Asymptotic Equivalence of Hadwiger’s Conjecture and its Odd Minor-Variant

Raphael Steiner *

September 7, 2021

Abstract

Hadwiger’s conjecture states that every $K_t$-minor free graph is $(t - 1)$-colorable. A qualitative strengthening of this conjecture raised by Gerards and Seymour, known as the Odd Hadwiger’s conjecture, states similarly that every graph with no odd $K_t$-minor is $(t - 1)$-colorable. For both conjectures, their asymptotic relaxations remain open, i.e., whether an upper bound on the chromatic number of the form $C t$ for some constant $C > 0$ exists.

We show that if every graph without a $K_t$-minor is $f(t)$-colorable, then every graph without an odd $K_t$-minor is $2f(t)$-colorable. Using this, the recent $O(t \log \log t)$-upper bound of Delcourt and Postle [1] for the chromatic number of $K_t$-minor free graphs directly carries over to the chromatic number of odd $K_t$-minor-free graphs. This (slightly) improves a previous bound of $O(t (\log \log t)^2)$ for this problem by Delcourt and Postle.

1 Introduction

Given a number $t \in \mathbb{N}$, a $K_t$-expansion is a graph consisting of vertex-disjoint trees $(T_s)_{s=1}^{t}$ and exactly one connecting edge between any pair of trees $T_s, T_{s'}$ for distinct $s, s' \in \{1, \ldots, t\}$. A graph is said to contain $K_t$ as a minor or to contain a $K_t$-minor if it admits a subgraph which is a $K_t$-expansion. Hadwiger’s conjecture, which may well be seen as one of the most central open problems in graph theory, states the following relation between minor containment and the chromatic number of graphs.

Conjecture 1 (Hadwiger 1943, [3]). If $G$ is a graph which does not contain $K_t$ as a minor, then $\chi(G) \leq t - 1$.

Hadwiger’s conjecture and many variations of it have been studied in past decades, a very good overview of the developments and partial results until about 2 years ago is given in the survey article [5] by Seymour. The best known asymptotic upper bound on the chromatic number of $K_t$-minor free graphs for a long time remained of magnitude $O(t \sqrt{\log t})$, as proved independently by Kostochka [13] and Thomason [20] in 1984. However, recently there has been progress. First, in 2019, Norine, Postle and Song [14] broke the $t \sqrt{\log t}$ barrier by proving an upper bound of the form $O(t (\log t)^{\beta})$ for any $\beta > \frac{1}{2}$. Subsequently, there have been several significant improvements of this bound and related results [15], [17], [18]. The following state of the art-bound was proved recently by Delcourt and Postle in [1].

*Institute of Theoretical Computer Science, ETH Zurich, Switzerland, raphaelmario.steiner@inf.ethz.ch. This work was supported by an ETH Postdoctoral Fellowship.
Theorem 1. The maximum chromatic number of $K_t$-minor free graphs is bounded from above by a function in $O(t \log \log t)$.

A strengthening of Hadwiger’s conjecture to so-called odd minors was conjectured by Gerards and Seymour in [1]. A $K_t$-expansion $H$ certified by a corresponding collection of vertex-disjoint trees $(T_i)_{i=1}^t$ is said to be odd, if there exists an assignment of two colors $\{1, 2\}$ to the vertices of $H$ in such a way that every edge contained in one of the trees $T_i$ with $i \in \{1, \ldots, t\}$ is bichromatic (i.e., has different colors at its endpoints), while every edge joining two distinct trees is monochromatic (i.e., has the same color at its endpoints).

Finally, we say that a graph contains $K_t$ as an odd minor or that it contains an odd $K_t$-minor if it contains a subgraph which is an odd $K_t$-expansion.

Conjecture 2 (Gerards and Seymour [1]). If $G$ is a graph which does not contain $K_t$ as an odd minor, then $\chi(G) \leq t - 1$.

The maximum chromatic number of odd $K_t$-minor free graphs is bounded from above by a function in $O(t \log \log t)$-colorable. A shorter proof for the same result was given by Kawarabayashi in [8].

Subsequently, an asymptotical improvement of this upper bound to $O(t(\log \log t)^\beta)$ for any $\beta > \frac{1}{2}$ was achieved by Norine and Song in [16]. This was improved further to $O(t(\log \log t)^6)$ by Postle in [19]. Very recently the exponent of the log log $t$ factor was further improved by Delcourt and Postle in [1], resulting in an $O(t(\log \log t)^2)$-bound. Many further results on odd $K_t$-minor free graphs are known, we refer to [6, 7, 9, 10, 11, 12] for some additional references.

The purpose of this note is to show that asymptotically, the maximum chromatic number of $K_t$-minor free graphs and the maximum chromatic number of odd $K_t$-minor free graphs differ at most by a multiplicative factor of 2.

Theorem 2. Let $t \in \mathbb{N}$ and let $f(t)$ be an integer such that every graph not containing $K_t$ as a minor is $f(t)$-colorable. Then every graph not containing $K_t$ as an odd minor is $2f(t)$-colorable.

Theorem 2 has a very simple proof, given in Section 2 below. It is useful in the sense that any progress made towards a better asymptotic upper bound on the chromatic number of $K_t$-minor free graphs carries over, without further work and only at the prize of a constant multiplicative factor, to odd $K_t$-minor free graphs. In particular, Theorem 1 together with Theorem 2 directly yields the following (slight) asymptotical improvement of the $O(t(\log \log t)^2)$-upper bound on the chromatic number of odd $K_t$-minor free graphs by Delcourt and Postle.

Corollary 3. The maximum chromatic number of odd $K_t$-minor free graphs is bounded from above by a function in $O(t \log \log t)$.

2 Proof of Theorem 2

The proof is based on the following lemma.

Lemma 4. Let $G$ be a graph. Then there exists $n \in \mathbb{N}$ and a partition of $V(G)$ into non-empty sets $X_1, \ldots, X_n$ such that the following hold:

- for every $1 \leq i \leq n$, the graph $G[X_i]$ is bipartite and connected,
- for every $1 \leq i < j \leq n$, either there are no edges in $G$ between $X_i$ and $X_j$, or there exist $u_1, u_2 \in X_i$ and $v \in X_j$ such that $u_1v, u_2v \in E(G)$ and $u_1$ and $u_2$ lie on different sides of the bipartition of $G[X_i]$. 

2
Proof. Let us define the partition \(X_1, X_2, \ldots \) of \(V(G)\) inductively as follows:

Suppose that for some integer \(i \geq 1\), all the sets \(X_k\) with \(1 \leq k < i\) have been defined already, and do not yet form a partition, i.e., \(\bigcup_{1 \leq k < i} X_k \neq V(G)\). We now choose \(X_i\) as an inclusion-wise maximal set among all subsets \(X \subseteq V(G) \setminus \bigcup_{1 \leq k < i} X_k\) which satisfy that \(G[X]\) is a bipartite and connected graph. Note that \(X_i \neq \emptyset\), since for every vertex \(x \in V(G) \setminus \bigcup_{1 \leq k < i} X_k\), the graph \(G[[x]]\) is bipartite and connected.

Since we are adding a non-empty set to our collection of pairwise disjoint subsets of \(V(G)\) at each step, the above procedure eventually yields a partition \(X_1, \ldots, X_n\) of \(V(G)\) for some \(n \in \mathbb{N}\). By definition, we have that \(G[X_i]\) is bipartite and connected for \(i = 1, \ldots, n\), and hence what remains to show is the second property of the partition stated in the lemma.

So let \(i, j\) be given such that \(1 \leq i < j \leq n\), and suppose that there exists at least one edge \(e \in E(G)\) between \(X_i\) and \(X_j\). Denote \(e = uv\) with \(u \in X_i\) and \(v \in X_j\). Let \(\{A; B\}\) be the unique bipartition of \(G[X_i]\). We claim that \(v\) must have a neighbor \(u_1 \in A\) and a neighbor \(u_2 \in B\), which then yields the statement claimed in the lemma. Indeed, suppose not, and suppose w.l.o.g. that \(v\) is not adjacent to any vertex in \(A\) (the case that \(v\) has no neighbor in \(B\) is of course symmetric). Then also the graph \(G[X_i \cup \{v\}]\) is bipartite and connected: It is connected since \(G[X_i]\) is connected and because of the edge \(uv\), and it is bipartite since \(\{A \cup \{v\}; B\}\) forms its unique bipartition. However, putting \(X := X_i \cup \{v\} \subseteq V(G) \setminus \bigcup_{1 \leq k < i} X_k\), this contradicts the definition of \(X_i\) as an inclusion-wise maximal subset of \(V(G) \setminus \bigcup_{1 \leq k < i} X_k\) inducing a bipartite and connected subgraph.

We can now easily deduce Theorem 2.

Proof of Theorem 2. Let \(t \in \mathbb{N}\) and suppose that \(f(t)\) is an integer such that every \(K_t\)-minor free graph is \(f(t)\)-colorable. Let \(G\) be any given graph without an odd \(K_t\)-minor, and let us prove that \(\chi(G) \leq 2f(2t)\).

We apply Lemma 4 to \(G\) and obtain a partition \(X_1, \ldots, X_n\) of \(V(G)\) with properties as stated in the lemma. Let \(H\) be defined as the graph with vertex-set \(\{1, \ldots, n\}\) and which has an edge between distinct vertices \(i\) and \(j\) if and only if there exists at least one edge in \(G\) between \(X_i\) and \(X_j\). It follows from the statement of the lemma that for every edge \(ij \in E(H)\) with \(i < j\), there exist \(u_1, u_2 \in X_i\) and \(v \in X_j\) such that \(u_1v, u_2v \in E(G)\) and \(u_1, u_2\) are on different sides of the bipartition of \(G[X_i]\).

We claim that \(\chi(G) \leq 2\chi(H)\). To see this, let \(c_H : \{1, \ldots, n\} \to \{1, \ldots, \chi(H)\}\) be a proper coloring of \(H\), and for every \(i \in \{1, \ldots, n\}\) let \(c_i : X_i \to \{1, 2\}\) be a proper coloring of the bipartite graph \(G[X_i]\). It now follows directly from the definition of \(H\) that the coloring \(c_G\) of \(G\) with color set \(\{\chi(H)\} \times \{1, 2\}\), defined by \(c_G(x) := (c_H(i), c_i(x))\) for every \(x \in X_i\) and \(i \in \{1, \ldots, n\}\), is a proper coloring of \(G\). Therefore, \(\chi(G) \leq 2\chi(H)\).

Next, we will show that \(\chi(H) \leq f(t)\) by proving that \(H\) does not contain \(K_t\) as a minor. Suppose towards a contradiction that \(H\) contains a subgraph which is a \(K_t\)-expansion, i.e., there exist vertex-disjoint trees \(\{T_s\}_{s=1}^t\) contained in \(H\) and for every pair \(\{s, s'\} \subseteq \{1, \ldots, t\}\) an edge \(e(s, s') \in E(H)\) with endpoints in \(T_s\) and \(T_{s'}\).

For every fixed \(s \in \{1, \ldots, t\}\), let us consider the subgraph \(G_s := G \bigcup_{e \in V(T_s)} X_e\) of \(G\). This is a connected graph because \(G[X_i]\) is connected for every \(i \in V(T_s)\), since \(T_s\) is connected, and since by definition of \(H\) for every edge \(ij \in E(T_s)\) there exists at least one connecting edge between \(X_i\) and \(X_j\) in \(G\). In particular, \(G_s\) contains a spanning tree \(T^G_s\) which has the property that \(T^G_s[X_i]\) forms a spanning tree of \(G[X_i]\), for every \(i \in V(T_s)\).

The trees \(\{T^G_s\}_{s=1}^t\) in \(G\) defined as above are pairwise vertex-disjoint. Let us denote by \(c : \bigcup_{s=1}^t V(T^G_s) \to \{1, 2\}\) a 2-color-assignment obtained by piecing together proper 2-colorings of the individual trees \(\{T^G_s\}_{s=1}^t\).
Claim. For every pair $\{s, s'\} \subseteq \{1, \ldots, t\}$, there exists an edge $f(s, s') \in E(G)$ with endpoints in $T^G_s$ and $T^G_{s'}$, such that $f(s, s')$ is monochromatic with respect to the coloring $c$.

Subproof. By assumption, there exists $e(s, s') \in E(H)$ which connects a vertex in $i \in V(T_s)$ to a vertex $j \in V(T_{s'})$. Possibly after relabelling assume w.l.o.g. $1 \leq i < j \leq n$. Then the second property of the partition $X_1, \ldots, X_n$ guaranteed by Lemma 3 yields the existence of vertices $u_1, u_2 \in X_i \subseteq V(T^G_s)$ and $v \in X_j \subseteq V(T^G_{s'})$ such that $f_1 := u_1v, f_2 := u_2v \in E(G)$, and such that $u_1$ and $u_2$ lie on different sides of the unique bipartition of $G[X_i]$. Since $u_1, u_2 \in X_i \subseteq V(T^G_s)$ and since by our choice of $T^G_s$ the graph $T^G_s[X_i]$ forms a spanning tree of $G[X_i]$, it follows that $u_1$ and $u_2$ also must be on different sides in the unique bipartition of $T^G_s[X_i]$. In particular, $c(u_1) \neq c(u_2)$, which implies that $c(u_r) = c(v)$ for some $r \in \{1, 2\}$. Now the edge $f_r \in E(G)$ connects the vertex $u_r$ in $T^G_s$ with the vertex $v$ in $T^G_{s'}$, and is monochromatic with respect to $c$. This proves the subclaim with $f(s, s') := f_r$.

It follows directly from the previous claim that the union of the vertex-disjoint trees $(T^G_s)_{s=1}^t$, joined by the edges $f(s, s')$ for every pair $\{s, s'\} \subseteq \{1, \ldots, t\}$, forms an odd $K_t$-expansion contained in $G$. This contradicts our initial assumption that $G$ does not contain $K_t$ as an odd minor. Hence, our initial assumption was wrong, and we have established that $H$ is $K_t$-minor free. Now it follows that $\chi(G) \leq 2\chi(H) \leq 2f(t)$, as required. This concludes the proof of the theorem.

References

[1] M. Delcourt and L. Postle. Reducing Linear Hadwiger’s Conjecture to Coloring Small Graphs. arXiv preprint, arXiv: 2108.01633, 2021.

[2] J. Geelen, B. Gerards, B. Reed, P. Seymour, A. Vetta. On the odd-minor variant of Hadwiger’s conjecture. Journal of Combinatorial Theory, Series B, 99(1), 20–29, 2009.

[3] H. Hadwiger. Über eine Klassifikation der Streckenkomplexe. Vierteljschr. Naturforsch. Ges. Zürich, 88, 133–143, 1943.

[4] T. Jensen and B. Toft. Graph coloring problems, Wiley, pp. 115, 1995.

[5] P. Seymour. Hadwiger’s conjecture. In Open Problems in mathematics, 417–437, Springer, 2016.

[6] D. Kang and S. Oum. Improper colourings of graphs with no odd clique minor. Combina-
torics, Probability and Computing, 28(5), 740–754, 2019.

[7] K. Kawarabayashi. A weakening of the odd Hadwiger’s conjecture. Combinatorics, Probabil-
ity and Computing, 17(6), 815–821, 2008.

[8] K. Kawarabayashi. Note on coloring graphs without odd $K_k$-minors. Journal of Combinatorial Theory, Series B, 99(4), 728–731, 2009.

[9] K. Kawarabayashi. The odd Hadwiger’s conjecture is “almost” decidable. arXiv preprint, arXiv:1508.04053, 2015.

[10] K. Kawarabayashi and B. Reed. Fractional coloring and the odd Hadwiger’s conjecture. European Journal of Combinatorics, 29(2), 411–417, 2008.
[11] K. Kawarabayashi, B. Reed and P. Wollan. The graph minor algorithm with parity conditions. In Proceedings IEEE 52nd Annual Symposium on Foundations of Computer Science, pp. 27–36, 2011.

[12] K. Kawarabayashi and Z. Song. Some remarks on the odd Hadwiger’s conjecture. Combinatorica, 27(4), 429–438, 2007.

[13] A. V. Kostochka. Lower bound on the Hadwiger number of graphs by their average degree. Combinatorica, 4, 307–316, 1984.

[14] S. Norine, L. Postle and Z. Song. Breaking the degeneracy barrier for coloring graphs with no $K_t$ minor. arXiv preprint, [arXiv:1910.09378], 2019.

[15] S. Norine and L. Postle. Connectivity and choosability of graphs with no $K_t$ minor. arXiv preprint, [arXiv:2004.10367], 2020.

[16] S. Norine and Z. Song. A new upper bound on the chromatic number of graphs with no odd $K_t$ minor. Combinatorica, 2021. [https://doi.org/10.1007/s00493-021-4390-3].

[17] L. Postle. Further progress towards Hadwiger’s conjecture. arXiv preprint, [arXiv:2006.11798], 2020.

[18] L. Postle. An even better density increment theorem and its application to Hadwiger’s conjecture. arXiv preprint, [arXiv:2006.14945], 2020.

[19] L. Postle. Further progress towards the list and odd versions of Hadwiger’s conjecture. arXiv preprint, [arXiv:2010.05999], 2020.

[20] A. Thomason. An extremal function for contractions of graphs. Mathematical Proceedings of the Cambridge Philosophical Society, 95, 261–265, 1984.