DIFFEOMORPHISMS, ISOTopies AND BRAID MONODROMY
FACTORIZATIONS OF PLANE CUSPIDAL CURVES

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Abstract. In this paper we prove that there is an infinite sequence of pairs of plane cuspidal curves \( C_{m,1} \) and \( C_{m,2} \), such that the pairs \((\mathbb{C}P^2, C_{m,1})\) and \((\mathbb{C}P^2, C_{m,2})\) are diffeomorphic, but \( C_{m,1} \) and \( C_{m,2} \) have non-equivalent braid monodromy factorizations. These curves give rise to the negative solutions of "Dif=Def" and "Dif=Iso" problems for plane irreducible cuspidal curves. In our examples, \( C_{m,1} \) and \( C_{m,2} \) are complex conjugated.

Up to our knowledge the following natural questions were open till now: does an existence of a diffeomorphism \( \mathbb{C}P^2 \to \mathbb{C}P^2 \) transforming an algebraic curve \( C_1 \) in an algebraic curve \( C_2 \) imply the existence of an equisingular deformation ("Dif=Def") or of a diffeotopy ("Dif=Iso") between them? We show that the response to the both questions is in the negative. In our examples the curves are cuspidal and irreducible (note that for nodal curves the responses are in the positive, see [H]).

Let \( C \subset \mathbb{C}P^2 \) be a plane algebraic curve of degree \( \deg C = d \). The embedding \( C \subset \mathbb{C}P^2 \) is determined, up to diffeotopy, by the so-called braid monodromy of \( C \) (see [KT]). The latter depends on a choice of generic homogeneous coordinates in \( \mathbb{C}P^2 \) and can be seen as a factorization of the full twist \( \Delta_2^d \) in the normal semi-group \( B_d^+ \) of quasi-positive braids on \( d \) strings (\( \Delta_2^d \) is the so-called Garside element; \( \Delta_2^d = (X_1 \cdots X_{d-1})^d \) in standard generators \( X_1, \ldots, X_{d-1} \) of \( B_d^+ \)). If the only singularities of \( C \) are ordinary cusps and nodes (cuspidal curve), then this factorization can be written as follows

\[
\Delta_2^d = \prod_i \rho_i^{-1} \sigma_i^s \rho_i, \quad s_i \in (1,2,3), \rho_i \in B_d,
\]

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where \( s_i = 1, 2, 3 \) correspond to branch points, nodes and cusps respectively, and \( B_d \) is the braid group.

Let

\[
h = g_1 \cdot \ldots \cdot g_r
\]

be such a factorization in \( B_d^+ \). The transformations which replace in (2) two neighboring factors \( g_i \cdot g_{i+1} \) by \( (g_i g_{i+1} g_i^{-1}) \cdot g_i \) or \( g_{i+1} (g_i^{-1} g_i g_{i+1}) \) are called Hurwitz moves.

For \( z \in B_d \), we denote by

\[
h_z = z^{-1} g_1 z \cdot z^{-1} g_2 z \cdot \ldots \cdot z^{-1} g_r z
\]

and say that \( h_z \) is obtained from (2) by simultaneous conjugation by \( z \). Two factorizations are called Hurwitz and conjugation equivalent if one can be obtained from the other by a finite sequence of Hurwitz moves followed by a simultaneous conjugation. If two factorizations are Hurwitz and conjugation equivalent then they are said to be of the same braid factorization type. As is known, any two factorizations of the form (1) corresponding to the same algebraic curve \( C \subset \mathbb{CP}^2 \) are of the same braid factorization type (see, for example, [MT]).

The notion of braid monodromy factorization extends word by word from the algebraic case to the case of so called semi-algebraic or Hurwitz curves (details can be found, for example, in [KT]).

If \( C_1, C_2 \subset \mathbb{CP}^2 \) are two projective cuspidal plane curves of the same braid factorization type, then ([KT], Theorem 7.1) there exists a diffeotopy \( F_t : \mathbb{CP}^2 \to \mathbb{CP}^2 \) such that \( F_1(C_1) = C_2. \)

The aim of this note is to prove the following theorems.

\(^1\) Actually, in [KT] it is proved the existence of symplectic isotopy. It is sufficient to rescale the isotopy constructed there.
Theorem 1. There are two sequences of plane irreducible cuspidal curves, \( \{C_{m,1}\} \) and \( \{C_{m,2}\} \), \( m \geq 5 \), such that the pairs \( (\mathbb{CP}^2, C_{m,1}) \) and \( (\mathbb{CP}^2, C_{m,2}) \) are diffeomorphic, but \( C_{m,1} \) and \( C_{m,2} \) are not isotopic and have different braid factorization types.

Theorem 2. Let \( X \) be a surface of general type with ample canonical class \( K \). Suppose that there is no homeomorphism \( h \) of \( X \) such that \( h^*[K] = -[K] \), \( [K] \in H^2(X; \mathbb{Q}) \). Then the moduli space of \( X \) consists of at least 2 connected components corresponding to \( X \) and \( \overline{X} \) (the bar states for reversing of complex structure, \( J \mapsto -J \)), and for any \( m \geq 5 \) these two connected components are distinguished by the braid factorization types of the branch curves of generic coverings \( f_m : X \to \mathbb{CP}^2 \) and \( \overline{f}_m : \overline{X} \to \mathbb{CP}^2 \) given by \( mK \) and \( m\overline{K} \), respectively.

These two theorems will be proved simultaneously. Note also that the curves from Theorem 1 constructed below are complex conjugated and belong to different components of the space of curves of given degree with given types of singularities.

Proof. By Bombieri theorem, the map \( X \to \mathbb{CP}^{r_m}, r_m = \dim H^0(X, mK) - 1 \), given by \( mK \) is an imbedding if \( m \geq 5 \). Let \( m \geq 5 \) and denote by \( X_m \) the image of \( X \) under the imbedding in \( \mathbb{CP}^{r_m} \) given by \( mK \), by \( pr_m : \mathbb{CP}^{r_m} \to \mathbb{CP}^2 \) a linear projection generic with respect to \( X_m \), by \( f_m = pr_m|_{X_m} \) the restriction of \( pr_m \) to \( X_m \), and by \( C_{m,1} \subset \mathbb{CP}^2 \) the branch curve of \( f_m \).

We identify \( X_m \) and \( \overline{X}_m \) as sets. Thus, the branched covering \( f_m : X_m \to \mathbb{CP}^2 \) can be considered as well as a holomorphic covering \( f_m : \overline{X}_m \to \mathbb{CP}^2 \), also branched along \( C_{m,1} \). Consider the standard complex conjugation \( c : \mathbb{CP}^2 \to \mathbb{CP}^2 \), in homogeneous coordinates it is given by \( c^*(x_i) = \overline{x}_i \). The composition \( \overline{f}_m = c \circ f_m : \overline{X}_m \to \mathbb{CP}^2 \) is a holomorphic generic covering with branch curve \( C_{m,2} = c(C_{m,1}) \). By construction, we have

\[ \overline{f}_m^*(\Lambda) = -f_m^*(\Lambda) = -m[K], \]

where \( \Lambda \in H^2(\mathbb{CP}^2, \mathbb{Q}) \) is the class of the projective line in \( \mathbb{CP}^2 \).
The set of generic coverings $f$ of $\mathbb{CP}^2$ branched along a cuspidal curve $C$ is in one-to-one correspondence with the set of epimorphisms from the fundamental group $\pi_1(\mathbb{CP}^2 \setminus C)$ to the symmetric groups $S_{\deg f}$ (up to inner automorphisms) satisfying some additional properties (see [K]). By Theorem 3 in [K], for $C_{m,1}$ (respectively, $C_{m,2}$) there exists only one such an epimorphism $\varphi_m : \pi_1(\mathbb{CP}^2 \setminus C_{m,1}) \to S_{\deg f_m}$ (respectively, $\overline{\varphi}_m$).

Assume that there is an isotopy $F_t : \mathbb{CP}^2 \to \mathbb{CP}^2$ such that $F_1(C_{m,1}) = C_{m,2}$. The epimorphism

$$\varphi_{m,t} : \pi_1(\mathbb{CP}^2 \times [0,1] \setminus \{(F_t(C_{m,1}), t)\}) \to S_{\deg f_m}$$

defines a 5-manifold $Y$ (with boundary) as a generic covering of $f_{m,t} : Y \to \mathbb{CP}^2 \times [0,1]$. This manifold has a natural structure of a locally trivial fibration over $[0,1]$. Note that by Theorem 3 in [K], over $t = 0$ the covering coincides with $f_m$ and over $t = 1$ with $\overline{f}_m$.

This fibration provides a homeomorphism $h$ of $X$ such that $h^*[K] = -[K]$. Thus, the isotopy $F_t$ can not exist. (Note that, in fact, $Y$ is smooth and $h$ can be made smooth everywhere except a finite number of points.)

If we assume that the branch curves $C_{m,1}$ $C_{m,2}$ have the same braid factorization type, then Theorem 7.1 in [KT] would imply the existence of an isotopy $F_t : \mathbb{CP}^2 \to \mathbb{CP}^2$ such that $F_1(C_{m,1}) = C_{m,2}$. This completes the proof of Theorem 2.

To prove the existence of curves satisfying Theorem 1, consider the arrangement of nine lines $L = L_1 \cup \cdots \cup L_9$ in $\mathbb{CP}^2$ dual to the nine inflection points of a smooth cubic $C$ in the dual plane. If $C$ is a cubic given by $x_1^3 + x_2^3 + x_3^3 = 0$, then the lines $L_1, \ldots, L_9$ are given by equations

$$L_1 = \{x_1 - x_3 = 0\}, \quad L_2 = \{x_1 - \mu^2 x_3 = 0\}, \quad L_3 = \{x_1 + \mu x_3 = 0\},$$
$$L_4 = \{x_2 - \mu^2 x_3 = 0\}, \quad L_5 = \{x_2 - x_3 = 0\}, \quad L_6 = \{x_2 + \mu x_3 = 0\},$$
$$L_7 = \{x_1 + \mu x_2 = 0\}, \quad L_8 = \{x_1 - \mu^2 x_2 = 0\}, \quad L_9 = \{x_1 - x_2 = 0\},$$

where $\mu = e^{\pi i/3}$. 
Consider an affine surface $S \subset \mathbb{C}^4$ given in $\mathbb{C}^4$ by equations

$$w_1^5 = l_1 l_2 l_3 l_4 l_5 l_9,$$

$$w_2 = l_1 l_3 l_4 l_6 l_7 l_8 l_9,$$

where $l_i = 0$ are the equations of $L_i$ in the chart $\{x_4 \neq 0\}$. Denote by $X$ the minimal desingularization of the normalization of the projective closure of $S$ in $\mathbb{C}P^4$.

As is shown in [KK], $X$ is a rigid surface of general type with $K_X^2 = 333$, the topological Euler characteristic $e(X) = 111$, and such that $X$ has no anti-holomorphic automorphisms. Its universal covering is the complex ball. By Mostow rigidity [M], any homeomorphism $h$ of $X$ is homotopic to a holomorphic or anti-holomorphic automorphism of $X$. Thus, $X$ satisfies the conditions of Theorem 2.

Consider a generic covering $f_m : X_m \to \mathbb{C}P^2$ given by $m$-canonical class and let $C_{m,1} \subset \mathbb{C}P^2$ be the branch curve of $f_m$. Calculations in [K], page 1155, give rise to

1. $\deg f_m = 333 m^2$;
2. $C_{m,1}$ is a plane cuspidal curve of $\deg C_{m,1} = 333 (3m + 1)$;
3. the geometric genus $g_m$ of $C_{m,1}$ equals $g_m = 333 (3m + 2)(3m + 1)/2 + 1$;
4. $C_{m,1}$ has $c = 111 (36m^2 + 27m + 5)$ ordinary cusps.

As is proved above, the curves $C_{m,1}$ and $C_{m,2} = \overline{C}_{m,1}$ give a sequence of curves satisfying Theorem 1. (Note that other examples of such curves can be obtained in a similar way if we take for $X$ a fake projective plane, i.e., a surface of general type with $p_g = q = 0$ and $K^2 = 9$; the universal covering of such surfaces is again the complex ball and, as is shown in [KK], they have no anti-isomorphisms either.)

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