(Non-)Anomalous D-brane and O-plane couplings: the normal bundle

Ben Craps $^{1,2}$, Frederik Roose $^3$

Instituut voor theoretische fysica
Katholieke Universiteit Leuven, B-3001 Leuven, Belgium

ABSTRACT

The direct string computation of anomalous D-brane and orientifold plane couplings is extended to include the curvature of the normal bundle. The normalization of these terms is fixed unambiguously. New, non-anomalous gravitational couplings are found.

1 Aspirant FWO, Belgium
2 E-mail: Ben.Craps@fys.kuleuven.ac.be
3 E-mail: Frederik.Roose@fys.kuleuven.ac.be

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1 Introduction

The Wess-Zumino part of the effective D-brane action has the following form, which can be derived from anomaly cancellation arguments \[1, 2\]:

\[
S_{WZ} = \frac{T_p}{\kappa} \frac{\mathcal{C}^i}{p+1} \wedge e^{2\pi \alpha'} F^i + B^i \wedge \sqrt{\hat{A}(R_T)/\hat{A}(R_N)} .
\] (1.1)

Here \( R_T \) and \( R_N \) are the curvatures of the tangent and normal bundles of the D-brane worldvolume, and \( \hat{A} \) denotes the A-roof genus:

\[
\sqrt{\frac{\hat{A}(R_T)}{\hat{A}(R_N)}} = 1 + \frac{(4\pi^2 \alpha')^2}{384 \pi^2} (\text{tr } R_T^2 - \text{tr } R_N^2) + \frac{(4\pi^2 \alpha')^4}{294912 \pi^4} (\text{tr } R_T^2 - \text{tr } R_N^2)^2 \\
+ \frac{(4\pi^2 \alpha')^4}{184320 \pi^4} (\text{tr } R_T^4 - \text{tr } R_N^4) + \ldots
\] (1.2)

For the other symbols and our conventions we refer to Ref. \[3\]. Here we will be interested in the gravitational part of these interactions, in particular in the part involving the normal bundle.

Similar interactions have been proposed for orientifold planes, which do not have a gauge field on their worldvolumes \[4, 5\]:

\[
S_{WZ} = -2^{p-4} \frac{T_p}{\kappa} \frac{\mathcal{C}^i}{p+1} \wedge \sqrt{\frac{L(R_T/4)}{L(R_N/4)}} .
\] (1.3)

The Hirzebruch polynomials \( L \) are given by:

\[
\sqrt{\frac{L(R_T/4)}{L(R_N/4)}} = 1 - \frac{(4\pi^2 \alpha')^2}{768 \pi^2} (\text{tr } R_T^2 - \text{tr } R_N^2) + \frac{(4\pi^2 \alpha')^4}{1179648 \pi^4} (\text{tr } R_T^2 - \text{tr } R_N^2)^2 \\
- \frac{7(4\pi^2 \alpha')^4}{1474560 \pi^4} (\text{tr } R_T^4 - \text{tr } R_N^4) + \ldots
\] (1.4)

Until a few months ago, all the evidence in favour of the gravitational terms in Eq. (1.1) and Eq. (1.3) consisted of duality \[6\] and anomaly arguments. Recently there have been various more explicit checks.

In Ref. \[3\] the presence of the four-form gravitational couplings involving the tangent bundle was verified by explicit disc and crosscap computations in string theory.

All the couplings in Eq. (1.1) and Eq. (1.3) were computed in Ref. \[3\] by factorizing various one-loop amplitudes in the RR channel. In particular, the 8-form gravitational coupling on O-planes was corrected and the topological origin of these terms was clarified. However, being less direct than the disc/crosscap computations, the method that was used does not allow, for instance, to determine conclusively the normalization of the normal bundle interaction. The reason is that their argument depends on formal manipulations of vanishing amplitudes.
In Ref. [7] the direct method of Ref. [3] was extended to include all the tangent bundle couplings present in Eq. (1.1) and Eq. (1.3). Agreement was found with the relevant results of Ref. [5]. Moreover it was shown that the integral determining the normalization of the 4-form coupling computed in Ref. [3] is in fact a factor two smaller than thought before, thus solving the apparent discrepancy between the string and supergravity calculations in the latter paper.

To summarize, the project of determining the anomalous gravitational D-brane and O-plane couplings by computing string scattering amplitudes seems to be almost completed. Only a direct computation of the normal bundle couplings has been missing. Besides having the conceptual advantage of being direct and unrelated to the earlier checks (which relied on anomaly arguments), it would also allow one to fix the normalizations in an unambiguous way.

The aim of the present paper is to provide this direct computation of the normal bundle couplings. In fact, it will be shown that they can be extracted rather easily from the computations in Refs [3, 7]. In those papers the restriction was made at some point to graviton polarizations and momenta purely along the brane/plane. This simplified the integrals to be performed. The main point of the present paper is that this restriction need not be made: analogous simplifications occur when restricting to small momenta, leaving the polarizations and momenta otherwise arbitrary.

The result will be that all anomalous gravitational D-brane and O-plane couplings are reproduced. In particular, also the normal bundle couplings come with the expected coefficients.

It turns out that the WZ-terms of brane and plane effective actions also contain non-anomalous, eightform gravitational terms. As far as we know, these have not been noted before.

2 The two graviton amplitude

In a recent paper [3] the amplitude for two gravitons and one RR potential in the presence of a D-brane was calculated. With the conventions and normalizations adopted in that paper the amplitude was shown to be

\[
A = \frac{\kappa^2 T_\mu^\alpha \alpha^2}{4\sqrt{2}(p-3)! \pi^2} \epsilon^{\alpha_1 \ldots \alpha_{p-4} \beta_1 \ldots \beta_4} c^{\alpha_1 \ldots \alpha_{p-3}} k_3 k_4 k_3 k_4 \]

\[
\left( (k_3 \cdot S \cdot \zeta_{3\beta_1}) (k_3 \cdot k_4) (\zeta_{3\beta_2} \cdot \zeta_{4\beta_1}) - (k_3 \cdot \zeta_{3\beta_1}) (k_4 \cdot k_4) (\zeta_{3\beta_2} \cdot \zeta_{4\beta_1}) \right)
\]

\[
\left. \epsilon^{\alpha_1 \ldots \alpha_{p-4} \beta_1 \ldots \beta_4} c^{\alpha_1 \ldots \alpha_{p-3}} k_3 k_4 k_3 k_4 \right\}
\]

\[
\int_{|z_3|,|z_4|>1} d^2 z_3 d^2 z_4 \left( |z_3|^2 - 1 \right)^{k_3 \cdot S \cdot k_3} \left( |z_4|^2 - 1 \right)^{k_4 \cdot k_4} \left( |z_3|^2 - 2 k_3 \cdot S \cdot k_3 - 2 k_3 \cdot S \cdot k_4 - 2 k_3 \cdot k_4 - 2 k_4 \cdot k_4 \right) \left( z_3 \bar{z}_4 - 1 \right)^2 \left( z_3 \bar{z}_4 - \bar{z}_3 z_4 \right)^2 .
\]

Restricting to small momenta, the momenta in the exponents can be put to zero. As has been shown recently in Ref. [7], the integral then correctly evaluates to
\[ \frac{2\pi^2}{3}, \text{ which is half the result given in [3].} \] For comparison with supergravity it will be useful to expand the kinematical prefactor in quantities that contain only longitudinal or transverse polarisations and momenta. The amplitude is thus found to be

\[ A_{\text{string}} = \sqrt{2} \pi^2 \kappa^2 T_p \alpha'^2 \frac{\epsilon^{\alpha_1 \ldots \alpha_{p-3} \beta_1 \beta_4} c_{\alpha_1 \ldots \alpha_{p-3}} k_{3 \beta_1} k_{4 \beta_3}}{6 (p-3)!} \]

\[ \left[ (k_4^\parallel \cdot \zeta_3^\parallel) (k_3^\parallel \cdot \zeta_4^\parallel) - (k_4^\parallel \cdot k_3^\parallel)(\zeta_3^\parallel \cdot \zeta_4^\parallel) - (k_4^\perp \cdot \zeta_3^\perp) (k_3^\perp \cdot \zeta_4^\perp) + (k_4^\perp \cdot k_3^\perp)(\zeta_3^\perp \cdot \zeta_4^\perp) \right], \]

that is, there is a split into ‘tangent’ and ‘normal’ pieces relative to the D-brane worldvolume.

Let us now check that with the adopted normalizations the string amplitude reproduces the result as expected from the D-brane action. Taking the Wess-Zumino action Eq. (1.1) the field theory amplitude evaluates to

\[ A_{\text{sugra}} = \sqrt{2} \pi^2 \kappa^2 T_p \alpha'^2 \frac{\epsilon^{\alpha_1 \ldots \alpha_{p-3} \beta_1 \beta_4} c_{\alpha_1 \ldots \alpha_{p-3}} k_{3 \beta_1} k_{4 \beta_3}}{6 (p-3)!} \]

\[ \left[ (k_4^\parallel \cdot \zeta_3^\parallel) (k_3^\parallel \cdot \zeta_4^\parallel) - (k_4^\parallel \cdot k_3^\parallel)(\zeta_3^\parallel \cdot \zeta_4^\parallel) - (k_4^\perp \cdot \zeta_3^\perp) (k_3^\perp \cdot \zeta_4^\perp) + (k_4^\perp \cdot k_3^\perp)(\zeta_3^\perp \cdot \zeta_4^\perp) \right], \]

which exactly coincides with the string theory expression Eq. (2.5). From the string theory point of view it is rather remarkable how the boundary state encodes the different ways in which the tangent and normal bundle curvatures appear.

Let us now combine the above observations and the analysis in Ref. [3] for the orientifold case. Considerations in a recent paper [5] suggest that the D-brane Wess-Zumino action Eq. (1.1) should be replaced by Eq. (1.3) for orientifolds. First, there is a factor \(-2^{p-4}\) in front of the orientifold action. This accounts for the charge of orientifold planes in the corresponding D-brane charge units. For the O9-plane this factor \(-32\) goes into the normalization of the cross-cap state. The relative factor \(-\frac{1}{2}\) between the two-graviton pieces in Eq. (1.2) and Eq. (1.4) is reflected by the same proportionality factor between the integral in the first formula of this section and the corresponding one for the orientifold plane [3, 7]. When considering lower-dimensional planes, we can clearly repeat the above analysis. The normalization of the cross-cap state acquires the appropriate powers of 2, and the kinematical prefactor in the string amplitude nicely splits up into tangent and normal pieces.

### 3 The four graviton amplitude

In Ref. [7] the four graviton amplitude was computed for the special case of momenta and polarizations along the brane/plane. In this section we extend his computation
to the general case. We will find that the anomalous couplings (1.1) are correctly reproduced. On top of those we find extra gravitational couplings, which cannot be factorized in terms of the curvatures of the tangent and the normal bundles.

The scattering amplitude of four gravitons and one RR-potential in the presence of a (high enough) D-brane should reproduce the last two terms of Eq. (1.2). The first one of those is the easier: the computation is essentially the ‘square’ of the one in the previous section. We will not repeat it here; the result is that this term is precisely reproduced by the part of the string amplitude for which the gravitons are contracted in pairs, such that the kinematical factor (and the integral for small momenta!) factorizes between two pairs of gravitons. (For momenta and polarizations along the brane, this part of the amplitude corresponds to the first line of Eq. (47) in Ref. [7].)

The last term of Eq. (1.2) should be reproduced by the non-factorizable parts of the amplitude (corresponding to the last three lines of Eq. (47) in Ref. [7]). It will turn out that this part of the amplitude indeed reproduces the anomalous term, but it does more: it also describes new, non-anomalous gravitational interactions on D-branes.

Without loss of generality, we concentrate on one of the terms (the others can be obtained by permuting the gravitons and antisymmetrizing in polarizations and momenta of each graviton). Normalized as in Ref. [3] it reads

\[
\frac{\sqrt{2}T_p \pi^4 k^4 \alpha'^4}{128(p - 7)!} \zeta_{\alpha_1 \cdots \alpha_{p - 7} \mu_1 \cdots \mu_8} c_{\alpha_1 \cdots \alpha_{p - 7}} \zeta_{\mu_1} k_{\mu_2} \cdots \zeta_{\mu_7} k_{\mu_8}
\]

\[
\times \left\{ \frac{14\pi^8}{45} \left[ (\zeta_1 \cdot S \cdot k_2)(\zeta_2 \cdot k_3)(\zeta_3 \cdot k_4)(\zeta_4 \cdot k_1) + (\zeta_1 \cdot k_2)(\zeta_2 \cdot S \cdot k_3)(\zeta_3 \cdot k_4)(\zeta_4 \cdot k_1) \\
+ (\zeta_1 \cdot k_2)(\zeta_2 \cdot k_3)(\zeta_3 \cdot S \cdot k_4)(\zeta_4 \cdot k_1) + (\zeta_1 \cdot k_2)(\zeta_2 \cdot k_3)(\zeta_3 \cdot k_4)(\zeta_4 \cdot S \cdot k_1) \right] \\
+ \frac{2\pi^8}{45} \left[ (\zeta_1 \cdot S \cdot k_2)(\zeta_2 \cdot S \cdot k_3)(\zeta_3 \cdot S \cdot k_4)(\zeta_4 \cdot k_1) \\
+ (\zeta_1 \cdot S \cdot k_2)(\zeta_2 \cdot S \cdot k_3)(\zeta_3 \cdot k_4)(\zeta_4 \cdot S \cdot k_1) \\
+ (\zeta_1 \cdot S \cdot k_2)(\zeta_2 \cdot k_3)(\zeta_3 \cdot S \cdot k_4)(\zeta_4 \cdot S \cdot k_1) \\
+ (\zeta_1 \cdot k_2)(\zeta_2 \cdot S \cdot k_3)(\zeta_3 \cdot S \cdot k_4)(\zeta_4 \cdot S \cdot k_1) \right] \right\} 
\]

(Actually, it can be seen from Eq. (47) in Ref. [7] that this should be the structure: the number of B_{ij}’s equals the number of left-right contractions giving rise to the integral. Compare with Eqs (2.9) and (2.10) in Ref. [3].) We refer to Ref. [7] for the numerical factors inside the braces, which result from evaluating a four (complex) dimensional integral analogous to the one appearing at the beginning of the previous section.

Writing, as in the previous section,

\[
\zeta_i \cdot k_j = \zeta_i^\parallel \cdot k_j^\parallel + \zeta_i^\perp \cdot k_j^\perp \\
\zeta_i \cdot S \cdot k_j = \zeta_i^\parallel \cdot k_j^\parallel - \zeta_i^\perp \cdot k_j^\perp 
\]
it is clear that the expression in braces contains

\[ \frac{64\pi^8}{45} (\zeta_1^\parallel \cdot k_2^\parallel) (\zeta_2^\parallel \cdot k_3^\parallel) (\zeta_3^\parallel \cdot k_4^\parallel) (\zeta_4^\parallel \cdot k_1^\parallel) , \]  
(3.9)

the part derived in Ref. \[7\], and

\[ - \frac{64\pi^8}{45} (\zeta_1^\perp \cdot k_2^\perp) (\zeta_2^\perp \cdot k_3^\perp) (\zeta_3^\perp \cdot k_4^\perp) (\zeta_4^\perp \cdot k_1^\perp) , \]  
(3.10)

the analogous expression for the normal bundle. Thus the anomalous D-brane couplings involving a gravitational eightform are indeed seen in an explicit string scattering computation.

However, unlike the other terms in the scattering amplitude, Eq. (3.7) contains terms which are not accounted for by the anomalous gravitational couplings in Eq. (1.1), such as

\[ \frac{8\pi^8}{15} (\zeta_1^\perp \cdot k_2^\perp) (\zeta_2^\parallel \cdot k_3^\parallel) (\zeta_3^\parallel \cdot k_4^\parallel) (\zeta_4^\parallel \cdot k_1^\parallel) . \]  
(3.11)

The other terms can be obtained from this one by obvious permutations, and by taking three ‘transversal’ factors rather than one (the latter terms have an extra minus sign).

For these ‘extra’ terms not to interfere with anomaly cancellation arguments, they (or rather, their Chern-Simons forms \[1\]) had better be invariant under coordinate transformations leaving the brane invariant, i.e. coordinate transformations that do not mix directions tangent and perpendicular to the brane worldvolume. Indeed, in terms of linearized curvature twoforms \[R\], these terms can be written schematically as

\[ \text{tr} \left( P_\parallel R P_\perp R P_\perp R P_\perp R \right) , \]  
(3.12)

where \( P_\parallel \) and \( P_\perp \) are constant matrices projecting on indices along and perpendicular to the brane, respectively. This is the derivative of

\[ \text{tr} \left( P_\parallel \omega P_\perp R P_\perp R P_\perp R \right) , \]  
(3.13)

where \( \omega \) is the (linearized) spin connection. Under coordinate transformations that do not mix directions tangent and perpendicular to the brane, the variation of \( \omega \) is a block-diagonal matrix, such that the variation of Eq. (3.13) vanishes indeed.

For orientifolds, the above analysis goes through practically unaltered. The only difference, except for the explicit factor \(-2^{p-4}\) in Eq. (1.3), is a change in the integrals providing the factors \( \frac{14\pi^8}{45} \) and \( \frac{2\pi^8}{45} \) in Eq. (3.7). These numbers are multiplied by \(-\frac{7}{8}\) \[4\], such that the eightform gravitational couplings differ only by a global factor \( 7.2^{p-7} \) from the D-brane ones.
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