Bulk viscosities of a cold relativistic superfluid: color-flavor locked quark matter

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We consider the phonon contribution to the bulk viscosities $\zeta_1$, $\zeta_2$ and $\zeta_3$ of a cold relativistic superfluid. We assume the low temperature $T$ regime and that the transport properties of the system are dominated by the phonons. We use kinetic theory in the relaxation time approximation and the low energy effective field theory of the corresponding system. The parametric dependence of the bulk viscosity coefficients is fixed once the equation of state is specified, and the phonon dispersion law to cubic order in momentum is known. We first present a general discussion, valid for any superfluid, then we focus on the color-flavor locked superfluid because all the parameters needed in the analysis can be computed in the high density limit of QCD, and also because of the possible astrophysical applications. For the three independent bulk viscosity coefficients we find that they scale with the temperature as $\zeta_i \sim 1/T$, and that in the conformal limit only the third coefficient $\zeta_3$ is non-zero.

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I. INTRODUCTION

Superfluidity was first discovered in $^4$He when this was cooled at temperatures below 2.17 K [1, 2]. Later on it was found that various quantum liquids, both bosonic and fermionic, exhibit the same property. The phenomenon is due to the appearance of a quantum condensate, which spontaneously breaks a global $U(1)$ symmetry.

Relativistic superfluids might be realized in the interior of neutron stars where the temperature is low and the energy scale of the particles is sufficiently high. In particular, in the inner crust of neutron stars the attractive interaction between neutrons can lead to the formation of a BCS condensate, the system then becomes superfluid. Moreover, if deconfined quark matter is present in the core of neutron stars it will very likely be in a color superconducting phase [3]. Quantum Chromodynamics (QCD) predicts that at asymptotically high densities quark matter is in the color-flavor locked phase (CFL) [4]. In this phase up, down and strange quarks of all three colors pair forming a difermion condensate that is antisymmetric in color and flavor indices. The order parameter breaks the baryonic number $U(1)_B$ symmetry spontaneously, and therefore CFL quark matter is a superfluid as well.

If superfluidity occurs in the interior of compact stars, it should be possible to find signatures of its presence through a variety of astrophysical phenomena. For example, the most natural explanation for the sudden spin-up of pulsars [5], the so-called glitches, relies on the existence of a superfluid component in the interior of the star, rotating much faster than the outer solid crust. After the unpinning of the superfluid vortices, there is a transfer of angular momentum from the interior of the star to the outer crust, giving rise to the pulsar glitch.

Another possibility to detect or discard the presence of relativistic superfluid phases consists in studying the evolution of the r-mode oscillations of compact stars [6]. R-modes are non-radial oscillations of the star with the Coriolis force acting as the restoring force. They provide a severe limitation on the rotation frequency of the star through coupling to gravitational radiation (GR) emission. When dissipative phenomena damp these r-modes the star can rotate without losing angular momentum to GR. If dissipative phenomena are not strong enough, these oscillations will grow exponentially and the star will keep slowing down until some dissipation mechanism is able to damp the r-mode oscillations. Therefore, the study of r-modes is useful in constraining the stellar structure and can be used to rule out some specific matter phases. For such studies it is necessary to consider in detail all the dissipative processes and to compute the corresponding transport coefficients.

On the theoretical ground, there is renewed interest in the topic of this manuscript, as it has been recently proposed an holographic model to describe relativistic superfluidity [7, 8]. With the holographic techniques one intends to model a strongly interacting gauge theory with a weakly interacting gravity dual, having in this way a prescription to study many properties of the gauge system, such as its transport coefficients.

There are different formulations of the hydrodynamical equations governing a relativistic superfluid when there is no dissipation [6, 14]. They were derived as relativistic generalizations of Landau’s two-fluid model of non-relativistic superfluid dynamics [1, 2]. In the non-dissipative limit it is possible to show that all these approaches are equivalent. The dissipative terms which enter into the relativistic hydrodynamical equations of one the above approaches have been derived in Ref. [15]. As it occurs in the non-relativistic case, for a relativistic superfluid one can define a thermal conductivity, $\kappa$, a shear viscosity, $\eta$, and four bulk viscosity coefficients, $\zeta_1, \zeta_2, \zeta_3, \zeta_4$. While $\zeta_2$ has the same meaning
as the bulk viscosity of a normal fluid, \( \zeta_1, \zeta_3 \) and \( \zeta_4 \) refer to dissipative process that lead to entropy production only in the presence of a space-time dependent relative motion between the superfluid and the normal fluid components [1, 2].

This paper is devoted to determine the phonon contribution to the bulk viscosity coefficients of a cold relativistic superfluid. The superfluid phonon is the Goldstone mode associated with the spontaneous breaking of a \( U(1) \) symmetry. Being a massless mode, at low temperatures it gives the leading contribution to both the thermodynamics and transport phenomena. The contribution of other quasiparticle degrees of freedom, which typically have an associated non-zero energy gap \( E \), is suppressed when \( T \ll E \) and we will assume that we are in such a temperature regime. This work is somehow a follow up of Ref. [17], where similar computations were done for a non-relativistic superfluid, focusing on ultracold Fermi gases close to the unitary limit.

We perform our computation in the framework of kinetic theory, with the use of the relaxation time approximation (RTA). We present a general discussion regarding the phonon contribution to the transport coefficients for a generic relativistic superfluid. The basic ingredient in our derivation is the low energy effective field theory associated to the superfluid phonon, which is essentially dictated by symmetry considerations [18, 19]. The effective Lagrangian is presented as the typical expansion in derivatives of the Goldstone field, and with it one can derive the phonon dispersion law and all the scattering rates that are needed for the computation of the transport coefficients to a given accuracy. It turns out that the scaling laws for the transport coefficients depend on the coefficients of the effective field theory, which have to be matched with the microscopic theory.

After the presentation of the discussion of the bulk viscosities for a generic relativistic superfluid, in the last part of the present paper we focus on the color-flavor locked superfluid. At asymptotically high density, and thus weak coupling limit, all the coefficients of the phonon effective field theory can be computed from QCD. A study of the shear viscosity and of the bulk viscosity coefficient \( \zeta_2 \) due to phonons has been already presented in Refs. [20, 21]. The contribution of kaons to \( \zeta_2 \) has been studied in Ref. [22, 23]. The contribution to thermal conductivity due to phonons was first estimated in Ref. [24], and recently computed in Ref. [25].

This paper is organized as follows. In Section II we review the hydrodynamical equations for a relativistic superfluid, when effects of dissipation are included. In Section III we present the effective field theory for the superfluid phonon, at the leading and next-to-leading order, following the work of Ref. [18, 19]. In Section IV we derive the expressions for the phonon contribution to the bulk viscosity coefficients. Based on these formulas, and on the scattering amplitudes for the pertinent processes, we present a general discussion on the scaling behavior of these transport coefficients in Section V. Section VI is devoted to the study of the CFL superfluid. We present our conclusions in Section VII. We leave for the Appendix the proof that one needs the phonon dispersion law beyond linear order to obtain the first non-trivial contribution to the bulk viscosities. Throughout we use natural units, \( \hbar = c = k_B = 1 \), and metric conventions \((+,−,−,−)\).

II. HYDRODYNAMICS OF A RELATIVISTIC SUPERFLUID

The hydrodynamical equations of a relativistic superfluid have been derived using different formulations [9–14]. Here we will use the one derived by Son [14]. The superfluid properties of a system arise from the spontaneous breaking of a continuous \( U(1) \) symmetry, with the appearance of a Goldstone mode. Since hydrodynamics is an effective field theory valid at long time and long length scales, the standard fluid variables couple to the Goldstone field.

The hydrodynamical equations for the superfluid take the form of conservation laws for both the current, \( n^{\mu} \), and energy-momentum tensor, \( T^{\mu\nu} \), of the system

\[
\partial_\mu n^{\mu} = 0 \, , \quad \partial_\mu T^{\mu\nu} = 0 \, .
\]  
(1)

One further adds the Josephson equation, which describes the dynamical evolution of the Goldstone field, \( \varphi \), or phase of the condensate

\[
u^{\mu}\partial_\mu\varphi + \mu = 0 \, ,
\]  
(2)

where \( \mu \) is the chemical potential of the system [26].

In this formulation of the superfluid hydrodynamics there is a clear interpretation of both the current and the stress-energy tensor as being due to the sum of the normal fluid part and the coherent motion of the condensate (the superfluid), as these are expressed as

\[
T^{\mu\nu} = (\rho + P) u^{\mu} u^{\nu} - Pg^{\mu\nu} + V^2 \partial^{\mu} \varphi \partial^{\nu} \varphi \, ,
\]  
(3)

\[
n^{\mu} = n_0 u^{\sigma} - V^2 \partial^{\mu} \varphi \, ,
\]  
(4)
where $u^\mu$ is the hydrodynamical velocity, $\rho$ stands for the energy density, $P$ is the pressure, and $V$ is a variable proportional to the quantum condensate. The energy density obeys the relation $\rho = ST + n_0\mu - P$, where $S$ is the entropy of the system.

The dissipative terms associated to the above hydrodynamical equations have been constructed in \[13\], showing that in the non-relativistic limit they correspond to those appearing in Landau’s two-fluid model. To this end it is better to write the hydrodynamical equations in terms of the new variable

$$u^\mu = -(\partial^\mu \varphi + \mu u^\mu) .$$  \hfill (5)

Then, one can show that in the non-relativistic limit the spatial component of this four vector becomes the counterflow velocity of Landau’s equations, $w = m(v_s - v_n)$, with the identification $v_s = -\frac{\Sigma}{\rho}$ \[13\].

In order to construct the dissipative terms in the fluid equations one has to define a comoving frame. One possible choice is that in the frame where $u_\mu = (1, 0, 0, 0)$ the particle current and energy are fixed as

$$n^\mu = (\bar{n}, V^2 \nabla \varphi) , \quad T^{00} = \epsilon .$$  \hfill (6)

Now one imposes some restrictions to the possible dissipative terms that might be added to the current and energy-momentum tensor, $n^\mu$ and $T^{\mu\nu}$, respectively. In particular, one can choose constraints similar to those of the Eckart frame for the normal fluid hydrodynamics, $n^\mu = 0$, and $u_\mu T^{\mu\nu}_n = 0$ .

Dissipative terms are taken into account as follows. First, one modifies the Josephson equation

$$u^\mu \partial_\nu \varphi = -\mu - \chi ,$$  \hfill (7)

with

$$\chi = -\zeta_3 \partial_\mu (V^2 u^\mu) - \zeta_4 \partial_\mu u^\mu .$$  \hfill (8)

One also adds dissipative terms to the energy-momentum tensor, which take the form

$$T^{\mu\nu}_d = \kappa (\Delta^{\mu\gamma} u^\mu + \Delta^{\mu\gamma} u^n) \left( \partial_\gamma T + Tu^\delta \partial_\delta u_\gamma \right) + \eta \Delta^{\mu\gamma} \Delta^{\nu\delta} \left( \partial_\delta u_\gamma + \partial_\gamma u_\delta \right) + \frac{2}{3} \eta \partial_\alpha u^\alpha ,$$  \hfill (9)

where $\Delta^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$. In principle, more terms respectful with the symmetries of the problem are possible, but those are neglected, assuming that they are small. One simply retains six transport coefficients, namely the shear viscosity $\eta$, the thermal conductivity $\kappa$, and $\zeta_1$, $\zeta_2$, $\zeta_3$, $\zeta_4$ which are the four bulk viscosity coefficients. In the non-relativistic limit the expressions of the bulk viscosity coefficients agree with those introduced by Khalatnikov \[2\], up to some mass factors \[13\].

According to the Onsager symmetry principle \[1\, 2\, 16\] the transport coefficients satisfy the relation $\zeta_1 = \zeta_4$, while the requirement of positive entropy production imposes that $\kappa, \eta, \zeta_2, \zeta_3$ are positive and that $\zeta_1^2 \leq \zeta_2 \zeta_3$.

The friction forces due to bulk viscosities can be understood as drops, with respect to their equilibrium values, in the main driving forces acting on the normal and superfluid components. These forces are given by the gradients of the pressure $P$ and the chemical potential $\mu$. One can write in the comoving frame

$$P = P_{eq} - \zeta_1 \text{div} (V^2 w) - \zeta_2 \text{div} u ,$$  \hfill (10)

$$\mu = \mu_{eq} - \zeta_3 \text{div} (V^2 w) - \zeta_1 \text{div} u ,$$  \hfill (11)

where $P_{eq}$ and $\mu_{eq}$ are the equilibrium pressure and chemical potential. Note that the drops in the driving forces proportional to $\zeta_1$ and $\zeta_2$ lead to entropy production only when $\text{div} w \neq 0$. Indeed in case $\text{div} w = 0$, the entropy production rate is given by \[2\, 15\]

$$R = \zeta_2 (\text{div} u)^2 .$$  \hfill (12)

Therefore, when $\text{div} w = 0$, in the Josephson equation \[7\] one has that $\chi = -\zeta_1 \partial_\mu u^\mu$, but such a term does not lead to dissipation.

III. EFFECTIVE FIELD THEORY FOR THE PHONON OF A RELATIVISTIC SUPERFLUID

All the phonon properties that are needed for the evaluation of the bulk viscosities can be extracted from the low energy effective field theory associated to the relativistic superfluid. This has been constructed in Refs. \[18\, 19\], and we review it here.
The effective field theory for the Goldstone field is constructed as a power expansion over derivatives and over fields, $\sim \partial^n \phi^m$, allowing only the terms that respect the underlying symmetries of the system one is considering. The coefficients appearing in the low energy Lagrangian can be in principle computed from the microscopic theory, through a standard matching procedure.

Let us review the form of the effective field theory. Consider a relativistic system that experiences the spontaneous breaking of a $U(1)$ symmetry. If $\Theta$ is the phase of the field that gets an expectation value in its ground state $\Psi = |\Psi| e^{-i\Theta}$, then $\Theta$ is related to the Goldstone field $\varphi$ by the relation $\Theta = \mu_0 t - \varphi$, where $\mu_0$ is the zero temperature chemical potential [20], and $t$ is the time.

At the lowest order in $\Theta$, the Lagrangian must be a function of

$$\mathcal{X} = \frac{1}{2} g^{\mu\nu} \partial_\mu \Theta \partial_\nu \Theta,$$  \hspace{1cm} (13)

where $g^{\mu\nu}$ is the metric tensor. Since we are not interested in curved space-time we can rewrite

$$\mathcal{X} = \frac{1}{2} \left( \mu_0^2 - 2 \mu_0 \partial_\mu \varphi + \partial_\mu \varphi \partial^\mu \varphi \right).$$  \hspace{1cm} (14)

The Lagrangian for the Goldstone field at lowest order (LO) is expressed as

$$\mathcal{L}_{\text{LO}} = P \sqrt{2\mathcal{X}},$$  \hspace{1cm} (15)

where $P$ is the zero-temperature pressure of the system one is considering. The reason why the Lagrangian takes this form is that the effective action of the theory at its minimum for constant classical field configurations has to be equal to the pressure.

One can expand the pressure function around $\mu_0$ finding in this way the Lagrangian

$$\mathcal{L}_{\text{LO}} = \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{c_s^2}{2} (\partial_t \varphi)^2 - g_3 \partial_\mu \varphi (\partial_\nu \varphi \partial^\nu \varphi) + g_4 (\partial_\nu \varphi \partial^\nu \varphi)^2 + \cdots,$$  \hspace{1cm} (16)

where we have rescaled the Goldstone field so as to have a kinetic term properly normalized. We have also neglected a piece proportional to $\partial_\mu \varphi$, only relevant for vortex configurations. The constants $c_s, g_3, g_4$ can be written in terms of different ratios of derivatives of the pressure. In particular, $c_s$ is seen to agree with the zero temperature speed of sound of the system

$$c_s^2 = \frac{1}{\mu_0} \left( \frac{\partial P}{\partial \mu_0} \right) \left( \frac{\partial^2 P}{\partial \mu_0^2} \right)^{-1},$$  \hspace{1cm} (17)

while the coupling constants are expressed as

$$g_3 = \frac{1}{2\mu_0} \left( \frac{\partial^2 P}{\partial \mu_0^2} - \frac{\partial P}{\partial \mu_0} \frac{\partial P}{\partial \mu_0} \right) \left( \frac{\partial^2 P}{\partial \mu_0^2} \right)^{-3/2},$$

$$g_4 = \frac{1}{8\mu_0^2} \left( \frac{\partial^2 P}{\partial \mu_0^2} - \frac{\partial P}{\partial \mu_0} \frac{\partial P}{\partial \mu_0} \right) \left( \frac{\partial^2 P}{\partial \mu_0^2} \right)^{-2}.$$  \hspace{1cm} (18)

At next-to-leading order (NLO) there are four different structures, but two of them vanish in flat space-time. For simplicity, we will not write them down. At NLO one has

$$\mathcal{L}_{\text{NLO}} = \frac{f_1(\mu_0^2)}{\mu_0^2} \partial_\mu \varphi \partial^\mu \varphi + \frac{f_2(\varphi)}{\varphi} (\partial_\mu \varphi)^2 \partial \mu \Theta)^2,$$  \hspace{1cm} (19)

and we have re-defined the functions that appear in Ref. [19] so as to make $f_1$ and $f_2$ dimensionless functions.

We will mainly be concerned with the effect that the NLO terms have in modifying the phonon dispersion law. To this end, it is enough to expand the Lagrangian up to terms which are quadratic in $\varphi$. We assume that the functions $f_{1,2}$ that appear in Eq. (19) are analytical, and that they can be expanded in a Taylor series around $\mu_0^2$. Thus, after the necessary re-scaling to have a normalized kinetic term, one finds

$$\mathcal{L}_{\text{LO}} + \mathcal{L}_{\text{NLO}} = \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{c_s^2}{2} (\partial_t \varphi)^2 + m_1 (\partial_\mu \varphi \partial^\mu \varphi) + m_2 (\partial_\nu \varphi \partial^\nu \varphi)^2 + \cdots$$  \hspace{1cm} (20)

where

$$m_1 = \frac{f_1(\mu_0^2)}{\partial^2 P/\partial \mu_0^2},$$

$$m_2 = \frac{f_2(\mu_0^2)}{\partial^2 P/\partial \mu_0^2},$$  \hspace{1cm} (21)
which leads to the following modification of the dispersion equation for the phonon
\[ \omega^2 - c_s^2 q^2 + 2m_1\omega^2(\omega^2 - q^2) + 2m_2(\omega^2 - q^2)^2 = 0. \] (22)

Assuming that \( \omega, q \ll \sqrt{\frac{\omega^2}{\mu_0^2}} \), then one finds the dispersion law
\[ \epsilon_q = c_s q + Bq^3 + \cdots \] (23)

where
\[ B = \left( c_s(1 - c_s^2)m_1 - \frac{1}{c_s}(1 - c_s^2)^2m_2 \right). \] (24)

The NLO Lagrangian also introduces corrections to the phonon self-couplings, that we will not explicitly write down here as they will only be necessary if we were interested in computing corrections to the leading order behavior of the transport coefficients.

It is also interesting to discuss the conformal invariance of the phonon effective field theory. If the macroscopic system is conformal invariant, this symmetry should be inherited by the low energy effective field theory. At LO, this information is encoded in the explicit form of the pressure of the system, and correspondingly, in the speed of sound that should take the value of \( 1/\sqrt{3} \). At NLO, conformal invariance puts some constraints on the form of the functions \( f_1 \) and \( f_2 \), which can only depend on powers of the ratio \( \mathcal{N}/\mu_0^2 \). If conformal invariance is not a symmetry of the system, the functions \( P, f_1 \) and \( f_2 \) will depend on the scale responsible for the breaking of that symmetry.

It is important to stress the general character of the present discussion, valid for any relativistic superfluid, independent of the fact that the system is strongly coupled. Indeed in the above derivation one only employs the symmetries of the system. As an application one may think to the neutron superfluid which is believed to exist in the interior of neutron stars, or to the CFL superfluid that we will treat more carefully in the last part of the present manuscript.

**IV. PHONON CONTRIBUTION TO THE FREQUENCY DEPENDENT BULK VISCOSITIES**

At very low temperatures phonons give the leading thermal contribution to all the thermodynamic properties of the superfluid. In the hydrodynamic regime, phonons also give the leading contribution to the transport coefficients entering into the two-fluid equations. Khalatnikov developed the kinetic theory associated to these degrees of freedom for a non-relativistic superfluid [2]. One can also construct the transport theory associated to these Goldstone mode excitations of a relativistic superfluid [22, 29], for example using a gravity analogue model, showing that in the non-relativistic limit one recovers the theory of Khalatnikov [27].

For the computation of the bulk viscosity coefficients we will use the method suggested by Khalantnikov for superfluid \(^4\)He, and recently used for the cold Fermi gas in the unitarity limit [17]. In this approach one studies the dynamical evolution of the phonon number density \( N_{\text{ph}} \), and relates the variation of this quantity with the transport coefficients of the system. It has been shown in Ref. [17], that this approach is equivalent to solving the phonon transport equation in the relaxation time approximation. Note that with the RTA one obtains the correct parametric behavior of the transport coefficients, but one cannot fix with precision the numerical factor in front of these quantities. In order to obtain such a factor, it is necessary to solve numerically the corresponding Boltzmann equation, a rather complicated task. For the astrophysical applications we have in mind, getting the correct parametric behavior of all the transport coefficients is good enough, as this fixes the time scales of different physical processes.

Consider a perturbation to the system that changes the number of phonons per unit volume \( N_{\text{ph}} \). Collisional processes will tend to restore the equilibrium phonon number density. The dynamical evolution of \( N_{\text{ph}} \) can be extracted from the transport equation obeyed by the phonon distribution function [22]. If we further use the relaxation time approximation to deal with the collision term, and assume that the speed of sound is constant, then we obtain that
\[ \partial_t(u^\nu N_{\text{ph}}) = -\frac{\delta N_{\text{ph}}}{\tau_{\text{rel}}}, \] (25)

where \( \delta N_{\text{ph}} \) measures the departure of the phonon density from its equilibrium value. The relaxation time is given by
\[ \frac{1}{\tau_{\text{rel}}} \sim \frac{\Gamma_{\text{ph}}}{N_{\text{ph}}}, \] (26)

where \( \Gamma_{\text{ph}} \) is the rate of change of the number of phonons, to be specified later on.
From now on we will work in the comoving frame \( u^\mu = (1, 0, 0, 0) \) to simplify the treatment. In equilibrium, the number density of phonons is a function of the total current density, \( \bar{n} = n_0 + V^2 \mu_0 \), and of the entropy \( S \). Since we are interested in astrophysical applications we shall assume that the perturbation is periodic, so that there is a dependence on time of the sort \( \sim \exp(i \omega c t) \), with typical frequencies \( \omega_c \), as measured in the comoving frame. For astrophysical applications \( \omega_c \) is of the order of the frequency of rotation of the star.

Since the phonon number density depends on the current density and on the entropy, we have that

\[
\delta N_{\text{ph}} = \frac{\partial N_{\text{ph}}}{\partial \bar{n}} \delta \bar{n} + \frac{\partial N_{\text{ph}}}{\partial S} \delta S.
\] (27)

The hydrodynamic equations take the form

\[
u^\mu \partial_\mu \bar{n} = -\bar{n} \partial_\mu u^\mu - \partial_\mu (V^2 \omega^\mu),
\] (28)

\[
u^\mu \partial_\mu S = -S \partial_\mu u^\mu,
\] (29)

and therefore in the comoving frame one can rewrite the phonon number variation as

\[
\delta N_{\text{ph}} = \frac{\tau_{\text{rel}}}{1 - i \omega_c \tau_{\text{rel}}} \left\{ \left( N_{\text{ph}} - \bar{n} \frac{\partial N_{\text{ph}}}{\partial \bar{n}} - S \frac{\partial N_{\text{ph}}}{\partial S} \right) \text{div} \mathbf{u} - \frac{\partial N_{\text{ph}}}{\partial \bar{n}} \text{div} (V^2 \mathbf{w}) \right\}.
\] (30)

At this point one studies how the variation of the phonon density alters the equilibrium values of the pressure and chemical potential \[ 2 \]. In this way one can identify the different bulk viscosity coefficients and if we define

\[
I_1 = \frac{\partial N_{\text{ph}}}{\partial \bar{n}}, \quad I_2 = N_{\text{ph}} - S \frac{\partial N_{\text{ph}}}{\partial S} - \bar{n} \frac{\partial N_{\text{ph}}}{\partial \bar{n}},
\] (31)

we obtain that

\[
\zeta_1 = -\frac{\tau_{\text{rel}}}{1 - i \omega_c \tau_{\text{rel}}} \frac{T}{N_{\text{ph}}} I_1 I_2,
\] (32)

\[
\zeta_2 = \frac{\tau_{\text{rel}}}{1 - i \omega_c \tau_{\text{rel}}} \frac{T}{N_{\text{ph}}} I_2^2,
\] (33)

\[
\zeta_3 = \frac{\tau_{\text{rel}}}{1 - i \omega_c \tau_{\text{rel}}} \frac{T}{N_{\text{ph}}} I_1^2,
\] (34)

and therefore \( \zeta_1^2 = \zeta_2 \zeta_3 \), meaning that one of the relations for positive entropy production is saturated.

For \( \omega_c = 0 \) and in the non-relativistic limit we recover the expressions of the bulk viscosity coefficients that were obtained by Khalatnikov \[ 2 \]. For non-vanishing frequencies we obtain bulk viscosities that have both real and imaginary parts, as it occurs when there are periodic perturbations in time in the system, see \[ 1 \]. For the computation of the dissipated energy in the system, only the real part matters (see for example \[ 28 \]). One can see that the various coefficients attain their maximum value when \( \omega_c = \frac{1}{c_s} \).

At this point it is easy to show that for phonons with a linear dispersion law all the bulk viscosity coefficients vanish, independent of whether the system is conformal invariant or not, see the Appendix for a short proof. This is in agreement with the results obtained for \(^4\text{He} \[ 30 \], and for cold Fermi atoms close to the unitary limit \[ 17 \]. Then we consider the first correction to the phonon dispersion law given by the cubic term, in the form given by Eq. \( 23 \).

After defining the dimensionless parameter

\[
x = \frac{BT^2}{c_s},
\] (35)

we compute the various quantities evaluated with the equilibrium phonon distribution function to the leading order in \( x \). In this way we obtain that the number of phonons per unit volume is given by

\[
N_{\text{ph}} = \frac{T^3}{2 \pi^2 c_s^3} \left( \Gamma(3) \zeta(3) - x \Gamma(6) \zeta(5) + \mathcal{O}(x^2) \right),
\] (36)

while the entropy turns out to be

\[
S_{\text{ph}} = \frac{T^3}{6 \pi^2 c_s^3} \left( \Gamma(5) \zeta(4) - 3x \Gamma(7) \zeta(6) + \mathcal{O}(x^2) \right),
\] (37)
where $\Gamma(z)$ and $\zeta(z)$ stand for the Gamma and Riemann zeta functions, respectively.

For the evaluation of the functional derivatives that appear in Eq. (31) we use the same strategy followed in Ref. [17]. We use as independent variables the temperature $T$ and the zero temperature chemical potential $\mu_0$. Using a Maxwell relation, and the phonon contribution to the chemical potential, we arrive at the expressions (see Ref. [17] for more details)

$$I_1 = \frac{60}{\pi c_s^2 \bar{n}^2} T^5 \left( \pi^2 \zeta(3) - 7 \zeta(5) \right) \left( c_s \frac{\partial B}{\partial \bar{n}} - B \frac{\partial c_s}{\partial \bar{n}} \right),$$

and

$$I_2 = -\frac{20}{\pi c_s^2 \bar{n}^2} T^5 \left( \pi^2 \zeta(3) - 7 \zeta(5) \right) \left( 2Bc_s + 3\bar{n} \left( c_s \frac{\partial B}{\partial \bar{n}} - B \frac{\partial c_s}{\partial \bar{n}} \right) \right).$$

For a conformal invariant non-relativistic superfluid one expects that $\zeta_1$ and $\zeta_2$ vanish, while there is no constraint on $\zeta_3$ [31]. For the ultracold Fermi superfluid in the unitarity limit this is what one indeed finds [17]. One can expect that these results hold for relativistic superfluids, as well. In the following we shall explicitly prove that this is what happens.

Let us study under which conditions in a conformally invariant system the phonon contribution to the first and second bulk viscosity coefficients vanish. From the expressions in Eqs. (33) and (34) this is equivalent to asking that $I_2 = 0$. In a conformally invariant system the speed of sound is fixed, $c_s^2 = 1/3$. Further, since $\mu_0$ is the only energy scale it follows that by dimensional analysis $B \propto c_s/\mu_0^2$ and $\bar{n} \sim \mu_0^3$. Upon considering these scaling behaviors in Eq. (39), and in the case that $B = ac_s/\mu_0^2$ one finds that $I_2$ vanishes.

In Section VI we will discuss the CFL superfluid, where we find that for massless quarks, and neglecting the running of the gauge coupling constant, the parameter $B$ has the dependence expected in a conformal invariant system, and thus the only non-vanishing bulk viscosity turns out to be $\zeta_3$.

V. ZERO-FREQUENCY TEMPERATURE DEPENDENCE OF RELAXATION TIME AND BULK VISCOSITIES OF A COLD RELATIVISTIC SUPERFLUID

In the previous Section we have derived the expressions of the three independent bulk viscosity coefficients. In order to determine their actual temperature dependence we now estimate the temperature dependence of the relaxation time for the pertinent process. We shall consider the situation where $\omega_c = 0$: the non-static case can be easily deduced as well. For the time being the only relevant assumption we will make is that the contribution of other quasiparticles to transport phenomena is thermally suppressed. Therefore, we shall restrict to the situation where $T$ is much less than any energy gap of the system.

The computation of the phonon contribution to the bulk viscosities presents different subtleties. As we saw in Section III the phonon self-interactions are described by a low energy effective field theory, whose form is the same for all the systems that share the same global symmetries. However, the coefficients that appear in the low energy Lagrangian depend on the system considered, as their values have to be matched with the corresponding microscopic theory. As we saw, the bulk viscosities turn out to depend on the coefficient $B$ that appears in the phonon dispersion law, see Eqs. (38) and (39). The transport coefficients also depend on the relaxation time of the collisional process responsible of the corresponding dissipative phenomena. For bulk viscosity those are collisions that change the number of phonons. It turns out that the sign of $B$ decides whether some collisional processes are kinematically allowed or not, and ultimately, this affects how the relaxation time associated to bulk viscosity scales with $T$.

For a dispersion law where $B$ is positive, the so-called Beliaev process [32] which describes the decay of one phonon into two, $\varphi \rightarrow \varphi \varphi$, is kinematically allowed. The decay rate associated to this process is given by

$$\Gamma_{\text{ph}} = \int dP \, dK \, dQ \, |\mathcal{M}(P, Q, K)|^2 n_{\text{eq}}(p_0) (1 + n_{\text{eq}}(q_0)) (1 + n_{\text{eq}}(k_0)) (2\pi)^4 \delta^4(P - K - Q),$$

where $P^\mu = (p_0, \mathbf{p})$ is the four momentum and we have defined the integration measure as

$$\int dP = \int \frac{d^3p}{(2\pi)^3} 2\Theta(p_0) \delta(p_0^2 - \epsilon_p^2),$$

and $n_{\text{eq}}$ is the Bose-Einstein equilibrium distribution function. The squared of the scattering amplitude at LO is expressed as

$$|\mathcal{M}(P, Q, K)|^2 = g_s^2 (p_0 \cdot Q + q_0 \cdot P + k_0 \cdot K \cdot Q \cdot P)^2,$$
written in terms of the momenta of the phonons and of the three phonons self-coupling constant defined in Eq. (38). Even without an explicit evaluation of the above integral, one can infer that \( \Gamma_{\text{ph}} \sim g_3^2 T^3 / c_s^3 \), and that the relaxation time scales with the temperature as \( \tau_{\text{rel}} \sim c_s^3 / g_3^2 T^3 \).

As we have already shown in the previous Section, for a conformally invariant system, \( \zeta_1 = \zeta_2 = 0 \), whereas \( \zeta_3 \neq 0 \). Upon substituting the scaling behavior of \( \tau_{\text{rel}} \), of \( I_1 \) and \( I_2 \) given in Eq. (38) and (39), of \( \mathcal{N}_{\text{ph}} \) given in Eq. (39), in Eqs. (31) one finds that \( \zeta_3 \sim T^3 \). When conformal symmetry is broken, it follows that the three independent bulk viscosity coefficients do not vanish and have the same temperature dependence, i.e. \( \zeta_i \sim T^3 \).

Including NLO corrections both in the dispersion law \( \epsilon_p \), and in the self-coupling \( g_3 \), corrects this leading order behavior with terms of order \( B^2 T^2 / c_s^3 \), which at low temperatures are negligible.

For a phonon dispersion law that curves downward, \( B < 0 \), the Beliaev process is not kinematically allowed, and one has then to consider the five phonons collisional processes. The decay rate for these processes is very different with respect to the \( \varphi \rightarrow \varphi \varphi \) process. Further, as we discuss below, collisions with large or small scattering angle also have a rather different decay rate.

The decay rate of the process \( \varphi \varphi \rightarrow \varphi \varphi \varphi \) is given by

\[
\Gamma_{\text{ph}} = \int \prod_{i=1,\ldots,5} dP_i |\mathcal{M}_{5\varphi}(P_1, P_2, P_3, P_4, P_5)|^2 n_{eq}(p_0^1) n_{eq}(p_0^2) (1 + n_{eq}(p_0^3)) (1 + n_{eq}(p_0^4)) (1 + n_{eq}(p_0^5)) (2\pi)^4 \delta^4(P_1 + P_2 - P_3 - P_4 - P_5),
\]

where \( \mathcal{M}_{5\varphi} \) is the scattering matrix for the process considered.

After a careful analysis, one realizes that this rate has a very different behavior if one considers large or small angle collisions. Let us consider the Feynman diagram reported in Fig. 1. The square of the scattering matrix behaves as

\[
|\mathcal{M}_{5\varphi}(P_1, P_2, P_3, P_4, P_5)|^2 = g_3^2 g_4^2 F_3^2(P_1, -P_3, -Q) F_4^2(P_2, Q, -P_4, -P_5)
\]

where \( Q^{\mu} = (q_0, q) = (p_{10} - p_{30}, p_1 - p_3) \) is the momentum transfer, and \( F_3 \) and \( F_4 \) are polynomial of the phonon momenta that we shall not specify here, but that can be easily inferred from the Lagrangian (16). If one considers a collision with large angle scattering, naive power counting suggests that \( \Gamma_{\text{ph}} \sim g_3^2 g_4^2 T^{16} / c_s^{12}, \) and thus \( \tau_{\text{rel}} \sim c_s^3 / (g_3^2 g_4^2 T^{13}) \), which turns out to be a rather large value for low temperatures.

Processes associated to small angle scattering behave very differently. Let us start by considering phonons with a linear dispersion law. Then, the scattering matrix associated to a small angle collision behaves as

\[
|\mathcal{M}_{5\varphi}(P_1, P_2, P_3, P_4, P_5)|^2 = g_3^2 g_4^2 F_3^2(P_1, -P_3, -Q) F_4^2(P_2, Q, -P_4, -P_5) / (4 c_s^{14} p_0^2 p_3^2 (1 - \cos \theta_{13})^2),
\]

where \( \theta_{13} \) is the angle between the vectors \( p_1 \) and \( p_3 \). In the limit where \( \theta_{13} \to 0 \) this process presents a collinear singularity, which needs to be regularized. This might be done by considering a phonon dispersion law that includes a cubic term, as then in the denominator of Eq. (35) one has a term proportional to \( B^2 \), which does not vanish in the \( \theta_{13} \to 0 \) limit.

Let us stress here that while we have based our discussion on the Feynman diagram in Fig. 1 there are other channels for five phonons collisions that also present similar collinear singularities. The naive power counting associated to small angle collisions then suggests that \( \Gamma_{\text{ph}} \sim g_3^2 g_4^2 T^{12} / (c_s^3 B^2) \), and thus a relaxation time scaling as \( \tau_{\text{rel}} \sim c_s^3 B^2 / (g_3^2 g_4^2 T^{10}) \),
which is a much faster process than that corresponding to large angle collisions. Because in the computation of the transport coefficients one has to consider first the collisions with the shorter relaxation time, one then concludes that in this case, and when conformal symmetry is broken, all the bulk viscosity coefficients scale with the temperature as $\zeta_i \sim \frac{1}{T}$. Corrections to this leading behavior would be very suppressed, as discussed earlier.

Summarizing, bulk viscosities in the cold regime of relativistic superfluid strongly depend on the form of the phonon dispersion law. If conformal symmetry is unbroken, one should have $\zeta_1 = \zeta_2 = 0$, while the three bulk viscosity coefficients are non-zero when conformal symmetry is broken. When $B > 0$ the process $\varphi \rightarrow \varphi \varphi$ is kinematically allowed and gives the dominant contribution to bulk viscosity. In this case one may expect a rather small value for the bulk viscosities, with scaling $\zeta_i \sim T^3$. When $B < 0$ one has to consider the processes $\varphi \varphi \rightarrow \varphi \varphi \varphi$ which is dominated by small angle scatterings. These processes give scalings $\zeta_i \sim 1/T$. Let us finally remark that while our discussion has been done for relativistic superfluids, it is also valid for non-relativistic superfluids.

VI. BULK VISCOSITIES FOR THE CFL PHASE DUE TO SUPERFLUID PHONONS

At asymptotically high densities quark matter is believed to be in the color-flavor locked phase [4]. In this phase up, down and strange quarks of all three colors form zero-momentum spinless Cooper pairs. The CFL pairing pattern spontaneously breaks both the flavor chiral and the baryonic number $U(1)_B$ symmetries, leaving a discrete $Z_2$ symmetry. There are eight (pseudo) Goldstone modes associated to the chiral group, and one exact massless Goldstone boson associated to the breaking of the $U(1)_B$, which is responsible for the superfluid properties of the system. All gluons are massive, and all quarks are gapped. All the properties of CFL quark matter can be computed from QCD in the high density limit [3].

In this Section we derive the bulk viscosity coefficients of CFL quark matter using the strategy discussed in the first part of the manuscript.

A. Conformal limit

As shown in Section III, the effective low energy Lagrangian for the Goldstone boson can be obtained from the pressure [18], and for the CFL quark matter at asymptotic densities we have that

$$P[\mu_0] = \frac{3}{4\pi^2} \mu_0^4,$$

where $\mu_0$ is the quark chemical potential. At very high $\mu_0$, where the coupling constant is small $g(\mu_0) \ll 1$, the effects of interactions and the effects of Cooper pairing are subleading and have been neglected in Eq. (46). Also, at very high densities one might neglect the quark masses, as $m_q \ll \mu_0$.

The values of all the coefficients which appear in the phonon Lagrangian at LO, Eq. (16), can now be extracted as explained in Section III, and are given by

$$c_2^s = \frac{1}{3}, \quad g_3 = \frac{\pi}{9\mu_0^2}, \quad g_4 = \frac{\pi^4}{108\mu_0^4}.$$

It is possible to compute the phonon dispersion law at NLO for massless quarks by matching with QCD at finite density [33, 34]. In this way one obtains that

$$\epsilon_p = c_s p \left(1 - \frac{11}{540} \frac{p^2}{\Delta^2} + O\left(\frac{p^4}{\Delta^4}\right)\right),$$

where $\Delta$ is the CFL gap. In the asymptotic high density limit, the gap can be computed from QCD [36]

$$\Delta \simeq \mu_0 g^{-5} \exp\left(-\frac{3\pi^2}{4g}\right) b_0,$$

where $g$ is the QCD gauge coupling constant, and $b_0 = 512\pi^4 (\frac{3}{2})^{5/2} \exp\left(-\frac{\pi^2 + 4}{8}\right)$ [37, 38].

For sufficiently large chemical potentials, one can neglect the running of the coupling constant [39]. Then one can consider that the gap $\Delta \propto \mu_0$, that is, the gap only depends linearly on the chemical potential and therefore does not
break conformal invariance. Moreover in the regime we consider the total particle density is \( \bar{n} \approx \frac{3m_s^2}{\pi^2} \), consistent with the approximation to the CFL pressure, and being in the regime \( T \ll \mu_0 \). Since \( \bar{n} \propto \mu_0^2 \) and \( \Delta \propto \mu_0 \), it follows that

\[
I_1 = -\frac{60}{7\epsilon_s^2 \pi^2} T^6 \left( \pi^2 \zeta(3) - 7\zeta(5) \right) \frac{2Bc_s}{3\bar{n}}, \quad I_2 = 0,
\]

where, according with Eq. (55), the coefficient of the cubic term in momentum is given by

\[
B = -\frac{11c_s}{540\Delta^2}.
\]

Since \( B < 0 \), we have to consider that the processes giving the leading contributions to the bulk viscosity are small angle five phonons collisions. The estimate of the relaxation time for these processes is given in Sec. V and turns out to be

\[
\tau_{\text{rel}} \sim \frac{c_s^3 \mu_0^1 \Delta^2}{T^9}.
\]

We consider first the case \( \omega_s = 0 \). From Eqs. (50) we conclude that \( \zeta_1 = \zeta_2 = 0 \), while the third bulk viscosity does not vanish and depends parametrically on the physical scales of the problem as

\[
\zeta_3 \sim \frac{1}{T} \frac{\mu_0^5}{c_s \Delta^8}.
\]

We remark that these are only approximated results that arises in the \( g \ll 1 \) limit, after neglecting the running of the QCD gauge coupling constant and the effect of the strange quark mass. These are good approximations in the regime of high density \([39]\). For non-vanishing values of \( \omega_s \), one can accordingly find the value of the only non-vanishing transport coefficient, given the expressions of the relaxation time and zero-frequency viscosity, see Eq. (53).

**B. Including scale breaking effects due to the strange quark mass**

The bulk viscosities are sensitive to scale breaking effects. The quantum scale anomaly breaks the conformal symmetry introducing, through dimensional transmutation, the quantum scale \( \Lambda_{\text{QCD}} \). One could then compute \( g \)-corrections to the pressure of quark matter, which would then modify the different coefficients of the phonon effective field theory, introducing terms proportional to the QCD beta function. In the very high \( \mu_0 \) limit, when \( g(\mu_0) \ll 1 \), we expect this to be a rather negligible effect, as mentioned in the previous Section. Also consider that for large values of the chemical potential the gap parameter, \( \Delta \), does not break scale invariance because in this case one has \( \Delta \approx \mu_0 \).

The inclusion of quark mass effects leads to the breaking of scale invariance. In the CFL phase, the three light quarks participate in the pairing process, and therefore the largest effect comes from the strange quark mass \( m_s \). However, one cannot consider arbitrarily large values of the strange quark mass, because otherwise the CFL phase becomes chromo-magnetically unstable \([43]\). Therefore, we shall take \( m_s^2 \ll 2\Delta \mu_0 \), which is the threshold value under which the CFL phase is stable. Since \( m_s \ll \mu_0 \) we will consider the leading order corrections in \( m_s^2/\mu_0^2 \). After imposing the constraints of electrical neutrality and beta equilibrium of quark matter, the first correction of the order \( m_s^2/\mu_0^2 \) to the pressure reads \([3]\)

\[
P[\mu_0] = \frac{3}{4\pi^2} \left( \mu_0^4 - \mu_0^2 m_s^2 \right).
\]

Using Eqs. (17) and (18) we find that to the order \( m_s^2/\mu_0^2 \) the coefficients of the LO phonon Lagrangian are corrected due to this scale breaking effect as

\[
c_s^2 = c_{s,0} \left( 1 - \frac{m_s^2}{3\mu_0^2} \right), \quad g_3 = g_{3,0} \left( 1 + \frac{m_s^2}{4\mu_0^2} \right), \quad g_4 = g_{4,0} \left( 1 + \frac{m_s^2}{3\mu_0^2} \right),
\]

where the \( c_{s,0}, g_{3,0} \) and \( g_{4,0} \) refer to the values obtained when \( m_s = 0 \). Thus, the leading order effect of the strange quark mass in the phonon Lagrangian is to modify the velocity of the phonon, that is the speed of sound, and a finite renormalization of the cubic and quartic phonon self-couplings.

The phonon dispersion law at cubic order including quark mass corrections has not yet been computed in the literature, as it is subtle. Because a mass term in the QCD Lagrangian couples the particle and antiparticle degrees of freedom, that computation requires the knowledge of the antiquark propagator in the CFL phase, exactly as it occurs
in the computation of the CFL meson masses \[40,42\]. In any case, the correction of the coefficient of the cubic term in momentum in the phonon dispersion law will be of the form

\[ B = B_0 \left( 1 + \tilde{b}_s \frac{m_s^2}{\mu_0^2} \right), \tag{56} \]

with \( \tilde{b}_s \) a dimensionless constant. Since we are only interested in determining the scaling laws and not the numerical factors, we leave the computation of this constant for a future publication.

Upon substituting the expression of Eq. \([45] \) and \([56] \) in Eqs. \([48] \) and \([48] \), one finds

\begin{align*}
I_1 &= -\frac{60}{16\pi T^5} \left( \pi^2 \zeta(3) - 7\zeta(5) \right) \frac{2B_0 c_s,0}{3\tilde{n}_0} \left( 1 + \left( 2\tilde{b}_s + 1 \right) \frac{m_s^2}{\mu_0^2} \right), \\
I_2 &= -\frac{20}{16\pi T^5} \left( \pi^2 \zeta(3) - 7\zeta(5) \right) 2B_0 c_s,0 \left( \frac{5}{3} - \tilde{b}_s \right) \frac{m_s^2}{\mu_0^2}. \tag{58} \end{align*}

The scale breaking terms affect the relaxation times as well, because the phonon self-couplings are modified for \( m_s \neq 0 \), as shown in Eqs. \([55] \). A naive power counting analysis suggests that

\[ \tau_{\text{rel}} \sim \tau_{\text{rel},0} \left( 1 + \left( 2\tilde{b}_s - \frac{1}{3} \right) \frac{m_s^2}{\mu_0^2} \right). \tag{59} \]

We discuss now the values of the bulk viscosities for the \( \omega_c = 0 \) case, the non-static case is then easily deduced as well. As we are only interested in the scaling laws associated to the bulk viscosity coefficients, we conclude that to order \( m_s^2/\mu_0^2 \)

\[ \zeta_1 \sim \frac{1}{T} \frac{\mu_0^9}{c_s,0} \frac{m_s^2}{\mu_0^2}, \quad \zeta_2 \sim \frac{1}{T} \frac{\mu_0^{12}}{c_s,0} \frac{m_s^4}{\mu_0^2}. \tag{60} \]

Further, taking into account that \( \Delta \propto \mu_0 \), one obtains the dependence of the different coefficients with the basic physical scales of the problem, \( T, \mu_0, m_s \). In this way one finds \([\zeta_1] = \frac{m_s^2}{T\mu_0^4}, [\zeta_2] = \frac{m_s^4}{T} \) and \([\zeta_3] = \frac{1}{T\mu_0^6} \). Note that the scaling behavior of \( \zeta_2 \) was previously found in Ref. \[21\]. There the phonon Boltzmann equation was solved numerically, using a variational method. In that article it was assumed a phonon spectrum linear in momentum, and the interactions were extracted from the LO Lagrangian only, neglecting all phonon physics at NLO. In the linearization of the transport equation, the variations of the speed of the phonon with the total density were neglected, and that is why a non-zero result was obtained, even if the whole computation was carried out with the LO phonon physics.

**VII. CONCLUSIONS**

Superfluidity arises at low temperatures after the spontaneous breaking of a global \( U(1) \) symmetry. It is possible to determine the expression of the low energy Lagrangian of the Goldstone mode by using symmetry arguments. However, the coefficients of the effective field theory are not fixed by the symmetries of the system; they have to be matched with the microscopic field theory, and thus depend on the physics at the very short scales.

In this paper we have used the effective field theory describing the low energy properties of a superfluid system, together with kinetic theory in the relaxation time approximation, to obtain the contribution of the superfluid phonon to the bulk viscosity coefficients. We have first presented a general discussion on the low energy Lagrangian of the system. Then, we have determined the scaling behavior of the bulk viscosities with \( T \). In order to do this we have estimated the relaxation time of the processes responsible for bulk viscosity. The allowed processes depend on the sign of the correction to the linear dispersion law of the phonon. If the correction to the linear dispersion law curves upward, one phonon can decay into two, but this process is kinematically forbidden in the opposite case. If the correction to the linear dispersion laws curves downward, then the most important process is \( \varphi \varphi \to \varphi \varphi \varphi \) which is dominated by small angle scatterings. We have presented a general discussion on how the relaxation times behave in both cases in Section \[\text{V} \].

In the derivation of the transport coefficients we have neglected the contribution of other quasiparticles. This approximation is justified in the situation where \( T \) is much less that any other energy gap of the system. If not, one should take into account the contribution of other quasiparticles as well.

We have used the general expressions of the transport coefficients for a relativistic superfluid to determine the phonon contribution to the bulk viscosity coefficients of CFL quark matter. In order to do this we have determined
the scaling of the relevant relaxation times. Let us stress that with the relaxation time approximation we can only obtain the correct parametric behavior of the transport coefficients, but this should be enough to get their correct scales to be used, for example, in the study of the r-mode oscillations of a quark star. Previous analysis for the r-modes of a CFL quark star have been carried out in Refs. [44–46], but there the transport coefficients that we have studied as well as the mutual friction that operates in rotating superfluids [47] were ignored. Let us also point out that the knowledge of the bulk viscosities is relevant for the evaluation of the cavity conditions in quark stars [48].

It would be interesting to study the phonon contribution to the bulk viscosities of a neutron superfluid applying the general expressions derived in the present paper, as that contribution has been ignored in astrophysical applications. It might also be interesting to study the phonon contribution to transport phenomena with holographic techniques, as this should be independent of whether the underlying microscopic theory is strongly or weakly coupled.

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Appendix A: Phonons with a linear dispersion law

For phonons with a linear dispersion law one can write

\[ N_{ph} = 3 \frac{\Gamma(3) \zeta(3)}{\Gamma(5) \zeta(4)} S_{ph}, \]  

(A1)

meaning that \( N_{ph} \equiv N_{ph}(S) \) and the phonon density is independent of \( \bar{n} \) at this order. From Eqs. (32,33,34) it follows that for phonons with a linear dispersion law \( \zeta_1 = \zeta_2 = \zeta_3 = 0 \). Note that this result is true for any form of the speed of sound, \( c_s \equiv c_s(\mu_0) \). The fact that for phonons with a linear dispersion law \( N_{ph} \) is independent of \( \bar{n} \) can also be explicitly proven. By the chain rule we have that

\[ \frac{\partial N_{ph}}{\partial \bar{n}} = \frac{\partial N_{ph}}{\partial T} \frac{\partial T}{\partial \bar{n}} + \frac{\partial N_{ph}}{\partial \mu_0} \frac{\partial \mu_0}{\partial \bar{n}}, \]  

(A2)

that we can rewrite as

\[ \frac{1}{3N_{ph}} \frac{\partial N_{ph}}{\partial \bar{n}} = \frac{1}{T} \left( \frac{\partial T}{\partial \bar{n}} \right)_S - \frac{1}{c_s} \left( \frac{\partial c_s}{\partial \bar{n}} \right)_S. \]  

(A3)

From the value of the entropy with a linear phonon dispersion law we have that

\[ c_s = \left( \frac{\Gamma(5) \zeta(4)}{6\pi^2} \right)^{1/3} \frac{T}{S^{1/3}}, \]  

(A4)

therefore

\[ \frac{1}{c_s} \left( \frac{\partial c_s}{\partial \bar{n}} \right)_S = \frac{1}{T} \left( \frac{\partial T}{\partial \bar{n}} \right)_S, \]  

(A5)

and it follows that \( \frac{\partial N_{ph}}{\partial \bar{n}} = 0 \).

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