An extended tanh-method and its application to the soliton breaking equation

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Abstract. In this paper, the soliton breaking equation is studied by using an extended tanh method and obtained some new exact soliton solutions which include Fan’s solutions.

1. Introduction

In recent years, the nonlinear partial differential equations (NPDEs) are widely used to describe many important phenomena and dynamic processes in physics, mechanics, chemistry, chemical kinetics, geochemistry and biology. There are a huge variety of methods available for constructing exact solutions of nonlinear PDEs. Some of the important methods are the inverse scattering method [1], Hirota’s method [2], Backlund transformation [3], F-expansion method [4], homogeneous balance method [5] and Jacobi elliptic function method [6], the variational iteration method[7], the homotopy perturbation method [8], Exp-function method [9,10], Adomian decomposition method [11].

Most recently, Wazwaz A.M has proposed a new extended tanh-function method [12] to study the Kuramoto-Sivashinsky equation, the Kawahara equation, the modified forms of Degasperis-Procesi equation and Camassa-Holm equation. In the paper, we used a new extended tanh-function method which was presented by Wazwaz in [12] to investigate the soliton breaking equation [13]

\[ u_{xx} - 4u_{x}u_{xy} - 2u_{y}u_{xx} - u_{xyy} = 0 \] (1)

and obtained some new exact soliton traveling solutions.

The organization of the paper is as follows. In Section 2, a new extended tanh-function method is presented. In Section 3, we investigate the soliton breaking equations using the new extended tanh-function method and obtain some new exact soliton traveling solutions. Finally conclusion and discussion are given in Section 4.

2. The new extended tanh-function method

The new modified extended tanh technique is based on the priori assumption that the travelling wave solutions can be expressed in terms of the tanh function. The main steps for using the tanh method:

1. We first consider a general form of nonlinear equation

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2. To find the travelling wave solution of Eq. (2), we introduce the wave variable  
\[ \xi = x + \alpha y + \lambda t \]
so that Eq (2) can be converted to an ODE

\[ Q(u, u_x, u_y, u_{xx}, u_{yy}, \cdots) = 0 \]  

(3)

3. If all terms of the resulting ODE (3) contain derivatives in \( \xi \), then by integrating this equation, and by considering the constant of integration to be zero, we obtain a simplified ODE.

4. We then introduce a new independent variable
\[ Y = \tanh(\mu \xi) \text{ or } Y = \tan(\mu \xi) \]  

(4)

that leads to the change of derivatives:
\[ \frac{dY}{d\xi} = \mu(1 \pm Y^2) \]  

(5)

Where \( \mu \) be a real parameter.

5. The new extended tanh-function method admits the use of the finite expansion

\[ Q(\xi) = S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{l=1}^{N} b_l Y^{-l} \]  

(6)

where \( M \) and \( N \) are positive integer, in most cases, that will be determined. Expansion (6) reduces to Wazwaz’s extended tanh-function method [12] when \( M=N \) and reduces to standard tanh method for \( b_l = 0(1 \leq l \leq N) \). The parameter \( M \) and \( N \) are usually obtained by balancing the linear terms of positive and negative highest order in the resulting equation with the positive and negative highest order nonlinear terms. Substituting (5) and (6) into the ODE (3) yields an equation in powers of \( Y \) that will lead to the determination of the parameters \( a_k (k = 0, 1, 2, \cdots, m), b_l (l = 1, 2, \cdots, N), c, \mu \).

3. Applications
The (2+1)-soliton breaking equation is given by

\[ u_{xt} - 4u_x u_y - 2u_y u_{xx} - u_{xxy} = 0 \]  

(7)

In order to apply the previous extended tanh-function method, we use the transformations

\[ u(x, y, t) = U(\xi) \quad \xi = x + \alpha y + \lambda t \]  

(8)

Then Eq (8) can be converted to the ODE

\[ \lambda U_{\xi\xi} - 4\alpha U_{\xi} U_{\xi\xi} - 2\alpha U_{\xi} U_{\xi\xi} - U_{\xi\xi\xi\xi} = 0 \]  

(9)

Integrate (9) and take the integral constant be zero, we obtain

\[ \lambda U_{\xi} - 3\alpha U_{\xi}^2 - \alpha U_{\xi\xi\xi} = 0 \]  

(10)

Balancing the order (positive and negative) of \( U_{\xi\xi\xi} \) with the order (positive and negative) of \( U_{\xi}^2 \) in Eq (10). This in turn gives
\[ M + 3 = 2(M+1) \]  

(11)
\[ -N - 3 = -2(N+1) \]

So that \( M = N = 1 \). This give the solution in the form

\[ U(\xi) = S(Y) = a_0 + a_1 Y + b_1 Y^{-1} \tag{12} \]

First, we consider \( \frac{dY}{d\xi} = \mu(1 - Y^2) \) and using the MATHEMATICA, we have

\[ U_\xi = \frac{\partial S(Y)}{\partial Y} \frac{\partial Y}{\partial \xi} = -Y^2 \mu a_1 + \mu(a_1 + b_1) + \frac{\mu b_1}{Y^2}; \tag{13} \]

\[ U_{\xi\xi} = \frac{\partial U_\xi}{\partial Y} \frac{\partial Y}{\partial \xi} = 2Y^3 \mu^2 a_1 - 2Y\mu^2 a_1 - \frac{2\mu^2 b_1}{Y}; \]

\[ U_{\xi\xi\xi} = \frac{\partial U_{\xi\xi}}{\partial Y} \frac{\partial Y}{\partial \xi} = -2\mu^3 (a_1 + b_1) + 8\mu^3 a_1 Y^2 - 6\mu^3 a_1 Y^4 + \frac{6\mu^3 b_1}{Y^2} + \frac{8\mu^3 b_1}{Y^2}; \tag{14} \]

Substituting Eq (13)—(14) into equation (10), using the MATHEMATICA, we obtain a system of algebraic equations, for \( a_0, a_1, b_1, \alpha, \lambda, \mu \) of the following form:

- \( Y^4 \) coeff: \( 6\alpha \mu a_1 - 6\alpha \mu^2 a_1^2 \);
- \( Y^2 \) coeff: \( -\lambda \mu a_1 - 8\alpha \mu^3 a_1 + 6\alpha \mu^2 a_1^2 + 6\alpha \mu^2 a_1 b_1 \);
- \( Y^0 \) coeff: \( \lambda \mu a_1 + 2\alpha \mu^3 a_1 - 3\alpha \mu^2 a_1^2 + \lambda \mu b_1 + 2\alpha \mu^3 b_1 - 12\alpha \mu^2 a_1 b_1 - 3\alpha \mu^2 b_1^2 \);
- \( Y^{-2} \) coeff: \( -\lambda \mu b_1 - 8\alpha \mu^3 b_1 + 6\alpha \mu^2 b_1^2 + 6\alpha \mu^2 a_1 b_1 \);
- \( Y^{-4} \) coeff: \( 6\alpha \mu b_1 - 3\alpha \mu^2 b_1^2 \);

Solving the system of the algebraic equations with the aid of MATHEMATICA we can distinguish three cases namely:

- Case 1:
  \[ \lambda = 4\alpha \mu^2, \quad a_1 = 2\mu, \quad b_1 = 0 \tag{15} \]
- Case 2:
  \[ \lambda = 4\alpha \mu^2, \quad a_1 = 0, \quad b_1 = 2\mu \tag{16} \]
- Case 3:
  \[ \lambda = 16\alpha \mu^2, \quad a_1 = 2\mu, \quad b_1 = 2\mu \tag{17} \]

according to Eq (15), (16), (17), the solutions to Eq (8) read

\[ u(x,t) = a_0 + 2\mu \tanh[\mu(x + \alpha y + 4\alpha \mu^2 t)] \tag{18} \]

\[ u(x,t) = a_0 + 2\mu \coth[\mu(x + \alpha y + 4\alpha \mu^2 t)] \tag{19} \]

\[ u(x,t) = a_0 + 2\mu \tanh[\mu(x + \alpha y + 16\alpha \mu^2 t)] + 2\mu \tanh^{-1}[\mu(x + \alpha y + 16\alpha \mu^2 t)] \tag{20} \]

With \( a_0, \mu \) and \( \lambda \) being arbitrary constants.
It is worth noting that the new exact soliton traveling wave solutions equation (20) is not obtained by Fan[13] by the tanh-method. If take \( 4\alpha \mu^2 = -1 \), then (18) and (19) are solutions of paper in [13] and (20) is a new soliton solution.

Similarly, if we choose

\[
\frac{dY}{d\xi} = \mu(1 + Y^2) \tag{21}
\]

Substituting Eq (13)—(14) and (21) into equation (10), we obtain

Case 1:

\[
\dot{\lambda} = -4\alpha \mu^2, \quad a_i = -2\mu, \quad b_i = 0, \tag{22}
\]

Case 2:

\[
\dot{\lambda} = -4\alpha \mu^2, \quad a_i = 0, \quad b_i = -2\mu, \tag{23}
\]

Case 3:

\[
\dot{\lambda} = -16\alpha \mu^2, \quad a_i = -2\mu, \quad b_i = 2\mu \tag{24}
\]

These yield three triangular for exact solutions

\[
u(x,t) = a_0 - 2\mu \tan[\mu(x + \alpha y - 4\alpha \mu^2 t)] \tag{25}
\]

\[
u(x,t) = a_0 - 2\mu \tanh^{-1}[\mu(x + \alpha y - 4\alpha \mu^2 t)] = a_0 - 2\mu \cot[\mu(x + \alpha y - 4\alpha \mu^2 t)] \tag{26}
\]

\[
u(x,t) = a_0 - 2\mu \tan[\mu(x + \alpha y - 16\alpha \mu^2 t)] + 2\mu \tan^{-1}[\mu(x + \alpha y - 16\alpha \mu^2 t)] = a_0 - 2\mu \tan[\mu(x + \alpha y - 16\alpha \mu^2 t)] + 2\mu \cot[\mu(x + \alpha y - 16\alpha \mu^2 t)] \tag{27}
\]

With \( a_0, \mu \) and \( \dot{\lambda} \) being arbitrary constants.

It is worth noting that the new exact soliton travelling wave solutions equation (27) is not obtained by Fan[13] by the tanh-method. If take \( 4\alpha \mu^2 = -1 \), then (25) and (26) are solutions of paper in [13] and (27) is a new soliton solution.

4. Concluding remark

The new extended tanh method in this paper was effectively used for analytic treatment of the soliton breaking equation. Some of the entirely new travelling solutions are obtained to the soliton breaking equation.

Most recently, J.H. He and S.D. Zhu have proposed a new effective Exp-method to obtain solutions of nonlinear partial differential equations. Obviously, it is very significance work to compare Exp-method with the extended tanh method and this is our task research in the future.

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