REPRESENTATION GROWTH OF THE HEISENBERG GROUP OVER $O[x]/(x^n)$

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Abstract. We present a conjectured formula for the representation zeta function of the Heisenberg group over $O[x]/(x^n)$ where $O$ is the ring of integers of some number field. We confirm the conjecture for $n \leq 3$ and raise several questions.

1. Introduction

Let $G$ be a finitely generated torsion-free nilpotent group (or a $T$-group for short). Two complex representations $\rho$ and $\sigma$ of $G$ are called twist-equivalent if there exists a 1-dimensional representation $\lambda$ of $G$ such that $\rho = \lambda \otimes \sigma$. Twist-equivalence is an equivalence relation on the set of finite dimensional irreducible complex representations of $G$ and its classes are called twist-isoclasses. The numbers $r_n(G)$ of twist-isoclasses of dimension $n$ are finite for all $n$, cf. [3, Theorem 6.6]. The representation zeta function of $G$ is defined to be the Dirichlet generating function $\zeta_G(s) := \sum_{n=1}^{\infty} \frac{r_n(G)}{n^s}$, where $s$ is a complex variable. The sequence $(r_n(G))$ grows polynomially and thus $\zeta_G(s)$ converges on a complex half-plane $\text{Re}(s) > \alpha$, cf. [8, Lemma 2.1]. The infimum of such $\alpha$ is the abscissa of convergence $\alpha(G)$ of $\zeta_G(s)$ which gives the precise degree of polynomial growth; i.e., $\alpha(G)$ is the smallest value such that $\sum_{n=1}^{N} r_n(G) = O(N^{\alpha(G)+\epsilon})$ for every $\epsilon \in \mathbb{R}_{>0}$.

Let $H$ be the Heisenberg group scheme associated to the Heisenberg $\mathbb{Z}$-Lie lattice of strict upper-triangular $3 \times 3$ matrices. For every ring $R$, the group $H(R)$ is isomorphic to the group of upper-unitriangular $3 \times 3$ matrices over $R$. If $R$ is torsion-free finitely generated over $\mathbb{Z}$, then $H(R)$ is a $T$-group of nilpotency class 2 and Hirsch length $3 \cdot \text{rk}_\mathbb{Z}(R)$. When $R = O$ is the ring of integers of a number field $K$, the zeta function of $H(O)$ is

\begin{equation}
\zeta_{H(O)}(s) = \frac{\zeta_K(s-1)}{\zeta_K(s)} = \prod_{\mathfrak{p} \in \text{Spec}(O)} \frac{1 - |O/\mathfrak{p}|^{-s}}{1 - |O/\mathfrak{p}|^{1-s}},
\end{equation}

where $\zeta_K(s)$ is the Dedekind zeta function of $K$, $\mathfrak{p}$ ranges over the nonzero prime ideals of $O$. This is proved in [4] for $K = \mathbb{Q}$, in [2] for quadratic number fields, and in [8, Theorem B] for arbitrary number fields. The zeta function $\zeta_{H(O)}(s)$ has abscissa of convergence $\alpha(H(O)) = 2$, which is independent of $K$, and may be meromorphically continued to the whole complex plane.

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In this paper, we consider the Heisenberg group over rings of the form $\mathcal{O}[x]/(x^n)$. If $n = 1$ then it is the Heisenberg group over $\mathcal{O}$. The zeta function of $H(\mathcal{O}[x]/(x^2))$ was computed in [7] Example 6.5. We compute the zeta function of $H(\mathcal{O}[x]/(x^n))$ for $n = 3$.

**Organization and notation.** In Section 2, we recall formulae of local representation zeta functions in terms of $p$-adic integrals. The zeta function for the case $n = 3$ is computed in Section 3. We conclude in Section 4 with several questions and conjectures.

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2. Preliminaries

2.1. Local representation zeta functions. The group $H(\mathcal{O}[x]/(x^n))$ is a $T$-group of nilpotency class 2 and Hirsch length $3n \cdot \text{rk}_Z(\mathcal{O})$. The zeta function $\zeta_{H(\mathcal{O}[x]/(x^n))}(s)$ has an Euler factorization (cf. [8] Proposition 2.2)

$$
\zeta_{H(\mathcal{O}[x]/(x^n))}(s) = \prod_{p \in \text{Spec}(\mathcal{O})} \zeta_{H(\mathcal{O}_p[x]/(x^n))}(s),
$$

where $p$ ranges over the nonzero prime ideals in $\mathcal{O}$ and $\mathcal{O}_p$ is the completion of $\mathcal{O}$ at $p$. The local factors $\zeta_{H(\mathcal{O}_p[x]/(x^n))}$ are rational in $|\mathcal{O}/p|^{-s}$ and almost all of them satisfy a functional equation (cf. [8] Theorem A).

The $\mathcal{O}$-lattice associated to $H(\mathcal{O}[x]/(x^n))$ has the following presentation; see [8] Section 2.4:

$${\langle} x_0, x_1, \cdots, x_{n-1} \mid y_0, y_1, \cdots, y_{n-1} \mid z_0, z_1, \cdots, z_{n-1} \rangle = \begin{cases} 
  z_{i+j} & \text{if } i + j < n, \\
  0 & \text{otherwise}.
\end{cases}$$

The associated commutator matrix with respect to the chosen $\mathcal{O}$-basis is defined by

$$
\mathcal{R}_n(Y) = \begin{pmatrix}
0 & Q_n(Y) \\
-Q_n(Y)^t & 0
\end{pmatrix},
$$

where

$$
Q_n(Y) = \begin{pmatrix}
Y_1 & Y_2 & Y_3 & \cdots & Y_{n-1} & Y_n \\
Y_2 & Y_3 & \cdots & Y_n & 0 \\
Y_3 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
Y_{n-1} & Y_n & \cdots & \cdots & \cdots & 0 \\
Y_n & 0 & \cdots & \cdots & \cdots & 0
\end{pmatrix} \in \text{Mat}_n(\mathcal{O}[Y_1, \cdots, Y_n]).
$$

Fix a nonzero prime ideal $p$ and denote $\mathcal{O} := \mathcal{O}_p$. Let $q := |\mathcal{O}/p|$ be the residue field cardinality and $p$ its characteristic. Let $W_n(\mathcal{O}) = \mathcal{O}^n \setminus p^n$. Set

$$
(2.1) \quad \mathcal{Z}_p(\rho, \tau) := \int_{(u,\gamma) \in p \times W_n(\mathcal{O})} |u|^\tau \prod_{j=1}^n \frac{||F_j(\gamma) \cup F_{j-1}(\gamma)u^2||^\rho}{||F_{j-1}(\gamma)||^\rho} d\mu,
$$
where the additive Haar measure $\mu$ on $\mathfrak{o}^{n+1}$ is normalized such that $\mu(\mathfrak{o}^{n+1}) = 1$, and

$$F_j(Y) = \{ f \mid f = f(Y) \text{ a principal } 2j \times 2j \text{ minor of } R_n(Y) \},$$

$$\|H(X,Y)\| = \min\{ v_p(h(X,Y)) \mid h \in H \}$$

for a finite set $H \subset \mathfrak{o}[X,Y]$.

The local factor $\zeta_{H_2([x]/(x^n))}(s)$ can be expressed in terms of the $p$-adic integral \[2.1\] as the following (cf. [8, Corollary 2.11]):

$$\zeta_{H_2([x]/(x^n))}(s) = 1 + (1 - q^{-1})^{-1}Z_p(-s/2, ns - n - 1).$$

### 2.2. Auxiliary lemmas.

**Lemma 2.1.** The following identities hold in the field of formal Laurent series $\mathbb{Q}((a, b, c))$.

1. $$\sum_{(X,Y)\in\mathbb{N}^2} a^X b^Y c^{\min\{X,Y\}} = \frac{abc(1-ab)}{(1-abc)(1-a)(1-b)}.$$ 
2. $$\sum_{(X,Y)\in\mathbb{N}^2} a^X b^Y c^{\min\{X,2Y\}} = \frac{abc(1-a+ac-a^2bc)}{(1-a)(1-b)(1-a^2bc^2)}.$$ 
3. $$\sum_{(X,Y)\in\mathbb{N}^2} a^X b^Y c^{\min\{X,Y\}+\min\{X,2Y\}} = \frac{abc^2(1-a+ac-abc-a^2bc+3)bc^3}{(1-a)(1-b)(1-a^2bc^2)(1-a^4bc^2)}.$$ 
4. $$\sum_{(X,Y,Z)\in\mathbb{N}^3} a^X b^Y c^Z d^{\min\{X,Y+2Z\}} e^{\min\{X,2Y+4Z\}} = \frac{1}{1-abd^2} \frac{1}{1-b} \frac{1}{1-c}.$$ 

**Proof.** The identity (1) is from [9, Lemma 2.2]. We present the proofs of (2) and (3) while (4) is proven similarly.

For (2), consider the case $X \leq Y$ and let $Y = X + Y'$ with $Y' \in \mathbb{N}_0$. Then

$$\sum_{(X,Y)\in\mathbb{N}^2} a^X b^Y c^{\min\{X,Y\}} = \sum_{(X,Y')\in\mathbb{N}^2} a^{X+Y'} b^{Y'} c^{X} = \frac{abc}{1-abc} \frac{1}{1-b}.$$ 

Consider the case $X > Y$ and let $X = Y + X'$ with $X' \in \mathbb{N}$. Then

$$\sum_{(X,Y)\in\mathbb{N}^2} a^X b^Y c^{\min\{X,2Y\}} = \sum_{(X',Y)\in\mathbb{N}^2} a^{X'+Y} b^Y c^{\min\{X'+2Y\}}$$

$$= \sum_{(X',Y)\in\mathbb{N}^2} a^{X'} (abc)^Y c^{\min\{X',Y\}}$$

$$= \frac{a^{2bc^2}(1-a^2bc)}{(1-a^2bc^2)(1-a)(1-abc)}$$

by (1). Hence

$$\sum_{(X,Y)\in\mathbb{N}^2} a^X b^Y c^{\min\{X,2Y\}} = \frac{abc}{1-abc} \frac{1}{1-b} + \frac{a^{2bc^2}(1-a^2bc)}{(1-a^2bc^2)(1-a)(1-abc)}.$$ 

$$= \frac{abc(1-a+ac-a^2bc)}{(1-a)(1-b)(1-a^2bc^2)}.$$
For (3), first consider the case \( X \leq Y \) and let \( Y = X + Y' \) with \( Y' \in \mathbb{N}_0 \). Then

\[
\sum_{(X,Y) \in \mathbb{N}^2} a^X b^Y c^{\min(X,Y)} c^{\min(X,2Y)} = \sum_{(X,Y) \in \mathbb{N} \times \mathbb{N}_0} a^X b^{X+Y'} c^X = \frac{abc^2}{1 - abc^2} \frac{1}{1 - b}.
\]

Consider now the case \( X > Y \) and let \( X = X' + Y \) with \( X' \in \mathbb{N} \). Then, by (1)

\[
\sum_{(X,Y) \in \mathbb{N}^2} a^X b^Y c^{\min(X,Y)} c^{\min(X,2Y)} = \sum_{(X',Y) \in \mathbb{N}^2} a^{X'} b^{X'} c^{\min(X'+Y,2Y)} = \sum_{(X',Y) \in \mathbb{N}^2} a^{X'} (abc^2)^{\min(X',Y)} = \frac{a^2 bc^3 (1 - a^2 bc^2)}{(1 - a^2 bc^3)(1 - a)(1 - abc^2)}.
\]

Hence

\[
\sum_{(X,Y) \in \mathbb{N}^2} a^X b^Y c^{\min(X,Y)} c^{\min(X,2Y)} = \frac{abc^2}{1 - abc^2} + \frac{a^2 bc^3 (1 - a^2 bc^2)}{(1 - a^2 bc^3)(1 - a)(1 - abc^2)} = \frac{abc^2(1 - a + ac - abc - a^2 bc^3 + a^3 b^2 c^3)}{(1 - a)(1 - b)(1 - abc^2)(1 - a^2 bc^3)}.
\]

\( \square \)

**Lemma 2.2.**

\[
I := \int_{x,y \in \mathfrak{p}} |x|^{3s-4} |x, y^3|^{-s} d\mu = (1 - q^{-1}) \frac{q^2 t^2 (1 + q^3 t^2 - q^3 t^3 + q^6 t^4 - q^6 t^5 - q^8 t^7)}{(1 - q^3 t^3)(1 - q^8 t^8)},
\]

where \( t := q^{-s} \).

**Proof.** Write \( I = I_1 + I_2 \) where

\[
I_1 := \int_{x, y \in \mathfrak{p}} |x|^{3s-4} |x, y^3|^{-s} d\mu,
\]

\[
I_2 := \int_{x, y \in \mathfrak{p}} |x|^{3s-4} |x, y^3|^{-s} d\mu.
\]

**Computation of I₁.** Let \( y = xy_1 \) with \( y_1 \in \mathfrak{o} \). Then

\[
I_1 = \int_{x \in \mathfrak{p}} |x|^{3s-4} |x|^{-s+1} d\mu = (1 - q^{-1}) \frac{q^2 t^2}{1 - q^2 t^2}.
\]

**Computation of I₂.** Let \( x = xy_1 \) with \( x_1 \in \mathfrak{p} \). Then

\[
I_2 = \int_{x_1, y \in \mathfrak{p}} |x_1|^{3s-4} |y|^{2s-3} |x_1, y^2|^{-s} d\mu.
\]
Since $\mu(\{(x, y) \in \mathbb{P}^2 \mid v(x_1) = X, v(y) = Y\}) = (1-q^{-1})^2q^{-X-Y}$, one has by Lemma 2.1 (2)

\[
I_2 = (1-q^{-1})^2 \sum_{(X, Y) \in \mathbb{N}^2} q^{-X-Y}q^{(-3s+4)X}q^{(-2s+2)Y}q^{s \min\{X, Y\}}
\]

\[
= (1-q^{-1})^2 \sum_{(X, Y) \in \mathbb{N}^2} q^{(-3s+3)X}q^{(-2s+2)Y}q^{s \min\{X, Y\}}
\]

\[
= (1-q^{-1})^2 q^{-3s+3-2s+2+2s}(1-q^{-3s+3}+q^{-3s+3+s}-q^{-6s+6-2s+2+2s})
\]

\[
= (1-q^{-1})^2 q^{5t^2(1-q^3t^3+q^4t^4-q^6t^5-q^8t^7)}
\]

\[
= (1-q^{-1})^2 \frac{q^{5t^2(1+q^3t^2-q^3t^3+q^6t^4-q^5t^4-q^8t^7)}}{(1-q^3t^3)(1-q^8t^8)}.
\]

Hence

\[
I = I_1 + I_2 = (1-q^{-1})^2 \frac{q^{2t^2(1+q^3t^2-q^3t^3+q^6t^4-q^6t^5-q^8t^7)}}{(1-q^3t^3)(1-q^8t^8)}.
\]

\[
\square
\]

2.3. The zeta function of $H(O[x]/(x^2))$. In this case $\tau = vs - 2 - 1 = 2s - 3$ and $F_0(Y) = 1, F_1(Y) = \{X^2, Y^2\}, F_2(Y) = \{Y^4\}$. We have

\[
Z := Z_p(-s/2, 2s - 3) = \int_{y=(x,y)\in W_2(o)} |u|^{2s-3}||u, x, y||^{-s}|ux, uy, y^2||^{-s}d\mu.
\]

It is computed in [7, Example 6.5] that

\[
\zeta_{H(O[x]/(x^2))}(s) = 1 + (1-q^{-1})^{-1}Z = \frac{(1-t)(1-q^2t^2)}{(1-qt)(1-q^4t^2)}, \text{ where } t := q^{-s}.
\]

Hence

\[
(2.3) \quad \zeta_{H(O[x]/(x^2))}(s) = \frac{\zeta_{K}(s-1)\zeta_{K}(2s-3)}{\zeta_{K}(s)\zeta_{K}(2s-2)}.
\]

3. The zeta function of $H(O[x]/(x^3))$

In this case, $\tau = 3s - 4$ and

\[
F_0(Y) = 1,
\]

\[
F_1(Y) = \{X^2, Y^2, Z^2\},
\]

\[
F_2(Y) = \{Z^4, Y^2Z^2, (XZ - Y^2)^2\},
\]

\[
F_3(Y) = \{Z^6\}.
\]

Set

\[
A := \|z^2, yz, xz - y^2\|^{-s},
\]

\[
B := \|z^2, yz, xz - y^2, xu, yu, uz\|^{-s},
\]

\[
C := \|z^3, z^2u, yzu, (xz - y^2)u\|^{-s}.
\]

Then

\[
\zeta_{H(O[x]/(x^3))}(s) = 1 + (1-q^{-1})^{-1}Z(s),
\]

\[
\square
\]
where
\[ Z := Z_p(-s/2, 3s - 4) = \int_{u \in \mathfrak{p}, y \in W_3(\mathfrak{o})} |u|^{3s-4} ABC d\mu. \]

Write \( Z = Z_1 + Z_2 + Z_3 \) where
\[ Z_1 := \int_{u \in \mathfrak{p}, z \in W_1(\mathfrak{o})} |u|^{3s-4} ABC d\mu, \]
\[ Z_2 := \int_{u, z \in p} |u|^{3s-4} ABC d\mu, \]
\[ Z_3 := \int_{u, y, z \in \mathfrak{p}} |u|^{3s-4} ABC d\mu. \]

3.1. Computation of \( Z_1 \). Since \( z \in W_1(\mathfrak{o}) \), it follows that \( A = B = C = 1 \). Hence
\[ Z_1 = (1 - q^{-1}) \int_{u, z \in \mathfrak{p}} |u|^{3s-4} d\mu = (1 - q^{-1})^2 \frac{q^{3s-1}}{1 - q^{s+1}}. \]

3.2. Computation of \( Z_2 \). Since \( y \in W_1(\mathfrak{o}) \) and \( z \in \mathfrak{p} \), it follows that \( xz - y \in W_1(\mathfrak{o}) \) and \( A = 1, B = 1, C = \|u, z^3\|^{-s} \). Thus
\[ Z_2 = (1 - q^{-1}) \int_{u, z \in \mathfrak{p}} |u|^{3s-4} \|u, z^3\|^{-s} d\mu. \]

It follows from Lemma 2.32 that
\[ Z_2 = (1 - q^{-1})^2 \frac{q^{2t^2}(1 + q^3 t^2 - q^3 t^3 + q^6 t^4 - q^6 t^5 - q^8 t^7)}{(1 - q^3 t^3)(1 - q^8 t^5)}. \]

3.3. Computation of \( Z_3 \). In this case, \( x \in W_1(\mathfrak{o}), y, z \in \mathfrak{p} \), whence
\[ A = \|z^2, yz, xz - y^2\|^s, \]
\[ B = \|u, z^2, yz, xz - y^2\|^{-s}, \]
\[ C = \|z^3, z^2u, yzu, (xz - y)^2 u\|^{-s}. \]

Write \( Z_3 = Z_{31} + Z_{32} \), where
\[ Z_{31} := \int_{u, y, z \in \mathfrak{p}} |u|^{3s-4} ABC d\mu, \]
\[ Z_{32} := \int_{u, y, z \in \mathfrak{p}} |u|^{3s-4} ABC d\mu. \]

3.3.1. Computation of \( Z_{31} \). Let \( y = zy_1 \) with \( y_1 \in \mathfrak{p} \). Then
\[ A = \|z^2, xz - y_1^2 z^2\|^s = |z|^s \|z, x - y_1^2 z\|^s = |z|^s \text{ (since } x - y_1^2 z \in W_1(\mathfrak{o})\), \]
\[ B = \|u, z\|^s, \]
\[ C = |z|^{-s} \|u, z^2\|^{-s}. \]

Thus
\[ Z_{31} = q^{-1} (1 - q^{-1}) \int_{u, z \in \mathfrak{p}} |z| |u|^{3s-4} \|u, z\|^{-s} \|u, z^2\|^{-s} d\mu. \]
Since $\mu\{ (u, z) \in p \mid v(u) = X, v(z) = Y \} = (1 - q^{-1})^2 q^{-X-Y}$, Lemma 2.1 (3) implies that

$$Z_{31} = q^{-1}(1 - q^{-1}) \int_{u, z \in p} |u|^{3s-4} |z| \|u, z\|^{-s} \|u, z^2\|^{-s} d\mu$$

$$= q^{-1}(1 - q^{-1})^3 \sum_{(X,Y) \in \mathbb{N}^2} q^{-X-Y} q^{(-3s+4)X} q^{-Y} q^{s \min(X,Y)} q^{s \min(X,2Y)}$$

$$= q^{-1}(1 - q^{-1})^3 \sum_{(X,Y) \in \mathbb{N}^2} q^{(-3s+3)X} q^{-2Y} q^{s \min(X,Y)} q^{s \min(X,2Y)}$$

$$= (1 - q^{-1})^3 t(1 - q^2 t^3 + q^4 t^2 - qt^4 - q^2 t^5 + q^3 t^6)$$

$$= (1 - q^{-2})(1 - q t)(1 - q^3 t^3)(1 - q^4 t^3).$$

3.3.2. Computation of $Z_{32}$. Let $z = yz_1$ with $z_1 \in o$. We have

$$A = |y|^s \|yz_1, xz_1 - y\|^s,$$

$$B = \|u, y^2 z_1, y(xz_1 - y)\|^{-s},$$

$$C = |y|^{-s} \|y^2 z_1, yz_1 u, u(xz_1 - y)\|^{-s}.$$

Thus

$$Z_{32} = \int_{x \in W_1(o)} |u|^{3s-4} |y| A_1 BC_1 d\mu.$$

Write $Z_{32} = Z_{321} + Z_{322}$, where

$$Z_{321} := \int_{x \in W_1(o)} |u|^{3s-4} |y| A_1 BC_1 d\mu,$$

$$Z_{322} := \int_{x \in W_1(o)} |u|^{3s-4} |y| A_1 BC_1 d\mu.$$

Computation of $Z_{321}$. Since $z_1 \in W_1(o)$, it follows that $xz_1 - y \in W_1(o)$ and so

$$Z_{321} = (1 - q^{-1})^2 \int_{u, y \in p} |u|^{3s-4} |y| \|u, y\|^{-s} \|u, y^2\|^{-s} d\mu = (q - 1) Z_{31}.$$

Computation of $Z_{322}$. Write $Z_{322} = Z_{322a} + Z_{322b}$, where

$$Z_{322a} := \int_{x \in W_1(o)} |u|^{3s-4} |y| A_1 BC_1 d\mu,$$

$$Z_{322b} := \int_{x \in W_1(o)} |u|^{3s-4} |y| A_1 BC_1 d\mu.$$

Computation of $Z_{322a}$. Let $y = z_1 y_1$ with $y_1 \in p$. We have

$$A_1 = |z_1|^s \quad \text{(since } x - y_1 \in W_1(o)),$$

$$B = \|u, y_1 z_1^2\|^{-s},$$

$$C_1 = |z_1|^{-s} \|y_1^2 z_1, u\|^{-s}.$$
Thus
\[ Z_{322a} = (1 - q^{-1}) \int_{u,y,z} |u|^3 s - 4 |y_1| |z_1| ^2 BC_2 d\mu. \]

Since \( \mu \{ (u, y_1, z_1) \in S \mid v(u) = X, v(y_1) = Y, v(z_1) = Z \} = (1 - q^{-1})^3 q^{-X - Y - Z}, \) one has
\[ Z_{322a} = (1 - q^{-1})^4 \sum_{(X,Y,Z) \in \mathbb{N}^3} q^{-X - Y - Z} q^{|-3s+4|X} q^{-Y} q^{-2Z} q^s \min \{X,Y+2Z\} q^s \min \{X,2Y+4Z\} \]
\[ = (1 - q^{-1})^4 \sum_{(X,Y,Z) \in \mathbb{N}^3} q^{(-3s+3)X} q^{-2Y} q^{-3Z} q^s \min \{X,Y+2Z\} q^s \min \{X,2Y+4Z\}. \]

One now can apply Lemma 2.1 (4) with \( a = q^{-3s+3}, b = q^{-2}, c = q^{-3} \) and \( d = q^s \) to obtain \( Z_{322a}. \) We record the result in the Appendix.

Computation of \( Z_{322b}. \) Let \( z_1 = y_{22} \) with \( z_2 \in o. \) We have
\[ A_1 = |y|^s \langle y_{22}, y_{22} - 1 \rangle^s, \]
\[ B = \|u, y^3 z_2, y^2 (y_{22} - 1)\|^s, \]
\[ C_1 = |y|^{-s} \| y^4 z_2^2, y^2 u, u(y_{22} - 1)\|^s. \]

Thus
\[ Z_{322b} = \int_{x \in W_1} |u|^{3s-4} |y|^2 A_2 BC_2 d\mu = Z_{322b_1} + Z_{322b_2}, \]

where
\[ Z_{322b_1} := \int_{x \in W_1} |u|^{3s-4} |y|^2 A_2 BC_2 d\mu, \]
\[ Z_{322b_2} := \int_{x \in W_1} |u|^{3s-4} |y|^2 A_2 BC_2 d\mu. \]

Computation of \( Z_{322b_1}. \) Since \( z_2 \in p, \) it follows that \( x z_2 - 1 \in W_1. \) Thus \( A_2 = 1, B = \|u, y^2\|^s \) and \( C_2 = \|y^4 z_2^2, u\|^s. \) It's now easy to compute
\[ Z_{322b_1} = (1 - q^{-1}) \int_{u,y,z} |u|^{3s-4} |y|^2 BC_2 d\mu. \]

Since \( \mu \{ (u, y, z_2) \in p^3 \mid v(u) = X, v(y) = Y, v(z_2) = Z \} = (1 - q^{-1})^3 q^{-X - Y - Z}, \) one has
\[ Z_{322b_1} = (1 - q^{-1})^4 \sum_{(X,Y,Z) \in \mathbb{N}^3} q^{-X - Y - Z} q^{(-3s+4)X} q^{-2Y} q^s \min \{X,Y+2Z\} q^s \min \{X,4Y+3Z\} \]
\[ = (1 - q^{-1})^4 \sum_{(X,Y,Z) \in \mathbb{N}^3} q^{(-3s+3)X} q^{-3Y} q^{-2Z} q^s \min \{X,Y+2Z\} q^s \min \{X,4Y+3Z\}. \]

One needs first to compute \( \sum_{(X,Y,Z) \in \mathbb{N}^3} q^X b^Y c^Z d^{X+2Y+3Z} \) similarly to Lemma 2.1 (4) and then apply to \( a = q^{-3s+3}, b = q^{-3}, c = q^{-1} \) and \( d = q^s \) to obtain \( Z_{322b_1}. \) The result is recorded in the Appendix.
Computation of $Z_{322b2}$. The equation $xz_2 \equiv 1 \mod p$ has $q - 1$ roots $(a_1, a_2) \in (F_q^*)^2$. We have

$$Z_{322b2} = \int_{x, y \in p} (u, y) \in p |u|^{3s-4} |y|^2 A_2 B C_2 d\mu$$

$$= \sum_{(a_1, a_2) \in (F_q^*)^2} \int_{x, y \in p} (a_1, a_2 + p \cdot a) |u|^{3s-4} |y|^2 A_2 B C_2 d\mu$$

$$= (q - 1)(q - 2)J_1 + (q - 1)J_2,$$

where

$$J_1 := \int_{x, y \in p} (a_1, a_2 + p \cdot a) |u|^{3s-4} |y|^2 A_2 B C_2 d\mu,$$

$$J_2 := \int_{x, y \in p} (a_1, a_2 + p \cdot a) |u|^{3s-4} |y|^2 A_2 B C_2 d\mu.$$

In computing $J_1$, notice that in this case $xz_1 \not\equiv 1 \mod p$, and so $A_2 = 1, B = \|u, y^2\|^{-s}$ and $C_2 = \|u, y^2\|^{-s}$, and thus we have

$$J_1 = q^{-2} \int_{u, y \in p} |u|^{3s-4} |y|^2 B C_2 d\mu.$$

Since $\mu\{u, y \in p^2 | v(u) = X, v(y) = Y\} = (1 - q^{-1})q^{-X-Y}$, one has

$$J_1 = q^{-2} (1 - q^{-1})^2 \sum_{(X, Y) \in \mathbb{N}^2} q^{-X-Y} q^{(-3s+4)X - 2Y} q^s \min\{X, 2Y\} q^{s \min\{X, 4Y\}}$$

$$= q^{-2} (1 - q^{-1})^2 \sum_{(X, Y) \in \mathbb{N}^2} q^{(-3s+3)X - 3Y} q^s \min\{X, 2Y\} q^{s \min\{X, 4Y\}}.$$

We first need to compute $\sum_{(X, Y) \in \mathbb{N}^2} a X^p B^Y c^{\min\{X, 2Y\}} d^{\min\{X, 4Y\}}$ similarly to Lemma 2.11 (3) and then apply with $a = q^{-3s+3}$, $b = q^{-3}$ and $c = q^s$ to obtain $J_1$. We record $J_1$ in the Appendix.

In computing $J_2$, notice that in this case, on each coset $(a_1, a_2) + p^2$ we have $xz_2 \equiv 1 \mod p$. We change variable $v = xz_2 - 1 \in p$. Then $A_2 = \|y, v\|^2, B = \|u, y^2, y^2 v\|^{-s}$, $C_2 = \|y^4, yu, uv\|^{-s}$ and

$$J_2 = q^{-1} \int_{u, y, v \in p} |u|^{3s-4} |y|^2 A_2 B C_2 d\mu.$$

Since $\mu\{u, y, v \in p | v(u) = X, v(y) = Y, v(v) = Z\} = (1 - q^{-1})^3 q^{-X-Y-Z}$, one has

$$J_2 = q^{-1} (1 - q^{-1})^3 \sum_{(X, Y, Z) \in p} q^{(-3s+3)X - 3Y - Z} q^{-s \min\{Y, Z\}}$$

$$\times q^{s \min\{X, 3Y + 2Z\}} q^{s \min\{X + Y, X + Z, 4Y\}}.$$

Again computing $\sum_{(X, Y, Z) \in \mathbb{N}^3} a X^p B^Y c^{Z} d^{\min\{X, Z\}} e^{\min\{X, 2Y + Z\}} d^{\min\{X, 3Y + 2Z\}} e^{\min\{X + Y, X + Z, 4Y\}}$ and then applying for $a = q^{-3s+3}, b = q^{-3}, c = q^{-1}$ and $d = q^s$ yields $J_2$ which we record in the Appendix.
obtain the residue field cardinality, we get
\[
\zeta_{H(x^3)}(s) = 1 + (1 - q^{-1})^{-1} \mathcal{Z} = \frac{(1 - t)(1 - q^2t^2)(1 - q^3t^3)}{(1 - qt)(1 - q^4t^2)(1 - q^5t^3)}.
\]
Hence
\[
(3.1) \quad \zeta_{H(O[x]/(x^n))}(s) = \frac{\zeta_K(s - 1)}{\zeta_K(s)} \cdot \frac{\zeta_K(2s - 3)}{\zeta_K(2s - 2)} \cdot \frac{\zeta_K(3s - 5)}{\zeta_K(3s - 4)}.
\]

4. Open questions

4.1. Heisenberg group scheme. Formulae (1.1), (2.3) and (3.1) agree in the following conjectured formula.

Conjecture 4.1. The representation zeta function of $H(O[x]/(x^n))$ is
\[
(4.1) \quad \zeta_{H(O[x]/(x^n))}(s) = \prod_{i=1}^{n} \frac{\zeta_K(is - 2i + 1)}{\zeta_K(is - 2i + 2)}.
\]

If Conjecture (1.1) holds, then the zeta functions $\zeta_{H(O[x]/(x^n))}(s)$ shares uniform analytic properties with $\zeta_{H(O)}(s)$ as suggested by the following conjecture.

Conjecture 4.2. Let $R$ be a ring which is torsion-free finitely generated over $\mathbb{Z}$. Then the representation zeta function $\zeta_{H(R)}(s)$ has the following properties:

1. Its abscissa of convergence is $\alpha(H) = 2$.
2. It can be analytically continued to the whole complex plane. The continued zeta function has no singularities on the line $\text{Re}(s) = 2$, apart from a simple pole at $s = \alpha(H)$.

Recall that the representation zeta function of $H(\mathbb{Z}_p)$ is
\[
\zeta_{H(\mathbb{Z}_p)}(s) = 1 + (1 - p^{-1})^{-1} \int_{x \in \mathbb{Z}_p, \|x\|_p \leq 1} |x|^{s-2} d\mu = \frac{1 - p^{-s}}{1 - p^{1-s}}.
\]

When computing the zeta function for $H(O_p)$, one just needs to replace $p$ by $q = |O/p|$ the residue field cardinality, $\mathbb{Z}_p$ by $O_p$, and replace $p$-adic norm $\|.|$ by $p$-adic norm $\|.|_p$ to obtain
\[
\zeta_{H(O_p)}(s) = 1 + (1 - q^{-1})^{-1} \int_{x \in O_p} |x|^{s-2} d\mu = \frac{1 - q^{-s}}{1 - q^{1-s}}.
\]

Question 4.3. Can one define the domain $W$, a valuation and norm $\|.|_n$ on $O_p[x]/(x^n)$ compatible with the $p$-adic norm $\|.|_p$ such that the zeta function of $H(O_p[x]/(x^n))$ can be computed as follows
\[
\zeta_{H(O_p[x]/(x^n))}(s) = 1 + (1 - q^{-1})^{-1} \int_{W} |x|_n^{s-2} d\mu?
\]

By expanding the conjectured formula (4.1) for the local zeta function, we get
\[
(4.2) \quad \zeta_{H(O_p[x]/(x^n))}(s) = \sum_{I \subseteq [n-1]} f_I(q^{-1}) \prod_{i \in I} \frac{q^{2n-2i-1-(n-i)s}}{1 - q^{2n-2i-1-(n-i)s}},
\]
where $I$ runs over all subsets of $[n - 1]_0 := \{0, 1, \ldots, n - 1\}$ and $f_I(q^{-1}) = (1 - q^{-1})^{\left|I\right|}$. The formula (4.2) looks similar to [8, (1.12)]. However, we have been unable to mimic the inductive proof of [8, Theorem C] to yield (4.2).
4.2. **Unipotent group schemes.** Once we understand the zeta function of $H(O_p[x]/(x^n))$, the next step is the following.

Let $Λ$ be a finitely generated free and torsion-free $O$-Lie lattice of nilpotency class $c$ and $O$-rank $h$. If $c > 2$ we assume that $Λ' := [Λ, Λ] ⊆ cΛ$. This enables us to associate to $Λ$ a unipotent group scheme $G := G_Λ$ (cf. [8, Section 2.1.2]). The group $G(O)$ is a $T$-group of nilpotency class $c$ and Hirsch length $h \cdot [K : Q]$. The Heisenberg group scheme $H$ is an example of such a unipotent group scheme.

The zeta function $ζ_{G(O)}(s)$ is the Euler product

$$ζ_{G(O)}(s) = \prod_{p \in \text{Spec}(O)} ζ_{G(O_p)}(s)$$

ranging over all nonzero prime ideals $p$ of $O$; cf. [8]. There exists a finite set $S$ of prime ideals such that for each $p \notin S$, the local zeta function $ζ_{G(O_p)}(s)$ is a rational function in $q^{-s}$ and satisfies a functional equation upon inversion of $q$; see [8, Theorem A]. Moreover, each such local representation zeta function can be expressed in terms of a $p$-adic integral; cf. [8, Corollary 2.11].

It is tempting to investigate the zeta function of $G(O[x]/(x^n))$ when $n$ tends to $∞$. As we have seen for the Heisenberg group scheme, if Conjecture 4.1 holds then

$$\lim_{n \to ∞} ζ_{H(O[x]/(x^n))}(s) = \prod_{i=1}^{∞} ζ_K((is - 2i + 1)/(is - 2i + 2)).$$

It is natural to ask the following.

**Question 4.4.** Let $G$ be a unipotent group scheme as above.

1. Is $G(O[[x]])$ twist-rigid, that is, is $r_n(G(O[[x]]))$ finite for every $n \in \mathbb{N}$? Does $G(O[[x]])$ have polynomial representation growth?

2. If (1) has positive answers then is it true that

$$ζ_{G(O[[x]])}(s) := \sum_{n=1}^{∞} r_n(G(O[[x]]))n^{-s} = \lim_{n \to ∞} ζ_{G(O[x]/(x^n))}(s)?$$

It is proved in [11, Theorem A] that the zeta function $ζ_{G(O)}(s)$ has rational abscissa of convergence $α(G)$, which is independent on the number field $K$, and can be analytically continued to $\text{Re}(s) > α(G) − δ(G)$ for some $δ(G) > 0$. In the spirit of Conjecture 4.2, we formulate the following.

**Conjecture 4.5.** Let $R$ be a ring which is finitely generated torsion-free $O$-module. Then the representation zeta function of $G(R)$ has the following properties:

1. The abscissa of convergence $α(G)$ of $ζ_{G(R)}(s)$ is independent of $R$.

2. It can be meromorphically continued to the half-plane $\text{Re}(s) > α(G) − δ(G)$ for some $δ(G) > 0$, where $δ(G)$ is independent of $K$.

4.3. **Topological representation zeta functions.** Topological zeta functions offer a way to define a limit as $p \to 1$ of families of $p$-adic zeta functions. Let $G$ be a unipotent group scheme defined as in Section 4.2. In [1], Rossmann introduces and studies topological representation zeta functions associated to unipotent group schemes $G$. Informally, we define the topological representation zeta function $ζ_{G,\text{top}}(s)$ to be the constant term of $ζ_{G(\mathbb{Z}_p)}(s)$ as a series in $p^{-1}$. 
Example 4.6. Consider the Heisenberg group scheme $H$. Expanding $p^z = (1 + (p - 1))z$ into a series in $p - 1$, we obtain $\zeta_{H}(\mathbb{Z}_p)(s) = \frac{s}{s - 1} + O(p - 1)$ and hence $\zeta_{H,\text{top}}(s) = \frac{s}{s - 1}$.

Let $\mathbb{Q}[\varepsilon_n] = \mathbb{Q}[x]/(x^n)$ and for a $\mathbb{Q}$-algebra $\mathfrak{g}$, let $\mathfrak{g}[\varepsilon_n] = \mathfrak{g} \otimes_{\mathbb{Z}} \mathbb{Q}[\varepsilon_n]$ regarded as a $\mathfrak{g}$-Lie lattice. Let $H[\varepsilon_n]$ denote the group attached to $\mathfrak{g}[\varepsilon_n]$; cf. [6, Section 7]. Then $H[\varepsilon_n](\mathbb{Z}_p) = H(\mathbb{Z}_p[x]/(x^n))$. Hence

$$\zeta_{H[\varepsilon_2],\text{top}}(s) = \frac{s(2s - 2)}{(s - 1)(2s - 3)} = \frac{2s}{2s - 3},$$

$$\zeta_{H[\varepsilon_3],\text{top}}(s) = \frac{s(2s - 2)(3s - 4)}{(s - 1)(2s - 3)(3s - 5)} = \frac{2s(3s - 4)}{(2s - 3)(3s - 5)}.$$  (4.3) (4.4)

An algorithm to compute topological representation zeta functions is implemented in [5]. Notice that Rossmann’s method computes topological representation zeta function directly without first computing the corresponding $p$-adic representation zeta function. Formulae (4.3) and (4.4) are consistent with computation results in [5]. Notice also that [5] can only compute the topological representation zeta function of $H[\varepsilon_n]$ up to $n = 3$ as we have done here for their $p$-adic representation zeta functions. Conjecture 4.1 suggests the following analogue for topological representation zeta functions of $H[\varepsilon_n]$.

Conjecture 4.7. The topological representation zeta function of $H[\varepsilon_n]$ is

$$\zeta_{H[\varepsilon_n],\text{top}}(s) = \prod_{i=1}^{n} \frac{is - 2i + 2}{is - 2i + 1}.$$  (4.7)

Remark 4.8. All Questions in [6, Section 7], except Question 7.3 which is not yet known, have positive answers for $G = H[\varepsilon_n]$ with $n \leq 3$.

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5. Appendix

\[ Z_{322a} = \frac{(1 - q^{-1})^4}{q^{15}} \left( \frac{q^{-2t}}{(1-q^{-2})(1-q^{-4})(1-t)(1-q^t)} + \frac{pt^2}{q^{10t^7}} + \frac{q^4t^3}{q^{13t^9}} \right) + \frac{(1-q^t)(1-q^{2t^2})(1-q^{4t^3})}{q^{15t^5}} \]

\[ Z_{322b} = \frac{(1 - q^{-1})^4}{q^{36}} \left( \frac{t}{(q-1)(1-q^{-5})(1-t)} + \frac{q^{2t^2}}{(1-q^{-1})(1-t)(1-t^2)(1-t^6)} + \frac{q^7t^4}{q^{11t^6}} + \frac{1}{(1-q^{-1})(1-t^2)(1-q^t)q^{10t^2}} \right) + \frac{(1-q^{-1})(1-q^{2t^2})(1-q^{4t^3})(1-q^{6t^4})}{q^{5t^7}} + \frac{(1-q^t)(1-q^{2t^2})(1-q^{4t^3})(1-q^{6t^4})}{q^{15t^15}} \]

\[ J_1 = \frac{q^{-2}(1 - q^{-1})^2}{(1-q^{-1})(1-t) + \frac{q^{3t^2}}{(1-t)(1-t^2)q^4} + \frac{q^6t^4}{(1-q^t)(1-t^2)q^8} + \frac{q^{10t^6}}{(1-q^t)(1-q^2t^4)q^{12t^9}} \]

\[ J_2 = \frac{q^{-1}(1 - q^{-1})^3}{(1-q^{-1})(1-q^{-1})(1-q^{-1}) + \frac{q^{-2t^2}}{(1-q^{-1})(1-q^{-1})q^{-1}} + \frac{q^2t^2}{q^{10t^5}} + \frac{q^{-4t}}{q^{14t^9}}} + \frac{(1-q^t)(1-q^{2t^2})(1-q^{4t^3})(1-q^{6t^4})}{q^{15t^7}} + \frac{(1-q^t)(1-q^{2t^2})(1-q^{4t^3})(1-q^{6t^4})}{q^{12t^5}} + \frac{(1-q^t)(1-q^{2t^2})(1-q^{4t^3})(1-q^{6t^4})}{q^{12t^5}} \]

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