Adaptive Morley FEM for the von Kármán equations with optimal convergence rates

Carsten Carstensen∗ and Neela Nataraj†

August 22, 2019

Abstract

The adaptive nonconforming Morley finite element method (FEM) approximates a regular solution to the von Kármán equations with optimal convergence rates for sufficiently fine triangulations and small bulk parameter in the Dörfler marking. This follows from the general axiomatic framework with the key arguments of stability, reduction, discrete reliability, and quasiorthogonality of an explicit residual-based error estimator. Particular attention is on the nonlinearity and the piecewise Sobolev embeddings required in the resulting trilinear form in the weak formulation of the nonconforming discretisation. The discrete reliability follows with a conforming companion for the discrete Morley functions from the medius analysis. The quasiorthogonality also relies on a novel piecewise $H^1$ a priori error estimate and a careful analysis of the nonlinearity.

Keywords: von Kármán equations, adaptivity, finite element method, a posteriori error estimate, piecewise $H^1$ a priori, Morley finite element, nonconforming finite element method, companion operator, medius analysis, axioms of adaptivity, optimal convergence rate, discrete reliability, quasiorthogonality

AMS Classification: 65N30, 65N12, 65N50

1 Introduction

This paper establishes the optimal convergence rates of an adaptive finite element method for the nonconforming Morley approximation to a regular solution of the semilinear von Kármán equations in a bounded polygonal Lipschitz domain $\Omega$ in the plane. The mathematical model describes the deflection $u$ of very thin elastic plates by a semi-linear system of fourth-order partial differential equations: For a given load function $f \in L^2(\Omega)$, seek $u, v \in H^2_0(\Omega)$ such that

$$\Delta^2 u = [u, v] + f \quad \text{and} \quad \Delta^2 v = -\frac{1}{2}[u, u] \quad \text{in} \ \Omega. \quad (1.1)$$

Here and throughout the paper, $\Delta^2$ denotes the biharmonic operator with $\Delta^2 \varphi = \varphi_{xxxx} + 2\varphi_{xxyy} + \varphi_{yyyy}$ and $[\cdot, \cdot]$ denotes the von Kármán bracket with $[\eta, \chi] = \eta_{xx} \chi_{yy} + \eta_{yy} \chi_{xx} - 2\eta_{xy} \chi_{xy} = \text{cof}(D^2\eta) : D^2 \chi$ for the co-factor matrix $\text{cof}(D^2\eta)$ of $D^2\eta$ (the colon $:$ denotes the scalar product between $2 \times 2$ matrices) for smooth functions $\varphi, \eta, \chi$ and their partial derivatives $\varphi_{xx}$ etc.

The existence of solutions, regularity, and bifurcation phenomena are discussed in [2, 24, 29] and the references therein. The weak solutions $u, v \in H^2_0(\Omega)$ to the von Kármán equations (1.1) belong to $H^2_0(\Omega) \cap H^{2+\gamma}(\Omega)$ with the index of elliptic regularity $1/2 < \gamma \leq 1$ determined by the interior angles of the polygonal boundary $\partial\Omega$ with $\gamma = 1$ if $\Omega$ is convex [4].

The major challenges in the numerical analysis of (1.1) are the non-linearity and the higher-order nature of the equations. The papers [10, 33–35] study the approximation and error bounds for regular solutions to (1.1) for conforming, mixed, and hybrid FEMs. Meanwhile, nonconforming FEMs [31], a $C^0$ interior penalty method [5], and discontinuous Galerkin FEMs [19] have been investigated and [18] suggests an abstract framework for $a$ priori and $a$ posteriori error control applicable to the von Kármán equations.

∗Department of Mathematics, Humboldt-Universität zu Berlin, 10099 Berlin, Germany. Distinguished Visiting Professor, Department of Mathematics, Indian institute of Technology Bombay, Powai, Mumbai-400076. Email: cc@math.hu-berlin.de

†Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India. Email: neela@math.iitb.ac.in
Given the reduced elliptic regularity on nonconvex polygons with \( \gamma < 1 \), the convergence for a quasi-uniform triangulation with (maximal) mesh-size \( h_{\text{max}} \) is not better than \( h_{\text{max}}^\gamma \) and adaptive mesh-refining is mandatory. Little is known in the literature about adaptive finite element methods (FEMs) and their convergence rates for semilinear problems. For particular strictly monotone and Lipschitz continuous operators, the residuals are similar to their linear relatives with unique exact (resp. discrete solutions) and optimal convergence rates are known \([11, 14, 26]\). Besides the \( p \)-Laplacian \([1]\), there are merely convergence proofs (but no optimal rates) for semilinear second-order problems \([27]\). The other known results are for eigenvalue problems for the Laplacian or the bi-Laplacian with a perturbation of the right-hand side \( f := \lambda u \) in the exact problem replaced by \( f_\delta := \lambda u - \lambda_\delta u_\delta \) on the discrete level. Since \( f - f_\delta := \lambda u - \lambda_\delta u_\delta \) is a higher-order perturbation, the axioms of adaptivity \([11, 21]\) lead to optimal convergence rates for sufficiently small mesh-sizes.

This paper provides the first rate-optimal adaptive algorithm for the von Kármán equations \([1, 11]\). In the absence of further structural information (e.g., the Rayleigh-Ritz principle for symmetric eigenvalue problems in \([12, 17]\) or small loads as in \([29]\)) the mesh-size has to be sufficiently small to guarantee the existence of a unique discrete solution \( \Psi \in \mathcal{V}(T) \) close to \( \Psi \in \mathcal{V} \). This is achieved in a pre-asymptotic step of the proposed adaptive algorithm AMFEM by uniform mesh refinements until the mesh-size is smaller than or equal to some input parameter \( \delta \). The standard adaptive FEM with solve, estimate, mark, and refine applies thereafter with a bulk parameter \( \theta \) in the Dörfler marking \([3, 11, 21, 22, 36]\). This paper presents an adaptive algorithm AMFEM and establishes optimal convergence rates for all sufficiently small positive \( \delta \) and \( \theta \).

For a triangle \( T \) of area \( |T| \), the Morley finite element approximation \( \Psi_M := (u_{\text{Mor}}, v_M) \) to \([1]\) leads to a volume residual \( \mu(T) := |T| \|f + [u_{\text{Mor}}, v_M]_H^2(V(T)) + \|u_{\text{Mor}}, u_M\|_L^2(V(T))^1/2 \). Despite the mesh-size factor \( h_T^2 = |T| \), this volume contribution is not obviously of higher order: The von Kármán equations bracket \([u_{\text{Mor}}, v_M]_{\text{pw}} \) with the subscript \( \text{pw} \) for the piecewise action of the derivatives involves partial derivatives like \( \partial_{\text{pw}} u_{\text{Mor}} / \partial x^2 \partial_{\text{pw}} v_M / \partial y^2 \). Its \( L^2 \) norm remains uncontrolled in terms of the discrete functions and their piecewise \( H^2 \) norm, while solely the \( L^2 \) norm of \( h_T \partial_{\text{pw}} u_{\text{Mor}} / \partial x^2 \partial_{\text{pw}} v_M / \partial y^2 \) is controlled via an inverse estimate. This heuristic argument indicates that, in contrast to the mentioned eigenvalue analysis, the nonlinear term \( \mu(T) \) in the Kármán equations cannot be regarded as a higher-order term and absorbed in the analysis. This results in additional difficulties in the quasiorthogonality \([11, 21]\). The remedy in this paper carefully exploits the precise derivatives of these nonlinearities on the discrete level and requires an \( \text{a priori} \) error estimate for the piecewise \( H^1 \) norm of the error \( \Psi - \Psi_M \).

The outline of the remaining parts of this paper reads as follows. Section 2 recalls some known preliminaries about the analysis of the von Kármán equations and tools from the medius analysis of the Morley FEM. Section 3 presents the \( \text{a priori} \) error estimates with a novel piecewise \( H^1 \) norm error estimate. Section 4 recalls the explicit residual-based error estimator from \([18]\) and introduces the adaptive algorithm AMFEM for the nonconforming Morley FEM. Section 5 gives details and proofs of stability, reduction, discrete reliability, and quasiorthogonality. This and \([11, 21]\) guarantee the optimal convergence rates of the proposed adaptive Morley FEM. The outline is restricted to two space dimensions for the von Kármán equations that are intrinsically two-dimensional, but the arguments extend to higher space dimensions as well.

Standard notation of Lebesgue and Sobolev spaces, their norms, and \( L^2 \) scalar products applies throughout the paper such as the abbreviations \( \| \cdot \|_p \) for \( \| \cdot \|_{L^p(\Omega)} \) and \( \| \cdot \|_{m,p} \) for \( \| \cdot \|_{W^{m,p}(\Omega)} \) and the local (resp. piecewise) version \( \| \cdot \|_{m,p,\omega} := \| \cdot \|_{W^{m,p}(\Omega, \text{loc})} \) (resp. \( \| \cdot \|_{m,p,\omega,\text{pw}} \)) for \( \omega \subset \Omega \) etc. and for the related seminorms. The notation \( H^s(\Omega) \) (resp. \( L^p(\Omega) \)) denotes the product space \( H^s(\Omega) \times H^s(\Omega) \) (resp. \( L^p(\Omega) \times L^p(\Omega) \)) for \( s \in \mathbb{R} \) (resp. \( 1 \leq p < \infty \)). The triple norm \( \| \cdot \| := \| \cdot \|_{H^2(\Omega)} \) is the energy norm and \( \| \cdot \|_{\text{pw}} := \| \cdot \|_{H^2(\Omega)} \) is its piecewise version with the piecewise Hessian \( D_{\text{pw}}^2 \).

The notation \( A \leq B \) abbreviates \( A \leq CB \) for some positive generic constant \( C \), which depends only on the initial triangulation \( T_{\text{init}} \) and on the regular solution \( \Psi \); \( A \approx B \) abbreviates \( A \lesssim B \lesssim A \).

2 Morley FEM for the von Kármán equations

The first subsection is devoted to the mathematical model of the von Kármán equations and is followed by a subsection on triangulations and discrete spaces. The third subsection recalls interpolation and enhancement for Morley functions and the fourth collects further preliminaries.
2.1 The von Kármán equations

Given \( f \in L^2(\Omega) \) in a bounded polygonal Lipschitz domain \( \Omega \subset \mathbb{R}^2 \), the weak formulation of the von Kármán equations \((1.1)\) seeks \( u, v \in H^2_0(\Omega) \) such that

\[
\begin{align*}
    a(u, \varphi_1) + b(u, v, \varphi_1) + b(v, u, \varphi_1) &= (f, \varphi_1)_{L^2(\Omega)} \quad \text{for all } \varphi_1 \in H^2_0(\Omega), & (2.1a) \\
    a(v, \varphi_2) - b(u, u, \varphi_2) &= 0 \quad \text{for all } \varphi_2 \in H^2_0(\Omega). & (2.1b)
\end{align*}
\]

Here and throughout the paper, for all \( \eta, \chi, \varphi \in H^2_0(\Omega) \) and all \( \Xi = (\xi_1, \xi_2), \Theta = (\theta_1, \theta_2), \Phi = (\varphi_1, \varphi_2) \in \mathcal{V} := H^2_0(\Omega) \times H^2_0(\Omega) \) (endowed with the product norm also denoted by \( \| \cdot \| \)) set

\[
\begin{align*}
    a(\eta, \chi) &:= \int_\Omega D^2 \eta : D^2 \chi \, dx, & \text{and } b(\eta, \chi, \varphi) &:= -\frac{1}{2} \int_\Omega [\eta, \chi] \varphi \, dx, \\
    A(\Theta, \Phi) &:= a(\theta_1, \varphi_1) + a(\theta_2, \varphi_2), \\
    B(\Xi, \Theta, \Phi) &:= b(\xi_1, \theta_2, \varphi_1) + b(\xi_2, \theta_1, \varphi_1) - b(\xi_1, \theta_1, \varphi_2), \\
    F(\Phi) &:= (f, \varphi_1)_{L^2(\Omega)}.
\end{align*}
\]

The boundedness and ellipticity properties \((10, 30)\) read, for all \( \Theta, \Phi \in \mathcal{V} \),

\[
A(\Theta, \Phi) \leq \| \Theta \| \| \Phi \|, \quad A(\Theta, \Theta) = \| \Theta \|^2, \quad B(\Xi, \Theta, \Phi) \leq \|\Xi\| \| \Theta \| \| \Phi \|.
\]

The trilinear form \( b(\cdot, \cdot, \cdot) \) is symmetric in first two variables and so is \( B(\cdot, \cdot, \cdot) \). The vector form of \((2.1)\) seeks \( \Psi = (u, v) \in \mathcal{V} \) with \( N(\Psi) = 0 \) for the nonlinear function \( N : \mathcal{V} \to \mathcal{V}' \),

\[
N(\Psi; \Phi) := A(\Psi, \Phi) + B(\Psi, \Psi, \Phi) - F(\Phi) = 0 \quad \text{for all } \Phi \in \mathcal{V}. \tag{2.2}
\]

**Theorem 2.1 (Regularity [4, 32])**. Given any \( f \in H^{-1}(\Omega) \) with norm \( \| f \|_{-1} = \| f \|_{H^{-1}(\Omega)} \), there exists at least one solution \( \Psi \) to \((2.2)\) and any such \( \Psi \) belongs to \( \mathcal{H}^{2, \gamma}(\Omega) \cap \mathcal{V} \) for some elliptic regularity index \( \gamma \in (1/2, 1] \) with \( \| \Psi \| \leq \| f \|_{-1} \) and \( \| \Psi \|_{H^{2, \gamma}(\Omega)} \leq \| f \|_{-1}^\beta + \| f \|_{-1} \) for all \( \Phi \in \mathcal{V} \).

A solution \( \Psi \) to \((2.2)\) is called a regular solution if the Frechét derivative \( DN(\Psi) \in L(\mathcal{V}; \mathcal{V}') \) of \( N \) at \( \Psi \) is an isomorphism. It is also known \([29]\) that for sufficiently small \( f \), the solution is unique and is a regular solution; but this paper aims at a local approximation of an arbitrary regular solution.

The Fréchet derivative \( DN(\Psi) = A(\cdot, \cdot, \cdot) + 2B(\Psi, \cdot, \cdot) \) of the operator \( N \) at the regular solution \( \Psi \) is an isomorphism and this is equivalent to an inf-sup condition

\[
0 < \beta := \inf_{\Theta \in \mathcal{V}} \sup_{\Phi \in \mathcal{V}} \left( \frac{A(\Theta, \Phi) + 2B(\Psi, \Theta, \Phi)}{\| \Theta \|_{-1} \| \Phi \|_{-1}} \right).
\]

Given a regular solution \( \Psi \in \mathcal{V} \) to \((2.1)\) and any \( g \in H^{-1}(\Omega) := H^1_0(\Omega)^* \) (with duality brackets \( \langle \cdot, \cdot \rangle_{H^{-1}(\Omega) \times H^1_0(\Omega)} \) the dual linearised problem seeks \( \zeta \in \mathcal{V} \) with

\[
\begin{align*}
    A(\Phi, \zeta) + 2B(\Psi, \Phi, \zeta) &= \langle g, \Phi \rangle_{H^{-1}(\Omega) \times H^1_0(\Omega)} \quad \text{for all } \Phi \in \mathcal{V}. \tag{2.4}
\end{align*}
\]

This problem is well-posed \([10, 30]\) and satisfies, for the same elliptic regularity index \( \gamma \in (1/2, 1] \) as in Theorem \(2.1\) that

\[
\| \zeta \| \leq \| \zeta \|_{H^{2, \gamma}(\Omega)} \leq \| g \|_{-1}. \tag{2.5}
\]

(In fact, \( \gamma \) is the same as for the biharmonic operator \([4]\) and is unique throughout the paper).

2.2 Triangulations and discrete spaces

Let \( T_{\text{init}} \) be a regular triangulation of the polygonal domain \( \Omega \subset \mathbb{R}^2 \) into triangles in the sense of Ciarlet \([6, 23]\). Each triangle \( T \) associates one of its edges \( \text{Eref}(T) \) as its reference edge and \( T_{\text{init}} \) satisfies the initial condition (IC) in the sense that for each interior edge \( E = \partial T_+ \cap \partial T_- \) shared by the two neighbouring triangles \( T_+ \) it holds either \( \text{Eref}(T_+) = E = \text{Eref}(T_-) \) or \( \text{Eref}(T_+) \neq E \neq \text{Eref}(T_-) \). Any refinement is defined by successive bisections of refinement edges, where the refined triangles inherit the refinement edges.
Figure 1: Possible refinements of a triangle $T$ in one level within the NVB. The dashed lines indicate the refinement edges of the sub-triangles as in [3, 37].

according to Figure 1. The newest vertex bisection (NVB) is described in any space dimension in [37] and known to generate shape-regular triangulations: In 2D there are at most $8/\text{int}$ different interior angles possible in any triangle $T \in \mathcal{T}$. If there exists a finite number of successive bisections that start with $\mathcal{T}_{\text{init}}$ (resp. $\mathcal{T}$) and end with a regular triangulation $\mathcal{T}$ (resp. $\mathcal{T}$), then $\mathcal{T}$ is called a admissible triangulation (resp. $\mathcal{T}$ is called admissible refinement of $\mathcal{T}$). $\mathcal{T} = \mathcal{T}(\mathcal{T}_{\text{init}})$ is the set of all admissible triangulations and $\mathcal{T}(\mathcal{T})$ is the set of all admissible refinements of $\mathcal{T} \in \mathcal{T}$. This paper concerns two very different subsets of admissible refinements of $\mathcal{T}$. Given any $0 < \delta < 1$, let $\mathcal{T}(\delta)$ be the set of all triangulations $\mathcal{T}$ with mesh-size $h_T := |T|^{1/2} = \delta$ for all triangles $T \in \mathcal{T}$ with area $|T|$. Given any $N \in \mathbb{N}$, let $\mathcal{N}(N)$ be the set of all triangulations $\mathcal{T}$ with at most $|T| \leq N + |\mathcal{T}_{\text{init}}|$ triangles (the counting measure $|\cdot|$ describes the cardinality here, but denotes the euclidean length or the area at other places).

Given any $T \in \mathcal{T}$, let $P_k(T)$ denote the piecewise polynomials of degree at most $k \in \mathbb{N}_0$. The mesh-size $h_T \in P_0(T)$ is defined by $h_T := |T|^{1/2} \approx \text{diam}(T)$ in any triangle $T \in \mathcal{T}$ of area $|T|$. Let the $L^2$ projection $P_k$ onto the space of piecewise polynomials $P_k(T)$ of degree at most $k$ act componentwise on vectors or matrices. The oscillations of $f$ in $T$ read $\text{osc}_m(f, T) := \|h_T(f - \Pi_m f)\|$ for $m \in \mathbb{N}_0$. The associated nonconforming Morley finite element space $M(T)$ reads

$$M(T) := \left\{ v_M \in P_2(T) \mid \begin{array}{l} v_M \text{ is continuous at the interior vertices and vanishes at the} \\ \text{vertices of } \partial \Omega; \quad \mathcal{P}_w v_M \text{ is continuous at the midpoints} \\ \text{of interior edges and vanishes at the midpoints of boundary edges} \end{array} \right\}. $$

The discrete space in the von Kármán equations is $\mathcal{V}(T) := M(T) \times M(T)$. For all scalars $\eta, \chi, \varphi \in H^2_0(\Omega) + M(T)$ and all vectors $\Xi = (\xi_1, \xi_2), \Theta = (\theta_1, \theta_2), \Phi = (\varphi_1, \varphi_2) \in H^2_0(\Omega) + \mathcal{V}(T)$, define the discrete bilinear, linear, trilinear forms by

$$a_{pw}(\eta, \chi) := \sum_{K \in \mathcal{T}} \int_K D^2_{pw} \eta : D^2_{pw} \chi \, dx \quad \text{and} \quad b_{pw}(\eta, \chi, \varphi) := -\frac{1}{2} \sum_{K \in \mathcal{T}} \int_K [\eta, \chi]_{pw} \varphi \, dx, $$

$$A_{pw}(\Theta, \Phi) := a_{pw}(\theta_1, \varphi_1) + a_{pw}(\theta_2, \varphi_2), \quad F(\Phi) := \sum_{K \in \mathcal{T}} \int_K f \varphi_1 \, dx \quad \text{and} \quad B_{pw}(\Xi, \Theta, \Phi) := b_{pw}(\xi_1, \theta_2, \varphi_1) + b_{pw}(\xi_2, \theta_1, \varphi_1) - b_{pw}(\xi_1, \theta_1, \varphi_2).$$

Notice that $B_{pw}(\bullet, \bullet, \bullet)$ is well-defined (by the global Sobolev embedding $H^2_0(\Omega) \hookrightarrow L^\infty(\Omega)$ for the last component) and symmetric with respect to the first two arguments, i.e., $B_{pw}(\Xi, \Theta, \Phi) = B_{pw}(\Theta, \Xi, \Phi)$ for all $\Xi, \Theta, \Phi \in \hat{\mathcal{V}}$. The Morley FEM seeks $\Psi_M = (u_M, v_M) \in \mathcal{V}(T)$ such that, for all $\Phi_M \in \mathcal{V}(T)$,

$$N_i(\Psi_M; \Phi_M) := A_{pw}(\Psi_M, \Phi_M) + B_{pw}(\Psi_M, \Psi_M, \Phi_M) - F(\Phi_M) = 0. \quad (2.6)$$

The norm on $\hat{\mathcal{V}} := \mathcal{V} \cup \mathcal{V}(T)$ is defined by $\|\Phi\|_{\hat{\mathcal{V}}} := (\|\varphi_1\|^2_{L^2} + \|\varphi_2\|^2_{L^2})^{1/2}$ for all $\Phi = (\varphi_1, \varphi_2) \in \hat{\mathcal{V}}$ with $\|\varphi_j\|^2_{L^2} := a_{pw}(\varphi_j, \varphi_j)$ and $\|\Phi\|^2_{L^2, pw} := \|\varphi_1\|^2_{L^2, pw} + \|\varphi_2\|^2_{L^2, pw}$ with $\|\varphi_j\|^2_{L^2, pw} = \sum_{K \in \mathcal{T}} (\|\varphi_j\|^2_{L^2, K} + \|D\varphi_j\|^2_{L^2, K})$, defines the piecewise $H^1$ norm for $j = 1, 2$.

2.3 Interpolation and enhancement

Given $T \in \mathcal{T}$ with the maximal mesh-size $h_{\text{max}}$ and its refinement $\hat{T} \in \mathcal{T}(\mathcal{T})$, define the Morley interpolation operator $I_M : H^2_0(\Omega) + M(T) \rightarrow M(T)$ for any $\hat{\nu} \in H^2_0(\Omega) + M(\hat{T})$ through the degrees of freedom

$$(I_M \hat{\nu})(z) = \hat{\nu}(z) \quad \text{for any vertex } z \quad \text{and} \quad \int_E \frac{\partial I_M \hat{\nu}}{\partial E} \, ds = \int_E \frac{\partial \hat{\nu}}{\partial E} \, ds \quad \text{for any edge } E \text{ of } T.$$
Lemma 2.2 (Morley interpolation [13, 22, 25, 28]). The Morley interpolation operator satisfies (a) the integral mean property \( D_{pw}^2 \Pi_0 = \Pi_0 D_{pw}^2 \) of the Hessian, (b) the approximation and stability property

\[
\| h_K^{-2} \| (1 - \Pi_0 v) \|_{\pi,T} + \| D_{pw} (1 - \Pi_0 v) \|_{\pi,T} \leq \| D_{pw}^2 (1 - \Pi_0 v) \|_{\pi,T} \leq \| D_{pw}^2 \Pi_0 \|_{\pi,T} - \| D_{pw}^2 \Pi_0 \|_{\pi,T} \text{ for } K \in T
\]

and \( \Pi_0 \in \Pi_0^2 (\Omega) \), and (c) \( \Pi_0 D_{pw}^2 (1 - \Pi_0 v) \|_{\pi,T} \leq \| v \|_{H^{2+\gamma}(\Omega)} \text{ for all } v \in \Pi_0^2 (\Omega) \cap H^{2+\gamma}(\Omega). \)

Giv en a vertex \( z \in \Omega \) in \( T \in \mathcal{T} \), the closure of its patch \( \omega_z := \text{int} (\cup T(z)) \) covers the neighbouring triangles \( \mathcal{T}(z) \) in \( T \) with the vertex \( z \). Given any triangle \( K \in \mathcal{T} \) with its set of vertices \( N(K) \), its patch is \( \Omega(K) := \{ z \in \Omega \mid z \in T \} \) and \( \mathcal{E}(\Omega(K)) \) denotes the set of edges \( E \) in \( T \) with \( \text{dist}(E, K) = 0 \).

Lemma 2.3 (Companion operator [8, 20, 23]). Given any \( T \in \mathcal{T} \) there exists an enrichment or companion operator \( E_M : \mathcal{M}(T) \rightarrow \Pi_0^2 (\Omega) \) such that any \( v_M \in \mathcal{M}(T) \) satisfies

\[
(a) \quad \Pi_0 (v_M - E_M v_M) = 0, \quad (b) \quad \Pi_0 D_{pw}^2 (v_M - E_M v_M) = 0,
\]

and \( \Pi_0 D_{pw}^2 (1 - \Pi_0 v_M) \|_{\pi,T} \leq \min_{v \in \Pi_0^2 (\Omega)} \| v - v_M \|_{\pi,T}, \quad \text{and} \quad (c) \quad \Pi_0 D_{pw}^2 (v_M - v) \|_{\pi,T} \leq \min_{v \in \Pi_0^2 (\Omega)} \| v - v_M \|_{\pi,T} \|_{H^{2+\gamma}(\Omega)}. \)

Remark 2.1 (consequence). It is an immediate consequence of Lemma 2.3 and Lemma 2.2c in the end that

\[
\| I_M v_M - E_M I_M v_M \|_{\pi,T} \leq \min_{w \in V_M} \| I_M v_M - w \|_{\pi,T} \leq \| v - I_M v_M \|_{\pi,T} \leq \| v_M - E_M v_M \|_{\pi,T} \| v_{H^{2+\gamma}(\Omega)}. \quad (2.7)
\]

holds for any \( v \in \Pi_0^2 (\Omega) \cap H^{2+\gamma}(\Omega) \). A similar estimate can be found in [17, Eq. (3.10)].

Remark 2.2 (extension). The enrichment or a companion operator \( E_M \) is designed in [25] first in terms of the HCT FEM to derive \( J_1 v_M \in H_0^2 (\Omega) \). In the second step, a linear combination of the squares of the cubic bubble-functions \( h_T^2 \) on the triangle \( T \in \mathcal{T} \) is added to define \( J_2 v_M \in H^2 (\Omega) \) with the prescribed integral means to deduce \( \Pi_0 (v_M - J_2 v_M) = 0 \). This can be extended to piecewise polynomials \( p_M (T) h_T^2 \) (rather than \( \mathbb{R} h_T^2 \) for \( m = 0 \) to design some \( J_{2+m} v_M \in H_0^2 (\Omega) \) with \( \Pi_0 (v_M - J_{2+m} v_M) = 0 \) for any \( m \in \mathbb{N}_0 \); cf. [15] for details of a corresponding design for Crouzeix-Raviart finite element schemes.

2.4 Preliminaries

This subsection collects some preliminary lemmas from earlier contributions.

Lemma 2.4 (Bounds for \( A_{pw}(\bullet, \bullet) \) from Lem. 4.2, 4.3, [11, Lem. 4.6]). If \( \Phi, \chi \in H_0^2 (\Omega) \cap H^{2+\gamma}(\Omega) \) and \( X_M = \nabla \phi \) then (a) \( A_{pw}(\Phi, E_M X_M - X_M) \leq h_T^2 \| \Phi \|_{H^{2+\gamma}(\Omega)} \| X_M \|_{\pi,T} \), and (b) \( A_{pw}(\Phi, X_M - \chi) \leq h_T^2 \| \Phi \|_{H^{2+\gamma}(\Omega)} \| X_M - \chi \|_{H^{2+\gamma}(\Omega)}. \)

The discrete analogs of the global Sobolev embeddings are of frequent relevance and easily derived with the companions for Morley functions; related results are known [8, Lem. 3.7] for \( C^0 \) functions.

Lemma 2.5 (discrete embeddings). Given \( T_{\text{init}} \) and \( 1 \leq p < \infty \), there exist constants \( C_{\text{dec}}, C_{\text{deb}} > 0 \) such that such that any \( \nabla v \in H_0^2 (\Omega) + M(\mathcal{T}) \) satisfies (a) \( \| \nabla v \|_{\infty,p} + \| \nabla v \|_{1,p,\pi,T} \leq C_{\text{dec}} \| \nabla v \|_{\pi,T} \) and any \( v_M \in \mathcal{M}(T) \cup H_0^2 (\Omega) \) satisfies (b) \( \| v_M \|_4 \leq C_{\text{deb}} \| v_M \|_{1,2,\pi,T}. \)

Proof. The proof of (a) is included as Lemma 4.7 in [18]. The proof of (b) for \( v_M \in \mathcal{M}(\mathcal{T}) \) is based on the following modification of Lemma 2.3b. Lemma 2.3c for \( K \in \mathcal{T} \) and the shape-regularity in step two show

\[
h_K^2 \| v_M - E_M v_M \|_{\pi,T} + \| v_M - E_M v_M \|_{1,2,\pi,T} \leq h_K^2 \| v_M \|_{\pi,T} \| E_M v_M \|_{1,2,\pi,T} \leq \| h_T v_M \|_{\pi,T} \| E_M v_M \|_{1,2,\pi,T} \leq \| v_M \|_{1,2,\pi,T}.
\]
Theorem 2.7

A priori error analysis

The function $E_{MV}$ is a piecewise polynomial with respect to some refinement $\mathcal{T}$ of $\mathcal{T}$ defined by the subdivision of each $K \in \mathcal{T}$ into three sub-triangles that consist of one edge and the center mid($K$) of $T$ as in the HCT FEM. Then $\|v_M - E_{MV}\|_{\infty} = \|v_M - E_{MV}\|_{L^\infty(K)}$ for at least one $K \in \mathcal{T}$ and $\hat{K} \subset K$ for some $K \in \mathcal{T}$ with $3|\hat{K}| = |K|$. An inverse estimate for the polynomial $w := \hat{v}_M|_K - E_{MV}|_K \in P_0(\hat{K})$ reads

$$\|w\|_{\infty, K} \leq C_{inv}|\hat{K}|^{-1/2}\|w\|_{L^2, \hat{K}} \leq \sqrt{3}C_{inv}|K|^{-1/2}\|w\|_{L^2, K}$$

with a universal constant $C_{inv}$ (for the shape-regularity of $K \in \mathcal{T}$ implies that of $\hat{K}$). This and (2.3) show

$$\|v_M - E_{MV}\|_{\infty} \leq \sqrt{3}C_{inv}\|v_M - E_{MV}\|_2 \leq \|v_M\|_{1,2,pw}.$$  (2.9)

The global Sobolev imbedding $H^1_0(\Omega) \hookrightarrow L^4(\Omega)$ is continuous (this implies the other assertion for $v_M \in H^1_0(\Omega)$). Hence $\|E_{MV}\|_4 \leq \|E_{MV}\|_{1,2} \leq \|E_{MV}\|_{1,2}$ from the Friedrichs inequality in the second step. This, triangle inequalities, the Hölder inequality, and (2.9) lead to

$$\|v_M\|_4 \leq \|E_{MV}\|_4 + \|v_M - E_{MV}\|_4 \leq \|E_{MV}\|_{1,2} \leq \|v_M\|_{1,2,pw} + \|v_M - E_{MV}\|_{1,2,pw}.$$  (2.10)

This and a final application of (2.3) conclude the proof of (b).

The following bounds apply frequently in the analysis of this paper; those are based on Lemma 2.5 and the global Sobolev imbedding $H^{2+\gamma}(\Omega) \hookrightarrow W^{2,4}(\Omega)$ (the latter requires $\gamma > 1/2$).

Lemma 2.6 (bounds for $B_{pw}(\bullet, \bullet, \bullet)$). If $\Phi \in H^{2+\gamma}(\Omega)$, $\Theta, \chi, \zeta \in H^2(v(T))$, and $\hat{x}_M \in V(\mathcal{F})$ for $\mathcal{T} \in \mathcal{V}(\mathcal{T})$, then (a) $B_{pw}(\Phi, \Theta, \chi, \zeta) \leq \|\Theta\|_{pw}\|\chi\|_{pw}$ and (b) $B_{pw}(\Phi, \Theta, \hat{x}_M) \leq \|\Phi\|_{H^{2+\gamma}(\Omega)}\|\Theta\|_{pw}\|\hat{x}_M\|_{1,2,pw}$.

Proof. (a) The definition of $b_{pw}(\Phi, \Theta, \chi)$, piecewise Hölder inequalities, and Lemma 2.5 in the last step show for scalar test functions $\phi, \theta, \chi \in H^2_0(\Omega) + M(\mathcal{T})$

$$2|b_{pw}(\phi, \theta, \chi)| = |\sum_{K \in \mathcal{T}} \int_{K} \phi \theta \chi \, dx| \leq \|\phi\|_{pw}\|\theta\|_{pw}\|\chi\|_{pw} \leq C_{de}\|\phi\|_{pw}\|\theta\|_{pw}\|\chi\|_{pw}.$$  (2.10)

The same arguments show in (b) with $\phi \in H^2_0(\Omega) \cap H^{2+\gamma}(\Omega)$ and $\hat{x}_M \in M(\mathcal{T})$ that

$$2|b_{pw}(\phi, \theta, \hat{x}_M)| \leq \|\phi\|_{2,4}\|\theta\|_{pw}\|\hat{x}_M\|_4 \leq C_5C_{de}\|\phi\|_{H^{2+\gamma}(\Omega)}\|\theta\|_{pw}\|\hat{x}_M\|_{1,2,pw}$$

with the operator norm $C_5$ of the continuous global Sobolev imbedding $H^{2+\gamma}(\Omega) \hookrightarrow W^{2,4}(\Omega)$ and Lemma 2.5b (with $\mathcal{F}$ replacing $\mathcal{T}$). The application of the above estimates in $B(\bullet, \bullet, \bullet)$ concludes the proof.

Recall the discrete analog to (2.3) at the regular solution $\tilde{v}_M$ as key in the a priori error analysis.

Theorem 2.7 (discrete inf-sup [18]). Given a regular solution $\Psi \in H^{2+\gamma}(\Omega) \cap \mathcal{V}$ to (2.2), there exist $\delta_1 > 0$ and $\beta_1 > 0$ such that $\mathcal{T} \in \mathcal{T}(\delta_1)$ implies

$$\beta_1 \leq \inf_{\Theta_M \in \mathcal{V}(\mathcal{T}), \Phi_M \in \mathcal{V}(\mathcal{T})} \sup_{\|\Theta_M\|_{pw}=1, \|\Phi_M\|_{pw}=1} (A_{pw}(\Theta_M, \Phi_M) + 2B_{pw}(\Psi, \Theta_M, \Phi_M)).$$  (2.10)

3 A priori error analysis

The a priori energy norm estimates for the Morley FEM for (2.2) can be found in [18, 31]. The subsequent theorem adds a new piecewise $H^1$ semi-norm a priori error estimate.

Theorem 3.1 (a priori). Given a regular solution $\Psi$ to (2.2), there exist $\eta_0 > 0$ (without loss of generality $\delta_0 < 1$) such that, for all $\mathcal{T} \in \mathcal{T}($), (a) there exists a unique solution $\Psi_M \in \mathcal{V}(\mathcal{T})$ to (2.6) with $\|\Psi - \Psi_M\|_{pw} \leq \eta_0$; the solutions $\Psi = (u, v)$ and $\Psi_M$ satisfy

$$(b) \|\Psi - \Psi_M\|_{pw} \leq \|\Psi - I_M\Psi\|_{pw} + \text{osc}_1(f + [u, v], \mathcal{T}) + \text{osc}_1([u, v], \mathcal{T}) \leq h^2_{max}\|\Psi\|_{H^{2+\gamma}(\Omega)},$$

$$(c) \|\Psi - \Psi_M\|_{1,2,pw} \leq h^2_{max}\|\Psi - \Psi_M\|_{pw} + \text{osc}_m(f, \mathcal{T})$$

for each $m \in \mathbb{N}$.
The second inequality in (3.3) is the stability from Lemma 2.3.d with the first inequality in the proof of Theorem 3.1. The proof is by a careful analysis of the perturbations from the nonconforming Lemma 2.3.a shows that \( \rho_I \) and \( \Pi \) with a well-known estimate for some indices \( a \) for some indices \( a \) and so Lemma 2.2.a in (b)) proves that \( \chi = (\chi_1, \chi_2) \) has a piecewise second order derivative \( \partial^2 \rho \) with integral mean zero over each triangle in \( T \). To exploit the consequences, observe that \( B_{pm}(\Psi, \chi, \zeta) \) is a sum of terms of the form

\[
I := \int_\Omega (\zeta_k \partial_{ab} \psi_j) \partial_{cd,pw} \chi \, dx = \int_\Omega (\lambda_{k,abj} - \Pi_0 \lambda_{abj}) \partial_{cd,pw} \chi \, dx \leq F_1 F_2
\]

for some indices \( a, b, c, d, j, k, \ell \in \{1, 2\} \) and the abbreviation \( \lambda_{k,abj} := \zeta_k \partial_{ab} \psi_j \in H^2(\Omega) \) with piecewise integral means \( \Pi_0 \lambda_{abj} \) (the undisplayed sign does not play a role in the estimates below). The integral \( I \) allows for a Cauchy inequality with one factor

\[
F_1 := \| \lambda_{k,abj} - \Pi_0 \lambda_{abj} \|_2 \leq \| \zeta_k \partial_{ab} \psi_j - \Pi_0 \zeta_k \partial_{ab} \psi_j \|_2
\]

(with the inequality from the \( L^2 \) orthogonality). A triangle inequality, \( \| \Pi_0 \zeta_k \|_{\infty} \leq \| \zeta_k \|_{\infty} \), and \( \| \zeta_k - \Pi_0 \zeta_k \|_{L^2} \leq h_{\max} \| \chi \|_{L^2} \) show

\[
F_1 \leq \| (\zeta_k - \Pi_0 \zeta_k) \partial_{ab} \psi_j \|_2 + \| \zeta_k \partial_{ab} \psi_j - \Pi_0 \partial_{ab} \psi_j \|_2 \leq \| \zeta_k \|_{W^{1,\infty}(\Omega)} \| \partial_{ab} \psi_j \|_{L^2} + h_{\max} \| \chi \|_{H^2(\Omega)}
\]

with a well-known estimate \( \| \zeta - \Pi_0 \zeta \|_{L^2} \leq h_{\max} \| \zeta \|_{H^2(\Omega)} \) for \( \zeta := \zeta_k \partial_{ab} \psi_j \in H^2(\Omega) \) in the last step. This and the global Sobolev embedding \( H^{2+}(\Omega) \hookrightarrow W^{1,\infty}(\Omega) \) conclude the analysis for the first factor in the upper bound of the prototype term

\[
F_1 \leq h_{\max} \| \Psi \|_{H^{2+}(\Omega)} \| \zeta \|_{H^{2+}(\Omega)}.
\]

In case (a), the second factor \( F_2 := \| \partial_{cd,pw} \chi \|_{L^2} \leq \| \chi \|_{p} = \| E_M \rho_M - \rho_M \|_{L^2} \) is controlled with Lemma 2.3.d and \( v = 0 \) as \( F_2 \leq \| \pi_M \rho \|_{L^2} \). In the other case (b), \( F_2 \leq \| \chi \|_{p} = \| \Psi - I_M \Psi \|_{p} \). Those two estimates and the previously displayed estimate for \( F_1 \) result in an upper bound of \( F_1 F_2 \geq | I | \). The evaluation of all those contributions of type \( I \) concludes the proof of (a) and (b).

**Proof of Theorem 3.1** The proof is by a careful analysis of the perturbations from the nonconforming functions with the companion operators in a duality argument. The first step is the definition and the isolation of the crucial term \( E_M \rho_M \in \mathcal{V} \): Let \( \rho_M := I_M \Psi - \Psi_M \in \mathcal{V}(\Omega) \) and recall \( E_M \) of Lemma 2.3 (acting componentwise). Triangle inequalities show that

\[
\| \Psi - \Psi_M \|_{L^2} \leq \| \Psi - I_M \Psi \|_{L^2} + \| \rho_M - E_M \rho_M \|_{L^2} + \| E_M \rho_M \|_{L^2}.
\]

In (3.1) \( \rho_M - E_M \rho_M = (I_M - 1)E_M \rho_M \) and so Lemma 2.3.a in (b) (in a global version) proves the first inequality in

\[
h_{\max}^2 \| \Psi - I_M \Psi \|_{L^2}^2 \leq \| \Psi - I_M \Psi \|_{L^2}^2 = \| \Psi - I_M \Psi \|_{L^2}^2 - \| \rho_M \|_{L^2}^2.
\]

In (3.2) \( \rho_M - E_M \rho_M = (I_M - 1)E_M \rho_M \) and so Lemma 2.3.a in (b) (in a global version) proves the second inequality in

\[
h_{\max}^{-1} \| \rho_M - E_M \rho_M \|_{L^2} \leq \| \rho_M - E_M \rho_M \|_{L^2} \leq \| \Psi - I_M \Psi \|_{L^2}.
\]

The second inequality in (3.3) is the stability from Lemma 2.3.d with \( v = 0 \). The last inequality in (3.3) follows from the Pythagoras theorem in (3.2). This concludes the first step in which (3.1), (3.3) show

\[
\| \Psi - \Psi_M \|_{L^2} \leq h_{\max} \| \Psi - \Psi_M \|_{L^2} + \| E_M \rho_M \|_{L^2}.
\]
The second step focuses on the last term in (3.4) with a duality argument with the solution $\zeta \in V$ to the linearized equation (2.4) with $g := -\Delta M \rho_M \in L^2(\Omega)$ ($E_M$ acts componentwise on $\rho_M$ and the Laplacian acts componentwise on $E_M \rho_M \in H^1_0(\Omega)$ and the regularity $\zeta \in H^{2,\gamma}(\Omega)$ from (2.5). Recall the linearised operator $A(\cdot, \cdot)$ of (2.4) and its piecewise analog $A_{pw}(\cdot, \cdot)$ that replaces $A(\cdot, \cdot)$ (resp. $B(\cdot, \cdot, \cdot)$) in (2.4) by $A_{pw}(\cdot, \cdot)$ (resp. $B_{pw}(\cdot, \cdot, \cdot)$). This and elementary operations like an integration by parts and elementary algebra eventually lead to

$$
\|E_M \rho_M\|_{L^2(\Omega)} = (\nabla E_M \rho_M, \nabla E_M \rho_M)_{L^2(\Omega)} = (g, E_M \rho_M)_{L^2(\Omega)} = A(\rho_M, \zeta),
$$

$$
= A_{pw}(E_M \rho_M - \rho_M, \zeta) + 2B_{pw}(\Psi, E_M \rho_M - \rho_M, \zeta) + A_{pw}(I_M \Psi - \Psi, \zeta) + 2B_{pw}(\Psi, I_M \Psi - \Psi, \zeta) + A_{pw}(E_M \rho_M - \rho_M, \zeta) + 2B_{pw}(\Psi, E_M \rho_M - \rho_M, \zeta) + A_{pw}(\Psi - \Psi_M, \zeta) + A_{pw}(\Psi - \Psi_M, \zeta) = : T_1 + \cdots + T_8.
$$

The eight terms $T_1, \cdots, T_8$ are controlled in step three of the proof. Lemma 2.4.a shows

$$
T_1 := A_{pw}(E_M \rho_M - \rho_M, \zeta) \leq \|\rho_M\|_{L^\infty} \|\zeta\|_{H^{2,\gamma}(\Omega)} \leq \|\rho_M\|_{L^\infty} \|\zeta\|_{H^{2,\gamma}(\Omega)}
$$
with (3.4) in the end. Lemma 2.4.a and (3.5) imply

$$
T_2 + T_3 = 2B_{pw}(\Psi, E_M \rho_M - \rho_M, \zeta) + 2B_{pw}(\Psi, I_M \Psi - \Psi, \zeta) \leq \|\rho_M\|_{L^\infty} \|\zeta\|_{H^{2,\gamma}(\Omega)} \leq \|\rho_M\|_{L^\infty} \|\zeta\|_{H^{2,\gamma}(\Omega)}.
$$

Lemma 2.2.a shows $A_{pw}(\Psi - I_M \Psi, \eta_M) = 0 = A_{pw}(\eta_M, \zeta - I_M \zeta)$ for all $\eta_M \in V_M$. Consequently,

$$
T_4 + T_5 = A_{pw}(I_M \Psi - \Psi, \zeta - I_M \zeta) + A_{pw}(\Psi - \Psi_M, \zeta - I_M \zeta) = 0.
$$

The boundness of $A_{pw}(\cdot, \cdot)$ and (2.7) result in

$$
T_6 := A_{pw}(\Psi - \Psi_M, I_M \zeta - E_M I_M \zeta) \leq \|\rho_M\|_{L^\infty} \|\zeta\|_{H^{2,\gamma}(\Omega)}.
$$

Lemma 2.4.a shows $I_M E_M \zeta = \zeta_M$ for $\varphi_M \in M(T)$ and Lemma 2.3.c shows $A_{pw}(\Psi_M, (1 - I_M) E_M I_M \zeta) = 0$. This and (2.2) (resp. (2.6)) in the end lead to

$$
T_7 := A_{pw}(\Psi - \Psi_M, E_M I_M \zeta - I_M \zeta) = A_{pw}(\Psi, E_M I_M \zeta - I_M \zeta) - A_{pw}(\Psi_M, I_M \zeta) = F(\rho_M E_M I_M \zeta - I_M \zeta) - B_{pw}(\Psi, I_M E_M I_M \zeta) + B_{pw}(\Psi_M, I_M \zeta).
$$

Lemma 2.4.b shows $A_{pw}(\zeta) = 0$ for $\zeta := E_M I_M \zeta - I_M \zeta = (1 - I_M) E_M I_M \zeta$. Since the piecewise second derivatives of $\Psi_M$ are piecewise constants, $B_{pw}(\Psi_M, \Psi_M, \Theta) = 0$ follows. This and elementary algebra (with the symmetry of $B_{pw}$ in the first two components) imply

$$
T_7 + T_8 = F(\Theta) + B_{pw}(\Psi - \Psi_M, \Psi - \Psi_M, E_M I_M \zeta) + 2B_{pw}(\Psi - \Psi_M, \Psi - \Psi_M, E_M I_M \zeta).
$$

The right-hand side consists of three terms $T_9 + T_{10} + T_{11}$ estimated in the sequel. Recall $A_{pw}(\Theta) = 0$ and apply Lemma 2.4.b to verify

$$
T_9 := F(\Theta) = (f - \Pi_0 f, \Theta)_{L^2(\Omega)} \leq \|\Theta\|_{L^\infty} \|\Theta\|_{L^2(\Omega)} \leq \|\rho_M\|_{L^\infty} \|\zeta\|_{H^{2,\gamma}(\Omega)}
$$
with $\Pi_0 \Theta = 0$ for $\Theta := E_M I_M \zeta - I_M \zeta = (1 - I_M) E_M I_M \zeta$. From (2.7) and Theorem 3.1.b show

$$
T_{10} := B_{pw}(\Psi - \Psi_M, \Psi - \Psi_M, E_M I_M \zeta) \leq \|\Psi\|_{H^{2,\gamma}(\Omega)} \|\Psi - \Psi_M\|_{L^2(\Omega)} \leq \|\zeta\|_{H^{2,\gamma}(\Omega)} \|\Psi - \Psi_M\|_{L^2(\Omega)} \|E_M I_M \zeta\|_{L^2(\Omega)}.
$$

The stability $\|E_M I_M \zeta\|_{L^2(\Omega)} \leq \|\zeta\|_{L^2(\Omega)}$ from Lemma 2.3.d and a proves $\|E_M I_M \zeta\|_{L^2(\Omega)} \leq \|\zeta\|_{H^{2,\gamma}(\Omega)}$. Lemma 2.6.b shows for $\zeta - E_M I_M \zeta \in V$ that

$$
T_{11} := 2B_{pw}(\Psi, \Psi - \Psi_M, \zeta - E_M I_M \zeta) \leq \|\Psi\|_{H^{2,\gamma}(\Omega)} \|\Psi - \Psi_M\|_{L^2(\Omega)} \leq \|\zeta\|_{H^{2,\gamma}(\Omega)} \|\Psi - \Psi_M\|_{L^2(\Omega)} \|E_M I_M \zeta\|_{L^2(\Omega)}.
$$

A triangle inequality, Lemma 2.2.b-c, and (2.7) lead to

$$
\|\zeta - E_M I_M \zeta\|_{L^2(\Omega)} \leq \|\zeta - I_M \zeta\|_{L^2(\Omega)} + \|I_M \zeta - E_M I_M \zeta\|_{L^2(\Omega)} \leq h^{1+\gamma}\|\zeta\|_{H^{2,\gamma}(\Omega)}.
$$
The combination of the aforementioned estimates with $1/2 < \gamma \leq 1$ results in

$$T_7 + T_8 = T_9 + T_{10} + T_{11} \leq h_T^\gamma \left( \|\Psi - \Psi_M\|_{pw} + \|\Psi\|_{\gamma(\Omega)} \right).$$

Step four is the conclusion of the proof. The estimates for $T_1$ to $T_8$ lead to

$$\|E_M\|_{1,2} \leq h_T^\gamma (1 + \|\Psi\|_{\gamma(\Omega)}) \left( \|\Psi - \Psi_M\|_{pw} + \|\Psi\|_{\gamma(\Omega)} \right).$$

Recall (2.4) with $\|\zeta\|_{\gamma(\Omega)} \leq \|g\|_{-1} \leq \|E_M\|_{1,2}$, so that

$$\|E_M\|_{1,2} \leq h_T^\gamma (1 + \|\Psi\|_{\gamma(\Omega)}) \left( \|\Psi - \Psi_M\|_{pw} + \|\Psi\|_{\gamma(\Omega)} \right).$$

Recall Theorem 2.1 and write $\|\Psi\|_{H^{2+\gamma}(\Omega)} \leq 1$ (so the constants depend on $\|f\|_{-1}$). This, the previous estimate, and (3.4) conclude the proof for $m = 0$. For $m \in \mathbb{N}$, utilise a companion operator $E_M$ outlined in Remark 2.2 with all the properties of Lemma 2.3 plus the higher-order orthogonality $\Pi_m (v_M - E_M v_M) = 0$ for all $v_M \in M(T)$. This allows in $T_0$ the extra orthogonality

$$F(\Theta) = (f - \Pi_m f, \Theta)_{L^2(\Omega)} \leq \|\Psi - \Psi_M\|_{pw} \leq h_T^\gamma \|\Psi\|_{\gamma(\Omega)}.$$ 

This modification enables the proof for general $m \in \mathbb{N}$; further details are omitted.

4 Adaptive mesh-refinement and the axioms of adaptivity

In the remainder of this paper, $\Psi = (u, v) \in \mathcal{V}$ is a fixed regular solution to (2.2) called the exact solution. Theorem 3.1.a leads to $e_0, \delta_0 > 0$ such that any triangulation $T \in \mathcal{T}(\delta_0)$ leads to a unique discrete solution $\Psi_M \in \mathcal{V}(T)$ to (2.6) with $\|\Psi - \Psi_M\|_{pw} \leq e_0$ and this $\Psi_M = (u_M, v_M)$ is called the discrete solution.

4.1 A posteriori error analysis

Given the discrete solution $\Psi_M = (u_M, v_M) \in \mathcal{V}(T)$ to (2.6) define $\eta(T, K) \geq 0$ as the square root of

$$\eta^2(T, K) := \frac{1}{|K|} \left( \|u_M - v_M\|_{L^2(K)} + \|u_M - u_M\|_{L^2(K)} + \left\| \Pi_M \|f\|_{L^2(\Omega)} \right\|_{L^2(\Omega)} \right).$$

Recall the notation from Subsection 2.2 and let $E(K)$ denote the three edges of a triangle $K \in T$ with area $|K|$. The jump $\{\bullet\}_E$ across an interior edge $E = \partial T_s \cap \partial T_r \in \mathcal{E}(\Omega)$ with tangential vector $\tau_E$ and normal vector $n_E$ is the difference of the respective traces on $E$ from the two neighbouring triangles $T_s$ that form the edge patch $\mathcal{P}_E := \text{int}(T_s \cup T_r)$. The jump $\{\bullet\}_E$ along a boundary edge $E \in \mathcal{E}(\partial \Omega)$ is simply the trace from the attached triangle $T_r = \tau_E E$; the contribution of the missing jump partner is zero. Like any other operator, $\{\bullet\}_E$ acts componentwise in $\|D^2_{pw} u_M\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)}$ etc. Given any subset $M \subseteq T$ of $T \in \mathcal{T}(\delta_0)$, its contribution $\eta(T, M) \geq 0$ is the square root of the sum

$$\eta^2(T, M) := \sum_{K \in M} \eta^2(T, K) \quad \text{and} \quad \eta(T, T) := \eta(T, M),$$

abbreviates the contribution of all triangles (by convention $\eta(T, \emptyset) := 0$). Recall the oscillations $\text{osc}_T^2(f, T) = \left\| h_T^2 (1 - \Pi_0) f \right\|_{L^2(\Omega)}$ of $f \in L^2(\Omega)$ for the mesh-size factor $h_T \in \mathcal{P}_0(T)$.

**Theorem 4.1** (a posteriori [18]). Given the exact solution $\Psi$ and $e_0$ and $\delta_0$ from Theorem 3.1.a, there exist positive constants $C_{\text{rel}}$ and $C_{\text{eff}}$ (which depend on $\mathcal{T}_{\text{init}}$ and on $\Psi$, $e_0$, $\delta_0$) so that, for all $T \in \mathcal{T}(\delta_0)$, the discrete solution $\Psi_M \in \mathcal{V}(T)$ and the error estimator $\eta(T)$ from (4.1)-(4.2) satisfy

$$C_{\text{rel}} \|\Psi - \Psi_M\|_{pw} \leq \eta(T) \leq C_{\text{eff}} \left( \|\Psi - \Psi_M\|_{pw} + \text{osc}_T(f, T) \right).$$

The proofs can be found in [18]; an alternative proof of the reliability (the first inequality in Theorem 4.1) follows in Corollary 4.2. This paper is thus self-contained for efficiency (the second inequality in Theorem 5.1 is not exploited in this paper (except for Remark 4.6).
4.2 Adaptive Morley finite element algorithm

Recall that the initial triangulation $T_\text{init}$ satisfies the initial condition (IC) for its reference edges from Subsection 2.2. Recall $\Psi$ and $\epsilon_0, \delta_0 > 0$ from Theorem 5.1a.

Adaptive algorithm (AMFEM).

**Input:** Initial triangulation $T_\text{init}$ with IC, $0 < \delta \leq \delta_0 < 1$, and $0 < \theta \leq 1$

**Compute** $T_\ell$ by uniform refinements of $T_\text{init}$ such that $T_0 \in \mathcal{T}(\delta)$

for $\ell = 0, 1, \ldots$

**Compute** discrete solution $\Psi_\ell = (u_\ell, v_\ell) \in \mathcal{V}(T_\ell)$ with $\|\Psi - \Psi_\ell\|_{pw} \leq \epsilon_0$

**Select** $M_\ell \subseteq T_\ell$ of (almost) minimal cardinality with

$$\theta \eta^2(\ell)(M_\ell) \leq \eta^2(\ell) := \eta^2(T_\ell) \tag{4.3}$$

**Compute** $T_{\ell+1} := \text{Refine}(T_\ell, M_\ell)$

**Output:** $T_\ell$, $\Psi_\ell$, and $\eta_\ell$ for $\ell \in \mathbb{N}_0$

Throughout this paper, $T_\ell$, $\Psi_\ell$, and $\eta_\ell$ will refer to the output of this adaptive algorithm and $\eta_\ell(K) := \eta(T_\ell, K)$ for all $K \in T_\ell$ etc. Some comments are in order before the axioms of adaptivity are reviewed.

**Remark 4.1 (exact solve).** The main idealisation of this paper is the assumption on exact solve in AMFEM. An optimal practical algorithm (optimal also with respect to the total run time of the overall algorithm with multilevel methods and nested iteration etc.) has to overcome further difficulties beyond the scope of this paper. An iterative solver has to be employed in practice and the termination of which has to be monitored. The computed approximation $\tilde{\Psi}(T_\ell)$ to the error estimators $\eta_\ell$ are based on computed approximations $\Psi_\ell$ to the discrete solution $\Psi$ and the error $\|\Psi - \Psi_\ell\|_{pw}$ has to be controlled. A practical termination criterion reads $\|\Psi_\ell - \Psi\|_{pw} \leq \kappa \|\Psi_\ell\|_{pw}$ for a small positive constant $\kappa$. This could be guaranteed e.g. by some Newton-Kantorovic theorem in the finite-dimensional nonlinear discrete problem. A perturbation analysis enables optimal convergence rates in the general case as in [16, 26].

In the absence of additional information on $\Psi$, the discrete problem may have multiple solutions and the selection of one in AMFEM is less clear. Moreover, $\epsilon_0, \delta_0 > 0$ from Theorem 5.1a exist but are not easy to quantify in general. The proposed version of AMFEM has an initial phase with uniform mesh-refining steps monitored with the input parameter $\delta$. One reason to choose $\delta > 0$ small is that $0 \leq \delta \leq \delta_0$ resolves the nonlinearity in the sense that it guarantees the existence of a unique discrete solution near $\Psi$.

**Remark 4.2 (input parameter).** The optimal convergence rates follow from Theorem 4.2 below under the conditions $0 < \delta, \theta \ll 1$ sufficiently small. The choice of the bulk parameter $\theta < 1/(1 + \Lambda_1^2 \Lambda_2)$ below is independent of $\delta \leq \delta_0$ (but depends on $\Psi, \delta_0, \epsilon_0$). Several arguments in the analysis of (A3)-(A4) below require $\delta > 0$ to be very small (possibly much smaller than $\delta_0$) and it is conjectured that this is not a technical artefact.

**Remark 4.3 (marking).** Recall the sum convention (4.2) for the meaning of the bulk criterion (4.3). A greedy algorithm for the computation of a subset $M^*_\ell$ with $\theta \eta^2(\ell)(M^*_\ell) \leq \eta^2(\ell)$ and minimal cardinality $|M^*_\ell|$ may first sort the triangles in $T_\ell$ according to the size of its estimator contribution $\eta_\ell(K)$. Quick sort may lead to superlinear computational costs and is circumvented in [36] by computing a subset $M_\ell$ of almost minimal cardinality $|M_\ell|$ with (4.3) and $|M_\ell| \leq C_{am}|M^*_\ell|$ for a universal constant $C_{am} \geq 1$.

**Remark 4.4 (refine).** The procedure $\text{Refine}$ specifies the newest vertex bisection (NVB) with completion (to avoid hanging nodes etc.). The output $T_{\text{init}} := \text{Refine}(T_{\text{init}}, M) \in \mathcal{T}(T_{\text{init}})$ is the smallest refinement of $T_{\text{init}}$ with NVB of Figure 1 and $M \in T_{\text{init}} \setminus T_{\text{init}}$. The initial condition of $T_{\text{init}}$ carries over to the first triangulation $T_0$ because of the uniform refinements with NVB. More details may be found in [3, 37].

4.3 Axioms of Adaptivity

Recall the 2-level notation: Each triangulation $T \in \mathcal{T}(\delta)$ (resp. its refinement $\tilde{T} \in \mathcal{T}(\tilde{T}))$ leads to a unique discrete solution $\Psi_T = (u_T, v_T) \in \mathcal{V}(T)$ (resp. $\tilde{\Psi}_T = (\tilde{u}_T, \tilde{v}_T) \in \mathcal{V}(\tilde{T})$) to (2.6) with $\|\Psi - \Psi_T\|_{pw} \leq \epsilon_0$ (resp. $\|\Psi - \tilde{\Psi}_T\|_{pw} \leq \epsilon_0$). This defines the (global) distance

$$\delta(T, \tilde{T}) := \|\Psi_T - \Psi_{\tilde{T}}\|_{pw}$$
of $T \in \mathcal{T}$ and its refinement $\tilde{T} \in \mathcal{T}(T)$ as a global non-negative real number. Recall the definition (4.1) of $\eta(T, K)$ for all $K \in T \in \mathcal{T}(\delta)$ and specify, for fixed $T \in \mathcal{T}$ and its fixed refinement $\tilde{T} \in \mathcal{T}(T)$,

$$\eta(K) := \eta(T, K) \quad \text{and} \quad \tilde{\eta}(T) := \eta(\tilde{T}, T) \quad \text{for} \quad K \in T \quad \text{and} \quad T \in \tilde{T}.$$

and adapt the sum conventions (4.2) for the short-hand notation $\eta$ and $\tilde{\eta}$ in the axioms (A1)-(A3) with universal constants $\Lambda_1, \Lambda_2,$ and $\Lambda_3$

(A1) Stability.

$$|\eta(T \cap \widetilde{T}) - \eta(T \cap \tilde{T})| \leq \Lambda_1 \delta(T, \tilde{T}).$$

(A2) Reduction.

$$\tilde{\eta}(T \setminus \tilde{T}) \leq 2^{-1/4} \eta(T \setminus \tilde{T}) + \Lambda_2 \delta(T, \tilde{T}).$$

(A3) Discrete Reliability.

$$\sum_{k \geq 1} \delta^2(T_k, \tilde{T}_k) \leq \Lambda_3 \eta^2(T \setminus \tilde{T}).$$

(A4) Quasioptogonality.

$$\sum_{k \geq 1} \delta^3(T_k, \tilde{T}_k) \leq \Lambda_4 \eta^3(T \setminus \tilde{T}).$$

The notation in the axiom (A4) solely concerns the outcome $T_\ell$ of AMFEM with a universal constant $\Lambda_4$ and already asserts that the left-hand side is a converging sum.

The subsequent section provides the proofs of all those four axioms and then allows the application of the abstract theorem for optimal convergence rates. Recall the definitions of $\mathcal{T}(\delta)$ for $0 < \delta < 1$ and $\mathcal{T}(N)$ for $N \in \mathbb{N}_0$ in Subsection 2.2 and define the set

$$\mathcal{T}(T_0, N) := \{T \in \mathcal{T}(T_0) : |T| \leq N + |T_0|\}$$

of all admissible refinements of $T_0$ with at most $N \in \mathbb{N}_0$ extra triangles.

**Theorem 4.2** (optimal rates in adaptive FEMs [11, 21]). Suppose (A1)-(A4), $0 < \theta < \theta_0 := 1/(1 + \Lambda_1^2 \Lambda_3)$ in AMFEM with output $(T_\ell)_{\ell \in \mathbb{N}_0}$ and $(\eta_\ell)_{\ell \in \mathbb{N}_0}$ and let $s > 0$. Then (there exist equivalence constants in)

$$\sup_{\ell \in \mathbb{N}_0} (1 + |T_\ell| - |T_0|)^s \eta_\ell \approx \sup_{N \in \mathbb{N}_0} (1 + N)^s \min_{T \in \mathcal{T}(T_0, N)} \eta(T, N)$$

(4.4)

with the minimum $\min_{T \in \mathcal{T}(T_0, N)} \eta(T, N)$ of all $\eta(T)$ with $T \in \mathcal{T}(T_0, N)$ for $N \in \mathbb{N}_0$.

**Proof.** The formulation of this theorem is a simplified version of the results in [11, 21] based on the seminal paper [36] for the special case $T_0 \equiv T_{\text{init}}$ (leave out the uniform refinement steps in the beginning). To enable unique discrete solutions near a regular solution $\Psi$, the present algorithm (AMFEM) involves the computation of $T_0$ and then runs a standard adaptive algorithm. Consequently, the analysis of the standard adaptive algorithm in [11, 21] applies and requires the axioms (A1)-(A4) to hold solely for $T \in \mathcal{T}(T_0)$ to guarantee (4.4). As a consequence, the equivalence constants (behind the notation $\approx$) in (4.4) depend on all parameters $\delta, \theta, T_0$, and $s$.

The point of this paper is the verification of (A1)-(A4) for small positive $\delta < 1$ to prove the main result of optimal rates.

**Theorem 4.3** (optimal rates in (AMFEM)). Given a regular solution $\Psi$ to (2.2) and an initial triangulation $T_{\text{init}}$, there exist positive constants $\delta, \theta < 1$ such that the algorithm (AMFEM) runs for all $0 < \delta \leq \delta_0$ and $0 < \theta \leq \theta_0$ with an output $(T_\ell)_{\ell \in \mathbb{N}_0}$ and $(\eta_\ell)_{\ell \in \mathbb{N}_0}$ that satisfies (4.4) for all $s > 0$ with equivalence constants (behind the notation $\approx$), which depend on $\Psi, T_{\text{init}}, \delta, \theta$, and $s$ but are independent of $\delta$ and $\theta$.

The proof is based on Theorem 4.2 and will be completed in Subsection 5.6 below.

**Remark 4.5** (pre-asymptotic range). The convergence rate is an intrinsically asymptotic concept and does not deteriorate if $\delta$ or $\theta$ in (AMFEM) are chosen far too small. The computational costs and the overall pre-asymptotic range, however, crucially depend on $\delta$ and may become larger and larger as $\delta$ approaches zero. In case of a regular solution close to a bifurcation point (with multiple solutions of small difference) the restrictions $\delta \leq \delta_0 < \min\{\delta_1, \delta_1\}$ from Theorem 2.7 and 3.1 may already enforce $\delta$ to be very small.

**Remark 4.6** (nonlinear approximation). The equivalence (4.4) asserts optimal convergence rates (for $s > 0$ is arbitrary) in terms of the error estimators. The efficiency in Theorem 4.1 transforms this to rate optimality with respect to nonlinear approximation classes [3] of the total error $\|\Psi - \Psi_M\|_{\text{pw}} + \text{osc}_0(f, T)$. 


5 Proofs

This section verifies (A1)-(A4) and Theorem 4.3. Throughout this section, 0 < δ ≤ δ₀ < 0 with δ₀, ɛ₀ > 0 from Theorem 4.1 and the 2-level notation of (the beginning of) Subsection 4.2 applies to ℋ ∈ ℰ(δ), ℋ ∈ ℰ(δ), Π = (um, vm) ∈ ℰ(δ) with ∥Π − Π₉₈∥₂ ≤ ɛ₀, Π₉₈ = (π₉₈, π₉₈) ∈ ℰ(δ) with ∥Π − Π₉₈∥₂ ≤ ɛ₀, δ := δ(ℋ, •), and δ := δ(ℋ, •); whereas ℋ, ℋ, and δ := δ(ℋ) etc. refer to the output of AMFEM.

5.1 Proof of stability (A1)

The proofs of (A1) and (A2) rely on triangle and Cauchy inequalities plus one lemma.

Lemma 5.1 (discrete jump control [21, Lem. 5.2]). There exists a universal constant C_{γ,ε}, which depends on the shape regularity in ℋ and the degree k ∈ ℕ₀, such that any ℋ ∈ ℰ and g ∈ P_k(ℋ) with its jumps

\[ [g]_E = \begin{cases} (g|_{T+})_E - (g|_{T-})_E & \text{for } E ∈ ℰ(T) \text{ with } E = \partial T_+ ∩ \partial T_-, \\ (g|_{E})_T & \text{for } E ∈ ℰ(∂Ω) \cap ℰ(K) \end{cases} \]

across any side E ∈ ℰ (i.e., with respect to ℋ ∈ ℰ) satisfy

\[ \sum_{K ∈ ℋ} |K|^{1/2} \sum_{E ∈ ℰ(K)} ∥[g]_E∥^2_{L^2(E)} ≤ C_{γ,ε} ∥g∥^2_{2}. \]

\[ \Box \]

Theorem 5.2 (Stability (A1)). (A1) holds for all ℋ ∈ ℰ(δ₀) and all ℋ ∈ ℰ(δ).

Proof. The proof follows [11, 21, 22] for linear second-order problems with focus on the nonlinear contributions. The definitions of δ(T ∩ ℋ) and η(T ∩ ℋ) in Subsection 4.3 and a first reverse triangle inequality in ℝᵐ with the number m := |ℋ ∩ ℋ| of triangles in ℋ ∩ ℋ lead to

\[ |δ(T ∩ ℋ) - η(T ∩ ℋ)|^2 ≤ \sum_{K ∈ ℋ ∩ ℋ} (δ(K) - η(K))^2. \]

For K ∈ ℋ ∩ ℋ, each of the terms δ(K) and η(K) allows a second and third reverse triangle inequality in ℝₘ and L²(K) or L²(ℋ) for E ∈ ℰ(K). This and \( ∥D^2(δM − VM)|_{E}|L^2(E) ≤ ∥D^2(δM − VM)|_{E} \) etc. with the Frobenius matrix norm \( |||⋅||| \) in ℝᵐ⁻² result in

\[ (δ(K) − η(K))^2 ≤ |K|^2 |||δM − VM|||₂,K − |||UM − VM|||₂,K + |K|^2 |||δM − VM|||₂,K \]

\[ + |K|^{1/2} \sum_{E ∈ ℰ(K)} (|||D^2(UM − VM)|{L²(E)}|||₂,K^2 + |||D^2(UM − VM)|{L²(E)}|||₂,K^2). \]

(5.1)

Triangle, Cauchy-Schwarz inequalities, and an inverse estimate (here indeed an equality for \( ∥D^2vM|||₂,K \) is \( |K|^{1/2} \) times the Frobenius norm of the constant Hessian \( D^2vM)K \) show

\[ |||δM − VM|||₂,K − |||UM − VM|||₂,K ≤ |||UM − VM|||₂,K + |||UM − VM|||₂,K \]

\[ ≤ |K|^{-1/2} |||D^2(UM − VM)|{L²(K)}|||₂,K + |K|^{-1/2} |||D^2(UM − VM)|{L²(K)}|||₂,K \]

\[ ≤ |K|^{-1/2} |||D^2(UM − VM)|{L²(K)}|||₂,K + |K|^{-1/2} |||D^2(UM − VM)|{L²(K)}|||₂,K. \]

This proves an estimate for the first term on the right-hand side of (5.1).

\[ |K|^2 |||δM − VM|||₂,K − |||UM − VM|||₂,K ≤ |K|(|D^2pw(VM)|{L²(K)}|||₂,K + |D^2pw(VM)|{L²(K)}|||₂,K(VM − ΨM))|||₂,K. \]

The substitution of VM (resp. VM) by UM (resp. UM) provides an analog inequality. The sum of those two estimates and the sum over all K ∈ ℋ ∩ ℋ with |K| ≤ hₘax shows

\[ \sum_{K ∈ ℋ ∩ ℋ} |K|^2 \left( |||δM − VM|||₂,K + |||UM − VM|||₂,K \right) \]

\[ ≤ \sum_{K ∈ ℋ ∩ ℋ} 2|K|(|||D^2pw(VM)|{L²(K)}|||₂,K + |D^2pw(VM)|{L²(K)}|||₂,K(VM − ΨM))|||₂,K \]

\[ ≤ 2hₘax(|||ΨM|||₂pw + |||ΨM|||₂pw(VM − ΨM)|{L²(ℋ)}|||₂,K \leq 4hₘaxM^{1/2}(ℋ, ℋ). \]
with the abbreviation $M := \|\Psi\| + \epsilon_0$.

The analysis of the jump terms in (5.1) is the same as in [13, 25]. With the substitution of $T$ by $\widehat{T}$, Lemma 5.1 applies (componentwise) to the jump contributions $D^2_{\text{pw}}(\Psi_M - \Psi) \in P_0(\widehat{T}; \mathbb{R}^{2 \times 2})$ in the sum of (5.1) over all $K \in \mathcal{T} \cap \widehat{\mathcal{T}}$. This proves (A1) with $\Lambda^2_{\text{C}} := C_{\text{C}}^2 + 4M^2|\Omega|$.

\[\Box\]

**Remark 5.1 (volume terms).** Subsection 5.2 revisits the above proof for the volume terms $\mu^2(K)$ in $\eta^2(K)$,

$$\mu^2(K) := |K|^2 \left( \|u_M, v_M\| + f \|h^T\|_K + \|u_M, u_M\|_2K \right)^2$$

for all $K \in \mathcal{T}$.

The formula defines the volume contributions $\mu^2(T)$ in $\eta^2(T)$ with the substitution of $u_M, v_M, K$ by $\widehat{u_M}, \widehat{v_M}, T$ for $T \in \widehat{\mathcal{T}}$. The proof of (A1) shows the refined estimate

$$|\widehat{\mu}(T \cap \widehat{\mathcal{T}}) - \mu(T \cap \mathcal{T})| \leq 2h_{\text{max}}M\delta(T, \widehat{T})$$

with an adaptation of the sum convention (4.3) to define $\mu(T \cap \widehat{\mathcal{T}})$ resp. $\widehat{\mu}(T \cap \widehat{\mathcal{T}})$.

**5.2 Proof of reduction (A2)**

The triangle $T \in \widehat{\mathcal{T}} \setminus \mathcal{T}$ is included in exactly one $K \in \mathcal{T}$ in the NVB refinement and $T \subseteq K$ proves $|T| \leq |K|/2$ to generate the reduction factor $2^{-1/4}$ displayed in (A2).

**Theorem 5.3 (Reduction (A2)).** (A2) holds for all $T \in \mathcal{T}(\partial K)$ and all $\widehat{T} \in \mathcal{T}(T)$.

**Proof.** Given any triangle $K \in \mathcal{T} \setminus \widehat{\mathcal{T}}$, the square of the error estimator for the $m \geq 2$ finer triangles $T \in \widehat{\mathcal{T}}(K) := \{T \in \widehat{\mathcal{T}} : T \subset K\}$ reads

$$\eta^2(\mathcal{T}(K)) = \sum_{T \in \mathcal{T}(K)} \left( |T|^2 \left( \|u_M, v_M\| + f \|h^T\|_K + \|u_M, u_M\|_2K \right)^2 \right) + |T|^{1/2} \sum_{F \in \partial(T)} \left( \|D^2u_M\|_{\Gamma F} \|\tau F\|_{L^2(F)}^2 + \|D^2v_M\|_{\Gamma F} \|\tau F\|_{L^2(F)}^2 \right).$$

Various triangle inequalities (in Lebesgue and Euclidean norms) show $\eta^2(\mathcal{T}(K)) \leq S_1 + S_2$ for

$$S_1 := \sum_{T \in \mathcal{T}(K)} \left( |T|^2 \left( \|u_M, v_M\| + f \|h^T\|_K + \|u_M, u_M\|_2K \right)^2 \right) + |T|^{1/2} \sum_{F \in \partial(T)} \left( \|D^2u_M\|_{\Gamma F} \|\tau F\|_{L^2(F)}^2 + \|D^2v_M\|_{\Gamma F} \|\tau F\|_{L^2(F)}^2 \right) \leq 2^{-1/2} \eta^2(K).$$

The proof of this utilises $|T|^{1/2} \leq 2^{-1/2}|K|^{1/2}$ etc. and a careful rearrangement of the jumps (that vanish over edges $E \in \mathcal{E}(T)$ inside $K$ and sum up to the $L^2$ contribution along $\partial K$) and the volume contribution. The second term

$$S_2 := \sum_{T \in \mathcal{T}(K)} \left( |T|^{1/2} \left( \|u_M, v_M\| + f \|h^T\|_K + \|u_M, u_M\|_2K \right)^2 \right) + |T|^{1/2} \sum_{F \in \partial(T)} \left( \|D^2u_M - u_M\|_{\Gamma F} \|\tau F\|_{L^2(F)}^2 + \|D^2v_M - v_M\|_{\Gamma F} \|\tau F\|_{L^2(F)}^2 \right)$$

is analysed as in the previous subsection. The arguments eventually prove

$$\sum_{T \in \mathcal{T}(K)} |T|^{1/2} \left( \|u_M, v_M\| + f \|h^T\|_K + \|u_M, u_M\|_2K \right)^2 \leq |K|M^2\|D^2_{\text{pw}}(\Psi_M - \Psi)\|_2K$$

and the analog estimate with $v_M$ (resp. $\widehat{v_M}$) substituted by $u_M$ (resp. $\widehat{u_M}$). The analysis of the jump terms is the same as in [13, 25] and Lemma 5.1 (applied to $\widehat{T}$ rather than $T$) eventually leads to (A2) with $\Lambda^2_{\text{C}} := C_{\text{C}}^2 + 2M^2|\Omega|$. \[\Box\]
Remark 5.2 (assumptions). The restriction to \( \mathcal{T} \in \mathbb{T}(\delta_0) \) in (A1)-(A2) guarantees the definition of the error estimators via the discrete solution through Theorem [5.1]. This is exclusively for notational consistency: (A1)-(A2) hold for any \( \Psi_M \in V(\mathcal{T}) \) and \( \hat{\Psi}_M \in V(\hat{\mathcal{T}}) \) and solely \( \Lambda_1, \Lambda_2 \) depend on a universal upper bound \( 2M \) for \( \|\Psi_M\|_{pw} + \|\hat{\Psi}_M\|_{pw} \).

Remark 5.3 (volume terms). Subsection [5.4] revisits the above arguments solely for the volume terms \( \mu^2(K) \) in \( \eta^2(K) \) for \( K \in \mathcal{T} \) (resp. \( \hat{\mu}^2(T) \) in \( \hat{\eta}^2(T) \) for \( T \in \hat{\mathcal{T}} \)) from Remark [5.1]. With an adaptation of the sum convention (4.2) for \( \mu \) and \( \hat{\mu} \), the proof of (A2) shows

\[
\hat{\mu}(\hat{T} \setminus T) \leq 2^{-1/2} \mu(T \setminus \hat{T}) + 2^{1/2} h_{\max} M \delta(T, \hat{T}).
\] (5.3)

5.3 Proof of discrete reliability (A3)

The parameters \( \delta_j \) and \( \Lambda_j \) in the following version of (A3) depend on the regular solution \( \Psi \) and its regularity in Theorem [3.1] on \( \delta_0, \delta_1 \) (resp. \( \delta_1, \beta_1 \)) from Theorem [3.1] (resp. Theorem [2.7]), on \( \mathcal{T}_{\text{init}} \) and \( \Omega \) with the regularity index \( \gamma \).

Theorem 5.4 (discrete reliability (A3)). There exists positive \( \delta_3 \leq \min\{\delta_0, \beta_1\} \) and \( \Lambda_3 \) such that \( \delta^2(T, \hat{T}) \leq \Lambda_3 \eta^2(T \setminus \hat{T}) \) holds for any \( T \in \mathbb{T}(\delta_3) \) with refinement \( \hat{T} \in \mathbb{T}(\hat{T}) \).

Proof. Given any refinement \( \hat{T} \in \mathbb{T}(\mathcal{T}) \) of \( T \in \mathcal{T} \), the interpolation operator \( I_M \) of Lemma [2.2] maps \( M(T) \rightarrow M(\hat{T}) \). The converse operation relies on a discrete Helmholtz decomposition. This paper follows [20] with a right-inverse \( I_M E_M \). The key idea is first to compute the companion operator \( E_M + \Lambda \psi_M \) for some \( \psi_M \in M(T) \) and second to apply the interpolation operator \( I_M \) of Lemma [2.2] on the finer triangulation \( \hat{T} \) (rather than \( T \)). This leads to \( I_M E_M : M(T) \rightarrow M(\hat{T}) \) with \( I_M(I_M E_M) = 1 \in M(T) \).

A modification of this idea is performed in [20, Def 6.9], [25] to define an operator \( J_\gamma : P_2(K) \rightarrow HCT(K) + P_2(K) \) for each \( K \in \mathcal{T} \) such that \( \psi_M^* := \hat{I}_M(J_\gamma \psi_M) \in \mathcal{V}(\hat{T}) \) is well defined [20, Lem. 6.14] and satisfies [20, Thm. 6.19] that

\[
C_{\gamma}^{-1} \|\Psi_M - \Psi_M^*\|_{pw} \leq \sum_{E \in \mathcal{E} \setminus \hat{\mathcal{E}}} \left| \omega_E \right|^{1/2} \left\| \left[ D^2 \Psi_M \right]_{E} \right\|_{L^1(E)}^{2} \leq \eta(T \setminus \hat{T})
\] (5.4)

with the mesh-size factor \( |\omega_E|^{1/2} \approx \text{diam}(E) \) for any edge \( E \) with its edge-patch \( \omega_E \) (area \( |\omega_E| \)) and some universal constant \( C_1 \) (that depends solely on \( \mathcal{T}_{\text{init}} \)). This restricts the sum over all edges \( E \) in (5.4) to those, which are coarse but not fine. The estimate (5.4) and a triangle inequality imply

\[
\delta(T, \hat{T}) \leq \|\Psi_M - \Psi_M^*\|_{pw} + C_1 \eta(T \setminus \hat{T}).
\] (5.5)

It remains to control \( \|\hat{\Psi}_M - \hat{\Psi}_M^*\|_{pw} \) for the Morley function \( \hat{\Psi}_M - \hat{\Psi}_M^* \in \mathcal{V}(\hat{T}) \). The discrete stability in Theorem [2.7] leads to some \( \hat{\psi}_M \in \mathcal{V}(\hat{T}) \) with \( \|\hat{\psi}_M\|_{pw} \leq 1/\beta_1 \) and

\[
\|\hat{\Psi}_M - \hat{\Psi}_M^*\|_{pw} \leq DN_{\beta}(\Psi, \hat{\Psi}_M - \hat{\psi}_M, \hat{\psi}_M) \leq C_2 \left\| \Psi-M - \Psi^*_M \right\|_{pw} \leq C_1 C_2 \eta(T \setminus \hat{T})
\] (5.6)

The boundedness in Lemma [2.6] and \( C_2 := (1 + 2 \sqrt{2} C_{de} \|\Psi\|)/\beta_1 \) show

\[
DN_{\beta}(\Psi, \Psi_M - \Psi^*_M, \hat{\psi}_M) \leq C_2 \left\| \Psi-M - \Psi^*_M \right\|_{pw} \leq C_1 C_2 \eta(T \setminus \hat{T})
\]

with (5.4) in the last step. The combination of this with (5.5)-(5.6) proves

\[
\delta(T, \hat{T}) \leq DN_{\beta}(\Psi, \Psi_M - \Psi_M^*, \hat{\psi}_M) + C_1 (1 + C_2) \eta(T \setminus \hat{T}).
\] (5.7)

Recall that \( \Psi_M \in \mathcal{V}(\mathcal{T}) \) (resp. \( \hat{\Psi}_M \in \mathcal{V}(\hat{T}) \)) solves the discrete problem with respect to \( T \) (resp. \( \hat{T} \)) in Lemma [2.2] and shows \( A_{pw}(\Psi_M, \hat{\Psi}_M) = A_{pw}(\Psi_M, I_M \hat{\psi}_M) \) and \( \text{elementary algebra} \) with the symmetry of \( B_{pw}(\bullet, \bullet, \bullet) \) in the first two variables lead to

\[
DN_{\beta}(\Psi, \Psi_M - \Psi_M^*, \hat{\psi}_M) = B_{pw}(2\Psi - \Psi_M - \Psi_M^*, \hat{\Psi}_M - \Psi_M, \hat{\psi}_M) + F(\hat{\psi}_M - I_M \hat{\psi}_M) - B_{pw}(\Psi_M, \Psi_M - \Psi_M^*, \hat{\psi}_M) \leq C_3 \eta(T \setminus \hat{T}).
\] (5.8)
The a priori error estimate \( \| \Psi - \tilde{\Psi}_M \|_{pw} \), \( \| \Psi - \Psi_M \|_{pw} \leq C(\gamma, \Psi) h_{max}^\gamma \) from Theorem \( 5.1 \) in terms of the maximal mesh-size \( h_{max} \) and Lemma \( 5.6 \) a result in

\[
B_{pw}(2\Psi - \tilde{\Psi}_M - \Psi_M, \Psi_M, \tilde{\Psi}_M, \tilde{\Psi}_M) = \sqrt{\delta} C_{dea} C(\gamma, \Psi) h_{max}^\gamma \delta(T, \tilde{T}).
\]

The last two contributions in \( (5.8) \) are volume residuals with the test function \( \tilde{\Psi}_M - I_M \tilde{\Psi}_M \), which vanishes a.e. in each \( K \in T \cap \tilde{T} \). This and Lemma \( 5.2 \) imply

\[
F(\tilde{\Psi}_M - I_M \tilde{\Psi}_M) = B_{pw}(\Psi_M, \Psi_M, \tilde{\Psi}_M) - B_{pw}(\Psi_M, \Psi_M, \tilde{\Psi}_M, \tilde{\Psi}_M) \leq \beta_1 \| \Psi_M - I_M \tilde{\Psi}_M \|_{pw}
\]

\[
\times \left( \sum_{K \in T \cap \tilde{T}} |K|^2 \left( \| f + \left[ u_M, v_M \right] \|_{E,K}^2 + \| [u_M, u_M] \|_{E,K}^2 \right) \right)^{1/2} \leq \eta(T \setminus \tilde{T})
\]

with \( \| \Psi_M - I_M \tilde{\Psi}_M \|_{pw} \leq \| \tilde{\Psi}_M \|_{pw} \leq 1/\beta_1 \) in the last step. These estimates control the right-hand side in \( (5.8) \). The resulting estimate and \( (5.7) \) lead to \( C_3 \equiv 1 \) with

\[
\left( 1 - \sqrt{\delta} C_{dea} \beta_1 C(\gamma, \Psi) h_{max}^\gamma \right) \delta(T, \tilde{T}) \leq C_3 \eta(T \setminus \tilde{T}). \tag{5.9}
\]

The estimate \( (5.9) \) holds for all triangulations in \( \mathcal{T}(\min\{\delta_0, \delta_1\}) \) and the particular choice \( \delta_3 = \min\{\delta_0, \delta_1\} \), \( (2\sqrt{2} C_{dea} C(\gamma, \Psi) / \beta_1)^{-1/2} \) proves \( (A3) \) with \( \Lambda_3 := 4C_3^2 \).

\( \square \)

The discrete reliability \( (A3) \) implies reliability of the error estimators.

Corollary 5.5 (reliability). Given the exact solution \( \Psi \) and \( \delta_3 \) from Theorem \( 5.4 \) the discrete solution \( \Psi_M \in V(T) \) for \( T \in \mathcal{T}(\delta_3) \) satisfies \( \| \Psi - \Psi_M \|_{pw} \leq \Lambda_3 \eta^2(T) \).

\( \square \)

Proof. Given \( T^{(k)} := T \in \mathcal{T}(\delta_k) \), define a sequence of uniform refinements by \( T^{(k+1)} = \text{REFINE}(T^{(k)}) \) for any \( k \in \mathbb{N} \). Let \( \tilde{T} := T^{(k)} \) for the parameter \( k \in \mathbb{N} \) and notice that the maximal mesh-size in \( \tilde{T} \) tends to zero as \( k \to \infty \). Hence Theorem \( 5.1 \) guarantees convergence of \( \| \Psi - \tilde{\Psi}_M \|_{pw} \to 0 \) as \( k \to \infty \). On the other hand, Theorem \( 5.4 \) shows \( \| \Psi_M - \tilde{\Psi}_M \|_{pw} \leq \Lambda_3 \eta^2(T) \). Since the upper bound does not depend on \( k \in \mathbb{N} \), this and a triangle inequality shows the assertion in the limit as \( k \to \infty \).

5.4 Preliminaries to the proof of quastrigraphonality

The quastrigraphonality is always subtle for nonconforming schemes and requires a careful analysis of the quadratic nonlinear contributions. The proof departs with two preliminary lemmas formulated in the (2-level) notation of \( (A1)-(A3) \). Recall the notation \( \mu \) resp. \( \mu \) in \( 5.2 \) \( 5.3 \) for the volume contributions of the error estimator \( \eta \) resp. \( \tilde{\eta} \) and adapt the sum convention \( (5.2) \) with \( \mu(T) = \sum_{K \in T} \mu(K) \) etc. Recall that \( h_T \in P_0(T) \) is the mesh-size in \( T \) with \( h_{max} := \max h_T \leq \delta_0 \) with \( \delta_0 \) and \( \epsilon_0 \) from Theorem \( 3.1 \). Suppose \( T \in \mathcal{T}(\delta_0) \) and \( T \in \mathcal{T}(\delta_0) \) throughout this section.

Lemma 5.6. The bound \( M := \| \Psi \| + \epsilon_0 \) satisfies

\[
\mu^2(T \setminus \tilde{T}) \leq 4 \mu^2(T) - 4 \mu^2(T) + 8 h_{max} M \delta(T, \tilde{T}) \left( \tilde{\mu}(\tilde{T}) + \mu(T) \right) + 24 h_{max}^2 M^2 \delta^2(T, \tilde{T}).
\]

Proof. Recall \( 5.3 \) and deduce

\[
\tilde{\mu}^2(T \setminus \tilde{T}) \leq 3/4 \mu^2(T \setminus \tilde{T}) + 6 h_{max}^2 M^2 \delta^2(T, \tilde{T}).
\]

This is equivalent to

\[
2^{-2} \mu^2(T \setminus \tilde{T}) + \mu^2(T) \leq \mu^2(T \setminus \tilde{T}) - \mu^2(T \setminus \tilde{T}) + 6 h_{max}^2 M^2 \delta^2(T, \tilde{T}).
\]

Recall \( 5.2 \) and the binomial formula to derive

\[
\mu^2(T \setminus \tilde{T}) - \mu^2(T \setminus \tilde{T}) \leq 2 h_{max} M \delta(T, \tilde{T}) \left( \tilde{\mu}(T \setminus \tilde{T}) + \mu(T \setminus \tilde{T}) \right).
\]

The combination of the previous two displayed estimates concludes the proof. \( \square \)
Lemma 5.7. There exists a constant $C_{q_0}$ (depending on $\Psi$, the constants in Theorem 5.7 and $T_{ini}$) such that

$$A_{pw}(\Psi - \bar{\Psi}_M, \Psi_M - \bar{\Psi}_M) \leq C_{q_0}^{1/2} \|\Psi - \bar{\Psi}_M\|_{pw} (\mu(T \setminus \hat{T}) + h_{\hat{T}}^{\gamma} \delta(T, \hat{T})).$$

Proof. Recall $D_{pw}^2 I_M = \Pi_0 D_{pw}^2$ from Lemma 2.2, set $\bar{\Phi}_M := \hat{I}_M(\Psi - \bar{\Psi}_M)$, and evaluate the discrete equations on the coarse (resp. fine) level to derive

$$A_{pw}(\Psi - \bar{\Psi}_M, \Psi_M - \bar{\Psi}_M) = A_{pw}(\Psi_M, \Psi) - A_{pw}(\bar{\Psi}_M, \Psi) - A_{pw}(\bar{\Psi}_M, \Psi_M) - A_{pw}(\bar{\Psi}_M, \hat{I}_M)$$

$$= F((I_M - \hat{I}_M)(\Psi - \bar{\Psi}_M)) - B_{pw}(\Psi_M, \Psi_M, I_M - \hat{I}_M)(\Psi - \bar{\Psi}_M)) + B_{pw}(\bar{\Psi}_M, \hat{I}_M, \bar{\Psi}_M, \Psi_M, \hat{I}_M) =: S_3 + S_4.$$ 

The definitions of $F_{pw}(\bullet)$, $B_{pw}(\bullet, \bullet, \bullet, \bullet, \bullet)$, the Cauchy inequality, and $(1 - I_M)\bar{\Phi}_M = 0$ a.e. in $K \in T \cap \hat{T}$ prove (with the vector $(f + [u_M, v_M], -\frac{1}{2}[u_M, u_M]) \in L^2(\Omega; \mathbb{R}^2)$ and the scalar product $\cdot$ in $\mathbb{R}^2$) that

$$S_3 := F((I_M - \hat{I}_M)(\Psi - \bar{\Psi}_M)) - B_{pw}(\Psi_M, \Psi_M, I_M - \hat{I}_M)(\Psi - \bar{\Psi}_M))$$

$$= \sum_{K \in T \cap \hat{T}} \int_K (f + [\mu_M, \nu_M], -\frac{1}{2}[\mu_M, \nu_M]) \cdot (I_M - \hat{I}_M)(\Psi - \bar{\Psi}_M) \ dx$$

$$\leq \mu(T \setminus \hat{T}) \left( \sum_{K \in T \cap \hat{T}} h_K^2 \|\bar{\Psi}_M - \bar{\Psi}_M\|_{L^2(\Omega)}^2 \right)^{1/2} \leq \mu(T \setminus \hat{T}) \|\Psi - \bar{\Psi}_M\|_{pw}$$

with Lemma 2.2 in the final step. The triangle inequality and Lemma 2.6 a-b (with $\hat{T}$ replacing $T$) show

$$S_4 := B_{pw}(\bar{\Psi}_M, \bar{\Psi}_M, \hat{I}_M) - B_{pw}(\Psi_M, \Psi_M, \bar{\Psi}_M) = B_{pw}(\bar{\Psi}_M - \bar{\Psi}_M, \bar{\Psi}_M, \bar{\Psi}_M)$$

$$= \|\Psi - \bar{\Psi}_M\|_{pw} \|\Psi - \bar{\Psi}_M\|_{pw} \|\Psi - \bar{\Psi}_M\|_{pw} \|\bar{\Phi}_M\|_{pw} + \|\bar{\Psi}_M\|_{H^{\gamma+n}(\Omega)} \|\bar{\Phi}_M\|_{1,2,pw}.$$ 

Lemma 2.2 a implies $\|\bar{\Phi}_M\|_{pw} \leq \|\bar{\Psi}_M\|_{pw}$. The triangle inequality, Lemma 2.6 a, b, and Theorem 5.1 c (with respect to $\hat{T}$ rather than $T$) result in

$$|\bar{\Phi}_M|_{1,2,pw} = |\bar{\Psi}_M - \bar{\Psi}_M|_{1,2,pw} \leq |\Psi - \hat{I}_M\Psi|_{1,2,pw} + |\Psi - \bar{\Psi}_M|_{1,2,pw} \leq h_{\max} \|\Psi - \bar{\Psi}_M\|_{pw} + h_{\max} \|\Psi - \bar{\Psi}_M\|_{pw} \leq h_{\max} \|\Psi - \bar{\Psi}_M\|_{pw}.$$ 

Consequently, $S_4 \leq h_{\max} \|\Psi - \bar{\Psi}_M\|_{pw} \delta(T, \hat{T})$. This concludes the proof. \hfill \Box

5.5 Proof of quasistochasticity (A4)

The proof of (A4) employs general arguments from the axioms and deduces (A4). Throughout this section, let $\mathcal{T}$, $\Psi$, and $\eta$ denote the output of AMFEM and abbreviate $\delta_{k,k+1} := \delta(T_k, T_{k+1})$ for $k \in \mathbb{N}_0$ and

(A4)$_{q_0}$ Quasistochasticity with $\varepsilon > 0$. There exists $0 < \Lambda_{A_{q_0}} < \infty$ such that

$$\sum_{k=\ell}^{\ell+m} \delta_{k,k+1}^2 \leq \Lambda_{A_{q_0}} \eta_k^2 + \varepsilon \sum_{k=\ell}^{\ell+m} \eta_k^2 \text{ holds for all } \ell, m \in \mathbb{N}_0. \quad (5.10)$$

Theorem 5.8 (Quasistochasticity). For any $\varepsilon > 0$ there exist positive $\delta \leq \delta_3$ and $\Lambda_{A_{q_0}}$ such that $\mathcal{T}_0 \in \mathbb{T}(\delta)$ implies (5.10).

Proof. Given a positive $\varepsilon$ we may and will assume without loss of generality that

$$0 < \varepsilon \leq \min\{1, \varepsilon_0, 2^{q_0/2}C_{q_0}^{1/2}A_3^{1/2}, 8A_1\}. \quad (5.11)$$

Select a positive maximal $\delta$ with $\delta \leq \min\{\delta_0, \delta_1\}$ and

$$\max\left\{2^{q_0}(C_{q_0}A_3M\delta)^{3/2}, 2^{q_0}C_{q_0}A_3\delta^{\gamma}, 1536 C_{q_0}A_3M^2 \delta^2\right\} \leq \varepsilon \quad (5.12)$$
and suppose the maximal mesh-sizes are bounded by the maximal mesh-size \( h_0 \) of \( \mathcal{T}_0 \in \mathcal{T}(\delta) \). Throughout the proof, abbreviate \( e_k := \| \Psi - \Psi_k \|_{\text{pw}} \), \( \delta_{k+1} := \| \Psi_{k+1} - \Psi_k \|_{\text{pw}} \), and adopt the notation of \( \mu \) and \( \tilde{\mu} \) to \( \mathcal{T}_k \), e.g., \( \mu_k^2(\mathcal{K}) := |\mathcal{K}|[(f + |u_k|, v_0)]_k^2 + \| u_k, u_0 \|_{H_0^1}^2 \) for all \( K \in \mathcal{T}_k \), \( k \in \mathbb{N}_0 \). Some algebra and Lemma 5.6 with \( \mathcal{T}_k \) and \( \mathcal{T}_{k+1} \) (replacing \( \mathcal{T} \) and \( \tilde{\mathcal{T}} \)) show

\[
\delta_{k+1}^2 + e_k^2 - e_{k+1}^2 = 2A_{\text{pw}}(\Psi - \Psi_{k+1}, \Psi_k - \Psi_{k+1}) \leq 2C_{\text{pw}}^1 e_k(\mu_k(\mathcal{T}_k \setminus \mathcal{T}_{k+1}) + h_0^2 \delta_{k+1}^2).
\]

Weighted arithmetic-geometric mean inequalities prove

\[
\frac{3}{4} \delta_{k+1}^2 + \left( 1 - \varepsilon 2^{-4} \Lambda_3^{-1} - 4C_{\text{pw}} h_0^2 \right) e_k^2 - e_{k+1}^2 \leq \varepsilon^{-1.2} C_{\text{pw}}^1 \Lambda_3 \mu_k^2(\mathcal{T}_k \setminus \mathcal{T}_{k+1}).
\]

With \( \mathcal{T}_k \) and \( \mathcal{T}_{k+1} \) (replacing \( \mathcal{T} \) and \( \tilde{\mathcal{T}} \)) and the abbreviations \( \mu_k := \mu(\mathcal{T}_k) \) etc., Lemma 5.6 reads

\[
\mu_k^2(\mathcal{T}_k \setminus \mathcal{T}_{k+1}) \leq 4 \mu_k^2 - 4 \mu_{k+1}^2 + 2^{1/2} M h_0 \delta_{k,k+1}(\mu_k^2 + \mu_{k+1}^2) + 2^{1/2} M^2 h_0^2 \delta_{k,k+1}.
\]

This and an arithmetic-geometric-mean inequality shows that the right-hand side in (5.13) is

\[
\leq \varepsilon^{-1.2} C_{\text{pw}}^1 \Lambda_3 (\mu_k^2 - \mu_{k+1}^2) + \varepsilon^{-2} 2^{1/2} C_{\text{pw}}^1 \Lambda_3^2 M^2 h_0^2 \left( \mu_{k+1}^2 + \mu_k^2 \right) + \left( 2^{1/2} + \varepsilon^{-1.3} 84 C_{\text{pw}}^1 \Lambda_3 M^2 h_0^2 \right) \delta_{k,k+1}.
\]

The combination with (5.13) simplifies for \( h_0 \leq \delta \) with (5.12) and leads, for any \( k \in \mathbb{N}_0 \), to

\[
2^{-2} \delta_{k+1}^2 + (1 - \varepsilon 2^{-3} \Lambda_3^{-1}) e_k^2 - e_{k+1}^2 \leq \varepsilon^{-1.2} C_{\text{pw}}^1 \Lambda_3 (\mu_k^2 - \mu_{k+1}^2) + \varepsilon^{-2} (\mu_k^2 + \mu_{k+1}^2).
\]

The sum over all \( k = \ell, \ell + 1, \ldots, \ell + m \) leads to terms \( \delta_{k,k+1}^2 - e_k^2 \) on the left-hand and \( \mu_k^2 - \mu_{k+1}^2 \) on the right-hand side with a telescoping sum. The term \( \mu_k^2 + \mu_{k+1}^2 \) arises twice in the upper bound and results in \( (2^{-3} - \varepsilon^{-1.2} C_{\text{pw}}^1 \Lambda_3) \mu_{k+m+1}^2 \leq 0 \) from (5.11); the other extreme term on the left-hand side is \( (1 - \varepsilon 2^{-3} \Lambda_3^{-1}) e_{k+1}^2 \geq 0 \) from (5.11). This, \( \mu_k \leq \eta_k \), and Corollary 5.5 with \( e_k \leq \Lambda_3 \eta_k^2 \) lead to

\[
\frac{1}{4} \sum_{k=\ell}^{\ell+m} \delta_{k+1}^2 \leq \Lambda_3 (1 + \varepsilon^{-1.2} C_{\text{pw}}^1) \eta_k^2 + \frac{\varepsilon}{4} \sum_{k=\ell}^{\ell+m} \eta_k^2.
\]

This concludes the proof of (A4) with \( \Lambda_{4(e)} = 4 \Lambda_3 (1 + \varepsilon^{-1.2} C_{\text{pw}}^1) \).

The refinement rules in AMFEM, (A1)-(A2), and (A4) for small \( \varepsilon \) imply (A4).

**Corollary 5.9 (Quasiorthogonality).** Given any \( 0 < \theta < \theta_0 := 1/(1 + \Lambda_2^2 \Lambda_3) \), there exists a positive \( \delta_4 \leq \delta_3 \) such that \( \mathcal{T}_0 \in \mathcal{T}(\delta_4) \) implies (A4).

**Proof.** Given any \( \theta < \theta_0 \) in AMFEM, (A1)-(A2) and [21], Thm. 4.1 lead to positive parameters \( \theta_{12} < 1 \) and \( \Lambda_{12} \) in (A12) undisplayed in this paper. Any choice of \( \varepsilon < (1 - \theta_{12})/\Lambda_{12} \) leads in Theorem 5.8 to some \( \delta_4 > 0 \) so that \( \mathcal{T}_0 \in \mathcal{T}(\delta_4) \) implies (A4) \( \varepsilon \). This and [21], Thm. 3.1 imply (A4).\]
Acknowledgements

The research of the first author has been supported by the Deutsche Forschungsgemeinschaft in the Priority Program 1748 under the project “foundation and application of generalized mixed FEM towards nonlinear problems in solid mechanics” (CA 151/22-2). The second author thanks the hospitality of the Humboldt-Universität zu Berlin in August 2017 when this work was initiated. The finalization of this paper has been supported by DST SERB MATRICS grant MTR/2017/000199 and SPARC project (id 235) entitled the mathematics and computation of plates.

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