Stability and Thermodynamics of Brane Black Holes

E. Abdalla, B. Cuadros-Melgar, A.B. Pavan 1, C. Molina 2

1Instituto de Física, Universidade de São Paulo, C.P.66.318, CEP 05315-970, São Paulo, Brazil
2Escola de Artes, Ciências e Humanidades, Universidade de São Paulo Av. Arlindo Bettio 1000, CEP 03828-000, São Paulo-SP, Brazil
E-mail: eabdalla@fma.if.usp.br, bertha@fma.if.usp.br, alan@fma.if.usp.br, cmolina@fma.if.usp.br

Abstract. We consider scalar and axial gravitational perturbations of black hole solutions in brane world scenarios. We show that perturbation dynamics is surprisingly similar to the Schwarzschild case with strong indications that the models are stable. Quasinormal modes and late-time tails are discussed. We also study the thermodynamics of these scenarios verifying the universality of Bekenstein’s entropy bound as well as the applicability of ‘t Hooft’s brickwall method.

1. Introduction
Recent developments on higher–dimensional gravity resulted in a number of interesting theoretical ideas such as the brane world concept, a string inspired model where the Standard Model fields are confined to a three dimensional hypersurface, the brane, while gravity propagates in the full spacetime, the bulk. The simplest models in this context, proposed by Randall and Sundrum [1], describe our world as a domain wall embedded in a $Z_2$-symmetric five dimensional anti–de Sitter (AdS) spacetime. In these scenarios it is natural to ask whether matter trapped on the brane can undergo gravitational collapse and still be described by a Schwarzschild type metric as in 4 dimensions. The most natural generalization corresponds to a black string infinite in the fifth dimension, whose induced metric on the brane is purely Schwarzschild [2]. However, the corresponding Kretschmann scalar diverges at the AdS horizon at infinity turning this solution into a physically unsuitable object. A Gregory-Laflamme instability [3] near the AdS horizon has been argued to conjecture the existence of a black cigar solution with a finite extension along the extra dimension. Such a solution has been found by Casadio et al. [4, 5] using the projected Einstein equations on the brane derived by Shiromizu et al. [6]. It has the desired “pancake” horizon structure ensuring a non-singular behavior in the curvature and Kretschmann scalars at least until the order of the multipole expansion considered there. In fact, this solution belongs to a class of black hole solutions found later by Bronnikov et al. [7], who also classified the brane black holes thus obtained in two families according to the horizon order.
In this work we are interested in the study of black holes from the point of view of a brane observer, as ourselves. Firstly, we study some aspects of the black hole thermodynamics, constructed from the Bekenstein-Hawking formula \cite{8, 9}. We explicitly compute both the one-loop correction to this formula using the ’t Hooft’s brickwall method \cite{10} and the entropy bound originally proposed by Bekenstein \cite{11} in order to enforce the generalized second law of thermodynamics (GSL). Secondly, we consider the response of a brane black hole perturbation which should represent some damped oscillating signal representing the so-called quasinormal modes (QNM). They are important because they dominate in the intermediate late-time decay of a perturbation and depend only on the parameters of a black hole being, therefore, the “footprints” of this structure.

2. Brane Black Hole Solutions

The vacuum Einstein equations in 5 dimensions, when projected on a 4-dimensional spacetime lead to the gravitational equation given by \cite{6}

\[ R^{(4)}_{\mu\nu} = \Lambda_4 g^{(4)}_{\mu\nu} - E_{\mu\nu} \]  

where $\Lambda_4$ is the brane cosmological constant, and $E_{\mu\nu}$ is proportional to the (traceless) projection on the brane of the 5-dimensional Weyl tensor.

The only combination of the Einstein equations in a brane world that can be written unambiguously without specifying $E_{\mu\nu}$ is their trace \cite{4, 5, 7},

\[ R^{(4)} = 4\Lambda_4. \]  

In order to obtain four dimensional solutions of Eq. (2), we choose the spherically symmetric form of the 4-dimensional metric given by

\[ ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \]  

We relax the condition $A(r) = B(r)$, which is accidentally verified in four dimensions but, in fact, there is no reason for it to continue to be valid in this scenario.

We will center our attention in the black hole type solutions obtained by one of the following algorithms $BH1$ and $BH2$ stated below \cite{7}.

First Algorithm ($BH1$): Specify a function $A(r)$, positive and analytical in a neighborhood of the event horizon $\mathcal{R} [r]$, in such a way that $4A + rA' > 0$ in $\mathcal{R} [r]$, and $A \approx (r - r_h)^{1/s}$, $s \in \mathbb{N}$, as $r \to r_h$. Then $B(r)$ is given by the general solution of (2) with vanishing brane cosmological constant,

\[ B(r) = \frac{2Ae^{3\Gamma}}{r(4A + rA')^2} \left[ \int_{r_h}^{r} (4A + rA')(2 - r^2\mathcal{R})e^{-3\Gamma} dr + C \right], \quad \Gamma(r) = \int \frac{A'}{4A + rA'} dr. \]  

where $C$ is an integration constant. For $C \geq 0$ we have a black hole metric.

Second Algorithm ($BH2$): Specify a function $A(r)$, positive and analytical in a neighborhood of the event horizon $\mathcal{R} [r]$, in such a way that $4A + rA' > 0$ in $\mathcal{R} [r]$, and $A \approx (r - r_h)^{2/s}$ as $r \to r_h$, $s$ being an odd positive integer. Then $B(r)$ is again given by (4). The black hole metric appears when $C = 0$. 

12th Conference on Recent Developments in Gravity (NEB XII) IOP Publishing
Journal of Physics: Conference Series 68 (2007) 012043 doi:10.1088/1742-6596/68/1/012043

2
A case in point of the first algorithm is the solution with the metric element \( A(r) \) having the usual form of a Schwarzschild black hole found by Casadio, Fabbri, and Mazzacurati (CFM solution) \([4, 5]\) given by

\[
A(r) = 1 - \frac{2M}{r}, \quad B(r) = \frac{(1 - \frac{2M}{r})(1 - \frac{M\gamma}{2r})}{1 - \frac{3M}{2r}},
\]

where \( \gamma \) is an integration constant. The event horizon is localized at \( r_h = 2M \) and the singularity at \( r = 3M/2 \) instead of \( r = 0 \). Notice that the Schwarzschild solution is recovered with \( \gamma = 3 \). In this work we are restricted to the case when \( \gamma \leq 4 \).

Another interesting example of this algorithm is the metric with zero Schwarzschild mass \([7]\) given by

\[
A(r) = 1 - \frac{h^2}{r^2}, \quad h > 0, \quad B(r) = \left(1 - \frac{h^2}{r^2}\right) \left(1 + \frac{C - h}{\sqrt{2r^2 - h^2}}\right),
\]

whose horizon is at \( r = h \). The singularity occurs at \( r = h/\sqrt{2} \). This example shows that in the brane world context a black hole may exist without matter and without mass, only as a tidal effect from the bulk gravity. However, there is a special situation when \( h^2 \) can be related to a 5-dimensional mass, namely, \( C = h \). In this case Eq. (6) is the induced metric of a 5-dimensional Schwarzschild black hole, as described in \([12]\), where the chosen background was ADD-type.

3. Black Hole Thermodynamics

In order to study the thermodynamical properties of the brane black holes generated by the BH1 and BH2 algorithms, we use the following expressions of the metric coefficients near the horizon

\[
A(r) = A_2(r - r_h)^2 + O(r - r_h)^3, \quad B(r) = B_3C + B_4(r - r_h)^2 + B_5(r - r_h)^3 + O((r - r_h)^4),
\]

for BH2 algorithm, where \( C \) is the family parameter.

We will show here the calculation for the BH2 family, which turns out to be more interesting, since the metric coefficients expansion (7) is different from the standard one.

We first consider the issue of the entropy bound. The surface gravity at the event horizon is given by

\[
\kappa = \sqrt{A_2B_3C}
\]

Let us consider an object with rest mass \( m \) and proper radius \( R \) descending into a BH2 black hole. The constants of motion associated to \( t \) and \( \phi \) are \([13]\)

\[
E = \pi_t, \quad J = -\pi_\phi,
\]

where

\[
\pi_t = g_{tt} \dot{t}, \quad \pi_\phi = g_{\phi\phi} \dot{\phi} \quad \text{also} \quad m^2 = -\pi_\mu \pi^\mu.
\]

For simplicity we just consider the equatorial motion of the object, \( i.e. \), \( \theta = \pi/2 \). The quadratic equation for the conserved energy \( E \) of the body coming from (9)-(10) is given by

\[
r^2 E^2 + A_2(r - r_h)^2(J^2 + m^2r^2) = 0.
\]
The gradual approach to the black hole must stop when the proper distance from the body’s center of mass to the black hole horizon equals $R$, the body’s radius,

$$\int_{r_h}^{r_h+\delta(R)} \frac{dr}{\sqrt{B(r)}} = R. \tag{12}$$

The energy at the point of capture $r = r_h + \delta$ is given by

$$E_{\text{cap}} \approx \sqrt{A_2(J^2 + m^2 r_h^2)} \delta \frac{r_h}{r_h} \tag{13}$$

This energy is minimal for $J = 0$, such that $E_{\text{min}} = \sqrt{A_2 m \delta}$.

From the First Law of Black Hole Thermodynamics we know that

$$dM = \kappa^2 dA \tag{14}$$

where $A_r$ is the rationalized area ($\text{Area}/4\pi$), and $dM = E_{\text{min}}$ is the change in the black hole mass due to the assimilation of the body. Thus, using (8) we obtain

$$dA_r = 2mR. \tag{15}$$

Assuming the validity of the GSL, $S_{BH}(M + dM) \geq S_{BH}(M) + S$, we derive an upper bound to the entropy $S$ of an arbitrary system of proper energy $E$,

$$S \leq 2\pi E R. \tag{16}$$

This result coincides with that obtained for the purely 4-dimensional Schwarzschild solution, and it is also independent of the black hole parameters [11]. It shows that the bulk does not affect the universality of the entropy bound.

Let us find now the quantum corrections to the classical BH entropy. We consider a massive scalar field $\Phi$ in the background of a BH2 black hole satisfying the massive Klein-Gordon equation,

$$(\Box - m^2) \Phi = 0. \tag{17}$$

In order to quantize this scalar field we adopt the Statistical Mechanical approach using the partition function $Z$, whose leading contribution comes from the classical solutions of the euclidean lagrangian that leads to the Bekenstein-Hawking formula. In order to compute the quantum corrections due to the scalar field we use the ’t Hooft’s brick wall method, which introduces an ultraviolet cutoff near the horizon, such that $\Phi(r) = 0$ at $r = r_h + \varepsilon$, and an infrared cutoff very far away from the horizon, $\Phi(r) = 0$ at $r = L \gg M$.

Thus, using the black hole metric (3) and the Ansatz $\Phi = e^{-iEt}Y_{\ell m}(\theta, \phi)$, Eq.(17) turns out to be

$$\frac{E^2}{A} R + \sqrt{\frac{B}{A}} \frac{1}{r^2} \partial_r \left( r^2 \sqrt{AB} \partial_r R \right) - \left[ \frac{\ell(\ell + 1)}{r^2} + m^2 \right] R = 0. \tag{18}$$

Using a first order WKB approximation with $R(r) \approx e^{iS(r)}$ in (18) and taking the real part of this equation we can obtain the radial wave number $K \equiv \partial_r S$ as being,

$$K = B^{-1/2} \left[ \frac{E^2}{A} - \left( \frac{\ell(\ell + 1)}{r^2} + m^2 \right) \right]^{1/2}. \tag{19}$$
Now we introduce the semiclassical quantization condition,

$$\pi n_r = \int_{r_h + \varepsilon}^{L} K(r, \ell, E) \, dr \, .$$  \hfill (20)

In order to compute the entropy of the system we first calculate the Helmholtz free energy $F$ of a thermal bath of scalar particles with temperature $1/\beta$,

$$F = \frac{1}{\beta} \int d\ell (2\ell + 1) \int dn_r \ln(1 - e^{-\beta E}) \, .$$ \hfill (21)

Following 't Hooft’s method, the leading divergent contribution to $F$ (with $\varepsilon \to 0$) is

$$F_\varepsilon = -\frac{r_h^2 \pi^3}{45 \beta^4} (A_2)^{-3/2} \left( \frac{B_3 C}{\varepsilon^2} \right) \, .$$ \hfill (22)

The corresponding entropy is then,

$$S_\varepsilon = \beta^2 \frac{\partial F}{\partial \beta} = \frac{4r_h^2 \pi^3 (A_2)^{-3/2}}{45 (B_3 C)^{1/2} \varepsilon^3} \, .$$ \hfill (23)

Using the value of the Hawking temperature $T_H = 1/\beta = \kappa/2\pi$, the proper thickness $\alpha = \int_{r_h}^{r_h + \varepsilon} \frac{dr}{\sqrt{A(r)B(r)}} \approx \varepsilon/\sqrt{B_3 C}$, and the horizon area $\text{Area} = 4\pi r_h^2$ we obtain

$$S_\varepsilon = \frac{\text{Area}}{360 \pi \alpha^2} \, ,$$ \hfill (24)

which is the same quadratically divergent correction found by 't Hooft [10] for the Schwarzschild black hole and by Nandi et al. [14] for the CFM brane black hole. Thus, we see that the correction is linearly dependent on the area.

The calculation of the entropy bound and entropy quantum correction for the BH1 black hole is similar and leads to the same results shown in (16) and (24).

4. Perturbative Dynamics: Matter and Gravitational Perturbations

We consider a massless scalar field with decomposition $\Psi(t, r, \theta, \phi) = R(t, r)Y_{\ell, m}(\theta, \phi)$ obeying

$$-\frac{\partial^2 R_\ell}{\partial t^2} + \frac{\partial^2 R_\ell}{\partial r^2} = V_{sc}(r(r_*))R_\ell \, ,$$ \hfill (25)

with the tortoise coordinate

$$\frac{dr_*(r)}{dr} = \frac{1}{\sqrt{A(r)B(r)}} \, .$$ \hfill (26)

and effective potential

$$V_{sc} = A(r)\frac{\ell(\ell + 1)}{r^2} + \frac{1}{2r} \left[ A(r) B'(r) + A'(r) B(r) \right] \, .$$ \hfill (27)

In order to address the gravitational perturbation we consider a first order perturbation of $R_{\alpha\beta} = -E_{\alpha\beta}$. In general, the gravitational perturbations depend on the tidal perturbations,
namely, $\delta E_{\alpha\beta}$. Since the complete bulk solution is not known, we shall use the simplifying assumption $\delta E_{\alpha\beta} = 0$. This assumption can be justified at least in a regime where the perturbation energy does not exceed the threshold of the Kaluza-Klein massive modes. Analysis of gravitational shortcuts [15] also supports this simplification showing that gravitational fields do not travel deep into the bulk. On the other hand, since we ignore bulk back-reaction, the developed perturbative analysis should not describe the late-time behavior of gravitational perturbations. Within such premises we obtain the gravitational perturbation equation

$$\delta R_{\alpha\beta} = 0.$$ (28)

We will consider axial perturbations in the brane geometry following the treatment in [16]. They are given by an equation of motion with the form given in (25) with the effective potential

$$V_{\text{grav}}(r) = A(r)\left(\frac{(\ell + 2)(\ell - 1)}{r^2} + \frac{2A(r)B(r)}{r^2} - \frac{1}{2r}\left[A(r)B'(r) + A'(r)B(r)\right]\right).$$ (29)

In order to analyze quasinormal mode phase and late-time behavior of the perturbations, we apply a numerical characteristic integration scheme based in the light-cone variables $u = t - r_*$ and $v = t + r_*$. In addition, to check some results obtained in “time–dependent” approach we employ the semi-analytical WKB-type method [17, 18]. Both approaches show good agreement for the fundamental overtone which is the dominating contribution in the signal for intermediate late-time.

At a qualitative level we have observed the usual picture in the perturbative dynamics for all fields and geometries considered here. After the initial transient regime, it follows the quasinormal mode phase and finally a power-law tail. In contrast to the 5-dimensional model in [19], in the present context we do not observe Kaluza-Klein massive modes in the late-time behavior of the perturbations. This is actually expected, since our treatment for gravitational perturbations neglects the back-reaction from the bulk.

Although the effective potentials may be non-positive definite for certain choices of the parameters, we do not observe unbounded solutions. Furthermore, the perturbative late-time tails have power-law behavior (in one case an oscillatory decay with power-law envelope).

4.1. CFM Black Hole

A basic feature of the effective potentials $V_{\text{sc}}^{\text{CFM}}$ and $V_{\text{grav}}^{\text{CFM}}$ is that they are not positive definite. It is no longer obvious that the scalar and gravitational perturbations will be stable. However, as we can see from Fig.1 the (scalar and gravitational) perturbative dynamics is always stable.

For large values of $\ell$ an analytical expression for the quasinormal frequencies can be obtained

$$\text{Re}(\omega_n) = \frac{\ell}{3\sqrt{3M}}, \quad \text{Im}(\omega_n) = \sqrt{\frac{2\ell}{3M^2}}\left(n + \frac{1}{2}\right).$$ (30)

Moreover, with the initial data having compact support the asymptotic potential has a late-time power-law tail

$$R^{\text{CFM}}_{\ell} \sim t^{-(2\ell+3)}.$$ (31)

This is a strong indication that the models are indeed stable.
4.2. Zero Mass Black Hole

Again, the effective potentials can be non-positive definite for specific choices of parameters, as illustrated in Fig. 2. Except for $C = h$, an explicit expression for the tortoise coordinate was not found. Nevertheless, the numerical integration is possible. With the choice $C = h$ we recover some results considered in [12]. Our results show that the dynamics of the scalar and axial gravitational perturbations is always stable.

Analytical expressions for the quasinormal frequencies for the scalar and gravitational
perturbations can be obtained in the limit of large multipole index $\ell$

$$\text{Re}(\omega_n) = \frac{\ell}{2h}, \quad \text{Im}(\omega_n) = \sqrt{\frac{\ell}{2h}} \left( n + \frac{1}{2} \right).$$  \hspace{1cm} (32)

The tail contribution to the scalar decay can be analytically treated, at least in the limit where $r \gg h$

$$R_{zm}^\ell \sim t^{-(2\ell+3)} \text{ with } 0 < C < h \text{ or } C > h,$$  \hspace{1cm} (33)

$$R_{zm}^\ell \sim t^{-(2\ell+4)} \text{ with } C = h.$$  \hspace{1cm} (34)

As observed in the CFM model, for the non-extreme “zero mass” model the scalar perturbation decays as a power-law tail suggesting that the model is stable.

The qualitative picture of the field evolution changes drastically when the extreme case $(C = 0)$ is considered. If $\ell = 0$, we observe the usual power-law tail dominating the late-time decay. But when $\ell > 0$, the simple power-law tail is replaced by an oscillatory decay with a power-law envelope,

$$R_{zm}^\ell \sim t^{-3/2} \sin (\omega_\ell \times t) \text{ with } C = 0 \text{ and } \ell > 0.$$  \hspace{1cm} (35)

The angular frequency $\omega_\ell$ for large times approaches a constant and it is well approximated by a linear function of $\ell$. This result implies that the dominating contributions in the late-time decay are the modes with $\ell > 0$, i.e., the power-law enveloped oscillatory terms.

**Acknowledgments**

This work was partially supported by Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) and Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).

**References**

[1] L. Randall and R. Sundrum, *Phys. Rev. Lett.* 83, 3370 (1999); *Phys. Rev. Lett.* 83, 4690 (1999).

[2] A. Chamblin, S.W. Hawking, H.S. Reall, *Phys. Rev.* D61, 065007 (2000).

[3] R. Gregory and R. Laflamme, *Phys. Rev. Lett.* 70, 2837 (1993).

[4] R. Casadio, A. Faibbri, and L. Mazzacurati, *Phys. Rev.* D65, 084040 (2002).

[5] R. Casadio and L. Mazzacurati, *Mod. Phys. Lett.* A18, 651 (2003).

[6] T. Shiromizu, K. Maeda and M. Sasaki, *Phys. Rev.* D62, 024012 (2000).

[7] K.A. Bronnikov, H. Dehren, and V.N. Melnikov, *Phys. Rev.* D68, 024025 (2003).

[8] J. D. Bekenstein, *Phys. Rev.* D7, 949 (1973).

[9] S. W. Hawking, *Commun. Math. Phys.* 43, 199 (1975).

[10] G. ‘t Hooft, *Nucl. Phys.* B256, 727 (1985).

[11] J. D. Bekenstein, *Phys. Rev.* D23, 287 (1981).

[12] P. Kanti and R. Konoplya, *Phys. Rev.* D73, 044002 (2006).

[13] B. Carter, *Phys. Rev.* 174, 1559 (1968); R. Hojman and S. Hojman, *Phys. Rev.* D15, 2724 (1977); B. Linet, *Gen. Rel. Grav.* 31, 1609 (1999); S. Hod, *Phys. Rev.* D61, 024023 (2000); ibid. *Phys. Rev.* D61, 024018 (2000); J. D. Bekenstein and A.E. Mayo, *Phys. Rev.* D61, 024020 (2000); Bin Wang, Elcio Abdalla, *Phys. Rev.* D62, 044030 (2000); Weigang Qiu, Bin Wang, Ru-Keng Su, Elcio Abdalla, *Phys. Rev.* D64, 027503 (2001).

[14] K. Nandi, Y.-Z. Zhang, A. Bhadra and P. Mitra, gr-qc/0506023.

[15] Elcio Abdalla, Bertha Cuadros-Melgar, Sze-Shiang Feng, Bin Wang, *Phys. Rev.* D65, 083512 (2002); Elcio Abdalla, Adenauer G. Casali, Bertha Cuadros-Melgar, *Int. J. Theor. Phys.* 43, 801 (2004).

[16] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Oxford University Press (New York, 1983).

[17] B. F. Schutz and C. M. Will, *Astrophys. J.* 291, L33 (1985).

[18] S. Iyer and C. M. Will, *Phys. Rev.* D35, 3621 (1987); R. A. Konoplya, *Phys. Rev.* D68, 024018 (2003).

[19] Sanjeev S. Seahra, Chris Clarkson, Roy Maartens, *Phys. Rev. Lett.* 94, 121302 (2005).