PROJECTIVE CLOSURES OF AFFINE MONOMIAL CURVES

JOYDIP SAHA, INDRANA TH SENGUPTA, AND PRANJAL SRIVASTAVA

ABSTRACT. We study the projective closures of three important families of affine monomial curves in dimension 4, namely the Backelin curve, the Bresinsky curve and the Arslan curve, in order to explore possible connections between syzygies and the arithmetic Cohen-Macaulay property.

1. INTRODUCTION

Let \( \mathbb{N} \) denote the set of nonnegative integers and \( k \) denote a field. Let \( r \geq 3 \) and \( \mathbf{n} = (n_1, \ldots, n_r) \) be a sequence of \( r \) distinct positive integers with \( \gcd(\mathbf{n}) = 1 \). Let us assume that the numbers \( n_1, \ldots, n_r \) generate the numerical semigroup \( \Gamma(n_1, \ldots, n_r) = \{ \sum_{j=1}^{r} z_j n_j \mid z_j \in \mathbb{N} \} \) minimally, that is, if \( n_i = \sum_{j=1}^{r} z_j n_j \) for some non-negative integers \( z_j \), then \( z_j = 0 \) for all \( j \neq i \) and \( z_i = 1 \). Let \( \eta : k[x_1, \ldots, x_r] \to k[t] \) be the mapping defined by \( \eta(x_i) = t^{n_i}, 1 \leq i \leq r \). Let \( p(n_1, \ldots, n_r) = \ker(\eta) \). Let us assume that \( n_r > n_i \) for all \( i < r \), for the sequence \( \mathbf{n} = (n_1, \ldots, n_r) \). We fix \( n_0 = 0 \). We define the semigroup \( \Gamma(n_1, \ldots, n_r) \subset \mathbb{N}^2 \), ( \( \mathbb{N} \) is the set of all non-negative integers) generated by \( \{ (n_i, n_r - n_i) \mid 0 \leq i \leq r \} \). Let us denote \( p(n_1, \ldots, n_r) \) be the kernel of \( k \)-algebra map \( \eta^H : k[x_0, \ldots, x_r] \to k[s, t], \eta^H(x_i) = t^{n_i} \). Then homogenization of the ideal \( p(n_1, \ldots, n_r) \) with respect to the variable \( x_0 \) is \( p(n_1, \ldots, n_r) \). Thus the projective curve \( \{ (a^{n_r} : a^{n_r-n_1} b^{n_1} : \ldots : b^{n_r}) \} \in \mathbb{P}_k^r \mid a, b \in k \} \) is the projective closure of the affine curve \( C(n_1, \ldots, n_r) := \{ (b^{n_1}, \ldots, b^{n_r}) \in \mathbb{A}_k^r \mid b \in k \} \), and we denote it by \( C(n_1, \ldots, n_r) \). We say that the projective curve \( C(n_1, \ldots, n_r) \) is arithmetically Cohen-Macaulay if the vanishing ideal \( p(n_1, \ldots, n_r) \) is a Cohen-Macaulay ideal.

Let \( \beta_i(p(n_1, \ldots, n_r)) \) denotes the \( i \)-th Betti number of the ideal \( p(n_1, \ldots, n_r) \). Therefore, \( \beta_i(p(n_1, \ldots, n_r)) \) denotes the minimal number of generators of \( p(n_1, \ldots, n_r) \). For a given \( r \geq 3 \), let \( \beta_i(r) = \text{lub}(\beta_i(p(n_1, \ldots, n_r))) \), where the lub is taken over all the sequences of positive integers \( n_1, \ldots, n_r \). Herzog [9] proved that, \( \beta_1(3) \) is 3 and it follows easily that \( \beta_2(3) \) is a finite integer as well. Bresinsky [3] (and [4]) defined a class of monomial curves in \( \mathbb{A}^4 \) and proved that \( \beta_1(4) = \infty \). He used this observation to prove that \( \beta_1(r) = \infty \), for every \( r \geq 4 \). We call this family the Bresinsky curve. Subsequently, it has been proved in [12] that all the higher Betti numbers of Bresinsky curve are also not bounded above by a fixed integer. F. Arslan introduced the curve \( \Gamma_{Ah} = \langle h(h+1), h(h+1)+1, (h+1)^2, (h+1)^2+1 \rangle, \) for \( h \geq 2 \), in \( \mathbb{A}^4 \), and proved the Cohen-Macaulayness of tangent cone of the curve at the origin (see [1]). Subsequent to this, Fröberg et al. presented the numerical semigroup \( \langle s, s+3, s+3n+1, s+3n+2 \rangle, \) for \( n \geq 2, r \geq 3n+2, s = r(3n+2) + 3, \) in \( \mathbb{A}^4 \), and proved the unboundedness of the type of this numerical semigroup (see [7]). This class of semigroups was first proposed by Backelin. All three families have one strong resemblance that all are monomial curves in the affine 4 space, and

2020 Mathematics Subject Classification. Primary 13D02, 13F55, 13P10, 13P20.

Key words and phrases. Monomial curves, Gröbner bases, Betti numbers.

The first author thanks NBHM, Government of India for post-doc fellowship at ISI kolkata.

The second author is the corresponding author.
there is no upper bound on the minimal generating set of the defining ideals of these curves. Recently, Herzog and Stamate [11] have proved that the projective closure of the Bresinsky curve is not arithmetically Cohen-Macaulay, but the projective closure of the Arslan curve is arithmetically Cohen-Macaulay, using a Gröbner basis criterion. In [14], Stamate has proved that the Betti sequence of the affine Bresinsky curve and affine Arslan curve are \((1, 4h, 8h - 4, 4h - 3)\) and \((1, 2h + 2, 4h, 2h - 1)\), respectively.

This paper is devoted to the study of the projective closures of three families: The Backelin curve, the Bresinsky curve and the Arslan curve. We have computed the syzygies of the projective closures of all three families and have proved that the Betti sequences of the affine curves and their projective closures remain the same for the Backelin curve and the Arslan curve, and they differ for the Bresinsky curve. In fact, the last Betti number of the projective closure of the Bresinsky curve is \(1\). Two main questions that form the background of this work are the following: (1) To find a suitable sufficient condition on an affine monomial curve so that its projective closure is arithmetically Cohen-Macaulay; (2) to find a suitable sufficient condition on an affine monomial curve so that the Betti sequence of the affine monomial curve is the same as the Betti sequence of its projective closure. This paper is attempt to understand these questions through three very important and interesting classes of curves mentioned above. Computations with [6] have helped us understand the structure of the syzygies for most of the examples.

This paper has been arranged in the following order: First, we study the affine Backelin curve, its Gröbner basis and use that to prove that its projective closure is arithmetically Cohen-Macaulay, with the help of the Gröbner basis criterion proved in (Theorem 2.2, [11]). We then compute the syzygies of the projective closure and the Hilbert series of the Backelin curve. Next, we compute the syzygies of the projective closure of the Bresinsky curve and finally the syzygies of the projective closure of the Arslan curve.

2. The Backelin Curve and Its Gröbner Basis

Let us first begin with a description of Backelin’s example of monomial curves in the affine space \(\mathbb{A}^4\). Backelin defined the numerical semigroups \((s, s + 3, s + 3n + 1, s + 3n + 2)\), for \(n \geq 2, r \geq 3n + 2\) and \(s = r(3n + 2) + 3\). In [7], it has been shown that the type of such semigroups are not bounded by the embedding dimension.

**Notation 2.1.** We fix some notations. For \(n \geq 2, r \geq 3n + 2\) and \(s = r(3n + 2) + 3\),
\[
\Gamma_{nr} := \Gamma(s, s + 3, s + 3n + 1, s + 3n + 2);
\]
\[
\overline{\Gamma}_{nr} := \overline{\Gamma}(s, s + 3, s + 3n + 1, s + 3n + 2);
\]
\[
\mathcal{P}_{nr} := \mathcal{P}(s, s + 3, s + 3n + 1, s + 3n + 2);
\]
\[
\overline{\mathcal{P}}_{nr} := \overline{\mathcal{P}}(s, s + 3, s + 3n + 1, s + 3n + 2), \text{ therefore } \overline{\mathcal{P}}_{nr} = (\mathcal{P}_{nr})^H \text{ with respect to the variable } x_0;
\]
\(B_{nr}\) denotes the affine Backelin curve defined by the numerical semigroup \(\Gamma_{nr}\):
\(\overline{B}_{nr}\) denotes the projective closure of the Backelin curves;
\(G(I)\) denotes the unique minimal generating set of a monomial ideal \(I\) in a polynomial ring over a field.
Theorem 2.2. (Gastinger) Let $A = k[x_1, \ldots, x_r]$ be the polynomial ring, $I \subset A$ the defining ideal of a monomial curve defined by natural numbers $a_1, \ldots, a_r$, whose greatest common divisor is 1. Let $J$ be an ideal contained in $I$. Then $J = I$ if and only if $\dim_k A/\langle J + (x_i) \rangle = a_i$, for some $i$; equivalently $\dim_k A/\langle J + (x_i) \rangle = a_i$ for any $i$.

Proof. See in [8].

Let $I \subset k[x_1, \ldots, x_r]$ be a monomial ideal, then it has unique minimal generating set, which we denote by $G(I)$.

Theorem 2.3. The defining ideal $\mathfrak{P}_{nr}$ of monomial curve associated to $\Gamma_{nr}$ is minimally generated by following binomials:

\[
\begin{align*}
  f_1 &= x_2x_3^2 - x_1x_4^3, \\
  f_{(2,i)} &= x_1^{n-i}x_3^{i-1} - x_2^{n-i+1}x_4^{3i-2}, 1 \leq i \leq n; \\
  f_{(3,j)} &= x_1^{-n+3+j}x_2^{1-j} - x_3^{2+3j}x_4^{-1-3j}, 0 \leq j \leq n-1; \\
  f_{(4,j)} &= x_1^{-2n+3+j}x_2^{2n-j} - x_3^{3j+1}x_4^{r+1-3j}, 0 \leq j \leq n-1; \\
  f_5 &= x_1^{-n+2}x_2^n - x_3^r; \\
  f_6 &= x_2^{n+1}x_3 - x_1^{n+2}; \\
  f_7 &= x_2^{n+1} - x_1^{n-1}x_3x_4.
\end{align*}
\]

Proof. Let $J_{nr} = \langle \{f_1, f_{(2,i)}, f_{(3,j)}, f_{(4,j)}, f_5, f_6, f_7 \mid 1 \leq i \leq n, 0 \leq j \leq n - 1 \} \rangle$, and

\[
\mathfrak{A}_{nr} = \{x_1x_4^3 \} \cup \{x_4^{r+2}, x_2x_4^{2n}, x_2^{n+1} \} \cup \{x_2^{n-i+1}x_4^{3i-2} \mid 1 \leq i \leq n \} \\
\quad \cup \{x_1^{-n+3+j}x_2^{n-1-j}, x_1^{-2n+3+j}x_2^{2n-j} \mid 0 \leq j \leq n - 1 \} \cup \{x_3 \}.
\]

Then, $J_{nr} + (x_3) = \langle \mathfrak{A}_{nr} \rangle$ and it can be verified easily that $\mathfrak{A}_{nr}$ is minimal. First we note that $J_{nr} \subset \mathfrak{P}_{nr}$. We use the standard result: If $I = \langle G(I) \rangle$ is a monomial ideal in $A = k[x_1, \ldots, x_r]$, then $A/I$ is a $k$-vector space whose basis consists of the images of monomials, which are not divisible by any elements of $G(I)$. Therefore $A/J_{nr} + (x_3)$ is the vector space over $k$ and basis consists of the images of monomials, which are not divisible by any elements of $\mathfrak{A}_{nr}$. These are listed below:

\[
\begin{align*}
  S_1 &= \{x_1^\alpha : 0 \leq \alpha \leq r + 1 \}, \\
  S_2 &= \{x_2^\beta : 1 \leq \beta \leq 2n \}, \\
  S_3 &= \{x_3^\gamma : 1 \leq \gamma \leq r + 1 \}, \\
  S_4 &= \{x_1^\alpha x_2^\beta : 1 \leq \alpha \leq r - \beta + 1, 1 \leq \beta \leq n - 1 \}, \\
  S_5 &= \{x_1^\alpha x_2^\beta : 1 \leq \alpha \leq r - \beta + 2, n + 1 \leq \beta \leq 2n \}, \\
  S_6 &= \{x_1^\alpha x_3 : 1 \leq \alpha \leq r + 1 \}, \\
  S_7 &= \{x_1^\alpha x_3^\beta : 1 \leq \alpha \leq n - 1 \}, \\
  S_8 &= \{x_2^\beta x_3^\gamma : 1 \leq \beta \leq n - 1, 1 \leq \gamma \leq 3(n - \beta) \}, \\
  S_9 &= \{x_1^\alpha x_2^\beta x_3^\gamma : 1 \leq \alpha \leq n - 1, 1 \leq \beta \leq n - 1, 1 \leq \gamma \leq 2 \}, \\
  S_{10} &= \{x_1^\alpha x_2 x_3 : n \leq \alpha \leq r - \beta + 1, 1 \leq \beta \leq n - 1 \}.
\end{align*}
\]
$S_{11} = \{x_1^\alpha x_2^{n-\alpha} : 1 \leq \alpha \leq r - n + 2\}$.

It is clear from the expressions of the elements of $S_i$ that all the sets $S_i$, $1 \leq i \leq 11$, are pairwise disjoint. Therefore the cardinality of this basis is,

$$
\sum_{i=1}^{11} |S_i| = (r + 2) + 2n + (r + 1) + \frac{(n-1)}{2}(2r-n+2) + \frac{n}{2}(2r-3n+3) + (r+1)
$$

$$
+ (n-1) + \frac{3}{2}n(n-1) + 2(n-1)^2 + (r-n+2) + \frac{n-1}{2}(2r-3n+4)
$$

$$
= 3nr + 3n + 2r + 4.
$$

Hence $\dim_k (A/J_r + \langle x_3 \rangle) = 3nr + 3n + 2r + 4 = s + 3n + 1$ and by Theorem 2.2 it follows that $J_r = \mathfrak{P}_r$.

To show the minimality of the generating set, we consider the homomorphism

$$
\theta : k[x_1, x_2, x_3, x_4] \rightarrow k[x_1, x_2, x_3, x_4]
$$

$$
\theta(x_i) = x_i, \text{ for } i = 1, 2, 4;
$$

$$
\theta(x_3) = 0.
$$

We note that, if $f$ is a generator of $J_r$ or if $f = \sum_{g_i \in S} p_i g_i$, where $S$ is a generating set of $J_r$, then $\theta(f) \in \mathfrak{A}_r$. This implies that $\theta(f) = \sum_{g_i \in S} \theta(p_i)\theta(g_i)$, which gives a contradiction because $m \nmid m'$ for any pair of monomials $m, m' \in \mathfrak{A}_r$. Therefore, $J_r$ is a minimal generating set of $\mathfrak{P}_r$.

Let us denote the above generating set of $\mathfrak{P}_r$ by $\mathcal{G}_r$, i.e.,

$$
\mathcal{G}_r = \{f_1, f_{(2,i)}, f_{(3,j)}, f_{(4,j)}, f_5, f_6, f_7 | 1 \leq i \leq n, 0 \leq j \leq n - 1\}.
$$

**Lemma 2.4.** Let $g = x_1^{r+2} - x_2 x_4^r$ be a polynomial in $k[x_1, \ldots, x_4]$. Suppose $G_r = (\mathcal{G}_r \setminus \{f_{(3,n-1)}\}) \cup \{g\}$. Then $G_r$ is also a generating set for the defining ideal $\mathfrak{P}_r$.

**Proof.** Follows from the relation $g = f_{(3,n-1)} + x_4^{r-3n+2} \cdot f_{(2,n)}$. 

**Theorem 2.5.** Let us consider the degree reverse lexicographic monomial order induced by $x_1 > x_2 > x_3 > x_4$ on $k[x_1, \ldots, x_4]$. Then, $G_r$ is a Gröbner basis of the defining ideal $\mathfrak{P}_r$, with respect to the above order.

**Proof.** We consider each $S$-polynomial and show that it reduces to zero upon division by $G_r$.

$$
(1) \quad S(f_1, f_{(2,1)}) = x_2^{n+1} x_4 x_3 - x_1^n x_4^3 = x_4(x_2^{n+1} x_3 - x_1^n x_4^2) = x_4 f_6.
$$

$$
S(f_1, f_{(2,i)}) = x_2^{n-i+2} x_4^{3i-2} - x_1^{n-i+1} x_3^{3i-4} x_4^3
$$

$$
= -x_4^3(x_2^{n-i+1} x_3^{3i-4} - x_1^{n-i} x_4^{3i-5})
$$

$$
= -x_4^3 f_{(2,i-1)}, \text{ for } 2 \leq i \leq n.
$$
We consider two cases for $S(f, f_{(3,n-2)})$:

(a) For $i = n - 1$,

$$S(f, f_{(4,n-1)}) = x_3^{3n+1}x_4^{r+1-3n+3} - x_1^{r-n+3}x_2^n x_3^3$$
$$= (x_3^{3n-1} - x_2 x_4^{n-2}) x_4^{r-3n+5} - (x_1^{r+2} - x_2 x_4^r) x_4^3$$
$$= f_{(2,n)} x_4^{r-3n+5} - g x_4^3.$$

(b) For $0 \leq i \leq n - 2$,

$$S(f, f_{(4,i)}) = x_3^{3i+4} x_4^{r+1-3i} - x_1^{r-2n+4+i} x_2^{2n-i-1} x_4^3$$
$$= (x_3^{3n-1} - x_2 x_4^{n-2}) x_4^{r-3n+4} - (x_1^{r^n+3} x_2^n - x_3 x_4^{r-1}) x_2 x_4^3$$
$$= f_{(2,n)} x_4^{r-3n+4} - f_{(3,0)} x_2 x_4^3.$$

(4) $S(f_1, g) = x_2^2 x_3^2 x_4^r - x_1^{r+3}$. Since $\gcd(Lt(f_1), Lt(g)) = 1$, so $S(f_1, g) \rightarrow G_{nr} 0$.

(5) $S(f_1, f_5) = -x_1^{r-n+3} x_2^{n-1} x_4^3 + x_4^{r+2} x_3^2 = -(x_1^{r-n+3} x_2^n - x_3 x_4^{r-1}) x_4^3 = -f_{(3,0)} (x_4^3)$.

(6) $S(f_1, f_6) = x_1^n x_2^2 x_4^2 - x_1 x_2^n x_4^2 = (x_1^{n-1} x_2^3 - x_2^n x_4) x_1 x_4^2 = f_{(2,1)} (x_1 x_4^2)$.

(7) $S(f_1, f_7) = -x_1 x_2^{n+1} x_4^3 + x_1^{2n-1} x_3 x_4^n$.

$$= (x_1^{n-1} x_3 - x_2^n x_4) (x_1 x_3 x_4 + x_1 x_3 x_4^n)$$
$$= f_{(2,1)} (x_1 x_3 x_4 + x_1 x_3 x_4^n).$$

(8) For $i < j$, we have

$$S(f_{(2,i)}, f_{(2,j)}) = x_1^{j-i} x_2^{n-j} x_3^{3j-i-2} x_4^{n-i+1} x_3^{2j-i} x_4^{i-2}$$
$$= (x_2 x_3^3 - x_1 x_4^3) \left( \sum_{l=0}^{j-i-1} x_1 x_2^{(n-i)-l} x_3^{3(j-i)-3(l+1)} x_4^{3i+(3l-2)} \right)$$
$$= f_1 \left( \sum_{l=0}^{j-i-1} x_1 x_2^{(n-i)-l} x_3^{3(j-i)-3(l+1)} x_4^{3i+(3l-2)} \right).$$

(9) $S(f_{(2,i)}, f_{(3,j)}) = x_3^{3(i+j)+1} x_4^{r-1-3j} - x_1^{r-2} x_2^{n+1+i} x_4^{2n-i-j} x_4^{3i-2}$. We consider three separate cases:
(a) For \( i + j = n \),
\[
S(f_{(2,i)}, f_{(3,j)}) = x_3^{3n+1}x_4^{r-3}x_2^{3i-2} - x_1^{r-n+3}x_2^{3i-2} = (x_1^{r-n+3}x_2^{3} - x_2^{2}x_4^{r-1})(-x_2x_4^{3i-2}) + (x_3^{3n-1} - x_2x_4^{3n-2})(x_3^{2}x_4^{r-1-3j}) = f_{(3,i)}(-x_2x_4^{3i-2}) + f_{(2,n)}(x_2^{2}x_4^{r-1-3j}).
\]

(b) For \( i + j > n \),
\[
S(f_{(2,i)}, f_{(3,j)}) = f_{(3,(i+j)-n-1)}(-x_1x_4^{3i-2}) + f_{(2,n)}(x_3^{3(i+j)-3n+2}x_4^{r-3i-3j}) + f_1(x_3^{3(i+j)-3n-1}x_4^{r+3n-3j-3})
\]

(c) For \( i + j < n \),
\[
S(f_{(2,i)}, f_{(3,j)}) = f_{(4,i+1)}x_4^{3i-2}.
\]

(10) \( S(g, f_{(2,i)}) = x_1^{r-n+2+i}x_2^{n-i+1}x_4^{3i-2} - x_2x_4^{3i-1}x_4^{r} = f_{(3,i-1)}x_2x_4^{3i-2}, \) for \( 1 \leq i \leq n - 1 \). For \( i = n \), we note that \( \text{gcd}(Lt(g), Lt(f_{(2,n)}) = 1 \), and hence \( S(f_{(2,n)}, g) \rightarrow G_{nr} \).

(11)\( S(f_{(2,i)}, f_{(4,j)}) = x_3^{3(i+j)}x_4^{r+1-3j} - x_1^{r-3n+3+i+j}x_2^{3n-i-j+1}x_4^{3i-2} \). We consider four separate cases:

(a) For \( l = i + j \leq n - 2 \),
\[
S(f_{(2,i)}, f_{(4,j)}) = (x_1^{r-n+2+l}x_2^{n-l} - x_3^{3l-1}x_4^{r-3l+2})(-x_3x_4^{3i-1}) + (x_2^{2n+1} - x_1^{2n-1}x_3x_4)(-x_1^{r-3n+l+3}x_2^{n-l}x_4^{3i-2}) = f_{(3,i-1)}(-x_3x_4^{3i-1}) + f_7(-x_1^{r-3n+l+3}x_2^{n-l}x_4^{3i-2}).
\]

(b) For \( l = i + j = n - 1 \),
\[
S(f_{(2,i)}, f_{(4,j)}) = (x_2^{2n+1} - x_1^{2n-1}x_3x_4)(-x_1^{r-2n+2}x_2x_4^{3i-2}) + (x_1^{r+1}x_2 - x_3^{3n-4}x_4^{r-3n+5})(-x_3x_4^{3i-1}) = f_7(-x_1^{r-2n+2}x_2x_4^{3i-2}) + f_{(3,n-2)}(-x_3x_4^{3i-1}).
\]

(c) For \( l = i + j = n - 1 \),
\[
S(f_{(2,i)}, f_{(4,j)}) = (x_1^{r+2} - x_2x_4^{r})(-x_3x_4^{3i-1}) + (x_2^{2n+1} - x_1^{2n-1}x_3x_4)(-x_1^{r-2n+3}x_4^{3i-2}) + (x_3^{3n-1} - x_2x_4^{3n-2})(x_3^{2}x_4^{r+1-3j}) = g(-x_3x_4^{3i-1}) + f_7(-x_1^{r-2n+3}x_4^{3i-2}) + f_{(2,n)}(x_3x_4^{r+1-3j}).
\]

(d) For \( i + j \geq n + 1 \)
\[
S(f_{(2,i)}, f_{(4,j)}) = f_{(4,(i+j)-n-1)}(-x_1x_4^{3i-2}) + f_{(2,n)}(x_3^{3(i+j)-3n+1}x_4^{r+1-3j}) + f_1(x_3^{3(i+j)-3n-2}x_4^{r-1-3j+3n}).
\]
(12) \[ S(f(2,i), f_5) = x_3^{3i-2} x_4^{r+2} - x_1^{r-2n+2+i} x_2^{2n-i+1} x_4^{3i-2} \]
\[ = - (x_1^{r-2n+i+2} x_2^{2n-i+1} - x_3^{3i-2} x_4^{r+4-3i}) x_4^{3i-2} \]
\[ = - f_{(4,i-1)} x_4^{3i+1}, \quad 1 \leq i \leq n. \]

(13) For \( 3 \leq i \leq n, \)
\[ S(f(2,i), f_6) = x_2^{2n-i+2} x_4^{3i-2} - x_1^{2n-i} x_4^{3i-2} \]
\[ = (x_1^{n-1} x_3^n - x_2^n x_4^n) (-x_1^{n-i+1} x_4^{3i-4}) \]
\[ - (x_2 x_3^3 - x_1 x_4^3) \sum_{l=1}^{i-2} (x_1^{n-i+l} x_2^{(n-l)} x_3^{3i-(7+3(l-1))} x_4^{l}) \]
\[ = f_{(2,1)} (-x_1^{n-i+1} x_2^{2n} x_3^{3i-4} - x_2^{n-i+2} x_4^{3i-3}) \]
\[ - f_1 \sum_{l=1}^{i-2} (x_1^{n-i+l} x_2^{(n-l)} x_3^{3i-(7+3(l-1))} x_4^{l}). \]

For \( i = 2, \)
\[ S(f(2,2), f_6) = x_2^{2n} x_4^4 - x_1^{2n-2} x_4^3 x_3^4 \]
\[ = - (x_1^{n-1} x_3^n - x_2^n x_4^n) (x_1^{n-1} x_4^2 x_3^2 + x_2^n x_4^2) \]
\[ = - f_{(2,1)} (x_1^{n-1} x_4^2 x_3^2 + x_2^n x_4^2 x_3^2). \]

For \( i = 1, \)
\[ S(f(2,1), f_6) = (x_2^{2n+1} x_4 - x_1^{2n-1} x_4 x_3) \]
\[ = (x_2^{2n+1} - x_1^{2n-1} x_4 x_3) x_4 \]
\[ = f_7 x_4. \]

(14) \[ S(f(2,i), f_7) = - x_1^{n-i} x_2^{3n-i+2} x_4^{3i-2} + x_1^{3n-i-1} x_4 x_3^{3i}, \quad 1 \leq i \leq n. \]

Since \( \gcd(Lt(f(2,i)), Lt(f_7)) = 1, \) we have \( S(f(2,i), f_7) \xrightarrow{\text{tor}} 0. \)

(15) \[ S(f(3,i), f(3,j)) = - x_3^{2+3i} x_4^{r-1-3i} x_1^{j-i} + x_2^{j-i} x_3^{2+3j} x_4^{r-1-3j} \]
\[ = (x_2 x_3^3 - x_1 x_4^3) \sum_{l=0}^{j-i-1} (x_1 x_2^{l(j-i)-(l+1)} x_3^{3j-(1+3l)} x_4^{r-3j+(-1+3l)}) \]
\[ = f_1 \sum_{l=0}^{j-i-1} (x_1 x_2^{l(j-i)-(l+1)} x_3^{3j-(1+3l)} x_4^{r-3j+(-1+3l)}), \quad i < j. \]

(16) \[ S(f(3,i), f_7) = x_2^{n-i} x_4^r - x_1^{n-1-i} x_3^{2+3i} x_4^{r-1-3i} \]
\[ = - (x_1^{n-i-1} x_3^{3i+2} - x_2^{n-i} x_4^{3i+1}) x_4^{r-1-3i} \]
\[ = - f_{(2,i+1)} x_4^{r-1-3i}. \]
(17) \( S(f_{(3,i)}, f_{(4,j)}) = x_1^{n+1-j} x_3^{j+1} x_4^{r+1-3j} - x_2^{n-j+1+i} x_3^{2+i} x_4^{r-1-3i} \). We consider three separate cases:

(a) For \( i = j \),

\[
S(f_{(3,i)}, f_{(4,i)}) = (x_1 x_3^3 - x_1 x_4^3)(-x_2^n x_3^{3i-1} x_4^{r-1-3i} + (x_1 x_3^2 - x_2 x_4) x_1 x_3^{3i-1} x_4^{r-3i+1} + f_1(-x_2^n x_3^{3i-1} x_4^{r-1-3i}) + f_2 x_1 x_3^{3i-1} x_4^{r-3i+1}.
\]

(b) For \( i < j \), take \( l = j - i \)

\[
S(f_{(3,i)}, f_{(4,j)}) = (x_1^{n-l} x_3^{3l-1} - x_2^{n-l+1} x_4^{3l-2}) x_3^{3i+2} x_4^{r+1-3j} = f_2 x_3^{3i+2} x_4^{r+1-3j}.
\]

(c) For \( i > j \) and \( l = j - i \)

\[
S(f_{(3,i)}, f_{(4,j)}) = f_2 x_1^{(i-j)+1} x_3^{3j-1} x_4^{r+1-3j} + (-f_1) \sum_{l=0}^{i-j} x_1 x_3^{n-j+i-l} x_3^{3i-1-3l} x_4^{r-1-3i+3l}.
\]

(18) \( S(f_{(3,i)}, f_5) = x_1^{i+1} x_4^{r+2} - x_2^{i+1} x_3^{3i} x_4^{r-1-3i} = -(x_2 x_3^3 - x_1 x_4^3) \left( \sum_{l=0}^{i} x_1 x_2^{2-l} x_3^{3i-3l} x_4^{r-3i+3l} \right) = -f_1 \left( \sum_{l=0}^{i} x_1 x_2^{2-l} x_3^{3i-3l} x_4^{r-3i+3l-1} \right), 0 \leq i \leq n - 2. \)

(19) \( S(f_{(3,i)}, f_6) = x_1^{r+3+i} x_4^2 - x_2^{2+i} x_3^{3i} x_4^{r-1-3i} = (x_1^{i+2} - x_2 x_4)(x_1^{i+1} x_4^2) - (x_2 x_3^3 - x_1 x_4^3) \left( \sum_{l=0}^{i} x_1 x_2^{1+3(i-l)} x_3^{3l} x_4^{r-3i+3l} \right) = g x_1^{i+1} x_4^2 - f_1 \left( \sum_{l=0}^{i} x_1 x_2^{1+i-(i-l)} x_3^{3(i-l)} x_4^{r-3i+3l-1} \right), 0 \leq i \leq n - 2. \)

(20) \( S(f_{(3,i)}, f_7) = x_1^{n+1+i} x_3 x_4 - x_2^{n+i+2} x_3^{2+i} x_4^{r-1-3i} = (x_1^{i+2} - x_2 x_4)(x_1^{n+i} x_3 x_4) + (x_2 x_3^3 - x_1 x_4^3) \left( \sum_{l=0}^{i-2} x_1 x_2^{n+i-l} x_3^{3l} x_4^{r-3i+3l} \right) = g(x_1^{n+i} x_3 x_4) + f_1(-x_1 x_2 x_3 x_4^{r-1}) - f_1 \left( \sum_{l=0}^{i-2} x_1 x_2^{n+i-(1-l)} x_3^{3l} x_4^{r-3i+3l} \right), 0 \leq i \leq n - 1. \)

(21) \( S(g, f_{(4,0)}) = -(x_2^{2n+1} - x_1^{2n-1} x_3 x_4) x_4^r \)
\[ \begin{align*}
S(g, f_{(4,1)}) &= (x_1^{n-1}x_3^2 - x_2^n x_4)(x_1^{n-i}x_3^{3i-1}x_4^{r+1-3i} + x_2 x_4^{r-1}) \\
&= f_{(2,1)}(x_1^{n-i}x_3^{3i-1}x_4^{r+1-3i} + x_2 x_4^{r-1}).
\end{align*} \]

\[ \begin{align*}
S(g, f_{(4,i)}) &= x_3^{3i+1}x_4^{r+1-3i} - x_1^{i-2}x_3^{n+1}x_2^{2n-1}x_4^r \\
&= (x_1^{n-1}x_3^2 - x_2^n x_4)(x_1^{n-i}x_3^{3i-1}x_4^{r+1-3i} + x_2 x_4^{r-1}) \\
&+ (x_2 x_3^3 - x_1 x_4^3)(\sum_{l=0}^{i-2} x_1^{n-i+l}x_2^{n-1-l}x_3^{3i-4-3l}x_4^{r+2-3i+3l}) \\
&= f_{(2,1)}(x_1^{n-i}x_3^{3i-1}x_4^{r+1-3i} + x_2 x_4^{r-1}) \\
&+ f_1(\sum_{l=0}^{i-2} x_1^{n-i+l}x_2^{n-1-l}x_3^{3i-4-3l}x_4^{r+2-3i+3l}), \quad i \geq 2.
\end{align*} \]

(22) \( S(g, f_5) = x_1 x_4^{r+2} - x_2 x_3 x_4^r = (x_2^{n+1} x_3 - x_1 x_4^n)(-x_4^r) = f_6(-x_4^r). \)

(23) \( S(g, f_6) = -x_2^{n+2} x_3 x_4^r + x_1^{r+n+2} x_4^2 \rightarrow G \) 0, since \( \gcd(Lt(g), Lt(f_6)) = 1. \)

(24) \( S(g, f_7) \rightarrow G \) 0, since \( \gcd(Lt(g), Lt(f_7)) = 1. \)

(25) \( S(f_{(4,i)}, f_{(4,j)}) = x_2^{j-i} x_3^{3j+1} x_4^{r+1-3j} - x_1^{j-i} x_3^{3i+1} x_4^{r+1-3i} \\
= (x_2 x_3^3 - x_1 x_4^3)(\sum_{l=0}^{j-i-1} x_1^l x_2^{(j-i)-(l+1)} x_3^{3j-(2+3l)} x_4^{r-3j+(1+3l)}) \\
= f_1(\sum_{l=0}^{j-i-1} x_1^l x_2^{(j-i)-(l+1)} x_3^{3j-(2+3l)} x_4^{r-3j+(1+3l)}), \quad i < j.
\]

(26) \( S(f_{(4,i)}, f_5) = x_2^{n-i} x_4^r - x_1^{n-i} x_3^{3i+2} x_4^{r+1-3i} \\
= (x_1^{n-i-1} x_3^{3i+2} - x_2^{n-i} x_4^{r+1-3i})(-x_4^r) \\
= f_{(2,i+1)}(-x_4^r), \quad 0 \leq i \leq n - 1.
\]

(27) We consider two cases for \( S(f_{(4,i)}, f_6): \)

(a) For \( 0 \leq i < n - 1, \)

\( S(f_{(4,i)}, f_6) = x_1^{r-n+3+i} x_2^{n-i} x_3^{3i+2} x_4^{r+1-3i} \\
= (x_1^{r-n+3+j} x_2^{n-i-j} x_3^{2+3j} x_4^{r-1-3j} x_4^2) \\
= x_4^2 f_{(3,i)}. \)

(b) For \( i = n - 1, \)

\( S(f_{(4,i)}, f_6) = x_1^{r+2} x_4^2 - x_3^{3n-1} x_4^{r-3n+4} \\
= (x_1^{r+2} - x_2 x_4^r) x_4^2 - (x_3^{3n-1} - x_2 x_4^{3n-2}) x_4^{r-3n+4} \)
For $0 \leq i \leq n - 1$,
\[
S(f_{4,i}, f_7) = x_1^{r+2+i}x_3x_4 - x_2^{1+i}x_3^{3i+1}x_4^{r+1-3i} = (x_1^{r+2} - x_2x_4^r)(x_1^i x_3 x_4)
\]
\[
- (x_2x_3^3 - x_1x_4^3)(\sum_{l=0}^{i-1} x_1^l x_2^{i-l} x_3^{3i+(-3l-2)} x_4^{r-3i+1+3l})
\]
\[
= g(x_1^i x_3 x_4) - f_1(\sum_{l=0}^{i-1} x_1^l x_2^{i-l} x_3^{3i+(-3l-2)} x_4^{r-3i+1+3l}).
\]

(30) $S(f_5, f_6) = x_1^{r+2} x_4^2 - x_2x_4^{r+2} = (x_1^{r+2} - x_2x_4^r)x_4^2 = gx_4^2.$

(31) $S(f_5, f_7) = -x_2^{n+1}x_4^{r+2} + x_1^{r+n+1}x_3^2 x_4$
\[
= (x_1^{-1} x_3^2 - x_2^2 x_4^4)(x_1^{r+2} x_4 + x_2x_4^{r+1}) + (x_1^{r-n+3} x_2^{n-1} - x_3^{2} x_4^{r-1}) x_1^{-1} x_2x_4^2
\]
\[
= f_{(2,1)}(x_1^{r+2} x_4 + x_2x_4^{r+1}) + f_{(3,0)} x_1^{-1} x_2 x_4^2.
\]

Each $S$-polynomial reduces to zero, therefore, by Buchberger’s Criterion, $G_{nr}$ is a Gröbner basis of $\mathcal{Q}_{nr}$, with respect to the degree reverse lexicographic monomial order $>\text{induced by} x_1 > x_2 > x_3 > x_4$.

**Corollary 2.6.** Let us consider the degree reverse lexicographic monomial order on $k[x_1, \ldots, x_4]$, induced by $x_1 > x_2 > x_3 > x_4$. Then, with respect to this order,
\[
G(\text{in}_<(\mathcal{Q}_{nr})) = \{x_1^{r+2}, x_2x_3^3, x_3 x_4^{r-n+2}, x_2^{n+1} x_3, x_2^{n-1} x_3, x_2^{2n+1}\} \cup \{x_1^{n-i}x_3^{3i-1} | 1 \leq i \leq n\} \cup \{x_1^{r-n+3+j} x_2^{n-1-j} | 0 \leq j \leq n-2\} \cup \{x_1^{r-2n+3+j} x_2^{2n-l} | 0 \leq l \leq n-1\}.
\]

**Proof.** Follows from Theorem 2.5

We now use two theorems from [11], written below, in order to determine the arithmetic Cohen-Macaulayness of the projective closure.

**Lemma 2.7.** Let $I$ be an ideal in $A = k[x_1, \ldots, x_r]$ and $I^H \subset A[x_0]$ its homogenization with respect to the variable $x_0$. Let $<\text{be any reverse lexicographic monomial order on} A$ and $<\text{the reverse lexicographic monomial order on} A[x_0]$ extended from $A$ such that $x_i > x_0$ for all $i$.

If $\{f_1, \ldots, f_n\}$ is the reduced Gröbner basis for $I$ w.r.t $<$, then $\{f_1^H, \ldots, f_n^H\}$ is the reduced Gröbner basis for $I^H$ w.r.t $<\text{, and } \text{in}_{<}(I^H) = (\text{in}_{<}(I))A[x_0]$.

**Proof.** See Lemma 2.1 in [11].

**Theorem 2.8.** Let $n : n_1, \ldots, n_r$ be a sequence of positive integers with $n_r > n_i$ for all $i < n$. Let $<\text{ any reverse lexicographic order on} A = k[x_1, \ldots, x_r]$ such that $x_i > x_r$ for all $1 \leq i < r$ and $<\text{the induced reverse lexicographic order on} A[x_0]$, where $x_n > x_0$. Then the following conditions are equivalent:

(i) The projective monomial curve $C(n_1, \ldots, n_r)$ is arithmetically Cohen-Macaulay.
We now use the algorithm 2.6 of [2]. Rearranging the generator of \( \mathcal{P} \) is a Cohen-Macaulay ideal.

\[ \text{(ii) in}_{<0}((p(n_1, \ldots, n_r))^H) \] (homogenization with respect to \( x_0 \)) is a Cohen-Macaulay ideal.

\[ \text{(iii) in}_{<}(p(n_1, \ldots, n_r)) \] is a Cohen-Macaulay ideal.

\[ \text{(iv) } x_r \text{ does not divide any element of } G(\text{in}_{<}(p(n_1, \ldots, n_r))). \]

**Proof.** See Theorem 2.2 in [11].

**Theorem 2.9.** The projective closure \( \overline{\mathcal{P}}_{nr} \) of the Backelin curve is arithmetically Cohen-Macaulay.

**Proof.** From Corollary 2.6, we see that \( x_4 \) does not divide any element of \( G(\text{in}_{<}(\mathcal{P}_{nr})) \). The proof follows from Theorem 2.8.

## 3. Hilbert Series of the Backelin Curves

In this section, we compute the Hilbert series of the Backelin curve, using Algorithm 2.6 of [2].

**Lemma 3.1.** Let \( I \subseteq k[x_1, \ldots, x_n] \) be a graded ideal and \( < \) a monomial order on \( k[x_1, \ldots, x_n] \). Then \( k[x_1, \ldots, x_n]/I \) and \( k[x_1, \ldots, x_n]/\text{in}_{<}(I) \) have the same Hilbert function, i.e.

\[ H(k[x_1, \ldots, x_n]/I, i) = H(k[x_1, \ldots, x_n]/\text{in}_{<}(I), i) \]

for all \( i \).

**Proof.** See Corollary 6.1.5 in [10].

For a tuple \( A = (a_1, \ldots, a_n) \in \mathbb{N}^n \), we write \( x^A := x_1^{a_1} \cdots x_n^{a_n} \), and \( |A| \) denotes the total degree of the monomial \( x^A \).

**Lemma 3.2.** Let \( I = (x^{A_1}, \ldots, x^{A_l}) \subseteq k[x_1, \ldots, x_n] \) be a monomial ideal. Let \( p(I) \) denote the numerator of the Hilbert series of \( k[x_1, x_2, \ldots, x_n]/I \). Then

\[ p(I) = p(x^{A_1}) - \sum_{i=2}^{l} t^{A_{i-1}}p(x^{A_{i-1}:x^{A_i}}), \]

where \( p(x^{A_1}, \ldots, x^{A_{i-1}:x^{A_i}}) \) denotes the numerator of the Hilbert Series of the ideal \( (x^{A_1}, \ldots, x^{A_{i-1}} : x^{A_i}) \).

**Proof.** See Corollary 2.3 in [2].

**Theorem 3.3.** The numerator of the Hilbert series of the defining ideal \( \mathcal{P}_{nr} \) of the Backelin curve is

\[ 1 - nt^{r+2} - 2t^{r+3} + (3n + 4)t^{r+4} - (2n + 2)t^{r+5} - t^{2n+3} + 2t^{2n+2} - t^{2n+1} + t^{n+4} + t^{n+3} \]

\[ - t^{n+2} - t^{n+1} - t^4 - \sum_{i=2}^{n} (t^{n+2i-1} + t^{n+2i-1} - 2t^{n+2i}). \]

**Proof.** From Corollary 2.6, we have, in \( \text{in}_{<}(\mathcal{P}_{nr}) \) is generated by the set

\[ \{x_2x_3, x_1^{-i}x_3^{i-1}, x_1^{-r-3+3j}x_2^{-n-1-j}, x_1^{-r-2n+3j}x_2^{-n-1-j}, x_1^{-r-n+2n}x_2, x_1^{-r-n-2-n}x_3, x_1^{-r-n+2n+1}x_3, x_1^{-r-n-2-n+1} \mid 1 \leq i \leq n, 0 \leq j \leq n-1 \}. \]

We now use the algorithm 2.6 of [2]. Rearranging the generator of \( \text{in}_{<}(\mathcal{P}_{nr}) \), so that they are in ascending lexicographic order on the reversed set of variables \( x_4 > x_3 > x_2 > x_1 \), we have

\[ G(\text{in}_{<}(\mathcal{P}_{nr})) = \{x_1^{r+n+2}, x_1^{r+n+1}x_2, \ldots, x_1^{r-n+3}x_2^{n-1}, x_1^{r-n+2}x_2^{n+1}, \ldots, x_1^{r-n-3}x_2^{2n}, x_2^{2n+1}, \}

\[ x_1^{-n+2}x_2^{n+1}x_3, x_1^{-n+2}x_2^{n+1}x_3, x_1^{-2}x_3, x_1^{-1}x_3, x_1x_3, x_1^{-3}x_3, x_3^{-3} \}. \]
By Lemma 3.1, the Hilbert series of $k[x_1, x_2, x_3, x_4]/I$ is equal to the Hilbert series of $\frac{k[x_1, x_2, x_3, x_4]}{in_c(\mathfrak{P}_m)}$. Therefore, it is sufficient to compute the Hilbert series of the latter. Let $I$ denote the monomial ideal $in_c(\mathfrak{P}_m)$ and $p(I)$ is the numerator of the Hilbert series of $in_c(\mathfrak{P}_m)$. Now using Lemma 3.2 to the ideal $I$, we have,

$$p(I) = p(x_1^{r+2}, x_1^{r+1}x_2, \ldots, x_1^{r-n+3}x_2^{n-1}, x_1^{r-n+2}x_2^{n+1}, x_1^{r-n+2}x_2^{n+1}, x_2^{n}x_3, x_2^{n+1}x_3, x_3^{n-1}x_3, x_3^{n-2}x_3, \ldots, x_3^{3n-1}).$$

Let

$$A_1 = x_1^{r+2}, A_2 = x_2^{2+n}, A_3 = x_3^{r-n+2}x_2^{n+1}, A_4 = x_2^{n+1}x_3, A_5 = x_2x_3^3,$$

$$B_i = x_1^{n-i}x_2^{n-i-1}, 0 \leq i \leq n - 2;$$

$$C_j = x_1^{2n+3-j}x_2^{2n-j}, 0 \leq j \leq n - 1;$$

$$D_1 = x_1^{n-1}x_3^2, D_l = x_1^{n-l}x_3^{3l-1}, 2 \leq l \leq n.$$

After arranging them in the ascending lexicographic order on the reversed set of variables $x_1 > x_2 > x_3$, we have

$$in_c(\mathfrak{P}_m) = \langle A_1, B_{n-2}, \ldots, B_0, C_{n-1}, \ldots, C_0, A_2, A_3, A_4, D_1, A_5, D_2, \ldots, D_n \rangle.$$ 

Using Lemma 3.2 we now have,

$$p(I) = p(A_1) - t^{r+2}p(A_1 : B_{n-2}) - t^{r+2}\sum_{i=0}^{n-3} p(A_1, B_{n-2}, \ldots, B_{i+1} : B_i)$$

$$- t^{r+3}p(A_1, B_{n-2}, \ldots, B_0 : C_{n-1}) - t^{r+3}\sum_{j=0}^{n-2} p(A_1, B_{n-2}, \ldots, B_0, C_{n-1}, \ldots, C_{j+1} : C_j)$$

$$- t^{2n+1}p(A_1, B_{n-2}, \ldots, B_0, C_{n-1}, \ldots, C_0 : A_2)$$

$$- t^{r+3}p(A_1, B_{n-2}, \ldots, B_0, C_{n-1}, \ldots, C_0, A_2 : A_3)$$

$$- t^{n+2}p(A_1, B_{n-2}, \ldots, B_0, C_{n-1}, \ldots, C_0, A_2, A_3 : A_4)$$

$$- t^{n+1}p(A_1, B_{n-2}, \ldots, B_0, C_{n-1}, \ldots, C_0, A_2, A_3, A_4 : D_1)$$

$$- t^4p(A_1, B_{n-2}, \ldots, B_0, C_{n-1}, \ldots, C_0, A_2, A_3, A_4, D_1 : A_5)$$

$$- \sum_{l=2}^{n} t^n+2l-1p(A_1, B_{n-2}, \ldots, B_0, C_{n-1}, \ldots, C_0, A_2, A_3, A_4, D_1, A_5, D_2, \ldots, D_{l-1} : D_l);$$

where

(i) $p(A_1) = p(x_1^{r+2}) = (1 - t^{r+2});$

(ii) $p(A_1 : B_{n-2}) = p(x_1) = (1 - t);$

(iii) $p(A_1, B_{n-2}, \ldots, B_{i+1} : B_i) = p(x_1) = (1 - t);$

(iv) $p(A_1, B_{n-2}, \ldots, B_0 : C_{n-1}) = p(x_1) = (1 - t);$

(v) $p(A_1, B_{n-2}, \ldots, B_0, C_{n-1}, \ldots, C_{j+1} : C_j) = p(x_1) = (1 - t);$
(vi) \( p(A_1, B_{n-2}, \ldots, B_0, C_{n-1}, \ldots, C_0 : A_2) = p(x_1^{r-2n+3}) = (1 - t^{r-2n+3}); \)

(vii) \( p(A_1, B_{n-2}, \ldots, B_0, C_{n-1}, \ldots, C_0, A_2 : A_3) = p(x_1, x_2) = (1 + t^2 - 2t); \)

(viii) \( p(A_1, B_{n-2}, \ldots, B_0, C_{n-1}, \ldots, C_0, A_2, A_3 : A_4) = p(x_1^{r-n+2}, x_1^{r-2n+3}x_2^{n-1}, x_2^n) = (1 - t^n - nt^{r-n+2} + nt^{r-n+3}); \)

(ix) \( p(A_1, B_{n-2}, \ldots, B_0, C_{n-1}, \ldots, C_0, A_2, A_3, A_4 : D_1) = p(x_2^{n+1}, x_1^{r-2n+3+j}x_2^{n-j}) = (1 - t^{n+1} - (n + 1)t^{r-n+3} + (n + 1)t^{r-n+4}), \) where \( 0 \leq j \leq n; \)

(x) \( p(A_1, B_{n-2}, \ldots, B_0, C_{n-1}, \ldots, C_0, A_2, A_3, A_4, D_1 : A_5) = p(x_1^{n-1}, x_2^n) = (1 - t^{n-1} - t^n + t^{2n-1}); \)

(xi) \( p(A_1, B_{n-2}, \ldots, B_0, C_{n-1}, \ldots, C_0, A_2, A_3, A_4, D_1, A_5, D_2, \ldots, D_{t-1} : D_t) = p(x_1, x_2) = (1 + t^2 - 2t). \)

Therefore,
\[
p(I) = (1 - t^{r+2}) - (n - 1)t^{r+2}(1 - t) - nt^{r+3}(1 - t) - t^{2n+1}(1 - t^{r-2n+3}) - t^{r+3}(1 + t^2 - 2t)
- t^{n+2}(1 - t^n - nt^{r-n+2} + nt^{r-n+3}) - t^{n+1}(1 - t^{n+1} - (n + 1)t^{r-n+3} + (n + 1)t^{r-n+4})
- t^4(1 - t^{n-1} - t^n + t^{2n-1}) - (1 + t^2 - 2t)(\sum_{i=2}^{n} t^{n+2i-1}),
\]
which is the same as
\[
p(I) = 1 - t^{r+2} - (n - 1)t^{r+2} + (n - 1)t^{r+3} - nt^{r+3} + nt^{r+4} - t^{2n+1} + t^{r+4} - t^{r+3} - t^{r+5} + 2t^{r+4}
- t^{n+2} + t^{2n+2} + nt^{r+4} - nt^{r+5} - t^{n+1} + t^{2n+2} + (n + 1)t^{r+4} - (n + 1)t^{r+5} - t^4 + t^{n+3}
+ t^{n+4} - t^{2n+3} - \sum_{i=2}^{n} (t^{n+2i-1} + t^{n+2i+1}2t^{n+2i}).
\]

Therefore,
\[
p(I) = 1 - nt^{r+2} - 2t^{r+3} + (3n + 4)t^{r+4} - (2n + 2)t^{r+5} - t^{2n+3} + 2t^{2n+2} - t^{2n+1} + t^{n+4} + t^{n+3}
- t^{n+2} - t^{n+1} + t^4 - \sum_{i=2}^{n} (t^{n+2i-1} + t^{n+2i+1} - 2t^{n+2i}).
\]

Hence, \( \frac{p(I)}{(1 - t)^2} \) is the Hilbert Series of the ring \( k[x_1, x_2, x_3, x_4]/I \).

4. Syzygies of the Affine and Projective Backelin Curve

In this section, we compute the minimal free resolution of the defining ideal of the projective closure of the Backelin curve by computing the syzygies explicitly.

**Notation 4.1.** Let us define the following matrices, which would be useful for writing the syzygies:

(a) \( \mathbf{C}^1_0 = (f^H_1, f^H_{(2,1)}; f^H_6, f^H_{(2,2)}; \ldots, f^H_{(2,[\frac{n}{2}])}; f^H_7, f^H_{(2,[\frac{n}{2}]+1)}; \ldots, f^H_{(2, n)}; g^H_1, f^H_{(3, n-2)}; \ldots, f^H_{(3, 0)}; f^H_{(4, n-1)}; \ldots, f^H_{(4, 0)}; f^H_{5}) \)
such that,

\[ R = \begin{bmatrix}
0_{n \times n} & 0_{n \times (n+1)} & T_{n \times (n-2)} & T_{n \times (n+1)} & T_{n \times (n+2)} & T_{n \times (n-1)} & T_{n \times 4}
\end{bmatrix}_{(3n+4) \times (6n+5)} \]

- \( r_{1,1} = -x_{4}^{r-1} \cdot x_{0} \),
- \( r_{1,2+j} = -x_{4}^{1+3j} \cdot x_{4}^{3n-3j} \cdot x_{0} \), where \( 0 \leq j \leq n - 2 \),
- \( r_{1,n+1} = -x_{0}^{2+3i} \cdot x_{4}^{r-3i-4} \), where \( 0 \leq i \leq n - 3 \),
- \( r_{1,2n+1+j} = -x_{2}^{j+1} \cdot x_{4}^{3n-5-3j} \), where \( 0 \leq j \leq n - 2 \),
- \( r_{1,3n+2+j} = -x_{3}^{j} \cdot x_{4}^{3n-4-3j} \), where \( 0 \leq j \leq n - 2 \),
- \( r_{1,4n} = -x_{4}^{n-2} \cdot x_{2}^{j} \),
- \( r_{1,6n+3} = -x_{4}^{n-1} \),
- \( r_{1,6n+5} = -x_{2}^{j} \),

\( r_{1,l} = 0 \), for \( 0 \leq j \leq n - 2 \), \( 0 \leq i \leq n - 3 \) and \( l \in \{1, \ldots, 6n+5\} \setminus \{1, 2+j, n+2+i, 2n+1+j, 3n+2+j, 4n, 6n+3, 6n+5\} \).

\[ T_{n \times (n-2)}^{i} = \begin{bmatrix}
0_{(n-2) \times (n-2)} & T_{n \times (n-2)}^{i,1} & T_{n \times (n-2)}^{i,2} & T_{n \times (n-2)}^{i,3}
\end{bmatrix} \]

\[ T_{n \times (n-2)}^{i,1} = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & -x_{3}^{i}
0 & 0 & \cdots & 0 & 0 & 0
0 & 0 & \cdots & 0 & -x_{3}^{i} & x_{1}
0 & 0 & \cdots & -x_{3}^{i} & x_{1} & 0
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots
0 & -x_{3}^{i} & x_{1} & 0 & 0 & 0
-x_{3}^{i} & x_{1} & 0 & 0 & 0 & 0
\end{bmatrix} \]

\[ T_{n \times (n-2)}^{i,2} = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & 0
0 & 0 & \cdots & 0 & 0 & -x_{3}^{i}
0 & 0 & \cdots & 0 & -x_{3}^{i} & x_{1}
0 & 0 & \cdots & -x_{3}^{i} & x_{1} & 0
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots
0 & -x_{3}^{i} & x_{1} & 0 & 0 & 0
-x_{3}^{i} & x_{1} & 0 & 0 & 0 & 0
\end{bmatrix} \]

\[ T_{n \times (n-2)}^{i,3} = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & 0
0 & 0 & \cdots & 0 & 0 & 0
0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix} \]

\[ T_{n \times (n+1)}^{i} = \begin{bmatrix}
T_{n \times (n-2)}^{i,1} & T_{n \times (n-2)}^{i,2} & T_{n \times (n-2)}^{i,3}
\end{bmatrix} \]
\[ T_{\frac{n}{2}+2 \times \frac{n}{2}+1}^{ii,1} = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
-x_1^{2n+4} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix},
\]

\[ T_{\frac{n}{2}+2 \times \frac{n}{2}}^{ii,2} = \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 \times \frac{2}{2} \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & -x_1 x_2 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & -x_1 x_2 & 0 & 0 & 0
\end{bmatrix},
\]

\[ T_{\frac{n}{2}+2 \times \frac{n}{2}+1}^{ii,3} = \begin{bmatrix}
0 & -x_1^{n+5} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & -x_1^2 \\
0 & 0 & 0 & 0 & \cdots & 0 & -x_1^2 x_2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & -x_1^2 x_2 & 0 & 0 & 0
\end{bmatrix},
\]

\[ T_{\frac{n}{2} \times \frac{n}{2}+1}^{ii,4} = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
x_2 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix},
\]

\[ T_{\frac{n}{2} \times (n+2)}^{iii} = \begin{bmatrix}
-x_1^{r-2n+3} x_2^n & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & -x_1^{r-2n+3} x_2^{n-1} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\
0 & 0 & -x_1^{r-2n+3} x_2^{n-1} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0
\end{bmatrix},
\]

\[ T_{\frac{n}{2} \times (n-1)}^{iv} = \begin{bmatrix}
x_1 x_2 & \cdots & x_1 x_2^{n+1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \ddots & \ddots
\end{bmatrix},
\]

\[ T_{n \times 4} = \begin{bmatrix}
x_1 x_2 & -x_2 x_3 & -x_1^{n+3} x_2 x_3 & 0 \\
-x_1 x_2 & x_3 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\[ S_{(n+2) \times n}^{i} = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 0 \\
x_2 x_3 & 0 & 0 & \cdots & 0
\end{bmatrix}.
\]
\[
\begin{align*}
\bullet S^{\text{ii}}_{(n+2)\times(n+1)} &= \\
&= \begin{bmatrix}
 x_1^2 x_4^{r-3n+4} x_0 & 0 & 0 & \cdots & 0 & x_0 x_4^{r-3n+5} & -x_3^2 \\
 0 & 0 & 0 & \cdots & 0 & 0 & x_1 \\
 0 & 0 & 0 & \cdots & 0 & -x_2 & 0 \\
 0 & 0 & 0 & \cdots & -x_2 & x_1 & 0 \\
 0 & 0 & 0 & \cdots & \vdots & \vdots & \vdots \\
 0 & -x_2 & x_1 & 0 & 0 & 0 & 0 \\
 -x_2^2 & x_1 & 0 & 0 & 0 & 0
\end{bmatrix} \\

\bullet S^{\text{iii}}_{(n+2)\times(n+1)} &= \\
&= \begin{bmatrix}
 x_1 & 0 & 0 & \cdots & 0 \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & 0 & \cdots & 0 \\
 0 & 0 & 0 & \cdots & 0
\end{bmatrix} \\

\bullet S^{\text{iv}}_{(n+2)\times(n+1)} &= \\
&= \begin{bmatrix}
 0 & 0 & -x_4 & x_2 & 0 & \cdots & 0 \\
 0 & 0 & -x_4 & x_2 & 0 & \cdots & 0 \\
 -x_4 & -x_3 x_4 & 0 & 0 & \cdots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix} \\

\bullet S^{\text{v}}_{(n+2)\times(n+1)} &= \\
&= \begin{bmatrix}
 0 & 0 & \cdots & 0 & 0 \\
 0 & 0 & \cdots & 0 & 0 \\
 0 & 0 & \cdots & 0 & -x_4 \\
 0 & 0 & \cdots & -x_4 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & -x_4 & 0 & 0 & 0 \\
 0 & -x_4 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0
\end{bmatrix} \\

\bullet S^{\text{vi}}_{(n+2)\times(n-1)} &= \\
&= \begin{bmatrix}
 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & x_4^2 & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & -x_2 x_4 & 0
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\bullet P^{\text{i}}_{(n+1)\times n} &= \\
&= \begin{bmatrix}
 x_1 & \cdots & 0 & 0 & 0 \\
 0 & \cdots & 0 & -x_2 & x_1 \\
 0 & \cdots & -x_2 & x_1 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & -x_2 & x_1 & 0 & 0 \\
 0 & x_1 & 0 & 0 & 0 \\
 x_1 & 0 & 0 & 0 & 0
\end{bmatrix} \\

\bullet P^{\text{ii}}_{(n+1)\times(n+1)} &= \\
&= \begin{bmatrix}
 x_1 & \cdots & 0 & 0 & 0 \\
 0 & \cdots & 0 & 0 & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & 0 & 0 & 0 \\
 x_2 & 0 & 0 & 0 & 0 \\
 x_2 & 0 & 0 & 0 & 0
\end{bmatrix} \\

\bullet P^{\text{iii}}_{(n+1)\times(n+1)} &= \\
&= \begin{bmatrix}
 0 & 0 & 0 & \cdots & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & 0 & 0 & \cdots & 0 \\
 x_2 & 0 & 0 & \cdots & 0 \\
 x_2 & 0 & 0 & \cdots & 0
\end{bmatrix}
\]
\[ \bullet P^\text{iv}_{(n+1)\times(n+2)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & x_3 & 0 \\ 0 & 0 & \cdots & x_3 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ -x_4 & x_3 & 0 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \bullet P^\text{v}_{(n+1)\times(n-1)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & x_4 \\ 0 & 0 & \cdots & x_4 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & x_4 & 0 & 0 & 0 \\ x_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ (c) \mathcal{C}^3_h := \mathcal{C}^3_{(6n+5)\times(3n+2)} = \begin{bmatrix} X^{(n+1)\times(n+1)} & Y^{(n+1)\times(n-1)} & 0^{(n+1)\times(n+2)} \\ X'^{(n-1)\times(n+1)} & Y'^{(n-1)\times(n-1)} & 0^{(n-1)\times(n+2)} \\ X^{n\times(n+1)} & 0^{n\times(n+1)} & Z^{n\times(n+2)} \\ X'^{n\times(n+1)} & Y'^{n\times(n-1)} & Z'^{n\times(n+2)} \end{bmatrix}, \]

where

\[ \bullet X^{(n+1)\times(n+1)} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & x_2 & 0 \\ 0 & 0 & 0 & \cdots & -x_4 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & -x_4 & \cdots & 0 & 0 & 0 \\ -x_4 & 0 & 0 & \cdots & 0 & -x_3 & 0 \end{bmatrix} \]

\[ \bullet Y^{(n+1)\times(n-1)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & x_3 & 0 \\ 0 & 0 & \cdots & x_3 & x_4 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & x_3 & 0 & 0 & 0 & 0 \\ x_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \bullet X'^{(n-1)\times(n+1)} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & x_3^2 & 0 \\ 0 & 0 & \cdots & x_3^2 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & x_3^2 & 0 & 0 & 0 & 0 \\ x_3^2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \bullet Y'^{(n-1)\times(n+1)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & x_3^2 \\ 0 & 0 & \cdots & x_3^2 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & x_3^2 & 0 & 0 & 0 \\ x_3^2 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ \bullet X'^{n\times(n+1)} = \begin{bmatrix} 0 & \cdots & 0 & 0 \end{bmatrix} \]
$$\begin{align*}
\mathcal{L}_{n \times (n+2)}'' &= \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & -x_2 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & -x_2 & x_4^3 & 0 & 0 & 0 \\
0 & 0 & \cdots & -x_2 & x_4^3 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & -x_2 & x_4^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
-x_2 & x_4^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -x_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & x_4 \\
\end{bmatrix}\\
\mathcal{X}_{n \times (n+1)}'' &= \begin{bmatrix}
0 & \cdots & 0 & 0 & x_4^{-3n+4}x_0 \\
0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 \\
0 & \cdots & 0 & -x_4^{-3n+4}x_0 & 0 \\
\end{bmatrix}\\
\mathcal{Y}_{n \times (n+1)}'' &= \begin{bmatrix}
0 & \cdots & 0 & 0 & x_1 & 0 & 0 & 0 \\
0 & \cdots & 0 & x_1 & -x_4^3 & 0 & 0 & 0 \\
0 & \cdots & x_1 & -x_4^3 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & x_1 & -x_4^3 & 0 & 0 & 0 & 0 & 0 \\
x_1 & -x_4^3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}\\
\mathcal{L}_{(n+1) \times (n+1)}'' &= \begin{bmatrix}
0 & \cdots & 0 & 0 & x_1 & 0 & 0 & 0 \\
0 & \cdots & 0 & x_1 & -x_2 & 0 & 0 & 0 \\
0 & \cdots & x_1 & -x_2 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & x_1 & -x_2 & 0 & 0 & 0 & 0 & 0 \\
x_1 & -x_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}\\
\mathcal{Y}_{(n+1) \times (n+1)}'' &= \begin{bmatrix}
0 & \cdots & 0 & x_2 & 0 & -x_4 \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & 0 & -x_4^{-2n+3} & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}\\
\mathcal{X}_{(n-1) \times (n+1)}'' &= \begin{bmatrix}
0 & \cdots & 0 & x_1 & 0 & 0 \\
0 & \cdots & 0 & x_1 & -x_2 & 0 & 0 \\
0 & \cdots & x_1 & -x_2 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & x_1 & -x_2 & 0 & 0 & 0 & 0 \\
x_1 & -x_2 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}\\
\mathcal{Y}_{(n-1) \times (n+1)}'' &= \begin{bmatrix}
0 & \cdots & 0 & x_2 \\
0 & \cdots & 0 & 0 & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 \\
\end{bmatrix}\\
\mathcal{X}_{4 \times (n+1)}'' &= \begin{bmatrix}
0 & 0 \cdots & 0 \\
0 & 0 \cdots & 0 \\
0 & x_2 & 0 \cdots & 0 \\
0 & 0 \cdots & 0 \\
\end{bmatrix}
\end{align*}$$
We can check that
\[ Z_{4 \times (n+2)}^{vi} = \begin{bmatrix}
0 & 0 & \cdots & 0 & x_1^{r-2n+3} & x_3 \\
-x_1^2 & 0 & \cdots & 0 & 0 & -x_1^2 \\
0 & 0 & \cdots & -x_4 & 0 & 0 \\
x_4 & 0 & \cdots & 0 & 0 & x_1^{n-1}
\end{bmatrix}. \]

Let us recall the Buchsbaum Eisenbud acyclicity criterion.

**Lemma 4.2.** Let \( R \) be a Noetherian ring, and
\[ F. : 0 \to F_s \xrightarrow{\phi_s} F_{s-1} \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to 0, \]
a complex of finite free \( R \)-modules. Set \( r_i = \sum_{j=i}^{s} (-1)^{j-i} \text{rank} F_j \). Then the following statements are equivalent:
- (a) \( F. \) is acyclic;
- (b) \( \text{grade} \ I_{r_i}(\phi_i) \geq i \) for \( i = 1, \ldots, s \).

**Proof.** See Theorem 1.4.13 in [5].

**Notation 4.3.** Given an \( n \times m \) matrix \( A \) with entries \( a_{ij} \), a minor of \( A \) is denoted by
\[ M = [r_1 \ r_2 \ \cdots \ r_p | c_1 \ c_2 \ \cdots \ c_p] = \det \begin{bmatrix}
ar_{r_1c_1} & a_{r_1c_2} & \cdots & a_{r_1c_p} \\
\vdots & \vdots & \cdots & \vdots \\
ar_{r_pc_1} & a_{r_pc_2} & \cdots & a_{r_rpc_p}
\end{bmatrix}, \]
where \( \{r_1, \ldots, r_p\} \subseteq \{1, \ldots, n\} \) and \( \{c_1, \ldots, c_p\} \subseteq \{1, \ldots, m\} \).

**Theorem 4.4.** For \( n \geq 4 \), a minimal graded free resolution of the defining ideal of the projective closure of the Backelin curve \( \Gamma_{nr} \) is given by
\[ \overline{\mathcal{R}}_h : 0 \to R^{3n+2} \xrightarrow{\epsilon^i_h} R^{6n+5} \xrightarrow{\epsilon^i_h} R^{3n+4} \xrightarrow{\epsilon^i_h} R \to R/\mathcal{P}_{nt} \to 0, \]
where the matrices \( \mathcal{C}^i_h, 1 \leq i \leq 3 \) are defined in notation 4.1. Moreover, for \( n \geq 4 \), a minimal graded free resolution of the defining ideal of the Backelin curve \( \Gamma_{nr} \) is given by,
\[ \mathcal{R}_h : 0 \to R^{3n+2} \xrightarrow{\epsilon^i_h} R^{6n+5} \xrightarrow{\epsilon^i_h} R^{3n+4} \xrightarrow{\epsilon^i_h} R \to R/\mathcal{P}_{nt} \to 0, \]
where the matrices \( \mathcal{C}^i_h, 1 \leq i \leq 3 \), are obtained by evaluating \( x_0 = 1 \) in \( \mathcal{C}^i_h, 1 \leq i \leq 3 \).

**Proof.** We can check that
\[ \overline{\mathcal{R}}_h : 0 \to R^{3n+2} \xrightarrow{\epsilon^i_h} R^{6n+5} \xrightarrow{\epsilon^i_h} R^{3n+4} \xrightarrow{\epsilon^i_h} R \to R/\mathcal{P}_{nt} \to 0 \]
is a chain complex by calculating \( \mathcal{C}^i_h \circ \mathcal{C}^{i+1}_h = 0, 1 \leq i \leq 2 \). To prove the exactness of the above chain complex, we use the Buchsbaum–Eisenbud acyclicity criterion (see Lemma 4.2). Let \( r_i \) be the \( i^{th} \) expected rank of \( \mathcal{C}^i_h \). Then,
- \( r_1 = \text{rank}(\mathcal{C}^1_h) - \text{rank}(\mathcal{C}^0_h) = (3n + 4) - (6n + 5) + (3n + 2) = 1; \)
- \( r_2 = \text{rank}(\mathcal{C}^2_h) - \text{rank}(\mathcal{C}^1_h) = (6n + 5) - (3n + 2) = 3n + 3; \)
- \( r_3 = \text{rank}(\mathcal{C}^3_h) = 3n + 2. \)

We need to show that \( \text{grade}(I_{r_i}(\mathcal{C}^i_h)) \geq i, 1 \leq i \leq 3 \), where \( I_{r_i}(\mathcal{C}^i_h) \) denotes the ideal generated by the \( r_i \times r_i \) minors of the matrix \( \mathcal{C}^i_h \).
Theorem 5.1. We study the syzygies of the projective closure of the Bresinsky curve and show that all the Betti numbers are unbounded functions of the defining ideal of these curves. Bresinsky proved that the minimal number of generators of \( I_h \) is \( 3 \). Let us consider the following polynomials:

\[
\mathcal{C}_h^{[21]} := [23 \ldots (3n + 4) \mid 1 \ldots (3n + 1) (6n + 3) (6n + 4)]
\]

\[
= -x_1^{-r-1} + x_3 \] with all the Betti numbers are unbounded functions of the defining ideal of these curves. Bresinsky proved that the minimal number of generators of \( I_h \) is \( 3 \). Let us consider the following polynomials:

\[
\mathcal{C}_h^{[22]} := [1 3 4 \ldots 3n + 4 \mid 2n (2n + 2) \ldots (3n - 1) (3n + 2) (4n + 1) \ldots (6n + 1) (6n + 4) (6n + 5)]
\]

\[
= x_3^{n-3} x_4 (x_2 x_4 - x_1 x_3^3) (x_3^{n-2} x_4^{r-3n+4} x_0 - x_1^{r-n+2} x_2^{n-1})
\]

We see that any two irreducible components in the factorization of \( \mathcal{C}_h^{[21]}, \mathcal{C}_h^{[22]} \) are coprime. Therefore \( \mathcal{C}_h^{[21]}, \mathcal{C}_h^{[22]} \) forms a regular sequence of length 2. Hence \( \text{grade}(I_{r_i}(\mathcal{C}_h^i)) \geq 2 \).

(2) We take the following minors from \( \mathcal{C}_h^3 \):

\[
\mathcal{C}_h^{[31]} := [1 (2n + 1) \ldots (3n - 2) (4n + 3) \ldots 5n (5n + 1) (5n + 3) (5n + 4) \ldots (6n + 1) (6n + 4) \mid 1 2 \ldots (3n + 2)]
\]

\[
= x_2^{n+1} - x_1^{2n-1} x_2 x_4;
\]

\[
\mathcal{C}_h^{[32]} := [1 2 \ldots 2 n (3n + 1) (3n + 2) (3n + 3) \ldots 4n 1 2 \ldots (3n + 2)] = x_3^{n-2} - x_2 x_3^{n-1} x_4^{3n-2};
\]

\[
\mathcal{C}_h^{[33]} := [3n (3n + 1) \ldots (6n + 2) (6n + 5) \mid 1 2 \ldots (3n + 2)] = x_1^{2r+4}.
\]

The leading terms of \( \mathcal{C}_h^{[31]}, \mathcal{C}_h^{[32]} \) and \( \mathcal{C}_h^{[33]} \), with respect to the negative degree reverse lexicographic ordering induced by \( x_1 < x_2 < x_3 < x_4 < x_0 \), are \( x_2^{2n+1}, x_3^{2n-2} \) and \( x_1^{2r+4} \), which are mutually coprime. Therefore, by Lemma 2.2 in [13], \( \mathcal{C}_h^{[31]}, \mathcal{C}_h^{[32]}, \mathcal{C}_h^{[33]} \) forms a regular sequence of length 3 and hence \( \text{grade}(I_{r_i}(\mathcal{C}_h^i)) \geq 3 \). This proves that the complex \( \mathcal{R}_n \) is exact and its minimality follows from the fact that all entries of the matrices \( \mathcal{C}_h^i, 1 \leq i \leq 3 \), lie in the homogeneous maximal ideal \( \langle x_0, \ldots, x_4 \rangle \). Hence \( \mathcal{R}_n \) is a minimal graded free resolution of \( \mathcal{P}_n \).

For the affine case we proceed as follows: We observe that \( \tilde{\mathcal{C}}_h \) is obtained by putting \( x_0 = 1 \) in the entries of \( \mathcal{C}_h \), hence it follows that \( \tilde{\mathcal{C}}_h^i \circ \mathcal{C}_h^{i+1} = 0, 1 \leq i \leq 2 \). After putting \( x_0 = 1 \) in the above minors, all the minors form a regular sequence and from Lemma 4.2 we have the exactness of \( \mathcal{R}_h \). We also have that \( \mathcal{R}_h \) is a minimal graded free resolution of \( \mathcal{P}_n \) since all the entries of the matrices \( \mathcal{C}_h^i, 1 \leq i \leq 3 \), lie in the homogeneous maximal ideal \( \langle x_0, \ldots, x_4 \rangle \).

\[ \square \]

5. SYZYGIES OF THE PROJECTIVE Closure OF THE BRESINSKY CURVE

In [3], Bresinsky defined the family of curves \( B_h \), for \( h \geq 2 \), defined by the family of numerical semigroups \( \Gamma((2h - 1)2h, (2h - 1)(2h + 1), 2h(2h + 1), 2h(2h + 1) + 2h - 1) \). Let \( \Omega_h \) denote the defining ideal of these curves. Bresinsky proved that the minimal number of generators \( \mu(\Omega_h) \) is an unbounded function of \( h \). Gröbner basis and syzygies of the Bersinsky curve have been computed and it has been proved that all the Betti numbers are unbounded functions of \( h \) in [12]. Here, we study the syzygies of the projective closure of the Bersinsky curve and show that all the Betti numbers are not unbounded function of \( h \). Note that, the last Betti number is 1 for the projective closure of the Bersinsky curve, but the projective curve is not Cohen-Macaulay.

**Theorem 5.1.** Consider the degree reverse lexicographic monomial order induced by \( x_1 > x_2 > x_3 > x_4 \), in the polynomial ring \( \mathbb{K}[x_1, x_2, x_3, x_4] \). For \( h \geq 2 \), let us consider the following polynomials:
$p_1 = x_2x_3 - x_1x_4$

$p_2 = x_2^{2h} - x_3^{2h-1}$

$p_{(3,j)} = x_1^{j+1}x_3^{2h-j} - x_2^j x_4^{2h-j}, \ 0 \leq j \leq 2h - 1$

$p_{(4,i)} = x_1^{i+1}x_2^{2h-i} - x_3^{i-1}x_4^{2h-i}, \ 1 \leq i \leq 2h$

$p_5 = x_3^{4h} - x_2^{2h-1}x_4^{2h+1}$

$p_{(6,i)} = x_1^{2i+1}x_2^{2h-2-i}x_4^{2+i} - x_3^{2h+1+i}, \ 0 \leq i \leq 2h - 3$

$p_7 = x_1x_2^{2h-1}x_4 - x_3^{2h}$

$p_8 = x_1^{2h}x_4 - x_3^{2h-1}$.

Let

$\mathfrak{G}_h = \{p_1, p_2, p_5, p_7, p_8\} \cup \{p_{(3,j)} | 0 \leq j \leq 2h - 1\} \cup \{p_{(4,i)} | 1 \leq i \leq 2h\} \cup \{p_{(6,i)} | 0 \leq i \leq 2h - 3\}$.

The set $\mathfrak{G}_h$ is a Gröbner basis of $\mathfrak{Q}_h$, with respect to the given monomial order.

**Proof.** We proceed by Buchberger’s algorithm. We compute each $S$-polynomial and show that it reduces to zero upon division by $\mathfrak{G}_h$.

(1) $S(p_1, p_2) = x_3^{2h} - x_1x_2^{2h-1}x_4 = -p_7$.

(2) For $0 \leq j < 2h - 1$,

$S(p_1, p_{(3,j)}) = x_4^{j+1}x_3^{2h-l} - x_1^{2+j}x_4x_3^{2h-j-1} = -x_4(x_1^{j+1}x_3^{2h-j} - x_2^j x_4^{2h-j})$

$= -x_4p_{(3,j+1)}$.

$S(p_1, p_{(3,2h-1)}) = x_2^{2h}x_4 - x_1^{2h+1}x_4 = x_4(x_2^{2h} - x_3^{2h-1}) - x_4(x_1^{2h+1} - x_3^{2h-1})$

$= x_4p_2 - x_4p_{(4,2h)}$.

(3) For $1 \leq i < 2h$,

$S(p_1, p_{(4,i)}) = x_3^ix_4^{2h-i} - x_1^{i+2}x_2^{2h-i-1}x_4 = -x_4(x_1^{i+2}x_2^{2h-i-1} - x_3^ix_4^{2h-i-1})$

$= -x_4p_{(4,i+1)}$;

$S(p_1, p_{(4,2h)}) = x_2x_3^{2h} - x_1^{2h+2}x_4$.

We have $\gcd(\text{Lt}(p_1), \text{Lt}(p_{(4,2h)})) = 1$, therefore, $S(p_1, p_{(4,2h)}) \rightarrow_{\mathfrak{G}_h} 0$.

(4) $S(p_1, p_5) = x_2^{2h}x_4^{2h+1} - x_1x_3^{4h-1}x_4 = (x_2^{2h} - x_3^{2h-1})x_4^{2h+1} - (x_1x_3^{2h} - x_4^{2h})x_3^{2h-1}x_4$

$= p_2x_4^{2h+1} - p_{(3,0)}x_3^{2h-1}x_4$.

(5) $S(p_1, p_{(6,i)}) = x_3^{2h+2+i}x_2^{3+i} - x_1^{3+i}x_2^{2h-3-i}x_4^{3+i} = -p_{(6,i+1)}, \ 0 \leq i \leq 2h - 4$

$= -p_8, \ i = 2h - 3$.

(6) $S(p_1, p_7) = x_3^{2h+1} - x_1^2x_2^{2h-2}x_4^{2} = -p_{(6,0)}$. 

PROJECTIVE CLOSURES OF AFFINE CURVES 21
(7) \( S(p_1, p_8) = x_2 x_3^{4h} - x_1^{2h+1} x_4^{2h+1} \)
\( = x_3^{4h-1}(x_2 x_3 - x_1 x_4) + (x_1^{2h} - x_4^{2h}) x_3^{2h-1} x_4 - x_4^{2h+1}(x_1^{2h+1} - x_3^{2h-1}) \)
\( = x_3^{4h-1} p_1 + p(3,0) x_3^{2h-1} x_4 - x_4^{2h+1} p(4,2h). \)

(8) \( S(p_2, p(3,l)) \rightarrow \phi_h 0, \text{ since } \gcd(Lt(p_2), Lt(p(3,l))) = 1. \)

(9) \( S(p_2, p(4,i)) = x_2^i x_3^{4h-1} x_4^{2h-1} - x_1^{i+1} x_3^{2h-1} \)
\( = (x_2^i x_3^{4h-1} - x_2^i x_4^{2h-1})(-x_1^{i-1}) + (x_2 x_3 - x_1 x_4)(\sum_{l=0}^{i-2} x_1^i x_2^{-i-l} x_3^{2h-i-l} x_4^{2h-i+l}) \)
\( = p(3,1)(-x_1^{i-1}) + p_1(\sum_{l=0}^{i-2} x_1^i x_2^{-i-l} x_3^{2h-i-l} x_4^{2h-i+l}), \quad 2 \leq i \leq 2h \)
\( S(p_2, p(4,1)) = -(x_2^2 x_3^{2h-1} - x_2 x_4^{2h-1}) = -p(3,1). \)

(10) \( S(p_2, p_5) \rightarrow \phi_h 0, \text{ since } \gcd(Lt(p_2), Lt(p(5))) = 1. \)

(11) \( S(p_2, p(6,i)) = x_2^{2+i} x_3^{2h+1+i} - x_1^{2+i} x_3^{2h-1} x_4^{2+i} \)
\( = -(x_2 x_3 - x_1 x_4)(\sum_{l=0}^{i+1} x_1^i x_2^{-i-l} x_3^{2h+i-l} x_4^l) \)
\( = -p_1(\sum_{l=0}^{i+1} x_1^i x_2^{-i-l} x_3^{2h+i-l} x_4^l), \quad 0 \leq i \leq 2h - 3. \)

(12) \( S(p_2, p_7) = x_2 x_3^{2h} - x_1 x_3^{2h-1} x_4 = (x_2 x_3 - x_1 x_4)x_3^{2h-1} = p_1 x_3^{2h-1}. \)

(13) \( S(p_2, p_8) = x_2^{2h} x_3^{4h-1} - x_1^{2h} x_3^{2h-1} x_4^{2h} = (x_2 x_3 - x_1 x_4)(\sum_{l=0}^{2h-1} x_1^l x_2^{2h-1-l} x_3^{4h-2-l} x_4^l) \)
\( = p_1(\sum_{l=0}^{2h-1} x_1^l x_2^{2h-1-l} x_3^{4h-2-l} x_4^l). \)

(14) \( S(p(3,i), p(3,j)) = x_2^j x_3^{j-i} x_4^{2h-j} - x_1^{j-i} x_2^i x_4^{2h-i} \)
\( = (x_2 x_3 - x_1 x_4)(\sum_{r=0}^{j-i-1} x_1^r x_2^{j-i-r} x_3^{j-i-2-r} x_4^{2h-j+r}) \)
\( = p_1(\sum_{r=0}^{j-i-1} x_1^r x_2^{j-i-r} x_3^{j-i-2-r} x_4^{2h-j+r}). \)
(15) Let us consider three cases separately for $S(p_{(3,j)}, p_{(4,i)})$.

(a) For $j = i$,
\[ S(p_{(3,j)}, p_{(4,i)}) = -x_2^{2h} x_4^{-i} + x_3^{2h-1} x_4^{-i} = -\left(x_2^{2h} - x_3^{2h-1}\right) x_4^{-i} = p_2(-x_4^{2h-i}). \]

(b) For $j < i$,
\[ S(p_{(3,j)}, p_{(4,i)}) = x_3^{2h-j+i-1} x_4^{-i} - x_1 x_2^{2h-i+j} x_4^{-j} = -x_4^{2h-i} \left(x_2^{i-j} x_4^{2h-i+j} x_4^{-i-j} - x_3^{2h+i-j-1}\right) = -x_4^{2h-i} p_{(6,i-j-2)}. \]

(c) For $j > i$,
\[ S(p_{(3,j)}, p_{(4,i)}) = x_1^{j-i} x_2^{2h+i-j-1} x_4^{-i} - x_2^{2h-i+j} x_4^{-i} = (x_2^{2h} - x_3^{2h-1})(-x_2^{j} x_4^{-j}) \]
\[ = (x_2 x_3 - x_1 x_4) \left(\sum_{k=0}^{j-i-1} x_1 x_2^{j-i-1-k} x_3^{2h-2-k} x_4^{2h-j+k}\right) = p_2(-x_2^{j-i} x_4^{-j}) - p_1 \left(\sum_{k=0}^{j-i-1} x_1 x_2^{j-i-1-k} x_3^{2h-2-k} x_4^{2h-j+k}\right). \]

(16) $S(p_{(3,j)}, p_5) = x_1^{1+j} x_2^{2h-1} x_4^{-1} - x_2^{2h+j} x_4^{-j}$

\[ = (x_1 x_2^{2h-1} x_4^{-1} - x_3^{2h}) x_1 x_4^{-1} - (x_2 x_3 - x_1 x_4) \left(\sum_{k=0}^{j-1} x_1 x_2^{j-1-k} x_3^{2h+j-1-k} x_4^{2h-j+k}\right) \]
\[ = p_7 x_1 x_4^{2h} - p_1 \left(\sum_{k=0}^{j-1} x_1 x_2^{j-1-k} x_3^{2h+j-1-k} x_4^{2h-j+k}\right). \]

(17) Let us consider four cases separately for $S(p_{(3,j)}, p_{(6,i)})$.

(a) For $j = i$,
\[ S(p_{(3,j)}, p_{(6,i)}) = x_3^{4h+1} - x_1 x_2^{2h-2} x_4^{2h+2} = (x_3^{4h} - x_2^{2h-1} x_4^{2h+1}) x_3 + (x_2 x_3 - x_1 x_4) x_2^{2h-2} x_4^{2h+1} = p_5 x_3 + p_1 x_2^{2h-2} x_4^{2h+1}. \]

(b) For $j = i + 1$,
\[ S(p_{(3,j)}, p_{(6,i)}) = x_1^{j-i-1} x_3^{4h+1+i-j} - x_2^{2h-2-i+j} x_4^{2h-j+i-2} = p_5. \]
(c) For $j > i + 1$,

$$S(p_{(3,j)} , p_{(6,i)}) = x_1^{j-i-1} x_3^{4h+1+i-j} - x_2^{2h-2-i+j} x_4^{2h-j+i-2}$$

$$= (x_1 x_3^{2h-j} - x_2^{2h})(x_1^{j-i-2} x_3^{2h+1+i-j}) - (x_2^{2h} - x_3^{2h-1})(x_2^{2h-j+i+2})$$

$$- (x_2 x_3 - x_1 x_4)(\sum_{k=0}^{j-i-3} x_1^k x_2^{j-i-3-k} x_3^{2h-2-k} x_4^{2h-j+i+2+k})$$

$$= p_{(3,0)} x_1^{j-i-2} x_3^{2h+1+i-j} - p_2 x_2^{2h-j+i+2}$$

$$- p_1(\sum_{k=0}^{j-i-3} x_1^k x_2^{j-i-3-k} x_3^{2h-2-k} x_4^{2h-j+i+2+k}).$$

(18) Let us consider two cases separately for $S(p_{(3,j)} , p_7)$.

(a) For $0 < j \leq 2h - 1$,

$$S(p_{(3,j)} , p_7) = -x_2^{2h+j-1} x_4^{2h-j+1} + x_1^j x_3^{4h-j}$$

$$= (x_1 x_3^{2h-j} - x_2^{2h}) x_1^{j-1} x_3^{2h-l} + (x_2^{2h} - x_3^{2h-1})(-x_2^{j-1} x_4^{2h-j+1})$$

$$- (x_2 x_3 - x_1 x_4)(\sum_{k=0}^{j-2} x_1^k x_2^{j-2-k} x_3^{2h-2-k} x_4^{2h-j+1+k})$$

$$= p_{(3,0)} x_1^{j-1} x_3^{2h-l} + p_2(-x_2^{j-1} x_4^{2h-j+1})$$

$$- p_1(\sum_{k=0}^{j-2} x_1^k x_2^{j-2-k} x_3^{2h-2-k} x_4^{2h-j+1+k}).$$

(b) For $j = 0$, we have $S(p_{(3,0)} , p_7) = x_3^{4h} - x_2^{2h-1} x_4^{2h+1} = p_5$.

(19) $S(p_{(3,j)} , p_8) = x_3^{6h-j-1} - x_1^{2h-j-1} x_2^{4h-j}$

$$= (x_3^{4h} - x_2^{2h-1} x_4^{2h+1}) x_3^{2h-j-1} + (x_2 x_3 - x_1 x_4)(\sum_{k=0}^{2h-j-2} x_1^k x_2^{2h-2-k} x_3^{2h-j-2-k} x_4^{2h+1+k})$$

$$= p_5 x_3^{2h-j-1} + p_1(\sum_{k=0}^{2h-j-2} x_1^k x_2^{2h-2-k} x_3^{2h-j-2-k} x_4^{2h+1+k}).$$

(20) $S(p_{(4,i)} , p_{(4,j)}) = x_j^{i-1} x_3^{j-i} x_4^{2h-j} - x_j^{i-1} x_3^{j-i} x_4^{2h-i}$
Let us consider four cases separately for \( S(p_{(4,i)}, p_j) \).

(a) For \( i = j \),

\[
S(p_{(4,i)}, p_{(6,j)}) = x_2 x_3^{2h+1+i} - x_1 x_4^{2+i} x_3^{j-i} x_4^{2h-i} \\
= (x_2 x_3 - x_1 x_4)(x_2 x_3^{2h+i} + x_1 x_3^{2h+i-1} x_4) + (x_1 x_3^{2h} - x_4^{2h}) x_1 x_3^{i-1} x_4^2 \\
= p_1(x_2 x_3^{2h+i} + x_1 x_3^{2h+i-1} x_4) + p_{(3,0)} x_1 x_3^{i-1} x_4^2.
\]

(b) For \( i < j \),

\[
S(p_{(4,i)}, p_{(6,j)}) = x_2^{j+2-i} x_3^{2h+1+j} - x_1^{1+j-i} x_3^{i-1} x_4^{2h-i+j+2} \\
= (x_2 x_3 - x_1 x_4)(\sum_{l=0}^{j-i+1} x_1^{l} x_2^{j+1-l} x_3^{2h+j-l} x_4^{l}) \\
+ (x_1 x_3^{2h} - x_4^{2h}) x_1^{1+j-i} x_3^{i-1} x_4^{j-i+2} \\
= p_1(\sum_{l=0}^{j-i+1} x_1^{l} x_2^{j+1-l} x_3^{2h+j-l} x_4^{l}) + p_{(3,0)} x_1^{1+j-i} x_3^{i-1} x_4^{j-i+2}.
\]

(c) For \( i \geq j + 2 \),

\[
S(p_{(4,i)}, p_{(6,j)}) = x_1^{i-j-1} x_3^{2h+1+j} - x_2^{i-j-2} x_3^{i-1} x_4^{2h-i+j+2} \\
= (x_1 x_3^{2h} - x_4^{2h}) x_1^{i-j-2} x_3^{i+1} - (x_2 x_3^{i-j-3} x_3^{i-2-l} x_4^{2h-i+j+2+l}) \\
= p_{(3,0)} x_1^{i-j-2} x_3^{1+j} - p_1(\sum_{l=0}^{i-j-3} x_1^{l} x_2^{i-j-3-l} x_3^{i-2-l} x_4^{2h-i+j+2+l}).
\]

(d) For \( i = j + 1 \),

\[
S(p_{(4,i)}, p_{(6,j)}) = x_2 x_3^{2h+1+j} - x_3^{j} x_4^{2h+1} = (x_2 x_3 - x_1 x_4) x_3^{2h+j} - (x_1 x_3^{2h} - x_4^{2h}) x_3^j x_4 \\
= p_1 x_2^{2h+j} - p_{(3,0)} x_3^j x_4.
\]

(23) \( S(p_{(4,1)}, p_7) = x_3^2 x_1 - x_4^2 = p_{(3,0)} \).

\[
S(p_{(4,i)}, p_7) = (x_1 x_3^{2h} - x_4^{2h}) x_1^{i-1} + (x_2 x_3 - x_1 x_4)(\sum_{l=0}^{i-2} x_1^{l} x_2^{i-2-l} x_3^{i-2-l} x_4^{2h-i+1+l})
\]
\[ p_{(3,0)}x_1^{i-1} + p_1 \left( -\sum_{l=0}^{i-2} x_1^l x_2^{i-2-l} x_3^{i-2-l} x_4^{2h-i+1+l} \right), \text{ for } 1 < i \leq 2h. \]

(24) \[ S(p_{(4,i)}, p_8) = x_2^{2h-i} x_3^{4h-1} - x_1^{2h-i-1} x_3^{i-1} x_4^{4h-i} \]
\[ = (x_2 x_3 - x_1 x_4) \left( \sum_{l=0}^{2h-i-1} x_1^l x_2^{2h-i-1-l} x_3^{4h-2-l} x_4^l \right) + (x_1 x_3^{2h} - x_4^{2h}) x_1^{2h-i-1} x_3^{i-1} x_4^{2h-i} \]
\[ = p_1 \left( \sum_{l=0}^{2h-i-1} x_1^l x_2^{2h-i-1-l} x_3^{4h-2-l} x_4^l \right) + p_{(3,0)} x_1^{2h-i} x_3^{i-1} x_4^{2h-i}, \text{ for } 1 \leq i < 2h. \]

(25) \[ S(p_{(4,2h)}, p_8) = x_1 x_3^{4h-1} - x_3^{2h-1} x_4^{2h} \]
\[ = (x_1 x_3^{2h} - x_4^{2h}) x_3^{2h-1} \]
\[ = p_{(3,0)} x_3^{2h-1}. \]

(26) \[ S(p_{(6,i)}, p_{(6,j)}) = x_2^{-i} x_3^{2h+1+i} - x_1^{-i} x_3^{2h+1+i} x_4^{-i} \]
\[ = (x_2 x_3 - x_1 x_4) \left( \sum_{l=0}^{j-i-1} x_1^l x_2^{-i-l} x_3^{2h+j-l} x_4^l \right) \]
\[ = p_1 \left( \sum_{l=0}^{j-i-1} x_1^l x_2^{-i-l} x_3^{2h+j-l} x_4^l \right), \text{ for } i < j. \]

(27) \[ S(p_{(6,i)}, p_7) = x_1^{1+i} x_4^{1+i} x_3^{2h} - x_1^{1+i} x_3^{2h+1+i} = -(x_2 x_3 - x_1 x_4) \left( \sum_{l=0}^{i} x_1^l x_2^{1-l} x_3^{2h+i-l} x_4^l \right) \]
\[ = -p_1 \left( \sum_{l=0}^{i} x_1^l x_2^{1-l} x_3^{2h+i-l} x_4^l \right). \]

(28) \[ S(p_7, p_8) = x_2^{2h-1} x_3^{4h-1} - x_1^{2h-1} x_3^{4h} x_4^{-1} = (x_2 x_3 - x_1 x_4) \left( \sum_{l=0}^{2h-2} x_1^l x_2^{2h-2-l} x_3^{4h-2-l} x_4^l \right) \]
\[ = p_1 \left( \sum_{l=0}^{2h-2} x_1^l x_2^{2h-2-l} x_3^{4h-2-l} x_4^l \right). \]
All the $S$-polynomials reduce to zero upon division by $\mathfrak{S}_h$. Therefore, $\mathfrak{S}_h$ is a Gröbner basis of $\Omega_h$ with respect to the degree reverse lexicographic monomial order induced by $x_1 > x_2 > x_3 > x_4$ on $k[x_1, x_2, x_3, x_4]$. □

Given $g \in k[x_1, x_2, x_3, x_4]$, let $g^H \in k[x_0, x_1, x_2, x_3, x_4]$ denote the homogenization of $g$, with respect to the indeterminates $x_0$.

**Lemma 5.2.** For $h \geq 2$, let $\overline{\mathfrak{S}_h} := \{g^H \mid g \in \mathfrak{S}_h\}$. Then $\overline{\mathfrak{S}_h}$ is a Gröbner basis of the homogenized ideal $\overline{\Omega_h}$, with respect to the degree reverse lexicographic monomial order induced by $x_1 > x_2 > x_3 > x_4 > x_0$ on $k[x_0, x_1, x_2, x_3, x_4]$.

**Proof.** Follows from Theorems 5.1 and Theorem 2.7 □

**Lemma 5.3.** The ideal $\overline{\Omega_h}$, for $h \geq 2$, is generated by the following polynomials:

- $p_1^H = x_2 x_3 - x_1 x_4$;
- $p_2^H = x_2^2 - x_0 x_3 x_4$;
- $p_{(3,j)}^H = x_1^{j+1} x_3 - x_0 x_2^2 x_4^{2j-2i}$; $0 \leq j \leq 2h - 1$;
- $p_{(4,i)}^H = x_1^{i+1} x_2^{2h-i} - x_0 x_3 x_4^{2h-i}$; $1 \leq i \leq 2h$;
- $p_5^H = x_3^h - x_2 x_4^{h-1}$.

**Proof.** From Lemma 5.2, $\overline{\mathfrak{S}_h} = \{p_1, p_2, p_5, p_7, p_8\} \cup \{p_{(3,j)} \mid 0 \leq j \leq 2h - 1\} \cup \{p_{(4,i)} \mid 1 \leq i \leq 2h\} \cup \{p_{(6,i)} \mid 0 \leq i \leq 2h - 3\}$, is a Gröbner basis of $\overline{\Omega_h}$. For $0 \leq i \leq 2h - 3$,

$$p_{(6,i)}^H = x_2^{2i} x_3^{2h-2i} x_4^{i+1} - x_3^{2h+1+i}$$

$$= p_2^H x_3^{2i} - p_1^H \sum_{l=0}^{1+i} x_1^l x_2^{2h-1-l} x_3^{1+l} x_4^l.$$

- $p_7^H = x_1 x_2^{2h-1} x_4 - x_3^{2h} x_0 = x_3(x_2^{2h} - x_3^{2h-1} x_0) - x_2^{2h-1}(x_2 x_3 - x_1 x_4)$

$$= p_2^H x_3 - p_1^H x_2^{2h-1}.$$ 

- $p_8^H = x_1^{2h} x_4^{2h-1} x_0$

$$= x_3^{2h} x_0^{2h} - x_0 x_3^{2h-1} - (x_2 x_3 - x_1 x_4) \left( \sum_{l=0}^{2h-1} x_3^l x_2^{2h-1-l} x_4^{2h-1-l} \right)$$

$$= x_3^{2h} p_2^H - p_1^H \left( \sum_{l=0}^{2h-1} x_3^l x_2^{2h-1-l} x_4^{2h-1-l} \right).$$

The set $\{p_1^H, p_2^H, p_{(3,j)}^H, p_{(4,i)}^H, p_5^H \mid 0 \leq j \leq 2h - 1, 1 \leq i \leq 2h\}$, for $h \geq 2$, generates the elements $p_{(6,i)}^H, p_7^H, p_8^H$. Therefore, it follows that $\{p_1^H, p_2^H, p_{(3,j)}^H, p_{(4,i)}^H, p_5^H \mid 0 \leq j \leq 2h - 1, 1 \leq i \leq 2h\}$ generates $\overline{\Omega_h}$, for $h \geq 2$. □

**Notation 5.4.** For writing the syzygies, we define the following matrices for $h \geq 2$:
(1) $\mathcal{B}_h^1 := \begin{pmatrix} p_{1}^{H} & p_{2}^{H} & p_{3(0)}^{H} & \cdots & p_{3(2h-1)}^{H} & p_{4(1)}^{H} & \cdots & p_{4(2h)}^{H} & p_{5}^{H} \end{pmatrix}_{1 \times (4h+3)}$

(2) $\mathcal{B}_h^2 = \begin{bmatrix}
A'_{(2h+1) \times (2h+1)} & 0_{(2h+1) \times (2h-1)} & B'_{(2h+1) \times (6h-1)} & 0_{(2h+1) \times (2h-1)} & D'_{(2h+2) \times 6} \\
A'_{(2h+1) \times (2h+1)} & B'_{(2h+1) \times (2h-1)} & 0_{(2h+1) \times ((6h-1))} & C''_{(2h+1) \times (2h+1)} & D''_{(2h+1) \times 6}
\end{bmatrix}
$

where,

- $R_{1 \times 8h-2}^B = [r_{1j}]$, with $r_{1,3+l} = x_0 x_2^{l-1} x_4^{2h-l-1}, 1 \leq l \leq 2h - 1$.
- $r_{1,2h+1+l} = x_0 x_3^{l-1} x_4^{2h-l-1}, 1 \leq l \leq 2h - 1$.
- $r_{1,4h+1+l} = x_1 x_3^{2h-l-1}, 1 \leq l \leq 2h - 1$.
- $r_{1,6h-1+l} = x_1 x_3^{2h-l-1}, 1 \leq l \leq 2h - 1$.
- $r_{1,j} = 0, j \in \{1, \ldots, 8h-2\} \setminus \{3+l, 2h+1+l, 4h+1+l, 6h-1+l\}$, for $1 \leq l \leq 2h-1$.

- $A'_{(2h+1) \times (2h+1)} = \begin{bmatrix}
x_0 x_4 & x_1^2 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & -x_1 & 0 & 0 & \cdots & 0 \\
0 & x_0 & x_3 & -x_1 & 0 & \cdots & 0 \\
0 & 0 & 0 & x_3 & \ddots & \ddots & 0 \\
& \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & \cdots & \ddots & \ddots & -x_1 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & x_3
\end{bmatrix}$

- $B'_{(2h+1) \times (6h-1)} = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
-x_2 & 0 & 0 & \cdots & 0 \\
x_4 & -x_2 & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \cdots & 0 \\
& \vdots & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & x_4 & -x_2 \\
0 & 0 & 0 & 0 & x_4
\end{bmatrix}$

- $\begin{bmatrix}
x_2^{2h} - x_0 x_3^{2h-1} & x_1 x_2^{2h-1} & x_0 x_2^{2h-1} & x_2^{2h-1} x_4 & x_3^{4h-1} - x_4^{2h-1} - x_3^{2h-1} - x_3 x_4 & x_2^{2h-1} \\
-x_2 x_3 + x_1 x_4 & -x_1 x_4 & -x_0 x_3 & -x_3 x_4 & -x_4^{2h-1} & -x_3^{2h-1} \\
0 & -x_0 & 0 & x_2^{2h} & x_3^{2h-1} x_4 & x_2^{2h-1} x_4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & -x_1 & 0 & 0 \\
0 & -x_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
-x_3 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}$

- $A''_{(2h+1) \times (2h+1)} = \begin{bmatrix}
0 & 0 & 0 & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
-x_3 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}$
\[
\begin{align*}
\bullet \quad & B'_{(2h+1)\times(2h-1)} = \\
= & \begin{bmatrix}
-x_1 & 0 & \cdots & \cdots & 0 \\
x_2 & -x_1 & \ddots & \ddots & 0 \\
0 & x_2 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & x_2 & -x_1 \\
0 & 0 & \cdots & 0 & x_2 \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix} \\
\bullet \quad & C'_{(2h+1)\times(2h-1)} = \\
= & \begin{bmatrix}
-x_3 & 0 & \cdots & \cdots & 0 \\
x_4 & -x_3 & \ddots & \ddots & 0 \\
0 & x_4 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & x_4 & -x_3 \\
0 & 0 & \cdots & 0 & x_4 \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix} \\
\bullet \quad & D'_{(2h+1)\times 6} = \\
= & \begin{bmatrix}
0 & x_4 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & x_3 & 0 & 0 & 0 \\
0 & 0 & 0 & -x_1 & -x_2 & -x_0
\end{bmatrix}
\end{align*}
\]

\[
(3) \quad \mathcal{B}_h^3 = \begin{bmatrix}
P_{(2h+1)\times(2h-1)} & Q_{(2h+1)\times(2h+2)} \\
P'_{(2h-1)\times(2h-1)} & Q'_{(2h-1)\times(2h+2)} \\
P''_{(2h-1)\times(2h-1)} & Q''_{(2h-1)\times(2h+2)} \\
0_{6\times(2h-1)} & Q_{6\times(2h+2)}
\end{bmatrix}
\begin{bmatrix}
R'_{(8h+4)\times 1} & R''_{(8h+4)\times 1}
\end{bmatrix}, \text{ such that,}
\]

\[
\begin{align*}
\bullet \quad & P_{(2h+1)\times(2h-1)} = \\
= & \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & x_3 \\
0 & 0 & 0 & \cdots & 0 & 0 \\
-x_2 & 0 & 0 & \cdots & 0 & 0 \\
x_4 & -x_2 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & x_4 & -x_2 & 0 \\
0 & 0 & 0 & 0 & x_4 & -x_2
\end{bmatrix} \\
\bullet \quad & P'_{(2h-1)\times(2h-1)} = \\
= \begin{bmatrix}
0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & x_1 \\
0 & \cdots & 0 & 0
\end{bmatrix}
\end{align*}
\]
\[
\begin{align*}
\text{• } P''_{(2h-1) \times (2h-1)} &= \begin{bmatrix}
x_1 & 0 & 0 & \cdots & 0 & 0 \\
-x_3 & x_1 & 0 & \cdots & 0 & 0 \\
0 & -x_3 & x_1 & \ddots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & 0 \\
0 & 0 & 0 & -x_3 & x_1 & 0 \\
0 & 0 & 0 & 0 & -x_3 & 0 \\
\end{bmatrix} \\
\text{• } P'''_{(2h-1) \times (2h-1)} &= \begin{bmatrix}
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 0 \\
0 & \cdots & 0 & -x_3 \\
\end{bmatrix} \\
\text{• } Q_{(2h+1) \times (2h+2)} &= \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & -x_1 & 0 \\
0 & 0 & \cdots & 0 & -x_3 & -x_4 & 0 & 0 \\
0 & 0 & \cdots & 0 & x_0 & 0 & 0 & -x_3^{2h-1}x_4 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \\
\text{• } Q'_{(2h-1) \times (2h+2)} &= \begin{bmatrix}
-x_3 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
x_4 & -x_3 & 0 & \ddots & 0 & -x_3 & -x_4 & 0 \\
0 & x_4 & -x_3 & \ddots & 0 & x_0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & x_4 & -x_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x_4 & 0 & 0 & x_3 \\
\end{bmatrix} \\
\text{• } Q''_{(2h-1) \times (2h+2)} &= \begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & x_0 & x_3^{2h} \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{bmatrix} \\
\text{• } Q'''_{(2h-1) \times (2h+2)} &= \begin{bmatrix}
x_1 & 0 & 0 & \cdots & 0 & x_2 & 0 & 0 \\
-x_2 & x_1 & 0 & \ddots & 0 & 0 & 0 & 0 \\
0 & -x_2 & x_1 & \ddots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & -x_2 & x_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -x_2 & 0 & 0 & 0 \\
\end{bmatrix} \\
\text{• } Q''''_{(2h-1) \times (2h+2)} &= \begin{bmatrix}
0 & 0 & \cdots & 0 & x_1 & x_0 & -x_4^{2h} \\
0 & 0 & \cdots & -x_1 & -x_2 & 0 & 0 \\
0 & 0 & \cdots & 0 & -x_2 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & x_2 & 0 \\
0 & 0 & \cdots & 0 & 0 & -x_1 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\end{bmatrix} \\
\text{• } Q^{(iv)}_{6 \times (2h+2)} &= \begin{bmatrix}
0 & 0 & \cdots & 0 & x_1 & x_0 & -x_4^{2h} \\
0 & 0 & \cdots & -x_1 & -x_2 & 0 & 0 \\
0 & 0 & \cdots & 0 & -x_2 & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & x_2 & 0 \\
0 & 0 & \cdots & 0 & 0 & -x_1 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\end{align*}
\]
\[ R'_{(8h+4) \times 1} = \left[ R'_{(l,1)} \right] \text{, such that,} \]
\[ R'_{(8h-1,1)} = x_3^{2h}, \]
\[ R'_{(4h+1+i,1)} = -x_2^{2h-i}x_4^{i+1}, \]
\[ R'_{(8h,1)} = x_3^{2h-1}x_4, \]
\[ R'_{(6h-1+j,1)} = x_3^{2h-1-j}x_4^{j+1}, \]
\[ R'_{(1,1)} = x_4^{2h}, R'_{(8h+3,1)} = x_0, \]
\[ R'_{(8h+4,1)} = -x_2, \]
\[ R'_{(l,1)} = 0, \text{ for } l \in [8h+4] \setminus \{8h-1, 4h+1+i, 8h, 6h-1+j, 1, 8h+3, 8h+4\}, \]
\[ 0 \leq i \leq 2h-2, 1 \leq j \leq 2h-2. \]
\[
R''_{(8h+4) \times 1} = \left[ R''_{(l,1)} \right], \text{ such that,} \]
\[ R''_{(8h,1)} = x_3^{2h}, \]
\[ R''_{(3,1)} = -x_2^{2h-1}x_4, \]
\[ R''_{(4h+1+i,1)} = -x_2^{2h-1-i}x_3^ix_4, \]
\[ R''_{(6h-1+j,1)} = x_3^{2h-j}x_4^j, \]
\[ R''_{(1,1)} = x_3x_4^{2h-1}, \]
\[ R''_{(8h+2,1)} = x_0, \]
\[ R''_{(8h+4,1)} = -x_1, \]
\[ R''_{(l,1)} = 0, \text{ for } l \in [8h+4] \setminus \{8h, 3, 4h+1+i, 6h-1+j, 1, 8h+2, 8h+4\} \text{ for } \]
\[ 1 \leq i \leq 2h-2, 1 \leq j \leq 2h-2. \]

(4) \[ \mathcal{B}_h^4 := \left( \Gamma_{ij} \right)_{(4h+3) \times 1}, \text{ such that, } \Gamma_{i1}, 1 \leq i \leq 4h+3, \text{ are defined as follows:} \]
\[ \Gamma_{(4h-1,1)} = x_3^{2h}, \]
\[ \Gamma_{(m,1)} = -x_2^{2h-m}x_4^m, \text{ for } 1 \leq m \leq 2h-1, \]
\[ \Gamma_{(4h-2,1)} = -x_3^{2h-1}x_4, \]
\[ \Gamma_{(2h-1+n,1)} = x_3^{2h-1-n}x_4^{n+1}, \text{ for } 1 \leq n \leq 2h-2, \]
\[ \Gamma_{(4h,1)} = -x_4^{2h}, \]
\[ \Gamma_{(4h+1,1)} = -x_0, \]
\[ \Gamma_{(4h+2,1)} = -x_1, \]
\[ \Gamma_{(4h+3,1)} = x_2, \]
\[ \Gamma_{(i,4h+1)} = 0, \text{ for } i \in \{1, \ldots, 8h+4\} \text{ and } \]
\[ i \notin \{4h-1, m, 4h-2, 2h-1+n, 4h, 4h+1, 4h+2, 4h+3\}, \]
\[1 \leq m \leq 2h - 1, 1 \leq n \leq 2h - 2.\]

**Theorem 5.5.** For \(h \geq 2\), a minimal graded free resolution of the homogenized ideal \(\mathfrak{I}_h\), which is the defining ideal of the projective closure of the Bresinsky curve, over the polynomial ring \(k[x_0, x_1, x_2, x_3, x_4]\), is given by

\[
\mathfrak{B}_h : 0 \rightarrow R \xrightarrow{\mathfrak{B}_h^4} R^{4h+3} \xrightarrow{\mathfrak{B}_h^3} R^{8h+4} \xrightarrow{\mathfrak{B}_h^2} R^{4h+3} \xrightarrow{\mathfrak{B}_h^1} R \rightarrow R/\mathfrak{I}_h \rightarrow 0,
\]

where the matrices \(\mathfrak{B}_i, 1 \leq i \leq 4\) have been defined above in notation 5.4.

**Proof.** One can check that

\[
\mathfrak{B}_h : 0 \rightarrow R \xrightarrow{\mathfrak{B}_h^4} R^{4h+3} \xrightarrow{\mathfrak{B}_h^3} R^{8h+4} \xrightarrow{\mathfrak{B}_h^2} R^{4h+3} \xrightarrow{\mathfrak{B}_h^1} R \rightarrow R/\mathfrak{I}_h \rightarrow 0
\]

is a chain complex by calculating \(\mathfrak{B}_h^i\mathfrak{B}_h^{i+1} = 0, 1 \leq i \leq 3\). To prove the exactness, we use Lemma 4.2. Let \(r_i\) be the \(i^{th}\) expected rank of \(\mathfrak{B}_h^1\). Then \(r_1 = (4h + 3) - (8h + 4) + (4h + 3) - 1 = 1, r_2 = 8h + 4 - (4h + 3) + 1 = 4h + 2, r_3 = 4h + 3 - 1 = 4h + 2, r_4 = 1\). We need to show that \(\text{grade}(I_{r_i}(\mathfrak{B}_h^1)) \geq i, 1 \leq i \leq 4\), where \(I_{r_i}(\mathfrak{B}_h^1)\) denotes the ideal generated by the \(r_i \times r_i\) minors of the matrix \(\mathfrak{B}_h^1\).

We take \(p_1^H \in I_{r_1}(\mathfrak{B}_h^1)\), and we have \(I_{r_1}(\mathfrak{B}_h^1) \geq 1\).

We take

- \(\mathfrak{L}_h^{[21]} := [1 3 4 \ldots (4h + 3)(4h + 1) \ldots (8h - 2) 8h (8h + 2) (8h + 3) (8h + 4)]
- \(= x_4^{4h-1}(-x_2^{2h} + x_3^{2h-1}x_0)(x_3^{4h} - x_2^{2h-1}x_4^{2h+1}).\)
- \(\mathfrak{L}_h^{[22]} := [2 \ldots (4h + 3)(1 3 \ldots 4h (8h + 1) (8h + 2) (8h + 4)]
- \(= -x_2^{2h-2}(-x_2x_3 + x_1x_4)(-x_1x_2^{2h-1}x_4 + x_3^{2h}x_0)(x_0x_2^{2h}).\)

The gcd of any two prime factors of \(\mathfrak{L}_h^{[21]}\) and \(\mathfrak{L}_h^{[22]}\) is 1, therefore, \(\mathfrak{L}_h^{[21]}, \mathfrak{L}_h^{[22]}\) forms a regular sequence. Hence, \(\text{grade}(I_{r_2}(\mathfrak{B}_h^2)) \geq 2\).

We take

- \(\mathfrak{L}_h^{[31]} := [1 (4h) \ldots (6h - 3) \ldots (8h - 3)(8h - 1)(8h)(8h + 3)(8h + 4)
- \[1 2 \ldots (4h + 1)(4h + 3)] = x_4^{4h+2}\)
- \(\mathfrak{L}_h^{[32]} := [1 \ldots 4h (8h + 1)(8h + 2)1 \ldots (4h + 2)] = x_4^{6h}(-x_2^{2h} + x_3^{2h-1}x_0)\)
- \(\mathfrak{L}_h^{[33]} := [3 \ldots (2h + 1)(4h + 1)(6h) \ldots 8h (8h + 3)1 \ldots (4h + 2)]
- \(= x_4^{6h-3}x_4 - x_0(\sum_{i=0}^{2h-1} x_4 x_3^{4h-2-i} x_3^{2h-i} x_4) - x_0 x_1^{2h-1}x_2^{2h-1}x_3^{2h-1}x_4^{2h-1}\)
- \(+ x_0 x_1^{2h-1}x_2^{2h-1}x_3^{2h} x_4^{2h-1} + x_0 x_1^{2h-2}x_3^{2h-1}x_4^{2h-2}\).

Consider the ideal \(\langle \mathfrak{L}_h^{[31]}, \mathfrak{L}_h^{[32]} \rangle\); its primary decomposition is \(\langle x_4^{4h+2}, x_4^{6h} \rangle \cap \langle x_4^{4h+2}, (-x_2^{2h} + x_3^{2h-1}x_0) \rangle\). Hence, the associated primes are \(\langle x_1, x_4 \rangle, x_1, (-x_2^{2h} + x_3^{2h-1}x_0)\). We observe that \(\mathfrak{L}_h^{[33]} \notin \langle x_1, x_4 \rangle\) and \(\mathfrak{L}_h^{[33]} \notin \langle x_1, (-x_2^{2h} + x_3^{2h-1}x_0) \rangle\). If \(\mathfrak{L}_h^{[33]} \notin \langle x_1, x_4 \rangle\), then \(x_0 x_1^{2h-2} x_3^{2h} \in \langle x_1, x_4 \rangle\), which is a contradiction, and if \(\mathfrak{L}_h^{[33]} \notin \langle x_1, (-x_2^{2h} + x_3^{2h-1}x_0) \rangle\), then again \(x_0 x_1^{2h-2} x_3^{2h} \notin \langle x_1, (-x_2^{2h} + x_3^{2h-1}x_0) \rangle\).
A cone of a monomial curve at origin, using Gr"obner basis. He introduced the following family of functions of $x$ induced by the closure of the Arslan curve $A$. The set $I = \{x_0, \ldots, x_4\}$, for $1 \leq i \leq 4$, belong to the homogeneous maximal ideal $\langle x_0, \ldots, x_4 \rangle$. Hence, $\mathcal{B}_h$ is a minimal free resolution of the defining ideal of the projective closure of the Bresinsky curve $\Omega_h$.}

\section{Syzygies of the Projective Closure of the Arslan Curve}

In [1] Arslan gave a necessary and sufficient criterion for the Cohen-macaulayness of the tangent cone of a monomial curve at origin, using Gr"obner basis. He introduced the following family of curves $\mathcal{A}_h$, for $h \geq 2$, defined by the family of numerical semigroups $\Gamma(h(h+1), h(h+1)+1, (h+1)^2, (h+1)^2+1)$, which we call the Arslan curve. In this section, we find the syzygies of the projective closure of the Arslan curve, i.e., $\Omega_h$ and show that all the Betti numbers are unbounded functions of $h$.

\begin{lemma}
Let us consider the following polynomials, for $h \geq 2$:

\begin{align*}
w &= x_2x_3 - x_1x_4; \\
g_i &= x_1^ix_3^{h-i+1} - x_2^{i+1}x_4^{h-i}, \text{ for } 0 \leq i \leq h-1; \\
g_h &= x_2^{h+1} - x_1^hx_3; \\
g_j &= x_1^{j+1}x_2^{h-j} - x_3x_4^{h-j}x_0, \text{ for } 0 \leq j \leq h.
\end{align*}

The set $\mathcal{A}_h = \{w, g_i, g_j \mid 1 \leq i, j \leq h\}$ is a Gröbner basis of the defining ideal of the projective closure of the Arslan curve $\mathcal{A}_h$, with respect to the degree reverse lexicographic monomial order induced by $x_1 > x_2 > x_3 > x_4 > x_0$ on the polynomial ring $k[x_0, x_1, x_2, x_3, x_4]$.

\begin{proof}
We use Proposition 5.2 in [11] and Theorem 2.7.
\end{proof}

\begin{notation}
For $h \geq 2$, let $\mathcal{I}_h := p(h(h+1), h(h+1)+1, (h+1)^2, (h+1)^2+1)$ and $\mathcal{J}_h := p(h+1), h(h+1)+1, (h+1)^2, (h+1)^2+1)$.

For writing the syzygies, we first define the following matrices for $h \geq 2$:

1. $\mathcal{A}_{\mathcal{I}_h} := (w \ g_0 \ \ldots \ g_h \ g_0 \ \ldots \ g_h)$

2. $\mathcal{A}_{\mathcal{B}_{\mathcal{I}_h}} := A_{\mathcal{I}_h}^2 \begin{bmatrix} A_{(h+1)^2 \times (h+1)}^2 & B_{h \times h}^2 & C_{h \times h}^2 & 0_{h \times h} \end{bmatrix}$ where,

\begin{align*}
A_{(h+1)^2 \times (h+1)}^2 &= \begin{bmatrix} x_0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
x_1 & 0 & \cdots & 0 \end{bmatrix} \\
B_{h \times h}^2 &= \begin{bmatrix} 0_{(h+1) \times h} \\
0_{(h+1) \times h} \\
\vdots & \vdots \\
0_{(h+1) \times h} \end{bmatrix} \\
C_{h \times h}^2 &= \begin{bmatrix} 0_{h \times h} \\
0_{h \times h} \\
\vdots & \vdots \\
0_{h \times h} \end{bmatrix} \end{align*}

\end{notation}
\[ \begin{bmatrix}
-x_1 & 0 & \cdots & \cdots & 0 \\
-x_3 & -x_1 & \ddots & \ddots & \vdots \\
0 & x_3 & \ddots & 0 & 0 \\
\vdots & \ddots & \ddots & -x_1 & 0 \\
0 & 0 & x_3 & -x_1 & 0 \\
0 & 0 & \cdots & 0 & -x_3 \\
\end{bmatrix} \]

\[ \begin{bmatrix}
-x_2 & 0 & \cdots & 0 \\
x_4 & -x_2 & \ddots & 0 \\
0 & x_4 & \ddots & 0 \\
\vdots & \ddots & \ddots & -x_2 \\
0 & 0 & \cdots & x_4 \\
\end{bmatrix} \]

\[ \begin{bmatrix}
-x_2 & -x_1 & 0 & 0 & \cdots & 0 \\
0 & x_2 & -x_1 & 0 & \cdots & 0 \\
0 & 0 & x_2 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & -x_1 \\
0 & 0 & \cdots & 0 & x_2 \\
\end{bmatrix} \]

\[ \begin{bmatrix}
-x_3 & 0 & \cdots & 0 \\
x_4 & -x_3 & \ddots & 0 \\
0 & x_4 & \ddots & 0 \\
\vdots & \ddots & \ddots & -x_3 \\
0 & 0 & \cdots & x_4 \\
\end{bmatrix} \]

such that,
\[ r_{1.1} = 0, l \in \{1, \ldots, 4h+1\} \setminus \{2+i, h+2+j, 2h+1, 2h+2+j, 3h+1, 3h+2+i\}, \]
\[ 0 \leq i \leq h - 1, 0 \leq j \leq h - 2; \]
\[ r_{1.2+i} = x_3^i x_4^{h-i-1}, 0 \leq i \leq h - 1; \]
\[ r_{1.2h+2+j} = x_2^{j+1} x_4^{h-j-1}, 0 \leq j \leq h - 2; \]
\[ r_{1.2h+1} = x_2^h, \]
\[ r_{1.2h+2+j} = x_3^j x_4^{h-j}, 0 \leq j \leq h - 2; \]
\[ r_{1.3h+1} = x_3^{h-1} x_4; \]
\[ r_{1.3h+2+i} = x_1^{i+1} x_2^{h-i-1}, 0 \leq i \leq h - 1. \]

(3) \[ \mathcal{Y}_h^3 := \begin{bmatrix}
0_{(h+1) \times (h-1)} & \begin{bmatrix}
B_{(h+1) \times (h-1)}^3 \\
A_{h \times (h-1)}^3 \\
A_{h \times (h-1)}^3 \\
0_{h \times (h-1)} \\
\end{bmatrix} \\
\end{bmatrix} \]

where,
\[ R = [r_{1,1} \ r_{2,1} \ \cdots \ r_{1,4h+1}]. \]
It is clear from the factorization that\[ D_h = \ldots \]

**Proof.** We use Lemma 4.2. Let\[ \mathbb{A}_h \](1,1) = −x₂x₃ + x₁x₄, \quad C(2,1) = −x₂x₄, \quad C(h+1,1) = x₃^2, \quad C(h+2,1) = x₀x₄,\n\[ C(2h+1,1) = −x₁x₂, \quad C(2h+2,1) = −x₃x₀, \]
\[ C(3h+1,1) = x₁^2, \quad C(3h+2,1) = x₂^2, \quad C(4h+1,1) = −x₁x₃, \]
\[ C(i,i) = 0, \quad i \in \{1, \ldots, (4h+1)\} \setminus \{1, 2, h+1, h+2, 2h+1, 2h+2, 3h+1, 3h+2, 4h+1\}.\]

**Theorem 6.3.** For \( h \geq 2 \), a graded free minimal resolution of \( \mathcal{O}_h \), the defining ideal of the projective closure of the Arslan curve \( \mathbb{A}_h \), is

\[ R \rightarrow R^{2h-1} \mathcal{O}_h \rightarrow R^{2h+3} \mathcal{O}_h \rightarrow R \rightarrow R/\mathcal{O}_h \rightarrow 0, \]

where the matrices \( \mathcal{O}_h \), 1 ≤ i ≤ 3, are defined in notation 6.2.

**Proof.** We use Lemma [4.2]. Let \( r_i \) be the \( i \)th expected rank of \( \mathbb{A}_h \). Then \( r_1 = (2h+3) - (4h+1) + (2h - 1) - 1 = 1, \quad r_2 = 4h+1 - (2h - 1) = 2h+2, \quad r_3 = 2h - 1. \) We need to show that grade\( (I_{r_i}(\mathbb{A}_h)) \) ≥ i, 1 ≤ i ≤ 3.

We take the following minors from \( \mathbb{A}_h^2 \):

- \( \mathcal{O}_h^{[21]} := [1 \ 3 \ \cdots \ (2h+3)|1 \ \cdots \ (2h+2)] = (x_2^{h+1} - x_1^h x_3) (-x_3^{-1} + x_2 x_4) x_3^{-1}, \)
- \( \mathcal{O}_h^{[22]} := [2 \ \cdots \ (2h+3)|1 \ 2 \ (2h+2) \ \cdots \ (4h+1)] = (x_2 x_3 - x_1 x_4) (-x_1 x_2^2 + x_1^h x_0) x_4^{-1}. \)

It is clear from the factorization that \( \mathcal{O}_h^{[21]}, \mathcal{O}_h^{[22]} \) form a regular sequence; hence grade\( (I_{r_2}(\mathbb{A}_h^2)) \) ≥ 2.

We take following minors from \( \mathbb{A}_h^3 \):

- \( \mathcal{O}_h^{[31]} := [2 \ \cdots \ h \ (2h+2) \ \cdots \ (3h+1) \ | 1 \ \cdots \ (2h-1)] = x_1^{h+1} x_3^{-1} - x_3^{2h-1} x_0 = x_3^{-1} (x_1 x_2^2 - x_3 x_0); \)
- \( \mathcal{O}_h^{[32]} := [2 \ \cdots \ h \ | \ 2h \ \cdots \ (2h-1)] = x_1^{h+1} x_3^{-1} - x_2 x_4^h = x_2^{-1} (x_3^h - x_2 x_4^h); \)
- \( \mathcal{O}_h^{[33]} := [(h+2) \ \cdots \ (2h+1) \ | \ (3h+2) \ \cdots \ 4h \ | \ 1 \ \cdots \ (2h-1)] = x_1 x_2^h - x_1^{-1} x_4^h x_0 = x_1^{-1} (x_1 x_2^2 - x_1^h x_0). \)
It can be seen that the primary decomposition of the ideal \( \langle D_{[31]}^h, D_{[32]}^h \rangle \) is

\[
\langle x_3^{h-1}, x_2^{h-1} \rangle \cap \langle x_3^{h-1}, x_3^{h+1} - x_2^{h+1} x_4^{2h} \rangle \cap \langle x_1^{h+1} - x_3 x_0, x_0 x_2^{h-1} \rangle \cap \langle x_1^{h+1} - x_3 x_0, x_3^{h+1} - x_2^{h+1} x_4^{2h} \rangle.
\]

Therefore, the associated primes of the ideal \( \langle D_{[31]}^h, D_{[32]}^h \rangle \) are,

\[
\langle x_3, x_2 \rangle, \langle x_3, x_3^{h+1} - x_2^{h+1} x_4^{2h} \rangle, \langle x_1^{h+1} - x_3 x_0, x_2 \rangle, \langle x_1^{h+1} - x_3 x_0, x_3^{h+1} - x_2^{h+1} x_4^{2h} \rangle.
\]

It is easy to verify that the polynomial \( D_{[33]}^h \) does not belong to any associated primes of the ideal \( \langle D_{[31]}^h, D_{[32]}^h \rangle \). Therefore, \( \langle D_{[31]}^h, D_{[32]}^h, D_{[33]}^h \rangle \) forms a regular sequence. Hence \( \text{grade}(I_{D_{[3]}^h}(A^h)) \geq 3 \). Minimality of the resolution follows from the fact that all the entries of the matrices \( M_i, 1 \leq i \leq 3 \) belong to the homogeneous maximal ideal \( \langle x_0, \ldots, x_4 \rangle \), and hence \( M_h \) is a minimal free resolution of \( \overline{J_h} \).

\[\square\]