GRAVITATING VORTICES WITH POSITIVE CURVATURE

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Abstract. We give a complete solution to the existence problem for gravitating vortices with positive topological constant $c > 0$. Our main result establishes the existence of solutions provided that a GIT stability condition for an effective divisor on the Riemann sphere is satisfied. To this end, we use a continuity path starting from Yang’s solution with $c = 0$, and deform the coupling constant $\alpha$ towards 0. A salient feature of our argument is a new bound $S_{\omega} \geq c$ for the curvature of gravitating vortices, which we apply to construct a limiting solution along the path via Cheeger-Gromov theory.

1. Introduction

This work is concerned with the existence of Abelian vortices on a compact Riemann surface $\Sigma$ with back-reaction on the metric. The vortex equation

$$i\Lambda_{\omega} F_h + \frac{1}{2}(|\phi|_h^2 - \tau) = 0,$$  \hspace{1cm} (1.1)

for a Hermitian metric $h$ on a line bundle $L$ over $\Sigma$ with section $\phi \in H^0(\Sigma, L)$, is a generalization of the equations on $\mathbb{R}^2$ which were introduced in 1950 by Ginzburg and Landau [15] in the theory of superconductivity. Abelian vortices have been extensively studied in the mathematics literature after the seminal work of Jaffe and Taubes [17, 29] on the Euclidean plane, and Witten [30] on the 2-dimensional Minkowski spacetime. A complete answer to the existence of solutions of (1.1) for the case of compact Riemann surfaces was established independently by Noguchi, Bradlow, and García-Prada [7, 13, 24].

Following these classical works, a question which has recently emerged is whether a solution of (1.1) produces a back-reaction on the Riemannian metric $g$ on $\Sigma$ (with Kähler form $\omega$). Back to the original motivation for the vortex equation in theoretical physics, this question is very natural, as it accounts for a mathematical explanation of gravitational effects on the vortex. A concrete proposal for gravitating vortices was put forward by the first author jointly with Álvarez-Cónsul and García-Prada in [2], in the form of the following coupled equations

$$i\Lambda_{\omega} F_h + \frac{1}{2}(|\phi|_h^2 - \tau) = 0,$$

$$S_{\omega} + \alpha(\Delta_{\omega} + \tau)(|\phi|_h^2 - \tau) = c, \hspace{1cm} (1.2)$$

where $\alpha, \tau \in \mathbb{R}$ are non-negative constants, $\Lambda_{\omega}$ is the trace operator, and the Laplacian is positive-definite by convention. The constant $c$ in the second equation in (1.2) is topological, as it is given by the following formula

$$c = \frac{2\pi(\chi(\Sigma) - 2\alpha \tau N)}{\text{Vol}_{\omega}}, \hspace{1cm} (1.3)$$

with $N = \int_{\Sigma} c_1(L)$ and $\text{Vol}_{\omega} := \int_{\Sigma} \omega$.

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The gravitating vortex equations (1.2) are fundamental, in the following sense: firstly, they admit a Hamiltonian interpretation [1, 4] akin to the existence problem for Kähler-Einstein metrics, where algebro-geometric stability obstructions appear on compact Kähler manifolds with $c_1 > 0$. Secondly, being a particular case of the Kähler-Yang-Mills equations [1], the coupled system (1.2) is motivated by the question of understanding moduli spaces for smooth polarised varieties equipped with vector bundles. In the present setup, the gravitating vortex equations provide an analytical approach for the moduli space parametrizing Riemann surfaces equipped with an effective divisor. Finally, for $c = 0$ in (1.3), the system (1.2) has a physical interpretation, because it is equivalent to the Einstein-Bogomol’nyi equations on a Riemann surface [2, 31]. Solutions of this last set of equations are known in the physics literature as Nielsen-Olesen cosmic strings [23], and describe a special class of solutions of the Abelian Higgs model coupled with gravity in four dimensions. In this setup, $\tau > 0$ is a symmetry breaking parameter in the theory, $\alpha/2\pi$ equals the gravitational constant, and $\phi$ represents physically the Higgs field.

The existence problem for gravitating vortices was first studied by Yang in the case $c = 0$ [31, 32], who proved a general existence result which shows that the location of the zeros of $\phi$ plays an important role to global existence. For $c \geq 0$, the first two authors jointly with Álvarez-Cónsul and García-Prada found a new obstruction to the existence of solutions of (2.2), establishing a relation with Geometric Invariant Theory for these equations [2, 4]. For $c < 0$, existence and uniqueness of solutions has been established in [4] in genus greater than one for a suitable range of the coupling constant $\alpha$, depending only on the topology of the surface and the line bundle. For $c > 0$, the analytical techniques in [4, 31] do not apply, and the existence problem has hitherto remained open. The main goal of the present paper is to provide a complete solution of the existence problem in the case $c > 0$.

To state our main result, we make a basic observation about the system (1.2) which plays an important role in the present work: any solution of (1.2) satisfies (see Lemma 4.9)

$$S_g \geq c.$$  \hfill (1.4)

Furthermore, by (1.3), the condition $c \geq 0$ implies that

$$\Sigma \cong \mathbb{P}^1,$$

and, in this setup, it is equivalent to the following constraint in the coupling constant

$$\alpha \in [0, \frac{1}{\tau N}].$$  \hfill (1.5)

**Theorem 1.1.** Suppose $\alpha \in (0, \frac{1}{\tau N})$ and that $\tau > 2N$ is satisfied. Let $D = \sum_j n_j p_j$ be the effective divisor on $\mathbb{P}^1$ corresponding to a pair $(L, \phi)$. Then, the gravitating vortex equations on $(\mathbb{P}^1, L, \phi)$ have solutions with coupling constant $\alpha$ provided that $D$ is GIT polystable for the canonical linearized $\text{SL}(2, \mathbb{C})$-action on the space of effective divisors.

The condition $\tau > 2N$ was first obtained in [7, 13, 24], and is equivalent to the existence of solutions of the vortex equation (1.1). Our main result is a converse of [4, Theorem 1.3], by the first two authors jointly with Álvarez-Cónsul and García-Prada, which established the GIT polystability of the divisor $D$ assuming the existence of gravitating vortices. Thus, combined with this result, Theorem 1.1 provides a complete solution of the existence problem for gravitating vortices with $c > 0$.

Applying Yang’s existence result for the case $c = 0$ ($\alpha = \frac{1}{\tau N}$) [31, 32], the statement of Theorem 1.3 holds true for the semi-open interval $(0, \frac{1}{\tau N})$. As a matter of fact, our method of proof of Theorem 1.1 exploits a continuity path starting from Yang’s solution with $\alpha = \frac{1}{\tau N}$, and deforming the coupling constant $\alpha$ towards 0. To prove in Proposition 3.1 that the existence of solutions of (1.2) is an open condition along the path, we distinguish two cases. The case when the support of the divisor $D$ has more than two points follows easily by application of the implicit function theorem (see Lemma 3.2). Our proof breaks down in the strictly polystable case (corresponding
to $D = \frac{N}{2} p_1 + \frac{N}{2} p_2$ due to the presence of symmetries. Motivated by this, in Definition 3.5 we introduce a notion of extremal pair, which provides an analogue for the gravitating vortex equations of the familiar notion of extremal metric in Kähler geometry. With this definition at hand, the proof of the strictly polystable case follows by a Lebrun-Simanca type argument [19] (see Lemma 3.10), combining the $\alpha$-Futaki invariant, introduced in [4], with a Matsushima-Lichnerowicz type theorem for the gravitating vortex equations (see [3, Theorem 3.6]).

As for closedness, the $C^0$ estimate for the Kähler potentials along the path is obstructed by the GIT polystability of the divisor $D$ [4, Theorem 1.3]. To tackle this problem, in Section 2.2 we relate (1.2) to a different set of equations that we call the Riemannian gravitating vortex equations (see Definition 2.8). These new equations get rid of the dependence on the line bundle at the cost of introducing singularities. Then, we derive an a priori $C^1$ estimate for the scalar curvature of a solution of (2.7) in Section 4.1, using which we obtain a Cheeger-Gromov limit. The diffeomorphisms involved in taking the limit are not necessarily holomorphic, but we use a slice theorem and the uniqueness of almost complex structure on $S^2$ to promote them to a sequence of holomorphic automorphisms of $\mathbb{P}^1$ (see Lemma 4.14). A delicate analysis of the Green’s functions along the sequence allows us to show that the amended Cheeger-Gromov limit is a solution of the Riemannian gravitating vortex equations. Thus, the limit divisor is polystable by [4] and must then be inside the SL(2, $\mathbb{C}$)-orbit of $D$.

It is interesting to observe that the estimates in the proof of our main result work even as we approach $\alpha \to 0$, producing a solution of the gravitating vortex equations on $\mathbb{P}^1$ with $\alpha = 0$. Since the system (1.2) decouples in this limit, [4, Theorem 1.3] does not apply and we are not able to conclude that the limiting divisor lies in the SL(2, $\mathbb{C}$)-orbit of $D$. The striking difference between the existence of gravitating vortices and the existence of (simply) vortices is that the latter does not impose any stability condition on the divisor. In the other extreme of the interval, when $\alpha \to \frac{\sqrt{3}}{\tau N}$ (and hence $c \to 0$), our estimates collapse and we are not able to provide new information about Yang’s solutions [31, 32].

We expect that the methods introduced in the present paper can be used to prove compactness of the moduli space of gravitating vortices with $c > 0$ (with moving complex structure and divisor, and fixed coupling constant $\alpha$) and the existence of a continuous surjective map from the moduli space onto the space of binary quantics $S^N \mathbb{P}^1 / \text{SL}(2, \mathbb{C})$. Consequently, for the case $c > 0$, [4, Conjecture 5.6] about the moduli space of gravitating vortices would reduce to solve the uniqueness problem. We speculate that the family of moduli spaces as $\alpha \to \frac{\sqrt{3}}{\tau N}$ may yield a method for understanding the more difficult moduli space of solutions of the Einstein-Bogomol’nyi equations (where $c = 0$), which plays a key role in the physical theory of cosmic strings [31, 32]. We leave these interesting perspectives for future investigations.

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2. THE GRAVITATING VORTEX EQUATIONS WITH $c > 0$

In this section we recall the definition of the gravitating vortex equations introduced in [2], state our main theorem, provide a Riemannian characterization of the equations, and establish a regularity result.

2.1. Gravitating vortices and main result. Let $\Sigma$ be a compact connected Riemann surface of arbitrary genus, $L$ a holomorphic line bundle over $\Sigma$, and $\phi$ a global holomorphic section of $L$. We will assume that $\phi$ is not identically zero, and hence

$$ N = \int_{\Sigma} c_1(L) > 0. \quad (2.1) $$
Fix real constants $\tau > 0$ and $\alpha \geq 0$, called the \textit{symmetry breaking parameter} and the \textit{coupling constant}, respectively.

**Definition 2.1.** The gravitating vortex equations, for a Kähler metric on $\Sigma$ with Kähler form $\omega$ and a Hermitian metric $h$ on $L$, are

\begin{equation}
\begin{aligned}
i\Lambda_\omega F_h + \frac{1}{2}(|\phi|_h^2 - \tau) &= 0, \\
S_\omega + \alpha(\Delta_\omega + \tau)(|\phi|_h^2 - \tau) &= c.
\end{aligned}
\end{equation}

Solutions of these equations will be called gravitating vortices.

In (2.2), $F_h$ is the curvature 2-form of the Chern connection of $h$, $\Lambda_\omega F_h \in C^\infty(\Sigma)$ is its contraction with $\omega$, $|\phi|_h \in C^\infty(\Sigma)$ is the pointwise norm of $\phi$ with respect to $h$, $S_\omega$ is the scalar curvature of $\omega$ (as usual, Kähler metrics will be identified with their associated Kähler forms), and $\Delta_\omega$ is the Laplace operator for the metric $\omega$, given by

$$
\Delta_\omega f = 2i\Lambda_\omega \bar{\partial}\partial f,
$$

for $f \in C^\infty(\Sigma)$. Notice that in this convention, $\Delta_\omega$ equals the Hodge Laplacian $\Delta_g$ of the Riemannian metric $g$ associated to $\omega$.

The constant $c \in \mathbb{R}$ is topological, and is explicitly given by

$$
c = \frac{2\pi(\chi(\Sigma) - 2\alpha \tau N)}{\text{Vol}_\omega},
$$

with $\text{Vol}_\omega := \int_\Sigma \omega$, as can be deduced by integrating (2.2) over $\Sigma$.

Given a fixed Kähler metric $\omega$, the first equation in (2.2), that is,

$$
i\Lambda_\omega F_h + \frac{1}{2}(|\phi|_h^2 - \tau) = 0,
$$

is the vortex equation for a Hermitian metric $h$ on $L$. The existence of solutions of (2.4), often called vortices, was established independently by Noguchi, Bradlow, and García-Prada.

**Theorem 2.2** ([7, 13, 14, 24]). For every fixed Kähler form $\omega$ there exists a solution $h$ of the vortex equation (2.4) if and only if

$$
\tau \cdot \frac{\text{Vol}_\omega}{2\pi} > 2N,
$$

in which case the solution is unique.

When $\alpha > 0$, finding a solution of the vortex equation (2.4) is not enough to solve the equations in Definition 2.1. As mentioned in Section 1, the existence problem for gravitating vortices has been studied in [2, 4, 31, 32] when $c \leq 0$. The main goal of the present paper is to provide a complete solution of the existence problem in the case $c > 0$.

In order to present our main result, we state next the main results in [4, 31, 32] in a way that is useful for the present paper. Observe first that the existence of gravitating vortices for $c \geq 0$ forces the topology of the surface to be that of the 2-sphere, because $N > 0$ implies $\chi(\Sigma) > 0$ by (2.3). Thus, up to biholomorphism, we can assume $\Sigma$ to be the Riemann sphere $\mathbb{P}^1$. Consider the effective divisor $D = \sum_j n_j p_j$ determined by the pair $(L, \phi)$. In this setup, the condition $c = 0$ in (2.3) is equivalent to $\alpha = \frac{1}{\pi N}$.

**Theorem 2.3** (Yang’s Existence Theorem). Assume (2.5) and $\alpha = \frac{1}{\pi N}$. Then, there exists a solution of the gravitating vortex equations (2.2) on $(\mathbb{P}^1, L, \phi)$ if one of the following conditions holds

1. $D = \frac{N}{\tau} p_1 + \frac{N}{\tau} p_2$, where $p_1 \neq p_2$ and $N$ is even. In this case the solution admits a $T^2$-symmetry.
(2) \( n_j < \frac{N}{2} \) for all \( j \).

The \( T^2 \)-symmetry of the solution in (1) will be made more explicit in Section 3.3. As observed in [2], these conditions have a precise algebro-geometric meaning in the context of Mumford’s Geometric Invariant Theory (GIT) [20], as a consequence of the following result.

**Proposition 2.4 ([20, Ch. 4, Proposition 4.1]).** Consider the space of effective divisors on \( \mathbb{P}^1 \) with its canonical linearised \( \text{SL}(2, \mathbb{C}) \)-action. Let \( D = \sum_j n_j p_j \) be an effective divisor, for finitely many different points \( p_j \in \mathbb{P}^1 \) and integers \( n_j > 0 \) such that \( N = \sum_j n_j \). Then

1. \( D \) is stable if and only if \( n_j < \frac{N}{2} \) for all \( j \).
2. \( D \) is strictly polystable if and only if \( D = \frac{N}{2} p_1 + \frac{N}{2} p_2 \), where \( p_1 \neq p_2 \) and \( N \) is even.
3. \( D \) is unstable if and only if there exists \( p_j \in D \) such that \( n_j > \frac{N}{2} \).

In the sequel, we will refer to the canonical linearised \( \text{SL}(2, \mathbb{C}) \)-action (on \( \mathbb{P}^1 \) or \( S^N(\mathbb{P}^1) \)) simply as ‘the \( \text{SL}(2, \mathbb{C}) \)-action’. Using Proposition 2.4, we can reformulate the numerical conditions in Yang’s existence theorem as the GIT polystability of the effective divisor \( D = \sum_j n_j p_j \), where “GIT polystable” means that either conditions (1) or (2) of Proposition 2.4 are satisfied. A converse for Yang’s existence theorem was obtained in [4] for arbitrary values of the topological constant \( c \).

**Theorem 2.5 ([4]).** If \( (\mathbb{P}^1, L, \phi) \) admits a solution of the gravitating vortex equations with coupling constant \( \alpha > 0 \), then (2.5) holds and the divisor \( D \) is polystable for the \( \text{SL}(2, \mathbb{C}) \)-action.

Our main result, which we will state next, provides an extension of Yang’s existence theorem for gravitating vortices on the Riemann sphere and also a converse for Theorem 2.5 when \( c > 0 \).

**Remark 2.7.** In the weak coupling limit \( \alpha \to 0 \) the equations (2.2) decouple and, by Theorem 2.2, the existence of solutions reduces to the numerical condition (2.5). Potentially, one could use the solution with \( \alpha = 0 \) to start the continuity method and yield an alternative proof of Theorem 2.6. Nonetheless, in this limit the automorphism group of the solution ‘jumps’ and our argument does not work.

### 2.2. Regularity and the Riemannian viewpoint.

In order to address the proof of Theorem 2.6, it will be useful to look at equations (2.2) from the point of view of Riemannian geometry. This will be used for the construction of a weak solution of the equations along the continuity path using the Cheeger-Gromov convergence theory.

Consider the 2-sphere \( S^2 \) and fix an unordered tuple of points with multiplicities

\[
\sum_j n_j p_j \in S^N(S^2),
\]

and total degree \( \sum_j n_j = N > 0 \).
Definition 2.8. A solution of the Riemannian gravitating vortex equations on \((S^2, \sum_j n_j p_j)\) with coupling constant \(\alpha > 0\) is given by a triple \((g, \eta, \Phi)\) such that

1. \(g\) is a smooth Riemannian metric on \(S^2\),
2. \(\eta\) is a smooth closed real 2-form on \(S^2\) such that \(\int_{S^2} \eta = 2\pi N\),
3. \(\Phi \in C^\infty(S^2)\) is a non-negative function \(\Phi \geq 0\), called the state function, vanishing precisely at the \(p_j \in S^2\), and such that \(\log \Phi \in L^1_{loc}(S^2)\),

which satisfy the following system of equations

\[
\begin{align*}
\eta + \frac{1}{2} (\Phi - \tau) \text{vol}_g &= 0, \\
S_g + \alpha (\Delta_g + \tau) (\Phi - \tau) &= c, \\
\Delta_g \log \Phi &= (\tau - \Phi) - 4\pi \sum_j n_j \delta_{p_j}.
\end{align*}
\]

(2.7)

Here, \(\delta_{p_j}\) denotes the Dirac delta function at the point \(p_j\). The last equation has to be understood in the distributional sense. Our interest in the system (2.7) is provided by the following basic result.

Lemma 2.9. Any solution \((g, \eta, \Phi)\) of the Riemannian gravitating vortex equations (2.7) on \((S^2, \sum_j n_j p_j)\) with coupling constant \(\alpha\), determines a tuple \((\Sigma, L, \phi, \omega, h)\) (unique up to rescaling of \(\phi\) and \(h\)), where \(\Sigma = (S^2, J)\) is the Riemann surface of genus zero determined by the conformal class of \(g\), \(L\) is a holomorphic line bundle with \(\int_{\Sigma} c_1(L) = N\), \(\phi \in H^0(\Sigma, L)\) with fixed divisor \(D = \sum_j n_j p_j\), and \((\omega, h)\) is a solution of the gravitating vortex equations with coupling constant \(\alpha\) such that \(|\phi|_h^2 = \Phi\) and \(\eta = iF_h\). Conversely, any such tuple determines a solution \((g, \eta, \Phi)\) of (2.7) on \((S^2, \sum_j n_j p_j)\) with coupling constant \(\alpha\).

Proof. Given \((\Sigma, L, \phi, \omega, h)\) as in the statement, we notice that the smooth function \(|\phi|_h^2\) satisfies the Poincaré-Lelong Formula

\[i\partial \bar{\partial} \log |\phi|_h^2 = -iF_h + 2\pi \sum_j n_j [p_j],\]

where \([p_j]\) denotes the current of integration over the divisor \(p_j \in \Sigma\). Thus, \(\Phi = |\phi|_h^2\) satisfies

\[\Delta_g \log \Phi = 2\Lambda_\omega \left(i\partial \bar{\partial} \log \Phi\right) = (\tau - \Phi) - 4\pi \sum_j n_j \delta_{p_j}.
\]

However, \((\Sigma, L, \phi, \omega, h)\) and \((\Sigma, L, \varepsilon \phi, \omega, |\varepsilon|^{-2} h)\) (with \(\varepsilon \in \mathbb{C}^*\)) correspond to the same \((g, \eta, \Phi)\). Taking a holomorphic coordinate \(z\) around \(p_j\), we have that \(\log \Phi\) differs from \(\log |z|^{2n_j}\) by a smooth function, and therefore \(\log \Phi \in L^1_{loc}(S^2)\).

For the converse, we construct the tuple \((\Sigma, L, \phi, \omega, h)\) from the data \((g, \eta, \Phi)\). The orientation together with \(g\) on \(S^2\) gives a complex structure \(J\), making \(S^2\) into a Riemann surface. Take \(L\) to be the line bundle defined by the effective divisor \(\sum_j n_j p_j\), and \(\phi\) to be a defining section of this divisor (there is an ambiguity of global multiple of a nonzero phase). By the first equation in (2.7) the form \(\eta\) is of type \((1, 1)\), and furthermore it has integral \(2\pi N = 2\pi c_1(L)\), by assumption. Therefore, \(-i\eta\) can be realized as the curvature form of some Hermitian metric \(h\) on \(L\). Since \(\log \Phi\) and \(\log |\phi|_h^2\) satisfies the same distributional quation, \(\log \Phi - \log |\phi|_h^2\) is constant on \(S^2\), i.e. \(\Phi = \gamma |\phi|_h^2\). By rescaling \(h\) by a factor of \(\gamma\), we see that \(\Phi = |\phi|_h^2\). If we choose \(\varepsilon \phi\) instead of \(\phi\) as the defining section of \(L\), for \(\varepsilon \in \mathbb{C}^*\), the condition \(\Phi = |\phi|_h^2\) fixes the Hermitian metric to be \(|\varepsilon|^{-2} h\).

\[\square\]
To finish this section, we prove a regularity result for solutions of the Riemannian gravitating vortex equations, used in Section 4. Let us spell out precisely the notion of weak solution of (2.7) that we will use. With our application in mind, we assume that the conformal class of the metric solution induces the standard almost complex structure $J_0$ in $\mathbb{P}^1$. Given a Kähler metric of class $C^{1,\beta}$ on $\mathbb{P}^1$ with volume $\nu = \text{Vol}(S^2, g) = \int_{S^2} \omega$, by fixing a smooth Kähler metric $g_0$ with the same volume we can write 

$$g = e^g g_0$$

for a $C^{1,\beta}$-function on $\mathbb{P}^1$. Using the identities

$$e^g S_g = S_{g_0} + \frac{1}{2} \Delta_{g_0} \varphi, \quad e^g \Delta_g = \Delta_{g_0},$$

we can interpret the system (2.7) in the distributional sense as follows

$$\eta + \frac{1}{2}(\Phi - \tau) e^\varphi \text{vol}_{g_0} = 0,$$

$$S_{g_0} + \frac{1}{2} \Delta_{g_0} \varphi + \alpha(\Delta_{g_0} + \tau e^\varphi)(\Phi - \tau) = c e^\varphi,$$

for $(\varphi, \eta, \Phi)$ of class $C^{1,\beta}$. Our notion of weak solution is precisely in this sense.

**Lemma 2.10.** Assume that $(g, \eta, \Phi)$ is a weak solution of (2.7) of class $C^{1,\beta}$, for $0 < \beta < 1$, on $(S^2, \sum_j n_j p_j)$ with coupling constant $\alpha$, such that the Riemann surface of genus zero determined by the conformal class of $g$ is the Riemann sphere $\mathbb{P}^1 = (S^2, J_0)$. Then, $(g, \eta, \Phi)$ is a smooth solution in the sense of Definition 2.8.

**Proof.** Denote $\nu = \text{Vol}(S^2, g) = \int_{S^2} \omega$, where $\omega = \text{vol}_g = e^\varphi \text{vol}_{g_0}$ is the Kähler form of $g$. For a choice of $p_j$, consider the Green’s function $G_j$ solving the distributional equation

$$dd^c G_j = [p_j] - \frac{1}{\nu} \omega$$

with the normalization $\int_{S^2} G_j \omega = 0$ (see section 3.3). Since $\omega$ is of class $C^{1,\beta}$ on isothermal coordinates on $\mathbb{P}^1$, it follows that $G_j$ is of class $C^{3,\beta}$ away from $p_j$. Consider now $G = \sum_j n_j G_j$ solving the distributional equation

$$\Delta_{g_0} G = \frac{N}{\nu} e^\varphi - \sum_j n_j e^\varphi(p_j) \delta_{p_j},$$

(2.10)

Combining this formula with the last equation in (2.7), it follows that $v = \log \Phi - 4\pi G$ is a solution of the distributional equation

$$\Delta_{g_0} v = (\tau - \Phi) e^\varphi - 4\pi N \frac{1}{\nu} e^\varphi.$$

(2.11)

By assumption, the right hand side is of class $C^{1,\beta}$, and therefore by the Schauder estimates it follows that $v \in C^{3,\beta}(S^2)$. By construction of $G$, choosing a holomorphic coordinate $z$ on $\mathbb{P}^1$ centered at $p_j$, it follows that

$$G - \frac{1}{4\pi} \log |z|^{2n_j} \in C^{3,\beta}$$

locally around this point, and therefore $\Phi = e^{\varphi + 4\pi G} \in C^{3,\beta}(S^2)$. Considering now the second equation in (2.7), it follows from the Schauder estimates that $\varphi$ (and hence $g$) is of class $C^{3,\beta}$ and, by the first equation in (2.7), $\eta$ is also of class $C^{3,\beta}$. The proof now follows by induction, iterating the previous argument. □
3. The method of continuity and openness

In this section we introduce the continuity path that we will use for the proof of Theorem 2.6, and prove that the existence of solutions of the gravitating vortex equations is an open condition for the coupling constant, via a Lebrun-Simanca type argument.

3.1. Continuity method. Let \( L \) a holomorphic line bundle over the Riemann sphere \( \mathbb{P}^1 \) with degree \( N \) (see (2.1)), and \( \phi \) a global holomorphic section of \( L \). We assume that \( \phi \) is not identically zero, and denote by
\[ D = \sum_j n_j p_j \in S^N(\mathbb{P}^1) \]
the corresponding divisor. We consider the gravitating vortex equations (2.2) on \( (\mathbb{P}^1, L, \phi) \) with coupling constant \( \alpha \in [0, \frac{1}{\tau N}] \), symmetry breaking parameter \( \tau > 0 \), and unknowns given by pairs \( (\omega, h) \) such that \( \int_{\mathbb{P}^1} \omega = 2\pi \). As in (2.3), we define
\[ c_\alpha = 2 - 2\alpha \tau N \geq 0, \]
and further assume that (see Theorem 2.2)
\[ \tau > 2N. \tag{3.1} \]

We will use the method of continuity, where the continuity path is simply equations (2.2), with the coupling constant \( \alpha \in [0, \frac{1}{\tau N}] \) playing the role of the continuity parameter, i.e.,
\[ i\Lambda_\omega F_h + \frac{1}{2}(|\phi|^2_h - \tau) = 0, \]
\[ S_\omega + \alpha(\Delta_\omega + \tau)(|\phi|^2_h - \tau) = c_\alpha. \tag{3.2} \]

At \( \alpha = \frac{1}{\tau N} \) the equations (3.2) admit a smooth solution given by Yang’s existence theorem (see Theorem 2.3), provided that \( D \) is GIT polystable for the \( SL(2, \mathbb{C}) \)-action on the space of effective divisors. The goal of this section is to prove that the following set is open
\[ S = \{ \alpha \in (0, \frac{1}{\tau N}] \text{ such that (3.2) has a smooth solution} \}. \tag{3.3} \]

**Proposition 3.1.** Assume that \( D \) is GIT polystable for the \( SL(2, \mathbb{C}) \)-action on the space of effective divisors. Then, the set \( S \) is non-empty and open.

The proof of Proposition 3.1 has two distinguished cases, corresponding to the stable and strictly polystable cases in Theorem 2.3. The strictly polystable case requires some additional tools due to the presence of symmetries, and we postpone its proof until Section 3.3.

**Lemma 3.2.** Assume that \( D \) is GIT polystable for the \( SL(2, \mathbb{C}) \)-action on the space of effective divisors. Then, if \( \phi \) vanishes at more than two points, the set \( S \) is open.

**Proof.** Let \( \alpha > 0 \) and fix a pair \((\omega_0, h_0)\). Consider the operator
\[ T_\alpha = (T^0_\alpha, T^1_\alpha) : C^\infty(\mathbb{P}^1) \times C^\infty(\mathbb{P}^1) \to C^\infty(\mathbb{P}^1) \times C^\infty(\mathbb{P}^1), \tag{3.4} \]
given by
\[ T^0_\alpha(u, f) = i\Lambda_\omega F_h + \frac{1}{2}|\phi|^2_h - \frac{\tau}{2}, \]
\[ T^1_\alpha(u, f) = -S_\omega - \alpha \Delta_\omega |\phi|^2_h + 2\alpha \tau i\Lambda_\omega F_h, \]
where \((\omega, h) = (\omega_0 + dd^c u, e^{2f} h_0)\). By the proof of [4 Lemma 6.3], the linearization of \( T_\alpha \) at \((u, f)\) satisfies
\[ \delta T^0_\alpha(\dot{u}, \dot{f}) = L^0_\alpha(\dot{u}, \dot{f}) + J\eta_{\dot{u}}d(T^0_\alpha(u, f)), \]
\[ \delta T^1_\alpha(\dot{u}, \dot{f}) = L^1_\alpha(\dot{u}, \dot{f}) + (d(T^1_\alpha(u, f)), d\dot{u})_\omega, \]
where \( L_\alpha = L_{\alpha, u, f} = (L_\alpha^0, L_\alpha^1) \) is the linear differential operator
\[
L_\alpha : C^\infty(\mathbb{P}^1) \times C^\infty(\mathbb{P}^1) \to C^\infty(\mathbb{P}^1) \times C^\infty(\mathbb{P}^1),
\]
declared by
\[
L_\alpha^0(\dot{u}, \dot{f}) = d^*(d\dot{f} + \eta \cdot \iota F_h) + (\phi, -J\eta_A \cdot dA\phi + f\phi)_h,
\]
\[
L_\alpha^1(\dot{u}, \dot{f}) = P^* \mathbb{P} \dot{u} - 4\alpha i\Lambda_\omega(dA\phi, -J\eta_A \cdot dA\phi + f\phi)_h
\]
\[-2\alpha i\Lambda_\omega d((d\dot{f} + \eta \cdot \iota F_h)\phi)_h) + 2\alpha \tau d^*(d\dot{f} + \eta \cdot \iota F_h).
\]
Here \( P^* \mathbb{P} \) is, up to a multiplicative constant factor, the LiCHnerowicz operator of the Kähler manifold \((\mathbb{P}^1, \omega)\), \( J \) is the standard almost complex structure on \(\mathbb{P}^1\), \( A \) is the Chern connection of \( h \), and \( \eta \) denotes the \( \omega \)-Hamiltonian vector field associated to \( u \). Moreover, the operator \( L_\alpha \) satisfies (see [4, Lemma 6.3])
\[
\langle (\dot{u}, 4\alpha \dot{f}), L_\alpha(\dot{u}, \dot{f}) \rangle_{L^2} = \|L_\eta \cdot J\|_{L^2}^2 + 4\alpha \|d\dot{f} + \eta \cdot \iota F_h\|^2_{L^2} + 4\alpha \|J\eta_A \cdot dA\phi - f\phi\|^2_{L^2}
\]
\[+ 4\alpha \langle (J\eta_A \cdot (id\dot{f} + \eta \cdot \iota F_h)), T^0_\alpha(\dot{u}, \dot{f}) \rangle_{L^2},
\]
where the \( L^2 \) products are taken with respect to the fixed metric \( \omega \). Assume now that \((\omega_0, h_0)\) is a smooth solution of (3.2) for \( \alpha_0 \in (0, \frac{1}{\tau}) \). Then, the linearization of \( T_{\alpha_0} \) at \((0, 0)\) for \( \alpha = \alpha_0 \) equals
\[
\delta T_{\alpha_0} = \delta_{(0,0)} T_{\alpha_0} = L_{\alpha_0}
\]
and the operator \( L_{\alpha_0} \) is self-adjoint. To characterize the kernel of \( L_{\alpha_0} \), let \( \text{Aut}(\mathbb{P}^1, L, \phi) \) denote the group of automorphisms of \((\mathbb{P}^1, L, \phi)\), given by automorphisms of \( L \) covering an element in \( \text{Aut}(\mathbb{P}^1) = \text{PGL}(2, \mathbb{C}) \) and preserving \( \phi \). Then, formula (3.7) implies that (see [4, Lemma 6.3])
\[
A^+ \eta_h + i\dot{f} \in \text{Lie Aut}(\mathbb{P}^1, L, \phi)
\]
for any \((\dot{u}, \dot{f}) \in \ker L_{\alpha_0}\). Here, \( A^+ \eta_h \) denotes the horizontal lift of \( \eta_h \) to the total space of \( L \) using the Chern connection \( A \) of \( h_0 \) and \( 1 \) denotes the canonical vertical vector field on \( L \). Now, by assumption, \( \phi \) vanishes at more than two points and therefore element in \( \text{Aut}(\mathbb{P}^1, L, \phi) \) must project to the identity in \( \text{PGL}(2, \mathbb{C}) \). Since \( \phi \neq 0 \), it follows that \( \text{Aut}(\mathbb{P}^1, L, \phi) = \{1\} \) (see [4, Section 4.1]), and therefore
\[
\ker L_{\alpha_0} = \mathbb{R} \times \{0\}.
\]
The result now follows by application of the implicit function theorem in a Sobolev completion of \((C^\infty(X)/\mathbb{R}) \times C^\infty(X)\). Smoothness of the solution follows easily bootstrapping in (3.2). \qed

3.2. Futaki invariant and extremal pairs. In the strictly polystable case, that is, for \( D = \frac{N}{2}p_1 + \frac{N}{2}p_2 \), Theorem 2.3 implies that the solution at \( \alpha = \frac{1}{\lambda} \) has an \( T^2 \)-symmetry, and hence the proof of Lemma 3.3 does not apply. In order to address the proof of Proposition 3.3 in this case in Section 3.3, we need two theoretical devices: the \( \alpha \)-Futaki invariant for the gravitating vortex equations, introduced in [4], and a notion of extremal pair. These are analogues for the gravitating vortex equations of the familiar notions of Futaki invariant and extremal metric in Kähler geometry, respectively.

To start, let us consider the situation that \( \phi \in H^0(\mathbb{P}^1, L) \) vanishes at exactly two points. We make the identification \( L = \mathcal{O}_{\mathbb{P}^1}(N) \), with \( N := c_1(L) > 0 \), and fix homogeneous coordinates \([x_0, x_1]\) on \( \mathbb{P}^1 \) such that
\[
\phi \cong x_0^{N-\ell} x_1^\ell,
\]
with \( 0 < \ell < N \) (the case \( \ell = N/2 \) corresponds to the strictly polystable case in Theorem 2.3). Here, we identify \( H^0(\mathbb{P}^1, L) \cong \mathbb{C}^N(\mathbb{C}^2)^* \) with the space of degree \( N \) homogeneous polynomials in the coordinates \( x_0, x_1 \), so it is a \( \text{GL}(2, \mathbb{C}) \)-representation, where \( g \in \text{GL}(2, \mathbb{C}) \) maps a polynomial \( p(x_0, x_1) \) into the polynomial \( p(g^{-1}(x_0, x_1)) \). Denote by \( \rho \) the canonical \( \text{GL}(2, \mathbb{C}) \)-linearization of \( L \) induced by the \( \text{GL}(2, \mathbb{C}) \)-representation \( H^0(\mathbb{P}^1, L) \). Note that an element in the centre, \( \lambda \in \mathbb{C}^* \subset \text{GL}(2, \mathbb{C}) \), acts via \( \rho \) on \( L \) by fibrewise multiplication by \( \lambda^{-N} \).
Lemma 3.3 ([1]). Let $\phi \in H^0(\mathbb{P}^1, L)$ as in [5,8]. Then $\text{Aut}(\mathbb{P}^1, L, \phi)$ is given by the image of the standard maximal torus $\mathbb{C}^* \times \mathbb{C}^* \subset \text{GL}(2, \mathbb{C})$ under the morphism

$$\rho_\ell : \mathbb{C}^* \times \mathbb{C}^* \twoheadrightarrow \text{Aut}(\mathbb{P}^1, L)$$

defined by

$$\rho_\ell(\lambda_0, \lambda_1) = \lambda_0^{N-\ell} \lambda_1^\ell \rho(\lambda_0, \lambda_1)$$

where $\lambda_0^{N-\ell} \lambda_1^\ell$ acts on $L$ by multiplication on the fibres.

By the Matsushima-Lichnerowicz type theorem for the gravitating vortex equations (see [3, Theorem 3.6]), the group of isometries of the solution in Theorem 2.3 for $\ell = \frac{N}{2}$ corresponds to the maximal compact (see Lemma 3.3)

$$K = S^1 \times S^1 \subset \text{Aut}(\mathbb{P}^1, L, \phi).$$

Note here that the projection of $K$ onto $\text{Aut}(\mathbb{P}^1) = \text{PGL}(2, \mathbb{C})$ is a circle.

To introduce the $\alpha$-Futaki invariant, we fix $\alpha > 0$ and $\tau > 2N$. Denote by $B$ the space of pairs $(\omega, h)$ consisting of a Kähler form $\omega$ on $\mathbb{P}^1$ with volume $2\pi$, and a Hermitian metric $h$ on $L$. Define a map

$$\mathcal{F}_{\alpha, \tau} : \text{Lie Aut}(\mathbb{P}^1, L, \phi) \rightarrow \mathbb{C},$$

by the following formula, for all $y \in \text{Lie Aut}(\mathbb{P}^1, L, \phi)$, where $(\omega, h) \in B$:

$$(\mathcal{F}_{\alpha, \tau}, y) = 4i\alpha \int_{\mathbb{P}^1} A_h y \left( i\Lambda_\omega F_h + \frac{1}{2} |\phi|^2_h - \frac{\tau}{2} \right) \omega - \int_{\mathbb{P}^1} \varphi \left( S_\omega + \alpha \Delta_\omega |\phi|^2_h - 2i\alpha \tau \Lambda_\omega F_h \right) \omega.$$  (3.10)

Here, $A_h$ is the Chern connection of $h$ on $L$, $A_h y \in C^\infty(\mathbb{P}^1, i\mathbb{R})$ is the vertical projection of $y$ with respect to $A_h$, and the complex valued function $\varphi$ on $\mathbb{P}^1$ is defined as follows. Let $\tilde{y}$ be the holomorphic vector field on $\mathbb{P}^1$ covered by $y$ and $A^1_h \tilde{y}$ its horizontal lift to a vector field on the total space of $L$ given by the connection $A_h$. Therefore, $y$ has a decomposition

$$y = A_h y + A^1_h \tilde{y}$$  (3.11)

into its vertical and horizontal components. Then $\varphi := \varphi_1 + i \varphi_2 \in C^\infty(\mathbb{P}^1, \mathbb{C})$ is determined by the unique decomposition,

$$\tilde{y} = \eta_{\varphi_1} + J \eta_{\varphi_2}$$  (3.12)

associated to the Kähler form $\omega$ (see [19]), where $\eta_{\varphi_j}$ is the Hamiltonian vector field of the function $\varphi_j \in C^\infty(\mathbb{P}^1)$ (here we assume $\int_{\mathbb{P}^1} \varphi_j \omega = 0$), for $j = 1, 2$, and $J$ is the almost complex structure of $\mathbb{P}^1$. Note that the previous decomposition uses the fact that $\mathbb{P}^1$ is simply connected.

The non-vanishing of $\mathcal{F}_{\alpha, \tau}$ provides an obstruction to the existence of gravitating vortices.

Proposition 3.4 ([1]). The map $\mathcal{F}_{\alpha, \tau}$ is independent of the choice of $(\omega, h) \in B$. It is a character of the Lie algebra $\text{Lie Aut}(\mathbb{P}^1, L, \phi)$, that vanishes identically if there exists a solution of the gravitating vortex equations (2.2) on $(\mathbb{P}^1, L, \phi)$ with coupling constant $\alpha$, symmetry breaking parameter $\tau$, and volume $2\pi$.

Next, we introduce the relevant notion of extremal pair. Let $\omega$ be a Kähler form on $\mathbb{P}^1$ and $h$ a Hermitian metric on $L$. Associated with the pair $(\omega, h)$, we consider a vector field

$$\zeta_{\alpha, \tau}(\omega, h) := i (i\Lambda_\omega F_h + \frac{1}{2} |\phi|^2_h - \frac{\tau}{2})1 + A^1_h \eta_{\alpha, \tau}$$

on the total space of $L$, where $1$ denotes the canonical vertical vector field on $L$ and $\eta_{\alpha, \tau}$ is the Hamiltonian vector field of the smooth function

$$S_\omega + \alpha \Delta_\omega |\phi|^2_h - 2i\alpha \tau \Lambda_\omega F_h.$$  (3.13)

Note that the vector field $\zeta_{\alpha, \tau}(\omega, h)$ is $\mathbb{C}^*$-invariant (actually it belongs to the extended gauge group determined by $(\omega, h)$, in the sense of [4]).
Definition 3.5. The pair \((\omega, h)\) is extremal if

\[
\zeta_{\alpha, \tau}(\omega, h) \in \text{Lie Aut}(\mathbb{P}^1, L, \phi),
\]

that is, the vector field \(\zeta_{\alpha, \tau}(\omega, h)\) is holomorphic and preserves \(\phi\).

Of course, solutions of the gravitating vortex equations \((2.2)\) correspond, precisely, to extremal pairs \((\omega, h)\) such that \(\zeta_{\alpha, \tau}(\omega, h) = 0\). More generally, the existence of an extremal pair with \(\zeta_{\alpha, \tau}(\omega, h) \neq 0\) is an obstruction to the existence of solutions of the gravitating vortex equations. This follows from Proposition 3.6 because \(\zeta_{\alpha, \tau}(\omega, h) \neq 0\) implies

\[
(\mathcal{F}_{\alpha, \tau}, \zeta_{\alpha, \tau}(\omega, h)) < 0,
\]

as can be shown by applying formula \((3.10)\) using \((\omega, h)\) to \(y = \zeta_{\alpha, \tau}(\omega, h)\) (cf. [1, Proposition 4.2]). The upshot of the previous abstract discussion is the following useful result.

Proposition 3.6. Assume that the divisor \(D = (N - \ell)p_1 + \ell p_2\) induced by \(\phi\) is strictly polystable, that is, \(\ell = N^2\). Then, \(\mathcal{F}_{\alpha, \tau} = 0\) and, consequently, any extremal pair \((\omega, h)\) is a solution of the gravitating vortex equations \((2.2)\).

Proof. The vanishing of \(\mathcal{F}_{\alpha, \tau}\) follows from \([4, \text{Lemma 4.6}]\), which implies that

\[
\langle \mathcal{F}_{\alpha, \tau}, y \rangle = 2\pi i \alpha (2N - \tau)(2\ell - N),
\]

where

\[
y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \text{Lie Aut}(\mathbb{P}^1, L, \phi) \subset \mathfrak{gl}(2, \mathbb{C})
\]

(and similarly for the other generator). The statement now follows from the fact that, if \((\omega, h)\) is an extremal pair with \(\zeta_{\alpha, \tau}(\omega, h) \neq 0\), then \((3.14)\) holds. \(\square\)

To finish this section, we give an example of extremal pair which is not a solution of the gravitating vortex equations.

Example 3.7. Consider \(L = \mathcal{O}_{\mathbb{P}^1}(1)\) on \(\mathbb{P}^1\), with \(\phi = x_0\) (see \((3.8)\)). Let \(\omega_{FS}\) be the Fubini–Study Kähler metric on \(\mathbb{P}^1\), normalized so that \(\int_{\mathbb{P}^1} \omega_{FS} = 2\pi\). Consider also the Fubini–Study Hermitian metric \(h_{FS}\) on \(\mathcal{O}_{\mathbb{P}^1}(1)\). We choose coordinates \(z = \frac{x_1}{x_0}\), so that \(\phi = 1\) and

\[
\omega_{FS} = \frac{idz \wedge d\bar{z}}{(1 + |z|^2)^2}, \quad h_{FS} = \frac{1}{1 + |z|^2} = |\phi|^2_{h_{FS}}.
\]

Using now that \(i\Lambda_{\omega_{FS}} F_{h_{FS}}\) and \(S_{\omega_{FS}}\) are constant, the Hamiltonian vector field corresponding to \((3.13)\), with \(\omega = \omega_{FS}\), equals the Hamiltonian vector field of the smooth function

\[
\alpha \Delta_{\omega_{FS}} |\phi|_{h_{FS}}^2 = 2\alpha \frac{1 - |z|^2}{1 + |z|^2},
\]

which turns out to be

\[
v = 4\alpha i z \frac{\partial}{\partial z}.
\]

Using now the equalities

\[
\partial \left( i \Lambda_{\omega_{FS}} F_{h_{FS}} + \frac{1}{2} |\phi|^2_{h_{FS}} - \tau \right) = i \partial |\phi|^2_{h_{FS}} = \frac{i}{8\alpha} i_{\omega_{FS}} \omega_{FS} = -\frac{1}{8\alpha} i_{\omega_{FS}} F_{h_{FS}},
\]

it follows from \([4, \text{Lemma 4.1}]\) that \((\omega_{FS}, h_{FS})\) is an extremal pair for \((\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1), x_0)\), provided that \(\alpha = 1/8\). Taking \(2 < \tau < 8\), we have \(\alpha = 1/8 \in (0, \frac{1}{\tau})\), and therefore in this case \(c > 0\).
Remark 3.8. One can compare the definition of extremal pair for the Kähler–Yang–Mills equations in [1, Definition 4.1] with Definition 3.9 via the process of dimensional reduction described in [2, Section 3.2]. Under this comparison, the former definition corresponds to the latter only for $\alpha = 1/4$, but clearly the notion of extremal pair for the Kähler–Yang–Mills equations can be generalized by considering a modification of the vector field $\zeta_\alpha$ (see [1, (4.136)]), with the Hermite–Yang–Mills term multiplied by $a \in \mathbb{R}_{>0}$.

3.3. The strictly polystable case. We address now Proposition 3.1 in the case that $\phi$ vanishes at exactly two points, with multiplicity $N_\phi$. The proof will follow by adaptation of the LeBrun-Simanca argument for extremal Kähler metrics [19] (cf. [1]).

Let $(\omega_0, h_0)$ be a solution of the gravitating vortex equations (2.2) on $(\mathbb{P}^1, L, \phi)$ with coupling constant $\alpha_0$. As already mentioned in Section 3.2, the group of isometries of the solution corresponds to the maximal compact (see Lemma 3.3)

$$K = S^1 \times S^1 \subset \mathbb{C}^* \times \mathbb{C}^* = \text{Aut}(\mathbb{P}^1, L, \phi).$$

Note that the projection of $K$ onto PGL(2, $\mathbb{C}$) is a circle. Given $k \geq 1$, the Hilbert space of $L^2$ functions on $\mathbb{P}^1$ with $k$ distributional derivatives in $L^2$, and let

$$L^2_k(\mathbb{P}^1)^K \subset L^2_k(\mathbb{P}^1)$$

be the closed subspaces of $K$-invariant functions. Given $\alpha \in \mathbb{R}$, the maps $T_\alpha$ and $L_\alpha = L_{\alpha,0,0}$ of Lemma 3.2 induce well-defined maps:

$$T_\alpha: V \to L^2_k(\mathbb{P}^1)^K \times L^2_{k+2}(\mathbb{P}^1)^K,$$

$$L_\alpha: L^2_{k+4}(\mathbb{P}^1)^K \times L^2_{k+4}(\mathbb{P}^1)^K \to L^2_k(\mathbb{P}^1)^K \times L^2_{k+2}(\mathbb{P}^1)^K,$$

where

$$V \subset L^2_{k+4}(\mathbb{P}^1)^K \times L^2_{k+4}(\mathbb{P}^1)^K.$$ (3.18)

is a neighborhood of the origin and $T_\alpha$ is $C^1$ with Fréchet derivative at $\alpha = \alpha_0$ given by $L_{\alpha_0}$.

Let $d^*$ and $G$ be the formal adjoint of the de Rham differential and the Green operator of the Laplacian for the fixed metric $\omega_0(\cdot, J_0 \cdot)$, respectively. Then for any symplectic form $\omega$ and any $\eta$ in the Lie algebra Lie $H_\omega$ of Hamiltonian vector fields over $(\mathbb{P}^1, \omega)$ we have

$$d(Gd^*(\eta, \omega)) = \eta,\omega.$$ (3.19)

As the image of the Green operator is perpendicular to the constants, the Hamiltonian function $f = Gd^*(\eta, \omega)$ is ‘normalized’ for the volume form $\omega_0$, that is, $\int_X f\omega_0 = 0$.

For each $(u, f) \in V$, we define a linear map

$$P_{(u,f)} = (P^0_u, P^1_f): \mathbb{R} \times \text{Lie } K \to L^2_k(\mathbb{P}^1)^K \times L^2_{k+2}(\mathbb{P}^1)^K,$$

$$(t, v) \mapsto (Gd^*(p(v), \omega) + t, \theta_h v),$$ (3.20)

where $p: K \to \text{PGL}(2, \mathbb{C})$ is the natural projection, while

$$(\omega, h) = (\omega_0 + dd^*u, e^{2f}h_0).$$

The map $P_{(u,f)}$ attaches to a vector field $v \in \text{Lie } K$ its vertical part $\theta_h v$, calculates the normalized Hamiltonian function of the vector field $p(v)$ over $(X, \omega)$, and adds an extra parameter $t$ which accounts for the fact that Hamiltonian functions are only determined up to a constant.

Here is the key link between extremal pairs and the linearization of the gravitating vortex equations. The proof is analogue to the proof of [1, Lemma 4.8], and it is therefore omitted.

Lemma 3.9. Let $(u, f) \in V$.

1. $P_{(u,f)}$ is injective.
(2) If $\hat{T}_\alpha(u,f) \in \text{Im } P_{(u,f)}$, then $(\omega, h)$ is an extremal pair.
(3) $\text{Im } P_0 \subset \ker L_\alpha$, with equality if $h_0$ is a solution of the vortex equation \((2.4)\) with respect to $\omega_0$.

Let $\langle \cdot, \cdot \rangle_{\omega_0}$ be the $L^2$-inner product on $L^2_k(\mathbb{P}^1)^K \times L^2_{k+2}(\mathbb{P}^1)^K$. Arguing as in \cite{[1]} Section 4.4 it is easy to see that the orthogonal projectors onto $\text{Im } \hat{P}_{(u,f)}$, denoted $\Pi_{(u,f)}: L^2_k(\mathbb{P}^1)^K \times L^2_{k+2}(\mathbb{P}^1)^K \rightarrow L^2_k(\mathbb{P}^1)^K \times L^2_{k+2}(\mathbb{P}^1)^K$, vary smoothly with $(u,f) \in V$ and, furthermore, the origin has an open neighbourhood $V_0 \subset V$ such that the following holds (cf. \cite{[19]}, (5.3)):

$$\ker(\text{Id} - \Pi_{(u,f)}) = \ker(\text{Id} - \Pi_0) \circ (\text{Id} - \Pi_{(u,f)}).$$

(3.21)

For any pair of non-negative integers $(l,m)$, let $I_{l,m} \subset L^2_l(\mathbb{P}^1)^K \times L^2_m(\mathbb{P}^1)^K$ be the orthogonal complement of $\text{Im } P_0$. Define

$$\mathcal{W} = V_0 \cap I_{k+4,k+4}.$$

Note that, under the assumptions in the last part of Lemma \ref{lem:3.9}, the subspace $\mathcal{W}$ is perpendicular to $\ker L_\alpha$. Define a LeBrun–Simanca map \cite{[19], §5}

$$T_\alpha: \mathcal{W} \longrightarrow I_{k,k+2}$$

$$(u,f) \longmapsto (\text{Id} - \Pi_0) \circ (\text{Id} - \Pi_{(u,f)}) \circ \hat{T}_\alpha(u,f).$$

(3.22)

Given $(\hat{u}, \hat{f}) \in I_{k+4,k+4}$, the directional derivative of $T_{\alpha_0}$ at the origin in the direction $(\hat{u}, \hat{f})$ is

$$\delta_\alpha T_{\alpha_0}(\hat{u}, \hat{f}) = (\text{Id} - \Pi_0) \circ L_{\alpha_0}(\hat{u}, \hat{f}).$$

(3.23)

where we have used that $(\omega_0, h_0)$ is a solution of the gravitating vortex equations.

We can now prove the main result of the present section, which combined with Lemma \ref{lem:3.2} concludes the proof of Proposition \ref{prop:3.1}. Recall that the section $\phi$ determines an effective divisor $D$ on $\mathbb{P}^1$.

**Lemma 3.10.** Assume that $D$ is GIT polystable for the $\text{SL}(2, \mathbb{C})$-action on the space of effective divisors. Then, if $\phi$ vanishes at exactly two points the set $S$ in \((3.3)\) is open.

**Proof.** Let $(\omega_0, h_0)$ be a solution of the gravitating vortex equations \((2.2)\) on $(\mathbb{P}^1, L, \phi)$ with coupling constant $\alpha_0$. Since the map $T_\alpha$ depends linearly on $\alpha$, it can be viewed as a $C^1$ map $T: \mathbb{R}^2 \times \mathcal{W} \rightarrow I_{k,k+2}$, whose the Fréchet derivative at the origin with respect to $u$ and $f$ for $\alpha = \alpha_0$ is $\delta_\alpha T_{\alpha_0} = (\text{Id} - \Pi_0) \circ L_{\alpha_0}$, by \((3.23)\). Since $h_0$ solves the vortex equation \((2.4)\) with respect to $\omega_0$, Lemma \ref{lem:3.2} applies and $(\text{Id} - \Pi_0) \circ L_{\alpha_0}$ is an isomorphism. Therefore, by the implicit function theorem, there exists an open neighbourhood $U$ of $\alpha_0$ such that for all $\alpha \in U$ there exists a pair $(u,f) \in I_{k+4,k+4}$ such that

$$T_\alpha(u,f) \in \ker (\text{Id} - \Pi_0) \circ (\text{Id} - \Pi_{(u,f)}),$$

and therefore $T_\alpha(u,f) \in \text{Im } P_{(u,f)}$ by \((3.24)\). Hence the pair $(\omega, h)$ determined by $(u,f)$ is extremal with coupling constant $\alpha$, by Lemma \ref{lem:3.9}. Finally, $(\omega, h)$ is a solution of the gravitating vortex equations by Proposition \ref{prop:3.1} (smoothness follows by bootstrapping in \((2.2)\)). \hfill $\square$

4. A PRIORI ESTIMATES AND CLOSEDNESS

The goal of this section is to prove that the existence of solutions of the gravitating vortex equations with $c > 0$ is a closed condition for the coupling constant, concluding the proof of Theorem 2.6. For this, in the present section we shift to the Riemannian point of view in Section 2.2 using systematically Lemma 2.4.
4.1. Scalar curvature and state function estimate. In this section we establish \textit{a priori} estimates for the state function and the scalar curvature of a solution of (2.7). To this end, we first prove two basic estimates about the state function $\Phi$.

**Lemma 4.1.** Let $(g, \eta, \Phi)$ be a smooth solution of (2.7). Then

- $0 \leq \Phi \leq \tau$
- $\frac{1}{2\pi} \int_{S^2} \Phi \text{vol}_g = \tau - 2N$

**Proof.** The first item follows from [4, Lem. 6.5] combined with Lemma 2.9, and it is also transparent from the maximum principle applied to the third equation in (2.7). The second item follows immediately from integrating the first equation in (2.7). \hfill $\square$

The following result is key for our method of proof for Theorem 2.6. In the case $c > 0$, which is most relevant for the present work, it implies that the Kähler metric of the gravitating vortices must have positive curvature.

**Proposition 4.2.** For any solution $(\eta, g, \Phi)$ to the system (2.7), the curvature of the metric is bounded from below: $S_g \geq c$, where $c$ is the topological constant defined in (2.3). As a consequence, the first eigenvalue $\lambda_1(\Delta_g) \geq c$.

**Proof.** It is derived by simply combining the first and third equations in (2.7). Since

$$\tau - \Phi = \Delta_g \log \Phi = \frac{\Delta_g \Phi}{\Phi} + \frac{\left| \nabla \Phi \right|^2}{\Phi^2}$$

away from the vanishing points $\{p_j\}$ of $\Phi$, we have

$$S_g = c + \alpha \tau (\tau - \Phi) - \alpha \Delta_g \Phi$$

$$= c + \alpha \frac{\left| \nabla \Phi \right|^2}{\Phi} + \alpha \tau (\tau - \Phi) - \alpha \Phi (\tau - \Phi)$$

$$= c + \alpha \frac{\left| \nabla \Phi \right|^2}{\Phi} + \alpha (\tau - \Phi)^2 \quad (4.1)$$

Because the terms $S_g, c, \alpha (\tau - \Phi)^2$ are smooth functions on $S^2$, it follows the term $\frac{\left| \nabla \Phi \right|^2}{\Phi}$ is also smooth (and obviously nonnegative) on $S^2$. The proposition follows. \hfill $\square$

The proof of the previous result singles out the term $\frac{\left| \nabla \Phi \right|^2}{\Phi}$. Our next task is to obtain an estimate for this quantity. For this, it will be useful to have a more geometric characterization in terms of the original unknowns for the gravitating vortex equations (see Lemma 2.9).

**Lemma 4.3.** With the notation of Lemma 2.9,

$$\frac{\left| \nabla \Phi \right|^2}{\Phi} = |\nabla^{1,0} \phi|^2.$$

**Proof.** Choose a local holomorphic coordinate $z$ and a local holomorphic frame $e$ of $L$, so that $\phi = f e$. Then, $\Phi = |f|^2 h$ where $h = h(e, e)$ is the Hermitian metric. Then $\Phi_z = f h (f_z + f (\log h)_z)$, thus

$$\left| \nabla \Phi \right|^2 = |f|^2 h^2 |f_z + f (\log h)_z|^2 g^{zz} = \Phi |\nabla^{1,0} \phi|^2.$$

\hfill $\square$
It is interesting to observe that the quantity $|\nabla^{1,0}\phi|^2$ appears very naturally in the moment map interpretation of the gravitating vortex equations, for the action of the extended gauge group (see [21, Formula (3.8)]). In fact, the infinite-dimensional symplectic geometry singles out

$$S_\omega - \alpha|\nabla^{1,0}\phi|^2_h - \alpha r\Lambda_\omega F_h = c',$$

with $c' \in \mathbb{R}$, as the second equation in [222] (of course, the corresponding systems of PDE are equivalent). In the physical case $c = 0$, the function $|\nabla^{1,0}\phi|^2$ appears in the trace of the stress-energy tensor of the Einstein field equations (see [31]).

To obtain an estimate for $|\nabla^{1,0}\phi|$, let us recall a general formula from Kähler geometry. For the benefit of the reader, a short proof is given.

**Proposition 4.4** (Weitzenböck Formula). Let $(M, \omega)$ be a Kähler manifold, and $L \to M$ be a holomorphic line bundle equipped with a Hermitian metric $h$ and a holomorphic section $s$. Then $\nabla s = \nabla^{1,0}s$ satisfies the following equality:

$$-\Delta|\nabla s|^2 = |\nabla\nabla s|^2 + |F_h|^2|s|^2 + (\text{Ric}_\omega - 2iF_h)(\nabla s, \nabla s) - |\nabla s|^2\Lambda_\omega iF_h - 2\text{Re} \left\langle \nabla(\Lambda_\omega iF_h), \nabla|s|^2 \right\rangle$$

**Proof.** Let $e$ be a local holomorphic frame of $L$, and let $s = fe$, suppose $h(e, e) = h$, then $|s|^2 = |f|^2h$, and $\nabla s = (\partial f + fh^{-1}\partial h)e$, thus

$$|\nabla s|^2 = f_i\bar{f}_jg^{ij}h + f_i\bar{f}_h g^{ij} + f_i\bar{f}_h g^{ij}$$

Using K-coordinates for the closed $(1,1)$-form $iF_h$ illustrated in [27, Prop. 2.2] for simplicity of calculation at a point, we get

$$-\Delta|\nabla s|^2 = f_{ik\bar{j}}g^{ij}h + f_i\bar{f}_j g^{ij} + f_i\bar{f}_h g^{ij}$$

Denote $iF_h = i\Theta_{k\bar{l}}dz_i \wedge d\bar{z}_j$, we know $h_{k\bar{l}} = -h\Theta_{k\bar{l}}$ and $h_{k\bar{l}} = -h\Theta_{k\bar{l}}$, and $g^{k\bar{l}}\Theta_{k\bar{l}} = (\Lambda_\omega iF_h)$. Plugging those terms in the above expression yields the formula in the proposition.

Assume now that $(g, \eta, \Phi)$ is a smooth solution of (2.7), and take $(\omega, h)$ as in Lemma 2.9. Then, using that $\text{Ric}_\omega = S_\omega$, $s = \phi$ and $iF_h = \eta = \frac{1}{2}(\tau - \Phi)\omega$, we derive the following:

**Proposition 4.5.** Given a smooth solution $(\eta, g, \Phi)$ of the system (2.7),

$$-\Delta g|\nabla^{1,0}\phi|^2 = |\nabla\nabla\Phi|^2 + \frac{1}{4}\phi(\tau - \Phi)^2 + \left(\alpha|\nabla^{1,0}\phi|^2 + c + (\tau - \Phi) \left(\alpha(\tau - \Phi) - \frac{3}{2}\right) + \Phi \right)|\nabla^{1,0}\phi|^2$$

**Proof.** Directly plugging in equation (1.3), we get

$$-\Delta g|\nabla^{1,0}\phi|^2 = |\nabla\nabla\Phi|^2 + \frac{1}{4}\phi(\tau - \Phi)^2 + (S_g - (\tau - \Phi))|\nabla^{1,0}\phi|^2|\nabla^{1,0}\phi|^2 + |\nabla\Phi|^2$$

(4.3)

The proposition follows by replacing $S_g$ using formula (1.1).

We are ready to prove our estimate for $|\nabla\Phi|^2$.

**Corollary 4.6** (derivative estimate). For any solution $(\eta, g, \Phi)$ of the system (2.7), we have

$$\frac{|\nabla\Phi|^2}{\Phi} \leq \frac{1}{\alpha} \left(\frac{3\tau}{2} - c\right)$$
Proof. By Proposition 4.5 at the maximum of \( b = |\nabla^{1,0}\phi|^2 \) we have
\[
0 \geq -\Delta_g b \geq \left( ab + c + (\tau - \Phi) \left[ \alpha(\tau - \Phi) - \frac{3}{2} \right] + \Phi \right) b \]
and therefore, since \( b \geq 0 \),
\[
b \leq \alpha^{-1}(-c + (\tau - \Phi)(3/2 - \alpha(\tau - \Phi))) - \Phi) \leq \frac{3\tau}{2} - c.
\]
The proof follows from Lemma 4.3. \( \square \)

Remark 4.7. Observe that existence of solutions of (2.7) implies that \( \tau > 2N > 2 \) (see Lemma 2.9, Theorem 2.2, and Theorem 2.5). Combined with the condition \( \alpha > 0 \), it implies that \( \tau > \frac{2}{\pi c} \).

The combination of equation (4.1) and Corollary 4.6 implies the main result of this section.

Theorem 4.8. For any solution \((\eta, g, \Phi)\) of the system (2.7), we have:

- (scalar curvature estimate)
  \[ c \leq S_g \leq \frac{(3 + 2\alpha\tau)\tau}{2} \]

- (state function estimate)
  \[
  -\frac{\tau^2}{4} \leq -\Delta_g \Phi = \frac{|\nabla\Phi|^2}{\Phi} - \Phi(\tau - \Phi) \leq \frac{1}{\alpha} \left( \frac{3\tau}{2} - c \right)
  \]  \hspace{1cm} (4.4)

To finish this section, we prove a higher order estimate for the scalar curvature of the conformal metric \( k = e^{2\alpha\Phi} g \). For this, note that we have the formula
\[
S_k = e^{-2\alpha\Phi}(S + \alpha\Delta_g \Phi) = e^{-2\alpha\Phi}(c + \alpha\tau(\tau - \Phi)).
\]  \hspace{1cm} (4.5)

Lemma 4.9. For any solution \((\eta, g, \Phi)\) of the system (2.7), we have:

- \( ce^{-2\alpha\tau} \leq S_k \leq c + \alpha\tau^2 \),
- \( |\nabla_k S_k|_k^2 \leq \frac{3}{2} \alpha\tau^2 (2c + 2\alpha\tau^2 + \tau)^2 \).

Proof. The proof follows from Lemma 4.1 and Corollary 4.6, combined with
\[
|\nabla_k S_k|_k^2 = e^{-2\alpha\Phi}|dS_k|_g^2 = \alpha^2 (2c + 2\alpha\tau(\tau - \Phi) + 2)^2 e^{-6\alpha\Phi} |\nabla g \Phi|_g^2 \leq \frac{3}{2} \alpha\tau^2 (2c + 2\alpha\tau^2 + \tau)^2.
\]  \hspace{1cm} (4.6)

\( \square \)

4.2. Cheeger-Gromov convergence for gravitating vortices. Let us take a sequence \((\omega_n, h_n)\) of solutions of the gravitating vortex equations with coupling constant
\[
\alpha_n \to \alpha \in (0, \frac{1}{\pi N})
\]
and fixed volume \( \text{Vol}_{\omega_n} = 2\pi \). Let \((\eta_n, g_n, \Phi_n)\) be the corresponding solution to Riemannian gravitating vortex equations (2.7). Our next goal is to construct a limiting solution \((g'_\infty, \eta'_\infty, \Phi'_\infty)\) of (2.7), as we approach the boundary of our continuity path (4.2). In this section, we start by constructing a limiting metric \( k'_\infty \) which, as we will see in Section 4.4, is related to \( g'_\infty \) by conformal rescaling. Firstly, let us study the compactness of the family \( g_n \) by using the \textit{a priori} estimates proved in Section 4.1. Let \( k_n = e^{2\alpha_n\Phi_n} g_n \) be the auxiliary Riemannian metric, for which the curvature and covariant derivative of the curvature are uniformly bounded by Lemma 4.9. Without further explanation, for the simplicity of the statements, any of the convergence below is subsequential convergence.
Lemma 4.10. By Cheeger-Gromov compactness, there exist a sequence of diffeomorphisms \( \varphi_n \) on \( S^2 \) such that \( \varphi_n^* k_n \to k_\infty \) in \( C^{2,\beta} \) as tensors as \( n \to \infty \).

Proof. By Lemma 4.1, the volume is bounded along the sequence, as
\[
2\pi \leq \text{Vol}_{k_n} \leq 2\pi e^{-2\alpha_i},
\]
and the diameter of \( k_n \) is bounded from above by Bonnet’s diameter estimate \[24\] Chapter 6, Section 4, Lemma 21]:
\[
\text{diam}(S^2, k_n) \leq \frac{\pi}{\sqrt{c_n e^{-2\alpha_i}}} =: D_n.
\]
Furthermore, using the Relative Volume Comparison theorem \[25\] Chapter 9, Section 1, Lemma 36], by comparing \( k_n \) with \( \mathbb{R}^2 \), the volume ratio is bounded from below:
\[
\frac{\text{Vol}_{k_n} B(p, r)}{\pi r^2} \geq \frac{\text{Vol}_{k_n} B(p, D_n)}{\pi D_n^2} \geq \frac{2}{D_n^2}, \quad \forall p \in S^2, r \in (0, D_n],
\]
and thus Cheeger-Gromov-Taylor’s Theorem \[8\] Theorem 4.7] gives an estimate on the lower bound of the injectivity radius:
\[
\text{inj}(S^2, k_n) \geq i_0
\]
for \( i_0 > 0 \) independent of \( n \). By Theorem 4.8 and Lemma 4.9, we have uniform bounds for \( S_k \) and \( |\nabla k_n, S_{k_n} k_n| \) along the sequence, and therefore the statement follows by Cheeger-Gromov compactness \[16\]. \( \square \)

Remark 4.11. Notice that, in the previous proof, \( i_0 \) could go to 0 if \( c \) goes to 0, so the estimate here collapses in the case \( c \to 0 \). However, ‘openness’ at \( \alpha = \frac{1}{4\pi} \) (see Proposition 3.1) means that we only need to restrict to the case \( c \geq c' > 0 \).

The key disadvantage of Lemma 4.10 is that the \( \varphi_n \) are only known to be diffeomorphisms of \( S^2 \), which might not respect the almost complex structure \( J_0 \) on \( \mathbb{P}^1 \) (in particular, the limit metric \( k_\infty \) may not be compatible with \( J_0 \)). Thus, even if we prove that Cheeger-Gromov convergence provides a limiting solution of \[22\], the points \( p_j \) move along the sequence to \( p_{j,n} = \varphi_n^{-1}(p_j) \), and we lose track of the divisor \( D = \sum n_j p_j \) as \( n \to \infty \). To remedy this shortcoming, we prove that the sequence of diffeomorphisms in Cheeger-Gromov convergence can be chosen to be holomorphic on \( \mathbb{P}^1 \). To achieve this goal, we need some preparatory material, which is probably folklore, but does not seem to appear in the existing literature.

Proposition 4.12. Any \( C^{k,\beta} \) \( (k \in \mathbb{N}_+, \beta \in (0,1) ) \) almost complex structure on \( S^2 \) is the pull-back of the standard almost complex structure by a \( C^{k+1,\beta} \) diffeomorphism.

Proof. Let \( J_0 \) be the standard almost complex structure on \( S^2 \), i.e. \( (S^2, J_0) = \mathbb{P}^1 \) and let \( a_0 \) be the smooth structure underlying \( J_0 \). Let \( g \) be a \( C^{k,\beta} \) \( (k \geq 0, \beta \in (0,1)) \) metric on \( (S^2, a_0) \) defining the given \( C^{k,\beta} \) almost complex structure \( J \). By Chern-Bers’ Theorem \[5\] \[9\], there exists an atlas of isothermal coordinates for \( J \) (obtained by solving the Beltrami equation) which is of class \( C^{k+1,\beta} \) with respect to \( a_0 \). The Uniformisation Theorem implies there exists a biholomorphic map \( f : (S^2, J) \to (S^2, J_0) \), i.e. \( f^* J_0 = J \). The map \( f \) is of course \( C^\infty \) with respect to the smooth structures of the left and the right manifolds, but only of class \( C^{k+1,\beta} \) with respect to \( a_0 \). \( \square \)

Our next goal is to prove the following result. We are grateful to Yohsuke Imagi for conversations about the proof.

Lemma 4.13. Let \( J_i \) be a sequence of \( C^{2,\beta} \) almost complex structures on \( S^2 \) converging in \( C^{2,\beta} \) sense to another almost complex structure \( J \) on \( S^2 \). Then, there exists a sequence of \( C^{3,\beta} \) diffeomorphisms \( f_i : S^2 \to S^2 \) such that
Proposition 4.12. By hypothesis, $J$

Proof. Let us denote by $\Sigma$ the Riemann surface $(S^2, J)$. We use the notation in the proof of Proposition 4.12. By hypothesis, $J$ is of class $C^{2,\beta}$ on $(S^2, a_0)$. We shall work in the smooth structure $a$ induced by $J$, where the almost complex structure is a smooth tensor. Consider the elliptic operator

$$\Omega^0(T^{1,0}\Sigma) \xrightarrow{\partial} \Omega^0(1,0\Sigma). \quad (4.7)$$

Since $\Sigma$ has genus zero we have that $H^0(1,0\Sigma) = 0$, and therefore by Hodge theory

$$\Omega^0(1,0\Sigma) = \text{Im} \bar{\partial} \oplus H^0(T^{1,0}\Sigma) = \text{Im} \bar{\partial}.$$  

Thus, (4.7) is an isomorphism. Here we have used that $\ker \bar{\partial}^* = H^0(T^{1,0}\Sigma)$ and $\ker \bar{\partial} = \Omega^0(1,0\Sigma)$, which follows by dimensional reasons.

Let $J$ be the space of almost complex structures on $\Sigma$, whose tangent space at $0$ with $\Omega = J^*\Omega$. Consider the $C^1$ map between Banach spaces (for the $C^{2,\beta}$ completions)

$$\Omega^0(T^{1,0}\Sigma) \longrightarrow J \quad (4.8)$$

where $f_v$ denotes the flow of $v$ (identified with a real vector field on $\Sigma$) at time 1. Then, the Fréchet differential of (4.8) coincides with (4.7), and therefore by the inverse function theorem it is invertible. Given now our sequence $J_i$ as in the statement, by taking isothermal coordinates for $J$ of class $C^{3,\beta}$ on $(S^2, a_0)$, we obtain that the $J_i$ are also of class $C^{2,\beta}$ on $\Sigma$, and converge to $J$ in $C^{2,\beta}$ sense. Then, there exists a sequence of $C^{2,\beta}$ vector fields $v_i$ on $\Sigma$ converging to 0, such that

$$J_i = f_{v_i}^*J.$$  

By standard ODE theory, the diffeomorphisms $f_{v_i}$ are of class $C^{3,\beta}$ on $\Sigma$. Taking isothermal coordinates for $J$ of class $C^{3,\beta}$ on $(S^2, a_0)$, it follows that the $f_{v_i}$ satisfy the conditions in the statement. \[\square\]

We are now ready to prove the main result of this section, which provides an amended Cheeger-Gromov convergence in Lemma 4.14. We will see, the family of diffeomorphism $\varphi_n$ in the Cheeger-Gromov convergence in Lemma 4.10 differs from a family of holomorphic automorphism $\sigma_n \in SL(2, \mathbb{C})$ by some $C^{3,\beta}$ controlled diffeomorphisms. We use the same notation as in Lemma 4.10.

Lemma 4.14. There exist a sequence $\sigma_n \in SL(2, \mathbb{C})$ and a $C^{2,\beta}$ Kähler metric $k'_\infty$ on $\mathbb{P}^1$ with volume $2\pi$, such that $\sigma_n^*k_n := k'_n \rightarrow k'_\infty$ in $C^{2,\beta}$ as tensors as $n \rightarrow \infty$.

Proof. By assumption, the almost complex structure $J_{g_n}$ is fixed to be $J_0$ along the sequence. Let $\varphi_n$ be the sequence of diffeomorphisms in Lemma 4.10 which satisfy

$$k_n = \varphi_n^*k_n \rightarrow_{C^{2,\beta}} k'_\infty \quad (4.9)$$

$$\varphi_n^*J_n = \varphi_n^*J_0 \rightarrow_{C^{2,\beta}} J'_\infty.$$  

By Proposition 4.12 and Lemma 4.13, there exists $C^{3,\beta}$ diffeomorphisms $\psi_n$ and $f$ on $S^2$ such that

$$\psi_n^*\varphi_n^*J_0 = J'_\infty = f^*J_0$$

$$\psi_n \rightarrow_{C^{3,\beta}} \text{Id}.$$  

(4.10)

We conclude that

$$\tilde{\sigma}_n := \varphi_n \circ \psi_n \circ f^{-1} \in \text{Aut}(S^2, J_0) = \text{PGL}(2, \mathbb{C}).$$  

(4.11)
Let $\Sigma, J$ be a compact Riemann surface, let $\omega_0$ be a Kähler form on $\Sigma$ with volume $\nu$, and $dd^c = -dJd = 2i\partial\bar{\partial}$, then there exists a (uniquely determined) Green’s function $G_{\omega_0}(\cdot, \cdot)$ satisfying

$$dd^c G_{\omega_0}(\cdot, Q) = [Q] - \frac{1}{\nu} \omega_0,$$

(4.12)

for all $Q \in \Sigma$. We notice that locally the Green’s function is asymptotic to $\frac{1}{n} \log |z|^2$ (the Green’s function on $\mathbb{C}$) where $z$ is a local holomorphic coordinate of $\Sigma$ centered at $Q$. The dependence of $G_\omega$ on the Kähler form $\omega$ in a fixed cohomology class is as follows: if we take another Kähler metric $\omega = \omega_0 + dd^c \lambda$ with the normalization $\int_{\Sigma} \lambda(\omega + \omega_0) = 0$, then [18] Proposition 1.3, Chapter II shows

$$G_\omega(P, Q) = G_{\omega_0}(P, Q) - \frac{1}{\nu} \left( \lambda(P) + \lambda(Q) \right)$$

(4.13)

As a consequence

$$\sup_{P, Q \in \Sigma} G_\omega \leq \sup_{P, Q \in \Sigma} G_{\omega_0} + 2 \sup_{P \in \Sigma} \lambda$$

Proposition 4.15. Let $(\Sigma, J_0, \omega_0)$ be a Riemann surface with a $C^3$ Kähler metric $\omega_0$, for $\Lambda > 1$ define

$$K_\Lambda = \{ \omega = \omega_0 + dd^c \lambda | \lambda \in C^{2, \beta}, \Lambda^{-1} \omega_0 \leq \omega \leq \Lambda \omega_0 \}$$

(1) Then there exists a constant $K = K(\omega_0, \beta, \Lambda) > 0$ such that $\forall \omega \in K_\Lambda$,

$$\sup_{P, Q \in \Sigma} G_\omega \leq K$$

(2) If $\omega_i \in [\omega_0]$ converges to $\omega_0$ in $C^3$ sense, and $Q_i \rightarrow Q$, then $G_{\omega_i}(\cdot, Q_i)$ converges in $C^{2, \beta}$ sense to $G_{\omega_0}(\cdot, Q)$ on any compact subset away from $Q$.

Proof. (1). For $\omega = \omega_0 + dd^c \lambda \in K_\Lambda$ with the normalization $\int_{\Sigma} \lambda \omega_0 = 0$, a standard elliptic estimate shows $||\lambda||_{L^\infty} \leq C(\omega_0)(\Lambda - 1)$. Since $\lambda = \lambda' - \frac{1}{2\nu} \int_{\Sigma} \lambda \omega$ satisfies the normalization condition $\int_{\Sigma} \lambda(\omega_0 + \omega) = 0$, we have

$$||\lambda||_{C^0} \leq \frac{1}{2} C(\Lambda, \omega_0).$$

Therefore, the desired upper bound of $G_\omega$ is obtained then since $G_{\omega_0}$ is bounded above.

(2). In the case $\omega_i = \omega_0 + dd^c \lambda_i'$ satisfies $||\omega_i - \omega_0||_{C^{2, \beta}} \rightarrow 0$, using the normalization $\int_{\Sigma} \lambda_i' \omega_0 = 0$,

$$||\lambda_i'||_{C^{2, \beta}} \leq C(\omega_0)||\omega_i - \omega_0||_{C^{2, \beta}} \rightarrow 0.$$
Therefore, the $C^{2,\beta}$ convergence away from $Q$ following from $||\lambda''||_{C^{2,\beta}} \to 0$ and the convergence property of $G_{w_0}$ under the convergence of its poles.

We go back now to the situation of our interest. With the notation of Lemma 4.14 we want to control the function $\log \Phi_n'$, for $\Phi_n' = \sigma_n' \Phi_n$, as $n \to \infty$. We will use the structural equation

$$\Delta_k \log \Phi_n' = (\tau - \Phi_n') e^{-2\alpha_n \Phi_n} - 4\pi \sum_j n_j \delta_{p_j,n}',$$  

(4.14)

which can be easily derived from the last equation in (2.7). For simplicity, denote $\nu_n' = \text{Vol}_{k_n'}$ and $p_j,n'$ = $\sigma_n^{-1}(p_j)$. By Lemma 4.1, $2\pi \leq \nu_n' \leq 2\pi e^{2\alpha_n \tau}$. Consider the solution $G_n'$ of the following Green’s function equation

$$\Delta_{k_n} G_n' = \frac{N}{\nu_n'} - \sum_j n_j \delta_{p_j,n'},$$  

(4.15)

given by by summing up the Green’s functions with a simple pole $G_{\omega_{k_n}}(\cdot, p_j,n') := G_{n,j}'$, i.e.

$$G_n' = \sum_j n_j G_{n,j}',$$

where $G_{n,j}'$ satisfies

$$dd^c G_{n,j}' = [p_j,n'] - \frac{1}{\nu_n'} \omega_{k_n'}$$  

(4.16)

with the normalization condition $\int_{\mathbb{P}^1} G_{n,j}' \omega_{k_n'} = 0$ for each $j$ (see Equation (4.12)). Taking the difference of the equations (4.14) and (4.15) and denoting $\nu_n' = \log \Phi_n' - 4\pi G_n'$, we obtain

$$\Delta_{k_n} \nu_n' = (\tau - \Phi_n') e^{-2\alpha_n \Phi_n} - \frac{4\pi N}{\nu_n'}.$$  

(4.17)

The uniform $C^0$ bound of the ‘right hand side’ given by Lemma 4.1, together with the $C^{2,\beta}$ bounded coefficient of $\Delta_{k_n}$ actually enable us to get the estimate

$$C_1 \leq \nu_n' - \frac{1}{\nu_n'} \int_{\mathbb{P}^1} \nu_n' \text{vol}_{k_n'} \leq C_2.$$  

(4.18)

This bound, in particular, implies that the oscillation of $\nu_n' = \log \Phi_n' - 4\pi G_n'$ is bounded above by $C_2 - C_1$. To get the estimate on $\nu_n'$ and $\Phi_n'$, we need some estimates on $G_n'$ (which resembles the singularities of $\Phi_n'$).

Now we pass to a subsequence (still use $n$ as its index) such that $p_j,n' = \sigma_n^{-1}(p_j) \to p_j,\infty$ (where the $p_j,\infty$'s need not be distinct), and we denote $D_\infty' = \sum_j n_j p_j,\infty$. Let $G_\infty'$ be the Green’s function for the divisor $D_\infty'$ under the limit $C^{2,\beta}$ metric $k_\infty'$ and $\nu_\infty' = \text{Vol}_{k_\infty'}$, i.e.

$$\Delta_{k_\infty'} G_\infty' = \frac{N}{\nu_\infty'} - \sum_j n_j \delta_{p_j,\infty}.$$  

(4.19)

then we have the following convergence:

**Proposition 4.16** (estimates on Green’s functions).

1. There exists $K' > 0$ (independent of $n$) such that

   $$\sup_{\mathbb{P}^1} G_n' \leq K'.$$

2. $G_n'$ converges to $G_\infty'$ in $C^{2,\beta}$ sense on any compact subset $\Omega \subset \mathbb{P}^1 \setminus D_\infty'$.

1Actually, Remark 3.3 of [8] gives an explicit upper bound: $G_n' \leq 24\pi^{diam^2(p_j, k_n')} / \text{vol}(p_j, k_n')$. 
Proposition 4.17. There exists constant $C_5 > 0$ such that
\[ |\log \Phi'_n - 4\pi G'_n|_{C^0(P^1)} \leq C_5. \]

Proof. By Lemma 4.1 there exists $x_0 \in P^1$ such that $\Phi'_n(x_0) = \tau - 2N$. Therefore, it follows that for all $x \in P^1$,
\[ \log \Phi'_n(x) - 4\pi G'_n(x) \geq \log \Phi'_n(x_0) - 4\pi G'_n(x_0) - (C_2 - C_1) \]
\[ \geq \log(\tau - 2N) - 4\pi K' - (C_2 - C_1) := C_3, \]
where we use the uniform upper bound of $G'_n$ in Proposition 4.10. This inequality implies the following lower bound of $\log \Phi'_n$ in terms of $G'_n$:
\[ \log \Phi'_n \geq 4\pi G'_n + C_3. \]

Finally, the oscillation bound (4.18) together with Lemma 4.1 imply
\[ \log \Phi'_n \leq 4\pi G'_n + \log \tau + \frac{1}{\nu_n} \int_{\nu'_n} -4\pi G'_n \, \text{vol}_{k'_n} + C_2 = 4\pi G'_n + \log \tau + C_2. \]

\[ \square \]

4.4. Construction of the limit solution. We are now ready to prove that the sequence constructed in Lemma 4.14 converges to a smooth solution of the Riemannian gravitating vortex equations. Note that the points $p_j$ move to $p'_j = \sigma_n^{-1}(p_j)$. Recall our notation $D'_\infty$ before the appearance of Equation (4.19).

Proposition 4.18. The sequence $(g'_n, \eta'_n, \Phi'_n) = \sigma'_n(g_n, \eta_n, \Phi_n)$ converges (up to taking a subsequence) as $n \to \infty$ in $C^{1,\beta}$ sense to a smooth solution $(g'_\infty, \eta'_\infty, \Phi'_\infty)$ of (2.7) on $(S^2, D'_\infty)$, with coupling constant $\alpha$ and symmetry breaking parameter $\tau$.

Proof. It follows from the third equation in (2.7) that
\[ \Delta_{k'_n} \Phi'_n = -\frac{\nabla_{k'_n} \Phi'_n \cdot \nu'_n}{\Phi'_n} + \Phi'_n(\tau - \Phi'_n)e^{-2\alpha_n\Phi'_n} \]
\[ = \sigma_n^* \left( -\frac{\nabla \Phi_n \cdot \nu_n}{\Phi_n} e^{-2\alpha_n\Phi_n} + \Phi_n(\tau - \Phi_n)e^{-2\alpha_n\Phi_n} \right) \]
away from the divisor $(\sigma_n)^{-1}(D)$, and since all the terms on the left and the right are smooth, this equation holds globally on $P^1$. 

Proof. Applying Proposition 4.15 to the present situation where $k'_n = \sigma_n^* k_n$ is a family of Kähler metric (by rescaling some constant bounded above and below (more precisely, consider $\frac{2\pi}{n} k'_n$), which does not affect the estimate on Green’s function) in the same Kähler class as $k'_\infty$. We know that $k'_n$ converges to $k'_\infty$ in $C^{2,\beta}$ and thus Proposition 4.19 shows
\[ \sup_{P^1} G_{\omega^k_n} \leq K. \]

Thus follows
\[ \sup_{P^1} G'_n \leq K' := NK. \]

Moreover, according to Proposition 4.15 on any compact subset away from $D'_\infty$ the function $G'_n$ converges to $G'_\infty$ in $C^{2,\beta}$ sense.

We conclude this section with the following uniform estimate on the state function.

Proposition 4.19. There exists constant $C_5 > 0$ such that
\[ |\log \Phi'_n - 4\pi G'_n|_{C^0(P^1)} \leq C_5. \]
Since \( k'_n = \sigma'_n k_n \) converges to \( k'_\infty \) in \( C^{2,\beta} \) sense, and the ‘right hand side’ of the above equation is uniformly bounded in \( C^0 \), we conclude that \( \Phi'_n \) is uniformly bounded in \( C^{1,\beta} \) by the standard \( W^{2,p} \) estimate. Therefore, \( g'_n = e^{-2\alpha_n \Phi'_n} k'_n \) converges to \( g'_\infty \) in \( C^{1,\beta} \) sense, and \( \eta'_n = \frac{1}{2} (\tau - \Phi'_n) \text{vol}_{g'_n} \) converges to \( \eta'_\infty \) in \( C^{1,\beta} \) sense. In one sentence,

\[
\sigma'_n (g_n, \Phi_n, \eta_n) \to_{C^{1,\beta}} (g'_\infty, \Phi'_\infty, \eta'_\infty). \tag{4.26}
\]

Looking back into Equation (4.17) and using \( ||v'_n||_{C^0(\mathbb{P})} \leq C_5 \) from Proposition 4.17, we obtain that the ‘right hand side’ is uniformly bounded in the \( C^\beta \) sense on \( \mathbb{P}^1 \), and thus the Schauder estimate implies

\[
||v'_n||_{C^{2,\beta}} \leq C_6. \tag{4.27}
\]

Hence, \( v'_n \) has a \( C^{2,\beta} \) limit \( v'_\infty \) as \( n \to \infty \), and \( v'_\infty \) satisfies (in the classical sense on \( \mathbb{P}^1 \))

\[
\Delta k'_\infty v'_\infty = (\tau - \Phi'_\infty) e^{-2\alpha \Phi'_\infty} - \frac{4\pi N}{v'_\infty}. \tag{4.28}
\]

Using Proposition 4.16, we know that \( \Phi'_n = e^{\nu_n + 4\pi G'_n} \) converges to \( \Phi'_\infty = e^{\nu_\infty + 4\pi G'_\infty} \) in \( C^{2,\beta} \) on any compact subset \( \Omega \subset \mathbb{P}^1 \setminus D'_\infty \) (and the convergence is \( C^{1,\beta} \) on \( \mathbb{P}^1 \)). Combining Equation 4.28 with (4.19), we conclude \( \log \Phi'_\infty = v'_\infty + 4\pi G'_\infty \) that the equation

\[
\Delta k'_\infty \log \Phi'_\infty = (\tau - \Phi'_\infty) e^{-2\alpha \Phi'_\infty} - 4\pi \sum_j n_j \delta_{p_j,\infty}, \tag{4.29}
\]

is satisfied in the classical sense on \( \mathbb{P}^1 \setminus D'_\infty \), and in the distributional sense on \( \mathbb{P}^1 \) (see (2.41)) and likewise, for the un-rescaled metric \( g'_\infty \),

\[
\Delta g'_\infty \log \Phi'_\infty = (\tau - \Phi'_\infty) - 4\pi \sum_j n_j \delta_{p_j,\infty}. \tag{4.30}
\]

The convergence of \( \Phi'_n \) to \( \Phi'_\infty \) implies \( g'_n \) converges to \( g'_\infty \) in \( C^{2,\beta} \) sense away from \( D'_\infty \) and \( C^{1,\beta} \) sense on \( \mathbb{P}^1 \). The consequence is that \( S_{g'_n} \to \lim_{n \to \infty} S_{g'_n} \) on \( \mathbb{P}^1 \setminus D'_\infty \), and it follows simply from Equation (2.7) that on \( \mathbb{P}^1 \setminus D'_\infty \),

\[
S_{g'_\infty} + \alpha (\Delta g'_\infty + \tau) (\Phi'_\infty - \tau) = c. \tag{4.31}
\]

Moreover, the equation

\[
\eta'_\infty + \frac{1}{2} (\Phi'_\infty - \tau) \text{vol}_{g'_\infty} = 0 \tag{4.32}
\]

is satisfied in classical sense on \( \mathbb{P}^1 \). The regularities of the data is:

\[
\begin{align*}
\eta'_\infty &\in C^{1,\beta}(\mathbb{P}^1) \\
g'_\infty &\in C^{1,\beta}(\mathbb{P}^1) \cap C^{2,\beta}_{\text{loc}}(\mathbb{P}^1 \setminus D'_\infty) \\
\Phi'_\infty &\in C^{1,\beta}(\mathbb{P}^1) \cap C^{2,\beta}_{\text{loc}}(\mathbb{P}^1 \setminus D'_\infty).
\end{align*} \tag{4.33}
\]

Smoothness of the solution follows by a direct application of Lemma 2.10.

With the previous results at hand, we are ready for the proof of our main result.

**Proof of Theorem 2.6.** By Proposition 4.18, we obtain a set of smooth limit data

\[
(\eta'_\infty, g'_\infty, \Phi'_\infty, D'_\infty) = \lim_{n \to \infty} \sigma'_n (\eta_n, g_n, \Phi_n, D)
\]

□
which is solution to (2.7). By Lemma 2.3 we have a solution \((\omega_\alpha', h_\alpha')\) of (2.2) with divisor \(D_\alpha'\) on \(\mathbb{P}^1\). Here \(\omega_\alpha' = g_\alpha'(J_\alpha', \cdot)\), and \(h_\alpha'\) is the Hermitian metric with curvature form \(\eta_\alpha'\). Theorem 2.5 implies then \(D_\alpha'\) is polystable in the GIT sense. The divisor \(D_\alpha'\) is in the orbit closure of \(D = \sum_j n_j p_j \in S^N(\mathbb{P}^1)\) under the \(\text{SL}(2, \mathbb{C})\)-action on \(S^N(\mathbb{P}^1)\). Thus, uniqueness of polystable orbits inside one orbit closure (following general GIT), implies \(D_\alpha' \in \text{SL}(2, \mathbb{C}) \cdot D\) since we assume \(D\) is polystable. This veriﬁes that

\[
D_\alpha' = \sigma^*(D)
\]

for some \(\sigma \in \text{SL}(2, \mathbb{C})\). Finally, \((\sigma^{-1})^* (\omega_\alpha', h_\alpha')\) gives the solution of the gravitating vortex equation with parameters \(\alpha\), and \(\tau\) and with the holomorphic section being the deﬁning section of the divisor \(D\). Therefore, the set \(S\) in (3.3) is closed and by Proposition 3.1 is also open, and therefore \(S = (0, \frac{1}{\tau N})\).

To ﬁnish this section, we make some comments about the limit \(\alpha \to 0\). Notice that the estimates in Lemma 4.9 for \(S_k\) and \(|\nabla_k S_k| k\) are still valid for \(\alpha \to 0^+\). Actually,

\[
S_k \to 2,
\]

\[
|\nabla_k S_k| k \to 0.
\]

in \(C^0\) sense as \(\alpha \to 0^+\).

For any \(\alpha_n \to 0^+\), let \((\omega_{\alpha_n}, h_{\alpha_n})\) be a solution to the gravitating vortices equations on \((\mathbb{P}^1, L, \phi)\). Then, the argument in Lemma 4.13 applies. More precisely, there exists a sequence of automorphism \(\sigma_n \in \text{SL}(2, \mathbb{C})\) such that

\[
\sigma_n^* \omega_{k_{\alpha_n}} := \omega_{h_{\alpha_n}^0} \longrightarrow_{C^{2,\beta}} \omega_{FS}.
\]

Since \(i F_{h_n^0} \leq \frac{1}{2} (\tau - \Phi_n') \omega_{k_{\alpha_n}}\) are uniformly bounded (with respect to \(\omega_{FS}\)), the metric \(h_n^0\) differs from \(h_{\omega_{FS}}\) by some factor \(e^{2f_n^0}\) with \(f_n^0\) being \(C^{1,\beta}\) bounded (for any \(\beta \in (0, 1)\)). The sections \(\phi_n^0 := \sigma_n^* \phi \in H^0(L)\) are a family of holomorphic sections, which measured under the family of Hermitian metrics \(h_n^0 := \sigma_n^* h_{\alpha_n}\) are uniformly bounded, i.e.

\[
|\phi_n^0|^2_{h_{\alpha_n}^0} \leq \tau.
\]

Thus, the norms measured in a ﬁxed Hermitian metric \(|\phi_n^0|_{h_{\omega_{FS}}}\) are also uniformly bounded. Because of the holomorphicity, we can take a subsequence \(\phi_n^0\) such that \(\phi_n^0 \to \phi_{\infty}^0\) in \(C^\infty\) sense.

By taking subsequence we obtain that \(f_n^0 \to f_{\infty}^0\), and therefore \(h_{\infty}^0\) is a \(C^{1,\beta}\) weak solution to the vortex equation

\[
i F_{h_{\infty}^0} + \frac{1}{2} (|\phi_{\infty}^0|^2_{h_{\infty}^0} - \tau) \omega_{FS} = 0.
\]

(4.36)

Standard regularity for Abelian vortices implies that \(h_{\infty}^0\) is smooth. By the integral condition

\[
\int_{\mathbb{P}^1} (\tau - |\phi_{\infty}^0|^2_{h_{\infty}^0}) \omega_{FS} = 4\pi N
\]

and the numerical condition \(\tau > 2N\), we conclude that \(\phi_{\infty}^0\) is a nonzero section of \(L\).

The gravitating vortex equations decouple at \(\alpha = 0\) and therefore in the limit \(\alpha \to 0\) we are not able to conclude that \(D_{\infty} \in \text{SL}(2, \mathbb{C}) \cdot D\). As stated before, the striking difference between gravitating vortices and Abelian vortices is that the existence of the latter does not impose any stability condition on the divisor.
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