Spin 1/2 properties of massless particles of any spin

Alexander Gersten and Amnon Moalem
Department of Physics, Ben Gurion University of the Negev, Beer-Sheva, Israel

Abstract. In previous papers we have derived wave equations for massless particles of any spin. In this paper we found that all these equations, are equations of the helicity operator and are similar in form to the spin one-half equation. In these equations the helicity operators are reducible representations of the helicity operator of spin one-half. The helicity of massless particles of any spin can have only projections in the forward and backward direction of the momentum. In order to have equations with this property, subsidiary conditions have to be imposed. We have developed a first quantized equations for massless particles of any spin for which the subsidiary conditions were implicitly included in the main equation. The helicity operators of these equations are reducible representations of the spin one half helicity operator. A new feature of this paper is to use this reducibility to present higher spin equation with its lower spin equivalent. As an application we recover the bi-quaternionic form of Maxwell’s equations (spin 1 equation in a form of spin one-half equation).

Key words: wave equations, massless particles, any spin, helicity equations, bi-quaternions.

1. Introduction
In our previous papers [5], [6], [7] we have derived wave equations for massless particles of any spin. In this paper we found that all these equations, are equations of the helicity operator and are similar in form to the spin one-half equation. In these equations the helicity operators are reducible representations of the helicity operator of spin one-half. Moreover, for any spin, there are only two helicity equations similar to the spin one-half equations.

The wavefunction of particle with spin \( s \) should describe the \( 2s + 1 \) components of the spin. For free particles the helicity is the projection of the spin in the direction of the momentum. For massless particles only the helicities in the direction and opposite to the direction of motion are different from zero, all other components vanish [1], [2], [3]. Therefore for spins \( s > \frac{1}{2} \) in addition to the \( 2s + 1 \) equations for the spin components, one has to add \( 2s - 1 \) conditions to eliminate the vanished helicities, these are the subsidiary conditions. Our helicity equations include implicitly the subsidiary conditions.

In the following we will work in the flat Minkowski space of special relativity with coordinates \( x_\mu, \mu = 0, 1, 2, 3 \) and use the short hand notation for functions,

\[ \psi \equiv \psi(x_\mu). \]

In Section 2 we present the Maxwell’s equations as wave equations of the helicity operator. In Section 3 we generalize this result for any spin. In Section 4 we give the basic properties of quaternions and bi-quaternions and we show how bi-quaternions can be used in the description of the helicity operators.
In Section 5 We derive the wave bi-quaternionic Maxwell’s equations. Here the Helicity operator is the same as the spin one-half helicity operator, the difference is the form of the wave function which is a bi-quaternion.

In Section 6 we summarize our results.

2. Maxwell’s equations as wave equations of the helicity operator

Maxwell’s equations in the covariant form [4] in Gaussian system units are,

\[ \partial_\mu F^{\mu \nu} = \frac{4\pi}{c} j^\nu, \quad \partial_\mu \tilde{F}^{\mu \nu} = 0, \quad \nu = 0, 1, 2, 3, \]  

where \( \partial_\mu = \frac{1}{c} \partial_\mu \) and the antisymmetric tensor \( F^{\mu \nu} \) and its dual \( \tilde{F}^{\mu \nu} \) are defined via the electric and magnetic fields \( \mathbf{E} \) and \( \mathbf{B} \) respectively as,

\[
(F^{\mu \nu}) = \begin{pmatrix}
0 & -E_x & -E_y & -E_z \\
E_x & 0 & -B_z & B_y \\
E_y & B_z & 0 & -B_x \\
E_z & -B_y & B_x & 0 \\
\end{pmatrix}, \quad (1)
\]

\[
\tilde{F}^{\mu \nu} = \frac{1}{2} \epsilon^{\mu \nu \alpha \beta} F_{\alpha \beta} = \begin{pmatrix}
0 & -B_x & -B_y & -B_z \\
B_x & 0 & E_z & -E_y \\
B_y & -E_z & 0 & E_x \\
B_z & E_y & -E_x & 0 \\
\end{pmatrix}, \quad (2)
\]

where \( \epsilon^{\mu \nu \alpha \beta} \) is the totally antisymmetric tensor \( (\epsilon^{0123} = 1, \quad \epsilon^{\mu \nu \alpha \beta} = -\epsilon^{\nu \mu \alpha \beta}) \). The sum of the \( F^{\mu \nu} \) and \( i\tilde{F}^{\mu \nu} \) is the self-dual antisymmetric tensor, it depends only on the combination,

\[
\Psi = (\mathbf{E} + i\mathbf{B}), \quad (4)
\]

\[
(F^{\mu \nu} + i\tilde{F}^{\mu \nu}) = \begin{pmatrix}
0 & -E_x - iB_z & -E_y - iB_y & -E_z - iB_z \\
E_x + iB_z & 0 & iE_z - B_z & B_y - iE_y \\
E_y + iB_y & B_z - iE_z & 0 & iE_x - B_x \\
E_z + iB_z & iE_y - B_y & B_x - iE_x & 0 \\
\end{pmatrix}, \quad (5)
\]

\[
= \begin{pmatrix}
0 & -\Psi_x & -\Psi_y & -\Psi_z \\
\Psi_x & 0 & i\Psi_y & -i\Psi_x \\
\Psi_y & -i\Psi_z & 0 & i\Psi_x \\
\Psi_z & i\Psi_y & -i\Psi_x & 0 \\
\end{pmatrix}, \quad (6)
\]

The self-dual Maxwell equations take the form,

\[
\partial_\mu \left( F^{\mu \nu} + i\tilde{F}^{\mu \nu} \right) = - (\alpha_\mu)^{\nu k} \partial_\mu \Psi_k = \frac{4\pi}{c} j^\nu, \quad \mu = 0, x, y, z, = 0, 1, 2, 3 \quad k = x, y, z, = 1, 2, 3 \]  

where,

\[
\alpha_0 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad \alpha_1 = \begin{pmatrix}
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -i \\
0 & 0 & i & 0 \\
\end{pmatrix}, \quad (7)
\]
\[
\alpha_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}
\] (8)

and the \( \Psi_k \) is one of the three components of the wave function,

\[
\Phi = \begin{pmatrix} \Psi_x \\ \Psi_y \\ \Psi_z \end{pmatrix} = \begin{pmatrix} E_x + iB_x \\ E_y + iB_y \\ E_z + iB_{zz} \end{pmatrix}.
\] (9)

Multiplying Eq. (7) by \( i\hbar c \), where \( \hbar \) is the reduced Planck constant, we obtain,

\[
(\alpha_\mu)^{\nu_k} (i\hbar c \partial_\mu) \Psi_k = -i\hbar c \frac{4\pi}{c} j^\nu.
\] (10)

Replacing \( i\hbar \partial_\mu \) by their wave energy-momentum operators, Eq. (10) takes the form,

\[
(\alpha_0 E - c\alpha \cdot p) \Phi = -i\hbar 4\pi J,
\] (11)

where \( J \) is the column of the four vector \( j^\nu \). Maxwell’s equations without sources (the photon equation) are,

\[
(\alpha_0 E - c\alpha \cdot p) \Phi = 0.
\] (12)

The structure of the \( \alpha \) matrices is,

\[
\alpha_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & (I_0) \\ 0 & 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & (S_x) \\ 0 & 0 & 0 & 0 \\ 0 & 0 \end{pmatrix},
\] (13)

\[
\alpha_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & (S_y) \\ 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}
\] (14)

\[
S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\] (15)

where \( S_i \) are the spin 1 matrices with the properties,

\[
[S_x, S_y] = iS_z, \quad [S_z, S_x] = iS_y, \quad [S_y, S_z] = iS_x, \quad S^2 = 2I_0.
\] (16)

Substituting Eqs. (13-15) in Eq. (12) we obtain,

\[
-cp \cdot \Psi = 0,
\] (17)

\[
(I_0 E - cS \cdot p) \Psi = 0.
\] (18)
Above

\[ \Psi = \begin{pmatrix} \Psi_x \\ \Psi_y \\ \Psi_z \end{pmatrix} = \begin{pmatrix} E_x + iB_x \\ E_y + iB_y \\ E_z + iB_z \end{pmatrix}. \quad (19) \]

Eq. (18) is the unconstrained helicity equation and Eq. (17) is the subsidiary condition (the perpendicularity condition) which eliminates the zero helicity component.

The \( \alpha \) matrices satisfy the same relations as the Pauli matrices, namely

\[ (\alpha_1)^2 = (\alpha_2)^2 = (\alpha_3)^2 = (\alpha_0)^2 = \alpha_0, \quad (20) \]
\[ \alpha_1 \alpha_2 = i\alpha_3 \ (\text{and cyclic permutation}), \quad (21) \]
\[ \alpha_k \alpha_l = -\alpha_l \alpha_k, \quad \text{for } k \neq l, \ k,l = 1,2,3, \quad (22) \]

therefore the \( \alpha \) matrices form a reducible representation of the Pauli matrices. Furthermore

\[ (\alpha_0 E - c \alpha \cdot p) (\alpha_0 E + c \alpha \cdot p) = (E^2 - c^2 p^2) \alpha_0, \quad (23) \]

As a result we obtain for the photon the two helicity equations:

\[ [E \alpha_0 - c p \cdot \alpha] \psi_R = 0, \quad \text{Helicity}/s = 1. \quad (24) \]
\[ [E \alpha_0 + c p \cdot \alpha] \psi_L = 0, \quad \text{Helicity}/s = -1. \quad (25) \]

One should note that both equations are equivalent to the free Maxwell’s equations.

3. Generalization to any spin

In previous papers we have generalized the helicity equations to any spin \([5],[6],[7]\). We have used the angular momentum basis of the \( D^{(s-1/2,1/2)} \) representation of the Lorentz group. This basis is the sum of the bases of spins \( s \) and spin \( s - 1 \) with

\[ (2s + 1) + (2s - 1) = 4s \text{ components}. \]

In this basis the wavefunction is,

\[ \Phi^{(4s)} = \begin{pmatrix} \psi_{s-1}^{(2s-1)} \\ \vdots \\ \psi_s^{(2s+1)} \\ \psi_{-s}^{(2s+1)} \\ \vdots \\ \psi_{-s}^{(2s-1)} \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \psi_{s}^{(2s+1)} \\ \vdots \\ \psi_{-s}^{(2s+1)} \end{pmatrix}. \quad (26) \]

The advantage of this formalism, as we have shown in Refs. \([6],[7]\) is that by equating to zero the spin \((s - 1)\) components of the wavefunction \( \psi_{(2s-1)}^{(4s)} \), the \( 2s - 1 \) subsidiary conditions (needed to eliminate the non-forward and non-backward helicities) are automatically satisfied.

The equations for any spin have the form

\[ \left[ \hat{E} \Gamma_0^{(4s)} - c \Gamma^{(4s)} \cdot \hat{p} \right] \Phi^{(4s)}_R = 0, \quad (27) \]
where the \((4s \times 4s)\) gamma matrices have the same multiplication table as the Pauli matrices, therefore they are reducible representation of the Pauli matrices. The equations have built in subsidiary conditions, which are needed to eliminate the non-forward and non-backward helicities. By equating to zero the spin \(s-1\) components of the wave function, the \(2s-1\) subsidiary conditions (needed to eliminate the non-forward and non-backward helicities), are automatically satisfied. For spins \(\frac{1}{2}\) and spin 1 they are the \(\sigma\) and the \(\alpha\) matrices respectively. For spin 2 we have found,

\[ \left[ \hat{E} \Gamma_0^{(4s)} + c \Gamma^{(4s)} \cdot \hat{p} \right] \Phi_L^{(4s)} = 0, \]  

where \(\Phi_L^{(4s)}\) are the \((4s \times 4s)\) gamma matrices have the same multiplication table as the Pauli matrices, therefore they are reducible representation of the Pauli matrices. The equations have built in subsidiary conditions, which are needed to eliminate the non-forward and non-backward helicities. By equating to zero the spin \(s-1\) components of the wave function, the \(2s-1\) subsidiary conditions (needed to eliminate the non-forward and non-backward helicities), are automatically satisfied. For spins \(\frac{1}{2}\) and spin 1 they are the \(\sigma\) and the \(\alpha\) matrices respectively. For spin 2 we have found,

\[ \Gamma_x^{(8)} = \frac{1}{2} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & \sqrt{3} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{\sqrt{3}}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\sqrt{3} \\ \sqrt{3} & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & -1 & 0 & -\sqrt{3} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{\sqrt{3}}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & -\sqrt{3} & 0 & 0 & 0 & -\sqrt{3} & 0 & -1 \\ 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \]  

\[ \Gamma_y^{(8)} = \frac{i}{2} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & -\sqrt{3} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{\sqrt{3}}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\sqrt{3} \\ \sqrt{3} & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 1 & 0 & -\sqrt{3} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & \sqrt{3} & 0 & \sqrt{3} & 0 \\ 0 & \sqrt{3} & 0 & 0 & 0 & \sqrt{3} & 0 & -1 \\ 0 & 0 & \sqrt{3} & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \]  

\[ \Gamma_z^{(8)} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ -\sqrt{3} & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{3} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \]  

and \(\Gamma_0^{(8)}\) is the \((8 \times 8)\) unit matrix.

4. Quaternions, bi-quaternions and Paulions
The quaternions usually are written in terms of three complex numbers as,

\[ Q = c_0 1 + c_1 i + c_2 j + c_3 k, \]

\[ i^2 = j^2 = k^2 = -1, \quad ij = -ji, \quad ik = -ki, \quad kj = -jk, \quad ijk = -1, \]
where the $c_\mu$ are real coefficients. Let us rewrite Eq. (32) as,

\[ Q = c_0 q_0 + c_1 q_1 + c_2 q_2 + c_3 q_3, \]  
(34)

\[ q_0 \equiv 1, \quad q_1 \equiv i, \quad q_2 \equiv j, \quad q_3 \equiv k. \]  
(35)

The conjugate quaternion is defined as,

\[ Q^C = c_0 q_0 - c_1 q_1 - c_2 q_2 - c_3 q_3. \]  
(36)

Using Eqs. (32-36) we obtain,

\[ Q^C Q = q_0 \left( c_0^2 + c_1^2 + c_2^2 + c_3^2 \right). \]  
(37)

Therefore quaternions are suitable for the metric $[1, 1, 1, 1]$. In Minkowski’s space of special relativity we will use the metric $[1, -1, -1, -1]$. Wave equations for massless particles can be derived by factorizing (the Klein-Gordon operator),

\[ p_0^2 - p_1^2 - p_2^2 - p_3^2 = 0, \quad p_0 = E/c \]  
(38)

The quaternions are not suitable for factorizing this equation. The factorization can be done with bi-quaternions, for which the coefficients $c_\mu$ in Eq. (32) or Eq.(35) can be complex. The bi-quaternions suitable for the Minkowski metric, which can factorize Eq.(38), are

\[ R = c_0 r_0 + c_1 r_1 + c_2 r_2 + c_3 r_3, \]  
(39)

where the $c_\mu$ are real coefficients and their relation to quaternions is,

\[ r_0 = q_0 = 1, \quad r_1 = i q_1, \quad r_2 = i q_2, \quad r_3 = i q_3. \]  
(40)

The conjugate is,

\[ R^C = c_0 r_0 - c_1 r_1 - c_2 r_2 - c_3 r_3, \]  
(41)

and

\[ R^C R = RR^C = r_0 \left( c_0^2 - c_1^2 - c_2^2 - c_3^2 \right). \]  
(42)

which allows factorization of Eq.(38). We will call these special bi-quaternions as Paulions as they have a matrix representation in terms of Pauli matrices,

\[ \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  
(43)

\[ R_\sigma = c_0 \sigma_0 + c_1 \sigma_x + c_2 \sigma_y + c_3 \sigma_z, \quad R_\sigma^C = c_0 \sigma_0 - c_1 \sigma_x - c_2 \sigma_y - c_3 \sigma_z, \]  
(44)

and

\[ R_\sigma R_\sigma^C = R_\sigma^C R_\sigma = \left( c_0^2 - c_1^2 - c_2^2 - c_3^2 \right) \sigma_0. \]

The two component massless neutrino equations can be derived from the decomposition,

\[ (E^2 - c^2 p^2) \sigma_0 \psi = [E \sigma_0 - c \sigma \cdot p] [E \sigma_0 + c \sigma \cdot p] \psi = 0, \]  
(45)
where $E$ is the energy and $\mathbf{p}$ the momentum operators.

From Eq. (45) two helicity equations (for positive energies) are obtained,

$$[E\sigma_0 - c\mathbf{p}\cdot\sigma] \psi_R = 0, \quad \text{Helicity}/s = 1. \quad (46)$$

$$[E\sigma_0 + c\mathbf{p}\cdot\sigma] \psi_L = 0, \quad \text{Helicity}/s = -1. \quad (47)$$

5. The bi-quaternionic Maxwell’s equations

In Section 2 the free Maxwell’s equations were converted to Eq.(12),

$$\left(\alpha_0 E - c\mathbf{a} \cdot \mathbf{p}\right) \Phi = 0. \quad \Phi = \begin{pmatrix} 0 \\ \Psi_x \\ \Psi_y \\ \Psi_z \end{pmatrix} = \begin{pmatrix} 0 \\ E_x + iB_x \\ E_y + iB_y \\ E_z + iB_z \end{pmatrix},$$

where the $\alpha$ matrices are given by Eq.(8).

One can check that the equation,

$$\left(\alpha_0 E - c\mathbf{a} \cdot \mathbf{p}\right) (-\alpha_1 \Psi_x - \alpha_2 \Psi_y - \alpha_3 \Psi_z) = 0, \quad \Psi = \begin{pmatrix} \Psi_x \\ \Psi_y \\ \Psi_z \end{pmatrix} = \begin{pmatrix} E_x + iB_x \\ E_y + iB_y \\ E_z + iB_z \end{pmatrix}, \quad (48)$$

is also equivalent to Maxwell’s equations. The difference between Eq.(12) and Eq.(48) is the way the wave function is presented. In Eq.(12) the wave function is a vector with complex coefficients, while in Eq.(48) the wave function has a property of a bi-quaternion, namely as the $\alpha_i$ matrices form a reducible representation of of the Pauli $\sigma_i$ matrices, Eq. (48) is a reducible representation of the following bi-quaternionic Maxwell’s equations,

$$\left(\sigma_0 E - \mathbf{a} \cdot \mathbf{p}\right) (-\mathbf{a} \cdot \Psi) = 0. \quad (49)$$

Eq. (49) is similar to the spin one-half massless neutrino equation Eq. (46), namely the helicity operator is the same as for the spin one-half equation, but the wave function is different, it is a bi-quaternion.

It is worthwhile to note that Maxwell has derived his early 20 equations using quaternions [8]. In modern presentations a different form of Maxwell’s original equations is used [4]. Our work is related to modern developments.

6. Summary and conclusions

First quantized equations for massless particles of any spin $s$ were developed. The helicities of massless particles of any spin can have only projections in the forward and backward direction of the momentum. In order to have equations with this property, subsidiary conditions have to be imposed. A formalism has been developed in which the subsidiary conditions were implicitly included in the main equation. The resulting equations for the two helicities are,

$$\left[\hat{E}\Gamma_0^{(4s)} - c\hat{\mathbf{p}}\cdot\mathbf{\Gamma}^{(4s)}\right] \Phi_R^{(4s)} = 0, \quad \left[\hat{E}\Gamma_0^{(4s)} + c\hat{\mathbf{p}}\cdot\mathbf{\Gamma}^{(4s)}\right] \Phi_L^{(4s)} = 0,$$

and they are similar to the spin one-half massless (neutrino) equations,

$$[E\sigma_0 - c\mathbf{p}\cdot\sigma] \psi_R = 0, \quad [E\sigma_0 + c\mathbf{p}\cdot\sigma] \psi_L = 0,$$
The helicity operator in Eq. (50) for any spin, has (multiple) eigenvalues for forward and backward momentum projections only, and is accordingly a reducible representation of the spin one-half helicity operator.

As an application we have recovered the bi-quaternionic form of Maxwell’s equations Eq. (49), similar to the spin one-half massless neutrino equations Eq. (51),

\[(\sigma_0 E - c\sigma \cdot p)(-\sigma \cdot \Psi) = 0,\]

namely the helicity operator is the same as for the spin one-half equation, but the wave function is a bi-quaternion.

Additional work is needed to understand the mechanism of presenting the spin equations with a lower spin equivalent equations.

References
[1] Wigner E.P., On Unitary Representations of the Inhomogeneous Lorentz Group. *Annals of Mathematics*, **40**, 1939, pp.149-204
[2] Weinberg S. *The Quantum Theory of Fields. Vol. I.*, Cambridge University Press, New York (1995), pp. 55-91
[3] LIU ChangLi and GE FengJun, arXiv:1403.2698v4 [physics.gen-ph] 28 Apr. 2014
[4] Jackson J.D., *Classical Electrodynamics, Third Edition*, John Wiley & sons, New York (1999), p.781
[5] Gersten A. and Moalem A., *Journal of Physics*: Conference Series **330** (2011) 012010.
[6] Gersten A. and Moalem A., *Journal of Physics*: Conference Series **437** (2013), 012019.
[7] Gersten A. and Moalem A., *Journal of Physics*: Conference Series **615** (2015) 012011.
[8] Maxwell, James Clerk, *A treatise on electricity and magnetism*, Oxford, Clarendon Press (1873).