\section*{Infinite Products Involving Binary Digit Sums}

Samin Riasat

\textbf{Abstract} Let \((u_n)_{n\geq0}\) denote the Thue-Morse sequence with values \(\pm1\). The Woods-Robbins identity below and several of its generalisations are well-known in the literature

\[\prod_{n=0}^{\infty} \left(\frac{2n+1}{2n+2}\right)^{u_n} = \frac{1}{\sqrt{2}}.\]

No other such product involving a rational function in \(n\) and the sequence \(u_n\) seems to be known in closed form. To understand these products in detail we study the function

\[f(b,c) = \prod_{n=1}^{\infty} \left(\frac{n+b}{n+c}\right)^{u_n}.\]

We prove some analytical properties of \(f\). We also obtain some new identities similar to the Woods-Robbins product.

\section{1 Introduction}

Let \(s_k(n)\) denote the sum of the digits in the base-\(k\) expansion of the non-negative integer \(n\). Although we only consider \(k = 2\), our results can be easily extended to all integers \(k \geq 2\). Put \(u_n = (-1)^{s_2(n)}\). In other words, \(u_n\) is equal to 1 if the binary expansion of \(n\) has an even number of 1’s, and is equal to \(-1\) otherwise. This is the so-called Thue-Morse sequence with values \(\pm1\). We study infinite products of the form

\[f(b,c) := \prod_{n=1}^{\infty} \left(\frac{n+b}{n+c}\right)^{u_n}.\]
The only known non-trivial value of \( f \) (up to the relations \( f(b, b) = 1 \) and \( f(b, c) = 1/f(c, b) \)) seems to be
\[
f\left(\frac{1}{2}, 1\right) = \sqrt{2},
\]
which is the famous Woods-Robbins identity [8, 9]. Several infinite products inspired by this identity were discovered afterwards (see, e.g., [5, 7]), but none of them involve the sequence \( u_n \). In this paper we compute another value of \( f \), namely,
\[
f\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{3}{2}.
\]
In Sect. 2.1 we look at properties of the function \( f \) and introduce a related function \( h \). In Sect. 3 we study the analytical properties of \( h \). In Sect. 4 we try to find infinite products of the form \( \prod R(n)^{u_n} \) admitting a closed form value, with \( R \) a rational function.

This paper forms the basis for the paper [3]. While the purpose of [3] is to compute new products of the forms \( \prod R(n)^{u_n} \) and \( \prod R(n)^{t_n} \), \( t_n \) being the Thue-Morse sequence with values 0, 1, we restrict ourselves in this paper to studying products of the form \( \prod R(n)^{u_n} \) in greater depth.

## 2 General Properties of \( f \) and a New Function \( h \)

We start with the following result on convergence.

**Lemma 1.** Let \( R \in \mathbb{C}(x) \) be a rational function such that the values \( R(n) \) are defined and non-zero for integers \( n \geq 1 \). Then, the infinite product \( \prod R(n)^{u_n} \) converges if and only if the numerator and the denominator of \( R \) have same degree and same leading coefficient.

**Proof.** See [3], Lemma 2.1. \( \square \)

Hence \( f(b, c) \) converges for any \( b, c \in \mathbb{C} \setminus \{-1, -2, -3, \ldots\} \). Using the definition of \( u_n \) we see that \( f \) satisfies the following properties.

**Lemma 2.** For any \( b, c, d \in \mathbb{C} \setminus \{-1, -2, -3, \ldots\} \),

1. \( f(b, b) = 1 \),
2. \( f(b, c)f(c, d) = f(b, d) \),
3. \( f(b, c) = \left(\frac{c+1}{b+1}\right) f\left(\frac{b}{2}, \frac{c}{2}\right) f\left(\frac{c+1}{2}, \frac{b+1}{2}\right) \).

**Proof.** The only non-trivial claim is part 3. To see why it is true, note that \( u_{2n} = u_n \) and \( u_{2n+1} = -u_n \), so that
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\[ f(b, c) = \prod_{n=1}^{\infty} \left( \frac{n+b}{n+c} \right)^{u_n} \]
\[ = \frac{1+c}{1+b} \prod_{n=1}^{\infty} \left( \frac{2n+b}{2n+c} \right)^{u_n} \prod_{n=1}^{\infty} \left( \frac{2n+1+c}{2n+1+b} \right)^{u_n} \]
\[ = \frac{1+c}{1+b} \prod_{n=1}^{\infty} \left( \frac{n+b/2}{n+c/2} \right)^{u_n} \prod_{n=1}^{\infty} \left( \frac{n+1+c}{n+1+b} \right)^{u_n} \]
\[ = \left( \frac{c+1}{b+1} \right) f \left( \frac{b}{2}, \frac{c}{2} \right) f \left( \frac{c+1}{2}, \frac{b+1}{2} \right) \]

as desired. □

One can ask the natural question: is \( f \) the unique function satisfying these properties? What if we impose some continuity/analyticity conditions?

Using the first two parts of Lemma 2 we get

\[ f(b, c) f(d, e) = \frac{f(b, c) f(c, d) f(d, c) f(d, c)}{f(c, d) f(d, c)} = \frac{f(b, e) f(d, c)}{f(c, c)} = f(b, e) f(d, c). \]

Hence the third part may be re-written as

\[ f(b, c) = \frac{f \left( \frac{b}{2}, \frac{b+1}{2} \right)}{f \left( \frac{c}{2}, \frac{c+1}{2} \right)} \cdot \frac{h(b)}{h(c)}. \] (1)

This motivates the following definition.

**Definition 1.** Define the function

\[ h(x) := f \left( \frac{x}{2}, \frac{x+1}{2} \right). \] (2)

Then Eqs. (1) and (2) give the following result.

**Lemma 3.** For any \( b, c \in C \ \{ -1, -2, -3, \ldots \} \),

\[ f(b, c) = \frac{c+1}{b+1} \cdot \frac{h(b)}{h(c)}. \] (3)

So understanding \( f \) is equivalent to understanding \( h \), in the sense that each function can be completely evaluated in terms of the other. Moreover, taking \( c = b + \frac{1}{2} \) in Eq. (3) and then using Eq. (2) gives the following result.

**Lemma 4.** The function \( h \) defined by Eq. (2) satisfies the functional equation

\[ h(x) = \frac{x+1}{x+\frac{1}{2}} h \left( \frac{x+1}{2} \right) h(2x). \] (4)
Again one may ask: is $h$ the unique solution to Eq. (4)? What about monotonic/continuous/smooth solutions?

An approximate plot of $h$ is given in Fig. 1 with the infinite product truncated at $n = 100$.

![Approximate plot of $h(x)$](image)

**Fig. 1** Approximate plot of $h(x)$

### 3 Analytical Properties of $h$

The following lemma forms the basis for the results in this section.

**Lemma 5.** For $b, c \in (-1, \infty)$,

1. if $b = c$, then $f(b, c) = 1$.
2. if $b > c$, then
   \[
   \left( \frac{c + 1}{b + 1} \right)^2 < f(b, c) < 1.
   \]
3. if $b < c$, then
   \[
   1 < f(b, c) < \left( \frac{c + 1}{b + 1} \right)^2.
   \]

**Proof.** Using Lemma [2] it suffices to prove the second statement.

Let $b > c > -1$ and put

\[
a_n = \log \left( \frac{n + b}{n + c} \right), \quad S_N = \sum_{n=1}^{N} a_n u_n, \quad U_N = \sum_{n=1}^{N} u_n.
\]
Note that \( a_n \) is positive and strictly decreasing to 0. Using \( s_2(2n) + s_2(2n+1) \equiv 1 \pmod{2} \) it follows that \( U_n \in \{-2,-1,0\} \) and \( U_n \equiv n \pmod{2} \) for each \( n \). Using summation by parts,

\[
S_N = a_{N+1}U_N + \sum_{n=1}^{N} U_n(a_n - a_{n+1}).
\]

So \(-2a_1 < S_N < 0\) for large \( N \). Exponentiating and taking \( N \to \infty \) gives the desired result. \( \square \)

Lemmas 3-5 immediately imply the following results.

**Theorem 1.** \( h(x)/(x+1) \) is strictly decreasing on \((-1, \infty)\) and \( h(x)(x+1) \) is strictly increasing on \((-1, \infty)\).

**Theorem 2.** For \( b, c \in (-1, \infty) \), \( f(b, c) \) is strictly decreasing in \( b \) and strictly increasing in \( c \).

**Theorem 3.** For \( x \in (-2, \infty) \),

\[
1 < h(x) < \left( \frac{x+3}{x+2} \right)^2.
\]

We now give some results on differentiability.

**Theorem 4.** \( h(x) \) is smooth on \((-2, \infty)\).

*Proof.* Recall the definition of \( h \):

\[
h(x) = \prod_{n=1}^{\infty} \left( \frac{2n+x}{2n+1+x} \right)^{u_n}.
\]

Then taking \( b = x/2 \) and \( c = (x+1)/2 \) in Eqs. (5) shows that the sequence \( S_n \) of smooth functions on \((-2, \infty)\) converges pointwise to \( \log h \).

Differentiating with respect to \( x \) gives

\[
S_N' = \sum_{n=1}^{N} \frac{u_n}{(2n+x)(2n+1+x)} = \sum_{n=1}^{N} u_n \left( \frac{1}{2n+x} - \frac{1}{2n+1+x} \right).
\]

Hence
\[ |S'_N - S'_M| \leq \sum_{n=M+1}^{N} \left( \frac{1}{2n+x} - \frac{1}{2n+1+x} \right) \]
\[ \leq \sum_{n=M+1}^{N} \left( \frac{1}{2n-1+x} - \frac{1}{2n+1+x} \right) \]
\[ = \frac{1}{2M+1+x} - \frac{1}{2N+1+x} \]
\[ < \frac{1}{2M-1} \to 0 \]
as \( M \to \infty \), for any \( x \in (-2, \infty) \) and \( N > M \). Thus \( S'_n \) converges uniformly on \((-2, \infty)\), which shows that \( \log h \), hence \( h \), is differentiable on \((-2, \infty)\).

Now suppose that derivatives of \( h \) up to order \( k \) exist for some \( k \geq 1 \). Note that
\[
S^{(k+1)}_N = (-1)^k k! \sum_{n=1}^{N} u_n \left( \frac{1}{(2n+x)^{k+1}} - \frac{1}{(2n+1+x)^{k+1}} \right).
\]
As before,
\[
\left| S^{(k+1)}_N - S^{(k+1)}_M \right| \leq k! \sum_{n=M+1}^{N} \left( \frac{1}{(2n+x)^{k+1}} - \frac{1}{(2n+1+x)^{k+1}} \right) \]
\[ \leq k! \sum_{n=M+1}^{N} \left( \frac{1}{(2n-1+x)^{k+1}} - \frac{1}{(2n+1+x)^{k+1}} \right) \]
\[ = \frac{k!}{(2M+1+x)^{k+1}} - \frac{k!}{(2N+1+x)^{k+1}} \]
\[ < \frac{k!}{(2M-1)^{k+1}} \to 0 \]
as \( M \to \infty \), for any \( x \in (-2, \infty) \) and \( N > M \). Hence \( S^{(k+1)}_n \) converges uniformly on \((-2, \infty)\), i.e., \( h^{(k)} \) is differentiable on \((-2, \infty)\).

Therefore, by induction, \( h \) has derivatives of all orders on \((-2, \infty)\). \( \Box \)

**Theorem 5.** Let \( a \geq 0 \). Then
\[
\log h(x) = \log h(a) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left( \sum_{n=2}^{\infty} u_n \left( \frac{1}{(n+a)^k} \right) \right) (x-a)^k,
\]
for \( x \in [a-1, a+1] \).

**Proof.** Let \( H(x) = \log h(x) \). By Theorem 4
\[
H^{(k+1)}(x) = (-1)^k k! \sum_{n=2}^{\infty} \frac{u_n}{(n+x)^{k+1}}.
\]
Hence
\[ |H^{(k+1)}(x)| \leq k! \sum_{n=2}^{\infty} \frac{1}{|n+x|^{k+1}} \leq k! \sum_{n=2}^{\infty} \frac{1}{(n+a-1)^{k+1}} \]

for \( x \in [a - 1, a + 1] \). So by Taylor’s inequality, the remainder for the Taylor polynomial for \( H(x) \) of degree \( k \) is absolutely bounded above by

\[ \frac{1}{k+1} \left( \sum_{n=2}^{\infty} \frac{1}{(n+a-1)^{k+1}} \right) |x-a|^{k+1} \]

which tends to 0 as \( k \to \infty \), since \( a \geq 0 \) and \( |x-a| \leq 1 \). Therefore \( H(x) \) equals its Taylor expansion about \( a \) for \( x \) in the given range. \( \Box \)

4 Infinite Products

Recall that

\[ f(b, c) = \prod_{n=1}^{\infty} \left( \frac{n+b}{n+c} \right)^{u_n} \]

From Lemma 2 we see that

\[ \prod_{n=0}^{\infty} \left( \frac{(n+b)(n+b+\frac{1}{2})(n+c)}{(n+c)(n+c+\frac{1}{2})(n+\frac{3}{2})} \right)^{u_n} = \frac{c+1}{b+1} \]

for any \( b, c \neq -1, -2, -3, \ldots \), and if \( b, c \neq 0, -1, -2, \ldots \), then

\[ \prod_{n=0}^{\infty} \left( \frac{(n+b)(n+b+\frac{1}{2})(n+c)}{(n+c)(n+c+\frac{1}{2})(n+\frac{3}{2})} \right)^{u_n} = 1. \]

Some interesting identities can be obtained from Eqs. (6)–(7). For example, in Eq. (6), taking \( c = (b+1)/2 \) gives

\[ \prod_{n=1}^{\infty} \left( \frac{(n+b)(n+b+\frac{1}{2})(n+c)}{(n+b+\frac{3}{4})(n+c+\frac{1}{2})} \right)^{u_n} = \frac{b+3}{2(b+1)} \]

while taking \( b = 0 \) gives

\[ \prod_{n=1}^{\infty} \left( \frac{(n+\frac{1}{2})(n+c)}{(n+c)(n+\frac{1}{2}+\frac{1}{2})} \right)^{u_n} = c+1 \]

for any \( b, c \neq -1, -2, -3, \ldots \).

We now turn our attention to the functional equation Eq. (4). Recall that it reads

\[ h(x) = \frac{x+\frac{1}{2}}{x+\frac{1}{2}} h \left( x + \frac{1}{2} \right) h(2x). \]
Taking $x = 0$ gives
\[ h(0) = \frac{2}{3} h \left( \frac{1}{2} \right) h(0). \]

Since $1 < h(0) < 9/4$ by Theorem 3, cancelling $h(0)$ from both sides gives $h(1/2) = 3/2$. This shows that
\[ \prod_{n=0}^{\infty} \left( \frac{4n + 3}{4n + 1} \right)^{u_n} = 2. \] (10)

Next, taking $x = 1/2$ in Eq. (4) gives
\[ h \left( \frac{1}{2} \right) = \frac{3}{4} h(1)^2 \]
hence $h(1) = \sqrt{2}$ (since $1 < h(1) < 16/9$ by Theorem 3) and we recover the Woods-Robbins product
\[ \prod_{n=0}^{\infty} \left( \frac{2n + 2}{2n + 1} \right)^{u_n} = \sqrt{2}. \] (11)

Similarly, taking $x = -1/2$ in Eq. (4) gives
\[ h \left( -\frac{1}{2} \right) = \frac{1}{2} h(0) h(-1) = \frac{1}{2} f \left( 0, \frac{1}{2} \right) f \left( -\frac{1}{2}, 0 \right) = \frac{1}{2} f \left( -\frac{1}{2}, 1/2 \right), \]
i.e.,
\[ \prod_{n=1}^{\infty} \left( \frac{(4n - 1)(2n + 1)}{(4n + 1)(2n - 1)} \right)^{u_n} = \frac{1}{2}. \] (12)

Taking $x = 1$ in Eq. (4) gives
\[ h(1) = \frac{4}{5} h \left( \frac{3}{2} \right) h(2) \]
hence $h(3/2) h(2) = 5\sqrt{2} / 4$ and this gives
\[ \prod_{n=0}^{\infty} \left( \frac{(4n + 3)(2n + 2)}{(4n + 5)(2n + 3)} \right)^{u_n} = \frac{1}{\sqrt{2}}. \] (13)

Taking $x = 3/2$ in Eq. (4) and using the previous result gives
\[ h(2)^2 h(3) = \frac{3}{\sqrt{2}} \]
which is equivalent to
\[ \prod_{n=0}^{\infty} \left( \frac{(2n + 2)(n + 1)}{(2n + 3)(n + 2)} \right)^{u_n} = \frac{1}{\sqrt{2}}. \] (14)
Eqs. (10)–(14) can also be combined in pairs to obtain other identities.

5 Concluding Remarks

The quantity \( h(0) \approx 1.62816 \) appears to be of interest \[1, 4\]. It is not known whether its value is irrational or transcendental. We give the following explanation as to why \( h(0) \) might behave specially in a sense.

Note that the only way non-trivial cancellation occurs in the functional equation Eq. (4) is when \( b = 0 \). Likewise, non-trivial cancellation occurs in Eq. (1) or property 3 in Lemma 2 only for \((b, c) = (0, 1/2)\) and \((1/2, 0)\). That is, the victim of any such cancellation is always \( h(0) \) or \( h(0)^{-1} \). So one must look for other ways to understand \( h(0) \).

Using the only two known values \( h(1/2) = 3/2 \) and \( h(1) = \sqrt{2} \), the following expressions for \( h(0) \) can be obtained from Theorem 5:

- By taking \( x = 0 \) and \( a = 1 \),
  \[
  h(0) = \sqrt{2} \exp \left( - \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=2}^{\infty} \frac{u_n}{(n+1)^k} \right).
  \]

- By taking \( x = 1 \) and \( a = 0 \),
  \[
  h(0) = \sqrt{2} \exp \left( \sum_{k=1}^{\infty} \left( - \frac{1}{k} \right)^k \sum_{n=2}^{\infty} \frac{u_n}{n^k} \right).
  \]

- By taking \( x = 0 \) and \( a = 1/2 \),
  \[
  h(0) = \frac{3}{2} \exp \left( \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=2}^{\infty} \frac{u_{2n+1}}{(2n+1)^k} \right).
  \]

- By taking \( x = 1/2 \) and \( a = 0 \),
  \[
  h(0) = \frac{3}{2} \exp \left( \sum_{k=1}^{\infty} \left( - \frac{1}{k} \right)^k \sum_{n=2}^{\infty} \frac{u_{2n}}{(2n)^k} \right).
  \]

The Dirichlet series
\[
\sum_{n=0}^{\infty} \frac{u_n}{(n+1)^k} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{u_n}{n^k}
\]
appearing in the above expressions were studied by Allouche and Cohen \[2\].

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