COACTIONS OF HOPF $C^*$-ALGEBRAS ON CUNTZ-PIMSNER
ALGEBRAS

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Abstract. Unifying two notions of an action and coaction of a locally com-
 pact group on a $C^*$-correspondence $(X, A)$ we introduce a notion of coaction
 $(\sigma, \delta)$ of a Hopf $C^*$-algebra $S$ on $(X, A)$, and show that $(\sigma, \delta)$ naturally induces
 a coaction $\zeta$ of $S$ on the associated Cuntz-Pimsner algebra $\mathcal{O}_X$ under an ap-
 propriate assumption that is automatically satisfied in a familiar situation. If
 $S$ is a reduced Hopf $C^*$-algebra arising from a multiplicative unitary, we con-
 struct the (reduced crossed product) correspondence $(X \rtimes _{\sigma} S, A \rtimes _{\delta} S)$ by the
 coaction $(\sigma, \delta)$ and prove that it has a representation on the reduced crossed
 product $\mathcal{O}_X \rtimes _{\zeta} S$. Moreover if this representation is covariant (a couple of
 equivalent conditions for this will be given), the crossed product $C^*$-algebra
 $\mathcal{O}_X \rtimes _{\zeta} S$ is shown to be isomorphic to the Cuntz-Pimsner algebra $\mathcal{O}_X \rtimes _{\sigma} S$.

1. Introduction

This paper was inspired by [7, 12] which show that an action or coactio
 of a locally compact group $G$ on a $C^*$-correspondence $(X, A)$ induces an actio
 or coaction of $G$ on the associated Cuntz-Pimsner algebra $\mathcal{O}_X$, and that the crossed
 product of $\mathcal{O}_X$ by this induced action or coaction can be realized as a Cuntz-Pimsner
 algebra. We generalize these results to the context of Hopf $C^*$-algebras [2], that
 is, we show that essentially the same results can be obtained if group actions or
 coactions are replaced by Hopf $C^*$-algebra coactions.

The Cuntz-Pimsner algebra $\mathcal{O}_X$ associated to a $C^*$-correspondence $(X, A)$ is the
 $C^*$-algebra generated by $X$ and $A$ through the universal covariant representation
 of $(X, A)$. The study on Cuntz-Pimsner algebras was initiated in [22] and the
 construction was extended in [14] to include in particular topological graph $C^*$-
 algebras.

Related to a study of group actions or coactions on directed graph $C^*$-algebras,
 there are results [17, Corollary 2.5] and [10, Theorem 2.4] which in the Cuntz-
 Pimsner algebra context can be described as follows. A function $\epsilon$ from the edges
 of a directed graph $E$ to a discrete group $G$ defines a coaction $(\sigma^\epsilon, \delta^\epsilon)$ of $G$ on
 the $C^*$-correspondence $(X(E), A)$ formed in [13], and also defines a skew product
 graph $E \times _\epsilon G$. It was then proved that the coaction $(\sigma^\epsilon, \delta^\epsilon)$ induces a coaction $\zeta^\epsilon$
 of $G$ on the graph $C^*$-algebra $\mathcal{O}_{X(E)}$ associated to $E$, and the crossed product by
 the coaction $\zeta^\epsilon$ is isomorphic to the skew product graph $C^*$-algebra, namely

$$\mathcal{O}_{X(E)} \times _{\zeta^\epsilon} G \cong \mathcal{O}_{X(E \times _\epsilon G)}.$$
These results have been extended in a number of ways. See [4, Remark 5.11], [7, Theorem 2.10], [9, Theorem 3.1], and [12, Theorem 4.4] for example.

Especially [7] and [12] generalize the results mentioned above on graph C*-algebras to Cuntz-Pimsner algebras. They start from an action or coaction of a locally compact group \( G \) on a C*-correspondence \( (X,A) \). (For the definition of actions and coactions of \( G \) on \( (X,A) \) we refer to [5].) In [7], it was shown that if \((\gamma,\alpha)\) is an action of an amenable group \( G \) on \( (X,A) \), then \((\gamma,\alpha)\) induces an action \(\beta\) of \( G \) on the Cuntz-Pimsner algebra \( \mathcal{O}_X \), and the crossed product by this action \(\beta\) is again a Cuntz-Pimsner algebra:

\[
\mathcal{O}_X \rtimes_{\beta} G \cong \mathcal{O}_{X \rtimes_{\alpha} G},
\]

where \( X \rtimes_{\alpha} G \) is the C*-correspondence over \( A \rtimes_{\alpha} G \) arising from the action \((\gamma,\alpha)\) ([5, Proposition 3.2]). The study in [12] also shows that a nondegenerate coaction \((\sigma,\delta)\) of \( G \) on \( (X,A) \) satisfying a certain invariance condition induces a coaction \(\zeta\) of \( G \) on \( \mathcal{O}_X \), and that under the hypothesis of Cuntz-Pimsner covariance, the crossed product by this coaction \(\zeta\) is again a Cuntz-Pimsner algebra:

\[
\mathcal{O}_X \rtimes_{\zeta} G \cong \mathcal{O}_{X \rtimes_{\sigma} G},
\]

where \( X \rtimes_{\sigma} G \) is the C*-correspondence over \( A \rtimes_{\sigma} G \) arising from the coaction \((\sigma,\delta)\) ([5, Proposition 3.9]).

In this paper we show that the passage from a group action or coaction on \( (X,A) \) to an action or coaction on \( \mathcal{O}_X \) can be extended nicely to the context of a Hopf C*-algebra coaction (Theorem 3.7). We prove an isomorphism analogous to (1.1) and (1.2) (Theorem 5.10), and also improve (or extend) the main results of [7, 12] (Remark 5.6 and Corollary 5.11). To this end we shall unify the separate notions of group actions and nondegenerate group coactions on \( (X,A) \) into a single notion of Hopf C*-algebra coactions on \( (X,A) \) (Definition 3.1). We also construct the reduced crossed product correspondences which are the C*-correspondences arising from Hopf C*-algebra coactions (Theorem 4.6).

Hopf C*-algebras arising from multiplicative unitaries are dealt with in [2] generalizing Kac algebras [6] and compact quantum groups [25, 26], and the study plays an important role in the development of locally compact quantum groups [19]. The notion of reduced crossed products by Hopf C*-algebra coactions is also established in [2] and the Takesaki-Takai duality is generalized. Basically, locally compact groups can be regarded as a special class of Hopf C*-algebras defined by multiplicative unitaries, and group actions and coactions are generalized as Hopf C*-algebra coactions. It is thus natural to try to extend the procedure of obtaining group actions or coactions on \( \mathcal{O}_X \) out of group actions or coactions on \( (X,A) \) and also the isomorphisms (1.1) and (1.2) to the context of Hopf C*-algebra coactions.

We now describe the organization of the paper and our results. In Section 2, we review from [5, Chapter 1] and [4, Appendix A] some results on multiplier correspondences that will be used throughout the paper. We also recall from [14, 2] basic definitions and facts on Cuntz-Pimsner algebras and reduced crossed products by Hopf C*-algebra coactions on C*-algebras. One thing we should note here is that in [5] Hilbert \( A \)-\( B \) bimodules were treated, whereas we are concerned only with C*-correspondences \((X,A)\) of right Hilbert \( A \)-modules \( X \) equipped with nondegenerate left actions \( \varphi_A \) by \( A \).
In Section 3, we introduce in Definition 3.1 a notion of coaction \((\sigma, \delta)\) of a Hopf \(C^*\)-algebra \(S\) on a \(C^*\)-correspondence \((X, A)\) generalizing both an action and nondegenerate coaction of a locally compact group on \((X, A)\). Our definition actually agrees with the notion of action of a locally compact group \(G\) on \((X, A)\) when \(S\) is the commutative Hopf \(C^*\)-algebra \(C_0(G)\), which will be shown in Theorem A.6, and we believe that this justifies our definition. We prove in Theorem 3.7 that if \((\sigma, \delta)\) is a coaction of \(S\) on \((X, A)\) and the ideal \(J_X\) is weakly \(\delta\)-invariant (Definition 3.5), then \((\sigma, \delta)\) induces a coaction \(\zeta\) of \(S\) on the associated Cuntz-Pimsner algebra \(O_X\). As an application, we construct an example of coactions on crossed products by \(\mathbb{Z}\) arising from cocycles. In our upcoming paper [16] more interesting construction of coactions of compact quantum groups [26] on directed graph \(C^*\)-algebras will be considered.

Section 4 is devoted to the construction of the reduced crossed product correspondence \((X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})\) for a coaction \((\sigma, \delta)\) of a reduced Hopf \(C^*\)-algebra \(S\) defined by a multiplicative unitary. Our construction reduces to the crossed product correspondence in the sense of [5] if the Hopf \(C^*\)-algebra coaction under consideration comes from a group action or nondegenerate group coaction on \((X, A)\) (Corollary B.7 and Remark 4.9). The point of our construction of \((X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})\) is that the left action \(\varphi_{A \rtimes_\delta \widehat{S}}: A \rtimes_\delta \widehat{S} \to \mathcal{L}(X \rtimes_\sigma \widehat{S})\) can be defined without requiring any universal property of the crossed product \(A \rtimes_\delta \widehat{S}\).

In Section 5, we prove in our situation an isomorphism corresponding to (1.1) and (1.2). Along the way we give the answer to the question raised in [12, Remark 4.5], that is, we prove that Theorem 4.4 of [12] still holds without the hypothesis of the Cuntz-Pimsner covariance of \((j_X, j_A)\) (see Remark 5.6). The \(C^*\)-correspondence \((X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})\) has a canonical representation on the reduced crossed product \(O_X \rtimes_\zeta \widehat{S}\) by the induced coaction \(\zeta\) (Proposition 5.4). We provide in Theorem 5.7 a couple of equivalent conditions that this canonical representation is covariant, which is automatic particularly when \(J_X = A\) or the left action \(\varphi_A\) is injective. Under this condition, we prove our main theorem (Theorem 5.10):

\[O_X \rtimes_\zeta \widehat{S} \cong O_{X \rtimes_\sigma \widehat{S}}.\]

Applying this to coactions of the commutative Hopf \(C^*\)-algebra we can extend Theorem 2.10 of [7] (see Corollary 5.11).

Finally, we provide two appendices which contain, in addition to a justification on our definition of coactions and construction of \((X \rtimes_\sigma \widehat{S}, A \rtimes_\delta \widehat{S})\), some related results: Theorem A.4 generalizes [1, Corollary 3.4], and Proposition B.3 gives a \(C^*\)-correspondence analogue of the well-known fact that \(\mathcal{L}_A(A \otimes \mathcal{H}) = M(A \otimes \mathcal{K}(\mathcal{H}))\) for a \(C^*\)-algebra \(A\) and a Hilbert space \(\mathcal{H}\).

2. Preliminaries

In this section, we review some definitions and properties related to multiplier correspondences, Cuntz-Pimsner algebras, and reduced crossed products by Hopf \(C^*\)-algebra coactions. Our references include [5, 4, 14, 2]. We also fix some notations and provide several lemmas, corollaries, and remarks that will be used in the subsequent sections.
2.1. $C^*$-correspondences. Let $A$ be a $C^*$-algebra. For two (right) Hilbert $A$-modules $X$ and $Y$, we denote by $\mathcal{L}(X,Y) = \mathcal{L}_A(X,Y)$ the Banach space of all adjointable operators from $X$ to $Y$, and by $\mathcal{K}(X,Y) = \mathcal{K}_A(X,Y)$ the closed subspace of $\mathcal{L}(X,Y)$ generated by the operators $\theta_{\xi,\eta}$:

$$
\theta_{\xi,\eta}(\eta') = \xi \cdot \langle \eta, \eta' \rangle_A \quad (\xi \in Y, \eta, \eta' \in X).
$$

We simply write $\mathcal{L}(X)$ and $\mathcal{K}(X)$ when $X = Y$; in this case, $\mathcal{L}(X)$ becomes a maximal unital $C^*$-algebra containing $\mathcal{K}(X)$ as an essential ideal.

A $C^*$-correspondence over a $C^*$-algebra $A$ is a Hilbert $A$-module $X$ equipped with a homomorphism $\varphi_A : A \to \mathcal{L}(X)$, called the left action. We use the notation $(X,A)$ of [12] to refer to a $C^*$-correspondence $X$ over $A$. We say that $(X,A)$ is nondegenerate if $\varphi_A$ is nondegenerate, that is, $\varphi_A(A)X = X$.

Every $C^*$-algebra $A$ has the natural structure of a $C^*$-correspondence over itself, called the identity correspondence (p. 368 of [14]). More generally, an automorphism $\varphi \in \text{Aut}(A)$ defines a $C^*$-correspondence as given in [22, Examples (3)] which we call a $\varphi$-identity correspondence and denote by $A(\varphi)$.

2.2. Multiplier correspondences. Throughout the paper, we restrict ourselves to nondegenerate $C^*$-correspondences, which in particular allows us to consider their multiplier correspondences, a generalization of multiplier $C^*$-algebras.

Let $(X,A)$ be a $C^*$-correspondence, and let $M(X) := \mathcal{L}(A,X)$. The multiplier correspondence of $X$ is the $C^*$-correspondence $M(X)$ over the multiplier algebra $M(A)$ with the Hilbert $M(A)$-module operations

\begin{equation}
(2.1) \quad m \cdot a := ma, \quad \langle m, n \rangle_{M(A)} := m^*n
\end{equation}

and the left action

\begin{equation}
(2.2) \quad \varphi_{M(A)}(a)m := \overline{\varphi(a)m}
\end{equation}

for $m, n \in M(X)$ and $a \in M(A)$, where $\overline{\varphi}$ is the strict extension of the nondegenerate homomorphism $\varphi_A$ and $ma, m^*n, \varphi_{M(A)}(a)m$ mean the compositions $m \circ a, m^* \circ n, \varphi_{M(A)}(a) \circ m$, respectively. The identification of $X$ with $K(A,X)$, in which each $\xi \in X$ is regarded as the operator $A \ni a \mapsto \xi \cdot a \in X$, gives an embedding of $X$ into $M(X)$:

$$
X \cong K(A,X) \subseteq M(X).
$$

We will always regard $X$ as a subspace of $M(X)$ through this embedding. Note that $K(M(X)) \subseteq M(K(X))$ nondegenerately.

The strict topology on $M(X)$ is the locally convex topology such that a net $\{m_i\}$ in $M(X)$ converges strictly to 0 if and only if for $T \in K(X)$ and $a \in A$, the nets $\{Tm_i\}$ and $\{m_i \cdot a\}$ both converge in norm to 0. It can be shown that $M(X)$ is the strict completion of $X$. Note that for a $\varphi$-identity correspondence $A(\varphi)$, the multiplier correspondence $M(A(\varphi))$ coincides with the $\overline{\varphi}$-identity correspondence $M(A(\overline{\varphi}))$ and the strict topology on $M(A(\varphi))$ is the usual strict topology on the multiplier algebra $M(A)$.

Let $(X,A)$ and $(Y,B)$ be $C^*$-correspondences. A pair

$$(\psi, \pi) : (X,A) \to (M(Y), M(B))$$

of a linear map $\psi : X \to M(Y)$ and a homomorphism $\pi : A \to M(B)$ is called a correspondence homomorphism if
For a correspondence homomorphism. Moreover, every nondegenerate correspondence homomorphism \( \psi, \pi \) : \((X, A) \rightarrow (B, B)\) into an identity correspondence \((B, B)\) is called a representation of \((X, A)\) on \(B\) and denoted simply by \((\psi, \pi) : (X, A) \rightarrow B\).

A correspondence homomorphism \( \psi, \pi : (X, A) \rightarrow (M(Y), M(B)) \) is said to be nondegenerate if

\[
\psi(X) \cdot B = Y \quad \text{and} \quad \pi(A) \overline{B} = B.
\]

In this case, \((\psi, \pi)\) extends uniquely to a strictly continuous correspondence homomorphism

\[
(\overline{\psi}, \overline{\pi}) : (M(X), M(A)) \rightarrow (M(Y), M(B))
\]

([5, Theorem 1.30]). Note that if \((\psi, \pi)\) is injective, then so is \((\overline{\psi}, \overline{\pi})\).

The next lemma will be used in Example 3.10 to construct a coaction of a Hopf \(C^*\)-algebra on a \(\varphi\)-identity correspondence.

**Lemma 2.1.** Let \( \varphi_A \in \text{Aut}(A) \) and \( \varphi_B \in \text{Aut}(B) \) be automorphisms on \(C^*\)-algebras \(A\) and \(B\), and \( \pi : A \rightarrow M(B) \) be a nondegenerate homomorphism. Let \( v \in M(B) \) be a unitary such that

\[
v \pi(\varphi_A(a)) = \overline{\varphi_B}(\pi(a)) v \quad (a \in A).
\]

Define

\[
\psi(a) := v \pi(a) \quad (a \in A).
\]

Then \((\psi, \pi) : (A(\varphi_A), A) \rightarrow (M(B(\varphi_B)), M(B))\) is a nondegenerate correspondence homomorphism. Moreover, every nondegenerate correspondence homomorphism from \((A(\varphi_A), A)\) into \((M(B(\varphi_B)), M(B))\) is of this form.

**Proof.** For \(a, a' \in A\),

\[
\psi(\varphi_A(a)a') = v \pi(\varphi_A(a)a') = v \pi(\varphi_A(a)) \pi(a') = \overline{\varphi_B}(\pi(a)) v \pi(a') = \overline{\varphi_B}(\pi(a)) \psi(a')
\]

and \(\langle \psi(a), \psi(a') \rangle_{M(B)} = \pi(a)^* v^* v \pi(a') = \pi((a, a')_A)\). Hence \((\psi, \pi)\) is a correspondence homomorphism, and obviously nondegenerate.

To see the converse, suppose that a nondegenerate correspondence homomorphism \((\psi, \pi)\) is given. Consider its strict extension \((\overline{\psi}, \overline{\pi})\). Let \(v = \overline{\psi}(1_{M(A)})\). Since \((\overline{\psi}, \overline{\pi})\) is a correspondence homomorphism, we have

\[
v^* v = \langle v, v \rangle_{M(B)} = \overline{\psi}(1_{M(A)}, 1_{M(A)}) = 1_{M(B)}.
\]

We also have

\[
vv^* (\psi(a) b) = \overline{\psi}(1_{M(A)}) (\overline{\psi}(1_{M(A)}), \psi(a) b)_{M(B)} = \overline{\psi}(1_{M(A)})(\pi(a) b = \overline{\psi}(a) b)
\]

for \(a \in A\) and \(b \in B\) so that \(vv^* = 1_{M(B)}\). Hence \(v\) is a unitary in \(M(B)\). Finally,

\[
v \pi(\varphi_A(a)) = \psi(\varphi_A(a)) = \psi(\varphi_A(1_{M(A)})) = \overline{\varphi_B}(\pi(a)) v.
\]

This completes the proof. \(\square\)
A correspondence homomorphism \((\psi, \pi) : (X, A) \to (M(Y), M(B))\) determines a (unique) homomorphism \(\psi^{(1)} : \mathcal{K}(X) \to \mathcal{K}(M(Y)) \subseteq M(\mathcal{K}(Y))\) such that
\[
\psi^{(1)}(\theta_{\xi, \eta}) = \psi(\xi)\psi(\eta)^* \quad (\xi, \eta \in X)
\]
(see for example [15, Definition 2.4] and the comment below it). If \((\psi, \pi)\) is nondegenerate, then so is \(\psi^{(1)}\); it is straightforward to verify that
\[
\psi(T\xi) = \overline{\psi^{(1)}(T)}\psi(\xi), \quad \overline{\psi^{(1)}(mn^*)} = \overline{\psi(m)}\overline{\psi(n)}^*
\]
for \(T \in \mathcal{L}(X), \xi \in X,\) and \(m, n \in M(X)\). The first relation of (2.3) shows that \(\overline{\psi^{(1)}}\) is injective whenever \(\psi\) is injective.

2.3. Tensor product correspondences. In this paper, the tensor products of Hilbert modules always mean the exterior ones ([20, pp. 34–35]). The tensor products of \(C^\ast\)-algebras are the minimal ones.

Let \((X_1, A_1)\) and \((X_2, A_2)\) be \(C^\ast\)-correspondences. We will identify
\[
\mathcal{K}(X_1 \otimes X_2) = \mathcal{K}(X_1) \otimes \mathcal{K}(X_2)
\]
via the identification \(\theta_{\xi_1, \eta_1} \otimes \eta_2 = \theta_{\xi_1, \eta_1} \otimes \theta_{\xi_2, \eta_2}\); hence \(M(\mathcal{K}(X_1 \otimes X_2)) = M(\mathcal{K}(X_1) \otimes \mathcal{K}(X_2))\). Equipped with the left action
\[
\varphi_{A_1 \otimes A_2} = \varphi_{A_1} \otimes \varphi_{A_2},
\]
the tensor product \(X_1 \otimes X_2\) becomes a \(C^\ast\)-correspondence over \(A_1 \otimes A_2\), called the tensor product correspondence.

Let \((\psi_i, \pi_i) : (X_i, A_i) \to (M(Y_i), M(B_i))\) \((i = 1, 2)\) be two correspondence homomorphisms. Then there exists a (unique) correspondence homomorphism
\[
(\psi_1 \otimes \psi_2, \pi_1 \otimes \pi_2) : (X_1 \otimes X_2, A_1 \otimes A_2) \to (M(Y_1 \otimes Y_2), M(B_1 \otimes B_2))
\]
such that \((\psi_1 \otimes \psi_2)(\xi_1 \otimes \xi_2) = \psi_1(\xi_1) \otimes \psi_2(\xi_2)\). If \((\psi_i, \pi_i)\)'s are both nondegenerate, then \((\psi_1 \otimes \psi_2, \pi_1 \otimes \pi_2)\) is also nondegenerate ([5, Proposition 1.38]).

Remark 2.2. It can be easily checked that
\[
\overline{(\psi_1 \otimes \psi_2)^{(1)}} = \overline{\psi_1^{(1)}} \otimes \overline{\psi_2^{(1)}}
\]
for two nondegenerate correspondence homomorphisms \((\psi_1, \pi_1)\) and \((\psi_2, \pi_2)\).

2.4. Cuntz-Pimsner algebras. For a \(C^\ast\)-correspondence \((X, A)\), define
\[
J_X := \varphi_A^{-1}(\mathcal{K}(X)) \cap \{a \in A : ab = 0 \text{ for } b \in \ker \varphi_A\}.
\]
The ideal \(J_X\) is characterized as the largest ideal of \(A\) mapped injectively into \(\mathcal{K}(X)\) by \(\varphi_A\).

A representation \((\psi, \pi)\) of \((X, A)\) is said to be covariant if
\[
\psi^{(1)}(\varphi_A(a)) = \pi(a) \quad (a \in J_X)
\]
([14, Definition 3.4]). We denote by \((k_X, k_A)\) the universal covariant representation of \((X, A)\) which can be shown to be injective ([14, Proposition 4.9]). The Cuntz-Pimsner algebra \(\mathcal{O}_X\) is the \(C^\ast\)-algebra generated by \(k_X(X)\) and \(k_A(A)\). Note that the embedding \(k_A : A \to \mathcal{O}_X\) is nondegenerate by our standing assumption that \((X, A)\) is nondegenerate. From the universality of \((k_X, k_A)\), if \((\psi, \pi)\) is a covariant representation of \((X, A)\) on \(B\), there exists a unique homomorphism \(\psi \times \pi : \mathcal{O}_X \to B\) called the integrated form of \((\psi, \pi)\) such that
\[
\psi = (\psi \times \pi) \circ k_X \quad \text{and} \quad \pi = (\psi \times \pi) \circ k_A.
\]
A representation \((\psi, \pi)\) of \((X, A)\) is said to admit a gauge action if there exists an action \(\beta\) of the unit circle \(\mathbb{T}\) on the \(C^*\)-subalgebra generated by \(\psi(X)\) and \(\pi(A)\) such that \(\beta_z(\psi(\xi)) = z\psi(\xi)\) and \(\beta_z(\pi(a)) = \pi(a)\) for \(z \in \mathbb{T}, \xi \in X,\) and \(a \in A\). The universal covariant representation \((k_X, k_A)\) admits a gauge action. The gauge invariant uniqueness theorem [14, Theorem 6.4] asserts that an injective covariant representation \((\psi, \pi)\) admits a gauge action if and only if \(\psi \times \pi\) is injective.

2.5. \(C\)-multiplier correspondences. Let \((X, A)\) be a \(C^*\)-correspondence, \(C\) be a \(C^*\)-algebra, and \(\kappa: C \to M(A)\) be a nondegenerate homomorphism. The \(C\)-multiplier correspondence \(M_C(X)\) of \(X\) and the \(C\)-multiplier algebra \(M_C(A)\) of \(A\) are defined by

\[
M_C(X) := \{ m \in M(X) : \varphi_{M(A)}(\kappa(C))m \cup m \cdot \kappa(C) \subseteq X \},
\]

\[
M_C(A) := \{ a \in M(A) : \kappa(C)a \cup a \cdot \kappa(C) \subseteq A \}.
\]

Under the restriction of the operations (2.1) and (2.2), \((M_C(X), M_C(A))\) becomes a \(C^*\)-correspondence.

**Notations 2.3.** By \(M_A(X)\) we mean the \(A\)-multiplier correspondence determined by \(\kappa = \text{id}_A:\)

\[
M_A(X) = \{ m \in M(X) : \varphi_A(A)m \in X \}.
\]

We also mean by \(M_A(K(X))\) the \(A\)-multiplier algebra determined by the left action \(\varphi_A\), that is,

\[
M_A(K(X)) = \{ m \in M(K(X)) : \varphi_A(A)m \cup m\varphi_A(A) \subseteq K(X) \}.
\]

Note that every \(C\)-multiplier correspondence \(M_C(X)\) is contained in \(M_A(X)\), and that \(K(M_A(X)) \subseteq K(M_C(X))\) ([4, Lemma A.9.(3)]).

The \(C\)-strict topology on \(M_C(X)\) is the locally convex topology whose neighborhood system at 0 is generated by the family \(\{ m : ||\varphi_{M(A)}(\kappa(c))m|| \leq \epsilon \} \) and \(\{ m : ||m \cdot \kappa(c)|| \leq \epsilon \} \) (\(c \in C, \epsilon > 0\)). The \(C\)-strict topology is stronger than the relative strict topology on \(M_C(X)\), and \(M_C(X)\) is the \(C\)-strict completion of \(X\). Likewise, the \(C\)-strict topology on \(M_C(A)\) is the locally convex topology defined by the family of seminorms \(||\kappa(c)\cdot || + || \kappa(c)|| \) (\(c \in C\)).

**Remark 2.4.** Let \((X, A)\) be a \(C^*\)-correspondence and \(M_C(X)\) be the \(C_i\) multiplier correspondence determined by a nondegenerate homomorphism \(\kappa_i: C_i \to M(A)\) \((i = 1, 2)\). It is clear that if \(\kappa_1(C_1)\) is nondegenerately contained in \(M(\kappa_2(C_2))\) \((\subseteq M(A))\), then \(M_{C_1}(X) \subseteq M_{C_2}(X)\) and the \(C_i\)-strict topology on \(M_{C_i}(X)\) is stronger than the relative \(C_2\)-strict topology. In particular, \(M_C(X) \subseteq M_A(X)\) and the \(C\)-strict topology is stronger than the relative \(A\)-strict topology.

For a not necessarily nondegenerate correspondence homomorphism we still have an extension theorem [4, Proposition A.11] which states as follows. Let \((\psi, \pi): (X, A) \to (M_D(Y), M_D(B))\) be a correspondence homomorphism, where \((M_D(Y), M_D(B))\) is a \(D\)-multiplier correspondence determined by a nondegenerate homomorphism \(\kappa_D: D \to M(B)\). Assume that \(\kappa_C: C \to M(A)\) and \(\lambda: C \to M(\kappa_D(D))\) \((\subseteq M(B))\) are nondegenerate homomorphisms such that \(\pi(\kappa_C(c)a) = \lambda(c)\pi(a)\) for \(c \in C\) and \(a \in A\). Then \((\psi, \pi)\) extends uniquely to a \(C\)-strict to \(D\)-strictly continuous correspondence homomorphism

\[
(\overline{\psi}, \overline{\pi}): (M_C(X), M_C(A)) \to (M_D(Y), M_D(B)),
\]

where \((M_C(X), M_C(A))\) is the \(C\)-multiplier correspondence determined by \(\kappa_C\).
Remarks 2.5. (1) If \((\psi, \pi)\) is nondegenerate, then every \(C\)-strict to \(D\)-strictly continuous extension of \((\psi, \pi)\) coincides with the restriction of its usual strict extension.

(2) Suppose that \(\overline{\psi}_i : M_{C_i}(X) \rightarrow M_{D_i}(Y)\) are \(C_i\)-strict to \(D_i\)-strictly continuous extensions \((i = 1, 2)\). If \(M_{C_1}(X) \subseteq M_{C_2}(X)\) and \(M_{D_1}(Y) \subseteq M_{D_2}(Y)\) and if the \(C_1\)-strict (\(D_1\)-strict, respectively) topology is stronger than the relative \(C_2\)-strict (\(D_2\)-strict, respectively) topology, then \(\overline{\psi}_1 = \overline{\psi}_2|_{M_{C_1}(X)}\).

We frequently need the following special form of [4, Proposition A.11].

**Theorem 2.6** ([4, Corollary A.14]). Let \((\psi, \pi) : (X, A) \rightarrow B\) be a representation such that \(\pi(A)\overline{B} = B\). Then

(i) \((\psi, \pi)\) extends uniquely to an \(A\)-strictly continuous correspondence homomorphism

\[ (\overline{\psi}, \overline{\pi}) : (M_{A}(X), M(A)) \rightarrow M_{A}(B), \]

where \(M_{A}(B)\) is the \(A\)-multiplier algebra determined by \(\pi\).

(ii) \(\psi(1) : \mathcal{K}(X) \rightarrow B\) extends uniquely to an \(A\)-strictly continuous homomorphism \(\psi(1) : M_{A}(\mathcal{K}(X)) \rightarrow M_{A}(B)\); moreover, \(\psi(1) = \overline{\psi}^{(1)}|_{\mathcal{K}(M_{A}(X))}\), that is, \(\psi^{(1)}(mn^*) = \overline{\psi}(m)\overline{\psi}(n)^*\) for \(m, n \in M_{A}(X)\).

**Notations 2.7.** Let \((X, A)\) be a \(C^*\)-correspondence and \(C\) be a \(C^*\)-algebra. Consider the representation \((k_X \otimes \text{id}_{C}, k_{AX} \otimes \text{id}_{C}) : (X \otimes C, A \otimes C) \rightarrow \mathcal{O}_{X} \otimes C\). Since \(k_X \otimes \text{id}_{C}\) is nondegenerate, the representation extends to an \((A \otimes C)\)-strict extension by Theorem 2.6.(i). Throughout the paper, we mean by \(k_X \otimes \text{id}_{C}\) the \((A \otimes C)\)-strict extension

\[ (2.4) \quad k_X \otimes \text{id}_{C} : M_{A \otimes C}(X \otimes C) \rightarrow M_{A \otimes C}(\mathcal{O}_{X} \otimes C) \]

and by \(M_{A \otimes C}(\mathcal{O}_{X} \otimes C)\) the \((A \otimes C)\)-multiplier algebra determined by \(k_{AX} \otimes \text{id}_{C}\). On the other hand, the \(C\)-multiplier correspondence \((M_{C}(X \otimes C), M_{C}(A \otimes C))\) is the one determined by the left action \(\varphi_{M(A \otimes C)}\) composed with the embedding \(C \hookrightarrow M(A \otimes C)\) onto the last factor.

For an ideal \(I\) of a \(C^*\)-algebra \(B\), let

\[ M(B; I) := \{ m \in M(B) : mB \cup Bm \subseteq I \}. \]

By [11, Lemma 2.4.(i)], \(M(B; I)\) is the strict closure of \(I\) in \(M(B)\).

**Lemma 2.8.** Let \((X, A)\) be a \(C^*\)-correspondence. Then

\[ J_{M_{A}(X)} \subseteq M(A; J_{X}). \]

**Proof.** We need to show that \(AJ_{M_{A}(X)} \subseteq J_{X}\). By definition, we have

\[ \varphi_{A}(AJ_{M_{A}(X)}) \subseteq \varphi_{A}(A)\mathcal{K}(M_{A}(X)) \subseteq \varphi_{A}(A)\mathcal{K}(X) \subseteq \mathcal{K}(X). \]

We also have

\[ AJ_{M_{A}(X)} \ker \varphi_{A} \subseteq J_{M_{A}(X)} \ker \varphi_{M(A)} = 0. \]

Consequently, \(AJ_{M_{A}(X)} \subseteq J_{X}\). \(\square\)

The next lemma, contained in the proof of [12, Lemma 2.5], will be useful in proving Theorem 3.7, Proposition 5.5, and Theorem 5.7.
Lemma 2.9. Let \((X, A)\) be a \(C^*\)-correspondence and \(C\) be a \(C^*\)-algebra. Then
\[
(2.5) \quad (k_X \otimes \text{id}_C)^{(1)} \circ \varphi_{M(A \otimes C)} = k_A \otimes \text{id}_C
\]
holds on \(M(A \otimes C; J_X \otimes C)\), that is, the diagram
\[
\begin{array}{ccc}
M_A \otimes C (K(X \otimes C)) & \xrightarrow{\varphi_{M(A \otimes C)}} & M_A \otimes C (O_X \otimes C) \\
\downarrow & & \downarrow \\
M(A \otimes C; J_X \otimes C) & \xrightarrow{k_A \otimes \text{id}_C} & M_A \otimes C (O_X \otimes C)
\end{array}
\]
commutes.

Proof. By definition, the vertical map makes sense and is \((A \otimes C)\)-strictly continuous. Theorem 2.6.(ii) also says that the extension \((k_X \otimes \text{id}_C)^{(1)}\) indicated by the lower right arrow is \((A \otimes C)\)-strictly continuous since \(k_A \otimes \text{id}_C\) is nondegenerate. Hence the composition on the left of (2.5) is well-defined on \(M(A \otimes C; J_X \otimes C)\) and \((A \otimes C)\)-strictly continuous. On the other hand, the horizontal map is the restriction of the usual strict extension \(k_A \otimes \text{id}_C\) and hence \((A \otimes C)\)-strictly continuous. Since (2.5) is valid on \(J_X \otimes C\), the conclusion now follows by \((A \otimes C)\)-strict continuity and the fact that \(J_X \otimes C\) is \((A \otimes C)\)-strictly dense in \(M(A \otimes C; J_X \otimes C)\).  

Recall from [3, Definition 12.4.3] the following terminology. Let \(A\) and \(C\) be \(C^*\)-algebras and \(J\) be a closed subspace of \(A\). The triple \((J, A, C)\) is said to satisfy the slice map property if the space
\[
F(J, A, C) = \{x \in A \otimes C : (\text{id} \otimes \omega)(x) \in J \text{ for } \omega \in C^*\}
\]
equals \(J \otimes C\), the norm closure of the algebraic tensor product \(J \otimes C\) in \(A \otimes C\).

Remarks 2.10. (1) If \(J\) is an ideal of \(A\), then \((J, A, C)\) satisfies the slice map property if and only if the sequence
\[
0 \longrightarrow J \otimes C \longrightarrow A \otimes C \longrightarrow (A/J) \otimes C \longrightarrow 0
\]
is exact; this is the case if \(A\) is locally reflexive or \(C\) is exact (see below [3, Definition 12.4.3]).

(2) Let \(\mathcal{H}\) be a Hilbert space. If \(C\) is a nondegenerate subalgebra of \(\mathcal{L}(\mathcal{H})\), then \(F(J, A, C)\) equals the norm closure of the following space
\[
\{x \in A \otimes C : (\text{id} \otimes \omega)(x) \in J \text{ for } \omega \in \mathcal{L}(\mathcal{H})_*\}.
\]

Corollary 2.11. Let \((X, A)\) be a \(C^*\)-correspondence and \(C\) be a \(C^*\)-algebra. Suppose that \((J_X, A, C)\) satisfies the slice map property. Then
\[
J_{X \otimes C} = J_X \otimes C.
\]
Furthermore,
\[
J_{M_A \otimes C (X \otimes C)} \subseteq M(A \otimes C; J_X \otimes C)
\]
and the injective representation
\[
(k_X \otimes \text{id}_C, k_A \otimes \text{id}_C) : (M_A \otimes C (X \otimes C), M(A \otimes C)) \to M_A \otimes C (O_X \otimes C)
\]
is covariant.
Proof. We always have \( J_X \otimes C \supseteq J_X \otimes C \) as shown in the first part of the proof of [12, Lemma 2.6]. We thus only need to show the converse \( J_X \otimes C \subseteq F(J_X, A, C) = J_X \otimes C \). But, this can be done in the same way as the second part of the proof of [12, Lemma 2.6], and then the first assertion of the proposition follows. Lemma 2.8 then verifies the second assertion on the inclusion. Finally, since \( \varphi_{M(A \otimes C)} \) maps \( J_{M(A \otimes C)}(X \otimes C) \) into \( K(M(A \otimes C)(X \otimes C)) \) on which
\[
(k_X \otimes \text{id})^{(1)} = k_X \otimes \text{id}^{(1)}
\]
by Theorem 2.6.(ii), the representation must be covariant by Lemma 2.9. \( \Box \)

Corollary 2.12. Under the same hypothesis of Corollary 2.11, the injective representation
\[
(k_X \otimes \text{id}_C, k_A \otimes \text{id}_C) : (X \otimes C, A \otimes C) \rightarrow \mathcal{O}_X \otimes C
\]
is covariant and the integrated form \( (k_X \otimes \text{id}_C) \times (k_A \otimes \text{id}_C) : \mathcal{O}_X \otimes C \rightarrow \mathcal{O}_X \otimes C \) is a surjective isomorphism.

Proof. Note that \( J_X \subseteq J_{M(A)} \) since \( J_X \) is an ideal of \( M(A) \) and is mapped injectively into \( K(X) \subseteq K(M(A)(X)) \) by \( \varphi_{M(A)} \). Hence \( (k_X \otimes \text{id}_C, k_A \otimes \text{id}_C) \) is covariant by Corollary 2.11. The integrated form is clearly surjective. Since \( (k_X, k_A) \) admits a gauge action, and hence so does \( (k_X \otimes \text{id}_C, k_A \otimes \text{id}_C) \), the integrated form must be injective by [14, Theorem 6.4]. \( \Box \)

2.6. Reduced and dual reduced Hopf \( C^* \)-algebras. By a Hopf \( C^* \)-algebra we mean a pair \((S, \Delta)\) of a \( C^* \)-algebra \( S \) and a nondegenerate homomorphism \( \Delta : S \rightarrow M(S \otimes S) \) called the comultiplication of \( S \) satisfying
\[
\begin{align*}
(1) & \quad \Delta \otimes \text{id} \circ \Delta = \text{id} \otimes \Delta \circ \Delta; \\
(2) & \quad \Delta(S)(1_{M(S)} \otimes S) = S \otimes S = \Delta(S)(S \otimes 1_{M(S)}).
\end{align*}
\]
Let \( G \) be a locally compact group. Then \((C_0(G), \Delta_G)\) is a Hopf \( C^* \)-algebra with the comultiplication \( \Delta_G(f)(r,s) = f(rs) \) for \( f \in C_0(G) \) and \( r,s \in G \). The full group \( C^* \)-algebra \( C^*_v(G) \) equipped with the comultiplication given by \( r \mapsto r \otimes r \) for \( r \in G \) is also a Hopf \( C^* \)-algebra. The same is true for the reduced group \( C^* \)-algebra \( C^*_r(G) \) such that the canonical surjection \( \lambda : C^*_r(G) \rightarrow C^*_v(G) \) is a morphism in the sense of [2] (see for example [24, Example 4.2.2]).

Let \( \mathcal{H} \) be a Hilbert space. A unitary operator \( V \) acting on \( \mathcal{H} \otimes \mathcal{H} \) is said to be multiplicative if it satisfies the pentagonal relation \( V_1 V_2 V_3 V_4 V_5 = V_1 V_2 \). For each functional \( \omega \in \mathcal{L}(\mathcal{H})_* \), define the operators \( L(\omega) \) and \( \rho(\omega) \) in \( \mathcal{L}(\mathcal{H}) \) by
\[
L(\omega) = \omega \otimes \text{id}(V), \quad \rho(\omega) = \text{id} \otimes \omega(V).
\]
The reduced algebra \( S_V \) and the dual reduced algebra \( \hat{S}_V \) are defined as the following norm closed subspaces of \( \mathcal{L}(\mathcal{H}) \):
\[
S_V = \{L(\omega) : \omega \in \mathcal{L}(\mathcal{H})_*\}, \quad \hat{S}_V = \{\rho(\omega) : \omega \in \mathcal{L}(\mathcal{H})_*\}.
\]
They are known to be nondegenerate subalgebras of \( \mathcal{L}(\mathcal{H}) \) ([2, Proposition 1.4]).

A multiplicative unitary \( V \) acting on \( \mathcal{H} \otimes \mathcal{H} \) is said to be regular if the norm closure of \( \overline{\{\text{id} \otimes \omega : \omega \in \mathcal{L}(\mathcal{H})_*\}} \) equals \( \mathcal{K}(\mathcal{H}) \), where \( \Sigma \) is the flip operator on \( \mathcal{H} \otimes \mathcal{H} \). Both \( S_V \) and \( \hat{S}_V \) then become Hopf \( C^* \)-algebras with the comultiplications
\[
\Delta_V(s) = V(s \otimes 1)V^*, \quad \hat{\Delta}_V(x) = V^*(1 \otimes x)V
\]
for \( s \in S \) and \( x \in \hat{S} \) ([2, Theorem 3.8]).

Let \( G \) be a locally compact group. For the regular multiplicative unitaries \( W_G \) and \( \hat{W}_G \) acting on \( L^2(G) \otimes L^2(G) \) by
\[
(W_G \xi)(r, s) = \xi(r, r^{-1} s), \quad (\hat{W}_G \xi)(r, s) = \xi(sr, s)
\]
for \( \xi \in C_c(G \times G) \) and \( r, s \in G \), it can be shown that \( S_{W_G} = C^*_r(G) = \hat{S}_{W_G} \) as Hopf \( C^* \)-algebras. Also, if \( \mu_G \) and \( \check{\mu}_G \) denote the nondegenerate embeddings \( C_0(G) \hookrightarrow \mathcal{L}(L^2(G)) \) given by
\[
(\mu_G(f) h)(r) = f(r) h(r), \quad (\check{\mu}_G(f) h)(r) = f(r^{-1}) h(r)
\]
for \( h \in C_c(G) \), they give isomorphisms from the Hopf \( C^* \)-algebra \( C_0(G) \) onto \( \hat{S}_{W_G} \) and \( S_{\hat{W}_G} \), respectively (see for example [24, Example 9.3.11]).

### 2.7. Reduced crossed products.

By a coaction of a Hopf \( C^* \)-algebra \((S, \Delta)\) on a \( C^* \)-algebra \( A \) we mean a nondegenerate homomorphism \( \delta : A \to M(A \otimes S) \) such that

(i) \( \delta \) satisfies the coaction identity: \( \delta \otimes \text{id} \circ \delta = \text{id} \otimes \Delta \circ \delta \);

(ii) \( \delta \) satisfies the coaction nondegeneracy: \( \delta(A)(1_{M(A)} \otimes S) = A \otimes S \).

Let \( V \) be a regular multiplicative unitary acting on \( \mathcal{H} \otimes \mathcal{H} \). Let \( \delta \) be a coaction of \( S_V \) on \( A \). We denote by \( \iota_{S_V} \) the inclusion map \( S_V \hookrightarrow M(K(\mathcal{H})) \) and by \( \delta_i \) the following composition
\[
\delta_i := \text{id}_A \otimes \iota_{S_V} \circ \delta : A \to M(A \otimes K(\mathcal{H})).
\]
We write \( 1_{M(A)} \otimes \hat{S}_V \) for the image of \( \hat{S}_V \) under the canonical embedding \( \hat{S}_V \hookrightarrow M(A \otimes K(\mathcal{H})) \). The reduced crossed product \( A \rtimes_{\delta} \hat{S}_V \) of \( A \) by the coaction \( \delta \) is defined to be the following norm closed subspace of \( M(A \otimes K(\mathcal{H})) \):
\[
A \rtimes_{\delta} \hat{S}_V = \delta_i(A)(1_{M(A)} \otimes \hat{S}_V).
\]
It is actually a \( C^* \)-algebra ([2, Lemma 7.2]).

**Remark 2.13.** In the literature, the reduced crossed product \( A \rtimes_{\delta} \hat{S}_V \) is usually defined as a subalgebra of \( \mathcal{L}_A(A \otimes \mathcal{H}) \) which can be identified with \( M(A \otimes K(\mathcal{H})) \). For the arguments concerning multiplier correspondences and the relevant strict topologies, it seems to be more convenient to work with \( M(A \otimes K(\mathcal{H})) \) rather than \( \mathcal{L}_A(A \otimes \mathcal{H}) \). This naturally leads us to regard \( A \rtimes_{\delta} \hat{S}_V \) as a subalgebra of \( M(A \otimes K(\mathcal{H})) \).

Let \( G \) be a locally compact group and \( A \) be a \( C^* \)-algebra. It is well-known that there exists a one-to-one correspondence between actions of \( G \) on \( A \) and coactions of \( C_0(G) \) on \( A \): to each action \( \alpha \) there corresponds a coaction \( \delta^\alpha \), and to a coaction \( \delta \) there corresponds an action \( \alpha^\delta \) such that
\[
\delta^\alpha(a)(r) = \alpha_r(a), \quad \alpha^\delta_r(a) = \delta(a)(r)
\]
for \( a \in A \) and \( r \in G \). Moreover, if \( \alpha : G \to \text{Aut}(A) \) is an action, then the reduced crossed product \( A \rtimes_{\alpha, r} G \) coincides with the crossed product \( A \rtimes_{\check{\delta}^\delta} \hat{S}_{\hat{W}_G} \) by the coaction
\[
\check{\delta}^\delta = \text{id}_A \otimes \check{\mu}_G \circ \delta^\alpha : A \to M(A \otimes S_{\hat{W}_G})
\]
The crossed product $A$ when viewed as subalgebras of $M(A \otimes K(H))$ (see for example [24, Chapter 9]). We will freely use these facts in the proof of Corollary 5.11, Theorem A.6, and Corollary B.7 with no further explanation.

A nondegenerate coaction of $G$ on a $C^*$-algebra $A$ is an injective coaction $\delta$ of the Hopf $C^*$-algebra $C^*(G)$ on $A$ ([5, Definition A.21]). Let
\[
(2.9) \quad \delta_\lambda := \text{id}_A \otimes X \circ \delta : A \to M(A \otimes C^*_\lambda(G)) = M(A \otimes \hat{S}_{W_\lambda}).
\]
The crossed product $A \rtimes_\delta G$ by $\delta$ is defined to be the reduced crossed crossed product $A \rtimes_\delta \hat{S}_{W_\lambda}$ by $\delta_\lambda$.

3. Coactions of Hopf $C^*$-algebras on $C^*$-correspondences

In this section, we define a coaction $(\sigma, \delta)$ of a Hopf $C^*$-algebra on a $C^*$-correspondence $(X, A)$ and prove that $(\sigma, \delta)$ induces a coaction on the associated Cuntz-Pimsner algebra $\mathcal{O}_X$ under a certain invariance condition (Theorem 3.7). Recall that the $C^*$-correspondences considered in this paper are always nondegenerate.

**Definition 3.1.** A coaction of a Hopf $C^*$-algebra $(S, \Delta)$ on a $C^*$-correspondence $(X, A)$ is a nondegenerate correspondence homomorphism
\[
(\sigma, \delta) : (X, A) \to (M(X \otimes S), M(A \otimes S))
\]
such that
(i) $\delta$ is a coaction of $S$ on the $C^*$-algebra $A$,
(ii) $\sigma$ satisfies the coaction identity: $\sigma \otimes \text{id}_S \circ \sigma = \text{id}_X \otimes \Delta \circ \sigma$,
(iii) $\sigma$ satisfies the coaction nondegeneracy:
\[
\varphi_{M(A \otimes S)}(1_{M(A)} \otimes S) \sigma(X) = X \otimes S.
\]

Note that the strict extensions $\sigma \otimes \text{id}_S$ and $\text{id}_X \otimes \Delta$ in (ii) are well-defined because the tensor product of two nondegenerate correspondence homomorphisms is also nondegenerate ([5, Proposition 1.38]).

**Remark 3.2.** It should be noted that
\[
\sigma(X) \cdot (1_{M(A)} \otimes S) = X \otimes S,
\]
which follows by the same argument of [5, Remark 2.11.(1) and (2)]. We then have $\sigma(X) \subseteq M_S(X \otimes S) \subseteq M_{A \otimes S}(X \otimes S)$.

**Remark 3.3.** Let $G$ be a locally compact group and $(X, A)$ be a $C^*$-correspondence. We show in Theorem A.6 that every action of $G$ on $(X, A)$ in the sense of [5, Definition 2.5] determines a coaction of the Hopf $C^*$-algebra $C_0(G)$ on $(X, A)$, and one can define in this way a one-to-one correspondence between actions of $G$ on $(X, A)$ and coactions of $C_0(G)$ on $(X, A)$. On the other hand, the nondegenerate coaction of $G$ [5, Definition 2.10] is by definition the coaction $(\sigma, \delta)$ of the Hopf $C^*$-algebra $C^*(G)$ on $(X, A)$ such that $\delta$ is injective. Definition 3.1 thus unifies the notions of actions and nondegenerate coactions of locally compact groups on a $C^*$-correspondences.

By Proposition 2.27 (Proposition 2.30, respectively) of [5], an action (nondegenerate coaction, respectively) of a locally compact group $G$ on $(X, A)$ determines an action (coaction, respectively) of $G$ on $\mathcal{K}(X)$, which can be generalized as the next proposition shows. Recall that we identify $\mathcal{K}(X_1 \otimes X_2) = \mathcal{K}(X_1) \otimes \mathcal{K}(X_2)$ for two Hilbert modules $X_1$ and $X_2$. 


Proposition 3.4. Let \((\sigma, \delta)\) be a coaction of a Hopf C*-algebra \(S\) on a C*-correspondence \((X, A)\). Then the nondegenerate homomorphism

\[
\sigma^{(1)} : \mathcal{K}(X) \to M(\mathcal{K}(X \otimes S)) = M(\mathcal{K}(X) \otimes S)
\]

is a coaction of \(S\) on \(\mathcal{K}(X)\). If \(\delta\) is injective then so is \(\sigma^{(1)}\). Moreover, the left action \(\varphi_A\) is \(\delta\)-\(\sigma^{(1)}\) equivariant: \(\sigma^{(1)} \circ \varphi_A = \varphi_A \otimes \text{id}_S \circ \delta\).

Proof. Let \(S = (S, \Delta)\). For \(\xi, \eta \in X\), we have

\[
(\sigma \otimes \text{id}_S)^{(1)}(\sigma(\xi)\sigma(\eta)^*) = (\sigma \otimes \text{id}_S)(\sigma(\xi))(\sigma(\eta))^*
\]

which verifies the coaction identity of \(\sigma\). It then follows by Remark 2.2 that

\[
(\sigma^{(1)} \otimes \text{id}_S \circ \sigma^{(1)})(\theta_{\xi,\eta}) = (\sigma \otimes \text{id}_S)^{(1)}(\sigma(\xi)\sigma(\eta)^*)
\]

by the second relation of (2.3) and the coaction nondegeneracy of \(\sigma\). This shows the coaction equivariancy of \(\sigma^{(1)}\), and thus \(\sigma^{(1)}\) is a coaction. For the assertion on the injectivity of \(\sigma^{(1)}\), see the comment below [14, Lemma 2.4].

The first relation of (2.3) and the fact that \((\sigma, \delta)\) is a correspondence homomorphism yield

\[
\overline{\sigma^{(1)}}(\varphi_A(a)) \sigma(\xi) = \sigma(\varphi_A(a) \xi) = \varphi_{M(A \otimes S)}(\delta(a)) \sigma(\xi).
\]

for \(a \in A\) and \(\xi \in X\). Multiplying by \(1_{M(A)} \otimes s\) on both end sides from the rights gives

\[
\overline{\sigma^{(1)}}(\varphi_A(a))(\sigma(\xi) \cdot (1_{M(A)} \otimes s)) = \varphi_{M(A \otimes S)}(\delta(a))(\sigma(\xi) \cdot (1_{M(A)} \otimes s))
\]

which leads to \(\overline{\sigma^{(1)}}(\varphi_A(a)) = \varphi_{M(A \otimes S)}(\delta(a))\) by the coaction nondegeneracy of \(\sigma\). But \(\varphi_{M(A \otimes S)} = \varphi_A \otimes \text{id}_S\) by definition, and thus the \(\delta\)-\(\sigma^{(1)}\) equivariancy of \(\varphi_A\) follows.

Definition 3.5. Let \((\sigma, \delta)\) be a coaction of a Hopf C*-algebra \(S\) on a C*-correspondence \((X, A)\). We say that the ideal \(J_X\) is weakly \(\delta\)-invariant if

\[
\delta(J_X)(1_{M(A)} \otimes S) \subseteq J_X \otimes S.
\]
Remark 3.6. The coaction nondegeneracy of \( \delta \) implies that \( J_X \) is weakly \( \delta \)-invariant if and only if \( \delta(J_X)(A \otimes S) \subseteq J_X \otimes S \), that is, 
\[
\delta(J_X) \subseteq M(A \otimes S; J_X \otimes S).
\]

Modifying the proof of [12, Proposition 3.1] one can prove the following theorem which will be used in Section 5. We provide a detailed proof for the reader’s convenience.

Theorem 3.7. Let \( (\sigma, \delta) \) be a coaction of a Hopf C*-algebra \( S \) on a C*-correspondence \( (X, A) \) such that the ideal \( J_X \) is weakly \( \delta \)-invariant. Then the representation 
\[
(k_X \otimes \text{id}_S \circ \sigma, k_A \otimes \text{id}_S \circ \delta) : (X, A) \to M_{A \otimes S}(O_X \otimes S)
\]
is covariant, and its integrated form \( \zeta := (k_X \otimes \text{id}_S \circ \sigma) \times (k_A \otimes \text{id}_S \circ \delta) \) is a coaction of \( S \) on \( O_X \) such that the diagram
\[
\begin{align*}
(X, A) \xrightarrow{(\sigma, \delta)} & \quad (M_{A \otimes S}(X \otimes S), M(A \otimes S)) \\
(k_X, k_A) \downarrow & \quad \downarrow (k_X \otimes \text{id}_S, k_A \otimes \text{id}_S) \\
O_X & \quad \zeta \\
& \quad M_{A \otimes S}(O_X \otimes S)
\end{align*}
\]
commutes. Moreover, if \( \delta \) is injective then so is \( \zeta \).

Proof. Let us first prove that \( (k_X \otimes \text{id}_S \circ \sigma, k_A \otimes \text{id}_S \circ \delta) \) is covariant, that is,
\[
(k_X \otimes \text{id}_S \circ \sigma)^{(1)} \circ \varphi_A = k_A \otimes \text{id}_S \circ \delta
\]
on \( J_X \). Since \( \sigma(X) \subseteq M_{A \otimes S}(X \otimes S) \) and thus \( \sigma^{(1)}(\mathcal{K}(X)) \subseteq \mathcal{K}(M_{A \otimes S}(X \otimes S)) \), we have
\[
(k_X \otimes \text{id}_S \circ \sigma)^{(1)} = (k_X \otimes \text{id}_S)^{(1)} \circ \sigma^{(1)}
\]
on \( \mathcal{K}(X) \) by Theorem 2.6.(ii). We then have
\[
(k_X \otimes \text{id}_S \circ \sigma)^{(1)} \circ \varphi_A = (k_X \otimes \text{id}_S)^{(1)} \circ \sigma^{(1)} \circ \varphi_A = (k_X \otimes \text{id}_S)^{(1)} \circ \varphi_A
\]
on \( J_X \) since \( \sigma^{(1)} \circ \varphi_A = \varphi_{A \otimes S} \circ \delta \) on \( J_X \) by [11, Lemma 3.3]. Hence, the requirement that \( (k_X \otimes \text{id}_S \circ \sigma, k_A \otimes \text{id}_S \circ \delta) \) be covariant amounts to that
\[
(k_X \otimes \text{id}_S)^{(1)} \circ \varphi_{A \otimes S} \circ \delta = k_A \otimes \text{id}_S \circ \delta
\]
on \( J_X \). By Remark 3.6, this equality will follow if we show that
\[
(k_X \otimes \text{id}_S)^{(1)} \circ \varphi_{A \otimes S} = k_A \otimes \text{id}_S
\]
on \( M(A \otimes C; J_X \otimes C) \). But, this is the content of Lemma 2.9, and therefore the representation \( (k_X \otimes \text{id}_S \circ \sigma, k_A \otimes \text{id}_S \circ \delta) \) is covariant.

We now show that \( \zeta \) is a coaction of \( (S, \Delta) \) on \( O_X \). Since
\[
(1_{M(O_X)} \otimes S)\zeta(k_X(X)) = k_A \otimes \text{id}_S(1_{M(A)} \otimes S)k_X \otimes \text{id}_S(\sigma(X)) = k_X \otimes \text{id}_S(\varphi_{M(A \otimes S)}(1_{M(A)} \otimes S) \sigma(X)) = k_X(X) \otimes \delta,
\]
we have
\[ (k_X(X)^*)(1_M(Ω_X) ⊗ S) = \left( (1_M(Ω_X) ⊗ S)ζ(k_X(X)) \right)^* = k_X(X)^* ⊗ S. \]
We also have \( ζ(k_X(X))(1_M(Ω_X) ⊗ S) = k_X(X) ⊗ S \). From these and the coaction nondegeneracy of \( δ \), we can deduce that \( ζ \) satisfies the coaction nondegeneracy.

The coaction nondegeneracy of \( ζ \) implies \( ζ(Ω_X) ⊆ M_{A⊗S}(Ω_X ⊗ S) \), and then we have the diagram \( (3.2) \).

To make it more accessible, we first note the followings. Let \( x ∈ A ⊗ S \) and \( m ∈ M_{A⊗S}(Ω_X ⊗ S) \). Then

\[
(ζ ⊗ id_S)((k_A ⊗ id_S)(x) m) = k_A ⊗ id_S ⊗ id_S((δ ⊗ id_S)(x)) ζ ⊗ id_S(m),
\]

\[
(id_{Ω_X} ⊗ Δ)((k_A ⊗ id_S)(x) m) = k_A ⊗ id_S ⊗ id_S((id_A ⊗ Δ)(x)) id_{Ω_X} ⊗ Δ(m),
\]

and similarly for \( (ζ ⊗ id_S)(m (k_A ⊗ id_S)(x)) \) and \( (id_{Ω_X} ⊗ Δ)(m (k_A ⊗ id_S)(x)) \). From these relations and also the nondegeneracy of \( δ ⊗ id_S \) and \( id_A ⊗ Δ \), we deduce that the restrictions

\[ ζ ⊗ id_S, \quad id_{Ω_X} ⊗ Δ : M_{A⊗S}(Ω_X ⊗ S) → M_{A⊗S⊗S}(Ω_X ⊗ S ⊗ S) \]

are \((A ⊗ S)\)-strict to \((A ⊗ S ⊗ S)\)-strictly continuous (cf. [4, Lemma A.5]). Therefore the following compositions

\[ ζ ⊗ id_S ⊗ k_X ⊗ id_S, \quad id_{Ω_X} ⊗ Δ ⊗ k_X ⊗ id_S : M_{A⊗S}(X ⊗ S) → M_{A⊗S⊗S}(Ω_X ⊗ S ⊗ S). \]

are \((A ⊗ S)\)-strict to \((A ⊗ S ⊗ S)\)-strictly continuous. Similarly, both maps

\[ σ ⊗ id_S, \quad id_X ⊗ Δ : M_{A⊗S}(X ⊗ S) → M_{A⊗S⊗S}(X ⊗ S ⊗ S) \]

are \((A ⊗ S)\)-strict to \((A ⊗ S ⊗ S)\)-strictly continuous, and hence so are the maps

\[ k_X ⊗ id_S ⊗ id_S ⊗ σ ⊗ id_S, \quad k_X ⊗ id_S ⊗ id_S ⊗ id_X ⊗ Δ : M_{A⊗S}(X ⊗ S) → M_{A⊗S⊗S}(Ω_X ⊗ S ⊗ S). \]

Since the equalities

\[ \frac{ζ ⊗ id_S ⊗ k_X ⊗ id_S}{k_X ⊗ id_S ⊗ id_S ⊗ id_X ⊗ Δ} = \frac{ζ ⊗ id_S ⊗ k_X ⊗ id_S}{id_{Ω_X} ⊗ Δ ⊗ k_X ⊗ id_S}, \]

hold on \( X ⊗ S \) which is \((A ⊗ S)\)-strictly dense in \( M_{A⊗S}(X ⊗ S) \) and since \( σ(X) ⊆ M_{A⊗S}(X ⊗ S) \), we now have

\[ ζ ⊗ id_S ⊗ ζ ⊗ k_X = ζ ⊗ id_S ⊗ k_X ⊗ id_S ⊗ σ = \frac{ζ ⊗ id_S ⊗ k_X ⊗ id_S ⊗ σ}{id_{Ω_X} ⊗ Δ ⊗ k_X ⊗ id_S}. \]

by the \((A ⊗ S)\)-strict to \((A ⊗ S ⊗ S)\)-strict continuity of the maps of \((3.3) \) and \((3.4) \)
and also by the coaction identity of \( σ \). Thus \( ζ \) satisfies the coaction identity.
For the last assertion of the theorem, assume that $\delta$ is injective. We only need to show by [14, Theorem 6.4] that the injective covariant representation $(\overline{k_X \otimes \text{id}_S} \circ \sigma, \overline{k_A \otimes \text{id}_S} \circ \delta)$ admits a gauge action. Let $\beta : \mathbb{T} \to Aut(O_X)$ be the gauge action. Note that for each $z \in \mathbb{T}$, the strict extension $\overline{\beta_z \otimes \text{id}_S}$ on $M(O_X \otimes S)$ maps $M_{A \otimes S}(O_X \otimes S)$ onto itself. Then the composition

\[
(3.5) \quad (\overline{\beta_z \otimes \text{id}_S} \circ \overline{k_X \otimes \text{id}_S}, \overline{\beta_z \otimes \text{id}_S} \circ \overline{k_A \otimes \text{id}_S}) : (M_{A \otimes S}(X \otimes S), M(A \otimes S)) \to M_{A \otimes S}(O_X \otimes S).
\]

gives a representation which is $(A \otimes S)$-strictly continuous. Since the equalities

\[
\overline{\beta_z \otimes \text{id}_S} \circ \overline{k_X \otimes \text{id}_S}(m) = z \overline{k_X \otimes \text{id}_S}(m), \quad \overline{\beta_z \otimes \text{id}_S} \circ \overline{k_A \otimes \text{id}_S}(n) = \overline{k_A \otimes \text{id}_S}(n)
\]

are valid for $m \in X \otimes S$ and $n \in A \otimes S$, and the representation (3.5) is $(A \otimes S)$-strictly continuous, the above equalities still hold for $m \in M_{A \otimes S}(X \otimes S)$ and $n \in M(A \otimes S)$. Since $\sigma(X) \subseteq M_{A \otimes S}(X \otimes S)$, it thus follows that

\[
\overline{\beta_z \otimes \text{id}_S} \circ \overline{k_X \otimes \text{id}_S} \circ \sigma = z \overline{k_X \otimes \text{id}_S} \circ \sigma,
\]

and similarly that $\overline{\beta_z \otimes \text{id}_S} \circ \overline{k_A \otimes \text{id}_S} \circ \delta = \overline{k_A \otimes \text{id}_S} \circ \delta$. This proves that the restrictions of $\overline{\beta_z \otimes \text{id}_S}$ to $\zeta(O_X)$ ($z \in \mathbb{T}$) define a gauge action of $\mathbb{T}$ on $\zeta(O_X)$, which establishes the theorem.

**Definition 3.8.** We call $\zeta$ in Theorem 3.7 the coaction *induced* by $(\sigma, \delta)$.

**Remarks 3.9.**

1. Let $G$ be a locally compact group. If $(\sigma, \delta)$ is a coaction of $C_0(G)$ on $(X, A)$, then $\overline{\delta(J_X)(1_{M(A) \otimes S})} = J_X \otimes S$ by [7, Lemma 2.6.(a)] and Theorem A.6. Hence, $J_X$ is automatically weakly $\delta$-invariant in this case.

2. Replacing in the diagram (3.2) the $(A \otimes S)$-multiplier correspondences by the $S$-multiplier correspondences $(M_S(X \otimes S), M_S(A \otimes S))$ and $M_S(O_X \otimes S)$, we can regard $(\overline{k_X \otimes \text{id}_S}, \overline{k_A \otimes \text{id}_S})$ as the $S$-strict extension by Remarks 2.5.(2).

We close this section with the following example which will be continued in Example 4.12 and 5.13.

**Example 3.10.** Let $\delta$ be a coaction of a Hopf $C^*$-algebra $(S, \Delta)$ on a $C^*$-algebra $A$ and $\varphi \in Aut(A)$. Let $v$ be a cocycle for the coaction $\delta$, that is, $v \in M(A \otimes S)$ is unitary such that

\[
v_{1\delta} \delta \otimes \text{id}_S(v) = \text{id}_A \otimes \Delta(v)
\]

([2, Definition 0.4]). Suppose that

\[
(3.6) \quad v \delta(\varphi(a)) = \overline{\varphi \otimes \text{id}_S(\delta(a))} v
\]

for $a \in A$. Define $\sigma : A(\varphi) \to M(A \otimes S(\varphi \otimes S)) = M(A(\varphi) \otimes S)$ by

\[
\sigma(a) := v \delta(a) \quad (a \in A).
\]

Then $(\sigma, \delta)$ is a coaction of $S$ on the $\varphi$-identity correspondence $(A(\varphi), A)$. Indeed, it is a correspondence homomorphism by Lemma 2.1. The computation

\[
\sigma \otimes \text{id}_S(\sigma(a)) = v_{1\delta} \delta \otimes \text{id}_S(v \delta(a)) = v_{1\delta} \delta \otimes \text{id}_S(v) \delta \otimes \text{id}_S(\delta(a)) = \text{id}_A \otimes \Delta(v) \text{id}_A \otimes \Delta(\delta(a)) = \text{id}_A \otimes \Delta(\sigma(a))
\]
verifies the coaction identity of $\sigma$. The coaction nondegeneracy of $\delta$ gives
\[
\begin{align*}
(1_{M(A)} \otimes S)\sigma(A) &= (1_{M(A)} \otimes S) v\delta(A) = (1_{M(A)} \otimes S) v\delta(\varphi(A)) \\
&= (1_{M(A)} \otimes S) (\varphi \otimes \text{id}_S \delta(A)) v \\
&= \varphi \otimes \text{id}_S ((1_{M(A)} \otimes S) \delta(A)) v = (A \otimes S)v = A \otimes S
\end{align*}
\]
so that $\sigma$ satisfies coaction nondegeneracy. Hence $(\sigma, \delta)$ is a coaction.

Since $J_{A(\varphi)} = A$, it follows by Theorem 3.7 that $(\sigma, \delta)$ induces a coaction $\zeta$ of $S$ on $O_{A(\varphi)}$ which is isomorphic to the crossed product $A \rtimes_\varphi Z$. An isomorphism $O_{A(\varphi)} \cong A \rtimes_\varphi Z$ can be given as follows. Let $(\pi, u)$ be the canonical covariant representation of the $C^*$-dynamical system $(A, Z, \varphi)$ on $M(A \rtimes_\varphi Z)$. Define $\psi : A(\varphi) \to A \rtimes_\varphi Z$ by
\[
\psi(a) = u^* \pi(a) \quad (a \in A(\varphi)).
\]
It can be easily checked that $(\psi, \pi)$ is a covariant representation of $(A(\varphi), A)$ on $A \rtimes_\varphi Z$. Furthermore, the integrated form $\psi \times \pi : O_{A(\varphi)} \to A \rtimes_\varphi Z$ is a surjective isomorphism. We will identify in this way the $C^*$-algebras $O_{A(\varphi)} = A \rtimes_\varphi Z$ as well as the universal covariant representations $(k_X, k_A) = (\psi, \pi)$.

We now describe $\zeta$ on the canonical generators of $A \rtimes_\varphi Z$. Theorem 3.7 says that
\[
\zeta(\pi(a)) = \pi \otimes \text{id}_S(\delta(a)),
\]
\[
\zeta(u^* \pi(a)) = \zeta(\psi(a)) = \psi \otimes \text{id}_S(\sigma(a)) = (u^* \otimes 1_{M(S)}) \pi \otimes \text{id}_S(\psi \delta(a))
\]
for $a \in A$. Note that $\zeta(u^*) = (u^* \otimes 1_{M(S)}) \pi \otimes \text{id}_S(v)$. Hence we have
\[
\zeta(\pi(a)u^n) = \pi \otimes \text{id}_S(\delta(a))(u^* \otimes 1_{M(S)}) \pi \otimes \text{id}_S(v)^{-n}.
\]
for $a \in A$ and $n \in \mathbb{Z}$.

4. Reduced crossed product correspondences $X \rtimes_{\sigma} \hat{S}$

This section is devoted to constructing the reduced crossed product correspondence $(X \rtimes_{\sigma} S, A \rtimes_\kappa \hat{S})$ from a coaction $(\sigma, \kappa)$ on $(X, A)$ of a reduced Hopf $C^*$-algebra $S$ defined by a regular multiplicative unitary.

Recall that the Toeplitz algebra $T_X$ is the $C^*$-algebra generated by $i_X(X)$ and $i_A(A)$, where $(i_X, i_A)$ is the universal representation of $(X, A)$ ([14]). The following lemma is a Toeplitz algebra analogue of Corollary 2.12.

**Lemma 4.1.** Let $(X, A)$ be a $C^*$-correspondence and $C$ be a $C^*$-algebra. Then the injective representation
\[
(i_X \otimes \text{id}_C, i_A \otimes \text{id}_C) : (X \otimes C, A \otimes C) \to T_X \otimes C
\]
gives rise to an isomorphism from $T_X \otimes C$ onto $T_X \otimes C$.

**Proof.** By the universal property of the Toeplitz algebra $T_X \otimes C$, there exists a homomorphism $\Psi : T_X \otimes C \to T_X \otimes C$ such that $\Psi(i_X \otimes C)(\xi \otimes c) = i_X(\xi) \otimes c$ and $\Psi(i_A \otimes C)(a \otimes c) = i_A(a) \otimes c$ for $\xi \in X$, $a \in A$, and $c \in C$. Clearly, $\Psi$ is surjective.

To see that $\Psi$ is injective, we first note that $(i_X \otimes \text{id}_C, i_A \otimes \text{id}_C)$ admits a gauge action. Thus, we only need to show by [14, Theorem 6.2] that the space
\[
I' = \{ x \in A \otimes C : (i_A \otimes \text{id}_C)(x) \in (i_X \otimes \text{id}_C)^{(1)}(K(X \otimes C)) = i_X^{(1)}(K(X)) \otimes C \}
\]
is zero. But for \( x \in I' \) and \( \omega \in C^* \), applying the slice map \( \text{id}_{1\otimes \omega} \) to \( (i_A \otimes \text{id}_C)(x) \) yields

\[
(\text{id}_{1\otimes \omega})((i_A \otimes \text{id}_C)(x)) = i_A((\text{id}_A \otimes \omega)(x)) \in i^{(1)}(\mathcal{K}(X)),
\]

which implies by [14, Theorem 6.2] that \( (\text{id}_A \otimes \omega)(x) = 0 \). Therefore, \( x = 0 \) as desired.

In what follows, for \( c \in C \) and \( \omega \in C^* \), we denote by \( \omega c \) and \( \omega \) the functionals on \( C \) given by

\[
(\omega c)(b) = \omega(cb), \quad (c\omega)(b) = \omega(bc) \quad (b \in C).
\]

**Proposition 4.2.** Let \((X, A)\) be a \(C^*\)-correspondence, \( C \) be a \(C^*\)-algebra, and \( \omega \in C^* \). Then the slice map \( \text{id}_X \otimes \omega : X \otimes C \to X \) extends uniquely to a strictly continuous linear map

\[
\text{id}_X \otimes \omega : M(X \otimes C) \to M(X)
\]

between the two multiplier correspondences.

**Proof.** Uniqueness assertion will follow immediately once we show the existence of a strict extension of \( \text{id}_X \otimes \omega \) since \( X \otimes C \) is strictly dense in \( M(X \otimes C) \).

By Lemma 4.1, \( X \otimes C \) can be embedded isometrically into \( T_X \otimes C \). Restricting to \( X \otimes C \) the slice map \( \text{id}_{1\otimes \omega} \) on \( T_X \otimes C \), we thus obtain a norm continuous extension \( \text{id}_X \otimes \omega : X \otimes \omega \to X \) of \( \text{id}_X \otimes \omega \).

We claim that the map \( \text{id}_X \otimes \omega \) just obtained is strictly continuous. Indeed, let \( \{x_i\} \) be a net in \( X \otimes C \) converging strictly to an \( x \in X \otimes C \), \( T \in \mathcal{K}(X) \), and \( a \in A \). Factor \( \omega \) into \( \omega_1 c_1 \) or \( c_2 \omega_2 \) for some \( \omega_1, \omega_2 \in C^* \) and \( c_1, c_2 \in B \). (The Hewitt-Cohen factorization theorem allows us to do this; see for example [23, Proposition 2.33].)

By norm continuity, we have

\[
T(\text{id}_X \otimes (\omega_1 c_1))(y) = (\text{id}_X \otimes \omega_1)((T \otimes c_1)y) \quad (y \in X \otimes C).
\]

Hence the net \( \{T(\text{id}_X \otimes (\omega_1 c_1))(x_i)\} = \{(\text{id}_X \otimes \omega_1)((T \otimes c_1)x_i)\} \) in \( X \) converges to \( (\text{id}_X \otimes \omega_1)((T \otimes c_1)x) = T(\text{id}_X \otimes \omega_1 c_1)(x) \) again by norm continuity. Similarly, \( \{\text{id}_X \otimes c_2 \omega_2 \}(x_i) \cdot a \) converges to \( (\text{id}_X \otimes c_2 \omega_2)(x) \cdot a \), which proves our claim.

By standard argument on continuous extensions (for example, see [18, Proposition 7.2]), \( \text{id}_X \otimes \omega \) extends strictly to all of \( M(X \otimes C) \).

**Remark 4.3.** It is easy to see that \( \text{id}_X \otimes \omega \) on \( M(X \otimes C) \) is norm bounded with \( ||\text{id}_X \otimes \omega|| \leq ||\omega|| \).

In the rest of this section and the next one, we restrict our attention to coactions of reduced Hopf \(C^*\)-algebras defined by regular multiplicative unitaries.

**Notations 4.4.** Until the end of Section 5, we will denote by \( \mathcal{H} \) the Hilbert space on the two-fold tensor product of which a regular multiplicative unitary \( V \) acts. To simplify notation, we often write \( S \) and \( \tilde{S} \) for the “reduced” and “dual reduced” Hopf \(C^*\)-algebras \( S_V \) and \( \tilde{S}_V \) defined by \( V \), respectively.

Let \( (\sigma, \delta) \) be a coaction of \( S \) on \( (X, A) \) and \( \iota_S : S \hookrightarrow M(\mathcal{K}(\mathcal{H})) \) be the inclusion map. Similarly to \( \delta \), given in (2.8), we denote by \( \sigma \), the composition

\[
\sigma = \text{id}_X \otimes \iota_S \circ \sigma
\]
where $\text{id}_X \otimes \iota_S$ is the strict extension. Evidently, $(\sigma, \delta)$ is a nondegenerate correspondence homomorphism:

\[
\begin{array}{c}
(X, A) \\
\downarrow \downarrow
\end{array}
\xrightarrow{(\sigma, \delta)}
\begin{array}{c}
(M(X \otimes K(H)), M(A \otimes K(H))) \\
\downarrow \downarrow
\end{array}
\xrightarrow{\text{id}_X \otimes \iota_S, \text{id}_A \otimes \iota_S}
\begin{array}{c}
(M(X \otimes S), M(A \otimes S))
\end{array}
\]

If $B$ is a $C^*$-algebra, $x \in \hat{S}$, and $s \in S$, we simply write $1_{M(B)} \otimes x$ and $1_{M(B)} \otimes s$ for the elements $1_{M(B)} \otimes \iota_S(x)$ and $1_{M(B)} \otimes \iota_S(s)$ in $M(B \otimes K(H))$, respectively.

The next lemma generalizes [2, Lemma 7.2]. The proof is not significantly different, but we provide it here for the reader’s convenience.

**Lemma 4.5.** Let $(\sigma, \delta) : (X, A) \to (M(X \otimes S), M(A \otimes S))$ be a coaction of $S$ on a $C^*$-correspondence $(X, A)$. Then the norm closures in $M(X \otimes K(H))$ of the subspaces $\sigma(X) \cdot (1_{M(A)} \otimes \hat{S})$ and $\varphi_{M(A \otimes K(H))}(1_{M(A)} \otimes \hat{S}) \sigma(X)$ coincide.

**Proof.** Let us show that each of the subspaces is contained in the norm closure of the other. Let $S = S_V$ be the reduced Hopf $C^*$-algebra obtained from a regular multiplicative unitary $V \in \mathcal{L}(H \otimes H)$. Set $\mathcal{K} = K(H)$. For $\xi \in X$ and $\omega \in \mathcal{L}(H)_+$, let $m$ be the following element of $\varphi_{M(A \otimes K(H))}(1_{M(A)} \otimes \hat{S}) \sigma(X)$:

\[
m = \varphi_{M(A \otimes X)}(1_{M(A)} \otimes \rho(\omega)) \sigma_i(\xi) = (1_{M(K(H))} \otimes \text{id}_{\mathcal{K}} \otimes \omega(V)) \circ \sigma_i(\xi).
\]

Write $V_{23} = 1_{M(A)} \otimes V \in M(A \otimes \mathcal{K} \otimes \mathcal{K})$. Similarly, $\sigma(\xi)_{12} = \sigma(\xi) \otimes 1_{M(S)} \in M(X \otimes S \otimes S)$ and $\sigma_i(\xi)_{12} = \text{id}_X \otimes \iota_S \otimes \iota_S(\sigma(\xi))_{12} \in M(X \otimes S \otimes S)$. Consider a net $\{v_i\}_i$ in $X \otimes S$ strictly converging to $\sigma(\xi)$. Since $S$ is a nondegenerate subalgebra of $M(\mathcal{K})$, we can see that the net $\{\text{id}_X \otimes \iota_S(v_i) \otimes 1_{M(\mathcal{K})}\}_i$ in $M(X \otimes \mathcal{K} \otimes \mathcal{K})$ converges strictly to $\sigma(\xi)_{12}$. Hence we deduce from Proposition 4.2 that

\[
m = \text{id}_X \otimes \text{id}_{\mathcal{X}} \otimes \omega(\varphi_{M(A \otimes \mathcal{K} \otimes \mathcal{K})}(V_{23}) \sigma_i(\xi)_{12}).
\]

We then have

\[
m = \text{id}_X \otimes \text{id}_{\mathcal{K}} \otimes \omega((\varphi_{M(A \otimes \mathcal{K} \otimes \mathcal{K})}(V_{23}) \sigma_i(\xi)_{12} \cdot V_{23})^* \cdot V_{23})
\]

\[
= \text{id}_X \otimes \text{id}_{\mathcal{K}} \otimes \omega(\text{id}_X \otimes \iota_S \otimes \iota_S(\text{id}_X \otimes \Delta_V(\sigma(\xi))) \cdot V_{23})
\]

\[
= \text{id}_X \otimes \text{id}_{\mathcal{K}} \otimes \omega(\text{id}_X \otimes \iota_S \otimes \iota_S(\sigma_i \otimes \iota_S(\sigma(\xi)))) \cdot V_{23})
\]

\[
= \text{id}_X \otimes \text{id}_{\mathcal{K}} \otimes \omega(\sigma_i \otimes \iota_S(\sigma(\xi)) \cdot V_{23})
\]

again by Proposition 4.2 and also by the definition of $\Delta_V$ in (2.6) and the coaction identity of $\sigma$. Write $\omega = \omega' \iota_S$. Then

\[
m = \text{id}_X \otimes \text{id}_{\mathcal{K}} \otimes \omega'((\varphi_{M(A \otimes \mathcal{K} \otimes \mathcal{K})}(1_{M(A)} \otimes 1_{M(\mathcal{K})} \otimes \iota_S) \sigma_i(\xi) \otimes \iota_S(\sigma(\xi))) \cdot V_{23}).
\]

Since $(\sigma_i \otimes \iota_S, \delta_i \otimes \iota_S)$ is a correspondence homomorphism,

\[
m = \text{id}_X \otimes \text{id}_{\mathcal{K}} \otimes \omega'((\sigma_i \otimes \iota_S(\varphi_{M(A \otimes \mathcal{K})}(1_{M(A)} \otimes \iota_S) \sigma(\xi)) \cdot V_{23}).
\]

The coaction nondegeneracy of $\sigma$ then implies that $m$ belongs to the space

\[
M = \text{id}_X \otimes \text{id}_{\mathcal{K}} \otimes \omega'((\sigma_i \otimes \iota_S)(X \otimes S) \cdot V_{23}).
\]
in which the elements $\text{id}_X \otimes \text{id}_{\mathcal{X}} \otimes \omega'((\sigma_i(\xi') \otimes s') \cdot V_{23})$ for $\xi' \in X$ and $s' \in S$ are linearly dense by Remark 4.3. But

$$\text{id}_X \otimes \text{id}_{\mathcal{X}} \otimes \omega'((\sigma_i(\xi') \otimes s') \cdot V_{23}) = \text{id}_X \otimes \text{id}_{\mathcal{X}} \otimes \omega'(\sigma_i(\xi')_{12} \cdot V_{23}) = \sigma_i(\xi') \cdot (1_{M(A)} \otimes \rho(\omega's')) \\
\in \sigma_i(X) \cdot (1_{M(A)} \otimes \hat{S}),$$

and therefore $m \in M \subseteq \sigma_i(X) \cdot (1_{M(A)} \otimes \hat{S})$.

For the converse, let $m' = \sigma_i(\xi) \cdot (1_{M(A)} \otimes \rho(\omega s))$. Then

$$m' = \sigma_i(\xi) \cdot (1_{M(A)} \otimes \text{id}_{\mathcal{X}} \otimes \omega(V)) = \text{id}_X \otimes \text{id}_{\mathcal{X}} \otimes \omega(\sigma_i(\xi)_{12} \cdot ((1_{M(A)} \otimes 1_{M(\mathcal{X})} \otimes s)V_{23})) \\
= \text{id}_X \otimes \text{id}_{\mathcal{X}} \otimes \omega((\sigma_i \otimes \iota_S)(\xi \otimes s) \cdot V_{23})$$

so that $m'$ is an element of the space $M' = \text{id}_X \otimes \text{id}_{\mathcal{X}} \otimes \omega((\sigma_i \otimes \iota_S)(X \otimes S) \cdot V_{23})$.

By coaction nondegeneracy and the fact that $(\sigma_i \otimes \iota_S, \delta_i \otimes \iota_S)$ is a correspondence homomorphism, we have

$$M' \subseteq \text{id}_X \otimes \text{id}_{\mathcal{X}} \otimes \omega((\varphi_{M(A \otimes S)})(1_{M(A)} \otimes X) \cdot V_{23}) \subseteq \text{id}_X \otimes \text{id}_{\mathcal{X}} \otimes \omega((\varphi_{M(A \otimes S \otimes \mathcal{X})})(1_{M(A)} \otimes 1_{M(\mathcal{X})} \otimes S)\sigma_i \otimes \iota_S(\sigma(X)) \cdot V_{23}).$$

Since

$$\sigma_i \otimes \iota_S(\sigma(X)) = \text{id}_X \otimes \iota_S \otimes \iota_S(\sigma \otimes \text{id}_{S}(\sigma(X))) = \text{id}_X \otimes \iota_S \otimes \iota_S(\text{id}_X \otimes \Delta(\sigma(X))) \\
= \varphi_{M(A \otimes \mathcal{X} \otimes \mathcal{X})}(1_{M(A)} \otimes V_{23}) \sigma_i(X)_{12} \cdot V_{23}$$

by coaction identity and strict continuity, we then have

$$M' \subseteq \text{id}_X \otimes \text{id}_{\mathcal{X}} \otimes \omega((\varphi_{M(A \otimes S \otimes \mathcal{X})})(1_{M(A)} \otimes 1_{M(\mathcal{X})} \otimes S)V_{23})\sigma_i(X)_{12} \subseteq \varphi_{M(A \otimes \mathcal{X})}(1_{M(A)} \otimes \hat{S})\sigma_i(X).$$

Consequently, $m' \in M' \subseteq \varphi_{M(A \otimes \mathcal{X})}(1_{M(A)} \otimes \hat{S})\sigma_i(X)$.

For a coaction $(\sigma, \delta)$ of $S$ on $(X, A)$, we denote by $X \times_\sigma \hat{S}$ the norm closure of the subspaces considered in Lemma 4.5:

$$X \times_\sigma \hat{S} := \sigma_i(X) \cdot (1_{M(A)} \otimes \hat{S}) = \varphi_{M(A \otimes \mathcal{K}(S))}(1_{M(A)} \otimes \hat{S})\sigma_i(X).$$

With the restriction of the operations on $(M(X \otimes \mathcal{K}(H)), M(A \otimes \mathcal{K}(H)))$, it becomes a $C^*$-correspondence:

**Theorem 4.6.** Let $(\sigma, \delta)$ be a coaction of a reduced Hopf $C^*$-algebra $S$ on a $C^*$-correspondence $(X, A)$. Then $(X \times_\sigma \hat{S}, A \times_\delta \hat{S})$ is a nondegenerate $C^*$-correspondence such that the inclusion

$$(X \times_\sigma \hat{S}, A \times_\delta \hat{S}) \hookrightarrow (M(X \otimes \mathcal{K}(H)), M(A \otimes \mathcal{K}(H)))$$

is a nondegenerate correspondence homomorphism. The left action $\varphi_A$ is injective if $\varphi_A$ is injective. Also,

$$\mathcal{K}(X \times_\sigma \hat{S}) = \mathcal{K}(X) \times_{\sigma(1)} \hat{S},$$
where \( \sigma^{(1)} \) is the coaction in Proposition 3.4, and
\[
\varphi_{A \times_{S} A}(\delta_{i}(a)(1_{M(A)} \otimes x)) = \sigma^{(1)}(\varphi_{A}(a))(1_{M(K(X))} \otimes x)
\]
for \( a \in A \) and \( x \in \widehat{S} \).

**Proof.** Set \( \mathcal{X} = \mathcal{K}(\mathcal{H}) \). The first assertion is clearly equivalent to saying that the following three conditions are satisfied:

(i) \( X \times_{\sigma} S \) is a Hilbert \( (A \times_{S} S) \)-module with respect to the operations on the Hilbert \( M(A \otimes \mathcal{X}) \)-module \( M(X \otimes \mathcal{X}) \):
\[
\langle \sigma_{i}(X) \cdot (1_{M(A)} \otimes \widehat{S}), \sigma_{i}(X) \cdot (1_{M(A)} \otimes \widehat{S}) \rangle_{M(A \otimes \mathcal{X})} \subseteq X \times_{\sigma} \widehat{S};
\]
(ii) the Hilbert \( (A \times_{S} \widehat{S}) \)-module \( X \times_{\sigma} \widehat{S} \) becomes a nondegenerate \( C^{*} \)-correspondence such that
\[
\varphi_{A \times_{S} A} = \varphi_{M(\mathcal{X})}|_{A \times_{S} A}:
\]
(iii) the inclusion \( (X \times_{\sigma} \widehat{S}, A \times_{S} \widehat{S}) \hookrightarrow (M(X \otimes \mathcal{X}), M(A \otimes \mathcal{X})) \) is a nondegenerate correspondence homomorphism:
\[
(X \times_{\sigma} \widehat{S}) \cdot (A \otimes \mathcal{X}) = X \otimes \mathcal{X}, \quad (A \times_{S} \widehat{S}) \cdot (A \otimes \mathcal{X}) = A \otimes \mathcal{X}.
\]

The condition (i) is clearly satisfied since \( (\sigma_{i}, \delta_{i}) \) is a correspondence homomorphism and \( (1_{M(A)} \otimes \widehat{S}) \delta_{i}(A)(1_{M(A)} \otimes \widehat{S}) \) is contained in \( \delta_{i}(A)(1_{M(A)} \otimes \widehat{S}) \). Lemma 4.5 shows that
\[
\varphi_{M(A \otimes \mathcal{X})}(1_{M(A)} \otimes \widehat{S}) \sigma_{i}(X) \cdot (1_{M(A)} \otimes \widehat{S}) = \sigma_{i}(X) \cdot (1_{M(A)} \otimes \widehat{S}).
\]
Since \( \varphi_{A} \) is nondegenerate, this equality combined with the following
\[
\varphi_{M(A \otimes \mathcal{X})}(\delta_{i}(A)) \sigma_{i}(X) = \sigma_{i}(\varphi_{A}(A)X)
\]
gives (ii). Since \( S \) and \( \widehat{S} \) are both nondegenerate subalgebras of \( M(\mathcal{X}) \), we have
\[
(X \times_{\sigma} \widehat{S}) \cdot (A \otimes \mathcal{X}) = \sigma_{i}(X) \cdot (A \otimes \mathcal{X}),
\]
and similarly \( (A \times_{S} \widehat{S}) \cdot (A \otimes \mathcal{X}) = A \otimes \mathcal{X} \). This verifies (iii), and the first assertion of the theorem is established. Since \( \varphi_{A \times_{S} A} \) is the restriction of \( \varphi_{A} \otimes \text{id}_{\mathcal{X}} \) which is injective if \( \varphi_{A} \) is, the assertion on the injectivity of \( \varphi_{A \times_{S} A} \) follows.

As in the computation (3.1), but using Lemma 4.5 instead of coaction nondegeneracy, we can deduce the equality \( \mathcal{K}(X \times_{\sigma} \widehat{S}) = \mathcal{K}(X) \times_{\sigma^{(1)}} \widehat{S} \). Finally,
\[
\varphi_{A \times_{S} A}(\delta_{i}(a)(1_{M(A)} \otimes x)) = \varphi_{A} \otimes \text{id}_{\mathcal{X}} \circ \text{id}_{A} \otimes \text{id}_{S}(\delta(a))(1_{M(K(X))} \otimes x)
\]
\[
= \text{id}_{\mathcal{K}(X)} \otimes \text{id}_{S} \circ \varphi_{A} \otimes \text{id}_{S}(\delta(a))(1_{M(K(X))} \otimes x)
\]
\[
= \text{id}_{\mathcal{K}(X)} \otimes \text{id}_{S} \circ \sigma^{(1)}(\varphi_{A}(a))(1_{M(K(X))} \otimes x)
\]
\[
= \sigma^{(1)}(\varphi_{A}(a))(1_{M(K(X))} \otimes x),
\]
in the third step of which we use the $\delta$-$\sigma^{(1)}$ equivariance of $\varphi_A$ obtained in Proposition 3.4. This completes the proof. \hfill $\square$

**Definition 4.7.** We call the $C^*$-correspondence $(X \rtimes_\sigma \hat{S}, A \rtimes_\delta \hat{S})$ in Theorem 4.6 the *reduced crossed product correspondence* of $(X, A)$ by the coaction $(\sigma, \delta)$ of $S$.

**Remark 4.8.** We require no universal property of the crossed product $A \rtimes_\delta \hat{S}$ to define the left action $\varphi_{A \rtimes_\delta \hat{S}} : A \rtimes_\delta \hat{S} \to \mathcal{L}(X \rtimes_\sigma \hat{S})$. It is just the restriction of $\varphi_{M(A \otimes K(H))}$.

**Remark 4.9.** For an action $(\gamma, \alpha)$ of a locally compact group $G$ on $(X, A)$, one can form the crossed product correspondence $(X \rtimes_{\gamma,r} G, A \rtimes_{\alpha,r} G)$ by [5, Proposition 3.2]. We will see in Corollary B.7 that it is isomorphic to the reduced crossed product correspondence $(X \rtimes_{\gamma\hat{G}} \hat{S}_G, A \rtimes_{\delta\hat{G}} \hat{S}_G)$, where $(\sigma_{\hat{G}}, \delta_{\hat{G}})$ is the coaction of the Hopf $C^*$-algebra $S_{\hat{G}}$ given in (B.3). On the other hand, if $(\sigma, \delta)$ is a nondegenerate coaction of $G$ on $(X, A)$ ([5, Definition 2.10]) and if $\sigma_\lambda := \text{id}_X \otimes \lambda \circ \sigma$ as (2.9), then the crossed product correspondence by $(\sigma, \delta)$ in the sense of [5, Proposition 3.9] is just the reduced crossed product correspondence by the coaction $(\sigma_{\lambda}, \delta_{\lambda})$ of the Hopf $C^*$-algebra $S_{\hat{G}}$. Construction in Theorem 4.6 thus extends both of the crossed product correspondences by actions and nondegenerate coactions of locally compact groups on $C^*$-correspondences.

As [12, Remark 2.7], we have the following, the proof of which is routine.

**Corollary 4.10.** Let $(\sigma, \delta)$ be a coaction of $S$ on $(X, A)$. Then the map

$$(j^\sigma_X, j^\delta_A) : (X, A) \to (M(X \rtimes_\sigma \hat{S}), M(A \rtimes_\delta \hat{S}))$$

defined by

$$j^\sigma_X(\xi) \cdot c := \sigma_\xi(c), \quad j^\delta_A(a)c := \delta_a(c)$$

for $\xi \in X$, $a \in A$, and $c \in A \rtimes_\delta \hat{S}$ is a nondegenerate correspondence homomorphism such that $j^\sigma_X(X) \subseteq M_{A \rtimes_\delta \hat{S}}(X \rtimes_\sigma \hat{S})$.

**Remark 4.11.** Since $(X \rtimes_\sigma \hat{S}, A \rtimes_\delta \hat{S})$ is by Theorem 4.6 a nondegenerate $C^*$-correspondence, Theorem 2.6(i) assures us that the universal covariant representation $(k_{X \rtimes_\sigma \hat{S}}, k_{A \rtimes_\delta \hat{S}})$ extends to an $(A \rtimes_\delta \hat{S})$-strictly continuous representation

$$(k_{X \rtimes_\sigma \hat{S}}, k_{A \rtimes_\delta \hat{S}}) : (M_{A \rtimes_\delta \hat{S}}(X \rtimes_\sigma \hat{S}), M(A \rtimes_\delta \hat{S})) \to M_{A \rtimes_\delta \hat{S}}(\mathcal{O}_{X \rtimes_\sigma \hat{S}}).$$

Combining this and Corollary 4.10, we have a representation

$$(k_{X \rtimes_\sigma \hat{S}} \circ j^\sigma_X, k_{A \rtimes_\delta \hat{S}} \circ j^\delta_A) : (X, A) \to M_{A \rtimes_\delta \hat{S}}(\mathcal{O}_{X \rtimes_\sigma \hat{S}}).$$

**Example 4.12.** Let $(\sigma, \delta)$ be a coaction of $S$ on a $\varphi$-identity correspondence $(A(\varphi), A)$ arising from a cocycle $\nu$ for $\delta$ satisfying the relation (3.6) as considered in Example 3.10. By Theorem 4.6, we can form the reduced crossed product correspondence $(A(\varphi) \rtimes_\sigma \hat{S}, A \rtimes_\delta \hat{S})$. Let $v_i = \text{id}_A \otimes \iota_S(\nu)$. Since the multiplication of $v_i$ from the left gives an isomorphism from the Hilbert module $A \rtimes_\delta \hat{S}$ onto $A(\varphi) \rtimes_\sigma \hat{S}$, we can regard the $C^*$-correspondence $A(\varphi) \rtimes_\sigma \hat{S}$ as the Hilbert module $A \rtimes_\delta \hat{S}$ with the left action

$$\varphi_{A \rtimes_\delta \hat{S}}(c) d = v_i^* \varphi \otimes \text{id}_{K(H)}(c)v_i d$$
Lemma 5.1. Let $\varphi: A \times_S \hat{S}$ be a reduced Hopf $C^*$-algebra and vector $d$ in the Hilbert module $A \times_S \hat{S}$. Note that $\varphi_{A \times_S \hat{S}}$ is injective.

If we take $v = 1_{M(A \otimes S)}$, then (3.6) reduces to
\[ \delta \circ \varphi = \varphi \otimes \id_{\hat{S}} \circ \delta, \]
and then $\varphi_{A \times_S \hat{S}}$ maps $A \times_S \hat{S}$ onto itself:
\[ \varphi_{A \times_S \hat{S}} \left( \delta_i(a)(1_{M(A)} \otimes x) \right) = \varphi \otimes \id_{\hat{S}} \left( \delta_i(a)(1_{M(A)} \otimes x) \right) \]
for $a \in A$ and $x \in \hat{S}$. Hence $\varphi_{A \times_S \hat{S}}$ defines an automorphism $\varphi \times \id$ on $A \times_S \hat{S}$ such that
\[ (\varphi \times \id)(\delta_i(a)(1_{M(A)} \otimes x)) = \delta_i(\varphi a)(1_{M(A)} \otimes x) \]
for $a \in A$ and $x \in \hat{S}$. We thus see that $A(\varphi) \times_S \hat{S}$ is the $(\varphi \times \id)$-identity correspondence $A \times_S \hat{S}(\varphi \times \id)$ in the case $v = 1_{M(A \otimes S)}$.

5. Reduced crossed products

In this section, we prove our main result (Theorem 5.10) which states that under a reasonable condition, $O_X \times_S \hat{S} \cong O_X \times_S \hat{S}$. Throughout this section, we simply write $\mathcal{X} = \mathcal{K}$. As before, the following representation
\[ \left( k_X \otimes \id_{\mathcal{X}}, k_A \otimes \id_{\mathcal{X}} \right): (M_{A \otimes \mathcal{X}}(X \otimes \mathcal{X}), M(A \otimes \mathcal{X})) \to M_{A \otimes \mathcal{X}}(O_X \otimes \mathcal{X}) \]
will play an important role. Recall that $k_X \otimes \id_C$ denotes the $(A \otimes C)$-strict extension to $M_{A \otimes C}(X \otimes C)$.

Lemma 5.1. Let $(X, A)$ be a $C^*$-correspondence. Let $S$ be a reduced Hopf $C^*$-algebra and $\iota_S: S \hookrightarrow M(\mathcal{X})$ be the inclusion. Then the following diagram of the relative strictly continuous extensions
\[
\begin{array}{ccc}
(M_{A \otimes S}(X \otimes S), M(A \otimes S)) & \xrightarrow{\id_X \otimes \iota_S, \id_A \otimes \iota_S} & (M_{A \otimes \mathcal{X}}(X \otimes \mathcal{X}), M(A \otimes \mathcal{X})) \\
(k_X \otimes \id_S, k_A \otimes \id_S) & \downarrow & (k_X \otimes \id_X, k_A \otimes \id_X) \\
M_{A \otimes S}(O_X \otimes S) & \xrightarrow{\id_{O_X} \otimes \iota_S} & M_{A \otimes \mathcal{X}}(O_X \otimes \mathcal{X})
\end{array}
\]
commutes.

Proof. By [4, Proposition A.11], we see that the upper and lower horizontal maps are $(A \otimes S)$-strict to $(A \otimes \mathcal{X})$-strictly continuous. Hence the two compositions in (5.1) are $(A \otimes S)$-strict to $(A \otimes \mathcal{X})$-strictly continuous. Since the diagram commutes on $(X \otimes S, A \otimes S)$, the conclusion follows by strict continuity. \hfill \Box

Corollary 5.2. Let $(\sigma, \delta)$ be a coaction of $S$ on $(X, A)$ such that $J_X$ is weakly $\delta$-invariant. Then
\[ \sigma_i(X) \subseteq M_{A \otimes \mathcal{X}}(X \otimes \mathcal{X}) \]
so that
\[ X \times_{\sigma} \hat{S} \subseteq M_{A \otimes \mathcal{X}}(X \otimes \mathcal{X}). \]

Also,
\[ k_X \otimes \id_{\mathcal{X}}(\sigma_i(\xi)) = \zeta_i(k_X(\xi)), \quad k_A \otimes \id_{\mathcal{X}}(\delta_i(a)) = \zeta_i(k_A(a)) \]
for $\xi \in X$ and $a \in A$. 

Proposition 5.4. Let \((\sigma, \delta)\) be a coaction of \(S\) on \((X, A)\) such that \(J_X\) is weakly \(\delta\)-invariant. Then, the restriction of \((k_X \otimes \id_{\mathcal{X}}, k_A \otimes \id_{\mathcal{X}})\) to \((X \rtimes_{\sigma} \hat{S}, A \rtimes_{\delta} \hat{S})\) defines an injective representation

\[
(k_X \rtimes_{\sigma} \id_{\hat{S}}, k_A \rtimes_{\delta} \id_{\hat{S}}) : (X \rtimes_{\sigma} \hat{S}, A \rtimes_{\delta} \hat{S}) \to \mathcal{O}_X \rtimes_{\zeta} \hat{S}
\]

such that

\[
k_X \rtimes_{\sigma} \id_{\hat{S}}(\sigma_i(\xi) \cdot (1_{M(A)} \otimes x)) = \zeta(k_X)(1_{M(\mathcal{O}_X)} \otimes x),
\]

\[
k_A \rtimes_{\delta} \id_{\hat{S}}(\delta_i(a)(1_{M(A)} \otimes x)) = \zeta(k_A)(1_{M(\mathcal{O}_X)} \otimes x),
\]

\[
k_X \rtimes_{\sigma} \id_{\hat{S}}(\varphi_{M(A \otimes \mathcal{X})}(1_{M(A)} \otimes x)\sigma_i(\xi)) = (1_{M(\mathcal{O}_X)} \otimes x)\zeta(k_X(\xi))
\]

for \(\xi \in X\), \(x \in \hat{S}\), and \(a \in A\).

Proof. Since \(X \rtimes_{\sigma} \hat{S} \subseteq M_{A \otimes \mathcal{X}}(X \otimes \mathcal{X})\), the restriction

\[
(k_X \rtimes_{\sigma} \id_{\hat{S}}, k_A \rtimes_{\delta} \id_{\hat{S}}) := \left(\overline{k_X \otimes \id_{\mathcal{X}}}_{|X \rtimes_{\sigma} \hat{S}}, \overline{k_A \otimes \id_{\mathcal{X}}}_{|A \rtimes_{\delta} \hat{S}}\right)
\]

is an injective representation of \((X \rtimes_{\sigma} \hat{S}, A \rtimes_{\delta} \hat{S})\) on \(M_{A \otimes \mathcal{X}}(\mathcal{O}_X \otimes \mathcal{X})\). Using the equalities (5.2), we have

\[
k_X \rtimes_{\sigma} \id_{\hat{S}}(\sigma_i(\xi) \cdot (1_{M(A)} \otimes x)) = \overline{k_X \otimes \id_{\mathcal{X}}(\sigma_i(\xi) \cdot (1_{M(A)} \otimes x))} = \overline{k_X \otimes \id_{\mathcal{X}}(\sigma_i(\xi)) \cdot \overline{k_A \otimes \id_{\mathcal{X}}(1_{M(A)} \otimes x)}} = \zeta(k_X(\xi))(1_{M(\mathcal{O}_X)} \otimes x)
\]

for \(\xi \in X\) and \(x \in \hat{S}\), and similarly for \(k_A \rtimes_{\delta} \id_{\hat{S}}\). The last equality of (5.3) can be seen similarly. \(\square\)

For an action \((\gamma, \alpha)\) of an amenable group \(G\) on \((X, A)\), it was proved in [7, Proposition 2.7] that \(J_X \rtimes_{\alpha, r} G = J_X \rtimes_{\gamma, r} G\). We prove a partial analogue of this in the next proposition.

Proposition 5.5. Let \((\sigma, \delta)\) be a coaction of \(S\) on \((X, A)\) such that \(J_X\) is weakly \(\delta\)-invariant. Then

\[
\delta_i(J_X)(1_{M(A)} \otimes \hat{S}) \subseteq J_{X \rtimes_{\sigma} \hat{S}}.
\]
Proof: Since the representation \((k_X \rtimes_{\sigma} \mathrm{id}_S, k_A \rtimes_{\delta} \mathrm{id}_S)\) is injective, it suffices to show that
\[
  k_A \rtimes_{\delta} \mathrm{id}_S(\delta_i(J_X)(1_{M(A)} \otimes \widehat{S})) \subseteq (k_X \rtimes_{\sigma} \mathrm{id}_S)^{(1)}(\mathcal{K}(X \rtimes_{\sigma} \widehat{S}))
\]
by [14, Proposition 3.3]. For this, let us first note the following. By Theorem 2.6.(ii), we have
\[
  (k_X \otimes \mathrm{id}_X)^{(1)} = \overline{k_X \otimes \mathrm{id}_X}^{(1)}
\]
on \(\mathcal{K}(M_{A \odot \mathcal{K}}(X \otimes \mathcal{K}'))\). Hence
\[
  \overline{(k_X \otimes \mathrm{id}_X)^{(1)}(\mathcal{K}(X \rtimes_{\sigma} \widehat{S}))} = \overline{(k_X \otimes \mathrm{id}_X)^{(1)}(\mathcal{K}(X \rtimes_{\sigma} \widehat{S}))} = (k_X \rtimes_{\sigma} \mathrm{id}_S)^{(1)}(\mathcal{K}(X \rtimes_{\sigma} \widehat{S}))
\]
by Remark 5.3 and Proposition 5.4.

In much the same way as the calculation (4.1) in the proof of Theorem 4.6, we see that
\[
  \delta_i(J_X)(1_{M(A)} \otimes \widehat{S}) \subseteq M(A \otimes \mathcal{K}; J_X \otimes \mathcal{K})
\]
since \(J_X\) is weakly \(\delta\)-invariant. It then follows by Proposition 5.4, Lemma 2.9, and the above equality that
\[
  k_A \rtimes_{\delta} \mathrm{id}_S(\delta_i(J_X)(1_{M(A)} \otimes \widehat{S}))
  = k_A \otimes \mathrm{id}_X(\delta_i(J_X)(1_{M(A)} \otimes \widehat{S}))
  = (k_X \otimes \mathrm{id}_X)^{(1)}(\varphi_{M(A \odot \mathcal{K})}(\delta_i(J_X)(1_{M(A)} \otimes \widehat{S}))
  = (k_X \otimes \mathrm{id}_X)^{(1)}(\varphi_A(J_X))(1_{M(\mathcal{K}(X))} \otimes \widehat{S})
  \subseteq (k_X \otimes \mathrm{id}_X)^{(1)}(\mathcal{K}(X) \rtimes_{\sigma(1)} \widehat{S})
  = (k_X \rtimes_{\sigma} \mathrm{id}_S)^{(1)}(\mathcal{K}(X) \rtimes_{\sigma} \widehat{S}),
\]
where the third and last step come from the \(\delta\)-\(\sigma(1)\) equivariance of \(\varphi_A\) and the equality \(\mathcal{K}(X) \rtimes_{\sigma(1)} \widehat{S} = \mathcal{K}(X) \rtimes_{\sigma} \widehat{S}\), respectively. This establishes the proposition. \(\square\)

By Proposition 5.5, we have \(J_X \rtimes_{\delta} \widehat{S} = A \rtimes_{\delta} \widehat{S}\) if \(J_X = A\).

Remark 5.6. Recall from Definition 3.1 and Lemma 3.2 of [11] that a nondegenerate correspondence homomorphism \((\psi, \pi) : (X, A) \to (M(Y), M(B))\) is Cuntz-Pimsner covariant if \(\psi(X) \subseteq M_B(Y)\) and \(\pi(J_X) \subseteq M(B; J_Y)\). Corollary 4.10 and Proposition 5.5 then assure us that the representation \((j_X^\mathcal{Y}, j_A^\mathcal{Y})\) is always Cuntz-Pimsner covariant since (5.4) is obviously equivalent to
\[
  j_A^\mathcal{Y}(J_X) \subseteq M(A \rtimes_{\delta} \widehat{S}; J_X \rtimes_{\delta} \widehat{S}).
\]
Therefore, Theorem 4.4 of [12] can be improved as follows: if \((\sigma, \delta)\) is a non-degenerate coaction of a locally compact group \(G\) on \((X, A)\) such that \(\delta(J_X) \subseteq M(A \otimes C^*(G); J_X \otimes C^*(G))\), then we always have \(O_X \rtimes_{\zeta} G \cong O_X \rtimes_{\sigma} G\).

Theorem 5.7. Let \((\sigma, \delta)\) be a coaction of a reduced Hopf \(C^*\)-algebra \(S\) on a \(C^*\)-correspondence \((X, A)\) such that \(J_X\) is weakly \(\delta\)-invariant. Then the following conditions are equivalent:

(i) The representation \((k_X \rtimes_{\sigma} \mathrm{id}_S, k_A \rtimes_{\delta} \mathrm{id}_S) : (X \rtimes_{\sigma} \widehat{S}, A \rtimes_{\delta} \widehat{S}) \to O_X \rtimes_{\zeta} \widehat{S}\)
is covariant.
(ii) The ideal $J_{X \ltimes_s \hat{S}}$ is contained in $M(A \otimes \mathcal{K}; J_X \otimes \mathcal{K})$.

(iii) The product $J_{X \ltimes_s \hat{S}}(\ker \varphi_A \otimes \mathcal{K})$ is zero.

Proof. (i) $\Leftrightarrow$ (ii): Suppose (i). Since $(k_X \ltimes_s id_{\hat{S}}, k_A \ltimes_s id_{\hat{S}})$ is injective, we have

$$J_{X \ltimes_s \hat{S}} = (k_X \ltimes_s id_{\hat{S}})^{-1}(\ker \hat{\varphi}_{A \ltimes_s \hat{S}})$$

by the comment below [15, Proposition 5.14]. The same reason shows

$$J_{M_\alpha \otimes \mathcal{K}(X \otimes \mathcal{K})} = (k_X \otimes id_{\mathcal{K}})^{-1}(\ker \hat{\varphi}_{A \otimes \mathcal{K}(X \otimes \mathcal{K}))}$$

since $\mathcal{K}$ is nuclear and hence $(k_X \otimes id_{\mathcal{K}}, k_A \otimes id_{\mathcal{K}})$ is covariant by Corollary 2.11. It thus follows that $J_{X \ltimes_s \hat{S}} \subseteq J_{M_\alpha \otimes \mathcal{K}(X \otimes \mathcal{K})}$ by Remark 5.3 and Proposition 5.4. But, the latter is contained in $M(A \otimes \mathcal{K}; J_X \otimes \mathcal{K})$ again by Corollary 2.11. This proves (i) $\Rightarrow$ (ii). Conversely, suppose (ii). Restricting the equality (2.5) of Lemma 2.9 to the subalgebra $J_{X \ltimes_s \hat{S}}$, we can write

$$k_X \ltimes_s id_{\hat{S}} = (k_X \ltimes_s id_{\hat{S}})^{(1)} \circ \varphi_{A \ltimes_s \hat{S}};$$

which verifies (ii) $\Rightarrow$ (i).

(ii) $\Leftrightarrow$ (iii): Assuming (ii) we have

$$J_{X \ltimes_s \hat{S}}(\ker \varphi_A \otimes \mathcal{K}) = J_{X \ltimes_s \hat{S}}(A \otimes \mathcal{K})(\ker \varphi_A \otimes \mathcal{K}) \subseteq (J_X \otimes \mathcal{K})(\ker \varphi_A \otimes \mathcal{K}) = 0,$$

and hence we get (iii). Finally, we always have

$$\varphi_{A \otimes \mathcal{K}}(A \otimes \mathcal{K}) J_{X \ltimes_s \hat{S}} \subseteq \varphi_{A \otimes \mathcal{K}}(A \otimes \mathcal{K}) K(X \ltimes_s \hat{S}) \subseteq K(X \otimes \mathcal{K})$$

by Remark 5.3. Since $\ker \varphi_{A \otimes \mathcal{K}} = \ker(\varphi_A \otimes id_{\mathcal{K}}) = \ker \varphi_A \otimes \mathcal{K}$ by the exactness of $\mathcal{K}$, (iii) implies

$$((A \otimes \mathcal{K})J_{X \ltimes_s \hat{S}}) \ker \varphi_{A \otimes \mathcal{K}} = (A \otimes \mathcal{K})(J_{X \ltimes_s \hat{S}}(\ker \varphi_A \otimes \mathcal{K})) = 0.$$ 

Therefore $(A \otimes \mathcal{K})J_{X \ltimes_s \hat{S}} \subseteq J_{X \otimes \mathcal{K}}$. But $J_{X \otimes \mathcal{K}} = J_X \otimes \mathcal{K}$ by Corollary 2.11, which proves (iii) $\Rightarrow$ (ii). \hfill $\square$

From (ii) and (iii) of Theorem 5.7, we immediately have the following corollary.

**Corollary 5.8.** Let $(\sigma, \delta)$ be a coaction of $S$ on $(X, A)$ such that $J_X$ is weakly $\delta$-invariant. If either $J_X = A$ or $\varphi_A$ is injective, then $(k_X \ltimes_s id_{\hat{S}}, k_A \ltimes_s id_{\hat{S}})$ is covariant.

**Corollary 5.9.** Let $(\sigma, \delta)$ be a coaction of $S$ on $(X, A)$ such that $\delta$ is trivial, that is, $\delta(a) = a \otimes 1_{M(S)}$ for $a \in A$. If the triple $(J_X, A, \hat{S})$ satisfies the slice map property, then $(k_X \ltimes_s id_{\hat{S}}, k_A \ltimes_s id_{\hat{S}})$ is covariant. Moreover, $J_{X \ltimes_s \hat{S}} = J_X \otimes \hat{S}$.

**Proof.** Evidently, $J_X$ is weakly $\delta$-invariant in this case. Hence the representation $(k_X \ltimes_s id_{\hat{S}}, k_A \ltimes_s id_{\hat{S}})$ on $O_X \ltimes_{\xi} \hat{S}$ makes sense by Proposition 5.4. To show that it is covariant, we may use the equivalent condition (iii) of Theorem 5.7. First note that $\varphi_{A \ltimes_s \hat{S}} = \varphi_{A \otimes \hat{S}} = \varphi_A \otimes id_{\hat{S}}$ since $\delta$ is trivial. Then

$$\ker \varphi_{A \ltimes_s \hat{S}} = \ker(\varphi_A \otimes id_{\hat{S}}) = \ker \varphi_A \otimes \hat{S};$$

by Remarks 2.10, and hence

$$J_{X \ltimes_s \hat{S}}(\ker \varphi_A \otimes \mathcal{K}) = (J_{X \ltimes_s \hat{S}}(\ker \varphi_A \otimes \hat{S}))(1_{M(A)} \otimes \mathcal{K}) = 0.$$
since \( \hat{S} \) nondegenerately acts on \( \mathcal{K} \). Therefore \((k_X \rtimes_\sigma id_{\hat{S}}, k_A \rtimes_\delta id_{\hat{S}})\) is covariant.

Let \( \omega \in \mathcal{L}(\mathcal{H})_* \) and \( T \in \mathcal{K} \). Applying the slice map \( id_A \otimes (\omega T) \) to \( J_X \rtimes_\sigma \hat{S} \) and then multiplying \( a \in A \) yields

\[
a(id_A \otimes (\omega T))(J_X \rtimes_\sigma \hat{S}) = (id_A \otimes \omega)((a \otimes T)J_X \rtimes_\sigma \hat{S}) \subseteq J_X,
\]

in which the last inclusion is due to the equivalent condition (ii) of Theorem 5.7. We thus have \((id_A \otimes \omega)(J_X \rtimes_\sigma \hat{S}) \subseteq J_X\), and conclude by Remarks 2.10.(2) that \( J_X \rtimes_\sigma \hat{S} \subseteq F(J_X, A, \hat{S}) = J_X \otimes \hat{S} \). The converse follows from Proposition 5.5. \( \square \)

We now state and prove our main theorem.

**Theorem 5.10.** Let \((\sigma, \delta)\) be a coaction of a reduced Hopf \( C^*\)-algebra \( S \) on a \( C^*\)-correspondence \((X, A)\) such that \( J_X \) is weakly \( \delta\)-invariant. Suppose that the representation \((k_X \rtimes_\sigma id_{\hat{S}}, k_A \rtimes_\delta id_{\hat{S}})\) is covariant. Then the integrated form

\[
(k_X \rtimes_\sigma id_{\hat{S}}) \times (k_A \rtimes_\delta id_{\hat{S}}) : O_X \rtimes_\sigma \hat{S} \to O_X \rtimes_\delta \hat{S}
\]

is a surjective isomorphism.

**Proof.** Set \( \Psi = (k_X \rtimes_\sigma id_{\hat{S}}) \times (k_A \rtimes_\delta id_{\hat{S}}) \). Note that the embedding \( k_A \rtimes_\delta id_{\hat{S}} \) is clearly nondegenerate, and hence \( \Psi \) is also nondegenerate.

We claim that \( \Psi(O_X \rtimes_\sigma \hat{S}) \) contains all the elements of the form

\[
(1_{M(O_X)} \otimes x)\zeta(k_X(\xi_1) \cdots k_X(\xi_n)k_X(\eta_1) \cdots k_X(\eta_m))(1_{M(O_X)} \otimes y)
\]

for nonnegative integers \( m \) and \( n \), vectors \( \xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_m \in X \), and \( x, y \in \hat{S} \). This will prove that \( \Psi \) is surjective ([14, Proposition 2.7]). Since

\[
\zeta((k_A(A))(1_{M(O_X)} \otimes \hat{S}) \subseteq \Psi(O_X \rtimes_\sigma \hat{S})
\]

by (5.3) of Proposition 5.4, we only show by considering adjoints that

\[
(1_{M(O_X)} \otimes x)\zeta(k_X(\xi_1) \cdots k_X(\xi_n)) \in \Psi(O_X \rtimes_\sigma \hat{S}),
\]

for positive integers \( n \), vectors \( \xi_1, \ldots, \xi_n \in X \), and \( x \in \hat{S} \). We now proceed by induction on \( n \). For \( n = 1 \), (5.5) follows from the last equality of (5.3). Suppose that (5.5) is true for an \( n \). Let \( \xi, \xi_1, \ldots, \xi_n \) be \( n + 1 \) vectors in \( X \) and \( x \in \hat{S} \). Take an element \( C \in O_X \rtimes_\sigma \hat{S} \) such that

\[
\Psi(C) = (1_{M(O_X)} \otimes x)\zeta(k_X(\xi_1)k_X(\xi_2) \cdots k_X(\xi_n)).
\]

By Remark 4.11, we have

\[
\overline{k_X \rtimes_\sigma \hat{S}(j_{O_X}^\omega(\xi))}(M_A \rtimes_S(O_X \rtimes_\sigma \hat{S})).
\]

We claim that

\[
\overline{\Psi(k_X \rtimes_\sigma \hat{S}(j_{O_X}^\omega(\xi))} = j_{O_X}^\delta(k_X(\xi)),
\]

where \( j_{O_X}^\delta : O_X \to M(O_X \rtimes_\delta \hat{S}) \) is the canonical homomorphism such that \( j_{O_X}^\delta(c)D = \zeta(c)D \) for \( c \in O_X \) and \( D \in O_X \rtimes_\delta \hat{S} \). In fact, for

\[
v = \Psi(k_A \rtimes_S(\delta_1(a)(1_{M(A)} \otimes x))) = \zeta(k_A(a))(1_{M(O_X)} \otimes x),
\]

\[
\overline{\Psi(k_X \rtimes_\sigma \hat{S}(j_{O_X}^\omega(\xi))} = j_{O_X}^\delta(k_X(\xi)),
\]

since \( k_X \rtimes_\sigma id_{\hat{S}}, k_A \rtimes_\delta id_{\hat{S}} \) is covariant.
we have

\[
\Psi(k_X \times_S S(j_X^\gamma(\xi))) = \Psi(k_X \times_S S(j_X^\gamma(\xi))k_{A \times_S S}(\delta_\alpha(1_{M(A)} \otimes x)))
\]

which verifies the equality (5.6) since \(k_A \rtimes \delta \text{id}_S\) is nondegenerate. It is now obvious that the product \(C \overline{k_X \times_S S(j_X^\gamma(\xi))} \in O_{X \times_S S}\) is such that

\[
\Psi(C \overline{k_X \times_S S(j_X^\gamma(\xi))) = (1_{M(S_X \otimes x)} \otimes x)\zeta_k(k_X(\xi_1) \cdots k_X(\xi_n)k_X(\xi),)
\]

Consequently, the statement (5.5) is true for all positive integer \(n\), and hence \(\Psi\) is surjective.

Let \(\beta : T \to Aut(O_X)\) be the gauge action. The strict extensions \(\overline{\beta}_z \otimes \text{id}_X\) are automorphisms on \(M(O_X \otimes X)\). We have

\[
\overline{\beta}_z \otimes \text{id}_X(\zeta_k(\beta_z(k_X(\xi))) = \zeta_k(\beta_z(k_X(\xi))) = z\zeta_k(k_X(\xi)),
\]

in the same way as the last part of the proof of Theorem 3.7. Therefore,

\[
\overline{\beta}_z \otimes \text{id}_X((k_X \times_\sigma \text{id}_S)(\sigma_\beta(1_{M(A)} \otimes x))) = \overline{\beta}_z \otimes \text{id}_X(\zeta_k(k_X(\xi)))(1_{M(S_X \otimes x)} \otimes x)
\]

and similarly

\[
\overline{\beta}_z \otimes \text{id}_X((k_X \times_\sigma \text{id}_S)(\delta_\alpha(1_{M(A)} \otimes x))) = (k_A \times_\delta \text{id}_S)(\delta_\alpha(1_{M(A)} \otimes x)).
\]

This proves that \(\overline{\beta}_z \otimes \text{id}_X\) maps the subalgebra \(O_X \times_\zeta S\) onto itself, and the injective covariant representation \((k_X \times_\sigma \text{id}_S, k_A \times_\delta \text{id}_S)\) admits a gauge action. We thus conclude by [14, Theorem 6.4] that \(\Psi\) is injective as well, which completes the proof.

Applying Theorem 5.10 to the case that \(S\) is commutative, we can extend Theorem 2.10 of [7] as the next corollary shows, the proof of which will be given in Appendix B. Let \((\gamma, \alpha)\) be an action of a locally compact group \(G\) on \((X, A)\). By Theorem A.6, \((\gamma, \alpha)\) defines a coaction \((\sigma^\gamma, \delta^\alpha)\) of \(C_0(G)\) on \((X, A)\), which in turn induces a coaction \(\zeta\) of \(C_0(G)\) on \(O_X\) by Theorem 3.7 and Remarks 3.9.(1). Let \(\beta^\zeta\) be the action of \(G\) on \(O_X\) corresponding to the coaction \(\zeta\). As in the formula (3) of [7], define

\[
(k_X \times_\gamma G, k_A \times_\alpha G) : (X \times_{\gamma, r} G, A \times_{\alpha, r} G) \to O_X \times_\beta^\zeta, r G
\]

by

\[
(k_X \times_\gamma G)(f)(r) = k_X(f(r)), \quad (k_A \times_\alpha G)(g)(r) = k_A(g(r))
\]

for \(f \in C_c(G, X),\) \(g \in C_c(G, A),\) and \(r \in G\), which will be seen to be a representation of \((X \times_{\gamma, r} G, A \times_{\alpha, r} G)\) on the \(C^*\)-algebra \(O_X \times_\beta^\zeta, r G\).
Corollary 5.11. Let \((\gamma, \alpha)\) be an action of a locally compact group \(G\) on \((X, A)\). If the representation \((k_X \rtimes_\gamma G, k_A \rtimes_\alpha G)\) is covariant, then its integrated form \((k_X \rtimes_\gamma G) \times (k_A \rtimes_\alpha G) : \mathcal{O}_{X \rtimes_\gamma G} \to \mathcal{O}_X \rtimes_\beta \delta G\) is a surjective isomorphism.

If \(S\) is defined by an amenable regular multiplicative unitary in the sense of [2], then we have the following:

Proposition 5.12. Let \((\sigma, \delta)\) be a coaction on \((X, A)\) of a reduced Hopf C*-algebra \(S\) defined by an amenable regular multiplicative unitary such that \(J_X\) is weakly \(\delta\)-invariant. If \(A\) is nuclear (or exact, respectively), then the same is true for \(\mathcal{O}_X \rtimes_\gamma S\).

Proof. If \(A\) is nuclear (or exact, respectively), then so is \(A \rtimes_\delta S\) by [21, Theorem 3.4] (or by [21, Theorem 3.13], respectively). Hence, the Toeplitz algebra \(T_{X \rtimes_\gamma S}\) is nuclear by [14, Corollary 7.2] (or exact by [14, Theorem 7.1], respectively). Since nuclearity or exactness passes to quotients, it suffices to show that the representation \((k_X \rtimes_\sigma id, k_A \rtimes_\delta id)\) gives rise to a surjection from \(T_{X \rtimes_\gamma S}\) onto \(\mathcal{O}_X \rtimes_\gamma S\).

The proof of this then goes parallel to the one given in the proof of Theorem 5.10 using the embedding \((\hat{i}_{X \rtimes_\sigma S}, \hat{i}_{A \rtimes_\delta S})\) of \((X \rtimes_\sigma S, A \rtimes_\delta S)\) into \(T_{X \rtimes_\gamma S}^{-}\) instead of \((k_X \rtimes_\delta S, k_A \rtimes_\delta S)\) used in there. \(\square\)

Example 5.13. Let \((\sigma, \delta)\) be the coaction of \(S\) on \((A(\varphi), A)\) considered in Example 3.10 arising from a cocycle \(v\) satisfying (3.6). We have seen in Example 4.12 that \(A(\varphi) \rtimes_\sigma S\) is the Hilbert module \(A \rtimes_\delta \hat{S}\) with the left action (4.2). By Corollary 5.8 and Theorem 5.10, \(\mathcal{O}_{A(\varphi)} \rtimes_\gamma \hat{S}\) is then the Cuntz-Pimsner algebra \(\mathcal{O}_{A(\varphi) \rtimes_\gamma S}\) since \(J_{A(\varphi)} = A\) and \(\varphi_A\) is injective.

If \(v = 1_{M(A \otimes S)}\), then we see from Example 3.10 and 4.12 that
\[
\mathcal{O}_{A(\varphi) \rtimes_\gamma S} = \mathcal{O}_{A \rtimes_\delta \hat{S}}(\varphi \rtimes id) = (A \rtimes_\delta \hat{S}) \rtimes_\varphi id \mathbb{Z}
\]
and
\[
\mathcal{O}_{A(\varphi)} \rtimes_\gamma \hat{S} = (A \rtimes_\delta \mathbb{Z}) \rtimes_\gamma \hat{S},
\]
and then have a surjective isomorphism
\[
\Psi : (A \rtimes_\delta \hat{S}) \rtimes_\varphi id \mathbb{Z} = \mathcal{O}_{A(\varphi) \rtimes_\gamma S} \to \mathcal{O}_{A(\varphi)} \rtimes_\gamma \hat{S} = (A \rtimes_\delta \mathbb{Z}) \rtimes_\gamma \hat{S}.
\]

We now describe \(\Psi\) on the canonical generators of the double crossed products \((A \rtimes_\delta \hat{S}) \rtimes_\varphi id \mathbb{Z}\) and \((A \rtimes_\delta \mathbb{Z}) \rtimes_\gamma \hat{S}\). As the covariant representation \((\pi, \alpha)\) given in Example 3.10, let \((\bar{\pi}, \bar{\alpha})\) be the canonical covariant representation of the \(C^*\)-dynamical system \((A \rtimes_\delta \hat{S}, \mathbb{Z}, \varphi \rtimes id)\) on \(M((A \rtimes_\delta \hat{S}) \rtimes_\varphi id \mathbb{Z})\). Let
\[
d_1 = k_{A(\varphi)} \rtimes_\gamma S(\delta_\varphi(a) \cdot (1_{M(A)} \otimes x)) \in \mathcal{O}_{A(\varphi) \rtimes_\gamma S},
\]
\[
d_2 = \bar{\alpha}^{\hat{\pi}}(\delta_\varphi(a)(1_{M(A)} \otimes x)) \in (A \rtimes_\delta \hat{S}) \rtimes_\varphi id \mathbb{Z},
\]
\[
d_3 = \zeta_\gamma(k_{A(\varphi)}(a))(1_{M(\mathcal{O}_{A(\varphi)})} \otimes x) \in \mathcal{O}_{A(\varphi)} \rtimes_\gamma \hat{S},
\]
\[
d_4 = \zeta_\gamma(u^*\pi(a))(1_{M(A \rtimes_\varphi \mathbb{Z})} \otimes x) \in (A \rtimes_\delta \mathbb{Z}) \rtimes_\gamma \hat{S}.
\]
We then have \(d_1 = d_2\) in (5.7), and \(d_3 = d_4\) in (5.8). Since \(\Psi(d_1) = d_3\), we may write \(\Psi(d_2) = d_4\). Note that \(\Psi(\bar{\alpha}(\bar{u})) = \zeta_\gamma(\bar{u})\). Therefore
\[
\Psi(\bar{\alpha}^{\hat{\pi}}(\delta_\varphi(a)(1_{M(A)} \otimes x))) = \zeta_\gamma(\bar{u}^{\hat{\pi}}(\bar{u}^\gamma(a))(1_{M(A \rtimes_\varphi \mathbb{Z})} \otimes x)),
\]
or equivalently, by the fact that $(\pi, u)$ and $(\bar{\pi}, \bar{u})$ are covariant representations,

$$\Psi(\bar{\pi}(\delta_i(a)(1_{M(A)} \otimes x)) \bar{\mu}^n) = \zeta_i(\pi(a)u^n)(1_{M(A \times_s Z)} \otimes x)$$

for $a \in A$, $x \in \hat{S}$, and $n \in \mathbb{Z}$.

**APPENDIX A. COACTIONS OF $C_0(G)$ ON $(X, A)$**

We show in Theorem A.6 that there exists a one-to-one correspondence between actions of a locally compact group $G$ on $(X, A)$ in the sense of [5] and coactions of the commutative Hopf $C^*$-algebra $C_0(G)$ on $(X, A)$.

Let $(X, A)$ be a nondegenerate $C^*$-correspondence as before, and $G$ be a locally compact Hausdorff space. We denote by $C_b(G, M(X)_s)$ the Banach space of all bounded continuous functions from $G$ to $M(X)$ endowed with the strict topology, and by $C_b(G, X)$ the closed subspace of $C_b(G, M(X)_s)$ consisting of functions with values in $X$ which are also continuous with respect to the norm topology on $X$. We denote by $C_c(G, X)$ the subspace of $C_b(G, X)$ of all compactly supported functions; $C_0(G, X)$ is the norm closure of $C_c(G, X)$.

Note that if $X = A$, then $C_b(G, M(X)_s)$ becomes a $C^*$-algebra under the usual point-wise operations. In this case,

$$C_b(G, M(X)_s) = M(X \otimes C_0(G))$$

([1, Corollary 3.4]). We first generalize this in Theorem A.4 to nondegenerate $C^*$-correspondences, which will allows us to prove the bijective correspondence between actions and coactions mentioned above.

**Proposition A.1.** $(C_b(G, M(X)_s), C_b(G, M(A)_s))$ is a $C^*$-correspondence with respect to the following point-wise operations

$$m \cdot l(r) = m(r) \cdot l(r),$$

$$\langle m, n \rangle_{C_b(G, M(A)_s)}(r) = \langle m(r), n(r) \rangle_{M(A)},$$

$$\varphi_{C_b(G, M(A)_s)}(l) m(r) = \varphi_{M(A)}(l(r)) m(r)$$

for $m, n \in C_b(G, M(X)_s)$, $l \in C_b(G, M(A)_s)$, and $r \in G$.

**Proof.** The only part requiring proof is that the functions on (A.1) are strictly continuous. We prove this only for the function $\varphi_{C_b(G, M(A)_s)}(l) m$. The others can be handled in the same way. Let $\{r_i\}$ be a net in $G$ converging to an $r \in G$, $T' \in K(X)$, and $a \in A$. Write $\varphi = \varphi_{C_b(G, M(A)_s)}$. Factor $T = T' \varphi_A(a')$ for some $T' \in K(X)$ and $a' \in A$, which is possible by the Hewitt-Cohen factorization theorem (see for example [23, Proposition 2.33]) since the left action $\varphi_A$ is nondegenerate. Then the difference

$$T(\varphi(l)m)(r_i) - T(\varphi(l)m)(r) = (T' \varphi_A(a'l(r_i)) m(r_i) - T' \varphi_A(a'l(r)) m(r_i))$$

$$+ (T \varphi_{M(A)}(l(r)) m(r_i) - T \varphi_{M(A)}(l(r)) m(r))$$

converges to 0 by the strict continuity of both $l$ and $m$ and also by the boundedness of $m$. On the other hand, $(\varphi(l)m)(r_i) \cdot a - (\varphi(l)m)(r) \cdot a$ converges to 0 evidently. Hence $\varphi(l)m$ is strictly continuous. $\square$
It is clear that \((C_0(G, X), C_0(G, A))\) is also a \(C^*\)-correspondence with respect to the restriction of operations (A.1).

We call a correspondence homomorphism \((\psi, \pi) : (X, A) \to (Y, B)\) an isomorphism if both \(\psi\) and \(\pi\) are bijective. In this case, \((X, A)\) and \((Y, B)\) are said to be isomorphic. The next corollary is an easy consequence of Corollary 2.12.

**Corollary A.2.** The \(C^*\)-correspondence \((C_0(G, X), C_0(G, A))\) and the tensor product correspondence \((X \otimes C_0(G), A \otimes C_0(G))\) are isomorphic.

**Lemma A.3.** With respect to the operations (A.1), the followings are satisfied:

1. \(\varphi_{C_0(G,M(A)_n)} \left( C_0(G,M(A)_n) \right) C_0(G,X) = C_0(G,X)\);
2. \(C_0(G,X) \cdot C_0(G,M(A)_n) = C_0(G,X)\);
3. \(C_0(G,M(X)_n) \cdot C_0(G,A) = C_0(G,X)\).

**Proof.** On each of (i) and (ii), the right-handed space is evidently contained in the left. Since \((C_0(G,X), C_0(G,A))\) is isomorphic to the nondegenerate \(C^*\)-correspondence \((X \otimes C_0(G), A \otimes C_0(G))\), the Hewitt-Cohen factorization theorem shows that the same is true for (iii). For the converse, let \(l \in C_0(G,M(A)_n)\) and \(x \in C_0(G,X)\), and write \(x = \varphi_{C_0(G,A)}(f) y\) for some \(f \in C_0(G,A)\) and \(y \in C_0(G,X)\). Then

\[
\varphi_{C_0(G,M(A)_n)}(l) x = \varphi_{C_0(G,A)}(lf) y \in C_0(G,X),
\]

which proves (i). Similarly write \(x = z \cdot g\) for \(z \in C_0(G,X)\) and \(g \in C_0(G,A)\). Then \(x \cdot l = y \cdot (gl) \in C_0(G,X)\), which verifies (ii). Finally, the triangle inequality verifies that the functions in the left-handed space of (iii) are continuous, which gives (iii).

From now on, we identify \(C_0(G,X) = X \otimes C_0(G)\) as well as \(C_b(G,M(A)_n) = M(A \otimes C_0(G))\). The next theorem generalizes [1, Corollary 3.4].

**Theorem A.4.** The map

\[
(\psi, \text{id}) : (C_b(G,M(X)_n), C_b(G,M(A)_n)) \to (M(X \otimes C_0(G)), M(A \otimes C_0(G))
\]

given by

\[
\psi(m) \cdot f = m \cdot f
\]

for \(m \in C_b(G,M(X)_n)\) and \(f \in A \otimes C_0(G)\) is an isomorphism.

**Proof.** By Lemma A.3, we can apply [5, Proposition 1.28] to see that \((\psi, \text{id})\) is an injective correspondence homomorphism. It thus remains to show that \(\psi\) is surjective. Let \(n \in M(X \otimes C_0(G))\). For each \(r \in G\), define \(m_n(r) : A \to X\) and \(m^*_n(r) : X \to A\) by

\[
m_n(r)(a) := (n \cdot (a \otimes \phi_r))(r), \quad m^*_n(r)(\xi) := (n^*(\xi \otimes \phi_r))(r),
\]

where \(\phi_r \in C_c(G)\) such that \(\phi_r \equiv 1\) on a neighborhood of \(r\). It is immaterial which \(\phi_r\) we take to define \(m_n(r)\) and \(m^*_n(r)\) as long as \(\phi_r \equiv 1\) near \(r\). Since

\[
\langle n \cdot (a \otimes \phi_r), \xi \otimes \phi_r \rangle_{A \otimes C_0(G)} = \langle a, m^*_n(r)(\xi) \rangle_A
\]

by evaluating at \(r\), and thus obtain a function \(m_n : G \to M(X)\) with \(m_n(r)^* = m^*_n(r)\). By definition, \(\|m_n(r)\| \leq \|n\|\) for \(r \in G\), and hence \(m_n\) is bounded. To see that \(m_n\) is strictly continuous, let \(\{r_i\}\) be a net in \(G\) converging to an \(r \in G\), \(a \in A\), and \(\xi, \eta \in X\). Evidently, \(\{m_n(r_i) \cdot a\}\) converges to \(m_n(r) \cdot a\). The same is true for the net \(\{m_n(r_i)^*\xi\},\)
and hence \( \{ \theta_{n,ξ}m_n(r) \} = \{ \theta_{n,m_n(r)}ξ \} \) converges to \( \theta_{n,m_n(r)}ξ = \theta_{n,ξ}m_n(r) \), and consequently \( \{ Tm_n(r) \} \) converges to \( Tm_n(r) \) for \( T \in K(X) \). Therefore \( m_n \in C_b(G,M(X)_α) \). Finally, we have
\[
(n \cdot (a \otimes φ_r))(r) = m_n(r) \cdot a = (m_n \cdot (a \otimes φ_r))(r)
\]
for \( a \in A \) and \( r \in G \), which shows \( ψ(m_n) = n \).

In what follows, we identify \( C_b(G,M(X)_α) = M(X \otimes C_0(G)) \).

**Corollary A.5.** The \( C_0(G) \)-multiplier correspondence \( M_{C_0(G)}(X \otimes C_0(G)) \) coincides with \( C_b(G,X) \).

**Proof.** Evidently, \( M_{C_0(G)}(X \otimes C_0(G)) \supseteq C_b(G,X) \). For the converse, let \( m \in M_{C_0(G)}(X \otimes C_0(G)) \). Let \( r \in G \), and take a \( φ_r \in C_c(G) \) such that \( φ_r \equiv 1 \) on a neighborhood \( U \) of \( r \). Since the function \( φ_M(A \otimes C_0(G))(1_M(A) \otimes φ_r)m \) belongs to \( X \otimes C_0(G) \) and agrees with \( m \) on \( U \), we see that \( m \) is continuous at \( r \). This proves the converse.

Let \( Aut(X,A) \) be the group of isomorphisms from \( (X,A) \) onto itself. Recall from [5, Definition 2.5] that an action of a locally compact group \( G \) on \( (X,A) \) is a homomorphism \( (γ,α) : G \to Aut(X,A) \) such that for each \( ξ \in X \) and \( a \in A \), the maps
\[
G \ni r \mapsto γ_r(ξ) \in X, \quad G \ni r \mapsto α_r(a) \in A
\]
are both continuous.

**Theorem A.6.** If \( (γ,α) \) is an action of a locally compact group \( G \) on \( (X,A) \), then there exists a coaction \( (σ^γ,δ^α) \) of \( C_0(G) \) on \( (X,A) \) such that
\[
(A.2) \quad σ^γ(ξ)(r) = γ_r(ξ), \quad δ^α(a)(r) = α_r(a)
\]
for \( ξ \in X \), \( a \in A \), and \( r \in G \). Moreover, the formulas in \( (A.2) \) define a one-to-one correspondence between actions of \( G \) on \( (X,A) \) and coactions of \( C_0(G) \) on \( (X,A) \).

**Proof.** It is well-known that \( δ^α \) is a coaction of \( C_0(G) \) on \( A \) (see for example [24, Chapter 9]). By Corollary A.5, the first formula in \( (A.2) \) defines a map
\[
σ^γ : X \to M_{C_0(G)}(X \otimes C_0(G)) \subseteq M(X \otimes C_0(G)).
\]
By definition, \( (γ_r,α_r) \) is a correspondence homomorphism for \( r \in G \), that is,
\[
\begin{align*}
(i) & \quad γ_r(φ_A(a)ξ) = φ_A(α_r(a))γ_r(ξ); \\
(ii) & \quad (γ_r(ξ),γ_r(η))_A = α_r((ξ,η)_A)
\end{align*}
\]
for \( r \in G \), which is equivalent to
\[
\begin{align*}
(i) & \quad σ^γ(φ_A(a)ξ) = φ_M(A \otimes C_0(G))(δ^α(a))σ^γ(ξ); \\
(ii) & \quad (σ^γ(ξ),σ^γ(η))_M(A \otimes C_0(G)) = δ^α((ξ,η)_A).
\end{align*}
\]
Hence \( (σ^γ,δ^α) \) is a correspondence homomorphism. Let \( ξ \in X \), \( φ \in C_c(G) \), and \( ε > 0 \). Take a neighborhood \( U \) of the neutral element of \( G \) such that \( ||γ_r(ξ) - ξ|| < ε \) for \( r \in U \). Choose a finite subcover \( \{ U_r \} \) of the support of \( φ \), and a partition of unity \( \{ ϕ_i \} \) subordinate to \( \{ U_r \} \). One can easily check that
\[
||ξ \otimes φ - \sum_i φ_M(A \otimes C_0(G))(1_M(A) \otimes ϕ_i φ)σ^γ(γ_r^{-1}(ξ))|| < ε,
\]
where \( σ^γ(γ_r^{-1}(ξ)) = γ_r(ξ) \)}
which proves that $\varphi_{M(A \otimes C_0(G))}(1_{M(A)} \otimes C_0(G)) \sigma^\gamma(X) \supseteq X \otimes C_0(G)$. The opposite inclusion is obvious, and hence $\sigma^\gamma$ satisfies the coaction nondegeneracy. For the coaction identity of $\sigma^\gamma$, let $\text{ev}_r : C_0(G) \to \mathbb{C}$ be the evaluation at $r \in G$. It then suffices to show that

$$\text{id}_X \otimes \text{ev}_r \otimes \text{ev}_r \circ (\sigma^\gamma \otimes \text{id}_{C_0(G)}) \circ \sigma^\gamma = \text{id}_X \otimes \text{ev}_r \otimes \text{ev}_s \circ \sigma^\gamma$$

for $r, s \in G$ since the strict extensions $\text{id}_X \otimes \text{ev}_r \otimes \text{ev}_s$ on $M(X \otimes C_0(G) \otimes C_0(G))$ correspond to the evaluations $m(r, s)$ for $m \in C_b(G \times G, M(X)_s)$, and hence separate the points of $M(X \otimes C_0(G) \otimes C_0(G))$. Note that on $X \otimes C_0(G)$

$$\text{id}_X \otimes \text{ev}_r \otimes \text{ev}_r \circ (\gamma \otimes \text{id}_{C_0(G)}) = (\text{id}_X \otimes \text{ev}_r \circ \sigma^\gamma) \otimes \text{id}_X \otimes \text{ev}_s$$

and

$$\text{id}_X \otimes \text{ev}_r \otimes \text{ev}_r \circ (\text{id}_X \otimes \Delta_G) = \text{id}_X \otimes (\text{ev}_r \otimes \text{ev}_s \circ \Delta_G) = \text{id}_X \otimes \text{ev}_{rs}.$$ 

Also note that if $(\psi_i, \pi_i) : (X_i, A_i) \to (M(X_{i+1}), M(A_{i+1}))$ are nondegenerate correspondence homomorphism $(i = 1, 2)$, then $\psi_2 \circ \psi_1 = \psi_2 \circ \psi_1$. We thus have

$$\text{id}_X \otimes \text{ev}_r \otimes \text{ev}_r \circ (\sigma^\gamma \otimes \text{id}_{C_0(G)}) \circ \sigma^\gamma = \text{id}_X \otimes \text{ev}_r \otimes \text{ev}_r \circ (\sigma^\gamma \otimes \text{id}_{C_0(G)}) \circ \sigma^\gamma$$

$$= \text{id}_X \otimes \text{ev}_r \circ \sigma^\gamma \circ (\text{id}_X \otimes \text{ev}_s) \circ \sigma^\gamma$$

$$= \text{id}_X \otimes \text{ev}_s \circ \sigma^\gamma \circ \text{id}_X \otimes \text{ev}_s \circ \sigma^\gamma$$

$$= \gamma_r \circ \gamma_s = \gamma_{rs}$$

$$= \text{id}_X \otimes \text{ev}_{rs} \circ \sigma^\gamma$$

$$= \text{id}_X \otimes \text{ev}_r \otimes \text{ev}_s \circ (\text{id}_X \otimes \Delta_G) \circ \sigma^\gamma$$

$$= \text{id}_X \otimes \text{ev}_r \otimes \text{ev}_s \circ \text{id}_X \otimes \Delta_G \circ \sigma^\gamma.$$ 

This establishes the first part of the theorem.

To prove the remaining part, let $(\sigma, \delta)$ be a coaction of $C_0(G)$ on $(X, A)$. Note that $\sigma(\xi) \in M_{C_0(G)}(X \otimes C_0(G)) = C_0(X, A)$. For each $r \in G$, define $(\gamma^\sigma_r, \alpha^\delta_r) : (X, A) \to (X, A)$ by

$$\gamma^\sigma_r(\xi) = \sigma(\xi)(r), \quad \alpha^\delta_r(a) = \delta(a)(r) \quad (\xi \in X, a \in A).$$

Then $(\gamma^\sigma_r, \alpha^\delta_r)$ is an injective correspondence homomorphism since $\alpha^\delta_r$ is injective and $(\gamma^\sigma_r, \alpha^\delta_r)$ is the composition $(\text{id}_X \otimes \text{ev}_r \circ \sigma, \text{id}_X \otimes \text{ev}_r \circ \delta)$ of two correspondence homomorphism. Reversing the order of the above computation leading to the coaction identity of $\sigma^\gamma$ shows that $\gamma^\sigma_r \circ \gamma^\sigma_s = \gamma_{rs}$ for $r, s \in G$, which also proves that $\gamma^\gamma_r$ is surjective. Consequently, $(\gamma^\gamma, \alpha^\delta)$ is an action of $G$ on $(X, A)$. It is now obvious that (A.2) gives a one-to-one correspondence between actions and coactions. This completes the proof. 

**Appendix B. Reduced crossed product correspondences** $X \rtimes_{\sigma} \hat{S}_W$ 

It is well-known that $\mathcal{L}_A(A \otimes \mathcal{H}) = M(A \otimes K(\mathcal{H}))$ for a $C^*$-algebra $A$ and a Hilbert space $\mathcal{H}$. We generalize this in Proposition B.3 to a nondegenerate $C^*$-correspondence:

$$\mathcal{L}_A(A \otimes \mathcal{H}, X \otimes \mathcal{H}) = M(X \otimes K(\mathcal{H})).$$
Using this, we show in Corollary B.7 that the construction of Theorem 4.6 reduces to the crossed product correspondence $(X \times_{\gamma,\tau} G, A \times_{\alpha, r} G)$ in the sense of [5] when the coaction under consideration comes from an action $(\gamma, \alpha)$ of $G$ on $(X, A)$.

Since we always want to use the left Haar measure, our reduced crossed product correspondence in this commutative case must be regarded as the one by a coaction of the Hopf $C^*$-algebra $S_{\hat{W}_G}$ defined by the multiplicative unitary $\hat{W}_G$.

We first clarify the $C^*$-correspondence $(\mathcal{L}_A(A \otimes \mathcal{H}, X \otimes \mathcal{H}), \mathcal{L}_A(A \otimes \mathcal{H}))$ in the next lemma whose proof is trivial, and so we omit it.

**Lemma B.1.** Let $\mathcal{H}$ be a Hilbert space. Then $(\mathcal{L}_A(A \otimes \mathcal{H}, X \otimes \mathcal{H}), \mathcal{L}_A(A \otimes \mathcal{H}))$ is a $C^*$-correspondence with respect to the following operations

\[(B.1)\quad m \cdot l = m \circ l, \quad (m, n)_{\mathcal{L}_A(A \otimes \mathcal{H})} = m^* \circ n, \quad \varphi_{\mathcal{L}_A(A \otimes \mathcal{H})} = \varphi_{\mathcal{M}(A \otimes \mathcal{K}(\mathcal{H}))}\]

for $m, n \in \mathcal{L}_A(A \otimes \mathcal{H}, X \otimes \mathcal{H})$ and $l \in \mathcal{L}_A(A \otimes \mathcal{H})$.

Note that $(\mathcal{K}_A(A \otimes \mathcal{H}, X \otimes \mathcal{H}), \mathcal{K}_A(A \otimes \mathcal{H}))$ is also a $C^*$-correspondence with the restriction of the operations on $(B.1)$.

**Lemma B.2.** There exists an isomorphism

\[(\psi_0, \text{id}): (\mathcal{K}_A(A \otimes \mathcal{H}, X \otimes \mathcal{H}), \mathcal{K}_A(A \otimes \mathcal{H})) \rightarrow (X \otimes \mathcal{K}(\mathcal{H}), A \otimes \mathcal{K}(\mathcal{H}))\]

such that $\psi_0(\theta_{\xi \otimes h, a \otimes k}) = \xi \cdot a^* \otimes \theta_{h, k}$ for $\xi \in X, a \in A$, and $h, k \in \mathcal{H}$.

**Proof.** Let $\xi_i \in X, a_i \in A$, and $h_i, k_i \in \mathcal{H}$ for $i = 1, \ldots, n$. We claim that the norm of the operator $\sum_{i=1}^n \theta_{\xi_i \otimes h_i, a_i \otimes k_i}$ agrees with that of $\sum_{i=1}^n \xi_i \cdot a_i^* \otimes \theta_{h_i, k_i}$, which will proves that $\psi_0$ is well-defined and isometric. For this, we may assume that $h_i$’s as well as $k_i$’s are mutually orthonormal. Then

\[
\left\| \sum_{i=1}^n \theta_{\xi_i \otimes h_i, a_i \otimes k_i} \right\|^2 = \left\| \sum_{i,j=1}^n \theta_{a_i \otimes k_j, \xi_i \otimes h_j} \theta_{\xi_j \otimes h_j, a_j \otimes k_i} \right\|
\]

\[
= \left\| \sum_{i,j=1}^n \theta_{\xi_i \otimes h_j, \xi_j \otimes h_j} a_i \otimes k_j \right\|
\]

\[
= \left\| \sum_{i=1}^n \theta_{\xi_i \otimes h_i, a_i \otimes k_i} \right\|
\]

By [8, Lemma 2.1], the last of the above equations is equal to the norm of the following product of two positive $n \times n$ matrices

\[
\left( \langle a_i \xi_i, a_i \xi_i \rangle_A \otimes k_i, a_j \langle \xi_j, \xi_j \rangle_A \otimes k_j \right)^{1/2} \left( \langle a_i \otimes k_i, a_j \otimes k_j \rangle_A \right)^{1/2}
\]

which is diagonal by orthogonality. Let

\[b_i = \langle \xi_i \cdot a_i^*, \xi_i \rangle_A \quad (i = 1, \ldots, n).
\]

Then

\[
\left\| \sum_{i=1}^n \theta_{\xi_i \otimes h_i, a_i \otimes k_i} \right\|^2 = \max_{i=1,\ldots,n} \left\| \left( \langle \xi_i \cdot a_i^*, \xi_i \rangle_A \langle \xi_i \cdot a_i^*, \xi_i \rangle_A \right)^{1/2} (a_i^* a_i)^{1/2} \right\|
\]

\[
= \max_{i=1,\ldots,n} \left\| (b_i b_i)^{1/2} (a_i^* a_i)^{1/2} \right\|.
\]
On the other hand,
\[
\left\| \sum_{i=1}^{n} \xi_i \cdot a_i^* \otimes \theta_{h_i,k_i} \right\|^2 = \left\| \sum_{i,j=1}^{n} \langle \xi_i \cdot a_i^* \otimes \theta_{h_i,k_i}, \xi_j \cdot a_j^* \otimes \theta_{h_j,k_j} \rangle_{A \otimes K(H)} \right\|
\]
\[
= \left\| \sum_{i,j=1}^{n} \langle \xi_i \cdot a_i^*, \xi_j \cdot a_j^* \rangle_{A \otimes K(h_i,k_i)} \right\|
\]
\[
= \max_{i=1,\ldots,n} \left\| \langle \xi_i \cdot a_i^*, \xi_i \rangle_{A} a_i^* \right\| = \max_{i=1,\ldots,n} \left\| b_i a_i^* \right\|
\]
again by orthonormality. Our claim then follows since
\[
\left\| (b_i^* b_i)^{1/2} (a_i^* a_i)^{1/2} \right\|^2 = \left\| (a_i^* a_i)^{1/2} b_i^* b_i (a_i^* a_i)^{1/2} \right\|
\]
\[
= \left\| b_i (a_i^* a_i)^{1/2} b_i^* \right\|^2 = \left\| b_i a_i^* b_i^* \right\|^2 = \left\| b_i a_i^* \right\|^2.
\]

What is left is now to show that \((\psi_0, \text{id})\) is a correspondence homomorphism. Let \(a, b, b' \in A, h, k, k' \in H, \xi, \xi' \in X\), and \(T \in K(H)\). Then,
\[
\psi_0\left( \varphi_{K(A \otimes H)}(a \otimes T) \eta_{\otimes h, b \otimes k} \right) = \psi_0(\theta_{\otimes h, b \otimes k}(a \otimes T))
\]
\[
= \varphi_A(a) \eta \cdot b^* \otimes \theta_{h,k}
\]
\[
= \varphi_{A \otimes K(H)}(a \otimes T)(\xi \cdot b^* \otimes \theta_{h,k})
\]
\[
= \varphi_{A \otimes K(H)}(a \otimes T) \psi_0(\theta_{\otimes h, b \otimes k})
\]
and
\[
\left( \psi_0(\theta_{\otimes h, b \otimes k}), \psi_0(\theta_{\otimes h', b' \otimes k'}) \right)_{A \otimes K(H)} = \left\langle \xi \cdot b^* \otimes \theta_{h,k}, \xi' \cdot b'^* \otimes \theta_{h',k'} \right\rangle_{A \otimes K(H)}
\]
\[
= \left\langle \xi \cdot b^*, \xi' \cdot b'^* \right\rangle_{A \otimes K(H)} \left\langle \theta_{\otimes h, b \otimes k}, \theta_{\otimes h', b' \otimes k'} \right\rangle_{K(H)}
\]
\[
= \left\langle \theta_{\otimes h, b \otimes k}, \theta_{\otimes h', b' \otimes k'} \right\rangle_{K(A \otimes H)}.
\]
This proves that \((\psi_0, \text{id})\) is a correspondence homomorphism. \(\square\)

In the next proposition, we identify \(K_A(A \otimes H, X \otimes H) = X \otimes K(H)\). Note that for \(m \in \mathcal{L}_A(A \otimes H, X \otimes H)\) and \(f \in A \otimes K(H) = K_A(A \otimes H)\), the right action \(m \cdot f\) defines an element of \(X \otimes K(H)\).

**Proposition B.3.** There exists an isomorphism
\[
(\psi, \text{id}) : \left( \mathcal{L}_A(A \otimes H, X \otimes H), \mathcal{L}_A(A \otimes H) \right) \to \left( M(X \otimes K(H)), M(A \otimes K(H)) \right)
\]
such that
\[
\psi(m) \cdot f = m \cdot f
\]
for \(m \in \mathcal{L}_A(A \otimes H, X \otimes H)\) and \(f \in A \otimes K(H)\).

**Proof.** With respect to the operations on (B.1), it can be easily seen that
(i) \(K_A(A \otimes H, X \otimes H) \cdot \mathcal{L}_A(A \otimes H) = K_A(A \otimes H, X \otimes H)\);
(ii) \(\varphi_{\mathcal{L}_A(A \otimes H)}(\mathcal{L}_A(A \otimes H)) \cdot K_A(A \otimes H, X \otimes H) = K_A(A \otimes H, X \otimes H)\);
(iii) \(\mathcal{L}_A(A \otimes H, X \otimes H) \cdot K_A(A \otimes H) = K_A(A \otimes H, X \otimes H)\).
By [5, Proposition 1.28], $(\psi, \pi)$ is thus an injective correspondence homomorphism. To see that $\psi$ is surjective, let $n \in M(X \otimes \mathcal{K}(\mathcal{H}))$. Take a net $\{x_i\}$ in $X \otimes \mathcal{K}(\mathcal{H})$ strictly converging to $n$. Then the limits $\lim_i x_i h$ and $\lim_i x_i^* k$ clearly exist for $h \in A \otimes \mathcal{H}$ and $k \in X \otimes \mathcal{H}$. Define $m_n : A \otimes \mathcal{H} \to X \otimes \mathcal{H}$ and $m_n^* : X \otimes \mathcal{H} \to A \otimes \mathcal{H}$ by

$$m_n h = \lim_i x_i h, \quad m_n^* k = \lim_i x_i^* k.$$  

We see from

$$\langle m_n h, k\rangle_A = \lim_i \langle x_i h, k\rangle_A = \lim_i \langle h, x_i^* k\rangle_A = \langle h, m_n^* k\rangle_A$$

that $m_n \in \mathcal{L}_A(A \otimes \mathcal{H}, X \otimes \mathcal{H})$ with the adjoint $m_n^*$. It is now obvious that $\psi(m_n) = n$, which completes the proof. \hfill \Box

From now on, we identify $\mathcal{L}_A(A \otimes \mathcal{H}, X \otimes \mathcal{H}) = M(X \otimes \mathcal{K}(\mathcal{H}))$.

**Remark B.4.** Let $\mu_G : C_0(G) \to \mathcal{L}(L^2(G))$ be the embedding in (2.7). The strict extension $\text{id}_X \otimes \mu_G$ then embeds $M(X \otimes C_0(G))$ into $M(X \otimes \mathcal{K}(L^2(G)))$ such that if $m \in C_0(G, M(X)_*)$ and $h \in C_c(G, A) \subseteq A \otimes L^2(G)$, then

$$(\text{id} \otimes \mu_G(m)h)(r) = m(r) \cdot h(r) \quad (r \in G)$$

by strict continuity.

Let $(\gamma, \alpha)$ be an action of a locally compact group $G$ on $(X, A)$. The **crossed product correspondence** $(X \rtimes_{\gamma, r} G, A \rtimes_{\alpha, r} G)$ is the completion of the $C_c(G, A)$-bimodule $C_c(G, X)$ such that

$$(x \cdot f)(r) = \int_G x(s) \cdot \alpha_s(f(s^{-1}r)) \, ds,$$

$$\langle x, y \rangle_{A \rtimes_{\alpha, r} G}(r) = \int_G \alpha^{-1}_s(\langle x(s), y(sr)\rangle_A) \, ds,$$

$$(\varphi_{A \rtimes_{\alpha, r} G}(f) x)(r) = \int_G \varphi_A(f(s)) \gamma_s(x(s^{-1}r)) \, ds$$

for $x, y \in C_c(G, X)$, $f \in C_c(G, A)$, and $r \in G$ ([5, Proposition 3.2]).

**Remark B.5.** The algebraic tensor product $X \otimes C_c(G)$ is dense in $X \rtimes_{\gamma, r} G$. This is because $X \otimes C_c(G)$ is $L^1$-norm dense in $C_c(G, X)$ and the crossed product norm on $C_c(G, A)$ is dominated by its $L^1$-norm.

The proof of the next theorem is only sketched.

**Theorem B.6.** Let $(\gamma, \alpha)$ be an action of a locally compact group $G$ on a $C^*$-correspondence $(X, A)$. Then, there exists an injective correspondence homomorphism $(\psi, \pi_\alpha) : (X \rtimes_{\gamma, r} G, A \rtimes_{\alpha, r} G) \to (\mathcal{L}_A(A \otimes L^2(G), X \otimes L^2(G)), \mathcal{L}_A(A \otimes L^2(G)))$ such that

$$(\psi_\gamma(x) h)(r) = \int_G \gamma^{-1}_r(x(s)) \cdot h(s^{-1}r) \, ds,$$

$$(\pi_\alpha(f) h)(r) = \int_G \alpha^{-1}_r(f(s)) h(s^{-1}r) \, ds$$

for $x \in C_c(G, X)$, $f \in C_c(G, A)$, $h \in C_c(G, A)$, and $r \in G$. 

Proof. It is well-known that $\pi_\alpha$ gives a nondegenerate embedding.

For each $x \in C_c(G, X) \subseteq X \times_{\gamma, r} G$, define $\rho_\gamma(x) : C_c(G, X) \to C_c(G, A)$ by

$$(\rho_\gamma(x)k)(r) = \int_G \Delta(r^{-1})\langle \gamma_s^{-1}(x(sr^{-1})), k(s)\rangle_A \, ds$$

for $k \in C_c(G, X)$ and $r \in G$, where $\Delta$ is the modular function of $G$. A routine computation yields

$$(\psi_\gamma(x)h, k)_A = (h, \rho_\gamma(x)k)_A, \quad \rho_\gamma(x)\psi_\gamma(y) = \pi_\alpha((x, y)_{A \times_{\alpha, \gamma} G})$$

for $h \in C_c(G, A) \subseteq A \otimes L^2(G)$, $k \in C_c(G, X) \subseteq X \otimes L^2(G)$, and $x, y \in C_c(G, X)$. This shows that $\psi_\gamma$ and $\rho_\gamma$ both extend continuously to all of $X \times_{\gamma, r} G$, and $\psi_\gamma(x) \in \mathcal{L}_A(A \otimes L^2(G), X \otimes L^2(G))$ for $x \in X \times_{\gamma, r} G$ with the adjoint $\psi_\gamma(x)^* = \rho_\gamma(x)$.

The second relation in (B.2) gives one of the requirement that $(\psi_\gamma, \pi_\alpha)$ be a correspondence homomorphism, that is,

$$\langle \psi_\gamma(x), \psi_\gamma(y) \rangle_{\mathcal{L}_A(A \otimes L^2(G))} = \pi_\alpha((x, y)_{A \times_{\alpha, \gamma} G})$$

for $x, y \in X \times_{\gamma, r} G$. For the other one, let $a \in A$ and $\phi \in C_c(G) \subseteq C^*_r(G)$. Since the strict extension $\overline{\pi_\alpha}$ embeds $A$ into $M(A \times_{\alpha, r} G)$ such that $(\overline{\pi_\alpha}(a)h)(r) = \alpha_r^{-1}(a)h(r)$, we deduce that

$$(\varphi_{\mathcal{L}_A(A \otimes L^2(G))}(\overline{\pi_\alpha}(a))k)(r) = \varphi_A(\alpha_r^{-1}(a))k(r)$$

for $k \in C_c(G, X)$ and $r \in G$. Similarly,

$$(\varphi_{\mathcal{L}_A(A \otimes L^2(G))}(\overline{\pi_\alpha}(\phi))k)(r) = \int_G \phi(s)k(s^{-1}r) \, ds$$

for $k \in C_c(G, X)$ and $r \in G$. An easy computation then verifies

$$\langle \psi_\gamma(\varphi_{A \times_{\alpha, r} G}(a \otimes \phi)x), h \rangle(r) = (\varphi_{\mathcal{L}_A(A \otimes L^2(G))}(\overline{\pi_\alpha}(a \otimes \phi))\psi_\gamma(x)h)(r)$$

for $h \in C_c(G, A) \subseteq A \otimes L^2(G)$ and $r \in G$, which gives

$$\psi_\gamma(\varphi_{A \times_{\alpha, r} G}(a \otimes \phi)x) = \varphi_{\mathcal{L}_A(A \otimes L^2(G))}(\overline{\pi_\alpha}(a \otimes \phi))\psi_\gamma(x).$$

We then have the same equality for $f \in A \times_{\alpha, r} G$ in place of $a \otimes \phi$. Therefore $(\psi_\gamma, \pi_\alpha)$ is an injective correspondence homomorphism.

Let $(\gamma, \alpha)$ be an action of $G$ on $(X, A)$, and $(\sigma^\gamma, \delta^\alpha)$ be the corresponding coaction. Define

$$(B.3) \quad \sigma^\gamma_G = \id_X \otimes \mu_G \circ \sigma^\gamma, \quad \delta^\alpha_G = \id_X \otimes \mu_G \circ \delta^\alpha,$$

where $\mu_G : C_0(G) \to S_{\mathcal{W}_G}$ is the Hopf $C^*$-algebra isomorphism given in (2.7). Then $(\sigma^\gamma_G, \delta^\alpha_G)$ is a coaction of $S_{\mathcal{W}_G}$ on $(X, A)$. In the next corollary, we regard $\sigma^\gamma_G(X) = \id_X \otimes \iota_{S_{\mathcal{W}_G}}(\sigma^\gamma_G(X))$ as a subspace of $\mathcal{L}_A(A \otimes L^2(G), X \otimes L^2(G))$.

**Corollary B.7.** Let $(\gamma, \alpha)$ be an action of a locally compact group $G$ on $(X, A)$. Then $(\psi_\gamma, \pi_\alpha)$ in Theorem B.6 gives an isomorphism from $(X \times_{\gamma, r} G, A \times_{\alpha, r} G)$ onto $(X \times_{\sigma^\gamma_G} S_{\mathcal{W}_G}, A \times_{\delta^\alpha_G} S_{\mathcal{W}_G})$ such that

$$(B.4) \quad \psi_\gamma(\xi \otimes \phi) = \sigma^\gamma_G(\xi) \cdot (1_M(A) \otimes \phi), \quad \pi_\alpha(a \otimes \phi) = \delta^\alpha_G(a)(1_M(A) \otimes \phi)$$

for $\xi \in X, a \in A,$ and $\phi \in C_c(G)$. 

Proof. We only need to prove that $\psi_\gamma$ satisfies the first equality in (B.4) and gives a surjection onto $X \rtimes_{\sigma_G^\gamma} \widehat{S(W_G)}$. Let $\xi \in X$ and $\phi \in C_c(G)$. We see from Remark B.4 that
\[
(\sigma_G^\gamma(\xi) h)(r) = \gamma^{-1}_r(\xi) \cdot h(r)
\]
for $h \in C_c(G, A)$ and $r \in G$. Hence
\[
(\psi_\gamma(\xi \otimes \phi) h)(r) = \gamma^{-1}_r(\xi) \cdot \int_G \phi(s) h(s^{-1} r) ds = (\sigma_G^\gamma(\xi)) ((1_{M(A)} \otimes \phi) h))(r),
\]
which shows the first equality in (B.4). Since $X \rtimes C_c(G)$ is dense in $X \rtimes_{\gamma, r} G$ by Remark B.5 and $\psi_\gamma$ is isometric, we must have $\psi_\gamma(X \rtimes_{\gamma, r} G) = X \rtimes_{\sigma_G^\gamma} \widehat{S(W_G)}$. $\square$

We now provide a proof of Corollary 5.11.

Proof of Corollary 5.11. Let
\[
\zeta_G = \text{id}_X \otimes \mu_G \circ \zeta.
\]
Clearly, $\zeta_G$ is the coaction of $\widehat{S(W_G)}$ on $O_X$ induced by $(\sigma_G^\gamma, \delta_G^\gamma)$. Define a representation
\[
(k_X \rtimes_{\gamma} G, k_A \rtimes_{\gamma} G) : (X \rtimes_{\gamma, r} G, A \rtimes_{\alpha, r} G) \to O_X \rtimes_{\beta, r} G
\]
to be the composition as indicated in the following diagram:

\[
\begin{array}{c}
(X \rtimes_{\gamma, r} G, A \rtimes_{\alpha, r} G) \\
(k_X \rtimes_{\gamma} G, k_A \rtimes_{\gamma} G) | \\
O_X \rtimes_{\beta, r} G
\end{array}
\xymatrix{
\ar[r]^{(\psi_\gamma, \pi_\alpha)} & \ar[r]^{(\psi_\gamma, \pi_\alpha)} & \ar[r]^{(\psi_\gamma, \pi_\alpha)} & O_X \rtimes_{\zeta_G} \widehat{S(W_G)}.}
\]

By definition ((5.3) and (B.4)), we have
\[
(k_X \rtimes_{\gamma} G)(f)(r) = k_X(f(r)), \quad (k_A \rtimes_{\alpha} G)(g)(r) = k_A(g(r))
\]
for $f \in C_c(G, X), g \in C_c(G, A)$, and $r \in G$. The conclusion then follows by Theorem 5.10. $\square$

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