Rigid rotor as a toy model for Hodge theory

Saurabh Gupta\textsuperscript{1,a}, R.P. Malik\textsuperscript{1,2,b}

\textsuperscript{1}Physics Department, Centre of Advanced Studies, Banaras Hindu University, Varanasi 221 005, U.P., India
\textsuperscript{2}DST Centre for Interdisciplinary Mathematical Sciences, Faculty of Science, Banaras Hindu University, Varanasi 221 005, India

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Abstract We apply the superfield approach to the toy model of a rigid rotor and show the existence of the nilpotent and absolutely anticommuting Becchi–Rouet–Stora–Tyutin (BRST) and anti-BRST symmetry transformations, under which, the kinetic term and the action remain invariant. Furthermore, we also derive the off-shell nilpotent and absolutely anticommuting (anti-) co-BRST symmetry transformations, under which, the gauge-fixing term and the Lagrangian remain invariant. The anticommutator of the above nilpotent symmetry transformations leads to the derivation of a bosonic symmetry transformation, under which, the ghost terms and the action remain invariant. Together, the above transformations (and their corresponding generators) respect an algebra that turns out to be a physical realization of the de Rham cohomological operators of differential geometry. Thus, our present model is a toy model for the Hodge theory.

1 Introduction

The model of a rigid rotor, over the centuries, has played a pivotal role in providing the key theoretical insights into the dynamics of the classical as well as quantum systems. In particular, in the realm of atomic, molecular and nuclear physics, the contribution of the model of a rigid rotor has been enormous (as far as deep understanding of many physical phenomena is concerned). In our present investigation, we show that the above model presents a tractable toy model for the Hodge theory where the symmetry transformations (and their corresponding generators) provide a physical realization of the de Rham cohomological operators of differential geometry.

In our present endeavor, we begin with an appropriate first-order Lagrangian (FOL) for a particle of mass $m = 1$ that is constrained to move on a circle of radius $a$ (i.e. the model of a rigid rotor). This Lagrangian is given below (see, e.g., [1] for a detailed discussion on its appropriateness)

$$L_f = p_r \dot{r} + p_\theta \dot{\theta} - \frac{p_\theta^2}{2r^2} + \lambda(r-a),$$

(1)

where $r$ and $\theta$ are the polar coordinates, $\dot{r} = (dr/dt)$, $\dot{\theta} = (d\theta/dt)$ are the generalized velocities, $p_r$ and $p_\theta$ are the corresponding canonical momenta and $\lambda$ is the Lagrange multiplier. All these variables are function of the evolution parameter $t$. It can be seen that $\Pi_\lambda \approx 0$ and $(r-a) \approx 0$ are the two first-class constraints on the theory [1] where $\Pi_\lambda$ is the momentum corresponding to the Lagrange multiplier variable $\lambda$. The existence of the first-class constraint, as is well-known [2, 3], is the signature of a gauge theory. In fact, the following infinitesimal local gauge transformations (\(\delta g\)) [1]

$$\delta_g p_r = f(t), \quad \delta_g \lambda = \dot{f}(t),$$

$$\delta_g r = \delta_g \theta = \delta_g p_\theta = 0,$$

are generated by the above first-class constraints, under which we can verify that the FOL transforms (with the infinitesimal gauge parameter $f(t)$) as

$$\delta_g L_f = \frac{d}{dt} \left( f(t) [r(t) - a] \right).$$

(3)

Thus, the action $S = \int (dt L_f)$, corresponding to the above Lagrangian, remains invariant under the infinitesimal local gauge symmetry transformations (2). The conserved charge, that emerges from the above symmetry transformations (due to the Noether theorem), is nothing but the constraint $(r-a)$.

\textsuperscript{a}e-mail: guptasaurabh4u@gmail.com
\textsuperscript{b}e-mail: malik@bhu.ac.in

We slightly differ from the FOL of [1] for the sake of brevity and algebraic convenience.
One of the most intuitive approaches to quantize a gauge theory is the Becchi–Rouet–Stora–Tyutin (BRST) formalism where the gauge parameter is replaced by (anti-) ghost fields. We take up the Lagrangian (1) and demonstrate, in our present paper, that its (anti-) BRST invariant version (cf. (4) below) is endowed with a set of six continuous symmetry transformations which act infinitesimally on the variables of the theory. In fact, we establish that the algebra of the continuous symmetry operators is exactly same as the algebra obeyed by the cohomological operators of differential geometry. We also demonstrate that the latter algebra is respected by the conserved charges that generate the above continuous symmetry transformations. Thus, we prove that the rigid rotor is a toy model for the Hodge theory where all the cohomological operators of differential geometry are identified with the continuous symmetries and their corresponding generators.

A couple of mathematical properties, that are associated with the key concepts of BRST formalism, are the nilpotency and the absolute anticommutativity of the (anti-) BRST symmetry transformations (and their corresponding generators). The geometrical origin and interpretation of the above properties are provided by the superfield formalism [4, 5] where the horizontality condition (HC) plays a decisive role. The latter condition (i.e. (HC)) physically implies that the curvature tensor of a given gauge theory is unaffected by the presence of the Grassmannian variables invoked in the superfield description of the BRST formalism [4, 5]. The (anti-) BRST transformations, for the gauge field and (anti-) ghost fields, are determined by utilizing the HC within the framework of superfield formalism. The components of the “gauge” potential of the rigid rotor possess some unusual properties, however. For instance, the components of this potential transform in a completely different manner (cf. (2), (5), (6)). To obtain such type of unusual transformations, within the framework of the superfield formulation, is a challenging problem for us. It is satisfying to state that we have accomplished this goal of obtaining the (anti-) BRST transformations for the “gauge” components and (anti-) ghost variables of the rigid rotor by making some specific choices in the application of the HC.

The main motivations for carrying out our present investigations are as follows. First, we have already proposed, in our earlier works [6–10], the field theoretic models for the Hodge theory in the case of 2D (non-) Abelian 1-form and 4D Abelian 2-form gauge theories. Thus, it is an interesting task for us to propose a simple toy model for the Hodge theory where there are almost no mathematical complications and the continuous symmetry transformations of the theory are found to be completely transparent. Second, for the present toy model, the components of the “gauge” potential transform in a completely different manner (see, e.g., [1]). It is, therefore, an interesting endeavor to exploit the superfield formalism to obtain such a kind of symmetry transformations within its geometrical framework. Finally, a new model for the Hodge theory is always an exciting development because its proof requires a variety of continuous symmetry transformations. The ensuing operator algebra of the above continuous symmetry transformations turns out to be reminiscent of the algebra obeyed by the de Rham cohomological operators of differential geometry. In other words, we establish a perfect analogy between some aspects of mathematics of the differential geometry and the continuous symmetries of the Lagrangian of our present toy model of a rigid rotor.

The contents of our present paper are organized as follows. In Sect. 2, we discuss the off-shell nilpotent and absolutely anticommuting (anti-) BRST symmetry transformations within the frameworks of Lagrangian formalism and superfield approach. This is followed, in Sect. 3, by our discussion about the nilpotent and absolutely anticommuting dual(co)-BRST and anti-co-BRST symmetry transformations in the realm of Lagrangian formulation. Our subsequent Sect. 4 deals with the derivation of a bosonic symmetry transformation. In Sect. 5, we discuss the ghost and discrete symmetry transformations in the theory. We demonstrate, in Sect. 6, the similarity between the algebra of the de Rham cohomological operators and continuous symmetry transformations (and corresponding generators). Finally, we make some concluding remarks and point out a few future directions in Sect. 7.

In our Appendix, we capture some of the key mathematical properties of the (anti-) BRST charges in the superfield formalism.

2 Nilpotent and absolutely anticommuting (anti-) BRST symmetries: two approaches

In this section, we discuss about the off-shell nilpotent (anti-) BRST symmetries in the Lagrangian as well as the superfield formulation. First, in Sect. 2.1, we dwell on the completely different type of transformations associated with the components of the “gauge” potential. Later on, in Sect. 2.2, we capture these unusual bit of transformations within the framework of the geometrical superfield formalism [4, 5] with judicious choices.

2.1 (Anti-) BRST symmetries: Lagrangian description

We begin with the following (anti-) BRST invariant first-order appropriate Lagrangian for the rigid rotor (see, e.g., [1] for details)

$$L_b = p_r \dot{r} + p_\theta \dot{\theta} - \frac{p_\theta^2}{2r^2} + \lambda(r - a) + b(\dot{\lambda} - p_r) + \frac{\dot{b}^2}{2} - i \dot{C} \dot{C} - i \dot{C} C,$$

(4)