Bose-Einstein Condensation in the
Relativistic Ideal Bose Gas

M. Grether,\textsuperscript{a} M. de Llano,\textsuperscript{b} and George A. Baker, Jr.\textsuperscript{c}
\textsuperscript{a}Facultad de Ciencias, Universidad Nacional Autónoma de México
04510 México, DF, Mexico
\textsuperscript{b}Texas Center for Superconductivity, University of Houston, Houston, TX
77204, USA and Instituto de Investigaciones en Materiales
Universidad Nacional Autónoma de México, 04510 México, DF, Mexico*
\textsuperscript{c}Theoretical Division, Los Alamos National Laboratory
Los Alamos, NM 87545, USA

February 1, 2008

Abstract

The Bose-Einstein condensation (BEC) critical temperature in a relativistic ideal
Bose gas of identical bosons, with and without the antibosons expected to be pair-
produced abundantly at sufficiently hot temperatures, is exactly calculated for all boson
number-densities, all boson point rest masses, and all temperatures. The Helmholtz
free energy at the critical BEC temperature is lower with antibosons, thus implying
that omitting antibosons always leads to the computation of a metastable state.

PACS # 03.75.Kk; 05.30.Jp; 47.75.+f

Since its theoretical prediction by Einstein in 1925 based on the work in 1924 by Bose on
photons, and after languishing for many decades as a mere academic exercise in textbooks,
BEC has been observed in the laboratory in laser-cooled, magnetically-trapped ultra-cold bosonic atomic clouds of $^{87}$Rb [1], $^{7}$Li [2], $^{23}$Na [3], $^1$H [4], $^{85}$Rb [5], $^4$He [6], $^{41}$K [7], $^{133}$Cs [8](a), $^{174}$Yb [8](b), and $^{52}$Cr [9]. Previously, BEC in a gas of excitons had also been reported [10]. More recently, BEC has been seen as well in fermionic atomic gases of $^{40}$K [11] and $^6$Li [12] as a result of some of the fermions presumably Cooper-pairing [13] into bosons. The role of hole Cooper pairs accounted for or not, along with the usual particle Cooper pairs, has been explored [14]-[17] with striking implications for many-electron superconducting states within a BEC scenario. Their effects in neutral-fermion many-particle systems are yet to be investigated. It is the analogy of hole pairs with antibosons which has piqued our interest in the problem to be dealt with in this paper. Although it is hard to imagine a many-boson system whose constituent bosons do not disintegrate into their components at the high temperatures where antiboson production becomes substantial, it may shed light on the dynamics of many-fermion systems where both particle- and hole-Cooper-pairing can occur at all temperatures. Moreover, recent specific-heat measurements [18] in TlCuCl$_3$ suggest a magnon BEC with a relativistic dispersion relation.

In early papers [19]-[21] on the relativistic ideal boson gas (RIBG) explicit BEC critical transition temperature $T_c$-formulae were derived for both the nonrelativistic and ultra-relativistic limits and specific-heat anomalies at $T_c$ were studied. In addition, Refs. [20]-[21] considered all space dimensions $d > 0$ and delved into the relation between $d$ and various critical exponents. Antiboson production, however, was not accounted for. The first papers to include both bosons and anti-bosons appear to be Refs. [22]-[23] where high-temperature expansions for the various thermodynamic functions (pressure, particle-number density, entropy, specific heats, etc.) were derived. Extensive numerical work in $d$ dimensions that does not rely such high-temperature expansions was reported in Refs. [24]-[25]. In an elegant treatment [26] with inverse Mellin transforms the specific heat anomaly of the RIBG at its BEC $T_c$ was found to be washed out when pair-production was included. The relationship between the BEC of the RIBG and spontaneous-symmetry breaking was explored in Refs. [23]-[27]; see also the rather complete Ref. [28], esp. Sec. 2.4. The so-called BCS-[29]-to-Bose crossover scenario (see Ref. [17] and refs. therein), and even the pseudogap concept [30] of
superconductors, first seemed to have appeared in quark physics in Ref. [31]; for a review see Ref. [32]. More recently, two “crossovers” have been identified [33] in an interacting fermion gas where pairing into bosonic Cooper pairs [13] can occur to form a relativistic superfluid as an example of the BCS-Bose crossover followed by a Bose-to-RIBG/BEC-crossover where both anti-bosons as well as bosons dominate the thermodynamics. A fully relativistic detailed study [34] of these crossovers at zero temperature has also appeared.

In this Letter we exhibit, as a function of boson number density, exact BEC transition temperatures for the RIBG gas of identical bosons with and without antibosons in 3D. The system with both kinds of bosons always has the higher \( T_c \), i.e., is the system with the first BEC singularity that appears as it is cooled. This suggests that the Helmholtz free energy might be lower and thus correspond thermodynamically to the stable system as opposed to a metastable system for the lower-\( T_c \) system. It is then calculated and indeed found to be lower at all densities for the complete problem with both bosons and antibosons, when compared to the problem without antibosons. This implies that the omission of antibosons will not lead to stable states.

The number of bosons \( N \) of mass \( m \) that make up an ideal boson gas in \( d \) dimensions (without antibosons) is \( N = \sum_k n_k \equiv \sum_k [e^{\beta(E_k - \mu(T))} - 1]^{-1} \) where \( \beta = 1/k_B T \), \( k_B \) is the Boltzmann constant and \( \mu(T) \) is the boson chemical potential. Here, the total energy of each boson is

\[
|E_k| \equiv \sqrt{c^2\hbar^2k^2 + m^2c^4} \tag{1}
\]

\[
= mc^2 + \frac{\hbar^2k^2}{2m} + O(k^4) \quad \text{if } \hbar k \ll mc^2 \quad \text{NR} \tag{2}
\]

\[
= \hbar k [1 + \frac{1}{2}(mc/\hbar k)^2 + O(k^{-4})] \quad \text{if } \hbar k \gg mc^2 \quad \text{UR} \tag{3}
\]

where \( k \) is the boson wavenumber, \( m \) its rest mass and \( c \) is the speed of light. The two limits refer to the nonrelativistic (\( NR \)) and ultrarelativistic (\( UR \)) extremes. For a cubic box of side length \( L \) in the continuous limit the sums over the \( d \)-dimensional wavevector \( k \) become integrals as \( \sum_k \to (L/2\pi)^d \int d^dk \). At the BEC critical transition temperature \( T_c \), \( \mu(T_c) = mc^2 \)
and the boson number density can be expressed as

\[ n \equiv \frac{N}{L^d} = \frac{1}{(2\pi)^d} \int d^d k \frac{1}{\exp[\beta_c (|E_k| - mc^2)] - 1} \]  

(4)

where \( \beta_c \equiv 1/k_B T_c \). In the nonrelativistic (NR) extreme (2) inserted into (4) leaves \( n = (2\pi)^{-d} \int d^d k [\exp[\beta_c (\hbar^2 k^2/2m)] - 1]^{-1} \). Putting \( d^d k = [2\pi^{d/2}/\Gamma(d/2)]k^{d-1} dk \) when integrating over terms independent of angles gives an expression for \( n \) as function of \( T_c \) in terms of the Bose function \[ g_\sigma(z) \] of \( z \equiv \exp(\mu/k_B T_c) \) which for \( z = 1 \) diverges, namely \( g_\sigma(1) \to \infty \) when \( \sigma \leq 1 \), but becomes the Riemann Zeta function \( \zeta(\sigma) < \infty \) when \( \sigma > 1 \). Here \( \Gamma(\sigma) \) is the gamma function. Solving for the critical temperature then gives

\[ k_B T_c^{NR-B} = \frac{2\pi\hbar^2}{m} \left[ \frac{\zeta(d/2)}{\zeta(d)} \right]^{2/d} \]  

(5)

where the superscript \( NR-B \) stands for the nonrelativistic limit with bosons (\( B \)) but no antibosons (\( \overline{B} \)). In 3D this reduces to the familiar textbook result \( k_B T_c^{NR-B} \approx 3.31\hbar^2 n^{2/3}/m \) since \( \zeta(3/2) \approx 2.612 \). In the ultrarelativistic (UR) extreme the leading term of (3) inserted into (4) leads to \( T_c = 0 \) for all \( d \leq 1 \) since then \( g_d(1) \) diverges. However, for \( d > 1 \)

\[ k_B T_c^{UR-B} = \left[ \frac{\hbar^d c^{d-1} \pi^{d/2} \Gamma(d/2)}{\Gamma(d) \zeta(d)} \right]^{1/d} n^{1/d} \]  

(6)

which in 3D becomes \( k_B T_c^{UR-B} = h\pi^{2/3}[n/\zeta(3)]^{1/3} \approx 2.017hcn^{1/3} \) as \( \zeta(3) \approx 1.20206 \). In 2D \( T_c^{UR-B} \neq 0 \) unlike the common instance with quadratic dispersion where \( T_c \) vanishes because \( g_1(1) \) diverges; specifically \( k_B T_c^{UR-B} = hc[2\pi n/\zeta(2)]^{1/2} \approx 1.954hcn^{1/2} \) since \( \zeta(2) = \pi^2/6 \).

At sufficiently high temperatures such that \( k_B T > mc^2 \) boson-antiboson pair-production occurs abundantly; this has been stressed by Huang [36]. The total energy \( E_k \) of each particle always satisfies \( E_k^2 = c^2\hbar^2 k^2 + m^2 c^4 \) so that \( E_k = \pm |E_k| \) where \( |E_k| \) is given by (11) and with the + sign referring to bosons and the − sign to antibosons. Instead of \( N = \sum_k n_k \) the \textit{complete} number equation is now [22]

\[ N - \overline{N} \equiv \sum_k (n_k - \overline{n}_k) \]

\[ = \sum_k \left[ \frac{1}{\exp[\beta(|E_k| - \mu)] - 1} - \frac{1}{\exp[\beta(|E_k| + \mu)] - 1} \right] \]  

(7)
where \( n_k, (\overline{n}_k) \) is the average number of bosons (antibosons) in the state of energy \( \pm |E_k| \), respectively, at a given temperature \( T \) and \( N (\overline{N}) \) is their respective total number at that temperature. Since \( n_k, (\overline{n}_k) > 0 \) for all \( k \) and \( E_0 = mc^2 \), the chemical potential must be bounded by \(-mc^2 \leq \mu \leq mc^2\). Instead of \( N \) constant one must now impose the constancy of \( N - \overline{N} \) to extract the correct BEC critical temperature, say, \( T^{BB}_c \) referring to both bosons \((B)\) and antibosons \((\overline{B})\). Since \( \mu(T^{BB}_c) = mc^2 \) (7) becomes

\[
n \equiv \frac{(N - \overline{N})}{L^d} = \frac{2\pi^{d/2}}{\Gamma(d/2)(2\pi)^d} \int_0^\infty dk k^{d-1} \frac{\sinh(\beta_c mc^2)}{\cosh \left( \beta_c \sqrt{c^2 \hbar^2 k^2 + m^2 c^4} \right) - \cosh(\beta_c mc^2)}. \tag{8}
\]

This is an exact expression for the BEC \( T_c \) of an ideal Bose gas at any temperature as it includes both bosons and antibosons; it is consistent with Eq. (13) of Ref. [24].

At low enough temperatures such that \( k_B T_c \ll mc^2 \) antibosons can be neglected and (8) simplifies to, say, \( T^{NR-B\overline{B}}_c \) which is precisely (5) as expected. In the opposite extreme \( k_B T_c \gg mc^2 \) (8) leads to the limiting expression, say,

\[
k_B T^{UR-B\overline{B}}_c = \left[ \frac{\hbar^d d^{d-2} \Gamma(d/2)(2\pi)^d}{4m \pi^{d/2} \Gamma(d) \zeta(d-1)} \right]^{1/(d-1)} n^{1/(d-1)} \tag{9}
\]

that sharply differs from (6). In 3D this becomes \( k_B T^{UR-B\overline{B}}_c = (3\hbar^3 c/m)^{1/2} n^{1/2} \), a result apparently first reported in Ref. [22]. This novel relation has suggested [18] itself experimentally as a magnon BEC in specific-heat measurements in \( TlCuCl_3 \).

As functions of the dimensionless boson number-density \( \hbar^3 n/m^3 c^3 \) Fig. 1 displays the behavior of the exact \( T^{BB}_c \) (in units of \( mc^2/k_B \)) numerically extracted from (8) for \( d = 3 \) (thick full curve labeled “exact \( B\overline{B}^n \)” compared with the nonrelativistic limit \( T^{NR-B\overline{B}}_c \) from (5) (dashed line labeled “NR-\( B\overline{B}^n \)” and with the ultrarelativistic \( T^{UR-B\overline{B}}_c \) just stated (full thin line labeled “UR-\( B\overline{B}^n \)”)). Fig. 2 shows how, at sufficiently high densities \( n \) and/or sufficiently small boson rest mass \( m \), the exact \( T^{BB}_c \) (again in units of \( mc^2/k_B \), full curve labeled “exact \( B\overline{B}^n \)” is clearly the first BEC singularity encountered as the many-boson system is cooled, compared with the “later” BEC singularity in the system without antibosons at \( T^B_c \) (dashed curve labeled “exact \( B^\prime \)” extracted numerically from (4). It is then tempting to speculate that the boson gas with both kinds of bosons will be the more stable, i.e., have a lower
Helmholtz free energy at all critical temperatures, at any fixed $\hbar^3 n/m^3 c^3$. This will now be shown to be the case indeed.

The exact Helmholtz free energy per unit volume $V = L^3$ for the boson-antiboson 3D mixture, when $T = T_c \equiv 1/k_B \beta_c$ and $\mu = mc^2$, is

$$F_{\text{exact}}^{BB}(T_c, V)/V = nmc^2 + (k_B T_c/2\pi^2) \int_0^\infty dk k^2 \{ \ln[1 - \exp(\beta_c [mc^2 - |E_k|])] \
+ \ln[1 - \exp(-\beta_c [mc^2 + |E_k|])] \}$$

(10)

where $|E_k|$ is given by (1) and $n = (N - \bar{N})/V$, and $T_c$ is extracted numerically from (8) for each value of $n$. If $k_B T_c << mc^2$ antibosons can be neglected entirely and $F_{\text{exact}}^{BB}(T_c, V)$ is just (8) but without the second log term and for which $T_c$ is now extracted numerically from (11) instead of from (8) for each value of $n$. Figure 3 is the difference between the two free energies and more clearly shows why $F_{\text{exact}}^{BB}(T_c, V)$ is always lower. The inset figure shows the behavior of these two free energies, each evaluated at their appropriate $T_c$ value; by inspection, both curves correspond as they should to positive pressures $P$ since $P = -(\partial F/\partial V)_{N,T}$. Figure 3 proves the speculation advanced.

In summary, based on exact numerical calculations of BEC $T_c$s in a 3D RIBG with and without the antibosons expected to be pair-produced at higher and higher temperatures, the highest critical $T_c$ is that associated with the RIBG system with both kinds of bosons taken into account. At lower $n$ and/or larger $m$ the higher $T_c$ merges smoothly from above onto the lower $T_c$ of the RIBG system with antibosons neglected. Comparing the associated Helmholtz free energies shows that the RIBG with both kinds of bosons has lower values for all $n$ and $m$ and thus substantiates the initial suspicion that the RIBG system with no antibosons is metastable with respect to the one with both kinds of bosons.

**Acknowledgments** MdeLl thanks E.H. Lieb for a discussion on the contents of this paper and for his encouragement; he also acknowledges support from UNAM-DGAPA-PAPIIT (Mexico) #IN108205 and CONACyT (Mexico) #41302-F. The work of F.J. Sevilla in the initial phases of this research is appreciated. We thank E.Ya. Sherman, L. He, and E. Babaev for calling Refs. [18], [31][32], and [34], respectively, to our attention. This work was supported in part by the U. S. Energy Department at the Los Alamos National Laboratory
Figure 1. BEC $T_c$s (in units $mc^2/k_B$) as function of boson number-density $n$ expressed in dimensionless form as $\hbar^3 n/m^3 c^3$. Thick curve labeled “exact $B\overline{B}$” is exact numerical result of (8) that corresponds to BEC $T_c$ in a RIBG with both bosons $B$ and anti-bosons $\overline{B}$. Thin full straight line labeled “UR-$B\overline{B}$” is the ultrarelativistic limit (9) for $d = 3$ with both kinds of bosons. Dashed straight line labeled “NR-$B\overline{B}$” is its corresponding nonrelativistic limit (5) for $d = 3$ and tends asymptotically to the “exact $B\overline{B}$” curve at smaller $n$.

Figure 2. Same as Fig. 1 comparing exact $B\overline{B}$ RIBG $T_c$ extracted from (8) against that of exact $B$ from (4).
Figure 3. Difference between exact Helmholtz free energy density $F/V$ (in units of $nmc^2$ which is the total rest-mass energy density) (full curve labeled “exact $B\bar{B}$”) and that without anti-bosons (dashed curve labeled “exact $B$”) using same horizontal axes as in Figs. 1 and 2. The inset figure displays the two free energies.

under Contract # DE-AC 52-06NA25396.

*Permanent address.

References

[1] M.H. Anderson, J.R. Ensher, M.R. Wieman, and E.A. Cornell, Science 269, 198 (1995).

[2] C.C. Bradley, C.A. Sackett, J.J. Tollett, and R.G. Hulet, Phys. Rev. Lett. 75, 1687 (1995).

[3] K.B. Davis, M.O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, and W. Ketterle, Phys. Rev. Lett. 75, 3969 (1995).

[4] D.G. Fried, T.C. Killian, L. Willmann, D. Landhuis, S.C. Moss, D. Kleppner, and T.J. Greytak, Phys. Rev. Lett. 81, 3811 (1998).
[5] S.L. Cornish, N.R. Claussen, J.L. Roberts, E.A. Cornell, and C.E. Wieman, Phys. Rev. Lett. 85, 1795 (2000).

[6] F. Pereira Dos Santos, J. Léonard, Junmin Wang, C.J. Barrelet, F. Perales, E. Rasel, C.S. Unnikrishnan, M. Leduc, and C. Cohen-Tannoudji, Phys. Rev. Lett. 86, 3459 (2001).

[7] G. Mondugno, G. Ferrari, G. Roati, R.J. Brecha, A. Simoni, and M. Inguscio, Science 294, 1320 (2001).

[8] (a) T. Weber, J. Herbig, M. Mark, H.C. Nagel, and R. Grimm, Science 299, 232 (2003); (b) Y. Takasu, K. Maki, K. Komori, T. Takano, K. Honda, M. Kumakura, T. Yabuzaki, and Y. Takahashi, Phys. Rev. Lett. 91, 040404 (2003).

[9] A. Griesmaier, J. Werner, S. Hensler, J. Stuhler, and T. Pfau, Phys. Rev. Lett. 94, 160401 (2005).

[10] J.-L. Lin and J.P. Wolfe, Phys. Rev. Lett. 71, 1222 (1993).

[11] M. Greiner, C.A. Regal, and D.S. Jin, Nature 426, 537 (2003).

[12] M.W. Zwierlein, C.A. Stan, C.H. Schunck, S.M.F. Raupach, S. Gupta, Z. Hadzibabic, and W. Ketterle, Phys. Rev. Lett. 91, 250401 (2003).

[13] L.N. Cooper, Phys. Rev. 104, 1189 (1956).

[14] V.V. Tolmachev, Phys. Lett. A 266, 400 (2000).

[15] M. Fortes, M.A. Solís, M. de Llano, and V.V. Tolmachev, Physica C 364, 95 (2001).

[16] M. de Llano and V.V. Tolmachev, Physica A 317, 546 (2003).

[17] S.K. Adhikari, M. de Llano, F.J. Sevilla, M.A. Solís, and J.J. Valencia, Physica C 453, 37 (2007).
[18] E.Ya. Sherman, P. Lemmens, B. Busse, A. Oosawa, and H. Tanaka, Phys. Rev. Lett. 91, 057201 (2003) and refs. therein.

[19] P.T. Landsberg and J. Dunning-Davies, Phys. Rev. 138, A1049 (1965).

[20] R. Beckmann, F. Karsch, and D.E. Miller, Phys. Rev. Lett. 43, 1277 (1979).

[21] R. Beckmann, F. Karsch, and D.E. Miller, Phys. Rev. A 25, 561 (1982).

[22] H.E. Haber and H.A. Weldon, Phys. Rev. Lett. 46, 1497 (1981).

[23] H.E. Haber and H.A. Weldon, Phys. Rev. D 25, 502 (1982).

[24] S. Singh and P.N. Pandita, Phys. Rev. A 28, 1752 (1983).

[25] S. Singh and R.K. Pathria, Phys. Rev. A 30, 442 (1984); Phys. Rev. A 30, 3198 (1984).

[26] H.O. Frota, M.S. Silva, and S. Goulart Rosa, Jr., Phys. Rev. A 39, 830 (1989).

[27] J.I. Kapusta, Phys. Rev. D 24, 426 (1981).

[28] J.I. Kapusta and C. Gale, Finite Temperature Field Theory: Theory and Applications, 2nd Ed. (Cambridge University Press, Cambridge, UK, 2006).

[29] J. Bardeen, L.N. Cooper and J.R. Schrieffer, Phys. Rev. 108, 1175 (1957).

[30] T. Timusk and B. Statt, Rep. Prog. Phys. 62, 61 (1999).

[31] E. Babaev, Phys. Rev. D 62, 074020 (2000).

[32] E. Babaev, Int. J. Mod. Phys. A 16, 1175 (2001).

[33] Y. Nishida and H. Abuki, Phys. Rev. D 72, 096004 (2005).

[34] L. He and P. Zhuang, Phys. Rev. D 75, 096003 (2007).

[35] R.K. Pathria, Statistical Mechanics, 2nd Ed. (Pergamon, Oxford, 1996), App. D.

[36] K. Huang, Statistical Mechanics, 2nd Ed. (Wiley, NY, 1987) see pp. 154-7 and esp. p. 294.
The graph shows a plot of $k_B T / mc^2$ versus $\hbar^3 n/m^3 c^3$. The curves labeled 'NR-BB', 'exact BB', and 'UR-BB' are depicted on the graph. The y-axis represents the scaled temperature, while the x-axis represents the scaled density. The graph illustrates the relationship between these two variables for different approximations.
$k_B T / mc^2$

$\hbar^3 n / m^3 c^3$
\[
\frac{F_{\text{exact B}} - F_{\text{exact BB}}}{V_n m c^2}
\]

![Graph showing the relationship between two variables with axes labeled as \( \hat{h}^3 n/m^3 c^3 \) and \( F/V_n m c^2 \).]