Maximum Uniquely Partitionable Multisets

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Abstract

For positive integers \( n \) and \( m \), consider a multiset of non-empty subsets of \([m]\) such that there is a unique partition of these subsets into \( n \) partitions of \([m]\). We constructively determine the maximum possible size \( g(n, m) \) of such a multiset in the regime \( n \geq 2^{m-1} - O(2^{m}) \). For \( n \leq 2^{m-1} - 1 \), we show that \( g(n, m) \geq \Omega(\frac{nm}{\log n}) \). In the regime \( n = 2^m \) for any \( c \in (0, 1) \), this lower bound simplifies to \( \Omega(n^c) \), and we show a matching upper bound \( g(n, m) \leq O(\frac{n}{c} \log(\frac{1}{c})) \) that is optimal up to a factor of \( \log(\frac{1}{c}) \). We also provide a graph-theoretic motivation of the problem and suggest some questions for future work.

1 Introduction

In extremal combinatorics, one often studies questions on maximum families. A typical approach is to fix a set \( A \) and consider a family of elements of \( A \), say \( \mathcal{F} \), for which a certain property \( P \) holds, and then attempt to find the size of the largest such family. Some examples are listed below.

- Let \( A \) be the subsets of \([n] = \{1, \ldots, n\}\) of size \( r \leq \frac{n}{2} \). The property \( P \) states that the pairwise intersection of any two sets in a family \( \mathcal{F} \) is non-empty. The famous Erdős–Ko–Rado theorem states that the size of \( \mathcal{F} \) is bounded by \( \binom{n}{r} - 1 \). \([2]\)

- Let \( A \) be the set of binary codewords of length \( n \). The property \( P \) states that the pairwise Hamming distance between any two codewords in a family \( \mathcal{F} \) is at least \( 2\delta \). Graham and Sloane provide lower bounds on the size of such families in \([6]\).

- Let \( A \) be the subsets of \([n] = \{1, \ldots, n\}\). Let \( L \) be a set of integers. Fix \( k \geq 2 \). Let \( P \) be the property that \( \mathcal{F} \) is \( k \)-wise \( L \)-intersecting: the cardinality of the intersection of any \( k \) sets in \( \mathcal{F} \) lies in \( L \). We can also have the further restriction that sets in \( \mathcal{F} \) must have the same size. This has been extensively studied in works like \([5, 4, 1]\).

In this paper, we study the maximum size of a multiset that admits a certain partition. This problem was originally motivated by considering the minimum number of edges required to specify a certain \( m \)-partite graph. For \( n, m \in \mathbb{Z}^+ \), we ask the following question:

**Question 1.** What is the size of the largest multiset of non-empty subsets of \([m] = \{1, \ldots, m\}\), say \( \mathcal{F} = \{C_1, \ldots, C_k\} \) such that there exists a unique partition \( \mathcal{P} = \{A_1, \ldots, A_n\} \) of \([k]\) with the property that \( \sqcup_{j \in A_i} C_j = [m] \) is a partition of \([m]\) for all \( i = 1, \ldots, n \)?
Interestingly, determining \( g(n,m) \) is equivalent to determining the minimum number of edges required to uniquely specify a certain multipartite graph. Let \( G \) be an \( m \)-partite graph on \( nm \) vertices such that every partition set has exactly \( n \) vertices where vertices in the \( i \)th partition are labelled \( v_{i,1}, \ldots, v_{i,n} \) for each \( i \in [m] \). Suppose \( G \) satisfies the following conditions:

(i) \( G \) contains exactly \( n \) connected components such that each component contains \( m \) vertices with one vertex from each of the \( m \) partition sets. So each component is of the form \( \{v_{1,j_1}, v_{2,j_2}, \ldots, v_{m,j_m}\} \).

(ii) Each of the \( n \) components are fully connected. So \( G \) contains \( n \) pairwise disjoint cliques, each with \( m \) vertices.

So \( G \) contains \( n \) copies of \( K_m \) where each copy of \( K_m \) has exactly one vertex in each of the \( m \) partition sets. Suppose we are given the labels of each vertex of \( G \). Then, for \( n, m \in \mathbb{Z}^+ \), we ask the following question:

**Question 2.** Let \( H \) be a labelled graph on \( nm \) vertices \( \{v_{ij} : (i,j) \in [m] \times [n]\} \) such that there exists a unique graph \( G \) which satisfies the two properties mentioned above and contains \( H \) as a subgraph. Let \( f(n,m) \) be the smallest possible number of edges in any such \( H \). What is \( f(n,m) \)?

We will show that in fact \( f(n,m) = nm - g(n,m) \), and in this paper we will focus on \( g(n,m) \). In Section 3.1, we show the equivalence between determining the size of a maximum multiset and determining the minimum number of edges required to uniquely specify the \( m \)-partite graph above. We then prove a simple result that \( g(2,m) = m + 1 \) (see Theorem 15) and introduce the Subset Criterion (see Theorem 16), a condition that a partitionable multiset must satisfy. We then show the following main results.

### 1.1 Regime of large \( n \)

(i) **Theorem 23** (Section 3.2): For \( n \geq 2^{m-1} - 1 \), we prove that

\[
g(n,m) = 2^{m-1} - 1 + n
\]

(ii) **Theorem 39** (Section 3.5) Borrowing some techniques from Section 3.3, we prove that when \( m \geq 6 \) and \( 2^{m-1} - 2^{\lceil \frac{m-1}{2} \rceil + 1} + 3 \leq n \leq 2^{m-1} - 1 \),

\[
g(n,m) = \left\lfloor \frac{2^{m-1} - 1 - n}{2} \right\rfloor + 2n
\]

### 1.2 Regime of smaller \( n \)

(i) **Theorem 28** (Section 3.3): For \( n \leq 2^{m-1} - 1 \), we prove the following lower bound.

\[
g(n,m) \geq \frac{n(m+1)}{\log_2(n+1) + 2}
\]

In the regime \( n = 2^{cm} \) for some \( c \in (0,1) \), this is equivalent to

\[
g(n,m) \geq \Omega\left(\frac{n}{c}\right)
\]
Theorem 32 (Section 3.4): For \( n = 2^m \) with \( c \in (0,1) \) and \( n \leq 2^{m-1} - 1 \), we prove the following upper bound that matches the lower bound up to a \( \log_2 \left( \frac{1}{c} \right) \) factor:

\[
g(n, m) \leq \frac{n}{c} (6 - 3.2 \log_2(c)) = O \left( \frac{n}{c} \log \left( \frac{1}{c} \right) \right)
\]

2 Preliminaries

Definition 3 (Partitionable Multiset). Fix \( n, m \in \mathbb{Z}^+ \). \( \mathcal{F} \) is a partitionable multiset if it is a multiset of non-empty subsets of \( [m] = \{1, \ldots, m\} \), say \( \mathcal{F} = \{C_1, \ldots, C_k\} \), such that there exists a partition \( \mathcal{P} = \{A_1, \ldots, A_n\} \) of \( [k] \) with the property that \( \bigcup_{j \in A_i} C_j = [m] \) is a partition of \( [m] \) for all \( i = 1, \ldots, n \).

Henceforth, a partition \( \mathcal{P} \) of a multiset \( \mathcal{F} \), will always refer to a partition that satisfies the above property. While a partitionable multiset is allowed to contain multiple copies of the same set, the only set that can repeat without violating the existence of a unique partition is \( [m] \) itself.

Definition 4 (Reassignment of a Partition). Fix \( n, m \in \mathbb{Z}^+ \). Let \( \mathcal{F} = \{C_1, \ldots, C_k\} \) be a partitionable multiset with partitions. If \( \mathcal{P} \) and \( \mathcal{P}' \) are distinct partitions, not just formed by relabelling some of the groups, then call \( \mathcal{P}' \) a reassignment of \( \mathcal{P} \).

Example 5. In Figure 2, \( \mathcal{P}' \) is a reassignment of \( \mathcal{P} \).

Definition 6 (Uniquely Partitionable Multiset). A partitionable multiset is uniquely partitionable if the partition \( \mathcal{P} = \{A_1, \ldots, A_n\} \) corresponding to it is unique, up to a trivial renaming of the partition sets.

For any \( n, m \in \mathbb{Z}^+ \), a uniquely partitionable multiset always exists. Namely, the multiset \( \mathcal{F} \) containing \( n \) copies of \( [m] \) is uniquely partitionable.

Example 7. Below are two examples of \( \mathcal{F} \) that are uniquely partitionable and not uniquely partitionable respectively.

![Example Diagram](image.png)

Figure 1: This corresponds to the case \( (2, 4) \). \( \mathcal{F} \) contains the 5 sets \( C_1 = \{1, 2\}, C_2 = \{3, 4\}, C_3 = \{1\}, C_4 = \{2, 3\} \) and \( C_5 = \{4\} \). It is clear that there is only one possible partition \( \mathcal{P} = \{A_1, A_2\} \) here with \( A_1 = \{1, 2\} \) and \( A_2 = \{3, 4, 5\} \). Therefore, this is a uniquely partitionable multiset. In fact, it also turns out to be maximum.
Figure 2: This corresponds to the case (3, 5). $\mathcal{F}$ contains the 9 sets $C_1 = \{1, 2\}, C_2 = \{3, 4\}, C_3 = \{5\}, C_4 = \{1, 4\}, C_5 = \{3, 5\}, C_6 = \{2\}, C_7 = \{1, 5\}, C_8 = \{2, 3\}$ and $C_9 = \{4\}$. There are two possible partitions. $\mathcal{P} = \{A_1, A_2, A_3\}$ with $A_1 = \{1, 2, 3\}, A_2 = \{4, 5, 6\}$ and $A_3 = \{7, 8, 9\}$ is a valid partition, but so is $\mathcal{P}' = \{A_1', A_2', A_3'\}$ with $A_1' = \{1, 5, 9\}, A_2' = \{4, 8, 3\}$ and $A_3' = \{7, 2, 6\}$. Thus, $\mathcal{F}$ is partitionable but not uniquely so.

**Definition 8** (Maximum Uniquely Partitionable Multiset). Fix $n, m \in \mathbb{Z}^+$. A uniquely partitionable multiset $\mathcal{F}$ is called maximum if for all partitionable multisets $\mathcal{F}'$, we have $|\mathcal{F}'| \leq |\mathcal{F}|$.

For brevity, we will refer to the above as maximum multisets. We know that $g(n, m)$ is the cardinality of a maximum multiset. Any partitionable $\mathcal{F}$ must satisfy $|\mathcal{F}| \leq nm$, and at least one uniquely partitionable multiset exists, so a maximum multiset exists for each pair $(n, m)$. Therefore, $g(n, m)$ is well defined.

**Definition 9** (Groups of a Partition). Let $\mathcal{P} = \{A_1, \ldots, A_n\}$ be a partition corresponding to $\mathcal{F}$. Then $\{C_j : j \in A_i\}$ is a group of $\mathcal{P}$ corresponding to $A_i$ for each $i$. We will abuse notation to say that $A_i$ itself is a group corresponding to $\mathcal{P}$.

**Definition 10** (Group Size). Let $\mathcal{P} = \{A_1, \ldots, A_n\}$ be a partition of $\mathcal{F}$. Then $d_i = |A_i| \geq 1$ is the size of the group $A_i$ for each $i$.

**Example 11.** In Figure 1, the group $A_1$ has size 2 (its 2 components are $\{1, 2\}$ and $\{3, 4\}$) and the group $A_2$ has size 3 (its 3 components are $\{1\}, \{2, 3\}$ and $\{4\}$).

**Definition 12** (Proper subsets induced by a Group). Let $\mathcal{F} = \{C_1, \ldots, C_k\}$ be a multiset and let $\mathcal{P} = \{A_1, \ldots, A_n\}$ be a partition of $\mathcal{F}$. Fix a group $A_i$. The proper subsets induced by $A_i$ are all the non-empty proper subsets of $[m]$ that can be formed by taking unions over some sets in $A_i$.

**Example 13.** In Figure 1, consider the group $A_2$. $A_2$ induces the proper subsets $\{1\}, \{2, 3\}, \{4\}, \{1, 2, 3\}, \{1, 4\}$ and $\{2, 3, 4\}$. A size $d$ group induces $2^d - 2$ proper subsets.

### 3 Determining $g(n, m)$

#### 3.1 Introductory Results

**Theorem 14** (Graph-Multiset Equivalence).

\[ f(n, m) = nm - g(n, m) \] (1)
Proof. We first discuss the equivalence of the graph theoretic interpretation of the problem formulated in Section 1. By the constraints of the problem, each vertex in the $m$-partite graph $G$ has degree $m - 1$, so $|E(G)| = \frac{nm(m-1)}{2}$. We want to find $f(n, m)$, the size of smallest subset of $E(G)$ such that $G$ can be uniquely determined. The equivalence with the size of the maximum multiset $g(n, m)$ is due to the following reason. Suppose we have a minimum subset $S \subseteq E(G)$ that uniquely specifies $G$. Each copy of $K_m$ must have a certain number of components determined by $S$. Let the $i$th copy have $d_i$ components. If $S$ is indeed minimum, we need no more than $m - d_i$ edges to specify these components. Identify the components of each $K_m$ with the subsets of $[m]$ to construct a partitionable multiset $F$, and identify the $i$th copy of $K_m$ with the $i$th group in a partition $P$. It is easy to see that $P$ is unique if and only if $S$ uniquely determines $G$. Further, \[
 |S| = \sum_{i=1}^{n} m - d_i = nm - \sum_{i=1}^{n} d_i = nm - |F| \]
So minimizing $|S|$ is equivalent to maximizing $|F|$, and $f(n, m) = nm - g(n, m)$. \hfill \square

We may now focus on $g(n, m)$ for the rest of the paper.

**Theorem 15.** Suppose $n = 2$. Then for all $m \in \mathbb{Z}^+$, \[ g(2, m) = m + 1 \]  

Proof. For each $m$, with $n = 2$, we can construct a uniquely partitionable multiset of size $m + 1$. Namely, let $F = \{C_1, \ldots, C_m, C_{m+1}\}$ with $C_i = \{i\}$ for $i \in [m]$ and $C_{m+1} = [m]$. It is clear that there is only one valid partition $P = \{[m], \{m + 1\}\}$. Therefore, $g(2, m) \geq m + 1$. We will show that we can do no better.

For $m = 1, 2$ the result holds by directly checking so assume $m \geq 3$. Consider any maximum multiset $F$ corresponding to $(2, m)$ with partition $P = \{A_1, A_2\}$. Suppose $A_1$ has less than $m$ groups. Then there must be at least one component in $A_1$ that has at least 2 elements. Let $C_1$ be a component in $A_1$ that contains at least 2 elements, say 1 and 2. Replace $C_1$ in $A_1$ with two subsets $C_1^{(a)} = C_1 \setminus \{1\}$ and $C_1^{(b)} = \{1\}$ to create a partition $P' = \{A_1', A_2'\}$ of a new partitionable multiset $F'$. The sets in $A_1$ and $A_1'$ are the same except for $C_1$ that has been split into two parts. The sets in $A_2$ and $A_2'$ are the same.

In $P'$, the group $A_2'$ must contain some sets with the elements 1 and 2, and we claim that 1 and 2 must lie in different components of $A_2'$. Suppose by contradiction that $A_2'$ has a component $C_2$ that contains both 1 and 2. Then we claim that $F'$ must have a unique partition. If $F'$ admits a reassignment of $P'$, then $C_1^{(a)}$ and $C_1^{(b)}$ must be assigned to different groups because otherwise such a reassignment would induce a reassignment of $P$ as well, a contradiction. So in fact $F'$ is uniquely partitionable with $|F'| > |F|$, contradicting the maximality of $F$.

Since 1 and 2 lie in different components of $A_2'$, combine these two components to form a single component $C_2$ resulting in a multiset $F''$ with the induced partition $P''$. In any partition of $F''$, since both groups need to have sets that contain the elements 1 and 2, $C_1^{(a)}$ and $C_1^{(b)}$ must lie in
the same group because they must be opposite $C_2$. So no reassignment of $P''$ is possible because it would induce a reassignment of the original $P$ as well. It follows that $F''$ has the same number of subsets as $F$ and is also uniquely partitionable, and thus maximum. Further, the number of subsets in the first group $A_1''$ of $F''$ is 1 less than $|A_1|$. We can repeat this as many times as required to construct a partitionable multiset with the same number of elements and $m$ subsets in the first group. This implies that the second group would have a single set $[m]$. So indeed, $g(2, m) = m + 1$. 

Theorem 16 (Subset Criterion). Suppose $F$ is a uniquely partitionable multiset with respect to $n, m \in \mathbb{Z}^+$. Let $P$ be the unique partition corresponding to $F$. Let $d_1, \ldots, d_n$ be the sizes of each group in $P$. Then we have that

$$\sum_{i=1}^{n} (2^{d_i} - 2) \leq 2^m - 2 \quad (3)$$

Proof. We will count the number of proper subsets of $[m]$ formed by each group. For any group $i$, there are $d_i \geq 1$ components. Therefore, by combining components, we can produce $2^{d_i} - 2$ proper subsets of $[m]$. Suppose the same proper subset of $[m]$ can be formed using only components of group $i$ and also using only components of group $j$ (with $i \neq j$). Then we can move the vertices forming that proper subset of $[m]$ in group $i$ to group $j$ and vice versa. This contradicts the fact that $P$ is a unique partition of $F$.

Each group must contribute $2^{d_i} - 2$ new proper subsets. The total number of such subsets is given by $\sum_{i=1}^{n} (2^{d_i} - 2)$. This must be less than or equal to $2^m - 2$, the number of proper subsets of $[m]$ because otherwise the Pigeonhole principle tells us that some two groups must have at least one common proper subset. The desired result follows. 

Remark 17. The Subset Criterion gives a necessary condition for $P$ to be the partition of a uniquely partitionable $F$, but it is not a sufficient condition.

3.2 Determining $g(n, m)$ when $n \geq 2^{m-1} - 1$

Lemma 18. Suppose $m \in \mathbb{N}$ and $n \geq 2^{m-1} - 1$. Then

$$g(n, m) = g(2^{m-1} - 1, m) + n - 2^{m-1} + 1 \quad (4)$$

Proof. Consider any maximum multiset $F$ and its unique partition $P$. We claim that there are at most $2^{m-1} - 1$ groups in $P$ of size greater than 1. Suppose there are $k$ groups of size 2 or more for some $k \in [n]$. We need to show that $k \leq 2^{m-1} - 1$.

Let $d_1, \ldots, d_k$ be the sizes of the $k$ groups with size at least 2. By assumption, $d_i \geq 2$ for all $i \in [k]$. By the Subset Criterion, we have

$$2^m - 2 \geq \sum_{i=1}^{k} (2^{d_i} - 2) \geq \sum_{i=1}^{k} (4 - 2) = 2k$$

So we must have that $2k \leq 2^m - 2$, so it follows that $k \leq 2^{m-1} - 1$. Therefore, for any $n \geq 2^{m-1} - 1$, $P$ must have at least $n - 2^{m-1} + 1$ groups of size 1, so we must have at least $n - 2^{m-1} + 1$ copies of $[m]$ in $F$. For the remaining $2^{m-1} - 1$ groups, we can introduce at most $g(2^{m-1} - 1, m)$ sets. So we have that $g(n, m) \leq g(2^{m-1} - 1, m) + n - 2^{m-1} + 1$. 

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Further, if we have a maximum multiset corresponding to \((2^{m-1}-1, m)\), we can always construct a uniquely partitionable multiset for \((n, m)\) by adding \(n - (2^{m-1} - 1)\) groups, each with the single set \([m]\). Therefore, \(g(n, m) \geq g(2^{m-1} - 1, m) + n - 2^{m-1} + 1\). The desired result follows. \(\square\)

**Remark 19.** Lemmas 20 and 21 below hold for all \(n \leq 2^{m-1} - 1\) which will prove useful later. However, here in Section 3.2, we will only make use of these lemmas when \(n = 2^{m-1} - 1\).

**Lemma 20.** Suppose \(n \leq 2^{m-1} - 1\). Then we have that

\[
g(n, m) \geq 2n \tag{5}
\]

*Proof.* We will construct a uniquely partitionable multiset \(\mathcal{F}\) with each group in the partition \(\mathcal{P}\) having size 2. Specify each group \(A_i\) in \(\mathcal{P}\) (for each \(i \in [n]\)) by a pair of proper subsets \((C_1^{(i)}, C_2^{(i)})\) of \([m]\) such that \(C_1^{(i)} \cup C_2^{(i)} = [m]\). There are \(\frac{2^m - 2}{2} = 2^{m-1} - 1\) such pairs, so we can specify up to \(2^{m-1} - 1\) groups, each of size 2, without repeating any proper subsets. The resultant multiset is \(\mathcal{F} = \{C_j^{(i)} : i \in [n], j \in \{1, 2\}\}\). We claim that \(\mathcal{P}\) is the only possible partition.

If there was any reassignment of the components to form a different collection of groups, we must still have exactly two components in each group. If not, then some group must have at least 3 components, so some other group must have exactly one component, which must be \([m]\). But no group had a single component of size \(m\) before the reassignment because we were working only with proper subsets. For any component \(C\) in group of size 2, it must be paired with its complement, which was precisely the original partition \(\mathcal{P}\). So we have that \(g(n, m) \geq 2n\). \(\square\)

**Lemma 21.** Suppose \(n \leq 2^{m-1} - 1\). Then there exists a maximum multiset \(\mathcal{F}\) with a unique partition \(\mathcal{P}\) such that each of the \(n\) groups in \(\mathcal{P}\) has size at least 2.

*Proof.* Suppose we have a maximum multiset \(\mathcal{F}_1\) with a unique partition \(\mathcal{P}_1\) such that at least one group has size 1. Then we must have at least one group with size at least 3. If not, every group has size at most 2 so the number of sets in \(\mathcal{F}\) is less than the number of sets in the case when every group has size 2, a contradiction to Lemma 20.

Let \(A_1\) and \(A_2\) be the groups with size 1 and size \(l \geq 3\) respectively. So \(A_1\) has only a single component \([m]\). Let the \(l\) components of \(A_2\) be \(C_1^{(b)}, C_2^{(b)}, C_3^{(b)}, \ldots, C_l^{(b)}\). Then reduce the number of sets in \(A_2\) to combine \(C_1(b)\) and \(C_2(b)\). Then we can specify \(A_1\) with 2 sets by breaking \([m]\) up into the corresponding components \(C_1^{(a)}\) and \(\cup_{i=2}^{l} C_i^{(a)}\).

We claim that this new multiset \(\mathcal{F}_2\) with its partition \(\mathcal{P}_2\) is still uniquely partitionable. Suppose by contradiction that there exists another partition \(\mathcal{P}_2'\) for \(\mathcal{F}_2\). We will show that we can then obtain a new partition \(\mathcal{P}_1'\) for \(\mathcal{F}_1\), contradicting the fact that \(\mathcal{P}_1\) was unique.

Any such \(\mathcal{P}_2'\) would have to involve dividing the sets of \(A_1\) into different groups, because otherwise we directly have a reassignment of \(\mathcal{P}_1\) as well. So we have that \(\cup_{i=2}^{l} C_i^{(a)}\) is paired with components from some other groups: let us call the union of these components \(K\).

1. Move \(C_1^{(a)}\) to where \(K\) is.
2. Combine the sets \(\cup_{i=2}^{k} C_i^{(a)}\) and \(\cup_{i=2}^{k} C_i^{(a)}\) and split the sets \(C_1^{(b)}\) and \(C_2^{(b)}\).
3. Move \(K\) to wherever \(C_1^{(b)}\) is.
4. Move \(C_1^{(b)}\) to wherever \(C_1^{(a)}\) was.
Since $A_1$ is now unaffected and $C_1^{(b)}, C_2^{(b)}$ are separated, this induces a reassignment of $P_1$, a contradiction. It follows that the newly formed multiset $F_2$ must also be uniquely partitionable and maximum as we have not changed the number of subsets. We have now eliminated the group of size 1. We can repeat this process as many times as required to obtain a maximum multiset with every group having size at least 2.

\begin{equation}
\text{Lemma 22. For any } m \in \mathbb{Z}^+,
\end{equation}

\begin{equation}
g(2^{m-1} - 1, m) = 2(2^{m-1} - 1) = 2^m - 2
\end{equation}

\text{Proof. By Lemma 20, we have that } g(2^{m-1} - 1, m) \geq 2(2^{m-1} - 1). \text{ Suppose by contradiction that there exists a maximum multiset } F \text{ with partition } P \text{ such that } |F| > 2(2^{m-1} - 1). \text{ By Lemma 21, we may assume without loss in generality that every group has size at least 2. We now show that in fact we cannot have any group with size at least 3. Suppose there exists at least one group with size at least 3. Then}

\begin{align*}
\sum_{i=1}^{2^{m-1}-1} (2^{d_i} - 2) &\geq 2^3 - 2 + \sum_{i=1}^{2^{m-2}} (2^2 - 2) \\
&= 2^m + 2
\end{align*}

This contradicts the Subset Criterion and completes the proof.

\begin{equation}
\text{Theorem 23. Suppose } m \in \mathbb{N} \text{ and } n \geq 2^{m-1} - 1. \text{ Then}
\end{equation}

\begin{equation}
g(n, m) = 2^{m-1} - 1 + n
\end{equation}

\text{Proof.}

\begin{align*}
g(n, m) &= g(2^{m-1} - 1, m) + (n - 2^{m-1} + 1) \text{ by Lemma 18} \\
&= 2(2^{m-1} - 1) + (n - 2^{m-1} + 1) \text{ by Lemma 22} \\
&= 2^{m-1} - 1 + n
\end{align*}

Note that for $m = 1$, $2^{m-1} - 1 = 0$ and for $m = 2$, $2^{m-1} - 1 = 1$, so Theorem 23 covers $g(n, 1)$ and $g(n, 2)$ for all $n$. Additionally, for $m = 3$, $2^{m-1} - 1 = 3$, and $g(2, 3) = 4$ by Theorem 15 and $g(1, 3) = 3$, so $g(n, 3)$ has been determined for all $n$. So we have a complete characterization for $m \leq 3$. 8
3.3 Lower Bounds for $g(n, m)$ when $n \leq 2^{m-1} - 1$

The focus of this section is to show a lower bound for $g(n, m)$. In the next section, we will provide close upper bounds in a large regime.

Definition 24. Fix $2 \leq k \leq m$. Let $P_k(m)$ be the largest possible integer $N$ such that there exists a uniquely partitionable multiset $F$ corresponding to $(N, m)$ with a partition $P$ that contains $N$ groups of size $k$ each.

- Observe that our results in Lemma 18 imply $P_2(m) = 2^{m-1} - 1$.
- It is easy to see that $P_k(m) \geq P_{k+1}(m)$. We just observe that for any uniquely partitionable multiset $F$ corresponding to $(P_{k+1}(m), m)$ with $P_{k+1}(m)$ groups of size $k+1$, combing any two components of every group produces a uniquely partitionable multiset $F'$ corresponding to $(P_{k+1}(m), m)$ with every group of size $k$.

Lemma 25. For any set $S$ with $N$ elements, there exist a set of $2^N - 1$ proper subsets of $S$ with each set containing at least $\lceil \frac{N}{2} \rceil$ elements such that no two sets are complements of each other.

Proof. If $N$ is odd, this follows by simply counting all possible subsets of size $\lceil \frac{N}{2} \rceil$ or greater because the sum of elements in any two such sets is more than $N$. If $N$ is even, we can construct a family with all proper subsets of $S$ with size $\frac{N}{2} + 1$ or larger. For the sets of size $\frac{N}{2}$, we can pair them up with their complements and add exactly one set from each set to our family. This must also have size $2^N - 1$. The desired result follows.

Theorem 26. Choose an integer $k$ such that $2 \leq k < m$ and $\frac{m}{k-1} \geq 2$. Then
\[ P_k(m) \geq 2^{\lceil \frac{m}{k-1} \rceil - 1} - 1 \quad (8) \]

Proof. For simplicity, assume that $k - 1$ divides $m$. The proof for other cases remains the same.

1. Divide $[m]$ into $k - 1$ sets: $S_1 = \{1, \ldots, \frac{m}{k-1}\}, S_2 = \{\frac{m}{k-1} + 1, \ldots, \frac{2m}{k-1}\}, \ldots, S_{k-1} = \{(k - 2)\frac{m}{k-1} + 1, \ldots, m\}$.

2. There are at least $l = 2^{k-1} - 1$ proper subsets of each $S_i$ with size at least $\lceil \frac{m}{2(k-1)} \rceil \geq 1$ by Lemma 25. For each $i \in [k - 1]$, label the sets $X_1^{(i)}, \ldots, X_l^{(i)}$.

3. Construct $l$ tuples of sets of the form $(X_j^{(1)}, \ldots, X_j^{(k-1)})$ for each $j \in [l]$.

4. We will now construct $l$ groups each of size $k$. Choose any tuple $(X^{(1)}, \ldots, X^{(k-1)})$. Let $C = \cup_{i=1}^{k-1} X^{(i)}$. So $C$ contains at least one element in $S_i$ for each $1 \leq i \leq k - 1$. For each $i$, let $C_i = S_i \setminus C$ be the remaining elements in $S_i$. Then construct a group with the sets $\{C, C_1, \ldots, C_{k-1}\}$. Call $C$ the central component of this group. Repeat this process for each tuple to create $l = 2^{\frac{m}{k-1} - 1} - 1$ groups of size $k$.

This construction clearly induces a partition $P$ for a multiset $F$ corresponding to $(2^{\frac{m}{k-1} - 1}, m)$ with every group of size $k$. We claim $F$ is uniquely partitionable.
Suppose by contradiction that \( \mathcal{P} \) admits a reassignment \( \mathcal{P}' \). We claim that each group in \( \mathcal{P}' \) must contain a central component. If not, then by the Pigeonhole principle, there exists some group with at least two central components \( C = \bigcup_{i=1}^{k-1} X_j^{(i)} \) and \( C' = \bigcup_{i=1}^{k-1} X_j^{(i)} \). However, by construction, both \( X_j^{(1)} \) and \( X_j^{(1)} \) contain at least \( \left\lceil \frac{m}{2(k-1)} \right\rceil \) elements in \( S_1 \). If \( \frac{m}{k-1} \) is odd, then \( C \) and \( C' \) must have a non-trivial intersection, a contradiction. If \( \frac{m}{k-1} \) is even, if either \( X_j^{(1)} \) or \( X_j^{(1)} \) has at least \( \frac{m}{2(k-1)} + 1 \) elements, then once again \( C \) and \( C' \) must have a non-trivial intersection. If both \( X_j^{(1)} \) and \( X_j^{(1)} \) have exactly \( \frac{m}{2(k-1)} \) elements, the only way they have an empty intersection is if \( X_j^{(1)} \cap X_j^{(1)} = S_1 \). But by construction, we chose a family of proper subsets of \( S_1 \) such that no subset in the family has a complement in the family. Therefore, each group must have exactly one central component.

Consider any central component \( C \). Assume that originally in \( \mathcal{P} \), \( C \) was in the group \( \{C, C_1, \ldots, C_{k-1}\} \). Since \( C \) does not contain all the elements in \( S_1 \) by construction, it requires at least one set \( C'_1 \subseteq S_1 \) in the same group even in \( \mathcal{P}' \). It cannot have more than one subset that is a subset of \( S_1 \), because by the Pigeonhole principle, this would mean some other central component has no subset of \( S_1 \) in its group in \( \mathcal{P}' \). So in fact \( C \) has exactly one set \( C'_1 \subseteq S_1 \) in its group in \( \mathcal{P}' \). In fact, \( C'_1 = C_1 \) because \( C'_1 \) must contain all the elements of \( S_1 \) not in \( C \) without having an intersection with \( C \), so we must have \( C'_1 = S_1 \setminus C = C_1 \). Repeating the same argument for \( i = 2, \ldots, k-1 \), \( C \) must be in the same group in both \( \mathcal{P} \) and \( \mathcal{P}' \). Repeating the argument for every central component, we have that \( \mathcal{P} = \mathcal{P}' \). It follows that \( \mathcal{F} \) is uniquely partitionable.

When \( k-1 \) does not divide \( m \), the same construction works. If \( r \) is the remainder when \( m \) is divided by \( k-1 \), just add the same \( r-1 \) integers into every central component. \( \square \)

**Theorem 27.** Choose an integer \( k \) such that \( 2 \leq k < m \) and \( \frac{m}{k-1} \geq 2 \). Then for any \( n \leq 2\left\lfloor \frac{m}{k-1} \right\rfloor - 1 \),

\[
g(n, m) \geq kn \tag{9}
\]

**Proof.** This follows immediately from Theorem 26. If \( n \leq 2\left\lfloor \frac{m}{k-1} \right\rfloor - 1 \), we can construct a uniquely partitionable multiset \( \mathcal{F} \) corresponding to \((n, m)\) with every group having size \( k \), so \( |\mathcal{F}| = kn \leq g(n, m) \). \( \square \)

We now choose the optimal value of \( k \) to obtain some bounds for any \( n \leq 2^{m-1} - 1 \).

**Theorem 28** (Lower Bound). Suppose \( 1 \leq n \leq 2^{m-1} - 1 \). Then

\[
g(n, m) \geq \frac{n(m+1)}{\log_2(n+1) + 2} \tag{10}
\]

In other words, \( g(n, m) = \Omega\left( \frac{mn}{\log n} \right) \) when \( n \leq 2^{m-1} - 1 \).

**Proof.** From Theorem 27, we need to choose \( k \) such that \( n \leq 2\left\lfloor \frac{m}{k-1} \right\rfloor - 1 \). So for a certain choice of \( k \), it suffices if the following condition holds:

\[
n \leq 2^{\frac{m-k-2}{k-1}} - 1 = 2^{\frac{m+1}{k-1}} - 1 \iff k \leq \left\lfloor \frac{m+1}{2 + \log_2(n+1)} \right\rfloor + 1
\]
It follows that
\[
g(n, m) \geq n \left( \left\lfloor \frac{m + 1}{\log_2(n + 1) + 2} \right\rfloor + 1 \right) \\
\geq \frac{n(m + 1)}{\log_2(n + 1) + 2}
\]

3.4 Upper Bounds for \( g(n, m) \) when \( n \leq 2^{m-1} - 1 \)

The natural question to ask is how close to optimal the lower bound from Section 3.3 is. We provide a upper bound in a large regime, namely, when \( n = 2^{cm} \) for some \( c \in (0, 1) \) which matches the lower bound in Theorem 28 up to a \( \log_2(\frac{1}{c}) \) factor.

Lemma 29. Let \( H_2(x) \) be the binary cross entropy function.
\[
H_2(x) = -x \log_2(x) - (1 - x) \log_2(1 - x)
\]

Suppose \( k \geq 2 \). Then
\[
\sum_{i=0}^{\left\lfloor \frac{m}{k} \right\rfloor} \binom{m}{i} \leq 2^{mH_2(\frac{1}{k})}
\]

Theorem 30. For any \( m \) and \( 3 \leq k \leq \frac{m}{2} \),
\[
P_k(m) \leq 2^{mH_2(\frac{1}{k})}
\]

Proof. Consider a uniquely partitionable multiset \( F \) corresponding to \( (P_k(m), m) \) with partition \( P \) where every group has size \( k \). Note that each group must have at least one subset with at most \( \left\lfloor \frac{m}{k} \right\rfloor \) elements. If \( F \) is uniquely partitionable, the only subset it can contain multiple times is \( \left\lfloor \frac{m}{k} \right\rfloor \), so each subset with size at most \( \left\lfloor \frac{m}{k} \right\rfloor \) appears at most once in \( F \). Therefore,
\[
|F| \leq \sum_{i=0}^{\left\lfloor \frac{m}{k} \right\rfloor} \binom{m}{i} \\
\leq 2^{mH_2(\frac{1}{k})} \text{ by Lemma 29}
\]

Lemma 31. Let \( k \geq 2 \) be an integer such that \( n > 2^{mH_2(\frac{1}{k})} \). Then \( g(n, m) \leq kn \).

Proof. Suppose by contradiction that \( g(n, m) > kn \). Consider a maximum multiset \( F \) corresponding to \( (n, m) \) with partition \( P \) and group sizes \( d_1, \ldots, d_n \). For all \( i \in [n] \), let \( r_i = \max(0, d_i - (k-1)) \). Note that
\[
\sum_{i=1}^{n} r_i \geq n
\]
because otherwise $g(n, m) \leq kn$. Consider any group with positive $r_i$. So we know $d_i = k + r_i - 1$. We claim that this group must contain at least $r_i$ subsets with size at most $\lceil \frac{n}{k} \rceil$. If not, this group must contain at most $r_i - 1$ subsets with size at most $\lceil \frac{m}{k} \rceil$, so at least $k$ subsets must have size at least $\lceil \frac{m}{k} \rceil + 1$. It follows that the sum of the cardinalities of the sets in the group is at least

$$k \left( \lceil \frac{m}{k} \rceil + 1 \right) \geq km - k + 1 + k = m + 1$$

This is a contradiction. Therefore, each group must contain at least $r_i$ subsets with size at most $\lceil \frac{m}{k} \rceil$. Since a uniquely partitionable multiset must contain at most $2^{mH_2(\frac{1}{k})}$ subsets with size at most $\lceil \frac{m}{k} \rceil$, it follows that $n \leq \sum_{i=1}^n r_i \leq 2^{mH_2(\frac{1}{k})}$, a contradiction.

**Theorem 32 (Upper Bound).** Let $c \in (0, 1)$ and $n = 2^{cm}$ with $1 \leq n \leq 2^{m-1} - 1$. Then

$$g(n, m) \leq \frac{n}{c} (6 - 3.2 \log_2(c)) \quad (14)$$

**Proof.** We will make use of Lemma 31 and find $k$ such that $n = 2^{cm} > 2^{mH_2(\frac{1}{k})}$. Note that if $k \geq 3$, $H_2(\frac{1}{k}) < 2 \log_2(k)$. So it suffices to find an integer $k \geq 3$ such that $\frac{2 \log_2(k)}{k} \leq c$, or equivalently $\frac{\ln(k)}{k} \leq \frac{c \ln 2}{2}$. Let $\epsilon = \frac{c \ln 2}{2} \in (0, \frac{\ln 2}{2})$. We claim that the following choice of $k$ suffices:

$$k = \left\lceil \frac{1.6}{\epsilon} \ln \left( \frac{1}{\epsilon} \right) \right\rceil$$

We note that $k \geq 3$ because:

$$\frac{1.6}{\epsilon} \ln \left( \frac{1}{\epsilon} \right) \geq \frac{3.2}{\ln 2} \ln \left( \frac{2}{\ln 2} \right) \geq 4$$

We want to show that $\frac{\ln k}{k} \leq \epsilon$. Since $\frac{\ln x}{x}$ is decreasing for $x \geq 3$, it suffices to show that

$$\frac{\ln \left( \frac{1.6}{\epsilon} \ln \left( \frac{1}{\epsilon} \right) \right)}{1.6} \ln \left( \frac{1}{\epsilon} \right) \leq \epsilon \iff \ln \left( \frac{1.6}{\epsilon} \ln \left( \frac{1}{\epsilon} \right) \right) \leq 1.6 \ln \left( \frac{1}{\epsilon} \right)$$

$$\iff 0.6 \ln \left( \frac{1}{\epsilon} \right) - \ln \left( \frac{1}{\epsilon} \right) - \ln(0.6) \geq 0$$

Define $h(x) = 0.6 \ln(x) - \ln(\ln(x)) - \ln(1.6)$. It suffices to show $h(x)$ is positive on $(0, \infty)$. The only critical point of $h(x)$ is at $x = e^{\frac{5}{3}}$. It is easy to verify that $h''(e^{\frac{5}{3}}) > 0$. It follows that $h(x)$ is always positive. It follows that

$$g(n, m) \leq n \left\lceil \frac{1.6}{\epsilon} \ln \left( \frac{1}{\epsilon} \right) \right\rceil$$

$$= n \left\lceil \frac{3.2}{c \ln 2} \ln \left( \frac{2}{c \ln 2} \right) \right\rceil$$

$$\leq n \left( \frac{3.2}{c} - \frac{\log_2(c)}{c} - \frac{\ln(2)}{c \ln 2} + 1 \right)$$

$$\leq \frac{n}{c} (6 - 3.2 \log_2(c))$$

$\square$
Remark 33. This upper bound of \( \frac{n}{c} (6 - 3.2 \log_2(c)) = O\left(\frac{n}{c} \log\left(\frac{1}{c}\right)\right) \) is in the regime \( n = 2^m \) for any \( c \in (0, 1) \). If we look at our lower bound from Theorem 28 in this regime, since \( \log_2(n) = cm \), we have that \( g(n, m) \geq \Omega\left(\frac{nm}{\log_2(n)}\right) = \Omega\left(\frac{n}{c}\right) \). Therefore, our bounds are tight up to a \( \log_2\left(\frac{1}{c}\right) \) factor.

3.5 Determining \( g(n, m) \) when \( 2^{m-1} - O\left(2^m\right) \leq n \leq 2^{m-1} - 1 \)

In this section, we obtain exact results for \( g(n, m) \) when \( 2^{m-1} - 2^{|\mathbb{Z}/2}\| + 3 \leq n \leq 2^{m-1} - 1 \).

Lemma 34. Suppose \( n \leq 2^{m-1} - 1 \). We can pick a maximum multiset \( \mathcal{F} \) with partition \( \mathcal{P} \) such that each of the \( n \) groups in \( \mathcal{P} \) has at least size 2 (by Lemma 21). Suppose for some \( 1 \leq k \leq n \), there are \( k \) groups with size at least \( d \). Then

\[
k(2^{d-1} - 2) \leq 2^{m-1} - 1 - n
\]

Proof.

\[
\sum_{i=1}^{n} (2^{d_i} - 2) \geq k(2^d - 2) + (n - k)(2^2 - 2)
\]

\[
= 2n + (2^d - 4)k
\]

By the Subset Criterion, \( 2n + (2^d - 4)k \leq 2^m - 2 \). Rearranging this completes the proof. \( \square \)

Lemma 35. Consider any \( n, k, m \in \mathbb{Z}^+ \) with \( k \leq n \). Suppose there exists a uniquely partitionable multiset \( \mathcal{F}_1 \) (not necessarily maximum) corresponding to \((k, m)\) such that each of the groups in the partition \( \mathcal{P}_1 \) have size 3. Suppose we can introduce \( n - k \) groups of size 2 such that none of the proper subsets induced by the \( n - k \) groups appeared in the first \( k \) groups. Then the \( n = k + (n - k) \) groups form a uniquely partitionable multiset \( \mathcal{F}_2 \) with partition \( \mathcal{P}_2 \) corresponding to \((n, m)\).

Proof. Suppose by contradiction that the construction is not uniquely partitionable. Consider a reassignment of \( \mathcal{P}_2 \), say \( \mathcal{P}_2' \). Suppose \( \mathcal{P}_2' \) contains a group of size 4 or more. Since the total number of subsets in the multiset must be preserved, we must have a component \( C \) that was originally in a size 3 group which is now in a size 1 or 2 group or originally in a size 2 group which is now in a size 1 group.

Since we never had a size 1 group (a group with the single component \([m]\) initially), a reassignment cannot contain a size 1 group. So \( C \) must have been in a size 3 group and is now in a size 2 group. But that means \( C^c \) was present as a single component in the original assignment \( \mathcal{P}_2 \). \( C^c \) could not have been in a size 3 group because otherwise \( \mathcal{P}_1 \) consisting of the original \( k \) groups would not be unique. It could not have been in a size 2 group because we introduced size 2 groups in a manner such that each component was not a proper subset induced by the size 3 groups. It follows that no such \( C \) exists. Thus, every group in \( \mathcal{P}_2' \) has size 3 or 2.

Pick a component \( C \) that was earlier in a size 3 group. If it is now in a size 2 group, then \( C^c \) must exist as a single component. By the same argument as above, \( C^c \) cannot exist. So \( C \) must be in a size 3 group again. So \( \mathcal{P}_2' \) has at least \( k \) size 3 groups, and cannot have more than \( k \) such groups because the number of subsets must be preserved. In fact, the assignment of the components to the \( k \) size 3 groups must be the same because \( \mathcal{P}_1 \) containing the original \( k \) groups was a unique partition.
The remaining \( n - k \) groups must all be of size 2. No reassignment of just size 2 groups is possible because every group must be of the form \((A, A^c)\). It follows that no reassignment \( \mathcal{P}'_2 \) is possible. Therefore, \( \mathcal{P}_2 \) is unique and so \( \mathcal{F}_2 \) is uniquely partitionable.

**Lemma 36.** Suppose \( n, m \in \mathbb{Z}^+ \) with \( n \leq 2^m - 1 \). Let \( k = \lfloor \frac{2^m - 1 - n}{2} \rfloor \). Suppose there exists a uniquely partitionable multiset \( \mathcal{F} \) with partition \( \mathcal{P} \) corresponding to \( k \) groups such that each of the groups has size 3. Then

\[
g(n, m) = 2n + k
\]

**Proof.** If \( k \) groups each have size 3, that induces a total of \( k(2^3 - 2) = 6k \) proper subsets. It follows that we have \( 2^m - 2 - 6k \) remaining proper subsets that occur in pairs. We want to ensure that the remaining \( n - k \) groups can be specified with size 2. We can construct at least \( \frac{2^m - 2 - 6k}{2} \) pairs, so we need to show that this is at least as large as \( n - k \) which is clearly true.

\[
\frac{2^m - 2 - 6k}{2} \geq n - k \iff k \leq \frac{2^m - 1 - n}{2}
\]

By Lemma 35, this results in a uniquely partitionable multiset. So it follows that

\[
g(n, m) \geq 3k + 2(n - k) = 2n + k
\]

Suppose by contradiction that \( g(n, m) > 2n + k \). Consider a maximum multiset \( \mathcal{F} \) with partition \( \mathcal{P} \) corresponding to \( (n, m) \). By Lemma 21, we can assume that each of the \( n \) groups has size at least 2. By plugging in \( d = 3 \) into Lemma 34, we can choose the maximum multiset \( \mathcal{F} \) with every group in \( \mathcal{P} \) at least of size 2 with at most \( k \) groups of size 3 or more. So we must have at least \( n - k \) groups of size 2. In this partition \( \mathcal{P} \), let \( k_i \) be the number of groups of size \( i \), for \( i \in [m] \). Assuming \( n > 1 \), we can assume that \( k_m = 0 \). By Lemma 21, we can assume that \( k_1 = 0 \). We also know that \( k_2 \geq n - k \). Therefore, we want to maximize

\[
\sum_{i=2}^{m-1} ik_i
\]

subject to the constraints that

\[
\sum_{i=2}^{m-1} k_i = n
\]

\[
\sum_{i=2}^{m} k_i 2^i \leq 2^m - 2 + 2n
\]

where the second constraint is obtained using the Subset Criterion inequality below.

\[
\sum_{i=2}^{m} k_i (2^i - 2) \leq 2^m - 2
\]

We will show that we can do no better than setting \( k_2 = n - k \) and \( k_3 = k \). This will contradict the fact that \( g(n, m) < 2n + k \). We know that \( k_2 \geq n - k \). So we leave \( n - k \) size 2 groups untouched for the rest of this argument. Initially, let the remaining \( k \) groups all be size 3. To
improve our optimization objective in Equation 16, we need to increase the size of some of these
$k$ groups, while perhaps reducing the size of some other groups in this set of $k$ groups to ensure
that we meet the constraint in Equation 18. It does not matter whether we can actually achieve
a uniquely partitionable multiset corresponding to these new size assignments, since even if we
could, if the optimization objective is not higher than our baseline, it does not result in a multiset
with larger size.

We can consider the right hand side of Equation 18 to be the total units available and the left hand side to be the number of units initially used. Since we start off with $n - k$ size 2 groups and $k$ groups of size 3, we have at most 2 units for the following reason:

$$2^m - 2 + 2n - [2^2(n - k) + 2^3k] = 4 \left( \frac{2^m - 1 - n}{2} \right) - 4k$$

$$\le 4(k + 0.5) - 4k$$

$$= 2$$

Assume that we have a maximum multiset $F$ that is better than our baseline of $n - k$ size 2
groups and $k$ size 3 groups. Let $B$ be the set of the $k$ groups that are initialized to size 3. Then
some groups in $B$ will have their sizes increased to 4 or more, while other groups in $B$ have to
move from size 3 to 2. In the optimal setting, nothing needs to go to size 1 by Lemma 21.

Suppose any group in $B$ ends up with size $i \ge 4$. Then the cost of changing a group from size
3 to size $i$ is $2^i - 2^3 = 2^i - 8$. Even if we make use of the 2 units available to us in the beginning,
the cost is $2^i - 10$. To offset this, some size 3 groups in $B$ need to be converted to size 2 groups.
The gain in units from such a conversion is $2^3 - 2^2 = 4$. Therefore, to offset the conversion of
a size 3 group to a size $i$ group, we need to convert at least $\lceil \frac{2^i - 10}{4} \rceil = 2^{i-2} - 3$ size 3 groups to size 2. Replacing the size 3 group with a size $i$ group increases our optimization objective in Equation
16 by $i - 3$, while replacing size 3 groups with size 2 groups reduces our objective by $2^{i-2} - 3$. For
such a change to ever benefit us, we need $i - 3 > 2^{i-2} - 3$, which never holds for $i \ge 4$. It follows
us that we were at the optimal solution initially, so $g(n, m) = 2n + k$.

We need to know when the assumption in Lemma 36 holds. For a fixed $(n, m)$ with $n \le 2^{m-1}-1$,
we want to construct $k = \lceil \frac{2^{m-1}-1-n}{2} \rceil$ size 3 groups to form a uniquely partitionable multiset $F$
corresponding to $(k, m)$. Clearly, we can do this when $k \le P_3(m)$.

**Lemma 37.** Consider any $m \in \mathbb{Z}^+$. By definition of $P_3(m)$, we can construct a uniquely partitionable multiset $F$ corresponding to $(P_3(m), m)$ such that the partition $P$ contains $P_3(m)$ groups
of size 3. Consider any $n \in \mathbb{Z}^+$ such that $2^{m-1}-1-2P_3(m) \le n \le 2^{m-1}-1$. Let $k = \lceil \frac{2^{m-1}-1-n}{2} \rceil$.
Then

$$g(n, m) = 2n + k$$

**Proof.** The condition on $n$ ensures that $k \le P_3(m)$. So pick a uniquely partitionable multiset $F$
corresponding to $(k, m)$ that contains $k$ size 3 groups. The result then follows by Lemma 36.

**Theorem 38.** Suppose $m \ge 6$. Then there exists a uniquely partitionable multiset $F$ corresponding
to $(2^{\lceil \frac{m}{2} \rceil} - 2, m)$ containing $2^{\lceil \frac{m}{2} \rceil} - 2$ groups of size 3. In other words, $P_3(m) \ge 2^{\lceil \frac{m}{2} \rceil} - 2$.

**Proof.** By plugging in $k = 3$ into Theorem 26 (Section 3.3), we only obtain a bound of the form
$P_3(m) \ge 2^{\lceil \frac{m}{2} \rceil} - 1$. However, using the same argument with some modification, we can get
an improvement with a factor of 2. Once again, we may assume $k - 1 = 2$ divides $m$ with the other case proved in a similar manner. The construction works the same way as in Theorem 26, with one key difference. We do not restrict ourselves to proper subsets of $S_1 = \{1, \ldots, \frac{m}{2}\}$ and $S_2 = \{\frac{m}{2} + 1, \ldots, m\}$ with size at least $\lceil \frac{m}{2}\rceil - 1 = \lceil \frac{m}{2}\rceil$. We can in fact consider all $2^{\frac{m}{2}} - 2$ proper subsets of $S_1$ and $S_2$. We only have to be careful with how we pair them up.

For every such proper subset $X \subseteq S_1$, there must be a proper subset $X' \subseteq S_1$ such that $X \cup X' = S_1$. List all of them as $X_1, X'_1, \ldots, X_l, X'_l$, where $l = \frac{2^{\frac{m}{2}} - 2}{2^{\frac{m}{2}} - 1}$. Similarly, list the $2^{\frac{m}{2}} - 2$ proper subsets of $S_2 = \{\frac{m}{2} + 1, \ldots, 2m\}$ as $Y_1, Y'_1, \ldots, Y_l, Y'_l$.

Pair the subsets into tuples as follows: $(X_1, Y'_1), (X'_1, Y_1), (X_2, Y'_2), (X'_2, Y_2)$ and so on, finally looping back by pairing $(X_l, Y'_l)$. In other words, for $1 \leq i \leq l$, we construct the pairs $(X_i, Y'_i), (X'_i, Y_{i+1})$ where we identify $Y_{i+1}$ with $Y_1$. So we have $2^{\frac{m}{2}} - 2$ pairs such that for any two pairs $(X, Y)$ and $(X', Y')$, $X \cup Y \neq (X' \cup Y')^c$ and $X \cup Y \neq X' \cup Y'$. (To be precise, for this property to hold, we would need $l \geq 2$, which is true when $m \geq 6$). Just like in Theorem 26, the construction of the central components is done by taking the union of each tuple, and for each central component $C$, $C_1 = S_1 \setminus C$ and $C_2 = S_2 \setminus C$ are defined in the same way. So we have constructed $2l = 2^{\frac{m}{2}} - 2$ groups which induces a partition $\mathcal{P}$ for a multiset $\mathcal{F}$ corresponding to $(2^{\frac{m}{2}} - 2, m)$ with every group of size 3. We claim that $\mathcal{F}$ is uniquely partitionable.

Suppose by contradiction that $\mathcal{P}$ admits a reassignment $\mathcal{P}'$. We claim there is no group of size 2 even in the reassignment. Suppose there was a size 2 group, which must be of the form $(Z, Z^c)$.

- Suppose $Z$ was originally of the form $C_1$ for some $C_1$ in the original construction. Then $Z \subseteq S_1$. So it follows that $S_2 \subseteq Z^c$. But no component in our original construction contained all of $S_2$, so $Z^c$ could not have been present in the original assignment, a contradiction. For the same reason, $Z$ could not have originally been of the form $C_2$.

- Suppose $Z$ was originally a central component $C$. By our method of pairing during the construction, $C^c$ is never introduced. So a size 2 group is impossible in a reassignment $\mathcal{P}'$.

If we have a group with size 4 or more in $\mathcal{P}'$, then we must have a group with size 2 or less, which is impossible. So even in the reassignment, every group must have size 3.

Pick any component in the new assignment $\mathcal{P}'$ that was originally a central component $C$. Since it must belong to size 3 group, $C$ must be paired with two other components even in $\mathcal{P}'$. We argue that $\mathcal{P}'$ cannot assign two central components to the same group.

Suppose by contradiction that $\mathcal{P}'$ contains some group with two central components. Since all groups have size 3, by the pigeonhole principle, there must be some group without any central components. This group must have either two proper subsets of $S_1$ and one proper subset of $S_2$ or vice versa. Without loss in generality suppose it has only one proper subset of $S_2$. Then since the other two components are subsets of $S_1$, there must be some element in $S_2$ that is not covered by these three components, a contradiction. It follows that $C$ must be paired with two components that are not central components.

$C$ does not include some elements in $S_1$. Apart from central components, only components of the form $C_1$ have elements in $S_1$. So we must choose $C_1$ such that $(S_1 \setminus C) \subseteq C_1$. In fact, we must have $C_1 = S_1 \setminus C$, because if $C_1$ had any additional elements, then it would have a non-trivial intersection with $C$, which is impossible. Similarly, we must have $C_2 = S_2 \setminus C$. But by construction, $C$ was in the same group even before the reassignment. It follows that no reassignment $\mathcal{P}'$ is possible, so $\mathcal{F}$ is uniquely partitionable. \qed
**Theorem 39.** Suppose \( m \geq 6 \) and \( n \in \mathbb{Z}^+ \) satisfies

\[
2^{m-1} + 3 - 2\lfloor \frac{m}{2} \rfloor + 1 \leq n \leq 2^{m-1} - 1 \tag{19}
\]

Then we have that

\[
g(n, m) = 2n + \left\lfloor \frac{2^{m-1} - 1 - n}{2} \right\rfloor \tag{20}
\]

**Proof.** Note that since \( P_3(m) \geq 2\lfloor \frac{m}{2} \rfloor - 2 \) by Theorem 38,

\[
n \geq 2^{m-1} - 1 - 2(2\lfloor \frac{m}{2} \rfloor - 2) \geq 2^{m-1} - 1 - 2P_3(m)
\]

The result then follows by the conclusion of Lemma 37. \( \square \)

## 4 Conclusion and Future Work

We have computed \( g(n, m) \) explicitly for \( n \geq 2^{m-1} - O(2^m) \). Further, for \( n \leq 2^{m-1} \), we provide a lower bound for \( g(n, m) \) of the form \( \Omega(\frac{nm}{\log n}) \). In the regime \( n = 2^{cm} \) for any \( c \in (0, 1) \), this lower bound simplifies to \( \Omega\left(\frac{n}{c}\right) \), and we provide a matching upper bound up to a \( \log\left(\frac{1}{c}\right) \) factor, thus showing that our bounds are near optimal for most values of \( n \). It would be interesting to see sharp bounds for \( g(n, m) \) when \( n \) is a polynomial in \( m \). Further, this problem admits some very natural extensions. Some such problems are listed below.

1. For any \( k \leq g(n, m) \), how many uniquely partitionable multisets of size \( k \) exist?

2. Is there an efficient algorithm to determine whether a multiset \( F \) is partitionable, and if so, output how many distinct permutations exist?

3. Given a uniquely partitionable multiset \( F \), is there a polynomial time algorithm to find the largest uniquely partitionable multiset \( F' \) that can be formed from \( F \) by splitting some of its subsets?

4. We saw how determining the size of a maximum multiset is equivalent to determining the smallest set of edges that can uniquely specify a labelled graph. More generally, we may consider the cardinality of such a minimum edge set for any graph family that enjoys certain properties. For example, for \( n \geq 4 \), if we know that \( G = C_n \), the cycle graph on \( n \) vertices labelled from 1 to \( n \), then we require at least \( n - 2 \) edges to uniquely specify \( G \) (and we can also easily do so with \( n - 2 \) edges).

A natural question to ask is given that \( G \) is a graph belonging to a certain family of labelled graphs \( \mathcal{G} \), what is the minimum number of edges required to uniquely specify \( G \)? In this example, \( \mathcal{G}_n = \{G : G \text{ is a cycle graph on } n \text{ vertices}\} \). In this paper, we considered a family of \( m \)-partite graphs on \( nm \) vertices with certain additional structure. It would be interesting to see what other graph families yield interesting results under such analysis.
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