Cube Polygon: A New Modified Euler Method to Improve Accuracy of Ordinary Differential Equation (ODE)

S.Nooraida¹*, M.M.Y.Nurhafizah², M.S.Anis³, A.M.M.Fahmi², A.W.F.Syarul¹, A.M.Iliana¹

¹Faculty of Computer, Media and Technology Management, TATI University College, 24000 Kemaman Terengganu, Malaysia. Tel./Fax +6-09-8601000/+6-09-8635863, Email: nooraida@tatiuc.edu.my, fahmy@tatiuc.edu.my, iliana@tatiuc.edu.my
²Faculty of Science and Technology Defence, Universiti Pertahanan Nasional Malaysia, Kem Sungai Besi, 57000 Kuala Lumpur, Malaysia. Tel./Fax +6-03-90513400/+6-03-90513028, Email: n.moziyana@gmail.com, fahmi@upnm.edu.my
³Faculty of Engineering, Universiti Pertahanan Nasional Malaysia, 26600 Pekan, Pahang, Malaysia. Tel./Fax +6-03-90513400/+6-03-90513028, Email: anis@upnm.edu.my

Abstract. Ordinary Differential Equation (ODEs) problems can solve using Euler method. Euler method suitable for first-order numerical and it is an effective and easy method. The objective of this paper is to suggest a new scheme for better accuracy. The accuracy is determined by comparing with exact solution using average concept. This paper proposes Cube Polygon as modified Euler method to improved accuracy and complexity. A set of simulation were carried out to demonstrate the accuracy of the proposed method. Testing has been done into SCILAB 6.0 by recording the maximum error. Results indicate that Cube Polygon provide more accurate results and reducing complexity for both smaller and higher step size.

1. Introduction

In order to solve complex problems in the field of engineering technology, various intensive developments in the formation of intact numerical methods are carried out. The numerical method is the method for obtaining the nearest approximation for the solution of various problems that can be described in the form of derivative equations. The main objective of this study is to propose a one-step method that uses the mean concept to get better accuracy. In practice, the Euler method is used in solving the Ordinary Differential Equation (ODE) problem. Euler's method is the easiest one-step method. It is a popular method in solving the Ordinary Differential Equation (ODE). In 1978, Euler proposed a method for solving Initial Value Problem (IVP). This Euler method is easy to implement and low computing costs [1]. This is because the Euler method is an iterative type. This advantage makes the Euler method the basis of the prototype development of more complex and sophisticated methods. Although Euler's method was able to provide a simple solution but the approximate solution provided was less accurate. This is because the number of errors increases during generation at each step [2]. Thus, the performance of the Euler method is not as good as other methods such as Runge Kutta, Adams-Bashforth and others. Therefore, the proposed new Modified Euler scheme will be developed to improve the accuracy and complexity.
This paper proposed a new scheme for modified Euler using average concept. The accuracy is determined with comparing the minimum error to exact solution. Cube Polygon (PC) is the name for the new proposed scheme. This proposed scheme is enhancements of other scheme develop by previous researchers Zulzamri [3] and Nurhafizah [4]. Zulzamri’s method referring as Polygon (P) used average of arithmetic mean for two points. It shows the improvement in term of accuracy and speed compared to Euler Method [5]. In previous study by Nurhafizah, [3] uses concept of Zulzamri [4] but choose Harmonic mean and Contra Harmonic mean in the equation.

Thus, this paper suggest a modified Euler method using the Cube mean concept. The method was constructed by extracting the (P) method in [6] and using Cube mean in the equation. Propose scheme is compared using three linear ODE between Polygon and Polygon Harmonic (PH) with the exact solution. The accuracy of the proposed method is demonstrated by the comparing error results of with step sizes of h 0.1, 0.01 and 0.001.

2. Methodology

In order to ensure research objective is achieved, research methodology that serves as a guide to ensure research will be done smoothly as per plan. The first phase is the development of the proposed scheme. This phase is produced after researching the idea of pre-research from a previous study of Euler's method, Modified Euler and the concept of mean. In this phase, authors developed a new scheme by combining Euler (R) and mean (T) methods. The original Euler method with general formula is chosen as the basis for developing a proposed method. Cube mean was selected in developing this proposed method. The general formula for the Cube mean for the two numbers resulted of new Modified Euler scheme. Figure 1 shows how the proposed method is developed.

![Figure 1](#)

The second phase focuses on the design and development of new Modified Euler scheme. In this phase, the new Modified Euler scheme be compared to the complexities obtained from previous studies. Comparisons are concentrated among new Modified Euler scheme towards existing and stable Modified Euler methods. Existing and stable Modified Euler methods selected are P. This new Modified Euler scheme will convert to the program code first. The program code to be used is the programming language of SCILAB 6.0. The next phase is an scheme testing phase. This phase will test the complexity of new Modified Euler scheme. 3 ODE numerical tests will be used. The result of this phase is the result of the accuracy of the proposed method.

The last phase will focus on analysis and discussion will be conducted in this study. In this phase, all the results of the test involving presentation methods in particular complexity will be recorded comparable. The comparison is based on the proposed method with the P method. Analysis was also made on the capabilities of the new Modified Euler scheme in solving problems in the
different size of step size \( h \). The results of the fifth phase testing will be analyzed and discussed especially in the complexity stage after comparisons are made.

2.1 Scheme Development

The new scheme suggested in this paper is naming as Cube Polygon. The scheme is based on the Cube Mean in P where as P is a well-known technique for improving Euler method. Euler methods used in [7] and [8] were extracted as basis for the scheme. The improvement to the Euler method has been done by using Cube mean within two coordinate points of function. Equation (1) is a basic formula of the Euler method.

\[
y_{n+1} = y_n + \Delta hf(t_0, y_0).
\]  

Equation (2) was modified using the average concept.

\[
y_{n+1} = y_n + hf\left(\frac{x_n + (x_n + h)}{2}\right)\left(\frac{y_n + (y_n + f(x_n, y_n))}{2}\right).
\]  

The proposed average in Cube mean for the two midpoints is written as equation (3).

\[
y_{n+1} = y_n + hf\left(\frac{(x_n)^3 + (x_n + h)^3}{2}\right)\left(\frac{(y_n)^3 + (y_n + hf(x_n, y_n))^3}{2}\right).
\]

Modifications to the slope of the function at estimated midpoints of \((x_0, y_0)\) and \((x_1, y_1)\) would improve the stability and accuracy of Euler.

3. Results and Discussion

Three sets of first-order ODEs were tested and recorded. Three sets of linear ODEs being testing using three different step sizes: 0.1, 0.01 and 0.001 to obtain the result of PC. These maximum error for the whole cycle sets were tested for accuracy. Table 1 illustrates the exact solutions for the ODEs.

Table 1. Set of Problem First Order ODEs

| Equation | Exact Solution | Initial Values | Interval of Integration |
|----------|----------------|----------------|-------------------------|
| \( y' = -0.2y \) | \( e^{(0.2)x} \) | \( y(0)=0 \) | \( 0 \leq x \leq 20 \) |
| \( y' = -0.7y \) | \( e^{(0.7)x} \) | \( y(0)=1 \) | \( 0 \leq x \leq 20 \) |
| \( y' = -0.5y \) | \( e^{(0.5)x} \) | \( y(0)=1 \) | \( 0 \leq x \leq 20 \) |

The comparison results between Polygon (P), Polygon Harmonic (PH) and Cube Polygon (CP) methods illustrates as in Table 2. Relative error based on [9] was calculated as:

\[
\text{Error} = \frac{|E-v|}{E}, \quad E_v = \text{Exact_value} \text{ and } E_v = \text{Euler’s_modified_value}
\]

Table 2. Results of Errors Using Various Step Size \( h \)

| Method | Problem 1 | Problem 2 | Problem 3 |
|--------|-----------|-----------|-----------|
| Step Size | Polygon | Polygon Harmonic | Polygon Cube |
| \( h = 0.001 \) | 0.038709 | 0.386952 | 0.035409 |
| \( h = 0.01 \) | 0.035409 | 0.043272 | 0.040015 |
| \( h = 0.1 \) | 0.043272 | 0.042978 | 0.034254 |
| \( h = 0.001 \) | 0.038709 | 0.386952 | 0.035409 |
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| \( h = 0.01 \) | 0.035409 | 0.043272 | 0.040015 |
| \( h = 0.1 \) | 0.043272 | 0.042978 | 0.034254 |
The results for each method were compared to the exact solution using maximum error. It is evident from Table 2 that the proposed Cube Polygon (PC) method provides more accurate result than Polygon (P) and Polygon Harmonic (PH) at higher and smaller $h$ sizes.

Problem 1 demonstrated that the results of the PC method is better than P and PH at $h=0.1$ with a Problem 2 demonstrated that the maximum error for PC is 0.071049, P method is 0.157179 and PH 0.271899 for $h = 0.1$. For $h=0.01$ PC scored 0.080536, P is 0.034254 compared to P 0.038709 and PH 0.042978.

Problem 2 demonstrated that the maximum error for PC is 0.071049, P method is 0.157179 and PH 0.271899 for $h = 0.1$. For $h=0.01$ PC scored 0.080536, P is 0.0156441 and PH is 0.276992. Then if $h=0.001$ PC scored 0.081461, P scored 0.157179 and PH 0.277489 respectively.

Problem 3 demonstrated that the PC contribute more accuracy with maximum error 0.062749 while P is 0.098525 and PH 0.142049 at $h=0.1$. at $h=0.01$ PC still give better accuracy with maximum error 0.069690, P is 0.104785 and PH is 0.147547. At $h=0.001$ maximum error for PC is 0.070371 and for P and PH are 0.105400 and 0.14088.

It can be conclude that the proposed CP method can be an alternative method for modified Euler. This is because it provides more accurate result than P and PH in solving ODEs at higher step size. Thus, it can reduce the complexity.

4. Conclusion

A new method using modified Euler, naming as Cube Polygon is the main finding of this study. In testing phase, Cube Polygon is being compared to Polygon scheme with the exact solution using SCILAB 6.0. In this study Cube Polygon has been demonstrated and provided solutions that are similar to exact solutions at small step size and also at higher step size. Using smaller step size will give a higher accuracy and higher step size would reduce complexity and processing time. As a conclusion, Cube Polygon can be used as an alternative scheme to solve ODE problems.

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References

[1] J. Kopriva, Semi-Analytical Purposes and Simulation. PhD Thesis, Brno University Of Technology. 2013, pp.38
[2] S. Fadugba, B. Ogunrinde and T. Okunlola, Euler's Method for Solving Initial Value Problems InOrdinary Differential Equations, ThePacific Journal of Science Technology Vol 13, Issue 2, 2012, pp.152- 158
[3] Z. Salleh, Ordinary Differential Equations (ODE) Using Euler's Technique and Scilab Programming, Mathematical Models and Methods in Modern Science Volume 20, Issue 4 2012, pp. 264-269
[4] N.M.M. Yusop, M.K. Hasan, M. Wook, M.F.M. Amran, S.R. Ahmad, Comparison New Scheme Modified Euler Based On Harmonic-Polygon Approach For Solving Ordinary Differential Equation, Journal of Telecommunication, Electronic and Computer Engineering, Volume 9, Issue 2-11, 2017, pp. 29-32
[5] Nooraida.S, Nurhafiza.M.M.Y, Syahrul,F,Anis.S.M. Cube Arithmetic: Improving Euler Method for Ordinary Differential Equation Using Cube Mean, Indonesian Journal of Electrical Engineering and Computer Science, Vol. 11, No. 3, September 2018, pp. 1109-1113

[6] Zarina, B.I., Mohamed, S., Khairil, I. and Zanariah, M., Block Method for Generalised Multistep Adams and Backwards Differentiation Formulae In Solving First Order ODEs. MATEMATIKA, 2005. Volume 21, Issue 1, pp. 25-11.

[7] Zaib M.S, Sania.Q, Asif.A.S, Muhammad S.C, A Modified ODE Solver for Autonomous Initial Value Problems, Mathematical Theory and Modeling, Volume 4, Issue 3, 2014, pp. 80-85

[8] N.M.M. Yusop and M.K. Hasan, Development of New Harmonic Euler Using Nonstandard Finite Difference Technique for Solving Stiff Problems, Jurnal Teknologi Volume 77, Issue 20, 2015, pp. 19–24

[9] R. Jaiswal, A. A. Pathan, Study of Numerical Analysis – Differential Equation International Journal of Advanced Research in Computer Science and Software Engineering, Volume 5, Issue 10, 2015, pp. 709-712