Quantum forced oscillator via Wigner transform

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Abstract
In this paper we review the basic results concerning the Wigner transform and critically discuss the problem of the reversibility. Then we completely solve the harmonic/inverted forced quantum oscillator in such a framework; eventually, the tunnel effect for the forced inverted oscillator is discussed.

Keywords
Wigner transform · Quantum oscillator · Tunnel effect · Semiclassical limit · Schrödinger equation

Mathematics Subject Classification 44A15 · 81xx · 81S30

1 Introduction
The formulation of quantum mechanics in the phase space has been a topic of great interest since Wigner’s groundbreaking paper [45]. In such a treatment the explanation of quantum-classical correspondence seems to be very spontaneous and many attempts have been made with the aim of shedding light on quantum mechanics from a classical perspective [8, 20, 31].

Let \( \psi(x, t) \), \( x \in \mathbb{R}^d \), be the solution to the time-dependent Schrödinger equation

\[
\begin{cases}
i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V \psi, \\
\psi(x, 0) = \psi_0(x), \quad \psi(\cdot, t) \in L^2(\mathbb{R}^d, dx).
\end{cases}
\] (1)

The Wigner (or Wigner–Slizard\(^1\)) transform is defined in the phase-space \((x, \xi) \in \mathbb{R}^{2d}\) as follows

\[ \psi(x, t) = \frac{1}{(2\pi\hbar)^{d/2}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \frac{\partial}{\partial \xi} \left( \frac{1}{2\pi\hbar} \right)^d \int_{\mathbb{R}^d} e^{i\xi \cdot \xi'} \psi(\xi', t) \, d\xi' \, d\xi. \] (2)

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\(^1\) In fact, the expression of such a transformation was introduced by Wigner; however, it was found by L. Slizard and E. Wigner for a different purpose some years before the publication of the paper [45].
\[ W^h(x, \xi, t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi \cdot y} \psi \left( x + \frac{1}{2} \hbar y, t \right) \overline{\psi \left( x - \frac{1}{2} \hbar y, t \right)} dy \]

and it solves to the Wigner equation

\[
\frac{\partial W^h}{\partial t} = -\frac{1}{m} \xi \cdot \nabla_x W^h + \nabla_x V \cdot \nabla_\xi W^h + \mathcal{R}(W^h) = \{ W^h, \mathcal{H} \} + \mathcal{R}(W^h) \tag{2}
\]

where \( \mathcal{H} = \frac{1}{2m} \xi^2 + V \) is the classical Hamiltonian function, \( \{ \cdot, \cdot \} \) denotes the Poisson’s brackets and \( \mathcal{R}(W^h) \) is a remainder term depending on the unknown function \( W^h \) that formally goes to zero in the semiclassical limit \( \hbar \to 0 \). Thus, as Wigner himself pointed out, in the semiclassical limit the function \( W^h \) solves the classical Liouville equation

\[
\frac{\partial W^h}{\partial t} = \{ W^h, \mathcal{H} \}.
\]

Furthermore, recalling that in the standard notation of quantum mechanics the probability density in position space is given by \( |\psi(x, t)|^2 \), another advantage of using the Wigner representation is related to the evaluation of expectation values for physical observable; in fact

\[
|\psi(x, t)|^2 = \int_{\mathbb{R}^d} W^h(x, \xi, t) d\xi
\]

and thus the probability density can be obtained by means of the solution \( W^h \) to the Wigner equation (2). Similarly, the probability density in the momentum space follows, too.

Thus, at first glance, it seems that the program of studying the semiclassical limit of quantum mechanics by means of the classical Liouville equation is very promising; unfortunately, the residual term \( \mathcal{R} \) usually contains derivatives of the unknown function \( W^h \) of higher than first order, and so rigorous treatment of the complete Eq. (2) is a formidable task. Many attempts, both theoretical and numerical, have been made [3, 11, 13, 16, 32, 34, 37, 38, 40, 44] in order to find an approximate solution to (2) but some gaps in the theory are still open when the potential \( V \) is not a quadratic function with respect to the spatial variable. Indeed, Eq. (2) is in general impossible to be analytically solved and only approximated solutions \( \tilde{W}^h \) may be obtained; in such a case a first problem arises: is \( \tilde{W}^h \) the Wigner transform of a square integrable wave-function \( \tilde{\psi} \)? and secondly: if \( \tilde{W}^h \) approximates, with respect to some norm, the exact solution \( W^h \) to (2) corresponding to the wave-function \( \psi \), then does \( \tilde{\psi} \) approximate \( \psi \) in the \( L^2 \) space? As discussed in Remark 7, as far as we known a fully satisfactory answer to these crucial questions has not yet been given.

However, when \( V \) is a quadratic function with respect to the spatial variable then the remainder term \( \mathcal{R} \) in (2) is exactly zero and thus the Wigner equation exactly reduces to the classical Liouville equation which admits an explicit solution. The quadratic potential case basically reduces to the harmonic/inverted oscillator and in such a case Eq. (1) has been widely studied obtaining the explicit expression of the propagator using, for instance, the Feynman path integrals [15, 22, 33, 35], or by solving the classical Liouville equation [4, 19], or by using the Heisenberg picture [6]; see also [43] for a review. A special attention has been paid in order to study the tunnel effect in the case of inverted oscillator [4, 6, 19].

In this paper we briefly review the Wigner transform and wavefunction dynamics in the Wigner representation. For simplicity we restrict our attention to the one-dimensional case; extension to higher dimensions is fairly straightforward (see, e.g., [17]). We then focus our attention on the case of second-degree potentials with respect to the spatial variable of the form \( V(x, t) = \gamma x^2 + Q(t)x \), where \( \gamma \) is a constant factor (in principle, one could similarly treat the case of \( \gamma(t) \) depending on \( t \) but we do not dwell on that problem here). In such a case
an explicit solution $W^h(x, \xi, t)$ to the classical Liouville equation is given. Next we apply our results to the special case where the initial wavefunction $\psi_0(x)$ has a Gaussian shape; in that case the tunnel effect for the inverted oscillator is studied in detail. We should remark that most of the results in this paper have already been obtained in previous works; our aim is to collect all these results in a self-consistent paper by adding some new results concerning, for example, Wigner continuity or the explicit solution in the Wigner representation for the forced harmonic/inverted oscillator.

The paper is organized as follows. In Sect. 2 we recall the definition of Wigner transform and collect some fundamental properties in Sect. 2.1; these results are not new and have already been demonstrated in the cited articles; however, for the sake of completeness, brief proofs are given. In Sect. 2.2 we discuss a problem that is not usually covered: the continuity of the Wigner transform. In Sect. 2.3 we discuss the problem of reversibility of the Wigner transform; in fact, an important and, in some respects, open problem consists in checking if a real-valued function defined in the phase-space is the Wigner transform of a pure/mixed state or not. In Sect. 2.4 we collect some important examples of Wigner transform of wave-functions $\psi(x, t)$.

In Sect. 3 we study the Schrödinger equation in the Wigner representation. As we have already discussed above the resulting equation is, in general, quite difficult to deal because the remainder term $R(W^h)$ consists of a formal sum of derivatives of the unknown function $W^h$ of any order. In Sect. 3.2 we consider a couple of particular Schrödinger equations: the case where the potential is given by a Dirac’s delta and the case of nonlinear Schrödinger equation; in both cases the resulting equation in the phase-space is an integro-differential equation.

In Sect. 4 we consider the case of the forced harmonic/inverted oscillator. Such a problem may be treated by solving the Schrödinger equation, as done in standard textbooks (see, e.g., [43]). In fact, in the case of a forced oscillator one could reduce it to the unforced one by means of an appropriate change of variable as done by Husimi [24], and then one treats the latter model by means of the same arguments discussed in Remark 5 because in the case of harmonic potential equation (21) simply reduces to (22). However, as we can see in Sect. 4.2, the treatment in the phase-space is much simpler because the classical Liouville equation reduces to a simple system of two ODEs.

Finally, in Sect. 5 we apply the results obtained in the previous section to the study of the dynamics of a Gaussian wave-function. The detailed study of the tunnel effect is given in Sect. 5.4.

A couple of short appendices with technical calculations complete the paper.

2 The Wigner transform

2.1 Definition and main properties

Let $\varphi, \phi \in L^2(\mathbb{R}, dx)$, then we define a new function in phase space $(x, \xi) \in \mathbb{R}^2$ as follows

$$\tilde{W}^h(x, \xi) := \left[ \mathcal{W}^h(\varphi, \phi) \right](x, \xi) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi y} \varphi \left( x + \frac{1}{2} \hbar y \right) \overline{\phi \left( x - \frac{1}{2} \hbar y \right)} dy.$$

The map $\mathcal{W}^h(\varphi, \phi)$ is bilinear since

$$\mathcal{W}^h(\lambda_1 \varphi_1 + \lambda_2 \varphi_2, \phi) = \lambda_1 \mathcal{W}^h(\varphi_1, \phi) + \lambda_2 \mathcal{W}^h(\varphi_2, \phi)$$
and 
\[ \tilde{\mathcal{W}}^h(\psi, \mu_1 \phi_1 + \mu_2 \phi_2) = \mu_1 \tilde{\mathcal{W}}^h(\psi, \phi_1) + \mu_2 \tilde{\mathcal{W}}^h(\psi, \phi_2) \]
for any \( \psi, \phi_1, \phi_2, \phi, \phi_1, \phi_2 \in L^2(\mathbb{R}, dx) \) and any \( \lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C} \). Furthermore, 
\[ \tilde{\mathcal{W}}^h(\psi, \phi) = \tilde{\mathcal{W}}^h(\phi, \psi). \] (3)

**Definition 1** Let \( \psi \in L^2(\mathbb{R}, dx) \), we define the (semiclassical) Wigner transform of the pure state \( \psi \) the function
\[ W^h(x, \xi) := \mathcal{W}^h(x, \xi) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iy\xi} \phi \left( x + \frac{1}{2} \hbar y \right) \varphi \left( x - \frac{1}{2} \hbar y \right) dy. \] (4)

We restrict our attention to the one-dimensional model where \( d = 1 \); in the case of higher dimension \( d > 1 \) then the numerical pre-factor \( (2\pi)^{-1} \) must be replaced by \( (2\pi)^{-d} \), \( x, \xi \) and \( y \) belong to \( \mathbb{R}^d \) and the product \( \xi \cdot y \) must be replaced by the scalar product \( \xi \cdot y \).

Since \( \tilde{\mathcal{W}}^h(\psi, \phi) \) then it is quite obvious to observe that the Wigner transform is not linear; i.e.,
\[ \mathcal{W}^h(\lambda \psi) = \lambda^2 \mathcal{W}^h(\psi) \quad \text{and} \quad \mathcal{W}^h(\psi_1 + \psi_2) \neq \mathcal{W}^h(\psi_1) + \mathcal{W}^h(\psi_2) \]
and the map \( \mathcal{W}^h \) is not a linear operator. Thus, if we denote \( D = \mathcal{W}^h[ L^2(\mathbb{R}, dx) ] \) the image of \( L^2(\mathbb{R}, dx) \) via the Wigner transform it follows that the set \( D \) is not a linear space (see Remark 4).

If \( \hat{\rho} \) is a density matrix that represents a mixed quantum state then the Wigner representation is given by
\[ \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iy\xi} \left( x + \frac{1}{2} \hbar y \right) \hat{\rho} \left( x - \frac{1}{2} \hbar y \right) dy. \] (5)

In the case of pure quantum state then \( \hat{\rho} = |\psi \rangle \langle \psi| \) and thus (5) takes the form (4). This paper mostly concerns the Wigner transform of pure states.

Here, we recall some basic facts about the Wigner transform:

i. The Wigner transform is not an injective map because \( \mathcal{W}^h(\psi) = \mathcal{W}^h(e^{i\theta} \psi) \) for any real-valued parameter \( \theta \) independent of \( x \). In fact, in Theorem 2 we’ll prove that if \( \mathcal{W}^h(\psi) = \mathcal{W}^h(\phi) \) for some \( \psi, \phi \in L^2(\mathbb{R}, dx) \) then \( \psi = \lambda \phi \) for some terms \( \lambda \) independent of \( x \) and such that \( |\lambda| = 1 \).

ii. \( \mathcal{W}^h \) is a real-valued function, indeed it directly comes from (3) and since \( \mathcal{W}^h(\psi) = \tilde{\mathcal{W}}^h(\psi, \phi) \). We must remark that \( \mathcal{W}^h(x, \xi) \) may be always a non-negative function (see, e.g., Example 2.4.2) or it may also take negative values (see, e.g., Example (2.4.1)). In fact, Hudson [23] proved that the Wigner transform \( \mathcal{W}^h(x, \xi) \) of a wave-function \( \varphi(x) \in L^2 \) is always non-negative if, and only if, \( \varphi(x) = e^{-\frac{1}{2}(ax^2 + bx + c)} \), \( \forall a > 0 \), is a Gaussian-type function (we should also mention that Janssen [25] extended such an analysis to non-\( L^2 \) functions).

iii. If \( \varphi(x) \) is an even/odd function, i.e. \( \varphi(-x) = \pm \varphi(x) \) then \( \mathcal{W}^h(-x, \xi) = \mathcal{W}^h(x, -\xi) \).

iv. The dependence on \( \hbar \) is as follows \( \mathcal{W}^h(x, \xi) = \frac{1}{\hbar} W^1 \left( x, \frac{\xi}{\hbar} \right) \).

v. For any \( \alpha, \beta > 0 \) it follows that
\[ \mathcal{W}^h(\alpha \varphi, \alpha \xi) = \alpha \left[ \mathcal{W}^h(T_\alpha \varphi) \right] (x, \alpha \xi) \]
and
\[
\left[ W(x, \beta \xi) \right](x, \beta \xi) = \frac{1}{\beta} \left[ W(T_{\beta^{-1}} \phi) \right](\beta x, \xi)
\]
where \( T_{\alpha} \phi(x) = \phi(\alpha x) \).

vi. If a function \( u(x, \xi; \hbar) \) is the Wigner transform of a function \( \phi \) depending on \( \hbar \) of the form
\[
\phi(x; \hbar) = \left[ \beta(h) T_{\alpha(h)} \phi \right](x) = \beta(h) \phi[\alpha(h)x],
\]
for some function \( \phi \in L^2(\mathbb{R}, dx) \) independent of \( h \) and some functions \( \alpha(h) \) and \( \beta(h) \) such that \( \alpha(h) \neq 0 \) and \( \beta(h) \neq 0 \) for any \( h \), then \( u(x, \xi; \hbar) \) may be written as
\[
u(x, \xi; \hbar) = \left[ W(\beta(h) T_{\alpha(h)} \phi) \right](x, \xi) = \frac{\beta^2(h)}{\alpha(h)} \left[ W(\phi) \right]\left(\alpha(h)x, \frac{\xi}{\alpha(h)}\right)
\]
for some function \( w(x, \xi) \) independent of \( h \).

Here, we collect some properties of the Wigner transform \([5, 10, 18, 21, 28, 39]\). To this end let \( S := S(\mathbb{R}) \subset L^2(\mathbb{R}, dx) \) be the Schwartz space of rapidly decreasing smooth functions.

**Lemma 1** Let \( \phi \in S \) and let \( W^h = W^h(\phi) \), then
\[
\int_{\mathbb{R}} W^h(x, \xi) d\xi = |\phi(x)|^2 \text{ and } \int_{\mathbb{R}} W^h(x, \xi) dx = \frac{2\pi}{h} \left| \hat{\phi}\left(\frac{\xi}{h}\right) \right|^2
\]
where \( \hat{\phi} \) is the Fourier transform of \( \phi \).

**Proof** We remark that
\[
W^h(x, \xi) = \hat{\nu}_h(\xi; x) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iy\xi} \nu_h(y; x) dy
\]
where \( \hat{\nu}_h \) is the Fourier transform of
\[
\nu_h(y; x) := \phi\left(x + \frac{1}{2} h y\right) \overline{\phi\left(x - \frac{1}{2} h y\right)}.
\]

Then it follows that
\[
\int_{\mathbb{R}} W^h(x, \xi) d\xi = \int_{\mathbb{R}} \hat{\nu}_h(\xi; x) e^{i\xi \cdot 0} d\xi = \nu_h(0; x) = |\phi(x)|^2.
\]

In order to prove the second statement let
\[
\int_{\mathbb{R}} W^h(x, \xi) dx = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-iy\xi} \phi\left(x + \frac{1}{2} h y\right) \overline{\phi\left(x - \frac{1}{2} h y\right)} dx dy
\]
\[
= \frac{1}{2\pi h} \int_{\mathbb{R}^2} e^{-i(u-v)\xi/h} \phi(u) \overline{\phi(v)} du dv
\]
\[
= \frac{2\pi}{h} \hat{\nu}_h\left(\frac{\xi}{h}\right) \hat{\phi}\left(\frac{\xi}{h}\right)
\]
where we set \( u = x + \frac{1}{2} h y \) and \( v = x - \frac{1}{2} h y \). \( \square \)
Remark 1 From Lemma 1 it follows that the expectation value $\langle a \rangle$ of a classical observable $a = a(x, \xi)$ is given by

$$\langle a \rangle = \langle \psi, A\psi \rangle = \int_{\mathbb{R} \times \mathbb{R}} a(x, \xi) W^h(x, \xi) \, dx \, d\xi$$

where $A$ is the linear operator associated to the classical observable and where we adopt the notation:

$$\langle f, g \rangle := \langle f, g \rangle_{L^2} = \int_{\mathbb{R}} f(x) g(x) \, dx, \quad f, g \in L^2(\mathbb{R}, dx).$$

Lemma 2 Let $\varphi \in S$ and let $W^h = W^h(\varphi)$, then

$$\int_{\mathbb{R} \times \mathbb{R}} W^h(x, \xi) \, dx \, d\xi = \|\varphi\|^2_{L^2(\mathbb{R}, dx)}.$$

Proof From Lemma 1 immediately follows that

$$\int_{\mathbb{R} \times \mathbb{R}} W^h(x, \xi) \, dx \, d\xi = \int_{\mathbb{R}} |\varphi(x)|^2 \, dx = \|\varphi\|^2_{L^2(\mathbb{R}, dx)}.$$

Remark 2 The previous Lemma does not prove that $W^h(x, \xi)$ belongs to $L^1(\mathbb{R} \times \mathbb{R}, dx \, d\xi)$. Indeed, in Example 2.4.1, we consider a function $\varphi \in L^2(\mathbb{R}, dx)$ such that its Wigner transform does not belongs to $L^1(\mathbb{R} \times \mathbb{R}, dx \, d\xi)$.

Lemma 3 Let $\varphi_1, \varphi_2 \in S$ and let $W^h_j = W^h(\varphi_j), j = 1, 2$, then

$$|\langle \varphi_1, \varphi_2 \rangle_{L^2(\mathbb{R}, dx)}|^2 = 2\pi \hbar \langle W^h_1, W^h_2 \rangle_{L^2(\mathbb{R} \times \mathbb{R}, dx \, d\xi)}.$$

In particular, if $W^h = W^h(\varphi), \varphi \in S$, then $W^h \in L^2(\mathbb{R} \times \mathbb{R}, dx \, d\xi)$ and

$$\|W^h\|_{L^2(\mathbb{R} \times \mathbb{R}, dx \, d\xi)} = \sqrt{\frac{1}{2\pi \hbar}} \|\varphi\|^2_{L^2(\mathbb{R}, dx)}.$$

Proof Recalling that

$$\frac{1}{2\pi} \int_{\mathbb{R}} e^{iux} \, dx = \delta(u)$$

is the Dirac’s $\delta$ distribution, then a straightforward calculation gives that

$$\langle W^h_1, W^h_2 \rangle_{L^2(\mathbb{R} \times \mathbb{R}, dx \, d\xi)} = \frac{1}{(2\pi)^2} \int_{\mathbb{R} \times \mathbb{R}} dx \, d\xi \int_{\mathbb{R}} dy_1 \int_{\mathbb{R}} dy_2 e^{i(y_1-y_2)\xi} \times \varphi_1(x - \frac{1}{2}h_y) \varphi_1(x + \frac{1}{2}h_y) \varphi_2(x + \frac{1}{2}h_y_2) \varphi_2(x - \frac{1}{2}h_y_2)$$

$$= \frac{1}{2\pi \hbar} |\langle \varphi_1, \varphi_2 \rangle_{L^2(\mathbb{R}, dx)}|^2$$

 proclaimed.
Remark 3. Let $p \in [1, +\infty]$, and let
\[ L^p_R(\mathbb{R} \times \mathbb{R}, dx d\xi) := \{ w \in L^p(\mathbb{R} \times \mathbb{R}, dx d\xi) : w(x, \xi) \text{ is real-valued} \} . \]

Then, by means of the previous Lemma we can conclude that the Wigner transform maps the Schwartz space $S$ to the Hilbert space $L^2_R(\mathbb{R} \times \mathbb{R}, dx d\xi)$:
\[ \varphi(x) \in L^2(\mathbb{R}, dx) \rightarrow \left[ W^h(\varphi) \right](x, \xi) \in L^2_R(\mathbb{R} \times \mathbb{R}, dx d\xi) . \]

In fact, $W^h(\varphi) \in L^2_R \cap L^\infty_R$; indeed:

Lemma 4. Let $\varphi \in S$ and let $W^h = W^h(\varphi)$, then
\[ \| W^h \|_{L^\infty(\mathbb{R} \times \mathbb{R})} \leq \frac{1}{\pi \hbar} \| \varphi \|_{L^2(\mathbb{R}, dx)}^2 . \]

Proof. The proof is quite simple:
\[
\left| W^h(x, \xi) \right| \leq \frac{1}{2\pi} \int_\mathbb{R} \left| \varphi \left( x + \frac{1}{2}\hbar y \right) \right| \left| \varphi \left( x - \frac{1}{2}\hbar y \right) \right| dy
\]
\[
\leq \frac{1}{2\pi} \left\{ \int_\mathbb{R} \left| \varphi \left( x + \frac{1}{2}\hbar y \right) \right|^2 dy \right\}^{1/2} \left\{ \int_\mathbb{R} \left| \varphi \left( x - \frac{1}{2}\hbar y \right) \right|^2 dy \right\}^{1/2}
\]
\[
= \frac{1}{\hbar \pi} \left\{ \int_\mathbb{R} |\varphi(u)|^2 du \right\}^{1/2} \left\{ \int_\mathbb{R} |\varphi(v)|^2 dv \right\}^{1/2} = \frac{1}{\pi \hbar} \| \varphi \|_{L^2(\mathbb{R}, dx)}^2
\]

where we set $u = x + \frac{1}{2}\hbar y$ and $v = x - \frac{1}{2}\hbar y$. \qed

2.2 Continuity of the Wigner transform

We have seen that the Wigner transform maps the space $S$ in the space $L^2_R(\mathbb{R} \times \mathbb{R}, dx d\xi)$. We prove now that such a map is continuous; in particular the following estimates hold true.

Theorem 1. Let $w_j = W^h(\varphi_j)$, where $\varphi_j \in S$ and $j = 1, 2$, then
\[ \| w_1 - w_2 \|_{L^2(\mathbb{R} \times \mathbb{R}, dx d\xi)} \leq C \inf_{\theta \in \mathbb{R}} \| e^{i\theta} \varphi_1 - \varphi_2 \|_{L^2(\mathbb{R}, dx)} \]
where
\[ C = \frac{1}{\sqrt{\pi \hbar}} \left( \| \varphi_1 \|_{L^2(\mathbb{R}, dx)} + \| \varphi_2 \|_{L^2(\mathbb{R}, dx)} \right) . \]

Proof. Let us denote
\[ \varphi_{1, \pm} := \varphi_{1, \pm}(x, y) = e^{i\theta} \varphi_1 \left( x \pm \frac{1}{2}\hbar y \right) , \quad \varphi_{2, \pm} := \varphi_{2, \pm}(x, y) = \varphi_2 \left( x \pm \frac{1}{2}\hbar y \right) , \]
where $\theta \in \mathbb{R}$ is independent of $x$ and $y$, and let
\[ \phi_{\pm} := \phi_{\pm}(x, y) = \varphi_{1, \pm}(x, y) - \varphi_{2, \pm}(x, y) , \]

\[ \sum_{\pm} \| \phi_{\pm} \|_{L^2(\mathbb{R} \times \mathbb{R}, dx d\xi)}^2 \leq C \inf_{\theta \in \mathbb{R}} \| e^{i\theta} \varphi_1 - \varphi_2 \|_{L^2(\mathbb{R}, dx)}^2 + \| \varphi_1 - \varphi_2 \|_{L^2(\mathbb{R}, dx)}^2 . \]
then
\[ |w_1(x, \xi) - w_2(x, \xi)| = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi y} \left[ \varphi_{1,+} - \varphi_{2,+} \right] dy \]
\[ = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\xi y} \left[ (\varphi_{1,+} - \varphi_{2,+})\varphi_{1,-} + \varphi_{2,+}(\varphi_{1,-} - \varphi_{2,-}) \right] dy \]
\[ = [\mathcal{F}_y(\varphi_{1,\cdot})](x, \xi) + [\mathcal{F}_y(\varphi_{2,\cdot})](x, \xi) \]

where \( \mathcal{F}_y(f) \) denotes the Fourier transform (with respect to \( y \)) of a function \( f \).

Furthermore, recalling that \( \| \mathcal{F}_y f \|_{L^2} = \frac{1}{\sqrt{2\pi}} \| f \|_{L^2} \) then
\[
\| w_1(x, \xi) - w_2(x, \xi) \|_{L^2(\mathbb{R} \times \mathbb{R}, dx \, d\xi)}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} dx \, d\xi |w_1(x, \xi) - w_2(x, \xi)|^2
\]
\[
= \int_{\mathbb{R}} dx \| w_1(x, \cdot) - w_2(x, \cdot) \|_{L^2(\mathbb{R}, d\xi)}^2
\]
\[
= \int_{\mathbb{R}} dx \left[ \mathcal{F}_y(\varphi_{1,\cdot}) \right](x, \cdot) + \left[ \mathcal{F}_y(\varphi_{2,\cdot}) \right](x, \cdot) \|_{L^2(\mathbb{R}, d\xi)}^2
\]
\[
= \frac{1}{2\pi} \int_{\mathbb{R}} dx \| (\varphi_{1,\cdot}) - (\varphi_{2,\cdot}) \|_{L^2(\mathbb{R}, dy)}^2
\]
\[
\leq \frac{1}{\pi} \int_{\mathbb{R}} dx \| (\varphi_{1,\cdot}) - (\varphi_{2,\cdot}) \|_{L^2(\mathbb{R}, dy)}^2 + \frac{1}{\pi} \int_{\mathbb{R}} dx \| (\varphi_{1,\cdot}) - (\varphi_{2,\cdot}) \|_{L^2(\mathbb{R}, dy)}^2.
\]

We restrict now our attention to the first integral
\[
\int_{\mathbb{R}} dx \| (\varphi_{1,\cdot}) - (\varphi_{2,\cdot}) \|_{L^2(\mathbb{R}, dy)}^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} dy \| (\varphi_{1,\cdot}) - (\varphi_{2,\cdot}) \|_{L^2(\mathbb{R}, dy)}^2
\]
\[
= \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \left| e^{i\theta} \varphi_1 \left( x + \frac{1}{2} \hbar y \right) - \varphi_2 \left( x + \frac{1}{2} \hbar y \right) \right|^2 \left| e^{i\theta} \varphi_1 \left( x - \frac{1}{2} \hbar y \right) \right|^2.
\]

By means of the change of variable \( x \to z = x + \frac{1}{2} \hbar y \) the integral above becomes
\[
\int_{\mathbb{R}} dz \left| e^{i\theta} \varphi_1 (z) - \varphi_2 (z) \right|^2 \int_{\mathbb{R}} dy \left| e^{i\theta} \varphi_1 (z - \hbar y) \right|^2 = \frac{1}{\hbar} \| e^{i\theta} \varphi_1 - \varphi_2 \|_{L^2}^2 \| \varphi_1 \|_{L^2}^2.
\]

The other integral can be similarly treated proving thus the Theorem. \( \square \)

We can also prove that

**Lemma 5** We have that
\[
\| w_1 - w_2 \|_{L^\infty(\mathbb{R} \times \mathbb{R}, dx \, d\xi)} \leq C \inf_{\theta \in \mathbb{R}} \| e^{i\theta} \varphi_1 - \varphi_2 \|_{L^2(\mathbb{R}, dx)}
\]

where \( C = \frac{1}{\pi \hbar} \left[ \| \varphi_1 \|_{L^2(\mathbb{R}, dx)} + \| \varphi_2 \|_{L^2(\mathbb{R}, dx)} \right] \).

**Proof** Estimate (11) immediately follows from (9); indeed:
\[
|w_1(x, \xi) - w_2(x, \xi)| \leq \frac{1}{2\pi} \int_{\mathbb{R}} \left[ |\varphi_{1,+} - \varphi_{2,+}| + |\varphi_{1,-} - \varphi_{2,-}| \right] dy
\]
\[
\leq \frac{1}{2\pi} \left[ \| \varphi_{1,+}(x, \cdot) - \varphi_{2,+}(x, \cdot) \|_{L^2(\mathbb{R}, dy)} \| \varphi_{1,-}(x, \cdot) - \varphi_{2,-}(x, \cdot) \|_{L^2(\mathbb{R}, dy)} + \| \varphi_{1,-}(x, \cdot) - \varphi_{2,-}(x, \cdot) \|_{L^2(\mathbb{R}, dy)} \right] \| \varphi_{2,+}(x, \cdot) \|_{L^2(\mathbb{R}, dy)}
\]

where, for instance,
Quantum forced oscillator via Wigner transform

\[
\left\| \varphi_{2,+}(x, \cdot) \right\|_{L^2(\mathbb{R}, dy)} = \left[ \int_{\mathbb{R}} \left| \varphi_{2,+}(x, y) \right|^2 dy \right]^{1/2} = \left[ \int_{\mathbb{R}} \left| \varphi_2 \left( x + \frac{1}{2} \hbar y \right) \right|^2 dy \right]^{1/2} = \left[ \frac{2}{\hbar} \int_{\mathbb{R}} \left| \varphi_2(z) \right|^2 dz \right]^{1/2} = \frac{\sqrt{2}}{\hbar} \left\| \varphi_2 \right\|_{L^2(\mathbb{R}, dx)}.
\]

\[\square\]

2.3 Reversibility of the Wigner transform

If \( \varphi \in S \) is a positive real-valued function then the transformation \( \mathcal{W}^h(\varphi) \) is invertible; indeed, from Lemma 1 we can obtain the absolute value of the wave-function:

\[
\varphi(x) = |\varphi(x)| = \sqrt{\int_{\mathbb{R}} W^h(x, \xi) d\xi}.
\]

In general, since the wave-function \( \varphi \) is not a real-valued function or it does not has a definite sign, the Wigner transform can be inverted as follows.

Lemma 6 Let \( \varphi \in S \) and let \( W^h = \mathcal{W}^h(\varphi) \), and let \( x^* \) be such that \( \int_{\mathbb{R}} W^h(x^*, \xi) d\xi \neq 0 \). Then \( \varphi(x^*) \neq 0 \) and

\[
\varphi(x) = \frac{1}{\varphi(x^*)} \int_{\mathbb{R}} W^h \left( \frac{x + x^*}{2}, \xi \right) e^{i(x-x^*)\xi/\hbar} d\xi. \tag{12}
\]

Proof Since \( \int_{\mathbb{R}} W^h(x^*, \xi) d\xi = |\varphi(x^*)|^2 \) then \( \varphi(x^*) \neq 0 \) immediately follows from the assumption. For any real-valued function \( f(x) \) it follows that

\[
\int_{\mathbb{R}} W^h \left[ f(x), \xi \right] e^{ix\xi} d\xi = \frac{1}{2\pi} \int_{\mathbb{R}^2} d\xi d\eta e^{i(x-y)\xi} \varphi \left( f(x) + \frac{1}{2} \hbar y \right) \overline{\varphi \left( f(x) - \frac{1}{2} \hbar y \right)}
\]

since (7). In particular, if we set \( f(x) = \frac{1}{2} \hbar x + x^* \) then, for any \( x \in \mathbb{R} \), it follows that

\[
\int_{\mathbb{R}} W^h \left( \frac{1}{2} \hbar x + x^*, \xi \right) e^{ix\xi} d\xi = \varphi(\hbar x + x^*)\overline{\varphi(x^*)}.
\]

If we call \( z = \hbar x + x^* \), i.e. \( x = \frac{z-x^*}{\hbar} \), then we obtain

\[
\overline{\varphi(x^*)} \varphi(z) = \int_{\mathbb{R}} W^h \left( \frac{z + x^*}{2}, \xi \right) e^{i(z-x^*)\xi/\hbar} d\xi
\]

from which (12) follows. \[\square\]

If we denote \( \theta^* \) the phase of \( \varphi(x^*) \) then (12) takes the form

\[
\varphi(x) = \frac{e^{i\theta^*}}{\sqrt{|\int_{\mathbb{R}} W^h(x^*, \xi) d\xi|}} \int_{\mathbb{R}} W^h \left( \frac{x + x^*}{2}, \xi \right) e^{i(x-x^*)\xi/\hbar} d\xi. \tag{13}
\]

Thus, the Wigner transform \( \mathcal{W}^h \) is invertible up to a phase factor. From this result and by making use of the continuity arguments proved in Sect. 2.2 we can extend the Wigner transform from \( S \) to the whole Hilbert space \( L^2(\mathbb{R}, dx) \) and conclude that:
Theorem 2 The map
\[ W^h : L^2(\mathbb{R}, dx) \to \mathcal{D} := W^h \left( L^2(\mathbb{R}, dx) \right) \subseteq L^2(\mathbb{R} \times \mathbb{R}, dxd\xi) \]
is, up to a phase factor independent of \( x \), a one-to-one map and the inverse map \( [W^h]^{-1} \) is given by (13) where \( x^* \) is any value such that \( \int_{\mathbb{R}} W^h(x^*, \xi)d\xi \neq 0 \).

In fact, \( \mathcal{D} \subseteq L^2(\mathbb{R}) \) and a question arises: what conditions on a real-valued square integrable function \( w(x, \xi) \in L^2(\mathbb{R} \times \mathbb{R}, dxd\xi) \) must be satisfied for it to be a Wigner function (of a pure state), i.e. \( w \in \mathcal{D} \)? A first answer to this question was given by [21]; they stated a set of necessary and sufficient conditions for a function \( w \in L^2(\mathbb{R} \times \mathbb{R}, dxd\xi) \) to be a Wigner transform (of a non necessarily pure state): that is \( w(x, \xi) \) must be normalized, in the sense that \( \int_{\mathbb{R}^2} w(x, \xi)dxd\xi = 1 \), and furthermore
\[ (w, W^h(\phi))_{L^2(\mathbb{R} \times \mathbb{R}, dxd\xi)} \geq 0, \forall \phi \in L^2(\mathbb{R}, dx). \] (14)

As pointed out by Narcowich and O’Connel [36] such a set of conditions is not very satisfactory because (14) is rather difficult to check from a practical point of view; indeed, they proposed a second set of conditions. Namely, let \( \tilde{w}(u, v) \) be the symplectic Fourier transform of \( w(x, \xi) \); then \( w \in L^2(\mathbb{R} \times \mathbb{R}, dxd\xi) \) is a Wigner transform function if, and only if, \( \tilde{w}(0, 0) = 1 \) and \( \tilde{w}(u, v) \) is continuous and it is of \( \hbar \)-positive type (see [26, 29, 30] for a definition of \( \hbar \)-positive type). Unfortunately, such a criterion cannot distinguish between Wigner functions associated with pure states or mixed states. Tatarski [41] introduced a necessary and sufficient condition for a function \( w \in L^2(\mathbb{R} \times \mathbb{R}, dxd\xi) \) to describe a pure quantum state; in particular, he proved that \( w \in \mathcal{D} \), that is \( w = W^h(\varphi) \) for some \( \varphi \in L^2 \), if, and only if,
\[ \frac{\partial^2}{\partial x_1 \partial x_2} \ln \left[ Q(x_1, x_2) \right] = 0 \] (15)
where
\[ Q(x_1, x_2) := \int_{\mathbb{R}} e^{i\xi(x_1-x_2)/\hbar} w \left( \frac{x_1-x_2}{2}, \xi \right) d\xi. \]
If (15) is satisfied then the wave-function \( \varphi(x) \) can be recovered from \( w(x, \xi) \) by (13), up to a phase factor independent of \( x \).

Finally, we should also mention the results by [14] where the authors make use of the notion of Narkowich–Wigner spectrum in order to characterize the Wigner functions of pure states.

2.4 Examples

Let us consider some examples of computation of Wigner transform of normalized functions \( \varphi \in L^2(\mathbb{R}, dx) \).

2.4.1 Example 1.

Let \( \varphi(x) = \frac{1}{\sqrt{2R}} \chi_{[-R, +R]}(x) \) where \( \chi \) is the characteristic function, i.e. \( \chi_A(x) = 1 \) if \( x \in A \) and \( \chi_A(x) = 0 \) if \( x \notin A \), and \( R > 0 \) is fixed. Then
\[ W^h(x, \xi) = \frac{1}{4\pi R} \int_{\mathbb{R}} e^{-i\xi y} \chi_{[-R, +R]} \left( x + \frac{1}{2}\hbar y \right) \chi_{[-R, +R]} \left( x - \frac{1}{2}\hbar y \right) dy. \]
is an even function with respect to $x$. We assume, for argument’s sake, that $x \geq 0$. Hence

$$W^h(x, \xi) = 0 \quad \text{if} \quad x \geq R$$

and

$$W^h(x, \xi) = \frac{1}{4\pi R} \int_{-2(R-x)/h}^{2(R-x)/h} e^{-i\xi y} dy = \frac{1}{2\pi R\xi} \sin \left( \frac{2\xi}{h}(R-x) \right) \quad \text{if} \quad x \leq R.$$ 

In conclusion

$$W^h(x, \xi) = \frac{1}{2\pi R\xi} \sin \left( \frac{2\xi}{h}(R-|x|) \right) \chi_{[-R,+R]}(x).$$

One should remark (see Appendix A) that

$$W^h \in L^1(\mathbb{R} \times \mathbb{R}, dx d\xi). \quad (16)$$

2.4.2 Example 2.

Let $\varphi(x) := \frac{1}{\sqrt{\pi}} e^{-x^2/2}$; then a straightforward calculation gives that

$$W^h(x, \xi) = \frac{1}{\pi h} e^{-\left(\frac{x^2+\xi^2}{\hbar^2}\right)}.$$ 

Now, let $\varphi_\alpha(x) = \sqrt{\frac{\alpha^2}{\pi}} e^{-\alpha^2 x^2/2} = \sqrt{\alpha} \varphi(\alpha x)$. Then

$$W^h_\alpha(x, \xi) = [\mathcal{W}^h(\varphi_\alpha)](x, \xi) = \mathcal{W}^h(\sqrt{\alpha} T_\alpha \varphi)(x, \xi) = \alpha \left[ \mathcal{W}^h(T_\alpha \varphi) \right](x, \xi)$$

$$= \mathcal{W}^h(\varphi) \left( \frac{x}{\alpha}, \frac{\xi}{\alpha} \right) = \frac{1}{\pi h} e^{-a^2 x^2 - \frac{\xi^2}{a^2 h^2}}.$$ 

In particular, if $\alpha = 1/\sqrt{\hbar}$ then

$$\varphi(x) = \frac{1}{\sqrt{\hbar} \pi} e^{-x^2/2\hbar}$$ 

has Wigner transform

$$W^h(x, \xi) = \frac{1}{\pi \hbar} e^{-\frac{x^2+\xi^2}{\hbar}}.$$ 

2.4.3 Example 3.

Let

$$\varphi(x) = \exp \left[ -\frac{1}{2} \left( (a_1 + ia_2)x^2 + (b_1 + ib_2)x + (c_1 + ic_2) \right) \right]$$ 

be the Gaussian-type function considered by Hudson [23], where $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$ and $a_1 > 0$. By means of a straightforward calculation it turns out that $W^h(x, \xi) = [\mathcal{W}^h(\varphi)](x, \xi)$ is still a real-valued and positive Gaussian-type function given by

$$W^h(x, \xi) = \exp \left[ -\frac{4\hbar^2(a_1^2 + a_2^2)x^2 + 8a_2\hbar \xi + 4\hbar^2(a_1b_1 + a_2b_2)x + b_2^2\hbar^2 + 4a_1c_1\hbar^2 + 4b_2\hbar \xi + 4\xi^2}{4a_1^2\hbar^2} \right] \frac{1}{\hbar \sqrt{\pi a_1}}.$$
2.4.4 Example 4.

Let \( \phi_n(x) = H_n(x)e^{-x^2/2} \) where \( H_n(x) \) is the \( n \)-th Hermite polynomial; then it is not hard to see [18] that its Wigner transform (for \( \hbar = 1 \)) is given by

\[
W^1_n(x, \eta) = \left[ \mathcal{W}^1(\phi_n) \right](x, \xi) = \frac{(-1)^n}{\pi} e^{-\eta^2/2} L_n \left[ 2(x^2 + \eta^2) \right]
\]

where \( L_n \) is the \( n \)-th Laguerre polynomial. Then

\[
W^1_n(x, \xi) = \frac{(-1)^n}{\pi} e^{-\eta^2/2} L_n \left[ 2x^2 + \frac{\eta^2}{\hbar^2} \right]
\]

Remark 4 We recall that \( L_0(z) = 1 \) and \( L_1(z) = 1 - x \); thus the function defined as

\[
w(x, \xi) = C_0 W^0_0(x, \xi) + C_1 W^1_1(x, \xi)
\]

\[
= \frac{1}{\hbar} e^{-\left( x^2 + \frac{\xi^2}{\hbar^2} \right)} \left[ C_0 - C_1 \left( 1 - \left( 2x^2 + \frac{2\xi^2}{\hbar^2} \right) \right) \right]
\]

\[
= \frac{C_0}{2\hbar} e^{-\left( x^2 + \frac{\xi^2}{\hbar^2} \right)} \left[ 1 + 2x^2 + \frac{2\xi^2}{\hbar^2} \right],
\]

where \( C_1 = \frac{1}{2} C_0 > 0 \), is such that \( w(x, \xi) > 0 \) everywhere. If \( w \in \mathcal{D} \) then, by the Hudson’s argument, \( w(x, \xi) \) must be of the form discussed in the Example 2.4.3. Because this is not the case then \( w \notin \mathcal{D} \) and thus we can conclude that the set \( \mathcal{D} \) is not closed to linear combinations.

2.4.5 Example 5. Wavefunction of the free Schrödinger equation

Let us consider the free linear Schrödinger equation \( i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2} \frac{\partial^2 \psi}{\partial x^2} \) with normalized initial condition

\[
\psi_0(x; \hbar) = \sqrt{\frac{2}{\pi}} e^{-x^2/2};
\]

then it is well known that [42]

\[
\psi(x, t; \hbar) = \left[ U(t) \psi_0 \right](x; \hbar) = \int_{\mathbb{R}} K_0(x - y, t) \psi_0(y; \hbar) dy
\]

\[
= \sqrt{\frac{2}{\pi}} \sqrt{i - 4\hbar t} e^{-i \frac{x^2}{16\hbar^2}} e^{-\frac{x^2(1 - 4\hbar t)}{16\hbar^2 t^2}}
\]

where the kernel \( K_0 \) is given by

\[
K_0(z, t) = \sqrt{\frac{1}{4\pi i \hbar t}} e^{\frac{z^2}{4\hbar t}}.
\]

A straightforward calculus gives that the Wigner transform \( W^\hbar = \mathcal{W}^\hbar(\psi) \) of the wavefunction \( \psi(x, t; \hbar) \) is given by

\[
W^\hbar(x, \xi, t) = \frac{1}{\pi \hbar} e^{-\frac{\xi^2}{2\hbar^2}} e^{-2(x-2\xi t)^2}.
\]
2.4.6 Example 6. Stationary solution for a singular potential

Let us consider the equation

\[ i\hbar \frac{\partial \psi}{\partial t} = -\hbar^2 \frac{\partial^2 \psi}{\partial x^2} + \gamma \delta \psi \]  \hspace{1cm} (17)

where \( \delta \) is the Dirac’s delta distribution supported at \( x = 0 \). We look for a stationary solution of the form \( \psi(x, t; \hbar) = e^{-iEt/\hbar} \varphi(x; \hbar) \) for some \( E \in \mathbb{R} \). It is well known [2] that exactly one stationary solution occurs only if \( \gamma < 0 \) and it is given by

\[ E = -\frac{\gamma^2}{4\hbar^2} \quad \text{and} \quad \varphi(x) = \sqrt{\kappa} e^{-\kappa |x|}, \quad \kappa = \frac{\sqrt{|E|}}{\hbar} = \frac{|\gamma|}{2\hbar^2} \]

By means of a straightforward calculation it turns out that the Wigner’s transform is an even function with respect \( x \) and \( \xi \) both,

\[ W^\hbar(x, \xi) = W^\hbar(-x, -\xi) = W^\hbar(x, \xi), \]

given by

\[ W^\hbar(x, \xi) = \frac{\hbar \kappa^2}{\xi \pi (\kappa^2 \hbar^2 + 4\xi^2)} e^{-2\kappa |x|} \left[ \cos \left( \frac{2|x|\xi}{\hbar} \right) \xi + \kappa \hbar \sin \left( \frac{2|x|\xi}{\hbar} \right) \right]. \]  \hspace{1cm} (18)

2.4.7 Example 7. Stationary soliton

Let us consider the Gross–Pitaevskii equation

\[ i\hbar \frac{\partial \psi}{\partial t} = -\hbar^2 \frac{\partial^2 \psi}{\partial x^2} + \nu |\psi|^2 \psi, \quad \nu \in \mathbb{R}, \]

where we look for a stationary solution of the form \( \psi(x, t; \hbar) = e^{-iEt/\hbar} \varphi(x; \hbar) \) for some \( E \in \mathbb{R} \) and a real-valued function \( \varphi \). Then \( \varphi(x; \hbar) \) is a normalized solution to the equation

\[ -\hbar^2 \varphi'' + \nu \varphi^3 = E \varphi, \]

that is it is given by \( \varphi(x) = A \text{sech}(Bx) \) where [12]

\[ A = \frac{\sqrt{-\nu}}{\sqrt{8\hbar}}, \quad B = -\frac{\nu}{4\hbar^2} \quad \text{and} \quad E = -\frac{\nu^2}{16\hbar^2} \]

provided that \( \nu < 0 \). In conclusion:

\[ \psi(x, t; \hbar) = \frac{\sqrt{-\nu}}{\sqrt{8\hbar}} \text{sech} \left( \frac{-\nu}{4\hbar^2} x \right) e^{i\nu t/16\hbar^3}. \]

Its Wigner transform does not depend on time and it is given by

\[ W^\hbar(x, \xi; \hbar) = \frac{1}{\hbar} \frac{\sin \left( \frac{2|x|\xi}{\hbar} \right)}{\sinh \left( \frac{\nu x}{2\hbar^2} \right) \sinh \left( \frac{4\pi \xi}{\nu} \hbar \right)}, \]  \hspace{1cm} (19)

by means of a straightforward calculation [27] (see Appendix B for details).

3 Schrödinger equation in the Wigner representation

Here we deal with the one-dimensional semiclassical linear Schrödinger equation

\[ \begin{cases} i\hbar \frac{\partial \psi}{\partial t} = H \psi \\ \psi(x, 0; \hbar) = \psi_0(x; \hbar) \end{cases} \quad \text{where} \quad H := -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x, t) \]  \hspace{1cm} (20)
where $\hbar \ll 1$ is a semiclassical parameter; for argument’s sake we choose the units such that $2m = 1$. Equation (20) is a one-dimensional linear Schrödinger equation and, under some suitable assumptions on the real-valued potential $V(x, t)$, it has a global solution $\psi(x, t; \hbar)$ and the conservation of the norm $\|\psi(\cdot, t; \hbar)\|_{L^2} = \|\psi(0; \hbar)\|_{L^2}$ holds true.

In order to consider a different approach we make use of the (semiclassical) Wigner transform $W^h(x, \xi, t; h)$ of $\psi(x, t; h)$ defined as:

$$W^h(x, \xi, t; h) = \frac{1}{2\pi} \int e^{-iy\xi} \psi \left(x + \frac{1}{2} \hbar y, t; h\right) \overline{\psi \left(x - \frac{1}{2} \hbar y, t; h\right)} dy.$$ 

Hereafter, we simply denote

$$W^h(x, \xi, t) := W^h(x, \xi, t; h).$$

In the Wigner’s representation the Schrödinger equation (20) takes the following form.

**Theorem 3** [Time-dependent Schrödinger equation in the Wigner representation]

Assume that $V(x, t)$ is a smooth real-valued function. The Wigner transform $W^h(x, \xi, t)$ satisfies to the following formal equation

$$\begin{cases} 
\partial W^h / \partial t = -2\xi \partial W^h / \partial x + \partial V \partial W^h / \partial \xi + \sum_{m=1}^{\infty} \frac{(-1)^m \hbar^{2m}}{2^m (2m)!} \partial_{x^{2m+1}} W^h \\
W^h(x, \xi, 0) = W^0_{\psi}(x, \xi) = \left[H^h(\psi_0)\right](x, \xi)
\end{cases} \quad (21)$$

**Remark 5** In the semiclassical limit where $\hbar \to 0$ the dominant term of Eq. (21) is formally given by

$$\frac{\partial W^h}{\partial t} = -2\xi \partial W^h / \partial x + \partial V \partial W^h / \partial \xi$$

(22)

and it coincides with the classical Liouville equation. If the potential $V(x)$ is independent of $t$ let $\mathcal{H}(q, p) = p^2 + V(q)$ be the classical Hamiltonian (where $2m = 1$) and let

$$\begin{cases} 
\dot{p} = \partial \mathcal{H} / \partial q \\
\dot{q} = -\partial \mathcal{H} / \partial p
\end{cases} \quad \text{with initial condition} \quad \begin{cases} 
p(0) = \xi \\
q(0) = x
\end{cases} \quad (23)$$

Then, the solution to (22) is given by

$$W(x, \xi, t) = W^0_{\psi}(q(t), p(t)) = W^0_{\psi} \left[S^t(x, \xi)\right]$$

where $(q(t), p(t)) = S^t(x, \xi)$ is the Hamiltonian flux associated to (23).

**Theorem 4** [Time-independent Schrödinger equation in the Wigner representation] Let $\psi(x, t; h) = \varphi(x; h)e^{-iEt/\hbar}$, where $\varphi \in L^2(\mathbb{R}, dx)$, be a stationary solution to the time-dependent Schrödinger equation

$$E \varphi = H \varphi, \quad \text{where} \quad H := -\frac{\hbar^2}{2m} \partial^2 / \partial x^2 + V(x),$$

and where $V(x)$ is a smooth real-valued function independent of $t$. Then the Wigner transform $W^h(x, \xi) = \left[H^h(\varphi)\right](x, \xi)$ is independent of $t$ and it satisfies to the following formal equation

$$E W^h = -\frac{\hbar^2}{4} \partial^2 W^h / \partial x^2 + \left[\xi^2 + V\right] W^h + \sum_{m=1}^{\infty} \frac{(-1)^m \hbar^{2m}}{2^m (2m)!} \partial_{x^{2m}} W^h$$

(24)
under the constrain
\[
0 = -2ξ \frac{\partial W^h}{\partial x} + dV \frac{\partial W^h}{\partial ξ} + \sum_{m=1}^{\infty} \frac{(-1)^m h^{2m} d^{2m+1} V}{2^{m}(2m+1)!} \frac{\partial^{2m+1} W^h}{\partial ξ^{2m+1}}.
\]  

Remark 6 We may remark that the systems of equations (21) and (24–25) are only formal because we make no statement about the convergence of the series. In fact, when the potential \(V\) is a polynomial of finite degree with respect to \(x\) then this problem does not arise.

Remark 7 One may, in principle, solve equations (21), or (24–25), by means of some approximation or numerical methods. This approach presents some problems: assume to find an approximate solution \(\tilde{W}^h\) to (21) such that \(\|W^h - \tilde{W}^h\| \ll 1\) with respect some norm, where \(W^h\) is the exact solution to (21). Some questions occur:

i. is the approximate solution \(\tilde{W}^h\) the Wigner transform of a pure state? This point was discussed in Sect. 2.3; but the criteria proposed by Narcowich and O’Connel, or by Tatarski, are rather difficult to be implemented in explicit models.

ii. and in the affirmative case, what can we say about the difference \(\|\psi - \phi\|_{L^2}\), where \(\psi = [\mathcal{W}^h]^{-1}(W^h)\) and \(\phi = [\mathcal{W}^h]^{-1}(\tilde{W}^h)\)? In fact, we have proved in Theorem 1 the continuity of the Wigner transform; but the continuity of the inverse is, as far as we know, an open problem: i.e., if \(\varphi_i = [\mathcal{W}^h]^{-1}(w_i)\) where \(w_i \in \mathcal{D}, i = 1, 2\), we would require that

\[
\inf_{\theta} \|e^{i\theta \varphi} \psi_1 - \varphi_2\|_{L^p} \leq C \|w_1 - w_2\|_{L^q}
\]

for some \(p, q\) and \(r\), and some positive constant \(C\) depending on \(\hbar\). However, some results hold true; for instance, let \(a(x)\) be a bounded classical observable, let \(<a> = \langle \varphi_j (\cdot), a(\cdot) \varphi_j (\cdot) \rangle, j = 1, 2\), be the expectation value of the classical observable \(a(x)\) on the quantum state described by the wave-function \(\varphi_j\). Then

\[
| < a >^1 - < a >^2 | = \left| \int_{\mathbb{R}} a(x) \left[ |\varphi_1(x)|^2 - |\varphi_2(x)|^2 \right] dx \right|
= \left| \int \int_{\mathbb{R}^2} a(x) \left[ w_1(x, \xi) - w_2(x, \xi) \right] d\xi dx \right|
\leq \|a\|_{L^\infty(\mathbb{R}, d\xi)} \|w_1 - w_2\|_{L^1(\mathbb{R} \times \mathbb{R}, d\xi d\xi)}
\]

3.1 Proofs of Theorems 3 and 4

Let us denote
\[
\psi_\pm := \psi \left( x \pm \frac{1}{2} \hbar y, t; \hbar \right) \text{ and } V_\pm = V \left( x \pm \frac{1}{2} \hbar y, t \right).
\]  

Then, from the Schrödinger equation (20) it follows that
\[
i \hbar \dot{\psi}_+ = -\hbar^2 \psi_+'' + V_+ \psi_+
\]  

and
\[
i \hbar \dot{\psi}_- = -\hbar^2 \psi_-'' + V_- \psi_-
\]
since the potential \( V \) is assumed to be a real-valued function. From these two equations it follows that

\[
\hbar \int_{\mathbb{R}} \dot{\psi}_+ \bar{\psi}_- e^{-i y \xi} \, dy = -\hbar^2 \int_{\mathbb{R}} \psi''_+ \bar{\psi}_- e^{-i y \xi} \, dy + \int_{\mathbb{R}} \psi_+ \bar{\psi}_- V e^{-i y \xi} \, dy \quad (29)
\]

and

\[
-\hbar \int_{\mathbb{R}} \psi_+ \bar{\psi}_- e^{-i y \xi} \, dy = -\hbar^2 \int_{\mathbb{R}} \psi''_+ \bar{\psi}_- e^{-i y \xi} \, dy + \int_{\mathbb{R}} \psi_+ \bar{\psi}_- V e^{-i y \xi} \, dy \quad (30)
\]

If we take the difference between (29) and (30), and then the sum, it follows that

\[
2\pi i \hbar \frac{\partial W}{\partial t} = i \hbar \int_{\mathbb{R}} \left( \psi_+ \bar{\psi}_- + \psi_- \bar{\psi}_+ \right) e^{-i y \xi} \, dy = -\hbar^2 F_1 + F_2 \quad (31)
\]

and

\[
i \hbar \int_{\mathbb{R}} \left( \psi_+ \bar{\psi}_- - \psi_- \bar{\psi}_+ \right) e^{-i y \xi} \, dy = -\hbar^2 F_3 + F_4 \quad (32)
\]

where

\[
F_1 = \int_{\mathbb{R}} \left( \psi''_+ \bar{\psi}_- - \psi_+ \bar{\psi}''_+ \right) e^{-i y \xi} \, dy,
F_2 = \int_{\mathbb{R}} \left( \psi_+ \bar{\psi}_- - \psi_- \bar{\psi}_+ \right) e^{-i y \xi} \, dy,
F_3 = \int_{\mathbb{R}} \left( \psi''_+ \bar{\psi}_- + \psi_+ \bar{\psi}''_+ \right) e^{-i y \xi} \, dy,
F_4 = \int_{\mathbb{R}} \left( \psi_+ \bar{\psi}_- + \psi_- \bar{\psi}_+ \right) e^{-i y \xi} \, dy.
\]

By means of straightforward calculations on can check that

\[
F_1 = \frac{4\pi i \xi}{\hbar} \frac{\partial W}{\partial x} \quad (33)
\]

and that

\[
F_3 = \pi \frac{\partial^2 W}{\partial x^2} - \frac{4\pi}{\hbar^2} \xi^2 W \quad (34)
\]

Concerning the other two terms \( F_2 \) and \( F_4 \) we have that

\[
F_2 = 4\pi \sum_{m=0}^{\infty} \frac{1}{(2m + 1)!} \frac{\partial^{2m+1} V}{\partial x^{2m+1}} \left( \frac{i}{2\hbar} \right)^{2m+1} \frac{\partial^{2m+1} W}{\partial \xi^{2m+1}} \quad (35)
\]

\[
F_4 = 4\pi \sum_{m=0}^{\infty} \frac{1}{(2m)!} \frac{\partial^{2m} V}{\partial x^{2m}} \left( \frac{i}{2\hbar} \right)^{2m} \frac{\partial^{2m} W}{\partial \xi^{2m}} \quad (36)
\]

from the formal power series expansion

\[
V_\pm = V \left( x \pm \frac{1}{2} \hbar y, t \right) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \frac{\partial^\ell V(x, t)}{\partial x^\ell} \left( \pm \frac{1}{2} \hbar y \right)^\ell.
\]
For instance

\[ F_2 = \int_{\mathbb{R}} (V_+ - V_-) \psi_+ \overline{\psi}_- e^{-i\xi y} dy \]

\[ = 2 \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \frac{\partial^{2m+1} V(x, t)}{\partial x^{2m+1}} \left( \frac{i}{2} \hbar \right)^{2m+1} \int_{\mathbb{R}} y^{2m+1} \psi_+ \overline{\psi}_- e^{-i\xi y} dy \]

\[ = 2 \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \frac{\partial^{2m+1} V(x, t)}{\partial x^{2m+1}} \left( \frac{i}{2} \hbar \right)^{2m+1} \frac{\partial^{2m+1}}{\partial \xi^{2m+1}} \int_{\mathbb{R}} \psi_+ \overline{\psi}_- e^{-i\xi y} dy \]

\[ = 4\pi \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \frac{\partial^{2m+1} V(x, t)}{\partial x^{2m+1}} \left( \frac{i}{2} \hbar \right)^{2m+1} \frac{\partial^{2m+1}}{\partial \xi^{2m+1}} W^{\hbar} . \]

Thus, from (31), (33) and (35) Theorem 3 follows. Similarly, by noticing that

\[ i\hbar \int_{\mathbb{R}} \left( \psi_+ \overline{\psi}_- - \psi_+ \overline{\psi}_- \right) e^{-i\xi y} dy = E \int_{\mathbb{R}} (\psi_+ \overline{\psi}_- + \psi_+ \overline{\psi}_-) e^{-i\xi y} dy = 4\pi EW^{\hbar} \]

when \( \psi(x, t; \hbar) = e^{-iEt/\hbar} \phi(x; \hbar) \), then \( W^{\hbar} \) is independent of \( t \) and from (32), (34) and (36) Theorem 4 follows.

The proofs are thus completed.

### 3.2 Schrödinger equations with singular potential or with a nonlinear potential in the Wigner representation

Here we consider the cases where the potential \( V \) is a singular function, namely a Dirac’s delta potential, or where a nonlinear potential occurs, namely we consider the Gross–Pitaevskii equation. In both cases one can write a formal equation to the Wigner representation of the wave-function.

#### 3.2.1 Schrödinger equation with a Dirac’s delta potential in the Wigner representation

We premise the following result.

**Lemma 7** Let \( V(x) = \gamma \delta_{x_0}(x) \) be a Dirac’s delta distribution supported at the point \( x = x_0 \). Then

\[ F_2 = \frac{4\gamma i}{\hbar} \int_{\mathbb{R}} W^{\hbar}(x, \xi) \sin \left[ \frac{2(x_0 - x)(\xi' - \xi)}{\hbar} \right] d\xi' \]

\[ F_4 = \frac{4\gamma}{\hbar} \int_{\mathbb{R}} W^{\hbar}(x, \xi) \cos \left[ \frac{2(x_0 - x)(\xi' - \xi)}{\hbar} \right] d\xi' \]

**Proof** Indeed (let us denote \( \psi(x, t; \hbar) \) by \( \psi(x) \) for sake of simplicity),

\[ F_2 = \int_{\mathbb{R}} (V_+ - V_-) \psi_+ \overline{\psi}_- e^{-i\xi y} dy \]

\[ = \gamma \int_{\mathbb{R}} \left[ \delta_{x_0} \left( x + \frac{1}{2} \hbar y \right) - \delta_{x_0} \left( x - \frac{1}{2} \hbar y \right) \right] \psi \left( x + \frac{1}{2} \hbar y \right) \overline{\psi} \left( x - \frac{1}{2} \hbar y \right) e^{-i\xi y} dy \]

\[ = \frac{4\gamma i}{\hbar} \int \left[ (\psi(x_0) \overline{\psi}(2x - x_0)) e^{-i2\xi(x_0 - x)/\hbar} \right] \]

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and similarly
\[ F_4 = \frac{4\gamma}{\hbar} \Re \left[ \psi(x_0) \psi(2x - x_0) e^{-i2\xi (x_0 - x)/\hbar} \right]. \]

Now, recalling Lemma 6 then
\[ \psi(x_0) \psi(2x - x_0) = \int_{\mathbb{R}} W^h(x, \xi') e^{i2(x_0 - x)\xi'/\hbar} d\xi'. \]

Therefore we can conclude that
\[ F_2 = \frac{4\gamma i}{\hbar} \Im \left\{ \int_{\mathbb{R}} W^h(x, \xi') e^{i2(x_0 - x)(\xi' - \xi)/\hbar} d\xi' \right\} \]
\[ = \frac{4\gamma i}{\hbar} \int_{\mathbb{R}} W^h(x, \xi') \sin \left[ \frac{2(x_0 - x)(\xi' - \xi)}{\hbar} \right] d\xi' \]
and
\[ F_4 = \frac{4\gamma}{\hbar} \Re \left\{ \int_{\mathbb{R}} W^h(x, \xi') e^{i2(x_0 - x)(\xi' - \xi)/\hbar} d\xi' \right\} \]
\[ = \frac{4\gamma}{\hbar} \int_{\mathbb{R}} W^h(x, \xi') \cos \left[ \frac{2(x_0 - x)(\xi' - \xi)}{\hbar} \right] d\xi'. \]

Thus the time-dependent Schrödinger
\[ i\hbar \frac{\partial \psi}{\partial t} = -\hbar^2 \frac{\partial^2 \psi}{\partial x^2} + \gamma \delta_{x_0} \psi \]
takes the form of integro-differential equation
\[ \frac{\partial W^h}{\partial t} = -2\xi \frac{\partial W^h}{\partial x} + \frac{2\gamma}{\pi \hbar^2} \int_{\mathbb{R}} W^h(x, \xi') \sin \left[ \frac{2(x_0 - x)(\xi' - \xi)}{\hbar} \right] d\xi' \]  
(39)
and the time-independent Schrödinger takes the form of integro-differential equation
\[ EW^h = -\frac{\hbar^2}{4} \frac{\partial^2 W^h}{\partial x^2} + \xi^2 W^h + \frac{\gamma}{\hbar \pi} \int_{\mathbb{R}} W^h(x, \xi') \cos \left[ \frac{2(x_0 - x)(\xi' - \xi)}{\hbar} \right] d\xi' \]  
(40)
with the constrain
\[ 0 = -2\xi \frac{\partial W^h}{\partial x} + \frac{2\gamma}{\pi \hbar^2} \int_{\mathbb{R}} W^h(x, \xi') \sin \left[ \frac{2(x_0 - x)(\xi' - \xi)}{\hbar} \right] d\xi'. \]  
(41)

**Remark 8** One can easily check that (18) satisfies (40–41).

### 3.2.2 Gross–Pitaevskii equation in the Wigner representation

The Gross–Pitaevskii equation has the form
\[ \begin{cases} 
    i\hbar \frac{\partial \psi}{\partial t} = H \psi + \nu |\psi|^2 \psi \\
    \psi(x, 0; \hbar) = \psi_0(x; \hbar)
\end{cases} \]
where \( H := -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x, t), \nu \in \mathbb{R}. \)  
(42)
In such a case the Wigner transform of Eq. (42) satisfies to the formal integro-differential equation

\[
\frac{\partial W^h}{\partial t} = -2\xi \frac{\partial W^h}{\partial x} + \sum_{m=0}^{+\infty} \frac{(-1)^m \hbar^{2m}}{2^{2m}(2m+1)!} \frac{\partial^{2m+1} W^h}{\partial x^{2m+1}} Y_-(x, t) \int_{\mathbb{R}} W^h(x, \xi', t) d\xi'
\]

By means of a straightforward (formal) calculation one obtains that (31) becomes

\[
2\pi \hbar i \frac{\partial W^h}{\partial t} = -\hbar^2 F_1 + F_2 + F_5
\]

where \(F_1\) and \(F_2\) are defined in (33) and (35). Concerning the term \(F_5\) we make use of the shortened notation (26), thus

\[
F_5 = v \int_{\mathbb{R}} e^{-iy\xi} \left[ |\psi_+|^2 \psi_+ \overline{\psi_-} - |\psi_-|^2 \psi_- \right] dy
\]

where we set

\[
Y_\pm(x, y, t; \hbar) = \left| \psi \left( x + \frac{1}{2} \hbar y, t; \hbar \right) \right|^2 \pm \left| \psi \left( x - \frac{1}{2} \hbar y, t; \hbar \right) \right|^2
\]

\[
= \int_{\mathbb{R}} W^h \left( x + \frac{1}{2} \hbar y, \xi', t \right) \pm W^h \left( x - \frac{1}{2} \hbar y, \xi', t \right) d\xi'.
\]

Then, the formal power series expansion \(W^h(x \pm \frac{1}{2} \hbar y, \xi', t) = \sum_{\ell=0}^{+\infty} \hbar^\ell \frac{\partial^\ell W^h}{\partial x^\ell}(\pm \frac{1}{2} \hbar y)\) yields to

\[
Y_-(x, y, t; \hbar) = \sum_{m=0}^{+\infty} \hbar^{2m+1} \frac{y^{2m+1}}{(2m+1)!2^{2m}} \frac{\partial^{2m+1} W^h}{\partial x^{2m+1}} \int_{\mathbb{R}} W^h(x, \xi', t) d\xi
\]

Therefore,

\[
F_5 = v \int_{\mathbb{R}} e^{-iy\xi} \sum_{m=0}^{+\infty} \hbar^{2m+1} \frac{y^{2m+1}}{(2m+1)!2^{2m}} \frac{\partial^{2m+1} W^h}{\partial x^{2m+1}} \int_{\mathbb{R}} W^h(x, \xi', t) d\xi \overline{\psi_+} \psi_- dy
\]

\[
= v \sum_{m=0}^{+\infty} \hbar^{2m+1} \frac{y^{2m+1}}{(2m+1)!2^{2m}} \left[ \int_{\mathbb{R}} W^h(x, \xi', t) d\xi \right] \int_{\mathbb{R}} y^{2m+1} e^{-iy\xi} \overline{\psi_+} \psi_- dy
\]

\[
= 2\pi \hbar i v \sum_{m=0}^{+\infty} \frac{(-1)^m \hbar^{2m}}{(2m+1)!2^{2m}} \left[ \frac{\partial^{2m+1} W^h}{\partial x^{2m+1}} \int_{\mathbb{R}} W^h(x, \xi', t) d\xi' \right] \left[ \frac{\partial^{2m+1} W^h}{\partial x^{2m+1}} \right] \int_{\mathbb{R}} W^h(x, \xi, t) d\xi
\]

from which (43) follows.

Similarly, the case of time-independent Gross–Pitaevskii equation can be treated. That is, if the potential \(V(x)\) does not depend on \(t\) and if \(\psi(x, t; \hbar) = e^{-iEt/\hbar} \varphi(x; \hbar)\) is a solution to the time-independent Gross-Pitaevskii equation \(E \varphi = H \varphi + v|\varphi|^2 \varphi\) then its Wigner
transform $W^\hbar = \mathcal{W}^\hbar(\phi)$ is a function independent of $t$ which satisfies the equation

$$EW^\hbar = -\hbar^2 \partial^2 W^\hbar \frac{4}{4} \partial x^2 + \sum_{m=0}^\infty \left(\frac{(-1)^m \hbar^{2m} \partial^{2m} W^\hbar}{2^{2m}(2m)!} \partial \xi^{2m} \partial x^{2m}\right)$$

$$\left[ V(x) + \nu \int_{\mathbb{R}} W^\hbar(x, \xi') d\xi' \right]$$

under the constrain

$$0 = -2\xi \partial W^\hbar \frac{4}{4} \partial x + \sum_{m=0}^\infty \left(\frac{(-1)^m \hbar^{2m+1} W^\hbar}{2^{2m+1}(2m+1)!} \partial \xi^{2m+1} \partial x^{2m+1}\right)$$

$$\left[ V(x) + \nu \int_{\mathbb{R}} W^\hbar(x, \xi') d\xi' \right]$$

Indeed, the constrain (45) immediately follows from (43) and from the fact that $W^\hbar$ must be independent of $t$. In order to prove (44) one must remark that (32) becomes

$$4\pi EW^\hbar = -\hbar^2 F_3 + F_4 + F_6$$

where $F_3$ and $F_4$ are defined by (34) and (36), and where

$$F_6 = \nu \int_{\mathbb{R}} e^{-i\xi y} Y_+(x, y; \hbar) \psi \overline{\psi} d\xi$$

$$= 2\pi \nu \sum_{m=0}^\infty \left(\frac{(-1)^m \hbar^{2m+1}}{(2m)!} \partial \xi^{2m+1} \partial x^{2m+1}\right)$$

$$\left[ V(x) + \nu \int_{\mathbb{R}} W^\hbar(x, \xi') d\xi' \right]$$

from which (44) follows.

4 Harmonic oscillator via the Wigner transform

Here we apply the Wigner transform to the case of real-valued potentials $V$ given by polynomials of second degree with respect to $x$; indeed, if $V$ is a polynomial with degree $r > 2$ then (21) becomes a PDE quite hard to solve.

In Sect. 4.1 we consider, at first, the eigenvalue problem for the harmonic oscillator in the Wigner representation, following the results given by §3.6.1 [39]. In Sect. 4.2 we consider then the solution to the time-dependent Schrödinger equation in the Wigner representation when the potential has the form $V(x, t) = \gamma x^2 + Q(t)x$ where $\gamma \in \mathbb{R}$ is a fixed constant and $Q(t)$ is any function depending on time. In fact, the case where $\gamma = \gamma(t)$ depends on time may be similarly treated but we don’t dwell here on such a problem.

4.1 Time-independent Schrödinger equation in the Wigner representation

If we look for a stationary solution $\psi(x, t) = e^{-iEt/\hbar}\psi(x)$, for $E \in \mathbb{R}$, then $W^\hbar$ is actually $t$-independent. In the harmonic oscillator model the potential $V(x) = \omega^2 x^2$ is a polynomial independent of $t$ of second degree with respect to $x$; from Theorem 4 then $W^\hbar \in L^2_{\mathbb{R}}(\mathbb{R} \times \mathbb{R}, dx d\xi)$ is the solution to the eigenvalue problem

$$EW^\hbar = -\frac{\hbar^2}{4} \partial^2 W^\hbar \frac{4}{4} \partial x^2 + \left[ \xi^2 + V(x) \right] W^\hbar - \frac{\hbar^2}{8} d^2 V \frac{\partial^2 W^\hbar}{\partial \xi^2}$$

(46)
under the constraint

\[ 0 = -2\xi \frac{\partial W^h}{\partial x} + \frac{\partial V}{\partial x} \frac{\partial W^h}{\partial \xi}. \]  

Equation (47) implies that the solution \( W^h \) has the form

\[ W^h(x, \xi) = f(y), \quad y = \xi^2 + V(x), \]

where \( f \) is a real-valued function. From Eq. (46) it turns out that \( f \) must satisfy to the equation (where \( f' \) and \( f'' \) respectively denote the first and second derivatives of \( f(y) \) with respect to its argument \( y \)):

\[
Ef = \left[ -\frac{\hbar^2}{4} \left( \frac{dV}{dx} \right)^2 - \frac{\hbar^2}{2} \xi^2 \frac{d^2V}{dx^2} \right] f'' - \frac{\hbar^2}{2} \frac{d^2V}{dx^2} f' + yf
\]

that is

\[
\tau \frac{d^2v}{d\tau^2} + (b - t) \frac{dv}{d\tau} - av = 0
\]

where

\[ b = 1 \text{ and } a = \frac{1}{2} - \frac{1}{2} \frac{\mathcal{E}}{\sqrt{\lambda}} \]

and

\[ f(y) = e^{-\tau/2} v(\tau), \quad \tau = 2\sqrt{\lambda} y, \quad \lambda = \omega^{-2}\hbar^{-2} \text{ and } \mathcal{E} = E\omega^{-2}\hbar^{-2}. \]

The general solution to (48) is a linear combination of the two Kummer’s functions \( M \) and \( U \) [1]:

\[ v(\tau) = C_1 M(a, b, \tau) + C_2 U(a, b, \tau). \]

Recalling also that

\[ U(a, b, \tau) \sim \frac{\Gamma(b - 1)}{\Gamma(a)} \tau^{1-b} \text{ as } \tau \to 0^+ \]

then \( C_2 = 0 \); furthermore, recalling that

\[ M(a, b, \tau) \sim \frac{\Gamma(b)}{\Gamma(a)} e^{\tau} \tau^{a-b} + e^{\pm i\pi a} \frac{\tau^{-a}}{\Gamma(b-a)} \text{ as } \tau \to +\infty \]

then \( f(y) \) belongs to \( L^2(\mathbb{R}^+) \) if \(-a = n \in \mathbb{N} \), that is

\[ -\frac{1}{2} + \frac{1}{2} \frac{\mathcal{E}}{\sqrt{\lambda}} \in \mathbb{N} \text{ that is } \mathcal{E}_n = \frac{2n + 1}{\hbar\omega}, \quad n = 0, 1, \ldots. \]

In particular, for \( \mathcal{E} = \mathcal{E}_n \) then

\[ f(y) = C_1 e^{-\tau/2} M(-n, 1, \tau) = C_1 e^{-y/\hbar\omega} n! L_n(2y/\hbar\omega) \]

where \( L_n \) is the \( n \)-th Laguerre polynomial. Hence,

\[ E = E_n = \omega^2 \hbar^2 \mathcal{E}_n = (2n + 1)\omega\hbar \]

(49)
and

\[ W^h(x, \xi) = Ce^{-\frac{\gamma^2 + \omega^2 x^2}{\hbar \omega}} L_n \left( \frac{2}{\hbar \omega} (\xi^2 + \omega^2 x^2) \right) \]

where \( C \) is a normalization constant. That is, we have proved that the time-independent Schrödinger equation has a real-valued stationary solution \( W^h \in L^2(\mathbb{R} \times \mathbb{R}, dx \, d\xi) \) when the energy \( E \) is given by (49). However, we should also check that \( W^h \) is the Wigner transform of a pure state. In fact, \( W^h \in \mathcal{D} \) because it is, up to a multiplication factor, the Wigner transform of the function \( \varphi_n \left( \frac{i}{\hbar} x \right) \) (see Example 2.4.4).

### 4.2 Time-dependent Schrödinger equation in the Wigner representation

Assume that \( V(x, t) = \gamma x^2 + Q(t) x \); then, from Theorem 3 it follows that \( W^h \in L^2(\mathbb{R} \times \mathbb{R}, dx \, d\xi) \) is the solution to the Cauchy problem

\[
\begin{align*}
\frac{\partial W^h}{\partial t} &= -2\xi \frac{\partial W^h}{\partial x} + \left[ 2\gamma x + Q(t) \right] \frac{\partial W^h}{\partial \xi} \\
W^h(x, \xi, 0) &= \left[ W^h(\psi_0) \right](x, \xi) = W_0^h(x, \xi).
\end{align*}
\tag{50}
\]

**Theorem 5** Let \( \gamma > 0 \) and let

\[
\begin{align*}
X(x, \xi, t) &= a_1(t)x + a_2(t)\xi + a_3(t) \\
\Xi(x, \xi, t) &= b_1(t)x + b_2(t)\xi + b_3(t)
\end{align*}
\]

where

\[
\begin{align*}
a_1(t) &= \cos(2\sqrt{\gamma}t), \quad a_2(t) = -\frac{1}{\sqrt{\gamma}} \sin(2\sqrt{\gamma}t), \quad a_3(t) = -\frac{1}{\sqrt{\gamma}} \int_0^t Q(t') \sin(2\sqrt{\gamma}t')dt' \\
b_1(t) &= \sqrt{\gamma} \sin(2\sqrt{\gamma}t), \quad b_2(t) = \cos(2\sqrt{\gamma}t), \quad b_3(t) = \int_0^t Q(t') \cos(2\sqrt{\gamma}t')dt'.
\end{align*}
\]

Then the solution to the Cauchy problem (50) is given by

\[
W^h(x, \xi, t) = W_0^h\left[ X(x, \xi, t), \Xi(x, \xi, t) \right].
\tag{51}
\]

**Proof** We look for solution to Eq. (50) of the form

\[
W^h(x, \xi, t) = G[f(x, \xi, t), g(x, \xi, t)]
\tag{52}
\]

where \( G(f, g) \) is a real-valued function and where \( f \) and \( g \) are linear function with respect to \( x \) and \( \xi \); i.e.

\[
f(x, \xi, t) = C_1(t)x + C_2(t)\xi + C_3(t) \quad \text{and} \quad g(x, \xi, t) = C_4(t)x + C_5(t)\xi + C_6(t)
\]

Thus, by substituting (52) in (50) it turns out that \( C_i(t), i = 1, \ldots, 6 \), are solutions to the following ODEs

\[
\begin{align*}
\dot{C}_1 &= 2\gamma C_2 \\
\dot{C}_2 &= -2C_1 \\
\dot{C}_3 &= Q(t)C_2 \\
\dot{C}_4 &= 2\gamma C_5 \\
\dot{C}_5 &= -2C_4 \\
\dot{C}_6 &= Q(t)C_5
\end{align*}
\quad \Rightarrow \quad \begin{align*}
C_1 &= c_1 \cos(2\sqrt{\gamma}t) + c_2 \sin(2\sqrt{\gamma}t) \\
C_2 &= \frac{1}{\sqrt{\gamma}} \left[ -c_1 \sin(2\sqrt{\gamma}t) + c_2 \cos(2\sqrt{\gamma}t) \right] \\
C_3 &= \int_0^t Q(t')C_2(t')dt' + c_3 \\
C_4 &= c_4 \cos(2\sqrt{\gamma}t) + c_5 \sin(2\sqrt{\gamma}t) \\
C_5 &= \frac{1}{\sqrt{\gamma}} \left[ -c_4 \sin(2\sqrt{\gamma}t) + c_5 \cos(2\sqrt{\gamma}t) \right] \\
C_6 &= \int_0^t Q(t')C_5(t')dt' + c_6
\end{align*}
\]

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where \(c_i, i = 1, 2, \ldots, 6\), are integration constants.

In order to satisfy to the initial condition it follows that the function \(G\) is such that

\[ W_0^h(x, \xi) = G(u, v) \]

where

\[
\begin{align*}
  u &= f(x, \xi, 0) = c_1 x + \frac{c_2}{\sqrt{\gamma}} \xi + c_3 \\
  v &= g(x, \xi, 0) = c_4 x + \frac{c_5}{\sqrt{\gamma}} \xi + c_6.
\end{align*}
\]

Inverting such a relation, provided that \(c_1 c_5 \neq c_2 c_4\), it follows that

\[
\begin{align*}
  x &= \frac{(u - c_3)c_5 - (v - c_6)c_2}{c_1 c_5 - c_2 c_4} \\
  \xi &= -\frac{(u - c_3)c_4 - (v - c_6)c_1}{c_1 c_5 - c_2 c_4} \sqrt{\gamma}.
\end{align*}
\]

and thus

\[ G(u, v) = W_0^h \left( \frac{(u - c_3)c_5 - (v - c_6)c_2}{c_1 c_5 - c_2 c_4}, -\frac{(u - c_3)c_4 - (v - c_6)c_1}{c_1 c_5 - c_2 c_4} \sqrt{\gamma} \right). \]

From this fact and from (52), then Theorem 5 follows. If \(c_1 c_5 = c_2 c_4\) then the same result can be obtained considering the limit \(c_1 c_5 \to c_2 c_4\). \(\square\)

**Remark 9** The case where \(\gamma < 0\) holds true by simply recalling that \(\cos(i\theta) = \cosh(\theta)\) and \(\sin(i\theta) = i \sinh(\theta)\). The case \(\gamma = 0\) holds true by simply taking the limit \(\gamma \to 0\) in (51).

**Remark 10** The Hamiltonian flux \((q(t), p(t)) = S^t(x, \xi)\) associated to the Hamiltonian \(H(p, q, t) = p^2 + V(q, t)\), where \(V(q, t) = \gamma q^2 + Q(t)q\), is the solution to the Hamiltonian system (where \(2m = 1\))

\[
\begin{align*}
  \dot{q} &= -2p = -\frac{\partial H}{\partial p} \\
  \dot{p} &= 2\gamma q + Q(t) = \frac{\partial H}{\partial q} = \frac{\partial H}{\partial q} \quad \text{and} \quad \begin{cases} q(0) = x \\
  p(0) = \xi \end{cases}
\end{align*}
\]

and it is given

\[
\begin{align*}
  q(t) &= \cos(2\sqrt{\gamma}t)x - \frac{1}{\sqrt{\gamma}} \sin(2\sqrt{\gamma}t)\xi - \frac{1}{\sqrt{\gamma}} \int_0^t Q(t') \sin(2\sqrt{\gamma}(t - t')) dt' \\
  &= X(x, \xi, t) + [a_2(t)b_3(t) - b_2(t)a_3(t) - a_3(t)] \\
  p(t) &= \sqrt{\gamma} \sin(2\sqrt{\gamma}t)x + \cos(2\sqrt{\gamma}t)\xi + \int_0^t Q(t') \cos(2\sqrt{\gamma}(t - t')) dt' \\
  &= \Xi(x, \xi, t) + [a_1(t)b_3(t) - b_1(t)a_3(t) - b_3(t)].
\end{align*}
\]

When the potential \(V\) is **independent** of \(t\), i.e. \(Q(t) = \text{constant}\), then we have that

\[
W_0^h(x, \xi, t) = \left[ W_0^h \circ S^t \right](x, \xi),
\]

since \(x(t) = X(x, \xi, t)\) and \(p(t) = \Xi(x, \xi, t)\) in such a case, and thus (54) agrees with Theorem 5. We must remark that (54) does not hold true in general when the potential \(V\) actually **depends** on \(t\), i.e. when \(Q(t)\) is not a constant function.
5 Dynamics of a Gaussian wave-function

Now, we apply Theorem 5 to the study of the solution \( \psi(x, t) \) to the Schrödinger equation

\[
\begin{aligned}
\frac{i\hbar}{\partial t} \frac{\partial \psi}{\partial x} &= -\hbar^2 \frac{\partial^2 \psi}{\partial x^2} + \left[ \gamma x^2 + Q(t) \right] \psi \\
\psi(x, 0) &= \psi_0(x)
\end{aligned}
\]  

(55)

where the initial wavefunction has a Gaussian shape:

\[
\psi_0(x) = \frac{1}{\sqrt{\pi \hbar}} e^{-(x-a)^2/2\hbar} e^{i p_0 x/\hbar}
\]  

(56)

such that \( \|\psi_0\|_{L^2(\mathbb{R}, dx)} = 1 \) and

\[
<x>^0 = \langle \psi_0, x \psi_0 \rangle = a \quad \text{and} \quad <p>^0 = -i\hbar \langle \psi_0, \nabla \psi_0 \rangle = p_0.
\]  

(57)

Its Wigner transform is (Example 2.4.3)

\[
W^h_0(x, \xi) = [W^h(\psi_0)](x, \xi) = \frac{1}{\pi \hbar} e^{-(x-a)^2/\hbar} e^{-\xi^2/2\hbar}. \tag{58}
\]

Theorem 6 Let \( \psi_0 \) given by (56); let \( a_j(t) \) and \( b_j(t) \), \( j = 1, 2, 3 \), defined in Theorem 5; let

\[
\begin{aligned}
A(t) &:= a_j^2(t) + b_j^2(t) \\
B(x, t) &:= 2a_2(t)[a_1(t)x + a_3(t) - a] + 2b_2(t)[b_1(t)x + b_3(t) - p_0] \\
v(t) &:= -b_2(t)[a_3(t) - a] + a_2(t)[b_3(t) - p_0]
\end{aligned}
\]

then

\[
|\psi(x, t)|^2 = \frac{1}{\sqrt{\pi \hbar A(t)}} e^{-\frac{(x-e(t))^2}{2\hbar A(t)}} \tag{59}
\]

and

\[
\psi(x, t) = \frac{1}{\sqrt{\pi \hbar A(t)}} e^{i \left[ \theta^* - \frac{\hbar (x_e(t))}{2\hbar A(t)} \right]} e^{-\frac{(x-e(t))^2}{2\hbar A(t)}} \tag{60}
\]

for some phase \( \theta^* \) independent of \( x \).

Proof From Theorem 5 it follows that

\[
W^h(x, \xi, t) = \frac{1}{\pi \hbar} e^{-[\gamma x^2 + Q(t)]/\hbar} e^{-\|\psi_0\|^2/\hbar}
\]

\[
= \frac{1}{\pi \hbar} e^{-[A(t)\xi^2 + B(x, t)\xi + C(x, t)]/\hbar} \tag{61}
\]

where \( A(t) \) and \( B(x, t) \) has been previously defined and where

\[
C(x, t) := [a_1(t)x + a_3(t) - a]^2 + [b_1(t)x + b_3(t) - p_0]^2
\]

A straightforward calculation gives that

\[
|\psi(x, t)|^2 = \int_{\mathbb{R}} W^h(x, \xi, t) d\xi = \frac{1}{\sqrt{\pi \hbar A(t)}} e^{-\frac{4C(x, t)A(t) - B^2(x, t)}{4A(t)\hbar}}
\]

\[
= \frac{1}{\sqrt{\pi \hbar A(t)}} e^{-\frac{d_1(t)x^2 + d_2(t)x + d_3(t)}{A(t)\hbar}}, \tag{62}
\]

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where (let us omit the dependence on the variable $t$)

\[
\begin{align*}
    d_1 & := (a_2b_1 - a_1b_2)^2 = 1 \\
    d_2 & := 2(a_2b_1 - a_1b_2)(-b_2a_3 + b_2a + b_3a_2 - a_2p_0) = -2v(t) \\
    d_3 & := (-b_2a_3 + b_2a + b_3a_2 - a_2p_0)^2 = v^2(t)
\end{align*}
\]

since $a_2b_1 - a_1b_2 = -1$. Hence, (59) follows.

In order to prove (60) we apply Lemma 5 with $x^* = 0$, then it follows that

\[
\psi(2x, t) = e^{i\mathcal{A}(t)/\hbar}e^{d_3(t)/2A(t)\hbar} \int_{\mathbb{R}} W^h(x, \xi, t) e^{i2\xi\hbar t}d\xi
\]

\[
= \frac{1}{\sqrt{\pi\hbar A(t)}} e^{i\mathcal{A}(t)/\hbar} e^{-\left(\frac{2x^2}{\hbar}\right)^2}
\]

for some phase factor $\theta^*$ independent of $x$. Hence (60) follows. □

**Remark 11** A straightforward calculation gives that

\[
<x> = \langle \psi(x, t), x\psi(x, t) \rangle = v(t).
\]

We remark that $v(t)$ coincides with the function $q(t)$ associated to the classical flux: $(q(t), p(t)) = S'(a, p_0)$ discussed in Remark 10; this fact agrees with the Ehrenfest Theorem for quadratic Hamiltonians [9].

Now, we are going to apply Theorem 6 to different cases in order to get an explicit expression of the probability density $|\psi(x, t)|^2$, and of the wavefunction $\psi(x, t)$ too in the simplest cases. The simplest cases (from the free problem to the harmonic oscillator) have been already known (see, e.g. [43]). Eventually, we consider the case of the forced harmonic (when $\gamma > 0$)/inverted (when $\gamma < 0$) oscillator where $V(x, t) = \gamma x^2 + [\lambda + b \cos(\Omega t)]x$ for some $\gamma$, $\lambda$, $b$, $\Omega \in \mathbb{R}$.

### 5.1 Free and Linear Stark potential

In such a case $V(x) = \lambda x$ for some $\lambda \in \mathbb{R}$ and thus Eq. (50) takes the form

\[
\frac{\partial W^h}{\partial t} = -2\xi \frac{\partial W^h}{\partial x} + \lambda \frac{\partial W^h}{\partial \xi}
\]

From (51) the real-valued general solution to this equation has the form (where $Q(t) = \lambda$ and where we take the limit $\gamma \to 0$)

\[
W^h(x, \xi, t) = W^h_0 \left( x - \lambda t^2 - 2\xi t, \xi + \lambda t \right) = \frac{1}{\pi \hbar} e^{-\left(x-\lambda t^2-2\xi t-\lambda t\right)^2/\hbar} e^{-\left(\xi+\lambda t-p_0\right)^2/\hbar}.
\]

Since

\[
A = 4t^2 + 1, \quad B = -4t(x - a - \lambda t^2) + 2\lambda t - 2p_0 \quad \text{and} \quad v = a + 2p_0t - \lambda t^2
\]

and then

\[
|\psi(x, t)|^2 = \frac{1}{4\pi\hbar^2} e^{-\frac{(a^2)(a^2) - 2p_0a^2}{\hbar(4t^2+1)}}
\]

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and

\[ \psi(x, t) = \frac{1}{\sqrt{\pi \hbar}} e^{\left[ \theta^* + \frac{(x^2 - 2\lambda^2 - 2a)x + p_0 + \lambda x}{\hbar(4t^2 + 1)} \right]} e^{-\frac{(x + \lambda^2 - 2p_0 t)^2}{2\hbar(4t^2 + 1)}} \]

for some \( \theta^* \) depending on \( t \). If \( \lambda = 0 \) then the solution obtained agrees with well known results (see Example 2.4.5).

### 5.2 Harmonic oscillator

In such a case \( V(x) = \omega^2 x^2 \) for some \( \omega \neq 0 \) and thus Eq. (50) takes the form

\[ \frac{\partial W^\hbar}{\partial t} = -2\xi \frac{\partial W^\hbar}{\partial x} + 2\omega^2 x \frac{\partial W^\hbar}{\partial \xi} \]

(63)

From (51) the real-valued general solution to this equation has the form

\[ W^\hbar(x, \xi, t) = \frac{1}{\pi \hbar} \exp \left[ -(A(t)\xi^2 + B(x, t)\xi + C(x, t))/\hbar \right] \]

where

\[ A(t) = \frac{1}{\omega^2} \sin^2 2\omega t + \cos^2 2\omega t \]

\[ B(x, t) = -\frac{2}{\omega} (x \cos 2\omega t - a) \sin 2\omega t + 2(\omega x \sin 2\omega t - p_0) \cos 2\omega t \]

\[ C(x, t) = (x \cos 2\omega t - a)^2 + (\omega x \sin 2\omega t - p_0)^2 \]

and

\[ v(t) = a \cos(2\omega t) + \frac{p_0}{\omega} \sin(2\omega t), \]

then

\[ |\psi(x, t)|^2 = \left[ \pi \hbar \left( \frac{1}{\omega^2} \sin^2(2\omega t) + \cos^2(2\omega t) \right) \right]^{-1/2} \exp \left[ -\frac{(x - v(t))^2}{\hbar \left( \frac{1}{\omega^2} \sin^2(2\omega t) + \cos^2(2\omega t) \right)} \right]. \]

**Remark 12** The inverted oscillator model may be similarly treated by simply recalling that \( \cos(i\theta) = \cosh(\theta) \) and \( \sin(i\theta) = i \sinh(\theta) \).

### 5.3 Forced oscillator

Let

\[ V(x, t) = \gamma x^2 + Q(t)x \text{ where } Q(t) = [\lambda + b \cos(\Omega t)] \]

for some \( \gamma, b, \lambda \in \mathbb{R} \); it is the potential of a forced harmonic oscillator (when \( \gamma > 0 \)) or of a forced inverted oscillator (when \( \gamma < 0 \)). We focus our attention here to the forced inverted oscillator, where \( \gamma < 0 \); the forced harmonic oscillator where \( \gamma > 0 \) can be similarly treated, one has only to separately consider the non-resonant case, where \( 4\gamma \neq \Omega^2 \), and the resonant case, where \( 4\gamma = \Omega^2 \).
Equation (50) takes the form
\[ \dot{W}^h = -2\xi \frac{\partial W^h}{\partial x} + \left[ 2\gamma x + \lambda + b \cos(\Omega t) \right] \frac{\partial W^h}{\partial \xi}, \] (64)
and it has solution (51). In particular, let \( \gamma = -\omega^2, \omega > 0 \), then a straightforward calculation gives that
\[ a_1(t) = b_2(t) = \cosh(2\omega t), \quad a_2(t) = -\frac{1}{\omega} \sinh(2\omega t), \quad b_1(t) = \omega \sinh(2\omega t) \]
and
\[ a_3(t) = \frac{\lambda}{2\omega^2} \left[ 1 - \cosh(2\omega t) \right] - b \frac{\Omega \sinh(2\omega t) \sin(\Omega t) + 2\omega [\cosh(2\omega t) \cos(\Omega t) - 1]}{\omega(4\omega^2 + \Omega^2)}, \]
\[ b_3(t) = \frac{\lambda}{2\omega} \sinh(2\omega t) + b \frac{2\omega \cos(\Omega t) \sin(2\omega t) + \Omega \sin(\Omega t) \cosh(2\omega t)}{(4\omega^2 + \Omega^2)}. \]
In such a case
\[ v(t) = -\frac{\lambda [\cosh(2\omega t) - 1] - 2a\omega^2 \cosh(2\omega t) - 2\omega p_0 \sinh(2\omega t)}{2\omega^2} + \frac{2b(\cos(\Omega t) - \cosh(2\omega t))}{(\Omega^2 + 4\omega^2)}. \]

5.4 Tunnel effect for the inverted oscillator

The question we consider is quite simple [4, 6, 19]: suppose that the quantum wave-function has a Gaussian shape such that \( \langle x \rangle_0 = a \) and \( \langle p \rangle_0 = p_0 \); then, assuming that \( a < 0 \) and \( p_0 > 0 \), we would compute the probability
\[ P(t) = \int_{-\infty}^{0} |\psi(x, t)|^2 dx \]
to find the quantum particle in the left-hand semi-axis \( x < 0 \) when \( t \) goes to infinity for different values of the energy. In fact, \( P(t) \) in the framework of quantum mechanics represents the probability to find the particle in the interval \((-\infty, 0)\) at the instant \( t \).

Initially we consider the case on the inverted oscillator, and then the case of the forced inverted oscillator.

5.4.1 Tunnel effect for the undriven inverted oscillator

Let us consider an initial wavefunction of the shape (56) where we assume, for argument’s sake, again \( 2m = 1 \); then
\[ E_q = \langle \psi_0, H \psi_0 \rangle = \frac{1}{2} (1 - \omega^2) \hbar + \frac{p_0^2}{2} - \omega^2 a^2 \]
is the energy in quantum mechanics.

Let \( v(t) \) and \( A(t) \) be the functions introduced in Theorem 6. The classical motion of the particle associated to the initial condition \( v(0) = a \) and \( \dot{A}(0) = \frac{B_0}{m} = 2p_0 \) has energy given by
\[ E_c = \frac{p^2}{2m} + V(x) = \frac{p_0^2}{2} - \omega^2 a^2. \]
When $p_0 = p_{\text{crit}} := |\omega a|$ then the energy $E_c = 0$ is equal to the top of the potential $V(x) = -\omega^2 x^2$. If $p_0 < p_{\text{crit}}$ then the energy $E_c < 0$ is less than the top of the potential and we classically expect that a particle, initially at $x = a < 0$ and moving forward, exhibits an inversion motion. Finally, if $p_0 > p_{\text{crit}}$ then the energy $E_c > 0$ is bigger than the top of the potential and we classically expect that a particle, initially moving forward from $x = a < 0$, passes the barrier and keeps going without reversing the motion.

Classically we thus expect that $P(t)$, which initially is close to the value 1 (for $\hbar$ small enough), is always bigger that $\frac{1}{2}$ and goes to 1 when $t$ goes to $+\infty$ if the energy is less that the barrier top; on the other hand, when the energy is bigger than the barrier top we expect that $P(t)$ takes the value $\frac{1}{2}$ at some instant $t$ and then it goes to 0 when $t$ goes to $+\infty$.

Because of the tunnel effect such a picture is quite different from a quantum mechanical point of view. Indeed, since $|\psi(x,t)|^2$ is given by (59) then a straightforward calculation gives that

$$P(t) = \frac{1}{2} \left[ 1 - \text{erf}\left( \frac{v(t)}{\sqrt{\hbar A(t)}} \right) \right]$$

(65)

In particular, it easily follows that

$$P(t) = \frac{1}{2} - \frac{1}{2} \text{erf}\left[ \frac{p_0 \sinh(2\omega t) + a\omega \cosh(2\omega t)}{\sqrt{\hbar} \sqrt{\sinh^2(2\omega t) + \omega^2 \cosh^2(2\omega t)}} \right]$$

(66)

and

$$P_{\infty} := \lim_{t \to +\infty} P(t) = \frac{1}{2} - \frac{1}{2} \text{erf}\left[ \frac{p_0 + a\omega}{\sqrt{\hbar} \sqrt{1 + \omega^2}} \right].$$

(67)

One must remarks that we recover the classical picture in the limit $\hbar \to 0^+$ (see Fig. 1).

---

**Fig. 1** Plot of the function $P(t)$ for $p_0 < p_{\text{crit}}$ (full line), $p_0 = p_{\text{crit}}$ (dot line) and $p_0 > p_{\text{crit}}$ (broken line); for argument’s sake we choose $a = -5$, $\omega = 1$ and $\hbar = 1$.
We can collect these comments as follows recalling that $\langle x \rangle^t = v(t)$:

- if $p_0 < p_{\text{crit}}$ then
  
  \[
  \lim_{t \to +\infty} \langle x \rangle^t = -\infty
  \]
  
  and $\frac{1}{2} < P_\infty < 1$;

- if $p_0 = p_{\text{crit}}$ then
  
  \[
  \lim_{t \to +\infty} \langle x \rangle^t = 0
  \]
  
  and $P_\infty = \frac{1}{2}$;

- if $p_0 > p_{\text{crit}}$ then
  
  \[
  \lim_{t \to +\infty} \langle x \rangle^t = +\infty
  \]
  
  and $0 < P_\infty < \frac{1}{2}$.

5.4.2 Tunnel effect for the driven inverted oscillator

If $V(x, t) = -\omega^2 x^2 + [\lambda + b \cos(\Omega t)] x$ then one can prove that

\[
P_\infty = \frac{1}{2} - \frac{1}{2} \text{erf} \left[ \frac{(-\lambda + 2a\omega^2 + 2p_0\omega)(\Omega^2 + 4\omega^2) - 4b\omega^2}{2\sqrt{\hbar}\omega\sqrt{1 + \omega^2}(\Omega^2 + 4\omega^2)} \right]
\]

\[
= \frac{1}{2} - \frac{1}{2} \text{erf} \left[ \frac{-\lambda}{2\sqrt{\hbar}\omega\sqrt{1 + \omega^2}} + \frac{a\omega + p_0}{\sqrt{\hbar}\sqrt{1 + \omega^2}} - \frac{2b\omega}{(\Omega^2 + 4\omega^2)\sqrt{\hbar}\sqrt{1 + \omega^2}} \right]
\]

In such a case the critical value for $p_0$ takes the form

\[
p_{\text{crit}} = \frac{\lambda(\Omega^2 + 4\omega^2) + 4b\omega^2 - 2a\omega(\Omega^2 + 4\omega^2)}{2\omega(\Omega^2 + 4\omega^2)},
\]

i.e.

\[
\begin{cases} 
  < p_{\text{crit}} & \text{then } \frac{1}{2} < P_\infty < 1 \\
  = p_{\text{crit}} & \text{then } P_\infty = \frac{1}{2} \\
  > p_{\text{crit}} & \text{then } 0 < P_\infty < \frac{1}{2}
\end{cases}
\]

A Proof of (16)

Let

\[
W^\hbar(x, \xi) = \frac{1}{2\pi R^2} \sin \left[ \frac{2\xi}{\hbar}(R - |x|) \right] \chi[-R, +R](x).
\]

Then

\[
I := \|W^\hbar(\cdot, \cdot)\|_{L^1(\mathbb{R} \times \mathbb{R}, dx d\xi)} = \frac{1}{2\pi R} \int_{\mathbb{R}} d\xi \int_{-R}^{R} dx \frac{1}{|\xi|} \left| \sin \left[ \frac{2\xi}{\hbar}(R - |x|) \right] \right|.
\]

We are going to prove that this integral diverges. To this end let us assume, for argument’s sake, that $R = 1$ and $\hbar = 2$; thus we have to compute the integral

\[
I = \frac{1}{\pi} \int_{0}^{+\infty} d\xi \int_{0}^{R} dx \frac{1}{\xi} |\sin \left[ \xi(1 - x) \right]|.
\]
For any fixed $\xi \geq \frac{3}{4}\pi$ and any positive integer $n = 0, 1, 2 \ldots$, let

$$A_{n,\xi} := \left\{ x \in [0, +1] : a_n := 1 - \frac{(n + \frac{3}{4})\pi}{\xi} \leq x \leq b_n := 1 - \frac{(n + \frac{1}{4})\pi}{\xi} \right\},$$

where $a_n \geq 0$ provided that

$$0 \leq n \leq N(\xi) := \left\lceil \frac{\xi - \frac{3}{4}\pi}{\frac{\pi}{2}} \right\rceil$$

where $\lceil y \rceil$ denote the integer part of $y$ and where $N(\xi) \geq 0$ for $\xi \geq \frac{3}{4}\pi$. Furthermore, the measure of the interval $A_{n,\xi}$ is $|A_{n,\xi}| = \frac{\pi}{2\xi}$ for any $n \leq N(\xi)$ and

$$|\sin[\xi(1-x)]| \geq \frac{1}{\sqrt{2}}, \forall x \in A_{n,\xi}.$$

Let $B_{\xi} := \bigcup_{n=0}^{N(\xi)} A_{n,\xi}$ with measure $|B_{\xi}| = [N(\xi) + 1] \frac{\pi}{2\xi} \sim \frac{1}{2}$ for large $\xi$. Hence, the integral

$$I \geq \frac{1}{\sqrt{2\pi}} \int_{\frac{3}{4}\pi}^{+\infty} d\xi \frac{1}{\xi} \int_{B_{\xi}} dx = \frac{1}{\sqrt{2\pi}} \int_{\frac{3}{4}\pi}^{+\infty} d\xi \frac{1}{\xi} |B_{\xi}|$$

diverges.

**B Proof of (19)**

Let $\alpha = -\frac{1}{2}\gamma > 0$ and let

$$\psi(x, t; \hbar) = \frac{\sqrt{\alpha}}{2\hbar^2} \text{sech} \left( \frac{\alpha}{2\hbar^2} x \right) e^{i\alpha^2 t/4\hbar^3};$$

its Wigner’s transform is

$$W^h = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iy\xi} \psi \left( x + \frac{1}{2} \hbar y, t \right) \psi \left( x - \frac{1}{2} \hbar y, t \right) dy = \frac{\alpha}{8\pi \hbar^2} \int_{\mathbb{R}} e^{-iy\xi} \text{sech} \left[ \frac{\alpha}{2\hbar^2} \left( x + \frac{1}{2} \hbar y \right) \right] \text{sech} \left[ \frac{\alpha}{2\hbar^2} \left( x - \frac{1}{2} \hbar y \right) \right] dy$$

If we set $a = \frac{\alpha}{2\hbar^2}$ and $b = \frac{\alpha}{4\hbar}$ we have to deal with the integral

$$I = \int_{\mathbb{R}} e^{-iy\xi} \text{sech}(ax + by)\text{sech}(ax - by) dy = 4 \int_{0}^{+\infty} \cos(y\xi) \frac{1}{\cos(\beta) + \cosh(\delta y)} dy$$
where $\beta = 2axi$ and $\delta = 2b$, and where we make use of the integral transform (6) §1.9 page 30 [7]. Thus

$$W^h = \frac{\alpha}{8\pi \hbar^2} \frac{\sin(ax\xi/b)}{b \sinh(2ax) \sinh(\pi \xi/2b)} = \frac{1}{\hbar} \sin \left( \frac{x \xi^2}{\hbar} \right) \sinh \left( \frac{\alpha x}{\hbar} \right) \sinh \left( \frac{\pi \xi}{2a} \right)$$

proving (19).

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