The existence of full dimensional invariant tori for an almost-periodically forced nonlinear beam equation

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Abstract. In this paper, we prove the existence of full dimensional invariant tori for a non-autonomous, almost-periodically forced nonlinear beam equation with a periodic boundary condition via KAM theory.

Keywords: Infinite-dimensional Hamiltonian system; KAM theory; Non-autonomous beam equation; Full dimensional invariant tori.

1 Introduction

Presently, there have been many remarkable results on the beam equations via KAM theory, see [4,6–9,21]. In these papers, the authors proved the existence of quasi-periodic solutions for nonlinear beam equations, which is to say, the persistence of finite dimensional invariant tori for linear equations. In this paper, we will discuss the existence of almost-periodic solutions for a nonlinear beam equation. As to almost-periodic solutions, via KAM theory, Bourgain [3] considered the nonlinear Schrödinger equation \( \sqrt{-1} u_t - u_{xx} + Mu + f(|u|^2)u = 0 \) with a periodic boundary condition, where \( M \) is a random Fourier multiplier. He proved that, for appropriate \( M \), the above equation has invariant tori of full dimension. While for fixed \( M \), by extracting parameters based on Birkhoff normal form, in [10] it has been proved that the equation admits a family of small-amplitude full-dimensional invariant tori. Niu and Geng [13] obtained almost periodic solutions for the case of higher dimensional beam equations.

The above equations do not depend on the forced terms. Physically, it means that there is no external force acting when the string is at rest, tending to distort its equilibrium of \( u \equiv 0 \). As to the case with a forced term, Zhang-Si [23] proved the existence of quasi-periodic solutions for the quasi-periodically forced nonlinear wave equation

\[ u_{tt} = u_{xx} - \mu u - \varepsilon \phi(t) h(u), \]  

(1.1)

where \( \mu > 0 \). If \( \mu = 0 \), the wave equation is completely resonant and we cannot directly extract parameters as usual. To overcome this difficulty, Yuan [22] has proved that (1.1) with \( \mu = 0 \) and \( \phi(t) \equiv 1 \) has some special solutions depending only on the space variable \( x \), and regarded

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such solutions as ‘parameters’. Along Yuan’s idea, Si [19] studied the quasi-periodic solutions for non-autonomous quasi-periodically forced nonlinear wave equation

\[ u_{tt} - u_{xx} + \phi(t, \varepsilon)u^3 = 0 \]

with periodic boundary conditions via infinite-dimensional KAM theory developed by Kuksin [12]. Besides, by the method of Lyapunov-Schmidt decomposition, Berti and Procesi [1] showed the existence of small amplitude quasi-periodic solutions with two frequencies \( \omega = (\omega_1, \omega_2) = (\omega_1, 1 + \varepsilon) \) for completely resonant wave equation with periodic forcing

\[
\begin{align*}
\frac{u_{tt}}{\sqrt{1 - \psi(t)}} - u_{xx} + mu + \phi(t)|u|^2u &= \varepsilon g(t), \\
u(t, x) &= u(t, x + 2\pi),
\end{align*}
\]

where the nonlinear forced term \( f(\omega_1 t, u) = a(\omega_1 t)u^{2d+1} + O(u^{2d+2}), \ d \in \mathbb{N} = \{1, 2, \cdots\}, \) is \( 2\pi/\omega_1 \)-periodic in time. By the similar method in [22], Rui and Si [15] turned the inhomogeneous Schrödinger equation

\[ \sqrt{-1}u_t - u_{xx} + \psi(t)(a_1 u + a_2 \bar{u}) = 0, \ t \in \mathbb{R}, \ x \in \mathbb{T}^1 \]

under periodic boundary conditions, the existence of almost-periodic solutions is discussed by Rui, Liu and Zhang [16]. Later on, based on reducibility via an improved KAM method, Rui-Liu [17] focused on almost-periodic solutions for the linear wave equation with almost-periodic forcing

\[ u_{tt} = u_{xx} - mu - \psi(\omega t, x)u, \ t \in \mathbb{R}, \ x \in [0, \pi] \]

subject to periodic boundary conditions. Both of the above equations with almost-periodic forcing are linear. A natural question is that whether or not there are some almost-periodic solutions for the nonlinear HPDEs with almost-periodic forcing. The aim of this paper is to discuss the existence of almost-periodic solutions.

In this paper, we consider the following nonlinear beam equation with almost-periodic forcing

\[ u_{tt} + u_{xxxx} + mu + \psi_0(\omega t) + \psi_1(\omega t)u + \psi_2(\omega t)u^2 + \psi_3(\omega t)u^3 = 0, \ t \in \mathbb{R}, \ x \in [0, 2\pi], \tag{1.2} \]

subject to the periodic boundary condition

\[ u(t, 0) = u(t, 2\pi), \ t \in \mathbb{R}, \tag{1.3} \]

where \( m > 0, \) and \( \psi_l(\omega t) \) \((l = 0, 1, 2, 3)\) are almost-periodic in time.

Notice that (1.2) is inhomogeneous if \( \psi_0(\omega t) \neq 0, \) therefore, it is easy to see that \( u \equiv 0 \) is not a solution of (1.2) + (1.3). Instead of the method in [19] to deal with inhomogeneous terms, we directly begin with the initial system. Moreover, since the forced terms are almost-periodic, the main difficulty is to deal with the infinitely many frequencies. Following the idea of Ruian Liu [17], we add some conditions on the almost-periodic forcing such that we can choose properly finite frequencies at each KAM step to deal with small divisors.
To state our conditions on the almost-periodic forcing, we introduce some notations.

Consider the frequencies $\omega$ of the almost-periodic function $\psi_l(\omega t)$ ($l = 0, 1, 2, 3$), and denote $\theta = \omega t \in \mathbb{T}^\infty$, where $\mathbb{T}^\infty$ is the infinite-dimensional torus. Let $O$ denote the closed set $[0, 1]^{\infty}$, that is

$$O = \{ \xi : \xi = (\xi_1, \xi_2, \cdots), \xi_j \in [0, 1], j = 1, 2, \cdots \}.$$ 

$\psi_l(\omega t)$, $l = 0, 1, 2, 3$ are almost-periodic functions. In this paper, we regard $\omega = (\omega_1, \cdots)$ as parameters. Since the parameter set $O$ is of finite dimension, we explain the positive measure in the sense of Remark 1.2. Actually, it is enough to assume the parameter set is of finite dimension at every KAM iteration. We write

$$\omega = (\omega_1, \omega_{i_2}, \cdots, \omega_{i_n}, \omega_{i_{n+1}}, \cdots) =: (\omega^n, \omega'_n) \in O^n \times O'_n \equiv O,$$

where $(i_1, i_2, i_3, \cdots)$ is a rearrangement of $(1, 2, 3, \cdots)$ and $O^n = [0, 1]^n$.

We choose a sequence $\{b_v\}$ satisfying $b_0 = b \geq 1$, $b_{v+1} > b_v$, and $b_v \in \mathbb{Z}^+$, $v = 0, 1, \cdots$. Let

$$I_v = \{ i_j : i_j \leq b_v, i_j \in \mathbb{Z}^+ \}, \; v = 0, 1, 2, \cdots,$$

thus, $I_\infty := \lim_{v \to \infty} I_v = \mathbb{Z}^+$. Denote

$$\omega^{b_0}_1 = (\omega_{i_1}, \cdots, \omega_{i_{b_0}}), \quad \omega^{b_{v+1}}_1 = (\omega_{i_{b(v+1)}}, \cdots, \omega_{i_{b(v+1)+1}}), \quad \omega^{b_v}_1 = (\omega_{i_1}, \cdots, \omega_{i_{b_v}});$$

$$\theta^{b_0}_1 = (\theta_{i_1}, \cdots, \theta_{i_{b_0}}), \quad \theta^{b_{v+1}}_1 = (\theta_{i_{b(v+1)}}, \cdots, \theta_{i_{b(v+1)+1}}), \quad \theta^{b_v}_1 = (\theta_{i_1}, \cdots, \theta_{i_{b_v}});$$

$$J^{b_0}_1 = (J_{i_1}, \cdots, J_{i_{b_0}}), \quad J^{b_{v+1}}_1 = (J_{i_{b(v+1)}}, \cdots, J_{i_{b(v+1)+1}}), \quad J^{b_v}_1 = (J_{i_1}, \cdots, J_{i_{b_v}}),$$

and let $[\psi]$ denotes the average of $\psi(\theta)$ on $\theta$, where $J_{i_k}$ is the action variable corresponding to angle variable $\theta_{i_k}$ later.

Throughout the paper, we assume that the small parameter $\varepsilon$ satisfies $0 < \varepsilon \ll 1$ and that the following assumptions (H1) and (H2) hold.

(H1) The functions $\psi_l(\omega t)$, $l = 0, 1, 2, 3$ are real analytic and almost-periodic in $t$ with frequencies $\omega$.

(H2) For $0 < \rho \leq 1$, $\psi_l(\theta) = \sum_{j=0}^{\infty} \varepsilon^{(1+\rho)^j} \psi^{(b_j)}_l(\theta^{b_j}_1)$, $l = 0, 1, 2, 3$ are absolutely convergent, $\psi_0(\theta) \neq 0$ and there exists an absolute constant $C_0$ such that

$$|\psi^{(b_j)}_l(\theta^{b_j}_1)| \leq C_0, \quad |\partial_{\theta^j_l} \psi^{(b_j)}_l(\theta^{b_j}_1)| \leq C_0, \quad j' \in I_0,$$

$$|\partial_{\theta^j_l} \psi^{(b_j)}_l(\theta^{b_j}_1)| \leq C_0, \quad j' \in I_j \setminus I_{j-1}, \quad l = 0, 1, 2, 3, \quad j = 1, 2, \cdots,$$

where $|\cdot|$ denotes the sup-norm on $\mathbb{T}^{b_j}$ and $\partial_{\theta^j_l}$, $f$ denotes the partial derivative of $f$ with respect to $\theta^j_l$.

**Theorem 1.1** Assume that the beam equation (1.2) with periodic boundary condition (1.3) satisfies the conditions (H1) and (H2). For $m > 0$ and $0 < \rho \leq 1$, there exists a positive measure Cantor-like subset $O^* \subset O$ such that for each $\omega = (\omega_1, \omega_2, \cdots)_{i_j \in I_\infty} \in O^*$, the beam equation (1.2) + (1.3) has an almost-periodic solution of the form

$$u(t, x) = \sum_{j \geq 0} q_j(\omega t) \cos(j x) \sqrt{\mu_j},$$
where \( \mu_j = \sqrt{j^4 + m} \), \( q_j(\omega t) \), \( j = 0, 1, \cdots \) are almost-periodic in \( t \) with frequencies \( \omega \) and
\[
\sup_{\theta \in \mathbb{T}} \| q(\theta) \|_{\ell^p} = O(\varepsilon^{\frac{1}{2} - \frac{1}{p^*}}) \text{ with } p > 0.
\]

**Remark 1.2** Let the set \( \mathcal{O} = [0, 1]^{\infty} \) with probability measure. We say \( \mathcal{O}^* \) is a set of large measure in \( \mathcal{O} \) if there exists a real number \( \gamma > 0 \) and \( 0 < \varepsilon \ll 1 \) such that the following inequality holds
\[
\text{meas}(\mathcal{O} \setminus \mathcal{O}^*) \leq C \varepsilon^\gamma,
\]
where meas is the standard probability measure on \([0, 1]\) and \( C > 0 \) is an absolute constant.

**Remark 1.3** Theorem 1.1 still holds for \( \rho > 1 \). In this case, set \( \varepsilon_v = \varepsilon^{\frac{1}{2}(1 + \frac{1}{p^*})} \), then we obtain a almost-periodic solution with the same form as that in Theorem 1.1 and \( \varepsilon \) satisfies
\[
\sup_{\theta \in \mathbb{T}} \| q(\theta) \|_{\ell^p} = O(\varepsilon^{\frac{1}{2} - \frac{1}{p^*}}).
\]

Here is the outline of the proof of Theorem 1.1. Consider the perturbed beam equation
\[
u_{tt} + u_{xxxx} + mu + \psi_0(\omega t) + \psi_1(\omega t)u + \psi_2(\omega t)u^2 + \psi_3(\omega t)u^3 = 0, \quad t \in \mathbb{R}, \quad x \in [0, 2\pi], \theta = \omega t.
\]
It is easy to check that the above equation corresponds to a Hamiltonian equation
\[
\begin{align*}
\dot{u} &= \frac{\partial H}{\partial v} = v \\
\dot{v} &= -\frac{\partial H}{\partial u} = -(u_{xxxx} + mu + \psi_0(\theta) + \psi_1(\theta)u + \psi_2(\theta)u^2 + \psi_3(\theta)u^3)
\end{align*}
\]
with the Hamiltonian
\[
H = \frac{1}{2} \langle v, v \rangle + \frac{1}{2} \langle Au, u \rangle + \int_0^{2\pi} \psi_0(\theta) u dx + \frac{1}{2} \int_0^{2\pi} \psi_1(\theta) u^2 dx \\
&\quad + \frac{1}{3} \int_0^{2\pi} \psi_2(\theta) u^3 dx + \frac{1}{4} \int_0^{2\pi} \psi_3(\theta) u^4 dx,
\]
where \( A = \partial_{xxxx} + m \). By Fourier transformation, the above Hamiltonian can be turned into the infinite dimensional Hamiltonian system with Hamiltonian
\[
H = N + P(\theta, z, \bar{z}, \omega),
\]
with \( z, \bar{z} \in \ell^{a,p}(\mathbb{C}) := \{ z = (z_0, z_1, z_2, \cdots) : z_i \in \mathbb{C}, |z_0|^2 + \sum_{j \geq 1} |z_j|^2 j^{2p} e^{2ja} < \infty \} \) for \( a > 0 \) and \( p > 0 \). Since the forced term \( \psi_l(\theta)(l = 0, 1, 2, 3) \) is an almost-periodic function with infinite frequencies \( \omega = (\omega_1, \cdots) \), we have to face the problem that how to treat infinite frequencies in the procedure of constructing almost-periodic solutions. But through observation, note that the forced terms still have some good ‘properties’ which take part in proving Theorem 1.1. \( \psi_l(\theta) \) can be written as the sum of infinite functions with each one depending on a finite dimensional vector \( \theta_l^j(j = b_0, b_1, \cdots) \). In detail,
\[
\psi_l(\theta) = \psi_{l0}^1(\theta_{l0}^0) + \varepsilon(1 + \rho)^n \psi_{l1}^1(\theta_{l1}^0) + \varepsilon(1 + \rho)^2 \psi_{l2}^1(\theta_{l2}^0) \cdots.
\]
According to the property of \( \psi_l(\theta) \) (\( l = 0, 1, 2, 3 \)), the perturbation \( P(\theta, z, \bar{z}) \) of the Hamiltonian \( H = N + P \) can also be written as
\[
P(\theta, z, \bar{z}) = \sum_{n \geq 0} \varepsilon(1 + \rho)^n \tilde{P}(\theta_{1n}^0, z, \bar{z}, \omega_{1n}^0).
\]
Therefore, we will construct the almost-periodic solution as follows. Firstly, we split the perturbation $P$ into two parts, that is

$$P(\theta, z, \bar{z}) = \tilde{P}^{b_0}(\theta^{b_0}, z, \bar{z}, \omega^{b_0}) + \sum_{n \geq 1} \varepsilon^{(1 + \rho)^n} \tilde{P}^{b_n}(\theta^{b_n}, z, \bar{z}, \omega^{b_n}),$$

where the former is the part depending only on $z, \bar{z}$ and the finite dimensional vectors $\theta^{b_0}, \omega^{b_0}$, while, the latter is the part not depending on $\theta^{b_0}$. Secondly, split $\tilde{P}^{b_0}(\theta^{b_0}, z, \bar{z}, \omega^{b_0})$ into two parts, $\tilde{P}^{b_0} = (\tilde{P}^{b_0})^{\text{low}} + (\tilde{P}^{b_0})^{\text{high}},$ where

$$(\tilde{P}^{b_0})^{\text{low}} = \sum_{\gamma, \kappa \in \mathbb{N}^1, |\gamma|_1 + |\kappa|_1 \leq 2} (\tilde{P}^{b_0})^{\gamma \kappa}(\theta^{b_0}, \omega^{b_0}) z^\gamma \bar{z}^\kappa$$

is the part of low order, and

$$(\tilde{P}^{b_0})^{\text{high}} = \sum_{\gamma, \kappa \in \mathbb{N}^1, |\gamma|_1 + |\kappa|_1 \geq 3} (\tilde{P}^{b_0})^{\gamma \kappa}(\theta^{b_0}, \omega^{b_0}) z^\gamma \bar{z}^\kappa$$

is the part of high order. Expand $(\tilde{P}^{b_0})^{\gamma \kappa}(\theta^{b_0}, \omega^{b_0})$ into Fourier series

$$(\tilde{P}^{b_0})^{\gamma \kappa}(\theta^{b_0}, \omega^{b_0}) = \sum_{k \in \mathbb{Z}^{b_0}} (\tilde{P}^{b_0})^{\gamma \kappa}_k e^{\sqrt{\text{I}(k, \theta^{b_0})}}.$$

As the standard KAM procedure, we will remove all non-normalized terms

$$\sum_{|\gamma|_1 + |\kappa|_1 \leq 2} \sum_{|k| + |\gamma - \kappa| \neq 0} (\tilde{P}^{b_0})^{\gamma \kappa}_k e^{\sqrt{\text{I}(k, \theta^{b_0})}} z^\gamma \bar{z}^\kappa$$

in $(\tilde{P}^{b_0})^{\text{low}}$ by a symplectic transformation $\Phi_0$. After the first step, we obtain the new Hamiltonian

$$H_1 = N_1 + P_1$$

where the new perturbation $P_1$ can be written in the form

$$P_1 = \hat{P}_1(\theta^{b_1}, z, \bar{z}, \omega^{b_1}) + \sum_{n \geq 2} \varepsilon^{(1 + \rho)^n} \tilde{P}^{b_n}(\theta^{b_n}, z, \bar{z}, \omega^{b_n}) \circ \Phi_0.$$

It is easy to check that $\hat{P}_1(\theta^{b_1}, z, \bar{z}, \omega^{b_1})$ depends only on $z, \bar{z}$ and the finite dimensional vectors $\theta^{b_1}, \omega^{b_1}$. As the first step, we will split $\hat{P}_1$ into $\hat{P}_1^{\text{low}}$ and $\hat{P}_1^{\text{high}}$ and remove the non-normalized terms in $\hat{P}_1^{\text{low}}$. In the end, after infinite transformations, we obtain a non-degenerate normal form

$$H^\infty = \langle \omega, J \rangle + \langle \Omega, z \bar{z} \rangle + \sum_{|\gamma|_1 + |\kappa|_1 \geq 3} \hat{P}^{\gamma \kappa}(\theta, \omega) z^\gamma \bar{z}^\kappa,$$

where $\hat{\Omega}_j$ is close to the eigenvalue $\mu_j$ of operator $A$. Finally, basing on the above normal form, we get the almost-periodic solution in Theorem 1.1.

In the process of removing the non-normalized terms, the following non-resonant conditions are needed,

$$|\langle k, \omega^{b_n} \rangle + \langle l, \Omega_v \rangle| \geq \frac{\alpha_v(l)2}{(1 + v^2)(|k| + 1)^{2b_v + 2}}.$$
for any \((k, l) \in \mathbb{Z}^{b_v} = \{(k, l) \neq 0, |l| \leq 2\} \subset \mathbb{Z}^{b_v} \times \mathbb{Z}^\infty\), where \(v\) represents the iteration step. It is known that the standard non-resonant conditions depend on the dimension of torus. In the present paper, the dimension of torus increases with the iteration. Therefore, the non-resonant conditions depend on iteration steps.

The rest of this paper is organized as follows. In Section 2, we introduce the Hamiltonian setting of \((1.2)+(1.3)\). Sections 3–5 are devoted to the proof of Theorem 1.1. Some technical lemmas are listed in Appendix.

2 Hamiltonian setting

The system \((1.2)+(1.3)\) can be written as a Hamiltonian system

\[
\begin{align*}
\dot{u} &= \frac{\partial H}{\partial v} = v \\
\dot{v} &= -\frac{\partial H}{\partial u} = -\left(u_{xxxx} + mu + \psi_0(\omega t) + \psi_1(\omega t)u + \psi_2(\omega t)u^2 + \psi_3(\omega t)u^3\right)
\end{align*}
\]

with the Hamiltonian

\[
H = \frac{1}{2} \langle v, v \rangle + \frac{1}{2} \langle Au, u \rangle + \int_0^{2\pi} \psi_0(\omega t)udu + \frac{1}{2} \int_0^{2\pi} \psi_1(\omega t)u^2dx + \frac{1}{3} \int_0^{2\pi} \psi_2(\omega t)u^3dx + \frac{1}{4} \int_0^{2\pi} \psi_3(\omega t)u^4dx,
\]

where \(A = \partial_{xxxx} + m\). The eigenvalues of the operator \(A\) with the periodic boundary condition are \(\mu_j^2 = j^4 + m, j \in \mathbb{Z}\), corresponding eigenfunction \(\phi_j(x) \in L^2[0, 2\pi]\)

\[
\phi_j(x) = \begin{cases} \\
\frac{1}{\sqrt{\pi}} \cos jx, & j > 0, \\
\frac{-1}{\sqrt{\pi}} \sin jx, & j < 0, \\
\frac{1}{\sqrt{2\pi}} & j = 0.
\end{cases}
\]

In order to avoid the double eigenvalues, we restrict ourselves to find some solutions which are even in \(x\). \(\{\phi_j : j \geq 0\}\) is a complete orthogonal basis of a subspace in \(L^2[0, 2\pi]\).

We introduce coordinates \(q = (q_0, q_1, q_2, \cdots)\) and \(\chi = (\chi_0, \chi_1, \chi_2, \cdots)\) by the following relations

\[
u(t, x) = \sum_{j \geq 0} \frac{q_j(t)}{\sqrt{\mu_j}} \phi_j(x), \quad v(t, x) = \sum_{j \geq 0} \sqrt{\mu_j} \chi_j(t) \phi_j(x).
\]

The coordinates are taken from some real Hilbert space

\[
\ell^{a,p} = \ell^{a,p}(\mathbb{R}) := \{q = (q_0, q_1, q_2, \cdots) : q_j \in \mathbb{R}, j \geq 0\}
\]

with norm

\[
\|q\|_{a,p}^2 = |q_0|^2 + \sum_{j \geq 1} |q_j|^2 j^{2p} e^{2ja} < \infty.
\]

In the sequel, we assume that \(a \geq 0\) and \(p > 0\).
We introduce a pair of action-angle variables \((J, \theta) \in \mathbb{R}^\infty \times T^\infty\) such that the Hamiltonian is autonomous. We then obtain the Hamiltonian
\[
H = \langle \omega, J \rangle + \frac{1}{2} \sum_{j \geq 0} \mu_j \left( \chi_j^2 + q_j^2 \right)
\]
\[
+ \int_0^{2\pi} \psi_0(\theta) \sum_{j \geq 0} \frac{q_j(t)}{\sqrt{\mu_j}} \phi_j(x) dx + \frac{1}{2} \int_0^{2\pi} \psi_1(\theta) \left( \sum_{j \geq 0} \frac{q_j(t)}{\sqrt{\mu_j}} \phi_j(x) \right)^2 dx
\]
\[
+ \frac{1}{3} \int_0^{2\pi} \psi_2(\theta) \left( \sum_{j \geq 0} \frac{q_j(t)}{\sqrt{\mu_j}} \phi_j(x) \right)^3 dx + \frac{1}{4} \int_0^{2\pi} \psi_3(\theta) \left( \sum_{j \geq 0} \frac{q_j(t)}{\sqrt{\mu_j}} \phi_j(x) \right)^4 dx
\]
with equations of motions
\[
\dot{\theta} = \omega, \quad \dot{J} = -\frac{\partial H}{\partial \theta}, \quad \dot{q}_j = \frac{\partial H}{\partial \chi_j}, \quad \dot{\chi}_j = -\frac{\partial H}{\partial q_j}, \quad j \geq 0
\]
with respect to the symplectic structure \(d\theta \wedge dJ + \sum_{j \geq 0} dq_j \wedge d\chi_j\). We introduce complex coordinates
\[
z_j = \frac{1}{\sqrt{2}} (q_j - \sqrt{-1} \chi_j), \quad \bar{z}_j = \frac{1}{\sqrt{2}} (q_j + \sqrt{-1} \chi_j), \quad j \geq 0,
\]
which are in the complex Hilbert space \(\ell^{a,p}(\mathbb{C})\). Here
\[
\ell^{a,p}(\mathbb{C}) := \{ z = (z_0, z_1, z_2, \cdots) : z_j \in \mathbb{C}, j \geq 0 \}
\]
with finite norm
\[
\| z \|^{2}_{a,p} = | z_0 |^2 + \sum_{j \geq 1} | z_j |^2 j^{2a} e^{2ja} < \infty.
\]
We then obtain the Hamiltonian
\[
H = \sum_{j=1}^{\infty} \omega_{ij} J_{ij} + \sum_{j \geq 0} \mu_j z_j \bar{z}_j + G_1 + G_2 + G_3 + G_4 \quad (2.2)
\]
and the symplectic structure \(d\theta \wedge dJ + \sqrt{-1} \sum_{j \geq 0} d\bar{z}_j \wedge dz_j\), where
\[
G_1 = \sqrt{\pi} \psi_0(\theta) \frac{z_0 + \bar{z}_0}{\sqrt{\mu_0}}, \quad G_2 = \frac{1}{4} \psi_1(\theta) \sum_{j \geq 0} \frac{(z_j + \bar{z}_j)^2}{\mu_j}, \quad (2.3)
\]
\[
G_3 = \frac{\sqrt{2}}{12} \psi_2(\theta) \sum_{i \neq j \neq k \neq l \neq 0, i, j, k, l \geq 0} \frac{G_{ijkl}^3}{\mu_i \mu_j \mu_k \mu_l} (z_i + \bar{z}_i)(z_j + \bar{z}_j)(z_k + \bar{z}_k)(z_l + \bar{z}_l), \quad (2.4)
\]
\[
G_4 = \frac{1}{16} \psi_3(\theta) \sum_{i \neq j \neq k \neq l \neq 0, i, j, k, l \geq 0} \frac{G_{ijkl}^4}{\mu_i \mu_j \mu_k \mu_l} (z_i + \bar{z}_i)(z_j + \bar{z}_j)(z_k + \bar{z}_k)(z_l + \bar{z}_l), \quad (2.5)
\]
\[
G_{ijkl}^3 = \int_0^{2\pi} \phi_i \phi_j \phi_k \phi_l dx, \quad G_{ijkl}^4 = \int_0^{2\pi} \phi_i \phi_j \phi_k \phi_l dx.
\]
It is easy to prove that \(G_{ijkl}^3 = 0\) unless \(i \pm j \pm l = 0\), and \(G_{ijkl}^4 = 0\) unless \(i \pm j \pm k \pm l = 0\).
According to the assumption (H2) on $\psi_l(\theta)$ ($l = 0, 1, 2, 3$), we split $G$ with $G = G_1 + G_2 + G_3 + G_4$ into infinitely many parts, that is $G = \sum_{n \geq 0} e^{(1+p)n} \tilde{P}^{bn}$, where $\tilde{P}^{bn}$ only depends on $z, \bar{z}, \omega^{bn}$ and angle variables $\theta_1^{bn}$. Then we just need to treat finitely many frequencies at each step. More precisely, we write (2.2) as
\[
H = \langle \omega, J \rangle + \sum_{j \geq 0} \mu_j z_j \bar{z}_j + \sum_{n \geq 0} e^{(1+p)n} \tilde{P}^{bn}
\] (2.6)
with
\[
\tilde{P}^{bn} = \sqrt{\frac{n}{m}} \psi_0^{bn}(\theta_1^{bn})(z_0 + \bar{z}_0) + \frac{1}{4} \psi_1^{bn}(\theta_1^{bn}) \sum_{j \geq 0} \frac{1}{\mu_j} (z_j^2 + 2z_j \bar{z}_j + \bar{z}_j^2)
\]
\[+ \frac{\sqrt{2}}{12} \psi_2^{bn}(\theta_1^{bn}) \sum_{i \pm j \pm l = 0, i, j, l \geq 0} G^3_{ijl} \mu_i \mu_j \mu_l (z_i + \bar{z}_i)(z_j + \bar{z}_j)(z_l + \bar{z}_l)
\]
\[+ \frac{1}{16} \psi_3^{bn}(\theta_1^{bn}) \sum_{i \pm j \pm k \pm l = 0, i, j, k, l \geq 0} G^4_{ijkl} \mu_i \mu_j \mu_k \mu_l (z_i + \bar{z}_i)(z_j + \bar{z}_j)(z_k + \bar{z}_k)(z_l + \bar{z}_l),
\] (2.7)
where $\psi_l^{bn}(\theta_1^{bn}), l = 0, 1, 2, 3$ are defined in the assumption (H2).

We define the Hamiltonian vector field of a Hamiltonian $Q$,
\[X_Q = (Q_J, -Q_\theta, -\sqrt{-1}Q_z, \sqrt{-1}Q_{\bar{z}}).
\]
To obtain the analyticity of $X_G$, it is convenient to introduce coordinates
\[w = (\cdots, w_{-1}, w_0, w_1, \cdots) \in \ell^a_{b,p}
\]
by setting $z_j = w_j$, $\bar{z}_j = w_{-j}$, where $\ell^a_{b,p}$ consists of all bi-infinite sequences with finite norm
\[\|w\|_{a,p}^2 = |w_0|^2 + |w_{-0}|^2 + \sum_{|j| \geq 1} \sum_{|l| \geq 1} |w_j|^2 |l|^2 \epsilon_{2p} |l|^a.
\]
Substituting $z_j = w_j$ and $\bar{z}_j = w_{-j}$ into (2.3), (2.4) and (2.5), then we obtain
\[G_1 = \sqrt{\pi} \psi_0(\theta) \frac{w_0 + w_{-0}}{\mu_0}, \quad G_2 = \frac{1}{4} \psi_1(\theta) \sum_{j \geq 0} \frac{(w_j + w_{-j})^2}{\mu_j},
\]
\[G_3 = \frac{\sqrt{2}}{12} \psi_2(\theta) \sum_{i \pm j \pm l = 0, i, j, l \in \mathbb{Z}} G^3_{ijl} \mu_i \mu_j \mu_l w_i w_j w_l,
\]
\[G_4 = \frac{1}{16} \psi_3(\theta) \sum_{i \pm j \pm k \pm l = 0, i, j, k, l \in \mathbb{Z}} G^4_{ijkl} \mu_i \mu_j \mu_k \mu_l w_i w_j w_k w_l.
\]

**Lemma 2.1** ([22, Lemma 2]) For $a \geq 0$ and $p > \frac{1}{2}$, the space $\ell^a_{b,p}$ is a Hilbert algebra with respect to convolution of sequences, and
\[\|q \ast h\|_{a,p} \leq c \|q\|_{a,p} \|h\|_{a,p}\]
with a constant $c$ depending only on $p$.

**Lemma 2.2** For $a \geq 0$, and $p > 0$, the Hamiltonian vector field $X_H$ is real analytic as a map from some neighbourhood of the origin in $\ell^a_b$ into $\ell^{a+2}_b$, with $\|X_H\|_{a,p+2} \leq C\varepsilon$ uniformly in $\theta \in \mathbb{T}^\infty$, where $C$ is a positive constant.

**Proof** Set $\tilde{w}_j = \frac{1}{\sqrt{\mu_j}}(|w_j| + |w_{-j}|)$. By the assumption (H2) and $G_{ijkl}^4 = \int_0^{2\pi} \phi_i \phi_j \phi_k \phi_l dx$, we have

$$|\partial_{w_i} G_4| \leq \frac{1}{16} |\psi_3(\theta)| \sum_{i+j+k+l, i, j, k, l \in \mathbb{Z}} \frac{|G_{ijkl}^4|}{\sqrt{\mu_i \mu_j \mu_k \mu_l}} |w_i w_j w_k|$$

$$\leq \frac{C\varepsilon}{\sqrt{\mu_l}} \sum_{i+j+k+l} \tilde{w}_i \tilde{w}_j \tilde{w}_k$$

$$= \frac{C\varepsilon}{\sqrt{\mu_l}} (\tilde{w} * \tilde{w} * \tilde{w})_l.$$ 

If $w \in \ell^a_b$, then $\tilde{w} \in \ell^{a+1}_b$, and for $p > 0$, the latter is a Hilbert algebra by Lemma 2.1. Therefore, $\tilde{w} * \tilde{w} * \tilde{w}$ also belongs to $\ell^{a+1}_b$, and $(G_4)_w \in \ell^{a+2}_b$ with

$$\|G_4\|_{a,p+2} \leq C\varepsilon \|\tilde{w} * \tilde{w} * \tilde{w}\|_{a,p+1} \leq C\varepsilon \|w\|_{a,p}^3.$$ 

Similarly,

$$\|G_3\|_{a,p+2} \leq C\varepsilon \|w\|_{a,p}^2.$$ 

We also obtain that

$$\|G_1\|_{a,p+2} = \|\partial_{w_0} G_1\|^2 + \|\partial_{w_{-0}} G_1\|^2 = 2 \left| \frac{\sqrt{\pi} \psi_0(\theta)}{\sqrt{\mu_0}} \right|^2 \leq C\varepsilon^2;$$

and

$$|\partial_{w_j} G_2| = \frac{1}{2} |\psi_1(\theta)| \frac{w_j + w_{-j}}{\mu_j} \leq C\varepsilon \frac{w_j}{\sqrt{\mu_j}}.$$ 

Hence,

$$\|G_2\|_{a,p+2} \leq C\varepsilon \|\tilde{w}\|_{a,p+1} \leq C\varepsilon \|w\|_{a,p}.$$ 

The proof of Lemma 2.2 is complete. \(\square\)

Let

$$H = N + P = \sum_{j=0}^{\infty} \omega_j J_j + \sum_{j \geq 0} \Omega_j(\omega) z_j \bar{z}_j + P(\theta, z, \bar{z}, \omega)$$

(2.8)

be a Hamiltonian defined on a phase space $\mathcal{P}^{a,p} := \mathbb{T}^\infty \times \mathbb{R}^\infty \times \ell^a \times \ell^a$, with the normal form $N = \sum_{j=1}^{\infty} \omega_j J_j + \sum_{j \geq 0} \Omega_j(\omega) z_j \bar{z}_j$, and the perturbation $P(\theta, z, \bar{z}, \omega) = \sum_{n \geq 0} \varepsilon^{(1+p)^n} \hat{P}^{b_n}$, where $\omega = (\omega_1, \omega_2, \cdots)$ and $\Omega(\omega) = (\Omega_0, \Omega_1, \cdots)$ represent respectively, the tangent and normal frequencies. Assume $P$ is analytic with respect to $\theta$, $z$, $\bar{z}$ and Lipschitz continuous in $\omega \in \mathcal{O}$. Note that, in this paper, we regard the tangent frequencies $\omega$ as parameters. When the perturbation vanishes, it is clear that $\mathcal{T}_0^{\infty} := \mathbb{T}^\infty \times \{0\} \times \{0\} \times \{0\}$ depending on $\omega$ are infinitely dimensional invariant tori. Whether can these tori persist if $P$ is sufficiently small? In the following, we will prove that most of them can survive by the KAM method. Firstly, we introduce some norms and notations.
The complex neighbourhood of torus $T_0^\infty$ is defined by
\[ D(s, r) : |\text{Im}\theta| < s, \quad |J| < r^2, \quad ||z||_{a,p} < r, \quad ||\bar{z}||_{a,p} < r, \]
where $| \cdot |$ denotes the sup-norm for complex vectors, and weighted phase space norms are defined by
\[ |W|_r = |W|_{r,a,p+2} = |X| + \frac{1}{r^2}|Y| + \frac{1}{r}||U||_{a,p+2} + \frac{1}{r}||V||_{a,p+2}, \quad (2.9) \]
for $W = (X, Y, U, V) \in \mathcal{P}^{a,p+2}$.

Furthermore, we assume that the Hamiltonian vector field $X_P$ is real analytic on $D(s, r)$ for some positive $s, r$ uniformly in $\omega \in \mathcal{O}$ with finite norm $|X_P|_{r,D(s,r) \times \mathcal{O}} = \sup_{D(s,r) \times \mathcal{O}} |X_P|_r$, and that the same holds for its Lipschitz semi-norm
\[ |X_P|_{\text{lip}, \mathcal{O}} = \sup_{\omega, \bar{\omega} \in \mathcal{O}, \omega \neq \bar{\omega}} \frac{|\Delta_{\omega, \bar{\omega}} X_P|_r}{|\omega - \bar{\omega}|}, \quad |X_P|_{\text{lip}, D(s,r) \times \mathcal{O}} = \sup_{D(s,r)} |X_P|_{r, \mathcal{O}}, \]
where $\Delta_{\omega, \bar{\omega}} X_P = X_P(\cdot, \omega) - X_P(\cdot, \bar{\omega})$. Fixing $-2 \leq \delta \leq 0$, the Lipschitz semi-norm of the frequencies $\Omega(\omega)$ are defined by
\[ |\Omega|_{\text{lip}, \mathcal{O}} = \sup_{\omega, \bar{\omega} \in \mathcal{O}, \omega \neq \bar{\omega}} \sup_{j \geq 0} \frac{j^{-\delta} |\Delta_{\omega, \bar{\omega}} \Omega_j|}{|\omega - \bar{\omega}|}. \]

For $\lambda \geq 0$, define
\[ |X_P|_{r, *}^\lambda = |X_P|_{r, * + \lambda |X_P|_{r, *}}, \]
where $*$ represents a set of variables (for example, $D(s, r) \times \mathcal{O}$), the symbol ‘$\ast$’ will always be used in this role and never have the meaning of exponentiation. Moreover, we introduce the notations
\[ \langle l \rangle_2 = \max(1, |\sum j^2 l_j|), \quad \mathcal{Z}^{b_v} = \{(k, l) \neq 0, |l| \leq 2\} \subset \mathcal{Z}^{b_v} \times \mathcal{Z}^\infty. \]

3 Iteration lemma and its proof

To state and prove the iterative lemma, we introduce some iterative constants and notations. Let $\varepsilon, s, r$ and $\rho$ be positive. Let $v \geq 0$ be the $v$-th KAM step, and set

1. $\varepsilon_0 = \varepsilon^{\frac{1}{2}}$, and $\varepsilon_{v+1} = \varepsilon_v^{1+\frac{\rho}{2}}, \rho > 0, \varepsilon_1 \leq 1$;
2. $\alpha_v = \varepsilon_v^{\frac{1+\rho}{2}}, M_v + 1 = (M_1 + 1)(2 - 2^{-v+1}) (v \geq 1), M_0 = 0, M_1 = \varepsilon_0^{1-\frac{\rho}{2}}, \lambda_v = \frac{\alpha_v}{M_v+1};$
3. $\sigma_{v+1} = \frac{\alpha_v}{2}, s_{v+1} = s_v - 6\sigma_v, s_0 = s$ as initial value, fix $\sigma_0 = s_0/24 \leq 1/20$ so that $s_0 > s_1 > \cdots \geq s_0/2$;
4. $r_v = (1 - \tau_v)r_0, \text{ with } \tau_0 = 0, r_0 = r, \tau_v = (1^2 + \cdots + v^{-2})/(2 \sum_{j=1}^\infty j^{-2}) (v \geq 1), \text{ and } d_v = \frac{1}{4}(r_v - r_{v+1}) = r_0/[8(v+1)^2 \sum_{j=1}^\infty j^{-2}], r_0 > r_1 > \cdots \geq r_0/2$;
5. $D_v = D(s_v, r_v) = \{(\theta, J, z, \bar{z}) \in \mathbb{C}^\infty/2\pi \mathbb{Z}^\infty \times \mathbb{C}^\infty \times \ell^{a,p} \times \ell^{a,p} : \text{Im}\theta < s_v, |J| < r_v^2, ||z||_{a,p} < r_v, ||\bar{z}||_{a,p} < r_v\};$
\[ \mathcal{O}_v^* = \{ \omega : \omega = (\omega_i, \omega_{i^v}, \omega_{i^v}, \omega_{(i^v+1)}, \ldots) \in \mathcal{O}_v^* \times \mathcal{O}_{b_v}, \ i_j \in \mathbb{Z}^+ \}, \]

where \( \mathcal{O}_{b_v} \) is the closed set of sequences \( \omega_{b_v} = (\omega_{i_{b_v+1}}, \omega_{i_{b_v+2}}, \ldots) \) with \( j \in [0, 1], j = i_{b_v+1}, \ldots \), and

\[ \mathcal{O}_{b_v} = \mathcal{O}^v \setminus \bigcup_{k,l} \mathcal{R}_{kl}^v, \]

\[ \mathcal{O}_v = \{ \omega_{b_v} : (\omega_{b_v}, \omega_{b_v}) \in \mathcal{O}_{v-1}^* \} \subset [0, 1]^{b_v}, \quad \mathcal{O}_{v-1} = \mathcal{O}, \]

\[ \mathcal{R}_{kl}^v = \{ \omega_{b_v} \in \mathcal{O}^v : |\langle k, \omega_{b_v} \rangle + \langle l, n_{v} \rangle | < \frac{\alpha_{v}(l)2}{(1+v^2)(|l| + 1)^{2n_v+2}} \}, \quad (k, l) \in \mathcal{Z}_{b_v}. \]

### 3.1 Iterative lemma

We have obtained the Hamiltonian (2.6) of (2.1), which is of the form (2.8) with the normal frequencies \( \Omega = (\mu_0, \mu_1, \mu_2, \ldots) \) and the perturbation \( P = \sum_{n \geq 0} \varepsilon^{(1+\rho)n} \bar{P}^{bn} \). By Lemma 2.2 and the assumption (H1), it follows that \( X_{\bar{G}} \) is real analytic in \( D(s, r) \) for some positive \( s, r \) uniformly in \( \omega \in \mathcal{O} \), Lipschitz continuous in \( \omega \in \mathcal{O} \) and \( |\Omega|^{lip} = 0. \)

Assume that at the \( n \)-th step of scheme, a Hamiltonian

\[ H_n = N_v + P_v = N_v + \hat{P}^v(\theta_{b_v}, z, z, \omega_{b_v}) + \sum_{n \geq v+1} \varepsilon^{(1+\rho)n} \hat{P}^{bn}(\theta_{b_v}^{bn}, z, z, \omega_{b_v}^{bn}) \circ \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{v-1}, \]

is considered as a small perturbation of some normal form \( N_v \). Split \( \hat{P}_v \) into two parts, that is \( \hat{P}_v = (\hat{P}_v)^{low} + (\hat{P}_v)^{high} \), where the low-degree terms \( (\hat{P}_v)^{low} \), denoted by \( \varepsilon_v R_{b_v} \) in the following, is defined by

\[ \varepsilon_v R_{b_v} = (\hat{P}_v)^{low} = \sum_{|\gamma| + |\kappa| \leq 2} \hat{P}^{\gamma \kappa}(\theta_{b_v}, \omega_{b_v}) z^{\gamma} \bar{z}^{\kappa}, \]

\[ R_{b_v} = R^{00b_v} + (R^{10b_v}, z) + (R^{01b_v}, \bar{z}) + (R^{20b_v}, z, z) + (R^{11b_v}, z, \bar{z}) + (R^{02b_v}, \bar{z}, \bar{z}) \]

\[ = \sum_{k \in \mathbb{Z}^{b_v}} R_k^{00b_v} e^{\sqrt{-1}(k, \theta_{b_v})} + \sum_{j \geq 0, k \in \mathbb{Z}^{b_v}} (R_k^{10b_v} z_j + R_k^{01b_v} \bar{z}_j) e^{\sqrt{-1}(k, \theta_{b_v})} \]

\[ + \sum_{i,j,k \geq 0, k \in \mathbb{Z}^{b_v}} (R_{kij}^{20b_v} z_i z_j + R_{kij}^{11b_v} z_i \bar{z}_j + R_{kij}^{02b_v} \bar{z}_i \bar{z}_j) e^{\sqrt{-1}(k, \theta_{b_v})} \]

and the high-degree terms \( (\hat{P}_v)^{high} \) of \( \hat{P}_v \) by

\[ (\hat{P}_v)^{high} = \sum_{|\gamma| + |\kappa| \geq 3} \hat{P}^{\gamma \kappa}(\theta_{b_v}, \omega_{b_v}) z^{\gamma} \bar{z}^{\kappa}. \]

The product \( z^{\gamma} \bar{z}^{\kappa} \) denotes \( \prod_n z_0^{\gamma_n} \bar{z}_n^{\kappa_n} \), where \( \gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n, \ldots) \), \( \kappa = (\kappa_0, \kappa_1, \ldots, \kappa_n, \ldots) \) with finitely many non-zero components and \( \gamma_n, \kappa_n \in \mathbb{N} \).

**Lemma 3.1** Suppose that \( H_v = N_v + P_v \) (\( v \geq 0 \)) is given on \( D_v \times \mathcal{O}_v^* \), where

\[ N_v = \langle \omega_{b_v}, J_{b_v} \rangle + \sum_{n \geq v+1} \langle \omega_{b_v}^{bn}, J_{b_v}^{bn} \rangle + \sum_{j \geq 0} \Omega_{aj}(\omega_{b_v}^{j+1}) z_j \bar{z}_j \]
is a normal form satisfying
\[ |\langle k, \omega^b \rangle + \langle l, \Omega_v \rangle| \geq \frac{\alpha_v \langle l \rangle}{(1 + v^2)(|k| + 1)(2v + 2)}, \quad (k, l) \in \mathcal{Z}_v^b, \]
\[ |\Omega_v|^{lip}_{\delta, \Omega^*_v} \leq M_v, \quad (3.1) \]
\[ \Omega_0 = \mu_j = \sqrt{j^4 + m}, \quad \Omega_{v_j} = \Omega_0 + \sum_{s=0}^{\nu-1} \xi^s [B_{jj}^{11b}], \quad \Omega_v = (\Omega_{v_0}, \Omega_{v_1}, \Omega_{v_2}, \ldots), \quad \nu \geq 1 \quad (3.2) \]

with \( B_{jj}^{11b} \) defined in Section 3.2 and \( P_v \) satisfies
\[ P_v = \hat{P}_v + \sum_{n \geq v+1} \epsilon^{(1+\rho)n} \hat{P}^{b0}(\theta^{b0}_1, z, \tilde{z}, \omega^{b0}_1) \circ \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_{v-1} \]
with
\[ \hat{P}_v = \epsilon_v R^{b0}(\theta^{b0}, z, \tilde{z}, \omega^{b0}) + \hat{P}^{b0}(\rho^{b0}, z, \tilde{z}, \omega^{b0}), \]
\[ |X_{\hat{P}^{b0}}|_{r_v, D(s_v, r_v) \times \Omega_v^*} \leq \epsilon_0 + \sum_{j=1}^{v} \frac{1}{j} \epsilon_j \quad (3.3) \]
\[ |X_{R^{b0}}|_{r_v, D(s_v, r_v) \times \Omega_v^*} \leq \frac{1}{2}, \quad |X_{\hat{P}^{b0}}|_{r_v, D(s_v, r_v) \times \Omega_v^*} \leq \epsilon_v. \quad (3.4) \]

Then there exists a Lipschitz family of real analytic symplectic coordinate transformations \( \Phi_0 : D_{v+1} \times \mathcal{O}_v^* \to D_v \) and a closed subset
\[ \mathcal{O}_{v+1}^* = \{(\omega^{b_{v+1}}, \omega^{b_{v+1}}') : \omega^{b_{v+1}} \in \mathcal{O}_v^{b_{v+1}} \} \]
of \( \mathcal{O}_v^* \), where
\[ \mathcal{O}_v^{b_{v+1}} = \mathcal{O}_v^{n+1} \setminus \left( \bigcup_{k,l} \mathcal{R}_{kl}^{v+1} \right), \]
\[ \mathcal{O}_v^{n+1} = \{ \omega^{b_{v+1}} : (\omega^{b_{v+1}}, \omega^{b_{v+1}}') \in \mathcal{O}_v^* \} \subset [0, 1]^{b_{v+1}}, \]
\[ \mathcal{R}_{kl}^{v+1} = \left\{ \omega^{b_{v+1}} \in \mathcal{O}_v^{n+1} : |\langle k, \omega^{b_{v+1}} \rangle + \langle l, \Omega_v \rangle| < \frac{\alpha_{v+1}(l)}{1 + (v + 1)^2(|k| + 1)(2v + 2)} \right\}, \quad (k, l) \in \mathcal{Z}_v^{b+1}, \]
such that for \( H_{v+1} = H_v \circ \Phi_v = N_{v+1} + P_{v+1} \) the same assumptions are satisfied with \( v + 1 \) in place of \( v \).

**Remark 3.2** The assumption (H2) and Lemma 2.2 imply that \( \hat{P}^{b_{v+1}} \) in (3.7) satisfy
\[ |X_{\hat{P}^{b_{v+1}}} |_{r_v, D(s_v, r_v)} \leq C \quad (n \geq 0). \]
Here and later, the letter \( C \) denotes suitable (possibly different) constants which are independent of iteration steps.
3.2 Solving homological equations

The coordinate transformation $\Phi_v$ is obtained as the time-1-map $X^t_{\mathcal{F}_v}|_{t=1}$ of the Hamiltonian vector field $X_{\mathcal{F}_v}$, where $\mathcal{F}_v$ has a similar expression of $R^b_v$,

$$
\mathcal{F}_v(\theta^b_v, z, \bar{z}, \omega^b_v) = \varepsilon_v F_v
$$

$$
= \varepsilon_v F^{00b}_v + \varepsilon_v \{F^{10b}_v, z\} + \varepsilon_v \{F^{01b}_v, \bar{z}\} + \varepsilon_v \{F^{02b}_v, z, \bar{z}\} + \varepsilon_v \{F^{11b}_v, z, \bar{z}\} + \varepsilon_v \{F^{12b}_v, z, \bar{z}\}
$$

$$
= \varepsilon_v \sum_{0 \neq k \in \mathbb{Z}^b} F^{00b}_k e^{-T(\theta^b_v)} + \varepsilon_v \sum_{j \geq 0, k \in \mathbb{Z}^b} (F^{10b}_{kj} z_j + F^{01b}_{kj} \bar{z}_j) e^{-T(\theta^b_v)}
$$

$$
+ \varepsilon_v \sum_{i,j \geq 0, k \in \mathbb{Z}^b, |k|+|i-j| \neq 0} (F^{20b}_{kij} z_i z_j + F^{11b}_{kij} z_i \bar{z}_j + F^{02b}_{kij} \bar{z}_i \bar{z}_j) e^{-T(\theta^b_v)}.
$$

By Taylor’s formula, we have

$$
H_{v+1} := H_v \circ \Phi_v
$$

$$
= N_v + \varepsilon_v \{N_v, F_v\} + \varepsilon_v^2 \int_0^1 (1 - t) \{\{N_v, F_v\}, F_v\} \circ X^t_{\mathcal{F}_v} dt
$$

$$
+ \varepsilon_v R^b_v + \varepsilon_v^2 \int_0^1 \{R^b_v, F_v\} \circ X^t_{\mathcal{F}_v} dt
$$

$$
+ \hat{P}^{high}_v + \varepsilon_v \{\hat{P}^{high}_v, F_v\} + \varepsilon_v^2 \int_0^1 (1 - t) \{\{\hat{P}^{high}_v, F_v\}, F_v\} \circ X^t_{\mathcal{F}_v} dt
$$

$$
+ (P_v - \hat{P}_v) \circ \Phi_v.
$$

Then we obtain the modified homological equation

$$
\varepsilon_v \{N_v, F_v\} + \varepsilon_v R^b_v + \varepsilon_v \{\hat{P}^{high}_v, F_v\}^\text{low} = N_{v+1} - N_v.
$$

(3.6)

If the homological equation is solved, then the new perturbation term $P_{v+1}$ can be written as

$$
P_{v+1} = \hat{P}^{high}_v + \varepsilon_v \{\hat{P}^{high}_v, F_v\}^\text{high}
$$

(3.7)

$$
+ \varepsilon_v^2 \int_0^1 (1 - t) \{N_v + \hat{P}_v^{high}, F_v\} \circ X^t_{\mathcal{F}_v} dt
$$

(3.8)

$$
+ \varepsilon_v^2 \int_0^1 \{R^b_v, F_v\} \circ X^t_{\mathcal{F}_v} dt + (P_v - \hat{P}_v) \circ \Phi_v.
$$

(3.9)

Note that the terms in (3.7) have at least three normal variables. The terms in (3.7) will be left since they have no effect on the tori. To make the terms in (3.8) and (3.9) smaller with KAM iteration, different from the standard strategy in J. Pöschel [14], we shrink the analytic radius of $z$ more slowly than $|14|$ such that the final analytic radius of $z$ will be $r_0/2$ instead of 0 (see the expression of the iterative constant $r_v$). Thus we can obtain a non-degenerate normal form in the end.

To solve the homological equation (3.6), we should know the term $\{\hat{P}^{high}_v, F_v\}^\text{low}$ exactly. Let $\hat{P}^{high}_{v0} = \hat{P}^{high}_{v0} + \hat{P}^{high}_{v1}$, where

$$
\hat{P}^{high}_{v0} = \sum_{|\gamma|+|\kappa|=3} \hat{P}^{\gamma\kappa}(\theta^b_v, \omega^b_v) z^\gamma \bar{z}^\kappa,
$$

$$
\hat{P}^{high}_{v1} = \sum_{|\gamma|+|\kappa|\geq 4} \hat{P}^{\gamma\kappa}(\theta^b_v, \omega^b_v) z^\gamma \bar{z}^\kappa.
$$
Set $F_v = F^0_v + F^1_v + F^2_v$, where

$$F^0_v = \sum_{0 \neq k \in \mathbb{Z}^{b_v}} F_k^{00b_v} e^{\sqrt{-1}(k, \theta^{b_v})},$$

$$F^1_v = \sum_{j \geq 0, k \in \mathbb{Z}^{b_v}} (F_{k+j}^{10b_v} z_j + F_{k+j}^{01b_v} \bar{z}_j) e^{\sqrt{-1}(k, \theta^{b_v})},$$

$$F^2_v = \sum_{i,j \geq 0, k \in \mathbb{Z}^{b_v}, |i-j| \neq 0} (F_{k+i+j}^{20b_v} z_i z_j + F_{k+i+j}^{11b_v} z_i \bar{z}_j + F_{k+i+j}^{02b_v} \bar{z}_i \bar{z}_j) e^{\sqrt{-1}(k, \theta^{b_v})}.$$

Denote $W^{b_v} = \{\hat{P}_v^\text{high}, F_v\}^\text{low}$, then by a direct calculation, we obtain

$$W^{b_v} = \sqrt{-1} \sum_{j \geq 0} \left( \partial_{z_j} \hat{P}_v^\text{high} \partial_{\bar{z}_j} F^1_v - \partial_{\bar{z}_j} \hat{P}_v^\text{high} \partial_{z_j} F^1_v \right) = \{\hat{P}_v^\text{high}, F^1_v\},$$

which is degree two in variables $(z, \bar{z})$. Write $W^{b_v} = (W^{b_v})^0 + (W^{b_v})^1 + (W^{b_v})^2$. Then we can easily get

$$W^{b_v} = (W^{b_v})^2 = \langle W^{20b_v} z, z \rangle + \langle W^{11b_v} z, \bar{z} \rangle + \langle W^{02b_v} \bar{z}, \bar{z} \rangle. \quad (3.10)$$

Let $(R^{b_v})^2$ be of the same form as $(W^{b_v})^2$ and set $B^{b_v} = (R^{b_v})^2 + (W^{b_v})^2$. More precisely,

$$B^{b_v} = \langle B^{20b_v} z, z \rangle + \langle B^{11b_v} z, \bar{z} \rangle + \langle B^{02b_v} \bar{z}, \bar{z} \rangle$$

with

$$B^{20b_v} = R^{20b_v} + W^{20b_v}, \quad B^{11b_v} = R^{11b_v} + W^{11b_v}, \quad B^{02b_v} = R^{02b_v} + W^{02b_v}.$$

By the definition of $F_v$ and $b_v < b_v + 1$, it implies that $\{ \sum_{j \geq b_v + 1} \omega_j J_j, F_v \} = 0$. Moreover, it is easy to see that $F_v$ and $\hat{P}_v^{b_v}$ $(n \geq 0)$ are independent of $J$. Therefore, (3.10) is equivalent to the following homological equations:

$$\left\{ \begin{array}{l}
\sqrt{-1} \langle k, \omega^{b_v} \rangle F_k^{00b_v} = R_k^{00b_v}, \quad k \neq 0, \\
\sqrt{-1} \langle (k, \omega^{b_v}) + \Omega v \rangle F_{k+j}^{10b_v} = R_{k+j}^{10b_v}, \\
\sqrt{-1} \langle (k, \omega^{b_v}) - \Omega v \rangle F_{k+j}^{01b_v} = R_{k+j}^{01b_v}, \\
\sqrt{-1} \langle (k, \omega^{b_v}) + \Omega v \rangle F_{k+i}^{20b_v} = R_{k+i}^{20b_v} + W_{k+i}^{20b_v} = B_{k+i}^{20b_v}, \\
\sqrt{-1} \langle (k, \omega^{b_v}) - \Omega v \rangle F_{k+i}^{11b_v} = R_{k+i}^{11b_v} + W_{k+i}^{11b_v} = B_{k+i}^{11b_v}, \quad |k| + |i| - j \neq 0, \\
\sqrt{-1} \langle (k, \omega^{b_v}) - \Omega v \rangle F_{k+i}^{02b_v} = R_{k+i}^{02b_v} + W_{k+i}^{02b_v} = B_{k+i}^{02b_v}, \\
\end{array} \right.$$

with $\varepsilon_v N_v \triangleq N_v + 1 - N_v = \varepsilon_v [R^{00b_v}] + \frac{1}{2\pi} \int_{\mathbb{R}^2} R^{00b_v} (\theta^{b_v}, \omega^{b_v}) d\theta^{b_v}$

will be omitted from $H_{v+1}$ in the following since it dose not affect the dynamics of the Hamiltonian vector field $X_{H_{v+1}}$ and $[B^{11b_v}]$ is defined analogously.

Concerning the estimate of $X_{F_v}$, we have the following lemma.

**Lemma 3.3** Suppose that uniformly on $O^*_v$, 

$$|\langle k, \omega^{b_v} \rangle + \langle l, \Omega_v \rangle| \geq \frac{\alpha_v (l_2)}{(1 + v^2)(|k| + 1)^{2b_v+2}}, \quad (k, l) \in \mathbb{Z}^{b_v}.$$
Then the linearized equation (3.6) has a solution \( F_v \) satisfying
\[
|X_{F_v}|_{r_v,D(s_v-\sigma_v,r_v-\rho_v)\times \mathcal{O}_v^*} \leq C\alpha_v^{-2}\sigma_v^{-1}A_v^2|X_{R_{b_v}}|_{r_v,D(s_v,r_v)\times \mathcal{O}_v^*}
\]
with \( A_v = (\frac{16(2b_v+3)}{e})^{4b_v+6}\sigma_v^{-(5b_v+6)} \).

**Proof** Consider the term \( F^{10b_v} \), we note that \( R_{10b_v} \) is an analytic map in \( \ell^{a,p+2} \) with a Fourier series expansion whose coefficients \( R_{k_{b_v}} = (R_{k_0}^{10b_v}, R_{k_1}^{10b_v}, \ldots) \) satisfy
\[
\sum_{k \in \mathbb{Z}^{b_v}} \|R_{k_{b_v}}^{10b_v}\|_{a,p+2}^2e^{2|k|s_v} \leq 2^{b_v}\|R_{10b_v}\|_{a,p+2,D(s_v)}^2,
\]
where \( D(s_v) = \{ |\text{Im}\theta_b| < s_v \} \) and \( \|R_{10b_v}\|_{a,p+2,D(s_v)} = \sup_{D(s_v)} \|R_{10b_v}\|_{a,p+2} \). According to the small divisor assumptions, we easily get
\[
|F_{k_{10b_v}}^{10b_v}| \leq \alpha_v^{-1}(1 + v^2)(|k| + 1)^{2b_v+2}|R_{k_{10b_v}}|
\]
and
\[
\|F_{k_{10b_v}}^{10b_v}\|_{a,p+2}^2 = \|F_{k_{10b_v}}^{10b_v}\|_{a,p+2}^2 + \sum_{j \geq 1} \|F_{k_{10b_v}}^{10b_v}\|_{a,p+2}^2e^{2j2^{a}}\leq \alpha_v^{-1}(1 + v^2)(|k| + 1)^{2b_v+2}\|R_{k_{10b_v}}\|_{a,p+2}^2
\]
uniformly on \( \mathcal{O}_v^* \). From (3.11), (3.12) and Lemma 7.3 in Appendix, it follows that
\[
\|F_{10b_v}\|_{a,p+2,D(s_v-\sigma_v)} \leq \sum_{k \in \mathbb{Z}^{b_v}} \|F_{k_{10b_v}}\|_{a,p+2}e^{2|k|(s_v-\sigma_v)}
\]
\[
\leq \sqrt{\sum_{k \in \mathbb{Z}^{b_v}} \|R_{k_{10b_v}}\|_{a,p+2}^2e^{2|k|s_v}} \sqrt{\sum_{k \in \mathbb{Z}^{b_v}} \alpha_v^{-1}(1 + v^2)(|k| + 1)^{2b_v+2}}e^{-2|k|\sigma_v}
\]
\[
\leq \alpha_v^{-1}A_v\|R_{10b_v}\|_{a,p+2,D(s_v)}
\]
and
\[
\|\partial_\theta F_{10b_v}\|_{a,p+2,D(s_v-\sigma_v)} \leq \alpha_v^{-1}A_v\|R_{10b_v}\|_{a,p+2,D(s_v)}.
\]
Since \( R_{10b_v} = (R_{b_v})_z \) \( \forall z \in \mathbb{C} \), we can easily get
\[
\|R_{10b_v}\|_{a,p+2,D(s_v)} \leq r_v|X_{R_{b_v}}|_{r_v,D(s_v,r_v)},
\]
and
\[
|X_{(F^{10b_v},z)}|_{r_v,D(s_v-\sigma_v,r_v)\times \mathcal{O}_v} \leq C\alpha_v^{-1}A_v|X_{R_{b_v}}|_{r_v,D(s_v,r_v)\times \mathcal{O}_v}.
\]
To estimate the Lipschitz semi-norm of \( F^{10b_v} \), let \( \delta_{k_{10b_v}} = \langle k, \omega_{b_v} \rangle + \Omega_{aj} \) and \( \Delta = \Delta_{\omega_{b_v},c_{b_v}} \) for \( \omega_{b_v}, c_{b_v} \in \mathcal{O}_v^* \), then we have
\[
\Delta F_{10b_v} = -\frac{\sqrt{-1} \Delta R_{10b_v}}{\delta_{k_{10b_v}}(\omega_{b_v})} + \frac{\sqrt{-1} R_{10b_v}(\omega_{b_v}) \Delta \delta_{k_{10b_v}}}{\delta_{k_{10b_v}}(\omega_{b_v})\delta_{k_{10b_v}}(\omega_{b_v})}.
\]
The small divisor assumptions imply that
\[
|\Delta F_{10b_v}| \leq (\alpha_v^{-1}(1 + v^2)(|k| + 1)^{2b_v+2}\|R_{k_{10b_v}}\|(|k|\Delta \omega_{b_v} | + |\Delta \Omega_{aj}|j^{-2}) + \alpha_v^{-1}(1 + v^2)(|k| + 1)^{2b_v+2}\|R_{k_{10b_v}}|.
\]
on $O^*_v$. Hence,
\[
\|\Delta F_k^{10b_v}\|_{a,p+2} \leq (\alpha_v^{-1}(1 + v^2)(|k| + 1)^{2b_v+2})^2 \|R_k^{10b_v}\|_{a,p+2} (|k| \Delta \omega_v^b | + |\Delta \Omega_v| - \delta) \\
+ \alpha_v^{-1}(1 + v^2)(|k| + 1)^{2b_v+2} \|\Delta R_k^{10b_v}\|_{a,p+2}.
\]
Summing up the Fourier series as (3.13), we have
\[
\|\Delta F^{10b_v}\|_{a,p+2,D(s_v - \sigma_v)} \leq \alpha_v^{-2} A_v \|R^{10b_v}\|_{a,p+2,D(s_v)} (|\Delta \omega_v^b | + |\Delta \Omega_v| - \delta) \\
+ \alpha_v^{-1} A_v \|\Delta R^{10b_v}\|_{a,p+2,D(s_v)}.
\]
Dividing by $|\omega_v^b - \tilde{\omega}_v^b|$ and taking the supremum over $\omega_v^b \neq \tilde{\omega}_v^b$ in $O_v^*$, we obtain
\[
\|F^{10b_v}\|_{a,p+2,D(s_v - \sigma_v)} \leq \alpha_v^{-1} A_v \left(\frac{M_v + 1}{\alpha_v}\right) \|R^{10b_v}\|_{a,p+2,D(s_v)} + \|R^{10b_v}\|_{a,p+2,D(s_v)}^{lip}
\]
and
\[
\|\partial \theta F^{10b_v}\|_{a,p+2,D(s_v - \sigma_v)} \leq \alpha_v^{-1} A_v \left(\frac{M_v + 1}{\alpha_v}\right) \|R^{10b_v}\|_{a,p+2,D(s_v)} + \|R^{10b_v}\|_{a,p+2,D(s_v)}^{lip}
\]
in view of (3.11). Then, we have
\[
|X_{(F^{10b_v},z)}|_{r_v, D(s_v - \sigma_v, r_v)}^{lip} \leq C\alpha_v^{-1} A_v \left(\frac{M_v + 1}{\alpha_v}\right) \|X_{R^{b_v}}|_{r_v, D(s_v, r_v)} \times O_v^* + |X_{R^{b_v}}|_{r_v, D(s_v, r_v)}^{lip} \times O_v^*).
\]
Thus, together with (3.14), we arrive at
\[
|X_{(F^{10b_v},z)}|_{r_v, D(s_v - \sigma_v, r_v)}^{\lambda_v} \leq C\alpha_v^{-1} A_v |X_{R^{b_v}}|_{r_v, D(s_v, r_v)}^{\lambda_v} \times O_v^*.
\]
(3.15)
For the term $F^{01b_v}$, the same estimate as (3.15) can be obtained, thus, we get
\[
|X_{F_v} |_{r_v, D(s_v - \sigma_v, r_v)}^{\lambda_v} \leq C\alpha_v^{-1} A_v |X_{R^{b_v}}|_{r_v, D(s_v, r_v)}^{\lambda_v} \times O_v^*.
\]
(3.16)
Before considering the term $F^{11b_v}$, we should get the estimate of the term $B^{b_v}$. Recall that
\[
B^{b_v} = (R^{b_v})^2 + (W^{b_v})^2,
\]
and
\[
(W^{b_v})^2 = \{\tilde{P}^{high}, F_v\}^{low} = \{\tilde{P}^{high}, F_v\}^{1}.
\]
Then by (3.3), (3.4), (3.16) and the generalized Cauchy inequality, we obtain
\[
|X_{B^{b_v}}|_{r_v, D(s_v - 2\sigma_v, r_v)}^{\lambda_v} \leq |X_{(R^{b_v})^2}|_{r_v, D(s_v, r_v)}^{\lambda_v} \times O_v^* + |X_{(W^{b_v})^2}|_{r_v, D(s_v - 2\sigma_v, r_v)}^{\lambda_v} \times O_v^* \\
\leq C|X_{R^{b_v}}|_{r_v, D(s_v, r_v)}^{\lambda_v} \times O_v^* + C\sigma_v^{-1} |X_{\tilde{P}^{high}}|_{r_v, D(s_v, r_v)}^{\lambda_v} \times O_v^* |X_{F_v} |_{r_v, D(s_v, r_v)}^{\lambda_v} \times O_v^* \\
\leq C|X_{R^{b_v}}|_{r_v, D(s_v, r_v)}^{\lambda_v} \times O_v^* \left(1 + \alpha_v^{-1} \sigma_v^{-1} A_v |X_{\tilde{P}^{high}}|_{r_v, D(s_v, r_v)}^{\lambda_v} \times O_v^*\right) \\
\leq C\alpha_v^{-1} \sigma_v^{-1} A_v |X_{R^{b_v}}|_{r_v, D(s_v, r_v)}^{\lambda_v} \times O_v^*.
\]
(3.17)
By the generalized Cauchy inequality, we have
\[
\|B^{11b_v}\|_{a,p+2,D(s_v - 2\sigma_v)} \leq \frac{1}{r_v} \|(B^{b_v})_z\|_{a,p+2,D(s_v - 2\sigma_v, r_v - d_v)} \leq |X_{B^{b_v}}|_{r_v, D(s_v - 2\sigma_v, r_v - d_v)} \times O_v^*.
\]
where $\| \cdot \|_{a,p+2,p,D(s_v-2\sigma_v)}$ is the operator norm of bounded linear operators from $\ell^a,p$ to $\ell^{a,p+2}$. This is equivalent to that $\tilde{B}^b_{ij} = (v_iB^i_{11b_v}w_j)$ is a bounded linear operator of $\ell^2$ into itself with the operator norm $\| \tilde{B}^b_{ij} \|_{D(s_v-2\sigma_v)} = \| B^i_{11b_v} \|_{a,p+2,p,D(s_v-2\sigma_v)}$, where $v_i, w_j$ are certain weights (see [14]). Expanding $\tilde{B}^b_{ij}$ into its Fourier series and as before, we know that

$$\sum_{k \in \mathbb{Z}^2} \| \tilde{B}^b_{ij} \|_{2^k|s_v-2\sigma_v|} \leq 2^b \| \tilde{B}^b_{ij} \|_{D(s_v-2\sigma_v)}.$$

By the small divisor assumptions and $|i^2 - j^2| = |i-j|(i+j)$, we find that the corresponding coefficient $\tilde{F}^b_{ij} = (\tilde{F}^b_{ij})$ satisfies the following estimate

$$|\tilde{F}^b_{ij}| \leq \frac{\alpha^{-1}(1 + v^2)(|k| + 1)^{2b_v+2}}{|i-j|} \tilde{B}^b_{ij}, \quad |k| + |i-j| \neq 0,$$

while $\tilde{B}^b_{0ij} = 0$. Lemma 7.2 in Appendix implies that

$$\| \| \tilde{F}^b_{ij} \| \| \leq 3\alpha^{-1}(1 + v^2)(|k| + 1)^{2b_v+2} \| \tilde{B}^b_{ij} \|$$

uniformly on $\mathcal{O}^s_v$. Summing up the Fourier series as before,

$$\| \tilde{F}^b_{ij} \|_{D(s_v-3\sigma_v)} \leq \sum_{k \in \mathbb{Z}^2} \| \tilde{F}^b_{ij} \|_{c^k|s_v-3\sigma_v|} \leq 3\alpha^{-1}A_v \| \tilde{B}^b_{ij} \|_{D(s_v-2\sigma_v)}$$

and

$$\| \partial \tilde{F}^b_{ij} \|_{D(s_v-3\sigma_v)} \leq \sum_{k \in \mathbb{Z}^2} |k| \| \tilde{F}^b_{ij} \|_{c^k|s_v-3\sigma_v|} \leq 3\alpha^{-1}A_v \| \tilde{B}^b_{ij} \|_{D(s_v-2\sigma_v)}.$$

Thus,

$$\| F^{11b_v} \|_{a,p+2,p,D(s_v-3\sigma_v)} \leq 3\alpha^{-1}A_v \| X_{B^b_v} \|_{r_v,D(s_v-2\sigma_v),r_v-d_v}$$

and

$$\| \partial F^{11b_v} \|_{a,p+2,p,D(s_v-3\sigma_v)} \leq 3\alpha^{-1}A_v \| X_{B^b_v} \|_{r_v,D(s_v-2\sigma_v),r_v-d_v}.$$

Finally, we have

$$|X(F^{11b_v},\omega)|_{r_v,D(s_v-3\sigma_v),r_v-d_v} \leq C\alpha^{-1}A_v \| X_{B^b_v} \|_{r_v,D(s_v-2\sigma_v),r_v-d_v} \times \mathcal{O}^s_v. \quad (3.18)$$

To obtain the estimate of Lipschitz semi-norm, let $\delta_{kij} = (k, \omega) + \Omega_{vi} - \Omega_{vj}$, then we have

$$\Delta \tilde{F}^b_{ij} = -\frac{\sqrt{-1} \Delta \tilde{B}^b_{ij}}{\delta_{kij}(\omega^b_v)} + \frac{\sqrt{-1} \tilde{B}^b_{ij}(\omega^b_v) \Delta \delta_{kij}}{\delta_{kij}(\omega^b_v) \delta_{kij}(\omega^b_v)},$$

which implies that

$$|\Delta \tilde{F}^b_{ij}| \leq \left( \alpha^{-1}(1 + v^2)(|k| + 1)^{2b_v+2} \right)^2 \frac{|\tilde{B}^b_{ij}|}{|i-j|} \left( |k| |\Delta \omega| + 2|\Delta \Omega_v| \right)$$

$$+ \alpha^{-1}(1 + v^2)(|k| + 1)^{2b_v+2} \frac{|\Delta \tilde{B}^b_{ij}|}{|i-j|}.$$
In the same way as (3.18), we have
\[ \|X_{(F^{11v}_z, z)}\|_{\text{lip}}^{\lambda_v} \leq C_\alpha^{-1} A_v \left( \frac{M_v + 1}{\alpha_v} \right) |X_{B^{11v}_v}|_{r_v, D(s_u - 2\sigma_v, r_v - d_v) \times \mathcal{O}_v^*} + |X_{B^{11v}_v}|_{r_v, D(s_u - 2\sigma_v, r_v - d_v) \times \mathcal{O}_v^*} \],
which, together with (3.17) and (3.18), leads to
\[ |X_{(F^{11v}_z, z)}|_{r_v, D(s_u - 3\sigma_v, r_v - d_v) \times \mathcal{O}_v^*} \leq C_\alpha^{-1} A_v |X_{B^{11v}_v}|_{r_v, D(s_u - 2\sigma_v, r_v - d_v) \times \mathcal{O}_v^*} \leq C_\alpha^{-2} \sigma_v^{-1} A_v^2 |X_{B^{11v}_v}|_{r_v, D(s_u, r_v) \times \mathcal{O}_v^*}. \] (3.19)

For the other terms of $F_v$, the same estimates or even better ones than (3.15) and (3.19) can be obtained. Thus, we finally get the estimate of the Hamiltonian vector field $X_{F_v}$
\[ |X_{F_v}|_{r_v, D(s_u - 3\sigma_v, r_v - d_v) \times \mathcal{O}_v^*} \leq C_\alpha^{-2} \sigma_v^{-1} A_v^2 |X_{R^{11v}_v}|_{r_v, D(s_u, r_v) \times \mathcal{O}_v^*}. \] (3.20)

The proof of Lemma 3.3 is completed. \( \square \)

Note that $|X_{R^{11v}_v}|_{r_v, D(s_u, r_v) \times \mathcal{O}_v^*} \leq \frac{1}{2}$. Then using the generalized Cauchy inequality, we get
\[ \frac{1}{\sigma_v} |X_{F_v}|_{r_v, D(s_u - 3\sigma_v, r_v - d_v) \times \mathcal{O}_v^*}, |DX_{F_v}|_{r_v, D(s_u - 4\sigma_v, r_v - 2d_v) \times \mathcal{O}_v^*} \leq C \varepsilon_v^{-1} \frac{1}{4^\rho} \] (3.21)

by $\varepsilon_v^{-1} A_v^2 |X_{R^{11v}_v}|_{r_v, D(s_u, r_v) \times \mathcal{O}_v^*} \leq C$ as $\varepsilon \ll 1$, where we require $d_v / r_v \geq \sigma_v$ which is fulfilled by setting $\sigma_0 \leq 1/20$, moreover $C$ is an absolute constant independent of $v$ and $\varepsilon$. Here we use the operator norm
\[ |L|_{r, s} = \sup_{W \neq 0} \frac{|LW|_{r, s, a, p + 2}}{|W|_{s, a, p}} \] (3.22)

with $|L|_{r, a, p + 2}$ defined in (2.9), and $|L|_{s, a, p}$ defined analogously. Then the flow $X_{F_v}^t$ of the vector field $X_{F_v}$ exists on $D(s_u - 4\sigma_v, r_v - 2d_v)$ for $-1 \leq t \leq 1$ and takes this domain into $D(s_u - 3\sigma_v, r_v - d_v)$. By Lemma A.4 in [14] and (3.21), we obtain
\[ |X_{F_v}^t - id|_{r_v, D(s_u - 3\sigma_v, r_v - d_v) \times \mathcal{O}_v^*} \leq C \varepsilon_v^{-1} \frac{1}{4^\rho} \] (3.23)
for $-1 \leq t \leq 1$. Similarly, the flow takes $D(s_u - 5\sigma_v, r_v - 3d_v)$ into $D(s_u - 4\sigma_v, r_v - 2d_v)$ and by generalized Cauchy inequality, we also have
\[ |DX_{F_v}^t - Id|_{r_v, D(s_u - 5\sigma_v, r_v - 3d_v) \times \mathcal{O}_v^*} \leq C \frac{1}{\sigma_v} |X_{F_v}|_{r_v, D(s_u - 3\sigma_v, r_v - d_v) \times \mathcal{O}_v^*} \leq C \varepsilon_v^{-1} \frac{1}{4^\rho} \] (3.24)
for $-1 \leq t \leq 1$.

### 3.3 The new Hamiltonian

From (3.5) and (3.6) we get the new Hamiltonian $H_v \circ \Phi_v = N_{v+1} + P_{v+1}$ with
\[ N_{v+1} = N_v + \varepsilon_v \sum_{j \geq 0} [B^{11v}_{jj}] z_j \bar{z}_j := N_v + \varepsilon_v (\tilde{\Omega}_v, z\bar{z}) \] (3.25)
and

\[ P_{v+1} = \varepsilon_v^2 \int_0^1 (1-t) \{ \{ N_v, F_v \}, F_v \} \circ X_{F_v}^t \, dt \]

\[ + \varepsilon_v^2 \int_0^1 \{ R^{b_v}, F_v \} \circ X_{F_v}^t \, dt + \varepsilon_v^2 \int_0^1 (1-t) \{ \{ \hat{P}^{high}_v, F_v \}, F_v \} \circ X_{F_v}^t \, dt + (P_v - \hat{P}_v) \circ \Phi_v \]

\[ + \hat{P}^{high}_v + \varepsilon_v \{ \hat{P}^{high}_v, F_v \}^{high}. \]

Denote

\[ \Omega_{(v+1)j}(\omega^{b_v}) = \Omega_{vj}(\omega^{b_{v-1}}) + \varepsilon_v [B^{11b_v}_{jj}](\omega^{b_v}). \]

Thus, we obtain the new normal form

\[ N_{v+1} = \langle \omega^{b_{v+1}}, J^{b_{v+1}} \rangle + \sum_{n \geq v+2} \langle \omega^{b_n}_1, J^{b_n}_1 \rangle + \langle \Omega_{v+1}, z \bar{z} \rangle. \]

Let

\[ P_{v+1} = \hat{P}_{v+1} + \sum_{n \geq v+2} \varepsilon^{(1+n)} \hat{P}^{b_n}_0 \circ \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_v \]  \hspace{1cm} (3.26)

with

\[ \hat{P}_{v+1} = \varepsilon_v \int_0^1 \{ \{ N_v, F_v \}, F_v \} \circ X_{F_v}^t \, dt \]

\[ + \varepsilon_v \int_0^1 (1-t) \{ \{ \hat{P}^{high}_v, F_v \}, F_v \} \circ X_{F_v}^t \, dt \]

\[ + \varepsilon_v \left( \hat{P}^{b_{v+1}}_0 \circ \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_v \right)^{high}. \]  \hspace{1cm} (3.27)

and

\[ \varepsilon_v \int_0^1 \{ \{ N_v, F_v \}, F_v \} \circ X_{F_v}^t \, dt + \varepsilon_v \int_0^1 \{ R^{b_v}, F_v \} \circ X_{F_v}^t \, dt \]

\[ + \varepsilon_v \left( \int_0^1 (1-t) \{ \{ \hat{P}^{high}_v, F_v \}, F_v \} \circ X_{F_v}^t \, dt \right)^{low} \]

\[ + \varepsilon_v \left( \hat{P}^{b_{v+1}}_0 \circ \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_v \right)^{low}. \]

Since \( \hat{P}_v \) and \( F_v \) depend on \( z, \bar{z}, \theta^{b_v} \) and \( \omega^{b_v} \). Moreover, \( \hat{P}^{b_{v+1}} \) depends on \( z, \bar{z}, \theta^{b_{v+1}} \) and \( \omega^{b_{v+1}} \). It is easy to check that \( \hat{P}_{v+1} \) depends on \( z, \bar{z}, \theta^{b_{v+1}} \) and \( \omega^{b_{v+1}} \). Thus, in order to remove the non-normalized terms in (\( \hat{P}_{v+1} \))^{low} in the next step, we only need to treat finite frequencies.

### 3.4 Estimate of the new norm form and new perturbation

The aim of this section is to estimate the new normal form \( N_{v+1} \) and the new perturbation \( P_{v+1} \) in (3.26). Now we consider \( R^{b_v} \) and we will prove that (3.4) is fulfilled with \( v+1 \) in place of \( v \). Concerning \( R^{b_{v+1}} \), we have

\[ X_{R^{b_{v+1}}} = \varepsilon_v \int_0^1 (1-t)(X_{F_v}^t)^* [X_{\{N_v,F_v\},F_v}] \, dt + \varepsilon_v \int_0^1 (X_{F_v}^t)^* [X_{R^{b_v},F_v}] \, dt \]

\[ + \varepsilon_v \int_0^1 (1-t) \{ \{ \hat{P}^{high}_v, F_v \}, F_v \} \circ X_{F_v}^t \, dt \]

\[ + \varepsilon_v \left( \hat{P}^{b_{v+1}}_0 \circ \Phi_0 \circ \Phi_1 \circ \cdots \circ \Phi_v \right)^{low}. \] \hspace{1cm} (3.29)
In the following, we will estimate every part of $X_{R^{b_{v+1}}}$, and we wish to show that $R^{b_{v+1}}$ satisfies (3.1) with $v + 1$ in place of $v$. By the generalized Cauchy estimate, we obtain

$$|X_{\{\nu, F_v\}}|_{r_v, D(s_0 - 4\sigma_v, r_v - 2d_v)} \leq C\sigma_v^{-1} |X_{F_v}|_{r_v, D(s_0 - 3\sigma_v, r_v - d_v)} \times \mathcal{O}_v^* \tag{3.30}$$

Following the same lines as (12) in [14], we obtain that for any vector field $Y$,

$$|(X_{F_v}^t)^* Y|_{r_v, D(s_0 - 6\sigma_v, r_v - 4d_v)} \leq 2 |Y|_{r_v, D(s_0 - 5\sigma_v, r_v - 3d_v)} \times \mathcal{O}_v^* \quad (0 \leq t \leq 1). \tag{3.31}$$

Therefore, by the generalized Cauchy inequality and (3.4), (3.30), we get

$$|(X_{F_v}^t)^* [X_{\{\nu, F_v\}}, X_{F_v}]|_{r_v, D(s_0 - 6\sigma_v, r_v - 4d_v)} \times \mathcal{O}_v^* \leq C\sigma_v^{-1} |X_{\{\nu, F_v\}}|_{r_v, D(s_0 - 6\sigma_v, r_v - 4d_v)} \times \mathcal{O}_v^* \tag{3.32}$$

and

$$|(X_{F_v}^t)^* [X_{R^{b_{v+1}}}, X_{F_v}]|_{r_v, D(s_0 - 6\sigma_v, r_v - 4d_v)} \times \mathcal{O}_v^* \leq 2 |X_{R^{b_{v+1}}}, X_{F_v}]|_{r_v, D(s_0 - 5\sigma_v, r_v - 3d_v)} \times \mathcal{O}_v^* \tag{3.33}$$

In the same way as (3.32) (3.33) and by (3.3), we have

$$X_{f_0 t} [(T^t, \nu, D(s_0 - 6\sigma_v, r_v - 4d_v)} \times \mathcal{O}_v^* \leq C\sigma_v^{-1} |X_{\nu, D(s_0 - 6\sigma_v, r_v - 4d_v)} \times \mathcal{O}_v^* \tag{3.34}$$

Then (3.34) implies that

$$X_{f_0 t} [(T^t, \nu, D(s_0 - 6\sigma_v, r_v - 4d_v)} \times \mathcal{O}_v^* \leq C\sigma_v^{-2} \left( |X_{F_v}|_{r_v, D(s_0 - 3\sigma_v, r_v - d_v)} \times \mathcal{O}_v^* \right)^2 \tag{3.35}$$

By repeatedly applying (3.31) to $\tilde{P}^{b_{v+1} \Phi_0 \Phi_1 \cdots \Phi_{v}}$, we have

$$X_{\tilde{P}^{b_{v+1} \Phi_0 \Phi_1 \cdots \Phi_{v}}} \leq 2^{v+3} C |X_{\tilde{P}^{b_{v+1}}}|_{r_0, D(s_0, r_0)} \times \mathcal{O}_v^* \tag{3.36}$$

Thus (3.36) and Remark 3.2 imply that

$$x_{\tilde{P}^{b_{v+1} \Phi_0 \Phi_1 \cdots \Phi_{v}}} \leq 2^{v+3} C \tag{3.37}$$

So altogether we obtain

$$X_{R^{b_{v+1}}} \leq C \varepsilon_v^{1 - \frac{1}{2\rho}} \sigma_v^{-2} \left( \alpha_v^{-2} A_v^2 \right)^2 + C \varepsilon_v \sigma_v^{-1} \left( \alpha_v^{-2} A_v^2 \right)^2 + 2^{v+3} C \varepsilon_v^{-1} \varepsilon_v^{(1 + \rho) + 1} \leq C \left( \varepsilon_v^{1 - \frac{1}{2\rho}} \left( \alpha_v^{-1} A_v \right)^4 + \varepsilon_v \right)$$

$$\leq C \varepsilon_v^{1 - \frac{1}{2\rho}} \leq \frac{1}{2}$$
as \( \varepsilon \) small enough. Then \( |X_{R^{v+1}}|_{\Omega_v}^{\lambda_v} \) is as required.

Moreover, in the same way as (3.36), we obtain

\[
|X_{\sum_{n \geq v+2} \varepsilon^{(1+\rho)^n} \frac{\partial^k}{\partial x_1^k} \phi_1(x, r_v) \phi_1(x, r_v)}|_{r_v, D(s_v-6\sigma_v, r_v-4d_v)} \leq 2^{v+3} C \sum_{n \geq v+2} \varepsilon^{(1+\rho)^n} |X_{\phi_1}|_{r_0, D(s_0, r_0)} \leq \frac{1}{2} \varepsilon^{v+1}
\]

(3.39)

provided that \( \varepsilon \) is small enough. Then (3.38) and (3.39) imply that (3.4) is fulfilled with \( v+1 \) in place of \( v \).

We now check that \( X_{\hat{\phi}^{v+1}} \) satisfy (3.3) with \( v+1 \) in place of \( v \).

\[
|X_{\hat{\phi}^{v+1}}|_{r_v, D(s_v-4\sigma_v, r_v-2d_v)} \leq C \varepsilon^{v+1} |X_{\hat{\phi}^{v+1}}|_{r_v, D(s_v, r_v)} + |X_{\phi_1}|_{r_v, D(s_v-4\sigma_v, r_v-2d_v)} \leq C(\alpha_v^{-1} \sigma_v^{-1} A_v)^2 |X_{\hat{\phi}^{v+1}}|_{r_v, D(s_v, r_v)}
\]

(3.40)

Recalling (3.28), by (3.3), (3.20), (3.34), (3.35), (3.36), (3.37), (3.40) and the fact that

\[
|X_{\hat{\phi}^{v+1}}|_{r_0, D(s_0, r_0)} \leq C
\]

for \( v \geq 0 \), we obtain

\[
|X_{\phi_1}|_{r_v, D(s_v-4\sigma_v, r_v-2d_v)} \leq C \varepsilon^{v+1} |X_{\hat{\phi}^{v+1}}|_{r_v, D(s_v, r_v)} \leq C \varepsilon^{v+1}
\]

Moreover, \( (3.17) \) and \( (3.25) \) imply that

\[
|\hat{\Omega}_v|_{-\delta, \Omega_{v+1}}^{\lambda_v} \leq |X_{B^{v+1}}|_{r_v, D(s_v-2\sigma_v, r_v-d_v)} \leq \varepsilon^{v+1}.
\]

(3.41)

Thus, by (3.17), the Lipschitz semi-norm of the new frequencies on \( \Omega_{v+1} \) is bounded by

\[
|\hat{\Omega}_{v+1}|_{-\delta, \Omega_{v+1}}^{\lambda_v} \leq M_v + \varepsilon_v |\hat{\Omega}_v|_{-\delta, \Omega_{v+1}}^{\lambda_v} \leq M_v + \frac{(M_v+1)\varepsilon_v}{\alpha_v} |X_{B^{v+1}}|_{r_v, D(s_v-2\sigma_v, r_v-d_v)} \leq (M_v+1)(1+\varepsilon_v^{-1}) - 1
\]

as required. This completes the proof of the iteration lemma. \( \square \)
4 Convergence of transformations

To apply iterative lemma with \( v = 0 \), set \( N_0 = N \), \( P_0 = P \), \( s_0 = s \), \( r_0 = r \). Choosing \( \tilde{P}_0 = \varepsilon \tilde{P}_0 \), then (3.3) and (3.4) with \( v = 0 \) are satisfied by Lemma 2.2. The small divisor conditions are satisfied by setting \( \mathcal{O}_*^0 = \{(\omega_{b_0}, \omega_{b_0}') : \omega_{b_0} \in \mathcal{O}_{b_0}^0 \} \), where \( \mathcal{O}_{b_0}^0 = [0, 1]^{b_0} \setminus \bigcup_{k,l} \mathcal{R}_{kl}^0 \), and \( \mathcal{R}_{kl}^0 = \{\omega_{b_0} \in [0, 1]^{b_0} : |\langle k, \omega_{b_0} \rangle + \langle l, \Omega_0 \rangle| < \frac{\alpha_0(1/2)}{|k|+|l|}|\omega_{b_0}|^2 \} \). Hence, using the iterative lemma, we obtain a sequence of transformations \( \Phi^v = \Phi_0 \circ \cdots \circ \Phi_{v-1} : D_v \times \mathcal{O}_{v-1}^* \rightarrow D_0 \) for \( v \geq 1 \), and \( H \circ \Phi^v = N_v + P_v \). We now prove the convergence of \( \Phi^v \). From (3.23) and (3.24), we obtain

\[
\frac{1}{\sigma_v} |\Phi_v - id|_{r,v,D_{v+1} \times \mathcal{O}_v^*}^\lambda_v, |D\Phi_v - I|_{r,v,D_{v+1} \times \mathcal{O}_v^*}^\lambda_v \leq C\varepsilon_v^{1-\frac{k}{2}}. \tag{4.1}
\]

We note that the operator norm \( |\cdot|_{r,s} \) defined in (3.22) satisfies \( |AB|_{r,s} \leq |A|_{r,r} |B|_{s,s} \) for \( r \geq s \). For \( v \geq 1 \), by the chain rule and using (4.4), we get

\[
|D\Phi^v|_{r_0,r_v,D_v \times \mathcal{O}_v^*} \leq \prod_{\mu=0}^{v-1} |D\Phi^v|_{r_\mu,r_\mu,D_{\mu+1} \times \mathcal{O}_\mu^*} \leq \prod_{\mu=0}^{\infty} (1 + \frac{1}{2^\mu+2}) \leq 2
\]

and

\[
|D\Phi^v|_{r_0,r_v,D_v \times \mathcal{O}_v^*} \leq \sum_{\mu=0}^{v-1} |D\Phi^v|_{r_\mu,r_\mu,D_{\mu+1} \times \mathcal{O}_\mu^*} \prod_{0 \leq j \leq v-1, j \neq \mu} |D\Phi^v|_{r_j,r_j,D_{j+1} \times \mathcal{O}_j^*}
\]

\[
\leq 2 \sum_{\mu=0}^{v-1} |D\Phi^v - id|_{r_\mu,r_\mu,D_{\mu+1} \times \mathcal{O}_\mu^*} \leq C\varepsilon_0^{1-\frac{k}{2}}
\]

for sufficiently small \( \varepsilon \).

Thus, we have

\[
|\Phi^{v+1} - \Phi^v|_{r_0,D_{v+1} \times \mathcal{O}_v^*} \leq |D\Phi^v|_{r_0,r_v,D_v \times \mathcal{O}_v^*} |\Phi_v - id|_{r_v,D_{v+1} \times \mathcal{O}_v^*} \leq 2 |\Phi_v - id|_{r_v,D_{v+1} \times \mathcal{O}_v^*},
\]

\[
|\Phi^{v+1} - \Phi^v|_{r_0,D_{v+1} \times \mathcal{O}_v^*} \leq |D\Phi^v|_{r_0,r_v,D_v \times \mathcal{O}_v^*} |\Phi_v - id|_{r_v,D_{v+1} \times \mathcal{O}_v^*}
\]

\[
+ |D\Phi^v|_{r_0,r_v,D_v \times \mathcal{O}_v^*} |\Phi_v - id|_{r_v,D_{v+1} \times \mathcal{O}_v^*} \leq C\varepsilon_0^{1-\frac{k}{2}} |\Phi_v - id|_{r_v,D_{v+1} \times \mathcal{O}_v^*} + 2 |\Phi_v - id|_{r_v,D_{v+1} \times \mathcal{O}_v^*},
\]

which together with (4.1) implies that

\[
|\Phi^{v+1} - \Phi^v|_{r_0,D_{v+1} \times \mathcal{O}_v^*} \leq C\varepsilon_0^{1-\frac{k}{2}} + 2 \frac{\lambda_0}{\lambda_v} |\Phi_v - id|_{r_v,D_{v+1} \times \mathcal{O}_v^*} \leq C\varepsilon_0^{1-\frac{k}{2}} \varepsilon_v^{1-\frac{k}{2}}.
\]

Therefore, the \( \Phi^v \) converges uniformly on \( \bigcap_{v \geq 0} (D_v \times \mathcal{O}_v^*) = D_* \times \mathcal{O}^* \) to a Lipschitz continuous family of real analytic torus embeddings \( \Phi^\infty : \mathbb{T}^\infty \times \mathcal{O}^* \rightarrow \mathcal{P}^{a,p+2} \), where \( D_* = D(s_0/2, r_0/2) \) and \( \mathcal{O}^* = \bigcap_{v \geq 0} \mathcal{O}_v^* \) and

\[
|\Phi_v - id|_{r_0,D_v \times \mathcal{O}^*} \leq C\varepsilon_0^{1-\frac{k}{2}} \varepsilon_v^{1-\frac{k}{2}}.
\]

Thus, at the end of iteration, we obtain the Hamiltonian \( H^\infty \) of the transformed Hamiltonian system, that is
\[
H^\infty = \langle \omega, J \rangle + \langle \tilde{\Omega}, z \tilde{z} \rangle + \sum_{|\gamma| + |\sigma| \geq 3} \hat{P}^{\gamma \sigma}(\theta, \omega) z^{\gamma} \tilde{z}^{\sigma}, \tag{4.2}
\]

where \(\omega = (\omega_1, \omega_2, \cdots) \in \mathcal{O}^*, J = (J_1, J_2, \cdots), i_j \in \mathcal{I}_\infty\) and \(\tilde{\Omega}_j\) is close to \(\mu_j\). It is easy to see that the transformed Hamiltonian system has a solution

\[
\theta = \omega t + \text{const. (mod } 2\pi), \quad z = \tilde{z} = 0.
\]

Therefore, it is easy to obtain that for each \(\omega = (\omega_1, \omega_2, \cdots) \in \mathcal{O}^*\), the beam equation \([1.2] + [1.3]\) has an almost-periodic solution of the form

\[
u(t, x) = \sum_{j \geq 0} \frac{g_j(\omega t) \cos(jx)}{\sqrt{\mu_j}}
\]

where \(g_j(\omega t), \ j = 0, 1, 2 \cdots\), are almost-periodic in \(t\) with frequencies \(\omega\) and \(\|q\|_{a,p+2} = O(\varepsilon^{\frac{1}{2} - \frac{1}{2}p})\).

## 5 Measure estimate

At the \(v\)-th KAM step, we have to exclude the following resonant sets

\[
\mathcal{R}^v = \bigcup_{k,l} \mathcal{R}_{kl}^v,
\]

where

\[
\mathcal{R}_{kl}^v = \left\{ \omega^{b_v} \in \mathcal{O}^* : |\langle k, \omega^{b_v} \rangle + |\langle l, \Omega_v \rangle | < \frac{\alpha_v \langle l \rangle_2}{(1 + v^2)(|k| + 1)^{2b_v + 1}} \right\},
\]

\[
\mathcal{O}^* = \left\{ \omega^{b_v} : (\omega^{b_v}, \omega^{b_v}_r) \in \mathcal{O}^*_{v-1} \right\} \subset [0, 1]^{b_v}
\]

with \((k, l) \in \mathcal{Z}^{b_v}\) and \(\mathcal{O}^*_{v-1} = \mathcal{O}\). Here, \(\omega^{b_v}\) and \(\Omega_v\) are defined and Lipschitz continuous on \(\mathcal{O}^*_{v-1}\). Throughout all the iteration steps, we obtain a decreasing sequence of Cantor-like parameter sets \(\mathcal{O} \supset \mathcal{O}_0^* \supset \mathcal{O}_1^* \supset \cdots\). Hence, in the limit, we finally get a parameter set \(\mathcal{O}^* = \bigcap_{v=0}^\infty \mathcal{O}_v^*\).

**Lemma 5.1** Let the set \(\mathcal{O} = [0, 1]^\infty\) with probability measure. Then the parameter set \(\mathcal{O}^*\) obtained above satisfies

\[
\text{meas}(\mathcal{O} \setminus \mathcal{O}^*) \leq C \varepsilon^{\frac{1}{\pi} \mu},
\]

where meas is the standard probability measure on \([0, 1]\) and \(C > 0\) is an absolute constant.

**Proof** In view of (3.2) and (3.41), we can easily get

\[
|\Omega_v - \Omega_0|_{-\delta} = \sup_{j \geq 0} \left| \sum_{s=0}^{v-1} \varepsilon_s [B^1_{jj}]_s \right| \leq \sum_{s=0}^{v-1} \varepsilon_s \left| X_{B^{1,s}_{r,s}} \right|_{\mathcal{O}_s^*} \leq C \varepsilon_0^{1 - \frac{\mu}{\pi}} < \alpha_0 = \varepsilon_0^{\frac{1}{\pi} \mu}.
\]

Moreover, as \(\|\langle l, \Omega_0 \rangle\|_2 \rightarrow 1\) with \(\langle l \rangle_2 \rightarrow \infty\), there exists a positive constant \(\beta \gg 6\alpha_0 > 0\) such that \(\|\langle l, \Omega_0 \rangle\| > \beta \langle l \rangle_2\). Thus

\[
|\langle l, \Omega_v - \Omega_0 \rangle| \leq |l|_{\delta} |\Omega_v - \Omega_0|_{-\delta} \leq \langle l \rangle_2 |\Omega_v - \Omega_0|_{-\delta} \leq \alpha_0 \langle l \rangle_2.
\]
where $|l|_\delta = \sum |l_j|^{\delta}$ and

$$|\langle l, \Omega_v \rangle| > |\langle l, \Omega_0 \rangle| - |\langle l, \Omega_v - \Omega_0 \rangle| \geq (\beta - \alpha_0)\langle l \rangle_2.$$  

Case 1. When $|k| \leq \frac{\beta\langle l \rangle_2}{4}$, 

$$|\langle k, \omega^{b_v} \rangle + \langle l, \Omega_v \rangle| \geq |\langle l, \Omega_v \rangle| - |k||\omega^{b_v}| \geq (\beta - \alpha_0)\langle l \rangle_2 - \frac{1}{4}\beta\langle l \rangle_2 > 2\alpha_0\langle l \rangle_2 \geq \alpha_v\langle l \rangle_2,$$

then $\mathcal{R}^v_{kl}$ is empty.

Case 2. When $|k| > \frac{\beta\langle l \rangle_2}{4}$, let

$$g_0^v(\omega^{b_v}) = \langle k, \omega^{b_v} \rangle, \quad g_1^v(\omega^{b_v}) = \langle k, \omega^{b_v} \rangle \pm \Omega_v,$$

$$g_2^v(\omega^{b_v}) = \langle k, \omega^{b_v} \rangle \pm (\Omega_v + \Omega_v), \quad g_3^v(\omega^{b_v}) = \langle k, \omega^{b_v} \rangle + \Omega_v - \Omega_v \quad (i \neq j),$$

where

$$\Omega_{0j} = \mu_j = \sqrt{j^4 + m}, \quad \Omega_{vj} = \sqrt{j^4 + m + O(\varepsilon_0^{1-\frac{\chi}{2}})}.$$

Choosing a vector $y^{b_v} \in \{1, -1\}^{b_v}$ such that $\langle k, y^{b_v} \rangle = |k|$, then we obtain

$$\left| \frac{d}{dt} g_0^v(\omega^{b_v} + ty^{b_v}) \right| = |\langle k, y^{b_v} \rangle| = |k| > 0,$$

$$\left| \frac{d}{dt} g_1^v(\omega^{b_v} + ty^{b_v}) \right| \geq |\langle k, y^{b_v} \rangle| - O(\varepsilon_0^{1-\frac{\chi}{2}}) \geq \frac{1}{3}|k| > 0,$$

$$\left| \frac{d}{dt} g_2^v(\omega^{b_v} + ty^{b_v}) \right| \geq |\langle k, y^{b_v} \rangle| - O(\varepsilon_0^{1-\frac{\chi}{2}}) \geq \frac{1}{3}|k| > 0,$$

$$\left| \frac{d}{dt} g_3^v(\omega^{b_v} + ty^{b_v}) \right| \geq |\langle k, y^{b_v} \rangle| - O(\varepsilon_0^{1-\frac{\chi}{2}}) \geq \frac{1}{3}|k| > 0$$

for sufficiently small $\varepsilon$. Furthermore,

$$\text{card}\{l : \langle l \rangle_2 \leq \frac{4|k|}{\beta} \} \leq \text{card}\{l : |l|_1 < \frac{8|k|}{\beta} \} \leq C \left(\frac{|k|}{\beta} \right)^2.$$

If we exclude the measure $\sum_{v \geq 0} \sum_{0 \neq k \in \mathbb{Z}^{b_v}} \sum_{\langle l \rangle_2 < \frac{4|k|}{\beta}} \frac{6\alpha_v\langle l \rangle_2}{|k|(1 + v^2)(|k| + 1)^{2b_v + 2}}$ along some direction, accordingly, exclude the full measure along other directions, then such a residual set is a subset of $\mathcal{O}^*$. First, the excluded measure of the fixed direction satisfies

$$\sum_{v \geq 0} \sum_{0 \neq k \in \mathbb{Z}^{b_v}} \sum_{\langle l \rangle_2 < \frac{4|k|}{\beta}} \frac{6\alpha_v\langle l \rangle_2}{|k|(1 + v^2)(|k| + 1)^{2b_v + 2}} \leq \sum_{v \geq 0} \sum_{0 \neq k \in \mathbb{Z}^{b_v}} \frac{C\alpha_v}{|k|(1 + v^2)(|k| + 1)^{2b_v + 2}} \left(\frac{|k|}{\beta} \right)^3 \leq \sum_{v \geq 0} \frac{C\alpha_v}{\beta^3(1 + v^2)} \leq C\alpha_0^3 = C\varepsilon_0^{\frac{1}{2}}\rho.$$
by the convergence of \( \sum_{0 \neq k \in \mathbb{Z}^{b_v}} \frac{|k|^2}{(|k| + 1)^{2b_v + 2}} \), where \( C \) is an absolute constant independent of \( v, \varepsilon \).

Therefore, we have
\[
\text{meas}(\mathcal{O} \setminus \mathcal{O}^*_v) \leq C \varepsilon \frac{1}{\sigma^v}. 
\]

The proof of lemma 5.1 is complete. □

Lemma 5.1 shows that the total measure of all excluded parameter sets can be as small as we wish, and we finally get a Cantor-like parameter set \( \mathcal{O}^*_v = \bigcap_{v \geq 0} \mathcal{O}^*_v \). This completes the proof of Theorem 1.1. □

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7 Appendix

Lemma 7.1 [3] For \( \sigma > 0 \) and \( v > 0 \), the following inequalities hold true:
\[
\sum_{k \in \mathbb{Z}^n} e^{-2|k|\sigma} \leq \frac{1}{\sigma^n} (1 + e)^n, \\
\sum_{k \in \mathbb{Z}^n} e^{-2|k|\sigma} |k|^v \leq \left( \frac{v}{e} \right)^v \frac{1}{\sigma^{v+n}} (1 + e)^n.
\]

Lemma 7.2 [31] If \( A = (A_{ij}) \) is a bounded linear operator on \( \ell^2 \), then also \( B = (B_{ij}) \) with
\[
B_{ij} = \frac{|A_{ij}|}{|i - j|}, \quad i \neq j,
\]
and \( B_{ii} = 0 \) is a bounded linear operator on \( \ell^2 \), and \( \|B\| \leq \frac{\pi}{\sqrt{3}} \|A\| \).

Lemma 7.3 For \( k \in \mathbb{Z}^{b_v} \), we have that
\[
\sqrt{2^{b_v}} \sum_{k \in \mathbb{Z}^{b_v}} |k|^2 (1 + v^2) (|k| + 1)^{2b_v + 2} e^{-2|k|\sigma_v} \leq \left( \frac{16(2b_v + 3)}{e} \right)^{4b_v + 6} \frac{1}{\sigma^v}.
\]
Proof Since $2^{9b_v + 8}(1 + v^2)^4 \cdot 4^{b_v} \leq 4^{8b_v + 12}$, and using Lemma 7.1, we obtain

$$2^{b_v} \sum_{k \in \mathbb{Z}^{b_v}} |k|^4[(1 + v^2)(|k| + 1)^{2b_v + 2}]^4 e^{-2|k|\sigma_v} \leq 2^{b_v}(1 + v^2)^4 \sum_{k \in \mathbb{Z}^{b_v}} |k|^4(|k| + 1)^{8b_v + 8} e^{-2|k|\sigma_v}$$

$$\leq 2^{9b_v + 8}(1 + v^2)^4 \left(\frac{8b_v + 12}{e}\right)^{8b_v + 12} \frac{1}{\sigma_v^{9b_v + 12}} (1 + e)^{b_v}$$

$$\leq 2^{9b_v + 8}(1 + v^2)^4 \left(\frac{8b_v + 12}{e}\right)^{8b_v + 12} \frac{1}{\sigma_v^{9b_v + 12}} 4^{b_v}$$

$$\leq \left(\frac{16(2b_v + 3)}{e}\right)^{8b_v + 12} \frac{1}{\sigma_v^{9b_v + 12}}.$$ 

Thus

$$\sqrt{2^{b_v} \sum_{k \in \mathbb{Z}^{b_v}} |k|^2[(1 + v^2)(|k| + 1)^{2b_v + 2}]^4 e^{-2|k|\sigma_v}} \leq \left(\frac{16(2b_v + 3)}{e}\right)^{4b_v + 6} \frac{1}{\sigma_v^{5b_v + 6}}. \quad \Box$$

Data Availability

The data that supports the findings of this study are available within this article.

References

[1] M. Berti, M. Procesi, Quasi-periodic solutions of completely resonant forced wave equations, Comm. Partial Differential Equations 31 (2006) 959–985.

[2] N.N. Bogolyubov, Yu.A. Mitropolskii, A.M. Samoilenko, Methods of Accelerated Convergence in Nonlinear Mechanics, Springer, New York, 1976 (Russian original: Naukova Dumka, Kiev 1969).

[3] J. Bourgain, On invariant tori of full dimension for 1D periodic NLS, J. Funct. Anal. 229 (2005) 62–94.

[4] J. Chang, Y. Gao, Y. Li, Quasi-periodic solutions of nonlinear beam equation with prescribed frequencies, J. Math. Phys 56 (5) (2015) 437–450.

[5] W. Craig, C. E. Wayne, Newton’s method and periodic solutions of nonlinear wave equations, Comm. Pure Appl. Math. 46 (1993) 1409–1498.

[6] L. H. Eliasson, B. Grébert, S. B. Kuksin, KAM for the nonlinear beam equation, Geom. Funct. Anal. vol 26 (2016) 1588–1715.

[7] J. Geng, J. You, KAM tori of Hamiltonian perturbations of 1D linear beam equations, J. Math. Anal. Appl. 277 (2003) 104–121.

[8] J. Geng, J. You, A KAM theorem for Hamiltonian partial differential equations in higher dimensional spaces, Comm. Math. Phys. 262 (2006) 343–372.

[9] J. Geng, J. You, KAM tori for higher dimensional beam equations with constant potentials, Nonlinearity 19 (2006) 2405–2423.
[10] J. Geng, Invariant tori of full dimension for a nonlinear Schrödinger equation, J. Differential Equations 252 (2012) 1–34.

[11] L. Jiao, Y. Wang, The construction of quasi-periodic solutions of quasi-periodic forced Schrödinger equation, Comm. Pure Appl. Anal. 8 (2009) 1585–1606.

[12] S. B. Kuksin, Nearly integrable infinite-dimensional Hamiltonian systems, Lecture Notes in Math. vol. 1556, Springer, Berlin, 1993.

[13] H. Niu, J. Geng, Almost periodic solutions for a class of higher dimensional beam equations, Nonlinearity 20 (2007) 2499–2517.

[14] J. Pöschel, A KAM-Theorem for some nonlinear partial differential equations, Ann. Sc. Norm. Sup. Pisa 23 (1996) 119–148.

[15] J. Rui, J. Si, Quasi-periodic solutions for quasi-periodically forced nonlinear Schrödinger equations with quasi-periodic inhomogeneous terms, Phys. D 286-287 (2014) 1–31.

[16] J. Rui, B. Liu, J. Zhang, Almost periodic solutions for a class of linear Schrödinger equations with almost periodic forcing, J. Math. Phys. 57, 092702 (2016).

[17] J. Rui, B. Liu, Almost-periodic solutions of an almost-periodically forced wave equation, J. Math. Anal. Appl. 451 (2017) 629–658.

[18] Y. Shi, J. Xu, X. Xu, Quasi-periodic solutions of generalized Boussinesq equation with quasi-periodic forcing, Discrete Contin. Dyn. Syst. B 22 (2017) 2501–2519.

[19] J. Si, Quasi-periodic solutions of a non-autonomous wave equations with quasi-periodic forcing, J. Differential Equations, 252 (2012) 5274–5360.

[20] Y. Wang, Quasi-periodic solutions of a non-autonomous quasi-periodically forced nonlinear beam equation, Commun. Nonlinear Sci. Number. Simulat. 17 (2012) 2682–2700.

[21] X. Xu, J. Geng, KAM tori for higher dimensional beam equation with a fixed constant potential, Sci. China, Ser. A: Math. 52 (2009) 2007-2018.

[22] X. Yuan, Quasi-periodic solutions of completely resonant nonlinear wave equations, J. Differential Equations, 230 (2005) 213–274.

[23] M. Zhang, J. Si, Quasi-periodic solutions of nonlinear wave equations with quasi-periodic forcing, Phys. D 228 (2009) 2185–2215.

[24] M. Zhang, Quasi-periodic solutions of two dimensional Schrödinger equations with quasi-periodic forcing, Nonlinear Anal. 135 (2016) 1–34.