Algebraic roots of Newtonian mechanics: correlated dynamics of particles on a unique worldline

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Abstract

In the development of the old ideas of Stueckelberg–Wheeler–Feynman on the ‘one-electron Universe’, we study the purely algebraic dynamics of the ensemble of (two kinds of) identical point-like particles. These are represented by the (real and complex conjugate) roots of a generic polynomial system of equations that implicitly defines a single ‘worldline’. The dynamics includes events of ‘merging’ of a pair of particles modelling the annihilation/creation processes. Correlations in the location and motion of the particles-roots relate, in particular, to the Vieta formulas. After a special choice of the inertial-like reference frame, the linear Vieta formulas guarantee that, for any worldline, the law of (non-relativistic) momentum conservation is identically satisfied. Thus, the general structure of Newtonian mechanics follows from the algebraic properties of a worldline alone. A simple example of, unexpectedly rich, ‘polynomial dynamics’ is retraced in detail and illustrated via an animation (available from stacks.iop.org/JPhysA/46/175206/mmedia).

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(Some figures may appear in colour only in the online journal)

1. Introduction: on the ‘one-electron Universe’ conjecture

One of the most beautiful and striking ideas dealing with the foundations of theoretical physics is, perhaps, the Wheeler–Feynman conjecture on the so-called one-electron Universe. In his famous telephone call to Feynman [1], J Wheeler said: ‘Feynman, I know why all electrons have the same charge and the same mass. . . . Because, they are all the same electron!’ In fact, this conjecture based on the notion of a set of particles located on a single worldline easily explains the property of identity of elementary particles of one kind, the processes of annihilation/creation of a pair of ‘particle–antiparticle’ (in which one treats a ‘positron’ as an ‘electron’ running backwards in time [2]), etc. Unfortunately, the ‘one-electron Universe’
Figure 1. Generic worldline, numerous point-like ‘particles’ (at \(s = s_2\)) and creation (at \(s = s_1\)) or annihilation (at \(s = s_3\)) events.

paradigm had not been fully realized; one of the reasons for this is its failure to explain the particle–antiparticle asymmetry; other more essential reasons will be revealed below.

It should be noted that the Wheeler conjecture on the ‘one-electron Universe’ and, especially, on ‘a positron as moving backwards in time electron’ had been in fact implicitly initiated by the pioneering works of Stueckelberg [3, 4]. He assumed the existence of worldlines of general type (forbidden in the canonical STR) that contain the segments corresponding to the superluminal velocities of particles’ movements (figure 1). Then, the corresponding (hyper)plane of equal values of the time-like coordinate \(s = s_2\) intersects a worldline at a (generally, great) number of points. Physically, these form an ensemble of identical particles located on a single worldline.

If, in the course of time, the coordinate \(s\) is assumed to increase monotonically, then a pair of particles can appear at a particular instant \(s = s_1\) (or disappear at \(s = s_3\)). These events model the processes of creation (annihilation) of a pair ‘particle–antiparticle’.

Most likely, Stueckelberg [3] himself considered \(s\) as a \textit{fourth coordinate} and did not assume it to be a real physical evolution parameter. As to the latter, he introduced a time-like parameter \(\lambda\), which monotonically increases along the trajectory and is proportional to the proper time \(\tau\) of a particle. After this, all equations of the theory can be formally represented in a relativistic invariant form. On the other hand, segments of the trajectory corresponding to the opposite increments of \(\tau\) and \(\lambda\), namely to \(d\tau/d\lambda < 0\), were regarded as representing the backwards-in-time motion of an antiparticle.

However, \(\lambda\)-parametrization is in fact parametrization for the history of the \textit{individual} particle, which is in conflict with the concept of the ‘one-electron Universe’. In order to preserve the ensemble of identical particles on a unique worldline, one should consider only \(s\) as the ‘true’ time. Then, however, velocities of ‘particles’ should also be measured with respect to the parameter \(s\) and are necessarily superluminal at some segments of their history (and even infinite at the annihilation points, see below). Stueckelberg himself fully comprehended this difficulty [3, p 592].

Subsequently, numerous approaches exploiting Stueckelberg’s ideas (including his specific interpretation of the wavefunction, action functional and Lagrangian, etc) came to be known as \textit{parametrized relativistic theories} (see, e.g., [5] and references therein). In most part of them, the additional time-like parameter had been treated as a \textit{Lorentz invariant evolution parameter} or even as \textit{absolute Newtonian time} [6–8]. Nonetheless, the ultimate

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1 Remarkably, in [9], the invariance of Stueckelberg’s action with respect to the \textit{Galilean transformations} had been proved.
physical meaning of the variable \( s \) is still unclear. One way or another, multiple ‘particles’ on a single worldline related to \( \text{one and the same value of } s \) are not causally connected and cannot be simultaneously detected by an observer.

These and similar considerations reveal a lot of problems which arise under one’s attempts to realize the ‘one-electron Universe’ conjecture. However, the Stueckelberg–Wheeler–Feynman idea is too attractive to be abandoned at once. On account of the above-mentioned Galilei-invariance of Stueckelberg’s construction, at the first step it seems quite natural to consider a purely non-relativistic picture of processes represented in figure 1.\(^2\) The Galilean–Newtonian picture is the one that we accept in the paper and which allows for a self-consistent realization of the ‘one-electron Universe’ conjecture. A preliminary sketch of possible ways to the relativization of the scheme can be found in [10].

Specifically, our main goal throughout the paper is to obtain the correlated dynamics of identical point-like particles from the purely algebraic properties of a single worldline and without any resort to the Lagrangian structure or other standard constituents of physics. In this regard, our approach is quite different and much more radical than those of Stueckelberg and Wheeler–Feynman.

In order to realize the ‘one-electron Universe’ paradigm analytically, instead of the definition of a worldline in a habitual parametric form (and in the simplest parametrization \( x_0 = s \))

\[
x_a = f_a(s) \quad (a = 1, 2, 3),
\]

we define it (which is widely accepted for curves in mathematics) in an implicit form, i.e. through a system of three algebraic equations

\[
F_a(x_1, x_2, x_3, s) = 0.
\]

Then again, for any value of the time-like coordinate \( s \), one generally has a whole set \((N)\) of real roots of this system, which define a correlated kinematics \( x_a = f_a^{(k)}(s) \) of the ensemble of identical point-like singularities on a unique worldline\(^3\).

Multiple properties and ‘events’ related to particle-like formations defined by the roots of the system (2) are considered in section 2 and illustrated therein by a rather simple example. We restrict ourselves to plane motion and to a polynomial form of two generating functions in (2). Particularly, we take into account not only real roots but complex conjugate roots as well: the latter turn out to have an independent physical sense and correspond to other kinds of particle-like formations.

In section 3, a short excursus into the methods of the mathematical investigation of the solutions of system (2) (in the 2D case) of a generic polynomial type is undertaken. In the main, these methods make use of the so-called resultants of two polynomials. We demonstrate, in particular, that the Vieta formulas, well known for a single polynomial equation, naturally arise in the 2D case too. Quite remarkably (in the key section 4), the latter not only ensure the correlations between the positions and dynamics of different particles in the ensemble but also reproduce the generic structure of Newtonian mechanics and, in particular, lead to the satisfaction of the law of momentum conservation (in the special inertial-like ‘reference frames’!)

Section 5 contains concluding remarks on the motivations and actual developments of the presented scheme. The paper also contains an important appendix. Therein, the surprisingly

\(^2\) Despite the generally accepted belief in the indissoluble connection of the annihilation/creation processes with relativistic structures.

\(^3\) In the STR, equations defining a worldline, at least in the simplest parametrization (1), do not contain any trace of relativistic structure as well. It is therefore admissible to preserve the relativistic term ‘worldline’ in the considered Galilean–Newtonian picture.
rich dynamics defined by the simple polynomial system introduced in section 2 is retraced in detail, with the help of numerical calculations and graphical representations of the results. One can also see an impressive animation of the dynamics in the supplementary data (available from stacks.iop.org/JPhysA/46/175206/mmedia).

2. Two kinds of point-like particles: algebraic kinematics

Consider for simplicity the case of plane motion and a curve defined implicitly through a system of two independent polynomial equations with real coefficients

\[ F_1(x_1, x_2, s) = 0, \quad F_2(x_1, x_2, s) = 0, \]

where \( s \in \mathbb{R} \) is the particular coordinate which, in addition, plays the role of the evolution parameter; its variations will be assumed monotonic. As was argued in the introduction, one can think of \( s \) as being a Newtonian-like absolute global time.

Restriction by a polynomial form of functions \( F_1 \) and \( F_2 \) is motivated by the fact that only in this case is one able to obtain the complete set of solutions to (3). Moreover, the roots are then explicitly linked via the Vieta formulas. This results in the identical satisfaction of the law of momentum conservation (see section 4 below).

It should be particularly emphasized that we consider the system (3) as the only one whose properties we shall study throughout the paper: we do not intend to supplement it with any additional equations or statements of a physical or mathematical nature which do not explicitly follow from (3).

For any \( s \), the system (3) generally has a finite (\( N \)) number of roots \( \{x_{k1}, x_{k2}\}, k = 1, 2, \ldots, N \). These define the positions of \( N \) identical point-like particles at the instant \( s \) on a 2D trajectory curve

\[ F(x_1, x_2) = 0, \]

whose form can be obtained from (3) after the elimination of \( s \) and which, generally, consists of a number of disconnected (on \( \mathbb{R}^2 \)) components. With monotonic growth of time \( s \), particles move along the trajectory curve with arbitrary velocities, and their number is (almost always) preserved.

However, at particular discrete instants of \( s \), say, at \( s = s_0 \), a pair of the real roots of (3) turn into one multiple root and then become a pair of complex conjugate roots. Consequently, the corresponding pair of particles merge (collide) at \( s = s_0 \) at some point \( \{x^*_{1}, x^*_{2}\} \) and then disappear from the real slice of space. Such an ‘event’ can serve as a model of the annihilation process. Conversely, at another instant a pair of the real roots can appear, modelling the process of pair creation.

It should be noted nevertheless that one cannot ignore the formations which correspond to the complex conjugate roots of (3) and ‘live’ in the complex extension of real space. This fact will become evident in the key section 4, while at the moment we only remark that such formations can be depicted with respect to equal real parts of their coordinates.

From this viewpoint, a pair of complex conjugate roots corresponds to a composite particle that consists of two parts coinciding at \( \mathbb{R}^2 \) but possessing opposite additional ‘tails’ represented by imaginary parts of coordinates. For brevity, we shall call particle-like formations represented by the real roots of the system (3) ‘R-particles’, and by complex conjugate pairs of roots ‘C-particles’.

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4 We suspect that generalization to the physical 3D case will only be technically more complicated but no problem of principal character will arise during it.
We can now present a simple example of the issues exposed above. Let us take the functions \( F_1 \) and \( F_2 \) in (3), say, in the following (randomly selected) form:

\[
F_1(x, y, s) = -2x^3 + y^3 + sx + sy + y + 2 = 0,
\]
\[
F_2(x, y, s) = -x^3 - 2x^2y + s + 3 = 0.
\]

Eliminating \( s \), one obtains the trajectory (on the real space slice) which consists of three disconnected components (figure 2). Then, via elimination of \( y \), one reduces the system (6) to a single polynomial equation \( P(x, s) = 0 \) of degree \( N = 9 \) in \( x \) and with coefficients depending on \( s \). The latter evidently allows for full analysis and numerical calculations.

As a result, one obtains that, for any \( s \), there exist precisely nine solutions of the system, some of them being real while others are complex conjugate. Analysing condition (5) (or, equivalently, the structure of discriminant of the polynomial \( P(x, s) \)), one concludes that there are six ‘events’ which correspond to the following (approximate) values of global time \( s \):

\[-97.3689; -4.0246; -3; -2.7784; -2.7669; +2932.49.\]

Some of these relate to annihilation (merging) events whereas others relate to creations of a pair. In the appendix (and in the animation file (available as supplementary data at stacks.iop.org/JPhysA/46/175206/mmedia); see the representative frame in figure A3 in the appendix), one can find many details of the surprisingly rich dynamics, including the processes of annihilation of two R-particles.
accompanied by the birth of a composite C-particle and vice versa. One also observes that the created C-‘quantum’ travels between two disconnected branches of the real trajectory, arrives at the second branch and when there gives rise to a divergent pair of real R-particles (creation of a pair), see also figure 2. Remarkably, this strongly resembles the process of exchange of quanta specific to QFT.

To conclude, let us obtain the expression for velocities of individual particles with respect to global time \( s \). Introducing the canonical parametrization for an individual R-particle as \( x_A = x_A(s) \) and then taking the total derivative with respect to \( s \) (denoted by a ‘dot’) in (3), one obtains

\[
0 = \dot{F}_A + \frac{\partial F_A}{\partial x_B} \dot{x}_B, \tag{7}
\]

whence it follows

\[
\dot{x}_C = -R_A^B \dot{F}_A, \tag{8}
\]

where \( R_A^B \) is the inverse matrix,

\[
R_A^B \frac{\partial F_A}{\partial x_B} = \delta_B^A. \tag{9}
\]

Comparing (8) with condition (5), one concludes that at the instants of annihilation/creation, the velocities of both particles involved in the process are necessarily infinite\(^5\). In the framework of the Galilean–Newtonian picture assumed throughout the paper, this property cannot cause any objection.

3. Resolving a system of polynomial equations: resultants and eliminants

Let us concentrate now on the procedure of resolution of the system of polynomial equations (3) of generic type (below we have made obvious redesignations \( x_1 \mapsto x, x_2 \mapsto y \)),

\[
 \begin{align*}
 F_1(x, y, s) &= [a_{n,0}(s)x^n + a_{n-1,1}(s)x^{n-1}y + \cdots + a_{0,n}(s)y^n] + \cdots + d_{0,0}(s) = 0, \\
 F_2(x, y, s) &= [b_{m,0}(s)x^m + b_{m-1,1}(s)x^{m-1}y + \cdots + b_{0,m}(s)y^m] + \cdots + b_{0,0}(s) = 0. \tag{10}
 \end{align*}
\]

Only forms of the highest (\( n \) and \( m \), respectively) and the least orders are written out in (10). Both polynomials are assumed to be functionally independent and irreducible, while all the coefficients \( \{a_{i,j}(s), b_{i,j}(s)\} \) depend on the evolution parameter \( s \) and take values in the field of real numbers \( \mathbb{R} \).

Rather surprisingly, not much is known about the properties of solutions to a nonlinear system of polynomial equations. Of course, some results can be taken from those for the one-dimensional case. For example, it is easy to demonstrate that all the roots \( \{x_0, y_0\} \) of such a system are either real (\( x_0 \) and \( y_0 \) both together) or both entering in complex conjugate pairs. However, even the problem of explicit determination of the full number of solutions of (10) over \( \mathbb{C} \) from, say, the properties of coefficients and degrees of the polynomials \( F_1 \) and \( F_2 \) is far from being completely resolved (in contrast to the one-dimensional case) [11].

In practical calculations, however, it is quite possible to determine this number and evaluate approximately all the roots of the system (10), for both real and complex conjugates. To do this, the most convenient method is perhaps the method of resultants [12, 13]. To be precise, let \( \{x_0, y_0\} \) be a solution to (10); then, for \( y = y_0 \) fixed, both equations (10) on \( x \) should have a common root \( x = x_0 \). The necessary and sufficient condition for this is well known:

\[
R_s(y) = g_N(s)y^N + g_{N-1}(s)y^{N-1} + \cdots + g_0(s) = 0, \tag{11}
\]

\( \text{This can also be seen just from figure 1, since at such instants one obviously has } \mathrm{d}s = 0, \mathrm{d}x_A \neq 0. \)
where \( R[F_1(x), F_2(y), x] = R_s(y) \) is the resultant of two polynomials \( F_1, F_2 \) via \( x \) taken at the condition \( y = y_0 \) (for simplicity the index 0 is omitted in (11) and below). The structure of the resultant (which in this case is often called eliminant) is represented by the determinant of a Sylvester matrix (see, e.g., [12, 13]). Coefficients \( g_I(s) \) depend on \( [a_i, j(s), b_i, j(s)] \).

Analogously, one can exchange the coordinates, and after the elimination of \( y \) arrive at the dual condition

\[
R_s(x) = f_K(s)x^K + f_{K-1}(s)x^{K-1} + \cdots + f_0(s) = 0. \tag{12}
\]

When the coefficients \( a_{0,m}, a_{1,m}, b_{0,m} \) and \( b_{0,m} \) are all nonzero, the leading terms in (11) and (12) are, as a rule\(^6\), of equal degree \( K = N = mn \) (see, e.g., [11, 14, 15]). Then, all their \((N = mn)\) solutions over \( \mathbb{C} \) can be numerically evaluated and put in correspondence with each other to obtain \( N \) solutions \( \{x_k(s), y_k(s)\}, k = 1, 2, \ldots, N \), of the initial system (10). If some of the above four coefficients turn to zero, the number of solutions can be less than (or equal to) the maximal possible value \( mn \). Nonetheless, in this case all the solutions can still be (approximately) obtained through seeking of eliminants with the help of a computational software program.

In order to illustrate the above presented procedure, consider the following system of equations (closely related to the previous system (6), see section 4 below):

\[
\begin{align*}
F_1 &= -2x^3 + y^2 + 6x^2x^2 + 3xy^2 - (6s^4 - s)x + (1 + s + 3s^2)y \\
&\quad + 2s^6 + s^2 + s + 2 = 0, \\
F_2 &= -x^3 - 2x^2y + (3s^2 - 2s)x^2 + 4s^2xy - (3s^4 - 4s^3)x - 2s^4y \\
&\quad + 5s^6 - 2s^5 + s + 3 = 0.
\end{align*}
\tag{13}
\]

Using the computer algebra system ‘Mathematica 8’, we easily find the eliminant \( R_s(y) \) and come to the equation

\[
R_s(y) = 17y^9 + 153x^8 + \cdots = 0. \tag{14}
\]

Analogously, we obtain the dual condition

\[
R_s(x) = -17x^9 + 153x^8 + \cdots = 0. \tag{15}
\]

The sets of nine solutions of equations (14) and (15) can now be obtained and put in one-to-one correspondence with each other to form nine solutions of the system (13). For example, at \( s = 1 \) the system has one real solution \( \{x \approx 2.3079, y \approx -0.4848\} \) (defining the position of one R-particle) and four pairs of complex conjugate roots (corresponding to four C-particles).

4. Vieta’s formulas and the law of momentum conservation

We are now ready to consider the most important issue of the present publication, namely the correlations of different roots and the related particles’ dynamics. Firstly, these correlations follow from the Vieta formulas, which are well known for the case of a single polynomial equation.

Specifically, as we have seen in the previous section, any system of two polynomial equations (10) can be reduced to a pair of dual equations (11) and (12) for eliminants—polynomials in one variable each (\( x \) or \( y \), respectively). Thus, we have demonstrated that Vieta’s formulas naturally arise in the 2D case too.

Below we consider the generic case when the degrees of both eliminants are equal, \( K = N \). Then, the first and simplest of the Vieta formulas (linear in roots) look as follows:

\[
\begin{align*}
NX(s) &:= x_1(s) + x_2(s) + \cdots + x_N(s) = -f_{N-1}(s)/f_N(s), \\
NY(s) &:= y_1(s) + y_2(s) + \cdots + y_N(s) = -g_{N-1}(s)/g_N(s). 
\end{align*}
\tag{16}
\]

\(^6\) Precisely, in the case when the numerical coefficient given by any of equal resultants \( R[F_i^n(1, y), F_i^n(1, y), y] = R[F_i^n(x, 1), F_i^n(x, 1), x] \) is nonzero, \( F_i^n \) and \( F_i^n \) being forms of the highest degrees (\( n \) and \( m \), respectively) in (10).
Obviously, quantities \( X(s), Y(s) \) can be regarded as coordinates of the centre of mass of the closed system of \( N \) identical (and, therefore, of equal masses \( m_1 = m_2 = \cdots = m_N \)) point-like particles with coordinates represented by the roots \( s_k(s), y_k(s) \), \( k = 1, 2, \ldots, N \), of the system (10) and varying in time \( s \).

An important fact here is that complex conjugate roots also enter the left-hand part of the conditions (16) though their imaginary parts cancel and do not contribute to the centre-of-mass coordinates. This observation makes it obvious that such roots cannot be regarded as ‘unphysical’; in contrast, they should be treated as a second type of particle-like formations (C-particles) which ‘appear/disappear’ in the processes of creation/annihilation of real R-particles and ‘move’ in the complex extension of space between the components of the trajectory of the latter. Only real parts of these complex conjugate roots contribute to the centre-of-mass coordinates (and to the total momentum, see below) and, on the other hand, can be visualized in the physical space. We have exemplified such a visualization in the previous section. As to the imaginary parts of such roots, they could be responsible for internal phases and corresponding frequencies of C-particles [16, 17]; however, their true meaning is vague at the present stage of consideration. Note also that the effective mass of a C-particle is in fact twice as great as that of an R-particle since any C-particle is represented by a pair of complex conjugate roots (and thus by their equal real parts on the physical space slice).

The right-hand part of equations (16) (dependent on \( s \)) indicate that, generally, the centre of mass of the closed ‘mechanical’ system of the R- and C-particles does not, generally, move uniformly and rectilinearly. However, one can treat this contradiction with Newtonian mechanics as a manifestation of the non-inertial nature of the reference frame being chosen. One has therefore the right to perform a coordinate transformation to another frame which would model the inertial properties of matter (recall that we assume only one single ‘worldline’ to exist which represents ‘all particles in the Universe’).

In fact, it is easier to just find the distinguished reference frame in which the centre of mass is at rest. To do this, let us return to the eliminants (11), (12) and get rid of the terms of the \((N - 1)\)th degree, setting

\[
x = \tilde{x} - (N - 1)f_{N-1}(s)/f_N(s), \quad y = \tilde{y} - (N - 1)g_{N-1}(s)/g_N(s).
\]

Now one can rewrite the system (10) in the new variables as

\[
\tilde{F}_1(\tilde{x}, \tilde{y}, s) = 0, \quad \tilde{F}_2(\tilde{x}, \tilde{y}, s) = 0
\]

and consider it as describing the same closed ‘mechanical’ system of \( N \) particles in the centre-of-mass reference frame. Indeed, equations on the eliminants (11), (12) in the new variables take the form

\[
\tilde{R}_1(\tilde{x}) = f_N(s)\tilde{x}^N + 0 + \cdots + \tilde{f}_0(s) = 0, \quad \tilde{R}_2(\tilde{y}) = g_N(s)\tilde{y}^N + 0 + \cdots + \tilde{g}_0(s) = 0
\]

and, according to Vieta’s formulas (16), one obtains

\[
N\tilde{X}(s) := \tilde{x}_1(s) + \tilde{x}_2(s) + \cdots + \tilde{x}_N(s) = 0, \quad N\tilde{Y}(s) := \tilde{y}_1(s) + \tilde{y}_2(s) + \cdots + \tilde{y}_N(s) = 0.
\]

Differentiating then (20) with respect to the evolution parameter \( s \) one obtains the law of conservation of the projections \( P_x, P_y \) of total momentum for a closed system of identical ‘interacting’ particles defined by equations (18):

\[
P_x := \tilde{x}_1(s) + \tilde{x}_2(s) + \cdots + \tilde{x}_N(s) = 0, \quad P_y := \tilde{y}_1(s) + \tilde{y}_2(s) + \cdots + \tilde{y}_N(s) = 0
\]

(the ‘tilde’ is omitted for simplicity).

If necessary, one can now transfer to another inertial reference frame using a Galilei transformation, say, \( y \mapsto y', x \mapsto x - Vs \), \( V = \text{constant} \) in which the centre of mass will move uniformly and rectilinearly with velocity \( V \); specifically, one obtains \( X(s) = Vs, Y(s) = 0 \).
Repeating now the procedure of differentiation, one obtains from (21) a universal constraint on instantaneous accelerations of interacting identical particles:
\[
\ddot{x}_1(s) + \ddot{x}_2(s) + \cdots + \ddot{x}_N(s) = 0, \quad \ddot{y}_1(s) + \ddot{y}_2(s) + \cdots + \ddot{y}_N(s) = 0, \tag{22}
\]
which, for the simplest case of a system of two particles, leads to Newton’s third law together with the definition of the forces of mutual interaction (provided the equal masses are set unit, \(m_1 = m_2 = 1\)):
\[
f_x^{(21)} := m_1 \ddot{x}_1, \quad f_x^{(12)} := m_2 \ddot{x}_2, \quad f_y^{(21)} := m_1 \ddot{y}_1, \quad f_y^{(12)} := m_2 \ddot{y}_2; \tag{23}
\]
\[
f_x^{(21)} + f_x^{(12)} = \ddot{x}_1 + \ddot{x}_2 \equiv 0, \quad f_y^{(21)} + f_y^{(12)} = \ddot{y}_1 + \ddot{y}_2 \equiv 0. \tag{24}
\]
Essentially, for two particles the whole system of Newton’s mechanics may be completely recovered (though a concrete form of the forces’ laws themselves is not fixed by the equations of the worldline (18)).

Let us now return to illustrating the general construction presented above of the model of the ‘mechanical’ system consisting of \(N = 9\) point-like particles and defined by the equations of the worldline (13). Since the terms of degree 8 = \(N - 1\) in the eliminants (14), (15) are nonzero and corresponding coefficients, moreover, depend on the time parameter \(s\), the total momentum is not conserved. Thus, (13) represents the worldline in a non-inertial reference frame. In order to make a transition to the centre-of-mass frame, one has to perform, according to (17), the transformation of coordinates of the form
\[
x = \tilde{x} + s^2, \quad y = \tilde{y} + s. \tag{25}
\]
In the new variables, eliminants (14), (15) take the form
\[
R_x(y) = 17y^9 + 0 + (35 + 33s)y^7 + \cdots = 0, \tag{26}
\]
\[
R_y(x) = -17x^9 + 0 + (4s - 4)x^7 + \cdots = 0, \tag{27}
\]
whereas the defining system (13) turns out to be exactly the system of equations (6) (already examined in section 2). It is now not difficult to check that the total momentum of all nine particles defined by the latter is the same at every instant \(s\) and, precisely, equal to zero. Thus, equations (6) and (13) represent in fact the same ensemble of identical particles in the inertial centre-of-mass reference frame and in a non-inertial one, respectively.

It is worth noting that, besides the simplest linear Vieta formulas (16), there exist other nonlinear ones, the highest of which, say, looks as follows:
\[
x_1(s)x_2(s)\cdots x_N(s) = f_0(s)/f_N(s), \quad y_1(s)y_2(s)\cdots y_N(s) = g_0(s)/g_N(s). \tag{28}
\]
In principle, it is possible to find a transformation of coordinates that will nullify a number of terms in the eliminants; in this case one would have, apart from the centre of mass and the related total momentum conservations, other combinations of roots (and their derivatives) which would preserve their values in time (‘nonlinear integrals of motion in the framework of Newtonian mechanics’).

To conclude the section, let us say some words about the law of energy conservation. In the framework of the nonrelativistic mechanics under consideration (and in contrast to the law of conservation of momentum), this law requires the potential energy to be taken into account. At present we are not aware whether the concrete form of the latter can be determined from the algebraic equations of the worldline alone.

Nonetheless, there exist some hints that the structure of the forces’ laws can indeed be encoded in the general properties of the worldline. For instance, as far back as 1836, C F Gauss made an interesting observation on the roots \(z_k\), \(k = 1, 2, \ldots, N\), of a single polynomial equation \(F(z) = 0\) of a general form (see, e.g., [13, chapter 1]). These define a set
of identical particles located at the corresponding points of the $\mathbb{C}$-plane. Consider now any root $z_0$ of the derivative polynomial equation $F'(z) = 0$ (which does not coincide with a (multiple) root of the initial equation). Then, it corresponds to a libration point (point of equilibrium) for the resultant field of radial forces produced by all the roots $\{z_k\}$, under the condition that these forces be inversely proportional to the distance, $f_k \propto 1/|z - z_k|$ (and effective ‘charges’ of the sources are all equal).

Unfortunately, we were unable to find an analogue of this remarkable property in the 3D case. However, the example indicates that even in the 2D case (and in the 3D one as well) the roots of the derivative equations for eliminants (11), (12), namely $R'_x(y) = 0, R'_y(x) = 0$, define in fact a new (third) kind of particle-like formations whose dynamics can be correlated with others in a quite nontrivial way. We intend to consider this issue in a forthcoming publication.

5. Conclusion

Stueckelberg–Wheeler–Feynman’s conjecture about identical particles moving along a unique worldline looks attractive not only from the ‘philosophical’ viewpoint. It easily solves, say, the paradox that point-like particles can meet at some points of the physical 3D space (even for a 2D space the codimension of such an event is zero!). Moreover, the very condition that all such particles-copies belong to the same curve turns out to be a rigid restriction which requires a strongly correlated dynamics of these copies that reproduces in fact the process of physical interactions.

We have demonstrated that any generic system of polynomial equations like (10) completely defines a single ‘worldline’ and an ensemble of identical pointlike particles located on it. Their dynamics with respect to the evolution parameter $s$ reproduces (via Vieta’s formulas) a generic structure of the Newtonian mechanics. After the choice of a special (inertial) reference frame, the dynamics obeys the law of momentum conservation (for the closed system of two kinds (R- and C-) of particle-like formations represented by real and complex conjugate roots, respectively). This looks like an important indication of the purely algebraic origins of the structure of (Galilean–Newtonian) mechanics and of physical interactions in general.

Of course, many problems of principal character, including those of particle–antiparticle asymmetry and transition to relativistic description, still remain unsolved. As the first natural step to necessary relativization of the theory, one has to explicitly introduce into the scheme an observer and consider the process of detection of the (R- and C-) particles governed by the retardation equation. At this step the fundamental constant, velocity of light $c$, naturally enters the theory.

Remarkably, such a procedure has been partially realized for a worldline defined in an ordinary parametric form but allowing for superluminar velocities [18, 19]. On a complexified spacetime background, a similar procedure was exploited in our works [17, 20]. In both approaches, the retardation equation possesses a (generally, great) number of roots that define the positions and dynamics of identical particle-like ‘copies’.

Generally, a conjecture on the complex geometry of (extended) spacetime results in a number of intriguing consequences and natural connections with the Kerr-type solutions of the Einstein–Maxwell electrovacuum equations, with R Penrose’s twisters and models of extended particles. For many decades, the conjecture has been elaborated by E T Newman and A Ya Burinskii et al. In particular, in [21, 22] and [23, 24], remarkable particle-like and string-like structures related to a ‘complex worldline’ and to the introduction of a ‘complex time’ parameter were discovered. On the other hand, in [25] the structure of multiparticle Kerr–Schild solutions was obtained by making use of twistor methods and the Kerr theorem.
to the concept of the dimerous electron pre-elements only when they emit a quantum-like signal. In this way one naturally comes physical particles could be detectable only at discrete instants of merging of two or more real particles, at the present stage of investigation this, of course, seems premature. Moreover, of different roots and, presumably, for their ability to form a sort of (stable) cluster quantum interference phenomena, one has a nontrivial amplification field of Lienard–Wiechert type. It is especially interesting that this field undergoes an amplification at the points of merging (annihilation/creation) of a pair of particles, so that one has a nontrivial caustic locus which can be naturally regarded as a set of quantum-like signals perceived by an external observer [17, 20].

As to the identification of the considered point-like formations (matter pre-elements) with real particles, at the present stage of investigation this, of course, seems premature. Moreover, physical particles could be detectable only at discrete instants of merging of two or more pre-elements only when they emit a quantum-like signal. In this way one naturally comes to the concept of the dimerous electron [17], which was found to be especially useful in the geometric explanation of the quantum interference phenomena.

Generally, at first one could make an attempt to find the reasons for the ‘attraction’ of different roots and, presumably, for their ability to form a sort of (stable) cluster which could really represent elementary particles, nuclei, etc. At present this still looks like a barely achievable dream, though the results obtained herein give essential support to the realization of the program.

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Appendix

Making use of the general procedure described in section 3, let us examine in detail the dynamics defined by the polynomial system of equations [6], namely the following one:

\[
\begin{align*}
F_1(x, y, s) &= -2x^3 + y^3 + sx + sy + y + 2 = 0, \\
F_2(x, y, s) &= -x^3 - 2x^2y + s + 3 = 0.
\end{align*}
\]  

(A.1)

The trajectory curve of particle-like formations represented by real roots of this system, following from the elimination of the evolution parameter \( s \), is defined by the equation

\[x^4 + 3x^3y + 2x^2y^2 - 2x^3 + y^3 + 3x + 2y - 2 = 0\]  

(A.2)

and consists of three disconnected components (see figure 2).

The full expressions for equations on eliminants \( R_1(x) \) and \( R_2(y) \) of the system (A.1) are as follows (compare with (26)):

\[
\begin{align*}
R_1(x) &= -17x^9 + (-4 + 4x)x^7 + (3s + 25)x^6 + (4s^2 + 12 + 16s)x^5 \\
&+ (-3s^2 - 18s - 27)x^4 + 27sx^3 + s^3 + 9s^2 + 27 = 0; \\
R_2(y) &= 17y^9 + (35 + 33s)y^7 + (-6s + 52)y^6 + (15s^2 + 34s + 19)y^5 \\
&+ (40 + 8s - 16s^2)y^4 + (49s + 11s^2 - s^3 + 113)y^3 + (-50s - 12 - 18s^3 - 72s^2)y^2 \\
&+ (148s^2 + 28s^3 + 208s + 48)y - 64 + s^4 - 48s^2 - 5s^3 - 96s = 0.
\end{align*}
\]  

(A.3)

One obtains from (A.3) that at any instant \( s \) the system (A.1) has nine solutions, some of them composed of complex conjugate pairs; besides, since the terms of eighth degree are absent,
the total momentum of the two types (R- and C-) of particles represented by real and complex conjugate roots is permanently equal to zero (the centre of mass reference frame). Values of \( s \) that determine singular points for the solutions of (A.1) related to the annihilation/creation events correspond to multiple roots\(^7\) of the equations (A.3), or common roots of the two systems of equations

\[
\begin{align*}
R_x(x) &= 0, \\
R'_x(x) &= 0,
\end{align*}
\tag{A.4}
\]
and

\[
\begin{align*}
R_y(y) &= 0, \\
R'_y(y) &= 0,
\end{align*}
\tag{A.5}
\]
where the ‘prime’ denotes differentiation with respect to \( x \) or \( y \), respectively.

Computing now resultants of the two polynomials in (A.4) or (A.5), which are in fact the so-called discriminants of equations (A.3), one verifies that these two have common factors, so that critical values of parameter \( s \) are obtained from the real roots of the equation

\[
R_{\text{common}}(s) = (s + 3)^3(−1030 738 720 704 832 − 2585 288 646 749 952s \\
− 2876 632 663 642 944s^2 − 3915 728 526 452 064s^3 \\
− 6758 379 899 260 446s^4 − 117 671 742 918 602s^5 \\
− 12 435 143 753 367s^6 − 4976 985 600s^7 \\
+ 1769 472s^8 ) = 0.
\tag{A.6}
\]
Equation (A.6) has an obvious root \( s = −3 \) of multiplicity 3, and 13 other roots of which only five turn out to be real. Thus, the system (A.3) defines six critical values of the parameter \( s \) at which some merging of roots and related particle-like formations take place. Approximate critical values of \( s \) have been written out in section 2 and will be reproduced below. Corresponding coordinates of the points of merging are then readily obtained from the eliminants’ equations (A.3).

Consider now a graphical representation of the successive dynamics of roots of the system (A.1) at different values of the time parameter \( s \). To begin with, let us agree about the notation in the figures. Three disconnected branches of the trajectory (A.2) are denoted as A, B, C. Circles denote the positions of real roots (particles of the type R), and squares real parts of complex conjugate roots (particles of the type C), which are assumed thus to be located both at one and the same space point. Arrows designate the direction of motion of roots under the positive increment of the parameter \( s \). The roots are numbered in order to follow their successive dynamics and transmutations. By a grey cross or circle with corresponding inscriptions \( s_k, k = 1, 2, \ldots \), one denotes the positions and instants of the annihilation or creation events, respectively. Finally, by dotted lines some segments of the projection of trajectories of complex conjugate roots onto the real plane are denoted, for a visual representation of the dynamics of the corresponding C-particles.

In figure A1(a), one sees that the real roots 1 and 2 move towards one another along the first branch of the trajectory C, up to their annihilation at \( s_1 ≈ −97.3689 \).

Figure A1(b) represents the intervening situation, when the above roots become complex conjugate and are under transition to the other branch B when they are expected to give rise to a new pair of R-particles, at \( s_2 ≈ −4.025 \). Note that one pair of complex conjugate roots is off the depicted space in figures A1(a) and (b) so that only seven roots are represented therein.

\(^7\) In order to determine these, one could use the explicit condition (5). We, however, prefer another, more visual, method of discriminants.
Figure A1. Disposition of roots of the system (A.1): (a) at $s \approx -162.37$; (b) at $s \approx -39.025$ (after first annihilation); (c) at $s \approx -3.725$ (after first pair creation); (d) at $s = -3$ (double merging); (e) at $s \approx -2.9$; (f) at $s \approx -2.768$ (second pair creation).
Figure A2. Disposition of roots of the system (A.1) at $s \approx 1432.49$ (after third annihilation).

Figure A3. A representative frame from the supplementary animation file ‘1eUniverse.gif’ (1898 036 bytes, available from stacks.iop.org/JPhysA/46/175206/mmedia).
In figure A1(c), one sees that the considered roots 1 and 2 give rise to a pair of real R-particles (1 and 2) at branch B of the trajectory. The root 3 moves towards real root (1) and will merge with the latter at \(s_3 = -3\). Note that the third pair of complex conjugate roots (8 and 9) appears in the space of vision so that the full number of roots \((N = 9)\) is depicted here and in the subsequent figures.

In figure A1(d), a peculiar situation of double merging is presented at \(s_3 = -3\) (recall that this is the exceptional root of multiplicity 3 of the equation for ‘events’ (A.6)). At this instant, besides the annihilation of two real R-particles (1 and 3) one has the merging of two complex conjugate pairs of roots (6,7 and 8,9) which takes place in the space exterior to the real trajectory (i.e. in the complex extension of the ‘physical’ 3D space). In contrast to the merging of real particles, such an event is not accompanied by the annihilation of a pair: in what follows, the merged pairs deviate from one another, without any modification of their structure (see figure A1(e)).

In figure A1(e), one observes only one real root (2) while one pair of complex conjugate roots (6 and 7), after divergence with the other pair (8 and 9), moves towards branch B of the trajectory where it will give rise to a pair of real roots (6 and 7) at the next moment \(s_4 \approx -2.78\).

In figure A1(f), the two created real particles (6 and 7) move in opposite directions along the branch B of the trajectory. At the next moment, annihilation of roots (2 and 7) at \(s_5 \approx -2.77\) is expected. The pair of complex conjugate roots moves towards the third branch A of the trajectory (to be seen at the next figure) which at the moment is still ‘empty’.

In figure A2, the disposition of roots is presented at a much greater scale. After the annihilation of roots 2 and 7 only one real R-particle (6) survives on branch B. The pair of roots 8 and 9 moves (precisely, in complex extension of space) towards the third, ‘empty’ branch of the trajectory A where the third pair creation is expected at the future moment \(s \approx 2932.49\). After this last event, there exist two real particles at branch A, one real particle at branch B and three pairs of complex conjugate roots (three C-particles). From now on, no other merging events exist: the dynamics is in fact over.

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