Spin Structures on Kleinian Manifolds

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Abstract. We derive the topological obstruction to spin–Klein cobordism. This result has implications for signature change in general relativity, and for the $N = 2$ superstring.

1. Introduction

In this paper we study topological obstructions to the global existence of spin–Klein structure on four–manifolds. In particular, we wish to determine the topological obstructions to spin–Klein cobordism (in analogy with the recently found obstructions to spin–Lorentz cobordism, see \[1, 2, 3\]).

Let $M$ be any four–manifold. We will say that a metric on $M$ is of Kleinian signature if it has signature $(-+++)$, and we will say that $M$ admits spin structure if we are able to globally define a fibre bundle over $M$ with structure group $\text{Spin}(2, 2)$ (where $\text{Spin}(2, 2)$ is the double cover of $\text{SO}(2, 2)$). It follows \[4\] that the obstruction to spin structure is the vanishing of the second Stiefel–Whitney class $w_2$, and we therefore need only understand the obstruction to putting a globally non–singular Klein metric on $M$.

Let $\{\Sigma_1, \Sigma_2, \ldots, \Sigma_n\}$ be an arbitrary collection of three–manifolds, then a cobordism for $\{\Sigma_1, \Sigma_2, \ldots, \Sigma_n\}$ is a four–manifold $M$ with $\partial M \cong \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n$ (where $\cup$ denotes disjoint union). We are interested in cobordisms that possess more structure. In particular, we shall say that $M$ is a spin–Klein cobordism for $\{\Sigma_1, \Sigma_2, \ldots, \Sigma_n\}$ if and only if $M$ is a cobordism for...
\{\Sigma_1, \Sigma_2, \ldots, \Sigma_n\}, and \(M\) admits both spin structure and a globally defined non–singular metric of Kleinian signature.

One motivation for this work is the recent interest in spacetimes in which ‘signature change’ is allowed [5, 6]. These signature change studies have centered upon cosmological solutions to the classical Einstein field equations in which the metric changes signature from \((+++\)) to \((-+++)\). It could be argued that this is too limiting, and more general signature transitions, for example those from \((-+++\)) to \((-+++)\) should be considered [7]. Our cobordism result is relevant to those cases in which the spacetime metric changes to a Kleinian signature.

Our result may also prove to be of importance to the \(N = 2\) superstring. The Weyl anomaly for this theory cancels provided that the string propagates in a four dimensional target space. If the worldsheet is chosen to be of Lorentzian signature then the target space must have Kleinian signature [8].

2. Kleinian metric homotopy

In order to understand the topology of Kleinian metrics, we first need to understand the obstruction to the existence of global Kleinian structure. Indeed, we have from [9]

\textbf{Lemma 1.} Let \(M\) be a smooth four–manifold without boundary. Then \(M\) admits a globally defined (non–singular) Kleinian metric if and only if there exists a globally defined (non–singular) field of 2–planes.

A field of 2–planes may be defined in the following way. Let \(G_{2,4} \cong S^2 \times S^2\) denote the set of 2–planes in \(\mathbb{R}^4\), then a field of 2–planes is just a section of the fibre bundle which has fibre \(G_{2,4}\). If the section is everywhere non–vanishing then the field of 2–planes is non–singular, otherwise the field of 2–planes is singular. These definitions provide a natural generalization of a (non–singular/singular) vector field, which is a (non–vanishing/vanishing) section of a sphere bundle. We will use the expressions ‘field of 2–planes’ and ‘plane field’ interchangeably.

Each singularity of a singular plane field has associated to it an index. Suppose that \(x\) is a point in our manifold \(M\) at which some plane field \(P\)
becomes singular, and let $S^3(x)$ denote a little sphere about $x$. Then the index indicates the homotopy type of the map (defined by $P$) from $S^3(x)$ to $G_{2,4}$. However, such homotopy classes are in one–to–one correspondence with elements of $\pi_3(G_{2,4}) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Thus the index of $P$ at $x$ is classified by a pair of integers.

Let $M$ be an oriented compact four–manifold. Let $H$ denote the free abelian group $H^2(M,\mathbb{Z})/\text{torsion subgroup}$, and let $S$ denote the intersection pairing on $H$ defined by the cup–product (ie. $S$ defines a map from $H \otimes H$ to $\mathbb{Z}$ by taking the cup–product of elements in $H$ and evaluating them on the fundamental orientation class of $M$). Define the coset $W \subseteq H/2H$ by $w \in W$ if $S(w,x) = S(x,x)$ mod 2 for all $x \in H$. Finally, let $\Omega$ denote the set of integers $\{S(w,w) | w \in W\}$. We now recall an important theorem of Hirzebruch and Hopf [10] (also see [11]).

**Theorem 1.** Let $M$ be an oriented compact four–manifold without boundary. Then $M$ has a field of 2–planes with finite singularities. The total index of such a field is given by a pair of integers $(a, b)$. The following integers and only these, occur as the index for some plane field on $M$:

$$a = \frac{1}{4} (\alpha - 3\sigma - 2\chi), \quad b = \frac{1}{4} (\beta - 3\sigma + 2\chi)$$

where $\alpha, \beta \in \Omega$, $\chi = \chi(M)$ denotes the Euler number of $M$, and $\sigma = \sigma(M)$ denotes the Hirzebruch signature of $M$.

Now suppose we are given a cobordism $M$ for some collection of closed, orientable three–manifolds $\{\Sigma_1, \Sigma_2, \ldots, \Sigma_n\}$. Then we can always form the double of $M$, denoted $2M$, in the usual way [12]. The double has no boundary, and its Euler number is given by $\chi(2M) = 2\chi(M)$. It also satisfies $\sigma(2M) = 0$ (since the two ‘halves’ of the double will have opposite orientations). Furthermore, we know from [13] that

$$\alpha, \beta = \sigma(2M) \mod 8.$$ 

Combining these results with Theorem 1 we see that the total index of any plane field on $2M$ has the following form:

$$\text{Total Index} = (2n - \chi(M), 2m + \chi(M))$$
where $m, n \in \mathbb{Z}$.

Now push all of the singularities in the plane field on $2M$ over to ‘one of the halves’ of $2M$. Then we have constructed a non–singular plane field on $M$ (by taking the singularity free half of $2M$), and by construction, the degree of the map from $\partial M$ to $G_{2,4}$ (defined by the plane field) must be

$$(2n - \chi(M), 2m + \chi(M)) \quad \text{for } m, n \in \mathbb{Z}.$$  

Recalling that a plane field defines a Klein metric up to homotopy, we shall call this degree the *Klein kink* of the metric with respect to $\partial M$ (in analogy with the Lorentz kink [3]). Thus we have shown

**Lemma 2.** Let $M$ be any orientable manifold with $\partial M \cong \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n \neq \emptyset$. Then there are always globally defined non–singular plane fields on $M$ and they must have Kleinian kink on $\partial M$ equal to

$$(2n - \chi(M), 2m + \chi(M)) \quad \text{for } m, n \in \mathbb{Z}.$$  

We can now combine Lemma 2 with the spin obstruction to get our main result.

**3. Obstruction to spin–Klein cobordism**

To begin, recall [1] the result that for manifolds $M$ with boundary we have the formula

$$(u(\partial M) + \chi(M)) = \hat{w}_2 \mod 2,$$  

where the mod 2 Kervaire semi–characteristic $u(\partial M)$ is defined as

$$u(\partial M) = \dim_{\mathbb{Z}_2} \left( H_0(\partial M; \mathbb{Z}_2) \oplus H_1(\partial M; \mathbb{Z}_2) \right) \mod 2,$$  

and $\hat{w}_2$ is defined in terms of the second Stiefel–Whitney class $w_2$ as

$$\hat{w}_2 = \begin{cases} 0 & \text{iff } w_2[c] = 0 \text{ for all } 2\text{--cycles } c \in H_2(M) \\ 1 & \text{iff } w_2[c] = 1 \text{ for some } 2\text{--cycle } c \in H_2(M). \end{cases}$$

Now, suppose we are given a collection of three–manifolds $\{\Sigma_1, \Sigma_2, \ldots, \Sigma_n\}$ and a field of 2–planes (ie. a Kleinian metric) on a collar neighbourhood of
Then the field of $2$–planes will define a map from a collar neighbourhood of $\Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n$ to $G_{2,4}$ and the homotopy type of the map will be indexed by the Klein kink, denoted $\text{kink}(\Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n; g_K)$, of the form $(k_1, k_2) \in \mathbb{Z} \times \mathbb{Z}$ (we call $k_1$ and $k_2$ the first and the second component of the Klein kink respectively). We then have

**Theorem 2.** Let $\{\Sigma_1, \Sigma_2, \ldots, \Sigma_n\}$ be some collection of closed orientable three–manifolds and let $\text{kink}(\Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n; g_K) = (k_1, k_2)$ be the degree of the map from $\Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_n$ to $G_{2,4}$ defined by some plane field on the collar neighbourhood of the $\Sigma_i$. Then there exists a spin–Klein cobordism $M$ for the $\Sigma_i$ (in the sense that the degree of the map from $\partial M \to G_{2,4}$ is also $(k_1, k_2)$) if and only if

$$(u(\partial M) + k_i) = 0 \mod 2,$$

where $k_i$ is either of $k_1$ or $k_2$.

**Proof.** Suppose such a spin–Klein cobordism $M$ exists. Then $w_2 = 0$, and so $u(\partial M) = \chi(M) \mod 2$. Furthermore, by Lemma 2 we know that

$$\text{kink}(\partial M; g_K) = (2n - \chi, 2m + \chi).$$

We see that the parity of $2n - \chi$ or $2m - \chi$ is determined solely by $\chi$ (ie. $k_i = \chi \mod 2$), and hence $(u(\partial M) + k_i) = 0 \mod 2$.

Conversely, suppose $(u(\partial M) + k_i) = 0 \mod 2$. Take any globally defined Klein metric $g_K$ on $M$ then we have

$$k_1 = k_2 = \chi \mod 2.$$

Hence $(u(\partial M) + \chi) = 0 \mod 2$ and so $w_2 = 0$. Thus $M$ is spin–Klein. ■
4. Examples and Conclusions

Some examples of our obstruction to Spin–Klein cobordism are as follows:

Example 1. Let $S^2$ be the two–sphere and $D^2$ a closed two–disk, then we may put a Kleinian metric $g_K$ on the product manifold $M \cong S^2 \times D^2$ by simply taking the product metric induced by a signature $(++)$ metric on $S^2$ and a signature $(-\ldots)$ metric on $D^2$. The boundary of $M$ is then given by $\partial M \cong S^1 \times S^2$, and the induced metric on $\partial M$ is non–singular with signature $(-+\ldots)$. Since the Klein kink of $g_K$ on $\partial M$ could also be calculated using intersection theory (by counting the number of degenerate points), it follows that $kink(\partial M; g_K) = (0,0)$. Hence, since $u(S^1 \times S^2) = 0$, we have that $S^2 \times D^2$ is a spin–Klein cobordism for $S^1 \times S^2$ with zero Klein kink.

Example 2. Take $\mathbb{R}^4$ with ordinary flat metric $g_K$ of signature $(-\ldots++)$, and let $M \cong D^4$ be some closed 4–ball in $\mathbb{R}^4$. Then $M$ is itself a Klein manifold with boundary $\partial M \cong S^3$. In this case, we can clearly identify the plane field $P$ (corresponding to $g_K$) with a pair of non–vanishing vectors $\{v_1, v_2\}$ which span $P$. In other words, the section $P$ of $G_{2,4}$ reduces to a section of the Stiefel manifold $V_{2,4}$. It follows that

$$kink(\partial M; g_K) = (kink(\partial M; v_1), kink(\partial M; v_2))$$

where $kink(\partial M; v_1) = kink(\partial M; v_2) = 1$ denote the kink of the respective vector field on $\partial M$. Hence, since $u(S^3) = 1$, we see that $D^4$ is an example of a spin–Klein cobordism for $S^3$ with $kink(S^3; g_K) = (1,1)$.

Example 3. As was noted in the introduction, our result may be applied to spacetimes that change signature from $(-++\ldots)$ to $(-\ldots++)$. Consider a spacetime $M \cong M_L \cup M_K$ with $\partial M_L = \partial M_K = \Sigma \neq \emptyset$, such that the metric $g_L$ on $M_L$ is of signature $(-++\ldots)$ and the metric $g_K$ on $M_K$ is of signature $(-\ldots++)$. We require that the two metrics agree on $\Sigma$, and hence the induced metric on $\Sigma$ is of signature $(-++\ldots)$. Now, restricting our attention to $(M_K, g_K)$, it follows that $kink(\partial M_K; g_K) = (0,0)$ and hence, from our theorem, $M_K$ has spin structure if and only if $u(\partial M_K) = 0$. Thus, since $u(S^3) = 1$, we see that it is not possible for ‘bubbles’ of Kleinian signature to form across an $S^3$ surface whilst maintaining spin structure.
Finally, we note that there is probably a generalization of this work to non–orientable manifolds in the form of a topological obstruction to \textit{pin}–Klein cobordism (cf. [4]). In analogy with the Lorentz case, this will presumably require us to pass to non–oriented plane fields (ie. sections of the bundle of non–oriented planes).

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