Complete gradient expanding Ricci solitons with finite asymptotic scalar curvature ratio

Huai-Dong Cao\textsuperscript{1} · Tianbo Liu\textsuperscript{1} · Junming Xie\textsuperscript{1}

Received: 18 March 2022 / Accepted: 31 October 2022 / Published online: 24 December 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract
Let \((M^n, g, f)\), \(n \geq 5\), be a complete gradient expanding Ricci soliton with nonnegative Ricci curvature \(Rc \geq 0\). In this paper, we show that if the asymptotic scalar curvature ratio of \((M^n, g, f)\) is finite (i.e., \(\limsup_{r \to \infty} Rr^2 < \infty\)), then the Riemann curvature tensor must have at least sub-quadratic decay, namely, \(\limsup_{r \to \infty} |Rm| r^\alpha < \infty\) for any \(0 < \alpha < 2\).

1 Introduction
This is a sequel to the earlier paper [1] by the first and the second authors in which curvature estimates were obtained for 4-dimensional complete gradient expanding Ricci solitons. By scaling the metric \(g\) if necessary, we shall assume throughout the paper that a gradient expanding Ricci soliton \((M^n, g, f)\) satisfies the equation
\[
Rc + \nabla^2 f = -\frac{1}{2} g,
\]
where \(Rc\) and \(\nabla^2 f\) denote the Ricci tensor of \(g\) and the Hessian of the potential function \(f \in C^\infty(M)\), respectively.

For any 4-dimensional complete gradient expanding Ricci soliton \((M^4, g, f)\) with non-negative Ricci curvature \(Rc \geq 0\), it was shown in [1] that there exists a constant \(C > 0\) such that, for any \(0 \leq a < 1\), the following curvature estimate hold on \(M^4\),
\[
|Rm| \leq \frac{C}{1 - a} \cdot R^a.
\]
Moreover, if the scalar curvature $R$ has at most polynomial decay, then

$$|Rm| \leq CR \text{ on } M^4.$$  

(1.2)

On the other hand, if the asymptotic scalar curvature ratio of $(M^4, g)$ is finite, i.e.,

$$\limsup_{r \to \infty} Rr^2 < \infty,$$

then $(M^4, g)$ has finite asymptotic curvature ratio

$$A := \limsup_{r \to \infty} |Rm|r^2 < \infty.$$  

(1.3)

As an application, it follows from the above result and the work of Chen and Deruelle [2] that any 4-dimensional complete noncompact non-flat gradient expanding Ricci soliton with nonnegative Ricci curvature and finite asymptotic scalar curvature ratio must have a $C^{1,\alpha}$ asymptotic cone structure at infinity, for any $\alpha \in (0, 1)$.

We remark that recent progress on curvature estimates for 4-dimensional gradient Ricci solitons has been led by the work of Munteanu and Wang [3], in which they proved that any complete gradient shrinking soliton with bounded scalar curvature $R$ must have bounded Riemann curvature tensor $Rm$. More significantly, they showed that the Riemann curvature tensor is controlled by the scalar curvature by $|Rm| \leq CR$ so that if the scalar curvature $R$ decays at infinity so does the curvature tensor $Rm$. Moreover, by exploring the differential equation $\Delta f R = R - 2|Rc|^2$ satisfied by shrinking solitons and combining with the scalar curvature lower bound of Chow et al. [4], they showed that the scalar curvature $R$ in fact must decay quadratically if $R$ goes to zero at infinity. It then follows that the curvature tensor $Rm$ must decay quadratically, hence the 4D shrinking soliton is asymptotically conical. Their curvature estimate, together with the uniqueness result of Kotschwar and Wang [5], has played a crucial role in the recent advance of classifying 4-dimensional complete gradient Ricci solitons, as well as in the classification of complex 2-dimensional complete gradient Kähler-Ricci solitons with scalar curvature going to zero at infinity by Conlon et al. [6]. See [7] for an extension, and also [8–10] for similar curvature estimates in the steady soliton case.

In [11], via the Moser iteration and a tour de force of integral estimates, Munteanu and Wang also obtained the curvature estimate for higher dimensional gradient shrinking Ricci solitons. Precisely, they showed that if the Ricci curvature of an $n$-dimensional ($n \geq 5$) complete gradient shrinking Ricci soliton goes to zero at infinity, then its Riemann curvature tensor $Rm$ must also go to zero at infinity. Furthermore, based on $|Rm| \to 0$ at infinity and the fact that the curvature tensor of Ricci shrinkers satisfy the differential inequality $\Delta f |Rm| \geq |Rm| - c|Rm|^2$, they were able to show that $Rm$ has to decay quadratically at infinity by using the maximum principle argument.

In this paper, inspired by the work of Munteanu and Wang [11], we investigate curvature estimates for higher dimensional gradient expanding Ricci solitons with nonnegative Ricci curvature and finite asymptotic scalar curvature ratio. Our main result is the following

**Theorem 1.1** Let $(M^n, g, f)$, $n \geq 5$, be an $n$-dimensional complete gradient expanding Ricci soliton with nonnegative Ricci curvature $Rc \geq 0$ and finite asymptotic scalar curvature ratio

$$\limsup_{r \to \infty} Rr^2 < \infty,$$

(1.4)

where $r = r(x)$ is the distance function to a fixed base point $x_0 \in M$. Then $(M^n, g, f)$ has finite $\alpha$-asymptotic curvature ratio for any $0 < \alpha < 2$,

$$A_\alpha := \limsup_{r \to \infty} |Rm|r^\alpha < \infty.$$  

(1.5)
Furthermore, there exist constant $C > 0$ depending on $n$ and the geometry of $(M^n, g, f)$, sequences $\{r_j\} \to \infty$ and $\{\alpha_j\} \to 2$ such that

$$|Rm|(x) \leq C(r(x) + 1)^{-\alpha_j}$$

for any $x \in M \setminus B(x_0, r_j + 1)$.

We point out that, compared to the shrinking case, there are several essential differences in the expanding case. First of all, certain integrals in the key integral estimate, which were good terms in the shrinking case, turned into potential trouble terms (see Remark 3.2) and they prevented us from obtaining the pointwise $Rm$ decay by merely assuming the Ricci curvature goes to zero at infinity. Secondly, in the expanding case, the assumption of $Rc \geq 0$ is essential to ensure a uniform lower bound for the Sobolev constant of unit geodesic balls $B_x(1)$ for all $x \in M$ (see Lemmas 2.9 and 2.8), or a uniform non-collapsing estimate for $B_x(1)$ (see Lemma 2.7), that is crucial for the Moser iteration to work. Finally, the corresponding differential inequality for $|Rm|$ in the expanding case becomes

$$\Delta f |Rm| \geq -|Rm| - c|Rm|^2,$$

from which the maximum principle argument does not seem to work for getting the quadratic decay as in the shrinking case, or any improved decay for $Rm$, even knowing $Rm$ goes to zero at infinity. Nevertheless, by adapting the integral estimates in [11] and using the Moser iteration, we are able to obtain the sub-quadratic decay (1.5) for $|Rm|$ under the assumption of finite asymptotic scalar curvature ratio (1.4).

**Remark 1.1** The same proof can be used to show that if the rate of decay for the scalar curvature $R$ is in the order of $\alpha$, with $0 < \alpha \leq 2$, then $Rm$ would have sub-$\alpha$ decay (see Theorem 4.2).

**Remark 1.2** Unlike the works of Munteanu and Wang [3, 11] for shrinking Ricci solitons, it seems that the best one can hope to prove in the expanding case is for $Rm$ to have the same decay rate as assumed for the Ricci curvature (or the scalar curvature). It remains an interesting question if one can improve the arbitrary sub-quadratic decay for $Rm$ in Theorem 1.1 to the quadratic decay.

## 2 Preliminaries

In this section, for the reader’s convenience, we fix the notations and collect several known results about gradient expanding Ricci solitons that we shall need later. Throughout the paper, we denote by

$$Rm = \{R_{ijkl}\}, \quad Rc = \{R_{ik}\}, \quad R$$

the Riemann curvature tensor, the Ricci tensor, and the scalar curvature of the metric $g = g_{ij}dx^idx^j$ in local coordinates $(x^1, \cdots, x^n)$, respectively.

**Lemma 2.1** (Hamilton [12]) Let $(M^n, g, f)$ be a complete gradient expanding Ricci soliton satisfying Eq. (1.1). Then

$$R + \Delta f = -\frac{n}{2},$$

$$\nabla_i R = 2R_{ij} \nabla_j f.$$
\[ R + |\nabla f|^2 = -f + C_0 \]

for some constant \( C_0 \).

Moreover, replacing \( f \) by \( f - C_0 \), we can normalize the potential function \( f \) so that

\[ R + |\nabla f|^2 = -f. \]

In the rest of the paper, we shall always assume this normalization.

Furthermore, by setting

\[ F = -f + \frac{n}{2}, \tag{2.1} \]

the expanding soliton equation (1.1) becomes

\[ \nabla^2 F = Rc + \frac{1}{2}g. \tag{2.2} \]

From (2.2), Lemma 2.1 and the normalization of \( f \), we have

\[ \nabla R = -2Rc(\nabla F, \cdot), \quad |\nabla F|^2 = F - R - \frac{n}{2}, \tag{2.3} \]

\[ \Delta F = R + \frac{n}{2} \quad \text{and} \quad \Delta_f F = F \quad \text{(i.e.,} \quad \Delta_f f = f - \frac{n}{2} \text{)}, \tag{2.4} \]

where \( \Delta_f =: \Delta - \nabla f \cdot \nabla \) is the weighted Laplace operator.

Next, we have the following well-known fact about the asymptotic behavior of the potential function of a complete non-compact gradient expanding soliton with nonnegative Ricci curvature (see, e.g., Lemma 5.5 in [13] or Lemma 2.2 in [2]).

**Lemma 2.2** Let \((M^n, g, f)\) be a complete noncompact gradient expanding Ricci soliton satisfying Eq. (1.1) and with nonnegative Ricci curvature \( Rc \geq 0 \). Then there exist some constants \( c_1 > 0 \) such that, outside some compact subset of \( M^n \), the function \( F = -f + n/2 \) satisfies the estimates

\[ \frac{1}{4}(r(x) - c_1)^2 \leq F(x) \leq \frac{1}{4}(r(x) + 2\sqrt{F(x_0)})^2, \tag{2.5} \]

where \( r(x) \) is the distance function from a base point in \( M^n \). In particular, \( F \) is a strictly convex exhaustion function achieving its minimum at its unique interior point \( x_0 \), which we shall take as the base point, and the underlying manifold \( M^n \) is diffeomorphic to \( \mathbb{R}^n \).

Another useful fact is the boundedness of the scalar curvature of a gradient expanding soliton with nonnegative Ricci curvature (see, e.g., Ma and Chen [14]).

**Lemma 2.3** Let \((M^n, g, f)\) be a complete noncompact gradient expanding Ricci soliton with nonnegative Ricci curvature \( Rc \geq 0 \). Then its scalar curvature \( R \) is bounded from above, i.e., \( R \leq R_0 \) for some positive constant \( R_0 \). Moreover, \( R > 0 \) everywhere unless \((M^n, g, f)\) is the Gaussian expanding soliton.

Note that, under the assumption of \( Rc \geq 0 \) (or even \( Rc \geq (\epsilon - \frac{1}{2})g \) for some constant \( \epsilon > 0 \)), the potential function \( F(x) \) defined by (2.1) grows quadratically hence is proportional to \( r^2(x) \), the square of the distance function, from above and below at large distance. In the rest of the paper, we denote by

\[ D(r) = \{ x \in M : F(x) \leq r \}. \]
By the Bishop volume comparison, we know that the volume $V(r)$ of $D(r)$ satisfies
\[
V(r) \leq cr^n. \tag{2.6}
\]

We now collect several well-known differential identities on the curvatures $R, \text{Rc}$ and $\text{Rm}$ that we shall use later.

**Lemma 2.4** Let $(M^n, g, f)$ be a complete gradient expanding Ricci soliton satisfying Eq. (1.1). Then, we have
\[
\Delta f R = -R - 2|\text{Rc}|^2,
\]
\[
\Delta f R_{ik} = -R_{ik} - 2R_{ijkl}R_{jl},
\]
\[
\Delta f \text{Rm} = -\text{Rm} + \text{Rm} \ast \text{Rm},
\]
\[
\nabla_l R_{ijkl} = \nabla_j R_{ik} - \nabla_i R_{jk} = -R_{ijkl} \nabla_l F,
\]
where, on the RHS of the third equation, $\text{Rm} \ast \text{Rm}$ denotes the sum of a finite number of terms involving quadratics in $\text{Rm}$.

Based on Lemma 2.4, one can easily derive the following differential inequalities (see also [3, 8] for the shrinking and steady ones):

**Lemma 2.5** Let $(M^n, g, f)$ be a complete gradient expanding Ricci soliton satisfying Eq. (1.1). Then
\[
\Delta f |\text{Rc}|^2 \geq 2|\nabla \text{Rc}|^2 - 2|\text{Rc}|^2 - 4|\text{Rm}||\text{Rc}|^2,
\]
\[
\Delta f |\text{Rm}|^2 \geq 2|\nabla \text{Rm}|^2 - 2|\text{Rm}|^2 - c|\text{Rm}|^3,
\]
\[
\Delta f |\text{Rm}| \geq -|\text{Rm}| - c|\text{Rm}|^2.
\]
Here $c > 0$ is some universal constant depending only on the dimension $n$.

Also, from [1] we have the following differential inequalities on the covariant derivative $\nabla \text{Rm}$ of the curvature tensor (see [3] for the shrinking case).

**Lemma 2.6** Let $(M^n, g, f)$ be a complete gradient expanding Ricci soliton satisfying Eq. (1.1). Then
\[
\Delta f |\nabla \text{Rm}|^2 \geq 2|\nabla^2 \text{Rm}|^2 - 3|\nabla \text{Rm}|^2 - c|\text{Rm}||\nabla \text{Rm}|^2
\]
and
\[
\Delta f |\nabla \text{Rm}| \geq -\frac{3}{2}|\nabla \text{Rm}| - c|\text{Rm}||\nabla \text{Rm}|.
\]

In [15], Carrillo and Ni proved the following non-collapsing result for gradient expanding soliton with nonnegative Ricci curvature.

**Lemma 2.7** (Carrillo and Ni [15]) Let $(M^n, g, f)$ be a complete gradient expanding Ricci soliton with nonnegative Ricci curvature. Then there exists a constant $\kappa > 0$ such that if $|\text{Rc}| \leq 1$ on a unit geodesic ball $B(x_0, 1)$ centered at $x_0$, then
\[
V(x_0, 1) \geq \kappa,
\]
where $V(x_0, 1)$ denotes the volume of $B(x_0, 1)$.

**Remark 2.1** In [15], the authors only stated the non-collapsing result for shrinking Ricci solitons (see Corollary 4.2 in [15]). But Lemma 2.7 holds similarly because of their logarithmic Sobolev inequality for expanding solitons with nonnegative Ricci curvature (see Theorem 5.2 in [15]).
Concerning the volume growth, Hamilton [16] obtained the following result (see also Proposition 9.46 in [17]).

**Lemma 2.8** (Hamilton [16]) Let \((\mathcal{M}^n, g, f)\) be any \(n\)-dimensional complete noncompact gradient expanding Ricci soliton with nonnegative Ricci curvature. Then it must have positive asymptotic volume ratio. Namely, for any base point \(x_0 \in \mathcal{M}^n\),

\[
\nu_{\mathcal{M}} := \lim_{r \to \infty} \frac{V(x_0, r)}{r^n} > 0,
\]

(2.7)

where \(V(x_0, r)\) denotes the volume of the geodesic ball \(B(x_0, r)\).

Finally, we shall need the following well-known result about Sobolev inequality on manifolds with nonnegative Ricci curvature and positive asymptotic volume ratio; see Yau [18], and the very recent work of Brendle [19] for a sharp version.

**Lemma 2.9** Let \((\mathcal{M}^n, g)\) be an \(n\)-dimensional complete manifold with nonnegative Ricci curvature \(Rc \geq 0\) and positive asymptotic volume ratio \(\nu_{\mathcal{M}} > 0\). Then there exists a constant \(C_s > 0\) such that, for any compact domain \(\Omega \subset \mathcal{M}\) and any positive smooth function \(\varphi\) with compact support in \(\Omega\),

\[
C_s \left(\int_{\Omega} \varphi^{\frac{n}{n+1}}\right)^{\frac{n+1}{n}} \leq \int_{\Omega} |\nabla \varphi|.
\]

### 3 The integral estimate

In this section, we prove a crucial integral curvature estimate needed in the proof of Theorem 1.1. First of all, we note that the assumptions of \(Rc \geq 0\) and the finite asymptotic scalar curvature ratio (1.4) imply that

(i) \(0 \leq R \leq R_0\).

(ii) \(\nabla_i \nabla_j F \geq \frac{1}{2} g_{ij}\).

(iii) \(|Rc| \leq R \leq \frac{C}{r}\), for some constant \(C > 0\).

Next, following [11], we define the cut-off function \(\phi\) with support in \(D(r)\) by

\[
\phi(x) = \begin{cases} \frac{1}{r} (r - F(x)) & \text{if } x \in D(r) \\ 0 & \text{if } x \in \mathcal{M}\backslash D(r) \end{cases}
\]

(3.1)

so that

\(\nabla \phi = -\nabla F / r\) and \(\Delta \phi = -\Delta F / r\) on \(D(r)\).

Also, for any large number \(p > 0\) to be chosen later, we let \(q = 2p\) and pick \(r_0 > 0\) sufficiently large so that

\(F \geq p^5\) and \(|Rc| \leq \frac{1}{p^5}\) on \(M\backslash D(r_0)\).

(3.2)

In the rest of the paper, we shall use the following conventions.

- \(C\): a positive constant that may depend on the geometry of \(D(r_0)\).
- \(c\): a positive constant depending only on the dimension \(n\) and \(R_0\).
- \(c(p)\): a positive constant depending on \(p, c\) and \(C\).
In addition, those constants may change from line to line.
Now we are ready to state our key integral curvature estimate.

**Proposition 3.1** Let \((M^n, g, f)\) be an \(n\)-dimensional complete gradient expanding Ricci soliton with nonnegative Ricci curvature \(Rc \geq 0\) and finite asymptotic scalar curvature ratio

\[
\lim \sup_{r \to \infty} Rr^2 < \infty.
\]

Then, for any constant \(a > 0\), there exists a constant \(c \geq 1\) such that if \(p > a + R_0 + \frac{n}{2} + c\) we have

\[
[1 - p^{-1}(a + R_0 + \frac{n}{2} + c)] \int_M |Rm|^p F^a \phi^q \leq c(p),
\]

where \(c(p)\) is in the order of \(p^p\).

**Remark 3.1** Proposition 3.1 actually holds for gradient expanding Ricci solitons with finite asymptotic Ricci curvature ratio, i.e., \(\lim \sup_{r \to \infty} |Rc| r^2 < \infty\), without assuming \(Rc \geq 0\). Indeed, as we shall see below in the proof of Proposition 3.1, the condition of \(Rc \geq 0\) is basically used only to guarantee that geodesic balls have at most polynomial (Euclidean) volume growth. However, note that any complete Riemannian manifold with quadratic Ricci curvature decay from below has polynomial volume growth (see [20])\(^1\). Meanwhile, under the finite asymptotic Ricci curvature ratio assumption, all the relevant properties or differential inequalities concerning the potential function \(F\) would still hold outside a compact set. Hence, the nonnegative Ricci assumption in Proposition 3.1 is not essential.

We shall divide the proof of Proposition 3.1 into several lemmas and adapt the arguments in [11].

**Lemma 3.1** Let \((M^n, g, f)\) be an \(n\)-dimensional complete gradient expanding Ricci soliton with nonnegative Ricci curvature \(Rc \geq 0\). Suppose \(p > a + R_0 + \frac{n}{2} + 1\), then

\[
[1 - p^{-1}(a + R_0 + \frac{n}{2} + c)] \int_M |Rm|^p F^a \phi^q \leq 4 \int_M |\nabla Rc|^2 |Rm|^{p-1} F^{a+1} \phi^q + cp^2 \int_M |\nabla F|^2 |Rm|^{p-3} F^{a-1} \phi^q + c(p).
\]

**Remark 3.2** In Lemma 3.1, the first two terms on the right hand side of the inequality are different from the shrinking case in [11]. Also note that Lemma 3.1 does not require the finite asymptotic scalar curvature ratio assumption.

**Proof** Since \(\Delta F \leq R_0 + n/2\) by (2.4) and Lemma 2.3, by integration by parts, we have

\[
-(R_0 + \frac{n}{2}) \int_M |Rm|^p F^a \phi^q \leq -\int_M (\Delta F) |Rm|^p F^a \phi^q = \int_M \nabla F \cdot \nabla (|Rm|^p) F^a \phi^q + a \int_M |Rm|^p |\nabla F|^2 F^{a-1} \phi^q
\]

\(^1\) See also Corollary 4.11 in the very recent work of Chan et al. [21].
\[ + q \int_M |Rm|^p F^a \phi^{q-1} \nabla F \cdot \nabla \phi \leq \int_M \nabla F \cdot \nabla(|Rm|^p F^a \phi^q) + a \int_M |Rm|^p F^a \phi^q, \]

where in the last inequality we have used the fact \(|\nabla F|^2 < F\) from (2.3) and \(\nabla \phi = -\nabla F / r\).

It then follows from the second Bianchi identity, as in [11], that

\[ -(a + R_0 + \frac{n}{2}) \int_M |Rm|^p F^a \phi^q \leq \int_M \nabla F \cdot \nabla(|Rm|^p) F^a \phi^q \]

\[ = p \int_M (\nabla_h F \cdot \nabla_h R_{ijkl}) R_{ijkl} |Rm|^{p-2} F^a \phi^q \]

\[ = 2p \int_M (\nabla_h F \cdot \nabla_l R_{ijkl}) R_{ijkl} |Rm|^{p-2} F^a \phi^q. \]

Performing integration by parts again, we obtain

\[ -(a + R_0 + \frac{n}{2}) \int_M |Rm|^p F^a \phi^q \leq -2p \int_M R_{ijkl} (\nabla_h \nabla_l F) R_{ijkl} |Rm|^{p-2} F^a \phi^q \]

\[ -2p \int_M (R_{ijkl} \nabla_h F)(\nabla_l R_{ijkl}) |Rm|^{p-2} F^a \phi^q \]

\[ -2p \int_M (R_{ijkl} \nabla_h F) R_{ijkl} \nabla_l (|Rm|^{p-2}) F^a \phi^q \]

\[ -2pa \int_M |R_{ijkl} \nabla_l F|^2 |Rm|^{p-2} F^{a-1} \phi^q \]

\[ + \frac{4p^2}{r} \int_M |R_{ijkl} \nabla_l F|^2 |Rm|^{p-2} F^a \phi^{q-1}. \]

Since \(Rc \geq 0\) implies \(\nabla_i \nabla_j F \geq \frac{1}{2} g_{ij}\), it follows that

\[ -2p \int_M R_{ijkl}(\nabla_h \nabla_l F) R_{ijkl} |Rm|^{p-2} F^a \phi^q \leq -p \int_M |Rm|^p F^a \phi^q. \]

Thus, by also using the last equality in Lemma 2.4, we get

\[ [p - (a + R_0 + \frac{n}{2})] \int_M |Rm|^p F^a \phi^q \]

\[ \leq -2p \int_M (R_{ijkl} \nabla_h F)(\nabla_l R_{ijkl}) |Rm|^{p-2} F^a \phi^q \]

\[ + 2p \int_M |R_{ijkl} \nabla_l F|^2 |Rm|^{p-2} F^a \phi^q \]

\[ + \frac{4p^2}{r} \int_M |R_{ijkl} \nabla_l F|^2 |Rm|^{p-2} F^a \phi^{q-1} \]

\[ = I + II + III. \]

For the first term, by using Lemma 2.4 again, we have

\[ I = -2p \int_M (R_{ijkl} \nabla_h F) R_{ijkl} (\nabla_l |Rm|^{p-2}) F^a \phi^q \]

\[ = 2p \int_M (\nabla_j R_{ik} - \nabla_i R_{jk}) R_{ijkl} (\nabla_l |Rm|^{p-2}) F^a \phi^q. \]
that (\[\nabla_j R_{ik} - \nabla_i R_{jk}\]) R_{ijkl}(\nabla_i |Rm|) |Rm|^{p-3} F^a \phi^q
\]
\leq 4p^2 \int_M |\nabla Rc| |\nabla Rm| |Rm|^{p-2} F^a \phi^q
\leq p \int_M |\nabla Rc|^2 |Rm|^{p-1} F^{a+1} \phi^q + 4p^3 \int_M |\nabla Rm|^2 |Rm|^{p-3} F^{a-1} \phi^q.

On the other hand, by Lemma 2.4, we have

\[II = 2p \int_M |R_{ijkl} \nabla_i F|^2 |Rm|^{p-2} F^a \phi^q\]
\[= 2p \int_M |\nabla_i R_{jk} - \nabla_j R_{ik}|^2 |Rm|^{p-2} F^a \phi^q\]
\[\leq 8p \int_M |\nabla Rc|^2 |Rm|^{p-2} F^a \phi^q\]
\[\leq c p \int_M |\nabla Rc| |\nabla Rm| |Rm|^{p-2} F^a \phi^q\]
\[\leq p \int_M |\nabla Rc|^2 |Rm|^{p-1} F^{a+1} \phi^q + c p \int_M |\nabla Rm|^2 |Rm|^{p-3} F^{a-1} \phi^q.
\]

Finally, since \(|\nabla F|^2 \leq F \leq r| D(r)\), by Lemma 2.4 and Young’s inequality,

\[III = \frac{4p^2}{r} \int_M |R_{ijkl} \nabla_i F|^2 |Rm|^{p-2} F^a \phi^{q-1}\]
\[\leq 4p^2 \int_M |R_{ijkl} \nabla_i F|^2 |\nabla i R_{ijkl} \nabla_i F|^{\frac{2p}{p+1}} |Rm|^{p-2} F^{a-1} \phi^{q-1}\]
\[\leq 16p^2 \int_M |\nabla Rc|^{\frac{2p}{p+1}} |Rm|^{\frac{a+p-1}{p+1}} F^{a-1} \phi^{q-1}\]
\[= 16p^2 \int_M |\nabla Rc|^{\frac{2p}{p+1}} |Rm|^{\frac{a+p-1}{p+1}} F^{a-1} + \frac{1}{p+1} \phi^{q-1}\]
\[= 16p^2 \int_M (|\nabla Rc|^2 |Rm|^{p-1} F^{a+1} \phi^q)^{\frac{p}{p+1}} \cdot (F^{a-2p} \phi^{p-1})^{\frac{p+1}{p+1}}\]
\[\leq 2p \int_M |\nabla Rc|^2 |Rm|^{p-1} F^{a+1} \phi^q + c(p) \int_M F^{a-2p} \phi^p\]
\[\leq 2p \int_M |\nabla Rc|^2 |Rm|^{p-1} F^{a+1} \phi^q + c(p).
\]

Here, in the last inequality, we have used the assumption \(p > a + R_0 + \frac{n}{2} + 1\) and the fact that \((M, g)\) has at most Euclidean volume growth to deduce that \(\int_M F^{a-2p} \leq c\).

Therefore,

\[\left[p - (a + R_0 + \frac{n}{2})\right] \int_M |Rm|^p F^a \phi^q\]
\[\leq 4p \int_M |\nabla Rc|^2 |Rm|^{p-1} F^{a+1} \phi^q\]
\[+ c p^3 \int_M |\nabla Rm|^2 |Rm|^{p-3} F^{a-1} \phi^q + c(p).
\]

This completes the proof of Lemma 3.1. 

\[\square\]
Remark 3.3 In the proof of Lemma 3.1, as well as the proofs of Lemmas 3.2 and 3.3 below, the constant $c(p)$ could be in the order of $p^p$ after applying Young’s inequality.

Lemma 3.2 Let $(M^n, g, f)$ be an $n$-dimensional complete gradient expanding Ricci soliton with nonnegative Ricci curvature $Rc \geq 0$ and finite asymptotic scalar curvature ratio

$$\lim sup_{r \to \infty} Rr^2 < \infty.$$ 

Suppose $p > a + \frac{n}{2} + 1$, then

$$2 \int_M |\nabla Rc|^2 |Rm|^{p-1} F^{a+1} \phi^q \leq c p^3 \int_M |\nabla Rm|^2 |Rm|^{p-3} F^{a-1} \phi^q$$

$$+ \frac{c}{p^2} \int_M |Rm|^p F^a \phi^q + c(p).$$

Proof First of all, by Lemma 2.5 and direct computations, we obtain

$$\Delta_f(|Rc|^2 |Rm|^{p-1}) = (\Delta_f |Rc|^2) |Rm|^{p-1} + |Rc|^2 \Delta_f (|Rm|^{p-1})$$

$$+ 2 \nabla (|Rc|^2) \cdot \nabla (|Rm|^{p-1})$$

$$\geq 2 |\nabla Rc|^2 |Rm|^{p-1} - 2p |Rc|^2 |Rm|^{p-1} - cp |Rc|^2 |Rm|^p$$

$$- 4p |\nabla Rc||\nabla Rm||Rc||Rm|^{p-2}.$$ 

Consequently,

$$2 \int_M |\nabla Rc|^2 |Rm|^{p-1} F^{a+1} \phi^q$$

$$\leq \int_M \Delta_f (|Rc|^2 |Rm|^{p-1}) F^{a+1} \phi^q$$

$$+ 2p \int_M |Rc|^2 |Rm|^{p-1} F^{a+1} \phi^q$$

$$+ cp \int_M |Rc|^2 |Rm|^p F^{a+1} \phi^q$$

$$+ 4p \int_M |\nabla Rc||\nabla Rm||Rc||Rm|^{p-2} F^{a+1} \phi^q$$

$$= I + II + III + IV.$$ 

On one hand, using the quadratic decay of $Rc$ and Young’s inequality, we have

$$II = 2p \int_M |Rc|^2 |Rm|^{p-1} F^{a+1} \phi^q$$

$$\leq cp \int_M |Rm|^{p-1} F^{a-1} \phi^q$$

$$\leq \frac{1}{p^2} \int_M |Rm|^p F^a \phi^q + c(p) \int_M F^{a-p} \phi^q$$

$$\leq \frac{1}{p^2} \int_M |Rm|^p F^a \phi^q + c(p),$$

where, in the last inequality, we have $\int_M F^{a-p} < c$ due to (2.6) and $p > a + \frac{n}{2} + 1$.

Moreover, by the quadratic decay of $Rc$ and (3.2), we get

$$III = cp \int_M |Rc|^2 |Rm|^p F^{a+1} \phi^q$$
\[ \leq \frac{c}{p^2} \int_M |Rm|^p F^a \phi^q + C. \]

On the other hand, since \( \Delta_f u = \Delta u - \nabla f \cdot \nabla u = \Delta u + \nabla F \cdot \nabla u \), by integration by parts, we have

\[
I = \int_M \Delta_f(|Rc|^2 |Rm|^{p-1}) F^{a+1} \phi^q
\]

\[
= \int_M \Delta(|Rc|^2 |Rm|^{p-1}) F^{a+1} \phi^q
\]

\[
+ \int_M \nabla F \cdot \nabla(|Rc|^2 |Rm|^{p-1}) F^{a+1} \phi^q
\]

\[
= \int_M |Rc|^2 |Rm|^{p-1} \Delta(F^{a+1} \phi^q)
\]

\[
+ \frac{q}{r} \int_M |Rc|^2 |Rm|^{p-1} |\nabla F|^2 F^{a+1} \phi^{q-1}
\]

\[
- \int_M |Rc|^2 |Rm|^{p-1} [\Delta F + (a + 1) F^{-1} |\nabla F|^2] F^{a+1} \phi^q
\]

\[
\leq \int_M |Rc|^2 |Rm|^{p-1} \Delta(F^{a+1} \phi^q)
\]

\[
+ 2p \int_M |Rc|^2 |Rm|^{p-1} F^{a+1} \phi^{q-1}
\]

\[= I_A + I_B.\]

Here, we have used the facts that \( |\nabla F|^2 \leq F \leq r \) on \( D(r) \) and \( \Delta F = R + \frac{n}{2} \geq 0 \).

Now, by direct computations, we have

\[
\Delta(F^{a+1} \phi^q)
\]

\[
= \Delta(F^{a+1}) \phi^q + F^{a+1} \Delta(\phi^q) + 2 \nabla F^{a+1} \cdot \nabla \phi^q
\]

\[
\leq [(a + 1) F^{a} \Delta F + a(a + 1) F^{a-1} |\nabla F|^2] \phi^q
\]

\[
+ F^{a+1} [q \phi^{q-1} \Delta \phi + q(q - 1) \phi^{q-2} |\nabla F|^2]
\]

\[
\leq c p^2 F^a \phi^q + 4 p^2 F^a \phi^{q-2}
\]

\[
\leq c p^2 F^a \phi^{q-2},
\]

where we have used the facts that \( \nabla F \cdot \nabla \phi \leq 0 \), \( \Delta \phi \leq 0 \), \( \Delta F \leq R_0 + n/2 \), \( |\nabla F|^2 \leq F \), and \( F |\nabla \phi|^2 \leq 1 \).

Hence, by Young’s inequality, the quadratic decay of \( Rc \) and (3.2), we obtain

\[
I_A = \int_M |Rc|^2 |Rm|^{p-1} \Delta(F^{a+1} \phi^q)
\]

\[
\leq c p^2 \int_M |Rc|^2 |Rm|^{p-1} F^a \phi^{q-2}
\]

\[
\leq \frac{c}{p^2} \int_M |Rm|^{p-1} \phi^{q-2} + C
\]

\[
\leq \frac{1}{p^2} \int_M |Rm|^p F^a \phi^q + c(p) \int_M F^{a-p} \phi^{q-2p} + C
\]

\[
\leq \frac{1}{p^2} \int_M |Rm|^p F^a \phi^q + c(p).
\]
Similarly,

\[ I_B = 2p \int_M |Rc|^2 |Rm|^p - 3 F^{a-1} \phi^q \]
\[ \leq cp \int_M |Rm|^{p-1} F^{a-1} \phi^q \]
\[ \leq \frac{1}{p^2} \int_M |Rm|^p F^{a-1} \phi^q + c(p) \int_M F^{a-1} \phi^{q-p} \]
\[ \leq \frac{1}{p^2} \int_M |Rm|^p F^{a} \phi^q + c(p). \]

Finally,

\[ IV = 4p \int_M \nabla Rc(Rm) |Rc| |Rm|^{p-2} F^{a+1} \phi^q \]
\[ \leq 4p^3 \int_M |\nabla Rm|^2 |Rm|^{p-3} F^{a-1} \phi^q \]
\[ + \frac{1}{p} \int_M |\nabla Rc|^2 |Rc|^2 |Rm|^{p-1} F^{a+3} \phi^q \]
\[ \leq 4p^3 \int_M |\nabla Rm|^2 |Rm|^{p-3} F^{a-1} \phi^q \]
\[ + \frac{c}{p} \int_M |\nabla Rc|^2 |Rm|^{p-1} F^{a+1} \phi^q. \]

By combining the above estimates, we have completed the proof of Lemma 3.2.

\[ \square \]

**Lemma 3.3** Let \((M^n, g, f)\) be an n-dimensional complete gradient expanding Ricci soliton with nonnegative Ricci curvature \(Rc \geq 0\). Suppose \(p > a + \frac{n}{2} + 1\), then

\[ 2 \int_M |\nabla Rm|^2 |Rm|^{p-3} F^{a-1} \phi^q \leq \frac{c}{p^5} \int_M |Rm|^p F^{a} \phi^q + c(p). \]

**Remark 3.4** Note that, like Lemma 3.1, Lemma 3.3 does not require the finite asymptotic scalar curvature ratio assumption either.

**Proof** First of all, note that

\[ 2|\nabla Rm|^2 \leq \Delta |Rm|^2 + \nabla F \cdot \nabla |Rm|^2 + 2|Rm|^2 + c|Rm|^3. \]

Therefore, by integration by parts, we have

\[ 2 \int_M |\nabla Rm|^2 |Rm|^{p-3} F^{a-1} \phi^q \]
\[ \leq \int_M (\Delta |Rm|^2) |Rm|^{p-3} F^{a-1} \phi^q \]
\[ + \int_M (\nabla F \cdot \nabla |Rm|^2) |Rm|^{p-3} F^{a-1} \phi^q \]
\[ + 2 \int_M |Rm|^{p-1} F^{a-1} \phi^q \]
\[ + c \int_M |Rm|^p F^{a-1} \phi^q. \]
\[
\leq -(a - 1) \int_M (\nabla F \cdot |\nabla Rm|^2) |Rm|^{p-3} F^{a-2} \phi^q \\
+ \frac{q}{r} \int_M (\nabla F \cdot |\nabla Rm|^2) |Rm|^{p-3} F^{a-1} \phi^{q-1} \\
+ \int_M (\nabla F \cdot |\nabla Rm|^2) |Rm|^{p-3} F^{a-1} \phi^q \\
+ 2 \int_M |Rm|^{p-1} F^{a-1} \phi^q \\
+ c \int_M |Rm|^p F^{a-1} \phi^q \\
= I + II + III + IV + V.
\]

It follows from integration by parts, \( \Delta F \leq R_0 + n/2 \), \( |\nabla F|^2 \leq F \) and (3.2) that

\[
I = -(a - 1) \int_M (\nabla F \cdot |\nabla Rm|^2) |Rm|^{p-3} F^{a-2} \phi^q \\
= - \frac{2(a - 1)}{p - 1} \int_M (\nabla F \cdot |\nabla Rm|^{p-1}) F^{a-2} \phi^q \\
= \frac{2(a - 1)}{p - 1} \int_M |Rm|^{p-1}[\Delta F + (a - 2) F^{-1} |\nabla F|^2] F^{a-2} \phi^q \\
- \frac{2(a - 1)q}{(p - 1)r} \int_M |Rm|^{p-1} |\nabla F|^2 F^{a-2} \phi^{q-1} \\
\leq c \int_M |Rm|^{p-1} F^{a-1} \phi^q + C \\
\leq \frac{1}{p^2} \int_M |Rm|^p F^a \phi^q + c(p) \int_M F^{a-p} \phi^q + C \\
\leq \frac{1}{p^2} \int_M |Rm|^p F^a \phi^q + c(p),
\]

where, in the last two inequalities, we have again used Young’s inequality, (2.6), and \( p > a + \frac{n}{2} + 1 \).

Similarly, for \( r \geq 1 \), as \( \Delta F = R + \frac{n}{2} > 0 \) and \( |\nabla F|^2 \leq F \leq r \) over \( D(r) \),

\[
II + III = \frac{q}{r} \int_M (\nabla F \cdot |\nabla Rm|^2) |Rm|^{p-3} F^{a-1} \phi^{q-1} \\
+ \int_M (\nabla F \cdot |\nabla Rm|^2) |Rm|^{p-3} F^{a-1} \phi^q \\
\leq (2p + 1) \int_M (\nabla F \cdot |\nabla Rm|^2) |Rm|^{p-3} F^{a-1} \phi^{q-1} \\
= \frac{2(2p + 1)}{p - 1} \int_M (\nabla F \cdot |\nabla Rm|^{p-1}) F^{a-1} \phi^{q-1} \\
= - \frac{2(2p + 1)}{p - 1} \int_M |Rm|^{p-1}[\Delta F + (a - 1) F^{-1} |\nabla F|^2] F^{a-1} \phi^{q-1} \\
+ \frac{2(2p + 1)(q - 1)}{(p - 1)r} \int_M |Rm|^{p-1} |\nabla F|^2 F^{a-1} \phi^{q-2} \\
\leq \frac{2(2p + 1)(q - 1)}{(p - 1)} \int_M |Rm|^{p-1} F^{a-1} \phi^{q-2}
\]
\[
\leq \frac{1}{p^5} \int_M |Rm|^p F^a \phi^q + c(p) \int_M F^{a-p} \phi^{q-2p}
\leq \frac{1}{p^5} \int_M |Rm|^p F^a \phi^q + c(p).
\]

On the other hand, by Young’s inequality, (2.6), and \( p > a + \frac{n}{2} + 1 \), we get

\[
IV = 2 \int_M |Rm|^{p-1} F^{a-1} \phi^q \leq \frac{1}{p^5} \int_M |Rm|^p F^a \phi^q + c(p),
\]

and

\[
V = c \int_M |Rm|^p F^{a-1} \phi^q \leq \frac{c}{p^5} \int_M |Rm|^p F^a \phi^q + C \leq \frac{1}{p^5} \int_M |Rm|^p F^a \phi^q + c(p),
\]

where we have used (3.2) in deriving the first inequality for \( V \).

Combining all the estimates above, the proof of Lemma 3.3 is completed.

Now we can conclude the proof of Proposition 3.1.

**Proof** For any constant \( a > 0 \), let \( p > a + R_0 + \frac{n}{2} + c \) for some constant \( c \geq 1 \). Then, by combining Lemmas 3.1, 3.2 and 3.3, Proposition 3.1 follows immediately.

\[ \square \]

**4 The proof of main theorem**

In this section, we use the integral estimate in Sect. 3 and the De Giorgi-Nash-Moser iteration to prove our main result on the pointwise decay estimate of the curvature tensor \( Rm \) as stated in the introduction (see also Theorem 1.1).

**Theorem 4.1** Let \((M^n, g, f)\) be an \( n \)-dimensional complete gradient expanding Ricci soliton with nonnegative Ricci curvature \( Rc \geq 0 \) and finite asymptotic scalar curvature ratio

\[
\limsup_{r \to \infty} Rr^2 < \infty,
\]

Then \((M^n, g, f)\) has finite \( \alpha \)-asymptotic curvature ratio for any \( 0 < \alpha < 2 \),

\[
A_\alpha := \limsup_{r \to \infty} |Rm|^r \alpha < \infty.
\]

Furthermore, there exist constant \( C > 0 \) depending on \( n \) and the geometry of \((M^n, g, f)\), sequences \( \{r_j\} \to \infty \) and \( \{\alpha_j\} \to 2 \) such that

\[
|Rm|(x) \leq C(r(x) + 1)^{-\alpha_j}
\]

for any \( x \in M \setminus B(x_0, r_j + 1) \).
**Proof** As in Munteanu and Wang [11], we now combine Proposition 3.1 and the De Giorgi-Nash-Moser iteration to obtain the pointwise curvature tensor decay estimate.

First of all, for any \( p > 0 \) large and \( a > 0 \) such that \( p > a + R_0 + \frac{n}{2} + c > a + \frac{n}{2} + 1 \), by Proposition 3.1 we have

\[
\int_M |Rm|^p F^a \phi^q \leq c(p).
\]

Since the cut-off function \( \phi \geq \frac{1}{2} \) on \( D(r/2) \) by (3.1), it follows that

\[
\int_{D(r/2)} |Rm|^p F^a \leq c(p)
\]

for \( r > r_0 \) arbitrarily large. Hence,

\[
\int_M |Rm|^p F^a \leq c(p).
\]

Note that if we define

\[
I(r) := \int_{D(r)} |Rm|^p F^a,
\]

then clearly \( I(r) \) is increasing in \( r \) and

\[
\lim_{r \to \infty} I(r) = \int_M |Rm|^p F^a \leq c(p).
\]

Thus, for any fixed \( p > 0 \) large there exists a constant \( r_p > r_0 \) such that

\[
\int_{M \setminus D(r_p)} |Rm|^p F^a \leq 1.
\]

Therefore, by Lemma 2.2, we have

\[
\int_{B(x,1)} |Rm|^p \leq c (r(x) + 1)^{-2a}
\]

for any \( x \in M \setminus D(r_p + 1) \).

Next, we apply the Moser iteration to get the pointwise decay estimate for \( Rm \) from (4.2). We start by deriving an inequality satisfied by \( \Delta |Rm|^2 \).

From Lemma 2.5, we note that

\[
\Delta_f |Rm|^2 \geq 2|\nabla Rm|^2 - 2|Rm|^2 - c|Rm|^3.
\]

Also, by using the Cauchy-Schwarz inequality and Kato’s inequality, we have

\[
\nabla F \cdot \nabla |Rm|^2 = 2|Rm| |\nabla F| |Rm| \leq \frac{1}{2} |Rm|^2 |\nabla F|^2 + 2|\nabla Rm|^2.
\]

Thus,

\[
\Delta |Rm|^2 \geq 2|\nabla Rm|^2 - 2|Rm|^2 - c|Rm|^3 - \nabla F \cdot \nabla |Rm|^2
\]

\[
\geq -2|Rm|^2 - c|Rm|^3 - \frac{1}{2} (F - R - \frac{n}{2}) |Rm|^2
\]

\[
\geq -c(F + |Rm|) |Rm|^2
\]

\[
= -u |Rm|^2,
\]

\( u \) is a constant.
where \( u := c (F + |Rm|) \).

By Lemmas 2.8 and 2.9, or by the Sobolev inequality in [22] together with the non-collapsing estimate of Carrillo and Ni in Lemma 2.7, we know that the Sobolev inequality holds on the unit geodesic ball \( B_x(1) \), with the Sobolev constant \( C_s \) independent of \( x \in M \). Therefore, by applying the Moser iteration (see [23] or [24]), we have

\[
|Rm|(x) \leq C_0 \left( \int_{B_x(1)} u^n + 1 \right)^{\frac{1}{p}} \left( \int_{B_x(1)} |Rm|^p \right)^{\frac{1}{p}},
\]

where \( C_0 > 0 \) depends only on \( n \) and \( C_s \). Note that, by (4.2) and the Bishop volume comparison, we have

\[
\int_{B_x(1)} |Rm|^n \leq \left( \int_{B_x(1)} |Rm|^p \right)^{\frac{n}{p}} \text{Vol}(B_x(1)) \leq c \left( r(x) + 1 \right)^{-2a_n \frac{n}{p}}
\]

for any \( x \in M \setminus D(r_\rho + 1) \). Hence,

\[
\int_{B_x(1)} u^n = c \int_{B_x(1)} (F + |Rm|)^n \
\leq c \int_{B_x(1)} F^n + c \int_{B_x(1)} |Rm|^n \
\leq c (r(x) + 1)^{2n}.
\]

(4.4)

Now, for \( p > 0 \) large, we take

\[
a = p - \left( \frac{n}{2} + R_0 + c + 1 \right).
\]

(4.5)

Then, by (4.2)–(4.4), we have

\[
|Rm|(x) \leq C_0 \left( \int_{B_x(1)} u^n + 1 \right)^{\frac{1}{p}} \left( \int_{B_x(1)} |Rm|^p \right)^{\frac{1}{p}} \
\leq C_0 (r(x) + 1)^{-\frac{2(a-n)}{p}}
\]

for \( x \in M \setminus D(r_\rho + 1) \).

On the other hand, for any \( \alpha \in (0, 2) \), then for \( p \) sufficiently large we have

\[
a - n = 1 - \frac{n}{2} + R_0 + c + 1 + n \geq \frac{\alpha}{2}.
\]

Now, for \( \alpha, \ p \) and \( a \) as above, by (4.6) we have

\[
|Rm|(x) \leq C_0 (r(x) + 1)^{-\frac{2(a-n)}{p}}
\]

for any \( x \in M \setminus D(r_\rho + 1) \).

Furthermore, note also that we have \( r_\rho \to \infty \) as \( p \to \infty \). Thus, if we take \( p = j \in \mathbb{N} \) and set

\[
\alpha_j = \frac{2(a-n)}{p} = 2 - \frac{3n + 2R_0 + 2c + 2}{j} \to 2.
\]
then there exists a sequence \( \{ r_j \} \to \infty \) such that

\[
|Rm|(x) \leq C_0 (r(x) + 1)^{-\alpha_j}
\]

for any \( x \in M \setminus D(r_j + 1) \).

This completes the proof of Theorem 4.1. \( \square \)

In fact, as we mentioned in Remark 1.1, the same proof can be used to prove the following more general curvature decay estimate.

**Theorem 4.2** Let \((M^n, g, f)\) be an \(n\)-dimensional complete gradient expanding Ricci soliton with nonnegative Ricci curvature \( R_c \geq 0 \) and finite \( \alpha_0 \)-asymptotic curvature ratio for any \( 0 < \alpha_0 \leq 2 \),

\[
\limsup_{r \to \infty} R^{\alpha_0} < \infty.
\]

Then, \((M^n, g, f)\) has finite \( \alpha \)-asymptotic curvature ratio for any \( 0 < \alpha < \alpha_0 \),

\[
A_\alpha := \limsup_{r \to \infty} |Rm|^r^\alpha < \infty.
\]

Furthermore, there exist constant \( C > 0 \) depending on \( n \) and the geometry of \((M^n, g, f)\), sequences \( \{ r_j \} \to \infty \) and \( \{ \alpha_j \} \to \alpha_0 \) such that

\[
|Rm|(x) \leq C (r(x) + 1)^{-\alpha_j}
\]

for any \( x \in M \setminus B(x_0, r_j + 1) \).

**Proof** For any \( \alpha_0 \in (0, 2] \), let \( \epsilon := \frac{\alpha_0}{2} \), then \( \epsilon \in (0, 1] \). For \( p > a + R_0 + \frac{n}{2} + 1 \), by following the same argument as in Lemma 3.1 and using Lemma 2.4, we have

\[
\left[ 1 - p^{-1}(a + R_0 + \frac{n}{2}) \right] \int_M |Rm|^p F^a \phi^q \leq 4 \int_M |\nabla R_c|^2 |Rm|^{p-1} F^a+\epsilon \phi^q \\
+ cp^2 \int_M |\nabla Rm|^2 |Rm|^{p-3} F^{a-\epsilon} \phi^q \\
+ c(p).
\]

Also, note that the same argument as in the proofs of Lemmas 3.2 and 3.3 give us the following: if \( \epsilon p > a + \frac{n}{2} + 1 \), then we have

\[
2 \int_M |\nabla R c|^2 |Rm|^{p-1} F^{a+\epsilon} \phi^q \leq cp^3 \int_M |\nabla Rm|^2 |Rm|^{p-3} F^{a-\epsilon} \phi^q \\
+ \frac{c}{p^5} \int_M |Rm|^p F^a \phi^q + c(p),
\]

and

\[
2 \int_M |\nabla Rm|^2 |Rm|^{p-3} F^{a-\epsilon} \phi^q \leq \frac{c}{p^5} \int_M |Rm|^p F^a \phi^q + c(p).
\]

By combining the estimates above, we see that if \( \epsilon p > a + \frac{n}{2} + 1 \) and \( p > a + \frac{n}{2} + R_0 + c \) then we have

\[
\left[ 1 - p^{-1}(a + \frac{n}{2} + R_0 + c) \right] \int_M |Rm|^p F^a \phi^q \leq c(p).
\]
As in the proof of Theorem 4.1, for any fixed \( p > 0 \) large, there exists a constant \( r_p > r_0 \) such that
\[
\int_{M \setminus D(r_p)} |Rm|^p F^a \leq 1.
\]
Therefore, by Lemma 2.2, for any \( x \in M \setminus D(r_p + 1) \), we get
\[
\int_{B_x(1)} |Rm|^p \leq c (r(x) + 1)^{-2a}.
\]
For any \( p > 0 \) large, we take
\[
a = \epsilon p - \left( \frac{n}{2} + R_0 + c + 1 \right).
\]
Then by following the same proof as in Theorem 4.1, we have
\[
|Rm|(x) \leq C_0 \left( \int_{B_x(1)} u^n + 1 \right)^{\frac{1}{p}} \left( \int_{B_x(1)} |Rm|^p \right)^{\frac{1}{p}} \leq C_0 (r(x) + 1)^{-\frac{2(a-n)}{p}}
\]
for \( x \in M \setminus D(r_p + 1) \).
We note that for any \( \alpha \in (0, \alpha_0) \), when \( p \) is sufficiently large, we have
\[
\frac{a - n}{p} = \epsilon - \frac{n}{2} + R_0 + c + 1 + \frac{n}{p} \geq \frac{\alpha}{2}.
\]
Now, for \( \alpha, \ p, \ a \) as above and any \( x \in M \setminus D(r_p + 1) \), by (4.8) we obtain
\[
|Rm|(x) \leq C_0 r(x) + 1)^{-\frac{2(a-n)}{p}}
\]
Moreover, as in the proof of Theorem 4.1, if we take \( p = j \in \mathbb{N} \) and set
\[
\alpha_j = \frac{2(a - n)}{p} = \alpha_0 - \frac{3n + 2R_0 + 2c + 2}{j} \to \alpha_0,
\]
then there exists a sequence \( \{r_j\} \to \infty \) such that
\[
|Rm|(x) \leq C_0 (r(x) + 1)^{-\alpha_j}
\]
for any \( x \in M \setminus D(r_j + 1) \).
This completes the proof of Theorem 4.2.

Acknowledgements We would like to thank Ovidiu Munteanu and Jiaping Wang for their interests in this work and their helpful comments and suggestions. We are also grateful to the referee for the careful reading of our paper and for providing valuable suggestions which led to a simpler version of Lemma 3.1 and a more streamlined proof of Lemmas 3.2 and 3.3 than in the previous version.

Data availability Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

Declaration

Conflict of interest There is no conflict of interest to disclose.

 Springer
References

1. Cao, H.-D., Liu, T.: Curvature estimates for four-dimensional complete gradient expanding Ricci solitons. J. Reine Angew. Math. **790**, 115–135 (2022)
2. Chen, C.-W., Deruelle, A.: Structure at infinity of expanding Ricci soliton. Asian J. Math. **19**(5), 933–950 (2015)
3. Munteanu, O., Wang, J.: Geometry of shrinking Ricci solitons. Compos. Math. **151**(12), 2273–2300 (2015)
4. Chow, B., Lu, P., Yang, B.: Lower bounds for the scalar curvatures of noncompact gradient Ricci solitons. Comptes Rendus Math. Acad. Sci. Paris **349**(23–24), 1265–1267 (2011)
5. Kotschwar, B., Wang, L.: Rigidity of asymptotically conical shrinking gradient Ricci solitons. J. Differ. Geom. **100**, 55–108 (2015)
6. Conlon, R. J., Deruelle, A., Sun, S.: Classification results for expanding and shrinking gradient Kähler-Ricci solitons. Preprint at arXiv:1904.00147 (2019)
7. Cao, H.-D., Ribeiro, R., Zhou, D.: Four-dimensional complete gradient shrinking Ricci solitons. J. Reine Angew. Math. **778**, 127–144 (2021)
8. Cao, H.-D., Cui, X.: Curvature estimates for four-dimensional gradient steady solitons. J. Geom. Anal. **30**(1), 511–525 (2020)
9. Chan, P.Y.: Curvature estimates for steady Ricci solitons. Trans. Am. Math. Soc. **372**(12), 8985–9008 (2019)
10. Cao, H.-D.: On curvature estimates for four-dimensional gradient Ricci solitons. Mat. Contemp. **49**, 87–139 (2022)
11. Munteanu, O., Wang, J.: Conical structure for shrinking Ricci solitons. J. Eur. Math. Soc. (JEMS) **19**(11), 3377–3390 (2017)
12. Hamilton, R.S.: The Formation of Singularities in the Ricci Flow, Surveys in Differential Geometry (Cambridge, MA, 1993), 2, 7–136. International Press, Cambridge (1995)
13. Cao, H.-D., Catino, G., Chen, Q., Mantegazza, C., Mazzieri, L.: Bach-flat gradient steady Ricci solitons. Calc. Var. Partial Differ. Equ. **49**(1–2), 125–138 (2014)
14. Ma, L., Chen, D.: Remarks on complete non-compact gradient Ricci expanding solitons. Kodai Math. J. **33**, 173–181 (2010)
15. Carrillo, J.A., Ni, L.: Sharp logarithmic Sobolev inequalities on solitons and applications. Commun. Anal. Geom. **17**(4), 721–753 (2009)
16. Hamilton, R.S.: Lectures on Ricci flow. Clay Summer School at MSRI, Oakland (2005)
17. Chow, B., Lu, P., Ni, L.: Hamilton’s Ricci Flow. Graduate Studies in Mathematics, vol. 77. American Mathematical Society, Providence; Science Press Beijing, New York (2006)
18. Yau, S.-T.: Survey on partial differential equations in differential geometry. Seminar on Differential Geometry, vol. 102, pp. 3-71. Princeton Univ. Press, Princeton (1982)
19. Brendle, S.: Sobolev inequalities in manifolds with nonnegative curvature. Preprint at arXiv:2009.13717v4 (2020)
20. Cheeger, J., Gromov, M., Taylor, M.: Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. J. Differ. Geom. **17**(1), 15–53 (1982)
21. Chan, P.Y., Ma, Z., Zhang, Y.: Volume growth estimates of gradient Ricci solitons. Preprint at arXiv:2202.13302 (2022)
22. Saloff-Coste, L.: Uniformly elliptic operators on Riemannian manifolds. J. Differ. Geom. **36**(2), 417–450 (1992)
23. Li, P.: Geometric Analysis. Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge (2012)
24. Li, P.: Lecture Notes on Geometric Analysis, in 'Lecture Notes Series 6 - Research Institute of Mathematics and Global Analysis Research Center’. Seoul National University, Seoul (1993)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.