The limiting behavior of some infinitely divisible exponential dispersion models

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Abstract

Consider an exponential dispersion model (EDM) generated by a probability µ on [0, ∞) which is infinitely divisible with an unbounded Lévy measure ν. The Jorgensen set (i.e., the dispersion parameter space) is then \( \mathbb{R}^+ \), in which case the EDM is characterized by two parameters: \( \theta_0 \) the natural parameter of the associated natural exponential family and the Jorgensen (or dispersion) parameter \( t \). Denote by \( EDM(\theta_0, t) \) the corresponding distribution and let \( Y_t \) is a r.v. with distribution \( EDM(\theta_0, t) \). Then if \( \nu((x, \infty)) \sim -\ell \log x \) around zero we prove that the limiting law \( F_0 \) of \( Y_t^{-t} \) as \( t \to 0 \) is of a Pareto type (not depending on \( \theta_0 \)) with the form \( F_0(u) = 0 \) for \( u < 1 \) and \( 1 - u^{-\ell} \) for \( u \geq 1 \). Such a result enables an approximation of the distribution of \( Y_t \) for relatively small values of the dispersion parameter of the corresponding EDM. Illustrative examples are provided.

Keywords: Exponential dispersion model; infinitely divisible distributions; limiting distributions; natural exponential family.

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1 Introduction

Let \( \{F_t : 0 < t < t_0 \leq \infty\} \) be a family of distributions associated with positive r.v.'s \( \{Y_t\} \) and Laplace transforms (LT's) \( L_t(u) = E(e^{-uY_t}) \). Also let \( Y_0 \) be a r.v. with distribution \( F_0 \). Bar-Lev and Enis (1987, Theorem 1) showed that \( Y^{-t} \overset{D}{\rightarrow} Y_0 \) as \( t \to 0 \) (where \( \overset{D}{\rightarrow} \) designates a convergence in distribution) iff

\[
\lim_{t \to 0} L_t(u^{1/t}) = \bar{F}_0(u) = 1 - F_0(u) \tag{1}
\]

at all continuity points of \( F_0 \). As Bar-Lev and Enis indicated, such a result can be viewed as a "centralization" problem in the following sense. In many cases the limiting distribution of \( Y_t \) as \( t \to 0 \) is degenerate. A measurable transformation \( g_t(Y_t) \) is then sought whose limiting distribution is non-degenerate. Accordingly, their Theorem 1 suggests a consideration of \( g_t(Y_t) = Y_t^{-t} \) (or, equivalently, of \( -t \ln Y_t \)) whose limiting distribution is non-degenerate (provided that (1) is satisfied). Bar-Lev and Enis presented several examples which satisfy (1). However, these examples heavily depend on the explicit (and relatively 'nice') form of \( L_t \).

A natural question then arises: Can one delineate subclasses of distributions which satisfy (1), regardless of the explicit form of \( L_t \)? Indeed, in this note we provide such a subclass, namely a subclass of exponential dispersion models (EDM's) generated by a probability \( \mu \) on \([0, \infty)\) which is infinitely divisible of type 1 (c.f., Jorgensen, 1987, 1997, 2006, and Letac and Mora, 1990). For such a subclass, the EDM is characterized by two parameters: \( \theta_0 \) the natural parameter of the associated natural exponential family (NEF) and the Jorgensen (or, equivalently, the dispersion parameter) \( t \in \mathbb{R}^+ \). We denote such a subclass of distributions by \( EDM(\theta_0, t) \) and prove that if \( Y_t \) has a distribution in \( EDM(\theta_0, t) \) then \( Y_t^{-t} \overset{D}{\rightarrow} Y_0 \) as \( t \to 0 \), where the distribution \( F_0 \) of \( Y_0 \) is of a Pareto type (not depending on \( \theta_0 \)) with the form

\[
F_0(u) = \begin{cases} 
0, & \text{if } u < 1, \\
1 - u^{-\ell}, & \text{if } u \geq 1,
\end{cases}
\]

for some \( \ell > 0 \). Such a result enables an approximation of the distribution of \( Y_t \) for relatively small values of the dispersion parameter of the corresponding EDM.

This note is organized as follows. In Section 2 we first introduce some preliminaries on NEF’s and EDM’s and then present our main result. Section 3 is devoted to some examples.
2 Preliminaries and the main result

We first briefly introduce some preliminaries related to NEF’s and their associated EDM’s. Let $\mu$ be a probability measure on $\mathbb{R}$. Assume that the effective domain of $\mu$ has a nonempty interior, i.e.,

$$\Theta \doteq \text{int } D = \left\{ \theta \in \mathbb{R} : L(\theta) = \int_{\mathbb{R}} e^{-\theta x} \mu(dx) < \infty \right\} \neq \emptyset.$$ 

The NEF generated by $\mu$ is then given by the set of probabilities

$$\left\{ P(\theta, \mu)(dx) = \frac{e^{-\theta x}}{L(\theta)} \mu(dx), \ \theta \in D \right\}.$$

Note that since $\mu$ is a probability measure then $0 \in D$. The Jorgensen set is defined by

$$\Lambda = \left\{ t \in \mathbb{R}^+ : L^t \text{ is a LT of some measure } \mu_t \text{ on } \mathbb{R} \right\},$$

whereas the corresponding EDM is the class of probabilities

$$\left\{ P(\theta, t, \mu_t)(dx) = \frac{e^{-\theta x}}{L^t(\theta)} \mu_t(dx), \ \theta \in D, t \in \Lambda \right\}, \quad (2)$$

Note that the class of EDM’s is abundant since any probability measure with a LT generates an EDM. An EDM is therefore parameterized by the two parameters $(\theta, t) \in D \times \Lambda$, where $\theta$ is the natural parameter of the corresponding NEF and $t$ is termed the dispersion parameter. EDM’s have a variety of applications in various areas, in particular, in generalized linear models (replacing the normal model as the error model distribution) and actuarial studies. Note that if $\mu$ is infinitely divisible then the Jorgensen set (or, equivalently, the dispersion parameter space) is $\Lambda = \mathbb{R}^+$. Also note that if $Y_t$ is a r.v. with distribution in (2) then its LT is given by

$$E(e^{-sY_t}) = \left( \frac{L(\theta + s)}{L(\theta)} \right)^t. \quad (3)$$

Now we consider the case where $\mu$ is infinitely divisible law of type 1 concentrated on $\mathbb{R}^+$. By this we mean that there exists an unbounded positive measure $\nu$ on $(0, \infty)$ such that for $\theta \geq 0$ one has $L(\theta) = \int_0^\infty e^{-\theta x} \mu(dx) = e^{k(\theta)}$ with

$$k(\theta) = -\int_0^\infty (1 - e^{-\theta x}) \nu(dx) \text{ and } \int_0^\infty \min(1, x) \nu(dx) < \infty.$$
ν is the Lévy measure of μ. The Lévy process associated with such a μ is sometimes called a pure jump subordinator (in this respect, of Lévy measures for NEF’s, see also Kokonendji and Khoudar, 2006). Note that this implies that limₜ→∞ k(θ) = −∞ since ν is unbounded and therefore limₜ→∞ L(θ) = 0 and μ({0}) = 0. We are now ready to present our main result relating to the limiting distribution of Y⁰ as t → 0.

Proposition 1 Assume that μ is an infinitely divisible probability measure of type 1. Also assume that

\[ G(x) = \int_x^\infty (1 - e^{-\theta x}) \nu(dy) = \nu((x, \infty)) \]

is such that

\[ \lim_{x \to 0} \frac{G(x)}{\log x} = -\ell \]

for some \( \ell > 0 \). Let \( \theta_0 \geq 0 \), then

\[ \lim_{t \to 0} \left( \frac{L(\theta_0 + u^{1/t})}{L(\theta_0)} \right)^t = \begin{cases} 1, & \text{if } u < 1, \\ u^{-\ell}, & \text{if } u \geq 1, \end{cases} \]

implying by (1) that \( Y_i \overset{D}{\to} Y_0 \) as \( t \to 0 \), where the distribution \( F_0 \) of \( Y_0 \) is given by

\[ F_0(u) = \begin{cases} 0, & \text{if } u < 1, \\ 1 - u^{-\ell}, & \text{if } u \geq 1, \end{cases} \]

Proof. We first prove (4) for \( \theta_0 = 0 \). For this, we give another presentation of \( k \) in terms of \( G \) which is obtained by an integration by parts. For \( \epsilon > 0 \) consider the Stieltjes integral

\[ k(\theta) = -\int_{\epsilon^+}^\infty (1 - e^{-\theta x}) \nu(dx) = (e^{-\theta \epsilon} - 1)G(\epsilon) - \theta \int_\epsilon^\infty e^{-\theta x} G(x)dx. \]

Since for \( \epsilon \to 0 \) we have \( (e^{-\theta \epsilon} - 1) \sim -\theta \epsilon \) and \( G(\epsilon) \sim -\ell \log \epsilon \) we get

\[ k(\theta) = \lim_{\epsilon \to 0} k(\epsilon) = -\theta \int_0^\infty e^{-\theta x} G(x)dx. \]

If \( 0 < u < 1 \) we have \( \lim_{t \to 0} u^{1/t} = 0 \). Since μ is a probability \( L(0) = 1 \) and thus \( \lim_{t \to 0} L(u^{1/t})t = 1 \). If \( u = 1 \) we also have that \( \lim_{t \to 0} L(1)^t = 1 \). If \( u > 1 \), We fix an arbitrary \( 0 < \epsilon < \ell \). By the definition of \( \ell \) there exists
0 < \eta < 1 \text{ such that if } 0 < x < \eta \text{ then } -(\ell - \varepsilon) \log x < G(x) < -(\ell + \varepsilon) \log x.

We now use (6) for writing

\[ \left| k(\theta) + \theta \int_\eta^\infty e^{-\theta x} G(x) dx - \ell \theta \int_0^\eta e^{-\theta x} \log x dx \right| < -\varepsilon \theta \int_0^\eta e^{-\theta x} \log x dx \]  

and observing that

\[ \theta \int_\eta^\infty e^{-\theta x} G(x) dx \leq G(\eta) \theta \int_\eta^\infty e^{-\theta x} dx = G(\eta) e^{-\theta \eta} \rightarrow 0. \]  \hspace{1cm} (8)

We need now the following evaluation. For \eta > 0 we have

\[ \lim_{\theta \rightarrow \infty} \frac{1}{\log \theta} \int_0^\eta e^{-\theta x} \log x dx = -1. \]  \hspace{1cm} (9)

To prove (9), we obtain, by a change of variable to \( v = \theta x \), that

\[ \frac{1}{\log \theta} \int_0^\eta e^{-\theta x} \log x dx = \frac{1}{\log \theta} \int_0^{\theta \eta} e^{-v} \log v dv - \int_0^{\theta \eta} e^{-v} dv \rightarrow_{\theta \rightarrow \infty} 0 - 1, \]  \hspace{1cm} (10)

where the last term on right hand side of (10) follows since \( \int_0^\infty e^{-v} \log v dv \) converges and \( \int_0^\infty e^{-v} dv = 1 \). We now divide both sides of (7) by \log \theta and let \theta \rightarrow \infty. From (8) and (9) we get that for all \varepsilon > 0

\[ -\ell - \varepsilon \ell \leq \liminf_{\theta \rightarrow \infty} \frac{1}{\log \theta} k(\theta) \leq \limsup_{\theta \rightarrow \infty} \frac{1}{\log \theta} k(\theta) \leq -\ell + \varepsilon \ell, \]

i.e., \( \lim_{\theta \rightarrow \infty} \frac{1}{\log \theta} k(\theta) = -\ell \). Applying this to \( \theta = u^{1/t} \) with a fixed \( u > 1 \) and letting \( t \rightarrow 0 \), we get

\[ \lim_{t \rightarrow 0} tk(u^{1/t}) = -\ell \log u, \quad \lim_{t \rightarrow 0} (L(u^{1/t}))^t = \frac{1}{u^t}, \]

which is the desired result. Finally suppose that \( \theta_0 > 0 \) and denote \( k_{\theta_0}(\theta) = k(\theta_0 + \theta) - k(\theta_0) \). Trivially,

\[ k_{\theta_0}(\theta) = -\int_0^\infty (1 - e^{-\theta x}) \nu_{\theta_0}(dx) \text{ with } \nu_{\theta_0}(dx) = e^{-\theta_0 x} \nu(dx) \]

and consider

\[ G_{\theta_0}(x) = \nu_{\theta_0}((x, \infty)) = \int_x^\infty e^{-\theta_0 y} \nu(dy) \]

\[ = e^{-\theta_0 x} G(x) - \theta_0 \int_x^\infty e^{-\theta_0 y} G(y) dy \]

\[ = k_x(\theta_0) + G(x) \]  \hspace{1cm} (11)

\[ k_x(\theta_0) \]  \hspace{1cm} (12)

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Line (11) is obtained by an integration by parts, whereas line (12) uses (5). We see easily from (12) that \( \lim_{x \to 0} G_{\theta_0}(x)/\log x = -\ell \). Therefore we are in the same situation as in the proof of the first part with \( \theta_0 = 0 \), and thus the proof is completed.

The following remark is useful to obtain a generalization of the three examples presented in Section 3 for any \( \ell > 0 \).

**Remark 1** Suppose that \((\mu_t)_{t>0}\) is a family of infinitely divisible distributions with \( \mu_t * \mu_s = \mu_{t+s} \), where \( t, s > 0 \). Assume that \( \mu_1 \) fulfills the premises of Proposition 1 with \( G(x) \sim -\ell \log x \). Then obviously for any fixed \( t > 0 \), \( \mu_t \) also fulfills such premises with \( t\ell \) replacing the role of \( \ell \). In all of the examples below, we have \( \ell = 1 \) and this remark shows how to get from them other examples with arbitrary \( \ell > 0 \).

### 3 Examples

**Example 1** If \( \mu(dx) = e^{-x}1_{(0,\infty)}(x)dx \) the corresponding infinitely divisible family is the gamma family with scale parameter 1. The Lévy measure here is \( \nu(dx) = e^{-x}1_{(0,\infty)}(x)\frac{dx}{x} \) and \( \ell = 1 \).

**Example 2** A discrete example is

\[
\nu(dx) = \sum_{n=1}^{\infty} \delta_{1/n}.
\]

We have \( k(\theta) = \sum_{n=1}^{\infty} \frac{1}{n}(1 - e^{-\theta/n}) \), \( G(x) = \sum_{n=1}^{[1/x]} 1/n \sim -\log x \) if \( x \to 0 \) and \( \ell = 1 \). The probability \( \mu \) is the distribution of \( \sum_{n=1}^{\infty} X_n/n \), where \( X_n \) is Poisson distributed with mean \( 1/n \) and the \( (X_n)_{n=1}^{\infty} \) are independent.

**Example 3** Utilizing Example 2.2 in [1], consider the infinitely divisible distribution \( \mu \) on \((0,\infty)\) with Laplace transform defined on \( \theta \geq 0 \) given by

\[
\theta + 1 - \sqrt{\theta^2 + 2\theta}.
\]

Note that the densities of the corresponding EDM are Bessel densities related to a symmetric random walk (see Feller, 1971, pp. 60-61).

The related Lévy measure of \( \mu \) is

\[
\nu(dx) = \frac{\text{1}_F(1/2; 1; -2x)1_{(0,\infty)}(x)}{x} \frac{dx}{x},
\]  
(13)
where _1F_1(a; b; z) is the so-called confluent entire function defined by \( \sum_{n=0}^{\infty} \frac{(a)_n}{n!(b)_n} z^n \), where \((a)_0 = 1\) and \((a)_{n+1} = (a+n)(a)_n\) define the Pochhammer symbol \((a)_n\). Therefore \( \nu \) has a density equivalent to \( 1/x \) when \( x \to 0 \) which implies that \( G(x) \approx -\log x \). Proposition 1 is thus satisfied with \( \ell = 1 \). To check the correctness of (13) observe that if \( k(\theta) = \log(\theta + 1 - \sqrt{\theta^2 + 2\theta}) \) then

\[
k'(\theta) = -\int_0^{\infty} e^{-\theta x} x \nu(dx) = -\frac{1}{\sqrt{\theta + 2}} \frac{1}{\sqrt{\theta}} = -\int_0^{\infty} e^{-\theta x} f(x) dx \times \int_0^{\infty} e^{-\theta y} g(y) dy,
\]

where

\[
f(x) = e^{-2x} \frac{1}{\sqrt{\pi x}} 1_{(0,\infty)}(x), \quad g(y) = \frac{1}{\sqrt{\pi y}} 1_{(0,\infty)}(y).
\]

Therefore the density of \( x \nu(dx), x > 0 \), is given by the convolution product \( f * g \), where by a change of variable \( y = tx \) and employing a Taylor expansion, one gets

\[
\int_0^x f(y)g(x-y)dy = \frac{1}{\pi} \int_0^1 \frac{e^{-2xt}}{\sqrt{t(1-t)}} dt = _1F_1(1/2; 1; -2x).
\]

Let us fix \( t > 0 \). Recall (see [2]) that for \( \theta > 0 \) the function \( \theta + 1 - \sqrt{\theta^2 + 2\theta} \) is the Laplace transform of the density \( f_t(x) = \frac{t}{x} e^{-x} I_t(x) 1_{(0,\infty)}(x) \) where the Bessel function \( I_t(x) \) is

\[
I_t(x) = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n + 1 + t)} x^{2n+t}.
\]

We fix now \( \ell > 0 \). If \( Y_t \) has density \( f_t \) this implies that the density of \( U = Y_t^{-\ell} \) is

\[
g_t(u) = \ell t^2 u^{2t+1} e^{-\frac{1}{u}} I_{\ell t}(\frac{1}{u}) 1_{(0,\infty)}(u).
\]

It would be quite delicate to prove directly from this last formula that when \( t \to 0 \) the law \( g_t(u) du \) converges to the Pareto law

\[
1_{(1,\infty)}(u) \frac{\ell du}{u^{\ell+1}}
\]

as shown by our Proposition.
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