\[ I = 2 \text{ Pion scattering length with improved actions on anisotropic lattices} \]

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Abstract

\[ \pi\pi \text{ scattering length in the } I = 2 \text{ channel is calculated within quenched approximation using improved gauge and improved Wilson fermion actions on anisotropic lattices. The results are extrapolated towards the chiral, infinite volume and continuum limit. This result improves our previous result on the scattering length. In the chiral, infinite volume and continuum limit, we obtain } a_0^{(2)} m_\pi = -0.0467(45), \text{ which is consistent with the result from Chiral Perturbation Theory, the experiment and results from other lattice calculations.} \]

\textbf{Key words: } \pi\pi \text{ scattering length, lattice QCD, improved actions.}  
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1 Introduction

Low-energy \( \pi\pi \) scattering experiment is a good testing ground for our understanding of the low-energy structure of Quantum Chromodynamics (QCD). Chiral Perturbation Theory \cite{1}, Roy equations \cite{2}, dispersion relations \cite{3} and other theoretical methods have been used in the study of low-energy \( \pi\pi \) scattering. Lattice QCD is a genuine non-perturbative method which can handle hadron-hadron scattering at low-energies. In recent years, owing to the progress in computing facilities and the use of improved actions, there have been several studies on pion pion \cite{4,5,6,7,8,9}, pion nucleon \cite{10}, pion kaon \cite{11} and kaon nucleon \cite{12} scattering in quenched lattice QCD. Very recently, CP-PACS collaboration even calculated pion pion scattering phase shift in the \( I = 2 \) channel using their unquenched configurations \cite{13,14}. In this letter, we report our recent quenched lattice result on the \( \pi\pi \) scattering length in the \( I = 2 \) channel. The result in this letter is an improvement over our previous result on the scattering length \cite{5}.
The lattice gauge action used in our study is the tadpole improved action on anisotropic lattices [15,16]. Using this gauge action together with improved Wilson fermion actions, glueball and hadron spectra have been studied within quenched approximation [15,16,17,18,19,20,21]. Configurations generated from the gauge action have also been utilized to calculate the the scattering length for $\pi\pi$ scattering in the $I = 2$ channel [5], $KN$ scattering in the $I = 1$ channel [12] and the $K\pi$ scattering in the $I = 3/2$ channel [11]. In this letter, we update our result on the $\pi\pi$ scattering length in the $I = 2$ channel. The number of gauge field configurations has been increased. A more general chiral extrapolation method is used. Otherwise, the basic procedure of the calculation is similar to that adopted in Ref. [5].

2 Numerical calculation of the scattering length

Configurations used in this calculation are generated using the pure gauge action for $6^340, 8^340$ and $10^350$ lattices with the gauge coupling $\beta = 1.9, 2.2, 2.4, 2.6$ and $3.0$. The spatial lattice spacing $a_s$ is roughly between 0.1fm and 0.4fm while the spatial physical size of the lattice ranges from 0.7fm to 4.0fm. For each set of parameters, several hundred (typically 512) decorrelated gauge field configurations are used to measure the fermionic quantities. Statistical errors are analyzed using the jack-knife method. Single pion, and rho mass values are obtained from the plateau of their corresponding effective mass plots with the fitting interval being automatically chosen by minimal $\chi^2$ per degree of freedom.

In order to calculate the elastic scattering lengths for hadron-hadron scattering on the lattice, or the scattering phase shifts in general, one uses Lüscher’s formula which relates the exact energy level of two hadron states in a finite box to the elastic scattering phase shift in the continuum. For $\pi\pi$ scattering at zero relative three momentum, this formula amounts to a relation between the exact energy $E_{\pi\pi}^{(I)}$ of the two pion system with vanishing relative momentum in a finite box of size $L$ with isospin $I$, and the corresponding scattering length $a_0^{(I)}$ in the continuum. This formula reads [22]:

$$E_{\pi\pi}^{(I)} - 2m_\pi = -\frac{4\pi a_0^{(I)}}{m_\pi L^3} \left[ 1 + c_1 \left( \frac{a_0^{(I)}}{L} \right) + c_2 \left( \frac{a_0^{(I)}}{L} \right)^2 \right] + O(L^{-6}), \quad (1)$$

where $c_1 = -2.837297$, $c_2 = 6.375183$ are numerical constants and $m_\pi$ is the mass of the pion. In this letter, we focus on the $\pi\pi$ scattering length $a_0^{(2)}$ in the isospin $I = 2$ channel.

To measure the hadron mass values $m_\pi, m_\rho$ and to extract the energy shift
\( \delta E_{\pi \pi}^{(2)} \), one constructs the correlation functions from the corresponding one meson and two meson operators in the appropriate symmetry channel. In this letter, we used the same operators as those in Ref. [5]. Numerically, it is more advantageous to construct the ratio of the correlation functions:

\[
\mathcal{R}^{I=2}(t) = \frac{C^{I=2}_{\pi \pi}(t)}{(C_{\pi}(t)C_{\pi}(t))},
\]

where \( C^{I=2}_{\pi \pi}(t) \) is the two-pion correlation function in the \( I = 2 \) channel and \( C_{\pi}(t) \) is the one-pion correlation function. One then uses the fitting function:

\[
\mathcal{R}^{I=2}(t) \propto e^{-\delta E_{\pi \pi}^{(2)} t} \sim 1 - \delta E_{\pi \pi}^{(2)} t,
\]

to determine the energy shift \( \delta E_{\pi \pi}^{(2)} \). Usually, the linear form is sufficient. However, for small lattices, the exponential form should be used. Similar situation was also observed by the JLQCD and CP-PACS collaboration in their calculations [6,9].

Two pion correlation function, or equivalently, the ratio \( \mathcal{R}^{I=2}(t) \) is constructed from quark propagators, which are obtained using the Multi-mass Minimal Residue algorithm with wall sources. Periodic boundary condition is applied to all three spatial directions while in the temporal direction, Dirichlet boundary condition is utilized.

After obtaining the energy shifts \( \delta E_{\pi \pi}^{(2)} \) from the two-pion correlation functions, the values of \( \delta E_{\pi \pi}^{(2)} \) are substituted into Lüscher’s formula to solve for the scattering length \( a_0^{(2)} \) for all parameter sets that have been simulated. From these results, attempts are made to perform an extrapolation towards the chiral, infinite volume and continuum limit.

### 3 Extrapolations of the scattering length

The chiral extrapolations of physical quantities are performed as discussed in Ref. [5]. The only difference is that, for all physical quantities, since we now have more bare quark mass values available, we tried to use a quadratic extrapolation in the bare quark mass rather than a linear extrapolation as used in Ref. [5]. For each set of parameters, we have calculated fermionic quantities with 8 different values of \( \kappa \). The largest value of \( \kappa \), which corresponds to lowest pion mass for each simulation point is chosen such that the solution of the quark propagator can be obtained within a reasonable number of Minimal Residue iterations (typically 600 to 800). The lowest pion to rho mass ratio for our simulation points are between 0.66 and 0.74. For the scattering length
$a_0^{(2)}$, we use the quantity [5] $F = a_0^{(2)}m_\rho^2/m_\pi$, which in the chiral limit is given by the current algebra value: [23]

$$F \equiv \frac{a_0^{(2)}m_\rho^2}{m_\pi} = -\frac{1}{16\pi} \frac{m_\rho^2}{f_\pi^2} \sim -1.364 \ .$$  \hfill (4)

Here the final numerical value is obtained by substituting in the experimental values for $m_\rho \sim 770\text{MeV}$ and $f_\pi \sim 93\text{MeV}$. Chiral perturbation theory with loops yields almost the same numerical value for factor $F$ [2] in the $I = 2$ channel. Note that the factor $F$ can be calculated on the lattice with good precision without the lattice calculation of meson decay constants. We have adopted a quadratic functional form in chiral extrapolation for all physical quantities and the fitting range of the extrapolation is self-adjusted by the program to yield a minimal $\chi^2$ per degree of freedom.

![Fig. 1. Infinite volume extrapolation for the quantity $F = a_0^{(2)}m_\rho^2/m_\pi$ obtained from our simulation results at $\beta = 3.0$, 2.6, 2.4, 2.2 and 1.9. The straight line represents the linear extrapolation in $1/L^3$. The extrapolated results are shown as solid square with the corresponding error at $L^{-3} = 0$. The values of $F$ after the chiral extrapolations are used to extrapolated towards the infinite volume limit by a function linear in $1/L^3$, as suggested by Lüscher’s formula. This extrapolation is shown in Fig 1 for all five values of $\beta$. The extrapolated results thus obtained for the factor $F$ are then used for the continuum limit extrapolation.](image-url)
Finally, an continuum limit extrapolation is performed to eliminate the finite lattice spacing errors. Since we have used the tadpole improved clover Wilson action, all physical quantities differ from their continuum counterparts by terms that are proportional to the spatial lattice spacing \( a_s \). The physical value of \( a_s \) in terms of the hadronic scale \( r_0 \) for each value of \( \beta \) can be found from Ref. [16,21]. The result of the continuum extrapolation is shown in Fig. 2 where the results from the chiral and infinite volume extrapolation discussed above are indicated as data points (open squares with error bars) in the plot for all 5 values of \( \beta \) that have been simulated. The straight line shows the extrapolation towards the \( a_s = 0 \) limit and the final extrapolated result is also shown as a solid square. For comparison, the corresponding result from two-loop chiral perturbation theory [2] is also shown near \( a_s = 0 \) as the open hexagon. The current algebra result and the one-loop chiral perturbation result are quite close to the two-loop chiral result numerically. To avoid crowdedness, they are not shown in the figure. The experimental result from E865 Collaboration [24] is shown as the shaded region near \( a_s = 0 \). The height of the shaded region designates the error for the experimental result. It is evident that these results are compatible within error bars.

To summarize, we obtain from the continuum extrapolation the following result: \( F = a_0^{(2)} m_{\rho}^2 / m_\pi = -1.41(14) \). If we substitute in the physical meson mass values, we obtain the \( \pi\pi \) scattering length in the \( I = 2 \) channel as: \( a_0^{(2)} m_\pi = -0.0467(45) \), which is to be compared with the current algebra value of \(-0.046 \) [23], one-loop chiral perturbation theory result of \(-0.042 \) [1] and the two-loop result of \(-0.0444(10) \) [2]. Our result is also consistent with the experimental result from E865 Collaboration which is: \( a_0^{(2)} m_\pi = -0.036(9) \) [24]. The result reported in this letter is also consistent with our previous result with the error reduced by a factor of two or so due to higher statistics and lattices.
with finer lattice spacing. This result also agrees with other lattice studies reported recently [6,9]. Their final result for $a_0 m_\pi$ is: $a_0^{(2)} m_\pi = -0.0410(69)$, which is also compatible with our result within error.

4 Conclusions

In this letter, we report our lattice result on pion-pion scattering lengths in isospin $I = 2$ channel, obtained using quenched lattice QCD. Simulations are performed on lattices with various sizes, ranging from 0.7fm to about 4fm and with five different values of lattice spacing with several hundred gauge field configurations. Quark propagators are measured with 8 different valence quark mass values. These enable us to explore the chiral limit, the finite volume and the finite lattice spacing errors in a systematic fashion. The lattice result for the scattering length is extrapolated towards the chiral, infinite volume and continuum limit where a result consistent with the experiment, Chiral Perturbation Theory and previous lattice results is found.

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