We present a formal theory of contextuality for a set of random variables grouped into different subsets (contexts) corresponding to different, mutually incompatible conditions. Within each context the random variables are jointly distributed, but across different contexts they are stochastically unrelated. The theory of contextuality is based on the analysis of the extent to which some of these random variables can be viewed as preserving their identity across different contexts when one considers all possible joint distributions imposed on the entire set of the random variables. We illustrate the theory on three systems of traditional interest in quantum physics (and also in non-physical, e.g., behavioral studies). These are systems of the Klyachko-type, Bell-type, and Leggett-Garg-type. Listed in this order, each of them is formally a special case of the previous one. For each of them we derive necessary and sufficient conditions for contextuality while allowing for experimental errors and contextual biases or signaling. Based on the same principles that underly these derivations we also propose a measure for the degree of contextuality and compute it for the three systems in question.

**Keywords:** CHSH inequalities; contextuality; EPR/Bohm paradigm; Klyachko inequalities; Leggett-Garg inequalities; measurement bias; measurement errors; probabilistic couplings; signaling.
1. INTRODUCTION

A deductive mathematical theory is bound to begin with definitions and/or axioms, and one is free not to accept them. We propose a certain definition of contextuality which may or may not be judged “good.” Ultimately, its utility will be determined by whether it leads to fruitful mathematical developments and interesting applications. Our definition applies to situations where contextuality is traditionally investigated in quantum physics: Klyachko–Can–Binicioglu–Shumovsky-type systems of measurements [1], Einstein–Podolsky–Rosen–Bohm–Bell-type systems [2–6], and Suppes–Zanotti–Leggett–Garg-type systems [7, 8]. In the absence of measurement errors and of what we call “inconsistency,” our contextuality criteria (necessary and sufficient conditions) coincide with the traditional inequalities. But our criteria also apply to situations with measurement errors, contextual biases, and signaling. Moreover, the logic of constructing the criteria of contextuality leads to a natural quantification of the degree of contextuality in the three types of systems considered.

The notion of probabilistic contextuality is usually understood to be about “sewing together” random variables recorded under different conditions. That is, it is viewed as answering the question: given certain sets of jointly distributed random variables, can a joint distribution be found for their union? The key aspect and difficulty in answering this questions is that different sets of random variables generally pairwise overlap, share some their elements. In Ernst Specker’s [9] well-known example with three magic boxes containing gems, which we present here in probabilistic terms, we have three binary random variables, $A, B, C$, that can only be recorded in pairs,

$$X = (A, B), \ Y = (B, C), \ Z = (A, C).$$  \hspace{1cm} (1)

That is, the joint distribution of $A$ and $B$ in $X$ is known, and the same is true for the components of $Y$ and $Z$. We ask whether there is a joint distribution of all three of them, $(A, B, C)$, that agrees with the distributions of $X, Y$, and $Z$ as its (2-component) marginals. In Specker’s example the boxes are magically rigged so that (assuming $A, B, C$ attain values $+1/-1$, denoting the presence/absence of a gem in the respective box)

$$\Pr[A = -B] = 1, \ \Pr[-B = C] = 1, \ \Pr[C = -A] = 1,$$  \hspace{1cm} (2)

which, obviously, precludes the existence of a jointly distributed $(A, B, C)$. We say then that the system of random variables (1) exhibits contextuality.

On a deeper level of analysis, however, contextuality is better to be presented as a problem of determining identities of the random variables recorded under different conditions. That is, it answers the question: is this random variable (under this condition), say, $A$ in $X$, “the same as” that one (under another condition), say, $A$ in $Y$, or is the former at least “as close” to the latter as their distributions in the two pairs allow?

This deeper view is based on the principle we dubbed Contextuality-by-Default, developed through a series of recent publications [10–17]. According to this principle, any two random variables recorded under different (i.e., mutually exclusive) conditions (treatments) are labeled by these conditions and considered stochastically unrelated (defined on different sample spaces, possessing no joint distribution). For a detailed analysis of the notion of stochastic unrelatedness within the framework of the Kolmogorovian probability theory, see Refs. [18–20]. Thus, in Specker’s example with the magic boxes, we need to denote the observed three pairs of random variables not as in (1), but as

$$X = (A_X, B_X), \ Y = (B_Y, C_Y), \ Z = (A_Z, C_Z).$$  \hspace{1cm} (3)

Of course, any other unique labeling making random variables in one context distinct from random variables in another context would do as well.

Stochastically unrelated random variables can always be coupled (imposed a joint distribution upon). This can generally be done in multiple ways, and no couplings are privileged a priori. For Specker’s example, one constructs a random 6-tuple

$$S = (A_X, B_X, B_Y, C_Y, A_Z, C_Z)$$  \hspace{1cm} (4)

such that its 2-marginals $X, Y, Z$ in (3) are consistent with the observed probabilities. In particular, they should satisfy

$$\Pr[A_X = -B_X] = 1, \ \Pr[-B_Y = C_Y] = 1, \ \Pr[C_Z = -A_Z] = 1.$$  \hspace{1cm} (5)

Such a coupling $S$ can be constructed in an infinity of ways. To match this representation with Specker’s original meaning, we have to impose additional constraints on the possible couplings. Namely, we have to require that $S$ in (4) be constructed subject to the following “identity hypothesis”:

$$\Pr[A_X = A_Z] = \Pr[B_X = B_Y] = \Pr[C_Y = C_Z] = 1.$$  \hspace{1cm} (6)
Such a coupling $S$, as we have already determined, does not exist, and we can say that the system of the random variables $S$ exhibits contextuality with respect to the identity hypothesis $[6]$. One might wonder whether this re-representation of the problem is useful. Aren’t the questions

“Let me see if I can ‘sew together’ $(A, B)$, $(B, C)$, and $(A, C)$ into a single $(A, B, C)$,”

and

“Let me see if I can put together $(A_X, B_X)$, $(B_Y, C_Y)$, and $(A_Z, C_Z)$ into a single $S$ in (4) under the identity hypothesis $[6]$.”

aren’t they one and the same question in two equivalent forms? Clearly, they are. But there are two (closely related) advantages of the second formulation:

1. It can be readily generalized by replacing the perfect identities in $[6]$ with less stringent or altogether different constraints; and

2. for any given constraint, if a coupling satisfying it does not exist, this approach allows one to gauge how close one can get to satisfying it, i.e., one has a principled way for constructing a measure for the degree of contextuality the system exhibits.

To illustrate these interrelated points on Specker’s example, observe that the identity hypothesis $[6]$ cannot be satisfied if the system is “inconsistently connected,” i.e., if the marginal distribution of, say, $A_X$ is not the same as that of $A_Z$. This may happen if the magic boxes somehow physically communicate (e.g., the gem can be transposed from one of the boxes being opened to another), and the probability of finding a gem in the first box ($A = 1$) is affected differently by the opening of the second box (i.e., in context $X$) and of the third box (in context $Z$). $A_X$ and $A_Z$ may have different distributions also as a result of (perhaps magically induced) errors in correctly identifying which of the two open boxes contains a gem: e.g., when boxes $i$ and $j$ are open ($i < j$), one may with some probability erroneously see/record the gem contained in the $i$th box as being in the $j$th box. We may speak of “signaling” between the boxes in the former case, and of “contextual measurement biases” in the latter. In either case, the requirement $[6]$ cannot be satisfied for $A_X$ and $A_Z$, and to determine this one does not even have to look at the observed distributions of $(A_X, B_X)$, $(B_Y, C_Y)$, and $(A_Z, C_Z)$. However, in either of the two cases one can meaningfully ask: what is the maximum possible value of $\Pr[A_X = A_Z]$ that is consistent with the distributions of $A_X$ and $A_Z$ (and analogously for $B_X, B_Y$ and $C_Y, C_Z$), and are these maximum possible values consistent with the observed distributions of $(A_X, B_X)$, $(B_Y, C_Y)$, and $(A_Z, C_Z)$?

We will proceed now to formulate these ideas in a more rigorous way.

2. SYSTEMS, RANDOM BUNCHES, AND CONNECTIONS

Let $X = (A, B, C, \ldots)$ be a (generalized) sequence\(^1\) of jointly distributed random variables, called components of $X$. We will refer to $X$ as a (random) bunch. Let $\mathcal{S}$ be a set of random bunches

$$X = (A_X, A_X', A_X'', \ldots), Y = (B_Y, B_Y', B_Y'', \ldots), Z = (C_Z, C_Z', C_Z'', \ldots), \ldots$$

(of arbitrary cardinalities), with the property that they are pairwise componentwise stochastically unrelated. The term means that no component of one random bunch is jointly distributed with any component of another.

Remark 1. Intuitively, each random bunch corresponds to certain conditions under which (or contexts in which) the components of the bunch are jointly recorded; and the conditions corresponding to different random bunches are mutually exclusive. Jan-Åke Larsson proposed (personal communication, November 2014) to simply call bunches “contexts.”

Any pair $\{A, B\}$ such that $A$ and $B$ are components of two distinct random bunches in $\mathcal{S}$ is called a (simple) connection. A set $\mathcal{C}$ of pairwise disjoint connections is called a simple set of (simple) connections.

---

\(^1\) A sequence is an indexed set, and “generalized” means that the indexing is not necessarily finite or countable. We try to keep the notation simple, omitting technicalities. A sequence of random variables that are jointly distributed is a random variable, if the latter term is understood broadly, as anything with a well-defined probability distribution, to include random vectors, random sets, random processes, etc.
Remark 2. Intuitively, a connection indicates a pair of random variables \( A \) and \( B \) that represent “the same” physical property, because of which, ideally, they should be “one and the same” random variable in different contexts. However, the distributions of \( A \) and \( B \) may be different due to signaling (from other random variables in their contexts) or due to contextual measurement biases. Note that the elements \( A \) and \( B \) of a connection never co-occur, i.e., they possess no joint distribution, and their “identity” therefore can never be verified by observation.

Together, \((S, C)\) form a system (of measurements, or of random bunches). Without loss of generality, we can assume that \( S \) contains no “non-participating” bunches, i.e., each random bunch \( X \) has at least one component \( A \) that belongs to some connection \( \{A, B\} \) in \( C \). In particular, if set \( C \) is finite, then so is set \( S \) (even if the number of components in some of the bunches in \( S \) is not finite).

Example 3 (Klyachko-system). A Klyachko-system \([1]\) consists of five pairs of binary (±1) random variables,

\[
S = \{(V_1, W_2), (V_2, W_3), (V_3, W_4), (V_4, W_5), (V_5, W_1)\}.
\]  

Abstracting away from the physical meaning, the schematic picture below shows five radius-vectors, each corresponding to a distinct physical property represented by a random binary variable. They can only be recorded in pairs \( S \), and each of these pairs corresponds to vertices connected by an edge of the pentagram. In accordance with the Contextuality-by-Default principle, we label each variable both by index \( i \) and each of these pairs corresponds to vertices connected by an edge of the pentagram. In accordance with the Contextuality-by-Default principle, we label each variable both by index \( i \) and the context, defined by which of the two pairs it enters. We use notation \( V_i \) in one of these pairs and \( W_i \) in another. For instance, \( i = 2 \) is used to label \( V_2 \) in the pair \( (V_2, W_3) \) and \( W_2 \) in the pair \( (V_1, W_2) \). With this notation, the simple set of the connections of interest in this system is

\[
C = \{(V_1, W_1), (V_2, W_2), (V_3, W_3), (V_4, W_4), (V_5, W_5)\}.
\]

In the ideal Klyachko-system, each recorded pair, say, \( (V_1, W_2) \), can attain values (+1, −1), (−1, +1), (−1, −1), but not (+1, +1). In our analysis, however, we allow for experimental errors, so that the “pure” Klyachko-system is a special case of a more general system in which \( \Pr[V_1 = +1, W_2 = +1] \) may be non-zero. In the ideal Klyachko-system the probabilities are computed in accordance with the principles of quantum mechanics, so that the distribution of \( (V_i, W_j) \) in \( S \) is determined by the angle between the radius vectors \( i \) and \( j \), and the distributions \( V_i \) is always the same as the distribution of \( W_i \) (\( i = 1, \ldots, 5 \)). In our analysis, however, we allow for “signaling” between the detectors and/or for “contextual measurement biases,” so that, e.g., \( V_1 \) in \( (V_1, W_2) \) and \( W_1 \) in \( (V_5, W_1) \) may have different distributions.

Remark 4. There is no “traditional” contextual notation for the Klyachko system, but one can think of a variety of alternatives to our \( V - W \) scheme, e.g., denoting the \( i \)th measurement in the context of being conjoint with the \( j \)th measurement by \( A_j^i \).

Example 5 (Bell-systems). A Bell-system \([2, 6]\) consists of four pairs of binary (±1) random variables,

\[
S = \{(V_1, W_2), (V_2, W_3), (V_3, W_4), (V_4, W_1)\}.
\]  

Again we abstract away from the physical meaning, involving spins of entangled particles. In the schematic picture below each direction (1 or 3 in one particle and 2 or 4 in another) corresponds to a binary random variable. They can be recorded in pairs \{1, 3\} \times \{2, 4\}, so each random variable participates in two contexts, and is denoted either \( V_i \) or \( W_i \) (\( i \in \{1, 2, 3, 4\} \)) accordingly. The simple set of connections of interest is

\[
C = \{(V_1, W_1), (V_2, W_2), (V_3, W_3), (V_4, W_4)\}.
\]
Here, \((V_i, W_j)\) may attain all four possible values \((\pm 1, \pm 1)\), but in the ideal system with space-like separation between the recordings of \(V_i\) and \(W_j\), the distribution of \(V_i\) is always the same as that of \(W_i\) \((i = 1, \ldots, 4)\). However, we allow for the possibility that the measurements are time-like separated (so that direct signaling is possible), as well as for the possibility that the results of the two measurements are recorded by someone who may occasionally make errors and be contextually biased. Thus, one may erroneously assign \(+1\) to, say, \(V_1\) more often than to \(W_1\).

Remark 6. The contextual notation for the Bell-systems adopted in our previous papers \([10–17]\) is \((A_{ij}, B_{ij})\), \(i, j \in \{1, 2\}\), where \(A\) and \(B\) refer to measurements on the first and second particles, respectively. The first index refers to one of the two \(A\)-measurements (1 or 2), the second index refers to one of the two \(B\)-measurements (1 or 2). So the non-contextual (misleading) notation for \((A_{ij}, B_{ij})\) would be \((A_i, B_j)\). In relation to our present notation, \(A_{11}\) corresponds to \(V_1\), and \(A_{12}\) (the same property in another context) to \(W_1\); \(A_{21}\) corresponds to \(W_3\), and \(A_{22}\) (the same property in another context) to \(V_3\); and analogously for \(B_{ij}\).

Example 7 (Leggett-Garg-system). A Leggett-Garg (LG)-system \([7, 8]\) consists of three pairs of binary \((\pm 1)\) random variables,

\[\mathcal{S} = \{(V_1, W_2), (V_2, W_3), (V_3, W_1)\}.\]

The three random variables are recorded in pairs, the logic of the notation being otherwise the same as above. The simple set of connections of interest in this system is

\[\mathcal{C} = \{(V_1, W_1), (V_2, W_2), (V_3, W_3)\}.\]

In the Leggett-Garg paradigm proper \([7]\), the three measurements are made at three moments of time, fixed with respect to some zero point, as shown in the schematic picture below. This is, however, only one possible physical meaning, and we can think of any three identifiable measurements performed two at a time.

Again, we allow for the possibility of signaling (which is predicted by the laws of quantum mechanics in some cases, e.g., for pure initial states, as shown in Ref. \([22]\)), i.e., earlier measurements may influence later ones. And, again, we allow for measurement errors and contextual biases: knowing, e.g., that \(W_2\) is preceded by \(V_1\) and is not followed by another measurement, and that \(V_2\) is followed by \(W_2\) and is not preceded by another measurement, may lead one to record \(V_2\) and \(W_2\) differently even if they are identically distributed “in reality.”

Remark 8. The contextual notation for the LG-systems adopted in Ref. \([15, 17]\) is \((Q_{ij}, Q_{ji})\), \(i, j \in \{1, 2, 3\}\) \((i < j)\), where the first index refers to the earlier of the two measurements. Thus, \(Q_{12}\) corresponds to \(V_1\) in our present notation, and \(Q_{13}\) (the same property in another context) to \(W_1\); \(Q_{23}\) corresponds to \(V_2\), and \(A_{21}\) (the same property in another context) to \(W_2\); and analogously for \(Q_{31}\) and \(Q_{32}\) (resp., \(V_3\) and \(W_3\)). In Ref. \([22]\) the notation used is \(Q_i^{(i;j)}\), where the superscript indicates the context and the subscript the physical property.

3. CONTEXTUALITY

This section contains our main definitions: of a maximal connection, of (in)consistent connectedness, and of (non)contextuality.
3.1. Couplings

Definition 9. A coupling of a set of random variables $X,Y,Z,...$ is a random bunch $(X^*,Y^*,Z^*,...)$ (with jointly distributed components), such that

$$X^* \sim X, Y^* \sim Y, Z^* \sim Z, \ldots,$$  \hfill (14)

where $\sim$ stands for “has the same distribution as.” In particular, a coupling $S$ for $\mathcal{S}$ is a random bunch coupling all elements (random bunches) of $\mathcal{S}$.

Example 10. For two binary ($\pm 1$) random variables $A,B$, any random bunch $(A^*,B^*)$ with the distribution

$$r_{11} = \Pr [A^* = +1, B^* = +1],$$
$$r_{10} = \Pr [A^* = +1, B^* = -1],$$
$$r_{01} = \Pr [A^* = -1, B^* = +1],$$
$$r_{00} = \Pr [A^* = -1, B^* = -1],$$

such that

$$r_{11} + r_{10} = \Pr [A = +1],$$
$$r_{11} + r_{01} = \Pr [B = +1],$$

is a coupling.

Remark 11. It is a simple but fundamental theorem of Kolmogorov’s probability theory [6, 8, 20, 21] that a coupling $(X^*,Y^*,Z^*,\ldots)$ of $X,Y,Z,\ldots$ exists if and only if there is a random variable $R$ and a sequence of measurable functions $(f_X,f_Y,f_Z,\ldots)$, such that

$$X \sim f_X (R), Y \sim f_Y (R), Z \sim f_Z (R), \ldots$$  \hfill (17)

In quantum mechanics, $R$ is referred to as a hidden variable. (In John Bell’s pioneering work [2], he considers the question of whether such a representation exists for four binary random variables $A_1,A_2,B_1,B_2$ with known distributions of $(A_i,B_j)$, $i,j \in \{1,2\}$. He imposes no constraints on $R$, but it is easy to see that the existence of some $R$ in his problem is equivalent to the existence of an $R$ with just 16-values.)

3.2. Maximally Coupled Connections and Consistent Connectedness

Definition 12. A coupling $(A^*,B^*)$ of a connection $\{A,B\} \in \mathcal{C}$ is called maximal if

$$\Pr [A^* = B^*] \geq \Pr [A^{**} = B^{**}]$$  \hfill (18)

for any coupling $(A^*,B^*)$ of $\{A,B\}$.

Definition 13. A system $(\mathcal{S},\mathcal{C})$ is consistently connected (CC) if $A \sim B$ in any connection $\{A,B\} \in \mathcal{C}$. Otherwise the system $(\mathcal{S},\mathcal{C})$ is inconsistently connected (not CC).

Remark 14. In physics, the CC condition is sometimes referred to as “no-signaling” [23, 25], the term we are going to avoid because then non-CC systems should be referred to as “signaling,” and the latter term has strong connotations making its use in our technical meaning objectionable to physicists. Inconsistent connectedness may be due to signaling in the narrow physical meaning, but it may also indicate measurement biases due to context (so that one measures $A$ differently when one also measures $B$ than when one also measures $C$). We make no distinction between “ideal” random variables and those measured “incorrectly.”

Lemma 15. In a CC system, a maximal coupling $(A^*,B^*)$ for any connection $\{A,B\} \in \mathcal{C}$ exists, and in this coupling

$$\Pr [A^* = B^*] = 1.$$  \hfill (17)

If the system is not CC, and $A \not\sim B$ in a connection $\{A,B\}$, then in a maximal coupling $(A^*,B^*)$, if it exists, $\Pr [A^* = B^*] < 1$.

Proof. Obvious. \hfill $\Box$

We focus now on binary systems, in which all components of the random bunches are binary $(\pm 1)$ variables. The three systems mentioned in the opening section, Klyachko, LG, and Bell-type ones, are binary. (A generalization to components with finite but arbitrary numbers of values is straightforward.)
Lemma 16. For a connection \( \{A, B\} \) with binary \((\pm 1)\) \(A, B\) and

\[
p = \Pr [A = 1] \geq \Pr [B = 1] = q, \tag{19}
\]

a maximal coupling \((A^*, B^*)\) exists, and its distribution is

\[
r_{11} = q
\]

\[
r_{10} = p - q
\]

\[
r_{01} = 0
\]

\[
r_{00} = 1 - p
\]

where

\[
r_{ab} = \Pr [A^* = 2a - 1, B^* = 2b - 1]. \tag{21}
\]

Proof. For given values \(p \geq q\), the maximum possible value of \(r_{11}\) is \(\min(p, q) = q\), and the maximum possible value of \(r_{00}\) is \(\min(1 - p, 1 - q) = 1 - p\); these values are attained in distribution \((20)\), with \(r_{01}, r_{10}\) determined uniquely.

Remark 17. In a maximal coupling \((A^*, B^*)\) of two binary random variables \(A, B\) the expectation

\[
\langle A^* B^* \rangle = 2(r_{11} + r_{00}) - 1 \tag{22}
\]

attains its maximum possible value

\[
\langle A^* B^* \rangle = 1 - 2(p - q). \tag{23}
\]

Remark 18. In some cases it is more convenient to speak of the minimum value of \(\Pr [A^* \neq B^*] = r_{10} + r_{01}\) rather than the maximum value of \(\Pr [A^* = B^*] = r_{11} + r_{00}\). This minimum can be presented as

\[
\Pr [A^* \neq B^*] = p - q = \frac{1}{2} (\langle A \rangle - \langle B \rangle). \tag{24}
\]

Remark 19. In a maximal coupling \((A^*, B^*)\) of two binary random variables \(A, B\),

\[
\Pr [A^* = B^*] = 1 \tag{25}
\]

if and only if \(A \sim B\), i.e.,

\[
p = \Pr [A = 1] = \Pr [B = 1] = q. \tag{26}
\]

3.3. Contextuality

Definition 20. Let maximal couplings exist for all connections in \(\mathcal{C}\). A system \((\mathcal{S}, \mathcal{C})\) is noncontextual if there exists a coupling \(S\) for \(\mathcal{S}\) in which all 2-marginals \((A^*, B^*)\) that couple the connections in \(\mathcal{C}\) are maximal couplings. If such a coupling \(S\) does not exist, the system is contextual.

Remark 21. In particular, if the system is CC, it is noncontextual if and only if there is a coupling \(S\) for \(\mathcal{S}\) in which \(\Pr [A^* = B^*] = 1\) for all connections \(\{A, B\} \in \mathcal{C}\). This comes very close to the traditional use of the term.

Example 22. Let system \((\mathcal{S}, \mathcal{C})\) consist of random bunches \((A, C), (B, D)\),

with all components binary \((\pm 1)\), and a single connection \(\{A, B\}\). To determine if the system is contextual, we consider all possible couplings for \(A, B, C, D\), i.e., all possible random bunches \((A^*, B^*, C^*, D^*)\)
such that \((A^*, C^*) \sim (A, C)\) and \((B^*, D^*) \sim (B, D)\). Denoting, for \(a, b, c, d \in \{0, 1\}\),

\[
\begin{align*}
  s_{ac} &= \Pr[A = 2a - 1, C = 2c - 1], \\
  t_{bd} &= \Pr[B = 2b - 1, D = 2d - 1], \\
  u_{abcd} &= \Pr[A^* = 2a - 1, B^* = 2b - 1, C^* = 2c - 1, D^* = 2d - 1],
\end{align*}
\]

the distributional equations \((A^*, C^*) \sim (A, C)\) and \((B^*, D^*) \sim (B, D)\) translate into the following 8 equations for 16 probabilities \(u_{abcd}\):

\[
\begin{align*}
  \sum_{b=0}^{1} \sum_{d=0}^{1} u_{abcd} &= s_{ac}, \quad a, c \in \{0, 1\}, \\
  \sum_{a=0}^{1} \sum_{c=0}^{1} u_{abcd} &= t_{bd}, \quad b, d \in \{0, 1\}.
\end{align*}
\]

The requirement that the coupling for \(\{A, B\}\) be maximal translates into the additional four equations

\[
\sum_{c=0}^{1} \sum_{d=0}^{1} u_{abcd} = r_{ab} = \Pr[A^* = 2a - 1, B^* = 2b - 1],
\]

where, in view of Lemma (16), \(r_{ab}\) is given by (20), with the same meaning of \(p, q\) and the same convention \(p = \Pr[A = 1] \geq \Pr[B = 1] = q\). The problem of contextuality therefore reduces to one of determining whether the system of 12 equations (28)-(29)-(30) for the 16 unknown \(u_{abcd} \geq 0\) has a solution. The answer in this case can be shown to affirmative, so the system considered is noncontextual.

Definition 20 is sufficient for all subsequent considerations in this paper, but we note that it can be extended to situations when maximal couplings for connections do not necessarily exist. Let us associate with each connection for \(A, B\) a supremal number

\[
p_{AB} = \sup_{\text{all couplings } (A^*, B^*)} \Pr[A^* = B^*].
\]

**Definition 23 (extended).** A system \((\mathcal{S}, \mathcal{C})\) is noncontextual (contextual) if there exists (resp., does not exist) a sequence of couplings \(S_1, S_2, \ldots\) for \(\mathcal{S}\) in which \(\Pr[A^* = B^*]\) for all connections \(\{A, B\}\) in \(\mathcal{C}\) uniformly converge to the corresponding supremal numbers \(p_{AB}\).

### 3.4. Measure of Contextuality for Binary Systems with Finite Simple Sets of Connections

We will assume that in the binary systems we are dealing with the simple set of connections \(\mathcal{C}\) is finite:

\[
\mathcal{C} = \{\{A_i, B_i\} : i \in \{1, \ldots, n\}\}.
\]

**Lemma 24.** Given a finite simple set of connections \(\{\{A_i, B_i\} : i \in \{1, \ldots, n\}\}\) in a binary system, the respective couplings in the set \(\{(\hat{A}_i^*, \hat{B}_i^*) : i \in \{1, \ldots, n\}\}\) are all maximal if and only if

\[
\sum_{i=1}^{n} \Pr[A_i^* \neq B_i^*] = \frac{1}{2} \sum_{i=1}^{n} |\langle A_i^* \rangle - \langle B_i^* \rangle|.
\]

**Proof.** Immediately follows from Lemma (16) and Remark (18). \(\square\)

**Notation.** We denote

\[
\Delta_0 (\mathcal{C}) = \frac{1}{2} \sum_{i=1}^{n} |\langle A_i^* \rangle - \langle B_i^* \rangle|,
\]

and this quantity is to play a central role in the subsequent computations.
**Definition 25.** Let $\Delta_{\min}(\mathcal{S}, \mathcal{C})$ for a system with $\mathcal{C} = \{\{A_i, B_i\} : i \in \{1, \ldots, n\}\}$ be the infimum for

$$\sum_{i=1}^{n} \Pr[A_i^* \neq B_i^*]$$

across all possible couplings $S$ for $\mathcal{S}$.

This is another quantity to play a central role in subsequent computations.

**Theorem 26.** For a binary system with a finite simple set of connections, the value $\Delta_{\min}(\mathcal{S}, \mathcal{C})$ is achieved in some coupling $S$, and

$$\Delta_{\min}(\mathcal{S}, \mathcal{C}) \geq \Delta_0(\mathcal{C}). \quad (34)$$

The system is noncontextual if and only if

$$\Delta_{\min}(\mathcal{S}, \mathcal{C}) = \Delta_0(\mathcal{C}). \quad (35)$$

**Proof.** That $\Delta_{\min}(\mathcal{S}, \mathcal{C})$ is an achievable minimum follows from the fact that any coupling $S$ is described by a system of linear inequalities relating to each other $\Pr[A^* = a, B^* = b, C^* = c, \ldots]$ for all possible values $(a, b, c, \ldots)$ of all the random variables involved (the union of the components of all random bunches in $\mathcal{S}$). $\Delta_{\min}(\mathcal{S}, \mathcal{C})$ being a linear combinations of these probabilities, its infimum has to be a minimum. (34) and (35) are obvious.

This theorem allows one to construct a convenient definition for the degree of contextuality in binary systems with finite number of connections.

**Definition 27.** In a binary system $(\mathcal{S}, \mathcal{C})$ with a finite simple set of connections the **degree of contextuality** is

$$CNTX(\mathcal{S}, \mathcal{C}) = \Delta_{\min}(\mathcal{S}, \mathcal{C}) - \Delta_0(\mathcal{C}) \geq 0. \quad (36)$$

In the subsequent sections of this paper we show how this definition of contextuality applies to Klyachko, LG, and Bell-systems.

**Remark 28.** Clearly, a measure of contextuality could also be constructed as $(1 + \Delta_{\min}(\mathcal{S}, \mathcal{C})) / (1 + \Delta_0(\mathcal{C})) - 1$, $(\Delta_{\min}(\mathcal{S}, \mathcal{C}) - \Delta_0(\mathcal{C})) / (\Delta_{\min}(\mathcal{S}, \mathcal{C}) + \Delta_0(\mathcal{C}))$, and in a variety of other ways. The only logically necessary aspect of the definition is that $CNTX(\mathcal{S}, \mathcal{C})$ is zero when $\Delta_{\min}(\mathcal{S}, \mathcal{C}) = \Delta_0(\mathcal{C})$ and positive otherwise. The simple difference is chosen because it has been shown to have certain desirable properties [15], but this choice is not critical for the present paper.

### 3.5. Conventions

#### 3.5.1. Abuse of language

To simplify notation we adopt the following convention: in a coupling $(X^*, Y^*)$ of two random variable $X,Y$ we drop the asterisks and write simply $(X, Y)$. In other words, we take two random variables $X,Y$ and form a new random variable $Z = (X, Y)$ as if $X,Y$ in $Z$ were the same random variables as $X,Y$ separately taken. Of course, they are not, as $X,Y$ are stochastically unrelated, whereas $(X^*, Y^*)$ are jointly distributed by definition. The abuse of language thus introduced is common, if not universally accepted in quantum physics, and we too conveniently resorted to it when discussing Specker’s magic boxes in our introductory section.

#### 3.5.2. Functions $s_{\text{even}}$ and $s_{\text{odd}}$

We will make use of the following notation. For any finite sequence of real numbers $(a_i : i \in \{1, \ldots, n\})$ we denote by

$$s_{\text{even}}(a_i : i \in \{1, \ldots, n\}) = s_{\text{even}}(a_1, \ldots, a_n) = \max_{\text{even number of } \pm} \sum_{i=1}^{n} (\pm a_i), \quad (37)$$
where each ± should be replaced with + or −, and the maximum is taken over all combinations of the signs containing an even number of −’s.

Analogously,

\[ s_{\text{odd}}(a_1 : i \in \{1, \ldots, n\}) = s_{\text{odd}}(a_1, \ldots, a_n) = \max_{\text{odd number of } -'s} \sum_{i=1}^{n} (\pm a_i). \quad (38) \]

4. KLYACHKO-SYSTEMS

4.1. Main Theorem

**Theorem 29** (contextuality measure and criterion for Klyachko-systems). In a Klyachko-system \((\mathcal{G}, \mathcal{C})\), with

\[ \mathcal{G} = \{(V_i, W_{i\oplus 5}) : i \in \{1, \ldots, 5\}\}, \mathcal{C} = \{(V_i, W_i) : i \in \{1, \ldots, 5\}\}, \quad (39) \]

where \(\oplus_5\) stands for addition modulo 5,

\[ \Delta_0(\mathcal{C}) = \frac{1}{2} \sum_{i=1}^{5} |\langle V_i \rangle - \langle W_i \rangle|, \quad (40) \]

\[ \Delta_{\text{min}}(\mathcal{G}, \mathcal{C}) = \frac{1}{2} \max(\Delta_0(\mathcal{C}), s_{\text{odd}}(\langle V_i W_{i\oplus 5} \rangle : i \in \{1, \ldots, 5\}) - 3). \quad (41) \]

Consequently, the degree of contextuality in the Klyachko-system is

\[ \text{CNTX}(\mathcal{G}, \mathcal{C}) = \frac{1}{2} \max \left(0, s_{\text{odd}}(\langle V_i W_{i\oplus 5} \rangle : i \in \{1, \ldots, 5\}) - 3 - \sum_{i=1}^{5} |\langle V_i \rangle - \langle W_i \rangle| \right), \quad (42) \]

and the system is noncontextual if and only if

\[ s_{\text{odd}}(\langle V_i W_{i\oplus 5} \rangle : i \in \{1, \ldots, 5\}) \leq 3 + \sum_{i=1}^{5} |\langle V_i \rangle - \langle W_i \rangle|. \quad (43) \]

**Proof.** The computer-assisted proof is based on Lemma 30 below, and its details, omitted here, are analogous to those in the proofs of Theorems 3-6 in Appendix of Ref. 17. \(\square\)

**Lemma 30.** The necessary and sufficient condition for the connection couplings \(\{(V_i, W_i) : i \in \{1, \ldots, 5\}\}\) to be compatible with the observed pairs \(\{(V_i, W_{i\oplus 5}) : i \in \{1, \ldots, 5\}\}\) is

\[ s_{\text{odd}}(\langle V_i W_{i\oplus 5} \rangle, \langle V_i W_i \rangle : i \in \{1, \ldots, 5\}) \leq 8, \quad (44) \]

which can be equivalently written as

\[ s_{\text{odd}}(\langle V_i W_{i\oplus 4} \rangle : i \in \{1, \ldots, 5\}) + s_{\text{even}}(\langle V_i W_i \rangle : i \in \{1, \ldots, 5\}) \leq 8, \]

\[ s_{\text{even}}(\langle V_i W_{i\oplus 4} \rangle : i \in \{1, \ldots, 5\}) + s_{\text{odd}}(\langle V_i W_i \rangle : i \in \{1, \ldots, 5\}) \leq 8. \]

**Remark 31.** The compatibility in the formulation of the lemma means the existence of a coupling \(S\) for \(\mathcal{G}\) with given marginals \(\{(V_i, W_i) : i \in \{1, \ldots, 5\}\}\) and given (coupled) connections \(\{(V_i, W_{i\oplus 5}) : i \in \{1, \ldots, 5\}\}\).

4.2. Special cases

In a CC Klyachko-system, \(\Delta_0(\mathcal{C}) = 0\), the criterion for noncontextuality acquires the form

\[ s_{\text{odd}}(\langle V_i W_{i\oplus 5} \rangle : i \in \{1, \ldots, 5\}) \leq 3. \quad (45) \]
If, in addition, the Klyachko-exclusion is satisfied, i.e., in every \( \langle V_i \oplus_5 W_i \rangle \),

\[
\Pr[V_i = 1, W_i = 1] = 0,
\]

then we have

\[
\langle V_i W_i \rangle = 1 - 2(p_i + p_i \oplus_5 1),
\]

where

\[
p_i = \Pr[V_i = 1] = \Pr[W_i = 1], \quad i \in \{1, \ldots, 5\}.
\]

Since

\[
-\sum_{i=1}^{5} \langle V_i W_i \rangle \leq s_{\text{odd}} (\langle V_i W_i \rangle : i \in \{1, \ldots, 5\}) \leq 3,
\]

and since

\[
-\sum_{i=1}^{5} \langle V_i W_i \rangle = 2 \sum_{i=1}^{5} (p_i + p_i \oplus_5 1) - 5 = 4 \sum_{i=1}^{5} p_i - 5,
\]

we get

\[
\sum_{i=1}^{5} p_i \leq 2.
\]

This is the Klyachko inequality, which has been derived in Ref. [1] as a necessary condition for noncontextuality. As it turns out (we omit the simple proof), this condition is also necessary.

**Theorem 32.** In a CC Klyachko-system with Klyachko exclusion, (45) is equivalent to (51).

5. **BELL-SYSTEMS**

5.1. **Main Theorem**

**Theorem 33** (contextuality measure and criterion for Bell-systems). In a Bell-system \((\mathcal{S}, \mathcal{C})\), with

\[
\mathcal{S} = \{(V_i, W_i) : i \in \{1, \ldots, 4\}\}, \quad \mathcal{C} = \{(V_i, W_i) : i \in \{1, \ldots, 4\}\},
\]

where \(\oplus_4\) stands for addition modulo 4,

\[
\Delta_0 (\mathcal{C}) = \frac{1}{2} \sum_{i=1}^{4} |\langle V_i \rangle - \langle W_i \rangle|,
\]

and

\[
\Delta_{\text{min}} (\mathcal{S}, \mathcal{C}) = \frac{1}{2} \max (\Delta_0 (\mathcal{C}), s_{\text{odd}} (\langle V_i W_i \rangle : i \in \{1, \ldots, 4\}) - 2).
\]

Consequently, the degree of contextuality in the Bell-system is

\[
\text{CTX} (\mathcal{S}, \mathcal{C}) = \frac{1}{2} \max \left(0, s_{\text{odd}} (\langle V_i W_i \rangle : i \in \{1, \ldots, 4\}) - 2 - \sum_{i=1}^{4} |\langle V_i \rangle - \langle W_i \rangle|\right),
\]

---

2 This means that the directions in the 3D real Hilbert space are chosen strictly in accordance with [1], with no experimental errors, signaling, or contextual biases involved. In this case, the values \((+1, +1)\) for paired measurements \((V_i, W_i)\) are excluded by the quantum theory, to this is what we refer to as "Klyachko exclusion."
and the system is noncontextual if and only if
\[ s_{\text{odd}} (\langle V_i W_{i+1} \rangle : i \in \{1, \ldots, 4\}) \leq 2 + \sum_{i=1}^{4} |\langle V_i \rangle - \langle W_i \rangle|. \] (56)

Proof. The computer-assisted proof is based on Lemma 34 below, and its details, omitted here, can be found in Ref. [17], Appendix and Theorems 3 and 5.

Lemma 34. The necessary and sufficient condition for the connections \{\langle V_i, W_i \rangle : i \in \{1, \ldots, 4\}\} to be compatible with the observed pairs \{(V_1, W_3), (V_1, V_4), (V_2, W_4), (W_2, V_3)\} is
\[ s_{\text{odd}} (\langle V_i W_{i+1} \rangle, \langle V_i W_i \rangle : i \in \{1, \ldots, 4\}) \leq 6, \] (57)
which can be equivalently written as
\[ s_{\text{odd}} (\langle V_i W_{i+1} \rangle : i \in \{1, \ldots, 4\}) + s_{\text{even}} (\langle V_i W_i \rangle : i \in \{1, \ldots, 4\}) \leq 6, \] (58)

5.2. Special case

In a CC Bell-system, \(\Delta_0 (\mathcal{C}) = 0\), the criterion for noncontextuality acquires the form
\[ s_{\text{odd}} (\langle V_i W_{i+1} \rangle : i \in \{1, \ldots, 4\}) \leq 2, \] (59)
which is the standard CHSH inequalities [4–6], presentable in a more familiar way as
\[ -2 \leq \langle V_1 W_2 \rangle + \langle V_2 W_3 \rangle + \langle V_3 W_4 \rangle - \langle V_4 W_1 \rangle \leq 2, \]
\[ -2 \leq \langle V_1 W_2 \rangle + \langle V_2 W_3 \rangle - \langle V_3 W_4 \rangle + \langle V_4 W_1 \rangle \leq 2, \]
\[ -2 \leq \langle V_1 W_2 \rangle - \langle V_2 W_3 \rangle + \langle V_3 W_4 \rangle + \langle V_4 W_1 \rangle \leq 2, \]
\[ -2 \leq - \langle V_1 W_2 \rangle + \langle V_2 W_3 \rangle + \langle V_3 W_4 \rangle + \langle V_4 W_1 \rangle \leq 2. \] (60)

6. LG-SYSTEMS

6.1. Main Theorem

Theorem 35 (contextuality measure and criterion for LG-systems). In an LG-system \((\mathcal{G}, \mathcal{C})\), with
\[ \mathcal{G} = \{(V_i, W_{i+1}) : i \in \{1, 2, 3\}\}, \mathcal{C} = \{(V_i, W_i) : i \in \{1, 2, 3\}\}, \] (61)
where \(\oplus_3\) stands for addition modulo 3,
\[ \Delta_0 (\mathcal{C}) = \frac{1}{2} \sum_{i=1}^{3} |\langle V_i \rangle - \langle W_i \rangle|, \] (62)
\[ \Delta_{\text{min}} (\mathcal{G}, \mathcal{C}) = \frac{1}{2} \max (\Delta_0 (\mathcal{C}), s_{\text{odd}} (\langle V_i W_{i+1} \rangle : i \in \{1, 2, 3\}) - 1). \] (63)
Consequently, the degree of contextuality in the LG-system is
\[ \text{CNTX} (\mathcal{G}, \mathcal{C}) = \frac{1}{2} \max \left( 0, s_{\text{odd}} (\langle V_i W_{i+1} \rangle : i \in \{1, 2, 3\}) - 1 - \sum_{i=1}^{3} |\langle V_i \rangle - \langle W_i \rangle| \right), \] (64)
and the system is noncontextual if and only if
\[ s_{\text{odd}} (\langle V_i W_{i+1} \rangle : i \in \{1, 2, 3\}) \leq 1 + \sum_{i=1}^{3} |\langle V_i \rangle - \langle W_i \rangle|. \] (65)
Lemma 36. The necessary and sufficient condition for the connections \{(V_i, W_i) : i \in \{1, 2, 3\}\} to be compatible with the observed pairs \{(V_i, W_{i\oplus 1}) : i \in \{1, 2, 3\}\} is
\[ s_{odd} (\{V_i W_{i\oplus 1}, V_i W_i \mid i \in \{1, 2, 3\}\} \leq 4, \tag{66} \]
which can be equivalently written as
\[ s_{odd} (\{V_i W_{i\oplus 1} \mid i \in \{1, 2, 3\}\}) + s_{even} (\{V_i W_i \mid i \in \{1, 2, 3\}\} \leq 4, \tag{67} \]
\[ s_{even} (\{V_i W_{i\oplus 1} \mid i \in \{1, 2, 3\}\}) + s_{odd} (\{V_i W_i \mid i \in \{1, 2, 3\}\} \leq 4. \]

6.2. Special cases

If the LG-system is a CC-system, \(\Delta_0 (\mathcal{C}) = 0\), the criterion for noncontextuality acquires the form
\[ s_{odd} (\{V_i W_{i\oplus 1} \mid i \in \{1, 2, 3\}\} \leq 1. \tag{68} \]
This can be written in the more familiar (Suppes-Zanotti’s form \[8\] as
\[ -1 \leq \langle V_2 W_2 \rangle + \langle V_3 W_3 \rangle + \langle V_3 W_2 \rangle \leq 1 + 2 \max (\langle V_1 W_2 \rangle, \langle V_2 W_3 \rangle, \langle V_3 W_1 \rangle). \tag{69} \]

Remark 37. In the temporal version of LG-systems (the Leggett-Garg paradigm proper), \(V_1\) and \(W_1\) are the results of the first measurement in both \((V_1, W_2)\) and \((V_3, W_1)\). They therefore cannot be influenced by later measurements (no signaling back in time). Consequently, if there are no contextual measurement biases, \(V_1 \sim W_1\), and
\[ \Delta_0 (\mathcal{C}) = \frac{1}{2} \sum_{i=2}^{3} |\langle V_i \rangle - \langle W_i \rangle|. \tag{70} \]
Including \(|\langle V_1 \rangle - \langle W_1 \rangle|\), however, does not hurt, and it allows one to accommodate cases with non-temporal measurements and biased measurements (when knowing whether variable 1 will be paired with 2 or with 3 changes the way one measures 1).

7. COMPARING THE SYSTEMS

The main theorems regarding our three systems, Theorems 29, 33, and 35, strongly suggest the following generalization, which we formulate as a conjecture.

Remark 38. As of mid-November 2014, we have a partial proof of this conjecture and of the supporting lemma that follows.

Conjecture 39. Let \((\mathcal{G}, \mathcal{C})\) be a system with
\[ \mathcal{G} = \{(V_i, W_{\pi(i)}) \mid i \in \{1, \ldots, n\}\}, \tag{71} \]
where \(\pi(\{1, \ldots, n\})\) is a complete permutation of \(\{1, \ldots, n\}\),\(^3\) and
\[ \mathcal{C} = \{(V_i, W_i) \mid i \in \{1, \ldots, n\}\}. \tag{72} \]
Then
\[ \Delta_0 (\mathcal{C}) = \frac{1}{2} \sum_{i=1}^{n} |\langle V_i \rangle - \langle W_i \rangle|, \tag{73} \]

\(^3\) A complete permutation is one without fixed points, i.e., \(\pi(i) \neq i, i = 1, \ldots, n\).
and
\[
\Delta_{\min}(\mathcal{E}, \mathcal{C}) = \frac{1}{2} \max \left( \Delta_0(\mathcal{E}), s_{\text{odd}}\left(\langle V_iW_{\pi(i)} : i \in \{1, \ldots, n\}\rangle - n + 2\right)\right).
\]
(74)

Consequently, the degree of contextuality in this system is
\[
\text{CNTX}(\mathcal{E}, \mathcal{C}) = \frac{1}{2} \max \left(0, s_{\text{odd}}\left(\langle V_iW_{\pi(i)} : i \in \{1, \ldots, n\}\rangle - n + 2 - \sum_{i=1}^{n} |(V_i) - (W_i)|\right)\right),
\]
(75)

and the system is noncontextual if and only if
\[
s_{\text{odd}}\left(\langle V_iW_{\pi(i)} : i \in \{1, \ldots, n\}\rangle\right) \leq n - 2 + \sum_{i=1}^{n} |(V_i) - (W_i)|.
\]
(76)

The corresponding generalization of the supporting lemmas is

**Conjecture 40.** The necessary and sufficient condition for the connections \(\{\langle V_i, W_i \rangle : i \in \{1, \ldots, n\}\}\) to be compatible with the observed pairs \(\{\langle V_i, W_{\pi(i)} \rangle : i \in \{1, \ldots, n\}\}\) is

\[
s_{\text{odd}}\left(\langle V_iW_{\pi(i)} \rangle, \langle V_iW_i \rangle : i \in \{1, \ldots, n\}\right) \leq 2n - 2,
\]
(77)

which can be equivalently written as

\[
s_{\text{odd}}\left(\langle V_iW_{\pi(i)} \rangle : i \in \{1, \ldots, n\}\right) + s_{\text{even}}\left(\langle V_iW_i \rangle : i \in \{1, \ldots, n\}\right) \leq 2n - 2,
\]
(78)

It is easy to see that the criterion and measure for the LG-system (Theorem 33) is a special case of those for the Bell-system (Theorem 35) which in turn is a special case of those for the Klyachko system. Specifically, by putting \(\langle V_5W_1 \rangle = 1\) in the Klyachko-system, and assuming in addition that \(W_5 \sim V_5\), so that \(\langle V_5W_5 \rangle = 1\) in the maximal coupling, we can replace \(W_5\) in \(\langle V_4W_5 \rangle\) with \(W_1\) and obtain the Bell-system. By putting then \(\langle V_4W_1 \rangle = 1\) in the Bell-system, and \(W_4 \sim V_4\), so that \(\langle V_4W_4 \rangle = 1\) in the maximal coupling, can replace \(W_4\) in \(\langle V_3W_4 \rangle\) with \(W_1\) and obtain the LG-system.

It is easy to see how this pattern generalizes. First of all, any complete permutation \(\pi\) can be replaced, by appropriate renaming, with an addition modulo \(n\), making the observed pairs \(\langle V_1, W_2 \rangle, \ldots, \langle V_{n-1}, W_n \rangle, \langle V_n, W_1 \rangle\). Let the LG, Bell, and Klyachko systems be designated as systems of order 3, 4, and 5, respectively, and the system in Conjecture 39 as an \((n)\)-system. By putting \(\langle V_nW_1 \rangle = 1\) in the \((n)\)-system, and assuming that \(W_n \sim V_n\), so that \(\langle V_nW_n \rangle = 1\) in the maximal coupling, we replace \(W_n\) in \(\langle V_{n-1}W_n \rangle\) with \(W_1\) and obtain an \((n-1)\)-system.

8. CONCLUSION

We have presented a theory of (non)contextuality in purely probabilistic terms abstracted away from physical meaning. The computational aspects of the theory are confined to finite systems with binary components, but it is easily generalizable to deal with components attaining arbitrary finite numbers of values. As the components of the bunches get more complex and/or connections get longer than pairs, the generalizations become less unique. This issue is not addressed in this paper.

The basis for the theory is the principle of Contextuality-by-Default, which has philosophical and mathematical consequences. Mathematically, it leads to revamping (while remaining within its confines) of the Kolmogorovian probability theory, with more emphasis than usual emphasis on stochastic unrelatedness. A reformulated theory may even avoid the notion of a sample space altogether [20].

Philosophically, it elucidates the difference between “ontic” and “epistemic” aspects of contextuality (perhaps even of probability theory generally). It is clear, among other things, that the choice of connections is not an “ontic” property of a system.

The theory also offers pragmatic advantages: it allows for non-CC-systems (whether due to signaling or due to context-dependent measurement biases) and for experimental/computational errors in the analysis (including statistical analysis) of experimental data.

The theory has predecessors in the literature. The idea of labeling differently random variables in different contexts and considering the probability with which they can be equal to each other if coupled has been prominently used by Larsson [29] and Svozil [27].
Acknowledgments

This work is supported by NSF grant SES-1155956 and AFOSR grant FA9550-14-1-0318. We have benefited from collaboration with Jan-Åke Larsson, J. Acacio de Barros, and Gary Oas, as well as from discussions with Samson Abramsky, Guido Bacciagaluppi, and Andrei Khrennikov. Jan-Åke Larsson, among other things, influenced our choice of terminology related to “signaling” and sequences of jointly distributed random variables. An abridged version of this paper was presented at the Purdue Winer Memorial Lectures in November 2014.

[1] A.A. Klyachko, M.A. Can, S. Binicioglu, and A.S. Shumovsky (2008). A simple test for hidden variables in spin-1 system. Phys. Rev. Lett. 101, 020403.
[2] J. Bell (1964). On the Einstein-Podolsky-Rosen paradox. Physics 1: 195-200.
[3] J. Bell (1966). On the problem of hidden variables in quantum mechanics. Rev. Modern Phys. 38: 447-453.
[4] J.F. Clauser, M.A. Horne, A. Shimony, R.A. Holt (1969). Proposed experiment to test local hidden- variable theories. Phys. Rev. Lett. 23: 880–884.
[5] J.F. Clauser, M.A. Horne (1974). Experimental consequences of objective local theories. Phys. Rev. D 10: 526–535.
[6] A. Fine (1982). Hidden variables, joint probability, and the Bell inequalities. Phys. Rev. Lett. 48: 291–295.
[7] A.J. Leggett & A. Garg (1985). Quantum mechanics versus macroscopic realism: Is the flux there when nobody looks? Phys. Rev. Lett. 54: 857–860.
[8] P. Suppes, M. Zanotti (1981). When are probabilistic explanations possible? Synthese 48: 191–199.
[9] E. Specker (1960). Die Logik Nicht Gleichzeitig Entscheidbarer Aussagen. Dialectica 14: 239–246 (English translation by M.P. Seevinck available as arXiv:1103.4537).
[10] E.N. Dzhafarov, J.V. Kujala (2013). All-possible-couplings approach to measuring probabilistic context. PLoS ONE 8(5): e61712. doi:10.1371/journal.pone.0061712.
[11] E.N. Dzhafarov, J.V. Kujala (2014). No-Forcing and No-Matching theorems for classical probability applied to quantum mechanics. Found. Phys. 44: 248–265.
[12] E.N. Dzhafarov, J.V. Kujala (2014). Embedding quantum into classical: contextualization vs conditionalization. PLoS One 9(3): e92818. doi:10.1371/journal.pone.0092818.
[13] E.N. Dzhafarov, J.V. Kujala (in press). Contextuality is about identity of random variables. Phys. Scripta (available as arXiv:1405.2116).
[14] E.N. Dzhafarov, J.V. Kujala (in press). Random variables recorded under mutually exclusive conditions: Contextuality-by-Default. Advances in Cognitive Neurodynamics IV.
[15] J.A. de Barros, E.N. Dzhafarov, J.V. Kujala, G. Oas (2014). Measuring observable quantum contextuality. arXiv:1406.3088.
[16] E.N. Dzhafarov, J.V. Kujala (2014). Probabilistic contextuality in EPR/Bohm-type systems with signaling allowed. arXiv:1406.0243.
[17] E.N. Dzhafarov, J.V. Kujala (2014). Generalizing Bell-type and Leggett-Garg-type inequalities to systems with signaling. arXiv:1407.2886.
[18] E.N. Dzhafarov, J.V. Kujala (2014). A qualified Kolmogorovian account of probabilistic contextuality. Lecture Notes in Computer Science 8369: 201–212.
[19] E.N. Dzhafarov, J.V. Kujala (2013). Order-distance and other metric-like functions on jointly distributed random variables. Proc. Amer. Math. Soc. 141(9): 3291–3301.
[20] E.N. Dzhafarov, J.V. Kujala (2013). Probability, random variables, and selectivity. arXiv:1312.2239.
[21] E.N. Dzhafarov, J.V. Kujala (2010). The Joint Distribution Criterion and the Distance Tests for selective probabilistic causality. Frontiers in Quantitative Psychology and Measurement. 1:151 doi: 10.3389/fpsyg.2010.00151.
[22] G. Bacciagaluppi (2014). Leggett-Garg inequalities, pilot waves and contextuality. arXiv:1409.4104.
[23] J. Cereceda (2000). Quantum mechanical probabilities and general probabilistic constraints for Einstein–Podolsky–Rosen–Bohm experiments. Found. Phys. Lett. 13: 427–442.
[24] Li. Masanes, A. Acin, & N. Gisin (2006). General properties of nonsignaling theories. Phys. Rev. A 73: 012112.
[25] S. Popescu & D. Rohrlich (1994). Quantum nonlocality as an axiom. Found. Phys. 24: 379–385.
[26] J.-Å. Larsson (2022). A Kochen-Specker inequality. Europhys. Lett., 58: 799–805.
[27] K. Svozil (2012). How much contextuality? Natural Computing 11, 261-265.