Abstract

We show that the recently derived (q-) discrete form of the Painlevé VI equation can be related to the discrete P_{III}, in particular if one uses the full freedom in the implementation of the singularity confinement criterion. This observation is used here in order to derive the bilinear forms and the Schlesinger transformations of both q-P_{III} and q-P_{VI}. 

1. Introduction.

In a recent paper [1], the first two authors obtained a first order system of $q$-difference equations the continuous limit of which was the sixth Painlevé equation. This completed the list of discrete forms of the Painlevé transcendents, since a discrete $P_{VI}$ was the only one that was missing. The method used for obtaining this mapping was to consider a $q$-difference version of the monodromy preserving deformation theory of linear differential equations. (We recall the strong connection of the continuous Painlevé equations [2] to the latter approach.) The $q$-difference system that was called $q$-$P_{VI}$ is the following:

$$
\frac{yy}{xx} = \frac{a_3a_4(x - q^n b_1)(x - q^n b_2)}{(x - b_3)(x - b_4)} \quad (1.1a)
$$

$$
\frac{x}{\bar{x}} = \frac{b_3b_4(y - q^n a_1)(y - q^n a_2)}{(y - a_3)(y - a_4)} \quad (1.1b)
$$

where $x = x(n), \bar{x} = x(n+1), \underline{x} = x(n-1)$ and $a_1, \ldots, b_4$ are constants constrained by

$$
a_1a_2/a_3a_4 = q b_1 b_2/\underline{b}_3 \underline{b}_4.
$$

This system is integrable in the sense that it results from the compatibility of linear $q$-difference equations. Moreover, the singularity confinement property of (1.1) was verified explicitly.

In [4], the remaining authors, while missing the connection with $P_{VI}$, identified equation (1.1) as presumably integrable in relation with the discrete $P_{III}$, which was called $d$-$P_{III}$ (but should have been called $q$-$P_{III}$ for reasons that will become obvious). Indeed, as was shown in [4], the first form of the discrete $P_{III}$ ever obtained:

$$
\frac{x}{\bar{x}} = \frac{cd(x - a)(x - b)}{(x - c)(x - d)},
$$

with $c, d$ constant and $a, b$ proportional to $\lambda^n$, is associated to a linear problem of $q$-difference type.

In section 2 we will show the precise relation between (1.1) and (1.2). The method used for the derivation of $q$-$P_{VI}$ is the “singularity confinement” discrete integrability criterion. In section 3 we present the bilinearization of both $q$-$P_{III}$ and $q$-$P_{VI}$ equations: the former is obtained through the use of four $\tau$-functions while for the latter 8 $\tau$-functions are necessary. Section 4 is devoted to the Schlesinger transformations of both equations.

2. Derivation of the $q$-$P_{VI}$ equation from singularity confinement.

We start with the $q$-$P_{III}$ equation written in this general form (where $a, b, c, d, g$ may $a$ priori all depend on $n$):

$$
\frac{x}{\bar{x}} = \frac{g(x - a)(x - b)}{(x - c)(x - d)} \quad (2.1)
$$

In order to obtain the precise integrable form of this equation we use the singularity confinement approach. This method is based on the observation that, for an integrable mapping, the singularities that appear at some iteration, due to the particular initial
conditions chosen, i.e. the ‘movable’ singularities, disappear after some iterations: they are “confined”. Based on this idea we can ask when a divergence appears in the iteration of (2.1). This happens whenever \(x\) passes through one of the roots of the denominator. Let us first consider the case where \(\bar{x} = \bar{c}\) while \(\bar{x}\) is free. We find thus that \(x\) diverges and \(\bar{x}\). Iterating further we would have found that the \(x\)’s obtained do not in general depend on the free quantity \(\bar{x}\). The only way to restore this dependence, i.e. to confine the singularity, at this stage, is to balance the singularity of \(x\) by a singularity of the rhs of (2.1). Thus \(\bar{x}\) must be equal to a root of the denominator. The assumption \(\bar{x} = \bar{c}\) is not acceptable because it corresponds to a periodic singularity and is thus in contradiction with the requirement that the singularity be movable. We are left with the choice \(\bar{x} = \bar{d}\) and we have \(\bar{c}\bar{d} = \bar{g}\). In a symmetric way, starting from \(\bar{x} = \bar{d}\) we find \(\bar{c}\bar{d} = \bar{g}\). Thus the first condition for confinement is

\[
\frac{\bar{c}}{\bar{d}} = \frac{c}{d} \quad (2.2)
\]

and \(c/d\) is periodic of period two.

This is not, however, the only source of divergence: \(x\) may be equal to one of the roots of the numerator. This would lead to a diverging \(\bar{x}\) unless the divergence is balanced by an appropriate vanishing factor. Let us start with the case \(\bar{x} = \bar{a}\). This leads to \(x = 0\). In order to obtain a finite \(\bar{x}\) we must have \(\bar{x} = \bar{b}\). (As previously, the confining condition \(\bar{x} = \bar{a}\) is not acceptable because it corresponds to a non-movable singularity). We find that we need \(a\bar{b} = gab/cd = \bar{a}\bar{b}\) where the last equation is obtained from the case \(\bar{x} = \bar{b}\). Thus the second confinement condition writes:

\[
\frac{a}{b} = a/b \quad (2.3)
\]

So \(a/b\) is also periodic of period two. In addition, using the expression of \(g\) above we also have

\[
a\bar{b}\bar{c}d = ab\bar{c}d \quad (2.4)
\]

We remark at this stage that the two conditions (2.2-3) relate \(n\)’s of the same parity. Thus, the parameters with even and odd indices are only related by (2.4).

If one introduces the simplifying assumption that the ratios \(c/d\) and \(a/b\) are not just periodic but strictly constant (as was done in [4]), one finds simply \(q\text{-P}_{III}\). A dependent variable transformation \(x(n) \rightarrow x(n)c(n)/C\) allows one to replace \(c(n)\) by the constant \(C\) (which we will denote simply by \(c\) from now on). Since \(c/d\) is a constant, \(d\) is now also a constant. From (2.4) we find that \(a\) and \(b\) are both proportional to \(\lambda^n\) for some \(\lambda\). However, this simplification is unnecessary. Proceeding in full generality, we first use the dependent variable transformation (for even and odd indices separately) and impose \(\bar{c} = \bar{c}\) and thus \(\bar{d} = \bar{d}\). We now find that (2.4) becomes \(a\bar{b}\bar{c}d = abc\bar{d}\) and the same equation at step \(n + 1\) is \(a\bar{b}\bar{c}d = abc\bar{d}\) i.e.

\[
a/a = \bar{b}/b \quad (2.5)
\]
Combining the latter with (2.3) implies that the ratio $\frac{a}{a} = \frac{b}{b}$ must be a parity independent constant. Calling this ratio $q$ we find that both $a$ and $b$ are proportional to $q^n$. We have in fact $a_{2n} = q^n a_e$, $a_{2n+1} = q^n a_o$ (thereby breaking formally the symmetry between even and odd) and the analogous expression for $b$. Moreover, (2.4) now writes:

$$qa_e b_e c_o d_o = a_o b_o c_e d_e$$

(2.6)

If $q = \lambda^2$ and $a_o = \lambda a_e$, $b_o = \lambda b_e$, we recover the $q$-P$_{\text{III}}$ case. As a last step we separate the even and odd $x$’s calling, for instance, the odd ones $y$ with the redefinition $x(2n) \to x(n)$, $x(2n + 1) \to y(n)$, and similarly for the $(a, b, c, d)$ which at odd $n$’s will now be called $(p, r, s, t)$. Because of the redefinition of $n$, what was previously called $\vec{x}$ is now just $\vec{\tau}$. We find two coupled equations for $x$ and $y$:

$$yy = \frac{st(x - a)(x - b)}{(x - c)(x - d)}$$

(2.7a)

$$\vec{\tau} x = \frac{cd(y - p)(y - r)}{(y - s)(y - t)}$$

(2.7b)

where $c, d, s, t$ are constants and $a, b, p, r$ are proportional to $q^n$. The constraint (2.4) now becomes:

$$prcd = qabst$$

(2.8)

We recognize immediately in (2.7), with the appropriate renaming of the coefficients, the system (1.1) that defines the $q$-P$_{\text{VI}}$ equation. Thus, $q$-P$_{\text{VI}}$ is indeed contained in an embryonic state in $q$-P$_{\text{III}}$.

3. BILINEAR FORMS FOR $q$-P$_{\text{III}}$ AND $q$-P$_{\text{VI}}$.

Let us start with the bilinearization of $q$-P$_{\text{III}}$. We expect $x$ to be given by a rational expression involving the $\tau$-functions. The latter being entire functions, the divergences of $x$ will be associated to the vanishing of $\tau$-fuctions which appear at the denominator of $x$. In order to proceed, we need the precise singularity structure of the equation [5].

From the previous section, we know that a singularity appears when $x$ passes through a root of the denominator. Then $x$ diverges at the next step and the singularity becomes confined at the subsequent step through the requirement that $x$ passes through the other root. Two singularity patterns exist, $\{c, \infty, d\}$ and $\{d, \infty, c\}$. The existence of these two singularity patterns suggests the introduction of two $\tau$-functions $F, G$:

$$x = c(1 + \frac{FG}{F G}) = d(1 + \frac{FG}{F G}).$$

(3.1)

The singularity patterns described above correspond to the cases where either $F$ or $G$ passes through zero. Starting from this ansatz the bilinear form of $q$-P$_{\text{III}}$ was given in [5]:

$$czFG - dF G + (c - d)FG = 0$$

(3.2a)
\[
\frac{cd}{c-d}(cFG - dGF) + (c-a)(d-b)FG + c(d-b)FG + d(c-a)GF = 0 \quad (3.2b)
\]

The first equation comes just from equating the two expressions of \(x\). However, this is not the simplest form. In fact, as we have seen, two ‘potential’ singularities exist when \(x\) passes through a root of the numerator. The patterns are \(\{a,0,\overline{b}\}\) and \(\{b,0,\overline{a}\}\) and we remark that no singularity actually develops. This suggests the introduction of two more, auxiliary \(\tau\)-functions \(J, K\) in the following manner:

\[
x = c(1 + \frac{FG}{FG}) = d(1 + \frac{FG}{FG}) = \frac{JK}{FG} \quad (3.3a)
\]

\[
\frac{1}{x} = \frac{1}{a}(1 + \frac{JK}{JK}) = \frac{1}{b}(1 + \frac{JK}{JK}) = \frac{FG}{JK}. \quad (3.3b)
\]

These equations in turn imply the \(q\)-P\(_{\text{III}}\) equation (1.2) for \(x\). By equating the expressions of \(x\) and \(1/x\) we have four equations for the four \(\tau\)-functions and thus the bilinearization is trivially obtained:

\[
c(FG + \overline{FG}) = d(FG + \overline{FG}) = JK \quad (3.4a)
\]

\[
\frac{1}{a}(JK + \overline{JK}) = \frac{1}{b}(JK + \overline{JK}) = FG \quad (3.4b)
\]

In order to obtain the bilinear form for \(q\)-P\(_{\text{VI}}\) we start with the same remark as for the nonlinear variable: separating odd and even \(x\)’s we introduced a new variable \(y\). Similarly, the \(\tau\)-functions \(F\) and \(G\) (and also \(J, K\) if we are talking about (3.4)) are split into even and odd. Let us keep the old names for the even ones, and for the odd introduce the new \(\tau\)-functions \(M, N, P, Q\). We look now for \(x\) and \(y\):

\[
x = c(1 + \frac{MN}{FG}) = d(1 + \frac{MN}{FG}) = \frac{JK}{FG} \quad (3.5a)
\]

\[
y = s(1 + \frac{FG}{MN}) = t(1 + \frac{FG}{MN}) = \frac{PQ}{MN} \quad (3.5b)
\]

and the analogous expressions for \(1/x, 1/y\). However, the expressions are slightly more complicated if we leave full generality to \(c, d, s\) and \(t\). By rescaling either \(x\) or \(y\) it is always possible to set

\[
\begin{align*}
    cd &= st, \\
    qab &= pr,
\end{align*} \quad (3.6)
\]

which we will assume from now on. Rewriting (3.2a) for both even and odd indices we find:

\[
\begin{align*}
    cMN - dMN + (c-d)FG &= 0 \quad (3.7a) \\
    sFG - tFG + (s-t)MN &= 0
\end{align*}
\]

while (3.2b) becomes:

\[
\frac{cd}{s-t}(sFG - tFG) + (c-a)(d-b)FG + c(d-b)MN + d(c-a)MN = 0
\]
\[
\frac{st}{c-d}(cMN - dN M) + (s-p)(t-r)M N + s(t-r)FG + t(s-p)FG = 0 \tag{3.7b}
\]

Similarly (3.4) can be transposed to the case of the 8 \( \tau \)-functions:

\[
c(FG + MN) = d(FG + MN) = JK
\]

\[
s(MN + F G) = t(MN + FG) = PQ
\]

\[
\frac{1}{a}(JK + P Q) = \frac{1}{b}(JK + PQ) = FG
\]

\[
\frac{1}{p}(P Q + J K) = \frac{1}{r}(P Q + JK) = MN \tag{3.8}
\]

where (3.6) has been assumed. This completes the bilinearization of \( q \)-P\(_{VI} \).

4. Miura and Schlesinger transforms for \( q \)-P\(_{III} \) and \( q \)-P\(_{VI} \).

In this section we shall derive the Schlesinger transformations for \( q \)-P\(_{III} \) and \( q \)-P\(_{VI} \). In the continuous case the Schlesinger are particular auto-Bäcklund transformations. As such they relate solutions of the same equation. However, the Schlesinger transformations relate solutions corresponding to the same monodromy data except for \textit{integer} differences in the monodromy exponents [2]. In the discrete case the relation of the Schlesinger transformations to monodromy exponents is not always clear. However, we can use an analogy with the continuous case. If one uses the proper parametrisation of the equation the Schlesinger transformations can be shown to be associated to simple changes of the parameters. The discrete case can be analysed in the same spirit. By using the proper parametrisation, one can identify, among the auto-Bäcklund transformations, those which correspond to simple changes of the parameters and which can thus be dubbed Schlesinger’s. As in the previous section, we shall start with \( q \)-P\(_{III} \) and then generalize the results to \( q \)-P\(_{VI} \) that is viewed as an asymmetric \( q \)-P\(_{III} \).

In order to find the Miura transform of \( q \)-P\(_{III} \), we use the standard ‘trick’ introduced in [6]. Starting from the variable \( x \) expressed in terms of \textit{two} \( \tau \)-functions \( F \) and \( G \), we introduce a new variable \( u \) which can be expressed in terms of only \textit{one} \( \tau \)-function, say \( G \). We remark that \( (x-c) \) is proportional to \( \overline{F}/F \) while \( (x-d) \) contains \( F/\overline{F} \). Upshifting the last object and multiplying by the first allows to get rid of \( F \) entirely. We thus find a first relation between \( x \) and \( u \):

\[
u = (x-c)(\overline{\tau} - d) \tag{4.1}
\]

where \( u \) is given by \( u = cd\overline{G}/G\overline{G} \). This is the first half of the Miura. Eliminating \( F \) between the two bilinear equations (3.2) leads to a (hexalinear) equation for \( G \) which can also be expressed in \( u \). The latter would be the ‘modified’ equation of \( q \)-P\(_{III} \) (‘modified’ in the sense of the relation that exists between the Korteweg-de Vries equation and the modified Korteweg-de Vries equation). However, it is simpler to obtain the second half of the Miura and proceed from there. As we have shown in [6], the complement of the
Miura involves a rational expression which must be homographic in both $u$ and $\underline{u}$. We find readily:

$$x = \frac{uu/\lambda - u - \underline{u} + cd - ab}{-u/d - u/c + c + d - a - b}$$  \hspace{1cm} (4.2)

Eliminating $x$ and $\underline{x}$ between (4.1), (4.2) and its upshift leads to an equation for $u, \underline{u}, \overline{u}$. This equation is, after a change of variables, a discrete form of $q$-PV equation (although not all the parameters of a $q$-PV are present). Introducing $U = u - cd$, we find:

$$(U\overline{U} - \lambda^2abcd)(U\overline{U} - abcd) = \frac{cd(U + bd)(U + \lambda ac)(U + ad)(U + \lambda bc)}{U + cd}$$ \hspace{1cm} (4.3)

In order to define a different Miura we can introduce the quantity:

$$w = (1/x - 1/a)(1/\overline{x} - 1/b)$$ \hspace{1cm} (4.4)

(we recall that $\overline{b} = b(n + 1) = \lambda b(n)$) which depends only on the $\tau$-function $K$ but not $J$. The second half of the Miura, analogous to (4.2), is:

$$x = \frac{-aw/\lambda - bw\lambda + 1/a + 1/b - 1/c - 1/d}{abw\overline{w} + 1/ab - 1/cd - w\lambda - \overline{w}/\lambda}$$ \hspace{1cm} (4.5)

Again eliminating $x$ leads to an equation for $w$. Introducing $W = w - 1/\overline{a}\overline{b}$, we find for this equation:

$$(W\overline{W} - \frac{1}{\lambda^2abcd})(W\overline{W} - \frac{1}{abcd}) = \frac{(W + 1/\lambda bd)(W + 1/ac)(W + 1/\lambda bc)(W + 1/ad)}{\lambda abW + 1}$$ \hspace{1cm} (4.6)

The quantity $\tilde{W} = U/\lambda abcd$ satisfies an equation, obtained from (4.2), namely:

$$(\tilde{W}\overline{\tilde{W}} - \frac{1}{\lambda^2abcd})(\tilde{W}\overline{\tilde{W}} - \frac{1}{abcd}) = \frac{(\tilde{W} + 1/\lambda ac)(\tilde{W} + 1/bd)(\tilde{W} + 1/\lambda bc)(\tilde{W} + 1/ad)}{\lambda ab\tilde{W} + 1}$$ \hspace{1cm} (4.7)

Equation (4.7) has the same form as (4.6) provided one introduces the parameters

$$\tilde{a} = a\sqrt{\lambda}, \hspace{0.2cm} \tilde{b} = b/\sqrt{\lambda}, \hspace{0.2cm} \tilde{c} = c\sqrt{\lambda}, \hspace{0.2cm} \tilde{d} = d/\sqrt{\lambda}$$ \hspace{1cm} (4.8)

We define

$$\tilde{w} = \tilde{W} + 1/\lambda \tilde{a}\tilde{b} = u/\lambda abcd$$ \hspace{1cm} (4.9)

and

$$\tilde{x} = \frac{-\tilde{a}\tilde{w}/\lambda - \tilde{b}\tilde{w}\lambda + 1/\tilde{a} + 1/\tilde{b} - 1/\tilde{c} - 1/\tilde{d}}{\tilde{a}\tilde{b}\tilde{w}\overline{\tilde{w}} + 1/\tilde{a}\tilde{b} - 1/\tilde{c}\tilde{d} - \tilde{w}\lambda - \overline{\tilde{w}}/\lambda}$$ \hspace{1cm} (4.10)

Given this definition of $\tilde{x}$ and since $\tilde{W}$ satisfies (4.7) it follows that

$$\tilde{w} = (1/\tilde{x} - 1/\tilde{a})(1/\tilde{x} - 1/\tilde{b})$$ \hspace{1cm} (4.11)
and therefore $\tilde{x}$ satisfies $q$-$P_{III}$ with parameters $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$. The transformation from $x$ to $\tilde{x}$ through (4.1), (4.8), (4.9) and (4.10) defines an auto-Bäcklund transformation for $q$-$P_{III}$. In this case this is indeed a Schlesinger transformation which we denote by $S^a_c$.

(The convention used here is to give explicitly the parameters associated to $x$, rather than $\pi$, in (4.1) and (4.4).) The inverse transformation $(S^a_c)^{-1}$ can be obtained by defining $w$ through (4.4), $\tilde{u} = w^{\lambda abcd}$ and finally $\tilde{x}$ through the analog of (4.2).

In a similar way we can introduce the transformations $S^b_c$, $S^a_d = (S^b_c)^{-1}$ and $S^b_d = (S^a_c)^{-1}$. They correspond to multiplying the two parameters which appear explicitly by $\sqrt{\lambda}$ while dividing the two others by the same quantity. These are the most elementary Schlesinger transformations. Using them we can construct further Schlesinger transformations that act separately on $\{a, b\}$ or $\{c, d\}$. For instance the product $S^a_c S^a_d$ corresponds to $a \rightarrow a\lambda$, $b \rightarrow b/\lambda$, $c \rightarrow c$, $d \rightarrow d$.

We turn now to $q$-$P_{VI}$ and its Schlesinger transforms. Given its relation to $q$-$P_{III}$ it is straightforward to find a Miura transformation analogous to (4.1). Note however that we need a two-component Miura:

\begin{equation}
\begin{aligned}
 u &= (x - c)(y - t), \quad v = (y - s)(\pi - d) \\
\end{aligned}
\end{equation}

and its second half:

\begin{equation}
\begin{aligned}
 x &= \frac{uv - sdu - ctv + st(cd - ab)}{-su - tv + st(c + d - a - b)} \\
 y &= \frac{uv - sdu - ctv + cd(st - pr)}{-du - cv + cd(s + t - p - r)}
\end{aligned}
\end{equation}

Eliminating $x$ and $y$ between (4.12) and (4.13) leads to an equation for $u, v, \pi$ while eliminating $\pi$ and $y$ from (4.12), (4.13) and the upshift of (4.13a) leads to an equation for $u, v, \pi$. We introduce, just as in the case of $q$-$P_{III}$, the new variables $U = u - ct$, $V = v - sd$ and we obtain:

\begin{equation}
\begin{aligned}
(UV - abst)(UV - prcd) &= \frac{sd(U + cr)(U + bt)(U + cp)(U + at)}{U + ct} \\
(VU - prcd)(VU - q^2 abst) &= \frac{ct(V + qsb)(V + rd)(V + qas)(V + pd)}{V + sd}
\end{aligned}
\end{equation}

In parallel to (4.4) we introduce the Miura:

\begin{equation}
\begin{aligned}
 w &= (1/x - 1/a)(1/y - 1/r), \quad z = (1/y - 1/p)(1/\pi - 1/b) \\
\end{aligned}
\end{equation}

where now $\bar{b} = qb$, the second half being:

\begin{equation}
\begin{aligned}
 x &= \frac{-qw - p\bar{z} + q(1/a + 1/b - 1/c - 1/d)}{prw\bar{z} - qw/b - p\bar{z}/a + q(1/ab - 1/cd)} \\
 y &= \frac{-aw - qbz + (1/p + 1/r - 1/s - 1/t)}{qabwz - aw/p - qbz/r + (1/pr - 1/st)}
\end{aligned}
\end{equation}
The quantities \( W = w - 1/\omega r \) and \( Z = z - 1/qpb \) satisfy the system:

\[
(WZ - q/prcd)(WZ - 1/qabs) = \frac{(W + 1/\omega r)(W + 1/dr)(W + 1/as)(W + 1/cr)}{pb(W + 1/\omega r)}
\]

\[
(WZ - 1/qabs) - (WZ - 1/qprcd) = \frac{(Z + 1/pd)(Z + 1/qbt)(Z + 1/pc)(Z + 1/qbs)}{qar(Z + 1/qpb)}
\]

In analogy to the \( q - \text{P}_{III} \) case we introduce \( \tilde{W} = U/\omega abs \) and \( \tilde{Z} = V/\omega prcd \), (recall that \( q = \lambda^2 \)), and obtain:

\[
(\tilde{W}\tilde{Z} - q/prcd)(\tilde{W}\tilde{Z} - 1/qabs) = \frac{d(\tilde{W} + \lambda/dr)(\tilde{W} + 1/\omega s)(\tilde{W} + 1/pd)(\tilde{W} + 1/\omega s)}{\lambda ab(t + \omega c/\omega abs)}
\]

\[
(\tilde{W}\tilde{Z} - 1/qabs) - (\tilde{W}\tilde{Z} - 1/qprcd) = \frac{t(\tilde{Z} + 1/\omega ct)(\tilde{Z} + 1/\omega pc)(\tilde{Z} + 1/\omega bt)(\tilde{Z} + 1/\omega c)}{\lambda prd(Z + s/\omega rpc)}
\]

This system has the same form as (4.17) provided one introduces the parameters:

\[
\tilde{a} = \sqrt{aps/c}, \quad \tilde{b} = \frac{1}{\lambda}\sqrt{brs/c}, \quad \tilde{c} = \sqrt{pca}, \quad \tilde{d} = \sqrt{adt/p}
\]

\[
\tilde{p} = \lambda\sqrt{apc/s}, \quad \tilde{r} = \sqrt{brs/c}, \quad \tilde{s} = \lambda\sqrt{acs/p}, \quad \tilde{t} = \frac{1}{\lambda}\sqrt{pdt/a}
\]

We remind the reader that in section 3 we fixed the relative scaling of \( x \) and \( y \) so that \( cd = st \) and \( qab = pr \) (equation 3.6). It is easy to check that these relations are satisfied by the tilded parameters above. We define

\[
\tilde{w} = \tilde{W} + 1/\omega r = u/\omega ab, \quad \tilde{z} = \tilde{Z} + 1/qpb = v/\omega prcd
\]

and \( \tilde{x}, \tilde{y} \) by the equivalent of (4.16) where \( w, z \) and all the parameters are tilded. Again, since \( \tilde{W}, \tilde{Z} \) satisfy (4.18) we have

\[
\tilde{w} = (1/\tilde{x} - 1/\tilde{a})(1/\tilde{y} - 1/\tilde{r}), \quad \tilde{z} = (1/\tilde{y} - 1/\tilde{p})(1/\tilde{x} - 1/\tilde{b})
\]

and \( \tilde{x}, \tilde{y} \) satisfy \( q - \text{P}_{VI} \) with ‘tilded’ parameters. The transformation from \( x, y \) to \( \tilde{x}, \tilde{y} \) is an auto-Bäcklund transformation \( B_{apcs}^{\alpha\nu} \) for \( q - \text{P}_{VI} \) that is not a Schlesinger. (The convention again is that \( a \) are associated to \( x \) in \( w, p \) to \( y \) in \( z, c \) is associated to \( x \) in \( u \) and \( s \) to \( y \) in \( v \)). Starting from that auto-Bäcklund \( B_{apcs}^{\alpha\nu} \) it is straightforward to obtain a Schlesinger transformation. In fact \( (B_{apcs}^{\alpha\nu})^2 \) is such a transformation where

\[
a \rightarrow a\lambda, \quad b \rightarrow b/\lambda, \quad c \rightarrow c\lambda, \quad d \rightarrow d/\lambda, \quad p \rightarrow p\lambda, \quad r \rightarrow r/\lambda, \quad s \rightarrow s\lambda, \quad t \rightarrow t/\lambda
\]
However this is not the simplest Schlesinger transformation one can construct starting from $B$. In fact, through the product of auto-Bäcklund transformations $B_{ct}^r B_{cs}^p$ we obtain the parameters:

$$
\hat{a} = \sqrt{\frac{abd}{c}}, \quad \hat{b} = \sqrt{\frac{abc}{d}}, \quad \hat{c} = \sqrt{\frac{bcd}{a}}, \quad \hat{d} = \sqrt{\frac{acd}{b}} \quad (4.24)
$$

$$
\hat{p} = \frac{1}{\lambda} \sqrt{\frac{prt}{s}}, \quad \hat{r} = \lambda \sqrt{\frac{prs}{t}}, \quad \hat{s} = \frac{1}{\lambda} \sqrt{\frac{rst}{p}}, \quad \hat{t} = \lambda \sqrt{\frac{pst}{r}}
$$

Next we perform the transformation, $X = \sqrt{abcd/\hat{x}}, Y = \sqrt{prst/\hat{y}}$ and obtain $q$-$P_{VI}$ for the variables $X, Y$. The resulting transformation is a Schlesinger corresponding to parameters:

$$
a \rightarrow a, \quad b \rightarrow b, \quad c \rightarrow c, \quad d \rightarrow d, \quad p \rightarrow p\lambda, \quad r \rightarrow r/\lambda, \quad s \rightarrow s\lambda, \quad t \rightarrow t/\lambda \quad (4.25)
$$

Further Schlesinger transformations can be obtained through combinations of the appropriate $B$’s.

5. Conclusion

In this paper we have undertaken a study of the discrete $P_{VI}$ ($q$-$P_{VI}$) equation that was derived in [1]. We have shown that this equation can be viewed as a kind of ‘asymmetric’ $q$-$P_{III}$. Indeed, its form is the most general one compatible with the singularity confinement requirement when applied to a $q$-$P_{III}$-type ansatz. Using this analogy we were able to obtain, in close parallel, results for the $q$-$P_{III}$ and $q$-$P_{VI}$. In particular we gave a simple bilinearization for both equations, involving four $\tau$-functions in the $q$-$P_{III}$ and eight $\tau$-functions in the $q$-$P_{VI}$ case. By deriving the appropriate Miura transformations for both equations we obtained their Schlesinger transformations.

Several open questions remain at this stage. For example, the structure of the space of particular solutions of $q$-$P_{VI}$ has only been touched upon in [1]. Rational solutions have not been considered at all. The coalescence chain of $q$-$P_{VI}$ which is expected to contain new discrete Painlevé equations has not been worked out yet. We hope to return to these questions in future publications.

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References.

[1] M. Jimbo and H. Sakai, A $q$-analog of the sixth Painlevé equation, preprint Kyoto-Math 95-16.

[2] M. Jimbo, T. Miwa, Physica D2 (1981) 407.

[3] B. Grammaticos, A. Ramani and V. Papageorgiou, Phys. Rev. Lett. 67 (1991) 1825.

[4] A. Ramani, B. Grammaticos and J. Hietarinta, Phys. Rev. Lett. 67 (1991) 1829.

[5] A. Ramani, B. Grammaticos and J. Satsuma, Jour. Phys. A 28 (1995) 4655.

[6] A. Ramani and B. Grammaticos, Jour. Phys. A 25 (1992) L633.