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Buckling and Pre Stressed Vibration Analysis of Laminated Plates Using New Shear Deformation

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Abstract. Buckling stresses and natural frequencies of cross-ply laminated composite plates are analyzed using new higher order shear deformation theory. The new displacement field proposed by J.L. Mantari et al depends on a parameter ‘‘m’’, whose value is determined so as to give results closest to the three dimensions buckling solutions. A set of fundamental dynamic equations of a two-dimensional higher-order theory for rectangular laminated plates made of elastic and orthotropic materials is derived through Hamilton’s principle, these equations of motion are applied to solve the vibration and pre stressed free vibration problems of laminated composite plates after buckling solution. These equations are solved using Navier-type for simply supported boundary conditions. It was observed that this theory gives results close to three dimensions elasticity solutions and those obtained by existing shear deformation plate theories.

1. Introduction
Composites laminated plates are very important structural element and have wide engineering applications so that many researchers have searched for static and dynamic behavior of laminated plates using various methods such as experimental, approximate and exact methods, which are 3-D and 2-D elasticity plate solutions [1]. Used new two hyperbolic displacement field models HPSDT\textsuperscript{1} and HPSDT\textsuperscript{2} to investigate the natural frequency and buckling load for orthotropic simply supported cross ply plate, these two models give accurate results for thick plate compared with other researches, while [2] developed a new HS\textsuperscript{1}DT of laminated plate from 3-D elasticity bending solutions by using an inverse method, to study the static, buckling and free vibration of beam and plate, he obtained results close to 3-D elasticity solutions more than other shear deformation theories, a unified formalization derived by [3] to study the buckling of HS\textsuperscript{1}DT isotropic and orthotropic plate he found that HS\textsuperscript{1}DT models can be classified as elastic gradient of Mindline’s plate model by adding shear strain gradient. Other researchers used finite element method to predict buckling of laminated plate, a new quadrilateral flat element developed [4], to study the free vibration and buckling behavior of laminated plate, this element give numerical solution even with badly shaped element because it calculate membrane, bending and geometric stiffness matrices by integrating along the boundaries of this element, buckling of laminated plate using FE [5], investigated model with an efficient C\textsuperscript{0} based on higher order zigzag theory. The C\textsuperscript{0} continuity is compensated in the stiffness matrix calculations by using penalty parameter approach. A new finite element formulation [6] developed to study static, free vibration and buckling analyses of laminated composite plates this formulation used combination of node based smoothing discrete shear gap method with the higher-order shear deformation plate theory, it is very simple and efficient when programmed and it is used for linear assumption only. Finite
element analysis [7] used non uniform B spline function and third shear deformation theory as basic to study the static, dynamic and buckling response of laminated plate.

2. Displacement Field

In the present work, a new higher order displacement field in which the displacement of the middle surface expanded as a combination of exponential trigonometric function of the thickness coordinate and the transverse displacement taken to be constant through the thickness is developed. The displacement field of the new higher order theory of laminated composite plate is:

\[
\begin{align*}
\mathbf{u}(x, y, z) &= u(x, y) - z \left( \frac{\partial u}{\partial x} + f(z) \theta_1(x, y) \right) \\
\mathbf{v}(x, y, z) &= v(x, y) - z \left( \frac{\partial v}{\partial y} + f(z) \theta_2(x, y) \right) \\
\mathbf{w}(x, y, z) &= w(x, y)
\end{align*}
\]

Where, \( u(x, y), v(x, y), w(x, y), \theta_1(x, y), \theta_2(x, y) \) are the five unknown functions of middle surface of the plate. While \( f(z) \) represents shape functions determining the distribution of the transverse shear strains and stresses along the thickness.

With the same procedure developed by [8] for free boundary conditions at the top and bottom surfaces of the plate, the new proposed displacement field is:

\[
\begin{align*}
\mathbf{u}(x, y, z) &= u(x, y) + z \left( \frac{\partial \theta_1}{\partial x} - \frac{\partial w}{\partial x} + \sin \frac{\pi z}{h} e^{m \cos \left( \frac{\pi z}{h} \right)} \theta_1 \right) \\
\mathbf{v}(x, y, z) &= v(x, y) + z \left( \frac{\partial \theta_2}{\partial y} - \frac{\partial w}{\partial y} + \sin \frac{\pi z}{h} e^{m \cos \left( \frac{\pi z}{h} \right)} \theta_2 \right) \\
\mathbf{w}(x, y, z) &= w(x, y) - m_0
\end{align*}
\]

Where: \( f(z) = \sin \frac{\pi z}{h} e^{m \cos \left( \frac{\pi z}{h} \right)} + yz, y = \frac{\pi m}{h}, m = \text{constant} \) (7)

For small strains, the strain-displacement relations take the form:

\[
\begin{align*}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{\partial w}{\partial x} = \frac{1}{2} \gamma_{xx} \varepsilon_{yy} &= \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} = \frac{1}{2} \gamma_{yy} \varepsilon_{zz} &= \frac{\partial w}{\partial z} = \frac{1}{2} \gamma_{zz} \varepsilon_{xy} &= \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \right) = \frac{1}{2} \gamma_{xy} \varepsilon_{yx} &= \frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y} \right) = \frac{1}{2} \gamma_{yx}
\end{align*}
\]

(8-13)

The strain associated with the displacement field by substituting equations (4,5,6) into equations (8-13) to give:

\[
\begin{align*}
\varepsilon_{xx} &= \varepsilon_{xx}^0 + \varepsilon_{xx}^1 + \sin \frac{\pi z}{h} e^{m \cos \left( \frac{\pi z}{h} \right)} \varepsilon_{xx}^2 \\
\varepsilon_{yy} &= \varepsilon_{yy}^0 + \varepsilon_{yy}^1 + \sin \frac{\pi z}{h} e^{m \cos \left( \frac{\pi z}{h} \right)} \varepsilon_{yy}^2 \\
\gamma_{xy} &= \gamma_{xy}^0 + \gamma_{xy}^1 + \sin \frac{\pi z}{h} e^{m \cos \left( \frac{\pi z}{h} \right)} \gamma_{xy}^2 \\
\gamma_{yx} &= \gamma_{yx}^0 + \gamma_{yx}^1 + \sin \frac{\pi z}{h} e^{m \cos \left( \frac{\pi z}{h} \right)} \gamma_{yx}^2
\end{align*}
\]

(14-17)

Where:

\[
\begin{align*}
\varepsilon_{xx} &= \left\{ \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial w}{\partial x} \right\} = \left\{ \varepsilon_{1x}, \varepsilon_{1y}, \varepsilon_{2x} \right\} = \left\{ \frac{m \partial \theta_1}{h}, \frac{m \partial \theta_2}{h}, \frac{\partial^2 w}{\partial x^2} \right\} \\
\varepsilon_{yy} &= \left\{ \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial y} \right\} = \left\{ \varepsilon_{1x}, \varepsilon_{1y}, \varepsilon_{2y} \right\} = \left\{ \frac{m \partial \theta_1}{h}, \frac{m \partial \theta_2}{h}, \frac{\partial^2 w}{\partial y^2} \right\} \\
\gamma_{xy} &= \left\{ \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} - 2 \frac{\partial^2 w}{\partial x \partial y} \right\} = \left\{ \varepsilon_{1x}, \varepsilon_{1y}, \partial^2 \right\} = \left\{ \frac{\partial \theta_1}{\partial x}, \frac{\partial \theta_2}{\partial x} \right\}
\end{align*}
\]

(19-23)
3. Hamilton’s Principles

The equation of motion of the new higher order theory will be derived using the dynamic version of the principle of virtual displacements [9].

\[
0 = \int_0^1 \delta \mathbf{U} + \delta \mathbf{V} - \delta \mathbf{K} \, d\tau
\]  

The virtual work done by applied forces \( \delta \mathbf{V} \) is:

\[
\delta \mathbf{U} = \int \left[ \sum_{n=1}^{N-1} \int \sigma_i \left( 1, z, \sin \frac{\pi x}{h} \cos \left( \frac{\pi x}{h} \right) \right) \right] d\tau \]  

The virtual strain energy \( \delta \mathbf{V} \) is:

\[
\delta \mathbf{V} = \int_0^1 \left[ \sum_{n=1}^{N-1} \int \sigma_i \left( 1, z, \sin \frac{\pi x}{h} \cos \left( \frac{\pi x}{h} \right) \right) \right] d\tau \]  

The virtual energy of the plate is:

\[
The virtual work done by applied forces \( \delta \mathbf{V} \) is:

\[
\delta \mathbf{V} = -\int \mathbf{N} \cdot \frac{\partial^2 \mathbf{V}}{\partial x^2} \, d\tau \]  

4. Equations of Motion

The Euler-Lagrange is obtained by substituting equations (27 – 28) into equation (24) and then setting the coefficient of \( \frac{\partial^2 \mathbf{V}}{\partial x^2} \) over \( \Omega_0 \) of equation (24) to zero separately, this give five equations of motion as follows:

\[
\delta \mathbf{u}: \frac{\partial \mathbf{N}_1}{\partial x} + \frac{\partial \mathbf{N}_6}{\partial y} = 0 \]  

\[
\delta \mathbf{v}: \frac{\partial \mathbf{N}_2}{\partial x} + \frac{\partial \mathbf{N}_6}{\partial y} = 0 \]  

\[
\delta \mathbf{w}: \frac{\partial \mathbf{M}_1}{\partial x} + \frac{\partial \mathbf{M}_2}{\partial y} + 2 \frac{\partial \mathbf{M}_6}{\partial x} \frac{\partial \mathbf{N}_6}{\partial y} = 0 \]  

\[
\delta \mathbf{\Theta}_1: \frac{\partial \mathbf{Q}_1}{\partial x} + \frac{\partial \mathbf{Q}_2}{\partial y} + \frac{\partial \mathbf{Q}_3}{\partial x} + \frac{\partial \mathbf{Q}_4}{\partial y} - \mathbf{M}_1 = 0 \]  

\[
\delta \mathbf{\Theta}_2: \frac{\partial \mathbf{Q}_1}{\partial x} + \frac{\partial \mathbf{Q}_2}{\partial y} + \frac{\partial \mathbf{Q}_3}{\partial x} + \frac{\partial \mathbf{Q}_4}{\partial y} - \mathbf{M}_2 = 0 \]  

The result forces are given by:

\[
\begin{align*}
\mathbf{N}_1 &= \sum_{k=1}^{N} \int_{x_k}^{x_{k+1}} \sigma_1 \, dx \\
\mathbf{N}_2 &= \sum_{k=1}^{N} \int_{x_k}^{x_{k+1}} \sigma_2 \, dx \\
\mathbf{N}_6 &= \sum_{k=1}^{N} \int_{x_k}^{x_{k+1}} \sigma_6 \, dx \\
\mathbf{M}_1 &= \sum_{k=1}^{N} \int_{x_k}^{x_{k+1}} \sigma_1 f(x) \, dx \\
\mathbf{M}_2 &= \sum_{k=1}^{N} \int_{x_k}^{x_{k+1}} \sigma_2 f(x) \, dx \\
\mathbf{P}_1 &= \sum_{k=1}^{N} \int_{x_k}^{x_{k+1}} \sigma_1 f(x) \, dx \\
\mathbf{P}_2 &= \sum_{k=1}^{N} \int_{x_k}^{x_{k+1}} \sigma_2 f(x) \, dx \\
\mathbf{Q}_1 &= \sum_{k=1}^{N} \int_{x_k}^{x_{k+1}} \sigma_1 f(x) \, dx \\
\mathbf{Q}_2 &= \sum_{k=1}^{N} \int_{x_k}^{x_{k+1}} \sigma_2 f(x) \, dx
\end{align*}
\]  

(34-36)
The plane stress reduced stiffness $Q_{ij}$ is:

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, \quad Q_{11} = \frac{E_2}{1-\nu_{12}\nu_{21}}, \quad Q_{66} = G_{12}, \quad Q_{44} = G_{23}, \quad Q_{55} = G_{13}$$

(37)

Where: $G_{12}$, $G_{23}$ and $G_{13}$= shear modulus of plate in planes 12, 23 and 13 respectively. While $E_1$ and $E_2$= Young's modulus in directions 1 and two of plate. Also $\nu_{12}$ and $\nu_{21}$ are poisson’s ratio in directions 12 and 21 respectively.

From the constitutive relation of the lamina, the transformed stress-strain relation of an orthotropic lamina in a plane state of stress is:

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} Q_{11} \alpha_{12} \alpha_{16} & Q_{12} \beta_{22} \beta_{26} & Q_{16} \gamma_{16} \gamma_{16} \\ Q_{12} \alpha_{12} \beta_{12} & Q_{22} \gamma_{22} \gamma_{26} & Q_{26} \gamma_{26} \gamma_{26} \\ Q_{16} \gamma_{16} \gamma_{16} & Q_{26} \gamma_{26} \gamma_{26} & Q_{66} \gamma_{16} \gamma_{26} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{bmatrix}, \quad \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \begin{bmatrix} Q_{11} \alpha_{12} \alpha_{16} & Q_{12} \beta_{22} \beta_{26} & Q_{16} \gamma_{16} \gamma_{16} \\ Q_{12} \beta_{12} \gamma_{12} & Q_{22} \gamma_{22} \gamma_{26} & Q_{26} \gamma_{26} \gamma_{26} \\ Q_{16} \gamma_{16} \gamma_{16} & Q_{26} \gamma_{26} \gamma_{26} & Q_{66} \gamma_{16} \gamma_{26} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{bmatrix}$$

(38)

The force results are related to the strains by the relations:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \end{bmatrix} = \begin{bmatrix} A_{11}A_{12}A_{16} & B_{11}B_{12}B_{16} & E_{11}E_{12}E_{16} \\ A_{12}A_{22}A_{26} & B_{12}B_{22}B_{26} & E_{12}E_{22}E_{26} \\ A_{16}A_{26}A_{66} & B_{16}B_{26}B_{66} & E_{16}E_{26}E_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{bmatrix} + \begin{bmatrix} D_{11}D_{12}D_{16} & F_{11}F_{12}F_{16} & H_{11}H_{12}H_{16} \\ D_{12}D_{22}D_{26} & F_{12}F_{22}F_{26} & H_{12}H_{22}H_{26} \\ D_{16}D_{26}D_{66} & F_{16}F_{26}F_{66} & H_{16}H_{26}H_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{bmatrix}$$

(39)

$$\begin{bmatrix} M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} B_{11}B_{12}B_{16} & D_{11}D_{12}D_{16} & F_{11}F_{12}F_{16} \\ B_{12}B_{22}B_{26} & D_{12}D_{22}D_{26} & F_{12}F_{22}F_{26} \\ B_{16}B_{26}B_{66} & D_{16}D_{26}D_{66} & F_{16}F_{26}F_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{bmatrix} + \begin{bmatrix} E_{11}E_{12}E_{16} & G_{11}G_{12}G_{16} & I_{11}I_{12}I_{16} \\ E_{12}E_{22}E_{26} & G_{12}G_{22}G_{26} & I_{12}I_{22}I_{26} \\ E_{16}E_{26}E_{66} & G_{16}G_{26}G_{66} & I_{16}I_{26}I_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{bmatrix}$$

(40)

$$\begin{bmatrix} P_1 \\ P_2 \\ P_6 \end{bmatrix} = \begin{bmatrix} A_{44} & A_{45} & A_{51} \\ A_{45} & A_{55} & A_{51} \\ A_{51} & A_{55} & A_{66} \end{bmatrix} \begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{zx} \end{bmatrix} + \begin{bmatrix} L_{44} & L_{45} & L_{51} \\ L_{45} & L_{55} & L_{51} \\ L_{51} & L_{55} & L_{66} \end{bmatrix} \begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{zx} \end{bmatrix}$$

(41)

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} + \begin{bmatrix} J_{44} & J_{45} \\ J_{45} & J_{55} \end{bmatrix} \begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{bmatrix}$$

(42)

Where: $A_{ij} = \int_0^h Q_{ij} \, dz \quad i = (1,2,4,5,6)$

(44)

$$B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij} = \int_0^h Q_{ij} (z, x, y) \, dz$$

(45)

$$J_{ij} = \int_0^h Q_{ij} \left( \nu \frac{z^2}{h^2} \sin \left( \frac{\pi z}{h} \right) \sin \left( \frac{\pi y}{h} \right) e^{m \cos \left( \frac{\pi z}{h} \right)} \right) \, dz \quad i = (1,2,6)$$

(46)

$$L_{i1} = \int_0^h Q_{i1} \left( -m \sin^2 \left( \frac{\pi y}{h} \right) \cos \frac{\pi z}{h} \right) \, dz \quad i = (4,5)$$

(47)

5. Navier’s Solution

In Navier's method the generalized displacements are expanded in a double trigonometric series in terms of unknown parameters. The choice of the function in the series is restricted to those which satisfy the boundary conditions of the problem as shown in figure 1. Substitution of the displacement expansion into the governing equations should give a set of algebraic equation among the parameter of the expansion. Simply supported boundary conditions are satisfied by assuming the following form of displacements [9]:

$$u(x, y, t) = \sum_{m=1}^\infty \sum_{n=1}^\infty Umn \cos (\alpha x) \sin (\beta y) \, e^{\lambda \lambda t}$$

(48)

$$v(x, y, t) = \sum_{m=1}^\infty \sum_{n=1}^\infty Vmn \sin (\alpha x) \cos (\beta y) \, e^{\lambda \lambda t}$$

(49)

$$w(x, y, t) = \sum_{m=1}^\infty \sum_{n=1}^\infty Wmn \sin (\alpha x) \sin (\beta y) \, e^{\lambda \lambda t}$$

(50)

$$\theta_1(x, y, t) = \sum_{m=1}^\infty \sum_{n=1}^\infty \theta_{1mn} \cos (\alpha x) \sin (\beta y) \, e^{\lambda \lambda t}$$

(51)

$$\theta_2(x, y, t) = \sum_{m=1}^\infty \sum_{n=1}^\infty \theta_{2mn} \sin (\alpha x) \cos (\beta y) \, e^{\lambda \lambda t}$$

(52)
Where: \( \alpha = \frac{m\pi}{h} \), \( \beta = \frac{n\pi}{h} (U_{mn}, V_{mn}, W_{mn}, \theta_{1mn}, \theta_{2mn}) \), are arbitrary constants.

The Navier solution exists if the following stiffnesses are zero, \( A_{16} = B_{16} = D_{16} = E_{16} = F_{16} = H_{16} = A_{26} = B_{26} = D_{26} = E_{26} = F_{26} = H_{26} = A_{45} = J_{45} = L_{45} = 0 \)

\[
\begin{align*}
\text{at } y=0 \text{ and } y=b, \\
u_0 = w_0 = \frac{\partial w_0}{\partial x} = 0 \\
N_{xy} = M_{xy} = 0
\end{align*}
\]

**Figure 1.** Boundary conditions for simply supported plate.

### 6. Eigenvalue Problem

The equations of motion (29-33) can be expressed in terms of displacements by substituting the force and moment resultants from equations (39 and 47) and substituting equations (48-52), the following eigenvalue equation is obtained:

\[
\begin{bmatrix}
c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\
c_{22} & c_{23} & c_{24} & c_{25} & c_{33} \\
c_{33} & c_{34} & c_{35} & c_{44} & c_{45} \\
c_{55} & & & &
\end{bmatrix}
\begin{bmatrix}
d_1 \\
d_2 \\
d_3 \\
d_4 \\
d_5
\end{bmatrix} = 0
\]

Where: \( \{d_{ij}\} = \{U_{mn}, V_{mn}, W_{mn}, \theta_{1mn}, \theta_{2mn}\} \)

And the stiffness element of \( C_{ij} \) are given in appendix, from which we can obtain the critical buckling load butting \( \omega = 0 \) or getting natural frequencies for the plate by using \( N_x = 0 \).

### 7. Vibration of Plate Under Initial Stress

In this article the effect of initial in plane stress on the natural frequency of plate is obtained.

Again the eigenvalue problem for free vibration problem is solved but with stiffness matrix for the plate contains the influence of the in plane load (\( N_x \)), as shown below:

\[
\begin{bmatrix}
m_{11} & m_{12} & m_{13} & m_{14} & m_{15} \\
m_{22} & m_{23} & m_{24} & m_{25} & m_{33} \\
m_{33} & m_{34} & m_{35} & m_{44} & m_{45} \\
m_{55} & & & &
\end{bmatrix}
\begin{bmatrix}
d_1 \\
d_2 \\
d_3 \\
d_4 \\
d_5
\end{bmatrix} +
\begin{bmatrix}
c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\
c_{22} & c_{23} & c_{24} & c_{25} & c_{33} \\
c_{33} & c_{34} & c_{35} & c_{44} & c_{45} \\
c_{55} & & & &
\end{bmatrix}
\begin{bmatrix}
d_m \\
d_{m+1} \\
d_{m+2} \\
d_{m+3} \\
d_{m+4}
\end{bmatrix} = 0
\]

### 8. Results and Discussions

In this section some design parameters such as thickness ratio, orthotropy ratio, number of layers and aspect ratio effects on buckling and free vibration behavior of laminated composite plates are studied using Mantari’s displacement function and solved by Matlab14 program.
8.1. Optimum 'm' Value
Mantari’s function used \( m=0.5 \) to give minimum difference for transverse displacement static solution and free vibration characteristics between 2-D and 3-D solution while in present work , \( m=0.05 \) is chosen to give accurate results for the critical buckling load so that we can use this same displacement model to predict buckling analysis for laminated simply supported cross ply (thick and thin) plate. Varying of buckling load with value of ‘m’ is drawn in figure 2 for \([0/90/90/0]\) square plate with \((a/H=10)\)

![Figure 2. Changing dimensionless Ncr with 'm' value.](image)

8.2. Buckling Results
Many design parameters are changed and compared with other published results , such as thickness ratio as shown in table(1) for \([0°/90°/90°/0°]\) plate which give good agreement with other researches, also changing orthotropy ratio \((E_1/E_2)\) for used material are listed in this table ,it is noticed that more close results are for \( m=0.05 \) while those for \( m=0.5 \) are a little over predict, while changing thickness ratio for another lamination \([0°/90°]\) are obtained in table(2) for \( m=0.05 \) and also they are in good agreement with other displacement field proposed by other researchers, buckling load for different lamination scheme are listed in table (3) again this function give accurate results compared with 3-D elasticity solution and effect of aspect ratio for the laminated plate on critical buckling load is studied and obtained in table(4) which give the same behaviour and close to those obtained by other researchers. Buckling mode is obtained for \( m=1, n=1 \) for all cases except for those written.

Material used in present work is: \( E_1/E_2 = open, G_{12}=G_{13}=0.6E_2 \) (Gpa), \( G_{23}=0.5E_2 \) (Gpa), \( \nu_{12}=\nu_{13}=0.25 \) and \( N_{cr}=(N\pi^2/E_2H^3) \)

8.3. Vibration of Plate under Initial Stress Results
Vibration analysis worked by [8] is used in present work but with adding initial in plane stress to investigate the validity of his function for such case. Results are obtained in tables 5 to 8 for different thickness ratio \((5,10,20)\), different orthotropy ratio \((E_1/E_2=20,40)\) and various number of layers, from which it seems that the fundamental frequency decrease when increasing the value of compressive
stress until the lowest natural frequency vanished when in plane stress reaches the critical buckling stress, which proved by other researchers.

Table 1. Normalized critical uni-axial buckling loads (Ncr) with various thickness ratios (a/h) and orthotropic ratio for simply supported cross-ply [0°/90°/90°/0°] square plate.

| a/h | References | 3  | 10  | 20  | 30  | 40  |
|-----|------------|----|-----|-----|-----|-----|
| 5   | Present work m=.5 | 4.615 | 7.272 | 9.618 | 11.168 | 12.307 |
|     | Present work m=.05 | 4.585 | 7.182 | 9.449 | 10.939 | 12.029 |
|     | Present work m=0 | 4.584 | 7.178 | 9.441 | 10.927 | 12.016 |
|     | Ref.[2] | 4.530 | 7.115 | 9.355 | 10.820 | 11.894 |
|     | PSDPT Ref.[10] | 4.5458 | 7.1554 | 9.4218 | 10.908 | 11.997 |
| 10  | Present work m=.5 | 5.459 | 10.028 | 15.462 | 19.931 | 23.693 |
|     | Present work m=.05 | 5.449 | 9.984 | 15.341 | 19.722 | 23.394 |
|     | Present work m=0 | 5.449 | 9.982 | 15.335 | 19.711 | 23.378 |
|     | Ref.[2] | 5.387 | 9.919 | 15.250 | 19.596 | 23.231 |
|     | Ref.[11] | 5.294 | 9.762 | 15.019 | 19.304 | 22.881 |
|     | PSDPT Ref.[10] | 5.3933 | 9.9405 | 15.298 | 19.674 | 23.340 |
| 20  | Present work m=.5 | 5.723 | 11.122 | 18.458 | 25.359 | 31.864 |
|     | Present work m=.05 | 5.7211 | 11.108 | 18.413 | 25.269 | 31.716 |
|     | Present work m=0 | 5.7210 | 11.107 | 18.411 | 25.264 | 31.708 |
|     | Ref.[2] | 5.657 | 11.049 | 18.345 | 25.182 | 31.607 |
|     | PSDPT Ref.[10] | 5.6590 | 11.056 | 18.363 | 25.216 | 31.659 |
| 50  | Present work m=.5 | 5.802 | 11.476 | 19.542 | 27.528 | 35.430 |
|     | Present work m=.05 | 5.8023 | 11.474 | 19.534 | 27.510 | 35.400 |
|     | Present work m=0 | 5.80231 | 11.4740 | 19.5343 | 27.509 | 35.398 |
|     | Ref.[2] | 5.738 | 11.418 | 19.478 | 27.451 | 35.336 |
|     | PSDPT Ref.[10] | 5.7383 | 11.419 | 19.482 | 27.457 | 35.346 |
| 100 | Present work m=.5 | 5.814 | 11.529 | 19.709 | 27.871 | 36.013 |
|     | Present work m=.05 | 5.8141 | 11.528 | 19.707 | 27.867 | 36.005 |
|     | Present work m=0 | 5.81411 | 11.5285 | 19.7072 | 27.866 | 36.004 |
|     | Ref.[2] | 5.749 | 11.473 | 19.653 | 27.812 | 35.949 |
|     | PSDPT Ref.[10] | 5.7499 | 11.473 | 19.654 | 27.814 | 35.952 |

Table 2. Normalized critical uni-axial buckling loads (Ncr) with various thickness ratios (a/h) for simply supported cross-ply [0°/90°] square plate, (E1/E2=40).

| References | a/h | 10  | 20  | 50  | 100 |
|------------|-----|-----|-----|-----|-----|
| Present work | 11.616 | 12.602 | 12.910 | 12.955 |
| Ref.[5] | 11.310 | 12.427 | 12.800 | 12.873 |
| Van et al.[4] | 11.360 | 12.551 | 12.906 | 13.039 |
| Chakrabarti and Sheikh, [12] | 11.349 | 12.510 | 12.879 | 12.934 |
| Reddy and Phan, HSDT [13] | 11.563 | 12.577 | 12.895 | 12.942 |

Table 3. Normalized critical uni-axial buckling loads (Ncr) with various modular ratios and lamination scheme for simply supported cross-ply square plate with thickness ratio (a/h = 10).

| References | Lamination | 3 | 10 | 20 | 30 | 40 |
|------------|------------|----|----|----|----|----|
| Present | 0°/90°/0° | 5.4451 | 9.8697 | 14.9169 | 18.9012 | 22.1459 |
| Ref.[5] | 5.2284 | 9.6259 | 14.6458 | 18.6158 | 21.8527 |
| 3-D Elasticity [11] | 5.3044 | 9.7621 | 15.0191 | 19.3040 | 22.8807 |
| Ferreira et al. [14] | 5.3869 | 9.8601 | 14.9746 | 19.0175 | 22.3070 |
| Fiedler et al. [15] | 5.3210 | 9.7180 | 14.7320 | 18.6902 | 21.9097 |
Reddy and Phan, [13] & 5.3933 & 9.406 & 15.2980 & 19.6740 & 22.3400 & 5.4645 & 10.1415 & 15.8513 & 20.6592 & 24.7757 & 5.3255 & 9.8943 & 15.5177 & 20.2477 & 24.2896 & 5.3552 & 10.0148 & 15.6527 & 20.4352 & 24.5929 & 5.4096 & 10.1500 & 16.0080 & 20.9990 & 25.3080 &  \\
Present & 0/90° & 0/90°/0° & 5.4645 & 10.1415 & 15.8513 & 20.6592 & 24.7757 & 5.3255 & 9.8943 & 15.5177 & 20.2477 & 24.2896 & 5.3552 & 10.0148 & 15.6527 & 20.4352 & 24.5929 & 5.4096 & 10.1500 & 16.0080 & 20.9990 & 25.3080 &  \\
Ref.[5] & 3-D Elasticity [11] & 5.3255 & 9.9603 & 15.6527 & 20.4663 & 24.5929 & 5.3552 & 10.0148 & 15.6527 & 20.4352 & 24.5024 & 5.3255 & 9.9603 & 15.6527 & 20.4663 & 24.5929 & 5.3552 & 10.0148 & 15.6527 & 20.4352 & 24.5024 &  \\
Ferreira et al. [14] & 5.3522 & 10.0148 & 15.6805 & 20.4352 & 24.5024 & 5.3522 & 10.0148 & 15.6805 & 20.4352 & 24.5024 & 5.3522 & 10.0148 & 15.6805 & 20.4352 & 24.5024 & 5.3522 & 10.0148 & 15.6805 & 20.4352 & 24.5024 &  \\
Reddy and Phan, [13] & 5.4096 & 10.1500 & 16.0080 & 20.9990 & 25.3080 & 5.4096 & 10.1500 & 16.0080 & 20.9990 & 25.3080 & 5.4096 & 10.1500 & 16.0080 & 20.9990 & 25.3080 & 5.4096 & 10.1500 & 16.0080 & 20.9990 & 25.3080 &  \\

Table 4. Normalized critical uni-axial buckling loads (Ncr) with various aspect ratios (a/b) for simply supported cross-ply [0°/90°], square plate (E1/E2=40).

| a/b | References | a/H = 5 | a/H = 10 | a/H = 20 | a/H = 50 | a/H = 100 |
|-----|------------|--------|----------|----------|----------|-----------|
| .5  | Present work | 8.848  | 18.488   | 25.856   | 29.151   | 29.693    |
|     | Ref.[5]     | 8.739  | 18.347   | 25.746   | 29.087   | 29.657    |
| 1   | Present work | 12.029 | 23.394   | 31.716   | 35.400   | 36.005    |
|     | Ref.[5]     | 11.858 | 23.134   | 31.517   | 35.278   | 35.923    |
| 2   | Present work | 16.681 | 48.119   | 93.579   | 113.210  | 115.335   |
|     | (m=3,n=1)   | 15.000 | 47.368   | 92.847   | 112.813  | 115.029   |
|     | (m=2,n=1)   |        |          |          |          |           |
|     | (m=1,n=1)   |        |          |          |          |           |

Table 5. Non dimensional natural frequency under various Ratio of Ncr ratio with various thickness ratios (a/h) for simply supported cross-ply [0°/90°], square plate and orthotropic ratio E1/E2 =20.

| Ratio of Ncr | a/H = 5 | a/H = 10 | a/H = 20 |
|--------------|--------|----------|----------|
| 0            | 7.940  | 8.892    | 9.216    |
| 0.1          | 7.543  | 8.439    | 8.744    |
| 0.2          | 7.123  | 7.960    | 8.245    |
| 0.3          | 6.677  | 7.451    | 7.714    |
| 0.4          | 6.199  | 6.904    | 7.144    |
| 0.5          | 5.680  | 6.309    | 6.524    |

Table 6. Non dimensional natural frequency under various Ratio of Ncr with various thickness ratios (a/h) for simply supported cross-ply [0°/90°], square plate and orthotropic ratio E1/E2 =40.

| Ratio of Ncr | a/H = 5 | a/H = 10 | a/H = 20 |
|--------------|--------|----------|----------|
| 0            | 7.940  | 8.892    | 9.216    |
| 0.1          | 7.543  | 8.439    | 8.744    |
| 0.2          | 7.123  | 7.960    | 8.245    |
| 0.3          | 6.677  | 7.451    | 7.714    |
| 0.4          | 6.199  | 6.904    | 7.144    |
| 0.5          | 5.680  | 6.309    | 6.524    |

Table 7. Non dimensional natural frequency under various Ratio of Ncr with various thickness ratios (a/h) for simply supported cross-ply [0°/90°], square plate and orthotropic ratio E1/E2 =20.
shells, or for other classical boundary conditions. Present theory accuracy may be checked by using it to buckling of cylindrical and spherical shells, or for other classical boundary conditions.

Table 8. Non dimensional natural frequency under various Ratio of N_y with various thickness ratios (a/h) for simply supported cross-ply [0°/90°] square plate and orthotropic ratio E1/E2 = 40.

| Ratio of N_y | a/h = 5 | a/h = 10 | a/h = 20 |
|-------------|--------|----------|----------|
| 0           | 9.321  | 10.652   | 11.133   |
| 0.1         | 8.826  | 10.112   | 10.563   |
| 0.2         | 8.377  | 9.542    | 9.961    |
| 0.3         | 7.863  | 8.935    | 9.32     |
| 0.4         | 7.321  | 8.84     | 8.623    |
| 0.5         | 6.716  | 7.577    | 7.884    |
| 0.6         | 6.061  | 6.797    | 7.058    |
| 0.7         | 5.326  | 5.915    | 6.120    |
| 1           | 0      | 0        | 0        |

9. Conclusions

Buckling analysis of laminated composite plates is carried out by using new higher order shear deformation theory for elastic composite plates proposed by J.L. Mantari et al. The new displacement field depends on a parameter “m”, whose value is determined so as to give results closest to the 3D elasticity solutions. In present work, value of “m=.05” gives results close to 3D elasticity solutions and other solutions methods worked by other researchers, while for static and free vibration analysis investigated by [8] used “m=.5” which gave the results close to 3D elasticity and it is used in present work to predict its validity to solve free vibration plate under initial in plane stress problem. Using this new displacement field does not change the buckling mode of simply supported cross-ply plate. Present theory accuracy may be checked by using it to buckling of cylindrical and spherical shells, or for other classical boundary conditions.

Appendix

\[
C_{11} = -A_{11} \alpha^2 - A_{66} \beta^2, \quad C_{12} = -A_{12} \alpha \beta - A_{66} \alpha \beta, \quad C_{13} = B_{11} \alpha^2 + B_{12} \alpha \beta^2 + 2B_{66} \alpha \beta^2
\]
\[
C_{14} = -B_{11} \frac{m \pi}{h} \alpha^2 - E_{11} \alpha^2 - B_{66} \frac{m \pi}{h} \beta^2 - E_{66} \beta^2
\]
\[
C_{15} = -B_{12} \frac{m \pi}{h} \alpha \beta - E_{12} \alpha \beta - B_{66} \frac{m \pi}{h} \beta \alpha - E_{66} \beta \alpha
\]
\[
C_{21} = C_{12}, \quad C_{22} = -A_{22} \beta^2 - A_{66} \alpha^2, \quad C_{23} = B_{12} \alpha^2 \beta + B_{22} \beta^3 + 2B_{66} \alpha^2 \beta
\]
\[
C_{24} = -B_{12} \frac{m \pi}{h} \alpha \beta - E_{12} \alpha \beta - B_{66} \frac{m \pi}{h} \beta \alpha - E_{66} \beta \alpha
\]
\[
C_{25} = -E_{22} \beta^2 - B_{66} \frac{m \pi}{h} \alpha^2 - E_{66} \alpha^2 - B_{22} \frac{m \pi}{h} \beta^2
\]
\[
C_{31} = C_{13}, \quad C_{32} = C_{23}
\]
\[
C_{33} = -D_{11} \alpha^4 - 2D_{12} \alpha^2 \beta^2 - D_{22} \beta^4 - 4D_{66} \alpha^2 \beta^2
\]
\[
C_{34} = D_{11} \frac{m \pi}{h} \alpha^3 + D_{12} \frac{m \pi}{h} \alpha \beta^2 + F_{12} \alpha \beta^2 + F_{11} \alpha^3 + 2D_{66} \frac{m \pi}{h} \beta \alpha^2 + 2F_{66} \alpha \beta^2
\]
\[
C_{35} = D_{12} \frac{m \pi}{h} \alpha^2 \beta + F_{12} \alpha \beta^2 + D_{22} \frac{m \pi}{h} \beta^3 + F_{22} \beta^3 + 2D_{66} \frac{m \pi}{h} \alpha^2 \beta + 2F_{66} \alpha \beta
\]
\[
C_{41} = C_{14}, \quad C_{42} = C_{24}, \quad C_{43} = C_{34}
\]
\[
C_{44} = -D_{11} \frac{m \pi}{h^2} \alpha^2 - 2F_{11} \frac{m \pi}{h} \alpha \beta^2 - D_{66} \frac{m \pi}{h^2} \beta^2 - 2F_{66} \frac{m \pi}{h} \beta \alpha^2 - H_{11} \alpha^2 - H_{66} \beta^2 - A_{55} \frac{m^2 \pi^2}{h^2}
\]
\[-2L_{55} \]
\[
C_{45} = -D_{12} \frac{m \pi}{h^2} \alpha \beta - 2F_{12} \frac{m \pi}{h} \alpha \beta - D_{66} \frac{m \pi}{h^2} \alpha \beta - 2F_{66} \frac{m \pi}{h} \alpha \beta - H_{12} \alpha \beta - H_{66} \alpha \beta
\]
\[
C_{51} = C_{15}, \quad C_{52} = C_{25}, \quad C_{53} = C_{35}, \quad C_{54} = C_{45}
\]
\[
C_{55} = -D_{22} \frac{m^2 \pi^2}{h^2} \beta^2 - 2F_{22} \frac{m \pi}{h} \beta^2 - D_{66} \frac{m^2 \pi^2}{h^2} \alpha^2 - 2F_{66} \frac{m \pi}{h} \alpha^2 - H_{22} \beta^2 - H_{66} \alpha^2 - A_{44} \frac{m^2 \pi^2}{h^2} \\
- 2L_{44} \frac{m \pi}{h} - L_{44}
\]

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