Analysis of Nonlinear Spring Arm for Improved Performance of Vibrational Energy Harvesting Devices

D Mallick¹, A Amann², S Roy¹

¹Micropower-Nanomagnetics Group, Tyndall National Institute, University College Cork, Cork, Ireland
²School of Mathematical Sciences, University College Cork, Cork, Ireland

Email: saibal.roy@tyndall.ie

Abstract. Recently, a number of attempts have been made to increase the operational bandwidth of the energy harvesting devices. Nonlinear mechanisms are one of them. In this paper, we report design and analytical formulation of stretching strain of an electromagnetic energy harvester on FR4 material under large deformation of the spring arms. It is found that nonlinearity has an inverse square dependence on thickness of the arms. Numerical solution of a monostable Duffing oscillator that governs the dynamics of such a large deformed nonlinear energy harvester showed that with decrease of load resistance, the average power output increases, where the output response depends strongly on the input force. For small input acceleration, the desired large amplitude vibration does not come into play and the response becomes linear. However, for higher input acceleration nonlinearity appears and the operational bandwidth increases, at the same time, output power level also increases.

1. Introduction

The field of Vibration-based Energy Harvesting (VEH) has been recognized as an area of intensive research for the past few years [1-4]. Most of the researches in the field of VEH involved linear resonating structures. The problem with linear systems is that the response drops significantly as the external excitation frequency shifts from the natural frequency of the structure. The mismatch of frequency is inevitable, as in the majority of practical scenarios the environmental vibration is either frequency varying or random in nature. Thus over the last few years a significant amount of research is focused on obtaining broadband response of VEH devices while increasing the output power.

2. Overview of Conventional Methods to widen the bandwidth

In this section, a brief overview of some of the commonly used methods for obtaining wideband response has been outlined. Among the state-of-the-art techniques, resonant frequency tuning [5-8], multimodal energy harvesting [9-10], and more recently nonlinear energy harvesting [11-17] have emerged as the popular solution.

Nonlinearity is generally introduced in VEH through nonlinear stiffness [11-17]. The dynamics of a general nonlinear oscillator can be described as:

$$\ddot{x} = -\frac{dU(x)}{dx} - \gamma \dot{x} + f(t)$$

(1)

Where x is the position of the oscillator; γ is the viscous damping coefficient; f(t) is the force input by the ambient vibration and U(x) is the potential function. For an electromagnetic (EM) generator, γ
also includes the damping caused by electromagnetic coupling. A common approach is to model the potential energy function $U(x)$ in a quadratic form \[16\] as

$$U(x) = -\frac{1}{2} \alpha x^2 + \frac{1}{4} \beta x^4.$$ 

The potential function $U(x)$ is symmetric and bistable for $\alpha > 0$, and monostable for $\alpha \leq 0$. For the bistable case, two minima at $x_m = \pm \sqrt{\alpha / \beta}$ are separated by a barrier at $x = 0$. For the monostable case, $\beta > 0$ indicates a hardening response whereas $\beta < 0$ is a softening response. Bistable nonlinear oscillator has been implemented by snap through mechanism \[15\], or on a piezomagnetoelastic beam through a magnetic attractive force \[11\] or repulsive force \[16\]. The advantage of using bistability is that it can be easily driven to the high amplitude branch with excitation from either amplitude periodic force or low amplitude stochastic force. On the other hand monostable nonlinear oscillators have been realized through nonlinear magnetic levitation \[12\], on a piezoelectric beam with magnetic end masses \[14\] etc. The monostable energy harvester can only work in the condition with slow and stepped frequency sweep \[21\]. However monostable oscillation appears naturally in a system, whereas to obtain bistable oscillation one needs to accommodate a special configuration.

3. Design and Static Analysis of Nonlinear Stretching of the Spring Arm

In this work, we report an EM Energy Harvester that utilizes nonlinear stretching strain to obtain wideband response. The designed harvester is shown in Figure 1(a). The oppositely polarized magnets are placed on the top of the central platform (10 mm$^2$ square) of the vibrating structure. Under the influence of external vibrations, the magnets move up and down and flux lines are cut by a conducting coil assembled above it. As a result, voltage is induced into the coil. We have chosen FR4 (standard PCB material) as the structural material of the vibrating structure. Nonlinear springs on FR4 could be a novel approach for wideband and low frequency applications due to its low Young’s Modulus (Y=22 GPa) and particularly for EM harvesters as conducting coils can be easily routed on FR4 laminates \[18\]. The length (L), width (W), thickness (d) of the cantilever arms are 9.2 mm, 0.8 mm, 0.2 mm.

![Figure 1](image-url) (a) Components of the proposed nonlinear energy harvester prototype. (b) Deformed structure in the first vibrating mode.

Large deflection appears when the proof mass deflection is comparable to, or larger than, the thickness of the spring arms. This results into stretching in addition to bending. The first vibrating mode of the device is shown in Figure 1(b).

The components of the spring force due to bending and stretching are obtained separately. For analytical modeling of the nonlinear spring force, a simplified model of fixed-fixed cantilever beam is used as shown in Figure 2(a). The contribution due to bending of the beam can be approximated by using the Euler Bernoulli’s beam equation for the vertical deflection of the neutral axis $u(x)$ along the beam is given by

$$\frac{d^4 u}{dx^4} = 0$$ \hspace{2cm} (2)

For a simple bending of the beam, $u(x)$ can be defined along the length of the beam. As seen from Figure 2(a), the boundary conditions for the problem are

$$u(0) = 0, \quad u'(0) = 0, \quad u(L) = d_f, \quad u'(L) = 0$$
Solving equation (2) for given boundary conditions, we obtain an expression for the deflection along the length \( x \) of the beam as

\[
u(x) = \frac{d_f}{l^2} \left[ 3x^2 - \frac{2x^3}{l} \right]
\]  

(3)

Now, the strain energy due to the bending component of the deformation,

\[
E_b(d_f) = \frac{Y}{24} d^3 W \int_0^l \left( \frac{\partial^2 u}{\partial x^2} \right)^2 dx = \frac{1}{2} Y \frac{d^4 W}{l^3} d_f^2
\]

(4)

Hence the component \( F_b \) of spring reaction force arising due to bending is given by

\[
F_b = -\frac{\partial E_b}{\partial d_f} = -Y \frac{d^3}{l^3} W d_f
\]

(5)

The contribution due stretching of the beam can be modeled using the same Figure 2(a). Stretching component of the strain can be approximated by Pythagoras theorem:

\[
Strain(v) = \frac{\Delta L}{L} \approx \frac{\sqrt{L^2 + (df)^2} - L}{L} = \sqrt{1 + \left( \frac{df}{L} \right)^2} - 1 \approx \frac{1}{2} \left( \frac{df}{L} \right)^2
\]

(6)

Then the energy stored due to this stretching component of strain can be written as:

\[
E_S = \int \frac{1}{2} Y d W \sigma^2 dL = \frac{1}{2} Y \left( \frac{d^2}{l^2} \right)^2 dWL = \frac{1}{2} Y \frac{dW}{l^3} d_f^4
\]

(7)

In a similar way, the component \( F_s \) of spring reaction force arising due to stretching is given by

\[
F_s = -\frac{\partial E_s}{\partial d_f} = -\frac{1}{2} Y \frac{d}{l^3} W d_f^3
\]

(8)

So, combining both the components, total spring reaction force is obtained analytically.

\[
F = F_b + F_s = -Y \frac{d^3}{l^3} W \left[ d^3 df + \frac{1}{2} d^2 d_f^3 \right] = -Y \frac{d^3}{l^3} W d^4 \left[ \frac{df}{d} + \frac{1}{2} \left( \frac{df}{d} \right)^3 \right]
\]

(9)

The above equation is similar to the well-known nonlinear spring force \( F = k x + k_n x^3 \). When the ratio \( \frac{df}{d} \) is less than one, the cubic nonlinearity is insignificant. When the tip deflection \( d_i \) is only greater than the thickness of the arm \( d \), or in other words, \( \frac{df}{d} \) is greater than one, then only effect of nonlinearity comes into play. Our proposed device exploits this cubic nonlinearity due to large deformation and results in wider bandwidth. The variation of the spring force with deflection is shown on Figure 2(b). The same result is validated by finite element analysis using COMSOL. Up to almost 200 \( \mu m \) of deflection, the stiffness is in the linear region. Beyond that, stiffness becomes nonlinear. The Duffing Potential \( U = \frac{1}{2} k x^2 + \frac{1}{4} k_n x^4 \) has monostable configuration [Figure 2(b) inset], which implies the hardening effect due to large deformations. From equation (9), the relation between linear stiffness constant \( k = \frac{Y W d^3}{l^3} \) and nonlinear stiffness constant \( k_n = \frac{Y W d}{2l^3} \) is

\[
k_n = \frac{1}{2d^2}
\]

(10)

For our design, the values of \( k \) and \( k_n \) are found to be 180.81 N/m and \( 2.25 \times 10^9 \) N/m³ and resonant frequency is \( f_0 = \frac{\omega_0}{2\pi} = 169 \) Hz.
4. Numerical Investigation of the Monostable Duffing Oscillator for EM Energy Harvesting

The governing equation for an electromechanical oscillator with restoring force \( \frac{du}{dx} \), base excitation \( \ddot{Z} \), and electromagnetic conversion mechanism [19] is given by

\[
m\ddot{x} + c\dot{x} + \frac{du}{dx} + \gamma I = -m\ddot{Z}
\]

(11)

As discussed in the previous section, the restoring force is of the form \( F = \frac{du}{dx} = kx + k_n x^3 \). So, substituting this in equation (11), we get the monostable Duffing oscillator equation

\[
m\ddot{x} + 2\gamma\omega_0 \dot{x} + kx + k_n x^3 + \gamma I = -m\omega^2 Z \sin \omega t
\]

(12)

Where, \( m \) is the moving mass, \( \rho \) is the damping co-efficient, \( z=Z\sin \omega t \) is the sinusoidal external perturbation, \( \omega \) is the external frequency. Since, we are considering EM transduction the external conversion circuit can be modeled using the following equation

\[
LI + R I - \gamma \dot{x} = 0
\]

(13)

Where \( \gamma \) is the electromagnetic coupling co-efficient \( R \) is the total resistance which includes coil resistance \( R_C \) and load resistance \( R_L \). \( L \) is the electromagnetic inductance. We have solved equation (12) and (13) numerically using 4\(^{th}\) order Runge Kutta method. The power dissipated in the electrical domain for the electromechanical oscillator is given by \( P(t) = \frac{v^2}{R} \hat{x}^2 \) and consequently we have calculated the average power as \( P_{avg} = \frac{1}{T} \int_0^T P(t) dt \). Spectra of average power for different load resistances are plotted in Figure 3(a). In our design we have chosen a copper wire wound coil having wire diameter 32 \( \mu \)m and 450 turns. The coil resistance is \( R_C = 60 \Omega \). We have also considered large input acceleration of \( \omega^2 Z = 20g \) (\( g=10m/s^2 \)). The value of \( m, \gamma \) and \( \rho \) are found to be \( 16\cdot10^{-5} \text{ kg}, 2.25\cdot10^{-3} \text{ Wb/m} \) and 0.1.

![Figure 3: (a) Power spectra for different values of load resistances and fixed \( \omega^2 Z=20g \). (b) Power spectra for different input accelerations and fixed \( R_L=200\Omega \).]

It is seen that as we lower the load resistance, total average power increases. This is because lowering the load increases the electrical damping and as a result more power is dissipated. It can be seen that there are two steady states available in that hysteresis width of the spectrum. So, in energy harvesting application the main concern is to keep the output in the high amplitude branch for longer time. The effect of external force can be seen from Figure 3(b). In this case we have kept load resistance fixed at \( R_L=200\Omega \). For smaller input force, the nonlinearity disappears and also the average power becomes low. The effect of nonlinear stretching becomes significant only with large force and it was seen in Figure 2(b) that the large deformation occurred for such high input force. Large amplitude deformation is responsible for increased average power. However, if we increase the input force too much then in addition to the main resonance several secondary resonances also appear increasing the overall loss within the system. Similar phenomenon was addressed by Parlitz et al [20].

Figure 4 shows the variation of jump frequency on up-sweep with load resistance and input acceleration. For small accelerations, there is no jump phenomena indicated by the blue surface of the curve. The jump frequency appears as the driving force increases and the surface become more reddish. Also for the red region, jump frequency increases with increasing load resistance. This is because increase of load reduces the total damping. As a result jump frequency increases. However
very low damping region is avoided in our analysis as such region includes more complex phenomenon such as sub harmonic resonances.

5. Conclusions
In this work, we have discussed the design and stretching strain based nonlinearity of an EM Energy Harvester. FR4 is chosen as the device material due to its suitable mechanical properties and ease of integratability with copper coils. An analytical formulation of large deformation of the spring arms has been done by calculating separately the contributions from bending and additional stretching, which results is cubic nonlinearity. The results are verified with the Finite Element Analysis. It is found that nonlinearity has an inverse square dependence on thickness of the arms. Also, numerical investigation of a monostable Duffing oscillator has been done. Results showed that with decrease of load resistance, the average power output increases, where the output response depends strongly on the input force. For small input acceleration, the desired large amplitude vibration does not come into play and the response becomes linear. However, for higher input acceleration nonlinearity appears and the operational bandwidth increases, at the same time, output power level also rises.

Acknowledgement
This work is financially supported by Science Foundation Ireland (SFI) Principal Investigator (PI) project on ‘Vibration Energy Harvesting’ grant no SFI-11/PI/1201.

References
[1] Kulkarni S, Roy S, O’Donnell T, Beeby S, Tudor J 2006 Journ. Appl. Phys. 99:08P511.
[2] Kulkarni S et al. 2008 Sensors and Actuators A 145–146:336–342.
[3] Beeby S P et al. 2007 J. Micromech. Microeng. 17:1257–1265.
[4] Koukharenko E et al. 2006 Microsystem technologies 12:1071-1077.
[5] Roundy S and Zhang Y 2005 Proc. SPIE, 5649:373-84.
[6] Leland E S and Wright P K 2006 Smart Mater. Struct., 15:1413-20.
[7] Challa V R, Prasad M G, Shi Y and Fisher F T 2008 Smart Mater. Struct., 17:015035.
[8] Peters C et al. 2009 J Micromech. Microeng, 9:094004.
[9] Sari I, Balkan T and Kulah H 2008 Sens. Actuators. A, 145-146:405-13.
[10] B Yang et al. 2009 J. Micromech. Microeng., 19:035001.
[11] Erturk A, Hoffmann J and Inman D J 2009 Appl. Phys. Lett., 94:254102.
[12] Mann B P and Sims N D 2009 J. Sound Vib., 319:515-30.
[13] Marinkovic B and Koser H 2009 Appl. Phys. Lett., 94:103505.
[14] Stanton S C, McGeehe C C and Mann B P 2010 Physica D, 239:640-53.
[15] Ramlan R, Brennan M J, Mace B R and Kovacic I 2010 Nonlinear Dyn., 59:545-58.
[16] Gammaitoni L, Neri I and Vocca H 2009 Appl. Phys. Lett., 94:164102.
[17] Cottone F, Vocca H and Gammaitoni L 2009 Phys. Rev. Lett., 102:080601.
[18] Hatipoglu G and Urey H Smart Mater. Struct. 2010 19:015022.
[19] R L Harne and K W Wang 2013 Smart Mater. Struct. 22:023001.
[20] Parlitz U, Lauterborn W 1985 Physics Letters 107A:8.
[21] Tang L et al. 2010 Journal of Intelligent Material Systems and Structures 21: 1867.

Figure 4: Variation of Jump Frequency on Up-Sweep with Load resistance and input acceleration. The blue portion implies there is no jump. As the curve becomes red, nonlinear phenomenon becomes prominent and jump frequency appears with high input force.