A Kind of Novel Analytical Approximate Solution to the Duffing Oscillator

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Abstract. In this paper, a kind of novel analytical approximate solution was constructed to the Duffing oscillator via a modified homotopy analysis method (HAM), namely the piecewise perturbation method (PPM). The validity and efficient of the obtained PPM solutions were investigated by comparing with the traditional HAM solution. The main advantage of the PPM is that it possesses higher accuracy and faster convergence rate than the traditional HAM.

1. Introduction
Nonlinear differential equations are known to describe a wide variety of real world problems. Thus, to solve these equations, several analytical methods have been suggested and the most famous one is the homotopy analysis method (HAM) devised by Liao [1]. As we know, it provides a convenient way to adjust and control the convergence region and rate of the series solutions and has been successfully used to solve a lot of nonlinear problems in science and engineering [2-5]. Recently, based on Liao’s HAM, we proposed a modified HAM called the piecewise perturbation method (PPM), which was employed to find more effective analytical approximations expressed by exponential function for the one-loop soliton solution to the Vakhnenko equation [6].

In the spirit of Ref. [6], the main aim of this work is to investigate the validity of PPM in constructing periodic analytical approximate solutions through the well-known Duffing oscillator [7,8]

\[ \ddot{y} + y + \varepsilon y^3 = 0, \quad y(0) = 1, \quad \dot{y}(0) = 0, \]

where the dot means the derivative with respect to the time \( t \). Note that, Liao applied HAM to investigate the frequency of the problem (1) with initial condition \( y(0)=a \) instead of \( y(0)=1 \) in Ref. [1], and discuss a more general nonlinear oscillation problem including the problem (1) in Ref. [9]. Kirkinis considered the problem (1) with \( \varepsilon \to 0^+ \) based on the renormalization group method [7], Liang recovered Kirkinis’s asymptotic expansion by means of the homotopy analysis method and proved Kirkinis’s solution is optimal [8].
The rest of the paper is arranged as follows. In Section 2, the HAM approximate solution given in Ref. [8] is recalled. Section 3 is devoted to construct the PPM approximate solution. In Section 4, comparisons are made between the HAM and PPM approximate solutions to show the advantages of the PPM. Conclusions are provided in Section 5.

2. HAM approximate solution

In this section, we recall the HAM approximate solution, see [8] for reference. By introducing the transformation

\[ y(t) = u(\tau) \]

\[ \tau = \omega t \]

in (1), (1) becomes

\[ \omega^2 u''(\tau) + u(\tau) + \varepsilon u'(\tau) + \varepsilon u(\tau) = 0, \quad u(0) = 1, \quad u'(0) = 0, \tag{2} \]

where the prime means the derivative with respect to \( \tau \). Then using the HAM by setting \( \varepsilon = 2 \) for instance and choosing the initial guess

\[ u_0(\tau) = \cos(\tau), \tag{3} \]

the HAM approximate solution for problem (1) is obtained as follows:

\[ y(t) = u_0(\omega t) + u_1(\omega t) + u_2(\omega t) + \cdots, \tag{4} \]

where

\[ u(\tau) = u_0(\tau) + u_1(\tau) + u_2(\tau) + \cdots \tag{5} \]

and

\[ \omega = \omega_0 + \omega_1 + \omega_2 + \cdots \tag{6} \]

with

\[ u_1(\tau) = \frac{1}{40} c_0 \cos(\tau) - \frac{1}{40} c_0 \cos(3\tau), \]

\[ u_2(\tau) = \left( \frac{1}{40} c_0 + \frac{39}{1600} c_0^2 \right) \cos(\tau) - \left( \frac{1}{40} c_0 + \frac{1}{1600} c_0^2 \right) \cos(3\tau) + \frac{1}{1600} c_0^2 \cos(5\tau), \tag{7} \]

\[ u_3(\tau) = \left( \frac{1}{40} c_0 + \frac{39}{800} c_0^2 + \frac{809}{64000} c_0^3 \right) \cos(\tau) - \left( \frac{1}{40} c_0 + \frac{1}{20} c_0^2 + \frac{32000}{809} c_0^3 \right) \cos(3\tau) \]

\[ + \left( \frac{1}{800} c_0^2 + \frac{1}{800} c_0^3 \right) \cos(5\tau) - \frac{1}{64000} c_0^3 \cos(7\tau), \cdots \]

and

\[ \omega_0 = \frac{\sqrt{10}}{2}, \quad \omega_1 = \frac{3\sqrt{10}}{800} c_0, \quad \omega_2 = \frac{3\sqrt{10}}{64000} c_0(800 + 797 c_0), \cdots \tag{8} \]

To determine the optimal value of the convergence-control parameter \( c_0 \) for the HAM approximate solution (4), one used the averaged residual error suggested in Ref. [10]. Here, the averaged residual error of a HAM approximation \( Y(t; c_0) \) to the problem (1) on the interval \([0,d]\) is defined by

\[ E(c_0) = \frac{1}{n} \sum_{j=0}^{n} \left[ Y(t_j; c_0) + Y(t_j; c_0) + 2Y(t_j; c_0) \right]^2, \tag{9} \]

where \( d=20, n=20 \) and \( t_j = jd/n = j \) are considered throughout this work. Then the optimal value of \( c_0 \) can be determined by solving the equation

\[ \frac{\partial E(c_0)}{\partial c_0} = 0. \tag{10} \]

After obtaining the optimal values of \( c_0 \) for the \( r^\text{th} \)-order HAM approximation expressed by (4), the optimal \( r^\text{th} \)-order HAM approximations \( U_r(t; \tau) \) and \( Y_r(t; \tau) \) with respect to \( y(t) \) and \( u(\tau) \) are also determined. For instance, by solving (10), the optimal values of \( c_0 \) in the \( r^\text{th} \)-order HAM approximation expressed by (4) are \(-0.9763023212, -1.029483878, -1.031268436, \) for \( r=1,2,3 \) respectively. Thus, with respect to \( u(\tau) \), we obtain
(i) the optimal 1st-order HAM approximation
\[ U^{[1]}(\tau) = 0.975592442 \cos(\tau) + 0.024407558 \cos(3\tau), \]  

(ii) the optimal 2nd-order HAM approximation
\[ U^{[2]}(\tau) = 0.974359334 \cos(\tau) + 0.02497826752 \cos(3\tau) + 0.0006623981594 \cos(5\tau), \]  

(iii) the optimal 3rd-order HAM approximation
\[ U^{[3]}(\tau) = 0.974050504 \cos(\tau) + 0.02530923058 \cos(3\tau) + 0.000623128570 \cos(5\tau) + 0.000171371602 \cos(7\tau). \]

3. Novel PPM approximate solution

In what follows, we will regard the above optimal HAM approximation \( U^{[r]}(\tau) \) as a new initial guess to (2), then use it to obtain the other finite-order HAM approximations, i.e. the PPM approximations.

To guarantee the consistency of symbols, we denote the new initial guess by \( U_0^{[r]}(\tau) \), i.e.
\[ U_0^{[r]}(\tau) = U^{[r]}(\tau). \]

Based on Ref. [8], we construct the zeroth-order deformation equation
\[ (1-q)L_0[\phi(\tau; q) - U_0^{[r]}(\tau)] = q c_0 N[\phi(\tau; q)], \]  

where \( q \in [0,1] \) is an embedding parameter, \( c_0 \) is a convergence-control parameter which can be used to adjust and control the convergence region and rate of the resulting series solution [1], \( \phi(\tau; q) \) is an unknown function with respect to \( \tau \) and \( q \), \( L \) and \( N \) are linear and nonlinear operators chosen by
\[ L[\phi(\tau; q)] = \frac{\partial^2}{\partial \tau^2} \phi(\tau; q) + \phi(\tau; q), \]
\[ N[\phi(\tau; q)] = \Omega^2(q) \frac{\partial^2}{\partial \tau^2} \phi(\tau; q) + \phi(\tau; q) + 2\phi(\tau; q)^3. \]

When \( q=0 \) and \( q=1 \), from the zero-order deformation (15) we have \( \phi(\tau; 0) = U_0^{[r]}(\tau) \) and \( \phi(\tau; 1) = u(\tau) \).

Thus, as \( q \) increases from 0 to 1, the solution \( \phi(\tau; q) \) varies from the new initial guess \( U_0^{[r]}(\tau) \) to the exact solution \( u(\tau) \).

Then expanding \( \Omega(q) \) and \( \phi(\tau; q) \) in Taylor series at \( q=0 \),
\[ \phi(\tau; q) = U_0^{[r]}(\tau) + \sum_{k=1}^{+\infty} U_k^{[r]}(\tau) q^k, \]
\[ \Omega(q) = \omega_0 + \sum_{k=1}^{+\infty} \omega_k q^k, \]
and by differentiation from (15), we gain the higher-order deformation equation
\[ L[u_k^{[r]}(\tau)] = \chi_k L[u_{k-1}^{[r]}(\tau)] + \frac{c_0}{(k-1)!} \frac{\partial^{k-1} N[\phi(\tau; q)]}{\partial q^{k-1}} \bigg|_{q=0}, \]  

with the initial conditions
\[ u_k^{[r]}(0) = 0, \quad u_k^{[r]'}(0) = 0. \]

Solving the higher-order deformation equation (18) and eliminating the secular term \( \tau \sin \tau \), \( u_k^{[r]}(\tau) \) and \( \omega_k \) will be determined uniquely one by one. Note that equation (18) is linear, and therefore it can be solved easily, especially by means of symbolic computation software such as Maple, Matlab, Mathematica and so on. Then assuming \( c_0 \) is so properly chosen that the series (17) converge at \( q=1 \), a new series solution for (2) with the initial guess \( U_0^{[r]}(\tau) \) can be obtained in the form
\[ u(\tau) = U_0^{[r]}(\tau) + \sum_{k=1}^{+\infty} u_k^{[r]}(\tau), \quad \omega = \omega_0 + \sum_{k=1}^{+\infty} \omega_k. \]

Finally, we obtain the following series solution for problem (1)
\[
y(t) = u_0^r(\omega t) + u_1^r(\omega t) + u_2^r(\omega t) + \cdots,
\]
where \(u\) and \(\omega\) are given in (20). Correspondingly, we get the \([m,r]^{th}\)-order PPM approximation
\[
y^{[m,r]}(t) = u_0^r(\Omega^{[m-1]} t) + \sum_{k=1}^{m} u_k^r(\Omega^{[m-1]} t),
\]
for nonlinear problem (1), where \(Q^{[m-1]} = \omega_0 + \omega_1 + \ldots + \omega_{m-1}\).

4. Results and discussions

In this section, the resulting PPM solutions are compared with the HAM one by solving the same number of the higher-order deformation equations. For instance, by solving four higher-order deformation equations, and using the optimal HAM approximations given in (11)-(13) as the new initial guesses, the \([1,3]^{th}\)-order, \([2,2]^{th}\)-order, \([3,1]^{th}\)-order PPM approximations can be obtained.

Consequently, by solving the averaged residual error (9) and (10) with respect to the corresponding approximations, we have

(i) the optimal \(4^{th}\)-order HAM approximation
\[
Y_{1}\{t\} = 0.9740629852\cos(\omega t) + 0.02527934614\cos(3\omega t) + 0.000641686780\cos(5\omega t) + 0.0000155448550\cos(7\omega t) + 0.000004375280234\cos(9\omega t)
\]
with \(\omega = 1.569098106\);

(ii) the optimal \([1,3]^{th}\)-order PPM approximation
\[
Y_{1,3}\{t\} = 0.9740560206\cos(\omega t) + 0.02528712832\cos(3\omega t) + 0.0006402377317\cos(5\omega t) + 0.0001618866077\cos(7\omega t) + 4.143149296 \cdot 10^{-7}\cos(9\omega t) + 1.017129529 \cdot 10^{-8}\cos(11\omega t) + 1.892757873 \cdot 10^{-10}\cos(13\omega t) + 2.202960033 \cdot 10^{-12}\cos(15\omega t) + 1.539496732 \cdot 10^{-14}\cos(17\omega t) + 5.979710042 \cdot 10^{-17}\cos(19\omega t) + 1.095566744 \cdot 10^{-19}\cos(21\omega t)
\]
with \(\omega = 1.569106623\);

(iii) the optimal \([2,2]^{th}\)-order PPM approximation
\[
Y_{2,2}\{t\} = 0.9740565315\cos(\omega t) + 0.02528682619\cos(3\omega t) + 0.0006401148275\cos(5\omega t) + 0.0001611117222\cos(7\omega t) + 4.062667529 \cdot 10^{-7}\cos(9\omega t) + 9.612380840 \cdot 10^{-9}\cos(11\omega t) + 2.167905583 \cdot 10^{-10}\cos(13\omega t) + 3.877010533 \cdot 10^{-12}\cos(15\omega t) + 5.732277341 \cdot 10^{-14}\cos(17\omega t) + 6.057125550 \cdot 10^{-16}\cos(19\omega t) + 5.125503127 \cdot 10^{-18}\cos(21\omega t) + 3.025637684 \cdot 10^{-20}\cos(23\omega t) + 1.131544941 \cdot 10^{-22}\cos(25\omega t)
\]
with \(\omega = 1.569112844\);

(iv) the optimal \([3,1]^{th}\)-order PPM approximation
\[
Y_{3,1}\{t\} = 0.9740564229\cos(\omega t) + 0.02528674921\cos(3\omega t) + 0.0006403954002\cos(5\omega t) + 0.0001601744009\cos(7\omega t) + 4.070537415 \cdot 10^{-7}\cos(9\omega t) + 8.317266408 \cdot 10^{-9}\cos(11\omega t) + 1.514719001 \cdot 10^{-10}\cos(13\omega t) + 2.460191352 \cdot 10^{-12}\cos(15\omega t) + 2.971094079 \cdot 10^{-14}\cos(17\omega t) + 3.087735518 \cdot 10^{-16}\cos(19\omega t) + 2.315934125 \cdot 10^{-18}\cos(21\omega t)
\]
with \(\omega = 1.569111563\).

The corresponding optimal values of the parameter \(c_0\) and the minimal values of the averaged residual error \(E(c_0)\) in terms of (23)-(26) are given in Table 1. By comparing the averaged residual
errors, it is concluded that all the PPM approximations (24)-(26) is better than the HAM approximation (23), and among them the optimal \([1,3]^{th}\)-order PPM approximation (24) has a smallest error, which is about \(10^6\) smaller than that of the optimal 4th-order HAM approximation (23).

Related plots are shown in Figure 1. From Figure 1 it is seen that they are all agree well with the numerical one given by the Runge-Kutta-Fehlberg 4-5 technique.

Table 1. The optimal values of the convergence-control parameter \(c_0\) and the minimal values of the averaged residual error \(E(c_0)\) in terms of (23)-(26).

| \(c_0\)                             | \(E(c_0)\)        |
|------------------------------------|-------------------|
| 4th-order HAM approximation        | -1.028753818      | 1.565189786E-8 |
| [1,3]^{th}-order PPM approximation | -1.048611349      | 2.766903060E-14|
| [2,2]^{th}-order PPM approximation | -1.001380204      | 5.838786088E-13|
| [3,1]^{th}-order PPM approximation | -.9970361641      | 1.050282547E-11|

Figure 1. Plots of different approximations expressed by (23)-(26).

5. Conclusions
Some novel analytical approximate solutions have been successfully constructed to the Duffing oscillator using the piecewise perturbation method. Compared to the traditional HAM solution, the obtained PPM solutions possess higher accuracy and faster convergence rate, as shown in Table 1. It indicates that the PPM is powerful and efficient in finding approximate solutions for nonlinear differential equations.

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