Nonlinear interaction of acoustic waves with a spheroidal particle: radiation force and torque effects

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The nonlinear interaction of a time-harmonic acoustic wave with an anisotropic particle gives rise to the radiation force and torque effects. These phenomena are at the heart of the acoustofluidics technology, where microparticles such as cells and microorganisms are acoustically manipulated. We present a theoretical model considering a generic acoustic beam interacting with a subwavelength spheroidal particle in a nonviscous fluid. Concise analytical expressions of the radiation force and torque are obtained in the scattering dipole approximation. The radiation force is given in terms of a gradient and scattering force; while the radiation torque has two fundamental contributions, namely, the momentum arm and acoustic spin (spin-torque effect). As a practical example, we use the theory to describe the interaction of two crossed plane waves and a prolate spheroidal particle. The results reveal the particle is transversely trapped in a pressure node and is axially pushed by the radiation force. Also, the momentum arm aligns the particle in the axial direction. At certain specific positions, only the spin-torque occurs. Our findings are remarkably consistent with finite-element simulations. The success of our model enables its use as an investigation tool for the manipulation of anisotropic microparticles in acoustofluidics.

I. INTRODUCTION

The behavior of microparticles under an ultrasonic acoustic wave has been extensively analyzed in micro-acoustofluidic devices [1, 2]. Notable examples include separation of circulating tumor cells [3], cell and microparticle patterning [1, 5], assess the membrane elasticity of cell by acoustic deformation [6, 7], and selective acoustic tweezer [8]. The nonlinear wave-particle interaction gives rise to the acoustic radiation forces and torques phenomena [9]. The careful control of these effects enables particle handling in micrometer-sized cavities or microchannels with a plethora of applications in biotechnology and analytical chemistry.

The acoustic radiation forces and torques are commonly investigated considering isotropic particles, i.e., with a spherical shape. In reality, the morphology of most cells and other microorganisms have a degree of asymmetry. Prominent examples of acoustofluidic systems for manipulation of asymmetric particles include glass fibers [10], *Escherichia coli* bacterium [11], red blood cells [12], microfibers [13], and alumina microdisks [14]. Other experiments have been performed in acoustic levitation systems in air [15, 17]. Understanding how acoustical forces and torques develop on anisotropic microparticles is key to dynamic analysis, as well as to devise new applications of acoustofluidic methods. Additionally, these phenomena seem to have a crucial role in the propelling mechanisms of microswimmers under an ultrasound field [18, 19].

At first glance, the available alternative to model the wave interaction with anisotropic particles is the use of numerical techniques, such as the finite element [10, 14, 20–22] and boundary element methods, Born approximation [23], numerical quadrature [25, 26], and T-matrix approach [27]. In general, numerical methods demand high-performance computing and high memory usage for three-dimensional simulations. Moreover, it is also challenging to determine the behavior of the wave-particle system as one or more parameters vary continuously.

Early studies involving anisotropic particles dealt with the radiation torque problem on circular disks [28, 31]. Some other investigations have been surveyed in Ref. [32]. More recently, efforts have been devoted to describing the acoustic radiation force [33, 34] and torque [35–37] on spheroidal particles. These analyses rely on the partial-wave expansion of the acoustic fields. In this method, the expansion coefficients of the incoming beam (beam-shape coefficients) should be known a priori [35–37]. Numerical schemes can also be employed to compute the beam-shape coefficients [38–40], and even experimental methods [41]. Regardless these studies, the ultrasonic waves produced in acoustofluidic devices have a complex spatial form (structured waves), and the corresponding beam-shape coefficients are generally unknown. In the case of isotropic particles, the radiation force caused by structured waves can be easily computed using the Gorkov’s theory [50], which requires only the incident pressure and fluid velocity, either analytically or numerically. A similar approach was developed for the radiation torque on a spherical particle [51].

The purpose of this article is to model the radiation force and torque imparted on a subwavelength spheroid...
by an arbitrary acoustic beam. Our approach is based on the general expressions of the radiation force [34] and torque [36, 37] that depend on the expansion coefficients of the incident and scattering waves, e.g., the beamshape and scattering coefficients. These expressions are exact to the dipole moment for the scattering wave. The scattering coefficients are obtained through the boundary conditions on the particle surface [34]. Also, the relationship between the beam-shape coefficients and the incident acoustic fields (pressure and fluid velocity) is established. Strikingly, the final expressions of the radiation force and torque are given in terms of the incoming fields evaluated at the geometric center of the particle. Despite being developed for a rigid spheroid, the theory can be readily adapted to accommodate the elasticity and absorption of the particle, as well as the surrounding fluid viscosity. These effects have already been investigated for isotropic particles [52, 53].

Our model is used for the analysis of a spheroidal particle interacting with two crossed plane waves at right angle. This structured beam forms a transverse standing wave and an axial (perpendicular) traveling wave. The model predicts the radiation force pushes the particle to a transverse pressure node or antinode. The radiation torque aligns the particle at a node in broadside orientation. In contrast, the orientation in an antinode is along the axis of the standing wave. Additionally, the axial radiation force pushes the particle in the direction parallel to the pressure node. Lastly, the model predictions are verified against finite-element (FE) simulations. We find an excellent agreement between the theoretical and numerical results.

II. THEORETICAL MODEL

A. Governing equations

Consider a fluid of infinite extension characterized by an ambient density \( \rho_0 \), adiabatic speed of sound \( c_0 \), and compressibility \( \beta_0 = 1/\rho_0 c_0^2 \). Our analysis is restricted to acoustic fields of time harmonic dependence \( e^{-i\omega t} \), with \( \omega = 2\pi f \) and \( f \) being the angular and linear frequencies, respectively. The corresponding wavenumber is \( k = \omega/c_0 = 2\pi/\lambda \), where \( \lambda \) is the acoustic wavelength. The acoustic pressure and fluid velocity are expressed using the complex-phase representation, \( p(r, t) = p(r)e^{-i\omega t} \) and \( v(r, t) = v(r)e^{-i\omega t} \), respectively. In Cartesian coordinates, the position vector and fluid velocity are \( r_i e_i \) (with \( i = x, y, z \)) and \( v = v_i e_i \), where the \( e_i \) is the Cartesian unit vector. The summation over repeated indexes is automatically assumed hereafter. We also adopt the notation \( (r_x, r_y, r_z) = (x, y, z) \).

In the inviscid limit, the wave dynamics is modeled by the well-known linear acoustic equations

\[
\begin{align*}
v &= -\frac{i}{\rho_0 c_0 k} \nabla p, \quad (1a) \\
(\nabla^2 + k^2) p &= 0, \quad (1b) \\
\nabla \times v &= 0. \quad (1c)
\end{align*}
\]

The time-dependent term \( e^{-i\omega t} \) is omitted for simplicity. Equation (1c) is obtained by taking the rotational of Eq. (1a).

In the presence of an inclusion, such as a particle, an incident wave is scattered by the inclusion. The boundary condition for the scattered pressure \( p_{sc} \) at the farfield is the Sommerfeld radiation condition. In spherical coordinates \( (r, \theta, \varphi) \), this means

\[
\lim_{r \to \infty} r (\partial_r - i k) p_{sc} = 0. \quad (2)
\]

The radiation condition singles out only the solution which represents “outgoing” waves.

Another boundary condition comes from considering the particle as a rigid body. In this case, the normal component of the total velocity of the fluid, i.e. incident plus scattering contributions, should vanish on the particle surface \( S_0 \),

\[
n \cdot v \big|_{r \in S_0} = 0. \quad (3)
\]

Here \( n \) is the outward normal unit-vector on \( S_0 \).

B. Prolate spheroidal particle

A prolate spheroidal particle with a major and minor semiaxis denoted by \( a \) and \( b \), is placed in the wave path of an arbitrary incoming beam as depicted in Fig. [4]. Its
geometric center defines the particle coordinate system \((x, y, z)\). The particle surface is conveniently described in prolate spheroidal coordinates whose connection the Cartesian system is expressed by

\[
x = \frac{d}{2} \sqrt{\left(\xi^2 - 1\right)\left(1 - \eta^2\right)} \cos \varphi, \\
y = \frac{d}{2} \sqrt{\left(\xi^2 - 1\right)\left(1 - \eta^2\right)} \sin \varphi, \\
z = \frac{d}{2} \xi \eta,
\]

where \(\xi \geq 1\) is the spheroidal radial coordinate, \(-1 \leq \eta \leq 1\), and \(0 \leq \varphi \leq 2\pi\) is azimuth angle, with the interfocal distance being

\[
d = 2\sqrt{a^2 - b^2}.
\]

The particle surface is given by

\[
\xi = \xi_0 = \left[1 - \left(\frac{b}{a}\right)^2\right]^{-1/2},
\]

where \(\xi_0\) is regarded as the geometric parameter of the particle. The particle orientation follows the \(z\)-direction, \(d = d\mathbf{e}_z\). Note that we recover a sphere of radius \(r_0\) by setting

\[
d \to 0, \quad \xi_0 \to \infty, \quad \frac{\xi_0 d}{2} \to r_0.
\]

The particle dynamics is better analyzed in a fixed laboratory system \((x', y', z')\). In this reference frame, the particle orientation angle \(\alpha\) is expressed by \(d \cdot \mathbf{e}_z = d \cos \alpha\).

**C. Acoustic scattering**

The incoming and scattered pressure can be expressed by the partial wave expansion in spherical coordinates \((r, \theta, \varphi)\),

\[
p_{in} = p_0 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} j_n(kr) Y_n^m(\theta, \varphi),
\]

\[
p_{sc} = p_0 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{nm} s_{nm} h_n(kr) Y_n^m(\theta, \varphi),
\]

in which \(p_0\) is the peak pressure, \(j_n\) is the \(n\)th-order spherical Bessel function, \(h_n\) is the spherical Hankel function of the first-type, and \(Y_n^m\) is the spherical harmonics of \(n\)-order and \(m\)-degree. The quantity \(a_{nm}\) is known as the beam-shape coefficient. The scattering coefficients \(s_{nm}\) can be obtained from the boundary condition in Eq. (3).

As we advance to consider a particle much smaller than the wavelength, i.e., the so-called long-wavelength limit, only the monopole \((n = 0)\) and dipole \((n = 1)\) modes of the multipole expansion in [34] suffice to describe the acoustic scattering [54]. We define the smallness parameter of the particle as

\[
\epsilon = \frac{\xi_0 kd}{2} = ka \ll 1.
\]

We shall use \(\epsilon\) as the expansion parameter in the long-wavelength approximation.

Using the partial wave expansion in spheroidal coordinates \((\xi, \eta, \varphi)\), one can show that the monopole and dipole scattering coefficients of a rigid spheroid are given by [34]

\[
s_{00} = -\frac{ie^3}{3} f_{00} - \frac{\epsilon^6}{9} f_{00}^2, \\
\]

\[
s_{10} = \frac{ie^3}{6} f_{10} - \frac{\epsilon^6}{36} f_{10}^2, \\
\]

\[
s_{1, -1} = s_{11} = \frac{ie^3}{12} f_{11} - \frac{\epsilon^6}{144} f_{11}^2,
\]

with \(i\) being the imaginary unit. The dipole coefficients of the modes perpendicular to the axial direction \((s_{1,1} \text{ and } s_{1,1})\) are degenerated due to the axial symmetry of the particle. The scattering factors \(f_{00}, f_{10}, \text{ and } f_{11}\) of Ref. [34] are re-written here as

\[
f_{00} = 1 - \xi_0^{-2}, \\
\]

\[
f_{10} = \frac{2}{3 \xi_0^2} \left[ \frac{\xi_0}{\xi_0^2 - 1} - \ln \left( \frac{\xi_0 + 1}{\sqrt{\xi_0^2 - 1}} \right) \right]^{-1}, \\
\]

\[
f_{11} = \frac{8}{3 \xi_0^2} \left[ \frac{2 - \xi_0^2}{\xi_0 (\xi_0 - 1)} + \ln \left( \frac{\xi_0 + 1}{\sqrt{\xi_0^2 - 1}} \right) \right]^{-1}.
\]

These functions depend solely on the geometric factor \(\xi_0\).

To recover the scattering coefficients of a spherical particle, we take the limit \(\xi_0 \to \infty\) in [11], which yields

\[
f_{00} = 1, \quad f_{10} = 1, \quad f_{11} = 2.
\]

It immediately follows from [10] that the dipole scattering coefficients of a sphere are all degenerated, as expected,

\[
s_{10} = s_{1, -1} = s_{11}, \quad \text{(sphere)}.
\]

**D. Multipole expansion of the incident beam**

The beam-shape coefficients of an incident beam is obtained as follows. Multiplying Eq. (8a) by \(Y_n^{m*}\), integrating the result over the unit-sphere, and using the orthogonal relation of the spherical harmonics

\[
\int_0^{2\pi} \int_0^\pi Y_n^m(\theta, \varphi) Y_n^{m'}(\theta, \varphi) \sin \theta \, d\theta \, d\varphi = \delta_{nm} \delta_{mm'},
\]

(with \(\delta_{nm}\) being the Kronecker delta function), we find [44]

\[
a_{nm} = \frac{1}{p_0 j_n(kr)} \int_0^\pi \int_0^{2\pi} p_n(kr, \theta, \varphi) Y_n^{m*}(\theta, \varphi) \sin \theta \, d\theta \, d\varphi,
\]

(14)
where asterisk denotes complex conjugate. The beam-shape coefficient can be calculated for any radial distance $kr$. In particular, one can relate them with the acoustic pressure evaluated in the origin of the particle frame, $r = 0$. We anticipate here that the acoustic radiation force depends on the beam-shape coefficients to the quadrupole approximation \cite{51}. Therefore, we truncate the Taylor expansion of the incident pressure around $r = 0$ as

$$p_{\text{in}}(r) = p_{\text{in}}(0) + r_i \partial_i p_{\text{in}}|_{r=0} + \frac{1}{2!} r_i r_j \partial_i \partial_j p_{\text{in}}|_{r=0}. \quad (15)$$

with $i, j = x, y, z$. Before proceeding, we need the asymptotic expansion of the spherical Bessel function for $r \to 0$, in which $\Gamma(n)$ is the gamma function. Replacing Eqs. (15) and (16) into Eq. (14) and evaluating the integrals with $r_x = r \sin \theta \cos \varphi$, $r_y = r \sin \theta \sin \varphi$, and $r_z = r \cos \theta$, we find beam-shape coefficient up to the quadrupole order,

$$a_{00} = \frac{4\pi}{p_0} p_{\text{in}}, \quad (17a)$$
$$a_{10} = 2i \sqrt{3} \frac{\rho_0 c_0}{p_0} v_{\text{in},z}, \quad (17b)$$
$$a_{1\pm1} = i \sqrt{6} \frac{\rho_0 c_0}{p_0} (\mp v_{\text{in},x} + i v_{\text{in},y}), \quad (17c)$$
$$a_{20} = -i \sqrt{5} \frac{\rho_0 c_0}{kp_0} (\partial_x v_{\text{in},x} + \partial_y v_{\text{in},y} - 2 \partial_z v_{\text{in},z}), \quad (17d)$$
$$a_{2\pm1} = i \sqrt{30} \frac{\rho_0 c_0}{kp_0} (\mp \partial_z v_{\text{in},x} + i \partial_z v_{\text{in},y}), \quad (17e)$$
$$a_{2\pm2} = i \sqrt{15} \frac{\rho_0 c_0}{2kp_0} (\partial_x v_{\text{in},x} - \partial_y v_{\text{in},y} \mp 2i \partial_z v_{\text{in},y}). \quad (17f)$$

In this derivation, we assumed the fluid flow is irrotational–see Eq. (1c). We remark the acoustic fields of (17) are evaluated at $r = 0$.

### E. Acoustic radiation force

As the linear momentum flux carried by a wave is exchanged with the particle, the radiation force (a time-average quantity over the wave period $2\pi/\omega$) appears on the particle \cite{17},

$$\mathbf{F}_{\text{rad}} = -\int_{S_1} \text{Re} \left[ \left( \frac{\beta_0 |p|^2}{4} - \frac{\rho_0 |v|^2}{4} \right) \mathbf{I} + \frac{\rho_0}{2} \mathbf{v} \mathbf{v}^* \right] \cdot e_r \, dS, \quad (18)$$

where $S_1$ is a spherical surface that encloses the particle, $\mathbf{I}$ is the unit tensor given in Eq. (C2), and $dS$ is the area element. The combination of two vectors as $\mathbf{v} \mathbf{v}^*$ forms a dyad, i.e., a second-rank tensor–see details in Appendix [C]. The total pressure and fluid velocity are $p = p_{\text{in}} + p_{\text{sc}}$ and $\mathbf{v} = v_{\text{in}} + v_{\text{sc}}$, respectively.

In the long-wavelength limit $\epsilon \ll 1$, the Cartesian components of the acoustic radiation force exerted on a spheroidal particle are expressed by \cite{34}

$$F_x + i F_y = \frac{i E_0}{2k^2} \left[ \sqrt{\frac{2}{3}} \left( a_{00} a_{1,-1} (s_{11}^* + s_{11} + 2s_{00}s_{11}) + a_{00} a_{11} (s_{00} + s_{11}^* + 2s_{00}s_{11}) \right) \right. \right.$$  

$$+ \sqrt{\frac{2}{5}} \left( a_{10} a_{21} s_{10} + a_{2,-1} a_{10}^* s_{10} + \sqrt{2} [a_{11} a_{22} s_{11} + a_{2,1} a_{11}^* s_{11}^*] \right) + \sqrt{\frac{2}{15}} \left( a_{20}^* s_{11} a_{1,-1} + a_{20} a_{11}^* s_{11}^* \right) \right], \quad (19a)$$

$$F_z = \frac{E_0}{k^2} \text{Im} \left[ \frac{2}{\sqrt{15}} a_{10} a_{20} s_{10} + \frac{1}{\sqrt{5}} (a_{21} a_{10}^* s_{10} + a_{11} a_{21}^* s_{11}^*) + \frac{1}{\sqrt{3}} a_{00} a_{10}^* (s_{11} + s_{11}^* + 2s_{00}s_{10}) \right], \quad (19b)$$

where the asterisk denotes complex conjugation, and $E_0 = \beta_0 p_0^2/2$ is the characteristic energy density of the incident wave.

To obtain the radiation force as a function of the incoming acoustic fields, we replace the beam-shape coefficients from (19) by (17). We also use the Helmholtz equation for the incoming pressure $\partial_i \partial_i p_{\text{in}} = -k^2 p_{\text{in}}$.
Accordingly, the radiation force is given by

\[
\mathbf{F}^{\text{rad}} = -\pi a^3 \text{Re} \left[ \mathbf{e}_i \cdot \nabla \mathbf{r} \right] = \mathbf{F}^{\text{grad}} + \mathbf{F}^{\text{sc}}.
\]

The scattering force results from the self-interaction of the scattering wave, whereas the gradient force comes from the incoming and scattering wave interaction. Furthermore, the scattering force is much weaker than the gradient one by a factor of \( \epsilon^3 \). The monopole tensors (\( \mathbf{Q}^{\text{grad}} \) and \( \mathbf{Q}^{\text{sc}} \)) and dipole vectors (\( \mathbf{D}^{\text{grad}} \) and \( \mathbf{D}^{\text{sc}} \)) are expressed by

\[
\mathbf{Q}^{\text{grad}} = \frac{2}{3} f_{00} p_{in}(0) \mathbf{e}_i \mathbf{e}_i,
\]

\[
\mathbf{Q}^{\text{sc}} = -\frac{f_{00}}{9} p_{in}(0) \left[ (2 f_{00} + f_{11}) (e_x e_x + e_y e_y) + 2(f_{00} + f_{10}) e_z e_z \right],
\]

\[
\mathbf{D}^{\text{grad}} = \frac{f_{11}}{2} \left[ v_{in,x}(0) e_x + v_{in,y}(0) e_y \right] - f_{10} \mathbf{v}_{in,z}(0) e_z,
\]

\[
\mathbf{D}^{\text{sc}} = -\frac{1}{6} \left[ \frac{f_{11}}{4} \left[ v_{in,x}(0) e_x + v_{in,y}(0) e_y \right] + f_{10} \mathbf{v}_{in,z}(0) e_z \right].
\]

Note that \( \mathbf{e}_i \mathbf{e}_j \) is a dyad, which forms the Cartesian basis of a second-rank tensor. It is straightforward to show from (20) that the gradient force is expressed by

\[
\mathbf{F}^{\text{grad}} = -\nabla U(0),
\]

\[
U = \pi a^3 \left[ \frac{\beta_0 f_{00}}{9} |p_{in}|^2 - \frac{\rho_0}{2} \left( \frac{f_{11}}{2} (|v_{in,x}|^2 + |v_{in,y}|^2) + f_{10} |v_{in,z}|^2 \right) \right].
\]

Considering an standing wave, the corresponding amplitude \( p_{in} \) is a real-valued function. Consequently, \( \nabla p_{in} \) and \( \mathbf{Q}^{\text{sc}} \) are also real-valued quantities. We thus conclude that \( \text{Re}[\mathbf{Q}^{\text{sc}} \cdot \nabla p_{in}] = 0 \) and \( \text{Re}[\mathbf{D}^{\text{sc}} \cdot \nabla \mathbf{v}_{in}] = 0 \), so only the gradient force remains.

On the contrary, for traveling waves, which possess complex amplitude, say, a plane wave \( e^{i k \cdot r} \), the gradient operator becomes \( \nabla \rightarrow i \mathbf{k} \), then \( \mathbf{F}^{\text{grad}} = 0 \).

### F. Acoustic radiation torque

The acoustic radiation torque exerted on the particle is given by [31]

\[
\mathbf{\tau}^{\text{rad}} = -\frac{1}{2} \text{Re} \int_{S_1} \mathbf{r} \times \rho_0 \mathbf{v}^* \cdot \mathbf{e}_r \, dS.
\]
Hence, the corresponding pressure field is
\[ \text{we introduce the (clockwise) rotation matrix around the direction.} \]

antinode regarding the particle center in the transverse direction.
\[ = \text{clockwise rotation matrix around the direction.} \]

We see that the wave interference sets in a standing wave. In Fig. 3, we sketch wave-particle interaction. The wave vectors in the laboratory frame are
\[ \begin{align*}
  k'_x &= \frac{k\sqrt{2}}{2} (e_{x'} + e_{z'}), \\
  k'_y &= \frac{k\sqrt{2}}{2} (-e_{x'} + e_{z'}). 
\end{align*} \]

Hence, the corresponding pressure field is
\[ p_{in} = \frac{p_0}{2} \left[ e^{ik'_x (r' + h')} + e^{ik'_y (r' + h')} \right], \]

\[ = p_0 \cos \left[ \frac{k}{\sqrt{2}} (x' + h) \right] e^{i k z'}. \]

We see that the wave interference sets in a standing wave along the transverse direction \((x')\) axis. The offset vector \(h' = he_{x'}\) gives the position of the nearest pressure antinode regarding the particle center in the transverse direction.

To calculate the wave vectors in the particle system, we introduce the (clockwise) rotation matrix around the \(y'\) axis,
\[ R_{y'}(\alpha) = \begin{pmatrix}
  \cos \alpha & 0 & -\sin \alpha \\
  0 & 1 & 0 \\
  \sin \alpha & 0 & \cos \alpha
\end{pmatrix}. \]

From (27), we obtain the wave vectors in the particle system as
\[ \begin{align*}
  k_1 &= R_{y'}(\alpha) k'_1, \\
  k_2 &= R_{y'}(\alpha) k'_2
\end{align*} \]

Also, the offset parameter becomes \(h = h \cos \alpha e_x + \sin \alpha e_z\). We thus obtain the incident pressure in the particle frame,
\[ p_{in} = \frac{p_0}{2} \left[ e^{ik_1 (r + h)} + e^{ik_2 (r + h)} \right], \]

\[ = \frac{p_0}{2} \left[ e^{ik([x+z] \cos \alpha - (x-z) \sin \alpha + h)/\sqrt{2}} \right. \]

\[ + e^{-ik([x-z] \cos \alpha + (x+z) \sin \alpha + h)/\sqrt{2}} \]. \]

The related fluid velocity is readily calculated by inserting this equation into Eq. (14),
\[ v_{in} = \frac{p_0}{2p_0 c_0 k} \left[ k_1 e^{ik_1 (r + h)} + k_2 e^{ik_2 (r + h)} \right]. \]

The linear momentum flux of the incoming beam is derived by replacing Eq. (32) into (25a),
\[ P(0) = \frac{E_0}{4} \text{Re} \left[ k_1 k_1 h + k_2 k_2 h + k_3 k_4 e^{i(k_1 - k_2) h} \right. \]

\[ + k_2 k_1 e^{-i(k_1 - k_2) h}]. \]
π/κ

When the particle is trapped in a pressure node (\(h\) noting that by substituting Eqs. (33) and (35) into Eq. (26), and fluid velocity polarizabilities are also shown for in a node and antinode are illustrated in Fig. 4. The local (spin down).

The incoming beam alongside the particle orientations \(2\pi \), and in counterclockwise manner at \((\text{spin up})\), and in clockwise direction at \((\text{spin down})\). The particle spins in clockwise direction at \(\alpha = \pi/2\) or antinode \((h = 0)\), the radiation torque becomes

\[
\tau^\text{rad} = -\tau^\text{rad}_\text{node} = -\frac{\pi a^3}{4} E_0 \alpha \sin 2\alpha \, e_y'.
\]

These torques are caused exclusively by the momentum arm effect. At the end, the particle will be aligned to broadside orientation \((\alpha = 0)\) or in a pressure node, and parallel to the standing wave axis \((\alpha = \pi/2)\) in a pressure antinode.

Another interesting field position is \(h = h_\pm = \pm 2\pi /4k\). In this case, only the spin-induced torque arises,

\[
\tau^\text{spin}_\pm = ±\frac{3}{48} \pi a^3 E_0 \alpha^2 e_y'.
\]

This torque sets the particle to rotate around the minor axis. The particle spins in clockwise direction at \(h = h_+\) (spin up), and in counterclockwise manner at \(h = h_-\) (spin down).

The incoming beam alongside the particle orientations in a node and antinode are illustrated in Fig. 4. The local fluid velocity polarizabilities are also shown for \(h = h_\pm\). In this case, the velocities rotate circularly.

C. Radiation force

The transformation that furnishes the radiation force in the laboratory frame is

\[
F^\text{rad'} = R_{y'}(-\alpha) F^\text{rad}.
\]

Here the matrix \(R_{y'}(-\alpha)\) defined in Eq. (29) represents a counterclockwise rotation by an angle \(\alpha\) around the \(y'\)-axis.

In the particle frame, the gradient force is derived by substituting Eqs. (31) and (32) into (22). After some straightforward calculations, we arrive at

\[
F^\text{grad} = \frac{\epsilon}{12\sqrt{2}} \pi a^2 E_0 \left[8f_{00} + 3(f_{11} - 2f_{10})\cos 2\alpha\right] \\
\sin(2\sqrt{2}k) \left(\cos \alpha e_x + \sin \alpha e_z\right).
\]

Inserting Eq. (40) into (39), we obtain the transverse radiation force in the laboratory frame,

\[
F^\text{grad'} = \epsilon \pi a^2 E_0 \Phi_{ac} \sin(2\sqrt{2}k) e_{x'},
\]

\[
\Phi_{ac} = \frac{1}{12 \sqrt{2}} \left[8f_{00} + 3(f_{11} - 2f_{10})\cos 2\alpha\right].
\]

The sign of the acoustophoretic factor \(\Phi_{ac}\) determines whether the particle will be trapped in a pressure node \((\Phi_{ac} > 0)\) or antinode \((\Phi_{ac} < 0)\). Moreover, the radiation force varies linearly with frequency, \(\epsilon \sim \omega\). We will analyze to the axial radiation force later.

We plot the acoustophoretic factor \(\Phi_{ac}\) versus \(\xi_0\) in Fig. 5. No significant difference is noted as the orientation changes from 0 to \(\pi/2\). The factor is positive; thus, the particle will be transversely trapped in a pressure node. As the particle approaches a spherical shape \(\xi_0 \rightarrow \infty\), then \(\Phi_{ac} \rightarrow \sqrt{2}/3\). In the other extreme, as \(\xi_0 \rightarrow 1\), we have \(\Phi_{ac} \rightarrow 0\). Hence, no transverse radiation force is produced on a thin line particle.

Now we take a closer look at the axial radiation force as the particle is trapped in a pressure node. This force is in fact the scattering force defined in (20). Again, we insert Eqs. (31) and (32) into (20) and set \(h = \pi /\sqrt{2k}\)
the axial radiation force in the laboratory frame, \( Q \) asymptotically approaches \( Q_{\text{app}} \).

Applying the rotation operator \( R \) by (scattering force of Eq. (43a)).

\[
\pi a = \frac{\pi a^2 \varepsilon^4 E_0}{24 \sqrt{2}} \left[ 4 f_{10}^2 + (f_{11}^2 - 4 f_{10}^2) \cos 2\alpha \right] \cos \alpha.
\]

We see this connection, we multiply Eq. (15b) of [61] by \( \rho \) and set the parameters, in the paper’s notation, to \( \rho = 1/2, \kappa = -1 \) (rigid sphere) and \( \gamma = \pi/4 \).

\[
F^\text{scat} = \frac{\pi a^2 \varepsilon^4 E_0}{24 \sqrt{2}} \left[ 4 f_{10}^2 + (f_{11}^2 - 4 f_{10}^2) \cos 2\alpha \right] \cos \alpha.
\]

\[
F^\text{rad} = \frac{\pi a^2 \varepsilon^4 E_0}{48 \sqrt{2}} \left[ 4 f_{10}^2 + (f_{11}^2 - 4 f_{10}^2) \cos 2\alpha \right] \cos \alpha.
\]

The quantity \( Q_{\text{rad}} \) is referred to as the dimensionless radiation force efficiency. We remark the scattering force is related to the linear momentum flux carried away by the scattered waves. This connection can be understood from the radiation force relationship with the scattering cross-section for a rigid spheroid as obtained in Refs. [61]. In turn, the scattering cross-section of a rigid spheroid scales as \( \pi a^2 (ka)^4 \) [60], which is the same dependence seen in the scattering of Eq. (43a).

In Fig. 6 we plot the radiation force efficiency divided by \( (ka)^4 \) versus with \( \xi_0 \). The efficiency is positive and asymptomatically approaches \( Q_{\text{rad}}/(ka)^4 \approx 1/6\sqrt{2} \) as \( \xi_0 \to \infty \). Also, it goes to zero as \( \xi_0 \to 1 \). We see that \( Q_{\text{rad}} \) is positive with no significant difference as the orientation changes from 0 to \( \pi/2 \). Thereby, the force pushes the particle along the direction of the traveling component of the incoming wave.

An important aspect of Eq. (43a) is the possibility to recover the axial radiation force imparted to a rigid spherical particle as obtained in Ref. 61. By replacing the scattering factors of [12] for a spherical particle of radius \( r_0 \) into (43a), we encounter \( F^\text{sphe} = \pi r_0^2 (kr_0)^4 E_0/6\sqrt{2} \). This result agrees with Ref. 61. To see this connection, we multiply Eq. (15b) of 61 by \( \pi r_0^2 E_0/4 \), and set the parameters, in the paper’s notation, to \( \rho = 1/2, \kappa = -1 \) (rigid sphere) and \( \gamma = \pi/4 \).

### TABLE I. Parameters used in the finite-element simulations in Comsol at room temperature and pressure.

| Parameter                      | Value               |
|--------------------------------|---------------------|
| Spheroidal particle            |                     |
| Major semiaxis \((a)\)        | 47.14 \(\mu m\)    |
| Minor semiaxis \((b)\)        | 30.71 \(\mu m\)    |
| Geometric parameter \((\xi_0)\)| 1.3181              |
| Size parameter \((\epsilon)\)  | 0.2                 |
| Medium (water)                 |                     |
| Density \((\rho_0)\)           | 999.66 \(kg \cdot \text{m}^{-3}\) |
| Speed of sound \((c_0)\)       | 1481 \(m \cdot \text{s}^{-1}\) |
| Compressibility \((\beta_0)\)  | 0.456 \(\text{GPa}^{-1}\) |
| Domain radius                  | 5\(a = 235.7 \mu m\) |
| Radius of \(S_1\)             | \(\alpha + b/4 = 54.82 \mu m\) |
| Maximum element size inside \(V_1\) | \(b/18 = 1.706 \mu m\) |
| Maximum element size outside \(V_1\) | \(\lambda/48 = 30.71 \mu m\) |
| PML depth (15 layers)          | 2\(a = 94.28 \mu m\) |
| Acoustic wave                  |                     |
| Pressure peak \((p_0)\)        | 100 \(kPa\)        |
| Acoustic energy density \((E_0)\)| 2.28 \(J \cdot \text{m}^{-3}\) |
| Linear frequency \((f)\)       | 1 \(\text{MHz}\)    |
| Wavenumber \((k)\)            | 4242.5 \(m^{-1}\)  |
| Wavelength \((\lambda)\)      | 1.481 \(\text{mm}\)  |
| Computer system                |                     |
| CPU                            | Xeon E5-2690 3.00GHz |
| Operating system               | Linux               |
| Maximum memory usage           | \(\sim 128 \text{GB}\) |
| Computational time per dataset | \(\sim 20 \text{min}\) |

### D. Finite-element simulations

We performed FE simulations of the wave-particle interaction in water. A set of full 3D simulations were devised in Comsol Multphysics (Comsol, Inc., USA). The FE model used in our simulations is outlined as follows. A spherical region of radius \(R\) is defined as the simulation domain. The particle is placed at the center of this region. We set a spherical surface \(S_0\) as the integration surface to compute the radiation force and torque as prescribed in Eqs. [18] and [22]. The volume between the particle surface \(S_0\) and \(S_1\) is denoted by \(V_1\). To mimic the Sommerfeld radiation condition given in Eq. [2] for
the scattered waves, we use the perfect matched layer (PML). The rigid boundary condition for the particle as given in Eq (3) is assumed. The sketch of the FE model is displayed in Fig. 7. The incident beam is set as the background pressure in the particle frame. The total pressure and fluid velocity fields are then computed and used to calculate the radiation force and torque. The physical parameters were inspired on those reported for acoustofluidic devices [62].

To verify the correctness of the numerical solutions, we performed some convergence tests by varying some parameter such as the domain radius, mesh density, and PML depth. The parameter variation is set up to the PML. The simulational parameters are presented in Table I. The simulational parameters are presented in Table I. More details on convergence analysis to radiation force problems can be found in [20]. Typical accuracy achieved in our tests is less than 1% for effects of order $O(1)$ and $O(\epsilon^3)$.

In our simulations, $\delta/b \ll 1$, viscous torque can be discarded. In the simulations, $\delta/b = 10^{-2}$, and thus, we may neglect viscous torque effects here.

In Fig. 8, we plot the transverse radiation force $F_{x^\text{rad}}$ as a function of the scaled distance $kh$. The particle (gray ellipse) has a geometric parameter of $\xi_0 = 1.3181$ with orientation $\alpha = 0$. The simulative parameters are presented in Table I.

\begin{align}
\text{NRMSE}(\%) &= \frac{100}{x_{\text{max}} - x_{\text{min}}} \sqrt{\frac{1}{N} \sum_{n=1}^{N} \left( x_n^{\text{theory}} - x_n^{\text{num}} \right)^2},
\end{align}

in which $N$ is the number of sampling points, $x_{\text{max}} = \max\{x_n^{\text{theory}}\}$ and $x_{\text{min}} = \min\{x_n^{\text{theory}}\}$.

Importantly, both the theoretical and numerical models consider the inviscid approximation. Although, actual fluids may support shear stress within the particle boundary layer, which may cause viscous torques [63]. When the boundary layer $\delta = (2\mu_0/\rho_0\omega)^{1/2}$ ($\mu_0$ is the dynamic viscosity of the fluid), is much smaller than the particle size, say $\delta/b \ll 1$, viscous torque can be discarded. In our simulations, $\delta/b = 10^{-2}$, and thus, we may neglect viscous torque effects here.

In Fig. 8, we show how the radiation torque varies as a function of the scaled distance $kh$. The particle (gray ellipse) has a geometric parameter of $\xi_0 = 1.3181$, which corresponds to the maximum amplitude of the dipole-difference factor seen in Fig. 2 and consequently the maximum of the radiation torque in Eq (36). The radiation force is evaluated with the equations of (41). We see the particle will be trapped in the pressure node $kh = \pi/\sqrt{2} = 0.71\pi$. Both visual inspection and NRMSE = 0.412% indicate the theoretical result is in remarkable agreement with the FE data.

The axial radiation force ($F_{z^\text{rad}}$) in a pressure node versus particle orientation is depicted in Fig. 9. This force was calculated with the equations of (43). The NRMSE = 5.15% is one order of magnitude above the error of the transverse force in Fig. 8. This is so because the scattering force is more sensitive to numerical errors, since it is much weaker than the transverse counterpart. The maximum and minimum force is experienced by the particle as its orientation is $\alpha = 0, \pi/2$, respectively.

In Fig. 9 we show how the radiation torque varies with the particle orientation in a pressure node. The torque is evaluated using Eq. (37), and is caused by the momentum arm only. As previously discussed, the preferential orientation of the particle is $\alpha = 0$, i.e., the particle will be aligned perpendicularly to the standing
FIG. 9. The axial radiation force \( F_{\text{sc}} \) at a pressure node versus particle orientation. The particle (gray ellipses) has a geometric parameter of \( \xi_0 = 1.3181 \). The simulational parameters are surveyed in Table I.

FIG. 10. The theoretical and numerical results of the radiation torque versus orientation angle \( \alpha \). The particle (gray ellipses) has a geometric parameter of \( \xi_0 = 1.3181 \) and is trapped in a pressure node. The simulational parameters are summarized in Table I.

wave axis. Again, we find an excellent agreement between the theory and numerical results, provided that NRMSE = 0.57%.

The amplitude of the spin-torque is shown in Fig. 11 versus the particle position \( kh \). The particle orientation is fixed to \( \alpha = 0 \). Equation (36) is used to calculate the radiation torque. The theoretical and numerical data are in good match with NRMSE = 4.51%. This error is ten times larger than that of the momentum arm torque in Fig. 10. Possibly, because the spin-torque is a weaker phenomenon and then is more sensitive to undesired wave reflections from the PML.

FIG. 11. The theoretical and numerical spin-torque on the spheroidal particle with a fixed orientation \( \alpha = 0 \). The particle (gray ellipse) has a geometric parameter of \( \xi_0 = 1.3181 \) and its position varies from an antinode to a node. The simulational parameters are shown in Table I.

scribe these phenomena considering subwavelength particles and an incoming wave of arbitrary character. The main contribution of our work are the laconic expressions of the radiation force in Eqs. (20) and (22), and radiation torque in Eq. (26). In these expressions, the acoustic fields can be described either analytically, numerically, or experimentally. Moreover, the particle anisotropy allows the rise of the radiation torque from the momentum arm and acoustic spin of the incoming wave. Whereas, the radiation force is composed of the gradient and scattering forces similarly to the spherical particle case [51].

Our theory is successfully employed to analyze the interaction between a rigid spheroidal particle and two plane waves crossing at right angle. This composed beam is one of the simplest waves to possess acoustic spin [55]. When the particle is trapped in a pressure node, the induced radiation torque is solely due to the momentum arm caused by linear momentum flux. However, as the particle is placed at \( \pi \sqrt{2}/4k \) along the transverse direction (i.e., the \( x \) axis), a clockwise spin-torque is activated. The spin flips as the particle changes place to \(-\pi \sqrt{2}/4k\). Importantly, the spin-torque is \( (ka)^3 \) weaker than the momentum arm contribution. The most prominent effect of the acoustic radiation force is to trap the particle in a transverse pressure node. The axial radiation force in a pressure node follows the traveling wave direction. The theoretical results are verified against FE simulations based on a full three-dimensional model. An excellent agreement is found between theory and numerical experiments.

The next level to be considered in our model is to bring in the elastic properties of particles. These features will convey the theory closer to acoustofluidic experiments with cells and other microorganisms. Moreover, energy absorption by the particle can be as well considered through complex wave numbers. The absorption is known to enhance the acoustic radiation force [52].

IV. CONCLUDING REMARKS

The acoustic radiation force and torque are the by-products of the nonlinear interaction of an incoming wave with a spheroidal particle. We develop a theory to de-
We turn to the acoustic spin term. The $O(\epsilon)$ flux is \( \mathbf{P} \cdot \mathbf{e}_z = \frac{\rho_0}{2} \text{Re} \mathbf{v}_{in,y}^* \mathbf{v}_{in,z} \), which allows us to re-write the $O(\epsilon^3)$ term in the right-hand side of Eq. \( \text{B2} \).

\[
\frac{\rho_0}{2} \text{Re} \mathbf{v}_{in,y}^* \mathbf{v}_{in,z} = -\left( \mathbf{e}_x \times (\mathbf{P} \cdot \mathbf{e}_z)_x \right)_y .
\]

We turn to the acoustic spin term. The $O(\epsilon^5)$ term of
Eq. (B2) can be written as

$$\rho_0 \text{Im} \left( \frac{v_{in,y}^* v_{in,z}}{v_{in,x}^* v_{in,z}} \right) = \frac{\rho_0}{2} \text{Im} \left( \frac{(v_{in,x}^* v_{in})_x}{(v_{in,x}^* v_{in})_y} \right) = \omega S_x$$

(B4)

with $S_z$ being the transverse spin density. By combining Eqs. (B3), (B4) into Eq. (B2), we arrive at

$$\tau_{\text{rad}} = -\pi \alpha^3 \left[ \chi [e_z \times P(0) \cdot e_z] - \frac{\alpha^2}{24} \chi S_z(0) \right].$$

(B5)

where $\chi = f_{11} - 2f_{10}$.

---

**Appendix C: Basic properties of dyads**

Let $a = a_i e_i$, $b = b_i e_i$, and $c = c_i e_i$ be vectors with $a_i, b_i, c_i \in \mathbb{C}$. The dyad $D$ is defined as the product

$$D = ab,$$

\(D_{ij} = a_i b_j, \quad i,j \in \{x,y,z\},\)

(C1b)

which is a second-rank tensor. Note that $ab \neq ba$. Dyads follow the distributive rule $a(b + c) = ab + ac$. The unit dyad is

$$I = e_x e_x + e_y e_y + e_z e_z = e_i e_i.$$  

(C2)

The pre- and post-dot product are defined, respectively, by

$$c \cdot ab = (c \cdot a)b = c_i a_j b_j,$$

(C3a)

$$ab \cdot c = a(b \cdot c) = b_i c_j a_j e_j.$$  

(C3b)

We also have

$$a \cdot I = I \cdot a = a.$$  

(C4)
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