An extension of the standard multifractional Brownian motion

M. Ait Ouahra, H. Ouahhabi, A. Sghir and M. Mellouk

1Mohammed First University. Faculty of Sciences Oujda. Department of Mathematics. Stochastic and Deterministic Modelling Laboratory. B.P. 717. Morocco.

2Department of Statistics. College of Business and Economics. United Arab Emirates University.

3MAP5, CNRS UMR 8145. University of Paris. 45, Street of Saints-Pres 75270. Paris Cedex 6. France.

Abstract

In this paper, firstly, we generalize the definition of the bifractional Brownian motion $B^{H,K}(t; t \geq 0)$, with parameters $H \in (0,1)$ and $K \in (0,1]$, to the case where $H$ is no longer a constant, but a function $H(.)$ of the time index $t$ of the process. We denote this new process by $B^{H,(.)K}$. Secondly, we study its time regularities, the local asymptotic self-similarity and the long-range dependence properties.

Key words: Gaussian process; Self similar process; Fractional Brownian motion; Bifractional Brownian motion; Multifractional Brownian motion; Local asymptotic self-similarity.

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1. Introduction

In recent years, the famous fractional Brownian motion $B^H := \left( B^H_t ; t \geq 0 \right)$, (fBm for short), with Hurst parameter $H \in (0,1)$, has considerable interest due to its applications in various scientific areas including: telecommunications, finance, turbulence and image processing, (see for examples: Addison and Ndumu [1], Cheridito [10], Comegna et al. [12], Samorodnitsky and Taqqu [25] and Taqqu [26]). The fBm was firstly introduced by Kolmogorov [19], and was later made popular by Mandelbrot and Van Ness [23]. It is the only centered and self-similar Gaussian process with stationary increments and covariance function:

$$R^H(t, s) := \mathbb{E}(B^H_t B^H_s) = \frac{1}{2} \left( t^{2H} + s^{2H} - |t-s|^{2H} \right), \quad \forall t, s \geq 0.$$ 

For large details about fBm, we refer to [13], [17], [20] and [24].

$^1$Corresponding author: ouahra@gmail.com
The increments of $B_{t}^{H,K}$ are only independents in the case of the standard Brownian motion: $(H = \frac{1}{2}, K = 1)$, and they are not stationary for any $K \in [0, 1]$, except the case of the fBm: $(K = 1)$, however, $B_{t}^{H,K}$ is quasi-helix in the sense of Kahane [18]:

\[
(1) \quad 2^{-K}|t - s|^{2HK} \leq \mathbb{E}\left( B_{t}^{H,K} - B_{s}^{H,K} \right)^{2} \leq 2^{1-K}|t - s|^{2HK}, \quad \forall s,t \geq 0.
\]

Moreover, according to [17], if we put: $\sigma_{t}^{2} := \mathbb{E}\left( B_{t}^{H,K} - B_{0}^{H,K} \right)^{2}$, then

\[
\lim_{\varepsilon \to 0} \frac{\sigma_{t}^{2}(t)}{2HK} = 2^{1-K}, \quad t > 0,
\]

therefore, the small increments of $B_{t}^{H,K}$ are approximately stationary. For the large increments, Maejima and Tudor [22] have proved that, when $h \to +\infty$, the sequence of increments process:

\[
\left( B_{t+h}^{H,K} - B_{t}^{H,K} ; t \geq 0 \right)
\]

converges modulo a constant, in the sense of the finite dimensional distributions, to the fBm $(B_{t}^{H,K} : t \geq 0)$ with Hurst parameter $HK$. This result can be interpreted like the bfBm has stationary increments for large increments. The key ingredient used in [22] is a decomposition in law of the bfBm presented by Lei and Nualart [20] as follows:

Let $W := \left( W_{\theta} ; \theta \geq 0 \right)$ be a standard Brownian motion independent of $B_{t}^{H,K}$. For any $K \in (0, 1)$, let $X^{K} := \left( X_{t}^{K} ; t \geq 0 \right)$ the centred Gaussian process defined by:

\[
X_{t}^{K} := \int_{0}^{+\infty} \left( 1 - e^{-\theta t} \right) \theta^{-(\frac{1+K}{2})} dW_{\theta},
\]

with the covariance function:

\[
\mathbb{E}\left( X_{t}^{K} X_{s}^{K} \right) = \frac{\Gamma(1-K)}{K} \left[ t^{K} + s^{K} - (t + s)^{K} \right], \quad \forall t,s \geq 0.
\]

$\Gamma$ is the well known Gamma function.

The authors in [20] showed by setting: $X_{t}^{H,K} := X_{t^{2HK}}^{K}$, that:

\[
(2) \quad \left( C_{1}(K)X_{t}^{H,K} + B_{t}^{H,K} ; t \geq 0 \right) \overset{d}{=} \left( C_{2}(K)B_{t}^{H,K} ; t \geq 0 \right),
\]

where $C_{1}(K) = \sqrt{\frac{2^{-K}}{\Gamma(1-K)}}$, $C_{2}(K) = 2^{\frac{1-K}{2}}$ and $\overset{d}{=}$ means equality of all finite dimensional distributions. The second application of (2) given in [22] is that the long-range dependence, (LRD for short), of the process $B_{t}^{H,K}$ depends on the value of the product $HK$:

- Long-memory: for every $a \in \mathbb{N}$:
  \[
  \sum_{n \geq 0} \text{cor}_{B_{t}^{H,K}}(a, a+n) = +\infty, \quad \text{if} \quad 2HK > 1,
  \]

- Short-memory: for every $a \in \mathbb{N}$:
  \[
  \sum_{n \geq 0} \text{cor}_{B_{t}^{H,K}}(a, a+n) < +\infty, \quad \text{if} \quad 2HK \leq 1.
  \]

where

\[
\text{cor}_{B_{t}^{H,K}}(a, a+n) := \mathbb{E}\left[ \left( B_{a+1}^{H,K} - B_{a}^{H,K} \right) \left( B_{a+n+1}^{H,K} - B_{a+n}^{H,K} \right) \right].
\]

This result was appeared also in Remark 7 by Russo and Tudor [23].
Now, we are ready to introduce our new process: Since the model of the fBm $B^H$ may be restrictive for different phenomena due to the fact that all its interesting properties are governed by the Hurst parameter $H$, this gave the motivation to Benassi et al. [6] and Lévy-Vhel and Peltier [21] to introduce, independently, a new model to generalize the fBm: It’s the multifractional Brownian motion, (mBm for short). Contrarily to the fBm, the almost sure Hölder exponent of the mBm is allowed to vary along the trajectory, a useful feature when one needs to model processes whose regularity evolves in time, such as Internet traffic or images. The definition of the mBm in [21] is based on the moving average representation of the fBm, where the constant Hurst parameter $H$ is substituted by a functional $H(.)$ as follows:

$$B^H_t = \frac{1}{\Gamma(H(t)+\frac{1}{2})} \left( \int_0^t (t-u)^{H(t)-\frac{1}{2}} - (-u)^{H(t)-\frac{1}{2}} \right) W(du)$$

$$+ \int_0^t (t-u)^{H(t)+\frac{1}{2}} W(du), \quad t \geq 0,$$

where $H(.) : [0, \infty) \mapsto [\mu, \nu] \subset (0, 1)$ is a Hölder continuous function of exponent $\beta > 0$, and $W$ is a standard Brownian motion on $\mathbb{R}$.

The authors in [6] defined the mBm by means of the harmonisable representation of the fBm as follows:

$$\tilde{B}^H_t = \int_{\mathbb{R}} e^{i\xi-t} \tilde{W}(d\xi), \quad t \geq 0,$$

where $\tilde{W}(\xi)$ is the Fourier transform of the series representation of white noise with respect to an orthonormal basis of $L^2(\mathbb{R})$. From these definitions, it’s easy to that mBm is a zero mean Gaussian processes whose increments are in general neither independents nor stationary. It is proved by Cohen [11] that the two representations of mBm are equivalent, up to a multiplicative deterministic function. This function is explicitly given by Boufoussi et al. [6]. Moreover, in Ayache et al. [5], the covariance function of the standard mBm: (i.e. the variance a time 1 is 1), has been deduced from its harmonisable representation as follows:

$$E(B^H_t B^H_s) = D(H(t), H(s)) \left[ t^{H(t)+H(s)} + \frac{s^{H(t)+H(s)} - |t-s|^{H(t)+H(s)}}{2} \right],$$

where

$$D(x, y) := \frac{\sqrt{(2x+1)(2y+1)} \sin(\pi x) \sin(\pi y)}{2 \Gamma(x + y + 1) \sin \left( \frac{\pi(x+y)}{2} \right)}.$$

Clearly, if $H(.) \equiv H$ a constant in $(0, 1)$, $D(H, H) = \frac{1}{2}$, and we find the covariance function of the fBm $B^H$, the zero mean Gaussian process with stationary increments.

In the same spirit as [6] and [21], since all the properties of the bBm $B^{H,K}$ is governed by the unique number $HK$, we introduce in this note a generalization of $B^{H,K}$, by substituting to the parameter $H$ in the covariance function $R^{H,K}$, a Hölder function $H(.) : [0, \infty) \mapsto [\mu, \nu] \subset (0, 1)$ with exponent $\beta > 0$. More precisely:

**Definition 1.** We define a new centred Gaussian process, starting from zero, and denoted by $B^{H(.)K} : (B^{H(.)K}_t ; t \geq 0)$, by the covariance function:

$$R^{H(.)K}(t, s) := \left( D(H(t), H(s)) \right)^K \left[ (t^{H(t)+H(s)} + s^{H(t)+H(s)})^K - |t-s|^{(H(t)+H(s))K} \right].$$
Remark 1. Clearly, when \( K = 1 \), \( B^{H(.),K} \) is a standard mBm. When \( H(.) \equiv H \) a constant in \((0,1)\), \( B^{H(.),K} \) is a bfBm with parameters \( H \in (0,1) \) and \( K \in (0,1) \).

2. The existence of \( B^{H(.),K} \)

In this section we prove the existence of our process by using the same argument used in \[17\] for the bfBm.

Proposition 1. For any \( K \in (0,1) \) and \( H(.) : [0, \infty) \rightarrow [\mu, \nu] \subset (0,1) \) a Hölder continuous function, the covariance function \( R^{H(.),K} \) appeared in Definition 1 is positive-definite.

Proof. We assume that \( K \in (0,1) \) since the special case \( K = 1 \) is evident. We use the following identity:

\[
t^K = \frac{K}{\Gamma(1-K)} \int_0^\infty (1 - e^{-tx}) \gamma x^{-1-K} dx, \quad \forall t \geq 0,
\]

where \( \Gamma \) is the gamma function. For any \( c_1, ..., c_n \in \mathbb{R} \), we have:

\[
\sum_{i=1}^n \sum_{j=1}^n c_i c_j R^{H(.),K}(t_i,t_j)
\]

\[
= \frac{K}{\Gamma(1-K)} \int_0^\infty \sum_{i=1}^n \sum_{j=1}^n c_i c_j \left[ -xe^{D(H(t_i),H(t_j)) \left( t_i^{H(t_i)+H(t_j)} + t_j^{H(t_i)+H(t_j)} \right)} 
\right.
\]

\[
+ e^{D(H(t_i),H(t_j)) \left( t_i^{H(t_i)+H(t_j)} + t_j^{H(t_i)+H(t_j)} \right)} \right] x^{-1-K} dx
\]

\[
= \frac{K}{\Gamma(1-K)} \int_0^\infty \sum_{i=1}^n \sum_{j=1}^n c_i c_j e^{D(H(t_i),H(t_j)) \left( t_i^{H(t_i)+H(t_j)} + t_j^{H(t_i)+H(t_j)} \right) - \left| t_i-t_j \right| H(t_i)+H(t_j)}
\]

\[
\times \left[ e^{D(H(t_i),H(t_j)) \left( t_i^{H(t_i)+H(t_j)} + t_j^{H(t_i)+H(t_j)} - \left| t_i-t_j \right| H(t_i)+H(t_j) \right)} - 1 \right] x^{-1-K} dx.
\]

We know by \[5\] that \( D(H(t), H(s)) \left( t^{H(t)+H(s)} + s^{H(t)+H(s)} - |t-s| H(t)+H(s) \right) \) is positive-definite, then so is:

\[
e^{D(H(t), H(s)) \left( t^{H(t)+H(s)} + s^{H(t)+H(s)} - |t-s| H(t)+H(s) \right)} - 1, \quad \forall x \geq 0,
\]

which gives the proof of the proposition. \( \square \)

Remark 2. It’s easy to see the following link between the covariance function of our process \( B^{H(.),K} \), the mBm \( B^{H(.),K} \) and a transformation of the process \( X^K \):

\[
KD \left( H(t), H(s) \right)^K \frac{DF(H(t), H(s))}{\Gamma(1-K)} \text{ cov} \left( X^K_{H(t)+H(s)}, X^K_{H(t)+H(s)} \right)
\]

\[
= \frac{KD \left( H(t), H(s) \right)^K}{DF(H(t), H(s),K)} \text{ cov} \left( B^H_{t}, B^H_{s} \right), \quad \forall s, t \geq 0
\]

Therefore, under the assumption of independence, when \( H(.) \equiv H \) a constant in \((0,1)\), we find easily (2) the decomposition in law of the bfBm. However, in the functional case \( H(.) \), since we cannot separate the variables \( s \) and \( t \) in \( D \left( H(t), H(s) \right) \), then we cannot deduce a decomposition in law of our process \( B^{H(.),K} \).
3. Regularities of the trajectories of $B^{H(.), K}$

In this section, we deal with the regularities of the trajectories of $B^{H(.), K}$. We follow the same method used in the case of the mBm, (see [9]). For this, we need the following regularity of the bBm $B^{H,K}$ with respect to the constant parameter $H$. We use (2) the decomposition in law of the bBm $B^{H,K}$.

**Proposition 2.** Let $[a, b) \subset [0, \infty)$ and $[a, \gamma) \subset (0, 1]$, and consider $B^{H,K}$ a bBm with parameters $H \in [a, \gamma]$ and $K \in [0, 1]$. Then, there exists a finite positive constant $C(\alpha, \gamma, K)$ such that, for all $H, H' \in [a, \gamma]$, we have:

$$
\sup_{t \in [a, b]} \mathbb{E}\left( B^{H,K}_t - B^{H',K}_t \right)^2 \leq C(\alpha, \gamma, K)|H - H'|^2.
$$

**Proof.** Using (2) and the elementary inequality: $(a - b)^2 \leq 2a^2 + 2b^2$, we obtain:

$$
\mathbb{E}\left( B^{H,K}_t - B^{H',K}_t \right)^2 \leq 2C^2(\gamma)\mathbb{E}\left( B^{H,K}_t - B^{H',K}_t \right)^2 + 2C^2(\gamma)\mathbb{E}\left( X^{H,K}_t - X^{H',K}_t \right)^2.
$$

In view of Lemma 3.1 in [9], (see also [21]), we know that:

$$
\mathbb{E}\left( B^{H,K}_t - B^{H',K}_t \right)^2 \leq C(\alpha, \gamma, K)|H - H'|^2.
$$

where

$$
C(\alpha, \gamma, K) = 4 \sup_{t \in [a, b]} \left( \int_0^1 \frac{1 - \cos(\theta t)}{\theta^{2\gamma + 1}} (\log(\theta))^2 \, d\theta + \int_1^\infty \frac{1}{\theta^{2\gamma + 1}} (\log(\theta))^2 \, d\theta \right) < +\infty.
$$

Now, let us deal with the process $X^{H,K}$. We have by the Itô’s isometry:

$$
\mathbb{E}\left( X^{H,K}_t - X^{H',K}_t \right)^2 = \int_0^\infty \left( e^{-\theta^2 H'} - e^{-\theta^2 H} \right)^2 \theta^{-(1 + K)} \, d\theta.
$$

Without loss of generality, we suppose that $H < H'$.

Making use of the theorem on finite increments for the function $x \mapsto e^{-\theta^2 x}$ for $x \in (H, H')$, there exists $\xi \in (H, H')$ such that:

$$
\mathbb{E}\left( X^{H,K}_t - X^{H',K}_t \right)^2 = 4|H - H'|^2 \frac{\alpha^2 \log^2(t)}{4t^4 \gamma} \int_0^\infty e^{-2\theta^2 \xi} \theta^{1-K} \, d\theta.
$$

- If $t \leq 1$, then $|t \log(t)| \leq e^{-1}$, and:

$$
\mathbb{E}\left( X^{H,K}_t - X^{H',K}_t \right)^2 \leq \frac{1}{(e\alpha)^2} |H - H'|^2 \int_0^\infty e^{-2\theta^2 \xi} \theta^{1-K} \, d\theta.
$$

Then

$$
\mathbb{E}(X^{H,K}_t - X^{H',K}_t)^2 \leq C(\alpha, \gamma, K)|H - H'|^2,
$$

where

$$
C(\alpha, \gamma, K) = \frac{1}{(e\alpha)^2} \sup_{t \in [a, b]} \left( \int_0^1 e^{-\theta^2 \xi} \theta^{1-K} \, d\theta + \int_1^\infty e^{-\theta^2 \xi} \theta^{1-K} \, d\theta \right) < +\infty.
$$

- If $t \geq 1$, we obtain:

$$
\mathbb{E}\left( X^{H,K}_t - X^{H',K}_t \right)^2 \leq C(\alpha, \gamma, K)|H - H'|^2.
$$

where

$$
C(\alpha, \gamma, K) = \left[ \sup_{t \in [a, b]} (4t^4 \gamma) \right] \left[ \sup_{t \in [a, b]} \left( \int_0^\infty e^{-2\theta^2 \xi} \theta^{1-K} \, d\theta \right) \right] < +\infty.
$$
Finally,
\begin{align}
(4) \quad \mathbb{E}\left( X^H_{t} - X^H_{t'} \right)^2 \leq C_3(\alpha, \gamma, K)|H - H'|^2,
\end{align}
where
\[ C_3(\alpha, \gamma, K) = \max \left( C_2(\alpha, \gamma, K) ; C_3(\alpha, \gamma, K) \right). \]
Consequently, by combining (3) and (4), we conclude the lemma. \hfill \square

**Remark 3.**

1. A similar result is obtained by Ait Ouahra and Sghir [9], (Lemma 3.2), for the sub-fractional Brownian motion \( S^H \) with parameter \( H \in (0, 1) \). It’s a continuous centred Gaussian process, starting from zero, with covariance function:
\[ \mathbb{E}(S^H_t S^H_s) = t^H + s^H - \frac{1}{2} \left( (t + s)^H + |t - s|^H \right). \]

2. In the case of fBm, (i.e. \( K = 1 \)), a similar result is given, independently, in [9], by using the moving average representation of fBm, and in [21], by using the harmonisable representation of fBm.

We turn now our interest to the study of the time regularities of our process.

**Theorem 1.** Let \( H(.) : [0, \infty) \rightarrow [\mu, \nu] \subset (0, 1) \) be a Hölder continuous function with exponent \( \beta > 0 \) and \( \sup H(t) < \beta \). Then for all \( \alpha, \gamma, K \), there exists a finite positive constant \( C(\mu, \nu, K) \) such that:
\[ \mathbb{E}\left( B^H_{H_{t}} - B^H_{H_{s}} \right)^2 \leq C(\mu, \nu, K)|t - s|^{2(H(t) \vee H(s))K}, \]
where \( C(\mu, \nu, K) \) is a finite positive constant.

**Proof.** By the elementary inequality \( (a + b)^2 \leq 2a^2 + 2b^2 \), we have:
\[ \mathbb{E}\left( B^H_{H_{t}} - B^H_{H_{s}} \right)^2 \leq 2\mathbb{E}\left( B^H_{H_{t}} - B^H_{H_{s}} \right)^2 + 2\mathbb{E}\left( B^H_{H_{t}} - B^H_{H_{s}} \right)^2. \]
By virtue of (1) and Proposition 2 and the fact that \( H(.) : [0, +\infty] \rightarrow [\mu, \nu] \subset (0, 1) \), we get:
\begin{align}
\mathbb{E}\left( B^H_{H_{t}} - B^H_{H_{s}} \right)^2 & \leq 2^{2-K}|t - s|^{2H(t)K} + 2C(\mu, \nu, K)|H(t) - H(s)|^2.
& \leq 2^{2-K}|t - s|^{2H(t)K} + 2C(\mu, \nu, K)|t - s|^2 \beta.
\end{align}
Since \( \sup H(t) < \beta \) and \( KH(t) \in (0, 1) \), we deduce that:
\[ |t - s|^2 \beta \leq |t - s|^{2KH(t)}. \]
Thus
\[ \mathbb{E}\left( B^H_{H_{t}} - B^H_{H_{s}} \right)^2 \leq C_4(\mu, \nu, K)|t - s|^{2KH(t)}, \]
where \( C_4(\mu, \nu, K) = 2^{2-K} + 2C(\mu, \nu, K). \)
Since the roles of \( t \) and \( s \) are symmetric, we obtain the desired result. \hfill \square

To prove Theorem 2, we need the following classical lemma.

**Lemma 1.** Let \( Y \) be a real centred Gaussian random variable. Then for all real \( \alpha > 0 \), we have:
\[ \mathbb{E}|Y|^\alpha = c(\alpha)\left( \mathbb{E}|Y|^2 \right)^{\frac{\alpha}{2}}, \]
where \( c(\alpha) = \frac{2^\frac{\alpha}{2} \Gamma(\frac{\alpha + 1}{2})}{\Gamma(\frac{\alpha}{2})}. \)
Theorem 2. Let $H(.) : [0, \infty) \mapsto [\mu, \nu] \subset (0, 1)$ be a Hölder continuous function with exponent $\beta > 0$ and $\sup_{t \geq 0} H(t) < \beta$. Then, there exists $\delta > 0$, and for any integer $m \geq 1$, there exist $M_m > 0$, such that:

$$\mathbb{E}\left(B_{t}^{H(t),K} - B_{s}^{H(s),K}\right)^{m} \geq M_{m}|t - s|^{m(H(t) \land H(s))K},$$

for all $s, t \geq 0$ such that $|t - s| < \delta$.

Proof. Using the elementary inequality: $(a + b)^{2} \geq \frac{1}{4}a^{2} - b^{2}$, we obtain:

$$\mathbb{E}\left(B_{t}^{H(t),K} - B_{s}^{H(s),K}\right)^{2} = \mathbb{E}\left(B_{t}^{H(t),K} - B_{s}^{H(s),K} + B_{s}^{H(s),K} - B_{s}^{H(s),K}\right)^{2} \geq \frac{1}{2}\mathbb{E}\left(B_{t}^{H(t),K} - B_{s}^{H(s),K}\right)^{2} - \mathbb{E}\left(B_{s}^{H(s),K} - B_{s}^{H(s),K}\right)^{2}.$$

Moreover, by using (1) and Proposition 2, we obtain:

$$\mathbb{E}\left(B_{t}^{H(t),K} - B_{s}^{H(s),K}\right)^{2} \geq \frac{1}{2\Gamma(K)|t - s|^{2H(t)K} - \mathbb{E}\left(B_{s}^{H(s),K} - B_{s}^{H(s),K}\right)^{2} \geq \frac{1}{2|t - s|^{2H(t)K}} - C(\mu, \gamma, K)|t - s|^{2\beta}$$

$$= |t - s|^{2H(t)K}\left(\frac{1}{\frac{1}{2\Gamma(K)} - C(\mu, \gamma, K)|t - s|^{2(\beta - H(t)K)}}\right).$$

Since $KH(t) < \beta$, we can choose $\delta$ small enough such that for all $s, t \geq 0$, and $|t - s| < \delta$, we have:

$$\frac{1}{\frac{1}{2\Gamma(K)} - C(\mu, \gamma, K)|t - s|^{2(\beta - H(t)K)} > 0.}$$

Indeed, it suffices to choose $\delta < \left(\frac{1}{\frac{1}{2\Gamma(K)} C(\mu, \gamma, K)}\right.\left.\wedge 1\right)^{\eta}$, where: $\eta = \left(\frac{1}{2}\left(\beta - K \sup_{t \geq 0} H(t)\right)^{-1}\right.$.

Finally, we get:

$$\mathbb{E}\left(B_{t}^{H(t),K} - B_{s}^{H(s),K}\right)^{2} \geq M|t - s|^{2H(t)K}, \text{ for all } |t - s| < \delta,$$

where $M = \left(\frac{1}{\frac{1}{2\Gamma(K)} - C(\mu, \gamma, K)\delta}\right)$.

Since $B^{H(\cdot),K}$ is a Gaussian process, then by Lemma 1 and the fact that the roles of $t$ and $s$ are symmetric, we obtain the desired result.  \hfill \Box

Remark 4. It is well known by Berman \cite{7} that, for a jointly measurable zero-mean Gaussian process $X := (X(t) : t \in [0, T])$ with bounded variance, the variance condition:

$$\int_{0}^{T} \int_{0}^{T} \left(\mathbb{E}|X_{t} - X_{s}|^{2}\right)^{-\frac{1}{2}} ds dt < +\infty,$$

is sufficient for the local time $L(t, x)$ of $X$ to exist on $[0, T]$ almost surely and to be square integrable as a function of $x$:

$$\int_{\mathbb{R}} L^{2}\left([a, b], x\right) dx < +\infty, \quad ([a, b] \subset [0, +\infty]).$$

For more informations on local time, the reader is referred to \cite{19,16,27} and the references therein. The natural question is to study the local non-determinism property.
for our process to prove the joint continuity of local time. For future work, we plan to study this question.

4. LOCAL ASYMPTOTIC SELF SIMILARITY PROPERTY

The dependence of $H(.)$ with respect to the time $t$ destroys all the invariance properties that we had for the fBm. For example the mBm is no more self-similar, nor with stationary increments. However, the authors in [21] showed that with the condition that $H(.)$ is $\beta$–Hölder continuous with exponent $\beta > 0$ and $\sup_{t \in \mathbb{R}^+} H(t) < \beta$, the mBm is locally asymptotically self-similar, (LASS for short), in the following sense:

$$\lim_{\rho \to 0^+} \left( \frac{B_{H(t)+\rho u}}{\rho^{H(t)}} - \frac{B_{H(t)}}{\rho^{H(t)}} ; \ u \geq 0 \right) \overset{d}{\longrightarrow} \left( B_{B_{H(t)}^u}^H ; \ u \geq 0 \right),$$

where $B_{H(t)}^u$ is a fBm with Hurst parameter $H(t)$, and $\overset{d}{\longrightarrow}$ stands for the convergence of finite dimensional distributions. Some authors use the term localizability for locally asymptotically self-similarity, (see Falconer [14],[15]).

Our process is another example of Gaussian process who loses the self similarity property when $H$ depend on $t$. However, we show in the following result that it is LASS. Before we deal with the proof of our result, we need the following lemma proved by Ait Ouahra et al. [2], (see Theorem 2.6).

**Lemma 2.** Let $B_{H,K}^H$ a bfBm with parameters $K \in (0,1)$ and $H \in (0,1)$. Then

$$\mathbb{E} \left( \frac{B_{t+\rho u}^{H(K)} - B_t^{H}}{\rho^{H(K)}} ; \ u \geq 0 \right) \overset{d}{\longrightarrow} 2^{1-K},$$

Now, we are ready to state and prove our result.

**Proposition 3.** Consider $H(.)$ a $\beta$–Hölder continuous function with exponent $\beta > 0$ such that $\sup_{t \geq 0} H(t) < \beta$, then $B_{H(t),K}^H$ is LASS:

$$\lim_{\rho \to 0^+} \left( \frac{B_{H(t)+\rho u}^{H(t),K} - B_{H(t)}^{H(t),K}}{\rho^{H(t)K}} ; \ u \geq 0 \right) \overset{d}{\longrightarrow} \left( 2^{1-K} B_{B_{H(t)}^u}^{H(t)K} ; \ u \geq 0 \right),$$

where $B_{H(t)}^{H(t)K}$ is a fBm with the Hurst parameter $H(t)K$.

**Proof.** We use the same arguments used in [21] in case of the mBm, (see Proposition 5). We prove the convergence in distribution by showing the following two statements:

(5) $$\mathbb{E} \left( \frac{B_{t+\rho u}^{H(t),K} - B_t^{H(t),K}}{\rho^{H(t)K}} \right) \overset{d}{\longrightarrow} 0,$$

(6) $$\mathbb{E} \left( \frac{B_{t+\rho u}^{H(t),K} - B_t^{H(t),K}}{\rho^{H(t)K}} \right)^2 \overset{d}{\longrightarrow} \sigma_t^2,$$

where

$$\sigma_t^2 = 2^{1-K} \text{Var} \left( \frac{B_{t+\rho u}^{H(t)K} - B_t^{H(t)K}}{\rho^{H(t)K}} \right) = 2^{1-K},$$
and $B^{H(t)K}$ is a fBm with the Hurst parameter $H(t)K$.
We deal with (6) since (5) is obvious. We have:

\[
\mathbb{E}\left[ \frac{B_{t+pu}^{H(t)K} - B_t^{H(t)K}}{\rho^{H(t)K}} \right]^2 = \mathbb{E}\left[ \frac{B_{t+pu}^{H(t)K} - B_t^{H(t)K}}{\rho^{H(t)K}} \right]^2 + 2\mathbb{E}\left[ \frac{B_{t+pu}^{H(t)K}}{\rho^{H(t)K}} - B_t^{H(t)K} \right] \left( \frac{B_{t+pu}^{H(t)K}}{\rho^{H(t)K}} - B_t^{H(t)K} \right)
\]

In view of Proposition 2, and the fact that $H(.)$ is $\beta$-Hölder continuous function, we have:

\[
\mathbb{E}\left[ \frac{B_{t+pu}^{H(t)K} - B_t^{H(t)K}}{\rho^{H(t)K}} \right]^2 \leq C(K)|H(t + pu) - H(t)|^2 \leq C'(K)\rho^{2(\beta - KH(t))}.
\]

Since $K \sup_{t} H(t) < \beta$, we get:

\[
\mathbb{E}\left[ \frac{B_{t+pu}^{H(t)K} - B_t^{H(t)K}}{\rho^{H(t)K}} \right]^2 \longrightarrow 0 \text{ as } h \rightarrow 0.
\]

In view of Lemma 1, we know that:

\[
\mathbb{E}\left[ \frac{B_{t+pu}^{H(t)K} - B_t^{H(t)K}}{\rho^{H(t)K}} \right]^2 \underset{\rho \to 0}{\longrightarrow} 2^{1-K}.
\]

Now, by Schwartz’s inequality, (1) and Proposition 2, we have:

\[
\mathbb{E}\left[ \frac{B_{t+pu}^{H(t)K} - B_t^{H(t)K}}{\rho^{H(t)K}} \right]^2 \leq \mathbb{E}\left[ \frac{B_{t+pu}^{H(t)K} - B_t^{H(t)K}}{\rho^{H(t)K}} \right]^2 \left( \frac{B_{t+pu}^{H(t)K} - B_t^{H(t)K}}{\rho^{H(t)K}} \right) \leq C\rho^{\beta - H(t)K} \rightarrow 0, \text{ since } K \sup_{t} H(t) < \beta.
\]

Hence, we deduce that:

\[
\mathbb{E}\left[ \frac{B_{t+pu}^{H(t)K} - B_t^{H(t)K}}{\rho^{H(t)K}} \right]^2 \underset{\rho \to 0}{\longrightarrow} 2^{1-K}.
\]

Consequently the LASS property is proved.
5. LONG RANGE DEPENDENCE

The long range dependence, (LRD for short), and long memory are synonymous notions. LRD measures long-term correlated processes. LRD is a characteristic of phenomena whose autocorrelation functions decay rather slowly. The presence and the extent of LRD is usually measured by the parameters of the process. Most of the definitions of LRD appearing in literature for stationary process are based on the second-order properties of a stochastic process. Such properties include asymptotic behavior of covariances, spectral density, and variances of partial sums. The specialness of LRD is a connection between long memory and stationarity, (see for example [25] in case of fBm and [22] and [24] in case of fbBm). For our process $B^{H(.),K}$, we use the same arguments used in [5] in case of standard mBm. Of course, the definitions must be adapted in our case since mBm and our extension does not have stationary increments, (see for example [3]):

**Definition 2.**

a) Let $Y$ be a second order process. $Y$ is said to have a LRD if there exists a function $\alpha(s)$ taking values in $(-1,0)$ such that:

$$\forall s \geq 0, \text{ cor}_Y(s, s + h) \approx h^{\alpha(s)} \text{ as } h \text{ tends to } +\infty,$$

where $\text{cor}_Y(s, t) := \frac{\text{cov}_Y(s, t)}{\sqrt{\mathbb{E}(Y^2(s))\mathbb{E}(Y^2(t))}}$.

b) Let $Y$ be a second-order process. $Y$ is said to have a LRD if:

$$\forall s \geq 0, \forall \delta \geq 0, \sum_{k=0}^{+\infty} |\text{cor}_Y(s, s + k\delta)| = +\infty.$$

In the next propositions, we prove some results about covariances and correlations of our process and its increments. In the sequel, we denote $f(t) \approx g(t)$ when $t$ if there exist $0 < c < d < +\infty$ such that for all sufficiently large $t$; $c \leq \frac{f(t)}{g(t)} \leq d$. We put:

$\text{cov}(t, s) := R^{H(.),K}(t, s)$ the covariance function of our process $B^{H(.),K}$ and $\text{cor}(t, s)$ its correlation function.

A) Asymptotic behaviour of the covariance and the correlation of $B^{H(.),K}$:

**Proposition 4.** When $t$ tends to infinity, and for all fixed $s \geq 0$, we have:

i) $K(H(t) + H(s)) < 1 \Rightarrow \text{cov}(t, s) \approx t^{H(t) + H(s))(K-1)}$.

ii) $K(H(t) + H(s)) > 1 \Rightarrow \text{cov}(t, s) \approx t^{K(H(t) + H(s))(H-1)}$.

iii) $K(H(t) + H(s)) < 1 \Rightarrow \text{cor}(t, s) \approx t^{-H(t)}$.

iv) $K(H(t) + H(s)) > 1 \Rightarrow \text{cor}(t, s) \approx t^{-KH(s)-1}$.

**Proof.** i) and ii) follows from a Taylor expansion of:

$$\left(t^{H(t)+H(s)} + s^{H(t)+H(s)}\right)^K - |t-s|^{(H(t)+H(s))K},$$

we obtain:

$$\text{cov}(t, s) = R^{H(.),K}(t, s) \approx K(D(H(t), H(s)))^K \left[s^{H(t)+H(s)}t^{H(t)+H(s))(K-1)}

+ (H(t) + H(s))st^{K(H(t)+H(s))(H-1)}, \text{ as } t \to \infty,$$

where the leading term is:

$$K(D(H(t), H(s)))^K s^{H(t)+H(s)}t^{H(t)+H(s))(K-1)} \text{ if } K(H(t) + H(s)) < 1,$$
and

\[ K(D(H(t), H(s)))^K (H(t) + H(s))^{s t^{K(H(t)+H(s))−1}} \quad \text{if} \quad K(H(t) + H(s)) > 1. \]

(Recall that \( H(t) + H(s) \) and \( D(H(t), H(s))^K \) are bounded).

iii) and vi): Using once again a Taylor expansion of:

\[
\text{cov}(t, s) := (2D(H(t), H(s)))^K \left( t^{H(t)+H(s)} + s^{H(t)+H(s)} \right)^K - |t - s|(H(t)+H(s))K_{KH(t)},
\]

where the leading term in this case is:

\[
K(D(H(t), H(s)))^K s^{H(t)+(1-K)H(s)} t^{-H(t)+(K-1)H(s)} \quad \text{if} \quad K(H(t) + H(s)) < 1,
\]

and

\[
K(D(H(t), H(s)))^K (H(t) + H(s)) t^{KH(s)-1} s^{1-KH(s)} \quad \text{if} \quad K(H(t) + H(s)) > 1.
\]

Since both \(-H(t)\) and \(KH(s) - 1\) belong to \((-1, 0)\) for all \(t, s\), we have the following result:

**Corollary 1.** For all admissible \(H(t)\), our process \(B^{H(.),K}\) has LRD in the sense of Definition 2)b). If, for all \(s\), \(K(H(t) + H(s)) > 1\) for all sufficiently large \(t\), then \(B^{H(.),K}\) has LRD in the sense of Definition 2)a), with functional LRD exponent: \(\alpha(s) = KH(s) - 1\).

**B) Asymptotic behaviour of the covariance and the correlation of the increments of \(B^{H(.),K}\):**

In the following results, to simplify the notation, let us denote:

\[
L(s, t) := \max \{H(t) + H(s), H(t + 1) + H(s), H(t) + H(s + 1), H(t + 1) + H(s + 1)\}
\]

**Proposition 5.** Let \(Y(t) = B^{H(t+1),K}(t + 1) - B^{H(t),K}_t\). Then, when \(t\) tends to infinity, and for all fixed \(s \geq 0\) such that the four quantities: \(H(t) + H(s), H(t + 1) + H(s), H(t) + H(s + 1), \) and \(H(t + 1) + H(s + 1)\) are all different, we have:

i) \(KL(s, t) < 1 \Rightarrow \text{cov}_Y(t, s) \approx t^{L(s,t)(K-1)}\),

ii) \(KL(s, t) > 1 \Rightarrow \text{cov}_Y(t, s) \approx t^{KL(s,t)-1}\),

iii) \(KL(s, t) < 1 \Rightarrow \text{cor}_Y(t, s) \approx t^{-K \text{max}(H(t), H(t+1))}\),

iv) \(KL(s, t) > 1 \Rightarrow \text{cor}_Y(t, s) \approx t^{K \text{max}(H(s), H(s+1))-1}\).

**Proof.**

i) and ii): By definition, we have:

\[
\text{cov}_Y(t, s) = \text{cov}(t + 1, s + 1) - \text{cov}(t + 1, s) - \text{cov}(t, s + 1) + \text{cov}(t, s).
\]

Applying the Taylor expansion to each covariance, we obtain:

- if \(KL(s, t) < 1\), from Proposition 4, it follow that:

\[
\text{cov}_Y(t, s) \approx t^{L(s,t)(K-1)}.
\]

- if at least one of \(K(H(t) + H(s)); K(H(t + 1) + H(s)); K(H(t) + H(s + 1))\) and \(K(H(t + 1) + H(s + 1))\) is greater than one, the order of \(\text{cov}_Y(t, s)\) will be the maximum of these value, since they all differ. More precisely, denoting \((t', s')\) the couple where the maximum of \(H(t) + H(s); H(t + 1) + H(s); H(t) + H(s + 1)\) and \(H(t + 1) + H(s + 1)\) is attained, we get:

\[
\text{cov}_Y(t, s) = K(D(H(t), H(s)))^K (H(t') + H(s'))^{s' t'^{K(H(t') + H(s'))-1}} + o(t'^{K(H(t') + H(s'))-1}).
\]
(iii) and iv): Again, this is simply obtained using Proposition 5 and the fact that 
\[ E(Y^2(t)) = O(t^{2K_{\text{max}}(H(t),H(t+1))}) \] 
if \( H(t) \) and \( H(t+1) \) differ (otherwise cancelation occur and the leading term is different). The exponent in the case where 
\( KL(s,t) > 1 \) results from the identity:
\[
\max \left( H(t) + H(s), H(t+1) + H(s), H(t) + H(s+1), H(t+1) + H(s+1) \right) 
- \max \left( H(t); H(t+1) \right) = \max(H(s), H(s+1)).
\]

**Corollary 2.** For all admissible \( H(t) \), our process \( B^{H(.),K} \) has LRD in the sense of Definition 2)b). If, for all \( s, KL(s,t) > 1 \), for all sufficiently large \( t \), the increments of \( B^{H(.),K} \) have LRD in the sense of Definition 2). As well as in the sense of Definition 2)a), with functional long range dependence exponent \( \alpha(s) = K_{\text{max}}(H(s),H(s+1)) - 1 \).

**Proof.** Obviously, both \( K_{\text{max}}(H(s),H(s+1)) - 1 \) and \( -K_{\text{max}}(H(t),H(t+1)) \) belong to \((-1;0)\). \( \square \)

**Conclusion and Outlook:**

(i) If we can prove the local non-determinism property for our process, (see Berman [8]), then Theorem 2 will be interesting to prove the existence and the Hölder regularities of the local time of our process, (see [9] in case of the mBm).

(ii) A response of the problem of the decomposition in law of our process appeared in Remark 2 will be useful to generalize a popular results for the bfBm like the existence and the Hölder regularities of its local time, (see [3] in case of bfBm and [9] in case of mBm).

(iii) It will be interesting to study a general case of Gaussian process of the form \( B^{H(.),K(.)} \) where both the parameters \( H \) and \( K \) depend on the time \( t \).

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