On a method of evaluation of zeta-constants based on one number theoretic approach

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Abstract

New formulas for approximation of zeta-constants were derived on the basis of a number-theoretic approach constructed for the irrationality proof of certain classical constants. Using these formulas it’s possible to approximate certain zeta-constants and their combinations by rational fractions and construct a method for their evaluation.

1 Introduction

The main purpose of this paper is to derive new formulas for approximation of zeta-constants, that is the values of the Riemann zeta function \( \zeta(n) \) with \( n \geq 2 \), \( n \) – integers, on the basis of a method which is used in the irrationality proofs.

The problem of constructing the methods for approximation and calculation of values of the Riemann zeta function at points of different arithmetic nature, and especially of zeta-constants, has been considered by many authors (we recall the works: [1]–[9]). In [10] a new method for approximation of these constants by rational fractions was presented. This method is constructed on the basis of the Hermite-Beukers approach (see [11]–[13]), which the latter one applied to prove the irrationality of zeta-constants \( \zeta(2) \) and \( \zeta(3) \) using two especially selected polynomials \( P_n(x) \) and \( Q_n(x) \), \( n \geq 1 \):

\[
P_n(x) = \frac{1}{n!} \left( \frac{d}{dx} \right)^n (x^n(1-x)^n), \quad Q_n(x) = (1-x)^n. \tag{1}
\]

Various versions of the Beukers method, its modifications and generalizations are widely used in the study of values of various functions on the irrationality (see, for example, [14]–[19]).

In studying the values of zeta-constants on the basis of Hermite-Beukers’ approach and in constructing rational approximations to them the problem arises to derive explicit formulas with coefficients of polynomials participating in integrals. For the case of two polynomials first such explicit formulas with coefficients in the combinations with zeta-constants depending on coefficients of polynomials in the most general canonical form were derived in [10]. The method from [10] gives a possibility to approximate zeta-constants and some of their combinations by enough simple expressions from rational fractions with coefficients of the polynomials [1] and calculate them effectively. However for
the study of zeta-constants $\zeta(n)$, $n \geq 2$, in full volume, it’s necessary to be able to derive such formulas for the case of $n, n \geq 2$, polynomials.

The present paper continues the study begun in [10]. New explicit formulas for approximation of the values of zeta-constants with participation of three polynomials are obtained. The possibility to get such formulas provide combinatorial lemmas 2.1 and 2.2. An algorithm for calculation of zeta-constants based on new formulas is described. One should mention that method from [10] assumes that both polynomials are needed to provide a reasonably good approximation (like in the Beukers method, see [12], [13]). The present method with three polynomials opens up a new choice opportunity between two ways: 1) all three polynomials will provide a convergence rate; 2) two polynomials will provide a convergence rate, and by manipulating the coefficients of the third polynomial, one can reduce the number of calculated rational fractions. Taking into account such a possibility, we will consider the third polynomial in the most general canonical form throughout the paper.

Further we use the following standard notations: a generalized harmonic number $H_n^{(m)}$ of order $m$ is the following sum for positive integers $n$ and $m$:

$$H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m}, \quad H_n^{(1)} = H_n, \quad H_0^{(m)} = 0.$$  (2)

The properties and methods for evaluation of harmonic numbers are widely studied: see, for example, [20], [21].

2 Auxiliary lemmas

For $|t| < 1$:

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots = \sum_{k=0}^{\infty} t^k.$$  (3)

Hence

$$I_3 = \int_0^1 \int_0^1 \int_0^1 \frac{dxdydz}{1-xyz} = \int_0^1 \int_0^1 \int_0^1 \sum_{k=0}^{\infty} x^ky^kz^k dxdydz = \sum_{k=0}^{\infty} \left( \int_0^1 x^k dx \right) \left( \int_0^1 y^k dy \right) \left( \int_0^1 z^k dz \right) = \sum_{k=0}^{\infty} \frac{1}{(k+1)^3} = \zeta(3).$$

Lemma 2.1. For any $r_1, r_2, r_3 \geq 0; s \geq 3$; the following relation holds:

$$I(r_1, r_2, r_3) = \int_0^1 \cdots \int_0^1 \frac{x_1^{r_1}x_2^{r_2}x_3^{r_3}}{1-x_1x_2x_3\cdots x_s} dx_1dx_2dx_3\cdots dx_s = \sum_{k=0}^{\infty} \frac{1}{(r_1 + k + 1)(r_2 + k + 1)(r_3 + k + 1)(k + 1)^{s-3}}.$$  (3)
Proof. For $0 < x_1, \ldots, x_s < 1$; $0 < x_1 \cdot x_2 \cdot \cdots \cdot x_s < 1$; we have:

$$\frac{1}{1 - x_1 \cdots x_s} = \sum_{k=0}^{\infty} (x_1 \cdots x_s)^k,$$

$$I(r_1, r_2, r_3) = \int_0^1 \cdots \int_0^1 \frac{x_1^{r_1} x_2^{r_2} x_3^{r_3}}{1 - x_1 x_2 x_3 \cdots x_s} dx_1 dx_2 dx_3 \cdots dx_s$$

$$= \sum_{k=0}^{\infty} \int_0^1 x_1^{k+r_1} dx_1 \int_0^1 x_2^{k+r_2} dx_2 \int_0^1 x_3^{k+r_3} dx_3 \int_0^1 x_4^k dx_4 \cdots \int_0^1 x_s^k dx_s$$

$$= \sum_{k=0}^{\infty} \frac{1}{r_1 + k + 1} \cdot \frac{1}{r_2 + k + 1} \cdot \frac{1}{r_3 + k + 1} \cdot \frac{1}{(r + k + 1)^{s-1}}.$$

Lemma 2.2. For any integer $s \geq 1$ and for any $r \geq 1; k \geq 0$, the following identities hold:

$$\frac{1}{(r + k + 1)(k + 1)^s} = \sum_{j=1}^{s} \frac{(-1)^j}{r^j(k + 1)^{s+1-j} + \frac{(-1)^s}{r^s(r + k + 1)},}$$

$$\frac{1}{(r + k + 1)^2(k + 1)^s} = \sum_{j=1}^{s} \frac{(-1)^j}{r^{j+1}(k + 1)^{s+1-j} + \frac{(-1)^s}{r^{s+1}(r + k + 1)}},$$

$$\frac{1}{(r + k + 1)^3(k + 1)^s} = \sum_{j=1}^{s} \frac{(-1)^j}{2r^{j+2}(k + 1)^{s+1-j} + \frac{(-1)^s}{2r^{s+2}(r + k + 1)}},$$

Proof. The formulas (4)–(6) are proved by induction on $s$. Let’s prove, for example, (6). For $s = 1$ it’s possible to verify directly that the equality is true

$$\frac{1}{(r + k + 1)^3(k + 1)} =$$

$$\frac{1}{r^3(k + 1)} - \frac{1}{r^3(r + k + 1)} - \frac{1}{r^2(r + k + 1)^2} - \frac{1}{r(r + k + 1)^3}$$

(7)

Let (6) be true for $s \leq n$, and

$$\frac{1}{(r + k + 1)^3(k + 1)^n} = \sum_{j=1}^{n} \frac{(-1)^j}{2r^{j+2}(k + 1)^{n+1-j} + (-1)^n \frac{n(n + 1)}{2r^{n+2}(r + k + 1)}}$$
\[+ ( -1)^n \frac{n}{(r + k + 1)^2} + ( -1)^n \frac{1}{r^n(r + k + 1)^3}. \]  

(8)

Let’s prove that (6) is true also for \( s = n + 1 \). We have from (8)

\[
\frac{1}{(r + k + 1)^3(k + 1)^n+1} = \frac{1}{k + 1} \sum_{j=1}^{n} \frac{(-1)^{j-1} j(j + 1)}{2r^{j+2}(k + 1)^{n+1-j}} + \frac{(-1)^{n+1} n(n + 1)}{2r^{n+2}(k + 1)(r + k + 1)} + \frac{( -1)^n}{r^{n+1}} \frac{1}{2r^n(r + k + 1)^2} + \frac{(-1)^n}{r^{n+1}} \frac{1}{(k + 1)(r + k + 1)^3}.
\]

Substituting in the last expression (4), (5) for \( s = 1 \) and (7) we find

\[
\frac{1}{(r + k + 1)^3(k + 1)^n+1} = \frac{1}{k + 1} \sum_{j=1}^{n} \frac{(-1)^{j-1} j(j + 1)}{2r^{j+2}(k + 1)^{n+1-j}} + \frac{(-1)^{n+1} n(n + 1)}{2r^{n+2}(k + 1)(r + k + 1)} + \frac{( -1)^n}{r^{n+1}} \frac{1}{2r^n(r + k + 1)^2} + \frac{(-1)^n}{r^{n+1}} \frac{1}{(k + 1)(r + k + 1)^3} \]

\[= \sum_{j=1}^{n} \frac{(-1)^{j-1} j(j + 1)}{2r^{j+2}(k + 1)^{n+1-j}} + \frac{(-1)^n}{r^{n+1}} \frac{n(n+1) + n + 1}{r^{n+3}(k + 1)} \]

\[+ \frac{(-1)^{n+1}}{r^{n+3}(r + k + 1)} + \frac{( -1)^n}{r^{n+1}(r + k + 1)^2} + \frac{(-1)^{n+1}}{r^{n+1}} \frac{1}{r(r + k + 1)^3} \]

that is the formula (6) for \( s = n + 1 \).
3 Main formula

Let \( a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n, c_0, c_1, \ldots, c_n, n \geq 1 \), be arbitrary numbers, and let \( P_n(x), Q_n(x) \) and \( T_n(x) \) be the polynomials:

\[
P_n(x) = a_0 + a_1 x + \cdots + a_n x^n, \quad Q_n(x) = b_0 + b_1 x + \cdots + b_n x^n, \quad T_n(x) = c_0 + c_1 x + \cdots + c_n x^n.
\]

Multiply the left and right sides of the relation (3) to \( a \) let \( r \) we divide the double sums in they are not equal, that is, for example, the sum in which the summation indices are equal and the sum in which \( r \) are not equal, that is, for example, \( r_1 = r_2 = r_3 = 0, 1, \ldots, n; s \geq 3 \). We get

\[
I_s = I_s(n) = \sum_{r_1=0}^{n} \sum_{r_2=0}^{n} \sum_{r_3=0}^{n} a_{r_1} b_{r_2} c_{r_3} I(r_1, r_2, r_3)
= \int_0^1 \cdots \int_0^1 \frac{P_n(x_1)Q_n(x_2)T_n(x_3)}{1 - x_1 x_2 x_3 \ldots x_s} dx_1 dx_2 dx_3 \ldots dx_s
= \sum_{k=0}^{\infty} \frac{1}{(k+1)^{s-3}} \sum_{r_1=0}^{n} \sum_{r_2=0}^{n} \sum_{r_3=0}^{n} \frac{a_{r_1} b_{r_2} c_{r_3}}{(r_1 + k + 1)(r_2 + k + 1)(r_3 + k + 1)}.
\]  

(9)

Separating in (3) the terms with \( r_1 = r_2 = r_3 = 0 \), then with \( r_1 = r_2 = 0 \), \( r_3 \neq 0; r_2 = r_3 = 0, r_1 \neq 0 \), \( r_1 = r_3 = 0, r_2 \neq 0 \), and finally with \( r_1 = 0, r_2 \neq 0, r_3 \neq 0; r_2 = 0, r_1 \neq 0, r_3 \neq 0; r_3 = 0, r_1 \neq 0, r_2 \neq 0 \), we find

\[
I_s = a_0 b_0 c_0 \zeta(s) + a_0 b_0 \sum_{k=0}^{\infty} \frac{1}{(k+1)^{s-1}} \sum_{r_3=1}^{n} \frac{c_{r_3}}{r_3 + k + 1}
+ a_0 c_0 \sum_{k=0}^{\infty} \frac{1}{(k+1)^{s-1}} \sum_{r_2=1}^{n} \frac{b_{r_2}}{r_2 + k + 1} + b_0 c_0 \sum_{k=0}^{\infty} \frac{1}{(k+1)^{s-1}} \sum_{r_1=1}^{n} \frac{a_{r_1}}{r_1 + k + 1}
+ a_0 \sum_{k=0}^{\infty} \frac{1}{(k+1)^{s-2}} \sum_{r_2=1}^{n} \sum_{r_3=1}^{n} \frac{b_{r_2} c_{r_3}}{(r_2 + k + 1)(r_3 + k + 1)}
+ b_0 \sum_{k=0}^{\infty} \frac{1}{(k+1)^{s-2}} \sum_{r_1=1}^{n} \sum_{r_3=1}^{n} \frac{a_{r_1} c_{r_3}}{(r_1 + k + 1)(r_3 + k + 1)}
+ c_0 \sum_{k=0}^{\infty} \frac{1}{(k+1)^{s-2}} \sum_{r_1=1}^{n} \sum_{r_2=1}^{n} \frac{a_{r_1} b_{r_2}}{(r_1 + k + 1)(r_2 + k + 1)(r_3 + k + 1)}.
\]

(10)

We divide the double sums in \( r_{\xi} \), \( \xi = 1, 2, 3 \); on the right side (10) into two sums: the sum in which the summation indices are equal and the sum in which they are not equal, that is, for example,
That is using the relations

\[ \sum_{r_1=1}^{n} \sum_{r_2=1}^{n} \frac{a_{r_1} b_{r_2}}{(r_1 + k + 1)(r_2 + k + 1)} \]

\[ = \sum_{r_1=1}^{n} \frac{a_{r_1} b_{r_1}}{(r_1 + k + 1)^2} + \sum_{r_1, r_2=1}^{n} \frac{a_{r_1} b_{r_2}}{r_2 - r_1} \left( \frac{1}{r_1 + k + 1} - \frac{1}{r_2 + k + 1} \right). \] \tag{11}

In a triple sum over \( r_\zeta \) from (10), we also separate terms with the same indices, using the relations

\[ + \sum_{r_1, r_2, r_3=1}^{n} \frac{a_{r_1} b_{r_2} c_{r_3}}{(r_1 + k + 1)^2(r_2 + k + 1)(r_3 + k + 1)} \]

\[ + \sum_{r_1, r_2, r_3=1}^{n} \frac{a_{r_2} b_{r_3} c_{r_1} + a_{r_3} b_{r_1} c_{r_1} + a_{r_1} b_{r_2} c_{r_1}}{(r_1 + k + 1)^2(r_2 + k + 1)(r_3 + k + 1)} \]

That is

\[ + \sum_{r_1, r_2=1}^{n} \frac{a_{r_2} b_{r_2} c_{r_2} + a_{r_1} b_{r_2} c_{r_1} + a_{r_2} b_{r_1} c_{r_1}}{(r_1 + k + 1)^2(r_2 + k + 1)} \]

\[ - \sum_{r_1, r_2, r_3=1}^{n} \frac{a_{r_1} b_{r_2} c_{r_3}}{(r_1 + k + 1)(r_3 - r_2)(r_3 + k + 1)} \left( \frac{1}{r_3 + k + 1} - \frac{1}{r_2 + k + 1} \right). \]

We get from here

\[ + \sum_{r_1, r_2=1}^{n} \frac{a_{r_2} b_{r_2} c_{r_2} + a_{r_1} b_{r_2} c_{r_1} + a_{r_2} b_{r_1} c_{r_1}}{(r_1 + k + 1)^2(r_2 + k + 1)} \]
From (13) we have with
\[
\sum_{r_1, r_2, r_3=1 \atop r_1 \neq r_2 \neq r_3}^{\infty} \frac{a_{r_1} b_{r_2} c_{r_3}}{(r_3 - r_2)(r_2 - r_1)} \left( \frac{1}{r_2 + k + 1} - \frac{1}{r_1 + k + 1} \right)
\]
\[
+ \sum_{r_1, r_2, r_3=1 \atop r_1 \neq r_2 \neq r_3}^{\infty} \frac{a_{r_1} b_{r_2} c_{r_3}}{(r_3 - r_2)(r_3 - r_1)} \left( \frac{1}{r_3 + k + 1} - \frac{1}{r_1 + k + 1} \right) .
\]
(12)

Substituting (11), (12) into (10) we find:

\[
I_s = a_0 b_0 c_0 \zeta(s) + \sum_{k=0}^{\infty} \sum_{r_1=1}^{\infty} \frac{a_0 b_0 c_{r_1} + a_0 c_0 b_{r_1} + b_0 c_0 a_{r_1}}{(k+1)^{s-1}(r_1 + k + 1)}
\]
\[
+ \sum_{k=0}^{\infty} \sum_{r_1=1}^{\infty} \frac{a_0 b_1 c_{r_1} + b_0 a_{r_1} c_{r_1} + c_0 a_{r_1} b_{r_1}}{(k+1)^{s-2}(r_1 + k + 1)^2}
\]
\[
- \sum_{k=0}^{\infty} \sum_{r_1, r_2=1}^{\infty} \frac{a_0 b_{r_1} c_{r_2} + b_0 a_{r_2} c_{r_1} + c_0 a_{r_1} b_{r_2}}{(k+1)^{s-2}(r_2 - r_1)} \left( \frac{1}{r_2 + k + 1} - \frac{1}{r_1 + k + 1} \right)
\]
\[
+ \sum_{k=0}^{\infty} \sum_{r_1, r_2=1}^{\infty} \frac{a_{r_1} b_{r_2} c_{r_1}}{(k+1)^{s-3}(r_1 + k + 1)^3} + \sum_{k=0}^{\infty} \sum_{r_1, r_2=1}^{\infty} \frac{a_{r_1} b_{r_2} c_{r_2} + a_{r_1} b_{r_2} c_{r_1} + a_{r_2} b_{r_1} c_{r_1}}{(k+1)^{s-3}(r_1 + k + 1)^2(r_2 - r_1)}
\]
\[
+ \sum_{k=0}^{\infty} \sum_{r_1, r_2=1}^{\infty} \frac{a_{r_1} b_{r_2} c_{r_2}}{(k+1)^{s-3}(r_2 - r_1)^2} \left( \frac{1}{r_2 + k + 1} - \frac{1}{r_1 + k + 1} \right)
\]
\[
+ \frac{1}{(r_2 - r_3)(r_2 - r_1)(r_2 + k + 1)} + \frac{1}{(r_1 - r_3)(r_1 - r_2)(r_1 + k + 1)} .
\]
(13)

From (13) we have with \( s = 3 \):

\[
I_3 = a_0 b_0 c_0 \zeta(3) + \sum_{k=0}^{\infty} \sum_{r_1=1}^{\infty} \frac{a_0 b_0 c_{r_1} + a_0 c_0 b_{r_1} + b_0 c_0 a_{r_1}}{(k+1)^{2}(r_1 + k + 1)}
\]
\[
+ \sum_{k=0}^{\infty} \sum_{r_1=1}^{\infty} \frac{a_0 b_{r_1} c_{r_1} + b_0 a_{r_1} c_{r_1} + c_0 a_{r_1} b_{r_1}}{(k+1)(r_1 + k + 1)^2}
\]
\[
- \sum_{k=0}^{\infty} \sum_{r_1, r_2=1}^{\infty} \frac{a_0 b_{r_1} c_{r_2} + b_0 a_{r_2} c_{r_1} + c_0 a_{r_1} b_{r_2}}{(k+1)(r_2 - r_1)} \left( \frac{1}{r_2 + k + 1} - \frac{1}{r_1 + k + 1} \right)
\]
Let’s introduce auxiliary notations:

\[ S_{00r_1} = a_0b_0c_{r_1} + a_0c_0b_{r_1} + b_0c_0a_{r_1}, \quad S_{0r_1r_1} = a_0b_{r_1}c_{r_1} + b_0a_{r_1}c_{r_1} + c_0a_{r_1}b_{r_1}, \]

\[ S_{0r_1r_2} = a_0b_{r_1}c_{r_2} + b_0a_{r_1}c_{r_2} + c_0a_{r_1}b_{r_2}, \quad S_{r_1r_1r_2} = a_{r_1}b_{r_1}c_{r_2} + a_{r_1}c_{r_1}b_{r_2} + b_{r_1}c_{r_1}a_{r_2}. \]

Transforming the sums from (14) with use of lemma 2.2, we find that:

1) for the second term on the right-hand side of (14) the following relation holds:

\[
\sum_{k=0}^{\infty} \sum_{r_1=1}^{n} \frac{S_{00r_1}}{(k+1)^2(r_1+k+1)} = \zeta(2) \sum_{r_1=1}^{n} \frac{S_{00r_1}}{r_1^2} - \sum_{r_1=1}^{n} \frac{S_{00r_1} \mathcal{H}_{r_1}}{r_1^2}; \tag{15}
\]

2) for the third term of the same part of (14) the following equality holds:

\[
\sum_{k=0}^{\infty} \sum_{r_1=1}^{n} \frac{S_{0r_1r_1}}{(k+1)(r_1+k+1)^2} = \sum_{r_1=1}^{n} \frac{S_{0r_1r_1} \mathcal{H}_{r_1}}{r_1^2} - \sum_{r_1=1}^{n} \frac{S_{0r_1r_1}}{r_1^2} \left( \zeta(2) - H_{r_1}^{(2)} \right); \tag{16}
\]

3) besides, we have:

\[
\sum_{k=0}^{\infty} \sum_{r_1=1}^{n} \frac{a_{r_1}b_{r_1}c_{r_1}}{(r_1+k+1)^3} = \sum_{r_1=1}^{n} a_{r_1}b_{r_1}c_{r_1} \left( \zeta(3) - H_{r_1}^{(3)} \right); \tag{17}
\]

4) as well we find:

\[
\sum_{k=0}^{\infty} \sum_{r_1=1}^{n} \frac{S_{0r_1r_2}}{r_2 - r_1} \left( \frac{1}{(k+1)(r_2+k+1)} - \frac{1}{(k+1)(r_1+k+1)} \right)
\]

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where
\[ I = \sum_{r_1, r_2 = 1 \atop r_1 \neq r_2}^n S_{0r_1r_2} \left( \frac{H_{r_2}}{r_2} - \frac{H_{r_1}}{r_1} \right); \]  
(18)

5) and finally we notice, that
\[
\sum_{k=0}^{\infty} \sum_{r_1, r_2 = 1 \atop r_1 \neq r_2}^n \frac{S_{r_1r_1r_2}}{(r_1 + k + 1)^2(r_2 - r_1)} = \sum_{r_1, r_2 = 1 \atop r_1 \neq r_2}^n S_{r_1r_1r_2} \left( -H_r^{(2)} - 2\psi(r_1) + \frac{1}{2} \right). 
\]  
(19)

Substituting (15)–(19) into (14) and doing the obvious transformations, we get

\[ I_3 = \zeta(3) \sum_{r_1=0}^n a_{r_1} b_{r_1} c_{r_1} + \zeta(2) \left( \sum_{r_1, r_2 = 1 \atop r_1 \neq r_2}^n \left( \frac{S_{00r_1} - S_{0r_1r_2}}{r_1} + \frac{S_{r_1r_1r_2}}{r_1^2} \right) \right) \]
\[ + \sum_{r_1, r_2, r_3 = 1 \atop r_1 \neq r_2 \neq r_3}^n \left( -a_{r_1} b_{r_1} c_{r_1} H_r^{(3)} + \left( \frac{S_{00r_1}}{r_1} - \frac{S_{0r_1r_2}}{r_2 - r_1} \right) H_r^{(2)} + \frac{S_{0r_1r_2}}{r_2 - r_1} \left( \frac{H_{r_2}}{r_2} - \frac{H_{r_1}}{r_1} \right) \right) \]
\[ - \frac{S_{0r_1r_2}}{r_2 - r_1} \left( \frac{H_{r_2}}{r_2} - \frac{H_{r_1}}{r_1} \right) \left( \frac{1}{r_2 + k + 1} - \frac{1}{r_1 + k + 1} \right) \]
\[ + \sum_{k=0}^{\infty} \sum_{r_1=1}^n \left( \frac{a_{r_1} b_{r_1} c_{r_1}}{(k + 1)(r_1 + k + 1)^3} \right) + \sum_{k=0}^{\infty} \sum_{r_1, r_2 = 1 \atop r_1 \neq r_2}^n \left( \frac{S_{r_1r_1r_2}}{(r_2 - r_1)^2(k + 1)} \right) \]
\[ + \sum_{k=0}^{\infty} \sum_{r_1, r_2 = 1 \atop r_1 \neq r_2}^n \left( \frac{a_{r_1} b_{r_2} c_{r_1}}{k + 1} \right) \frac{1}{(r_3 - r_2)(r_3 - r_1)(r_3 + k + 1)} \]

where \( H_r^{(n)} \), \( \xi = 1, 2, 3; m = 1, 2, 3 \), are harmonic numbers, that is sums, defined by (2). Similarly, we find for \( s = 4 \) from (13), that

\[ I_4 = a_0 b_0 c_0 \zeta(4) + \sum_{k=0}^{\infty} \sum_{r_1=1}^n \frac{S_{00r_1}}{(k + 1)^3(r_1 + k + 1)} + \sum_{k=0}^{\infty} \sum_{r_1=1}^n \frac{S_{0r_1r_1}}{(k + 1)^2(r_1 + k + 1)^2} \]
\[ - \sum_{k=0}^{\infty} \sum_{r_2 = 1 \atop r_1 \neq r_2}^n \frac{S_{0r_1r_2}}{(k + 1)^2(r_2 - r_1)} \left( \frac{1}{r_2 + k + 1} - \frac{1}{r_1 + k + 1} \right) \]
\[ + \sum_{k=0}^{\infty} \sum_{r_1=1}^n \frac{a_{r_1} b_{r_1} c_{r_1}}{(k + 1)(r_1 + k + 1)^3} + \sum_{k=0}^{\infty} \sum_{r_1, r_2 = 1 \atop r_1 \neq r_2}^n \frac{S_{r_1r_1r_2}}{(r_2 - r_1)^2(k + 1)} \]
\[ + \sum_{k=0}^{\infty} \sum_{r_1, r_2 = 1 \atop r_1 \neq r_2}^n \frac{a_{r_1} b_{r_2} c_{r_1}}{k + 1} \frac{1}{(r_3 - r_2)(r_3 - r_1)(r_3 + k + 1)} \]
Transforming the sums from (21) on the basis of the lemma 2.2, we get:

\[
I_4 = a_0 b_0 c_0 \zeta(4) + \zeta(3) \sum_{r_1=1}^{n} \frac{S_{00r_1} - a_{r_1} b_{r_1} c_{r_1}}{r_1} + \zeta(2) \star 
\]

\[
+ \sum_{r_1, r_2 = 1}^{n} \frac{2S_{0r_1r_2} - S_{00r_1} - a_{r_1} b_{r_1} c_{r_1}}{r_1^2} - \frac{S_{0r_1r_2}}{r_2 - r_1} \left( \frac{1}{r_2} - \frac{1}{r_1} \right) + \frac{S_{r_1r_1r_2}}{r_1 \left( r_2 - r_1 \right)} 
\]

\[
+ \sum_{r_1 = 1}^{n} \frac{S_{00r_1} - 2S_{0r_1r_2} + a_{r_1} b_{r_1} c_{r_1}}{r_1^2} H_{r_1} + \sum_{r_1 = 1}^{n} \frac{a_{r_1} b_{r_1} c_{r_1} - S_{0r_1r_2} H_{r_1}^{(2)}}{r_1^2} 
\]

\[
+ \sum_{r_1, r_2 = 1}^{n} \frac{S_{r_1r_1r_2}}{r_2 - r_1} \left( \frac{H_{r_2}}{r_2 - r_1} - \frac{H_{r_1}}{r_1} \right) 
\]

\[
+ \sum_{r_1, r_2 = 1}^{n} \frac{S_{r_1r_1r_2}}{r_2 - r_1} \left( \frac{H_{r_2}}{r_2 - r_1} - \frac{H_{r_1}}{r_1} \right) + \sum_{r_1, r_2, r_3 = 1}^{n} \frac{a_{r_1} b_{r_2} c_{r_3} \star}{r_1 \neq r_2 \neq r_3} 
\]

\[
+ \frac{H_{r_1}}{r_1 (r_1 - r_2)(r_1 - r_3)} + \frac{H_{r_2}}{r_2 (r_2 - r_1)(r_2 - r_3)} + \frac{H_{r_3}}{r_3 (r_3 - r_1)(r_3 - r_2)} \right) .
\]  (22)

For any \( s \) \( \geq 5 \) we transform the terms (13) on the basis of the lemma 2.2.
Carrying out algebraic calculations similar to the above ones, we get for \( s \) \( \geq 5 \):

\[
I_s = a_0 b_0 c_0 \zeta(s) + \left( \sum_{r_1=1}^{n} \frac{S_{00r_1}}{r_1} \right) \zeta(s - 1) 
\]

\[
+ \left( \sum_{r_1, r_2 = 1}^{n} \frac{S_{0r_1r_2} - S_{00r_1}}{r_1^2 r_2 - r_1} \left( \frac{1}{r_2} - \frac{1}{r_1} \right) \right) \zeta(s - 2) 
\]

\[
+ \sum_{j=3}^{s-4} (-1)^{j-1} \left( \sum_{r_1, r_2, r_3 = 1}^{n} \frac{S_{00r_1} - (j - 1)S_{0r_1r_2}}{r_1^j} \left( \frac{j - 2}{r_2 - r_1} \right) \right) 
\]

\[
+ \frac{S_{r_1r_1r_2}}{r_2 - r_1} \left( \frac{j - 2}{r_1^j - 1} + \frac{1}{r_2 - r_1} \left( \frac{1}{r_2^j - 1} - \frac{1}{r_1^j - 1} \right) \right) - \frac{S_{0r_1r_2}}{r_2 - r_1} \left( \frac{1}{r_2^j - 1} - \frac{1}{r_1^j - 1} \right) 
\]

\[
+ \frac{S_{00r_1}}{r_1} - \frac{S_{0r_1r_2}}{r_2 - r_1} \left( \frac{1}{r_2 - r_1} - \frac{1}{r_1} \right) \right) \right) .
\]  (22)
\[ + a_{r_1} b_{r_2} c_{r_3} \left( \frac{1}{(r_3 - r_2)(r_3 - r_1)r_3^{s-2}} + \frac{1}{(r_2 - r_3)(r_2 - r_1)r_2^{s-2}} + \frac{1}{(r_1 - r_3)(r_1 - r_2)r_1^{s-2}} \right) \zeta(s-j) + \]

\[ (-1)^{s-3} \sum_{r_1, r_2, r_3 = 1}^{n} \left( \frac{s}{r_1^{s-1}} - \frac{s}{r_1^{s-1}} \right) - \frac{S_{r_1 r_2}}{r_2 - r_1} \left( \frac{s - 5}{r_2^{s-4}} + \frac{1}{r_2 - r_1} \left( \frac{1}{r_2^{s-5}} - \frac{1}{r_1^{s-5}} \right) \right) \]

\[ - a_{r_1} b_{r_2} c_{r_3} \left( \frac{1}{(r_3 - r_2)(r_3 - r_1)r_3^{s-5}} + \frac{1}{(r_2 - r_3)(r_2 - r_1)r_2^{s-5}} + \frac{1}{(r_1 - r_3)(r_1 - r_2)r_1^{s-5}} \right) \zeta(3) \]

\[ + (-1)^{s-2} \sum_{r_1, r_2, r_3 = 1}^{n} \left( \frac{s - 2}{r_1^{s-2}} \right) - \frac{S_{r_1 r_2}}{r_2 - r_1} \left( \frac{s - 3}{r_2^{s-3}} + \frac{1}{r_2 - r_1} \left( \frac{1}{r_2^{s-4}} - \frac{1}{r_1^{s-4}} \right) \right) \]

\[ + a_{r_1} b_{r_2} c_{r_3} \left( \frac{1}{(r_3 - r_2)(r_3 - r_1)r_3^{s-4}} + \frac{1}{(r_2 - r_3)(r_2 - r_1)r_2^{s-4}} + \frac{1}{(r_1 - r_3)(r_1 - r_2)r_1^{s-4}} \right) \zeta(2) \]

\[ + (-1)^{s-2} \sum_{r_1, r_2, r_3 = 1}^{n} \left( \frac{s - 3}{r_1^{s-1}} \right) + \frac{S_{r_1 r_2}}{(r_2 - r_1)r_1^{s-2}} \frac{H_{r_1}}{(r_2 - r_1)^{s-2}} + \frac{H_{r_2}}{(r_2 - r_1)^{s-2}} + \frac{H_{r_3}}{(r_2 - r_1)^{s-2}} \]

\[ S_{r_1 r_2} \frac{H_{r_2}}{(r_2 - r_1)^{s-3}} - \frac{H_{r_3}}{(r_2 - r_1)^{s-3}} + a_{r_1} b_{r_2} c_{r_3} \left( \frac{H_{r_1}}{(r_1 - r_3)(r_1 - r_2)r_1^{s-3}} \right) \]

\[ + (-1)^{s-3} \sum_{r_1, r_2 = 1}^{n} \left( \frac{S_{r_1 r_2} - (s - 3)a_{r_1} b_{r_2} c_{r_1}}{r_1^{s-2}} - \frac{S_{r_1 r_2}}{(r_2 - r_1)r_1^{s-3}} \right) H_{r_1}^{(2)} \]

\[ 11 \]
Thus we have proved the following theorem:

**Theorem 3.1.** Let $P_n(x)$, $Q_n(x)$ and $T_n(x)$ be three polynomials of degree $n$, $n \geq 1$:

\[
P_n(x) = a_0 + a_1 x + \cdots + a_n x^n,
\]

\[
Q_n(x) = b_0 + b_1 x + \cdots + b_n x^n;
\]

\[
T_n(x) = c_0 + c_1 x + \cdots + c_n x^n;
\]

$a_0, a_1, \ldots, a_n$, $b_0, b_1, \ldots, b_n$, $c_0, c_1, \ldots, c_n$ — arbitrary numbers. Define the integral $I_s$, $s \geq 3$, as

\[
I_s = I_s(n) = \int_0^1 \cdots \int_0^1 \frac{P_n(x_1)Q_n(x_2)T_n(x_3)}{1-x_1x_2x_3 \cdots x_s} dx_1 dx_2 dx_3 \cdots dx_s.
\]

Then the following relations hold:

\[
I_3 = A_{s-2,3}\zeta(3) - A_{s-2,2}\zeta(2) - A_{s-2},
\]

\[
I_4 = A_{s-3,4}\zeta(4) + A_{s-3,3}\zeta(3) - A_{s-3,2}\zeta(2) - A_{s-3},
\]

\[
I_s = A_{1,s}\zeta(s) + A_{1,s-1}\zeta(s-1) + \cdots + A_{1,3}\zeta(3) - A_{1,2}\zeta(2) - A_1,
\]

where for $s = 5, 6, 7, \ldots$

\[
A_{s-2,3} = \sum_{r=1}^{n} a_r b_r c_r,
\]

\[
A_{s-2,2} = \sum_{r=1}^{n} \sum_{l=0}^{r-1} \frac{S_{srl} - S_{irl}}{r-l},
\]

\[
A_{s-2} = \sum_{r=1}^{n} a_r b_r c_r H_r^{(3)} - \sum_{r=1}^{n} \sum_{l=0}^{r-1} (S_{srl} - S_{irl}) \left( \frac{H_r^{(2)} - H_l^{(2)}}{r-l} + \frac{H_r - H_l}{(r-l)^2} \right)
\]

\[
+ \sum_{r=2}^{n} \sum_{l=1}^{r-1} (S_{srl} + S_{irl}) \left( \frac{H_r}{(r-i)(i-l)} + \frac{H_r}{(r-i)(r-l)} + \frac{H_l}{(l-i)(l-r)} \right),
\]

\[
A_{s-3,4} = ab_0c_0,
\]

\[
A_{s-3,3} = \sum_{r=1}^{n} \frac{S_{s0r} - a_r b_r c_r}{r},
\]

\[
A_{s-3,2} = \sum_{r=1}^{n} a_r b_r c_r + S_{s00r} - \frac{2S_{00r}}{r^2} + \frac{H_r H_l}{r(l-r)}.
\]
\[
\sum_{r=2}^{n} \sum_{l=1}^{r-1} \left( \frac{S_{0rl} + S_{0lr}}{r-l} \left( \frac{1}{r} - \frac{1}{r-l} \right) - \frac{1}{r-l} \left( \frac{S_{rul} - S_{lur}}{r} \right) \right), \tag{32}
\]

\[
A_{s-3} = \sum_{r=1}^{n} \left( \frac{2S_{0rr} - a_r b_r c_r - S_{00r}}{r^2} H_r + \frac{S_{0rr} - a_r b_r c_r H_r^{(2)}}{r^2} - \frac{a_r b_r c_r H_r^{(3)}}{r} \right) - \sum_{r=2}^{n} \sum_{l=1}^{r-1} \left( \frac{S_{rrl} - S_{ltr}}{(r-l)^2} \left( \frac{H_r}{r} - \frac{H_l}{l} \right) + \frac{1}{r-l} \left( \frac{S_{rul} H_r}{r^2} - S_{ltr} H_l \right) \right) + \frac{1}{r-l} \left( \frac{H_r}{r} - \frac{H_l}{l} \right), \tag{33}
\]

\[
A_{1,s} = a_0 b_0 c_0, \quad A_{1,s-1} = \sum_{r=1}^{n} \frac{S_{00r}}{r}. \tag{34}
\]

\[
A_{1,s-2} = \sum_{r=1}^{n} \frac{S_{0rr} - S_{00r}}{r^2} - \sum_{r=2}^{n} \sum_{l=1}^{r-1} \left( \frac{S_{0lr} + S_{0rl}}{r-l} \left( \frac{1}{r} - \frac{1}{r-l} \right) \right), \tag{35}
\]

for \( j = 3, 4, \ldots, s-2 : \)

\[
A_{1,s-j} = (-1)^{j-1} \left( \sum_{r=1}^{n} \frac{S_{00r} - (j-1)S_{0rr} + \frac{(j-2)(j-1)}{2} a_r b_r c_r}{r^j} \right) - \sum_{r=2}^{n} \sum_{l=1}^{r-1} \left( \frac{S_{ltr} - S_{rul}}{(r-l)^2} \left( \frac{1}{r^j} - \frac{1}{(r-l)^j} \right) + \frac{j-2}{r-l} \left( \frac{S_{ltr}}{r^{j-2}} - \frac{S_{rul}}{r^{j-1}} \right) \right) - \sum_{r=2}^{n} \sum_{l=1}^{r-1} \sum_{i=0}^{l-1} \left( S_{ltr} + S_{rul} \right) \left( \frac{H_r}{r^{j-2}(i-r)(i-l)} + \frac{H_l}{(r-l)(i-r)(l-i)} \right), \tag{36}
\]

\[
A_1 = (-1)^{s-1} \left( \sum_{r=1}^{n} \frac{S_{00r} + (s-2)S_{0rr} - \frac{(s-2)(s-3)}{2} a_r b_r c_r}{r^{s-1}} H_r \right) + \frac{(s-3)a_r b_r c_r - S_{0rr} H_r^{(2)}}{r^{s-2}} + \frac{a_r b_r c_r H_r^{(3)}}{r^{s-3}} - \left( \sum_{r=2}^{n} \sum_{l=1}^{r-1} \frac{1}{r-l} \left( \frac{H_r^{(2)} S_{ltr}}{l^{s-3}} - \frac{H_l^{(2)} S_{rul}}{r^{s-3}} \right) \right) + \frac{s-3}{r-l} \left( \frac{H_l S_{ltr}}{r^{s-2}} - \frac{H_r S_{rul}}{l^{s-2}} \right) + \frac{S_{ltr} - S_{rul}}{(r-l)^2} \left( \frac{H_r}{r^{s-3}} - \frac{H_l}{l^{s-3}} \right). \tag{37}
\]

\( 13 \)
\[-\sum_{r=2}^{n} \sum_{l=1}^{r-1} \sum_{i=0}^{l-1} \left( S_{irl} + S_{ilr} \right) \left( \frac{H_i}{r^3(i-r)(i-l)} + \frac{H_l}{l^3(i-l)(l-r)} \right), \tag{37} \]

where for \(0 \leq \mu \leq n; 0 \leq \nu \leq n; 0 \leq \lambda \leq n;\)

\[ S_{\mu \nu \lambda} = a_\mu b_\nu c_\lambda + b_\mu c_\nu a_\lambda + c_\mu a_\nu b_\lambda, \tag{38} \]

\(H_n^{(m)}\) are harmonic numbers defined by (2).

4 New formulas for approximation of zeta-constants

Note, the theorem 3.1 formulas have the most general form, the coefficients \(a_0, a_1, \ldots, a_n; b_0, b_1, \ldots, b_n; c_0, c_1, \ldots, c_n\) are arbitrary numbers.

To approximate effectively a zeta-constant \(\zeta(s), s \geq 3,\) on the basis of formulas (28)–(38), one should select such polynomials (24)–(26) which could provide a smallness of the absolute value of integral (27), for example of the order of \(2^{-Cn}, C = const \geq 1.\) We assume that polynomials (24), (25), (26) are chosen in such a way that \(I_s\) is small,

\[ I_s = I_s(n) = \theta_s(n) = \theta_s. \tag{39} \]

Then we have the system:

\[
\begin{align*}
A_{1,s} \zeta(s) + A_{1,s-1} \zeta(s-1) + \cdots + A_{1,3} \zeta(3) &= A_{1,2} \zeta(2) + A_{1} + \theta_s \\
A_{2,s-1} \zeta(s-1) + \cdots + A_{2,3} \zeta(3) &= A_{2,2} \zeta(2) + A_{2} + \theta_{s-1}, \\
&\vdots \\
A_{s-2,3} \zeta(3) &= A_{s-2,2} \zeta(2) + A_{s-2} + \theta_3
\end{align*}
\]

where the coefficients \(A_{i,j}\) are defined by (29)–(38) for \(s - 2\) values \(I_r, r = 3, 4, \ldots, s.\) The value \(\zeta(s)\) is expressed by the ratio of determinants:

\[ \Delta = A_{1,s} A_{2,s-1} \cdots A_{s-2,3}, \quad \zeta(s) = \frac{\Delta_s}{\Delta}, \]

wherein

\[ \Delta_s = \begin{vmatrix}
A_{1,2} \zeta(2) + A_{1} + \theta_s & A_{1,s-1} & \cdots & \cdots & A_{1,3} \\
A_{2,2} \zeta(2) + A_{2} + \theta_{s-1} & A_{2,s-1} & \cdots & \cdots & A_{2,3} \\
& & \cdots & \cdots & \cdots \\
A_{s-2,2} \zeta(2) + A_{s-2} + \theta_3 & 0 & \cdots & \cdots & A_{s-2,3}
\end{vmatrix} = \Delta_s^{(1)} + \Delta_s^{(2)} + \Delta_s^{(3)}, \]

where
\[ \Delta^{(1)}_s = \begin{vmatrix} A_1 \zeta(2) & A_{1,s-1} & \cdots & A_{1,3} \\ A_2 \zeta(2) & A_{2,s-1} & \cdots & A_{2,3} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s-2} \zeta(2) & 0 & \cdots & A_{s-2,3} \end{vmatrix}, \quad \Delta^{(2)}_s = \begin{vmatrix} A_1 & A_{1,s-1} & \cdots & A_{1,3} \\ A_2 & A_{2,s-1} & \cdots & A_{2,3} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s-2} & 0 & \cdots & A_{s-2,3} \end{vmatrix}, \quad \Delta^{(3)}_s = \begin{vmatrix} \theta & A_{1,s-1} & \cdots & A_{1,3} \\ \theta & A_{2,s-1} & \cdots & A_{2,3} \\ \vdots & \vdots & \ddots & \vdots \\ \theta & 0 & \cdots & A_{s-2,3} \end{vmatrix}. \]

Thus,

\[ \zeta(s) = \frac{\Delta_s}{\Delta} = \frac{1}{\Delta} \sum_{\nu=1}^{s-2} (A_{\nu,2} \zeta(2) + A_{\nu} + \theta_{s+1-\nu}) \Delta_{\nu s}, \]

where \( \Delta_{\nu s} \) — the corresponding algebraic complements, i.e.

\[ \zeta(s) = \frac{\zeta(2)}{\Delta} \sum_{\nu=1}^{s-2} A_{\nu,2} \Delta_{\nu s} + \frac{1}{\Delta} \sum_{\nu=1}^{s-2} A_{\nu} \Delta_{\nu s} + \frac{1}{\Delta} \sum_{\nu=1}^{s-2} \theta_{s+1-\nu} \Delta_{\nu s}. \quad (40) \]

## 5 Approximation

Now we estimate the values of \( \theta_s, s \geq 3 \), from (39), (40) for especially chosen polynomials. Like in [10], as main polynomials providing the approximation accuracy we take two polynomials: the shifted Legendre polynomial as \( P_n(x) \), the binomial polynomial as \( Q_n(x) \). The third polynomial \( T_n(x) \) will be written in the canonical form. We have

**Lemma 5.1.** Let

\[ P_n(x) = \frac{1}{n!} \left( \frac{d}{dx} \right)^n (x^n(1-x)^n) = a_0 + a_1 x + \cdots + a_n x^n, \quad (41) \]

\[ a_r = \frac{(-1)^r (n+r)!}{(r!)^2 (n-r)!}, \quad (42) \]

\[ Q_n(x) = (1-x)^n = b_0 + b_1 x + \cdots + b_n x^n; \quad (43) \]

\[ b_r = (-1)^r \frac{n!}{r!(n-r)!}. \quad (44) \]

Let

\[ T_n(x) = c_0 + c_1 x + \cdots + c_n x^n, \quad c^* = \max_{0 \leq i \leq n} |c_i|; \quad (45) \]

Then for the integral
\[ I_s = I_s(n) = \int_0^1 \cdots \int_0^1 \frac{P_n(x_1)Q_n(x_2)T_n(x_3)}{1-x_1x_2x_3 \cdots x_s} \, dx_1 \, dx_2 \, dx_3 \cdots \, dx_s, \quad s \geq 3, \]

the following estimate is valid
\[ |I_s| \leq \frac{c^*}{2^n}. \quad (46) \]

**Proof.** From (43), (44) we have for \( k \geq 1 \)
\[ \int_0^1 x^k Q_n(x) \, dx = \int_0^1 x^k (1-x)^n \, dx = B(k+1, n+1) = \frac{k!n!}{(k+n+1)!}, \]
where \( B(x, y) — \) the Euler beta function (see, for example, [22], [23]), \( B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \), where \( \Gamma(x + 1) = x\Gamma(x) \) is the Euler gamma function. At the same time, from (41), (42) we find by integration by parts:
\[ \int_0^1 x^k P_n(x) \, dx = \frac{1}{n!} \int_0^1 x^k \frac{d}{dx} \left( \frac{d^{n-1}}{dx^{n-1}} (x^n (1-x)^n) \right) \, dx = \]
\[ = (-1)^n \frac{k(k-1) \cdots (k-n+1)}{n!} B(k+1, n+1) = (-1)^n \frac{k(k-1) \cdots (k-n+1)}{(k+1) \cdots (k+n)(k+n+1)}, \]
for \( k \geq n \). If \( k < n \), then (see [10] for details)
\[ \int_0^1 x^k P_n(x) \, dx = 0. \]
Thus, we have with the selected \( P_n(x) \) and \( Q_n(x) \):
\[ \int_0^1 x^k Q_n(x) \, dx = B(k+1, n+1), \]
\[ \int_0^1 x^k P_n(x) \, dx = \begin{cases} 0, & \text{if } k \leq n-1 \\ (-1)^n \frac{k(k-1) \cdots (k-n+1)}{n!} B(k+1, n+1), & \text{if } k \geq n, \end{cases} \]
\[ \int_0^1 x^k T_n(x) \, dx = \sum_{i=0}^n \frac{c_i}{k+1+i} = T(k,n). \]
Consequently
\[ I_s = I_s(n) = \int_0^1 \cdots \int_0^1 \frac{P_n(x_1)Q_n(x_2)T_n(x_3)}{1-x_1x_2x_3 \cdots x_s} \, dx_1 \, dx_2 \, dx_3 \cdots \, dx_s = \]
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\[
I_s = \sum_{k=0}^{\infty} \int_0^1 \cdots \int_0^1 (x_1x_2 \ldots x_s)^k P_n(x_1)Q_n(x_2)T_n(x_3)dx_1dx_2dx_3 \ldots dx_s = \\
= (-1)^n \sum_{k=n}^{\infty} \frac{k(k-1) \ldots (k-n+1)}{n!} \frac{B^2(k+1,n+1)T(k,n)}{(k+1)^{s-3}}.
\] (47)

We find from (47):

\[
|I_s| \leq c^* \sum_{j=0}^{\infty} \frac{(n+j)!B^2(n+1+j,n+1)}{(n+1+j)^{s-3}} \\
\leq c^* \sum_{j=0}^{\infty} \frac{B(n+1+j,n+1)}{(2n+j+1)(n+1+j)^{s-3}} \frac{\Gamma^2(n+j+1)}{\Gamma(2n+j+1)\Gamma(j+1)}.
\] (48)

Since (see, for example, [22])

\[
\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \gamma)\Gamma(\beta - \gamma)} = \prod_{\nu=0}^{\infty} \left(1 + \frac{\gamma}{\alpha + \nu}\right) \left(1 - \frac{\gamma}{\beta + \nu}\right),
\]

then

\[
\frac{\Gamma^2(n+j+1)}{\Gamma(2n+j+1)\Gamma(j+1)} = \prod_{\nu=0}^{\infty} \left(1 - \left(\frac{n}{n+1+j+\nu}\right)^2\right) < 1
\]

for any \( n \geq 1 \). Consequently from (48) for any \( s \geq 3 \) follows the estimate

\[
|I_s| \leq c^* \sum_{j=0}^{\infty} \frac{B(n+1+j,n+1)}{(2n+j+1)(n+1+j)^{s-3}} \leq c^*\frac{2}{\Gamma(\frac{1}{2},n)} \leq c^* 2^{-2n},
\]

that is the estimate (46).

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