Continuous-discrete robust UKF for nonlinear systems with parameter uncertainties

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Abstract

This paper addresses continuous-discrete filtering problems for parametric uncertain nonlinear systems. We developed a Robust Unsecented Kalman Filter (RUKF) for discrete-time nonlinear systems with parameter uncertainties in a previous work. The RUKF is more effective than the conventional UKF for uncertain systems. However, the previous RUKF is a discrete time filtering algorithm and it cannot be directly applied to a continuous-discrete filtering problem. So, we modify the predictive step of the RUKF in order to deal with the continuous-discrete filtering problem. The validities of the proposed methods are illustrated in Monte Carlo simulations.

1 Introduction

The Kalman Filter (KF) provides optimal state estimates for linear dynamic systems with Gaussian process and measurement noises. For nonlinear systems, the Extended Kalman Filter (EKF) has long been the de facto standard. In recent years, the effectiveness of Unscented Kalman Filter (UKF) has been attracting attention. The UKF uses a deterministic sampling technique called Unscented Transformation (UT) [1, 2].

It is well known that the uncertainties of the system degrade the performance of the Kalman filter, and previous researches have improved the robustness of the filters [3, 4]. These robust filters are less sensitive to deviations, but they require specific structures for the model and they also assume the parameter uncertainties must be norm-bounded. So they are not easy to be implemented in practice. Another approach commonly used in the engineering fields is considering a joint estimation problem [5], where the uncertain parameters are treated as new states in the joint estimation problem. The augmented system equation has extra state dimensions, however the measurement dimension does not change. So, the local observability of the augmented system cannot be satisfied sometime and the stability of the filter cannot be ensured. The Desensitized Kalman Filter (DKF) [6] is a robust filtering algorithm that is based on the principles of desensitized optimal control. The DKF does not require any structures to be specified or any extra state dimensions, and it is a promising robust estimator. However, calculation of DKF’s optimal gain is more complicated than that of the standard KF, and the method of designing the weight matrix is heuristic.

We developed the Robust UKF (RUKF) in our previous research [7]. We approximated nonlinear equations by the Unscented Statistical Linearization (USL) [8] and analyzed the influence of the parameter uncertainties on estimation errors.

The RUKF only requires the knowledge of the nominal value and variance matrices of uncertain parameters, so it is easy to implement in practice. Furthermore, the proposed RUKF does not need the augment state dimension, the calculation cost of unscented transformation is same as the conventional UKF. The RUKF can be calculated in a standard UKF framework and the weight matrix can be designed systematically in contrast to DKF, so it is easy to apply in real systems.

However, the previous RUKF is a discrete time filtering algorithm and it cannot be directly applied to a continuous-discrete filtering problem, where the dynamic equations are modeled as continuous-time stochastic processes, and the measurement equations are modeled as discrete time stochastic processes. So far as the authors know, little study has dealt with a continuous-discrete filtering with parameter uncertainties. Therefore, we modify the predictive step of RUKF in order to deal with the continuous-discrete filtering problem. We derive the predictive step of continuous-discrete RUKF (CD-RUKF) based on the continuous-discrete UKF [9]. The CD-RUKF is like the RUKF for discrete time systems and it can be applied if the nominal value and variance matrices of uncertain parameters are known.

The validities of the proposed methods are illustrated in Monte Carlo simulations.

2 Preliminaries

In this section, we briefly introduce the UKF and the USL algorithms to define the notations used in the following sections.
2.1 Unscented Kalman Filter

Consider a discrete time nonlinear stochastic system without parameter uncertainties,

\[ x_k = f(x_{k-1}) + w_{k-1}, \]
\[ y_k = h(x_k) + v_k, \]

where the vector \( x_k \in \mathbb{R}^n, y_k \in \mathbb{R}^m \) are states and measurements, \( w_k \in \mathbb{R}^n, v_k \in \mathbb{R}^m \) are process noise and measurement noise. Stochastic noise \( w_k \) and \( v_k \) are uncorrelated zero-mean Gaussian white sequence and their covariances are

\[ \mathbb{E}[w_k \cdot w_k^T] = Q_k, \quad \mathbb{E}[v_k \cdot v_k^T] = R_k, \quad \mathbb{E}[w_k \cdot v_k^T] = 0. \]

Furthermore, we also assume that state vector \( x_k \) is also Gaussian and its mean and covariance matrix are given as

\[ \mathbb{E}[x_k] = \hat{x}_{k|k}, \quad \mathbb{E}[(x_k - \hat{x}_{k|k}) \cdot (x_k - \hat{x}_{k|k})^T] = P_k^{xx}. \]

The basic UKF is a two-step estimator whose predictive step is given as following.

Define weights \( W_i \) as

\[ W_i^{(m)} = \frac{\lambda_{ukf}}{n + \lambda_{ukf}}, \]
\[ W_i^{(c)} = 2(n + \lambda_{ukf}) \]
\[ W_i^{(m)} = W_i^{(c)} - 1 + \alpha_{ukf}^2 \lambda_{ukf} + \beta_{ukf}, \]

where \( i = 1, \ldots, 2n \) and \( \lambda_{ukf} = \alpha_{ukf}^2 \lambda_{ukf} + \beta_{ukf} - n \) is a tuning parameter. The parameter \( \alpha_{ukf}, 0 \leq \alpha_{ukf} < 1 \), controls the size of the sigma point distribution and it should ideally be a small number and \( \kappa_{ukf} \) is a secondary scaling parameter which is usually set to either 0 or 3 \( - n \). For Gaussian, \( n + \lambda_{ukf} = 3 \) and \( \beta_{ukf} = 2 \) are optimal choices [10]. Generate \( 2n + 1 \) sigma points \( X_{k-1|k} \) as

\[ X_{k-1|k} = \hat{x}_{k-1|k-1} \times 1_{1 \times (2n+1)} + \sum_{i=0}^{2n} \gamma_{ukf} \sqrt{P_{xx}^{k-1|k-1}} \] \( \sqrt{\sum_{i=0}^{2n} \gamma_{ukf} \sqrt{P_{xx}^{k-1|k-1}}} \),

where \( \gamma_{ukf} = \sqrt{n + \lambda_{ukf}} \) and \( \sqrt{P_{xx}^{k-1|k-1}} \) is Cholesky square root.

Transform the sigma points \( X_{k-1|k-1} \) using state equation (1) and compute the predictive mean and covariance as following,

\[ X_{k|k-1,i} = f(X_{k-1|k-1,i}), \]
\[ \hat{x}_{k|k-1} = \sum_{i=0}^{2n} W_i^{(m)} X_{k|k-1,i}, \]
\[ P_{k|k-1}^{xx} = \sum_{i=0}^{2n} W_i^{(c)} (X_{k|k-1,i} - \hat{x}_{k|k-1}) \]
\[ (X_{k|k-1,i} - \hat{x}_{k|k-1})^T + Q_{k-1}. \]

Regenerate sigma point \( X_{k|k-1,i} \) using \( (\hat{x}_{k|k-1}, P_{k|k-1}^{xx}) \) obtained by (7) and (8), transform the sigma points \( X_{k|k-1} \) using measurement equation (2) and compute measurement mean and covariance as follows:

\[ \mathbb{E}[y_{k}] = h(X_{k|k-1}), \]
\[ \hat{y}_{k|k-1} = \sum_{i=0}^{2n} W_i^{(m)} y_{k|k-1,i}, \]
\[ P_{k|k-1}^{yy} = \sum_{i=0}^{2n} W_i^{(c)} (y_{k|k-1,i} - \hat{y}_{k|k-1}) \]
\[ (y_{k|k-1,i} - \hat{y}_{k|k-1})^T + R_k, \]
\[ P_{k|k-1}^{xy} = \sum_{i=0}^{2n} W_i^{(c)} (X_{k|k-1,i} - \hat{x}_{k|k-1}) \]
\[ (y_{k|k-1,i} - \hat{y}_{k|k-1}) \]

And, an update step of the UKF is given by the classical KF update, that is,

\[ K_k = P_{k|k-1}^{yy} (P_{k|k-1}^{yy})^{-1}, \]
\[ \hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k (y_k - \hat{y}_{k|k-1}), \]
\[ P_{k|k}^{xx} = P_{k|k-1}^{xx} - K_k P_{k|k-1}^{yy} K_k^T. \]

2.2 Unscented statistical linearization

In this section, we briefly introduce the USL[8]. Consider following nonlinear dynamics without parameter uncertainties,

\[ x_k = f(x_{k-1}) + w_{k-1} \]

where the vector \( x_k \in \mathbb{R}^n \) is a states vector and \( w_k \in \mathbb{R}^n \) is a process noise. Stochastic noise \( w_k \) is uncorrelated zero-mean Gaussian white sequence and its covariance matrix is \( \mathbb{E}[w_k \cdot w_k^T] = Q_k \). Furthermore, we also assume that state vector \( x_k \) is also Gaussian and its covariance matrix is \( \mathbb{E}[(x_k - \hat{x}_{k|k}) \cdot (x_k - \hat{x}_{k|k})^T] = P_k^{xx} \) where \( \hat{x}_{k|k} = \mathbb{E}[x_k] \).

We consider the following optimization problem to approximate the nonlinear system \( f(x_{k-1}) \) in (16) around \( \hat{x}_{k|k-1} \) as

\[ J_k = \arg \min_{F_k, b_k} \mathbb{E} \left[ \| f(x_{k-1}) - \{ F_k (x_{k-1} - \hat{x}_{k|k-1}) + b_k \} \|^2 \right]. \]

Differentiate \( J_k \) with respect to coefficient matrix \( F_k \) and drift term \( b_k \), and set each value to 0, we can get optimal solutions as

\[ F_k = \mathbb{E}[f(x_{k-1}) \hat{x}_{k|k-1}^T] \left( P_k^{xx} \right)^{-1}, \]
\[ b_k = \mathbb{E}[f(x_{k-1})], \]

where we defined \( \hat{x}_{k-1} := x_{k-1} - \hat{x}_{k-1|k-1} \). Furthermore, we can also approximate coefficient matrix \( F_k \)
and drift term $b_{k-1}$ by using Unscented Transformation (UT) as

$$F_{k-1} \approx \left\{ \sum_{i=0}^{2n} W_i f(X_i) (X_i - \hat{x}_{k-1|k-1})^T \right\} \{P^x_{k-1}\}^{-1},$$

(20)

$$b_{k-1} \approx \sum_{j=0}^{2n} W_j f(X_j),$$

(21)

where $X_i$ are sigma points sampled from $\mathcal{N}(\hat{x}_{k-1|k-1}, P^x_{k-1})$. And optimal design method of weight matrix $W_i$ for Gaussian distribution is in [8].

Then, the nonlinear system (16) can be approximated as

$$x_k \approx F_{x,k-1} \hat{x}_{k-1} + b_{k-1} + w_{k-1}. \quad (22)$$

The USL inherits good properties of the UT, that is, we do not need calculate a Jacobian matrix. Furthermore, we can adopt this linearization method for discontinuous systems which are not differentiable.

### 3 Discrete-time RUKF

#### 3.1 Derivation of RUKF

In this section, we briefly introduce the discrete-time RUKF algorithms[7] for preparations of deriving a continuous-discrete RUKF.

Consider a discrete time nonlinear stochastic system with parameter uncertainties:

$$x_k = f(x_{k-1}, p_{k-1}) + g(x_{k-1}) u_{k-1} + w_{k-1}, \quad (23)$$

$$y_k = h(x_k) + v_k,$$

(24)

where the vector $p_k \in \mathbb{R}^l$ represents parameter uncertainties and $u_k$ is control input. Furthermore, we assume that mean and covariance matrix of unknown parameter vector $p_k$ are known as $p^\text{nom}_k$ and $P^p_{k-1}$.

Consider the following optimization problem [8] to approximate a nonlinear system $f(x_{k-1}, p_{k-1})$ in (23) instead of calculating Taylor expansion:

$$J_{k-1} = \arg \min_{F_{k-1}, G_{k-1}, d_{k-1}} \mathbb{E} \{ \| f(x_{k-1}, p_{k-1}) + g(x_{k-1}) u_{k-1} - (F_{k-1} \hat{x}_{k-1} + G_{k-1} \hat{p}_{k-1} + d_{k-1}) \|^2 \}.$$

(25)

where $\hat{x}_{k-1} := x_{k-1} - \hat{x}_{k-1}$ and $\hat{p}_{k-1} := p_{k-1} - p^\text{nom}_{k-1}$.

Differentiate $J_{k-1}$ with respect to each coefficient $F_{k-1}, G_{k-1}, d_{k-1}$ and set each value to zero, we can get optimal solutions as

$$F_{k-1} = \mathbb{E} \left\{ f(x_{k-1}, p_{k-1}) + g(x_{k-1}) u_{k-1} \right\} \hat{x}_{k-1}^T \quad (26)$$

$$G_{k-1} = \mathbb{E} \left\{ f(x_{k-1}, p_{k-1}) + g(x_{k-1}) u_{k-1} \right\} \hat{p}_{k-1}^T \quad (27)$$

$$d_{k-1} = \mathbb{E} \{ f(x_{k-1}, p_{k-1}) + g(x_{k-1}) u_{k-1} \}.$$

(28)

Here, coefficient matrices $F_{k-1}, G_{k-1}$ and drift term $d_{k-1}$ can be approximated by using UT as

$$F_{k-1} \approx \left\{ \sum_{i=0}^{2n} W_{x,i} \left\{ f(X_i, p^\text{nom}_{k-1}) + g(X_i) u_{k-1} \right\} \right\} \{P^x_{k-1}\}^{-1}, \quad (29)$$

$$G_{k-1} \approx \left\{ \sum_{j=0}^{2n} W_{p,j} \left\{ f(\hat{x}_{k-1}, p^\text{nom}_{k-1}) + g(\hat{x}_{k-1}) u_{k-1} \right\} \right\} \{P^p_{k-1}\}^{-1}, \quad (30)$$

$$d_{k-1} \approx \sum_{i=0}^{2n} W_{x,i} \left\{ f(X_i, p^\text{nom}_{k-1}) + g(X_i) u_{k-1} \right\}, \quad (31)$$

where $\mathcal{P}_j$ are sigma points sampled from $\mathcal{N}(p^\text{nom}_{k-1}, P^p_{k-1})$. Therefore, the nonlinear system (23) can be approximated as

$$x_k \approx F_{k-1} \hat{x}_{k-1} + G_{k-1} \hat{p}_{k-1} + d_{k-1} + w_{k-1}. \quad (32)$$

Using $\hat{x}_{k|k-1} \approx \mathbb{E} \{ f(x_{k-1}, p_{k-1}) + g(x_{k-1}) u_{k-1} \}$, the prediction error $\tilde{x}_{k|k-1}$ can be given as

$$\tilde{x}_{k|k-1} = x_k - \hat{x}_{k|k-1} \approx F_{k-1} \hat{x}_{k-1} + G_{k-1} \hat{p}_{k-1} + w_{k-1}. \quad (33)$$

Furthermore, the prediction error covariance matrix can be given as

$$\mathbb{E} (\tilde{x}_{k|k-1} \tilde{x}_{k|k-1}^T) \approx F_{k-1} P^x_{k-1} F_{k-1}^T + G_{k-1} P^p_{k-1} G_{k-1}^T + Q_{k-1}, \quad (34)$$

where we assume state and parameter are uncorrelated, that is, $\mathbb{E}[\tilde{x}_{k-1} \hat{p}_{k-1}^T] = 0$ and $\mathbb{E}[\hat{x}_{k-1} \tilde{x}_{k-1}^T] = 0$ are satisfied. From equation (34), we can confirm that the influence of parameter uncertainties is calculated as $G_{k-1} P^p_{k-1} G_{k-1}^T$.

Let define $P^x_{x,k|k-1}$ and $P^\text{prob}_{k|k-1}$ as

$$P^x_{x,k|k-1} := F_{k-1} P^x_{k-1} F_{k-1}^T + Q_{k-1}, \quad (35)$$

$$P^\text{prob}_{k|k-1} := G_{k-1} P^p_{k-1} G_{k-1}^T, \quad (36)$$

where $P^x_{x,k|k-1}$ is almost same values $P^x_{k|k-1}$ given by (8).

Then we can redefine the prediction error covariance matrix as (37) instead of $P^x_{k|k-1}$ given by (8):

$$P^x_{x,k|k-1} := P^x_{x,k|k-1} + P^\text{prob}_{k|k-1} \quad (37)$$

The RUKF does not need to calculate Jacobian matrix because it uses equation (30) to calculate the influence of parameter uncertainties.

#### 3.2 Extension to matrix form of RUKF

We provide a matrix form of prediction error covariance matrix of RUKF (37) in this subsection. The
matrix form of the RUKF is useful to derive the continuous-discrete RUKF.

The covariance matrix $F_{\text{RUKF}}^{x}$ of (37) can be written in a matrix form as follows [9]:

$$X_{mtx,k-1} = E\left[ \begin{array}{c} \hat{x}_{k-1|k-1} \\ \vdots \\ \hat{x}_{k-1|k-1} \end{array} \right] + \gamma_{k} rukf \left[ \left[ \sqrt{P_{xx,k-1}} \right] \right],$$

(38)

$$X_{mtx,k|k-1} = f(X_{mtx,k-1|k-1}, P_{xx,k-1}),$$

(39)

$$p_{xx,ukf} = X_{mtx,k|k-1} W_{x} X_{mtx,k|k-1} + Q_{k-1},$$

(40)

where $W_{x}$ is matrix form of weight of sigma points given by

$$\omega_{x} = \left[ \begin{array}{c} W_{0}^{(m)} \\ \vdots \\ W_{2n}^{(m)} \end{array} \right],$$

(41)

$$W_{x} = (I - [\omega_{x} \ldots \omega_{x}] \times \text{diag} \left( W_{0}^{(c)} \ldots W_{2n}^{(c)} \right)) \times (I - [\omega_{x} \ldots \omega_{x}] \right)^{T},$$

(42)

The term of the parameter uncertainties $P_{\text{RUKF}}$ can be transformed by the same way as

$$P_{mtx,k-1} = \left[ \begin{array}{c} p_{nom}^{(m)} \\ \vdots \\ p_{nom}^{(m)} \end{array} \right] + \tau \sqrt{P_{pp,k-1}} \sqrt{P_{pp,k-1}},$$

(43)

$$\chi_{mtx,k-1}^{(p)} = f(\hat{x}_{k-1|k-1}, P_{mtx,k-1}),$$

(44)

$$P_{k|k-1}^{(p)} = \chi_{mtx,k|k-1}^{(p)} W_{p} \left( \chi_{mtx,k|k-1}^{(p)} \right)^{T},$$

(45)

where matrix form of weight $W_{p}$ is the same definition as (42).

4 Continuous-discrete time RUKF

Consider continuous-discrete time nonlinear stochastic systems without parameter uncertainties:

$$dx(t) = \left\{ f(x(t), p) + g(x(t))u(t) \right\} dt + L(t) db(t),$$

(46)

$$y_{k} = h(x_{k}) + v_{k},$$

(47)

where $x(t) \in \mathbb{R}^{n}$ is state; $b(t)$ is a Brown motion with Diffusion matrix $Q_{c}(t)$; $y_{k} \in \mathbb{R}^{m}$ is measurement; $v_{k}$ is a zero mean Gaussian noise with covariance matrix $R_{k}$; $y_{k}, v_{k}$ are same as in the discrete-time systems. We assume parameter vector $p \sim \mathcal{N}(p_{nom}, P_{pp})$ is time independent in this section.

We omit (t) to simplify equations except for the case where it is necessary to write it. Using $\Delta t$ which is sufficiently close to zero, we can approximate state dynamics (46) by

$$x(t + \Delta t) - x(t) \approx \left\{ F \hat{x} + G \hat{p} + E[f(x, p) + g(x) dt] \right\} \Delta t + L(t) \Delta b(t) + o(\Delta t),$$

(48)

where $\Delta b(t) \sim \mathcal{N}(0, Q_{c}(t) \Delta t)$ and $o(\Delta t)$ is a function such that $\frac{o(\Delta t)}{\Delta t} \rightarrow 0$ when $\Delta t \rightarrow 0$. We apply the USL to continuous nonlinear dynamics (46) around $(\hat{x}, p_{nom})$ and obtain the following approximated linear equation,

$$f(x, p) + g(x)u \approx F \hat{x} + G \hat{p} + \mathbb{E}[f(x, p) + g(x) dt],$$

(49)

where $\Delta t = x - \hat{x}$ and $\hat{p} = p - p_{nom}$.

We define a predictive error $\tilde{x} = x(t + \Delta t) - \hat{x}(t + \Delta t)$ and obtain

$$\tilde{x} \approx x(t) + \left\{ (F \hat{x} + G \hat{p} + \mathbb{E}[f(x, p) + g(x) dt]) \Delta t + L \Delta b \right\} \Delta t,$$

(50)

Furthermore, we can approximate the prediction error covariance matrix as

$$P^{-}(t + \Delta t) = E[\tilde{x} \cdot (\tilde{x})^{T}],$$

$$\approx \mathbb{E} \left[ (F \hat{x} + G \hat{p} + L \Delta b)(F \hat{x} + G \hat{p} + L \Delta b)^{T} \right],$$

$$= FP_{xx} F^{T} + GP_{pp} G^{T} + LQ_{c} L^{T}. (51)$$

Then, we can derive prediction equations by matrix forms as the same way [9],

$$\tilde{X}_{mtx}(t + \Delta t) = X_{mtx}(t) + f(\tilde{X}_{mtx}(t), p_{nom}) \Delta t + o(\Delta t),$$

(52)

$$m^{-}(t + \Delta t) = \tilde{X}_{mtx}(t + \Delta t) \Delta m,$$

(53)

$$P^{-}(t + \Delta t) = \tilde{X}_{mtx}(t + \Delta t) W_{x} \tilde{X}_{mtx}(t + \Delta t)^{T} + \left\{ \chi_{mtx}^{(p)} W_{p} \left( \chi_{mtx}^{(p)} \right)^{T} + LQ_{c} L^{T} \right\} \Delta t,$$

(54)

where $X_{mtx}$ is a matrix form of the sigma points $\chi_{i}$ and $W_{x}, W_{p}$ are matrix forms of weight $W_{x}, W_{p}$ in section 3.2.

The differential equation of the predictive mean $m(t)$ is same as conventional UKBF [9] because (53) does not include any term of the parameter uncertainties. So, we only derive the differential equation of the predictive covariance matrix. Substitute (52) into (54) and retaining only first-order terms results in

$$P^{-}(t + \Delta t) = \left\{ X_{mtx}(t) + f(\tilde{X}_{mtx}(t), p_{nom}) \Delta t \right\} W_{x} \left\{ \star \right\}^{T} + \left\{ \chi_{mtx}^{(p)} W_{p} \left( \chi_{mtx}^{(p)} \right)^{T} + LQ_{c} L^{T} \right\} \Delta t + o(\Delta t),$$

$$= X_{mtx}(t) W_{x} \tilde{X}_{mtx}(t)^{T} + f(\tilde{X}_{mtx}(t), p_{nom}) \Delta t + \left\{ \chi_{mtx}^{(p)} W_{p} \left( \chi_{mtx}^{(p)} \right)^{T} + LQ_{c} L^{T} \right\} \Delta t + o(\Delta t).$$

(55)

We obtain the following differential equation of the predictive covariance matrix by dividing by $\Delta t$ and taking
the limit $\Delta t \to 0$,
\[
\frac{dP_{\text{UKF}}}{dt} = f(X_{\text{mtz}}(t), p_{\text{nom}})W_xX_{\text{mtz}}(t)^T
+ X_{\text{mtz}}(t)W_xf(X_{\text{mtz}}(t), p_{\text{nom}})^T
+ X(p)_{\text{mtz}}W_p[X(p)_{\text{mtz}}]^T + LQ_xL^T,
\]
where we define a state vector $x := [x \ \dot{x} \ \ddot{x}]^T$. We assume a discrete-time linear measurement equation as follow:
\[
y_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x_k + v_k,
\]
and the control input $u$ is changed according to time as $u = 10(0 \leq t \leq 8), u = 10\sin(10t)(8 < t \leq 12)u = -10(12 < t)$. We set the initial conditions as
\[
x_0 = \begin{bmatrix} 0 \\ \pi \\ 0 \\ \pi \end{bmatrix}, \dot{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, P_0 = 10I_{4 \times 4}.
\] (57)

Physical values of the nominal system are summarized in Table 1.

| Parameter      | Value       |
|---------------|------------|
| $M$           | Math of Cart | 0.5 [kg] |
| $m$           | Math of Pendulum | 1.0 [kg] |
| $l$           | Length of Pendulum | 1.0 [m] |
| $B$           | Friction Coefficient | 0.5 [kg/s] |
| $C$           | Viscous Friction Coefficient | 0.3 [kg m$^2$/s] |

We assume that the friction coefficient $B$ and the viscous friction coefficient $C$ have uncertainties and we sampled $B$ and $C$ as
\[
B \sim N(0.5, 0.1), \quad C \sim N(0.3, 0.05).
\] (58)

And true values of $B$ and $C$ do not change during one simulation.

We compare the following filters:

**PUKF**: Conventional UKF which uses true values of the parameters (58).

**NUKF**: Conventional UKF which uses nominal values of the parameters $B = 0.5$ and $C = 0.3$.

**RUKF**: Proposed robust UKF.

Table 2 shows the performance comparison for a 50-run Monte Carlo simulation. The simulation was performed over time period 20s using Euler-Maruyama scheme with $\Delta t = 0.001$ s, and sampling period are fixed to $\Delta t_o = 0.2$ s. In this simulation condition with a long sampling period $\Delta t_o = 0.2$ s, discrete-time UKFs can not realize accurate estimation.

We can confirm that estimation accuracy of $x_2$ of the RUKF is superior to that of the NUKF. On the other hand, estimation accuracy of $x_3$ of the RUKF is almost same as that of the NUKF. Therefor, we can judge that our proposed method is effective for uncertain system.

Fig. 2 shows time series of estimation error of state $x_2$ and $x_4$ of each filter. We can confirm estimation error of $x_4$ of the NUKF is degraded when the control input change occurs ($t = 8$).

Fig. 1: cart pendulum system

Nonlinear dynamics of the cart pendulum system is given by
\[
f(x, p) =
\begin{bmatrix}
  x_2 \\
  \{ -B(J + m \ddot{l})x_2 + Cml \cos(x_3)x_4 + \\
  (J + m \dddot{l})m \sin(x_3)x_2^3 - m^2l^2g \sin(x_3) \cos(x_3) \}/\alpha(x) \\
  x_4 \\
  \{ Bml \cos(x_3)x_4 - C(m + M)x_4 \\
  -m^2l^2g \sin(x_3) \cos(x_3)x_2^3 \}/\alpha(x)
\end{bmatrix},
\]

\[
g(x)u =
\begin{bmatrix}
  0/(J + m \dddot{l})/\alpha(x) \\
  0
\end{bmatrix} u,
\]

\[
\alpha(x) = J(m + M) + Mml^2 + m^2l^2 \sin^2(x_3),
\]
Table 2: Simulation results - RMSE of each filter

|   | PUKF | NUKF | RUKF |
|---|------|------|------|
| $x_1$ | 0.047 | 0.047 | 0.047 |
| $x_2$ | 0.090 | 0.544 | 0.477 |
| $x_3$ | 0.048 | 0.048 | 0.048 |
| $x_4$ | 0.129 | 0.299 | 0.300 |

Fig. 2: Estimation error of state $x_2$ and $x_4$

6 Conclusions

In this research, we addressed the continuous-discrete problem for parametric uncertain nonlinear systems with nonlinear measurements.

First, we derived a matrix form of discrete-time RUKF. Then, we analyzed the effect of the parameter uncertainties of continuous-time dynamics. Finally, we extended the discrete-time RUKF to the continuous-discrete RUKF by using the matrix form.

The validity of the method was illustrated in a numerical example. The estimation accuracy of CD-RUKF is better than the conventional UKBF if the nonlinear system has parameter uncertainties.

Our continuous-discrete RUKF can be calculated in the conventional UKBF framework like a previous work, so we can implement it easily to real applications. The CD-RUKF can be applied even when the joint estimation method cannot be used since the observability or the stability of the augmented nonlinear system is lost. This property is very useful for real applications whose measurable states are often restricted.

Furthermore, the update rule of the CD-RUKF is same as that of the discrete-time RUKF, so we can apply the update rule of the adaptive RUKF [11] to achieve more accurate estimations.

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