BOSONIC PHYSICAL STATES IN $N = 1$ SUPERGRAVITY?

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ABSTRACT

It is argued that states in $N = 1$ supergravity that solve all of the constraint equations cannot be bosonic in the sense of being independent of the fermionic degrees of freedom. (Based on a talk given by Miguel Ortiz at the 7th Marcel Grossmann Meeting.)

The canonical quantization of supergravity involves the solution of a number of constraint equations restricting the form of physical wave functionals, each of which corresponds to a symmetry of the theory. The constraint equations for $N = 1$ were discussed by D'Eath in 1984, where it was shown that in principle it is sufficient to solve the two supersymmetry constraints in order to obtain completely gauge invariant wave functionals (this assumes the absence of anomalies in the operator algebra). The reason for this is that the bracket of the two supersymmetry constraints yields the familiar Hamiltonian and momentum constraints that are present because of the diffeomorphism invariance of the theory. In a subsequent paper which was published this year, D'Eath used this simplifying feature to argue that explicit solutions of the quantum constraints can be found, and that these have the special property that they are bosonic. From this result, it has been argued by D'Eath that supergravity is a finite theory. However, this latter claim is dependent upon the existence of purely bosonic solutions.

In response to Ref. 2, we demonstrated that states solving the supergravity constraints cannot be independent of the fermionic variables, and must almost certainly consist of an infinite product of Grassman valued fields. In this short paper, we review one argument presented there which shows that a bosonic state of the kind discussed by D'Eath cannot solve the supergravity constraints, and we shall attempt to update some of the arguments to clarify what is meant in our work by a bosonic state.

The Lagrangian for $N = 1$ supergravity is

$$\mathcal{L} = \frac{1}{16\kappa^2} \varepsilon^{\mu
u\rho\sigma} \varepsilon_{abcd} E^a_{\mu} E^b_{\nu} R^c_{\rho} R^d_{\sigma} - \frac{1}{2} \varepsilon^{\mu
u\rho\sigma} \left( \bar{\psi}_\mu E^a_{\nu} \bar{\sigma}_a D_\rho \psi_\sigma - D_\rho \bar{\psi}_\mu E^a_{\nu} \bar{\sigma}_a \psi_\sigma \right),$$  \hspace{1cm} (1)

in terms of Weyl spinor gravitino fields $\psi_{A\mu}$ and $\bar{\psi}_{A'\mu}$ and a vierbein field $E^a_{\mu}$. The conventions we use are the same as in Ref. 4.

To work in the canonical formalism, we must choose a polarisation, and here we follow Ref. 1 in working in the holomorphic representation, which uses state functionals depending on $\epsilon^a_i$ and $\psi_{Ai}$. 

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As mentioned above, it is in principle sufficient to look at the supersymmetry constraints, provided that we ensure that any state functional is a Lorentz invariant combination of its arguments. These constraints take the relatively simple form:

\[ S^A F = \left[ -\varepsilon^{ijk} e^a_i e^{a'}_j \sigma^A \sigma^{A'} (D_j \psi_{Ak}) + \frac{\hbar k^2}{2} \sigma^{aA'} \psi_{Ai} \left( \frac{\delta}{\delta e^a_i} \right) \right] F[e^a_i, \psi_{Ai}] = 0 \]  

(2)

and

\[ S^A F = \left[ D_j \frac{\delta}{\delta \psi_{Aj}} + \frac{\hbar k^2}{2} \frac{\delta}{\delta \psi_{Bj}} D_{BA'} \sigma^{aA'} \sigma^{a'} \frac{\delta}{\delta e^a_i} \right] F[e^a_i, \psi_{Ai}] = 0 \ , \]

(3)

where the \( \sigma \) and \( \sigma' \) are sigma matrices and \( D_{AA'}ij \) is a function of \( e^a_i \). The connection for the covariant derivative \( D_j \) and other details can be found in Ref. 4.

The important feature of the constraints (2) and (3) is that every term in (3) involves derivatives with respect to the gravitino field, so any state that is independent of \( \psi_{Ai} \),

\[ \frac{\delta F[e^a_i, \psi_{Ai}]}{\delta \psi_{Ai}} = 0 \ , \]

(4)

automatically solves (3). Thus if a solution of the form \( F[e^a_i, \psi_{Ai}] = F^{(0)}[e^a_i] \) can be found to Eq. (2), it is a physical state. We now present a simple scaling argument from Ref. 4 which shows that there is no such solution of (2).

Multiplying by \( F^{-1} \) and integrating over an arbitrary continuous spinor test function \( \bar{\epsilon}(x) \), the constraint (2) becomes

\[ \int d^3x \bar{\epsilon}(x) \left[ -\varepsilon^{ijk} e^a_i (x) \sigma_a (D_j \psi_k(x)) + \frac{\hbar k^2}{2} \psi_i(x) \sigma^{aA'} \left( \frac{\delta \ln F^{(0)}[\epsilon]}{\delta e^a_i(x)} \right) \right] = 0 \ , \]

(5)

which must be satisfied for all \( \bar{\epsilon}(x) \), \( e^a_i(x) \), and \( \psi_k(x) \). Let the integral in Eq. (5) be \( I \), and let \( I' = I + \Delta I \) be the integral when \( \bar{\epsilon}(x) \) is replaced by \( \bar{\epsilon}(x) e^{-\phi(x)} \) and \( \psi_i(x) \) is replaced by \( \psi_i(x)e^{\phi(x)} \), where \( \phi(x) \) is a scalar function. Since \( \bar{\epsilon}(x)\psi_i(x) \) is unchanged, the second term (with the functional derivative) cancels in the difference between \( I' \) and \( I \), so that we must have

\[ \Delta I = -\int d^3x \varepsilon^{ijk} e^a_i(x) \bar{\epsilon}(x) \sigma_a \psi_k(x) \partial_j \phi(x) = 0 \ . \]

(6)

Notice that \( \Delta I \) is independent of the state \( F^{(0)}[\epsilon] \). Clearly, it is possible to choose the arbitrary fields \( \bar{\epsilon}(x) \), \( \phi(x) \), \( e^a_i(x) \) and \( \psi_k(x) \) such that (6) is nonvanishing; therefore no physical state is bosonic in the sense of Eq. (4).

The constraint equation (2) can in principle be solved using the method of characteristics. In this approach, the value of the wave functional is specified on an appropriate subspace of the configuration space \{\( e^a_i \), \( \psi_{Ai} \)\} and the constraint (2) is used to calculate the value throughout the rest of the space. By the scaling argument above, such a construction can never yield a state which is independent of \( \psi_{Ai} \) throughout configuration space; it would thus be necessary to check independently that the state was a solution of the \( S^A \) constraint (3), which was not carried out in Ref. 2.

We now show directly why the method of characteristics fails to produce bosonic solutions. Assume that at some fixed configuration \( e^{a(0)}(x) \) the functional \( F[e^{(0)}, \psi_{Ai}] \) is independent of \( \psi_{Ai} \).

An infinitesimal supersymmetry transformation changes \( e^{a(0)}(x) \) to

\[ e^{a(0)} + \delta e^a_i (0) = e^{a(0)} - \frac{i k^2}{2} \varepsilon^{ijk} \sigma^{aA'} \psi_{Ai} \ , \]

(7)
leaving $\psi_{A_i}$ unchanged. From (2) we see that the value of $F$ at the transformed configuration is

$$F[e^{(0)} + \delta e, \psi] = F[e^{(0)}] - \int d^3x \bar{\epsilon}_{A'} \epsilon^{ijk} e^a_i \bar{\sigma}_a A' A D_j \psi_{Ak} F .$$  \hfill (8)

Scaling $\psi$ and $\bar{\epsilon}$ as before leaves $\delta e$ unchanged and thus gives

$$F[e^{(0)} + \delta e, e^{\phi(x)} \psi] = F[e^{(0)}] - \int d^3x e^{-\phi(x)} \bar{\epsilon}_{A'} \epsilon^{ijk} e^a_i \bar{\sigma}_a A' A D_j (e^{\phi(x)} \psi_{Ak}) F .$$  \hfill (9)

The right hand sides of (8) and (9) will not be equal in general, which shows that the state $F$ cannot be independent of $\psi$ in the full configuration space of the theory.

We conclude that any solution of the constraint (2) arrived at using the method of characteristics cannot be bosonic in the sense we have defined. Therefore only a very small subclass of solutions of this constraint can also be solutions of (3); Ref. 4 presents arguments that any true physical state must contain an infinite product of Grassmann fields.

**Acknowledgements**

This short paper draws heavily on material contained in Ref. 4 and should be regarded as a summary of certain aspects of that work. We thank Peter D’Eath and Hermann Nicolai for helpful discussions.

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