CLASS VII₀ SURFACES WITH $b₂$ CURVES

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Abstract. We give an affirmative answer to a conjecture of Ma. Kato, namely that every compact complex surface $S$ in Kodaira’s class VII₀ having $b₂(S) > 0$ and $b₂(S)$ rational curves, admits a global spherical shell.

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A surface is a compact complex manifold of dimension 2. We denote by $b_i(S)$ the $i$-th Betti number of $S$. 

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0. Introduction

Kodaira’s class $VII_0$, which consists of minimal compact complex surfaces $S$ having $b_1(S) = 1$, is not completely understood so far. In fact, only the case when $b_2(S) = 0$ is completely classified, by the work of Kodaira [13], Inoue [11], Bogomolov [1], Li-Yau-Zheng [14] and Teleman [18].

For $b_2(S) > 0$ a construction method and thus a large subclass of surfaces has been introduced by Kato [12]. These are exactly the minimal surfaces with $b_2 > 0$ containing global spherical shells. (See the next section for the definition.) One can show that a surface $S$ of class $VII_0$ has at most $b_2(S)$ rational curves on it and that if moreover $S$ admits a global spherical shell, then there are exactly $b_2(S)$ rational curves on $S$. Ma. Kato conjectured that the converse should be true as well. Important progress towards this conjecture was made by I. Nakamura [15], [16], who showed that if $S$ has $b_2(S)$ rational curves, then their configuration is that of the curves of a surface with global spherical shells and $S$ is a deformation of a blown up Hopf surface.

This paper is devoted to the proof of Kato’s conjecture:

**Main Theorem:** If $S$ is a surface of class $VII_0$ with $b_2(S) > 0$ and with $b_2(S)$ rational curves, then $S$ admits global spherical shells.

At present all known surfaces $S$ of class $VII_0$ with $b_2(S) > 0$ contain global spherical shells. In fact, by making additional assumptions on $S$ like the existence of a homologically trivial divisor \([8]\) or of two cycles of rational curves \([13]\) or of a holomorphic vector field \([6], [7]\), it was shown that $S$ contains global spherical shells. The present paper makes a further step in this direction.

The paper is organized as follows. Section 1 is preparatory. We recall some facts on surfaces with global spherical shells and on surfaces $S$ of class $VII_0$ with $b_2(S) > 0$ rational curves. Such surfaces were called special by Nakamura \([16]\). We also prove a fact we shall need later, namely that if $S$ is special, then the canonical bundle of a suitable finite ramified covering of $S$ is numerically divisorial.

Using the knowledge of the configuration of rational curves on a special surface, we prove in section 2 the existence of a logarithmic 1-form twisted by a flat line bundle. Passing to a finite ramified covering we get a global twisted holomorphic vector field. The twisting is again by some flat line bundle. This induces a true holomorphic vector field on the universal covering $\tilde{S}$ of $S$.

In section 3 we prove that this holomorphic vector field is completely integrable and that the universal covering of the complement of the curves of $S$ is isomorphic to $\mathbb{H} \times \mathbb{C}$, where $\mathbb{H}$ denotes the complex half plane.

Section 4 is devoted to the computation of the action of the fundamental group on $\mathbb{H} \times \mathbb{C}$.

This allows us to recover in section 5 the contracting rigid germ of holomorphic map which gave birth to our surface $S$. Using the work of C. Favre \([9]\) which classifies such germs, we are able to conclude.
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1. Preliminaries

We start by recalling some definitions and known facts. A compact complex surface $S$ is said to belong Kodaira’s class $\text{VII}$ if $b_1(S) = 1$ and to class $\text{VII}_0$ if it is moreover minimal. Surfaces of class $\text{VII}_0$ with $b_2 = 0$ have been completely classified, see for instance [18]. In this paper we deal with the case $b_2 > 0$. It is well known then, that the Kodaira dimension of $S$ is negative and that the algebraic dimension vanishes. In particular, $S$ has finitely many irreducible curves.

At present the only known surfaces of class $\text{VII}_0$ with $b_2 > 0$ contain global spherical shells (GSS). A global spherical shell is a neighborhood $V$ of $S^3 \subset \mathbb{C}^2 \setminus \{0\}$ which is holomorphically embedded in the surface $S$ such that $S \setminus V$ is connected. All surfaces with GSS may be constructed by a procedure due to Ma. Kato [12]. As a consequence they all have exactly $b_2(S)$ rational curves; some of them admit an elliptic curve as well. Ma. Kato also made the following

**Conjecture** If $S$ is a class $\text{VII}_0$ surface with $b_2 > 0$ rational curves, then $S$ admits a GSS.

Following I. Nakamura [16] we shall call a class $\text{VII}_0$ surface **special**, if it has $b_2 > 0$ rational curves. Since special surfaces admitting homologically trivial divisors or with all rational curves organized in one or two cycles have been shown to admit GSS, [8], [14], [3], [12], we concentrate our attention on the remaining ones. We shall call them **special surfaces of intermediate type**.

In [16], Nakamura proved that the configuration of the rational curves of a special surface of intermediate type is the same as that of a surface with GSS with the same $b_2$. In particular the dual graph of such a configuration is connected and contains a cycle to which some trees are attached. Nakamura also showed that these surfaces are deformations of blown-up primary Hopf surfaces, in particular that their fundamental group is isomorphic to $\mathbb{Z}$. Furthermore, he proposed a line of attack to Kato’s conjecture. However, one of his conjectures proved to be not correct, see [19].

**Notation** We denote by $D$ the maximal reduced divisor of a special surface $S$, by $M(S)$ the intersection matrix of the rational curves of $S$ and set $k(S) := \sqrt{|\det M(S)|} + 1$. (It is well known that $\sqrt{|\det M(S)|}$ is the index of the subgroup generated by classes of curves in $H^2(S, \mathbb{Z})$ and thus it is an integer). Moreover we denote by $\tilde{S}$ the universal cover of $S$ and by $\tilde{D}$ the preimage of $D$ in $\tilde{S}$. Then $\tilde{D}$ is the universal cover of $D$.

In this section we show that after passing to a ramified covering and a resolution of singularities, we may suppose that the anticanonical bundle of our surface is
numerically divisorial, i.e. there exists a flat line bundle $L$ on $S$ and a divisor $D_K$ such that $K_S^{-1} \cong L \otimes \mathcal{O}(D_K)$.

Since under our assumptions $\pi_1(S) \cong \mathbb{Z}$, it is easy to see that the flat line bundles are parametrized by $\mathbb{C}^* \cong \text{Hom}(\pi_1(S), \mathbb{C}^*) \cong \text{Pic}^0(S)$. We shall often write $L^\lambda$ for the line bundle corresponding to the complex number $\lambda \in \mathbb{C}^*$.

**Lemma 1.1.** Let $S$ be a special surface of intermediate type. Then there exist a positive integer $m$, a flat line bundle $L$ and an effective divisor $D_m$ such that $(K_S \otimes L)^{\otimes m} = \mathcal{O}(-D_m)$.

**Proof:** Since the cohomology classes associated to the rational curves of $S$ generate $H^2(S, \mathbb{Q})$, there always exist $m \in \mathbb{N}^*$, $L \in \text{Pic}^0(S)$ and a divisor $D_m$ such that $(K_S \otimes L)^{\otimes m} = \mathcal{O}(-D_m)$.

We have only to check that $D_m \geq 0$. Let $D_m = D_+ - D_-$ with $D_+, D_- \geq 0$ and $D_+.D_- \geq 0$. The adjunction formula implies that $D_mC \leq 0$ for any irreducible curve $C$ on $S$. Hence

$$0 \geq D_m.D_- = (D_+ - D_-).D_- \geq -D_-^2 \geq 0.$$ 

But then $D_- = 0$, since $S$ does not admit homologically trivial divisors. 

**Definition 1.2.** The smallest possible $m \in \mathbb{N}^*$ for which a decomposition $(K_S \otimes L)^{\otimes m} = \mathcal{O}(-D_m)$ as in Lemma 1.1 exists, will be called the **index** of the surface $S$ and denoted by $m(S)$. When $m(S) = 1$ we denote $D_K = D_1$ and call this the **numerically anticanonical divisor** of $S$.

Notice that $D_m$ is unique when $S$ is of intermediate type. The following proposition will enable us to reduce the proof of the Main Theorem to the case of special surfaces of index 1. We have formulated it for simplicity for special surfaces of intermediate type, but the general case can be proved similarly.

**Proposition 1.3.** Let $S$ be a special surface of intermediate type with index $m := m(S) > 1$. Then there exists a diagram

$$
\begin{array}{ccc}
S' & \xrightarrow{\pi'} & Z' \\
\downarrow{\pi} & & \downarrow{\rho'} \\
T & & Z \\
\downarrow{\rho} & & \downarrow{\rho} \\
Z & & T' \\
\end{array}
$$

where

- $(Z, \pi, S)$ is a $m$-fold cyclic ramified covering space of $S$, branched over $D_m$,
- $(T, \rho, Z)$ is the minimal desingularization of $Z$. 

• \((T, c, S')\) is the contraction of the (possible) exceptional curves of the first kind,
• \(S'\) is a special surface with action of the group of \(m\)-th roots of unity \(\mathbb{U}_m\), with index \(m(S') = 1\),
• \(Z'\) is the quotient space of \(S'\) by \(\mathbb{U}_m\),
• \((T', \rho', Z')\) is the minimal desingularization of \(Z'\),
• \((T', c', S)\) is the contraction of the (possible) exceptional curves of the first kind, such that the restriction over \(S \setminus D\) is commutative, i.e.
\[
\theta := \pi \circ \rho \circ c^{-1} = \rho' \circ \rho'^{-1} \circ \pi' : S' \setminus D' \to S \setminus D.
\]
Moreover, if \(S'\) has a GSS, then \(S\) has a GSS as well.

Proof: We have
\[
(K_S \otimes L)^\otimes m = \mathcal{O}(-D_m).
\]
Let \(X\) be the total space of the line bundle \(K^{-1} \otimes L^{-1}\). We choose an open trivialisation covering \(U = (U_i)\) for \(K\) and \(L\) with local coordinates \((z_i^1, z_j^2)\), defining cocycles \((k_{ij})\) and \((g_{ij})\) of \(K\) and \(L\) such that \(D_m \cap U_i = \{f_i = 0\}\) and
\[
k_{ij}^m g_{ij}^m \frac{f_i}{f_j} = 1.
\]
If \(\zeta_i\) is the fiber variable of \(X\) over \(U_i\), the equations \(\zeta_i^m = f_i(z)\) fit together and define an analytic subspace \(Z \subset X\). It is easy to see that \(\mathbb{U}_m\) acts holomorphically and effectively on \(Z\) and that \(Z/\mathbb{U}_m = S\). Let \(\pi : Z \to S\) be the projection on \(S\). The ramified covering \((Z, \pi, S)\) is branched exactly over \(\text{supp}(D_m)\). The local meromorphic 2-forms
\[
\omega_i = \frac{dz_i^1 \wedge dz_j^2}{\zeta_i}
\]
yield a twisted meromorphic 2-form \(\omega\) on \(X\), hence on \(Z\), for
\[
\omega_i = \frac{dz_i^1 \wedge dz_j^2}{\zeta_i} = \frac{k_{ij}^{-1} dz_1^j \wedge dz_2^j}{(k_{ij} g_{ij})^{-1} \zeta_j} = g_{ij} \frac{dz_1^j \wedge dz_2^j}{\zeta_j} = g_{ij} \omega_j.
\]
Now, let \((T, \rho, Z)\) be the minimal desingularization of \(Z\). This includes of course the normalization of \(Z\). Notice that the normalization is connected by the minimality of \(m\). Set \(H = \rho^* L\), then \(\tau = \rho^* \omega\) is a twisted meromorphic 2-form on \(T\) which does not vanish and has a non trivial polar divisor \(E\) and \(K_T \otimes H = \mathcal{O}(-E)\). Finally, in order to obtain \(S'\), we contract all exceptional curves of the first kind. It is clear that the index \(S'\) is one. As before \(\mathbb{U}_m\) acts holomorphically on \(S'\) and \(S' \setminus D'\) is a covering manifold of \(S \setminus D\).

The quotient \(Z' = S'/\mathbb{U}_m\) is a normal surface. The desingularization \((T', \rho', Z')\) yields \(S\) after contraction of the exceptional curves of the first kind. Since \(S\) has no non-constant meromorphic functions, the same holds for \(S'\). Applying then the classification of Kodaira, \(S'\) is a K3 surface, a torus or a surface of class \(\text{VII}_0\). The first two cases have a trivial canonical bundle, hence \(S'\) belongs to class \(\text{VII}_0\). Moreover \(b_2(S') > 0\), because it contains a cohomologically non trivial divisor.

We shall now show that if \(S\) has a GSS, then \(S'\) will also have this property. We may choose a GSS \(S^3 \subset V \subset \mathbb{C}^2 \setminus \{0\}\) which after embedding in \(S\) cuts only one
curve $C$ of our surface $S$. For a suitable choice of global coordinates $(z_1, z_2)$ on $V$, the intersection $C \cap V$ is given by the equation $z_1 = 0$. Moreover $K_S|_V$ and $L|_V$ are trivial on $V$. Let $\zeta$ be the fiber coordinate of $X$ over $V$. Suppose $D_m \cap V = n(C \cap V)$. The pull-back $\pi^{-1}(S^3)$ of the sphere $S^3$ to $Z$ is given by the equations

$$|z_1|^2 + |z_2|^2 = 1, \quad z_1^n = \zeta^m.$$ 

Let $d : = \text{gcd}(n, m)$ and $n' = n/d$, $m' = m/d$. Then there are $d$ irreducible components around $\pi^{-1}(S^3)$ in $Z$ which will become disjoint after normalization. We denote by $\Sigma_1, \ldots, \Sigma_d$ the corresponding components of $\pi^{-1}(S^3)$. Let us choose $\Sigma_1$ with equations

$$|z_1|^2 + |z_2|^2 = 1, \quad z_1^{n'} = \zeta^{m'}.$$ 

This component is normalized by the map

$$(t, z_2) \mapsto (tm', z_2, tm').$$

The pull-back $\Sigma_1'$ of $\Sigma_1$ to $T$ will be given in these coordinates by the equation

$$|t|^{2m'} + |z_2|^2 = 1.$$ 

We check now that $T \setminus \Sigma_1'$ is connected. Let $P, Q \in T \setminus \Sigma_1'$ two points in a neighborhood of $\Sigma_1' \cap \rho^{-1}(D)$ which find themselves on different sides of $\Sigma_1'$. Their projections $\pi(\rho(P)), \pi(\rho(Q))$ on $S$ may be connected by a path avoiding $S^3$ and $D$. We may lift this path to a path $\gamma$ in $Z$ which connects $\rho(P)$ to some $Q' \in \pi^{-1}(\pi(\rho(Q)))$. In order to ensure that $Q'$ and $\rho(Q)$ coincide, it is enough to let the initial path in $S$ turn the needed number of times around the components of $D$. By further lifting $\gamma$ to $T$ we get the looked for connectedness. For $m' > 1$,

$$\Sigma_1' = \{(t, z) \in \mathbb{C}^2 \mid |t|^{2m'} + |z_2|^2 = 1\}$$

is no longer a sphere, but remains the border of a bounded Stein domain of $\mathbb{C}^2$. Noticing that $\Sigma_1'$ may be approximated by a strictly pseudoconvex hypersurface (for example of equation $\epsilon|t|^2 + |t|^{2m'} + |z_2|^2 = 1$) it is possible to use the same arguments as in [2] in order to get a contracting holomorphic germ and hence a GSS on $S'$.

In particular, when $S$ has a GSS, $S'$ has exactly $b_2(S') = -K_Z^2$ rational curves. But since for a special surface $S$, the dual graph of the curves is the same as for some surface with a GSS and since the intersection matrix of the curves of $S'$ depends only on this graph, we get that our $S'$ also has $b_2(S') = -K_Z^2$, rational curves. Thus $S'$ is special under the weaker condition that $S$ is special (of intermediate type). Finally, it is not difficult to show (cf. [4]), that quotients of GSS surfaces by the actions of finite cyclic groups of automorphisms remain GSS surfaces. Thus if $S'$ has a GSS, then $S$ will also have one.

2. Existence of a twisted logarithmic 1-form

In this section we shall consider a special surface $S$ of intermediate type and prove that it always admits a twisted logarithmic 1-form. Let $D = D_{\text{max}} = \sum_{i=1}^n C_i$ be the maximal reduced divisor of $S$. 

\[\square\]
Lemma 2.1. If $S$ has index $m(S) = 1$, then the numerically anticanonical divisor satisfies $D_{-K} > D$. In particular, for every flat line bundle $L$ on $S$, $H^0(S, K_S \otimes L) = 0$.

Proof: Suppose that $D_{-K}$ does not contain all the curves of $S$. Since by [16] the maximal divisor $D$ is connected, there will exist an irreducible curve $C$ which is not contained in $D_{-K}$ but such that $\text{supp}(D_{-K}) \cap C \neq \emptyset$. Then $K_S.C = -D_{-K}.C < 0$ which gives a contradiction to the adjunction formula. Thus $D_{-K} \geq D$. But the equality $D_{-K} = D$ would imply $D^2 = -b_2(S)$ which may happen only on Inoue-Hirzebruch surfaces by [16]. But Inoue-Hirzebruch surfaces are not of intermediate type, so in our case we have $D_{-K} > D$. \hfill \Box

Lemma 2.2. For every $L \in \text{Pic}^0(S)$ we have $\Gamma(S, \Omega^1 \otimes L) = 0$.

Proof: We may suppose $L \neq 0$. Using the exact sequence of sheaves

$$0 \to d\mathcal{O}(L) \to \Omega^1 \otimes L \to \Omega^2 \otimes L \to 0$$

we get

$$\Gamma(\Omega^1 \otimes L) = \Gamma(d\mathcal{O}(L)).$$

Take now $\omega \in \Gamma(d\mathcal{O}(L))$ and denote by $\tilde{\omega}$ its pull-back on the universal covering $\tilde{S}$ of $S$. Then $g^*\tilde{\omega} = \lambda \tilde{\omega}$ where $g$ is a generator of $\pi_1(S) \simeq \mathbb{Z}$ and $\lambda$ is the twisting factor which corresponds to $L \in \text{Pic}^0(S) \simeq \mathbb{C}^*$. Let $f$ be a primitive of $\tilde{\omega}$ and $c \in \mathbb{C}$ such that $f \circ g = \lambda f + c$. Replacing $f$ by $h := f + \frac{c}{\lambda - 1}$, we get $h \circ g = \lambda h$ which means that $h$ induces a section in $\Gamma(S, \mathcal{O}(L))$. By our assumption on $S$, $h$ has then to be the zero section and thus $\omega = 0$.

Lemma 2.3. A non-trivial twisted logarithmic 1-form on $S$ is always closed and has poles along each curve of $S$.

Proof: Let $0 \neq \omega \in \Gamma(S, \Omega^1(\log D) \otimes L)$ for a flat line bundle $L$. Then $d\omega \in \Gamma(S, \Omega^2(D) \otimes L)$. If $d\omega \neq 0$, then its associated divisor $\Gamma$ satisfies $0 \leq -\Gamma \leq D$ which contradicts Lemma 2.1. Thus $d\omega = 0$. By Lemma 2.2 the pole divisor $D_\infty$ of $\omega$ is non-trivial. Then $D_\infty$ must contain the cycle of rational curves of $S$, otherwise one could write $\omega$ as a non-trivial logarithmic 1-form in a neighborhood $V$ of $D_\infty$ and since the dual graph of $D_\infty$ would be contractible, $\omega$ would be holomorphic on $V$ by [17]. So let now $C_1$ be an irreducible component of $D_\infty$ and suppose there exists a rational curve $C_2$ not contained in $D_\infty$ such that $0 \neq C_1 \cap C_2 = \{p\}$. Choose coordinates $(z_1, z_2)$ locally around $p$ such that $C_i = \{z_i = 0\}, i = 1, 2$, and write

$$\omega = \alpha_1 \frac{dz_1}{z_1} + \alpha_2 dz_2$$

in these coordinates. Since

$$0 = d\omega = \frac{\partial \alpha_1}{\partial z_2} dz_2 \wedge \frac{dz_1}{z_1} + \frac{\partial \alpha_2}{\partial z_1} dz_1 \wedge dz_2,$$

we see that $\alpha_1$ must have the form

$$\alpha_1(z_1, z_2) = \beta(z_1) + z_1 z_2 \gamma(z_1, z_2).$$
If \( \beta(0) \neq 0 \), the restriction of \( \omega \) to \( C_2 \) would have as only pole a simple pole in \( p \), which is impossible. Therefore \( \beta(0) = 0 \) and
\[
\omega = \left( \frac{\beta}{z_1} + z_2 \gamma \right) dz_1 + \alpha z_2
\]
has no pole along \( C_1 \), which gives a contradiction. Thus \( D_{\infty} = D \). \( \square \)

**Lemma 2.4.** Two logarithmic 1-forms \( \omega_1, \omega_2 \) on \( S \) twisted by flat line bundles \( L_1, L_2 \) are necessarily linearly dependent.

**Proof:** If \( \omega_1, \omega_2 \) were not linearly dependent, their exterior product \( \omega_1 \wedge \omega_2 \in \Gamma(\mathcal{O}_S, \Omega^2(\log D) \otimes L_1 \otimes L_2) \) would be non-identically zero. This contradicts Lemma 2.4. \( \square \)

**Lemma 2.5.** If \( L \) is a flat line bundle on \( S \) and \( \Gamma(\mathcal{O}_S, \Omega^1(\log D) \otimes L^{-1}) = 0 \), then the morphism
\[
H^2(\mathcal{O}_S(L)) \to H^2(\mathcal{O}_S(L))
\]
induced by restriction is bijective.

**Proof:** We may suppose \( L \) to be non-trivial, since otherwise the conclusion holds by our hypotheses on \( S \).

The long exact cohomology sequence of the diagram
\[
\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{C}_D(L) & \rightarrow & \mathcal{O}_D(L) & \rightarrow & \bigoplus_{i=1}^n \Omega_{C_i} \otimes L & \rightarrow & 0 \\
& & \uparrow \text{restr} & & \uparrow \text{restr} & & \uparrow \text{restr} & & \\
0 & \rightarrow & \mathcal{C}_S(L) & \rightarrow & \mathcal{O}_S(L) & \rightarrow & d\mathcal{O}_S(L) & \rightarrow & 0 \\
\end{array}
\]
gives
\[
\begin{array}{cccccc}
H^1(\mathcal{O}_D(L)) & \rightarrow & H^1(\bigoplus_{i=1}^n \Omega_{C_i} \otimes L) & \rightarrow & H^2(\mathcal{C}_D(L)) & \rightarrow & H^2(\mathcal{O}_D(L)) \\
H^1(\mathcal{O}_S(L)) & \rightarrow & H^1(d\mathcal{O}_S(L)) & \rightarrow & H^2(\mathcal{C}_S(L)) & \rightarrow & H^2(\mathcal{O}_S(L)) \\
\end{array}
\]
Using the identities
\[
H^2(\mathcal{O}_S(L)) = 0, \quad H^0(\mathcal{O}_S(L)) = 0, \quad H^2(\mathcal{O}_D(L)) = 0, \quad H^0(\mathcal{O}_D(L)) = 0
\]
and the theorem of Riemann-Roch, we get \( H^1(\mathcal{O}_S(L)) = 0 \) and \( H^1(\mathcal{O}_D(L)) = 0 \). Thus our task comes to showing the bijectivity of the morphism
\[
H^1(d\mathcal{O}_S(L)) \rightarrow H^1(\bigoplus_{i=1}^n \Omega_{C_i} \otimes L).
\]

On the other hand the commutative triangle
\[
\begin{array}{ccc}
\bigoplus_{i=1}^n \Omega_{C_i} \otimes L & \rightarrow & \Omega^1(L), \\
\uparrow \text{restr} & & \uparrow \text{restr} \\
d\mathcal{O}_S(L) & \rightarrow & \\
\end{array}
\]
the long exact cohomology sequence associated to

\[ 0 \to d\mathcal{O}_S(L) \to \Omega^1_S(L) \to \Omega^2_S(L) \to 0 \]

and the fact that \( H^0(\Omega^2_S(L)) = 0 \), allow us to get the commutative triangle

\[
\begin{array}{ccc}
H^1(\bigoplus_{i=1}^n \Omega_{C_i} \otimes L) & \to & H^1(\mathcal{O}_S(L)) \\
\downarrow & & \downarrow \\
H^1(\Omega^1(L)) & \to & H^1(\Omega^1(L))
\end{array}
\]

where the horizontal arrow is an isomorphism. Thus, we have only to prove that the morphism

\[ (3) \quad H^1(\Omega^1(L)) \to H^1(\bigoplus_{i=1}^n \Omega_{C_i} \otimes L) \]

is bijective. We examine the dimensions first.

\[
\dim H^1(\bigoplus_{i=1}^n \Omega_{C_i} \otimes L) = \sum_{i=1}^n \dim H^1(\Omega_{C_i} \otimes L) = n
\]

and

\[
\dim H^1(\Omega^1(L)) = -\chi(\Omega^1(L)) + h^0(\Omega^1(L)) + h^2(\Omega^1(L)) = -\chi(\Omega^1(L)) = -\chi(\Omega^1) = b_2,
\]

since we know that \( h^0(\Omega^1(L)) = h^0(\Omega^1(L^{-1})) = 0 \). By assumption \( b_2 = n \), so it is enough to show only the surjectivity of the morphism \((3)\). In order to do so, let us compute the kernel \( \mathcal{N} \) of the surjective morphism

\[ \Omega^1_S \to \bigoplus_{i=1}^n \Omega_{C_i}. \]

Locally around a point \( C_1 \cap C_2 \), where \( C_i = \{ z_i = 0 \} \), this morphism is given by

\[ f_1 dz_1 + f_2 dz_2 \mapsto (f_1(z_1,0)dz_1, f_2(0,z_2)dz_2). \]

Thus a section of the kernel must have the form \( z_2g_1dz_1 + z_1g_2dz_2 \) where \( g_1, g_2 \) are holomorphic functions. Therefore we get the duality

\[ \mathcal{N} \otimes \Omega^1_S(\log D) \xrightarrow{\wedge} \Omega^2_S \]

\[ (z_2g_1dz_1 + z_1g_2dz_2, \frac{h_1}{z_1}dz_1 + \frac{h_2}{z_2}dz_2) \xrightarrow{(g_1h_2 - g_2h_1)dz_1 \wedge dz_2}, \]

proving that

\[ \mathcal{N} \simeq \Omega^1_S(\log D) \wedge \Omega^2_S. \]

In order to finish the proof we consider now the long exact cohomology sequence of
and use the fact that $H^2(N \otimes L) = H^0(\Omega^1_S(\log D) \otimes L^{-1})^* = 0$ which is ensured by hypothesis. □

**Theorem 2.6.** Let $S$ be a special surface of intermediate type and $k := k(S)$, then there is a choice of a generator of $\pi_1(S) \simeq \mathbb{Z}$ such that $S$ admits a closed logarithmic 1-form twisted by the flat line bundle $L^k$.

**Proof:** By [16] there exists a surface $S'$ with a GSS such that the dual graph of the maximal reduced divisor $D'$ of $S'$ coincides with that of $D$. By [3] and [2], Thm. 1.2.24, we may choose the generator of $\pi_1(S') \simeq \pi_1(D')$ such that $\Gamma(S', \Omega^1(\log D') \otimes L^k) \neq 0$. We further fix the generator of $\pi_1(S) \simeq \pi_1(D) \simeq \pi_1(S') \simeq \pi_1(D')$ to be the same as above. We may now suppose that $\Gamma(S, \Omega^1(\log D) \otimes L^{1/k}) = 0$, otherwise we change the generator of $\pi_1(S)$. We shall identify sections of sheaves on $S$ twisted by $L^k$ with sections of the pullback-sheaves on the universal cover $\tilde{S}$ of $S$ which respect the representation $\rho : \pi_1(S) \to \mathbb{C}$ defining $L^k$. We start with the exact sequence

$$0 \to d\mathcal{O}(L^k) \to d\mathcal{O}(\log D) \otimes L^k \to \bigoplus_{i=1}^n \mathcal{O}_{\tilde{C}_i} \otimes L^k \to 0 \quad (4)$$

where $\tilde{C}_i$ are the normalisations of the curves $C_i \subset S$ and with a certain element

$$a \in \Gamma(S, \bigoplus_{i=1}^n \mathcal{O}_{\tilde{C}_i} \otimes L^k) = \Gamma(\tilde{S}, \bigoplus_{i=-\infty}^\infty \mathcal{O}_{\tilde{C}_i})^\rho$$

to be defined below. Seen as an element in

$$\Gamma(\tilde{S}, \bigoplus_{i=-\infty}^\infty \mathcal{O}_{\tilde{C}_i}) \simeq \bigoplus_{i=-\infty}^\infty \Gamma(\tilde{S}, \mathcal{O}_{\tilde{C}_i}),$$

$a$ becomes a vector in $\mathbb{C}^\mathbb{Z}$ with the property $a_{i+n} = ka_i$ for all $i \in \mathbb{Z}$. Choose a non-trivial element

$$\omega' \in \Gamma(S', \Omega^1(\log D') \otimes L^k)$$

and put

$$a_i := \int_{\gamma_i} \omega',$$

where $\gamma_i$ is a small path around $C_i' \subset \tilde{D}'$ and we denote again by $\omega'$ the pull-back of $\omega'$ to the universal cover $\tilde{S}'$ of $S'$. The Camacho-Sad formula for the foliation defined by $\tilde{\omega}'$ gives

$$C_i^2 = - \sum_{i \neq j, C_i \cap C_j \neq \emptyset} \frac{a_i}{a_j}.$$
see \[\text{[7]}\]. Moreover we have seen in \[\text{[7]}\] that \(\omega'\) may be chosen such that \(a \in \mathbb{Z}[1/k]^{\mathbb{Z}}\). For such a choice let further \(U_j\) be small neighborhoods of the curves \(C_j\) on \(\tilde{S}\) and consider divisors \(D_j\) in these neighborhoods of the form
\[
D_j := k'^n (a_j C_j + \sum_{C_i \not\sim C_j \not=} a_i C_i),
\]
for some \(n \in \mathbb{N}\) which is sufficiently large. Since \(D_j \cdot C_j = 0\), \(D_j\) is the zero divisor of some holomorphic function \(f_j \in \mathcal{O}(U_j)\). Put then
\[
\omega_j := k'^n \frac{df_j}{f_j} \in \Gamma(U_j, d\mathcal{O}(\log D)).
\]
One may choose the functions \(f_j \in \mathcal{O}(U_j)\) such that \(g^* \omega_{j+n} = k \omega_j\) holds for all \(j \in \mathbb{Z}\); here \(g \in \pi_1(S)\) denotes the “positive” generator of \(\pi_1(S)\). There exist local coordinate functions \(z_j, z_i\) which define the curves \(C_j, C_i\) such that around \(C_i \cap C_j\) we have
\[
\omega_j = a_j \frac{dz_j}{z_j} + a_i \frac{dz_i}{z_i}
\]
and around points of \(C_j\) away from any other compact curve
\[
\omega_j = a_j \frac{dz_j}{z_j}.
\]
We look now at the situation on \(U_i \cap U_j\). Here \(\omega_j - \omega_i\) is a closed holomorphic 1-form and thus exact if \(U_i \cap U_j\) is simply connected, which we shall always suppose; in fact, \(\omega_j - \omega_i\) has the form
\[
a_j \frac{dg_j}{g_j} + a_i \frac{dg_i}{g_i},
\]
with \(g_i, g_j \in \mathcal{O}^*(U_i \cap U_j)\).
Let \(\mathcal{U}\) be the covering of \(\tilde{S}\) which consists of the open sets \(U_i\) and of \(\tilde{S} \setminus \tilde{D}\). We set \(\omega = 0\) on \(\tilde{S} \setminus \tilde{D}\). Then we get a cocycle \((\omega_j - \omega_i)_{j,i} \in H^1(\mathcal{U}, d\mathcal{O}(L^k))\) which represents the image of \(a\) through the canonical connecting homomorphism associated to the sequence \(4\). We follow this image further through the isomorphism
\[
H^1(d\mathcal{O}(L^k)) \cong H^2(\mathbb{C}(L^k))
\]
which comes from the short exact sequence
\[
0 \to \mathbb{C}(L^k) \to \mathcal{O}(L^k) \to d\mathcal{O}(L^k) \to 0.
\]
In order to do this we pass to a finer covering \(\mathcal{V} = (V_\nu)_\nu\) which has the property that the sets \(V_{\mu\nu} := V_\mu \cap V_\nu\) are simply connected. Let \(\gamma\) be the refinement map and set \(\omega_\nu := \omega_{\gamma(\nu)}|_{V_\nu}\) for \(V_\nu \subset U_{\gamma(\nu)}\). Since \(\omega_\nu - \omega_\mu\) is exact on \(V_{\mu\nu}\), we may choose primitives \(f_{\nu\mu}\). This is done such that
\[
f_{\nu\mu} = 0 \quad \text{if} \quad \gamma(\mu) = \gamma(\nu) \quad \text{and}
\]
\[
f_{\nu\mu} = f_{\gamma(\nu)\gamma(\mu)} \quad \text{if} \quad \gamma(\mu) \neq \gamma(\nu) \quad \text{and} \quad \gamma(\mu), \gamma(\nu) \in \mathbb{Z}.
\]
The cocycle 

\[ ((f_{\mu \nu} - f_{\lambda \nu} + f_{\lambda \mu})|_{V_{\lambda \mu \nu}})_{\lambda \mu \nu} \]

is the looked for element in \( H^2(V, \mathbb{C}(L^k)) \). But we remark now that the trace of this cocycle on \( \tilde{D} \) is zero and thus its image in \( H^2(D \cap V, \mathbb{C}(L^k)) \) by the restriction morphism will be zero as well. One sees this by checking the different possibilities for \( \lambda, \mu, \nu \), such that \( V_{\lambda \mu \nu} \cap \tilde{D} \neq \emptyset \):

- when \( \gamma(\lambda) = \gamma(\mu) = \gamma(\nu) \), all primitives are zero
- when \( \gamma(\lambda) = \gamma(\mu) \neq \gamma(\nu) \), one gets again \( f_{\mu \nu} - f_{\lambda \nu} + f_{\lambda \mu}|_{V_{\lambda \mu \nu}} = 0 \).

Notice that at least two elements among \( \gamma(\lambda), \gamma(\mu), \gamma(\nu) \) must be equal.

Now Lemma 2.5 shows that the class of our cocycle in \( H^2(S, \mathbb{C}(L^k)) \) vanishes. Hence there exists a non-trivial element \( \omega \) in \( \Gamma(S, d\mathcal{O}(\log D) \otimes L^k) \) which maps onto \( a \in \Gamma(S, \bigoplus_{i=1}^{\alpha} \mathcal{O}_{\tilde{C}_i} \otimes L^k) \).

**Corollary 2.7.** The foliation associated to \( \omega \in \Gamma(S, \Omega^1(\log D) \otimes L^k) \) has simple singularities and rational Camacho-Sad indices with respect to the curves of \( S \).

**Proof:** If \( \omega \) is obtained as in the main part of the proof of Theorem 2.6 then the associated Camacho-Sad indices are \( \frac{\omega}{\partial g_j} \), i.e. the same as those associated to \( \omega' \) on \( S' \), and thus rational. But if \( \omega \in \Gamma(S, \Omega^1(\log D) \otimes L^1/k) \), where the orientation on \( \pi_1(S) \) and \( \pi_1(S') \) is chosen to be the same, we have to reconsider the Camacho-Sad equations of the components \( C_i \) of \( D \). Let \( \mathcal{F} \) denote the foliation associated to \( \omega \). Lemma 2.3 implies that the irreducible components of \( D \) are invariant for \( \mathcal{F} \). Let \( C_1, C_2 \) be two such components which intersect at \( p \) and \( (z_1, z_2) \) local coordinate functions around \( p \) such that \( C_i = \{ z_i = 0 \} \) for \( i = 1, 2 \). We may write \( \omega \) around \( p \) as

\[
\omega = g_1 \frac{dz_1}{z_1} + g_2 \frac{dz_2}{z_2}
\]

where \( g_1, g_2 \) are holomorphic functions in \( z_1, z_2 \). Consider now small paths \( \gamma_i \) turning around \( C_i \), contained say in the local curve \( z_{3-i} = c_{3-i} \), for two constants \( c_1, c_2 \in \mathbb{C} \). Since \( \omega \) is closed the integrals

\[
\int_{\gamma_1} \omega = 2\pi i \text{Res}_{z_1=0}(\omega|_{z_2=c_2}) = 2\pi i g_1(0, c_2)
\]

\[
\int_{\gamma_2} \omega = 2\pi i \text{Res}_{z_2=0}(\omega|_{z_1=c_1}) = 2\pi i g_2(c_1, 0)
\]

are independent of \( c_1, c_2 \). Moreover, since \( \omega \) has true poles of order one along \( C_1 \) and \( C_2 \), both integrals are non-zero; in particular \( g_1(0, 0) \neq 0 \neq g_2(0, 0) \) and \( p \) is a simple singularity for \( \mathcal{F} \). Now the foliation \( \mathcal{F} \) is defined locally around \( p \) by the kernel of the form \( z_2 g_1 dz_1 + z_1 g_2 dz_2 \) and the Camacho-Sad index of \( \mathcal{F} \) with respect to \( C_2 \) is by definition

\[
\text{CS}(\mathcal{F}, C_2, p) := \text{Res}_{z_1=0} \left( \frac{\partial}{\partial z_2} \left( \frac{z_2 g_1}{z_1 g_2} \right) \right) |_{C_2} = \text{Res}_{z_1=0} \left( - \frac{g_1(z_1, 0)}{z_1 g_2(z_1, 0)} \right) = - \frac{g_1(0, 0)}{g_2(0, 0)} = - \int_{\gamma_2} \frac{\omega}{\gamma_1} \int_{\gamma_1} \omega
\]
and similarly for $C_1$. A consequence of this is that all Camacho-Sad indices of $\mathcal{F}$ along the curves of $S$ are non-zero.

Now recall that the maximal divisor $D$ of $S$ consists of a cycle $\sum_{i=1}^{m} C_i$ of rational curves to which a non zero number of trees of rational curves is attached. It is easy to see that the Camacho-Sad indices associated to the curves of the trees are all rational. Indeed, if $B_{i,1}, \ldots, B_{i,m_i}$ is the tree with root $C_i$, and if $-B_{i,j}^2 = b_{i,j}$, we get:

$$CS(\mathcal{F}, B_{i,m_i}, B_{i,m_i} \cap B_{i,m_i-1}) = -b_{i,m_i}$$

for the top,

$$CS(\mathcal{F}, B_{i,m_i-1}, B_{i,m_i} \cap B_{i,m_i-1}) = -\frac{1}{b_{i,m_i}},$$

$$CS(\mathcal{F}, B_{i,m_i-1}, B_{i,m_i-1} \cap B_{i,m_i-2}) = -b_{i,m_i-1} + \frac{1}{b_{i,m_i}}$$

and so on. Setting as in [7] $b_i := -CS(\mathcal{F}, C_i, C_i \cap B_{i,1}), d_i := -C_i^2 - b_i$ and $\alpha_i := -CS(\mathcal{F}, C_i, C_i-1 \cap C_i),$ $i = 1, \ldots, m$, we get the equations

$$\alpha_i + \frac{1}{\alpha_{i+1}} = d_i, \text{ for } 1 \leq i \leq m - 1$$

and

$$\alpha_m + \frac{1}{\alpha_1} = d_m.$$

Each $\alpha_i$ is the solution of a quadratic equation with rational coefficients and since we know already that a rational solution exists by working with $\omega'$ on $S'$, the other solution has to be rational as well.

Corollary 2.8. A special surface of intermediate type of index 1 possesses a non-trivial holomorphic vector field $\theta$ twisted by some flat line bundle and its vanishing divisor $D_\theta$ contains all the curves of $D$ except perhaps the summits of the trees. In particular $\theta$ vanishes on all the curves of the cycle of $D$.

Proof: Let $S$ be a special surface of intermediate type and let $D_{-K}$ be the (unique) numerical anticanonical divisor on $S$. This means that there exists a torsion factor $\kappa \in \mathbb{C}^*$ such that

$$\mathcal{O}(D_{-K}) = K^{-1}_S \otimes L^\kappa.$$

Let now $0 \neq \omega \in \Gamma(S, \Omega^1(\log D) \otimes L^k)$ and $Z$ the subspace of the intersection points of the curves of $S$. By Corollary 2.4 this is exactly the space of singularities of the associated foliation to $\omega$. Thus $\omega$ induces an exact sequence

$$0 \to \mathcal{O}(-D) \otimes L^{1/k} \to \Omega^1 \to L^{\kappa} \otimes \mathcal{J}_Z \otimes \mathcal{O}(D - D_{-K}) \to 0$$

and by duality we get a non-trivial section

$$\theta \in \Gamma(S, \Theta_S \otimes L^{\kappa})$$

vanishing on $D_\theta := D_{-K} - D$. We know by Lemma 2.4 that $D_\theta > 0$. Let now $C_1 \not\subset \text{Supp}(D_\theta)$ such that $C_1$ intersects an irreducible curve $C_2 \subset D_\theta$. Since $\theta$ defines the same foliation as $\omega$ the curve $C_1$ is $\theta$-invariant. On the other hand since $C_1 \cap C_2$ is a singularity of this foliation and $\theta$ vanishes on $C_2$ the vanishing order of the restriction of $\theta$ to $C_1$ is at least two. But then since $C_1$ is rational this order
is exactly two and $C_1$ cannot intersect another curve of $D$. Thus $C_1$ is the top of a tree of rational curves. This implies our statement. □

3. THE UNIVERSAL COVERING OF THE COMPLEMENT OF THE CURVES

From now on we shall consider a special surface $S$ of intermediate type which admits a numerical anticanonical divisor. We have seen that in this case $S$ possesses a non-trivial twisted logarithmic 1-form $\omega$ with twisting factor $k = k(S)$ and a non-trivial twisted holomorphic vector field $\theta$ with twisting factor, say $\lambda \in \mathbb{C}^*$. The case $\lambda = 1$, i.e. $\theta$ is a holomorphic vector field on $S$, was considered and completely understood in [6] and [7]. In this section we prove that in the general case, as in case $\lambda = 1$, the universal covering of $S \setminus D$ is isomorphic to $\mathbb{H} \times \mathbb{C}$, where $\mathbb{H}$ denotes the complex half-plane.

Let $U$ be a small open neighborhood of $D$, such that $D$ is a deformation retract of $U$. We have

$$\pi_1(U) = \pi_1(D) = \pi_1(S) = \mathbb{Z}$$

and we denote as before by $g$ a generator of this group. There is a fundamental domain $U_0$ for the action of $\mathbb{Z}$ in the inverse image $\tilde{U}$ of $U$ in the universal cover $\tilde{S}$ of $S$, such that the border of $U_0$ in $\tilde{U}$ cuts $\tilde{D}$ in a component $C_0$ and in its translated $g(C_0)$ along a circle $S^1$. Set $Y_0 := \bigcup_{\nu \geq 0} g^{\nu}(U_0)$. We keep the notation $\omega$ for the logarithmic 1-form one gets on $\tilde{S}$.

**Lemma 3. 1.** There is a normalization of $\omega$ such that the representation

$$\rho : \pi_1(\tilde{S} \setminus \tilde{D}) \to \mathbb{C}$$

$$\gamma \mapsto \int_\gamma \omega$$

has as image $2\pi i \mathbb{Z}[\frac{1}{k}] \subset \mathbb{C}$. Furthermore, one can choose this normalization such that $\rho(\pi_1(Y_0 \setminus \tilde{D})) = 2\pi i \mathbb{Z}$.

**Proof:** Since $\tilde{S}$ is simply connected, the group $\pi_1(\tilde{S} \setminus \tilde{D})$ is generated by small paths around the irreducible components of $\tilde{D}$. Thus keeping the notations of the previous section, we see that $\rho(\pi_1(\tilde{S} \setminus \tilde{D}))$ is a $\mathbb{Z}[\frac{1}{k}]$-module generated by

$$2\pi i a_0 = \int_{\gamma_0} \omega, \ldots, 2\pi i a_{n-1} = \int_{\gamma_{n-1}} \omega,$$

where $\gamma_0, \ldots, \gamma_{n-1}$ are small paths around the curves $C_0, \ldots, C_{n-1}$ in $U_0$.

Using Corollary 2.7 and the way we computed the Camacho-Sad indices, we see that we can normalize the form $\omega$ such that the numbers $a_0, \ldots, a_{n-1}$ are non-zero integers with g.c.d. equal to 1; so $\rho(\pi_1(\tilde{S} \setminus \tilde{D}))$ is free of rank 1 as $\mathbb{Z}[\frac{1}{k}]$-module.

In a similar way the group $\rho(\pi_1(Y_0 \setminus \tilde{D}))$ is generated as a $\mathbb{Z}$-module by small paths around the irreducibles components of $\tilde{D}$ which meet $U_0$ and we get the announced result. □
In what follows we suppose $\omega$ to be normalized such that $\rho(\pi_1(Y_0 \setminus \tilde{D})) = 2\pi i \mathbb{Z}$. This is the same as to say that $a_0, \ldots, a_{n-1}$ are non-zero integers with g.c.d. equal to 1.

Let $A$ be a fundamental domain for the action of $\mathbb{Z}$ on $\tilde{S}$ and $X_0 := \bigcup_{j \geq 0} g^j(A)$. Translating by $g$, we may suppose that $\tilde{D} \cap X_0 \subset Y_0$. We remark that after such a translation we have

$$\rho(\pi_1((Y_0 \cup X_0) \setminus \tilde{D})) = \rho(\pi_1(Y_0 \setminus \tilde{D})) = 2\pi i \mathbb{Z}.$$ 

Fix $z_0 \in U_0$. We define a holomorphic function $f$ on $(Y_0 \cup X_0) \setminus \tilde{D}$ by

$$f(z) = \exp\left(\int_{z_0}^z \omega + \frac{1}{k-1} \int_{z_0}^{g(z_0)} \omega\right).$$

One verifies easily that $f$ is well defined and that

$$(\dagger) \quad f(g(z)) = f^k(z)$$

for $z \in (Y_0 \cup X_0) \setminus \tilde{D}$.

Let $C$ be the smooth part in $\tilde{D}$ of an irreducible component of $\tilde{D} \cap Y_0$. Since $\omega$ is a closed logarithmic 1-form, an easy computation shows that one can extend $f$ meromorphically across $C$ such that $\tilde{C}$ belongs to the zero or to the polar set of the extension of $f$. By replacing $\omega$ with $-\omega$ if necessary, one finds at least one preimage on $\tilde{S}$ of a component of the cycle of $D$ in the zero-set of $f$. This means that the connected components of the polar set of $f$ are exceptional divisors in $Y_0$. Thus the polar set is empty, see \cite{17}, and $f$ vanishes on $\tilde{D} \cap Y_0$. In particular, we see that the integers $a_0, \ldots, a_{n-1}$ are positive.

**Lemma 3. 2.** $|f(z)| < 1$ for any $z \in Y_0 \cap X_0$.

*Proof:* Remark first that one can extend the function $|f|$ to the whole of $\tilde{S}$. The extended function is still denoted by $|f|$. Then the statement of the Lemma can be rephrased by saying that the image of $|f|$ is the interval $[0, 1[$. Suppose now that this is not so. Then $|f|$ has as image $[0, \infty[$ since $|f| \circ g = |f|^k$. We consider the real hypersurface $\tilde{H} := |f|^{-1}(1)$, which is $\pi_1(S)$-invariant and thus descends to a compact real hypersurface $H$ on $S$. Obviously, $H$ is a (compact) leaf of the foliation defined by $\Re \omega$. Remark that the morphism $\pi_1(H) \to \pi_1(S)$ is non-trivial. Otherwise the connected components of $\tilde{H}$ would be compact and their intersection with $f^{-1}(1)$ would give compact analytic curves in the complement of $\tilde{D}$, which is absurd. By passing to a finite unramified covering of $S$, we may even suppose that $\pi_1(H) \to \pi_1(S)$ is surjective.

Next we prove that $\omega$ must have non-vanishing periods on $\tilde{H}$. If not, consider a $\pi_1(S)$-invariant neighborhood $\tilde{V}$ of $\tilde{H}$ which is the preimage of a neighborhood $V$ of $H$ in $\tilde{S}$ and on which $\omega$ has no periods; thus $\omega|_{\tilde{V}}$ is exact. We define the following holomorphic function on $\tilde{V}$:

$$h(z) := \int_{z_0}^z \omega + \frac{1}{k-1} \int_{z_0}^{g(z_0)} \omega.$$
where $z_0 \in \tilde{V}$ is a fixed base point, $z \in \tilde{V}$ and integration is done along paths in $\tilde{V}$. We have $h(g(z)) = kh(z)$ for all $z \in \tilde{V}$. The function $h$ does not take the value 0, because the set $\{h = 0\}$ would then be a $\pi_1(S)$-invariant analytic curve giving rise to a compact curve in the complement of $D$ in $S$, which is absurd. Hence one may consider the 1-form $d(\log|h|)$ which is closed and descends to a closed non-twisted form on $\tilde{V}$. This form obviously defines the same foliation as $\Re\omega$. But this means that this foliation has trivial holonomy. This implies that the leaves near $H$ are also compact. In particular they are contained in $V$. The inverse images of these leaves are leaves of the foliation defined by $|f|$ on $\tilde{S}$. On the other hand they are completely contained in $\tilde{V}$ and intersect all sets $g^\nu(\tilde{S} \setminus X_0)$, $\nu \in \mathbb{Z}$. But the relation $f(g(z)) = f^k(z)$ and the choice of $V$ imply that

$$\bigcap_{\nu \in \mathbb{Z}} |f|(g^\nu(\tilde{V} \setminus X_0)) = \{1\},$$

giving a contradiction.

Thus $\omega|_{\tilde{H}}$ has non-vanishing periods. The next point is to show that $f$ may be extended holomorphically to $U_1 := |f|^{-1}(1, \infty]$.

In order to do this we consider the sets $U_\alpha := |f|^{-1}(1, \infty]$ for $\alpha \geq 1$ and

$$M := \{\alpha > 1 | f \text{ admits a holomorphic extension to } U_\alpha\}.$$

Then $M$ is non-empty, since $U_\alpha \subset X_0$ for $\alpha > \max_{z \in \partial X_0} |f(z)|$. Furthermore, the set $M$ is closed in $]1, \infty[$ since $U_{\alpha_0} = \bigcup_{\alpha > \alpha_0} U_\alpha$ for each $\alpha_0$. It remains to check that $M$ is open as well. Let $\alpha \in M$. There exists some $\nu \in \mathbb{N}$ such that $U_\alpha \subset g^{-\nu}(X_0)$.

On $g^{-\nu}(X_0)$ one can extend $f^{k^\nu}$ holomorphically. Take now a finite open covering $V_1, \ldots, V_\mu$ of $(\partial U_\alpha) \setminus X_0$ such that each $V_i$ intersects $U_\alpha$ and that on $V_i$ a $k^\nu$-th root of $f^{k^\nu}$ is defined. Consider next on each $V_i$ that $k^\nu$-th root of $f^{k^\nu}$ which coincides with $f$ on $V_i \cap U_\alpha$. This gives an extension of $f$ to

$$U_\alpha \cup X_0 \cup \bigcup_{i=1}^\mu V_i.$$  

Remark that this set will contain some $U_\beta$ with $\beta < \alpha$.

We finish the proof of the Lemma by considering a path $\gamma \subset U_1$ such that $\int_\gamma \omega \neq 0$, which is possible since $\omega$ has non-vanishing periods on $H$. But for $\nu$ sufficiently large

$$\int_{g^{-\nu}\gamma} \omega = \frac{\int_\gamma \omega}{k^\nu},$$

cannot be a multiple of $2\pi i$, which is incompatible with the definition of $f$ on the whole of $U_1$. 

**Proposition 3.3.** The holomorphic vector field $\tilde{\theta} \in \Gamma(\tilde{S}, \tilde{\Theta}_{\tilde{S}})$ induced by $\theta$ is completely integrable on $\tilde{S}$. 


We consider the integrability of $\tilde{\theta}$ along a leaf $F$ of the foliation. It will be enough to show that one can find a local integration radius which is uniform for all $F$. We set $A_0 := X_0 \cup Y_0 \setminus g(X_0 \cup Y_0)$. The set $A_0$ intersects $\hat{D}$ along a curve $C_0$ and its translated by $g$, $C_n$. There is a constant $c \in [0, 1]$ such that $F \subset |f|^{-1}(c)$. Using the relation $|f| \circ g = |f|^k$ again and Lemma 3.2 we see that $\lim_{\nu \to \infty} |f| \circ g^\nu = 0$ uniformly on compact sets, hence there exists a $\nu_0 \in \mathbb{N}$ such that $F \cap \bigcup_{\nu \geq \nu_0} g^\nu(A_0) = \emptyset$. On the other side there exists a $\nu_1 \in \mathbb{N}$ such that $F \cap \bigcup_{\nu \geq \nu_1} g^{-\nu}(A_0) \subset \hat{U}$. By passing to a suitable translation we may assume that $\nu_1 = 0$. We may further assume that around $\hat{U} \cap \partial A_0$ we have coordinate functions $(z_0, z_1)$ such that $C_0 = \{z_0 = 0\}$,

$$\omega = a_0 \frac{dz_0}{z_0}$$

with $a_0 \in \mathbb{N}^*$ and

$$\tilde{\theta} = \alpha(z_0, z_1)z_0^s \frac{\partial}{\partial z_1}$$

with $s \in \mathbb{N}^*$ and $\alpha$ a nowhere vanishing holomorphic function. Here we have $f(z_0, z_1) = z_0^{a_0}$. Thus

$$(z_0, z_1) \in F \cap \partial\left(\bigcup_{\nu \geq 0} g^\nu(A_0)\right)$$

implies $|z_0|^{a_0} = c$. But since $F \cap \bigcup_{\nu \geq 0} g^\nu(A_0)$ is contained in the compact set $\bigcup_{\nu=0}^{\nu_0} g^\nu(A_0)$, the integration radius of $\tilde{\theta}$ at points of

$$F \cap \bigcup_{\nu \geq 0} g^\nu(A_0)$$

is at least as large as the integration radius of $\tilde{\theta}$ at

$$F \cap \partial\left(\bigcup_{\nu=0}^{\nu_0} g^\nu(A_0)\right)$$

and this is the minimal integration radius of $\alpha(z_0, z_1)z_0^s \frac{\partial}{\partial z_1}$ at points $(z_0, z_1)$ with $|z_0|^{a_0} = c$. Looking now at $\tilde{\theta}$ on

$$F \cap \partial\left(\bigcup_{\nu \geq \nu_0}^{\nu_0} g^\nu(A_0)\right)$$

for $r \in \mathbb{N}$, we see by applying $g^r$, that the integration radius at these points will be at least as large as the minimal integration radius of $\lambda^r \alpha(z_0, z_1)z_0^s \frac{\partial}{\partial z_1}$ at points $(z_0, z_1) \in F \cap \partial\left(\bigcup_{\nu \geq 0} g^\nu(A_0)\right)$ with $|z_0|^{a_0} = c^k$. But the sequence of velocities

$$|\lambda|^r c^{sk^{r/a_0}} \sup |\alpha|$$

is obviously bounded and thus there is an uniform integration radius for $\tilde{\theta}$ on $F$. $\Box$

The kernel ker $\rho$ defines a covering

$$\pi : X' \to \tilde{S} \setminus \hat{D}.$$  

One checks immediately that the $\mathbb{Z}$-action induced by $g$ on $\pi_1(\tilde{S} \setminus \hat{D})$ stabilizes ker $\rho$ and thus induces a $\mathbb{Z}$-action on $X'$. We denote again by $g$ a lift of $g$ on $X'$. 
Thus we get an action of the semi-direct product $\mathbb{Z} \rtimes \mathbb{Z}[\frac{1}{2}]$ on $X'$ whose quotient is $S \setminus \hat{D}$. Let $\omega' = \pi^*\omega$ and $\phi : X' \to \mathbb{C}$ be a primitive of $\omega'$ on $X'$, such that $\exp(\phi)$ and $f \circ \pi$ coincide on a connected component of the $\pi$-preimage of $(Y_0 \cup X_0) \setminus \hat{D}$. Since $f \circ g = k \phi$, the image of $\phi$ is invariant under the action of the multiplicative group $\{k^\nu | \nu \in \mathbb{Z}\}$; this image is also invariant under the action of the additive group $2\pi i \mathbb{Z}[\frac{1}{2}]$. Since $f$ takes its values in the unit disk by Lemma 3.2, we see now that $\phi(X')$ must coincide with the left half plane $\mathbb{H}_l := \{w \in \mathbb{C} | \Re(w) < 0\}$. The function $\phi : X' \to \mathbb{H}_l$ is a holomorphic surjection. The connected components of its fibers are leaves of the inverse image foliation induced by $\mathcal{F}$. For the proof of the next Proposition the reader is referred to [4], Prop. 2.2. Note that the Camacho-Sad indices of our foliation here, are positive integers, just as in [4].

**Proposition 3.4.** The fibers of $\phi$ are connected.

**Corollary 3.5.** The foliation defined by $\theta$ on $\hat{S} \setminus \hat{D}$ has no closed leaf.

*Proof:* If $F$ were a closed leaf on $\hat{S} \setminus \hat{D}$, its preimage $\pi^{-1}(F)$ in $X'$ would also be closed. But since $\rho_{\pi_1(F)}$ is trivial, the group $\mathbb{Z}[\frac{1}{2}]$ operates on $X'$ by permuting components of $\pi^{-1}(F)$. Using the preceeding Proposition, the non-discreteness of the $2\pi i \mathbb{Z}[\frac{1}{2}]$-orbits in $\mathbb{H}_l$ and the fact that $\phi$ is a submersion, we get a contradiction.

**Lemma 3.6.** The fibers of $\phi$ are isomorphic to $\mathbb{C}$.

*Proof:* Since $\tilde{\theta}$ is completely integrable, there is a holomorphic $\mathbb{C}$-action on $\tilde{S} \setminus \tilde{D}$. The fixed point set of a non-trivial element of $\mathbb{C}$ is a closed analytic subset of $\tilde{S} \setminus \tilde{D}$ which is a union of leaves of the foliation. By the preceeding Corollary such a fixed point set cannot have dimension 1. Hence it is either empty or the whole space $\tilde{S} \setminus \tilde{D}$. We must exclude the second case. In this situation the $\mathbb{C}$-action factorizes to a $\mathbb{C}_*^*$-action. Thus all fibers of $\phi$ are isomorphic to $\mathbb{C}_*^*$. Moreover the $\mathbb{C}_*^*$-action lifts to $X'$ making $\phi : X' \to \mathbb{H}_l$ into a principal $\mathbb{C}_*^*$-bundle. But such a bundle over $\mathbb{H}_l$ is always trivial. We may therefore see $\phi$ as the first factor projection

$$X' \simeq \mathbb{H}_l \times \mathbb{C}_*^* \to \mathbb{H}_l.$$ 

If $(w, z)$ denote coordinate functions on $\mathbb{H}_l \times \mathbb{C}_*^*$ the pull-back of the vector field has the form $\alpha(w)z \frac{\partial}{\partial z}$ for some $\alpha \in \mathcal{O}^*(\mathbb{H}_l)$. We now consider generators $f_g, f_\gamma$ of the groups $\mathbb{Z}$ and $\mathbb{Z}[\frac{1}{2}]$ acting on $\mathbb{H}_l \times \mathbb{C}_*^*$

By passing to a double covering of $S$ we may assume that $f_g$ acts on the $\mathbb{C}_*^*$-fibers by homotheties. Suppose that the same holds for $f_\gamma$. Then we have

$$f_g(w, z) = (kw, \beta(w)z)$$
$$f_\gamma(w, z) = (w + 2\pi i, \gamma(w)z)$$

for some $\beta, \gamma \in \mathcal{O}^*(\mathbb{H}_l)$. The compatibility with the pulled-back vector field implies

$$\alpha(kw) = \lambda \alpha(w), \alpha(w + 2\pi i) = \alpha(w)$$

for all $w \in \mathbb{H}_l$. The second relation implies that $\alpha$ is the composition of a function $u$ on the punctured unit disc with the exponential. Then the first relation translates into

$$u(\zeta^k) = \lambda u(\zeta), \text{ for all } \zeta \in \Delta^*;$$
(by $\Delta$ we denoted the unit disk in $\mathbb{C}$). But this implies $\lambda = 1$. Hence an effective $\mathbb{C}^*$-action on $S$, which is excluded by [10].

When $f_\gamma$ is composed with an inversion a similar argument applies, working with $\alpha^2$ instead of $\alpha$ for instance.

Thus the $\mathbb{C}$-action is effective and the fibers of $\phi$ are isomorphic to $\mathbb{C}$.

Using now the lift of the $\mathbb{C}$-action on $X'$, we get a $\mathbb{C}$-principal bundle structure on $X'$ over $\mathbb{H}_l$. Again, such a bundle is holomorphically trivial. In conclusion we have proven the following

**Theorem 3.7.** The universal covering of $S \setminus D$ is isomorphic to $\mathbb{H}_l \times \mathbb{C}$.

### 4. The action of the fundamental group

We consider a system of holomorphic coordinates $(w, z)$ in $\mathbb{H}_l \times \mathbb{C} \simeq \tilde{S} \setminus \tilde{D}$. The integrable vector field induced here by $\theta$ has no zeros and is therefore constant on each fiber of the projection of $\mathbb{H}_l \times \mathbb{C}$ on $\mathbb{H}_l$. Consequently, this vector field is of the form $\alpha(w) \frac{\partial}{\partial z}$ on $\mathbb{H}_l \times \mathbb{C}$, where $\alpha \in O^*(\mathbb{H}_l)$. Conjugating by the automorphism

$$(w, z) \mapsto (w, \alpha^{-1} \cdot z),$$

one gets $\alpha \equiv 1$.

Let $\gamma \in \pi_1(\tilde{S} \setminus \tilde{D})$ be a closed path in $\tilde{S} \setminus \tilde{D}$ with $p(\gamma) = 2\pi i$.

Denoting by $g_\gamma$ the automorphism of $\mathbb{H}_l \times \mathbb{C} \simeq \tilde{S} \setminus \tilde{D}$ corresponding to $\gamma$, we have:

$$g_\gamma(w, z) = (w + 2\pi i, z + f_\gamma(w))$$

$$g(w, z) = (kw, \lambda z + f_g(w)).$$

The case $\lambda = 1$ was treated in [1]. From now on we shall therefore assume that $\lambda \in \mathbb{C} \setminus \{0, 1\}$.

The automorphism $g_\gamma$ generates an action of $\mathbb{Z}$ on $\mathbb{H}_l \times \mathbb{C}$, which induces a holomorphic $\mathbb{C}$-principal bundle

$$\mathbb{H}_l \times \mathbb{C}/\mathbb{Z} \to \mathbb{H}_l/\mathbb{Z} \simeq \Delta^*.$$}

The triviality of this bundle proves the existence of a holomorphic function $h : \mathbb{H}_l \to \mathbb{C}$ such that

$$h(w + 2\pi i) - h(w) = f_\gamma(w)$$

and conjugation by $(w, z) \mapsto (w, z + h(w))$ gives us the new form

$$g_\gamma(w, z) = (w + 2\pi i, z).$$

In what follows we suppose therefore that $f_\gamma \equiv 0$.

We have

$$g \circ g_\gamma \circ g^{-1} = g_\gamma^k,$$

which gives a group isomorphism
\[ <g_\gamma, g> \simeq \mathbb{Z} \times \mathbb{Z}[\frac{1}{k}] \simeq \pi_1(S \setminus D) \]
on the one hand, and the \(2\pi i\)-periodicity of the function \(f_g\) on the other hand.

Factorizing by \(\exp : \mathbb{H}_l \to \Delta^*, \ w \mapsto e^w =: \zeta\), gives a Laurent series expansion
\[
f_g(w) = \sum_{m \in \mathbb{Z}} a_m e^{mw} = \sum_{m \in \mathbb{Z}} a_m \zeta^m.
\]
A conjugation by
\[
(w, z) \mapsto (w, z + \beta(w))
\]
where \(\beta\) is a \(2\pi i\)-periodic function on \(\mathbb{H}_l\), does not change the form of \(g_\gamma\), but replaces \(f_g\) by
\[
w \mapsto f_g(w) + \beta(kw) - \lambda \beta(w).
\]
Let
\[
h(\zeta) := \sum_{m \in \mathbb{Z}} a_m \zeta^m, \quad h_+(\zeta) := \sum_{m > 0} a_m \zeta^m.
\]
The series \(\sum_{l=0}^{\infty} \lambda^{-(l+1)}h_+(\zeta^k)\) converges uniformly on compact sets in \(\Delta^*\). To see this it is enough to write \(h_+(\zeta) = \zeta(\zeta^{-1}h_+(\zeta))\) and to remark that \(\zeta^{-1}h_+(\zeta)\) is holomorphic in 0. Let
\[
\chi(\zeta) := \sum_{l=0}^{\infty} \lambda^{-(l+1)}h_+(\zeta^k).
\]
We have
\[
\lambda \chi(\zeta) - \chi(\zeta^k) = h_+(\zeta).
\]
One sets \(\beta(w) := \chi(e^w) + \frac{a_0}{\lambda - 1}\) and gets
\[
f_g(w) + \beta(kw) - \lambda \beta(w) = \sum_{m < 0} a_m e^{mw}.
\]
We can therefore suppose that \(f_g(w) = h(e^w)\), where \(h \in \mathcal{O}(\mathbb{P}_1(\mathbb{C}) \setminus \{0\})\) and \(h(\infty) = 0\). One still has the possibility to conjugate with \((w, z) \mapsto (w, z + \chi(e^w))\), where \(\chi \in \mathcal{O}(\mathbb{P}_1(\mathbb{C}) \setminus \{0\})\).

For a function
\[
h(\zeta) = \sum_{m < 0} a_m \zeta^m
\]
and \(l \in \mathbb{N}^*\), one considers
\[
h_l(\zeta) := \sum_{k^l|m, m < 0} a_m \zeta^m
\]
and
\[
h_0 := h.
\]
Each \(h_l\) is of the form \(h_l(\zeta) = f_l(\zeta^k)\), for a certain function \(f_l\). Let \(\chi\) be the formal series \(-\sum_{l \geq 1} \lambda^{-l-1}f_l\). Formally one has:
Remark that if
\[ h(\zeta) + \chi(\zeta^k) - \lambda \chi(\zeta) = (h(\zeta) - f_1(\zeta^k)) + \lambda(f_1(\zeta) - f_2(\zeta^k)) + \lambda^2(f_2(\zeta) - f_3(\zeta^k)) + \ldots \]
and each term contains in its Laurent series expansion in \( \zeta \) only terms \( b_m \zeta^m \) with \( k \nmid m \).

For \( 0 < R < 1 \), \( l \geq 1 \) and \( |\zeta| \geq R \) one has:
\[
|f_l(\zeta)| \leq \sum_{k|l, m < 0} |a_m|R^m \leq \sum_{k|l, m < 0} |a_m|R^{m+k-1} \leq R^{k-1} \sum_{m \leq 0} |a_m|R^m,
\]
which shows that the series defining \( \chi \) is uniformly convergent on compact sets of \( \mathbb{P}_1(\mathbb{C}) \setminus \{0\} \). One gets the following normal form for \( h \):

\[
h(\zeta) = \sum_{m < 0, k|l \atop k \nmid m} a_m \zeta^m.
\]

Remark that if \( h \) is of this form, any modification of \( h(\zeta) \) by conjugation gives

\[
h(\zeta) + \chi(\zeta^k) - \lambda \chi(\zeta)\]
which is in normal form if and only if the function \( \zeta \mapsto \chi(\zeta^k) - \lambda \chi(\zeta) \) is identically zero. To see it, write the power series expansions of these functions:

\[
\chi(\zeta) = \sum_{m < 0} b_m \zeta^m, \quad -\chi(\zeta^k) + \lambda \chi(\zeta) = \sum_{m < 0} c_m \zeta^m.
\]

If there is \( c_r \neq 0 \) for \( r \in \mathbb{Z}_- \), then \( k \nmid r \), \( \lambda b_r = c_r \), \( \lambda b_{kr} = c_{kr} + b_r = \lambda^{-1} c_r \), \( \lambda^3 b_{k^2 r} = \lambda^2 c_{k^2 r} + \lambda^2 b_{kr} = c_r \), \( \ldots \) and the series \( \sum_{m < 0} b_m \zeta^m \) would not be convergent on \( \mathbb{P}_1(\mathbb{C}) \setminus \{0\} \), which one verifies by putting \( \zeta = |\lambda| \) or \( \zeta = 1 \). A contradiction!

We are now in the following normalized situation:

**Proposition 4.1.** The action of the fundamental group \( \pi_1(S \setminus D) \approx \mathbb{Z} \ltimes \mathbb{Z}[1/k] \) on the universal cover \( \mathbb{H}_l \times \mathbb{C} \) of \( S \setminus D \) is generated by the two automorphisms:

\[
g_\gamma(w, z) = (w + 2\pi i, z)
\]
\[
g(w, z) = (kw, \lambda z + f_g(w))
\]

with \( f_g(w) = h \circ \exp(w) = H \circ \exp(-w) \) where

\[
H(\zeta) = \sum_{m > 0 \atop k|l \atop k \nmid m} A_m \zeta^m,
\]

\( A_m = a_{-m} \).

The elements \( g_{l,n} := g^{-n} \circ g_\gamma \circ g^n \) for \( n \in \mathbb{N}, \ l \in \mathbb{Z} \) form a subgroup \( \Gamma \) of \( \pi_1(S \setminus D) \) isomorphic to \( \mathbb{Z}^{1/k} \). Explicitly we have

\[
g_{l,n}(w, z) = (w + \frac{2\pi il}{k^n}, z + \sum_{j=0}^{n-1} \lambda^{-j-1}(f_g(k^j w) - f_g(k^j w + \frac{2\pi il}{k^n}))).
\]
with \( f_g(w) = H \circ \exp(-w) \) where \( H \) is of the form
\[
H(\zeta) = \sum_{m>0 \atop k|m} A_m \zeta^m.
\]
We know that \( \Gamma \) acts properly discontinuously. Therefore \( H \) is non-trivial. One verifies easily that if \( n \geq m \), \( g_{p,n} \circ g_{q,m} = g_{p+qk-n,m} \).

For \( l = 1 \), let
\[
G_{n,j}(\zeta) = H(\zeta^{k^j}) - H(\zeta^{k^j} \exp(-2\pi i k^{j-n})), \quad 0 \leq j < n,
\]
\[
F_n(\zeta) = \sum_{0 \leq j < n} \lambda^{-j-1} G_{n,j}(\zeta).
\]

With these notations and putting \( \zeta = \exp(-w) \), we have
\[
\sum_{j=0}^{n-1} \lambda^{-j-1} (f_g(k^j w) - f_g(k^j w + \frac{2\pi i}{k^{n-j}})) = \sum_{j=0}^{n-1} \lambda^{-j-1} G_{n,j}(\zeta) = F_n(\zeta),
\]
and
\[
g_{1,n}(w, z) = (w + \frac{2\pi i}{k^n}, z + F_n(\zeta)).
\]

The rest of this section is devoted to the proof of the following

**Theorem 4.2.** The function \( H \) is a non-constant polynomial.

This theorem generalizes the similar statement of [7] whose proof we owe A. Borichev. In what follows we use the notations:
\( r \mathbb{T} \) (resp. \( r \mathbb{D} \)) the circle (resp. the open disc) of radius \( r > 0 \) and \( K \) the compact set \( \{ z \in \mathbb{C} \mid 3 \leq |z| \leq 3^k \} \). The different circles \( r \mathbb{T} \) are equipped with the normalized Lebesgue measure \( dm(\zeta) \), for which \( \int_{r \mathbb{T}} dm(\zeta) = 1 \).

If \( f \) is holomorphic on \( 3 \mathbb{D} \), then wind\((f)\) denotes the number of zeros of \( f \) in \( 3 \mathbb{D} \).

The proofs of the following lemmas are left to the reader; see however [7] for the case \( \lambda = 1 \).

**Lemma 4.3.** Let \( f \) be a holomorphic function on the closed unit disc \( 3^k \mathbb{D} \) such that for all \( z \in K \mid f(z) \mid \geq 1 \). Then one has the relations
\[
\int_{3^k \mathbb{T}} \ln |f(\zeta)|^{\text{wind}(f)} \, dm(\zeta) = \int_{3^k \mathbb{T}} \ln |f(\zeta)|^{\text{wind}(f)} \, dm(\zeta),
\]
\[
\int_{3^k \mathbb{T}} \ln |f(\zeta)| \, dm(\zeta) = \int_{3^k \mathbb{T}} \ln |f(\zeta)| \, dm(\zeta) - (k-1)\text{wind}(f) \ln 3.
\]

**Lemma 4.4.** For all \( a, b \in \mathbb{C} \), such that \( |a| > 2 \) and \( |b| < 1 \), one has
\[
\ln |a + b| \geq \ln(|a|) - |b|.
\]
Lemma 4.5. Let $f$ be a holomorphic function defined in a neighborhood of the closed disc $3\mathbb{D}$. We set

$$A = \int_{3T} \ln^+ |f(\zeta)| \, dm(\zeta).$$

Hence there is a constant $C > 0$, independent of $f$, for which

$$\ln \int_{2T} |f(\zeta)| \, dm(\zeta) \leq CA.$$

(Here $\ln^+ := \max(\ln, 0)$.)

Lemma 4.6. Let $f = \sum_{s \geq 0} \hat{f}(s)z^s$ be a holomorphic function in a neighborhood of the disc $2\mathbb{D}$. Then, for all $s \in \mathbb{N}$,

$$|\hat{f}(s)| = |\int_{2T} f(\zeta)\zeta^{-s}dm(\zeta)| \leq 2^{-s} \int_{2T} |f(\zeta)| \, dm(\zeta).$$

Lemma 4.7. For all $n \geq 0$ we have

$$\lambda F_{n+1}(\zeta) = G_{n+1,0}(\zeta) + F_n(\zeta^k).$$

Lemma 4.8. For every $\nu \geq 1$,

$$\hat{F}_n(\nu k^{n-1}) = \lambda^{-n} A_\nu (1 - \exp(-2\pi i\nu/k)).$$

In particular,

$$|\hat{F}_n(\nu k^{n-1})| \geq |\lambda|^{-n} \frac{|A_\nu|}{k}.$$  

Proof of Theorem 4.2:

Since $\Gamma$ acts properly discontinuously, the images $g_{1,n}(K)$ of the compact $K$ tend to infinity. Since the first component of $g_{1,n}(w, z)$ converges, this implies that the sequence $(|F_n|)$ converges uniformly on $K$ to $+\infty$.

Since $\mathcal{H}$ is $C^1$ on $K$, there is a constant $c > 0$ independent of $n$ for which

$$\beta_n := \sup_{\zeta \in K} |G_{n,0}(\zeta)| = \sup_{\zeta \in K} |H(\zeta) - H(\zeta \exp(-2\pi ik^{-n}))| \leq ck^{-n}.$$

Fix a positive integer $N$ such that

\begin{align*}
\text{(*)} & \quad \text{for all } \zeta \in K, \ n \geq N, \ |F_n(\zeta)| \geq 2 \\
\text{(**)} & \quad \lambda^{-1} \sum_{n > N} \beta_n \leq 1.
\end{align*}

Set $W := \text{wind}(F_N)$ and $F(\zeta) := F_N(\zeta^k)$. Using (\text{*}), we get $\text{wind}(F) = kW$. We recall that $\lambda F_{N+1}(\zeta) = F(\zeta) + G_{N+1,0}(\zeta)$. Combine now Rouché’s Theorem applied to $F$ and $G_{N+1,0}$ with the inequalities (\text{*}) and (\text{**}) to obtain $\text{wind}(F_{N+1}) = kW$. By induction, one shows that for $p \in \mathbb{N}$,

$$\text{wind}(F_{N+p}) = k^p W.$$
The lemmas 4.3 and 4.7 imply
\[
\int_{3T} \ln |F_N(\zeta)| \, dm(\zeta) = \int_{3T} \ln |F_N(\zeta)| \, dm(\zeta) - (k - 1)W \ln 3
\]
\[
= \int_{3T} \ln |F_N(\zeta^k)| \, dm(\zeta) - (k - 1)W \ln 3
\]
\[
= \int_{3T} \ln |\lambda F_{N+1}(\zeta) - G_{N+1,0}(\zeta)| \, dm(\zeta) - (k - 1)W \ln 3.
\]
Applying lemma 4.4 and the inequalities (\dagger) and (\dagger\dagger), one gets
\[
\int_{3T} \ln |F_N(\zeta)| \, dm(\zeta) \geq \int_{3T} \ln |\lambda F_{N+1}(\zeta) - G_{N+1,0}(\zeta)| \, dm(\zeta) - (k - 1)W \ln 3
\]
\[
\geq \int_{3T} \ln |\lambda F_{N+1}(\zeta)| \, dm(\zeta) - \beta_{N+1} - (k - 1)W \ln 3,
\]
and furthermore
\[
\int_{3T} \ln |\lambda F_{N+1}(\zeta)| \, dm(\zeta) \leq \int_{3T} \ln |F_N(\zeta)| \, dm(\zeta) + \beta_{N+1} + (k - 1)W \ln 3.
\]
Induction gives the inequalities
\[
\int_{3T} \ln |F_{N+p}(\zeta)| \, dm(\zeta) \leq \int_{3T} \ln |F_N(\zeta)| \, dm(\zeta) + \sum_{N<s \leq N+p} \beta_s
\]
\[
+ (k - 1)W \ln 3 \sum_{0 \leq s < p} k^s - p \ln |\lambda| \leq C + Wk^p \ln 3 - p \ln |\lambda|,
\]
for a certain constant \( C > 0 \) independent of \( p \in \mathbb{N} \).

Lemma 4.5 and the inequalities (\dagger) and (\dagger\dagger) give the existence of a constant \( C_1 > 0 \) independent of \( p \) such that
\[
\ln \int_{2T} |F_{N+p}(\zeta)| \, dm(\zeta) \leq C_1 + C_1 k^p.
\]
If \( H \) is not a polynomial, one can find an integer \( \nu \) verifying
\[
\nu > C_1 k^{1-N/\ln 2}, \quad \text{and} \quad A_\nu \neq 0.
\]
But now, on the one hand Lemma 4.8 gives
\[
\ln |\widehat{F}_{N+p}(\nu k^{N+p-1})| \leq \ln \int_{2T} |F_{N+p}(\zeta)| \, dm(\zeta) - \nu k^{N+p-1} \ln 2
\]
\[
\leq C_1 k^p - \nu k^{N+p-1} \ln 2
\]
\[
= C_1 - (\nu k^{N-1} \ln 2 - C_1)k^p.
\]
On the other hand, by Lemma 4.8
\[
|\widehat{F}_{N+p}(\nu k^{N+p-1})| \geq |\lambda|^{-N-p} A_\nu/k \geq 0,
\]
and therefore \( \ln |\widehat{F}_{N+p}(\nu k^{N+p-1})| \geq C_2 - C_3 p \) for some constants \( C_2 \) and \( C_3 \) independent of \( p \). This is a contradiction. \( \blacksquare \)
5. The Contracting Germ

We have seen that the action of the group $\pi_1(S \setminus D)$ on $\mathbb{H}_l \times \mathbb{C}$ is generated by the two automorphisms

$$
\begin{align*}
\gamma & \mapsto (w + 2\pi i, z) \\
g & \mapsto (kw, \lambda z + H(e^{-w})).
\end{align*}
$$

where $H(\zeta) = \sum_{m=1}^{s} A_m \zeta^m$ is a polynomial in normal form, i.e. $A_m = 0$ for all $m > 0$ with $k|m$ and $A_s \neq 0$.

Let $l := \left\lceil \frac{s}{k} \right\rceil + 1$. We will conjugate our group by

$$
\phi(w, z) = (w, z + \lambda^{-1} \sum_{m=1}^{l-1} A_m e^{-mw}).
$$

This has no effect on $g\gamma$, but

$$
\phi \circ g \circ \phi^{-1}(w, z) = (kw, \lambda z + Q(e^{-w})),
$$

where $Q(\zeta) = H(\zeta) - \sum_{m=1}^{l-1} A_m \zeta^m + \lambda^{-1} \sum_{m=1}^{l-1} A_m \zeta^{mk}$ is a polynomial of degree $s$ with

$$
\zeta^{\min(k,l)} | Q(\zeta).
$$

Iterating this procedure if necessary, we end up with a polynomial $Q$ of degree $s$ such that $\zeta^l | Q(\zeta)$. Let $Q(\zeta) := \sum_{m=1}^{s} b_m \zeta^m$ and $d := \gcd \{k, m | b_m \neq 0\}$.

We conjugate now with $\phi(w, z) = (dw, z)$:

$$
\phi \circ g \circ \phi^{-1}(w, z) = (w + 2\pi id, z)
$$

and

$$
\phi \circ g \circ \phi^{-1}(w, z) = (kw, \lambda z + Q(e^{-w/d})).
$$

One verifies directly that the group generated is the same as the group $G'$ generated by

$$(w, z) \mapsto (w + 2\pi i, z)$$

$$(w, z) \mapsto (kw, \lambda z + Q(e^{-w/d})).$$

Let now

$$l' := \left\lceil \frac{s}{kd} \right\rceil + 1$$

and

$$Q'(\zeta) := \sum_{m=1}^{s} b_m \zeta^{m/d}.$$

Using the inequality $d[x/d] < \lfloor x \rfloor + 1, d \in \mathbb{N}^*, x \in \mathbb{R}$ and the fact that the indices of the non-vanishing coefficients of $Q'$ are divisible by $d$, one verifies easily that $\zeta^{l'} | Q'(\zeta)$. We conjugate now with $(w, z) \mapsto (w, e^{l'w} z)$ and the generators of $G'$ become

$$(w, z) \mapsto (w + 2\pi i, z)$$

and

$$(w, z) \mapsto (kw, \lambda z + Q(e^{-w/d})).$$
(w, z) ↦ (kw, λe^{v(k-1)w}z + P(e^w)),

where $P$ is the polynomial defined by

$$P(ξ) := ξ^{l'k}Q'(ξ^{-1}).$$

Remark that deg$P ≤ l'(k - 1)$ and that $P(0) = 0$. Let

$$l'(k-1)$$

$$P(ξ) = \sum_{m=1}^{l'(k-1)} c_mξ^m. $$

We have g.c.d.$\{k, m \mid c_m ≠ 0\} = 1.$

This relation implies that the contracting germ

$$f : Δ^* \times \mathbb{C} \to Δ^* \times \mathbb{C},$$

$$f(ξ, z) := (ξ^k, λξ^{l'(k-1)}z + P(ξ))$$

is locally injective around $(0, 0)$: If $f(ξ_1, z_1) = f(ξ_0, z_0)$, then $ξ_1 = ξ_0$ and $λξ_0^{l'(k-1)}z_0 + P(ξ_0) = λξ_1^{l'(k-1)}z_1 + P(ξ_1)$. Put $ε := ξ_1/ξ_0$. One has $ε^k = 1$ and

$$z_1 = ε^{l'} [z_0 + λ^{-1}ξ_0^{l'(k-1)} \sum_{m=1}^{l'(k-1)} c_mξ_0^m (1 - ε^m)].$$

If $ε = 1$, one has $z_1 = z_0$. Otherwise take $m_0$ the smallest index such that $c_{m_0} ≠ 0$ and $ε^{m_0} ≠ 1$. The existence of such an index is ensured by the relation g.c.d.$\{k, m \mid c_m ≠ 0\} = 1.$ We write now

$$z_1 = ε^{l'} [z_0 + λ^{-1}ξ_0^{l'(k-1)+m_0} (c_{m_0} (1 - ε^{m_0}) + \sum_{m=m_0+1}^{l'(k-1)} c_mξ_0^{m-m_0} (1 - ε^m))]$$

and we see that for $z_0$ and $ξ_0$ sufficiently small, $z_1$ stays away from 0. The local injectivity follows now directly.

By Proposition 1.2.8 of [3], we get that $f$ is a defining germ for a minimal GSS surface $S'$ whose maximal divisor we denote by $D'$. One can verify that the quotient of $\mathbb{H}_l \times \mathbb{C}$ by the action of $G'$ is the same as the one of $Δ^* \times \mathbb{C}$ by the equivalence relation $u_1 ∼ u_2 :$ there exist $n_1, n_2 ∈ \mathbb{N}$ such that $f^{n_1}(u_1) = f^{n_2}(u_2)$.

It follows by construction that $S \setminus D$ and $S' \setminus D'$ are isomorphic. Since the intersection matrices of $D$ and of $D'$ are negative-definite and neither $D$ nor $D'$ contain exceptional curves of the first kind, this isomorphism is extendable to an isomorphism of $S$ onto $S'$. This ends the proof of the Main Theorem. □

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