A nonlinear inverse problem of the Korteweg-de Vries equation

Shengqi Lu\textsuperscript{1} · Miaochao Chen\textsuperscript{2} · Qilin Liu\textsuperscript{3}

Abstract In this paper we prove the existence and uniqueness of solutions of an inverse problem of the simultaneous recovery of the evolution of two coefficients in the Korteweg-de Vries equation.

Keywords Korteweg-de Vries equation · Inverse problem · Integral overdetermination · Two coefficients

Mathematics Subject Classification 35K55 · 49K20 · 82D55

1 Introduction

We consider the following nonlinear Korteweg-de Vries (KdV) equation for an unknown scalar function \( u = u(x, t) \), \( x \in \mathbb{R}, t \in \mathbb{R} \):

\[ \frac{
abla}{u} + u \frac{
abla}{u} u + u^3 = 0 \]

Communicated by Ari Laptev.

\textsuperscript{1} Department of Mathematics and Physics, Sanjiang University, Nanjing 210012, People’s Republic of China

\textsuperscript{2} School of Applied Mathematics, Chaohu University, Hefei 238000, People’s Republic of China

\textsuperscript{3} School of Mathematics, Southeast University, Nanjing 211189, People’s Republic of China
\[ u_t + uu_x + u_{xxx} + \alpha(t)u = f(t)g(x), \quad x \in \Omega, \quad t \in (0, T), \quad (1.1) \]
\[ u(x, 0) = u_0(x), \quad x \in \Omega = (0, 1), \quad (1.3) \]

with the condition of integral overdeterminations:
\begin{equation}
\int_{\Omega} u(x, t)w_1(x)dx = \phi_1(t), \quad \int_{\Omega} u(x, t)w_2(x)dx = \phi_2(t), \quad t \in [0, T]. \quad (1.4)
\end{equation}

Here \( T \) is a given positive constant, \( u_0, g, w_1, w_2, \phi_1 \) and \( \phi_2 \) are known functions, \( \alpha \) and \( f \) are two coefficients to be determined by (1.4). Here it should be noted that all functions are real.

We will use the Hilbert space \( H^k \): \( \{ f \in H^k; f, f', \ldots, f^{(k)} \in L^2 \} \), and \( H_{per}^k(\Omega) := \{ u \in H^k; u(x) = u(x + 1) \text{ for all } x \in \mathbb{R} \} \).

When \( f(t) = \alpha(t) = 0, u_0 \in H_{per}^2(\Omega) \), the existence and uniqueness of solutions for the problem (1.1)–(1.3) was proved by Temam [1] and later by Bona and Smith [2].

In this paper, we want to study the nonlinear inverse problem consists of finding a set of the functions \{\( u, \alpha, f \)\} satisfying (1.1)–(1.4). This kind of inverse problem of incompressible Naiver–Stokes equations and Ginzburg–Landau equations in superconductivity has been studied in [3–5].

Throughout this paper, we will assume

(H1) \( u_0 \in H_{per}^2(\Omega), \)

(H2) \( \alpha, f \in C[0, T], \)

(H3) \( g \in H_{per}^2(\Omega), \)

(H4) \( w_1, w_2 \in H_{per}^3(\Omega), \)

(H5) \( \phi_1, \phi_2 \in C^1[0, T], \quad \int_{\Omega} u_0w_1(x)dx = \phi_1(0), \quad \int_{\Omega} u_0w_2(x)dx = \phi_2(0), \quad \phi_2g_{10} - \phi_1g_{20} \neq 0 \text{ for all } t \in [0, T], \)

(H6) \( \int_{\Omega} g(x)w_1(x)dx = g_{10} \neq 0, \quad \int_{\Omega} g(x)w_2(x)dx = g_{20} \neq 0. \)

**Remark 1.1** (H5) implies that \( w_1 \) and \( w_2 \) be linearly independent.

Let us explain what a solution to the direct problem (1.1)–(1.3) is.

**Definition 1.1** Assume that (H1), (H2) and (H3) hold true. A function \( u \) in \( L^\infty(0, T; H_{per}^2(\Omega)) \cap C([0, T]; L^2(\Omega)) \) is called a solution to (1.1)–(1.3) if (1.2) and (1.3) hold true everywhere and (1.1) holds true in the following sense,
\begin{align*}
-\int_{0}^{T} \int_{\Omega} (u\phi_t + u_{xx}\phi_x)dxdt &+ \int_{0}^{T} \int_{\Omega} (uu_x + \alpha u - fg)\phi dxdt \\
&= \int_{\Omega} (u_0\phi(x, 0) - u(x, T)\phi(x, T))dx \tag{1.5}
\end{align*}
for all \( \phi \in L^\infty(0, T; H_{per}^1(\Omega)). \)
By a similar proof as that in [6], we can prove an existence and uniqueness result of the direct problem (1.1)–(1.3) and we here omit the details.

**Theorem 1.1** Let (H1)–(H3) be satisfied. Then there exists a unique solution $u$ satisfying

$$u \in L^\infty(0, T; H^2_\text{per}(\Omega)) \cap C([0, T], L^2(\Omega)), \quad u_t \in L^\infty(0, T; H^{-1}(\Omega)), \quad \forall T > 0.$$  

Based on Theorem 1.1, we can define the nonlinear operator

$$A : C([0, T]) \times C([0, T]) \to C([0, T]) \times C([0, T])$$

acting on every vector $\chi = \{\alpha(t), f(t)\}$ as follows:

$$[A(\chi)](t) = \{[A_1(\alpha, f)](t), [A_2(\alpha, f)](t)\}, \quad t \in [0, T]$$

where

$$[A_1(\alpha, f)](t) = \frac{1}{\phi_2 g_{10} - \phi_1 g_{20}} \left\{ \phi'_1 g_{20} - \phi'_2 g_{10} + g_{10} \int_{\Omega} \left( \frac{1}{2} u^2 w'_2 + uw''_2 \right) dx \right\},$$

$$[A_2(\alpha, f)](t) = \frac{1}{\phi_2 g_{10} - \phi_1 g_{20}} \left\{ \phi'_1 \phi_2 - \phi_1 \phi_2' + \phi_1 \int_{\Omega} \left( \frac{1}{2} u^2 w'_2 + uw''_2 \right) dx \right\},$$

and $u$ has been already found as the unique solution of the system (1.1)–(1.3). We proceed to study the operator equation of the second kind over the space $C([0, T]) \times C([0, T])$:

$$\chi = A\chi \quad (1.8)$$

Here the space $C([0, T]) \times C([0, T])$ is equipped with the norm

$$\|u\|_{C([0, T]) \times C([0, T])} = \|u_1\|_{C([0, T])} + \|u_2\|_{C([0, T])},$$

where $u_1$ and $u_2$ are the components of the vector $u$ and

$$\|u_i\|_C = \sup_{t \in [0, T]} |\exp\{-\gamma t\}u_i(t)|,$$

with $\gamma = a$ a positive constant to be determined.

Let us explain what a solution to the inverse problem (1.1)–(1.4) is.

**Definition 1.2** Let (H1) and (H3)–(H6) hold true. A vector function

$$(u, \alpha, f) \in L^\infty(0, T; H^2_\text{per}(\Omega)) \cap C([0, T]; L^2(\Omega)) \times C([0, T]) \times C([0, T])$$
is called a solution to the inverse problem (1.1)–(1.4) if (1.2), (1.3) and (1.4) hold true everywhere and (1.1) holds true in the sense of (1.5).

An interrelation between the inverse problem (1.1)–(1.4) and the nonlinear equation (1.8) from the view point of their solvability is revealed in the following assertion.

**Theorem 1.2** Let (H1)–(H6) be satisfied. Then the following assertions are valid:

(a) if the inverse problem (1.1)–(1.4) is solvable, then so is the Eq. (1.8).

(b) if Eq. (1.8) has a solution, then there exists a solution of the inverse problem (1.1)–(1.4).

**Proof** The proof is the same as that of [3, pp. 285–288], however, for the reader’s convenience, we present the proof.

(a) Let the inverse problem (1.1)–(1.4) possess a solution, say \( \{u, \alpha, f\} \). Now testing (1.1) by \( w_i \) (\( i = 1, 2 \)) and using (1.4), we see that

\[
\phi_i'(t) - \frac{1}{2} \int_{\Omega} u^2 w_i' dx - \int_{\Omega} u w_i''' dx + \alpha \phi_i(t) = f(t) g_{i0}.
\]

Solving the above system and using (H6) and (H5) we conclude that \( \alpha \) and \( f \) solve the Eq. (1.8).

(b) We suppose that Eq. (1.8) has a solution, say \( \{\alpha, f\} \). By Theorem 1.1 on the unique solvability of the direct problem we are able to recover \( u \) as the solution of (1.1)–(1.3) associated with \( \{\alpha, f\} \), so that it remains to be shown that the function \( u \) satisfies the over-determination condition (1.4), which follow from (1.6), (1.7), and (1.8) immediately. This provides support for the view that \( \{u, \alpha, f\} \) is just a solution of the inverse problem (1.1)–(1.4).

This completes the proof.

Now we are in a position to state our main result.

**Theorem 1.3** Let (H1)–(H6) be satisfied and \( T \) be small enough. Then there exists a unique solution \( \{u, \alpha, f\} \) to the inverse problem (1.1)–(1.5) with \( (\alpha, f) \in D \) where

\[
D := \left\{ (\alpha, f) \in C([0, T]) \times C([0, T]) \left| \| (\alpha, f) \|_C \leq \frac{1}{NT} e^{-\gamma T} \right. \right\}
\]

and \( N \) is a large integer to be determined in the following calculations.

We will use \( \| \cdot \| := \| \cdot \|_{L^2(\Omega)} \).

In the next Sect. 2, we give some preliminaries to the proof of Theorem 1.3, which is given in the final Sect. 3.

**2 Preliminaries**

Let \( u \) be the unique solution of the problem (1.1)–(1.3) when \( (\alpha, f) \in D \) is given. We will prove some a priori estimates for \( u \).
Lemma 2.1 Let $N \geq 2$. Then

$$\|u(\cdot, t)\| \leq 2\|u_0\| + \|g\| =: a_1, \quad \forall t \in [0, T].$$

(2.1)

Here $Te^{\gamma T} \|\alpha\|_C \leq \frac{1}{2}$ and $Te^{\gamma T} \|f\|_C \leq \frac{1}{2}$.

Proof Multiplying (1.1) by $u$ and integrating by parts lead to

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \alpha \|u\|^2 = f(t) \int_{\Omega} gu \, dx \leq |f| \|g\| \|u\|,$$

and thus

$$\|u\| \frac{d}{dt} \|u\| \leq |\alpha| \|u\|^2 + |f| \|g\| \|u\|.$$

Dividing both sides by $\|u\|$, we get

$$\frac{d}{dt} \|u\| \leq |\alpha| \|u\| + |f| \|g\|.$$

Integrating this inequality gives

$$\|u\| \leq \|u_0\| + Te^{\gamma T} \|\alpha\|_C \|u\|_{L^\infty(0, T; L^2(\Omega))} + \|g\| Te^{\gamma T} \|f\|_C$$

$$\leq \|u_0\| + \frac{1}{2} \|u\|_{L^\infty(0, t; L^2(\Omega))} + \frac{1}{2} \|g\|$$

which gives (2.1). \qed

Lemma 2.2 Let $N \geq 8$. Then

$$\|u_x(\cdot, t)\|^2 \leq \frac{9}{4} (2\|u_0\| + \|g\|)^{10/3} + \frac{11}{4} (2\|u_0\| + \|g\|)^3$$

$$+ \frac{3}{4} \|g'\|^2 + \frac{3}{4} \|g\|_{L^\infty(2\|u_0\| + \|g\|)^2} + 6\|u_0\|^2 + 2\|u_0\|_{L^3}^3 =: a_2^2.$$  

(2.2)

Here $Te^{\gamma T} \|\alpha\|_C \leq \frac{1}{8}$ and $Te^{\gamma T} \|f\|_C \leq \frac{1}{8}$.

Proof Let $I_1(u) := \int_{\Omega} \left( u_x^2 - \frac{1}{3} u^3 \right) \, dx$, then from [6, pp. 261–262] we have

$$\frac{d}{dt} I_1(u) = \alpha(t) \int_{\Omega} (u^3 - 2u_x^2) \, dx + f(t) \int_{\Omega} (2g'u_2 - gu^2) \, dx$$

$$\leq 2|\alpha(t)| \int_{\Omega} u_x^2 \, dx + |\alpha(t)| \left| \int_{\Omega} u^3 \, dx \right|$$

$$+ |f(t)| \int_{\Omega} u_x^2 \, dx + |f(t)| (\|g'\|^2 + \|g\|_{L^\infty} \|u\|^2).$$

(2.3)
Also from [6, pp. 262] we estimate

\[ | \int_{\Omega} u^3 \, dx | \leq \| u_x \|^2 + \frac{3}{4} \| u \|^{10/3} + \| u \|^3. \] (2.4)

From (2.3) and (2.4) we deduce that

\[ \frac{d}{dt} I_1(u) \leq (3|\alpha(t)| + |f(t)|) \int_{\Omega} u_x^2 \, dx + |\alpha(t)| \left( \frac{3}{4} \| u \|^{10/3} + \| u \|^3 \right) + |f(t)| (\| g' \|^2 + \| g \|_{L^\infty} \| u \|^2). \]

Integrating this inequality gives

\[ \int_{\Omega} u_x^2 \, dx \leq (3\| \alpha \|_C + \| f \|_C) T e^{\gamma T} \| u_x \|_{L^\infty(0, T; L^2(\Omega))} \]

\[ + T e^{\gamma T} \| \alpha \|_C \left[ \frac{3}{4} (2\| u_0 \| + \| g \|)^{10/3} + (2\| u_0 \| + \| g \|)^3 \right] \]

\[ + T e^{\gamma T} \| f \|_C \left[ \| g' \|^2 + \| g \|_{L^\infty} (2\| u_0 \| + \| g \|)^2 \right] + \frac{1}{3} \left| \int_{\Omega} u^3 \, dx \right| + I_1(u_0) \]

\[ \leq \frac{1}{2} \| u_x \|^2_{L^\infty(0, T; L^2(\Omega))} + \frac{1}{8} \left[ \frac{3}{4} (2\| u_0 \| + \| g \|)^{10/3} + (2\| u_0 \| + \| g \|)^3 \right] \]

\[ + \frac{1}{8} [\| g' \|^2 + \| g \|_{L^\infty} (2\| u_0 \| + \| g \|)^2] + \frac{1}{3} \left| \int_{\Omega} u^3 \, dx \right| + I_1(u_0) \]

which gives (2.2) by (2.4) and (2.1).

**Lemma 2.3** Let \( N \geq 8 \). Then there exists a positive constant \( C_1 \) depending only on \( \| u_0 \|_{H^2} \) and \( \| g \|_{H^2} \) such that

\[ \| u_{xx} (\cdot, t) \|^2 \leq C_1. \] (2.5)

Here \( T e^{\gamma T} \| \alpha \|_C \leq \frac{1}{8} \) and \( T e^{\gamma T} \| f \|_C \leq \frac{1}{8} \). We used the embedding inequality

\[ \| u \|_{L^\infty} \leq C_0 (\| u \| + \| u_x \|) \leq C_0 (a_1 + a_2) = : a_3, \]

and

\[ \left| \int_{\Omega} u^3 \, dx \right| \leq a_2^2 + \frac{3}{4} a_1^{10} + a_3^3 = : a_4, \]

and

\[ C_1 := 2I_2(u_0) + \frac{55}{12} a_3 a_2^2 + \frac{1}{4} \left[ \| g'' \|^2 + \frac{5}{3} \| g \|_{L^\infty} a_1^2 + \frac{25}{9} \| g \|_{L^\infty} a_1^2 + \frac{5}{9} \| g \|_{L^\infty} a_4 \right]. \]
Proof Let $I_2(u) := \int_{\Omega} \left( u_{xx}^2 - \frac{5}{3} uu_x^2 + \frac{5}{36} u^4 \right) dx$, we use the idea in [6, pp. 275]. Testing (1.1) by

$$L_2(u) := 2u_{xxxx} + \frac{5}{3} u^2_x + \frac{10}{3} uu_{xx} + \frac{5}{9} u^3.$$ 

We obtain

$$\frac{d}{dt} I_2(u) = \int_{\Omega} u_t L_2(u) dx,$$

$$\alpha(t) \int_{\Omega} \left( -2u_{xx}^2 + 5uu_x^2 - \frac{5}{9} u^4 \right) dx = -\int_{\Omega} \alpha u L_2(u) dx,$$

$$f(t) \int_{\Omega} \left( 2g''u_{xx} + \frac{5}{3} g^2 u_x^2 + \frac{10}{3} gu_{xx} + \frac{5}{9} g u^3 \right) dx = \int_{\Omega} f g L_2(u) dx,$$

and

$$\int_{\Omega} (uu_x + uu_{xx}) L_2(u) dx$$

$$= \int_{\Omega} (uu_x + uu_{xx}) \left( 2u_{xxxx} + \frac{5}{3} u^2_x + \frac{10}{3} uu_{xx} + \frac{5}{9} u^3 \right) dx$$

$$= \int_{\Omega} \left[ (-u_x^2 - uu_{xx}) 2u_{xxx} + \frac{5}{3} uu_x^3 + \frac{10}{3} u^2 u_x u_{xx} + \left( \frac{5}{3} u_x^2 + \frac{10}{3} uu_{xx} + \frac{5}{9} u^3 \right) u_{xxx} \right] dx$$

$$= \int_{\Omega} \left[ \left( -\frac{u_x^2}{3} + \frac{4}{3} uu_{xx} + \frac{5}{9} u^3 \right) u_{xxx} + \frac{5}{3} uu_x^3 + \frac{5}{3} u^2 ((u_x)^2)_x \right] dx$$

$$= \int_{\Omega} \left[ \left( \frac{2}{3} uu_x^2 + \frac{2}{3} (u_{xx})^3 \right) - \frac{5}{3} u^2 u_x u_{xx} - \frac{5}{3} uu_x^3 \right] dx$$

$$= -\frac{5}{3} \int_{\Omega} (u^2 u_x u_{xx} + uu_x^3) dx$$

$$= -\frac{5}{3} \int_{\Omega} (-uu_x^3 + uu_x^3) dx = 0.$$ 

Thus we have

$$\frac{d}{dt} I_2(u) = \alpha(t) \int_{\Omega} \left( -2u_{xx}^2 + 5uu_x^2 - \frac{5}{9} u^4 \right) dx$$

$$+ f(t) \int_{\Omega} \left( 2g''u_{xx} + \frac{5}{3} g^2 u_x^2 + \frac{10}{3} gu_{xx} + \frac{5}{9} g u^3 \right) dx$$

$$\leq 2(\alpha(t) + |f(t)|) \int_{\Omega} u_{xx}^2 dx + 5|\alpha(t)| \|u\|_{L^\infty} \|u_x\|^2 + \frac{5}{9} |\alpha(t)| \int_{\Omega} u^4 dx$$

$$+ |f(t)| \left[ \|g''\|^2 + \frac{5}{3} \|g\|_{L^\infty} \|u_x\|^2 + \frac{25}{9} \|g\|_{L^\infty} \|u\|^2 + \frac{5}{9} \|g\|_{L^\infty} \|u\|_L^3 \right].$$

(2.6)
Now estimating as Lemma 2.2 gives (2.5).

Lemma 2.4 Let \( u_i \) \((i = 1, 2)\) be the corresponding solutions of the problem (1.1)–(1.3) with data \( \alpha_i, f_i \) \((i = 1, 2)\), then there exists a positive constant \( C_2 \) depending only on \( \| u_0 \|_{H^2} \) and \( \| g \|_{H^2} \) such that

\[
\sup_{[0,t]} \| (u_1 - u_2)(\cdot, \tau) \| \leq \frac{4}{\gamma - 2C_2} e^{\gamma t}[\| f_1 - f_2 \| c \| g \| + (2 \| u_0 \| + \| g \|) \| \alpha_1 - \alpha_2 \| c].
\]

(2.7)

Here we take \( N \geq 8 \) and \( \gamma > 2C_2 \) and \( C_2 \geq \| u_{1x} + u_{2x} \|_{L^\infty(\Omega \times (0,T))} \). We used \( \int_\Omega u_{1x} dx = \int_\Omega u_{2x} dx = 0 \), and the embedding inequality

\[
\| u_{1x} + u_{2x} \|_{L^\infty} \leq C_0 \| u_{1xx} + u_{2xx} \| \leq 2C_0 \sqrt{C_1} =: C_2.
\]

Proof From (1.1) we obtain

\[
(u_1 - u_2)_t + \frac{1}{2} \left( u_1^2 - u_2^2 \right)_x + (u_1 - u_2)_{xxx} + \alpha_1 (u_1 - u_2) + (\alpha_1 - \alpha_2) u_2 = (f_1 - f_2) g.
\]

(2.8)

Multiplying (2.8) by \( u_1 - u_2 \) and integrating by parts, and noting that

\[
\frac{1}{2} \int_\Omega (u_1^2 - u_2^2)_x (u_1 - u_2) dx
\]

\[
= \frac{1}{2} \int_\Omega (u_1 + u_2)_x(u_1 - u_2)^2 dx + \frac{1}{2} \int_\Omega (u_1 + u_2)(u_1 - u_2)_x(u_1 - u_2) dx
\]

\[
= \frac{1}{2} \int_\Omega (u_1 + u_2)_x(u_1 - u_2)^2 dx - \frac{1}{4} \int_\Omega (u_1 + u_2)_x(u_1 - u_2)^2 dx
\]

\[
= \frac{1}{4} \int_\Omega (u_1 + u_2)_x(u_1 - u_2)^2 dx,
\]

we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega (u_1 - u_2)^2 dx = -\frac{1}{4} \int_\Omega (u_1 + u_2)_x(u_1 - u_2)^2 dx - \alpha_1(t) \int_\Omega (u_1 - u_2)^2 dx
\]

\[
+ (f_1(t) - f_2(t)) \int_\Omega g(u_1 - u_2) dx - (\alpha_1(t) - \alpha_2(t)) \int_\Omega u_2(u_1 - u_2) dx
\]

\[
\leq \frac{1}{4} C_2 \int_\Omega (u_1 - u_2)^2 dx + |\alpha_1(t)| \int_\Omega (u_1 - u_2)^2 dx
\]

\[
+ \| g \| \| f_1(t) - f_2(t) \| \| u_1 - u_2 \| + \| u_2 \| \| \alpha_1(t) - \alpha_2(t) \| \| u_1 - u_2 \|,
\]
which gives
\[
\frac{d}{dt} \|u_1 - u_2\| \leq C_2 \|u_1 - u_2\| + |\alpha_1(t)||u_1 - u_2| + \|g\| |f_1(t) - f_2(t)| + \|u_2||\alpha_1(t) - \alpha_2(t)|.
\]

Integrating the above inequality, we derive
\[
\|(u_1 - u_2)(\cdot, t)\| \leq C_2 \int_0^t \|(u_1 - u_2)(\cdot, \tau)\| d\tau + \int_0^t |\alpha_1(\tau)| \|u_1 - u_2\| \|e\| d\tau
\]
\[
+ \|g\| \int_0^t |f_1(\tau) - f_2(\tau)| d\tau + \|u_2\| \|_{L_\infty} \int_0^t |\alpha_1(\tau) - \alpha_2(\tau)| d\tau
\]
\[
\leq C_2 \int_0^t \|(u_1 - u_2)(\cdot, \tau)\| d\tau + \frac{1}{2} \sup_{[0, t]} \|(u_1 - u_2)(\cdot, \tau)\|
\]
\[
+ \frac{1}{\gamma} e^{\gamma t} \|(f_1 - f_2)\|_C \|g\| + \|u_2\| \|_{L_\infty} \|\alpha_1 - \alpha_2\|_C,
\]

which leads to
\[
\sup_{[0, t]} \|(u_1 - u_2)(\cdot, \tau)\| \leq 2C_2 \int_0^t \sup_{[0, \tau]} \|(u_1 - u_2)(\cdot, \xi)\| d\tau
\]
\[
+ \frac{2}{\gamma} e^{\gamma t} \left[\|f_1 - f_2\|_C \|g\| + (2\|u_0\| + \|g\|) \|\alpha_1 - \alpha_2\|_C\right].
\]

An application of Gronwall’s inequality gives (2.7).

\[\square\]

3 Proof of Theorem 1.3

Lemma 3.1 Let (H1)–(H6) be satisfied. If T is small enough, then A maps D onto itself.

Proof Using Lemma 2.1, we have
\[
\|Ax\|_C \leq \frac{1}{\|\phi_2 g_{10} - \phi_1 g_{20}\|_C} \left[\|\phi_1 \|_C \|\phi_2\|_C + \|\phi_2\|_C \|\phi_1\|_C + (\|\phi_2\|_C + |g_{20}|)\right]
\]
\[
\times \left(\frac{1}{2} \|u\|_\infty \|w_1\|_\infty + \|u\| \|w_1''\| \right)
\]
\[
+ (\|\phi_1\|_C + |g_{10}|) \left(\frac{1}{2} \|u\|_\infty \|w_2\|_\infty + \|u\| \|w_2''\| \right) + \|\phi_1\|_C \|g_{20}\| + \|\phi_2\|_C \|g_{10}\|
\]
\[
\leq \frac{1}{\|\phi_2 g_{10} - \phi_1 g_{20}\|_C} \left[\|\phi_1 \|_C \|\phi_2\|_C + \|\phi_2\|_C \|\phi_1\|_C \right].
\]
Lemma 3.2 Let (H1)–(H6) be satisfied. Then there exists a positive constant $C_3$ such that if $\gamma = 3C_2 + C_3$ and $T$ is small enough, then the operator $A$ is a contraction mapping in the ball $D$.

Proof Using Lemmas 2.1 and 2.4, we have

$$
\|A\chi_1 - A\chi_2\| \leq \frac{e^{-\gamma t}}{\|\phi_2 g_{10} - \phi_1 g_{20}\|c} \times \left[ \left( \frac{1}{2} \|\phi_2\|c + |g_{20}| \right) \|u_1 + u_2\|L^2 \|w_1''\|L^\infty + (\|\phi_2\|c + |g_{20}|) \|w_1''\| \right] \|u_1 - u_2\|
$$

$$
+ \left( \frac{1}{2} \|\phi_1\|c + |g_{10}| \right) \|u_1 + u_2\|L^2 \|w_2''\|L^\infty + (\|\phi_1\|c + |g_{10}|) \|w_2''\| \right] \|u_1 - u_2\|
$$

$$
\leq \frac{e^{-\gamma t}}{\|\phi_2 g_{10} - \phi_1 g_{20}\|c} \left( \left( \frac{1}{2} \|\phi_2\|c + |g_{20}| \right) (4\|u_0\| + 2\|g\|) \|w_1''\|L^\infty + (\|\phi_2\|c + |g_{20}|) \|w_1''\| \right) \|u_1 - u_2\|
$$

$$
+ \left( \frac{1}{2} \|\phi_1\|c + |g_{10}| \right) \|w_2''\| \right] \|u_1 - u_2\|
$$

$$
\leq \frac{C_3}{\gamma - 2C_2} \|\chi_1 - \chi_2\| \leq \frac{C_3}{C_2 + C_3} \|\chi_1 - \chi_2\|c.
$$

This proves Lemma 3.2. □

Proof of Theorem 1.3 The proof of Theorem 1.3 follows easily from Lemma 3.1 and Lemma 3.2 by employing the contraction mapping principle. □

Acknowledgements The authors are indebted to the referee for careful reading of the paper and many nice suggestions. This paper is supported by the key project of university natural science of Anhui province (No. KJ2017A453), the University Teaching Research Foundation of Anhui province (No. 2016 jyxm0693).

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

1. Temam, R.: Sur un problème non linéaire. J. Math. Pures Appl. 48, 159–172 (1969)
2. Bona, J.L., Smith, R.: The initial value problem for the Korteweg-de Vries equation. Philos. Trans. Roy. Soc. Lond. Ser. A 278, 555–604 (1975)
3. Prilepko, A.I., Orlovsky, D.G., Vasin, I.A.: Methods for Solving Inverse Problems in Mathematical Physics. Marcel Dekker, Inc., New York (1999)
4. Fan, J., Jiang, S.: Well-posedness of an inverse problem of a time-dependent Ginzburg–Landau model for superconductivity. Commun. Math. Sci. 3(3), 393–401 (2005)
5. Fan, J., Nakamura, G.: Local solvability of an inverse problem to the density-dependent Navier–Stokes equations. Appl. Anal. 87(10–11), 1255–1265 (2008)
6. Temam, R.: Infinite-Dimensional Dynamical Systems in Mechanics and Physics, 2nd edn. Springer-Verlag, Berlin (1997)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.