Concavity and convexity of several maps involving Tracy-Singh products, Khatri-Rao products, and operator-monotone functions of positive operators

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ABSTRACT: We establish concavity and convexity theorems for a number of operator-valued maps involving Tracy-Singh products and Khatri-Rao products of positive operators on a Hilbert space. Operator means serve as useful tools for some convexity results. We also investigate certain maps dealing with positive operator-monotone functions. In this case, the concavity and the convexity of such maps are examined through suitable integral representations of the operator-monotone functions on the unit interval with respect to finite Borel measures.

KEYWORDS: positive operator, Tracy-Singh product, Khatri-Rao product, operator-monotone function, Bochner integration

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INTRODUCTION

This paper focuses on concavity and convexity of certain maps dealing with Tracy-Singh products and Khatri-Rao products of operators. Such operator products are generalizations of famous matrix products in the literature, namely, the Kronecker product, the Hadamard product, the Tracy-Singh product, and the Khatri-Rao product.

Recall that the Kronecker product is defined for two matrices $A = [a_{ij}]$ and $B$ of arbitrary sizes resulting in a block matrix

$$A \otimes B = [a_{ij}B],$$

The Hadamard product is defined for two matrices $A$ and $B$ of the same size

$$A \circ B = [a_{ij}b_{ij}].$$

Concavity and convexity properties of several matrix-valued maps involving Kronecker products and Hadamard products were collected in Refs. 1–3. As a generalization of the Kronecker product, the Tracy-Singh product is defined for partitioned matrices $A = [A_{ij}]$ and $B = [B_{ij}]$ by

$$A \ast B = [(A_{ij} \ast B_{ij})],$$

The work of Al-Zhour extends some results of Ando $^{1}$ to Tracy-Singh products of positive definite matrices. The Khatri-Rao product $^{6}$, as a generalized Hadamard product, for $A = [A_{ij}]$ and $B = [B_{ij}]$ in the same block-matrix form, is defined by

$$A \circ B = [A_{ij} \circ B_{ij}].$$

In functional analysis aspect, the tensor product of Hilbert space operators can be viewed as an infinite-dimensional extension of the Kronecker product. Mond and Pečarić $^{7}$ extended the matrix results of Ando $^{1}$ to Hilbert space operators and obtained concavity/convexity theorems associated with positive operator-monotone functions. Ref. 8 extended the notion of tensor product for operators and Tracy-Singh product for matrices to the Tracy-Singh product for Hilbert space operators, and supply its algebraic and order properties. Analytic properties of the Tracy-Singh product were discussed in Ref. 9. Ref. 10 introduced the Khatri-Rao product of Hilbert space operators and gave a relationship between the Khatri-Rao product and the Tracy-Singh product of two operators via isometric selection operators.

In this study, we investigate concavity and convexity of certain maps related to Tracy-Singh products and Khatri-Rao products of operators. The main tools we use are operator means and suitable integral representations of certain operator-monotone functions.
functions. Our results in this paper generalize the results known so far for Tracy-Singh and Khatri-Rao products of matrices and tensor products of operators. Furthermore, we develop new concavity/convexity theorems.

PRELIMINARIES ON TRACY-SINGH AND KHATRI-RAO PRODUCTS

Throughout this paper, let \( \mathcal{H}, \mathcal{H}', \mathcal{K} \) and \( \mathcal{K}' \) be complex Hilbert spaces. When \( \mathcal{X} \) and \( \mathcal{Y} \) are Hilbert spaces, the symbol \( \mathbb{B}(\mathcal{X}, \mathcal{Y}) \) stands for the algebra of bounded linear operators from \( \mathcal{X} \) into \( \mathcal{Y} \), and when \( \mathcal{X} = \mathcal{Y} \), we write \( \mathbb{B}(\mathcal{X}) \) instead of \( \mathbb{B}(\mathcal{X}, \mathcal{X}) \). The cone of positive operators on \( \mathcal{H} \) is denoted by \( \mathbb{B}(\mathcal{H})^+ \). For self-adjoint operators \( A \) and \( B \) on the same space, the situation \( A \geq B \) means that \( A – B \) is positive. Denote the set of all positive invertible operators on \( \mathcal{H} \) by \( \mathbb{B}(\mathcal{H})^{++} \). If \( A \in \mathbb{B}(\mathcal{H})^{++} \), we write \( A \succ 0 \). The identity operator and the zero operator are denoted by \( I \) and \( 0 \), respectively.

To define the Tracy-Singh product and the Khatri-Rao product for operators, we decompose

\[
\mathcal{H} = \bigoplus_{j=1}^{n} \mathcal{H}_j, \quad \mathcal{H}' = \bigoplus_{i=1}^{m} \mathcal{H}'_i,
\]

\[
\mathcal{K} = \bigoplus_{l=1}^{p} \mathcal{K}_l, \quad \mathcal{K}' = \bigoplus_{k=1}^{q} \mathcal{K}'_k,
\]

where all \( \mathcal{H}_j, \mathcal{H}'_i, \mathcal{K}_l \) and \( \mathcal{K}'_k \) are Hilbert spaces. For each \( j \), let \( U_j : \mathcal{H}_j \rightarrow \mathcal{H} \) be the canonical embedding

\[
(0, \ldots, 0, x_j, 0, \ldots, 0) \rightarrow x_j.
\]

Similarly, for each \( i \), let \( V_i : \mathcal{H}'_i \rightarrow \mathcal{H}' \) be the canonical embedding. For each \( i \) and \( k \), let \( P_i : \mathcal{H}' \rightarrow \mathcal{H}'_i \) and \( Q_k : \mathcal{K}' \rightarrow \mathcal{K}'_k \) be the orthogonal projections. Each \( A \in \mathbb{B}(\mathcal{H}, \mathcal{H}') \) and \( B \in \mathbb{B}(\mathcal{K}, \mathcal{K}') \) can be expressed uniquely as operator matrices

\[
A = [A_{ij}]_{i,j=1}^{m,n}, \quad B = [B_{kl}]_{k,l=1}^{p,q},
\]

where \( A_{ij} = P_i U_j \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i) \) and \( B_{kl} = Q_k V_l \in \mathbb{B}(\mathcal{K}_l, \mathcal{K}_k) \) for each \( i, j, k, l \).

**Definition 1** Let \( A = [A_{ij}]_{i,j=1}^{m,n} \in \mathbb{B}(\mathcal{H}, \mathcal{H}') \) and \( B = [B_{kl}]_{k,l=1}^{p,q} \in \mathbb{B}(\mathcal{K}, \mathcal{K}') \). We define the Tracy-Singh product of \( A \) and \( B \) to be the bounded linear operator

\[
A \boxtimes B = \bigoplus_{j=1}^{n} \bigoplus_{i=1}^{m} \mathcal{H}_j \otimes \mathcal{H}_i \rightarrow \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} \mathcal{H}'_i \otimes \mathcal{H}'_j.
\]

When \( m = p \) and \( n = q \), we define the Khatri-Rao product of \( A \) and \( B \) to be the bounded linear operator

\[
A \square B = [A_{ij} \otimes B_{kl}]_{i,j=1}^{n,m} \rightarrow \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} \mathcal{H}_i \otimes \mathcal{H}_j.
\]

**Lemma 1** (Refs. 8, 9) Let \( A, B, C, D \) be compatible operators. Then

(i) The map \( (A, B) \rightarrow A \otimes B \) is bilinear and jointly continuous.

(ii) \((A \otimes B)(C \otimes D) = (AC) \otimes (BD)\).

(iii) If \( A \) and \( B \) are invertible, then \( (A \otimes B)^{-1} = A^{-1} \otimes B^{-1} \).

(iv) If \( A \) and \( B \) are positive, then \( (A \otimes B)^{\alpha} = A^{\alpha} \otimes B^{\alpha} \) for any \( \alpha > 0 \).

(v) If \( A \succ 0 \) and \( B \succ 0 \), then \( A \otimes B \succ C \otimes D \succ 0 \).

(vi) If \( A \succ 0 \) and \( B \succ 0 \), then \( A \otimes B \succ \mathbb{B} \).

**Lemma 2** (Ref. 9) Let \( A \in \mathbb{B}(\mathcal{H}) \) and \( B \in \mathbb{B}(\mathcal{H}) \). If \( A \succ 0 \) and \( B \succ 0 \), then \( A \otimes B \succ \mathbb{B} \).

**Lemma 4** (Ref. 10) There are isometries \( Z_1 \) and \( Z_2 \) such that

\[
A \square B = Z_1^*(A \square B)Z_2
\]

for all \( A \in \mathbb{B}(\mathcal{H}, \mathcal{H}') \) and \( B \in \mathbb{B}(\mathcal{K}, \mathcal{K}') \). For the case \( \mathcal{H} = \mathcal{H}' \) and \( \mathcal{K} = \mathcal{K}' \), we have \( Z_1 = Z_2 := Z \).

**Lemma 5** The Khatri-Rao product of operators is jointly continuous.

*Proof:* It follows from (1) and the continuity of the Tracy-Singh product (Lemma 1). □

For each \( i = 1, \ldots, k \), let \( \mathcal{H}_i \) and \( \mathcal{H}'_i \) be Hilbert spaces and decompose

\[
\mathcal{H}_i = \bigoplus_{r=1}^{n} \mathcal{H}_{i,r}, \quad \mathcal{H}'_i = \bigoplus_{s=1}^{m} \mathcal{H}'_{i,s},
\]

where all \( \mathcal{H}_{i,r} \) and \( \mathcal{H}'_{i,s} \) are Hilbert spaces. For a finite number of operator matrices \( A_i \in \mathbb{B}(\mathcal{H}_i, \mathcal{H}'_i) \) for \( i = 1, \ldots, k \), we use the following notations,

\[
\bigoplus_{i=1}^{k} A_i = (A_1 \square A_2) \square \cdots \square A_{k-1} \square A_k,
\]

\[
\bigodot_{i=1}^{k} A_i = ((A_1 \square A_2) \square \cdots \square A_{k-1}) \square A_k.
\]
Lemma 6 There are isometries $Z_1$ and $Z_2$

$$\bigotimes_{i=1}^{k}A_i = Z_1^* \left[ \bigotimes_{i=1}^{k}A_i \right] Z_2$$

for any $A_i \in \mathbb{B}(\mathcal{H}_i, \mathcal{H}_i')$, $i = 1, \ldots, k$. If $\mathcal{H}_i$ and $\mathcal{H}_i'$ are the same space for all $i$, then $Z_1 = Z_2 := Z$.

Proof: We proceed by induction on $k$. If $k = 2$, the property (2) is true by Lemma 4. Suppose that there exist isometries $R_1$ and $R_2$ such that

$$\bigotimes_{i=1}^{k-1}A_i = R_1^* \left[ \bigotimes_{i=1}^{k-1}A_i \right] R_2.$$

By Lemma 4, there are isometries $S_1, S_2$ such that

$$\left( \bigotimes_{i=1}^{k-1}A_i \right) \square A_k = S_1^* \left[ \bigotimes_{i=1}^{k-1}A_i \right] S_2.$$

Then

$$\bigotimes_{i=1}^{k}A_i = \left( \bigotimes_{i=1}^{k-1}A_i \right) \square A_k = S_1^* [ \bigotimes_{i=1}^{k-1}A_i ] S_2.$$

Set $Z_1 = (R_1 \square I) S_1$ and $Z_2 = (R_2 \square I) S_2$. Then $Z_1$ and $Z_2$ are isometries. When $\mathcal{H}_i = \mathcal{H}_i'$ for all $i = 1, \ldots, k$, we have $Z_1 = Z_2$ from the construction. □

CONVEXITY AND CONVEXITY
In this section, we provide concavity and convexity theorems related to Tracy-Singh products of operators. First of all, recall the following terminologies:

Definition 2 A function $f: (0, \infty) \to (0, \infty)$ is said to be operator-monotone if $f[A] \succ f[B]$ whenever $A \succ B > 0$. Here, $f[A]$ is the (continuous) functional calculus of $f$ defined on the spectrum of $A$.

Definition 3 Let $\mathcal{H}_1, \ldots, \mathcal{H}_k, \mathcal{H}$ be Hilbert spaces. For each $i = 1, \ldots, k$, let $E_i$ be a convex subset of $\mathbb{B}(\mathcal{H}_i)$. A function $\phi: E_1 \times \cdots \times E_k \to \mathbb{B}(\mathcal{H})$ is said to be concave if

$$\phi((1-t)A_1 + tB_1, \ldots, (1-t)A_k + tB_k) \leq (1-t)\phi(A_1, \ldots, A_k) + t\phi(B_1, \ldots, B_k)$$

for any $A_i, B_i \in E_i$ ($i = 1, \ldots, k$) and $t \in (0, 1)$. A function $\phi$ is convex if $-\phi$ is concave. A map between two convex sets is said to be affine if it preserves convex combinations.

Recall that, for each $t \in (0, 1)$, the $t$-weighted harmonic mean and the $t$-weighted geometric mean of $A, B \in \mathbb{B}(\mathcal{H})^{++}$ is defined respectively by

$$A \triangledown tB = \left[ (1-t)A^{-1} + tB^{-1} \right]^{-1},$$

$$A \#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^{t/2}A^{1/2}.$$

For arbitrary $A, B \in \mathbb{B}(\mathcal{H})^+$, we define the $t$-weighted geometric mean of $A$ and $B$ to be

$$A \#_t B = \lim_{\epsilon \to 0} (A + \epsilon I) \#_1 (B + \epsilon I),$$

where the limit is taken in the strong-operator topology.

Lemma 7 (Ref. 11) For each $t \in [0, 1]$, the map $(A, B) \mapsto A \triangledown tB$ is concave on $\mathbb{B}(\mathcal{H})^{++} \times \mathbb{B}(\mathcal{H})^{++}$. The next lemma gives an integral representation of operator-monotone functions on $(0, \infty)$ in terms of Borel measures on $[0, 1]$.

Lemma 8 (Ref. 12) Let $f: (0, \infty) \to (0, \infty)$ be an operator-monotone function. Then there is a finite Borel measure $\mu$ on $[0, 1]$ such that

$$f(x) = \int_0^1 1_\Gamma(x \, d\mu(t)), \quad x > 0.$$ (3)

Theorem 1 Let $f: (0, \infty) \to (0, \infty)$ be an operator-monotone function. If $\phi_1: \mathbb{B}(\mathcal{H})^{++} \to \mathbb{B}(\mathcal{H})^{++}$ and $\phi_2: \mathbb{B}(\mathcal{H})^{++} \to \mathbb{B}(\mathcal{H}')^{++}$ are concave maps, then the maps

$$(A, B) \mapsto f[\phi_1(A) \otimes \phi_2(B)^{-1}] \cdot (I \otimes \phi_2(B))$$

are concave on $\mathbb{B}(\mathcal{H})^{++} \times \mathbb{B}(\mathcal{H}')^{++}$. Proof: Let $A \in \mathbb{B}(\mathcal{H})^{++}$ and $B \in \mathbb{B}(\mathcal{H})^{++}$. Then $\phi_1(A) > 0$ and $\phi_2(B) > 0$. Lemma 1 implies that $f[\phi_1(A) \otimes \phi_2(B)^{-1}]$ is concave for any $(A, B) \in \mathbb{B}(\mathcal{H})^{++} \times \mathbb{B}(\mathcal{H}')^{++}$. By Lemma 8, there is a finite Borel measure $\mu$ on $[0, 1]$ such that (3) holds. Using Bochner integration, we have

$$\int_0^1 \left[ \left( \int_0^1 (\phi_1(A) \otimes \phi_2(B)^{-1}) \cdot (I \otimes \phi_2(B)) \right) \cdot \phi_1(A) \otimes \phi_2(B)^{-1} \right] \cdot (I \otimes \phi_2(B))$$

for any $A, B \in E_i$ ($i = 1, \ldots, k$) and $t \in (0, 1)$. A function $\phi$ is convex if $-\phi$ is concave. A map between two convex sets is said to be affine if it preserves convex combinations.

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For each \( t \in [0, 1] \), by Lemma 1 we obtain
\[
\{(I \otimes I) !, (\phi_1(A) \otimes \phi_2(B)^{-1}) \} \cdot (I \otimes \phi_2(B)) = \left[ (1 - t)(I \otimes I) + t(\phi_1(A) \otimes \phi_2(B)^{-1}) \right]^{-1} \cdot (I \otimes \phi_2(B))
\]
\[
= \left[ (I \otimes \phi_2(B)^{-1}) \right]^{-1} \\
\cdot \left[ (1 - t)(I \otimes I) + t(\phi_1(A) \otimes \phi_2(B)) \right]^{-1} \\
= \left[ (1 - t)(I \otimes \phi_2(B)^{-1}) + t(\phi_1(A) \otimes I) \right]^{-1} \\
= (I \otimes \phi_2(B)) !, (\phi_1(A) \otimes I).
\]

Since the weighted harmonic mean is concave (Lemma 7), so is the map
\[(A, B) \mapsto \{(I \otimes I) !, (\phi_1(A) \otimes \phi_2(B)^{-1}) \} \cdot (I \otimes \phi_2(B)).\]
Thus the map (4) is concave. Similarly, the map (5) is concave.

**Theorem 1**


equiv \quad \end{align*}

Theorem 1 implies the concavity of the map
\[
(A, B) \rightarrow g[\phi_1(A) \otimes \phi_2(B)^{-1}] \cdot (I \otimes \phi_2(B))
\]
\[
= \left[ (I \otimes \phi_2(B)^{-1}) \cdot f[\phi_1(A)^{-1} \otimes \phi_2(B)] \right]^{-1}
\]
\[
= \left[ f[\phi_1(A)^{-1} \otimes \phi_2(B)] \cdot (I \otimes \phi_2(B)^{-1}) \right]^{-1}.
\]
Thus the map (6) is convex. Similarly, the map (7) is convex.

**Theorem 2**

\[
Let f : (0, \infty) \rightarrow (0, \infty) be an operator-monotone function. If \( \phi_1 : B(\mathcal{X})^{++} \rightarrow B(\mathcal{X})^{++} \) is a concave map and \( \phi_2 : B(\mathcal{X})^{++} \rightarrow B(\mathcal{X}')^{++} \) is an affine map, then the maps
\[
(A, B) \rightarrow f[\phi_1(A)^{-1} \otimes \phi_2(B)] \cdot (I \otimes \phi_2(B)),
\]
\[
(A, B) \rightarrow f[\phi_2(B) \otimes \phi_1(A)^{-1}] \cdot (\phi_2(B) \otimes I)
\]
are convex on \( B(\mathcal{X})^{++} \times B(\mathcal{X}')^{++} \).

**Proof:** By Lemma 8, there is a finite Borel measure \( \mu \) on \([0, 1]\) such that (3) holds. Then
\[
\int_0^1 \left\{ (I \otimes I) !, (\phi_1(A)^{-1} \otimes \phi_2(B)) \right\} (I \otimes \phi_2(B)) \, d\mu(t).
\]
For each \( t \in [0, 1] \), it follows from Lemma 1 that
\[
\left\{ (I \otimes I) !, (\phi_1(A)^{-1} \otimes \phi_2(B)) \right\}
\]
\[
\left[ (1 - t)(I \otimes I) + t(\phi_1(A)^{-1} \otimes \phi_2(B)) \right]^{-1}
\]
\[
\left[ (1 - t)(I \otimes I) + t(\phi_1(A) \otimes \phi_2(B)^{-1}) \right]^{-1}
\]
\[
\left[ (I \otimes \phi_2(B)) \left[ (1 - t)(I \otimes \phi_2(B)) + t(\phi_1(A) \otimes I) \right] \right]^{-1}.
\]
are concave on \( B(\mathcal{X})^{++} \times B(\mathcal{X}')^{++} \).

The concavity of the map \((A, B) \rightarrow (1 - t)(I \otimes \phi_2(B)) + t(\phi_1(A) \otimes I)\) and the affinity of the map \((A, B) \rightarrow I \otimes \phi_2(B)\) together yield the convexity of the map
\[
(A, B) \rightarrow
\]
\[
(I \otimes \phi_2(B)) \left[ (1 - t)(I \otimes \phi_2(B)) + t(\phi_1(A) \otimes I) \right]^{-1} (I \otimes \phi_2(B))
\]
\[
= \left\{ (I \otimes I) !, (\phi_1(A)^{-1} \otimes \phi_2(B)) \right\} (I \otimes \phi_2(B)).
\]
Hence the map (8) is convex. Similarly, the map (9) is convex.

**Corollary 2** The maps
\[
(A, B) \rightarrow I \otimes (B \log [B]) - \log [A] \otimes B,
\]
\[
(A, B) \rightarrow (A \log [A]) \otimes I - A \otimes \log [B]
\]
are convex on \( B(\mathcal{X})^{++} \times B(\mathcal{X}')^{++} \).
Theorem 3. Ref. 1 to the case of Tracy-Singh product of Khatri-Rao and Tracy-Singh products. Established by using the concavity theorems for Tracy-Singh products of operators are established. In this section, we present concavity theorems for Khatri-Rao products of operators.

Example 2. Let \( \phi_1 : B(\mathcal{H})^{++} \to B(\mathcal{H})^{++} \) be a concave map and \( \phi_2 : B(\mathcal{H})^{++} \to B(\mathcal{H})^{++} \) an affine map. For any \( 0 \leq p \leq 1 \), we have by Theorem 2 that the maps

\[
(A, B) \mapsto [\phi_1(A)^{-1} \otimes \phi_2(B)]^p \cdot (I \otimes \phi_2(B)),
\]

\[
(A, B) \mapsto [\phi_2(B) \otimes \phi_1(A)^{-1}]^p \cdot (\phi_2(B) \otimes I)
\]

are convex on \( B(\mathcal{H})^{++} \times B(\mathcal{H})^{++} \).

We mention that the maps (5), (7), (9) and (11) are extensions of results discussed in Ref. 7.

Concavity Theorems for Tracy-Singh and Khatri-Rao Products

In this section, we present concavity theorems for Tracy-Singh products of operators. Concavity theorems for Khatri-Rao products of operators are established by using the concavity theorems for Tracy-Singh products and the connection between the Khatri-Rao and Tracy-Singh products.

The next result generalizes Corollary 6.2 of Ref. 1 to the case of Tracy-Singh product of operators.

Theorem 3. Let \( 0 \leq p_i \leq 1 \), \( i = 1, \ldots, k \), be such that \( \sum_{i=1}^{k} p_i \leq 1 \). Then the map

\[
(A_1, \ldots, A_k) \mapsto \bigotimes_{i=1}^{k} A_i^{p_i}
\]

is concave on \( B(\mathcal{H})^{++} \times \cdots \times B(\mathcal{H})^{++} \).

Proof: We proceed by induction on \( k \). Clearly, the map \( A_1 \mapsto A_i^{p_i} \) is concave. Suppose the assertion is generally true for the case \( k-1 \). If \( p_k = 0 \), then the map becomes

\[
(A_1, \ldots, A_k) \mapsto ((A_1 \otimes A_2) \otimes \cdots \otimes A_{k-1}) \otimes I,
\]

which is concave. If \( p_k = 1 \), then \( p_i = 0 \) for all \( i = 1, \ldots, k-1 \) and the map is clearly concave. Now suppose \( 0 < p_k < 1 \). By the induction assumption, the map

\[
\phi(A_1, \ldots, A_{k-1}) = \bigotimes_{i=1}^{k-1} A_i^{p_i/(1-p_k)}
\]

is concave. By applying Theorem 1 with \( f(x) = x^{p_k} \), the map

\[
(A_1, \ldots, A_k) \mapsto f(\phi(A_1, \ldots, A_{k-1})^{-1} \otimes A_k) \cdot (\phi(A_1, \ldots, A_{k-1}) \otimes I)
\]

is concave. We obtain the concavity of the map (12),

\[
f(\phi(A_1, \ldots, A_{k-1})^{-1} \otimes A_k) \cdot (\phi(A_1, \ldots, A_{k-1}) \otimes I)
\]

\[
= \phi(A_1, \ldots, A_{k-1})^{-1-p_k} \otimes A_k^{p_k} \cdot (\phi(A_1, \ldots, A_{k-1}) \otimes I)
\]

\[
= \phi(A_1, \ldots, A_{k-1})^{-1-p_k} \otimes A_k^{p_k} = \bigotimes_{i=1}^{k} A_i^{p_i}.
\]

A special case of Theorem 3 is when \( k = 2 \).

Corollary 3. For each \( r \in (0, 1) \), the map

\[
(A, B) \mapsto A^{1-r} \otimes B^r
\]

is concave on \( B(\mathcal{H})^+ \times B(\mathcal{H})^+ \).

Proof: Theorem 3 implies that the map (13) is concave on \( B(\mathcal{H})^{++} \times B(\mathcal{H})^{++} \). Since the Tracy-Singh product is jointly continuous (Lemma 1), this map is also concave on \( B(\mathcal{H})^+ \times B(\mathcal{H})^+ \).

Next, we develop concavity theorems for Khatri-Rao products of operators.

Theorem 4. Let \( 0 \leq p_i \leq 1 \), \( i = 1, \ldots, k \), be such that \( \sum_{i=1}^{k} p_i \leq 1 \). Then the map

\[
(A_1, \ldots, A_k) \mapsto \bigotimes_{i=1}^{k} A_i^{p_i}
\]

is concave on \( B(\mathcal{H})^{++} \times \cdots \times B(\mathcal{H})^{++} \).

Proof: From Lemma 6, the map \( X \mapsto Z^*XZ \), taking the Tracy-Singh product \( \bigotimes_{i=1}^{k} A_i \) into the Khatri-Rao product \( \bigotimes_{i=1}^{k} A_i \), is linear and preserves positivity. Recall that the composition between a linear map and a concave map results in a concave map. Since the map \( (A_1, \ldots, A_k) \mapsto \bigotimes_{i=1}^{k} A_i^{p_i} \) is concave by Theorem 3, we have the concavity of the map is concave. We obtain the concavity of the map from (12), since

\[
(A_1, \ldots, A_k) \mapsto Z^*\left( \bigotimes_{i=1}^{k} A_i^{p_i} \right) Z = \bigotimes_{i=1}^{k} A_i^{p_i}.
\]
Corollary 4 For each \( r \in (0, 1) \), the map
\[
(A, B) \mapsto A^{-r} \otimes B^r,
\]
is concave on \( \mathbb{B}(\mathscr{H})^+ \times \mathbb{B}(\mathscr{H})^+ \).

Proof: It follows from Theorem 4 when \( k = 2 \) together with the continuity of the Khatri-Rao product, Lemma 5.

Convexity Theorems for Tracy-Singh and Khatri-Rao Products

In this section, we establish convexity theorems for Tracy-Singh products and Khatri-Rao products of operators. Weighted arithmetic/geometric/harmonic means of operators serve as useful tools.

Lemma 9 (Ref. 13) Let \( A_i, B_i \in \mathbb{B}(\mathscr{H})^+ \), \( 1 \leq i \leq k \). Then
\[
\bigotimes_{i=1}^k (A_i)_{#_i} \bigotimes_{i=1}^k (B_i)_{#_i} = \bigotimes_{i=1}^k (A_i, #_i, B_i).
\]

Theorem 5 Let \( \phi_i \), \( i = 1, \ldots, k \), be a concave map from \( \mathbb{B}(\mathscr{H})^{++} \) to \( \mathbb{B}(\mathscr{H}')^{++} \). Then the map
\[
(A_1, \ldots, A_k) \mapsto \bigotimes_{i=1}^k \phi_i(A_i)^{-1}
\]
is convex on \( \mathbb{B}(\mathscr{H})^{++} \times \cdots \times \mathbb{B}(\mathscr{H}_k)^{++} \).

Proof: Let \( t \in [0, 1] \). By continuity, we may assume that \( A_i \) and \( B_i \) are positive invertible operators. Applying Lemmas 1 and 9 and the arithmetic-geometric means inequality for operators, we have
\[
\bigotimes_{i=1}^k \phi_i((1-t)A_i + tB_i)^{-1}
\]
\[
\leq \bigotimes_{i=1}^k ((1-t)\phi_i(A_i) + t\phi_i(B_i))^{-1}
\]
\[
\leq \bigotimes_{i=1}^k (\phi_i(A_i) \#_i \phi_i(B_i))^{-1}
\]
\[
= \bigotimes_{i=1}^k \phi_i(A_i)^{-1} \#_i \bigotimes_{i=1}^k \phi_i(B_i)^{-1}
\]
\[
\leq (1-t) \bigotimes_{i=1}^k \phi_i(A_i)^{-1} + t \bigotimes_{i=1}^k \phi_i(B_i)^{-1}.
\]
Hence the map (15) is convex.

Corollary 5 Let \( 0 < p_i \leq 1 \), \( i = 1, \ldots, k \). Then the map
\[
(A_1, \ldots, A_k) \mapsto \bigotimes_{i=1}^k A_i^{-p_i}
\]
is convex on \( \mathbb{B}(\mathscr{H})^{++} \times \cdots \times \mathbb{B}(\mathscr{H}_k)^{++} \).

Proposition 1 Let \( 0 \leq p_i \leq 1 \), \( i = 1, \ldots, k \), and \( 1 \leq q \leq 2 \) be such that \( \sum_{i=1}^k p_i = q-1 \). Then the map
\[
(A_1, \ldots, A_{k+1}) \mapsto \bigotimes_{i=1}^k A_i^{-p_i} \otimes A_i^{q}
\]
is convex on \( \mathbb{B}(\mathscr{H})^{++} \times \cdots \times \mathbb{B}(\mathscr{H}_k)^{++} \).

Proof: By Theorem 3, the map
\[
(A_1, \ldots, A_{k+1}) \mapsto \bigotimes_{i=1}^k A_i^{p_i} \otimes A_i^{2-q}
\]
is concave on \( \mathbb{B}(\mathscr{H})^{++} \times \cdots \times \mathbb{B}(\mathscr{H}_k)^{++} \). Clearly, the map
\[
(A_1, \ldots, A_{k+1}) \mapsto \bigotimes_{i=1}^k A_i^{p_i} \otimes A_i^{q}
\]
is affine. It follows from Lemma 1 that the map
\[
(A_1, \ldots, A_{k+1}) \mapsto \bigotimes_{i=1}^k A_i^{p_i} \otimes A_i^{2-q}
\]
is convex.

Theorem 6 For each \( r \in (0, 1) \), the maps
\[
(A, B) \mapsto A^{-r} \otimes B^{1+r},
\]
\[
(A, B) \mapsto A^{1+r} \otimes B^{-r}
\]
are convex on \( \mathbb{B}(\mathscr{H})^{++} \times \mathbb{B}(\mathscr{H})^{++} \).

Proof: The convexity of the map (16) follows from Proposition 1. By continuity, we may assume that \( A \) and \( B \) are invertible. Lemma 1 implies that
\[
A^{1+r} \otimes B^{-r} = (A^r \otimes B^{-r})(A \otimes I) = (A \otimes B^{-1})^r(A \otimes I).
\]
It follows from Lemmas 1 and 8 that
\[
A^{1+r} \otimes B^{-r} = \int_0^1 ((I \otimes I) r (A \otimes B^{-1})) d\mu(r)(A \otimes I)
\]
\[
= \int_0^1 (1-t)(I \otimes I) + t(A \otimes B^{-1})^{-1} d\mu(t)(A \otimes I)
\]
\[
= \int_0^1 (1-t)(I \otimes I) + t(A^{-1} \otimes B) d\mu(t)(A \otimes I)
\]
\[
= \int_0^1 (A \otimes I) [(1-t)(A \otimes I) + t(I \otimes B)]^{-1} d\mu(t)(A \otimes I).
\]
Since the map $A \mapsto A^{-1}$ is convex and the map $(A, B) \mapsto (1-t)(A \boxtimes I) + t(I \boxtimes B)$ is affine, the map $(A, B) \mapsto (I \boxtimes B)((1-t)(A \boxtimes I) + t(I \boxtimes B))^{-1}(I \boxtimes B)$ is convex. Thus the map $(A, B) \mapsto A^{1+r} \boxtimes B^{-r}$ is convex. \hfill \Box

**Proposition 2** Let $\phi_i$, $i = 1, \ldots, k$, be concave maps from $B(\mathcal{H})_{++}$ to $B(\mathcal{H})_{++}$. Then the map

$$(A_1, \ldots, A_k) \mapsto \bigoplus_{i=1}^{k} \phi_i(A_i)^{-1}$$

is convex on $B(\mathcal{H})_{++} \times \cdots \times B(\mathcal{H})_{++}$.

**Proof:** It follows from Lemma 6 and Theorem 5. \hfill \Box

**Corollary 6** Let $0 < p_i \leq 1$ for each $i = 1, \ldots, k$. Then the map

$$(A_1, \ldots, A_k) \mapsto \bigoplus_{i=1}^{k} A_i^{-p_i}$$

is convex on $B(\mathcal{H})_{++} \times \cdots \times B(\mathcal{H})_{++}$.

**Proof:** It follows from Proposition 2 by putting $\phi_i(A_i) = A_i^{p_i}$ for each $i$. \hfill \Box

**Proposition 3** For each $r \in (0, 1)$, the maps

$$(A, B) \mapsto A^{-r} \boxtimes B^{1+r},$$

$$(A, B) \mapsto A^{1+r} \boxtimes B^{-r}$$

are convex on $B(\mathcal{H})_{++} \times B(\mathcal{H})_{++}$.

**Proof:** It follows from Lemma 4 and Theorem 6. \hfill \Box

Recall that the Moore-Penrose inverse of an operator $T \in B(\mathcal{H}, \mathcal{H}')$ is the operator $T^\dagger \in B(\mathcal{H}', \mathcal{H})$ satisfying the conditions $TT^\dagger T = T$, $T^\dagger TT^\dagger = T$, $(TT^\dagger)^* = TT^\dagger$, and $(T^\dagger T)^* = T^\dagger T$. It is well known that $T^\dagger$ exists if and only if the range of $T$ is closed.

**Lemma 10 (Ref. 15)** Let

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in B(\mathcal{H}_1 \oplus \mathcal{H}_2)$$

be a self-adjoint operator. Suppose that $T_{11}$ has a closed range. Then $T \geq 0$ if and only if $T_{11} \geq 0$, $T_{21} = T_{11}^T T_{12}$ and $T_{22} \geq T_{12}^T T_{12}$.

Recall that for any interval $J$, a continuous function $f : J \to \mathbb{R}$ is convex if and only if $f(x + h) + f(x - h) - 2f(x) \geq 0$ for all $x \in J$ and $h > 0$ such that $x \pm h \in J$.

**Theorem 7** Let $A \in B(\mathcal{H})^+$ and $B \in B(\mathcal{H})^+$ have closed ranges. Then the operator-valued function

$$(\phi : [-1, 1] \to B(\bigoplus_{i=1}^{n} \mathcal{H}_i \oplus \mathcal{H}_i)), \quad \phi(t) = A^{1+t} \boxtimes B^{-1-t} + A^{-1-t} \boxtimes B^{1+t}$$

is convex on $[-1, 1]$, decreasing on $[-1, 0]$, increasing on $[0, 1]$, attains minimality at $t = 0$, and attains maximality at $t = -1, 1$.

**Proof:** Let $s \in [-1, 1]$ and $t > 0$ be such that $s \pm t \in [-1, 1]$. Consider the operator matrices

$$T_1 = \begin{bmatrix} A_{1+s} & A_{1+s-t} \\ A_{1+s-t} & A_{1+s} \end{bmatrix}, \quad T_2 = \begin{bmatrix} A_{1-s} & A_{1-s-t} \\ A_{1-s-t} & A_{1-s} \end{bmatrix},$$

$$T_3 = \begin{bmatrix} B_{1+s-t} & B_{1+s} \\ B_{1+s} & B_{1+s-t} \end{bmatrix}, \quad T_4 = \begin{bmatrix} B_{1-t} & B_{1-s} \\ B_{1-s} & B_{1-t} \end{bmatrix}.$$

Note that

$$A_{1+s} = (AA^\dagger A)^{1+s+t} A^{-t} = A^{1+s+t} (A^{1+s+t})^* A^{1+s},$$

$$A_{1+s-t} = A^{-t} (AA^\dagger A)^{1+s+t} A^{-t} = A^{-t} (A^{1+s+t})^* A^{1+s}.$$

We have by Lemma 10 that $T_i$ is positive for all $i = 1, 2, 3, 4$. By the monotonicity of Khatri-Rao product, Lemma 3, we have that the operator $X = T_1 \boxtimes T_2 + T_3 \boxtimes T_4$ is

$$\begin{bmatrix} A_{1+s} \boxtimes B_{1-s} + A_{1-s} \boxtimes B_{1+s} & A_{1+s} \boxtimes B_{1-s} + A_{1-s} \boxtimes B_{1+s} \\ A_{1+s} \boxtimes B_{1-s} + A_{1-s} \boxtimes B_{1+s} & A_{1+s} \boxtimes B_{1-s} + A_{1-s} \boxtimes B_{1+s} \end{bmatrix},$$

which is positive. Similarly, the operator $Y$,

$$\begin{bmatrix} A_{1+s-t} \boxtimes B_{1-s} + A_{1-s-t} \boxtimes B_{1+s} & A_{1+s-t} \boxtimes B_{1-s} + A_{1-s-t} \boxtimes B_{1+s} \\ A_{1+s-t} \boxtimes B_{1-s} + A_{1-s-t} \boxtimes B_{1+s} & A_{1+s-t} \boxtimes B_{1-s} + A_{1-s-t} \boxtimes B_{1+s} \end{bmatrix},$$

is also positive. It follows that

$$0 \leq X + Y = \begin{bmatrix} \phi(s+t) + \phi(s-t) & 2\phi(s) \\ 2\phi(s) & \phi(s+t) + \phi(s-t) \end{bmatrix},$$

$$= U \begin{bmatrix} \phi(s+t) + \phi(s-t) + 2\phi(s) \\ \phi(s+t) + \phi(s-t) - 2\phi(s) \end{bmatrix} U^*,$$

where

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$
Corollary 7 Let \( A \in \mathbb{B}(\mathcal{H})^+ \) and \( B \in \mathbb{B}(\mathcal{H})^+ \) have closed ranges. Then the parameterization
\[
\psi : [0, 1] \to \mathbb{B}\left(\bigoplus_{i=1}^{n} \mathcal{H}_i \otimes \mathcal{H}_i\right),
\]
\[
\psi(t) = A^t \boxplus B^{1-t} + A^{1-t} \boxplus B^t
\]
is convex on \([0, 1]\), decreasing on \([0, 1/2]\), increasing on \([1/2, 1]\), attains minimality at \( t = 1/2 \), and attains maximality at \( t = 0, 1 \).

Proof: Let \( f : [0, 1] \to [-1, 1] \) be defined by \( f(t) = 2t - 1 \). Then \( \psi = \phi \circ f \) where \( \phi \) is given by (18). Now, the desired results follow from Theorem 7 by using \( f([0, 1]) = [-1, 1] \), \( f([0, 1/2]) = [-1, 0] \), \( f([1/2, 0]) = [0, 1] \), and \( f(1/2) = 0 \).

As a consequence, we obtain an operator version of the arithmetic-geometric mean inequality as follows.

Corollary 8 Let \( A \in \mathbb{B}(\mathcal{H})^+ \) and \( B \in \mathbb{B}(\mathcal{H})^+ \) have closed ranges. For any \( t \in [1/2, 1] \), we have
\[
2(A^{1/2} \boxplus B^{1/2}) \leq A^t \boxplus B^{1-t} + A^{1-t} \boxplus B^t \leq A \boxplus B,
\]
where \( \boxplus \) denotes the Khatri-Rao sum \(^{16}\) defined by \( A \boxplus B = A \boxplus I + I \boxplus B \).

We mention that Theorem 5, Corollary 5, and Proposition 1 generalize the matrix results involving Tracy-Singh products provided in Ref. 5.

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REFERENCES
1. Ando T (1979) Concavity of certain maps on positive definite matrices and applications to Hadamard products. Linear Algebra Appl 26, 203–41.
2. Bhatia R (2007) Positive Definite Matrices, Princeton University Press, Princeton, New Jersey.
3. Matharu JS, Aujla JS (2009) Hadamard product versions of the Chebysheff and Kantorovich inequalities. J Inequal Pure Appl Math 10, 1–6.
4. Tracy DS, Singh RP (1972) A new matrix product and its applications in partitioned matrix differentiation. Stat Neerl 26, 143–57.
5. Al-Zhour Z (2014) Several new inequalities on operator means of non-negative maps and Khatri-Rao products of positive definite matrices. J King Saud Univ Sci 26, 21–7.
6. Khatri CG, Rao CR (1968) Solutions to some functional equations and their applications to characterization of probability distributions. Sankhya 30, 167–80.
7. Mond B, Pečarić JE (1998) Operator convex functions of several variables. Soochow J Math 24, 239–54.
8. Ploymukda A, Chansangiam P, Lewkeeratiyutkul W (2018) Algebraic and order properties of Tracy-Singh products for operator matrices. J Comput Anal Appl 24, 656–64.
9. Ploymukda A, Chansangiam P, Lewkeeratiyutkul W (2018) Analytic properties of Tracy-Singh products for operator matrices. J Comput Anal Appl 24, 665–74.
10. Ploymukda A, Chansangiam P (2016) Khatri-Rao products of operator matrices acting on the direct sum of Hilbert spaces. J Math 2016, 1–7.
11. Hiai F, Petz D (2014) Introduction to Matrix Analysis and Applications, Springer, New Delhi.
12. Chansangiam P, Lewkeeratiyutkul W (2015) Khatri-Rao sum of Hilbert space operators. J Math Sci Technol 40, 595–601.
13. Xuan NQ, Sheng L (2008) Positive semi-definite matrices of adjointable operators on Hilbert C*-modules. Linear Algebra Appl 428, 992–1000.
14. Caradus SR (1978) Generalized Inverses and Operator Theory, Queen's Papers in Pure and Applied Mathematics 50, Queen's University, Kingston.
15. Xu Q, Sheng L (2008) Positive semi-definite matrices of adjointable operators on Hilbert C*-modules. Linear Algebra Appl 428, 992–1000.