GLOBAL STRONG SOLUTION TO THE TWO DIMENSIONAL NONHOMOGENEOUS INCOMPRESSIBLE HEAT CONDUCTING NAVIER-STOKES FLOWS WITH VACUUM

YONGFU WANG
School of Economic Mathematics
Southwestern University of Finance and Economics
Chengdu 611130, China

(Communicated by Shouhong Wang)

Abstract. In this paper, we prove the unique global strong solution for the two-dimensional (2D) nonhomogeneous incompressible heat conducting Navier-Stokes flows when the initial density can contain vacuum states, as long as the initial data satisfies some compatibility condition. Furthermore, our main result improves all the previous results where the initial density is strictly positive. The main ingredient of the proof is to use some critical Sobolev inequality of logarithmic type, which were originally due to Brezis-Gallouet in [3] and Brezis-Wainger in [4], some regularity properties of Stokes system and some delicate energy estimates for nonhomogeneous incompressible heat conducting flows.

1. Introduction and main results. The motion of an incompressible viscous, heat conducting fluid is governed by the following nonhomogeneous incompressible Navier-Stokes system ([24]):

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \text{div}(2\mu \mathbb{D}(u)) + \nabla P &= 0, \\
c_v \left[ \partial_t (\rho \theta) + \text{div}(\rho u \theta) \right] - \kappa \Delta \theta &= 2\mu |\mathbb{D}(u)|^2 \\
\text{div} u &= 0,
\end{align*}
\]

where \( u = (u^1, u^2) \), \( \rho, \theta \) and \( P \) denote the velocity, density, absolute temperature and pressure, respectively. The constant viscosity coefficient \( \mu > 0 \). The positive constants \( c_v \) and \( \kappa \) as the heat capacity and the coefficient of heat conduction, respectively.

The deformation tensor \( \mathbb{D}(u) \) denotes as

\[
\mathbb{D}(u) = \frac{1}{2} \left( \nabla u + (\nabla u)^\text{tr} \right).
\]

2010 Mathematics Subject Classification. Primary: 35Q35, 35B65; Secondary: 76D03.

Key words and phrases. Nonhomogeneous Navier-Stokes equations, heat conducting flows, strong solutions, global regularity, vacuum.

The author is partially supported by NSFC grant 11801460.
In this paper, we focus on the system (1) with the initial condition
\[(ρ, u, θ)(0, x) = (ρ_0, u_0, θ_0)(x), \quad x \in Ω,\] (2)
and the boundary condition
\[u = 0, \quad \frac{∂θ}{∂n} = 0, \quad \text{on } ∂Ω,\] (3)
where \(n\) is unit outward normal to \(∂Ω\), and \(Ω\) is bounded smooth domain in \(\mathbb{R}^2\).

If there is no temperature field, i.e., \(θ ≡ 0\), (1) reduces to nonhomogeneous Navier-Stokes equations. There is huge literature about the well-posedness of solution to nonhomogeneous Navier-Stokes equations in multi-dimensional when initial density \(ρ_0\) is bounded away from 0. We will recall some results for multi-dimensional nonhomogeneous Navier-Stokes equations. Indeed, Antontsev and Kazhikov [1, 20] established the global existence of weak solutions. Later, Antontsev, Kazhikov and Monakhov [2] obtained the local existence and uniqueness of strong solutions. Furthermore, they proved that the local strong solution is global in two dimensions. When the initial density allows vacuum, the global existence of weak solution to nonhomogeneous Navier-Stokes equations was established by Simon [29], see also [24]. Choe and Kim [6] constructed a local strong solution as long as the initial data satisfied some compatibility conditions. Huang-Wang [17, 15] and Zhang [34] established the global existence of strong solutions on bounded domain of \(\mathbb{R}^3\), when the initial data satisfied some suitable small conditions. Recently, He, Li and Lü [11] obtain the global existence of strong solutions in \(\mathbb{R}^3\) to nonhomogeneous Navier-Stokes equations with density-dependent viscosity and vacuum, provided that the initial velocity is suitably small. However, the global existence of strong or smooth solutions is still an open problem in three dimensions space. Therefore, it is important and reasonable to study the mechanism of blowup and structure of possible singularities of strong solutions to the nonhomogeneous Navier-Stokes equations. If \(T^* < ∞\) is the maximal time of existence of a strong solution, Kim [21] proved the Serrin type criterion, namely,
\[\lim_{T \to T^*} ∥u∥_{L^r(0,T;L^s_w)} = ∞,\] (4)
where \(r\), \(s\) satisfy \(\frac{2}{n} + \frac{n}{r} = 1\), \(n < r ≤ ∞\), \(n\) is the dimension of the domain and \(L^r_w\) is weak \(L^r\) space.

We go back to the nonhomogeneous heat conduction system (1). The local existence of strong solutions was proved by Choe and Kim [5]. However, the global existence of strong solutions for heat conducting viscous incompressible fluids (1)-(3) is still open to two dimensions case. Therefore, one question came out naturally, wether the local strong solutions blows up in finite time. Similar arguments as [21], one can get the same criterion (4) for two dimensional nonhomogeneous incompressible heat conducting Navier-Stokes flows. In particular, for two dimensions case, it says that \(∥u∥_{L^2(0,T;L^∞)}\) is uniformly bounded, then the local solution is global. Indeed, this is the main aim of this paper.

Throughout this paper, for \(1 ≤ r ≤ ∞\) and integer \(k ≥ 0\), we denote the standard Lebesgue and Sobolev spaces as follows,
\[\int f dx = \int Ω f dx,\]
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and
\[ L^r = L^r(\Omega), \ W^{k,r} = W^{k,r}(\Omega), \ H^k = W^{k,2}(\Omega), \ D^{k,r} = \{ u \in L^1_{loc} | \nabla^k u \in L^r \}, \]
\[ H^1_0 = \{ u \in H^1 | u = 0 \text{ on } \partial \Omega \}, \ H^2_0 = \{ u \in H^2 | \nabla u \cdot n = 0 \text{ on } \partial \Omega \}. \]

Set the material derivative of \( f \) as follows.
\[ \dot{f} := f_t + u \cdot \nabla f. \]

Then, the strong solutions to the initial boundary value problem (1)-(3), are defined as follows.

**Definition 1.1.** (Strong solutions) \((\rho, u, \theta)\) is called a strong solution to (1)-(3) in \( \Omega \times (0,T) \), if for some \( r_0 > 2 \),
\[
\begin{align*}
\rho &\geq 0, \ \rho \in C([0,T]; W^{1,r_0}), \ \rho_t \in C([0,T]; L^{r_0}), \\
u &\in C([0,T]; H^1_0 \cap H^2) \cap L^2 (0,T; D^{2,r_0}), \\
\theta &\geq 0, \ \theta \in C([0,T]; H^2_0) \cap L^2 (0,T; D^{2,r_0}), \\
(\sqrt{\rho u}, \sqrt{\rho \theta}) &\in L^\infty (0,T; L^2), \ (u_0, \theta_0) \in L^2 (0,T; D^{1,2}),
\end{align*}
\]
and \((\rho, u, \theta)\) satisfies (1) a.e. in \( \Omega \times (0,T) \).

Our main result of this paper can be stated as follows.

**Theorem 1.2.** For constant \( r > 2 \), assume that the initial data \((\rho_0, u_0, \theta_0)\) satisfy
\[
\rho_0 \geq 0, \ \theta_0 \geq 0, \ \rho_0 \in W^{1,r}, \ u_0 \in H^1_0 \cap H^2, \ \theta_0 \in H^2_0, \ \text{div} u_0 = 0, \ (6)
\]
and the compatibility conditions
\[
- \text{div}(2\mu D(u_0)) + \nabla P_0 = \sqrt{\rho_0} g_1, \quad (7)
\]
and
\[
- \kappa \Delta \theta_0 - 2\mu |D(u_0)|^2 = \sqrt{\rho_0} g_2, \quad (8)
\]
for some \( P_0 \in D^{1,2} \) and \( g_1, g_2 \in L^2 \). Then there exists a global strong solution \((\rho, u, \theta)\) of the system (1)-(3).

**Remark 1.** Along the same arguments as [5], we can obtain the local existence of strong solutions with vacuum to the system (1)-(3) in a two dimensional bounded domain. Therefore, the maximal time \( T^* \) is well-defined. We will verify Theorem 1.2 by the contradiction arguments, and obtain the local strong solution does not blow up in finite time. Throughout this paper, we will concentrate on establishing global estimate for the density, velocity and temperature field. Furthermore, the approach can also be adapted to deal with the periodic case. In particular, it would be interesting to study the Cauchy problem in \( \mathbb{R}^2 \).

**Remark 2.** Indeed, for the two dimensions case, the key ingredient of the analysis here is a critical Sobolev inequality of logarithmic type, which was stated in Lemma 4 in the next section. The general version in multi-dimension \( n \geq 2 \) is as follows,
\[
\| f \|^2_{L^2(0,T; L^q)} \leq C \left( 1 + \| f \|^2_{L^2(0,T; W^{1,q})} \ln(e + \| f \|^2_{L^2(0,T; W^{1,q})}) \right),
\]
for \( q > n \). The energy inequality tells us \( \| u \|_{L^2(0,T; H^1)} \) is uniformly bounded. This together with logarithmic Sobolev inequality gives the uniform estimates of \( \| u \|_{L^2(0,T; L^\infty)} \). However, in higher-dimensional case \( n \geq 3 \), it seems to be difficult to obtain the estimates \( \| u \|_{L^2(0,T; W^{1,q})} \), which is the main obstacle to extend the results in multi-dimension \( n \geq 3 \).
Remark 3. We would like to mention the works on the well-posedness theory and blowup criteria for multi-dimensional incompressible magnetohydrodynamic flows (see [16, 23, 25, 34]) and references therein. Furthermore, we also would like to refer the blowup criteria for compressible heat conducting flows in [7, 8, 9, 12, 13, 14, 18, 19, 30, 31, 32, 33] and references therein.

The rest of this paper is organized as follows. In section 2, we will recall some known facts and elementary inequalities that will be used later. We will prove the Theorem 1.2 in Section 3.

2. Preliminaries. In this section, we will recall some elementary lemmas and inequalities that will be used later. We start with the local existence of strong solutions, which can be established in the same manner as [5].

Proposition 1. (Local existence of strong solutions) Assume that the initial data \((\rho_0, u_0, \theta_0)\) satisfy (6)-(8). Then there exist a positive constant \(T_0\) and unique strong solution \((\rho, u, \theta)\) to the initial boundary value problem (1)-(3) on \(\Omega \times (0, T_0)\).

Next, we will introduce the so-called Gagliardo-Nirenberg inequality, its proof can be founded in [26, 27].

Lemma 2.1. Let \(\Omega\) be a bounded domain in \(\mathbb{R}^2\) with smooth boundary \(\partial \Omega\) of class \(C^m\) and let \(u \in W^{m,p}(\Omega) \cap L^q(\Omega)\) where \(1 \leq r, q \leq \infty\). For any integer \(j, 0 \leq j < m\) and any \(\frac{m}{m-j-2} \leq \varrho \leq 1\) we have
\[
\|D^j u\|_{L^p} \leq \|u\|_{W^{m,r}}^{\varrho} \|u\|_{L^q}^{1-\varrho},
\]
provided that
\[
\frac{1}{p} = \frac{j}{2} + \varrho \left(\frac{1}{r} - \frac{m}{2}\right) + (1 - \varrho)\frac{1}{q},
\]
and \(m - j - \frac{2}{r}\) is not a nonnegative integer. If \(m - j - \frac{2}{r}\) is a nonnegative integer (9) holds with \(\varrho = \frac{2}{m}\).

Remark 4. Indeed, the inequality (9) was originated from the reference [26] by L. Nirenberg in 1959. In particular, the following inequality will be used in the next section.
\[
\|u\|_{L^4} \leq C\|u\|^\frac{3}{4}_{H^1,1} \|u\|_{L^2}^{\frac{1}{4}}.
\]

Some regularity results for the Stokes equations were sated as follows. For its proof, refer to [10].

Lemma 2.2. Let \(\Omega\) be a bounded domain of \(\mathbb{R}^2\) of class \(C^{1,1}\). For any \(r \in (1, \infty)\), if \(F \in L^r\), there exists some positive constant \(C\) depending only on \(r\) such that the unique weak solution \((u, P) \in H^1_0 \times H^1\) to the following Stokes system
\[
\begin{aligned}
-\text{div}(2\mu \mathbf{D}(u)) + \nabla P &= F, &\text{in }\Omega, \\
div u &= 0, &\text{in }\Omega, \\
u &= 0, &\text{on }\partial\Omega,
\end{aligned}
\]
satisfy
\[
\|u\|_{W^{2,r}} \leq C\|F\|_{L^r} + C\|u\|_{H^1}.
\]
Finally, in order to obtain the bound of \( \|u\|_{L^2(0,T;L^\infty)} \), we introduce a critical Sobolev inequality of logarithmic type, which were originally due to Brezis-Gallouet in [3] and Brezis-Wainger in [4], and extended by Ozawa in [28]. For its proof, refer to [16, 22, 30].

**Lemma 2.3.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \), and \( f \in L^2(0,T;H^1 \cap W^{1,p}) \) with \( p > 2 \). Then there exists a constant \( C \) depending only on \( p \) such that

\[
\|f\|_{L^2(0,T;L^\infty)}^2 \leq C \left( 1 + \|f\|_{L^2(0,T;H^1)}^2 \ln(1 + \|f\|_{L^2(0,T;W^{1,p})}^2) \right).
\]

(12)

3. **Proof of the main results.** In view of Proposition 1, there exists \( T^* > 0 \) such that \((\rho, u, \theta)\) be the strong solution to (1)-(3) on \( \Omega \times (0,T^*) \) with initial data \((\rho_0, u_0, \theta_0)\) satisfying (6)-(8). We will establish some necessary a priori bounds for strong solutions \((\rho, u, \theta)\) and extend the local strong solutions beyond \( T^* \).

First, due to \( \text{div} u = 0 \), we establish the following upper uniform estimates of the density.

**Lemma 3.1.** There exists a positive constant \( C \) such that

\[
\rho \geq 0,
\]

and

\[
\sup_{0 \leq T < T^*} \|\rho\|_{L^1 \cap L^\infty} \leq C,
\]

(14)

Here and after, \( C_0 \) and \( C \) denote generic positive constants depending only on \( \mu, c_\nu, T^* \) and the initial data.

**Proof.** Indeed, since \( \text{div} u = 0 \), we obtain \( \|\rho\|_{L^1 \cap L^\infty} = \|\rho_0\|_{L^1 \cap L^\infty} \).

The particle path can be defined as follows:

\[
\begin{aligned}
\frac{d}{ds} y(s; x, t) &= u(y, s) \\
y(t; x, t) &= x.
\end{aligned}
\]

(15)

It follows from the continuity equation that the density can be expressed by the formula

\[
\rho(x, t) = \rho_0 (y(0; x, t)),
\]

which together with \( \rho_0 \geq 0 \) implies (13).

Hence, we complete the proof of Lemma 3.1. \( \square \)

Next, we will give the standard energy estimates as follows.

**Lemma 3.2.** Under the assumptions in Theorem 1.2, one has

\[
\sup_{0 \leq t \leq T} \left( \|\sqrt{\rho} u\|_{L^2}^2 + \|\rho \theta\|_{L^1} \right) + \int_0^T \|\nabla u\|_{L^2}^2 dt \leq C,
\]

(16)

for any \( 0 \leq T < T^* \).

**Proof.** Using the standard maximum principle in [8] to (1) to together with \( \theta_0 \geq 0 \), we obtain

\[
\inf_{\Omega \times [0,T]} \theta(x, t) \geq 0.
\]
Integrating (1) with the spatial variable \(x\) and integrating by parts gives

\[
c_v \frac{d}{dt} \int \rho \theta dx = 2\mu \int |\nabla(u)|^2 dx
\]
\[
= \frac{\mu}{2} \int (\partial_i u^j + \partial_j u^i)^2 dx
\]
\[
= \mu \int |\nabla u|^2 dx + 2\mu \int \partial_i u^j \partial_j u^i dx
\]
\[
= \mu \int |\nabla u|^2 dx. \tag{17}
\]

Multiplying (1) by \(u\) and integrating the resulting equation over \(\Omega\) yield to

\[
\frac{1}{2} \frac{d}{dt} \int \rho |u|^2 dx + \mu \int |\nabla u|^2 dx = 0. \tag{18}
\]

Thus, adding (18) multiplied by 2 to (17), we obtain

\[
\frac{d}{dt} \int (\rho |u|^2 + c_v \rho \theta) dx + \mu \int |\nabla u|^2 dx = 0,
\]

this together with Gronwall’s inequality gives (16).

Thus, we complete the proof of Lemma 3.2.

The key estimates on \(\nabla u\) will be given in the following lemma. To prove that, we will make use of critical Sobolev inequality of logarithmic type, which was stated in Lemma 2.3.

**Lemma 3.3.** Under the assumptions in Theorem 1.2, it holds that for \(0 \leq T < T^*\),

\[
\sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 + \int_0^T \|\sqrt{\rho u} t\|_{L^2}^2 dt \leq C. \tag{19}
\]

**Proof.** Multiplying the momentum equation (1) by \(u_t\), integrating the resulting equation over \(\Omega\), integrating by parts and using Young’s inequality yield to

\[
\frac{\mu}{2} \frac{d}{dt} \int |\nabla u|^2 dx + \int \rho |u_t|^2 dx = -\int \rho u \cdot \nabla u \cdot u_t dx
\]
\[
\leq \frac{1}{2} \int \rho |u_t|^2 dx + C \int \rho |u|^2 |\nabla u|^2 dx
\]
\[
\leq \frac{1}{2} \int \rho |u_t|^2 dx + C \|u\|_{L^\infty}^2 \int |\nabla u|^2 dx,
\]

which implies that

\[
\mu \frac{d}{dt} \int |\nabla u|^2 dx + \int \rho |u_t|^2 dx \leq C \|u\|_{L^\infty}^2 \int |\nabla u|^2 dx. \tag{20}
\]

Let

\[
\Phi(t) = e + \sup_{\tau \in [0,t]} \|\nabla u(\tau)\|_{L^2}^2 + \int_0^t \|\sqrt{\rho u_t}\|_{L^2}^2 d\tau. \tag{21}
\]

Then, it follows from (20) and Gronwall’s inequality that, for every \(0 \leq s \leq T < T^*\), one has

\[
\Phi(T) \leq C \Phi(s) \exp \left\{ C \int_s^T (\|u(\tau)\|_{L^\infty}^2) d\tau \right\}. \tag{22}
\]

Now, we will estimate the term \(\|u\|_{L^2(s,T;L^\infty)}\).
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Using Lemma 2.3, we obtain that
\[
\|u\|_{L^2(s,T;W^{1,3})}^2 \leq C\left\{1 + \|\nabla u\|_{L^2(s,T;H^1)}^2 \ln \left(e + \|\nabla u\|_{L^2(s,T;W^{1,3})}^2\right)\right\}
\]
\[
\leq C_0 \left\{1 + \|\nabla u\|_{L^2(s,T;L^2)}^2 \ln (C\Phi(T))\right\},
\]
where we have used Poincaré inequality and the following facts

\[
\|u\|_{W^{1,3}} \leq C\|\nabla u\|_{L^3} + C\|u\|_{L^3}
\]
\[
\leq C\|u\|_{W^{2,\frac{6}{5}}} + C\|u\|_{H^1}
\]
\[
\leq C\|\rho\|_{L^\infty} + C\|\nabla u\|_{L^2}
\]
\[
\leq C\|\rho u\|_{L^\frac{6}{5}} + C\|\rho u \cdot \nabla u\|_{L^\frac{6}{5}} + C\|\nabla u\|_{L^2}
\]
due to (9) and (11). This together with Poincaré inequality and Hölder inequality gives

\[
\|u\|_{L^2(s,T;W^{1,3})} \leq C\|\nabla u\|_{L^2(s,T;L^2)} + C\|\sqrt{\rho} \|_{L^2(s,T;L^2)}^2 \|\nabla u\|_{L^2(s,T;L^1)} \|\nabla u\|_{L^\infty(s,T;L^2)}
\]\n\[
\leq C\|\nabla u\|_{L^2(s,T;L^2)} + C\|\sqrt{\rho} \|_{L^2(s,T;L^2)}^2 + C\|\nabla u\|_{L^2(s,T;H^1)} \|\nabla u\|_{L^\infty(s,T;L^2)}
\]
\[
\leq C\|\nabla u\|_{L^2(s,T;L^2)}^2 + C\|\sqrt{\rho} \|_{L^2(s,T;L^2)}^2 + C\|\nabla u\|_{L^2(s,T;L^2)} \|\nabla u\|_{L^\infty(s,T;L^2)}^2.
\]

The combination of (22) with (23) gives

\[
\Phi(T) \leq C\Phi(s) (C\Phi(T))^{\frac{\|\nabla u\|_{L^2(s,T;L^2)}^2}{C_0}}.
\]

In view of the energy estimate (16), we can choose s close enough to \(T^*\), such that

\[
\lim_{T \to T^*} \frac{C_0 \|\nabla u\|_{L^2(s,T;L^2)}^2}{2} \leq \frac{1}{2},
\]
then for \(s < T < T^*\) yield to

\[
\Phi(T) \leq C\Phi^2(s) < \infty.
\]

This finishes the proof of Lemma 3.2. \(\Box\)

**Remark 5.** Unfortunately, the critical Sobolev inequality of logarithmic type (12) cannot be adapted to deal with the 2D Cauchy problem, and it seems difficult to estimate \(\|u\|_{L^2(\mathbb{R}^2)}\).

We improve the regularity of the temperature \(\theta\) as follows.

**Lemma 3.4.** Under the assumptions in Theorem 1.2, it holds that for \(0 \leq T < T^*\),

\[
\sup_{0 \leq t \leq T} \|\sqrt{\rho} \theta\|_{L^2}^2 + \int_0^T \|\nabla \theta\|_{L^2}^2 dt \leq C.
\]
Proof. Multiplying (1) by $\theta$ and integrating the resulting equation over $\Omega$, one has
\[ c_v \frac{d}{dt} \int \rho \theta^2 dx + 2\kappa \int |\nabla \theta|^2 dx \leq C \int |\nabla u|^2 \theta dx \]
\[ \leq C \|\nabla u\|_{L^2}^2 \|\theta\|_{L^2} \]
\[ \leq C \|\nabla u\|_{H^1} \|\nabla u\|_{L^2} \|\theta\|_{L^2} \]
\[ \leq C \left( \|\nabla u\|_{L^2} + \|\nabla^2 u\|_{L^2} \right) \|\nabla u\|_{L^2} \|\theta\|_{L^2} \]
\[ \leq C \left( 1 + \|\nabla^2 u\|_{L^2} \right) \|\theta\|_{L^2} \]
\[ \leq C \|\nabla^2 u\|_{L^2}^2 + C \varepsilon \|\theta\|_{L^2}^2 + C, \tag{25} \]
where we have used (19) and Hölder inequality.

Next, we will estimate $\|\nabla^2 u\|_{L^2}^2$ and $\|\theta\|_{L^2}$, respectively. It follows from (9), (19), Poincaré inequality and Young’s inequality that
\[ \|\nabla^2 u\|_{L^2}^2 \leq \int \rho |u_t|^2 dx + C \|\nabla^2 u\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \]
\[ \leq C \int \rho |u_t|^2 dx + C \|\nabla u\|_{H^1} \|\nabla u\|_{L^2} \|\nabla u\|_{H^1} \|\nabla u\|_{L^2} + C \]
\[ \leq C \int \rho |u_t|^2 dx + C \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} \]
\[ \leq C \int \rho |u_t|^2 dx + \frac{1}{2} \|\nabla^2 u\|_{L^2}^2 + C, \tag{26} \]
which implies
\[ \|\nabla^2 u\|_{L^2}^2 \leq \int \rho |u_t|^2 dx + C. \]
Furthermore, it follows from the Poincaré inequality that
\[ \|\theta - \bar{\theta}\|_{L^2} \leq C \|\nabla \theta\|_{L^2}, \]
where $\bar{\theta}$ denotes the average of $\theta$, namely, $\bar{\theta} = \frac{1}{|\Omega|} \int \theta dx$. Therefore, one has
\[ \|\theta\|_{L^2} \leq C \|\nabla \theta\|_{L^2} + C \|\theta\|_{L^2}. \tag{27} \]
On the one hand, we obtain
\[ |\bar{\theta}| \int \rho dx \leq \int \rho \theta dx + \int \rho (\theta - \bar{\theta}) dx \]
\[ \leq C + \|\nabla \theta\|_{L^2}, \]
due to Poincaré inequality and (16).

This together with (27) gives
\[ \|\theta\|_{L^2} \leq C \|\nabla \theta\|_{L^2} + C. \tag{28} \]

Together with (25), (26) and (28), choosing $\varepsilon$ suitably small, one has
\[ c_v \frac{d}{dt} \int \rho \theta^2 dx + \kappa \int |\nabla \theta|^2 dx \leq C \int \rho |u_t|^2 dx + C, \]
this together with (19) implies that (24).

Thus, we complete the proof of Lemma 3.4.
The boundedness of $\|\sqrt{\rho u_t}\|_{L^2}$ and $\|\nabla \theta\|_{L^2}$ will be proved as follows. The proofs are similar as Appendix A in [13] to the compressible case. To convenience to the readers, we sketch the detail proofs as follows.

**Lemma 3.5.** Under the assumptions in Theorem 1.2, it holds that for $0 \leq T < T^*$,

$$\sup_{0 \leq t \leq T} (\|\sqrt{\rho u_t}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) + \int_0^T (\|\nabla u_t\|_{L^2}^2 + \|\sqrt{\rho \theta_t}\|_{L^2}^2) dt \leq C. \quad (29)$$

**Proof.** Firstly, applying the operator $\partial_t$ to (1) yields to

$$\rho u_{tt} + \rho u \cdot \nabla u_t - \mu \Delta u_t = -\nabla P_t - \rho u_t - (\rho u)_t \cdot \nabla u,$$

which multiplied the above equality by $u_t$, using integrating by parts over $\Omega$ and (1)\textsubscript{1}, leads to

$$\frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 dx + \mu \int |\nabla u_t|^2 dx$$

$$= - \int \rho_t |u_t|^2 dx - \int (\rho u_t)^t \cdot \nabla u \cdot u_t dx$$

$$= \int \text{div}(\rho u_t) u_t^2 dx + \int \text{div}(\rho u) u \cdot \nabla u \cdot u_t dx - \int \rho u_t \cdot \nabla u \cdot u_t dx$$

$$= N_1 + N_2 + N_3. \quad (30)$$

We estimate each term on the right hand side of (30) as follows.

It follows from (9), (19), (24), (26), integrating by parts, Poincaré inequality and Hölder inequality that

$$N_1 \leq C \int \rho |u| |u_t| |\nabla u_t| dx$$

$$\leq C \|u\|_{L^6} \|\sqrt{\rho u_t}\|_{L^3} \|\nabla u_t\|_{L^2}$$

$$\leq C \|u\|_{H^1} \|\sqrt{\rho u_t}\|_{L^2} \|u_t\|_{L^6} \|\nabla u_t\|_{L^2}$$

$$\leq C \|\nabla u\|_{L^2} \|\sqrt{\rho u_t}\|_{L^2} \|\nabla u_t\|_{L^2}$$

$$\leq \frac{\mu}{6} \|\nabla u_t\|_{L^2}^2 + C \|\sqrt{\rho u_t}\|_{L^2}^2. \quad (31)$$

$$N_2 \leq \int (\rho |u| |\nabla u|^2 |u_t| + \rho |u|^2 |\nabla^2 u| |u_t| + \rho |u|^2 |\nabla u| |\nabla u_t|) dx$$

$$\leq C \|u\|_{L^6} \|\nabla u\|_{L^6}^2 \|\sqrt{\rho u_t}\|_{L^2}$$

$$+ \frac{1}{2} \int \|\nabla u_t\|_{L^2}^2 + \|\nabla u_{tt}\|_{L^2} + \|\nabla u\|_{L^6} \|\nabla u_t\|_{L^2}$$

$$\leq C \|u\|_{H^1} \|\nabla u\|_{L^6} \|\sqrt{\rho u_t}\|_{L^2}$$

$$+ C \|u\|_{H^1} \|\nabla^2 u\|_{L^2} \|u_t\|_{H^1} + C \|u\|_{H^1} \|\nabla u\|_{H^1} \|\nabla u_t\|_{L^2}$$

$$\leq C \left( \|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2} \right) \|\sqrt{\rho u_t}\|_{L^2} + C \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2}$$

$$+ C \left( \|\nabla u\|_{L^2} + \|\nabla^2 u\|_{L^2} \right) \|\nabla u_t\|_{L^2}$$

$$\leq \frac{\mu}{6} \|\nabla u_t\|_{L^2}^2 + C \|\sqrt{\rho u_t}\|_{L^2}^2 + C \|\sqrt{\rho u_t}\|_{L^2}^2 + C$$

$$\leq \frac{\mu}{6} \|\nabla u_t\|_{L^2}^2 + C \|\sqrt{\rho u_t}\|_{L^2}^2 + C,$$
and

\[ N_3 \leq \int \rho |u_t|^2 |\nabla u| \, dx \]
\[ \leq \| \sqrt{\rho} u_t \|_{L^4}^2 \| \nabla u \|_{L^2} \]
\[ \leq C \| \sqrt{\rho} u_t \|_{L^2}^\frac{1}{2} \| u_t \|_{L^6}^\frac{3}{2} \]
\[ \leq C \| \sqrt{\rho} u_t \|_{L^2}^\frac{1}{2} \| u_t \|_{H^1}^\frac{3}{2} \]
\[ \leq C \| \sqrt{\rho} u_t \|_{L^2}^\frac{1}{2} \| \nabla u_t \|_{L^2}^\frac{3}{2} \]
\[ \leq \frac{\mu}{6} \| \nabla u_t \|_{L^2}^2 + C \| \sqrt{\rho} u_t \|_{L^2}^2. \]  

(33)

Hence, the combination of (30)-(33) gives

\[ \frac{d}{dt} \int \rho |u_t|^2 \, dx + \mu \int |\nabla u_t|^2 \, dx \leq C \| \sqrt{\rho} u_t \|_{L^2}^4 + C, \]

which together with Gronwall’s inequality and (19) yield to

\[ \sup_{0 \leq t \leq T} \| \sqrt{\rho} u_t \|_{L^2}^2 + \int_0^T \| \nabla u_t \|_{L^2}^2 \, dt \leq C. \]  

(34)

Next, we will estimate the boundedness of \( \| \nabla \theta \|_{L^2}^2 \).

Indeed, multiplying (1)_3 by \( \theta_t \) and integrating the resulting equation over \( \Omega \) leads to

\[ \frac{\kappa}{2} \frac{d}{dt} \int |\nabla \theta|^2 \, dx + c_v \int \rho \theta_t^2 \, dx = -c_v \int \rho u \cdot \nabla \theta \, dx + 2\mu \int |\mathcal{D}(u)|^2 \theta \, dx \]
\[ = J_1 + J_2. \]  

(35)

It follows from Hölder inequality, Sobolev embedding inequality, (9), (19), (26), (28) and (34) that

\[ J_1 \leq C \| u \|_{L^\infty} \| \sqrt{\rho} u_t \|_{L^2} \| \nabla \theta \|_{L^2} \]
\[ \leq C \| u \|_{W^{1,\infty}} \| \sqrt{\rho} \theta_t \|_{L^2} \| \nabla \theta \|_{L^2} \]
\[ \leq C \left( \| u \|_{L^3} + \| \nabla u \|_{L^3} \right) \| \sqrt{\rho} \theta_t \|_{L^2} \| \nabla \theta \|_{L^2} \]
\[ \leq C \left( \| u \|_{H^1} + \| \nabla u \|_{H^1} \right) \| \sqrt{\rho} \theta_t \|_{L^2} \| \nabla \theta \|_{L^2} \]
\[ \leq C \left( 1 + \| \nabla^2 u \|_{L^2} \right) \| \sqrt{\rho} \theta_t \|_{L^2} \| \nabla \theta \|_{L^2} \]
\[ \leq C \left( 1 + \| \sqrt{\rho} u_t \|_{L^2} \right) \| \sqrt{\rho} \theta_t \|_{L^2} \| \nabla \theta \|_{L^2} \]
\[ \leq \frac{C_v}{2} \| \sqrt{\rho} \theta_t \|_{L^2}^2 + C \| \nabla \theta \|_{L^2}^2. \]  

(36)
Furthermore, due to (9), (19), (28) and (34), one has

\[ J_2 = 2\mu \frac{d}{dt} \int |\mathcal{D}(u)|^2 \theta dx - 2\mu \int (|\mathcal{D}(u)|^2)_t \theta dx \]

\[ \leq 2\mu \frac{d}{dt} \int |\mathcal{D}(u)|^2 \theta dx + C \int |\nabla u| |\nabla u_t| \theta dx \]

\[ \leq 2\mu \frac{d}{dt} \int |\mathcal{D}(u)|^2 \theta dx + C \|\nabla u\|_{L^4} \|\nabla u_t\|_{L^2} \|\theta\|_{L^4} \]

\[ \leq 2\mu \frac{d}{dt} \int |\mathcal{D}(u)|^2 \theta dx + C \|\nabla u\|_{H^1} \|\nabla u_t\|_{L^2} \|\theta\|_{H^1} \]  \quad (37)

\[ \leq 2\mu \frac{d}{dt} \int |\mathcal{D}(u)|^2 \theta dx + C \left( \|\nabla u\|_{L^2} + \|\nabla^2 u\|_{L^2} \right) \|\nabla u_t\|_{L^2} \left( \|\nabla \theta\|_{L^2} + C \right) \]

\[ \leq 2\mu \frac{d}{dt} \int |\mathcal{D}(u)|^2 \theta dx + C \left( 1 + \|\nabla^2 u\|_{L^2} \right) \|\nabla u_t\|_{L^2} \left( \|\nabla \theta\|_{L^2} + C \right) \]

\[ \leq 2\mu \frac{d}{dt} \int |\mathcal{D}(u)|^2 \theta dx + C \|\nabla u_t\|_{L^2}^2 + C \|\nabla \theta\|_{L^2}^2 + C. \]  \quad (38)

Substituting (36) and (37) into (35), one has

\[ \frac{d}{dt} \left( \kappa |\nabla \theta|^2 - 4\mu |\mathcal{D}(u)|^2 \theta \right) dx + c_v \int \rho \theta_t^2 dx \leq C \|\nabla u_t\|_{L^2}^2 + C \|\nabla \theta\|_{L^2}^2 + C. \]  \quad (39)

Noticing that

\[ 4\mu \int |\mathcal{D}(u)|^2 \theta dx \leq C \|\nabla u\|_{H^1}^2 \|\nabla \theta\|_{L^2} \]

\[ \leq C \left( \|\nabla u\|_{L^2} \right) \|\nabla u_t\|_{L^2} \left( \|\nabla \theta\|_{L^2} + C \right) \]

\[ \leq C \left( 1 + \|\nabla u\|_{L^2} \right) \left( \|\nabla \theta\|_{L^2} + C \right) \]

\[ \leq \frac{\kappa}{2} \|\nabla \theta\|_{L^2}^2 + C, \]

due to (9), (19), (26), (28) and (34).

This together with Gronwall’s inequality, (38) indicates

\[ \sup_{0 \leq t \leq T} \|\nabla \theta\|_{L^2}^2 + \int_0^T \|\sqrt{\rho} u_t\|_{L^2}^2 dt \leq C. \]  \quad (39)

Thus, the desired (29) follows from (34) and (39). This finishes the proof of Lemma 3.5.

Next, the following lemma will deal with the higher order estimates of temperature field \( \theta \).

**Lemma 3.6.** Under the assumptions in Theorem 1.2, it holds that for \( 0 \leq T < T^* \),

\[ \sup_{0 \leq t \leq T} \|\sqrt{\rho} u_t\|_{L^2} + \int_0^T \|\nabla \theta_t\|_{L^2}^2 dt \leq C. \]  \quad (40)

**Proof.** Indeed, applying the operator \( \partial_t \) to (1)_3 and a series of direct computations yield to

\[ c_v \left( \rho \theta_t + \rho u \cdot \nabla \theta \right) - \kappa \Delta \theta_t \]

\[ = c_v \left( \text{div}(\rho u) \theta_t + \text{div}(\rho u) u \cdot \nabla \theta - \rho u_t \cdot \nabla \theta \right) + 2\mu \left( |\mathcal{D}(u)|^2 \right)_t. \]  \quad (41)
Multiplying (41) by \( \theta_i \) integrating over \( \Omega \) yield

\[
\frac{c_v}{2} \frac{d}{dt} \int \rho \theta_i^2 \, dx + \kappa \int \nabla \theta_i^2 \, dx = c_v \int \text{div}(pu_i) \theta_i^2 \, dx - c_v \int pu_i \cdot \nabla \theta_i \, dx \\
+ c_v \int \text{div}(pu_i) u \cdot \nabla \theta_i \, dx + 2\mu \int (|\mathfrak{S}(u)|^2) \theta_i \, dx = \sum_{i=1}^{4} N_i.
\]

By virtue of view of Hölder inequality, Poincaré inequality, (9), (19), (26) and (29), one has

\[
N_1 \leq C \int \rho |u|u_i|\nabla \theta_i| \, dx \\
\leq C \|u\|_{L^\infty} \|\sqrt{\rho} \theta_i\|_{L^2} \|\nabla \theta_i\|_{L^2} \\
\leq C \|u\|_{H^2} \|\sqrt{\rho} \theta_i\|_{L^2} \|\nabla \theta_i\|_{L^2} \\
\leq C (\|\nabla u\|_{L^2} + \|\nabla^2 u\|_{L^2}) \|\sqrt{\rho} \theta_i\|_{L^2} \|\nabla \theta_i\|_{L^2} \\
\leq C (1 + \|\sqrt{\rho} u_i\|_{L^2}) \|\sqrt{\rho} \theta_i\|_{L^2} \|\nabla \theta_i\|_{L^2} \\
\leq C \|\sqrt{\rho} \theta_i\|^2_{L^2} + \frac{\kappa}{8} \|\nabla \theta_i\|^2_{L^2},
\]

\[
N_2 \leq C \|\sqrt{\rho} u_i\|_{L^2} \|\nabla \theta_i\|_{L^*} \|\theta_i\|_{L^4} \\
\leq C \|\nabla \theta_i\|^2_{L^2} \|\nabla \theta_i\|^2_{H^1} \|\theta_i\|^2_{L^2} \|\theta_i\|^2_{H^2} \\
\leq C \|\nabla \theta_i\|^2_{L^2} \left(\|\nabla^2 \theta_i\|^2_{L^2} + \|\nabla \theta_i\|^2_{L^2}\right) \left(\|\nabla \theta_i\|^2_{L^2} + C\right)^2 \\
\leq C \|\nabla \theta_i\|^2_{L^2} + \frac{\kappa}{8} \|\nabla \theta_i\|^2_{L^2} + C,
\]

\[
N_3 \leq C \int \left(\rho |u| |\nabla u| |\nabla \theta| \theta_i + \rho |u|^2 |\nabla^2 \theta| \theta_i + \rho |u|^2 |\nabla \theta| |\nabla \theta| \theta_i \right) \, dx \\
\leq C \|u\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla \theta\|_{L^2} \|\theta_i\|_{L^6} + C \|u\|^2_{L^2} \|\nabla^2 \theta\|_{L^2} \|\theta_i\|_{L^6} \\
+ C \|u\|^2_{L^2} \|\nabla \theta\|_{L^2} \|\nabla \theta\|_{L^2} \\
\leq C \|u\|^2_{H^2} \|\nabla u\|^2_{H^1} \|\nabla^2 \theta\|_{L^2} \|\nabla \theta\|_{L^2} (\|\nabla \theta_i\|_{L^2} + 1) \\
+ C \|u\|^2_{H^2} \|\nabla^2 \theta\|_{L^2} \|\nabla \theta_i\|_{L^2} + 1) + C \|u\|^2_{H^2} \|\nabla \theta\|_{L^2} \|\nabla \theta_i\|_{L^2} \\
\leq C (\|\nabla \theta_i\|_{L^2} + 1) (\|\nabla^2 \theta\|_{L^2} + 1) \\
\leq C \|\nabla^2 \theta\|^2_{L^2} + \frac{\kappa}{8} \|\nabla \theta_i\|^2_{L^2} + C,
\]

and

\[
N_4 \leq C \int |\nabla u| |\nabla u_i| |\theta_i| \, dx \\
\leq C \|\nabla u\|_{L^4} \|\nabla u_i\|_{L^2} \|\theta_i\|_{L^4} \\
\leq C \|\nabla u\|_{H^1} \|\nabla u_i\|_{L^2} \|\theta_i\|_{H^1} \\
\leq C \|\nabla u_i\|^2_{L^2} + \frac{\kappa}{8} \|\nabla \theta_i\|^2_{L^2} + C.
\]
\[
\|\nabla^2 \theta\|_{L^2}^2 \leq C \left( \|\rho_t\|_{L^2}^2 + \|\rho u \cdot \nabla \theta\|_{L^2}^2 + \|\nabla u^2\|_{L^2}^2 \right) \\
\leq C \|\sqrt{\rho} \theta_t\|_{L^2}^2 + C \|\rho u\|_{L^\infty}^2 \|\nabla \theta\|_{L^2}^2 + C \|\nabla u\|_{L^4}^4 \\
\leq C \|\sqrt{\rho} \theta_t\|_{L^2}^2 + C \|\rho u\|_{L^2}^2 \|\nabla \theta\|_{L^2}^2 + C \|\nabla u\|_{L^4}^4 \\
\leq C \|\sqrt{\rho} \theta_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\nabla u\|_{H^1} + C \\
\leq C \|\sqrt{\rho} \theta_t\|_{L^2}^2 + C,
\]
due to (9), (26) and (29).

Substituting (43)-(47) into (42) leads to
\[
c^* \frac{d}{dt} \int \rho \theta^2 dx + \kappa \int |\nabla \theta|^2 dx \leq C \|\nabla u_t\|_{L^2}^2 + C \|\sqrt{\rho} \theta_t\|_{L^2}^2 + C,
\]
which combined with Gronwall’s inequality and (29) that (40).

This finishes the proof of Lemma 3.6.

Finally, we will give the second spatial derivatives of \( u \) and the \( L^q \)-estimate \((q > 2)\) of the first spatial derivative of \( \rho \).

**Lemma 3.7.** Under the assumptions in Theorem 1.2, and let \( q > 2 \) be defined in Theorem 1.2, we have that
\[
\sup_{0 \leq t \leq T} (\|\nabla u\|_{H^1} + \|\nabla \theta\|_{H^1} + \|\rho\|_{W^{1,q}}) + \int (\|\nabla^2 u\|_{L^q}^2 + \|\nabla^2 \theta\|_{L^q}^2) dt \leq C,
\]
for any \( 0 \leq T < T^* \).

**Proof.** First, due to (26) and (29) yield to
\[
\|\nabla^2 u\|_{L^2} \leq C \|\rho u_t\|_{L^2} + C \leq C. \tag{49}
\]

Next, it follows from (47) and (40) that
\[
\|\nabla^2 \theta\|_{L^2}^2 \leq C \|\sqrt{\rho} \theta_t\|_{L^2}^2 + C \leq C. \tag{50}
\]

Furthermore, \( |\nabla \rho|^r (2 \leq r \leq q) \) satisfies
\[
(|\nabla \rho|^r)_t + \text{div}(\nabla \rho^r u) + r |\nabla \rho|^{r-2} (\nabla \rho)^{tr} \mathbf{D}(u)(\nabla \rho) = 0,
\]
which together with integrating by parts in \( \Omega \) implies
\[
\frac{d}{dt} \|\nabla \rho\|_{L^r} \leq C \|\nabla u\|_{L^\infty} \|\nabla \rho\|_{L^r}. \tag{51}
\]

Next, we estimate term \( \|\nabla u\|_{L^\infty} \). In view of Lemma 2.1, Lemma 2.2, Poincaré inequality, (29) and (49) gives to
\[
\|\nabla u\|_{L^\infty} \leq \|\nabla u\|_{W^{1,3}} \leq C \|\nabla u\|_{L^2} \\
\leq C \|\nabla u\|_{W^{1,3}} + C \|\nabla u\|_{L^2} \\
\leq C (\|\nabla u\|_{L^3} + \|\nabla^2 u\|_{L^3}) + C \\
\leq C (\|\nabla u\|_{H^1} + \|\rho u_t\|_{L^3}) + C \\
\leq C \|u_t\|_{H^1} + C \|u\|_{L^\infty} \|\nabla u\|_{L^3} + C \\
\leq C \|u_t\|_{H^1} + C \|u\|_{H^3} \|\nabla u\|_{L^3} + C \\
\leq C \|\nabla u_t\|_{L^2} + C.
\]
Combining (51) with (52), we get
\[
\frac{d}{dt} \| \nabla \rho \|_{L^r} \leq C \left( \| \nabla u_t \|_{L^2}^2 + 1 \right) \| \nabla \rho \|_{L^r}.
\]  
(53)

This together with (29), Gronwall's inequality and taking \( r = q \) yield to
\[
\sup_{0 \leq t \leq T} \| \nabla \rho \|_{L^q} \leq C.
\]  
(54)

Furthermore, it follows from (9), (11), (19), (40), (49) and (50) that
\[
\int_0^T \| \nabla^2 u \|_{L^q}^2 \, dt \leq C \int_0^T \left( \| \rho u_t \|_{L^q}^2 + \| \rho u \cdot \nabla u \|_{L^q}^2 \right) \, dt + C
\]
\[
\leq C \int_0^T \left( \| u_t \|_{H^1}^2 + \| u \|_{L^\infty}^2 \| \nabla u \|_{L^q}^2 \right) \, dt + C
\]
\[
\leq C \int_0^T \left( \| u_t \|_{H^1}^2 + \| u \|_{H^2}^2 \| \nabla u \|_{H^1}^2 \right) \, dt + C
\]
\[
\leq C \int_0^T \| \nabla u_t \|_{L^2}^2 \, dt + C
\]
\[
\leq C,
\]  
(55)

and
\[
\int_0^T \| \nabla^2 \theta \|_{L^q}^2 \, dt \leq C \int_0^T \left( \| \rho \theta_t \|_{L^q}^2 + \| \rho u \cdot \nabla \theta \|_{L^q}^2 + \| \nabla \theta \|_{L^q}^2 \right) \, dt
\]
\[
\leq C \int_0^T \left( \| \theta_t \|_{H^1}^2 + \| u \|_{L^\infty}^2 \| \nabla \theta \|_{L^q}^2 + \| \nabla \theta \|_{L^q}^2 \right) \, dt
\]
\[
\leq C \int_0^T \left( \| \theta_t \|_{H^1}^2 + \| u \|_{H^2}^2 \| \nabla \theta \|_{H^1}^2 + \| \nabla \theta \|_{H^2}^2 \| \nabla u \|_{L^q}^2 \right) \, dt
\]
\[
\leq C \int_0^T \| \nabla \theta_t \|_{L^2}^2 \, dt + C
\]
\[
\leq C.
\]  
(56)

We obtain the desired (48) from (49), (50), (54), (55) and (56).
Therefore, we complete the proof of Lemma 3.7.

\[\square\]

**Remark 6.** In particular, for the convenience of the readers, we give a detailed proof to the regularity of terms \( \rho u \) and \( \rho \theta \) from the estimates on \( \sqrt{\rho u_t} \) and \( \rho \theta_t \) as follows.

It follows from (19), (24), (42) and (48), we obtain
\[
\| \rho u \|_{L^2}^2 = \| \rho u_t + \rho u \cdot \nabla u \|_{L^2}^2
\]
\[
\leq C \left( \| \rho u_t \|_{L^2}^2 + \| \rho u \cdot \nabla u \|_{L^2}^2 \right)
\]
\[
\leq C \left( \| u_t \|_{L^2}^2 + \| u \|_{L^\infty}^2 \| \nabla u \|_{L^2}^2 \right)
\]
\[
\leq C \left( \| \nabla u_t \|_{L^2}^2 + \| u \|_{H^2}^2 \| \nabla u \|_{H^1}^2 \right)
\]
\[
\leq C \left( \| \nabla u \|_{L^2}^2 + \| \nabla^2 u \|_{L^2}^2 \right) \leq C,
\]
and
\[
\|\rho\dot{\theta}\|_{L^2}^2 = \|\rho\dot{\theta} + \rho u \cdot \nabla \theta\|_{L^2}^2 \\
\leq C (\|\rho \dot{\theta}\|_{L^2}^2 + \|u\|_{L^\infty}^2 \|\nabla \theta\|_{L^2}^2) \\
\leq C (\|\rho \dot{\theta}\|_{L^2}^2 + \|u\|_{H^2}^2 \|\nabla \theta\|_{L^2}^2) \leq C.
\]

**Proof of Theorem 1.2** In view of Proposition 1, there exists a $T^* > 0$ such that the boundary value problem (1)-(3) has a unique strong solution $(\rho, u, \theta)$ on $\Omega \times (0, T^*)$. Next, we will extend the local strong solution to all times.

Set
\[ T^* = \sup \{ T \mid (\rho, u, \theta) \text{ is a strong solution on } \Omega \times (0, T) \}. \]

With the aid of the standard embedding $L^\infty(\tau, T; H^1) \cap H^1(\tau, T; L^2) \hookrightarrow C([\tau, T]; L^q)$, for any $q \in (2, \infty)$, and Lemma 3.2-Lemma 3.5 and Lemma 3.7, we obtain
\[ \nabla u, \nabla \theta \in C([\tau, T]; L^2 \cap L^q), \]
and
\[ \rho \in C([0, T]; W^{1,q}). \]

Finally, we claim that
\[ T^* = \infty. \]

Otherwise, $T^* < \infty$, it follows from Lemma 3.2-Lemma 3.7 that $(\rho, u, \theta)(x, T^*) = \lim_{t \to T^*} (\rho, u, \theta)(x, t)$ satisfy the conditions imposed on the initial data at the time $t = T^*$. Furthermore, one has $\left( \rho \dot{u}, \rho \dot{\theta} \right) \in C([0, T^*]; L^2)$, which implies
\[ \left( \rho \dot{u}, \rho \dot{\theta} \right) (x, T^*) = \lim_{t \to T^*} \left( \rho \dot{u}, \rho \dot{\theta} \right) (x, t) \in L^2. \]

Hence, one has
\[ (-\mu \Delta u - (\mu + \lambda)\nabla \div u + \nabla P)_{|t=T^*} = \sqrt{\rho}(x, T^*) g_1(x, T^*), \]
and
\[ (-\kappa \Delta \theta - 2\mu |\nabla u|^2)_{|t=T^*} = \sqrt{\rho}(x, T^*) g_2(x, T^*), \]
where
\[ g_1(x) = \begin{cases} 
\rho^{-\frac{1}{2}}(x, T^*) (\rho \dot{u})(x, T^*), & \text{for } x \in \{ x | \rho(x, T^*) > 0 \}, \\
0, & \text{for } x \in \{ x | \rho(x, T^*) = 0 \},
\end{cases} \]
and
\[ g_2(x) = \begin{cases} 
c_v \rho^{-\frac{1}{2}}(x, T^*) (\rho \dot{\theta})(x, T^*), & \text{for } x \in \{ x | \rho(x, T^*) > 0 \}, \\
0, & \text{for } x \in \{ x | \rho(x, T^*) = 0 \},
\end{cases} \]
satisfying $(g_1, g_2) \in L^2$ due to (29) and (38). Thus, $(\rho, u, P)(x, T^*)$ satisfy compatibility conditions (7) and (8). Therefore, we can extend the local strong solution beyond $T^*$ by taking $(\rho, u, \theta)(x, T^*)$ as the initial data and applying the Proposition 1. This contradicts the assumption of $T^*$. This completes the proof of Theorem 1.2.
Acknowledgments. The author would like to thank the editors and referees for their helpful suggestions and careful reading which has improved the presentation of this paper.

REFERENCES

[1] S. A. Antontsev and A. V. Kazhikov, *Mathematical Study of Flows of Nonhomogeneous Fluids*, Lecture Notes, Novosibirsk State University, Novosibirsk, U.S.S.R., 1973.

[2] S. A. Antontsev, A. V. Kazhikov and V. N. Monakhov, *Boundary Value Problems in Mechanics of Nonhomogeneous Fluids*. North-Holland Publishing Co., Amsterdam, 1990.

[3] H. Brézis and T. Gallouet, *Nonlinear Schrödinger evolution equations*, *Nonlinear Anal.*, 4 (1980), 677–681.

[4] H. Brézis and S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities*, *Comm. Partial Differential Equations*, 5 (1980), 773–789.

[5] Y. Cho and H. Kim, *Existence result for heat-conducting viscous incompressible fluid with vacuum*, *J. Korean Math. Soc.*, 45 (2008), 645–681.

[6] H. J. Choe and H. Kim, *Strong solutions of the Navier-Stokes equations for nonhomogeneous incompressible fluids*, *Comm. Partial Differential Equations*, 28 (2003), 1183–1201.

[7] L. Du and Y. Wang, *A blowup criterion for viscous, compressible, and heat-conductive magnetohydrodynamic flows*, *J. Math. Phys.*, 56 (2015), 091503, 20 pp.

[8] J. Fan, S. Jiang and Y. Ou, *A blow-up criterion for compressible viscous heatconductive flows*, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27 (2010), 337–350.

[9] D. Fang, R. Zi and T. Zhang, *A blow-up criterion for two dimensional compressible viscous heat-conductive flows*, *Nonlinear Anal.*, 75 (2012), 3130–3141.

[10] G. P. Galdi, *An Introduction to the Mathematical Theory of Navier-Stokes Equations: Linearized Steady Problems*, Vol. 1, Springer-Verlag, New York, 1994.

[11] C. He, J. Li and B. Lü, *On the cauchy problem of 3D nonhomogeneous Navier-Stokes equations with density-dependent viscosity and vacuum*, preprint, *arXiv:1709.05608*, (2017).

[12] X. Hu and D. Wang, *Global existence and large-time behavior of solutions to the three-dimensional equations of compressible magnetohydrodynamic flows*, *Arch. Ration. Mech. Anal.*, 197 (2010), 203–238.

[13] X. Huang and J. Li, *Serrin-type blowup criterion for viscous, compressible, and heat conducting Navier-Stokes and Magnetohydrodynamic flows*, *Comm. Math. Phys.*, 324 (2013), 147–171.

[14] X. Huang, J. Li and Y. Wang, *Serrin-type blowup criterion for full compressible Navier-Stokes system*, *Arch. Rational Mech. Anal.*, 207 (2013), 303–316.

[15] X. Huang and Y. Wang, *Global strong solution of 3D inhomogeneous Navier-Stokes equations with density-dependent viscosity*, *J. Differential Equations*, 259 (2015), 1606–1627.

[16] X. Huang and Y. Wang, *Global strong solution to the 2D nonhomogeneous incompressible MHD system*, *J. Differential Equations*, 254 (2013), 511–527.

[17] X. Huang and Y. Wang, *Global strong solution with vacuum to the two-dimensional density-dependent Navier-Stokes system*, *SIAM J. Math. Anal.*, 46 (2014), 1771–1788.

[18] X. Huang and Z. Xin, *On formation of singularity for non-isentropic Navier-Stokes equations without heat-conductivity*, *Discrete Contin. Dyn. Syst.*, 36 (2016), 4477–4493.

[19] S. Jiang and Y. Ou, *A blow-up criterion for compressible viscous heat-conductive flows*, *Acta Math. Sci. Ser. B (Engl. Ed.)*, 30 (2010), 1851–1864.

[20] A. V. Kajikov, *Resolution of boundary value problems for nonhomogeneous viscous fluids*, *Dokl. Akad. Nauk.*, 216 (1974), 1008–1010.

[21] H. Kim, *A blow-up criterion for the nonhomogeneous incompressible Navier-Stokes equations*, *SIAM J. Math. Anal.*, 37 (2006), 1417–1434.

[22] H. Kozono, T. Ogawa and Y. Taniuchi, *The critical Sobolev inequalities in Besov spaces and regularity criterion to some semi-linear evolution equations*, *Math. Z.*, 242 (2002), 251–278.

[23] H.-L. Li, X. Xu and J. Zhang, *Global classical solutions to 3D compressible magnetohydrodynamic equations with large oscillations and vacuum*, *SIAM J. Math. Anal.*, 45 (2013), 1356–1387.

[24] P.-L. Lions, *Mathematical Topics in Fluid Mechanics*, Vol. 2., Compressible Models, The Clarendon Press, Oxford University Press, New York, 1998.
[25] B. Lü, Z. Xu and X. Zhong, Global existence and large time asymptotic behavior of strong solutions to the Cauchy problem of 2D density-dependent magnetohydrodynamic equations with vacuum, *J. Math. Pures Appl.*, **108** (2017), 41–62.

[26] L. Nirenberg, *On elliptic partial differential equations*, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **13** (1959), 115–162.

[27] A. Novotný and I. Straškraba, *Introduction to the Mathematical Theory of Compressible Flow*. Oxford Lecture Series in Mathematics and its Applications, Vol. 27, Oxford University Press, Oxford, 2004.

[28] T. Ozawa, *On critical cases of Sobolev’s inequalities*, *J. Funct. Anal.*, **127** (1995), 259–269.

[29] J. Simon, *Nonhomogeneous viscous incompressible fluids: Existence of velocity, density, and pressure*, *SIAM J. Math. Anal.*, **21** (1990), 1093–1117.

[30] Y. Sun, C. Wang and Z. Zhang, A Beale-Kato-Majda criterion for three dimensional compressible viscous heat-conductive flows, *Arch Ration. Mech. Anal.*, **201** (2011), 727–742.

[31] Y. Wang, One new blowup criterion for the 2D full compressible Navier-Stokes system, *Nonlinear Anal. Real World Appl.*, **16** (2014), 214–226.

[32] Z. Xin, Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density, *Comm. Pure Appl. Math.*, **51** (1998), 229–240.

[33] Z. Xin and W. Yan, On blowup of classical solutions to the compressible Navier-Stokes equations, *Comm. Math. Phys.*, **321** (2013), 529–241.

[34] J. Zhang, Global well-posedness for the incompressible Navier-Stokes equations with density-dependent viscosity coefficient, *J. Differential Equations*, **259** (2015), 1722–1742.

Received August 2019 for publication.

E-mail address: wyf12470163.com