Lorentz violation and black-hole thermodynamics

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Abstract

We consider nonstandard photons from nonbirefringent modified Maxwell theory and discuss their propagation in a fixed Schwarzschild spacetime background. This particular modification of Maxwell theory is Lorentz-violating and allows for maximal photon velocities differing from the causal speed $c$ of the asymptotic background spacetime. In the limit of geometrical optics, light rays from modified Maxwell theory are found to propagate along null geodesics in an effective metric. We observe that not every Lorentz-violating theory with multiple maximal velocities different from the causal speed $c$ modifies the notion of the event horizon, contrary to naive expectations. This result implies that not every Lorentz-violating theory with multiple maximal velocities necessarily leads to a contradiction with the generalized second law of thermodynamics.

Key words: Lorentz violation, geometrical optics, black-hole thermodynamics

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1. Introduction

Recently, Dubovsky and Sibiryakov [1] investigated a specific Lorentz-violating theory, based on the ghost-condensate model of Ref. [2], which allows for different maximal velocities of different species of particles in a preferred reference frame. The authors of Ref. [1] demonstrated that this, in turn, implies the possibility of constructing a perpetuum mobile of the second kind (which relies on heat transfer from a cold body to a hot body, without other changes). The basic idea is that different particles in the theory considered have a different notion of the event horizon of a Schwarzschild black hole. Making use of the quantum-mechanical Hawking effect, the conclusion is that particles with different maximal velocities measure different effective black-hole masses and temperatures, which, in principle, allows for the construction of a perpetuum mobile [1].

Subsequently, Eling et al. [3] proposed a classical mechanism for Lorentz-violating theories with multiple propagation speeds, which also indicates a conflict between the generalized second law of thermodynamics and such Lorentz-violating theories. (The suggested mechanism [3] is similar to Penrose’s energy-extraction mechanism for a rotating black hole, which relies on the existence of the so-called ergosphere region. For the nonrotating black hole in the Lorentz-violating theory considered there is also an accessible ergosphere region.) Eling et al. [3] and Jacobson and Wall [4] further suggest that the violation of the generalized second law of thermodynamics [5, 6, 7, 8] may be a general feature of Lorentz-symmetry-breaking theories with different maximal velocities for different species of particles.

However, the argument of Eling et al. [3] is limited to Lorentz-violating

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1^Recall that, in a Lorentz-invariant theory, the standard Schwarzschild horizon occurs at \( r = 2G_NM/c^2 \), where \( c \) is the unique maximal attainable velocity of all particles. Naively, Lorentz-noninvariant particles with \( v_{\text{max}} < c \) would then have an effective horizon outside the one of the Lorentz-invariant particles.
theories, for which test particles propagating in the background Schwarzschild spacetime (with mass parameter $M$) perceive effective metrics which are again simply Schwarzschild but with different masses and horizons. In this article, we will consider a particular Lorentz-violating modification of Maxwell theory, which allows for a broader class of effective metrics. We introduce an additional free parameter “four-vector” $\xi^\mu$ which determines the nature of the resulting effective metric for the photon field.

It turns out that the effective metric from parameters $\xi^\mu$ is not necessarily a Schwarzschild metric with a different mass. Other possibilities are the original metric itself (i.e., unchanged $M$) or even a non-Ricci-flat metric with an event horizon at the original Schwarzschild radius $r = 2 G_N M/c^2$. Since the latter possibilities exclude the construction of the particular type of perpetuum mobile considered in Refs. [1, 3], we conclude that not every Lorentz-violating theory with different maximal velocities of different particle species necessarily leads to a manifest contradiction with black-hole thermodynamics.

Our main result agrees with that of Sagi and Bekenstein [9] obtained in a different theory. These authors considered black-hole solutions in a Lorentz-violating tensor-vector-scalar theory of gravity, which implies different propagation velocities of light and gravitational waves. In this particular tensor-vector-scalar theory of gravity, the construction of a classical perpetuum mobile along the lines of Ref. [3] can also be excluded and the violation of the second law as suggested in Ref. [1] can be avoided if the conjectured equality of effective graviton radiation temperature and photon Hawking temperature holds. The Lorentz violation studied in the present article is of a different type and allows for different maximal velocities between photons and matter, in contrast to the tensor-vector-scalar theory of gravity, where all matter propagation is described by the same physical metric.

The possibility of constructing Lorentz-violating theories which do not contradict the generalized second law of thermodynamics [5, 6, 7, 8] may
suggest a super selection rule for the Lorentz-violating parameters of an explicit Lorentz-violating theory, as it seems reasonable to assume that a theory must not allow for the construction of a perpetuum mobile. (For us, an “explicitly Lorentz-violating theory” is a theory where the Lorentz-violating parameters already occur at the level of the action, such as in the Standard Model Extension action of Ref. [10].) However, the coupling of explicitly Lorentz-violating theories to gravity remains an open problem [11], which must be solved before we can fully understand the interplay of Lorentz violation and curved spacetime and are able to draw definitive conclusions on the subject.

2. Theory: General aspects

2.1. Units and conventions

Electromagnetism (standard and nonstandard) is, first, described with rationalized MKSA units and, then, with natural units in order to get \( c = G_N = k_B = \hbar = 1 \). Spacetime indices are denoted by Greek letters and correspond to the standard spherical coordinates \( t, r, \theta, \phi \), while local Lorentz indices are denoted by Latin letters and run from 0 to 3. The symbol \( \eta_{\mu \nu} \) stands for the flat-spacetime Minkowski metric and \( g_{\mu \nu} \) for the curved-spacetime metric, both with signature \((+,-,-,-)\). The absolute value of the determinant of the metric is abbreviated as \( g = | \det g_{\mu \nu} | \). The vierbein is denoted \( e^{\mu}_{\ a} = g^{\mu \nu} \eta_{ab} e_{\ b}^\nu \) and obeys the relations \( e^{\mu}_{\ a} e_{\ b}^\mu = \delta^b_a, \ e^{\mu}_{\ a} e_{\ a}^\mu = \delta_\mu^\nu, \) and \( g_{\mu \nu} = e_{\ a}^\mu e_{\ b}^\nu \eta_{ab} \). Finally, covariant derivatives are written as \( D_{\mu} \).

2.2. Nonbirefringent modified Maxwell theory in flat spacetime

Modified Maxwell theory in flat spacetime is a generalized \( U(1) \) gauge theory with a Lagrange density which consists of the standard Maxwell term and an additional Lorentz-violating bilinear term. Specifically, the flat-spacetime photonic Lagrange density reads:

\[
\mathcal{L}_{\text{modMax, flat}} = -\frac{1}{4} F_{ab} F_{cd} \eta^{ac} \eta^{bd} - \frac{1}{4} \kappa^{abcd} F_{ab} F_{cd}, \tag{2.1}
\]
in terms of the standard Maxwell field strength $F_{ab} \equiv \partial_a A_b - \partial_b A_a$. The Lorentz-violating “tensor” $\kappa^{abcd}$ has the same symmetries as the Riemann curvature tensor, as well as a double-trace condition:

$$\kappa^{abcd} = \kappa^{[ab][cd]}, \quad \kappa^{abcd} = \kappa^{cdab}, \quad \kappa^{ab}_{\ ab} = 0.$$ (2.2)

Under the simplest assumption usually discussed in the literature, $\kappa^{abcd}$ is constant and has 19 independent parameters. More generally, $\kappa^{abcd}(x)$ could be an arbitrary, but fixed, “tensor field,” effectively corresponding to more than 19 parameters, possibly an infinite number.

The following Ansatz [12] reduces modified Maxwell theory to the non-birefringent sector:

$$\kappa^{abcd} = \frac{1}{2} \left( \eta^{ac} \tilde{\kappa}^{bd} - \eta^{ad} \tilde{\kappa}^{bc} + \eta^{bd} \tilde{\kappa}^{ac} - \eta^{bc} \tilde{\kappa}^{ad} \right),$$ (2.3)

in terms of a symmetric and traceless matrix $\tilde{\kappa}^{ab}$. Later on, it will be convenient to employ the following decomposition of $\tilde{\kappa}^{ab}$:

$$\tilde{\kappa}^{ab} = \kappa \left( \xi^a \xi^b - \eta^{ab} \xi^c \xi^c / 4 \right),$$ (2.4a)

$$\kappa \equiv \frac{4}{3} \tilde{\kappa}^{ab} \xi^a \xi^b,$$ (2.4b)

relative to a normalized parameter four-vector $\xi^a$ with $\xi_a \xi^a = 1$ or $-1$, corresponding to the timelike or spacelike case, respectively. The discussion of geometrical optics for modified Maxwell theory to be given in Sec. 3 is valid for general parameter functions $\kappa = \kappa(x)$. However, we choose the parameter $\kappa$ to be spacetime independent for the three concrete cases discussed in Sec. 4.

From (2.1), the modified Maxwell equation in flat spacetime is given by

$$\partial_a \left( F^{ab} + \kappa^{abcd} F_{cd} \right) = 0.$$ (2.5)

The electromagnetic theory (2.1) allows for maximal photon velocities different from $c = 1$, which clearly indicates the breaking of Lorentz invariance.
See, e.g., Refs. [12, 13, 14, 15] for further details of the simplest version of modified Maxwell theory in Minkowski spacetime and physical bounds on its 19 spacetime-independent parameters.

2.3. Nonbirefringent modified Maxwell theory coupled to gravity

The vierbein formalism is particularly well-suited for describing Lorentz-violating theories, since it allows to distinguish between local Lorentz and general coordinate transformations [11] and to set the torsion identically to zero. A minimal coupling procedure then yields the following Lagrange density for the photon part of the action:

\[ L_{\text{modMax}} = -\sqrt{g} \left( \frac{1}{4} F_{\mu\nu} F_{\rho\sigma} g^\mu\rho g^\nu\sigma + \frac{1}{4} \kappa^{\mu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \right), \]

(2.6a)

\[ \kappa^{\mu\rho\sigma} \equiv \kappa^{abcd} e^a_{\mu} e^b_{\nu} e^c_{\rho} e^d_{\sigma}. \]

(2.6b)

Here, and in the following, the numbers \( \kappa^{abcd}(x) \) are considered to be fixed parameters, with no field equations of their own. For the sake of simplicity, the dependence on \( x^\mu \) will be suppressed in the following, \( \kappa^{abcd}(x) \equiv \kappa^{abcd} \), unless stated otherwise.

The purely gravitational part of the Lagrange density is given by the standard Einstein–Hilbert term [16],

\[ L_{\text{grav}} = \sqrt{g} \frac{1}{16\pi} R, \]

(2.7)

with the Ricci curvature scalar \( R \) from the metric \( g_{\mu\nu} \). The complete action is then

\[ \mathcal{S} = \int d^4 x \left( L_{\text{grav}} + L_{\text{modMax}} \right), \]

(2.8)

where the integration domain still needs to be specified.
2.4. Field equations

The variational principle for variations of the action (2.8) with respect to the gauge-field $A_\mu(x)$ results in the following Euler–Lagrange equation of the photon coupled to gravity:

$$D_\mu \left( F^{\mu\nu} + \kappa \mu\nu\alpha\beta F_{\alpha\beta} \right) = 0,$$

(2.9)

with spacetime-covariant derivative $D_\mu$. Similarly, variation of the action (2.8) with respect to the vierbein $e^\mu_a(x)$ leads to the following Einstein equation:

$$G^{\mu\nu} = 8\pi T^{\mu\nu}_{\text{vierbein}} \equiv 8\pi \frac{\delta \mathcal{F}_{\text{modMax}}}{\delta e^\mu_a} \epsilon^{\nu a},$$

(2.10)

where $G^{\mu\nu} \equiv R^{\mu\nu} - (1/2) R g^{\mu\nu}$ denotes the standard Einstein tensor.

Even though the explicit expression for the energy-momentum tensor is not needed in this article, we give it for completeness:

$$T^{\mu\nu}_{\text{vierbein}} = - \left( F^{\mu\rho} F^{\nu\sigma} g_{\rho\sigma} + \kappa^{\mu\rho\sigma} F^\nu F_{\rho\sigma} \right)$$

$$- \frac{1}{4} g^{\mu\nu} \left( F^{\rho\sigma} F_{\rho\sigma} + \kappa^{\rho\sigma\alpha\beta} F_{\alpha\beta} F_{\rho\sigma} \right),$$

(2.11a)

$$\tilde{T}^{\mu\nu}_{\text{vierbein}} = - \left( 1 - (\kappa/2) \xi^\rho \xi_\rho \right) F^{\mu\rho} F^{\nu\sigma} g_{\rho\sigma} + \kappa \left( F^{\nu\rho} F^\mu_{\alpha\beta} \xi^\sigma \xi^\rho + F^{\nu\sigma} F^\rho_{\alpha\beta} \xi^\mu \xi^\rho \right)$$

$$- \frac{1}{4} g^{\mu\nu} \left( 1 - (\kappa/2) \xi^\rho \xi_\rho \right) F_{\alpha\beta} F^{\alpha\beta} + 2\kappa F^{\alpha\beta} F_{\alpha\gamma} \xi_{\beta\gamma} \right),$$

(2.11b)

where the first expression holds for the general modified Maxwell theory and the second (distinguished by a tilde) for the nonbirefringent sector defined by (2.3)–(2.4). Clearly, the relevant expression (2.11b) contains an asymmetric part for generic $\xi^\nu$ with $F^{\mu\nu} \xi^\nu \neq 0$.

Since the energy-momentum tensor $T^{\mu\nu}_{\text{vierbein}}$ of explicitly Lorentz-violating theories is, in general, neither symmetric nor covariantly conserved [11], it is, a priori, not clear that theory (2.8) is well defined or that it has nontrivial solutions at all. Since it will be sufficient for our purpose to treat the photon
as a test particle described by the modified Maxwell equation (2.9) in a
given spacetime background, we may safely ignore this issue. However, it is
not difficult to show that theory (2.8) has nontrivial solutions for both the
gravitational and photonic fields.

3. Effective metric for modified-Maxwell-theory photons

3.1. Photonic part of the action

With the particular nonbirefringent Ansatz (2.3)–(2.4), the photonic La-
grange density (2.6) simplifies to

$$
\mathcal{L}_{\text{modMax}} = -\sqrt{g} \left( 1 - \frac{1}{2} \kappa(x) \xi^\rho \xi_\rho \right) \frac{1}{4} F_{\mu\nu} F_{\rho\sigma} \tilde{g}^{\mu\rho} \tilde{g}^{\nu\sigma},
$$

(3.1)

for $\xi^\rho \xi_\rho = g_{\rho\sigma} \xi^\rho \xi^\sigma = \pm 1$ and the following effective metric and inverse metric:

$$
\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) - \frac{\kappa(x)}{1 + \kappa(x) \xi^\rho \xi_\rho / 2} \xi_\mu(x) \xi_\nu(x),
$$

(3.2a)

$$
\tilde{g}^{\mu\nu}(x) = g^{\mu\nu}(x) + \frac{\kappa(x)}{1 - \kappa(x) \xi^\rho \xi_\rho / 2} \xi^\mu(x) \xi^\nu(x),
$$

(3.2b)

which can be verified to satisfy $\tilde{g}^{\mu\nu} \tilde{g}_{\nu\rho} = \delta^\mu_{\rho}$.

As will be demonstrated in the next subsection, the effective metric is a
useful mathematical tool to describe the propagation of nonstandard photons
obeying the field equation (2.9) in a given spacetime background. However,
unless explicitly stated otherwise, all lowering or raising of indices is under-
stood to be performed by contraction with the original background metric
g_{\mu\nu} or its inverse $g^{\mu\nu}$.

\footnote{Note that (3.1) still contains the square root of the determinant of the original spacetime metric $g_{\mu\nu}$. For the explicit cases to be discussed in Sec. 4, it turns out that $\sqrt{g}$ is simply proportional to $\sqrt{\tilde{g}}$ and that $\kappa$ is spacetime independent. This then implies that (3.1) for these special cases equals, up to an over-all constant, the standard Maxwell action in terms of the effective metric (3.2).}
3.2. Dispersion relation and geometrical optics

The Lagrange density (3.1) is proportional to a Lagrange density for a
standard photon moving in the effective gravitational field (3.2). We will
now demonstrate that photon trajectories (in the geometrical-optics approxi-
mination) are indeed given by null geodesics in this effective metric.

Consider a plane-wave Ansatz,

\[ A_\mu(x) = C_\mu(x) e^{iS(x)}, \quad (3.3) \]

in the Lorentz gauge \( D_\mu A^\mu = 0 \). As usual, we define the wave vector to be
normal to surfaces of equal phase,

\[ k_\mu \equiv \partial_\mu S. \quad (3.4) \]

Inserting Ansatz (3.3) into the field equation (2.9) gives the following disper-
sion relation:

\[ k_\mu k_\nu g^{\mu \nu} = -\frac{\kappa}{1 - \kappa \xi_\rho \xi^\rho} \left( g^{\mu \nu} \xi_\mu k_\nu \right)^2, \quad (3.5a) \]

or equivalently

\[ k_\mu k_\nu \tilde{g}^{\mu \nu} = 0. \quad (3.5b) \]

The right-hand side of (3.5a), with an over-all factor \( \kappa \), determines the change
in the photon dispersion relation due to the Lorentz-violating part in the
Lagrange density (2.6). The vector

\[ \tilde{k}_\mu \equiv \tilde{g}^{\mu \nu} k_\nu = \dot{x}^\mu \quad (3.6) \]

is tangent to geodesics \( x^\mu(\lambda) \) with respect to the effective metric \( \tilde{g}_{\mu \nu} \). Here,
and in the following, an overdot denotes differentiation with respect to the
affine parameter \( \lambda \).
In order to avoid obvious difficulties with causality, we intend to restrict our considerations to a subset of theories without spacelike photon trajectories. As discussed above, the tangent vector to a photon path is given by (3.6). The condition then reads

$$\dot{x}^\mu g_{\mu\nu} \dot{x}^\nu = k_\mu \tilde{g}^{\mu\nu} g_{\nu\alpha} \tilde{g}^{\alpha\beta} k_\beta \geq 0.$$ 

(3.7)

Using definition (3.2b), we find

$$\kappa(x) + \frac{1}{2} \left( \kappa(x) \right)^2 \xi^\rho \xi_\rho \geq 0,$$

(3.8)

which is satisfied by $\kappa(x) \geq 0$ or $\kappa(x) \leq -2$ for timelike $\xi^\mu$ and by $0 \leq \kappa(x) \leq 2$ for spacelike $\xi^\mu$. Since we are only interested in small deformations of standard electrodynamics, we restrict our considerations to

$$0 \leq \kappa(x) \ll 1,$$

(3.9)

which ensures the causality of the theory for both choices of $\xi^\mu$, at least as far as the signal-propagation velocity is concerned.

In the above discussion, we have neglected derivatives of the amplitude $C^\mu(x)$ of Ansatz (3.3) and a term involving the Ricci tensor (the typical length scale of $A^\mu$ is assumed to be much smaller than the length scale of the spacetime background). See, e.g., Refs. [16, 17] for further discussion of the geometrical-optics approximation.

In the next section, three different examples of modified Maxwell theory in a Schwarzschild spacetime background will be presented. Up to now, $\kappa(x)$ was allowed to be a function of the coordinates, but for the following three cases we take $\kappa$ to be a constant, with values in the range (3.9).

### 4. Three modified Maxwell theories in a Schwarzschild background

In this section, the spacetime metric is considered to be fixed to that of a Schwarzschild black hole,

$$ds^2 = (1 - 2M/r) dt^2 - (1 - 2M/r)^{-1} dr^2 - r^2 d\Omega^2,$$

(4.1)
with parameter $M$ interpreted as the central mass.

The Lorentz-violating parameters $\xi^\mu$ of the following three examples (Cases 1–3) are introduced by hand and are, for the moment, not provided by an underlying theory. But the Case–1 background field $\xi^\mu$ can be obtained from the ghost-condensate model of Ref. [2] and, in Appendix A, we demonstrate that it is also possible to obtain the Case–2 background field $\xi^\mu$ as the solution of a dynamic model [Case–3 is more difficult, as will be explained in Footnote 7 of the appendix].

4.1. Case 1: Ricci-flat effective metric with changed horizon

As a first example, consider nonbirefringent modified-Maxwell-theory photons moving in the standard Schwarzschild background \( (4.1) \). For the parameters $\xi^\mu$ entering (3.1), we choose

$$\xi^\mu \bigg|_{\text{Case 1}} = \left( \frac{1}{1 - \frac{2M}{r}}, -\sqrt{\frac{2M}{r}}, 0, 0 \right)$$

and take constant $\kappa$ with $0 < \kappa \ll 1$ (the $\kappa = 0$ case corresponds to standard Maxwell theory). Defining two auxiliary parameters

$$\epsilon \equiv \frac{\kappa}{1 - \xi^\rho \xi^\rho}, \quad \chi \equiv \frac{\epsilon}{1 + \epsilon} > 0,$$

the corresponding effective background (3.2a) reads

$$d\tilde{s}^2 \bigg|_{\text{Case 1}} = \left( 1 - \frac{2M}{r} - \chi \right) dt^2 - 2\chi \frac{\sqrt{2Mr}}{r - 2M} dt dr$$

$$- \frac{r}{r - 2M} \left( 1 + \chi \frac{2M}{r - 2M} \right) dr^2 - r^2 d\Omega^2.$$

From the effective metric (4.4) at $r \gg 2M$, the maximal photon velocity is found to be given by $v_{\gamma, \text{max}} = \sqrt{1 - \chi}$, which, for small but positive $\chi$, is less than the maximal velocity $c = 1$ of Lorentz-invariant matter, according to (4.1).

An appropriate coordinate transformation,

$$dt = \sqrt{1 + \epsilon} dT + \chi \frac{\sqrt{2Mr}}{r - 2M} \frac{1}{1 - \frac{2M}{r} - \chi} dr,$$

(4.5)
reveals that (4.4) is just a standard Schwarzschild background, but with a rescaled mass:
\[
d\tilde{s}^2_{\text{Case } 1} = \left(1 - 2\tilde{M}/r\right) dT^2 - \left(1 - 2\tilde{M}/r\right)^{-1} dr^2 - r^2 d\Omega^2, \quad (4.6a)
\]
\[
\tilde{M} \equiv M (1 + \epsilon). \quad (4.6b)
\]
Therefore, it is clear that the event horizon of these modified-Maxwell-theory photons is different from the standard Schwarzschild event horizon at \( r = r_{\text{Schw}} \equiv 2M \). Specifically, these nonstandard photons would have a horizon at \( r = 2\tilde{M} = 2M (1 + \epsilon) \), which lies outside the horizon of standard Lorentz-invariant matter (protons, neutrons, and electrons) at \( r = 2M \). A theory consisting of these Lorentz-violating photons and standard Lorentz-invariant matter (protons, neutrons, and electrons) would suffer the same difficulties concerning the generalized second law of thermodynamics as described in Refs. [1, 3, 4].

At the level of a Gedankenexperiment, let us try to give a concrete realization of the very simple entropy-reducing process discussed in Ref. [4]. The theory we consider is modified quantum electrodynamics (QED) with an action containing, first, the standard Dirac terms [18, 19] for protons, neutrons, and electrons, and, second, the Case–1 modified-Maxwell-theory term, explicitly given by (3.1), (3.2), and (4.2), for constant \( \kappa \) at a fixed nonzero value \( 0 < \kappa \ll 1 \). Calling these modified-Maxwell-theory photons “A–type particles” and these standard fermions (protons, neutrons, and electrons)

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3 The effective background (4.4) transformed to the Lemaitre reference frame agrees with the effective metric obtained in Ref. [1] for a minimally coupled scalar field interacting with the ghost condensate.

4 These standard-model fermions may also have standard Lorentz-invariant strong interactions so that nuclei can be formed, which are needed for the walls of the box used in the Gedankenexperiment and the rope attached to the box. Otherwise, the box walls have to be made of frozen hydrogen and a frozen-hydrogen chain needs to be fabricated (in order to replace the rope), all within the possibilities of a Gedankenexperiment operating at \( T < 14 \) K and normal pressure.
“B–type particles,” we can just quote the authors of Ref. [4]: “To violate the heat version of the second law, one can suspend a B–box containing thermal A–radiation on a B–rope, and lower the box past the A–horizon. The A–heat can then be discarded in the black hole, thus converting it into work at infinity, without changing the exterior appearance of the black hole nor using up any exterior fuel.” Two crucial ingredients here are, first, the existence of A–B interactions and, second, the possibility that certain incoming positive-energy A–photons fall into the negative-energy states of the “ergoregion” \( r \in (2M, 2\tilde{M}) \), which can be described by the effective Schwarzschild metric (4.6) transformed to regular coordinates such as Kruskal coordinates [16]. More specifically, it is possible to repeat the detailed calculation of Ref. [7] to show that the final increase of the black-hole entropy is strictly less than the initial entropy of the A–radiation dropped in. In this way, the generalized second law of thermodynamics would be violated for the modified QED theory with Case–1 modified-Maxwell-theory photons (A–type particles) and standard Lorentz-invariant fermions (B–type particles).

4.2. Case 2: Non-Ricci-flat effective metric with unchanged horizon

Again, consider the Schwarzschild spacetime background (4.1) in the coordinate patch \( r > 2M \), but now take

\[
\xi^\mu|_{\text{Case 2}} = \left(0, \sqrt{1 - 2M/r}, 0, 0\right)
\]

(4.7)

and constant \( \kappa \) with \( 0 < \kappa \ll 1 \). The effective line element (3.2a) for the photon field is then given by

\[
\tilde{ds}^2|_{\text{Case 2}} = (1 - 2M/r) dt^2 - \frac{1}{1 - \eta} (1 - 2M/r)^{-1} dr^2 - r^2 d\Omega^2,
\]

(4.8)

in terms of the auxiliary Lorentz-violating parameter

\[
\eta \equiv \frac{\kappa}{1 - \xi^{\rho} \xi_{\rho}\kappa/2} = \frac{\kappa}{1 + \kappa/2} > 0.
\]

(4.9)

From the effective metric (4.8) at \( r \gg 2M \), the maximal radial photon velocity is found to be given by \( v_{\gamma, \text{rad, max}} = \sqrt{1 - \eta} < 1 \), for small but
positive $\eta$. The maximal tangential photon velocity is not affected by the Lorentz violation, $v_{\gamma, \text{tang, max}} = 1$.

Although the resulting effective metric (4.8) still has a coordinate singularity at $r = 2M$, it is no longer a Schwarzschild solution: the effective Ricci scalar does not vanish and is, in fact, given by

$$R[\tilde{g}] = -2\eta/r^2. \quad (4.10)$$

However, the temperature (derived from the periodicity of the Euclidean time variable $\tau \equiv i t$) is given by $T = (8\pi M)^{-1}$, which equals the Hawking temperature of a Schwarzschild black-hole of mass $M$ in standard general relativity, $T_H = (8\pi M)^{-1}$. That is, the effective background for the photons appears to have the same temperature as the standard Schwarzschild black hole, irrespective of the Lorentz-violating parameter $|\eta| < 1$. For this particular Lorentz-violating theory, there is no obvious conflict with the generalized second law of thermodynamics and it is not possible to construct a perpetuum mobile of the second kind, at least along the lines suggested by Refs. [1, 3].

In order to get some insight into the possible meaning of the non-Ricci-flat effective metric (4.8), let us seek a perfect-fluid matter source which would give rise to this metric by the Einstein equation $\tilde{G}^\alpha_\beta \equiv G^\alpha_\beta [\tilde{g}] = 8\pi T^\alpha_\beta$. The energy-momentum tensor is $T^\alpha_\beta = (\rho + P)u^\alpha u_\beta - P g^\alpha_\beta$ and the comoving four-velocity of the fluid is assumed to be $u^\alpha = \left(1 - 2M/r \right)^{-1/2}, 0, 0, 0 \right)$. Since the Einstein tensor corresponding to the metric (4.8) is $\tilde{G}^\alpha_\beta = \text{diag}(\eta/r^2, \eta/r^2, 0, 0)$, we obtain the energy density $8\pi \rho = \tilde{G}^t_t = \eta/r^2$ and the isotropic pressure $24\pi P = -\tilde{G}^r_r = -\eta/r^2$. Interestingly, the required perfect fluid corresponds to some type of modulated “dark energy,” with constant equation-of-state parameter $w \equiv P/\rho = -1/3$.

\textsuperscript{5}In Sec. 5.4 we will also show by explicit calculation that the expected photon horizon at $r = 2M/(1 - \eta) > 2M$ does not occur.
4.3. Case 3: Ricci-flat effective metric with unchanged horizon

It is even possible to have a situation where the effective metric is again the Schwarzschild metric (4.1) with the same mass $M$, just as if the Lorentz violation were not present (but it is and gives rise to measurable effects, as will be shown in Sec. 5.3). This happens, for instance, if the Lorentz-violating tensor $\tilde{\kappa}_{\mu\nu}$ is spatially isotropic in the coordinate patch $r > 2M$:

$$\xi^\mu \big|_{\text{Case 3}} = \left( (1 - 2M/r)^{-1/2}, 0, 0, 0 \right), \quad (4.11)$$

taking, again, constant $\kappa$ with $0 < \kappa \ll 1$. Now, the effective line element (3.2a) is

$$d\tilde{s}^2 \big|_{\text{Case 3}} = \frac{1}{1 + \epsilon} (1 - 2M/r) \, dt^2 - (1 - 2M/r)^{-1} \, dr^2 - r^2 d\Omega^2, \quad (4.12)$$

with $\epsilon$ defined by (4.3). From the effective metric (4.12) at $r \gg 2M$, the maximal photon velocity is found to be given by $v_{\gamma, \max} = 1/\sqrt{1 + \epsilon} < 1$, for small but positive $\epsilon$. The effective metric (4.12) becomes, of course, the original Schwarzschild metric by rescaling $t = \sqrt{1 + \epsilon} \, \tilde{t}$.

Observe that this equivalence of metrics only means that light rays in the effective metric follow the same paths as standard photons in the original Schwarzschild metric. But, in the observer's reference frame, these light rays travel with a velocity different from $c$ since the dispersion relation is still given by (3.5). In fact, consider a stationary observer with four-velocity $u^\mu = \left( (1 - 2M/r)^{-1/2}, 0, 0, 0 \right)$, in whose local inertial coordinate system the vierbein has components

$$e_t^0 = \sqrt{1 - 2M/r} = (e_r^1)^{-1}, \quad e_\theta^2 = r, \quad e_\phi^3 = r \sin \theta, \quad (4.13)$$

with the other components vanishing. The observer measures the frequency $\omega = g_{\mu\nu} k^\mu u^\nu = k^0$, where the wave vector $k^\mu$ satisfies (3.5). One readily obtains the quadratic dispersion relation

$$\omega(k)^2 = \frac{1 - \kappa/2}{1 + \kappa/2} \, |k|^2 = \frac{1}{1 + \epsilon} \, |k|^2, \quad (4.14)$$
for wave vector $\mathbf{k} \equiv (k^1, k^2, k^3)$. With $\kappa > 0$, it is then clear that light rays travel with maximal velocity

$$v_{\gamma, \text{max}} = \sqrt{\frac{1 - \kappa/2}{1 + \kappa/2}} < 1,$$

(4.15)

which agrees with the previous value given a few lines below (4.12). Of course, the dispersion relation is more complicated than (4.14) for nonisotropic cases, e.g., the choices (4.2) and (4.7) for the parameters $\xi^\mu$.

As will be demonstrated in Sec. 5, photons traveling in an effective background described by (4.12) have measurable properties different from standard photons moving in a Schwarzschild background, even though the background (4.12) is related to a standard Schwarzschild background by a mere coordinate transformation. This reflects the fact that theory (2.6) is not invariant under general coordinate transformations. This is especially relevant if the modified-Maxwell-theory photon is also coupled to standard Lorentzinvariant matter. An observer with standard measuring devices still perceives a standard Schwarzschild background described by line element (4.11). It is the presence of the two metrics (4.1) and (4.12), which cannot be brought to a standard Schwarzschild background simultaneously, that gives rise to the measurable properties derived in the next section.

5. Gravitational redshift and bending of light

In this section, we will calculate the gravitational redshift and the bending of light for the three modified Maxwell theories of Sec. 4, there called Cases 1–3. Furthermore, we will explicitly demonstrate that the effective metric (4.8) of Case 2 does not have a photon horizon at the naively expected value $r = 2M/(1 - \eta)$. We start, however, with a brief discussion of the horizons present in these theories.
5.1. Horizons

Just as for the standard Schwarzschild geometry, the coordinate \( t \) of the Case–1 line element becomes spacelike and \( r \) becomes timelike for \( r < 2\tilde{M} \). A photon emitted from position \( r < 2\tilde{M} \) must always travel forward in “time,” i.e., to decreasing \( r \) and can, therefore, never leave the region \( r < 2\tilde{M} \). This picture is confirmed by studying the geodesic paths of photons for the inside region \( r < 2\tilde{M} \). Hence, \( r = 2\tilde{M} \) is a genuine event horizon for Case 1.

A similar argument holds for Cases 2–3 if, on the one hand, \( \tilde{M} \) is replaced by \( M \) and if, on the other hand, the following assumptions for the parameters \( \xi^\mu \) in the interior region \( r < 2M \) are made:

\[
\left. \xi^\mu \right|_{\text{Case 2, interior}} = \left( 0, \sqrt{2M/r - 1}, 0, 0 \right), \tag{5.1a}
\]

\[
\left. \xi^\mu \right|_{\text{Case 3, interior}} = \left( 1/\sqrt{2M/r - 1}, 0, 0, 0 \right). \tag{5.1b}
\]

With these additional assumptions, the \( r = 2M \) surface is an event horizon for Case 2 and Case 3. Independently of these additional assumptions, we will show in the next subsection that this \( r = 2M \) surface is an infinite-redshift horizon for a distant observer.

Adding standard Lorentz-invariant matter to the modified-Maxwell-theory photons, there is then an outer horizon for the Case–1 photons and an inner one for the matter particles (the ergoregion in between these horizons is crucial for the apparent violation of the generalized second law of thermodynamics, as discussed in the last paragraph of Sec. 4.1). The horizons for Case–2 and Case–3 modified-Maxwell-theory photons coincide with the horizon for standard matter.

5.2. Redshift

To discuss the redshift, we solve explicitly for a light ray described by a tangent vector \( k_\mu \), which obeys

\[
k_\mu \tilde{g}^{\mu\nu} k_\nu = 0, \quad \tilde{g}^{\mu\nu} k_\nu \tilde{D}_\mu k_\lambda = 0, \tag{5.2}
\]
where $\tilde{\nabla}_\mu$ denotes covariant differentiation with respect to the effective metric $\tilde{g}_{\mu\nu}$. We assume that the hypothetical measuring device is built of standard matter (protons, neutrons, and electrons), so that no additional Lorentz violation is introduced by the measuring process, at least to leading order in $\kappa$. The frequency measured by an observer at spacetime point $P_i$ with arbitrary four-velocity $u_\mu$ is then given by

$$\omega_{P_i} = k_\mu g^{\mu\nu} u_\nu \bigg|_{P_i}. \quad (5.3)$$

In the following, we restrict the discussion to a stationary observer with $u_\mu = t_\mu / (t_\nu t_\nu)$, where $t_\mu$ is the timelike Killing field of the observer Schwarzschild background.

A straightforward calculation for all three cases of Sec. 5 then shows that the redshift equals that of standard photons:

$$\frac{\omega_1}{\omega_2} \bigg|_{\text{Case 1,2,3}} = \sqrt{\frac{1 - 2G_N M / (r_2 c^2)}{1 - 2G_N M / (r_1 c^2)}}, \quad (5.4)$$

where $G_N$ and $c$ have been restored temporarily. However, (5.4) is only valid for $P_i$ lying outside the corresponding photon horizon, i.e., $r_i > 2\tilde{M}$ for Case 1 and $r_i > 2M$ for Case 2 and Case 3.

Now, consider the following Gedankenexperiment. A massive gamma-ray source is falling towards the Schwarzschild singularity and emits photons isotropically (in its rest frame). An observer near infinity, $r_2 \gg 2M$, measures then the following redshift:

$$z \approx 1 / \sqrt{1 - 2M/r_1} - 1, \quad (5.5)$$

where $r_1$ is the emission point of the gamma-ray photon. (We neglect possible additional Doppler-shift effects, in order to simplify the discussion.) For the Cases 2–3, the redshift (5.5) goes to infinity when the source approaches $r_1 = 2M$. For Case 1, however, the maximal redshift an observer at infinity will find is

$$z = 1 / \sqrt{1 - 2M / (2M(1 + \epsilon))} - 1 = \sqrt{(1 + \epsilon) / \epsilon} - 1, \quad (5.6)$$
which is large but finite for small but positive $\epsilon$.

5.3. Bending of light

It is well known that standard light rays propagating in the exterior region of the Schwarzschild spacetime experience deflection by the central mass. We now analyze this deflection process for light rays obeying the modified Maxwell equation (2.9).

For all the effective metrics of Sec. 4, we still find a timelike and a rotational Killing field, given by $t^\mu = (\partial/\partial t)^\mu$ and $\psi^\mu = (\partial/\partial \phi)^\mu$, respectively. Also, just as for the usual Schwarzschild metric, suitable rotations of the coordinate system are able to confine a geodesic to the equatorial plane $\theta = \pi/2$.

For Case 2 and Case 3, the effective metric is still diagonal. For a null geodesic $x^\mu(\lambda)$ in these backgrounds, it is then possible to identify the following constants of motion [16]:

$$E = \tilde{g}_{\mu \nu} t^\mu \tilde{k}^{\nu} = \tilde{g}_{tt} \dot{t}, \quad (5.7a)$$

$$L = -\tilde{g}_{\mu \nu} \psi^\mu \tilde{k}^{\nu} = r^2 \dot{\phi}, \quad (5.7b)$$

where $\dot{x}^\mu(\lambda) = \tilde{k}^\mu$ denotes the tangent vector. Making use of these constants of motion, the geodesic equation in these effective backgrounds reduces to

$$0 = \frac{E^2}{\tilde{g}_{tt} \tilde{g}_{rr}} + \dot{r}^2 - \frac{L^2}{r^2 \tilde{g}_{rr}}. \quad (5.8)$$

The equation for the spatial orbit of the light ray is then given by

$$\frac{d\phi}{dr} = \frac{L}{r^2} \frac{1}{\sqrt{-E^2/(\tilde{g}_{rr} \tilde{g}_{tt}) + L^2/(r^2 \tilde{g}_{rr})}}. \quad (5.9)$$

The deflection angle $\delta \phi$ is found to be

$$\delta \phi \equiv -\pi + 2 \int_{r_0}^{\infty} dr \frac{d\phi}{dr}, \quad (5.10)$$

where the distance of closest approach, $r_0$, is given by

$$\left. \frac{dr}{d\phi} \right|_{r=r_0} = 0. \quad (5.11)$$
Inserting the corresponding values for Case 2 and using definition (4.9) for $\eta$, one finds
\[
\left. \frac{d\phi}{dr} \right|_{\text{Case 2}} = \frac{L}{r^2 \sqrt{1 - \eta}} \frac{1}{\sqrt{E^2 - L^2 (1 - 2M/r) / r^2}},
\]
which is $1/\sqrt{1 - \eta}$ times the standard expression. The total deflection angle is, therefore, changed compared to what would be expected for standard Lorentz-invariant photons by the same factor,
\[
\delta \phi |_{\text{Case 2}} = \left. \frac{4G_NME}{Lc^2} \right|_{\text{standard}}^{\eta} = \frac{1}{\sqrt{1 - \eta}} \delta \phi |_{\text{standard}},
\]
where $G_N$ and $c$ have been restored temporarily. Recall that $E$ and $L$ have the dimension of length and length square, respectively, which is consistent with the standard definition [16] of the apparent impact parameter, $b \equiv L/E$.

Similarly, one finds for Case 3:
\[
\left. \frac{d\phi}{dr} \right|_{\text{Case 3}} = \frac{L}{r^2 \sqrt{E^2(1 + \epsilon) - L^2 (1 - 2M/r) / r^2}},
\]
with $\epsilon$ defined by (4.3). The corresponding integral (5.10) can be evaluated to first order in $M$. We then obtain for the deflection angle
\[
\delta \phi |_{\text{Case 3}} \approx \sqrt{1 + \epsilon} \delta \phi |_{\text{standard}},
\]
with terms of order $(G_NME/(Lc^2))^2$ neglected.

Case 1 is more subtle. Of course, the calculation in background (4.6) yields $\delta \phi = 4G_N\tilde{M}\tilde{E}/(c^2\tilde{L})$. But, now, $\tilde{E}$ and $\tilde{L}$ are the constants of motion in the background (4.6), which still have to be related to the physically measurable quantities $E_{\text{phys}}$ and $L_{\text{phys}}$, which a stationary observer would measure at infinity (the observer being stationary with respect to the original Schwarzschild background). The required relation is given by
\[
E_{\text{phys}} = \tilde{E}/\sqrt{1 + \epsilon}, \quad L_{\text{phys}} = \tilde{L},
\]
with terms of order $(G_NME/(Lc^2))^2$ neglected.
so that
\[ \delta \phi \bigg|_{\text{Case 1}} \approx \frac{(1 + \epsilon)^{3/2} 4G_N ME_{\text{phys}}}{c^2 L_{\text{phys}}} = (1 + \epsilon)^{3/2} \delta \phi_{\text{standard}}. \] (5.17)

Here, we have, again, neglected terms of order \( \left( G_N ME_{\text{phys}} / (L_{\text{phys}} c^2) \right)^2 \).

For all three cases considered, the deflection of the particular modified-Maxwell-theory photons is different from that of a standard Lorentz-invariant photon.\(^6\) Still, the impact of the Lorentz violation on the issue of black-hole thermodynamics depends on the case discussed, with Case 1 leading to an apparent contradiction with the generalized second law of thermodynamics (cf. the discussion at the end of Sec. 4.1) but Case 2 and Case 3 not.

5.4. Case 2: Outbound photon

For the effective metric (4.4) of Case 1, a maximal speed of light \( v_{\gamma, \text{max}} = c / \sqrt{1 + \epsilon} \) is associated with a modified horizon \( r = \tilde{r}_{\text{Schw}} \equiv (1 + \epsilon) r_{\text{Schw}} = (1 + \epsilon) 2M \); see also Footnote 1. It is tempting to generalize this statement, i.e., to reason that a modified maximal speed of photons always corresponds to a modified Schwarzschild horizon. By the same reasoning, also a nontrivial modification of the horizon for the effective metric (4.8) of Case 2 would be expected.

The maximal velocity is, however, direction dependent. For particles moving “parallel” to the parameter four-vector \( \xi^\mu \) of Case 2, the maximal achievable velocity is \( \sqrt{1 - \eta} \). Nontrivial effects could then be expected for \( r = 2M / (1 - \eta) \), analogous to those in Refs. 1, 3. But it will be shown in the following that a photon which starts from \( r = 2M / (1 - \eta/2) < 2M / (1 - \eta) \) can escape, on a geodesic path, to arbitrary large coordinate \( r \) in a finite time.

---

\(^6\)As there is only one type of photon in nature, it may be more appropriate to compare, in a Gedankenexperiment, the nonstandard deflection of a possible modified-Maxwell-theory photon and the standard deflection of a Lorentz-invariant proton at ultrarelativistic energy.
t. Hence, \( r = 2M/(1 - \eta) \) is not a horizon for the particular type of photons considered and the intuitive reasoning proves incorrect. The Schwarzschild horizon at \( r_{Schw} = 2M \) persists, manifesting itself through the vanishing of the timelike Killing field at \( r = r_{Schw} \equiv 2M \). Both Lorentz-invariant matter (protons, neutrons, and electrons) and our particular Lorentz-violating photons would still have an event horizon at \( r = r_{Schw} \).

The tangent vector for a radial outgoing geodesic \( \dot{x}^\mu(\lambda) \) with dimensionless affine parameter \( \lambda \) is given by the following differential equation:

\[
\dot{x}^\mu = E \left( \frac{r}{r - 2M}, \sqrt{1 - \eta}, 0, 0 \right), \tag{5.18}
\]

for \( r \geq 2M/(1 - \eta/2) \). Here, \( E \) is again the energy-like constant of motion and the Lorentz-violating parameter \( \eta \) is assumed to be small but nonzero, \( 0 < \eta \ll 1 \). For a geodesic starting at \( t = 0 \) from position

\[
r_{start} \equiv r_{Schw}/(1 - \eta/2) = 2M/(1 - \eta/2), \tag{5.19a}
\]

the integrated path is explicitly given by

\[
x^\mu(\lambda) = \left( E \tau(\lambda), E \lambda \sqrt{1 - \eta} + r_{start}, 0, 0 \right), \tag{5.19b}
\]

\[
\tau(\lambda) = \lambda + \frac{2M}{E \sqrt{1 - \eta}} \ln \left( 1 + \frac{E \lambda \sqrt{1 - \eta}}{r_{start} - r_{Schw}} \right), \tag{5.19c}
\]

for parameter \( \lambda \in [0, \infty) \). Clearly, arbitrary large positions \( r \) can be reached in a finite time \( t \) and there is no event horizon at \( r = 2M/(1 - \eta) \).

6. Conclusion

In this article, we studied the propagation of nonstandard photons in a given gravitational background and introduced a general method to describe the geometrical-optics approximation of nonbirefringent modified Maxwell theory in a given curved-spacetime background. With the nonbirefringent Ansatz \( (2.3) - (2.4) \) for the parameters \( \kappa^{\mu\nu\rho\sigma} \) in \( (2.6) \), one can, in fact, describe all phenomena of the geometrical-optics approximation by making
use of an effective metric $\tilde{g}_{\mu\nu}(x)$, where the photons follow null geodesics of this effective metric. Standard Lorentz-invariant particles (e.g., protons, neutrons, and electrons) still propagate according to the usual equations of motion with the original spacetime metric $g_{\mu\nu}(x)$.

Several choices of the parameters $\kappa^{\mu\nu\rho\sigma}$ have been presented in Sec. 4, for example, a choice which rescales the mass value for the effective Schwarzschild metric (Case 1) or a choice which generates an effective metric with negative Ricci scalar (Case 2). We find that, in general, a maximal photon propagation velocity smaller than the causal velocity $c$ influences the bending of light by the Schwarzschild mass (Sec. 5.3), but has not necessarily an effect on the notion of the event horizon.

Lorentz-violating theories with effective metrics that modify the event horizon (Case 1 of Sec. 4) appear to allow for the construction of a perpetuum mobile of the second kind [1, 3], bringing them in conflict with the generalized second law of thermodynamics [5, 6, 7, 8]. Other Lorentz-violating theories such as those of Cases 2–3 of Sec. 4 avoid such difficulties and are examples of theories which incorporate Lorentz violation but do not appear to violate the generalized second law of thermodynamics. Hence, we have shown that not every Lorentz-violating theory with multiple maximal velocities is in apparent conflict with black-hole thermodynamics.

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A. Dynamic model for Case–2 Lorentz Violation

The effective metrics discussed in Sec. 4 have not been obtained from an underlying theory but were introduced by hand to show that not every Lorentz-violating dispersion relation with maximal velocities different from
the causal velocity $c$ leads to modifications of the black-hole horizon and corresponding difficulties with black-hole thermodynamics. In this appendix, we present a toy model which yields one of the proposed effective metrics as a solution of the field equations. The particular toy model presented here is directly inspired by the model studied in Ref. [2] (related models have been studied in Refs. [20, 21] and other references therein).

Consider a gravitating scalar field $\phi$ described by the following action:

$$S = \int d^4x \left( L_{\text{grav}} + \sqrt{g} m^4 V \right), \quad (A.1a)$$

$$V = (X + 1)^2, \quad (A.1b)$$

$$X \equiv g^{\mu\nu} (\partial_{\mu}\phi) (\partial_{\nu}\phi), \quad (A.1c)$$

with the pure-gravity Lagrange density $L_{\text{grav}}$ given by (2.7) and a noncanonical mass dimension $-1$ for the scalar field $\phi$, so that $X$ is dimensionless.

Two remarks are in order. First, observe that, with definition (A.1b), the sign of the quadratic kinetic term of the $\phi$ field in (A.1a) is standard (non-ghostlike), contrary to the case of the model of Ref. [2]. Second, the explicit potential $V(X)$ in (A.1b) can be generalized, provided that there remains a minimum at $X = -1$ with

$$V(-1) = V'(-1) = 0, \quad V''(-1) > 0, \quad (A.2)$$

where the prime denotes differentiation with respect to $X$.

The field equations from (A.1) are

$$G^{\mu\nu} = -4\pi m^4 \left( 2V' \partial^\mu \phi \partial^\nu \phi + V g^{\mu\nu} \right), \quad (A.3a)$$

$$D_\mu (V' \partial^\mu \phi) = 0, \quad (A.3b)$$

where $G^{\mu\nu}$ denotes the Einstein tensor defined under (2.10) and $D_\mu$ the covariant derivative.
The flat spacetime solution of the field equations (A.3) has
\[ g_{\mu\nu} = \eta_{\mu\nu} \equiv \text{diag}(1, -1, -1, -1), \] (A.4a)
\[ \phi = n_\mu x^\mu, \] (A.4b)
in terms of Cartesian coordinates and a constant spacelike vector \( n_\mu \), that is, \( n_\mu n_\nu \eta^{\mu\nu} = -1 \). This vector \( n_\mu \) may be the source of observable anisotropy (Lorentz violation) if the scalar field \( \phi \) is coupled to standard matter; cf. Ref. [14] for experimental bounds in the photon sector.

The Schwarzschild metric and any \( \phi \) configuration with \( X \equiv \partial_\mu \phi \partial^\mu \phi = -1 \) also solve the field equations (A.3). Together with the Schwarzschild metric (4.1) in standard spherical coordinates, an explicit solution for the scalar field \( \phi \) over the coordinate patch \( r > 2M \) is given by
\[ g_{\mu\nu} = \text{diag} \left( [1 - 2M/r, -1 - 2M/r]^{-1}, -r^2, -r^2 \sin^2 \theta \right), \] (A.5a)
\[ \bar{\phi} = 2M \ln \left( \frac{\sqrt{r} + \sqrt{r - 2M}}{\sqrt{M}} \right) + \sqrt{r (r - 2M)}, \] (A.5b)
where \( 2M \) multiplied by \( G_N/c^2 \) is the length parameter of the solution. The field configuration (A.5b) solves the field equation (A.3b) outside the horizon and its gradient is identical to the Case–2 background field [4.7],
\[ \xi_\mu \big|_{\text{Case 2}} = \partial_\mu \bar{\phi}, \] (A.6)
with nonvanishing radial component \( \partial_r \bar{\phi} = 1/\sqrt{1 - 2M/r} \).

Note that throughout most of the present article (the only exception occurring in Sec. 5.1), we do not make any assumptions on the form of \( \xi_\mu \) inside the Schwarzschild horizon. In particular, the derivation of the maximal redshift horizon in Sec. 5.2 and the calculation of the outbound photon in Sec. 5.4 are independent of the \( \xi_\mu \) configuration at \( r \leq 2M \).

Now add to the action (A.1) the photon action from (2.6) using (2.3)–(2.4) and replace \( \xi_\mu \) there by \( \partial_\mu \phi \). For \( \phi = \bar{\phi} \) from (A.5b), this then reproduces, in
the geometrical optics approximation, the Case–2 model discussed in Sec 4.2 of the main text.\(^7\)

The model (A.1) just serves the purpose of a proof of principle and shows that there can be dynamic Lorentz-violating theories with the horizon structure discussed in this article. It is certainly not a serious phenomenological model and may have difficulties with causality and stability, in addition to the obvious non-renormalizability. At the classical level, though, where most of the considerations of our article apply, it provides an example of a theory yielding one of the suggested background configurations $\xi_\mu$.

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\(^7\)It appears to be difficult to obtain the Case–3 model as the solution of a pure scalar theory, because its parameters $\xi^\mu$ cannot be written as the gradient of a scalar field. If, however, we restrict ourselves to spherically symmetric fields, we can use the scalar theory (A.1) also for the Case–3 model. The idea is to interpret the spherically symmetric fields as belonging to a $(1 + 1)$–dimensional reduced theory (possibly coming from a higher-dimensional gauge field theory)\(^22\) and to use the Levi–Civita symbol $\epsilon^{mn}$ normalized by $\epsilon^{01} = 1$. Specifically, the dynamic Lorentz-violating parameters in spherical coordinates are given by $\xi^\mu \big|_{\text{Case } 3} = (\xi^0, \xi^1, 0, 0)$ with $\xi^m = \epsilon^{mn} \partial_n \phi$ for the gradient $(\partial_0, \partial_1) \equiv (\partial_t, \partial_r)$. 

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