ON THE INSTABILITY OF EIGENVALUES

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Abstract. This is the proceeding of a talk given in Workshop on Differential Geometry and its applications at Alexandru Ioan Cuza University Ia¸ si, Romania, September 2–4, 2009. I explain how positive commutator estimates help in the analysis of embedded eigenvalues in a geometrical setting. Then, I will discuss the disappearance of eigenvalues in the perturbation theory and its relation with the Fermi golden rule.

1. Introduction

Let $\mathbb{H} := \{(x, y) \in \mathbb{R}^2, y > 0\}$ be the Poincaré half-plane and we endow it with the metric $g := y^{-2}(dx^2 + dy^2)$. Consider the group $\Gamma := \text{PSL}_2(\mathbb{Z})$. It acts faithfully on $\mathbb{H}$ by homographies, from the left. The interior of a fundamental domain of the quotient $\mathbb{H}/\Gamma$ is given by $X := \{(x, y) \in \mathbb{H}, |x| < 1, x^2 + y^2 > 1\}$. Let $\mathcal{H} := L^2(X, y)$ be the set of $L^2$ integrable function acting on $X$, with respect to the volume element $dx dy/y^2$. Let $C_\infty^b(X)$ be the restriction to $X$ of the smooth bounded functions acting on $\mathbb{H}$ which are $C_\infty$-valued and invariant under $\Gamma$. The (non-negative) Laplace operator is defined as the closure of $\Delta := -y^2(\partial_x^2 + \partial_y^2)$, on $C_\infty^b(X)$.

It is a (unbounded) self-adjoint operator on $L^2(X)$. Using Eisenstein series, for instance, one sees that its essential spectrum is given by $[1/4, \infty)$ and that it has no singularly continuous spectrum, with respect to the Lebesgue measure. It is well-known that $\Delta$ has infinitely many eigenvalues accumulating at $+\infty$ and that every eigenspace is of finite dimension. We refer to [5] for an introduction to the subject.

We consider the Schrödinger operator $H_\lambda := \Delta + \lambda V$, where $V$ is the multiplication by a bounded, real-valued function and $\lambda \in \mathbb{R}$. We focus on an eigenvalue $k > 1/4$ of $\Delta$ and assume that the following hypothesis of Fermi golden rule holds true. Namely, there is $c_0 > 0$ so that:

\[
\lim_{\varepsilon \to 0^+} PV \overline{P} \text{Im}(H_0 - k + i\varepsilon)^{-1} \overline{P} VP \geq c_0 P,
\]

in the form sense and where $P := P_k$, the projection on the eigenspace of $k$, and $\overline{P} := 1 - P$. As $P$ is of finite dimension, the limit can be taken in the weak or in the strong sense. At least formally, $\overline{P} \text{Im}(H_0 - k + i\varepsilon)^{-1} \overline{P}$ tends to the Dirac mass $\pi \delta_k(\overline{P}H_0)$. Therefore, the potential $V$ couples the eigenspace of $k$ and $\overline{P}H_0$ over $k$ in a non-trivial way. This is a key assumption in the second-order perturbation theory of embedded eigenvalues, e.g., [13], and all the art is to prove that it implies there is $\lambda_0 > 0$ that $H_\lambda$ has no eigenvalue in a neighborhood of $k$ for $\lambda \in (0, |\lambda_0|)$.
In [4], one shows that generically the eigenvalues disappear under the perturbation of a potential (or of the metric) on a compact set. In this note, we are interested about the optimal decay at infinity of the perturbation given by a potential. Using the general result obtained in [3] and under a hypothesis of Fermi golden rule, one is only able to cover the assumption $VL^3 = o(1)$, as $y \to +\infty$, where $L$ denotes the operator of multiplication by $L: (x,y) \mapsto 1 + \ln(y)$. We give the main result:

**Theorem 1.1.** Let $k > 1/4$ be an $L^2$-eigenvalue of $\Delta$. Suppose that $VL = o(1)$, as $y \to +\infty$ and that the Fermi golden rule (1.1) holds true, then there is $\lambda_0 > 0$, so that $H_\lambda$ has no eigenvalue in a neighborhood of $k$, for all $\lambda \in (0, |\lambda_0|)$. Moreover, if $VL^{1+\varepsilon} = o(1)$, as $y \to +\infty$ for some $\varepsilon > 0$, then $H_\lambda$ has no singularly continuous spectrum.

We believe that the hypothesis $VL = o(1)$ is optimal in the scale of $L$. In our approach, we use the Mourre theory, see [1, 12] and establish a positive commutator estimate.

## 2. Idea of the proof

Standardly, for $y$ large enough and up to some isometry $U$, see for instance [6, 9, 10] the Laplace operator can be written as

\begin{equation}
\tilde{\Delta} = (-\partial_r^2 + 1/4) \otimes P_0 + \tilde{\Delta}(1 \otimes P_0^+)\end{equation}

on $C_c^\infty((c, \infty), dr) \otimes C^\infty(S^1)$, for some $c > 0$ and where $P_0$ is the projection on constant functions and $P_0^+ := 1 - P_0$. The Friedrichs extension of the operator $\tilde{\Delta}(1 \otimes P_0^+)$ has compact resolvent.

Then, as in [9, 10], we construct a conjugate operator. One chooses $\Phi \in C_c^\infty(\mathbb{R})$ with $\Phi(x) = x$ on $[-1, 1]$, and sets $\Phi_T(x) := \Upsilon \Phi(x/\Upsilon)$, for $\Upsilon \geq 1$. Let $\tilde{\chi}$ be a smooth cut-off function being 1 for $r$ big enough and 0 for $r$ being close to $c$. We define on $C_c^\infty((c, \infty) \times S^1)$ a micro-localized version of the generator of dilations:

\begin{equation}
S_{\Upsilon, 0} := \tilde{\chi} \left( (\Phi_T(-i\partial_r)r + r\Phi_T(-i\partial_r)) \otimes P_0 \right) \tilde{\chi}.
\end{equation}

The operator $\Phi_T(-i\partial_r)$ is defined on the real line by $\mathcal{F}^{-1}\Phi_T(\cdot)\mathcal{F}$, where $\mathcal{F}$ is the unitary Fourier transform. We also denote its closure by $S_{\Upsilon, 0}$ and it is self-adjoint. In [6] for instance, one does not use a micro-localization and one is not able to deal with really singular perturbation of the metric as in [9, 10].

Now, one obtains

\begin{equation}
[\partial_r^2, \tilde{\chi}(\Phi_T r + r\Phi_T)\tilde{\chi}] = 4\tilde{\chi}\partial_r \Phi_T \tilde{\chi} + \text{remainder}.
\end{equation}

Using a cut-off function $\tilde{\mu}$ being 1 on the cusp and 0 for $y \leq 2$, we set

\begin{equation}
S_\Upsilon := U^{-1}S_{\Upsilon, 0}U \tilde{\mu}
\end{equation}

This is self-adjoint in $L^2(X)$. Now by taking $\Upsilon$ big enough, one can show, as in [9, 10] that given an interval $\mathcal{J}$ around $k$, there exist $\varepsilon_\Upsilon > 0$ and a compact operator $K_\Upsilon$ such that the inequality

\begin{equation}
E_{\mathcal{J}}(\Delta)[\Delta, iS_\Upsilon]E_{\mathcal{J}}(\Delta) \geq \left( 4 \inf(\mathcal{J}) - \varepsilon_\Upsilon \right) E_{\mathcal{J}}(\Delta) + E_{\mathcal{J}}(\Delta) K_\Upsilon E_{\mathcal{J}}(\Delta)
\end{equation}
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holds in the sense of forms, and such that \( \varepsilon \Upsilon \) tends to 0 as \( \Upsilon \) goes to infinity. Here, \( E_J(\cdot) \) denotes the spectral measure above the interval \( J \).

Now, we apply \( \overline{P} \) to the left and right of (2.5). Easily one has \( \overline{P} E_J(\Delta) = \overline{P} E_J(\Delta \overline{P}) \).

We get:

\[
\overline{P} E_J(\Delta) \left[ \overline{P} \Delta, i \overline{P} S_\Upsilon \overline{P} \right] E_J(\overline{P} \Delta \overline{P}) \geq (4 \inf_{J} \varepsilon \Upsilon - \varepsilon \Upsilon \Upsilon) \overline{P} E_J(\Delta \overline{P}) \overline{P} + \overline{P} E_J(\Delta) K \Upsilon E_J(\overline{P} \Delta \overline{P}) \overline{P}
\]

One can show that \( \overline{P} S_\Upsilon \overline{P} \) is self-adjoint in \( \overline{P} L^2(X) \) and that \( \left[ \overline{P} \Delta, \overline{P} S_\Upsilon \overline{P} \right] \) extends to a bounded operator.

We now shrink the size of the interval \( J \). As \( \overline{P} \Delta \) has no eigenvalue in \( J \), then the operator \( \overline{P} E_J(\Delta) K \Upsilon E_J(\overline{P} \Delta \overline{P}) \overline{P} \) tends to 0 in norm. Therefore, by shrinking enough, one obtains a smaller interval \( J \) containing \( k \) and a constant \( c > 0 \) so that

\[
(2.6) \quad \overline{P} E_J(\Delta) \left[ \overline{P} \Delta, i \overline{P} S_\Upsilon \overline{P} \right] E_J(\overline{P} \Delta \overline{P}) \geq c \overline{P} E_J(\Delta \overline{P}) \overline{P}
\]

holds true in the form sense on \( \overline{P} L^2(X) \). At least formally, the positivity on \( \overline{P} L^2(X) \) of the commutator \( \left[ H_\lambda, i \overline{P} S_\Upsilon \overline{P} \right] \), up to some spectral measure and to some small \( \lambda \), should be a general fact and should not rely on the Fermi golden rule hypothesis.

We now try to extract some positivity on \( \overline{P} L^2(X) \). First, we set

\[
R_\varepsilon := \left( (H_0 - k)^2 + \varepsilon^2 \right)^{-1/2}, \quad R_\varepsilon \overline{P} \text{ and } F_\varepsilon := \overline{R}_\varepsilon^2.
\]

Note that \( \varepsilon R_\varepsilon^2 = \text{Im} (H_0 - k + i\varepsilon)^{-1} \) and that \( R_\varepsilon \) commutes with \( P \). Using (1.1), we get:

\[
(2.7) \quad (c_1/\varepsilon) P \geq \overline{P} V \overline{P} F_\varepsilon \overline{P} V \overline{P} \geq (c_2/\varepsilon) P,
\]

for \( \varepsilon_0 > \varepsilon > 0 \).

We follow an idea of [2], which was successfully used in [8, 11] and set

\[
B_\varepsilon := \text{Im}(\overline{R}_\varepsilon^2 V \overline{P}).
\]

It is a finite rank operator. Observe now that we gain some positivity as soon as \( \lambda \neq 0 \):

\[
(2.8) \quad P[H_\lambda, i \lambda B_\varepsilon] P = \lambda^2 P V F_\varepsilon V P \geq (c_2 \lambda^2/\varepsilon) P.
\]

It is therefore natural to modify the conjugate operator \( S_\Upsilon \) to obtain some positivity on \( \overline{P} L^2(X) \). We set

\[
(2.9) \quad \hat{S}_\Upsilon := \overline{P} S_\Upsilon \overline{P} + \lambda \theta B_\varepsilon.
\]

It is self-adjoint on \( D(S_\Upsilon) \) and is diagonal with respect to the decomposition \( \overline{P} L^2(X) \oplus \overline{P} L^2(X) \).

Here \( \theta > 0 \) is a technical parameter. We choose \( \varepsilon \) and \( \theta \), depending on \( \lambda \), so that \( \lambda = o(\varepsilon) \), \( \varepsilon = o(\theta) \) and \( \theta = o(1) \) as \( \lambda \) tends to 0. We summarize this into:

\[
(2.10) \quad |\lambda| \ll \varepsilon \ll \theta \ll 1, \text{ as } \lambda \text{ tends to 0}.
\]
With respect to the decomposition $\overline{PE}_\mathcal{J}(\Delta) \oplus PE_\mathcal{J}(\Delta)$, as $\lambda$ goes to 0, we have
\[
E_\mathcal{J}(\Delta) \left[ \lambda V, i\mathcal{P}S_\mathcal{T}\mathcal{P} \right] E_\mathcal{J}(\Delta) = \begin{pmatrix} O(\lambda) & O(\lambda) \\ O(\lambda) & 0 \end{pmatrix},
\]
\[
E_\mathcal{J}(\Delta)[\lambda, i\lambda \theta B_x]E_\mathcal{J}(\Delta) = \begin{pmatrix} 0 & O(\lambda \theta_{\varepsilon}^{-1/2}) \\ O(\lambda \theta_{\varepsilon}^{-1/2}) & 0 \end{pmatrix},
\]
and
\[
E_\mathcal{J}(\Delta)[\lambda V, i\lambda \theta B_x]E_\mathcal{J}(\Delta) = \begin{pmatrix} O(\lambda^2 \theta_{\varepsilon}^{-3/2}) & O(\lambda^2 \theta_{\varepsilon}^{-3/2}) \\ O(\lambda^2 \theta_{\varepsilon}^{-3/2}) & \lambda^2 \theta F_{\varepsilon} \end{pmatrix}.
\]

Now comes the delicate point. Under the condition (2.11) and by choosing $\mathcal{I}$, slightly smaller than $\mathcal{J}$, we use the previous estimates and a Schur Lemma to deduce:
\[
E_\mathcal{I}(H_\lambda)[H_\lambda, i\hat{S}_\mathcal{T}]E_\mathcal{I}(H_\lambda) \geq \frac{c\lambda^2 \theta}{\varepsilon} E_\mathcal{I}(H_\lambda),
\]
for some positive $c$ and as $\lambda$ tends to 0.

We mention that only the decay of $V L$ is used to establish the last estimate. In fact, one uses that $[V, i\hat{S}_\mathcal{T}](\Delta + 1)^{-1}$ is a compact operator. Now it is a standard use of the Mourre theory to deduce Theorem 1.1 and refer to [1], see [9, 10] for some similar application of the theory. For the absence of eigenvalue, one relies on the fact that given an eigenfunction $f$ of $H_\lambda$ w.r.t. an eigenvalue $\kappa \in \mathcal{I}$, one has:
\[
\langle f, [H_\lambda - \kappa, i\hat{S}_\mathcal{T}]f \rangle = 0.
\]
Then, one applies $f$ on the right and on the left of (2.12) and infers that $f = 0$ thanks to the fact that the constant $c\lambda^2 \theta$ is non-zero.

In [9, 10], we prove that the $C_0$-group $(e^{i\hat{S}_\mathcal{T}}t)_{t \in \mathbb{R}}$ stabilizes the domain $\mathcal{D}(H_\lambda) = \mathcal{D}(\Delta)$. By perturbation, we prove that this is also the case for $(e^{i\hat{S}_\mathcal{T}}t)_{t \in \mathbb{R}}$. Thanks to this property, we can expand the commutator of (2.13) in a legal way. This is known as the Virial theorem in the Mourre Theory, see [1, 12].

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