ON THE IMPLEMENTATION OF A FINITE VOLUMES SCHEME WITH MONOTONE TRANSMISSION CONDITIONS FOR SCALAR CONSERVATION LAWS ON A STAR-SHAPED NETWORK

SABRINA FRANCESCA PELLEGRINO

ABSTRACT. In this paper we validate the implementation of the numerical scheme proposed in [3] by Andreianov, Coclite and Donadello, called ACD, which describes the evolution of traffic at a junction consisting of \( m \) incoming and \( n \) outgoing arcs. Incoming arcs are parametrized by \( x \in \mathbb{R}_- \) and numbered by the index \( i \in I = \{1, \ldots, m\} \), while outgoing arcs are parametrized by \( x \in \mathbb{R}_+ \) and numbered by the index \( j \in J = \{m + 1, \ldots, m + n\} \) in such a way that the junction is always located at \( x = 0 \). We denote the generic arc by \( \Omega_h, h \in H = \{1, \ldots, m + n\}, \) and the network by \( \Gamma = \bigcup_{h \in H} \Omega_h \).

We describe the evolution of traffic on each arc by the Lighthill-Whitham-Richards (LWR) model

\[
\rho_{h,t} + f_h(\rho_h)_x = 0, \quad \text{for } t > 0, x \in \Omega_h, h \in H,
\]

where \( \rho_h \) is the density and \( f_h \) is the flux on the \( h \)-th arc. We assume that the arcs have a common maximal density \( \rho_{\max} > 0 \) and the fluxes are all bell-shaped (unimodal), Lipschitz and non-degenerate nonlinear i.e.

\[(F) \quad \text{for all } h \in H, f_h \in \text{Lip}(\{0, \rho_{\max}\}; \mathbb{R}_+) \text{ with } \|f'_h\|_{L^\infty} \leq L_h, f_h(0) = 0 = f_h(\rho_{\max}), \text{ and there exists } \rho_{h, c} \in (0, \rho_{\max}) \text{ such that } f'_h(\rho) (\rho_{h, c} - \rho) > 0 \text{ for a.e. } \rho \in [0, \rho_{\max}].\]

\[(NLD) \quad \text{for all } h \in H, f'_h \text{ is not constant on any non-trivial subinterval of } [0, \rho_{\max}].\]

We augment (1) with the initial conditions

\[
\rho_h(0,x) = \rho_{h,0}(x), \quad x \in \Omega_h,
\]

where \( \rho_{h,0} \in L^1(\Omega_h; [0, \rho_{\max}]), h \in H \). We also impose the conservation of the total density at the junction, i.e. for a.e. \( t \in \mathbb{R}_+ \)

\[
\sum_{i \in I} f_i(\rho_i(t,0^-)) = \sum_{j \in J} f_j(\rho_j(t,0^+)).
\]

Notice that the previous equation makes sense as the assumption \((F)\) ensures the existence of strong traces [14] [16].

In [3] the authors prove the well-posedness of solutions obtained as vanishing viscosity limits for the Cauchy problem [1] [2]. Their result relies upon the explicit characterization of the class of admissible weak solution at the junction in terms of vanishing viscosity germ, see [1] [3] [5]. It represents a generalized study of the model investigated in [8], in which the authors establish the existence of weak solutions as limit of vanishing viscosity approximations. Such results are relevant in the perspective of a theoretical analysis of PDEs on networks, in particular, they allow the extension of the analogy between vanishing viscosity and numerical scheme both in the network case. Furthermore, their analysis is applicable to general junction solvers enjoying the order-preservation property, see for instance [10].

The aim of this paper is to validate the implementation of the numerical scheme proposed in [3] by comparison with an explicit solution here computed and with the solutions of Riemann problems both for merge, divide and 2-2 cases. Moreover, we show the consistency of the scheme with respect to the case of a network with no discontinuity at the junction, namely taking the same flux on each arc, and finally a convergence analysis is

\textbf{Key words and phrases.} finite volumes scheme, networks, scalar conservation laws, transmission conditions, Riemann solver at the junction.

The author acknowledges the support of the Région Bourgogne Franche-Comté, projet 2017-2020 “Analyse mathématique et simulation numérique d’EDP issues de problèmes de contrôle et du trafic routier” and of the Université de Bourgogne Franche-Comté, projet Chrysalide 2017 “Contrôle, analyse numérique et applications d’équations hyperboliques sur un réseau”. The author is member of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).
also performed. These results are used in the validation of the finite volumes scheme with point constraints at the junction introduced in [9].

The paper is organized as follows. In Section 2 we briefly recall the main theoretical results for ACD. In Section 3 we compute an explicit solution for the problem in the case of a merge consisting of two incoming and one outgoing arcs. In Section 4 we present the numerical scheme. Finally, Section 5 is devoted to the validation of the scheme.

2. WELL-POSEDNESS OF ACD IN THE FRAME OF ADMISSIBLE SOLUTIONS

The well-posedness for the general Cauchy problem (1), (2), (3) is established in [3] in the frame of admissible solutions.

We recall some definitions.

Definition 2.1. A function $\rho_h \in L^\infty(\mathbb{R}_+ \times \Omega_h; [0, \rho_{\text{max}}])$, $h \in H$, is a weak solution of (5) if

- for every $k \in [0, \rho_{\text{max}}]$ and every nonnegative test function $\phi \in C^\infty(\mathbb{R}_+ \times \Omega_h; \mathbb{R})$ with compact support
  $$\int_0^\infty \int_{\Omega_h} \left[ |\rho_h - k| \partial_t \phi + \text{sign}(\rho_h - k) \left( (f_h(\rho_h) - f_h(k)) \partial_x \phi \right) \right] \, dx \, dt + \int_{\Omega_h} |\rho_{h,0}(x) - k| \phi(0, x) \, dx \geq 0;$$
- for a.e. $t > 0$, it holds
  $$\sum_{i \in I} f_i(\rho_i(t, 0-)) = \sum_{j \in J} f_j(\rho_j(t, 0+)).$$

We remind the formulation of the Bardos-LeRoux-Nédélec boundary condition for conservation laws in terms of the Godunov numerical flux (see [7,11]), which will be useful for the definition of admissible solution at the junction.

Definition 2.2. The Godunov flux related to a flux $f$ satisfying (F) is the function which associates to any couple $(a, b) \in [0, \rho_{\text{max}}]^2$ the value $f(\rho(t, 0-)) = f(\rho(t, 0^+))$ (the equality holds due to the Rankine-Hugoniot condition, where $\rho$ is the Kruzkov [12] entropy solution to the Riemann problem

$$\begin{cases} 
\partial_t \rho + \partial_x f(\rho) = 0, & t > 0, \ x \in \mathbb{R}, \\
\rho(0, x) = \begin{cases} a & \text{if } x < 0, \\
b & \text{if } x > 0, 
\end{cases} & x \in \mathbb{R},
\end{cases}$$

see Figure 7. The analytical expression of the Godunov flux is given by

$$G(a, b) = \begin{cases} 
\min_{s \in [a, b]} \{ f(s) \} & \text{if } a \leq b, \\
\max_{s \in [a, b]} \{ f(s) \} & \text{if } a \geq b.
\end{cases}$$

![Figure 1. The Godunov flux.](image)

We denote by $G_h$ the Godunov flux associated with the flux $f_h$, $h \in H$.

Consider the initial boundary value problem

$$\begin{cases} 
\partial_t \rho + \partial_x f(\rho) = 0, & \text{for } (t, x) \in \mathbb{R}_+ \times \mathbb{R}_-, \\
\rho(t, 0) = \rho_h(t), \\
\rho(0, x) = \rho_h(x),
\end{cases}$$

and assume $\rho$ is a Kruzkov entropy solution in the interior of the half plane $\mathbb{R}_+ \times \mathbb{R}_-$. Then $\rho$ satisfies the boundary condition in the sense of Bardos-LeRoux-Nédélec (see [7]) if and only if $f(\rho(t, 0-)) = G(\rho(t, 0-), \rho_h(t))$. 
Fix an initial condition \( \tilde{\rho}_0 = (\rho_{1,0}, \ldots, \rho_{m+n,0}) \in L^\infty(\mathbb{R}^+ \times \Gamma; [0, \rho_{\text{max}}]) \). We look for a function \( \tilde{\rho} = (\rho_{1}, \ldots, \rho_{m+n}) \) such that for every \( h \in H \), \( \rho_h \in L^\infty(\mathbb{R}^+ \times \Omega_h; [0, \rho_{\text{max}}]) \) is a weak entropy solution of

\[
\begin{align*}
\partial_t \rho_h + \partial_x f_h(\rho_h) &= 0, & \text{on} & \, \, [0, T] \times \Omega_h, \\
\rho_h(t,0) &= \rho_h(t), & \text{on} & \, \, [0, T], \\
\rho_h(0,x) &= \rho_{h,0}(x), & \text{on} & \, \, \Omega_h,
\end{align*}
\]

where \( \tilde{v} : \mathbb{R}^+ \to [0, \rho_{\text{max}}] \) is to be fixed in the sequel in order to ensure the conservation at the junction.

The condition (4) is equivalent to ask for the traces \( \rho_h(t,0) \) to satisfy the boundary condition in the sense of Bardos-LeRoux-Nédélec

\[
\begin{align*}
\partial_t \rho_h + \partial_x f_h(\rho_h) &= 0, & \text{on} & \, \, [0, T] \times \Omega_h, \\
\rho_h(t,0) &= \rho_h(t), & \text{on} & \, \, [0, T], \\
\rho_h(0,x) &= \rho_{h,0}(x), & \text{on} & \, \, \Omega_h,
\end{align*}
\]

In order to describe the solution of (1) we postulate (see [1],[6])

\[
\begin{align*}
\rho_h(t,0) &= \rho_h(t), & \text{for a.e.} & \, \, t \in \mathbb{R}^+, & \text{for all} \, \, h \in H.
\end{align*}
\]

The criterion for the choice of \( p \) is equivalent to the condition (4), indeed, due to (6) and (7), we can express (4) in the following way

\[
\begin{align*}
\sum_{i \in I} G_i (p_i(t,0^-), p(t)) &= \sum_{j \in J} G_j (p(t), p_j(t,0^+)), & \text{for a.e.} & \, \, t > 0.
\end{align*}
\]

We can now give the definition of admissible solution.

**Definition 2.3.** Given an initial condition \( \tilde{\rho}_0 \in L^\infty(\Gamma; [0, \rho_{\text{max}}]) \), we say that \( \tilde{\rho} = (\rho_{1}, \ldots, \rho_{m+n}) \) in \( L^\infty(\mathbb{R}^+ \times \Gamma; [0, \rho_{\text{max}}]) \) is an admissible solution for the Cauchy problem at the junction (1), if there exists a function \( p \in L^\infty(\mathbb{R}^+ \times [0, \rho_{\text{max}}]) \) such that, for any \( h \in H \), \( \rho_h \) is a weak solution of (5) with \( \rho_h, \, h \in H \), and such that \( \tilde{p} \) fulfill (4).

The authors provide a characterization of vanishing viscosity limits for the problem (1) in terms of \( m + n \) Dirichlet problems on \( \Omega_h \), \( h \in H \) coupled by a transmission condition at the junction. To this aim, they introduce the vanishing viscosity germ (see [1],[4],[5]), which can be identified by the set of all possible stationary admissible solutions to (1) that are constant on each road of \( \Gamma \).

The main result of [3] is summarized in the following theorem.

**Theorem 2.4** (Theorem 3.1 in [3]). For any given initial condition \( \tilde{\rho}_0 = (\rho_{0,1}, \ldots, \rho_{0,m+n}) \in L^\infty(\mathbb{R}^+ \times \Gamma; [0, \rho_{\text{max}}]) \) the problem (1) admits a unique admissible solution \( \tilde{\rho} \) in \( L^\infty(\mathbb{R}^+ \times \Gamma; [0, \rho_{\text{max}}]) \).

Moreover, if \( \tilde{\rho}_1 \) and \( \tilde{\rho}_0 \) are two admissible solutions corresponding respectively to the initial condition \( \tilde{\rho}_0 \) and \( \tilde{\rho}_0 \), then for all \( M > 0 \) and \( t < M/L \), where \( L = \max \{ \| f_h \|_{L^\infty([0, \rho_{\text{max}}])} \mid h = 1, \ldots, m + n \} \),

\[
\begin{align*}
\sum_{i=1}^m \int_{-(M-Lt)}^0 |\rho_i(t,x) - \rho_i'(t,x)| \, dx + & \sum_{j=m+1}^{m+n} \int_0^M |\rho_j(t,x) - \rho_j'(t,x)| \, dx \\
\leq & \sum_{i=1}^m \int_{-M}^0 |\rho_{i,0}(x) - \rho_{i,0}(x)| \, dx + \sum_{j=m+1}^{m+n} \int_0^M |\rho_{j,0}(x) - \rho_{j,0}(x)| \, dx.
\end{align*}
\]

In particular, the map that associates to \( \tilde{\rho}_0 \) the unique corresponding admissible profile \( \tilde{\rho}(t) \), is non-expansive with respect to the \( L^1 \) distance for all \( t > 0 \).

3. AN EXPLICIT ADMISSIBLE SOLUTION AT A MERGE

In this section we compute an explicit solution for the problem consisting of two incoming and one outgoing arcs. We consider

\[
f(\rho) = f_h(\rho) = \rho(1 - \rho)
\]

as the flux for each arc. As initial condition, we choose

\[
\begin{align*}
\rho_{1,0}(x) &= \chi_{[-1/2,0]}(x), & \rho_{2,0}(x) &= 3/4 \chi_{[-1/4,0]}(x), & \rho_{3,0}(x) &= 0.
\end{align*}
\]

The exact solution is obtained by an explicit analysis of the wave-front interactions, with computer assisted computation of the front slopes and interaction times.

At time \( t = 0 \), let \( p_1 \approx 0.85 \) be the solution of

\[
G_1(1, p_1) + G_2(3/4, p_1) = G_3(p_1, 0),
\]

then, on \( \Omega_1 \) a rarefaction \( \mathcal{R}_{O,1} \) starts from \( O(0,0) \) and its values are given by

\[
\mathcal{R}_{O,1} = \frac{1}{2} \left( 1 - \frac{x}{t} \right), \quad \text{for} \quad -t \leq x \leq -\frac{\sqrt{2}}{2} t.
\]
On $\Omega_2$ starts the backward shock $\mathcal{S}_{O,2}$ given by
\[
\mathcal{S}_{O,2} : \dot{x}(t) = \sigma \left( \frac{3}{4}, p_1 \right), \quad x(0) = 0.
\]
On $\Omega_3$ a rarefaction starts from $O(0,0)$ and its values are given by
\[
\mathcal{R}_{O,3} = \frac{1}{2} \left( 1 - \frac{x}{t} \right), \quad \text{for } 0 \leq x \leq t.
\]
On $\Omega_2$, let $C(x_C,t_C)$ be the point where the shock $\mathcal{S}_{B,2} : x(t) = -\frac{1}{6} + \frac{t}{4}$ originated from $B(-1/4,0)$ interacts with the shock $\mathcal{S}_{O,2}$. As a result, from $C$ starts a shock given by
\[
\mathcal{S}_{C,2} : \dot{x}(t) = \sigma (0, p_1), \quad x(t_C) = x_C,
\]
and reaches the junction $x = 0$ at time $t = t_F = 3/2$ that corresponds to the time at which the second incoming arc becomes empty. On $\Omega_1$, in $D(-1/2,1/2)$, the stationary shock $\mathcal{S}_{A,1}$ originated from $A(-1/2,0)$ interacts with the rarefaction $\mathcal{R}_{O,1}$. As a result, from $D$ starts a shock $\mathcal{S}_{D,1}$ given by
\[
\mathcal{S}_{D,1} : \dot{x}(t) = \sigma (0, \mathcal{R}_{O,1}(t,x(t))), \quad x(1/2) = -1/2.
\]
Let $E(x_E,t_E)$ be the intersection between $\mathcal{S}_{D,1}$ and $x(t) = -\sqrt{2}/2 \cdot t$. From this point starts a forward shock $\mathcal{S}_{E,1} : \dot{x}(t) = \sigma (0, p_1), \quad x(t_E) = x_E$.

Let $p_2 = 1/2$ be the solution of
\[
G_1(p_1,p_2) + G_2 (0,p_2) = G_3 \left( p_2, \frac{1}{2} \right), \quad \text{for } t > t_F.
\]
Therefore, a rarefaction appears on $\Omega_1$:
\[
\mathcal{R}_{F,1}(t,x) = \frac{1}{2} \left( 1 - \frac{x}{t - t_F} \right), \quad \text{for } -\frac{\sqrt{2}}{2} (t - t_F) < x \leq 0.
\]
Let $G$ be the point where $\mathcal{S}_{E,1}$ and $\mathcal{R}_{F,1}$ interact. From this point starts a forward shock $\mathcal{S}_{G,1}$, with left state $\rho = 0$, which reaches the junction at time $t_H \approx 2.75$, then $\Omega_1$ is empty. Finally, on $\Omega_3$ at time $t_H$ starts a shock which interacts with the rarefaction $\mathcal{R}_{O,3}$ generating the additional shock $\mathcal{S}_{H,3} : \dot{x}(t) = \sigma (0, \mathcal{R}_{O,3}(t,x(t))), \quad x(t_H) = 0$.

4. Finite volumes numerical scheme

We fix a constant space step $\Delta x$. For $\ell \in \mathbb{Z}$ and $h \in \mathbb{H}$, we set $x^h_{\ell} = \ell \Delta x$. We define the cell centers $x^h_{\ell + \frac{1}{2}} = (\ell + \frac{1}{2}) \Delta x$ for $\ell \in \mathbb{Z}$ and consider the uniform spatial mesh on each $\Omega_h$,
\[
\bigcup_{\ell \in \mathbb{Z}} (x^\ell_1, x^\ell_{1+1}), \quad i \in \mathbb{Z}, \quad \bigcup_{j \in \mathbb{J}} (x^j_1, x^j_{1+1}), \quad j \in \mathbb{J},
\]
so that the position of the junction $x = 0$ corresponds to $x^0_1$ for each edge. Then we fix a constant time step $\Delta t$ satisfying the CFL condition
\[
\Delta t \max_h \{ L_h \} \leq \frac{\Delta x}{2},
\]
and for $s \in \mathbb{N}$ we define the time discretization $t^s = s \Delta t$. At each time $t^s$, $p^{h,s}_{\ell + \frac{1}{2}}$ represents an approximation of the main value of the solution on the interval $[x^h_{\ell}, x^h_{\ell+1}]$, $\ell \in \mathbb{Z}$, along the $h$-th arc. We initialize the scheme by discretizing the initial conditions
\[
p^{h,0}_{\ell + \frac{1}{2}} = \frac{1}{\Delta x} \int_{x^h_{\ell}}^{x^h_{\ell+1}} \rho^0_h(x) dx,
\]
for all $h \in \mathbb{H}$ and for $\ell \leq -1$ if $h \in \mathbb{I}$, $\ell \geq 0$ if $h \in \mathbb{J}$.

For each $s \in \mathbb{N}$, at all cell interfaces $x^h_{\ell}$ with $h \neq 0$ we consider the standard Godunov flux $G_h$ corresponding to the flux $f_h$. At the junction $x^0_1$ we take on each arc $\Omega_h$ the Godunov flux corresponding to the admissible solution of the Riemann problem at the junction, defined as in [3], which we compute by means of a two-step procedure:

(i) find
\[
p^* \in [0, \rho_{\max}] \quad \text{s.t.} \quad \sum_{i \in I} G_i(p^{-s}_{\ell - \frac{1}{2}}, p^*) = \sum_{j \in J} G_j(p^s, p^j_{\ell + \frac{1}{2}}),
\]
We observe good agreement between the exact and the numeric solution. The parameters for the simulations are $\Delta x = 10^{-4}$ and $\Delta t = 0.5 \Delta x = 0.5 \times 10^{-4}$.  

5. Validation of the numerical scheme  

We propose here to validate the numerical scheme (13)-(14)-(15) making a comparison with the explicit solution to (1) constructed in Section 3. The setup for the simulation is as follows.  

We consider a network consisting of three edges and one junction, for the whole network, for the incoming and for the outgoing arcs at a given time $t$. Figure 2 depicts the relative $L^1$-error at time $t = 2.4$ computed in Section 5.  

Table 1. Relative $L^1$-error at time $t = 2.4$ computed in Section 5.  

| Number of cells per arc | $E_{L^1}^{s,:}$ | Rate of convergence | $E_{L^1}^{s,\Gamma}$ | Rate of convergence | $E_{L^1}^{s,J}$ | Rate of convergence |
|-------------------------|-----------------|---------------------|---------------------|---------------------|-----------------|---------------------|
| 60                      | $6.5374 \times 10^{-2}$ | -                   | $2.1928 \times 10^{-1}$ | -                   | $1.4155 \times 10^{-2}$ | -                   |
| 120                     | $3.4281 \times 10^{-2}$ | 0.9313              | $1.1380 \times 10^{-1}$ | 0.9464              | $7.8554 \times 10^{-3}$ | 0.8496              |
| 600                     | $7.6754 \times 10^{-3}$ | 0.9302              | $2.4933 \times 10^{-2}$ | 0.9441              | $1.9468 \times 10^{-3}$ | 0.8625              |
| 1200                    | $4.8890 \times 10^{-3}$ | 0.8800              | $1.6393 \times 10^{-2}$ | 0.8831              | $1.0579 \times 10^{-3}$ | 0.8660              |
| 6000                    | $1.9875 \times 10^{-3}$ | 0.7721              | $7.1933 \times 10^{-3}$ | 0.7575              | $2.5294 \times 10^{-4}$ | 0.8738              |
| 12000                   | $1.6804 \times 10^{-3}$ | 0.7034              | $6.3143 \times 10^{-3}$ | 0.6815              | $1.3587 \times 10^{-4}$ | 0.8774              |

(ii) compute

$$P_{t+\frac{1}{2}}^{h,s} = P_{t+\frac{1}{2}}^{h,s} - \frac{\Delta t}{\Delta x} \left( F_{t+\frac{1}{2}}^{h,s} - F_{t}^{h,s} \right),$$

where

$$F_{t}^{h,s} = \begin{cases} G_h \left( \rho_{t+\frac{1}{2}}^{h,s}, \rho_{t+\frac{1}{2}}^{i,h} \right), & \text{if } h \in I \text{ and } \ell \leq -1 \text{ or } h \in J \text{ and } \ell \geq 1, \\ G_h \left( \rho_{t+\frac{1}{2}}^{h,i}, \rho_{t+\frac{1}{2}}^{h,s} \right), & \text{if } h \in I \text{ and } \ell = 0, \\ G_h \left( \rho_{t+\frac{1}{2}}^{1}, \rho_{t+\frac{1}{2}}^{h,s} \right), & \text{if } h \in J \text{ and } \ell = 0. \end{cases}$$

The choice of the Godunov’s flux is motivated by the fact that all admissible stationary solutions are exact solutions for such scheme. However, one can use any other numerical flux that is monotone, consistent and Lipschitz continuous.

A convergence result for the scheme (13)-(14)-(15) can be found in [3].
5.2. Riemann problem for a 1-2 divide. We consider a network consisting of three edges and one junction, with one incoming and two outgoing arcs. We consider \([-\tfrac{1}{2}, 0]\) as domain of computation for the incoming arc and \([0, \tfrac{1}{2}]\) for the outgoing ones, and we take a normalized flux \(f(\rho) = \rho (1 - \rho)\) for each arc. We consider as initial conditions for the Riemann problems \(\vec{\rho}_0, a = (\tfrac{1}{4}, 2/3, 2/5)\) and \(\vec{\rho}_0, b = (3/4, 1/3, 4/5)\), respectively.

Figure 2 shows a qualitative comparison between the numerically computed solution \(\tilde{\rho}\) and the explicitly one \(\rho_\Delta\) at times \(t = 1.2\) and \(t = 2.4\).

5.3. Riemann problem for 2-2 network. We consider here a junction with two incoming and two outgoing arcs. We consider \([-1/6, 0]\) as domain of computation for the incoming arcs and \([0, 1/6]\) for the outgoing ones,
Figure 3. With reference to the Riemann problems test for a 2-1 merge: comparison between the explicit solution and the numeric approximation for the Riemann problems at time \( t = 1/2 \). First line shows the comparison of the profiles of solution along each arc corresponding to the initial condition \( \bar{\rho}_0, a \); while last line refers to the initial datum \( \bar{\rho}_0, b \).

Figure 4. With reference to the Riemann problems for a 1-2 divide: comparison between the explicit solution and the numerical approximation for the Riemann problems at time \( t = 1/2 \). First line shows the comparison of the profiles of solution along each arc corresponding to the initial condition \( \bar{\rho}_0, a \); while last line refers to the initial datum \( \bar{\rho}_0, b \).

and we take a normalized flux \( f(\rho) = \rho (1 - \rho) \) for each arc. As initial condition for the Riemann problems on the network we choose \( \bar{\rho}_0, a = (1/4, 1/5, 2/3, 5/6) \) and \( \bar{\rho}_0, b = (3/4, 1/5, 1/3, 1/6) \).

Also in this case, in Figures 5 and 6 we find a good agreement between the exact and the numerical solutions. The parameters for the simulation are \( \Delta x = 10^{-4} \) and \( \Delta t = 0.5 \Delta x = 0.5 \times 10^{-4} \).

5.4. A network with no discontinuity at junction. In this section we consider a network consisting of an arc without any discontinuity at the junction, namely, we assume that the flux on the incoming arc coincides with the flux on the outgoing arc, and therefore, this setting is equivalent to the case of a single arc. This simulation exploits the idea consisting in solving two scalar conservation laws on half-space coupled by an
ad hoc transmission condition at the interface [1]. We remark that, in the case of discontinuous flux, the transmission condition can be interpreted in terms of a flux constraint at the interface [2].

We apply the scheme to two different domains, one including the junction located at $x = 0$, and one not. More in details, we choose $[-1/2, 1/2]$ and $[0, 1]$ as domain of computation, and $f(\rho) = \rho(1 - \rho)$ as flux along the arcs. We consider the initial densities $\rho_0^1 = 0.75\chi_{[-1/4,0]}$ and $\rho_0^2 = 0.75\chi_{[0,1/4]}$ for the first simulation, and $\rho_0^2 = 0.75\chi_{[0,1/4]}$ for the second one. The parameters of computation are $\Delta x = 10^{-4}$ and $\Delta t = 0.5 \Delta x = 0.5 \times 10^{-4}$.

Figure 7 displays the comparison between the profiles of solutions in the two simulations at times $t = 2/5$ and $t = 3/4$. We can observe the same qualitative behavior shifted of $|x| = 1/2$. 
(a) The profile of the solution of the first simulation at time $t = 2/5$ for $x \in [-1/2, 1/2]$.

(b) The profile of the solution of the second simulation at time $t = 2/5$ for $x \in [0, 1]$.

(c) The profile of the solution of the first simulation at time $t = 3/4$ for $x \in [-1/2, 1/2]$.

(d) The profile of the solution of the second simulation at time $t = 3/4$ for $x \in [0, 1]$.

**Figure 7.** With reference to the case of a network with no discontinuity at the junction: comparison between the two simulations at two different times.

**References**

[1] Andreianov, B., and Cancès, C. On interface transmission conditions for conservation laws with discontinuous flux of general shape. *J. Hyperbolic Differ. Equ.*, 12, 2 (2015), 343–384.

[2] Andreianov, B., and Cancès, C. The Godunov scheme for scalar conservation laws with discontinuous bell-shaped flux functions. *Applied Mathematics Letters* 25, 11 (2012), 1844 – 1848.

[3] Andreianov, B., Cancès, C., and Donadello, C. Well-posedness for vanishing viscosity solutions of scalar conservation laws on a network. *Discrete and Continuous Dynamical Systems A* 37, 11 (2017), 5913–5942.

[4] Andreianov, B., Karlsen, K. H., and Risebro, N. H. On vanishing viscosity approximation of conservation laws with discontinuous flux. *Netw. Heterog. Media* 5, 3 (2010), 617–633.

[5] Andreianov, B., Karlsen, K. H., and Risebro, N. H. A theory of $L^1$-dissipative solvers for scalar conservation laws with discontinuous flux. *Archive for Rational Mechanics and Analysis* 201, 1 (2011), 27–86.

[6] Andreianov, B., and Mitrović, D. Entropy conditions for scalar conservation laws with discontinuous flux revisited. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 32, 6 (2015), 1307–1335.

[7] Bardos, C., Leroux, A. Y., and Nédélec, J. C. First order quasilinear equations with boundary conditions. *Communications in Partial Differential Equations* 4, 9 (1979), 1017–1034.

[8] Coclite, G. M., and Garavello, M. Vanishing viscosity for traffic on networks. *SIAM Journal on Mathematical Analysis* 42, 4 (2010), 1761–1783.

[9] Dal Santo, E., Donadello, C., Pellegrino, S. F., and Rosini, M. D. Representation of capacity drop at a road merge via point constraints in a first order traffic model. *To appear*.

[10] Dal Santo, E., Rosini, M. D., Dymski, N., and Benyahia, M. General phase transition models for vehicular traffic with point constraints on the flow. *Mathematical Methods in the Applied Sciences* 40, 18 (2017), 6623–6641.

[11] Dubois, F., and Floch, P. L. Boundary conditions for nonlinear hyperbolic systems of conservation laws. *Journal of Differential Equations* 71, 1 (1988), 93 – 122.

[12] Kružikov, S. N. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)* 81 (123) (1970), 228–255.

[13] Lighthill, M. J., and Whitham, G. B. On kinematic waves. II. A theory of traffic flow on long crowded roads. *Proc. Roy. Soc. London. Ser. A.* 229 (1955), 317–345.

[14] Panov, E. Y. Existence of strong traces for quasi-solutions of multidimensional conservation laws. *J. Hyperbolic Differ. Equ.* 4, 4 (2007), 729–770.

[15] Richards, P. I. Shock waves on the highway. *Operations Res.* 4 (1956), 42–51.

[16] Vasseur, A. Strong traces for solutions of multidimensional scalar conservation laws. *Arch. Ration. Mech. Anal.* 160, 3 (2001), 181–193.