The difference and ratio of the fractional matching number and the matching number of graphs

Ilkyoo Choi∗  Jaehoon Kim†  Suil O‡

December 24, 2015

Abstract

Given a graph $G$, the matching number of $G$, written $\alpha'(G)$, is the maximum size of a matching in $G$, and the fractional matching number of $G$, written $\alpha'_f(G)$, is the maximum size of a fractional matching of $G$. In this paper, we prove that if $G$ is an $n$-vertex connected graph that is neither $K_1$ nor $K_3$, then $\alpha'_f(G) - \alpha'(G) \leq \frac{n-2}{6}$ and $\frac{\alpha'_f(G)}{\alpha'(G)} \leq \frac{3n}{2n+2}$. Both inequalities are sharp, and we characterize the infinite family of graphs where equalities hold.

1 Introduction

For undefined terms, see [5]. Throughout this paper, $n$ will always denote the number of vertices of a given graph. A matching in a graph is a set of pairwise disjoint edges. A perfect matching in a graph $G$ is a matching in which each vertex has an incident edge in the matching; its size must be $n/2$, where $n = |V(G)|$. A fractional matching of $G$ is a function $\phi : E(G) \to [0, 1]$ such that for each vertex $v$, $\sum_{e \in \Gamma(v)} \phi(e) \leq 1$, where $\Gamma(v)$ is the set of edges incident to $v$, and the size of a fractional matching $\phi$ is $\sum_{e \in E(G)} \phi(e)$. Given a graph $G$, the matching number of $G$, written $\alpha'(G)$, is the maximum size of a matching in $G$, and

∗Department of Mathematical Sciences, KAIST, Daejeon, South Korea, ilkyoo@kaist.ac.kr research partially supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) (NRF-2015R1C1A1A02036398)
†School of Mathematics, University of Birmingham, Edgbaston, Birmingham, United Kingdom, kimJS@bham.ac.uk research partially supported by the European Research Council under the European Union’s Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreements no. 306349 (J. Kim)
‡Department of Mathematics, Simon Fraser University, Burnaby, BC, V5A 1S6, osuilo@sfu.ca. research partially supported by an NSERC Grant of Bojan Mohar
the fractional matching number of $G$, written $\alpha_f'(G)$, is the maximum size of a fractional matching of $G$.

Given a fractional matching $\phi$, since $\sum_{e \in \Gamma(v)} \phi(e) \leq 1$ for each vertex $v$, we have that $2 \sum_{e \in E(G)} \phi(e) \leq n$, which implies $\alpha_f'(G) \leq n/2$. By viewing every matching as a fractional matching it follows that $\alpha_f'(G) \geq \alpha'(G)$ for every graph $G$, but equality need not hold. For example, the fractional matching number of a $k$-regular graph equals $n/2$ by setting weight $1/k$ on each edge, but the matching number of a $k$-regular graph can be much smaller than $n/2$. Thus it is a natural question to find the largest difference between $\alpha_f'(G)$ and $\alpha'(G)$ in a (connected) graph.

In Section 3 and Section 4, we prove tight upper bounds on $\alpha_f'(G) - \alpha'(G)$ and $\alpha_f'(G)/\alpha'(G)$, respectively, for an $n$-vertex connected graph $G$, and we characterize the infinite family of graphs achieving equality for both results. As corollaries of both results, we have upper bounds on both $\alpha_f'(G) - \alpha'(G)$ and $\alpha_f'(G)/\alpha'(G)$ for an $n$-vertex graph $G$, and we characterize the graphs achieving equality for both bounds.

Our proofs use the famous Berge–Tutte Formula [1] for the matching number as well as its fractional analogue. We also use the fact that there is a fractional matching $\phi$ for which $\sum_{e \in E(G)} \phi(e) = \alpha_f'(G)$ such that $f(e) \in \{0, 1/2, 1\}$ for every edge $e$, and some refinements of the fact. We can prove both Theorem 6 and Theorem 8 with two different techniques, and for the sake of the readers we demonstrate each method in the proofs of Theorem 6 and Theorem 8.

2 Tools

In this section, we introduce the tools we used to prove the main results. To prove Theorem 6, we use Theorem 1 and Theorem 2. For a graph $H$, let $o(H)$ denote the number of components of $H$ with an odd number of vertices. Given a graph $G$ and $S \subseteq V(G)$, define the deficiency $\text{def}(S)$ by $\text{def}(S) = o(G - S) - |S|$, and let $\text{def}(G) = \max_{S \subseteq V(G)} \text{def}(S)$. Theorem 1 is the famous Berge–Tutte formula, which is a general version of Tutte’s 1-factor Theorem [4].

**Theorem 1 ([1]).** For any $n$-vertex graph $G$, $\alpha'(G) = \frac{1}{2} (n - \text{def}(G))$.

For the fractional analogue of the Berge–Tutte formula, let $i(H)$ denote the number of isolated vertices in $H$. Given a graph $G$ and $S \subseteq V(G)$, let $\text{def}_f(S) = i(G - S) - |S|$ and $\text{def}_f(G) = \max_{S \subseteq V(G)} \text{def}_f(S)$. Theorem 2 is the fractional version of the Berge–Tutte Formula. This is also the fractional analogue of Tutte’s 1-Factor Theorem saying that $G$ has a fractional perfect matching if and only if $i(G - S) \leq |S|$ for all $S \subseteq V(G)$ (implicit in Pulleyblank [2]), where a fractional perfect matching is a fractional matching $f$ such that $2 \sum_{e \in E(G)} f(e) = n$. 

2
Theorem 2 ([3] See Theorem 2.2.6). For any \( n \)-vertex graph \( G \), \( \alpha'_f(G) = \frac{1}{2}(n - \text{def}_f(G)) \).

When we characterize the equalities in the bounds of Theorem 6 and Theorem 8, we need the following proposition. Recall that \( G[S] \) is the graph induced by a subset of the vertex set \( S \).

Proposition 3 ([3] See Proposition 2.2.2). The following are equivalent for a graph \( G \).
(a) \( G \) has a fractional perfect matching.
(b) There is a partition \( \{ V_1, \ldots, V_n \} \) of the vertex set \( V(G) \) such that, for each \( i \), the graph \( G[V_i] \) is either \( K_2 \) or Hamiltonian.
(c) There is a partition \( \{ V_1, \ldots, V_n \} \) of the vertex set \( V(G) \) such that, for each \( i \), the graph \( G[V_i] \) is either \( K_2 \) or Hamiltonian graph on an odd number of vertices.

Theorem 4 and Observation 5 are used to prove Theorem 8.

Theorem 4 ([3] See Theorem 2.1.5). For any graph \( G \), there is a fractional matching \( f \) for which
\[
\sum_{e \in E(G)} f(e) = \alpha'_f(G)
\]
such that \( f(e) \in \{0, 1/2, 1\} \) for every edge \( e \).

Given a fractional matching \( f \), an unweighted vertex \( v \) is a vertex with \( \sum_{e \in \Gamma(v)} f(e) = 0 \), and a full vertex \( v \) is a vertex with \( f(vw) = 1 \) for some vertex \( w \). Note that \( w \) is also a full vertex. An \( i \)-edge \( e \) is an edge with \( f(e) = i \). Note that the existence of an 1-edge guarantees the existence of two full vertices. A vertex subset \( S \) of a graph \( G \) is independent if \( E(G[S]) = \emptyset \), where \( G[S] \) is the graph induced by \( S \).

Observation 5. Among all the fractional matchings of an \( n \)-vertex graph \( G \) satisfying the conditions of Theorem 4, let \( f \) be a fractional matching with the greatest number of edges \( e \) with \( f(e) = 1 \). Then we have the following:
(a) The graph induced by the \( \frac{1}{2} \)-edges is the union of odd cycles. Furthermore, if \( C \) and \( C' \) are two disjoint cycles in the graph induced by \( \frac{1}{2} \)-edges, then there is no edge \( uu' \) such that \( u \in V(C) \) and \( u' \in V(C') \).
(b) The set \( S \) of the unweighted vertices is independent. Furthermore, every unweighted vertex is adjacent only to a full vertex.
(c) \( \alpha'(G) \geq w_1 + \sum_{i=1}^{\infty} ic_i, \alpha'_f(G) = w_1 + \sum_{i=1}^{\infty} (\frac{2i+1}{2})c_i, \) and \( n = w_0 + 2w_1 + \sum_{i=1}^{\infty} (2i + 1)c_i \), where \( w_0, w_1, \) and \( c_i \) are the number of unweighted vertices, the number of 1-edges, and the number of odd cycles of length \( 2i + 1 \) in the graph induced by \( \frac{1}{2} \)-edges in \( G \), respectively.

Proof. (a) The graph induced by the \( \frac{1}{2} \)-edges cannot have a vertex with degree at least 3 since \( \sum_{e \in \Gamma(v)} f(e) \leq 1 \) for each vertex \( v \). Thus the graph must be a disjoint union of paths.
or cycles. If the graph contains a path or an even cycle, then by replacing weight 1/2 on each edge on the path or the even cycle with weight 1 and 0 alternatively, we can have a fractional matching with the same fractional matching number and more edges with weight 1, which contradicts the choice of $f$. Thus the graph induced by the $\frac{1}{2}$-edges is the union of odd cycles. If there is an edge $uv$ such that $u \in V(C)$ and $v \in V(C')$, where $C$ and $C'$ are two different odd cycles induced by some $\frac{1}{2}$-edges, then $f(uv) = 0$, since $\sum_{e \in \Gamma(x)} f(e) \leq 1$ for each vertex $x$. By replacing weights 0 and 1/2 on the edge $uv$ and the edges on $C$ and $C'$ with weight 1 on $uv$, and 0 and 1 on the edges in $E(C)$ and $E(C')$ alternatively, not violating the definition of a fractional matching, we have a fractional matching with the same fractional matching number with more edges with weight 1, which is a contradiction. Thus we have the desired result.

(b) If two unweighted vertices $u$ and $v$ are adjacent, then we can put a positive weight on the edge $uv$, which contradicts the choice of $f$. If there exists an unweighted vertex $x$, which is not incident to any full vertex, then $x$ must be adjacent to a vertex $y$ such that $f(yy_1) = \frac{1}{2}$ and $f(yy_2) = \frac{1}{2}$ for some vertices $y_1$ and $y_2$. By replacing the weights 0, 1/2, and 1/2 on $xy$, $yy_1$, and $yy_2$ with 1, 0, 0, respectively, we have a fractional matching with the same fractional matching number with more edges with weight 1, which is a contradiction.

(c) By the definitions of $w_0$, $w_1$, and $c_i$, we have the desired result.

3 Sharp upper bound for $\alpha'_f(G) - \alpha'(G)$

What are the structures of the graphs having the maximum difference between the fractional matching number and the matching number in an $n$-vertex connected graph? The graphs may have big fractional matching number and small matching number. So, by the Berge–Tutte Formula and its fractional version, they may have a vertex subset $S$ such that almost all of the odd components of $G - S$ have at least three vertices in order to get $S$ to have small fractional deficiency and big deficiency. This is our idea behind the proof of Theorem 6.

**Theorem 6.** For $n \geq 5$, if $G$ is a connected graph with $n$ vertices, then $\alpha'_f(G) - \alpha'(G) \leq \frac{n-2}{6}$, and equality holds only when either

(i) $n = 5$ and either $C_5$ is subgraph of $G$ or $K_2 + K_3$ is a subgraph of $G$, or

(ii) $G$ has a vertex $v$ such that the components of $G - v$ are all $K_3$ except one single vertex.

**Proof.** Among all the vertex subsets with maximum deficiency, let $S$ be the largest set. By the Berge-Tutte Formula, $\alpha'(G) = \frac{1}{2}(n - \text{def}(S))$, and by the choice of $S$, all components of $G - S$ have an odd number of vertices. Let $x$ be the number of isolated vertices of $G - S$, and let $y$ be the number of other components of $G - S$. This implies $n \geq |S| + x + 3y$. If $S = \emptyset$, then $\alpha'(G) \in \{ \frac{n}{2}, \frac{n-1}{2} \}$, depending on the parity of $n$. In this case, $\alpha'_f(G) - \alpha'(G) \leq \frac{n-2}{6}$. If $S \neq \emptyset$, then $\alpha'(G) < \frac{n}{2}$, and $\alpha'_f(G) - \alpha'(G) \leq \frac{n-2}{6}$.
Figure 1: All 5-vertex graphs in Theorem 6 (i) and Theorem 8 (i)

Figure 2: All graphs in Theorem 6 (ii) and Theorem 8 (ii)
\[
\frac{n}{2} - \frac{n-1}{2} = \frac{1}{2} \leq \frac{n^2}{6}, \text{ since } n \geq 5. \text{ Now, assume that } S \text{ is non-empty.}
\]

**Case 1:** \( x = 0 \). Since \( \text{def}_f(G) \geq 0 \), \(|S| \geq 1\), and \( n \geq |S| + 3y \), we have

\[
\alpha'_f(G) - \alpha'(G) = \frac{1}{2}(n - \text{def}_f(G)) - \frac{1}{2}(n - \text{def}(G)) = \frac{1}{2}(\text{def}(S) - \text{def}_f(G))
\]

\[
\leq \frac{1}{2}(y - |S| - 0) \leq \frac{1}{2}\left(\frac{n - |S|}{3} - |S|\right) = \frac{n - 4|S|}{6} \leq \frac{n - 4}{6} < \frac{n - 2}{6}.
\]

**Case 2:** \( x \geq 1 \). Since \( n \geq |S| + x + 3y \), \(|S| \geq 1\), and \( x \geq 1 \), we have

\[
\alpha'_f(G) - \alpha'(G) = \frac{1}{2}(n - \text{def}_f(G)) - \frac{1}{2}(n - \text{def}(G)) = \frac{1}{2}(\text{def}(S) - \text{def}_f(G))
\]

\[
\leq \frac{1}{2}(x + y - |S| - (x - |S|)) \leq \frac{y}{2} = \frac{n - x - |S|}{6} \leq \frac{n - 2}{6}.
\]

Equality in the bound requires equality in each step of the computation. When \( n = 5 \), we conclude that (i) follows by Proposition 3. In Case 1, we cannot have equality, and in Case 2, we have \(|S| = 1\), \( x = 1 \), and \( n = |S| + x + 3y = 2 + 3y \). Since \( G \) is connected, the components of \( G - S \) are \( P_3 \) or \( K_3 \) except only one single vertex. If a component of \( G - S \) is a copy of \( P_3 \), then by choosing the central vertex \( u \) of the path, we have \( \text{def}(S \cup \{u\}) = o(G - (S \cup \{u\})) - |S \cup \{u\}| = o(G - S) - |S| \), yet \( |S \cup \{u\}| > |S| \), which contradict the choice of \( S \). Thus we have the desired result. \( \square \)

**Corollary 7.** For any \( n \)-vertex graph \( G \), we have \( \alpha'_f(G) - \alpha'(G) \leq \frac{n}{6} \), and equality holds only when \( G \) is the disjoint union of copies of \( K_3 \).

**Proof.** First, we show that if \( n \leq 4 \) and \( G \) is connected, then \( \alpha'_f(G) - \alpha'(G) \leq \frac{n}{6} \), and equality holds only when \( G = K_3 \). If \( n \leq 2 \), then \( G \in \{K_1, K_2\} \), which implies that \( \alpha'_f(G) - \alpha'(G) = 0 < n/6 \). If \( n = 3 \), then \( G \in \{P_3, K_3\} \). Note that \( \alpha'_f(P_3) - \alpha'(P_3) = 1 - 1 = 0 < 3/6 \) and \( \alpha'_f(K_3) - \alpha'(K_3) = 3/2 - 1 = 1/2 \leq 3/6 \). Furthermore, equality holds only when \( G = K_3 \). If \( n = 4 \), then either \( G = K_{1,3} \) or \( G \) contains \( P_4 \) as a subgraph. Since \( \alpha'_f(K_{1,3}) - \alpha'(K_{1,3}) = 1 - 1 = 0 < 4/6 \) and \( \alpha'_f(P_4) - \alpha'(P_4) = 2 - 2 = 0 < 4/6 \), we conclude that for any positive integer \( n \), \( \alpha'_f(G) - \alpha'(G) \leq \frac{n}{6} \). In fact, if \( n \geq 5 \), then by Theorem 6, the difference must be at most \( \frac{n^2 - 2}{6} \). Thus, for connected graphs, equality holds only when \( G = K_3 \).

Now, if we assume that \( G \) is disconnected, then \( G \) is the disjoint union of connected graphs \( G_1, \ldots, G_k \). Let \( |V(G_i)| = n_i \) for \( i \in [k] \). Since

\[
\alpha'_f(G) - \alpha'(G) = [\alpha'_f(G_1) + \cdots + \alpha'_f(G_k)] - [\alpha'(G_1) + \cdots + \alpha'(G_k)]
\]

\[
= [\alpha'_f(G_1) - \alpha'(G_1)] + \cdots + [\alpha'_f(G_k) - \alpha'(G_k)] \leq \frac{n_1}{6} + \cdots + \frac{n_k}{6} = \frac{n}{6},
\]

equality holds only when each \( G_i \) is a copy of \( K_3 \) for \( i \in [k] \). \( \square \)
4 Sharp upper bound for $\frac{\alpha'_f(G)}{\alpha'(G)}$

To prove the upper bound of Theorem 8, we still can use the Berge-Tutte formula and its fractional analogue. However, we provide an alternative way to prove the theorem.

**Theorem 8.** For $n \geq 5$, if $G$ is a connected graph with $n$ vertices, then $\frac{\alpha'_f(G)}{\alpha'(G)} \leq \frac{3n}{2n+2}$, and equality holds only when either

(i) $n = 5$ and either $C_5$ is a subgraph of $G$ or $K_2 + K_3$ is a subgraph of $G$, or

(ii) $G$ has a vertex $v$ such that the components of $G - v$ are all $K_3$ except one single vertex.

**Proof.** Among all the fractional matchings of an $n$-vertex graph $G$ with the size equal to $\alpha'_f(G)$, let $f$ be a fractional matching such that the number of edges $e$ with $f(e) = 1$ is maximized. We follow the notation in Observation 5.

Case 1: $w_0 = w_1 = 0$. Since $G$ is connected and $n \geq 5$, there exists only one $i$ such that $i \geq 2$ and $c_i$ is not zero, and $\alpha'(G) = ic_i \neq 0$. Then we have

$$\frac{\alpha'_f(G)}{\alpha'(G)} \leq \frac{(2i+1)c_i}{i c_i} = 1 + \frac{1}{2i} \leq \frac{5}{4}.$$  

Case 2: $w_0 \geq 1$ and $w_1 = 0$. By part (b) of Observation 5, this cannot happen.

Case 3: $w_0 = 0$ and $w_1 \geq 1$. Since $\sum_{i=1}^{\infty} c_i \leq \frac{n-2w_1}{3}$, by part (c) of Observation 5, we have

$$\frac{\alpha'_f(G)}{\alpha'(G)} \leq \frac{w_1 + \sum_{i=1}^{\infty} (\frac{2i+1}{2})c_i}{w_1 + \sum_{i=1}^{\infty} ic_i} = \frac{n-w_0}{n-w_0 - \sum_{i=1}^{\infty} c_i} = \frac{n}{n - \sum_{i=1}^{\infty} c_i} \leq \frac{n}{n - \frac{n-2w_1}{3}} = \frac{3n}{2n + 2w_1} \leq \frac{3n}{2n+2}.$$  

Case 4: $w_0 \geq 1$ and $w_1 \geq 1$. Since $\sum_{i=1}^{\infty} c_i \leq \frac{n-2w_1-w_0}{3}$, by part (c) of Observation 5, we have

$$\frac{\alpha'_f(G)}{\alpha'(G)} \leq \frac{n-w_0}{2(n-w_0 - \frac{n-2w_1-w_0}{3})} = \frac{n-w_0}{n-w_0 - \frac{n-2w_1-w_0}{3}} = \frac{3(n-w_0)}{2(n+w_1-w_0)} < \frac{3n}{2(n+w_1)} \leq \frac{3n}{2(n+1)}.$$  

Equality in the bound requires equality in each step of the computation; we only need to check Case 1 and Case 2. In Case 1, we have $i = 2$, which means that $n = 5$ and $G$ contains a copy of $C_5$. In Case 3, we have $w_1 = 1$ and $\sum_{i=1}^{\infty} c_i = \frac{n-2}{3}$, which means that the graph induced by the $\frac{1}{2}$-edges is the union of $K_3$. Thus $G$ has $K_2 + kK_3$ as a subgraph for some positive integer $k$. Note that there is an edge between the copy of $K_2$ and any copy of $K_3$ by part (b) of Observation 5. Also, there are no edges between any pair of two triangles by part (a) of Observation 5. Let $u$ and $v$ be the two vertices corresponding to the copy of $K_2$. If there are two different triangles $C$ and $C'$ in $G$ such that $u$ and $v$ are incident to $C$ and $C'$, respectively, then we have $\alpha'(G) > w_1 + c_1$, which implies that we cannot have equality in the first inequality in Case 3. Thus, we conclude that $G$ contains a copy of either $K_2 + K_3$ as a subgraph or a vertex $v$ such that the components of $G - v$ are all $K_3$ except only one single vertex. 

\[\Box\]
Corollary 9. For any $n$-vertex graph $G$ with at least one edge, we have $\frac{\alpha_f'(G)}{\alpha'(G)} \leq \frac{3}{2}$, and equality holds only when $G$ is the disjoint union of copies of $K_3$.

Proof. By the proof of Corollary 7 if $n \leq 4$ and $G$ is connected, then $\frac{\alpha_f'(G)}{\alpha'(G)} \leq \frac{3}{2}$, and equality holds only when $G = K_3$. If we assume that $G$ is disconnected, then $G$ is the disjoint union of connected graphs $G_1, \ldots, G_k$. Let $|V(G_i)| = n_i$ for $i \in [k]$. Without loss of generality, we may assume that $\frac{\alpha_f'(G_1)}{\alpha'(G_1)} \geq \frac{\alpha_f'(G_i)}{\alpha'(G_i)}$ for all $i \in [k]$. Then we have

$$\frac{\alpha_f'(G)}{\alpha'(G)} = \frac{\alpha_f'(G_1) + \cdots + \alpha_f'(G_k)}{\alpha'(G_1) + \cdots + \alpha'(G_k)} \leq \frac{\alpha_f'(G_1)}{\alpha'(G_1)} \leq \frac{3}{2},$$

and equality holds only when each $G_i$ is a copy of $K_3$ for $i \in [k]$.

References

[1] C. Berge, Sur le couplage maximum d’un graphe. *C. R. Acad. Sci. Paris* 247 (1958), 258–259.

[2] W.R. Pulleyblank, Minimum node covers and 2-bicritical graphs. *Math. Programming* 17 (1979), no. 1, 91–103.

[3] E.R. Scheinerman and D.H. Ullman, *Fractional Graph Theory: A Rational Approach to the Theory of Graphs* (Wiley & Sons, 2008).

[4] W. T. Tutte, The factorization of linear graphs. *J. London Math. Soc.* 22 (1947), 107–111.

[5] D.B. West, *Introduction to Graph Theory*, Prentice Hall, Inc., Upper Saddle River, NJ, 2001.