Non-compact Calabi-Yau Metrics from Nonlinear Realizations

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Abstract

We give a method to construct Calabi-Yau metrics on $G$-invariant vector bundles over Kähler coset spaces $G/H$ using supersymmetric nonlinear realizations with matter coupling. As a concrete example we discuss the $\mathbb{C}P^N$ model coupled with matter. The canonical line bundle is reproduced by the singlet matter and the cotangent bundle with a new non-compact Calabi-Yau metric which is not hyper-Kähler is obtained by the anti-fundamental matter.

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1 Introduction

The Einstein equation is a partial differential equation and so is very difficult to solve in general without symmetry. When the manifold has isometry large enough, it is often reduced to an algebraic equation or an ordinary differential equation (ODE) which is easy to solve. The Einstein manifolds satisfying $R_{\mu\nu} = h g_{\mu\nu}$ are solutions of the Einstein equation. For instance, homogeneous spaces (coset spaces) $G/H$ admit Einstein metrics with positive $h$, in which case the Einstein equation reduces to a set of algebraic equations.

The Calabi-Yau manifolds which are important ingredient for string compactification are Kähler-Einstein manifolds with vanishing $h$ in one of their definitions. They cannot be homogenous because homogeneous manifolds have positive $h$, so the next class expected to be solved is cohomogeneity one although the manifold is non-compact. Therefore the classification of Calabi-Yau manifolds of cohomogeneity one is important and interesting. A class is given by the Stenzel metrics on the cotangent bundles $T^* (G/H)$ with $G/H$ rank one coset spaces. In particular, in the case of $G/H = SO(N)/SO(N - 1) \simeq S^{N-1}$, it is the higher-dimensional deformed conifold, which includes the (six-dimensional) deformed conifold and the Eguchi-Hanson manifold as lower dimensional manifolds. Another class is given by the canonical line bundles over Kähler coset spaces $G/H$. For each of these models the Einstein equation has enough symmetry to be reduced to ODE.

A natural metric on a conifold different from was constructed and identified with the canonical line bundle over the quadric surface $Q^N = SO(N + 2)/[SO(N) \times U(1)]$. In this was generalized to Hermitian symmetric spaces (HSS) $G/H$ with classical groups $G$, using supersymmetric gauge theories. Generalized conifold with $E_6$ ($E_7$) symmetry, which is defined by $\Gamma_{ijk} \phi^i \phi^j = 0$ ($d_{\alpha\beta\gamma} \phi^\alpha \phi^\beta \phi^\gamma = 0$) with $\Gamma$ ($\Gamma$) the rank 3 (rank 4) symmetric tensor of $E_6$ ($E_7$) and with $\phi^i$ ($\phi^a$) the fundamental representation 27 (56), was constructed in [15]. It was identified with the canonical line bundle over the exceptional HSS $E_6/[SO(10) \times U(1)] (E_7/[E_6 \times U(1)])$. Later in [16] we directly constructed the canonical line bundle over arbitrary Kähler-Einstein coset space $M = G/H$ using the nonlinear realization method, and manifolds obtained in Refs. [12]–[15] were correctly reproduced for HSS.

In this paper, we generalize this class to $G$-invariant vector bundles over Kähler coset spaces $M = G/H$, using the nonlinear realization method with matter coupling. From the requirement of $G$-invariance, the dimensions of the vectors as fibers are restricted because they are matter fields belonging to a representation of $H$. The number of $G$-invariants...
composed of fields coincides with the cohomogeneity of the manifold. (It is also related with
the number of the so-called quasi-Nambu-Goldstone bosons [18].) When the matter belongs
to an irreducible representation of $H$, we have one $G$-invariant and the total manifold can
be cohomogeneity one, so the Ricci-flat condition is reduced to ODE to be solved. As a
concrete example, we work out the projective space $\mathbb{C}P^N$. The singlet matter reduces to
the line bundle considered previously, and matter in the anti-fundamental representation
provides the cotangent bundle $T^*\mathbb{C}P^N$ which is not hyper-Kähler. Our model gives new
finite nonlinear sigma model and therefore suggests a conformal field theory.

This paper is organized as follows. In Sec. 2, a brief review on supersymmetric nonlinear
realizations with matter coupling is given with emphasis on its relation with cohomogeneity
of the manifold. In Sec. 3, we apply this method to the $\mathbb{C}P^N$ model coupled with matter
and briefly discuss the singlet matter. A new metric on the cotangent bundle is discussed in
Sec. 4. Sec. 5 is devoted to summary and discussions. In Appendix A, we give a relation with
the hyper-Kähler Calabi metric on the cotangent bundle using the (hyper-)Kähler quotient.

## 2 Nonlinear Realizations with Matter

In this section, we review the nonlinear realization with matter coupling focusing on its
relation with our problem. It provides an easy procedure to construct the $G$-invariant metric
on $G/H$ as a nonlinear sigma model [17]. In supersymmetric nonlinear sigma models the
associated manifolds must be Kähler [19], and we need Kähler potentials $K(\varphi, \varphi^*)$ instead
of the metrics for manifestly supersymmetric Lagrangian:

\[ \mathcal{L} = \int d^4 \theta \, K(\varphi, \varphi^*) = -g_{ij}^* (\varphi, \varphi^*) \partial_\mu \varphi^i \partial^{*\mu} \varphi^{*j} + \cdots \]

with $g_{ij}^* \equiv \partial_i \partial_j K(\varphi, \varphi^*)$ the Kähler metric, where dots denote terms including fermionic
superpartners of $\varphi^i$. Here we have used the same letter $\varphi^i$ for chiral superfields and their
complex scalar components. The most general discussion for nonlinear realization on target
Kähler manifolds was given in [21] (see [22] as a review). They gave systematic method

to construct the Kähler potential invariant under $g \in G$ up to a Kähler transformation:

\[ K(\varphi, \varphi^*) \xrightarrow{g} K'(\varphi, \varphi'^*) = K(\varphi, \varphi^*) + \Lambda(g, \varphi) + \Lambda^*(g, \varphi^*) \]

The target manifolds can be compact homogeneous [23]–[28] or non-compact non-homogeneous [29]–[18].

\[ \text{Matter coupling was discussed in [21], [30], [31].} \]

\[ \text{We use the language of the four-dimensional } \mathcal{N} = 1 \text{ supersymmetry for superfields [20].} \]
We would like to construct manifolds of cohomogeneity one by adding matter to base manifolds $M$. Hence we consider compact homogenous Kähler coset spaces $M = G/H$ as base manifolds because non-compact manifolds constructed by nonlinear realizations are non-homogeneous so cohomogeneity greater than one. Compact Kähler coset spaces $G/H$ can be written as $G/H = G/[H_{s.s.} \times U(1)^r]$ with $H_{s.s.}$ the semi-simple subgroup in $H$ and $r \equiv \text{rank } G - \text{rank } H_{s.s.}$ [32]. There exists isomorphism $G/H \simeq G^C/\hat{H}$ where $G^C$ is the complexification of $G$ and $\hat{H}$ is its complex subgroup of which $H^C$ is a subgroup: $\hat{H} = H^C \oplus B$ with $B$ nilpotent generators. (We use the Calligraphic font for Lie algebras.) To construct supersymmetric Lagrangian, we use the complex coset spaces $G^C/\hat{H}$ because they are directly parameterized by complex coordinates as chiral superfields. The coset representative of $G^C/\hat{H}$ is given by $\xi = \exp(\varphi \cdot Z)$ with $\varphi^i$ chiral superfields and $Z_i$ complex generators in $G^C - \hat{H}$. It is transformed under $g \in G$ as

$$\xi \xrightarrow{g} \xi' \equiv \exp(\varphi' \cdot Z) = g\xi \hat{h}^{-1}(g, \xi), \quad (2.2)$$

where $\hat{h}' \in \hat{H}$ is a compensator needed to put $g\xi$ into an element of a coset representative as Figure 1.

**Figure 1:** The $G$-transformation law for $\xi$

The Kähler potential for Kähler $G/H$ invariant under $G$ up to a Kähler transformation is given by [21] [23] [24]

$$K = \sum_a c_a \log \det_{\eta_a} \xi^\dagger \xi, \quad (2.3)$$

with $a = 1, \cdots, r$ and $c_a$ real constants. We have used the fundamental representation for $\xi$. $\eta_a$ are projection operators satisfying $\hat{H} \eta = \hat{H} \eta$, $\eta^2 = \eta$ and $\eta^\dagger = \eta$, and $\det_{\eta}$ denotes the
determinant in the subspace projected by $\eta$. For $CP^N$ discussed in this paper in detail, the Kähler potential is just the one for the Fubini-Study metric: $K = c \log(1 + |\varphi|^2)$.

In the nonlinear realization method, the matter fields are defined as fields belonging to some representation of $H$ and their nonlinear $G$-transformation can be defined in the standard way [17]. They are usually considered as fermions in non-supersymmetric theory. In supersymmetric theories, matter also belong to chiral superfields so their bosonic degrees of freedom add non-compact directions to $M$. The total space becomes a $G$-invariant vector bundle over $M$ with bosonic matter as a fiber.

Let $\psi$ be matter chiral superfields belonging to the representation $\rho_0$ of $H$. Now we assume that this representation is irreducible. The nonlinear $G$-transformation of the matter fields is defined by

$$\psi \xrightarrow{g} \psi' = \rho_0(\tilde{h}'(g, \xi))\psi \tag{2.4}$$

using the compensator $\tilde{h}'$ in (2.2). The matter representation $(\psi, \rho_0)$ can be embedded into some representation $(\tilde{\psi}, \rho)$ of $G^C$ with simply adding zero components, like

$$\tilde{\psi} = \begin{pmatrix} \psi \\ 0 \end{pmatrix}, \quad \rho(\tilde{H}) = \begin{pmatrix} \rho_0(\tilde{H}) & \cdots \\ \vdots & \ddots & \vdots \end{pmatrix} \tag{2.5}$$

By a field redefinition using the coset representative $\xi$, matter chiral superfields $\chi$ defined by

$$\chi = \rho(\xi)\tilde{\psi} \tag{2.6}$$

is found to transform linearly under $G$:

$$\chi \xrightarrow{g} \chi' = \rho(\xi')\tilde{\psi}' = \rho(g\tilde{h}'^{-1})\rho(\tilde{h}')\tilde{\psi} = \rho(g)\chi \tag{2.7}$$

which are called the standard representation [17].

(One of) the $G$-invariant of matter fields can be found immediately from (2.4) as

$$X \equiv |\chi|^2 \equiv \chi^\dagger \chi = \tilde{\psi}^\dagger \rho(\xi^\dagger \xi)\tilde{\psi} = (\psi^\dagger, 0)\rho(\xi^\dagger \xi) \begin{pmatrix} \psi \\ 0 \end{pmatrix}. \tag{2.8}$$

The Kähler potential for the matter fields can be written as \footnote{When there exists a $G$-symmetric tensor for this representation, we have more $G$-invariants. For instance, $G = SO(N)$ has the invariant tensor $\delta_{ij}$ so real and imaginary parts of $\sum_i \chi^i \chi^i$ are also invariant. Such a case does not give us a manifold of cohomogeneity one.}

$$K_{\text{matter}} = f(X), \tag{2.9}$$
where \( f \) is an arbitrary function. The Kähler potential for the total space coupled with the matter fields can be written as
\[
K = \sum_a c_a \log \det \eta_a \xi_a^\dagger \xi_a + f(X) .
\] (2.10)
If we set \( \psi = 0 \), we have \( X = 0 \) and the Kähler potential reduces to the one of the Kähler coset \( G/H \). Therefore the total space is the vector bundle over \( G/H \) with the matter \( \psi \) fiber.

Before closing this section we make a comment on the importance of the irreducibility. If \( \rho_0 \) is reducible, it can be divided into \( n \)-irreducible sectors \( \rho_0^{(I)} \) of \( H \) with \( I(=1, \cdots , n) \) labeling each \( H \)-sector. In the same way, we can construct the matter in the standard representation \( \chi^{(I)} \). There exist at least \( n \) \( G \)-invariants \( X^{(I)} \equiv |\chi^{(I)}|^2 \). If two of \( \chi^{I} \)'s are in the same representation the inner product of them, \( Y^{(IJ)} \equiv \chi^{(I)} \chi^{(J)} \), is also invariant. The matter Kähler potential is an arbitrary function of the several variables:
\[
K_{\text{matter}} = f(X^{(1)}, \cdots , X^{(n)}; Y^{(IJ)} + Y^{(IJ)} + i Y^{(IJ)}, \cdots ) .
\] (2.11)
Since the number of \( G \)-invariants coincides with the cohomogeneity of the manifold, this case obviously gives us a manifold with cohomogeneity greater than \( n \). Therefore the Ricci-flat condition is still a partial differential equation although the number of variables is reduced.

### 3 Projective space \( \mathbb{C}P^N \) coupled with matter

We discuss \( \mathbb{C}P^N = SU(N + 1)/[SU(N) \times U(1)] = G/H \), which is parameterized by the fields \( \varphi \) belonging to the fundamental representation \( \mathbf{N} \) of \( SU(N) \). For the \( \mathbb{C}P^N \) model, the complex isotropy is given by \( \hat{H} = \left(\begin{array}{cc} U(1)^C & B \\
0 & SU(N)^C \end{array}\right) \) with \( B \) denoting \( N \) nilpotent generators. The coset representative of \( \mathbb{C}P^N \) is given by
\[
\xi = e^{\varphi Z} = \left(\begin{array}{cc} 1 & 0 \\
\varphi & 1_N \end{array}\right) , \quad \xi^{-1T} = \left(\begin{array}{cc} 1 & -\varphi^T \\
0 & 1_N \end{array}\right) .
\] (3.1)
As the matter fields coupled with \( \mathbb{C}P^N \), we can consider the singlet \( 1 \), the fundamental representation \( \mathbf{N} \) or the anti-fundamental representation \( \overline{\mathbf{N}} \) of \( SU(N) \) with suitable charges for \( U(1) \subset H \) immediately.\(^3\) The total space is the vector bundle over \( \mathbb{C}P^N \) with the matter fields as a fiber. The Kähler potential for the total space can be written as (the projection \( \eta \) in Eq. (2.3) is given by \( \eta = \text{diag} , (1, 0, \cdots , 0). )
\[
K = c \log (1 + |\varphi|^2) + f(X) ,
\] (3.2)
\(^3\)Of course, we can consider higher representations of \( SU(N) \).
where the $G$-invariant $X$ is constructed as \( \text{(2.8)} \) using the matter fields belonging to some representation of $H$. Here we summarize the matter coupling belonging to the following representations of $SU(N)$, 1) $\sigma \in 1$, 2) $\psi \in N$ and 3) $\psi \in \overline{N}$:

1. $\sigma \in 1$.

In this case, the matter field $\sigma$ can be promoted to the standard representation as

\[
\chi = \xi \begin{pmatrix} \sigma \\ 0 \end{pmatrix} = \sigma \begin{pmatrix} 1 \\ \varphi \end{pmatrix},
\]
and the invariant is calculated as

\[
X = |\chi|^2 = |\sigma|^2(1 + |\varphi|^2).
\]

This provides the (canonical) line bundle over $\mathbb{C}P^N$ \cite{10, 11, 15, 16}.

2. $\psi \in N$.

The standard representation is given by

\[
\chi = \rho(\xi) \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \xi \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \psi \end{pmatrix}.
\]

Hence the invariant $X = |\psi|^2$ is trivial, and this gives just the direct product of $\mathbb{C}P^N$ and the space of $\psi$.

3. $\psi \in \overline{N}$.

The standard representation for the matter field $\psi$ is given by

\[
\chi = \rho(\xi) \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \xi^{-1T} \begin{pmatrix} 0 \\ \psi \end{pmatrix} = \begin{pmatrix} -\varphi \cdot \psi \\ \psi \end{pmatrix}.
\]

Hence the invariant is calculated as

\[
X = |\psi|^2 + |\varphi \cdot \psi|^2.
\]

The total space is (topologically) the $\mathbb{C}^N$-bundle over $\mathbb{C}P^N$ or the cotangent bundle over $\mathbb{C}P^N$, $T^*\mathbb{C}P^N$.\(^4\) We discuss this case in detail, below.

After making some comments on the singlet matter $\sigma \in 1$ in the rest of this section, we will work out the anti-fundamental representation matter $\psi \in \overline{N}$ in detail in the next section.

\(^4\)This cotangent bundle does not have to be endowed with the hyper-Kähler metric.
Since the invariant for the singlet can be rewritten as
\[ X = |\sigma|^2 e^{K_0}, \quad K_0 \equiv \log(1 + |\varphi|^2), \] (3.8)
with \(K_0\) the Kähler potential for the base \(\mathbb{C}P^N\), the \(G\)-transformation law for \(\sigma\) is
\[ \sigma \to \sigma' = \sigma e^{-\Lambda(g, \varphi)}, \quad K_0 \to K'_0 = K_0 + \Lambda(g, \varphi) + \Lambda^*(g, \varphi^*) \] (3.9)
which is the one of the line bundle \(\mathcal{O}_{\mathbb{C}P^N}(-1)\). (Using the projection \(\eta\), \(\Lambda\) can be given by \(\Lambda = -\log \det \eta \hat{h}^{-1}\).)

If we define the function \(g(X) \equiv f(X) + c \log X\), the Kähler potential for the total space can be rewritten as
\[ K = c \log(1 + |\varphi|^2) + g(X) - c \log X = g(X) - \log \sigma - \log \sigma^*, \] (3.10)
where the last two terms can be eliminated by the Kähler transformation. Therefore we do not need to include the Kähler potential for the base space \(\mathbb{C}P^N\) itself into the one for the total space. Then the Ricci-flat metric and its Kähler potential was obtained in \([15, 16]\). We just quote the result here:
\[ K = g(X) = (\lambda X^{N+1} + b) \frac{1}{n+1} + b \frac{1}{n+1} \cdot I(b^{\frac{1}{n+1}}(\lambda X^{N+1} + b)^{\frac{1}{n+1}}; N+1), \] (3.11)
with \(b\) an integration constant, \(\lambda\) a constant related to \(N\), and \(I(y; n)\) the function defined by
\[
I(y; n) \equiv \int_y^\infty \frac{dt}{t^n - 1} = \frac{1}{n} \left[ \log (y - 1) - \frac{1 + (-1)^n}{2} \log (y + 1) \right] + \frac{1}{n} \sum_{r=1}^{\frac{n-1}{2}} \frac{2r\pi}{n} \cos \left( \frac{2r\pi}{n} \right) \cdot \log \left( y^2 - 2y \cos \frac{2r\pi}{n} + 1 \right) + \frac{2}{n} \sum_{r=1}^{\frac{n-1}{2}} \sin \left( \frac{2r\pi}{n} \right) \cdot \arctan \left( \frac{\cos \left( \frac{2r\pi}{n} \right) - y}{\sin \left( \frac{2r\pi}{n} \right)} \right). \] (3.12)

We note that the coordinate transformation \(\rho = \sigma^{N+1}\), with \(\varphi\) unchanged, is needed to get a regular metric at \(\sigma = 0\) \([14]\). Then the invariant in the new coordinates is \(X^{N+1} = |\rho|^2 e^{(N+1)K_0}\). We thus find that this is the canonical line bundle. In the limit of \(b \to 0\), the manifold becomes an orbifold \(\mathbb{C}^{N+1}/\mathbb{Z}_{N+1}\) \([10, 11, 15, 16]\). Therefore \(b\) regularizes the orbifold singularity.
4 New Calabi-Yau metric on $T^*\mathbb{C}P^n$

We work out the anti-fundamental representation $\psi \in \mathbb{N}$ in this section. The manifold is topologically $T^*\mathbb{C}P^n$ but will not be equipped with a hyper-Kähler metric except for $N = 1$. The relation with the hyper-Kähler metric on $T^*\mathbb{C}P^n$ will be discussed in Appendix. We denote coordinates of the total space by $z^\mu = (\varphi^i, \psi_{\bar{i}})$ ($i, \bar{i} = 1, \ldots, N$). We represent the differentiations with respect to these coordinates by a comma with indices of corresponding coordinates. The differentiations of $f$ can be calculated, to give

$$f,_{i} = f' \cdot \varphi_i \cdot \psi^*,$$

$$f,_{\bar{i}} = f' \cdot [\psi_i^* + \varphi^i (\varphi^* \cdot \psi^*)]$$

(4.1)

and

$$f,_{ij}^* = f' \cdot \varphi_i \varphi_j^* + f'' \cdot \varphi_i \psi_{\bar{j}}^* |\varphi \cdot \psi|^2,$$

$$f,_{i\bar{j}} = f' \cdot \psi_{\bar{i}} \varphi^j + f'' \cdot \psi_{\bar{i}} (\varphi^* \cdot \psi^*)[\psi_{\bar{j}} + \varphi^j (\varphi \cdot \psi)],$$

$$f,_{\bar{i}j} = f' \cdot (\delta_{ij} + \varphi^i \varphi^j) + f'' \cdot [\psi_i^* + \varphi^i (\varphi^* \cdot \psi^*)][\psi_{\bar{j}} + \varphi^j (\varphi \cdot \psi)],$$

(4.2)

where the prime denotes the differentiation with respect to the argument $X$. Using these expressions, components of the metric $g_{\mu\nu} = K,_{\mu\nu}^*$ can be written as

$$g_{\mu\nu} = \begin{pmatrix} g_{ij}^* & g_{i\bar{j}}^* \\ g_{\bar{i}j}^* & g_{\bar{i}\bar{j}}^* \end{pmatrix},$$

(4.3)

with each block being

$$g_{ij}^* = c \left[ \frac{\delta_{ij}}{1 + |\varphi|^2} - \frac{\varphi^{*i} \varphi^j}{(1 + |\varphi|^2)^2} \right] + f,_{ij}^* ,$$

$$g_{i\bar{j}}^* = c \frac{\varphi^{*i}}{1 + |\varphi|^2} + f,_{i\bar{j}}^* ,$$

$$g_{\bar{i}j}^* = f,_{i\bar{j}}^* .$$

(4.4)

Let us calculate the determinant of this metric. As vacuum expectation values, we can set $\langle \varphi \rangle = 0$ and $\langle \psi \rangle = (\epsilon, 0, \cdots, 0)^T$ without loss of generality. Then the determinant of the metric on this point $v$ is

$$\det g_{\mu\nu}^*|_v = c^N (f')^{N-1} [c + f'|\epsilon|^2](f' + f''|\epsilon|^2).$$

(4.5)

Any point on the manifold can be transformed onto this point $v$ by a $G$-transformation; We can take $\langle \varphi \rangle = 0$, because $\varphi$ parameterize homogeneous manifold $G/H$ and hence any $\varphi$ can be transformed to the origin by a $G$-transformation. Then, we can set $\langle \psi \rangle$ as one component using an $H$-transformation, because $\psi$ belongs to an irreducible representation of $H$. 

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Using $X|_v = |e|^2$ at $v$, we obtain

$$\det g_{\mu\nu}^* = c^N (f')^{N-1} (c + f' X)(f' + f'' X) \quad (4.6)$$

for the determinant of the metric at general points because $X$ is the $G$-invariant.

Using the determinant of the metric, the Ricci-form can be written as $R_{\mu\nu}^* \equiv -\partial_{\mu} \partial_{\nu} \log \det g_{\kappa\lambda}^*$. The Ricci-flat condition $R_{\mu\nu}^* = 0$ is in general a partial differential equation which is very difficult to solve. However, in this case, we can reduce it to ODE

$$(f')^{N-1}(c + f' X)(f' + f'' X) = \frac{a}{N+1}, \quad (4.7)$$

with $a$ a real constant, and the numerical factor is just for convenience.

We can solve this ODE easily. Setting

$$F(X) \equiv f'(X) X \quad (4.8)$$

we obtain

$$F^{N-1}(c + F)F' = \frac{a}{N+1} X^{1-N}. \quad (4.9)$$

This can be integrated, to give an algebraic equation

$$F^{N+1} + \frac{N+1}{N} cF^N = \frac{a}{2-N} X^{2-N} + b \quad (4.10)$$

for $N \neq 2$, or

$$F^3 + \frac{3}{2} cF^2 = a \log X + b \quad (4.11)$$

for $N = 2$, with $b$ being a real integration constant. In principle Eq. (4.10) can be solved numerically, but for lower $N$ we can solve it analytically.

First let us consider the $N = 1$ case of $T^* \mathbb{C}P^1$. The invariant is $X = |\psi|^2 (1 + |\varphi|^2)$ for $N = 1$, with $\varphi$ and $\psi$ one component. Eq. (4.10) reduces to

$$F^2 + 2cF = aX + b, \quad (4.12)$$

which can be solved

$$F(X) = f' X = -c \pm \sqrt{c^2 + b + aX}. \quad (4.13)$$
This can be integrated once again, to yield

\[
f(X) = -c \log X \pm \left[ 2 \sqrt{aX + r^2} + r \log \left( \frac{\sqrt{aX + r^2} - r}{\sqrt{aX + r^2} + r} \right) \right] \tag{4.14}
\]

with \( r \equiv \sqrt{e^2 + b} \), where we have not included an integration constant because it does not contribute to the metric. Therefore we obtain

\[
K = c \log(1 + |\varphi|^2) + f(X)
\]

\[
= 2 \sqrt{aX + r^2} + r \log \left( \frac{\sqrt{aX + r^2} - r}{\sqrt{aX + r^2} + r} \right) - c \log \psi - c \log \psi^* , \tag{4.15}
\]

where the last two terms can be eliminated by the Kähler transformation. Here we have chosen the plus sign in Eq. (4.14) for the positivity of the metric. Setting \( \varphi' = \psi \varphi \) the invariant is \( X = |\psi|^2 + |\varphi'|^2 \). Eq. (4.15) is the Kähler potential \(^{[33]}\) for the Eguchi-Hanson metric \(^{[9]}\) on \( T^* \mathbb{C}P^1 \). This is a hyper-Kähler manifold because any Ricci-flat Kähler manifold is a hyper-Kähler manifold in real four-dimensions [the holonomy is \( SU(2) \simeq Sp(1) \)] but it is not true for higher \( N \).

Next let us consider the \( N = 2 \) case of \( T^* \mathbb{C}P^2 \). For \( N = 2 \), third order Eq. (4.11) can be solved, to give

\[
F(X) = \begin{cases} 
-\frac{c}{2} + G^\frac{1}{3}_+ (X) + G^\frac{1}{3}_- (X) , \\
-\frac{c}{2} + \omega^2 G^\frac{1}{3}_+ (X) + \omega^3 G^\frac{1}{3}_- (X) , \\
-\frac{c}{2} + \omega^3 G^\frac{1}{3}_+ (X) + \omega^2 G^\frac{1}{3}_- (X), 
\end{cases} \tag{4.16}
\]

with \( \omega = e^{2\pi i/3} \) and

\[
G_\pm (X) \equiv \frac{1}{2} \left[ a \log X + b - \frac{c^3}{4} \pm \sqrt{(a \log X + b)^2 - \frac{c^3}{2} (a \log X + b)} \right]. \tag{4.17}
\]

When the discriminant is positive, the first one is the real solution. Otherwise the rest ones are needed. We thus obtain (for the positive discriminant)

\[
K = c \log(1 + |\varphi|^2) - \frac{c}{2} \log X + \int dXX^{-1} (G^\frac{1}{3}_+ + G^\frac{1}{3}_-) . \tag{4.18}
\]

This is no longer hyper-Kähler. We have obtained a Calabi-Yau (but not hyper-Kähler) metric on \( T^* \mathbb{C}P^2 \).

In the end, we consider the \( N = 3 \) case of \( T^* \mathbb{C}P^3 \). Eq. (4.10) reduces to

\[
F^4 + 4c' F^3 + aX^{-1} - b = 0 \tag{4.19}
\]
with \( c' = \frac{1}{3}c \). Then \( F \) has four solutions given by

\[
F(X) = \begin{cases} 
-c' \pm \frac{1}{2} \sqrt{\lambda - p} + \frac{1}{2} \sqrt{-\lambda - p \pm 2q(\lambda - p)^{-\frac{1}{2}}} , \\
-c' \pm \frac{1}{2} \sqrt{\lambda - p} - \frac{1}{2} \sqrt{-\lambda - p \pm 2q(\lambda - p)^{-\frac{1}{2}}} ,
\end{cases}
\]

with

\[
\lambda = -2c'^2 + \frac{1}{3} \left[ \frac{1}{2} \left( -q + \sqrt{q^2 + 4p^3} \right) \right]^\frac{1}{3} + \frac{1}{3} \left[ \frac{1}{2} \left( -q - \sqrt{q^2 + 4p^3} \right) \right]^\frac{1}{3} ,
\]

\[
p = -\frac{4}{3}(aX^{-1} + b) ,
\]

\[
q = -16(ac'^2X^{-1} + 2bc'^2 - 8c'^6) .
\]

The Kähler potential for the total space is obtained

\[
K = c \log(1 + |\varphi|^2) - \frac{c}{3} \log X + \int dXX^{-1}(\cdots) ,
\]

where dots denote the last two terms in \( F \) in the solution (4.20).

To write down the metric, we do not need the Kähler potential itself but its derivative \( f' \). We thus have derived the explicit expression for the metric for \( N = 2, 3, 4 \).

## 5 Summary and Discussions

We have given a method to construct a \( G \)-invariant metric on the vector bundle over Kähler \( G/H \) using matter coupling of the nonlinear realization. The dimension of the vector as fiber is restricted by the \( G \)-invariance of the manifold because the vector belongs to the \( H \)-representation. To solve the Ricci-flat condition, cohomogeneity one is essential in which case it is reduced to ODE to be solved easily. This requirement implies that the matter should belong to an irreducible representation of \( H \) at least. As a concrete example, we have discussed the matter coupling in the \( \mathbb{C}P^N \) model. The singlet matter has reproduced the canonical line bundle and the anti-fundamental representation has given a Calabi-Yau metric on \( T^*\mathbb{C}P^N \) which is not hyper-Kähler. We have given the explicit expression for the metric for \( N = 2, 3, 4 \) [Eq. (5.1) with (5.13), (5.16) and (5.20)] and the algebraic equation (5.10) for higher \( N \). A relation with hyper-Kähler Calabi metric on \( T^*\mathbb{C}P^N \) is given in Appendix A.

We could introduce the matter belonging to higher (irreducible) representation of \( H \) which would provide a new metric. Our model can be extended to other base manifolds for instance to the quadric surface \( Q^N = SO(N + 2)/[SO(N) \times U(1)] \). We will
reproduce the canonical line bundle over $Q^N$ and will get the Calabi-Yau metric on $T^*Q^N$ which is not hyper-Kähler.

Our model is a finite nonlinear sigma model because the beta function is proportional to the Ricci-form. For the $\mathbb{C}P^N$ model, $R_{\mu\nu} = hg_{\mu\nu}$ with positive $h$ holds and so it is not finite. Instead, there exists a mass gap and a gauge boson is dynamically induced in the large-$N$ limit [35]. We added matter into the $\mathbb{C}P^N$ model to make total finite. Investigating the relation with quantum properties of the $\mathbb{C}P^N$ model and our model is interesting. Looking for a conformal field theory corresponding to our finite model is also an important task.

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**A Relation with the hyper-Kähler Calabi metric**

The standard metric on $T^*\mathbb{C}P^N$ is the Calabi metric [10], which is a hyper-Kähler metric. Here, we discuss the relation between our metric and the Calabi metric using a (hyper-)Kähler quotient construction [36] for $T^*\mathbb{C}P^N$ [37, 38]. First prepare chiral superfields $\phi$ and $\chi$ belonging to $\mathbf{N} + \mathbf{1}$ and $\overline{\mathbf{N}} + \mathbf{1}$ of $SU(N+1)$, respectively. Let $V$ and $\sigma$ be auxiliary vector and chiral superfields, respectively, introduced as Lagrange multipliers to give constraints among $\phi$ and $\chi$. Consider a $U(1)$ gauge symmetry, given by

\[ V \rightarrow V' = V - i\Lambda + i\Lambda^\dagger, \quad \sigma \rightarrow \sigma' = e^{-i(1+\theta)}\sigma, \]

\[ \phi \rightarrow \phi' = e^{i\Lambda} \phi, \quad \chi \rightarrow \chi' = e^{iq\Lambda} \chi, \quad (A.1) \]

where $q$ is the $U(1)$-charge for $\chi$ relative to $\phi$. Then, the most general Lagrangian for these field contents can be written as \(^6\)

\[ \mathcal{L} = \int d^4\theta \left[ e^V \phi^\dagger \phi + f(e^V \chi^\dagger \chi) - cV \right] + \left[ \int d^2\theta \sigma(\phi \cdot \chi - \delta_{q,-1} b) + \text{c.c.} \right], \quad (A.2) \]

\(^6\)At first sight, one might consider that the most general Kähler potential should be $\mathcal{F}(e^V \phi^\dagger \phi, e^V \chi^\dagger \chi)$. However we can show that one variable in the arbitrary function can be linearized in the path integral formalism [34].
with \( f \) an arbitrary function. Here \( c \in \mathbb{R} \) is a Fayet-Iliopoulos parameter and \( b \in \mathbb{C} \) can be non-zero only when \( q = -1 \).

We discuss particular values of \( q \): 1) \( q = 0 \) and 2) \( q = -1 \).

1) The Kähler potential on \( T^* \mathbb{C}P^N \) given by \( \mathbb{C}P^N \) coupled with the matter belonging to \( \mathbb{N} \) corresponds to the case of \( q = 0 \) (and \( b = 0 \)):

\[
\mathcal{L}_{q=0} = \int d^4\theta \left[ e^V \phi^+ \phi + f(\chi^+ \chi) - cV \right] + \left[ \int d^2\theta \, \sigma(\phi \cdot \chi) + \text{c.c.} \right].
\]  

(A.3)

The equations of motion for \( V \) and \( \sigma \) read

\[
\frac{\partial \mathcal{L}}{\partial V} = e^V \phi^+ \phi - c = 0, \quad \frac{\partial \mathcal{L}}{\partial \sigma} = \phi \cdot \chi = 0,
\]

respectively. Eliminating \( V \) and solving the constraint, we obtain

\[
\mathcal{L}_{q=0} = \int d^4\theta \left[ c \log |\phi^\dagger \phi| + f(\chi^\dagger \chi) \right],
\]  

(A.5)

with a gauge fixing \( \phi^\dagger = 1 \):

\[
\phi = \left( \begin{array}{c} 1 \\ \varphi \end{array} \right), \quad \chi = \left( \begin{array}{c} -\varphi \cdot \psi \\ \psi \end{array} \right).
\]  

(A.6)

This is the Kähler potential considered in the above discussion, Eq. (3.2) with (3.7).

2) On the other hand, the hyper-Kähler metric on \( T^* \mathbb{C}P^N \) can be obtained by choosing \( q = -1 \):

\[
\mathcal{L}_{q=-1} = \int d^4\theta \left[ e^V \phi^\dagger \phi + f(e^{-V} \chi^\dagger \chi) - cV \right] + \left[ \int d^2\theta \, \sigma(\phi \cdot \chi - b) + \text{c.c.} \right].
\]  

(A.7)

We should impose the Ricci-flat condition to determine \( f \) after eliminating \( V \), but we know the answer \( f(X) = X \) because it is hyper-Kähler:

\[
\mathcal{L}_{q=-1,RF} = \int d^4\theta \left[ e^V \phi^\dagger \phi + e^{-V} \chi^\dagger \chi - cV \right] + \left[ \int d^2\theta \, \sigma(\phi \cdot \chi - b) + \text{c.c.} \right].
\]  

(A.8)

This is the hyper-Kähler quotient construction \[36\] for the hyper-Kähler Calabi metric \[10\] on \( T^* \mathbb{C}P^N \) \[37\] \[38\].\(^7\) The \( \mathcal{N} = 2 \) SUSY is enhanced to \( \mathcal{N} = 4 \) SUSY in two-dimensional space-time (\( \mathcal{N} = 2 \) SUSY in four-dimensional space-time) with \((V, \sigma) \mathcal{N} = 4 \) vector multiplet and \((\phi, \chi) \mathcal{N} = 4 \) hypermultiplets. \( c \) and \( b \) constitute the triplet of the FI parameters. After eliminating \( V \) and \( \sigma \) by their equations of motion, we obtain

\[
K = c \log |\phi|^2 + \sqrt{c^2 + 4|\phi|^2} |\chi|^2 - c \log \left( c + \sqrt{c^2 + 4|\phi|^2} |\chi|^2 \right),
\]  

(A.9)

\(^7\)Adding masses to hypermultiplets, the potential term for \( T^* \mathbb{C}P^N \) can be obtained \[38\].
with proper gauge fixing. (For $b = 0$ we have (A.6), and for $b \neq 0$ we should take an another gauge [37, 38].)

We thus have found the difference between our Calabi-Yau metric and the hyper-Kähler Calabi metric on $T^*\mathbb{C}P^N$ comes from the relative gauge $U(1)$-charge between $\chi$ and $\phi$.

For the general value of $q$, the Kähler potential after eliminating $V$ is

$$K = c \log \phi + h[(\phi^\dagger \phi)^{-q} \chi^\dagger \chi] , \quad (A.10)$$

with some function $h$ related with $f$. This can be found from the algebraic geometry point of view [39]: the argument $(\phi^\dagger \phi)^{-q} \chi^\dagger \chi$ of $h$ is the gauge invariant and remains after the integration over $V$. The construction of the Calabi-Yau metric with general $q$ is left for a future work.

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