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Schwinger-Dyson equations and flavor mixing

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Abstract. We briefly review some results on Green’s functions for mixed fermion fields and reinterpret them in terms of Schwinger–Dyson equations. Working with retarded propagators, we identify the self-energy operator for the mixing interaction, and find an exactly solvable expansion for the complete-fermion propagator. Finally, we show that an equation à la Bethe–Salpeter can be derived from Schwinger–Dyson equations, leading to the formal concept of single-particle bound state.

1. Introduction

The idea of flavor oscillations was first conceived by Bruno Pontecorvo \cite{1} in order to quantitatively explain the results of the Homestake experiment \cite{2}. While phenomena such as neutrino or $K^0$–$\bar{K}^0$ oscillations are currently experimentally well established \cite{3}, there are still open issues and loopholes in theoretical description of fermion mixing. The standard theoretical approach to this subject is based on quantum mechanics (QM) or on perturbative quantum field theory (QFT) \cite{4}. A problem with this approach is that flavor-states cannot be described as standard free asymptotic in—out states and hence one cannot use the LSZ formalism to directly relate Green functions with the $S$ matrix \cite{5}.

A non-perturbative QFT approach to mixing was first developed in Ref. \cite{6} where the non-trivial structure of the mixing transformation generator was discovered. As consequence, corrections to the standard Pontecorvo oscillation formula were found \cite{7}. This result was then extended to the case of many flavors in Ref. \cite{8}. Other developments can be found, e.g. in Ref. \cite{9, 10}. A pedagogical introduction to this subject can be found in Ref. \cite{11}.

In this work we re-interpret the known results on fermion mixing in QFT, in a new and quite illuminating way. In particular, with the help of Schwinger–Dyson (SD) equations, we find a formal analogy with the description of bound states via Bethe–Salpeter equation. This in turn will allow us to introduce the formal concept of single particle bound-state. Aforementioned analogy with the bound-state description will be instrumental in providing a new fresh view in the description of flavor fields. Although in the present paper we deal with Dirac fermions, we expect that similar results also holds for the case of Majorana fermions, see Ref. \cite{12} for a treatment of mixing of Majorana neutrinos in QFT.
The paper is organized as follows: in Section 2 we review some fundamentals on fermion mixing transformation in QFT and related flavor Fock space. In Section 3, following Ref. [7] and Ref. [8] we discuss the problem related to the definition of Green’s functions for flavor fields. Finally, in Section 4 we reinterpret previous results by means of SD equations and we introduce the concept of single-particle bound states.

2. Fermion mixing and flavor vacuum

Let us consider the fermion mixing Lagrangian:

\[ \mathcal{L}(x) = \bar{\nu}(x) \left( i\gamma^\mu \partial_\mu - M \right) \nu(x), \]  

with

\[ \nu(x) = \begin{bmatrix} \nu_e(x) \\ \nu_\mu(x) \end{bmatrix}, \]  

and

\[ M = \begin{bmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{bmatrix}. \]  

The field equations are then:

\[ (i\gamma^\mu \partial_\mu - M) \nu(x) = 0. \]  

These can be rewritten as two independent Dirac equations thanks to the Pontecorvo mixing transformation:

\[ \nu_e(x) = \nu_1(x) \cos \theta + \nu_2(x) \sin \theta, \]
\[ \nu_\mu(x) = -\nu_1(x) \sin \theta + \nu_2(x) \cos \theta, \]

with \( \tan 2\theta = 2m_{e\mu}/(m_\mu - m_e) \). \( \nu_e \) and \( \nu_\mu \) are called flavor fields while \( \nu_1 \) and \( \nu_2 \) are the mass fields, satisfying free Dirac equations:

\[ (i\gamma^\mu \partial_\mu - m_j) \nu_j(x) = 0, \quad j = 1, 2. \]  

The masses \( m_1 \) and \( m_2 \) are related to the original parameters as:

\[ m_e = m_1 \cos^2 \theta + m_2 \sin^2 \theta, \]
\[ m_\mu = m_1 \sin^2 \theta + m_2 \cos^2 \theta. \]  

Now \( \nu_1 \) and \( \nu_2 \) can be expanded as

\[ \nu_i(x) = \frac{1}{\sqrt{V}} \sum_{k,r} e^{ikx} \left[ u^r_{k,i}(t) \alpha^r_{k,i} + v^r_{-k,i}(t) \beta^r_{-k,i} \right], \quad i = 1, 2. \]  

These are truly asymptotic fields and the (mass) vacuum satisfies:

\[ \alpha^r_{k,i}[0]_{1,2} = \beta^r_{-k,i}[0]_{1,2} = 0. \]

1 Here we consider the system in a box of volume \( V \). This regularization scheme permits to avoid problems with inequivalent representation of field algebra [13, 14, 15]. One has to perform the limit \( V \to \infty \) at the the end of calculations.
Now we rewrite the Pontecorvo relations (5),(6) as:

\[ \nu_e(x) = G_\theta^{-1}(t) \nu_1(x) G_\theta(t), \quad (12) \]

\[ \nu_\mu(x) = G_\theta^{-1}(t) \nu_2(x) G_\theta(t), \quad (13) \]

where \( G_\theta(t) \), the mixing generator, is given by

\[ G_\theta(t) = \exp \left[ \theta \int \! \! d^3x \left( \nu_1^\dagger(x) \nu_2(x) - \nu_2^\dagger(x) \nu_1(x) \right) \right], \quad (14) \]

and in a finite volume limit is a unitary operator, preserving the canonical anticommutation relations. The equivalence between Eqs.(12),(13) and the Pontecorvo relations (5),(6) can be verified by using the Baker-Campbell-Hausdorff formula. The action of the mixing generator does not leave the vacuum |0\rangle invariant:

\[ |0(t)\rangle_{e, \mu} = G_\theta^{-1}(t) |0\rangle_{1,2}. \quad (15) \]

|0(t)\rangle_{e, \mu} is the vacuum for the flavor Fock space \( \mathcal{H}_{e, \mu} \) and is called flavor vacuum. Thanks to \( G_\theta(t) \) we can write at every time the annihilation operators of the flavor vacuum as

\[ \alpha_k^{r, \sigma}(t) \equiv G_\theta^{-1}(t) \alpha_{k, j}^r G_\theta(t), \quad (16) \]

\[ \beta_k^{r, \sigma}(t) \equiv G_\theta^{-1}(t) \beta_{k, j}^r G_\theta(t), \quad (\sigma, \bar{\sigma}) = (e, 1), (\mu, 2). \quad (17) \]

In terms of these operators we can expand the flavor fields \( \nu_e \) and \( \nu_\mu \):

\[ \nu_e(x) = \frac{1}{\sqrt{V}} \sum_{k, r} e^{ik \cdot x} \left[ u_{k,1}^r(t) \alpha_{k, e}(t) + v_{k,1}^r(t) \beta_{k, e}(t) \right], \quad (18) \]

\[ \nu_\mu(x) = \frac{1}{\sqrt{V}} \sum_{k, r} e^{ik \cdot x} \left[ u_{k,2}^r(t) \alpha_{k, \mu}(t) + v_{k,2}^r(t) \beta_{k, \mu}(t) \right]. \quad (19) \]

Choosing a Lorentz frame so that \( k = (0, 0, |k|) \), Eqs.(16),(17) explicitly read:

\[ \alpha_{k, e}(t) = \cos \theta \alpha_{k, 1}^r + \sin \theta \left( U_k(t) \alpha_{k, 2}^r + e^r V_k(t) \beta_{k, 1}^r \right), \quad (20) \]

\[ \alpha_{k, \mu}(t) = \cos \theta \alpha_{k, 2}^r - \sin \theta \left( U_k(t) \alpha_{k, 1}^r - e^r V_k(t) \beta_{k, 1}^r \right), \quad (21) \]

\[ \beta_{k, e}(t) = \cos \theta \beta_{k, 1}^r + \sin \theta \left( U_k(t) \beta_{k, 2}^r - e^r V_k(t) \alpha_{k, 2}^r \right), \quad (22) \]

\[ \beta_{k, \mu}(t) = \cos \theta \beta_{k, 2}^r - \sin \theta \left( U_k(t) \beta_{k, 1}^r + e^r V_k(t) \alpha_{k, 1}^r \right), \quad (23) \]

where \( e^r = (-1)^r \) and we defined

\[ U_k(t) \equiv u_{k,2}^r(t) u_{k,1}^r(t) = v_{k,1}^r(t) v_{k,2}^r(t), \]

\[ V_k(t) \equiv e^r u_{k,1}^r(t) v_{k,2}^r(t) = -e^r u_{k,2}^r(t) v_{k,1}^r(t). \quad (24) \]

Now it is possible to define flavor states as excitations of the flavor vacuum. For example a flavor neutrino state will be

\[ |\nu_{k, \sigma}(t)\rangle \equiv \alpha_{k, \sigma}^r(t) |0(t)\rangle_{e, \mu}, \quad \sigma = e, \mu. \quad (25) \]

This is an eigenstate of the (non-conserved) flavor charge:

\[ Q_\sigma(t) = \int \! \! d^3x \nu_\sigma^\dagger(x) \nu_\sigma(x) = \sum_{k, r} \left( \alpha_{k, \sigma}^r(t) \alpha_{k, \sigma}^r(t) - \beta_{k, \sigma}^r(t) \beta_{k, \sigma}^r(t) \right), \quad \sigma = e, \mu, \quad (26) \]

as one easily check.
3. Green’s functions for flavor fields and oscillation formula

Following Ref.[7] let us construct the propagator for flavor fields. The main problem resides in the choice of the vacuum. As shown in Ref.[16], this problem is equivalent to the choice of appropriate boundary conditions.

A first possibility is to consider the propagator constructed on the mass vacuum\(^2\) \(|0\rangle_{1,2}\):

\[
S_f(x, y) = \begin{bmatrix}
S^{\alpha\beta}_{ee}(x, y) & S^{\alpha\beta}_{e\mu}(x, y) \\
S^{\alpha\beta}_{\mu e}(x, y) & S^{\alpha\beta}_{\mu\mu}(x, y)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1,2(0|T \nu_e^{\alpha}(x) \nu_e^{\beta}(y)|0\rangle_{1,2} & 1,2(0|T \nu_e^{\alpha}(x) \nu_e^{\beta}(y)|0\rangle_{1,2} \\
1,2(0|T \nu_{\mu}^{\alpha}(x) \nu_e^{\beta}(y)|0\rangle_{1,2} & 1,2(0|T \nu_{\mu}^{\alpha}(x) \nu_e^{\beta}(y)|0\rangle_{1,2}
\end{bmatrix}.
\]

This can be explicitly written in terms of the propagators of mass fields:

\[
S_f(x, y) = \begin{bmatrix}
S^{\alpha\beta}_{ee}(x, y) \cos^2 \theta + S^{\alpha\beta}_{ee}(x, y) \sin^2 \theta & (S^{\alpha\beta}_{ee}(x, y) - S^{\alpha\beta}_{e\mu}(x, y)) \cos \theta \sin \theta \\
(S^{\alpha\beta}_{e\mu}(x, y) - S^{\alpha\beta}_{\mu e}(x, y)) \cos \theta \sin \theta & S^{\alpha\beta}_{\mu\mu}(x, y) \cos^2 \theta + S^{\alpha\beta}_{\mu\mu}(x, y) \sin^2 \theta
\end{bmatrix},
\]

where

\[
S^{\alpha\beta}_{j}(x, y) = i \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{k + m_j}{k^2 - m_j^2 + i\varepsilon}, \quad j = 1, 2.
\]

In Ref.[7] it was defined the amplitude of the process where a -flavor particle was created at \(t = 0\) and is observed unchanged at time \(t > 0\), as:

\[
\mathcal{P}_{ee}(k, t) = i u^*_{k,1} e^{i\omega_k t} S^{\alpha\beta}_{ee}(k, t) \gamma^0 u^*_{k,1},
\]

where \(S^{\alpha\beta}_{ee}(k, t)\) is the Fourier transform of the Wightman function

\[
S^{\alpha\beta}_{ee}(t, x; 0, y) = 1,2(0|\nu_e(t, x) \nu_e(0, y)|0\rangle_{1,2}.
\]

An explicit calculation gives us:

\[
\mathcal{P}^{\gamma\beta}_{ee}(k, t) = \cos^2 \theta + \sin^2 \theta |U_k|^2 e^{-i(\omega_2 - \omega_1)t}.
\]

However this result is unacceptable because

\[
\mathcal{P}^{\gamma\beta}_{ee}(k, 0^+) = \cos^2 \theta + \sin^2 \theta |U_k|^2 < 1.
\]

Therefore, in Ref.[7], it was proposed to consider the propagator

\[
G_f(x, y) = \begin{bmatrix}
G^{\alpha\beta}_{ee}(x, y) & G^{\alpha\beta}_{e\mu}(x, y) \\
G^{\alpha\beta}_{\mu e}(x, y) & G^{\alpha\beta}_{\mu\mu}(x, y)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
e,\mu(0|0)\nu_e^{\alpha}(x) \nu_e^{\beta}(y)|0(0)\rangle_{e,\mu} & e,\mu(0|0)\nu_e^{\alpha}(x) \nu_e^{\beta}(y)|0(0)\rangle_{e,\mu} \\
e,\mu(0|0)\nu_e^{\alpha}(x) \nu_e^{\beta}(y)|0(0)\rangle_{e,\mu} & e,\mu(0|0)\nu_e^{\alpha}(x) \nu_e^{\beta}(y)|0(0)\rangle_{e,\mu}
\end{bmatrix}.
\]

\(^2\) In this Section we use the same notation as in Ref.[7]. The propagator is thus defined as the vacuum expectation value of the time-ordered products of fields. Such a definition differs from the one of Ref. [17] just by a factor \(i\). However, in the next Section it will be useful to follow the convention of Ref.[17].
on the flavor vacuum. As anticipated, $G_f(x, y)$ differs from $S_f(x, y)$ just by boundary terms. For example, we consider the Fourier transform of $G_{ee}(x, y)$:

$$G_{ee}(k, t) = S_{ee}(k, t) + 2\pi i \sin^2 \theta \left[ |V_k|^2 (k^2 + m_2^2) \delta(k^2 - m_2^2) \right.\left. - |U_k||V_k| \sum_r \left( e^{i \nu_{e,k,2} r} \pi_{r,k,2} \delta(k_0 - \omega_2) + e^{i \nu_{e,k,2} r} \bar{\pi}_{r,k,2} \delta(k_0 + \omega_2) \right) \right].$$

By introducing, as before, the Wightman function $G_{ee}^\ast(k, t)$, we define

$$\mathcal{P}_{ee}^>(k, t) = i u_{k,1}^\dagger e^{i\omega_{k,1} t} G_{ee}^\ast(k, t) \gamma^0 u_{k,1}. \quad (36)$$

This can be evaluated explicitly:

$$\mathcal{P}_{ee}^>(k, t) = \cos^2 \theta + \sin^2 \theta \left( |U_k|^2 e^{-i(\omega_{k,2} - \omega_{k,1}) t} + |V_k|^2 e^{i(\omega_{k,1} + \omega_{k,2}) t} \right). \quad (37)$$

Now $\mathcal{P}_{ee}^>(k, t)$ satisfies the right initial condition:

$$\mathcal{P}_{ee}^>(k, 0^+) = 1. \quad (38)$$

Moreover

$$|\mathcal{P}_{ee}^>(k, t)|^2 + |\mathcal{P}_{ee}^>(k, t)|^2 + |\mathcal{P}_{ee}^>(k, t)|^2 + |\mathcal{P}_{ee}^>(k, t)|^2 = 1. \quad (39)$$

Here $\mathcal{P}_{ee}^>$, $\rho, \sigma = e, \mu$ are defined in a similar way as $\mathcal{P}_{ee}^>$. We can thus identify the different pieces in the Eq.(39) as flavor (un)-changing probabilities. The QFT oscillation formula can be explicitly derived [7]:

$$P_{\nu_{e} \rightarrow \nu_{\mu}}(k, t) = \sin^2(2\theta) \left[ |U_k|^2 \sin^2 \left( \frac{\omega_{k,1} - \omega_{k,2}}{2} t \right) + |V_k|^2 \sin^2 \left( \frac{\omega_{k,1} + \omega_{k,2}}{2} t \right) \right]. \quad (40)$$

Note that in the ultrarelativistic limit $|k| \rightarrow \infty$, $|U_k|^2 \rightarrow 1$ and $|V_k|^2 \rightarrow 0$, and we thus recover the standard quantum mechanical oscillation formula [4].

Tough the result in Eq.(40) could seem related with the use of the flavor vacuum, in Ref. [8] it was shown that it can be re-obtained by using retarded propagators both on flavor and mass vacua:

$$S^{\text{ret}}(t, x; 0, y) = \theta(t) \left. \langle 0 | \{ \nu_\rho(t, x), \gamma_\sigma(0, y) \} |0\rangle \right|_{1,2}, \quad (41)$$
$$G^{\text{ret}}(t, x; 0, y) = \theta(t) \langle 0 | \{ \nu_\rho(t, x), \gamma_\sigma(0, y) \} |0\rangle_{e,\mu}, \quad \rho, \sigma = e, \mu. \quad (42)$$

In fact these turns out to be equal and, passing to Fourier space, we get:

$$S^{\text{ret}}(k, t) = G^{\text{ret}}(k, t). \quad (43)$$

Defining

$$P_{\nu_\rho \rightarrow \nu_\sigma}(k, t) = \text{Tr} \left[ G^{\text{ret}}_{\sigma\rho}(k, t) G^{\text{ret}}_{\sigma\rho}(k, t) \right], \quad (44)$$

one can check that the probability (44) gives us the result (40). The oscillation formula (40) has thus to be interpreted as the consequence of the QFT treatment of mixing and not as directly related with the use of the flavor vacuum.
4. Schwinger-Dyson equations and single particle bound state

The partition function can be written down from the Lagrangian (1) thanks to the functional integral formalism:

$$Z[\eta, \bar{\eta}] = \int \mathcal{D}\eta \mathcal{D}\bar{\eta} e^{i\int dx (L(x) + \bar{\eta}(x)\nu(x) + \bar{\nu}(x)\eta(x))}.$$  

(45)

Now the fields $\nu(x) = (\nu_e(x), \nu_\mu(x))$ and the currents $\eta(x) = (\eta_e(x), \eta_\mu(x))$ are Grassmann variables. The partition function (45) can be explicitly written as:

$$Z[\eta, \bar{\eta}] = \mathcal{N} e^{-i\int dx dy (\bar{\eta}(x)S_F(x,y)\eta(y))},$$  

(46)

where $\mathcal{N}$ is a normalization factor, and $S_F(x,y)$ is a $2 \times 2$ matrix Green’s function, satisfying the delta-source equation:

$$(i\partial - M)S_F(x,y) = \delta^4(x-y)\mathbb{I}.$$  

(47)

Here $\mathbb{I}$ is the $2 \times 2$ identity matrix. It is well known that $S_F(x,y)$ is defined up to a solution of the homogeneous equation

$$(i\partial - M)S^0_F(x,y) = 0.$$

(48)

This is exactly the aforementioned freedom in the choice of boundary conditions. Moreover, from the observation that the functional integral of a functional derivative is zero

$$\int \mathcal{D}\eta \mathcal{D}\bar{\eta} \frac{\delta}{\delta \bar{\eta}(x)} e^{i\int dx' (L(x') + \bar{\eta}(x')\nu(x'))} = 0,$$

(49)

we find the SD equations for $Z[\eta, \bar{\eta}]$:

$$\left(\eta(x) + (i\partial - M)\frac{\delta}{\delta \bar{\eta}(x)}\right)Z[\eta, \bar{\eta}] = 0,$$

(50)

which explicitly read

$$\eta_e(x)Z[\eta, \bar{\eta}] = -(i\partial - m_e)\frac{\delta}{\delta \bar{\eta}_e(x)}Z[\eta, \bar{\eta}] + m_e \frac{\delta}{\delta \bar{\eta}_\mu(x)}Z[\eta, \bar{\eta}],$$  

(52)

$$\eta_\mu(x)Z[\eta, \bar{\eta}] = -(i\partial - m_\mu)\frac{\delta}{\delta \bar{\eta}_\mu(x)}Z[\eta, \bar{\eta}] + m_\mu \frac{\delta}{\delta \bar{\eta}_e(x)}Z[\eta, \bar{\eta}].$$  

(53)

Deriving respect to $\eta_e(y)$ and $\eta_\mu(y)$ and putting $\eta_\mu = \eta_e = \bar{\eta}_e = \bar{\eta}_\mu = 0$, we get:

$$(S^F_{ee})^{-1}(x-y) = (i\partial - m_e)\delta^4(x-y) - m_e \frac{\delta}{\delta \bar{\eta}_e(x)} \left( S^F_{ee} \right)^{-1}(0),$$  

(54)

$$(S^F_{\mu\mu})^{-1}(x-y) = (i\partial - m_\mu)\delta^4(x-y) - m_\mu \frac{\delta}{\delta \bar{\eta}_\mu(x)} \left( S^F_{\mu\mu} \right)^{-1}(0),$$  

(55)

$$m_\mu S^F_{\mu\mu}(x-y) = (i\partial - m_\mu)S^F_{\mu\mu}(x-y),$$  

(56)

$$m_e S^F_{ee}(x-y) = (i\partial - m_e)S^F_{ee}(x-y).$$  

(57)

It is straightforward to check that these equations are equivalent to the matrix equation (47). Therefore we can derive:

$$(i\partial - m_\mu)\left( S^F_{ee} \right)^{-1}(x-y) = (i\partial - m_\mu)(i\partial - m_e)\delta^4(x-y) - m^2_\mu \delta^4(x-y),$$  

(58)

$$(i\partial - m_e)\left( S^F_{\mu\mu} \right)^{-1}(x-y) = (i\partial - m_e)(i\partial - m_\mu)\delta^4(x-y) - m^2_e \delta^4(x-y).$$  

(59)
Passing to Fourier space we need to specify which propagator we are dealing with, i.e. the boundary conditions. These are determined by choosing a contour on the complex $p_0$-plane. Because of Eq.(43), we will deal with retarded propagators. Let us remember that, for free Dirac fields, the retarded propagator reads as:

$$S^\text{ret}_{\alpha\beta}(x-y) = (i\partial^\mu + m)_{\alpha\beta} \Delta^\text{ret}(x-y),$$

where $\Delta^\text{ret}(x-y)$ is the Klein-Gordon retarded propagator:

$$\Delta^\text{ret}(x-y) = \frac{1}{(2\pi)^4} \int d^4k \frac{e^{-i k \cdot (x-y)}}{k^2 - m^2},$$

and the contour is shown in figure 1. With this prescription in mind, we replace $G^\text{ret}(x,y)$ to $S^F(x,y)$ and we write the SD equations in momentum space:

$$(\not{k} - m_\mu)(G^\text{ret}_{e\mu})^{-1}(k) = (\not{k} - m_\mu)(\not{k} - m_e) - m^2_{e\mu},$$

$$(\not{k} - m_e)(G^\text{ret}_{\mu\mu})^{-1}(k) = (\not{k} - m_e)(\not{k} - m_\mu) - m^2_{e\mu},$$

introducing $G^\text{ret}_e(p) = (\not{k} - m_e)^{-1}$ and $G^\text{ret}_\mu = (\not{k} - m_\mu)^{-1}$, we can rewrite Eqs.(62),(63) as:

$$G^\text{ret}_{e\mu}(k) = G^\text{ret}_e(k) \left(1 - m^2_{e\mu} G^\text{ret}_e(k) G^\text{ret}_\mu(k)\right)^{-1},$$

$$G^\text{ret}_{\mu\mu}(k) = G^\text{ret}_\mu(k) \left(1 - m^2_{e\mu} G^\text{ret}_e(k) G^\text{ret}_\mu(k)\right)^{-1}.$$

These can be rewritten as sum of the series ($\hbar$-expansion):

$$G^\text{ret}_{e\mu}(k) = G^\text{ret}_e(k) + G^\text{ret}_e(k) \Sigma_e(k) G^\text{ret}_e(k) + \ldots,$$

$$G^\text{ret}_{\mu\mu}(k) = G^\text{ret}_\mu(k) + G^\text{ret}_\mu(k) \Sigma_\mu(k) G^\text{ret}_\mu(k) + \ldots,$$

where

$$\Sigma_e(k) = m^2_{e\mu} G^\text{ret}_\mu(k),$$

$$\Sigma_\mu(k) = m^2_{e\mu} G^\text{ret}_e(k).$$
These are the self-energy operators in the momentum space. In fact:

$$\Gamma_{ee}(k) = \Gamma_e(k) - m^2 e \Sigma^e_e(k),$$  \hspace{1cm} (70)

$$\Gamma_{\mu\mu}(k) = \Gamma_\mu(k) - m^2 \Sigma^\mu_\mu(k),$$  \hspace{1cm} (71)

where the $\Gamma$’s are two-point truncated functions (inverse propagators). These relations indicate that flavor-particles propagation is characterized by an infinite series of processes of flavor-oscillation in the vacuum.

Finally we introduce the formal but intriguing concept of single-particle bound state. The idea is to derive an equation à la Bethe–Salpeter for the flavor fields. Let us calculate:

\[
(\not{k} - m_e - \Sigma_e)(\not{k} - m_\mu - \Sigma_\mu) G^\text{rel}_{ee}(k) = m^2 e \Sigma^e_e(k) G^\text{rel}_{ee}(k) \\
- (\not{k} - m_e - \Sigma_e) \Sigma^\mu_\mu G^\text{rel}_{ee}(k).
\]  \hspace{1cm} (72)

By using that $G^\text{rel}_{ee}(k) = (\not{k} - m_e - \Sigma_e)^{-1}$, we get:

\[
(\not{k} - m_e - \Sigma_e)(\not{k} - m_\mu - \Sigma_\mu) G^\text{rel}_{e\mu}(k) = m^2 e \mu - (\not{k} - m_e - \Sigma_e) \Sigma^\mu_\mu G^\text{rel}_{e\mu}(k).
\]  \hspace{1cm} (73)

If we define the potential:

$$V_{e\mu}(k) = (\not{k} - m_e - \Sigma_e(k)) \Sigma_\mu(k),$$  \hspace{1cm} (74)

we derive a Bethe–Salpeter-like equation in the flavor-space:

\[
(\not{k} - m_e - \Sigma_e)(\not{k} - m_\mu - \Sigma_\mu) G^\text{rel}_{e\mu}(k) = m^2 e \mu - V_{e\mu} G^\text{rel}_{e\mu}(k).
\]  \hspace{1cm} (75)

This suggestive relation indicates that the flavor fields can be formally viewed as bound states in the abstract flavor-space, revealing their non-asymptotic nature. Therefore flavor particles cannot be described by LSZ asymptotic states. This is evident in the time-dependence of the flavor vacuum and flavor creation and annihilation operators (see Eqs.(15),(20)-(23)).

5. Conclusions

Structural richness of the phenomenon of fermion mixing in QFT can be appreciated by analyzing the propagator of flavor fields. It turns out that self-energy operator does not lead to a simple renormalization of the bare mass, but it regulates the phenomenon of oscillations in the vacuum (see Eq. (66)). Moreover, Schwinger–Dyson equations permit to derive a relation which in its structure resembles a Bethe–Salpeter equation describing bound states in the flavor space, rather than in spacetime as usual (cf. Eq. (75)). This non-perturbative nature of mixing is evidently at the origin of the time dependence of flavor annihilation and creation operators (cf. Eqs. (20)-(23)).

Generally the problem of the boundary conditions in the flavor-vacuum Green functions is rather complicated issue but fortunately in the present paper one could resort to a formal identity for retarded Green functions of mass and flavor vacua. There the ignorance of the flavor-vacuum boundary conditions was translated on the level of boundary conditions for mass vacuum where these are well understood. This fact was explicitly represented in the identity (43). Recently, in Ref. [16], a generalized generating functional for mixed-representations Green’s functions (i.e. Green’s functions on different vacua) was derived. Such functional permits to take into account automatically the boundary conditions thanks to a consistent extension of the standard generating functional for Green’s functions. A treatment of mixing with this technique was done in Ref. [18].
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