Aperiodic order, integrated density of states and the continuous algebras of John von Neumann

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Abstract. In [5] Lenz and Stollmann proved the existence of the integrated density of states in the sense of uniform convergence of the distributions for certain operators with aperiodic order. The goal of this paper is to establish a relation between aperiodic order, uniform spectral convergence and the continuous algebras invented by John von Neumann. We illustrate the technique by proving the uniform spectral convergence for random Schrödinger operators on lattices with finite site probabilities, percolation Hamiltonians and for the pattern-invariant operators of self-similar graphs.

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1 Introduction

First, let us recall the notion of uniform spectral convergence studied by Lenz and Stollmann \[4\], \[5\]. Let \( G \) be a simple, connected infinite graph with bounded vertex degrees, with vertex set \( V \) and edge set \( E \). Let \( \Delta_G : L^2(V) \to L^2(V) \) be the combinatorial Laplacian operator:

\[
\Delta_G f(x) = \deg(x)f(x) - \sum_{(x,y) \in E} f(y).
\]

A Følner-sequence in \( G \) is a sequence of finite subsets \( \{Q_n\}_{n=1}^{\infty} \subset V \) such that \( |\partial Q_n|/|Q_n| \to 0 \), where

\[
\partial Q_n := \{ p \in Q_n \mid \text{there exists } q \notin Q_n, (p,q) \in E \}.
\]

For a subset \( A \subset V \), let \( \Delta_A : L^2(A) \to L^2(A) \) be the Laplacian on the subgraph \( G_A \) spanned by the vertices of \( A \). If \( A \) is finite, then it is a positive, self-adjoint linear transformation. The normalized spectral distribution function of \( \Delta_A \) is defined the following way:

\[
N_{\Delta_A}(\lambda) := \{ \text{the number of eigenvalues of } A \text{ counted with multiplicities, not greater than } \lambda, \text{ divided by } |A| \}.
\]

We say that the integrated density of states exists in the sense of uniform convergence of the distributions (shortly: the uniform spectral convergence exists) if there is a monotone function \( N_{\Delta_G} \) such that for any Følner-sequence \( \{Q_n\}_{n=1}^{\infty} \) the functions \( N_{\Delta_{Q_n}} \) uniformly converge to \( N_{\Delta_G} \). It is important to note that weak spectral convergence, that is

\[
\lim_{n \to \infty} N_{\Delta_{Q_n}}(\lambda) = N_{\Delta_G}
\]

for all points of continuity of \( N_{\Delta_G} \) is generally much easier to prove (see the discussion in Section 7). Lenz and Stollmann proved the existence of universal spectral convergence in the case of aperiodic order for Laplacian-type operators defined by nice Delone-sets of \( \mathbb{R}^d \).

In our paper we study the same problem from a ring theoretical point of view. We introduce the spectral functions for the elements of continuous algebras defined by Goodearl \[2\] as a generalization of the original construction of John von Neumann \[7\]. These rings \( \mathcal{R} \) are complete metric rings equipped with rank functions. We shall see in Proposition \[3\] that if \( \{T_n\}_{n=1}^{\infty} \subset \mathcal{R} \) converge to \( T \in \mathcal{R} \), then the spectral functions of \( T_n \) converge uniformly to the spectral function of \( T \). We shall use this fact to prove uniform spectral convergence for random and deterministic operators. The examples considered in the paper are:
Random Schrödinger operators on lattices with finite site probabilities.

Percolation Hamiltonians.

Pattern-invariant operators on self-similar graphs.

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2 The Goodearl-von Neumann construction

In this section we recall Goodearl’s construction of rank rings \cite{2} via Bratteli diagrams. This is a generalization of the original construction of John von Neumann \cite{7}. Let $R_1, R_2, \ldots$ be finite dimensional algebras over the real numbers,

$$R_i := \bigoplus_{\alpha=1}^{k_i} \operatorname{Mat}_{n_{i,\alpha} \times n_{i,\alpha}}(\mathbb{R}).$$

Next we consider injective unital algebra homomorphisms $\phi_i : R_i \to R_{i+1}$. For each $i \geq 1$ we have $k_i$ nodes $\{a^i_\alpha\}_{\alpha=1}^{k_i}$. Also, for each pair $(a^i_\alpha, a^{i+1}_\beta)$, $1 \leq \alpha \leq k_i$, $1 \leq \beta \leq k_{i+1}$ we have weights $w_{\alpha,\beta}$ satisfying the following compatibility condition:

- For any $1 \leq \beta \leq k_{i+1}$:

$$n_{i+1,\beta} = \sum_{\alpha=1}^{k_i} w_{\alpha,\beta} n_{i,\alpha}.$$

Thus this system defines diagonal imbeddings $\phi_i : R_i \to R_{i+1}$.

One can associate Markov-transition probabilities to the weight system: $M(\beta, \alpha) := \frac{w_{\alpha,\beta} n_{i,\alpha}}{n_{i+1,\beta}}$. A rank function on the inductive limit is defined as follows. Let $p_{i,\alpha} > 0$ be real numbers satisfying the relations:

$$\sum_{\alpha=1}^{k_i} p_{i,\alpha} = 1 \quad (1)$$

For any $1 \leq \alpha \leq k_i$ $p_{i,\alpha} = \sum_{\beta=1}^{k_{i+1}} M(\beta, \alpha)p_{i+1,\beta} \quad (2)$
Then for each \( i \geq 1 \) one has a normalized rank function \( r_i : \mathcal{R}_i \to \mathbb{R} \), defined as

\[
r_i(A_{i,1} \oplus A_{i,2} \oplus \ldots \oplus A_{i,k_i}) := \sum_{\alpha=1}^{k_i} p_{i,\alpha} \frac{\text{Rank } A_{i,\alpha}}{n_{i,\alpha}}.
\]

Then if \( A = (A_{i,1} \oplus A_{i,2} \oplus \ldots \oplus A_{i,k_i}) \in \mathcal{R}_i \), \( r_i(A) = r_{i+1}(\phi_{i+1}(A)) \). Hence we defined a normalized rank function \( r \) on the inductive limit \( \lim_{\phi_i} \mathcal{R}_i \). Clearly the rank \( r \) satisfies the following conditions:

- \( r(A) \geq 0 \) for any \( A \in \lim_{\phi_i} \mathcal{R}_i \).
- \( r(1) = 1, r(0) = 0 \) and \( r(A) = 0 \) if and only if \( A = 0 \).
- \( r(A + B) \leq r(A) + r(B) \).
- \( r(AB) \leq r(A), r(AB) \leq r(B) \).

Therefore \( d(A, B) := r(A - B) \) defines a metric on the inductive limit ring. Its completion is the continuous regular ring \( \mathcal{R} \) and the rank \( r \) extends to \( \mathcal{R} \) satisfying the four conditions above.

**3 Spectral functions**

Let us consider the inductive limit system

\[
\mathcal{R}_1 \xrightarrow{\phi_1} \mathcal{R}_2 \xrightarrow{\phi_2} \ldots
\]

as in Section 2. For each \( i \geq 1 \) consider the vectorspace \( \bigoplus_{\alpha=1}^{k_i} \mathbb{R}^{n_{i,\alpha}} = V_i \). The algebra \( \mathcal{R}_i = \bigoplus_{\alpha=1}^{k_i} \text{Mat}_{n_{i,\alpha} \times n_{i,\alpha}}(\mathbb{R}) \) acts on \( V_i \) the natural way. Now we define a dimension function on the \( \mathcal{R}_i \)-submodules of \( V_i \). Note that any such \( \mathcal{R}_i \)-submodule \( L \) is in the form \( \bigoplus_{\alpha=1}^{k_i} L_{i,\alpha} \), where \( L_{i,\alpha} \subseteq \mathbb{R}^{n_{i,\alpha}} \) is a linear subspace. Then

\[
\dim_{\mathcal{R}_i} \left( \bigoplus_{\alpha=1}^{k_i} L_{i,\alpha} \right) := \sum_{\alpha=1}^{k_i} p_{i,\alpha} \frac{\dim L_{i,\alpha}}{n_{i,\alpha}}.
\]

Then of course, \( \dim_{\mathcal{R}_i}(V_i) = 1 \). Let \( T \in \mathcal{R}_i \), then we call a \( \mathcal{R}_i \)-submodule \( L \subseteq V_i \) a \( \lambda \)-space of \( T \) if \( \|T(v)\| \leq \lambda \|v\| \), for any \( v \in L \), where \( \|v\| \) is the Euclidean norm of \( v \).
Similarly, we call a \( R_i \)-submodule \( N \subseteq V_i \) a \( \lambda^+ \)-space of \( T \) if \( \|T(v)\| > \lambda\|v\| \), for any \( 0 \neq v \in N \). Now we define \( \sigma^i_T(\lambda) \) as the maximal dimension (with respect to \( \dim_{R_i} \)) of all the \( \lambda^- \)-spaces of \( T \) and \( \tilde{\sigma}^i_T(\lambda) \) as the maximal dimension of all the \( \lambda^+ \)-spaces of \( T \).

**Lemma 3.1** For any \( T \in R_i \), \( \sigma^i_T(\lambda) + \tilde{\sigma}^i_T(\lambda) = 1 \). If \( T = \bigoplus_{\alpha = 1}^{k_i} T_\alpha \) is a positive, self-adjoint element then \( \sigma^i_T(\lambda) \) equals to \( \sum_{\alpha = 1}^{k_i} p_{i,\alpha}N_{T_\alpha}(\lambda) \), where \( N_{T_\alpha} \) is the normalized spectral distribution function, defined in the Introduction.

**Proof.** Let \( S \in R_i \) be the positive self-adjoint matrix such that \( S^2 = T^*T \). Then \( \sigma^i_S(\lambda) = \sigma^i_T(\lambda) \) and \( \tilde{\sigma}^i_S(\lambda) = \tilde{\sigma}^i_T(\lambda) \). Indeed

\[
\|T(v)\| \leq \lambda\|v\| \iff \langle T(v),T(v) \rangle \leq \lambda^2\|v\|^2 \iff \langle T^*T(v),v \rangle \leq \lambda^2\|v\|^2 \iff \langle S^2(v),v \rangle \leq \lambda^2\|v\|^2 \iff \|S(v)\| \leq \lambda\|v\|.
\]

Hence in order to prove the equality \( \sigma^i_T(\lambda) + \tilde{\sigma}^i_T(\lambda) = 1 \) it is enough to suppose that \( T \) is positive self-adjoint. Clearly, \( \sigma^i_T(\lambda) + \tilde{\sigma}^i_T(\lambda) \leq 1 \) since the intersection of a \( \lambda^- \)-space and a \( \lambda^+ \)-space is always the zero vector. On the other hand, the eigenvectors of \( T \) corresponding to eigenvalues not greater than \( \lambda \) span a \( \lambda^- \)-space respectively the eigenvectors corresponding to eigenvalues that greater than \( \lambda \) span a \( \lambda^+ \)-space and the sum of their \( \dim_{R_i} \)-dimensions is \( 1 \).

**Lemma 3.2** For any \( T \in R_i \),

\[
\sigma^i_T(\lambda) = \sigma^{i+1}_{\phi_i(T)}(\lambda) \quad \text{and} \quad \tilde{\sigma}^i_T(\lambda) = \tilde{\sigma}^{i+1}_{\phi_i(T)}(\lambda).
\]

**Proof.** First note that \( \phi_i \) does not only map \( R_i \) into \( R_{i+1} \) but also it naturally associates \( R_{i+1} \)-submodules of \( V_{i+1} \) to \( R_i \)-submodules of \( V_i \) preserving their dimensions, that is for any \( R_i \)-submodule \( L \)

\[
\dim_{R_i}(L) = \dim_{R_{i+1}}(\phi_i(L)).
\]

Simply let \( P_L \) be the orthogonal projection onto \( L \), then we define \( \phi_i(L) \) as the range of \( \phi_i(P_L) \). If \( L \subseteq V_i \) is a \( \lambda^- \)-space of \( T \), then \( \phi_i(L) \) is a \( \lambda^- \)-space of \( \phi_i(T) \). Similarly, if \( N \subseteq V_i \) ...
is a $\lambda^+$-space of $T$, then $\phi_i(N)$ is a $\lambda^+$-space of $\phi_i(T)$. Hence the image of a maximal dimensional $\lambda^-$-space must be a maximal dimensional $\lambda^-$-space as well. □

By the previous lemma, we have spectral functions $\sigma_T(\lambda)$ and $\tilde{\sigma}_T(\lambda)$ on the inductive limit ring $\lim_{i} R_i$.

**Proposition 3.1** Let $\mathcal{R}$ be the rank closure of $\lim_{i} R_i$ as in Section 2. Suppose that $A_i \in R_i$, $A \in \mathcal{R}$ and $A_i \to A$ in the metric defined by the rank function. Then the spectral functions $\sigma_{A_i}$ uniformly converge to a function $\sigma_A$ and $\sigma_A$ does not depend on the choice of the sequence $\{A_i\}_{i=1}^\infty$. Also, for any $A, B \in R_i$; $\|\sigma_A - \sigma_B\|_\infty \leq d(A, B)$ where $\|\|_\infty$ is the $L^\infty$-norm.

**Proof.** First we need the following comparison lemma.

**Lemma 3.3** Let $T, S \in R_i$ such that $r_i(T - S) = \epsilon$. Then for any $\lambda \geq 0$,

$$|\sigma^i_T(\lambda) - \sigma^i_S(\lambda)| \leq \epsilon \quad \text{and} \quad |\tilde{\sigma}^i_T(\lambda) - \tilde{\sigma}^i_S(\lambda)| \leq \epsilon$$

**Proof.** Let $L$ be a $\lambda^-$-space for $T$. Note that $\dim_{R_i} \ker (T - S) = 1 - \epsilon$, since $r_i(T - S) = \dim_{R_i} \text{Im} (T - S)$. Also,

$$\dim_{R_i} \ker (T - S) + \dim_{R_i} L \leq 1 + \dim_{R_i} (\ker (T - S) \cap L).$$

That is

$$\dim_{R_i} (\ker (T - S) \cap L) \geq \dim_{R_i} L - \epsilon.$$

Clearly, $\ker (T - S) \cap L$ is a $\lambda^-$-space for $S$, thus our lemma follows. □

Now we finish the proof of our proposition. By [3], the spectral function $\sigma : \lim_{i} R_i \to L^\infty[0, \infty)$ is a Lipschitz-continuous map from our inductive limit ring to the Banach space of bounded functions. Hence $\sigma$ extends to the metric completion $\mathcal{R}$ in a unique continuous way. □

Later we shall use the following simple version of Lemma 3.3.

**Lemma 3.4** If $A, B$ are positive self-adjoint linear transformations in $\text{Mat}_{n \times n}(\mathbb{R})$ and $\frac{\text{Rank}(A-B)}{n} \leq \epsilon$, then for any $\lambda \geq 0$: $|N_A(\lambda) - N_B(\lambda)| \leq \epsilon.$
4 Random Schrödinger operators on lattices

Let $G_d$ be the $d$-dimensional standard lattice graph. For each vertex $p$ of $G_d$ we have independent, identically distributed random variables $X_{(p)}$ which take the real values $J = \{c_1, c_2, \ldots, c_k\}$ with probabilities $\{p_1, p_2, \ldots, p_k\}$, $p_i > 0$. Now let $\Omega = J^{G_d}$ the associated Bernoulli state space with the product measure. Thus for each $\omega \in \Omega$, we have an operator $\Delta_\omega$:

$$\Delta_\omega f(x) = \deg f(x) \omega(x) - \sum_{(x,y) \in E(G_d)} f(y).$$

**Theorem 1** For any Følner-sequence $\{Q_n\}_{n=1}^{\infty} \subset G_d$ the following holds: For almost all $\omega \in \Omega$, the spectral distribution functions $N_{\Delta_{Q_n}}$ converge uniformly to a function $N_{\Delta}$, where $N_{\Delta_{Q_n}}$ are the normalized spectral distribution functions of the truncated Laplacians $p_{Q_n} \Delta_{i_{Q_n}}$, where $p_{Q_n} : L^2(G_d) \to L^2(Q_n)$ is the standard projection operator and $i_{Q_n} : L^2(Q_n) \to L^2(G_d)$ is its adjoint, the imbedding operator. The function $N_{\Delta}$ does not depend on $\omega$.

**Proof.** First of all note that by adding a sufficiently large constant to the numbers $\{c_1, c_2, \ldots, c_k\}$ we may suppose that $\Delta_\omega$ is positive operator for all $\omega \in \Omega$. The proof shall be given in several steps.

**Step 1.** We construct a Bratteli system as in Section 2. Consider the cubes $C_i$ in $G_d$ with sidelong $2^i$. That is $C_i = \{0, 1, 2, \ldots, 2^i-1\}^d$. Now let $A_i$ be the finite set of configurations ($J$-valued functions) on $C_i$. Then $k_i = |A_i| = k^{2^d}$.

**Step 2.** Now we construct the weights $w_{\alpha,\beta}$. For each configuration $a_{i+1}^{i+1} \in A_{i+1}$ we associate $2^d$ (not necessarily different) elements of $A_i$. Namely, the cube $C_{i+1}$ is partitioned into $2^d$ dyadic cubes of sidelong $2^i$ and the configuration $a_{i+1}^{i+1}$ determines a configuration on each such subcubes. Then $w_{\alpha,\beta}$ is the number of occurences of the configuration $a_\alpha^i \in A_i$ in $a_{i+1}^{i+1}$.

**Step 3.** Now we define the parameters of our Bratelli diagrams. First of all $n_{i,\alpha} = 2^{i\alpha}$ for any $1 \leq \alpha \leq k_i$. Now let $p_{i,\alpha}$ be the probability of the configuration $a_{\alpha}^i$. That is

$$p_{i,\alpha} := \prod_{x \in C_i} P(a_{\alpha}^i(x)).$$
where of course $P(c_i) = p_i$. Thus, $\sum_{\alpha=1}^{k_i} p_{i,\alpha} = 1$ holds.

**Step 4.**

**Lemma 4.1** For any $1 \leq \alpha \leq k_i$

$$p_{i,\alpha} = \sum_{\beta=1}^{k_{i+1}} M(\beta, \alpha)p_{i+1,\beta}.$$

**Proof.** Let us consider the cube $C_{i+1}$ with sidelength $2^{i+1}$ that determines $2^d$ dyadic subcubes of sidelength $2^i$. Let us randomly choose a $J$-valued function on $C_{i+1}$ (the probability of having the value $c_i$ on the vertex $x$ is $p_i$). The expected value of the number of configurations of the type $a^i_\alpha$ on the $2^d$-subcubes is of course $2^d p_{i,\alpha}$. By the theorem of conditional expectations this value equals the sum $\sum_{\beta=1}^{k_{i+1}} p_{i+1,\beta}w_{\alpha,\beta}$, where $w_{\alpha,\beta}$ is the number of occurrences of the configuration $a^i_\alpha$ in $a^{i+1}_\beta$. Since $w_{\alpha,\beta} = 2^d M(\beta, \alpha)$, our Lemma follows. 

**Step 5.** The finite dimensional algebra $R_i$ is defined as $\bigoplus_{\alpha=1}^{k_i} \operatorname{Mat}_{2^d \times 2^d}(\mathbb{R})$, that is each element of $R_i$ is represented by $k_i 2^i$ by $2^d$ matrices. Each configuration $a^i_\alpha$ defines an element $M(a^i_\alpha) \in \operatorname{Mat}_{2^d \times 2^d}(\mathbb{R})$.

$$M(a^i_\alpha) v_x = \deg(x) P(a^i_\alpha(x)) v(x) - \sum_{y \in C_i \mid (x,y) \in G_d} v_y$$

where $v_x$ is the basis vector of $\mathbb{R}^{2^d}$ associated to the vertex $x \in C_i$ (for the sake of simplicity we identify matrices and linear transformations). Note that $\deg(x)$ is the degree of vertex $x$ in the subgraph spanned by $C_i$. Therefore we defined an element $\tilde{\Delta}_i \in R_i \subset \mathcal{R}$, where $\mathcal{R}$ is the metric completion of the inductive limit ring.

**Step 6.**

**Lemma 4.2** The elements $\{\tilde{\Delta}_i\}_{i=1}^{\infty}$ form a Cauchy-sequence in $\mathcal{R}$, hence the limit $\lim_{i \to \infty} \tilde{\Delta}_i = \tilde{\Delta}$ is well-defined.

**Proof.** We need to estimate the rank $r(\tilde{\Delta}_i - \tilde{\Delta}_j)$. First we partition $C_j$ into $2^{d(j-i)}$ dyadic subcubes. Any such subcube has boundary points and non-boundary points. Boundary points are those points which are adjacent to a point of another subcube. By our definition,

$$\tilde{\Delta}_i(v_x) = \tilde{\Delta}_j(v_x)$$
for any non-boundary point \( x \in C_j \). Hence \( r(\bar{\Delta}_i - \bar{\Delta}_j) \) is bounded above by \( \beta_{i,j} \), where

\[
\beta_{i,j} := \frac{\{ \text{the number of boundary points in } C_j \}}{|C_j|}.
\]

Since for large enough \( i \), \( \beta_{i,j} < \epsilon \) for any \( j > i \), the lemma follows.

**Step 7.** Now we turn to the proof of Theorem 1. Let \( \{Q_n\}_{n=1}^\infty \) be a Følner-sequence in \( G_d \). Let \( \omega \in \Omega \) be a configuration on \( G_d \). Pick a positive integer \( j \) and cover \( G_d \) by disjoint translates of our standard \( C_j \)-cube. That is \( G_d = \bigcup_{\vec{t}} (C_j + \vec{t}) \), where all the coordinates of the vectors \( \vec{t} \) are divisible by \( 2^j \). We compare the following two self-adjoint transformations

\[
\Delta_{Q_n} : L^2(Q_n) \to L^2(Q_n)
\]

given by \( \Delta_{Q_n} = p_{Q_n} \Delta_i Q_n \), and

\[
\Delta_{j_{Q_n}}^j : L^2(Q_n) \to L^2(Q_n)
\]

given by \( \sum_w p_{Q_n} \Delta\omega_{jw} \), where \( w \) runs through all the translated copies of \( C_i \) which are contained completely in the set \( Q_n \).

By the Følner-property, for any \( \epsilon > 0 \) there exist integers \( j_\epsilon, n_\epsilon \) such that

- \( \text{Rank} \left( \frac{\Delta_{Q_n} - \Delta_{j_{Q_n}}^j}{|Q_n|} \right) < \epsilon \).

and for any \( n > n_\epsilon \), \( \| s_{\bar{\Delta}_i} - s_{\bar{\Delta}_{j_\epsilon}} \|_\infty < \epsilon \) where \( \| \|_\infty \) denotes the \( L^\infty \)-norm.

Hence by Lemma 3.1 if \( n > n_\epsilon \)

\[
|N_{\Delta_{Q_n}}(\lambda) - N_{\Delta_{j_{Q_n}}^j}(\lambda)| < \epsilon,
\]

for any \( \lambda \geq 0 \).

**Step 8.** In order to finish the proof of our theorem it is enough to prove that for any \( \epsilon > 0 \)

\[
\limsup_{n \to \infty} \| s_{\bar{\Delta}_{j_\epsilon}} - N_{\Delta_{j_{Q_n}}^j} \|_\infty \leq \epsilon
\]

shall hold for almost all \( \omega \in \Omega \). The function \( N_{\bar{\Delta}} \) is defined as \( s_{\bar{\Delta}} \).

Again we consider the set of all configurations \( A_{j_\epsilon} \) on the cube \( C_{j_\epsilon} \). Then by Lemma 3.1

\[
s_{\bar{\Delta}_{j_\epsilon}}(\lambda) = \sum_{w_\alpha \in A_{j_\epsilon}} p_{j_\epsilon,\alpha} N_{w_\alpha}(\lambda),
\]
where $N_{\omega}^{\alpha}$ is the normalized spectral distribution function of the transformation $\Delta_{\omega}^{\alpha}$. On the other hand,

$$N_{\Delta_{Q_n}^{j_\alpha}}(\lambda) = \sum_{\omega_\alpha \in A_{j_\alpha}} R_{j_\alpha,\alpha}^n N_{\omega}^{\alpha}(\lambda),$$

where $R_{j_\alpha,\alpha}^n$ is the number of occurrences of the configuration $\omega_\alpha$ in the $C_{j_\alpha}$-translates contained by $Q_n$ divided by the number of all such translates. By the Theorem of Large Numbers, for almost all $\omega \in \Omega$: $\lim_{n \to \infty} R_{j_\alpha,\alpha}^n = P_{j_\alpha,\alpha}$ holds for all configuration $\omega_\alpha \in A_{j_\alpha}$. Hence the Theorem follows.

## 5 Percolation Hamiltonians

In this section we apply our approximation method to prove uniform spectral convergence in the case of bond- and site-percolation Hamiltonians. The pointwise spectral convergence was proved in this case in [3] and [8].

**The bond percolation Hamiltonian:** Again we work on the lattice $G_d$. For the edges $e \in E(G_d)$ we consider independent identically distributed random variables $X(e)$, such that

$$P(X(e) = 0) = p, \quad P(X(e) = 1) = 1 - p.$$

The associated Bernoulli-state space $\Omega^B = \prod_{e \in E(G_d)} \{0, 1\}$ is equipped with the standard product measure. Then each $\omega \in \Omega$ defines a subgraph $G^B_\omega \subset G_d$, where the vertex set is the lattice and $e \in E(G^B_\omega)$ if and only if $\omega(e) = 0$.

**Theorem 2** For all Følner-sequences $\{Q_n\}_{n=1}^\infty \subset G_d$ (in the original sense in the lattice) the following statement holds: For almost all $\omega \in \Omega$ the normalized spectral distribution functions $N_{\Delta_{Q_n}}$ uniformly converge to the integrated density of state function $N^B_{\Delta}$. The function $N^B_{\Delta}$ does not depend on $\omega$. The normalized spectral distribution functions $N_{\Delta_{Q_n}}$ are associated to the Laplacian operator of the subgraphs of $G^B_\omega$ spanned by the vertices of $Q_n$.

**The site percolation Hamiltonian:** In the case of site percolation Hamiltonian we consider independent identically distributed random variables (with the same distribution as above)
indexed by the vertices of the lattice $G_d$. Again, the state space $\Omega^S = \prod_{p \in G_d} \{0, 1\}$ is equipped with the product measure. For $\omega \in \Omega^S$ one has the subgraph $G^S_\omega \subset G^d$, where $e \in E(G^S_\omega)$ if and only if $\omega(p) = \omega(q) = 0$ for the endpoints of $e$.

**Theorem 3** For all Følner-sequences $\{Q_n\}_{n=1}^\infty \subset G_d$ the following statement holds: For almost all $\omega \in \Omega$ the normalized spectral distribution functions $N_{\Delta_{Q_n}}$ uniformly converge to the integrated density of state function $N^S_{\Delta}$. The function $N^S_\Delta$ does not depend on $\omega$. The normalized spectral distribution functions $N_{\Delta_{Q_n}}$ are associated to the Laplacian operator of the subgraphs of $G^S_\omega$ spanned by the vertices of $Q_n$.

**Proof.** (of Theorem 2, the proof of Theorem 3 is completely analogous)

**Step 1.** Let $A_i$ be the finite configuration space of all $\{0, 1\}$-valued functions on lattice edges in the $2^i$-cube $C_i$, $k_i = |A_i|$.

**Step 2.** For $a^i_\alpha \in A_i$ and $a^{i+1}_\beta \in A_{i+1}$ let $w_{\alpha, \beta}$ be the number of occurrences of the configuration $a^i_\alpha \in A_i$ in the $2^d$ subconfigurations determined by $a^{i+1}_\beta \in A_{i+1}$.

**Step 3.** For $a^i_\alpha \in A_i$, let $p_{i, \alpha}$ the configuration probability that is

$$p_{i, \alpha} := p^{T(a^i_\alpha)}(1 - p)^{S(a^i_\alpha)},$$

where $T(a^i_\alpha)$ is the number of zeroes in the configuration $a^i_\alpha$ and $S(a^i_\alpha)$ is the number of ones in the configuration $a^i_\alpha$.

**Step 4.**

$$p_{i, \alpha} = \sum_{\beta=1}^{k_{i+1}} M(\beta, \alpha) p_{i+1, \beta}.$$ 

follows from the conditional expectation argument as in the proof of Theorem 1.

**Step 5.** The finite dimensional algebra $R_i$ is again isomorphic to $\bigoplus_{\alpha=1}^{k_i} \text{Mat}_{2^d \times 2^d}(\mathbb{R})$. Each configuration $a^i_\alpha$ defines an element $M(a^i_\alpha) \in \text{Mat}_{2^d \times 2^d}(\mathbb{R})$ as the associated Laplacian operator $\Delta_{a^i_\alpha} : L^2(C_i) \to L^2(C_i)$. Hence again we defined an element $\bar{\Delta}_i \in R_i \subset R$.

**Step 6.** The fact that the elements $\{\bar{\Delta}_i\}_{i=1}^\infty$ form a Cauchy sequence in $R$ can be seen exactly the same way as in the proof of Theorem 1.

**Step 7.** Let $\{Q_n\}_{n=1}^\infty$ be a Følner-sequence in $G_d$. Let $\omega \in \Omega$ be a configuration on $G_d$. We consider the same translated copies as in the proof of Theorem 1. Let us compare the
following two self-adjoint transformations:

$$\Delta_{Q_n} : L^2(Q_n) \to L^2(Q_n)$$

where $\Delta_{Q_n}$ is the Laplacian of the subgraph of $G^B_\omega$ spanned by the vertices of $Q_n$ and

$$\Delta^j_{Q_n} : L^2(Q_n) \to L^2(Q_n)$$

given by $\sum_w \Delta_w$, where $\Delta_w$ is the Laplacian associated to the dyadic subcube $w$. Again, by the Følner-property for any $\epsilon > 0$ there exist integers $j_\epsilon, n_\epsilon$ such that

- $\frac{\text{Rank}(\Delta_{Q_n} - \Delta^j_{Q_n})}{|Q_n|} < \epsilon$.

- and for any $n > n_\epsilon$, $\|s_{\Delta} - s_{\Delta^j_{Q_n}}\|_\infty < \epsilon$.

Hence by Lemma 3.4 if $n > n_\epsilon$

$$|N_{\Delta_{Q_n}}(\lambda) - N_{\Delta^j_{Q_n}}(\lambda)| < \epsilon,$$

for any $\lambda \geq 0$.

Step 8. The final argument, that for any $\epsilon > 0$

$$\limsup_{n \to \infty} \|s_{\Delta^j_{Q_n}} - N_{\Delta^j_{Q_n}}\|_\infty \leq \epsilon$$

holds for almost all $\omega \in \Omega$ follows exactly the same way as in the case of the random Schrödinger operators.

6 Pattern-invariant operators on self-similar graphs

Let $G(V, E)$ be a connected infinite graph with bounded vertex degrees. A pattern-invariant operator $A : L^2(V) \to L^2(V)$ (given by its operator kernel $A : V \times V \to \mathbb{R}$) satisfies the following properties:

- $A(x, y) = 0$ if $d_G(x, y) > r_A$, where $d_G$ is the shortest path distance on the vertices.

The value $r_A$, the propagation of $A$, depends only on $A$.  

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A(x, y) = A(ψ(x), ψ(y)) if ψ is a graph isomorphism (there can be more than one) between the \( r_\mathcal{A} \)-ball around \( x \) and the \( r_\mathcal{A} \)-ball around \( ψ(x) \), mapping of course \( x \) to \( ψ(x) \).

Note that

\[
A(f)(x) = \sum_{y \in V} A(x, y)f(y),
\]

for \( f \in L^2(V) \). These sort of operators were considered in [4], [5], and [8].

Now we define self-similar graphs (there are plenty of definitions we choose one which fits our purposes). First we fix two positive integers \( d \) and \( k \). Let \( G_1 \) be a finite connected graph with vertex degree bound \( d \) with a distinguished subset \( S_1 \subset V(G_1) \), which we call the set of connecting vertices. Now we consider the graph \( \tilde{G}_1 \), which consists of \( k \) disjoint copies of \( G_1 \) with following additional properties:

- The graph \( G_1 \) is identified with the first copy.
- In each copy the vertices associated to a connecting vertex of \( G_1 \) is a connecting vertex of the graph \( \tilde{G}_1 \).

The graph \( G_2 \) is defined by adding some edges to \( \tilde{G}_1 \) such that both endpoints of these new edges are connecting vertices. The resulting graph should still have vertex degree bound \( d \). Finally the subset \( S_2 \subset V(G_2) \) is chosen as a subset of the connecting vertices of \( \tilde{G}_1 \) such that \( S_2 \cap V(G_1) = \emptyset \). That is \( G_1 \subset G_2 \) is a subgraph and vertices of \( G_2 \) which are not in \( G_1 \) are connecting vertices. Inductively, suppose that the finite graphs \( G_1 \subset G_2 \subset \ldots \subset G_n \) are already defined and the vertex degrees in \( G_n \) are not greater than \( d \). Also suppose that a set \( S_n \subset V(G_n) \) is given and \( S_n \cap V(G_{n-1}) = \emptyset \). Now the graph \( \tilde{G}_n \) consists of \( k \) disjoint copies of \( G_n \) and

- The graph \( G_n \) is identified with the first copy.
- In each copy the vertices associated to a connecting vertex of \( G_n \) is a connecting vertex of the graph \( \tilde{G}_n \).

Again, \( G_{n+1} \) is constructed by adding edges to \( \tilde{G}_n \) with endpoints which are connecting vertices, preserving the vertex degree bound condition. The set of connecting vertices \( S_{n+1} \)
is chosen as a subset of the connecting vertices of $\tilde{G}_n$, such that $S_{n+1} \cap V(G_n) = \emptyset$. The union of the graphs $\{G_n\}_{n=1}^\infty$ is a connected infinite graph with vertex degrees not greater than $d$. The graph $G$ is self-similar if $\lim_{n \to \infty} \frac{|S_n|}{|V(G_n)|} = 0$. In this case $\{V(G_n)\}_{n=1}^\infty$ form a Følner-sequence of $G$. Note that all Euclidean lattices is self-similar in our sense.

**Theorem 4** Let $G$ be a self-similar graph and $A$ be a self-adjoint pattern-invariant operator. Then there exists a real function $N_A$ such that for any Følner-sequence $\{Q_n\}_{n=1}^\infty \subset V(G)$, $N_{A_{Q_n}}$ uniformly converge to $N_A$, where $N_{A_{Q_n}}$ are the spectral distribution functions of the truncated operators $p_{Q_n} A i_{Q_n}$.

In this case we have the simplest possible Bratteli system (the one originally considered by John von Neumann):

$\text{Mat}_{l \times l}(\mathbb{R}) \overset{\phi_1}{\to} \text{Mat}_{k_1 \times k_1}(\mathbb{R}) \overset{\phi_2}{\to} \ldots$

That is $\mathcal{R}_i = \text{Mat}_{k_i \times l \times k_{i-1} \times l}(\mathbb{R})$, where $l = |V(G_1)|$. For each $i \geq 1$, $k_i = 1$ and each weight equals to $k$. Then $\tilde{\Delta}_i \in \mathcal{R}_i$ is just the operator $A_{G_i}$. The proof of Theorem 4 is completely analogous to the one of Theorem 1 and left for the interested reader.

### 7 A general conjecture

One can easily see that for our self-similar graphs, uniform spectral convergence holds even for the random Schrödinger operators and percolation Hamiltonians. For nice Markovian substitution systems our method works without any trouble. The most general graphs for which universal spectral convergence might hold are the following ones.

Let $G(V, E)$ be a connected infinite graph with bounded vertex degrees. We suppose that $G$ is an amenable graph, that is $G$ has Følner-sequence $\{F_n\}_{n=1}^\infty$. The $r$-pattern of a vertex $x \in V$ is the graph automorphism class of the rooted ball around $x$. That is $x, y \in V$ have the same $r$-pattern if there exists a graph isomorphism $\phi$ between the balls $B_r(x)$ and $B_r(y)$ such that $\phi(x) = y$. Denote by $P_r(G)$ the finite set of all possible $r$-patterns in $G$. We say that $G$ is an abstract quasicrystal graph if for any $\alpha \in P_r(G)$ there exists a frequency $P(\alpha)$ such that
For any Følner-sequence \( \{Q_n\}_{n=1}^\infty \):

\[
\lim_{n \to \infty} \frac{|Q_n^\alpha|}{|Q_n|} = P(\alpha),
\]

where \( Q_n^\alpha \subseteq Q_n \) is the set of vertices in \( Q_n \) having the \( r \)-pattern \( \alpha \).

The graphs considered by Lenz and Stollmann are such abstract quasicrystal graphs. Another interesting examples are the Cayley graphs of finitely generated amenable groups.

**Conjecture 1** If \( G \) is an abstract quasicrystal graph, then for the random Schrödinger operators, percolation Laplacians as well as self-adjoint pattern invariant operators associated to \( G \) the uniform spectral convergence exists.

Note that for the Laplacian operator the existence of the integrated density of states in the weak sense of convergence follows from the argument of Theorem 3. Even for the general pattern-invariant operators and the random operators the limit graph method yields the existence of the integrated density of states. The problem is to handle the jumps of the integrated density of states. One should prove that for each value \( \lambda \), where the integrated density of state function jumps the normalized dimensions of the eigenfunctions of the truncated operators on the Følner-sequence converge exactly to the size of the jump. The same problem emerges in the theory of \( L^2 \)-invariants in the so-called Approximation Theorem (see [3]).

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