},i\neq\alpha} = 0$ as leading order, where $\alpha = X$ or $Y$ and $i = X, Y, Z$. In direction of $Z$, namely, the direction of geodesic motion, the observed redshift takes form of

$$z_{\text{obs},Z} = \frac{\sqrt{1 + \varsigma^2} \pm \varsigma}{\sqrt{a^2(T, Z) + \varsigma^2} \pm \varsigma} - 1,$$

(7)

$$z_{\text{obs},i\neq Z} = 0,$$

(8)

where $\pm$ is used to describe forward and backward propagating light, respectively. From
Eq. (5), the amplitude of isotropic redshift \(|z| = \frac{1}{a(T,0)} - 1\) is used in distance-redshift relation. In order to obtain Eq. (6) in geodesic reference frame, we wish to express the redshift \(z_{\text{obs},i}\) in term of isotropic redshift \(\bar{z}_i\). Here, for simplicity, we calculate observed redshift \(z_{\text{obs},i}\) in direction of \(j\)-axis, separately, and expand them in term of \(\bar{z}^{(j)}_i \equiv \delta^{(j)}_i \left( \frac{1}{a(T,0)} - 1 \right)\). One can obtain

\[
z_{\text{obs},i} = z_{\text{obs},i}\big|_{\bar{z}=0} + \frac{\partial z_{\text{obs},i}}{\partial \bar{z}_j} \big|_{\bar{z}=0} \bar{z}_j + O(\bar{z}^2),
\]

where \(i, j = X, Y, Z\) and

\[
z_{\text{obs},i}\big|_{\bar{z}=0} = 0,
\]

\[
\frac{\partial z_{\text{obs},i}}{\partial \bar{z}} \big|_{\bar{z}=0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + \varsigma^2 \end{pmatrix}.
\]

In the direction of \(Z\), the redshift of forward and backward propagating lights are shown to be the same. It indicates that deviation of isotropic redshift caused by motional reference frame should be understood as redshift space distortion and quantified by cosmic shear. Differed from standard interpretation of redshift space distortion that is caused by density fluctuation, here, the space-time deformation accounts for the redshift space distortion.

In order to estimate parameter \(\varsigma\), using Eqs. (4), (5) and (11), we would extract cosmic shear via rewriting Eq. (9) as

\[
z_{\text{obs}} = \tilde{H}_0 d + \tilde{Q} d,
\]

where

\[
\tilde{H}_0 \equiv H_0 \left( \frac{2}{3} + \frac{1}{3(1 + \varsigma^2)} \right),
\]

\[
\tilde{Q} \equiv \begin{pmatrix} \frac{H_0 \varsigma^2}{3(1 + \varsigma^2)} \\ \frac{H_0 \varsigma^2}{3(1 + \varsigma^2)} \\ -\frac{2H_0 \varsigma^2}{3(1 + \varsigma^2)} \end{pmatrix},
\]

and \(d \equiv d_L \hat{r}\), the radial distance. The \(\tilde{Q}\) is cosmic shear and \(\tilde{H}_0\) is a modified Hubble constant which should be distinguished from \(H_0\). In observation, we suggest using \(\tilde{H}_0\) as local measured Hubble constant, because it corresponds to the isotropic part of observed redshift introduced in Eq. (5). As leading order terms of observed redshift are all proportional to distance from the calculation, there is no place for bulk flow.
We ansatz that the observed cosmic shear is at least partly caused by the motional statue of heliocentric reference frame. Thus, a constraint, that \( \sqrt{Q_{ij} \tilde{Q}^{ij}} \lesssim (\sqrt{Q_{ij} Q^{ij}})_{\text{obs}} \), should be satisfied. Here, we use \( \sqrt{Q_{ij} Q^{ij}} \) to quantify amplitude of cosmic shear, since cosmic shear is traceless. Using the observed value of cosmic shear from Ref. [40], we obtain constrained range of parameter \( |\varsigma| \lesssim 0.252 \).

With the value of parameter \( \varsigma \), we can estimate how much discrepancy of Hubble constant can be attributed from geodesic motion of heliocentric reference frame. Without model-independently corrected redshift with peculiar velocity, there is an upper bound of discrepancy of Hubble constant that might be caused by motional statue of the reference frame. We estimate the relative deviation as

\[
\frac{\tilde{H}_0}{H_0} - 1 = \sqrt{1 + \varsigma^2} - 1 \lesssim 3.1\%. \tag{15}
\]

The deviation could be narrowed, if one can extract isotropic part redshift from redshift space distortion, as it did in Ref. [40]. Differed from \( \bar{z} \), isotropic part of redshift \( \tilde{z} \) can be read from Eqs. (12) and (13), \( \tilde{z} \equiv (\frac{2}{3} + \frac{1}{3(1+\varsigma^2)}) \bar{z} \). And the distance-redshift relation can be rewritten as,

\[
d_L = \frac{\tilde{z}}{H_0} \left( 1 + \frac{1}{2} \left( 1 - q_0 - \frac{\varsigma^2}{1 + \varsigma^2} \right) \frac{\tilde{z}}{\frac{2}{3} + \frac{1}{3(1+\varsigma^2)}} \right)
- \frac{1}{6} \left( 1 - q_0 - 3q_0^2 + j_0 + \frac{2(1 - q_0) \varsigma^2}{1 + \varsigma^2} - 3 \left( \frac{\varsigma^2}{1 + \varsigma^2} \right)^2 \left( \frac{\tilde{z}}{\frac{2}{3} + \frac{1}{3(1+\varsigma^2)}} \right)^2 \right). \tag{16}
\]

In low redshift approximation \( \tilde{z} = \bar{H}_0 d_L \), where \( \bar{H}_0 \) is the local measured Hubble constant, as discussed above. We estimate the relative deviation between the local measured Hubble constant \( \bar{H}_0 \) and \( H_0 \) from CMB frame,

\[
\frac{\bar{H}_0}{H_0} - 1 = \frac{2}{3} \sqrt{1 + \varsigma^2} + \frac{1}{3\sqrt{1 + \varsigma^2}} - 1 \lesssim 1.1\%. \tag{17}
\]

Although, it seems to be difficult to alleviate Hubble tension. The effect of motional status of reference frame is closed to magnitude caused by local structure [35] and might not be neglected.

IV. CONCLUSIONS AND DISCUSSIONS

In this paper, we discussed that the discrepancy of Hubble constant may be related with motional status of reference frame. We introduced an adapted coordinate of a geodesic
observer in FLRW space-time, which is beyond the description of FLRW metric. With cosmic shear as an indicator of space-time deformation suggested by the geodesic reference frame, we estimated that there are about 1.1% or more riskily 3.1% discrepancy of Hubble constant that can be attributed from the geodesic motion of heliocentric reference frame. Although, it seems not able to provide alleviation for Hubble tension, the novel prospective of cosmic observation is more significant than the result itself. We showed that Hubble constant is reference frame dependent and the observable effect of motional status of an observer can not be neglected in the universe. It’s falsifiable and related with redshift space distortion closely. In the future, redshift survey from a motional satellite might be used to test general relativity.

Other key part of our model is estimation of the parameter $\varsigma$ from cosmic shear. The $\varsigma$ is independent of redshift in our calculation. Although, we adopted the value of cosmic shear that is measured at $z \lesssim 0.03$ [40], the $\varsigma$ is expected to remain the same in higher redshift.

One might have a question: why could it work, as deviation of redshift between heliocentric and CMB frame measured by peculiar velocity seems to be small, so that in higher redshift it can be neglected [9]. There are two points that we wish to clarify. Firstly, we provided other understanding of the peculiar velocity, at least the cosmic shear part. We used peculiar velocity of celestial objects to infer motional status of heliocentric reference frame with respect to CMB frame. Celestial objects and heliocentric reference frame undergoing relative motions with respect to CMB frame are conceptually different. And here, we focused on the effect of motion status of our reference frame on Hubble constant. Secondly, we considered the relative motion of reference frame in the framework of general relativity. It brings some new feature, as we have shown above. That is what the peculiar velocity which is only well-defined in special relativity [43] can not provide.

Fundamentally, the work is based on what we find in FLRW space-time that the adapted coordinates for different geodesic observers are not equivalent. It’s interesting on conceptual level. In Minkowski space-time, due to Lorentz symmetry, we can’t distinguish reference frames of static and inertial observers. In FLRW space-time, transformations between different geodesic observers turn to be a kind of asymptotic symmetry as $a \to 1$. And we can distinguish reference frames of different geodesic observers. That’s the reason that we can figure out the difference of Hubble constant in CMB and heliocentric frame, although, there might not be a real observer that is static with respect to CMB frame.
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