Option Pricing for Symmetric Lévy Returns with Applications

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Abstract

This paper considers options pricing when the assumption of normality is replaced with that of the symmetry of the underlying distribution. Such a market affords many equivalent martingale measures (EMM). However we argue (as in the discrete-time setting of [23]) that an EMM that keeps distributions within the same family is a “natural” choice. We obtain Black-Scholes type option pricing formulae for symmetric Variance-Gamma and symmetric Normal Inverse Gaussian models.

Keywords: Symmetric distribution, Lévy processes, equivalent martingale measure, risk-neutral pricing, option pricing, Variance Gamma process, Normal Inverse Gaussian process.

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1 Introduction

In the classical Black-Scholes model the stock price follows a geometric Brownian motion and the return process is a Brownian motion with drift. In some cases empirical evidence shows that a more general symmetric distribution is more appropriate for returns – see for example [28], [29], [10], [9], [18], [19].
In this work we replace the assumption of normality with that of symmetry, while retaining all other assumptions such as independence and stationarity of increments. This leads to returns that are symmetric Lévy processes; such stock prices are known as log-symmetric Lévy processes.

We adopt a classical approach to the definition of symmetry; a random variable \( Y \) has a symmetric distribution if for some \( \mu \), the location parameter, \( (Y - \mu) \) and \( -(Y - \mu) \) have the same distribution. In turn, a symmetric Lévy process is defined as having symmetric marginal distributions. This is easily shown to reduce to assuming that the Lévy measure is symmetric (about zero): \( \nu(A) = \nu(-A) \) for any Borel set \( A \subset \mathbb{R} \).

Our definition of symmetry differs from that in [9] where a Lévy market is said to be symmetric if a certain law before and after the change of measure through Girsanov’s theorem coincide.

The literature on option pricing with Lévy processes is vast – see for example, [2], [5], [4], [9], [25], [33]. In particular, it is well known that Lévy market models, barring the Brownian motion case, are incomplete ([33], p.77) and the choice of an EMM is not unique. In fact there are infinitely many EMMs to choose from and any selection is arbitrary and motivated by various other considerations. Among the most popular methods are the Esscher transform ([15], [21], [4]), minimum entropy martingale measure ([13], [14], [39]), minimal martingale measure ([12], [4]), minimax and minimal distance martingale measure ([17], variance-optimal martingale measure ([34], and mean-correcting martingale measure ([33], chapter 6).

In some cases the Esscher transform produces a continuum of EMMs that require further refinement on the selection by optimizing the relative entropy or some other utility function. [24]. In reality, it is hard to tell which measure the market chooses, and this topic requires further research.

The choice of EMM is important not only for obtaining the price of an option but also for calculating hedging parameters. By the change of numéraire formula, these parameters give probabilities of option exercise under different EMMs.

In the case of Lévy processes with symmetric marginal distributions there is a unique EMM within the same family of distributions as the real world distribution. If the process has a Brownian component, then the natural EMM is the same as in the classical case, obtained by changing the drift (location) parameter. If the process does not have a a Brownian component (and \( \mu < r \) – see Section 4, then the natural EMM is obtained by changing the variance (scale) parameter. In both cases, we obtain closed form option pricing formulae in which the normal distribution is replaced by other symmetric distributions. This is reminiscent of the suggestion made by McDonald [27]
in 1996, with the difference that in that paper arbitrage is possible whereas
our approach is arbitrage-free.

The search for a “natural” EMM under symmetry started in [23] in a
discrete-time setting. In this paper we extend the exploration to continuous-
time models. The main contributions of the present work can be summarized
as follows.

- The model for the stock price process is \( S_t = S_0 e^{Y_t} \), where \( Y_t \) is a
  symmetric Lévy process.
- As a Lévy process, \( Y_t \) is specified by the characteristic triplet \((\mu, c, \nu)\).
  As a random variable, it is described by the parameters \((\mu, \sigma^2, \psi)\) of
  the symmetric family of \( Y_1 \). We show how the two characterizations
  relate to each other.
- We construct an equivalent measure \( Q \) under which
  (1) the symmetric Lévy process \( Y_t \) remains a symmetric Lévy process;
  (2) the distribution of the Lévy process \( Y_t \) remains in the same sym-
      metric family of distributions as the real world distribution;
  (3) the discounted price process \( e^{-rt} S_t \) is a martingale.

We call such a change of measure a natural equivalent martingale
measure.
- We derive option pricing formulae under the natural EMM.

A brief account of Lévy processes and symmetric distributions necessary for
our purposes are given in Section 2. In Section 3, we give the construction of
a natural equivalent martingale measure for log-symmetric Lévy processes.
In Section 4 we consider the option pricing with a natural EMM. Section 5
contains applications of this approach to symmetric Variance Gamma and
Normal Inverse Gaussian models.

2 Preliminaries

2.1 Lévy Processes with Symmetric Marginal Distributions

A Lévy process \((Y_t)_{t \geq 0}\) on \(\mathbb{R}\) is a process with independent and stationary
increments. It is defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) endowed with a
complete filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) to which \(Y_t\) is adapted. \(Y_t\) has right-continuous
with left limits sample paths, and \(Y_t - Y_s\) is independent of \(\mathcal{F}_s\) and has the
same distribution as $Y_{t-s}$ for $0 \leq s < t$. A Lévy process is fully determined by its initial value, $Y_0$, here assumed to be nil, and the distribution of the increment over one unit time interval, $Y_1$. The distribution of $Y_t$ is infinitely divisible for any $t$, and its characteristic function satisfies

$$ \mathbb{E}(e^{iuY_t}) = \left( \mathbb{E}[e^{iuY_1}] \right)^t, \quad u \in \mathbb{R}. $$

By the Lévy-Khintchine representation,

$$ \mathbb{E}[e^{iuY_1}] = e^{\Lambda(u)}, $$

with the characteristic exponent

$$ \Lambda(u) = i\mu u - \frac{1}{2}c^2u^2 + \int_{\mathbb{R}} \left( e^{iuy} - 1 - iuy1_{\{|y| \leq 1\}} \right) \nu(dy), $$

where $\mu \in \mathbb{R}$, and $\nu$ is a Lévy measure satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}} (1 \wedge y^2)\nu(dy) < \infty$. The triplet $(\mu, c, \nu)$ is referred to as the characteristic triplet of $Y$.

We call a Lévy process $(Y_t)_{t \geq 0}$ symmetric if, for each $t \geq 0$, the random variable $Y_t$ is symmetric (about the location parameter $\mu$): $(Y_t - \mu)$ and $(\mu - Y_t)$ have the same distribution. By (2), this is easily seen to be equivalent to the random variable $Y_1$ being symmetric (about $\mu$), and by (3), to the Lévy measure $\nu$ being symmetric (about 0): for any Borel set $A \subset \mathbb{R}$, $\nu(-A) = \nu(A)$, where $-A = \{x \in \mathbb{R} : -x \in A\}$. In this case, the characteristic exponent $\Lambda$ can be written as (e.g. [32], p.263)

$$ \Lambda(u) = iu\mu - \frac{1}{2}c^2u^2 - 2\int_{0}^{\infty} (1 - \cos(uy))\nu(dy). $$

In what follows we assume that $Y_1$ has finite mean and variance (in fact, finite exponential moments). It is easy to see that, in this case, the mean of $Y_1$ is precisely $\mu$

$$ \mathbb{E}[Y_1] = \mu, $$

and the variance $\sigma^2$ is given by (e.g. [6], proposition 3.13)

$$ \sigma^2 = \text{Var}(Y_1) = c^2 + \int_{\mathbb{R}} y^2\nu(dy). $$

On the other hand, the random variable $Y_1$ has a symmetric distribution with location $\mu$ and scale $\sigma$. As such its characteristic function takes the form

$$ \phi_{Y_1}(u) = e^{iu\mu} \psi \left( \frac{\sigma^2}{2}u^2 \right), $$
where the function $\psi(u) : [0, \infty) \to \mathbb{R}$ is called the characteristic generator of the symmetric family (e.g. [11], p.32). $\psi$ is unique up to scaling, and if chosen such that $\psi'(0) = -1$, yields that $\mu$ and $\sigma^2$ are the mean and variance of $Y_1$ respectively. We denote by $S(\mu, \sigma^2, \psi)$ the distribution whose characteristic function is of the form (7).

A detailed account of the properties of symmetric distributions (also known as elliptical distributions) is given in Fang et al. [11].

2.2 Symmetric Lévy Processes and Marginals

The following proposition relates the characteristic triplet of a symmetric Lévy process $(Y_t)_{t \geq 0}$ to the parameters of the symmetric distribution of $Y_1$.

**Proposition 2.1.** Let $(Y_t)_{t \geq 0}$ be a symmetric Lévy process with characteristic triplet $(\mu, c, \nu)$. Then $Y_1$ has distribution $S(\mu, \sigma^2, \psi)$ where $\sigma^2$ is given by (6), and $\psi$ by

$$
\psi(v) = \exp \left\{ -\frac{c^2 v}{\sigma^2} - 2 \int_0^\infty \left( 1 - \cos(y \sqrt{2v/\sigma}) \right) \nu(dy) \right\},
$$

and $v = \frac{\sigma^2 u^2}{2}$. Furthermore, $Y_t$ has distribution $S(\mu t, \sigma^2 t, \psi_t)$ with

$$
\psi_t(v) = \left( \psi(v/t) \right)^t.
$$

**Proof.** The proof is a straightforward examination of the characteristic function. The form of $\psi_t$ is due to

$$
\mathbb{E}[e^{iuY_t}] = \left( \mathbb{E}[e^{iuY_1}] \right)^t = (\varphi_{Y_1}(u))^t = e^{i\mu t} \left( \psi \left( \frac{\sigma^2 t u^2}{2t} \right) \right)^t.
$$

2.3 Equivalent Change of Measure for Lévy Processes

In general, a Lévy process under an equivalent measure need not remain Lévy, as independence of increments may not be preserved. However, there is a class of equivalent measures under which it does.

**Theorem 2.2.** Let $Y_t$ be a Lévy process on $\mathbb{R}$ with characteristic triplet $(\mu, c, \nu)$ under $\mathbb{P}$. Let $\eta \in \mathbb{R}$ and a function $\phi$ be arbitrary such that

$$
\int_\mathbb{R} \left( e^{\phi(y)/2} - 1 \right)^2 \nu(dy) < \infty.
$$

Then
1. The limit

$$\lim_{\epsilon \downarrow 0} \left( \sum_{s \leq t, |\Delta Y_s| > \epsilon} \phi(\Delta Y_s) - t \int_{|y| > \epsilon} (e^{\phi(y)} - 1) \nu(dy) \right),$$

exists (uniformly in \(t\) on any bounded interval).

2. The process

$$D_t = \eta Y^c_t - \frac{\eta^2 c^2 t}{2} - \eta \mu t$$

$$+ \lim_{\epsilon \downarrow 0} \left( \sum_{s \leq t, |\Delta Y_s| > \epsilon} \phi(\Delta Y_s) - t \int_{|y| > \epsilon} (e^{\phi(y)} - 1) \nu(dy) \right),$$

where \(Y^c_t\) is the continuous part of \(Y_t\), defines a probability measure \(Q\) equivalent to \(P\) by

$$\frac{dQ}{dP}_{|\mathcal{F}_t} = e^{D_t}. \quad (10)$$

3. The process \(Y_t\) remains a Lévy process under \(Q\) with characteristic triplet \((\tilde{\mu}, c, \tilde{\nu})\), where

$$\tilde{\mu} = \mu + \int_{-1}^{1} y(\tilde{\nu} - \nu)(dy) + c^2 \eta$$

and

$$\tilde{\nu}(dy) = e^{\phi(y)} \nu(dy).$$

4. Conversely, any probability measure equivalent to \(P\) under which \(Y_t\) remains a Lévy process must be of the form \((10)\) and the characteristic triplet must be \((\tilde{\mu}, c, \tilde{\nu})\) as specified above.

**Proof.** This theorem is a direct consequence of Lemma 33.6 and Theorems 33.1 and 33.2 of Sato [32]. Statements 1. and 2. follow from Lemma 33.6. The Lévy property of the process \(Y_t\) under \(Q\) is a consequence of Theorem 33.2. The form of its characteristic triplet is given by Theorem 33.1. Statement 4. is given by Theorems 33.1 and 33.2.

Next we consider the case of a symmetric Lévy process \(Y_t\). In order that it remains symmetric under an equivalent measure \(Q\), we show that it is necessary and sufficient that the function \(\phi\) given in the above theorem be even. We also describe how the parameters of the symmetric family transform under the change of measure.
Theorem 2.3. Let $Y_t$ be a Lévy process on $\mathbb{R}$ with characteristic triplet $(\mu, c, \nu)$ under $\mathbb{P}$, and $Q$ be any “Lévy-preserving” equivalent change of measure as described in Theorem 2.2. Then $Y_t$ is symmetric, or equivalently the Lévy measure $\tilde{\nu}$ is symmetric, if and only if $\phi(-y) = \phi(y)$ $\nu$-a.e., and in this case, the $Q$-distribution of $Y_1$ is $S(\tilde{\mu}, \tilde{\sigma}^2, \tilde{\psi})$ where

\[
\tilde{\mu} = \mu + c^2 \eta, 
\]  
\[
\tilde{\sigma}^2 = c^2 + \int_{\mathbb{R}} y^2 e^{\phi(y)} \nu(dy), 
\]  
\[
\tilde{\psi}(v) = \exp \left\{ -\frac{c^2 v^2}{\tilde{\sigma}^2} - 2 \int_{0}^{\infty} (1 - \cos(y \sqrt{2v} / \tilde{\sigma})) e^{\phi(y)} \nu(dy) \right\}. 
\]

Proof. Since $\tilde{\nu}(dy) = e^{\phi(y)} \nu(dy)$, the evenness of $\phi$ is clearly equivalent to the symmetry of $\tilde{\nu}$. Furthermore, if $\tilde{\nu}$ and $\nu$ are both symmetric, then $\int_{-1}^{1} \tilde{y} (\tilde{\nu} - \nu)(dy) = 0$. Hence, $\tilde{\mu} = \mu + c^2 \eta$. The other two parameters are obtained from (11) and (13). \qed

3 The Natural Change of Measure

Consider a symmetric Lévy process $Y_t$ with $\mathbb{P}$-characteristic triplet $(\mu, c, \nu)$, and $\mathbb{P}$-distribution of $Y_1$, $S(\mu, \sigma^2, \psi)$. We call an equivalent measure $Q$ natural for $Y_t$ if $Y_t$ is Lévy under $Q$ and the $Q$-distribution of $Y_1$ belongs to a family of symmetric distributions with same characteristic generator, $\tilde{\psi} = \psi$. When searching for a natural change of measure, an interesting fact emerges; there is, up to a constant, a unique natural equivalent measure for each Lévy process. Furthermore, the specific change of measure takes a dichotomous form depending on whether or not a Brownian component is present. The next two theorems detail these facts. We start with the uniqueness result, then show existence.

Theorem 3.1. Let $Y_t$ be a symmetric Lévy process with $\mathbb{P}$-characteristic triplet $(\mu, c, \nu)$, and $\mathbb{P}$-distribution of $Y_1$, $S(\mu, \sigma^2, \psi)$. Suppose $Q$ is a natural change of measure.

1. If $c \neq 0$ (a Brownian component is present), then under $Q$, the characteristic triplet becomes $(\tilde{\mu}, c, \nu)$, where $\tilde{\mu} = \mu + c^2 \eta$, for some $\eta$. In this case, $c$ and $\nu$ remain unchanged.
If $c = 0$ (no Brownian component is present), then under $Q$ the characteristic triplet becomes $(\mu, 0, \tilde{\nu})$ where $\tilde{\nu}(A) = \int 1_A(\beta y)\nu(dy)$ for some $\beta > 0$. In this case, $\mu$ and $c$ remain unchanged.

**Proof.** Since we only consider equivalent measures $Q$ that preserve Lévy property, we denote the characteristic triplet under $Q$ with $(\tilde{\mu}, c, \tilde{\nu})$. By Theorem 2.2 we know that

$$\tilde{\nu}(dy) = e^{\phi(y)}\nu(dy)$$

for some $\phi$.

The proof of the Theorem uses Theorem 2.2 and an analytical lemma.

Using expressions (8) and (13), we can see that $Q$ is natural (ie $\psi = \tilde{\psi}$) if and only if for all $v > 0$, the function $\phi$ in (14) is even and satisfies the following integral equation

$$\int_0^\infty \left[ (1 - \cos(y\sqrt{2v}/\tilde{\sigma}))e^{\phi(y)} - (1 - \cos(y\sqrt{2v}/\sigma)) \right] \nu(dy) + \frac{c^2}{2} \left( \frac{1}{\tilde{\sigma}^2} - \frac{1}{\sigma^2} \right) = 0. \quad (15)$$

In Lemma 7.2 we show that

$$\lim_{v \to \infty} \int_0^\infty \frac{1}{v} \left[ (1 - \cos(y\sqrt{2v}/\tilde{\sigma}))e^{\phi(y)} - (1 - \cos(y\sqrt{2v}/\sigma)) \right] \nu(dy) = 0. \quad (16)$$

Hence by dividing by $v$ and taking limit in (15), we obtain that

$$\frac{c^2}{2} \left( \frac{1}{\tilde{\sigma}^2} - \frac{1}{\sigma^2} \right) = 0, \quad (17)$$

which reduces (15) to

$$\int_0^\infty \left[ (1 - \cos(y\sqrt{2v}/\tilde{\sigma}))e^{\phi(y)} - (1 - \cos(y\sqrt{2v}/\sigma)) \right] \nu(dy) = 0. \quad (18)$$

(1) Consider first the case $c \neq 0$. It follows from (17) that $\sigma^2 = \tilde{\sigma}^2$, and re-parameterizing (18) using $\omega = \frac{\sqrt{2v}}{\tilde{\sigma}} = \frac{\sqrt{2v}}{\sigma}$ we get that

$$\int_0^\infty (1 - \cos(\omega y))\tilde{\nu}(dy) = \int_0^\infty (1 - \cos(\omega y))\nu(dy), \quad \forall \omega > 0.$$

It is now, at least intuitively, clear that this implies that $\tilde{\nu} = \nu$. This is however not straightforward and requires a detailed proof. As it is a
technical matter, it is given in the Appendix in Lemma 7.1. Note that in this case, and since \( \tilde{\sigma} = \sigma \),
\[
\int_{0}^{\infty} y^2 \tilde{\nu}(dy) = \int_{0}^{\infty} y^2 \nu(dy).
\]

(2) Consider now the case \( c = 0 \). Clearly \( \tilde{\mu} = \mu + c^2 \eta = \mu \). Also, with \( \beta = \tilde{\sigma}/\sigma \), \( \lambda = \sqrt{2v/\tilde{\sigma}} \) and \( \nu_\beta(dy) = \nu(\frac{1}{\beta}dy) \), (18) becomes
\[
\int_{0}^{\infty} (1 - \cos(\lambda y)) \tilde{\nu}(dy) = \int_{0}^{\infty} (1 - \cos(\lambda y)) \nu_\beta(dy), \quad \forall \lambda > 0.
\]
Again, using Lemma 7.1 we get that \( \tilde{\nu} = \nu_\beta \) a.e.

**Theorem 3.2.** Let \( Y_t \) be a symmetric \( \tilde{\nu} \) a.e.

**Theorem 3.2.** Let \( Y_t \) be a symmetric Lévy process with \( \mathbb{P} \)-characteristic triplet \( (\mu, c, \nu) \), and \( \mathbb{P} \)-distribution of \( Y_1 \) \( S(\mu, \sigma^2, \psi) \).

(1) If \( c \neq 0 \) (a Brownian component is present), then for any \( \eta \) there is a natural change of measure \( Q \), such that the characteristic triplet becomes \( (\tilde{\mu}, c, \nu) \), where \( \tilde{\mu} = \mu + c^2 \eta \). In this case, the \( Q \)-distribution of \( Y_1 \) is \( S(\tilde{\mu}, \sigma^2, \psi) \).

(2) If \( c = 0 \) (no Brownian component is present), then for any \( \beta > 0 \) there is a natural change of measure \( Q \), such that the characteristic triplet becomes \( (\mu, 0, \tilde{\nu}) \), where \( \tilde{\nu}(A) = \int 1_A(\beta y)\nu(dy) \). In this case, the \( Q \)-distribution of \( Y_1 \) is \( S(\mu, \tilde{\sigma}^2, \psi) \), where \( \tilde{\sigma} = \beta \sigma \).

**Proof.** The proof immediately follows from Theorems 2.2 and 2.3 as well as from the proof of Theorem 3.1. It is also easy to check that the \( Q \)-distribution of \( Y_1 \) is as claimed.

**4 Option Pricing with a Natural EMM**

**4.1 Natural Equivalent Martingale Measures**

Let now \( S_t = S_0 e^{Y_t} \) be a model for stock prices, where \( Y_t \) is a symmetric Lévy process. According to the Fundamental Theorems of Mathematical Finance, options on stock are priced by using an EMM \( Q \), under which the discounted stock price process \( e^{-rt}S_t \), \( 0 < t \leq T \) is a martingale.

To the requirement that \( Q \) be a natural equivalent measure we now add the condition that it also be a martingale measure.
Theorem 4.1. (1) Let $Q$ be a natural EMM for a symmetric Lévy process, then the following relation must hold between the parameters of the $Q$-distribution of $Y_1$,

$$
\tilde{\mu} + \ln \psi\left(-\bar{\sigma}^2/2\right) = r. \tag{19}
$$

(2) If a Brownian component is present ($c \neq 0$) and $Q$ is a natural EMM, then

$$
\tilde{\mu} = r - \ln \psi\left(-\sigma^2/2\right). \tag{20}
$$

Further, such $Q$ exists and is unique.

(3) If a Brownian component is absent ($c = 0$) and $Q$ is a natural EMM, then $\bar{\sigma}^2$ is a root of equation

$$
\ln \psi\left(-\bar{\sigma}^2/2\right) = r - \mu. \tag{21}
$$

Further, such $Q$ exists if and only if the $\mu < r$, and when it exists it is unique.

Proof. Imposing the martingale property under $Q$ to $e^{-rt}S_t$ leads to the requirement that

$$
\mathbb{E}_Q [e^{Y_{t-s}}] = e^{r(t-s)}.
$$

On the other hand, since under the natural EMM $Y_1$ has distribution $S(\tilde{\mu}, \bar{\sigma}, \psi)$, we have

$$
\mathbb{E}_Q [e^{Y_1}] = e^{\tilde{\mu}}\psi\left(-\bar{\sigma}^2/2\right).
$$

Therefore the martingale property holds if and only if

$$
e^{\tilde{\mu}}\psi\left(-\bar{\sigma}^2/2\right) = e^r,
$$

which is equivalent to (19).

By the natural change of measure Theorem 3.1 if $c \neq 0$, only $\mu$ can be changed, consequently we obtain (20). If $c = 0$, only $\sigma^2$ can be changed, hence we obtain (21).

When $c = 0$, the requirement that $\mu < r$ immediately follows by Jensen’s inequality: $\mathbb{E}_Q [e^{Y_1}] \geq e^\mu$. \hfill \square

Remark 4.2 (Discrete time). In discrete time the natural EMM always exists, in contradiction with the continuous-time setting of Lévy process without Brownian component. Furthermore, the parameters satisfy (19) – for details, see [23]. In discrete time, the natural EMM always exists (and is unique) and is obtained by changing the location $\mu$ while keeping $\sigma^2$. 

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4.2 Option Pricing with a Natural EMM

According to the method of pricing by no arbitrage, the value of the option at time 0 is given by the expectation of the payoff function, i.e.,

\[ C_0 = e^{-rT}E_Q[(S_T - K)^+] \tag{22} \]

where \( T \) is the time to maturity, \( K \) is the strike price, and \( Q \) is an EMM. The above formula (22) is arbitrage-free even when \( Q \) is not unique (\[7\], \[35\] p.398).

In this section we write the option pricing formula (22) using change of numéraire (see \[16\], or \[22\], section 11.5), which gives

\[ C_0 = S_0Q_1(S_T > K) - e^{-rT}KQ(S_T > K), \tag{23} \]

where \( Q_1 \), defined by

\[ \frac{dQ_1}{dQ} = e^{-rT} \frac{S_T}{S_0}, \tag{24} \]

is the measure under which the process \( e^{rt}/S_t \) is a martingale.

4.2.1 Symmetric Lévy Returns with Brownian Component

Let \( Y_t \) be a symmetric Lévy process with \( \mathbb{P} \)-characteristic triplet \((\mu, c, \nu)\) and such that \( c \neq 0 \). Let \( S(\mu, \sigma^2, \psi) \) be the \( \mathbb{P} \)-distribution of \( Y_1 \) and \( Q \) be the natural EMM (for \( Y_t \)). Then, under \( Q \), \( Y_t \) remains a symmetric Lévy process with characteristic triplet \((\tilde{\mu}, c, \nu)\), and the distribution of \( Y_1 \) becomes \( S(\tilde{\mu}, \sigma^2, \psi) \), where

\[ \tilde{\mu} = r - \ln \psi(-\sigma^2/2). \]

Now, it is easy to see that \( Q_1 \) is also a natural EMM. Indeed, since \(-Y_t\) is a Lévy process with \( \mathbb{P} \)-characteristic triplets \((-\mu, c, \nu)\), and since the distribution of \(-Y_1\) is \( S(-\mu, \sigma^2, \psi) \), \( Q_1 \) is chosen so that

\[ \tilde{\mu}_1 = r + \ln \psi\left(-\frac{\sigma^2}{2}\right). \tag{25} \]

This choice is unique as the location parameter \((\tilde{\mu}_1)\) uniquely determines \( \eta \), which in turn specifies the equivalent measure. By the uniqueness of \( \tilde{\mu}_1 \), \( Q_1 \) is unique.

Under \( Q_1 \), \( Y_t \) is a symmetric Lévy process with marginals from the family \( S(\tilde{\mu}_1 t, \sigma^2 t, \psi_1) \).
Proposition 4.3. Denote by $F_T$ the $\mathbb{P}$-distribution function of the standardized variable $(Y_T - \mu T)/(\sigma \sqrt{T})$:

$$F_T(y) = \mathbb{P}(Y_T \leq \sigma \sqrt{T}y + \mu T).$$

Then

$$F_T(y) = \mathcal{Q}(Y_T \leq \sigma \sqrt{T}y + \tilde{\mu} T) = \mathcal{Q}_1(Y_T \leq \sigma \sqrt{T}y + \tilde{\mu}_1 T),$$

and the option pricing formula (23) becomes

$$C_0 = S_0 F_T \left( \frac{\ln \left( \frac{S_0}{K} \right) + \left( r + \ln \psi(-\sigma^2/2) \right) T}{\sigma \sqrt{T}} \right) - e^{-rT} K F_T \left( \frac{\ln \left( \frac{S_0}{K} \right) + \left( r - \ln \psi(-\sigma^2/2) \right) T}{\sigma \sqrt{T}} \right). \quad (26)$$

Proof. The first statement follows from the fact that the distribution of $(Y_T - \mathbb{E}[T])/(\sigma \sqrt{T})$ is the same for all three measures $\mathbb{P}$, $\mathcal{Q}$, and $\mathcal{Q}_1$; its characteristic function, under all three probabilities, is given by $(\psi \left( u^2/(2T) \right))^T$. Also, since $Y_T$ is symmetric about $\mu$, $F_T$ is symmetric about zero and $1 - F_T(a) = F_T(-a)$. (26) now follows by simple arithmetic.

4.2.2 Symmetric Lévy Returns without Brownian Component

Consider now the case when $Y_t$ is a symmetric Lévy process with $\mathbb{P}$-characteristic triplet $(\mu, 0, \nu)$. Suppose further that the interest rate $r$ is greater than the location parameter $\mu$. As before, let $S(\mu, \sigma^2, \psi)$ be the $\mathbb{P}$-distribution of $Y_1$ and $\mathcal{Q}$ be the natural EMM (for $Y_1$). Then the $\mathcal{Q}$-distribution of $Y_1$ becomes $S(\mu, \tilde{\sigma}^2, \psi)$, where $\tilde{\sigma}^2$ is the solution of the equation (21).

Unlike the case of symmetric Lévy returns with a Brownian component the EMM $\mathcal{Q}_1$ does not define a natural change of measure. However, in specific cases considered here (Variance Gamma and Normal Inverse Gaussian), we are able to identify the distributions of $(Y_T - \mu T)/(\tilde{\sigma} \sqrt{T})$ under $\mathcal{Q}$ and that of $(Y_T - \mu T)/(\tilde{\sigma}_1 \sqrt{T})$ under $\mathcal{Q}_1$. Denoting by $F_T$ and $F_T^1$ the respective cumulative distribution functions, we can write

$$C_0 = S_0 \mathcal{Q}_1(S_T > K) - e^{-rT} K \mathcal{Q}(S_T > K) \quad (27)$$

$$= S_0 \left[ 1 - F_T^1 \left( -\frac{\ln \left( \frac{S_0}{K} \right) + \mu T}{\tilde{\sigma}_1 \sqrt{T}} \right) \right] - e^{-rT} K F_T \left( \frac{\ln \left( \frac{S_0}{K} \right) + \mu T}{\tilde{\sigma} \sqrt{T}} \right).$$
5 Variance Gamma Model

Here the stock price is modelled as $S_t = S_0 e^{Y_t}$, where $Y_t$ is a Variance-Gamma (VG) process. The marginal distributions of the VG process was originally given in [25] in terms of special functions involving the modified Bessel function of the second kind and the degenerate hypergeometric function. In the special case of symmetric processes the marginals turn out to be symmetric Bessel distributions. This observation leads to elegant formulae.

5.1 Symmetric Variance Gamma Process

Denote by $\text{Bessel}(\mu, \sigma^2, \lambda)$ the Bessel distribution with mean $\mu$, variance $\sigma^2$ and shape parameter $\lambda$. A symmetric Bessel distribution has mean $\mu = 0$, and characteristic function ([20], p.51) of the form

$$\varphi(u) = \left(\frac{1}{1 + \frac{u^2 \sigma^2}{2\lambda}}\right)^{\lambda}.$$ 

The density function of the symmetric Bessel distribution is given by ([20], p.50)

$$f(x) = \sqrt{\frac{2\lambda}{\pi \sigma^2}} \left(\sqrt{\frac{\lambda x^2}{2\sigma^2}}\right)^{\lambda - \frac{1}{2}} \frac{1}{\Gamma(\lambda)} K_{\lambda - \frac{1}{2}} \left(2 \sqrt{\frac{\lambda x^2}{2\sigma^2}}\right),$$

(28)

where $K_w(.)$ is the modified Bessel function of the second kind. We always consider symmetric Bessel distribution shifted by $\mu$ but we will drop the word “shifted”. The symmetric Bessel distribution $\text{Bessel}(\mu, \sigma^2, \lambda)$ belongs to the family of symmetric distributions $S(\mu, \sigma^2, \psi)$ with characteristic generator

$$\psi(v) = \left(\frac{1}{1 + \frac{v^2 \sigma^2}{2\lambda}}\right)^{\lambda}.$$ 

(29)

We note that the kurtosis of symmetric Bessel distribution is $3 + \frac{3}{\lambda}$ and hence, the shape parameter $\lambda$ is related to the excess kurtosis by $\lambda = \frac{3}{\gamma}$. (since the excess kurtosis of a random variable $Y$ is $\gamma = \frac{E[(Y-\mu)^4]}{\sigma^4} - 3$).

A symmetric VG $Y_t$ has characteristic function

$$\mathbb{E} \left[ e^{iuY_t} \right] = e^{iu\mu t} \left(\frac{1}{1 + \frac{u^2 \sigma^2}{2\lambda t}}\right)^{\frac{1}{2}} = e^{iu\mu t} \left(\frac{1}{1 + \frac{u^2 \sigma^2}{2\lambda t}}\right)^{\lambda t},$$ 

(30)

in which we have employed $\lambda = \frac{1}{\kappa}$. By inspecting the characteristic function we have
Proposition 5.1. The marginals of a symmetric Variance Gamma process $Y_t$ is a symmetric Bessel distribution with mean $\mu_t$, variance $\sigma^2 t$ and shape parameter $\lambda_t$, i.e., $Y_t \sim \text{Bessel}(\mu_t, \sigma^2 t, \lambda_t)$, which belongs to the family of symmetric distributions $S(\mu_t, \sigma^2 t, \psi_t)$ where the characteristic generator is given by

$$\psi_t(v) = \left(\psi\left(\frac{v}{t}\right)\right)^{\lambda t}.$$  \hfill (31)

5.2 Option Pricing with Symmetric VG

5.2.1 Continuous Time

We determine the distributions of $Y_t$ under the EMMs $Q$ and $Q_1$. By Theorems 3.1 and 4.1, if $\mu < r$ the $Q$-distribution of $Y_1$ is symmetric Bessel $\text{Bessel}(\mu, \tilde{\sigma}^2, \psi)$ where by using (29) we get from (21)

$$\tilde{\sigma}^2 = 2\lambda(1 - e^{-(r-\mu)/\lambda}).$$ \hfill (32)

Under $Q_1$, the distribution of $Y_t$ is identified by the following.

Proposition 5.2. Denote by $f_{Y_t}^Q$ the $Q$-density of $Y_t \sim \text{Bessel}(\mu_t, \tilde{\sigma}^2_t, \lambda_t)$.

Then the density of $Y_t$ under $Q_1$, given by $e^{y-rt}f_{Y_t}^Q(y)$, is the density function of an asymmetric Bessel distribution.

Proof. From the change of numéraire defined by (24), it can be seen that the density of $Y_t$ under $Q_1$ is given by

$$e^{y-rt}f_{Y_t}^Q(y) = e^{y-\mu t\left(1 - \tilde{\sigma}^2\right)}\left(\frac{2\lambda}{\pi \tilde{\sigma}^2}\right)^{\lambda t-\frac{1}{2}} \frac{1}{\Gamma(\lambda t)} K_{\lambda t-\frac{1}{2}}\left(2\frac{\lambda(y-\mu t)^2}{2\tilde{\sigma}^2}\right).$$ \hfill (33)

Using the characteristic generator (31) for the symmetric Bessel distribution, it follows that

$$e^{y-rt} = e^{y-\mu t\left(1 - \frac{\tilde{\sigma}^2}{2\lambda}\right)^\lambda}.$$ \hfill (34)

Now, apply (34) and let $y^* = y - \mu t$, the expression in the second line of (33) becomes

$$e^{y^*\left(1 - \frac{\tilde{\sigma}^2}{2\lambda}\right)^\lambda}\left(\frac{2\lambda}{\pi \tilde{\sigma}^2}\right)^{\lambda t-\frac{1}{2}} \frac{1}{\Gamma(\lambda t)} K_{\lambda t-\frac{1}{2}}\left(\sqrt{2\frac{\lambda(y^*)^2}{\tilde{\sigma}^2}}\right).$$
Subsequently, let \( m = \lambda t - \frac{1}{2}, \ a = -\sqrt{\frac{\sigma^2}{2\pi}}, \) and \( b = \sqrt{\frac{\sigma^2}{2\pi}}, \) we obtain

\[
e^{y - rt f_{Y_t}(y)} = e^{y}(1 - a^2)^{m + \frac{1}{2}} \frac{1}{b\sqrt{\pi}} \left( \frac{|y^*|}{2b} \right)^m \frac{1}{\Gamma(m + \frac{1}{2})} K_m \left( \frac{|y^*|}{b} \right)
\]

\[
= \frac{(1 - a^2)^{m + \frac{1}{2}} |y^*|^m}{\sqrt{\pi}2mb^{m+1}\Gamma(m + \frac{1}{2})} e^{y} K_m \left( \frac{|y^*|}{b} \right).
\]

On closer observation, we immediately recognize that the density written in the form (35) is the density of an asymmetric Bessel function distribution ([20], p.50), which has the form

\[
f_Z(z) = \frac{|1 - a^2|m + \frac{1}{2}|z|^m}{\sqrt{\pi}2mb^{m+1}\Gamma(m + \frac{1}{2})} e^{-a^2} K_m \left( \frac{|z|}{b} \right).
\]

For completeness we give explicit expressions of the mean, variance, skewness and kurtosis of an asymmetric Bessel distribution ([20], p. 51).

Mean = \((2m + 1)ba(a^2 - 1)^{-1}\) (37)

Variance = \((2m + 1)b^2(a^2 + 1)(a^2 - 1)^{-2}\) (38)

Skewness = \(2a(a^2 + 3)(2m + 1)^{-1/2}(a^2 + 1)^{-3/2}\) (39)

Kurtosis = \(3 + 6(a^4 + 6a^2 + 1)(2m + 1)^{-1}(a^2 + 1)^{-2}\) (40)

where \(a = -\sqrt{\frac{\sigma^2}{2\pi}}, \ b = -a, \) and \(m = \lambda t - \frac{1}{2}.

Let \(Y_t^* = Y_t - \mu t,\) it follows from (35) that \(Y_t^*\) has an asymmetric Bessel distribution, denoted \(Bessel^1(\mu t, \tilde{\sigma}_1^2 t, \lambda t),\) where the mean \(\mu t\) and variance \(\tilde{\sigma}_1^2 t\) are given by (37) and (38) respectively. In particular,

\[
E[Y_t^*] = \mu_1 = 2\lambda(e^{(r - \mu)/\lambda} - 1), \quad (41)
\]

\[
Var(Y_t^*) = \tilde{\sigma}_1^2 = 2\lambda(e^{(r - \mu)/\lambda} - 1)(2e^{(r - \mu)/\lambda} - 1), \quad (42)
\]

in which we have employed (32). Therefore under \(Q_1, \ Y_t = Y_t^* + \mu t \sim Bessel^1(\mu t + \mu_1 t, \tilde{\sigma}_1^2 t, \lambda t).\)

Finally, denote by \(B_{1\lambda}(y)\) the cumulative distribution function of the standardized symmetric Bessel random variable \(\frac{Y_t - \mu}{\tilde{\sigma}_1 \sqrt{t}} \sim Bessel(0, 1, \lambda t)\) under \(Q,\) and by \(B_{1\lambda}^1(y)\) the cumulative distribution function of the standardized asymmetric Bessel random variable \(\frac{Y_t^* - \mu_1}{\tilde{\sigma}_1 \sqrt{t}} \sim Bessel^1(0, 1, \lambda t)\) under \(Q_1.\)
Proposition 5.3. Let $Y_t \sim Bessel(\mu t, \sigma^2 t, \lambda t)$, and $\mu < r$. Then the arbitrage-free price of a call option using natural EMM is given by
\[
C_0 = S_0 \left[ 1 - B_{1T}^1 \left( - \frac{\ln \left( \frac{S_t}{K} \right) + \mu T + \mu_1 T}{\tilde{\sigma}_1 \sqrt{T}} \right) \right]
- e^{-rT} KB_{\lambda T} \left( \frac{\ln \left( \frac{S_t}{K} \right) + \mu T}{\tilde{\sigma} \sqrt{T}} \right),
\] (43)
where $\mu_1 = 2\lambda \left( e^{(r-\mu)/\lambda} - 1 \right)$, $\tilde{\sigma}_1^2 = 2\lambda \left( e^{(r-\mu)/\lambda} - 1 \right) \left( 2e^{(r-\mu)/\lambda} - 1 \right)$ and $\tilde{\sigma}_2^2 = 2\lambda \left( 1 - e^{-(r-\mu)/\lambda} \right)$.

Remark 5.4. An option pricing formula for a general VG process was given in [26] and [25] (eq. 25). While [26] presented the formula as a double integral of elementary functions and obtained the price by numerical integration, [25] provided a closed form formula in terms of the special functions involving the modified Bessel function of the second kind and the degenerate hypergeometric function. For the symmetric case the formula is much simpler.

Remark 5.5. The shortcoming of the natural EMM approach in continuous time is that $\mu < r$. This is overcome by using discrete time, where natural EMM exists also when $\mu \geq r$, and is given in the next section.

5.2.2 Discrete Time

Let now $S_N = S_0 e^{Y_N}$ be a model for stock prices, where $Y_N = \sum_{n=1}^{N} \Delta Y_n$ is a symmetric VG process in discrete time. $\Delta Y_n$, $n = 1, \ldots, N$ are i.i.d. symmetric Bessel distribution $Bessel(\mu, \sigma^2, \lambda)$, which belongs to the symmetric family $S(\mu, \sigma^2, \psi)$, where $\psi$ is given in (29).

It is possible to chose both $Q$ and $Q_1$ as natural EMM’s that shift only the location parameter. Hence, the $Q$-distribution of $\Delta Y_n$ is $Bessel(\tilde{\mu}, \sigma^2, \lambda)$ with $\tilde{\mu} = r - \ln \psi \left( - \frac{\sigma^2}{2} \right)$. The $Q_1$-distribution of $\Delta Y_n$ is $Bessel(\tilde{\mu}_1, \sigma^2, \lambda)$ with $\tilde{\mu}_1 = r + \ln \psi \left( - \frac{\sigma^2}{2} \right)$. This is easily seen, see also [23]. Denote by $B_{\lambda N}(y)$ the cumulative distribution function of the standardized symmetric Bessel random variable $\frac{Y_N - \mu N}{\sigma \sqrt{N}}$.

Proposition 5.6. Let $\Delta Y_n$ follow a symmetric Bessel distribution $Bessel(\mu, \sigma^2, \lambda)$, then the arbitrage-free price of a call option with $N$ periods to expiration is
given by

\[ C_0 = S_0 B_{\lambda N} \left( \frac{\ln \left( \frac{S_0}{K} \right) + (r - \lambda \ln(1 - \frac{\sigma^2}{2\lambda})) N}{\sigma \sqrt{N}} \right) \]

\[ - e^{-rN} K B_{\lambda N} \left( \frac{\ln \left( \frac{S_0}{K} \right) + (r + \lambda \ln(1 - \frac{\sigma^2}{2\lambda})) N}{\sigma \sqrt{N}} \right). \]  

(44)

5.2.3 Numerical Comparisons

For comparisons, we approximate the distributions of the standardized Bessel random variables \((Y_T - \mu T)/\sigma \sqrt{T}\) (the time \(T\) is replaced by \(N\) in the discrete case) by the standard Normal, in other words, \(B_{\lambda T}\) by \(\Phi\). We also approximate the standardized asymmetric Bessel random variable that arises in the continuous time case by the standard Normal, because its distribution is only slightly negatively skewed and therefore it is negligible. We will assume this is the case (skewness is small) in our approximation. Moreover, recall that the shape parameter \(\lambda\) and the excess kurtosis \(\gamma\) of the symmetric Bessel distribution are related by \(\lambda = \frac{3}{\gamma}\). Thus, for each of the continuous time and discrete time cases, we obtain an easy to use Black-Scholes type formula for option pricing which gives correction that accounts for the excess kurtosis.

In the continuous time case, the generalized or modified Black-Scholes formula for log-symmetric VG model (VG-C) is given by

\[ C_0 \approx S_0 \Phi \left( \frac{\ln \left( \frac{S_0}{K} \right) + \mu T + \mu_1 T}{\tilde{\sigma}_1 \sqrt{T}} \right) - e^{-rT} K \Phi \left( \frac{\ln \left( \frac{S_0}{K} \right) + \mu T}{\tilde{\sigma} \sqrt{T}} \right), \]  

(45)

where \(\mu_1 = \frac{6}{\gamma} (e^{(r-\mu)\gamma/3} - 1)\), \(\tilde{\sigma}_1^2 = \frac{6}{\gamma} (e^{(r-\mu)\gamma/3} - 1)(2e^{(r-\mu)\gamma/3} - 1)\) and \(\tilde{\sigma}_2^2 = \frac{6}{\gamma} (1 - e^{-(r-\mu)\gamma/3})\). Note that the Black-Scholes formula is a special case of the generalized version (VG-C) \([45]\) when \(\gamma \to 0\) due to the followings:

\[ \mu_1 = \frac{6}{\gamma} (e^{(r-\mu)\gamma/3} - 1) \to 2(r - \mu), \]

\[ \tilde{\sigma}_1^2 = \frac{6}{\gamma} (e^{(r-\mu)\gamma/3} - 1)(2e^{(r-\mu)\gamma/3} - 1) \to 2(r - \mu), \]

\[ \tilde{\sigma}_2^2 = \frac{6}{\gamma} (1 - e^{-(r-\mu)\gamma/3}) \to 2(r - \mu). \]

And if \(2(r - \mu) = \sigma^2\), which is a constant as in the Black-Scholes model (Recall that under the risk-neutral measure \(Q\), the mean \(\mu = r - \frac{\sigma^2}{2}\) and
the volatility \( \sigma \) is a constant), then by using these results and some simple manipulations, it is easy to see that the generalized formula (VG-C) (45) is the exact Black-Scholes formula.

In the discrete time case, the modified Black-Scholes formula for log-symmetric VG model (VG-D) is given by

\[
C_0 \approx S_0 \Phi\left( \frac{\ln \left( \frac{S_0}{K} \right) + \left( r - \frac{3}{\gamma} \ln\left(1 - \frac{2\sigma^2}{6}\right) \right) N}{\sigma \sqrt{N}} \right) - e^{-rN} K \Phi\left( \frac{\ln \left( \frac{S_0}{K} \right) + \left( r + \frac{3}{\gamma} \ln\left(1 - \frac{2\sigma^2}{6}\right) \right) N}{\sigma \sqrt{N}} \right).
\]  

(46)

It can be seen that the Black-Scholes formula is a limit of the generalized version (VG-D) (46) for every \( N \) when \( \gamma \to 0 \) due to

\[
\frac{3}{\gamma} \ln \left(1 - \frac{\gamma \sigma^2}{6}\right) \to -\frac{\sigma^2}{2}.
\]

The classical Black-Scholes formula (BS) is considered robust in the sense that for small values of excess kurtosis \( \gamma \), it coincides with the modified Black-Scholes formulae in both continuous time and discrete time cases. However, even for the moderate values of \( \gamma \), the distinction between the modified Black-Scholes formulae (VG-C and VG-D) and BS is noticeable (see Figure 1), with the disagreement between VG-C and BS formulae being greater than the disagreement between VG-D and BS. The exact prices and percentage differences are represented in Table 1.

| Time to maturity (weeks) | 2   | 12  | 22  | 32  | 42  | 52  |
|--------------------------|-----|-----|-----|-----|-----|-----|
| BS formula               | 0.160 | 0.434 | 0.622 | 0.782 | 0.927 | 1.062 |
| VG-D formula             | 0.162 | 0.439 | 0.628 | 0.789 | 0.935 | 1.071 |
| Percentage difference    | 1.13 | 1.01 | 0.94 | 0.88 | 0.84 | 0.80 |
| VG-C formula             | 0.192 | 0.511 | 0.725 | 0.904 | 1.065 | 1.213 |
| Percentage difference    | 19.85 | 17.67 | 16.46 | 15.55 | 14.81 | 14.17 |

Table 1: Option prices and percentage differences obtained by VG-C, VG-D and BS formulae for log-Bessel distribution weekly returns, \( S_0 = K = 10, r = 0.06, \sigma = 0.19, \mu = 0.03, \gamma = 4 \)

**Remark 5.7.** We suggest to use the modified formulae for any distribution with a positive excess kurtosis.
Figure 1: Option prices and percentage differences obtained by VG-C, VG-D and BS formulae for log-Bessel distribution weekly returns, $S_0 = K = 10, r = 0.06, \sigma = 0.19, \mu = 0.03, \gamma = 4$

6 Normal Inverse Gaussian Model

6.1 The NIG Process and Distribution

In this section, the model is $S_t = S_0e^{Y_t}$, where $Y_t$ is a Normal inverse Gaussian (NIG) process. The NIG distributions were first introduced by Barndorff-Nielsen [1] as a subclass of the Generalized Hyperbolic distribution with parameter $\lambda = -\frac{1}{2}$. Denote by $NIG(\alpha, \beta, \delta, \mu)$ the NIG distribution, where $\mu$ is the location parameter, $\delta$ is the scale parameter, $\alpha$ is the shape parameter and $\beta$ is for skewness. The density of NIG distribution is given by (see e.g. [33] p.60, or [31])

$$f_Y(y) = \frac{\alpha}{\pi}e^{\delta\sqrt{\alpha^2-\beta^2}+\beta(y-\mu)} K_1\left(\frac{\alpha\delta}{\sqrt{1+(\frac{y-\mu}{\delta})^2}}\right) \frac{K_1\left(\alpha\delta\sqrt{1+(\frac{y-\mu}{\delta})^2}\right)}{\sqrt{1+(\frac{y-\mu}{\delta})^2}}, \quad (47)$$
where \( K_1 \) is the modified Bessel function of the third kind, \( y, \mu \in \mathbb{R}, \delta \geq 0 \) and \( 0 \leq |\beta| \leq \alpha \). The characteristic function of NIG distribution is

\[
\varphi(u) = e^{iu\mu} \frac{e^{\delta \sqrt{\alpha^2 - \beta^2}}}{e^{\delta \sqrt{\alpha^2 - (\beta + iu)^2}}}, \tag{48}
\]

and the mean, variance, skewness and kurtosis of NIG distribution are

\[
\text{Mean} = \mu + \frac{\beta \delta}{\sqrt{\alpha^2 - \beta^2}} \tag{49}
\]
\[
\text{Variance} = \frac{\delta \alpha^2}{(\sqrt{\alpha^2 - \beta^2})^3} \tag{50}
\]
\[
\text{Skewness} = \frac{3\beta}{\alpha(\delta \sqrt{\alpha^2 - \beta^2})^{\frac{1}{2}}} \tag{51}
\]
\[
\text{Kurtosis} = 3 \left( 1 + \frac{\alpha^2 + 4\beta^2}{\delta \alpha^2 \sqrt{\alpha^2 - \beta^2}} \right). \tag{52}
\]

The symmetric NIG Lévy processes have symmetric NIG marginals. The NIG distribution is symmetric when the skewness parameter \( \beta = 0 \). In this case, the density \( (47) \) of a symmetric NIG is

\[
f_Y(y) = \frac{\alpha}{\pi} e^{\alpha \delta} \frac{K_1 \left( \alpha \delta \sqrt{1 + \left( \frac{y - \mu}{\delta} \right)^2} \right)}{\sqrt{1 + \left( \frac{y - \mu}{\delta} \right)^2}}. \tag{53}
\]

The characteristic function \( (48) \) for symmetric NIG is

\[
\varphi(u) = e^{iu\mu} e^{-\alpha \delta \left( 1 - \sqrt{1 + (\frac{u}{\delta})^2} \right)}. \tag{54}
\]

It follows from equations \( (49) \), \( (50) \) and \( (52) \), respectively, that \( \mu \) is the mean, variance is \( \frac{\delta}{\alpha} \), skewness is \( \frac{3\beta}{\alpha \delta} \) and kurtosis is \( 3 + \frac{3\beta}{\alpha} \). We will denote the distribution of symmetric NIG by \( SNIG(\alpha, 0, \delta, \mu) \).

By an inspection of the characteristic function we can see that the characteristic generator of symmetric Normal inverse Gaussian distributions is given by

\[
\psi(v) = e^{\zeta(1 - \sqrt{1 + \frac{v}{\zeta^2}})}, \tag{55}
\]

where \( \zeta = \alpha \delta \).
6.2 Option Pricing with Symmetric NIG Model

6.2.1 Continuous Time

We determine the distributions of $Y_t$ under the EMM’s $Q$ and $Q_1$.

If $\mu < r$ there is a natural EMM and $Y_1$ is symmetric NIG distribution $SNIG(\alpha, 0, \delta, \mu)$, where $\frac{\delta}{\alpha} = \tilde{\sigma}^2$. where by using (55) we get from (21)

$$\tilde{\sigma}^2 = 2(r - \mu) - \left(\frac{r - \mu}{\alpha \sigma}\right)^2,$$

(56)

Consequently, $Y_t \sim SNIG(\alpha, 0, \delta t, \mu t)$ under $Q$ with mean $\mu t$ and variance $\tilde{\sigma}^2 t = \tilde{\delta} t / \alpha$.

Under $Q_1$, the distribution of $Y_t$ is identified in the following theorem.

**Proposition 6.1.** The density of $Y_t$ under $Q_1$, given by $e^{y-r} f_{Y_1}^Q(y)$, is the density function of an asymmetric NIG distribution, $Y_t \sim NIG(\alpha, 1, \delta t, \mu t)$.

**Proof.** Using (21) and (24)

$$e^{y-r} f_{Y_1}^Q(y) = e^{y-\mu t-\alpha \delta t+\delta t \sqrt{\alpha^2-1}} \alpha e^{\alpha \delta t} K_1 \left(\frac{\alpha \delta t \sqrt{1 + \left(\frac{y-\mu t}{\delta t}\right)^2}}{\sqrt{1 + \left(\frac{y-\mu t}{\delta t}\right)^2}} \right).$$

(57)

One can verify that (57) is the density function of an asymmetric NIG distribution (see (47)) with parameters $\alpha$ (unchange), $\beta = 1$, $\mu = \mu t$ and $\delta = \delta t$.

The mean and variance are given by (49) and (50), respectively.

$$\mathbb{E}[Y_1] = \mu_1 = \mu + \sqrt{\frac{\alpha^2}{\alpha^2 - 1}} \tilde{\sigma}^2,$$

(58)

$$\text{Var}(Y_1) = \tilde{\sigma}_1^2 = \left(\sqrt{\frac{\alpha^2}{\alpha^2 - 1}} \right)^3 \tilde{\sigma}^2,$$

(59)

where $\tilde{\sigma}^2$ is given by (56).

Denote by $F_{SNIG}(y)$, the cumulative distribution function of the standardized symmetric NIG random variable $\frac{Y_t-\mu t}{\tilde{\sigma} \sqrt{t}} \sim SNIG(\alpha, 0, \alpha, 0)$ under $Q$. 

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and denote by $F_{NIG}(y)$ the cumulative distribution function of the standardized asymmetric NIG random variable \( \frac{Y - \mu}{\tilde{\sigma} \sqrt{t}} \sim NIG(\alpha, 1, \alpha, 0) \) under $Q_1$, we obtain the explicit formula for option pricing with symmetric NIG process that can be written in terms of the cumulative distribution functions of standardized (symmetric and asymmetric) NIG.

**Proposition 6.2.** Let $Y_t \sim SNIG(\alpha, 0, \delta, \mu_t)$, and $\mu < r$. Then the arbitrage-free price of a call option using natural EMM is given by

\[
C_0 = S_0 \left[ 1 - F_{NIG} \left( -\frac{\ln \left( \frac{S_0}{K} \right) + \mu T}{\tilde{\sigma} \sqrt{T}} \right) \right] - e^{-rT} K F_{SNIG} \left( \frac{\ln \left( \frac{S_0}{K} \right) + \mu T}{\tilde{\sigma} \sqrt{T}} \right).
\]

(60)

where $\mu_1 = \mu + \sqrt{\frac{\alpha^2}{\alpha^2 - 1}} \tilde{\sigma}^2$, $\tilde{\sigma}^2 = \left( \sqrt{\frac{\alpha^2}{\alpha^2 - 1}} \right)^3 \tilde{\sigma}^2$ and $\tilde{\sigma}^2 = 2(\mu - r) - (\frac{r - \mu}{\alpha \tilde{\sigma}})^2$.

Discrete time model allows for natural EMM even when $\mu \geq r$, Remark 5.5.

### 6.2.2 Discrete Time

The stock price process in discrete time $S_N = S_0e^{Y_N}$ where $Y_N = \sum_{n=1}^{N} \Delta Y_n$ with $\Delta Y_n$, $n = 1, \ldots, N$ are i.i.d. $SNIG(\alpha, 0, \delta, \mu)$, which belongs to the symmetric family $S(\mu, \sigma^2, \psi)$ where $\sigma^2 = \frac{\delta^2}{\alpha^2}$ and $\psi$ is (55). Recalling that for a fixed $\sigma^2$, we can obtain two natural EMM’s $Q$ and $Q_1$ by changing only the location parameter $\mu$ so that $Y_N$ remains a symmetric NIG Lévy process. The $Q$-distribution of $\Delta Y_n$ is $SNIG(\alpha, 0, \delta, \bar{\mu})$ where $\bar{\mu} = r - \ln \psi( -\frac{\sigma^2}{2})$, and the $Q_1$-distribution of $\Delta Y_n$ is $SNIG(\alpha, 0, \delta, \tilde{\mu}_1)$ with $\tilde{\mu}_1 = r + \ln \psi( -\frac{\sigma^2}{2})$. By (55) and $\zeta = \alpha \delta = \alpha^2 \sigma^2$, we obtain

\[
\ln \psi( -\frac{\sigma^2}{2}) = \zeta \left( 1 - \sqrt{1 - \frac{\sigma^2}{\zeta}} \right) = \alpha^2 \sigma^2 - \alpha \sigma^2 \sqrt{\alpha^2 - 1}.
\]

(61)

Therefore, we obtain the following result for the exact option pricing formula with symmetric NIG process in discrete time.

**Proposition 6.3.** Let $\Delta Y_n$ follow a symmetric NIG distribution $SNIG(\alpha, 0, \delta, \mu)$, then the arbitrage-free price of a call option with $N$ periods to expiration is
given by

\[
C_0 = S_0 F_{SNIG} \left( \frac{\ln \left( \frac{S_0}{K} \right) + (r + \alpha^2 \sigma^2 - \alpha \sigma^2 \sqrt{\alpha^2 - 1}) N}{\sigma \sqrt{N}} \right) - e^{-r N} K F_{SNIG} \left( \frac{\ln \left( \frac{S_0}{K} \right) + (r - \alpha^2 \sigma^2 + \alpha \sigma^2 \sqrt{\alpha^2 - 1}) N}{\sigma \sqrt{N}} \right). \tag{62}
\]

### 6.3 Numerical Comparisons

For comparisons, we approximate the standardized symmetric NIG distribution by the standard Normal, in other words, \( F_{SNIG} \) by \( \Phi \). We also approximate the standardized asymmetric NIG distribution that arises in the continuous time case by the standard Normal, i.e., \( F_{NIG} \) by \( \Phi \), since it is only slightly positively skewed. Therefore we assume that the skewness is negligible. Moreover, recall that the shape parameter \( \zeta = \alpha \delta \) and the excess kurtosis \( \gamma \) of the symmetric NIG distribution are related by \( \gamma = \frac{3}{\zeta} \). Thus, for each of the continuous time and discrete time cases, we obtain an easy to use Black-Scholes type formula for option pricing which gives correction that accounts for the excess kurtosis.

In the continuous time case, the generalized or modified Black-Scholes formula for the log-symmetric NIG model (NIG-C) is given by

\[
C_0 \approx S_0 \Phi \left( \frac{\ln \left( \frac{S_0}{K} \right) + \mu_1 T}{\sigma_1 \sqrt{T}} \right) - e^{-r T} K \Phi \left( \frac{\ln \left( \frac{S_0}{K} \right) + \mu T}{\sigma_1 \sqrt{T}} \right), \tag{63}
\]

where \( \mu_1 = \mu + \sqrt{\frac{3}{3 - \gamma \sigma^2}} \sigma^2, \quad \sigma_1^2 = \left( \frac{3}{3 - \gamma \sigma^2} \right)^3 \sigma^2 \) and \( \sigma^2 = 2(r - \mu) - \frac{\gamma}{3} (r - \mu)^2 \), in which we have applied the fact that

\[
\frac{\alpha^2}{\alpha^2 - 1} = \frac{\alpha^2 \sigma^2}{\alpha^2 \sigma^2 - \sigma^2} = \frac{\alpha \delta}{\alpha \delta - \sigma^2} = \frac{3}{3 - \gamma \sigma^2}.
\]

Observe that, when \( \gamma \to 0 \), we have \( \sigma^2 \to 2(r - \mu) \) and \( \frac{3}{3 - \gamma \sigma^2} \to 1 \). Consequently, the Black-Scholes formula is a special case of the generalized version (NIG-C) \( (63) \) when \( \gamma \to 0 \) because

\[
\mu_1 = \mu + \sqrt{\frac{3}{3 - \gamma \sigma^2}} \sigma^2 \to \mu + 2(r - \mu),
\]

\[
\sigma_1^2 = \left( \frac{3}{3 - \gamma \sigma^2} \right)^3 \sigma^2 \to 2(r - \mu).
\]
And let $2(r - \mu) = \sigma^2$, which is a constant as in the Black-Scholes model (Recall that under the risk-neutral measure $Q$, the mean $\mu = r - \frac{\sigma^2}{2}$ and the volatility $\sigma$ is a constant), then by using these results and some simple manipulations, it is not hard to see that the generalized formula (NIG-C) (63) is the exact Black-Scholes formula.

In the discrete time case, the modified Black-Scholes formula for log-symmetric NIG model (NIG-D) is given by

$$C_0 \approx S_0 \Phi\left(\frac{\ln \left(\frac{S_0}{K}\right) + \left(r + \frac{3}{\gamma} \left(1 - \sqrt{1 - \frac{\gamma \sigma^2}{3}}\right) \right) N}{\sigma \sqrt{N}}\right) - e^{-r N} K \Phi\left(\frac{\ln \left(\frac{S_0}{K}\right) + \left(r - \frac{3}{\gamma} \left(1 - \sqrt{1 - \frac{\gamma \sigma^2}{3}}\right) \right) N}{\sigma \sqrt{N}}\right).$$

(64)

It can be seen that the Black-Scholes formula is a limit of the generalized version (NIG-D) (64) for every $N$ when $\gamma \to 0$ due to

$$\frac{3}{\gamma} \left(1 - \sqrt{1 - \frac{\gamma \sigma^2}{3}}\right) \to \frac{\sigma^2}{2}.$$

An example of the option price formulae plotted against the expiration time $T$ using similar set of parameter values as in the log-symmetric VG model is given below (see Figure 2). Again, it is evident that the distinction between the modified Black-Scholes formulae (NIG-C and NIG-D) and BS is noticeable even for this moderate values of $\gamma$ (see Figure 1). As in the previous model, the disagreement between NIG-C and BS formulae is greater than the disagreement between NIG-D and BS. The exact prices and percentage differences are represented in Table 2.
Figure 2: Option prices and percentage differences obtained by NIG-C, NIG-D and BS formulae for log-NIG distribution weekly returns, $S_0 = K = 10$, $r = 0.06$, $\sigma = 0.19$, $\mu = 0.03$, $\gamma = 4$

| Time to maturity (weeks) | 2   | 12  | 22  | 32  | 42  | 52  |
|--------------------------|-----|-----|-----|-----|-----|-----|
| BS formula               | 0.160 | 0.434 | 0.622 | 0.782 | 0.927 | 1.062 |
| NIG-D formula            | 0.162 | 0.439 | 0.628 | 0.789 | 0.935 | 1.071 |
| Percentage difference    | 1.14  | 1.01  | 0.94  | 0.89  | 0.85  | 0.81  |
| NIG-C formula            | 0.195 | 0.519 | 0.735 | 0.917 | 1.079 | 1.229 |
| Percentage difference    | 21.91 | 19.52 | 18.18 | 17.18 | 16.36 | 15.66 |

Table 2: Option prices and percentage differences obtained by NIG-C, NIG-D and BS formulae for log-NIG distribution weekly returns, $S_0 = K = 10$, $r = 0.06$, $\sigma = 0.19$, $\mu = 0.03$, $\gamma = 4$

7 Appendix

Lemma 7.1. Let $\nu$ and $\tilde{\nu}$ be two measure on $(0, +\infty)$ with finite second moments:

$$\kappa = \int_0^\infty y^2 \nu(dy) < +\infty \text{ and } \tilde{\kappa} = \int_0^\infty y^2 \tilde{\nu}(dy) < +\infty.$$
Let \( \beta = \sqrt{\kappa/\bar{\kappa}} \). If
\[
\int_0^\infty (1 - \cos(\omega y)) \tilde{\nu}(dy) = \int_0^\infty (1 - \cos(\beta \omega y)) \nu(dy), \quad \forall \omega > 0,
\]
then \( \tilde{\nu} = \nu_\beta \), where \( \nu_\beta \) is defined by
\[
\int g(y) \nu_\beta(dy) = \int g(\beta y) \nu(dy).
\]

**Proof.** To show that the two measures are identical we show that the Mellin transforms of certain associated probability distributions are the same. First we compute the Laplace transforms of the two sides of (65):
\[
\int_0^\infty e^{-\lambda \omega} \left( \int_0^\infty (1 - \cos(\beta \omega y)) \nu(dy) \right) d\omega = \int_0^\infty \left( \int_0^\infty e^{-\lambda \omega} (1 - \cos(\beta \omega y)) d\omega \right) \nu(dy) = \int_0^\infty \left( \frac{\beta^2 y^2}{\lambda(\lambda^2 + \beta^2 y^2)} \right) \nu(dy),
\]
and similarly for \( \tilde{\nu} \). Here we have used the identity
\[
\int_0^\infty e^{-\lambda \omega} (1 - \cos(\omega y)) d\omega = \frac{y^2}{\lambda(\lambda^2 + y^2)}.
\]
(65) becomes
\[
\int_0^\infty \frac{y^2}{\lambda^2 + y^2} \tilde{\nu}(dy) = \int_0^\infty \frac{\beta^2 y^2}{\lambda^2 + \beta^2 y^2} \nu(dy), \quad \forall \lambda > 0.
\]
Consider now the probability measures with support on \((0, +\infty)\), \( \tilde{n}(dy) = \frac{1}{\kappa} y^2 \tilde{\nu}(dy) \) and \( n(dy) = \frac{1}{\kappa} y^2 \nu_\beta(dy) \). Then the positive random variables \( X \) and \( \tilde{X} \) with respective distributions \( n \) and \( \tilde{n} \) satisfy, for all \( \lambda > 0 \),
\[
\mathbb{E} \left[ \frac{1}{\lambda^2 + X^2} \right] = \mathbb{E} \left[ \frac{1}{\lambda^2 + \tilde{X}^2} \right].
\]
In other words, the Mellin transforms of \( X^2 \) and \( \tilde{X}^2 \) are equal, which in turn implies that the laws of \( X \) and \( \tilde{X} \) are the same, that \( \tilde{n} = n \), and consequently that \( \tilde{\nu} = \nu_\beta \). \( \square \)

**Lemma 7.2.**
\[
\lim_{v \to \infty} \int_0^\infty \frac{1}{v} \left[ (1 - \cos(y \sqrt{2v}/\sigma)) e^{\phi(y)} - (1 - \cos(y \sqrt{2v}/\sigma)) \right] \nu(dy) = 0.
\]
Proof. Let 

\[ f_v(y) = \frac{1}{v} \left[ (1 - \cos(y\sqrt{2v}/\tilde{\sigma})) e^{\phi(y)} - (1 - \cos(y\sqrt{2v}/\sigma)) \right]. \]

Clearly, for any fixed \( y \), \( \lim_{v \to \infty} f_v(y) = 0 \). Using the inequality \( 1 - \cos(x) \leq x^2/2 \) we obtain

\[ |f_v(y)| \leq \frac{y^2}{\sigma^2} e^{\phi(y)} + \frac{y^2}{\tilde{\sigma}^2} = G(y). \]

\( G(y) \) is integrable with respect to \( \nu \), since the Lévy measures \( \tilde{\nu} \) and \( \nu \) satisfy

\[ \int_{\mathbb{R}} (1 \wedge y^2) \tilde{\nu}(dy) < \infty, \quad \text{and} \quad \int_{\mathbb{R}} (1 \wedge y^2) \nu(dy) < \infty; \]

and existence of variance implies

\[ \int_{\{|y| > 1\}} y^2 \tilde{\nu}(dy) < \infty, \quad \text{and} \quad \int_{\{|y| > 1\}} y^2 \nu(dy) < \infty. \]

The result follows by dominated convergence. \( \square \)

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