A note on the entire functions: theorems, properties and examples

Xiaodan Huo\textsuperscript{1, a, *},† Yufei Zhan\textsuperscript{2, b, *, †}

\textsuperscript{1}Math & Stats Department, Boston University, 02215, 808 Commonwealth Avenue, MA, United States.
\textsuperscript{2}Math Department, Peking University, 100871, No. 5 Yiheyuan Road, Haidian District, Beijing, China.

*Corresponding author e-mail: \textsuperscript{a}vivihu@bu.edu, \textsuperscript{b}1800010731@pku.edu.cn.
†These authors contributed equally.

Abstract. In this work, we summarized several fundamental theorems of the entire functions and explored how their zeros determine those functions. The background of this work is based on Elias M. Stein and Rami Shakarachi, and their lectures about functions that are holomorphic in the whole complex plane that once were taught at Princeton University. First, we introduced Jensen’s formula and gave detailed proof. This relates the values of a meromorphic function inside a disk with its boundary values on the circumference and with its zeros and poles\textsuperscript{[3, 4]}. Next, we studied the proof of Weierstrass infinite products and the definition of canonical factors. Last, Hadamard’s factorization theorem and a few main lemmas were introduced. We showed the proof of the Hadamard factorization theorem and solved its several applications through using examples. Hadamard's theory is well demonstrated in this article, and we showed the rigor and scientific validity of the theory through examples.

1. Introduction
Complex analysis is both a classical and beautiful subject. Most of the fundamental theorems of the complex analysis can be dated back into the 19th century or earlier \textsuperscript{[1, 2]}. The holomorphic functions in the whole complex plane were investigated here, and we studied several classic theorems to help prove them.

In the first section, we review Jensen’s formula. This relates the values of a meromorphic function inside a disk with its boundary values on the circumference and with its zeros and poles\textsuperscript{[3, 4]}. With this formula, we will derive the relationship between the growth of a holomorphic function and its number of zeros inside the disc. It characterizes the value of the logarithm of modulus of an entire function at the origin in terms of zeros and its integrals along the boundary of a disc\textsuperscript{[5, 6]}. The main idea is to consider its connection between zeros of an entire function and its average logarithm value to prove this formula.

In the second section, We summarize the investigation of infinite products and their several properties. We explored the order of growth of function \( f \) with its two given theorems. This could further help with explaining the upcoming content in Chapter five for Hadamard's theory\textsuperscript{[7, 8]}. In the third section, we give technical proof of Hadamard’s theory. Hadamard’s factorization theorem and a few main lemmas are introduced. Hadamard’s theory refined the result from Weierstrass, saying
that the functions of finite order could take the degree of the canonical factors to be constant, and $g$ is then a polynomial \cite{9,10}.

Complex number analysis is widely used in the fields of physics and mathematics to help analyze the motion patterns of light waves in electricity, etc. The formulas involved are used in a large number of proofs in the field of pure mathematics.

2. Main Works

2.1 Jensen's formula

In the text below, we denote by $D_R$ and $C_R$ the open disc and circle of radius $R$ centered at the origin. Also, if not mentioned, we will exclude the zero function, which is a trivial case.

**Theorem 2.1** (Jensen's formula): Let $\Omega$ be an open set that contains the closure of a disc $D_R$, and suppose that $f$ is holomorphic in $\Omega$, $f(0) \neq 0$, $f$ has finite zeros on $D_R$ and does not vanish on $C_R$. If $z_1, z_2, \ldots, z_N$ denotes the zeros of $f$ in $D_R$ (counted with multiplicities), then

$$
\log |f(0)| = \sum_{k=1}^{N} \log \left( \frac{|z_k|}{R} \right) + \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(Re^{i\theta})| \, d\theta.
$$

**Proof:**

We consider the function $g(z) = \frac{f(z)}{(z-z_1)(z-z_2)\cdots(z-z_N)}$. Since the denominator of $g(z)$ contains all of the zeros of $f(z)$, $g(z)$ has a definition on $\Omega - \{z_1, z_2, \ldots, z_N\}$, and is bounded around each zero of $f(z)$; therefore, those zeros are removable singularities, and $g(z)$ becomes a holomorphic function on $D_R$. Thus we have

$$
\log |f(0)| = \log |g(0)| + \sum_{k=1}^{N} \log \left( |z_k| \right).
$$

Now it’s sufficient to prove the theorem for holomorphic functions with no zeros inside the closure of $D_R$ and the linear function $z - w$.

1. Assume $g(z)$ is holomorphic and vanishes nowhere inside the closure of $D_R$. We have already known that we can define a function $h(z) = \log g(z)$ which is also holomorphic on $D_R$. Therefore, $|g(0)| = |e^{h(0)}| = e^{\text{Re}(h(0))}$, and since $\text{Re}(h(0))$ is a harmonic function on $D_R$, according to the mean value theorem of harmonic functions we have

$$
\log |g(0)| = \text{Re}(h(0)) = \frac{1}{2\pi} \int_{0}^{2\pi} \text{Re} \left( h(Re^{i\theta}) \right) \, d\theta = \frac{1}{2\pi} \int_{0}^{2\pi} \log |g(Re^{i\theta})| \, d\theta.
$$

2. Let $g(z) = z - w$, where $w$ is a complex number in $D_R$. Then, it suffices to prove

$$
\log R = \frac{1}{2\pi} \int_{0}^{2\pi} \log |Re^{i\theta} - w| \, d\theta = \log R + \frac{1}{2\pi} \int_{0}^{2\pi} \log \left| e^{i\theta} - \frac{w}{R} \right| \, d\theta,
$$

let $a = \frac{w}{R}$, the equation above is equivalent to

$$
\int_{0}^{2\pi} \log |1 - ae^{i\theta}| \, d\theta = 0.
$$

Consider the function $F(z) = 1 - az$. $F(z)$ is holomorphic and vanishes nowhere on the unit disc, so there exists a holomorphic function $H(z)$ defined on the unit disc such that $F(z) = e^{H(z)}$. By the mean value theorem, we have

$$
\int_{0}^{2\pi} \log |1 - ae^{i\theta}| \, d\theta = \int_{0}^{2\pi} \text{Re} \left( H(e^{i\theta}) \right) \, d\theta = 2\pi \text{Re}(H(0)) = 2\pi \log |F(0)| = 0,
$$

which finishes the proof of Jensen's formula.

Next, we consider the number of zeros of $f(z)$. Let $n(r)$ denote the number of zeros of $f(z)$ in the open disc $D_r$, where $0 < r < R$. We first prove a lemma.

**Lemma 2.2** Let $z_1, z_2, \ldots, z_N$ denote the zeros of $f$ in $D_R$, then
\[
\sum_{k=1}^{N} \log \left( \frac{R}{|z_k|} \right) = \int_0^R n(r) \frac{dr}{r}.
\]  

**Proof**: First, notice that
\[
\sum_{k=1}^{N} \log \left( \frac{R}{|z_k|} \right) = \sum_{k=1}^{N} \int_0^R \frac{\eta_k(r)}{r} \frac{dr}{r}.
\]  

Next, consider a characteristic function
\[
\eta_k(r) = \begin{cases} 
1, & \text{if } r > |z_k| \\
0, & \text{if } r \leq |z_k|.
\end{cases}
\]  

Therefore we have
\[
\sum_{k=1}^{N} \log \left( \frac{R}{|z_k|} \right) = \sum_{k=1}^{N} \int_0^R \eta_k(r) \frac{dr}{r} = \int_0^R \sum_{k=1}^{N} \eta_k(r) \frac{dr}{r}.
\]  

Also, in the disc \( D_r \), \( \sum_{k=1}^{N} \eta_k(r) = n(r) \), thus we have
\[
\sum_{k=1}^{N} \log \left( \frac{R}{|z_k|} \right) = \int_0^R n(r) \frac{dr}{r}.
\]  

From this lemma and Jensen’s formula, we conclude that
\[
\int_0^R n(r) \frac{dr}{r} = \frac{1}{2\pi} \int_0^{2\pi} \log(|f(Re^{i\theta})|) d\theta - \log|f(0)|.
\]  

Now, we will move to another topic about the order of entire functions, and we will soon find the use of this conclusion.

**Example:**
1. Give another proof of Jensen’s formula in the unit disc using the functions (called Blaschke factors)
   \[
   \psi_\alpha (z) = \frac{\alpha - z}{1 - \overline{\alpha}z}.
   \]  

**Proof:**
Suppose \( f \) is analytic on the disc and has zeros at \( z_1, z_2, \cdots, z_N \), counted with multiplicity.
\[
\psi_\alpha (z) \equiv \frac{\alpha - z}{1 - \overline{\alpha}z},
\]
\[
\psi_k (z) \equiv \psi_{z_k}(z).
\]

Hence \( g(z) = \frac{f(z)}{\psi_1(z)\psi_2(z)\cdots\psi_N(z)} \) is analytic on the unit disc and has no zeros.

Therefore,
\[
\log \left| \frac{f(0)}{\psi_1(0)\psi_2(0)\cdots\psi_N(0)} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{f(e^{i\theta})}{\psi_1(e^{i\theta})\psi_2(e^{i\theta})\cdots\psi_N(e^{i\theta})} \right| d\theta.
\]  

What’s more,
\[
\psi_k (0) = z_k.
\]  

And
\[
|\psi_k(e^{i\theta})| = \left| \frac{z_k - e^{i\theta}}{1 - \overline{z_k} e^{i\theta}} \right| = \left| \frac{z_k - e^{i\theta}}{e^{i\theta}(z_k - e^{i\theta})} \right| = 1.
\]  

Therefore,
\[
\log|f(0)| = \sum_{k=1}^{N} \log|z_k| + \frac{1}{2\pi} \int_0^{2\pi} \log(|f(Re^{i\theta})|) d\theta.
\]  

2. Prove that if \( f \) is holomorphic in the unit disc, bounded and not identically zero, and \( z_1, z_2, \cdots, z_n, \cdots \) are its zeros \((|z_k| < 1)\), then
\[
\sum_k (1 - |z_k|) < \infty. \tag{20}
\]

**Proof:**
By Jensen’s formula, we have for each \( R < 1, \)
\[
\sum_{|z_k|<R} \log \left| \frac{R}{z_k} \right| = \int_0^{2\pi} \log |f(Re^{i\theta})| \frac{d\theta}{2\pi} - \log |f(0)|. \tag{21}
\]

Because \( f \) is bounded, the right-hand side is bounded above by some constant \( M \) as \( R \) varies. Suppose now we fix \( R \) and let \( R' > R. \)
Hence,
\[
\sum_{|z_k|<R} \log \left| \frac{R'}{z_k} \right| \leq \sum_{|z_k|<R'} \log \left| \frac{R'}{z_k} \right| < M. \tag{22}
\]
Let \( R' \to 1, \) and get
\[
\sum_{|z_k|<R} \log \left| \frac{1}{z_k} \right| \leq M. \tag{23}
\]
Let \( R \to 1, \) and get
\[
\sum_k \log \left| \frac{1}{z_k} \right| \leq M. \tag{24}
\]

So
\[
\sum_k (1 - |z_k|) \leq \sum_k \log \left| \frac{1}{z_k} \right| \leq M < \infty. \tag{25}
\]

### 2.2 Growth of Order

The next proposition tells us the growth of the number of zeros of an entire function with finite order of growth.

**Definition 2.2.3** Suppose that \( f(z) \) is an entire function. We say \( f(z) \) has an order of growth \( \leq \rho \) for a positive number \( \rho, \) if there exists 2 positive numbers \( A, B, \) such that \(|f(z)| \leq Ae^{Bz^\rho} \) for all complex numbers \( z. \) Also, we define the order of growth of \( f(z) \) as \( \rho_f = \inf \rho. \)

**Theorem 2.2.4** If \( f \) is an entire function with an order of growth \( \leq \rho, \) then:
(i) There exists \( C > 0 \) such that \( n(r) \leq Cr^\rho \) for all sufficiently large \( r. \)
(ii) If \( f(0) \neq 0 \) and \( z_1, z_2, \ldots \) denotes the zeros of \( f, \) then for all \( s > \rho \) we have
\[
\int_1^\infty \frac{1}{|z_k|^s} < \infty. \tag{26}
\]

**Proof:** Recall the conclusion (12):
\[
\int_0^R n(x) \frac{dx}{x} = \frac{1}{2\pi} \int_0^{2\pi} \log(|f(Re^{i\theta})|) d\theta - \log|f(0)|. \tag{27}
\]
We choose \( R = 2r, \) then we have
\[
\int_r^{2r} n(x) \frac{dx}{x} \leq \frac{1}{2\pi} \int_0^{2\pi} \log(|f(Re^{i\theta})|) d\theta - \log|f(0)|. \tag{28}
\]
Also,
\[
\int_r^{2r} n(x) \frac{dx}{x} \geq n(r) \int_r^{2r} \frac{dx}{x} = n(r) \log 2, \tag{29}
\]
and by carefully choosing a positive number \( C', \) we have
\[
\frac{1}{2\pi} \int_0^{2\pi} \log(|f(Re^{i\theta})|) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \log Ae^{BR^\rho} d\theta \leq C'r^\rho, \tag{30}
\]
for all sufficiently large $r$. 

For the second part, since the zeros of $f$ converges nowhere on $\mathbb{C}$, it suffices to prove the condition that $|z_k| > 1$ for all $k$. Notice that

$$
\sum_{|z_k|>1} \frac{1}{|z_k|^s} = \sum_{j=0}^{\infty} \left( \sum_{|z| \leq |z_k| < 2^{j+1}} |z_k|^{-s} \right) \leq \sum_{j=0}^{\infty} 2^{-js} n(2^{j+1}) \leq c \sum_{j=0}^{\infty} 2^{-js} 2^{(j+1)s} \\
\leq c' \sum_{j=0}^{\infty} (2^{n-s})^j < \infty.
$$

(31)

Thus, the theorem is proved.

**Theorem 2.2.5** The order of growth of a polynomial is zero.

**Proof:**

Note that $|z^n| = \mathcal{O}(e^{\varepsilon n})$, $\forall \varepsilon > 0$.

Hence the order of $p(z)$ is zero.

### 2.3 Infinite Product

If $\sum |a_n| < \infty$, then the product $\prod_{n=1}^{\infty} (1 + a_n)$ converges. Moreover, the product converges to 0 if and only if one of its factors is 0.

**Proof:**

If $\sum |a_n| < \infty$, then $|a_n| \to 0$, when $n$ approaches to infinity. So we must have $|a_n| < 1/2$, for all large $n$.

The partial products,

$$
\prod_{n=1}^{N} (1 + a_n) = \prod_{n=1}^{N} e^{\log(1+a_n)} = e^{\sum_{n=1}^{N} \log(1+a_n)} = e^{\sum_{n=1}^{N} b_n} = e^{B_N}.
$$

(32)

By the power series expansion, if $|z| < 1/2$, then

$$
|\log(1+z)| = |z - \frac{z^2}{2} + \frac{z^3}{3} \cdots | \leq |z| \cdot |1 - \frac{z}{2} + \frac{z^2}{3} - \cdots | \\
\leq |z| \cdot (1 + |z| + |z|^2 + \cdots ) \\
\leq |z| \cdot \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^n \\
\leq 2|z|.
$$

(33)

Therefore, $|b_n| \leq 2|a_n|$. And as $N$ approaches to infinity, $B_N$ converges to a complex number, say $B$.

Because the exponential function is continuous, as $N \to \infty$, $e^{B_N} \to e^B$.

Also, if $1 + a_n \neq 0$ for all $n$, then the limit is non-zero because the exponential function is non-negative.

**Proposition 2.3.1**

Suppose $\{F_n\}$ is a sequence of holomorphic functions on the open set $\Omega$. If there exist constants $c_n > 0$ such that $\sum c_n < \infty$, and $|F_n(z) - 1| \leq c_n$ for all $z \in \Omega$, then

(i) The product $\prod_{n=1}^{\infty} F_n(z)$ converges uniformly in $\Omega$ to be a holomorphic function $F(z)$.

(ii) If $F(z)$ does not vanish for any $n$, then

$$
\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} F_n'(z) / F_n(z).
$$

(34)

**Proof:**

Write $F_n(z) = 1 + a_n(z)$ with $|a_n(z)| \leq c_n$.

Therefore, the product converges uniformly to a holomorphic function, denoted by $F(z)$. So the
first statement is proved.
Suppose that $K$ is a compact subset of $\Omega$, and let

$$G_N(z) = \prod_{n=1}^{N} F_n(z), \quad \text{(35)}$$

$G_N \to F$ uniformly in $\Omega$, so $G'_N \to F'$ uniformly in $K$.

$$|G'_N - F'|/G_N = \left|G'_N F - G_N F'\right|/G_N F \leq |F||G'_N - G_N| + |G_N||F' - F|/|G_N F|.$$  \quad \text{(36)}

Hence,

$$G'_N / G_N \to F' / F \quad \text{uniformly on } K.$$  \quad \text{(37)}

and because $K$ is an arbitrary compact subset of $\Omega$, the limit holds for every point of $\Omega$.

Moreover,

$$G'_N / G_N = \left(\prod_{n=1}^{N} F_n\right)' / \prod_{n=1}^{N} F_n = \sum_{n=1}^{N} F'_n / F_n \to \sum_{n=1}^{\infty} F'_n / F_n = F' / F.$$  \quad \text{(38)}

So we prove the second part of the proposition.

**Example**

We have infinite products

$$\pi \cdot \cot \pi z = \sum_{n=-\infty}^{\infty} \frac{1}{z+n} = \lim_{N \to \infty} \sum_{|n| \leq N} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2},$$ \quad \text{(39)}

and

$$\frac{\sin \pi z}{\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$ \quad \text{(40)}

**Proof:**

We prove the first formula by showing that both the left side and right side share the following three properties:

(i) $F(z + 1) = F(z)$ whenever $z$ is not an integer.

(ii) $F(z) = \frac{1}{z} + F_0(z)$, where $F_0$ is analytic near 0.

(iii) $F(z)$ has simple poles at the integers, and no other singularities.

Let $F(z) = \pi \cdot \cot \pi z$,

(i) $F(z + 1) = \pi \frac{\cos \pi (z+1)}{\sin \pi (z+1)} = \pi \frac{\cos \pi z}{\sin \pi z} = F(z)$

(ii) $\lim_{z \to 0} \frac{\pi z \cdot \cos \pi z}{\sin \pi z} = 0$, so $z=0$ is a simple pole. And $F(z) = \frac{1}{z} + F_0(z)$, where $F_0$ is analytic near 0.

(iii) $\sin \pi z = 0$ when $z = k(k \in \mathbb{Z})$. And since $\sin' \pi z \mid_{z=k} \neq 0$, $z = k$ are simple zeros. So $F(z)$ has simple poles at the integers, and no other singularities.

Then let $F(z) = \sum_{n=-\infty}^{\infty} \frac{1}{z+n} = \lim_{N \to \infty} \sum_{|n| \leq N} \frac{1}{z+n}$.

(i) \quad \sum_{|n| \leq N} \frac{1}{z+n+1} = \frac{1}{z+N+1} \sum_{|n| \leq N} \frac{1}{z+n} \quad \text{(41)}
Let \( N \) approach to infinity, \( F(z + 1) = F(z) \).

(ii) \[
F(z) = \sum_{n=-\infty}^{\infty} \frac{1}{z + n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.
\] (42)

So \( F(z) \equiv \frac{1}{z} + G_0(z) \), where \( G_0 \) is analytic near 0.

(iii) Since \( \frac{1}{F(z)} \) has simple zeros if and only if \( z = k(k \in \mathbb{Z}) \), \( F(z) \) has simple poles at the integers, and no other singularities.

Now, let
\[
\Delta(z) \equiv F(z) - \sum_{-\infty}^{\infty} \frac{1}{z + n}.
\] (43)

Therefore,

(i) \( \Delta(z + 1) = \Delta(z) \)

(ii) \( \Delta(z) = F_0(z) - \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \). \( \Delta \) is removable at the origin.

The singularities at all the integers are also removable because of the periodicity, which implies that \( \Delta \) is entire.

To prove the formula, we need to show that the function \( \Delta \) is bounded in the complex plane \( \mathbb{C} \).

We only need to do so in the strip \( |Re(z)| \leq \frac{1}{2} \) because of \( \Delta(z + k) = \Delta(z) \), where \( z \) is in the strip and \( k \in \mathbb{Z} \).

Because \( \Delta \) is holomorphic, \( \Delta \) is bounded in the rectangle which satisfies \( |Re(z)| \leq \frac{1}{2} \) and \( |Im(z)| \leq 1 \).

If \( |Im(z)| > 1 \) and \( z = x + iy \),
\[
\cot \pi z = i \cdot \frac{e^{-2\pi y} + e^{-2\pi ix}}{e^{-2\pi y} - e^{-2\pi ix}}.
\] (44)

Notice that,
\[
|\cot \pi z| = \left| i \cdot \frac{e^{-2\pi y} + e^{-2\pi ix}}{e^{-2\pi y} - e^{-2\pi ix}} \right| = \left| \frac{e^{-2\pi y} + e^{-2\pi ix}}{e^{-2\pi y} - e^{-2\pi ix}} \right| \leq \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}}.
\] (45)

So if \( |Im(z)| > 1 \), \( |\cot \pi z| \) is bounded.

On the other hand, if \( y > 1 \),
\[
\left| \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \right| = \left| \frac{1}{x + iy} + \sum_{n=1}^{\infty} \frac{2(x + iy)}{x^2 + y^2 - n^2 + 2ixy} \right|
\leq \frac{1}{\sqrt{x^2 + y^2}} + \sum_{n=1}^{\infty} \frac{|2(x + iy)|}{|x^2 + y^2 - n^2 + 2ixy|}
\leq C + C \sum_{n=1}^{\infty} \frac{y}{y^2 + n^2}.
\] (46)

Let \( x \mapsto yx \), considering
\[
\int_0^{\infty} \frac{y}{y^2 + x^2} dx = \int_0^{\infty} \frac{1}{1 + x^2} dx < \infty.
\] (47)

This integral is independent of \( y \) and shows that \( \sum_{n=1}^{\infty} \frac{y}{y^2 + n^2} \) is bounded.

To sum up the above, \( \Delta \) is bounded in \( \mathbb{C} \).

Applying Liouville’s theorem, \( \Delta \) is a constant.

Since \( \Delta \) is odd, \( \Delta \equiv 0 \).

Therefore,
\[
\pi \cdot \cot \pi z = \sum_{n=-\infty}^{\infty} \frac{1}{z + n} = \lim_{N \to \infty} \sum_{|n| \leq N} \frac{1}{z + n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.
\]

Now let
\[
G(z) = \frac{\sin \pi z}{\pi} \quad \text{and} \quad P(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).
\]

Due to the fact that \(\sum 1/n^2 < \infty\). And according to proposition 3.2, \(P(z)\) converges.

Notice that
\[
\frac{P'(z)}{P(z)} = z + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \quad \text{and} \quad \frac{G'(z)}{G(z)} = \pi \cdot \cot \pi z.
\]

Therefore,
\[
\left(\frac{P(z)}{G(z)}\right)' = \frac{P'(z)}{G(z)} \left[\frac{P(z)}{G(z)} - \frac{G'(z)}{G(z)}\right] = 0.
\]

So \(P(z) = cG(z)\).

As \(z\) approaches to 0, we have
\[
\lim_{z \to 0} \frac{G(z)}{z} = \lim_{z \to 0} \frac{\sin \pi z}{\pi z} = 1 \quad \text{and} \quad \lim_{z \to 0} \frac{P(z)}{z} = \lim_{z \to 0} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = 1.
\]

Hence \(c = 1\).

To sum up,
\[
\frac{\sin \pi z}{\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).
\]

**Example**

1. **Prove that if** \(|z| < 1\), then
\[
(1 + z)(1 + z^2)(1 + z^4)\cdots = \prod_{k=0}^{\infty} \left(1 + z^{2^k}\right) = \frac{1}{1 - z}.
\]

**Proof:**

For \(|z| \leq R < 1\), since
\[
\sum |z|^{2^k} \leq \sum R^{2^k} < \sum R^k = \frac{1}{1 - R} < \infty.
\]

The product \(\prod_{k=0}^{\infty} \left(1 + z^{2^k}\right)\) is convergent.

We will prove by induction that
\[
\prod_{k=0}^{N} \left(1 + z^{2^k}\right) = \sum_{j=0}^{2^{N+1} - 1} z^j.
\]

The base case \(N = 0\) is obvious, and the inductive step is
\[
\prod_{k=0}^{N+1} \left(1 + z^{2^k}\right) = \left(1 + z^{2^{N+1}}\right) \prod_{k=0}^{N} \left(1 + z^{2^k}\right)
\]
\[
= \left(1 + z^{2^{N+1}}\right) \sum_{j=0}^{2^{N+1} - 1} z^j
\]
\[
= \sum_{j=0}^{2^{N+1} - 1} z^j + z^{2^{N+1}} \sum_{j=0}^{2^{N+1} - 1} z^j
\]
\[
\sum_{j=0}^{2N+1-1} z^j + \sum_{j=2N+1}^{2N+2-1} z^j = \sum_{j=0}^{2N+2-1} z^j.
\]

When \( N \to \infty \),
\[
\sum_{j=0}^{2N+2-1} z^j \to \frac{1}{1-z}.
\]

Therefore,
\[
\prod_{k=0}^{\infty} \left(1 + z^{2^k}\right) = \lim_{N \to \infty} \prod_{k=0}^{N+1} \left(1 + z^{2^k}\right) = \lim_{N \to \infty} \sum_{j=0}^{2N+2-1} z^j = \frac{1}{1-z}.
\]

(2) **We define the Jacobi theta function as follows:**
\[
\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi n^2 \tau} e^{2\pi i n z},
\]
then, this infinite sum defines an entire function of order 2 for each \( \tau \) such that \( \Im(\tau) > 0 \).

**proof:** First, we will show that the sum converges. We fix \( \tau = p + it \), where \( t > 0 \). Notice that
\[
\sum_{n=1}^{\infty} e^{\frac{-\pi n^2}{2}} \leq \int_{0}^{\infty} e^{\frac{-\pi x^2}{2}} dx = \frac{1}{\sqrt{2\pi t}} < \infty.
\]

Also we have that for fixed \( z \), when \( n > \frac{4|z|}{\tau} \), \(-n^2 t + 2n|z| \leq -\frac{n^2 t}{2} \), thus
\[
|\Theta(z|\tau)| = |1 + \sum_{n=1}^{\infty} e^{\pi n^2 \tau} (e^{2\pi i n z} + e^{-2\pi i n z})| \leq 1 + \sum_{n=1}^{\infty} e^{\pi n^2 t} \cdot 2e^{2\pi n|z|}
\]
\[
= 1 + \sum_{n=1}^{\left\lfloor \frac{4|z|}{\tau} \right\rfloor} e^{\pi n^2 t} \cdot 2e^{2\pi n|z|} + \sum_{n=\left\lfloor \frac{4|z|}{\tau} \right\rfloor+1}^{\infty} e^{\pi n^2 t} \cdot 2e^{2\pi n|z|}
\]
\[
\leq 1 + 2 \sum_{n=1}^{\left\lfloor \frac{4|z|}{\tau} \right\rfloor} e^{\pi n^2 t} \cdot e^{2\pi n|z|} + 2 \sum_{n=1}^{\infty} e^{\frac{-\pi n^2 t}{2}}
\]
\[
\leq 1 + \frac{2}{\sqrt{2t}} + 2 \sum_{n=1}^{\left\lfloor \frac{4|z|}{\tau} \right\rfloor} e^{\pi n^2 t} \cdot e^{2\pi n|z|}.
\]

Therefore, the sum converges for each \( z \in \mathbb{C} \). Also, when \( 1 \leq n \leq \frac{4|z|}{\tau} \),
\[
\sum_{n=1}^{\left\lfloor \frac{4|z|}{\tau} \right\rfloor} e^{\pi n^2 t} \cdot e^{2\pi n|z|} \leq \left( \frac{4|z|}{\tau} + 1 \right) e^{-\pi t \cdot \frac{16|z|^2}{\tau}}.
\]

Thus, the order of \( \Theta(z|\tau) \) is not more than 2.

Finally, we take \( z = iy \), then whenever \( y = mt, m \in \mathbb{N} \), we have
\[ |\Theta(y|\tau)| \geq \sum_{n=1}^{4m} e^{-\pi n^2} e^{2\pi n y} \geq \frac{e^{16\pi y^2}}{\ell} = e^{\frac{16\pi |z|^2}{\ell}}, \]  

(64)

from which we deduce that the order of \( \Theta(z|\tau) \) is exactly 2.

(3). (Mittag – Leffler) Suppose that \( Q_n(z) = \sum_{k=1}^{N_n} e_k^n z^k \) are given polynomials, and \( \{a_n\}_{n=1}^{\infty} \) is a given complex sequence whose only limit is \( \infty \). Then, there exists a meromorphic function \( f \), whose only poles are \( \{a_n\}_{n=1}^{\infty} \), and \( f(z) - Q_n \left( \frac{1}{z - a_n} \right) \) is holomorphic near \( a_n \).

In other words, given poles and the order of each pole, we can construct a meromorphic function and have only had the required poles and orders of poles.

**proof:** We first rearrange \( \{a_n\} \) such that \( |a_1| \leq |a_2| \leq |a_3| \leq \ldots \). Since the only limit of \( \{a_n\}_{n=1}^{\infty} \) is \( \infty \), then for each \( n \) we can choose \( R_n > 0 \) such that \( |a_n| > R_n \) and \( |a_n| < R_n \), where \( |a_{n-k+1}| = |a_{n-k+2}| = \cdots = |a_n| \). Then, \( Q_n \left( \frac{1}{z - a_n} \right) \) is holomorphic inside \( D_R \). According to Runge’s approximation theorem, we can find a polynomial \( P_n(z) \) such that \( |Q_n \left( \frac{1}{z - a_n} \right) - P_n(z)| \leq \frac{1}{2^n} \forall z \in D_R \). Now, we define

\[
f(z) = \sum_{n=1}^{\infty} \left( Q_n \left( \frac{1}{z - a_n} \right) - P_n(z) \right).
\]  

(65)

Then, \( \forall N \in \mathbb{N}, \sum_{n=1}^{N} \left( Q_n \left( \frac{1}{z - a_n} \right) - P_n(z) \right) \) defines a meromorphic function in \( R_N \) whose poles are \( a_1, a_2, \ldots, a_N \), and satisfies that \( f(z) - Q_n \left( \frac{1}{z - a_n} \right) \) is holomorphic near each pole. Next,

\[
\left| \sum_{n=N+1}^{\infty} \left( Q_n \left( \frac{1}{z - a_n} \right) - P_n(z) \right) \right| \leq \sum_{n=N+1}^{\infty} \left| Q_n \left( \frac{1}{z - a_n} \right) - P_n(z) \right|
\]

\[
\leq \sum_{n=N+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^{N+1}}.
\]  

(66)

thus this part of sum is also holomorphic inside \( D_R \). Then, \( f(z) \) holds the required properties in each \( D_R \), and because \( R \) can be arbitrarily big, \( f(z) \) holds the required properties on \( \mathbb{C} \), and is exactly the function we are looking for.

### 2.4 Hadamard Factorization Theorem

Define canonical factors: \( E_0(z) \equiv 1 - z \) and \( E_k(z) \equiv (1 - z)e^{z^2 + \cdots + z^k} \), for \( k \geq 1 \).

**K** is an integer and called the degree of the canonical factors.

**Lemma 2.4.1** For some \( c > 0 \), if \( |z| \leq 1/2 \), then \( |1 - E_k(z)| \leq c|z|^{k+1} \).

**Proof:**

If \( |z| \leq 1/2 \), then \( 1 - z \equiv e^{\log (1-z)} \) and

\[
E_k(z) = e^{\log (1-z) + z^2 + \cdots + z^k} \equiv e^w.
\]  

(67)

According to Taylor expansion,

\[
\log(1 - z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots.
\]  

(68)

Hence,

\[
w = -\sum_{n=k+1}^{\infty} \frac{|z|^n}{n}.
\]  

(69)
Since $|z| \leq 1/2$,
\[
|w| = |z|^{k+1} \sum_{n=k+1}^{\infty} \frac{|z|^{n-k-1}}{n} = |z|^{k+1} \sum_{j=0}^{\infty} \sum_{n=k+1}^{\infty} \frac{1}{n} \leq |z|^{k+1} \sum_{j=0}^{\infty} \frac{1}{2} \leq 2|z|^{k+1}.
\]
(70)

From the equation, $|w| \leq 1$.
And according to Taylor expansion,
\[
|1 - E_k(z)| = |1 - e^w| = 1 - \left(1 + w + \frac{w^2}{2} + \cdots\right) \leq e^{|w|} \leq c|z|^{k+1}.
\]
(71)

**Theorem 2.4.2** If any sequence $\{a_n\}$ of complex numbers satisfies $|a_n| \to \infty$ as $n \to \infty$, there exists an entire function $f$ that only vanishes at all $z = a_n$. Any other such entire function is of the form $f(z)e^{g(z)}$, where $g$ is entire.

**Proof:**
Suppose that $m$ is the zero of order at the origin, and $a_1, a_2, a_3, \ldots$ are all non-zero.
Weierstrass product is defined by
\[
f(z) \equiv z^m \prod_{n=1}^{\infty} E_n(z/a_n).
\]
(72)

So $f$ is the entire function. And $f$ has a zero of order $m$ at 0, $f$ only vanishes at each $a_n$. $\forall R > 0$ ($R$ is fixed), and suppose $z \in \{|z| < R\}$.
(i) $|a_n| \leq 2R$,
Since $|a_n| \to \infty$, we have only finitely terms. The finite product vanishes at all $z = a_n(|z| = |a_n| < R)$.
(ii) $|a_n| \geq 2R$,
So $|z/a_n| \leq 1/2$
Implied by previous lemma,
\[
\left|1 - E_n \left(\frac{z}{a_n}\right)\right| \leq c \left|\frac{z}{a_n}\right|^{k+1} \leq \frac{c}{2^{n+1}}.
\]
(73)

Because
\[
\sum_{n}^{\infty} c2^{-n+1} < \infty \text{ and } \left|1 - E_n \left(\frac{z}{a_n}\right)\right| \leq \frac{c}{2^{n+1}},
\]
(74)

$\prod_{|a_n| \geq 2R} E_n(z/a_n)$ is a holomorphic function when $|z| < R$.
Since
\[
\left|\frac{z}{a_n}\right| \leq \frac{1}{2} \text{ and } E_n \left(\frac{z}{a_n}\right) \neq 0,
\]
(75)

Therefore, $f$, only vanishes at all $z = a_n$.
Then, if $f_1$ and $f_2$ are two entire functions that only vanishes at all $z = a_n$, then $f_1/f_2$ has removable singularities at all the points $a_n$.

Therefore, $f_1/f_2$ is entire and they vanish nowhere. Then $f_1(z)/f_2(z) = e^{g(z)}$. In other words, $f_1(z) = f_2(z)e^{g(z)}$, where $g$ is entire.

The proof of Weierstrass’s theorem is complete.

First, the Weierstrass’s function theorem of entire functions was proved. Next, Hadamard shows a more precise estimation for entire functions with finite orders, in which could be taken $g(z)$ as a polynomial and $E_k(z)$ up to a constant. Given by

**Theorem 2.4.3 (Hadamard’s function theorem):** Let $f$ be an entire function with finite order, order $\rho_0$ and $a_1, a_2, \ldots$ be zeros of $f$ whose limit is $\infty$. Then, we have the formula
\[
f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_k \left(\frac{z}{a_n}\right).
\]
where $m$ is the order of zero of $f$ at $z = 0$, $k$ is an integer such that $k \leq \rho_0 < k + 1$, and $g(z)$ is a polynomial with its degree $\leq k$.

Now, through introducing lemma:

**Lemma 2.4.4** Suppose $f$ is holomorphic in an open set $\Omega$ containing the closure of $D_R$. Let $u(z) = \text{Re}(f(z)), v(z) = \text{Im}(f(z))$, then

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(\xi) \frac{\xi + z}{\xi - z} d\theta + iv(0) \forall z \in D_R, \text{where} \xi = Re^{i\theta}. \quad (76)$$

**Proof:** From Cauchy’s Formula, therefore

$$f(z) = \frac{1}{2\pi i} \int_{|\xi| = R} \frac{f(\xi)}{\xi - z} d\xi \forall z \in D_R. \quad (77)$$

Let $\xi = Re^{i\theta}$, therefore

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\xi f(\xi)}{\xi - z} d\theta \forall z \in D_R. \quad (78)$$

On the other hand, since $|z| < R$, the function $\frac{f(\xi)}{\xi - z}$ is holomorphic in $D_R$, therefore

$$0 = \frac{1}{2\pi i} \int_{|\xi| = R} \frac{\xi f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i} \int_{|\xi| = R} \frac{\xi f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi} \int_0^{2\pi} \frac{\xi f(\xi)}{\xi - z} d\theta. \quad (79)$$

Hence $\frac{1}{2\pi} \int_0^{2\pi} \frac{zf(\xi)}{\xi - z} d\theta = 0$, so

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\xi f(\xi) + z f(\xi)}{\xi - z} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{\xi f(\xi)}{\xi - z} d\theta + \frac{i}{2\pi} \int_0^{2\pi} v(\xi) d\theta. \quad (80)$$

Recall the mean value formula of the harmonic functions

$$\frac{i}{2\pi} \int_0^{2\pi} v(\xi) d\theta = iv(0). \quad (81)$$

Therefore, the proof is complete.

**Lemma 2.4.5** Suppose $f$ is holomorphic in an open set $\Omega$ containing the closure of $D_R$. Denote $c_n$ as the $n$th coefficient of the Taylor's expansion

of $f$ at $z = 0$. Then we have $|c_n| \leq \frac{2}{R^n} \max_{|z|=R} \text{Re}(f(z) - f(0))$

**proof:** From Lemma 2.2

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} u(\xi) \frac{\xi + z}{\xi - z} d\theta + iv(0) = \frac{1}{2\pi} \int_0^{2\pi} u(\xi) \frac{1 + z}{1 - \frac{z}{\xi}} d\theta + iv(0). \quad (82)$$

inside $D_R$, $|z| < |\xi|$, so

$$\frac{1 + \frac{z}{\xi}}{1 - \frac{z}{\xi}} = \left(1 + \frac{z}{\xi}\right) \left(1 + \frac{z}{\xi} + \left(\frac{z}{\xi}\right)^2 + \cdots \right) = 1 + \sum_{n=1}^{\infty} \frac{2}{\xi^n} z^n. \quad (83)$$

Thus

$$f(z) = f(0) + \sum_{n=1}^{\infty} \left(\frac{1}{\pi} \int_0^{2\pi} \frac{u(\xi)}{\xi^n} d\xi\right) z^n. \quad (84)$$

So that $c_n = \frac{1}{\pi} \int_0^{2\pi} \frac{u(\xi)}{\xi^n} d\xi$. Since the function $u(z)$ is continuous in the closure of $D_R$, there exists $M > 0$ such that $|u(z)| < M \forall z \in D_R$. Notice that

$$\frac{1}{\pi} \int_0^{2\pi} \frac{M}{\xi^n} d\xi = 0 \forall n \geq 1. \quad (85)$$

Therefore,
\[
|c_n| = \left| \frac{1}{\pi} \int_0^{2\pi} \frac{M - u(\xi)}{\xi^n} \, d\xi \right| \leq \frac{1}{\pi R^n} \int_0^{2\pi} \left( M - u(Re^{it}) \right) \, dt = \frac{2}{R^n} (M - u(0)). \tag{86}
\]

Take \( M = \max_{|z|=R} u(z) \), the proof is then complete.

The proof of Hadamard’s theorem is divided into several steps.

**Step 1.** Take \( k \) as the integer \( \lfloor \rho_0 \rfloor \) \( \forall \mathbb{C} \), which implies the factor
\[
\prod_{n=1}^{\infty} E_k \left( \frac{z}{a_n} \right) = \prod_{n=1}^{\infty} e^{-\sum_{m=k+1}^{\infty} \left( \frac{1}{m} \right) \left( \frac{z}{a_n} \right)^{m}}
\]
converges on \( \mathbb{C} \). For this, it suffices to prove that \( \forall R > 0, E_k \) converges uniformly on \( D_R \). Choose a positive integer \( N \) such that \( a_n \leq 2R \, \forall \, n \leq N \) and \( a_n > 2R \, \forall \, n > N \); since the only limit of \( \{a_n\} \) is \( \infty \), there will exist the proper \( N \). Then, inside the disc \( D_R \),
\[
\log \left( \prod_{n=N+1}^{\infty} e^{-\sum_{m=k+1}^{\infty} \left( \frac{1}{m} \right) \left( \frac{z}{a_n} \right)^{m}} \right) = \left| \sum_{m=N+1}^{\infty} \sum_{m=k+1}^{\infty} \frac{1}{m} \left( \frac{z}{a_n} \right)^{m} \right|
\]
\[
\leq \sum_{n=N+1}^{\infty} \sum_{m=k+1}^{\infty} \frac{1}{m} \left| \frac{z}{a_n} \right|^m \leq \sum_{n=N+1}^{\infty} \frac{z}{a_n} \sum_{m=1}^{\infty} \frac{1}{2^m} \left| \frac{z}{a_n} \right|^{k+1}.
\tag{87}
\]
Since \( f(z) \) has finite order, according to the lemma we have proved,
\[
\sum_{n=N+1}^{\infty} \frac{1}{|a_n|^{k+1}} \leq \sum_{n=1}^{\infty} \frac{1}{|a_n|^{k+1}} < \infty.
\tag{88}
\]
Thus \( \prod_{n=1}^{\infty} E_k \left( \frac{z}{a_n} \right) \) converges uniformly on \( D_R \). Since \( R \) could be choose arbitrarily, the conclusion is complete.

**Step 2.** Now, according to Weierstrass’ theorem, \( f(z) \) must have the form
\[
f(z) = Z^m e^{g(z)} \prod_{n=1}^{\infty} E_k \left( \frac{z}{a_n} \right),
\tag{89}
\]
and \( g(z) \) is an entire function. If \( f(0) \neq 0 \), suppose that \( f(0) = 1 \); if \( f(0) = 0 \), use \( \frac{f(z)}{Z^m} \) to replace \( f \) instead.

Let \( Q_N(z) = \prod_{n=1}^{N} \left( 1 - \frac{z}{a_n} \right) \), then \( f_N(z) = \frac{f(z)}{Q_N(z)} \) defines a holomorphic function in the disc \( D_R \), thus
\[
g(z) = \log(f_N(z)) - \sum_{n=N+1}^{\infty} \log \left( 1 - \frac{z}{a_n} \right) - P_k,
\tag{90}
\]
where \( P_k \) is a polynomial with degree \( k \). Let \( g_N(z) = \log(f_N(z)) \), then \( g_N(0) = 0 \). Suppose
\[
g(z) = \sum_{m=0}^{\infty} C_m x^m k!,
\tag{91}
\]
\[
g_N(z) = \sum_{m=0}^{\infty} C_m(N) x^m m!,
\tag{92}
\]
\[
\sum_{n=N+1}^{\infty} \log \left( 1 - \frac{z}{a_n} \right) = \sum_{m=0}^{\infty} C_m(N) x^m m!,
\tag{93}
\]
and notice that \( C_m \) is independent with \( N \), then it suffices to prove that
\[
0 = C_m = \lim_{N \to \infty} (C_m(N) - C'_m(N)) = \lim_{N \to \infty} C_m(N) \forall m > k.
\tag{94}
\]
When \( |z| = 2R \) and \( n \leq N \).
Denote $M_f(R)$ as the maximum of $|f(z)|$ inside $D_R$. So that

$$\left|1 - \frac{z}{a_n}\right| = \left|\frac{a_n - z}{a_n}\right| \geq \frac{R}{|a_n|} \geq 1$$ (95)

Since the order of $f$ is $\rho_0$, then

$$\text{Re}(g_N(z)) = \log|f(z)| \leq \log\left(M_f(2R)\right) \leq c_1 + c_2(2R)^{\rho_0},$$

where $c_1, c_2$ are two constants. Recall Lemma 2.3, next

$$|C_m(N)| \leq \frac{2}{R^m} \max_{|z| = R} \text{Re}(g_N(z) - g_N(0)) \leq \frac{2}{R^m} (c_1 + c_2(2R)^{\rho_0}).$$

Let $N \to \infty$, then $R \to \infty$; since $m > \rho_0$, then

$$\lim_{N \to \infty} |C_m(N)| = 0.$$ (99)

This ends the proof of Hadamard’s function theorem.

**Theorem 2.4.6** If $F$ is an entire function and has a non-integral order $\rho$, then $F$ has infinitely many zeros.

**Proof:** We will deduce from contradiction. Suppose that $F$ has finitely many zeros, namely $a_1, a_2, \ldots, a_N$, instead. Assume that $k < \rho < k + 1$, where $k$ is an integer. Then, from Hadamard’s function theorem, we have

$$F(z) = z^m e^{g(z)} \prod_{n=1}^{N} \left(1 - \frac{z}{a_n}\right) e^{p_k(z/a_n)},$$

where $g(z)$ and $P_k(z)$ are all polynomials with degree $k$. Since the degree of a polynomial is 0, there exists $A_1 > 0, B_1 > 0$ such that

$$\left|z^m \prod_{n=1}^{N} \left(1 - \frac{z}{a_n}\right)\right| \leq A_1 e^{B_1|z|^k}.$$ (101)

Also, since $g(z) + \sum_{n=1}^{N} P_k(z/a_n)$ is a polynomial of degree $k$, we can find

$$A_2 > 0, B_2 > 0 \text{ such that } \left|e^{g(z)} \prod_{n=1}^{N} e^{p_k(z/a_n)}\right| \leq A_2 e^{B_2|z|^k}.$$ Thus,

$$|F(z)| = \left|z^m \prod_{n=1}^{N} \left(1 - \frac{z}{a_n}\right)e^{g(z)} \prod_{n=1}^{N} e^{p_k(z/a_n)}\right| \leq A_1 A_2 e^{(B_1 + B_2)|z|^k},$$ (102)

which implies that the order of $F$ is less than $k$, a contradiction.

**Theorem 2.4.7 (Pringsheim interpolation formula)** Suppose $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two given complex sequences, where $a_n$ are distinct complex numbers, whose only limit point is $\infty$ and $a_n = 0$. Then, there exists an entire function $F(z)$ such that $F(a_n) = b_n \forall n \in \mathbb{N}$.

**Proof:** Assume that $|a_1| < |a_2| < |a_3| < \ldots$. Let

$$F(z) = \frac{b_0}{E'(z)} E(z) + \sum_{k=1}^{\infty} \frac{b_k}{E'(a_k) z - a_k} \frac{E(z)}{z} \left(z - a_k\right)^{m_k},$$

where $E(z)$ is the Weierstrass product of $\{a_n\}_{n=1}^{\infty}$, and $m_k$ are positive integers. We declare that by carefully choosing $m_k$, $F(z)$ defines an entire function with the required properties.

**Step 1:** $\forall R > 0$, there exists $k_R$ such that $|a_k| > 2R \forall k \geq k_R$. Now we will prove that the sum $\sum_{k=k_R}^{\infty} b_k E(z) (z/a_k)^{m_k}$ converges uniformly on $D_R$ by choosing $m_k$...
carefully.
First, notice that since \( \{a_n\}_{n=1}^{\infty} \) are distinct, each \( a_n \) is a zero of \( E(z) \) of order 1. Thus there exists an entire function \( g_k(z) \) such that \( g_k(a_k) \neq 0 \). Thus we have

\[
E'(a_k) = g_k(a_k) = \lim_{z \to a_k} \frac{E(z)}{z - a_k} = -e^{\gamma_k} \prod_{n=1}^{\infty} \left( 1 - \frac{a_k}{a_n} \right) e^{\zeta_n/a_n}.
\]  

(109)

Then,

\[
\frac{E(z)}{E'(a_k)} = \frac{z (1 - \frac{a_k}{a_n}) e^{\gamma_k} \prod_{n=1}^{\infty} \left( 1 - \frac{a_k}{a_n} \right) e^{\zeta_n/a_n}}{-e^{\gamma_k} \prod_{n=1}^{\infty} \left( 1 - \frac{a_k}{a_n} \right) e^{\zeta_n/a_n}} = \prod_{n=1}^{\infty} \left( 1 - \frac{a_k}{a_n} \right) e^{\zeta_n/a_n}.
\]

(110)

For fixed \( k \), \( \exists N_k \) such that when \( n > N_k \) we have \( |a_n| > 2|a_k| \). Then,

\[
\prod_{n=1}^{\infty} \left( 1 - \frac{a_k}{a_n} \right) e^{\zeta_n/a_n} = \prod_{n=1}^{N_k} a_n - a_k \prod_{n=N_k+1}^{\infty} \left( 1 - \frac{a_k}{a_n} \right) e^{\zeta_n/a_n} = \sum_{n=1}^{N_k} a_n - a_k \prod_{n=N_k+1}^{\infty} \left( 1 - \frac{a_k}{a_n} \right) e^{\zeta_n/a_n}.
\]

(111)

Notice that

\[
\left| \prod_{n=N_k+1}^{\infty} \left( 1 - \frac{a_k}{a_n} \right) e^{\zeta_n/a_n} \right| \leq \prod_{n=N_k+1}^{\infty} e^{2 \zeta_n/a_n} \leq \prod_{n=N_k+1}^{\infty} e^{2 \zeta_n/a_n} \leq \sqrt{e},
\]

(112)

and

\[
\left| \prod_{n=1}^{N_k} a_n - a_k \right| \leq \prod_{n=1}^{N_k} \left| a_n - a_k \right|,\]

(113)

where we can choose \( m_{k_1} \) such that

\[
\prod_{n=1}^{N_k} \left| a_n - a_k \right| \leq \prod_{n=1}^{N_k} \left| a_n - a_k \right| \leq \frac{1}{2^{k_1}}.
\]

Next, when \( n \leq N_k \), \( |a_n - a_k| \geq \frac{n}{a_n} \), therefore

\[
\sum_{s=1}^{n} \frac{1}{s} \left( a_k / a_n \right) - \frac{1}{s} \left( z / a_n \right) \leq 2 \sum_{s=1}^{n} \frac{1}{s} \left| a_k / a_n \right|^s.
\]

(114)

Thus, we have

\[
\prod_{n=1}^{N_k} e^{\zeta_n/a_n} \leq \prod_{n=1}^{N_k} e^{2 \zeta_n/a_n} \leq \prod_{n=1}^{N_k} e^{2 \zeta_n/a_n} \leq \prod_{n=1}^{N_k} e^{2 \zeta_n/a_n} \leq 1/2^{k_2}.
\]

(115)

So there exists \( m_{k_2} \) such that

\[
\prod_{n=1}^{N_k} e^{2 \zeta_n/a_n} \leq 1/2^{k_2}.
\]

(116)

Finally, since \( |b_k| \leq |b_k|/|z-a_k| \),

\[
\left| \prod_{n=1}^{N_k} e^{2 \zeta_n/a_n} \right| \leq \prod_{n=1}^{N_k} e^{2 \zeta_n/a_n} \leq 1/2^{k_2}.
\]

(118)

Take \( m = m_{k_1} + m_{k_2} + m_{k_3} \), then

\[
\sum_{k=k_R}^{\infty} b_k \frac{E(z)}{z-a_k} \left( \frac{z}{a_k} \right)^m \leq \sum_{k=k_R}^{\infty} \frac{1}{2^{k_2}} = \frac{1}{2^{k_2}} \leq 1.
\]

(119)
Thus, this sum converges uniformly in $D_R$. Since we can choose $R$ arbitrarily, $F(z)$ defines an entire function.

**Step 2:** We have already proved that $E'(a_k) = \lim_{z \to a_k} \frac{E(z)}{z-a_k}$, and notice that $\frac{E(a_n)}{a_n-a_k} = 0 \forall n \neq k$. Thus we have

$$F(a_n) = \lim_{z \to a_k} \frac{b_k}{E'(a_k) z - a_k} \left( \frac{z}{a_k} \right)^{m_k} = b_n,$$

which finishes the proof.

3. Conclusion

In this paper, we summarize several fundamental theorems of entire functions. Through Jensen’s formula, we derive the identity linking the growth of an entire function and its number of zeros inside a disc with growing radius. Next, by introducing the theorem of infinite products, we go through some examples and find the infinite product using the zeros of some specific entire functions. Then, after defining the order of growth of an entire function, we come to Weierstrass’ theorem which gives the complete structure of the entire functions and a formula with a form of infinite products. When the function has finite growth of order, Hadamard’s theorem gives a more precise estimation of the product and can be treated as a perfect result of this problem. As a result to our research, we gives a proof of Mittag-Leffler’s Theorem on $\mathbb{C}$, and introduce the Pringsheim interpolation formula, which can be seen as a corollary of Hadamard’s theorem. In the future, since the research on entire functions had such beautiful results, we will move on to meromorphic functions and its specific properties, and will take a look at some special functions such as the $\Gamma$ function and Riemann’s $\zeta$ function. These future plans, we believe, will produce more useful results in complex analysis and other fields, such as the analytic number theory.

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