A review of some works in the theory of diskcyclic operators

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Abstract

In this paper we give a brief review concerning diskcyclic operators and then we provide some further characterizations of the diskcyclic operators on the separable Hilbert spaces. In particular, we show that if \( x \in \mathcal{H} \) has disk orbit under \( T \) that is somewhere dense in \( \mathcal{H} \) then the orbit of \( x \) under \( T \) need not be everywhere dense in \( \mathcal{H} \). We also show that the inverse and adjoint of a diskcyclic operator need not be diskcyclic. Moreover, we establish another diskcyclicity criterion and finally we give a sufficient condition for the somewhere density disk orbit to be everywhere dense.

Keywords: Hypercyclic operators, Supercyclic operators, Diskcyclic operators, Diskcyclicity criterion.

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1 Introduction.

In this paper, all Hilbert spaces are separable over the field \( \mathbb{C} \) of complex numbers. As usual, \( \mathbb{N} \) is the set of all non-negative integers, \( \mathbb{Z} \) is the set of all integers and \( \mathcal{B}(\mathcal{H}) \) is the space of all continuous linear operators on a Hilbert space \( \mathcal{H} \).

An operator \( T \) is called hypercyclic if there is some vector \( x \in \mathcal{H} \) such that \( \text{Orb}(T, x) = \{T^n x : n \in \mathbb{N}\} \) is dense in \( \mathcal{H} \), where such a vector \( x \) is called hypercyclic for \( T \). The first example of hypercyclic operator was given by Rolewicz

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in [12]. He proved that if $B$ is a backward shift on the Banach space $\ell^p(\mathbb{N})$ then $\lambda B$ is hypercyclic for any complex number $\lambda$ such that $|\lambda| > 1$. This leads us to the consideration of scaled orbits. Later, Hilden and Wallen in [9] undertook the concept of supercyclic operators. An operator $T$ is called supercyclic if there is a vector $x \in \mathcal{H}$ such that $COrb(T, x) = \{\lambda T^n x : \lambda \in \mathbb{C}, n \in \mathbb{N}\}$ is dense in $\mathcal{H}$, where $x$ is called supercyclic vector. For the more detailed information on both hypercyclicity and supercyclicity, [2, 6, 10] can be referred.

In the same spirit, since the operator $\lambda B$ is not hypercyclic whenever $|\lambda| \leq 1$, we are motivated to study the disk orbit. The diskcyclicity phenomenon was introduced by Zeana in her PhD thesis [14]. An operator $T$ is called diskcyclic if there is a vector $x \in \mathcal{H}$ such that the set $DOrb(T, x) = \{\alpha T^n x : n \geq 0, \alpha \in \mathbb{C}, |\alpha| \leq 1\}$ is dense in $\mathcal{H}$, where the vector $x$ is called diskcyclic for $T$. The diskcyclic criterion - a sufficient set of conditions for the diskcyclicity - was created by Zeana in [14] and it has been shown that the diskcyclicity criterion is a midway between the hypercyclicity and supercyclicity criterions.

The following diagram shows the relationship among the cyclic operators on a Hilbert space.

\[
\text{Hypercyclicity} \Rightarrow \text{Diskcyclicity} \Rightarrow \text{Supercyclicity}.
\]

It was known that the hypercyclic operators are strictly infinite dimensional phenomena however the supercyclic operators exist on both one-dimensional and infinite dimensional Hilbert spaces [8]. In [11] the first two authors of this paper proved the existence of the diskcyclic operators on one-dimensional Hilbert spaces.

This article consists of three sections. The Section 2 gives a brief review of some works on the diskcyclicity. Some characterizations of the diskcyclicity for the bilateral weighted shifts on $\ell^2(\mathbb{Z})$ will be described. We give an example of diskcyclic but non-hypercyclic operator and an example of supercyclic but non-diskcyclic operator.

In Section 3, we represent another equivalent version of the diskcyclicity criterion. We show that a unilateral backward weighted shift is hypercyclic if and only if it is diskcyclic. The diskcyclicity shares many structures with hypercyclicity and supercyclicity, nevertheless not all. Based on the previous works, an operator is hypercyclic (or supercyclic) if and only if its inverse is hypercyclic (or supercyclic respectively). Through Example 3.9 (Example 3.10, Example 3.11) we show that neither the adjoint nor the inverse of the diskcyclic operators need be diskcyclic. In addition, the somewhere density of the orbit of an operator (the cone generated by orbit) implies the everywhere density of the orbit of the operator (cone gen-
erated by orbit). However, we show that the somewhere density of the disk orbit does not imply to the everywhere density by giving the Counter-example 3.13.

Finally, in Corollary 3.10 we give a sufficient condition for somewhere density disk orbit to be everywhere dense and we also give some spectral properties of the diskcyclic operators.

2 Preliminaries.

We denote the disk orbit \{αT^n x : n ≥ 0, α ∈ C, |α| ≤ 1\} by \(\text{DO}_{\text{rb}}(T, x)\), the set of all diskcyclic vectors by \(\mathbb{D}C(T)\) and the set of all diskcyclic operators by \(\mathbb{D}C(\mathcal{H})\).

The following results are due to Zeana [14] unless otherwise stated. A necessary but not sufficient condition for the diskcyclicity is due to the following proposition.

**Proposition 2.1.** If \(x\) is a diskcyclic vector for \(T\) then we have

\[
\inf\{\|αT^n x\| : n ≥ 0, α ∈ [0, 1]\} = 0 \quad \text{and} \quad \sup\{\|T^n x\| : n ≥ 0\} = ∞.
\]

**Proof.** Since \(α \in [0, 1]\), then it is clear that \(\inf\{\|αT^n x\| : n ≥ 0, α ∈ [0, 1]\} = 0\).

Towards a contradiction, assume that

\[
\sup\{\|T^n x\| : n ≥ 0, α ∈ [0, 1]\} = m < ∞,
\]

and \(y ∈ \mathcal{H}\) such that \(\|y\| > m\). Since \(T ∈ \mathbb{D}C(\mathcal{H})\), then there exist sequences \(\{n_k\}\) in \(\mathbb{N}\) and \(\{α_k\}\) in \(C\); \(|α_k| ≤ 1\) such that \(α_kT^{n_k}x → y\). It follows that \(\|y\| ≤ m\) and this is a contradiction. \(\square\)

**Proposition 2.2.** If \(\{\mathcal{H}_i\}\) is a family of Hilbert spaces, \(T_i ∈ \mathcal{B}(\mathcal{H}_i)\) for all \(i\) and \(⊕T_i ∈ \mathbb{D}C(⊕\mathcal{H}_i)\), then \(T_i ∈ \mathbb{D}C(\mathcal{H}_i)\) for all \(i\).

**Proof.** Let \(y = (y_1, y_2, \ldots) ∈ ⊕\mathcal{H}_i\) and \(x = (x_1, x_2, \ldots) ∈ \mathbb{D}C(⊕T_i)\), then there exist sequences \(\{α_k\}\) in \(C\); \(|α_k| ≤ 1\) and \(\{n_k\}\) in \(\mathbb{N}\) such that \(α_k(⊕T_i)^{n_k}x → y\), as \(n_k → ∞\). It easily follows that \(α_kT_i^{n_k}x_i → y_i\) for all \(i\). \(\square\)

**Definition 2.3.** A bounded linear operator \(T : X → X\) is called disk transitive if for any pair \(U, V\) of nonempty open subsets of \(X\), there exist \(α ∈ C; 0 < |α| ≤ 1\), and \(n ≥ 0\) such that \(T^n(αU) ∩ V ≠ \emptyset\) or equivalently, there exist \(α ∈ C; |α| ≥ 1\), and \(n ≥ 0\) such that \(T^{-n}(αU) ∩ V ≠ \emptyset\).

**Proposition 2.4.** Let \(\mathcal{H}\) and \(\mathcal{K}\) be Hilbert spaces, \(T ∈ \mathcal{B}(\mathcal{H})\) and \(S ∈ \mathcal{B}(\mathcal{K})\). Assume that \(G : \mathcal{H} → \mathcal{K}\) is a bounded linear transformation with dense range and \(SG = GT\). If \(T ∈ \mathbb{D}C(\mathcal{H})\), then \(S ∈ \mathbb{D}C(\mathcal{K})\).
Proof. Let \( x \in \mathbb{D}C(T) \). Then we have
\[
\mathbb{D}Orb(S, Gx) = \{ \alpha S^n Gx : n \geq 0, \alpha \in \mathbb{C}, |\alpha| \leq 1 \} = \{ \alpha G T^n x : n \geq 0, \alpha \in \mathbb{C}, |\alpha| \leq 1 \} = G \{ \alpha T^n x : n \geq 0, \alpha \in \mathbb{C}, |\alpha| \leq 1 \} \supseteq G(\mathbb{H}).
\]
Since \( R(G) \) is dense in \( \mathbb{K} \), it follows that \( \mathbb{D}Orb(S, Gx) \) is dense in \( \mathbb{K} \). Thus \( S \in \mathbb{D}C(\mathbb{K}) \) with diskcyclic vector \( Gx \).

Proposition 2.5. Let \( T, S \in \mathcal{B}(\mathcal{H}) \) such that \( ST = TS \) and \( R(S) \) is dense in \( \mathcal{H} \). If \( x \in \mathbb{D}C(T) \), then \( Sx \in \mathbb{D}C(T) \).

Proof. Since \( x \in \mathbb{D}C(T) \), then
\[
\mathbb{D}Orb(T, Sx) = \{ \alpha T^n Sx : n \geq 0, \alpha \in \mathbb{C}, |\alpha| \leq 1 \} = \{ \alpha S T^n x : n \geq 0, \alpha \in \mathbb{C}, |\alpha| \leq 1 \} = S \{ \alpha T^n x : n \geq 0, \alpha \in \mathbb{C}, |\alpha| \leq 1 \} \supseteq S \{ \alpha T^n x : n \geq 0, \alpha \in \mathbb{C}, |\alpha| \leq 1 \} = S(\mathcal{H}) = R(S).
\]
Thus \( \mathbb{D}Orb(T, Sx) \) is dense in \( \mathcal{H} \) and hence \( Sx \in \mathbb{D}C(T) \).

From the last proposition one can easily deduce that there are many diskcyclic vectors if the operator has one diskcyclic vector.

Corollary 2.6. If \( x \) is a diskcyclic vector of \( T \), then \( T^n x \) is also a diskcyclic vector of \( T \) for all \( n \in \mathbb{N} \).

Proposition 2.7. Every diskcyclic operator on \( \mathcal{H} \) is disk transitive.

Proof. Let \( T \) be a diskcyclic operator. Then, by the previous corollary, it is clear that \( \mathbb{D}C(T) \) is a dense set. Assume that \( U \) and \( V \) are two open sets. Then there exist an \( \alpha \in \mathbb{C}; |\alpha| \leq 1 \) and a non-negative integer \( N \) such that \( \alpha T^N x \in U \). Hence we can find \( \lambda \in \mathbb{C}, |\lambda| \leq |\alpha| \) and \( n \geq N \), such that \( \lambda T^n x \in V \). Thus \( (\lambda/\alpha)T^{n-N} U \cap V \neq \phi \).

Proposition 2.8. Let \( T \in \mathbb{D}C(\mathcal{H}) \) and \( \{ B_k \} \) be a countable open basis for \( \mathcal{H} \). Then
\[
\mathbb{D}C(T) = \bigcap_k \left( \bigcup_{\alpha \in \mathbb{C}} \bigcup_{n \geq 1} T^{-n}(\alpha B_k) \right)
\]
is a dense \( G_\delta \) set.
Proof. We have \( x \in \mathbb{D}C(T) \) if and only if \( \{ \alpha T^n x : n \geq 0, |\alpha| \leq 1 \} \) is dense in \( \mathcal{H} \) if and only if for each \( k > 0 \), there exist \( \alpha \in \mathbb{C} ; |\alpha| \leq 1 \), and \( n \in \mathbb{N} \) such that
\[ \alpha T^n x \in B_k \text{ if and only if } x \in \bigcap_k \left( \bigcup_{\substack{\alpha \in \mathbb{C} \backslash \{0\}, \\ |\alpha| \geq 1}} \bigcup_n T^{-n}(\alpha B_k) \right). \]
Thus
\[ \mathbb{D}C(T) = \bigcap_k \left( \bigcup_{\substack{\alpha \in \mathbb{C} \backslash \{0\}, \\ |\alpha| \geq 1}} \bigcup_n T^{-n}(\alpha B_k) \right). \]
Since \( \mathbb{D}C(T) \) can be written as a countable intersection of open sets, then \( \mathbb{D}C(T) \) is a \( G_\delta \) set. Moreover, the density of \( \mathbb{D}C(T) \) follows from Corollary 2.6.

Corollary 2.9. Every vector in \( \mathcal{H} \) can be written as a sum of two diskcyclic vectors of a diskcyclic operator \( T \).

Proposition 2.10. Every disk transitive operator on \( \mathcal{H} \) is diskcyclic.

Proof. Let \( T \) be disk transitive and \( \{ B_k \} \) be a countable open basis for \( \mathcal{H} \). Then for each open set \( U \) and any \( k \in \mathbb{N} \), there exist \( n \in \mathbb{N} \) and \( \alpha \in \mathbb{C}, |\alpha| \geq 1 \) such that \( U \cap T^{-1}(\alpha B_k) \neq \emptyset \). It follows that
\[ \bigcup_{\alpha \in \mathbb{C} \backslash \{0\}, |\alpha| \geq 1} \bigcup_n T^{-n}(\alpha B_k) \]
is dense. By Proposition 2.8 and the Baire Category Theorem we have that \( \mathbb{D}C(T) \) is dense and, therefore \( T \) is diskcyclic.

The following important lemma is the main tool to prove many results.

Lemma 2.11. Let \( T \in \mathcal{B}(\mathcal{H}) \). The following statements are equivalent.

1. \( T \in \mathbb{D}C(\mathcal{H}) \).
2. \( T \) is disk transitive.
3. For each \( x, y \in \mathcal{H} \), there exist sequences \( \{ x_k \} \) in \( \mathcal{H} \), \( \{ n_k \} \) in \( \mathbb{N} \), and \( \{ \alpha_k \} \) in \( \mathbb{C} \); \( 0 < |\alpha_k| \leq 1 \) such that \( x_k \to x \) and \( T^{n_k} \alpha_k x_k \to y \).
4. For each \( x, y \in \mathcal{H} \) and each neighborhood \( W \) of the zero in \( \mathcal{H} \), there exist \( z \in \mathcal{H} \), \( n \in \mathbb{N} \), and \( \alpha \in \mathbb{C} \); \( 0 < |\alpha| \leq 1 \) such that \( x - z \in W \) and \( T^n \alpha z - y \in W \).

Proof. (1) \( \Leftrightarrow \) (2): Follow from Proposition 2.7 and Proposition 2.11.

(2) \( \Rightarrow \) (3): Let \( x, y \in \mathcal{H} \) and let \( B_k = \mathbb{B}(x, 1/k), B'_k = \mathbb{B}(y, 1/k) \) for all \( k \geq 1 \). From part (2) there exist sequences \( \{ n_k \} \) in \( \mathbb{N} \), \( \{ \alpha_k \} \) in \( \mathbb{C} \); \( 0 < |\alpha_k| \leq 1 \) for all
\( k \geq 1 \) and \( \{ x_k \} \) in \( \mathcal{H} \) such that \( x_k \in B_k \) and \( T^{nk}\alpha_k x_k \in B'_k \) for all \( k \geq 1 \). Then \( \| x_k - x \| < 1/k \) and \( \| T^{nk}\alpha_k x_k - y \| < 1/k \) for all \( k \geq 1 \).

(3) \( \Rightarrow \) (4): Follows immediately from part (3) by taking \( z = x_k \) for a large \( k \in \mathbb{N} \).

(4) \( \Rightarrow \) (2): Let \( U \) and \( V \) be two non-empty open subset of \( \mathcal{H} \). Let \( W \) be a neighborhood for zero, pick \( x \in U \) and \( y \in V \), so there exist \( z \in \mathcal{H} \), \( n \in \mathbb{N} \), \( \alpha \in \mathbb{C} \), \( |\alpha| \leq 1 \) such that \( x - z \in W \) and \( T^n\alpha z - y \in W \). It follows immediately that \( z \in U \) and \( T^n\alpha z \in V \).

\[ \square \]

**Proposition 2.12 (Diskcyclicity criterion).** Let \( T \in \mathcal{B}(\mathcal{H}) \) with the following properties.

1. There exist two dense sets \( X, Y \) in \( \mathcal{H} \) and right inverse of \( T \) (not necessary bounded) \( S \) such that \( S(Y) \subseteq Y \) and \( TS = I_Y \).

2. There is a sequence \( \{ n_k \} \) in \( \mathbb{N} \) such that
   
   \[ (a) \lim_{k \to \infty} \| S^{n_k} y \| = 0 \text{ for all } y \in Y; \]
   \[ (b) \lim_{k \to \infty} \| T^{n_k} x \| \| S^{n_k} y \| = 0 \text{ for all } x \in X, \ y \in Y. \]

Then \( T \in \mathbb{D}C(\mathcal{H}) \).

The characterizations of diskcyclicity for the bilateral weighted shifts on \( \ell^2(\mathbb{Z}) \) have been proved by Zeana in \cite{14} as shown in the following results.

**Theorem 2.13.** Let \( T \) be a bilateral forward weighted shift on the Hilbert space \( \mathcal{H} = \ell^2(\mathbb{Z}) \) with the weight sequence \( \{ w_n \}_{n \in \mathbb{Z}} \). Then the following statements are equivalent.

1. \( T \in \mathbb{D}C(\mathcal{H}) \).

2. For all \( q \in \mathbb{N} \),
   
   \[ (a) \limsup_{n \to \infty} \min_{q} \left\{ \prod_{k=h-n}^{h-1} w_k : |h| \leq q \right\} = \infty. \]
   \[ (b) \liminf_{n \to \infty} \max_{q} \left\{ \prod_{k=j}^{j+n-1} w_k : |h|, |j| \leq q \right\} = 0. \]

3. \( T \) satisfies the diskcyclicity criterion.

**Proof.** (1) \( \Rightarrow \) (2): The proof is similar to \cite{13} Theorem 3.1 and observe that the condition (a) holds from the fact that \( |\alpha| \leq 1 \).
(2) ⇒ (3): Let $X = Y$ be the manifold spanned by $\{e_n\}_{n \in \mathbb{Z}}$ and let $x = \sum_{|j| \leq q} x_j e_j$ and $y = \sum_{|j| \leq q} y_j e_j$. Assume that $B$ is the right inverse of $T$, then

$$Be_n = \frac{1}{w_{n-1}} e_{n-1},$$

and

$$\|T^n x\| \leq \max_q \left\{ \prod_{k=j}^{j+n-1} w_k : |j| \leq q \right\} \|x\|,$$

$$\|B^n y\| \leq \min_q \left\{ \prod_{k=h-n}^{h-1} w_k : |h| \leq q \right\} \|y\|.$$

Thus

$$\|T^n x\| \|B^n y\| \leq \max_q \left\{ \frac{\prod_{k=j}^{j+n-1} w_k}{\prod_{k=h-n}^{h-1} w_k} : |j|, |h| \leq q \right\} \|x\| \|y\|.$$

Let $\epsilon > 0$ and $q \in \mathbb{N}$. Assume there exists a positive integer $n > 2q$ satisfies

$$\prod_{k=-n}^{n-1} w_k < \epsilon$$

and

$$\prod_{k=-n}^{h-1} w_k > \frac{1}{\epsilon}$$

for all $|j|, |h| \leq q$. Then

$$\lim_{n \to \infty} \|T^n x\| \|B^n y\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \|B^n y\| = 0.$$

(3) ⇒ (1): It follows from Proposition 2.12.

**Corollary 2.14.** Let $T \in \mathcal{D}C(\ell^2(\mathbb{Z}))$ be a forward weighted shift with weight sequence $\{w_n\}_{n \in \mathbb{N}}$. Then there is a sequence $\{n_r\}$ in $\mathbb{N}$ such that

1. $\lim_{r \to \infty} \left( \prod_{k=1}^{n_r} \frac{1}{w_{-k}} \right) = 0,$

2. $\lim_{r \to \infty} \left( \prod_{k=1}^{n_r} w_k \right) \left( \prod_{k=1}^{n_r} \frac{1}{w_{-k}} \right) = 0.$

**Proof.** By Theorem 2.13 take $q = 0$, then $j = h = 0$. Hence

$$\limsup_{n \to \infty} \left( \prod_{k=-n}^{n-1} w_k \right) = \infty$$

and

$$\liminf_{n \to \infty} \left( \prod_{k=0}^{n-1} w_k \right) \left( \prod_{k=1}^{n} \frac{1}{w_{-k}} \right) = 0.$$

Thus there is a sequence $\{n_r\}$ in $\mathbb{N}$ such that

$$\lim_{r \to \infty} \left( \prod_{k=1}^{n_r} \frac{1}{w_{-k}} \right) = 0.$$
and
\[ \lim_{r \to \infty} \left( \prod_{k=0}^{n_r-1} w_k \right) \left( \prod_{k=1}^{n_r} \frac{1}{w_{-k}} \right) = 0. \]

Let \( \epsilon > 0 \), then there exists a positive integer \( m > 0 \) such that for all \( r > m \)
\[ \left| \left( \prod_{k=0}^{n_r-1} w_k \right) \left( \prod_{k=1}^{n_r} \frac{1}{w_{-k}} \right) \right| < \epsilon \left| \frac{w_0}{w_{n_r}} \right|. \]

Hence
\[ \left| \left( \prod_{k=1}^{n_r} w_k \right) \left( \prod_{k=1}^{n_r} \frac{1}{w_{-k}} \right) = \left| \left( \prod_{k=0}^{n_r-1} w_k \right) \left( \prod_{k=1}^{n_r} \frac{1}{w_{-k}} \right) \right| < \epsilon \frac{w_0}{w_{n_r}}. \]

Therefore
\[ \lim_{r \to \infty} \left( \prod_{k=1}^{n_r} \frac{1}{w_{-k}} \right) = 0 \quad \text{and} \quad \lim_{r \to \infty} \left( \prod_{k=1}^{n_r} w_k \right) \left( \prod_{k=1}^{n_r} \frac{1}{w_{-k}} \right) = 0. \]

\[ \Box \]

**Proposition 2.15.** Let \( T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \) be an invertible forward weighted shift with weight sequence \( \{w_n\}_{n \in \mathbb{N}} \). Then \( T \in \mathbb{D}C(\ell^2(\mathbb{Z})) \) if and only if there exists a sequence \( \{n_r\} \) in \( \mathbb{N} \) such that

1. \( \lim_{r \to \infty} \prod_{k=1}^{n_r} \frac{1}{w_{-k}} = 0; \)
2. \( \lim_{r \to \infty} \left( \prod_{k=1}^{n_r} w_k \right) \left( \prod_{k=1}^{n_r} \frac{1}{w_{-k}} \right) = 0. \)

**Proof.** The first part follows from Corollary 2.14.

Conversely, we will verify that \( T \) satisfies the diskcyclicity criterion. Let
\[ X = Y = \{ x \in \ell^2(\mathbb{Z}) : x \text{ has only finitely many non–zero coordinates} \}, \]
and let \( B \) be the inverse of \( T \). It is sufficient, by linearity, the triangle inequality, [4 Lemma 3.1] and [4 Lemma 3.3] to suppose that \( x = e_1 \) and \( y = e_0 \). Since
\[ \lim_{r \to \infty} \|B^{n_r}e_0\| = \lim_{r \to \infty} \prod_{k=1}^{n_r} \frac{1}{w_{-k}} = 0 \]
and
\[ \lim_{r \to \infty} \|T^{n_r}e_1\| \|B^{n_r}e_0\| = \lim_{r \to \infty} \left( \prod_{k=1}^{n_r} w_k \right) \left( \prod_{k=1}^{n_r} \frac{1}{w_{-k}} \right) = 0, \]
we reach the proof of the proposition. \[ \Box \]

We should note that a bilateral weighted shift operator is invertible if and only if there is a positive real number \( m \) such that \( |w_n| \geq m \) for all \( n \in \mathbb{Z} \). The following corollary shows that Proposition 2.15 still holds for some further general cases.
Corollary 2.16. Let $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be a bilateral forward weighted shift with the weight sequence $\{w_n\}_{n \in \mathbb{N}}$ and assume that there exists a positive integer $m > 0$ such that $w_n \geq m$ for all $n < 0$ (or for all $n > 0$). Then $T \in DC(\ell^2(\mathbb{Z}))$ if and only if there is a sequence $\{n_r\}$ in $\mathbb{N}$ such that

1. $\lim_{r \to \infty} \prod_{k=1}^{n_r} \frac{1}{w_{-k}} = 0$;
2. $\lim_{r \to \infty} \left( \prod_{k=1}^{n_r} w_k \right) \left( \prod_{k=1}^{n_r} \frac{1}{w_{-k}} \right) = 0$.

Proof. The proof of “if” part follows from Theorem 2.13 and it is similar to the proof of the second part of (1) [4, Theorem 4.1] and the proof of (2) [4, Theorem 4.1]. The “only if” part follows from Corollary 2.14.

Since the bilateral weighted backward shifts are unitarily equivalent to bilateral weighted forward shifts, then the above results were extended to backward shift operators and their proofs have been proved by similar steps.

Theorem 2.17. Let $T$ be a bilateral backward weighted shift on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{Z})$ with the weight sequence $\{w_n\}_{n \in \mathbb{Z}}$. Then the following statements are equivalent.

1. $T \in DC(\mathcal{H})$.
2. For all $q \in \mathbb{N}$;
   (a) $\limsup_{n \to \infty} \min_{q} \left\{ \prod_{k=h+1}^{h+n} w_k : |h| \leq q \right\} = \infty$;
   (b) $\liminf_{n \to \infty} \max_{q} \left\{ \frac{\prod_{k=h+1}^{j+n} w_k}{\prod_{k=h+1}^{h+n} w_k} : |h|, |j| \leq q \right\} = 0$.
3. $T$ satisfies the diskcyclicity criterion.

Corollary 2.18. Let $T \in DC(\ell^2(\mathbb{Z}))$ be a backward weighted shift with the weight sequence $\{w_n\}_{n \in \mathbb{N}}$. Then there is a sequence $\{n_r\}$ in $\mathbb{N}$ such that

1. $\lim_{r \to \infty} \left( \prod_{k=1}^{n_r} \frac{1}{w_k} \right) = 0$;
2. $\lim_{r \to \infty} \left( \prod_{k=1}^{n_r} w_{-k} \right) \left( \prod_{k=1}^{n_r} \frac{1}{w_k} \right) = 0$.

Proposition 2.19. Let $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be an invertible backward weighted shift with the weight sequence $\{w_n\}_{n \in \mathbb{N}}$. Then $T \in DC(\ell^2(\mathbb{Z}))$ if and only if there exists a sequence $\{n_r\}$ in $\mathbb{N}$ such that
1. \( \lim_{r \to \infty} \prod_{k=1}^{n_r} \frac{1}{w_k} = 0; \)

2. \( \lim_{r \to \infty} \left( \prod_{k=1}^{n_r} w_{-k} \right) \left( \prod_{k=1}^{n_r} \frac{1}{w_k} \right) = 0. \)

**Corollary 2.20.** Suppose \( T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \) is a bilateral backward weighted shift with the weight sequence \( \{w_n\}_{n \in \mathbb{N}} \) and there exists a positive integer \( m > 0 \) such that \( w_n \geq m \) for all \( n < 0 \) (or for all \( n > 0 \)). Then \( T \in DC(\ell^2(\mathbb{Z})) \) if and only if there is a sequence \( \{n_r\} \) in \( \mathbb{N} \) such that

1. \( \lim_{r \to \infty} \prod_{k=1}^{n_r} \frac{1}{w_k} = 0; \)

2. \( \lim_{r \to \infty} \left( \prod_{k=1}^{n_r} w_{-k} \right) \left( \prod_{k=1}^{n_r} \frac{1}{w_k} \right) = 0. \)

Now we will give an example of a diskcyclic operator which is not hypercyclic.

**Example 2.21.** Let \( T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \) be the bilateral forward weighted shift with the weight sequence

\[
    w_n = \begin{cases} 
        2 & \text{if } n \geq 0, \\
        3 & \text{if } n < 0. 
    \end{cases}
\]

Then \( T \) is diskcyclic but not hypercyclic.

**Proof.** By applying Corollary 2.16 and taking \( n_r = n \). Observe that

\[
    \lim_{n \to \infty} \prod_{k=1}^{n} \frac{1}{w_k} = \lim_{n \to \infty} \prod_{k=1}^{n} \frac{1}{3} = \lim_{n \to \infty} \frac{1}{3^n} = 0;
\]

and

\[
    \lim_{n \to \infty} \left( \prod_{k=1}^{n} w_k \right) \left( \prod_{k=1}^{n} \frac{1}{w_k} \right) = \lim_{n \to \infty} \left( \prod_{k=1}^{n} 2 \right) \left( \prod_{k=1}^{n} \frac{1}{3} \right) = \lim_{n \to \infty} \left( \frac{2^n}{3^n} \right) = 0.
\]

Thus by Corollary 2.16, \( T \) is diskcyclic. On the other hand, since for all increasing sequence \( n_r \) of positive integers

\[
    \lim_{r \to \infty} \left( \prod_{k=1}^{n_r} w_k \right) = \lim_{r \to \infty} \left( \prod_{k=1}^{n_r} 2 \right) = \lim_{n \to \infty} \left( 2^{n_r} \right) = \infty,
\]

from [1, Fact 1.5], then \( T \) is not hypercyclic. \( \square \)

The following example gives us an operator which is supercyclic but not diskcyclic.
Example 2.22. Let $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be a bilateral forward weighted shift with the weight sequence
\[
    w_n = \begin{cases} 
      \frac{1}{3} & \text{if } n \geq 0, \\
      \frac{1}{2} & \text{if } n < 0.
    \end{cases}
\]
Then $T$ is supercyclic but not diskcyclic.

Proof. By using Corollary 2.16 and taking $n_r = n$. Indeed we get
\[
    \lim_{n \to \infty} \left( \prod_{k=1}^{n} w_k \right) \left( \prod_{k=1}^{n} \frac{1}{w_{-k}} \right) = \lim_{n \to \infty} \left( \prod_{k=1}^{n} \frac{1}{3} \right) \left( \prod_{k=1}^{n} 2 \right) = \lim_{n \to \infty} \left( \frac{1}{3^n} \right) (2^n) = 0.
\]
From [4, Theorem 4.1], $T$ is supercyclic. However, for all increasing sequence $n_r$ of positive integers we have
\[
    \lim_{r \to \infty} \prod_{k=1}^{n_r} \frac{1}{w_{-k}} = \lim_{r \to \infty} \prod_{k=1}^{n_r} 2 = \lim_{r \to \infty} 2^{n_r} = \infty.
\]
Hence, by Corollary 2.16 $T$ is not diskcyclic.

The following results are noteworthy spectral properties of the diskcyclic operators which have been proved by Zeana in [14].

Proposition 2.23. Let $T \in \mathbb{D}C(\mathcal{H})$. Then $T^*$ has at most one eigenvalue with modules greater than 1.

Proof. Since $T \in SC(\mathcal{H})$, then $\sigma_p(T^*)$ contains at most one non-zero eigenvalue, let say $\lambda$. Hence there is a unit vector $z \in \mathcal{H}$ such that $T^*z = \lambda z$. Let $x \in \mathbb{D}C(T)$. Then it is easy to prove that
\[
    \{ |\langle \mu T^n x, z \rangle| \mid n \geq 0, \mu \in \mathbb{C}; |\mu| \leq 1 \} \text{ is dense in } \mathbb{R}^+ \cup \{0\}. \tag{1}
\]
Note that for all $n \geq 1$, $|\langle \mu T^n x, z \rangle| \leq |\mu|^n |\lambda|^n |\langle x, z \rangle|$. If we suppose $|\lambda| \leq 1$, then
\[
    |\langle \mu T^n x, z \rangle| \leq |\langle x, z \rangle|,
\]
which contradicts (1). Therefore we reach the desired result.

Corollary 2.24. Let $T \in \mathcal{B}(\mathcal{H})$. If $\sigma(T)$ has a connected component $\sigma$ such that $\sigma \subset B(0, 1)$, then $T \not\in \mathbb{D}C(\mathcal{H})$.

Proof. Towards a contradiction, suppose that a diskcyclic operator $T$ has a connected component $\sigma$ such that $\sigma \subset B(0, 1)$. Then, by Riesz decomposition Theorem, $T = T_1 \oplus T_2$ such that $\sigma(T_1) = \sigma$. It follows that $\mathbb{D}Orb(T_1, x)$ is bounded for all $x \in \mathcal{H}$ and hence $T_1$ can not be dense in $\mathcal{H}$, a contradiction to Proposition 2.22.
3 Main Results

We adjust the diskcyclicity criterion in order to obtain another version.

**Theorem 3.1** (Second Diskcyclicity Criterion). Let $T \in B(H)$. If there exists an increasing sequence of integers $\{n_k\}_{k \in \mathbb{N}}$ and two dense sets $D_1, D_2 \in H$ such that

(a) For each $y \in D_2$, there exists a sequence $\{x_k\}$ in $H$ such that $x_k \to 0$, and $T^{n_k}x_k \to y$,

(b) $\|T^{n_k}x\| \|x_k\| \to 0$ for all $x \in D_1$,

then $T$ is diskcyclic.

**Proof.** We will verify Proposition 2.12. Since for each $y \in D_2$ there exists a sequence $\{x_k\}$ in $H$ such that $T^{n_k}x_k \to y$, then there are maps $S$ which are right inverses of $T$ on $D_2$ such that $x_k = S^{n_k}y$ for some increasing sequence of integers $\{n_k\}_{k \in \mathbb{N}}$. Moreover, the condition (2) of Proposition 2.12 directly follows. □

**Proposition 3.2.** Both the diskcyclicity criteria are equivalent to each other.

**Proof.** If $T$ satisfies the diskcyclicity criterion then, by setting $S^{n_k}y = x_k$, we get the desired result. The other implication follows from Theorem 3.1. □

Now we illustrate the Second Diskcyclicity Criterion with the following example.

**Example 3.3.** Let $T = 2B$, where $B$ is the unilateral backward shift on $\ell^2(\mathbb{N})$. Then $T$ is diskcyclic.

**Proof.** Let $X = Y$ be the dense set in $\ell^2(\mathbb{N})$ such that all except finitely many coordinates of each element of $X$ are zero and let $n_k = k$. Then we will achieve conditions (a) and (b) of Theorem 3.1. For each $y \in Y$ assume that $x_k = (\frac{1}{2}F)^k y$, where $F$ is the unilateral forward shift on $\ell^2(\mathbb{N})$. Therefore, $\|x_k\| = \|\frac{1}{2^k}F^ky\| \to 0$. Moreover, $T^kx_k = (2B)^k(\frac{1}{2}F)^ky = y$. Hence (a) holds. Since $\|B^kx\| = 0$ eventually for a large enough $k$, then we have

$$\|T^kx\| \|x_k\| = \|2^kB^kx\| \|\frac{1}{2^k}F^ky\| = \|B^kx\| \|F^ky\| = 0.$$ 

Thus condition (b) holds. It follows that $T$ satisfies the Second Diskcyclicity Criterion and so $T$ is a diskcyclic operator. □

In general case we have the following example.

**Example 3.4.** Let $T = aB$, where $B$ is the unilateral backward shift on $\ell^2(\mathbb{N})$ and $|a| > 1$. Then $T$ satisfies the Second Diskcyclicity Criterion.
The next example shows that every multiple of the unilateral backward shift need not be diskcyclic.

**Example 3.5.** If $T = aB$, where $|a| \leq 1$, and $B$ is the unilateral backward shift on $\ell^2(\mathbb{N})$, then $T$ is not diskcyclic.

**Proof.** Since $T^n(e_n) = \frac{1}{|a|^n} e_{n-k}$, where $\{e_n\}_{n \in \mathbb{N}}$ is the canonical basis of $\ell^2(\mathbb{N})$, then $\|T^n(e_n)\| = \frac{1}{|a|^n}$. It follows that $\alpha \|T^n(x)\| \to 0$ for all $x \in \ell^2(\mathbb{N})$ and $\alpha \in \mathbb{C}$, $|\alpha| \leq 1$. Thus $\mathcal{D}Orb(T, x)$ is bounded and can not be dense in $\ell^2(\mathbb{N})$. \(\square\)

From Example 3.4, Example 3.5 and [10, Corollary 1.6] we can easily deduce the following corollary.

**Corollary 3.6.** A multiple of a unilateral backward shift on $\ell^2(\mathbb{N})$ is hypercyclic if and only if it is diskcyclic.

The following corollary follows directly from [2, Theorem 1.40] and Corollary 3.6

**Corollary 3.7.** If $B$ is a unilateral backward weighted shift with weight sequence $\{w_n\}$, then $B$ is diskcyclic if and only if

$$\limsup_{n \to \infty} (w_1, w_2, \ldots, w_n) = \infty.$$

The following example was shown to us by Nareen and Kilçman in [11].

**Example 3.8.** Let $T \in B(\mathbb{C})$ be defined by $T(x) = 2x$. Then $T$ is diskcyclic on $\mathbb{C}$.

It can be easily checked that the adjoint of the operator in Example 3.8 is diskcyclic. However, this property is not true in general as the following example.

**Example 3.9.** Let $T$ be defined on $\ell^2(\mathbb{Z})$ as in Example 2.21. Then $T$ is diskcyclic but its adjoint is not.

**Proof.** From Example 2.21 we have that $T$ is diskcyclic. Now we will show that $T^*$ is not diskcyclic. It is clear that $T^* e_n = Be_n$, where $B$ is the bilateral backward weighted shift with weight sequence

$$z_n = \begin{cases} 2 & \text{if } n > 0, \\ 3 & \text{if } n \leq 0. \end{cases}$$

Then for all increasing sequence $n_r$ of positive integers, we have

$$\lim_{r \to \infty} \prod_{k=1}^{n_r} \frac{1}{z_k} = \lim_{r \to \infty} \prod_{k=1}^{n_r} \frac{1}{2} = \lim_{r \to \infty} \frac{1}{2^{n_r}} = 0,$$

however,

$$\lim_{r \to \infty} \left( \prod_{k=1}^{n_r} z_{-k} \right) \left( \prod_{k=1}^{n_r} \frac{1}{z_k} \right) = \lim_{r \to \infty} \left( \prod_{k=1}^{n_r} 3 \right) \left( \prod_{k=1}^{n_r} \frac{1}{2} \right) = \lim_{r \to \infty} \left( \frac{3}{2} \right)^{n_r} = \infty.$$

From Corollary 2.20 it follows that $T^*$ is not diskcyclic. \(\square\)
It has been shown that an invertible $T \in B(H)$ is hypercyclic(or supercyclic) if and only if $T^{-1}$ is hypercyclic(or supercyclic respectively). However, this equivalence is not necessarily true for the diskcyclicity as in the following two examples.

**Example 3.10.** Let $T$ be defined on $\mathbb{C}$ as in Example 3.8 then $T^{-1}$ is not diskcyclic.

**Proof.** Since $T^{-1}x = \frac{1}{2}x$, then $D\text{Orb}(T^{-1}, y)$ is bounded for all $y \in \mathbb{C}$ and hence can not be dense in $\mathbb{C}$. It follows that $T^{-1}$ is not a diskcyclic operator. \hfill \Box

Here we recall that in the infinite dimensional spaces there are also diskcyclic operators whose inverses are not diskcyclic.

**Example 3.11.** Let $T$ be defined on $\ell^2(\mathbb{Z})$ as in Example 2.21. Then $T$ is diskcyclic, but its inverse is not.

**Proof.** Since $|w_n| \geq 2$ for all $n \in \mathbb{Z}$, then $T$ is invertable. The inverse of $T$ is the bilateral backward weighted shift with the weight sequence

$$z_n = \frac{1}{w_{n-1}} = \begin{cases} \frac{1}{2} & \text{if } n > 0, \\ \frac{1}{3} & \text{if } n \leq 0. \end{cases}$$

For all increasing sequence $n_r$ of positive integers, we have

$$\lim_{r \to \infty} \prod_{k=1}^{n_r} \frac{1}{z_k} = \lim_{r \to \infty} \prod_{k=1}^{n_r} 2 = \lim_{r \to \infty} 2^{n_r} = \infty.$$

By Corollary 2.20 we have that $T^{-1}$ is not a diskcyclic operator. \hfill \Box

There are diskcyclic operators, in which their inverses are also diskcyclic as in the next example.

**Example 3.12.** Let $F$ be a bilateral forward weighted shift on $\ell^2(\mathbb{Z})$ with weight sequence

$$w_n = \begin{cases} \frac{1}{2} & \text{if } n \geq 0, \\ 3 & \text{if } n < 0. \end{cases}$$

Then both $F$ and $F^{-1}$ are diskcyclic.

**Proof.** Since $|w_n| \geq \frac{1}{2}$ for all $n \in \mathbb{Z}$, then $F$ is invertable. Also we have

$$\lim_{n \to \infty} \prod_{k=1}^{n} \frac{1}{w_k} = \lim_{n \to \infty} \prod_{k=1}^{n} \frac{1}{3} = \lim_{n \to \infty} \frac{1}{3^n} = 0,$$

and

$$\lim_{n \to \infty} \left( \prod_{k=1}^{n} w_k \right) \left( \prod_{k=1}^{n} \frac{1}{w_k} \right) = \lim_{n \to \infty} \left( \prod_{k=1}^{n} \frac{1}{2} \right) \left( \prod_{k=1}^{n} \frac{1}{3} \right) = \lim_{n \to \infty} \frac{1}{2^n 3^n} = 0.$$
It follows from Corollary 2.16 that $F$ is diskcyclic. Moreover, the inverse of $F$ is the bilateral backward weighted shift $Bc_n = \frac{1}{w_{n-1}}e_{n-1}$ with weight sequence

$$z_n = \frac{1}{w_{n-1}} = \begin{cases} 2 & \text{if } n > 0, \\ \frac{1}{3} & \text{if } n \leq 0. \end{cases}$$

Since

$$\lim_{n \to \infty} \prod_{k=1}^{n} \frac{1}{z_k} = \lim_{n \to \infty} \prod_{k=1}^{n} \frac{1}{2} = \lim_{n \to \infty} \frac{1}{2^n} = 0,$$

and

$$\lim_{n \to \infty} \left( \prod_{k=1}^{n} \frac{1}{z_{-k}} \right) \left( \prod_{k=1}^{n} \frac{1}{z_k} \right) = \lim_{n \to \infty} \left( \prod_{k=1}^{n} \frac{1}{3} \right) \left( \prod_{k=1}^{n} \frac{1}{2} \right) = \lim_{n \to \infty} \frac{1}{2^n 3^n} = 0,$$

by Corollary 2.20, we have that $F^{-1}$ is diskcyclic. \(\square\)

Bourdon and Feldman in [3] proved that if $\operatorname{Orb}(T, x)$ (or $C\operatorname{Orb}(T, x)$) is somewhere dense, then $\operatorname{Orb}(T, x)$ (or $C\operatorname{Orb}(T, x)$ respectively) is everywhere dense set. However, if $D\operatorname{Orb}(T, x)$ is somewhere dense then the situation is different. It will be illustrated by the following example.

**Example 3.13.** Let $T$ be defined on $\mathbb{C}$ as in Example 3.8. Then $D\operatorname{Orb}(T, x)$ is somewhere dense in $\mathbb{C}$ but not everywhere dense in $\mathbb{C}$.

**Proof.** It is clear that $D\operatorname{Orb}(T, x) = \{ z : z \in \mathbb{C}, |z| \leq |x| \}$. Then we have

$$\left( \overline{D\operatorname{Orb}(T, x)} \right) = \{ z : z \in \mathbb{C}, |z| < |x| \} = \emptyset$$

for all $x \in \mathbb{C} \setminus \{0\}$. But we have that $\overline{D\operatorname{Orb}(T, x)} = \{ z : z \in \mathbb{C}, |z| \leq |x| \} \neq \mathbb{C}$. \(\square\)

The idea of the above example is obviously due to the fact that the disk orbit of any operator on $\mathbb{C}$ contains at least a non-trivial closed disk which can never be nowhere dense set. In other words, the disk orbit of any operator on $\mathbb{C}$ is somewhere dense in $\mathbb{C}$. Therefore, if the somewhere density of a disk orbit implied to the everywhere density of the disk orbit, any operator on $\mathbb{C}$ would be diskcyclic, which is a contradiction to Example 3.13.

The following lemma will be our main tool to prove the next theorem.

**Lemma 3.14.** If $T \in \mathcal{B}(\mathcal{H})$ is invertable and

$$A = \{ x \in \mathcal{H} : \| x - \alpha_i T^{n_i} y \| \to 0 \text{ for some increasing sequence } \{n_i\} \subset \mathbb{N} \text{ and } \alpha_i \in \mathbb{D} \}$$

is a non trivial subset of $\mathcal{H}$, then $A$ is an invariant closed subset of $\mathcal{H}$ under both $T$ and $T^{-1}$. 15
Proof. Let us choose \( x \in \mathcal{A} \), then
\[
\|\alpha_i T^{n_i} y - T(x)\| = \|T(\alpha_i T^{n_i} - 1 y - x)\| \\
\leq \|T\| \|\alpha_i T^{n_i} - 1 y - x\| \to 0.
\]
It follows that \( Tx \in \mathcal{A} \). By the same way
\[
\|T^{-1} x - \alpha_i T^{n_i} - 1 y\| = \|T^{-1} (x - \alpha_i T^{n_i} y)\| \\
\leq \|T^{-1}\| \|x - \alpha_i T^{n_i} y\| \to 0.
\]
Thus \( T^{-1} x \in \mathcal{A} \).

Now we shall show that \( \mathcal{H}\setminus \mathcal{A} \) is open. Let us choose \( v \in \mathcal{H}\setminus \mathcal{A} \), then there is an \( \epsilon > 0 \) such that for any large positive number \( N \), we have \( \|v - \alpha_i T^{n_i} y\| > \epsilon \) for all \( n \geq N \) and all \( \alpha_i \in \mathbb{D} \). Suppose that \( z \in B(v, \epsilon) = \{ x \in \mathcal{H} : \|v - x\| < \epsilon \} \). Then for every increasing sequence \( \{n_i\} \) of positive integers and \( \alpha_i \in \mathbb{D} \) we have
\[
\|v - \alpha_i T^{n_i} y\| = \|z - \alpha_i T^{n_i} y + v - z\| \\
\leq \|z - \alpha_i T^{n_i} y\| + \|v - z\| \\
\epsilon < \|v - \alpha_i T^{n_i} y\| - \|v - z\| \leq \|z - \alpha_i T^{n_i} y\|.
\]
Hence \( z \in \mathcal{H}\setminus \mathcal{A} \) and so \( B(v, \epsilon) \subseteq \mathcal{H}\setminus \mathcal{A} \). It follows that \( \mathcal{A} \) is an invariant closed set under both operators \( T \) and \( T^{-1} \).

**Theorem 3.15.** Let \( T \in B(\mathcal{H}) \) be an invertable operator, and let us suppose that we have the following properties.

1. Both \( T \) and \( T^{-1} \) are diskcyclic operators.
2. There is a vector \( y \) in \( \mathcal{H} \) such that \( \overline{\mathbb{D} \text{Orb}(T, y)} = \overline{\mathbb{D} \text{Orb}(T^{-1}, y)} = \mathcal{H} \).
3. There is a vector \( y \) in \( \mathcal{H} \) such that \( \overline{\mathbb{D} \text{Orb}(T, y) \cup \mathbb{D} \text{Orb}(T^{-1}, y)} = \mathcal{H} \).
4. Either \( T \) or \( T^{-1} \) is diskcyclic operator.

Then \( 1 \iff 2 \Rightarrow 3 \iff 4 \).

**Proof.** (1) \( \Rightarrow \) (2): Let \( T \) and \( T^{-1} \) be diskcyclic operators. Since the set of all diskcyclic vectors is dense \( G_\delta \), then by the Baire Category Theorem, we deduce that (1) \( \Rightarrow \) (2).

The implications (2) \( \Rightarrow \) (1), (2) \( \Rightarrow \) (3) and (4) \( \Rightarrow \) (3) are trivial.

(3) \( \Rightarrow \) (4): By hypothesis \( \overline{\mathbb{D} \text{Orb}(T, y) \cup \mathbb{D} \text{Orb}(T^{-1}, y)} = \mathcal{H} \), the Baire Category Theorem indicates that either \( \overline{\mathbb{D} \text{Orb}(T, y)} \) or \( \overline{\mathbb{D} \text{Orb}(T^{-1}, y)} \) has nonempty interior. If one of them, say \( \overline{\mathbb{D} \text{Orb}(T, y)} \) is nowhere dense, then \( \overline{\mathbb{D} \text{Orb}(T^{-1}, y)} = \mathcal{H} \), and therefore \( T^{-1} \) is diskcyclic. Otherwise, since \( \overline{\mathbb{D} \text{Orb}(T, y)} \) is somewhere dense, then we can find a set \( \mathcal{A} \) exactly as in Lemma 3.13 a positive integer \( p \), \( 0 \neq |\alpha_p| \leq 1 \) and \( \epsilon > 0 \) such that \( B(\alpha_p T^p y, \epsilon) \subseteq \overline{\mathbb{D} \text{Orb}(T, y)}. \) It follows that there
exist a large number $N$, an increasing sequence $n_i; n_i \geq p \geq 0$ and a sequence $\alpha_i \in \mathbb{D}$ such that 

$$\|\alpha_p T^p y - \alpha_i T^{n_i} y\| \to 0 \text{ for all } i > N.$$ 

Then we have 

$$\left\| \beta T^p y - \beta \frac{\alpha_i}{\alpha_p} T^{n_i} y \right\| \to 0 \text{ for all } i > N, \text{ and } \beta \in \mathbb{D}.$$ 

Thus we can find a sequence $\beta_i \in \mathbb{D}$ such that $\|\beta T^p y - \beta_i T^{n_i} y\| \to 0$ for all $i > N$. Since $A$ is invariant under both operators $T$ and $T^{-1}$, then $\beta y \in T^{-n}(A) \subseteq A$ and $T^n(\beta y) \in T^n(A) \subseteq A$ for all $n \in \mathbb{N}$. As a consequence, $\overline{\text{Orb}(T, y)} \subseteq A$. By the same way, we get $T^{-n}(\beta y) \in T^{-n}(A) \subseteq A$ and hence $\overline{\text{Orb}(T^{-1}, y)} \subseteq A$. Therefore, 

$$\overline{\text{Orb}(T, y)} \supseteq A = \overline{\text{Orb}(T, y) \cup \text{Orb}(T^{-1}, y)} = \mathcal{H}.$$ 

This concludes the proof.

\[\square\]

**Corollary 3.16.** Let $T$ be an invertible operator and, let $y \in \mathcal{H}$ such that $\overline{\text{Orb}(T, y)}$ is somewhere dense and $\overline{\text{Orb}(T, y) \cup \text{Orb}(T^{-1}, y)} = \mathcal{H}$. Then $\overline{\text{Orb}(T, y)}$ is everywhere dense in $\mathcal{H}$.

The following proposition gives us some characterizations of the spectrum of the diskcyclic operators.

**Proposition 3.17.** Let $T \in \mathbb{D}(\mathcal{H})$. Then we have the following properties.

1. $\sigma(T) \cup \partial(R\mathbb{D})$ is connected for some $R \geq 1$.

2. If $\alpha \in \sigma_p(T^*)$ then $\dim \ker(T^* - \alpha)^k = 1$ for all $k \geq 1$.

**Proof.** First we prove (1). Since $T \in SC(\mathcal{H})$ then, from [7, Proposition 3.1], we have $\sigma(T) \cup \partial(R\mathbb{D})$ is connected for some $R > 0$. Moreover, from Corollary 2.24 we have $\sigma(T) \notin (\mathbb{D})^o$. Thus $\sigma(T) \subset (\mathbb{C} \setminus \mathbb{D})$. Therefore, $\sigma(T) \cup \partial(R\mathbb{D})$ is connected for some $R \geq 1$. For part (2), the proof follows from [2] Proposition 1.26 and Proposition 1.27.

\[\square\]

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