Symbolic coding of linear complexity for generic translations of the torus, using continued fractions

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May 26, 2020

Abstract

In this paper, we prove that almost every translation of $\mathbb{T}^2$ admits a symbolic coding which has linear complexity $2n + 1$. The partitions are constructed with Rauzy fractals associated with sequences of substitutions, which are produced by a particular extended continued fraction algorithm in projective dimension 2. More generally, in dimension $d \geq 1$, we study extended measured continued fraction algorithms, which associate to each direction a subshift generated by substitutions, called $S$-adic subshift. We give some conditions which imply the existence, for almost every direction, of a translation of the torus $\mathbb{T}^d$ and a nice generating partition, such that the associated coding is a conjugacy with the subshift.

Keywords: symbolic dynamics, continued fractions, renormalization, Rauzy fractal, bounded remainder sets, $S$-adic system, $S$-adic subshift, Lyapunov exponents, torus translation, Pisot substitution conjecture

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1 Introduction

The first motivation of this paper is to find symbolic codings of translations of the torus $\mathbb{T}^d$ with low complexity. In dimension 1, every irrational translation of $\mathbb{T}^1$ admits a generating partition made of two intervals whose symbolic coding complexity is $n + 1$, generating the famous Sturmian words [29][32]. However, the endpoints of the intervals must be chosen carefully, since most partitions into two intervals lead to a symbolic coding of complexity $2n$ [17].

In higher dimension $d \geq 2$, a result of Chevallier [16] ensures that, for any minimal translation of $\mathbb{T}^d$, and for any generating partition of $\mathbb{T}^d$ with polygonal atoms, the corresponding symbolic coding has complexity in $\Omega(n^d)$. Hence, if we want to go below this bound, we will have to abandon the smooth shape of the atoms, while keeping their topological and measure-theoretic regularity to avoid trivial constructions: the partitions must still be generating, the atoms should be the closure of their interior, and their boundaries should have zero Lebesgue measure.

In the seminal paper [34], for the special case of the translation of $\mathbb{T}^2$ with vector $(\rho, \rho^2)$, where $\rho = 1.839286755214161...$ is the real root of $X^3 - X^2 - X - 1$, Rauzy constructed such a generating partition whose associated subshift is the Tribonacci subshift with complexity $2n + 1$ (see also [15]). This construction highly relies on the algebraic nature of the translation vector, which is witnessed in the self-similarity of the fractal generating partition.

Actually, Rauzy constructs a piecewise translation of a fundamental domain of the plane for the action of $\mathbb{Z}^2$, and the projection modulo $\mathbb{Z}^2$ of each piece forms an atom of the partition in $\mathbb{T}^2$: the translation is deduced from the partition. If a minimal translation of $\mathbb{T}^2$ is coded with such a liftable generating partition, the resulting complexity is at least $2n + 1$ [9] (this result was generalized in higher dimensions in [12] with the bound $dn + 1$). Hence, looking for generating partitions with complexity $2n + 1$ for translations of $\mathbb{T}^2$ seems to be a reasonable target.

Some known families of subshifts with complexity $2n + 1$ can be tried out. They are generated by continued fraction algorithms. The first candidate is the Arnoux-Rauzy algorithm. Unfortunately, the set of points where this algorithm can be iterated is too narrow; this set is known as the Rauzy gasket, see [8] for references. Another candidate is the continued fraction algorithm associated with the set of 3-interval exchange transformations. It is defined for almost every direction and produces subshifts with complexity $2n + 1$, but we know since [25] that almost all of them are weakly mixing. Thus, they cannot be conjugated to a translation on a torus.

Recently, Cassaigne introduced a continued fraction algorithm which has nice combinatorial properties and which is defined on the full space of parameters [3][14]. The first objective of this paper is to use this algorithm to construct, for almost every translation of $\mathbb{T}^2$, a regular generating partition whose coding has complexity $2n + 1$.

To this end, we develop a general framework for constructing Rauzy fractals out of infinite sequences of substitutions, and use them as the atoms of the generating partitions. Our approach is direct and provides an alternative to the “dual” construction of [11]. For this, we use particular topologies on $\mathbb{Z}^{d+1}$, introduced in [1], that we extend to the $S$-adic context.

We prove that when the sequences of substitutions are generated by an ergodic ex-
tended continued fraction algorithm whose second Lyapunov exponent is negative, the existence of a single parameter that fulfills the requirements to produce nice Rauzy fractals can be spread to obtain a set of good parameters of full measure.

As byproducts of those constructions, the atoms of the partitions provide bounded remainder sets; also, we get a renormalization scheme that relates the continued fraction algorithm to the first return map on some of the atoms.

2 Statement of the results and outline of the paper

Our main theorem is the following, we refer to Definition 1.

Theorem A. Lebesgue-almost every translation of $\mathbb{T}^2$ admits a nice generating partition whose symbolic coding has complexity $2n + 1$.

In order to prove it, we use the Cassaigne algorithm \cite{3, 14} and prove that it fulfills the hypotheses of Theorem B below.

Let $(X, s_0, \mu)$ be an extended measured continued fraction algorithm, see Definition 51. We assume that $(X, s_0, \mu)$ satisfies the Pisot condition, see Definition 60. Let $G_0 \subseteq X$ be the set of seed points, see Definition 64. The notations $P$, $\Lambda$, $e_0$, $v(x)$, and $\psi$ are defined in Sections 3.1 and 3.2.

Theorem B. Let $(X, s_0, \mu)$ be an extended measured continued fraction algorithm satisfying Pisot condition. Assume $G_0 \neq \emptyset$, then, for $\mu$-almost every point $x \in X$, there exists a translation $z \mapsto z + t_x$ on the torus $\mathbb{T}^d$ and a nice generating partition such that the associated symbolic coding is a measurable conjugacy with the subshift associated to $x$.

Moreover, we can take $t_x = \psi(e_0 - v(x))$ for a given isomorphism $\psi: P/\Lambda \to \mathbb{T}^d$.

We prove Theorem B by defining, for $\mu$-almost every point $x \in X$, a Rauzy fractal $R$, and by showing that it gives a nice generating partition of $\mathbb{T}^d$ whose symbolic coding corresponds to the subshift. This is done with Theorem C below, see Definition 38 for a definition of good directive sequence.

Theorem C. Let $s \in S^N$ be a good directive sequence. Then the Rauzy fractal $R(s)$ is a measurable fundamental domain of $P$ for the lattice $\Lambda$. It can be decomposed as a union $R(s) = \bigcup_{a \in A} R_a(s)$ which is disjoint in Lebesgue measure, and each piece $R_a(s)$ is the closure of its interior.

Moreover, the pieces $R_a(s)$, $a \in A$, of the Rauzy fractal induce a nice generating partition of the translation by $e_0 - v$ on the torus $P/\Lambda$, where $v$ is the unit vector of the direction of $s$. Its symbolic coding is a measurable conjugacy with the subshift associated to $s$.

Theorem C does not depend on a continued fraction algorithm. It is proven in Section 4. We introduce some topologies in Subsection 3.7 which play a central role in the proof of Theorem C. We prove that every good directive sequence gives a nice Rauzy fractal with all the wanted properties.

Theorem B is proven in Section 6. We first establish in Proposition 66 that the existence of a seed point implies that $\mu$-almost all points of $X$ are good. Then we use Theorem C.
Theorem [A] is proven in Section [8]. We first recall some facts about the algorithm and one of its invariant measures and the associated Lyapunov exponents. Then we consider a particular periodic point for $F$, and we prove that it is a seed point. Being a seed point is a decidable property for such periodic points, see Proposition [56]. This allows to apply Theorem [B].

3 Tools

3.1 Geometrical setting

Let $d \geq 1$ be an integer. In Section [5] we will work with continued fraction algorithms. To define them in dimension $d$ it is convenient to work in the $d + 1$-dimensional space $\mathbb{R}^{d+1}$, or rather its positive cone $\mathbb{R}_+^{d+1} \setminus \{0\}$. This is why we introduce some notations here.

Let $(e_i)_{0 \leq i \leq d}$ be the canonical basis of $\mathbb{R}^{d+1}$ (note the unusual numbering of dimensions). The space $\mathbb{R}^{d+1}$ is equipped with the classic norm $\|\cdot\|_1$ defined by $\|(y_0, \ldots, y_d)\|_1 = \sum_{i=0}^{d} |y_i|$. The space we are really interested in is $\mathbb{P}R^d = (\mathbb{R}_+^{d+1} \setminus \{0\}) / \mathbb{R}_+^*$, the set of positive directions.

For a vector $y \in \mathbb{R}_+^{d+1} \setminus \{0\}$, we denote by $[y] = \mathbb{R}_+^* y \in \mathbb{P}R^d$ the corresponding direction. Conversely, for every $x \in \mathbb{P}R^d$, we denote by $v(x) \in \mathbb{R}_+^{d+1}$ the unique representative of $x$ such that $\|v(x)\|_1 = 1$. And for every matrix $M \in M_{d+1}(\mathbb{R})$ and $x \in \mathbb{P}R^d$, we write $Mx = [Mv(x)]$ if $Mv(x) \in \mathbb{R}_+^{d+1} \setminus \{0\}$.

We define a distance on $\mathbb{P}R^d$, making it a metric space, by $d(x, y) = \|v(y) - v(x)\|_1$. Note that $\mathbb{P}R^d$ is thus isometric to the simplex $\Delta = \{y \in \mathbb{R}_+^{d+1} \mid \|y\|_1 = 1\}$. Open balls in $\mathbb{P}R^d$ are denoted $B(x, r)$.

Let $h$ denote the linear form on $\mathbb{R}_+^{d+1}$ defined by $h(y_0, \ldots, y_d) = \sum_{i=0}^{d} y_i$. Note that, when $y \in \mathbb{R}_+^{d+1}$, $h(y) = \|y\|_1$. Let $P$ be the hyperplane $\{y \in \mathbb{R}_+^{d+1} \mid h(y) = 0\}$. In the following we consider the lattice $\Lambda = P \cap \mathbb{Z}^{d+1} = \langle e_1 - e_0, \ldots, e_d - e_0 \rangle$. Let us denote by $\lambda$ the Lebesgue measure on $P$.

For $y \in \mathbb{R}_+^{d+1} \setminus \{0\}$, let $\pi_y$ denote the projection along $y$ onto $P$ (note that $y$ does not belong to $P$ as $\lambda(h(y) = \|y\|_1 > 0)$. This map sends a vector $z \in \mathbb{R}_+^{d+1}$ to $z - h(z) \frac{y}{h(y)}$. For $x \in \mathbb{P}R^d$, we also denote $\pi_x = \pi_{v(x)}$. Remark that $\Lambda \subseteq P$, so $\Lambda$ is preserved by every projection $\pi_y$.

We say that $y = (y_0, \ldots, y_d) \in \mathbb{R}_+^{d+1}$ has a totally irrational direction, or that $[y] \in \mathbb{P}R^d$ is a totally irrational direction, if $y_0, \ldots, y_d$ are linearly independent over $\mathbb{Q}$.

3.2 Translations on the torus

We define the $d$-dimensional torus as $P/\Lambda$, and let $q: P \to P/\Lambda$ denote the quotient map. We still denote by $\lambda$ the Lebesgue measure transported on $P/\Lambda$. Note that our definition of a torus differs from the usual one $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$, but they are isomorphic, in a non-canonical way that depends on a choice of a basis of $\Lambda$. To fix an isomorphism, let $L: P \to \mathbb{R}^d$ be the restriction to $P$ of the linear map which sends $(x_i)_{0 \leq i \leq d}$ to $(x_i)_{1 \leq i \leq d}$. Since $L(\Lambda) = \mathbb{Z}^d$, $L$ induces a map $\psi: P/\Lambda \to \mathbb{T}^d$ such that the following diagram

\[
\begin{array}{ccc}
\mathbb{R}^d & \to & \mathbb{R}^d / \mathbb{Z}^d \\
\downarrow & & \downarrow \\
\mathbb{P}R^d & \to & \mathbb{P}R^d / \mathbb{R}_+^*
\end{array}
\]
commutes:

\[
\begin{array}{ccc}
P & \xrightarrow{L} & \mathbb{R}^d \\
\downarrow & & \downarrow \\
P/\Lambda & \xrightarrow{\psi} & T^d
\end{array}
\]

Let \( t \in P/\Lambda \), and \( \hat{t} = (t_0, \ldots, t_d) \) \( \in P \) be a representative of \( t \). Then \( t \) is said to be a **totally irrational vector** if \( \hat{t} + e_0 \) has a totally irrational direction, i.e., if \( 1, t_1, \ldots, t_d \) are linearly independent over \( \mathbb{Q} \). Note that totally irrational vectors should not be confused with totally irrational directions.

For \( t \in P/\Lambda \), we consider the associated translation

\[
T_t = \left( \begin{array}{ccc}
P/\Lambda & \longrightarrow & P/\Lambda \\
\downarrow & & \downarrow \\
z & \longmapsto & z + t
\end{array} \right)
\]

We recall that a translation \( T_t \) is **minimal** if, and only if, \( t \) is a totally irrational vector \( \mathbb{Q} \).

If \( x \in \mathbb{P}\mathbb{R}^d_+ \) is a direction, we denote \( T_x = T_{\pi_x(e_0)+\Lambda} \).

Finally remark that any isomorphism \( \psi: P/\Lambda \rightarrow T^d \) preserves Lebesgue measure (up to a multiplicative constant) and totally irrational vectors. This will be used in Theorem B.

### 3.3 Words

For a fixed \( d \geq 1 \), we define the **alphabet** \( A \) as the finite set \( A = \{0, \ldots, d\} \). Its elements are called **letters**. A **finite word** is an element of the monoid \( A^* = \bigcup_{n \in \mathbb{N}} A^n \). The **length** of a finite word \( u \in A^n \) is denoted by \( |u| = n \). The set of non-empty words is the semigroup \( A^+ = \bigcup_{n \geq 1} A^n \). An **infinite word** is an element of \( A^\mathbb{N} \). A word can be finite or infinite.

The set of words \( A^* \cup A^\mathbb{N} \) is endowed with the topology of coordinatewise convergence.

When a word \( u \) can be written as a product of three words \( pfs \), \( p \) is called a **prefix** of \( u \), \( f \) is called a **factor** of \( u \), \( s \) is called a **suffix** of \( u \), and the length of \( p \) is called an **occurrence** of \( f \) in \( u \). The number of occurrences of a finite word \( f \) in a word \( u \) is denoted by \( |u|_f \).

The **complexity** of an infinite word \( u \) is the map \( p: \mathbb{N} \rightarrow \mathbb{N} \) which associates to any integer \( n \) the number of factors of \( u \) of length \( n \).

A **substitution** is an element \( \sigma \) of \( \text{hom}(A^*, A^*) \): for all finite words \( u, v \in A^* \), we have \( \sigma(uv) = \sigma(u)\sigma(v) \). A substitution is characterized by the images of letters. A **non-erasing substitution** is an element \( \sigma \) of \( \text{hom}(A^+, A^+) \): it is a substitution that maps every letter to non-empty words.

The **abelianization map** is the monoid morphism \( \text{ab}: A^* \rightarrow \mathbb{Z}^{d+1} \) such that \( \text{ab}(a) = e_a \) for every letter \( a \) in \( A \) (recall that \( (e_a)_{a \in A} \) is the canonical basis of \( \mathbb{R}^{d+1} \)). We use the same notation for the map from \( \text{hom}(A^*, A^*) \) to \( M_{d+1}(\mathbb{Z}) \) such that \( \text{ab}(\sigma) \text{ab}(w) = \text{ab}(\sigma(w)) \) for a substitution \( \sigma \) and a word \( w \). A substitution \( \sigma \) is said to be **unimodular** if \( |\det(\text{ab}(\sigma))| = 1 \).

The action of a non-erasing substitution \( \sigma \) can be extended to infinite words by the limit procedure:

\[
\sigma(u) = \lim_{p \to \infty} \sigma(p)
\]
An infinite word $u$ is a fixed point of $\sigma$ if $\sigma(u) = u$. An infinite word $u$ is a periodic point of $\sigma$ if there exists an integer $p \geq 1$ such that $\sigma^p(u) = u$.

For an integer $k \geq 1$, an infinite word $u$ is $k$-balanced if for any two factors $v, w$ of $u$ of the same length, and any $a \in A$, we have $||v|_a - |w|_a|| \leq k$. An infinite word is balanced if it is $k$-balanced for some integer $k$.

For an infinite word $u$, the (possibly undefined) frequency vector of $u$ is

$$\text{freq}(u) = \lim_{|p| \to \infty} \frac{\text{ab}(p)}{|p|} \in \mathbb{R}^{d+1}.$$ 

When this limit exists, we say that $u$ admits a frequency vector. This is in particular the case if $u$ is balanced (see Proposition 18) or if it is an element of a uniquely ergodic subshift.

For a non-empty finite word $w \in A^+$, we denote by $w^\omega$ the infinite word $\lim_{n \to \infty} w^n$.

Finally we define the shift map $T$ on $A^\mathbb{N}$ that maps an infinite word $u$ to its suffix $Tu$ such that $u = aTu$ with $a \in A$. Remark that with the coordinatewise topology, the shift map $T$ is continuous, and a subset $X \subseteq A^\mathbb{N}$ is called a subshift if $X$ is closed and shift-invariant.

The orbit of $u \in A^\mathbb{N}$ is the set $O(u) = \{T^n u \mid n \in \mathbb{N}\}$ and the subshift generated by $u$ is its orbit closure $\Omega_u = \overline{O(u)}$. To a finite factor $w$ of $u$ we associate the cylinder $[w] = \{x \in \Omega_u \mid \exists s \in \Omega_u, x = ws\}$.

### 3.4 Symbolic coding

A measured topological dynamical system is a triple $(X,T,\mu)$ such that $X$ is a compact topological space, $\mu$ is a finite Borel measure, and $T: X \to X$ is a $\mu$-almost everywhere continuous map such that $\mu(T^{-1}(B)) = \mu(B)$ for any Borel set $B$ of $X$.

Given a measured topological dynamical system $(X,T,\mu)$ and a measurable partition $(P_i)_{i \in I}$ of $X$, we associate the coding $\text{cod}: X \to I^\mathbb{N}$ defined by $\text{cod}(y) = (i_n)_{n \in \mathbb{N}}$ and $\forall n \in \mathbb{N}$, $T^n y \in P_{i_n}$. The map $\text{cod}$ is a symbolic coding of the system $(X,T,\mu)$ and the closure of $\text{cod}(X)$ defines a subshift over the alphabet $I$. A generating partition of the map $T$ is a measurable partition whose coding is injective $\mu$-almost everywhere.

The atoms of the partitions we will construct will not be smooth, but they will keep some topological and measure-theoretic regularity: a generating partition $(P_i)_{i \in I}$ of $X$ is regular if every set $\overline{P_i}$ is the closure of its interior and if the boundary of each $P_i$ has measure zero.

A measurable subset $A$ of $X$ is said to be a bounded remainder set for the map $T$ if there exists a constant $K$ such that, for $\mu$-almost every $x$ in $X$ and every integer $N$,

$$\left| \sum_{n=0}^{N-1} 1_A(T^n(x)) - N \frac{\mu(A)}{\mu(X)} \right| \leq K,$$

where $1_A$ is the indicator function of the subset $A$. As we shall see, the atoms of the generating partition we will construct are bounded remainder sets.

Now, let $T_x$ be a translation of the torus $P/A$. The triple $(P/A, T_x, \lambda)$ is a measured topological dynamical system, where $\lambda$ denotes the Lebesgue measure inherited from $P$. 


The generating partitions we will construct on \( P/\Lambda \) actually come from a piecewise translation of a measurable fundamental domain of \( P \) for the action of \( \Lambda \): a finite measurable partition \((P_i)_{i \in I}\) of \( P/\Lambda \) is said to be liftable with respect to the translation \( T_t: z \mapsto z + t \) of \( P/\Lambda \) if there exists:

- a measurable fundamental domain \( D \subseteq P \) for the action of \( \Lambda \)
- a measurable partition \((D_i)_{i \in I}\) of \( D \)
- some vectors \((t_i)_{i \in I}\) in \( P^I \)

such that for every \( i \) in \( I \):

- \( D_i + t_i \subseteq D \)
- \( q(D_i) = P_i \)
- \( q(t_i) = t \)

The map \( E = \begin{pmatrix} D & D \\ y & y + t_i \end{pmatrix} \) if \( y \in D_i \) is called a piecewise translation or a domain exchange, and is measurably conjugated to the translation \( T_t \) via the quotient map \( q : P \to P/\Lambda \).

**Definition 1.** A finite measurable partition \((P_i)_{i \in I}\) of \( P/\Lambda \) is said to be a nice generating partition with respect to the translation \( T_t: z \mapsto z + t \) of \( P/\Lambda \) if it is generating, regular, liftable, and every \( P_i \) is a bounded remainder set.

### 3.5 \( S \)-adic systems and \( S \)-adic subshifts

Let \( S \subseteq \hom(A^+, A^+) \) be a finite set of non-erasing substitutions on the alphabet \( A \).

An **\( S \)-adic system** is a shift-invariant subset of \( S^\mathbb{N} \). Note that we do not impose that \( S \)-adic systems are topologically closed. For instance, in Section 7.1 we will consider \( S = \{\tau_0, \tau_1\} \) and the \( S \)-adic system \( \{(s_k) \in S^\mathbb{N} \mid \text{each element of } S \text{ occurs infinitely often in } (s_k)\} \).

An element \( s = (s_k) \) of an \( S \)-adic system is called a directive sequence.

**Definition 2 (\( S \)-adic subshift).** Let \( s \) be a directive sequence. Then the **\( S \)-adic subshift associated with \( s \)** is the subshift \( \Omega_s \) defined as follows. Let first \( L \subseteq A^* \) be the language of all factors of finite words of the form \( s_{(0,n)}(a) \) for all \( n \in \mathbb{N} \) and \( a \in A \), where \( s_{(k,n)} = s_k \circ \cdots \circ s_{n-1} \). Then \( \Omega_s \) is the set of infinite words \( w \in A^\mathbb{N} \) such that all factors of \( w \) are in \( L \).

\( S \)-adic subshifts were introduced by Ferenczi [22], where he proves that every word of linear complexity is an element of some \( S \)-adic subshift in an \( S \)-adic system with some additional conditions. This notion has been used in many places thereafter. We refer to [20] and [10] for reference.

**Remark 3.** There are alternative ways to define subshifts from a directive sequence. One is to consider the set

\[ \Omega_s' = \bigcap_{n \in \mathbb{N}} \{ T^k s_{(0,n)}(w) \mid w \in A^\mathbb{N}, k \in \mathbb{N} \}. \]
Another way is to first define an infinite word \( u \) by starting from a fixed letter \( a \in A \) and taking a limit point of the sequence of finite words

\[
s_0(a), s_0(s_1(a)), s_0(s_1(s_2(a))), \ldots
\]

then consider the subshift generated by \( u \) (i.e., the smallest closed subset of \( A^\mathbb{N} \) invariant by the shift and containing \( u \)), which is a subset of \( \Omega_s \).

Here, we will let the directive sequence act on sequences of infinite words, each word representing a scale on which the corresponding substitution acts.

A word sequence is an element \( u = (u_n) \) of \( (A^\mathbb{N})^\mathbb{N} \).

Directive sequences act naturally on word sequences as follows:

\[
\begin{pmatrix}
S^\mathbb{N} \times (A^\mathbb{N})^\mathbb{N} \\
(s, u)
\end{pmatrix} \rightarrow \begin{pmatrix}
(A^\mathbb{N})^\mathbb{N} \\
N \rightarrow A^\mathbb{N}
\end{pmatrix}
\begin{pmatrix}
k \\
s_k(u_{k+1})
\end{pmatrix}
\]

**Definition 4.** A fixed point of a directive sequence \( s \) is a fixed point for the above action, that is, a word sequence \( u \) satisfying:

\[\forall k \in \mathbb{N}, \ s_k(u_{k+1}) = u_k.\]

Example [10] gives an example of fixed point of a directive sequence.

Directive sequences always admit fixed points. Indeed, choose a letter \( a \in A \) and for each \( n \in \mathbb{N} \), consider the word sequence \( u^{(n)} = (u_k^{(n)})_{k \in \mathbb{N}} \) defined by \( u_k^{(n)} = a^{\omega} \) when \( k \geq n \) and \( u_k^{(n)} = s_{[k,n]}(a^{\omega}) \) when \( k < n \), where \( s_{[k,n]} = s_k \circ \cdots \circ s_{n-1} \). Then let \( u \) be a limit point of this sequence of word sequences when \( n \) tends to infinity, in the compact space \( (A^\mathbb{N})^\mathbb{N} \) (with the coordinatewise topology). This \( u \) is a fixed point of \( s \).

Fixed points of a directive sequence are not unique in general.

This generalizes the notion of fixed point and the notion of periodic point for a single substitution \( \sigma \). Let \( \sigma^\omega \) denote the constant directive sequence with all terms equal to \( \sigma \). Similarly, for \( v \in A^\mathbb{N} \), let \( v^\omega \) denote the word sequence with all terms equal to \( v \).

**Lemma 5.** Let \( \sigma \in S \) be a substitution. We have

- \( v \in A^\mathbb{N} \) is a fixed point of \( \sigma \) if, and only if, \( v^\omega \in (A^\mathbb{N})^\mathbb{N} \) is a fixed point of \( \sigma^\omega \),

- if \( u \in (A^\mathbb{N})^\mathbb{N} \) is a fixed point of \( \sigma^\omega \), then \( u_0 \) is a periodic point of \( \sigma \).

**Proof.** The first point is clear. Let \( u \) be a fixed point of \( \sigma^\omega \), and for all \( n \in \mathbb{N} \), let \( a_n \in A \) be the first letter of \( u_n \). Then, the sequence \( a_n \) is periodic, with a period \( p \leq d+1 \) since \( a_n \) is completely determined by \( a_{n+1} \). Now, if \( \lim_{n \to \infty} |\sigma^n(a_0)| = \infty \), then \( \sigma^\omega_a(u_0) \) converges as \( n \) tends to infinity to the word \( u_0 = u_p \), so \( u_0 \) is a periodic point of \( \sigma \). Otherwise, we have for all \( n \in \mathbb{N} \), \( \sigma^n(a_0) = a_n \), and the directive sequence \( (Tu_n)_{n \in \mathbb{N}} \) is also a fixed point of \( \sigma^\omega \), where \( T: A^\mathbb{N} \to A^\mathbb{N} \) is the shift map. If we iterate the argument and take the least common multiple of the periods obtained, it gives a period \( p \) for which \( u_0 \) is a fixed point of \( \sigma^p \). \qed
Definition 6. For a fixed point $u \in (A^N)^N$ of a directive sequence $s$, we define the subshift $\Omega_u$ as the subshift $\Omega_{u_0}$, that is the smallest closed subset of $A^N$ invariant by the shift and containing $u_0$.

Definition 7. We say that a directive sequence $s \in S^N$ is primitive if

$$\forall k, \exists n \geq k, \forall a \in A, s_{[k,n]}(a) \text{ contains every letter.}$$

It is equivalent to $\forall k, \exists n \geq k, ab(s_{[k,n]}) > 0$.

Definition 8. We say that a directive sequence $s \in S^N$ is everywhere growing if for all $a \in A$, we have

$$\lim_{n \to \infty} \left| s_{[0,n]}(a) \right| = \infty.$$ 

It is equivalent to say that the 1-norm of each column of the matrix $ab(s_{[0,n]})$ tends to infinity.

Remark that if a directive sequence $s$ is primitive, then for all $k \in \mathbb{N}$, and all $a \in A$, we have $\left| s_{[k,n]}(a) \right| \to \infty$. In particular, $s$ is everywhere growing.

Proposition 9. Let $s \in S^N$ be a primitive directive sequence. Then the subshift $\Omega_s$ is minimal. In particular, for every fixed point $u \in (A^N)^N$ of $s$, we have $\Omega_u = \Omega_s$. Thus, $\Omega_u$ does not depend on the choice of the fixed point $u$.

Proof. Let $w$ and $w' \in \Omega_s$ be two words of the subshift. Let $p$ be a prefix of $w$. Then, there exists $n \in \mathbb{N}$ and $a \in A$ such that $p$ is a factor of $s_{[0,n]}(a)$. Using the primitivity, let $N \geq n$ such that $ab(s_{[N,N]}) > 0$. Now, take a factor $f$ of $w'$ of length at least $2 \max_{c \in A} \left| s_{[0,N]}(c) \right|$. There exists $k \in \mathbb{N}$ and $b \in A$ such that $f$ is a factor of $s_{[0,k]}(b)$. Necessarily, we have $k \geq N$, and $s_{[0,k]}(b)$ is a concatenation of words $s_{[0,N]}(c)$, for each letter $c$ of $s_{[N,k]}(b)$. Hence, there exists $c \in A$ such that $s_{[0,N]}(c)$ is a factor of $f$. Then, the letter $a$ appears in the word $s_{[n,N]}(c)$. So $p$ is a factor of $s_{[0,n]}(a)$ which is a factor of $s_{[0,N]}(c)$ which is a factor of $f$ which is a factor of $w'$. We conclude that for every $w, w' \in \Omega_s$, every prefix of $w$ is a factor of $w'$, thus the subshift $\Omega_s$ is minimal.

To end the proof, remark that for any fixed point $u$ of $s$, we have $u_0 \in \Omega_s$ since we have $\lim_{n \to \infty} s_{[0,n]}(a_n) = u_0$, where $a_n$ is the first letter of $u_n$. Hence, by minimality we get $\Omega_u = \Omega_s$. 

Let us give an example of a fixed point of a directive sequence. The reader will recognize that each $u_k$ is a Sturmian word, see $[32]$.

Example 10. Let $S = \{\tau_0, \tau_1\}$, with $\tau_0 = \begin{cases} 0 & \mapsto 0 \\ 1 & \mapsto 01 \end{cases}$, $\tau_1 = \begin{cases} 0 & \mapsto 10 \\ 1 & \mapsto 1 \end{cases}$, and let us consider the directive sequence $s = \tau_0 \tau_1 \tau_1 \tau_0 \tau_1 \tau_0 \tau_1 \tau_0 \tau_1 \tau_1 \tau_0 \tau_1 \tau_1 \tau_0 \tau_1 \tau_0 \tau_1 \tau_1 \tau_0 \tau_1 \tau_1 \tau_0 ...$. Then, there exists a fixed point...
\( u \in \{0,1\}^N \) of \( s \) beginning with

\[
\begin{align*}
    u_0 &= 01010010101010001010010010100101010101010... \\
    u_1 &= 1101101110110111011011101101110110110110... \\
    u_2 &= 1011011010110111011011101010110110110110... \\
    u_3 &= 001000100100010010001000100010001000100... \\
    u_4 &= 0100100100100100100100100100100100100100... \\
    u_5 &= 10100100101011010110110101101101011011010... \\
    u_6 &= 0010010001000100010001000100010001000100... \\
\end{align*}
\]

Fixed points encompass both the time and scale dynamics in a single object. In the context of this paper, the time dynamics will correspond to the action of the translation on the torus and the scale dynamics will correspond to the action of the continued fraction algorithm on the space of translations. Symbolically, the shift map on \( \Omega_{u_0} \) encodes the time dynamics, while shifting the fixed point \((u_k) \mapsto (u_{k+1})\) corresponds to accelerating the time dynamics.

In the example above, we can visualize how fixed points grasp the multi-scale structure of the dynamical system with the following alignment:

\[
\begin{align*}
    u_0 &= 01010010101010001010010010100101010101010... \\
    u_1 &= 1101101110110111011011101101110110110110... \\
    u_2 &= 1011011010110111011011101010110110110110... \\
    u_3 &= 001000100100010010001000100010001000100... \\
    u_4 &= 0100100100100100100100100100100100100100... \\
    u_5 &= 10100100101011010110110101101101011011010... \\
    u_6 &= 0010010001000100010001000100010001000100... \\
\end{align*}
\]

As we will see in section when the substitutions enjoy some recognizability properties, the scale dynamics corresponds to inducing on some atoms of the partition.

Rokhlin towers and ordered Bratteli diagrams are other combinatorial objects that account for the multi-scale structure of dynamical systems. An ordered Bratteli diagram \( \mathcal{B} \) can be associated to a directive sequence \((s_k)\) \([10]\). When \((s_k)\) is everywhere growing, the minimal infinite paths of \( \mathcal{B} \) are in bijective correspondence with the fixed points \((u_k)\) of \((s_k)\): the \( k \)th edge of the infinite path is encoded by the first letter of the word \( u_k \).

### 3.6 Matrices

To each substitution \( \sigma \) is associated a matrix \( ab(\sigma) \). To obtain precise results, we need to recall some facts about matrices. Recall that \( \mathbb{R}^{d+1} \) is equipped with the norm \( \|\cdot\|_1 \).
The operator norm of a matrix $M \in M_{d+1}(\mathbb{R})$ is defined by

$$\|M\|_1 = \sup_{v \in \mathbb{R}^{d+1}\setminus\{0\}} \frac{\|Mv\|_1}{\|v\|_1}.$$ 

Moreover we also define a semi-norm for a subspace $V$:

$$\left\| M \right\|_1 V = \sup_{v \in V \setminus \{0\}} \frac{\|Mv\|_1}{\|v\|_1}.$$ 

Finally we write $M > 0$ if every coefficient of $M$ is positive.

Given a directive sequence $s$, we define $M_k(s) = ab(s_k)$, denoted simply by $M_k$ when there is no ambiguity on what is the directive sequence. We use the classical notation $M_{[k,n)} = M_k \ldots M_{n-1}$.

A matrix $M \in M_{d+1}(\mathbb{R})$ is said to be Pisot if it has non-negative integer entries, its dominant eigenvalue is simple and all other eigenvalues have absolute values less than one. A substitution $\sigma$ is said to be Pisot if the matrix $ab(\sigma)$ is Pisot.

### 3.7 Topologies on the integer half-space and worms

Let us define the integer half-space: $\mathbb{H} = \{z \in \mathbb{Z}^{d+1} \mid h(z) \geq 0\}$ and for $i \geq 0$, $\mathbb{H}_i = \{z \in \mathbb{Z}^{d+1} \mid h(z) = i\} = \Lambda + ie_0$. The following two definitions are crucial in the rest of the paper.

**Definition 11.** For any fixed $x \in \mathbb{R}^d_+$, we define the topology $\mathcal{T}(x)$ on $\mathbb{H}$: a subset $V \subseteq \mathbb{H}$ is open if there exists an open set $U \subseteq P$ such that $V = \pi_x^{-1}(U) \cap \mathbb{H}$.

Remark that $\mathcal{T}(x)$ is the finest topology on $\mathbb{H}$ that makes $\pi_x : \mathbb{H} \to P$ continuous. It is metrizable if, and only if, $x \cap \mathbb{H} = \{0\}$, which is the case if $x$ is a totally irrational direction.

We introduce the notion of worm:

**Definition 12.** Given an infinite word $u \in A^\mathbb{N}$, its worm is the set

$$W(u) = \{ab(p) \mid p \text{ prefix of } u\} \subseteq \mathbb{N}^{d+1} \subseteq \mathbb{H}.$$ 

For a worm and a letter $a$ we can define the subsets

$$W_a(u) = \{ab(p) \mid pa \text{ prefix of } u\} = W(u) \cap (W(u) - e_a).$$

**Remark 13.** The subsets $W_a(u)$, $a \in A$, form a partition of $W(u)$: $W(u) = \sqcup_{a \in A} W_a(u)$.

An example of worm is depicted in Figure 1.

**Lemma 14 (tiling).** A worm $W$ tiles the integer half-space by translations: $\mathbb{H} = W \oplus \Lambda$.

Figure 2 shows an example of such a tiling by the worm $(01001)^\omega$, for $d = 1$. 

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Figure 1: Worm of the word $u = (01001)^\omega$. $W_0(u)$ in blue and $W_1(u)$ in red.

Figure 2: Tiling of $\mathbb{H}$ by a worm, by translation by the group $\Lambda = \langle e_1 - e_0 \rangle$.

Proof. The lattice $\Lambda$ acts on $\mathbb{H}$ by translation. The orbits of this action are the cosets $H_i$. A worm intersects each coset exactly once. \hfill \square

Lemma 15 (automatic balance). If a worm $W(u)$ has non-empty interior for some topology $T(x)$ with $x \in \mathbb{P}R_+^d$, then $\pi_x(W(u))$ is bounded.

Proof. Let $U$ be an open subset of $P$ such that $\emptyset \neq \pi_x^{-1}(U) \cap \mathbb{H} \subseteq W(u)$. Up to restricting it, we assume that $U$ is included in an open ball $B(0, M)$ for some $M > 0$.

Let us consider the translation by $\pi_x(e_0)$ modulo $\Lambda$ of the $d$-dimensional torus $P/\Lambda$:

$$T_x = T_{\pi_x(e_0) + \Lambda} = \begin{pmatrix} P/\Lambda \\ z \mapsto z + \pi_x(e_0) \end{pmatrix}$$

For any integer $i$, $\pi_x^{-1}(U)$ intersects $H_i$ if, and only if, $T_x^i(0)$ belongs to the open subset $U + \Lambda$ of the torus. The translation $T_x$ acts minimally on every orbit closure. By hypothesis, the open set $U + \Lambda$ intersects the orbit of 0, hence $T_x^i(0)$ belongs to $U + \Lambda$ for $i$ in a syndetic subset of $\mathbb{N}$: $\exists K \geq 1, \forall i \in \mathbb{N}, \exists 0 \leq k \leq K, T_x^{i+k}(0) \in U + \Lambda$. If, for each integer $i$, we denote by $w_i$ the single element of $W(u) \cap H_i$, we have $\|w_{i+1} - w_i\|_1 = 1$. Hence, any point of the worm $W(u)$ is at distance at most $K$ of a point of $\pi_x^{-1}(U)$ (see Figure 3).

Since the direction $x$ is in $\mathbb{P}R_+^{d+1}$, the projection $\pi_x$ is 2-Lipschitz. Hence, $\pi_x(W(u)) \subseteq B(0, M + 2K)$, which concludes the proof. \hfill \square

Lemma 16 (uniform automatic balance). If there exists a ball $B$ of $P$, a sequence of directions $x^{(n)} \in \mathbb{P}R_+^d$ such that $x^{(n)} \xrightarrow{n \to \infty} x^\infty$ with $x^\infty$ a totally irrational direction, and
Figure 3: A worm cannot escape too far between consecutive interior points

$a$ sequence of worms $W(u_n)$ such that $\forall n \in \mathbb{N}$, $\pi_x^{-1}(B) \cap \mathbb{H} \subseteq W(u_n)$, then $\pi_x(u_n)(W(u_n))$ is uniformly bounded for $n \in \mathbb{N}$.

Proof. Following the proof of the previous lemma, we consider the translation $T_x^\infty$ by $\pi_x^\infty(e_0) + \Lambda$ on the torus $P/\Lambda$. Let $M > 0$ and $p$ such that $B = B(p, M)$, and define $B' = B(p, M/2)$. Since $x^\infty$ is a totally irrational direction, the translation $T_x^\infty$ acts minimally on the whole torus $P/\Lambda$. Hence, there exists a constant $K$ such that for all $x \in B$, there exists $0 < i \leq K$ such that $T_i^\infty(x) \in B'$. If we take $n_0$ such that for all $n \geq n_0$, $d(x^n, x^\infty) \leq \frac{M}{2K}$, then we have that for all $y \in B$, there exists $0 < i \leq K$ such that for all $n \geq n_0$, $T_i^\infty(y) \in B$. Similarly, there exists $0 < j \leq K$ such that for all $n \geq n_0$, $T_j^\infty(y) \in B$. Hence, for all $n \geq n_0$, $\pi_x(u_n)(W(u_n)) \subseteq B(p, M + 2K)$. The result follows since $W(u_n)$ is bounded for $n < n_0$ by Lemma 15. □

The following lemma is useful to propagate non-emptiness of the interior.

Lemma 17. Let $W \subseteq \mathbb{H}$, let $x \in \mathbb{P}^d$ be a totally irrational direction, and let $M \in GL_{d+1}(\mathbb{Z}) \cap M_{d+1}(\mathbb{N})$. If $W$ has non-empty interior for the topology $T(x)$, then $MW$ has non-empty interior for the topology $T(Mx)$.

Proof. Let $U \subseteq P$ be a bounded non-empty open subset such that $\pi_x^{-1}(U) \cap \mathbb{H} \subseteq W$. We have $M \in GL_{d+1}(\mathbb{Z})$, so $M^{-1} \mathbb{Z}^{d+1} = \mathbb{Z}^{d+1}$, and $M^{-1} \mathbb{H}$ is a half-space

$$M^{-1} \mathbb{H} = \{ z \in \mathbb{Z}^{d+1} \mid \sum_{i=0}^{d} (Mz)_i \geq 0 \} = \{ z \in \mathbb{Z}^{d+1} \mid \sum_{i=0}^{d} \alpha_i z_i \geq 0 \},$$

for some coefficients $\alpha_i$. Since the matrix $M$ is non-negative and invertible, we have $\alpha_i > 0$ for all $i$, so $v(x)$ is in the half-space $\{ z \in \mathbb{R}^{d+1} \mid \sum_{i=0}^{d} \alpha_i z_i > 0 \}$. Hence, for every $t \in \mathbb{R}^{d+1}$, the intersection of the line $t + \mathbb{R} v(x)$ with the set

$$\{ z \in \mathbb{R}^{d+1} \mid \sum_{i=0}^{d} \alpha_i z_i \geq 0 \} \setminus \{ z \in \mathbb{R}^{d+1} \mid \sum_{i=0}^{d} z_i \geq 0 \}$$
is bounded. Using that moreover $U$ is bounded we get that
\[ L = \pi^{-1}_x(U) \cap M^{-1}H \backslash H \subseteq \mathbb{Z}^{d+1} \]
is a finite set. Moreover, we have $\pi^{-1}_x(\pi_x(L)) \cap M^{-1}H = L$ since $\pi_x$ is injective on $\mathbb{Z}^{d+1}$, and we have $M\pi^{-1}_x(U \setminus \pi_x(L)) = \pi^{-1}_M(\pi_M(M(U \setminus \pi_x(L))))$, so we have
\[
MW \supseteq M(\pi^{-1}_x(U) \cap H)
\supseteq M(\pi^{-1}_x(U) \cap M^{-1}H \setminus L)
= M(\pi^{-1}_x(U \setminus \pi_x(L))) \cap H
= \pi^{-1}_M(\pi_M(M(U \setminus \pi_x(L)))) \cap H.
\]
Finally $\pi_M(M(U \setminus \pi_x(L))) \neq \emptyset$ is open, so $MW$ has non-empty interior for $T(Mx)$.

We finish this subsection with a result that shows how to relate properties of the worm to some combinatorial properties of $u$:

**Proposition 18.** An infinite word $u \in A^\mathbb{N}$ is balanced if, and only, if there exists a direction $x \in \mathbb{R}^d$ such that $\pi_x(W(u))$ is bounded.

**Proof.** First of all, remark that a word $u$ is balanced if, and only if, there exists a constant $K$ such that, for any two factors $v, w$ of $u$, $\|ab(v)\|_1 = \|ab(w)\|_1 \implies \|ab(v) - ab(w)\|_1 \leq K$.

Assume that $\pi_x(W(u)) \subseteq B(0, L)$. First of all remark that for a finite word $w$ its length fulfills $|w| = h(ab(w)) = \|ab(w)\|_1$. Moreover if $p$ is a prefix of $u$, then we have $ab(p) - |p| v(x) = \pi_x(ab(p)) \in \pi_x(W(u))$, so we get $\|ab(p) - |p| v(x)\|_1 \leq L$.

Let $w$ be a factor of $u$ and let $p$ be a prefix of $u$ such that $pw$ is a prefix of $u$. Since $ab(w) = ab(pw) - ab(p)$ and $|w| = |pw| - |p|$ we obtain:
\[
\|ab(w) - |w| v(x)\|_1 \leq \|ab(pw) - |pw| v(x)\|_1 + \|ab(p) - |p| v(x)\|_1 \leq 2L.
\]
Thus if we consider two factors $w_1$ and $w_2$ of $u$ of the same length, we deduce:
\[
\|ab(w_1) - ab(w_2)\|_1 \leq \|ab(w_1) - |w_1| v(x)\|_1 + \|ab(w_2) - |w_2| v(x)\|_1 \leq 4L.
\]
Thus the word $u$ is balanced.

Now, assume that $u$ is balanced, and let $K$ such that for any two factors $v, w$ of $u$, $\|ab(v)\|_1 = \|ab(w)\|_1 \implies \|ab(v) - ab(w)\|_1 \leq K$. Let $p_n$ be the prefix of $u$ of length $n$.

For every $k \geq 1$, by cutting $p_n$ into $\left[ \frac{n}{k} \right] \frac{k}{N}$ parts of length $k$ and a remaining factor of length less than $k$, we get
\[
\left\| ab(p_n) - \frac{n}{k} ab(p_k) \right\|_1 \leq K \left( \frac{\left[ \frac{n}{k} \right]}{n} + \frac{\left[ \frac{k}{N} \right]}{N} \right) + \frac{k}{N} + k + \frac{1}{n} \left( \frac{n}{k} \right) - \frac{1}{N} \left( \frac{N}{k} \right) + k.
\]
Hence, for every $N \geq n \geq 1$ and every $k \geq 1$, we have
\[
\left\| \frac{1}{n} ab(p_n) - \frac{1}{N} ab(p_N) \right\|_1 \leq K \left( \frac{\left[ \frac{n}{k} \right]}{n} + \frac{\left[ \frac{k}{N} \right]}{N} \right) + \frac{k}{N} + k + \frac{1}{n} \left( \frac{n}{k} \right) - \frac{1}{N} \left( \frac{N}{k} \right) + k.
\]
\[
\leq \frac{2K}{k} + \frac{3k}{n}.
\]

Thus, by taking \( k = \lfloor \sqrt{n} \rfloor \), we see that \((\frac{1}{n} \ab(p_n))_{n \geq 1}\) is a Cauchy sequence, so it converges to some vector \( v \in \mathbb{R}^{d+1}_+ \) with \( \|v\|_1 = 1 \).

Now, for every \( n \in \mathbb{N} \) we have
\[
\|2 \ab(p_n) - \ab(p_{2n})\|_1 = \| \ab(p_n) - \ab(q_n)\|_1 \leq K,
\]
where \( q_n \) is such that \( p_{2n} = p_n q_n \). So, for all \( n \in \mathbb{N} \), we have
\[
\|\pi_v(\ab(p_n))\|_1 = \|\ab(p_n) - nv\|_1 \leq \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \|2 \ab(p_{2k}) - \ab(p_{2k+1})\|_1 \leq \sum_{k=0}^{\infty} \frac{K}{2^{k+1}} = K,
\]
since \( \lim_{k \to \infty} \frac{1}{2^k} \ab(p_{2k}) = nv \). Hence, we have \( \pi_v(W(u)) \subseteq B(0, K) \). \( \square \)

### 3.8 Worms and Dumont-Thomas numeration

In all the following we consider \( S \subseteq \hom(A^+, A^+) \) a finite set of unimodular substitutions on the alphabet \( A \). We give a definition of the Dumont-Thomas numeration, which is a generalization, for a finite set \( S \) of substitutions, of the one given for a single substitution in [18].

**Definition 19.** The Dumont-Thomas alphabet associated to \( S \) is defined as
\[
\Sigma = \{ \ab(p) \mid \exists \sigma \in S, \ \exists a, b \in A, \ \text{pb prefix of } \sigma(a) \} \subseteq \mathbb{Z}^{d+1}.
\]

Remark that it is a finite set, since \( S \) and \( A \) are finite.

And we introduce an automaton:

**Definition 20.** We call abelianized prefix automaton of the set of substitutions \( S \), the automaton \( A \) defined by

- \( \text{alphabet } \Sigma \times S \),
- \( \text{set of states } A \),
- \( \text{transition} \ a \xrightarrow{t, \sigma} b, \ \text{with} \ (a, t, \sigma, b) \in A \times \Sigma \times S \times A \ \text{if, and only if, there exist} \ u, v \in A^* \ \text{such that} \ \sigma(a) = ubv, \ \text{with} \ ab(u) = t. \)

We denotes by \( a \xrightarrow{t_0, s_0} \ldots \xrightarrow{t_n, s_n} b \in A \) if we have a path in the automaton: there exist states \( a = a_{n+1}, a_n, \ldots, a_1, a_0 = b \) such that for all \( 0 \leq k \leq n \), \( a_{k+1} \xrightarrow{t_k, s_k} a_k \) is a transition in the automaton.

**Example 21.** Let \( S = \{ \tau_0, \tau_1 \} \), with \( \tau_0 = \{ 0 \mapsto 0, 1 \mapsto 01 \} \), \( \tau_1 = \{ 0 \mapsto 10, 1 \mapsto 1 \} \). Then, the Dumont-Thomas alphabet is \( \Sigma = \{ 0, e_0, e_1 \} \), and the abelianized prefix automaton \( A \) is depicted in Figure [2].

For every word \( u \in \{0, 1\}^N \) we have the relations
\[
W_0(\tau_0(u)) = M_0 W_0(u) \sqcup M_0 W_1(u),
\]
\[
W_1(\tau_0(u)) = M_0 W_1(u) + e_0,
\]
\[
W_0(\tau_1(u)) = M_1 W_0(u) + e_1,
\]
\[
W_1(\tau_1(u)) = M_1 W_0(u) \sqcup M_1 W_1(u),
\]
where \( e_0 = (1, 0), \ e_1 = (0, 1), \ M_0 = \ab(\tau_0) \) and \( M_1 = \ab(\tau_1) \).
The automaton \( A \) is depicted in Figure 12 for the set of Cassaigne substitutions, and in Figure 11 for the set of Arnoux-Rauzy substitutions.

Remark that for every \( u \in A^N \), \( \sigma \in S \) and \( a \in A \), we have the following relation

\[
W_a(\sigma(u)) = \bigcup_{\substack{b \xrightarrow{t} a}} ab(\sigma)W_b(u) + t. \tag{1}
\]

If we iterate Equation (1), we get

**Lemma 22** (Dumont-Thomas numeration). Let \( s \in S^N \) be a directive sequence and consider a fixed point \( u \in (A^N)^N \) of \( s \). Let \( b_n \) be the first letter of the word \( u_n \). We assume that \( |s_{[0,n]}(b_n)| \xrightarrow{n \to \infty} \infty \). Then, for every \( a \in A \) we have

\[
W_a(u_0) = \bigcup_{n \in \mathbb{N}} \{ \sum_{k=0}^{n} M_{[0,k]} t_k | b_{n+1} \xrightarrow{t_{n:s_n}} \ldots \xrightarrow{t_{0:s_0}} a \}. \]

**Remark 23.** In the following we will use the fact that for every \( a, b \in A \), in the automaton, the number of paths \( b \xrightarrow{t_{-1:s_{-1}}} \ldots \xrightarrow{t_{k:s_k}} a \) is equal to \( (M_{[k,i]})_{a,b} \).

3.9 Rauzy fractals

We recall the following result, see [10, Theorem 5.7].

**Proposition 24.** Let \( s \in S^N \) be a directive sequence and let \( u \) be a fixed point of \( s \). Assume that \( \bigcap_{n \in \mathbb{N}} M_{[0,n]}(s)\mathbb{R}^d_+ = \{ x \} \) and that \( s \) is everywhere growing (see Definition 3). Then the subshift \( \Omega_s \) is uniquely ergodic, and for every word \( w \in \Omega_s \), we have \( \text{freq}(w) = v(x) \). In particular, if \( u \in (A^N)^N \) is a fixed point of \( s \), then \( \text{freq}(u_0) = v(x) \).

Since the matrix \( M_k(s) \) is invertible, remark that \( \bigcap_{n \in \mathbb{N}} M_{[0,n]}(s)\mathbb{R}^{d+1}_+ \) is a line if, and only if, \( \bigcap_{n \in \mathbb{N}} M_{[k,k+n]}(s)\mathbb{R}^{d+1}_+ \) is a line for every \( k \in \mathbb{N} \). When it is the case, we denote by \( v^{(k)} \in \mathbb{R}^{d+1} \) the vector such that \( \|v^{(k)}\|_1 = 1 \) and \( \mathbb{R}_+v^{(k)} = \bigcap_{i \geq k} M_{[k,i]}(s)\mathbb{R}^{d+1}_+ \).

**Definition 25.** Let \( u \in A^N \) be an infinite word admitting a frequency vector \( v = \text{freq}(u) \). We define \( R(u) \) as the closure of \( \pi_v W(u) \subseteq P \). For a letter \( a \in A \), we also define \( R_a(u) \) as the closure of \( \pi_v W_a(u) \). The set \( R(u) \) is called Rauzy fractal. It is a generalization of the classical notion, for a fixed point of a substitution, see [34, 32] for references.
Example 26. For $u = (01001)\omega$, we have $\text{freq}(u) = (3/5, 2/5)$. So we can define the Rauzy fractal by projecting on the hyperplane (i.e., line) $x + y = 0$, and we get a Rauzy fractal with only 5 points. See Figure 5.

Examples of non-substitutive Rauzy fractals are drawn in Figure 13 and in Figure 6.

Remark 27. If $s \in S^N$ is a directive sequence such that $\bigcap_{n \in \mathbb{N}} M_{(0,n)}(s) \mathbb{P} \mathbb{R}^+ = \{x\}$, with $x$ a totally irrational direction, then, by Lemma 30 and by Proposition 24, for every fixed point $u \in (A^n)^N$ of $s$, the infinite word $u_0$ admits $v(x)$ as frequency vector, hence we can define the Rauzy fractal $R(u_0)$. 

Figure 5: The Rauzy fractal $R(u)$ as the closure of the projection of the worm $W(u)$ on the hyperplane $P$. Example for $u = (01001)^\omega$, so $v = (3/5, 2/5)$.

Figure 6: Approximation of the Rauzy fractal of a directive sequence beginning with $c_1c_1c_0c_1c_0c_0c_0c_1c_1c_0c_1c_0c_1c_0c_0c_0c_1c_1c_0c_0c_1c_1c_0c_1c_1c_0c_0c_0c_0c_0c_0c_1c_0c_0$, where $c_0$ and $c_1$ are defined in Section 8.

Here, $v = (0.279291082100669\ldots, 0.1294709739854265\ldots, 0.5912379439139045\ldots)$. 

Remark 27. If $s \in S^N$ is a directive sequence such that $\bigcap_{n \in \mathbb{N}} M_{(0,n)}(s) \mathbb{P} \mathbb{R}^+ = \{x\}$, with $x$ a totally irrational direction, then, by Lemma 30 and by Proposition 24, for every fixed point $u \in (A^n)^N$ of $s$, the infinite word $u_0$ admits $v(x)$ as frequency vector, hence we can define the Rauzy fractal $R(u_0)$. 

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For a given directive sequence \( s \), we do not have uniqueness of fixed point \( u \) in general. But under some assumptions it defines a unique Rauzy fractal (see Proposition 29 below). When it is the case, we denote the Rauzy fractal by \( R(s) \).

Using Rauzy fractals, we can give a characterization of the interior of \( W_a(u) \) for the topology \( T(x) \), with the following lemma.

**Lemma 28.** For every open subset \( B \) of the plane \( \mathbb{P} \), for every totally irrational direction \( x \in \mathbb{P} \), for every infinite word \( u \in \mathbb{A} \), and for every letter \( a \in \mathbb{A} \) we have the equivalence between

1. \( H \cap \pi^{-1}_x(B) \subseteq W_a(u) \),
2. \( \forall b \in \mathbb{A} \setminus \{a\}, \forall t \in \Lambda \setminus \{0\}, B \cap R_b = \emptyset = B \cap (R + t) \),

where \( R_b \) is the closure of \( \pi_x W_b(u) \) and \( R \) is the closure of \( \pi_x W(u) \).

In particular, \( p \in H \) is in the interior of \( W_a(u) \) for the topology \( T(x) \) if, and only if,

\[
\pi_x(p) \notin \bigcup_{a \in \mathbb{A} \setminus \{a\}} R_a \cup \bigcup_{t \in \Lambda \setminus \{0\}} R + t.
\]

**Proof.** We have

\[
W_a(u) = H \setminus \left( \bigcup_{b \in \mathbb{A} \setminus \{a\}} W_b(u) \cup \bigcup_{t \in \Lambda \setminus \{0\}} W(u) + t \right),
\]

\( \pi_x \) is injective on \( H \), and \( \pi_x(H) \) is dense in \( P \), so we have the equivalences

\[
H \cap \pi_x^{-1}(B) \subseteq W_a(u) \iff B \cap \pi_x(H) \subseteq \pi_x(W_a(u))
\]

\[
\iff B \cap \pi_x \left( \bigcup_{b \in \mathbb{A} \setminus \{a\}} W_b(u) \cup \bigcup_{t \in \Lambda \setminus \{0\}} W(u) + t \right) = \emptyset
\]

\[
\iff B \cap \left( \bigcup_{b \in \mathbb{A} \setminus \{a\}} R_b \cup \bigcup_{t \in \Lambda \setminus \{0\}} R + t \right) = \emptyset.
\]

\( \square \)

The following proposition allows to show that the Rauzy fractal does not depend on the choice of a fixed point \( u \) of \( s \), and it gives a useful characterization, with left-infinite paths in the abelianized prefix automaton.

**Proposition 29.** Let \( s \in S^\mathbb{N} \). We assume that we have

- primitivity: \( \forall n \in \mathbb{N}, \exists k \geq n, M_{[n,k]} > 0 \),
- strong convergence: \( \sum_n \| \pi_v M_{[0,n]} \|_1 \) converges, for some vector \( v \).
Then, for every letter $a \in A$ and every fixed point $u \in (A^\mathbb{N})^\mathbb{N}$ of $s$, we have

$$R_a(u_0) = \left\{ \sum_{n=0}^{\infty} \pi_v(M_{[0,n]}t_n) \mid ... \xrightarrow{t_{n,s_n}} ... \xrightarrow{t_{0,s_0}} a \in A \right\}.$$  

In particular, the Rauzy fractal does not depend on the choice of the fixed point $u$ and is compact.

**Proof.** Let $u$ be a fixed point of $s$. We denote by $b_n$ the first letter of the word $u_n$. By Lemma [22] for every letter $a \in A$, we have the equality

$$W_a(u_0) = \bigcup_{n \in \mathbb{N}} \left\{ \sum_{k=0}^{n} M_{[0,k]}t_k \mid b_{n+1} \xrightarrow{t_{n,s_n}} ... \xrightarrow{t_{0,s_0}} a \right\}.$$  

Hence, we have

$$R_a = \bigcup_{n \in \mathbb{N}} \left\{ \sum_{k=0}^{n} \pi_v(M_{[0,k]}t_k) \mid b_{n+1} \xrightarrow{t_{n,s_n}} ... \xrightarrow{t_{0,s_0}} a \right\}.$$  

Let us show one inclusion. Let $... \xrightarrow{t_{n,s_n}} ... \xrightarrow{t_{0,s_0}} a$ be a left-infinite path in the automaton $A$. Let $\epsilon > 0$. By the strong convergence hypothesis, and using that the Dumont-Thomas alphabet $\Sigma$ is finite, there exists $n \in \mathbb{N}$ such that

$$\max_{t \in \Sigma - \Sigma} \|t\|_1 \sum_{k=n+1}^{\infty} \|\pi_vM_{[0,k]}\|_1 \leq \epsilon.$$  

Using primitivity, there exists a path $b_{N+1} \xrightarrow{t'_{N,s_N}} ... \xrightarrow{t'_{0,s_0}} a \in A$, with $t'_k = t_k$ for every $k \leq n$. We have

$$\left\| \sum_{k=0}^{N} \pi_v(M_{[0,k]}t_k') - \sum_{k=0}^{N} \pi_v(M_{[0,k]}t'_k) \right\|_1 \leq \max_{t \in \Sigma - \Sigma} \|t\|_1 \sum_{k=n+1}^{\infty} \|\pi_vM_{[0,k]}\|_1 \leq \epsilon.$$  

Since $\sum_{k=0}^{N} \pi_v(M_{[0,k]}t'_k) \in R_a$ and since $R_a$ is closed, we deduce the inclusion

$$\left\{ \sum_{n=0}^{\infty} \pi_v(M_{[0,n]}t_n) \mid ... \xrightarrow{t_{n,s_n}} ... \xrightarrow{t_{0,s_0}} a \right\} \subseteq R_a.$$  

Let us show the other inclusion. We have the inclusion

$$\bigcup_{n \in \mathbb{N}} \left\{ \sum_{k=0}^{n} \pi_v(M_{[0,k]}t_k) \mid b_{n+1} \xrightarrow{t_{n,s_n}} ... \xrightarrow{t_{0,s_0}} a \right\} \subseteq \left\{ \sum_{n=0}^{\infty} \pi_v(M_{[0,n]}t_n) \mid ... \xrightarrow{t_{n,s_n}} ... \xrightarrow{t_{0,s_0}} a \right\}$$  

because for every $n \in \mathbb{N}$, there exists a left-infinite path labeled by zeroes going to $b_n$ since for every $k \geq n$, $u_n = s_{(n,k)}(u_k)$. To end the proof, it remains to show that the set $\left\{ \sum_{n=0}^{\infty} \pi_v(M_{[0,n]}t_n) \mid ... \xrightarrow{t_{n,s_n}} ... \xrightarrow{t_{0,s_0}} a \right\}$ is compact. We define a natural distance on the set of left-infinite paths in the automaton $A$ by taking a distance $2^{-n}$ between two paths that coincide for the last $n$ transitions. This distance makes the set of left-infinite paths compact, and the map sending a left-infinite path $... \xrightarrow{t_{n,s_n}} ... \xrightarrow{t_{0,s_0}} a$ to the corresponding sum $\sum_{n=0}^{\infty} \pi_v(M_{[0,n]}t_n)$ is continuous. So we get the compactness. □
The primitivity hypothesis of Proposition 29 can be replaced by the hypothesis that \( v \) has a totally irrational direction. Indeed, we have the following lemma and remark.

**Lemma 30.** Let \( v \in \mathbb{R}_{+}^{d+1} \) having a totally irrational direction such that
\[
\bigcap_{n \in \mathbb{N}} M_{[0,n)} \mathbb{R}_{+}^{d+1} = \mathbb{R}_{+} v,
\]
then we have primitivity:
\[
\forall k \in \mathbb{N}, \exists n \geq k, \ M_{[k,n)} > 0.
\]

**Proof.** For all \( k \in \mathbb{N} \), we have
\[
\bigcap_{n \in \mathbb{N}} M_{[k,n)} \mathbb{R}_{+}^{d+1} = \mathbb{R}_{+} M_{[0,k)}^{-1} v,
\]
and \( M_{(0,k)} v \) has a totally irrational direction, so it is enough to prove the result for \( k = 0 \).

We have \( \bigcap_{n \in \mathbb{N}} M_{[0,n)} \mathbb{R}_{+}^{d+1} = \mathbb{R}_{+} v, \) with \( v \) having a totally irrational direction, so there exists \( n \in \mathbb{N} \) such that \( M_{[0,n)} \mathbb{R}_{+}^{d+1} \) does not meet the boundary \( \bigcup_{i=0}^{d} \mathbb{R}_{+}^{d} \times \{0\} \times \mathbb{R}_{+}^{d-i}, \) and this is equivalent to \( M_{[0,n)} > 0. \)

**Remark 31.** Let \( s \in S^{N} \) be a directive sequence, and \( v \in \mathbb{R}_{+}^{d+1}. \) If \( \sum \|\pi_v M_{[0,n)}\|_1 \) converges, then we have \( \bigcap_{n \in \mathbb{N}} M_{[0,n)} \mathbb{R}_{+}^{d+1} = \mathbb{R}_{+} v. \) Indeed, if we take such vector \( v \) with \( \|v\|_1 = 1, \) then we have
\[
\forall y \in \mathbb{R}_{+}^{d+1}, \quad \|M_{[0,n)} y - \|M_{[0,n)} y\|_1 v\|_1 \leq \|\pi_v M_{[0,n)}\|_1 \|y\|_1 \xrightarrow{n \to \infty} 0.
\]

**Corollary 32.** Let \( s \in S^{N} \) be a directive sequence such that the sum \( \sum_n \|\pi_x M_{[0,n)}(s)\|_1 \) converges for a totally irrational direction \( x \in \mathbb{P} \mathbb{R}^{d}. \)

Then, for every letter \( a \in A \) and every fixed point \( u \in (A^{N})^{N} \) of \( s, \) we have
\[
R_a(u_0) = \left\{ \sum_{n=0}^{\infty} \pi_x(M_{[0,n)} t_n) \mid \ldots \xrightarrow{t_n,s_n} \ldots \xrightarrow{t_0,s_0} a \in A \right\}.
\]

In particular, we have the properties:

- the Rauzy fractal \( R(s) \) and its pieces do not depend on the choice of a fixed point,
- \( R(s) \) is bounded,
- \( R(s) \) covers the plane: \( \bigcup_{t \in A} R(s) + t = \mathbb{P} \),
- \( R(s) \) has non-empty interior.

**Proof.** Thanks to Remark 31 and Lemma 30, we can apply Proposition 29, hence we deduce the formula, the fact that the Rauzy fractal and its pieces do not depend on the choice of a fixed point, and the boundedness of \( R(s) \). Now, for any fixed point \( u \in (A^{N})^{N} \) of \( s \) we have \( W(u_0) \oplus \Lambda = \mathbb{H}, \) and we have that \( \pi_x(\mathbb{H}) \) is dense in \( P \) since \( x \) is a totally irrational direction. Hence, we deduce that the union \( \bigcup_{t \in A} R(s) + t \) is dense in \( P. \) Since \( R(s) \) is bounded, this union is locally finite, thus locally closed. Hence we get the wanted covering. The last point is a consequence of the Baire category theorem: if the interior of \( R(s) \) was empty, then the interior of the countable union \( \bigcup_{t \in A} R(s) + t = \mathbb{P} \) would be empty, which is absurd. \( \square \)
Remark 33. We emphasis the fact the the interior of $R(s)$ does not correspond to the interior of $W(u_0)$ for the topology $T(x)$, where $u \in (A^\mathbb{N})^\mathbb{N}$ is a fixed point of $s$. For a totally irrational direction $x$, if an open set $O$ of $P$ is such that $\pi_x^{-1}(O) \cap \mathbb{H} \subseteq W_a(u_0)$, then $O$ is included in the interior of $R_a(u_0)$, but the converse is false in general.

### 3.10 Technical lemmas

This subsection contains technical lemmas that we use in our proofs.

The following lemma assumes an exponential convergence that implies the strong convergence of Proposition 29. It says that this exponential convergence is invariant by the shift of the directive sequence.

**Lemma 34.** Let $s$ be a directive sequence and let $x \in \mathbb{PR}_d$ be a direction. Then, for every $k \in \mathbb{N}$, we have the equality

$$\limsup_{n \to \infty} \frac{1}{n} \ln \left\| \pi_x M_{[0,n]} \right\|_{1} = \limsup_{n \to \infty} \frac{1}{n} \ln \left\| \pi_x M_{[k,k+n]} \right\|_{1},$$

where $x^{(k)} = M_{(0,k)}^{-1} x$. In particular, if we have that $\limsup_{n \to \infty} \frac{1}{n} \ln \left\| \pi_x M_{[0,n]} \right\|_{1} < 0$, then we have

$$\forall k \in \mathbb{N}, \exists C > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \left\| \pi_x M_{[k,k+n]} \right\|_{1} \leq e^{-nC}.$$

**Proof.** Let $N$ be the linear endomorphism of $P$ such that $\pi_x M_{[0,k]} = N \pi_x^{(k)}$. Remark that $N$ is invertible. We have the inequalities

$$\left\| \pi_x M_{[0,n+k]} \right\|_{1} \leq \left\| N \right\|_{1} \left\| \pi_x^{(k)} M_{[k,n+k]} \right\|_{1},$$

and

$$\left\| \pi_x^{(k)} M_{[k,k+n]} \right\|_{1} \leq \left\| N^{-1} \right\|_{1} \left\| \pi_x M_{[0,n+k]} \right\|_{1}.$$  

So we get the wanted equality

$$\limsup_{n \to \infty} \frac{1}{n} \ln \left\| \pi_x M_{[0,n]} \right\|_{1} = \limsup_{n \to \infty} \frac{1}{n} \ln \left\| \pi_x M_{[0,n+k]} \right\|_{1} = \limsup_{n \to \infty} \frac{1}{n} \ln \left\| \pi_x^{(k)} M_{[k,k+n]} \right\|_{1}.$$  

We deduce the second part of the lemma by taking

$$C = -\frac{1}{2} \limsup_{n \to \infty} \frac{1}{n} \ln \left\| \pi_x M_{[k,k+n]} \right\|_{1}.$$

\[\square\]

The remaining lemmas in this subsection are topology exercises and are not specific to our subject.

**Lemma 35.** Let $B$, $C$, $D$ be open subsets of $P$. If $D \subseteq \overline{B}$ and $D \subseteq \overline{C}$, then $D \subseteq \overline{B \cap C}$.

**Proof.** Let $x \in D$. Let $r_0 > 0$ small enough to have $B(x, r_0) \subseteq D$ (balls are assumed open in this proof). Let $r > 0$ such that $r \leq r_0$.

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• As $D \subseteq \overline{B}$, we get that $B(x, r) \subseteq D \subseteq \overline{B}$. If $B(x, r) \cap B = \emptyset$, then $x \notin \overline{B}$ which is absurd. So there exists $y \in B(x, r) \cap B$, and since these are open sets, there exists $r' > 0$ such that $B(y, r') \subseteq B(x, r) \cap B$.

• Also, $B(y, r') \subseteq B(x, r) \subseteq \overline{C}$ thus there exists $z$ and $t > 0$ such that $B(z, t) \subseteq B(y, r') \cap C$.

• Finally, for all $r$ small enough we have found $z \in B \cap C$ such that $d(x, z) < r$ (since $z \in B(x, r)$).

Therefore $x \in \overline{B \cap C}$.

The next technical lemmas are useful in the proof of Proposition 66.

Lemma 36. Let $H$ be a closed subset of a metric space $X$, and let $\mu$ be a finite measure on $X$ such that $\mu(H) = 0$. Then for every $\epsilon > 0$ there exists an open subset $O$ such that $\mu(O) \leq \epsilon$ and $H \subseteq O$.

Proof. For $n \geq 1$, let $H_n = \{ x \in X \mid d(x, H) < \frac{1}{n} \}$. We have $\bigcup_{n \geq 1} X \setminus H_n = X \setminus H$, because $H$ is closed. Thus, we have $\lim_{n \to \infty} \mu(X \setminus H_n) = \mu(X \setminus H) = \mu(X)$. Let $\epsilon > 0$. There exists $n \geq 1$ such that $\mu(H_n) \leq \epsilon$. Then, the open set $O = H_n$ suits.

Lemma 37. Let $X \subseteq \mathbb{P}R^d_+$ and let $\mu$ be a probability measure on $X$. Let $N \subseteq X$ be the set of non totally irrational directions of $X$. We assume that $\mu(N) = 0$. Then, for every $\epsilon > 0$ there exists an open set $O$ of $X$ such that $O$ contains all the non totally irrational directions and such that $\mu(O) \leq \epsilon$.

Proof. The set $N$ is the union of kernels of linear forms with rational coefficients. Thus it is a countable union of closed subsets. Let $(N_n)_{n \in \mathbb{N}}$ be closed subsets such that $N = \bigcup_{n \in \mathbb{N}} N_n$. Let $\epsilon > 0$. For every $n \in \mathbb{N}$, let $O_n$ be an open set given by Lemma 36 such that $\mu(O_n) \leq \frac{\epsilon}{2^{n+1}}$ and $N_n \subseteq O_n$. Then, the open set $O = \bigcup_{n \in \mathbb{N}} O_n$ satisfies what we want: we have $N \subseteq O$ and

$$\mu(O) \leq \sum_{n \in \mathbb{N}} \mu(O_n) \leq \sum_{n \in \mathbb{N}} \frac{\epsilon}{2^{n+1}} = \epsilon.$$

4 General conditions for the existence of nice Rauzy fractals

4.1 Statement

Definition 38. We say that a directive sequence $s \in S^N$ is good if

1. $\limsup_{n \to \infty} \frac{1}{n} \ln \left\| \pi_x M_{[0,n]} \right\|_1 < 0$, for some direction $x \in \mathbb{P}R^d_+$,

2. The direction $x$ is totally irrational,
3. There exists a fixed point \( u \in (A^N)^N \) of \( s \), an increasing sequence of integers \((k_n)_{n \in \mathbb{N}}\), and a positive radius \( r > 0 \) such that

\[
\forall n \in \mathbb{N}, \forall a \in A, \exists p \in P, \mathbb{H} \cap \pi_{x(k_n)}^{-1}(B(p,r)) \subseteq W_a(u_{k_n}),
\]

where \( x(k_n) = M_{0,k_n}^{-1} x \),

4. The sequence \( x(k_n) \) has a limit which is a totally irrational direction.

By Remark 31, the direction \( x \) is unique. We call it the direction of \( s \).

Remark that for a good directive sequence, the Rauzy fractal does not depend on the choice of a fixed point, is compact and has non-empty interior by Corollary 32. We recall Theorem C that will be proven in the rest of this section:

**Theorem C.** Let \( s \in S^N \) be a good directive sequence. Then the Rauzy fractal \( R(s) \) is a measurable fundamental domain of \( P \) for the lattice \( \Lambda \). It can be decomposed as an union \( R(s) = \bigcup_{a \in A} R_a(s) \) which is disjoint in Lebesgue measure, and each piece \( R_a(s) \) is the closure of its interior.

Moreover, the pieces \( R_a(s) \), \( a \in A \), of the Rauzy fractal induce a nice generating partition of the translation by \( e_0 - v \) on the torus \( P/\Lambda \), where \( v \) is the unit vector of the direction of \( s \). Its symbolic coding is a measurable conjugacy with the subshift associated to \( s \).

**Remark 39.** If we consider a directive sequence of the form \( \sigma^\omega \), where \( \sigma \) is an unimodular substitution, then we are back in the classical setting of the Rauzy fractal associated with a single substitution. In this sense, Theorem C gives a generalization of [1, Theorem 1.3.3].

The converse of Theorem C is true for directive sequences of the form \( \sigma^\omega \): if the subshift is conjugate to a translation on a torus, then \( \sigma^\omega \) is good, where \( \sigma \) is an irreducible Pisot unimodular substitution. See Subsection 10.5 for more details.

The Pisot substitution conjecture gives that for every irreducible Pisot unimodular substitution \( \sigma \), the directive sequence \( \sigma^\omega \) is good. See Subsection 10.5 for more details.

### 4.2 Proof of Theorem C

In all this subsection we assume that \( s \) is a good directive sequence, \( x^{(k)} \in \mathbb{P}^d \) is such that

\[
\{x^{(k)}\} = \bigcap_{n \geq k} M_{[k,n]} \mathbb{P}^d,
\]

and \( u \) is a fixed point of \( s \). We also denote \( x = x^{(0)} \). Remark that for all \( k \in \mathbb{N} \) we have \( x^{(k)} = M_{0,k}^{-1} x \). We denote by \( R^{(k)} = \pi_{x^{(k)}} W(u_k) \) and \( \forall a \in A, R^{(k)}_a = \pi_{x^{(k)}} W_a(u_k) \) the Rauzy fractal uniquely defined by the good directive sequence \((s_n)_{n \geq k}\).

#### 4.2.1 Step 1: proof that we have a topological tiling

**Lemma 40.** For every \( k \in \mathbb{N} \) and every \( a \in A \), the set \( W_a(u_k) \) has non-empty interior for \( T(x^{(k)}) \).
Proof. Consider $k \in \mathbb{N}$ and $a \in A$. We have by Equation (1)

$$W_a(u_k) = \bigcup_{b \xrightarrow{s_k \rightarrow a} \rightarrow} M_b W_b(u_{k+1}) + t.$$  

Now, if we assume that the interior of $W_b(u_{k+1})$ is not empty for $T(x^{(k+1)})$, for some $b \in A$ such that $a$ occurs in $s_k(b)$, then by Lemma [17] the interior of $M_b W_b(u_{k+1})$ is non-empty for $T(x^{(k)})$. So $W_a(u_k)$ also has non-empty interior.

Then we iterate the process: For $l > k$ we have

$$W_a(u_k) = \bigcup_{b \xrightarrow{s_{l-1} \rightarrow \ldots s_{k-1} \rightarrow s_k \rightarrow a} \rightarrow} M_{[k,l]} W_b(u_l) + \sum_{i=k}^{l-1} M_{[k,i]} t_i. \quad (2)$$

By hypothesis, for a fixed $k$ we can find $l \geq k$ where the interior of $W_b(u_l)$ is non-empty for all $b \in A$. Since $M_{[k,l]}$ is invertible, there exists at least one $b \in A$ such that the union in (2) is non-empty, and we deduce the result. \hfill \square

**Lemma 41.** For every $k \in \mathbb{N}$ and every $a \in A$, the interior of $W_a(u_k)$ is dense in $W_a(u_k)$ for $T(x^{(k)})$.

Now consider $U \subseteq P$ an open set such that $\pi_{x^{(k)}}^{-1}(U) \cap \mathbb{H}$ is the interior of $W_a(u_k)$. Then the set $U$ is dense in $R_a^{(k)}$.

**Proof.** Consider $m \in W_a(u_k)$ and $V$ open set containing $m$. We want to find an element of $V$ in the interior of $W_a(u_k)$. By Equation (2), $m$ belongs to a set of the following form for each $l \geq k$:

$$M_{[k,l]} W_b(u_l) + t$$

By Lemma [16] the sets $\pi_{x^{(k)}} W_b(u_{k_n})$ are uniformly bounded for $n \in \mathbb{N}$, thus we deduce with $\limsup_{n \rightarrow \infty} \frac{1}{n} \ln \| \pi_{x^{(k')}} W_{[0,n]} \|_1 < 0$ and Lemma [34] that the diameter of

$$\pi_{x^{(k)}} M_{[k,k_n]} W_b(u_{k_n})$$

is arbitrarily small for $n \in \mathbb{N}$ large enough, hence there exists $n \in \mathbb{N}$ such that we have the inclusion $M_{[k,k_n]} W_b(u_{k_n}) + t \subseteq V$. As this set has non-empty interior by Lemma [17] it follows that $V$ intersects the interior of $W_a(u_k)$. This proves that the interior of $W_a(u_k)$ is dense in $W_a(u_k)$.

Now, if $U$ is an open subset of $P$ such that $\pi_{x^{(k)}}^{-1}(U) \cap \mathbb{H}$ is the interior of $W_a(u_k)$, then the projection $U \cap \pi_{x^{(k)}}(\mathbb{H})$ is dense in $\pi_{x^{(k)}}(W_a(u_k))$ which is dense in $R_a^{(k)}$. Thus $U$ is dense in $R_a^{(k)}$. \hfill \square

**Lemma 42.**

- For every $t \in \Lambda \setminus \{0\}$, $R \cap (R + t)$ has empty interior.
- For $a \neq b \in A$, $R_a \cap R_b$ has empty interior.
Proof. We denote $\pi = \pi_{x(0)}$. By Lemma 14, we have $W(u_0) \cap (W(u_0) + t) = \emptyset$. Now consider $U \subseteq P$ an open set such that $\pi^{-1}(U) \cap \mathbb{H}$ is the interior of $W(u_0)$. Then, we have

$$
\pi^{-1}(U) \cap (\pi^{-1}(U) + t) \cap \mathbb{H} = \emptyset \implies U \cap (U + \pi(t)) \cap \pi(\mathbb{H}) = \emptyset
$$

$$
\implies U \cap (U + \pi(t)) = \emptyset,
$$

because $\pi(\mathbb{H})$ is dense in $P$ since the direction $x^{(0)}$ is totally irrational. Moreover, by Lemma 41, we have that $U$ is dense in the $R$. Then, by Lemma 35, the empty set $U \cap (U + t)$ is dense in the interior of $R \cap (R + t)$. We deduce that the interior of $R \cap (R + t)$ is empty.

For $a \neq b$, we have $W_a(u_0) \cap W_b(u_0) = \emptyset$. Let $U_a$ and $U_b$ be open subsets of $P$ such that $W_a(u_0) = \pi^{-1}(U_a) \cap \mathbb{H}$ and $W_b(u_0) = \pi^{-1}(U_b) \cap \mathbb{H}$. Then, by Lemma 35, the empty set $U_a \cap U_b$ is dense in the interior of $R_a \cap R_b$. We deduce that the interior of $R_a \cap R_b$ is empty.

4.2.2 Step 2: proof that the boundary has zero Lebesgue measure

For every $k \in \mathbb{N}$, we denote $v^{(k)} = v(x^{(k)})$ the unique vector such that $[v^{(k)}] = x^{(k)}$ and $\|v^{(k)}\|_1 = 1$. Remark that the direction $x^{(k)}$ being totally irrational, the numbers $v^{(k)}_a$ can not be equal to zero. Let us then define $g_k = \max_{a \in A} \frac{\lambda(H_k)}{v^{(k)}_a}$, $f_k = \max_{a \in A} \frac{\lambda(H_k)}{v^{(k)}_a}$. Let $N_k$ be the linear endomorphism of $P$ such that $N_k \circ \pi_{x^{(k+1)}} = \pi_{x^{(k)}} \circ M_k$. This map is well-defined since $M_k \pi_{x^{(k+1)}} = x^{(k)}$. Observe that $N_k$ is an invertible map.

Lemma 43. For all $l > k$, we have

$$
\det(N_{[k,l]}) = \frac{\det(M_{[k,l]})}{\|M_{[k,l]} v^{(l)}\|_1}.
$$

Proof. Consider two bases of $\mathbb{R}^{d+1}$ made by a basis of $P$ and $v^{(l)}$ for one, and by the same basis of $P$ and $v^{(k)}$ for the second one. Then we compute the matrix of the linear map $M_{[k,l]}$ in these bases. To do this we use the definition of $N_{[k,l]}$ and the fact that $M_{[k,l]} v^{(l)} = \|M_{[k,l]} v^{(l)}\|_1 v^{(k)}$. Thus we obtain

$$
\begin{pmatrix}
[N_{[k,l]}] & 0 \\
* & \|M_{[k,l]} v^{(l)}\|_1
\end{pmatrix}
$$

The matrix of change of basis is $\begin{pmatrix} \text{Id} & * \end{pmatrix}$. Then we compute the determinant of the matrix, and obtain the result.

Lemma 44. For every $k \in \mathbb{N}$, we have $g_k \leq g_{k+1}$ and $f_k \leq f_{k+1}$.

Proof. For every $k \in \mathbb{N}$ and $a \in A$, we have the equality

$$
R^{(k)}_a = \bigcup_{b \xrightarrow{t,\pi_k} a} N_k R^{(k+1)}_b + \pi_{v^{(k)}}(t).
$$
We deduce
\[
\left(\lambda(R^{(k)}_a)\right)_{a \in A} \leq \left(\sum_{b \in A} |\det N_b| \lambda(R^{(k+1)}_b)\right)_{a \in A} \\
= |\det N_k| M_k(\lambda(R^{(k+1)}_a))_{a \in A} \\
\leq |\det N_k| M_k g_{k+1}v^{(k+1)}.
\]
By Lemma 43 and using $|\det M_k| = 1$ we have
\[
\left(\lambda(R^{(k)}_a)\right)_{a \in A} \leq \frac{1}{||M_kv^{(k+1)}||_1} g_{k+1}M_k v^{(k+1)} = g_{k+1}v^{(k)},
\]
thus the sequence $g_k$ is nondecreasing. The proof is similar for the sequence $f_k$. \(\square\)

Let $I = \{k_n \mid n \in \mathbb{N}\}$, and let $a \in A$. For every $b \in A$, $k, r \in I$ and $l \in I$, let
\[
L_{b}^{k,l} = \{\pi_{v^{(k)}} \left(\sum_{i=k}^{l-1} M_{[k,i]} t_i\right) \in P \mid b \xrightarrow{t_{i-1},s_{i-1}} \ldots \xrightarrow{t_k,s_k} a\}
\]
\[
P_{b}^{k,l} = \{t \in L_{b}^{k,l} \mid N_{[k,l]} R_{b}^{(l)} + t \subseteq B(p_t, r)\},
\]
where $r > 0$ and $p_t \in P$ are such that $\mathbb{H} \cap \pi_{v^{(l)}}^{-1} B(p_t, r) \subseteq W_{a}(u_t)$. It is possible by definition of $k_n$, since the directive sequence is good (see Definition 38).

**Lemma 45.** There exists a uniform constant $C_a > 0$ such that for all $n \in I$, there exists $l_0$ such that for all $l > l_0$, $l \in I$, we have
\[
\sum_{b \in A} v_{b}^{(l)} \#P_{b}^{n,l} \geq C_a \sum_{b \in A} v_{b}^{(l)} \#L_{b}^{n,l}.
\]

**Proof.** Let $n \in I$. Let $l_0 \in \mathbb{N}$ such that $\forall l \geq l_0$ with $l \in I$, the diameter of $N_{[n,l]} R_{b}^{(l)}$ is less than $r/2$. It is possible, using Lemma 34, and because $R^{(l)}$, $l \in I$, are uniformly bounded by Lemma 16. Then, for every $t \in L_{b}^{k,l}$, if $N_{[n,l]} R_{b}^{(l)} + t$ meets $B(p_t, r/2)$, then it is included in $B(p_t, r)$. Thus, we have
\[
g_t \sum_{b \in A} v_{b}^{(l)} \#P_{b}^{n,l} \geq \sum_{b \in A} \lambda(R^{(l)}_b) \#P_{b}^{n,l} \geq \frac{\lambda(B(p_t, r/2))}{|\det(N_{[n,l]})|}.
\]
Moreover we have $\sum_{b \in A} v_{b}^{(l)} \#L_{b}^{n,l} = (M_{[n,l]} v^{(l)})_a = ||M_{[n,l]} v^{(l)}||_1 v^{(n)}_a$. We deduce by Lemma 43
\[
\sum_{b \in A} v_{b}^{(l)} \#P_{b}^{n,l} \geq \frac{1}{g_t} \frac{\lambda(B(p_t, r/2))}{|v^{(n)}_a|}.
\]
Since $1/g_t$ and $v^{(n)}_a$ converges to non-zero values, by hypothesis of total irrationality on the limit of the sequence of directions $x^{(n)}$, for $n \in I$, we deduce the result. \(\square\)
Proposition 46. There exists \( c < 1 \) such that, for every \( k \in I \), there exists \( l > k \) in \( I \) such that \( f_k \leq cf_l \), where \( I = \{ k_n \mid n \in \mathbb{N} \} \).

Proof. Let \( k \in I \). For every \( l > k \), we have

\[
R_a^{(k)} = \bigcup_{b \in A} \bigcup_{t \in L_b^{k,l}} N_{[k,l]}R_b^{(l)} + \sum_{i=k}^{l-1} \pi_v(b)(M_{[k,i]}t_i).
\]

Hence, we have

\[
\partial R_a^{(k)} \subseteq \bigcup_{b \in A} \bigcup_{t \in L_b^{k,l}} N_{[k,l]}\partial R_b^{(l)} + t.
\]

Now if \( l \in I \) and \( t \in P_{b}^{k,l} \), then \( N_{[k,l]}\partial R_b^{(l)} + t \) included in the interior of \( R_a^{(k)} \), thus we deduce

\[
\partial R_a^{(k)} \subseteq \bigcup_{b \in A} \bigcup_{t \in L_b^{k,l} \setminus P_{b}^{k,l}} N_{[k,l]}\partial R_b^{(l)} + t,
\]

Using the inclusion we deduce

\[
\lambda(\partial R_a^{(k)}) \leq |\text{det}(N_{[k,l]})| \sum_{b \in A} \lambda(\partial R_b^{(l)}) \#(L_b^{k,l} \setminus P_{b}^{k,l})
\]

\[
\leq |\text{det}(N_{[k,l]})| f_l \sum_{b \in A} v_b^{(l)} \#(L_b^{k,l} \setminus P_{b}^{k,l})
\]

\[
\leq |\text{det}(N_{[k,l]})| f_l (1 - C_a) \sum_{b \in A} v_b^{(l)} \#L_b^{k,l}
\]

\[
= (1 - C_a) f_l |\text{det}(N_{[k,l]})| \| M_{[k,l]}v^{(l)} \|_1 v_a^{(k)}
\]

\[
= (1 - C_a) f_l v_a^{(k)}.
\]

Thus \( f_k \leq (1 - C) f_l \), with \( C = \min_{a \in A} C_a \).

Lemma 47. For every \( a \in A \), we have \( \lambda(\partial R_a) = 0 \).

Proof. We deduce from Proposition 46 that there exists \( k \in \mathbb{N} \) such that \( f_k = 0 \). Then we have \( \lambda(\partial R_a^{(n)}) = 0 \) for every \( n \leq k \) and every \( a \in A \) by Lemma 44.

4.2.3 Step 3: proof that the translation is conjugate to the subshift

We refer to [1]. In the theorem that we recall below, the authors give conditions to prove that the translation by \( \pi_x(e_0) \) on the torus \( P/\Lambda \simeq \mathbb{T}^d \) is measurably conjugate to the subshift \( \Omega_w \) generated by a word \( w \in A^\mathbb{N} \). We check that each condition is satisfied for the word \( w = u_0 \):

- The boundedness is given by the Corollary 32
- Minimality of the subshift \( \Omega_{u_0} = \Omega_u \) is given by Proposition 5 since we have primitivity by Lemma 30.
• Boundaries of $R_a$, $a \in A$, have zero Lebesgue measure thanks to Lemma 47.

• Thanks Lemma 42, we have a topological tiling, and thanks to Lemma 47 the boundaries have zero Lebesgue measure. Hence we deduce that the union $\bigcup_{t \in \Lambda} R + t = P$ is disjoint in Lebesgue measure.

Thus, we can use the following theorem for $w = u_0$.

**Theorem 48.** [1, Theorem 2.3] Let $w \in A^\mathbb{N}$ be an infinite word, and let $x$ be a totally irrational direction. Let $R = \pi_x(W(w))$ and for all $a \in A$, $R_a = \pi_x(W_a(w))$. We assume that we have the following:

• the set $\pi_x(W(w))$ is bounded,

• the subshift $(\Omega_w, T)$ generated by $w$ is minimal,

• the boundaries of $R_a$, $a \in A$, have zero Lebesgue measure,

• the union $\bigcup_{t \in \Lambda} R + t = P$ is disjoint in Lebesgue measure.

Then there exists a Borel $T$-invariant measure $\mu$ such that the subshift $(\Omega_w, T, \mu)$ is measurably conjugate to the translation on the torus $(P/\Lambda, T_{\pi_x(e_0)}, \lambda)$.

The idea to prove this theorem is to show that the natural conjugacy that we have between the shift map on the orbit $O(w) = \{T^n w \mid n \in \mathbb{N}\}$ and the translation by $\pi_x(e_0)$ on the quotient $\mathbb{H}/\Lambda$, gives a measurable conjugacy after taking the closure. See Figure 7. Hence, we have:

**Remark 49.** The symbolic coding coming from the partition of the Rauzy fractal into pieces $R_a(s)$, $a \in A$ is a measurable conjugacy between the translation by $\pi_x(e_0)$ on the torus $P/\Lambda$ and the subshift $\Omega_s = \Omega_w$. In particular, it gives a generating partition.

![Figure 7: Commutative diagrams of the conjugacy between the shift $T$, the domain exchange $E$ and the translation on the quotient $T_{\pi_x(e_0)}$, before and after taking the closure](image)

Figure 8 shows the tiling of $P$ by the Rauzy fractal of Figure 6 for the lattice $\Lambda$. The vector $\pi_x(e_0)$ giving the translation in the quotient $P/\Lambda$ is also depicted.
Figure 8: Tiling of $P$ by a Rauzy fractal, giving the translation by $\pi_x(e_0)$ on the torus $P/\Lambda$

4.2.4 Conclusion

Now, we prove Theorem C. Starting from a good directive sequence, there exists a Rauzy fractal $R$ by Remark 27. By Lemma 42 we know that $R$ and $R + t, t \in \Lambda \setminus \{0\}$ have intersection of empty interior. Since $\bigcup_{t \in \Lambda} R + t = P$, by Corollary 32 we deduce that $R$ defines a topological tiling of the torus $P/\Lambda \simeq T^d$. By step 2 we know that the boundaries of $R$ and of the pieces $R_a$ have zero Lebesgue measures. Thus we deduce that up to a set of zero Lebesgue measure, $R$ defines a measurable fundamental domain of $P$ for the action of $\Lambda$. By Lemma 42 we know that the interior of the intersection of two pieces is empty. So such intersection $R_a \cap R_b$ is included in the boundary of $R_a$ and it has zero Lebesgue measure. Thus, we get that the union $R(s) = \bigcup_{a \in A} R_a(s)$ is disjoint in Lebesgue measure. By step 3, Theorem 48 and Remark 49 give the expected conjugacy.

Now, we prove that for every $a \in A$, $R_a$ is a bounded remainder set for the translation by $\pi_x(e_0)$ on the torus $P/\Lambda$. The Rauzy fractal being bounded, there exists a constant $K$ such that $R - R \subseteq B(0, K)$, where $R - R = \{p - q \mid p, q \in R\}$ is the set of differences. Let $u$ be a fixed point of $s$. By the previous conjugacy, for $\lambda$-almost every $z \in P/\Lambda$, there exists an infinite word $w \in \Omega_u$ such that $w$ is the coding of the orbit of $z$ by the translation for the measurable partition $(R_a)_{a \in A}$ of the torus $P/\Lambda$. Then, for every $a \in R_a$ we have the equality

$$\sum_{n=0}^{N-1} 1_{R_a}(T^n_{\pi_x(e_0)}(z)) = |p_N|_a,$$

where $p_N$ is the prefix of length $N$ of the word $w$. Since $w \in \Omega_u$, for every $N \in \mathbb{N}$ the word $p_N$ is a factor of $u_0$. Thus

$$ab(p_N) - Nv(x) = \pi_x(ab(p_N)) \in \pi_x(W(u_0) - W(u_0)) \subseteq R - R \subseteq B(0, K).$$

Hence, for $\lambda$-almost every $z \in P/\Lambda$ and for every $a \in A$ we get the inequality

$$\left| \sum_{n=0}^{N-1} 1_{R_a}(T^n_{\pi_x(e_0)}(z)) - Nv(x)_a \right| \leq K.$$

Now, by Birkhoff ergodic theorem, we get that for every $a \in A$, $v(x)_a = \frac{\lambda(R_a)}{\lambda(R)}$, so $R_a$ is a bounded remainder set.
Finally note that, by construction, \((R_a)_{a \in A}\) is a liftable partition of the torus. Altogether, we get that \((R_a)_{a \in A}\) is a nice generating partition.

5 Dynamics of continued fractions

5.1 Extended continued fraction algorithms

**Definition 50.** An extended continued fraction algorithm, denoted \((X, s_0)\), is the data of

- a subset \(X \subseteq \mathbb{PR}^d\),
- a finite alphabet \(A = \{0, \ldots, d\}\), with \(d \geq 1\),
- a finite set \(S \subseteq \text{hom}(A^+, A^+)\) of unimodular substitutions on the alphabet \(A\),
- a map \(s_0 : X \to S\) such that for all \(x \in X\), \(ab(s_0(x))^{-1}x \in X\).
- a map defined by
  \[
  F = \left( \begin{array}{cc}
  X & \longrightarrow \\
  x & \longrightarrow \quad \text{ab}(s_0(x))^{-1}x
  \end{array} \right).
  \]

We use the word extended to indicate that the algorithm uses substitutions. If we do not use the substitutions, we can retain their matrices \(ab(s_0(x))\) only, or even just the map \(F\). We then speak of a continued fraction algorithm, denoted \((X, F)\).

Given a continued fraction algorithm \((X, F)\), there are several possible choices for \(S\) and \(s_0\) to turn it into an extended continued fraction algorithm \((X, s_0)\). These choices do not yield the same associated subshifts, and not the same complexity function.

Moreover we define \(X_0\) as the biggest subset of \(X\) such that

- \(F^n\) is continuous on \(X_0\) for all \(n \in \mathbb{N}\),
- \(X \setminus X_0\) contains all the non totally irrational directions.

For \(x \in X\), we also denote \(s_i(x) = s_0(F^i(x)), i \geq 0\). The matrices associated to the substitutions are denoted by \(M_i = M_i(x) = ab(s_i(x))\), and we use the classical notation: if \(p \leq q\), \(M_{[p,q]}\) stands for the product of matrices \(M_p M_{p+1} \ldots M_{q-1}\). With the map \(s_0\) we can do some symbolic dynamics: it allows to define a map

\[
\begin{pmatrix}
  X \\
  x
\end{pmatrix} \quad \mapsto \quad s^N(x) = (s_n(x))_{n \in \mathbb{N}}.
\]

**Definition 51.** Let \((X, s_0)\) be an extended continued fraction algorithm, equipped with a measure \(\mu\). We say that \((X, s_0, \mu)\) is an extended measured continued fraction algorithm if

1. \(\mu\) is an ergodic \(F\)-invariant Borel probability measure,
2. The map \(s_0\) is measurable with respect to \(\mu\),
3. \(s_0\) is measurable with respect to \(\mu\),
3. \( \mu(X_0) = 1 \),

4. for all measurable \( Y \subseteq X \) we have \( \mu(Y) = 0 \implies \mu(F(Y)) = 0 \),

5. \( \exists \varepsilon > 0, \forall x \in X_0, \forall n \geq 1, \mu(F^n(\{ y \in X \mid M_{[0,n]}(y) = M_{[0,n]}(x) \})) > \varepsilon. \)

As above, if we are not interested in the particular choice of substitution, we will consider instead the measured continued fraction algorithm \((X, F, \mu)\).

Remark that for usual continued fraction algorithms, \(X \setminus X_0\) is an invariant set. And in this case, the ergodicity of \(\mu\) gives \(\mu(X \setminus X_0) = 1\) or \(\mu(X \setminus X_0) = 0\). Hence, the hypothesis \(\mu(X_0) = 1\) is equivalent to say that \(\mu\) is not supported only by \(X \setminus X_0\).

Now we give a criterion to prove that a map satisfies the hypotheses of Definition 51.

**Proposition 52.** Assume that we have a map

\[
F = \begin{pmatrix} \text{X} & \longrightarrow & \text{X} \\ \text{x} & \longmapsto & \text{ab}(s_0(x))^{-1}x \end{pmatrix}
\]

such that there exists a finite union \(H\) of rational hyperplanes of \(\mathbb{PR}_+^d\) that partition \(X \setminus H\) into a finite number of pieces \((X_i)_{i \in I}\) such that for every \(i \in I\),

- \(\text{ab}(s_0)\) is constant on \(X_i\),
- \(\text{(ab}(s_0(x))^{-1}X_i) \setminus H\) is a union of pieces: \(\text{(ab}(s_0(x))^{-1}X_i) \setminus H = \bigcup_{j \in J} X_j\) for some \(J \subseteq I\).

If \(\mu\) is a Borel ergodic probability measure on \(X\) such that

- for all \(i \in I\), \(\mu(X_i) > 0\),
- for every measurable subset \(Y\), we have \(\mu(Y) = 0 \implies \mu(F(Y)) = 0\).
- the measure of the set of non totally irrational directions is zero,

then \((X, s_0, \mu)\) is an extended measured continued fraction algorithm as defined in Definition 51.

Such a family \((X_i)_{i \in I}\) is sometimes called a Markov partition.

**Proof.** Let \(X_{\text{irr}} \subseteq X\) be the set of totally irrational directions of \(X\). With such hypotheses, the map \(F\) is continuous on \(X_{\text{irr}} \subseteq X \setminus H\), and the set \(X_{\text{irr}}\) is invariant by \(F\). So, for all \(n \in \mathbb{N}\), \(F^n\) is continuous on \(X_{\text{irr}}\), so \(X_0 = X_{\text{irr}}\). By hypothesis, we have \(\mu(X_0) = 1\). It remains to show the property

\[ \exists \varepsilon > 0, \forall x \in X_0, \forall n \geq 1, \mu(F^n(\{ y \in X \mid M_{[0,n]}(x) = M_{[0,n]}(y) \})) > \varepsilon. \]

Let us show that for all \(n \geq 1\), we have

\[ F^n(\{ y \in X \mid M_{[0,n]}(x) = M_{[0,n]}(y) \}) \supseteq M_{n-1}^{-1}(x)X_{i(F^n-x)} = F(X_{i(F^n-x)}), \]

where \(i(y) \in I\) is such that \(y \in X_{i(y)}\). It will ends the proof since the sets \(F(X_{i(F^n-x)})\) have positive measure

\[ \mu(F X_{i(F^n-x)}) = \mu(F^{-1}(F X_{i(F^n-x)})) \geq \mu(X_{i(F^n-x)}) > 0, \]

32
and there are finitely many of them.

The inclusion is equivalent to
\[ \{ y \in X \mid M_{[0,n)}(x) = M_{[0,n)}(y) \} \supseteq M_{[0,n-1)}(x)X_{i(F^{n-1}x)}. \]

We show the inclusion for every \( x \in X_0 \) by induction on \( n \).

Let \( y \in M_{[0,n-1)}(x)X_{i(F^{n-1}x)} \). If \( n = 1 \), we have \( y \in X_{i(x)} \), so \( M_0(x) = M_0(y) \).

Otherwise, by hypothesis, we have \( X_{i(F(x))} \subseteq F(X_{i(x)}) \), so
\[ M_0(x)X_{i(F(x))} \subseteq X_{i(x)}. \]

If we iterate this, we see that we have \( M_{[0,n-1)}(x)X_{i(F^{n-1}x)} \subseteq X_{i(x)} \). So, \( M_0(x) = M_0(y) \)
and we have \( F(y) \in M_{[0,n-2]}(F(x))X_{i(F^{n-1}x)} \). By induction hypothesis with \( x \) replaced with \( F(x) \), we get that \( M_{[0,n)}(x) = M_{[0,n)}(y) \).

\[ \square \]

The hypotheses of this proposition are true for the usual continued fraction algorithms of Brun, and Cassaigne. See Sections 7 and 8.

**Lemma 53.** Let \((X, s_0)\) be an extended continued fraction algorithm. We have
\[ \forall x \in X_0, \forall k \in \mathbb{N}, \exists r > 0, d(x, y) \leq r \Rightarrow \forall i \leq k, s_i(x) = s_i(y). \]

**Proof.** This is an obvious consequence of the definition of the set \( X_0 \), where \( s_n \) is continuous for every \( n \in \mathbb{N} \).

\[ \square \]

**Remark 54.** We have \( x = M_{[0,k]}F^k(x) \), so that \( v(x) \in \bigcap_{k \geq 0} M_{[0,k]}(x)\mathbb{R}^{d+1} \). If this cone is a line it follows that \( v(x) = v^{(0)} \), and more generally that \( v(F^k(x)) = v^{(k)} \).

### 5.2 Lyapunov exponents

Consider a dynamical system \((X, T)\) with a \( T \)-invariant Borel probability measure \( \mu \) on \( X \). A **cocycle** of the dynamical system \((X, T)\) is a map \( M : X \times \mathbb{N} \to GL_{d+1}(\mathbb{R}) \) such that
- \( M(x, 0) = \text{Id} \) for all \( x \in X \),
- \( M(x, n + m) = M(T^n(x), m)M(x, n) \) for all \( x \in X \) and \( n, m \in \mathbb{N} \).

We denote \( M(x, -n) = M(x, n)^{-1} \) for \( n > 0 \). Let \( \| \cdot \| \) be any norm on \( \mathbb{R}^{d+1} \).

**Theorem 55** (Oseledets). Let \((X, T)\) be a dynamical system and \( \mu \) be an invariant probability measure for this system. Let \( M \) be a cocycle of \((X, T)\) in \( GL_{d+1}(\mathbb{R}) \) such that the maps \( x \mapsto \ln \| M(x, 1) \|, x \mapsto \ln \| M(x, -1) \| \) are \( L^1 \)-integrable with respect to \( \mu \).

Then there exists a measurable set \( Z \subseteq X \) with \( \mu(Z) = 1 \) and measurable functions \( r, \theta_i \) from \( Z \) to \( \mathbb{R} \), such that for all \( x \in Z \) there is
- an integer \( r(x) \) with \( 0 < r(x) \leq d + 1 \)
- \( r(x) \) distinct numbers \( \theta_1(x) > \cdots > \theta_{r(x)}(x) \)
• a sequence of linear subspaces
\[ \mathbb{R}^{d+1} = E_1(x) \supseteq \cdots \supseteq E_r(x) \supseteq E_{r(x)+1}(x) = \{0\} \]
such that
\[ y \in E_i(x) \setminus E_{i+1}(x) \iff \lim_{n \to \infty} \frac{1}{n} \ln \|M(x,n)y\| = \theta_i(x) \]

If in addition \( \mu \) is an ergodic measure, then \( Z \) can be chosen so that the functions that map \( x \) to \( r(x), \theta_1(x), \ldots, \theta_r(x), \dim E_1(x) \ldots, \dim E_r(x) \) are constant on \( Z \). Then we denote \( \theta_i(x) \) by \( \theta_i(\mu) = \theta_i(T, \mu) \).

The numbers \( \theta_i(T, \mu), i = 1 \ldots m \) are called Lyapunov exponents of the cocycle \([31, 24]\). We also use the following formulas, see \([10, \text{Theorem 6.3}]\). Remark that in order to avoid confusion we denote the Lyapunov exponents of \( x \) by \( \theta_i(x) \) and by \( \theta_i(\mu) \) their value for almost all points with respect to the ergodic measure \( \mu \).

**Corollary 56.** In the ergodic case we have, for every \( x \in Z \),
\[ \theta_1(\mu) = \lim_{n \to \infty} \frac{1}{n} \ln \|M(x,n)\|, \quad \theta_2(\mu) = \lim_{n \to \infty} \frac{1}{n} \ln \|M(x,n)E_2(x)\| . \]

### 5.3 Lyapunov exponents for a continued fraction algorithm

In the following, since we use transpose of matrix, we consider the dual space of \( \mathbb{R}^{d+1} \). For a vector \( v \in \mathbb{R}^{d+1} \) we denote \( v^\circ \) the orthogonal in the dual space, i.e., the set of linear forms which vanish on \( v \).

**Remark 57.** For every \( x \in X \), we have the equality
\[ v(x)^\circ = \{ \varphi \circ \pi_x \mid \varphi \in (\mathbb{R}^{d+1})^* \}. \]

Indeed, we have \( \pi_x(v(x)) = 0 \) and \( \text{Im}(\pi_x) = P \) is a hyperplane.

**Lemma 58.** Let \((X, F)\) be a continued fraction algorithm.

- \( \forall y \in X, \forall n \in \mathbb{N}, \forall N \geq n, M_{(0,N)}(y) = M_{(0,n)}(y)M_{(0,N-n)}(F^n y) . \)
- \( \forall y \in X, \forall n \in \mathbb{N}, \forall N \geq n, \pi_y M_{(0,N)}(y) = \pi_y M_{(0,n)}(y)\pi F^n(y)M_{(0,N-n)}(F^n y) . \)
- The map
\[ \begin{pmatrix} X \times \mathbb{N} \\ (x, n) \end{pmatrix} \to \text{GL}_{d+1}(\mathbb{R}) \quad M(x, n) = M_{(0,n)}(x) \]
defines a cocycle.

**Proof.** For the first point, consider \( i \geq n \), then \( F^i(y) = F^{i-n}(F^n y) \), thus \( M_i(y) = M_{i-n}(F^n(y)) \). Then the third point is a consequence of the definition of a cocycle. It remains to prove the second point. It is a consequence of the first point and of the identity
\[ \pi_x M \pi_{M^{-1}x} = \pi_x M, \]
for every \( x \in \mathbb{P}^d \cap M \mathbb{P}^d \) and every matrix \( M \in \text{GL}_{d+1}(\mathbb{R}) \). This identity comes from the fact that \( M \pi_{M^{-1}x}(y) = M y - h(y)M v(M^{-1}x) \), and \( \pi_x(M v(M^{-1}x)) = 0 \).
Let now \((X, F, \mu)\) be a measured continued fraction algorithm, as defined in Section 5.

We use Theorem 55 and Corollary 56 for the cocycle defined in Lemma 58. Remark that the hypothesis of integrability is automatically satisfied since this cocycle takes only a finite numbers of values. In the following we consider the set \(Z\) given by Theorem 55 for this cocycle \((x, n) \mapsto M(x, n)\). In particular we have the following corollary.

**Corollary 59.** For every \(x \in Z\) and \(\varphi \in (\mathbb{R}^{d+1})^*\) we have

\[
\lim_{n \to \infty} \frac{1}{n} \ln \|\varphi M(x, n)\|_1 = \theta_i(x) \iff \varphi \in E_i(x) \setminus E_{i+1}(x),
\]

and we have

\[
\theta_1(x) = \theta_1(\mu) = \lim_{n \to \infty} \frac{1}{n} \ln \|M(x, n)\|_\infty,
\]

\[
\theta_2(x) = \theta_2(\mu) = \lim_{n \to \infty} \frac{1}{n} \ln \|M(x, n)\|_{E_2(x)}\|_\infty.
\]

**Definition 60.** A measured continued fraction algorithm \((X, F, \mu)\) is said to satisfy Pisot condition if for \(\mu\)-almost every point \(x\) we have \(\theta_1(x) > 0 > \theta_2(x)\), and \(\text{codim}(E_2(x)) = 1\).

**Lemma 61.** Let \(x \in Z\). Assume \(\text{codim}(E_2(x)) = 1\) and \(\bigcap_{n \geq 0} M(x, n)\mathbb{R}^{d+1} = \mathbb{R}^d v\). Then we have

\[
E_2(x) = v^o.
\]

**Proof.** Let \(y \in E_2(x)\) (a linear form, or a line vector), and let \(w = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^{d+1}\).

By Hölder inequality we have

\[
\|y M(x, n) w\| \leq \|y M(x, n)\| \|w\|_1.
\]

Let us denote \(w_n = \frac{M(x, n) w}{\|M(x, n) w\|_1}\), we obtain

\[
\frac{1}{n} \ln |yw_n| \leq \frac{1}{n} \ln \|y M(x, n)\|_\infty + \frac{1}{n} \ln \|w\|_1 - \frac{1}{n} \ln \|M(x, n) w\|_1.
\]

But we have

\[
\|M(x, n) w\|_1 = \sum_{i=0}^d |e_i^* M(x, n) w|.
\]

And for \(e_i^* \notin E_2(x)\) (it exists since \(d \geq 1\), we have \(\lim_{n \to \infty} \frac{1}{n} \ln \|e_i^* M(x, n)\|_\infty = \theta_1(x)\).

Moreover, we have \(\|e_i^* M(x, n)\|_\infty \leq |e_i^* M(x, n) w|\) so we have

\[
\liminf_{n \to \infty} \frac{1}{n} \ln \|M(x, n) w\|_1 \geq \lim_{n \to \infty} \frac{1}{n} \ln \|e_i^* M(x, n)\|_\infty = \theta_1(x).
\]

And we have \(\lim_{n \to \infty} \frac{1}{n} \ln \|y M(x, n)\|_\infty = \theta_i(x), i \geq 2\) with \(y \in E_i(x) \setminus E_{i+1}(x)\), so we get

\[
\limsup_{n \to \infty} \frac{1}{n} \ln |yw_n| \leq \theta_i(x) - \theta_1(x) < 0.
\]

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And we also have $w_n \xrightarrow{n \to \infty} v$ by hypothesis. We deduce that we have
\[ yv = \lim_{n \to \infty} yw_n = 0, \]
so $y \in v^\circ$. Since $\dim(E_2(x)) = \dim(v^\circ)$ we obtain that $E_2(x) = v^\circ$. \hfill \Box

**Lemma 62.** Let $x \in Z$. We assume that
\[ \bigcap_{n \geq 0} M_{[0,n)}(\mathbb{R}_+^{d+1}) = \mathbb{R}_+ v, \]
and that $\text{codim}(E_2(x)) = 1$. Then, we have
\[ \left\| \pi_v M_{[0,n)} \right\|_1 \leq (d+1) \left\| M_{[0,n)|E_2(x) \right\|_\infty. \]

**Proof.** Recall that $\pi_v$ is the projection on $P$ with respect to the direction $\mathbb{R}v$.
\[
\left\| \pi_v M_{[0,n)} \right\|_1 = \sup_{w, ||w||_1 \leq 1} \left\| \pi_v M_{[0,n)} w \right\|_1
\]
\[= \sup_{w, ||w||_1 \leq 1} \left| \sum_{i=0}^{d} e_i^* \pi_v M_{[0,n)} w \right|
\]
\[\leq \sum_{i=0}^{d} \sup_{w, ||w||_1 \leq 1} \left| e_i^* \pi_v M_{[0,n)} w \right| = \sum_{i=0}^{d} \left\| e_i^* \pi_v M_{[0,n)} \right\|_\infty.
\]

On the other hand, we have by Lemma 61 and Remark 57
\[ E_2(x) = \{ \varphi \circ \pi_v \mid \varphi \in (\mathbb{R}^{d+1})^* \} \subseteq (\mathbb{R}^{d+1})^*. \]

We conclude
\[ \left\| \pi_v M_{[0,n)} \right\|_1 \leq \left\| M_{[0,n)|E_2(x) \right\|_\infty \sum_{i=0}^{d} ||e_i^* \circ \pi_v||_\infty. \]

Since $||v||_1 = 1$, we have by definition $\pi_v(e_j) = e_j - v$, thus we deduce $e_i^* \circ \pi_v(e_j) = \delta_{i,j} - v_i$, and $||e_i^* \circ \pi_v||_\infty = \max(v_i, 1 - v_i)$. We deduce
\[ \left\| \pi_v M_{[0,n)} \right\|_1 \leq (d+1) \left\| M_{[0,n)|E_2(x) \right\|_\infty. \]

\hfill \Box

From Lemma 62 and Corollary 59 we deduce the following

**Corollary 63.** Let $x \in Z$. Assume that $\bigcap_{n \geq 0} M_{[0,n)} \mathbb{R}^d_+ = \{x\}$ and that $\text{codim}(E_2(x)) = 1$. Then we have the equality
\[ \lim_{n \to \infty} \frac{1}{n} \ln \left\| \pi_x M_{[0,n)} \right\|_1 = \theta_2(x). \]
Proof. By Lemma 61 and Remark 57, we have $E_2(x) = \{ \varphi \circ \pi_x \mid \varphi \in (\mathbb{R}^{d+1})^* \}$. Let $\varphi \in (\mathbb{R}^{d+1})^* \setminus \{0\}$ be a non-zero linear form such that $\varphi \circ \pi_x \in E_2(x) \setminus E_3(x)$. Then, using Lemma 62, we have the inequalities
\[
\| \varphi \circ \pi_x M_{[0,n)} \|_1 \leq \left\| \pi_x M_{[0,n)} \right\|_1 \| \varphi \|_1 \leq (d + 1) \left\| M_{[0,n)}^{t} | E_2(x) \right\|_\infty \| \varphi \|_1.
\]
We have $\lim_{n \to \infty} \frac{1}{n} \ln \| \varphi \circ \pi_x M_{[0,n)} \|_1 = \theta_2(x)$ since $\varphi \circ \pi_x \in E_2(x) \setminus E_3(x)$, and we have
\[
\lim_{n \to \infty} \frac{1}{n} \ln \left\| M_{[0,n)}^{t} | E_2(x) \right\|_\infty = \theta_2(x)
\]
by Corollary 59. Thus by squeeze theorem we get that the limit $\lim_{n \to \infty} \frac{1}{n} \ln \| \pi_x M_{[0,n)} \|_1$ exists and is equal to $\theta_2(x)$.

6 A lot of good points

The aim of this section is to prove that one seed point gives a set of full $\mu$-measure of good directive sequences (see Proposition 66). With this result, and with Theorem C, the proof of Theorem B will be easy.

6.1 Definitions and main result

Definition 64. We define the set of seed points $G_0$ as the set of points $x \in X$ such that
\begin{itemize}
  \item $x$ is a totally irrational direction,
  \item for every $n \in \mathbb{N}$, $F^n$ is continuous at $x$,
  \item $\limsup_{n \to \infty} \frac{1}{n} \ln \left\| \pi_x M_{[0,n)}(x) \right\|_1 < 0$,
  \item there exists a letter $a \in A$ and a fixed point $u \in (A^N)^N$ of $s(x)$ such that $W_a(u_0)$ has non-empty interior for the topology $T(x)$ (see Definitions 3, 11, and 12).
\end{itemize}
Definition 65. We define the set of good points
\[
G = \{ x \in X \mid s(x) \text{ is a good directive sequence} \},
\]
where a good directive sequence is defined in Definition 38.

Proposition 66. Let $\mu$ be an $F$-invariant ergodic probability measure on $X$ satisfying the Pisot condition (see Definition 60). If $G_0 \neq \emptyset$, then $\mu(G) = 1$.

Remark 67. For a periodic point $x$ (i.e., such that $F^p(x) = x$ for some $p \geq 1$), all the conditions to be a seed point, except the last one, are easily tested:
\begin{itemize}
  \item we have $\limsup_{n \to \infty} \frac{1}{n} \ln \left\| \pi_x M_{[0,n)}(x) \right\|_1 < 0$ if, and only if, $M_{[0,p)}$ is Pisot.
  \item we have that $x$ is a totally irrational direction if the matrix $M_{[0,p)}$ has an irreducible characteristic polynomial,
\end{itemize}
• we have the continuity of $F^n$ for every $n \in \mathbb{N}$ if, and only, if we have it for $0 < n \leq p$,
• the last property is automatic for an irreducible Pisot unimodular substitution if the Pisot substitution conjecture holds (see Subsection 10.5 for more details).

In the proof of Proposition 66, we need a variant of the notion of seed point:

**Definition 68.** We define $G_1$ as the set of $x \in X$ such that

- $x \in X_0$,
- $\limsup_{n \to \infty} \frac{1}{n} \ln \|\pi_x M_{[0,n)}(x)\|_1 < 0$,
- there exists a fixed point $u$ of $s(x)$ such that, for all $a \in A$, $W_a(u)$ has non-empty interior for the topology $T(x)$.

The definitions of $G_1$ and $G_0$ differ only by their last properties where we ask that the interior is not empty for every $a \in A$ rather than for one.

In the following, we use some more notations.

**Definition 69.** Let us define

$$Z_C = \{ x \in X | \forall n \in \mathbb{N}, \|\pi_x M_{[0,n)}(x)\|_1 \leq Ce^{-\frac{l}{2}} \}.$$ 

Let $B$ be a ball of positive radius in $P$ and let $C > 0$. For every $a \in A$, we define

$$G_{B,C}^a = Z_C \cap \{ x \in X | \pi_x^{-1}(B) \cap \mathbb{H} \subseteq W_a(x) \}$$

and

$$G_{B,C} = \bigcup_{a \in A} G_{B,C}^a.$$ 

### 6.2 Proof of Proposition 66

In all this subsection, we assume that $\mu$ is an $F$-invariant ergodic probability measure on $X$ satisfying the Pisot condition (see Definition 60). The strategy is to prove

$$G_0 \neq \emptyset \implies \mu(G_0) > 0 \implies \mu(G_1) > 0 \implies \mu(G) = 1.$$ 

In the following each step corresponds to one of these implications.

#### 6.2.1 Step 1: $G_0 \neq \emptyset \implies \mu(G_0) > 0$

**Lemma 70.** Let $x \in X$. If we have $\limsup_{n \to \infty} \frac{1}{n} \ln \|\pi_x M_{[0,n)}(x)\|_1 < 0$, then there exists $C_x > 0$ such that $x \in Z_{C_x}$.

**Proof.** Let $l = \limsup_{n \to \infty} \frac{1}{n} \ln \|\pi_x M_{[0,n)}(x)\|_1$. There exists $n_x \in \mathbb{N}$ such that for all $n \geq n_x$, we have

$$\|\pi_x M_{[0,n)}(x)\|_1 \leq e^{nl/2}.$$ 

If we take $C_x = \max(\max_{n<n_x} \|\pi_x M_{[0,n)}(x)\|_1e^{-nl/2}, 1, -\frac{2}{l})$, we have $x \in Z_{C_x}$. 

Remark that using this lemma, we have the equality
\[ G_1 = X_0 \cap \bigcup_{C > 0} \bigcup_{(B_k)_k \in A} \bigcap_{a \in A} G_{B_a, C}^n. \]

**Lemma 71.** We have
\[ \lim_{C \to \infty} \mu(Z_C) = 1. \]

**Proof.** Let \( Y = \{ x \in X \mid \limsup_{n \to \infty} \frac{1}{n} \ln \| \pi_x M_{[0, n]} \|_1 < 0 \} \). We have \( \mu(Y) = 1 \) by Corollary 63 and because \( \theta_2(F, \mu) < 0 \). And by Lemma 70 we have \( Y \subseteq \bigcup_{C > 0} Z_C \). Since \( Z_C \) is increasing with \( C \), we get \( \lim_{C \to \infty} \mu(Z_C) \geq \mu(Y) = 1. \)

**Lemma 72.** We have
\[ \forall x \in G_0, \exists C \in \mathbb{R}_+, \forall r > 0, \mu(B(x, r) \cap Z_C \cap X_0) > 0. \]

**Proof.** Let \( x \in G_0 \). Let \( \epsilon > 0 \) such that
\[ \forall n \geq 1, \mu(F^n(\{ y \in X_0 \mid M_{[0, n]}(x) = M_{[0, n]}(y) \})) \geq 2\epsilon. \]

This is given by our hypotheses on the measured continued fraction algorithm (see Definition 51).

Now let
\[ O_K = \{ x \in X \mid 1 < K \min_i (v(x)_i) \}. \]

We have \( \mu(\bigcup_{K > 1} O_K) = 1 \), by definition of a measured continued fraction algorithm (see Definition 51). So there exists \( K > 1 \) such that \( \mu(O_K) > 1 - \epsilon \).

By Lemma 71, there exists \( C' \geq 1 \) such that \( \mu(Z_{C'}) > 1 - \epsilon \). We choose the constant \( C = (K + 1)C_x C' \), where \( C_x \geq 1 \) is such that \( x \in Z_{C_x} \). Let \( r > 0 \). Thank to Remark 31, we can take \( n \in \mathbb{N} \) large enough such that \( M_{[0, n]}(x) X \) is included in \( B(x, r) \). Then, we take
\[ Y = M_{[0, n]}(x) (Z_{C'} \cap O_K) \cap \{ y \in X_0 \mid M_{[0, n]}(x) = M_{[0, n]}(y) \}. \]

By the previous inequalities, we have \( \mu(F^n(Y)) = \mu(M_{[0, n]}^{-1} Y) > 0 \). Using that we have a measured continued fraction algorithm (see Definition 51), we get that \( \mu(Y) > 0 \).

We have \( Y \subseteq X_0 \cap B(x, r) \) by construction. Let us show that we have the inclusion \( Y \subseteq Z_C \). Let \( y \in Y \). For all \( N \geq n \), we have by Lemma 58
\[
\left\| \pi_y M_{[0, N]}(y) \right\|_1 \leq \left\| \pi_y M_{[0, n]}(y) \right\|_1 \left\| \pi_{F^n(y)} M_{[0, N-n]}(F^n y) \right\|_1 \leq \left\| \pi_y M_{[0, n]}(x) \right\|_1 \left\| \pi_{F^n(y)} M_{[0, N-n]}(F^n y) \right\|_1
\]

because we have \( M_{[0, n]}(x) = M_{[0, n]}(y) \) by construction of \( Y \). Now, let us show that we have
\[ \left\| \pi_y M_{[0, n]}(x) \right\|_1 \leq (K + 1)C_x e^{-n/C}. \]

Recall that \( h : \mathbb{R}^{d+1} \to \mathbb{R} \) is the sum, i.e., the linear form such that for every \( w \in \mathbb{R}_+^{d+1} \), \( h(w) = \| w \|_1 \). Now, for every \( z \in X \), we have for all \( w \in \mathbb{R}^{d+1} \), \( \pi_z w = w - h(w)v(z) \). Let \( M = M_{[0, n]}(x) \). We have \( x \in Z_{C_x} \) and \( C \geq C_x \), so we have for every \( w \in \mathbb{R}^{d+1} \),
\[
\left\| Mw - h(Mw)v(x) \right\|_1 = \left\| \pi_x Mw \right\|_1 \leq C_x e^{-n/C} \| w \|_1.
\]
Let $y'$ such that $M y' = y$. The previous inequality applied with $w = v(y')$ gives
\[
\| M v(y') - h(Mv(y'))v(x) \|_1 \leq C_x e^{-n/C}.
\]
By triangular inequality, and using that $v(y) = \frac{Mv(y')}{h(Mv(y'))}$, we have
\[
\| \pi_y M w \|_1 = \| M w - h(Mw)v(y) \|_1 \\
\leq \| M w - h(Mw)v(x) \|_1 + \left| \frac{h(Mw)}{h(Mv(y'))} \right| \| M v(y') - h(Mv(y'))v(x) \|_1,
\]
so we get
\[
\| \pi_y M w \|_1 \leq C_x e^{-n/C} \left( \| w \|_1 + \left| \frac{h(Mw)}{h(Mv(y'))} \right| \right).
\]
Now, we have $y' \in O_K$, so $h(Mv(y')) \geq \frac{1}{K} \max \{ \| C \|_1 \mid C \text{ column of } M \}$, and we have $|h(Mw)| \leq \max \{ \| C \|_1 \mid C \text{ column of } M \} \| w \|_1$. Thus, we have
\[
\| w \|_1 + \left| \frac{h(Mw)}{h(Mv(y'))} \right| \leq (K + 1) \| w \|_1.
\]
We deduce that $\| \pi_y M_{(0,n)}(x) \|_1 \leq (K + 1) C_x e^{-n/C}$. And by construction of $Y$ we have $F^n(y) \in Z_C$, and we have $C \geq C'$, so we have
\[
\| \pi_{F^n(y)} M_{(0,N-n)}(F^n y) \|_1 \leq C' e^{-(N-n)/C}.
\]
We deduce that
\[
\| \pi_y M_{(0,N)} \|_1 \leq C e^{-N/C}.
\]
Hence, we get that $Y \subseteq B(x, r) \cap Z_C \cap X_0$ with $\mu(Y) > 0$, so $\mu(B(x, r) \cap Z_C \cap X_0) > 0$. □

The next lemma says that if $x$ is in $G_0$, then there exists a set of positive measure of points close to $x$ where the Rauzy fractals are close to each other for the Hausdorff distance $\delta$ in $P$ defined by
\[
\delta(A, B) = \max \left( \sup_{x \in A} d(x, B), \sup_{x \in B} d(A, x) \right),
\]
for every subsets $A, B$ of $P$.

**Lemma 73.** For all $x \in G_0$ and for all $\epsilon > 0$, there exists $C > 0$ and $V \subseteq B(x, \epsilon) \cap Z_C \cap X_0$ such that $\mu(V) > 0$ and $\forall y \in V$, $\forall a \in A$, $\delta(R_a(x), R_a(y)) \leq \epsilon$.

**Proof.** Let $x \in G_0$. Let $C \in \mathbb{R}_+$ given by Lemma 72. Let $k \in \mathbb{N}$ big enough to have
\[
\frac{C e^{-k/C}}{1 - e^{-1/C}} \max_{t \in \Sigma} \| t \|_1 \leq \frac{\epsilon}{3},
\]
where $\Sigma \subseteq \mathbb{Z}^{d+1}$ is the finite Dumont-Thomas alphabet for our $S$-adic system, see Definition 19. Then, we choose $R^{(k)} > 0$ small enough such that for all $y \in B(x, R^{(k)})$, we have
\[
\forall (t_i)_{i \leq k} \in \Sigma^{k+1}, \| \pi_x \left( \sum_{i=0}^{k} M_{(0,i)}(x) t_i \right) - \pi_y \left( \sum_{i=0}^{k} M_{(0,i)}(x) t_i \right) \|_1 \leq \frac{\epsilon}{3},
\]
for every subsets $A, B$ of $P$. □
It is possible because we compare the images by \( \pi_x \) and by \( \pi_y \) of the same element 
\[ \sum_{i=0}^{k} M_{[0,i]}(x) t_i \] that lives in a finite set.

Then, we take \( r > 0 \) given by Lemma \[ \text{Lemma 53} \] such that \( \|x - y\|_1 \leq 2r \implies \forall i \leq k, s_i(x) = s_i(y) \), and we can assume that \( r \leq R^{(k)} \) and \( r \leq \epsilon \) up to take the minimum of the three values, and we let \( V = B(x, r) \cap Z_C \cap X_0 \). Let’s show that the set \( V \) satisfy what we want. We have \( \mu(V) > 0 \) by Lemma \[ \text{Lemma 72} \].

Let \( y \in V \). We have convergence of the series \( \sum_{n=0}^{\infty} \|\pi_x M_{[0,n]}(y)\|_1 \) since \( y \in Z_C \). Hence, we can use Corollary \[ \text{Corollary 32} \] and we get that
\[
R_a(y) = \{ \sum_{n=0}^{\infty} \pi_y(M_{[0,n]}(y))t_n \mid \ldots \xrightarrow{t_n,s_n(y)} \ldots \xrightarrow{t_0,s_0(y)} a \in A \},
\]
and we get the same description for \( R_a(x) \).

Let \( p \in R_a(x) \), and let \( \ldots \xrightarrow{t_n,s_n(x)} \ldots \xrightarrow{t_0,s_0(x)} a \) be a left-infinite path in the abelianized prefix automaton \( A \) such that
\[
p = \sum_{n=0}^{\infty} \pi_x(M_{[0,n]}(x))t_n.
\]
We have \( s_i(x) = s_i(y) \) for all \( i \leq k \), and the matrices of substitutions of \( S \) are invertible, so we can take a left-infinite path \( \ldots \xrightarrow{t_n,s_n(y)} \ldots \xrightarrow{t_0,s_0(y)} a \) in the automaton \( A \) such that \( t_i = t'_i \) for all \( i \leq k \). This defines a point \( p' \in R_a(y) \) by
\[
p' = \sum_{n=0}^{\infty} \pi_y(M_{[0,n]}(y)t'_n).
\]
We have the inequalities
\[
\|p - p'\|_1 \leq \left\| \pi_x \left( \sum_{i=0}^{k} M_{[0,i]}(x)t_i \right) - \pi_y \left( \sum_{i=0}^{k} M_{[0,i]}(y)t_i \right) \right\|_1
+ \left\| \sum_{i=k+1}^{\infty} \pi_x \left( M_{[0,i]}(x)t_i \right) \right\|_1
+ \left\| \sum_{i=k+1}^{\infty} \pi_y \left( M_{[0,i]}(y)t'_i \right) \right\|_1.
\]
Then using that \( x \in Z_C \) and \( y \in Z_C \), we have
\[
\|p - p'\|_1 \leq \frac{\epsilon}{3} + \sum_{j=k+1}^{\infty} Ce^{-j/C} \|t_j\|_1
+ \sum_{j=k+1}^{\infty} Ce^{-j/C} \|t'_j\|_1
\]
\[
\leq \frac{\epsilon}{3} + 2Ce^{-k/C} \max_{t \in \Sigma} \|t\|_1
\]
\[
\leq \epsilon.
\]
By reverting the role of \( x \) and \( y \), we also show that for any point \( p \in R_a(y) \), there exists a point \( p' \in R_a(x) \) such that \( \|p - p'\|_1 \leq \epsilon \), so we get the wanted inequality
\[
\delta(R_a(x), R_a(y)) \leq \epsilon.
\]
\[ \square \]
Lemma 74. If $G_0 \neq \emptyset$ then $\mu(G_0) > 0$.

Proof. Let $x \in G_0$. Let $a \in A$, and let $u$ be a fixed point of $s(x)$, such that there exists an open ball $B_a = B(c_a, r_a)$ of positive radius $r_a > 0$ such that $\mathbb{H} \cap \pi^{-1}_x(B_a) \subseteq W_a(u)$. Then, by Lemma 28, we have that for all $b \in A \setminus \{a\}$ and $t \in \Lambda \setminus \{0\}$

$$B_a \cap R_b(x) = \emptyset = B_a \cap (R(x) + t).$$

We take the $C' > 0$ and $V \subseteq B(x, \epsilon) \cap Z_{C'} \cap X_0$ given by Lemma 73 for $\epsilon = r_a/2$. Let $B'_a = B(c_a, r_a/2)$ the open ball with half the radius of $B_a$ and same center. Let us show that for all $b \in A \setminus \{a\}$ and $t \in \Lambda \setminus \{0\}$ we have

$$\forall y \in V, B'_a \cap R_b(y) = \emptyset = B'_a \cap (R(y) + t).$$

If $b \in A \setminus \{a\}$, then we have for all $y \in V$ and for all $p \in R_b(y)$,

$$d(c_a, p) \geq d(c_a, R_b(x)) - d(R_b(x), p) \geq r_a - \delta(R_b(x), R_b(y)) \geq r_a - r_a/2 = r_a/2,$$

so $B'_a \cap R_b(y) = \emptyset$.

If $t \in \Lambda \setminus \{0\}$, then we have for all $y \in V$ and all $p \in R(y) + t$,

$$d(c_a, p) \geq d(c_a, R(x) + t) - d(R(x) + t, p) \geq r_a - \delta(R(x) + t, R(y) + t) \geq r_a/2,$$

so $B'_a \cap (R(y) + t) = \emptyset$.

By Lemma 28 we deduce that for every $y \in V$, we have the inclusion $\pi^{-1}_y(B'_a) \cap \mathbb{H} \subseteq W_a(y)$. We get that $V \subseteq G_{B'_a, C'}$, so the set $G_0$ has positive measure. \hfill \square

6.2.2 Step 2: $\mu(G_0) > 0 \implies \mu(G_1) > 0$

Lemma 75. We have $\mu(G_0) > 0 \implies \mu(G_1) > 0$.

Proof. If $\mu(G_0) > 0$, then by Poincaré recurrence theorem, we have

$$\mu \left( \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} F^{-k}G_0 \right) > 0.$$

Let $x \in \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} F^{-k}G_0$. Using $\limsup_{n \to +\infty} \frac{1}{n} \ln \left\| \pi_x M_{[0,n)}(x) \right\|_1 < 0$ and Remark 31 we deduce from Lemma 30 that there exists $n_0 \in \mathbb{N}$ such that $M_{[0,n_0)}(x) > 0$. Let $n \geq n_0$ such that $F^n(x) \in G_0$. Let $u$ be a fixed point of $s(x)$, and $a \in A$ a letter, such that $W_a(u_n)$ has non-empty interior for the topology $T(F^n x)$. For every $b \in A$, we have the equality

$$W_b(u) = \bigcup_{c \in [a_1, a_2]} M_{[0,n)}(x)W_c(u_n) + \sum_{k=0}^{n-1} M_{[0,k]}(x)t_k.$$

And thanks to $M_{[0,n)}(x) > 0$, we know that for every $b$, the letter $c = a$ appears in this union. The interior of $W_a(u_n)$ is non-empty for the topology $T(F^n x)$, so by Lemma 17 we have the non-emptiness of the interior of $W_b(u)$ for the topology $T(x)$, for every $b \in A$. By Lemma 34 and Lemma 70, there exists $C > 0$ such that $x \in Z_C$. We get that $x \in G_1$. So $G_1$ contains the set $\bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} F^{-k}G_0$ which has positive measure. \hfill \square
### 6.2.3 Step 3: $\mu(G_1) > 0 \implies \mu(G) = 1$

**Lemma 76.** The set $G$ is measurable and $F$-invariant.

**Proof.** Since we assume that $\mu$ is a Borel measure, the fact that $G$ is a measurable set is an exercise left to the reader. Let us show that $G$ is $F$-invariant. We check that every point of Definition 38 is invariant.

1. By Lemma 34, the limit $\lim_{n \to \infty} \frac{1}{n} \ln \| \pi_x M_{[0,n]}(x) \|_1$ is $F$-invariant.
2. The total irrationality of the direction is a $F$-invariant property since $F$ acts by integer invertible matrices.
3. Let us denote $x' = F(x) = M(x)^{-1} x$, and let $v^{(k)}(x') = v^{(k+1)}$, $s' = s(x')$, and $u'$ be the word sequence obtained by shifting $u$ by one, i.e., $u'_k = u_{k+1}$. Then $u'$ is a fixed point of $s'$, and $u_0 = s_0(x)u_1 = s_0(x)u'_0$. We deduce that the condition $v^{-1}_u(B(y,r)) \cap \mathbb{H} \subseteq W_a(u_k)$ can be written as $v^{-1}_u(B(y,r)) \cap \mathbb{H} \subseteq W_a(u'_{k+1})$.

Thus it suffices to replace $(k_n)_{n \in \mathbb{N}}$ by $(k_n+1)_{n \in \mathbb{N}}$ or by $(k_n+1)_{n \in \mathbb{N}}$, and we deduce that this property is preserved by $F$ and $F^{-1}$.
4. The last property is clearly preserved, and the limit is the same.

We conclude that $F^{-1}(G) = G$.

**Lemma 77.** If $\mu(G_1) > 0$, then $\mu(G) = 1$.

**Proof.** Let $(B_a)_{a \in A}$ be a family of balls of positive radius and $C > 0$ such that

$$\mu(\bigcap_{a \in A} C_{B_a,C}^a) > 0.$$

Using Lemma 37, let $O \subseteq X$ be an open set containing all the non totally irrational directions such that $\mu(O) < \mu(\bigcap_{a \in A} C_{B_a,C}^a)$.

First of all we claim that

$$\bigcap_{n_0 \in \mathbb{N}} \bigcup_{n \geq n_0} F^{-n} \left( \bigcap_{a \in A} C_{B_a,C}^a \cap X_0 \setminus O \right) \subseteq G$$

Indeed if $m$ is inside $\bigcap_{n_0 \in \mathbb{N}} \bigcup_{n \geq n_0} F^{-n} \left( \bigcap_{a \in A} C_{B_a,C}^a \cap X_0 \setminus O \right)$, then there exists infinitely many $n$ such that $F^n(m)$ belongs to $\bigcap_{a \in A} C_{B_a,C}^a \cap X_0 \setminus O$. This gives the third property of Definition 38. The last property follows from the fact that the set $O$ is open.

Now we apply the Poincaré recurrence theorem: We have $\mu(\bigcap_{a \in A} C_{B_a,C}^a \cap X_0 \setminus O) > 0$, thus $\mu$-almost every point of $\bigcap_{a \in A} C_{B_a,C}^a \cap X_0 \setminus O$ comes back to $\bigcap_{a \in A} C_{B_a,C}^a \cap X_0 \setminus O$. We deduce $\mu(G) > 0$. By Lemma 76 $G$ is an $F$-invariant set, thus by ergodicity we have $\mu(G) = 1$.

This ends the proof of Proposition 66.

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6.3 Proof of Theorem \[\text{B}\]

By hypothesis we can apply Proposition \[\text{66}\]. We deduce that \(\mu(G) = 1\). Now we apply Theorem \[\text{C}\] for each point of \(G\), and it gives that the Rauzy fractal induces a generating partition of the translation by \(e_0 - v(x)\) on the torus \(P/\Lambda\). And its symbolic coding is a measurable conjugacy with the subshift associated to \(x\). If \(\psi: P/\Lambda \rightarrow \mathbb{T}^d\) is an isomorphism, then we get that the subshift is measurably conjugate to the translation by \(\psi(e_0 - v(x))\) on the torus \(\mathbb{T}^d\).

7 Examples of continued fraction algorithms

Here we list some classical examples of continued fraction algorithms and we check if the hypotheses of Theorem \[\text{B}\] are fulfilled.

7.1 Classical continued fraction algorithm

The algorithm is defined on the whole \(X = \mathbb{P} \mathbb{R}^2_+\). Let \(S = \{\tau_0, \tau_1\}\), where

\[
\tau_0 = \begin{cases} 
0 &\mapsto 0 \\
1 &\mapsto 01
\end{cases}, \quad \tau_1 = \begin{cases} 
0 &\mapsto 10 \\
1 &\mapsto 1
\end{cases}
\]

Remark that this example is constructed on the same set \(S\) as in Example \[\text{10}\] and that the abelianization of the substitutions are

\[
M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
\]

We define the extended continued fraction algorithm as:

\[
s_0 = \begin{pmatrix} X \rightarrow S \\ [(x_0, x_1)] \rightarrow \begin{cases} 
\tau_0 &\text{if } x_0 \geq x_1 \\
\tau_1 &\text{if } x_0 < x_1
\end{cases}
\end{pmatrix}
\]

The associated continued fraction algorithm is:

\[
F = \begin{pmatrix} X \rightarrow X \\ [(x_0, x_1)] \rightarrow \begin{cases} 
[(x_0 - x_1, x_1)] &\text{if } x_0 \geq x_1 \\
[(x_0, x_1 - x_0)] &\text{if } x_0 < x_1
\end{cases}
\end{pmatrix}
\]

This algorithm is known as the additive continued fraction algorithm in dimension one, see \[\text{3}\].

Remark that with the change of coordinates \(x = \frac{x_0}{x_1}\), we obtain the map

\[
\begin{pmatrix} (0, +\infty) \rightarrow (0, +\infty) \\ x \mapsto \begin{cases} 
x - 1 &\text{if } x \geq 1 \\
\frac{1}{1-x} &\text{if } x < 1
\end{cases}
\end{pmatrix}
\]
There exists an ergodic invariant measure for this algorithm which is absolutely continuous with respect to Lebesgue measure, it density can be expressed $\frac{1}{x}$ in this coordinate system, but this measure has infinite volume. So we cannot apply our Theorem B.

The usual acceleration of this algorithm restricted to $(0,1)$ is given by the map \[ (0,1) \rightarrow (0,1) \] \[ x \mapsto \{\frac{1}{x}\} \]. This map has an invariant ergodic probability measure which is absolutely continuous with respect to Lebesgue measure, with density $\frac{1}{\log 2 \frac{1}{1+x}}$, see [4]. But it cannot be described with a finite number of matrices, so we cannot either apply our Theorem B with this acceleration.

However, this additive algorithm is very well known, and for every totally irrational direction, fixed points of the directive sequence $s(x)$ are constituted of Sturmian words. See [32] for more details. It could be shown that for every totally irrational direction $x \in \mathbb{PR}^1$, the directive sequence $s(x)$ is good. Hence, we deduce by Theorem C that for such direction $x$ there exists a generating partition of the translation by $e_0 - v(x)$ on torus $P/\Lambda \simeq \mathbb{T}^1$ whose symbolic coding is a measurable conjugacy with the subshift $\Omega_{s(x)}$. And we easily check that the set of $e_0 - v(x)$, for $x$ a totally irrational direction, is the set of totally irrational vectors of $P/\Lambda$. On the other hand, the complexity of the subshift $\Omega_{s(x)}$ is $p(n) = n + 1$. Thus, we get, for every irrational translation of $\mathbb{T}^1$, a generating partition whose symbolic coding has complexity $n + 1$.

This result is already well known. We know that Sturmian words have complexity $n + 1$, and that there exists a partition of the torus with two intervals whose symbolic coding is measurably conjugated to the subshift. See [32] for more details.

7.2 Brun algorithm

Now we give an example of a continued fraction algorithm which does not have associated substitutions (i.e., not an extended continued fraction algorithm). Let $X$ be $\mathbb{PR}^3$. For $\zeta \in S_3$ (the permutation group on the set $\{0, 1, 2\}$) we define $X_\zeta = \{[(x_0, x_1, x_2)] \in X \mid x_{\zeta(0)} < x_{\zeta(1)} < x_{\zeta(2)}\}$. Then we define the six matrices

\[
B_{012} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad B_{021} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_{120} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

\[
B_{102} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad B_{201} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_{210} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Then we define $M : X \rightarrow GL_3(\mathbb{Z})$ by $Mx = B_\zeta$ if $x \in X_\zeta$. If $x$ is in $X$ and not in some $X_\zeta$, then we extend the definition arbitrarily. The following will not depend on these choices.

The Brun algorithm is then defined, as all algorithm of continued fraction, by

\[
F = \left( \begin{array}{ccc} X & \rightarrow & X \\ x & \mapsto & (Mx)^{-1}x \end{array} \right).
\]

In other words, the algorithm subtracts from the largest coordinate the largest of the remaining ones.
Lemma 78. [3] The following function is a density function of an invariant probability measure for $F$:

$$
\frac{1}{2x_{\zeta(1)}(1 - x_{\zeta(1)})(1 - x_{\zeta(0)} - x_{\zeta(1)})} \quad x \in X_{\zeta}.
$$

Remark 79. For the invariant probability measure $\mu$ given by this lemma, we have

- $\mu(X_0) = 1$,
- for every $Y \subseteq X$ such that $\mu(Y) = 0$, we have $\mu(F(Y)) = 0$,
- $\exists \epsilon > 0, \forall x \in X_0, \forall n \in \mathbb{N}, \mu(F^n(\{y \in X \mid M_{[0,n]}(y) = M_{[0,n]}(x)\})) > \epsilon$.
- This measure is ergodic [33, 27].

Because $\mu$ is absolutely continuous with respect to Lebesgue, and by Proposition 52.

In other words, $(X, F, \mu)$ is a measured continued fraction algorithm.

General conditions that permit to check the Pisot condition for the Brun algorithm with the measure $\mu$ are given in [6].

As said at the beginning it is not an extended continued fraction algorithm, but we can extend it. In [26], some choices have been made to associate a finite set of substitutions to this algorithm. Denoting $b_{\zeta}$ the substitution with matrix $B_{\zeta}$ such that $b_{\zeta}(a)$ starts with $a$ for every letter $a \in \{0, 1, 2\}$, we find that

$$
b_{210}b_{021}b_{102} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 2 & 0 \\
\end{pmatrix}^3.
$$

Figure 9: Rauzy fractal of the directive sequence $(b_{210}b_{021}b_{102})^\omega$

Using Proposition 86, we can check that the interior of $W_0(u)$ is non-empty for the topology $T(x_0)$, where $u$ is a fixed point of the substitution $b_{210}b_{021}b_{102}$ and $x_0 = [v_0]$ with $v_0 = \text{freq}(u)$. Hence, we can check that $x_0 \in X$ is a seed point. Therefore, we can apply Theorem 13 and we get that for $\mu$-almost every point $x$ of $X$, the $S$-adic subshift associated to $x$ is measurably conjugate to a translation on the torus $P/\Lambda$. 

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7.3 Arnoux-Rauzy algorithm

The Arnoux-Rauzy extended continued fraction algorithm is defined by

\[
s_0 = \begin{pmatrix} X \mapsto S \\ [(x_0, x_1, x_2)] \mapsto \begin{cases} \ar_0 & \text{if } x_0 > x_1 + x_2 \\ \ar_1 & \text{if } x_1 > x_0 + x_2 \\ \ar_2 & \text{if } x_2 > x_0 + x_1 \end{cases} \end{pmatrix}
\]

where \( S = \{ \ar_0, \ar_1, \ar_2 \} \) with

\[
\ar_0 = \begin{cases} 0 \mapsto 0 \\ 1 \mapsto 10 \\ 2 \mapsto 20 \end{cases}, \quad \ar_1 = \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 1 \\ 2 \mapsto 21 \end{cases}, \quad \ar_2 = \begin{cases} 0 \mapsto 02 \\ 1 \mapsto 12 \\ 2 \mapsto 2 \end{cases}
\]

The associated continued fraction algorithm is

\[
F = \begin{pmatrix} X \mapsto X \\ [(x_0, x_1, x_2)] \mapsto \begin{cases} [(x_0 - x_1 - x_2, x_1, x_2)] & \text{if } x_0 > x_1 + x_2 \\ [(x_0, x_1 - x_0 - x_2, x_2)] & \text{if } x_1 > x_0 + x_2 \\ [(x_0, x_1, x_2 - x_1 - x_0)] & \text{if } x_2 > x_0 + x_1 \end{cases} \end{pmatrix}
\]

In other words, the algorithm subtracts from the largest coordinate the sum of the other ones. Here again we extend this definition to the boundaries of the sets in any choice. In this case, the set \( X \) is defined a posteriori as the subset of points of \( \mathbb{P}\mathbb{R}_+^2 \) from which \( F^n \) is defined for all \( n \in \mathbb{N} \), it is known as the Rauzy gasket. It is depicted in Figure 10.

![Figure 10: Rauzy gasket](image)

For this set \( S \) of substitutions, the Dumont-Thomas alphabet is \( \Sigma = \{ 0, e_0, e_1, e_2 \} \), and the automaton \( \mathcal{A} \) is depicted in Figure 11.

This algorithm has been well studied, see [5, 8]. In [6], some sufficient conditions for a measured continued fraction algorithm to satisfy Pisot condition are given. One of these conditions is independent of the ergodic measure. It is called Pisot property. They prove that Pisot property is satisfied for this algorithm.
Figure 11: Abelianized prefix automaton $A$ for the Arnoux-Rauzy substitutions

It appears that this algorithm has a lot of ergodic measures. One of them has been introduced in [7]. And this measure is a good candidate, but we haven’t check that it fulfils all the hypotheses needed in Definition 51.

8 Application: Cassaigne algorithm and two-dimensional translations

First we define the Cassaigne extended measured continued fraction algorithm, and show that it fulfills the hypotheses of Theorem B. Then we will prove Theorem A.

8.1 Description of the algorithm

The algorithm is defined on the whole $X = \mathbb{PR}^2$. Let $\mu$ be the measure on $X$ with density $\frac{1}{(1-x_0)(1-x_2)}$ with respect to the Lebesgue measure on $X$. Let $S = \{c_0, c_1\}$, where

$$c_0 = \begin{cases} 
0 \mapsto 0 \\
1 \mapsto 02 \\
2 \mapsto 1
\end{cases}, \quad c_1 = \begin{cases} 
0 \mapsto 1 \\
1 \mapsto 02 \\
2 \mapsto 2
\end{cases}$$

We define the extended continued fraction algorithm as:

$$s_0 = \left( \begin{array}{c}
X \\
[(x_0, x_1, x_2)]
\end{array} \right) \mapsto \left( \begin{array}{c}
S \\
\{c_0 \text{ if } x_0 \geq x_2, c_1 \text{ if } x_0 < x_2\}
\end{array} \right)$$

The associated continued fraction algorithm is:

$$F = \left( \begin{array}{c}
X \\
[(x_0, x_1, x_2)]
\end{array} \right) \mapsto \left( \begin{array}{c}
X \\
[(x_0 - x_2, x_2, x_1)] \text{ if } x_0 \geq x_2, \\
[(x_1, x_0, x_2 - x_0)] \text{ if } x_0 < x_2
\end{array} \right)$$

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The matrices associated to the substitutions $c_0$ and $c_1$ are:

\[
M_0 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.
\]

For this set $S$ of substitutions, the Dumont-Thomas alphabet is $\Sigma = \{0, e_0\}$, and the automaton $\mathcal{A}$ is depicted in Figure 12.

![Abelianized prefix automaton $A$ for $S = \{c_0, c_1\}$](image)

**Lemma 80.** $(X, s_0, \mu)$ is an extended measured continued fraction algorithm and satisfies the Pisot condition.

**Proof.** We refer to [3] for a proof of the $F$-invariance of the measure $\mu$. By [35] we know that Selmer algorithm is ergodic. Moreover we know that Cassaigne algorithm is conjugated to Selmer algorithm [14], thus we deduce the ergodicity of this measure. It is well known that for the Selmer algorithm the second Lyapunov exponent is strictly negative, with $x \mapsto \text{codim}(E_2(x))$ $\mu$-almost surely constant to 1, see [30]. Thus by conjugation we deduce $\theta_2(F, \mu) < 0$ and $\text{codim}(E_2(x)) = 1$ for $\mu$-almost every $x \in X$. This algorithm fulfills the condition of Proposition 52 since $\mu$ is absolutely continuous with respect to Lebesgue. Hence $(X, s_0, \mu)$ is an extended measured continued fraction algorithm.

The Figure 13 illustrates approximations of Rauzy fractals $R(x)$ obtained by choosing points $x \in \mathbb{P}R^d$ randomly for the Lebesgue measure and applying the Cassaigne algorithm to compute the directive sequence up to a certain integer $n$. We plot the set of points

\[
\left\{ \sum_{k=0}^{n} \pi_x(M_{(0,k)} t_k) \mid b \overset{t_n, s_n}{\longrightarrow} \ldots \overset{t_0, s_0}{\longrightarrow} a \right\}
\]

with a color depending on the letter $a$.

If the Rauzy fractal $R_{n+1}(x)$ associated to the point $F^{n+1}(x)$ is bounded and not too large (which occurs with high probability), then the Hausdorff distance between the approximation and the Rauzy fractal is at most some pixels. Rauzy fractals of this article have been drawn using the Sage mathematical software and the badic package. These are available here: [www.sagemath.org](http://www.sagemath.org) and [https://gitlab.com/mercaptop/badic](https://gitlab.com/mercaptop/badic).
8.2 There exists a seed point

We consider the substitution \( c_0 c_1 = \begin{cases} 0 \mapsto 02 \\ 1 \mapsto 01 \\ 2 \mapsto 1 \end{cases} \). We denote by \( u = (c_0 c_1)^\omega(0) \in A^\mathbb{N} \) its unique fixed point. Its abelianization is the matrix 
\[
M = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 0
\end{pmatrix}.
\]

This matrix has for characteristic polynomial \( X^3 - 2X^2 + X - 1 \) is irreducible over \( \mathbb{Q} \), with one eigenvalue greater than 1 and two other complex eigenvalues of modulus less than 1. We take \( \beta \) the one with negative imaginary part.

Thus this substitution is Pisot unimodular. Let \( x_0 \in \mathbb{P} \mathbb{R}^d \) be the class of a Perron eigenvector of \( \text{ab}(c_0 c_1) \). The goal is to prove that \( x_0 \) is a seed point:

**Proposition 81.** The point \( x_0 \) is in \( G_0 \) (see Definition 64).

We need to prove several lemmas first.

**Lemma 82.** The point \( x_0 \) is a totally irrational direction. The map \( F^n \) is continuous at \( x_0 \) for every \( n \in \mathbb{N} \).

**Proof.** The characteristic polynomial \( X^3 - 2X^2 + X - 1 \) is irreducible over \( \mathbb{Q} \) and splits with simple roots over the splitting field. Thus the Galois group acts transitively on the eigenvectors. Hence, if \( x_0 \) was not a totally irrational direction, it would give a rational non-zero vector in the left kernel of the matrix of eigenvectors. And this is absurd because this matrix is invertible. We deduce that \( x_0 \) is a totally irrational direction, thus we have the continuity of \( F^n \) at \( x_0 \) for every \( n \in \mathbb{N} \). \( \square \)
Here, there is a natural projection on $\mathbb{C}$ for which $M$ acts by multiplication by $\beta$:

**Lemma 83.** Consider the linear map $\phi$ from $\mathbb{R}^3$ to $\mathbb{C}$ given by $\phi(v) = e \cdot v$, for the line vector $e = (1, \beta^2 - \beta, \beta - 1)$. This map induces a bijection between $P$ and $\mathbb{C}$. For every $v \in \mathbb{R}^3$, we have $\phi(Mv) = \beta \phi(v)$ and $\phi(\pi_{x_0}v) = \phi(v)$.

**Proof.** Remark that the line vector $e$ is a left-eigenvector of $M$ for the eigenvalue $\beta$. Let $v \in \mathbb{R}^3$. We have $\phi(v) = e \cdot v$, so we have $\phi(Mv) = eMv = \beta ev = \beta \phi(v)$. And $x_0$ is the class of a right eigenvector of $M$ for an eigenvalue different of $\beta$, so we have $ex_0 = 0$, thus we get $\phi(\pi_{x_0}v) = \phi(v - h(v)v(x_0)) = \phi(v)$. Now we check that the rank of $\phi$ is 2, so its kernel is the vector space spanned by $x_0$, which intersect $P$ only at 0. Thus $\phi$ induces a bijection between $P$ and $\mathbb{C}$. \hfill $\square$

With Lemmas 22 and 83, we can project the worm $W(u)$ on the complex plane

$$\phi(W_u(u)) = \left\{ \sum_{k=0}^{n-1} t_k \beta^k \mid 0 \xrightarrow{t_{n-1}} \cdots \xrightarrow{t_0} a \right\} \subset \mathbb{C},$$

where $0 \xrightarrow{t_{n-1}} \cdots \xrightarrow{t_0} a$ denotes a path in the automaton $\phi(A)$ of the Figure 14 (we do not label the edges by the substitution since there is only one in this case). $\phi(A)$ is the image by $\phi$ of the abelianized prefix automaton $A$ for the substitution $c_0c_1$.

The Figure 15 shows the Rauzy fractal $R$ of the directive sequence $(c_0c_1)^\omega$ and its image $\phi(R)$ by $\phi$.

![Figure 15: Rauzy fractal of $(c_0c_1)^\omega$ in $P$ (left) and its projection by $\phi$ on $\mathbb{C}$ (right)](image)

The following lemma permits to find good bounding boxes (for example good disks) that contain the parts of the Rauzy fractal.
Lemma 84. If there exists \((O_a)_{a \in A}\) open subsets of \(\mathbb{C}\) and an integer \(n \in \mathbb{N}\) such that for every \(a \in A\) we have
\[
\bigcup_{b \in \mathbb{C}} \left( \beta^n O_b + \sum_{k=0}^{n-1} t_k \beta^k \right) \subseteq O_a,
\]
then for all \(a \in A\), we have \(\phi(R_a) \subseteq O_a\).

Proof. Let \(y \in \phi(R_a)\). By Corollary 32, there exists an infinite path \(\ldots \to t_{n-1} \to t_n \to a\) in the automaton \(\phi(A)\) of Figure 14 such that
\[
y = \sum_{k=0}^{\infty} t_k \beta^k.
\]
Let \(b \in A\) such that \(b \to t_{n-1} \to t_n \to a\) is a path in the automaton. Let \(l > 0\) be the distance between \(\beta^n O_b + \sum_{k=0}^{n-1} t_k \beta^k\) and the complement of \(O_a\). We denote by \(D\) the usual distance on \(\mathbb{C}\). Let \(k \in \mathbb{N}\) be large enough such that \(|\beta^k n|_{\text{max}_{c \in A} \sup_{x \in \phi(R)} D(z, \overline{O_c})} < l\), and let \(c \in A\) such that \(c \to t_{kn-1} \to \ldots \to t_n \to b \to t_{n-1} \to \ldots \to t_0 \to a\) is a path in the automaton. We have
\[
\sum_{p=kn}^{\infty} t_p \beta^{p-kn} \in \phi(R). \quad \text{So we have } D(\sum_{p=kn}^{\infty} t_p \beta^p, \beta^{kn} \overline{O_c}) < l.
\]
Thus, we have \(D(y, \beta^{kn} \overline{O_c} + \sum_{p=0}^{kn-1} t_p \beta^p) < l\). And we have the inclusion
\[
\beta^{kn} \overline{O_c} + \sum_{p=0}^{kn-1} t_p \beta^p \subseteq \beta^n \overline{O_b} + \sum_{k=0}^{n-1} t_k \beta^k
\]
by iterating \(k - 1\) times the inclusion of the hypothesis. So, we get that \(y\) is in \(O_a\). \(\square\)

Corollary 85. For the Rauzy fractal associated to \((c_0c_1)\) we have the inclusions
\[
\phi(R_0) \subseteq B(-0.19 - 0.15i, 0.75) =: O_0,
\]
\[
\phi(R_1) \subseteq B(0.5 - 0.6i, 0.655) =: O_1,
\]
\[
\phi(R_2) \subseteq B(0.865 + 0.123i, 0.566) =: O_2.
\]

Proof. We use Lemma 84 for \(n = 8\), and check the result by computer, see Figure 16. \(\square\)

In the following we denote \(z_a, r_a\) the center and the radius of the ball \(O_a\) for \(a = 0, 1, 2\). Now we can prove Proposition 81.

Proof of Proposition 81. We check all the conditions that show that \(x_0\) is a seed point:

1. By Lemma 82, the direction \(x_0\) is totally irrational and for all \(n \in \mathbb{N}\), \(F^n\) is continuous at \(x_0\).
2. We have
\[
\lim_{n \to \infty} \frac{1}{n} \ln \left\| \pi_{\theta_0, n}(x_0) \right\|_1 = \lim_{n \to \infty} \frac{1}{2n} \ln \left\| \pi_{\theta_0, ab(c_0c_1)^n} \right\|_1 = \frac{1}{2} \ln |\beta| < 0.
\]
\[52\]
We have $0 \not\in R_1 \cup R_2$ by Corollary 85 since $0 \not\in B(0.5 - 0.6i, 0.655) \cup B(0.865 + 0.123i, 0.566)$.

For $t \in \Lambda \\setminus \{0, e_1 - e_2, e_2 - e_1\}$, we check that we have $|\phi(t)| > 1.5 > \max_{a \in A} r_a + |z_a|$, so by Corollary 85 we get that $0 \not\in R + t$. For $t \in \{e_1 - e_2, e_2 - e_1\}$, we check that we have for all $a \in A$, $|z_a + \phi(t)| > r_a$, thus we have $0 \not\in R + t$.

We have $0 \not\in R_1 \cup R_2 \cup \bigcup_{t \in \Lambda \setminus \{0\}} R + t$, so by Lemma 28 we have that $0$ is in the interior of $W_0(u)$. In particular, the interior of $W_0(u)$ is non-empty. Furthermore, there exists a fixed point $w \in (A^N)^N$ of the directive sequence $s(x_0) = (c_0c_1)^\omega$ such that $w_0$ is the fixed point $u$ of the substitution $c_0c_1$.

Remark that the previous proof could be adapted to prove the semi-decidability of being a seed point for $F$-periodic points. But we have even the decidability.

**Proposition 86.** If $F$ is computable, then being a seed point is a decidable property for $F$-periodic points.

**Proof.** Let $x$ be an $F$-periodic point of period $p$. Total irrationality of the direction $x$ is equivalent to irreducibility of the characteristic polynomial of $M_{[0,p)}(x)$, and this can be checked algorithmically. The fact that $x$ is not a discontinuity point of $F^n$ can be checked since $F$ is computable, and it is enough to test it for $n \leq p$. The hypothesis that $\limsup_{n \to \infty} \|\pi_x M_{[0,n)}(x)\|_1 < 0$ is equivalent to check that the matrix $M_{[0,p)}$ is Pisot, and this is decidable. Then, if $u$ is a fixed point of $s(x)$, then $u_0$ is a periodic point for the substitution $s_{[0,p)}$, see Lemma 5. Then, we use [1, Theorem 5.12]. This theorem allows to describe the interior of $W_a(u_0)$ with a finite automaton, such that the interior is empty if, and only if, the language of the automaton is empty. Moreover this automaton is computable from $s_{[0,p)}$. And checking if an automaton has an empty language is decidable.
Figure 17: Proof that $0 \notin R_1 \cup R_2 \cup \bigcup_{t \in \Lambda \setminus \{0\}} R + t$ thanks to covering with balls

The computation in \cite{1} is done for the bi-infinite topology, but it is possible to use it to compute the interior for the topology $T(x_0)$, by adding a left infinite part to our worm. 

8.3 Proof of Theorem \cite{A}

We refer to \cite{14, Proposition 6} for the proof of the following result:

**Lemma 87.** Consider a directive sequence $s$ in $S^\infty$, where $S = \{c_0, c_1\}$. Assume that $s$ cannot be written as a finite sequence followed by an infinite concatenation of $c_0^2$ and $c_1^2$. Then $\Omega_s$ is minimal and has complexity $2^n + 1$.

Now we deduce the proof of Theorem \cite{A} with Lemmas \cite{80} and \cite{81} we can apply Theorem \cite{B} since $\mu$ is absolutely continuous with respect to the Lebesgue measure.

The map $x \mapsto t_x$ of Theorem \cite{B} is the map $x \mapsto \psi(e_0 - v(x))$, where $\psi: P/\Lambda \to \mathbb{T}^2$ is an isomorphism. Now remark that $(e_0 - \Delta) \cup (\Delta - e_0)$ form a cover of a measurable fundamental domain of $P$ for the action of $\Lambda$. Thus the set $\{\psi(e_0 - v(x)) \mid x \in G\} \cup \{\psi(v(x) - e_0) \mid x \in G\}$ is of full measure in $\mathbb{T}^2$. Hence, we get for Lebesgue-almost every translation of $\mathbb{T}^2$ a nice generating partition whose symbolic coding is conjugate to the subshift. With Lemma \cite{87} we deduce the result.

9 Renormalization schemes

In general, the first return map of a minimal torus translation on a bounded remainder set is close to be a torus translation \cite{21}. In the present case, we will see how selecting the atoms on which to induce explicitly leads to another torus translation, and how the induction process relates to the continued fraction algorithm.

We will focus on Cassaigne algorithm, and then explain how to adapt the reasoning to other algorithms.

Let us first look at the symbolic level. Let us consider a directive sequence $s = (s_k)$ starting with $c_0$, and let $u$ be one of its fixed points.
The word $u_0 = c_0(u_1)$ is the concatenation of the three finite words 0, 02 and 1. Those three words are \textit{return words} on the pair $\{0, 1\}$, i.e. any word in $\Omega_u$ starting with 0 or 1 can be written in a unique way as a concatenation of 0, 02 and 1, and 0 and 1 appear only at the first positions of those words. Hence, inducing the subshift $\Omega_{u_0}$ on the clopen set $[0] \cup [1] = \Omega_{u_0} \setminus [2]$ leads to a subshift isomorphic to $\Omega_{u_1}$, whose directive sequence is $(s_{k+1})$.

Now, assume that $s = (s_k)$ starts with $c_1$, and again let $u$ be one of its fixed points. In this case, the images of the letters by $c_1$ are not return words, and we have to look backwards: the reverse of the images of the letters by $c_1$, that is 1, 20 and 2 are return words on the pair $\{1, 2\}$. An option could be to reverse $c_1(1)$ in the definition of $c_1$ to be 20 as it will not change the continued fraction algorithm, but it will increase the complexity of the associated subshift, which we can not afford. Instead, we remark that inducing on $T([1]) \cup T([2]) = \Omega_u \setminus T([0])$, where $T$ denotes the shift map, leads again to a subshift isomorphic to $\Omega_{u_1}$, whose directive sequence is again $(s_{k+1})$.

All those remarks translate to the geometrical level, and we get the following renormalization scheme. To simplify the notations, we identify the Rauzy fractals $R_i(x) \subseteq P$ with their image by the projection $q: P \to P/\Lambda$. Let $T_x$ be a translation of the torus, and let $(R_0(x), R_1(x), R_2(x))$ be the associated partition by Rauzy fractals.

- \textit{(bottom type)} if $\lambda(R_0(x)) \leq \lambda(R_2(x))$, let $U = (P/\Lambda) \setminus R_2(x)$
- \textit{(top type)} if $\lambda(R_0(x)) > \lambda(R_2(x))$, let $U = (P/\Lambda) \setminus T_x(R_0(x))$

Then, the induced application $(T_x)_U$ is isomorphic to the translation $T_{F(x)}$, but it is defined on a smaller torus $P/\Lambda'$, with $U$ being a measurable fundamental domain of $P$ for the action of a lattice $\Lambda'$. The induced application $(T_x)_U$ can be renormalized to the translation $T_{F(x)}$ on the reference torus $P/\Lambda$: the linear map that sends $U$ to $R(F(x))$ is $N^{-1}$, where $N$ is the linear endomorphism of $P$ such that $N \circ \pi_{F(x)} = \pi_x \circ ab(s_0(x))$ that was introduced in subsection \ref{sect:renormalization}. And we have the relation $\Lambda' = N\Lambda$.

It is remarkable to see that this scheme is pretty similar to the famous Rauzy-Veech induction for interval exchange maps \cite{Rauzy} (we named the \textit{top} and \textit{bottom} types after the naming scheme of \cite{Rauzy}).

Figure \ref{fig:induction-renormalization} shows two steps of induction and renormalization starting from $v(x) = (0.256005715380561..., 0.286881483823029..., 0.457112800796410...)$... The corresponding directive sequence is $s(x) = c_1c_0c_1c_0c_1c_0c_0c_1c_0c_0c_1c_0c_0c_0c_1c_0c_1c_0c_0c_1c_1c_0c_0c_1c_1...$ It corresponds to the translation by

\[ t = (0.743994284619439..., -0.286881483823029..., -0.457112800796410...) \]

on the torus $P/\Lambda$. The figure shows the Rauzy fractals $R(x), R(F(x))$ and $R(F^2x)$, with $R_0$ in red, $R_1$ in green and $R_2$ in blue. The first line shows the decomposition $R = R_0 \cup R_1 \cup R_2$, and the second line shows the decomposition $R = (R_0 + \pi(c_0)) \cup (R_1 + \pi(c_1)) \cup (R_2 + \pi(c_2))$ after applying the domain exchange corresponding to the translation on the torus.

The Brun and Arnoux-Rauzy extended continued fraction algorithms also enjoy a similar renormalization scheme:

- For the Brun algorithm (with the substitutions as in \cite{Brun}), it suffice to induce on the complementary of the image of the second largest atom. Another choice of substitutions will lead to a different renormalization scheme.
Figure 18: Two steps of induction/renormalization. The sets of induction are outlined in black. Note that the pictures look flipped (and stretched) from one step to the next one, this is due to the fact that the renormalization matrices have negative determinant.

- For the Arnoux-Rauzy algorithm, it suffice to induce on the image of the largest atom.

In all cases, one renormalization step corresponds to applying one step of the continued fraction algorithm.

10 Remarks and open problems

10.1 Comments on the results of another paper

In [11] the authors prove two theorems on the same subject. Their Theorem 3.1 is in the same spirit as our Theorem C. In their case, they need some hypotheses such as irreducibility and balanceness for the directive sequences or coincidence conditions on the subshift. In our theorem these conditions are not assumed, and are replaced by our notion of good directive sequence. Theorem 3.3 of [11] is in the same spirit as our Theorem B. Here again, the hypotheses are not on the same objects.

10.2 Translation vectors vs directions

The link between a continued fraction algorithm and torus translations was done by associating to every direction $x \in \mathbb{PR}^d$, a translation vector of $\mathbb{T}^d$ via the composition:

$$\chi: \mathbb{PR}^d \xrightarrow{a} \Delta \xrightarrow{f} P \xrightarrow{q} P/\Lambda \xrightarrow{\psi} \mathbb{T}^d.$$
The map $f = \left( \Delta \quad \rightarrow \quad P \quad y \quad \mapsto \quad \pi_y(e_0) \right)$ is affine (and injective), which is why we could transport results holding for almost every direction to results holding for almost every torus translation. Note however that the map $\chi$ is not surjective, so that we had to project the simplex twice to cover all possible torus translations in dimension 2 in the proof of Theorem A (Section 8.3). If $t$ is an element of $T^2$, either $t$ or $-t$ is the image of some direction $x \in \mathbb{P}R^2_+$. Since the translation $T_t$ is conjugated to the translation $T_{-t}$, we got the result for almost every translation of $T^2$.

In higher dimensions, if we identify $T^d$ with the unit hypercube $[0, 1]^d$, the image of $\mathbb{P}R^d_+$ is the convex hull $S_d$ of $\{0, e_1, \ldots, e_d\}$, whose Lebesgue measure is only $1/d!$.

If $\alpha \in GL_d(\mathbb{Z})$ is an automorphism of $T^d$, $T_t$ is conjugated to $T_\alpha(t)$. As shown in [23], there exists an explicit finite family $\{\alpha_i\}_{0 \leq i < d!}$ of elements of $GL_d(\mathbb{Z})$ and a family $\{n_i\}_{0 \leq i < d!}$ of elements of $\mathbb{Z}^d$ such that $[0, 1]^d = \bigcup_{0 \leq i < d!} \alpha_i(S_d) + n_i$, that is,

$$T^d = \bigcup_{0 \leq i < d!} \alpha_i(\mathbb{P}R^d_+).$$

Such tiling is also known as Kuhn triangulation [28].

Therefore, if we want to go from a particular translation $T_t$ of $T^d$ to a projective direction and study its dynamics through continued fractions, it suffices to find to which atom $\alpha_i(S_d)$ of the triangulation it belongs, and to associate the direction $x = \chi^{-1}(\alpha_i^{-1}(t))$ (note that $\chi$ is injective, except on the finite set $\{[e_0], \ldots, [e_d]\}$).

### 10.3 Exceptional directions in Cassaigne algorithm

A natural question is to understand the set of directions where the conclusion of Theorem A is true. Our proof shows that it works for a subset of measure one in the set of totally irrational directions. Can we extend the result of Theorem A to all this set? It is not possible with our proof, but maybe we can use some other continued-fraction algorithm, or some unrelated method. Indeed, there are subshifts defined by the Cassaigne algorithm which are not balanced [2], so there are directions where we can not use this algorithm to construct symbolic codings of translations of $T^2$. More generally, one may ask whether some subshifts defined by the Cassaigne algorithm are weakly mixing, see [13].

To finish with Cassaigne algorithm we list some properties of exceptional directions. Let $x = [(x_0, x_1, x_2)] \in \mathbb{P}R^3_+$ be a direction and $s(x) = (s_k)$ be its associated directive sequence. We have equivalence between $\text{dim}_Q \text{span}(x_0, x_1, x_2) = 1$ and the fact that the sequence $(s_k)$ is ultimately constant. Moreover $\text{dim}_Q \text{span}(x_0, x_1, x_2) = 1$ if, and only if, $s(x)$ is not everywhere growing. And finally the property $\text{dim}_Q \text{span}(x_0, x_1, x_2) \leq 2$ is equivalent to the fact that $(s_k)$ can be written as the concatenation of a finite sequence followed by an infinite concatenation of $c_0^2$ and $c_1^2$ (even runs) [13, Lemma 1].
10.4 Higher dimensions

Another natural question is to generalize Theorem A for \( d \geq 3 \). For example we could be interested in the following set \( S \):

\[
\begin{align*}
  a &\mapsto a \\
  b &\mapsto c \\
  c &\mapsto d \\
  d &\mapsto ab
\end{align*}
\]

It seems that the complexity of the subshift is linear, but it is bigger than \( 3n + 1 \). Moreover we do not know actually if the other hypotheses are fulfilled.

10.5 Pisot substitution conjecture and converse of Theorem C for constant directive sequences

We say that a substitution is \textit{irreducible} if its matrix has an irreducible characteristic polynomial.

The \textit{Pisot substitution conjecture} states (or is equivalent to the fact) that the conclusion of our Theorem C is true for every directive sequence of the form \( \sigma^\omega \), with \( \sigma \) an irreducible Pisot unimodular substitution: the subshift is measurably conjugated to a translation on a torus.

But for such particular directive sequences, this is equivalent to being good.

\textbf{Lemma 88.} Let \( \sigma \) be an irreducible Pisot unimodular substitution such that for a periodic point \( u \in A^\mathbb{N} \), there exists a letter \( a \in A \) such that \( W_a(u) \) is not empty for the topology \( \mathcal{T}(x) \), where \( x \in \mathbb{P}^d \) is the class of a Perron eigenvector of \( ab(\sigma) \). Then, the directive sequence \( \sigma^\omega \) is good.

\textit{Proof.} We check that \( s = \sigma^\omega \) satisfies the four points of Definition 38:

1. We have \( \lim_{n \to \infty} \frac{1}{n} \| \pi_x M^{(0,n)} \|_1 = \lim_{n \to \infty} \frac{1}{n} \| \pi_x M^n \|_1 = \ln(|\beta|) \), where \( \beta \) is the second biggest eigenvalue of \( M = ab(\sigma) \) in absolute value. We have \( \ln(|\beta|) < 0 \) since \( \sigma \) is Pisot.

2. The direction \( x \) is totally irrational since \( \sigma \) is irreducible.

3. We have for all \( n \in \mathbb{N} \), \( x^{(n)} = x \), so \( \lim_{n \to \infty} x^{(n)} = x \) exists and is a totally irrational direction, so we have the fourth point.

4. Let us check the third point. Let \( u_0 \) be a periodic point of \( \sigma \) of period \( p \), such that there exists \( a \in A \) such that \( W_a(u_0) \) has non-empty interior for \( \mathcal{T}(x) \), and let \( u \in (A^\mathbb{N})^\mathbb{N} \) be the word sequence defined by

\[
u_n = \sigma^{p-(n \mod p)}(u_0),\]

where \( (n \mod p) \) is the remainder in the division of \( n \) by \( p \). We easily check that \( u \) is a fixed point of the directive sequence \( \sigma^\omega \). By Lemma 30 there exists \( n_0 \in \mathbb{N} \)
such that for all \( n \geq n_0 \), \( M_{0,n} = M^n > 0 \). We choose \( n \geq n_0 \) divisible by \( p \). Hence we have \( u_n = u_0 \). Now, for every \( b \in A \), we have the equality

\[
W_b(u_0) = \bigcup_{c \rightarrow b} c \mapsto \ldots \rightarrow c_{n-1} \rightarrow c_{n-2} \rightarrow \ldots \rightarrow c_{0} \rightarrow b
\]

By Lemma \([17]\) \( W_b(u_0) \) has non-empty interior for all \( b \in A \). Hence, we get the third point, with the sequence \((k_n)_{n \in \mathbb{N}} = (n)_{n \in \mathbb{N}}\).

In \([1,\text{Theorem 3.3}]\), they prove the following

**Theorem 89.** Let \( \sigma \) be an irreducible Pisot unimodular substitution. Then we have the equivalence between

- \( \sigma \) satisfies the Pisot substitution conjecture,
- there exists a periodic point \( u \in A^\mathbb{N} \) and a letter \( a \in A \) such that \( W_a(u) \) has non-empty interior for \( T(x) \),

where \( x \in \mathbb{PR}_d^+ \) is the class of a Perron eigenvector of \( ab(\sigma) \).

From this lemma and this theorem we deduce the following

**Corollary 90.** The converse of Theorem \([4]\) is true for directive sequences of the form \( \sigma^\omega \), where \( \sigma \) is an irreducible Pisot unimodular substitution. In other words, if \( \Omega_{\sigma^\omega} \) is measurably conjugated to a translation on a torus, then the directive sequence \( \sigma^\omega \) is good.

And we can restate the Pisot substitution conjecture as:

**Conjecture 91** (Reformulation of the Pisot substitution conjecture). For every irreducible Pisot unimodular substitution \( \sigma \), the directive sequence \( \sigma^\omega \) is good.

And a generalization of the Pisot substitution conjecture could be:

**Conjecture 92** (Generalization of the Pisot substitution conjecture). Let \( S \) be a set of unimodular substitutions. Let \( s \in S^\mathbb{N} \) be a directive sequence such that there exists a totally irrational direction \( x \in \mathbb{PR}_d^+ \) and a constant \( C > 0 \) such that for every \( k \) and \( n \in \mathbb{N} \), \( \| \pi_x M_{[k,k+n]} \|_1 \leq C e^{-C/n} \). Then, \( s \) is good.

The conjecture could be even more general:

**Conjecture 93** (Generalization of the Pisot substitution conjecture). Let \( S \) be a set of unimodular substitutions. Let \( s \in S^\mathbb{N} \) be a directive sequence such that there exists a totally irrational direction \( x \in \mathbb{PR}_d^+ \) such that \( \sum_n \| \pi_x M_{[k,k+n]} \|_1 \) converges uniformly in \( k \). Then the subshift associated to \( s \) is measurably conjugated to a translation on a torus.

11 Thanks

The authors would like to thank Mélodie Andrieu, Nicolas Bédaride, Jean-François Bertazzon, Julien Cassaigne, Paul Mercat, and Thierry Monteil for their help in preparing this article.
## Nomenclature

### Greek alphabet
- $\beta$ complex number, 50
- $\Delta$ simplex, 5
- $\delta$ Hausdorff distance, 40
- $\theta_2(x)$ Lyapunov exponent, 34
- $\Lambda$ integer lattice in $P$, 5
- $\lambda$ Lebesgue measure, 5
- $\mu$ Borel measure, 7
- $\pi_y, \pi_{v(x)}$ projection on $P$ along $y$ or $v(x)$, 5
- $\rho$ Tribonacci number, 3
- $\Sigma$ Dumont-Thomas alphabet, 16
- $\sigma$ generic substitution, 6
- $\sigma^\omega$ constant directive sequence, 9
- $\tau_0, \tau_1$ Sturmian substitutions, 44
- $\varphi$ linear form, 34
- $\phi$ linear map from $\mathbb{R}^3$ to $\mathbb{C}$, 51
- $\psi$ torus isomorphism from $P/\Lambda$ to $T^d$, 5
- $\Omega(n^d)$ growth rate, 3
- $\Omega_s$ $S$-adic subshift, 8
- $\Omega_u$ subshift generated by $u$, 7
- $w^\omega$ periodic word, 7

### Latin alphabet
- $A$ alphabet, 6
- $A^*$ abelianized prefix automaton, 16
- $A^N$ infinite words, 6
- $ab$ abelianization, 6
- $ar_0, ar_1, ar_2$ Arnoux Rauzy substitutions, 47
- $B$ Bratteli diagram, 11
- $B(x,r)$ ball in the projective space, 5
- $b_c$ substitution for Brun algorithm, 46
- $c_0, c_1$ Cassaigne substitutions, 48
- cod symbolic coding, 7
- $D$ distance on $\mathbb{C}$, 52
- $d$ dimension, 5
- $d(x,y)$ distance, 5
- $E$ domain exchange, 8
- $(e_i)_{0 \leq i \leq d}$ basis of $\mathbb{R}^{d+1}$, 5
- $E(x)$ Lyapunov spaces, 34
- $F$ continued fraction algorithm, 31
- $f_k$ measure of the boundary of the Rauzy fractal, 20
- freq($u$) frequency vector, 7
- $G$ good points, 37
- $G_0$ seed points, 37
\( G_1 \) auxiliary set related to the good points, 38
\( g_k \) measure of Rauzy fractal, 26
\( G_{B,C} \) seed set with explicit bound, 38
\( h \) sum of coordinates, 5
\( \mathbb{H} \) integer half-space, 12
\( \text{hom}(A^+,A^+) \) substitutions, 6
\( \text{hom}(A^+,A^+) \) non-erasing substitutions, 6
\( (k_n) \) integer sequence, 24
\( L \) linear map from \( P \) to \( \mathbb{R}^d \), 5
\( M_k(s) \) \( k \)-th matrix of \( s \), 12
\( M(k,n) \) product \( M_k \ldots M_{n-1} \), 12
\( M_{d+1}(\mathbb{R}) \) square matrices, 5
\( n \) integer, 3
\( N_k \) endomorphism of \( P \), 26
\( O(u) \) orbit, 7
\( P \) hyperplane where \( h \) cancels, 5
\( p(n) \) complexity function, 6
\( \mathbb{P}\mathbb{R}^+_d \) set of positive directions, 5
\( q \) quotient map \( P \to P/\Lambda \), 5
\( R(u), R(s) \) Rauzy fractal, 17
\( R_a(u) \) atom of the Rauzy fractal, 17
\( S \) finite set of substitutions, 8
\( s = s(x) \) directive sequence associated to \( x \), 31
\( s_0 \) extended continued fraction algorithm, 31
\( (s_k) \) directive sequence, 8
\( s[\ell, \ell) \) product of substitutions, 8
\( T \) shift map, 7
\( \mathcal{T}(x) \) topology on \( \mathbb{H} \), 12
\( \mathbb{T}^d \) torus, 5
\( T_t \) translation of the torus by vector \( t \), 6
\( T_x \) translation of the torus associated with the direction \( x \), 6
\( U \) open subset of \( P \), 12
\( u \) word, 6
\( u = (u_k) \) word sequence, usually fixed point, 9
\( V \) open subset of \( \mathbb{H} \) for some topology \( \mathcal{T}(x) \), 12
\( v(x) \) representative of \( x \) of norm 1, 5
\( v(k) \) vector, 26
\( W(u) \) worm, 12
\( W_a(u) \) subset of a worm, 12
\( X \) base set of a dynamical system, 7
\( x^{(k)} \) \( k \)-th element of a sequence of directions, 22
\( Z \) subset of measure one in Oseledec’s theorem, 33
\( Z_C \) set of points with explicit exponential convergence, 38

**Other symbols**

\([w]\) cylinder, 7
\([y]\) direction of the vector \( y \), 5
References

[1] S. Akiyama and P. Mercat. Yet another characterization of the Pisot substitution conjecture, 2018. Electronic preprint arXiv:1810.03500.

[2] M. Andrieu. PhD thesis, Aix-Marseille Université, 2020. In preparation.

[3] P. Arnoux and S. Labbé. On some symmetric multidimensional continued fraction algorithms. Ergodic Theory Dynam. Systems, 38(5):1601–1626, 2018.

[4] P. Arnoux and A. Nogueira. Mesures de Gauss pour des algorithmes de fractions continues multidimensionnelles. Ann. Sci. École Norm. Sup. (4), 26(6):645–664, 1993.

[5] P. Arnoux and G. Rauzy. Représentation géométrique de suites de complexité $2n+1$. Bull. Soc. Math. France, 119(2):199–215, 1991.

[6] A. Avila and V. Delecroix. Some monoids of Pisot matrices. In New trends in one-dimensional dynamics, volume 285 of Springer Proc. Math. Stat., pages 21–30. Springer, Cham, 2019.

[7] A. Avila, P. Hubert, and A. Skripchenko. Diffusion for chaotic plane sections of 3-periodic surfaces. Invent. Math., 206(1):109–146, 2016.

[8] A. Avila, P. Hubert, and A. Skripchenko. On the Hausdorff dimension of the Rauzy gasket. Bull. Soc. Math. France, 144(3):539–568, 2016.

[9] J.-F. Bertazzon. Fonction complexité associée à une application ergodique du tore. Bull. London Math. Soc., 44(6):1155–1168, 2012.

[10] V. Berthé and V. Delecroix. Beyond substitutive dynamical systems: $S$-adic expansions. In Numeration and substitution 2012, RIMS Kôkyûroku Bessatsu, B46, pages 81–123. Res. Inst. Math. Sci. (RIMS), Kyoto, 2014.

[11] V. Berthé, W. Steiner, and J. M. Thuswaldner. Geometry, dynamics, and arithmetic of $S$-adic shifts. Ann. Inst. Fourier (Grenoble), 69(3):1347–1409, 2019.

[12] N. Bédaride and J.-F. Bertazzon. Minoration of the complexity function associated to a translation on the torus. Monatsh. Math., 171(3-4):291–304, 2013.

[13] J. Cassaigne, S. Ferenczi, and A. Messaoudi. Weak mixing and eigenvalues for Arnoux-Rauzy sequences. Ann. Inst. Fourier (Grenoble), 58(6):1983–2005, 2008.
[14] J. Cassaigne, S. Labbé, and J. Leroy. A set of sequences of complexity $2n + 1$. In Combinatorics on words, volume 10432 of Lecture Notes in Comput. Sci., pages 144–156. Springer, Cham, 2017.

[15] N. Chekhova, P. Hubert, and A. Messaoudi. Propriétés combinatoires, ergodiques et arithmétiques de la substitution de Tribonacci. J. Théor. Nombres Bordeaux, 13(2):371–394, 2001.

[16] N. Chevallier. Coding of a translation of the two-dimensional torus. Monatsh. Math., 157(2):101–130, 2009.

[17] G. Didier. Combinatoire des codages de rotations. Acta Arith., 85(2):157–177, 1998.

[18] J.-M. Dumont and A. Thomas. Systèmes de numération et fonctions fractales relatifs aux substitutions. Theoret. Comput. Sci., 65:153–169, 1989.

[19] F. Durand. Combinatorics on Bratteli diagrams and dynamical systems. In Combinatorics, automata and number theory, volume 135 of Encyclopedia Math. Appl., pages 324–372. Cambridge Univ. Press, Cambridge, 2010.

[20] F. Durand, J. Leroy, and G. Richomme. Do the properties of an $S$-adic representation determine factor complexity? J. Integer Seq., 16(2):Article 13.2.6, 30, 2013.

[21] S. Ferenczi. Bounded remainder sets. Acta Arith., 61(4):319–326, 1992.

[22] S. Ferenczi. Rank and symbolic complexity. Ergodic Theory Dynam. Systems, 16(4):663–682, 1996.

[23] H. Freudenthal. Simplicialzerlegungen von beschränkter flachheit. Annals of Mathematics, 43(3):580–582, 1942.

[24] H. Furstenberg and H. Kesten. Products of random matrices. Ann. Math. Statist., 31:457–469, 1960.

[25] A. B. Katok and A. M. Stepin. Approximations in ergodic theory. Uspehi Math. Nauk, 22(5):81–106, 1967.

[26] S. Labbé. 3-dimensional continued fraction algorithms cheat sheets, 2015. Electronic preprint arXiv:1511.078399.

[27] J. C. Lagarias. The quality of the Diophantine approximations found by the Jacobi-Perron algorithm and related algorithms. Monatsh. Math., 115(4):299–328, 1993.

[28] C. W. Lee and F. Santos. Subdivisions and triangulations of polytopes. In C. D. Tóth, J. O’Rourke, and J. E. Goodman, editors, Handbook of Discrete and Computational Geometry, pages 415–447. CRC Press, 3rd edition, 2017.

[29] M. Morse and G. A. Hedlund. Symbolic dynamics ii. sturmian trajectories. American Journal of Mathematics, 62(1):1–42, 1940.

[30] K. Nakaishi. Strong convergence of additive multidimensional continued fraction algorithms. Acta Arith., 121(1):1–19, 2006.
[31] V. I. Oseledets. A multiplicative ergodic theorem: Lyapunov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.*, 19:197–231, 1968.

[32] N. Pytheas Fogg. *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.

[33] G. Rauzy. Échanges d’intervalles et transformations induites. *Acta Arithmetica*, 34(4):315–328, 1979.

[34] G. Rauzy. Nombres algébriques et substitutions. *Bull. Soc. Math. France*, 110(2):147–178, 1982.

[35] F. Schweiger. *Multidimensional continued fractions*. Oxford Science Publications. Oxford University Press, Oxford, 2000.

[36] S. Tabachnikov. *Billiards*. Number 1 in Panoramas et Synthèses. Société Mathématique de France, 1995.

[37] J.-C. Yoccoz. Interval exchange maps and translation surfaces. In *Homogeneous flows, moduli spaces and arithmetic*. Proceedings of the Clay Mathematics Institute summer school, Centro di Recerca Mathematica Ennio De Giorgi, Pisa, Italy, June 11–July 6, 2007, pages 1–69. Providence, RI: American Mathematical Society (AMS); Cambridge, MA: Clay Mathematics Institute, 2010.