Abstract. We prove a duality for factorization homology which generalizes both usual Poincaré duality for manifolds and Koszul duality for $\mathcal{E}_n$-algebras. The duality has application to the Hochschild homology of associative algebras and enveloping algebras of Lie algebras. We interpret our result at the level of topological quantum field theory.

CONTENTS

Introduction 2
Terminology 6
Acknowledgements 6
1. Review of reduced factorization homology 6
1.1. Zero-pointed manifolds 6
1.2. Reduced factorization homology 8
1.3. Exiting disks 9
1.4. Free and trivial algebras 10
1.5. Linear Poincaré duality 13
2. Filtrations 13
2.1. Main results 14
2.2. Non-unital algebras 18
2.3. Support for Theorem 2.1.4 (cardinality layers) 18
2.4. Free calculation 21
2.5. Support for Theorem 2.1.6 (convergence) 23
2.6. Support for Theorem 2.1.8 (Goodwillie layers) 24
2.7. Support for Theorem 2.1.8 (comparing cofiltrations) 24
3. Factorization homology of formal moduli problems 25
3.1. Formal moduli 25
3.2. Factorization homology with formal moduli coefficients 27
3.3. Case of finitely presented highly coconnective algebras 30
3.4. Comparing $\text{Artin}_n$ and $\text{FPres}_n$ 30
3.5. Resolving by $\text{FPres}_{\leq -n}$ 31
4. Hochschild homology of associative and enveloping algebras 33
4.1. Case $n = 1$ 33
4.2. Lie algebras 34
References 36

2010 Mathematics Subject Classification. Primary 55U30. Secondary 13D03, 14B20, 14D23.

Key words and phrases. Factorization homology. Topological quantum field theory. Derived algebraic geometry. Formal moduli problems. Topological chiral homology. Koszul duality. Operads. $\infty$-Categories. Goodwillie-Weiss manifold calculus. Goodwillie calculus of functors. Hochschild homology.

DA was partially supported by ERC adv.grant no.228082, and by the National Science Foundation under Award No. 0902639. JF was supported by the National Science Foundation under Award 1207758.
This paper arises from the following question: what is Poincaré duality in factorization homology? Before describing our solution, we give some background for this question. After Lurie in \cite{Lu2}, a factorization homology, or topological chiral homology, theory is a homology-type theory for $n$-manifolds; these theories are natural with respect to embeddings of manifolds and satisfy a symmetric monoidal generalization of the Eilenberg-Steenrod axioms for ordinary homology—see \cite{Fr3}. This relatively new theory has two particularly notable antecedents: the labeled or amalgamated configuration space models of mapping spaces of Salvatore \cite{Sa}, Segal \cite{Se4}, and Kallel \cite{Ka}, after \cite{Bo}, \cite{Mc}, \cite{May}, and \cite{Se1}; the algebraic approaches to conformal field theory of Beilinson & Drinfeld in \cite{BeD}, via factorization algebras, and of Segal in \cite{Se3}.

In the last few years, significant work has taken place in this subject in addition to the basic investigation of the foundations of factorization homology in \cite{Lu2}, \cite{Fr3}, and \cite{AFT2}. In algebraic geometry, Gaitsgory & Lurie use factorization algebras to prove Weil conjecture’s on Tamagawa numbers for algebraic groups in the function field case—see \cite{GL}. In mathematical physics, Costello has developed in \cite{Co} a renormalization machine for quantizing field theories; by work of Costello & Gwilliam in \cite{CG}, this machine outputs a factorization homology theory as a model for the observables in a perturbative quantum field theory. Their work gives an array of interesting examples of factorization homology theories and manifold invariants connected with quantum groups and knot and 3-manifold invariants.

The question of Poincaré duality finds motivation from all the preceding works. We focus on two veins. First, factorization homology theories are characterized by a monoidal generalization of the Eilenberg-Steenrod axioms for usual homology, so that factorization homology specializes to ordinary homology in the case the target symmetric monoidal category is that of chain complexes with direct sum. As such, it makes sense to ask that different values of factorization homology theories, valued in a general symmetric monoidal category, likewise enjoy a relationship specializing to that of Poincaré duality. From this perspective, the present work fills in the bottom middle corner in the following table of analogies.

| Ordinary/Generalized | Factorization | Physics |
|----------------------|---------------|---------|
| topological space $M$| $n$-manifold $X$| spacetime $M$ |
| abelian group $A$    | $n$-disk stack $X$| quantum field theory $Z$ |
| additivity $\sqcup$  | multiplicativity $\otimes$| locality $\otimes$ |
| homology $C_\ast(M, A)$| factorization homology $\int_M X$| observables $Z(M)$ |
| linearity $P_1C_\ast(M, A)$| Goodwillie calculus $P_\ast\int_M X$| perturbative observables $Z_{\text{pert}}(M)$ |
| Poincaré duality $C_\ast(M, A) \simeq C^\ast(M, A[n])$| Poincaré/Koszul duality $\int_X X^\wedge \simeq \left(\int_M T_x X[-n]\right)^\vee$|

Second, in the perspective espoused in the works \cite{BeD} and \cite{CG}, factorization homology theories are algebraic models for physical field theories. An essential notion in quantum field theory is that of a duality, in which different field theories give equivalent, or linearly dual, values on top dimensional manifolds; an especially important example is that of S-duality. In extended topological quantum field theory, as appears in the cobordism hypothesis after Baez & Dolan \cite{BaD} and Lurie \cite{Lu3}, there is likewise a fundamental duality: the duality of higher $n$-categories which appear as the values of the field theories on points.

While both of these perspectives augur for a notion of duality in factorization homology, they likewise both indicate that an essential ingredient is missing. In the first case, usual Poincaré duality is a relationship between usual homology and compactly supported cohomology—this suggests that a notion of compactly supported factorization cohomology is necessary. From the perspective of the
cohomology hypothesis in the 1-dimensional case, the factorization homology theories \( \int A \) is closely related to extended topological field theories \( Z \) whose value on a point is \( Z(*) = \text{Perf}_A \), and the value on the circle is \( Z(S^1) \simeq \int_{S^1} A \simeq \text{HH}_*(A) \), the Hochschild chains of \( A \). One would expect the dual field theory \( Z^\vee \) to take the value \( Z^\vee(S^1) = \text{HH}_*(A)^\vee \), the \( k \)-linear dual of the Hochschild chains of \( A \). However, there is in general no algebra \( B \) for which \( \text{HH}_*(B) \) is equivalent to \( \text{HH}_*(A)^\vee \). That is, the category \( Z^\vee(*) \) is not given by perfect modules for some other algebra. Under restrictive conditions, however, this sometimes happens: namely, the algebra \( \mathbb{D}A \), Koszul dual to \( A \), is a candidate. One can construct a natural map, which is an instance of our Poincaré/Koszul duality map,

\[
\text{HH}_*(\mathbb{D}A) \to \text{HH}_*(A)^\vee
\]

which is sometimes an equivalence; for instance, this is a formal exercise in the case \( A = \text{Sym}(V) \), for \( V \) a finite complex in strictly positive homological degrees. In general, we shall see, \( \text{HH}_*(A)^\vee \) is a type of completion of \( \text{HH}_*(\mathbb{D}A) \).

One can also take inspiration from S-duality: in Chern-Simons, the fundamental symmetry transforms the coupling constant by \( \hbar \sim -4\pi^2/\hbar \). Thus, the duality exchanges perturbative and non-perturbative parts of a field theory, a feature that fits suggestively with the nonconvergence in the inequality of forms the coupling constant by \( \hbar \).

Our two sources of motivation thus offer two pointers on where to look for Poincaré duality in factorization homology. The first says that the factorization homology with coefficients in \( A \) should be equivalent to some other construction, not factorization homology per se, but some cohomological variant. Our TQFT motivation suggests that the choice of coefficients for this factorizable generalization of cohomology should be related to the Koszul dual of \( A \) associated to a field theory which is no longer perturbative. Assembling these hints, one might conclude that such a Poincaré duality should relate the factorization homology with coefficients in \( A \) to a not necessarily perturbative form of factorization homology with coefficients related to the Koszul dual of \( A \), a generalization that one can contemplate after the originating works of [GiK] and [Pr].

There is, however, already a not necessarily perturbative form of factorization homology, defined in [Fr3]. Namely, for an \( n \)-manifold \( M \) and a scheme \( X \) whose structure sheaf \( \mathcal{O}_X \) is enhanced to form a sheaf of \( n \)-disk algebras, then one can define the factorization homology of \( M \) with coefficients in \( X \) as

\[
\int_M X = \Gamma\left(X, \int_M \mathcal{O}_X\right)
\]

the global sections over \( X \) of the sheaf obtained by computing the factorization homology of the structure sheaf. Intuitively, one can also think of this object as the functions on the space of maps from \( M \) to \( X \). We adopt the point of view, after Costello & Gwilliam, that factorization homology with coefficients in stacks over \( n \)-disk algebras offers a suitable generic model for the observables in a field theory which not necessarily perturbative.

There is, finally, just such a formal scheme associated to an \( n \)-disk algebra, lifting the functor of Koszul duality. Namely, there is a formal scheme \( \text{MC}_A \) whose ring of global functions is exactly the Koszul dual of \( A \). [Li14]: in characteristic zero, \( \text{MC}_A \) is a lift of the Maurer-Cartan space from Lie algebras to \( n \)-disk algebras. We now have all the ingredients necessary to state our main theorem; see Theorem 3.2.4.

**Theorem 0.0.1 (Poincaré/Koszul duality).** Let \( \overline{M} \) be a compact smooth \( n \)-dimensional cobordism with boundary partitioned as \( \partial\overline{M} \cong \partial_L \cup \partial_R \). For \( A \) an augmented \( n \)-disk algebra over a field \( k \) with \( \text{MC}_A \) the associated formal moduli functor of \( n \)-disk algebras, there is a natural equivalence

\[
\left(\int_{\overline{M} \setminus \partial_R} A\right)^\vee \simeq \int_{\overline{M} \setminus \partial_L} \text{MC}_A
\]

between the \( k \)-linear dual of the factorization homology of \( \overline{M} \setminus \partial_R \) with coefficients in \( A \) and the factorization homology of \( \overline{M} \setminus \partial_L \) with coefficients in the moduli functor \( \text{MC}_A \).
Note that if the boundary of the compact $n$-manifold $\overline{M}$ is empty, then the statement above reduces to the simpler expression of an equivalence
\[
\left( \int_{\overline{M}} A \right)^\vee \simeq \int_{\overline{M}} \text{MC}_A.
\]

This theorem coheres to our dual motivations. First, the result specializes to the dual of usual Poincaré duality by setting $A$ to be an algebra with respect to direct sum; in this case the left hand side becomes usual homology with coefficients in $A$, and the right hand side becomes usual cohomology with coefficients in an $n$-fold shift of $A$. Second, from the cobordism formulation one can see this result involves a duality for the extended topological field theories defined by the factorization homology with coefficients in $A$ and $\text{MC}_A$.

Given connectivity or coconnectivity hypotheses on the algebra $A$, one can replace the moduli problem $\text{MC}_A$ with its algebra of global sections $D^nA$, the Koszul dual of $A$. We have the following, combining Theorem 2.1.8, Theorem 2.1.6, and Proposition 3.3.2.

**Theorem 0.0.2.** Let $\overline{M}$ be a compact smooth $n$-dimensional cobordism with boundary partitioned as $\partial M \cong \partial_L \sqcup \partial_R$. Let $A$ be a finitely presented augmented $n$-disk algebra such that either:
- $A$ has a connected augmentation ideal;
- $A$ is an algebra over a field $k$ and the augmentation ideal of $A$ is $(-n)$-coconnective.

There is a natural equivalence
\[
\left( \int_{\overline{M} \setminus \partial_R} A \right)^\vee \simeq \int_{\overline{M} \setminus \partial_L} D^nA.
\]

This result is interesting even for dimension $n = 1$. The factorization homology of the circle is equivalent to Hochschild homology, so in this case we obtain a linear duality between the Hochschild homology of an associative algebra and either the Hochschild homology of the noncommutative moduli problem $\text{MC}_A$ or of the Koszul dual $A \simeq \text{Hom}_A(k, k)$. See Corollary [5.1.1]. This specialization is particularly comprehensible in the case where the algebra $A = U\mathfrak{g}$ is the enveloping algebra of a Lie algebra over a field of characteristic zero. In this case, we obtain a relation between the enveloping algebra of a Lie algebra $\mathfrak{g}$ and the formal moduli problem $\text{MC}_\mathfrak{g}$ controlled by the Lie algebra, i.e., whose values are given by the Maurer-Cartan space: $\text{MC}_\mathfrak{g}(R) = \text{MC}(\mathfrak{g} \otimes R) \simeq \text{Map}_{\text{Lie}}(TR[-1], \mathfrak{g})$. Then we have the following, which generalizes a result of Feigin & Tsygan in [FT].

**Theorem 0.0.3.** For $\mathfrak{g}$ a Lie algebra over a field of characteristic zero with associated formal moduli problem $\text{MC}_\mathfrak{g}$, there is a natural equivalence
\[
\text{HH}_*(U\mathfrak{g})^\vee \simeq \text{HH}_*(\text{MC}_\mathfrak{g}),
\]
between the dual of the Hochschild homology of the enveloping algebra of $\mathfrak{g}$ and the Hochschild homology of the moduli problem $\text{MC}_\mathfrak{g}$. If $\mathfrak{g}$ is finite-dimensional and concentrated in either homological degrees less than $-1$ or greater than 0, then there is an equivalence
\[
\text{HH}_*(U\mathfrak{g})^\vee \simeq \text{HH}_*(C^{\mathfrak{g}})
\]
between the dual of the Hochschild homology of the enveloping algebra and the Hochschild homology of Lie algebra cohomology.

**Remark 0.0.4.** All of our results are valid, with identical proofs, if smooth $n$-manifolds are replaced with $B$-structured smooth $n$-manifolds for any fibration $B \to BO(n)$. However, for simplicity of presentation we omit this notational clutter from most of the current work.

We now overview the contents of this paper, section by section.

In **Section 1**, we review the category $\mathcal{ZMfd}_n$ of zero-pointed $n$-manifolds and the factorization homology of zero-pointed manifolds from [AF1]. A zero-pointed manifold consists of a pointed topological space $M_*$, which is an $n$-manifold $M$ with an extra point $*$ and an extension of the topology of $M$ to $M_*$; the essential example is a space $\overline{M}/\partial \overline{M}$, the quotient of an $n$-manifold by its
boundary. This theory naturally incorporates functoriality for both embeddings and Pontryagin-Thom collapse maps of embeddings, a feature we employ in order to present a unified treatment of duality in homology/cohomology and algebra/coalgebra. More precisely, the zero-pointed theory provides additional functorialities for factorization homology with coefficients in an augmented $n$-disk algebra which, in particular, endows the factorization homology

$$\int_{(\mathbb{R}^n)^+} A$$

with the structure of an $n$-disk coalgebra, where $(\mathbb{R}^n)^+$ is the 1-point compactification of $\mathbb{R}^n$. Consequently, we arrive at a geometric presentation of an $n$-disk coalgebra structure on the $n$-fold iterated bar construction of an augmented $n$-disk algebra. We apply this to construct the Poincaré/Koszul duality map, which goes from factorization homology with coefficients in an $n$-disk algebra to factorization cohomology with coefficients in the Koszul dual $n$-disk coalgebra. We lastly recall a version of twisted Poincaré duality, which asserts that our duality map is an equivalence in the case of stable $\infty$-category with direct sum.

In Section 2, we introduce two (co)filtrations of factorization homology and cohomology. One comes from Goodwillie’s calculus of homotopy functors. A second comes from a cardinality filtration $\mathcal{D}isk_n^k$ of $\mathcal{D}isk_n$, which generalizes the Goodwillie-Weiss calculus filtration. We prove that the Poincaré/Koszul duality map exchanges the Goodwillie and the cardinality cofiltrations. That is, in this instance, we prove that Goodwillie calculus and Goodwillie-Weiss calculus are Koszul dual to one another. As a consequence, we obtain spectral sequences for factorization homology whose $E^1$ terms are identified as homologies of configuration spaces; for the circle, one of these spectral sequences generalizes the Bökstedt spectral sequence. Finally, in the case of our main theorem, we conclude that the Poincaré/Koszul duality map is an equivalence when the algebra $A$ is connected.

In Section 3, we introduce factorization homology with coefficients in a formal moduli problem. We prove that the Poincaré/Koszul duality map is an equivalence in the case of a $(-n)$-coconnective $n$-disk algebra over a field. We also prove an instance of Koszul duality proper, that there is equivalence between Artin $n$-disk algebras and finitely presented $(-n)$-coconnective $n$-disk algebras. Using this, we show our main theorem, that the moduli-theoretic Poincaré/Koszul duality map is an equivalence for any augmented $n$-disk algebra. We conclude by specializing these results to the case of associative algebras and Lie algebras in Section 4.

Remark 0.0.5. In this work, we use Joyal’s quasi-category model of $\infty$-category theory [Jo]. Boardman & Vogt first introduced these simplicial sets in [BV], as weak Kan complexes, and their and Joyal’s theory has been developed in great depth by Lurie in [LU1] and [LU2], our primary references; see the first chapter of [LU1] for an introduction. We use this model, rather than model categories or simplicial categories, because of the great technical advantages for constructions involving categories of functors, which are ubiquitous in this work. More specifically, we work inside of the quasi-category associated to this model category of Joyal’s. In particular, each map between quasi-categories is understood to be an iso- and inner-fibration; and (co)limits among quasi-categories are equivalent to homotopy (co)limits with respect to Joyal’s model structure.

We will also make use of topological categories, such as the topological category $\text{Mfld}_n$ of $n$-manifolds and embeddings. By a functor $\text{Mfld}_n \to \mathcal{C}$ from a topological category such as $\text{Mfld}_n$ to an $\infty$-category $\mathcal{C}$ we will always mean a functor $N\text{Sing} \text{Mfld}_n \to \mathcal{C}$ from the simplicial nerve of the simplicial category $\text{Sing} \text{Mfld}_n$ obtained by applying the singular functor $\text{Sing}$ to the hom spaces of the topological category.

The reader uncomfortable with this language can substitute the words “topological category” for “$\infty$-category” wherever they occur in this paper to obtain the correct sense of the results, but they should then bear in mind the proviso that technical difficulties may then abound in making the statements literally true. The reader only concerned with algebra in chain complexes, rather than spectra, can likewise substitute “pre-triangulated differential graded category” for “stable $\infty$-category” wherever those words appear, with the same proviso.
Terminology. [⊗-conditions] Throughout this document, we will use the letter \( C \) for a symmetric monoidal \( \infty \)-category, and \( \mathbb{1} \) for its symmetric monoidal unit. We will not distinguish in notation between it and its underlying \( \infty \)-category.

- We say \( C \) is \( \otimes \)-presentable if its underlying \( \infty \)-category is presentable and its symmetric monoidal structure \( \otimes \) distributes over small colimits separately in each variable. We say \( C \) is \( \otimes \)-stable-presentable if it is \( \otimes \)-presentable and its underlying \( \infty \)-category is stable.
- We say \( C \) is \( \otimes \)-cocomplete if its underlying \( \infty \)-category admits small colimits and its symmetric monoidal structure \( \otimes \) distributes over small colimits separately in each variable.
- We say \( C \) is \( \otimes \)-sifted cocomplete if its underlying \( \infty \)-category admits sifted colimits and its symmetric monoidal structure \( \otimes \) distributes over sifted colimits separately in each variable.

Acknowledgements. We thank Jacob Lurie for his great influence on our work; we use, in particular, his opuses [Lu1] and [Lu2] throughout. We thank Greg Arone for being generous with his ideas over several very helpful conversations on Goodwillie calculus. JF thanks Kevin Costello for offering many insights in many conversations over the years.

1. Review of reduced factorization homology

We recall some notions among zero-pointed manifolds and factorization (co)homology thereof, as established in [AF1].

1.1. Zero-pointed manifolds. We recall the extension of the symmetric monoidal topological category \( \mathbb{M}fld \) of topological \( n \)-manifolds and open embeddings among them (with compact-open topologies), to zero-pointed \( n \)-manifolds.

**Definition 1.1.1 (Zero-pointed manifolds).** An object of the symmetric monoidal topological category of zero-pointed manifolds \( \mathbb{Z}Mfld \) is a locally compact Hausdorff based topological space \( M^* \) for which the complement of the base point \( M := M^* \setminus * \) is a topological manifold. The topological space of morphisms \( \mathbb{Z}Emb(M^*, M'^*) \) consists of based maps \( f: M^* \to M'^* \) for which the restriction \( f|: f^{-1}M' \to M' \) is an open embedding, endowed with the compact-open topology. Composition is given by composing based maps. The symmetric monoidal structure is disjoint union. There is the full sub-symmetric monoidal topological category

\[
\mathbb{Z}Mfld_n \subset \mathbb{Z}Mfld
\]

consisting of those zero-pointed manifolds \( M^* \) for which \( M \) has dimension exactly \( n \). For a topological \( n \)-manifold \( M \), we denote by \( M^+ \) the zero-pointed manifold defined by \( M \) with a disjoint zero-point; \( M^+ \) is the zero-pointed manifold defined by the 1-point compactification of \( M \). We denote the full sub-symmetric monoidal categories of \( \mathbb{Z}Mfld_n \)

\[
\text{Disk}_{n,+} \subset \mathbb{Z}\text{Disk}_n \supset \text{Disk}_n^+
\]

which consist of wedge sums of \( \mathbb{R}^n_+ \), of \( \mathbb{R}^n_+ \) and \( (\mathbb{R}^n)^+ \), and of \( (\mathbb{R}^n)^+ \), respectively.

**Example 1.1.2 (Cobordisms).** Let \( \overline{M} \) be a cobordism, i.e., a compact manifold with partitioned boundary \( \partial \overline{M} = \partial_L \sqcup \partial_R \). The based topological space

\[
M^* := * \coprod_{\partial_L} (\overline{M} \setminus \partial_R)
\]

is a zero-pointed manifold.

Our results of this paper will make the following requirement of the topology around the zero-point: that it is is a conical singularity.
**Definition 1.1.3 (Conically finite).** We say a zero-pointed manifold \( M_* \) is *conically finite* if there is a smooth compact manifold \( \overline{M} \) with partitioned boundary \( \partial \overline{M} = \partial_L \sqcup \partial_R \) together with a based homeomorphism

\[
M_* \cong * \prod_{\partial_L} (\overline{M} \setminus \partial_R).
\]

The \( \infty \)-category of *conically finite* zero-pointed manifolds is the full sub-\( \infty \)-category

\[
\mathcal{Z}\text{Mfld}^{\text{fin}} \subset \mathcal{Z}\text{Mfld}
\]

consisting of the conically finite ones.

**Remark 1.1.4.** Not every zero-pointed manifold \( M_* \) is conically finite. The one-point compactification of an infinite genus surface illustrates this. For \( M \) a non-smoothable topological manifold, then \( M_* \) also illustrates this.

**Remark 1.1.5.** In the Definition 1.1.3 of conically finite, the smooth structure on \( \overline{M} \) and the homeomorphism \( M_* \cong * \prod_{\partial_L} (\overline{M} \setminus \partial_R) \) are not part of the datum of a conically finite zero-pointed manifold. This relieves us of pseudo-isotopy issues. For instance, for \( \overline{M} \) and \( M' \) two compact smooth manifolds with boundary whose interiors are homeomorphic, then the zero-pointed manifolds

\[
* \prod_{\partial M} \overline{M} \cong * \prod_{\partial M'} M'
\]

are equivalent in the \( \infty \)-category \( \mathcal{Z}\text{Mfld} \), and the set of equivalences (up to equivalence) is parametrized by the \( h \)-cobordisms of \( \partial \overline{M} \).

**Remark 1.1.6.** Consider the notation of Definition 1.1.3. A zero-pointed embedding from \(* \prod_{\partial_L} (\overline{M} \setminus \partial_R) \) to \(* \prod_{\partial_L} (\overline{M'} \setminus \partial_{R'}) \) is vastly different from an embedding from \( \overline{M} \) to \( M' \) that respect the partitioned boundaries in any sense.

We now catalogue some facts about \( \mathcal{Z}\text{Mfld}_n \) that are proven in [AF1]. Note first the continuous functor \( \text{Mfld}_n \to \text{Mfld}_{n+} \), given by \( M \mapsto M_* \), which is symmetric monoidal.

**Theorem 1.1.7 ([AF1]).** Let \( \mathcal{C} \) be a symmetric monoidal \( \infty \)-category.

- There is a contravariant involution \( \neg \) : \( \mathcal{Z}\text{Mfld}_n \cong \mathcal{Z}\text{Mfld}^{op}_n \), \( \neg \) which sends a zero-pointed manifold

\[
M_* \mapsto (M_*)^+ \setminus *,
\]

the 1-point compactification of \( M \) minus the original zero-point.

- There is a canonical equivalence of \( \infty \)-categories

\[
\text{Fun}^\otimes (\text{Mfld}_{n+}, \mathcal{C}) \xrightarrow{\cong} \text{Fun}^\otimes_{\text{aug}} (\text{Mfld}_n, \mathcal{C})
\]

to augmented symmetric monoidal functors from \( \text{Mfld}_n \).

Let \( i \) be a finite cardinality, and let \( M_* \) be a zero-pointed manifold. There is an open embedding \( \text{Conf}_i(M) \to (M_*)^{\wedge i} \) of the configuration space into the iterated smash product. We denote

\[
\text{Conf}_i(M_*) \quad \text{and} \quad \text{Conf}_i^\wedge(M_*)
\]

for locally compact Hausdorff topologies on the underlying set of \( \text{Conf}_i(M) \Pi * \): the first has the coarsest topology such for which the evident inclusion \( \text{Conf}_i(M) \Pi * \to (M_*)^{\wedge (1,...,i)} \) is continuous; the second has the finest topology such for which the collapse map \( (M_*)^{\wedge (1,...,i)} \to \text{Conf}_i(M) \Pi * \) is continuous.

**Proposition 1.1.8 ([AF1]).** For each finite cardinality \( i \) and each zero-pointed \( n \)-manifold \( M_* \), the based spaces \( \text{Conf}_i(M_*) \) and \( \text{Conf}_i^\wedge(M_*) \) exist and are zero-pointed \( (ni) \)-manifolds. Further, the given maps

\[
\text{Conf}_i(M_*) \to (M_*)^{\wedge i} \to \text{Conf}_i^\wedge(M_*)
\]

...
are morphisms of zero-pointed manifolds. They bear a canonical relation \( \text{Conf}_i(M) \cong \text{Conf}_i^+(M^-) \). If \( M \) is conically finite, then both \( \text{Conf}_i(M) \) and \( \text{Conf}_i^+(M^-) \) are conically finite; and for any abelian group \( A \) the singular homology \( H_q(\text{Conf}_i(M_n); A) \) vanishes for \( q > n\ell + (n - 1)(i - \ell) \), where \( \ell \) is the number of components of \( M \) and \( i \) is greater than \( \ell \).

**Example 1.1.9.** In the case \( M = M^+ \), then \( \text{Conf}_i(M) = \text{Conf}_i(M^+) \), and the coconnectivity statement of Proposition 1.1.8 follows by induction on \( i \) through the standard fibration sequence \( M \setminus \{x_1, \ldots, x_i\} \rightarrow \text{Conf}_{i+1}(M_n) \rightarrow \text{Conf}_i(M_n) \).

**Remark 1.1.10.** In Definition 1.1.3 we could not do without the smoothness assumption on \( M \), and still guarantee that conical finiteness of \( M \) implies conical finiteness of \( \text{Conf}_i^+(M^-) \). For instance, in the case that \( M \) is a compact topological \( n \)-manifold, then there is an obstruction to witnessing a based homeomorphism \( \text{Conf}_2(M) \cong \ast \amalg \mathbb{Z} \) with \( Z \) a topological manifold with boundary. This obstruction is that of a lift of the tangent classifier \( \tau_M : M \rightarrow \text{BTop}(n) \) to \( \text{BHomeo}(\mathbb{D}^n) \), and the map \( \text{BHomeo}(\mathbb{D}^n) \rightarrow \text{BTop}(n) \) is only known to be \( \sim \frac{3}{2} \)-connected.

**Remark 1.1.11 (B-structures).** In [AF1] we explain a theory of zero-pointed manifolds equipped with a \( B \)-structure, where \( B \rightarrow \text{BTop}(n) \) is a map of spaces. We will make use of this generalization for the simple case where \( B \rightarrow \text{BTop}(ni) \) is the classifying space of the block-sum homomorphism \( \Sigma \amalg \text{Top}(n) \rightarrow \text{Top}(ni) \), and the \( B(\Sigma \amalg \text{Top}(n)) \)-structured zero-pointed manifolds are of the form \( \text{Conf}_i(M_n) \) and \( \text{Conf}_i^+(M_n) \).

1.2. **Reduced factorization homology.** Theorem 1.1.7 justifies the following definitions.

**Definition 1.2.1 (Reduced factorization (co)homology).** Let \( \mathcal{C} \) be a symmetric monoidal \( \infty \)-category.

The \( \infty \)-categories of augmented \( n \)-disk algebras and of augmented \( n \)-disk coalgebras, respectively, are those of symmetric monoidal functors

\[
\mathcal{A}l_{\text{aug}}(\mathcal{C}) := \text{Fun}^\otimes(\text{Disk}_{n,+}, \mathcal{C}) \quad \text{and} \quad \mathcal{C} \mathcal{A}l_{\text{aug}}(\mathcal{C}) := \text{Fun}^\otimes(\text{Disk}_n^+, \mathcal{C}).
\]

Restrictions along the inclusions \( \text{Disk}_{n,+} \hookrightarrow \mathcal{Z} \mathfrak{m} d_{n} \hookleftarrow \text{Disk}_n^+ \) have (a priori partially defined) adjoints depicted in the diagram

\[
\begin{array}{ccc}
\text{Bar} : & \mathcal{A}l_{\text{aug}}(\mathcal{C}) & \overset{f}{\longrightarrow} \mathcal{F} \text{un}^\otimes(\mathcal{Z} \mathfrak{m} d_{n}, \mathcal{C}) & & \mathcal{C} \mathcal{A}l_{\text{aug}}(\mathcal{C}) : \text{cBar} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Disk}_{n,+} & \overset{\mathcal{D} \mathfrak{m} d_{n,+}}{\longrightarrow} & \text{Disk}_n^+ & \overset{\mathcal{D} \mathfrak{m} d_{n,+}}{\longrightarrow} & \text{Disk}_n^+ \\
\end{array}
\]

whose left and right composites are as depicted. Explicitly, for \( A \) an augmented \( n \)-disk algebra, \( C \) an augmented \( n \)-disk coalgebra, and \( M \) a zero-pointed \( n \)-manifold, the values of these adjoints are given as

\[
\int_M A := \text{colim} \left( (\text{Disk}_{n,+})_{/M} \rightarrow \text{Disk}_{n,+} \overset{A}{\longrightarrow} \mathcal{C} \right)
\]

and

\[
\int_M C := \lim' \left( (\text{Disk}_n^+)_{/M} \rightarrow \text{Disk}_n^+ \overset{C}{\longrightarrow} \mathcal{C} \right)
\]

which we refer to respectively as the factorization homology \( M \) with coefficients in \( A \), and as the factorization cohomology of \( M \) with coefficients in \( C \) — the latter which is hand-crafted to be contravariant, to fit prior examples.

**Theorem 1.2.2 (AF1).** Let \( \mathcal{C} \) be a symmetric monoidal \( \infty \)-category which admits sifted colimits. If \( M \) is conically finite, then \( \int_M A \) exists and \( \int A \) depicts a covariant functor to \( \mathcal{C} \) from conically finite zero-pointed \( n \)-manifolds. In addition, \( \int A \) depicts a symmetric monoidal functor from conically finite zero-pointed \( n \)-manifolds provided \( \mathcal{C} \) is \( \otimes \)-sifted cocomplete. The dual result holds for \( \int^C \).
There is a canonical comparison arrow between factorization homology and factorization cohomology.

**Theorem 1.2.3** (Poincaré/Koszul duality map ([AFT1])). Let $\mathcal{C}$ be a symmetric monoidal functor for which $\mathcal{C}$ admits sifted colimits and cosifted limits. Let $A$ be an $n$-disk algebra in $\mathcal{C}$. Let $M_*$ be a conically finite zero-pointed $n$-manifold. There is a canonical arrow in $\mathcal{C}$

\[ \int_{M_*} A \rightarrow \int_{M_*^c} \text{Bar } A \]

which is functorial in $M_*$ and $A$.

### 1.3. Exiting disks.

The slice $\infty$-category $\text{Disk}_{n,+}/M_*$ appears in the defining expression for factorization homology. We review a variant of this $\infty$-category, $\text{Disk}_+(M_*)$, of exiting disks in $M_*$, which offers several conceptual and technical advantages. Heuristically, objects of $\text{Disk}_+(M_*)$ are embeddings from finite disjoint unions of basics into $M$, while morphisms are isotopies of such to embeddings with some of these isotopies witnessing disks slide off to infinity where they are forgotten – disks are not allowed to slide in from infinity, unlike in $\text{Disk}_{n,+}/M_*$.

To define this $\infty$-category of exiting disks, we require some regularity near the zero-point. This is accomplished by the notion of a **stratified space** introduced in another work ([AFT1]).

**Definition 1.3.1.** Let $M_*$ be a zero-pointed manifold. By a **conically smooth** structure on $M_*$ we mean a structure as a stratified space on $M_*$ in the sense of [AFT1]. We say $M_*$ is **conically smoothable** if it admits a conically smooth structure. A **conically smooth** zero-pointed manifold is a zero-pointed manifold $M_*$ together with a conically smooth structure (that will not appear in its notation).

**Remark 1.3.2.** The unzipping construction of [AFT1] grants that a conically smooth structure on $M_*$ is a smooth manifold $\overline{M}$ with compact boundary together with a based homeomorphism $M_* \cong * \coprod_{\partial M} \overline{M}$.

For this subsection, fix a conically smooth zero-pointed manifold $M_*$. In [AFT2] we define, for each stratified space $X$, the $\infty$-category

\[ \text{Disk}(\text{Bsc})_{/X} \]

of finite disjoint unions of basics embedding into $X$. This is a stratified version of $\text{Disk}_{n/M_*}$.

**Definition 1.3.3** ($\text{Disk}_+(M_*)$). The $\infty$-category of exiting disks of $M_*$ is the full $\infty$-subcategory

\[ \text{Disk}_+(M_*) \subset \text{Disk}(\text{Bsc})_{/M_*} \]

consisting of those $V \hookrightarrow M_*$ whose image contains $*$. Explicitly, an object of $\text{Disk}_+(M_*)$ is a conically smooth open embedding $B \cup U \hookrightarrow M_*$ where $B \cong \mathbb{C}(L)$ is a cone-neighborhood of $* \in M_*$ and $U$ is abstractly diffeomorphic to a finite disjoint union of Euclidean spaces, and a morphism is an isotopy to an embedding among such. We use the notation

\[ \text{Disk}^+(M_*^+) := \text{Disk}_+(M_*)^{\text{op}}. \]

The unique zero-pointed embedding $* \hookrightarrow M_*$ induces the functor

\[ \text{Disk}_{n,+} = \text{Disk}_{n,+}/* \to \text{Disk}_{n,+}/M_* \]

**Theorem 1.3.4** ([AFT]).

1. The $\infty$-category $\text{Disk}_+(M_*)$ is sifted.
2. There is a final functor

\[ \text{Disk}_+(M_*) \rightarrow (\text{Disk}_{n,+}/M_*)/(\text{Disk}_{n,+}) \]

whose value on $(B \cup U \hookrightarrow M_*)$ is represented by $(U_+ \hookrightarrow M_*) \in \text{Disk}_{n,+}/M_*$. 

Consider the composite functor
\[ \text{Alg}_n^{\text{aug}}(\mathcal{C}) \to \text{Fun}_{\text{Disk}_{n,+}/M_*}(\mathcal{C}) \xrightarrow{\text{Lan}_*} \text{Fun}_{(\text{Disk}_{n,+}/M_*)/\text{Disk}_{n,+}} \to \text{Fun}(\text{Disk}_+(M_*), \mathcal{C}) : \]

the first arrow is restriction along the projection $\text{Disk}_{n,+}/M_* \to \text{Disk}_{n,+}$; the second arrow is left Kan extension along the quotient functor $\text{Disk}_{n,+}/M_* \to (\text{Disk}_{n,+}/M_*)/(\text{Disk}_{n,+})$; the third arrow is restriction along that asserted in Theorem 1.3.4.

**Notation 1.3.5.** Given an augmented $n$-disk algebra $A : \text{Disk}_{n,+} \to \mathcal{C}$, we will use the same notation $A : \text{Disk}_+(M_*) \to \mathcal{C}$ for the value of the functor (2) on $A$.

We content ourselves with this Notation 1.3.5 because of the immediate corollary of Theorem 1.3.4.

**Corollary 1.3.6.** Let $\mathcal{C}$ be a symmetric monoidal $\infty$-category whose underlying $\infty$-category admits sifted colimits. Let $A : \text{Disk}_{n,+} \to \mathcal{C}$ be an augmented $n$-disk algebra, and let $C : \text{Disk}_+ \to \mathcal{C}$ be an augmented $n$-disk coalgebra. There are canonical identifications in $\mathcal{C}$:

\[
\int_{M_*} A \simeq \colim_{(B \cup U \to M_*) \in \text{Disk}_+(M_*)} A(U_+),
\]

and

\[
\int_{M_*} C \simeq \lim_{(B \cup U \to M_*) \in \text{Disk}_+(M_*)} C(V^+).
\]

1.4. **Free and trivial algebras.** We give two procedures for constructing augmented $n$-disk algebras. In this subsection we fix a symmetric monoidal $\infty$-category $\mathcal{C}$ which is $\otimes$-presentable. From Corollary 3.2.3.5 of [Lu2], the $\infty$-category $\text{Alg}_n^{\text{aug}}(\mathcal{C})$ is presentable.

**Definition 1.4.1 (Top($n$)-modules).** Let $G$ be a topological group and let $\mathcal{C}$ be an $\infty$-category. The $\infty$-category of $G$-modules is the functor category

\[
\text{Mod}_G(\mathcal{C}) := \text{Fun}(BG, \mathcal{C})
\]

from the $\infty$-groupoid associated to $G$.

Throughout this work we will make serial use of the following classical result, the Kister-Mazur Theorem. Recall $\text{Top}(n)$, homeomorphisms of $\mathbb{R}^n$; endowed with the compact-open topology, it forms a topological group.

**Theorem 1.4.2 ([Ki]).** The inclusion of topological monoids $\text{Top}(n) \to \text{Emb}(\mathbb{R}^n, \mathbb{R}^n)$ is a weak homotopy equivalence of underlying topological spaces.

Write $\mathcal{C}_1//_{/1}$ for the $\infty$-category of objects $E \in \mathcal{C}$ equipped with a retraction onto the symmetric monoidal unit: $\text{id}_1 : 1 \to E \to 1$. Note that if $\mathcal{C}$ is stable, then there is a natural equivalence $\mathcal{C}_1//_{/1} \xrightarrow{\sim} \mathcal{C}$ sending an object $1 \to E \to 1$ to the cokernel of the unit $c\text{Ker}(1 \to E)$. Write $\text{Disk}_{n,+} \subseteq \text{Disk}_{n,+}$ for the full $\infty$-subcategory consisting of those zero-pointed Euclidean spaces with at most one non-base component. By Theorem 1.4.2 this full $\infty$-subcategory is initial among pointed $\infty$-categories under $\text{BTop}(n)$. In other words, there there is a canonical equivalence of $\infty$-categories

\[
\text{Fun}_1(\text{Disk}_{n,+} \subseteq \text{Disk}_{n,+}, \mathcal{C}) \xrightarrow{\sim} \text{Mod}_{\text{Top}(n)}(\mathcal{C}_1//_{/1})
\]

where the source is functors whose value on $*$ is a symmetric monoidal unit of $\mathcal{C}$; the target is $\text{Top}(n)$-modules in retractive objects over the unit.

**Definition 1.4.3 (Free).** Restriction along $\text{Disk}_{n,+} \subseteq \text{Disk}_{n,+}$ determines the solid arrow, referred to as the underlying $\text{Top}(n)$-module:

\[
(3) \quad \text{Alg}_n^{\text{aug}}(\mathcal{C}) \xrightarrow{\text{Mod}_{\text{Top}(n)}(\mathcal{C}_1//_{/1})}.\]
This forgetful functor preserves limits, and so there is a left adjoint, as depicted, referred to as the augmented free functor.

We next explain the following diagram of $\infty$-categories:

$$
\begin{array}{cccc}
\text{Disk}^{\leq 1}_{n,+} & \longrightarrow & \text{Disk}^{\text{finj}}_{n,+} & \longrightarrow & \text{Disk}^{\text{finj}}_{n,+} \\
\downarrow & & \downarrow & & \downarrow \\
\text{Fin}^{\leq 1}_n & \longrightarrow & \text{Fin}^{\text{finj}}_n & \longrightarrow & \text{Fin}^{\text{finj}}_n.
\end{array}
$$

$\text{Fin}_n$ is the category of based finite sets, $\text{Fin}^{\leq 1}_n$ is the full subcategory of those $I_+$ whose cardinality is bounded as $|I| \leq 1$, and $\text{Fin}^{\text{finj}}_n$ is the subcategory of those based maps $I_+ \to J_+$ for which the restriction $f_! : f^{-1}J \to J$ is injective – this latter subcategory is symmetric monoidal. The right vertical arrow is given by taking connected components, which is symmetric monoidal. The top left and top middle $\infty$-categories are defined so that the diagram consists of pullback squares – the top middle $\infty$-category is thus symmetric monoidal.

We next define the righthand functor $t_{\text{aug}} : \text{Fin}^\ast_n \to \text{Fin}^{\text{finj}}_n$. For a morphism $f : I_+ \to J_+$, let $J^f_+$ be the maximal subset of $J_+$ such that $f^{-1}(J^f_+) \to J_+$ is injective. Note that there is a projection map $I_+ \to f^{-1}(J^f_+)$ sending the complement of $f^{-1}(J^f_+)$ to the base point $+$. The functor $t_{\text{aug}}$ is defined to be the identity on objects; on morphisms, $t_{\text{aug}}$ sends a map $f : I_+ \to J_+$ to the composite map $I_+ \to f^{-1}(J^f_+) \to J_+$ given by the projection map $I_+ \to f^{-1}(J^f_+)$ followed by the map $f$. Note that $t_{\text{aug}}$ is the identity on the subcategory $\text{Fin}_n^{\text{finj}}$. After Theorem 1.4.2 the vertical arrows are conservative.

Lemma 1.4.4. Let $\mathcal{C}$ be a symmetric monoidal $\infty$-category. Restriction along the full inclusion $\text{Disk}^{\leq 1}_{n,+} \to \text{Disk}^{\text{finj}}_{n,+}$ implements an equivalence of $\infty$-categories

$$
\text{Fun}(\text{Disk}^{\text{finj}}_{n,+}, \mathcal{C}) \xrightarrow{\cong} \text{Mod}_{\text{Top}(n)}(\mathcal{C}_1//1).
$$

Proof. The inclusion $\text{Disk}^{\leq 1}_{n,+} \to \text{Disk}^{\text{finj}}_{n,+}$ realizes $\text{Disk}^{\text{finj}}_{n,+}$ as the free symmetric monoidal $\infty$-category on $\text{Disk}^{\leq 1}_{n,+}$ subject to the relation that $*$ is the symmetric monoidal unit. This follows from the corresponding statement for $\text{Fin}^{\leq 1}_n \to \text{Fin}^{\text{finj}}_n$, which is readily verified.

We make the next definition through Lemma 1.4.4.

Definition 1.4.5 (Cotangent space). Through the equivalence of Lemma 1.4.4, restriction along $t_{\text{aug}} : \text{Disk}^{\text{finj}}_{n,+} \to \text{Disk}^{\text{finj}}_{n,+}$ determines the solid arrow, referred to as the augmented trivial functor:

$$
\text{Alg}_{t_{\text{aug}}}(\mathcal{C}) \xleftarrow{t_{\text{aug}}} \text{Mod}_{\text{Top}(n)}(\mathcal{C}_1//1).
$$

This functor $t_{\text{aug}}$ has a left adjoint $L_{\text{aug}}$ that we refer to as the augmented cotangent space functor.

Intuition. In not homotopy coherent terms, the augmented $n$-disk algebra $t_{\text{aug}}V$ has $V$ as its underlying $\text{Top}(n)$-module and has trivial multiplication, which is to say that each multiplication map $V \otimes V \to V$ factors as the augmentation followed by the unit: $V \otimes V \to 1 \otimes 1 \simeq 1 \to V$.

The cotangent space and the free functor cancel each other.

Lemma 1.4.6. Let $\mathcal{C}$ be a $\otimes$-presentable symmetric monoidal $\infty$-category. There is a canonical equivalence of endofunctors of $\text{Mod}_{\text{Top}(n)}(\mathcal{C}_1//1)$:

$$
L_{\text{aug}} \circ F_{\text{aug}} \xrightarrow{\cong} \text{id}.
$$
Proof. The arrow is adjoint to the canonical arrow $id \to (t^{aug})_{|\text{disk}_{n,1}^\leq 1}$ between endofunctors of $\text{Mod}_{\text{Top}(n)}(\mathcal{C}_1//1)$. Because $\text{disk}_{n,+}^{\text{inj}} \to \text{disk}_{n,+}$ is an equivalence on underlying $\infty$-groupoids, this canonical arrow is an equivalence. 

\[\square\]

1.4.1. Stable case. In this section, fix a $\otimes$-stable-presentable symmetric monoidal $\infty$-category $\mathcal{C}$. Recall that $\mathcal{C}$ is naturally tensored over the $\infty$-category of pointed spaces. Using this structure, we define a functor \(\text{Mod}_{\text{Top}(n)}(\text{Spaces}_*) \times \text{Mod}_{\text{Top}(n)}(\mathcal{C}) \to \mathcal{C}\) by the following composite:

\[
\begin{array}{c}
\text{Mod}_{\text{Top}(n)}(\text{Spaces}_*) \times \text{Mod}_{\text{Top}(n)}(\mathcal{C}) \\
\downarrow \\
\text{Mod}_{\text{Top}(n)}(\text{Spaces}_*) \times \text{Mod}_{\text{Top}(n)}(\mathcal{C}) \\
\downarrow \\
\text{Mod}_{\text{Top}(n)}(\mathcal{C})
\end{array}
\]

where the second step is given by the tensoring operation $\text{Spaces}_* \times \mathcal{C} \to \mathcal{C}$; the third step is restriction along the diagonal map $\text{Top}(n) \to \text{Top}(n) \times \text{Top}(n)$; the last step is taking the coinvariants of the action by $\text{Top}(n)$. Dually, we define a functor

\[\text{Map}_{\text{Top}(n)}(-,-) : \text{Mod}_{\text{Top}(n)}(\text{Spaces}_*)^{\text{op}} \times \text{Mod}_{\text{Top}(n)}(\mathcal{C}) \to \mathcal{C}\]

by substituting the cotensor $(\text{Spaces}_*)^{\text{op}} \times \mathcal{C} \to \mathcal{C}$ for the tensor, and invariants for coinvariants.

In the case that $\mathcal{C}$ is stable there is an equivalence $\text{Ker}_{\text{aug}} : \mathcal{C}_1//1 \simeq \mathcal{C} : \nabla \oplus (-)$, and thereafter an equivalence

\[\text{Ker}_{\text{aug}} : \text{Mod}_{\text{Top}(n)}(\mathcal{C}_1//1) \simeq \text{Mod}_{\text{Top}(n)}(\mathcal{C}) : \nabla \oplus (-).
\]

Notation 1.4.7 ($F$ and $L$). In the case that $\mathcal{C}$ is stable, we denote

\[F : \text{Mod}_{\text{Top}(n)}(\mathcal{C}) \overset{\simeq}{\longrightarrow} \text{Mod}_{\text{Top}(n)}(\mathcal{C}_1//1) \overset{\text{aug}}{\longrightarrow} \text{Alg}_{\text{aug}}(\mathcal{C}),\]

\[L : \text{Alg}_{\text{aug}}(\mathcal{C}) \overset{\text{aug}}{\longrightarrow} \text{Mod}_{\text{Top}(n)}(\mathcal{C}_1//1) \overset{\simeq}{\longrightarrow} \text{Mod}_{\text{Top}(n)}(\mathcal{C}) : t.\]

In this stable case, let us make $F$ explicit, and recognize $L$. The underlying $\text{Top}(n)$-module of the value of $F$ on a $\text{Top}(n)$-module $V$ is

\[F(V) \simeq \bigoplus_{i \geq 0} \left( \text{Conf}_i^\text{fr}(\mathbb{R}^n_* \times_{\Sigma_i} \text{Top}(n)) \otimes V^{\otimes i} \right).
\]

Stability of $\mathcal{C}$ implies stability of $\text{Mod}_{\text{Top}(n)}(\mathcal{C})$. In the solid diagram among $\infty$-categories

\[
\begin{array}{c}
\text{Stab}(\text{Alg}_{\text{aug}}(\mathcal{C})) \\
\downarrow \alpha \\
\text{Alg}_{\text{aug}}(\mathcal{C}) \\
\downarrow L \\
\text{Mod}_{\text{Top}(n)}(\mathcal{C})
\end{array}
\]

there is a canonical filler, from the stabilization, as a colimit preserving functor. See §7.3.4 of [Lu2] or Proposition 2.23 of [Fr2], which state that the functor $\alpha$ is an equivalence, and so the horizontal functor in the above diagram witnesses $\text{Mod}_{\text{Top}(n)}(\mathcal{C})$ as a stabilization of $\text{Alg}_{\text{aug}}(\mathcal{C})$.  

\[\square\]
1.5. Linear Poincaré duality. In the case that the symmetric monoidal $\infty$-category is of the form $S^{\otimes}$, with underlying $\infty$-category $S$ stable and presentable, and whose symmetric monoidal structure is given by direct sum, factorization homology and factorization cohomology profoundly simplify. Here we state Poincaré duality in this simplified setting.

For $X$ a small $\infty$-category with a zero-object, we denote by $\mathbf{PShv}_*(X)$ the $\infty$-category of those (space-valued) presheaves on $X$ whose value on the zero-object is $*$.

**Definition 1.5.1** (Frame bundle). The frame bundle functor is the composition

$$\operatorname{Fr}: \mathcal{Z}\text{Mfld} \rightarrow \mathbf{PShv}_*(\mathcal{Z}\text{Mfld}) \rightarrow \mathbf{PShv}(\mathcal{B}\text{Top}(n))^{*/} \simeq \text{Mod}_{\text{Top}(n)=\text{Spaces}}(\text{Spaces}_*)$$

of the Yoneda embedding, followed by restriction along the full subcategory $\mathcal{D}\text{isk}_{n+1}^{\leq 1} \subset \mathcal{Z}\text{Mfld}$ – this subcategory is initial among $\infty$-categories under $\mathcal{B}\text{Top}(n)$ with a zero-object. Explicitly, $\operatorname{Fr}_{M_*}$ can be identified as the pointed space $\mathcal{Z}\text{Emb}(\mathbb{R}^n_+, M_*)$ with $\mathcal{B}\text{Top}(n)$-action given by precomposition by self-homeomorphisms of $\mathbb{R}^n$. **Remark 1.5.4** (With $B$-structures). We follow up on Remark 1.1.11. There is a version of Theorem 1.5.3 that is also true in the context of $B$-manifolds – it is stated and proved there. We will make use of this version as it applies to $B(\Sigma_i \sqcup \text{Top}(n))$-structured zero-pointed manifolds of the form $\text{Conf}_{i+}(M_*)$.

**Theorem 1.5.3** (Linear Poincaré duality [AF1]). Let $S$ be a stable and presentable $\infty$-category, and let $E$ and $F$ be $\text{Top}(n)$-modules in $S$. Let $M_*$ be a conically finite zero-pointed $n$-manifold. A morphism of $\text{Top}(n)$-modules $\alpha: (\mathbb{R}^n)^+ \otimes E \rightarrow F$ canonically determines a morphism in $S$

$$\alpha_{M_*}: \operatorname{Fr}_{M_*} \otimes_{\text{Top}(n)} E \rightarrow \text{Map}^{\text{Top}(n)}(\operatorname{Fr}_{M_*}, F).$$

Furthermore, if $\alpha$ is an equivalence then so is $\alpha_{M_*}$.

**Example 2.0.5**. Here are some standard examples of such entities.

- Let $\overline{M}$ be a smooth cobordism from $\partial_L =: \partial_L$ to $\partial_R =: \partial_R$. In this case, $M_+^* = \ast \bigsqcup \partial_R (\overline{M} \setminus \partial_L)$, and

$$\text{Conf}_{i+}(M_*) = \ast \bigsqcup_B \{f: \{1, \ldots, i\} \rightarrow \overline{M} \setminus \partial_R\}.$$
where \( B = \{ f \mid \emptyset \neq f^{-1}\partial M \} \), and
\[
\text{Conf}^i(M_\ast) = \bigtimes_{B'} \{ f: \{1, \ldots, i\} \to \overline{M} \setminus \partial \}
\]
where \( B' = \{ f \mid \emptyset \neq f^{-1}\partial M \text{ or } |f^{-1}| > 1 \text{ for some } x \in \overline{M} \} \).

- Write \( \text{Ch}_k \) for the \( \infty \)-category of chain complexes over a commutative ring \( k \). Likewise, write \( \text{Spec}_k \) the \( \infty \)-category of modules over a ring spectrum \( R \). Then \( \text{Ch}_k^\lor, \text{Ch}_k^\lor, \text{Spec}_k^\lor, \) and \( \text{Spec}_k^\lor \) are examples of such symmetric monoidal \( \infty \)-categories. In general, any such \( \mathcal{C} \) is symmetric monoidally tensored and cotensored over finite spaces.

- For \( S \) a stable presentable \( \infty \)-category, let \( E \in S \) be an object. The assignment \( A: U_+ \to E^{U_+} \) depicts an augmented \( n \)-disk algebra in \( S^\oplus \), its underlying object is (non-canonically) identified as \( E[-n] \simeq \Omega^n E \). Likewise, the assignment \( C: U_+ \to U_+ \otimes E \) depicts an augmented \( n \)-disk coalgebra in \( S^\oplus \). In general, any such \( \mathcal{C} \) is substantially more interesting, and also more involved, depending on the specifics of \( \mathcal{C} \).

2.1. Main results. Here we display the main results in this section and prove them based upon results developed in latter subsections.

2.1.1. Cardinality (co)filtration. We observe a natural filtration of factorization homology, and identify the filtration quotients; we do likewise for factorization cohomology.

Write \([M_*]\) for the set of connected components of \( M_* \), which we regard as a based set. For each finite cardinality \( i \), we will denote the subcategories of based finite sets
\[
(\text{Fin}_{\leq i})/[M_*] \subset (\text{Fin}_{\ast})/[M_*] \subset (\text{Fin}_{\ast})/[M_*]
\]
where an object of the middle is a surjective based map \( I_+ \to [M_*] \), and a morphism between two such is a surjective map over \([M_*]\); and where the left is the full subcategory consisting of those \( I_+ \to [M_*] \) for which the cardinality \(|I| \leq i \) is bounded. Taking connected components gives a functor
\[
[-]: \text{Disk}_+(M_*) \to (\text{Fin}_{\ast})/[M_*]
\]
to based finite sets over the based set of connected components of \( M_* \).

Definition 2.1.1 \((\text{Disk}_{\leq i}(M_*)). \) We define the \( \infty \)-category
\[
\text{Disk}_{\leq i}(M_*) := \text{Disk}_+(M_*)/[\text{Fin}_{\ast}]/[M_*] \subset \text{Disk}_+(M_*)
\]
and, for each finite cardinality \( i \), the full \( \infty \)-subcategory
\[
\text{Disk}_{\leq i}(M_*) := \text{Disk}_+(M_*)/[\text{Fin}_{\leq i}]/[M_*] \subset \text{Disk}_+(M_*)
\]
We denote the opposites:
\[
\text{Disc}_{\geq i} := (\text{Disc}^{-}_{\leq i}(M_*))^\circ \text{ and } \text{Disc}_{\leq i} := (\text{Disc}^{-}_{\leq i}(M_*))^\circ.
\]

Definition 2.1.2. Let \( i \) be a finite cardinality. We define the object of \( \mathcal{C} \)
\[
\tau_{\leq i} \int_{M_*} A := \colim_{(B, U_+ \to M_*) \in \text{Disk}_{\leq i}(M_*)} A(U_+) = \colim \left( \text{Disk}_{\leq i}(M_*) \to \text{Disk}_+(M_*) \xrightarrow{A} \mathcal{C} \right).
\]
We define the object of \( \mathcal{C} \)
\[
\tau_{\leq i} \int_{M_*} C := \lim_{(B, V_+ \to M_*) \in \text{Disk}_{\geq i}(M_*)} C(V_+) = \lim \left( \text{Disk}_{\geq i}(M_*) \to \text{Disk}_+(M_*) \xrightarrow{C} \mathcal{C} \right).
\]
We likewise define such objects for the comparison \( "\leq i" \) replaced by other comparisons among finite cardinalities, such as \( "\geq i" \) and \( "= i" \).
The following $\mathbb{Z}_{\geq 0}$-indexed sequence of fully faithful functors
\[
\cdots \longrightarrow \text{Disk}^{\leq i}_{+}(M_{*}) \longrightarrow \text{Disk}^{\leq i+1}_{+}(M_{*}) \longrightarrow \cdots \longrightarrow \text{Disk}^{\text{surj}}_{+}(M_{*})
\]
 witnesses $\text{Disk}^{\text{surj}}_{+}(M_{*})$ as a sequential colimit. There results a canonical sequence in $\mathcal{C}$
\[
\cdots \longrightarrow \tau^{\leq i-1} \int_{M_{*}} A \longrightarrow \tau^{\leq i} \int_{M_{*}} A \longrightarrow \cdots \longrightarrow \int_{M_{*}} A .
\]
Dually, there is a canonical sequence in $\mathcal{C}$
\[
\int_{M_{*}} C \longrightarrow \cdots \longrightarrow \tau^{\leq i} \int_{M_{*}} C \longrightarrow \tau^{\leq i-1} \int_{M_{*}} C \longrightarrow \cdots .
\]
There are likewise sequences with $\tau^{\leq -}$ replaced by $\tau^{\geq -}$.

**Lemma 2.1.3** (Cardinality convergence). *The morphism in $\mathcal{C}$ from the colimit of the cardinality sequence
\[
\tau^{\leq \infty} \int_{M_{*}} A \xrightarrow{\simeq} \int_{M_{*}} A
\]
is an equivalence. Likewise, the morphism in $\mathcal{C}$ to the limit
\[
\int_{M_{*}} C \xrightarrow{\simeq} \tau^{\leq \infty} \int_{M_{*}} C
\]
is an equivalence.*

**Proof.** Directly apply Corollary 1.3.6 and Lemma 2.3.2. \qed

The following main result of this subsection identifies the layers of the filtration \((6)\) and the cofiltration \((7)\) in terms of configuration spaces. Our proof relies on results developed in the coming subsections.

**Theorem 2.1.4** (Cardinality cokernels and kernels). *Each arrow in the cardinality filtration of factorization homology in \((6)\) belongs to a canonical cofiber sequence
\[
\tau^{\leq i-1} \int_{M_{*}} A \longrightarrow \tau^{\leq i} \int_{M_{*}} A \longrightarrow \text{Conf}^{-,fr}_{i}(M_{*}) \otimes_{\Sigma_{i} \text{Top}(n)} A^{\otimes i} .
\]
Likewise, each arrow in the cardinality cofiltration of factorization cohomology in \((7)\) belongs to a canonical fiber sequence
\[
\text{Map}_{\Sigma_{i} \text{Top}(n)}(\text{Conf}^{-,fr}_{i}(M_{*}), C^{\otimes i}) \longrightarrow \tau^{\leq i} \int_{M_{*}} C \longrightarrow \tau^{\leq i-1} \int_{M_{*}} C .
\]

**Proof.** We explain the cofiber sequence \((8)\). The argument concerning the fiber sequence \((9)\) is dual. From Lemma 2.3.5 with that notation, there is a cofiber sequence
\[
\tau^{\leq i-1} \int_{M_{*}} A \longrightarrow \tau^{\leq i} \int_{M_{*}} A \longrightarrow \text{colim}\left( \text{Disk}^{\leq i}_{+}(M_{*}) / \text{Disk}^{\leq i-1}_{+}(M_{*}) \right) \xrightarrow{j^{*}_{\text{red}} \otimes \text{red}} \mathcal{C}
\]
Through Lemma 2.3.3 the colimit expression is canonically identified as the colimit
\[
\text{colim} \left( \text{Disk}^{\leq 1}_{+}(\text{Conf}^{-,fr}_{i}(M_{*})) \xrightarrow{A^{\otimes i}} \mathcal{C} \right).
\]
Unwinding definitions, this colimit is $\text{Conf}^{-,fr}_{i}(M_{*}) \otimes_{\Sigma_{i} \text{Top}(n)} A^{\otimes i}$ – the tensor over based modules in spaces. \qed

15
2.1.2. **Goodwillie cofiltration.** We are about to apply Goodwillie’s calculus to functors of the form $\text{Alg}_{n}^{aug}(\mathcal{C}) \to \mathcal{C}$. This calculus was developed by Goodwillie in [Go] for application to functors from pointed spaces and has since been generalized; see [Lu2], [Ku1], and [Ku2]. We begin by recalling this formalism.

Let $i$ be a finite cardinality. Let $X$ and $Y$ be presentable $\infty$-categories, each with a zero-object. The $\infty$-category of polynomial functors of degree $i$ is the full $\infty$-subcategory of reduced functors

$$\text{Poly}_i(X, Y) \subset \text{Fun}_c(X, Y)$$

consisting of those that send strongly coCartesian $(i + 1)$-cubes to Cartesian $(i + 1)$-cubes. This inclusion admits a left adjoint

$$P_i : \text{Fun}_c(X, Y) \to \text{Poly}_i(X, Y)$$

implementing a localization. There is the full $\infty$-subcategory of homogeneous functors of degree $i$

$$\text{Homog}_i(X, Y) \subset \text{Poly}_i(X, Y)$$

consisting of those polynomial functors $H$ of degree $i$ for which $P_j H \simeq 0$ is the zero-functor provided $j < i$. Consequently, each reduced functor $F : X \to Y$ canonically determines a cofiltration of reduced functors

$$F \to P_{\infty} F \to \cdots \to P_i F \to P_{i-1} F \to \cdots$$

with each composite arrow $F \to P_i F$ the unit for the above adjunction, and with each kernel $\text{Ker}(P_i F \to P_{i-1} F)$ homogeneous of degree $i$ – here we have used the notation $P_{\infty} F := \lim\limits_i P_i F$ for the sequential limit of the cofiltration.

For a fixed zero-pointed $n$-manifold $M_n$, we apply this discussion to the functor $\int_{M} : \text{Alg}_{n}^{aug}(\mathcal{C}) \to \mathcal{C}/I_{/1} \cong \mathcal{C}$ – here we have used the identification of retractive objects over the unit of $\mathcal{C}$ with $\mathcal{C}$, as discussed in [1.3]. In particular, there is a canonical arrow between functors

$$\int_{M_n} \to P_{\infty} \int_{M_n}.$$ (10)

The next result identifies the layers of the Goodwillie cofiltration of factorization homology in terms of configuration spaces and the cotangent space. Our proof will make use of results developed in the coming subsections.

**Theorem 2.1.5.** There is a canonical fiber sequence among functors $\text{Alg}_{n}^{aug}(\mathcal{C}) \to \mathcal{C}$:

$$\text{Conf}_i(M_n) \otimes_{\Sigma_i(\text{Top}(n))} L(-)^{\otimes i} \to P_i \int_{M_n} \to P_{i-1} \int_{M_n}$$

for every $i$ a finite cardinality.

**Proof.** Through Lemma 2.6.1 the $i$-homogeneous layer of the Goodwillie cofiltration of the functor $\int_{M_n}$ is a symmetric functor of $i$-variables of $\text{Top}(n)$-modules. To identify this multi-variable functor we evaluate on free augmented $n$-disk algebras. Through Corollary 2.4.2 there is a canonical identification

$$\text{Ker}(P_i \int_{M_n} \mathbb{F}V \to P_{i-1} \int_{M_n} \mathbb{F}V) \simeq \text{Conf}_i^fr(M_n) \otimes_{\Sigma_i(\text{Top}(n))} V^{\otimes i}$$

functorially in the $\text{Top}(n)$-module $V$. This verifies the theorem in this free case because of the canonical equivalence $L\mathbb{F}V \simeq V$ of Lemma 1.4.6.

Unlike the cardinality (co)filtration of factorization (co)homology of the previous subsection, the Goodwillie cofiltration of factorization homology does not always converge. The next result provides an understood class of parameters for which the Goodwillie cofiltration converges.

We will reference the notion of a $t$-structure on $\mathcal{C}$, which is just a usual $t$-structure on the homotopy category of $\mathcal{C}$ (see Definition 1.2.1.4 of [Lu2]). We say that a $t$-structure on $\mathcal{C}$ is compatible with the
symmetric monoidal structure of \( \mathcal{C} \) if the restriction of the functor \( \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C} \) to \( \mathcal{C}_{\geq 0} \times \mathcal{C}_{\geq 0} \) factors through \( \mathcal{C}_{\geq 0} \), and if the unit \( 1 \in \mathcal{C}_{\geq 0} \) is connective.

**Theorem 2.1.6.** Suppose there exists a cocomplete t-structure on \( \mathcal{C} \) that is compatible with the symmetric monoidal structure. Then the value of the canonical arrow (11) evaluated on \( A \)

\[
\int_{M_*} A \xrightarrow{\sim} P_\infty \int_{M_*} A
\]

is an equivalence in \( \mathcal{C} \) provided one of the following criteria is satisfied:

- the augmentation ideal \( \text{Ker}(A \to 1) \) is connected (with respect to the given t-structure);
- the topological space \( M_* \) is connected and compact and the augmentation ideal \( \text{Ker}(A \to 1) \) is connective (with respect to the given t-structure).

**Proof.** Choose such a t-structure. There is a pair of equivalences in \( \mathcal{C} \)

\[
\text{Ker} \left( \int_{M_*} A \to P_\infty \int_{M_*} A \right) \xrightarrow{\sim} \lim_{\to} \text{Ker} \left( \int_{M_*} A \to P_i \int_{M_*} A \right) \xrightarrow{\sim} \lim_i \tau^{>i} \int_{M_*} A
\]

- (2) is because the formation of kernels commutes with sequential limits; (2) is Lemma 2.5.3.

We claim that the object \( \tau^{>i} \int_{M_*} A \) is i-connected under the named criteria. This claim implies \( \lim_i \tau^{>i} \int_{M_*} A \) is infinitely connected under the named criteria, and so completes the proof because \( \mathcal{C} \) is cocomplete with respect to the given t-structure.

We now prove the above mentioned claim. Since connectivity is preserved under colimits, it suffices to resolve the algebra \( A \) by free \( n \)-disk algebras. We can thus reduce to showing that \( \tau^{>i} \int_{M_*} F_{\text{aug}} V \) is i-connected, the case in which \( A \simeq F_{\text{aug}} V \) is the free augmented \( n \)-disk algebra on a \( \text{Top}(n) \)-module \( V \). By Corollary 2.3 we have a calculation of this truncation as

\[
\tau^{>i} \int_{M_*} F_{\text{aug}} V \simeq \bigoplus_{\ell \geq i} \text{Conf}_\ell(M_*) \bigotimes_{\Sigma_i \text{Top}(n)} V^{\otimes \ell}.
\]

So it is enough to show that \( \text{Conf}_i(M_*) \bigotimes_{\Sigma_i \text{Top}(n)} V^{\otimes i} \) is i-connected under the named criteria, for all \( i \). This follows from the definitional feature of t-structures that \( \mathcal{C}_{>i} \to \mathcal{C} \) is a colocalization and hence the essential image is closed under colimits – see §1.2.1 of [Lu2]. Apply this fact to the functor \( \text{Disk}^{<1}_{\text{fin}}(\text{Conf}_i(M_*)) \xrightarrow{j^{\text{fin}}_{\text{aug}} A^{\otimes i}} \mathcal{C} \) from the proof of Theorem 2.1.4 (which is in the notation of §2.8) supposing one of the criteria is satisfied.

\[\Box\]

**Remark 2.1.7.** A similar result is treated by Matsuoka in [Mat].

2.1.3. **Comparing cofiltrations.** We compare the cardinality and Goodwillie cofiltrations.

Recall the Poincaré duality arrow (1) of Theorem 1.2.1.

**Theorem 2.1.8.** The Poincaré duality arrow \( \int(-) \to \int(-)^\sim \text{Bar} \) extends to an equivalence of cofiltrations of functors \( \text{Alg}_{\text{aug}}(\mathcal{C}) \to \text{Fun}(\mathcal{Z}\text{Mfd}^{\text{fin}}_n, \mathcal{C}) \):

\[
\begin{align*}
(11) & \quad P_\bullet \int_{(-)} \xrightarrow{\sim} \tau^{\leq \bullet} \int_{(-)^\sim} \text{Bar}.
\end{align*}
\]

Specifically, for each smoothable compact \( n \)-manifold \( \overline{M} \) with partitioned boundary \( \partial\overline{M} = \partial_L \sqcup \partial_R \), each augmented \( n \)-disk algebra \( A \) in \( \mathcal{C} \), and each finite cardinality \( i \), there is an equivalence in \( \mathcal{C} \)

\[
P_i \int_{\overline{M} \smallsetminus \partial_L} A \xrightarrow{\sim} \tau^{\leq i} \int_{\overline{M} \smallsetminus \partial_L} \text{Bar}(A).
\]
Corollary 2.1.9. The Poincaré duality arrow $\int_{M_*} A \to \int_{M_*} \text{Bar} A$ of (1) canonically factors through an equivalence in $\mathcal{C}$

$$P_\infty \int_{M_*} A \xrightarrow{\sim} \int_{M_*} \text{Bar} A$$

from the limit of the Goodwillie cofiltration. This factorization is functorial in $M_*$ and $A$.

Proof of Theorem 2.1.8. Let $M_*$ be a zero-pointed $n$-manifold. Let $i$ be a finite cardinality. Corollary 2.7.3 asserts that the functor $\tau^{\leq i} \int_{M_*} \text{Bar}: \text{Alg}_{au}^n(\mathcal{C}) \to \mathcal{C}$ is polynomial of degree $i$. The morphism of cofiltrations $P_* \int_{M_*} \to \tau^{\leq i} \int_{M_*} \text{Bar}$ follows through the universal property of the Goodwillie cofiltration. There results a morphism of $i$-homogeneous layers:

$$\text{Ker}(P_i \int_{M_*} \to P_{i-1} \int_{M_*}) \to \text{Ker}(\tau^{\leq i} \int_{M_*} \text{Bar} \to \tau^{\leq i-1} \int_{M_*} \text{Bar})$$.

Through Theorem 2.1.4 and Theorem 2.1.5, this morphism is canonically equivalent to the morphism of functors

(12) $\text{Conf}^i(M_*) \times \times L(\leq i) \to \text{Map}^{i, \text{Top}(n)}(\text{Conf}^i(M_*)^\nu, \text{Bar}(-)^\nu(i))$.

This is the arrow of Corollary 2.7.3 applied to the $B(\text{Conf}^i(M_*))$-structured zero-pointed manifold $\text{Conf}^i(M_*)$ and so the arrow is an equivalence.

2.2. Non-unital algebras. In an ambient stable situation, there is a convenient equivalence between augmented algebras and non-unital algebras.

Recall the connected component functor $\text{Disk}_{n+, \text{nu}} \to \text{Fin}_n$, which is symmetric monoidal. We denote $\text{Fin}^\text{surj} \subset \text{Fin}_n$ for the subcategory of based finite sets and surjections among them; this is a symmetric monoidal subcategory.

Definition 2.2.1. $\text{Disk}_{n+, \text{nu}}^\text{surj}$ is the pullback $\infty$-category

$$\text{Disk}_{n+, \text{nu}}^\text{surj} := (\text{Disk}_{n+, \text{nu}})|_{\text{Fin}_n^\text{surj}}.$$

For $\mathcal{C}$ a symmetric monoidal $\infty$-category, the $\infty$-category of non-unital $n$-disk algebras in $\mathcal{C}$ is

$$\text{Alg}_{n}^{\text{nu}}(\mathcal{C}) := \text{Fun}^\otimes_0(\text{Disk}_{n+, \text{nu}}^\text{surj}, \mathcal{C})$$

the $\infty$-category of non-unital symmetric monoidal functors that respect terminal objects.

Proposition 2.2.2 (Proposition 5.4.4.10 of [L2]). There is an equivalence of $\infty$-categories $\text{Ker}^{\text{aug}}: \text{Alg}_{n}^{\text{aug}}(\mathcal{C}) \simeq \text{Alg}_{n}^{\text{nu}}(\mathcal{C}): \mathbb{1} \oplus (-)$ – the values of the left functor are depicted as $\text{Ker}^{\text{aug}} A: \mathbb{V}_{n+} \to \text{Ker}(A(\mathbb{V}_{n+}) \to A(*) \simeq \mathbb{1})$. This equivalence is natural among such $\mathcal{C}$.

2.3. Support for Theorem 2.1.4 (cardinality layers).

2.3.1. Configuration space quotients. Here we pull some results from [AFT1], [AFT2], and [AF1] to identify the cofiber of $\text{Disk}_{n}^{\leq i}(M_*) \to \text{Disk}_{n}^{\leq i}(M_*)$. For this subsection we fix a finite cardinality $i$.

In [AFT1] we construct, for each stratified space $X$, a stratified space $\text{Ran}_{\leq i}(X)$ whose points are finite subsets $S \subset X$ for which the map to connected components $S \to [X]$ is surjective; and we show that $\text{Ran}_{\leq i}(-)$ is continuously functorial among conically smooth embeddings among stratified spaces which are surjective on connected components. In [AFT2] we explain that the resulting functor

(13) $\text{Ran}_{\leq i}: \text{Disk}_{\text{nu}}^{\leq i}(\text{Bsc})/X \xrightarrow{\sim} \text{Bsc}/\text{Ran}_{\leq i}(X)$

is an equivalence.
Corollary 2.3.1. The functor \([13]\) restricts as an equivalence of \(\infty\)-categories:

\[
\Ran_{\leq i} : \Disk_{\leq i}^\leq(M_*) \xrightarrow{\sim} \Disk_{\leq i}^\leq(\Ran_{\leq i}(M_*)) .
\]

**Proof.** Let \(B \subset M_*\) be a basic neighborhood of \(*\). In [AFT1] it is shown that any conically smooth open embedding from a basic \(U \hookrightarrow M_*\) whose image contains \(*\) is canonically based-isotopic to one that factors through a based isomorphism \(U \cong B \hookrightarrow M_*\). We conclude that the projection from the slice

\[
(\Disk(\Bsc)/M_*)^B/ \xrightarrow{\sim} \Disk_+(M_*)
\]

is an equivalence of \(\infty\)-categories. Likewise, we conclude that the projection from the slice

\[
(\Disk_{\leq i}^\leq(\Ran_{\leq i}(M_*)))^\Ran_{\leq i}(B)/ \xrightarrow{\sim} \Disk_{\leq i}^\leq(\Ran_{\leq i}(M_*))
\]

is an equivalences of \(\infty\)-categories. The result follows from the equivalence [13]. \(\square\)

This identification affords the following essential consequence.

**Lemma 2.3.2.** The functor

\[
\Disk_{\leq i}^\surj(M_*) \longrightarrow \Disk_{\leq i}^\surj(M_*)
\]

is final.

**Proof.** The assertion follows from showing that the functor \(\Disk_{\leq i}^\surj(\Bsc)/M_* \longrightarrow \Disk(\Bsc)/M_*\) is final, since the pullback of a final functor is again final. Writing \(M_* \cong \bigvee_{i \in [M_*]} M_{i,*}\) as a wedge over its components, this functor is expressible as a product

\[
\Disk_{\leq i}^\surj(\Bsc)/M_* \cong \coprod_{i \in [M_*]} \Disk_{\leq i}^\surj(\Bsc)/M_{i,*} \longrightarrow \coprod_{i \in [M_*]} \Disk(\Bsc)/M_{i,*} \cong \Disk(\Bsc)/M_* .
\]

Since a product of final functors is final, we can reduce to the case of a factor, i.e., the case in which \(M_*\) is irreducible. Applying Quillen’s theorem A, it suffices to show the weak contractibility of the classifying space of \(\Disk_{\leq i}^\surj(\Bsc)/M_*\) for each \(V \in \Disk(\Bsc)/M_*\) for \(M_*\) connected.

We show this in two cases: \(V\) empty or nonempty. If \(V\) is empty, we have an identification from [AFT1] of this classifying space as

\[
\B\big(\Disk_{\leq i}^\surj(\Bsc)/M_*\big) \cong \Ran(M_*) ,
\]

the \(\Ran\) space of \(M_*\), which is weakly contractible. To see this, we refer to a now standard argument of [BeD]: since the \(\Ran\) space carries a natural H-space structure by taking unions of subsets, and the composition of the diagonal and the H-space multiplication is the identity, therefore its homotopy groups are zero. If \(V\) is nonempty, then the map \(V \hookrightarrow M_*\) is surjective on components hence defines an object of \(\Disk_{\leq i}^\surj(\Bsc)/M_*\). Consequently, \(\Disk_{\leq i}^\surj(\Bsc)/M_*\)^\(V/\) has an initial object in this case, hence this \(\infty\)-category has a weakly contractible classifying space. \(\square\)

The following is an essential result, a description of the layers in the cardinality filtration of the \(\infty\)-category \(\Disk_{\geq i}^\surj(M_*)\).

**Lemma 2.3.3.** There is a canonical cofiber sequence among \(\infty\)-categories

\[
\Disk_{\geq i}^\leq(M_*) \longrightarrow \Disk_{\geq i}^\leq(M_*) \longrightarrow \Disk_{\geq i}^\leq(\Conf_{\geq i}^\infty(M_*)\Sigma_i)
\]

for any \(i\) a finite cardinality.
Proof. We explain the diagram among \(\infty\)-categories

\[
\begin{array}{ccc}
\text{Disk}_{\leq i}(M_s) & \xrightarrow{\text{Eq. 2.3.1}} & \text{Disk}_{\leq i}(M_s) \\
& \downarrow & \\
\text{Disk}_{\leq i}(\text{Ran}_{\leq i}(M_s)) & \xrightarrow{\text{Eq. 2.3.1}} & \text{Disk}_{\leq i}(\text{Ran}_{\leq i}(M_s))
\end{array}
\]

The top horizontal functor is the one whose cofiber is being examined. The vertical equivalences are directly from Lemma 2.3.1, as indicated. A main result of [AFT1] gives that, for

\[
X \to Y \to Z
\]
a cofiber sequence of stratified spaces comprised of constructible maps, then there is a resulting cofiber sequence of \(\infty\)-categories

\[
\mathcal{Bsc}/X \to \mathcal{Bsc}/Y \to \mathcal{Bsc}/Z.
\]

Applying this to

\[
\text{Ran}_{\leq i}(M_s) \to \text{Ran}_{\leq i}(M_s) \to \text{Conf}_{i}(M_s),
\]

followed up by the same logic as in the proof of Lemma 2.3.1, gives that the bottom sequence in the diagram is a cofiber sequence.

\[\square\]

2.3.2. Reduced extensions. We explain a couple general maneuvers concerning left Kan extensions in the presence of zero-objects.

For \(K \xrightarrow{f} K'\) functor among small \(\infty\)-categories, and \(\mathcal{C}\) a presentable \(\infty\)-category, there is an adjunction among functor \(\infty\)-categories

\[
f_b : \mathcal{C}^K \rightleftarrows \mathcal{C}^{K'} : f^*
\]

with the right adjoint \(f^*\) given by precomposing with \(f\), and with the left adjoint given by left Kan extension.

Lemma 2.3.4. Let \(\mathcal{C}\) be a presentable \(\infty\)-category with a zero-object. Let \(\ast \xrightarrow{x} K\) be a pointed \(\infty\)-category. Denote the fiber \(\infty\)-category as in the sequence \(\text{Fun}_0(K, \mathcal{C}) \to \mathcal{C}^K \xrightarrow{x} \mathcal{C}\). Then the inclusion of the fiber admits a left adjoint \((-)^\text{red} : \mathcal{C}^K \to \text{Fun}_0(K, \mathcal{C})\) which fits into a cofiber sequence in \(\mathcal{C}^K\)

\[
x_2 x^* \to \text{id}_{\mathcal{C}^K} \to (-)^\text{red}.
\]

Proof. Because \(x^*\) preserves colimits, the universal morphism \(c\text{Ker}(x^* x_2 x^* \to x^*) \xrightarrow{\sim} x^* \text{Ker}(x_2 x^* \to \text{id})\) is an equivalence in \(\mathcal{C}\). Because the inclusion \(\ast \xrightarrow{x} K\) is fully faithful, the morphism \(x^* x_2 x^* \xrightarrow{\sim} x^*\) is an equivalence in \(\mathcal{C}\). So the values of the cofiber \(\text{cKer}(x_2 x^* \to \text{id})\) lie in \(\text{Fun}_0(K, \mathcal{C})\). Because \(0 \in \mathcal{C}\) is zero-pointed, then the value \(x_2(0) \in \mathcal{C}^K\) is the zero-functor. It follows that the endofunctor \(x_2 x^*\) restricts to the zero functor, on \(\text{Fun}_0(K, \mathcal{C})\); therefore \((-)^\text{red}\) restricts to the identity functor on \(\text{Fun}_0(K, \mathcal{C})\). Finally, the arrow \(\text{id} \to (-)^\text{red}\) witnesses \((-)^\text{red}\) as a left adjoint as claimed.

\[\square\]

Lemma 2.3.5. Let \(i : K_0 \to K\) be a fully faithful functor among \(\infty\)-categories, and consider the functor \(j : K \to K/K_0 := \amalg_{K_0} K\) to the cone, regarded as a pointed \(\infty\)-category. There is a cofiber sequence in the functor \(\infty\)-category \(\mathcal{C}^K\)

\[
i_2^* i^* \to \text{id}_{\mathcal{C}^K} \to j^* j_b^\text{red}
\]

for any \(\mathcal{C}\) a stable presentable \(\infty\)-category.
Theorem 2.4.1. Let \( V \otimes \) colimits. In this subsection we suppose further that the symmetric monoidal structure of \( C \) distributes over colimits. This calculation is a fundamental input to a number of our arguments.

Proof. There is a canonical fiber sequence of \( \infty \)-categories
\[
\text{Fun}^{\text{red}}(K/K_0, \mathcal{C}) \xrightarrow{i^*} \mathcal{C}^K \xrightarrow{i^*} \mathcal{C}^{K_0}.
\]
Because \( i^* \) preserves colimits, the universal morphism \( \text{cKer}(i^*i^* \rightarrow i^*) \xrightarrow{\simeq} i^* \text{cKer}(i^* \rightarrow \text{id}) \) is an equivalence in \( \mathcal{C}^{K_0} \). Because \( i^* \): \( K_0 \rightarrow K \) is fully faithful, the morphism \( i^*i^* \rightarrow i^* \) is an equivalence in \( \mathcal{C}^{K_0} \). It follows that the values of \( \text{cKer}(i^* \rightarrow \text{id}) \) lie in \( \text{Fun}_0(K/K_0, \mathcal{C}) \). Because \( 0 \in \mathcal{C} \) is a zero-object, the value \( i^*_0 \) is the zero-functor. It follows that \( i^*_0 \) restricts to the zero-functor on \( \text{Fun}_0(K/K_0, \mathcal{C}) \); therefore \( \text{cKer}(i^* \rightarrow \text{id}) \) restricts to the identity functor on \( \text{Fun}_0(K/K_0, \mathcal{C}) \). In summary, the endofunctor \( \text{cKer}(i^* \rightarrow \text{id}) : \mathcal{C}^K \rightarrow \mathcal{C}^K \) factors through \( \text{Fun}_0(K/K_0, \mathcal{C}) \xrightarrow{j^*} \mathcal{C}^K \), and the factoring functor \( \mathcal{C}^K \rightarrow \text{Fun}_0(K/K_0, \mathcal{C}) \) is left adjoint to \( j^* \). \( \Box \)

2.4. Free calculation. Here we give the calculation of the factorization homology of a free algebra. This calculation is a fundamental input to a number of our arguments.

In this subsection we suppose further that the symmetric monoidal structure of \( \mathcal{C} \) distributes over colimits.

For the next result, we point out that each \( \text{Top}(n) \)-module \( V \) in \( \mathcal{C} \) determines a \( \Sigma_i \text{Top}(n) \)-module \( V^{\otimes i} \) in \( \mathcal{C} \).

Theorem 2.4.1. Let \( \mathcal{C} \) be a \( \otimes \)-cocomplete symmetric monoidal \( \infty \)-category. Let \( V \in \text{Mod}_{\text{Top}(n)}(\mathcal{C}) \) be a \( \text{Top}(n) \)-module in retractive objects over the unit of \( \mathcal{C} \). Then there is a canonical identification of the factorization homology of the free augmented algebra on \( V \) in terms of configuration spaces of \( M \):
\[
\int_{M_*} \mathcal{Z}^{\text{aug}} V \simeq \bigoplus_{i \geq 0} \left( \Sigma_i \text{Top}(n) \right) \left( \text{Conf}_i(M_*) \otimes_{\Sigma_i \text{Top}(n)} \mathcal{C}^{\otimes i} \right) \cup_{\text{Top}(n)} \mathcal{C}^{\otimes i}.
\]

Proof. The term on the righthand side depicts a functor \( \mathcal{Z} \mathcal{M} \mathcal{f} \mathcal{d}_n \rightarrow \mathcal{C} \), naturally in \( V \). This functor canonically extends as a symmetric monoidal functor, because of the distribution assumption on the symmetric monoidal structure of \( \mathcal{C} \). It is manifest that the restriction to \( \text{Disk}_{n,+} \) of the righthand side satisfies the universal property of the free functor \( \mathcal{Z}^{\text{aug}} \). This proves the theorem for the case of \( M_* = \bigvee_j \mathbb{R}^n_+ \) a finite disjoint union of Euclidean spaces.

Let us explain the sequence of equivalences in \( \mathcal{C} \):
\[
\int_{M_*} \mathcal{Z}^{\text{aug}} V \simeq \begin{cases} \text{(1)} & \text{colim} \left( \text{Disk}_+ (M_*) \xrightarrow{\text{aug} V} \mathcal{C} \right) \\ \text{(2)} & \bigoplus_{i \geq 0} \text{colim} \left( \text{Conf}_i(M_*) \otimes_{\Sigma_i \text{Top}(n)} \mathcal{C}^{\otimes i} \right) \\ \text{(3)} & \bigoplus_{i \geq 0} \text{colim} \left( \text{Disk}^{\text{aug}}_+ (U_+) \xrightarrow{\mathcal{C}^{\otimes i}} \mathcal{C} \right) \\ \text{(4)} & \bigoplus_{i \geq 0} \text{colim} \left( \mathcal{X}^i \xrightarrow{\mathcal{C}^{\otimes i}} \mathcal{C} \right) \\ \text{(5)} & \bigoplus_{i \geq 0} \text{colim} \left( \text{Disk}^{\text{aug}}_+ (M_*) \xrightarrow{\mathcal{C}^{\otimes i}} \mathcal{C} \right) \\ \text{(6)} & \bigoplus_{i \geq 0} \text{colim} \left( \text{Disk}^{-1}_+ \left( \text{Conf}_i(M_*) \right) \xrightarrow{\mathcal{C}^{\otimes i}} \mathcal{C} \right) \end{cases}
\]

The equivalence (1) is the finality of the functor \( \text{Disk}_+ (M_*) \rightarrow (\text{Disk}_{n,+}/M_*)/\text{Disk}_{n,+} \) (Theorem 1.5.4). The equivalence (2) follows from the first paragraph, combined with the distribution of
∀ over sifted colimits (using Theorem 1.3.4). The equivalences (3) and (6) are the identifications $\overline{\text{Disk}}_{\Sigma,i}^+(\mathcal{M}_\ast) \simeq \overline{\text{Disk}}_{\Sigma,i}^+(\text{Conf}_1(\mathcal{M}_\ast))$ of Lemma 1.3.5.

Consider the $\infty$-category of arrows $\text{Fun}([1], \overline{\text{Disk}}_+^+(\mathcal{M}_\ast))$. Evaluation at 0 gives a functor $\text{ev}_0 : \text{Fun}([1], \overline{\text{Disk}}_+^+(\mathcal{M}_\ast)) \to \overline{\text{Disk}}_+^+(\mathcal{M}_\ast)$.

We denote the pullback $\infty$-category

$$\mathcal{X}^i := \overline{\text{Disk}}_{\Sigma,i}^+(\mathcal{M}_\ast) \times_{\text{Disk}_+^+(\mathcal{M}_\ast)} \text{Fun}([1], \overline{\text{Disk}}_+^+(\mathcal{M}_\ast)),$$

There is thus a functor $\mathcal{X}^i \xrightarrow{\text{ev}_0} \overline{\text{Disk}}_{\Sigma,i}^+(\mathcal{M}_\ast)$, which is a Cartesian fibration. For each object $e : \bigsqcup\mathbb{R}^n \to M$ in $\overline{\text{Disk}}_{\Sigma,i}^+(\mathcal{M}_\ast)$, the object $(e = e)$ is initial in the fiber $\infty$-category $\text{ev}_0^*e$. It follows that the functor $\mathcal{X}^i \xrightarrow{\text{ev}_0} \overline{\text{Disk}}_{\Sigma,i}^+(\mathcal{M}_\ast)$ is final. This explains the equivalence (5). The equivalence (4) is formal, because nested colimits agree with colimits.

\[ \square \]

**Corollary 2.4.2.** Let $\mathcal{C}$ be a $\otimes$-stable-presentable symmetric monoidal $\infty$-category, and let $V$ be a $\text{Top}(n)$-module in $\mathcal{C}$. For each finite cardinality $i$, the diagram in $\mathcal{C}$

$$\tau_i \int_{\mathcal{M}_\ast} \mathbb{F}^\text{aug} V \longrightarrow \int_{\mathcal{M}_\ast} \mathbb{F}^\text{aug} V \longrightarrow P_i \int_{\mathcal{M}_\ast} \mathbb{F}^\text{aug} V$$

can be canonically identified with the diagram

$$\bigoplus_{j > i} \left( \text{Conf}_{\Sigma,i}(\mathcal{M}_\ast) \times V^{\otimes j} \right) \longrightarrow \bigoplus_{j \geq 0} \left( \text{Conf}_{\Sigma,i}(\mathcal{M}_\ast) \times V^{\otimes j} \right) \longrightarrow \bigoplus_{j \leq i} \left( \text{Conf}_{\Sigma,i}(\mathcal{M}_\ast) \times V^{\otimes j} \right)$$

given by inclusion and projection of summands.

**Proof.** The identification of the left term can be seen by inspecting the definition of $\tau_i \int_{\mathcal{M}_\ast}$ and tracing the proof of Theorem 2.4.1. The identification of the right term follows by induction on $i$, for which both the base case and the inductive step are supported by Corollary 1.4.2 using the equivalence $L\mathbb{F}V \simeq V$ of Lemma 1.4.6. That the arrows are as claimed is manifest.

\[ \square \]

A basic feature of Koszul duality in general is that it exchanges free algebras and trivial algebras. We are about to reference the notion of a trivial augmented $n$-coalgebra, the definition of which is exactly dual to that of trivial algebras (see Definition 1.4.6).

**Lemma 2.4.3.** Let $\mathcal{C}$ be a $\otimes$-presentable symmetric monoidal $\infty$-category. Let $V$ be a $\text{Top}(n)$-module in $\mathcal{C}_{/ / 1}$ and consider the diagonal $\text{Top}(n)$-module $(\mathbb{R}^n)^+ \otimes V$ in $\mathcal{C}_{/ / 1}$. There is a canonical equivalence of augmented $n$-disk coalgebras in $\mathcal{C}$

$$\text{Bar}(\mathbb{F}^\text{aug} V) \simeq t_{\text{aug}_{\text{Alg}}}((\mathbb{R}^n)^+ \otimes V)$$

from the bar construction of the free augmented algebra to the trivial augmented coalgebra.

**Proof.** Let $J$ be a finite set and consider the action

$$\text{ZEmb}((\mathbb{R}^n)^+, \sqrt{(\mathbb{R}^n)^+}) \bigotimes_{j \in J} \int_{(\mathbb{R}^n)^+} \mathbb{F}^\text{aug} V \longrightarrow \int_{\bigvee_{j \in J}(\mathbb{R}^n)^+} \mathbb{F}^\text{aug} V \simeq \left( \int_{(\mathbb{R}^n)^+} \mathbb{F}^\text{aug} V \right)^{\otimes J}.$$

Through Theorem 2.4.1 using the pigeonhole principle, for each $i_1, \ldots, i_j > 1$ the restriction of this morphism to $\text{Conf}_{1}((\mathbb{R}^n)^+ \otimes V$ followed by the projection to $\bigotimes_{j \in J} \text{Conf}_{ij}((\mathbb{R}^n)^+ \otimes V^{\otimes i_j}$ is canonically equivalent to the zero morphism.

For each $\epsilon > 0$ consider the subspace $\text{Conf}_{ij}((\mathbb{R}^n)^+ \subset \text{Conf}_{1}((\mathbb{R}^n)^+$ consisting of those based maps $f : \{1, \ldots, i\} \to (\mathbb{R}^n)^+$, whose restriction $f_j : f^{-1}\mathbb{R}^n \to \mathbb{R}^n$ is injective and $x = y \in \mathbb{R}^n$ whenever $\|f(x) - f(y)| < \epsilon$ - the inclusion of this subspace is a based weak homotopy equivalence. Flows the vector field $x \mapsto x$ on $\mathbb{R}^n$ for infinite time witnesses a deformation retraction.
of $\text{Conf}_i^\star((\mathbb{R}^n)^+)\}$ onto $\ast$ provided $i > 1$. We conclude that $\text{Conf}_i((\mathbb{R}^n)^+)\}$ is weakly contractible for $i > 1$. Through Theorem 2.4.1 we arrive at a canonical identification
\[ 1 \oplus \text{Conf}_i((\mathbb{R}^n)^+) \otimes V \to \int_{(\mathbb{R}^n)^+}^{\text{aug}} V. \]

Combined with the above paragraph, we conclude that $\text{Bar}(\text{aug}V)$ is canonically identified as the trivial augmented $n$-disk coalgebra on the $\text{Top}(n)$-module $\text{Conf}_1((\mathbb{R}^n)^+) \otimes V = ((\mathbb{R}^n)^+ \otimes V)$. □

2.5. Support for Theorem 2.1.6 (convergence).

**Lemma 2.5.1.** Let $\mathcal{C}$ be a $\otimes$-sifted cocomplete symmetric monoidal $\infty$-category. The functors $\text{Alg}_{\text{aug}}^n(\mathcal{C}) \to \mathcal{C}$
\[ \tau_{> i} \int_{M_j}^\ast \quad \text{and} \quad \int_{M_j}^\ast \quad \text{and} \quad P_i \int_{M_j}^\ast \]
preserve sifted colimits for any $i$ a finite cardinality.

**Proof.** We first prove the statement for $\int_{M_j}^\ast$. Let $A : J \to \text{Alg}_{\text{aug}}^n(\mathcal{C})$ be a diagram of augmented $n$-disk algebras in $\mathcal{C}$, indexed by a shifted $\infty$-category. The canonical arrow $\text{colim}_{j \in J} \int_{M_j} A_j \to \int_{M_j} \text{colim}_{j \in J} A_j$ in $\mathcal{C}$ is a composition
\[ \text{colim}_{j \in J} \text{colim}_{U \in \text{Disk}^+_{M_j}} A_j(U) \cong \text{colim}_{U \in \text{Disk}^+_{M_j}} \text{colim}_{j \in J} A_j(U) \to \text{colim}_{U \in \text{Disk}^+_{M_j}} (\text{colim}_{j \in J} A_j)(U) \]
where the outer objects are in terms of the defining expression for factorization homology, the left equivalence is through commuting colimits, and the right arrow is a colimit of canonical arrows. Because of the hypotheses on $\mathcal{C}$, each arrow $\text{colim}_{j \in J} A_j(U) \to (\text{colim}_{j \in J} A_j)(U)$ is an equivalence if and only if it is for $U$ connected. This is the case provided the forgetful functor $\mathcal{E}_{\text{aug}}^n : \text{Alg}_{\text{aug}}^n(\mathcal{C}) \to \mathcal{C}$ preserves sifted colimits. This assertion is Proposition 3.2.3.1 of [Lu2].

Because $P_i$ is a left adjoint, it commutes with sifted colimits. The statement is thus true for $P_i \int_{M_j}^\ast$ after the first paragraph.

The functor $\tau_{> i} \int_{M_j}^\ast : \text{Alg}_{\text{aug}}^n(\mathcal{C}) \to \text{Fun}(\text{Disk}_{> i}(M_j), \mathcal{C})$ is a composition of two functors, the latter of which exists on the image of the first and it commutes with those sifted colimits on which it is defined. Colimits in the middle $\infty$-category are given objectwise, and so it is enough to show that the restriction $\text{Alg}_{\text{aug}}^n(\mathcal{C}) \to \text{Fun}(\text{Disk}_{> i}(M_j), \mathcal{C})$ preserves sifted colimits for each finite cardinality $j$. For $j = 1$, this follows from the first paragraph as the case $M_j = \mathbb{R}^n$. For $j$ general this follows because the functor $\otimes : \mathcal{C}^x \to \mathcal{C}$ preserves sifted colimits, by assumption. □

**Lemma 2.5.2** (Free resolutions). Let $\mathcal{C}$ be a $\otimes$-sifted cocomplete symmetric monoidal $\infty$-category. Consider the full $\infty$-subcategory $\text{SFree} \subset \text{Alg}_{\text{aug}}^n(\mathcal{C})$ which is the smallest containing the image of $\mathcal{F} : \text{Mod}_{\text{Top}(n)}(\mathcal{C}_1 \times 1) \to \text{Alg}_{\text{aug}}^n(\mathcal{C})$ that is closed under sifted colimits and equivalences. Then the inclusion $\text{SFree} = \text{Alg}_{\text{aug}}^n(\mathcal{C})$ is an equality. In other words, every augmented $n$-disk algebra is a sifted colimit of free augmented $n$-disk algebras.

**Proof.** We apply the Barr-Beck theorem. The forgetful functor $\text{Alg}_{\text{aug}}^n(\mathcal{C}) \to \mathcal{C}_1 \times 1$ is conservative and preserves sifted colimits (Proposition 3.2.3.1 of [Lu2]). The result then follows from Proposition 4.7.4.14 of [Lu2]. □

**Lemma 2.5.3.** Let $A$ be an augmented $n$-disk algebra in $\mathcal{C}$. The canonical sequence of arrows among functors $\text{Alg}_{\text{aug}}^n(\mathcal{C}) \to \mathcal{C}$
\[ \tau_{> i} \int_{M_j}^\ast \to \int_{M_j}^\ast \to P_i \int_{M_j}^\ast \]

\[ \tau_{> i} \int_{M_j}^\ast \to \int_{M_j}^\ast \to P_i \int_{M_j}^\ast \]
is a cofiber sequence for any \( i \) a finite cardinality.

**Proof.** Corollary 2.4.2 immediately gives the result for the case that \( A \) is free. Every augmented \( n \)-disk algebra is a sifted colimit of free algebras. Lemma 2.5.1 states that the left two functors commute with sifted colimits. Because \( P_i \) is a left adjoint, the functor \( P_i \int_{M_i} \) commutes with sifted colimits. \( \square \)

### 2.6. Support for Theorem 2.1.5 (Goodwillie layers)

For the next result we denote the diagonal functor as \( \text{diag}_i : \text{Mod}_{\text{Top}(n)}(\mathcal{C}) \to \text{Mod}_{\text{Top}(n)}(\mathcal{C})^{\times i} \).

**Lemma 2.6.1.** There is an equivalence of \( \infty \)-categories

\[
\text{Poly}_i\left( \text{Mod}_{\text{Top}(n)}(\mathcal{C})^{\times i}, \mathcal{C} \right) \xrightarrow{\sim} \text{Homog}_i(\text{Alg}_{\text{aug}}(\mathcal{C}), \mathcal{C})
\]

for each \( i \) a finite cardinality. It assigns to \( D \) the functor \( D(\text{diag}_i \circ L(\cdot)) \).

**Proof.** Theorem 6.1.4.7 of [Lu2] gives the following identification: there is a canonical fully faithful functor

\[
\text{Homog}_i(\text{Alg}_{\text{aug}}(\mathcal{C}), \mathcal{C}) \hookrightarrow \text{Fun}(\text{Stab}(\text{Alg}_{\text{aug}}(\mathcal{C}))^{\times i}, \mathcal{C})
\]

whose essential image consists of the \( \Sigma_i \)-invariant functors that preserve colimits in each variable. For \( \emptyset \) an operad, there is the general canonical equivalence of \( \infty \)-categories \( \text{Stab}(\text{Alg}_{\text{aug}}(\mathcal{C})) \simeq \text{Mod}_{\emptyset(1)}(\mathcal{C}) \) – see §7.3.4 of [Lu2]. Through the general equivalence above, the named expression depicts an inverse to this fully faithful functor on its essential image. \( \square \)

**Corollary 2.6.2.** Let \( M_* \) be a zero-pointed \( n \)-manifold, and let \( i \) be a finite cardinality. The functor

\[
\text{Conf}_i^* (M_*) \bigotimes_{\Sigma_i \text{Top}(n)} L(-)^{\otimes i} : \text{Alg}_{\text{aug}}(\mathcal{C}) \to \mathcal{C}
\]

is homogeneous of degree \( i \).

**Proof.** The coend \( \text{Conf}_i^* (M_*) \bigotimes_{\Sigma_i \text{Top}(n)} L(-)^{\otimes i} : \text{Mod}_{\text{Top}(n)}(\mathcal{C})^{\times i} \to \mathcal{C} \) preserves colimits and is \( \Sigma_i \)-invariant, so we can apply Lemma 2.6.1. \( \square \)

### 2.7. Support for Theorem 2.1.8 (comparing cofibrations)

In this subsection we fix a \( \otimes \)-stable-presentable symmetric monoidal \( \infty \)-category \( \mathcal{C} \).

We use the following result, which generalizes Lemma 2.4.3 away from the case of free algebras, at the level of \( \text{Top}(n) \)-modules.

**Theorem 2.7.1** (Corollary 2.29 of [Fr2]). There is a canonical equivalence between functors \( \text{Alg}_{\text{aug}}(\mathcal{C}) \to \text{Mod}_{\text{Top}(n)}(\mathcal{C}) \)

\[
(\mathbb{R}^n)^+ \otimes L(-) \simeq \text{cKer}_{\text{aug}}(\text{Bar}(\cdot)).
\]

**Corollary 2.7.2.** For each conically finite zero-pointed \( (n_i) \)-manifold \( P_* \), equipped with a \( \Sigma_i \text{Top}(n) \)-structure, there is a canonical equivalence of functors \( \text{Alg}_{\text{aug}}(\mathcal{C}) \to \mathcal{C} \)

\[
\text{Fr}_{P_*} \bigotimes_{\Sigma_i \text{Top}(n)} L(-)^{\otimes i} \xrightarrow{\sim} \text{Map}_{\Sigma_i \text{Top}(n)}(\text{Fr}_{P_*}, \text{Bar}(\cdot)^{\otimes i}).
\]

**Proof.** Theorem 2.7.1 gives a canonical equivalence among \( \Sigma_i \text{Top}(n) \)-modules in \( \mathcal{C} \):

\[
(\mathbb{R}^n_i)^+ \otimes L(-)^{\otimes i} \simeq ((\mathbb{R}^n_i)^+ \otimes L(-))^{\otimes i} \simeq (\text{cKer}_{\text{aug}}(\text{Bar}(\cdot)))^{\otimes i}.
\]

The result is a direct corollary of Theorem 1.5.3 according to the \( \mathcal{B}(\Sigma_i \text{Top}(n)) \)-structured version of Remark 1.5.4. \( \square \)
Corollary 2.7.3. For each finite cardinality \(i\), the functor \(\tau_{\leq i} \int_{\mathcal{M}_n} \text{Bar}: \text{Alg}_{\text{aug}}^n(\mathcal{C}) \rightarrow \mathcal{C}\) is polynomial of degree \(i\).

Proof. Through Corollary 2.7.2 Corollary 2.6.2 implies \(\text{Map}^{\Sigma_i \text{Top}(n)}(\mathcal{P}_n, \text{Bar}(\text{Aug})^\bigotimes)\) is homogeneous of degree \(i\). The result then follows by induction using the fibration sequence of Theorem 2.1.3.

\[ \square \]

3. Factorization homology of formal moduli problems

In this section, we will be concerned with a generalization of factorization homology which allows for a more general coefficient system, a moduli functor of \(n\)-disk algebras (as in the works on derived algebraic geometry [Lu4], [IV], and [Fr1]).

Definition 3.0.4 (Linear dual). Let \(\mathcal{C}\) be a symmetric monoidal \(\infty\)-category presentable. There is a functor \(\mathcal{C}^{\text{op}} \rightarrow \text{PShv}(\mathcal{C})\) given by \(c \mapsto \mathcal{C}((-, c, 1))\), morphisms to the symmetric monoidal unit. Should \(\mathcal{C}\) be \(\otimes\)-presentable, this functor canonically factors through the Yoneda embedding as a functor

\[ (-)^\vee : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C} \]

which we refer to as linear dual.

We choose to simplify and familiarize matters by restricting our generality.

Convention (Focus on \(\text{Ch}^0\)). Henceforward, we pick a field \(k\) and, unless otherwise stated, work over the background symmetric monoidal \(\infty\)-category \(\text{Ch}^0\) of chain complexes over \(k\) – its equivalences are quasi-isomorphisms. With tensor product it becomes a \(\otimes\)-stable-presentable symmetric monoidal \(\infty\)-category, and it is endowed with a standard t-structure. Our choice to work over a field is for the basic but fundamental property that the duality functor exchanges \(i\)-connected and \((-i)\)-coconnected objects.

Notation 3.0.5. Let \(i\) be an integer. We use the familiar notation \(\text{Ch}^\geq_{\mathcal{C}} \subset \text{Ch}_{\mathcal{C}} \supset \text{Ch}^\leq_{\mathcal{C}}\) for the full \(\infty\)-subcategories consisting of those chain complexes whose the degrees for whose non-zero homology is bounded as indicated.

We simplify the notation

\[ \text{Alg}_{\mathcal{C}}^{\mathcal{G}} := \text{Alg}_{\mathcal{G}}^{\mathcal{C}}(\text{Ch}_k) \quad \overset{\text{Prop 2.2.2}}{\sim} \quad \text{Alg}_{\mathcal{G}}^{\text{aug}}(\text{Ch}_k) \quad := \text{Alg}_{\mathcal{G}}^{\text{aug}}. \]

We denote the full \(\infty\)-subcategories \(\text{Alg}_{\mathcal{C}}^{\mathcal{G}, \geq_{\mathcal{C}}} \subset \text{Alg}_{\mathcal{G}}^{\mathcal{C}} \supset \text{Alg}_{\mathcal{G}}^{\mathcal{G}, \leq_{\mathcal{C}}}\) consisting of those non-unital \(n\)-disk algebras whose underlying chain complex lies in \(\text{Ch}^\geq_{\mathcal{C}}\) and \(\text{Ch}^\leq_{\mathcal{C}}\), respectively. We likewise denote the full \(\infty\)-subcategories \(\text{Alg}_{\mathcal{G}}^{\mathcal{G}, \geq_{\mathcal{C}}} \subset \text{Alg}_{\mathcal{G}}^{\mathcal{G}, \leq_{\mathcal{C}}}\) consisting of those augmented \(n\)-disk algebras \(A\) whose associated non-unital algebra lies in \(\text{Alg}_{\mathcal{G}}^{\mathcal{G}, \geq_{\mathcal{C}}} \subset \text{Alg}_{\mathcal{G}}^{\mathcal{G}, \leq_{\mathcal{C}}},\) respectively.

3.1. Formal moduli. We begin with a few essential notions from derived algebraic geometry of \(n\)-disk algebras.

Definition 3.1.1 (\(\text{Perf}_k\)). We denote the full \(\infty\)-subcategory \(\text{Perf}_k \subset \text{Ch}_k\) consisting of those chain complexes over \(k\) whose \(V\) for which the \(k\)-module \(\bigoplus_{q \in \mathbb{Z}} H_q V\) is finite rank over \(k\). We denote the intersection \(\text{Perf}_k^\geq := \text{Perf}_k \cap \text{Ch}_k^\geq\) consisting of those perfect complexes whose homology vanishes below dimension \(i\).

Definition 3.1.2 (\(\text{Triv}_n\) and \(\text{Artin}_n\)). The following is a sequence of full \(\infty\)-subcategories

\[ \text{Triv}_n \subset \text{Artin}_n \subset \text{Alg}_{\mathcal{G}}^{\mathcal{G}, \geq_{\mathcal{C}}, \geq 0} , \]

in which the first which is the essential image of \(\text{Mod}_{\text{Top}(n)}(\text{Perf}_k^\geq)\) under the functor that assigns a complex the associated trivial algebra – it consists of trivial connective non-unital \(n\)-disk algebras whose underlying \(\text{Top}(n)\)-module is a perfect chain complex. The \(\infty\)-category \(\text{Artin}_n\) of non-unital Artin \(n\)-disk algebras is the smallest full \(\infty\)-subcategory of \(\text{Alg}_{\mathcal{G}}^{\mathcal{G}, \geq_{\mathcal{C}}, \geq 0}\) that contains \(\text{Triv}_n\) and is closed under small extensions. That is:
If \( B \) is in \( \text{Artin}_n \), \( V \) is in \( \text{Triv}_n \), and the following diagram

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
k & \longrightarrow & V
\end{array}
\]

is a pullback square in \( \text{Alg}^{nu,\geq 0}_n \), then \( A \) is in \( \text{Artin}_n \).

**Remark 3.1.3.** In \([Lu4]\), Lurie uses the terminology small for an equivalent definition in the case of \( n \)-disk algebras with trivializations of the tangent bundles, i.e., \( \mathcal{E}_n \)-algebras.

**Definition 3.1.4** (Moduli functor). The \( \infty \)-category of formal \( n \)-disk moduli functors

\[
\text{Moduli}_n := \text{Fun}(\text{Artin}_n, \text{Spaces})
\]

is the \( \infty \)-category of copresheaves on non-unital Artin \( n \)-disk algebras.

**Remark 3.1.5.** To obtain a workable geometric theory of formal moduli functors such as that of \([Lu4]\), \([TV]\), or \([Hi]\), one should restrict to those presheaves that satisfy some gluing condition, such as preserving limits of small extensions (after \([Sc]\)). We will not use these conditions, so for simplicity of presentation we omit them.

**Example 3.1.6** (Formal spectrum). There is the composite functor

\[
\text{Spf}: (\text{Alg}^{>0,\text{aug}}_n)^{\text{op}} \longrightarrow \text{PShv}(\text{Alg}^{\text{aug},\geq 0}_n) \longrightarrow \text{Moduli}_n
\]

of Yoneda followed by restriction. Its values are given by \( \text{Spf}(A): R \mapsto \text{Alg}^{nu}_n(A, R) \), which we refer to as the formal spectrum of \( A \).

We choose to conceptually simplify our duality formalism by using linear duals and never consider coalgebras.

**Definition 3.1.7.** The Koszul duality functor is the composite

\[
\bar{\text{D}}_n: (\text{Alg}^{\text{aug}}_n)^{\text{op}} \xrightarrow{\text{Bar}^{\text{op}}} (c\text{Alg}^{\text{aug}}_n)^{\text{op}} \xrightarrow{(-)^\vee} \text{Alg}^{\text{aug}}_n
\]

which is \( \text{Bar} \) followed by linear dual.

**Definition 3.1.8.** For an Artin \( n \)-disk algebra \( R \) and an augmented \( n \)-disk algebra \( A \) over a field \( k \), the Maurer-Cartan space

\[
\text{MC}_A(R) := \text{Alg}^{\text{aug}}_n(\bar{\text{D}}_n R, A)
\]

is the space of maps from the Koszul dual of \( R \) to \( A \). The Maurer-Cartan functor \( \text{MC}: \text{Alg}^{\text{aug}}_n \rightarrow \text{Moduli}_n \) is the adjoint of the pairing

\[
\text{Artin}_n \times \text{Alg}^{\text{aug}}_n \longrightarrow \text{Alg}^{\text{aug,op}}_n \times \text{Alg}^{\text{aug}}_n \rightarrow \text{Spaces}
\]

where the first functor is \( \bar{\text{D}}_n \times \text{id} \) and the second functor is the mapping space; \( \text{MC} \) sends \( A \) to the functor \( \text{MC}_A \).

**Remark 3.1.9.** Our definition of the moduli functor \( \text{MC}_A \) has the same form as that given by Lurie in \([Lu4]\), there denoted \( \Psi A \). Our construction of the functor \( \bar{\text{D}}_n \), on the other hand, is somewhat different: we use the geometry of zero-pointed manifolds, whereas Lurie uses twisted arrow categories. To verify that these two constructions agree requires a relationship between twisted arrow categories and zero-pointed manifolds. We defer this problem to a separate work.

The formal spectrum functor \( \text{Spf} \) has a right adjoint.

**Definition 3.1.10** (Algebra of functions). We denote the \((\text{augmented}) n\text{-disk algebra of functions} \) functor \( \mathcal{O}: \text{Moduli}_n \longrightarrow (\text{Alg}^{\text{aug}}_n)^{\text{op}} \) that is given as

\[
\mathcal{O}(X) := \lim_{(\text{Spf}(R) \rightarrow X) \in ((\text{Artin}_n)^{\text{op}}/X)} R = \lim_{((\text{Artin}_n^{\text{op}}/X))^{\text{op}} \rightarrow \text{Artin}_n \rightarrow \text{Alg}^{\text{aug}}_n}
\]

26
In other words, the functor $\mathcal{O}$ is the right Kan extension of the inclusion $(\text{Artin}_n)^{\text{op}} \hookrightarrow (\text{Alg}^{\text{aug}}_n)^{\text{op}}$ along the functor $\text{Spf}$:

![Diagram](https://example.com/diagram.png)

The Maurer-Cartan functor is a lift of the duality functor $\mathbb{D}^n$. Namely, there is a canonical equivalence

$$\mathbb{D}^n A \simeq \mathcal{O}(\text{MC}_A)$$

between the Koszul dual of $A$ and the augmented $n$-disk algebra of functions on the Maurer-Cartan functor of $A$.

### 3.2. Factorization homology with formal moduli coefficients

We have a notion of factorization homology with coefficients in a formal $n$-disk moduli functor.

**Definition 3.2.1 (Factorization homology with formal moduli).** We extend factorization homology to formal moduli functors

![Diagram](https://example.com/diagram.png)

as the right Kan extension of $\int$ along $\text{Spf}$, denoted with the same symbol. Explicitly, the factorization homology of a zero-pointed $n$-manifold $M$ with coefficients in a formal $n$-disk moduli functor $X$ is

$$\int_M X := \lim_{(\text{Spf}(R) \to X) \in ((\text{Artin}_n^{\text{op}})^{\text{op}})\leftarrow} \int_{M_*} R = \lim \left( (\text{Artin}_n^{\text{op}})^{\text{op}} \to \text{Artin}_n \xrightarrow{\int_{M_*}} \text{Ch}_k \right).$$

**Remark 3.2.2.** One can think of $\int_M X$ as $\Gamma(X, \int_{M_*} \mathcal{O})$, the global sections of the sheaf on $X$ given by calculating the factorization homology of $M_*$ with coefficients in the structure sheaf of $X$. Importantly, the canonical arrow $\int_M X \to \int_{M_*} \mathcal{O}(X)$ is typically not an equivalence provided $X$ is not affine, and it can be regarded as a type of completion.

**Remark 3.2.3.** Unless a formal moduli functor $X$ is affine, factorization homology $\int_M X$ will typically not satisfy $\otimes$-excision. Consequently, the topological quantum field theory $\int_X X$ is typically not perturbative for non-affine $X$.

We now state our main theorem.

**Theorem 3.2.4 (Poincaré/Koszul duality for formal moduli).** Let $k$ be a field. Then there is a canonical equivalence of functors $\text{Alg}^{\text{aug}}_n \to \text{Fun}(\mathbb{Z}\text{Mfd}^{\text{fin}}_n, \text{Ch}_k)$,

$$\left( \int_{(-)} \right)^\vee \simeq \int_{(-)} \text{MC}_A.$$

Specifically, for each augmented $n$-disk algebra $A$ in chain complexes over $k$, and each $n$-dimensional cobordism $\overline{M}$ with boundary $\partial \overline{M} = \partial_L \bigsqcup \partial_R$, there is a canonical equivalence of chain complexes over $k$

$$\left( \int_{(\overline{M} \setminus \partial_L)} A \right)^\vee \simeq \int_{(\overline{M} \setminus \partial_R)} \text{MC}_A$$

to the linear dual of the factorization homology with coefficients in $A$, and the factorization homology with coefficients in the Maurer-Cartan moduli functor of $A$. 27
Remark 3.2.5. Let $A$ be an augmented $n$-disk algebra in chain complexes over $k$, and let $M$ be a closed $(n-d)$-manifold. There results the augmented $d$-algebras $\int_{M^\wedge (-)} A : \text{Disk}_{d,+} \to \text{Ch}_k$ and $\int_{M^\wedge (-)} MC_A : \text{Disk}_{d,+} \to \text{Ch}_k$. Our result specializes to a canonical equivalence of augmented $d$-algebras

$$D^d\left(\int_{M^\wedge \mathbb{R}^d_+} A\right) \simeq \int_{M^\wedge \mathbb{R}^d_+} MC_A$$

where $D^d$ is the Koszul dual functor of augmented $d$-algebras in chain complexes over $k$.

We now prove our main theorem, making use of the three results which will be developed in the coming subsections.

Proof of Theorem 3.2.4. Let $A$ be an augmented $n$-disk algebra in chain complexes over $k$, and let $M$ be a zero-pointed $n$-manifold. We explain the diagram of canonical equivalences in $\text{Ch}_k$, each natural in all of their arguments:

$$\prod\limits_{\text{Prop} 3.5.3} \left(\int_{M^\wedge (-)} A\right)^\vee \simeq \lim\limits_{\text{Thm 3.3.3}} \int_{M^\wedge (-)} MC_A \simeq \lim\limits_{\text{Def 3.2.1}} \left(\int_{M^\wedge (-)} F\right)^\vee \simeq \lim\limits_{\text{Thm 3.4.2}} \left(\int_{M^\wedge (-)} F\right)^\vee \simeq \lim\limits_{\text{Thm 3.3.3}} \int_{M^\wedge (-)} D^n F.$$  

By Proposition 3.5.3, we can calculate factorization homology with coefficients in $A$ as a colimit over finitely presented $(-n)$-coconnective $n$-disk algebras: $\text{colim}_{F \in (\text{FPres}_{<n}^{\leq})/A} \int_{M^\wedge} F \underset{\text{Def 3.2.1}}{\simeq} \int_{M^\wedge} A$. By Theorem 4.3.3 for a finitely presented $(-n)$-coconnective $n$-disk algebra $F$, there is a natural equivalence $\left(\int_{M^\wedge} F\right)^\vee \simeq \int_{M^\wedge} D^n F$. By Theorem 3.4.2, Koszul duality restricts to an equivalence $D^n : \text{FPres}_{<n}^{\leq} \simeq (\text{Artin}_n)^{op} : \mathbb{D}$ between finitely presented and Artin $n$-disk algebras. By definition $\text{Map}(\text{Spf}(R), MC_A) \simeq \text{Map}(D^n R, A)$ for $R$ Artin, and so this last equivalence gives an equivalence of slice categories $\text{FPres}_{<n}^{\leq}/A \simeq (\text{Artin}_n^{op})/MC_A$.

\[\square\]

3.3. Case of finitely presented highly coconnective algebras. We first prove the main result for the special case in which $A$ is a finitely presented $(-n)$-coconnective augmented $n$-disk algebra. The definition of finite presentation is exactly dual to our definition of Artin.

Recall from Definition 2.4.3 the free algebra functor $F : \text{Mod}_{\text{Top}(n)}(\text{Ch}_k) \to \text{Alg}_n^{\text{aug}}$. It is a particular feature of $n$-disk algebras, which we shall see in the proof of Theorem 3.3.3 that $FV$ lies in $\text{Alg}_{<n}^{\leq}$ provided $V \in \text{Perf}_{\mathbb{F}_k}^{\leq}$ and $i > 0$. This provokes the following definition.

Definition 3.3.1 (Free$_n$ and FPres$_{<n}^{\leq}$). We denote the full $\infty$-subcategories

$$\text{Free}_n^{<n} \subset \text{FPres}_n^{<n} \subset \text{Alg}_n^{\text{aug}, \leq, <n},$$

the first which is the essential image of $F|_{\text{Perf}_{\mathbb{F}_k}^{<n}}$, and the second which is the smallest such satisfying the following property:
Let

\[
\begin{array}{ccc}
FV & \longrightarrow & k \\
\downarrow & & \downarrow \\
A & \longrightarrow & B
\end{array}
\]

be a pushout square among augmented \((-n)\)-coconnective \(n\)-disk algebras in which \(V\) is a \(\text{Top}(n)\)-module in \(\text{Perf}_{k}^{\leq n}\) and \(A \in \text{FPres}_{k}^{\leq n}n\). Then \(B \in \text{FPres}_{k}^{\leq n}n\).

Before addressing this affine case, we require the following easy, but important, comparison of factorization homology and factorization cohomology in the stable setting.

**Proposition 3.3.2.** Let \(C\) be a \(\otimes\)-stable-presentable symmetric monoidal \(\infty\)-category. Let \(A\) be an augmented \(n\)-disk algebra in \(C\), and assume that the natural map

\[
(A^\vee)^{\otimes i} \xrightarrow{\simeq} (A^{\otimes i})^\vee
\]

is an equivalence for all \(i \geq 0\). The composition \(\text{Disk}_{n}^{+} \cong \text{Disk}_{n}^{\text{op}} \xrightarrow{A^\vee} C^{\text{op}} \xrightarrow{(-)^\vee} C\) canonically extends to an augmented \(n\)-disk coalgebra \(A^\vee: \text{Disk}_{n}^{+} \to C\). Furthermore, for any conically finite zero-pointed \(n\)-manifold \(M\), the canonical arrow

\[
\int_{M_*} A^\vee \xrightarrow{\simeq} \left(\int_{M_*} A\right)^\vee
\]

is an equivalence.

**Proof.** The first statement is clear, by hypothesis. Furthermore, the hypothesis on \(A\) supports the leftward arrow in the string of canonical equivalences in \(\text{Ch}_k\):

\[
\left(\int_{M_*} A\right)^\vee = \left(\lim_{U_+ \in \text{Disk}_{n}(M_*)} A(U_+)^\vee \xrightarrow{\simeq} \lim_{U_+ \in \text{Disk}_{n}(M_*)^{\text{op}}} A(U_+)^\vee \xrightarrow{\simeq} \lim_{U_+ \in \text{Disk}_{n}(M_*)^{\text{op}}} A^\vee(U_+) = \int_{M_*} A^\vee\right).
\]

\(\Box\)

We now turn to the affine case in the proof of our main theorem.

**Theorem 3.3.3.** Let \(k\) be a field. There is an equivalence of functors \(\text{FPres}_{n}^{\leq n} \to \text{Fun}(\mathcal{Z}\text{Mfld}_{n}^{\text{fin}}, \text{Ch}_k)\)

\[
\left(\int_{(-)}\right)^\vee \simeq \int_{(-)}^\vee \mathbb{D}^n.
\]

Specifically, for each finitely presented \((-n)\)-coconnective augmented \(n\)-disk algebra \(A\) in chain complexes over \(k\), and each \(n\)-dimensional \((-n)\)-coconnective augmented \(n\)-disk algebra \(A\), there is a canonical equivalence of chain complexes over \(k\)

\[
\left(\int_{(\mathcal{M}, \partial L)} A\right)^\vee \simeq \int_{(\mathcal{M}, \partial R)} \mathbb{D}^n A
\]

**Proof.** Fix a conically finite zero-pointed \(n\)-manifold \(M_*\), and a finitely presented \((-n)\)-coconnective augmented \(n\)-disk algebra \(A\). We will explain the diagram in \(\text{Ch}_k\)

\[
\left(\int_{M_*} A\right)^\vee \xrightarrow{(1)} \left(\int_{M_*} A\right)^\vee \xrightarrow{(2)} \left(\int_{M_*} \text{Bar}(A)\right)^\vee \xrightarrow{(3)} \left(\int_{M_*} \mathbb{D}^n A\right)^\vee \xrightarrow{(4)} \int_{M_*} \mathbb{D}^n A
\]

and that each arrow in it is an equivalence. The arrow (1) is the linear dual of the canonical map to the limit of the Goodwillie cofiltration. The arrow (2) is the linear dual of that from Theorem 2.1.8 which factors the Poincaré duality morphism and is an equivalence by that result. The arrow (3) is the linear dual of that of Proposition 3.3.2. The arrow (4) is the standard one.

We verify that (3) is an equivalence. Recall that since \(A\) is finitely presented, therefore \(LA\) is perfect. The equivalence of chain complexes \(\mathbb{D}^n A \simeq k \oplus ((\mathbb{R}^n)^+ \otimes LA)^\vee\) given by Theorem
implies $\mathbb{D}^n A$ too is perfect. We can thus apply Proposition \ref{prop:3.3.2} to conclude that (3) is an equivalence.

We next show (1) is an equivalence. For this we will show that the successive layers in the Goodwillie cofiltration become contractible through a range. Specifically, through Theorem \ref{thm:2.1.8} we will show
\[
\text{Conf}_t^i (M_s) \otimes_{\Sigma_i (\text{Top}(n))} (LA)^{\otimes i}
\]
is $(-i + \ell)$-coconnective, where $\ell$ is the number of components of $M_s$ and $i$ is sufficiently large.

For $V \in \text{Mod}_{\text{Top}(n)}(\text{Perf}_k^{\geq -r})$ then $LV \simeq V$ is $(-r)$-coconnective. Being a left adjoint, the cotangent space functor satisfies $L(k \amalg B) \simeq \text{Lk}_{LV} k \amalg LB \simeq \text{cKer}(V \to LB)$. We conclude that $LA$ is a $(-n)$-coconnective $\text{Top}(n)$-module, and thereafter that $(LA)^{\otimes i}$ is a $(-ni)$-coconnective $\Sigma_i \wr \text{Top}(n)$-module, whenever $A$ is in $\text{FPres}_k^{< -n}$.

Via factorization homology, the $\Sigma_i \wr \text{Top}(n)$-module $(LA)^{\otimes i}$ determines a $\oplus$-excisive symmetric monoidal functor $\mathcal{Z}(\text{fld}_{\Sigma_i (\text{Top}(n))}) \to \text{Ch}_k^{\oplus i}$ from conically finite zero-pointed $ni$-manifolds equipped with a $B(\Sigma_i \wr \text{Top}(n))$-structure on their tangent bundle. Explicitly, this functor is given by $W_s \mapsto \text{Fr}_{W_s} \otimes_{\Sigma_i (\text{Top}(n))} (LA)^{\otimes i}$. Through $\oplus$-excision, the value of this functor on $W_s$ is $(s-ni)$-coconnective whenever $W_s$ is $s$-coconnective. The zero-pointed $(ni)$-manifold $\text{Conf}_t(M_s)$ is equipped with a $B(\Sigma_i \wr \text{Top}(n))$-structure on its tangent bundle; and Proposition \ref{prop:1.1.8} explains that it is conically finite and coconnectivity equal to $nt + (n-1)(i-\ell)$, where $\ell$ is the number of components of $M_s$ and we assume $i > \ell$. The coconnectivity range implying equivalence (1) now follows by addition.

Lastly, we show (4) is an equivalence. It suffices to show that $\int_{M_s^-} \mathbb{D}^n A$ is connective and has finite rank homology groups over $k$ in all dimensions. To show this, we apply the cardinality filtration of factorization homology: since $\int_{M_s^-} \mathbb{D}^n A$ is a sequential colimit of the filtration $\tau^{\leq i} \int_{M_s^-} \mathbb{D}^n A$, we further reduce to showing that the layers of the filtration are connective and grow in connectivity with $i$. By Theorem \ref{thm:2.1.8} these cardinality layers are shifts of the duals of the Goodwillie layers of $\int_{M_s^-} A$. Consequently, their connectivities follows from the coconnectivities of the Goodwillie layers, which were computed in the proof of equivalence (1).

\[\square\]

3.4. Comparing Artin\textsubscript{n} and FP\textsubscript{res}\textsubscript{n}. We show that Koszul duality implements an equivalence between finitely presented $(-n)$-coconnective algebras and connective Artin algebras.

An identical argument to that for Lemma \ref{lem:2.4.3} replacing the Goodwillie cofiltration with the cardinality filtration gives the following dual result.

Lemma 3.4.1. Let $k$ be a connective commutative ring in spectra. Let $V \in \text{Mod}_{\text{Top}(n)}(\text{Perf}_k^{\geq 0})$ be a $\text{Top}(n)$-module in perfect connective $k$-modules. Then $\text{Bar}(tV)$ is a free augmented $n$-disk coalgebra. If $k$ is a field, then the Koszul dual $\mathbb{D}^n(k \oplus V)$ is a free augmented $n$-disk algebra.

We now have the following important duality result.

Theorem 3.4.2. Koszul duality restricts to a contravariant equivalence
\[
\mathbb{D}^n : \text{FPres}_k^{< -n} \simeq (\text{Artin}_n)^{\text{op}} ; \mathbb{D}^n
\]
between finitely presented $(-n)$-coconnective augmented $n$-disk algebras and Artin $n$-disk algebras.

Proof. Lemma \ref{lem:2.4.3} shows that $\mathbb{D}^n$ sends free algebras to trivial algebras. Inspecting further, $\mathbb{D}^n$ restricts as a functor $\text{Free}_k^{< -n} \to \text{Triv}_{\geq 0}$ – this uses that the linear dual of a coconnective object is connective, since we are working over a field $k$. Because $\mathbb{D}^n = (\text{Bar})^\vee : \text{Alg}_{\text{aug}}^{\text{aug}} \to (\text{Alg}_{\text{aug}}^{\text{aug}})^{\text{op}}$ is the composite of left adjoints, it preserves colimits. Inspecting the definitions of $\text{FPres}_k^{< -n}$ and $\text{Artin}_n$, we conclude that $\mathbb{D}^n : \text{FPres}_k^{< -n} \to (\text{Alg}_{\text{aug}}^{\text{aug}})^{\text{op}}$ canonically factors through $(\text{Artin}_n)^{\text{op}}$:
\[
\mathbb{D}^n : \text{FPres}_k^{< -n} \longrightarrow (\text{Artin}_n)^{\text{op}}.
\]
A further application of Lemma 3.4.3 and inspection of the definitions of \( \text{FPres}_n^{\leq-n} \) and \( \text{Artin}_{n} \), gives that this restricted functor is essentially surjective. We will now explain that this restricted functor is fully faithful.

For each pair of objects \( A, B \in \text{FPres}_n^{\leq-n} \) we must show that map of spaces

\[
\text{Map}_{\text{Alg}}(A, B) \xrightarrow{\mathcal{D}} \text{Map}_{\text{Alg}}(\mathcal{D}^n B, \mathcal{D}^n A)
\]

is an equivalence. Again, because \( \mathcal{D}^n \) preserves colimits, it is enough to consider the case that \( A = \mathcal{F} V \) is free on a \( \text{Top}(n) \)-module \( V \) in \( \text{Perf}_k^{\leq-n} \). For this case will explain the canonical factorization of the map \( 15 \) through equivalences:

\[
\begin{align*}
\text{Map}_{\text{Alg}}(\mathcal{F} V, B) & \approx \text{Map}_{\text{Top}(n)}(V, B) \\
& \approx \text{Map}_{\text{Top}(n)}(\mathcal{D}^n B, (\mathcal{D}^n)^{\vee}) \\
& \approx \text{Map}_{\text{Top}(n)}(\mathcal{D}^n B, (\mathcal{D}^n)^{\vee}) \\
& \approx \text{Map}_{\text{Top}(n)}(\mathcal{D}^n B, (\mathcal{D}^n)^{\vee}) \\
& \approx \text{Map}_{\text{Top}(n)}(\mathcal{D}^n B, (\mathcal{D}^n)^{\vee}) \\
& \approx \text{Map}_{\text{Top}(n)}(\mathcal{D}^n B, (\mathcal{D}^n)^{\vee})
\end{align*}
\]

The equivalence \( 1 \) is by the free-forgetful adjunction. The equivalence \( 2 \) is an application of the functor \( (\mathcal{D}^n)^{\vee} : \text{Mod}_{\text{Top}(n)}(\text{Ch}_k) \rightarrow \text{Mod}_{\text{Top}(n)}(\text{Ch}_k) \), which is an equivalence, because \( \text{Ch}_k \) is stable and linear dual implements an equivalence on finite chain complexes. The equivalence \( 3 \) is because the canonical map \( B \rightarrow \mathcal{D}^n \mathcal{D}^n B \) is an equivalence – this is Theorem 3.3.3. The equivalence \( 4 \) is from Theorem 2.7.1. The equivalence \( 5 \) is the cotangent space-trivial adjunction. The equivalence \( 6 \) is the identification of the dual of a free algebra from Lemma 2.4.3. It is straightforward to check that this composite equivalence agrees with the map \( 15 \).

From Lemma 3.4.1 \( (\mathcal{D}^n)^{\vee} \) restricts as an inverse to \( \mathcal{D}^n \) on \( \text{Triv}_k^{\leq-n} \). Because the forgetful functor \( \text{Alg}_n(\text{Ch}_k) \rightarrow \text{Mod}_{\text{Top}(n)}(\text{Ch}_k) \) is conservative, it follows that \( (\mathcal{D}^n)^{\vee} : (\text{Artin}_n)^{\vee} \rightarrow \text{FPres}_n^{\leq-n} \) is inverse to \( \mathcal{D}^n \).

\[ \square \]

**Corollary 3.4.3.** For \( A \in \text{FPres}_k^{\leq-n} \), the moduli problem \( \text{MC}_A \) is affine. Moreover, there is an equivalence

\[ \text{MC}_A \approx \text{Spf}(\mathcal{D}^n A). \]

### 3.5. Resolving by \( \text{FPres}_n^{\leq-n} \). The rest of this section is devoted to proving the last required result used in the proof of our main theorem. This is Proposition 3.5.3 which states that factorization homology with general coefficients is given by left Kan extension of factorization homology restricted to finitely presented augmented \( n \)-disk algebras whose augmentation ideal is \( (\mathcal{D}^n)^{\vee} \)-coconnective.

**Lemma 3.5.1 (Highly coconnected free resolutions).** Let \( A \) be an augmented \( n \)-disk algebra in \( \text{Ch}_k^{\leq-n} \). Consider the \( \infty \)-category \( (\text{Free}_{n}^{\leq-n})/A \) of augmented \( n \)-disk algebras over \( A \) which are free on \( (\mathcal{D}^n)^{\vee} \)-coconnective perfect \( \text{Top}(n) \)-modules in chain complexes over \( k \). The following are true.

1. The slice \( \infty \)-category \( (\text{Free}_{n}^{\leq-n})/A \) is sifted.
2. The canonical arrow

\[
\text{colim}_{\mathcal{F} \in (\text{Free}_{n}^{\leq-n})/A} \mathcal{F} V \rightarrow A
\]

is an equivalence in \( \text{Alg}_n^{\text{aug}} \).
Proof. For the first statement, we must show the diagonal map \((\text{Free}_n^{≤-n})/A \rightarrow (\text{Free}_n^{≤-n})/A \times (\text{Free}_n^{≤-n})/A\) is final. For this, we use Quillen’s Theorem A and argue that the iterated slice ∞-category
\[
((\text{Free}_n^{≤-n})/A)_{(\mathbb{F}V \rightarrow A, \mathbb{F}W \rightarrow A)}/
\]
has contractible classifying space for each pair of objects \((\mathbb{F}V \rightarrow A)\) and \((\mathbb{F}W \rightarrow A)\) of \((\text{Free}_n^{≤-n})/A\). This iterated slice ∞-category has an initial object, given by the universal arrow \(\mathbb{F}(V \oplus W) \rightarrow A\) determined by the map of underlying \(\text{Top}(n)\)-modules \(V \oplus W \rightarrow \mathbb{F}V \oplus \mathbb{F}W \rightarrow A + A \rightarrow A\).

Now, consider the smallest full ∞-subcategory \(\text{Free}_n^{≤-n} \subset \text{Alg}_n^{\text{nu}}\) that is closed under sifted colimits and contains the free non-unital \(n\)-disk algebras on perfect chain complexes which are \((-n)\)-coconnective. We prove the second statement in two steps.

- We prove the statement for \(A \in S\). Choose a sifted diagram \(J \xrightarrow{F} \text{Free}_n\) witnessing \(A\) as its colimit. Let \(V\) be a \(\text{Top}(n)\)-module in \(\text{Perf}_k\). There is the diagram of canonical maps among spaces
  \[
  \begin{array}{ccc}
  \colim_{j \in J} \text{Map}_{\text{Alg}_n}(\mathbb{F}V, F_j) & \xrightarrow{\simeq} & \text{Map}_{\text{Alg}_n}(\mathbb{F}V, \colim_{j \in J} F_j) \\
  \simeq & & \simeq \\
  \colim_{j \in J} \text{Map}_{\text{Mod}_{\text{Top}(n)}}(\mathbb{C}) (V, F_j) & \xrightarrow{\simeq} & \text{Map}_{\text{Mod}_{\text{Top}(n)}}(\mathbb{C}) (V, \colim_{j \in J} F_j)
  \end{array}
  \]
  and we now explain why each is an equivalence. The left vertical map is the free-forgetful adjunction. The top horizontal map is an equivalence because \(V\) is perfect. The left vertical map is an equivalence because the forgetful functor preserves sifted colimits (Proposition 3.2.3.1 of [Lu2]). We conclude that the top horizontal map is an equivalence of spaces. It follows that the canonical functor
  \[
  \colim_{j \in J} (\text{Free}_n)/F_j, \xrightarrow{\simeq} (\text{Free}_n)/A
  \]
  is an equivalence of ∞-categories.

- We now show \(S = \text{Alg}_n^{\text{nu}}\). Use the notation \(\overline{\text{Free}_n} \subset \text{Alg}_n^{\text{nu}}\) for the full ∞-subcategory consisting of the essential image of the free functor \(\mathbb{F}\). The existence of free resolutions (Lemma 2.5.2) implies the closure of \(\overline{\text{Free}_n}\) by sifted colimits equals \(\text{Alg}_n^{\text{nu}}\). The equality \(S = \text{Alg}_n^{\text{nu}}\) follows after verifying the inclusion \(\text{Free}_n \subset S\).


Because \(\mathbb{F}\) is a left adjoint, and because colimits in \(\text{Mod}_{\text{Top}(n)}(\text{Ch}_k) = \text{Fun}(\text{BTop}(n), \text{Ch}_k)\) are objectwise, it is enough to show the following statement:

Let \(\text{Perf}_k^{≤-n} \subset S' \subset \text{Ch}_k\) be the smallest full ∞-subcategory closed under sifted colimits that contains \(\text{Perf}_k^{≤-n}\). Then the inclusion \(S' \subset \text{Ch}_k\) is an equality.

Well, each \(V \in \text{Ch}_k^{≤-n}\) can be witnessed as a filtered colimit of perfect \((-n)\)-coconnective perfect chain complexes over \(k\), and so \(\text{Ch}_k^{≤-n} \subset S'\). The full ∞-subcategory \(\text{Ch}_k^{≤-n} \subset \text{Ch}_k\) is closed under direct sums, and it follows that the tensor \(K \otimes V \in S'\) for any simplicial set \(K\) whenever \(V \in \text{Ch}_k^{≤-n}\). In particular \(\Sigma^\ell V \in S'\) for every non-negative integer \(\ell\) provided \(V \in \text{Ch}_k^{≤-n}\). We conclude the inclusion \(\text{Ch}_k^{≤∞} \subset S'\) of chain complexes that are \(\ell\)-coconnective for some \(\ell\). Finally, every object \(V \in \text{Ch}_k\) is a filtered colimit of finite chain complexes over \(k\).

\[\square\]

Through Lemma 2.5.1, Lemma 3.5.1 gives the following result.

**Corollary 3.5.2.** Let \(A\) be an augmented \(n\)-disk algebra in \(\text{Ch}_k\). Then the canonical arrow
\[
\colim_{\mathbb{F}V \in (\text{Free}_n^{≤-n})/A} \int_{M_+} \mathbb{F}V \xrightarrow{\simeq} \int_{M_+} A
\]
is an equivalence in \(\text{Ch}_k\).
**Proposition 3.5.3.** Let $A$ be an augmented $n$-disk algebra in $\text{Ch}_k$. The canonical arrow
\[
\colim_{F \in (\text{Free}_n^{\leq -n})/A} \int_{M_*} F \xrightarrow{\sim} \int_{M_*} A
\]
is an equivalence in $\text{Ch}_k$.

**Proof.** We explain the sequence of equivalences in $\text{Ch}_k$:
\[
\begin{align*}
\colim_{F \in (\text{Free}_n^{\leq -n})/A} \int_{M_*} F \xrightarrow{\sim} \colim_{F \in (\text{FPres}_n^{\leq -n})/A} \int_{M_*} F \xrightarrow{\sim} \int_{M_*} A.
\end{align*}
\]
The arrows are the canonical ones, from restricting colimits. Corollary 3.5.2 states that the composite arrow is an equivalence. The result is verified upon showing that the left arrow is an equivalence.

Consider the commutative triangle of $\infty$-categories
\[
\begin{array}{ccc}
\text{(Free}_n^{\leq -n})/A & \xrightarrow{\int_{M_*}} & \text{Ch}_k \\
\downarrow & & \downarrow \\
(\text{FPres}_n^{\leq -n})/A & \xrightarrow{\int_{M_*}} & (\text{FPres}_n^{\leq -n})/A
\end{array}
\]
Let $F \to A$ be an object of $(\text{FPres}_n^{\leq -n})/A$. From Corollary 3.5.2, the canonical map
\[
\colim_{F \in (\text{Free}_n^{\leq -n})/F} \int_{M_*} F \xrightarrow{\sim} \int_{M_*} F
\]
is an equivalence. It follows that the above triangle is a left Kan extension, and thereafter that the left arrow of (16) is an equivalence.

\qed

4. Hochschild homology of associative and enveloping algebras

In the remainder of this paper, we detail the meaning and consequences of our main theorem in the 1-dimensional case, where it becomes a statement about usual Hochschild homology and where the Maurer-Cartan functor $\text{MC}$ reduces to the familiar Maurer-Cartan functors for associative and Lie algebras. We will first describe the general case of an associative algebra, then we will further specialize to the case of enveloping algebras of Lie algebras in characteristic zero.

4.1. Case $n = 1$. For an augmented associative algebra $A$, consider the Maurer-Cartan functor $\text{MC}_A$ from Definition 3.1.8 i.e., by considering $A$ as a framed 1-disk algebra. Our main theorem has the following consequence in dimension 1. This generalizes an essentially equivalent result for cyclic homology due to Feigin & Tsygan in [FT]; see also the operadic generalization of Getzler & Kapranov in [GeK].

**Corollary 4.1.1.** Let $A$ be an augmented associative algebra over a field $k$. There is an equivalence
\[
\text{HH}_*(A)^\vee \simeq \text{HH}_*(\text{MC}_A)
\]
between the dual of the Hochschild homology of $A$ and the Hochschild homology of the moduli functor $\text{MC}_A$. If $A$ is either connected and degreewise finite, or $(-1)$-coconnective and finitely presented, then there is an equivalence
\[
\text{HH}_*(A)^\vee \simeq \text{HH}_*(\text{DA})
\]
between the linear dual of the Hochschild homology of $A$ and the Hochschild homology of the Koszul dual of $A$. 

33
Remark 4.1.2. A different proof of this result can be given via Morita theory, after results of Lurie, using that Koszul duality exchanges quasi-coherent sheaves and ind-coherent sheaves; a treatment along these lines has also been given by Campbell in [Ca]. We briefly summarize: by Theorem 3.5.1 of [Lu4], Koszul duality for associative algebras extends to a Koszul duality for modules and an equivalence $\text{Perf}_A^R \simeq \text{Coh}_{\text{MC}_A}^L$ between perfect right $A$-modules and left coherent sheaves on $\text{MC}_A$. Coherent sheaves are, by definition, the dual to perfect sheaves; Hochschild homology is a symmetric monoidal functor and therefore preserves duals. Consequently, the Hochschild homology $\text{HH}_{+, (\text{Perf}_{\text{MC}_A}^R)}$ is the dual of $\text{HH}_{+, (\text{Perf}_A^R)}$. Corollary 4.1.1 then follows from showing formal descent for Hochschild homology of formal moduli problems, implying that the Hochschild homology of $\text{Perf}_{\text{MC}_A}$ agrees with that of $\text{MC}_A$.

4.2. Lie algebras. We will now specialize the preceding results in order to offer an interesting interpretation of our results in Lie theory over a field $k$ of characteristic zero. Our main duality result, Theorem 3.2.4, specializes to one in Lie theory, Corollary 4.2.3 which makes no reference to either $n$-disk algebras or factorization homology: there is a duality between the Hochschild homology of an enveloping algebra and the Hochschild homology of the associated Maurer-Cartan space.

Convention (Characteristic zero and framings). Henceforward, we make the following choices to more conveniently make the connection with Lie algebras. We fix a field $k$ of characteristic zero. We work with (augmented) $\mathcal{E}_n$-algebras in place of (augmented) $\mathcal{D}\text{isk}_n$-algebras; and likewise with framed zero-pointed $n$-manifolds in place of zero-pointed $n$-manifolds. These easy substitutions do not affect the preceding arguments. We use the superscript notation $\text{Artin}_n^\text{fr}$ for this framed modification of $\text{Artin}_n$.

There is the directed system of symmetric monoidal $\infty$-categories

\[
\text{colim} \left( \cdots \longrightarrow \mathcal{D}\text{isk}_{n,+}^\text{fr} \longrightarrow \mathcal{D}\text{isk}^\text{fr}_{n+1,+} \longrightarrow \cdots \right) \longrightarrow \text{Fin}_* \]

whose colimit is $\text{Fin}_*$, based finite sets with wedge sum – it is the symmetric monoidal envelope of the $\infty$-operad corepresenting augmented $\mathcal{E}_\infty$-algebras. Because we are working over characteristic zero, there is an equivalence $\text{CAlg}^\text{aug} \longrightarrow \text{Alg}^\text{aug}_{\mathcal{E}_\infty}$ between augmented commutative algebras and augmented $\mathcal{E}_\infty$-algebras in $\text{Ch}_k^\odot$.

Each resulting functor $\text{Alg}^\text{fr}_n \rightarrow \text{Alg}^\text{fr}_{n-1}$ preserves pullbacks and sends trivial non-unital $\mathcal{E}_n$-algebras to trivial non-unital $\mathcal{E}_{n-1}$-algebras, by inspection. There results an inverse system of $\infty$-categories

\[
\text{Artin} := \lim \left( \cdots \longrightarrow \text{Artin}^\text{fr}_n \longrightarrow \text{Artin}^\text{fr}_{n-1} \longrightarrow \cdots \right)
\]

and, through the above paragraph, the limit consists of the non-unital $\text{Artin}$ commutative algebras in $\text{Ch}_k^\odot$.

As in the $n$-disk algebra case, the Maurer-Cartan functor of a Lie algebra defines a covariant functor on commutative $\text{Artin}$ algebras. Recall the functor $T : \text{CAlg}^\text{aug} \rightarrow \text{Alg}_{\text{Lie}}^\text{op}$ sending an augmented commutative algebra $A$ to the relative tangent space of the augmentation map $A \rightarrow k$, together with its standard Lie algebra structure constructed in [SS]; see also [TV2] for an $\infty$-categorical treatment of this Lie algebra structure.

Definition 4.2.1. For a commutative $\text{Artin}$ algebra $R$ and a Lie algebra $\mathfrak{g}$ over a field $k$, the Maurer-Cartan space $\text{MC}_\mathfrak{g}(R)$ is the space of maps $\text{Alg}_{\text{Lie}}(TR, \mathfrak{g})$ from the tangent complex of $R$ to $\mathfrak{g}$. The Maurer-Cartan functor $\text{MC} : \text{Alg}_{\text{Lie}} \rightarrow \text{Moduli}_{\infty}^\text{fr}$ is the adjoint of the pairing

\[
\text{Artin} \times \text{Alg}_{\text{Lie}} \longrightarrow \text{Alg}_{\text{Lie}}^\text{op} \times \text{Alg}_{\text{Lie}} \longrightarrow \text{Spaces}
\]

where the first functor is $T \times \text{id}$ and the second functor is the mapping space.

We can now state our main application to Lie theory, relating the Maurer-Cartan functor and enveloping algebras.
Theorem 4.2.2. Let \( \mathfrak{g} \) be a Lie algebra over a field \( k \) of characteristic zero. Let \( \overline{M} \) be a compact framed \( n \)-manifold with partitioned boundary \( \partial \overline{M} = \partial_R \sqcup \partial_L \). There is a natural equivalence of chain complexes over \( k \)

\[
\left( \int_{\overline{M} \setminus \partial_R} U_n \mathfrak{g} \right)^\vee \simeq \int_{\overline{M} \setminus \partial_L} MC_\mathfrak{g}
\]

between the linear dual of the factorization homology with coefficients in the universal enveloping \( E_n \)-algebra of \( \mathfrak{g} \) and the factorization homology with coefficients in the Maurer-Cartan moduli functor of \( \mathfrak{g} \). In the case \( n = 1 \) and \( \overline{M} = S^1 \), this specializes as a natural equivalence of chain complexes over \( k \)

\[
HH_* (U \mathfrak{g})^\vee \simeq HH_* (MC_\mathfrak{g})
\]

between the linear dual of the Hochschild homology of the universal enveloping algebra of \( \mathfrak{g} \) and the Hochschild homology of the Maurer-Cartan moduli functor of \( \mathfrak{g} \).

The following proof relies on Lemma 4.2.4, which is to follow.

Proof of Theorem 4.2.2. We demonstrate a sequence of equivalences of chain complexes over \( k \):

\[
\left( \int_{M_*} U_n \mathfrak{g} \right)^\vee \cong \lim_{\ell \to \infty} \left( \int_{M_*} U_n \mathfrak{g} \right)^\vee
\]

\[
\cong \lim_{\ell \to \infty} \left( \int_{M_* \wedge (\mathfrak{R'}_\ell)^+} U_{n+\ell} \mathfrak{g} \right)^\vee
\]

\[
\cong \lim_{\ell \to \infty} \lim_{R \in \text{Artin}^n_{\mathfrak{MC} \mathfrak{g}} \wedge (M_+^{\wedge (\mathfrak{R'}_\ell)} + U_{n+\ell} \mathfrak{g})} \int_{M_+^{\wedge (\mathfrak{R'}_\ell)} + S}
\]

\[
\cong \int_{M^+} \mathfrak{MC} \mathfrak{g}
\]

The first equivalence is clear, as a sequential limit of a constant sequential diagram. The third equivalence is Theorem 3.2.4 applied to \( U_{n+\ell} \mathfrak{g} \). The fourth equivalence follows from Lemma 4.2.4. The fifth equivalence follows because each map of chain complexes \( \int_{M_+^{\wedge (\mathfrak{R'}_\ell)} + S} \) being an equivalence of chain complexes, because \( S \) is commutative thereby making \( \int_{\_} S \) based homotopy invariant.

To explain the second equivalence we compare the two functors \( \text{Mfld}^0_\mathfrak{n} \to \text{Ch}_k \) which are \( \int_{\_} U \mathfrak{g} \) and \( \int_{\_ \wedge (\mathfrak{R'}_\ell\mathfrak{R'})} U_{n+\ell} \mathfrak{g} \). By the Fubini theorem for factorization homology from [AFT2], it suffices to show the equivalence of the two algebras \( U_n \mathfrak{g} \) and \( \int_{(\mathfrak{R'}_\ell)} U_{n+\ell} \). This equivalence of algebras is implied by Theorem 5.11 of [PV3], which assures that the composite of the left adjoint functors \( \text{Alg}_{\text{Lie}} U_{n+\ell} \to \text{Alg}_{\text{aug}}^{\text{MC} \mathfrak{g}} \) is left adjoint to the restriction, and therefore agrees with \( U_n \).

We have the following pair of limiting cases of this equivalence, in the case where \( \mathfrak{MC} \mathfrak{g} \) is affine or \( \mathfrak{g} \) is highly connected.

Corollary 4.2.3. Let \( k \) be a field of characteristic zero. Let \( \mathfrak{g} \) be a Lie algebra over over \( k \). Suppose \( \mathfrak{g} \) is either

- \( n \)-connective and finite in each degree, or
- \( (−1) \)-coconnective and finite as a \( k \)-module.

Let \( \overline{M} \) be a compact \( n \)-manifold with partitioned boundary \( \partial \overline{M} = \partial_L \sqcup \partial_R \). Then there is a natural equivalence of chain complexes over \( k \)

\[
\left( \int_{\overline{M} \setminus \partial_R} U_n \mathfrak{g} \right)^\vee \simeq \int_{\overline{M} \setminus \partial_L} C^* \mathfrak{g}
\]
between the linear dual of the factorization homology with coefficients in the $E_n$-universal enveloping algebra of $g$ and the factorization homology with coefficients in the Lie algebra cochains of $g$. In the case $n = 1$ and $M = S^1$, this specializes as a natural equivalence of chain complexes over $k$

$$HH_\ast(Ug) \simeq HH_\ast(C^\ast g)$$

involving the Hochschild homologies of the universal enveloping algebra of $g$ and the Lie algebra cohomology of $g$.

The following result was used in the proof of Theorem 4.2.2. Our proof relies on essential results of [AF2], where it is shown that in characteristic zero Lie algebras are the inverse limit of $E_n$-algebras via enveloping algebras.

**Lemma 4.2.4.** Let $g$ be a Lie algebra. There is a canonical sequence of $\infty$-categories

$$\text{Artin}_{/MC_g} \xrightarrow{\simeq} \lim_{\ell \to \infty} (\text{Artin}_{/MC_{U\ell}}^{fr})$$

whose limit is commutative Artin algebras over $MC_g$.

**Proof.** For each $\ell$ there is a composite of copresheaves on $\text{Artin}

$$MC_g := \text{Map}_{E_\ell}(T, g) \xrightarrow{U_T} \text{Map}_{E_\ell}(U_T, U \ell g) \simeq \text{Map}_{E_\ell}(D^\ell, U \ell g)$$

in where the last equivalence follows from the equivalence $D^\ell \simeq U_T$ of functors $\text{Artin} \to \text{Alg}_{\text{aug}}$. Because we are working over a field of characteristic zero, the results of [AF2] give that the resulting map to the limit

$$MC_g \xrightarrow{\simeq} \lim_{\ell \to \infty} \text{Map}_{E_\ell}(D^\ell, U \ell g)$$

is an equivalence of copresheaves on $\text{Artin}$. Through the unstraightening construction ([Lm1]), this equivalence of copresheaves gives a diagram of $\infty$-categories

$$\text{Artin}_{/MC_g}^{fr} \xrightarrow{\simeq} \lim_{\ell \to \infty} (\text{Artin}_{/MC_{U\ell}}^{fr})$$

whose horizontal arrows are equivalences.

\[ \square \]

**References**

[AF1] Ayala, David; Francis, John. Zero-pointed manifolds. Preprint. Available at [http://arxiv.org/abs/1409.2857](http://arxiv.org/abs/1409.2857)

[AF2] Ayala, David; Francis, John. Chiral Lie theory. In preparation.

[AFT1] Ayala, David; Francis, John; Tanaka, Hiro Lee. Local structures on stratified spaces. Preprint. Available at [http://arxiv.org/abs/1409.0501](http://arxiv.org/abs/1409.0501)

[AFT2] Ayala, David; Francis, John; Tanaka, Hiro Lee. Factorization homology of stratified spaces. Preprint. Available at [http://arxiv.org/abs/1409.0848](http://arxiv.org/abs/1409.0848)

[BaD] Baez, John; Dolan, James Higher-dimensional algebra and topological quantum field theory. J. Math. Phys. 36 (1995), no. 11, 6073–6105.

[BeD] Beilinson, Alexander; Drinfeld, Vladimir. Chiral algebras. American Mathematical Society Colloquium Publications, 51. American Mathematical Society, Providence, RI, 2004.

[BFN] Ben-Zvi, David; Francis, John; Nadler, David. Integral transforms and Drinfeld centers in derived algebraic geometry. J. Amer. Math. Soc. 23 (2010), no. 4, 909–966.

[BV] Boardman, J. Michael; Vogt, Rainer. Homotopy invariant algebraic structures on topological spaces. Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin-New York, 1973.

[Bc] Bödigheimer, C.-F. Stable splittings of mapping spaces. Algebraic topology (Seattle, Wash., 1985), 174–187, Lecture Notes in Math., 1286, Springer, Berlin, 1987.

[Ca] Campbell, Jonathan. Derived Koszul duality and topological Hochschild homology. Preprint. Available at [http://arxiv.org/abs/1401.5147](http://arxiv.org/abs/1401.5147)

[Co] Costello, Kevin. Renormalization and effective field theory. Mathematical Surveys and Monographs, 170. American Mathematical Society, Providence, RI, 2011.
[CG] Costello, Kevin; Gwilliam, Owen. Factorization algebras in perturbative quantum field theory. Preprint. Available at http://www.math.northwestern.edu/~costello/renormalization

[FT] Feigin, Boris; Tsygan, Boris. Cyclic homology of algebras with quadratic relations, universal enveloping algebras and group algebras. K-theory, arithmetic and geometry (Moscow, 1984–1986), 210–239, Lecture Notes in Math., 1289, Springer, Berlin, 1987.

[Fr1] Francis, John. Derived algebraic geometry over \( \mathcal{E}_n \)-rings. Thesis (PhD) – Massachusetts Institute of Technology. 2008.

[Fr2] Francis, John. The tangent complex and Hochschild cohomology of \( \mathcal{E}_n \)-rings. Compos. Math. 149 (2013), no. 3, 430–480.

[Fr3] Francis, John. Factorization homology of topological manifolds. Preprint. Available at http://arxiv.org/abs/1206.5522

[GL] Gaitsgory, Dennis; Lurie, Jacob. Weil’s Conjecture for Function Fields I. Preprint. Available at http://www.math.harvard.edu/~lurie/

[GeK] Getzler, Ezra; Kapranov, Mikhail. Cyclic operads and cyclic homology. Geometry, topology, & physics, 167–201, Conf. Proc. Lecture Notes Geom. Topology, IV, Int. Press, Cambridge, MA, 1995.

[GiK] Ginzburg, Victor; Kapranov, Mikhail. Koszul duality for operads. Duke Math. J. 76 (1994), no. 1, 203–272.

[Gw] Goodwillie, Thomas. Calculus. III. Taylor series. Geom. Topol. 7 (2003), 645–711.

[Hi] Hinich, Vladimir. Formal stacks as dg-coalgebras. J. Pure Appl. Algebra 162 (2001), No. 2-3, 209–250.

[Jo] Joyal, André. Quasi-categories and Kan complexes. Special volume celebrating the 70th birthday of Professor Max Kelly. J. Pure Appl. Algebra 175 (2002), no. 1-3, 207–222.

[Ka] Kallel, Sadok. Spaces of particles on manifolds and generalized Poincaré dualities. Q. J. Math. 52 (2001), no. 1, 45–70.

[Ki] Kister, James. Microbundles are fibre bundles. Ann. of Math. (2) 80 (1964) 190–199.

[Ku1] Kuhn, Nicholas. Goodwillie towers and chromatic homotopy: an overview. Proceedings of the Nishida Fest (Kinosaki 2003), 245–279, Geom. Topol. Monogr., 10, Geom. Topol. Publ., Coventry, 2007.

[Ku2] Kuhn, Nicholas. Localization of André-Quillen-Goodwillie towers, and the periodic homology of infinite loopspaces. Adv. Math. 201 (2006), no. 2, 318–378.

[Lu1] Lurie, Jacob. Higher topos theory. Annals of Mathematics Studies, 170. Princeton University Press, Princeton, NJ.; University of Tokyo Press, Tokyo, 1997.

[Lu2] Lurie, Jacob. Higher algebra. Preprint dated September 14, 2014. Available at http://www.math.harvard.edu/~lurie/

[Lu3] Lurie, Jacob. On the classification of topological field theories. Current developments in mathematics, 2008, 129–280, Int. Press, Somerville, MA, 2009.

[Lu4] Lurie, Jacob. Derived algebraic geometry X: Formal moduli problems. Preprint. Available at http://www.math.harvard.edu/~lurie/

[Matsu] Matsumoto, Taku. Descent and the Koszul duality for locally constant factorisation algebras. Thesis (Ph.D.) – Northwestern University. 2014.

[May] May, J. Peter. The geometry of iterated loop spaces. Lectures Notes in Mathematics, Vol. 271. Springer-Verlag, Berlin-New York, 1972. viii+175 pp.

[Mc] McDuff, Dusa. Configuration spaces of positive and negative particles. Topology 14 (1975), 91–107.

[Pr] Priddy, Stewart. Koszul resolutions. Trans. Amer. Math. Soc. 152 (1970) 39–60.

[Qu] Quinn, Frank. Ends of maps. III. Dimensions 4 and 5. J. Differential Geom. 17 (1982), no. 3, 503–521.

[Sc] Salvatore, Paolo. Configuration spaces with summable labels. Cohomological methods in homotopy theory (Bellaterra, 1998), 375–395, Progr. Math., 196, Birkhäuser, Basel, 2001.

[Sc] Schlessinger, Michael. Functor of Artin rings. Trans. Amer. Math. Soc. 130 (1968) 208–222.

[SS] Schlessinger, Michael; Stasheff, James. The Lie algebra structure of tangent cohomology and deformation theory. J. Pure Appl. Algebra 38 (1985), no. 2-3, 313–322.

[Se1] Segal, Graeme. Configuration-spaces and iterated loop-spaces. Invent. Math. 21 (1973), 213–221.

[Se2] Segal, Graeme. The topology of spaces of rational functions. Acta. Math., 143 (1979), 39–72.

[Se3] Segal, Graeme. The definition of conformal field theory. Topology, geometry and quantum field theory, 421–577, London Math. Soc. Lecture Note Ser., 308, Cambridge Univ. Press, Cambridge, 2004.

[Se4] Segal, Graeme. Locality of holomorphic bundles, and locality in quantum field theory. The many facets of geometry, 164–176, Oxford Univ. Press, Oxford, 2010.

[Sm] Smale, Stephen. Generalized Poincaré’s conjecture in dimensions greater than four. Ann. of Math. (2) 74 (1961), 391–406.

[TV] Toën, Bertrand; Vezzosi, Gabriele. Homotopical algebraic geometry. II. Geometric stacks and applications. Mem. Amer. Math. Soc. 193 (2008), no. 902.

[We] Weiss, Michael. Embeddings from the point of view of immersion theory. I. Geom. Topol. 3 (1999), 67–101.
Department of Mathematics, Montana State University, Bozeman, MT 59717
E-mail address: david.ayala@montana.edu

Department of Mathematics, Northwestern University, Evanston, IL 60208
E-mail address: jnkf@northwestern.edu