Summary: Multigrid methods are popular for solving linear systems derived from discretizing PDEs. Local Fourier analysis (LFA) is a technique for investigating and tuning multigrid methods. P-multigrid is popular for high-order or spectral finite element methods, especially on unstructured meshes. In this paper, we introduce LFAToolkit.jl, a new Julia package for LFA of high-order finite element methods. LFAToolkit.jl analyzes preconditioning techniques for arbitrary systems of second order PDEs and supports mixed finite element methods. Specifically, we develop LFA of p-multigrid with arbitrary second-order PDEs using high-order finite element discretizations and examine the performance of Jacobi and Chebyshev smoothing for two-grid schemes with aggressive p-coarsening. A natural extension of this LFA framework is the analysis of h-multigrid for finite element discretizations or finite difference discretizations that can be represented in the language of finite elements. With this extension, we can replicate previous work on the LFA of h-multigrid for arbitrary order discretizations using a convenient and extensible abstraction. Examples in one, two, and three dimensions are presented to validate our LFA of p-multigrid for the Laplacian and linear elasticity.

MSC:

65M55 Multigrid methods; domain decomposition for initial value and initial-boundary value problems involving PDEs
65N30 Finite element, Rayleigh-Ritz and Galerkin methods for boundary value problems involving PDEs
65Txx Numerical methods in Fourier analysis

Keywords: local Fourier analysis; p-multigrid; high-order; finite elements

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References:

[1] Adams, M., Brezina, M., Hu, J., and Tuminaro, R., Parallel multigrid smoothing: Polynomial versus gauss-Seidel, J. Comput. Phys., 188 (2003), pp. 593-610, doi:10.1016/S0021-9991(03)00194-3. · Zbl 1022.65030
[2] Balay, S., Abhyankar, S., Adams, M. F., Brown, J., Brune, P., Buschelman, K., Dalcin, L., Dener, A., Eijkhout, V., Gropp, W. D., Karpeyev, D., Kansalak, D., Knepley, M. G., May, D. A., McInnes, L. C., Mills, R. T., Munson, T., Rupp, K., Sanan, P., Smith, B. F., Zampini, S., Zhang, H., and Zhang, H., PETSc Users Manual, Technical report ANL-95/11 - Revision 3.15, Argonne National Laboratory, 2021, https://www.mcs.anl.gov/petsc.
[3] Brandt, A., Multi-level adaptive solutions to boundary-value problems, Math. Comp., 31 (1977), pp. 333-390. · Zbl 0373.65054
[4] Brandt, A., Guide to multigrid development, in Multigrid Methods, Springer, 1982, pp. 220-312. · Zbl 0505.65037
[5] Brannick, J., Hu, X., Rodrigo, C., and Zikatanov, L., Local Fourier analysis of multigrid methods with polynomial smoothers and aggressive coarsening, Numer. Math. Theory Methods Appl., 8 (2015), pp. 1-21, doi:10.4208/nmtma.2015.w01si. · Zbl 1340.65290
[6] Briggs, W. L., Henson, V. E., and McCormick, S. F., A Multigrid Tutorial, 2nd ed., SIAM, 2000, doi:10.1137/1.9780898719505. · Zbl 0958.65128
[7] Brown, J., Efficient nonlinear solvers for nodal high-order finite elements in 3D, J. Sci. Comput., 45 (2010), pp. 48-63, doi:10.1007/s10915-010-9396-8. · Zbl 1203.65245
[8] Brown, J., Abdelfattah, A., Barra, V., Beams, N., Camier, J. S., Dobrev, V., Dudouit, Y., Ghaffari, L., Kolev, T., Medina, D., Pazner, W., Ratnayaka, T., Thompson, J. L., and Tomov, S., libCEED: Fast algebra for high-order element-based discretizations, J. Open Source Softw., 6 (2021), 2945, doi:10.21105/joss.02945.
[9] Brown, J., Barra, V., Beams, N., Ghaffari, L., Knepley, M., Moses, W., Shakeri, R., Stengel, K., Thompson, J. L., and Zhang, J., Performance Portable Solid Mechanics via Matrix-Free (p-)Multigrid, preprint, https://arxiv.org/abs/2204.01722, 2022.
[10] Brown, J., He, Y., and MacLachlan, S., Local Fourier analysis of balancing domain decomposition by constraints algorithms, SIAM J. Sci. Comput., 41 (2019), pp. S346-S369, doi:10.1137/18M1191373.
[11] Brown, J., He, Y., MacLachlan, S., Menickelly, M., and Wild, S. M., Tuning multigrid methods with robust optimization and local Fourier analysis, SIAM J. Sci. Comput., 43 (2021), pp. A109-A138, doi:10.1137/19M1308669.

[12] Davydov, D., Pelteret, J.-P., Arndt, D., Kronbichler, M., and Steinmann, P., A matrix-free approach for finite-strain hyperelastic problems using geometric multigrid, Internat. J. Numer. Methods Engrg., 121 (2020), pp. 2874-2895, doi:10.1002/nme.6336.

[13] Demkowicz, L., Oden, J. T., Rachowicz, W., and Hardy, O., Toward a universal \(h\) adaptive finite element strategy, Part 1. Constrained approximation and data structure, Comput. Methods Appl. Mech. Engrg., 77 (1989), pp. 79-112, doi:10.1016/0045-7825(89)90129-1.

[14] Deville, M. O., Fischer, P. F., and Mund, E. H., High-Order Methods for Incompressible Fluid Flow, Cambridge University Press, 2002. - Zbl 07303441

[15] Fischer, P., Min, M., Rathnayake, T., Dutta, S., Kolev, T., Dobrev, V., Camier, J.-S., Kronbichler, M., Warburton, T., Swirydowicz, K., and Brown, J., Scalability of high-performance PDE solvers, Int. J. High Perform. Comput. Appl., 34 (2020), doi:10.1177/1094342020915762.

[16] Gutknecht, M. H. and Röllin, S., The Chebyshev iteration revisited, Parallel Comput., 28 (2002), pp. 263-283, doi:10.1016/S0167-8191(01)00139-9. - Zbl 0984.68209

[17] He, Y. and MacLachlan, S., Two-level Fourier analysis of multigrid for higher-order finite-element discretizations of the Laplacian, Numer. Linear Algebra Appl., 27 (2020), e2285, doi:10.1002/nla.2285.

[18] Hughes, T. J., The Finite Element Method: Linear Static and Dynamic Finite Element Analysis, Courier Corporation, 2012.

[19] Kahl, K. and Kintscher, N., Automated local Fourier analysis (aLFA), BIT, 60 (2020), pp. 651-686, doi:10.1007/s10543-019-00797-w.

[20] Knoll, D. A. and Keyes, D. E., Jacobian-free Newton-Krylov methods: A survey of approaches and applications, J. Comput. Phys., 193 (2004), pp. 357-397, doi:10.1016/j.jcp.2003.08.010. - Zbl 1036.65045

[21] Kronbichler, M. and Ljungkvist, K., Multigrid for matrix-free high-order finite element computations on graphics processors, ACM Trans. Comput. (TOPC), 6 (2019), pp. 1-32, doi:10.1145/3322813.

[22] Kumar, P., Rodrigo, C., Gaspar, F. J., and Oosterlee, C. W., On local Fourier analysis of multigrid methods for PDEs with jumping and random coefficients, SIAM J. Sci. Comput., 41 (2019), pp. A1385-A1413, doi:10.1137/18M1173769.

[23] MacLachlan, S. P. and Oosterlee, C. W., Local Fourier analysis for multigrid with overlapping smoothers applied to systems of PDEs, Numer. Linear Algebra Appl., 27 (2020), e2285, doi:10.1002/nla.2285.

[24] Oden, J. T., Demkowicz, L., Rachowicz, W., and Westermann, T., Toward a universal \(h\) adaptive finite element strategy, Part 2. a posteriori error estimation, Comput. Methods Appl. Mech. Engrg., 77 (1989), pp. 113-180, doi:10.1016/0045-7825(89)90130-8. - Zbl 0723.73075

[25] Rachowicz, W., Oden, J. T., and Demkowicz, L., Toward a universal \(h\) adaptive finite element strategy part 3. design of \(h\) meshes, Comput. Methods Appl. Mech. Engrg., 77 (1989), pp. 181-212, doi:10.1016/0045-7825(89)90131-X. - Zbl 0723.73076

[26] Rittich, H., Extending and Automating Fourier Analysis for Multigrid Methods, Ph.D. thesis, Universität Wuppertal, Fakultät für Mathematik und Naturwissenschaften, 2018.

[27] Rodrigo, C., Gaspar, F. J., and Zikatanov, L. T., On the validity of the local Fourier analysis of multigrid methods, J. Comput. Math., 37 (2019), pp. 349-348, doi:10.4208/jcm.1803-m2017-0294. - Zbl 1449.65374

[28] Rüsch, E. M. and Patera, A. T., Spectral element multigrid. I. Formulation and numerical results, J. Sci. Comput., 2 (1987), pp. 389-406, doi:10.1007/BF01066505

[29] Stüben, K. and Trottenberg, U., Multigrid methods: Fundamental algorithms, model problem analysis and applications, in Multigrid Methods, Springer, 1982, pp. 1-176. - Zbl 0562.65071

[30] Thompson, J. L., LFAToolkit, https://github.com/jeremylt/LFAToolkit.jl, 2021.

[31] Trottenberg, U., Oosterlee, C. W., and Schüller, A., Multigrid, Academic Press, 2001.

[32] van der Vegt, J. J. and Rhebergen, S., Discrete Fourier Analysis of Multigrid Algorithms, Memorandum, Department of Applied Mathematics, University of Twente.

[33] Wienands, R. and Joppich, W., Practical Fourier Analysis for Multigrid Methods, CRC Press, 2004. - Zbl 1062.65045

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