UNIVERSAL QUANTUM SEMIGROUPOIDS

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Abstract. We introduce the concept of a universal quantum linear semigroupoid (UQSGd), which is a weak bialgebra that coacts on a (not necessarily connected) graded algebra $A$ universally while preserving grading. We restrict our attention to algebraic structures with a commutative base so that the UQSGds under investigation are face algebras (due to Hayashi). The UQSGd construction generalizes the universal quantum linear semigroups introduced by Manin in 1988, which are bialgebras that coact on a connected graded algebra universally while preserving grading. Our main result is that when $A$ is the path algebra $kQ$ of a finite quiver $Q$, each of the various UQSGds introduced here is isomorphic to the face algebra attached to $Q$. The UQSGds of preprojective algebras and of other algebras attached to quivers are also investigated.

1. Introduction

Broadly, this work initiates the study of universal quantum symmetries in the weak context, which is needed as weak Hopf algebras naturally arise in the theory of fusion categories [ENO05], in capturing symmetries in subfactor theory [NV00] (e.g., as remarked in [BCG22]), and in studying solutions of the dynamical Yang-Baxter equation [EN01]. The goal of this work is to examine the universal quantum symmetries of $N$-graded algebras, that are not necessarily connected, within the framework of weak bialgebra coactions. All algebraic structures here are $k$-linear, for $k$ an arbitrary base field, and we reserve $\otimes$ to mean $\otimes_k$. Consider the hypotheses below.

Hypothesis 1.1 $(A, A_0)$. Let $A$ be a locally finite, $N$-graded $k$-algebra $A$, that is, $A$ has $k$-vector space decomposition $\bigoplus_{i \in N} A_i$ with $A_i \cdot A_j \subseteq A_{i+j}$, and $\dim_k A_i < \infty$. Further suppose that the degree 0 component $A_0$, which is a finite-dimensional $k$-subalgebra of $A$, is a commutative and separable $k$-algebra. In particular, this implies that $A_0$ is a Frobenius algebra over $k$. We say that $A$ is connected if $A_0 = k$, although we do not assume that condition here.

The prototypical examples of algebras $A$ whose symmetries we will examine are path algebras of finite quivers. Throughout, we fix the following notation.

Notation 1.2 $(Q, Q_0, Q_1, s, t, kQ, e_i, p, q, a, b)$. Let $Q = (Q_0, Q_1, s, t)$ be a finite quiver (i.e., a directed graph), where $Q_0$ is a finite collection of vertices, $Q_1$ is a finite collection of arrows, and $s, t : Q_1 \to Q_0$ denote the source and target maps, respectively. We read paths of $Q$ from left-to-right. Let $kQ$ be the path algebra attached to $Q$, which is the $k$-algebra generated by $\{e_i\}_{i \in Q_0}$ and $\{p\}_{p \in Q_1}$, with multiplication given by $m(e_i \otimes e_j) = \delta_{i,j} e_i$ for $i, j \in Q_0$ and
that is connected. The path algebra \( kQ \) is \( \mathbb{N} \)-graded by path length, where for each \( \ell \in \mathbb{N} \), \( (kQ)_\ell = k(Q_\ell) \), where \( Q_\ell \) consists of paths of length \( \ell \) in \( Q \). We usually use the letters \( a, b \) to denote paths in \( Q \).

Using path algebras as the prototypical examples of not necessarily connected \( k \)-algebras \( A \) in Hypothesis 1.1 is apt because if \( A \) is generated by \( A_1 \) over \( A_0 \), then \( A \) is isomorphic to a quotient of some path algebra \( kQ \). Namely, \( A_0 \) is isomorphic to the path algebra on an arrowless quiver \( Q_0 \) with \( |Q_0| = \dim_k A_0 \). Further, path algebras are free structures in the sense that they are tensor algebras: \( kQ \cong T_{kQ_1}(kQ_1) \), where \( kQ_1 \) is a \( kQ_0 \)-bimodule; we will return to this fact later in the introduction. Moreover, interesting examples of graded quotients of path algebras include preprojective algebras [GP79, Rin98] and superpotential algebras (see, e.g., [BSW10]).

Before we study quantum symmetries of the algebras \( A \) in Hypothesis 1.1, let us recall various notions of a universal bialgebra coacting on \( A \) in the case when \( A \) is connected, which are all due to Manin [Man88] (for the case when \( A \) is quadratic). First, we make the standing assumption that will be used throughout this work often without mention.

**Hypothesis 1.3.** Let \( C \) be a monoidal category of corepresentations of an algebraic structure \( H \). If \( A \) is an algebra in \( C \) (i.e., if \( A \) is an \( H \)-comodule algebra), then we assume that each graded component \( A_i \) of \( A \) is an object in \( C \) (i.e., is an \( H \)-comodule). Namely, we assume that all coactions of \( H \) on \( A \) preserve grading, or are linear, in this work.

Consider the following universal bialgebras that coact on a connected algebra \( A \) as in Hypothesis 1.1, from either the left or right.

**Definition 1.4** (left UQSG, \( O^{\text{left}}(A) \); right UQSG, \( O^{\text{right}}(A) \)). [Man88, Chapter 4 and Sections 5.1–5.8] Let \( A \) be a \( k \)-algebra as in Hypothesis 1.1 that is connected.

(a) Let \( O := O^{\text{left}}(A) \) be a bialgebra for which \( A \) is a left \( O \)-comodule algebra via left \( O \)-comodule map \( \lambda^O : A \to O \otimes A \). We call \( O^{\text{left}}(A) \) the left universal quantum linear semigroup (left UQSG) of \( A \) if, for any bialgebra \( H \) for which \( A \) is a left \( H \)-comodule algebra via left \( H \)-comodule map \( \lambda^H : A \to H \otimes A \), there exists a unique bialgebra map \( \pi : O \to H \) so that \( \lambda^H = (\pi \otimes \text{Id}_A) \lambda^O \).

(b) Let \( O := O^{\text{right}}(A) \) be a bialgebra for which \( A \) is a right \( O \)-comodule algebra via right \( O \)-comodule map \( \rho^O : A \to A \otimes O \). We call \( O^{\text{right}}(A) \) the right universal quantum linear semigroup (right UQSG) of \( A \) if, for any bialgebra \( H \) for which \( A \) is a right \( H \)-comodule algebra via right \( H \)-comodule map \( \rho^H : A \to A \otimes H \), there exists a unique bialgebra map \( \pi : O \to H \) so that \( \rho^H = (\text{Id}_A \otimes \pi) \rho^O \).

Other appearances of bialgebras that coact linearly and universally on algebraic structures from one side include the universal bi/Hopf algebras that coact on (skew-)polynomial algebras in [RRT02, LT07, CFR09], and the universal bi/Hopf algebras that coact on a superpotential algebra (or, equivalently that preserve a certain multilinear form) in [DVL90, BDV13, CW19].

Ideally, a universal bialgebra should behave ring-theoretically and homologically like the algebra that it coacts on. But this is not the case even when the algebra is a polynomial ring in two variables; see Example 5.7(b). Namely, \( O^{\text{left}}(k[x,y]) \) is a non-Noetherian algebra of
infinite Gelfand–Kirillov (GK) dimension, whereas \( k[x, y] \) is Noetherian of GK-dimension 2. Towards the goal above, one can consider a ‘smaller’ universal bialgebra introduced by Manin, which coacts an algebra \( A \) universally from the left and right via ‘transposed’ coactions. Indeed, Manin inquired if such a universal bialgebra reflects the behavior of \( A \) (in the connected and quadratic case) in [AST91, Introduction].

**Definition 1.5** (transposed UQSG, \( O^{\text{trans}}(A) \)). (cf., [Man88, Section 5.10, Chapters 6 and 7]) Let \( A \) be a \( k \)-algebra as in Hypothesis 1.1 that is connected.

(a) Let \( H \) be a bialgebra for which \( A \) is a left \( H \)-comodule algebra via left \( H \)-comodule map \( \lambda_H^A \), and for which \( A \) is a right \( H \)-comodule algebra via right \( H \)-comodule map \( \rho_H^A \). We call \( A \) a transposed \( H \)-comodule algebra if, for the transpose of \( \rho_H^A \),

\[
(\rho_H^A)^T : (ev_A \otimes \text{Id}_H \otimes \text{Id}_A)(\text{Id}_{A^*} \otimes \rho_H^A \otimes \text{Id}_{A^*})(\text{Id}_{A^*} \otimes \text{coev}_A) : A^* \to H \otimes A^*,
\]

we obtain \( \lambda_H^A \) by identifying a basis of \( A \) with the dual basis of \( A^* \).

(b) Let \( O := O^{\text{trans}}(A) \) be a bialgebra for which \( A \) is a transposed \( O \)-comodule algebra via left \( O \)-comodule map \( \lambda^O \) and right \( O \)-comodule map \( \rho^O \). We call \( O^{\text{trans}}(A) \) the transposed universal quantum linear semigroup (transposed UQSG) of \( A \) if, for any bialgebra \( H \) for which \( A \) is a transposed \( H \)-comodule algebra via left \( H \)-comodule map \( \lambda_H \) and a right \( H \)-comodule map \( \rho_H \), there exists a unique bialgebra map \( \pi : O \to H \) so that \( \lambda_H = (\pi \otimes \text{Id}_A)\lambda^O \) and \( \rho_H = (\text{Id}_A \otimes \pi)\rho^O \).

Other instances of bialgebras that coact linearly and universally on algebraic structures in a transposed manner include the universal bi/Hopf algebras that coact on skew-polynomial algebras in [Tak90, AST91] (these are special cases of the construction in [Man88]), and the universal bi/Hopf algebras that coact on a superpotential algebras in [CWW19] (this is a generalization of the construction in [Man88]).

In order to study the quantum symmetries of an algebra \( A \) which satisfies Hypothesis 1.1, but is not necessarily connected, we use coactions of weak bialgebras, which are structures that have the underlying structure of an algebra and a coalgebra, with weak compatibility conditions between these substructures [Definition 2.1]. For a weak bialgebra \( H \), there are two important coideal subalgebras, \( H_s \) and \( H_t \), called the source and target counital subalgebras, that measure how far \( H \) is from being a bialgebra. Namely, \( H \) is a bialgebra if and only if both \( H_s \) and \( H_t \) are the ground field \( k \). These subalgebras are always separable and Frobenius (see Proposition 2.3(a)).

Since we are considering quantum symmetries of algebras \( A \) whose degree 0 components \( A_0 \) are commutative separable algebras, we will work within the framework of weak bialgebras with commutative counital subalgebras, which are the same as \( V \)-face algebras by [Sch98, Theorem 4.3] and [Sch03, Theorems 5.1 and 5.5]. Here, \( V \) is a finite set. A key example of a \( V \)-face algebra is the weak bialgebra \( \mathcal{I}(Q) \) attached to a finite quiver \( Q \), which was introduced by Hayashi in [Hay93, Hay96]. In this case, \( V = Q_0 \) and a presentation of \( \mathcal{I}(Q) \) is provided in Example 2.6. Next, we propose a conjecture, which is a modification of [Hay99, Proposition 2.1] that remains unproved.
**Conjecture 1.6.** Suppose that $k$ is algebraically closed. If $H$ is a finite-dimensional weak bialgebra with commutative counital subalgebras, then $H$ is isomorphic to a weak bialgebra quotient of $S_t(Q)$ for some finite quiver $Q$.

This is akin to the result that every finite-dimensional algebra over an algebraically closed field is isomorphic to a quotient of a path algebra of some finite quiver (see, e.g., [ASS06, Theorem II.3.7]).

Returning to the study of the quantum symmetries of algebras $A$ as in Hypothesis 1.1, we proceed by realizing such an algebra $A$ as a comodule algebra over a weak bialgebra $H$ (which will eventually have commutative counital subalgebras). For a weak bialgebra $H$, let $^H A$ (resp., $A^H$) denote the category of left (resp., right) $H$-comodule algebras. An example of an object in $^H A$ (resp., in $A^H$) is $H_t$ (resp., $H_s$) via comultiplication [Examples 2.15 and 2.16]. Moreover, if an algebra $A$ satisfying Hypothesis 1.1 belongs to $^H A$ (resp., $A^H$), then so does the subalgebra $A_0$ [Remark 3.2]. We are now ready to introduce various notions of a universal weak bialgebra coacting on $A$ which are the focus of our work.

**Definition 1.7** (left UQSGd, $O^{\text{left}}(A)$; right UQSGd, $O^{\text{right}}(A)$; trans. UQSGd, $O^{\text{trans}}(A)$). Let $A$ be $k$-algebra as in Hypothesis 1.1.

(a) Let $O := O^{\text{left}}(A)$ be a weak bialgebra so that $A \in O^A$ via left $O$-comodule map $\lambda^O$ with $O_0 \cong A_0$ in $O^A$. We call $O^{\text{left}}(A)$ the left universal quantum linear semigroupoid (left UQSGd) of $A$ if, for any weak bialgebra $H$ such that $A \in {}^H A$ via left $H$-comodule map $\lambda^H$ with $H_t \cong A_0$ in $^H A$, there exists a unique weak bialgebra map $\pi : O \to H$ so that $\lambda^H = (\pi \otimes \text{Id}_A) \lambda^O$.

(b) Let $O := O^{\text{right}}(A)$ be a weak bialgebra so that $A \in A^O$ via right $O$-comodule map $\rho^O$ with $O_0 \cong A_0$ in $A^O$. We call $O^{\text{right}}(A)$ the right universal quantum linear semigroupoid (right UQSGd) of $A$ if, for any weak bialgebra $H$ such that $A \in A^H$ via right $H$-comodule map $\rho^H$ with $H_s \cong A_0$ in $A^H$, there exists a unique weak bialgebra map $\pi : O \to H$ so that $\rho^H = (\text{Id}_A \otimes \pi) \rho^O$.

(c) Let $O := O^{\text{trans}}(A)$ be a weak bialgebra so that $A \in O^A$ and $A \in A^O$ so that $A$ is a transposed $O$-comodule algebra, and with $O_0 \cong A_0$ in $O^A$ and $O_0 \cong A_0$ in $A^O$. We call $O^{\text{trans}}(A)$ the transposed universal quantum linear semigroupoid (transposed UQSGd) of $A$ if, for any weak bialgebra $H$ such that $A \in {}^H A$ and $A \in A^H$ for which $A$ is a transposed $H$-comodule algebra, and with $H_t \cong A_0$ in $^H A$ and $H_s \cong A_0$ in $A^H$, there exists a unique weak bialgebra map $\pi : O \to H$ so that $\lambda^H = (\pi \otimes \text{Id}_A) \lambda^O$ and $\rho^H = (\text{Id}_A \otimes \pi) \rho^O$.

Discussion about these definitions is provided in Remarks 3.3, 3.5–3.8, 3.11–3.14; the most important observation is that, without the condition that the ‘base’ of the weak bialgebra is isomorphic to the ‘base’ of the comodule algebra, such universal weak bialgebras are not likely to exist [Remark 3.3]. This brings us to our main result.

**Theorem 1.8.** For a finite quiver $Q$, the UQSGds $O^{\text{left}}(kQ)$, $O^{\text{right}}(kQ)$, and $O^{\text{trans}}(kQ)$ of the path algebra $kQ$ exist, and each is isomorphic to Hayashi’s face algebra $S_t(Q)$ as weak bialgebras.
For example, if we take $A$ to be the (connected, graded) free algebra $k\langle t_1, \ldots, t_n \rangle$, i.e., the path algebra on the $n$-loop quiver $Q_{n\text{-loop}}$, then the UQSGds of $A$ are the classical UQSGs of Definitions 1.4 and 1.5, and

$$O^{\text{left}}(A) \cong O^{\text{right}}(A) \cong O^{\text{trans}}(A) \cong \mathcal{H}(Q_{n\text{-loop}});$$

see Example 4.20. But these isomorphisms need not hold if $A$ is a proper quotient of $k\langle t_1, \ldots, t_n \rangle$ [Example 5.7]. In general, we have the following results for UQSGds of graded quotient algebras of $kQ$.

**Proposition 1.9.** Let $I \subseteq kQ$ be a graded ideal which is generated in degree 2 or greater. If $O^*(kQ/I)$ exists, then we have an isomorphism of weak bialgebras,

$$O^*(kQ/I) \cong \mathcal{H}(Q)/I,$$

for some biideal $I$ of $\mathcal{H}(Q)$. Here, $*$ means ‘left’, ‘right’, or ‘trans’.

Finally, in the case when $I$ is generated in degree 2, i.e., when $kQ/I$ is quadratic [Definition 5.8], we establish a non-connected generalization of [Man88, Theorem 5.10]. The quadratic dual $(kQ/I)^!$ of the quadratic algebra $kQ/I$ is reviewed in Definition 5.8.

**Theorem 1.10.** If the quotient algebra $kQ/I$ is quadratic, then we have that

(a) $O^{\text{left}}(kQ/I) \cong O^{\text{right}}((kQ/I)^!)^{\text{op}},$

(b) $O^{\text{right}}(kQ/I) \cong O^{\text{left}}((kQ/I)^!)^{\text{op}},$

(c) $O^{\text{left}}(kQ/I) \cong O^{\text{right}}(kQ/I)^{\text{cop}},$

(d) $O^{\text{trans}}(kQ/I) \cong O^{\text{trans}}((kQ/I)^!)^{\text{op}},$

as weak bialgebras.

See Remark 5.14 for a discussion of universal quantum semigroupoids of $N$-homogeneous algebras; such algebras include quiver potential algebras.

The paper is organized as follows. We present background material and preliminary results on weak bialgebras, monoidal categories of corepresentations of weak bialgebras, and (examples of) comodule algebras over weak bialgebras in Section 2. We introduce the theory of universal quantum linear semigroupoids (of algebras as in Hypothesis 1.1) in Section 3, including Definition 1.7. Our main result, Theorem 1.8 on the UQSGds of path algebras, is established in Section 4. Examples and results about UQSGds of quotients of path algebras are presented in Section 5, including Proposition 1.9 and Theorem 1.10. We end by providing directions for future investigation on universal quantum linear groupoids (i.e., universal weak Hopf algebras) in Section 6.

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2. Preliminaries

In this section, we provide background material and preliminary results on weak bialgebras [Section 2.1], and on corepresentation categories of weak bialgebras and algebras within them [Section 2.2]. We end by providing crucial examples of comodule algebras over weak bialgebras [Section 2.3].

2.1. Weak bialgebras. To begin, recall that a k-algebra is a k-vector space A equipped with a multiplication map $m : A \otimes A \to A$ and unit map $u : k \to A$ satisfying associativity and unitality constraints. We reserve the notation 1 to mean $1 := u(1_k)$. A k-coalgebra is a k-vector space C equipped with a comultiplication map $\Delta : C \to C \otimes C$ and counit map $\varepsilon : C \to k$ satisfying coassociativity and counitality constraints. If $(C, \Delta, \varepsilon)$ is a coalgebra, we use sumless Sweedler notation and write $\Delta(c) = c_1 \otimes c_2$ for $c \in C$.

Definition 2.1. A weak bialgebra over $k$ is a quintuple $(H, m, u, \Delta, \varepsilon)$ such that

1. $(H, m, u)$ is a k-algebra,
2. $(H, \Delta, \varepsilon)$ is a k-coalgebra,
3. $\Delta(ab) = \Delta(a)\Delta(b)$ for all $a, b \in H$,
4. $\varepsilon(abc) = \varepsilon(ab_1)c(b_2) = \varepsilon(ab_2)c(b_1)$ for all $a, b, c \in H$,
5. $\Delta^2(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1)$.

The difference between a bialgebra and a weak bialgebra can be understood as a weakening of the compatibility between the algebra and coalgebra structures. In a weak bialgebra, we still have that comultiplication is multiplicative (e.g., condition (iii)), but the counit is no longer multiplicative and we do not necessarily have $\Delta(1) = 1 \otimes 1$ or $\varepsilon(1) = 1$. Instead, we have weak multiplicity of the counit (condition (iv)) and weak comultiplicativity of the unit (condition (v)).

Definition 2.2 $(\varepsilon_s, \varepsilon_t, H_s, H_t)$. Let $(H, m, u, \Delta, \varepsilon)$ be a weak bialgebra. We define the source and target counital maps, respectively as follows:

$\varepsilon_s : H \to H, \quad x \mapsto 1_1\varepsilon(x1_2)$
$\varepsilon_t : H \to H, \quad x \mapsto \varepsilon(1_1x)1_2$.

We denote the images of these maps as $H_s := \varepsilon_s(H)$ and $H_t := \varepsilon_t(H)$. We call $H_s$ the source counital subalgebra and $H_t$ the target counital subalgebra of $H$ (see Proposition 2.3).

These subalgebras have special properties that we will need below.

Proposition 2.3. Let $H$ and $K$ be weak bialgebras. The following statements hold.

(a) $H_s$ and $H_t$ are separable Frobenius (so, finite-dimensional) k-algebras.
(b) $\varepsilon_s(y) = y$ for $y \in H_s$, and $\varepsilon_t(z) = z$ for $z \in H_t$.
(c) If $y \in H_s$ and $z \in H_t$, then $yz = zy$.
(d) $\Delta(y) = 1_1 \otimes y1_2 = 1_1 \otimes 1_2y$ for $y \in H_s$, and $\Delta(z) = 1_1z \otimes 1_2 = z1_1 \otimes 1_2$ for $z \in H_t$.
(e) $H_s$ (resp., $H_t$) is a left (resp., right) coideal subalgebra of $H$. We also have that

$H_t = \{(\varphi \otimes \text{Id})\Delta(1) : \varphi \in H^*\}, \quad H_s = \{(\text{Id} \otimes \varphi)\Delta(1) : \varphi \in H^*\}$.

(f) $\varepsilon_t$ is an anti-isomorphism from $H_s$ to $H_t$, i.e. $H_s \cong H_t^{\text{op}}$ as k-algebras.
(g) $H$ is a bialgebra if and only if $\dim_k H_s = 1$, if and only if $\dim_k H_t = 1$. 

(h) Any nonzero weak bialgebra morphism $\alpha : H \to K$ preserves counital subalgebras, i.e., $H_s \cong K_s$ and $H_t \cong K_t$ as $k$-algebras.

**Proof.** (a) This follows from [BCJ11, Corollary 4.4] and [BNS99, Proposition 2.11].

(b), (c), (d), (e) These parts follow from [BNS99, Section 2.2] and [NV02, Propositions 2.2.1 and 2.2.2].

(f) This is an immediate consequence of [BCJ11, Propositions 1.15 and 1.18].

(g) This is standard, and follows from (f) and [Nik02, Definition 3.1, Remark 3.2], for instance.

(h) The result for weak Hopf algebras is provided in [NV02, Proposition 2.3.3], and we generalize this to weak bialgebras as follows. Write $\Delta(1_H) = \sum_{i=1}^n w_i \otimes z_i$ with $\{w_i\}_{i=1}^n$ and $\{z_i\}_{i=1}^n$ linearly independent. By part (e), $H_s = \text{span}_k\{w_i\}_{i=1}^n$. Using the linear independence of $\{w_i\}_{i=1}^n$, we have

$$n = \dim_k H_s.$$ (2.4)

Since $z_j = \varepsilon_t(z_j) \sum_{i=1}^n \varepsilon(w_i z_j) z_i$ and $\{z_i\}_{i=1}^n$ are linearly independent, we also have

$$\varepsilon(w_i z_j) = \delta_{i,j}. (2.5)$$

Therefore

$$\dim_k H_s \overset{(2.4)}{=} n \overset{(2.5)}{=} \sum_{i=1}^n \varepsilon(w_j z_j) = \varepsilon_H((1_H)_1 (1_H)_2) \overset{(e)}{=} \varepsilon_K((1_K)_1 (1_K)_2) = \dim_k K_s.$$ (2.6)

Here, (e) holds because the nonzero map $\alpha : H \to K$ is an algebra and a coalgebra map; that is, $1_K = u_K(1_k) = \alpha u_H(1_k) = \alpha(1_H)$ and

$$\varepsilon_K m_K \Delta_K(1_K) = \varepsilon_K m_K (\alpha \otimes \alpha) \Delta_H(1_H) = \varepsilon_K \alpha m_H \Delta_H(1_H) = \varepsilon_H m_H \Delta_H(1_H).$$

Moreover, since $\alpha$ is a coalgebra map,

$$\Delta(1_K) = \sum_{i=1}^n \alpha(w_i) \otimes \alpha(z_i).$$

By part (e), $K_s = \text{span}\{\alpha(w_i)\}$, i.e., $\alpha|_{H_s} : H_s \to K_s$ is a surjective algebra morphism. Thus, $\alpha|_{H_s}$ is bijective. The proof for target subalgebras is similar. See also [Sch03, Lemma 6.3] for an alternative proof. \hfill \Box

In this paper, the main weak bialgebras of interest are the following examples due to Hayashi, see, e.g., [Hay96, Example 1.1]. Recall Notation 1.2.

**Example 2.6** (Hayashi’s face algebra attached to a quiver). For a finite quiver $Q$, we define the weak bialgebra $F(Q)$ as follows. As a $k$-algebra,

$$F(Q) = \frac{k\langle x_{i,j}, x_{p,q} \mid i,j \in Q_0, p,q \in Q_1 \rangle}{(R)},$$

for indeterminates $x_{i,j}$ and $x_{p,q}$ with relations $R$, given by:

$$x_{p,q}x_{p',q'} = \delta_{t(p),s(p')} \delta_{t(q),s(q')} x_{p,q} x_{p',q'}, \quad \text{(2.7)}$$

$$x_{s(p),s(q)} x_{p,q} = x_{p,q} x_{t(p),t(q)}, \quad \text{(2.8)}$$

for all $p, p', q, q' \in Q_1$, and

$$x_{i,j} x_{k,l} = \delta_{i,k} \delta_{j,l} x_{i,j} \quad \text{(2.9)}$$
for all $i, j, k, \ell \in Q_0$. (In fact, (2.7) follows from (2.8) and (2.9).) Then $\mathcal{H}(Q)$ is a unital $k$-algebra, with unit given by

$$(2.10) \quad 1_{\mathcal{H}(Q)} = \sum_{i,j \in Q_0} x_{i,j}.$$ 

Let $k \geq 2$ and suppose that $p_1 p_2 \cdots p_k, q_1 q_2 \cdots q_k \in Q_k$, where each $p_i, q_i \in Q_1$. As shorthand, we define the symbols

$$(2.11) \quad x_{p_1 \cdots p_k, q_1 \cdots q_k} := x_{p_1, q_1} x_{p_2, q_2} \cdots x_{p_k, q_k}. $$ 

With this notation, as a vector space we can write

$$\mathcal{H}(Q) = \bigoplus_{i \geq 0} \bigoplus_{a, b \in Q_i} k x_{a, b}.$$ 

For $a, b \in Q_\ell$, the coalgebra structure is given by

$$(2.12) \quad \Delta(x_{a, b}) = \sum_{c \in Q_\ell} x_{a, c} \otimes x_{c, b} \quad \text{and} \quad \varepsilon(x_{a, b}) = \delta_{a, b}. $$ 

It can be checked that this structure makes $\mathcal{H}(Q)$ a weak bialgebra.

We record the following facts about $\mathcal{H}(Q)$.

**Proposition 2.13.** Let $Q$ be a finite quiver.

(a) For $p_1, \ldots, p_k, q_1, \ldots, q_k \in Q_1$,

$$\varepsilon(x_{p_1, q_1} \cdots x_{p_k, q_k}) = \left( \delta_{t(p_1), s(p_2)} \cdots \delta_{t(p_{k-1}), s(p_k)} \right) \left( \delta_{t(q_1), s(q_2)} \cdots \delta_{t(q_{k-1}), s(q_k)} \right) \delta_{p_1, q_1} \cdots \delta_{p_k, q_k}. $$

(b) For each $j \in Q_0$, define

$$a_j = \sum_{i \in Q_0} x_{i, j} \quad \text{and} \quad a'_j = \sum_{i \in Q_0} x_{j, i}. $$

Then $\{a_j\}_{j \in Q_0}$ and $\{a'_j\}_{j \in Q_0}$ are complete sets of primitive orthogonal idempotents in $\mathcal{H}(Q)$ called the ‘face idempotents’ (see [Hay93]).

(c) As $k$-vector spaces, $\mathcal{H}(Q)_s = \bigoplus_{j \in Q_0} k a_j$ and $\mathcal{H}(Q)_t = \bigoplus_{j \in Q_0} k a'_j$.

**Proof.** (a) The equation clearly holds for $k = 1$. We will show this for $k = 2$; the rest follows by induction:

$$\varepsilon(x_{p_1, q_1} x_{p_2, q_2}) \overset{(2.7)}{=} \varepsilon(\delta_{t(p_1), s(p_2)} \delta_{t(q_1), s(q_2)} x_{p_1, q_1} x_{p_2, q_2}) \overset{(2.11)}{=} \delta_{t(p_1), s(p_2)} \delta_{t(q_1), s(q_2)} \varepsilon(x_{p_1, q_1} x_{p_2, q_2}) \overset{(2.12)}{=} \delta_{t(p_1), s(p_2)} \delta_{t(q_1), s(q_2)} \delta_{p_1, q_1} \delta_{p_2, q_2} = \delta_{t(p_1), s(p_2)} \delta_{t(q_1), s(q_2)} \delta_{p_1, q_1} \delta_{p_2, q_2}.$$ 

(b) This is straightforward to check.

(c) We get $\varepsilon_s(x_{a, b}) = \delta_{a, b} \sum_{i \in Q_0} x_{i, t(a)}$ and $\varepsilon_t(x_{a, b}) = \delta_{a, b} \sum_{j \in Q_0} x_{s(b), j}$ for $a, b \in Q_\ell$. \(\square\)
2.2. Corepresentation categories of weak bialgebras. Here, we discuss the monoidal categories of corepresentations of weak bialgebras, and algebras within these categories.

**Definition 2.14.** A monoidal category \( \mathcal{C} = (\mathcal{C}, \otimes, 1, \alpha, \lambda) \) consists of: a category \( \mathcal{C} \); a bifunctor \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \); a natural isomorphism \( \alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \sim X \otimes (Y \otimes Z) \) for each \( X, Y, Z \in \mathcal{C} \); an object \( 1 \in \mathcal{C} \); and natural isomorphisms \( l_X : 1 \otimes X \sim X, r_X : X \otimes 1 \sim X \) for each \( X \in \mathcal{C} \), such that the pentagon and triangle axioms are satisfied (see [EGNO15, Equations 2.2, 2.10]).

An example of a monoidal category is \( \text{Vec}_k \), the category of finite-dimensional \( k \)-vector spaces, with \( \otimes = \otimes_k \), \( 1 = k \), and with the canonical associativity and unit isomorphisms. If \( H \) is a weak bialgebra, we can endow the category of right (or left) \( H \)-comodules with the structure of a monoidal category as follows.

**Example 2.15 ([BCJ11, Nil]).** For a weak bialgebra \( H = (H, m, u, \Delta, \varepsilon) \), the category \( M^H \) of right \( H \)-comodules can be given the structure of a monoidal category:

\[ M^H = (\text{Comod-}H, \otimes, 1 = H_s, \alpha = \alpha_{\text{Vec}_k}, 1, r). \]

Here, for \( M, N \in M^H \), the monoidal product of \( M \) and \( N \) is defined to be

\[ M \otimes N := \{ m \otimes n \in M \otimes N \mid m \otimes n = \varepsilon(m_{[1]}n_{[1]})m_{[0]} \otimes n_{[0]} \}. \]

The counital subalgebra \( H_s \) is naturally a right \( H \)-comodule since the image of \( \Delta|_{H_s} \) is a subspace of \( H_s \otimes H \), and so \( \Delta|_{H_s} \) can be viewed as a map \( H_s \to H_s \otimes H \). By [BCJ11, Theorem 3.1], \( H_s \) is the unit object of the monoidal category \( M^H \). By [BCJ11, Section 3], the monoidal category \( M^H \) has unit isomorphisms:

\[ l_M : H_s \otimes M \to M, \quad x \otimes m = \varepsilon(x_{[1]}m_{[1]})x_{[0]} \otimes m_{[0]} \mapsto \varepsilon(xm_{[1]})m_{[0]}, \]

\[ r_M : M \otimes H_s \to M, \quad m \otimes x = \varepsilon(m_{[1]}x_{[1]})m_{[0]} \otimes x_{[0]} \mapsto \varepsilon(m_{[1]}x)m_{[0]}, \]

for all \( M \in M^H \).

**Example 2.16.** Likewise, for a weak bialgebra \( H = (H, m, u, \Delta, \varepsilon) \), the category \( H^M \) of left \( H \)-comodules can be given the structure of a monoidal category:

\[ H^M = (H-\text{Comod}, \otimes, 1 = H_t, \alpha = \alpha_{\text{Vec}_k}, 1, r). \]

To the best of our knowledge, the details of the monoidal structure of this category are not explicitly stated in the literature, so we include them for the convenience of the reader. For \( M, N \in H^M \), the monoidal product of \( M \) and \( N \) is defined to be

\[ M \otimes N := \{ m \otimes n \in M \otimes N \mid m \otimes n = \varepsilon(m_{(-1)}n_{(-1)})m_{[0]} \otimes n_{[0]} \}. \]

The restriction of the coproduct \( \Delta|_{H_t} \), viewed as a map \( H_t \to H \otimes H_t \) makes \( H_t \) a left \( H \)-comodule which is the unit object of the monoidal category \( H^M \). Explicitly, the unit isomorphisms of \( H^M \) are given by:

\[ l_M : H_t \otimes M \to M, \quad x \otimes m = \varepsilon(x_{(-1)}m_{(-1)})x_{[0]} \otimes m_{[0]} \mapsto \varepsilon(xm_{(-1)})m_{[0]} \]

\[ r_M : M \otimes H_t \to M, \quad m \otimes x = \varepsilon(m_{(-1)}x_{(-1)})m_{[0]} \otimes x_{[0]} \mapsto \varepsilon(m_{(-1)}x)m_{[0]}, \]

for all \( M \in H^M \).

Now we turn our attention to algebras in monoidal categories.
Definition 2.17 ($\text{Alg}(\mathcal{C})$). Let $(\mathcal{C}, \otimes, 1, \alpha, \iota, r)$ be a monoidal category. An algebra in $\mathcal{C}$ is a triple $(A, m, u)$, where $A$ is an object in $\mathcal{C}$, and $m : A \otimes A \to A$, $u : 1 \to A$ are morphisms in $\mathcal{C}$, satisfying unitality and associativity constraints:

$$m(m \otimes \text{Id}) = m(\text{Id} \otimes m)\alpha_{A,A,A}, \quad m(u \otimes \text{Id}) = l_A, \quad m(\text{Id} \otimes u) = r_A.$$ 

A morphism of algebras $(A, m_A, u_A)$ to $(B, m_B, u_B)$ is a morphism $f : A \to B$ in $\mathcal{C}$ so that $fm_A = m_B \otimes_B (f \otimes f)$ and $fu_A = u_B$. Algebras in $\mathcal{C}$ and their morphisms form a category, which we denote by $\text{Alg}(\mathcal{C})$.

Algebras in $\text{Vec}_k$ are the same as $k$-algebras.

Now we consider algebras that have the structure of a comodule over a weak bialgebra $H$. There are two related notions: we can consider the objects in $\text{Alg}(\mathcal{M}^H)$ (or, $\text{Alg}(H\mathcal{M})$), or we can consider $k$-algebras (i.e., objects of $\text{Alg}(\text{Vec}_k)$) which are also right (or, left) $H$-comodules such that the algebra and comodule structures are compatible as done below.

Definition 2.18 ($H^A, A^H$). Let $H$ be a weak bialgebra.

(a) Consider the category $H^A$ of left $H$-comodule algebras defined as follows. The objects of $H^A$ are objects of $\text{Alg}(\text{Vec}_k)$,

$$(A, m_A : A \otimes A \to A, u_A : k \to A),$$

with $1_A := u_A(1_k)$, so that the $k$-vector space $A$ is a left $H$-comodule via

$$\lambda_A : A \to H \otimes A, \quad a \mapsto a[-1] \otimes a[0],$$

the multiplication map $m_A$ is compatible with $\lambda_A$ in the sense that

$$(ab)[-1] \otimes (ab)[0] = a[-1] b[-1] \otimes a[0] b[0] \quad \forall a, b \in A;$$

the unit map $u_A$ is compatible with $\lambda_A$ in the sense that

$$\lambda_A(1_A) \in H_s \otimes A.$$ 

The morphisms of $H^A$ are maps in $\text{Alg}(\text{Vec}_k)$ that are also $H$-comodule maps.

(b) Consider the category $A^H$ of right $H$-comodule algebras defined as follows. The objects of $A^H$ are objects of $\text{Alg}(\text{Vec}_k)$,

$$(A, m_A : A \otimes A \to A, u_A : k \to A),$$

with $1_A := u_A(1_k)$, so that the $k$-vector space $A$ is a right $H$-comodule via

$$\rho_A : A \to A \otimes H, \quad a \mapsto a[0] \otimes a[1],$$

the multiplication map $m_A$ is compatible with $\rho_A$ in the sense that

$$(ab)[0] \otimes (ab)[1] = a[0] b[0] \otimes a[1] b[1] \quad \forall a, b \in A;$$

the unit map $u_A$ is compatible with $\rho_A$ in the sense that

$$\rho_A(1_A) \in A \otimes H_t.$$ 

The morphisms of $A^H$ are maps in $\text{Alg}(\text{Vec}_k)$ that are also $H$-comodule maps.

The categories $H^A$ and $\text{Alg}(H^A)$ (likewise, $A^H$ and $\text{Alg}(A^H)$) are essentially the same.
Proposition 2.21. [WWW22, Theorem 4.4] There is an isomorphism of categories between \( \text{Alg}(\mathcal{M}^H) \) and \( A^H \), and between \( \text{Alg}(^H\mathcal{M}) \) and \( ^HA \).

In [WWW22, Theorem 4.5], the functors between \( \text{Alg}(\mathcal{M}^H) \) and \( A^H \) are given explicitly. For the isomorphism between \( \text{Alg}(^H\mathcal{M}) \) and \( ^HA \), the proof should be adjusted using the structures in Example 2.16 rather than Example 2.15.

2.3. Examples. Now we provide some examples of comodule algebras over weak bialgebras, which will be important in the rest of the paper.

Example 2.22. Consider Hayashi’s face algebra \( \mathcal{F}(Q) \) from Example 2.6. It is straightforward to check that the path algebra \( kQ \) belongs to \( \mathcal{F}(Q)A \) and to \( A^\mathcal{F}(Q) \) via the coactions:

\[
\lambda : kQ \to \mathcal{F}(Q) \otimes kQ \quad \rho : kQ \to kQ \otimes \mathcal{F}(Q)
\]

\[
e_j \mapsto \sum_{i \in Q_0} x_{j,i} \otimes e_i \\
q \mapsto \sum_{p \in Q_1} x_{p,q} \otimes p
\]

for \( j \in Q_0 \) and \( q \in Q_1 \). See [WWW22, Example 4.10] for verification that \( kQ \in A^\mathcal{F}(Q) \).

Example 2.23. Let \( Q_{\bullet \bullet} \) be the quiver with two vertices and no arrows. Let \( D \) be the algebra

\[
D = \frac{k(x,y)}{(x^2 = x, y^2 = y, xy = yx = 0)}
\]

so that \( 1_D = x + y \) (as an algebra, \( D \cong kQ_{\bullet \bullet} \)). Define a coproduct \( \Delta_D \) on \( D \) by

\[
\Delta_D(x) = x \otimes x + y \otimes y, \quad \Delta_D(y) = x \otimes y + y \otimes x
\]

and a counit \( \varepsilon_D \) by

\[
\varepsilon_D(x) = 1_k, \quad \varepsilon_D(y) = 0_k.
\]

One can verify that this makes \( D \) a bialgebra.

One can show that \( kQ_{\bullet \bullet} \) is a transposed \( D \)-comodule algebra [Definition 1.5(a)] under the left and right coactions:

\[
\lambda : kQ_{\bullet \bullet} \to D \otimes kQ_{\bullet \bullet}, \quad \rho : kQ_{\bullet \bullet} \to kQ_{\bullet \bullet} \otimes D, \quad e_1 \mapsto x \otimes e_1 + y \otimes e_2, \quad e_2 \mapsto y \otimes e_1 + x \otimes e_2; \quad e_1 \mapsto e_1 \otimes x + e_2 \otimes y, \quad e_2 \mapsto e_1 \otimes y + e_2 \otimes x.
\]

For our next example, we will need the following two lemmas. These lemmas are well-known, and their proofs are routine.

Lemma 2.24. If \( (H,m_H,u_H,\Delta_H,\varepsilon_H) \) and \( (K,m_K,u_K,\Delta_K,\varepsilon_K) \) are weak bialgebras, then \( H \oplus K \) is a weak bialgebra with the following structure for all \( h,g \in H,k,l \in K \):

- multiplication: \( (h,k)(g,l) := (hg,kl) \);
- unit: \( 1_{H \oplus K} := (1_H,1_K) \);
- comultiplication: \( \Delta_{H \oplus K}((h,k)) := (h_1,0) \otimes (h_2,0) + (0,k_1) \otimes (0,k_2) \);
- counit: \( \varepsilon_{H \oplus K}((h,k)) := \varepsilon_H(h) + \varepsilon_K(k) \).

We also have that

\[
(\varepsilon_{H \oplus K})_i(h,k) = ((\varepsilon_H)_i(h), (\varepsilon_K)_i(k)), \quad (\varepsilon_{H \oplus K})_s(h,k) = ((\varepsilon_H)_s(h), (\varepsilon_K)_s(k)).
\]

□
Lemma 2.25. Suppose that $V$ is a right $H$-comodule via
\[ \rho_H : V \to V \otimes H, \quad v \mapsto v_{[0]} \otimes v_{[1]} . \]
Then $V$ is a right $(H \oplus K)$-comodule via
\[ \rho : V \to V \otimes (H \oplus K), \quad v \mapsto v_{[0]} \otimes (v_{[1]}, 0). \]
Furthermore, if $V$ is a right $H$-comodule algebra via $\rho_H$, then $V$ is a right $(H \oplus K)$-comodule algebra via $\rho$. A similar statement holds for left $H$-comodules and left $H$-comodule algebras.
\[ \Box \]

Example 2.26. Let $Q_{\bullet \bullet}$ be the quiver with two vertices and no arrows, and recall the bialgebra $D$ defined in Example 2.23. A presentation of $D$ is given by
\[ D := \mathbb{k} \langle y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2} \rangle \]
with unit $1_D = y_{1,1} + y_{1,2}$.

Claim 1. $\mathbb{k}Q_{\bullet \bullet}$ is a left and right $(D \oplus D)$-comodule algebra via linear coactions.

Proof of Claim 1. The coalgebra structure is given by
\[ \Delta_D(y_{i,j}) = \sum_{k \in (Q_{\bullet \bullet})_0} y_{i,k} \otimes y_{k,j}, \quad \varepsilon_D(y_{i,j}) = \delta_{i,j}, \quad \text{for all } i, j \in (Q_{\bullet \bullet})_0. \]

With this presentation, $D$ left and right coacts linearly on $\mathbb{k}Q_{\bullet \bullet}$ via
\[ \mathbb{k}Q_{\bullet \bullet} \to D \otimes \mathbb{k}Q_{\bullet \bullet}, \quad e_i \mapsto \sum_{j \in (Q_{\bullet \bullet})_0} y_{i,j} \otimes e_j \]
\[ \mathbb{k}Q_{\bullet \bullet} \to \mathbb{k}Q_{\bullet \bullet} \otimes D, \quad e_i \mapsto \sum_{j \in (Q_{\bullet \bullet})_0} e_j \otimes y_{j,i}. \]

By Lemma 2.25 and Example 2.23, we have that the coactions
\[ \lambda : \mathbb{k}Q_{\bullet \bullet} \to (D \oplus D) \otimes \mathbb{k}Q_{\bullet \bullet}, \quad e_i \mapsto \sum_{j \in (Q_{\bullet \bullet})_0} (y_{i,j}, 0) \otimes e_j \]
\[ \rho : \mathbb{k}Q_{\bullet \bullet} \to \mathbb{k}Q_{\bullet \bullet} \otimes (D \oplus D), \quad e_i \mapsto \sum_{j \in (Q_{\bullet \bullet})_0} e_j \otimes (y_{j,i}, 0) \]
yield the claim.
\[ \Box \]

Claim 2. $(D \oplus D)_t \cong \mathbb{k}(Q_{\bullet \bullet})_0$ as algebras over $\mathbb{k}$.

Proof of Claim 2. Consider the morphism
\[ \psi : \mathbb{k}(Q_{\bullet \bullet})_0 \to (D \oplus D)_t, \quad e_1 \mapsto (1_D, 0), \quad e_2 \mapsto (0, 1_D). \]

First, we will show that as a $\mathbb{k}$-vector space, $(D \oplus D)_t = \text{Span}_\mathbb{k} \{(1_D, 0), (0, 1_D)\}$. By Lemma 2.24, we have
\[ (\varepsilon_{D\oplus D})_t(1_D, 0) = ((\varepsilon_D)_t(1_D), 0) = (1_D, 0), \quad (\varepsilon_{D\oplus D})_t(0, 1_D) = (0, (\varepsilon_D)_t(1_D)) = (0, 1_D). \]
Therefore, $\text{Span}_\mathbb{k} \{(1_D, 0), (0, 1_D)\} \subseteq (D \oplus D)_t$. To show the reverse inclusion, note that for $a, b \in D$ we have
\[ (\varepsilon_{D\oplus D})_t(a, b) = ((\varepsilon_D)_t(a), (\varepsilon_D)_t(b)) = (\varepsilon_D(a)1_D, \varepsilon_D(b)1_D), \]
where the last equality holds because $D$ is a bialgebra. Thus, $\mathbb{k}(Q_{\bullet \bullet})_0 \cong \dim(D \oplus D)_t$ as $\mathbb{k}$-vector spaces; here, $\dim(D \oplus D)_t = \dim \mathbb{k}(Q_{\bullet \bullet})_0 = 2$. It is also clear that $\psi$ preserves the unit and multiplication. Therefore, $\psi$ is an isomorphism of $\mathbb{k}$-algebras.
Claim 3. \( k(Q_{\bullet \bullet})_0 \not\cong (D \oplus D)_t \) as left \((D \oplus D)\)-comodules, where \( k(Q_{\bullet \bullet})_0 \) is a left \((D \oplus D)\)-comodule via Claim 1, and \((D \oplus D)_t \) is naturally a left \((D \oplus D)\)-comodule via comultiplication [Example 2.16].

**Proof of Claim 3.** By way of contradiction, suppose that we have an isomorphism \( \varphi : k(Q_{\bullet \bullet})_0 \rightarrow (D \oplus D)_t \) of left \((D \oplus D)\)-comodules. Explicitly, the comodule structures are given by

\[
\lambda : k(Q_{\bullet \bullet})_0 \rightarrow (D \oplus D) \otimes k(Q_{\bullet \bullet})_0, \quad e_i \mapsto \sum_{j \in (Q_{\bullet \bullet})_0} (y_{i,j}, 0) \otimes e_j,
\]

\[
\lambda_t := \Delta_{(D \oplus D)} : (D \oplus D)_t \rightarrow (D \oplus D) \otimes (D \oplus D)_t, \quad (1_D, 0) \mapsto (1_D, 0) \otimes (1_D, 0),
\]

\[
0, 1_D) \mapsto (0, 1_D) \otimes (0, 1_D).
\]

Since \((D \oplus D)_t = \text{Span}_k \{(1_D, 0), (0, 1_D)\}\), (see proof of Claim 2), we can write

\[
\varphi(e_i) = \alpha_i(1_D, 0) + \beta_i(0, 1_D),
\]

for some \( \alpha_i, \beta_i \in k \). Since \( \varphi \) is a left \((D \oplus D)\)-comodule map, \((\text{Id}_{(D \oplus D)} \otimes \varphi) \lambda = \lambda_t \varphi \). In particular,

\[
\sum_{j \in (Q_{\bullet \bullet})_0} (y_{i,j}, 0) \otimes (\alpha_j(1_D, 0) + \beta_j(0, 1_D)) = \sum_{j \in (Q_{\bullet \bullet})_0} (y_{i,j}, 0) \otimes \varphi(e_j)
\]

\[
= (\text{Id}_{(D \oplus D)} \otimes \varphi) \left( \sum_{j \in (Q_{\bullet \bullet})_0} (y_{i,j}, 0) \otimes e_j \right)
\]

\[
= (\text{Id}_{(D \oplus D)} \otimes \varphi) \lambda(e_i)
\]

\[
= \lambda_t(\alpha_i(1_D, 0) + \beta_i(0, 1_D))
\]

\[
= \alpha_i(1_D, 0) \otimes (1_D, 0) + \beta_i(0, 1_D) \otimes (0, 1_D).
\]

Notice that the left hand side is contained in \((D \oplus 0) \otimes (D \oplus D)_t\). Therefore, we must have that \( \beta_i = 0 \), since if not, the right hand side is not contained in \((D \oplus 0) \otimes (D \oplus D)_t\). Therefore, \( \varphi \) is not surjective and not an isomorphism of \((D \oplus D)\)-comodules. \(\square\)

### 3. Universal linear coactions on graded algebras

In this section, we introduce the notion of a weak bialgebra that coacts linearly and universally on a graded algebra \( A \) as in Hypothesis 1.1. The universal weak bialgebras coacting on \( A \) are defined below in Definitions 3.4 and 3.10 below; we call them *universal quantum linear semigroupoids*. Recall here that \( A \) is \( \mathbb{N} \)-graded \( k \)-algebra with \( \dim_k A_i < \infty \) for all \( i \in \mathbb{N} \), such that \( A_0 \) is a commutative, separable (so, Frobenius) \( k \)-algebra (we discuss how the assumptions on \( A_0 \) are used in Remarks 3.5 and 3.11 below). Moreover, we say that \( A \) is *connected* if \( A_0 = k \), and that \( A \) is *non-connected* otherwise.

To proceed, we reinterpret the standing assumption, Hypothesis 1.3 from the introduction, as follows.

**Hypothesis 3.1.** [\( \lambda, \lambda_i, \rho, \rho_i \)] Let \( H \) be a weak bialgebra, and recall the notion of a \( H \)-comodule algebra from Definition 2.18. From now on, we impose the assumptions below.

(a) Each left \( H \)-comodule algebra structure on \( A \) will be *linear* in the sense that, for the structure map \( \lambda := \lambda_A^H : A \rightarrow H \otimes A \), the restriction \( \lambda|_{A_i} := \lambda_i \) makes \( A_i \) a left \( H \)-comodule for each \( i \).
(b) Each right $H$-comodule algebra structure on $A$ will be linear in the sense that, for the structure map $\rho := \rho^H : A \to A \otimes H$, the restriction $\rho|_{A_i} := \rho_i$ makes $A_i$ a right $H$-comodule for each $i$.

Remark 3.2. If $H$ left coacts linearly via $\lambda$ on $A$, then $A_0$ is a left $H$-comodule algebra via $\lambda_0$. By [WWW22, Theorem 4.5], we can view $A_0$ as an object in the category $^H_A$ [Definition 2.18]. The analogous statement holds for right coactions.

Next, we discuss a naive notion of a weak bialgebra coacting universally on $A$, that is, by merely replacing 'bialgebra' with 'weak bialgebra' in the definition of a universal quantum linear semigroup [Definition 1.4]. This weak bialgebra fails to exist, even for an easy example of non-connected graded algebra $A$, as seen below.

Remark 3.3. Let $A$ be an algebra satisfying Hypothesis 1.1. Suppose that there exists a weak bialgebra $U := U(A)$ that left coacts on $A$ so that, for every weak bialgebra $H$ that left coacts on $A$, there exists a unique weak bialgebra map $\pi : U \to H$ so that $(\pi \otimes \text{Id}_A)\lambda^U = \lambda^H$. We will show that in general, such a weak bialgebra fails to exist.

Let $H$ be any nonzero weak bialgebra which left coacts on $A$. Then since there exists a weak bialgebra map $\pi : U \to H$, we have that $\dim_k U_i = \dim_k H_i$ by Proposition 2.3(h).

Now take $A = kQ_{\bullet\bullet}$ as in Section 2.3, which is a comodule algebra over both the bialgebra $D$ (Example 2.23) and also over the weak bialgebra $D \oplus D$ (Example 2.26). By the above, if we take $H = D$, then we have $\dim_k U_i = \dim_k D_i$, and so by Proposition 2.3(g), $\dim_k U_i = 1$. On the other hand, we can also substitute $H$ by $D \oplus D$ and have $\dim_k U_i = \dim_k (D \oplus D)_i$; by Claim 2 of Example 2.26, $\dim_k (D \oplus D)_i = \dim_k k(Q_{\bullet\bullet})_0 = 2$. Hence $1 = \dim_k U_i = 2$, which is a contradiction. Hence, $U(kQ_{\bullet\bullet})$ does not exist.

To remedy the non-existence issue in the remark above, we impose an extra hypothesis relating $A_0$ to the counital subalgebras of our universally coacting weak bialgebras. This is motivated by Claim 3 in Example 2.26. Our main result, Theorem 4.17 below, shows that with this additional hypothesis, for any path algebra $kQ$, there exists a universal weak bialgebra coacting on $kQ$.

Definition 3.4 (left UQSGd, $\mathcal{O}^{\text{left}}(A)$; right UQSGd, $\mathcal{O}^{\text{right}}(A)$). Take a $k$-algebra $A$ as in Hypothesis 1.1.

(a) Let $\mathcal{O} := \mathcal{O}^{\text{left}}(A)$ be a weak bialgebra that left coacts on $A$ with $A_0 \cong \mathcal{O}_i$ in $\mathcal{O}_i$, so that for any weak bialgebra $H$ that left coacts on $A$ with $A_0 \cong H_i$ in $^H_A$, there is a unique weak bialgebra map $\pi : \mathcal{O} \to H$ so that $(\pi \otimes \text{Id}_A)\lambda^\mathcal{O} = \lambda^H$. We refer to $\mathcal{O}^{\text{left}}(A)$ as the left universal quantum linear semigroupoid (left UQSGd) of $A$, and refer to its coaction on $A$ as universally base preserving.

(b) Let $\mathcal{O} := \mathcal{O}^{\text{right}}(A)$ be a weak bialgebra that right coacts on $A$ with $A_0 \cong \mathcal{O}_i$ in $\mathcal{O}_i$, so that for any weak bialgebra $H$ that right coacts on $A$ with $A_0 \cong H_i$ in $^H_A$, there is a unique weak bialgebra map $\pi : \mathcal{O} \to H$ so that $\text{Id}_A \otimes \pi \rho^\mathcal{O} = \rho^H$. We refer to $\mathcal{O}^{\text{right}}(A)$ as the right universal quantum linear semigroupoid (right UQSGd) of $A$, and refer to its coaction on $A$ as universally base preserving.

Here, the left (resp., right) $H$-coaction on $A_0$ is given by $\lambda_0$ (resp., $\rho_0$) as in Remark 3.2, and the left (resp., right) $H$-coaction on $H_i$ (resp., on $H_s$) is given by $\Delta_H$ as in Example 2.16.
We make several remarks about the definition above.

Remark 3.5. We use the assumption that $A_0$ is Frobenius and separable (from Hypothesis 1.1) in the definition above and in Definition 3.10 below. Namely, for any weak bialgebra $H$, the counital subalgebras $H_s$ and $H_t$ are Frobenius and separable $k$-algebras [Proposition 2.3(a)]. We do not need to require that $A_0$ is commutative for Definition 3.4.

Remark 3.6. Note that the notion of universally base preserving coaction is weaker than the naïve notion of a universal coaction discussed in Remark 3.3. Thus, the UQSGds in Definition 3.4 are more likely to exist than the universal weak bialgebras in Remark 3.3.

Remark 3.7. Observe that the universally base preserving condition takes a simple form when viewed through a categorical lens. By Proposition 2.21, we have categorical isomorphisms $\text{Alg}(H,M) \cong \text{Alg}^{H} A$ and $\text{Alg}(M^H) \cong A^H$, and by Examples 2.15 and 2.16, the unit objects of the monoidal categories $\text{Alg}(H,M)$, $\text{Alg}(M^H)$ are $H_t$, $H_s$, respectively. So, the requirement that $H_t \cong A_0$ in $H^A$ (resp., $H_s \cong A_0$ in $A^H$) is equivalent to requiring that $A_0$ is isomorphic to the unit object of the monoidal category $\text{Alg}(H,M)$ (resp., $\text{Alg}(M^H)$).

Remark 3.8. Definition 3.4 generalizes Definition 1.4, the notion of a one-sided UQSG (or, universal bialgebra that coacts from one side). Indeed, take $A$ a locally finite, connected $\mathbb{N}$-graded algebra and suppose that $O^{\text{left}}(A)$ exists. Then, $(O^{\text{left}}(A))_i = A_0 = k$, as $k$-vector spaces. So, $O^{\text{left}}(A)$ must also be a bialgebra by Proposition 2.3(g), and thus, we recover the left UQSG $O^{\text{left}}(A)$ of $A$ when $A$ is connected.

To generalize the transposed UQSG from Definition 1.5(b) to the weak bialgebra setting, we need the following definitions. First, recall the transposed coaction from Definition 1.5(a) which we reinterpret below.

Definition 3.9. Suppose that $H$ is a weak bialgebra coacting linearly on $A$ on the left and right via coactions $\lambda : A \to H \otimes A$ and $\rho : A \to A \otimes H$. Then for each $i$, $H$ coacts from the left and right on $A_i$ via the restrictions $\lambda_i$ and $\rho_i$. We call $A$ a transposed $H$-comodule algebra if for each $i$, there exists a basis $\{v^i_j\}_{1 \leq j \leq \dim A_i}$ for $A_i$ such that the coactions can be written in the following form:

$$
\begin{align*}
\lambda_i : A_i &\to H \otimes A_i \\
v^i_j &\mapsto \sum_{1 \leq k \leq \dim A_i} z^i_{j,k} \otimes v^i_k
\end{align*}
$$

$$
\begin{align*}
\rho_i : A_i &\to A_i \otimes H \\
v^i_j &\mapsto \sum_{1 \leq k \leq \dim A_i} v^i_k \otimes z^i_{k,j},
\end{align*}
$$

for some $z^i_{j,k} \in H$.

Definition 3.10 (transposed UQSGd, $O^{\text{trans}}(A)$). Let $O := O^{\text{trans}}(A)$ be a weak bialgebra such that $A$ is a transposed $O$-comodule algebra with $A_0 \cong O_1$ in $O_A$ and $A_0 \cong O_0$ in $A^O$, so that for any weak bialgebra $H$ for which $A$ is a transposed $H$-comodule algebra with $A_0 \cong H_0$ in $H^A$ and $A_0 \cong H_s$ in $A^H$, there exists a unique weak bialgebra map $\pi : O \to H$ such that $(\pi \otimes \text{Id}_A)\lambda^O = \lambda^H$ and $(\text{Id}_A \otimes \pi)\rho^O = \rho^H$. We call $O^{\text{trans}}(A)$ the transposed universal quantum linear semigroupoid (transposed UQSGd) of $A$.

Remark 3.11. We use the assumption that $A_0$ is commutative in Definition 3.10. Namely, by Proposition 2.3(f): $A_0 \cong H_t \cong H_s^{\text{op}} \cong A_0^{\text{op}}$ as $k$-algebras.
**Question 3.12** (P. Etingof). Can the assumption that $A_0$ is commutative be removed by altering Definition 3.10 (so that the results in the remainder of the paper are unaffected)?

**Remark 3.13.** For the same reasons as given in Remark 3.8, we can see that the above definition is a generalization of the transposed UQSG from Definition 1.5(b).

**Remark 3.14.** We only define the left/right/transposed UQSGd of $A$ up to weak bialgebra isomorphism, and it is unique (up to weak bialgebra isomorphism) if it exists.

Now we show that if the left and right UQSGd of $A$ exist and are isomorphic to each other, then the transposed UQSGd of $A$ exists and is isomorphic to the left (or right) UQSGd. To proceed, consider the following terminology.

**Definition 3.15.** Let $H$ be a weak bialgebra and let $\lambda: A \to H \otimes A$ be a left coaction. We call this coaction inner-faithful if, whenever $\lambda(A) \subseteq K \otimes A$ for some weak subbialgebra $K$, we must have that $K = H$. Right inner-faithful coactions are defined similarly.

**Lemma 3.16.** Take $A$ as in Hypothesis 1.1, and suppose that $O^{\text{left}}(A)$ exists.

(a) Suppose $H$ is a weak bialgebra that left coacts on $A$ with $A_0 \cong H_0$ in $H_A$. Then, $H$ coacts on $A$ inner-faithfully if and only if the weak bialgebra map $\pi: O^{\text{left}}(A) \to H$ (that arises from Definition 3.4(a)) is surjective.

(b) The weak bialgebra $O^{\text{left}}(A)$ left coacts on $A$ inner-faithfully.

Similar statements hold for right (resp., transposed) coactions and for the UQSGd $O^{\text{right}}(A)$ (resp., $O^{\text{trans}}(A)$).

**Proof.** (a) If $\pi$ is not surjective, then let $K := \text{im}(\pi)$ which is a proper weak subbialgebra of $H$. We get that $K$ left coacts on $A$ via $\lambda^K = (\pi \otimes \text{Id}_A)\lambda^{O^{\text{left}}(A)}: A \to K \otimes A$. Therefore, $H$ does not left coact on $A$ inner-faithfully.

Conversely, suppose that $H$ does not coact on $A$ inner-faithfully, and that there exists a proper weak subbialgebra $K$ of $H$ (via inclusion $\iota$) so that the coaction of $K$ on $A$ factors through $H$ on $A$. Then, $(\pi \otimes \text{Id}_A)\lambda^{O^{\text{left}}(A)} = \lambda^K = (\iota \otimes \text{Id}_A)\lambda^K$. Now the im($\pi$) consists of the coefficients of $\lambda^K$ in $K$. So, im($\pi$) cannot be $H$, and $\pi$ is not surjective.

(b) This follows from part (a) by taking $\pi = \text{Id}_{O^{\text{left}}(A)}$. \qed

**Proposition 3.17.** Suppose that $O^{\text{left}}(A)$ and $O^{\text{right}}(A)$ exist, and let $O(A) := O^{\text{left}}(A)$. Suppose that $O^{\text{left}}(A) \cong O^{\text{right}}(A)$ as weak bialgebras, and that their respective coactions on $A$ are transpose. Then $O^{\text{trans}}(A)$ exists, and $O^{\text{trans}}(A) \cong O(A)$ as weak bialgebras.

**Proof.** Assume that $O(A) := O^{\text{left}}(A)$ and $O^{\text{right}}(A)$ exist. For simplicity of proof, assume that $O(A) = O^{\text{right}}(A)$ as weak bialgebras (instead of using an isomorphism). Now, suppose that we have a weak bialgebra $H$ that left coacts and right coacts (via transposed coactions $\lambda^H: A \to H \otimes A$, $\rho^H: A \to A \otimes H$) with the property that $H_s \cong A_0$ in $H_A$ and $H_t \cong A_0$ in $H_A$. We will show that $O(A)$ satisfies the universal property described in Definition 3.10; therefore, $O^{\text{trans}}(A)$ exists and $O^{\text{trans}}(A) \cong O(A)$ as weak bialgebras.

Since $O(A) := O^{\text{left}}(A)$ and $O^{\text{right}}(A)$ exist, we have the following maps:

$\lambda^L : A \to O(A) \otimes A, \quad \pi^L : O(A) \to H, \quad \rho^R : A \to A \otimes O(A), \quad \pi^R : O(A) \to H$. 

with the property that $\mathcal{O}(A)$ left coacts on $A$ via $\lambda^L$, $\mathcal{O}^{\text{right}}(A)$ right coacts on $A$ via $\rho^R$, $\lambda^L$ and $\rho^R$ are transposed coactions, and $\pi^L$ and $\pi^R$ are the unique weak bialgebra maps satisfying the following equations:

$$(\pi^L \otimes \text{Id}_A)\lambda^L = \lambda^H, \quad (\text{Id}_A \otimes \pi^R)\rho^R = \rho^H.$$ 

We make the following definitions:

$$\lambda := \lambda^L : A \to \mathcal{O}(A) \otimes A, \quad \rho := \rho^R : A \to A \otimes \mathcal{O}(A), \quad \pi := \pi^L : \mathcal{O}(A) \to H.$$ 

We will prove that $\pi$ is the unique weak bialgebra map such that $(\text{Id}_A \otimes \pi)\rho = \rho^H$. This will imply that $\mathcal{O}(A)$ has the universal property of Definition 3.10, so we must have $\mathcal{O}(A) \cong O^{\text{trans}}(A)$ as weak bialgebras. In fact, since $\pi^R$ is the unique weak bialgebra map such that $(\text{Id}_A \otimes \pi^R)\rho = \rho^R$, it suffices to show that $\pi = \pi^R$.

Since $\lambda^H$ and $\rho^H$ are transposed coactions, for each $i$ there exists a basis $\{v^i_j\}_{1 \leq j \leq \dim A_i}$ of $A_i$ such that the restricted coactions can be written in the following form:

$$\lambda^H_i : A_i \to H \otimes A_i, \quad \rho^H_i : A_i \to A_i \otimes H$$

$$v^i_j \mapsto \sum_{1 \leq k \leq \dim A_i} z^{i}_{j,k} \otimes v^i_k$$

$$v^i_j \mapsto \sum_{1 \leq k \leq \dim A_i} v^i_k \otimes z^{i}_{k,j},$$

for some $z^{i}_{j,k} \in H$. Since $\{v^i_j\}$ is a basis for $A_i$ and the coactions $\lambda, \rho$ are transpose, we can write

$$\lambda_i : A_i \to \mathcal{O}(A) \otimes A_i$$

$$v^i_j \mapsto \sum_{1 \leq k \leq \dim A_i} y^{i}_{j,k} \otimes v^i_k$$

for some $y^{i}_{j,k} \in \mathcal{O}(A)$.

By the previous claims, we know that $(\pi \otimes \text{Id}_A)\lambda = \lambda^H$ and $(\text{Id}_A \otimes \pi^R)\rho = \rho^H$. Therefore, for each $v^i_j$ we have

$$\sum_{1 \leq k \leq \dim A_i} \pi(y^i_{j,k}) \otimes v^i_k = (\pi \otimes \text{Id}_A)\lambda_i(v^i_j) = \lambda^H_i(v^i_j) = \sum_{1 \leq k \leq \dim A_i} z^{i}_{j,k} \otimes v^i_k.$$ 

Since the $\{v^i_j\}$ are a basis for $A_i$, we know that $\pi(y^i_{j,k}) = z^{i}_{j,k}$ for each $i, j, k$. Similarly, for each $v^i_j$ we have

$$\sum_{1 \leq k \leq \dim A_i} v^i_k \otimes \pi^R(y^i_{j,k}) = (\text{Id}_A \otimes \pi^R)\rho_i(v^i_j) = \rho^H_i(v^i_j) = \sum_{1 \leq k \leq \dim A_i} v^i_k \otimes z^{i}_{k,j},$$

Since the $\{v^i_j\}$ are a basis for $A_i$, we know that $\pi^R(y^i_{j,k}) = z^{i}_{k,j}$ for each $i, j, k$. Therefore, for each $i, j, k$, we have

$$\pi^R(y^i_{j,k}) = z^{i}_{k,j} = \pi(y^i_{j,k}),$$

Since the coactions are inner-faithful by Lemma 3.16, $\mathcal{O}(A)$ is generated as an algebra by the $y^i_{j,k}$. (Else, there exists an algebra generator of $\mathcal{O}(A)$ not in the set $\{y^i_{j,k}\}$ and the proper weak subbialgebra generated by the $y^i_{j,k}$ coacts on $A$, contradicting the inner-faithfulness of the coaction of $\mathcal{O}(A)$ on $A$.) Finally, $\pi$ and $\pi^R$ are algebra maps, so we must have $\pi = \pi^R$, as desired. \[\square\]
4. Universal quantum linear semigroupoids of a path algebra

In this section, we will prove our main theorem, Theorem 4.17, constructing the left, right, and transposed UQSGds of the path algebra $kQ$ of a finite quiver $Q$. Furthermore, we will show that all three are isomorphic to Hayashi’s face algebra $\mathcal{F}(Q)$ (Example 2.6). Note that the coactions in the following hypothesis is a specific case of that in the Definition 3.9. In several of our results, we will assume one of the three hypotheses given below.

**Hypothesis 4.1.** Let $Q$ be a finite quiver, and let $(H, m, u, \Delta, \varepsilon)$ be a weak bialgebra. Consider the following formulas for each $j \in Q_0$ and each $q \in Q_1$:

$$
\lambda : kQ \to H \otimes kQ \quad \rho : kQ \to kQ \otimes H
$$

$$
e_j \mapsto \sum_{i \in Q_0} y_{j,i} \otimes e_i \quad e_j \mapsto \sum_{i \in Q_0} e_i \otimes y_{i,j}
$$

$$
q \mapsto \sum_{p \in Q_1} y_{p,q} \otimes p \quad q \mapsto \sum_{p \in Q_1} p \otimes y_{p,q}
$$

for some elements $y_{i,j} \in H_0$ and $y_{p,q} \in H_1$. We will consider three separate hypotheses in the sequel.

(a) Assume that $kQ$ is a left $H$-comodule algebra via a coaction of the form $\lambda$.

(b) Assume that $kQ$ is a right $H$-comodule algebra via a coaction of the form $\rho$.

(c) Assume that $kQ$ is a transposed $H$-comodule algebra via $\lambda$ and $\rho$.

The following results will be of use in this section.

**Lemma 4.2.** Let $H$ be a weak bialgebra which coacts on $kQ$ as in Hypothesis 4.1. Consider the following formulas for each $i, j, k \in Q_0$ and $p, q \in Q_1$:

$$
\Delta(y_{i,j}) = \sum_{k \in Q_0} y_{i,k} \otimes y_{k,j}, \quad \varepsilon(y_{i,j}) = \delta_{i,j} \cdot 1_k,
$$

$$
\Delta(y_{p,q}) = \sum_{r \in Q_1} y_{p,r} \otimes y_{r,q}, \quad \varepsilon(y_{p,q}) = \delta_{p,q} \cdot 1_k
$$

$$
y_{k,i}y_{k,j} = \delta_{i,j} y_{k,i}
$$

$$
y_{i,k}y_{j,k} = \delta_{i,j} y_{i,k}
$$

$$
y_{s(p),s(q)}y_{p,q} = y_{p,q}
$$

$$
y_{p,q}y_{(p),t(q)} = y_{p,q}
$$

(a) If Hypothesis 4.1(a) holds, then $H$ satisfies (4.3), (4.4), (4.6), (4.7), and (4.8).

(b) If Hypothesis 4.1(b) holds, then $H$ satisfies (4.3), (4.4), (4.5), (4.7), and (4.8).

(c) If Hypothesis 4.1(c) holds, then $H$ satisfies (4.3) to (4.8).

**Proof.** We will prove (a). The proof for (b) is similar and hence omitted, while (c) follows from (a) and (b).

The formulas for $\Delta$ and $\varepsilon$ follow from the coassociativity and counitality of $\lambda$. For example, for $i \in Q_0$,

$$
\sum_{j \in Q_0} \Delta(y_{i,j}) \otimes e_j = (\Delta \otimes \text{Id})\lambda(e_i) = (\text{Id} \otimes \lambda)\lambda(e_i)
$$

$$
= \sum_{k \in Q_0} y_{i,k} \otimes \lambda(e_k) = \sum_{j,k \in Q_0} y_{i,k} \otimes y_{k,j} \otimes \varepsilon(e_j)
$$

Since the $e_j$ are linearly independent, we have that $\Delta(y_{i,j}) = \sum_{k \in Q_0} y_{i,k} \otimes y_{k,j}$. Further, since $e_i = \text{Id}_{kQ}(e_i) = (\varepsilon \otimes \text{Id}_{kQ})\lambda(e_i) = \sum_{p \in Q_0} \varepsilon(y_{i,j})e_j$, we conclude that $\varepsilon(y_{i,j}) = \delta_{i,j} \cdot 1_k$. This proves (4.3); the proof for (4.4) is similar.
Next, we will use the fact that $kQ$ is a left $H$-comodule algebra. We have
\[
\sum_{k \in Q_0} \delta_{i,j} y_{i,k} \otimes e_k = \lambda(\delta_{i,j} e_i) \\
= \lambda(e_i e_j) \\
= \lambda(e_i) \lambda(e_j) \\
= \left( \sum_{k \in Q_0} y_{i,k} \otimes e_k \right) \left( \sum_{\ell \in Q_0} y_{j,\ell} \otimes e_{\ell} \right) \\
= \sum_{k,\ell \in Q_0} y_{i,k} y_{j,\ell} \otimes e_k e_{\ell} \\
= \sum_{k \in Q_0} y_{i,k} y_{j,k} \otimes e_k.
\]
Since the $e_k$ are linearly independent, we must have $\delta_{i,j} y_{i,k} = y_{i,k} y_{j,k}$, that is, (4.6) holds.

To show (4.7), notice that for $p \in Q_1$ we have
\[
\sum_{q \in Q_1} y_{p,q} \otimes q = \lambda(p) \\
= \lambda(e_{s(p)} p) \\
= \lambda(e_{s(p)}) \lambda(p) \\
= \left( \sum_{i \in Q_0} y_{s(p),i} \otimes e_i \right) \left( \sum_{q \in Q_1} y_{p,q} \otimes q \right) \\
= \sum_{i \in Q_0, q \in Q_1} y_{s(p),i} y_{p,q} \otimes e_i q \\
= \sum_{q \in Q_1} y_{s(p),q} y_{p,q} \otimes q.
\]
By linear independence of the set $\{q\}_{q \in Q_1}$, we have $y_{p,q} = y_{s(p),q} y_{p,q}$ for all $p, q \in Q_1$.

We use the relation $p = pe_{t(p)}$ for $p \in Q_1$ to prove (4.8) in the same manner as (4.7). □

The following proposition is a collection of identities that hold if we only assume the existence of a left $H$-coaction making $kQ$ an $H$-comodule algebra.

**Proposition 4.9.** Let $H$ be a weak bialgebra which coacts on $kQ$ as in Hypothesis 4.1(a). For each $j \in Q_0$, consider the elements
\[
\eta_j := \sum_{i \in Q_0} y_{i,j}, \quad \theta_j := \sum_{i \in Q_0} y_{j,i}.
\]
The following statements hold.

(a) For each $j \in Q_0$, $\eta_j$ and $\theta_j$ are non-zero elements of $H$.
(b) For each $j \in Q_0$, $\eta_j$ is an idempotent element of $H_s$.
(c) $\delta_{i,j} y_{i,k} \equiv H_s$, as left $H$-comodule algebras, then the following statements hold:
   (i) $\{y_{i,k}\}_{k \in Q_0}$ is a $k$-basis of $H_s$.
   (ii) $1_H = \sum_{i,j \in Q_0} y_{i,j}$.
   (iii) For each $k \in Q_0$, $\psi(e_k) = \theta_k$; hence $\theta_k \in H_s$.
   (iv) The set $\{\theta_i\}_{i \in Q_0}$ is a $k$-basis for $H_s$ of orthogonal idempotent elements.
   (v) The set $\{\eta_j\}_{j \in Q_0}$ is a $k$-basis for $H_s$ of orthogonal idempotent elements.
   (vi) For all $i, j, k, \ell \in Q_0$, $y_{i,j} y_{k,\ell} = \delta_{i,k} \delta_{j,\ell} y_{i,j}$.

**Proof.** (a) To show that $\eta_j$ is non-zero, we note that
\[
\varepsilon(\eta_j) = \varepsilon \left( \sum_{i \in Q_0} y_{i,j} \right) = 1.
\]
A similar calculation shows that $\varepsilon(\theta_j) = 1$, so $\theta_j$ is non-zero.
(b) Let \( j \in Q_0 \). Then
\[
\eta_j^2 = \sum_{i,k \in Q_0} y_{i,j} y_{k,j} \tag{4.6}
\]
so \( \eta_j \) is idempotent. Moreover, note that
\[
\sum_{i,j \in Q_0} y_{i,j} \otimes e_j = (\varepsilon_s \otimes \text{Id}_{kQ_0}) \lambda(1_{kQ_0}) = (\varepsilon \otimes \text{Id}_{kQ_0}) \lambda(1_{kQ_0}) = \sum_{i,j \in Q_0} \varepsilon_s(\sum_{i \in Q_0} y_{i,j}) \otimes e_j.
\]
Since the \( e_j \) are linearly independent, for each \( j \in Q_0 \) we have \( \eta_j := \sum_{i \in Q_0} y_{i,j} \in H_s \).

(c) Suppose that \( kQ_0 \cong H_t \) as left \( H \)-comodule algebras. Then there exists an algebra isomorphism \( \psi : kQ_0 \to H_t \) which is also a map of left \( H \)-comodules. Hence,
\[
(\text{Id}_H \otimes \psi)\lambda_0 = \lambda_{H_t} \psi,
\]
for \( \lambda_{H_t} = \Delta_{H_t}|_{H_t} \) by Example 2.16.

(i) Evaluating the left-hand side on \( 1_{kQ_0} \), we have
\[
(\text{Id} \otimes \psi)\lambda_0(1_{kQ_0}) = (\text{Id} \otimes \psi)\lambda_0(\sum_{i \in Q_0} e_i) = (\text{Id} \otimes \psi)(\sum_{i,j \in Q_0} y_{i,j} \otimes e_j)
\]
\[
= \sum_{j \in Q_0} (\sum_{i \in Q_0} y_{i,j}) \otimes \psi(e_j) = \sum_{j \in Q_0} \eta_j \otimes \psi(e_j).
\]
On the other hand, \( \psi \) is an algebra map, so \( \psi(1_{kQ_0}) = 1_H \). Thus, \( \Delta \psi(1_{kQ_0}) = 1_1 \otimes 1_2 \). Hence,
\[
1_1 \otimes 1_2 = \sum_{j \in Q_0} \eta_j \otimes \psi(e_j).
\]
Since the distinct \( e_j \) are linearly independent and \( \psi \) is an algebra isomorphism, the \( \psi(e_j) \) are also linearly independent. Now by Proposition 2.3(e), we conclude that the \( \{\eta_j\}_{j \in Q_0} \) span \( H_s \). By Proposition 2.3(f), we have \( \dim_k H_t = \dim_k H_s \). Therefore \( \dim_k H_s = \dim_k kQ_0 = |Q_0| \), so we have that \( \{\eta_j\}_{j \in Q_0} \) is a \( k \)-basis of \( H_s \).

(ii) For any \( k \in Q_0 \) we have
\[
\sum_{j \in Q_0} \eta_j \psi(e_k) \otimes \psi(e_j) \tag{4.11} = 1_1 \psi(e_k) \otimes 1_2 \tag{4.11} = \Delta \psi(e_k) \tag{2.3(d)} = (\text{Id}_H \otimes \psi)\lambda_0(e_k) = \sum_{j \in Q_0} y_{k,j} \otimes \psi(e_j).
\]
Since the \( \psi(e_j) \) are linearly independent, we have that
\[
\eta_j \psi(e_k) = y_{k,j} \tag{4.12}
\]
for each \( j, k \in Q_0 \).

By (4.11),
\[
1_H = 1_1 \varepsilon(1_2) = \sum_{j \in Q_0} \eta_j \varepsilon(\psi(e_j)).
\]

Next, consider the following calculation:
\[
\sum_{k \in Q_0} \eta_k \otimes \psi(e_k) \tag{4.11} = 1_1 \otimes 1_2 \tag{4.11} = \Delta(1_H) \tag{4.13} = \Delta(\sum_{j \in Q_0} \varepsilon(\psi(e_j)) \eta_j) \tag{4.3} = \Delta(\sum_{i,j \in Q_0} \varepsilon(\psi(e_j)) y_{i,j}) \tag{4.3} = \sum_{i,j,k \in Q_0} \varepsilon(\psi(e_j)) y_{i,k} \otimes y_{k,j}
\]

For each $k \in Q_0$, notice that

$$y_{k,j} = \eta_j \psi(e_k) = \eta_j \sum_{\ell \in Q_0} \varepsilon(\psi(e_{\ell})) y_{k,j} y_{\ell,k} = \sum_{i,j \in Q_0} \varepsilon(\psi(e_{\ell})) y_{i,j} y_{k,\ell}.$$

Multiplying both sides of the equation on the left by $y_{k,j}$ yields

$$y_{k,j} = (y_{k,j})^2 = \sum_{i,j \in Q_0} \varepsilon(\psi(e_{\ell})) y_{k,j} y_{i,j} y_{\ell,k} = \sum_{i,j \in Q_0} \varepsilon(\psi(e_{\ell})) \delta_{i,k} y_{k,j} y_{\ell,k}$$

for each $k \in Q_0$. Notice that

$$y_{k,j} \sum_{\ell \in Q_0} \varepsilon(\psi(e_{\ell})) y_{k,j} y_{\ell,k} = \sum_{i,j \in Q_0} \varepsilon(\psi(e_{\ell})) y_{i,j} y_{k,\ell}$$

Now for each $k \in Q_0$, we get

$$1_k = \varepsilon(y_{k,k})$$

$$= \varepsilon(y_{k,k} \psi(e_k))$$

$$= \varepsilon(y_{k,k} \psi(1_k))$$

$$= \varepsilon(y_{k,k} \psi(1_2 \psi(e_k)))$$

$$= \varepsilon(y_{k,k} \psi(\psi(e_k)))$$

Then

$$\psi \text{ alg. map } \eta = \sum_{i \in Q_0} \varepsilon(\psi(e_i)) \eta_i \psi(e_k)$$

$$= \varepsilon(\sum_{i \in Q_0} y_{k,k} y_{i,k}) \varepsilon(\psi(e_k))$$

Finally,

$$1_H = \sum_{j \in Q_0} \varepsilon(\psi(e_j)) \eta_j = \sum_{j \in Q_0} \eta_j = \sum_{i,j \in Q_0} y_{i,j}.$$ 

(iii) For each $k \in Q_0$, we have

$$\theta_k = \sum_{j \in Q_0} y_{k,j} \psi(e_k) = \sum_{j \in Q_0} \eta_j \psi(e_k) = (\sum_{i,j \in Q_0} y_{i,j}) \psi(e_k) = \psi(e_k).$$

(iv) Since $\{e_i\}_{i \in Q_0}$ is a $k$-basis of $\mathbb{k}Q_0$ of orthogonal idempotent elements and $\psi$ is an algebra isomorphism, $\{\psi(e_i)\}_{i \in Q_0}$ is a $k$-basis of orthogonal idempotents of $H_1$. By part (iii), the claim follows.
(v) Since \(\kappa \mathcal{Q}_0 \cong \nu H_s\) as algebras, by Proposition 2.3(f) we have \(H_s \cong \bigoplus_{i \in \mathcal{Q}_0} \kappa_i\). Let \(\{E_i\}_{i \in \mathcal{Q}_0}\) be a set of primitive idempotents of \(\bigoplus_{i \in \mathcal{Q}_0} \kappa_i\). By part (b), for each \(k \in \mathcal{Q}_0\), \(\eta_k\) is an idempotent of \(H_s\). Hence, \(\gamma(\eta_k) = \sum_{i \in I_k} E_i\) for some subset \(I_k\) of \(\mathcal{Q}_0\). Therefore,

\[
\sum_{k \in \mathcal{Q}_0} \sum_{i \in I_k} E_i = \sum_{k \in \mathcal{Q}_0} \gamma(\eta_k) = \gamma(\sum_{i \in \mathcal{Q}_0} y_{i,k}) \overset{(ii)}{=} \gamma(1) = 1 \bigoplus \kappa = \sum_{i \in \mathcal{Q}_0} E_i.
\]

As a result, we conclude that \(\gamma(\eta_k) = E_i\) for some \(i \in \mathcal{Q}_0\) and for \(k \neq j\), we have that \(\gamma(\eta_k) \neq \gamma(\eta_j)\). Thus, \(\{\eta_k\}_{k \in \mathcal{Q}_0}\) is a set of orthogonal idempotents of \(H_s\).

(vi) For each \(i, j, k, \ell \in \mathcal{Q}_0\), we have:

\[
y_{i,j} y_{k,\ell} = \eta_j \psi(e_i) \eta_k \psi(e_k) \overset{(4.12)}{=} \eta_j \theta_i \eta_k \theta_k = \theta_i \theta_k \eta_j \eta_k \overset{(iv), (v)}{=} \delta_{i,k} \delta_{j,\ell} \theta_i \eta_j \overset{(iii), (4.12)}{=} \delta_{i,k} \delta_{j,\ell} y_{i,j},
\]

where the third equality holds by parts (b), (iii), and Proposition 2.3(c). \(\square\)

The analogue of Proposition 4.9 for a weak bialgebra coaction on \(\mathbb{k}Q\) satisfying Hypothesis 4.1(b) also holds, and follows by a similar proof.

**Proposition 4.16.** Let \(H\) be a weak bialgebra which coacts on \(\mathbb{k}Q\) as in Hypothesis 4.1(b). For each \(j \in \mathcal{Q}_0\), consider the elements

\[
\eta_j := \sum_{i \in \mathcal{Q}_0} y_{i,j}, \quad \theta_j := \sum_{i \in \mathcal{Q}_0} y_{j,i}.
\]

The following statements hold.

(a) For each \(j \in \mathcal{Q}_0\), \(\eta_j\) and \(\theta_j\) are non-zero elements of \(H\).

(b) For each \(j \in \mathcal{Q}_0\), \(\theta_j\) is an idempotent element of \(H_s\).

(c) If \(\mathbb{k} \mathcal{Q}_0 \cong \phi H_s\) as right \(H\)-comodule algebras, then the following statements hold:

(i) \(\{\eta_k\}_{k \in \mathcal{Q}_0}\) is a \(\mathbb{k}\)-basis of \(H_s\).

(ii) \(1_H = \sum_{i,j \in \mathcal{Q}_0} y_{i,j}\).

(iii) For each \(k \in \mathcal{Q}_0\), \(\phi(e_k) = \eta_k\); hence \(\eta_k \in H_s\).

(iv) The set \(\{\eta_j\}_{j \in \mathcal{Q}_0}\) is a \(\mathbb{k}\)-basis for \(H_s\) of orthogonal idempotent elements.

(v) The set \(\{\theta_i\}_{i \in \mathcal{Q}_0}\) is a \(\mathbb{k}\)-basis for \(H_s\) of orthogonal idempotent elements.

(vi) For all \(i, j, k, \ell \in \mathcal{Q}_0\), \(y_{i,j} y_{k,\ell} = \delta_{i,k} \delta_{j,\ell} y_{i,j}\). \(\square\)

This brings us to the main result of the paper.

**Theorem 4.17.** Let \(Q\) be a finite quiver with path algebra \(\mathbb{k}Q\). Then the universal quantum linear semigroups \(\mathcal{O}_{\text{left}}(\mathbb{k}Q), \mathcal{O}_{\text{right}}(\mathbb{k}Q),\) and \(\mathcal{O}_{\text{trans}}(\mathbb{k}Q)\) exist, and they are each isomorphic to \(\mathcal{S}(Q)\) as weak bialgebras.

**Proof.** Consider the left and right coaction of \(\mathcal{S}(Q)\) on \(\mathbb{k}Q\) presented in Example 2.22. We will show in full detail that \(\mathcal{O}_{\text{left}}(\mathbb{k}Q) \cong \mathcal{S}(Q)\) as weak bialgebras (under Hypothesis 4.1(a)), and briefly discuss the proof that \(\mathcal{O}_{\text{right}}(\mathbb{k}Q) \cong \mathcal{S}(Q)\) as weak bialgebras (under Hypothesis 4.1(b)). Then we have that \(\mathcal{O}_{\text{left}}(\mathbb{k}Q) \cong \mathcal{S}(Q) \cong \mathcal{O}_{\text{right}}(\mathbb{k}Q)\) as weak bialgebras, and that the coactions of the left/right UQSGDs are transposed via Example 2.22. Hence, Proposition 3.17 yields \(\mathcal{O}_{\text{trans}}(\mathbb{k}Q) \cong \mathcal{S}(Q)\) as weak bialgebras (under Hypothesis 4.1(c)).

To proceed, we will show that \(\mathcal{S}(Q)\) satisfies the universal property of \(\mathcal{O}_{\text{left}}(\mathbb{k}Q)\). Indeed we have that \(\mathcal{S}(Q)\) is a weak bialgebra that left coacts on \(\mathbb{k}Q\) [Example 2.22]. Moreover,
Recall that we have elements \( \{ y_{i,j} \}_{i,j \in Q_0} \) and \( \{ y_{p,q} \}_{p,q \in Q_1} \) in \( H \), as well as idempotents \( \{ \eta_t \}_{t \in Q_0} \) in \( H_s \) and \( \{ \theta_t \}_{t \in Q_0} \) in \( H_t \), as in Lemma 4.2 and Proposition 4.9. Now consider the map \( \pi \) defined on the algebra generators of \( \mathcal{S}(Q) \) and extended multiplicatively and linearly:

\[
\pi : \mathcal{S}(Q) \to H \quad \text{defined by} \quad x_{i,j} \mapsto y_{i,j} \quad \text{for} \quad i, j \in Q_0, \quad x_{p,q} \mapsto y_{p,q} \quad \text{for} \quad p, q \in Q_1.
\]

We aim to show first that \( \pi \) is a weak bialgebra map (i.e., that \( \pi \) is an algebra map and a coalgebra map) satisfying \((\pi \otimes \text{Id}_{kQ}) \mathcal{S}(Q) = \mathcal{S}(H) \) and that \( \pi \) is the only such weak bialgebra map \( \mathcal{S}(Q) \to H \) with this property. This would achieve the result that \( \mathcal{S}^{\text{left}}(kQ) \cong \mathcal{S}(Q) \) as weak bialgebras.

To show that \((\pi \otimes \text{Id}_{kQ}) \mathcal{S}(Q) = \mathcal{S}(H) \), note that for \( i \in Q_0 \),

\[
(\pi \otimes \text{Id}_{kQ}) \mathcal{S}(Q)(e_i) = \langle \pi \otimes \text{Id}_{kQ} \rangle \left( \sum_{j \in Q_0} x_{i,j} \otimes e_j \right) = \sum_{j \in Q_0} y_{i,j} \otimes e_j = \mathcal{S}(H)(e_i).
\]

A similar calculation shows that \((\pi \otimes \text{Id}_{kQ}) \mathcal{S}(Q)(p) = \mathcal{S}(H)(p) \) for \( p \in Q_1 \). Since \( \pi \), \( \mathcal{S}(Q) \) and \( \mathcal{S}(H) \) are multiplicative, we must have \((\pi \otimes \text{Id}_{kQ}) \mathcal{S}(Q) = \mathcal{S}(H) \).

The unitality of \( \pi \) follows from the computation:

\[
\pi(1_{\mathcal{S}(Q)}) \overset{(2.10)}{=} \pi(\sum_{i,j \in Q_0} x_{i,j}) = \sum_{i,j \in Q_0} y_{i,j} \overset{4.9(c)(ii)}{=} 1_H.
\]

To prove that \( \pi \) is multiplicative, note that by Proposition 4.9(c)(vi), for all \( i,j,k,l \in Q_0 \),

\[
y_{i,j} y_{k,l} = \delta_{i,k} \delta_{j,l} y_{i,j}.
\]

By (4.7) and (4.8) in Lemma 4.2, we have

\[
y_{s(p), s(q)} y_{p,q} = y_{p,q} = y_{p,q} y_{t(q), t(p)}.
\]

So, we obtain that

\[
y_{p,q} y_{p', q'} \overset{(4.7),(4.8)}{=} y_{p,q} y_{t(p), t(q)} y_{s(p'), s(q')} y_{p', q'} = \delta_{l(p), s(p')} \delta_{l(q), s(q')} y_{p,q} y_{t(p), t(q)} y_{p', q'} \overset{(4.8)}{=} \delta_{l(p), s(p')} \delta_{l(q), s(q')} y_{p,q} y_{p', q'}.
\]

Now (2.7), (2.8) and (2.9) imply that \( \pi \) is multiplicative. Therefore, \( \pi \) is an algebra map.
Next, we will show that \( \pi \) is also a coalgebra map, i.e., that \( \Delta_H \pi = (\pi \otimes \pi) \Delta_{S(Q)} \) and \( \varepsilon_H \pi = \varepsilon_{S(Q)} \). We will prove this for \( x_{p,q} \) by induction on the length \( \ell \) of the paths \( p, q \in Q \). If \( \ell = 0,1 \), then the assertion holds by (4.3) and (4.4) in Lemma 4.2. Now take

\[
\begin{align*}
p &= p_1 \cdots p_{\ell-1} p_{\ell} \\
q &= q_1 \cdots q_{\ell-1} q_{\ell}
\end{align*}
\]

paths of length \( \ell \) with \( p_i, q_i \in Q \). Then, for \( \ell \geq 2 \):

\[
\begin{align*}
\Delta_H \pi(x_{p_1,q_1} \cdots x_{p_{\ell-1},q_{\ell-1}} x_{p_{\ell},q_{\ell}}) &= \Delta_H(y_{p_1,q_1} \cdots y_{p_{\ell-1},q_{\ell-1}} y_{p_{\ell},q_{\ell}}) \\
 &= \Delta_H(y_{p_1,q_1} \cdots y_{p_{\ell-1},q_{\ell-1}}) \Delta_H(y_{p_{\ell},q_{\ell}}) \\
 &= (\Delta_H \pi)(x_{p_1,q_1} \cdots x_{p_{\ell-1},q_{\ell-1}})(\Delta_H \pi)(x_{p_{\ell},q_{\ell}}) \\
 &= \pi \Delta_{S(Q)}(x_{p_1,q_1} \cdots x_{p_{\ell-1},q_{\ell-1}} x_{p_{\ell},q_{\ell}}),
\end{align*}
\]

where the first three equalities and the last equality hold because \( \pi, \Delta_H \), and \( \Delta_{S(Q)} \) preserve multiplication. Further, we have

\[
\begin{align*}
(\varepsilon_H \pi)(x_{p_1,q_1} \cdots x_{p_{\ell-1},q_{\ell-1}} x_{p_{\ell},q_{\ell}}) &= \varepsilon_H(y_{p_1,q_1} \cdots y_{p_{\ell-1},q_{\ell-1}} y_{p_{\ell},q_{\ell}}) \\
 &= \varepsilon_H(y_{p_1,q_1} \cdots y_{p_{\ell-1},q_{\ell-1}} 1_H y_{p_{\ell},q_{\ell}}) \\
 &= \varepsilon_H(y_{p_1,q_1} \cdots y_{p_{\ell-1},q_{\ell-1}} 11) \in H(12 y_{p_{\ell},q_{\ell}}) \\
&= \varepsilon_H(y_{p_{\ell},q_{\ell}}) \\
&= \varepsilon_H(y_{p_{\ell},q_{\ell}}) \\
&= \varepsilon_H(12 y_{p_{\ell},q_{\ell}}) \\
&= \varepsilon_H(12 y_{p_{\ell},q_{\ell}})
\end{align*}
\]

as desired. Therefore, we have shown that \( \pi \) is a map of weakbialgebras.
It remains to show that \( \pi \) is unique. Suppose that \( \pi' : \mathfrak{F}(Q) \to H \) is a weak bialgebra homomorphism such that \( (\pi' \otimes \text{Id}_{kQ}) \lambda^{\mathfrak{S}(Q)} = \lambda^H \). Let \( i \in Q_0 \). Then

\[
(\pi' \otimes \text{Id}_{kQ}) \lambda^{\mathfrak{S}(Q)}(e_i) = (\pi' \otimes \text{Id}_{kQ}) \left( \sum_{j \in Q_0} x_{i,j} \otimes e_j \right) = \sum_{j \in Q_0} \pi'(x_{i,j}) \otimes e_j
\]

while

\[
\lambda^H(e_i) = \sum_{j \in Q_0} y_{i,j} \otimes e_j.
\]

Since the \( e_j \) are linearly independent in \( kQ \), this implies that \( \pi'(x_{i,j}) = y_{i,j} \) for each \( i, j \in Q_0 \). By a similar argument, \( \pi'(x_{p,q}) = y_{p,q} \) for any \( p, q \in Q_1 \). Hence, \( \pi' \) and \( \pi \) agree on a set of algebra generators for \( \mathfrak{F}(Q) \), and since both \( \pi' \) and \( \pi \) are algebra homomorphisms, we have that \( \pi' = \pi \).

To show that \( \mathfrak{F}(Q) \) satisfies the universal property of \( O^{\text{right}}(kQ) \), one only needs to make the following adjustments to the proof above: assume Hypothesis 4.1(b) in place of Hypothesis 4.1(a) (i.e., replace the left coaction \( \lambda \) with the right coaction \( \rho \); replace \( \psi \) with an isomorphism \( \phi : kQ_0 \xrightarrow{\sim} H_s \) in \( A^H \); and employ Proposition 4.16 in place of Proposition 4.9 in the argument that \( \pi : \mathfrak{F}(Q) \to H \) is an algebra map. Then, the result for \( O^{\text{right}}(kQ) \) follows in a manner similar to that for \( O^{\text{left}}(kQ) \) above. \( \square \)

With Lemma 3.16, the following is a consequence of the theorem above.

**Corollary 4.19.** The weak bialgebra \( \mathfrak{F}(Q) \) coacts on \( kQ \) inner-faithfully. \( \square \)

We end this section with an example of our result above in the bialgebra case, thus obtaining a left/right/transposed UQSG as in Definitions 1.4 and 1.5.

**Example 4.20.** Suppose that \( Q \) is a finite quiver with \( |Q_0| = 1 \) and \( |Q_1| = n \) for some \( n \in \mathbb{N} \), that is, \( Q \) is the \( n \)-loop quiver. Here, \( kQ \) is isomorphic to the free algebra \( k \langle t_1, \ldots, t_n \rangle \). Now Theorem 4.17 implies that, as bialgebras,

\[
O^{\text{left}}(k \langle t_1, \ldots, t_n \rangle) \cong O^{\text{right}}(k \langle t_1, \ldots, t_n \rangle) \cong O^{\text{trans}}(k \langle t_1, \ldots, t_n \rangle) \cong \mathfrak{F}(Q_{\text{n-loop}}),
\]

where \( \mathfrak{F}(Q_{\text{n-loop}}) \) is defined in Example 2.6. Indeed, \( \dim_k(\mathfrak{F}(Q))_s = |Q_0| = 1 \) by Proposition 2.13(c), so all of the structures above are bialgebras by Proposition 2.3(g). Moreover, one can check that \( \mathfrak{F}(Q_{\text{n-loop}}) \) is isomorphic to the free algebra \( k \langle x_{i,j} \mid 1 \leq i, j \leq n \rangle \).

5. **Universal quantum linear semigroupoids of quotients of path algebras**

Let \( Q \) be a finite quiver and let \( I \) be a graded ideal of \( kQ \). In this section, we study the UQSGds of the quotient algebra \( kQ/I \), showing that if they exist, they are each a quotient of \( \mathfrak{F}(Q) \) [Proposition 5.4]. Moreover, we generalize a result of Manin by showing that a UQSGd of a quadratic quotient algebra is isomorphic to the opposite UQSGd of its quadratic dual [Theorem 5.10]. We also provide several examples. To start, we need a few well-known facts.

**Definition 5.1.** Let \((H, m, u, \Delta, \varepsilon)\) be a weak bialgebra. A biideal of \( H \) is a \( k \)-subspace \( I \subseteq H \) which is both an ideal and a coideal, that is: \( hI \subseteq I \) and \( Ih \subseteq I \) for any \( h \in H \); \( \Delta(I) \subseteq I \otimes H + H \otimes I \); and \( \varepsilon(I) = 0 \).

**Lemma 5.2.** The kernel of a weak bialgebra map is a biideal.
Proof. Let $\alpha : H \to K$ be a weak bialgebra map. Since the kernel of an algebra map is an ideal and the kernel of a coalgebra map is a coideal, ker $\alpha$ is a biideal. \hfill \square

**Lemma 5.3.** Suppose that $H$ is a weak bialgebra and that $I$ is a biideal. Then $H/I$ can be given the structure of a weak bialgebra as follows, for all $h, k \in H$: $m_{H/I}(h + I, (h + I)) = hk + I$; $1_{H/I} = 1_H + I$; $\Delta_{H/I}(h + I) = (h_1 + I) \otimes (h_2 + I)$; and $\varepsilon_{H/I}(h + I) := \varepsilon_H(h)$.

**Proof.** The structures given above make $H/I$ both an algebra and a coalgebra. A straightforward calculation verifies the compatibility conditions given in Definition 2.1. \hfill \square

**Proposition 5.4.** Let $Q$ be a finite quiver and let $I \subseteq kQ$ be a graded ideal which is generated in degree 2 or greater. If $O^*(kQ/I)$ exists (where $*$ means ‘left’, ‘right’, or ‘trans’), we have $O^*(kQ/I) \cong \delta_I(Q)/\ker \pi$.

**Remark 5.5.** If $I$ has generators in degree 0 or 1, then we can choose a smaller quiver $Q'$ and an ideal $I'$ of $kQ'$ such that $kQ'/I' \cong kQ/I$ as algebras and $I'$ is generated in degree 2 or greater.

**Proof of Proposition 5.4.** We will prove this statement for $O^\text{left}(kQ/I)$; the other statements follow similarly. By Lemma 5.2, it suffices to show that we have a weak bialgebra surjection $\pi : \delta_I(Q) \to O^\text{left}(kQ/I)$, in which case, $O^\text{left}(kQ/I) \cong \delta_I(Q)/\ker \pi$.

Let $O := O^\text{left}(kQ/I)$. For $i \in Q_0$ and $p \in Q_1$, let $\overline{e}_i$, $\overline{p}$ denote the images of $e_i$, $p$ in $kQ/I$ under the canonical quotient map $kQ \to kQ/I$ (regarding $p$ as an element of $kQ_1$). Since $I$ is generated in degree 2 or greater, $(kQ/I)_0 \cong kQ_0$ as algebras, and $\dim_k(kQ/I)_1 = \dim_k(kQ_1) = |Q|$. Hence, $\{\overline{e}_i\}_{i \in Q_0}$ is a basis of $(kQ/I)_0$ and $\{\overline{p}\}_{p \in Q_1}$ is a basis of $(kQ/I)_1$.

We can write $1_{kQ/I} = \sum_{i \in Q_0} \overline{e}_i$. Then we have a linear coaction

$$\lambda : kQ/I \to O \otimes kQ/I$$

$$\overline{e}_j \mapsto \sum_{i \in Q_0} y_{j,i} \otimes \overline{e}_i$$

$$\overline{p} \mapsto \sum_{p \in Q_1} y_{q,p} \otimes \overline{p}$$

for some elements $y_{i,j}, y_{p,q} \in O$. The result of Lemma 4.2(a) holds for this coaction. Namely, the proof is the same, except we replace $kQ$ with $kQ/I$, elements of the form $e_i$ for $i \in Q_0$ with $\overline{e}_i$, and arrows $p \in Q_1$ (regarded as elements of $kQ_1$) with $\overline{p}$, making use of the fact that these elements of $kQ/I$ still satisfy the fundamental relations $\overline{e}_i \overline{e}_j = \delta_{j,i} \overline{e}_i$, $\overline{e}_{\delta(p)} \overline{p} = \overline{p} \overline{e}_{\delta(p)}$, for $i \in Q_0, p \in Q_1$. Therefore, the results of Proposition 4.9(a),(b) also hold, since their proofs use the identities given in Lemma 4.2(a). Moreover, by the definition of a UQSGd, there exists a left $O$-comodule algebra structure on $(kQ/I)_0$ such that $O_1 \cong (kQ/I)_0$ in $Q^\text{A}$. Therefore, if we replace $kQ_0$ with $(kQ/I)_0$ in the statement and proof of Proposition 4.9(c), also replacing $e_i \in kQ_0$ with $\overline{e}_i \in (kQ/I)_0$, we obtain the same result.

Now, imitating the proof of Theorem 4.17, we define a map $\pi$ defined on the algebra generators of $\delta_I(Q)$ and extended multiplicativey and linearly:

$$\pi : \delta_I(Q) \to O \quad \text{defined by} \quad x_{i,j} \mapsto y_{i,j} \quad \text{for} \quad i, j \in Q_0, \quad x_{p,q} \mapsto y_{p,q} \quad \text{for} \quad p, q \in Q_1.$$ 

To show that $\pi$ is an algebra map, we can simply follow the proof for Theorem 4.17, since this proof only uses the results of Lemma 4.2(a) and Proposition 4.9. To show that $\pi$ is a coalgebra map, we again follow the proof for Theorem 4.17, replacing the paths $p = p_1 \cdots p_{\ell-1} p_\ell$...
and \( q = q_1 \cdots q_{c-1}q_c \) (for \( p_i, q_i \in Q_1 \)) with their images under the canonical quotient map \( kQ \to kQ/I \). This proof only uses the results of Lemma 4.2(a) and Proposition 4.9, the weak bialgebra structure of \( \mathcal{H}(Q) \), and the fact that \( \pi \) is multiplicative, so the result still holds. Therefore, \( \pi \) is a weak bialgebra map.

Finally, we will show that \( \pi \) is surjective. By Lemma 3.16, the coaction of \( \mathcal{O} \) on \( kQ/I \) is inner-faithful, and so \( \mathcal{O} \) is generated as a weak bialgebra by the \( y_{i,j} \) and \( y_{p,q} \) for \( i,j \in Q_0 \) and \( p,q \in Q_1 \). By the definition of \( \pi \) and the fact that \( \pi \) is a weak bialgebra map, we can see that \( \pi \) is surjective.

Every connected graded \( k \)-algebra which is finitely generated in degree one is isomorphic to \( kQ/I \) where \( Q \) is a finite quiver with \( |Q_0| = 1 \). For these algebras, we obtain the following immediate corollary.

**Corollary 5.6.** If \( Q \) is a finite quiver with \( |Q_0| = 1 \) and \( |Q_1| = n \), then \( \mathcal{O}^*(kQ/I) \) is a bialgebra quotient of the face algebra \( \mathcal{H}(Q_{n\text{-loop}}) \) from Example 4.20, where * means ‘left’, ‘right’, or ‘trans’.

The next example is a special case of Proposition 5.4, which describes the UQSGs explicitly as a quotient of \( \mathcal{H}(Q) \) when \( kQ/I \) is the polynomial ring \( k[t_1, \ldots, t_n] \).

**Example 5.7.** Let \( A = k[t_1, \ldots, t_n] \). We can describe \( A \) as a quotient of a path algebra \( kQ/I \) where, \( Q \) is a quiver with one vertex and \( n \) arrows \( t_1, \ldots, t_n \), and \( I = ([t_i, t_j])_{1 \leq i < j \leq n} \). Since \( A \) is connected graded, as noted in Remarks 3.8 and 3.13, the UQSGs of \( A \) are classical UQSGs (bialgebras).

(a) By [AST91, Theorem 1], we have that

\[ O^\text{trans}(kQ/I) = O^\text{trans}(k[t_1, \ldots, t_n]) \cong O(\text{Mat}_n(k)). \]

We will show that \( O^\text{trans}(kQ/I) \cong \mathcal{H}(Q)/\mathcal{I} \), where \( \mathcal{H}(Q) \) is Hayashi’s face algebra attached to an \( n \)-loop quiver from Example 4.20 and \( \mathcal{I} \) is the biideal of \( \mathcal{H}(Q) \) generated by the commutators \([x_{i,t}, x_{k,t}]\) for \( 1 \leq i, j, k, \ell \leq n \).

Let \( O := O^\text{trans}(A) \). Tracing through the two-sided version of Proposition 5.4, we have coactions

\[
\begin{align*}
\lambda : A &\to O \otimes A, & \lambda(t_i) &= \sum_{j=1}^{n} y_{t_i,t_j} \otimes t_j, \\
\rho : A &\to A \otimes O, & \rho(t_i) &= \sum_{j=1}^{n} t_j \otimes y_{t_j,t_i}.
\end{align*}
\]

Using the fact that \( \lambda(t_i)\lambda(t_j) = \lambda(t_j)\lambda(t_i) \) and \( \rho(t_i)\rho(t_j) = \rho(t_j)\rho(t_i) \) for all \( 1 \leq i, j \leq n \), one can show that all of the elements \( y_{t_i,t_j} \) commute in \( O \). Applying the coassociative and counital properties of \( \rho \) and \( \lambda \), we can obtain the coalgebra structure on \( O \), namely:

\[ \Delta(y_{t_i,t_j}) = \sum_{k=1}^{n} y_{t_i,t_k} \otimes y_{t_k,t_j}, \quad \text{and} \quad \varepsilon(y_{t_i,t_j}) = \delta_{i,j}. \]

Thus, we can see that this presentation of \( O \) agrees with the usual presentation of the bialgebra \( O(\text{Mat}_n(k)) \).

Now let \( \pi : \mathcal{H}(Q) \to O^\text{trans}(A) \) be the surjective weak bialgebra map given in Proposition 5.4, namely \( \pi(x_{t_i,t_j}) = y_{t_i,t_j} \). This is the canonical surjection of the free
algebra on $n^2$ generators onto the polynomial algebra in $n^2$ variables and hence its kernel $I$ is generated as a biideal by all commutators $[x_{t_i,t_j}, x_{t_k,t_\ell}].$

(b) The left UQSGd $O^{\text{left}}(A)$ and right UQSGd $O^{\text{right}}(A)$ are the `half quantum groups' described in the introduction (see, e.g., [CFR09]). Explicitly, one can check that $O^{\text{left}}(A)$ is the quotient of $\mathcal{H}(Q)$ by the biideal generated by

$$\{[x_{t_i,t_j}, x_{t_k,t_\ell}]\}_{1 \leq i,j,k \leq n} \quad \text{and} \quad \{[x_{t_i,t_j}, x_{t_k,t_\ell}] - [x_{t_k,t_\ell}, x_{t_i,t_j}]\}_{1 \leq i,j,k,\ell \leq n} \text{ with } j \neq \ell.$$ 

Similarly, $O^{\text{right}}(A)$ is the quotient of $\mathcal{H}(Q)$ by the biideal generated by

$$\{[x_{t_i,t_j}, x_{t_k,t_\ell}]\}_{1 \leq i,j,k \leq n} \quad \text{and} \quad \{[x_{t_i,t_j}, x_{t_k,t_\ell}] - [x_{t_i,t_\ell}, x_{t_k,t_j}]\}_{1 \leq i,j,k,\ell \leq n} \text{ with } i \neq k.$$ 

Hence, when $A$ is a proper quotient of $kQ$, we need not have $O^{\text{left}}(A) \cong O^{\text{trans}}(A) \cong O^{\text{right}}(A)$, in contrast with the path algebra case of Theorem 4.17.

Next we turn our attention to UQSGds of quadratic quotient algebras $kQ/I$. Consider the following terminology.

**Definition 5.8** ([GMV98, Section 2], [MV07, Section 1], [Gaw14]). Let $Q$ be a finite quiver and suppose $I$ is a graded ideal of the path algebra $kQ$.

(a) The *opposite quiver* $Q^{\text{op}}$ of $Q$ is defined to be the quiver formed by $(Q^{\text{op}})_0 = Q_0$ and $(Q^{\text{op}})_1 = Q_1^\ast$, where $Q_1^\ast$ is the arrow set consisting of reversed arrows of $Q_1$. For $p \in Q_1$, its reverse in $Q_1^\ast$ is denoted by $p^\ast$. If $a = p_1 \ldots p_\ell$ is a path of length $\ell$ in $Q$, then we let $a^\ast = p_\ell^\ast \ldots p_1^\ast \in Q^{\text{op}}$. If $f = \sum_i \alpha_i a_i$ is an element of $kQ$, the element $f^\ast \in kQ^{\text{op}}$ is defined to be $\sum_i \alpha_i a_i^\ast$.

(b) We identify $kQ^{\text{op}}$ with $(kQ_\ell)^\ast$ so that if $\{a_1, \ldots, a_d\}$ is the basis of $kQ_\ell$ consisting of paths of length $\ell$, then $\{a_1^\ast, \ldots, a_d^\ast\}$ is the dual basis.

(c) We call the quotient algebra $kQ/I$ *quadratic* if $I$ is generated by elements of $kQ_2$.

(d) The *quadratic dual* of the quadratic algebra $kQ/I$ is defined to be

$$(kQ/I)^\dagger = kQ^{\text{op}}/I^{\text{op}},$$

where $I^{\text{op}}$ is the ideal of $kQ^{\text{op}}$ generated by the orthogonal complement of the set $I_{\text{op}} := \{f^\ast \in kQ^{\text{op}} \mid f \in I \cap kQ_2\}$ in $kQ_2^{\text{op}}$.

**Remark 5.9.** As is our convention of Notation 1.2, we still read paths from left-to-right in $Q^{\text{op}}$. Hence, in $kQ^{\text{op}}$ we have

$$q^\ast p^\ast = (pq)^\ast$$

for $p, q \in Q$ (which is nonzero when $s(p^\ast) = t(p) = s(q) = t(q^\ast)$). Note that identifying $p \in kQ$ with $p^\ast \in kQ^{\text{op}}$ yields an anti-isomorphism of algebras and so $kQ^{\text{op}} \cong (kQ)^{\text{op}}$.

For the face algebras $\mathcal{H}(Q)$ and $\mathcal{H}(Q^{\text{op}})$ attached to $Q$ and $Q^{\text{op}}$, respectively, the map which sends $x_{a,b} \in \mathcal{H}(Q)$ to $x_{a^\ast,b^\ast} \in \mathcal{H}(Q^{\text{op}})$ is an anti-isomorphism of algebras and an isomorphism of coalgebras. As weak bialgebras, $\mathcal{H}(Q^{\text{op}}) \cong \mathcal{H}(Q)^{\text{op}}$.

The following theorem is a non-connected generalization of [Man88, Theorem 5.10].

**Theorem 5.10.** Let $Q$ be a finite quiver and suppose $I$ is an ideal such that $kQ/I$ is quadratic. Then, we have that
(a) \( O^\text{left}(kQ/I) \cong O^\text{right}((kQ/I)')^{\text{op}} \),
(b) \( O^\text{right}(kQ/I) \cong O^\text{left}((kQ/I)')^{\text{op}} \),
(c) \( O^\text{left}(kQ/I) \cong O^\text{right}(kQ/I)^{\text{cop}} \),
(d) \( O^\text{trans}(kQ/I) \cong O^\text{trans}((kQ/I)')^{\text{op}} \),
as weak bialgebras.

Proof. We will only provide the proofs of parts (a) and (c), as other parts will hold by similar arguments. To start, suppose that \( Q_1 = \{p_1, \ldots, p_n\} \). Then,

\[
I = \left\langle r_\alpha := \sum_{i,j=1}^n \text{with } t(p_i) = s(p_j) \ c^{[\alpha]}_{i,j} \ p_i \ p_j \right\rangle_{\alpha = 1, \ldots, m} \subseteq kQ_2
\]

for some scalars \( c^{[\alpha]}_{i,j} \). Moreover, we have

\[
I^{\perp}_{\text{op}} = \left\langle r^*_{\beta} := \sum_{i,j=1}^n \text{with } t(p_i') = s(p_j) \ d^{[\beta]}_{k,\ell} \ p_i' \ p_j' \right\rangle_{\beta = 1, \ldots, \mid Q_2 \mid - m} \subseteq kQ_2^{\text{op}}
\]

for some scalars \( d^{[\beta]}_{k,\ell} \). Here, \( \sum_{i,j=1}^n \text{with } t(p_i) = s(p_j) \ d^{[\beta]}_{i,j} \ c^{[\alpha]}_{i,j} = 0 \) for each pair \( \alpha, \beta \).

(a) By Proposition 5.4, we have that \( O^\text{left}(kQ/I) = \mathcal{H}(Q)/\mathcal{I} \) for some biideal \( \mathcal{I} \) of \( \mathcal{H}(Q) \), with the coalgebra structure induced by \( \mathcal{H}(Q) : \Delta(x_{p_i,p_k}) = \sum_{\alpha=1}^n x_{p_i,p_k} \otimes x_{p_k,p_i} \) and \( \varepsilon(x_{p_i,p_k}) = \delta_{i,k} \). We assert that

\[
\mathcal{I} = \left\langle \sum_{i,j,k,\ell=1}^n c^{[\alpha]}_{i,j} d^{[\beta]}_{k,\ell} x_{p_i,p_k} x_{p_j,p_\ell} \right\rangle_{\alpha = 1, \ldots, m, \beta = 1, \ldots, \mid Q_2 \mid - m}
\]

Namely, there exists a basis of \( kQ_2 \) consisting of elements \( \{r_\alpha\}_{\alpha = 1, \ldots, m} \) and \( \{s_\gamma\}_{\gamma = 1, \ldots, \mid Q_2 \mid - m} \) so that the evaluation \( \langle r^*_{\beta}, s_\gamma \rangle = \delta_{\beta,\gamma} \) for each \( \beta, \gamma = 1, \ldots, \mid Q_2 \mid - m \). Moreover, for each \( k, \ell \), we can write

\[
(p_k p_\ell) = \sum_{\gamma} d^{[\gamma]}_{k,\ell} s_\gamma + \sum_{\alpha} e^{[\alpha]}_{k,\ell} r_\alpha
\]

for some scalars \( e^{[\alpha]}_{k,\ell} \). (This can be checked by evaluation with \( r^*_{\beta} \).) Now, the left coaction of \( \mathcal{H}(Q) \) on \( kQ_2 \), given by \( p_i \mapsto \sum_k x_{p_i,p_k} \otimes p_k \) from Example 2.22, preserves the relation \( r_\alpha \) if and only if the following expression lies in \( O \otimes I \):

\[
\begin{aligned}
&\sum_{i,j,k,\ell=1}^n c^{[\alpha]}_{i,j} d^{[\beta]}_{k,\ell} x_{p_i,p_k} x_{p_j,p_\ell} \otimes p_k p_\ell \\
&= \sum_{i,j,k,\ell=1}^n c^{[\alpha]}_{i,j} x_{p_i,p_k} x_{p_j,p_\ell} \otimes p_k p_\ell \\
&= \sum_{i,j,k,\ell=1}^n c^{[\alpha]}_{i,j} d^{[\beta]}_{k,\ell} x_{p_i,p_k} x_{p_j,p_\ell} \otimes s_\gamma + \sum_{\alpha',i,j,k,\ell} c^{[\alpha']}_{i,j} e^{[\alpha]}_{k,\ell} x_{p_i,p_k} x_{p_j,p_\ell} \otimes r_{\alpha'}.
\end{aligned}
\]

Since \( \{s_\gamma\}_{\gamma} \cup \{r_{\alpha'}\}_{\alpha'} \) is a basis of \( kQ_2 \), we must have that

\[
\left\{ \sum_{i,j,k,\ell=1}^n c^{[\alpha]}_{i,j} d^{[\beta]}_{k,\ell} x_{p_i,p_k} x_{p_j,p_\ell} = 0 \right\}_{\alpha = 1, \ldots, m, \beta = 1, \ldots, \mid Q_2 \mid - m}
\]

are the generators of the relation space \( \mathcal{I} \) for \( O^\text{left}(kQ/I) \) as in Proposition 5.4.
On the other hand, by Example 2.22 we have a right coaction of $\mathcal{H}(Q^{\text{op}})$ on $\mathbb{k}Q^{\text{op}}$ given by $p_i^* \mapsto \sum_k p_k^* \otimes x_{p_k^*,p_i^*}$. By a similar argument as above, this coaction preserves the relations of $I_{\text{op}}$, if and only if

$$ \left\{ \sum_{i,j,k,\ell=1}^n d_{k,\ell}^{[\beta]} c_{i,j}^{[\alpha]} x_{p_j^*,p_k^*} x_{p_i^*,p_\ell^*} = 0 \right\}_{\eta=1,\ldots,m}^{\beta=1,\ldots,|Q_2|-m}$$

are the generators of the relation space of $O^{\text{right}}((\mathbb{k}Q/I)^\ell)$ as a quotient of $\mathcal{H}(Q^{\text{op}})$. Hence,

$$O^{\text{right}}((\mathbb{k}Q/I)^\ell) = \mathcal{H}(Q^{\text{op}})/\left\langle \sum_{i,j,k,\ell=1}^n c_{i,j}^{[\alpha]} d_{k,\ell}^{[\beta]} x_{p_j^*,p_k^*} x_{p_i^*,p_\ell^*} \right\rangle_{\alpha=1,\ldots,m}^{\beta=1,\ldots,|Q_2|-m},$$

with $\Delta(x_{p_i^*,p_j^*}) = \sum_{m=1}^n x_{p_i^*,p_m^*} \otimes x_{p_m^*,p_j^*}$ and $\varepsilon(x_{p_i^*,p_j^*}) = \delta_{i,j}$.

Now, the desired isomorphism from $O^{\text{left}}(\mathbb{k}Q/I)$ to $O^{\text{right}}((\mathbb{k}Q/I)^\ell)^{\text{op}}$ is obtained by sending $x_{p_i^*,p_k^*}$ to $x_{p_k^*,p_i^*}$.

(c) By Proposition 5.4, we have that $O^{\text{right}}(\mathbb{k}Q/I) = \mathcal{H}(Q)/\mathcal{I}'$ for some biideal $\mathcal{I}'$ of $\mathcal{H}(Q)$, with the coalgebra structure induced by $\mathcal{H}(Q) : \Delta(x_{p_k^*,p_i^*}) = \sum_{w=1}^n x_{p_k^*,p_w^*} \otimes x_{p_w^*,p_i^*}$, and $\varepsilon(x_{p_k^*,p_i^*}) = \delta_{i,k}$. Now, the right coaction of $\mathcal{H}(Q)$ on $\mathbb{k}Q$, given by $p_i \mapsto \sum_k p_k \otimes x_{p_k^*,p_i^*}$ from Example 2.22, preserves each relation $r_\alpha$ of $I$ if and only if

$$\mathcal{I}' = \left\langle \sum_{i,j,k,\ell=1}^n c_{i,j}^{[\alpha]} d_{k,\ell}^{[\beta]} x_{p_k^*,p_i^*} x_{p_i^*,p_\ell^*} \right\rangle_{\alpha=1,\ldots,m}^{\beta=1,\ldots,|Q_2|-m}.$$

Now, considering the presentation of $O^{\text{left}}(\mathbb{k}Q/I)$ from part (a), the desired isomorphism from $O^{\text{left}}(\mathbb{k}Q/I)$ to $O^{\text{right}}((\mathbb{k}Q/I)^\ell)^{\text{cop}}$ is obtained by sending $x_{p_i^*,p_k^*}$ to $x_{p_k^*,p_i^*}$. \square

**Example 5.12.** Let $A = \mathbb{k}Q$. Then $A' = \mathbb{k}Q^{\text{op}}/\langle \mathbb{k}(Q^{\text{op}})^2 \rangle$ where $\langle \mathbb{k}(Q^{\text{op}})^2 \rangle$ is the ideal of $\mathbb{k}Q^{\text{op}}$ generated by the space $\mathbb{k}(Q^{\text{op}})^2$. By the above theorem, we have

$$O^{\text{left}}(A) \cong O^{\text{right}}(A')^{\text{op}}$$

as weak bialgebras. Since, by Theorem 4.17, $O^{\text{left}}(\mathbb{k}Q) \cong \mathcal{H}(Q)$, we have that $O^{\text{right}}(A')^{\text{op}} \cong \mathcal{H}(Q)$. Further, $\mathcal{H}(Q)^{\text{op}} \cong \mathcal{H}(Q^{\text{op}})$, and so we conclude that

$$O^{\text{right}}(A') \cong \mathcal{H}(Q^{\text{op}}).$$

Similarly, we have that $O^{\text{left}}(A') \cong O^{\text{trans}}(A') \cong \mathcal{H}(Q^{\text{op}})$ as weak bialgebras.

We end with a family of concrete examples of UQSGds for quadratic quotient path algebras—namely, those for preprojective algebras.
Example 5.13. Let $Q$ be the extended type $A$ Dynkin quiver with $|Q_0| \geq 3$, and consider its double $\overline{Q}$ formed by adding $p^*$ for each $p \in Q_1$. For example, when $|Q_0| = 3$,

\[ Q = \begin{array}{c}
1 \\
\downarrow p_1 \\
2 \\
\uparrow p_2 \\
3 \\
\downarrow p_3
\end{array} \quad \overline{Q} = \begin{array}{c}
1 \\
\downarrow p_1 \\
2 \\
\uparrow p_2 \\
3 \\
\downarrow p_3
\end{array}
\]

The preprojective algebra on $Q$ is defined to be the $k$-algebra,

\[ \Pi_Q = k\overline{Q}/(\sum_{i \in Q_0} p_i p_i^* - \sum_{i \in Q_0} p_i^* p_i - 1). \]

(Here, we index the vertices $i$ by elements of $\mathbb{Z}/Q_0/\mathbb{Z}$.) By [Wei19, Section 3], we have that

\[ \Pi_Q \cong k\overline{Q}/(p_i p_i^* - p_i^* p_i - 1)_{i \in Q_0} \]

as $k$-algebras. Therefore, any path in $Q$ can be rewritten (in $\Pi_Q$) so that all of the nonstar arrows occur, followed by all of the star arrows. We omit the details here, but we have that, as weak bialgebras,

\[ \mathcal{O}^{\text{left}}(\Pi_Q) \cong \mathcal{H}(\overline{Q})/\mathcal{I}, \quad \mathcal{O}^{\text{right}}(\Pi_Q) \cong \mathcal{H}(\overline{Q})/\mathcal{J}, \quad \mathcal{O}^{\text{trans}}(\Pi_Q) \cong \mathcal{H}(\overline{Q})/(\mathcal{I} + \mathcal{J}), \]

for

\[ \mathcal{I} = \left\{ \frac{x_{pk,p} x_{p_k^*,p^{2}} - x_{p_k^*,p^{1}} x_{pk,p^{1}} + x_{pk,p^{1}} x_{p_k^*,p^{1}}}{x_{p_k^*,p^{1}} x_{pk,p^{1}} - x_{pk,p^{1}} x_{p_k^*,p^{1}}} \right\}_{i,k \in Q_0}, \]

\[ \mathcal{J} = \left\{ \frac{x_{p_k,p} x_{p_k^*,p_k^*} - x_{p_k^*,p_k^*} x_{p_k,p_k^*}}{x_{p_k^*,p_k^*} x_{p_k,p_k^*} - x_{p_k,p_k^*} x_{p_k^*,p_k^*}} \right\}_{i,k \in Q_0}. \]

Remark 5.14. The universal quantum semigroupoids analyzed in Theorem 5.10 generalize Manin’s universal semigroups for quadratic algebras [Man88], and likewise, universal quantum semigroupoids exist and can be constructed explicitly for $N$-homogeneous algebras. This, in turn, generalizes Chirvasitu–Walton–Wang’s universal quantum semigroupoids for such algebras [CWW19] to the non-connected setting. Prevalent non-connected $N$-homogeneous algebras include quiver (super)potential algebras, which are used frequently in cluster theory [DWZ08], in Donaldson–Thomas theory [MR10], and in other fields; we expect that the universal symmetries of these algebras will have similar applications.

6. FOR FURTHER INVESTIGATION: UNIVERSAL QUANTUM LINEAR GROUPOIDS

In this section, we consider weak Hopf algebras that coact universally (and linearly) on an algebra $A$ in Hypothesis 1.1, and propose directions for future research. First, let us recall the notion of a universal coacting Hopf algebra, prompted by [Man88, Chapter 7].
Definition 6.1 (UQG). Take $A$ as in Hypothesis 1.1 and further assume that $A$ is connected. Then a Hopf algebra is said to be a left (resp., right, transposed) universal quantum linear group (UQG) of $A$ if it satisfies the conditions of Definition 1.4(a) (or, Definition 1.4(b), Definition 1.5(b)) by replacing ‘bialgebra’ with ‘Hopf algebra’.

A general way of constructing a UQG from a UQSG is by taking the Hopf envelope as discussed briefly in [Man88, Section 7.5]. Other explicit constructions involve the quantum determinant (also known as the homological determinant), which is a (typically) central group-like element $D$ of a UQSG that depends on the UQSG coaction on $A$. Here, one takes a UQSG, say $\mathcal{O}^{\text{trans}}(A)$, and forms two Hopf algebras depending on whether the quantum determinant is trivial (i.e., equal to the unit) or is arbitrary:

$$O_{\text{SL}}^{\text{trans}}(A) := \mathcal{O}^{\text{trans}}(A)/(D - 1), \quad O_{\text{GL}}^{\text{trans}}(A) := \mathcal{O}^{\text{trans}}(A)[D^{-1}].$$

We refer to these universal Hopf algebras as UQGs of SL-type and of GL-type, respectively.

It is therefore natural to ask if this can be generalized to the framework of universal coacting weak Hopf algebras. We recall the definition of a weak Hopf algebra below.

Definition 6.2. A weak Hopf algebra is a sextuple $(H, m, u, \Delta, \varepsilon, S)$, where $(H, m, u, \Delta, \varepsilon)$ is a weak bialgebra and $S : H \to H$ is a $k$-linear map called the antipode that satisfies the following properties for all $h \in H$:

$$S(h_1)h_2 = \varepsilon_s(h), \quad h_1S(h_2) = \varepsilon_t(h), \quad S(h_1)h_2S(h_3) = S(h).$$

Note that if $H$ is a weak Hopf algebra, the following are equivalent: $H$ is a Hopf algebra; $\Delta(1) = 1 \otimes 1$; $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$ for all $x, y \in H$; $S(x_1)x_2 = \varepsilon(x)1$ for all $x \in H$; and $x_1S(x_2) = \varepsilon(x)1$ for all $x \in H$ [BNS99, page 5].

Now we define a universal weak Hopf algebra, similar to the manner that a UQG was defined above, for $A$ not necessarily connected.

Definition 6.3 (UQGd). Take $A$ as in Hypothesis 1.1. Then a weak Hopf algebra is said to be a left (resp., right, transposed) universal quantum linear groupoid of $A$ if it satisfies the conditions of part (a) (resp., (b), (c)) of Definition 1.7 by replacing ‘weak bialgebra’ with ‘weak Hopf algebra’.

This prompts the following series of questions.

Question 6.4. Take $A$ as in Hypothesis 1.1. In general:

1. When are the UQSGds $\mathcal{O}^{\text{left}}(A)$, $\mathcal{O}^{\text{right}}(A)$, $\mathcal{O}^{\text{trans}}(A)$ weak Hopf algebras?

2. What is a ‘weak Hopf envelope’ (of a UQGd)?

Pertaining to the SL-type and GL-type constructions:

3. What is the quantum determinant $D$ of the coaction of a UQGd $\mathcal{O}^{\text{left}}(A)$ (or, $\mathcal{O}^{\text{right}}(A)$, $\mathcal{O}^{\text{trans}}(A)$) on $A$?

4. Is $D$ invertible, and if so, are $\mathcal{O}^{\text{left}}(A)[D^{-1}]$, $\mathcal{O}^{\text{right}}(A)[D^{-1}]$, $\mathcal{O}^{\text{trans}}(A)[D^{-1}]$ weak Hopf algebras that coact on $A$ (universally) with arbitrary quantum determinant?
Are $\mathcal{O}^{\text{left}}(A)/(D - 1)$, $\mathcal{O}^{\text{right}}(A)/(D - 1)$, $\mathcal{O}^{\text{trans}}(A)/(D - 1)$ weak Hopf algebras that coact on $A$ (universally) with trivial quantum determinant?

Finally, as discussed in the introduction:

Ideally, a universal (weak) bi/Hopf algebra should behave ring-theoretically and homologically like the algebra that it coacts on.

This holds for transposed coactions on many connected graded algebras in Hypothesis 1.1; see, e.g., [AST91, WW16]. Likewise, the best candidates we have for the philosophy to hold for coactions on algebras in Hypothesis 1.1 are the transposed UQSGds [Definition 3.10] and the transposed UQGds (defined above). Naturally, we inquire:

**Question 6.5.** Take an algebra $A$ in Hypothesis 1.1. In general, does the ring-theoretic and homological behavior of the transposed UQSGd and transposed UQGds of $A$ reflect that of $A$? More specifically, if $A$ has one of the following properties,

- (a) finite Gelfand-Kirillov dimension/nice Hilbert series,
- (b) Noetherian/coherent,
- (c) domain/prime/semiprime,
- (d) finite global dimension/finite injective dimension,
- (e) skew (or twisted) Calabi-Yau,

the transposed UQSGd and transposed UQGds of $A$ satisfy the same property as well?

Pertinent articles include the work of Gaddis, Reyes, Rogalski, and Zhang on (non-connected) graded skew Calabi-Yau algebras [RRZ14, RR22, RR19, GR21].

**Remark 6.6.** After the first version of this article appeared, Question 6.5 was addressed in work of Calderón and Walton [CW] in the case that $A$ is the path algebra $kQ$ of a finite quiver. It was shown that many properties hold for $kQ$ if and only if they hold for its UQSGd $\mathcal{H}(Q)$, including: having finite dimension, having finite Gelfand–Kirillov dimension, being noetherian, being semiprime, and being Koszul. Graph-theoretic techniques were used to achieve these results.

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