Boundedness of normalization generalized differential operator of fractional formal

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Abstract

Many authors have considered and investigated generalized fractional differential operators. The main object of this present paper is to define a new generalized fractional differential operator \(T^{\beta,\tau,\gamma}\), which generalized the Srivastava-Owa operators. Moreover, we investigate of the geometric properties such as univalency, starlikeness, convexity for their normalization. Further, boundedness and compactness in some well known spaces, such as Bloch space for last mention operator also are considered. Our tool is based on the generalized hypergeometric function.

1 Introduction

The study of fractional operators (integral and differential) plays a vital and essential role in mathematical applied and mathematical analysis. To define a generalized fractional differential operators and study their properties is one of important areas of current ongoing research in the geometric function theory and its concerning fields. Many authors generalized fractional differential operators on previously known classes of analytic and univalent functions to discover and produce new classes and to investigate various interesting properties of new classes, for example, (see [1–5]). In addition, several interesting applications of special function based definition for generalized fractional differential operators can be found in [6–9].

let \(A\) denote the class of functions \(f(z)\) of the form:

\[
f(z) = z + \sum_{\kappa=2}^{\infty} a_{\kappa} z^{\kappa}
\]

which are analytic functions in the open unit disk \(U := \{z \in \mathbb{C} : |z| < 1\}\) and satisfy the condition \(f(0) = f'(0) - 1 = 0\). The subclass of \(A\), consisting of all univalent(or ono-to-one) functions \(f(z)\) in \(U\) is denoted by \(S\). A function \(f(z) \in A\) is a starlike functions of order \(\lambda\) \((0 \leq \lambda < 1)\), if and only if

\[
\Re\left\{ \frac{zf'(z)}{f(z)} \right\} > \lambda, \quad z \in U
\]
and denoted by $S^*_\lambda(m)$. Similarly, if $f(z) \in \mathcal{A}$ and satisfies the following inequality

$$\Re\left\{zf''(z) + 1\right\} > \lambda, \quad z \in U$$

for some $\lambda (0 \leq \lambda < 1)$, then $f$ is said to be convex function of order $\lambda$ and we denote this class by $K_\lambda(m)$.

**Theorem 1.** (Bieberbach’s Conjecture [10]) If the function $f(z)$ defined by (1.1) is in the class $S^*$ then $|a_\kappa| \leq \kappa$ for all $\kappa \geq 2$ and if it is in the class $K$ then $|a_\kappa| \leq 1$ for all $\kappa \geq 2$.

For the function $f(z)$ defined by (1.1) and $h(z) = z + \sum_{\kappa=2}^{\infty} b_\kappa z^\kappa$, the convolution (or Hadamard product) $f * h$ is given by

$$f * h(z) = z + \sum_{\kappa=2}^{\infty} a_\kappa b_\kappa z^\kappa$$

(1.4)

The operator $O^{\beta,\tau}_z$ is defined in terms of Riemann-Liouville fractional differential operator $D^{\beta-\tau}_z z$ as

$$O^{\beta,\tau}_z f(z) = \frac{\Gamma(\tau)}{\Gamma(\beta)} z^{1-\tau} D^{\beta-\tau}_z z^{\beta-1} f(z) \quad (z \in U),$$

(1.5)

This operator given by Tremblay [11]. Recently, Ibrahim [12] extended Tremblay’s operator in terms of Srivastava-Owa fractional differential of $f(z)$ of order $(\beta - \tau)$ and defined as follows

$$\mathcal{T}^{\beta,\tau}_z f(z) = \frac{\Gamma(\tau)}{\Gamma(\beta)} z^{1-\tau} D^{\beta-\tau}_z z^{\beta-1} f(z),$$

(1.6)

or, equivalent

$$\mathcal{T}^{\beta,\tau}_z f(z) = \frac{\Gamma(\tau)}{\Gamma(\beta)} \frac{d}{dz} \int_0^z (z - \zeta)^{-\beta} \zeta^{\tau-1} d\zeta \quad (z \in U).$$

(1.7)

Often, the generalized fractional differential operators and their applications associated with special functions (see [13]). So, the Fox-Wright $p\Psi_q$ generalization of the hypergeometric $p F_q$ function is considered one of the important special function in geometric function theory defined by

$$p\Psi_q[z] = p\Psi_q \left[ \begin{array}{c} (a_j, 1);_1 p; \\ (b_j, B_j);_q q; \end{array} \right] = \sum_{\kappa=0}^{\infty} \frac{\Pi_{j=1}^{p} \Gamma(a_j + \kappa A_j)}{\Pi_{j=1}^{q} \Gamma(b_j + \kappa B_j)(1)_{\kappa}} z^\kappa.$$
where
\[
\Delta = \frac{\prod_{j=1}^{q} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)}
\]
and \(\,_{p}F_{q}\) is the generalized hypergeometric function. The Pochhammer symbol denoted by \((\rho)_{\kappa}\) and defined as follows
\[
(\rho)_{\kappa} := \frac{\Gamma(\rho + \kappa)}{\Gamma(\rho)} = \left\{ \begin{array}{ll}
\rho(\rho + 1)(\rho + \kappa - 1) & (\kappa \in \mathbb{N}; \rho \in \mathbb{C}), \\
1 & (\kappa = 0; \rho \in \mathbb{C} \setminus \{0\}),
\end{array} \right.
\]
where \(\Gamma\) is the well-known Gamma function.

In the present paper, the generalized Tremblay operator \(T_{\beta,\tau,\gamma}^{z}\) of analytic function is defined. Also, the univalence properties of normalization generalized operator are investigated and proved. Further, the boundedness and compactness of the last operator are studied.

2 Background and Results

In this section, we consider the generalized Tremblay type fractional differential operator and then we determine the generalized fractional differential of some special functions. For this main purpose, we begin by recalling the Srivastava-Owa fractional differential operators of \(f(z)\) of order \(\beta\) defined by
\[
D_{z}^{\beta} f(z) := \frac{1}{\Gamma(1 - \beta)} \frac{d}{dz} \int_{0}^{z} (z - \zeta)^{-\beta} f(\zeta) d\zeta,
\]
where \(0 \leq \beta < 1\), and the function \(f(z)\) is analytic in simply-connected region of the complex \(z\)-plane containing the origin and the multiplicity of \((z - \zeta)^{-\beta}\) is removed by requiring \(\log(z - \zeta)\) to be real when \((z - \zeta) > 0\) (see [14, 15]). Then under the conditions of the above definition the Srivastava-Owa fractional differential of \(f(z) = z^{\kappa}\) is defined by
\[
D_{z}^{\beta} \{z^{\kappa}\} = \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa - \rho + 1)} z^{\kappa-\beta}.
\]
The theory of fractional integral and differential operators has found significant importance applications in various areas, for example [16]. Recently, many mathematicians have developed various generalized fractional differential of Srivastava-Owa type, for example, see [17, 18]. Further, we consider a new generalized Srivastava-Owa type fractional differential formulas which is recently appeared.

Definition 1. [1] The generalized Srivastava-Owa fractional differential of \(f(z)\) of order \(\beta\) is defined by
\[
D_{z}^{\beta,\gamma} f(z) := \frac{(\gamma + 1)^{\beta}}{\Gamma(1 - \beta)} \frac{d}{dz} \int_{0}^{z} (z^{\gamma+1} - \zeta^{\gamma+1})^{-\beta} \zeta^{\gamma} f(\zeta) d\zeta,
\]
where \(0 \leq \beta < 1\), \(\gamma > 0\) and \(f(z)\) is analytic in simply-connected region of the complex \(z\)-plane \(C\) containing the origin, and the multiplicity of \((z^{\gamma+1} - \zeta^{\gamma+1})^{-\beta}\) is removed by requiring \(\log(z - \zeta)\) to be real when \((z - \zeta) > 0\). In particular, the generalized Srivastava-Owa fractional differential of function \(f(z) = z^{\kappa}\) is defined by
\[
D_{z}^{\beta,\gamma} \{z^{\kappa}\} = \frac{(\gamma + 1)^{\beta-1} \Gamma(\frac{\kappa}{\gamma+1} + 1)}{\Gamma(\frac{\kappa}{\gamma+1} + 1 - \beta)} z^{(1-\beta)(\gamma+1)+\kappa-1}.
\]
Now, we present the generalized fractional operator of type fractional differential as follows:

**Definition 2.** The generalized fractional differential of $f(z)$ of two parameters $\beta$ and $\tau$ is defined by

$$
\mathcal{T}_z^{\beta,\gamma} f(z) := \frac{(\gamma + 1)^{\beta - \tau} \Gamma(\tau)}{\Gamma(\beta) \Gamma(1 - \beta + \tau)} \left( z^{1-\tau} \frac{d}{dz} \right) \int_0^z \frac{\zeta^{-\beta - 1} f(\zeta)}{(\zeta^{\gamma + 1} - \zeta^{\gamma + 1})^{\beta - \tau}} d\zeta,
$$

(2.4)

$$
(\gamma \geq 0; 0 < \beta \leq 1; 0 < \tau \leq 1; 0 \leq \beta - \tau < 1),
$$

where the function $f(z)$ is analytic in simple-connected region of the complex $z$-plane $\mathbb{C}$ containing the origin, and the multiplicity of $(z^{\gamma + 1} - \zeta^{\gamma + 1})^{-\beta - \tau}$ is removed by requiring $\log (z^{\gamma + 1} - \zeta^{\gamma + 1})$ to be non-negative when $(-\beta - \tau) > 0$.

**Remark 1.** For $f(z) \in \mathcal{A}$, we have

i- When $\gamma = 0$ in (2.4), is reduce to the classical known one [16].

ii- When $\tau = \beta$, then

$$
\mathcal{T}_z^{\beta,\beta,\gamma} f(z) = f(z).
$$

(2.5)

We investigate the generalized fractional differential of the function $f(z) = z^\nu; \nu \geq 0$.

**Theorem 2.** Let $0 \leq \beta - \tau < 1$ for some $0 < \beta \leq 1; 0 < \tau \leq 1$ and $\nu \in \mathbb{N}$, then we have

$$
\mathcal{T}_z^{\beta,\tau,\gamma} \{z^\nu\} = \frac{(\gamma + 1)^{\beta - \tau} \Gamma(\nu + \beta - 1 + 1) \Gamma(\tau)}{\Gamma(\nu + 1 + \beta + \gamma + 1) \Gamma(\beta)} z^{(1 - \beta - \gamma + \nu + 1)}. 
$$

(2.6)

**Proof.** Applying (2.4) in Definition 2 to the function $z^\nu$, we obtain

$$
\mathcal{T}_z^{\beta,\tau,\gamma} \{z^\nu\} = \frac{(\gamma + 1)^{\beta - \tau} \Gamma(\tau)}{\Gamma(\beta) \Gamma(1 - \beta + \tau)} \left( z^{1-\tau} \frac{d}{dz} \right) \int_0^z \frac{\zeta^{\gamma + \beta + \nu + 1} (z^{\gamma + 1} - \zeta^{\gamma + 1})^{\tau - \beta} d\zeta}{(\zeta^{\gamma + 1} - \zeta^{\gamma + 1})^{\beta - \tau}}.
$$

Let use the substitution $w := (\zeta^{\gamma + 1})^{\tau - 1}$ in this expression, then have

$$
\mathcal{T}_z^{\beta,\tau,\gamma} \{z^\nu\} = \frac{(\gamma + 1)^{\beta - \tau - 1} \Gamma(\tau)}{\Gamma(\beta) \Gamma(1 - \beta + \tau)} \left( z^{1-\tau} \frac{d}{dz} \right) \int_0^1 w^{\frac{\nu + \beta - 1}{\gamma + 1} (1 - w)^{\tau - \beta} d\zeta}
$$

(2.7)

$$
= \frac{\Gamma(\gamma + 1) (\nu + \beta + \gamma + v + 1) \Gamma(\tau)}{\Gamma(\beta) \Gamma(1 - \beta + \tau)} \Gamma(1 - \beta + \gamma + \nu + 1)
$$

$$
\times \left( z^{(1 - \beta - \gamma + \nu + 1)} \right) B\left( \frac{\nu + \beta - 1}{\gamma + 1} + 1, 1 - \beta + \tau \right),
$$

$$
= \frac{(\gamma + 1)^{\beta - \tau} \Gamma(\nu + 1) \Gamma(\tau)}{\Gamma(\nu + 1 + \beta + \gamma + 1) \Gamma(\beta)} z^{(1 - \beta - \gamma + \nu + 1)},
$$

where $B(\cdot, \cdot)$ in (2.7) is the Beta function defined by [ ]

$$
B(u, v) = \int_0^1 \eta^u (1 - \eta)^{v-1} d\eta \quad (u > 0; \nu > 0),
$$

(\Gamma(u) \Gamma(v))

$$
\Gamma(u + v) \quad (u, \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-).
$$

Thus, the proof Theorem 2 is completed. □
2.1 Applications by using special functions

In this subsection we replace the normalized of function $f(z)$ by some special functions in Theorem 2. Begin let recall generalized of Koebe function $f(z) = z/(z - 1)^{-\alpha}$ as follows:

**Theorem 3.** Let $f(z) = z(1 - z)^{-\alpha}$, $\alpha \geq 1$ and $z \in \mathbb{U}$, then

$$\mathbb{T}_z^{\beta,\tau,\gamma} f(z) = \frac{(\gamma + 1)^{\beta - \tau} \Gamma(\tau)}{\Gamma(\beta)} (1 - \beta + \tau)^{\gamma} \sum_{\kappa=1}^{\infty} \frac{(\alpha)_{\kappa}}{(1)_{\kappa}} z^{\kappa}.$$

**Proof.** Since

$$\mathbb{T}_z^{\beta,\tau,\gamma} \{ z(1 - z)^{-\alpha} \} = \mathbb{T}_z^{\beta,\tau,\gamma} \left( \sum_{k=1}^{\infty} \frac{(\alpha)_{\kappa}}{(1)_{\kappa}} z^{\kappa} \right)$$

then, by using Theorem 2 we obtain

$$= \frac{(\gamma + 1)^{\beta - \tau} \Gamma(\tau)}{\Gamma(\beta) \Gamma(1 - \beta + \tau)} \left( \int_0^z (1 - z)^{\gamma} \frac{d}{dz} \left( z^{\gamma + 1} - \zeta^{\gamma + 1} \right)^{\tau - \beta} \sum_{\kappa=1}^{\infty} \frac{(\alpha)_{\kappa}}{(1)_{\kappa}} \zeta^{\kappa} d\zeta \right).$$

$$= \frac{(\gamma + 1)^{\beta - \tau} \Gamma(\tau)}{\Gamma(\beta)} (1 - \beta + \tau)^{\gamma} \sum_{k=1}^{\infty} \frac{(\kappa + \alpha) \Gamma(\frac{k+\beta}{\gamma+1} + \frac{\beta}{\gamma+1} - 1)}{\Gamma(\alpha+1) \Gamma(\frac{\alpha+1}{\gamma+1} + 1 - \beta + \tau) \Gamma(1)_{\kappa}} \frac{z^{\kappa}}{z^{\kappa+1}}.$$

$$= \frac{\alpha(\gamma + 1)^{\beta - \tau} \Gamma(\tau)}{\Gamma(\beta)} (1 - \beta + \tau)^{\gamma} \sum_{k=0}^{\infty} \frac{(\kappa + \alpha + 1) \Gamma(\kappa + 1) \Gamma(\frac{\kappa+\beta}{\gamma+1} + \frac{\beta}{\gamma+1} + 1)}{\Gamma(\alpha+1) \Gamma(\frac{\alpha+1}{\gamma+1} + 1 - \beta + \tau) \Gamma(1)_{\kappa}} \frac{z^{\kappa}}{z^{\kappa+1}}.$$

$$= \frac{\alpha(\gamma + 1)^{\beta - \tau} \Gamma(\tau)}{\Gamma(\beta)} (1 - \beta + \tau)^{\gamma} \sum_{k=1}^{\infty} \frac{(\alpha + 1, 1, 1 + \frac{\beta}{\gamma+1}, \frac{\beta}{\gamma+1})}{(2, 1, 1 - \beta + \tau + \frac{\beta}{\gamma+1})} \left( \frac{(\alpha + 1, 1, 1 + \frac{\beta}{\gamma+1}, \frac{\beta}{\gamma+1})}{(2, 1, 1 - \beta + \tau + \frac{\beta}{\gamma+1})} \right) \frac{z^{\kappa}}{z^{\kappa+1}}.$$

Setting $\alpha = 2$ in Theorem 3, we obtain

**Corollary 1.** Let $f(z) = z(1 - z)^{-2}$, then

$$\mathbb{T}_z^{\beta,\tau,\gamma} f(z) = \frac{2(\gamma + 1)^{\beta - \tau} \Gamma(\tau)}{\Gamma(\beta)} (1 - \beta + \tau)^{\gamma} \sum_{k=1}^{\infty} \frac{(\alpha)_{\kappa}}{(1)_{\kappa}} z^{\kappa}.$$

Setting $\alpha = 1$, in Theorem 3 we obtain

**Corollary 2.** Let $f(z) = z(1 - z)^{-1}$ then we have

$$\mathbb{T}_z^{\beta,\tau,\gamma} f(z) = \frac{(\gamma + 1)^{\beta - \tau} \Gamma(\tau)}{\Gamma(\beta)} (1 - \beta + \tau)^{\gamma} \sum_{k=1}^{\infty} \frac{(\alpha)_{\kappa}}{(1)_{\kappa}} z^{\kappa}.$$

The following Theorems is seen to immediately follow from Theorems 2 and 3.
Theorem 4. Let $0 < \beta \leq 1$ and $0 < \tau \leq 1$ such that $0 \leq \beta - \tau < 1$. Then we have
\[
\begin{align*}
\mathcal{T}_z^{\beta, \tau, \gamma}\{z e^z\} &= \frac{(\gamma + 1)^{\beta - \tau} \Gamma(\tau)}{\Gamma(\beta)} z^{(1 - \beta + \tau)} z^{-\gamma + \beta - 1} (z^{\gamma + 1} + 1)^{1 - \beta + \tau} \sum_{\kappa=1}^{\infty} \frac{(a\zeta)^{\kappa}}{(1)_{\kappa}} d\zeta, \\
&= \frac{(\gamma + 1)^{\beta - \tau} \Gamma(\tau)}{\Gamma(\beta)} \left(\sum_{\kappa=0}^{\infty} \frac{\Gamma(\kappa + \beta - 1)}{\Gamma(\gamma + 1) + 1} \right) z^\kappa.
\end{align*}
\]

Theorem 5. Let $0 < \beta \leq 1$ and $0 < \tau \leq 1$ such that $0 \leq \beta - \tau < 1$. Then, we obtain
\[
\begin{align*}
\mathcal{T}_z^{\beta, \tau, \gamma}\{z \frac{\partial}{\partial z} F_1(\alpha, \lambda; z)\} &= \frac{(\gamma + 1)^{\beta - \tau} \Gamma(\tau)}{\Gamma(\beta) \Gamma(1 - \beta + \tau)} z^{(1 - \beta + \tau)} \sum_{\kappa=1}^{\infty} \frac{(a\zeta)^{\kappa}}{(1)_{\kappa}} B\left(\frac{\kappa + \beta - 1}{\gamma + 1} + 1, 1 - \beta + \tau\right) z^\kappa \\
&= \frac{\alpha(\gamma + 1)^{\beta - \tau} \Gamma(\tau)}{\lambda \Gamma(1 + \tau)} z^{(1 - \beta + \tau) + \gamma} \sum_{\kappa=0}^{\infty} \frac{(1)_{\kappa}(\alpha + 1)}{(\kappa + 1)_{\kappa}(1)_{\kappa}} B\left(\frac{\kappa + \beta - 1}{\gamma + 1} + 1, 1 - \beta + \tau\right) z^\kappa,
\end{align*}
\]
where $F_1(\alpha, \lambda; z)$ is the confluent hypergeometric Kummer function see [6] defined by
\[
F_1(\alpha, \lambda; z) := \frac{\Gamma(\lambda)}{\Gamma(\alpha) \Gamma(\lambda - \alpha)} \int_0^1 t^{\alpha - 1} (1 - t)^{\lambda - \alpha - 1} e^z dt,
\]
and $(a)_{\kappa}$, $(\lambda)_{\kappa}$ are given by [10].

Theorem 6. Let $0 < \beta \leq 1$ and $0 < \tau \leq 1$ such that $0 \leq \beta - \tau < 1$. Then, we obtain
\[
\begin{align*}
\mathcal{T}_z^{\beta, \tau, \gamma}\{z \frac{\partial}{\partial z} \Omega_{\alpha, \lambda, \rho}(z, s, r)\} &= \frac{(\gamma + 1)^{\beta - \tau} \Gamma(\tau)}{\Gamma(\beta) \Gamma(1 - \beta + \tau)} z^{(1 - \beta + \tau) + \gamma} \sum_{\kappa=1}^{\infty} \frac{(a\zeta)^{\kappa}}{(1)_{\kappa}} B\left(\frac{\kappa + \beta - 1}{\gamma + 1} + 1, 1 - \beta + \tau\right) z^\kappa
\end{align*}
\]
where extended general Hurwitz-Lerch Zeta function was introduced in [21] by
\[
\begin{align*}
\Omega_{\alpha, \lambda, \rho}(z, s, a) := \sum_{\kappa=0}^{\infty} \frac{(a\zeta)^{\kappa}}{(\rho)_{\kappa} (1)_{\kappa}} \frac{z^\kappa}{(\kappa + a)^s}.
\end{align*}
\]
where $\rho, a \in \mathbb{Z} \setminus \{0, -1, -2, \cdots\}$, $s \in \mathbb{C}, \Re(s) > 0$ when $|z| < 1$, for more details about this function see [20].
3 Normalization generalized operator

In this section we normalize the generalized operator $T^\beta,\tau,\gamma$ of type fractional differential of analytic univalent functions in $U$ and defined in two terms as follows:

Let the following conditions to be realized:

$$0 \leq \beta - \tau < 1, \quad \gamma \geq 0,$$

we defined the operator $\Theta^\beta,\tau,\gamma f(z) : A \to A$ by

$$\Theta^\beta,\tau,\gamma f(z) = \frac{z^{(\beta-\tau-1)}\Gamma(\frac{\beta}{\gamma+1} + 1 - \beta + \tau)\Gamma(\beta)}{(\gamma + 1)^{\beta-\tau}\Gamma(\frac{\beta}{\gamma+1} + \Gamma(\tau)} \{T^\beta,\tau,\gamma z f(z)\}$$

$$= z + \sum_{\kappa=2}^\infty \Phi^\beta,\tau,\gamma(\kappa) a_\kappa z^\kappa$$

where

$$\Phi^\beta,\tau,\gamma(\kappa) := \frac{\Gamma(\frac{\beta}{\gamma+1} + 1 - \beta + \tau)\Gamma(\frac{\kappa+\beta-1}{\gamma+1} + 1)}{\Gamma(\frac{\beta}{\gamma+1} + 1)\Gamma(\frac{\kappa+\beta-1}{\gamma+1} + 1 - \beta + \tau)}.$$

Not that

$$\Theta^\beta,\tau,\gamma f(0) = 0$$

Next we employ the well known method of convolution product two functions of analytic univalent functions for $\Theta^\beta,\tau,\gamma$ and define as follows:

$$\Theta^\beta,\tau,\gamma f(z) = z + \sum_{\kappa=2}^\infty \frac{\Gamma(\frac{\beta}{\gamma+1} + 1 - \beta + \tau)\Gamma(\frac{\kappa+\beta-1}{\gamma+1} + 1)}{\Gamma(\frac{\beta}{\gamma+1} + 1)\Gamma(\frac{\kappa+\beta-1}{\gamma+1} + 1 - \beta + \tau)} a_\kappa z^\kappa$$

Therefore,

$$\Theta^\beta,\tau,\gamma f(z) := \frac{\Gamma(\frac{\beta}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{\beta}{\gamma+1} + 1)} z_2 \Psi_1[z] * f(z)$$

Now we investigate to study the univalence properties for operator $\Theta^\beta,\tau,\gamma$.

**Theorem 7.** Let $f \in S$. If the following conditions satisfied

(i) For $0 \leq \beta < 0$, $\tau > 0$ and $\beta - \tau < 1$. 

7
(ii) \( \rho_i > 0, i = 1, \ldots, p \) and \( \lambda_j > 0, j = 1, \ldots, q; p \leq q + 1 \),
then the operator \( \Theta^{\beta,\tau,\gamma} f(z) \in S \) in open unite disk \( \Omega \).

\[
2 \Psi_1 \left[ (3,1), (1 + \frac{\beta}{\gamma+1}, \frac{1}{\gamma+1}); (1 - \beta + \tau + \frac{\beta}{\gamma+1}, \frac{1}{\gamma+1}); \right] + 2 \Psi_1 \left[ (2,1), (1 + \frac{\beta}{\gamma+1}, \frac{1}{\gamma+1}); (1 - \beta + \tau + \frac{\beta}{\gamma+1}, \frac{1}{\gamma+1}); \right] < 2 \left( \frac{\Gamma(\frac{\beta}{\gamma+1} + 1)}{\Gamma(\frac{\gamma+1}{\gamma+1} + 1 - \beta + \tau)} \right).
\]

**Proof.** Suppose the function \( f \in S \) and let

\[
\Theta^{\beta,\tau,\gamma} f(z) = z + \sum_{\kappa=2}^{\infty} w_{\kappa} z^\kappa
\]

be defined by equality (3.2), where where

\[
w_{\kappa} := \Phi_{\beta,\tau,\gamma}(\kappa) a_{\kappa}
\]

and the function \( \Phi_{\beta,\tau,\gamma}(\kappa) \) is defined by (3.3). To prove that the operator \( \Theta^{\beta,\tau,\gamma} \) preserves the class \( S \) we require the following sufficient condition

\[
\ell_1 := \sum_{\kappa=2}^{\infty} \kappa |w_{\kappa}| = \sum_{\kappa=2}^{\infty} \kappa \Phi_{\beta,\tau,\gamma}(\kappa) |a_{\kappa}| < 1,
\]

By using Remark (1), we give the estimate for the coefficients of an univalent function belong to \( S \) in \( \Omega \) also, by employ this estimate, we can get another estimate for \( \ell_1 \) in \( S \) as follows,

\[
\ell_1 = \sum_{\kappa=2}^{\infty} \kappa \Phi_{\beta,\tau,\gamma}(\kappa) |a_{\kappa}| \leq \sum_{\kappa=2}^{\infty} \kappa^2 \Phi_{\beta,\tau,\gamma}(\kappa) = \sum_{\kappa=2}^{\infty} \frac{(k)^2}{\kappa!} \Phi_{\beta,\tau,\gamma}(\kappa) \Gamma(k) \ell(k) < 1
\]

where

\[
\ell(k) = \frac{\Gamma(\frac{\kappa+\beta-1}{\gamma+1} + 1) (1)_{\kappa}}{\Gamma(\frac{\kappa+\beta}{\gamma+1} - 1) \Gamma(\frac{\kappa+\beta}{\gamma+1} + 1 - \beta + \tau)}
\]

The series in (3.6) is transformed into a sum of two terms by employing the following relation:

\[
\frac{k^2}{(1)_{k-1}} = \frac{k}{(1)_{k-1}} + \frac{1}{(1)_{k-2}}
\]
Depending on \((1)_\kappa = \kappa!\) and \((1)_{\kappa-1} = (\kappa - 1)!\), the estimate \(3.10\) becomes the next form:

\[
\ell_1 \leq \sum_{\kappa=2}^{\infty} \frac{\kappa^2}{(1)_\kappa} \ell(\kappa) = \sum_{\kappa=2}^{\infty} \left( \frac{1}{(1)_{\kappa-1}} + \frac{1}{(1)_{\kappa-2}} \right) \ell(\kappa)
\]

\[
= \sum_{\kappa=2}^{\infty} \frac{\ell(\kappa)}{(1)_{\kappa-1}} + \frac{\ell(\kappa)}{(1)_{\kappa-2}}
\]

\[
= \sum_{\kappa=2}^{\infty} \frac{\Gamma(\frac{\beta}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{\beta}{\gamma+1} + 1)} \frac{\Gamma(\frac{\kappa+\beta-1}{\gamma+1} + 1)}{\Gamma(\frac{\kappa+\beta-1}{\gamma+1} + 1 - \beta + \tau)(1)_{\kappa-1}}
\]

\[
+ \sum_{\kappa=2}^{\infty} \frac{\Gamma(\frac{\beta}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{\beta}{\gamma+1} + 1)} \frac{\Gamma(\frac{\kappa+\beta-1}{\gamma+1} + 1)}{\Gamma(\frac{\kappa+\beta-1}{\gamma+1} + 1 - \beta + \tau)(1)_{\kappa-2}}
\]

\[
= \frac{\Gamma(\frac{\beta}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{\beta}{\gamma+1} + 1)} \left( \sum_{\kappa=1}^{\infty} \frac{\Gamma(\frac{\kappa+\beta}{\gamma+1} + 1)}{\Gamma(\frac{\kappa+\beta}{\gamma+1} + 1 - \beta + \tau)(1)_{\kappa+1}} \right)
\]

\[
+ \sum_{\kappa=0}^{\infty} \frac{\Gamma(\frac{\kappa+\beta+1}{\gamma+1} + 1)}{\Gamma(\frac{\kappa+\beta+1}{\gamma+1} + 1 - \beta + \tau)(1)_{\kappa+2}}
\]

\[
= \frac{\Gamma(\frac{\beta}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{\beta}{\gamma+1} + 1)} \left( \sum_{\kappa=1}^{\infty} \frac{\Gamma(\frac{\kappa+\beta}{\gamma+1} + 1)}{\Gamma(\frac{\kappa+\beta}{\gamma+1} + 1 - \beta + \tau)(1)_{\kappa}} \frac{1}{(1)_\kappa} \right)
\]

then by employing the Fox-Wright function given by \((1.8)\), we can transform the estimate \(\ell_1\) at \(z = 1\) as follows

\[
\Psi \left[ \frac{(3, 1), (1 + \frac{\beta}{\gamma+1}, \frac{1}{\gamma+1})}{(1 - \beta + \tau + \frac{\beta}{\gamma+1}, \frac{1}{\gamma+1})} \right] + 2 \Psi \left[ \frac{(2, 1), (1 + \frac{\beta}{\gamma+1}, \frac{1}{\gamma+1})}{(1 - \beta + \tau + \frac{\beta}{\gamma+1}, \frac{1}{\gamma+1})} \right] - \frac{\Gamma(\frac{\beta}{\gamma+1} + 1)}{\Gamma(\frac{\beta}{\gamma+1} + 1 - \beta + \tau)} < \frac{\Gamma(\frac{\beta}{\gamma+1} + 1)}{\Gamma(\frac{\beta}{\gamma+1} + 1 - \beta + \tau)}\]

Hence,

\[
\Theta^{\gamma, \tau, \gamma} : S \rightarrow S.
\]

\[\square\]

**Theorem 8.** Let the condition 1 as the Theorem \(1.7\) is satisfied. If \(0 \leq \beta - \tau < 1\),

\[
2 \Psi \left[ \frac{(2, 1), (1 + \frac{\beta}{\gamma+1}, \frac{1}{\gamma+1})}{(1 - \beta + \tau + \frac{\beta}{\gamma+1}, \frac{1}{\gamma+1})} \right] \leq 2 \left( \frac{\Gamma(\frac{\beta}{\gamma+1} + 1)}{\Gamma(\frac{\beta}{\gamma+1} + 1 - \beta + \tau)} \right).
\]
then the operator maps a convex function $f(z)$ into a univalent function that is $\Theta^{\beta,\tau,\gamma}: K \rightarrow S$.

Proof. Assume that $f(z) \in K$ and let

$$
\Theta^{\beta,\tau,\gamma} f(z) = z + \sum_{\kappa=2}^{\infty} w_\kappa z^\kappa
$$

be given by (3.2) where

$$
w_\kappa := \Phi^{\beta,\tau,\gamma}(\kappa) a_\kappa
$$

and the function $\Phi^{\beta,\tau,\gamma}$ is given by (3.3). To proof that the operator $\Theta^{\beta,\tau,\gamma} z$ preserves the class $S$ we require the following sufficient condition

$$
\ell_2 := \sum_{\kappa=2}^{\infty} \kappa |w_\kappa| = \sum_{\kappa=2}^{\infty} \kappa \Phi^{\beta,\tau,\gamma}(\kappa) |a_\kappa| < 1.
$$

We know That the coefficient of a convex function belong to $S$ is $|a_\kappa| < 1$. So we can estimate $\ell_2$ as follows,

$$
\ell_2 = \sum_{\kappa=2}^{\infty} \kappa \Phi^{\beta,\tau,\gamma}(\kappa) |a_\kappa| \leq \sum_{\kappa=2}^{\infty} \kappa^2 \Phi^{\beta,\tau,\gamma}(\kappa) (3.9)
$$

where

$$
\ell(\kappa) = \frac{\Gamma(\frac{\beta+1}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{\beta}{\gamma+1} + 1)} \frac{\Gamma(\frac{\kappa+\beta-1}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{\kappa+\beta}{\gamma+1} + 1 - \beta + \tau)}
$$

and $(a)_\kappa$ is Pochhammer symbol, with the following relation

$$
\frac{\kappa}{(1)_\kappa} = \frac{1}{(1)_{\kappa-1}}.
$$

Since $(1)_\kappa = \kappa!$, then the estimate (3.9) has the next form

$$
\ell_2 \leq \sum_{\kappa=2}^{\infty} \frac{\kappa}{(1)_\kappa} \ell(\kappa) = \sum_{\kappa=2}^{\infty} \frac{1}{(1)_{\kappa-1}} \ell(\kappa) = \sum_{\kappa=2}^{\infty} \frac{\ell(\kappa)}{(1)_{\kappa-1}}
$$

$$
= \sum_{\kappa=2}^{\infty} \frac{\Gamma(\frac{\beta}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{\beta}{\gamma+1} + 1)} \frac{\Gamma(\frac{\kappa+\beta-1}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{\kappa+\beta}{\gamma+1} + 1 - \beta + \tau)} (1)_{\kappa-1}
$$

$$
= \frac{\Gamma(\frac{\beta}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{\beta}{\gamma+1} + 1)} \sum_{\kappa=1}^{\infty} \frac{\Gamma(\frac{\kappa+\beta}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{\kappa+\beta-1}{\gamma+1} + 1 - \beta + \tau)} (1)_{\kappa-1}
$$

$$
= \frac{\Gamma(\frac{\beta}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{\beta}{\gamma+1} + 1)} \left( \sum_{\kappa=1}^{\infty} \frac{\Gamma(\frac{\kappa+\beta}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{\kappa+\beta-1}{\gamma+1} + 1 - \beta + \tau)} (1)_{\kappa-1} \right)
$$

$$
= \frac{\Gamma(\frac{\beta}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{\beta}{\gamma+1} + 1)} \left( \sum_{\kappa=1}^{\infty} \frac{\Gamma(\frac{\kappa+\beta}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{\kappa+\beta-1}{\gamma+1} + 1 - \beta + \tau)} (1)_{\kappa-1} \right)
$$

$$
= \frac{1}{\Gamma(\frac{\beta}{\gamma+1} + 1)} \left( \sum_{\kappa=1}^{\infty} \frac{\Gamma(\frac{\kappa+\beta}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{\kappa+\beta-1}{\gamma+1} + 1 - \beta + \tau)} (1)_{\kappa-1} \right)
$$

$$
= \frac{1}{\Gamma(\frac{\beta}{\gamma+1} + 1)} \left( \sum_{\kappa=1}^{\infty} \frac{\Gamma(\frac{\kappa+\beta}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{\kappa+\beta-1}{\gamma+1} + 1 - \beta + \tau)} (1)_{\kappa-1} \right)
$$

10
which equivalents
\[
\frac{\Gamma\left(\frac{\beta}{1+\gamma} + 1 - \beta + \tau\right)}{\Gamma\left(\frac{\beta}{1+\gamma} + 1\right)} \sum_{\kappa=1}^{\infty} \frac{\Gamma(\kappa+2)\Gamma\left(\frac{\kappa+\beta}{1+\gamma} + 1\right)}{\Gamma\left(\frac{\kappa+\beta}{1+\gamma} + 1 - \beta + \tau\right)(1)_{\kappa}} \frac{1}{(1-\beta+\tau + \frac{\beta}{1+\gamma})_{\kappa}}.
\]
Therefore, by utilizing the Fox-Wright function, we transform the estimate \(\ell_1\) at \(z = 1\),
\[
\frac{\Gamma\left(\frac{\beta}{1+\gamma} + 1 - \beta + \tau\right)}{\Gamma\left(\frac{\beta}{1+\gamma} + 1\right)} \Psi_1 \left[\left(2,1\right),\left(1+\frac{\beta}{1+\gamma},\frac{1}{1+\gamma}\right)\right] - 1 < 1.
\]
Hence
\[
\Theta^{\beta,\tau,\gamma} : K \to S.
\]
By this the proof is completed. \(\square\)

4 Boundedness and compactness in Bloch space

In this section we characterize the boundedness and compactness of operator \(\Theta^{\beta,\tau,\gamma}\) given by (3.5) on weighted \(\mu\)-Bloch space \(B^\mu_w\). First, let recall the well known Bloch space \(B\) and weighted Bloch space \(B^\mu\) [13, 19] are defined respectively,

**Definition 3.** A holomorphic function \(f \in H(U)\) is said to be in Bloch space \(B\) whenever
\[
\|f\|_B = \sup_{z \in U} \left(1 - |z|^2\right) |f'(z)| < \infty.
\]

and the little Bloch space \(B_0\) is given as follows
\[
\lim_{|z| \to 1-} \left(1 - |z|^2\right) |f'(z)| = 0.
\]

**Definition 4.** Let \(w : [0,1) \to [0,\infty)\) and \(f\) be an analytic function on unit disk \(U\) is said to be in the weighted Bloch space \(B^\mu_w\) if
\[
\left(1 - |z|\right) |f'(z)| < h w(1 - |z|), \quad z \in U.
\]
for some \(h = h_f > 0\). Not that, if \(w = 1\) then \(B^\mu_w\) reduces to the classical Bloch space \(B\). Further, the weighted \(\mu\)-Bloch space \(B^\mu_w\), covering of all \(f \in B^\mu_w\) defined by
\[
\lim_{|z| \to 1-} \frac{(1 - |z|)\mu |f'(z)|}{w(1 - |z|)} = 0.
\]
and
\[
\|f\|_{B^\mu_w} = \sup_{z \in U} |f'(z)| \frac{(1 - |z|)\mu}{w(1 - |z|)} < \infty.
\]

It is easy to note that if an analytic function \(g(z) \in B^\mu_w\), then
\[
\sup_{z \in U} |kg(z)| \frac{(1 - |z|)\mu}{w(1 - |z|)} \leq c < \infty,
\]
where \(k\) is a positive number.
**Theorem 9.** Let $f$ be an analytic function on open unit disk $U$, and $\mathbb{B}_w^\mu; w: [0, 1) \to [0, \infty)$. Then

$$f \in \mathbb{B}_w^\mu \iff \Theta^{\beta,\tau,\gamma} f \in \mathbb{B}_w^\mu.$$  

**Proof.** Let suppose $f \in \mathbb{B}_w^\mu$, then by using equalities (3.5) and (4.1), we obtain

$$||\Theta^{\beta,\tau,\gamma} f||_{\mathbb{B}_w^\mu} = \sup_{z \in U} \left| \frac{\theta^{\beta,\tau,\gamma} f}{z} \left( \frac{1 - |z|^\mu}{w(1 - |z|)} + \sup_{z \in U} \left| \frac{\Gamma(\frac{1+\beta}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{1+\beta}{\gamma+1})} \Psi_1[z] \ast f(z) \right| \right) \right| \left( \frac{1 - |z|}{w(1 - |z|)} \right)^\mu$$

$$\leq \sup_{z \in U} \left| \frac{\theta^{\beta,\tau,\gamma} f}{z} \left( \frac{1 - |z|^\mu}{w(1 - |z|)} + \sup_{z \in U} \left| \frac{\Gamma(\frac{1+\beta}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{1+\beta}{\gamma+1})} \Psi_1[z] \ast f(z) \right| \right) \right| \left( \frac{1 - |z|}{w(1 - |z|)} \right)^\mu$$

by following the derivative of convolution due to Ruscheweyh (see [22], pp. 39), we have

$$||\Theta^{\beta,\tau,\gamma} f||_{\mathbb{B}_w^\mu} \leq c + \sup_{z \in U} \left| f'(z) \right| \left( \frac{1 - |z|^\mu}{w(1 - |z|)} \right) \leq 1$$

where $\Psi_1$ is given by (3.3) and $|z| < 1$. Hence $\Theta^{\beta,\tau,\gamma} f \in \mathbb{B}_w^\mu$. On the other hand, if $\Theta^{\beta,\tau,\gamma} f \in \mathbb{B}_w^\mu$, then

$$||\Theta^{\beta,\tau,\gamma} f||_{\mathbb{B}_w^\mu} = \sup_{z \in U} \left| (\theta^{\beta,\tau,\gamma} f)'(z) \right| \left( \frac{1 - |z|^\mu}{w(1 - |z|)} \right) < \infty.$$  

Since

$$\Theta^{\beta,\tau,\gamma} f(z) = \left( \frac{\Gamma(\frac{1+\beta}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{1+\beta}{\gamma+1})} \right) \Psi_1[z] \ast f(z),$$

then,

$$f(z) = \frac{\Gamma(\frac{1+\beta}{\gamma+1} + 1 - \beta + \tau)}{\Gamma(\frac{1+\beta}{\gamma+1})} \Psi_1[z] \ast \Theta^{\beta,\tau,\gamma} f(z)$$

is an analytic function on open unit disk $U$, we have the following

$$\sup_{z \in U} \left| f'(z) \right| \left( \frac{1 - |z|^\mu}{w(1 - |z|)} \right) \leq c_1 < \infty.$$
where \( U_{\frac{1}{2}} = \{ z : |z| < \frac{1}{2} \} \) and \( c_1 \) is a positive constant. Therefore by \( \Theta^{\beta, \tau, \gamma}f(z) \in \mathbb{B}^\mu_w \), we obtain

\[
\|f\|_{\mathbb{B}^\mu_w} = \sup_{z \in U_{\frac{1}{2}}} |f'(z)| \frac{(1 - |z|)^\mu}{w(1 - |z|)} 
\]

\[
\leq c_1 + \sup_{z \in U_{\frac{1}{2}}} \left( \frac{\Gamma(\frac{d}{\tau} + 1 - \beta + \tau)}{\Gamma(\frac{d}{\tau} + 1)} \frac{(1 - |z|)^\mu}{z^2 \Psi_1(z)} \right) \frac{(1 - |z|)^\mu}{w(1 - |z|)},
\]

\[
= c_1 + \sup_{z \in U_{\frac{1}{2}}} \left( \frac{\Gamma(\frac{d}{\tau} + 1 - \beta + \tau)}{\Gamma(\frac{d}{\tau} + 1)} \left( z^2 \Psi_1(z) \left( \Theta^{\beta, \tau, \gamma}f \right)' - \Theta^{\beta, \tau, \gamma}f(2 \Psi_1(z)^{\prime}) \right) \right) \frac{(1 - |z|)^\mu}{w(1 - |z|)},
\]

\[
\leq c_1 + \frac{2}{c} \sup_{z \in U_{\frac{1}{2}}} |\Theta^{\beta, \tau, \gamma}f| \left( \frac{1 - |z|)^\mu}{w(1 - |z|)} \right) + 2 \sup_{z \in U_{\frac{1}{2}}} \left| f * 2 \Psi_1(z) \frac{(1 - |z|)^\mu}{w(1 - |z|)} \right|
\]

\[
\leq c_1 + 2\|\Theta^{\beta, \tau, \gamma}f\|_{\mathbb{B}^\mu_w} + 2c_2 < \infty.
\]

where \( c_1 \) and \( \sup_{z \in U_{\frac{1}{2}}} \frac{|f * 2 \Psi_1(z)| (1 - |z|)^\mu}{w(1 - |z|)} \leq c_2 \) are positive constant, so \( f \in \mathbb{B}^\mu_w \). This completes the proof. \(\square\)

**Theorem 10.** Let \( f \) be an analytic function on open unit disk \( U \), and \( \mathbb{B}^\mu_w ; w : [0, 1) \rightarrow [0, \infty) \). Then

\( \Theta^{\beta, \tau, \gamma}f : \mathbb{B}^\mu_w \rightarrow \mathbb{B}^\mu_w \)

is compact.

**Proof.** If \( \Theta^{\beta, \tau, \gamma}f \) is compact, then it is bounded and by Theorem 9 it satisfies that \( f \in \mathbb{B}_w \) because \( \mathbb{B}_w \subset \mathbb{B}^\mu_w \). Let assume that \( f \in \mathbb{B}_w \), that \( (f_n)_{n \in \mathbb{N}} \subset \mathbb{B}^\mu_w \) be such that \( f_n \rightarrow 0 \) converges uniformly on \( U \) as \( n \rightarrow \infty \). Since \( (f)_{n \in \mathbb{N}} \) convergence uniformly on each compact \( U \), we have that there in \( \mathbb{N} > 0 \) such that for every \( n > N \) and every \( z \in U \), there is an \( 0 < \delta < 1 \), such that for every \( n \geq 1 \),

\[
\left| \frac{\Gamma(\frac{d}{\tau} + 1 - \beta + \tau)}{\Gamma(\frac{d}{\tau} + 1)} \left( z^2 \Psi_1(z) \right. \right. \left. \left. \star f_n(z) \right) \right| < \varepsilon
\]

where \( \delta < |z| < 1 \). Since \( \delta \) is arbitrary, then we can choose

\[
\left| \frac{(1 - |z|)}{(1 - |z|)^\mu} \right| < 1
\]
for all $\delta < |z| < 1$ and

$$
||\Theta^{\beta,\tau,\gamma}f_n||_{B_w} = \sup_{z \in U \setminus U_\delta} \left\{ \frac{\Gamma\left(\frac{\beta + 1 - \beta + \tau}{\tau + 1}\right)}{\Gamma\left(\frac{\beta + 1}{\tau + 1}\right)} \left( (z[\Psi_1[z] * f_n(z)])' \right) \right\} \left( 1 - |z| \right) w(1 - |z|),
$$

$$
= \sup_{z \in U \setminus U_\delta} \left\{ \frac{\Gamma\left(\frac{\beta + 1 - \beta + \tau}{\tau + 1}\right)}{\Gamma\left(\frac{\beta + 1}{\tau + 1}\right)} 2^{\Psi_1[z] * z f_n'(z)} \right\} \left( 1 - |z| \right) w(1 - |z|),
$$

$$
\leq \varepsilon + \sup_{z \in U \setminus U_\delta} \left\{ \frac{\Gamma\left(\frac{\beta + 1 - \beta + \tau}{\tau + 1}\right)}{\Gamma\left(\frac{\beta + 1}{\tau + 1}\right)} 2^{\Psi_1[z] * z f_n'(z)} \right\} \left( 1 - |z| \right) w(1 - |z|),
$$

$$
\leq \varepsilon + c||f_n||_{B_w}.
$$

(4.3)

Since for $f_n \to 0$ on $U$ we get $||f_n||_{B_w} \to 0$, and that $\varepsilon$ is an arbitrary positive number, by letting $n \to \infty$ in (4.3), we have that $\lim_{n \to \infty} ||\Theta^{\beta,\tau,\gamma}f_n||_{B_w} = 0$. Thus $\Theta^{\beta,\tau,\gamma}$ is compact.

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