Noise properties of the SET transistor in the co-tunneling regime

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Abstract. Zero-frequency spectral densities of current noise, charge noise, and their cross-correlation are calculated for the SET transistor in the co-tunneling regime. The current noise has a form expected for the uncorrelated co-tunneling events. Charge noise is created by the co-tunneling and also by the second-order transitions in a single junction. Calculated spectral densities determine transistor characteristics as quantum detector.

1. INTRODUCTION

Single-electron-tunneling (SET) transistor [1,2,3] is the natural measuring device for the potential quantum logic circuits based on the charge states of mesoscopic Josephson junctions [4,5,6]. This fact, together with the general interest to the problem of quantum measurement, motivates current discussions of the SET transistors as quantum detectors – see, e.g., [7]. Detector characteristics of the transistor are determined by its noise properties, and have been studied so far [8,9] in the regime of classical electron tunneling. The aim of this brief note is to calculate the transistor noise properties in the Coulomb blockade regime, when the current flows in it by the process of co-tunneling. An expected, and confirmed below, advantage of co-tunneling for quantum signal detection is a weaker back-action noise on the measured system produced by the SET transistor.

2. CO-TUNNELING IN THE SET TRANSISTOR

SET transistor [1] is a small conductor, typically a small metallic island, placed between two bulk external electrodes, that forms two tunnel junctions with these electrodes. Due to Coulomb charging of the island by tunneling electrons, the current $I$ through the structure depends on the island electrostatic potential controlled by an external gate voltage $V_g$. Sensitivity of the current $I$ to variations of the voltage $V_g$ makes it possible to measure this voltage, and is the basis for transistor operation as the detector.

When the bias voltage $V$ across the transistor is smaller than the Coulomb blockade threshold (dependent on $V_g$), the tunneling is suppressed by the Coulomb
charging energy required to transfer an electron in or out of the central electrode. In this regime, the current $I$ flows through the transistor only by the co-tunneling process that consists of two electron jumps across the two transistor junctions in the same direction (for review, see [10]). Energy diagram of the co-tunneling transitions is shown in Fig. 1. Transitions go via two virtual intermediate charge states with $n = \pm 1$ extra electrons on the island and large charging energies $E_{1,2}$. The energies $E_{1,2}$ are equal to the change of electrostatic energy due to the first electron jump in the first or the second junction, and depend on the voltages $V$, $V_g$, junction capacitances $C_{1,2}$, and capacitance $C_g$ that couples the gate voltage $V_g$:

$$E_1 = E_C \left( \frac{1}{2} + q_0 - \left( C_1 + \frac{C_g}{2} \right) \frac{V}{e} \right), \quad E_2 = E_C \left( \frac{1}{2} - q_0 - \left( C_2 + \frac{C_g}{2} \right) \frac{V}{e} \right). \quad (1)$$

Here $E_C$ is the characteristic charging energy of the transistor $E_C = e^2/C_\Sigma$; $C_\Sigma = C_1 + C_2 + C_g$, and $q_0$ is the charge (in units of electron charge $e$) induced by the gate voltage, $q_0 \equiv C_g V_g/e$. Equations (1) assume that the voltage $V_g$ is applied to the transistor symmetrically.

![Figure 1: Energy diagrams of the four co-tunneling transitions in the SET transistor. The arrows indicate electron jumps. The left and right diagrams show, respectively, the transitions through the charge states with $n = +1$ and $n = -1$ extra electrons on the central electrode of the transistor. The energy labels are the changes of electrostatic energy in the first electron jump of each co-tunneling process.](image)

The co-tunneling regime is realized for values of $V$ and $V_g$ that make the energies $E_j$, $j = 1,2$, sufficiently large, $E_j \gg T, \hbar G_{1,2}/C_\Sigma$, where $T$ is the temperature and $G_{1,2}$ are the tunnel conductances of the transistor junctions. The conductances $G_{1,2}$ are assumed to be small, $g_{1,2} \equiv \hbar G_{1,2}/2\pi e^2 \ll 1$. In this regime, one can neglect thermally-induced “classical” electron transitions of the first order in conductances $g_j$, and also neglect quantum broadening of the charge states by tunneling [11,12]. Transport characteristics of the transistor are determined then by the co-tunneling transitions of the second order in $g_j$. 
The gate voltage $V_g$ controls the co-tunneling current $I$ in the transistor through the dependence of the energies $E_j$ on the induced charge $q_0$. Viewed as the detector for measurement of small variations of $V_g$, the transistor is characterized by: (1) the linear response coefficient $\lambda = \delta \langle I \rangle / \delta V_g$, where $\delta \langle I \rangle$ is the change of the average current due to variation of $V_g$; (2) the spectral density $S_I$ of the current noise (output noise of the detector); (3) the spectral density $S_Q$ of the fluctuations of the charge $Q = e n$ on the central electrode of the transistor ("back-action" noise); and (4) cross-correlation $S_{IQ}$ between the current and charge noise. In general, the back-action noise is the fundamental property of a quantum detector responsible for localization of the measured system in the eigenstates of the measured observable. For the SET transistor, the measured observable is the voltage $V_g$, and the back-action noise originates from fluctuations of the charge $Q$ in the process of electron transfer through the transistor. The voltage $V_g$ is coupled to $Q$ in the transistor energy as $QV_gC_g/C_\Sigma$, where $Q = 0, \pm e$ in the co-tunneling regime. The fluctuations of $Q$ produce random force that acts on the system creating $V_g$, and lead to mutual decoherence of the states of this system with different values of $V_g$.

In the next section, the noise spectral densities are calculated in the zero-frequency limit for the SET transistor in the co-tunneling regime. The zero-frequency results determine the detector characteristics of the transistor in the frequency range below the characteristic frequency of electron tunneling.

3. NOISE CALCULATION

Transport properties of the transistor related to co-tunneling can be calculated by straightforward perturbation theory in tunneling in the second order in tunnel conductances $g_j$. The tunneling part of the transistor Hamiltonian is

$$H_T = \sum_{j=1,2} H_j^\pm,$$

where $H_j^\pm$ describe forward and backward electron transfer in the $j$th junction. The junction electrodes are assumed to have quasicontinuous density of states. The only nonvanishing free correlators of $H_j^\pm$ can be expressed in terms of the conductances $g_j$ (see, e.g., [10]):

$$\langle H_j^\pm(t) H_j^\mp(t') \rangle = g_j \int \frac{d\omega \omega e^{i\omega(t'-t)}}{1 - e^{-\omega/T}}.$$  \hspace{1cm} (2)

The zero-frequency spectral densities of current and charge noise are given
by the standard expressions. For instance,

\[ S_I = \frac{1}{\pi} \text{Re} \int_{-\infty}^{t} \text{dt}' \text{Tr}\{S(\text{t})I(\text{t})S(\text{t}, \text{t}')I(\text{t}')S(\text{t}')\rho_0\}, \]  

(3)

where \( S(\text{t}, \text{t}') = \mathcal{T} \exp\{-i \int_{\text{t}}^{\text{t}'} d\tau H_T(\tau)\} \) is the evolution operator of the transistor due to tunneling, \( \rho_0 \) is its unperturbed equilibrium density matrix, \( S(\text{t}) \equiv S(\text{t}, -\infty) \), and the current \( I \) can be calculated in either of the two junctions, e.g., \( I = ie(H_T^+ - H_T^-) \). The time dependence of all operators \( H_T \) is now due to both internal energies of the junction electrodes and the electrostatic charging energy of the transistor. The term \( -\langle I \rangle^2 \) is omitted in (3), since the average co-tunneling current \( \langle I \rangle \) is of the same order in \( g_j \) as \( S_I \). Therefore, \( \langle I \rangle^2 \) is of higher order than \( S_I \), and can be neglected in the perturbative calculation adequate for the co-tunneling regime. The same considerations apply to the spectral densities \( S_Q \) and \( S_{IQ} \) discussed below.

Expanding all evolution operators in (3) up to the second order in \( H_T^\pm \), and evaluating the averages with the help of Eq. (2), we find that \( S_I \) is given by the standard expression characteristic for uncorrelated tunneling:

\[ S_I = \frac{e^2}{2\pi} (\gamma^+ + \gamma^-). \]  

(4)

Here \( \gamma^\pm \) are the rates of forward and backward co-tunneling:

\[ \gamma^+ = \frac{2\pi g_1 g_2}{\hbar} \int \frac{d\omega_1}{1 - e^{-\omega_1/T}} \int \frac{d\omega_2}{1 - e^{-\omega_2/T}} \delta(\omega_1 + \omega_2 - eV) \left( \frac{1}{E_1 + \omega_1} + \frac{1}{E_2 + \omega_2} \right)^2, \]

and \( \gamma^- \) is given by the same expression with \( eV \) and \( E_j \) changed into \( -eV \) and \( E_j + eV \). The rates \( \gamma^\pm \) also determine the average co-tunneling current, \( \langle I \rangle = e(\gamma^+ - \gamma^-) \).

Expression for the spectral density \( S_Q \) of charge fluctuations is obtained from Eq. (3) by replacing the current operators with the charge operators \( Q = en \). To find \( S_Q \) in the co-tunneling regime one needs to expand the evolution operators in this expression up to the fourth order in the tunneling terms \( H_T \). The non-vanishing contribution to \( S_Q \) comes then from one particular choice of orders of expansion of different evolution operators:

\[ S_Q = -\frac{e^2}{\pi} \text{Re} \int_{-\infty}^{t} \text{dt}_1 \int_{-\infty}^{t} \text{dt}_2 \int_{-\infty}^{t} \text{dt}_3 \int_{-\infty}^{t} \text{dt}_4 \text{Tr}\{H_T(t_1)H_T(t_2)H_T(t_3)H_T(t_4)\rho_0\}. \]  

(5)

Taking the trace in Eq. (5) we get the two types of contributions to \( S_Q \). In one of them, the pairings of operators \( H_T \) belong to the two different junctions, while
in the other, all $H_T$’s belong to one junction. The first contribution describes fluctuations of the charge in the process of co-tunneling, and can be split into the two terms, $S^\pm$, associated with the forward and backward transitions. The second contribution $\bar{S}$ is created by the back-and-forth tunneling within the same junction. Accordingly, $S_Q$ can be written as

$$S_Q = \bar{S} + S^\pm,$$

where the different terms are found from Eq. (5) to be

$$\bar{S} = \sum_j (2\pi e g_j)^2 \frac{\hbar}{6} \left( \frac{1}{E_j^2} - \frac{1}{(E_j + eV)^2} \right)^2, \quad j, j' = 1, 2, \ j' \neq j,$$

$$S^+ = \hbar e^2 g_1 g_2 \int \frac{d\omega_1 \omega_1}{1 - e^{-\omega_1/T}} \frac{d\omega_2 \omega_2}{1 - e^{-\omega_2/T}} \delta(\omega_1 + \omega_2 - eV) \cdot \left( \frac{1}{(E_1 + \omega_1)^2} - \frac{1}{(E_2 + \omega_2)^2} \right)^2.$$

The last term $S^-$ is given by the same expression as $S^+$ with $eV$ and $E_{1,2}$ replaced, respectively, by $-eV$ and $E_{1,2} + eV$.

Finally, we calculate the correlator $S_{IQ}$ between the charge and current noise:

$$S_{IQ} = \frac{1}{2\pi} \int dt' \Tr \{ S^\dagger(t) I(t) S(t, t') Q(t') S(t') \rho_0 \}. \quad (7)$$

Similarly to Eq. (3), it is convenient to break the integral in (7) into the two parts, $t' < t$ and $t' > t$. Expanding then the evolution operators up to the third order in $H_T$, and evaluating averages using Eq. (2), we find:

$$S_{IQ} = ie^2 g_1 g_2 \int \frac{d\omega_1 \omega_1}{1 - e^{-\omega_1/T}} \frac{d\omega_2 \omega_2}{1 - e^{-\omega_2/T}} \left\{ \delta(\omega_1 + \omega_2 - eV) \cdot \left( \frac{1}{E_1 + \omega_1} - \frac{1}{E_2 + \omega_2} \right) \left( \frac{1}{E_1 + \omega_1 + E_2 + \omega_2} + \frac{1}{E_1 + eV + \omega_1} + \frac{1}{E_2 + eV + \omega_2} \right)^2 + \delta(\omega_1 + \omega_2 + eV) \cdot \left( \frac{1}{E_1 + \omega_1} - \frac{1}{E_2 + \omega_2} \right) \left( \frac{1}{E_1 + E_2 + \omega_1 + \omega_2} + \frac{1}{E_1 + eV + \omega_1} + \frac{1}{E_2 + eV + \omega_2} \right)^2 \right\}. \quad (8)$$

The correlator $S_{IQ}$ (8) is purely imaginary. From the perspective of the general theory of quantum linear detection [13] this fact means that the SET transistor in the co-tunneling regime is “symmetric” detector with the output noise and back-action noise uncorrelated at the classical level, since classically, the correlator between current and charge is given by the symmetrized correlation function which contains only the real part of $S_{IQ}$. Imaginary part of $S_{IQ}$
should be directly related to the linear response coefficient $\lambda$ of the transistor [13]. Comparison of Eq. (8) to the expression for $\lambda = \delta(I)/\delta V_g$ shows that, indeed, the two quantities are related: $(C_g/C_\Sigma) \text{Im}[S_{IQ}] = -\hbar \lambda/4\pi$.

4. RESULTS AND DISCUSSION

In this section, we calculate explicitly the noise spectral densities $S_Q$, $S_I$, and $S_{IQ}$ in various limits, and discuss the implications of the obtained relations for the characteristics of transistor as quantum detector. At small bias voltages $V$, $V \sim T \ll E_j$, the electron excitation energies $\omega_j$ can be neglected in comparison to energy barriers $E_j$, and expressions for the spectral densities obtained in the previous section are simplified:

$$S_Q = \frac{\hbar e^2}{6} \left( \frac{1}{E_1^2} - \frac{1}{E_2^2} \right)^2 \left\{ g_1 g_2 [(eV)^2 + (2\pi T)^2] eV \coth \frac{eV}{2T} + (g_1^2 + g_2^2) 4\pi^2 T^3 \right\},$$

$$S_I = \frac{g_1 g_2 e^3 V}{6\hbar} \left( \frac{1}{E_1} + \frac{1}{E_2} \right)^2 \left[ (eV)^2 + (2\pi T)^2 \right] \coth \frac{eV}{2T},$$

$$S_{IQ} = i \frac{g_1 g_2 e^3 V}{6} \left( \frac{1}{E_1} - \frac{1}{E_2} \right) \left( \frac{1}{E_1} + \frac{1}{E_2} \right)^2 \left[ (eV)^2 + (2\pi T)^2 \right].$$

A fundamental figure-of-merit of a quantum detector is “energy sensitivity” $\epsilon$ that is defined in terms of the output noise, back-action noise, and response coefficient of the detector - see, e.g., [13]. Qualitatively, energy sensitivity characterizes the amount of noise introduced by the detector into the measurement process, and is limited from below by $\hbar/2$ due to the quantum mechanical restrictions on the detector dynamics. In the case of SET transistor in the co-tunneling regime (when the current $I$ and charge $Q$ are classically uncorrelated), $\epsilon$ can be expressed directly through the three spectral densities $S_I$ (4), $S_Q$ (6), and $S_{IQ}$ (8), as follows:

$$\epsilon = \frac{\hbar}{2} \sqrt{\frac{S_I S_Q}{|S_{IQ}|}},$$

Equations (9) show that at small bias voltages the transistor energy sensitivity (10) is:

$$\epsilon = \frac{\hbar}{2} \left[ \coth \frac{eV}{2T} \left( \coth \frac{eV}{2T} + \left( \frac{g_1}{g_2} + \frac{g_2}{g_1} \right) 4\pi^2 T^3 \right) \right]^{1/2}. \quad (11)$$

The energy sensitivity (11) approaches the fundamental limit $\hbar/2$ at $eV \gg T$ [14].
At larger bias voltages \( eV \sim E_j \gg T \), temperature \( T \) can be neglected in equations for the noise spectral densities of the previous section and they give:

\[
S_Q = \hbar e^2 g_1 g_2 \left\{ \frac{(eV)^3}{6} \sum_j \frac{1}{(E_j(E_j + eV))^2} + \frac{4eV}{(E_1 + E_2 + eV)^2} - \frac{2eV(E_1 + E_2) + (eV)^2 + 2E_1 E_2}{(E_1 + E_2 + eV)^3} \sum_j \ln\left(1 + \frac{eV}{E_j}\right) \right\},
\]

\[
S_I = \frac{e^2 g_1 g_2}{\hbar} \left\{ \left( eV + \frac{2E_1 E_2}{E_1 + E_2 + eV} \right) \sum_j \ln\left(1 + \frac{eV}{E_j}\right) - 2eV \right\},
\]

\[
S_{IQ} = i e^2 g_1 g_2 \left\{ \frac{eV(E_1 - E_2)}{(E_1 + eV)(E_2 + eV)} \left( 1 + \frac{eV(E_1 + E_2 + eV)}{2E_1 E_2} \right) - \frac{E_1 - E_2}{E_1 + E_2 + eV} \sum_j \ln\left(1 + \frac{eV}{E_j}\right) \right\}.
\]

Figure 2 shows the zero-temperature energy sensitivity \( 10 \) as a function of the bias voltage calculated from these equations for the SET transistor with equal junction capacitances and several values of the charge \( q_0 \) induced by the gate voltage. The energy sensitivity reaches \( \hbar/2 \) at small bias voltages and diverges when the voltage approaches the Coulomb blockade threshold (see also Eq. \( 13 \) below). The non-monotonic behavior of \( \epsilon \) at small \( q_0 \) is caused by the fact that the correlator \( S_{IQ} \) vanishes for \( E_1 \rightarrow E_2 \).

Figure 2: Energy sensitivity, in units of \( \hbar \), of the SET transistor in the cotunneling regime.
The energy barriers $E_j$ decrease with increasing bias voltage $V$, and one or both of them disappear when $V$ approaches the Coulomb blockade threshold. At the threshold, the spectral densities (12) diverge. For generic values of the gate voltage, only one of the barriers is suppressed at the threshold, e.g., $E_1 \rightarrow 0$. In this case,

$$S_Q = \frac{h g_1 g_2 e^3 V}{6 E_1^2}, \quad S_I = \frac{g_1 g_2 e^3 V}{\hbar} \ln \left( \frac{eV}{E_1} \right), \quad S_{IQ} = -\frac{g_1 g_2 e^3 V}{2 E_1}.$$

The energy sensitivity (10) also slowly diverges:

$$\epsilon = \frac{\hbar}{2} \left[ \frac{2}{3} \ln \left( \frac{eV}{E_1} \right) \right]^{1/2}. \quad (13)$$

All these divergencies should be regularized by the finite width of the charge states created by tunneling [11,12]. Qualitatively, this means that for $E_1$ smaller than $\hbar G_2/C_X$, the logarithm in Eq. (13) should saturate at $\ln(1/g_2)$. Quantitative treatment of the threshold region is outside the scope of this work.

To summarize, we have studied noise properties of the SET transistor in the co-tunneling regime, and calculated its energy sensitivity as a detector. The energy sensitivity approaches $\hbar/2$ for small bias voltages, and slowly diverges at the Coulomb blockade threshold.

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[14] It should be noted, however, that this fact does not necessarily mean that the regime of relatively small bias voltages, $T \ll eV \ll E_j$, represents optimal operating point of the practical SET transistors. Other noise sources, not included in the model but present in realistic systems, make it important to have large absolute values of the output signal, the condition that is not fulfilled for the SET transistor biased deep inside the Coulomb blockade region.