We present a free-form reconstruction of the primordial power spectrum using Planck 2018 CMB temperature and polarisation data. We extend the modified Richardson-Lucy (MRL) algorithm to include polarisation and apply it to the CamSpec unbinned $C_\ell$s. Combined with a new regularisation technique inspired by the diffusion equation, we obtain a form of primordial power spectrum with features that improve the fit to each of TT, TE, and EE data simultaneously. The resulting features are consistent with the previous findings from the temperature-only analyses. We evaluate the statistical significance of the features in our reconstructions using simulated $C_\ell$s and find the data to be consistent with having a featureless primordial power spectrum. The machinery developed here will be a complimentary tool in the search for features in the primordial power spectrum with upcoming CMB surveys.

I. INTRODUCTION

The Structure of the Universe is seeded by the perturbations in the early universe. Statistical properties of these primordial perturbations are well summarised by their power spectrum. The simplest models of inflation predict the primordial (scalar) power spectrum to be nearly scale-invariant only with a small red tilt, characterised by its amplitude $A_s$ and tilt $n_s$ (see e.g. [1–3] for reviews). CMB observations to date are consistent with having a power-law primordial spectrum [4]. Precise estimates have been placed on these two parameters from the latest CMB surveys [5].

There are numerically physically well-motivated models of inflation which predict deviations away from the power-law spectrum, however. Among them are models with oscillations in the primordial power spectrum caused by, e.g. sharp features in the inflationary potential [6] and resonance [7] (see [8, 9] for reviews). CMB observations to date are consistent with having a power-law primordial spectrum [10]. Precise estimates have been placed on these two parameters from the latest CMB surveys [5].

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These works presented in the literature for studying deviations from the power-law spectrum. The first considers minimally parametric extensions to the power-law spectrum. Scale variation of the tilt $n_s$ has been studied by introducing a running and/or running of the running [4, 18, 19]. Template-based approaches have been used to look for oscillations in the power spectrum [20]. Power spectrum amplitudes at some fixed nodes (or ‘knots’) have been added as additional parameters for Bayesian analysis, making use of linear interpolations [21, 22] or cubic smoothing splines [23–25] in between the nodes. The dependence on the number of nodes and their locations have been thoroughly studied in [4, 19, 26–29]. Inclusion of principal modes from a PCA analysis on the Fisher matrix was studied in [30].

There have been two broad classes of approaches in the literature for studying deviations from the power-law spectrum. The first considers minimally parametric extensions to the power-law spectrum. Scale variation of the tilt $n_s$ has been studied by introducing a running and/or running of the running [4, 18, 19]. Template-based approaches have been used to look for oscillations in the power spectrum [20]. Power spectrum amplitudes at some fixed nodes (or ‘knots’) have been added as additional parameters for Bayesian analysis, making use of linear interpolations [21, 22] or cubic smoothing splines [23–25] in between the nodes. The dependence on the number of nodes and their locations have been thoroughly studied in [4, 19, 26–29]. Inclusion of principal modes from a PCA analysis on the Fisher matrix was studied in [30].

The second group of works in the literature takes quite the opposite approach; they reconstruct the functional form of the primordial power spectrum, either directly or by introducing a large number of degrees of freedom in the process. A suitable regularisation process is often required to control overfitting. Some early works using wavenumber bins include [31–34]. The cosmic inversion method [35–37] solves for the primordial power spectrum directly using differential equations derived from the cosmological perturbation theory. Free-form reconstructions can be obtained by maximising the likelihood subject to some roughness penalties on the function val-
ues [38], their derivatives [39, 43] (Tikhonov regularisation), or their second derivatives [4, 18, 19, 44] (analogous to smoothing splines). Localised features have been studied using wavelet basis functions [45, 46]. A direct inversion using Singular Value Decomposition on the transfer kernel [47] and Modified Richardson-Lucy algorithm (MRL) [48–57] can be grouped into this category.

We adopt MRL in this work. MRL is inspired by the Richardson-Lucy deconvolution algorithm first developed for Image Analysis applications [58, 59]. A real-world image is often the true image convolved with the imaging system’s point spread function (PSF). RL algorithm enables an iterative reconstruction of the underlying image given the observed image and the PSF. This technique has been first brought into the cosmological context in [60, 61]. It was then used to reconstruct the primordial power spectrum from WMAP data in [48] and the algorithm evolved further through [53, 54, 56, 57]. Instead of the PSFs which blur images, we have the CMB transfer functions that evolve and project the primordial perturbations. The adapted algorithm, dubbed MRL, solves the inverse problem of reconstructing the primordial power spectrum $P(k)$ from the observed CMB $C_{\ell}$s given the transfer functions. After a suitable regularisation which reduces the noise fitting, we obtain a form of primordial power spectrum which may contain hints of features.

In this work, we reconstruct the form of the primordial power spectrum using the CMB temperature and polarisation data from Planck using the MRL algorithm, extending our previous analyses on WMAP [53, 54] and Planck temperature data [56]. We first develop the necessary formalism to incorporate the TT, TE and EE datasets together for MRL. As the polarisation spectra of CMB tend to have lower signal-to-noise than the temperature counterparts, it is easier for MRL to amplify noises and/or cosmic variances during its iteration process compared to previous temperature-only analyses. To tackle this problem, we introduce a new regularisation term in the MRL iteration which simulates the diffusion equation; a feature will diffuse away unless it is strongly favoured by the data.

This paper is organised as follows. Sec. II outlines the reconstruction algorithm used in this work. We formulate how we incorporate polarisation data and regularise the results. The method is applied to a two-dimensional image for demonstration purposes. Sec. III contains the main results of this work. We find a form of regularised primordial power spectrum which gives a better fit to each of TT, TE and EE data and test its statistical significance. Sec. IV summarises the main conclusions of this work.

II. METHODOLOGY

A. Modified Richardson-Lucy deconvolution algorithm

The CMB angular power spectrum can be computed from the primordial power spectrum $P(k)$ by

$$C^XY_{\ell} = 4\pi \int d(ln k) \Delta^X_{\ell}(k) \Delta^Y_{\ell}(k) P(k),$$

(1)

where $\Delta^X_{\ell}(k)$ is the CMB transfer function obtained from Bolzmann solvers such as CAMB [62] and CLASS [63]. Here $X = T$ or $E$, which correspond to the CMB temperature and $E$-mode polarisation, respectively. The dimensionless power spectrum is parameterised using two parameters in the Planck analysis so that $P(k) = A_s(k/k_{\text{pivot}})^{n_s-1}$, where $A_s$ and $n_s$ indicate the scalar amplitude and tilt, respectively [4, 5, 64]. The pivot scale is set to $k_{\text{pivot}} = 0.05\text{Mpc}^{-1}$.

We discretise the $k$ space using a (one-dimensional) grid provided by CAMB so that

$$C^XY_{\ell} = \sum_k G^XY_{\ell k} P_k.$$

(2)

The matrix $G^XY_{\ell k}$ depends on the cosmological parameters and captures relevant information about the evolution and projection of the CMB anisotropies. Note that the integration weights for the $k$ grid (bins) are also included in $G_{\ell k}$.

The relation (2) can be used to reconstruct $P(k)$ from the observed $C_{\ell}$. However, due to the oscillatory nature of the integrals appearing in (2) and hence in $G_{\ell k}$, a direct inversion is non-trivial. We adopt a well-studied method to tackle this problem: the Modified Richardson-Lucy (MRL) algorithm [48]. In MRL, $P(k)$ is reconstructed from the data $C^\text{obs}_{\ell}$ iteratively through

$$P_{k}^{(i+1)} = P_{k}^{(i)} + P_{k}^{(i)} \sum_{\ell} G^\text{obs}_{\ell k} - C^\text{obs}_{\ell} \tilde{G}_{\ell k},$$

(3)

$$\tanh[Q_{\ell}^{(i)} (C^\text{obs}_{\ell} - C^\text{obs}_{\ell})],$$

where

$$C_{\ell}^{(i)} \equiv \sum_k G_{\ell k} P^{(i)},$$

(4)

$$Q_{\ell}^{(i)} \equiv \sum_{\ell'} \text{Cov}^{-1}(\ell, \ell') \left( C^\text{obs}_{\ell'} - C^\text{obs}_{\ell} \right),$$

(5)

$$\tilde{G}_{\ell k} \equiv G_{\ell k} / \sum_{\ell'} G_{\ell' k}.$$  

(6)

At the $i$th iteration, $P^{(i)}(k)$ is updated so that $C_{\ell}^{(i)}$ approaches closer to the observed data $C^\text{obs}_{\ell}$. The tanh term appearing in (4) provides weights according to the error bars in the data: the points with more significant deviations away from the data are weighted more than
others. By using a full covariance matrix we also include non-diagonal correlation information.

MRL provides a solution to the inverse problem \( \mathcal{L} \) in a non-linear manner, not necessarily minimising a particular loss function but improving the fit to the data as the iteration passes. We stop at the iteration when a desired property (such as convergence) is satisfied. In principle, MRL can find a solution to any noisy \( C_\ell \)'s by overfitting, so the reconstruction result should be interpreted with care. We do not propose the reconstructed spectrum as a new model better than the power spectrum, since having more degrees of freedom can always improve the fit to the data. Instead, we use the reconstruction to look for features that may be hiding in the data, aided with some statistical analysis using simulated data.

\[ C_{\ell_{\text{tot}}}^{\ell} = \left( \begin{array}{c} C_{\ell_{\text{min}}}^{TT} \\ \vdots \\ C_{\ell_{\text{max}}}^{TT} \\ \gamma C_{\ell_{\text{min}}}^{TE} \\ \vdots \\ \gamma C_{\ell_{\text{max}}}^{TE} \\ \gamma^2 C_{\ell_{\text{min}}}^{EE} \\ \vdots \\ \gamma^2 C_{\ell_{\text{max}}}^{EE} \end{array} \right) \quad , \quad \gamma G_{\ell_{\text{min}}}^{TT} \quad , \quad \gamma G_{\ell_{\text{max}}}^{TT} \quad , \quad \gamma G_{\ell_{\text{min}}}^{TE} \quad , \quad \gamma G_{\ell_{\text{max}}}^{TE} \quad , \quad \gamma^2 G_{\ell_{\text{min}}}^{EE} \quad , \quad \gamma^2 G_{\ell_{\text{max}}}^{EE} \right) \]

Note that \( \ell_{\text{min}} \) and \( \ell_{\text{max}} \) are not the same across TT, TE and EE in practice, so the blocks have varying sizes. We approximate the covariance matrix of the full TT-TEE then as a block diagonal matrix. In terms of the new concatenated matrices we have

\[ C_{\ell_{\text{tot}}}^{\ell} = \sum_k G_{\ell_{\text{tot}}}^{\ell} P_k . \]

With these concatenated matrices, the MRL update for \( P_k \) in \( \mathcal{L} \) now includes contributions from each of TT, TE and EE, with the latter two weighted by factors of \( \gamma \) and \( \gamma^2 \). Furthermore, the covariance matrix in \( Q_L \) contains cross-covariances between the TT, TE, and EE (assumed to be cosmic variance only for our analysis). This ensures that the tanh weights can still be significant if, for example, the TT fit is good but the correlated TE and EE counterparts deviate a lot from the data.

The parameter \( \gamma \) in \( Q_L \) is a free parameter which determines how heavily weighted the polarisation data is compared to the temperature one. Note that scaling both \( C_\ell \) and \( G_\ell \)'s by some \( \ell \) dependent factor does alter the reconstruction result, since MRL is inherently non-linear. We keep \( \gamma < 1 \), motivated by the fact that the Planck TT data has higher signal-to-noise than the TE and EE counterparts overall. We also found that the noisy EE data at high \( \ell \)s (often being negative) can sometimes cause numerical instabilities when not weighted down, since the fractional updates in \( \mathcal{L} \) can struggle when \( C_\ell^{(\ell)} \) crosses zero.

C. Regularisation

The MRL algorithm allows us to find a free-form primordial power spectrum which improves the fit to the observed data. It is a data-driven technique that can provide insights to where the primordial features may be. Meanwhile, as the iterations pass, the algorithm starts introducing sharp, high-frequency oscillations that ‘fit the
noise’ in the data. Regularising the result is therefore crucial when searching for more physically viable forms of the primordial power spectrum.

In our previous works, the reconstructed primordial power spectrum obtained from the MRL algorithm was smoothed by convolving it with a Gaussian kernel in the log space. On a discretised $k$ grid, this reads

$$P_{k}^{\text{Smooth}} = \frac{\sum_{k'} P_{k'} \exp \left(-\left(\frac{\log k' - \log k}{\Delta}\right)^2\right)}{\sum_{k'} \exp \left(-\left(\frac{\log k' - \log k}{\Delta}\right)^2\right)},$$

for a tunable smoothing scale $\Delta$.

This method provides a smooth power spectrum with features which improve the fit to data and regularises the result to prevent overfitting [5]. The smoothing scale can also be varied across the $k$ range to study features with different oscillation frequencies. We note that the smoothing is performed after the MRL iterations.

Incorporating the CMB polarisation data can act as a form of regularisation by itself. Overfitting the CMB TT $C_\ell$s by introducing large features in the primordial spectrum often results in a worse fit to the TE and EE data. However, we found that the primordial power spectrum reconstructed using MRL or other methods still tends to sacrifice the goodness of fit to the polarisation data to improve the temperature counterpart.

In this work, we look for primordial power spectra which can improve the fit to each of the TT, TE and EE data. One of the main challenges for achieving this via MRL was that the algorithm prioritises fitting the TT data over the TE and EE which have smaller signal-to-noises overall. We found that smoothing the reconstructed spectrum using [6] allows us to counter this effect. However, this smoothing method requires a choice of kernel which is, to some extent, arbitrary. The smoothing is also applied post-iterations. This way of smoothing the reconstructed primordial spectrum would depend on the value of $\Delta$, the width of smoothing, and also treat the whole range of the spectrum in a uniform way while we may have different forms of features at different wavenumbers.

To address this issue and to further improve the regularisation process, we developed a novel approach by directly including a smoothing term at each MRL iteration. Inspired by the diffusion (heat) equation

$$\frac{\partial f}{\partial t} = \kappa \frac{\partial^2 f}{\partial x^2},$$

which causes the peaks and troughs of $f(t)$ to diffuse over time, we modify the MRL iteration step as follows.

$$P^{(i+1)}_{k} = P^{(i)}_{k} + P^{(i)}_{k} \sum_{\ell} \frac{C_{\ell}^{\text{obs}} - C_{\ell}^{(i)}}{C_{\ell}^{(i)}} G_{\ell k} \tanh \left[Q^{(i)}_{\ell} \left(C_{\ell}^{\text{obs}} - C_{\ell}^{(i)}\right)\right] + \kappa \sum_{k'} D_{kk'} P^{(i)}_{k'}.$$  

The matrix $D_{kk'}$ here is a discrete version of the differential operator $\partial^2 / \partial k^2$. At each iteration, the MRL term in (11) pushes $P(k)$ towards a direction which improves the overall fit to data, while the regularisation term led by $\kappa$ acts to ‘diffuse’ the sharp fluctuations present in the spectrum. Therefore, $\kappa$ here acts as a regularisation parameter; the larger the $\kappa$, the smoother the reconstruction. Note that we retrieve the usual MRL when $\kappa = 0$.

We note that, while we are the first to include this regularisation term within the MRL iterations, this method of regularisation itself is far from new. Diffusion-based techniques have been used in various fields as a mean to regularise ill-posed inverse problems. The term $\sum_{k'} D_{kk'} P_{k'}$ may also appear in gradient descent updates of an optimisation problem if the loss function contains a $L_2$ roughness penalty of the derivative $P'(k)$.  

For a visual understanding of the reconstruction methods studied in this work, we tested the algorithms on a two-dimensional image deconvolution task. The corresponding inverse problem is $g(i) = \sum_j p(i,j) f(j)$ (cf. 2), where the original image $f(i)$ is convolved with a kernel $p(i,j)$ to give the observed image $g(i)$. We reconstruct the original image $f(i)$ from observation $g(i)$ given the kernel $p(i,j)$ in Figure 4.

In the absence of noise, the Richardson-Lucy (RL) deconvolution algorithm accurately reconstructs the original image from a blurred one. For the noisy image, both RL and MRL work well to recover the sharp features in the image. There is a moderate amplification of noise during this process. Our novel regularised MRL (RegMRL) method suppresses this noise by having a...
III. RESULTS AND DISCUSSION

A. Validation using simulated data

First of all, we validate the reconstruction method on some mock data. We create several forms of primordial power spectra with added features, project them to obtain the corresponding power spectra with added features, project them to obtain some mock data. We create several forms of primordial power spectra and regularised MRL reconstructions were applied on the mock Cℓs to recover the primordial power spectrum. Figure 2 shows the result.

We find that MRL can indeed recover the features present in the primordial power spectrum as it has previously been shown. Note that, in previous works, the reconstruction results shown here were smoothed afterwards using a Gaussian smoothing kernel to reduce the noise. Here, we use the diffusive regularisation method outlined in Section II and obtain the results shown on the right-hand side of Figure 2. The regularisation enables us to retrieve the features more clearly.

We note that, for the bottom two plots of Figure 2, the regularised MRL reconstruction results do not perfectly trace the true underlying P(k)’s. The reason is twofold; first, the diffusive smoothing term tends to smooth out high-frequency oscillations which do not contribute significantly to fitting the data. Regularised MRL reconstructs the dip around k = 7 × 10⁻³, for example, to be smaller than the true value for this reason. Second, the inherent degeneracy of the inverse problem may direct the algorithm to converge on another form of P(k) which gives near identical Cℓs. The wavenumbers (k) and the multipoles (ℓ) have a one-to-one correspondence under the Limber approximation (ℓ ≈ kηrec), but in reality, each k contributes to multiple ℓs that are near or larger than kηrec. A lack of power at a scale can therefore be compensated by excess powers on nearby scales. The third row of Figure 2 shows one such result; both MRL and regularised MRL results have a small dip around k = 2 × 10⁻², compensated by small oscillations around the true P(k) for before and after. The corresponding CℓTTs are within the fractional error of O(10⁻³) to true values. Extra fluctuations around the true feature are also small enough to avoid suppression from diffusion.

B. Reconstruction of P(k) from Planck CMB

We now apply the method outlined in the previous section to the CamSpec data specified earlier in Section II. Figure 3 shows the reconstructed primordial power spectra from the CamSpec TT, TE, and EE data using MRL. The power spectrum was initially set to the best-fit power-law spectrum given by P(k) = A_s(k/k_{pivot})^{ns−1}, where the scalar amplitude A_s and the spectral index ns was taken from the CamSpec TTTEEE best-fit parameters [55]. We note that the reconstructed result is mostly insensitive to this initial choice. MRL reconstructions pick up more and more features as the iteration goes on, improving the fit to the data. The result is a highly oscillatory curve which ‘overfits’ the noise, as shown on the top plot of Figure 3. Regularising the reconstruction results is therefore essential to obtain a spectrum that can give us some physical insight into the features.

The middle plot of Figure 3 contains the MRL reconstructions that are smoothed afterwards using a Gaussian kernel (9) with ∆ = 0.03. By smoothing out the very high-frequency oscillations (in k space) which are less likely to be physical, we get a clearer view of the features of relevant scales. Note that most of the prominent features from MRL are still present after the smoothing process, but oscillations in the k range of (10⁻², 10⁻¹) Mpc⁻¹ have been greatly reduced in amplitude. The smoothing scale shown here is chosen as a reference; in practice, we may vary its value and choose based on the chi-squared value from the data or simulations.

The bottom plot of Figure 3 has the regularised MRL reconstructions obtained via iterations given in [11] with κ = 0.1. We observe that some of the oscillations present in the MRL reconstructions diffuse away. The reconstructed curve mostly converges in 100 iterations, after which the remaining features have sufficiently high improvements to the overall fit to counterbalance the regularisation term which smooths them out. Compared to

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4 Note that the regularisation term is proportional to the Laplacian \( \nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2) \).

5 This may not always be the case in some Image Analysis applications for which having clear edges and/or pleasant-looking image matters.

6 Here, \( \eta_{rec} \) denotes conformal distance to the last scattering surface.
FIG. 1. The deconvolution algorithms discussed in this paper are tested on a two-dimensional image: Penzias, Wilson and their horn antenna. The image is then blurred using a Gaussian kernel (b). The Richardson-Lucy (RL) algorithm (c) accurately recovers the original image from (b). In the presence of additional noise (d), both RL and Modified Richardson-Lucy (MRL) recover most of the features present in the original image ((e) and (f)), albeit with a moderate amplification of noise. Regularised MRL (RegMRL) reduces these noisy parts with minimal loss of image sharpness through the diffusive regularisation method elaborated in the text. Note that the convolution kernel and noise variance information are known to the deconvolution algorithms shown here.
FIG. 2. Recovery of various primordial power spectra from mock data using MRL (left) and regularised MRL (right). Based on 4 different forms of power spectra with artificial features shown above (black), we compute the $C_\ell$'s and draw 100 realisations from them using full covariance matrices (see text for details). The reconstructed $P(k)$'s for each of these realisations are drawn in thin coloured lines. MRL successfully retrieves the underlying features but has relatively large fluctuation due to amplification of noise. The regularisation introduced in this work significantly reduces the noise and allows us to recover the features more clearly. Note that previous studies on MRL [54] did not use the noisy power spectra shown on the left; they were smoothed post-reconstruction using a Gaussian kernel. The $\ell$ axis is computed using Limber approximation as a rough guide to the scales; $\ell \approx k\eta_{\rm rec}$. 
the Gaussian smoothing case, we see that the low-\(k\) features are suppressed further, whilst more of the features at higher \(k\) survive.

The regularisation parameter \(\kappa\) is a free variable which controls the smoothness of the reconstructed power spectra. When \(\kappa\) is too high, however, the numerical procedures involved in the finite element method (discretisation) can become unstable. A rough bound for stability is \(\kappa < 0.5\) (after a suitable normalisation). We experimented with different values of \(\kappa\) on \(C_{\ell}\)s simulated using the CamSpec covariance matrix. We chose \(\kappa = 0.1\) as it is the largest number for which the iterations are reliably stable. We note that one can freely tune the amount of regularisation applied by changing \(\kappa\).

The features reconstructed in the power spectra plotted in [3] can be categorized in three parts – suppression and dip at the largest scales (\(k < 10^{-3}\) Mpc), broad features at the intermediate scales (\(k < 10^{-3} - 10^{-2}\) Mpc) and a combination of oscillatory features at different frequencies at the small scales (\(k < 10^{-2} - 10^{-1}\) Mpc). The features at the largest and intermediate scales were present since WMAP with similar significance. The reconstructed spectra at different iterations and smoothing obtained from Planck data have been shown to provide a better fit to WMAP data as well, compared to power law [50].

Several inflationary models have been proposed that effectively fit these ‘outliers’. An initial period of moderate fast-roll [63], punctuated inflation [69] or a kinetic dominated initial condition [70, 72] generate a suppression in power at the largest scales. Transition in the scalar field potential [6] or presence of a step [7, 8, 73, 75] generates a dip and a bump which can fit the \(\ell \sim 22\) outlier. These types of models may also produce sinusoidal oscillations or sharp features in the power spectrum. Wiggly Whipped Inflation (WWI) combines these two types of features and finds several candidates as local best fits to WMAP data [76–79]. Axion monodromy models [9–12] generate logarithmically spaced oscillations that fit the Planck data [76–79]. These types of models may also produce sinusoidal oscillations or sharp features in the power spectrum.

Figure 4 shows the \(\chi^2\) values obtained for the reconstructed spectra at each iteration, obtained using MRL, MRL with Gaussian kernel smoothing, and regularised MRL. The four plots correspond to the improvements in \(\chi^2\) with respect to the power-law \(P(k)\) for the combined, TT, TE, and EE datasets. We observe that all three methods improve the fit to each of the TT, TE, and EE \(C_{\ell}\)s during early iterations. As the iterations go on, however, they start to sacrifice the fit to the polarisation (TE and EE) \(C_{\ell}\)s in favour of the temperature (TT) counterpart. MRL reconstructions after 40 iterations have a worse fit to the polarisation data than the power-law spectrum. The algorithm then keeps on over-fitting the noise in the TT data in order to improve the total \(\chi^2\) further.

Smoothing MRL using a Gaussian kernel removes many rapid oscillations present in the spectrum obtained from MRL (Figure 3) that are less likely physical. However, the smoothing also increases \(\chi^2\). Part of this is because the smoothing is applied to soften the features present regardless of how much they affect \(\chi^2\). Regularised MRL addresses this issue by having the regularisation term within each iteration; only the features with significant improvements in the fit can remain. As can be seen from Figure 4, regularised MRL gives better \(\Delta \chi^2\) overall compared to smoothed MRL. The fit to EE data, in particular, benefits a lot from this new regularisation technique.

We note that there are forms of primordial power spectrum which improve the fit to each of TT, TE, and EE \(C_{\ell}\)s whilst remaining relatively smooth. For example, amongst the \(P(k)\)s reconstructed from regularised MRL, the fit to EE data is optimised at iteration 32 where \(\Delta \chi^2_{TT} = -2.60 \times 10^{-2}, \Delta \chi^2_{TE} = -8.74 \times 10^{-4}\), and \(\Delta \chi^2_{EE} = -2.33 \times 10^{-3}\).

In Figure 5, we show \(C_{\ell}\)s computed from the MRL reconstructions at different iterations for TT power spectrum, together with its residuals. Figures 7 and 8 are the plots for TE and EE power spectra, respectively, and are included in Appendix A. Note that \(D_{\ell}^{XY} = (\ell + 1)C_{\ell}^{XY} / 2\pi\). The unbinned \(C_{\ell}\) data points are shown in grey, while the binned \(C_{\ell}\)s with \(\Delta \ell = 50\) are plotted in red together with their error bars.

For \(\ell < 10\), we find that the MRL reconstructions tend to fit the low observed \(C_{\ell}^{TT}\)s and yield values lower than the \(\Lambda\)CDM best fit. In particular, the low quadrupole (\(\ell = 2\)) in the data drives the low-\(k\) primordial power spectrum values towards zero, as can be seen in Figure 3. Due to large cosmic variances in this multipole range, however, the statistical significance of these outliers needs to be assessed with care. Note that if the value of \(C_{\ell}\) is set to be larger than the best fit value, then the corresponding MRL reconstructions would no longer be suppressed at low-\(k\) as seen in Figure 3.

The dip around \(\ell = 22\), in particular, has been found in the WMAP first-year data release and is also present in Planck [87]. Planck has estimated the tension between
FIG. 3. Primordial power spectrum reconstructed from CamSpec TTTEEE data using MRL (top), MRL smoothed using a Gaussian kernel with $\Delta = 0.03$ (middle), and the regularised MRL with $\kappa = 0.1$ (bottom). We observe some common features within all three spectra but their respective amplitudes vary due to the differences in regularisation methods. Note the log scaling in both axes; the near-scale-invariant power spectrum appears as a straight line (black dashed). The $\ell$ axis is computed using Limber approximation as before.
the CMB power spectrum in $\ell \lesssim 40$ and best-fit $\Lambda$CDM model to be $2.5\sigma$-$3.0\sigma$ \[87\]. The reconstructed spectra show a dip around $k \sim 2 \times 10^{-3}/\text{Mpc}$. If a new physics is responsible for the outlier, it should also be present in the EE spectrum. However, given the low signal-to-noise ratio at the largest scale E-mode observation in Planck we are yet to verify this feature.

For $\ell \geq 30$, the MRL reconstructions show an increasing amount of fluctuation over iterations. These added fluctuations lie within $1\sigma$ of the binned $C_{\ell}$ data points. The largest deviations from the best fit $C_{\ell}s$ are located in the range $30 \leq \ell \leq 500$ and near $\ell = 750$. These roughly correspond to $2 \times 10^{-3} \leq k \leq 5 \times 10^{-2}/\text{Mpc}$ and $6 \times 10^{-2}/\text{Mpc}$ in $P(k)$ where large features are visible in the reconstructions (Figure 3). These deviations are also visible in the TE and EE spectra.

### C. Testing statistical significance

We performed a simulation-based analysis in order to assess the statistical significance of the features present in the reconstructed spectra. The regularised MRL reconstruction method was applied to 1000 sets of simulated $C_{\ell}^{TT}$, $C_{\ell}^{TE}$ and $C_{\ell}^{EE}$s, generated based on the power-law spectrum. Here, $C_{\ell}s$ were drawn similarly to the ones used for Figure 2. $C_{\ell}s$ under $\ell < 30$ were drawn from a chi-squared distribution with $2\ell + 1$ degrees of freedom, while $\ell > 30$ ones were drawn from a multivariate normal distribution using the full CamSpec covariance matrix. Cross-covariances between datasets (TT, TE and EE) were set to the cosmic variance values. Figure 6 contains the results.

We test the null hypothesis $H_0$: ‘data is consistent with a power-law $P(k)$’ against the alternative hypothesis $H_1$: ‘features in $P(k)$ are required to explain the CMB data’. We form our $p$-value statistic at each $k$ by comparing the features in $P(k)$ against the distribution of $P(k)$ values obtained from the power-law simulations. The 68% and 95% confidence intervals (CIs), approximately equal to the $1\sigma$ and $2\sigma$ bounds, are evaluated from the 1000 simulations. As can be seen in Figure 6 we do not find any features to lie outside the 95% CIs and therefore do not reject the null hypothesis; the data is consistent with having no features in $P(k)$.

Note that we treated each $k$ independently in this analysis without adjusting for the ‘look-elsewhere effect’ caused by the large number of $k$ points involved. This is in general more favourable for detecting features since only one of many $p$-values (for each $k$) needs to be significant. An adjusted statistic is required to incorporate the number of effectively independent tests that are being made and the probability of having at least one large $p$-value. In this analysis, however, this is not as crucial because we do not detect any statistically significant result even without adjusting for the look-elsewhere effect.

FIG. 4. Improvements in the chi-squared fit to CamSpec data for MRL, smoothed MRL and regularised MRL (‘regMRL’) reconstructions discussed in the main text. The four plots show normalised chi-squared improvements to the combined, TT, TE, and EE data. All three methods improve the fit to individual spectrum during early iterations, but later they sacrifice the fit to TE and EE $C_{\ell}s$ in favour of the TT counterpart. MRL (black) without smoothing or regularisation gives the best improvement to the overall fit, albeit with many rapid oscillations in the spectrum (Figure 3) which are less likely to be physical. Smoothing MRL using a Gaussian kernel (green dotted) removes these oscillations whilst worsening the fit to data. Regularised MRL (blue dashed) has a similar level of regularisation but achieves a better fit to the data.
IV. CONCLUSION

In this work, we produced free-form reconstructions of the primordial power spectrum $P(k)$ based on the CMB temperature and polarisation data from Planck. The MRL algorithm has been used to iteratively reconstruct $P(k)$ from the unbinned CamSpec $C_\ell$ data and the full covariance matrix. We introduced a new approach inspired by the diffusion equation to regularise this reconstruction process.

We focused on the forms of $P(k)$ that can improve the fit to the individual $C_\ell^{TT}$, $C_\ell^{TE}$ and $C_\ell^{EE}$ simultaneously, and not simply boosting the fit to one at the cost of worsening another. A form of $P(k)$ which achieves this was found, yielding normalised chi-squared improvements of $\Delta \tilde{\chi}_TT^2 = -2.60 \times 10^{-2}$, $\Delta \tilde{\chi}_TE^2 = -8.74 \times 10^{-4}$, and $\Delta \tilde{\chi}_EE^2 = -2.33 \times 10^{-3}$. It is also worth noting that the reconstructed result retains many of the features found in the previous temperature-only analyses.

We studied the statistical significance of the features found in the reconstructed $P(k)$ by comparing them with those from 1000 mock $C_\ell$s drawn from a power-law $P(k)$. Overall, we did not reject the null hypothesis; the data is consistent with a featureless power-law $P(k)$.  

FIG. 5. Temperature angular power spectra for the reconstructed primordial power spectra from regularised MRL with CamSpec unbinned data taken from [65]. Transfer functions for the projection are computed using CAMB [62] using the best-fit parameters to CamSpec TTTEEE. Shown below is the residual with respect to the $\Lambda$CDM best-fit $C_\ell$s. Note the transition from log scale to linear at $\ell = 30$ in the plot.
FIG. 6. Primordial power spectrum reconstructed from CamSpec TTTEEE data using regularised MRL after 50 iterations, compared with the best-fit power-law spectra. We apply the same algorithm to reconstruct $P(k)$ from 1000 sets of simulated $C_\ell$s based on the power-law spectrum and provide the 68% and 95% confidence intervals at each $k$. Below, we plot the reconstructed spectrum’s deviations from the median value obtained from simulated values, divided by the standard deviation $\sigma(k)$ at $k$, also obtained from simulations. The $\ell$ axis is computed using Limber approximation as a rough guide to the scales. The increased values of $\sigma(k)$ at high $k$s are due to the lack of polarisation data at $\ell > 2000$ and temperature at $\ell > 2500$. Overall, we do not find any statistically significant deviations from the power-law spectrum.

The free-form reconstruction of $P(k)$ studied in this work serves as a useful tool for the search for features in the primordial power spectrum. The iterative approach developed in this work is complementary to other means such as minimally parametric approaches and direct template fitting, and it hints us at where the potential features in $P(k)$ may be. Furthermore, the methodology was extended to be able to find primordial power spectra with features which improve the fit to TT, TE and EE simultaneously. This will benefit further from the forthcoming CMB surveys with increased polarisation sensitivities. We plan on extending our analysis to other datasets and to future CMB experiments.

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Appendix A: Reconstructed TE and EE angular power spectra from regularised MRL
FIG. 7. TE angular power spectra obtained for the reconstruction results specified in Figure 5.

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FIG. 8. EE angular power spectra obtained for the reconstruction results specified in Figure 5.

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