The elliptic quantum algebra $A_{q,p}(\widehat{sl}_n)$ and its bosonization at level one

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Abstract

We extend the work of Foda et al and propose an elliptic quantum algebra $A_{q,p}(\widehat{sl}_n)$. Similar to the case of $A_{q,p}(\widehat{sl}_2)$, our presentation of the algebra is based on the relation $RLL = LLR^*$, where $R$ and $R^*$ are $\mathbb{Z}_n$ symmetric R-matrices with the elliptic moduli chosen differently and a scalar factor is also involved. With the help of the results obtained by Asai et al, we realize type I and type II vertex operators in terms of bosonic free fields for $\mathbb{Z}_n$ symmetric Belavin model. We also give a bosonization for the elliptic quantum algebra $A_{q,p}(\widehat{sl}_n)$ at level one.

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1 Introduction

The investigation of the symmetries in quantum integrable models has been attracting a great deal of interests. Recently, the quantum affine algebra $U_q(\widehat{sl}_2)$ has been studied extensively and applied successfully to the XXZ model in the anti-ferromagnetic regime,
see [1] and the references therein. The R-matrix associated with the $XXZ$ model is the six-vertex model which is a trigonometric vertex model. Using the approach of free boson realization of the vertex operators, Jimbo et al obtained integral formulae for the correlation functions and the form factors for the $XXZ$ model.

It is well known that we can obtain the six-vertex model from the Baxter’s [2] eight-vertex model by taking a special limit. And we know that the $XYZ$ model, spin chain equivalent of the eight-vertex model, is a generalization of $XXZ$ model. Foda et al [3] proposed an elliptic extension of the quantum affine algebra $A_{q,p}(\widehat{sl}_2)$ as an algebra of symmetries for the eight-vertex model. The Kyoto group also conjecture that type I and type II vertex operators for the elliptic algebra $A_{q,p}(\widehat{sl}_2)$ can be found. So, it is an open problem to give a bosonic free fields realization for vertex operators of the elliptic algebra $A_{q,p}(\widehat{sl}_2)$. It is also an interesting problem to extend the elliptic algebra $A_{q,p}(\widehat{sl}_2)$ to a more general case $A_{q,p}(\widehat{sl}_n)$ which would play the role of the symmetry algebra in $Z_n$ Belavin model[11]. In this paper, we will study these problems.

It is now believed that the vertex operator approach [1,4] and the method of bosonization are very powerful to study correlation functions of solvable lattice models. It is firstly formulated for vertex models, and then extended to incorporate face models[5,6]. Lukyanov and Pugai [6] give successfully a bosonic realization of the vertex operators for ABF model [7]. This result is developped to to a more general case by Asai et al [8]. They give a bosonization of vertex operators for the $A_{n-1}^{(1)}$ face model [9], the ABF model being the case $n = 2$ with restricted condition. We know there is a face-vertex correspondence between $A_{n-1}^{(1)}$ face model and $Z_n$ Belavin [10] vertex model. When $n = 2$, $Z_n$ symmetric Belavin model reduces to the Baxter’s eight-vertex model. In our former work [11], we use the intertwiners of face-vertex correspondence, and obtained type I vertex operators for $Z_n$ Belavin vertex model with the help of the results obtained by Asai et al [8]. The correlation functions of $Z_n$ Belavin model are also obtained. In the present paper, continuing to extend the results in Ref.[8] and our former work in Ref.[11], we give a bosonization of type II vertex operators for $Z_n$ Belavin model. Using the Miki’s [12] construction, we obtain a bosonic realization for the elliptic algebra $A_{q,p}(\widehat{sl}_n)$ (which is first proposed in this paper) at level one. As a special case of $n=2$, this will give the bosonization for $A_{q,p}(\widehat{sl}_2)$ algebra at level one.

The paper is organized as follows. In Section 2 we introduce the $Z_n$ symmetric Belavin vertex model and give a definition for the elliptic algebra $A_{q,p}(\widehat{sl}_n)$. In Section 3 we define the vertex operators for the algebra $A_{q,p}(\widehat{sl}_n)$ at level one. Section 4 is devoted to the main results of this paper. Extending the work of Asai et al, we introduce another set of boson oscillators and obtain a bosonization of type II vertex operators for $A_{n-1}^{(1)}$ face model. Using face-vertex correspondence, we give a free boson realization of type I and type II vertex operators for $Z_n$ Belavin model. This gives a bosonization of the elliptic
algebra $A_{q,p}(\mathfrak{sl}_n)$ at level one. Finally, we give summary and discussions in Section 5. Appendix contains some detailed calculations.

2 The model and the elliptic algebra $A_{q,p}(\mathfrak{sl}_n)$

We first introduce some notations. Let $n \in \mathbb{Z}^+$ and $n \geq 2$, $w \in \mathbb{C}$ and $\text{Im} w \geq 0$, $r \geq n+2$ and take real value, $\tau \in \mathbb{C}$ with $\text{Im} \tau > 0$.

Define matrix $g, h, I_\alpha$ with elements take values $g_{ij} = \omega^i \delta_{ij}$, $\omega = e^{\frac{2\pi i}{n}}$, $h_{jk} = \delta_{j+1,k}$, $I_\alpha = I(\alpha_1, \alpha_2) = g^\alpha h^\alpha$.

The elliptic functions defined as

$$\theta \left[a \atop b \right](z, \tau) = \sum_{m \in \mathbb{Z}} \exp\{\pi i[(m+a)^2 \tau + 2(m+a)(z+b)]\},$$

$$\sigma_\alpha(z, \tau) = \sigma(\alpha_1, \alpha_2)(z, \tau) = \theta \left[ \frac{1}{2} + \frac{\alpha_1}{n} \atop \frac{1}{2} + \frac{\alpha_2}{n} \right](z, \tau).$$

Redefine $z \equiv vw$, $\tau \equiv rw$. The R-matrix of $Z_n$ symmetric R-matrix can be defined as

$$\bar{R}(vw, rw) = \frac{\sigma_0(w, rw)}{\sigma_0(vw + w, rw)} \sum_{\alpha} W_\alpha(vw, rw) I_\alpha \otimes I_{\alpha}^{-1},$$

where

$$W_\alpha(wv, rw) = \frac{\sigma_\alpha(vw + \frac{w}{n}, rw)}{n \sigma_\alpha(w, rw)}.$$  (2)

It is necessary to introduce other notations

$$R(v) \equiv R(vw, rw) = x^{2v(\frac{1}{n} - 1)} \frac{g_1(v)}{g_1(-v)} \bar{R}(vw, rw),$$

where $x = e^{\pi i w}$ and

$$g_1(v) = \left\{ \frac{x^{2v} \xi^2}{x^{2n} \xi^{2n}} \right\} \left\{ \frac{x^{2n+2r-2} \xi^{2n}}{x^{2r} \xi^{2n}} \right\}$$

(4)

where $\{z\} = (z; x^{2r}, x^{2n})$ and $(z; p_1, \ldots, p_n) \equiv \prod_{\{n_i\}=0}^{\infty}(1 - z p_1^{n_1} \cdots p_m^{n_m})$.  

3
The R-matrix $R(v, rw)$ have the following properties:

**Yang – Baxter equation:**

\[
R_{12}(v_1 - v_2)R_{13}(v_1 - v_3)R_{23}(v_2 - v_3) = R_{23}(v_2 - v_3)R_{13}(v_1 - v_3)R_{12}(v_1 - v_2)
\]

**Unitarity:**

\[
R_{12}(v)R_{21}(-v) = 1,
\]

**Cross – unitarity:**

\[
\sum_{jl} R_{kl}^{li}(v) R_{lj}^{kj}(-v) = \delta_{ii}\delta_{kk}.
\]

(5)

The parameters $v, w, r$ in our parameterization for $Z_n$ symmetric R-matrix in Eq.(3) are related to that of Foda et al [8] as follows: $q = e^{i\pi w} = x, \xi = x^v, p = x^{2r}$.

Define

\[
R^*(v) = \Delta_n^2(v) R(vw, (r - c)w),
\]

\[
\Delta_n(v) = -x^{\frac{2(n-2)}{n}} v \left(\frac{x^{2n-2+2v} x^{2n}}{(x^{2+2v} x^{2n}) (x^{2n-2+2v} x^{2n})} \right)
\]

(6)

**Definition:** Algebra $A_{q,p}(gl_n)$ is generated by $L_{ij}(v)$ satisfying the relation

\[
R^+(v_1 - v_2)L_1(v_1)L_2(v_2) = L_2(v_2)L_1(v_1)R^{++}(v_1 - v_2),
\]

(7)

where

\[
R^+(v) = R(v)\tau^{-1}(-v + \frac{1}{2}) , \quad R^{++}(v) = R^*(v)\tau^{-1}(-v + \frac{1}{2})
\]

\[
\tau(v) = x^{\frac{2(n-1)}{n}} v \left(\frac{x^{1+2v} x^{2n}}{(x^{2n-1+2v} x^{2n}) (x^{1+2v} x^{2n})} \right)
\]

(8)

By a standard argument based on the anti-symmetric fusion for $Z_n$ symmetry R-matrix [10,15], we find that the following quantum determinant belongs to the center of $A_{q,p}(gl_n)$:

\[
q - \det L(v) = \sum_{\sigma \in S_n} \text{sign}(\sigma) L_{1,\sigma(1)}(v - n)L_{2,\sigma(2)}(v - n + 1) \ldots L_{n,\sigma(n)}(v - 1)
\]

(9)

Therefore, we can impose the further relation $q - \det L(v) = q^\xi$ and define the quotient algebra

\[
A_{q,p}(sl_n) = A_{q,p}(gl_n)/(q - \det L(v) - q^\xi)
\]

(10)

**Remark:** For the case $n = 2$, we have $\Delta_n(v) = -1$. The algebra $A_{q,p}(sl_n)$ reduces to the original elliptic algebra $A_{q,p}(sl_2)$ proposed by Foda et al[3].

4
3 Bosonization of the algebra $A_{q,p}(\hat{sl}_n)$ at level one

In the following, we will mainly restrict our attention to the level one case, and we have $c = 1$. The $R^*(v)$ now becomes
\[ R^*(v) = \Delta^2_n(v)R(v, (r-1)w). \] (11)

The algebra relation remains as:
\[ R^+(v_1 - v_2)L_1(v_1)L_2(v_2) = L_2(v_2)L_1(v_1)R^{*+}(v_1 - v_2). \] (12)

We first introduce type I vertex operator corresponding to the half-column transfer matrix of $Z_n$ Belavin model [5,11,13], and type II vertex operator of $Z_n$ Belavin model which are expected to create the eigenstates of the transfer matrix. We denote the two types of vertex operators as:

- Vertex operator of type I: $\Phi_i(v)$,
- Vertex operator of type II: $\Psi^*_i(v)$.

These Vertex operators satisfy the Faddeev-Zamolodchikov (ZF) algebra.
\[
\Phi_j(v_2)\Phi_i(v_1) = \sum_{lk} R^j_{ik}(v_1 - v_2)\Phi_l(v_1)\Phi_k(v_2),
\] (13)
\[
\Psi^*_i(v_1)\Psi^*_j(v_2) = \sum_{lk} \Psi^*_k(v_2)\Psi^*_l(v_1)R^{*l}_{kj}(v_1 - v_2)\Delta^{-1}_n(v_1 - v_2),
\] (14)
\[
\Phi_i(v_1)\Psi^*_j(v_2) = \tau^{-1}(v_1 - v_2)\Psi^*_j(v_2)\Phi_i(v_1).
\] (15)

In the next section, we will give a q-boson free fields realization of the type I and type II vertex operators listed above.

Introduce Miki’s construction [12],
\[ L_{ij}(v) = \Phi_i(v)\Psi^*_j(v - \frac{1}{2}). \] (16)

Using relations of the ZF algebra Eq.(13)—Eq.(15), one can prove that the operator matrix $L$ constructed above satisfy the defining relation of the elliptic quantum algebra Eq.(7). Here the relation $\frac{\tau(v + \frac{1}{2})}{\tau(-v + \frac{1}{2})} = \Delta^{-1}_n(v)$ has been used.

Thus if the bosonization of the type I and type II vertex operators at level one satisfying Eq.(13)—Eq.(15) can be find, we can find a free boson realization of the elliptic algebra $A_{q,p}(\hat{sl}_n)$ at level one.
4 Bosonization for vertex operators

In this section, we will use face-vertex correspondence relation to obtain a bosonization of vertex operators for $Z_n$ Belavin model satisfying Eq.(13)—Eq.(15).

4.1 Bosonization for $A_{n-1}^{(1)}$ face model

We first give a brief review of the bosonization of type I vertex operators for $A_{n-1}^{(1)}$ face model [8], and then similarly construct the bosonization of type II vertex operators for $A_{n-1}^{(1)}$ face model.

Let $\varepsilon_\mu (1 \leq \mu \leq n)$ be the orthonormal basis in $R^n$, which are supplied with the inner product $<\varepsilon_\mu, \varepsilon_\nu> = \delta_{\mu\nu}$. Set

$$\tau_\mu = \varepsilon_\mu - \epsilon, \quad \epsilon = \frac{1}{n} \sum_{\mu=1}^{n} \varepsilon_\mu.$$ 

The type $A_{n-1}^{(1)}$ weight Lattice is the linear span of the $\tau_\mu : P = \sum_{\mu=1}^{n} Z \tau_\mu$. Let $\omega_\mu (1 \leq \mu \leq n - 1)$ be the fundamental weights and $\alpha_\mu (1 \leq \mu \leq n - 1)$ be the simple roots:

$$\omega_\mu = \sum_{\nu=1}^{\mu} \tau_\nu, \quad \alpha_\mu = \varepsilon_\mu - \varepsilon_{\mu+1}.$$ 

An ordered pair $(b, a) \in P^2$ is called admissible if and only if there exists $\mu (1 \leq \mu \leq n)$ such that $b - a = \tau_\mu$. An ordered set of four weights $(c\ b\ a)$ $\in P^4$ is called an admissible configuration around a face if and only if the pairs $(b, a), (c, b), (d, a)$ and $(c, d)$ are admissible. To each admissible configuration around a face, one can associate the Botlzmann weight for it $[2, 7, 9]$.

We introduce the zero mode operators $q_\mu, p_\mu (1 \leq \mu \leq n)$, which satisfy:

$$[p_\mu, i q_\nu] = <\varepsilon_\mu, \varepsilon_\nu> = \delta_{\mu\nu}.$$ 

Set

$$Q_{\tau_\mu} = q_\mu - \frac{1}{n} \sum_{t=1}^{n} q_t, \quad P_{\tau_\mu} = p_\mu - \frac{1}{n} \sum_{t=1}^{n} p_t + \frac{1}{\sqrt{r(r-1)}} w_\mu,$$

where $\{w_j\}$ are generic complex numbers, which ensures that the intertwiners $\tilde{\varphi}(v)_\mu^{(k)}$ and $\varphi(v)^{(k)}_{\mu, \nu}$ (see the following subsection) could exist. We can also reconstruct the zero mode operators $P_\alpha, Q_\alpha [8]$ indexed by $\alpha \in P$, which are $Z$-linear in $\alpha$ and satisfy

$$[P_\alpha, i Q_\beta] = <\alpha, \beta> \quad (\alpha, \beta \in P).$$
Now we consider the oscillator part. Define free bosonic oscillators $\beta^j_m \ (1 \leq j \leq n, m \in Z/\{0\})$ satisfying relations

$$[\beta^j_m, \beta^k_n] = \begin{cases} m \sum_{[m]} \sum_{[n]} \delta_{n+m,0}, & j = k \\ -m \sum_{[m]} \sum_{[n]} \delta_{n+m,0}, & j \neq k. \end{cases} \tag{17}$$

Here we have used the standard notation $[a]_x = \frac{a - x - n}{x - x}$. The bosonic oscillators should satisfy the constraint: $\sum_{j=1}^n x^{-2jm} \beta^j_m = 0$. One can check that the above constraint is compatible with the commutation relations Eq.(18). Similarly, we can also introduce another set of bosonic oscillators $\beta^j_m \ (1 \leq j \leq n, m \in Z/\{0\})$ which will be used to construct the type II vertex operators and are related to the original bosons by: $\beta^j_m = \frac{1}{(r-1)m} \beta^j_m$. Use $\beta^j_m, \beta^{j'}_m$, we define operators

$$S^j_m = (\beta^j_m - \beta^{j+1}_m)x^{-jm}, \quad \Omega^j_m = \sum_{k=1}^{j} x^{(j-2k+1)m} \beta^k_m; \quad S^{j'}_m = (\beta^{j'}_m - \beta^{j'+1}_m)x^{-jm}, \quad \Omega^{j'}_m = \sum_{k=1}^{j} x^{(j-2k+1)m} \beta^k_m. \tag{18, 19}$$

Define the bosonic Fock spaces $\mathcal{F}_{l,k} = c(\{\beta^j_1, \beta^{j'}_1, \cdots \}_{1 \leq j \leq n}) |l, k >$ and the vacuum vector

$$|l, k > = e^{i \sqrt{-r} Q_l - i \sqrt{r} Q_k} |0, 0 > \tag{20}$$

In the following part of this paper, we will construct the bosonic realization of the vertex operators defined in Eq.(13)—Eq.(15) on the space $\sum_{l,k \in P} \mathcal{F}_{l,k}$. For the generic $r$ (e.g. irrational number), we can introduce two type independent operators $\hat{K}$ and $\hat{L}$ with $\hat{K} = \sum_{\mu} \hat{k}_\mu \xi_\mu$, $\hat{L} = \sum_{\mu} \hat{l}_\mu \xi_\mu$, which act on the space $\sum_{l,k \in P} \mathcal{F}_{l,k}$ as

$$\hat{k}_\mu F_{l,k} = (w_\mu + k_\mu) F_{l,k}, \quad \text{if} \quad k = \sum_{\mu} k_\mu \xi_\mu$$

$$\hat{l}_\mu F_{l,k} = (w_\mu + l_\mu) F_{l,k}, \quad \text{if} \quad l = \sum_{\mu} l_\mu \xi_\mu$$

Let us introduce some basic operators acting on the space $\sum_{l,k \in P} \mathcal{F}_{l,k}$

$$\eta_j(v) = e^{-i \sqrt{-r} \Omega_m x^{-2vm}} e^{-\sum_{m \neq 0} \frac{1}{m} \Omega_m x^{-2vm}};$$

$$\xi_j(v) = e^{i \sqrt{-r} \Omega_m x^{-2vm}} e^{\sum_{m \neq 0} \frac{1}{m} \Omega_m x^{-2vm}};$$

$$\eta^{j'}_j(v) = e^{i \sqrt{-r} \Omega_m x^{-2vm}} e^{\sum_{m \neq 0} \frac{1}{m} \Omega_m x^{-2vm}};$$

$$\xi^{j'}_j(v) = e^{-i \sqrt{-r} \Omega_m x^{-2vm}} e^{-\sum_{m \neq 0} \frac{1}{m} \Omega_m x^{-2vm}}. \tag{21}$$
We can obtain the normal order relations which will be presented in Appendix. From those results, we have the following commutation relations

\[
\begin{align*}
\xi_j(v_1)\xi_{j+1}(v_2) &= -\left[\frac{v_1 - v_2 + \frac{1}{2}}{v_1 - v_2 - \frac{1}{2}}\right]\xi_{j+1}(v_2)\xi_j(v_1), \\
\xi_j(v_1)\eta_j(v_2) &= -\left[\frac{v_1 - v_2 + \frac{1}{2}}{v_1 - v_2 - \frac{1}{2}}\right]\eta_j(v_2)\xi_j(v_1), \\
\xi_j(v_1)\xi_j(v_2) &= \left[\frac{v_1 - v_2 - 1}{v_1 - v_2 + 1}\right]\xi_j(v_2)\xi_j(v_1),
\end{align*}
\]

where

\[
[v] = x^{v^2 - v(x^{2v}; x^{2r})(x^{2r}; x^{2v})(x^{2r}; x^{2v})} = \sigma\left(\frac{v}{r}, -\frac{1}{r}w\right) \times \text{const.}
\]

The bosonization of type I vertex operator in $A_{n-1}^{(1)}$ face model was given by Asai et al.[8].

\[
\phi_{\mu}(v) = \int \prod_{j=1}^{\mu-1} \frac{dx_{2v_j}}{2\pi i x_{2v_j}} \eta_1(v)\xi_1(v_1)\cdots\xi_{\mu-1}(v\mu-1) \prod_{j=1}^{\mu-1} f(v_j - v_{j-1}, \pi_{j\mu}),
\]

\[
\pi_{\mu} = \sqrt{r(r-1)}P_{\pi_{\mu}}, \quad \pi_{j\mu} = \pi_{\mu} - \pi_{\nu}, \quad v_0 = v, \quad f(v, w) = \left[\frac{v + \frac{1}{2} - w}{v - \frac{1}{2}}\right].
\]

The integration contours are simple closed curves around the origin satisfying

\[x|x^{2v_j-1}| < |x^{2v_j}| < x^{-1}|x^{2v_j-1}|
\]

We have also another set of relations

\[
\begin{align*}
\xi'_j(v_1)\xi'_{j+1}(v_2) &= -\left[\frac{v_1 - v_2 - \frac{1}{2}'}{v_1 - v_2 + \frac{1}{2}'}\right]\xi'_{j+1}(v_2)\xi'_j(v_1), \\
\xi'_j(v_1)\eta'_j(v_2) &= -\left[\frac{v_1 - v_2 - \frac{1}{2}'}{v_1 - v_2 + \frac{1}{2}'}\right]\eta'_j(v_2)\xi'_j(v_1), \\
\xi'_j(v_1)\xi_j(v_2) &= \left[\frac{v_1 - v_2 + 1'}{v_1 - v_2 - 1'}\right]\xi_j(v_2)\xi'_j(v_1),
\end{align*}
\]

where

\[
[v]' = x^{v^2 - v(x^{2v}; x^{2r-2})(x^{2r-2}; x^{2v-2})(x^{2r-2}; x^{2v-2})} = \sigma\left(\frac{v}{r-1}, -\frac{1}{(r-1)w}\right) \times \text{const}'.
\]
Construct type II vertex operators in $A_{n-1}^{(1)}$ face model as:

$$
\phi'_\mu(v) = \oint \frac{d^2x}{2\pi i x^2 v_j} \eta'_1(v) \xi'_1(v_1) \cdots \xi'_{\mu-1}(v_{\mu-1}) \prod_{j=1}^{\mu-1} f'(v_j - v_{j-1}, \pi_{j\mu}),
$$

where

$$
f'(v, w) = \frac{[v - \frac{1}{2} + w]'}{[v + \frac{1}{2}]'}.
$$

Here the integration contours are simple closed curves around the origin satisfying

$$
x|x^{2v_j}| < |x^{2v_j}| < x^{-1}|x^{2v_j-1}|
$$

Define the face Boltzmann weights for admissible configurations around a face

$$
\tilde{W}(a|v)^{\mu\nu} = 1,
$$

$$
\tilde{W}(a|v)^{\mu\nu} = \frac{[v + a_{\mu\nu}] [1]}{[v + 1][a_{\mu\nu}]}, \quad \mu \neq \nu,
$$

$$
\tilde{W}(a|v)^{\mu\nu} = \frac{[v|[\mu - 1]}{[v + 1][a_{\mu\nu}]}, \quad \mu \neq \nu.
$$

We also define another Boltzmann weights

$$
\tilde{W}'(a|v)^{\mu\nu} = 1,
$$

$$
\tilde{W}'(a|v)^{\mu\nu} = \frac{[v + a_{\mu\nu}] [1]}{[v + 1][a_{\mu\nu}]}, \quad \mu \neq \nu,
$$

$$
\tilde{W}'(a|v)^{\mu\nu} = \frac{[v|[\mu - 1]}{[v + 1][a_{\mu\nu}]}, \quad \mu \neq \nu.
$$

Following the results obtained by Asai et al in ref. [8], we have

$$
\phi_\mu(v_2)\phi_\nu(v_1) = r_1(v_2 - v_1) \sum_{\mu', \nu'} \phi_{\mu'}(v_1)\phi_{\nu'}(v_2)\tilde{W}(\pi|v_1 - v_2)^{\mu\nu}_{\mu'\nu'}.
$$

Using the method introduced by Asai et al in ref. [8] and noting that

$$
f'(v, \pi_{j\mu} - r) = \frac{[v - \frac{1}{2} + \pi_{j\mu} - r]'}{[v + \frac{1}{2}]'} = \frac{[v - \frac{1}{2} + \pi_{j\mu} - 1]'}{[v + \frac{1}{2}]'},
$$

$$
f'(v, \pi_{\mu\nu} + r) = -f(-v, \pi_{\mu\nu} + 1)|_{r \rightarrow r - 1},
$$

$$
f'(v, \pi_{j\mu} + r) = -f(-v, \pi_{j\mu} - 1)|_{r \rightarrow r - 1},
$$

$$
f'(v, \pi_{\mu\nu} - r) = -f(-v, \pi_{\mu\nu} - 1)|_{r \rightarrow r - 1}.
$$
we can derive the following commutation relations:

\[ \phi^{\prime}_{\mu}(v_1)\phi_{\nu}(v_2) = r^I_1(v_1 - v_2) \sum_{\mu', \nu'} \phi^{\prime}_{\mu'}(v_2)\phi_{\nu'}(v_1) \tilde{W}^{\mu'\nu'}_{\nu'\mu'}(\pi|v_1 - v_2), \tag{33} \]

\[ \phi_{\mu}(v_1)\phi^{\prime}_{\nu}(v_2) = \tau(v_1 - v_2)\phi_{\nu}(v_2)\phi_{\mu}(v_1), \tag{34} \]

where

\[ r_1(v) = \frac{2^{(r-1)(n-1)\nu}}{g_1(-v)} g_1(v), \quad r_1^{\prime}(v) = \frac{2^{(r-1)\nu}}{g_1(-v)} g_1(v). \]

One can see that \( \phi_{\mu} \), \( \phi^{\prime}_{\mu} \) are the intertwiners between the Fock space

\[ \phi_{\mu}(v) : F_{l,k} \longrightarrow F_{l,k+\tau_\mu}, \quad \phi^{\prime}_{\mu}(v) : F_{l,k} \longrightarrow F_{l+\tau_{\mu},k} \]

### 4.2 Face-vertex correspondence and modular transformation

It is convenient to introduce some notations. Define

\[
\tilde{R}^{(1)}(v) = \frac{\sigma_0(\frac{1}{r}, -\frac{1}{rw})}{\sigma_0(\frac{v+1}{r}, -\frac{1}{rw})} \sum_\alpha \sigma_\alpha \left( \frac{v}{r}, -\frac{1}{rw} \right) I_\alpha \otimes I_\alpha^{-1},
\]

\[
\tilde{R}^{(1)}(v) = \frac{\sigma_0(\frac{1}{r-1}, -\frac{1}{(r-1)w})}{\sigma_0(\frac{v+1}{r-1}, -\frac{1}{(r-1)w})} \sum_\alpha \sigma_\alpha \left( \frac{v}{r-1}, -\frac{1}{(r-1)w} \right) I_\alpha \otimes I_\alpha^{-1},
\]

\[
\varphi^{(k)}_{\mu,\tilde{K}}(v) = \theta^{(k)}(v + n < \tilde{K}, \epsilon_\mu >) + (n - 1) - \frac{1}{r},
\]

\[
\varphi^{(k)}_{\tilde{L},\mu}(v) = \theta^{(k)}(v + n < \tilde{L}, \epsilon_\mu >) + P_0 - \frac{1}{(r-1)w},
\]

\[
\theta^{(k)}(z, \tau) = \theta \left[ \begin{array}{c} -\frac{1}{n} \\ 0 \end{array} \right] (z, n\tau),
\]

Although we choose the different function \( \theta^{(k)}(v) \) from the ordinary one[15], the face-vertex correspondence relation is still survived

\[
\sum_{k,l} \tilde{R}^{(1)}(v_1 - v_2)_{ij}^{kl} \varphi^{(k)}_{\nu,\tilde{K}+\epsilon_\mu} (v_1) \varphi^{(l)}_{\mu,\tilde{K}} (v_2) = \sum_{\mu', \nu'} \tilde{W}(\tilde{K}|v_1 - v_2)_{\nu',\mu'}^{\mu,\nu} \varphi^{(i)}_{\mu',\tilde{K}} (v_1) \varphi^{(j)}_{\nu,\tilde{K}+\epsilon_\mu} (v_2)
\]

\[
\sum_{k,l} \tilde{R}^{(1)}(v_1 - v_2)_{ij}^{kl} \varphi^{(k)}_{\tilde{L},\mu} (v_1) \varphi^{(l)}_{\tilde{L}-\epsilon_\mu,\nu} (v_2) = \sum_{\mu', \nu'} \tilde{W}'(\tilde{L}|v_1 - v_2)_{\nu',\mu'}^{\mu,\nu} \varphi^{(i)}_{\tilde{L}-\epsilon_\mu,\nu} (v_1) \varphi^{(j)}_{\tilde{L},\mu} (v_2)
\]
For the generic number \( \{w_j\} \), we can introduce intertwiners \( \tilde{\varphi}_{\mu,\tilde{R}}(v) \) and \( \tilde{\varphi}_{L,\mu}^{(k)}(v) \) satisfying relations\([15]\)

\[
\sum_k \tilde{\varphi}_{\mu,\tilde{R}}^{(k)}(v) \varphi_{\nu,\tilde{R}}^{(k)}(v) = \delta_{\mu\nu}, \quad \sum_k \tilde{\varphi}_{L,\mu}^{(k)}(v) \varphi_{L,\nu}^{(k)}(v) = \delta_{\mu\nu}.
\]

So it is easy to find the following face-vertex correspondence relations

\[
\sum_{k,l} \tilde{R}^{(1)}(v_1 - v_2)_{kl} \tilde{\varphi}_{\mu,\tilde{R}}^{(k)}(v_1) \tilde{\varphi}_{\nu,\tilde{R} + \epsilon_{\mu}}^{(l)}(v_2) = \sum_{\mu',\nu'} \tilde{W}(\tilde{R}|v_1 - v_2)_{\mu'\nu'} \tilde{\varphi}_{\nu',\tilde{R} + \epsilon_{\nu'}}^{(i)}(v_1) \tilde{\varphi}_{\nu,\tilde{R}}^{(j)}(v_2),
\]

\[
\sum_{k,l} \tilde{R}^{(1)}(v_1 - v_2)_{kl} \tilde{\varphi}_{L,\mu}^{(k)}(v_1) \tilde{\varphi}_{L,\nu}^{(l)}(v_2) = \sum_{\mu',\nu'} \tilde{W}'(\tilde{L}|v_1 - v_2)_{\mu'\nu'} \tilde{\varphi}_{L,\nu'}^{(i)}(v_1) \tilde{\varphi}_{L,\mu}^{(j)}(v_2). \tag{36}
\]

Introduce the modular transformation

\[
\theta \left[ \frac{1}{2} + a \right] \left( \frac{z}{\tau}, -\frac{1}{\tau} \right) = \theta \left[ \frac{1}{2} + b \right] \left( \frac{z}{\tau} \right) \exp\{ \frac{z^2}{\tau} + a - b + 2ab \} \times \text{const.,} \tag{38}
\]

where the const. only depends on \( \tau \). Therefore, we can derive the following relations for \( Z_n \) symmetry R-matrices \( \tilde{R}^{(1)}(v) \) and \( \tilde{R}^{(1)}(v) \)

\[
(M \otimes M)\tilde{R}^{(1)}(v)(M^{-1} \otimes M^{-1}) = x^{\frac{2v(1-n)}{n}} P \tilde{R}(vw, rw) P, \tag{39}
\]

\[
(M \otimes M)\tilde{R}^{(1)}(v)(M^{-1} \otimes M^{-1}) = x^{\frac{2v(1-n)}{n(r-1)}} P \tilde{R}(vw, (r-1)w) P, \tag{40}
\]

where \( P \) is the permutation operator acting on the tensor space \( V \otimes V \) as \( P(e_i \otimes e_j) = e_j \otimes e_i \), and the matrix \( (M)_{ik} = \omega^{-ik} = \exp \{ -\frac{2i\pi}{n} l k \} \) which have the following properties

\[
MgM^{-1} = h^{-1}, \quad MhM^{-1} = g.
\]

### 4.3 Bosonization for \( Z_n \) Belavin model

Based on the bosonization for \( A_n^{(1)} \) face model and the face-vertex correspondence, we will construct the bosonization of two type vertex operators for \( Z_n \) Belavin model.

Firstly, define

\[
\Phi_j(v) = \sum_{i=1}^{n} \sum_{\mu=1}^{n} M_{ji} \phi_{\mu}^{(i)}(v) \varphi_{\mu,\tilde{R}}^{(i)}(-v), \tag{41}
\]

\[
\Psi_j(v) = \sum_{i=1}^{n} \sum_{\mu=1}^{n} M_{ji} \phi_{\mu}^{(i)}(v) \varphi_{L,\mu}^{(i)}(-v), \tag{42}
\]

\[11\]
which satisfy the commutation relations on the space $\sum_{l,k\in P} \oplus F_{l,k}$

\[ \Phi_j(v_2)\Phi_i(v_1) = \sum_{lk} R_{lk}^{ij}(v_1 - v_2)\Phi_l(v_1)\Phi_k(v_2), \quad (43) \]

\[ \Psi_i(v_1)\Psi_j(v_2) = \sum_{lk} \Psi_k(v_2)\Psi_l(v_1)R^{*ij}_{lk}(v_1 - v_2)\Delta_n^{-1}(v_1 - v_2), \quad (44) \]

\[ \Phi_i(v_1)\Psi_j(v_2) = \tau(v_1 - v_2)\Psi_j(v_2)\Phi_i(v_1). \quad (45) \]

Based on the anti-symmetric fusion for $Z_n$ symmetric R-matrix [10,15], we can define

\[ \Psi_j^*(v) = \Psi(1,...,j-1,j+1,...,n)(v) \equiv \sum_{\sigma \in S_{n-1}} \text{sign}(\sigma)\Psi_{\sigma(1)}(v + n - 1)\Psi_{\sigma(2)}(v + n - 2)....\Psi_{\sigma(n)}(v + 1), \]

and derive the following relations

\[ \Phi_j(v_2)\Phi_i(v_1) = \sum_{lk} R_{lk}^{ij}(v_1 - v_2)\Phi_l(v_1)\Phi_k(v_2), \quad (46) \]

\[ \Psi_i^*(v_1)\Psi_j^*(v_2) = \sum_{lk} \Psi_i^*(v_2)\Psi_l^*(v_1)R^{*ij}_{lk}(v_1 - v_2)\Delta_n^{-1}(v_1 - v_2), \quad (47) \]

\[ \Phi_i(v_1)\Psi_j^*(v_2) = \tau^{-1}(v_1 - v_2)\Psi_j^*(v_2)\Phi_i(v_1). \quad (48) \]

Therefore, we obtain the bosonic realization of vertex operators $\Phi_i(v)$ and $\Psi_i^*(v)$ defined in section 2. Moreover, we obtain the bosonic realization for $A_{q,p}(\hat{sl}_n)$ algebra at level one through Miki construction Eq.(16). Thus this algebra is self-consistent.

**Remark:** For the case $n=2$ ($\Delta_2 = -1$), the bosonic operators $\Phi_i(v)$ and $\Psi_i^*(v)$ become the type I and type II vertex operators of $A_{q,p}(\hat{sl}_2)$ at level one proposed by Foda et al [3].

**Discussions**

For $n=2$, the algebra $A_{q,p}(\hat{sl}_2)$ reduces to the original one $A_{q,p}(\hat{sl}_2)$ algebra which was first proposed by Foda et al [3]. The quantum affine algebra $U_q(\hat{sl}_2)$ is a degeneration algebra of $A_{q,p}(\hat{sl}_2)$ with $p = 0$ [3]. Unfortunately, due to the nontrivial scalar factor $\Delta_n(v)$ when $2 < n$, the relation between the elliptic algebra $A_{q,p}(\hat{sl}_n)$ and $U_q(\hat{sl}_n)$ is still an open problem.

As discussed in [8], vertex operators $\phi_{\mu}(v)$ in $A_{n-1}^{(1)}$ face model was the q-analog of the chiral primary fields of q-deformed W-algebras[14]. However, the vertex operators $\Phi_i(v)$ and $\Psi_j^*(v)$ in $Z_n$ Belavin model are reconstructed by the vertex operators of $A_{n-1}^{(1)}$ face model through the face-vertex correspondence relations. Moreover, the elliptic algebra $A_{q,p}(\hat{sl}_n)$ at
level one can be constructed by Miki construction. So, there exists some relations between the algebra \( A_{q,p}(sl_n) \) and q-deformed W-algebras.

Besides the degenerate algebra \( A_{q,0}(sl_n) \), there exists another degenerate algebra \( A_{h,\eta}(sl_n) \) [16], which can be obtained by the scaling limit \((v = \frac{\hbar}{\eta}, q = x, p = x^\eta, x \to 1)\) of the elliptic algebra \( A_{q,p}(sl_n) \). The resulted algebra \( A_{h,\eta}(sl_n) \) is some deformation of Yangian double \( DY(sl_n) \) [16,17]. In the scaling limit case the R-matrix entering the \( \Psi^*\Psi^* \) commutation relation would be interpreted as the S-matrix for soliton of affine Toda theory (In the case of \( n=2 \), it is related to sine-Gordon theory [17]). Moreover, the \( h \)-deformed W-algebra [18] would be obtained by the quantum Hamiltonian reduction from the degenerate algebra \( A_{h,\eta}(sl_n) \).

In our formulation, the algebra \( A_{q,p}(sl_n) \) is formulated in the framework of the “RLL” approach in terms of the L-operator. It would be of great importance to find an analogous Drinfeld currents for the algebra \( A_{q,p}(sl_n) \). For a special case \( A_{q,p}(sl_2) \) with the scaling limit, the Drinfeld currents for the algebra \( A_{h,\eta}(sl_2) \) was found to be the Gauss coordinates of the L-operator[16]. We expect that the same relation would be existed for the elliptic algebra \( A_{q,p}(sl_n) \).

**Appendix**

**A. The normal order relation for basic operator**

We list the relations as three series:

**Type I:**

\[
\eta_1(v_1)\eta_1(v_2) = x^{\frac{2(r-1)(n-1)v_1}{nr}}g_1(v_2 - v_1) : \eta_1(v_1)\eta_1(v_2) : \\
\eta_j(v_1)\xi_j(v_2) = x^{\frac{2(1-r)v_1}{r}}s(v_2 - v_1) : \eta_j(v_1)\xi_j(v_2) : \\
\xi_j(v_2)\eta_j(v_1) = x^{\frac{2(1-r)v_2}{r}}s(v_1 - v_2) : \xi_j(v_2)\eta_j(v_1) : \\
\xi_j(v_1)\xi_{j+1}(v_2) = x^{\frac{2(1-r)v_1}{r}}s(v_2 - v_1) : \xi_j(v_1)\xi_{j+1}(v_2) : \\
\xi_{j+1}(v_2)\xi_j(v_1) = x^{\frac{2(1-r)v_2}{r}}s(v_1 - v_2) : \xi_{j+1}(v_2)\xi_j(v_1) : \\
\xi_j(v_1)\xi_j(v_2) = x^{\frac{4(r-1)v_1}{r}}t(v_2 - v_1) : \xi_j(v_1)\xi_j(v_2) : \\
\xi_j(v_1)\xi_l(v_2) = : \xi_j(v_1)\xi_l(v_2) : \quad \text{if } |l - j| > 1, \\
\eta_j(v_1)\xi_l(v_2) = : \eta_j(v_1)\xi_l(v_2) : \quad \text{if } l \neq j.
\]

**Type II:**

\[
\eta'_1(v_1)\eta'_1(v_2) = x^{\frac{2(r-1)(n-1)v_1}{(r-1)n}}g'_1(v_2 - v_1) : \eta'_1(v_1)\eta'_1(v_2) :
\]

(49)
Type I with type II:

\[
\begin{align*}
\xi_j'(v_1)\eta_j'(v_2) &= x^{-2r_1}v_1 s(v_2 - v_1) \colon \xi_j'(v_1)\eta_j'(v_2) : \\
\eta_j'(v_2)\xi_j'(v_1) &= x^{-2r_1}v_2 s(v_1 - v_2) \colon \eta_j'(v_2)\xi_j'(v_1) : \\
\xi_j'(v_1)\xi_j'_{j+1}(v_2) &= x^{-2r_1}v_1 s(v_2 - v_1) \colon \xi_j'(v_1)\xi_j'_{j+1}(v_2) : \\
\xi_{j'}_{j+1}(v_2)\xi_j'(v_1) &= x^{-2r_2}v_2 s(v_2 - v_1) \colon \xi_{j'}_{j+1}(v_2)\xi_j'(v_1) : \\
\xi_j'(v_1)\xi_j'(v_2) &= x^{-2r_1}v_1 |'(v_2 - v_1) \colon \xi_j'(v_1)\xi_j'(v_2) : \\
\xi_j'(v_1)\xi_j'(v_2) &= : \xi_j'(v_1)\xi_j'(v_2) : \text{ if } |l - j| > 1, \\
\eta_j'(v_1)\xi_j'(v_2) &= : \eta_j'(v_1)\xi_j'(v_2) : \text{ if } l \neq j,
\end{align*}
\]

(50)

\[
\begin{align*}
\eta_j(v_1)\xi_j'(v_2) &= (x^{2v_1} - x^{2v_2}) \colon \xi_j'(v_2)\eta_j(v_1) : \\
\xi_j'(v_2)\eta_j(v_1) &= (x^{2v_2} - x^{2v_1}) \colon \eta_j(v_1)\xi_j'(v_2) : \\
\xi_j(v_1)\eta_j'(v_2) &= (x^{2v_1} - x^{2v_2}) \colon \xi_j(v_1)\eta_j'(v_2) : \\
\eta_j'(v_2)\xi_j(v_1) &= (x^{2v_2} - x^{2v_1}) \colon \eta_j'(v_2)\xi_j(v_1) : \\
\eta_j(v_1)\xi_j'(v_2) &= : \eta_j(v_1)\xi_j'(v_2) := \xi_j'(v_2)\eta_j(v_1) \text{ if } l \neq j, \\
\eta_j'(v_1)\xi_j'(v_2) &= : \eta_j'(v_1)\xi_j'(v_2) := \xi_j(v_2)\eta_j'(v_1) \text{ if } l \neq j, \\
\xi_j(v_1)\xi_j'(v_2) &= (x^{2v_1} - x^{2v_2}) \colon \xi_j(v_1)\xi_j'(v_2) : \\
\xi_{j'}_{j+1}(v_2)\xi_j(v_1) &= (x^{2v_2} - x^{2v_1}) \colon \xi_{j'}_{j+1}(v_2)\xi_j(v_1) : \\
\xi_j(v_1)\xi_j'(v_2) &= x^{-4v_1} \frac{1}{(1 - x^{2v_2 - v_1})(1 - x^{-1}x^{2v_2 - v_1})} \colon \xi_j(v_1)\xi_j'(v_2) : \\
\xi_j'(v_2)\xi_j(v_1) &= x^{-4v_2} \frac{1}{(1 - x^{2v_1 - v_2})(1 - x^{-1}x^{2v_1 - v_2})} \colon \xi_j'(v_2)\xi_j(v_1) : \\
\xi_j(v_1)\xi_j'(v_2) &= : \xi_j(v_1)\xi_j'(v_2) : \text{ if } |j - l| > 1, \\
\xi_j'(v_2)\xi_j(v_1) &= : \xi_j'(v_2)\xi_j(v_1) : \text{ if } |j - l| > 1, \\
\eta_j(v_1)\eta_j'(v_2) &= x^{2v_1 \sum_{j=0}^{n} \frac{x^{2n-j}x^{2v_2-v_1}; x^{2n}}{(x^jx^{2v_2-v_1}; x^{2n})}} \colon \eta_j(v_1)\eta_j'(v_2) : \\
\eta_j'(v_2)\eta_j(v_1) &= x^{2v_2 \sum_{j=0}^{n} \frac{x^{2n-j}x^{2v_1-v_2}; x^{2n}}{(x^jx^{2v_1-v_2}; x^{2n})}} \colon \eta_j'(v_2)\eta_j(v_1) : 
\end{align*}
\]

(51)

where

\[
\begin{align*}
g_1(v) &= \frac{\{x^{2n+2v}\} \{x^{2n+2r-2}x^{2n}\}}{\{x^{2n}x^{2n}\} \{x^{2r}x^{2r}\}}, \\
s(v) &= \frac{(x^{2r-1}x^{2v}; x^{2r})}{(x^{2v}; x^{2r})}, \\
t(v) &= (1 - x^{2v}) \frac{x^{2r-2}x^{2v}; x^{2r}}{(x^{2v}; x^{2r})},
\end{align*}
\]
using the exchange relation of \( \phi \)

\[
g'(v) = \left\{ x^{2n} \right\} v \left\{ x^{2n+2r-2} \right\} v', \quad \{ z \}' = (z; x^{2r-2}, x^{2n})
\]

\[
s'(v) = \frac{(x^{2r-1}x^{2r-2})}{(x^{-1}x^{2r-2})}, \quad t'(v) = (1 - x^{2r}) \frac{(x^{-2}x^{2r}; x^{2r-2})}{(x^{2r}x^{2r-2})}
\]

(52)

**B. Proof of the commutation relations Eq.(43)-Eq.(45)**

Notice that the commutation relation for zero mode operators, we have

\[
\phi_{\nu}(v_1)\tilde{\varphi}^{(k)}_{\mu,\bar{R}}(v_2) = \varphi^{(k)}_{\mu,\bar{R}}(v_2)\phi_{\nu}(v_1), \quad \phi'_{\nu}(v_1)\varphi^{(k)}_{\mu,\bar{R}}(v_2) = \tilde{\varphi}^{(k)}_{\mu,\bar{R}}(v_2)\phi'_{\nu}(v_1),
\]

\[
\phi_{\nu}(v_1)\tilde{\varphi}^{(k)}_{L,\mu}(v_2) = \varphi^{(k)}_{L,\mu}(v_2)\phi_{\nu}(v_1) , \quad \phi'_{\nu}(v_1)\varphi^{(k)}_{L,\mu}(v_2) = \tilde{\varphi}^{(k)}_{L,\mu}(v_2)\phi'_{\nu}(v_1).
\]

Define

\[
Z_j(v) = \sum_{\mu=1}^{n} \phi_{\mu}(v)\tilde{\varphi}^{(i)}_{\mu,\bar{R}}(-v), \quad Z'_j(v) = \sum_{\mu=1}^{n} \phi'_{\mu}(v)\tilde{\varphi}^{(i)}_{-L,\mu}(-v).
\]

Notice that

\[
\tilde{W}(\tilde{\pi}|v)^{\nu'\mu'}_{\nu\mu}|_{F_{i,k}} = \tilde{W}(\tilde{K}|v)^{\nu'\mu'}_{\nu\mu}|_{F_{i,k}}
\]

using the exchange relation of \( \phi_{\mu}(v) \) in Eq.(32) and the face-vertex correspondence by \( \tilde{\varphi}^{(k)}_{\mu,\bar{R}}(v) \) in Eq.(36), we have

\[
Z_j(v_2)Z_i(v_1) = \sum_{l,k} r_1(v_2 - v_1)(P \tilde{R}^{(1)}(v_1 - v_2)P)_{l,k}^{ij}Z_l(v_1)Z_k(v_2).
\]

Notice the properties of \( Z_n \) symmetry R-matrix under the modular transformation Eq.(39), we obtain

\[
Z_j(v_2)Z_i(v_1) = \sum_{l,k} (M^{-1} \otimes M^{-1} R(v_1 - v_2)M \otimes M)_{l,k}^{ij}Z_l(v_1)Z_k(v_2)
\]

Due to \( \Phi_j(v) = \sum_{i=1}^{n} M_{ji}Z_i(v) \), one can obtain Eq.(43). Using the same method and notice that

\[
\tilde{W}^{r}(\tilde{L}|v)^{\nu'\mu'}_{\nu'\mu} = \tilde{W}^{r}(\tilde{\pi}|v)^{\nu'\mu'}_{\nu'\mu}, \quad \frac{r'_1(v)}{r_1(-v)}|_{r \rightarrow r-1} = \Delta_n(v);
\]

\[
\tilde{W}^{r}(\tilde{L}|v)^{\nu'\mu'}_{\nu'\mu} |_{F_{i,k}} = \tilde{W}^{r}(\tilde{\pi}|v)^{\nu'\mu'}_{\nu'\mu} |_{F_{i,k}}
\]

we can obtain Eq.(44) and Eq.(45).
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