1. Introduction

It has been proved in [JS90] that the Axiom of Full Reflection at an $n$-Mahlo cardinal is equiconsistent with a $\Pi^1_n$-indescribable cardinal and in [JW94] that consistency of the Axiom of Full Reflection at a measurable cardinal follows from consistency of a coherent sequence of measures with a repeat point. It has been conjectured in [JW94] that the two principles are actually equiconsistent. However we prove that Full Reflection at a measurable cardinal can be obtained surprisingly from only one measure. Furthermore the method also generalizes to larger cardinals as strong or supercompact. Hence we can conclude that the Axiom of Full Reflection at large cardinals weaker than measurable, e.g. as $n$-Mahlo, does push the consistency strength up, but does not push the consistency strength up at a measurable or larger cardinals.

To state the main theorem let us review the basic definitions and facts. If $S$ is a stationary subset of a regular uncountable cardinal $\kappa$ then the trace of $S$ is the set...
\[
\text{Tr}(S) = \{\alpha < \kappa; \ S \cap \alpha \text{ is stationary in } \alpha\}
\]
and we say that \(S\) \textit{reflects at } \alpha \in \text{Tr}(S).\) If \(S\) and \(T\) are both stationary, we define
\[
S < T \text{ if for almost all } \alpha \in T, \ \alpha \in \text{Tr}(S)
\]
and say that \(S\) \textit{reflects fully} in \(T\). (Throughout the paper, “for almost all” means “except for a nonstationary set of points”). It can be proved that this relation is a well-founded partial ordering (see [JW94] or [J84]). The \textit{order } o(S) \textit{of a stationary set of regular cardinals} is defined as the rank of \(S\) in the relation <:
\[
o(S) = \sup\{o(T) + 1; \ T \subseteq \text{Reg}(\kappa) \text{ is stationary and } T < S\}.
\]
For a stationary set \(T\) such that \(T \cap \text{Sing}(\kappa)\) is stationary define \(o(T) = -1\). \textit{The order of } \kappa \textit{is then defined as}
\[
o(\kappa) = \sup\{o(S) + 1; S \subseteq \kappa \text{ is stationary}\}.
\]
Note that the order \(o(\kappa)\) provides a natural generalization of the Mahlo hierarchy: \(\kappa\) is exactly \(o(\kappa)\)-Mahlo if \(o(\kappa) < \kappa^+\) and greatly Mahlo if \(o(\kappa) \geq \kappa^+\).

We say that a stationary set \(S\) \textit{reflects fully at regular cardinals} if for any stationary set \(T\) of regular cardinals \(o(S) < o(T)\) implies \(S < T\).

\textbf{Axiom of Full Reflection at } \kappa. \textit{Every stationary subset of } \kappa \textit{reflects fully at regular cardinals.}

Notice that the axiom presents in a sense the maximal possible amount of reflection of stationary subsets of \(\kappa\) at regular cardinals.

Now we are able to state the main theorem:

\textbf{Theorem.} \textit{Let } \phi(\kappa) \textit{be one of the following principles:}

\begin{itemize}
\item[(i)] \(\kappa\) is measurable,
\item[(ii)] the Mitchell order of \(\kappa\) is \(\kappa^{++}\),
\item[(iii)] \(\kappa\) is \(n\)-strong,
\item[(iv)] \(\kappa\) is strong,
\item[(v)] \(\kappa\) is \(\kappa^{+n}\)-supercompact,
\item[(vi)] \(\kappa\) is supercompact.
\end{itemize}
Assume that $V$ satisfies GCH and $\phi(\kappa)$, then there is a model where GCH, the Axiom of Full Reflection at $\kappa$, and $\phi(\kappa)$ hold.

The case (ii) has been actually proved in [JW94]: it has been proved in the paper that if $\mathcal{U}$ is a coherent sequence of measures then there is a forcing notion $P_{\kappa+1}$ that preserves any repeat point of $\mathcal{U}$ on $\kappa$. If $o_{\mathcal{U}}(\kappa) = \kappa^{++}$ then there are $\kappa^{++}$ repeat points on $\kappa$ and it is not difficult to see that the Mitchell order of $\kappa$ is $\kappa^{++}$ in the generic extension by $P_{\kappa+1}$. Thus we will work only on cases (i) and (iii)-(vi).

2. Proof of the theorem

The proof should be self-contained, however a knowledge of [JW94] is helpful.

Assume that $V$ satisfies GCH and $j : V \rightarrow M$ is an elementary embedding such that $\text{crit}(j) = \kappa$ and $V \cap \kappa^M \subseteq M$. We will define a forcing $P_{\kappa+1}$ that will work in all cases (i),(iii)-(vi). $P_{\kappa+1}$ will be an Easton support iteration of $\langle Q_{\lambda} : \lambda \leq \kappa \rangle$; $Q_{\lambda}$ will be nontrivial only for $\lambda$ Mahlo, and in that case it will be an iteration (defined in $V(P_{\lambda})$) of length $\lambda^+$ with $< \lambda$ support of forcing notions shooting clubs through certain sets $X \subseteq \lambda$ always with the property that $X \supseteq \text{Sing}(\lambda)$. This will be guarantee $Q_{\lambda}$ to be essentially $< \lambda$-closed (i.e. it will have a $< \lambda$-closed dense subset). Consequently $Q_{\lambda}$ will be $\lambda^+$-c.c., $P_{\lambda}$ will be $\lambda$-c.c., and the factor iteration $P_{\lambda+1,\kappa+1}$ above $\lambda$ will be essentially $\lambda$-closed. Therefore $P_{\kappa+1}$ will preserve cardinals, cofinalities, and GCH.

Consider an iteration $Q$ of $\langle \text{CU}(\dot{X}_\alpha) : \alpha < l(Q) \rangle$ with $< \lambda$ support, where $\text{CU}(\dot{X}_\alpha)$ denotes the forcing shooting a club in $V(P_{\lambda} * Q \upharpoonright \alpha)$ through a subset $\dot{X}_\alpha$ of $\lambda$ containing $\text{Sing}(\lambda)$. In that case we say that $Q$ is an iteration of order 0. Since $Q \upharpoonright \alpha$ is essentially $< \lambda$-closed, conditions in $\text{CU}(\dot{X}_\alpha)$ can be taken in $V(P_{\lambda})$ rather than in $V(P_{\lambda} * Q \upharpoonright \alpha)$. So $Q$ can be considered to be a set of sequences of closed bounded subsets of $\lambda$ in $V(P_{\lambda})$. Since $P_{\lambda}$ is $\lambda$-c.c. there is an appropriate $P_{\lambda}$-name for $Q$ of cardinality $\lambda$ if $l(Q) < \lambda^+$, and of cardinality $\lambda^+$ if $l(Q) = \lambda^+$. Let $\tilde{Q}$ be another iteration of $\langle \text{CU}(\dot{Y}_\gamma) : \gamma < l(\tilde{Q}) \rangle$ of order 0. We say that $Q$ is a subiteration of $\tilde{Q}$ if there is a 1-1 function $\pi : l(Q) \rightarrow l(\tilde{Q}) : \alpha \mapsto \gamma_\alpha$ inducing an embedding of $Q$ into $\tilde{Q}$ such that $\dot{X}_\alpha$ is an equivalent name to $\dot{Y}_{\gamma_\alpha}$ with respect to the induced embedding of $Q \upharpoonright \alpha$ into $\tilde{Q}$. Notice that the sequence $\langle \gamma_\alpha : \alpha < l(Q) \rangle$ does not have to be increasing. Any $Q$-name can be considered to be a $\tilde{Q}$-name via the induced embedding; $\tilde{Q}$ is actually isomorphic to an iteration of order 0 in the form $Q * R$.

We will need to estimate (in $V(P_{\lambda})$) the number of iterations of order 0 and
length $< \lambda^+$. Each such iteration is a set of sequences with $< \lambda$ support of bounded subsets of $\lambda$. Therefore it is easy to see that the number is $\leq 2^\lambda = \lambda^+$. 

For any iteration $Q$ of order $\delta + 1$ we will define certain filters $F_{\lambda,\delta}^Q$ on $\lambda$ in $V(P_\lambda * Q)$. Simultaneously by induction on $\beta$ and $l(Q)$ we define $Q$ to be an iteration of order $\beta$ if it is an iteration of $\langle CU(\dot{X}_\alpha) ; \alpha < l(Q) \rangle$ with $< \lambda$-support such that $l(Q) < \lambda^+$ and for all $\alpha < l(Q)$:

$$P_\lambda * Q \upharpoonright \alpha \models "\text{Sing}(\lambda) \subseteq \dot{X}_\alpha \text{ and } \dot{X}_\alpha \in F_{\lambda,\delta}^{Q|\alpha}\text{ for all } \delta < \beta."$$

Let us call such an assignment $Q \mapsto F_{\lambda,\delta}^Q$ a filter system $F_{\lambda,\delta}$. $F_{\lambda,\delta}$ will be defined for all $\delta < \Theta(\lambda)$ where $\Theta(\lambda)$ will be specified later. The filter systems will have among others the property that $F_{\lambda,\delta}^{Q|\alpha} \subseteq F_{\lambda,\delta}^Q$.

$Q_\lambda$ is then defined in $V(P_\lambda)$ to be an iteration of length $\lambda^+$ such that for all $\alpha < \lambda^+$ $Q_\lambda \upharpoonright \alpha$ is an iteration of order $\Theta(\lambda)$, and all potential names for subsets of $\lambda$ are used cofinally many times.

It remains to find the filter systems $F_{\lambda,\delta}$ (working in $V(P_\lambda)$). We require that for any iteration $Q$ of order $\delta + 1$ the following is satisfied:

(i) If $Q'$ is a subiteration of $Q$ then

$$F_{\lambda,\delta}^{Q'} = F_{\lambda,\delta}^Q \cap V(P_\lambda * Q'),$$

(ii) $P_\lambda * Q \models "F_{\lambda,\delta}^Q \supseteq \text{Club}(\lambda)\text{ is a proper filter,}\)

$$\forall S \subseteq \text{Sing}(\lambda) \text{ stationary: } \text{Tr}(S) \in F_{\lambda,\delta}^Q,$$

$$\forall S \subseteq \lambda : (\exists \gamma < \delta : S \text{ is } F_{\lambda,\gamma}^Q\text{-positive}) \Rightarrow \text{Tr}(S) \in F_{\lambda,\delta}^Q,$$

(iii) $P_\lambda * Q \models "\forall S \subseteq \text{Reg}(\lambda) : (\forall \gamma < \delta : S \text{ is } F_{\lambda,\gamma}^Q\text{-thin}) \Rightarrow \kappa \setminus \text{Tr}(S) \in F_{\lambda,\delta}^Q."$

Moreover we require that

(iv) there is an iteration $Q$ of order $\delta + 1$, a $P_\lambda * Q$-name $\dot{X}$ for a subset of $\lambda$ and $p \ast q \in P_\lambda * Q$ so that

$$p \ast q \models_{P_\lambda * Q} "\dot{X} \text{ is } F_{\lambda,\gamma}^Q\text{-thin for all } \gamma < \delta," \text{ but }$$

$$p \ast q \models_{P_\lambda * Q} "\dot{X} \text{ is } F_{\lambda,\delta}^Q\text{-positive."}$$

By induction on $\delta$ choose a filter system $F_{\lambda,\delta}$ as long as there is such a filter system with properties (i)–(iv). Since the number of iterations $Q$ of length $< \lambda^+$
with $< \lambda$-support shooting closed unbounded subsets of $\lambda$ is $\leq \lambda^+$ and since $F_{\lambda,\delta}$ is by (iv) different from all $F_{\lambda,\gamma}$ ($\gamma < \delta$), this process must eventually stop after a number of steps $\Theta(\lambda) < \lambda^{++}$.

Apply this process by induction on all $\lambda < \kappa$ defining an iteration $P_\kappa$ below $\kappa$. Put $P_{\kappa+1} = (jP_\kappa) \upharpoonright (\kappa + 1)$. Note that $P_{\kappa+1} = P_\kappa \ast Q_\kappa$ where $Q_\kappa$ is an iteration of length $\kappa^+$ with $< \kappa$-support, given by certain filter systems $F_{\kappa,\delta}$ ($\delta < \Theta = \Theta(\kappa)$).

We claim that

$$V(P_{\kappa+1}) \models \text{“Full Reflection at } \kappa\text{”}$$

and the embedding $j$ can be in many cases lifted onto $V(P_{\kappa+1})$.

Let us define $F_j$ in $V$ similarly as in [JW94] to be a $\Theta$-th filter system on $\kappa$:

By induction on $l(Q)$ say that $Q$, an iteration of $\langle CU(\hat{X}_\alpha); \alpha < l(Q) \rangle$, is an iteration of order $\Theta + 1$ w.r.t. $F_j$ if it is an iteration of order $\Theta$ and for all $\alpha < l(Q)$

$$P_\kappa \ast Q \upharpoonright \alpha \models \text{“} \hat{X}_\alpha \in F_j^{Q|\alpha}. \text{”}$$

If $Q$ is an iteration of order $\Theta + 1$ w.r.t. $F_j$, $\hat{X}$ a $P_\kappa \ast Q$-name, $p \ast q \in P_\kappa \ast Q$, we define $p \ast q \models \text{“} \hat{X} \in F_j^{Q}. \text{”}$

(1) $$p \models_{jP_\kappa} \forall H \in \text{Gen}_j(Q, G^*) : q \in H \Rightarrow [H]^j \models_{jQ} \kappa \in j\hat{X}. \text{”}$$

Here $\text{Gen}_j(Q, G^*)$ is defined as follows: let $G^*$ be a $jP_\kappa$-generic filter over $V$, $G = G^* \upharpoonright P_\kappa$. Then $Q$ is obviously a subiteration of $Q_\kappa$ which gives a filter $H$ from $G^* \upharpoonright Q_\kappa$ that is $Q$-generic over $V[G]$. $\text{Gen}_j(Q, G^*)$ denotes the set of all filters $H$ obtained in this way. We can easily find many $H \in \text{Gen}_j(Q, G^*)$ such that $q \in H$: since $Q_\kappa$ is an iteration of order $\Theta$ such that all potential names are used cofinally many times we can find a sequence of ordinals $\langle \gamma_\alpha; \alpha < l(Q) \rangle$ inducing a subiteration embedding of $Q$ into $Q_\kappa$ such that all $\gamma_\alpha$'s are above any given $\beta < \kappa^+$; hence by a density argument there is $r \in G^*$ and such a sequence $\langle \gamma_\alpha; \alpha < l(Q) \rangle$ with the property that $r \upharpoonright \langle \gamma_\alpha; \alpha < l(Q) \rangle = q$.

Represent an $H \in \text{Gen}_j(Q, G^*)$ as $\langle C_\beta; \beta < l(Q) \rangle$ where $C_\beta$'s are the generic closed unbounded subsets of $\kappa$. $[H]^j$ is a sequence of length $j(l(Q))$ defined as follows

$$[H]^j(\gamma) = \begin{cases} C_\beta \cup \{\kappa\} & \text{if } j(\beta) = \gamma, \\ \emptyset & \text{otherwise.} \end{cases}$$

To prove that $[H]^j \in jQ$ all we need is to check inductively that $[H]^j \upharpoonright j(\beta) \models_{j(Q_1|\beta)} \text{“} \kappa \in j\hat{X}_\beta. \text{”}$ But this immediately follows from the assumption $P_\kappa \ast Q \upharpoonright \beta \models \text{“} \hat{X}_\beta \in F_j^{Q|\beta}. \text{”}$
Lemma 1. The filter system $F_{\kappa, \Theta} = F_j$ satisfies (i)--(iii) with $\delta = \Theta$.

Proof. (i) Let $Q, Q'$ be two iterations of order $\Theta + 1$; assume $\pi$ embeds $Q$ into $Q'$ via $\langle \alpha_\delta; \delta < l(Q) \rangle$ as a subiteration. Let $\tilde{X}$ be a $P_\lambda \ast Q$-name for a subset of $\lambda$.

Suppose $p \ast q \in P_\lambda \ast Q$, $p \ast q \models_{P_\lambda \ast Q} \exists \tilde{X} \in F^Q_j$." We want to prove that

$$p \ast p(q) \models_{P_\lambda \ast Q} \exists \tilde{X} \in F^Q_j".$$}

Let $G^* \ni p$ be $jP_\lambda$-generic over $V$, $H' \in \text{Gen}_j(Q', G^*)$, $H' \ni \pi(q)$. Then the embedding of $Q'$ into $(jP_\lambda)^\lambda$ induces via $\pi$ an embedding of $Q$ into $(jP_\lambda)^\lambda$ giving $H \in \text{Gen}_j(Q, G^*)$ such that $q \in H$. Moreover $j\pi$ embeds $jQ$ into $jQ'$ by elementarity, and $(j\pi)((H'])^j \geq [H']^j$. Since $[H']^j \models_{jQ} \exists \lambda \in j\tilde{X}"$ it follows that $[H']^j \models_{jQ'} \exists \lambda \in j\tilde{X}"$.

Now suppose $p \ast q' \in P_\lambda \ast Q', p \ast q' \models_{P_\lambda \ast Q'} \exists \tilde{X} \in F^Q_j"$. Let $q \in Q$ be such that $\pi(q)$ agrees with $q'$ on the set $\{\alpha_\delta; \delta < l(Q)\}$. We claim that $p \ast q \models_{P_\lambda \ast Q} \exists \tilde{X} \in F^Q_j"$. Let $G^* \ni p$ be $jP_\lambda$-generic over $V$, $H \in \text{Gen}_j(Q, G^*)$, and $q \in H$. We need to prove $[H']^j \models_{jQ} \exists \lambda \in j\tilde{X}"$. Suppose it is not true, then there is $\tilde{q} \leq [H']^j$ such that $\tilde{q} \models_{jQ} \exists \lambda \notin j\tilde{X}"$. Express $Q'$ as $Q \ast R$, and as above find a subiteration embedding of $Q'$ into $(jP_\lambda)^\lambda$ that extends the embedding of $Q$, giving $H' \in \text{Gen}_j(Q', G^*)$ such that $H' \upharpoonright Q = H$, and $q' \in H'$. In other words if $\pi_1: l(Q) \to \lambda^+$ embeds $Q$ into $(jP_\lambda)^\lambda$ then we obtain $\pi_2: l(Q') \to \lambda^+$ embedding $Q'$ into $(jP_\lambda)^\lambda$ such that $\pi_1(\delta) = \pi_2(\alpha_\delta)$ for $\delta < l(Q)$. Now $j\pi$ embeds $jQ$ into $jQ'$ via $j(\alpha_\delta; \delta < l(Q))$, thus $(j\pi)(\tilde{q}) \in jQ'$ and

$$\text{supp} ((j\pi)(\tilde{q})) \subseteq j(\{\alpha_\delta; \delta < l(Q)\}).$$

Moreover $\text{supp} ([H']^j) = j''l(Q'),$ if $\alpha < l(Q')$ then either $\alpha \in \{\alpha_\delta; \delta < l(Q)\}$, and then $(j\pi)(\tilde{q})(j\alpha)$ extends $[H']^j(j\alpha)$, or $\alpha \notin \{\alpha_\delta; \delta < l(Q)\}$, then $j(\alpha) \notin \text{supp} ((j\pi)(\tilde{q})).$ Consequently $(j\pi)(\tilde{q})$ and $[H']^j$ are compatible. But $[H']^j \models_{jQ'} \exists \lambda \in j\tilde{X}",$ while $(j\pi)(\tilde{q}) \models_{jQ'} \exists \lambda \notin j\pi\tilde{X}"$ - a contradiction.

(ii) Each $F^Q_j$ is obviously proper and contains $\text{Club}(\kappa)$. Let $P_\kappa \ast Q \models \exists \dot{S} \subseteq \kappa \text{ is } F^Q_{\kappa, \gamma} \text{-positive}$ for some $\gamma < \Theta$ (or $\dot{S} \subseteq \text{Sing}(\kappa)$ is stationary). We wish to prove that $P_\kappa \ast Q \models \exists \text{Tr}(\dot{S}) \in F^Q_j$. Assume towards a contradiction that $G^*$ is $jP_\kappa$-generic over $V$, $H \in \text{Gen}_j(Q, G^*)$, and $[H']^j \not\models_{jQ} \exists \kappa \in \text{Tr}(\dot{S}).$ So there is $H^* \ni [H']^j$ $jQ$-generic over $V[G^*]$ so that

$$V[G^* \ast H^*] \models \text{"S is nonstationary."}$$
Since \((jP_\kappa)_{\kappa+1}^\ast jQ\) is essentially \(\kappa\)-closed and \(Q_\kappa\) is \(\kappa^+\)-c.c. there is a sufficiently large \(\alpha < \kappa^+\), such that if \(G = G^* \upharpoonright P_\kappa\), \(\tilde{H} = G^* \upharpoonright (Q_\kappa \upharpoonright \alpha)\) then
\[
V[G \ast \tilde{H}] \models "S is nonstationary;"
\]
which is a contradiction with (i) as \(V[G \ast H] \models "S is \(F_{\kappa,\gamma}^H\)-positive"\) and \(Q\) is an subiteration of \(Q_\kappa \upharpoonright \alpha\) giving \(H\) from \(\tilde{H}\) (provided \(\alpha\) is large enough).

(iii) Assume that
\[
P_\kappa \ast Q \models \"\hat{S} \subseteq \text{Reg}(\kappa)\) and \(\forall \gamma < \Theta : \hat{S} \) is \(F_{\kappa,\gamma}^Q\)-thin."
\]
We want to prove that \(P_\kappa \ast Q \models \"\kappa \setminus \text{Tr} (\hat{S}) \in F_j^Q.\) Assume \(G^*\) is \(jP_\kappa\)-generic, \(H \in \text{Gen}_j(Q, G^*)\), \(H^* \ni [H]^j \ast jQ\)-generic over \(V[G^*]\) and \(V[G^* \ast H^*] \models \"\kappa \notin j(\kappa \setminus \text{Tr} (S))\)\) i.e. \(V[G \ast \tilde{H}] \models "S is stationary"\) where \(\tilde{H} = (G^*) \upharpoonright Q_\kappa\). But a club have been shot through \(\kappa \setminus S\) in the iteration \(Q_\kappa\) - a contradiction. \(\square\)

\textbf{Lemma 2.} Let \(Q_o\) be an iteration of \((CU(X_\alpha); \alpha < l(Q))\) of order \(\Theta\), \(\hat{X}\) a \(P_\kappa \ast Q_o\)-name for a subset of \(\kappa\), \(p \ast q \in P_\kappa \ast Q_o\). Then \(Q_o\) is an iteration of order \(\Theta + 1\) w.r.t. \(F_j\), and moreover if \(p \ast q \models \"\hat{X}\) is \(F_{\kappa,\gamma}^{Q_o}\)-thin for all \(\gamma < \Theta\)" then \(p \ast q \models \"\hat{X}\) is \(F_{\kappa,\gamma}^Q\)-thin."

\textit{Proof.} Assume towards a contradiction that \(p \ast q \models \"\hat{X}\) is \(F_{\kappa,\gamma}^{Q_o}\)-positive." Then we claim that the construction of filter systems \(F_{\kappa,\gamma}\) in \(M = \text{Ult}(V, U)\) could not stop at \(\Theta\). \(F_j\) cannot be constructed in \(M\), however we can construct its approximation.

Firstly define \(\tilde{F}_{\kappa,\Theta}\) as follows:

Let \(\tilde{F}_{\kappa,\Theta}^\emptyset (Q = \emptyset)\) be generated in \(V(P_\kappa)\) by all sets that should be there by (ii) and (iii), and by \(\check{X}_o\). Note that \(\check{X}_\alpha\) is forced to be in \(F_j^{Q_o \setminus \alpha}\) for all \(\alpha < l(Q)\) by the induction hypothesis. Hence \(\tilde{F}_{\kappa,\Theta}^\emptyset \subseteq F_j^\emptyset\) verifying that \(\tilde{F}_{\kappa,\Theta}^\emptyset\) is a proper filter. Similarly define \(\tilde{F}_{\kappa,\Theta}^{Q_k}\) for iterations \(Q\) of order \(\Theta + 1\) w.r.t. previously defined \(\tilde{F}_{\kappa,\Theta}^{Q_o}\).

We also have to make sure that \(\hat{X}_\alpha \in \tilde{F}_{\kappa,\Theta}^{Q_o \setminus \alpha}\) for all \(\alpha < l(Q_o)\). This filter system satisfies (ii) and (iii), clearly \(\tilde{F}_{\kappa,\Theta}^{Q'} \subseteq \tilde{F}_{\kappa,\Theta}^{Q}\) if \(Q'\) is a subiteration of \(Q\), however (i) does not have to hold. To achieve that define
\[
F_{\kappa,\Theta}^\emptyset = \bigcup \{\tilde{F}_{\kappa,\Theta}^{Q_o \setminus \alpha}; Q\) is an iteration of order \(\Theta + 1\) w.r.t. \(\tilde{F}_{\kappa,\Theta}\}\).
\]
Then for \(Q\) an iteration of order \(\Theta + 1\) w.r.t. previously defined \(F_{\kappa,\Theta}^{Q_o}\)'s by induction on \(l(Q)\) define
\[
F_{\kappa,\Theta}^Q = \bigcup \{\tilde{F}_{\kappa,\Theta}^{Q'} \cap V(P_\kappa \ast Q); Q'\) is an iteration of order \(\Theta + 1\) w.r.t. \(\tilde{F}_{\kappa,\Theta}\}\.
\]
such that $Q$ is an subiteration of $Q'$.

It is not difficult to see that such $Q'$ exists. We have constructed a filter system $F_{\kappa, \Theta}$ in $M$ that satisfies (i)–(iii). Moreover $Q_\alpha$ is an iteration of order $\Theta + 1$ w.r.t. $F_{\kappa, \Theta}$, and so (iv) holds for the $\mathcal{X}, p \ast q$ from the assumption of the lemma - a contradiction. □

Let $G \ast H$ be $P_\kappa \ast Q_\kappa$-generic over $V$.

**Lemma 3.** $V[G \ast H] \models \text{"Full Reflection holds up to } \kappa."$

**Proof.** For $\gamma < \Theta$ define $F^{H}_{\kappa, \gamma} = \bigcup_{\alpha < \kappa^+} F^{H|\alpha}_{\kappa, \gamma}$. We know that $F^{H}_{\kappa, \gamma} \supseteq \text{Club}(\kappa)$ is proper. By (i) if $S \in V[G \ast H \upharpoonright \alpha]$ is $F^{H|\alpha}_{\kappa, \gamma}$-positive then it is $F^{H}_{\kappa, \gamma}$-positive. Moreover by the construction $S \subseteq \text{Reg}(\kappa)$ is stationary iff $S$ is $F^{H}_{\kappa, \gamma}$-positive for some $\gamma < \Theta$ iff $S$ is $F^{H|\alpha}_{\kappa, \gamma}$-positive whenever $S \in V[G \ast H \upharpoonright \alpha]$. Let us firstly prove that $V[G \ast H] \models \text{"S < Reg(\kappa)"}$ for $S \subseteq \text{Sing}(\kappa)$ stationary in $V[G \ast H]$. Let $S \in V[G \ast H \upharpoonright \alpha]$ then $S$ is also stationary in this model, and so by (ii) $\text{Tr}(S) \in F^{H|\alpha}_{\kappa, \gamma}$ for all $\gamma < \Theta$, consequently a club has been shot through $\text{Sing}(\kappa) \cup \text{Tr}(S)$.

Now let $S \subseteq \text{Reg}(\kappa)$ be stationary, denote $\gamma_S$ to be the least $\gamma$ such that $S$ is $F^{H|\alpha}_{\kappa, \gamma}$-positive. The following claim completes the proof of Full Reflection at $\kappa$ in $V[G \ast H]$ (the proof for $\lambda < \kappa$ is identical).

**Claim.** Let $S, T \subseteq \text{Reg}(\kappa)$ be two stationary sets. Then $\gamma_S < \gamma_T$ iff $S < T$. Consequently $\gamma_S = \gamma_T$ iff $o(S) = o(T)$.

**Proof.** Let $S, T \in V[G \ast H \upharpoonright \alpha]$, $\gamma_S < \gamma_T$. Then $S$ is $F^{H|\alpha}_{\kappa, \gamma_S}$-positive, and so by (ii) $\text{Tr}(S) \in F^{H|\alpha}_{\kappa, \delta}$ for all $\delta > \gamma_S$. Thus $T \setminus \text{Tr}(S)$ is $F^{H|\alpha}_{\kappa, \delta}$-thin for all $\delta < \Theta$, so a club has been shot through $\kappa \setminus (T \setminus \text{Tr}(S))$, which means that $T \setminus \text{Tr}(S)$ is nonstationary in $V[G \ast H]$, i.e. $S < T$.

On the other hand assume that $S < T$, then necessarily $\gamma_S \leq \gamma_T$. By the definition of $\gamma_S$ the set $S$ is $F^{H|\alpha}_{\kappa, \delta}$-thin for all $\delta < \gamma_S$, and so by (iii) $\text{Tr}(S)$ is $F^{H|\alpha}_{\kappa, \gamma_S}$-thin. Since $T \setminus \text{Tr}(S)$ is nonstationary in $V[G \ast H]$, it must be $F^{H|\alpha}_{\kappa, \gamma_S}$-thin. Thus $T = (T \setminus \text{Tr}(S)) \cup \text{Tr}(S)$ is $F^{H|\alpha}_{\kappa, \gamma_S}$-thin proving $\gamma_S < \gamma_T$.

Finally if $\gamma_S = \gamma_T$ and say $o(S) < o(T)$ then there must be $S' < T$ such that $o(S) = o(S')$. By the fact proven above $\gamma_{S'} < \gamma_T = \gamma_S$, and so $S' < S$ - a contradiction. □ Claim, Lemma 3

Finally we need to prove that $P_{\kappa+1}$ preserves large cardinal properties of $\kappa$. Let us firstly consider measurability and supercompactness of $\kappa$. 

Lemma 4. Let $\lambda \geq \kappa$ be a cardinal such that

(i) $V \cap \lambda \in \in M \subseteq M$,
(ii) $\lambda^+ < j(\kappa) < j(\kappa^+) < \lambda^+$,
(iii) there is no Mahlo cardinal between $\kappa$ and $\lambda + 1$.

Then the embedding $j : V \to M$ can be extended to $j^{**} : V[G \ast H] \to M[G^* \ast H^*]$ in $V[G \ast H]$ so that $V[G \ast H] \cap \lambda \in M[G^* \ast H^*] \subseteq M[G^* \ast H^*]$.

Proof. By the definition of $P_{\kappa + 1}$ the forcing $jP_{\kappa + 1}$ factors as $P_{\kappa + 1} \ast R_o \ast j(Q_\kappa)$. So all we need is to find an $R_o \ast j(Q_\kappa)$-generic filter $H_o \ast H^*$ over $M[G \ast H]$ so that $p \ast q \in G \ast H$ implies $j(p \ast q) \in G \ast H \ast H_o \ast H^*$. The factor iteration $R_o = (jP_{\kappa + 1})_{\kappa + 1, j_\kappa}$ starts with a nontrivial forcing at the first Mahlo cardinal in $M$ above $\kappa$ which must be above $\lambda$. Consequently $R_o$ is essentially $\lambda$-closed in $M[G \ast H]$ as well as in $V[G \ast H]$. Let $D$ be a $\lambda$-closed dense subset of $R_o$. The number of dense subsets of $D$ in $M[G \ast H]$ is $j(\kappa^+)$ and the cardinality of $j(\kappa^+)$ in $V$ is just $\lambda^+$. Thus we have only $\lambda^+$ dense subsets of a forcing that is $\lambda$-closed in $V[G \ast H]$, and so it is easy to construct $H_o \in V[G \ast H]$ that is $R_o$-generic over $M[G \ast H]$. Obviously $p \in G$ implies $j(p) \in G^* = G \ast H \ast H_o$, thus $j$ extends to $j^* : V[G] \to M[G^*]$ in $V[G \ast H]$. It immediately follows from the $\kappa$-c.c. of $P_\kappa$ that $V[G] \cap \lambda \in M[G^*] \subseteq M[G^*]$. Next we need to find a filter $H^* \in V[G \ast H]$ that is $j^*(Q_\kappa)$-generic over $M[G^*]$, and such that $[H \restriction \alpha]^j \in H^*$ for all $\alpha < \kappa^+$.

It is easy to see that the number of antichains of $Q_\kappa$ (in $V[G]$) is only $\kappa^+$: if $A \subseteq Q_\kappa$ is an antichain, then $|A| \leq \kappa$, which implies that there is an $\alpha < \kappa^+$ such that $A \subseteq Q_\kappa \restriction \alpha$, the number of subsets of $Q_\kappa \restriction \alpha$ is only $\kappa^+$. By elementarity $M[G^*] \models \"the number of antichains in $j^*(Q_\kappa)$ is $j(\kappa^+)\"$. Moreover $M[G^*] \models \"j^*(Q_\kappa) is essentially $\lambda$-closed\". Let $D$ be a $\lambda$-closed dense open subset of $j^*(Q_\kappa)$, put

$$D = \{ A \in M[G^*] ; A \subseteq D \text{ is an antichain} \}.$$

Then $V[G \ast H] \models \"D is $\lambda$-closed, $|D| = |j(\kappa^+)| = \lambda^+\."$ Now we have to distinguish two cases: if $\lambda \geq \kappa^+$ then $[H]^j = \bigcup_{\alpha < \kappa^+}[H \restriction \alpha]^j$ is a good master condition in $j^*(Q_\kappa)$, and we can easily build up $H^* \in V[G \ast H]$ $j^*(Q_\kappa)$-generic over $M[G^*]$ such that $[H]^j \in H^*$. If $\lambda = \kappa$ then we have to be more careful. Let $\langle A_\alpha ; \alpha < \kappa^+ \rangle$ be an enumeration of $D$ in which each element of $D$ occurs cofinally many times. Construct a descending sequence of conditions $\langle q_\alpha ; \alpha < \kappa^+ \rangle \subseteq D$ with the following properties

(i) $q_\alpha \in j^*(Q_\kappa \restriction \alpha)$,
Lemma 5. Let \( j : V \to M \) be given by a \((\kappa, \lambda)\)-extender: \( \text{crit } (j) = \kappa, V \cap \kappa \mathcal{M} \subseteq M \), \( M = \{(jf)(a); a \in [\lambda]<\omega, f \in [\kappa][\alpha] V\}. \) Moreover assume that \( P \) is a notion of forcing such that \( M \models \{P \leq j(\kappa^+) \}, P \) has \( j(\kappa^+)\)-c.c., and \( P \) is \( \lambda\)-closed.” Then there is \( G \in V \) \( P\)-generic over \( M \).

Proof. (J. Zapletal) We can assume that \( P \subseteq j(\kappa^+) \). Let \( \langle f_\alpha; \alpha < \kappa^+ \rangle \) be an enumeration of all functions \( \kappa \to [\kappa^+]^\kappa \). Construct a sequence \( \langle p_\alpha; \alpha < \kappa^+ \rangle \) of conditions in \( P \) as follows: Put \( p_0 = 1 \). For limit \( \alpha \) get a lower bound of \( \langle p_\delta; \delta < \alpha \rangle \) using closedness of \( M \) and \( P \). For \( \alpha = \beta + 1 \) put \( X = \{(jf_\beta)(a); a \in [\lambda]<\omega, (jf_\beta)(a) \subseteq P \) is a maximal antichain\}. \( X \) is a set in \( M \) of cardinality \( \leq \lambda \), hence we can find \( p_{\beta+1} < p_\beta \) that meets all of those maximal antichains using closedness of \( P \) in \( M \).

By the chain condition the filter \( G \) generated by \( \langle p_\alpha; \alpha < \kappa^+ \rangle \) is \( P\)-generic over \( M \).

Let \( j : V \to M \) be \( \gamma \)-strong, i.e. \( \text{crit } (j) = \kappa, V_{\kappa+\gamma} \subseteq M, \gamma < j(\kappa) \). It is a
standard fact on extenders (see [Ka93]) that we can assume

$$M = \{(jf)(a); a \in [\lambda]^{<\omega}, f \in [\kappa]^{|\kappa|}V\},$$

where $\lambda = |V_{\kappa+\gamma}|^{+M} < j(\kappa)$.

Assume there is no Mahlo cardinal between $\kappa$ and $\lambda+1$. Let $P_{\kappa+1}$ be constructed from $j$, $jP_{\kappa+1} = P_{\kappa+1} \ast R_\text{o} \ast (jQ_\kappa)$, $G \ast H \ast P_{\kappa+1}$-generic over $V$. To construct $H_\text{o} \in V[G \ast H]$ $R_\text{o}$-generic over $M[G \ast H]$ consider an enumeration $\langle f_\alpha; \alpha < \kappa^+ \rangle$ of all functions in $V$ from $\kappa$ to $[P_\kappa]^\kappa$. Construct a descending chain $\langle p_\alpha; \alpha < \kappa^+ \rangle \subseteq R_\text{o}$ similarly as in the proof of lemma 5 so that $p_\alpha$ meets any maximal antichain $\subseteq R_\text{o}$ of the form $(jf_\alpha)(a)/G \ast H$ ($a \in [\lambda]^{<\omega}$). We only have to observe that $R_\text{o}$ is $\kappa$-closed in $V[G \ast H]$ and $\lambda$-closed in $M[G \ast H]$. The sequence $\langle p_\alpha; \alpha < \kappa^+ \rangle$ generates a filter $H_\text{o} \subseteq R$ generic over $M[G \ast H]$. Now $j : V \to M$ is lifted to $j^* : V[G] \to M[G^*]$ in $V[G \ast H]$, where $G^* = G \ast H \ast H_\text{o}$. The embedding $j^*$ is obviously again given by an $(\kappa, \lambda)$-extender.

To construct a $j^*Q_\kappa$-generic $M[G^*]$ filter $H^* \in V[G \ast H]$ consider an enumeration $\langle f_\alpha; \alpha < \kappa^+ \rangle$ of all functions from $\kappa$ into $[Q_\kappa]^\kappa$, each with cofinally many repetitions. We need $[H \upharpoonright \alpha]^j \in H^*$ for all $\alpha < \kappa^+$, hence construct a descending sequence $\langle p_\alpha; \alpha < \kappa^+ \rangle \subseteq j^*Q_\kappa$ so that

(i) $p_\alpha \in j^*(Q_\kappa \upharpoonright \alpha)$,

(ii) $p_\alpha \leq [H \upharpoonright \alpha]^j$,

(iii) $p_\alpha$ meets any maximal antichain $\subseteq j^*(Q_\kappa \upharpoonright \alpha)$ of the form $(j^*f_\alpha)(a)$ for an $a \in [\lambda]^{<\omega}$.

Since any maximal antichain in $j^*Q_\kappa$ is actually an antichain in $j^*(Q_\kappa \upharpoonright \alpha)$ for some $\alpha < \kappa^+$, the sequence generates a desired $H^* \in V[G \ast H]$ $j^*Q_\kappa$-generic over $M[G^*]$. Therefore $j^*$ is lifted to $j^{**} : V[G \ast H] \to M[G^* \ast H^*]$. Obviously $V[G \ast H] \cap M[G^* \ast H^*] \subseteq M[G^* \ast H^*]$ as $P_{\kappa+1}$ is $\kappa^+$-c.c. To prove that $j^{**}$ is $\gamma$-strong it is enough to show that $\mathcal{P}_{\gamma}^{V[G \ast H]}(\kappa^+) \subseteq M[G^* \ast H^*]$. For each $\delta < \gamma$ fix a bijection $\pi_\delta : \mathcal{P}_\delta(\kappa^+) \times P_{\kappa+1} \to \mathcal{P}_\delta(\kappa^+)$ that is in $M$ ($\mathcal{P}_0(\kappa^+) = \kappa^+$, $\mathcal{P}_{\delta+1} = \mathcal{P}(\mathcal{P}_\delta)$). We actually need $\langle \pi_\delta; \delta < \gamma \rangle \in M$. Then for each element $x$ of $\mathcal{P}_{\gamma}^{V[G \ast H]}(\kappa^+)$ use $\pi_\delta$’s to find a code in $\mathcal{P}_\gamma(\kappa^+) \subseteq M$ for its $P_{\kappa+1}$-name $\dot{x}$. Consequently the name $\dot{x}$ itself can be decoded in $M$, and so $x = i_{G \ast H}(\dot{x})$ is in $M[G \ast H] \subseteq M[G^* \ast H^*]$.

We say that $\kappa$ is strong if it is $\gamma$-strong for every $\gamma$. As in the case of a supercompact cardinal we can assume without loss of generality that there is no inaccessible cardinal above $\kappa$, and then use the same argument to find $P_{\kappa+1}$ that works for class
many γ’s preserving the strongness of κ. That concludes our proof of the main theorem.

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