THE 2-CATEGORY OF WEAK ENTWINING STRUCTURES

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Abstract. A weak entwining structure in a 2-category $K$ consists of a monad $t$ and a comonad $c$, together with a 2-cell relating both structures in a way which generalizes a mixed distributive law. A weak entwining structure can be characterized as a compatible pair of a monad and a comonad, in 2-categories generalizing the 2-category of comonads and the 2-category of monads in $K$, respectively. This observation is used to define a 2-category $\text{Entw}^w(K)$ of weak entwining structures in $K$. If the 2-category $K$ admits Eilenberg-Moore constructions for both monads and comonads and idempotent 2-cells in $K$ split, then there are pseudo-functors from $\text{Entw}^w(K)$ to the 2-category of monads and to the 2-category of comonads in $K$, respectively. The Eilenberg-Moore objects of the lifted monad and the lifted comonad are shown to be equivalent. If $K$ is the 2-category of functors induced by bimodules, then these Eilenberg-Moore objects are isomorphic to the usual category of weak entwined modules.

Introduction

Mixed distributive laws \cite{1} in a 2-category $K$ (or ‘entwining structures’, as they are called more often in the Hopf algebraic terminology), can be described in some equivalent ways \cite{8}. They are monads in the 2-category $\text{Cmd}(K)$ of comonads in $K$, equivalently, they are comonads in the 2-category $\text{Mnd}(K)$ of monads in $K$. Consequently, they can be regarded as 0-cells of a 2-category $\text{Entw}(K)$, defined to be isomorphic to $\text{Mnd}(\text{Cmd}(K)) \cong \text{Cmd}(\text{Mnd}(K))$.

If a 2-category $K$ admits Eilenberg-Moore constructions for monads, that is, the inclusion 2-functor $I : K \to \text{Mnd}(K)$ possesses a right 2-adjoint $J$, then the 2-functor $\text{Cmd}(J)$ takes a mixed distributive law of a monad $t$ and a comonad $c$ in $K$ to a comonad $J(t) \Rightarrow J(t)$, which is a lifting of $c$, cf. \cite{7}. Symmetrically, if $K$ admits Eilenberg-Moore constructions for comonads, that is, the inclusion 2-functor $I_* : K \to \text{Cmd}(K)$ possesses a right 2-adjoint $J_*$, then $\text{Mnd}(J_*)$ takes $(t, c)$ to a monad $J_*(c) \Rightarrow J_*(c)$, which is a lifting of $t$. If Eilenberg-Moore constructions in $K$ exist both for monads and comonads, then the 2-functors $J_*, \text{Cmd}(J)$ and $J_*, \text{Mnd}(J_*)$ are 2-naturally isomorphic. In particular, the lifted monad $\overrightarrow{t}$ and the lifted comonad $\overrightarrow{c}$ possess isomorphic Eilenberg-Moore objects, see \cite{7}. In the case when $K$ is the 2-category $\text{CAT} = \text{[Categories; Functors; Natural Transformations]}$, this is the category of $(t, c)$-bimodules, also called ‘entwined modules’.

In order to treat algebra extensions by weak bialgebras in \cite{3}, entwining structures were generalized to ‘weak entwining structures’ in \cite{5}. A weak entwining structure in a 2-category $K$ also consists of a monad $t$ and a comonad $c$, together with a 2-cell $tc \Rightarrow ct$, but the compatibility axioms with the unit of the monad and the counit of the comonad are weakened. We are not aware of any characterization of a weak
entwining structure as a monad or as a comonad in some 2-category. Instead, in this
note we observe that a weak entwining structure in an arbitrary 2-category $\mathcal{K}$ can be
described as a compatible pair of a comonad in a 2-category $\text{Mnd}'(\mathcal{K})$, which extends
$\text{Mnd}(\mathcal{K})$, and a monad in $\text{Cmd}^p(\mathcal{K}) \coloneqq \text{Mnd}'(\mathcal{K}_*)$ (where $(-)_*$ means the vertically
opposite 2-category). This observation is used to define in Section [1] a 2-category
$\text{Entw}^w(\mathcal{K})$, whose 0-cells are weak entwining structures in $\mathcal{K}$ and whose 1-cells and
2-cells are also compatible pairs of 1-cells and 2-cells, respectively, in $\text{Mnd}(\text{Cmd}^p(\mathcal{K}))$
and $\text{Cmd}(\text{Mnd}'(\mathcal{K}))$. By construction, the 2-category $\text{Entw}^w(\mathcal{K})$ comes equipped with
2-functors $A : \text{Entw}^w(\mathcal{K}) \to \text{Cmd}(\text{Mnd}'(\mathcal{K}))$ and $B : \text{Entw}^w(\mathcal{K}) \to \text{Mnd}(\text{Cmd}^p(\mathcal{K}))$.

If a 2-category $\mathcal{K}$ admits Eilenberg-Moore constructions for monads and idempo-
tent 2-cells in $\mathcal{K}$ split, then the 2-functor $J$ above factorizes through the inclusion
$\text{Mnd}(\mathcal{K}) \hookrightarrow \text{Mnd}'(\mathcal{K})$ and an appropriate pseudo-functor $Q : \text{Mnd}'(\mathcal{K}) \to \mathcal{K}$. The
image of a weak entwining structure $(t, c)$ under the pseudo-functor $\text{Cmd}(Q)A$ is a
‘weak lifting’ of $c$ for $t$, cf. [2]. Symmetrically, if $\mathcal{K}$ admits Eilenberg-Moore construc-
tions for comonads and idempotent 2-cells in $\mathcal{K}$ split, then there is a pseudo-functor
$Q_+ : \text{Cmd}^p(\mathcal{K}) \to \mathcal{K}$, such that $\text{Mnd}(Q_+)B$ takes a weak entwining structure $(t, c)$ to
a weak lifting of $t$ for $c$. If Eilenberg-Moore constructions in $\mathcal{K}$ exist both for mon-
ads and comonads and also idempotent 2-cells in $\mathcal{K}$ split, then we prove in Section [2]
that the pseudo-functors $J, \text{Cmd}(Q)A$ and $J\text{Mnd}(Q_+)B : \text{Entw}^w(\mathcal{K}) \to \mathcal{K}$ are pseudo-
naturally equivalent. In particular, for any weak entwining structure $(t, c)$, the weak
lifting of $t$ for $c$, and the weak lifting of $c$ for $t$, possess equivalent Eilenberg-Moore
objects.

As a motivating example, we can consider the 2-category $\mathcal{K}$ obtained as the image
of the bicategory $\text{BIM}_k = \text{[Algebras; Bimodules; Bimodule Maps]}$ (over a commu-
tative ring $k$) under the hom 2-functor $\text{BIM}_k(k, -) : \text{BIM}_k \to \text{CAT}$. A weak entwining
structure $((-) \otimes_R T, (-) \otimes_R C)$ in this 2-category is given by a $k$-algebra $R$, an $R$-ring
$T$, an $R$-coring $C$ and an $R$-bimodule map $C \otimes_R T \to T \otimes_R C$. In this case, we
obtain that the Eilenberg-Moore category of the weakly lifted comonad $(-) \otimes_R C$ (on
the category $M_T$ of $T$-modules) is isomorphic to the Eilenberg-Moore category of the
weakly lifted monad $(-) \otimes_R T$ (on the category $M^C$ of $C$-comodules), and it is iso-
morphic also to $\text{Entw}^w(\mathcal{K})(M_k, M_k), (-) \otimes_R T, (-) \otimes_R C)$, known as the category of
‘weak entwined modules’. In particular, if $R$ is a trivial $k$-algebra (i.e. $R = k$), we
re-obtain [4] Proposition 2.3.

**Notations.** We assume that the reader is familiar with the theory of 2-categories.
For a review of the occurring notions (such as a 2-category, a 2-functor and a 2-
adjunction, monads, adjunctions and Eilenberg-Moore construction in a 2-category)
we refer to the article [4].

In a 2-category $\mathcal{K}$, horizontal composition is denoted by juxtaposition and vertical
composition is denoted by $*$, 1-cells are represented by an arrow $\to$ and 2-cells are
represented by $\Rightarrow$.

For any 2-category $\mathcal{K}$, $\text{Mnd}(\mathcal{K})$ denotes the 2-category of monads in $\mathcal{K}$ as in [5] and
$\text{Cmd}(\mathcal{K}) \coloneqq \text{Mnd}(\mathcal{K}_*)$ denotes the 2-category of comonads in $\mathcal{K}$, where $(-)_*$ refers to
the vertical opposite of a 2-category. Throughout, we denote by $I : \mathcal{K} \to \text{Mnd}(\mathcal{K})$ the
inclusion 2-functor (with underlying maps $k \mapsto (k, k, k), V \mapsto (V, V), \omega \mapsto \omega$ on the
0-, 1-, and 2-cells, respectively). Its right 2-adjoint, if it exists, is denoted by $J$. The
inclusion 2-functor \( K \to \text{Cmd}(K) \) is denoted by \( I_\ast \) and its right 2-adjoint, whenever it exists, is denoted by \( J_\ast \).

If a 2-category \( K \) admits Eilenberg-Moore constructions for monads (i.e. the 2-functor \( J \) exists), then any monad \((k \xrightarrow{t} k, tt \xrightarrow{f} t, k \xrightarrow{\eta} t)\) in \( K \) determines a canonical adjunction \((k \xrightarrow{J} J(t)), J(t) \xleftarrow{\Theta} k, f \circ \Theta \Rightarrow J(t), k \xrightarrow{\eta} v f)\) such that \((t, \mu, \eta) = (v f, v c f, \eta), cf. \[3\] Theorem 2\]. Throughout, these notations are used for this canonical adjunction. For a monad \((t', \mu', \eta')\), the canonical adjunction is denoted by \((f', v', e', \eta')\), etc.

We say that in a 2-category \( K \) idempotent 2-cells split if, for any 2-cell \( V \overset{\Theta}{\Rightarrow} V \) in \( K \) such that \( \Theta \ast \Theta = \Theta \), there exist a 1-cell \( \tilde{V} \) and 2-cells \( V \overset{\psi}{\Rightarrow} \tilde{V} \) and \( \tilde{V} \overset{\Theta}{\Rightarrow} V \), such that \( \pi \ast e = \tilde{V} \) and \( e \ast \pi = \Theta \).

1. The 2-CATEGORY OF WEAK ENTWINING STRUCTURES

Consider a monad \((k \xrightarrow{t} k, tt \xrightarrow{f} t, k \xrightarrow{\eta} t)\) and a comonad \((k \xrightarrow{c} k, c \xrightarrow{\delta} cc, c \xrightarrow{\epsilon} k)\) in a 2-category \( K \) and a 2-cell \( tc \xrightarrow{\psi} ct \). The triple \((t, c, \psi)\) is termed a weak entwining structure provided that the following axioms in \[5\] hold.

\[
\begin{align*}
(1.1) & \quad \psi \ast \mu c = c \mu \ast t \psi \ast t \psi; \\
(1.2) & \quad \delta t \ast \psi = c \psi \ast \psi c \ast t \delta; \\
(1.3) & \quad \psi \ast \eta c = c \epsilon t \ast c \psi \ast c \eta c \ast \delta; \\
(1.4) & \quad \epsilon t \ast \psi = \mu \ast t \epsilon t \ast t \psi \ast t \eta c.
\end{align*}
\]

The most important difference between such a weak entwining structure and a usual entwining structure (i.e. mixed distributive law) is that in the weak case \((c, \psi)\) is no longer a 1-cell \( t \to t \) in \( \text{Mnd}(K) \) and \((t, \psi)\) is not a 1-cell \( c \to c \) in \( \text{Cmd}(K) \). Still, as it was observed in [2], \((t, \mu, \eta)\) is a monad and \((c, \delta, \epsilon)\) is a comonad in an extended 2-category of (co)monads in \( K \), recalled in the following theorem.

**Theorem 1.1** ([2], Corollary 1.4 and Theorem 3.5). For any 2-category \( K \), the following data constitute a 2-category, to be denoted by \( \text{Mnd}'(K) \).

- **0-cells** are monads \((k \xrightarrow{t} k, \mu, \eta)\) in \( K \).
- **1-cells** \((k \xrightarrow{t} k, \mu, \eta) \xrightarrow{(V, \psi)} (k' \xrightarrow{t'} k', \mu', \eta')\) are pairs, consisting of a 1-cell \( k \xrightarrow{V} k' \) and a 2-cell \( t' V \xrightarrow{\psi} V t \) in \( K \) such that

\[
(1.5) \quad V \mu \ast \psi t \ast t' \psi = \psi \ast \mu' t.
\]

- **2-cells** \((V, \psi) \xrightarrow{\omega t} (W, \phi)\) are 2-cells \( V \xrightarrow{\omega t} W \) in \( K \), satisfying

\[
(1.6) \quad \omega t \ast \psi = W \mu \ast \phi t \ast t' \omega t \ast t' \psi \ast t' \eta' V.
\]

Horizontal and vertical compositions are the same as in \( K \).

The 2-category \( \text{Mnd}'(K) \) contains \( \text{Mnd}(K) \) as a vertically full 2-subcategory.

Moreover, if \( K \) admits Eilenberg-Moore constructions for monads and idempotent 2-cells in \( K \) split, then the following maps determine a pseudo-functor \( Q : \text{Mnd}'(K) \to K \).

For a 0-cell \((t, \mu, \eta)\), \( Q(t, \mu, \eta) := J(t, \mu, \eta) \).
For a 1-cell \((t, \mu, \eta) \overset{(V, \psi)}{\rightarrow} (t', \mu', \eta')\), \(Q(V, \psi)\) is the unique 1-cell \(Q(t, \mu, \eta) \rightarrow Q(t', \mu', \eta')\) in \(K\) for which
\[
v'cQ(V, \psi) = \pi \ast Vv \ast \psi v \ast t'.
\]
(1.7)

For a 2-cell \((V, \psi) \overset{\gamma}{\Rightarrow} (W, \phi)\), \(Q(\omega)\) is the unique 2-cell \(Q(V, \psi) \Rightarrow Q(W, \phi)\) in \(K\) for which
\[
v'Q(\omega) = \pi \ast \omega v \ast t,
\]
where \(Vv \overset{\pi}{\Rightarrow} v'Q(V, \psi) \overset{\gamma}{\Rightarrow} Vv\) denote a chosen splitting of the idempotent 2-cell
(1.8)

\[Vv \ast \psi v \ast \eta' V : Vv \Rightarrow Vv,\]
for any 1-cell \((V, \psi)\) in \(\text{Mnd}^d(K)\).

For 1-cells \(t \overset{(V, \psi)}{\rightarrow} t' \overset{(V', \psi')}{\rightarrow} t''\) in \(\text{Mnd}^d(K)\), the coherence natural is 2-cell \(Q((V', \psi')(V, \psi)) \overset{\pi}{\Rightarrow} Q(V', \psi')Q(V, \psi)\) is the unique 2-cell \(\gamma\) for that 
\[v''\gamma = (v''Q((V', \psi')(V, \psi)) \overset{\pi}{\Rightarrow} V'Vv \overset{\pi}{\Rightarrow} V'V'Q(V, \psi) \overset{\pi}{\Rightarrow} v''Q(V', \psi')Q(V, \psi))\] (so
\[v''\gamma^{-1} = (v''Q(V', \psi')Q(V, \psi)) \overset{\pi}{\Rightarrow} V'v'Q(V, \psi) \overset{\pi}{\Rightarrow} V'Vv \overset{\pi}{\Rightarrow} v''Q((V', \psi')(V, \psi))).\]

With the convention of choosing a trivial splitting \(Vv \overset{\pi}{\Rightarrow} Vv \overset{\pi}{\Rightarrow} Vv\) whenever (1.9) is an identity 2-cell, the image of any identity 1-cell \(t \overset{(k, t)}{\rightarrow} t\) under \(Q\) becomes equal to the identity 1-cell \(Q(t)\). This convention also ensures that the composite pseudo-functor 
\[\text{Mnd}(K) \rightarrow \text{Mnd}^d(K) \overset{Q}{\rightarrow} K\] is equal to \(J\). The pseudo-natural isomorphism class of \(Q\) does not depend on the choice of the 2-cells \(\pi\) and \(v\).

For any 2-category \(K\), we put \(\text{Cmd}^\pi(K) := \text{Mnd}^d(K)_*\). Applying Theorem 1.1 to the 2-category \(K_*\), we conclude that whenever \(K\) admits Eilenberg-Moore constructions for comonads and idempotent 2-cells in \(K\) split, \(J_*\) extends to a pseudo-functor 
\[Q_* : \text{Cmd}^\pi(K) \rightarrow K.\]

After all these preparations, we are ready to construct a 2-category of weak entwining structures in any 2-category \(K\).

**Theorem 1.2.** For any 2-category \(K\), the following data constitute a 2-category, to be denoted by \(\text{Entw}^{\ast}(K)\).

- **0-cells** are triples \(((k \overset{t}{\rightarrow} k, \mu, \eta), (k \overset{c}{\rightarrow} k, \delta, \varepsilon), \psi)\), consisting of a monad \((k \overset{t}{\rightarrow} k, \mu, \eta)\), a comonad \((k \overset{c}{\rightarrow} k, \delta, \varepsilon)\) and a 2-cell \(tc \overset{\psi}{\Rightarrow} ct\) in \(K\), such that
  - \(t \overset{(c, \psi)}{\rightarrow} t, \delta, \varepsilon\) is a comonad in \(\text{Mnd}^d(K)\) and
  - \(c \overset{(t, \psi)}{\rightarrow} c, \mu, \eta\) is a monad in \(\text{Cmd}^\pi(K)\).

- **1-cells** \(((k \overset{t}{\rightarrow} k, \mu, \eta), (k \overset{c}{\rightarrow} k, \delta, \varepsilon), \psi)^{(W, \alpha, \beta)} \rightarrow ((k' \overset{t'}{\rightarrow} k', \mu', \eta'), (k' \overset{c'}{\rightarrow} k', \delta', \varepsilon'), \psi')\) are triples, consisting of a 1-cell \(k \overset{W}{\Rightarrow} k'\) and 2-cells \(tW \overset{\alpha}{\Rightarrow} Wt\) and \(Wc \overset{\beta}{\Rightarrow} c'\) in \(K\), such that
  - \(t \overset{(c, \psi)}{\rightarrow} t, \delta, \varepsilon\)^{(W, \alpha, \beta)} \(t' \overset{(c', \psi')}{\rightarrow} t', \delta', \varepsilon'\) is a 1-cell in \(\text{Cmd}(\text{Mnd}^d(K))\) and
  - \(c \overset{(t, \psi)}{\rightarrow} c, \mu, \eta\)^{(W, \alpha, \beta)} \(c' \overset{(t', \psi')}{\rightarrow} c', \mu', \eta'\) is a 1-cell in \(\text{Mnd}(\text{Cmd}^\pi(K))\).

- **2-cells** \((W, \alpha, \beta) \overset{\alpha}{\Rightarrow} (W', \alpha', \beta')\) are 2-cells \(W \overset{\alpha}{\Rightarrow} W'\) in \(K\), such that
  - \((W, \alpha, \beta) \overset{\alpha}{\Rightarrow} ((W', \alpha'), \beta')\) is a 2-cell in \(\text{Cmd}(\text{Mnd}^d(K))\) and
  - \((W, \beta, \alpha) \overset{\beta}{\Rightarrow} ((W', \beta'), \alpha')\) is a 2-cell in \(\text{Mnd}(\text{Cmd}^\pi(K))\).
Horizontal and vertical compositions are the same as in $K$.

Proof. In order to see that 0-cells in $\text{Entw}^u(K)$ are precisely the weak entwining structures, note that (1.1) expresses the requirement that $t \xrightarrow{(c, \psi)} t$ is a 1-cell in $\text{Mnd}^0(K)$ and (1.2) means that $c \xrightarrow{(t, \psi)} c$ is a 1-cell in $\text{Cmd}^e(K)$. Axiom (1.3) means that $(k, c) \xrightarrow{\beta} (t, \psi)$ is a 2-cell in $\text{Cmd}^e(K)$ and (1.4) holds if and only if $(c, \psi) \xrightarrow{\delta} (k, t)$ is a 2-cell in $\text{Mnd}^0(K)$. If these four conditions hold, then also $(t, \psi)(t, \psi) \xrightarrow{\omega} (t, \psi)$ is a 2-cell in $\text{Cmd}^e(K)$. That is,

$$
c\in\epsilon t \ast c\psi \ast c\mu \ast \psi t c \ast t\psi c \ast t\delta = c\epsilon t \ast c\psi \ast c\psi c \ast t\delta \ast \mu c
$$

Similarly, (1.1-1.4) imply that $(1.17)$ in $\text{Entw}(K)$.

By Theorem (1.1), a triple $(k \xrightarrow{W} k', t'W \xrightarrow{\beta} Wt, Wc \xrightarrow{\beta} c'W)$ is a 1-cell $((k \xrightarrow{\delta} k, \mu, \eta), (k' \xrightarrow{\epsilon} k', \nu, \zeta), (k' \xrightarrow{\delta} k'', \nu', \zeta'), (\epsilon, \zeta'))$ in $\text{Entw}(K)$ if and only if the following equalities hold.

(1.10) \(\alpha \ast \mu' W = W \mu \ast \alpha t \ast t' \alpha;\)

(1.11) \(\alpha \ast \eta' W = W \eta;\)

(1.12) \(\delta' W \ast \beta = c' \beta \ast \beta c \ast W \delta;\)

(1.13) \(c' W \ast \beta = W \epsilon;\)

(1.14) \(c' W \mu \ast c' \alpha \ast \psi' W t \ast t' \beta t \ast t' W \psi \ast t' W \eta c = \beta t \ast W \psi \ast \alpha c;\)

(1.15) \(c' W \ast e t \ast c' W \psi \ast c' \alpha c \ast \psi' W c \ast t' \beta c \ast t' W \delta = \beta t \ast W \psi \ast \alpha c.\)

The equality (1.10) is equivalent to saying that $t \xrightarrow{(W, \alpha)} t'$ is a 1-cell in $\text{Mnd}^0(K)$ and (1.12) is equivalent to $c \xrightarrow{(W, \beta)} c'$ being a 1-cell in $\text{Cmd}^e(K)$. The equality (1.15) means (after being simplified using (1.13)) that $(t', \psi')(W, \beta) \xrightarrow{\alpha} (W, \beta)(t, \psi)$ is a 2-cell in $\text{Cmd}^e(K)$ and (1.14) means (after being simplified using (1.11)) that $(W, \alpha)(c, \psi) \xrightarrow{\beta} (c', \psi')(W, \alpha)$ is a 2-cell in $\text{Mnd}^0(K)$. Conditions (1.10) and (1.11) mean that $(c \xrightarrow{(t, \psi)} c, \mu, \eta)$ $\xrightarrow{(W, \beta), \alpha}$ $(t', \psi') \xrightarrow{(c', \mu', \eta')}$ is a 2-cell in $\text{Mnd}(\text{Cmd}^e(K))$, while (1.12) and (1.13) express that $(t \xrightarrow{(c, \psi)} t, \delta, \epsilon)$ $\xrightarrow{(W, \beta), \alpha}$ $(t' \xrightarrow{(c', \psi')}, t', \delta', \epsilon')$ is a 2-cell in $\text{Mnd}(\text{Mnd}^0(K))$.

A 2-cell $W \xrightarrow{\epsilon} W'$ in $K$ is a 2-cell $(W, \alpha, \beta) \xrightarrow{\omega} (W', \alpha', \beta')$ in $\text{Entw}(K)$ if and only if

(1.16) \(\alpha' \ast t' \omega = \omega t \ast \alpha;\)

(1.17) \(\beta' \ast \omega c = c' \omega \ast \beta.\)

For any weak entwining structure $((k \xrightarrow{\delta} k, \mu, \eta), (k' \xrightarrow{\epsilon} k, \delta, \epsilon), (\epsilon, \delta))$ in $K$, the triple $(W = k, \alpha = t, \beta = c)$ satisfies the equalities (1.10-1.15). Hence it is an (identity) 1-cell in $\text{Entw}^u(K)$. The sets of 1-cells and 2-cells in $\text{Cmd}(\text{Cmd}^e(K))$ and $\text{Mnd}(\text{Cmd}^e(K))$ are closed under the horizontal composition in $K$ by Theorem (1.1). Therefore the
horizontal composite of 1-cells and 2-cells in \( \text{Entw}^w(\mathcal{K}) \) is a 1-cell and a 2-cell in \( \text{Entw}^w(\mathcal{K}) \), respectively.

For any 1-cell \((W, \alpha, \beta)\) in \( \text{Entw}^w(\mathcal{K}) \), the identity 2-cell \( W \xrightarrow{\text{id}} W \) in \( \mathcal{K} \) satisfies (1.16) and (1.17). Hence it is an (identity) 2-cell in \( \text{Entw}^w(\mathcal{K}) \). Since the sets of 2-cells in \( \text{Cmd}(\text{Mnd}^w(\mathcal{K})) \) and \( \text{Mnd}(\text{Cmd}^w(\mathcal{K})) \) are closed under the vertical composition in \( \mathcal{K} \) by Theorem 1.1, the vertical composite of 2-cells in \( \text{Entw}^w(\mathcal{K}) \) is a 2-cell in \( \text{Entw}^w(\mathcal{K}) \) again.

Associativity and unitality of the horizontal and vertical compositions in \( \text{Entw}^w(\mathcal{K}) \) and the interchange law follow by the respective properties of \( \mathcal{K} \).

From Theorem 1.2 we immediately deduce the existence of some 2-functors.

**Corollary 1.3.** For any 2-category \( \mathcal{K} \), the following assertions hold.

1. There is a 2-functor \( Y : \mathcal{K} \to \text{Entw}^w(\mathcal{K}) \), determined by the maps \( k \mapsto (I(k), I'(k), k) \), \( V \to (V, V, V) \) and \( \omega \mapsto \omega \) on the 0-, 1-, and 2-cells, respectively.

2. There is a 2-category isomorphism \( \Phi : \text{Entw}^w(\mathcal{K}) \cong \text{Entw}^w(\mathcal{K}_\ast)_\ast \), determined by the maps \( (t, c, \psi) \mapsto (c, t, \psi) \), \( (W, \alpha, \beta) \mapsto (W, \beta, \alpha) \) and \( \omega \mapsto \omega \) on the 0-, 1-, and 2-cells, respectively. In particular, for any weak entwining structures \((t, c, \psi) \) and \((t', c', \psi') \) in \( \mathcal{K} \), there is a category isomorphism \( \text{Entw}^w(\mathcal{K})((t, c, \psi), (t', c', \psi')) \cong \text{Entw}^w(\mathcal{K}_\ast)((c, t, \psi), (c', t', \psi')) \), which is 2-natural both in \((t, c, \psi) \) and \((t', c', \psi') \).

3. There is a 2-functor \( A : \text{Entw}^w(\mathcal{K}) \to \text{Cmd}(\text{Mnd}^w(\mathcal{K})) \), determined by the maps \( ((t, \mu, \eta), (c, \delta, \varepsilon), \psi) \mapsto (t \xrightarrow{(c, \psi)} t, \delta, \varepsilon) \), \( (W, \alpha, \beta) \mapsto ((W, \alpha), \beta) \) and \( \omega \mapsto \omega \) on the 0-, 1-, and 2-cells, respectively.

4. There is a 2-functor \( B : \text{Entw}^w(\mathcal{K}) \to \text{Mnd}(\text{Cmd}^w(\mathcal{K})) \), determined by the maps \( ((t, \mu, \eta), (c, \delta, \varepsilon), \psi) \mapsto (c \xrightarrow{(t, \psi)} c, \mu, \eta) \), \( (W, \alpha, \beta) \mapsto ((W, \beta), \alpha) \) and \( \omega \mapsto \omega \) on the 0-, 1-, and 2-cells, respectively.

In contrast to the case of usual entwining structures, there seems to be no reason to expect that the 2-functors \( A \) and \( B \) in Corollary 1.3 are isomorphisms.

2. Equivalence of Eilenberg-Moore Objects

If a 2-category \( \mathcal{K} \) admits Eilenberg-Moore constructions for both monads and comonads and idempotent 2-cells in \( \mathcal{K} \) split, then by Theorem 1.1 and Corollary 1.3, there are two pseudo-functors \( J: \text{Cmd}(\mathcal{Q}) \to \) and \( J\text{Mnd}(\mathcal{Q}) \to \) \( \text{Entw}^w(\mathcal{K}) \to \mathcal{K} \). The aim of this section is to prove that both are right biaadjoints of \( Y \) in Corollary 1.3, hence they are pseudo-naturally equivalent. Consequently, for any weak entwining structure \((t, c, \psi) \) in \( \mathcal{K} \), the monad \( \mathcal{Q}_\ast(c \xrightarrow{(t, \psi)} c) \) and the comonad \( \mathcal{Q}(t \xrightarrow{(c, \psi)} t) \) in \( \mathcal{K} \) possess equivalent Eilenberg-Moore objects.

Recall that any pseudo-functor \( Q : \mathcal{A} \to \mathcal{B} \) between 2-categories, induces a pseudo-functor \( \text{Cmd}(Q) : \text{Cmd}(\mathcal{A}) \to \text{Cmd}(\mathcal{B}) \) with underlying maps as follows. A comonad \( (A \xrightarrow{\delta} A, \delta, \varepsilon) \) in \( \mathcal{A} \) is taken to the comonad \( Q(A) \xrightarrow{Q(\delta)} Q(A) \), with comultiplication \( Q(c) \xrightarrow{Q(\delta)} Q(cc) \xrightarrow{\varepsilon} Q(c)Q(c) \) and counit \( Q(c) \xrightarrow{Q(\varepsilon)} Q(1_A) \xrightarrow{\varepsilon} 1_{Q(A)} \). A 1-cell \( (A \xrightarrow{\delta} A, \delta, \varepsilon) \xrightarrow{(\psi, \psi')} (A' \xrightarrow{\delta'} A', \delta', \varepsilon') \) in \( \text{Cmd}(\mathcal{A}) \) is taken to a pair consisting of the 1-cell
Proposition 2.1. Consider a 2-category $\mathcal{K}$ which admits Eilenberg-Moore constructions for monads and in which idempotent 2-cells split. Let $\omega = (\kappa, \mu, \eta)$, $(k \vdash k, \delta, \varepsilon, \psi)$ be weak entwining structure in $\mathcal{K}$. The following categories are isomorphic.

(i) The Eilenberg-Moore category $\text{Cmd}(\mathcal{K})(I_*(l), \text{Cmd}(Q)(t \xrightarrow{(c, \psi)} t, \delta, \varepsilon))$ of the co-monad $\mathcal{K}(l, Q(t \xrightarrow{(c, \psi)} t)) : \mathcal{K}(l, Q(t)) \to \mathcal{K}(l, Q(t))$;

(ii) the category $\text{Entw}^w(\mathcal{K})(Y(l), (t, c, \psi))$.

Moreover, these isomorphisms provide the 1-cell parts of a pseudo-natural isomorphism $\text{Cmd}(\mathcal{K})(I_*(\mathcal{K}), \text{Cmd}(Q)(\mathcal{K})A(\mathcal{K})) \cong \text{Entw}^w(\mathcal{K})(\mathcal{K}(\mathcal{K}), (-, -))$.

Proof. By (1.10, 1.13), the objects in the category $\text{Entw}^w(\mathcal{K})(Y(l), (t, c, \psi))$ are triples $(l \xrightarrow{\kappa} k, tW \xrightarrow{\delta} W, W \xrightarrow{\varepsilon} cW)$, such that $I(l) \xrightarrow{(W, \kappa)} t$ is a 1-cell in $\text{Mnd}(\mathcal{K})$, $I_*(l) \xrightarrow{(W, \kappa)} c$ is a 1-cell in $\text{Cmd}(\mathcal{K})$ and

$$(1.1)\quad c\varrho \ast \psi W * t\kappa = \kappa \ast \varrho.$$\\

Morphisms $(W, \varrho, \kappa) \to (W', \varrho', \kappa')$ in $\text{Entw}^w(\mathcal{K})(Y(l), (t, c, \psi))$ are 2-cells $W \xrightarrow{\omega} W'$ in $\mathcal{K}$, such that $(W, \varrho) \xrightarrow{\omega} (W', \varrho')$ is a 2-cell in $\text{Mnd}(\mathcal{K})$ and $(W, \kappa) \xrightarrow{\omega} (W', \kappa')$ is a 2-cell in $\text{Cmd}(\mathcal{K})$. We prove that the stated isomorphism is given by

$$\text{Entw}^w(\mathcal{K})(Y(l), (t, c, \psi)) \to \text{Cmd}(\mathcal{K})(I_*(l), \text{Cmd}(Q)((c, \psi, \delta, \varepsilon)),\)$$

$$(W, \varrho, \kappa) \mapsto (W', \varrho', \kappa') \mapsto \text{Cmd}(Q)((W, \varrho, \kappa) \xrightarrow{Q(\omega)} \text{Cmd}(Q)((W', \varrho', \kappa')).$$

If applying the convention of choosing trivial splittings of identity 2-cells, as described in Theorem 1.1, then when restricted to the 2-subcategory $\text{Mnd}(\mathcal{K})$ of $\text{Mnd}'(\mathcal{K})$, $Q$ is equal to $J$. Hence by [8, Theorem 2], there is a category isomorphism

$$(2.2)\quad \mathcal{K}(l, Q(t)) \to \text{Mnd}(\mathcal{K})(I(l), t), \quad V \xrightarrow{\omega} V' \mapsto (vV, v\xi V) \xrightarrow{v\omega} (vV', v\xi V');$$

$$\text{Mnd}(\mathcal{K})(I(l), t) \to \mathcal{K}(l, Q(t)), \quad (W, \varrho) \xrightarrow{\omega} (W', \varrho') \mapsto Q(W, \varrho) \xrightarrow{Q(\omega)} Q(W', \varrho').$$

We claim that there is a bijection also between 2-cells $(W, \varrho) \xrightarrow{\xi} (c, \psi)(W, \varrho)$ in $\text{Mnd}'(\mathcal{K})$, and 2-cells $Q(W, \varrho) \xrightarrow{\xi} Q(c, \psi)Q(W, \varrho)$ in $\mathcal{K}$, for any 1-cell $I(l) \xrightarrow{(W, \varrho)} t$ in $\text{Mnd}(\mathcal{K})$. Indeed, for a 2-cell $\kappa$ as described, $\xi := (Q(W, \varrho) \xrightarrow{Q(\xi)} Q((c, \psi)(W, \varrho)) \xrightarrow{\xi} Q(c, \psi)Q(W, \varrho))$ is a 2-cell in $\mathcal{K}$ as needed. Conversely, for a 2-cell $\xi$ as above, use the chosen splitting $cv \xrightarrow{c\xi} vQ(c, \psi) \xrightarrow{\xi} cv$ of the idempotent 2-cell (1.9) to construct a 2-cell $\kappa := \iota Q(W, \varrho) \ast \xi : W \Rightarrow cW$ in $\mathcal{K}$. It satisfies

$$(\kappa \ast \varrho) = \iota Q(W, \varrho) \ast \xi \ast v\xi Q(W, \varrho) = \iota Q(W, \varrho) \ast v\xi Q(c, \psi)Q(W, \varrho) \ast t\varrho \xi$$

$$(\kappa \ast \varrho) = \iota Q(W, \varrho) \ast \iota Q(W, \varrho) \ast c\varrho \ast \psi W \ast t\xi Q(W, \varrho) \ast t\varrho \xi = \iota Q(W, \varrho) \ast \psi W \ast t\kappa,$$

where the last equality follows by $\iota f \ast \iota \pi f \ast \psi = c\mu \ast \psi t \ast \eta tc \ast \psi \equiv \psi \ast \mu c \ast \eta tc = \psi$. Hence $\kappa$ is a 2-cell $(W, \varrho) \xrightarrow{(c, \psi)(W, \varrho)} \text{Mnd}'(\mathcal{K}))$, as required. In order to see that this correspondence $\kappa \leftrightarrow \xi$ is a bijection, note that by (1.8), $vQ(\iota Q(W, \varrho))$
is equal to the composite of \( \nu Q(c, \psi)Q(W, \varrho) \overset{Q(W, \varrho)}{\Rightarrow} cW \) and the chosen epi 2-cell \( cW \Rightarrow \nu Q((c, \psi)(W, \varrho)). \) That is, \( Q(\nu Q(W, \varrho)) \) is equal to the coherence iso 2-cell \( Q(c, \psi)Q(W, \varrho) \overset{\sim}{\Rightarrow} Q((c, \psi)(W, \varrho)). \) Hence starting with a 2-cell \( \xi \) and iterating both constructions, we re-obtain \( \xi. \) In the opposite order, applying both constructions to \( \kappa, \) by (1.8) we get \( \iota Q(W, \varrho) \ast \pi Q(W, \varrho) \ast \kappa. \) This is equal to \( \kappa \) by

\[
(2.3) \quad \iota Q(W, \varrho) \ast \pi Q(W, \varrho) \ast \kappa = c\varrho \ast \psi W \ast \eta cW \ast \kappa \overset{(2.1)}{=} \kappa.
\]

Next we show that \( Q(W, \varrho) \overset{Q(\kappa)}{\Rightarrow} Q((c, \psi)(W, \varrho)) \overset{\sim}{\Rightarrow} Q(c, \psi)Q(W, \varrho) \) is a coassociative coaction if and only if \( W \overset{\ast}{\Rightarrow} cW \) is coassociative, and it is counital if and only if \( \kappa \) is counital. Compose the coassociativity condition \( Q((c, \psi)\kappa) \ast Q(\kappa) = Q(\delta(W, \varrho)) \ast Q(\kappa) \) horizontally by \( \nu \) on the left and compose it vertically by the chosen mono 2-cell \( \nu Q((c, \psi)(c, \varphi)(W, \varrho)) \Rightarrow c\varepsilon W \) on the left. Applying (1.8), (2.1) and (2.3), the resulting equivalent condition can be written in the form

\[
c\varrho \ast \psi W \ast \psi cW \ast \eta cW \ast \delta W \ast \kappa \overset{(1.8)}{=} \delta W \ast c\varrho \ast \psi W \ast \eta cW \ast \kappa \overset{(2.1)}{=} \delta W \ast \kappa,
\]

hence starting with a 2-cell \( \xi \ast \kappa = c\varrho \ast \psi W \ast \psi cW \ast \eta cW \ast dW \ast \kappa. \) Since

\[
c\varrho \ast \psi W \ast \psi cW \ast \eta cW \ast \delta W \ast \kappa \overset{(2.2)}{=} \delta W \ast c\varrho \ast \psi W \ast \eta cW \ast \kappa \overset{(2.1)}{=} \delta W \ast \kappa,
\]

this proves that the coaction on \( Q(W, \varrho) \) is coassociative if and only if \( \kappa \) is so. By (2.2), (1.8) and (2.3), the counitality condition \( Q(\varepsilon(W, \varrho)) \ast Q(\kappa) = Q(W, \varrho) \) is equivalent to \( \varepsilon W \ast \kappa = W. \) Thus there is a bijection between the objects of \( \text{Cmd}(\text{Mnd}(I_\ast l), \text{Cmd}(Q)((c, \psi), \delta, \varepsilon)) \) and the objects of \( \text{Entw}^w(\mathcal{K})(Y(l), (t, c, \psi)), \) as stated.

One can see by similar steps that, for a 2-cell \( (W, \varrho) \overset{\sim}{\Rightarrow} (W', \varrho') \) in \( \text{Mnd}(\mathcal{K}), \) \( Q(\omega) \) is a morphism \( Q(W, \varrho) \rightarrow Q(W', \varrho') \) in \( \text{Cmd}(\mathcal{K})(I_\ast l), \text{Cmd}(Q)((c, \psi), \delta, \varepsilon)) \) if and only if \( \kappa' \ast \omega = c\varrho' \ast \psi W' \ast \eta cW' \ast c\omega \ast \kappa. \) Since

\[
c\varrho' \ast \psi W' \ast \eta cW' \ast c\omega \ast \kappa = c\varrho' \ast d\omega \ast \psi W \ast t\kappa \ast \eta W = c\omega \ast c\varrho \ast \psi W \ast t\kappa \ast \eta W \overset{(2.4)}{=} c\omega \ast \kappa,
\]

we conclude that \( Q(\omega) \) is a morphism of \( \mathcal{K}(l, Q(c, \psi))-\text{coalgebras} \) as needed, if and only if \( \omega = c\varrho \ast \psi W \ast \eta cW \ast c\omega \ast \kappa. \) Since

\[
(2.4) \quad \text{Entw}^w(Y(-), -) \rightarrow \text{Cmd}(\mathcal{K})(\text{Cmd}(Q)A(\text{cmd}(Q), \text{Cmd}(Q)A(-)),
\]

with 1-cell parts the functors induced by the pseudo-functor \( \text{Cmd}(Q)A \) and 2-cell parts provided by its pseudo-naturality isomorphisms. Recall that \( AY \) differs from \( \text{Cmd}(I_\ast l), I_\ast l \) by the inclusion 2-functor \( \text{Cmd}(\text{Mnd}(\mathcal{K})) \hookrightarrow \text{Cmd}(\text{Mnd}^\prime(\mathcal{K})). \) Since applying \( Q : \text{Mnd}^\prime(\mathcal{K}) \rightarrow \mathcal{K} \) after \( \mathcal{K} \overset{J_\ast}{\rightarrow} \text{Mnd}(\mathcal{K}) \hookrightarrow \text{Mnd}^\prime(\mathcal{K}) \) we obtain the identity functor \( J_\ast l = l \), it follows that \( \text{Cmd}(Q)A(\text{Y}(-), -) = I_\ast l, \) as pseudo-functors. Thus (2.4) is, in fact, a pseudo-natural transformation \( \text{Entw}^w(Y(-), -) \rightarrow \text{Cmd}(\mathcal{K})(I_\ast l, \text{Cmd}(Q)A(-)). \) Since we already proved that its 1-cells are isomorphisms, it is a pseudo-natural isomorphism, as stated.

**Theorem 2.2.** Let \( \mathcal{K} \) be a 2-category which admits Eilenberg-Moore constructions for both monads and comonads and in which idempotent 2-cells split. The following
diagram of pseudo-functors is commutative, up to a pseudo-natural equivalence.

\[
\begin{array}{ccc}
\text{Entw}^w(\mathcal{K}) & \longrightarrow & \text{Cmd}(	ext{Mnd}'(\mathcal{K})) \\
\downarrow B & & \downarrow \text{Cmd}(Q) \\
\text{Mnd}(	ext{Cmd}^n(\mathcal{K})) & \longrightarrow & \text{Mnd}(\mathcal{K}) \\
\end{array}
\]

In particular, for any weak entwining structure \((t, c, \psi)\) in \(\mathcal{K}\), the monad \(\text{Mnd}(Q_*)(c \mapsto t, \delta, \varepsilon)\) and the comonad \(\text{Cmd}(Q)(t \mapsto t, \mu, \eta)\) in \(\mathcal{K}\) possess equivalent Eilenberg-Moore objects.

**Proof.** The proof consists of showing that both \(J_*\text{Cmd}(Q)A\) and \(J\text{Mnd}(Q_*)B\) are right biadjoints of the 2-functor \(Y\) in Corollary [2.2](1). Then the claim follows by uniqueness of a biadjoint up to a pseudo-natural equivalence.

On one hand, there is a sequence of pseudo-natural isomorphisms

\[K(\cdot, J_*\text{Cmd}(Q)A(\cdot)) \cong \text{Cmd}(\mathcal{K})(I_*(-), \text{Cmd}(Q)A(-)) \cong \text{Entw}^w(\mathcal{K})(Y(-), -),\]

where the second isomorphism follows by Proposition [2.1](1).

On the other hand, applying Proposition [2.1](1) to the 2-category \(\mathcal{K}\) (in the third step) and using Corollary [1.3](2) (in the last step), we obtain a sequence of pseudo-natural isomorphisms

\[K(\cdot, J\text{Mnd}(Q_*)B(\cdot)) \cong \text{Mnd}(\mathcal{K})(I(-), \text{Mnd}(Q_*)B(-)) \cong \text{Cmd}(\mathcal{K}_*)(I(-), \text{Mnd}(Q_*)B(-)) \cong \text{Entw}^w(\mathcal{K}_*)(\Phi Y(-), \Phi(-)) \cong \text{Entw}^w(\mathcal{K})(Y(-), -).\]

\[\square\]

**Example 2.3.** Consider the 2-subcategory \(\mathcal{K}\) of \text{CAT}, whose 1-cells are functors induced by bimodules. Explicitly, 0-cells be module categories \(M_R\) for algebras \(R\) over a fixed commutative ring \(k\). The 1-cells \(\text{Mnd}(R) \to \text{Mnd}(R')\) be \(R\)-\(R'\) bimodules \(V\), i.e. functors \((-)\otimes_R V : M_R \to M_{R'}\). The 2-cells \(V \Rightarrow W\) be \(R\)-\(R'\) bimodule maps \(\omega : V \to W\), i.e. natural transformations \((-)\otimes_R V \Rightarrow (-)\otimes_R W\).

A weak entwining structure in \(\mathcal{K}\) is then a triple \((t := (-)\otimes_R T, c := (-)\otimes_R C, \psi := (-)\otimes_R \Psi)\), where \(R\) is a \(k\)-algebra, \(T\) is an \(R\)-ring (i.e. a monad \(T \Rightarrow R\) in \text{BIM}_k\), \(C\) is an \(R\)-coring (i.e. a comonad \(C \Rightarrow R\) in \text{BIM}_k\), and \(\Psi : C\otimes_R T \Rightarrow T\otimes_R C\) is an \(R\)-bimodule map such that the equalities [1.1][1.2][1.3] hold true.

In this particular 2-category \(\mathcal{K}\), the idempotent 2-cell [1.9] is given by an idempotent map. Taking its obvious splitting through its range, the associated pseudo-functor \(Q : \text{Mnd}'(\mathcal{K}) \to \mathcal{K}\) in Theorem [2.1] becomes a 2-functor. Hence the isomorphisms in Proposition [2.1] become 2-natural, so that the equivalent Eilenberg-Moore objects in Theorem [2.2] become isomorphic.

Under the minor restriction that \(R = k\), the monad \(\text{Mnd}(Q_*)B((-)\otimes_R T, (-)\otimes_R C, (-)\otimes_R \Psi)\) and the comonad \(\text{Cmd}(Q)A((-)\otimes_R T, (-)\otimes_R C, (-)\otimes_R \Psi)\) were described in [3] Section 2. It was shown in [4] Proposition 2.3 that their Eilenberg-Moore
categories are isomorphic to the category of so-called weak entwining structures. Using the constructions in the current paper, this category of weak entwining structures is nothing but \( \text{Entw}^w(\mathcal{K})(Y(k), ((-) \otimes_R T, ((-) \otimes_R C, ((-) \otimes_R \Psi)) \).

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