Reduced density matrices of oscillator systems

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Abstract

We study the evolution of an oscillator interacting via the most general bilinear coupling (with time-independent coefficients) with an “environment” consisting of a set of other harmonic oscillators. We are mainly interested in a possibility of using the Fokker-Planck equation to describe this evolution. Studying different interaction Hamiltonians, we show that unambiguous reduction to the Fokker-Planck equation is possible only within the framework of the so called rotating-wave approximation. As special cases we consider in detail the evolution of two coupled oscillators and relaxation of a charged oscillator in a uniform magnetic field.

1 Statement of the problem

This part is devoted to the multidimensional generalizations of the harmonic oscillator model. As a rule, these are various systems of interacting oscillators, some of which may be placed in external uniform electric and magnetic fields. More precisely this class of systems can be characterized by a Hamiltonian that is assumed to be in the form of the general inhomogeneous quadratic form

\[ \hat{H} = \frac{1}{2} \hat{q} \mathbf{B} \hat{q} + \mathbf{C} \hat{q}. \]  

(1.1)

Here, \( \hat{q} \) is conceived as a \( 2N \)-dimensional vector (\( N \) being the number of degrees of freedom) whose components are linear combinations of Cartesian coordinates and momenta conjugated to them. The most frequent choice is \( \hat{q} = (p_1, \ldots, p_N, x_1, \ldots, x_N) \), but in the presence of an external magnetic field it is more convenient to deal not with the canonical momenta \( p_j \) but with the kinetic momenta \( \pi_j = p_j - eA_j(x)/c \), where \( A(x) \) is the vector potential. Other choices are also possible, taking into account the concrete physical applications. For instance, the components of the vector \( \hat{q} \) may be bosonic annihilation and creation operators constructed from the coordinate and momentum operators, etc. \( \mathbf{B} \) is a symmetric \( 2N \times 2N \) matrix, which may depend on time, as also the \( 2N \)-dimensional vector \( \mathbf{C} \).

Multidimensional systems with Hamiltonian (1.1) were the subject of investigation in numerous papers: see, e.g., [1-3], [4-10] and the references therein. In the present paper we consider the case where the multidimensional vector \( \hat{q} \) may be split in two parts: \( \hat{q} = (\hat{Q}, \xi) \), where the vector \( \hat{Q} \) describes a small subsystem, whereas the vector \( \xi \) is related to a “thermostat”. In the simplest cases the evolution of a unidimensional harmonic oscillator coupled with a “thermostat” was studied, e.g., in [13, 17-38]. We investigate the most general quadratic “interaction Hamiltonians” and compare the results of this “microscopic” approach with different phenomenological models considered in reviews [13]. Emphasis is placed on application to the problem of harmonic oscillator relaxation, including the case where a (charged) oscillator is placed in a uniform magnetic field.

2 Phenomenological Fokker-Planck equation

The evolution of any closed system is governed by the quantum Liouville equation for the statistical operator \( \hat{\rho} \),

\[ i\hbar \partial \hat{\rho} / \partial t = \hat{H} \hat{\rho} - \hat{\rho} \hat{H}. \]  

(2.1)

For any quadratic Hamiltonian (1.1) this equation results in the linear equation for the average values of the \( 2N \)-vector \( \hat{q} \):

\[ \langle \dot{q} \rangle = -\Sigma \mathbf{B}(q) - \Sigma \mathbf{C}, \]  

(2.2)
where \( \langle q \rangle = \text{Tr}(\rho \hat{q}) \), and antisymmetric nondegenerate \( 2N \times 2N \) matrix \( \Sigma \) with \( c \)-number coefficients is defined via the commutation relations between the operators \( \hat{q}_a \),

\[
[\hat{q}_a, \hat{q}_b] = -i\hbar \Sigma_{\alpha \beta}, \quad \Sigma = \|\Sigma_{\alpha \beta}\|, \quad \alpha, \beta = 1, 2, \ldots 2N.
\]  

(2.3)

The operator equation (2.1) can be transformed into a partial differential equation upon the choice of some concrete representation for the statistical operator. For instance, in \([40]\). Eq. (2.7) assumes an especially compact form in terms of the \( 2 \times 2 \) matrix \( \rho \) (the tilde designates a transposed matrix). Then we obtain the following second-order equation for the density matrix \( \rho(x, x') \):

\[
\begin{align*}
\frac{i\hbar}{2} \frac{\partial \rho}{\partial t} = & \frac{\hbar^2}{2} b_1^{mn} \left( \frac{\partial^2 \rho}{\partial x'_m \partial x'_n} - \frac{\partial^2 \rho}{\partial x_m \partial x_n} \right) - i\hbar b_2^{mn} \left( x_m \frac{\partial \rho}{\partial x_m} + x'_n \frac{\partial \rho}{\partial x'_n} \right) \\
& + \frac{1}{2} b_3^{mn} \left( x_m x_n - x'_m x'_n \right) \rho - i\hbar \rho \text{Tr} \hat{b}_2 \\
& - i\hbar c_1^{nm} \left( \frac{\partial \rho}{\partial x_m} + \frac{\partial \rho}{\partial x'_m} \right) + c_2^{mn} (x_m - x'_m) .
\end{align*}
\]

(2.4)

Here \( c_1 \) and \( c_2 \) are \( N \)-dimensional components of the vector \( \mathbf{C} = (c_1, c_2) \). The disadvantage of Eq. (2.4) is the broken symmetry between the coordinates and momenta, which is inherent in the Hamiltonian. This symmetry can be restored if one proceeds from the complex density matrix to the real Wigner function

\[
W(p, x) = \int \rho(x + \xi/2, x - \xi/2) \exp(-ip\xi/\hbar) \text{d}\xi, 
\]

(2.5)

\[
\rho(x, x') = \int W \left( p, \frac{1}{2}(x + x') \right) \exp[ip(x - x')/\hbar] \text{d}p/(2\pi\hbar)^N.
\]

(2.6)

Applying transformation (2.3) to Eq. (2.4) we obtain an equivalent equation that is first order with respect to all the derivatives:

\[
\frac{\partial W}{\partial t} = (p_n b_2^{mn} + x_n b_1^{mn} + c_2^m) \frac{\partial W}{\partial p_m} - (p_n b_1^{nm} + x_n b_3^{nm} + c_1^m) \frac{\partial W}{\partial x_m}.
\]

(2.7)

This equation demonstrates the distinction and the advantage of the Wigner function for the description of quadratic quantum systems (other remarkable features of the Wigner function were discussed, e.g., in [40]). Eq. (2.7) assumes an especially compact form in terms of the \( 2N \)-vector \( q \),

\[
\frac{\partial W}{\partial t} = \frac{\partial}{\partial q_\alpha} \left[ \Sigma \mathbf{B} q + \Sigma \mathbf{C} \right]_\alpha W.
\]

(2.8)

Here the \( 2N \times 2N \) matrix \( \Sigma \), in accordance with Eq. (2.3), equals

\[
\Sigma = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix},
\]

(2.9)

\( I_N \) being the \( N \times N \) unit matrix.

Now let us suppose that we have a given linear equation for the first-order average values

\[
\langle q \rangle = \mathbf{A}(t) \langle q \rangle + \mathbf{K}(t),
\]

(2.10)

with an arbitrary matrix \( \mathbf{A}(t) \) and an arbitrary vector \( \mathbf{K}(t) \). The problem investigated in this section is whether it is possible to find an equation for the statistical operator or the Wigner function that would result in the given Eq. (2.10). It is trivial to check that Eq. (2.10) is the consequence of the equation

\[
\frac{\partial W}{\partial t} = -\frac{\partial}{\partial q_\alpha} \left[ (\mathbf{A} q + \mathbf{K})_\alpha W \right].
\]

(2.11)
However, although Eq. (2.11) preserves the normalization of the Wigner function,

\[ \int W(q) \, dq / (2\pi \hbar)^N = 1, \quad (2.12) \]

it does not agree, in general, with the fundamental quantum mechanical principle of the positive definiteness of the statistical operator and (this is almost the same) with the uncertainty relations. Indeed, let us consider the operator \( \hat{F} = \alpha_j \left( \hat{q}_j - \langle q_j \rangle \right) \) with arbitrary complex coefficients \( \alpha_j \). For any statistical operator, due to its nonnegative definiteness, the inequality \( \langle \hat{F}^\dagger \hat{F} \rangle \equiv \text{Tr} \left( \hat{\rho} \hat{F}^\dagger \hat{F} \right) \geq 0 \) must hold. Taking into account the structure of the operator \( \hat{F} \), we arrive at the conclusion on the nonnegative definiteness of the bilinear Hermitian form

\[ \alpha^* \Phi \alpha \equiv \alpha_j^* \Phi_{jk} \alpha_k, \quad \Phi_{jk} = \langle (\hat{q}_j - \langle q_j \rangle) (\hat{q}_k - \langle q_k \rangle) \rangle, \]

whose matrix \( \Phi \) is constructed from the centered second-order moments of the operators \( \hat{q}_j, j, k = 1, 2, \ldots, 2N \). It is convenient to distinguish even and odd parts of the matrix \( \Phi \). The symmetric part consists of the symmetrized second moments (covariances)

\[ M_{ij} = M_{ji} = \frac{1}{2} \langle \hat{q}_i \hat{q}_j + \hat{q}_j \hat{q}_i \rangle - \langle q_i \rangle \langle q_j \rangle. \quad (2.13) \]

The antisymmetric part, in accordance with Eq. (2.3), is expressed through the commutator matrix, so that

\[ \Phi = M - \frac{i\hbar}{2} \Sigma. \]

The elements of the matrix \( M \) are calculated in terms of the Wigner function as follows:

\[ M_{ij} = \int q_i q_j W(q) \, dq / (2\pi \hbar)^N - \langle q_i \rangle \langle q_j \rangle, \]

\[ \bar{q} = \langle q \rangle = \int q W(q) \, dq / (2\pi \hbar)^N. \]

Due to Eq. (2.11), the variance matrix satisfies the equation

\[ \dot{M} = A M + M \tilde{A}. \quad (2.14) \]

The first-order averages do not influence the variances in the case under study.

If the coefficients \( \alpha_j \) are chosen in such a way that the commutator \( [\hat{F}, \hat{F}^\dagger] \) is positive (the simplest example is \( \hat{F} = \hat{x} + i\hat{p} \)), then the operator \( \hat{F} \) coincides within a constant factor with a boson annihilation operator. Choosing the initial state to be the vacuum state for this operator, at the initial instant we have

\[ \text{Tr} \left( \hat{F}^\dagger \hat{F} \rho(0) \right) = \alpha^* \Phi(0) \alpha = 0. \]

Moreover, the matrix equalities \( \Phi(0) \alpha = \alpha^* \Phi(0) = 0 \) hold as well. For the chosen initial state we have at \( t > 0 \), due to Eq. (2.14),

\[ \alpha^* \Phi(t) \alpha = \alpha^* \left[ M(t) - \frac{i\hbar}{2} \Sigma \right] \alpha \]

\[ = \alpha^* \left[ A(0) M(0) + M(0) \tilde{A}(0) \right] t \alpha + O(t^2) \]

\[ = \frac{1}{2} i\hbar t \alpha^* \left[ A(0) \Sigma + \Sigma \tilde{A}(0) \right] \alpha + O(t^2). \quad (2.15) \]

For the Hamiltonian systems we have \( A = -\Sigma B \), \( \tilde{A} = B \Sigma \), and the linear with respect to time term disappears. But for an arbitrary matrix \( A \) the right-hand side of Eq. (2.15) can be negative. For example, in the model of a damped oscillator with the equations for the averages

\[ \dot{x} = p, \quad \dot{p} = -\omega_0^2 x - 2\gamma p, \quad (2.16) \]
the matrices entering Eq. (2.15) read
\[
\mathbf{A} = \begin{bmatrix} -2\gamma & -\omega_0^2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{A}\Sigma + \Sigma\mathbf{A} = \begin{bmatrix} 0 & -2\gamma \\ 2\gamma & 0 \end{bmatrix}.
\]
Then, for the initial vacuum state of operator \(\hat{F} = \hat{x} + i\hat{p}\), when \(\alpha = (i, 1)\), we obtain from Eq. (2.15)
\[
\alpha^*\Phi(t)\alpha = -2\hbar t\gamma + \mathcal{O}(t^2) < 0.
\]

Consequently, in the general case Eq. (2.11) is unacceptable. Thus we need more complicated generalizations. The simplest possibility is to add terms with second derivatives to the right-hand side of Eq. (2.11), i.e., to transform this equation into the Fokker-Planck equation:
\[
\frac{\partial W}{\partial t} = -\frac{\partial}{\partial q_\alpha} \left[ (\mathbf{A}q + \mathbf{K})_\alpha W \right] + \mathbf{D}_{\alpha\beta} \frac{\partial^2 W}{\partial q_\alpha \partial q_\beta},
\]
where the diffusion coefficients \(\mathbf{D}_{\alpha\beta} = D_{\beta\alpha}\), combined into a symmetric matrix \(\mathbf{D} = \| D_{\alpha\beta} \|\), may depend on time but do not depend on the coordinates. The new “diffusion” terms do not change the equation for the average values (2.10), moreover, they preserve the normalization (2.12). But they enable one to “save” the nonnegative definiteness of the statistical operator. Indeed, considering the evolution of the bilinear form \(\alpha^*\Phi(t)\alpha\), we obtain, instead of Eq. (2.15), the equation
\[
\alpha^*\Phi(t)\alpha = t\alpha^* \left( 2\mathbf{D} + \frac{i\hbar}{2} \left[ \mathbf{A}(0)\Sigma + \Sigma\mathbf{A}(0) \right] \right) \alpha + \mathcal{O}(t^2),
\]

since Eq. (2.14) is replaced by
\[
\dot{\mathbf{M}} = \mathbf{A}\mathbf{M} + \mathbf{M}\tilde{\mathbf{A}} + 2\mathbf{D}.
\]

Consequently, the necessary condition of the compatibility of Eq. (2.18) with the principles of quantum mechanics is the nonnegative definiteness of the matrix
\[
\mathbf{D}_* = \mathbf{D} + \frac{i\hbar}{4} \left[ \mathbf{A}\Sigma + \Sigma\mathbf{A} \right] \geq 0 \quad (2.20)
\]
\textit{at any instant of time.} Moreover, it can be proved [39, 41] that the condition \(\mathbf{D}_* \geq 0\) is sufficient as well.

Vector \(q\) was defined above as \(q = (p, x)\), and the matrix \(\Sigma\) had the explicit form given by Eq. (2.9). Let us make the time-independent transformation of the variables
\[
q' = Tq, \quad \text{det} T \neq 0, \quad \text{Im} T \neq 0.
\]

Then Eqs. (2.10) and (2.18) preserve their forms, provided the matrices \(\mathbf{A}\) and \(\mathbf{D}\) are replaced by the matrices \(\mathbf{A}' = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\) and \(\mathbf{D}' = \mathbf{T}\mathbf{D}\mathbf{T}\). The nonnegative definiteness of the matrix \(\mathbf{D}_*\) of Eq. (2.20) is equivalent to the nonnegative definiteness of the matrix \(\mathbf{D}_* = \mathbf{T}\mathbf{D}\mathbf{T}\). The latter has, in turn, again the form (2.20), if one replaces the matrix \(\Sigma\) by \(\Sigma' = \mathbf{T}\Sigma\mathbf{T}\), but this is just the transformation law of any matrix defined according to Eq. (2.3). This way we arrive at the important conclusion, that all the formulas containing the matrices \(\mathbf{A}, \mathbf{D}, \) and \(\Sigma\) are valid not only in the case where the components of the vector \(q\) coincide with the canonically conjugate momenta and Cartesian coordinates, but also in the general case where the components of the vector \(q\) are arbitrary Hermitian operators with \(c\)-number commutators, provided the matrix \(\Sigma\) is defined according to Eq. (2.3).

To transform Eq. (2.18) to an operator form that is independent on the concrete representation, one should take into account the following correspondence relations between the operators \(\hat{q}\tilde{\rho}, \hat{\rho}\tilde{q}\), and their Weyl symbols (they result from Eqs. (2.3), (2.4)):
\[
\tilde{q}\hat{\rho} \leftrightarrow \left( q - \frac{i\hbar}{2} \Sigma \frac{\partial}{\partial q} \right) W(q), \quad \tilde{\rho}q \leftrightarrow \left( q + \frac{i\hbar}{2} \Sigma \frac{\partial}{\partial q} \right) W(q),
\]
\[
qW(q) \leftrightarrow \frac{1}{2} (\tilde{q}\hat{\rho} + \hat{\rho}\tilde{q}), \quad \frac{\partial W}{\partial q} \leftrightarrow -\frac{i}{\hbar} \Sigma^{-1} (\tilde{q}\hat{\rho} - \hat{\rho}\tilde{q}).
\]
Making transformations (2.22) in Eq. (2.18) we arrive at the equation
\[
\frac{\partial \hat{\rho}}{\partial t} = \frac{i}{2\hbar} \left[ \hat{q} \Sigma^{-1} A \hat{\rho} + \hat{\rho} \Sigma^{-1} \hat{q} + \hat{q} \left( \Sigma^{-1} A + \Sigma^{-1} \hat{A} \right) \hat{\rho} \right] \\
+ \frac{i}{\hbar} \left[ \hat{q} \Sigma^{-1} K \hat{\rho} - \hat{\rho} \Sigma^{-1} \hat{K} \right] - \frac{1}{\hbar^2} \left[ \hat{q} \Sigma \hat{S} \hat{q} + \hat{\rho} \Sigma - 2 \hat{q} \hat{S} \hat{\rho} \right],
\]

(2.23)

where the symmetrical matrix
\[
S = -\Sigma^{-1} D \Sigma
\]

must satisfy a constraint equivalent to Eq. (2.20):
\[
S_* \equiv S - \frac{i\hbar}{4} \left[ \hat{A} \Sigma^{-1} + \Sigma^{-1} \hat{A} \right] \geq 0.
\]

(2.25)

Eqs. (2.21)-(2.25) hold for any vector \(q\) with a c-number commutator matrix \(\Sigma\) of (2.3).

Comparison of the elegant equation (2.18) for the Wigner function with the much more cumbersome equation (2.23) demonstrates once more the advantage of the Wigner representation for describing quantum systems with linear equations of motion for the averages.

There is an important difference between the quantum Fokker-Planck equation (2.18) and its classical counterpart. The classical Fokker-Planck equation contains, as a rule, second derivatives only with respect to momenta. However, such a simple set of the diffusion coefficients is unacceptable in the quantum case (although sometimes this incorrect equation was considered: see, e.g., [42]). This statement can be easily demonstrated on the example of system (2.16). Writing the matrix \(D\) in the form
\[
D = \begin{pmatrix}
D_p & D_{px} \\
D_{px} & D_x
\end{pmatrix}
\]

and taking into account Eq. (2.17) we obtain the matrix
\[
D_* = \begin{pmatrix}
D_p & D_{px} - i\hbar \gamma/2 \\
D_{px} + i\hbar \gamma/2 & D_x
\end{pmatrix}.
\]

The condition of its positive definiteness is given by the inequality [43]-[46]
\[
\det D_* \equiv D_p D_x - D_{px}^2 - \hbar^2 \gamma^2/4 \geq 0,
\]

(2.26)

whose violation leads to the violation of the uncertainty relations [43], [47]. For an arbitrary one-dimensional system (2.10) the condition \(D_* \geq 0\) is equivalent to the inequality [11]
\[
\det D \geq \hbar^2 (\text{Tr} A)^2/16.
\]

(2.27)

3 Fokker-Planck equation for a subsystem

In the preceding section we have shown that any given equation (2.10) may be considered as a consequence of some suitable Fokker-Planck equation for the Wigner function. The only problem is to select the set of diffusion coefficients satisfying the condition \(D_* \geq 0\). It is clear that by taking sufficiently large diffusion coefficients one can always satisfy this condition. The problem of finding “minimal admissible” diffusion coefficients for some simple systems (such as a one-dimensional harmonic oscillator or a two-dimensional isotropic oscillator in a uniform magnetic field) was investigated in [39], [48], [49].

Here we investigate the following problem. Suppose we have a large closed quantum system with \(N\) degrees of freedom, described by Hamiltonian (1.1). Let us split the vector \(q\) in two parts: \(q = (Q, \xi)\), where the \(n\)-dimensional vector \(Q\) describes a subsystem, while the vector \(\xi\) relates to a reservoir. The question is: what kind of equation describes the evolution of the subsystem if one performs an averaging over the variables of the reservoir?

The Wigner function of the whole system is given by the relation
\[
W(q, t) = \int G(q, q', t) W(q', 0) dq',
\]

(3.1)
where the propagator \( G(q, q', t) \) satisfies Eq. (2.3) and the initial condition \( G(q, q', 0) = \delta(q - q') \). Since Eq. (2.8) is first-order with respect to all the variables, its propagator is extremely simple:

\[
G(q, q', t) = \delta(q - q, t; q'),
\]

(3.2)

where the vector

\[
q, (t; q') = R(t)[q' - \Delta(t)]
\]

(3.3)
is the solution to the classical equation of motion (2.4), satisfying the initial condition \( q, (0; q') = q' \). Consequently, the \( 2N \times 2N \) matrix \( R(t) \) satisfies the equation

\[
\dot{R} = -\Sigma BR = \mathcal{A} R,
\]

(3.4)

and the initial condition \( R(0) = I_{2N} \). The vector \( \Delta(t) \) equals zero at \( t = 0 \). For \( t > 0 \) it is determined from the equations

\[
\dot{\Delta} = R^{-1} \Sigma C \equiv \Sigma \Delta R.
\]

(3.5)

These two forms are equivalent due to the identities

\[
\tilde{R}(t) \Sigma^{-1} R(t) \equiv \Sigma^{-1} \quad \tilde{R}(t) \Sigma \tilde{R}(t) \equiv \Sigma \quad R^{-1} \equiv \Sigma \Delta R^{-1},
\]

(3.6)

which follow from Eq. (3.4).

It is clear that the Heisenberg equation of motion for the operator \( \hat{q} \) coincides with Eq. (2.3), provided \( \{q\} \) is replaced by \( \tilde{q} \). The solution to this equation is given by Eq. (3.3) with caret over \( q \) and \( q' \). Therefore, the identities (3.6) mean nothing but the conservation of the commutation relations (2.3) in time, or, in other words, the canonicity of transformation (3.3) and the unitarity of the evolution operator.

Now let us proceed to the averaging of the total Wigner function over the reservoir variables \( \xi \). Our first assumption is that the initial total Wigner function is factorized:

\[
W(q, 0) = W_0(q)W_1(\xi).
\]

(3.7)
The second assumption concerns the initial Wigner function of the reservoir. We assume it to be Gaussian,

\[
W_1(\xi) = \hbar^M (\det \mathbf{F})^{-1/2} \exp \left[-\frac{1}{2}(\xi - \gamma)\mathbf{F}^{-1}(\xi - \gamma)\right],
\]

(3.8)

with some symmetric positive definite \( 2M \times 2M \) matrix \( \mathbf{F} \) and a \( 2M \)-vector \( \gamma \) \((M \text{ being the number of degrees of freedom of the reservoir})\). In particular, it may correspond to a mixed equilibrium state [8, 16, 50] or to a pure squeezed coherent state [7, 16, 51]. For any physically admissible Gaussian Wigner function the covariance matrix \( \mathbf{F} \) must satisfy a set of conditions expressing generalized uncertainty relations. The simplest among them is the inequality [22, 23] \( \det \mathbf{F} \geq (\hbar^2/4)^M \). Moreover, the parameter

\[
\mu = (\hbar/2)^M (\det \mathbf{F})^{-1/2} \leq 1
\]

(3.9)
characterizes “the degree of quantum mechanical purity” of the Gaussian state:

\[
\mu = \text{Tr} \rho^2 = \int W^2(\xi) \; d\xi / (2\pi\hbar)^M.
\]

(3.10)

To calculate the averaged Wigner function

\[
W_\xi(Q, t) = \int W(Q, \xi) d\xi / (2\pi\hbar)^M
\]

(3.11)
we split the matrix \( R \) and the vector \( \Delta \) into rectangular blocks in accordance with the decomposition \( q = (Q, \xi) \):

\[
R = \left[ \begin{array}{cc} R_{11} & R_{12} \\ R_{21} & R_{22} \end{array} \right], \quad \Delta = \left[ \begin{array}{c} \Delta Q \\ \Delta \xi \end{array} \right].
\]

(3.12)
Then Eqs. (3.1), (3.2), (3.7) lead to the integral

\[
W_\xi(Q, t) = \int \delta(Q - R_{11}[Q' - \Delta Q] - R_{12}[\xi' - \Delta \xi])
\times \delta(\xi - R_{21}[Q' - \Delta Q] - R_{22}[\xi' - \Delta \xi])
\times W_0(Q')W_1(\xi')dQ'd\xi'/ (2\pi\hbar)^M
\]

(3.13)
Finally we arrive at the same equation (3.1), but with the variables $Q$ and the most general quadratic interaction Hamiltonian $\mathcal{H}$. Let us illustrate the formulas of the preceding section, applying them to a very simple special model, “remaining part of the system” is Gaussian. However, the coefficients of the equation obtained in this way depend, as a rule, on time even for a time-independent Hamiltonian of the closed system.

4 Two coupled oscillators

Let us illustrate the formulas of the preceding section, applying them to a very simple special model, where both the “subsystem under study” and the “reservoir” are the harmonic oscillators with a single degree of freedom. This example admits exact solutions, in contrast to the more realistic situation of a reservoir with a very large number of degrees of freedom, where one has to make various simplifications and approximations to obtain a closed result that would be easy to analyze. Consider first the free Hamiltonian

$$H_0 = \frac{p_1^2}{2m_1} + \frac{1}{2} m_1 \omega_1^2 x_1^2 + \frac{p_2^2}{2m_2} + \frac{1}{2} m_2 \omega_2^2 x_2^2$$

and the most general quadratic interaction Hamiltonian

$$H_{\text{int}} = g_{pp} p_1 p_2 + g_{px} p_1 x_2 + g_{xp} x_1 p_2 + g_{xx} x_1 x_2.$$
We assume all the coefficients to be time-independent. In the notation of the preceding section we should write
\[ \mathbf{q} = (p_1, x_1, p_2, x_2), \quad \mathbf{Q} = (p_1, x_1), \quad \xi = (p_2, x_2). \]

Then the matrix \( \mathbf{A} = -\Sigma \mathbf{B} \) (see Eq. (3.4)) reads
\[
\mathbf{A} = \begin{pmatrix}
0 & -m_1 \omega_1^2 & -g_{xp} & -g_{xx} \\
-m_1^{-1} & 0 & g_{pp} & g_{px} \\
-g_{px} & -g_{xx} & 0 & -m_2 \omega_2^2 \\
g_{pp} & g_{px} & m_2^{-1} & 0
\end{pmatrix}.
\]

Its characteristic equation turns out to be biquadratic:
\[
\det(\mathbf{A} - i\omega \mathbf{I}) = \omega^4 - \left(\omega_1^2 + \omega_2^2 - 2\Delta\right) \omega^2 + \omega_1^2 \omega_2^2 + \Delta^2 - g = 0,
\]
where
\[
\Delta = \det \mathbf{A}_{12} = g_{pp} g_{xx} - g_{px} g_{xp}.
\]

The solutions of Eq. (4.4) are as follows:
\[
\omega_{\pm} = \frac{1}{\sqrt{2}} \left( \left\{ \begin{array}{c}
\frac{1}{2} \left( \omega_1^2 + \omega_2^2 \right) + \left( \omega_1^2 \omega_2^2 + \Delta^2 - g \right)^{1/2} + \Delta \\
\frac{1}{2} \left( \omega_1^2 + \omega_2^2 \right) - \left( \omega_1^2 \omega_2^2 + \Delta^2 - g \right)^{1/2} + \Delta
\end{array} \right\}^{1/2}
\right) \\
\pm \left\{ \left( \omega_1^2 + \omega_2^2 \right) - \left( \omega_1^2 \omega_2^2 + \Delta^2 - g \right)^{1/2} + \Delta \right\}^{1/2}
\]
(two other solutions are equal to \(-\omega_{\pm}\)).

Thus, the evolution of two coupled harmonic oscillators can be described explicitly for quite arbitrary quadratic interaction Hamiltonians with time-independent coefficients. If both frequencies given by Eq. (4.7) are real, then the particles perform harmonic oscillations. For certain parameters complex normal frequencies are possible. Then the motion becomes aperiodic. However, since any normal frequency \(\omega_+\) or \(\omega_-\) is accompanied by the frequency with the opposite sign, it is impossible to obtain damped oscillations of either particle. The coordinate and momentum of any oscillator will increase with time.

To illustrate this statement, let us consider first the case (which seems the most natural) of the interaction via the coordinates, where the only nonzero coefficient in Eq. (3.4) is \(g_{xx}\). Then \(\Delta = 0\), and for sufficiently strong coupling, when \(g > \omega_1^2 \omega_2^2\), we have the real frequency \(\omega_+\) and the pure imaginary frequency \(\omega_-\):
\[
\omega_+ \equiv \omega = \left\{ \left[ g + \frac{1}{4} \left( \omega_1^2 - \omega_2^2 \right)^2 \right]^{1/2} + \frac{1}{2} \left( \omega_1^2 + \omega_2^2 \right) \right\}^{1/2},
\]
\[
\omega_- = i\lambda, \quad \lambda = \left\{ \left[ g + \frac{1}{4} \left( \omega_1^2 - \omega_2^2 \right)^2 \right]^{1/2} - \frac{1}{2} \left( \omega_1^2 + \omega_2^2 \right) \right\}^{1/2}.
\]

Solving Eq. (3.4), we obtain the following formula for the matrix \(\mathbf{R}_{11}(t)\):
\[
\mathbf{R}_{11} = \begin{pmatrix}
\rho_+ \cos \omega t + \rho_- \cosh \lambda t & m_1 (-\omega \rho_+ \sin \omega t + \lambda \rho_- \sinh \lambda t) \\
\frac{1}{m_1} \left( \frac{\rho_+}{\omega} \sin \omega t + \frac{\rho_-}{\lambda} \sinh \lambda t \right) & \rho_+ \cos \omega t + \rho_- \cosh \lambda t
\end{pmatrix},
\]
where
\[
\rho_{\pm} = \frac{1}{2} \pm \frac{1}{4} \left( \omega_1^2 - \omega_2^2 \right) \left[ g + \frac{1}{4} \left( \omega_1^2 - \omega_2^2 \right)^2 \right]^{-1/2}.
\]

We see that the oscillations of the first particle actually increase. Its energy is derived from the nonpositive definite potential energy of the whole system, since the whole system turns out to be unstable when
Obtain, instead of Eq. (4.9), the expression following inequality must hold:

\[ \text{sign} (\omega^2 - \omega^2) > 0, \]

where \( \omega_+ \) is positive while the argument of the second square bracket is negative. Then the \( \omega \) signs. Of course, such a system is unstable, and it can be considered only as an extremely simplified case as well. Suppose for simplicity that \( m_1 = 1 \), \( m_2 = -1 \). Then \( g < 0 \). The square roots in Eq. (4.7) can be extracted if \( g = 2 \omega_0^2 \Delta \). In this case the imaginary parts of \( \omega_\pm \) differ from zero, provided \( \Delta < 0 \). Such a situation holds for the following relations between the coupling coefficients:

\[ g_{xp} = g_{pp}, \quad g_{xx} = -\omega_0^2 g_{pp}. \]

Consequently, the amplitude of the oscillations increases without bound. The drift matrix \( A \) depends on time as follows:

\[ A = \begin{pmatrix} \gamma \tanh \gamma t & -\omega_0^2 \\ 1 & \gamma \tanh \gamma t \end{pmatrix}. \]

An interesting model described by the Hamiltonian

\[ H = p_1 p_2 + \omega_0^2 x_1 x_2 + \gamma (x_2 p_2 - x_1 p_1) \]

was proposed for the first time by Bateman [54]. The Lagrangian

\[ L = \dot{x}_1 \dot{x}_2 - (\omega_0^2 + \gamma^2) x_1 x_2 + \gamma (x_1 \dot{x}_2 - \dot{x}_1 x_2) \]

was considered by Morse and Feshbach [55]. This system was investigated, e.g., in [56] (see also [13]). The equations of motion in this case read

\[ \dot{x}_1 + 2\gamma \dot{x}_1 + (\omega_0^2 + \gamma^2) x_1 = 0, \quad \dot{x}_2 + 2\gamma \dot{x}_2 + (\omega_0^2 + \gamma^2) x_2 = 0. \]

Consequently, at the classical level we have damping in the first mode and amplification in the second mode. Moreover, at the classical level both particles are completely independent due to Eq. (4.20). The quantum picture is more complicated, since the quantum behaviour is governed not by the second-order equations of motion, but by the Hamiltonian (1.15), in which the dynamical variables of both particles are entangled. (The nonunique correspondence between the equations of motion and the Lagrangians or Hamiltonians leading to them, as well as related ambiguities of quantization, were investigated in detail in [57]. It can be proved [58] that no Hamiltonian leading to Eq. (4.20) and coinciding with Eq. (1.1)
for $\gamma = 0$ exists.) In particular, the velocity of each particle is determined by the generalized momentum of the other particle:

$$\dot{x}_1 = p_2 - \gamma x_1, \quad \dot{x}_2 = p_1 + \gamma x_2.$$  

The two other Hamilton equations read

$$\dot{p}_1 = \gamma p_1 - \omega_0^2 x_2, \quad \dot{p}_2 = \gamma p_2 - \omega_0^2 x_1.$$  

In terms of the blocks of the matrix $R$ these equations can be rewritten as follows:

$$\dot{R}_{1k} = a R_{1k} + b R_{2k}, \quad \dot{R}_{2k} = b R_{1k} - a R_{2k} \quad (k = 1, 2) \quad (4.21)$$

where $a$ and $b$ are $2 \times 2$ matrices,

$$a = \begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix}, \quad b = \begin{bmatrix} 0 & -\omega_0^2 \\ 1 & 0 \end{bmatrix}.$$  

Eliminating the matrix $R_{2k}$ we obtain the second-order equation

$$\ddot{R}_{1k} - 2a \dot{R}_{1k} + (\gamma^2 + \omega_0^2) R_{1k} = 0. \quad (4.22)$$

Seeking its solution in the form $R_{1k}(t) = \exp(\Lambda t) R_{1k}$ we obtain the characteristic equation

$$\Lambda^2 - 2a \Lambda + (\gamma^2 + \omega_0^2) I = 0,$$

whose solution reads

$$\Lambda = a \pm i \omega_0 I.$$  

Taking into account the initial conditions

$$R_{11}(0) = I, \quad \dot{R}_{11}(0) = a, \quad R_{12}(0) = 0, \quad \dot{R}_{12}(0) = b,$$

we obtain finally the matrices

$$R_{11} = \cos \omega_0 t \begin{bmatrix} e^{\gamma t} & 0 \\ 0 & e^{-\gamma t} \end{bmatrix}, \quad R_{12} = \sin \omega_0 t \begin{bmatrix} 0 & -\omega_0 e^{\gamma t} \\ \omega_0^{-1} e^{-\gamma t} & 0 \end{bmatrix}.$$  

Consequently, after averaging over the state of the second particle we have a system with a damped coordinate, but with a momentum increasing in time without bound. Moreover, the momentum will no longer be related to the velocity. The drift matrix (3.19) in the Fokker-Planck equation for the averaged Wigner function of the first particle depends on time as follows:

$$A = \begin{bmatrix} \omega_0 \tan(\omega_0 t) + \gamma & 0 \\ 0 & \omega_0 \tan(\omega_0 t) - \gamma \end{bmatrix}. $$  

We see that the reduction of “Bateman’s mirror model” (4.18) does not lead to a damped quantum oscillator in the conventional meaning of this term.

## 5 Oscillator in a thermostat: weak coupling limit

We now proceed to a more realistic model, where the oscillator under study (its frequency will be denoted by $\omega_0$) is coupled to a large number of other oscillators with frequencies $\omega_i$. This model was the subject of investigations in many papers: see, e.g., [17]-[38]. A more comprehensive reference list can be found in the review [13]. We assume that each oscillator is described by the standard Hamiltonian

$$H_i = \frac{1}{2} (p_i^2 + \omega_i^2 x_i^2)$$

with unit mass (this can easily be achieved by rescaling the coordinates), while the quadratic interaction Hamiltonian is chosen in the most general form:

$$H_{\text{int}} = \sum_i (z_i p_i p_0 + v_i p_i x_0 + u_i x_i p_0 + g_i x_i x_0). \quad (5.1)$$
The coupling constants \( z_i, v_i, u_i, g_i \) are assumed to be time independent.

Equation (5.3) and its initial condition are equivalent to the following equations and initial conditions for the blocks of the matrix \( \mathbf{R} \) of (3.12):

\[
\begin{align*}
\dot{\mathbf{R}}_{11} &= \mathbf{A}_{11} \mathbf{R}_{11} + \mathbf{A}_{12} \mathbf{R}_{21}, & \mathbf{R}_{11}(0) &= \mathbf{I}, \quad (5.2) \\
\dot{\mathbf{R}}_{21} &= \mathbf{A}_{21} \mathbf{R}_{11} + \mathbf{A}_{22} \mathbf{R}_{21}, & \mathbf{R}_{21}(0) &= 0, \quad (5.3) \\
\dot{\mathbf{R}}_{12} &= \mathbf{A}_{11} \mathbf{R}_{12} + \mathbf{A}_{12} \mathbf{R}_{22}, & \mathbf{R}_{12}(0) &= 0, \quad (5.4) \\
\dot{\mathbf{R}}_{22} &= \mathbf{A}_{21} \mathbf{R}_{12} + \mathbf{A}_{22} \mathbf{R}_{22}, & \mathbf{R}_{22}(0) &= \mathbf{I}. \quad (5.5)
\end{align*}
\]

The \( 2 \times 2 \) matrix \( \mathbf{A}_{11} \) and the \( 2M \times 2M \) matrix \( \mathbf{A}_{22} \) (\( M \) is the number of oscillators in the reservoir) read

\[
\mathbf{A}_{11} = \begin{bmatrix} 0 & -\omega_0^2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{A}_{22} = \begin{bmatrix} 0 & -\text{diag}(\omega_1^2, \ldots, \omega_i^2, \ldots) \\ \text{diag}(1, \ldots, 1, \ldots) & 0 \end{bmatrix}. \quad (5.6)
\]

The matrices \( \mathbf{A}_{12} \) and \( \mathbf{A}_{21} \) are rectangular with the dimensions \( 2 \times 2M \) and \( 2M \times 2 \), respectively:

\[
\mathbf{A}_{12} = \begin{bmatrix} -v_i & \cdots & -v_i & \cdots & -g_i \cdots & -g_i \cdots \\ z_i & \cdots & z_i & \cdots & u_i \cdots & u_i \cdots \end{bmatrix}, \quad (5.7)
\]

\[
\mathbf{A}_{21} = \begin{bmatrix} -u_1 & -g_1 \\ \vdots & \vdots \\ -u_i & -g_i \\ \vdots & \vdots \\ z_1 & v_1 \\ \vdots & \vdots \\ z_i & v_i \\ \vdots & \vdots \end{bmatrix}. \quad (5.8)
\]

In this section we consider the case where all the elements of the interaction matrices \( \mathbf{A}_{12} \) and \( \mathbf{A}_{21} \) are small. Then we may use perturbation theory. In the zeroth approximation we obtain from Eq. (5.2)

\[
\mathbf{R}_{11}^{\!(0)}(t) = \exp (\mathbf{A}_{11} t). \quad (5.9)
\]

Putting this expression into the right-hand side of Eq. (5.3) we obtain the first-order solution for the matrix \( \mathbf{R}_{21} \),

\[
\mathbf{R}_{21}^{\!(1)}(t) = \exp (\mathbf{A}_{22} t) \int_0^t \exp (-\mathbf{A}_{22} \tau) \mathbf{A}_{21} \exp (\mathbf{A}_{11} \tau) \, d\tau. \quad (5.10)
\]

Then Eqs. (5.9) and (5.10) lead to the following first-order approximation for the matrix \( \mathbf{A} \) governing the evolution of the average values of the subsystem variables according to Eq. (2.18):

\[
\mathbf{A}^{\!(1)} = \mathbf{A}_{11} + \mathbf{A}_{12} \mathbf{R}_{21}^{\!(1)}(t) \left[ \mathbf{R}_{11}^{\!(0)}(t) \right]^{-1}. \quad (5.11)
\]

Taking into account Eqs. (5.9) and (5.10) and making the change of variable \( t - \tau = x \) in the integrand we arrive at the formula

\[
\mu \equiv \mathbf{A} - \mathbf{A}_{11} = \mathbf{A}_{12} \int_0^t \exp (\mathbf{A}_{22} x) \mathbf{A}_{21} \exp (-\mathbf{A}_{11} x) \, dx. \quad (5.12)
\]

The solutions to Eqs. (5.4) and (5.5) in the same approximation read

\[
\mathbf{R}_{22}^{\!(0)}(t) = \exp (\mathbf{A}_{22} t), \quad (5.13)
\]

\[
\mathbf{R}_{12}^{\!(1)}(t) = \exp (\mathbf{A}_{11} t) \int_0^t \exp (-\mathbf{A}_{11} \tau) \mathbf{A}_{12} \exp (\mathbf{A}_{22} \tau) \, d\tau. \quad (5.14)
\]
Then the diffusion matrix of the Fokker-Planck equation \((2.18)\), due to Eqs. \((3.21)\) and \((5.4)\), reads
\[
\mathbf{D} = \text{sym} \left( \mathbf{A}_{12} \mathbf{R}_{22}^{(0)}(t) \mathbf{FR}_{12}^{(1)}(t) \right) \\
= \text{sym} \left\{ \mathbf{A}_{12} \exp(\mathbf{A}_{22}t) \mathbf{F} \int_0^t \exp(\mathbf{A}_{22}\tau) \mathbf{A}_{12} \exp(-\mathbf{A}_{11}\tau) \ d\tau \exp(\mathbf{A}_{11}t) \right\}. \tag{5.15}
\]

The explicit forms of the matrices \(\mathbf{R}_{11}^{(0)}(t)\) and \(\mathbf{R}_{22}^{(0)}(t)\) are as follows:
\[
\exp(\mathbf{A}_{11}t) = \begin{pmatrix}
\cos \omega_0 t & -\omega_0 \sin \omega_0 t \\
\omega_0^{-1} \sin \omega_0 t & \cos \omega_0 t
\end{pmatrix}, \tag{5.16}
\]
\[
\exp(\mathbf{A}_{22}t) = \begin{pmatrix}
\text{diag}(\cos \omega_i t) & \text{diag}(\omega_0^{-1} \sin \omega_i t) \\
\text{diag}(\omega_0^{-1} \sin \omega_i t) & \text{diag}(\cos \omega_0 t)
\end{pmatrix}. \tag{5.17}
\]

If we choose the thermostat variance matrix to be
\[
\mathbf{F} = \begin{pmatrix}
\text{diag}(\omega_0^2 f_i) & 0 \\
0 & \text{diag}(f_i)
\end{pmatrix}, \tag{5.18}
\]
(in particular, \(\mathbf{F}\) may be an equilibrium variance matrix for the thermostat variables), then it describes the steady-state solution of Eq. \((2.14)\) in the absence of the interaction:
\[
\mathbf{F}(t) = \exp(\mathbf{A}_{22}t) \mathbf{F} \exp(\mathbf{A}_{22}^\dagger t) = \mathbf{F}. \tag{5.19}
\]

Consequently, one may interchange the matrices \(\exp(\mathbf{A}_{22}t)\) and \(\mathbf{F}\) in Eq. \((5.15)\) in accordance with the formula
\[
\exp(\mathbf{A}_{22}t) \mathbf{F} = \mathbf{F} \exp(-\mathbf{A}_{22}^\dagger t). \tag{5.20}
\]

Then Eq. \((5.15)\) assumes the form \((x = t - \tau)\)
\[
\mathbf{D} = \text{sym} \left\{ \int_0^t \exp(\mathbf{A}_{11}x) \mathbf{A}_{12} \exp(-\mathbf{A}_{22}x) \ dx \mathbf{F} \mathbf{A}_{12} \right\}. \tag{5.21}
\]

The explicit expressions for the matrix elements of \(2 \times 2\) matrix \(\mu_i\) of \((5.12)\) are as follows:
\[
\mu_{11} = \frac{1}{2} \sum_i \left[ - \Delta_i S_i^{(\pm)} + \omega_0^{-1} G_i S_i^{(-)} + \kappa_i C_i^{(\pm)} \right], \tag{5.22}
\]
\[
\mu_{12} = \frac{1}{2} \sum_i \left[ \omega_0 \kappa_i S_i^{(-)} + \omega_0 \Delta_i C_i^{(-)} + G_i C_i^{(\pm)} \right], \tag{5.23}
\]
\[
\mu_{21} = \frac{1}{2} \sum_i \left[ \omega_0^{-1} \kappa_i S_i^{(-)} - \omega_0^{-1} \Delta_i C_i^{(-)} - Z_i C_i^{(\pm)} \right], \tag{5.24}
\]
\[
\mu_{22} = \frac{1}{2} \sum_i \left[ - \Delta_i S_i^{(\pm)} - \omega_0 Z_i S_i^{(-)} - \kappa_i C_i^{(\pm)} \right]. \tag{5.25}
\]

We have introduced the notation
\[
S_i^{(\pm)} = \frac{\sin(\omega_i - \omega_0) t}{\omega_i - \omega_0} \pm \frac{\sin(\omega_i + \omega_0) t}{\omega_i + \omega_0}, \tag{5.26}
\]
\[
C_i^{(\pm)} = \frac{1 - \cos(\omega_i - \omega_0) t}{\omega_i - \omega_0} \pm \frac{1 - \cos(\omega_i + \omega_0) t}{\omega_i + \omega_0}. \tag{5.27}
\]

Other parameters are the bilinear combinations of the coupling constants:
\[
\Delta_i = g_i z_i - u_i v_i, \quad \kappa_i = \omega_i z_i v_i + g_i u_i / \omega_i, \quad G_i = \omega_i v_i^2 + g_i^2 / \omega_i, \quad Z_i = \omega_i z_i^2 + u_i^2 / \omega_i. \tag{5.28}
\]
Eq. [5.21] results in the following expressions for the elements of the diffusion matrix $D$:

$$
D_{11} = \frac{1}{2} \sum_i \omega_i f_i \left[ G_i S_i^{(+)} + \omega_0 \Delta_i S_i^{(-)} - \omega_0 \kappa_i C_i^{(-)} \right], \quad (5.29)
$$

$$
D_{22} = \frac{1}{2} \sum_i \omega_i f_i \left[ Z_i S_i^{(+)} + \omega_0^{-1} \Delta_i S_i^{(-)} + \omega_0^{-1} \kappa_i C_i^{(-)} \right], \quad (5.30)
$$

$$
D_{12} = \frac{1}{2} \sum_i \omega_i f_i \left[ -\kappa_i S_i^{(+)} + (2\omega_0)^{-1} (\omega_0^2 Z_i - G_i) C_i^{(-)} \right]. \quad (5.31)
$$

We see that in the general case both the drift matrix and the diffusion matrix have rather complicated time dependences. However, under certain conditions the formulas can be simplified, if we proceed to the *continuum limit*. This means that we assume the number of oscillators in the reservoir to be very large and the frequencies $\omega_i$ to be so close to each other that we may replace the sums by integrals over $d\omega_i \equiv d\omega$. Then we need to calculate integrals of the following type:

$$
\sigma_\pm = \int \varphi(\omega) \frac{\sin(\omega \pm \omega_0) t}{\omega \pm \omega_0} d\omega, \quad (5.32)
$$

$$
\sigma_\pm = \int \varphi(\omega) \frac{1 - \cos(\omega \pm \omega_0) t}{\omega \pm \omega_0} d\omega. \quad (5.33)
$$

From physical considerations it is clear that the elements of the matrices $\mu$ and $D$ are determined mainly by the terms in Eqs. (5.22)-(5.25) and (5.29)-(5.31) that correspond to frequencies near $\omega_0$, since the most effective interactions between the oscillator under study and the thermostat oscillators take place under resonance condition. Indeed, for a sufficiently smooth function $\varphi(\omega)$ and for $t \gg \omega_0^{-1}$, only points belonging to the domain $|\omega - \omega_0| \leq t^{-1}$ make a significant contribution to the integrals (5.32) and (5.33), due to the rapid oscillations of the trigonometric functions outside this domain. Thus, assuming that $\omega = \omega_0$ in all the functions except $\sin(\omega + \omega_0)t$, we may evaluate the integral $\sigma_+$ as follows:

$$
\sigma_+ \approx \frac{\varphi(\omega_0)}{2\omega_0} \int_{\omega_0}^{2\omega_0} \sin(\omega + \omega_0) t d\omega = \frac{\varphi(\omega_0)}{2\omega_0 t} [\cos(\omega_1 + \omega_0) t - \cos(\omega_2 + \omega_0) t].
$$

Consequently, we may neglect the value of $\sigma_+$ at $\omega_0 t \gg 1$. The same is true for the integral

$$
\int \varphi(\omega) \frac{\cos(\omega + \omega_0) t}{\omega + \omega_0} d\omega \sim \mathcal{O} \left( \frac{1}{\omega_0 t} \right).
$$

As concerns the integral $\sigma_-$, its value does not depend on time for $t \gg \omega_0^{-1}$, since making the substitutions $x = \omega - \omega_0$, $y = \omega t$, we obtain

$$
\sigma_- = \varphi(\omega_0) \int \frac{\sin xt}{x} dx = \varphi(\omega_0) \int \frac{\sin y}{y} dy = \pi \varphi(\omega_0). \quad (5.34)
$$

To evaluate the integral $\int \varphi(\omega) \frac{\cos(\omega - \omega_0) t}{\omega - \omega_0} d\omega$ we use the Taylor expansion $\varphi(\omega) = \varphi(\omega_0) + \varphi'(\omega_0)(\omega - \omega_0) + \cdots$, make the substitution $x = \omega - \omega_0$, and expand the limits of integration from $-\infty$ to $\infty$. Then the first integral vanishes, because the function $\cos(\omega t)/\omega$ is odd. The second integral decreases at least as $1/t$ when $t \to \infty$. Therefore, we may assume that the integrals $\delta_\pm$ do not depend on time at $\omega_0 t \gg 1$:

$$
\delta_\pm = \int \frac{\varphi(\omega)}{\omega - \omega_0} d\omega. \quad (5.35)
$$

In the case of the “minus” sign the principal value of the integral is implied: it is designated with the symbol $\int'$. We arrive at the following expressions for the matrix elements of the matrices $\mu$ and $D$ in the continuum limit:

$$
\mu_{11} = -\frac{1}{2} \pi \nu(\omega_0) \left[ \Delta(\omega_0) + \omega_0^{-1} G(\omega_0) \right] + \int' \frac{\omega \kappa(\omega) \nu(\omega)}{\omega^2 - \omega_0^2} d\omega, \quad (5.36)
$$

$$
\mu_{12} = \frac{1}{2} \pi \nu(\omega_0) \omega \kappa(\omega_0) + \int' \frac{\nu(\omega)}{\omega^2 - \omega_0^2} \left[ \omega_0^2 \Delta(\omega) + \omega G(\omega) \right] d\omega, \quad (5.37)
$$
\[ \mu_{21} = \frac{1}{2} \pi \nu(\omega_0) \omega_0^{-1} \kappa(\omega_0) - \int \frac{\nu'(\omega) \kappa(\omega)}{\omega^2 - \omega_0^2} \left[ \Delta(\omega) + \omega Z(\omega) \right] \, d\omega, \]  
(5.38)

\[ \mu_{22} = - \frac{1}{2} \pi \nu(\omega_0) \left[ \Delta(\omega_0) + \omega_0 Z(\omega_0) \right] - \int \frac{\omega \nu(\omega) \nu'(\omega)}{\omega^2 - \omega_0^2} \, d\omega, \]  
(5.39)

\[ D_{11} = \frac{1}{2} \pi \nu(\omega_0) f(\omega_0) \omega_0 \left[ \omega_0 \Delta(\omega_0) + G(\omega_0) \right] - \omega_0^2 \int \frac{\omega f(\omega) \kappa(\omega) \nu'(\omega)}{\omega^2 - \omega_0^2} \, d\omega, \]  
(5.40)

\[ D_{22} = \frac{1}{2} \pi \nu(\omega_0) f(\omega_0) \left[ \Delta(\omega_0) + \omega_0 Z(\omega_0) \right] + \int \frac{\omega f(\omega) \kappa(\omega) \nu'(\omega)}{\omega^2 - \omega_0^2} \, d\omega, \]  
(5.41)

\[ D_{12} = - \frac{1}{2} \pi \nu(\omega_0) \omega_0 f(\omega_0) \kappa(\omega_0) + \int \frac{\omega f(\omega) \nu'(\omega)}{\omega^2 - \omega_0^2} \left[ \omega^2 Z(\omega) - G(\omega) \right] \, d\omega. \]  
(5.42)

Here \( \nu(\omega) \) is the density of states function, whereas the functions \( Z(\omega), G(\omega), \) etc. are obvious generalizations of functions defined in Eq. (5.28).

In principle, the frequency dependences of the coupling constants can be chosen in such a way that the integrals in Eqs. (5.39)-(5.41) vanish. For instance, this is possible provided the corresponding combinations of functions \( \kappa, \nu, f, G, Z, \Delta, \omega, \) understood as functions of the argument \( x = \omega^2, \) do not change their values under the reflection in the point \( x_0 = \omega_0^2, \) and these functions (or at least the density of states) decrease sufficiently rapidly with distance from the point \( x_0 \) in both directions. In such a case all the coefficients \( \mu_{ik} \) and \( D_{ik} \) are determined by the values of the aforementioned functions at the point \( \omega_0. \) Moreover, the diffusion coefficients are proportional to the corresponding elements of the matrix \( \mu: \)

\[ D_{11} = -\omega_0^2 f_0 \mu_{11}, \quad D_{22} = -f_0 \mu_{22}, \quad D_{12} = -f_0 \mu_{12} = -\omega_0^2 f_0 \mu_{21}. \]  
(5.43)

We see that in the continuous weak coupling limit the reduced Wigner function of the oscillator obeys (under certain conditions) the Fokker-Planck equation (2.18) with time-independent coefficients at times \( t \gg \omega_0^{-1} \). Let us check, however, whether condition (2.27) is fulfilled. Since \( \text{Tr} A = \text{Tr} \mu, \) then due to Eq. (5.43) we must check the inequality

\[ \omega_0^2 f_0^2 \mu_{11} \mu_{22} - f_0^2 \mu_{12}^2 \geq \hbar^2 \left( \mu_{11} + \mu_{22} \right)^2 / 16. \]  
(5.44)

Taking into account Eqs. (5.39)-(5.38) and Eq. (5.28), we arrive at the inequality (all the functions are taken at the point \( \omega = \omega_0 \))

\[ (4f_0^2 \hbar^2) \left[ 2 \Delta^2 + \Delta(\omega Z + G/\omega) \right] \geq \left[ 2 \Delta + (\omega Z + G/\omega) \right]^2. \]  
(5.45)

It cannot be satisfied for an arbitrary choice of coupling constants. For instance, it is violated if \( \Delta = 0, \) or when any three of four coefficients \( g, z, u, v \) vanish.

This result seems paradoxical. Indeed, we started from the exact equation of motion for the density matrix of a closed system, found the exact solution to this equation, and after this we performed averaging over the thermostat degrees of freedom. Since the laws of quantum mechanics were not violated at any step, the reduced density matrix must be positive definite at any time for quite arbitrary coupling constants. On the other hand, if, for instance, \( z_i = u_i = v_i = 0, \) then inequality (5.45) does not hold. Hence, following the reasonings given in Sec. 2 we could obtain a nonpositive definite density matrix in the process of evolution!

This apparent contradiction is resolved in the following way. Inequality (2.27) is a necessary and sufficient condition ensuring that any density matrix that was positive definite at any instant of time will remain positive definite at all subsequent moments. But in the problem under study we have the selected instant \( t = 0: \) this is just the moment when the interaction with the thermostat was turned on. Since in the presence of the interaction with the environment the evolution of the oscillator density matrix is nonunitary, the set of density matrices \( \rho(t) \) arising from all initially admissible density matrices \( \rho(0) \) does not coincide with the set of all admissible density matrices. In particular, correct initial density matrices cannot turn into the specific ones that could become nonpositive definite at some instant of time in the case of violation of inequality (5.45). Therefore, there is no need to check conditions like (2.27) or (5.44), (5.45) when the density matrix of the subsystem is obtained by reduction of the exact density matrix of the closed system: the reduced density matrix (calculated with the proper accuracy) turns out to be positive definite automatically.
However, if the goal is the derivation of a self-consistent Fokker-Planck equation on the basis of a “microscopic” model of the oscillator interacting with a large reservoir, then we have to recognize that the underlying “microscopic” model cannot be quite arbitrary: its parameters must satisfy rather strong restriction \((5.45)\), in order to prevent the appearance of the unphysical solutions when this equation is applied to arbitrary initial states.

Returning to the analysis of Eqs. \((5.36)-(5.43)\), we notice that the matrix
\[
F_0 = \begin{pmatrix}
\omega_0^2 f_0 & 0 \\
0 & f_0
\end{pmatrix}
\] (5.46)
satisfies, due to Eq. \((5.43)\), the relation
\[
AF_0 + F_0 \tilde{A} + 2D = 0.
\] (5.47)
This means that \(F_0\) is the steady state solution to Eq. \((2.19)\) for the oscillator variance matrix, independently of the concrete values of the coefficients of the drift matrix \(\mu_{ik}\). In particular, if
\[
f_0 = \frac{\hbar}{2\omega_0} \coth \left( \frac{\hbar \omega_0}{2kT} \right),
\] (5.48)
then the matrix \((5.46)\) coincides with the equilibrium variance matrix of the oscillator. Consequently, the steady state solution of the Fokker-Planck equation with the coefficients \((5.36)-(5.42)\) is the equilibrium Wigner function of the oscillator, if the thermostat itself is in the equilibrium state. Moreover, we may imagine a situation with
\[
f_0 = \frac{\hbar}{2\omega_0} \sum_{i=0}^{n} \alpha_i \coth \left( \frac{\hbar \omega_0}{2kT_i} \right), \quad \sum_{i=1}^{n} \alpha_i = 1, \quad \alpha_i \geq 0,
\] (5.49)
Then, the oscillator under study exhibits relaxation to the Gaussian steady state with the variance matrix given by Eqs. \((5.46)\) and \((5.49)\). Such a situation may be realized when the thermostat consists of several large independent subsystems possessing their own temperatures (see, e.g., [59]). In this case \(n\) is the number of subsystems, and \(\alpha_i\) is the “weight” of each subsystem.

We see that the final steady state of the oscillator weakly interacting with a thermostat does not depend on the concrete values of the coupling constants, provided the thermostat was initially in an equilibrium or quasi-equilibrium (described by Eq. \((5.49)\) state. Now let us analyze possible forms of the drift matrix \(A\). Due to the property \(\mu_{12} = \omega_0^2 \mu_{21}\) its characteristic equation \(\det(A - \lambda I) = 0\) reads
\[
\lambda^2 - (\mu_{11} + \mu_{22}) \lambda + \omega_0^2 + \mu_{11} \mu_{22} - \mu_{12} \mu_{21} = 0.
\] (5.50)
The solutions to this equations can be written as \(\lambda_{1,2} = -\gamma \pm i\omega_*\), with
\[
\omega_* = \left[ \omega_0^2 - \frac{1}{4} (\mu_{11} - \mu_{22})^2 - \mu_{12} \mu_{21} \right]^{1/2},
\] (5.51)
\[
\gamma = -\frac{1}{2} \text{Tr} \mu = -\frac{1}{2} (\mu_{11} + \mu_{22}) = \frac{1}{4} \pi \nu(\omega_0) \left[ 2\Delta_0 + \omega_0 Z_0 + \omega_0^{-1} G_0 \right].
\] (5.52)
The last formula holds even when the integrals in Eqs. \((5.36)\) and \((5.39)\) are not equal to zero. Assuming the damping coefficient \(\gamma\) to be positive and comparing Eqs. \((5.52)\) and \((5.45)\), we may rewrite inequality \((5.45)\) as follows (recall that all functions are taken at \(\omega = \omega_0\)):
\[
(4f \omega/\hbar)^2 \Delta \geq 2\Delta + (\omega Z + G/\omega) = 4\gamma/[\pi \nu(\omega)].
\] (5.53)
In particular, for an equilibrium thermostat at zero temperature (when \(f = \hbar/2\omega\)) we obtain the relation
\[
2\Delta \geq \omega Z + G/\omega,
\] (5.54)
which due to Eq. \((5.28)\) turns into the inequality
\[
(u + v)^2 + (\omega z - g/\omega)^2 \leq 0.
\] (5.55)
Thus we arrive at a striking conclusion: the microscopic model of the interaction between the oscillator and the equilibrium reservoir at zero temperature, based on the general quadratic Hamiltonian (5.1), leads to the Fokker-Planck equation with time independent coefficients, valid for any physically admissible initial states of the oscillator, if and only if the coupling constants at the main oscillator frequency satisfy the restriction
\[ u_0 = -v_0, \quad z_0 = g_0/\omega_0^2. \]  
In this case Eqs. (5.36)-(5.39) yield
\[ \mu_{11} = \mu_{22}, \quad \mu_{12} = \mu_{21} = 0, \]  
so we have the unique drift matrix
\[ \mu = \begin{pmatrix} -\gamma & 0 \\ 0 & -\gamma \end{pmatrix} \]  
with the damping coefficient
\[ \gamma = \pi \nu (\omega_0) \Delta_0, \quad \Delta_0 = u_0^2 + g_0^2/\omega_0^2. \]  
Due to Eq. (5.57) the frequency \( \omega_* \) in formula (5.53) exactly equals the oscillator eigenfrequency \( \omega_0 \).

The equations of motion for the average values of the coordinate and momentum read
\[ \dot{p} = -\gamma p - \omega_0^2 x, \]  
\[ \dot{x} = p - \gamma x. \]  
One can see that consistent quantum mechanical consideration do not result in the conventional classical equations (2.16). To understand the origin of Eqs. (5.60) and (5.61), let us introduce the annihilation and creation operators
\[ a = \frac{\omega_0 \hat{x} + i \hat{\pi}}{\sqrt{2\hbar \omega_0}}, \quad a^\dagger = \frac{\omega_0 \hat{x} - i \hat{\pi}}{\sqrt{2\hbar \omega_0}}. \]  
which are the most natural for the description of a quantum oscillator. It turns out that precisely Eqs. (5.60), (5.61) lead to uncoupled equations for \( \hat{a} \) and \( \hat{a}^\dagger \):
\[ \dot{a} = -i \omega_0 \hat{a} - \gamma \hat{a}, \quad \dot{a}^\dagger = i \omega_0 \hat{a}^\dagger - \gamma \hat{a}^\dagger. \]  
Denoting the annihilation and creation operators for the thermostat oscillators by \( \hat{c}_i \) and \( \hat{c}_i^\dagger \) we may rewrite the interaction Hamiltonian (5.1) (for thermostat oscillators whose frequencies coincide with \( \omega_0 \)) as follows:
\[ \hat{H}_{\text{int}}(\omega_0) = \hbar \sum_{\omega_i = \omega_0} \left( \delta_0 \hat{a}^\dagger \hat{c}_i + \delta_0^* \hat{a} \hat{c}_i^\dagger \right), \]  
\[ \delta_0 = g_0/\omega_0 + i u_0, \quad \Delta_0 = |\delta_0|^2. \]  
Hamiltonian (5.64) was considered in almost every paper devoted to the models of a quantum damped oscillator (see, e.g., review [13]). But frequently it was chosen only because it is the simplest one. We have shown in fact that it is the only possible quadratic interaction Hamiltonian ensuring the validity of the resulting Fokker-Planck equation for any initial states taken at any initial times. This is probably related to the quantum nature of the interactions between the systems: each act of interaction must consist in the annihilation of a quantum in one system and its creation in another system. Precisely the Hamiltonian (5.64) expresses this property in the most distinct form.

Nonetheless we have no sufficient grounds for claiming that other interaction (quadratic) Hamiltonians should be excluded. But they will result in the Fokker-Planck equation with time-dependent drift and diffusion matrices describing nonexponential relaxation.

Recall that we assumed the integrals in Eqs. (5.36)-(5.42) to equal zero. What will happen if we abandon this assumption, but assume instead that the constraints (5.54) hold for all the coupling constants? In this case we have \( \kappa(\omega) \equiv 0 \) at all frequencies. Furthermore, \( G(\omega) \equiv \omega^2 Z(\omega) \equiv \omega \Delta(\omega) \). Consequently, the integral terms “survive” only in the off-diagonal elements of both matrices \( \mu \) and \( D \):
\[ \mu_{11} = \mu_{22} = -\gamma, \]  
\[ \mu_{12} = \int \frac{\nu(\omega) \Delta(\omega)}{\omega^2 - \omega_0^2} \left[ \omega_0^2 + \omega^2 \right] d\omega, \]
Hamiltonians:

In the previous section, because the Hamiltonian (6.2) can be expressed as a sum of two oscillator fields

The annihilation operators $\hat{a}, \hat{a}^\dagger$ satisfy the commutation relations

The frequencies are defined as follows:

In the continuous weak coupling limit the self-consistent equations of motion for the first-order average values of the operators $\hat{a}$ and $\hat{b}$ are given by Eq. (6.63), provided one replaces $\omega_0$ with $\omega_+$ and $\omega_-$. Furthermore, two different damping coefficients are possible: they are determined by the density of states and coupling constants at the frequencies $\omega_{\pm}$ (see Eqs. (5.59) and (5.64)):

### 6 Oscillator in a magnetic field. Weak coupling with a thermostat

We now consider, within the framework of the same scheme, the case where the subsystem under study is a two-dimensional isotropic oscillator with eigenfrequency $\omega_0$ and mass $m$ placed in a uniform magnetic field $\mathcal{H}$ characterized by the cyclotron frequency

The Hamiltonian of this subsystem reads

where $\hat{x}$ and $\hat{y}$ are the operators of the kinetic momentum projections, related to the canonical momentum $p$ and the vector potential $A$ in the usual way:

The problem of constructing the Fokker-Planck equation for this subsystem is reduced to that solved in the previous section, because the Hamiltonian (6.2) can be expressed as a sum of two oscillator Hamiltonians:

The annihilation operators

satisfy the commutation relations

The frequencies are defined as follows:

In the continuous weak coupling limit the self-consistent equations of motion for the first-order average values of the operators $\hat{a}$ and $\hat{b}$ are given by Eq. (6.63), provided one replaces $\omega_0$ with $\omega_+$ and $\omega_-$. Furthermore, two different damping coefficients are possible: they are determined by the density of states and coupling constants at the frequencies $\omega_{\pm}$ (see Eqs. (5.59) and (5.64)):

$$
\gamma_{\pm} = \pi \nu(\omega_{\pm})|\delta(\omega_{\pm})|^2.
$$

(5.65)

(the function $\gamma(\omega)$ is defined by Eq. (5.59) with $\omega_0$ replaced by $\omega$).

The presence of nonzero coefficients $\mu_{12}$ and $\mu_{21}$ may be interpreted as some kind of renormalization of mass and eigenfrequency of the main oscillator due to the interaction with the environment. However, inequality (5.24) is obviously violated at zero temperature, when $f = \hbar/2\omega$, since due to Eq. (5.65) coefficient $D_{12}$ is strictly negative for all temperatures (evidently, both functions $\nu(\omega)$ and $f(\omega)$ are positive). This example shows once more that the self-consistent Fokker-Planck equations with time-independent coefficients can be derived from microscopic models only in exceptional cases.
The relations inverse to Eqs. (6.3) and (6.6) read
\[\dot{\pi}_x = (\hbar/2\Omega)^{1/2} \left[ \omega_+ (\hat{a} + \hat{a}^\dagger) + \omega_- (\hat{b} + \hat{b}^\dagger) \right], \tag{6.10}\]
\[\dot{\pi}_y = i(\hbar/2\Omega)^{1/2} \left[ \omega_+ (\hat{a}^\dagger - \hat{a}) + \omega_- (\hat{b}^\dagger - \hat{b}) \right], \tag{6.11}\]
\[\dot{x} = i(\hbar/2m\Omega)^{1/2} \left[ \hat{a} - \hat{a}^\dagger + \hat{b} - \hat{b}^\dagger \right], \tag{6.12}\]
\[\dot{y} = (\hbar/2m\Omega)^{1/2} \left[ \hat{a} + \hat{a}^\dagger - \hat{b} - \hat{b}^\dagger \right]. \tag{6.13}\]
The average values of the coordinates and the kinetic momenta obey equations resulting from equations of the form of (5.6):
\[\dot{\pi}_x = -\alpha\pi_x + \omega\pi_y - m\omega_0^2 x + m\omega_0^2 y, \tag{6.14}\]
\[\dot{\pi}_y = -\omega\pi_x - \alpha\pi_y - m\omega_0^2 x - m\omega_0^2 y, \tag{6.15}\]
\[\dot{x} = m^{-1}\pi_x - m^{-1}\epsilon\pi_y - \eta x, \tag{6.16}\]
\[\dot{y} = m^{-1}\epsilon\pi_x + m^{-1}\pi_y - \eta y. \tag{6.17}\]
We have introduced the notation
\[\alpha = (\gamma_+\omega_+ + \gamma_-\omega_-)/\Omega, \quad \eta = (\gamma_+\omega_- + \gamma_-\omega_+)/\Omega, \quad \epsilon = (\gamma_- - \gamma_+)/\Omega. \tag{6.18}\]
The second-order equations of motion read
\[\ddot{x} + (\gamma_- + \gamma_+) \dot{x} - \omega\dot{y} + (\omega_0^2 + \gamma_-\gamma_+) x - (\gamma_-\omega_+ + \gamma_+\omega_-) y = 0, \tag{6.19}\]
\[\ddot{y} + (\gamma_- + \gamma_+) \dot{y} + \omega\dot{x} + (\omega_0^2 + \gamma_-\gamma_+) y + (\gamma_-\omega_+ + \gamma_+\omega_-) x = 0. \tag{6.20}\]
We see that “one-photon” interaction with a thermostat of the form of (5.64) results in coordinate-dependent forces perpendicular to the vector $\mathbf{r} = (x, y)$ and proportional to the damping coefficients. The necessity of introducing such forces was shown earlier in [39, 48, 49] within the framework of a phenomenological approach. As was demonstrated in these papers, if the second-order equations of motion (for a charged particle or oscillator placed in a uniform magnetic field) contain the term $-\gamma\dot{r}$, then it is impossible to satisfy simultaneously Eq. (2.19) with the equilibrium matrix $\mathcal{M}(T)$ for all temperatures (including $T = 0$) and condition (2.21), unless a force of the form $\mathbf{f} = [\hbar \times \mathbf{r}]$ is introduced. Now we have arrived at the same result on the base of the microscopic model. Moreover, the relation between the velocity and the kinetic momentum becomes much more complicated than in the conservative case (see Eqs. (6.16) and (6.17)).

It is clear from the preceding section that the steady state Wigner function at $t \to \infty$ coincides with the equilibrium distribution that was found in refs. [3] [4]. Since this distribution is Gaussian, it is completely determined (see Eq. (3.8)) by the equilibrium variance matrix

\[
\mathcal{M}^{(eq)} = \begin{pmatrix}
\mathcal{M}_\pi & 0 & 0 & \mathcal{M}_a \\
0 & \mathcal{M}_\pi & -\mathcal{M}_a & 0 \\
0 & -\mathcal{M}_a & \mathcal{M}_\rho & 0 \\
\mathcal{M}_a & 0 & 0 & \mathcal{M}_\rho \\
\end{pmatrix}, \tag{6.21}
\]

\[
\mathcal{M}_\pi = \frac{m\hbar\Omega}{4Q} \left[ \left( 1 + \frac{\omega^2}{\Omega^2} \right) \sinh \tilde{\Omega} - 2\frac{\omega}{\Omega} \sinh \tilde{\omega} \right], \tag{6.22}\]
\[
\mathcal{M}_\rho = \frac{\hbar}{m\Omega Q} \sinh \tilde{\Omega}, \tag{6.23}\]
\[
\mathcal{M}_a = \frac{\hbar}{2Q} \left[ \frac{\omega}{\Omega} \sinh \tilde{\Omega} - \sinh \tilde{\omega} \right], \tag{6.24}\]

where
\[Q(\beta) = \cosh \tilde{\Omega} - \cosh \tilde{\omega}, \quad \tilde{\Omega} = \frac{1}{2}\beta\hbar\Omega, \quad \tilde{\omega} = \frac{1}{2}\beta\hbar\omega, \tag{6.25}\]

and $\beta = 1/kT$ is the inverse temperature of the thermostat.
The drift matrix $A$ corresponding to Eqs. (6.14)-(6.17) reads
\[
A = \begin{pmatrix}
-\alpha & \omega & -m\omega_0^2 & m\omega_0^2 \\
-\omega & -\alpha & -m\omega_0^2 & -m\omega_0^2 \\
\gamma & -\gamma & -\gamma & 0 \\
\gamma & \gamma & \gamma & 0
\end{pmatrix}.
\] (6.26)

Putting this matrix into Eq. (6.47) with the matrix $A^{(eq)}$ instead of $F_0$ we obtain the diffusion matrix
\[
D = \begin{pmatrix}
D_x & 0 & 0 & D_\pi \\
0 & D_x & -D_\pi & 0 \\
0 & -D_\pi & D_\rho & 0 \\
D_\pi & 0 & 0 & D_\rho
\end{pmatrix}
\] (6.27)

with the following coefficients,
\[
D_x = \frac{m\hbar}{2\Omega} \left[ (\gamma+\omega_+^2 + \gamma_+\omega_-^2) \sinh \bar{\Omega} - (\gamma+\omega_+^2 - \gamma_+\omega_-^2) \sinh \bar{\omega} \right],
\] (6.28)
\[
D_\pi = \frac{\hbar}{2\Omega} \left[ (\gamma+\omega_+ - \gamma_+\omega_-) \sinh \bar{\Omega} - (\gamma+\omega_+ + \gamma_+\omega_-) \sinh \bar{\omega} \right],
\] (6.29)
\[
D_\rho = \frac{\hbar}{2m\Omega} \left[ (\gamma+\gamma_-) \sinh \bar{\Omega} - (\gamma+\gamma_-) \sinh \bar{\omega} \right].
\] (6.30)

In particular, at zero temperature ($\beta = \infty$) we obtain
\[
D_\pi^{(low)} = \frac{m\hbar}{2\Omega} (\gamma+\omega_+^2 + \gamma_+\omega_-^2), \\
D_\pi^{(low)} = \frac{\hbar}{2\Omega} (\gamma+\omega_+ - \gamma_+\omega_-), \\
D_\rho^{(low)} = \frac{\hbar}{2m\Omega} (\gamma+\gamma_-).
\] (6.31)

In the opposite, high-temperature, case ($\beta \to 0$) we have
\[
D_\pi^{(high)} = mkT\alpha, \\
D_\rho^{(high)} = \frac{kT\eta}{m\omega_0^2}, \\
D_\pi^{(high)} = -kT\epsilon.
\] (6.32)

It is noteworthy that all three diffusion coefficients remain nonzero even in the high temperature limit, which is usually identified with the quasiclassical limit. Recall that in classical statistical mechanics it is usually implied that the only nonzero diffusion coefficient is $D_x$.

Various sets of the diffusion coefficients compatible with inequality (2.20) and leading to an equilibrium steady state (with the variance matrix (6.23)) in the limit of infinitely small damping were constructed within the framework of the phenomenological approach in [39]. However, none of them had a structure similar to that given by Eqs. (6.26)-(6.31).

For example, only the coefficient $D_x$ was proportional to the temperature in the high-temperature limit, whereas the other diffusion coefficients decreased as $1/kT$, in contrast to Eq. (6.32). This difference is due to at least two causes. First, it was assumed in [39, 48, 49] that the elements $A_{11}$ and $A_{32}$ of the drift matrix must be zero, although the elements $A_{14}$ and $A_{23}$ could be nonzero. Eq. (6.26) shows that within the framework of the microscopic approach such a choice is impossible, since all these coefficients are proportional to the parameter $\epsilon$. Furthermore, in the aforementioned papers we admitted the possibility that some coefficients of the drift matrix (those related to the damping) could depend on temperature. In principle, such a possibility (i.e., the time dependence of the coupling constants in the interaction Hamiltonian) is not excluded within the framework of the microscopic approach as well. Then the high temperature limit of the diffusion matrix coefficients could be quite different from Eq. (6.32).

The expressions (6.26)-(6.31) are simplified in the special case of the equal damping coefficients, when $\gamma_+ = \gamma_- = \gamma_0$:
\[
A^{(0)} = \begin{pmatrix}
-\gamma_0 & \omega & -m\omega_0^2 & 0 \\
-\omega & -\gamma_0 & 0 & -m\omega_0^2 \\
\gamma & -\gamma & -\gamma & 0 \\
\gamma & \gamma & \gamma & 0
\end{pmatrix}.
\] (6.33)
\[ D^{(0)}_{\pi} = \frac{\hbar \gamma_0}{2m \omega^2} \left( (\omega^2 + 2\omega_0^2) \sinh \Omega - \omega \Omega \sinh \tilde{\omega} \right), \]  
\[ D^{(0)}_A = \frac{\hbar \gamma_0}{2m \omega} \left( \omega \sinh \tilde{\omega} - \Omega \sinh \tilde{\omega} \right), \]  
\[ D^{(0)}_\rho = \frac{\hbar \gamma_0}{m \Omega Q} \sinh \tilde{\omega}. \]  

If coefficient $\gamma_-$ tends to zero sufficiently rapidly as $\omega_- \to 0$, then the set of diffusion coefficients (6.28)-(6.30) possesses the finite limit for a free particle in a magnetic field, when $\omega_0 = \omega_- = 0, \Omega = \omega, \gamma_+(\omega_+ \equiv \omega) = \gamma$:

\[ D_\pi = \frac{1}{2} \gamma \hbar \omega \coth \tilde{\omega}, \]  
\[ D_A = \frac{1}{2} \gamma \hbar \coth \tilde{\omega}, \]  
\[ D_\rho = \frac{\gamma \hbar}{2m \omega} \coth \tilde{\omega}, \]  
\[ \alpha = \gamma, \quad \eta = 0, \quad \epsilon = -\gamma/\omega. \]  

In this case the operators $\hat{b}$ and $\hat{b}^\dagger$ become the integrals of motion, whose real and imaginary parts are connected with the center-of-orbit operators in a uniform magnetic field [60]-[62].

One should remember, however, that the results obtained in this section can be justified only under rather strong limitations imposed on the interaction Hamiltonian. First, it must be written in the specific form (5.64) at the resonant frequencies. Secondly, the off-resonance terms must ensure the disappearance of the integral terms in Eqs. (5.36)-(5.42). In particular, the density of states must decrease sufficiently rapidly as $\omega \to \infty$.

All these conditions are violated, for example, in the case where the role of a reservoir is played by a quantized electromagnetic field coupled to the oscillator by means of the standard interaction Hamiltonian in the dipole approximation:

\[ \hat{H}_{\text{int}} = -\frac{e}{m} (2\pi \hbar)^{1/2} \sum_{j,\sigma} \frac{(\pi \tau_{j,\sigma})}{\sqrt{\omega_j}} \left[ \hat{c}(\mathbf{k}_j, \sigma) + \hat{c}^\dagger(\mathbf{k}_j, \sigma) \right], \]  

where $\hat{c}(\mathbf{k}_j, \sigma)$ is the operator of annihilation of a photon with wave vector $\mathbf{k}_j$, frequency $\omega_j = c|\mathbf{k}_j|$, and polarization $\sigma$; $\tau_{j,\sigma}$ is the unit polarization vector perpendicular to the vector $\mathbf{k}_j$. The density of states is proportional to $\omega^2$ in this case, and the integrals in Eqs. (6.36)-(6.42) diverge. Consequently, in this case the radiation damping leads to nonexponential relaxation.

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