THE MAXIMAL NUMBER OF SINGULAR POINTS ON LOG DEL PEZZO SURFACES

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Abstract. We prove that a del Pezzo surface with Picard number one has at most four singular points.

1. INTRODUCTION

A log del Pezzo surface is a projective algebraic surface \( X \) with only quotient singularities and ample anticanonical divisor \(-K_X\).

Del Pezzo surfaces naturally appear in the log minimal model program (see, e.g., [7]). The most interesting class of del Pezzo surfaces is the class of surfaces with Picard number 1. It is known that a log del Pezzo surface of Picard number one has at most five singular points (see [8]). Earlier the author proved there is no log del Pezzo surfaces of Picard number one with five singular points [1]. In this paper we give another, simpler proof.

Theorem 1.1. Let \( X \) be a del Pezzo surface with log terminal singularities and Picard number is 1. Then \( X \) has at most four singular points.

Recall that a normal complex projective surface is called a rational homology projective plane if it has the same Betti numbers as the projective plane \( \mathbb{P}^2 \). J. Kollár [9] posed the problem to find rational homology \( \mathbb{P}^2 \)'s with quotient singularities having five singular points. In [4] this problem is solved for the case of numerically effective \( K_X \). Our main theorem solves Kollár's problem in the case where \(-K_X\) is negative.

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2. Preliminary results

We work over complex number field $\mathbb{C}$. We employ the following notation:

- $(-n)$-curve is a smooth rational curve with self intersection number $-n$.
- $K_X$: the canonical divisor on $X$.
- $\rho(X)$: the Picard number of $X$.

**Theorem 2.1** (see [8, Corollary 9.2]). Let $X$ be a rational surface with log terminal singularities and $\rho(X) = 1$. Then

\[
\sum_{P \in X} \frac{m_P - 1}{m_P} \leq 3,
\]

where $m_P$ is the order of the local fundamental group $\pi_1(U_P - \{P\})$ ($U_P$ is a sufficiently small neighborhood of $P$).

So, every rational surface $X$ with log terminal singularities and Picard number one has at most six singular points. Assume that $X$ has exactly six singular points. Then by (*) all singularities are Du Val. This contradicts the classification of del Pezzo surfaces with Du Val singularities (see, e.g., [3], [10]).

2.2. Thus to prove Theorem 1.1 it is sufficient to show that there is no log del Pezzo surfaces with five singular points and Picard number one. Assume the contrary: there is log del Pezzo surfaces with five singular points and Picard number one. Let $P_1, \ldots, P_5 \in X$ be singular points and $U_{P_i} \ni P_i$ small analytic neighborhood. By Theorem 2.1 the collection of orders of groups $\pi_1(U_{P_1} - P_1), \ldots, \pi_1(U_{P_5} - P_5)$ up to permutations is one of the following:

2.2.1. $(2, 2, 3, 3, 3), (2, 2, 2, 4, 4), (2, 2, 2, 3, n), \ n = 3, 4, 5, 6,$

2.2.2. $(2, 2, 2, 2, n), \ n \geq 2$.

**Remark 2.3.** According to the classification of del Pezzo surfaces with Du Val singularities we may assume that there is a non-Du Val singular point. The case 2.2.1 is discussed in [4, proof of Prop. 6.1]. Thus it is sufficient to consider case 2.2.2.

2.4. Notation and assumptions. Let $X$ be a del Pezzo surface with log terminal singularities and Picard number $\rho(X) = 1$. We assume that we are in case 2.2.2, i.e. the singular locus of $X$ consists of four points $P_1, P_2, P_3, P_4$ of type $A_1$ and one more non Du Val singular point.
with \(|\pi_1(U_{P_5} - P_5)| = n \geq 3\). Let \(\pi : \bar{X} \to X\) be the minimal resolution and let \(D = \sum_{i=1}^{n} D_i\) be the reduced exceptional divisor, where the \(D_i\) are irreducible components. Then there exists a uniquely defined an effective \(\mathbb{Q}\)-divisor \(D^\# = \sum_{i=1}^{n} \alpha_i D_i\) such that \(\pi^*(K_{\bar{X}}) \equiv D^\# + K_{\bar{X}}\).

**Lemma 2.5** (see, e.g., [13, Lemma 1.5]). Under the condition of 2.4, let \(\Phi : \bar{X} \to \mathbb{P}^1\) be a generically \(\mathbb{P}^1\)-fibration. Let \(m\) be a number of irreducible components of \(D\) not contain in any fiber of \(\Phi\) and let \(d_f\) be a number of \((-1)\)-curves contained in a fiber \(f\). Then

\[
(1) \quad m = 1 + \sum f(d_f - 1).
(2) \quad \text{If } E \text{ is a unique } (-1)\text{-curve in } f, \text{ then the coefficient } E \text{ in } f \text{ is at least two.}
\]

The following lemma is a consequence of the Cone Theorem.

**Lemma 2.6** (see, e.g., [13, Lemma 1.3]). Under the condition of 2.4, every curve on \(\bar{X}\) with negative selfintersection number is either \((-1)\)-curve or a component of \(D\).

**Definition 2.7.** Let \((Y,D)\) be a projective log surface. \((Y,D)\) is called the weak log del Pezzo surface if the pair \((Y,D)\) is klt and the divisor \(- (K_Y + D)\) is nef and big.

For example, in the above notation, \((\bar{X},D^\#)\) is a weak del Pezzo surface. Note that if \((Y,D)\) is a weak log del Pezzo surface with \(\rho(Y) = 1\) then divisor \(- (K_Y + D) = A\) is ample and \(Y\) has only log terminal singularities. Hence, \(Y\) is a log del Pezzo surface.

**Lemma 2.8** (see, e.g., [1, Lemma 2.9]). Suppose \((Y,D)\) is a weak log del Pezzo surface. Let \(f : Y \to Y'\) be a birational contraction. Then \((Y',D' = f_*D)\) is also a weak log del Pezzo surface.

### 3. Proof of the main theorem: the case where \(X\) has cyclic quotient singularities

In this section we assume that \(X\) has only cyclic quotient singularities.

The following lemma is very similar to that in [5]. For convenience of the reader we give a complete proof.

**Lemma 3.1.** Under the condition of 2.4, suppose \(P_5\) is a cyclic quotient singularity. Then there exists a generically \(\mathbb{P}^1\)-fibration \(\Phi : \bar{X} \to \mathbb{P}^1\) such that \(f \cdot D \leq 2\), where \(f\) is a fiber of \(\Phi\).
Proof. Let $\nu : \hat{X} \to X$ be the minimal resolution of the non Du Val singularities and let $E = \sum E_i$ be the exceptional divisor. By [12, Corollary 1.3] or [8, Lemma 10.4] we have $| -K_X| \neq \emptyset$. Take $B \in | -K_X|$. Then we can write
$$K_{\hat{X}} + \hat{B} = \nu^*(K_X + B) \sim 0,$$
where $\hat{B}$ is an effective integral divisor. We obviously have $\hat{B} \geq E$.

Run the MMP on $\hat{X}$. We obtain a birational morphism $\phi : \hat{X} \to \tilde{X}$ such that $\tilde{X}$ has only Du Val singularities and either $\rho(\tilde{X}) = 2$ and there is a generically $\mathbb{P}^1$-fibration $\psi : \tilde{X} \to \mathbb{P}^1$ or $\rho(\tilde{X}) = 1$. Moreover, $\phi$ is a composition
$$\hat{X} = X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} \ldots \xrightarrow{\phi_n} X_{n+1} = \tilde{X},$$
where $\phi_i$ is a weighted blowup of a smooth point of $X_{i+1}$ with weights $(1, n_i)$ (see [11]).

Assume that $\rho(\tilde{X}) = 1$, then every singular point on $\tilde{X}$ is of type $A_1$. By the classification of del Pezzo surfaces with Du Val singularities and Picard number one (see, e.g., [3], [10]) we have $\tilde{X} = \mathbb{P}^2$ or $\tilde{X} = \mathbb{P}(1, 1, 2)$.

Assume that $\rho(\tilde{X}) = 1$ and $\tilde{X} = \mathbb{P}(1, 1, 2)$. Since $\phi_*(\hat{B})$ has at most two components, we see that $\phi$ contracts at most two curves $K_1$ and $K_2$ such that $K_i$ is not component of $E$. Since $X$ has four singular points of type $A_1$, we see that $\tilde{X}$ has at least two singular points, a contradiction.

Assume that $\rho(\tilde{X}) = 1$ and $\tilde{X} = \mathbb{P}^2$. Since $\phi_*(\hat{B})$ has at most three components, we see that $\phi$ contracts at most three curves $K_1$, $K_2$ and $K_3$ such that $K_i$ is not component of $E$. Since $X$ has four singular points of type $A_1$, we see that $\tilde{X}$ has at least one singular point, a contradiction.

Therefore, $\rho(\tilde{X}) = 2$ and there is a generically $\mathbb{P}^1$-fibration $\psi : \tilde{X} \to \mathbb{P}^1$. Let $g : \hat{X} \to \tilde{X}$ be the minimal resolution of $\tilde{X}$. Let $\Phi' = \psi \circ \phi$ and let $f'$ be a fiber of $\Phi'$. Then $f' \cdot E \leq f' \cdot \hat{B} = -K_{\hat{X}} \cdot f' = 2$. Set $\Phi = \Phi' \circ g$. □

3.2. Let $f$ be a fiber of $\Phi$. By Lemma 3.1 we have the following cases:

3.2.1. $f$ meets exactly one irreducible component $D_0$ of $D$ and $f \cdot D_0 = 1$.

Let $L$ be a singular fiber of $\Phi$. By Lemma 2.5 (1) the fiber $L$ contains exactly one $(-1)$-curve $F$. By Lemma 2.5 (2) $F$ does not meet $D_0$. Then $F$ meets at most two components of $D$. Blowup one of the points of intersection $F$ and $D$. We obtain a surface $Y$. Let $h : Y \to Y'$ be
a contraction of all curves with selfintersection number at most \(-2\). Note that \(Y'\) has only log terminal singularities but not of type 2.2.2, a contradiction.

3.2.2. \(f\) meets exactly two irreducible components \(D_1, D_2\) of \(D\) and \(D_1 \cdot f = D_2 \cdot f = 1\).

By Lemma 2.5 (1) there exists a unique singular fiber \(L\) such that \(L\) has two \((-1)\)-curves \(F_1\) and \(F_2\). Note that one of these curves, say \(F_1\), meets \(D\) at one or two points. Blowup one of the points of intersection \(F_1\) and \(D\). We obtain a surface \(Y\). Let \(h: Y \to Y'\) be a contraction of all curves with selfintersection number at most \(-2\). Note that \(Y'\) has only log terminal singularities but not of type 2.2.2, a contradiction.

3.2.3. \(f\) meets exactly one irreducible component \(D_0\) of \(D\) and \(f \cdot D_0 = 2\). Let \(A\) be a connected component of \(D\) containing \(D_0\).

By Lemma 2.5 (1) every singular fiber of \(\Phi\) contains exactly one \((-1)\)-curve. Note that every singular fiber of \(\Phi\) either contains two connected components of \(A - D_0\) or the coefficient of a unique \((-1)\)-curve in this fiber is equal two. If a singular fiber \(L\) contains exactly one \((-1)\)-curve with coefficient two, then the dual graph of \(L\) is following:

\[
\begin{array}{ccc}
& -2 & \\
\circ & -1 & \\
& -2 & \\
\end{array}
\]

Since \(X\) has five singular points with orders of local fundamental groups \((2, 2, 2, 2, n)\), we see that \(\Phi\) has two singular fibers \(L_1, L_2\) of type (**)) and possibly one more singular fiber \(L_3\). Note that \(L_3\) contains both connected component of \(A - D_0\). Let \(\mu: \tilde{X} \to \mathbb{F}_n\) be the contraction of all \((-1)\)-curves in fibers of \(\Phi\), where \(\mathbb{F}_n\) is the Hirzebruch surface of degree \(n\) (rational ruled surface) and \(n = 0, 1\). Denote \(\tilde{D}_0 := \mu_* D_0\). Note that \(\tilde{D}_0 \sim 2M + kf\), where \(M^2 = -n\) and \(M \cdot f = 1\). Since we contract at most five curves that meet \(D_0\), and \(D_0^2 \leq -2\), we see that \(0 < \tilde{D}_0^2 \leq 3\). Hence, \(0 < -4n + 4k \leq 3\). This is impossible, a contradiction.

4. **Proof of the main theorem: the case where \(X\) has non-cyclic quotient singularity**

Under the condition of 2.4, assume \(X\) has a non-cyclic singular point, say \(P\). Then there is a unique component \(D_0\) of \(D\) such that \(D_0 \cdot (D - D_0) = 3\) (see [2]).

**Lemma 4.1.** There is a generically \(\mathbb{P}^1\)-fibration \(\Phi: \tilde{X} \to \mathbb{P}^1\) such that \(\Phi\) has a unique section \(D_0\) in \(D\) and \(D_0 \cdot f \leq 3\), where \(f\) is a fiber of \(\Phi\).
Proof. Recall that $P$ is not Du Val. Let $h : \tilde{X} \to \hat{X}$ be contract all curves in $D$ except $D_0$. Let $\hat{D}_0 = h_*(D_0)$ then $\hat{X}$ has seven singular points, $\rho(\hat{X}) = 2$ and there is $\nu : \hat{X} \to X$ such that $K_\hat{X} + a\hat{D}_0 = \nu^*K_X$. Note that $(\hat{X}, a\hat{D}_0)$ is a weak log del Pezzo. Let $R$ be the extremal rational curve different from $\hat{D}$. Let $\phi : \hat{X} \to \tilde{X}$ be the contraction of $\hat{R}$.

4.2. There are two cases:

4.2.1. $\rho(\tilde{X}) = 1$. Then, by Lemma 2.8 $\tilde{X}$ is a del Pezzo surface. If the number of singular points of $\hat{X}$ on $R$ is at most two, $\tilde{X}$ has at least five singular points and all points are cyclic quotients. Thus assume that there is at least three singular points of $\hat{X}$ on $R$, say $P_1, P_2, P_3$. Let $R_1 = \sum_i R_{1i}, R_2 = \sum_i R_{2i}$ and $R_3 = \sum_i R_{3i}$ be the exceptional divisors on $\hat{X}$ over $P_1, P_2$ and $P_3$, respectively. Let $\hat{R}$ is the proper transformation of $R$ on $\hat{X}$. Since $\hat{R}$ is not component of $D$, we see that $\hat{R}^2 \geq -1$. Indeed, this follows from Lemma 2.6. Note that matrix of intersection of component $\hat{R} + R_1 + R_2 + R_3$ is not negative definite. Hence, $\hat{R} + E_1 + E_2 + E_3$ can not be contracted, a contradiction.

4.2.2. $\tilde{X} = \mathbb{P}^1$. Let $g : \tilde{X} \to \hat{X}$ be the resolution of singularities. Then $\Phi = \phi \circ g : \tilde{X} \to \mathbb{P}^1$. Note that there is a unique horizontal curve $D_0$ in $D$. Let $f$ be a fiber of $\Phi$. Denote coefficient of $D_0$ in $D^2$ by $\alpha$. Then

$$0 > (K_{\tilde{X}} + D^2) \cdot f = -2 + \alpha(D_0 \cdot f).$$

Hence, $D_0 \cdot f < \frac{2}{\alpha}$. Since $P$ is not Du Val, we see that $\alpha \geq \frac{1}{2}$. Hence, $D_0 \cdot f \leq 3$.

\[ \square \]

By Lemma 2.8 (1) every singular fiber of $\Phi$ contains exactly one $(-1)$-curve. Let $B$ be the exceptional divisor corresponding to the non-cyclic singular point. Note that $B$ contains $D_0$.

4.3. Consider three cases.

4.3.1. $D_0 \cdot f = 1$. Then every singular fiber of $\Phi$ contains exactly one connected component of $B - D_0$. On the other hand, $B - D_0$ contains three connected component. Hence $X$ has at most four singular points, a contradiction.

4.3.2. $D_0 \cdot f = 2$. Let $F_1, F_2, F_3$ be a connected components of $B - D_0$. We may assume $F_1$ is $(-2)$-curve (see 2). Let $L_1$ be a singular fiber of $\Phi$. Assume that $L_1$ contains $F_1$. Then $L_1$ is type $(*)$ and $L_1$ contain $F_2$. Hence, $F_2$ is a $(-2)$-curve. Let $L_2$ be a singular fiber of $\Phi$. 

Assume that $L_2$ contains $F_3$ and let $E$ be a unique $(-1)$-curve in $L_2$. By blowing up the point of intersection $E$ and $F_3$, we obtain a surface $Y$. Let $h : Y \to Y'$ be a contraction of all curve with selfintersection number at most $-2$. Note that $Y'$ has only log terminal singularities but not of type $2.2.2$, a contradiction.

4.3.3. $D_0 \cdot f = 3$. Since every component of $D - B$ is a $(-2)$-curve, we see that every singular fiber of $\Phi$ contains a connected component of $B - D_0$. Note that $B - D_0$ contains three connected components. Hence $X$ has at most four singular points, a contradiction.

This completes the proof of Theorem 1.1.

REFERENCES

[1] G. N. Belousov, *Del Pezzo surfaces with log terminal singularities*, Mat. Zametki 83 (2008), no. 2, 170-180 (Russian) English translate Math. Notes.

[2] Brieskorn E. *Rationale Singularit"aten komplexer Fl"achen*, Invent. Math. 4 (1968), 336 – 358.

[3] Furushima M. *Singular del Pezzo surfaces and analytic compactifications of 3-dimensional complex affine space $C^3$*, Nagoya Math. J. 104 (1986), 1 – 28.

[4] D. Hwang, J. Keum *The maximum number of singular points on rational homology projective planes* arXiv:math.AG/0801.3021v3.

[5] P. Hacking Yu. Prokhorov *Smoothable del Pezzo surfaces with quotient singularities* Unpublished manuscript.

[6] Kawamata Y. *Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces*, Ann. of Math. 127 (1988), 93 – 163.

[7] Kawamata Y., Matsuda K. & Matsuki J. *Introduction to the minimal model program*, Adv. Stud. Pure Math. 10 (1987), 283 – 360.

[8] Keel S. & McKernan J. *Rational curves on quasi-projective surfaces*, Memoirs AMS 140 (1999), no. 669.

[9] Kollar J. *Is there a topological Bogomolov-Miyaoka-Yau inequality?*, Pure and Applied Math. Quarterly 4 No. 2 (2008).

[10] Miyanishi M. & Zhang D. -Q. *Gorenstein log del Pezzo surfaces of rank one*, J. Algebra. 118 (1988), 63 – 84.

[11] Morrison D. *The Birational Geometry of Surfaces with Rational Double Points*, Math. Ann. 271 (1985), 415-438.

[12] Prokhorov Yu. G. Verevkin A. B. *The Riemann-Roch theorem on surfaces with log terminal singularities* J. Math Sci. (N. Y.) 140 (2007), no. 2, 200-205.

[13] Zhang D.-Q. *Logarithmic del Pezzo surfaces of rank one with contractible boundaries*, Osaka J. Math. 25 (1988), 461 – 497.