Integrable Vector Perturbations of W-invariant Theories
and
their Quantum Group Symmetry

A. Babichenko

Racah institute of Physics, Hebrew University
Jerusalem, 91904, Israel

Abstract

Perturbations of $WD_n$ and $W_3$ conformal theories which generalize the (1, 2) perturbations of conformal minimal models are shown to be integrable by counting argument. $A_{2n-1,q}^{(2)}$ and $D_{4,q}^{(3)}$ symmetries of corresponding S-matrices are conjectured and proved by explicit construction of conserved nonlocal charges in the $WD_3$ case with the proper quantum group of symmetry.
1 Introduction

In the recent years a progress has been achieved in the understanding of integrable perturbations of two dimensional Conformal Field Theories (CFT) from the point of view of their current algebra and Factorizable Scattering Theory (FST) of perturbed massive model. It was realized from the very beginning [1], that the structure of the (1,2) perturbation of minimal CFT, corresponding on the classical level to Zhiber-Mikhailov-Shabat model, and its FST is more complicated then the structure of of (1,3) perturbation, which classically corresponds to Sine-Gordon model. After the work of Fateev and Zamolodchikov [2] it was not clear whether there exists some general description of FST and a symmetry of (1,2) perturbations of minimal models, because the discovered group of symmetry of FST turned out to be very different: $E_8, E_7$ and $E_6$ for $p=3,4$ and 6 minimal models correspondingly. Nevertheless the general solution with $A_{2q}^{1}(2)$ symmetry of FST of any (1,2) perturbed minimal model was found by Smirnov [3] which reproduces the previously known solutions as particular cases, however this reproduction is essentially nontrivial and is based on the properties of representations of $sl(2)_q$ at special values of $q$ equal to root of unity.

Along with integrable perturbations of conformal models with Virasoro and Kac-Moody algebra, perturbations of CFT with other additional symmetries and their FST were studied. In ref. [4] few integrable perturbations of CFT $Z_N$ parafermionic models were studied, which are the lowest minimal models of $W$-invariant theories. The (1,3) integrable perturbation of CFT was naturally generalized to $W$-invariant theories as a perturbation by the field corresponding to the adjoint representation of the algebra $A_n$, and their FST were constructed in [5]. In [4] and [5] the vacuum structure of the theory was conjectured to be in correspondence to the admissibility diagram of some Interaction Round the Face (IRF) models, and in [5] the S-matrix of the model was explicitly constructed by solving of Yang Baxter equation by use of $A_n$ invariant IRF Boltzmann weights. The integrability of such a perturbation of an adjoint type for $WX$-invariant theories constructed on an arbitrary Lie group $X$ ( treated as the coset construction $X_{p+1}/X_{p,1}$, see, for example, [6] ) is known for a long time, and some examples of corresponding FST for these models ($B_n, C_n, D_n$ and
few others) were discussed, for example, in few recent works of Gepner [7]. On the classical level the trivial reason for integrability of the adjoint type perturbation is expressed by the fact of correspondence of $\phi_{adj}$ to the maximal positive root of the algebra such that together with screening operators of the $WX$-invariant theory they form $X$-invariant Affine Toda Field Theory (ATFT) with imaginary coupling constant [8]. In this sense the (1,2) perturbation of Virasoro minimal models may be called vector type perturbation (with respect to $sl(2)$).

The natural question arrives: are there generalizations of (1,2) type integrable perturbations of CFT to W-invariant theories? Recently the existence of such perturbations was pointed out [9]. It was conjectured that, $W D_n^{(p)} + \phi_{vect}$ for $n \geq 3$ is integrable, where $\phi_{vect}$ is the primary field corresponding to the fundamental weight of vector representation of $D_n$ (the field $211...1|11...1$ in the notation of ref.[6] ). The hint for the integrability of such perturbed model is actually seen even on the “classical” level, since the perturbing field together with screening operators of the $WD_n$ give rise to the $B_n$ imaginary coupled ATFT. It was conjectured that the FST of this integrable model should have $A_{2n-1,q}^{(2)}$ symmetry. For $n = 2$ it was found that the corresponding analog of this integrable vector perturbation is $WA_2^{(p)} + (21|11)$, and $D_4^{(3)}$ symmetry of corresponding FST was conjectured. In this case the Hamiltonian of perturbed model completed by screening operators of $WA_2$ (or $W_3$ in usual notations) form the $G_2$ imaginary coupled ATFT.

Checks of integrability of $W_3$ and $WD_3$ which were done in [9], are exact for irrational values of central charge, since using the counting argument they did not take into account the summation over the root lattice in the formula of W characters, ignoring the highest null vectors. The suggestion of the symmetry of the FST theory in this work was done analyzing the $p \to \infty$ and is in complete correspondence with the non simply-laced duality of real coupled ATFT observed recently in refs [10],[11].

In this work we study in more details the above described vector perturbations of $WD_n$ (and $W_3$) theories trying to show explicitly the presence of $A_{2n-1,q}^{(2)}$ symmetry with the help of nonlocal currents of the model, and then discuss our attempt to obtain the S-matrix on the base of the $A_{2n-1,q}^{(2)}$-symmetric R-matrix built in the work [12]. In section 2 we start from more rigorous check of integrability by counting argument with exact calculation of $W$-characters at rational values of central charge (minimal models). After that (section
3) nonlocal charges for n=3 case with the algebra $A_{\frac{3}{5},q}^{(2)}$ are constructed explicitly and the role of the gradation chosen for the R-matrix (as a starting point for the S-matrix construction), which commute with obtained comultiplication, is discussed. In the last section we summarize the results and briefly discuss one other integrable perturbation of $WD_n$ model.

2 Vector perturbations and their integrability

Before starting with the hamiltonian of the vector perturbation for $W_3$ minimal models in the free field representation, recall the standard notations of primary fields (ref 6) for $WX_n^{(p)}$ minimal model. Primary field $\Phi_{{(i\vec{p})}}$ is characterized by the set of integers $(l_1, ..., l_r|l'_1, ..., l'_r)$, where $r$ is the rank of $X_n$. It can be written as

$$\Phi_{{(i\vec{p})}} = :e^{i\vec{\beta}\vec{\phi}(z)} :$$

$$\vec{\beta} = \sum_{i=1}^{r}(\alpha_+(1-l_i) + \alpha_-(1-l'_i))\vec{\omega}_i$$

where $\alpha_+ = \sqrt{\frac{p-1}{p+1}}$, $\alpha_- = -\sqrt{\frac{p+1}{p}}$, and $\vec{\omega}_i$ are fundamental weights of the algebra $X_n$.

Screening fields are: $e^{i\alpha_+\tilde{a}_i\phi(z)}$, where $\tilde{a}_i, i = 1, ..., r$, are positive roots of $X_n$.

Consider Hamiltonian for the perturbation of $W_3$ conformal theory by the operator $\Phi_{{(21\vec{p})}}$ which has the dimension $\frac{1}{3}(1 - \frac{4}{p+1})$ in the $(p, p + 1)$ unitary minimal model. This Hamiltonian may be written as

$$H = \lambda \int d^2 z : (e^{i\alpha_+\tilde{a}_1\phi(z)} + e^{i\alpha_+\tilde{a}_2\phi(z)} + e^{-i\alpha_+\tilde{a}_1\phi(z)}) :$$

The set of vectors $(\tilde{a}_1, \tilde{a}_2, -\tilde{a}_1)$ expressed, for example, in standard orthonormal basis $(e_1 - e_2, e_2 - e_3, 2/3e_1 - 1/3e_2 - 1/3e_3)$, obviously forms the set of roots of $G_2$ affine algebra.

This fact allows us to consider the above perturbed CFT as a good candidate to integrable model, which is the $G_2$ ATFT with the imaginary coupling constant, and its conserved currents should have spins equal to the exponents of $G_2$ (3,5) modulo its Coxeter number (6). But we should be convinced that the integrability survives also on the quantum level.
The simplest way to try to check quantum integrability is to check the working of so called counting argument [1], which is sufficient but not necessary condition of integrability. It says that, comparing dimensions of the representation of the $W$ algebra based on the perturbing operator as primary field, modulo derivatives, with those of unity operator on different Virasoro levels, we can see the existence of a conserved current of higher spin, and hence the integrability, if it turns out that the dimension of the former is less then the dimension of the latter. The dimensions of Verma moduli can be extracted from the characters of highest weight representations of corresponding $W$-algebra. Using the character formula (see, for example, ref [6]) for the "completely degenerate" representations of $W$-algebra

$$\chi(\vec{\Omega}, \vec{\Omega}') = \left[q(1/24) \prod_{i=1}^{\infty} (1 - q^i)\right]^{-r} \times \sum_{\hat{s} \in w} \sum_{\hat{\lambda} \in \Gamma_{\alpha}} \det(\hat{s}) q^{[\hat{p}\hat{s}\vec{\Omega} - (p+1)\vec{\Omega} + p(p+1)\hat{\lambda}]^2/2p(p+1)}$$

where $(\vec{\Omega}, \vec{\Omega}') = (\vec{\omega}, \vec{l}, \vec{\omega}', \vec{l}')$ is the primary field in the notations of (1), the sums run over the elements $\hat{s}$ of the Weyl group $w$ and the root lattice $\Gamma_{\alpha}$ of the Lie algebra of the rank $r$, and $p$ is the number of minimal model, we found (with a help of Mathematika) that according to their Virasoro levels $n = 2,\ldots,10$ the dimensions of Verma moduli of the perturbing field (21|11) modulo total derivatives are $(1, 1, 2, 2, 3, 4, 6, 6, 10)$ for $p = 4$, $(1, 1, 3, 3, 6, 7, 13, 15, 25)$ - for $p = 5$, $(1, 1, 3, 3, 6, 7, 13, 15, 26)$ - for $p \geq 6$. The same dimensions calculated for the operator of unity ((11|11)-field) are $(1, 1, 1, 1, 3, 1, 4, 4, 6)$ - for $p = 4$, and $(1, 1, 1, 1, 4, 2, 7, 7, 12)$ - for $p \geq 5$. We see, comparing the dimensions of levels which correspond to spin equal to 5, that there is a conserved current of that spin, as it should be according to our observation on the $G_2$ ATFT structure of the perturbed theory (5 is one of the exponents of $G_2^{(1)}$). The explicit form of the conserved charge

$$P_6(z, \bar{z}) = a : T^3 : + b : (\partial T)^2 : + c : W^2 : + d : W \partial T :$$

was argued in [9], where $T$ and $W$ are energy momentum tensor and generator of $W$-symmetry, and $a, b, c, d$ are some constants.
In the same way if we perturb $WD^{(p)}_{n,p}$ minimal theory by the operator $\Phi_{(21\ldots 1|11\ldots 1)}$ with the conformal dimension

\[
\Delta = \frac{1 + (np - (n - 1)(p + 1))^2 + \sum_{k=2}^{n-2} k^2}{2p(p+1)} \quad (n \geq 4)
\]

\[
\Delta = \frac{1 + (p - 2)^2}{2p(p+1)} \quad (n = 3)
\]

it easily can be seen, that the set of screening vertex operators together with the perturbing one forms the potential of imaginary coupled $B_n$ ATFT. The check of counting argument in this case by use of the formula (3) gives the following sequences of dimensions of Verma moduli at different Virasoro levels (spins) $2, \ldots, 11$: 1) $n = 3$ $(1, 1, 2, 1, 5, 4, 11, 11, 22, 26)$ -for the unity operator and $(2, 1, 5, 5, 12, 14, 28, 36, 64, 85)$ -for the perturbing one; 2) $n = 4$ $(1, 0, 3, 0, 5, 2, 11, 7, 22, 19)$ -for the unity operator and $(1, 2, 3, 5, 9, 13, 22, 33, 52, 77)$ for the perturbing operator; 3) $n = 5$ $(1, 0, 2, 1, 4, 2, 9, 7, 18, 18)$ -for the unity and $(1, 1, 4, 3, 9, 10, 21, 26, 48, 63)$ -for the perturbing operator. So we see the existence of conserved charges of spin 3 for each $n$ and even a charge of spin 5 for $n = 5$ case, in full correspondence with the exponents of $B_n$ ATFT.

As we mentioned in the introduction, the conjectured symmetries of the factorized S-matrices for these integrable field theories are expressed by the algebras which are dual to the corresponding non simply laced ATFT (with an imaginary coupling constant), i.e. $D_4^{(3)}$ for $G_2$ and $A_{2n-1}^{(2)}$ -for $B_n$. The presence of these symmetries in the perturbed models were argued in [9] on the basis of the analysis of the conserved (local) charge algebras. In the next section we shall build explicitly nonlocal charges in vector perturbed $WD^{(p)}_3$ theory which form the $A_{5,q}^{(2)}$ algebra and shall discuss its representation by fundamental solitons.

3 Algebra of nonlocal charges and its representation

Quantum deformed symmetries play the central role in the investigation of quantum integrable systems and give a powerful tool for construction of factorized exact S-matrix for the system provided the R-matrix corresponding to this quantum group symmetry is known. The way to see some certain quantum symmetry of an integrable perturbed CFT
explicitly in terms of its free field representation may lay only in the construction of nonlocal currents expressed in terms of these free fields, since only braiding relations of the nonlocal currents may give rise to some quantum deformed algebra. The construction of nonlocal charges with sl(2)\(_q\) algebra of symmetry in Sine-Gordon theory obtained as a perturbations of minimal unitary conformal models was done in the works [13]. There it was shown how this construction may be generalized to any ATFT with other group of symmetry, and in the work [14] the results of Smirnov for the S-matrix of (1, 2) perturbations of minimal models were reproduced from the point of view of nonlocal charges. In this section we shall follow [14] and construct nonlocal charges for the integrable model WD\(_3^{(p)}\) which form the algebra A\(_{s,q}^{(2)}\) and show how the change of gradation for R-matrix of this group of symmetry (from the homogeneous to the spin gradation in which we shall observe our quantum symmetry) will fix the dependence of the effective coupling constant on the bare coupling \(\beta\).

So, we are going to consider perturbed WD\(_3^{(p)}\) theory with the Hamiltonian

\[
H = \frac{\lambda}{2\pi} \int d^2 z \left( e^{-i\beta(\Phi_1 - \Phi_2)} + e^{-i\beta(\Phi_2 - \Phi_3)} + e^{-i\beta(\Phi_2 + \Phi_3)} + e^{i\beta\Phi_1} \right)
\]

\[
= \frac{\lambda}{2\pi} \int d^2 z \Phi_{\text{pert}}(z, \bar{z})
\]

(6)

where \(\Phi_i(z, \bar{z}) = \phi_i(z) + \bar{\phi}_i(\bar{z}), i = 1, 2, 3\) are free fields of WD\(_3\), \(\lambda\) is the coupling constant of the perturbation, and \(\beta = \sqrt{p/(p+1)}\). We briefly recall the method for constructing of nonlocal conserved charges of a perturbed CFT ([1],[13]). If we assume the existence of some conserved chiral currents \(J(z), \bar{J}(\bar{z})\) for nonperturbed CFT then for the perturbed currents we have the following Zamolodchikov’s equations up to the first order

\[
\bar{\partial}J(z, \bar{z}) = \lambda \oint \frac{d\omega}{2\pi i} \Phi_{\text{pert}}(\omega, \bar{z})J(z)
\]

\[
\partial\bar{J}(z, \bar{z}) = \lambda \oint \frac{d\bar{\omega}}{2\pi i} \Phi_{\text{pert}}(z, \bar{\omega})\bar{J}(\bar{z})
\]

(7)

If the operator product expansions (OPE) in these contour integrations have the form

\[
\Phi_{\text{pert}}(\omega, \bar{z})J(z) = \frac{\bar{h}(\omega, \bar{z})}{(\omega - \bar{z})^2} + \frac{\partial\omega\bar{f}(\omega, \bar{z})}{(\omega - \bar{z})} + \text{regular terms}
\]

(8)
\[ \Phi_{\text{pert}}(z, \bar{\omega}) \bar{J}(\bar{z}) = \frac{h(z, \bar{\omega})}{(\bar{\omega} - \bar{z})^2} + \frac{\bar{\partial}_\omega f(z, \bar{\omega})}{(\bar{\omega} - \bar{z})} + \text{regular terms} \]

then the Zamolodchikov’s equations take the form

\[ \bar{\partial}J(z, \bar{z}) = \partial \bar{H}(z, \bar{z}) \quad (9) \]
\[ \partial \bar{J}(z, \bar{z}) = \bar{\partial}H(z, \bar{z}) \]

where

\[ \bar{H}(z, \bar{z}) = \lambda (\bar{h}(z, \bar{z}) + \bar{f}(z, \bar{z})) \]
\[ H(z, \bar{z}) = \lambda (h(z, \bar{z}) + f(z, \bar{z})) \]

which means the existence of the conserved charges

\[ Q = \int \frac{dz}{2\pi i} J + \int \frac{d\bar{z}}{2\pi i} \bar{H} \quad (10) \]
\[ \bar{Q} = \int \frac{dz}{2\pi i} H + \int \frac{d\bar{z}}{2\pi i} \bar{J} \]

For the case under consideration the “maximal” set of currents which satisfy eqs. is

\[ J_1(z) = e^{\frac{i}{\beta} (\phi_2(z) - \phi_3(z))}; \quad \bar{H}_1(z, \bar{z}) = \lambda \frac{\beta^2}{\beta^2 - 1} e^{i(\frac{1}{\beta} - \beta)(\phi_2(z) - \phi_3(z))} e^{-i\beta(\bar{\phi}_2(\bar{z}) - \bar{\phi}_3(\bar{z}))} \quad (11) \]
\[ J_2(z) = e^{\frac{i}{\beta} (\phi_1(z) - \phi_2(z))}; \quad \bar{H}_2(z, \bar{z}) = \lambda \frac{\beta^2}{\beta^2 - 1} e^{i(\frac{1}{\beta} - \beta)(\phi_1(z) - \phi_2(z))} e^{-i\beta(\bar{\phi}_1(\bar{z}) - \bar{\phi}_2(\bar{z}))} \]
\[ J_3(z) = e^{-i\frac{2}{\beta} \phi_1(z)}; \quad \bar{H}_3(z, \bar{z}) = \lambda \frac{\beta^2}{\beta^2 - 2} e^{-2i(\frac{1}{\beta} - \beta)\phi_1(z)} e^{i\beta \bar{\phi}_1(\bar{z})} \]
\[ J_0(z) = e^{\frac{i}{\beta} (\phi_2(z) + \phi_3(z))}; \quad \bar{H}_0(z, \bar{z}) = \lambda \frac{\beta^2}{\beta^2 - 1} e^{i(\frac{1}{\beta} - \beta)(\phi_2(z) + \phi_3(z))} e^{-i\beta(\bar{\phi}_2(\bar{z}) + \bar{\phi}_3(\bar{z}))} \]
\[ \bar{J}_1(z) = e^{-\bar{z}(\bar{\phi}_2(z) - \bar{\phi}_3(z))}; \quad H_1(z, \bar{z}) = \frac{\beta^2}{\beta^2 - 1} e^{-i(\frac{1}{2} - \beta)(\bar{\phi}_2(z) - \bar{\phi}_3(z))} e^{i\beta(\phi_2(z) - \phi_3(z))} \quad (12) \]

\[ \bar{J}_2(z) = e^{-\bar{z}(\bar{\phi}_1(z) - \bar{\phi}_2(z))}; \quad H_2(z, \bar{z}) = \frac{\beta^2}{\beta^2 - 1} e^{-i(\frac{1}{2} - \beta)(\bar{\phi}_1(z) - \bar{\phi}_2(z))} e^{i\beta(\phi_1(z) - \phi_2(z))} \]

\[ \bar{J}_3(z) = e^{i\bar{z}\phi_1(z)}; \quad H_3(z, \bar{z}) = \frac{\beta^2}{\beta^2 - 1} e^{2i(\frac{1}{2} - \frac{\beta}{2})\bar{\phi}_1(z)} e^{-i\beta\phi_1(z)} \]

\[ \bar{J}_0(z) = e^{-\bar{z}(\bar{\phi}_2(z) + \bar{\phi}_3(z))}; \quad H_0(z, \bar{z}) = \frac{\beta^2}{\beta^2 - 1} e^{-i(\frac{1}{2} - \beta)(\bar{\phi}_2(z) + \bar{\phi}_3(z))} e^{i\beta(\phi_2(z) + \phi_3(z))} \]

with the spins of the charges \((10)\)

\[ s_i = -s_i = \frac{1}{\beta^2} - 1; \quad i = 0, 1, 2 \]

\[ s_3 = -s_3 = \frac{2}{\beta^2} - 1; \]

We should recall now that the quasi-chiral components \(\phi_i, \bar{\phi}_i\) of the Toda free fields \(\Phi_i\) commute only in the absence of perturbation \(\lambda = 0\), but in general their vertex operators obey certain braiding relations (see ref [13]). Skipping details we will give the result which easily can be checked by use of these braiding relations: the definition of topological charges

\[ T_1 = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} dx \partial_x (\Phi_2 - \Phi_3) \]

\[ T_2 = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} dx \partial_x (\Phi_1 - \Phi_2) \]

\[ T_3 = -2\frac{\beta}{2\pi} \int_{-\infty}^{\infty} dx \partial_x \Phi_1 \]

\[ T_0 = -2T_2 - T_1 - T_3 \]

leads to the following algebra of charges:

\[ [T_i, Q_j] = A_{ij}Q_j \]

\[ [T_i, \bar{Q}_j] = -A_{ij}\bar{Q}_j \]

\[ Q_i\bar{Q}_j - q^{-A_{ij}}\bar{Q}_j Q_i = \delta_{ij}a_i(1 - q^{2T_i}) \]
where

$$A_{ij} = \begin{pmatrix}
2 & 0 & -1 & 0 \\
0 & 2 & -1 & 0 \\
-1 & -1 & 2 & -2 \\
0 & 0 & -2 & 4
\end{pmatrix}$$

(16)

differs just by diagonal normalization from Cartan matrix of the affine Lie algebra $A_5^{(2)}$,

$$q = e^{-\frac{i\pi}{\beta^2}}$$

(17)

is the deformation parameter of quantum group, and

$$a_i = \frac{\lambda}{2\pi i} \left( \frac{\beta^2}{\beta^2 - 1} \right)^2; \quad i = 0, 1, 2$$

(18)

$$a_3 = \frac{\lambda}{2\pi i} \left( \frac{\beta^2}{\beta^2 - 2} \right)^2$$

(19)

The standard substitution ([13],[14])

$$Q_i = c_i e^{s_i \theta} E_i q^{H_i}$$

$$\bar{Q}_i = c_i e^{s_i \theta} F_i \bar{q}^{H_i}$$

$$H_i = T_i$$

$$c_i^2 = a_i (q_i^{-2} - 1)$$

$$q_i = q^{A_{ii}/2}$$

which introduces the spectral parameter ($\theta$) dependence, transforms the algebra (15) into the quantum affine Lie algebra $A_{5q}^{(2)}$ with Chevalley basis $E_i, F_i, H_i \ (i = 0, 1, 2, 3)$. 
\[ [H_i, H_j] = 0 \quad (21) \]
\[ [H_i, E_j] = A_{ij} E_j \]
\[ [H_i, F_j] = -A_{ij} F_j \]
\[ [E_i, F_j] = \delta_{ij} q^H_i q^{-H_i} \]

The fundamental representation of this algebra can be chosen the same as for the 
\( A_5^{(2)} \) and the basis for Cartan subalgebra in this representation can be taken as 
\( H_1 = \text{diag}(1, 0, 0, 0, 0, -1); H_2 = \text{diag}(0, 1, 0, 0, -1, 0); H_3 = \text{diag}(0, 0, 1, -1, 0, 0); H_0 = -2H_2 - H_1 - H_3. \)

The next natural step is the construction of the sixtet of fundamental soliton fields for the model which will form the representation of the above written algebra of nonlocal currents. One of possible candidates can be chosen as \( \psi_{i \pm} = e^{\pm \frac{i}{2} \phi_i} \), where \( i = 1, 2, 3 \), or another choice : \( \bar{\psi}_{i \pm} = e^{\pm \frac{i}{2} \bar{\phi}_i} \). It is easily can be checked by standard technique of conformal field theory that any of these two sets of fields possesses the correct topological charges.

\[ [T_{0}, \psi_{1 \pm}] = 0, \quad [T_{0}, \psi_{2 \pm}] = \pm \psi_{2 \pm}, \quad [T_{0}, \psi_{3 \pm}] = \pm \psi_{3 \pm} \quad (22) \]
\[ [T_{1}, \psi_{1 \pm}] = 0, \quad [T_{1}, \psi_{2 \pm}] = \pm \psi_{2 \pm}, \quad [T_{1}, \psi_{3 \pm}] = \mp \psi_{3 \pm} \]
\[ [T_{2}, \psi_{1 \pm}] = \pm \psi_{1 \pm}, \quad [T_{2}, \psi_{2 \pm}] = \mp \psi_{2 \pm}, \quad [T_{2}, \psi_{3 \pm}] = 0 \]
\[ [T_{3}, \psi_{1 \pm}] = \mp 2\psi_{1 \pm}, \quad [T_{3}, \psi_{2 \pm}] = 0, \quad [T_{3}, \psi_{3 \pm}] = 0 \]

Clearly each of the set of fields \( \psi, \bar{\psi} \) suffers from the ill-defined action of the part of the charges \( Q_{i}, Q_{b} \) because of the branch cuts under the contour integrals in part of the operator product expansions \( Q\psi \), and hence does not form the correct representation of the full algebra \( A_5^{(2)} \). But actually we need for our purposes here just the fields which will permit us to define braiding relations between currents and fundamental soliton fields which is compatible with the comultiplication structure of the revealed group \( A_5^{(2)} \). Using relations \( (22) \) one can show by the technique of the braiding relations of the vertices for fields \( \phi \) \([13]\), that for the fields \( \psi \) defined above, the following braiding relations are valid.
\[ J_i(x) \psi_j(\pm y) = q^{\tau_{ij}(\pm y)} J_i(x) \]
\[ \bar{J}_i(x) \psi_j(\pm y) = q^{-\tau_{ij}(\pm y)} \bar{J}_i(x) \]

for \( x < y \), and commutation of \( J_i(x) \) and \( \psi_j(\pm y) \) for \( x > y \), where \( \tau_{ij}(\pm y) \) are the topological charges of fields \( \psi_j(\pm y) \) with respect to \( T_i \), and the latter can be read from (22). Such braiding relations induce comultiplication

\[
\Delta(Q_i) = Q_i \otimes 1 + q H_i \otimes Q_i \tag{24}
\]
\[
\Delta(\bar{Q}_i) = \bar{Q}_i \otimes 1 + q H_i \otimes \bar{Q}_i
\]
\[
\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i
\]

which acts on the tensor product of two soliton states.

The natural further step in the direction to the construction of FST of the model is to try to find such an S-matrix for the fundamental particles (as an operator which acts in the tensor product of two vector spaces of representations) which possesses the symmetry revealed above and expressed by the algebra of nonlocal charges. That means the S-matrix should commute with the comultiplication (24)

\[
[S, \Delta(H_i)] = [S, \Delta(Q_i)] = [S, \Delta(\bar{Q}_i)] = 0 \tag{25}
\]

Introducing the notation \( \hat{S} = PS \), where \( P \) is the permutation matrix, these equations can be rewritten as

\[
\left[ \hat{S}(\theta_1, \theta_2), \Delta(H_i) \right] = 0 \tag{26}
\]
\[
\hat{S}(\theta_1, \theta_2) \left( e_i \otimes q^{\frac{H_i}{2}} + q^{\frac{H_i}{2}} \otimes e_i \right) = \left( q^{\frac{H_i}{2}} \otimes e_i + e_i \otimes q^{\frac{H_i}{2}} \right) \hat{S}(\theta_1, \theta_2)
\]
\[
\hat{S}(\theta_1, \theta_2) \left( f_i \otimes q^{\frac{H_i}{2}} + q^{\frac{H_i}{2}} \otimes f_i \right) = \left( q^{\frac{H_i}{2}} \otimes f_i + f_i \otimes q^{\frac{H_i}{2}} \right) \hat{S}(\theta_1, \theta_2)
\]

where \( \theta_1, \theta_2 \)-rapidities of the incoming particles,

\[
e_i = x_i E_i, \quad f_i = x_i^{-1} F_i, \quad x_i = x_i(\theta_j) = e^{s_i \theta_j} \tag{27}
\]
and the dependence of $e_i$ and $f_i$ on the rapidity ($\theta_1$ or $\theta_2$) is defined by their positions in tensor product (they depend on $\theta_1$ on the first place, and on $\theta_2$ on the second).

Such system of equations like (26) was solved with respect to $S$ (without unitarity and crossing symmetry conditions) by Jimbo [12] almost for all affine algebras of symmetry of R-matrix $S$, but he used another Cartan basis and his results were obtained in other gradation. His Cartan basis $h_i$ is connected to our basis $H_i$ by transformation

$$h_1 = H_1 - H_2, \quad h_2 = H_2 - H_3, \quad h_3 = H_2 + H_3, \quad h_0 = -2H_1$$

(28)

with corresponding change in Chevalley generators $E'_i, F'_i$, such that eq. (26) and (27) remain valid in this new basis. R-matrix was obtained in so called homogeneous gradation, when the spectral parameter dependence $x$ is introduced as a multiplier of only one of Chevalley generators, which correspond to the highest weight of $A_5$ in its odd component of Dynkin diagramm automorphism decomposition - the root number $3$ in our case. So in this homogeneous gradation

$$e_{3}^{hom} = xE_3', \quad f_{3}^{hom} = x^{-1}F_3', \quad x = x(\theta_2 - \theta_1)$$

(29)

$$e_{i}^{hom} = E'_i, \quad f_{i}^{hom} = F'_i, \quad i = 0, 1, 2$$

and R-matrix $S$ is a function of $x$. If we want to make use of Jimbo’s result for the $A^{(2)}_5$ R-matrix in our S-matrix construction, we should change homogeneous gradation into spin, (which, as we saw, was naturally dictated by the nonlocal charges of the system) by "gauge" transformation of the Jimbo’s solution:

$$\tilde{R}(x, k) = \sigma_{21} R(x, k) \sigma_{12}^{-1}$$

(30)

where

$$\sigma_{12} = x_0(\theta_1)^{-\sum_{i=1}^{3} h_{i}^{\alpha_i}} \otimes x_0(\theta_2)^{-\sum_{i=1}^{3} h_{i}^{\alpha_i}}$$

(31)

Using obvious relations
we have the following system of equations which fixes $a_i$ in (31):

\[
\begin{align*}
  x_0 - \frac{a_1 - a_2 - a_3}{2} &= x_0 \\
  x_0 - \frac{a_1 + a_2}{2} &= x_0 \\
  x_0 - \frac{a_1 + a_3}{2} &= x_0 \\
  x_0 - a_1 x &= x_1
\end{align*}
\]

Solving first 3 equations we have $a_1 = 4$, $a_2 = a_3 = 3$, and the last equation gives us important relation

\[ x = x_1 x_0^4 \] (34)

Since the dependence on the coupling constant $\beta$ enters Jimbo’s R-matrix only through its dependence on $x$, the relation (34) gives us the effective coupling constant $\xi$ as function of $\beta$:

\[ x = e^{\frac{2\pi\theta}{\xi}}, \quad \frac{2\pi}{\xi} = \frac{6}{\beta^2} - 5 \] (35)

We could now take ”gauged” R-matrix solution in the spin gradation and, multiplying it by some scalar function of $x$ and $k$, imply unitarity and crossing symmetry conditions, trying to construct the S-matrix. Let us make few comments on this way of S-matrix construction. We checked that Jimbo’s R-matrix solution, being gauged from the homogeneous to the spin gradation with constants $a_i$ found above, becomes crossing invariant, since the multiplying factor in the crossing relation exactly cancels by the ”gauge” factor.

If we would like now to fit the deformation parameter $k$ with the coupling constant by use of the crossing transformation for the R-matrix solution ($x \rightarrow -\frac{x}{k^2}$ corresponds to $\theta \rightarrow i\pi - \theta$), then in the notations
\[ x = e^{-i\pi a}, \quad a = \frac{2i\theta}{\xi}, \quad k = e^{-i\pi b} \]  

we get

\[ b = -\frac{\pi}{3\xi} + \frac{2m + 1}{6} \]  

where \( m \) is an arbitrary integer number, and choice of it in principle is crucial for the S-matrix pole structure. In addition there is well known CDD ambiguity in the solution of unitarity and crossing symmetry conditions for the scalar factor mentioned above. Both of these two ambiguities (CDD one and the integer \( m \) choice) was reduced in the work \[3\] in the case of \( A_2^{(2)} \) S-matrix construction for (1, 2)-perturbed Virasoro minimal models by comparison of the S-matrix for lightest breather-breather scattering, found by bootstrap from the kink-kink S-matrix, with the known solution for the real coupled Toda S-matrix \[15\]. Such a pattern for comparison exists in our case as well – there is the S-matrix solution for the real coupled \( A_{2n-1}^{(2)} \) Toda model \[10\], and we can try to do the same, but we will discuss this problem elsewhere.

4 Discussion

We have shown by counting argument that (21...11...1)-perturbations of \( WD_n \) theories (and (21|11) - of \( W_3 \)) are integrable, and it was not surprising after we realized their \( B_n \) (and \( G_2 \)) imaginary coupled ATFT structure on the ”classical level”. The conjectured \( A_{2n-1}^{(2)} \) symmetry of their quantum S-matrix has been proved by explicit construction of nonlocal charges with this quantum group symmetry and it was shown that there are a set of fields, which play the role of fundamental solitons in the sense that their braiding relations with nonlocal charges give rise to the correct comultiplication for this quantum group.

We saw that the R-matrix of the model, as a commutant of the found coproduct structure, should be connected to the known \( A_{2n-1}^{(2)} \) R-matrix Jimbo’s solution by change of gradation. As we pointed out, in the spin gradation, dictated by the revealed current structure, the Jimbo’s R-matrix becomes crossing invariant, which gives a hope to expect
that it can serve as a good basis for the S-matrix construction. In addition, the consistency condition of the gradation change fixed explicitly the S-matrix effective coupling constant dependence.

It should be emphasised here, that the considered model was not a minimal W model, since we started from the free bosonic representation with central charge \( c = n \) adding to it some screening operators and perturbation. The presense of the ”curvature” term of Feigin-Fucks minimal model construction will change all the picture drastically, but as we know from the examples of Sine-Gordon and ZMS models, the perturbation of considered minimal W-models probably can be obtained as a quantum group reduction of the S-matrix discussed here. However it seems that the most natural and the simplest way of the S-matrix construction for the perturbed minimal models can be done ([18]) on the base of the RSOS models, which are partialy studied for our groups of symmetry ([16],[17]).

The examples of integrable perturbations of W-invariant theories analyzed in this paper probably don’t exhaust all of them, which reveal more rich structure then in Virasoro case. It can be seen by comparison of Dynkin diagrams of affine Lie algebras with those of non affine Lie algebras of other type \( X \). Each case, when the former one can be obtained from the latter \( X \) by adding to \( X \) some combination of its fundamental weights, might be considered as a candidate for integrability of the perturbation of \( WX \) by the field corresponding to this specific combination of weights of \( X \). As examples we can mention here \((3,1,...1)\) perturbation of \( WD_n \) theories, which gives the \( A_{2n-1}^{(2)} \) Affine Toda theory (this perturbation is irrelevant for the unitary minimal models, but is relevant in some nonunitary minimal models),and, probably more attractive, example of vector perturbation of \( WB_n \) theories, which seems to give the \( A(0,2n)^{(4)} \) supersymmetric ATFT (with broken supersymmetry). But each of these cases requires separate detailed investigation.

5 Acknowledgements

I thank S.Elitzur and I.Vaysburd for helpful discussions and C.Efthimiou for useful comments on his work [14].
References

[1] A.B.Zamolodchikov *Advanced Studies in Pure Mathematics* **19** (1989) 641

[2] V.A.Fateev, A.B.Zamolodchikov *Int. Journal of Mod. Phys.* **A5** (1990) 1025

[3] F.A.Smirnov *Int. Journal of Mod. Phys.* **A6** (1991) 1407

[4] V.A.Fateev *Int. Journal of Mod. Phys.* **A6** (1991) 2109

[5] H.J. de Vega, V.A.Fateev *Int. Journal of Mod. Phys.* **A6** (1991) 3221

[6] V.A.Fateev, S.L.Lukyanov *Soviet Scientific Reviews* **A Phys. 15** (1990) 1

[7] D.Gepner *Foundations of Rational Quantum Field Theory, 1*; Preprint CALT-68-1825, 1992 and *Spectra of RSOS Soliton Theories*; Preprint CALT-68-1926, 1993

[8] T.Eguchi, S.K.Yang *Phys.Let.* **B224** (1989) 373; *Phys.Let.* **B235** (1990) 282

[9] I.Vaysburd *Integrable perturbations of \( W_n \) and WZW models*; Preprint SISSA Ref.19/94/FM, 1994

[10] G.W.Delius, M.T.Grisaru, D.Zanon *Nucl.Phys.* **B382** (1992) 365

[11] E.Corrigan, P.E.Dorey, R.Sasaki *On a generalized bootstrap principle*; Preprint YITP/U-93-09, 1993

[12] M.Jimbo *Com.Math.Phys.* **102** (1986) 537

[13] D.Bernard, A.LeClair *Com.Math.Phys.* **142** (1991) 99; G.Felder, A.LeClair *Int. Journal of Mod. Phys.* **A7,Suppl 1A**, (1992) 239

[14] C.Eftthimion *Nucl.Phys.* **B398** (1993) 697

[15] A.E.Arienstein, V.A.Fateev, A.B.Zamolodchikov *Phys.Lett.* **B87** (1979) 389

[16] A.Kuniba *Nucl.Phys.* **B355** (1991) 801
[17] S.O. Warnaar  *Algebraic construction of higher rank dilute A models*;
    Preprint of Melbourne Univ., 1994

[18] A. Babichenko  *Fundamental S-matrix for vector perturbed WD_n minimal models*;
    (in preparation)