SCHRODINGER EQUATION AND WAVE EQUATION ON FINITE GRAPHS

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Abstract. In this paper, we study the schrodinger equation and wave equation with the Dirichlet boundary condition on a connected finite graph. The explicit expressions for solutions are given and the energy conservations are derived. Applications to the corresponding nonlinear problems are indicated.

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1. INTRODUCTION

In this paper we study the schrodinger equation and wave equation with the Dirichlet boundary condition on a connected finite graph. This work is a complement to the paper [4]. To our best knowledge, this is the first paper on this direction.

A graph $G = (V, E)$ is a pair of the vertex-set $V$ and the edge-set $E$. Each edge is an unordered pair of two vertices. If there is an edge between $x$ and $y$, we write $x \sim y$. We assume that $G$ is local finite, i.e., there exists a constant $c > 0$ such that $\text{deg}(x) := \# \{y \in V : (x, y) \in E\} \leq c$ for all $x \in G$.

Let $S$ be a finite subset of $V$, the subgraph $G(S)$ generated by $S$ is a graph, which consists of the vertex-set $S$ and all the edges $x \sim y$, $x, y \in S$ as the edge set. The boundary $\delta S$ of the induced subgraph $G(S)$ consists of all vertices that are not in $S$ but adjacent to some vertex in $S$. We assume that the subgraph $G(S)$ is connected. In below, we write $(a, b) = \bar{a} \bar{b}$ for complex numbers $a$ and $b$. We now recall some facts from the book [1]. Sometimes people may like to write

\[ \bar{S} = S \cup \delta S. \]

For a function $f : S \cup \delta S \to \mathbb{C}$, let

\[ \nabla_{xy} f = f(y) - f(x) \]

for $y \sim x$. Then we recall that

\[ \int_{\bar{S}} f = \sum_{x \in S} f(x), \quad \|f\|^2 = \int_{\bar{S}} |f|^2 = \sum_{x \in \bar{S}} f(x) \overline{f(x)} \]

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\[ \| \nabla f \|^2 = \sum_{\{x,y \in S\}} |f(x) - f(y)|^2. \]

We say that \( f : S \cup \delta S \to \mathbb{C} \) satisfies the Neumann boundary condition if for all \( x \in \delta S \),
\[ \sum_{\{y \in S; y \sim x\}} (f(y) - f(x)) = 0. \]

Then the Laplacian operator can be written as
\[ (\Delta f)(x) = \sum_{y; y \sim x} (f(y) - f(x)) = \sum_{y; y \sim x} \nabla_{xy} f. \]

Also the Neumann condition can be written as
\[ \sum_{\{y \in S; y \sim x\}} \nabla_{xy} f = 0, \quad \forall x \in \delta S. \]

We say that \( f : S \cup \delta S \to \mathbb{C} \) satisfies the Dirichlet boundary condition if
\[ f(x) = 0 \quad \text{for all} \quad x \in \delta S. \]

With Neumann or Dirichlet boundary condition, the Laplacian operator on \( S \) has finite eigenvalues \( \lambda_j > 0 \) with the corresponding eigen-functions \( \phi_j(x) \) [1], i.e.,
\[ -\Delta \phi_j = \lambda_j \phi_j, \quad \text{in} \quad S, \quad \int_S |\phi_j|^2 = 1. \]

In short, we can write this as
\[ -\Delta = \sum_j \lambda_j I_j \]

where \( I_j \) is the projection on to the \( j \)-th eigenfunction \( \phi_j \) of the induced subgraph \( S \) (see p.145 in [1]).

Then we define the Schrödinger kernel as
\[ S_t(x, y) = \sum_j e^{-i\lambda_j t} \phi_j(x) \phi_j(y), \quad t \geq 0. \]

As in the heat kernel case we can write this as
\[ S_t = \sum_j e^{-i\lambda_j t} I_j. \]

Then
\[ S_t = e^{it\Delta} = I + it\Delta + \ldots, \quad S_0 = I. \]

For a function \( f : S \cup \delta S \to \mathbb{C} \), we define
\[ u(x, t) = \sum_y S_t(x, y)f(y), \quad x \in S, \quad t > 0. \]

Note that
\[ S_0(x, y) = I = \sum_j \phi_j(x) \phi_j(y). \]
Then for any \( f : S \cup \delta S \to \mathbb{C} \),
\[
u(x, 0) = S_0(x, y)f(y) = \sum_y \sum_j \phi_j(x)\phi_j(y)f(y) = f(x).
\]

Then we can directly verify that the function \( u \) satisfies the Schrödinger equation
\[
i\partial_t u(x, t) + \Delta u(x, t) = 0, \quad x \in S, \quad t > 0,
\]
with \( u(x, 0) = f(x) \) for \( x \in S \). We denote \( V(S) \) the space of functions \( f : S \cup \delta S \to \mathbb{C} \) satisfy the Dirichlet boundary condition and with the \( L^2 \) norm.

Then we show the following result.

**Theorem 1.** Assume that the function \( f : S \cup \delta S \to \mathbb{C} \) satisfies the Dirichlet boundary condition. Then there is a global solution \( u(t) : S \cup \delta S \to \mathbb{C} \), which can be expressed in (1), such that \( u \) satisfies the Schrödinger equation
\[
i\partial_t u(x, t) + \Delta u(x, t) = 0, \quad x \in S, \quad t > 0,
\]
with \( u(x, 0) = f(x) \) for \( x \in S \) and with the mass conservation
\[
\|u(t)\|^2 = \|f\|^2
\]
and the energy conservation
\[
\int_S |\nabla u(t)|^2 = \int_S |\nabla f|^2.
\]

The proof will be given in next section via the use of the spectrum of the Laplacian and the Green formula.

Similarly we consider the wave equation on \( \bar{S} \). Given two functions \( f \) and \( g : \bar{S} \to \mathbb{R} \) and both satisfy the Dirichlet condition. We consider the following wave equation
\[
(2) \quad \nu_{tt} = \Delta \nu(x, t), \quad x \in S, \quad t > 0,
\]
with the initial conditions \( \nu(x, 0) = f(x) \) and \( \nu_t(x, 0) = g(x) \) for \( x \in S \).

We have the following result.

**Theorem 2.** The solution to (2) is given by
\[
(3) \quad \nu(x, t) = \sum_j [I_j f(x) \cos(\sqrt{\lambda_j}t) + I_j g(x) \frac{\cos(\sqrt{\lambda_j}t)}{\sqrt{\lambda_j}}] \phi_j(x).
\]

Furthermore, we have that
\[
\int_S [\|\nabla \nu\|^2 + \nu_{tt}^2]dx = \int_S [\|\nabla f\|^2 + g^2]dx.
\]

We remark that the related Duhamel principle can be used to give solutions to the corresponding non-homogenous problems and we omit them here. We don’t consider problems on the infinite locally finite connected graphs \( V \) with symmetric weight as in [4] for the following reason. One can prove the (skew) self-adjoint property of \( \Delta (i\Delta) \) (as proved in [3]) which gives
the solutions to linear Schrödinger and wave equations on $L^2(V)$. However, for such general graph $V$, the Strichartz type equality for Schrödinger operator (or related interpolation inequality for wave operator) is missing for applications to nonlinear Schrödinger equations (for nonlinear wave equations).

The plan of this paper is below. In section 2, we prove Theorem 1. We prove Theorem 2 in section 3. We give some applications of our results in section 4.

2. Proof of Theorem 1

The importance of the boundary conditions above is the boundary term vanishing in the formula below.

**Theorem 3.** Assume that $f: S \cup \delta S \to \mathbb{C}$. Then we have

$$\int_S (\Delta f, f) = -\frac{1}{2} \int_S |\nabla f|^2 + \sum_{x \in S} \sum_{y \in \delta S} \overline{f(x)} \nabla_{xy} f.$$  \hspace{1cm} (4)

This can be verified directly. In fact, we can directly verify the following more general formula (see Theorem 2.1 in [2]) in a compact form.

**Theorem 4.** Assume that $f, g: S \cup \delta S \to \mathbb{C}$. Then we have

$$\int_S (\Delta f, g) = -\frac{1}{2} \sum_{x, y \in S} (\nabla_{xy} f, \nabla_{xy} g).$$  \hspace{1cm} (5)

We remark that the formula (5) is proved in [2] for real functions, but the complex case can be done by writing the complex function into the sum of real and imaginary parts.

For completeness, we give the proof of Theorem 4 below.

**Proof.** As remarked above, we need only prove the result for real functions.

We make the following computation

$$\int_S (\Delta f, g) = \sum_{x \in S} (\Delta f(x), g(x)) = \sum_{x \in S} \sum_{y \sim x} (\nabla_{xy} f, g(x))$$

$$= \sum_{x, y \in S; y \sim x} (\nabla_{xy} f, g(x)) + \sum_{x \in S} \sum_{y \in \delta S; y \sim x} (\nabla_{xy} f, g(x))$$

$$= \sum_{x, y \in S; x \sim y} (\nabla_{xy} f, g(y) - \nabla_{xy} g) + \sum_{x \in S} \sum_{y \in \delta S; y \sim x} (\nabla_{xy} f, g(x))$$

$$= -\sum_{x, y \in S; x \sim y} (\nabla_{xy} f, \nabla_{xy} g) + \sum_{x, y \in S; x \sim y} (\nabla_{xy} f, g(y))$$

$$\sum_{x \in S} \sum_{y \in \delta S; y \sim x} (\nabla_{xy} f, g(x)) =: A + B + C.$$

For the term $B$, we have

$$B = \sum_{x, y \in S; x \sim y} (\nabla_{xy} f, g(y))$$

$$\sum_{y \in S} (\sum_{x \in S; x \sim y} \nabla_{xy} f, g(y)) - \sum_{y \in S} (\sum_{x \in \delta S; x \sim y} \nabla_{xy} f, g(y))$$

$$= \sum_{y \in S} (-\Delta f(y), g(y)) + \sum_{y \in S} (\sum_{x \in \delta S; x \sim y} \nabla_{xy} f, g(y))$$

$$= -\int_S (\Delta f, g) + C.$$
(For the second term of the left hand side of the above equation, we change $x$ to $y$, $y$ to $x$, and see that this term is just $C$). It follows that

$$\int_S (\Delta f, g) = A + (-\int_S (\Delta f, g) + C) + C.$$  

Then we have

$$2\int_S (\Delta f, g) = A + C.$$ 

We then re-write this into (5). The proof is complete. □

For $f : S \cup \delta S \to \mathbb{C}$ satisfying the Dirichlet condition, the boundary term in Theorem 3 can be written as

$$-\sum_{x \in S} \sum_{y \in \delta S} (f(y) - f(x)) \nabla_{xy} f = -\sum_{x \in S} \sum_{y \in \delta S} |\nabla_{xy} f|^2,$$

which is real. This fact is useful in the proof of mass and energy conservation below.

We now use Theorem 3 (and Theorem 4) to prove Theorem 1.

Proof. Assume that $f$ satisfies either Neumann or Dirichlet boundary condition. Compute, via the use of the formula (4),

$$\frac{d}{dt} \|u(t)\|^2 = 2\text{Re}(u, u_t) = 2\text{Re}(u, i\Delta u) = 0.$$ 

here we have used implicitly the boundary condition which implies that $u_t = 0$ on $\delta S$. We then have the mass conservation

$$\|u(t)\|^2 = \|f\|^2.$$ 

Similarly we compute

$$\frac{d}{dt} \int_S |\nabla u(t)|^2 = 2\text{Re}(\nabla u, \nabla u_t).$$ 

Using the formula (5) for $f = u$ and $g = u_t$, we know that

$$\text{Re}(\nabla u, \nabla u_t) = -2\text{Re} \int_S (\Delta u, u_t).$$ 

Using the Schrodinger equation, we know that the term in the right side is

$$-2\text{Re} \int_S (\Delta u, i\Delta u),$$

which is zero too. Then

$$\frac{d}{dt} \|\nabla u(t)\|^2 = 0$$

and we then get the energy conservation

$$\|\nabla u(t)\|^2 = \|\nabla f\|^2.$$ 

The uniqueness of the solution follows from the mass conservation. This completes the proof. □
3. WAVE EQUATIONS ON FINITE GRAPHS

Recall that two functions $f$ and $g : \tilde{S} \to \mathbb{R}$ satisfy the Dirichlet condition. The problem under consideration is the following wave equation

$$(6) \quad u_{tt} = \Delta u(x, t), \quad x \in S, \ t > 0,$$

with the initial conditions $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$ for $x \in S$.

Here is the proof of Theorem 2.

Proof. Write the solution as $u(x, t) = \sum_j u_j(t) \phi_j(x)$, where $u_j(t) = I_j u(x, t)$. Using the initial conditions, we know that $u_j(0) = I_j f(x)$, $u_{jt}(0) = I_j g(x)$.

Inserting $u(x, t) = \sum_j u_j(t) \phi_j(x)$ into (6) we get that

$$u_{jtt} + \lambda_j u_j = 0.$$

Solve this equation we know that

$$u_j(t) = I_j f(x) \cos(\sqrt{\lambda_j} t) + I_j g(x) \cos(\sqrt{\lambda_j} t).$$

Hence we can write the solution in the form (3).

Note that

$$\frac{d}{dt} \int_S [\|\nabla u\|^2 + u_t^2] dx = 2 \int_S [\nabla u, \nabla u_t] + (u_{tt}, u_t)] dx.$$

Using the Green formula and (6) we then have

$$\int_S [(-\Delta u, u_t) + (u_{tt}, u_t)] dx = 0.$$

That is,

$$\frac{d}{dt} \int_S [\|\nabla u\|^2 + u_t^2] dx = 0.$$

This proves Theorem 2. □

4. DISCUSSIONS

The advantage of our formulation of Schrodinger equation and wave equation in finite graphs is that it gives us the local existence result of the corresponding nonlinear equations by the use of fixed point theorem. For example, there is a unique local in time solution $u(x, t)$ to the nonlinear Schrodinger equation

$$(7) \quad i \partial_t u(x, t) + \Delta u(x, t) = |u|^{p-1} u(x, t), \quad x \in S, \ t > 0,$$
with the initial data $u(x,0) = f(x)$ for $x \in S$ and with the Dirichlet boundary condition. Here $p > 1$. In fact, by Duhamel principle, we know that the problem is equivalent to the fixed point problem

$$u(x,t) = \sum_y S_t(x,y) f(y) + \int_0^t \sum_y S_{t-\tau}(x,y) |u|^{p-1} u(y,\tau) d\tau,$$

on the Banach space $C^0([0,T], V(S))$. Here $T > 0$ is sufficiently small.

Using the Nehari method, one can easily obtain the following.

**Theorem 5.** Given $p > 1$ and $V(x)$ a non-negative function on $S$. There exists a ground state solution $u \in V(S)$ to the problem

$$-\Delta u(x) + V(x)u(x) = |u|^{p-1} u(x), \quad u(x) > 0, \quad x \in S.$$

Recall here that a ground state is a minimizer of the functional

$$I(u) = -\frac{1}{2} \int_S (|\nabla u(x)|^2 + V(x)|u(x)|^2) - \frac{1}{p+1} \int_S |u(x)|^{p+1}$$

over the set

$$\{u \in V(S); u \neq 0, \int_S |u|^{p+1} = \int_S (|\nabla u(x)|^2 + V(x)|u(x)|^2)\}.$$

The minimizer exists and the proof is straightforward, so we omit it.

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