An efficient Monte Carlo method for utility-based pricing

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We propose an efficient numerical method, based on the Lambert function, for the computation and study of the reservation price as well as the value function in the case of illiquidity. Our theoretical results are illustrated by numerical simulations.

1. Introduction

We consider a continuous-time financial model with time horizon $T$. An economic agent will receive, at time $T$, $\lambda$ units of a derivative written on some non-traded stock $S$ and seek to evaluate the price she is willing to pay for that. This is the classical problem of pricing in mathematical finance, but as the investor is facing trading restriction, she wants to estimate the risk premium related to this lack of marketability. Indeed, in a complete market when $S$ can be dynamically traded, the derivative price is given by its discounted expected value under some martingale measure and the marketability risk premium is null. As this approach is not feasible when $S$ is not dynamically traded, we assume that the investor hedges her position on the derivative on $S$ by trading another "similar" liquid asset $P$. The asset $P$ must be thought of as a related index or as another stock from a closed industry sector. We follow [8] and use the reservation price concept in order to value the illiquidity of $S$. The reservation price is the price $p$ which leaves the agent indifferent between paying $p$ at time 0 and receiving $\lambda$ units of shares of the derivative on $S$ at time $T$, or doing nothing. This is a preference-based asking price. In the case of a Black-Scholes-like market, when the preferences of the agent are represented by some exponential utility function, this problem has been well-studied in the literature: see [7, 14, 2, 5, 4] and [6] and the references therein. It is possible to obtain a semi-closed formula for the reservation price using an expectation, which may thus be computed with the Monte Carlo method. Nevertheless, we observe that the Monte Carlo method leads to unreliable estimators suffering from very high variance (see Figures 3 and 7). Thus, our first motivation is to find an efficient numerical method for the computation of the reservation price.

An alternative method to Monte Carlo is to produce a polynomial approximation. This has been done for an expansion in terms of small $\lambda$ in [2, 5, 4] and in terms of the correlation $\rho$ between $P$ and $S$ in [13] and [12]. Thus, we first derive polynomial approximations in $\rho$ and show (see Figure
that these polynomial approximations can be extremely bad. We take the opportunity to make a correction on the error order of the expansions proposed by Monoyios, as the coefficients are defined using the minimal martingale measure and thus depend on the correlation.

Based on these observations, we propose a new and efficient numerical method. For this method, we adapt an idea from [4]: we remove the expectation that gives the reservation price of the most randomness as possible using the Lambert function. This allows splitting the reservation price into a deterministic part and a random part, the first one being preponderant in meaningful financial situations, while the second will help to provide a low variance estimator for the price (see Figures 3 and 7). For example, this variance may decrease from 5.66 to 0.017. These methods also provide some multiplicative decompositions for the value function. Note that these deterministic approximations remain semi-closed because they rely on the Lambert function.

We then apply the preceding results to the case of an agent willing to cover a long position on the non-traded stock. This is a very important issue in management sciences which is often referred to as the restricted stock problem or marketability problem: see [11, 9, 3], or for the real option non-traded stock. This is a very important issue in management sciences which is often referred to as the restricted stock problem or marketability problem. We analyse the lower bound of the reservation price with respect to the correlation and the reservation price and for the value function. We also provide an approximation of the optimal problem, see [5] and the references therein. We derive deterministic upper and lower bounds for to as the restricted stock problem or marketability problem: see [11, 9, 3], or for the real option non-traded stock. This is a very important issue in management sciences which is often referred to as the restricted stock problem or marketability problem.

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The paper is organized as follows. In Section 2 we present the financial model. In Section 3 we derive a polynomial approximation in the case of a long stock position. Section 4 provides the decomposition of the reservation price and the value function using the Lambert function. Then, Section 5 analyses the case of a long stock position, while Section 6 examines the short put case. Finally, Section 7 regroups the missing proofs of the paper.

2. The financial model

We consider a continuous time financial model with time horizon \( T \) and a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\), where the filtration \((\mathcal{F}_t)_{0 \leq t \leq T}\) satisfies the usual conditions (right continuous, \( \mathcal{F}_0 \) contains all null sets of \( \mathcal{F}_T = \mathcal{F} \)). We denote by \( S = (S_t)_{0 \leq t \leq T} \) (resp. \( P = (P_t)_{0 \leq t \leq T} \)) the observable price process of some non-traded (resp. dynamically traded) asset. We assume that

\[
\begin{align*}
    dS_t &= S_t \left( \mu dt + \eta dZ_t \right), \\
    dP_t &= P_t \left( \mu dt + \sigma dB_t \right),
\end{align*}
\]

where \( Z = (Z_t)_{0 \leq t \leq T} \) and \( B = (B_t)_{0 \leq t \leq T} \) are correlated standard Brownian motions under the historical probability \( \mathbb{P} \). The correlation is denoted by \( \rho \in (-1, 1) \) and, as usual, we will write that

\[
dZ_t = \rho dB_t + \sqrt{1 - \rho^2} dW_t,
\]

where \( W \) and \( B \) are independent Brownian motions under \( \mathbb{P} \). The constants \( \nu \) and \( \mu \) (resp. \( \eta \) and \( \sigma \)) respectively represent the expected return (resp. the volatility) of the non-traded and of the traded assets. There is also a riskless bond, with dynamic given by \( D_t = e^{rt} \), where \( r \) is the constant riskless interest rate. All discount values for any process \( Y = (Y_t)_{0 \leq t \leq T} \) will be denoted by \( \hat{Y} = (\hat{Y}_t)_{0 \leq t \leq T} \), where \( \hat{Y}_t = Y_t / D_t \).
The agent’s initial wealth is denoted by \( x \) and her trading strategy is denoted by \( \Pi = (\Pi_t)_{0 \leq t \leq T} \), where \( \Pi_t \) denotes the cash invested in the traded-asset \( P \) at time \( t \). The agent wealth, starting from an initial wealth \( x \) and following a self-financed strategy \( \Pi \), is denoted by \( X^{x,\Pi} = (X^{x,\Pi}_t)_{0 \leq t \leq T} \) and her discount wealth follows the dynamics (as the agent cannot invest in \( S \))

\[
dX^{x,\Pi}_t = \Pi_t \frac{d\tilde{P}_t}{\tilde{P}_t}.
\]

To value the amount the agent will accept to pay in order to receive the derivative at time \( T \), we use a utility-based price. The risk preferences of the agent are modeled thanks to an exponential utility function, which is very popular because of its separability properties. Indeed, in contrast to the power utility function, it gives an explicit representation for the reservation price. Let

\[
U(x) = -\frac{1}{\gamma} \exp(-\gamma x),
\]

where \( \gamma > 0 \) is the constant absolute risk aversion parameter of the agent.

Let \( V(x_0, s_0, \lambda, h) \) be the value function at time 0 for an agent that maximizes her expected utility of terminal wealth at time \( T \), if in addition she receives \( \lambda > 0 \) units of the European derivative \( H = h(S_T) \), where \( h \) is a measurable function whose assumptions will be specified below.

\[
V(x_0, s_0, \lambda, h) = \sup_{\Pi \in \mathcal{A}} \mathbb{E} U \left( X^{x_0,\Pi}_T + \lambda h(S_T) \right), \tag{1}
\]

where \( \mathcal{A} \) is the set of admissible trading strategies, i.e., the set of \( \Pi \) which are adapted process such that \( \int_0^T \Pi_t^2 ds < +\infty \) a.s. for all \( t \geq 0 \). This problem has been widely studied; please refer to the textbook [10] and the references therein. It is possible to find a semi-closed formula for the optimal trading strategy and for the value function. Two approaches can be used: the dual approach or the primal one. The primal approach (see [4] or [5]) leads to a non-linear Hamilton-Jacobi-Bellman equation that can be linearized using the Hopf-Cole transformation. Please refer to [15], where this transformation is called the distortion method. With a slight adjustment to Section 5 of [4], we determine that

\[
V(x_0, s_0, \lambda, h) = -\frac{1}{\gamma} e^{-\gamma x_0 e^{rT}} \frac{1}{\sqrt{2\pi \sigma^2 T}} \left( \mathbb{E} \left( \exp \left( -\lambda \gamma (1 - \rho^2) h \left( s_0 e^{(\nu - \eta \rho)T} e^{\eta \sqrt{T} N} \right) \right) \right) \right)^{\frac{1}{\lambda \gamma}}. \tag{2}
\]

where \( N \) is a standard Gaussian law. Note that the parameters of \( P \) appear only through the Sharpe ratio \( \frac{\mu - \frac{\sigma^2}{2}}{\sigma} \) of \( P \).

If \( h \) is bounded from below on \( (0, \infty) \), the asking reservation price \( p_h \) of \( \lambda > 0 \) units of the derivative \( H = h(S_T) \) is the amount, which leaves the agent indifferent between paying \( p_h \) at time 0 and receiving \( \lambda \) units of \( H \) at time \( T \), or doing nothing, i.e., \( p_h \) is a solution of the following equation:

\[
V(x_0 - p_h, s_0, \lambda, h) = V(x_0, s_0, 0, h).
\]

If \( h \) is bounded from above on \( (0, \infty) \), the selling reservation price of \( \lambda > 0 \) units of the derivative \( H = h(S_T) \) is the amount, which leaves the agent indifferent between receiving \( p_h^{sell} \) at time 0 and delivering \( \lambda \) units of \( H \) at time \( T \), or doing nothing, i.e.,

\[
V(x_0 + p_h^{sell}, s_0, \lambda, -h) = V(x_0, s_0, 0, -h). \tag{3}
\]
Thus, if $h$ is bounded on $(0, \infty)$, we easily determine that

$$p_{h}^{sell} = -p_{-h}. \quad (4)$$

Letting $\delta = \nu - \eta \frac{\mu - r}{\sigma}$, \((2)\) and \((3)\) lead to

$$p_{h} = -\frac{e^{-rT}}{\gamma (1 - \rho^2)} \ln \mathbb{E} \left( -\lambda \gamma (1 - \rho^2) h \left( s_{0} e^{(\delta - \eta \rho^2)T} e^{\eta \sqrt{T}N} \right) \right). \quad (5)$$

Note that

$$V(x_{0}, s_{0}, \lambda, h) = -\frac{1}{\gamma} \exp \left( -\gamma e^{\tau T} (x_{0} + p_{h}) - \frac{(\mu - r)^{2}}{2\sigma^{2}} T \right). \quad (6)$$

Thus, we will first study $p_{h}$ and obtain the results on $V$ using \((6)\). In the following, we will focus on a derivative $H = h(S_{T})$, with $h(x) = \zeta(x)1_{x \leq K}$, where $K \in [0, +\infty]$ and $\zeta$ is a measurable function, which is bounded from below on $[0, K]$. Choosing $K = \infty$ and $\zeta = id$, where $id(x) = x$, for all $x \in \mathbb{R}$, $H = S_{T}$ and this is the case of a long stock position. If $K \in [0, +\infty)$ and $\zeta(x) = (x - K)$, then $h(x) = -(K - x)_{+}$ (where $x_{+} = \max(x, 0)$) and as $p_{h}^{sell} = -p_{-(K-x)_{+}}$, it also includes the case of a short position on the put option $H = (K - S_{T})_{+}$.

We will denote by $V(x_{0}, s_{0}, \lambda)$ and $p$ the value function of $\lambda > 0$ units of the non-traded stock $S$ at time $T$ and its asking reservation price. Similarly, $V_{\text{put}}(x_{0}, s_{0}, \lambda, K)$ and $p_{\text{put}}$ will be respectively the value function of $\lambda$ units of a short position on a put option of strike $K$, written on the non-traded stock $S$ at time $T$, and its selling reservation price.

In the sequel, we will consider several market conditions. The first one presents a meaningful market situation that will be taken as a reference throughout the article. We also introduce two other market situations to observe the effect of a low initial price (see Table 2) and a long time horizon (see Table 3).

**Table 1.**

| $r$ | $\lambda$ | $T$ | $s_{0}$ | $\nu$ | $\eta$ | $\mu$ | $\sigma$ |
|-----|-----------|-----|---------|-------|-------|-------|-------|
| 0.1% | 2 | 0.25 | 100 | 20% | 30% | 10% | 20% |

**Table 2.**

| $r$ | $\lambda$ | $T$ | $s_{0}$ | $\nu$ | $\eta$ | $\mu$ | $\sigma$ |
|-----|-----------|-----|---------|-------|-------|-------|-------|
| 0.1% | 20 | 0.3 | 1 | 35% | 40% | 10% | 20% |

**Table 3.**

| $r$ | $\lambda$ | $T$ | $s_{0}$ | $\nu$ | $\eta$ | $\mu$ | $\sigma$ |
|-----|-----------|-----|---------|-------|-------|-------|-------|
| 0.1% | 10 | 10 | 100 | 30% | 30% | 5% | 10% |
3. Polynomial approximation

In this section, we search for some polynomial approximation for the reservation price that will be relevant in general market situations. Such an approach was already studied by many authors. Davis [2] achieved an expansion with respect to $\sqrt{1 - \rho^2}$ using ideas from Malliavin calculus, while Henderson [4] focused on an expansion on $\lambda$ and found a similar expression. Later, Monoyios [12, 13] gave a higher order expansion on $1 - \rho^2$ for the selling reservation price.

It is worth noting that the Monoyios’s expansions are not Taylor expansions in the sense that some dependence upon $\rho$ remains present in the coefficients, which are defined using the minimal martingale measure. As a consequence, we strongly believe that the error order provided by Monoyios (see Theorem 3 in [13]) may have been misestimated. Nevertheless, his approximation has proven to be reliable for the computation of the selling reservation price of a put option for a very small risk aversion (see Table 1 in [13]). We show in Figure 1 that this approximation remains correct for the computation of the asking reservation price of a stock for very small values of the risk aversion or very high correlation, but that this is no more the case elsewhere.

![Figure 1](image)

**Figure 1.:** Estimated asking reservation price of $\lambda S_T$ as function of the correlation in the market situation of Table 1 with $\gamma = 0.001$ and $\gamma = 0.5$ using Monte Carlo method in black (solid line) and with the fourth-order approximation of Monoyios in red (dashed line).

To complete the work of Davis and Monoyios, we propose below a Taylor expansion (when the correlation is close to one) of the asking reservation price of a non-traded stock. We will show that Taylor expansions remain extremely poor approximations when the correlation is not very close to one (see Figure 2).

Let $G = e^{\gamma \sqrt{T} N}$ and $g_c$ be the cumulant-generating function of $G$, i.e., $g_c(t) = \ln(\mathbb{E}(e^{tG}))$. Since $G$ is a log-normal random variable, $g_c$ exists only for negative values of $t$. Furthermore, as $G$ has moments of any order, the function $g_c$ is $C^\infty$ on $(-\infty, 0)$, with limits at $0^-$, for all derivatives. The function $g_c$ thus admits a Taylor expansion up to an arbitrary order $n > 0$ at $0^-$ and we have the following decomposition:

$$\forall t \in (-\infty, 0) \quad g_c(t) = \sum_{k=1}^{n} \frac{t^k}{k!} \chi_k + t^n \epsilon(t) \quad \text{with} \quad \lim_{t \to 0^-} \epsilon(t) = 0.$$
The coefficients $\chi_k$ are called the cumulants of the distribution of $G$. As a function of $\rho$, the reservation price can be written as follows (see (5) with $h = id$)

$$p(\rho) = -\frac{e^{-rT}}{\gamma(1 - \rho^2)} g_{\rho} \left(-\lambda \gamma (1 - \rho^2) s_0 e^{(\nu - \eta \frac{\mu - r}{\sigma} - \frac{\mu - r}{\sigma})T}\right)$$

$$= \sum_{k=0}^{n} c_k (1 - \rho)^k + (1 - \rho)^n \varepsilon(\rho)$$

with $\lim_{\rho \to 1^-} \varepsilon(\rho) = 0$, (7)

where the coefficients $(c_k)_{k>0}$ can be expressed in terms of the sequence $(\chi_k)_{k>0}$ for any arbitrary order $n \geq 0$. We have computed the coefficients in (7) up to order 4. As a first step, we present the following expression:

$$\chi_1 = e^{\frac{2}{3}T}$$

$$\chi_2 = e^{\eta^2 T} (e^{\eta^2 T} - 1)$$

$$\chi_3 = e^{\frac{3\eta^2 T}{2}} (e^{3\eta^2 T} - 3e^{\eta^2 T} + 2)$$

$$\chi_4 = e^{2\eta^2 T} (e^{6\eta^2 T} - 4e^{3\eta^2 T} - 3e^{2\eta^2 T} + 12e^{\eta^2 T} - 6)$$

$$\chi_5 = e^{\frac{5\eta^2 T}{2}} (e^{10\eta^2 T} - 5e^{6\eta^2 T} - 10e^{4\eta^2 T} + 20e^{3\eta^2 T} + 30e^{2\eta^2 T} - 60e^{\eta^2 T} + 24).$$

Using the notation $\hat{p} = \lambda e^{-rT} s_0 e^{(\nu - \eta \frac{\mu - r}{\sigma})T}$ and $\alpha = \eta \frac{\mu - r}{\sigma} T$, we then determine that:

$$c_0 = \hat{p}$$

$$c_1 = \alpha \hat{p} - \hat{p}^2 \gamma e^{rT} \chi_2$$

$$c_2 = \frac{1}{2} \hat{p} \alpha^2 - \frac{X_2}{2} \hat{p}^2 \gamma (4\alpha - 1)e^{rT} + \frac{2 \chi_3}{3} \hat{p}^3 \gamma^2 e^{2rT}$$

$$c_3 = \frac{1}{6} \hat{p} \alpha^3 - \chi_2 \hat{p}^2 \gamma (2\alpha - 1)e^{rT} + \frac{2 \chi_3}{3} \hat{p}^3 \gamma (3\alpha - 1)e^{2rT} - \frac{\chi_4}{3} \hat{p}^4 \gamma^3 e^{3rT}$$

$$c_4 = \frac{1}{24} \hat{p} \gamma^4 - \frac{X_2}{3} \hat{p}^2 \gamma^2 (4\alpha - 3)e^{rT} + \chi_3 \hat{p}^3 \gamma^2 (-2\alpha + 3\alpha^2 + \frac{1}{6}) e^{2rT}$$

$$- \frac{\chi_4}{6} \hat{p}^4 \gamma (8\alpha - 3)e^{3rT} + \frac{2 \chi_5}{15} \hat{p}^5 \gamma^4 e^{4rT}.$$  

Note that $\hat{p}$ corresponds to the expectation of $\lambda e^{-rT} S_T$ computed under the unique probability measure, which turns the discounted value of the traded asset $P$ into a martingale in the complete case, i.e., when $\rho = 1$. Indeed,

$$\frac{d\tilde{S}_T}{\tilde{S}_T} = \left(\nu - r - \eta \frac{\mu - r}{\sigma}\right) dt + \frac{\eta}{\sigma} d\tilde{P}_T.$$

We now perform numerical computations for the market situation specified in Table 1.
Figure 2.: In the market situation of Table 1 the estimated asking reservation price of $\lambda S_T$ as a function of $\rho$ using the Monte Carlo method is shown in black and Taylor expansions are displayed in red. The 99% confidence intervals of the Monte Carlo estimate are in grey.

Looking at Figure 2, Taylor expansions of all orders are unreliable approximations when the correlation is lower than 0.9, and higher orders of expansion lead to greater errors. Thus, Taylor expansions (and general polynomial expansions) do not give approximations of the asking reservation price that remains reliable on $(-1, 1)$, at least for reasonable market situations. They also do not provide an estimation of the error between the price and the expansion.

4. Lambert approach

The polynomial method has been shown to be irrelevant for our purpose. Thus, we now provide another approach, where we split the reservation price into a deterministic part and a random part. To do that, we adapt a very simple idea from [1]: we remove the expectation of the most randomness as possible using the Lambert function. From a numerical viewpoint, we will perform the Monte Carlo method only on the random part, and this will provide an estimator with a lower variance (see Figures 3 and 7). From a theoretical perspective, we will see that unlike the reservation price, the deterministic part can be studied as a function of $\rho$ (and $\gamma$).

We start with the decomposition of $L_{f,\beta}$, the Laplace transform of the random variable $f(e^{\eta \sqrt{T_N}})1_{e^{\eta \sqrt{T_N}} \leq \beta}$, where $\beta \in [0, +\infty]$ and $f$ is a measurable function that is bounded from below on $[0, \beta]$. Note that [1] studies the case where $\beta = \infty$ and $f = id$. As $f$ is bounded from below on $[0, \beta]$, $L_{f,\beta}$ is well defined for all $\theta > 0$ by

$$L_{f,\beta}(\theta) = \mathbb{E} \left( \exp \left( -\theta f \left( e^{\eta \sqrt{T_N}} \right) 1_{e^{\eta \sqrt{T_N}} \leq \beta} \right) \right).$$
Using the decomposition of the Laplace transform, we obtain the decomposition of the reservation price \( p_{\zeta,K} \) (and of the value function \( \tilde{V}_{\zeta,K} \)) of a derivative with payoff \( \zeta(S_T)1_{S_T \leq K} \), where \( K \in [0,\infty] \) and \( \zeta \) is a measurable function, that is bounded from below on \([0,K]\).

Indeed, let \( \hat{s}_0 = s_0 e^{(\delta - \frac{\sigma^2}{2}) t} \), \( \hat{K} = K / \hat{s}_0 \) and \( \hat{\zeta}(x) = \zeta(\hat{s}_0 x) \) for all \( x \geq 0 \). Then, \( \hat{\zeta} \) is bounded from below on \([0,\hat{K}]\) and there is a simple relation between \( L_{\hat{\zeta},\hat{K}} \) and \( p_{\zeta,K} \). Letting \( \tilde{\theta} = \lambda \gamma (1 - \rho^2) \) and using (5) with \( h = \zeta 1_{[0,K]} \), we obtain that

\[
\begin{align*}
    p_{\zeta,K} &= -\frac{e^{-\gamma T}}{\gamma(1 - \rho^2)} \ln \mathbb{E} \left( \exp \left( -\hat{\theta} \zeta (e^{nuT}) 1_{e^{nuT} \leq \hat{K}} \right) \right) \\
    &= -\frac{e^{-\gamma T}}{\gamma(1 - \rho^2)} \ln L_{\hat{\zeta},\hat{K}}(\tilde{\theta}).
\end{align*}
\]

(8)

We now propose a decomposition of \( L_{f,\beta}(\theta) \). To do that, we employ a kind of Taylor expansion of \( f \). At this stage, we keep general values for \( u, v \in \mathbb{R} \) with \( v > 0 \), as we use different values for \( u \) and \( v \), whether \( \beta = \infty \) or not. For all \( \theta > 0 \),

\[
\begin{align*}
    L_{f,\beta}(\theta) &= \mathbb{E} \left( \exp \left( -\theta \left( f(e^{\eta T z}) - u - ve^{\eta T z} \right) 1_{e^{\eta T z} \leq \beta} - \theta \left( u + ve^{\eta T z} \right) 1_{e^{\eta T z} \leq \beta} \right) \right) \\
    &= \int_{\mathbb{R}} \exp \left( -\theta \left( f(e^{\eta T z}) - u - ve^{\eta T z} \right) 1_{z \leq \frac{\ln \beta}{\eta}} - k_{\beta}(z) \right) \frac{dz}{\sqrt{2\pi}}.
\end{align*}
\]

(9)

where for all \( z \in \mathbb{R} \)

\[
    k(z) = \theta \left( u + ve^{\eta T z} \right) + \frac{z^2}{2}, \quad k_{\beta}(z) = k(z) 1_{z \leq \frac{\ln \beta}{\eta}} + \frac{z^2}{2} 1_{z > \frac{\ln \beta}{\eta}} = \theta \left( u + ve^{\eta T z} \right) 1_{z \leq \frac{\ln \beta}{\eta}} + \frac{z^2}{2}.
\]

(10)

Note that \( k_{+\infty} = k \). Moreover, \( k_{\beta} \) is continuous if and only if \( u = -v\beta \).

To ensure the smallest possible randomness in \( L_{f,\beta}(\theta) \), we seek to remove the integral of the maximum of \( e^{-k_{\beta}} \). Thus, we aim to minimize \( k_{\beta} \); to do that, we first study \( k \). Obviously, \( k \) is \( C^\infty \), and for all \( z \in \mathbb{R} \),

\[
k'(z) = \theta v \eta T e^{\eta T z} + z.
\]

As \( v > 0 \), for all \( z \geq 0 \), we have \( k'(z) \geq 0 \). Assume now that \( z < 0 \)

\[
k'(z) \geq 0 \iff \theta v \eta^2 T \geq -\eta^2 T e^{-\eta T z}.
\]

At this point, we need to determine the inverse function of \( x \mapsto xe^x \). Let

\[
    l : x \in (-1, +\infty) \mapsto xe^x \in \left( -\frac{1}{e}, +\infty \right).
\]

(11)

Then, \( l \) is a \( C^\infty \) strictly increasing one-to-one correspondence, and thus we can introduce the Lambert function as its inverse.

**Definition 4.1** The Lambert function \( W \) is the inverse function of \( l \):

\[
W : x \in \left( -\frac{1}{e}, +\infty \right) \mapsto l^{-1}(x) \in (-1, +\infty).
\]
It follows that $W$ has the following properties:

$$\forall x > -\frac{1}{e} \quad W(x)e^{W(x)} = x$$  \hspace{1cm} (12)

$$\forall x > -1 \quad W(xe^x) = x$$  \hspace{1cm} (13)

**Lemma 4.2** The Lambert function $W$ is strictly increasing $C^\infty$ and

$$\forall x > -\frac{1}{e} \quad W'(x) = \frac{1}{e^{W(x)} + x} = \frac{W(x)}{x(1 + W(x))}.$$  \hspace{1cm} (14)

Moreover, $W$ admits a Taylor expansion around 0

$$W(x) = x - x^2 + x^2 e(x) \text{ with } \lim_{x \to 0} e(x) = 0.$$  \hspace{1cm} (15)

Finally, $W(x) \sim_{x \to +\infty} \ln(x)$.

Note that Lemma 4.2 and (13) imply that $W(0) = 0$ and $W(x) > 0$, $\forall x > 0$.

We compute the minimum of $k_\beta$ in Lemma 7.1 in the appendix. This provides the following multiplicative decomposition of $L_{f,\beta}$, which extends the results of [1].

**Theorem 4.3** Let $\beta \in [0, +\infty]$, $\theta, v > 0$ and $f$ be a measurable function that is bounded from below on $[0, \beta]$. Let $u$ be any real number if $\beta = +\infty$ and $u = -v\beta$ if $\beta < +\infty$. Moreover, suppose that

$$\frac{W(\theta v\eta^2 T)}{\eta^2 T} + \frac{W^2(\theta v\eta^2 T)}{2\eta^2 T} \leq \theta v\beta.$$  \hspace{1cm} (16)

Then,

$$L_{f,\beta}(\theta) = L_\beta(\theta)I_{f,\beta}(\theta),$$

where,

$$L_\beta(\theta) = \exp \left( -\theta u + \frac{W(\theta v\eta^2 T)}{\eta^2 T} + \frac{W(\theta v\eta^2 T)^2}{2\eta^2 T} \right)$$

$$I_{f,\beta}(\theta) = \mathbb{E} \left( \exp \left[ -\theta f \left( \frac{W(\theta v\eta^2 T)}{\theta \eta^2 T} e^{\eta \sqrt{T} N} - u - \frac{W(\theta v\eta^2 T)}{\theta \eta^2 T} e^{\eta \sqrt{T} \eta} \right) \right] \phi_\beta(N, \theta) \right)$$

$$\phi_\beta(y, \theta) = \exp \left( -\frac{W(\theta v\eta^2 T)}{\eta^2 T} \left( e^{\eta \sqrt{T} y} - 1 - \eta \sqrt{T} y \right) \right) \exp \left( \frac{W(\theta v\eta^2 T)}{\eta^2 T} e^{\eta \sqrt{T} y} - \theta v\beta \right)_+, $$

with the convention that $(-\infty)_+ = 0$.

**Remark 4.4** We remark in $\phi_\beta$ regarding the difference between an exponential function and its Taylor expansion of order one in zero. If $u = f(0)$ and $v = f'(0)$, another Taylor expansion of order one in zero appears in $I_{f,\beta}$. This provides the intuition of why $I_{f,\beta}$ should be close to 1.
Theorem 4.3 provides the following decomposition of the reservation price and of the value function for a derivative with payoff \( \zeta(S_T)1_{S_T \leq K} \).

**Theorem 4.5** Let \( K \in [0, +\infty] \) and \( \zeta \) be a measurable function that is bounded from below on \([0, K] \). Let \( v > 0 \) and \( u = -vK/(s_0 e^{(\delta-\frac{\sigma^2}{2})T}) \) if \( K < +\infty \) or \( u \) be any real number if \( K = +\infty \). Moreover, suppose that

\[
W(\lambda\gamma(1-\rho^2)\nu_1^2T) + \frac{W^2(\lambda\gamma(1-\rho^2)\nu_2^2T)}{2\eta^2T} \leq \frac{K}{s_0}\lambda\gamma(1-\rho^2)ve^{-(\delta-\frac{\sigma^2}{2})T}. \tag{17}
\]

Then,

\[
p_{\zeta,K} = D_{\zeta,K} + A_{\zeta,K}, \tag{18}
\]

where

\[
D_{\zeta,K} = \lambda e^{-rT} u + \frac{e^{-rT}}{\gamma(1-\rho^2)} \left( \frac{W(\lambda\gamma(1-\rho^2)\nu_1^2T)}{\eta^2T} + \frac{W^2(\lambda\gamma(1-\rho^2)\nu_2^2T)}{2\eta^2T} \right) \tag{19}
\]

\[
A_{\zeta,K} = -\frac{e^{-rT}}{\gamma(1-\rho^2)} \ln \left( I_{\hat{K},K}(\lambda\gamma(1-\rho^2)) \right). \tag{20}
\]

where \( I_{\hat{K},K} \) is defined in Theorem 4.3 with \( \hat{K} = K/(s_0 e^{(\delta-\frac{\sigma^2}{2})T}) \) and \( \zeta(x) = \zeta(s_0 e^{(\delta-\frac{\sigma^2}{2})T} x) \), for all \( x \geq 0 \).

We call \( D_{\zeta,K} \) the deterministic part of \( p_{\zeta,K} \) and \( A_{\zeta,K} \) its random part. Moreover,

\[
V_{\zeta,K}(x_0, s_0, \lambda, \zeta 1_{[0,K]}) = V_{D_{\zeta,K}}(x_0, s_0, \lambda, \zeta 1_{[0,K]}) V_{A_{\zeta,K}}(s_0, \lambda, \zeta 1_{[0,K]}), \tag{21}
\]

where

\[
V_{D_{\zeta,K}}(x_0, s_0, \lambda, \zeta 1_{[0,K]}) = -\frac{1}{\gamma} \exp \left( -\gamma e^{rT}(x_0 + D_{\zeta,K}) - \frac{(\mu - r)^2}{2\sigma^2} T \right) \tag{22}
\]

\[
V_{A_{\zeta,K}}(s_0, \lambda, \zeta 1_{[0,K]}) = \exp \left( -\gamma e^{rT} A_{\zeta,K} \right). \tag{23}
\]

**Proof of Theorem 4.5.** The decomposition of \( p_{\zeta,K} \) is a direct consequence of (8) and Theorem 4.3. Then, the decomposition of the value function is obtained using (6).

**Remark 4.6** With the decomposition provided by (18) and (21), we perform a Monte Carlo method on the random part of the reservation prices or of the value function (see (20) and (23)). We will thus need to choose \( u \) and \( v \) such that \( A_{\zeta,K} \) is as small as possible, i.e., such that \( I_{\hat{K},K}(\lambda\gamma(1-\rho^2)) \) is close to 1: see Remark 4.4.

We call this the Lambert Monte Carlo method. We show below that it is indeed numerically efficient: see Figures 3 and 7.

**Remark 4.7** The deterministic approximation \( D_{\zeta,K} \) of \( p_{\zeta,K} \) can be understood as an asking reservation price. Indeed, if we assume that \( V_{D_{\zeta,K}}(x_0, s_0, \lambda, \zeta 1_{[0,K]}) \) is a value function, then (22) and \( D_{\zeta,K}|_{\lambda=0} = 0 \) (see (19)) show that

\[
V_{D_{\zeta,K}}(x_0 - D_{\zeta,K}, s_0, \lambda, \zeta 1_{[0,K]}) = -\frac{1}{\gamma} \exp \left( -\gamma x_0 e^{rT} - \frac{(\mu - r)^2}{2\sigma^2} T \right)
= V_{D_{\zeta,K}}(x_0, s_0, 0, \zeta 1_{[0,K]}).
\]
5. Long stock position

In this part, we focus on the case of a long stock position on the non-traded stock, which is, as already mentioned, of great importance in management science.

5.1. Decomposition of the reservation price and of the value function

The next theorem is a direct consequence of Theorem 4.5. Recall that \( p \) is the asking reservation price of \( \lambda > 0 \) units of \( S \) and \( V \) is the associated value function.

**Theorem 5.1** Let \( \bar{w} = W \left( s_0 e^{(\delta - \frac{\eta^2}{2})T} \eta^2 T \lambda \gamma (1 - \rho^2) \right) \). Then,

\[
p = D + A
\]

where

\[
D = \frac{e^{-rT}}{\gamma (1 - \rho^2)} \left( \frac{\bar{w}}{\eta^2 T} + \frac{\bar{w}^2}{2 \eta^2 T} \right)
\]

\[
A = -e^{-rT} \ln \mathbb{E} \left( \exp \left( -\frac{\bar{w}}{\eta^2 T} (e^{\eta^2 T N} - 1 - \eta^{2 N}) \right) \right).
\]

Moreover,

\[
V(x_0, s_0, \lambda) = V_D(x_0, s_0, \lambda) V_A(s_0, \lambda),
\]

where

\[
V_D(x_0, s_0, \lambda) = -\frac{1}{\gamma} \exp \left( -\gamma e^{rT} (x_0 + D) - \frac{(\mu - r)^2}{2 \sigma^2} T \right)
\]

\[
V_A(s_0, \lambda) = \exp \left( -\gamma e^{rT} A \right).
\]

**Remark 5.2** As \( e^x \geq x + 1 \) and \( \bar{w} \geq 0 \) (see [15]), \( A \geq 0 \) and the deterministic part \( D \) is a lower bound for \( p \).

**Proof of Theorem 5.1** Let \( \hat{s}_0 = s_0 e^{(\delta - \frac{\eta^2}{2})T} \) and \( \hat{\theta} = \lambda \gamma (1 - \rho^2) \). We choose in Theorem 4.5 \( K = +\infty \), \( \zeta = id \), \( u = 0 \) and \( v = \hat{s}_0 \). Then, (17) is obviously true. Moreover, \( \hat{K} = +\infty \) and \( \hat{\zeta}(x) = \hat{s}_0 x \), for all \( x \geq 0 \). Then, for all \( y \in \mathbb{R} \),

\[
\phi_{\hat{K}}(y, \hat{\theta}) = \exp \left( -\frac{\bar{w}}{\eta^2 e^{T \hat{\theta}}} \left( e^{\eta^2 T y} - 1 - \eta^{2 T y} \right) \right)
\]

\[
I_{\hat{\zeta}, \hat{K}}(\hat{\theta}) = \mathbb{E} \left( \phi_{\hat{K}} \left( N, \hat{\theta} \right) \exp \left( -\hat{\theta} \hat{\zeta} \left( \frac{\bar{w}}{\theta \hat{s}_0 \eta^2 T} e^{\eta^2 T N} - \frac{\bar{w}}{\theta \eta^2 T e^{\eta^2 T N}} \right) \right) \right)
\]

\[
= \mathbb{E} \left( \phi_{\hat{K}}(N, \hat{\theta}) \right).
\]

\[\square\]
5.2. Deterministic approximations of $p$

We have seen that $D$ provides a deterministic lower bound for the reservation price (see Remark 5.2). We also provide a deterministic upper bound for $p$ and study the quality of both bounds. To do that, we introduce $\theta : \rho \in (-1, 1) \mapsto \lambda \gamma (1 - \rho^2)$,

$$w : \rho \in (-1, 1) \mapsto W \left( s_0 \eta^2 T e^{(v - \eta \rho^2 + T - \frac{a^2}{2})}\theta(\rho) \right)$$  \hspace{1cm} (27)

$$d : \rho \in (-1, 1) \mapsto \frac{\lambda e^{-rT}}{\theta(\rho) \eta^2 T} \frac{w(\rho)}{2} \left( 1 + \frac{w(\rho)}{2} \right)$$  \hspace{1cm} (28)

$$a : \rho \in (-1, 1) \mapsto -\frac{\lambda e^{-rT}}{\theta(\rho)} \mathbb{E} \left( \exp \left( -\frac{w(\rho)}{\eta^2 T} \left( e^{\sqrt{T} N} - 1 - \eta \sqrt{T} N \right) \right) \right)$$  \hspace{1cm} (29)

Recalling (25) and (26), we determine that $D = d(\rho)$ and $A = a(\rho)$. We also set $G = g(\rho)$, where for all $\rho \in (-1, 1)$,

$$g(\rho) = d(\rho) + b(\rho) = \frac{\lambda e^{-rT}}{\theta(\rho) \eta^2 T} \frac{w(\rho)}{2} \left( e^{\frac{a^2}{2}} + \frac{w(\rho)}{2} \right).$$  \hspace{1cm} (30)

**Lemma 5.3** The functions $d$ and $g$ are $C^1$. We have that $d > 0$ and that $g > 0$. Moreover,

$$\lim_{\rho \to 1^-} d(\rho) = \lambda e^{-rT} s_0 e^{(v - \eta \rho^2 - \frac{a^2}{2})} \lim_{\rho \to 1^+} d(\rho) = \lambda e^{-rT} s_0 e^{(v + \eta \rho^2 - \frac{a^2}{2})}$$  \hspace{1cm} (31)

$$\lim_{\rho \to 1^-} g(\rho) = \lambda e^{-rT} s_0 e^{(v - \eta \rho^2 - \frac{a^2}{2})} \lim_{\rho \to 1^+} g(\rho) = \lambda e^{-rT} s_0 e^{(v + \eta \rho^2 - \frac{a^2}{2})}.$$  \hspace{1cm} (32)

We have that, for all $\rho \in (-1, 1)$,

$$\left( b(\rho) - \frac{\lambda e^{-rT}}{\theta(\rho)} \frac{w(\rho)^2}{2 \eta^2 T^2} e_2 \right) \leq a(\rho) \leq b(\rho)$$  \hspace{1cm} (33)

where $e_2 = e^{2\eta^2 T} - 2(1 + \eta^2 T) e^{\frac{a^2}{2}} + \eta^2 T + 1$. Moreover, $a$, $b$, $d$ and $g$ are bounded on $(-1, 1)$.

**Remark 5.4** Recall that $\hat{p} = \lambda e^{-rT} s_0 e^{(v - \eta \rho^2 - \frac{a^2}{2})}$ is an approximation at order 0 of $p$ near $\rho = 1^-$ (see [7]). Thus, unlike $d$, the upper bound $g$ is not biased for high positive correlation. This suggests that $g$ may be a better approximation for $p$ when $\rho \to 1$. This is confirmed in the numerical simulations below. Note that [31] shows that $\lim_{\rho \to 1^-} d(\rho) / p = e^{-\frac{a^2}{2}}$. Thus, small values of $\eta$ and $T$ reduce the bias between $d$ and $p$ for high positive correlations. We will see in (34) below that this observation can be extended to all values of the correlation.

**Theorem 5.5** We have that $D \leq p \leq G$. Letting

$$V_G(x_0, s_0, \lambda) = -\frac{1}{\gamma} \exp \left( -\gamma e^{rT}(x_0 + G) - \frac{(\mu - r)^2}{2\sigma^4} T \right).$$

Then,

$$V_D(x_0, s_0, \lambda) \leq V(x_0, s_0, \lambda) \leq V_G(x_0, s_0, \lambda).$$
Proof of Theorem 5.5. Recalling (24), \( p = D + A \). Thus, Remarks 5.2, (33) and (30) show that
\[
D \leq p \leq d(\rho) + b(\rho) = G.
\]
The last inequality then follows from (27) and (6).

Proposition 5.6. We determine that
\[
e^{-\eta^2 T} \leq \frac{1 + \varpi(\rho)}{e^{\eta^2 T} + \varpi(\rho)} \leq \frac{D}{p} \leq 1
\]
(34)
\[
1 \leq \frac{G}{p} \leq 1 + \frac{w(\rho)e_2}{2\eta^2 T + \eta^2 Tw(\rho)} \leq 1 + \frac{e_2}{\eta^2 T},
\]
(35)
where \( e_2 \) is defined in Lemma 5.3.

Remark 5.7. The bounds in (34) easily apply to \( A/p \) and show that \( A \) is small relative to \( p \):
\[
0 \leq \frac{A}{p} \leq \frac{e^{-\eta^2 T} - 1}{e^{-\eta^2 T} + \varpi(\rho)} \leq 1 - e^{-\eta^2 T}.
\]

Having bounds that only depend on \( \eta \) and \( T \) legitimizes the use of \( D \) and \( G \) as uniform approximations of the reservation price in terms of \( \rho \in (-1, 1) \). For example, if \( \eta^2 T \leq 0.04 \), then \( 0.98p \leq D \leq p \), and if \( \eta^2 T \leq 0.145 \), \( p \leq G \leq 1.02p \).

We will see that numerically, \( D \) provides a better approximation when \( \gamma \) (or \( \lambda s_0 \)) are large enough: see Figure 5. In contrast, \( G \) is more reliable for small values of \( \gamma \) (or \( \lambda s_0 \)): see Figure 4.

Remark 5.8. As in Proposition 5.6, it is possible to derive bounds on \( V_D \) and \( V_G \).
\[
1 \leq \frac{V_D}{V} \leq \exp\left(\frac{\varpi(\rho)}{(1 - \rho^2)\eta^2 T} \left( e^{\eta^2 T} - 1 \right) \right)
\]
\[
\exp\left(\frac{-\varpi^2(\rho)}{2(1 - \rho^2)\eta^4 T^2}e_2 \right) \leq \frac{V_G}{V} \leq 1.
\]

5.3. Numerical applications for the reservation price

We now provide numerical simulations to illustrate the quality of our deterministic approximations of the reservation price \( p \). We also plot the correlation \( \rho^* \) that minimizes \( D \), which will be studied in Section 5.5 below.
Figure 3.: Estimated asking reservation price of the $\lambda S_T$ as a function of $\rho$ in the market situation of Table 1 with $\gamma = 0.5$ in black, lower deterministic approximation $d$ in blue (dashed line) and upper deterministic approximation $g$ in purple (dotted line) with $10^4$ simulations per point. The vertical lines indicate the position of the minimum of $D$.

We see that the Lambert Monte Carlo estimator of the asking reservation price of $\lambda S_T$ does not suffer from high variance, as is the case for the Direct Monte Carlo estimator. For example, in the situation of Table 1 with $\gamma = 0.5$ and $\rho = 0$, the variance of the direct method (for $10^4$ simulations per point) is 5.66 and that of the Lambert method is 0.017. We see that $D$ and $G$ are indeed good approximations of $p$. We now propose other simulations in the market situations of Tables 2 and 3 in order to determine the quality of $D$ and $G$.

Figure 4.: Estimated asking reservation price of the $\lambda S_T$ as a function of $\rho$ in black in the market situation of Table 2 with $\gamma = 0.1$, lower deterministic approximation $d$ in blue (dashed line) and upper deterministic approximation $g$ in purple (dotted line) with $10^4$ simulations per point. The vertical lines indicate the position of the minimum of $D$. 

14
Lambert Monte Carlo method

Figure 5.: Estimated asking reservation price of the $\lambda S_T$ as a function of $\rho$ in black in the market situation of Table 3 with $\gamma = 15$, lower deterministic approximation $d$ in blue (dashed line) and upper deterministic approximation $g$ in purple (dotted line). The vertical lines indicate the position of the minimum of $D$.

We see that the upper deterministic approximation $G$ performs better when $\lambda s_0$ and $T$ are small (the situation of Table 2), while when $T$ and $\lambda s_0$ are large (the situation of Table 3), the lower bound $D$ is clearly more accurate. For the numerical values of Table 3, the Direct Monte Carlo method does not work because of rounding issues. Moreover, note that the minimizer of the reservation price in $\rho$ is indeed close to the minimum $\rho^*$ of the deterministic part $d$, even in the situation of Table 3, where $D$ is not a good approximation of $p$.

We now compute the deterministic bounds for $D/p$ and $G/p$ of Proposition 5.6 in order to quantify the precision of our deterministic approximations of $p$ and to determine how sharp they are.

We start with the lower bounds of $D/p$. In the market situation of Table 1, we have that $\exp (-\eta^2 T/2) = 0.9888$. The other bound in (34) is sharper (for $\gamma \in \{0.5, 4, 15\}$ and $\rho \in \{-0.8, -0.4, 0, 0.4, 0.8\}$, it is between 0.9910 and 0.9956). Thus, the reservation price is well-explained by $D$.

In the situation of Table 1, the two bounds for $D/p$ are indeed very close. This is not the case in the market situation of Table 3. Indeed, we have $\exp (-\eta^2 T/2) = 0.6376$, while the other bound in (34) is much tighter, as seen in Table 4.

We now compute the upper bounds of $G/p$ in (35). In the market situation of Table 1, we have that $1 + e_2/\eta^2 T = 1.017$. The other bound is between 1.0035 and 1.0106 (for $\gamma \in \{0.5, 4, 15\}$ and $\rho \in \{-0.8, -0.4, 0, 0.4, 0.8\}$). The bounds are again very close, and the reservation price is also well explained by $G$. In the market situation of Table 3, we find that $1 + e_2/\eta^2 T = 3.2112$ and the values of the other upper bound in (35) are greater than 2, even though $G$ is still a correct approximation.
5.4. Optimal strategy and value function
We focus now on the optimal strategy \((Π^{*,λ}(t, S_t))_{0 \leq t \leq T}\) for \([1]\), i.e., the strategy that maximizes the expected utility in \([1]\), when \(h = id\). We first recall the expression of the optimal trading strategy. As we consider an asking reservation price instead of selling reservation price (see (4)), the expected utility in (1), when \(h \in (0, s]\), the series expansion can be extremely imprecise for \(p\), especially when \(\gamma\) is not very small and \(|\rho|\) is not near 1. Thus, we propose to use in (36) our decomposition (24) of the reservation price:

\[
Π^{*,λ}(t, s) = e^{-r(T-t)} \frac{μ-r}{γσ^2} - \frac{ηρ}{σ} s \frac{∂p_t}{∂s}(ρ, s),
\]

where \(p_t\) is the dynamic version of \(p\), i.e., the solution of \(V_t(x + p_t, s, λ, id) = V_t(x_0, s, 0, id)\), where \(V_t(x, s, λ, id)\) is the value function at \(t\), if \(S_t = s\) and \(X_t = x\). For all \(t \in [0, T]\) and \(s \geq 0\), as for \([5]\), we determine that

\[
p_t(ρ, s) = -\frac{e^{-r(T-t)}}{γ(1 - ρ^2)} \ln \left( \exp \left( -\lambdaγ(1 - ρ^2) se \left( \frac{δ - ρ^2}{2} \right)(T-t) + η√T−N \right) \right).
\]

The computation of the optimal strategy \(Π^{*,λ}\) involves the partial derivative of \(p_t\). Therefore, numerical issues may appear because the Direct Monte Carlo method performs badly even for \(p\); see Figure 3. Monoyios (see Corollary 1 in [12] or Section 4.1.1 in [13]) addresses this issue by computing the series expansion of the partial derivative in (36). However, we have seen in Section 3 that the series expansion can be extremely imprecise for \(p\), especially when \(\gamma\) is not very small and \(|\rho|\) is not near 1. Thus, we propose to use in (36) our decomposition (24) of the reservation price:

\[
Π^{*,λ}(t, s) = e^{-r(T-t)} \frac{μ-r}{γσ^2} - \frac{ηρ}{σ} s \frac{∂w_t}{∂s}(ρ, s) - \frac{ηρ}{σ} s \frac{∂d_t}{∂s}(ρ, s),
\]

where for all \(s > 0\) and \(ρ \in (-1, 1)\)

\[
w_t(ρ, s) = W \left( sη^2(T-t)e^{(e-ηρs+ηρs}\frac{δ - ρ^2}{2})\right)\),
\]

\[
d_t(ρ, s) = -\frac{λe^{-r(T-t)}}{θ(ρ)η^2(T-t)} \left( w_t(ρ, s) + \frac{w_t^2(ρ, s)}{2} \right),
\]

\[
a_t(ρ, s) = -\frac{λe^{-r(T-t)}}{θ(ρ)} \ln \left( \exp \left( -\frac{w_t(ρ, s)}{η^2(T-t)} \left( e^{η√T−T}N - 1 - η√T−TN \right) \right) \right).
\]

Using (14), we determine that

\[
\frac{∂w_t}{∂s}(ρ, s) = \frac{w_t(ρ, s)}{s(1 + w_t(ρ, s))},
\]

\[
\frac{∂d_t}{∂s}(ρ, s) = \frac{λe^{-r(T-t)}}{θ(ρ)η^2(T-t)} \frac{∂w_t}{∂s}(ρ, s)(1 + w_t(ρ, s)) = \frac{λe^{-r(T-t)}}{θ(ρ)η^2(T-t)s}w_t(ρ, s).
\]
Thus, we propose to approximate $\left(\Pi^{\ast}(t, S_t), \lambda(t, S_t)\right)_{0 \leq t \leq T}$ with the deterministic strategy $\left(\Pi^D(t, S_t), \lambda(t, S_t)\right)_{0 \leq t \leq T}$ defined for all $t \in [0, T]$ and $s > 0$ by

$$
\Pi^D(t, s) = e^{-r(T-t)} \left( \frac{\mu - r}{\gamma \sigma^2} - \frac{\eta \rho}{\sigma} \frac{\partial h}{\partial s}(\rho, s) \right) 
= e^{-r(T-t)} \left( \frac{\mu - r}{\gamma \sigma^2} - \frac{\rho w_t(\rho, s)}{\sigma \eta \gamma (1 - \rho^2)(T-t)} \right).
$$

(37)

This strategy avoids taking the partial derivative and the related numerical issues. We do not provide any theoretical results regarding the quality of this approximation, but we show numerically in Figure 6 that both strategies are close.

![Estimated optimal strategy vs Deterministic strategy](image)

**Figure 6.** Graph of the two strategies under the situation of Table 2 with $\gamma = 0.1$, $x_0 = 0$ and $\rho = 0.6$. For the computation of the estimated optimal strategy, we use the Lambert Monte Carlo method with $10^4$ simulations per point.

It is also interesting to compare $E_D = E(U(X^x_0, \Pi^D))$ with the value function $V = V(x_0, s, \lambda, id)$: see [1]. Note that in the market situation of Table 1 when $x_0 = 80$ and $\gamma = 0.5$, the value function $V$ is very small as $V \leq V_G = -2.42.10^{-45}$. Thus, providing numerical approximations for $V$ is not meaningful. We thus turn to the market situation of Table 2 with $\gamma = 0.1$ and $x_0 = 0$.

| $\rho$  | $-0.8$ | $-0.5$ | $-0.2$ | $0.2$  | $0.5$  | $0.8$  |
|--------|--------|--------|--------|--------|--------|--------|
| $V$    | $-0.983$ | $-1.069$ | $-1.134$ | $-1.189$ | $-1.208$ | $-1.204$ |
| CI     | $[-0.983, -0.982]$ | $[-1.069, -1.068]$ | $[-1.135, -1.134]$ | $[-1.190, -1.189]$ | $[-1.209, -1.207]$ | $[-1.205, -1.203]$ |
| $V_D$  | $-1.035$ | $-1.122$ | $-1.188$ | $-1.245$ | $-1.207$ | $-1.263$ |
| $V_G$  | $-0.982$ | $-1.067$ | $-1.132$ | $-1.189$ | $-1.265$ | $-1.203$ |
| $E_D$  | $-0.985$ | $-1.070$ | $-1.140$ | $-1.187$ | $-1.206$ | $-1.206$ |
| $\text{SD}$ | $0.411$ | $0.547$ | $0.625$ | $0.626$ | $0.595$ | $0.475$ |

**Table 5:** Numerical values for $V$, $V_D$ and $V_G$ under the market situation of Table 2 when $\gamma = 0.1$ and $x_0 = 0$. $V$ is computed using the Lambert Monte Carlo method. CI is the confidence interval of $V$ at 99%. $E_D$ is the expected utility with the strategy (37) computed using the standard Monte Carlo method with $10^4$ simulations per point and a time step of $T/200$ in the Euler scheme (i.e., 200 rebalances of the strategy). The last line is the standard deviation of $E_D$. 

17
We see that the deterministic approximations of $V$ are reliable. Moreover, looking at Table 5, $V_G$ is much more precise than $V_D$. However, $E_D$, the expected utility computed with the strategy (37), provides an even better approximation. This is true in other market situations and confirms the relevance of the approximated optimal strategy. Note that the deterministic bounds given in Remark 5.8 provide less interesting approximations than in the case of the reservation price.

5.5. Variations of $d$

One may examine the problem of pricing and hedging from a different point of view. Assume that an agent has the choice between different hedging assets $P$ and chooses the one that minimizes the reservation price with the same level of rentability: i.e., $\mu$, $\sigma$ and $r$ are fixed, and she minimizes $p$ in $\rho$. Indeed, as $p$ is an asking price, the agent may want to pay as little as possible. Unfortunately, the variation and the minimum of the reservation price (as a function of $\rho$) are very difficult to study. However, this can be done for the deterministic part $d$, as stated in the next proposition, and the numerical applications of Section 5.3 suggest that $\rho^*$, the minimum of $d$, is in fact a good approximation of the real minimum of the reservation price.

Proposition 5.9 Let

$$\rho^* = \frac{\eta T}{W\left(\lambda \gamma s_0 \eta^2 T e^{(\nu - \frac{\eta^2}{2})T}\right)} \frac{\mu - r}{\sigma}.$$  

(38)

If $\rho^* \leq -1$, $d$ is increasing on $(-1, 1)$. If $\rho^* \geq 1$, $d$ is decreasing on $(-1, 1)$.

Now, assume that $-1 < \rho^* < 1$. Then, $d$ is decreasing on $(-1, \rho^*)$ and increasing elsewhere and

$$d(\rho^*) = d(0) - e^{-rT} (\mu - r)^2 T \frac{2}{2\gamma \sigma^2} \text{ and } g(\rho^*) = g(0) - e^{-rT} (\mu - r)^2 T \frac{2}{2\gamma \sigma^2}.$$  

Moreover,

$$\inf_{\rho \in [-1, 1]} p \geq d(0) - e^{-rT} \left(\frac{\mu - r}{\sigma}\right)^2 T$$  

(39)

$$\inf_{\rho \in [-1, 1]} V(x_0, s_0, \lambda) \geq -\frac{1}{\gamma} \exp \left(-\gamma e^{rT} (x_0 + d(0))\right).$$  

(40)

Remark 5.10 It is worth noting that even if $g$ and $d$ appear to be similar (see (28) and (30)), the study of the variations of $g$ is much more difficult. In particular, we are unable to find the minimum of $g$.

Since $d(0)$ is independent of $\mu$ and $\sigma$, the infima in (39) and (40) could also be taken in $\mu$ and $\sigma$ as long as they satisfy $|\frac{\mu - r}{\sigma}| \leq \frac{w(0)}{\eta}$; see (27) for the definition of $w$.

We show now numerically that even if the agent chooses to hedge herself with the asset with correlation $\rho^*$, it does not cripple the superhedging probability $P(X_T^{p(\rho^*)}, H^{\rho, \lambda} + \lambda S_T \geq 0)$ for $\Pi^{D, \lambda}$ defined in (37). To do that, we compute a Monte Carlo estimation (with $10^4$ simulations per point and a time step of $T/200$ in the Euler scheme, i.e., 200 rebalances of the strategy) of this probability in the market situations of Tables 1 and 2 for different values of $\rho$. 

Note that the value function and the reservation price share the same minimizer in $\rho$; see (4). Thus, it is clear that the market embedded with the correlation that minimizes the asking reservation price is always the worst choice for an agent who wants to maximize her expected utility.
Thus, we have an asymptote of $\rho$ as a function of the risk aversion.

In the next lemma, we present an asymptote of $\rho^*$ as a function of time.

**Lemma 5.11** We determine that

$$
\rho^* = \frac{\mu - r}{\sigma} \left( \frac{1}{\eta s_0 \lambda \gamma} + \left( \eta - \frac{\nu - \eta^2}{2 \eta s_0 \lambda \gamma} \right) T \right) + T \epsilon(T), \tag{41}
$$

where $\lim_{T \to 0^+} \epsilon(T) = 0$. Assume that $\nu \geq \eta^2/2$. Then,

$$
\lim_{T \to +\infty} \rho^* = \frac{\mu - r}{\sigma} \frac{\eta}{\nu - \eta^2}/2.
$$

The following table shows the efficiency of the first-order approximation in (41).

| $T$ | $\rho^*$ | Relative error |
|-----|---------|---------------|
| $0.01$ | $0.310$ | $<0.01\%$ |
| $0.1$  | $0.321$ | $0.15\%$ |
| $0.5$  | $0.357$ | $2.76\%$ |
| $0.8$  | $0.379$ | $5.87\%$ |
| $1$   | $0.391$ | $8.23\%$ |

**Table 8.** Relative error between $\rho^*$ and the first-order approximation in (41) in the market situation of Table 2 when $\gamma = 0.2$.

We finish this part with the study of the deterministic part as a function of the risk aversion. Recalling (25), the deterministic part of the Lambert decomposition is given by $D = \bar{d}(\gamma)$, where

$$
\bar{\theta} : \gamma \in (0, +\infty) \mapsto \lambda \gamma (1 - \rho^2)
$$

$$
\bar{w} : \gamma \in (0, +\infty) \mapsto W(s_0 \eta^2 T e^{(\delta - \frac{\eta^2}{2})T}) \tilde{\theta}(\gamma)
$$

$$
\bar{d} : \gamma \in (0, +\infty) \mapsto \frac{\lambda e^{-r T}}{\tilde{\theta}(\gamma) \eta^2 T \tilde{\theta}(\gamma)} \left( \bar{w}(\gamma) + \frac{\bar{w}^2(\gamma)}{2} \right).
$$

**Proposition 5.12** The function $\bar{d}$ is $C^1$, strictly decreasing and

$$
\lim_{\gamma \to 0^+} \bar{d}(\gamma) = \lambda e^{-r T} s_0 e^{(\delta - \frac{\eta^2}{2})T}, \quad \lim_{\gamma \to +\infty} \bar{d}(\gamma) = 0.
$$

Thus, $\bar{d}$ is bounded.
Remark 5.13 When the risk aversion goes to $+\infty$, $\check{d}$ goes to zero, the subhedging price of $\lambda S_T$.
When the risk aversion goes to 0, $\check{d}$ goes to $\check{p} e^{-\frac{\bar{w}^2}{2} T}$, where $\check{p}$ is the expectation of $\lambda S_T e^{-rT}$ under the minimal martingale measure.

6. Short put position

We now study the case of a short position on a put option with payoff $(K - S_T)^+$, where $K \in (0, \infty)$. Recall that $V_{\text{put}}(x_0, s_0, \lambda, K)$ and $p_{\text{put}}$ are respectively the value function of a short position on $\lambda$ units of a put option and its selling reservation price, i.e., $V_{\text{put}}(x_0, s_0, \lambda, K) = V(x_0, s_0, \lambda, -(K - x)_+)$ and $p_{\text{put}} = p_{\text{sell}}^{\text{ell}}(K - x)_+$; see [4].

Theorem 6.1 Let $\bar{w} = W\left(s_0 e^{(\delta - \frac{\mu^2}{2})T} \eta^2 T \lambda \gamma (1 - \rho^2) \right)$ and suppose that

$$K \geq \frac{1}{\lambda \gamma (1 - \rho^2)} \left( \frac{\bar{w}}{\eta^2 T} + \frac{\bar{w}^2}{2\eta^2 T} \right).$$

Then,

$$p_{\text{put}} = D_{\text{put}} + A_{\text{put}}$$

where

$$D_{\text{put}} = \lambda e^{-rT} K - \frac{e^{-rT}}{\gamma (1 - \rho^2)} \left( \frac{\bar{w}}{\eta^2 T} + \frac{\bar{w}^2}{2\eta^2 T} \right)$$

$$A_{\text{put}} = \frac{e^{-rT}}{\gamma (1 - \rho^2)} \ln E \left( \psi_K (N) \right)$$

where

$$\psi_K (y) = \exp \left( \frac{-\bar{w}}{\eta^2 T} \left( e^{\eta T y} - 1 - \eta T y \right) + \left( \frac{\bar{w}}{\eta^2 T} e^{\eta T y} - \lambda \gamma (1 - \rho^2) K \right)_+ \right).$$

Moreover,

$$V_{\text{put}}(x_0, s_0, \lambda, K) = V_{\text{D}}^{\text{put}}(x_0, s_0, \lambda, K) V_{\text{A}}^{\text{put}}(s_0, \lambda, K),$$

where

$$V_{\text{D}}^{\text{put}}(x_0, s_0, \lambda, K) = -\frac{1}{\gamma} \exp \left( -\gamma e^{rT} (x_0 - D_{\text{put}}) - \frac{(\mu - r)^2}{2\sigma^2} T \right)$$

$$V_{\text{A}}^{\text{put}}(s_0, \lambda, K) = \exp \left( \gamma e^{rT} A_{\text{put}} \right).$$

Proof of Theorem 6.1 Let $\check{s}_0 = s_0 e^{(\delta - \frac{\mu^2}{2})T}$ and $\check{\theta} = \lambda \gamma (1 - \rho^2)$. As $p_{\text{put}} = -p_{-(K - x)_+}$, we want to use Theorem 4.5 with $K < +\infty$, $\zeta(x) = x - K$, $v = \check{s}_0$ and $u = -v K / \check{s}_0 = -K$. Then, $\check{\zeta}(x) = \check{s}_0 x - K$. As (17) is equivalent to $\frac{\bar{w}}{\gamma T} + \frac{\bar{w}^2}{2\gamma T} \leq K \lambda \gamma (1 - \rho^2)$, it is true by (42). Thus, it
remains to prove that $I_{\hat{\zeta},K}(\hat{\theta}) = E(\psi_K(N))$. This follows from

$$\phi_K(y, \hat{\theta}) = \exp\left(-\frac{\bar{w}}{\eta^2 T} (e^{\eta \sqrt{T} y} - 1 - \eta \sqrt{T} y) + \left(\frac{\bar{w}}{\eta^2 T} e^{\eta \sqrt{T} y} - \hat{\theta} K\right)_{+}\right)$$

and

$$\hat{\zeta} \left(\frac{W(\hat{\theta} \eta^2 T)}{\hat{\theta} \eta^2 T} e^{\eta \sqrt{T} N}\right) - u - v \frac{W(\hat{\theta} \eta^2 T)}{\hat{\theta} \eta^2 T} e^{\eta \sqrt{T} N} = 0.$$

For the decomposition of the value function, (4) and (6) imply that

$$V^{\text{put}}(x_0, s_0, \lambda, K) = -\frac{1}{\gamma} \exp\left(-\gamma e^{\eta^2 T}(x_0 - p^{\text{put}}) - \frac{(\mu - r)^2}{2\sigma^2} T\right).$$

---

**Figure 7.** Computation of the selling reservation price $p^{\text{put}}$ as a function of $\rho$ in the market situation of Table 1 with $\gamma = 0.5$ and $K = 110$ with $10^4$ simulations per point.
As in the long stock position case, the Direct Monte Carlo estimator provides an estimator with high variance. Conversely, the Lambert Monte Carlo estimator exhibits low variance. For example, in the situation of Table 1, with $\gamma = 0.5$ and $x_0 = 0$, the variance is 4.745 for the direct method (for $10^4$ simulations per point) and 0.007 for the Lambert method.

We now propose numerical simulations for the value function. In the market situation of Table 1 when $x_0 = 75$, $\gamma = 0.5$ and $K = s_0 = 100$, $V_{\text{put}}$ and $V_{\text{put}}^D$ are large enough to be relevant as long as we work with correlation $\rho$ such that $|\rho| \leq 0.5$. Indeed, for higher values of $\rho$, the values of $V_{\text{put}}$ and $V_{\text{put}}^D$ are very small and the numerical estimations are not relevant (for example, if $\rho = 0.7$, $V \approx -1.9 \times 10^{-4}$ and $V_{\text{put}}^D = -1.8 \times 10^{-4}$).

We see that the deterministic approximation of $V_{\text{put}}$ is still reliable even if it is less faithful than that for the long stock position. Note that in the market situation of Table 2 with $\gamma = 0.1$ and $x_0 = 0$, the decomposition of $V_{\text{put}}$ cannot be computed because (42) does not hold.

**Conclusion and discussion**

We have proposed a decomposition of the reservation price and of the value function in the case of a long position on the non-traded asset and for a short position on a put option. This decomposition provides efficient numerical methods and allows construction of upper and lower bounds for the reservation price and the value function, as well as for the optimal strategy. One may ask if the...
same conclusions apply to the case of a long position on a call option. The decomposition of the asking reservation price of a long position on a call option can be obtained using similar methods (the details are not provided in the paper for sake of brevity). The Lambert Monte Carlo method provides good results for deep in- and out-of-the-money call options, but for at-the-money call options, it is not clearly better than when using the direct Monte Carlo method. An in-the-money (resp. at- or out-of-) call option of strike $K$ is an option, where $K > K^*$ (resp. $K < K_*$ or $K_* \leq K \leq K^*$), where

$$K_* = \frac{\bar{w}}{\lambda \gamma (1 - \rho^2) \eta^2 T} \quad \text{and} \quad K^* = s_0 e^{(\delta - \frac{\sigma^2}{2}) T},$$

see Theorem 5.1 for the definition of $\bar{w}$. We easily see that $K_* \leq K^*$. If $K < K_*$ or $K > K_*$, the deterministic part of the decomposition of the reservation price is a correct approximation of the price. This is illustrated in the figure below.

![Figure 9: Estimated asking reservation price of $\lambda(S_T - K)^+$ as a function of $K$ in the market situation of Table 1 with $\gamma = 0.5$ and $\rho = 0.8$, Lambert Monte Carlo method in black, Direct Monte Carlo method in blue (dashed line), and deterministic approximation in purple (dashed line). The price is simulated with $10^4$ simulations per point. The first vertical line is $K_*$ and the second one is $K^*$.](image)

7. Appendix

7.1. Proofs of Section 4

Proof of Lemma 4.2. First, $W$ is clearly $C^\infty$, being the inverse of $l$. Now, we differentiate (12) and obtain that for $x > -\frac{1}{e}$,

$$(W e^W)'(x) = 1 \iff W'(x) e^{W(x)} + W'(x) W(x) e^{W(x)} = 1$$

$$\iff W'(x) e^{W(x)} (1 + W(x)) = 1$$

$$\iff W'(x) = \frac{1}{e^{W(x)} + x} = \frac{W(x)}{x(1 + W(x))},$$

(43)
where we have used for the last equivalence the fact that \( \forall x > -\frac{1}{e} \), \( W(x) > -1 \) and (12). As \( \forall x > -\frac{1}{e} \), \( W'(x) > 0 \), \( W \) is strictly increasing. As \( W \) is \( C^\infty \), we determine that
\[
W(x) = W(0) + W'(0)x + \frac{W''(0)}{2}x^2 + x^2\epsilon(x) \quad \text{with} \quad \lim_{x \to 0} \epsilon(x) = 0.
\]
Using (13) and (14), \( W(0) = 0 \) and \( W'(0) = 1 \). Now differentiating (43), we determine that for \( x > -\frac{1}{e} \)
\[
W''(x) = -\frac{1 + W'(x)e^{W(x)}}{(x + e^{W(x)})^2}
\]
and \( W''(0) = -2 \). Finally, as \( \lim_{x \to +\infty} W(x) = +\infty \), (12) provides that
\[
\lim_{x \to +\infty} \frac{\ln(x)}{W(x)} = 1 + \lim_{x \to +\infty} \frac{\ln(W(x))}{W(x)} = 1.
\]

The minimal value of \( k_\beta \) (see (10)) is needed to prove Theorem 4.3. Fix \( \theta, v > 0 \). Let \( m \) be defined on \([0, \infty]\) by
\[
m(\beta) = \begin{cases} 
-\frac{W(\theta v\eta^2 T)}{\eta \sqrt{T}} & \text{if } \beta \in \left[ \frac{1}{\theta v} \left( \frac{W(\theta v\eta^2 T)}{\eta^2 T} + \frac{W^2(\theta v\eta^2 T)}{2\eta^2 T} \right), +\infty \right] \\
0 & \text{elsewhere}.
\end{cases}
\]

**Lemma 7.1** Let \( \beta \in [0, +\infty] \) and \( \theta, v > 0 \). Let \( u \) be any real number if \( \beta = +\infty \) and \( u = -v\beta \) otherwise. Then, \( k_\beta \) reaches its minimum value in \( m(\beta) \). Moreover,
\[
k_\beta(m(\beta)) = -\left( \theta v\beta - \frac{W(\theta v\eta^2 T)}{\eta^2 T} + \frac{W^2(\theta v\eta^2 T)}{2\eta^2 T} \right),
\]
\[
k_{+\infty}(m(+\infty)) = \frac{W(\theta v\eta^2 T)}{\eta^2 T} + \frac{W^2(\theta v\eta^2 T)}{2\eta^2 T} + \theta u.
\]

**Proof of Lemma 7.1** As \( v > 0 \), for \( z \geq 0 \), we have \( k'(z) > 0 \). Letting \( z < 0 \), as \( W \) is strictly increasing and using (13), we obtain that
\[
k'(z) \geq 0 \iff \theta v\eta^2 T \geq -\eta \sqrt{T} z e^{-\eta \sqrt{T} z}
\]
\[
\iff W(\theta v\eta^2 T) \geq W(-\eta \sqrt{T} z e^{-\eta \sqrt{T} z}) = -\eta \sqrt{T} z
\]
\[
\iff z \geq -\frac{W(\theta v\eta^2 T)}{\eta \sqrt{T}},
\]
which gives the variations of \( k \). Moreover, using (12),
\[
k\left( -\frac{W(\theta v\eta^2 T)}{\eta \sqrt{T}} \right) = \theta v e^{-W(\theta v\eta^2 T)} + \frac{W^2(\theta v\eta^2 T)}{2\eta^2 T} + \theta u
\]
\[
= \frac{W(\theta v\eta^2 T)}{\eta^2 T} + \frac{W^2(\theta v\eta^2 T)}{2\eta^2 T} + \theta u.
\]
As $k_{\infty} = k$, we determine that $k_{\beta}$ reaches its minimum value in $m(\infty)$ and that (45) holds true. We now study $k_{\beta}$ if $\beta < \infty$. As $u = -v\beta$, $k_{\beta}$ is continuous.

Assume first that $$\frac{W(\theta v^2 T)}{\eta v T} + \frac{W^2(\theta v^2 T)}{2\eta^2 T} \leq \theta v \beta.$$ We show that $k_{\beta}$ reaches its minimal value at $-\frac{W(\theta v^2 T)}{\eta v T}$. As $v, \theta > 0$, (12) implies that $$e^{-W(\theta v^2 T)} = \frac{W(\theta v^2 T)}{\theta v^2 T} \leq \beta.$$ (46)

As (46) is equivalent to $-\frac{W(\theta v^2 T)}{\eta v T} \leq \frac{\ln \beta}{\eta v T}$, (10) and (45) show that

$$k_{\beta} \left( -\frac{W(\theta v^2 T)}{\eta v T} \right) = k \left( -\frac{W(\theta v^2 T)}{\eta v T} \right) = \frac{W(\theta v^2 T)}{\eta v T} + \frac{W^2(\theta v^2 T)}{2\eta^2 T} - \theta v \beta.$$ (47)

As $k_{\beta} = k$ on $(-\infty, \frac{\ln \beta}{\eta v T})$, it also shows that $k_{\beta}$ is decreasing on $(-\infty, -\frac{W(\theta v^2 T)}{\eta v T})$ and increasing on $(-\frac{W(\theta v^2 T)}{\eta v T}, \frac{\ln \beta}{\eta v T})$. Now, on $\left( \frac{\ln \beta}{\eta v T}, +\infty \right)$, $k_{\beta}(z) = \frac{z^2}{2}$, we distinguish two cases. First, assume that $\beta \geq 1$. Then, $-\frac{W(\theta v^2 T)}{\eta v T} \leq 0 \leq \frac{\ln \beta}{\eta v T}$ and $k_{\beta}$ is increasing on $\left( \frac{\ln \beta}{\eta v T}, +\infty \right)$. Thus, $k_{\beta}$ reaches its minimum value in $-\frac{W(\theta v^2 T)}{\eta v T}$. If now $\beta < 1$, using (46) again, we find that $-\frac{W(\theta v^2 T)}{\eta v T} \leq \frac{\ln \beta}{\eta v T} \leq 0$. Therefore, $k_{\beta}$ is decreasing on $\left( \frac{\ln \beta}{\eta v T}, 0 \right)$ and increasing on $\left( 0, +\infty \right)$. Thus, 0 and $-\frac{W(\theta v^2 T)}{\eta v T}$ are two potential minima. However, using (47), we determine that

$$k_{\beta} \left( -\frac{W(\theta v^2 T)}{\eta v T} \right) \leq 0 = k_{\beta}(0),$$

and we conclude that $k_{\beta}$ reaches its minimal value at $-\frac{W(\theta v^2 T)}{\eta v T}$.

Suppose now that $\theta v \beta < \frac{W(\theta v^2 T)}{\eta v T} + \frac{W^2(\theta v^2 T)}{2\eta^2 T}$. This implies that $\beta < 1$. Indeed, using (12) and $1 + x \leq e^x$,

$$\beta < \frac{W(\theta v^2 T)}{\theta v^2 T} \left( 1 + \frac{W(\theta v^2 T)}{2} \right) = e^{-W(\theta v^2 T)} \left( 1 + \frac{W(\theta v^2 T)}{2} \right) \leq 1.$$ We want to show that $k_{\beta}$ reaches its minimum value at 0. Then, as $\beta < 1$, $k_{\beta}(0) = 0$, (44) is proven by recalling (47).

Assume first that $\beta \leq \frac{W(\theta v^2 T)}{\theta v^2 T} = e^{-W(\theta v^2 T)}$, then $\frac{\ln \beta}{\eta v T} \leq -\frac{W(\theta v^2 T)}{\eta v T}$. As $k_{\beta} = k$ on $(-\infty, \frac{\ln \beta}{\eta v T})$, $k_{\beta}$ is decreasing on $(-\infty, \frac{\ln \beta}{\eta v T})$. Moreover, on $\left( \frac{\ln \beta}{\eta v T}, +\infty \right)$, $k_{\beta}(z) = \frac{z^2}{2}$ and $k_{\beta}$ is decreasing on $\left( \frac{\ln \beta}{\eta v T}, 0 \right)$ and increasing on $\left( 0, +\infty \right)$. Thus, $k_{\beta}$ reaches its minimum value at 0. Assuming now that $\beta > \frac{W(\theta v^2 T)}{\theta v^2 T} = e^{-W(\theta v^2 T)}$, we determine that $-\frac{W(\theta v^2 T)}{\eta v T} < \frac{\ln \beta}{\eta v T} < 0$. Thus, as $k_{\beta} = k$ on $(-\infty, \frac{\ln \beta}{\eta v T})$, $k_{\beta}$ is decreasing on $(-\infty, -\frac{W(\theta v^2 T)}{\eta v T})$ and increasing on $\left( -\frac{W(\theta v^2 T)}{\eta v T}, \frac{\ln \beta}{\eta v T} \right)$. Moreover, on $\left( \frac{\ln \beta}{\eta v T}, +\infty \right)$, $k_{\beta}(z) = \frac{z^2}{2}$, and $k_{\beta}$ is decreasing on $\left( \frac{\ln \beta}{\eta v T}, 0 \right)$ and increasing on $\left( 0, +\infty \right)$. Thus, $-\frac{W(\theta v^2 T)}{\eta v T}$ and 0 are two potential minima. Using (47), we determine that

$$k_{\beta} \left( -\frac{W(\theta v^2 T)}{\eta v T} \right) > k_{\beta}(0)$$

and we conclude that $k_{\beta}$ reaches its minimum in 0. □
Proof of Theorem 4.3. Let $\beta \in [0, +\infty)$ and $\nu, v > 0$. Let $u$ be any real number if $\beta = +\infty$ and $u = -v\beta$ otherwise. Assume that (16) holds true. We set $\hat{\eta} = \eta\sqrt{T}$ and $\hat{w} = W(\theta v\hat{\eta}^2)$. Using (9), we have that

$$L_{f,\beta}(\theta) = \exp(-k_\beta(m(\beta))) \int_\mathbb{R} \exp \left[ -\theta \left( f(e^{\hat{\eta}^2}) - u - ve^{\hat{\eta}^2} \right)_y \, dz \sqrt{2\pi} \right]$$

(48)

Recall that when $\beta < +\infty$, we have chosen $u = -v\beta$. Thus, all $\beta \in [0, \infty]$, (16) and Lemma 7.1 imply that $m(\beta) = -\hat{w}/\hat{\eta}$. Using (44) and (45), we determine that

$$\exp(-k_\beta(m(\beta))) = \exp \left( -\left( \theta u + \frac{\hat{w}}{\hat{\eta}^2} + \frac{\hat{w}^2}{2\hat{\eta}^2} \right) \right) = L_{\beta}(\theta).$$

(49)

Then, (48) and (49) with the change of variable $z = y - m(\beta) = y + \hat{w}/\hat{\eta}$ and the fact that $e^{-\hat{w}} = \hat{w}/(\theta v\hat{\eta}^2)$ (see (12)) imply that

$$L_{f,\beta}(\theta) = L_{\beta}(\theta) \int_\mathbb{R} \exp \left[ -\theta \left( f \left( \frac{\hat{w}}{\theta v\hat{\eta}^2} e^{\hat{\eta}^2} \right) - u - v\frac{\hat{w}}{\theta v\hat{\eta}^2} e^{\hat{\eta}^2} \right) \right] \, dz \sqrt{2\pi}.$$ 

(50)

Letting $y \in \mathbb{R}$, we determine that

$$k_\beta \left( y - \frac{\hat{w}}{\hat{\eta}} \right) = \theta \left( u + ve^{\hat{\eta}^2} e^{-\hat{w}} \right) 1_{y \leq \frac{\ln \beta}{\hat{\eta}} + \frac{\hat{w}}{\hat{\eta}}} + \frac{(y - \frac{\hat{w}}{\hat{\eta}})^2}{2}$$

$$= \theta u \left( 1_{y \leq \frac{\ln \beta}{\hat{\eta}} + \frac{\hat{w}}{\hat{\eta}}} + \frac{\hat{w}^2}{2\hat{\eta}^2} \right) \right) \right) 1_{y > \frac{\ln \beta}{\hat{\eta}} + \frac{\hat{w}}{\hat{\eta}}} + \frac{y^2}{2}.$$ 

We remark that $\frac{\ln \beta}{\hat{\eta}} + \frac{\hat{w}}{\hat{\eta}} \geq 0$. This is trivially true if $\beta = +\infty$. If $\beta < +\infty$, as $u = -v\beta$

$$\frac{\ln \beta}{\hat{\eta}} + \frac{\hat{w}}{\hat{\eta}} \geq 0 \Leftrightarrow \beta e^{\hat{\eta}^2} = \beta \frac{\theta v^2}{\hat{w}} \geq 1 \Leftrightarrow \theta v\beta \geq \frac{\hat{w}}{\hat{\eta}^2},$$

which is true by (16). Thus,

$$k_\beta \left( y - \frac{\hat{w}}{\hat{\eta}} \right) = \theta u + \frac{\hat{w}}{\hat{\eta}^2} + \frac{\hat{w}^2}{2\hat{\eta}^2}$$

$$k_\beta \left( y - \frac{\hat{w}}{\hat{\eta}} \right) - k_\beta \left( -\frac{\hat{w}}{\hat{\eta}} \right) = \theta \left( 1_{y > \frac{\ln \beta}{\hat{\eta}} + \frac{\hat{w}}{\hat{\eta}}} e^{\hat{\eta}^2} \right) \right) \right) 1_{y > \frac{\ln \beta}{\hat{\eta}} + \frac{\hat{w}}{\hat{\eta}}} + \frac{y^2}{2}.$$ 

Assume that $\beta < +\infty$. Then,

$$y > \frac{\ln \beta}{\hat{\eta}} + \frac{\hat{w}}{\hat{\eta}} \Leftrightarrow e^{\hat{\eta}^2} > \theta e^{\hat{\eta}^2} \Leftrightarrow \frac{\hat{w}}{\hat{\eta}^2} e^{\hat{\eta}^2} > \theta v\beta.$$ 

Thus, as $u = -v\beta$

$$\left( \theta u + \frac{\hat{w}}{\hat{\eta}^2} e^{\hat{\eta}^2} \right) 1_{y > \frac{\ln \beta}{\hat{\eta}} + \frac{\hat{w}}{\hat{\eta}}} = \left( \frac{\hat{w}}{\hat{\eta}^2} \right).$$
The same obviously holds true when $\beta = +\infty$, with the convention $(-\infty)_+ = 0$. Thus, we determine that

$$\exp \left( - \left( k_\beta \left( y - \frac{\hat{w}}{\eta} \right) - k_\beta \left( \frac{\hat{\nu}}{\eta} \right) \right) \right) = \phi_\beta(y, \theta) e^{-\frac{y^2}{2}}.$$ 

Therefore, (50) implies that

$$L_{f,\beta}(\theta) = L_\beta(\theta) \int_\mathbb{R} \exp \left[ -\theta \left( f \left( \frac{\hat{w}}{\theta \nu \eta^2} e^{\eta y} \right) - u - \frac{\hat{w}}{\theta \eta^2} e^{\eta y} \right) 1_{y \leq \frac{m_\beta + \hat{\nu}}{\eta}} \right] \phi_\beta(y, \theta) e^{-\frac{y^2}{2}} \frac{dy}{\sqrt{2\pi}} = L_\beta(\theta) I_{f,\beta}(\theta).$$ 

\[\square\]

### 7.2. Proofs of Section 5

**Proof of Lemma 5.3.** As $w(\rho) > 0$ (see (15)), we determine that $d > 0$ and $g > 0$. Lemma 4.2 and $\theta(1) = 0$ imply that $d$ and $g$ are $C^1$ and that $w(\rho)$ goes to $0$ as $\theta(\rho) \eta^2 T \rho \to 1$. As $w(\rho) \to 0$, the first limits in (31) and (32) are proved. The case when $\rho$ goes to $-1^+$ is treated similarly. We deduce that $d$ and $g$ are bounded on $(-1, 1)$.

Using the Jensen inequality for the convex function $x \mapsto e^{-x}$, we obtain that

$$\mathbb{E} \left( \exp \left( - \frac{w(\rho)}{\eta^2 T} \left( e^{\eta \sqrt{T} N} - 1 - \eta \sqrt{T} N \right) \right) \right) \geq \exp \left( - \frac{w(\rho)}{\eta^2 T} \left( e^{\frac{\eta^2 T}{2}} - 1 \right) \right).$$

And so, using Remark 5.2 we determine that

$$0 \leq a(\rho) \leq \frac{\lambda e^{-\frac{\eta^2 T}{2}} w(\rho)}{\theta(\rho) \eta^2 T} \left( e^{\frac{\eta^2 T}{2}} - 1 \right) = b(\rho).$$

Then, as $e^{-x} \leq 1 - x + \frac{x^2}{2}$ for all $x \geq 0$, we have that

$$\mathbb{E} \left( \exp \left( - \frac{w(\rho)}{\eta^2 T} \left( e^{\eta \sqrt{T} N} - 1 - \eta \sqrt{T} N \right) \right) \right) \leq 1 - \frac{w(\rho)}{\eta^2 T} \left( e^{\frac{\eta^2 T}{2}} - 1 \right) + \frac{w(\rho)^2}{2\eta^4 T^2} e_2,$$

where

$$e_2 = \mathbb{E} \left( \left( e^{\eta \sqrt{T} N} - 1 - \eta \sqrt{T} N \right)^2 \right) = e^{2\eta^2 T} - 2(1 + \eta^2 T)e^{\frac{\eta^2 T}{2}} + \eta^2 T + 1.$$ 

Composing with the logarithm function and using the inequality $\ln(1 + x) \leq x$ yields

$$a(\rho) \geq \beta(\rho) - \frac{\lambda e^{-\frac{\eta^2 T}{2}} w(\rho)^2}{\theta(\rho) 2\eta^4 T^2} e_2,$$

which combined with $a(\rho) \geq 0$ yields in turn the lower bound for $a(\rho)$.
Finally, recalling (51), we determine that
\[
\lim_{\rho \to 1-} b(\rho) = \lambda e^{-rT} s_0 e^{(\nu - \eta \frac{\mu - r}{\sigma})T} e^{\frac{s^2 T}{2}} - 1
\]
\[
\lim_{\rho \to 1+} b(\rho) = \lambda e^{-rT} s_0 e^{(\nu + \eta \frac{\mu - r}{\sigma})T} e^{\frac{s^2 T}{2}} - 1.
\]

Since \( b \) is continuous on \((-1, 1)\) (recall that \( W \) is continuous), \( b \) is bounded, and since \( 0 \leq a \leq b \), \( a \) is also bounded.

**Proof of Proposition 5.9.** Lemma 5.3 and (24) show that \( D > 0 \), that \( D \leq p = D + A = D + B \) and that
\[
1 \leq \frac{p}{D} \leq 1 + \frac{e^{\frac{s^2 T}{2}} - 1}{1 + \frac{w(\rho)}{2}} \iff \frac{1 + \frac{w(\rho)}{2}}{e^{\frac{s^2 T}{2}} + w(\rho)} \leq \frac{D}{p} \leq 1.
\]

As \( w(\rho) \geq 0 \)
\[
\frac{1 + \frac{w(\rho)}{2}}{e^{\frac{s^2 T}{2}} + w(\rho)} = e^{-\frac{s^2 T}{2} - \frac{w(\rho)}{2}} e^{\frac{s^2 T}{2} + \frac{w(\rho)}{2}} \geq e^{-\frac{s^2 T}{2}}.
\]

Moreover, as \( p \leq G \) and \( G \geq 0 \), we determine that \( 1 \leq G/p \) and using (52),
\[
p = G + A - B \geq G - \frac{\lambda e^{-rT} w(\rho)^2}{\theta(\rho)} \frac{2 \eta^2 T^2 w_2}{2}.
\]

As \( p \geq D \), using (28), we determine that
\[
\frac{G}{p} \leq 1 + \frac{\lambda e^{-rT} w(\rho)^2}{\theta(\rho)} \frac{2 \eta^2 T^2 p}{2} \leq 1 + \frac{\lambda e^{-rT} w(\rho)^2}{\theta(\rho)} \frac{2 \eta^2 T^2 D}{2} = 1 + \frac{w(\rho)w_2}{2 \eta^2 T + \eta^2 T w(\rho)}.
\]

**Proof of Proposition 5.9.** Let \( \delta : \rho \in [-1, 1] \mapsto \nu - \eta \rho \frac{\mu - r}{\sigma} \). We easily determine that for all \( \rho \in (-1, 1) \),
\[
d(\rho) = \frac{\lambda e^{-rT}}{\eta^2 T \theta(\rho)} \left( \nu(\rho)(1 + w(\rho)) - \frac{\theta'(\rho)}{\theta(\rho)} w(\rho) \left( 1 + \frac{w(\rho)}{2} \right) \right).
\]

As \( s_0 \eta^2 T e^{(\delta(\rho) - \frac{s^2}{2})T} \theta(\rho) \geq 0 \), (14) implies that
\[
w'(\rho) = s_0 \eta^2 T e^{(\delta(\rho) - \frac{s^2}{2})T} \theta(\rho) \left( s_0 \eta^2 T e^{(\delta(\rho) - \frac{s^2}{2})T} \theta(\rho) \right) \left( \frac{\theta'(\rho)}{\theta(\rho)} - \frac{\eta \frac{\mu - r}{\sigma}}{T} \right)
\]
\[
= \frac{\theta'(\rho)}{\theta(\rho)} - \frac{\eta \frac{\mu - r}{\sigma}}{T} = \frac{\theta'(\rho)}{\theta(\rho)} - \frac{\frac{\theta'(\rho)}{\theta(\rho)} - \frac{\eta \frac{\mu - r}{\sigma}}{T}}{1 + \frac{1}{w(\rho)}}.
\]

28
It follows that
\[
d'(\rho) = \frac{\lambda e^{-rT}w(\rho)}{x_2^2T\theta(\rho)} \left( \frac{\theta'(\rho)}{\theta(\rho)} - \eta \frac{\mu - r}{\sigma} T - \theta'(\rho) \left( 1 + \frac{w(\rho)}{2} \right) \right) \\
= - \frac{\lambda e^{-rT}w(\rho)}{x_2^2T\theta(\rho)} \left( \eta \frac{\mu - r}{\sigma} T - \frac{\rho}{1 - \rho^2} w(\rho) \right).
\]

As \(w(\rho) > 0\) and \(\theta(\rho) > 0\) for all \(\rho \in (-1, 1)\), the sign of \(d'(\rho)\) is the sign of \(\frac{\rho}{1 - \rho^2} w(\rho) - \eta \frac{\mu - r}{\sigma} T\). Supposing first that \(\mu \geq r\), then for \(\rho \in (-1, 0)\), \(d'(\rho) \leq 0\). If now \(\rho \in (0, 1)\)
\[
d'(\rho) \geq 0 \iff W \left( s_0\eta^2 T e^{(\delta - \frac{\rho^2}{\nu^2}) T} \theta(\rho) \right) \geq \eta \frac{\mu - r}{\sigma} T \frac{1 - \rho^2}{\rho} \\
\iff s_0\eta^2 T e^{(\mu - \rho \sigma T \eta \frac{\mu - r}{\sigma} T)} \theta(\rho) \geq \eta \frac{\mu - r}{\sigma} T \frac{1 - \rho^2}{\rho} e^{\eta \frac{\mu - r}{\sigma} T} \\
\iff \lambda_1 s_0\eta^2 T e^{(\mu - \rho \sigma T \eta \frac{\mu - r}{\sigma} T)} \geq \eta \frac{\mu - r}{\sigma} T e^{\eta \frac{\mu - r}{\sigma} T} \\
\iff W \left( \lambda_1 s_0\eta^2 T e^{(\mu - \rho \sigma T \eta \frac{\mu - r}{\sigma} T)} \right) \geq \eta \frac{\mu - r}{\sigma} T \\
\iff \rho \geq \rho^*.
\]

For the second equivalence, we have composed by the reciprocal function of \(W\) (see (11); as for \(\rho \in (0, 1), \frac{\rho}{1 - \rho^2} \geq 0\), and for the fourth one by \(W\) (see (12)).

Suppose now that \(\mu \leq r\). Then, for \(\rho \in [0, 1)\), \(d'(\rho) \geq 0\). If now \(\rho \in (-1, 0)\), we prove as before that \(d'(\rho) \geq 0\) if and only if \(\rho \geq \rho^*\). The results for the variation of \(d\) are as follows.

Letting \(\kappa = \lambda_1 s_0\eta^2 T e^{(\mu - \rho \sigma T \eta \frac{\mu - r}{\sigma} T)} > 0\), then \(\rho^* = (\eta \frac{\mu - r}{\sigma} T) / W(\kappa)\). Assume now that \(-1 < \rho^* < 1\). We show that
\[
w(\rho^*) = W(\kappa)(1 - \rho^2) = w(0)(1 - \rho^2).
\]

Indeed,
\[
w(\rho^*) = W \left( \kappa e^{-\rho^* \frac{\mu - r}{\sigma} T} (1 - \rho^2) \right) = W \left( e^{-\frac{(\rho^*)^2}{W(\kappa)}} \kappa W^2(\kappa) - (\eta \frac{\mu - r}{\sigma} T)^2 \right) \\
= W \left( e^{-\frac{(\rho^*)^2}{W(\kappa)}} + W(\kappa) \right) \left( - \frac{(\eta \frac{\mu - r}{\sigma} T)^2}{W(\kappa)} + W(\kappa) \right) = - \frac{(\eta \frac{\mu - r}{\sigma} T)^2}{W(\kappa)} + W(\kappa) = W(\kappa)(1 - \rho^2),
\]

where we have used for the third equality (12), as \(\kappa > 0\), and for the fourth one, (13), as \(-1 < \rho^* < 1\), and thus \(- \frac{(\eta \frac{\mu - r}{\sigma} T)^2}{W(\kappa)} + W(\kappa) > 0\). It follows that
\[
d(\rho^*) = \frac{e^{-rT}}{\gamma^2T} W(\kappa) \left( 1 + \frac{w(\rho^*)}{2} \right) = \frac{e^{-rT}}{\gamma^2T} W(\kappa) \left( 1 + \frac{1}{2} W(\kappa) \left( 1 - \frac{(\eta \frac{\mu - r}{\sigma} T)^2}{W^2(\kappa)} \right) \right) \\
= \frac{e^{-rT}}{\gamma^2T} \left( W(\kappa) + \frac{1}{2} W^2(\kappa) - \frac{1}{2} \left( \eta \frac{\mu - r}{\sigma} T \right)^2 \right) = d(0) - e^{-rT} \frac{(\mu - r)^2}{2\gamma\sigma^2}.
\]
Using (29) and (53), we obtain that
\[ b(\rho^*) = \frac{\lambda e^{-T} w(\rho^*)}{\theta(\rho^*)} \left( e^{\frac{\nu^2}{2} T} - 1 \right) = \frac{e^{-T} w(0)}{\gamma} \left( e^{\frac{\nu^2}{2} T} - 1 \right) = b(0) \]

\[ g(\rho^*) = d(\rho^*) + b(\rho^*) = d(0) + b(0) = e^{-T} e^{\frac{\mu - r}{2} T} = g(0) = e^{-T} e^{\frac{\mu - r}{2} T} = g(0) = b(0). \]

Using (24), Remark 5.2 and (39), we determine that
\[ p = d(\rho) + a(\rho) \geq d(\rho^*) = d(0) + b(0) = d(0) - e^{-T} e^{\frac{\mu - r}{2} T} \]

and (39) follows. Then, (6) implies (40).

Proof of Lemma 5.11. We use the notation \( \epsilon \) for any function vanishing when \( T \to 0^+ \). Using Lemma 4.2, we obtain that
\[ W \left( \lambda \gamma^2 T s_0 e^{(\nu - \frac{\nu^2}{2}) T} \right) = \lambda \gamma^2 T s_0 \left( 1 + \left( \nu - \frac{\nu^2}{2} \right) T \right) \left( 1 - \lambda \gamma^2 T s_0 \right) + T \epsilon(T) \]

Using (38), we thus obtain that
\[ \rho^* = \frac{\mu - r}{\sigma} \frac{1}{\lambda \gamma s_0} \left( 1 + \left( \lambda \gamma^2 s_0 - \left( \nu - \frac{\nu^2}{2} \right) \right) T + T \epsilon(T) \right). \]

Now, as \( \nu \geq \frac{\nu^2}{2} \), the asymptotic equivalent in Lemma 4.2 implies that
\[ W(\lambda \gamma^2 T s_0 e^{(\nu - \frac{\nu^2}{2}) T}) \sim_{T \to +\infty} \left( \nu - \frac{\nu^2}{2} \right) T \] and \( \rho^* \sim_{T \to +\infty} \frac{\mu - r}{\sigma} \frac{\eta}{\nu - \frac{\nu^2}{2}}. \)

Proof of Proposition 5.12. Using (14), we see that
\[ \bar{\omega}'(\gamma) = \frac{\bar{\theta}'(\gamma)}{\bar{\theta}(\gamma)} \frac{\bar{\omega}(\gamma)}{1 + \bar{\omega}(\gamma)}. \]

The function \( \bar{d} \) is \( C^1 \) (see Lemma 4.2) and we easily ascertain that
\[ \bar{d}'(\gamma) = \frac{\lambda e^{-T} \bar{\omega}'(\bar{\gamma})}{\eta^2 T} \left( \bar{\omega}(\bar{\gamma}) \right) \left( 1 + \bar{\omega}(\bar{\gamma}) \right) - \frac{\bar{\theta}'(\gamma)}{\bar{\theta}^2(\gamma)} \left( \bar{\omega}(\gamma) + \frac{\bar{\omega}^2(\gamma)}{2} \right) \]
\[ = -\frac{\lambda e^{-T}}{2\theta^2(\rho)\eta^2 T} \bar{\omega}^2(\gamma) \bar{\theta}'(\gamma) < 0, \]

30
since $\bar{\theta} > 0$, $\bar{\theta} > 0$ and $\bar{w} > 0$ (recall (15)). Using Lemma 4.2, we determine that

$$
\bar{w}(\gamma) \sim_{\gamma \to 0^+} s_0 \eta^2 T e^{(\delta - \frac{\eta^2}{2})T} \bar{\theta}(\gamma) \quad \text{and} \quad \bar{w}(\gamma) \sim_{\gamma \to +\infty} \ln(\gamma).
$$

Thus,

$$
\bar{d}(\gamma) \sim_{\gamma \to 0^+} \lambda e^{-rT} s_0 e^{(\delta - \frac{\eta^2}{2})T} \left(1 + \frac{\bar{w}(\gamma)}{2}\right) \sim_{\gamma \to 0^+} \lambda e^{-rT} s_0 e^{(\delta - \frac{\eta^2}{2})T} \sim_{\gamma \to +\infty} \frac{\lambda e^{-rT} \ln^2(\gamma)}{2\bar{\theta}(\gamma) \eta^2 T} \sim_{\gamma \to +\infty} 0.
$$

\[\square\]

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