HEREDITARILY NORMAL MANIFOLDS OF DIMENSION > 1
MAY ALL BE METRIZABLE

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Abstract. P. J. Nyikos has asked whether it is consistent that every hereditarily normal manifold of dimension > 1 is metrizable, and proved it is if one assumes the consistency of a supercompact cardinal, and, in addition, that the manifolds are hereditarily collectionwise Hausdorff. We are able to omit these extra assumptions.

1. Nyikos’ Manifold Problem

For us, a manifold is simply a locally Euclidean topological space. Mary Ellen Rudin proved that MA + \sim CH implies every perfectly normal manifold is metrizable [17]. Hereditary normality (\(T_5\)) is a natural weakening of perfect normality; Peter Nyikos noticed that, although the Long Line and Long Ray are hereditarily normal non-metrizable manifolds, and indeed the only 1-dimensional non-metrizable connected manifolds [12], it is difficult to find examples of dimension > 1 (although one can do so with \(\diamondsuit\) [17] or CH [18]). He therefore raised the problem of whether it was consistent that there weren’t any [11], [12]. In a series of papers [13, 14, 15, 16] he was finally able to prove this from the consistency of a supercompact cardinal, if he also assumed that the manifolds were hereditarily collectionwise Hausdorff. We will demonstrate that neither of these extra assumptions is necessary:

Theorem 1.1. It is consistent that every hereditarily normal manifold of dimension > 1 is metrizable.

For a coherent Souslin tree \(S\) (see §2) PFA(\(S\)) is the statement [23] §4: If \(P\) is a proper poset that preserves \(S\) and if \(D_\alpha (\alpha < \omega_1)\) is a sequence of dense open subsets of \(P\) there is a filter \(G \subset P\) such that \(G \cap D_\alpha \neq \emptyset\) for all \(\alpha < \omega_1\). The notation PFA(\(S\))[\(S\)] is adopted in [8] to abbreviate that we are in a forcing extension by \(S\) of a model in which \(S\) was a coherent Souslin tree and in which PFA(\(S\)) held.

Theorem 1.2. It is a consequence of PFA(\(S\))[\(S\)] that every hereditarily normal manifold of dimension greater than 1 is metrizable.

\textsuperscript{1}Research supported by NSF grant DMS-1501506.
\textsuperscript{2}Research supported by NSERC grant A-7354.

Date: March 25, 2018.
2000 Math. Subj. Class. Primary 54A35, 54D15, 54D45, 54E35, 03E05, 03E35, 03E65; Secondary 54D20, 03E55.

Key words and phrases: hereditarily normal, manifold, metrizable, coherent Souslin tree, proper forcing, PFA(\(S\))[\(S\)], locally compact, \(P\)-ideal, perfect pre-image of \(\omega_1\), sequentially compact.
We will isolate some known (quotable) consequences of PFA(S)[S]. The first, rather easy, is that the bounding number \( b \) is greater than \( \omega_1 \) [9]. The next is the important P-ideal dichotomy.

**Definition 1.3.** A collection \( \mathcal{I} \) of countable subsets of a set \( X \) is a P-ideal if each subset of a member of \( \mathcal{I} \) is in \( \mathcal{I} \), finite unions of members of \( \mathcal{I} \) are in \( \mathcal{I} \), and whenever \( \{ I_n : n \in \omega \} \subseteq \mathcal{I} \), there is a \( J \in \mathcal{I} \) such that \( I_n - J \) is finite for all \( n \).

**PID is the statement:** For every P-ideal \( \mathcal{I} \) of countable subsets of some uncountable set \( A \) either

(i) there is an uncountable \( B \subseteq A \) such that \([B]^{\omega_1} \subseteq \mathcal{I} \), or else
(ii) the set \( A \) can be decomposed into countably many sets, \( \{ B_n : n \in \omega \} \), such that \( |B_n \cap I| = \emptyset \) for each \( n \in \omega \).

The consistency of \( \text{PID} \) does have large cardinal strength but for P-ideals on \( \omega_1 \) it does not – see the discussion at the bottom of page 6 in [23]. A statement similar to the \( \text{PID} \) for ideals on \( \omega_1 \) is the one we need; it also does not have large cardinal strength and is weaker than the \( \omega_1 \) version of the statement in [23, 6.2].

The statement \( P_{22} \) was introduced in [4]. For completeness, and to introduce the ideas we will need for another consequence of PFA(S)[S], we include a proof in §2 that it is a consequence of PFA(S)[S].

**\( P_{22} \) is the statement:** Suppose \( \mathcal{I} \) is a P-ideal on a stationary subset \( B \) of \( \omega_1 \). Then either

(i) there is a stationary \( E \subseteq B \) such that every countable subset of \( E \) is in \( \mathcal{I} \),

or (ii) there is a stationary \( D \subseteq B \) such that \([D]^{\omega_1} \cap \mathcal{I} = \emptyset \).

A space \( X \) is said to be \( \aleph_1 \)-collectionwise Hausdorff if the points of any closed discrete subset of cardinality at most \( \aleph_1 \) can be surrounded by pairwise disjoint open sets (separated). If a separable space is hereditarily \( \aleph_1 \)-collectionwise Hausdorff, then it can have no uncountable discrete subsets (known as having countable spread).

The next consequence of PFA(S)[S] is:

**\( CW \):** Normal, first countable spaces are \( \aleph_1 \)-collectionwise Hausdorff.

\( CW \) was first shown to be consistent in [21]; it was derived from \( V = L \) in [9], and was shown to be a consequence of PFA(S)[S] in [8]. In fact, it is shown in [8] that simply forcing with any Souslin tree will produce a model of \( CW \). Let us note now that \( CW \) implies that any hereditarily normal manifold is hereditarily \( \aleph_1 \)-collectionwise Hausdorff. Therefore \( CW \) implies that each separable hereditarily normal manifold has countable spread.

Our next axiom is our crucial new additional consequence of PFA(S)[S]:

**\( PPI^+ \):** every sequentially compact non-compact regular space contains an uncountable free sequence. Additionally, if the space is first countable, then it contains a copy of the ordinal space \( \omega_1 \).

Let \( GA \) denote the group (or conjunction) of hypotheses: \( b > \omega_1, CW, PPI^+ \) and \( P_{22} \). We have, or show, that each is a consequence of PFA(S)[S], and also establish the desired theorem. We show in [24] that \( GA \) is consistent (not requiring any large cardinals).
Theorem 1.4. GA implies that all hereditarily normal manifolds of dimension greater than one are metrizable.

We acknowledge some other historical connections. The statement PPI$^+$ is a strengthening of

PPI: Every first countable perfect pre-image of $\omega_1$ includes a copy of $\omega_1$.

PPI was proved from PFA by Fremlin [7], see also e.g. [3]. Another consequence of PFA(S)[S] relevant to this proof is

$$\sum^-_\omega: \text{In a compact } T_2, \text{countably tight space, locally countable subspaces of size } \aleph_1 \text{ are } \sigma\text{-discrete.}$$

$\sum^-$ was proved from MA $+$ $\sim$CH by Balogh [2], extending work of [20]. $\sum^-$ implies $b > \omega_1$; this follows from the result in [26, 2.4] where it is shown that $b = \aleph_1$ implies there is a compact hereditarily separable space which is not Lindelöf, since $\sum^-_\omega$ implies there is no such space. $\sum^-_\omega$ was shown to be a consequence of PFA(S)[S] in [5].

We will need the following consequence of GA which is a weaker statement than $\sum^-_\omega$. The key fact that PFA(S)[S] implies compact, separable, hereditarily normal spaces are hereditarily Lindelöf was first proven in [23, 10.6].

Lemma 1.5. GA implies that if $X$ is a hereditarily normal manifold then separable subsets of $X$ are Lindelöf and metrizable.

Proof. Let $Y$ be any separable subset of $X$ and assume that $Y$ is not Lindelöf. Recursively choose points $y_\alpha$, together with open sets $U_\alpha$, so that $y_\alpha \in Y \setminus \bigcup_{\beta < \alpha} U_\beta$, $y_\alpha \in U_\alpha$, and $U_\alpha$ is separable and compact. Define an ideal $\mathcal{I}$ of countable subset $a$ of $\omega_1$ according to the property that $a \in \mathcal{I}$ providing $\{y_\alpha: \alpha \in a\} \cap U_\beta$ is finite for all $\beta \in \omega_1$. Since $b > \omega_1$ we have that $\mathcal{I}$ is a P-ideal (see [23, 6.4]). To check this, assume that $\{a_\alpha: \alpha \in \omega\}$ are pairwise disjoint infinite members of $\mathcal{I}$. For each $\alpha$, fix an enumerating function $e_\alpha$ from $\omega$ onto $a_\alpha$. For each $\beta \in \omega_1$, there is a function $f_\beta \in \omega^\omega$ so that, for each $\alpha \in \omega$ and each $m > f_\beta(n)$, $e_\alpha(m) \notin U_\beta$. Using $b > \omega_1$, there is an $f \in \omega^\omega$ such that $f_\beta < f$ for each $\beta \in \omega_1$. For each $n$, let $F_n = \{e_\alpha(m): m < f(n)\}$. It follows that $a = \bigcup \{a_\alpha \setminus F_n: \alpha \in \omega\}$ meets each $U_\beta$ in a finite set. Thus $a \in \mathcal{I}$ and mod finite contains $a_\alpha$ for each $n$.

If $B$ is any subset of $\omega_1$ such that $[B]^\omega_0 \subset \mathcal{I}$, then $D = \{y_\beta: \beta \in B\}$ is discrete since $D \cap U_\beta$ is finite for each $\beta \in B$. By P$_{22}$ we must then have that there is an uncountable $B \subset \omega_1$ satisfying that $[B]^\omega_0 \cap \mathcal{I}$ is empty. Now let $A$ be the closure (in $X$) of $\{y_\beta: \beta \in B\}$. We check that $A$ is sequentially compact. Let $\{x_\alpha: \alpha \in \omega\}$ be any infinite subset of $A$, we show that there is a limit point in $A$. Since $X$ is first countable this shows that $A$ is sequentially compact. If $\{x_\alpha: \alpha \in \omega\} \cap \{y_\beta: \beta \in B\}$ is infinite, let $b \in [B]^\omega_0$ be chosen so that $\{x_\alpha: \alpha \in \omega\} \cap \{y_\beta: \beta \in b\}$ is infinite, and so has a limit point in the compact set $\overline{U_\beta}$. Otherwise, we may suppose that, for each $n$, there is an infinite $a_n \subset B$ such that $\{y_\alpha: \alpha \in a_n\}$ converges to $x_n$. Again, using that $b > \omega_1$, similar to the verification that $\mathcal{I}$ is a P-ideal, there must be a $\beta \in \omega_1$ such that $U_\beta \cap \{y_\alpha: \alpha \in a_n\}$ is infinite for infinitely many $n$. For any such $\beta$, there are infinitely many $n$ with $x_n \in U_\beta$. It again follows that $\overline{U_\beta}$ contains a limit of the sequence $\{x_\alpha\}$. To finish the proof, we apply PPI$^+$ to conclude that
either $A$ is compact or it contains a copy of $\omega_1$. Since $\omega_1$ contains uncountable discrete sets and $Y$ is separable, we must have that $A$ is compact. However, the final contradiction is that $A$ is not hereditarily Lindelöf and so it cannot be covered by finitely many Euclidean open subsets of $X$. □

The literature on non-metrizable manifolds has identified two main types of non-Lindelöf manifolds, literally called Type I and Type II. A manifold is Type II if it is separable and non-Lindelöf. Lemma [15] shows that there are no hereditarily normal Type II manifolds if GA holds. A manifold is said to be Type I, e.g. the Long Line, if it can be written as an increasing $\omega_1$-chain, $\{Y_\alpha : \alpha \in \omega_1\}$, where each $Y_\alpha$ is Lindelöf, open, and contains the closure of each $Y_\beta$ with $\beta < \alpha$. In this next definition, we use the set-theoretic notion of countable elementary submodels to help make a more strategic choice of a representation of our Type I manifolds. For a cardinal $\theta$, the notation $H(\theta)$ denotes the standard set-theoretic notion of the set of all sets that are hereditarily of cardinality less than $\theta$. These are commonly used as stand-ins for the entire set-theoretic universe to avoid issues with Gödel’s famous incompleteness theorems in arguments and constructions using elementary submodels. We refer the reader to any advanced book on set-theory for information about the properties of $H(\theta)$.

**Definition 1.6.** Suppose that $X$ is a non-metrizable manifold with dimension $n$. Let $B_X$ denote the collection of compact subsets of $X$ that are homeomorphic to the closed Euclidean $n$-ball $B^n$. A family $\{M_\alpha : \alpha \in \omega_1\}$ is an elementary chain for $X$ if there is a regular cardinal $\theta$ with $B_X \in H(\theta)$ so that for each $\alpha \in \omega_1$, $M_\alpha$ is a countable elementary submodel of $H(\theta)$ such that $B$ and each $M_\beta$ ($\beta < \alpha$) are members of $M_\alpha$. The chain is said to be a continuous chain if for each limit $\alpha \in \omega_1$, $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$.

Whenever $\{M_\alpha : \alpha \in \omega_1\}$ is an elementary chain for $X$, let $X(M_\alpha)$ denote the union of the collection $B_X \cap M_\alpha$.

Here is the main reason for our preference to use elementary submodels in this proof. Again the main ideas are from [15], but the proof using elementary submodels is much simpler. Throughout the paper the term component refers to the standard notion of connected component.

**Lemma 1.7.** Suppose that $X$ is a non-metrizable hereditarily normal manifold of dimension $n > 1$. Let $\theta$ be a large enough regular cardinal $\theta$ so that $B_X \in H(\theta)$ and let $M$ be a countable elementary submodel of $H(\theta)$ such that $B_X$ is a member of $M$. Then $X(M) = \bigcup(M \cap B_X)$ is an open Lindelöf subset of $X$ with the property that every component of the non-empty boundary, $\partial X(M)$, is non-trivial.

**Proof.** We let $B_X$ denote the family of all homeomorphic copies of the closed unit ball of $\mathbb{R}^n$ in $X$. As $X$ is a manifold this family is such that whenever $O$ is open in $X$ and $x \in O$ there is a $B \in B_X$ such that $x$ is in the interior of $B$ and $B \subseteq O$.

Let $Y$ denote the set $X(M)$. Since $Y$ is metrizable, and $X$ is not, $Y$ is a proper subset of $X$. Each member of $B_X \cap M$ is separable and hence $B \cap M$ is dense in $B$ whenever $B \in B_X \cap M$; it follows that $Y \cap M$ is a dense subset of $Y$.

We also note that $Y$ is open since if $B \in B_X \cap M$, then $B$ is compact and so is contained in the interior of a finite union of members of $B_X$. By elementary, there is a such a finite set in $B_X \cap M$. Similarly, if $B'$ is a finite subset of $B_X \cap M$, then, by elementary, each Lindelöf component of $X \setminus B'$ that meets $Y \cap M$ will
be a subset of $Y$. More precisely, if $C$ is such a component and if $y \in C \cap M$, then $M$ will witness that there is a countable collection of members of $\mathcal{B}_X$ that covers the component of $y$ in $X \setminus \partial X$. Also, we have that $X$ itself must have non-Lindel"of components since Lindel"of subsets of any manifold are metrizable while $X$, being locally connected, is the free union of its components.

Now we choose any $x$ in $\partial X(M) = \partial Y = Y \setminus Y$. Take any $B \in \mathcal{B}_X$ with $x$ in its interior. We assume, working towards a contradiction, that the component of $x$ in $\partial Y$ is $\{x\}$.

Since $\partial Y \cap B$ is compact and $\{x\}$ is a component of $\partial Y \cap B$, we can split the latter set into two relatively clopen sets $C$ and $D$, where $C$ is the union of all components of $\partial Y \cap B$ that meet the boundary of $B$ and $D$ is its complement. For now we allow for the possibility that $C = \emptyset$ but $D$ is not empty as it contains $x$.

We choose $W$ with $D \subseteq W$ and such that $\overline{W}$ is contained in the interior of $B$ and disjoint from $C$.

Note that $W$ and $\overline{W}$ are Lindel"of because $B$ is compact and hereditarily Lindel"of, being homeomorphic to the unit ball of $\mathbb{R}^n$. Since $\partial W$ and $\partial Y$ are disjoint the set $\partial W \cap Y$ is closed and hence compact. There is a finite subfamily $\mathcal{B}_1$ of $M \cap \mathcal{B}_X$ whose union contains $\partial W \cap Y$. The complement $W \setminus \mathcal{B}_1$ is a neighbourhood of $x$, so it meets $Y \cap M$. The component, $E$, of $x$ in this complement is Lindel"of but not contained in $Y$, therefore $E$ is not a component of $X \setminus \mathcal{B}_1$ which implies that $E \setminus \overline{W}$ is not empty. Since $\dim E > 1$, $x$ can not be a cut-point of $E$ and so it follows that $x$ is not the only point of $E \cap \partial Y \subset W \cap \partial Y$.

This means that we can choose disjoint open subsets of $W$, say $O_1$ and $O_2$, each also having compact non-empty intersection with $\partial Y$ and whose boundaries $\partial O_1$ and $\partial O_2$ miss $\partial Y$. Fix points $z_1 \in O_1 \cap \partial O_1$ and $z_2 \in O_2 \cap \partial O_2$. Now $Y \cap (\partial O_1 \cup \partial O_2)$ is compact and again can be covered by some $K$ where $K$ is equal to a union of some finite subfamily $\mathcal{B}_2$ of $M \cap \mathcal{B}_X$. Also since $Y \cap (\partial O_1 \cup \partial O_2)$ is disjoint from the boundary of $B$, we can ensure that $K$ is disjoint from the boundary of $B$. Since $K$ is a compact subset of $Y$, each component of $O_1 \setminus K$ and $O_2 \setminus K$ meets $Y \cap M$; so choose points $y_1$ and $y_2$ in $Y \cap M$ that are in the components in $O_1 \setminus K$ and $O_2 \setminus K$ of $z_1$ and $z_2$ respectively. Let $C_1$ and $C_2$ be the components in $X \setminus K$ of $y_1$ and $y_2$ respectively. Neither component is contained in $Y$ and so neither is Lindel"of. Thus, neither is contained in $B$ and so they both meet the (arcwise) connected boundary of $B$. Since components of $B \setminus K$ are path-connected, there is a path in $X \setminus K$ from $y_1$ to $y_2$. By elementarity there is such a path in $M$ and such a path would lie completely within $Y$ (the path is covered by a finite subfamily of $\mathcal{B}_X$ and one such family should be in $M$). However, $Y \cap (O_1 \setminus K)$ is clopen in $Y$ so there is no path in $Y$ that connects $y_1$ and $y_2$. This contradiction finishes the proof. \qed

This next corollary is the representation as a Type I sub-manifold that we require.

**Corollary 1.8.** Suppose that $X$ is a non-metrizable hereditarily normal manifold of dimension greater than 1. Then there is an increasing chain $\{Y_\alpha : \alpha \in \omega_1\}$ of open Lindel"of subsets satisfying that

1. for each $\alpha$, the boundary $\partial Y_\alpha$ is non-empty and contained in $Y_{\alpha+1}$,
2. for each $\alpha$, each component of $\partial Y_\alpha$ is non-trivial,
3. for limit $\alpha$, $Y_\alpha = \bigcup \{Y_\beta : \beta \in \alpha\}$.

Additionally, the union $\bigcup \{Y_\alpha : \alpha \in \omega_1\}$ is closed (and open) in $X$. 
Proof. Fix a continuous elementary chain \( \{ M_\alpha : \alpha \in \omega_1 \} \) for \( X \). Fix any \( \alpha \in \omega_1 \). By Lemma \( \ref{1.7} \), \( Y_\alpha = X(M_\alpha) \) is Lindelof with non-empty boundary, \( \partial X(M_\alpha) \), and each component in \( \partial X(M_\alpha) \) is non-trivial. By Lemma \( \ref{1.7} \), \( X(M_\alpha) \) is Lindelof, and so by elementarity, \( M_{\alpha+1} \cap B_X \) is a cover of \( X(M_\alpha) \). Finally, \( \bigcup \{ Y_\alpha : \alpha \in \omega_1 \} \) is closed because any \( x \in X \) that is in the closure will be in \( Y_\alpha \subset Y_{\alpha+1} \) for some \( \alpha \in \omega_1 \).

Now we are ready to give a proof of the main theorem. The clever topological ideas of the proof are taken from \( \cite{13} \) p189. A sketch of this proof appears in \( \cite{22} \). The main idea of the proof is to use \( \text{PPI}^+ \) to find copies of \( \omega_1 \) and, combined with Lemma \( \ref{1.8} \) to show that, in fact, there are copies of the Tychonoff plank in the space. It is easily shown that the Tychonoff plank is not hereditarily normal.

**Theorem 1.9.** The statement \( \text{GA} \) implies that each hereditarily normal manifold of dimension greater than 1 is metrizable.

**Proof.** Assume that \( X \) is a non-metrizable hereditarily normal manifold of dimension greater than 1. Let \( \{ Y_\alpha : \alpha \in \omega_1 \} \) be chosen as in Corollary \( \ref{1.8} \). For each \( \alpha \in \omega_1 \), choose any point \( x_\alpha \in \partial Y_\alpha \). It is immediate that \( \{ x_\alpha : \alpha \in \omega_1 \} \) is nowhere dense in \( X \). Also let \( \{ U_\alpha : \alpha \in \omega_1 \} \subset B_X \) be any sequence so that \( U_\alpha \subset Y_{\alpha+1} \) and \( x_\alpha \) is in the interior of \( U_\alpha \). We first show that if \( E \subset \omega_1 \) is stationary, then \( D = \{ x_\beta : \beta \in E \} \) is not discrete. For each limit \( \alpha \), using item (3) of Corollary \( \ref{1.8} \), there is a \( \beta < \alpha \) such that \( U_\alpha \cap Y_{\beta} \setminus D \) is not empty. By the pressing down lemma, there is a fixed \( \beta \) such that \( \beta = \beta_\alpha \) for uncountably many \( \alpha \in E \). Since \( Y_{\beta} \setminus D \) is separable, there are \( \alpha, \alpha' \in E \) such that \( U_\alpha \cap U_{\alpha'} \cap Y_{\beta} \) is not empty. The choice of the sequence of \( U_\alpha \)'s was (basically) arbitrary, and so it follows that \( D \) can not be separated. Since \( D \cup (X \setminus D) \) is first countable, and thus \( \aleph_1 \)-collectionwise Hausdorff, this shows that \( D \) cannot be discrete.

Define the ideal \( \mathcal{I} \) by \( a \in \mathcal{I} \) if \( a \in [\omega_1]^{\aleph_0} \) and, for all \( \beta \in \omega_1 \), \( \{ x_\alpha : \alpha \in a \} \cap U_\beta \) is finite. As before, \( \mathcal{I} \) is a P-ideal on \( \omega_1 \). If \( A \subset \omega_1 \) satisfies that \( [A]^{\aleph_0} \subset \mathcal{I} \), then \( D = \{ x_\alpha : \alpha \in A \} \) is discrete. Therefore there is no such stationary \( A \), and so by \( \text{P}_{22} \), there is a stationary subset \( A \subset \omega_1 \) such that \( [A]^{\aleph_0} \cap \mathcal{I} \) is empty. It again follows that \( X_A = \{ x_\alpha : \alpha \in A \} \) is sequentially compact. Let us choose, by applying \( \text{PPI}^+ \), a copy \( W \) of \( \omega_1 \) contained in \( X_A \). Let \( W = \{ w_\xi : \xi \in \omega_1 \} \) be the homeomorphic indexing of \( W \). For each \( \alpha \in \omega_1 \), Lemma \( \ref{1.5} \) implies that \( Y_{\alpha} \) is Lindelof and, by elementarity, contained in \( Y_{\alpha+1} \). Therefore, we have that, for each \( \alpha \), \( W \cap Y_\alpha \) is countable, and its closure is contained in \( Y_{\alpha+1} \). It follows that there is a cub \( C \subset \omega_1 \) satisfying that for each \( \gamma < \beta \) both in \( C \), the set \( \{ w_\beta : \gamma \leq \beta < \delta \} \) is contained in \( Y_\delta \setminus Y_\gamma \). Therefore \( \{ w_\gamma : \gamma \in C \} \) is another copy of \( \omega_1 \) with the property that \( w_\gamma \notin \partial Y_\gamma \) for each \( \gamma \in C \).

For each \( \gamma \in C \), apply Lemma \( \ref{1.8} \) so as to choose infinite compact connected \( K_\gamma \subset \partial Y_\gamma \) with \( w_\gamma \in K_\gamma \). Make another selection \( y_\gamma \in K_\gamma \setminus W \) arbitrarily. Now choose, for each \( \gamma \in C \), a basic set \( V_\gamma \in B_X \) so that \( y_\gamma \) is in the interior of \( V_\gamma \) and \( V_\gamma \subset X \setminus W \). Proceeding as we did with the sequence of \( \{ x_\alpha : \alpha \in \omega_1 \} \), there is a stationary set \( A_1 \subset C \) so that \( \{ y_\alpha : \alpha \in A_1 \} \) has sequentially compact closure. Since \( Y_{\alpha} \) is Lindelof and contains \( \{ y_\alpha : \alpha \in A_1 \} \) for each \( \gamma \in C \), it follows then that the closure of \( \{ y_\alpha : \alpha \in A_1 \} \) is compact and disjoint from \( W \) for each \( \gamma \in C \). Since \( X \) is first countable, this also implies that the closure of the entire set \( \{ y_\alpha : \alpha \in A_1 \} \) is disjoint from the closed set \( W \). Since \( X \) is normal, there is a continuous function \( f \) from \( X \) into \([0,1]\) such that \( f(W) = \{1\} \) and \( f(y_\alpha) = 0 \) for
all $\alpha \in A_1$. Note that $f[K_\alpha] = [0, 1]$ for each $\alpha \in A_1$. Finally, using $f$ we will show there is a non-normal subspace for our contradiction. For each $\alpha \in A_1$, choose, yet another, point $z_\alpha \in K_\alpha$ in such a way that the map $f$ restricted to $\{z_\alpha : \alpha \in A_1\}$ is one-to-one. Repeating the steps above, there is a stationary set $A_2 \subset A_1$ so that the closure of each countable subset of $\{z_\alpha : \alpha \in A_2\}$ is compact. Let $Z$ denote the closure of the set $\{z_\alpha : \alpha \in A_2\}$, and for each $r \in [0, 1]$, let $Z_r = f^{-1}(r) \cap Z$. We will use the following property of these subsets of $Z$. Consider any open set $U$ of $X$ that contains $Z_r \cap \partial Y_\gamma$ for any $r \in [0, 1]$ and $\gamma \in C_\omega$. Since $Z_r \cap Y_\gamma$ has compact closure, there is a $\beta < \gamma$ such that $Z_r \setminus Y_\beta$ is contained in $U$. By the pressing down lemma, given any open $U$ containing $Z_r \cap Y_\gamma$ for all any stationary set of $\gamma \in \omega_1$, there is a $\beta \in \omega_1$ such that $Z_r \setminus Y_\beta$ is contained in $U$.

Choose any $r \in [0, 1]$ such that $r$ is a complete accumulation point of $\{f(z_\alpha) : \alpha \in A_2\}$. Choose any sequence $\{r_n : n \in \omega\}$ converging to $r$ so that each $r_n$ is also a complete accumulation point of $\{f(z_\alpha) : \alpha \in A_2\}$. There is a common cub $C_\omega$ such that $Z_{r_n} \cap \partial Y_\gamma$ and $Z_r \cap \partial Y_\gamma$ is not empty for each $n \in \omega$ and $\gamma \in C_\omega$. Let $Z_r(C_\omega') = \{Z_r \cap \partial Y_\gamma : \gamma \in C_\omega\}$ where $C_\omega'$ is the set of relative limit points of $C_\omega$. Since $Z_r$ is closed in $Z$, it follows that $Z_r \setminus Z_r(C_\omega')$ is a closed subset of $Z \setminus Z_r(C_\omega')$. We also note that $H = Z \cap \bigcup\{\partial Y_\gamma : \gamma \in C_\omega\}$ is a closed subset of $Z$, and so $H \setminus Z_r(C_\omega')$ is a closed subset of $Z \setminus Z_r(C_\omega')$. We show that $Z_r \setminus Z_r(C_\omega')$ and $H \setminus Z_r(C_\omega')$ cannot be separated by disjoint open subsets of $Z \setminus Z_r(C_\omega')$. Since $Z_r \setminus Z_r(C_\omega')$ and $H \setminus Z_r(C_\omega')$ are disjoint, this will complete the proof. Suppose that $U$ is an open subset of $Z \setminus Z_r(C_\omega')$ that contains $H \setminus Z_r(C_\omega')$. By the above mentioned property of each $Z_{r_n}$, we have that there is a $\beta \in \omega_1$ such that $Z_{r_n} \setminus Y_\beta$ is contained in $U$ for each $n \in \omega$. Choose any $\beta < \gamma \in C_\omega \setminus C_\gamma'$. For each $n$, choose $z_n \in Z_{r_n} \setminus \partial Y_\gamma$. Since $Z \setminus \partial Y_\gamma$ is compact, let $z$ be any limit point of $\{z_n : n \in \omega\}$. By the continuity of $f$, $f(z) = r$ and so $z \in Z_r \cap \partial Y_\gamma$. In other words, $z \in Z_r \setminus Z_r(C_\omega')$, completing the proof that $H \setminus Z_r(C_\omega')$ and $Z_r \setminus Z_r(C_\omega')$ cannot be separated by open sets.

2. ON $\mathbf{P}_{22}$

As usual $S$ is a coherent Souslin tree. For us, it will be a full branching downward closed subtree of $\omega^{<\omega_1}$. Naturally it is a Souslin tree (no uncountable antichains) and has the additional property

for each $s \in S$ and $t \in \omega^{<\omega_1}$ with $\text{dom}(t) = \text{dom}(s)$, $t$ is in $S$ if and only if $\{\xi \in \text{dom}(s) : s(\xi) \neq t(\xi)\}$ is finite.

In a forcing argument using $S$ as the forcing poset, we will still use $s < s'$ to mean that $s \subset s'$, and so, $s'$ is a stronger condition. We will also use the more compact notation $o(s)$ to denote the order-type of $\text{dom}(s)$ for $s \in S$. Now we give a proof that our statement $\mathbf{P}_{22}$ is a consequence of $\text{PFA}(S)[S]$ following [23, 6.1].

Here is a simple standard fact about forcing with a Souslin tree that we will need repeatedly.

**Lemma 2.1.** Suppose that $S$ is a Souslin tree and $S \in M$ for some countable elementary submodel of any $H(\theta)$ $(\theta \geq \omega_2)$. If $\check{x}, \check{X} \in M$ are Souslin names, and $s \in S \setminus M$, then there is an $s' < s$ with $s' \in M$ such that

1. $s \Vdash \check{X} = \emptyset$ if and only if $s' \Vdash \check{X} = \emptyset$,
2. $s \Vdash \check{x} \in \check{X}$ if and only if $s' \Vdash \check{x} \in \check{X}$. 


Proof. The second item follows from the first (by simply considering the set \(\hat{X} \cap \{\hat{x}\}\)) so we consider any \(\hat{X}\) in \(M\). Since \(S\) is a ccc forcing and the set of conditions that decide the statement \(\hat{X} = \emptyset\) is dense and open, there is a \(\gamma \in M \cap \omega_1\) such that each element of \(S\), \(\gamma\) decides this statement. Therefore \(s \upharpoonright \gamma\) decides the statement and, since \(s\) is a stronger condition than \(s \upharpoonright \gamma\), they assign the same truth value to the statement. \(\square\)

Note, for example, Lemma 2.1 can be used to show that if \(\hat{E} \in M\) is an \(S\)-name of a subset of \(\omega_1\) and \(s \Vdash M \cap \omega_1 \subseteq \hat{E}\), then \(s \Vdash \hat{E}\) is stationary. To see this we can let \(\hat{X}\) denote the set of (ground model) cub subsets of \(\omega_1\) that are disjoint from \(\hat{E}\). Then, if \((s \upharpoonright M \cap \omega_1) \not\subseteq \hat{E}\), we have that for all cub \(C\) in \(M\), \(s \Vdash C \cap \hat{E}\) is not empty. So, if \(s' < s\) is in \(M\), we have that \(s'\) forces that \(\hat{X}\) is empty, and \(\hat{E}\) is stationary.

**Proposition 2.2.** Assume PFA(S) then \(S\) forces that \(P_{22}\) holds.

Proof. Let \(\hat{I}\) be the \(S\)-name for a \(P\)-ideal on a stationary subset \(B\) of \(\omega_1\) and assume that some \(s_0 \in S\) forces that \(\hat{I} \cap [E]^{\aleph_0} \neq \emptyset\) for all stationary sets \(E\). If \(s_0\) also forces that \(\hat{I}\) is a countable antichain, then using that \(S\) is homogeneous and the forcing maximum principle, we can assume that \(s_0\) is the root of \(S\) and just show that \(\hat{I}\) is not a counterexample. Fix any well-ordering \(\prec\) of \(H(\aleph_2)\).

**Claim 1.** For each countable elementary submodel \(M\) of \((H(\aleph_2), \prec)\) and each \(s \in S_M \cap \omega_1\), there is a set \(a(s, M)\) such that \(s \Vdash a(s, M) \in \hat{I}\) and \(s \Vdash a \subseteq \ast a(s, M)\) for all \(a \in M \cap \mathcal{N}\).

Proof of Claim 1: Since \(s\) forces that \(\hat{I}\) is a \(P\)-ideal, there is a \(\prec\)-minimal name \(\dot{a}\) such that \(1\) forces that each member of \(M \cap \hat{I}\) is a subset mod finite of \(\dot{a}\). Since \(S\) is ccc, there is a countable maximal antichain \(\{s_n : n \in \omega\}\) and a countable family \(\{a_n : n \in \omega\}\) of elements of countable subset of \(\omega_1\) such that, for each \(n\), \(s_n \Vdash \dot{a} = a_n\). Furthermore, \(s\) forces a value on each member of \(M \cap \hat{I}\). Let \(\mathcal{J}\) denote the countable family of sets forced by \(s\) to be members of \(M \cap \hat{I}\). Note that every member of \(\mathcal{J}\) is mod finite contained in every member of \(\{a_n : n \in \omega\}\). We may choose \(a(s, M)\) to be the \(\prec\)-minimal set that splits this \((\omega, \omega)\)-gap.

One change from \([23]\) is that we begin with a partition \(\mathcal{E} = \{E_s : s \in S\}\) of \(\omega_1\) by stationary sets so that, in addition, \(E_s \subseteq B\) for each \(s \in S\) other than the root \(\emptyset\). Thus \(\bigcup\{E_s : s \in S \setminus \{\emptyset\}\}\) contains \(\omega_1 \setminus B\). We also require that \(\text{dom}(s) < \delta\) for all limit \(\delta \in E_s\). Then we let \(\mathcal{P}\) be the collection of all mappings of the form \(p : \mathcal{M}_p \to S\), where

1. \(\mathcal{M}_p\) is a finitely chain of countable elementary submodels of \((H(\aleph_2), \mathcal{E}, \prec)\),
2. \(M \in \mathcal{M}_p\) and \(\delta = M \cap \omega_1 \in E_s\) implies \(s < p(M) \in S_\delta\),
3. \(M \in N \in \mathcal{M}_p\) implies \(a(p(M), M) \in N\).

We let \(p \leq q\) if,

4. \(\mathcal{M}_p \supset \mathcal{M}_q\) and \(q = p \upharpoonright \mathcal{M}_q\),
5. \(N \cap \omega_1 \in a(q(M), M)\) whenever \(N \cap \omega_1 \notin E_\emptyset,\ p(N) < q(M)\) with \(M \in N \in \mathcal{M}_q,\) and \(M \in \mathcal{M}_p \setminus \mathcal{M}_q\).

In order to apply PFA(S) to \(\mathcal{P}\), we have to show that \(\mathcal{P}\) is a proper poset that preserves that \(S\) is Souslin. Once we do, we let \(\mathcal{G}\) be a filter on \(\mathcal{P}\) that meets sufficiently many (no more than \(\omega_1\)) dense subsets to ensure that there is a cub \(C \subseteq \omega_1\) such that for each \(\delta \in C\), there is a \(p_\delta \in \mathcal{G}\) and an \(M_\delta \in \mathcal{M}_{p_\delta}\) with \(M_\delta \cap \omega_1 = \delta\). The role of the family \(\mathcal{E}\) is to ensure the next Claim.
Claim 2. Each $s_0 \in S$ forces that the $S$-name $\dot{E} = \{ \delta \in B : p_\delta(M_\delta) \in \dot{g} \}$ is a stationary subset of $\omega_1$, where $\dot{g}$ is the $S$-name of the generic branch through $S$.

Proof of Claim 2: It suffices to show that $s_0$ does not force that $\dot{E}$ is not stationary by finding an extension that forces $\dot{E}$ is stationary. Choose any $\delta \in C \cap E_{s_0}$. We have that $s = p_\delta(M_\delta)$ forces that $\delta = M_\delta \cap \omega_1$ is in $\dot{E}$. Also, since $\delta \in E_{s_0}$, we have, from the definition of $P$, that $s_0 < s$. By Lemma 2.1 as explained in the discussion immediately following it, we have that $s$ forces that $\dot{E}$ is stationary.

Claim 3. Each $s \in S$ forces that $[\dot{E}]^{\aleph_0} \subset \dot{I}$, where $\dot{E}$ is defined in Claim 2.

Proof of Claim 3: It suffices to show that if $\gamma \in \omega_1$ and $s \in S_\gamma$, then $s \Vdash \dot{E} \cap \gamma \in \dot{I}$. Recall that there is a $\delta > \gamma$ such that $s < p_\delta(M_\delta)$. By the definition of the ordering on $P$ (item (5)) we have that $\{ \gamma \in \dot{E} : p_\gamma(M_\gamma) < p_\delta(M_\delta) \text{ and } M_\gamma \notin M_\delta \}$ is contained in $a(p(M_\delta), M_\delta)$. Therefore, $p(M_\delta)$ forces that $\dot{E} \cap \delta \in \dot{I}$.

We finish the proof of the Proposition by proving that $S \times P$ is proper. Let $M$ be any countable elementary submodel of $H(\kappa)$ for some regular $\kappa > \omega_2$. We show that any pair $(s^1, q)$ where $s^1 \in S \setminus M$ and $M \cap H(\aleph_2) \in M_{p_\delta}$ is an $M$-generic condition for $S \times P$. Consider any dense open set $D$ of $S \times P$ that is a member of $M$. By extending the condition $(s^1, q)$ we can assume that $(s^1, q)$ is in $D$ and that there is some countable elementary submodel of $H(\kappa)$ containing $q$ but not $s^1$. It is useful to regard $D$ as an $S$-name $\dot{D}$ of a dense open subset of $P$ in the sense that if $(t, p) \in D$, then we interpret this as $t \Vdash p \in \dot{D}$.

It is evident from conditions (4) and (5) of the definition of $P$ that $q_0 = q \upharpoonright M \in M$ and that $q$ is an extension of $q_0$. Let $\delta = M \cap \omega_1$. Let $\{ M_1, \ldots, M_\ell \}$ be an increasing enumeration of $M_\delta \setminus M$. Of course $M_1 = M \cap H(\aleph_2)$. Let $\{ s_0, \ldots, s_m \}$ be any one-to-one list of the set $\{ s^1 \setminus \delta, q(M_1) \setminus \delta, \ldots, q(M_\ell) \setminus \delta \}$ so that $s_0 = s^1 \setminus \delta$. For each $1 \leq j \leq \ell$, let $M_j$ denote the value such that $s_{mj} = q(M_j) \setminus \delta$. Let $J$ denote those $1 \leq j \leq \ell$ such that $q(M_j) \setminus [\delta, M_j \cap \omega_1] \subset s^1$. To avoid trivialities, we can assume that we extended $(s^1, q)$ if necessary, so as to have that $J$ is not empty.

Since $S$ is a coherent Souslin tree, there is a $\delta \in M$ such that $s_0 \setminus [\delta, \delta] = s_1 \setminus [\delta, \delta]$ for each $i \leq m$. By increasing $\delta$ we can also ensure that $M \cap \omega_1 < \delta$ for each $M \in M_{q_0}$. Let $\bar{s}_i = s_i \setminus \delta$ for $i \leq m$, and notice that $\{ \bar{s}_0, \ldots, \bar{s}_m \} \in M_0$. For each $s \in S$ with $\delta \leq \text{dom}(s)$, let $\bar{s}_0 \oplus s$ denote the function $\bar{s}_0 \cup \delta \setminus [\delta, \text{dom}(s)]$; since $S$ is a coherent Souslin tree $\bar{s} \oplus s \in S$. Note that $J = \{ j < \ell : \bar{s}_0 \oplus p(M_j) < s^1 \}$. Also, define $J_B$ to be the set $\{ j \in J : M_j \cap \omega_1 \notin E_0 \}$.

Say that $(t, p) \in D$ is like $(s^1, q)$ providing

(1) there is a $M^p_0 \in M_p$ such that $\bar{s} \in M^p_0$ and $q_0 = p \upharpoonright M^p_0$,
(2) $M_p \setminus M^p_0$ has size $\ell$, enumerated as $\{ M^p_0, \ldots, M^p_{\ell-1} \}$ in increasing order,
(3) $\bar{s}_j \prec p(M^p_j)$ for $j < \ell$,
(4) $J = \{ j < \ell : \bar{s}_0 \oplus p(M^p_j) < t \}$,
(5) $J_B = \{ j \in J : M^p_j \cap \omega_1 \notin E_0 \}$.

Our proof that $S \times P$ is proper will depend on finding some $(t, p) \in D \cap M$ that is like $(s^1, q)$ and, in addition, is compatible with $(t_0, q)$. Of course this requires that $t < s^1$, but what else? Since $M_p \in M_0$ and $p < q_0$ we automatically have that $M_p \cup M_q$ is an $\varepsilon$-chain. The most difficult (and remaining) requirement is to ensure that if $p(M^p_k) < q(M_k)$ then $M^p_k \cap \omega_1$ must be in $a(q(M_k), M_0)$ if $M^p_k \notin E_0$. 
Interestingly, the values of $1 \leq j \leq \ell$ that we will have to worry about are exactly those values in $J_B$ (in most proofs it would be all values of $J$). This is because we must have that $p(M_j^p) < s_k$ and so $s_0 \oplus p(M_j^p) < s_0 < s^j$. Since also, $t < s^j$ and $t \in M_0$, we have that $s_0 \oplus p(M_j^p) < t$, which is the requirement that $j \in J$. One frequently troublesome aspect to these proofs is that the values of $k$ for which $p(M_j^p) < q(M_k)$ will be all $k$ such that $i_k = i_j$, not just values of $k$ in $J$. For easier reference in the remaining proof, let $a_k = a(q(M_k), M_k)$ for $1 \leq k \leq m$.

The set $L \subset D$ consisting of those pairs $(t, p)$ that are like $(s^j, q)$ is an element of $M$. For each $(t, p) \in L$, let $T_{t,p} = \langle t_0, t_1, \ldots, t_\ell \rangle$ be a re-naming of $\langle t, p(M_j^p), p(M_j^p) \rangle \ldots, p(M_j^p) \rangle$. Let $T(L)$ denote the set $\{ T_{t,p} : (t, p) \in L \}$, and for each $1 \leq j \leq m$, let $T(L)_j = \{ \tilde t \mid \tilde t = \langle t_0, t_1, \ldots, t_\ell \rangle \in T(L) \}$. Of course $T(L)_j$ is equal to $T(L)$. Since $D$ is an open subset of $S \times P$, let us note that if $\langle t_0, t_1, \ldots, t_{j-1} \rangle \in T(L)_j$, then $\langle t_0, t_1, \ldots, t_{j-1} \rangle \in T(L)_j$ for all $\tilde t_0 > t_0$.

Now we want to use $T(L)$ to define an $S$-name of a subset of $[\omega_1]^{\leq \ell}$. For $\tilde t = \langle t_0, t_1, \ldots, t_{j-1} \rangle \in T(L)_j$ ( $1 \leq j \leq \ell$), let $\Delta_\ell$ be the sequence $\langle \delta_1, \ldots, \delta_{j-1} \rangle$ where $\delta_k = \text{dom}(t_k)$. We define $\hat F_j$ to be the $S$-name consisting of all pairs $(t_0, \langle \delta_1, \ldots, \delta_{j-1} \rangle)$ for which there is a $\tilde t = \langle t_0, t_1, \ldots, t_\ell \rangle$ in $T(L)$ such that $\Delta_\ell = \langle \delta_1, \ldots, \delta_{j-1} \rangle$. In saying that $\hat F_j$ is an $S$-name we are adopting the standard abuse of notation that an element of the ground model can be used as an $S$-name for itself. By reverse induction on $\ell > k \geq 1$, we define $\hat F_k$. Having defined $\hat F_{k+1}$, we define $\hat F_k$. If $k + 1 \notin J_B$, then $\hat F_k = \hat F_{k+1}$. If $j = k + 1$ is in $J_B$, then $(t_0, \langle \delta_1, \ldots, \delta_k \rangle)$ is in $\hat F_k$ providing $t_0$ forces that the set

$$F_j(\langle \delta_1, \ldots, \delta_k \rangle) = \{ \gamma : (\exists \bar t_0)(\exists \bar \delta) (t_0, \bar \delta) \in F_j \text{ and } \langle \delta_1, \ldots, \delta_k, \gamma \rangle = \bar \delta \mid j \}$$

is stationary.

The next, somewhat standard, step is to prove that, for each $k < \ell$ with $k + 1 \in J$, $s^j \Vdash \Delta_{T, s^j} \upharpoonright k \in \hat F_k$. Again, this is by reverse induction on $\ell > k \geq 0$. Let $\gamma = \Delta_{T, s^j} = \langle \gamma_1, \ldots, \gamma_\ell \rangle$. Certainly, $s^j \Vdash \gamma \in \hat F_\ell$. We again take note of the fact that $\hat F_k \in M_0$ for each $0 \leq k \leq \ell$. Let $J_B = \{ j_1, \ldots, j_{\ell} \}$ be listed in increasing order. For $j \leq k \leq \ell$, we have that $s^j \Vdash \hat F_k = \hat F_{k-1}$. Now let $j = k + 1 = j_\ell$ and observe that $F_{\hat F_j}(\gamma \mid k)$ is a member of the model $M_{j_\ell}$, and that $\gamma_j = M_j \cap \omega_1$ is forced by $s^j$ to be an element of $F_{\hat F_j}(\gamma \mid k)$. We show that this means that $s^j$ forces that $F_{\hat F_j}(\gamma \mid k)$ is stationary. Within $M_j$, there is a maximal antichain (in fact a level) of $S$ with the property that each member of the antichain decides whether or not $F_{\hat F_j}(\gamma \mid k)$ is stationary. For each such node that decides that it is not stationary, there is a cub in $M_j$ that is forced to be disjoint. Since $\gamma_j$ is in every cub from $M_j$ and since $s^j$ forces that $\gamma_j$ is in $F_{\hat F_j}(\gamma \mid k)$, we have that it is forced to be stationary. This completes the inductive step that $s^j$ forces that $\gamma \mid k$ is in $\hat F_k$.

To complete the proof, we work our way back up from $\min(J_B)$ to $\max(J_B)$ in order to pick a suitable $(t, p) \in D \cap M$ that is compatible with $(s^j, q)$. Recall that the main requirement, once we know that $(t, p) \in L \cap M$, is to have that $\delta_j \in a_k$ for each $j \in J_B$ and each $1 \leq k \leq \ell$ with $i_j = i_k$, where $\Delta_{T, p} = \langle \delta_1, \ldots, \delta_\ell \rangle$. We begin with $j_0 = \min(J)$, and we note that $s^j$ forces that $F_{\hat F_{j_0-1}} \in M_0$ is non-empty. By Lemma 2.1, there is an $t_0 \in M \cap S$ with $t_0 < s^j$ that also forces $F_{\hat F_{j_0-1}}$ is not empty. By elementarity, there is a sequence $\delta_0 \in M_0$ such that $t_0 \Vdash \delta_0 \in F_{\hat F_{j_0-1}}$. By definition, $t_0 \Vdash F_{\hat F_{j_0}}(\delta_0)$ is stationary. Now we use our assumptions on $\hat F$ in order to
find a member of $\dot{F}_{j_0}(\delta_0)$ that is in $a_{j_0}$. This next step can seem a bit like sleight of hand. We have that $t_0 \models \dot{F}_{j_0}(\delta_0)$ is stationary, and so there is an extension (in $M_0$) of $t_0$ and an infinite set $a$ that is forced to be contained in $\dot{F}_{j_0}(\delta_0)$ and to be a member of $\bar{I}$. However, $t_0$ may be incomparable with $s_{i_0}$ and so $a$ is of no help in choosing a suitable element of $a_{j_0}$. The solution is to use that $S$ is coherent. Let $g$ be a generic filter for $S$ with $s^g \in g$. Since $S$ is coherent, the collection $s_{i_0} \oplus g = \{s \in S: (\exists t \in g) s < (s_{i_0} \oplus t)\}$ is also a generic filter for $S$ since it is an $\omega_1$-branch. The ideal $\bar{I}(s_{i_0})$ we get by interpreting the name $\bar{I}$ using the filter $s_{i_0} \oplus B$, is a $\mathcal{P}$-ideal satisfying that $[E]^{\aleph_0} \cap \bar{I}(s_{i_0})$ is non-empty for all stationary sets $E$. Also, the set $E = \text{val}_g(\dot{F}_{j_0}(\delta_0))$ is a stationary set. By elementarity, there is an infinite set $a \in M_0$ such that $a \in [E]^{\aleph_0}$ and $a \in \bar{I}(s_{i_0})$. Again by elementarity, and Lemma 2.4 there is a condition $t_1 \in M \cap g$ extending $t_0$ and satisfying that $t_0 \models a \in \dot{F}_{j_0}(\delta_0)$ and $s_{i_0} \oplus t_1 \models a \in \bar{I}$. Let us note that $a \in a_k$ for each $1 \leq k \leq m$ such that $i_{j_0} = k$. Therefore, we may choose a $\delta_{j_0} \in a \cap \forall \{a_k: i_k = i_{j_0}\}$. Next choose any sequence $\delta_1 \in M_0$ such that, by further extending $t_1$, we have that $t_1 \models \delta_1 \in \dot{F}_{j_0}$ and witnesses that $\delta_{j_0} \in \dot{F}_{j_0}(\delta_0)$.

We proceed in the same way to choose $\delta_{j_1}$ and an extension $t_2$ of $t_1$ so that $\delta_{j_1} \in a_k$ for each $k$ with $i_k = i_{j_1}$ and so that $t_2$ forces that there is a $\delta_2 \in \dot{F}_{j_2}$ witnessing that $\delta_{j_1} \in \dot{F}_{j_2}(\delta_1)$. Proceeding in this way we succeed in choosing $t_{\ell}$ in $M_0$ with $t_{\ell} \in s^\ell$ and a sequence $\delta_{\ell}$ satisfying that there is a $p \in M_0$ such that $(t_{\ell}, p) \in L$, $\Delta_{T_{\ell}, p} = (\delta_1, \ldots, \delta_{\ell}) = \delta_{\ell}$, and that $\delta_{j_n} \in a_k$ for each $1 \leq n \leq \ell$ and $1 \leq k \leq \ell$ such that $i_k = i_{j_n}$. Of course this means that $(t_{\ell}, p) \in D \cap M$ and $(t_{\ell}, p) \notin (s^\ell, q)$ as required.

\[\square\]

3. ON PPI$^+$

This first result is a reformulation of a classic result of Sapirovskii.

Lemma 3.1. Assume that $X$ is a sequentially compact non-compact space. Then either $X$ has a countable subset with non-compact closure or $X$ has an $\aleph_1$-sized subset $E$ and an open set $W$ containing the sequential closure of $E$ and such that $E$ has no complete accumulation point in $W$.

Proof. We may as well assume that countable subsets of $X$ have compact closure. Since $X$ is sequentially compact and not compact, it is not Lindelöf. Let $U$ be any open cover of $X$ that has no countable subcover and satisfies that the closure of each member of $U$ is contained in some other member of $U$. We begin an inductive construction by choosing any countable subset $U_0$ of $U$ and any point $x_0 \in X \setminus \bigcup U_0$. Suppose $\lambda < \omega_1$, and that we have chosen, for each $\alpha < \lambda$, a countable collection $U_\alpha \subset U$ and a point $x_\alpha \in X \setminus \bigcup U_\alpha$, so that $\{x_\beta: \beta < \alpha\} \cup \bigcup_{\beta < \alpha} U_\beta \subset U_\alpha$. Since $U$ has no countable subcover, this induction continues for $\omega_1$-many steps. We let $E$ be the sequence $\{x_\beta: \beta \in \omega_1\}$ and let $W$ be the union of the collection $\bigcup \{U_\alpha: \alpha \in \omega_1\}$. By construction we have that the closure of every countable subset of $E$ is contained in $W$. But also, for each $y \in W$, we have that there is an $\alpha \in \omega_1$ such that $y \in U_\alpha$ while $\bigcup U_\alpha \cap E = \{x_\beta: \beta < \alpha\}$ is countable.

\[\square\]

For the remainder of the section we have an $S$-name of a sequentially compact non-compact space $X$ which we may assume has base set $\theta$. According to Lemma 3.1 we will assume that either $\omega \subset \theta$ is forced (by 1) to be dense in $X$, or, that the
sequential closure of the points $\omega_1 \subset \theta$ are forced (by 1) to have no complete accumulation point in some open neighborhood $W$. In particular then, the sequential closure of $\omega_1$ itself contains no complete accumulation point of $\omega_1$.

In the first case, our application of PFA(S) will be simplified if we use the method sometimes called the cardinal collapsing trick. This is to show that we may again assume that we have an uncountable set, denoted $\omega_1$, so that the sequential closure is contained in an open set $\mathcal{W}$ in which $\omega_1$ has no complete accumulation point. It will be easier to remember if we call this the separable case. The simple countably closed poset $2^{<\omega_1}$ is $S$-preserving. We will work, for the separable case, in the forcing extension by $2^{<\omega_1}$ - a model in which CH holds. Just as we have in Lemma 3.1, we would like to show that there is an uncountable set $E$ and an open set $\mathcal{W}$ so that the sequential closure of $E$ is contained in $\mathcal{W}$ while having no complete accumulation points in $\mathcal{W}$. We certainly have that the forcing $2^{<\omega_1}$ preserves that $X$ is forced by $S$ to be sequentially compact and not compact. We briefly work in the forcing extension by $2^{<\omega_1} \times S$. Let $X$ denote the space obtained from the name $\mathcal{X}$. If the base set $\theta$ for $X$ is equal to $\varepsilon$ then, since it is forced to be countably compact, it is forced that $X$ has an uncountable set with no complete accumulation point at all. On the other hand if $X$ has cardinality greater than $\varepsilon$, we can fix any point $z$ of $X$ that is not in the sequential closure of $\omega$. Since $X$ is regular and $\omega$ is dense, the point $z$ has character $\omega_1$. Let $\{W_\alpha : \alpha \in \omega_1\}$ enumerate a neighborhood base for $z$ satisfying that the closure of $W_{\alpha+1}$ is contained in $W_\alpha$ for each $\alpha \in \omega_1$. For each $\alpha$, we may choose a point $x_\alpha$ from the sequential closure of $\omega$ so that $x_\alpha$ is in $W_\beta$ for all $\beta \leq \alpha$. Now we have that the uncountable set $E = \{x_\alpha : \alpha \in \omega_1\}$ satisfies that its sequential closure is contained in the open set $\mathcal{W} = X \setminus \{z\}$ and has no complete accumulation point in $\mathcal{W}$.

Next we choose an assignment of $S$-names of neighborhoods $\{\dot{U}(x, n) : x \in \theta, n \in \omega\}$, each of which is forced to have closure contained in $\mathcal{W}$. We may assume that 1 forces that these are regular descending and that $\omega_1 \cap \dot{U}(x, 0)$ is countable for all $x \in \theta$. These are chosen in the generic extension by $2^{<\omega_1}$ in the separable case. If we are also assuming that $X$ is forced to be first countable, then we assume that $\{\dot{U}(x, n) : n \in \omega\}$ is forced to form a neighborhood base for $x$.

3.1. the sequential structure. Since $S$ is ccc, it follows that if $\{\dot{x}_n : n \in \omega\}$ is a sequence of $S$-names and $1 \models \dot{x}_n \in X$ for each $n$, then there is an infinite $L \subset \omega$ such that $1 \models \{\dot{x}_n : n \in L\}$ is a converging sequence in $X$. To see this, recursively choose a mod finite descending sequence $\{L_\alpha : \alpha \in \gamma\}$ and conditions $\{s_\alpha : \alpha \in \gamma\}$ satisfying that $s_\alpha$ forces that $\{\dot{x}_n : n \in L_\beta\}$ (for $\beta < \alpha$) is not converging, while $\{\dot{x}_n : n \in L_\alpha\}$ is. Since the family $\{s_\alpha : \alpha \in \gamma\}$ is an antichain, this process must end.

Definition 3.2. Say that a sequence $\{\dot{x}_n : n \in \omega\}$ is an $S$-converging sequence in $X$ providing $1 \models \{\dot{x}_n : n \in \omega\}$ is a converging sequence (which includes, for example, constant sequences).

There is a well-known space in the study of sequential spaces, namely the space $S_\omega$ from [1]. This is the strongest sequential topology on the set of finite sequences of integers, $\omega^{<\omega}$, in which, for each $t \in \omega^{<\omega}$, the set of immediate successors, $\langle \check{t} \upharpoonright n : n \in \omega\rangle$, converges to $t$. If $T$ is any subtree of $\omega^{<\omega}$, we will consider $T$ to be topologized as a subspace of $S_\omega$. As usual, for $t \in T$, $T_t$ will denote the subtree
with root $t$ and consisting of all $t' \in T$ which are comparable with $t$. Also for $t \in T$, let $T^+_t$ denote the tree $\{t' \in \omega^{<\omega} : t \prec t' \in T\}$ i.e. the canonically isomorphic tree with root $\emptyset$.

Of particular use will be those $T \subset \omega^{<\omega}$ that are well-founded (that is, contain no infinite branch). Let $\text{WF}$ denote those downward closed well-founded trees $T$ with the property that every branching node has a full set of immediate successors. Such a tree will have a root, root($T$) (which need not be the root of $\omega^{<\omega}$) which is either the minimal branching node or, if there are no branching nodes, the maximum member of $T$. When discussing the topology on $T \in \text{WF}$ we ignore the nodes strictly below the root of $T$. The meaning of the rank of $T$ will really be the rank of $T_t$ where $t$ is the root of $T$. We use $\text{rk}(T)$ to denote the ordinal $\alpha \in \omega_1$ which is the rank of $T$. If $t \in T$ is a maximal node, then $\text{rk}(T_t) = 0$, and if root($T$) $\subset t \in T$, then $\text{rk}(T_t) = \sup(\text{rk}(T_{t'}) + 1 : t < t' \in T_t)$. We let $\text{WF}(\alpha) = \{T \in \text{WF} : \text{rk}(T) \leq \alpha\}$ and $\text{WF}(\langle \alpha \rangle) = \bigcup_{\beta < \alpha} \text{WF}(\beta)$.

If we have a Hausdorff space $X$ on a base set containing the set $\omega_1$ and we have a point $x$ in the sequential closure of $\omega_1$, then there is a $T \in \text{WF}$ and a function $y$ from max($T$) into $\omega_1$ such that there is a continuous extension of $y$ to all of $T$ such that $y(\text{root}(T)) = x$ (it does not matter what value $y(t)$ takes for $t < \text{root}(T)$).

Since our space $X$ is forced to be sequentially compact, we will be working with points in the sequential closure of $\omega_1$. In fact, we will only work with such function pairs $y, T$ that are forced by 1 to extend continuously to all of $T$. The goal is to try to make choices of points in $\check{X}$ that are, in a strong sense, not dependent on the generic filter for $S$.

For each $\alpha \in \omega_1$, let $Y_\alpha$ denote the set of all functions $y$ into $\omega_1$ where $\text{dom}(y)$ is the set of all maximal nodes of some $T \in \text{WF}(\alpha)$. We put $y \in Y_\alpha$ in $Y_\alpha$ providing 1 forces that $y$ extends continuously to all of $T_y$ as a function into $X$. We let $Y = \bigcup_{\alpha \in \omega_1} Y_\alpha$ and for $y \in Y$, we will abuse notation by letting $y$ also denote the name of the unique continuous extension of $y$ to all of $\{t \in T_y : \text{root}(T) \subset t\}$. More precisely, if needed, for each $t \in T_y \setminus \text{max}(T)$ with root($T$) $\subset t$, $y(t)$ can be used to denote the name that has the form $\{(s, \xi_s) : s \in S_\gamma\}$ where $\gamma$ is minimal such that each $s$ in $S_\gamma$ decides the value of $y(\xi_s)$ for each $t \leq s \in T_y$ in the continuous extension of $y$ and, of course, $s$ forces $y(t) = \xi_s \in \theta$ for each $s \in S_{\gamma}$. The minimality of $\gamma$ makes this choice canonical. Thus for $y \in Y$ and $\text{root}(T_y) \subset t \notin \text{max}(T_y)$, the sequence $\{y(t^{\prec n}) : n \in \omega\}$ is an $S$-converging sequence that is forced to converge to $y(t)$. Note also that if $y \in Y$ and $t \in T_y$, then $y \upharpoonright (T_y)_t \in Y$.

**Definition 3.3.** Say that $y_1$ and $y_2$ in $Y$ are equivalent, denoted $y_1 \equiv y_2$, providing $T^+_y = T^+_y$, and for each maximal $t \in T^+_y$, $y_1(\text{root}(T_{y_1}))$ equals $y_2(\text{root}(T_{y_2}))$.

Clearly if $y_1 \equiv y_2$, then $y_1(\text{root}(T_{y_1}))$ is the same name as $y_2(\text{root}(T_{y_2}))$. Now that we have identified our structure $Y$ we extend the notion to define a closure operator on any given finite power of $Y$ which will help us understand points in the sequential closure of $\omega_1$ in $\check{X}$. If $y \in Y$, we use $e(y)$ as an alternate notation for $y(\text{root}(T_y))$.

**Definition 3.4.** For each integer $n > 0$, and subset $B$ of $Y^n$ we similarly define the hierarchy $\{B^{(\alpha)} : \alpha \in \omega_1\}$ by recursion. In addition, we again (recursively) view each $\vec{b} \in B^{(\alpha)}$ as naming a point in $X^n$. The set $B$ will equal $B^{(0)}$. Naturally the point $e(\vec{b})$ named is the point of $X^n$ named coordinatewise by $\vec{b}$.
For limit $\alpha$, $B^{(\alpha)}$ (which could also be denoted as $B^{(<\alpha)}$) will equal $\bigcup_{\beta<\alpha} B^{(\beta)}$. The members of $B^{(\alpha+1)}$ for any $\alpha$, will consist of the union of $B^{(\alpha)}$ together with all those $\vec{b} = \langle y_i : i \in n \rangle \in (Y_{n+1})^n$ such that there is a sequence $\langle \vec{b}_k : k \in \omega \rangle$ so that

1. for each $k \in \omega$, $\vec{b}_k$ is a member of $B^{(\alpha)} \cap (Y_n)^n$,
2. for each $i \in n$ and $k \in \omega$, $(\vec{b}_k)_i \in Y$ is equivalent to $y_i \upharpoonright (T_{y_i})_{k-1}$, where $t_i$ is the root of $T_{y_i}$.

When $\vec{b}$ is constructed from a sequence $\{\vec{b}_k : k \in \omega\}$ as in this construction, we can abbreviate this by saying that $\{\vec{b}_k : k \in \omega\}$ $Y$-converges to $\vec{b}$. Also if we say that $\{\vec{b}_k : k \in L\}$ $Y$-converges to $\vec{b}$ for some infinite set $L$, we just mean by a simple re-enumeration of $\{\vec{b}_k : k \in L\}$.

For $n > 1$ we may view $Y^n$ as an $S$-sequential structure and for any $A \subset Y^n$, we say that $A^{(\omega_1)}$ is the sequential closure and is sequentially closed. Notice that this $S$-sequential structure on $Y^n$ is defined in the ground model.

The next lemma should be obvious.

**Lemma 3.5.** For each $A \subset Y$, 1 forces that $e[A^{(\omega_1)}]$ is a sequentially compact subset of $X$.

**Definition 3.6.** For each $S$-name $\dot{A}$ and $s \Vdash \dot{A} \subset Y^n$, we define the $S$-name $(\dot{A})^{(\omega_1)}$ according to the property that for each $s < t$ and $t \Vdash \dot{y} \in (\dot{A})^{(\omega_1)}$, there is a countable $B \subset Y^n$ such that $t \Vdash B \subset \dot{A}$ and $\dot{y} \in B^{(<\omega_1)}$.

For an $S$-name $\dot{A}$ and $s \Vdash \dot{A} \subset Y^n$, we will also interpret $e[(\dot{A})^{(\omega_1)}]$ in the forcing extension in the natural way as a subset of $X^n$. This may need some further clarification.

**Lemma 3.7.** Suppose that $\vec{y}$ is a member of $B^{(\alpha)}$ for some $B \subset Y^n$ and some $\alpha < \omega_1$. Also suppose that $\{s_i : i < \ell\} \subset S$ and that $W$ is an $S$-name for a neighborhood of $e(\vec{y})$. Then there is a $\vec{b} \in B$ such that for each $i < \ell$, there is an $s_i' \supset s_i$ forcing that $e(\vec{b}) \in W$.

**Proof.** We may suppose that $B$ is a countable set and we may proceed by induction on $\alpha$. Let $y_i$ be equal to $\vec{y}_i$ for $i < n$, and let $t_i$ denote the root of $T_{y_i}$. By the definition of $B^{(\alpha)}$, there is a sequence $\langle \vec{y}_k : k \in \omega \rangle$ with the property that $\vec{y}_k \in B^{(<\alpha)}$ for each $k$, such that, for each $i < n$ and each $k \in \omega$, $(\vec{y}_k) \upharpoonright (T_{y_i})_{k-1}$ is equivalent to $(\vec{b}_k)_i$. For each $i < \ell$, choose $s_i' \supset s_i$ so that $s_i'$ forces a value, $W_i$, on $W \cap e[B \cup \{\vec{y}_k : k \in \omega\}]$. Since this sequence, $\{e(\vec{y}_k) : k \in \omega\}$ is assumed to be $S$-converging to a point in $W_i$, there is a $k$ such that the point $e(\vec{y}_k)$ is in each $W_i$. Of course the result now follows by the induction hypothesis.

Note that the members of $Y_0$ have a singleton domain and for each $\alpha \in \omega_1$, let $e^{-1}(\alpha)$ denote the member of $Y_0$ that sends the minimal tree to the singleton $\alpha$. Our assumption that $\omega_1$ has no complete accumulation points in its sequential closure implies that no point is a member of every member of the family $\{\{e^{-1}(\alpha) : \alpha > \delta\}^{(\omega_1)} : \delta \in \omega_1\}$. That is, this family is a free filter of $S$-sequentially closed subsets of $Y$. By Zorn’s Lemma, we can extend it to a maximal free filter, $\mathcal{F}_0$, of $S$-sequentially closed subsets of $Y$. ’
3.2. A new idea in PFA(S). Now we discuss again the special forcing properties that a coherent Souslin tree will have. Assume that \( g \) is (the) generic filter on \( S \) viewed as a cofinal branch. For each \( s \in S \), \( o(s) \) is the level (order-type of domain) of \( s \) in \( S \). For any \( t \in S \), define \( s \oplus t \) to be the function \( s \cup t \upharpoonright [o(s), o(t)) \). Of course when \( o(t) \leq o(s) \), \( s \oplus t \) is simply \( s \). One of the properties of \( S \) ensures that \( s \oplus t \in S \) for all \( s, t \in S \). We similarly define \( s \oplus g \) to be the branch \( \{ s \oplus t : t \in g \} \).

**Definition 3.8.** Let \( bS \) denote the set of \( \omega_1 \)-branches of \( S \).

**Lemma 3.9.** In the extension \( V[g] \), \( bS = \{ s \oplus g : s \in S \} \). Furthermore, for each \( s \in S \), \( V[s \oplus g] = V[g] \).

The filter \( F_0 \) may not generate a maximal filter in the extension \( V[g] \) and so we will have to extend it. Looking ahead to the PFA(S) step, we would like (but probably can’t have) this (name of) extension to give the same filter in \( V[s \oplus g] \) as it does in \( V[g] \). We adopt a new approach. We will define a filter (of \( S \)-sequentially closed) subsets of the product structure \( Y^{bS} \). We try to make this filter somehow symmetric.

We introduce some notational conventions. Let \( S^{<\omega} \) denote the set of finite tuples \( \langle s_i : i < n \rangle \) for which there is a \( \delta \) such that each \( s_i \in S_\delta \). Our convention will be that they are distinct elements. We let \( \Pi_{\langle s_i : i < n \rangle} \) denote the projection from \( Y^{bS} \) to \( Y^n \) (which we identify with the product \( Y^{\langle s_i \oplus g : i < n \rangle} \)).

**Definition 3.10.** Suppose that \( \dot{A} \) is an \( S \)-name of a subset of \( Y^n \) for some \( n \), in particular, that some \( s \) forces this. Let \( s' \) be any other member of \( S \) with \( o(s') = o(s) \). We define a new name \( \dot{A}_s' \) (the \( (s, s') \)-transfer perhaps) which is defined by the property that for all \( \langle y_i \rangle_{i < n} \in Y^n \) and \( s < t \in S \) such that \( t \Vdash \langle y_i \rangle_{i < n} \in \dot{A} \), we have that \( s' \oplus t \Vdash \langle y_i \rangle_{i < n} \in \dot{A}_s' \).

**Lemma 3.11.** For any generic \( g \subseteq S \), \( \text{val}_{s \oplus g}(\dot{A}) = \text{val}_{s' \oplus g}(\dot{A}_s') \).

**Theorem 3.12.** There is a family \( \mathcal{F} = \{ (s^\alpha, \{ s_i^\alpha : i < n_\alpha \}, \dot{F}_\alpha) : \alpha \in \lambda \} \) where,

1. for each \( \alpha \in \lambda \), \( \{ s_i^\alpha : i < n_\alpha \} \in S^{<\omega} \), \( s^\alpha \in S \), \( o(s^\alpha) \leq o(s^\omega) \),
2. \( \dot{F}_\alpha \) is an \( S \)-name such that \( s^\alpha \Vdash \dot{F}_\alpha = (\dot{F}_\alpha)^{\omega_1} \subset Y^{n_\alpha} \),
3. for each \( s \in S\) and \( F \in \mathcal{F}_0 \), \( (s, \{ s_i \}, \dot{F}) \in \mathcal{F} \),
4. for each \( s \in S_{o(s^\alpha)} \), \( (s, \{ s_i^\alpha : i < n_\alpha \}, (\dot{F}_\alpha)^{s^\alpha}) \in \mathcal{F} \),
5. for each generic \( g \subseteq S \), the family \( \{ \Pi_{i < n_\alpha}^{-1}(\text{real}(\dot{F}_\alpha)) : s^0 \in g \} \) is finitely directed; we let \( \dot{F}_1 \) be the \( S \)-name for the filter base it generates,
6. for each generic \( g \subseteq S \) and each \( \langle s_i : i < n \rangle \in S^{<\omega} \), the family \( \{ \text{val}_g(\dot{F}_\alpha) : s^\alpha \in g \text{ and } \{ s_i \oplus g : i < n \} = \{ s_i^\alpha \oplus g : i < n_\alpha \} \} \) is a maximal filter on the family of \( S \)-sequentially closed subsets of \( Y^n \).

**Proof.** Straightforward recursion or Zorn’s Lemma argument over the family of “symmetric” filters (those satisfying (1)-(5)).

**Definition 3.13.** For any \( \langle s_i : i < \ell \rangle \in S^{<\omega} \), let \( \dot{F}_{\langle s_i : i < \ell \rangle} \) denote the filter on \( Y^\ell \) induced by \( \Pi_{\langle s_i : i < \ell \rangle}^{-1}(\dot{F}_1) \).

**Definition 3.14.** Let \( \dot{A} \) denote the family of all \( (s, \langle s_i : i < \ell \rangle, \dot{A}) \) satisfying that \( o(s) \geq o(s_0), \langle s_i : i < \ell \rangle \in S^{<\omega} \), and \( s \Vdash \dot{A} \in \dot{F}_{\langle s_i : i < \ell \rangle} \). As usual, for a family \( G \) of set, \( G^+ \) denotes the family of sets that meet each member of \( G \).
Lemma 3.15. For each \((s, \langle s_i : i < n \rangle, \hat{A}) \in A\), the object \((s, \langle s_i : i < n \rangle, \hat{A}(\omega_1))\) is in the list \(\mathcal{F}\).

In this next Lemma it is crucial that there are no dots on the sequence \(\langle y^M(s) : s \in S_0 \rangle\). The significance of there being no dots is that, regardless of the generic \(g \subseteq S\), we will have that \(e_g(y^M(g \cap M))\) is the limit of the same sequence from within \(M\).

Lemma 3.16. Suppose that \(M < H(\kappa)\) (for suitably big \(\kappa\)) is a countable elementary submodel containing \(Y, A\). Let \(M \cap \omega_1 = \delta\). There is a sequence \(\langle y^M(s) : s \in S_0 \rangle\) such that for every \((\check{s}, \langle s_i : i < n \rangle, \hat{A}) \in A \cap M\), and every \(s \in S_0\) with \(\check{s} < s\), there is a \(B \subseteq Y^\omega \cap M\) such that \(\langle y^M(s_i : i < n) : i < n \rangle \in B^{(s_0)}(\delta)\) and \(s \models B \subseteq A\).

Proof. Let \(\{\langle s^m, \langle s^m_i : i < n_m \rangle, \hat{A}_m \rangle : m \in \omega\}\) enumerate the family \(A \cap M\). Also, fix an enumeration, \(\{s^m : m \in \omega\}\), of \(S_0\). Let \(\{\alpha_m : m \in \omega\}\) be an increasing cofinal sequence in \(\delta\). At stage \(m\), we let \(\beta_m\) be large enough so that \(s^m \models [\beta_m, \delta) = s^m \cup [\beta_m, \delta)\) for all \(i < m\). Replace the list \(\{\langle s^i, s^i \cup \beta^i : i < \ell_j \rangle, \hat{A}_j \rangle : j < m\) with \(\beta^i\) belongs to the new list \(\{\langle s^i, s^i \cup \beta^i : i < \ell_j \rangle, \hat{A}_j \rangle : j < m\) so that for all \(i, j < m\) with \(s^j < s^i\) (from original list) the new list includes \(\langle s^i \cup \beta^i, s^i \cup \beta^i : i < \ell_j \rangle, \hat{A}_j \rangle\); and so that for all \(j < m\) in the new list \(s^j\) and \(s^j\) are all in \(S_0\). Nothing else is added to the new list (in particular, the new list is contained in \(A \cap M\)).

Now we have \(s^i \models (\hat{A}_j)^{s^i} \models \check{F}_{(s^i \models \hat{A}_j)}\) and so also have that \(s^i \models (\hat{A}_j)^{s^i} \models \check{F}_{(s^i \models \hat{A}_j)}\) (because they are essentially the same sets).

Let \(\Sigma = \{\sigma_k : k \in K\}\) lex-enumerate \(\{\langle s^i \cup \beta^i, s^i \cup \beta^i : i < \ell_j \rangle, \hat{A}_j \rangle : j < m\}\) in \(M\). Let \(\Pi_\Sigma\) be the projection map from \(Y^{bs^i}\) onto \(Y^{s^i}\). Let \(\Pi_\Sigma(\langle s^i \models \hat{A}_j \rangle)\) be defined by the equation \(\Pi_\Sigma(\langle s^i \models \hat{A}_j \rangle) = \Pi_{(s^i \models \hat{A}_j)}\). We consider the filter (name) \(\check{F}_\Sigma\).

For each \(j < L_m\) and \(i < m\) such that \(s^j < s^i\), it is forced by \(s^0\) that the set \(\Pi_{\Sigma(\langle s^i \models \hat{A}_j \rangle)}^{-1}(\langle \check{A}_j \rangle)^{s^i}_{s^0}(\check{\omega}^i)\) is a member of \(\check{F}_\Sigma\) and all are in \(M\). Select any \(\check{y}_m \in M \cap \Sigma^\omega\) with the property that \(\Pi_{\Sigma(\langle s^i \models \hat{A}_j \rangle)}(\check{y}_m) \in \check{A}_j(\check{\omega}^i)\) for all \(j < L_m\). Choose a sequence \(\{B_j : j < L_m\}\) of countable subsets of \(Y^{\check{\omega}^i}\) (in fact \(B_j \subseteq \check{\omega}^i\)) which are in \(M\) and satisfy that, for each \(j < L_m\), \(s^0 \models B_j \subseteq (\hat{A}_j)^{s_0}\) (where \(s^j < s^i\)), and so that \(\Pi_{\Sigma(\langle s^i \models \hat{A}_j \rangle)}(\check{y}_m) \in B_j^{(s^i)}\). Note that if \(s^j < s^i\), then \(s^j \models B_j \subseteq \hat{A}_j\).

If we now return to the “original” list, we have that for all \(i, j < m\) and \(s^j < s^i\), \(s^j \models \check{y}_m \models \langle s^j \cup s^0 \models \beta^i : k < \ell_j \rangle \in B_j^{(s^i)}\). Now suppose we have so chosen \(\check{y}_m\) for each \(m \in \omega\). We assert the existence of an infinite set \(\ell \subseteq \omega\) with the property that for all \(j, i \in \omega\), \(s^j\) forces that the sequence \(\chi_{y_m}(s^j \models \alpha_m \models s^i) : m \in \ell\) is defined and \(S\)-converging on a cofinite set. For each \(i\), \(y^M(s^i)\) is the \(S\)-name in \(Y\) which is equal to the limit of this \(S\)-converging sequence.

3.3. \(S\)-preserving proper forcing. Now we are ready to define our poset \(P\).

Recall that we have a fixed assignment \(\{U(x, n) : x \in \theta, n \in \omega\}\) of \(S\)-names of neighborhoods (regular descending for each \(x\)).
Definition 3.17. A condition $p \in \mathcal{P}$ consists of $(\mathcal{M}_p, S_p, m_p)$ where $\mathcal{M}_p$ is a finite $\varepsilon$-chain of countable elementary submodels of $(H(\kappa), \{U(x,m) : x \in \theta, m \in \omega\})$ for some suitable $\kappa$. We let $M_p$ denote the maximal element of $\mathcal{M}_p$ and let $\delta_p = M_p \cap \omega_1$. We require that $m_p$ is a positive integer and $S_p$ is a finite subset of $S_{\delta_p}$.

For $s \in S_p$ and $M \in \mathcal{M}_p$, we use both $s \upharpoonright M$ and $s \cap M$ to denote $s \upharpoonright (M \cap \omega_1)$. We require that the sequence $\{y^M(s) : s \in S_{M \cap \check{\omega}_1}\}$ is in each $M'$ whenever $M \in M'$ are both in $\mathcal{M}_p$. This can be made automatic if we use a fixed well-ordering of $H(\varepsilon)$ and define $\{y^M(s) : s \in S_{M \cap \check{\omega}_1}\}$ to be the minimal sequence satisfying Lemma 2.16.

It is helpful to simultaneously think of $S_p$ as inducing a finite subtree, $S^{1}_{p}$, of $S$ equal to $\{s \upharpoonright M : s \in S_p$, and $M \in M_p\}$.

For each $s \in S_p$ and each $M \in \mathcal{M}_p \setminus M_p$ we define an $S$-name $\check{W}_p(s \upharpoonright M)$ of a neighborhood of $e(y^M(s \upharpoonright M))$. It is defined as the name of the intersection of all sets of the form $\check{U}(e(y^M(s' \upharpoonright M'))), m_p)$ where $s' \in S_p$, $M' \in \mathcal{M}_p \cap M_p$, and $s \upharpoonright M \subset s' \upharpoonright M'$ and $e(y^M(s \upharpoonright M)) \in \check{U}(e(y^M(s' \upharpoonright M'))), m_p)$. We adopt the convention that $\check{W}_p(s \cap M)$ is all of $X$ if $s \cap M \notin S^{1}_{p}$.

The definition of $p < q$ is that $\mathcal{M}_q \subset \mathcal{M}_p$, $m_q \leq m_p$, $S_q \subset S^{1}_{p}$ and for each $s' \in S_p$ and $s \in S_q$, we have that $s'$ forces that $e(y^M(s \upharpoonright M)) \in \check{W}_q(s \upharpoonright M')$ whenever $M \in \mathcal{M}_p \setminus \mathcal{M}_q$ and $M'$ is the minimal member of $\mathcal{M}_q \cap (\mathcal{M}_q \setminus M)$.

It is a notational convenience, and worth noting, that we make no requirements on sets of the form $\check{U}(s, m_q)$ for $s \in S_q$.

Before we develop the important properties of $\mathcal{P}$ let us check that

Proposition 3.18. If $\mathcal{P}$ is $S$-preserving, then $\text{PFA}(S)$ implies that $S$ forces that $\check{X}$

(1) contains a free $\omega_1$-sequence, and

(2) if $\check{X}$ is first countable, contains a copy of $\omega_1$.

Proof. Let us first consider the easier non-separable case.

For any condition $q \in \mathcal{P}$, let $\mathcal{M}(q)$ denote the collection of all $M$ such that there exists a $p < q$ such that $M \in \mathcal{M}_p$. For each $\beta < \alpha \in \omega_1$, $s \in S_\alpha$, and $m \in \omega$, let

$$D(\beta, \alpha, s, m) = \{p \in \mathcal{P} : \exists s \in S_p \ s < s, m < m_p, \text{ and } \exists M \in \mathcal{M}_p) \ (\beta \in M, \alpha \notin M) \text{ or } \forall M \in \mathcal{M}(p) (\beta \in M \Rightarrow \alpha \in M)\}.$$

It is easily shown that each $D(\beta, \alpha, s, m)$ is a dense open subset of $\mathcal{P}$. Consider the family $\mathcal{D}$ of all such $D(\beta, \alpha, s, m)$, and let $G$ be a $\mathcal{D}$-generic filter. Let $\mathcal{M}_G = \{M : (\exists p \in G) \ M \in \mathcal{M}_p\}$ and let $C = \{M \cap \omega_1 : M \in \mathcal{M}_G\}$. Let $g \subset S$ be a generic filter. For each $\gamma \in C$ and $M \in \mathcal{M}_G$ with $M \cap \omega_1 = \gamma$, let $x_\gamma = e(y^M(g \cap M))$ (we omit the trivial proof that there is exactly one such $M$ for each $\gamma \in C$).

We show that the set $W = \{x_\gamma : \gamma \in C\}$ contains an uncountable free sequence of $X$, and that, if $X$ is first countable, $W$ is homeomorphic to the ordinal $\omega_1$. If $\gamma < \delta$ are both in $C$ then $x_\gamma$ and $x_\delta$ are distinct. To see this, let us note that since $x_\gamma \in M_\beta$ there is a $\beta \in M_\delta$ such that $U(x_\gamma, 0) \cap \omega_1 \subset \beta$. Also, the closure of $U(x_\gamma, 1)$ was assumed to be contained in $U(x_\gamma, 0)$. But now, $x_\delta = e(y^M(g \cap M_\delta))$ was chosen so as to be in the closure of $e(\hat{F} \cap M_\delta)$ for each $\hat{F} \in M_\delta$. In particular, $x_\delta$ is in the closure of $\omega_1 \setminus \beta$, and so it is not in $U(x_\gamma, 1)$. 
We may now define the map $f$ sending $x_\gamma$ to the ordinal o.t.(C ∩ $\gamma$) (the order type). It is certainly 1-to-1 and onto. Let $\{\xi_n : n \in \omega\} \subset \omega_1$ be strictly increasing with supremum $\xi$. For each $n$, let $f(x_{\xi_n}) = \xi_n$ and $f(x_\gamma) = \xi$. Fix any $m \in \omega$, set $s = g \cap S_\gamma$ and choose any $p \in G \cap D(0, \gamma, s, m)$. Since $\gamma \in C$, we may assume by extending $p$, that there is an $M_\gamma \in \mathcal{M}_p$ with $M_\gamma \cap \omega_1 = \gamma$. Let $\beta$ be the maximum element of $\{M \cap \omega_1 : M \in \mathcal{M}_p \cap \mathcal{M}_q\}$. We have that $f(x_\beta) < \xi$ and so there is an $n_0$ such that $\xi_n > f(x_\beta)$ for all $n > n_0$. Now for any $r < p$ with $r \in G$, and $M \in \mathcal{M}_r$ with $\beta < M \cap \omega_1 < \gamma$ we have that $s \restriction e(y^M(s \cap M)) \in \hat{U}(s \cap M, s, m)$. From this it follows that $x_\gamma \in U(x_{\gamma}, m)$ for all $n > n_0$. This shows that each limit point of $\{x_\gamma : n \in \omega\} \subset U(x_\gamma, m)$. In fact more is true: the set $\{x_\alpha : \beta < \alpha < \gamma\}$ is contained in $U(x_\gamma, m)$ and, because $U(x_\gamma, 0)$ is in $\mathcal{M}_{\gamma + 1}$, $U(x_\gamma, 0) \cap \{x_\alpha : \gamma < \alpha < \omega_1\}$ is empty. This shows that in all of $X$, $\{x_\alpha : \beta < \alpha < \gamma\}$ and $\{x_\alpha : \gamma < \alpha\}$ have disjoint closures. Since $\gamma$ was an arbitrary member of $C$, this shows that the closure of the full initial segment $\{x_\alpha : \alpha < \gamma\}$ is disjoint from the closure of $\{x_\alpha : \gamma < \alpha\}$.

Now we assume that $X$ is first countable. We have just given a proof using sequences that, since $\omega_1$ is also first countable, the inverse map, $f^{-1}$, is continuous. Since $\hat{X}$ is forced to be Hausdorff, $f^{-1}$ is also a homeomorphism.

Now we consider the separable case. We are working in the forcing extension by $2^{<\omega_1}$. For each $\alpha \in \omega_1$, let $E_\alpha$ denote the dense open subset of $2^{<\omega_1}$ of conditions that decide which member of $\hat{X}$ is equal to the chosen ordinal $\alpha$ of the copy of $\omega_1$ that is forced to have no complete accumulation points in $W$. In particular, for each $x \in \theta$ and $n \in \omega$, let $E(x, n)$ denote the dense open set of conditions in $2^{<\omega_1}$ that decide on the value of the name $\hat{U}(x, n)$ and that decide the countable set $\hat{U}(x, n) \cap \omega_1$. Fix a $2^{<\omega_1}$-name, $\mathcal{P}$ for our poset $\mathcal{P}$ as defined above. We are assuming that $2^{<\omega_1}$ forces that $\mathcal{P}$ is proper and $S$-preserving. By [10] (and see [23, 4.1]) it follows that the iteration $2^{<\omega_1} * \mathcal{P}$ is proper and $S$-preserving. The rest of the proof proceeds just as in the non-separable case. 

We prove a kind of density lemma.

**Lemma 3.19.** If $\mathcal{P} \in M$ for some countable $M \prec H(\mu)$, then for each $p \in \mathcal{P}$ and each $\alpha \in M \cap \omega_1$, such that $M \cap H(\alpha) \in \mathcal{M}_p$, there is an $M' \in M$ such that $\alpha \in M'$, $r = (\mathcal{M}_p \cup \{M', S_p, m_p\}) \in \mathcal{P}$, and $r < p$.

**Proof.** Let $M_0 = M \cap H(\alpha)$ and let $S_0 = \{s_i : i < \ell\}$ enumerate the set $\{s \cap M : s \in S_0\}$ in the lexicographic order. In this proof we adopt the convention that we will enumerate $S_q$ for any condition $q$ in increasing lexicographic order. Let $M^\dagger$ be the maximum element of $\mathcal{M}_p \cap M$ and set $S^\dagger = \{s \cap M^\dagger : s \in S_p\}$. We define $p \restriction M$ to be $(\mathcal{M}_p \cap M, S^\dagger, m_p)$. It is routine to verify that $p \prec p \restriction M$.

By increasing $\alpha$ we may assume that $s_i \restriction [\alpha, M \cap \omega_1] = s_j \restriction [\alpha, M \cap \omega_1]$ for all $i, j < \ell$ and that $\omega_1 \cap \bigcup(M_p \cap M) < \alpha$. For each $i < \ell$, let $\tilde{s}_i = s_i \restriction \alpha$ and let $\tilde{S} = \{\tilde{s}_i : i < \ell\}$.

It is easily checked that $r = ((M_0) \cup (\mathcal{M}_p \cap M), S_0, m_p)$ is in $\mathcal{P}$ and is an extension of $p \restriction M$. Notice that this implies that $S_\gamma$ is equal to $\tilde{S} \oplus s_0 = \{\tilde{s}_i \oplus s_0 : i < \ell\}$.

Define the $S$-name $\dot{A}$ as

$$\{s_q^\dagger, (y^M(s_q^\dagger) : i < \ell) : q < p \restriction M, M_q \cap M_q = M_q \cap M, m_q = m_p, S_q = \tilde{S} \oplus s_0^\dagger\}$$

This set $\dot{A}$ is a member of $M_0$ and, by virtue of $r$, $(s_0, (y^M(s_i) : i < \ell))$ is an element of $\dot{A}$. We show, by a density and elementary submodel argument, that we
have that \( s_0 \) forces that \( \dot{A} \) is in \( \tilde{F}^+_{(\bar{s}_i: i < \ell)} \). First of all, there is a dense subset of \( S \) each member of which decides the statement \( \exists F \in \tilde{F}_{(\bar{s}_i: i < \ell)} \) \( \dot{A} \cap F = \emptyset \). Since this dense set is in \( M_0 \), there is an \( \bar{F} \in M_0 \) such that \( s_0 \models \bar{F} \in \tilde{F}_{(\bar{s}_i: i < \ell)} \), and either \( s_0 \models \dot{A} \in \tilde{F}_{(\bar{s}_i: i < \ell)} \) or \( s_0 \models \dot{A} \cap \bar{F} = \emptyset \). However, it is clear that there is a \( \beta \in M_0 \) such that \( s_0 \upharpoonright \beta \models \bar{F} \in \tilde{F}_{(\bar{s}_i: i < \ell)} \) and so, by Lemma 3.16, \( \langle y^{M_0}(s_i) : i < \ell \rangle \in \bar{F} \). It follows then that there is an \( s \in M_0 \) below \( s_0 \) which also forces that \( \dot{A} \) is in \( \tilde{F}^+_{(\bar{s}_i: i < \ell)} \).

Well, what all this proves is that \( (s, \{ \bar{s}_i : i < \ell \}, \dot{A}) \) is a member of the collection \( \mathcal{A} \) and is in \( M \cap H(\kappa) \). Apply Lemma 3.16 and select \( B \subset M \cap Y^\kappa \) so that \( s_0 \models B \subset \dot{A} \) and \( s_0 \models \langle y^{M}(\bar{s}_i \oplus s_0) : i < \ell \rangle \in B^{(\delta + 1)} \). What this actually means (see Definition 3.24) is that there is a \( \langle \bar{b}_k : k \in \omega \rangle \) of elements of \( B^{(\delta)} \) which is \( S \)-converging coordinatewise to this element. Now, by Lemma 3.27 there is a \( \bar{b} \in B \) satisfying that each \( s \in S_p \) forces that \( e(\bar{b}) \) is in the product neighborhood \( W_p(s_0) \times \cdots \times W_p(s_i) \). This \( \bar{b} \in B \) is of course equal to \( \langle y^{M_p}(s^0_i) : i < \ell \rangle \) for some \( q \) as in the definition of \( \dot{A} \). It follows that \( M_q \) is the desired value for \( M^\prime \).

That was a warm-up. What we really should have proven is

**Lemma 3.20.** For all \((s, r) \in S \times \mathcal{P} \) such that \( s \notin M_r \), and any \( M_0 \in \mathcal{M}_r \) and \((s^0, \langle s^0_i : i < \ell_j \rangle, \dot{A}_j) \in \mathcal{A} \cap M_0 \) with \( s^0 \) \( s \), there is an \( \bar{a} \in Y^{\ell_j} \cap M_0 \) such that \( s \models \bar{a} \in \dot{A}_j \) and for each \( s^0_i \in S_r \), and \( i < \ell_j \), \( s^0_i \models e(\bar{a}_i) \in W_r((s^0_i \oplus s) \cap M_0) \).

**Proof.** Let \( s_0 = s \cap M_0 \). By definition of \( \mathcal{A} \), we have that \( s^0 \models \dot{A}_j \in \tilde{F}^+_{(s^0_i : i < \ell_j)} \).

By Lemma 3.16 there is a countable \( B \subset M_0 \) such that \( s_0 \models B \subset \dot{A}_j \) and \( s_0 \models \langle y^{M_0}(s^0_i \oplus s_0) : i < \ell_j \rangle \in B^{(\delta_0 + 1)} \), where \( \delta_0 = M_0 \cap \omega_1 \). Apply Lemma 3.27 to conclude there is a \( \bar{b} \in B \) satisfying that each \( s \in S_r \) forces that \( e(\bar{b}) \) is in the product neighborhood \( W_r(s^0_0 \oplus s_0) \times \cdots \times W_r(s^0_{\ell_j-1} \oplus s_0) \).

All we have to do now is to prove that

**Theorem 3.21.** The poset \( S \times \mathcal{P} \) is proper.

**Proof.** As usual, we assume that \( M \prec H(\mu) \) (for some suitably large \( \mu \)) is countable, and that \( M_0 = M \cap H(\kappa) \in \mathcal{M}_p \) for some condition \( p \). Let \( D \subset M \) be a dense subset of \( S \times \mathcal{P} \) and assume that \( (s^1, r) \in D \). We may assume that there is some elementary submodel \( M_r \) including \( r \), that \( s^1 \notin M_r \), and that \( s^1 \cap M_r \in S_r \). This means that for all \( s^1 \in M \) and \( x \in M_r \cap \theta \), \( s^1 \cap M_r \) forces a value on \( \dot{U}(x, m_r) \cap M_r \). Let \( \langle \dot{M}_i : i \in \dot{\ell} \rangle \) enumerate \( \mathcal{M}_r \cap M \) in increasing order. By Lemma 3.19 we can assume that \( \mathcal{M}_r \cap M \) is not empty, and let \( M^\dagger \) denote the maximum element. Let \( \alpha = M^\dagger \cap \omega_1 \) and \( \delta_0 = M_0 \cap \omega_1 \). We may additionally assume that \( s \models \langle \alpha, \delta_0 \rangle = s' \models \langle \alpha, \delta_0 \rangle \) for all \( s, s' \in S_r \).

The plan now is to find \( q \in M \cap \mathcal{P} \) so that \( (s^1, q) \in D \) and \( q \not\models r \). Usual argument will arrange that \( q < r \models M \) and, loosely speaking, that, for some easily chosen expansion \( S_r \) of \( S_r \), \( (M_q \cup M_r, S_r, q, m_r) \) will be a condition in \( \mathcal{P} \) extending \( q \). The challenge is to ensure that such a condition is also an extension of \( r \), which requires that, for each \( \dot{M} \in \mathcal{M}_r \), we have that \( s \models e(\dot{y}^{\dot{M}}(s' \cap \dot{M})) \in W_r(s' \cap M) \) for all \( s, s' \in S_r \). Some standard elementary submodels as side-conditions reasoning, together with Lemma 3.20 do the trick. We have applied Lemma 3.19 above so we
also have that \( s_i = \bar{s}_i \oplus s_0 \) for each \( i < \ell \). One thing we have gained is that in checking if \( q \not\models r \), we need only check on membership in sets of the form \( \bar{W}_r(\bar{s}_i \oplus \bar{s}_0) \).

Let us say that \( q \) is \textbf{like} \( r \) (or \( q \equiv r \)) providing

1. \( M_r \cap M_0 \) is an initial segment of \( M_q \),
2. \( M_q \setminus M_r = \{ M_0^s : i < \bar{\ell} \} \) has cardinality \( \bar{\ell} = |M_r \setminus M| \),
3. the tree structure \( (S_q^i, \prec, \oplus) \) is isomorphic to \( (S_r^j, \prec, \oplus) \),
4. \( \{ s_i^q : i < \bar{\ell} \} = \{ s \cap M_0^q : s \in S_q \} \) is also equal to \( \{ \bar{s}_i \oplus s_0^q : i < \bar{\ell} \} \).

For \( q \equiv r \), and \( k < \bar{\ell} \), let \( \langle s_i^{q,k} : i < \bar{\ell}_{q,k} \rangle \) be the set \( \{ s \cap M_{q,k}^k : s \in S_q \} \) ordered lexicographically. Also let \( \bar{y}_q^k \) denote the \( \bar{\ell}_{q,k} \)-tuple \( \langle y_{m,k}^k(s_i^{q,k}) : i < \bar{\ell}_{q,k} \rangle \). We have to do this because we want these to be members of \( Y_{\bar{\ell}_{q,k}} \). Thus we have the \( \bar{\ell} \)-tuple \( \langle \bar{y}_q^k : k < \bar{\ell} \rangle \) associated with each \( q \equiv r \). Of course, \( \bar{\ell}_{q,k} \) is equal to \( \bar{\ell}_{r,k} \) for \( q \equiv r \). Also, for each \( k < \bar{\ell} \), let \( i_k \) denote the index \( i \) with the property that \( s_i^{q,k} < s_i^q \).

Recursively define a collection of sets and names. First we have the S-name:

\[
\hat{Y}_k^\ell = \{ (s, \langle \bar{y}_m^q : k < \bar{\ell} \rangle) : (s, q) \in D \text{ and } q \equiv r \}.
\]

As usual, we have that \( \hat{Y}_k^\ell \in M_0 \) (\( q \equiv r \) can be described within \( M \)). Now define, for \( k \in \{ \bar{\ell} - 1, \bar{\ell} - 2, \ldots, 0 \} \) (in that order)

\[
(1) \quad \hat{A}(q, k) = \{ (s, \langle \bar{y}_m^q : j \leq k \rangle) : \hat{Y}_{k+1} \}
\]

\[
\text{and let } (k > 0)
\]

\[
\hat{y}_k = \{ (s, \langle \bar{y}_m^q : m < k \rangle) : s \models \hat{A}(q, k) \in \hat{F}_{\langle s_i^{q,k} : i < \bar{\ell}_{q,k} \rangle} \}.
\]

Thus \( \hat{A}(q, \bar{\ell} - 1) \) contains the “top” element of the sequence \( \langle \bar{y}_q^k : k < \bar{\ell} \rangle \). Of course \( s \models \hat{A}(q, k) \in \hat{F}_{\langle s_i^{q,k} : i < \bar{\ell}_{q,k} \rangle} \) is equivalent to \( (s, \langle s_i^{q,k} : i < \bar{\ell}_{q,k} \rangle, \hat{A}(q, k)) \) being a member of \( A \). We use the notation \( \hat{A}(q, k) \) rather than the more cumbersome \( \hat{A}(\langle \bar{y}_q^k : j < k \rangle) \) but let us note that the definition depends only on the parameters \( \langle \bar{y}_q^k : j < k \rangle \) and \( \hat{Y}_{k+1} \) in \( M_{q,k}^k \) (and the latter is an element of \( M_0 \)).

Clearly \( (s^q, \langle \bar{y}_q^k : k < \bar{\ell} \rangle) \in \hat{Y}_k^\ell \) and, for readability, let \( k = \bar{\ell} - 1 \). Recall that \( i = i_k \) was defined so that \( s_i^q = s^q \cap M_{q,k}^k \). We then have that \( s^q \models \bar{y}_q^k \in \hat{A}(r,k) \).

Now we show that \( (s^q, \langle s_i^{r,k} : i < \bar{\ell}_{r,k} \rangle, \hat{A}(r, k)) \) is in \( A \). First choose any \( \xi_k \in M_{r,k}^k \setminus M_{q,k}^k \) large enough so that all members of \( \{ s_i^{q,k} : i < \bar{\ell}_{r,k} \} \) agree on the interval \( [\xi_k, M_{r,k}^k \cap \omega_1) \). Let \( \bar{s}_i^r = s_i^r \upharpoonright \beta_k \) for each \( i < \bar{\ell}_{r,k} \). As discussed above, we have that \( \hat{A}(r,k) \) is an element of \( M_{r,k}^k \). By Lemma 2.1 there is a \( \gamma \in M_{r,k}^k \cap \omega_1 \) such that each \( s \in S_{\gamma} \) decides the statement \( \hat{A}(r,k) \in \hat{F}_{\langle s_i^{q,k} : i < \bar{\ell}_{q,k} \rangle} \). It follows from Lemma 3.10 that \( s^q \) forces that \( \bar{y}_q^k \) is a witness to the fact that \( \hat{A} \in \hat{F}_{\langle s_i^{q,k} : i < \bar{\ell}_{q,k} \rangle} \).

Then, by applying Lemma 2.1 again, there is a \( \beta_k \in M_{q,k}^k \) such that the tuple \( (s^q \upharpoonright \beta_k, \langle \bar{s}_i^r \downarrow : i < \bar{\ell}_{r,k} \rangle, \hat{A}(r,k)) \) is in \( A \cap M_{r,k}^k \). This now shows that (for this value of \( k \)) \( (s^q \upharpoonright \beta_k, \langle \bar{y}_m^r : m < k \rangle) \) is a member of \( \hat{Y}_k^\ell \).

Continuing this standard argument, walking down from \( s^q \), shows that, for each \( k < \bar{\ell} \), there is a \( \xi_k \in M_{r,k}^k \) such that \( (s^q \upharpoonright \xi_k, \langle \bar{y}_m^r : m < k \rangle) \) is a member of \( \hat{Y}_{k+1} \). Now we have that there is some \( \beta_0 \in M_0 \) such that \( (s^q \upharpoonright \beta_0, \emptyset) \) is a member of \( \hat{Y}_{1} \); and more importantly that \( (s^q \upharpoonright \beta_0, \emptyset, \hat{A}(r,0)) \) is in \( A \cap M_0 \). By Lemma 3.20 there is a \( \bar{y}_0 \in M_0 \) such that \( s^q \models \bar{y}_0 \in \hat{A}(r,0) \) and, for each \( s' \in S_{r} \) and \( i < \bar{\ell}_{r,0} \),
Unlike in the first step, it may happen that (we have that) \( s' \models \mathcal{A} (q_0, 1) \) for a suitable \( q_0 \equiv r \) such that \( \gamma_0 = M_0^{q_0} \cap \omega_1 \), then we have that \( s_0^{q_0} = s' \models r_0 \), and for each \( i < \ell_{r,0} \),

\[
s_i^{0} = s_i^{q_0} = \tilde{q}_i \models \gamma_0 + s_i^{q_0}.
\]

By elementarity, there is a \( \alpha_1 \in M_0 \) such that

\[
\left(s_1 \models \alpha_1, (s_i^0 : i < \ell_{r,0}), \tilde{A}(q_0, 1) \right) \in \mathcal{A}.
\]

We apply Lemma 3.20 again and obtain \( r \equiv q_1 \in M_0 \) such that \( s_1 \models \tilde{y}^{q_1} \in \tilde{A}(q_0, 1) \), and for each \( s' \in S_r \) and \( i < \ell_{r,1} \), \( s' \models e(\tilde{y}_1) \in W_r((s_i^{q_1} + s^1) \upharpoonright M_0) \).

Unlike in the first step, it may happen that \( (s_i^{q_1} + s^1) \models M_0 \) is not in \( S_r^1 \), which poses no problem since then \( W_r((s_i^{q_1} + s^1) \upharpoonright M_0) \) is all of \( \theta \). If \( i < \ell_{r,1} \) and \( i < \ell_{r,1} \), let \( s_i^1 = s_i^{q_1} \). So we again have that \( (s_i^1, (s_i^0 : i < \ell_{r,0}), \tilde{A}(q_1, 2)) \) is in \( \mathcal{A} \).

Well, we just repeat this argument for \( \ell \) steps until we find \( q = q_\ell \equiv r \) with the property that \( (s_\ell^1, q) \in D \) and, for each \( k < \ell \) and each \( s' \in S_r \), \( s' \models e(\tilde{y}^q)]((s_i^{q_k} + s^1) \upharpoonright M_0^q) \subseteq W_r((s_i^{q_k} + s^1) \upharpoonright M_0^q) \).

Again, for larger values of \( k \), it may happen that \( (s_i^{k} + s^1) \cap M_0 \) is not in \( S_r^1 \), and so \( W_r((s_i^{k} + s^1) \cap M_0) \) would simply equal \( \omega \).

\[ \square \]

4. On the consistency of GA

Shelah has defined the \( \kappa \)-p.i.c. (for “proper isomorphism condition”). The reason for this is that a countable support iteration of length \( \omega_2 \) of \( \aleph_2 \)-p.i.c.proper posets will (under CH) satisfy the \( \aleph_2 \)-chain condition, while just assuming that the factors themselves satisfy the \( \aleph_2 \)-chain condition does not guarantee the iteration will. A diamond sequence on \( \omega_2 \) will help us decide which proper \( \aleph_2 \)-p.i.c.posets of size \( \aleph_2 \) to use in such an iteration. The resulting iteration will have cardinality \( \aleph_2 \) and the objects which we want to consider in the extension will also have cardinality \( \aleph_2 \).

We have to show that given any such reflected object there is an appropriate \( \aleph_2 \)-p.i.c. proper poset of cardinality \( \aleph_2 \) which will introduce the required set and that this set is still appropriate in the final model. For this we use the method of Todorcevic [24] in which side conditions are finite sets (or matrices) of elementary submodels rather than the more common method in which side conditions are simple finite chains of elementary submodels. It is this change which is the key in making the posets strongly \( \aleph_2 \)-cc (and iterable), thus removing the need for large cardinals to prove the results.

**Definition 4.1 (19 Ch. VIII).** A poset \( P \) is said to satisfy the \( \kappa \)-pic if whenever we have a sufficiently large cardinal \( \lambda \), a well-ordering \( \prec \) of \( H(\lambda), i < j < \kappa \), two countable elementary submodels \( N_i \) and \( N_j \), \( \mathcal{H}(H(\lambda), \prec, \varepsilon) \) such that \( \kappa \) and \( P \) are in \( N_i \cap N_j \), \( i \in N_i \), \( j \in N_j \), \( N_i \cap i = N_j \cap j \), and suppose further that we are given \( p \in N_i \) and an isomorphism \( h : N_i \rightarrow N_j \) such that \( h(i) = j \) and \( h \) is the identity on \( N_i \cap N_j \) then there is a \( q \in P \) such that :

1. \( q < p \) and \( q < h(p) \) are both \( N_i \) and \( N_j \) generic,
2. if \( r \in N_i \cap P \) and \( q' < q \) there is a \( q'' < q' \) so that \( q'' < r \) if and only if \( q'' < h(r) \)
Proposition 4.2. A countable support iteration of length at most $\omega_1$ of $\aleph_2$-p.i.c. proper posets is again $\aleph_2$-p.i.c.. Furthermore if CH holds and the iteration has length at most $\aleph_2$ then the iteration satisfies the $\aleph_2$-cc.

Proposition 4.3. A proper poset of cardinality $\aleph_1$ satisfies the $\aleph_2$-p.i.c.

Lemma 4.4 ([CH]). If $P$ is a proper $\aleph_2$-p.i.c. poset and $G$ is $P$-generic over $V$ then $V[G] \models \mathfrak{c} = \omega_1$.

Following [24], for a countable elementary submodel $N$ of $H(\aleph_2)$, we let $\mathcal{N}$ be the transitive collapse, and we let $h_N : N \to \mathcal{N}$ be the collapsing map, i.e. $h_N(x) = \{h_N(y) : y \in x \cap N\}$.

Lemma 4.5. Suppose that $N_1, N_2$ are countable elementary submodels of $H(\aleph_2)$ such that $\mathcal{N}_1 = \mathcal{N}_2$, and let $h_{N_1,N_2}$ denote the map $h_{N_1} \circ h_{N_2}$. Then $h_{N_1,N_2}$ is the identity on $H(\aleph_1) \cap N_1$ and for each $A \in N_1$ with $A \subset H(\aleph_1)$, $A \cap N_1 = h_{N_1,N_2}(A) \cap N_2$.

Proof. It follows by $\varepsilon$-induction that each $x \in N_1 \cap H(\aleph_1)$, $x \subset N_1$ and so $x \in \mathcal{N}_1$. Therefore we also have, by $\varepsilon$-induction, that $h_{N_1}(x) = x = h_{N_1}^{-1}(x)$. 

A family $[\mathcal{N}]$ is an elementary matrix if, for some integer $n > 0$,

1. $[\mathcal{N}] = \{\mathcal{N}_1, \ldots, \mathcal{N}_n\}$
2. for each $1 \leq i \leq n$, $\mathcal{N}_i$ is a finite set of countable elementary submodels of $H(\omega_2)$
3. for each $1 \leq i \leq n$, $\mathcal{N}_i = \mathcal{N}_2$ for each pair $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{N}_i$
4. for each $1 \leq i < j \leq n$ and each $\mathcal{N}_i \in \mathcal{N}_i$, there is an $\mathcal{N}_j \in \mathcal{N}_j$ with $\mathcal{N}_i \in \mathcal{N}_j$.

It will be convenient to let $N \in [\mathcal{N}]$, for an elementary submodel $N$ of $H(\aleph_2)$, be an abbreviation for $N \in \mathcal{N}$ for some $\mathcal{N} \in [\mathcal{N}]$.

Lemma 4.6 ([CH]). If $\mathcal{I}$ is an $S$-name of a $\mathcal{P}$-ideal on $\omega_1$ such that $1$ forces that $\mathcal{I} \cap [E]^{\aleph_0}$ is not empty for all stationary sets $E \subset \omega_1$, then there is an $\mathcal{S}$-preserving $\aleph_2$-p.i.c.proper poset $\mathcal{P}$ of cardinality $2^{\omega_1}$ such that $\mathcal{P}$ forces that there is an $S$-name $\hat{E}$ of a stationary set with $1 \models [\hat{E}]^{\aleph_0} \subset \mathcal{I}$.

Lemma 4.7 ([CH]). If $\hat{X}$ is an $S$-name of a sequentially compact non-compact first countable space, then there is an $\mathcal{S}$-preserving $\aleph_2$-p.i.c. proper poset $\mathcal{P}$ of cardinality $2^{\aleph_1}$ such that $\mathcal{P}$ forces that there is an $S$-name $\{\hat{x}_\gamma : \gamma \in \omega_1\} \subset \hat{X}$ that is forced to contain an uncountable free sequence, and, if $\hat{X}$ is first countable, to be a homeomorphic copy of $\omega_1$.

The proofs are very similar with the same underlying idea in that we replace elementary chains from the original proofs with elementary matrices. The usage of elementary matrices is the device to make the poset satisfy the $\aleph_2$-p.i.c. The proof that the modified poset is proper and $\mathcal{S}$-preserving relies on the fact that CH guarantees that the key combinatorics take place within $H(\aleph_1)$ and so, by Lemma 4.6 no new arguments or constructions are required. Since it is newer, we sketch the proof of Lemma 4.7 and leave the proof of Lemma 4.6 to the interested reader. In actual fact, this method isn’t really needed for the consistency of $\mathcal{P}_{22}$ because the needed poset can be chosen to have cardinality $\mathfrak{c}$. The reason this is not true for $\mathcal{P}_{21}^+$ is that we must utilize the construction of the maximal filter of $S$-sequentially
closed sets which may have cardinality $2^{\omega_1}$. We simply indicate the modifications
needed to the proof of Lemma 3.21.

**Proof.** Let $\bar{X}$ be the $S$-name as formulated in the Lemma. By Lemma 3.11, we can
pass to a subspace and assume that either $\bar{X}$ is separable or that the set $\omega_1 \times \{0\}$
is a subset and is forced to not have a complete accumulation point. Since we are assuming
CH, we can, in either case, pass to an $S$-name of cardinality $\aleph_1$ for a
subspace that is still sequentially compact and not compact. With this space having
cardinality $\aleph_1$ it is clear that also in the separable case, we can assume that $\omega_1 \times \{0\}$
is a subspace with no complete accumulation point. We can now assume that the
base set for $\bar{X}$ is $\omega_1 \times \omega_1$ (i.e. any set that is a subset of $\mathbb{H}(\aleph_1)$). The entire topology
$\tau$ for $\bar{X}$ can be coded as a subset of $\mathbb{S}_1 \times \omega_1$ where $(s,m) \in \tau$ codes the
fact that $s$ forces that $(\gamma,\delta)$ is in $U_1(\alpha, \beta, m)$.

The family $\{Y_\alpha : \alpha \in \omega_1\}$ and $\mathbf{WF}$ as defined in \S 3.1 are already subsets of $\mathbb{H}(\aleph_1)$. We also fix a well-order
$\prec_{\omega_1}$ of $\mathbb{H}(\aleph_1)$.

Finally, with no changes, the family $\mathcal{A}$ as defined in Definition 3.14 is a subset
of $\mathbb{H}(\aleph_2)$. This family $\mathcal{A}$ was the key parameter in defining our poset $\mathcal{P}$. Lemma
3.16 holds for any $M \prec (\mathbb{H}(\aleph_2), \tau, \mathcal{A})$ (meaning $\tau \in M$ and $\mathcal{A}$ is a new term in the
language). The choice of the sequence $\{y^M(s) : s \in S_\delta\}$ from Lemma 3.16
will be the $\prec_{\omega_1}$-minimal such sequence.

**Claim 4.** Consider any set $\mathcal{N}$ of pairwise isomorphic countable elementary sub-
models of $(\mathbb{H}(\aleph_2), \tau, \mathcal{A})$; i.e. $\mathcal{N} = \mathbb{N}_\alpha$ for $\aleph_1, \aleph_1 \in \mathcal{N}$. Let $\delta = \omega_1 \cap \omega_1$ for any
$\mathcal{N} \in \mathcal{N}$. Let $\mathcal{N}_1, \mathcal{N}_2$ be elements of $\mathcal{N}$. We then have that the two sequences
$\{y^{\mathcal{N}_1}(s) : s \in S_\delta\}$ and $\{y^{\mathcal{N}_2}(s) : s \in S_\delta\}$ are the same.

**Proof of Claim 2.** To prove the claim, let $(\bar{s}, \{s_i : i < n\}, \bar{A})$ be any member of
$\mathcal{A} \cap \mathcal{N}_1$ and assume that $\bar{s} < s \in S_\delta$. Choose $B \subset Y \cap \mathcal{N}_1$ such that $\bar{s} \Vdash B \subset \bar{A}$
and $s \Vdash \langle y^{\mathcal{N}_1}(s \oplus s_i) : i < n\rangle \in B^{(1)}$. By Lemma 3.17, $h_{\mathcal{N}_1,\mathcal{N}_2}((\bar{s}, \{s_i : i < n\}, \bar{A}))$
is in $\mathcal{A} \cap \mathcal{N}_2$. Since $\mathcal{A} \subset \mathbb{H}(\aleph_1)$, we also have by Lemma 3.18, that $h_{\mathcal{N}_1,\mathcal{N}_2}(\bar{A}) \cap \mathcal{N}_2$
is equal to $\bar{A} \cap \mathcal{N}_1$. Therefore, we have that $s$ also forces that $B$ is a subset of
$h_{\mathcal{N}_1,\mathcal{N}_2}(\mathcal{A})$. Well this shows that $\langle y^{\mathcal{N}_1}(s \oplus s_i) : i < n\rangle$ satisfies this particular requirement of
$\langle y^{\mathcal{N}_2}(s \oplus s_i) : i < n\rangle$ with respect to $h_{\mathcal{N}_1,\mathcal{N}_2}((\bar{s}, \{s_i : i < n\}, \bar{A}))$.
Since $h_{\mathcal{N}_1,\mathcal{N}_2}$ is an isomorphism, this shows that $\langle y^{\mathcal{N}_1}(s) : s \in S_\delta\rangle$ works as a choice
for $\langle y^{\mathcal{N}_2}(s) : s \in S_\delta\rangle$, and so, indeed, they are the same. \hfill \Box

A condition $p \in \mathcal{P}$ consists of $([\mathcal{N}_p], S_p, m_p)$ where $[\mathcal{N}_p]$ is a elementary matrix
of submodels of $(\mathbb{H}(\aleph_2), \prec_{\omega_1}, \tau, \mathcal{A})$. We let $\delta_p = \omega_1 \cap \omega_1$ for any maximal $N \in [\mathcal{N}_p]$.
We require that $m_p$ be a positive integer and $S_p$ is a finite subset of $\mathcal{S}_p$.

For each $s \in S_p$ and each non-maximal $N \in [\mathcal{N}_p]$ we define an $S$-name $\bar{W}_p(s \upharpoonright N)$
of a neighborhood of $e(y^N(s \upharpoonright N))$. It is defined as the name of the intersection of all sets of the form $U(s \upharpoonright N', m_p)$ where $s' \in S_p$, non-maximal $N' \in [\mathcal{M}_p]$, and $s \upharpoonright N \subset s' \upharpoonright N'$ and $e(y^N(s \upharpoonright N)) \in U(s' \upharpoonright N')$. We adopt the convention that
$\bar{W}_p(s \cap N)$ is all of $X$ if $s \cap N \notin \mathcal{S}_p$.

The definition of $p < q$ is that each $N \in [\mathcal{N}_q]$ is a member of $[\mathcal{N}_p]$, $m_q \leq m_p$, $S_q \subset S_p$ and for each $s' \in S_p$ and $s \in S_q$, we have that $s'$ forces that $e(y^N(s \upharpoonright N)) \in \bar{W}_q(s \upharpoonright N')$ whenever $N \in [\mathcal{N}_p]$ and $N \notin [\mathcal{N}_q]$ and $N'$ is a minimal member
of \([\mathcal{N}_q] \setminus N\), which is itself not a maximal member of \([\mathcal{N}_q]\). Again we note that we make no requirements on sets of the form \(\hat{U}(s, m_q)\) for \(s \in S_q\).

Because of Claim \(\text{4.8}\) the proof that \(\mathcal{P}\) is proper and \(S\)-preserving proceeds exactly as in the proof of Theorem \(3.21\).

**Claim 5.** \(\mathcal{P}\) satisfies the \(\mathcal{N}_2\)-p.i.c.

**Proof of Claim** \(\text{5}\) Let \(\lambda\) be a sufficiently large cardinal, fix a well-ordering \(\prec\) of \(H(\lambda)\), let \(i < j < \omega_2\) be such that there are two countable elementary submodels \(N_i\) and \(N_j\) of \((H(\lambda), \prec, \in)\) such that \(\mathcal{P}\) is in \(N_i \cap N_j\), \(i < N_i\), \(j < N_j\), \(N_i \cap i = N_j \cap j\), and suppose further that we are given \(p \in \mathcal{P} \cap N_i\) and an isomorphism \(h : N_i \rightarrow N_j\) such that \(h(i) = j\) and \(h\) is the identity on \(N_i \cap N_j\). We must show that there is a \(q \in \mathcal{P}\) such that:

1. \(q < p\), \(q < h(p)\) and \(q\) is both \(N_i\) and \(N_j\) generic,
2. \(\text{if } r \in N_i \cap \mathcal{P}\) and \(q' < q\) there is a \(q'' < q'\) so that \(q'' < r\) if and only if \(q'' < h(r)\).

We first show that since \(\mathcal{P} \in N_i \cap N_j\), we also have that \(\{\prec_{\omega_1}, \tau, A\} \in N_i \cap N_j\). The reason is that the collection \(\{N : (\exists p \in \mathcal{P})N \in [\mathcal{N}_p]\}\) is in \(N_i \cap N_j\). It follows that \(N_i' = (\mathcal{N}_i \cap H(\omega_2), \prec_{\omega_1}, \tau, A)\) is an elementary submodel of \((H(\omega_2), \prec_{\omega_1}, \tau, A)\). \(N_i'\) defined similarly is as well. The definition of the \([\mathcal{N}_q]\) for \(q\) is canonical. Given that \([\mathcal{N}_p] = \{N_1, N_2, \ldots, N_n\}\), we set \([\mathcal{N}_q] = \{N_1 \cup h(N_1), \ldots, N_n \cup h(N_n), N_i', N_j'\}\). The existence of \(h\) ensures that \(N_i' = N_j'\). Since \([\mathcal{N}_p] \in N_i'\) and \(h([\mathcal{N}_p]) = [\mathcal{N}_{h(p)}] \in N_j'\) we have that \([\mathcal{N}_q]\) is an elementary matrix. Choose \(S_\varphi \subset S_{N_i \cap N_j}\) to be any finite set such that \(S_\varphi \cap h(S_p) \subset S_{N_i'}^k\). We already know that \(q\) is both \(N_i\) and \(N_j\) generic from the arguments in Theorem \(3.21\).

Finally let \(r \in N_i \cap \mathcal{P}\) and \(q' < q\) with \(q' \in \mathcal{P}\). We may assume, by symmetry, that there is a \(q'' < q'\) that is also below \(r\). Let \([\mathcal{N}_{q''}]\) be listed as \(\{N_1, \ldots, N_k\}\) and let \(1 < \ell \leq k\) be chosen so that \(N_i' \in N_{\ell'}\). For \(1 \leq m < \ell\), let \(N_{i,m} = N_i \cap N_{i,m}\). Of course we have that \(N \in N_{i,m}\) for each \(1 \leq i < \ell\) and each \(N \in h(N_{i,m})\). It is the easily verified that

\[
[N_q] = \{h(N_{i,1}') \cup N_1, \ldots, h(N_{i,\ell-1}') \cup N_{i,\ell-1}, N_i, \ldots, N_k\}
\]

is an elementary matrix, and so \(\hat{q} \in \mathcal{P}\) where \([N_q], S_{q''}, m_{q''}\), where

\[
[N_q] = \{h(N_{i,1}') \cup N_1, \ldots, h(N_{i,\ell-1}') \cup N_{i,\ell-1}, N_i, \ldots, N_k\}.
\]

It is immediate that \(\hat{q} < q''\), and so \(\hat{q} < q', \hat{r}\). We just have to show that \(\hat{q}\) is also below \(h(r)\). Since \(q'' < r\), we have that \([N_r] \in N_i\) is a submatrix of \(\{N_i \cap N_1, \ldots, N_i \cap N_{\ell-1}\}\), and so \([N_{h(r)}]\) is a submatrix of \([N_q]\).

This completes the proof.

**Definition 4.8.** The stationary set of ordinals \(\lambda \in \omega_2\) with uncountable cofinality is denoted as \(S_\varphi^\lambda\). The principle \(\Diamond(S_\varphi^\lambda)\) is the statement:

There is a family \(\{X_\lambda : \lambda \in S_\varphi^\lambda\}\) such that

1. for each \(\lambda \in S_\varphi^\lambda, X_\lambda \subset \lambda\),
2. for each \(X \subset \omega_2\), the set \(E_X = \{\lambda \in S_\varphi^\lambda : X \cap \lambda = X_\lambda\}\) is stationary.

**Theorem 4.9.** Assume \(2^{\aleph_0} = \aleph_1\) and \(\Diamond(S_\varphi^\lambda)\). There is a proper poset \(\mathbb{P}\) so that in the forcing extension by \(\mathbb{P}\) there is a coherent Souslin tree \(S\) such that, in the full forcing extension by \(\mathbb{P} \ast S\), the statement \(GA\) holds.
Proof. We construct a countable support iteration sequence \((\mathcal{P}_\alpha, \mathcal{Q}_\beta) : \alpha \leq \omega_2, \beta < \omega_2\). By induction, we assume that \(\mathcal{P}_\alpha\) is proper, has cardinality at most \(\aleph_2\), and that
\[
\models_{\mathcal{P}_\alpha} \mathcal{Q}_\alpha \text{ satisfies the } \aleph_2\text{-p.i.c.}
\]

Note that by Lemmas \([4.2\) and \([4.4\) we will have that, for each \(\alpha < \omega_2\), CH holds in the forcing extension by \(\mathcal{P}_\alpha\). We may assume that \(\mathcal{Q}_0\), and therefore \(\mathcal{P}_1\) is constructed so that there is a \(\mathcal{P}_1\)-name, \(\dot{S}\) of a coherent Souslin tree (henceforth we suppress the dot on the \(S\)). We further demand of our induction that, for \(\alpha \geq 1\)
\[
\models_{\mathcal{P}_\alpha} \mathcal{Q}_\alpha \text{ is } \mathcal{S}\text{-preserving.}
\]

For each ordinal \(0 < \alpha \in \omega_2 \setminus S^2\), we let \(\mathcal{Q}_\alpha\) denote the \(\mathcal{P}_\alpha\)-name of the standard Hechler poset for adding a dominating real. This ensures that \(b = \omega_2\) in the forcing extension by \(\mathcal{P}_{\omega_2}\).

For the rest of the construction, fix any function \(h\) from \(\omega_2\) onto \(H(\aleph_2)\). Also let \{\(X_\lambda : \lambda \in S^2\}\) be a \(\diamond\)\(S^2\)-sequence.

Now consider \(\lambda \in S^2\) and let \(x_\lambda = h[X_\lambda]\). We define \(\mathcal{Q}_\lambda\) according to cases:

1. if \(x_\lambda\) is the \(\mathcal{P}_\lambda\)-name of a \(\mathcal{P}\)-ideal on \(\omega_1\) such that
   \[1 \models [E]^{\aleph_0} \cap x_\lambda \text{ is not empty for all stationary sets } E \subset \omega_1\]
   then \(\mathcal{Q}_\lambda\) is the \(\mathcal{P}_\lambda\)-name of the poset from Theorem \([4.6\]

2. if \(x_\lambda\) is the \(\mathcal{P}_\lambda\)\(-S\)-name of a subset of \(\lambda \times \lambda \times \lambda\) so that if we define, for \(\xi, \eta \in \lambda, U(\xi, \eta)\) to be the \(\mathcal{P}_\lambda\)\(-S\)-name of the subset of \(\lambda\) such that
   \[\{\langle \xi, \eta \rangle \times U(\xi, \eta) = x_\lambda \cap \{\langle \xi, \eta \rangle \times \lambda\}\}
   \]
   i.e. for \((\langle p, s \rangle, \langle \xi, \eta, \gamma \rangle)\) in the set \(x_\lambda\), \((\langle p, s \rangle, \gamma)\) is in the name \(U(\xi, \eta)\), and \(\mathcal{P}_\lambda\)\(-S\) forces that the family \(\{U(\xi, \eta) : \eta \in \lambda\}\) is a local base for \(\xi\) in a sequentially compact regular topology on \(\lambda\), and that no finite subset of \(\{U(\xi, \eta) : \xi, \eta \in \lambda\}\) covers \(\lambda\), then \(\mathcal{Q}_\lambda\) is the \(\mathcal{P}_\lambda\)-name of the poset from Theorem \([4.7\]

3. in all other cases, \(\mathcal{Q}_\lambda\) is the \(\mathcal{P}_\lambda\)-name of the Cohen poset \(2^{<\omega}\).

Assume that \(\mathcal{I}\) is a \(\mathcal{P}_\omega\)\(-S\)-name of a \(\mathcal{P}\)-ideal on \(\omega_1\) satisfying that there is some \((p_0, s_0) \in \mathcal{P}_\omega \ast S\) forcing that \([E]^{\aleph_0} \cap \mathcal{I}\) is not empty for all stationary sets \(E \subset \omega_1\). The ideal of all countable subsets of \(\omega_1\) is also such an ideal, so we can find a \(\mathcal{P}_\omega \ast S\)-name \(\mathcal{J}\) such that \((p, s) \models \mathcal{J} = \mathcal{I}\), and \(1 \models [E]^{\aleph_0} \cap \mathcal{J}\) is not empty for all stationary sets \(E \subset \omega_1\). Let \(X \subset \omega_2\) be chosen to be the set of all \(\xi \in \omega_2\) with the property that there is a \(\mu < \omega_2\) such that \(h(\xi)\) is a \(\mathcal{P}_\mu\)\(-S\)-name with \(1 \models h(\xi) \in \mathcal{J}\). There is a cub \(C \subset \omega_2\) such that for each \(\mu < \mu' \in C\):

1. the collection \(\{h(\xi) : \xi \in X \cap \mu\}\) is a collection of \(\mathcal{P}_\mu\)\(-S\)-names,
2. for each countable subset \(\{\xi_n : n \in \omega\}\) of \(X \cap \mu\) there is a \(\xi < \mu'\) such that \(1 \models h(\xi_n)\) is almost contained \(h(\xi)\) for each \(n\).
3. every \(\mathcal{P}_\mu\)\(-S\)-name that is forced by \(1\) to be a member of \(\mathcal{J}\) is equivalent to a name in \(\{h(\xi) : \xi \in X \cap \mu'\}\).

Therefore, there is a \(\lambda \in E_X \cap C\) such that \(X_\lambda = X \cap \lambda\). We may of course assume that \(p_0 \in \mathcal{P}_\lambda\). Routine checking now shows that \(x_\lambda\) satisfies clause (1) in the definition of \(\mathcal{Q}_\lambda\). It follows that \(\mathcal{P}_\omega \ast S\) is a model of \(\mathcal{P}_{22}\).

Now suppose that we have a \(\mathcal{P}_\omega \ast S\)-name of a sequentially compact non-compact space. We note that \(\mathcal{P}_\omega \ast S\) forces that \(2^{\aleph_0} = \aleph_2\). Therefore, by Lemma \([3.1\), we
can pass to a name of a sequentially compact non-compact subspace which has cardinality at most $\omega_2$. In fact, we can assume this space has cardinality exactly $\omega_2$ by taking the free union with the Cantor space. Let $\tilde{Z}$ denote the $\mathcal{P}_{\omega_2} \ast S$-name of this space. Again, by Lemma 5.1 we can assume that each point of the space has a separable neighborhood. This means that, with re-indexing, we can assume that the base set for the space is the ordinal $\omega_2$ and $\{\dot{U}(\xi, \eta) : \xi, \eta \in \omega_2\}$ is the list of $\mathcal{P}_{\omega_2} \ast S$-names of the neighborhood bases of the points, and that no finite subcollection covers. We define $X$ to be the set of all those $\alpha \in \omega_2$ such that $h(\alpha)$ is a tuple of the form $((p, s), (\xi, \eta, \gamma))$, i.e. a $\mathcal{P}_{\omega_2} \ast S$-name of a member of $\omega_2 \times \omega_2 \times \omega_2$, where $(p, s) \models \gamma \in \dot{U}(\xi, \eta)$.

We again want to choose a $\lambda$ in $E_X \cap C$ for some special cub set $C$ and in this case it is much simpler to make use of uncountable elementary submodels. Let $\kappa$ be any regular cardinal greater than $2^{\omega_2}$, and let $\{M_\alpha : \alpha \in \omega_2\}$ be chosen so that, for each $\alpha \in \omega_2$:

1. $X, h$ and $\mathcal{P}_{\omega_2} \ast S$ are in $M_\alpha$
2. $\omega_1 \subset M_\alpha$ and $M_\alpha$ has cardinality $\aleph_1$,
3. for each $\beta < \alpha$, every countable subset of $M_\beta$ is an element of $M_\alpha$,
4. $M_\alpha$ is an elementary submodel of $H(\kappa)$,
5. if $\alpha$ is a limit ordinal, then $M_\alpha = \bigcup \{M_\beta : \beta < \alpha\}$.

Items (2) and (4) guarantee that $M_\alpha \cap \omega_2$ is an initial segment of $\omega_2$ – hence an ordinal. The chain $\{M_\alpha : \alpha \in \omega_2\}$ is a continuous chain because of item (5), and so $C = \{M_\alpha \cap \omega_2 : \alpha \in \omega_2\}$ is a closed and unbounded subset of $\omega_2$. Now we choose $\lambda \in E_X \cap C$, we can also choose $\lambda$ so that it is $M_\lambda$ with $M_\lambda \cap \omega_2$ being $\lambda$. Using items (1), (3) and (4) and the fact that $\lambda \in S^2_1$, it is now easy to show that $x_\lambda$ will satisfy the requirement (2) in the construction of $\dot{Q}_\lambda$. It then follows, as in the proof of Lemma 5.19, that $\mathcal{P}_{\lambda+1} \ast S$ will force the existence of the necessary $\omega_1$-sequence showing that $\tilde{Z}$ is not a counterexample to $\text{PPI}^+$.

\[\square\]

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