\(\beta\)-admissibility of observation and control operators for hypercontractive semigroups

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Abstract

We prove a Weiss conjecture on \(\beta\)-admissibility of control and observation operators for discrete and continuous \(\gamma\)-hypercontractive semigroups of operators, by representing them in terms of shifts on weighted Bergman spaces and using a reproducing kernel thesis for Hankel operators. Particular attention is paid to the case \(\gamma = 2\), which corresponds to the unweighted Bergman shift.

Keywords: Admissibility; semigroup system; dilation theory; Bergman space; hypercontractivity; reproducing kernel thesis; Hankel operator

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1 Introduction

We study infinite dimensional observation systems of the form

\[
\begin{align*}
\dot{x}(t) &= Ax(t), \\
y(t) &= Cx(t), \\
x(0) &= x_0 \in X,
\end{align*}
\]

where \(A\) is the generator of a strongly continuous semigroup \((T(t))_{t \geq 0}\) on a Hilbert space \(\mathcal{H}\) and \(C\) is a linear bounded operator from \(D(A)\), the domain of \(A\) equipped with the graph topology, to another Hilbert space \(\mathcal{Y}\). For the well-posedness of the system with respect to the output space \(L^2_{\beta}(0, \infty; \mathcal{Y}) := \{f : (0, \infty) \to \mathcal{Y} \mid f \text{ measurable}, \|f\|_{L^2_{\beta}}^2 := \int_0^\infty \|f(t)\|^2 t^{\beta} dt < \infty\}\) it is required that \(C\) is an \(\beta\)-admissible observation operator for \(A\), that is, there exists an \(M > 0\) such that

\[\|CT(\cdot)x_0\|_{L^2_{\beta}(0, \infty; \mathcal{Y})} \leq M\|x_0\|_{\mathcal{H}}, \quad x_0 \in D(A).\]

It is easy to show that \(\beta\)-admissibility implies the resolvent condition

\[\sup_{\lambda \in \mathbb{C}_+} (\Re \lambda)^{1+\beta} \|C(\lambda - A)^{-(1+\beta)}\| < \infty\] (1)
where \( C_+ \) denotes the open right half plane of \( C \). Whether or not the converse implication holds is commonly referred to as a **weighted Weiss conjecture**. For \( \beta = 0 \) the conjecture was posed by Weiss [22]. In this situation the conjecture is true for contraction semigroups if the output space is finite-dimensional, for right-invertible semigroup and for bounded analytic semigroups if \((-A)^{1/2}\) is 0-admissible. However, in general the conjecture is not true. We illustrate this in Figure 1.

For \( \beta \neq 0 \), there is much less known. In the situation \( \beta < 0 \), the weighted Weiss conjecture is true for bounded analytic semigroups if \((-A)^{1/2}\) is 0-admissible [8], but in general the weighted Weiss conjecture does not hold [24]. If \( \beta > 0 \), then the weighted Weiss conjecture is true for normal contraction semigroups and for the right-shift on \( L^2_{-\alpha}(0,\infty) \) for \( \alpha > 0 \) if \( (-A)^{1/2} \) is 0-admissible, see Figure 2. Again, in general the conjecture is not true. In Theorem 4.4 we show that the weighted Weiss conjecture holds if the dual of the cogenerator \( T^* \) of the semigroup \((T(t))_{t\geq 0}\) is \( \gamma \)-hypercontractive for some \( \gamma > 1 \). The proof is based on the fact that \( \gamma \)-hypercontractions are unitarily equivalent to the restriction of the backward shift to an invariant subspace of a weighted Bergman space, the Cayley transform between discrete-time and continuous-time systems and that the weighted Weiss conjecture holds for the backward shift to an invariant subspace of a weighted Bergman space [10]. In order to apply the results of [10] we first have to extend them to the vector-valued Bergman spaces.

Due to the fact that \( C \) is a \( \beta \)-admissible observation operator for \((T(t))_{t\geq 0}\) if and only if \( C^* \) is a \((-\beta)\)-admissible control operator for \((T^*(t))_{t\geq 0}\), where \( \beta \in (-1,1) \), the resolvent growth conditions for \( \beta \)-admissible control operators can be derived from those of \((-\beta)\)-admissible observation operators.

Beside continuous-time systems we also prove a discrete-time version of the Weiss conjecture. For \( T \in \mathcal{L}(\mathcal{H}) \), \( E \in \mathcal{L}(\mathcal{U},\mathcal{H}) \) and \( F \in \mathcal{L}(\mathcal{H},\mathcal{Y}) \) we consider the discrete time linear systems:

\[
x_{n+1} = Tx_n + Eu_{n+1}, \quad y_n = Fx_n \quad \text{with} \quad x_0 \in \mathcal{H}
\]

and \( u_n \in \mathcal{U}, \, n \in \mathbb{N} \). Here, \( \mathcal{H} \) is the state space, \( \mathcal{U} \) the input space and \( \mathcal{Y} \) is the output space.
| $\dim \mathcal{Y} < \infty$ | $\dim \mathcal{Y} \leq \infty$ |
|-----------------|-----------------|
| $(T(t))_{t \geq 0}$ normal contraction semigroup \[23\] | $(T(t))_{t \geq 0}$ analytic & bounded semigr. and $(-A)^{1/2}$ 0-admissible \[8\] |
| $T^* \gamma$-hypercontractive, $\gamma > 1$ (Thm. 4.4) | $T^* \gamma$-hypercontractive, $\gamma > 1$ (Thm. 4.4) |
| $(T(t))_{t \geq 0}$ right-shift on $L^2_{-\alpha}(0, \infty)$, $\alpha > 0$, \[10\] | |
| * Counterexample in general \[25\] |

Figure 2: Weighted Weiss conjecture: Case $\beta > 0$

of the system.

Let $\beta > -1$. By $\ell^2_\beta(\mathcal{U})$ we denote the sequence space

$$
\ell^2_\beta(\mathcal{U}) := \{\{u_n\}_n | u_n \in \mathcal{U} \text{ and } \|\{u_n\}_n\|_\beta^2 := \sum_{n=0}^{\infty} (1+n)^\beta |u_n|^2 < \infty\}.
$$

Clearly, $\ell^2_\beta(\mathcal{U})$ equipped with the norm $\| \cdot \|_\beta$ is a Hilbert space. Following \[8\] and \[23\], we say that $F$ is a $\beta$-admissible observation operator for $T$, if there exists a constant $M > 0$ such that

$$
\sum_{n=0}^{\infty} (1+n)^\beta \|FT^n x\|^2 \leq M \|x\|^2
$$

for every $x \in \mathcal{H}$.

To test whether a given observation operator is $\beta$-admissible, a frequency-domain characterization is convenient and, to this end, it is not difficult to show that $\beta$-admissibility of $F$ for $T$ implies the resolvent growth condition

$$
\sup_{z \in \mathbb{D}} (1-|z|^2)^{1+\beta/2} \|F(1-zT)^{-\beta-1}\|_{\mathcal{L}(\mathcal{H}, \mathcal{Y})} < \infty,
$$

(3)

where $\mathbb{D}$ is the open unit disc.

The question of whether the converse statement holds, commonly referred to as a (weighted) Weiss conjecture, is much more subtle. For $\beta = 0$, the conjecture is true if $T$ is a contraction and the output space $\mathcal{Y}$ is finite-dimensional \[9\]. It was shown by \[24, 23\] that for $T$ a normal contraction and finite-dimensional output spaces the weighted Weiss conjecture holds for positive $\beta$, but not in the case $\beta \in (-1, 0)$. Moreover, the weighted Weiss conjecture holds if $T$ is a Ritt operator and a contraction for $\beta > -1$ \[17\], but it is not true for general contractions if $\beta > 0$, see \[25\]. Recently, in \[10\] it was shown that the Weiss conjecture holds for the forward shift on weighted Bergman spaces. One aim of this paper is to show the Weiss conjecture for adjoint operators of $\gamma$-hypercontractions. We obtain a characterisation
of $\beta$-admissibility, $\beta > 0$, with respect to $\gamma$-hypercontractions ($\gamma > 1$) by characterising $\beta$-admissibility with respect to the shift operator on vector-valued weighted Bergman spaces. It is shown in [10] that in the case of a scalar-valued Bergman space, $\beta$-admissibility with respect to the shift operator can be characterised by the resolvent growth bound (3). We extend this analysis to the vector-valued setting.

We proceed as follows. In Section 2 we introduce and study $\gamma$-hypercontractive operators and $\gamma$-hypercontractive strongly continuous semigroups. In particular, $\gamma$-hypercontractions are unitarily equivalent to the restriction of the backward shift to an invariant subspace of a weighted Bergman space. Section 3 is devoted to the weighted Weiss conjecture for discrete-time systems. We first extend the result of [10] concerning the shift operator on a scalar-valued weighted Bergman space. Section 4 positive results concerning the weighted Weiss conjecture for continuous-time systems are given.

## 2 $\gamma$-hypercontractions

Let $\mathcal{H}$ be a Hilbert space. For $T \in \mathcal{L}(\mathcal{H})$, we define

$$M_T : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}), \quad M_T(X) = T^*XT.$$ 

**Definition 2.1** ([2], [4]). Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{L}(\mathcal{H})$, $\|T\| \leq 1$. Let $\gamma \geq 1$. We say that $T$ is a $\gamma$-hypercontraction, if for each $0 < r < 1$,

$$(1 - M_rT)^\gamma(I) \geq 0.$$ 

Note that the left hand side in the definition is well-defined in the sense of the usual holomorphic functional calculus, since $\sigma(1 - M_rT) \subset \mathbb{C}_+$. A 1-hypercontraction is of course just an ordinary contraction. If $T$ is a normal contraction, then it is easy to show by the usual continuous functional calculus that $T$ is also a $\gamma$-hypercontraction for each $\gamma \geq 1$. Moreover, all strict contractions are $\gamma$-hypercontractions, as the next result shows.

**Theorem 2.2.** Let $T \in \mathcal{L}(\mathcal{H})$ with $\|T\| < 1$. Then $T$ is a $\gamma$-hypercontraction for sufficiently small $\gamma > 1$.

**Proof:** Suppose that $\|T\| < 1$. Then $\|M_T\| < 1$, and $\sigma(1 - M_T)$ is bounded away from the negative real axis, so an analytic branch of the logarithm exists on some open set $\Omega \supseteq \sigma(1 - M_T)$. For $\gamma \geq 1$, define $f_\gamma(z) = \exp(\gamma \log z)$, analytic on $\Omega$.

Now $f_\gamma(z) \to z$ uniformly for $z$ in compact subsets of $\Omega$, and therefore $f_\gamma(1 - M_T)$, defined by the analytic functional calculus, converges to $1 - M_T$ in the norm on $\mathcal{L}(\mathcal{L}(\mathcal{H}))$ (see, e.g., [5 Thm. 3.3.3]).

Hence, in particular, $(1 - M_T)^\gamma(I) \to (1 - M_T)(I) = I - T^*T$ in norm in $\mathcal{L}(\mathcal{H})$ as $\gamma \to 1$.

Since $\|T\| < 1$, $\sigma((1 - M_T)(I))$ is strictly contained in the positive real axis, and thus for sufficiently small $\gamma > 1$ the spectrum of $(1 - M_T)^\gamma(I)$ is also strictly contained in the positive real axis, by continuity properties of the spectrum (see, e.g., [5 Thm. 3.4.1]).

Hence $(1 - M_T)^\gamma(I) \geq 0$ for all $\gamma$ sufficiently close to 1, and so $T$ is a $\gamma$-hypercontraction.

If $n \in \mathbb{N}$, then equivalently, $T \in \mathcal{L}(\mathcal{H})$ is an $n$-hypercontraction if and only if

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*k}T^k \geq 0$$
for all $1 \leq m \leq n$.
In particular, a Hilbert space operator $T$ is 2-hypercontractive if it satisfies

$$I - T^*T \geq 0$$

(that is, it is a contraction), and also

$$I - 2T^*T + T^*2T^2 \geq 0.$$  (4)

Note, that for $1 < \mu < \gamma$, the $\gamma$-hypercontractivity property implies $\mu$-hypercontractivity.
We are particularly interested in $\gamma$-hypercontractive operators as they are unitarily equivalent to the restriction of the backward shift to an invariant subspace of a weighted Bergman space, which we now define.

**Definition 2.3.** Let $D$ denote the open unit disk in the complex plane $C$. For $\alpha > -1$, the weighted Bergman space $A^2_\alpha(D,K)$, where $K$ is a Hilbert space, contains of analytic functions $f : D \to K$ for which

$$\|f\|_\alpha^2 = \int_D \|f(z)\|^2 dA_\alpha(z) < \infty,$$  (5)

where $dA_\alpha(z) = (1 + \alpha)(1 - |z|^2)^\alpha dA(z)$ and $dA(z) := \frac{1}{\pi} dx dy$ is area measure on $D$ for $z = x + iy$. We note that the norm $\|f\|_\alpha$ is equivalent to

$$\left(\sum_{n=0}^{\infty} \|f_n\|^2 (1 + n)^{-(1+\alpha)}\right)^{\frac{1}{2}},$$  (6)

where $f_n$ are the Taylor coefficients of $f$.

For each $\alpha > -1$, let $S_\alpha$ denote the shift operator on the weighted Bergman space $A^2_\alpha(D,K)$,

$$S_\alpha f(z) = zf(z) \quad (f \in A^2_\alpha(D,K))$$

The following theorem is a special case of Corollary 7 in [4]. For the case of integer $\gamma$, this was proved in [2].

**Theorem 2.4.** Let $\alpha > -1$. Let $\mathcal{H}$ be a Hilbert space and let $T \in \mathcal{L}(\mathcal{H})$ be an $\alpha + 2$-hypercontraction with $\sigma(T) \subset D$. Then $T$ is unitarily equivalent to the restriction of $S^*_\alpha$ to an invariant subspace of $A^2_\alpha(D,K)$, where $K$ is a Hilbert space.

Next we introduce the concept of $\gamma$-hypercontractive semigroups.

**Definition 2.5.** Let $(T(t))_{t \geq 0}$ be a strongly continuous contraction semigroup on a Hilbert space $\mathcal{H}$, with infinitesimal generator $A$. We call a $C_0$-semigroup $(T(t))_{t \geq 0}$ $\gamma$-hypercontractive if each operator $T(t)$ is a $\gamma$-hypercontraction.

In the following we assume that $(T(t))_{t \geq 0}$ is a strongly continuous contraction semigroup on a Hilbert space $\mathcal{H}$, with infinitesimal generator $A$. As in [21], the cogenerator $T := (A + I)(A - I)^{-1}$ exists, and is itself a contraction. Rydhe [20] studied the relation between $\gamma$-hypercontractivity of a strongly continuous contraction semigroup and its cogenerator. He proved that $T$ is $\gamma$-hypercontractive if every operator $T(t)$, $t \geq 0$, is $\gamma$-hypercontractive. Conversely, if every operator $T(t)$, $t \geq 0$, is $N$-hypercontractive for some $N \in \mathbb{N}$, then $T$ is
$N$-hypercontractive. However, by means of an example, Rydhe \cite{20} showed that for general $\gamma$-hypercontractivity this reverse implication is false. Clearly, if $A$ generates a contraction semigroup of normal operators, then the cogenerator of $(T(t))_{t \geq 0}$ is $\gamma$-hypercontractive for each $\gamma \geq 1$.

In particular 2-hypercontractivity can be characterized as follows, see \cite{20}. For completeness we include a more elementary proof, which also yields additional information.

**Proposition 2.6.** Let $(T(t))_{t \geq 0}$ be a strongly continuous contraction semigroup acting on a Hilbert space $H$. Then the following statements are equivalent.

1. $(T(t))_{t \geq 0}$ is 2-hypercontractive.
2. The function $t \mapsto \|T(t)x\|^2$ is convex for all $x \in H$.
3. 
   \[
   \Re(A^2y, y) + \|Ay\|^2 \geq 0 \quad (y \in D(A^2)).
   \] 
   or equivalently,
   \[
   \|(A + A^*)x\|^2 + \|Ax\|^2 \geq \|A^*x\|^2 \quad (y \in D(A) \cap D(A^*)).
   \]
4. The cogenerator $T$ is a 2-hypercontraction.

**Proof** We first prove that Part 1 and Part 2 are equivalent. Take $t \geq 0$ and $\tau > 0$. If $T(\tau)$ is a 2-hypercontraction, then, by (4) we have
   \[
   \langle T(t)x, T(t)x \rangle - 2\langle T(t + \tau)x, T(t + \tau)x \rangle + \langle T(t + 2\tau)x, T(t + 2\tau)x \rangle \geq 0,
   \]
or
   \[
   \|T(t + \tau)x\|^2 \leq \frac{1}{2} \left( \|T(t)x\|^2 + \|T(t + 2\tau)x\|^2 \right),
   \] 
   which is the required convexity condition.

Conversely, the convexity condition (8) implies that $T(\tau)$ is a 2-hypercontraction (take $t = 0$). Next we show that Part 2 are Part 3 equivalent. For $t > 0$ and $y \in D(A^2)$ we calculate the second derivative of the function $g : t \mapsto \|T(t)y\|^2$.

\[
g'(t) = \frac{d}{dt} \langle T(t)y, T(t)y \rangle = \langle AT(t)y, T(t)y \rangle + \langle T(t)y, AT(t)y \rangle.
\]

Similarly,
\[
g''(t) = \langle A^2T(t)y, T(t)y \rangle + 2\langle AT(t)y, AT(t)y \rangle + \langle T(t)y, A^2T(t)y \rangle.
\]

If $g$ is convex, then letting $t \to 0$ gives the condition (7).

Conversely, the condition (7) gives the convexity of $t \mapsto \|T(t)y\|^2$ for $y \in D(A^2)$, and by density this holds for all $y$.

Finally we show the equivalence of Part 3 and Part 4. We start with the condition (7) and calculate
\[
\langle (I - 2T^*T + T^*T^2)x, x \rangle
\]
for $x = (A - I)^2y$ (note that $(A - I)^{-2} : H \to H$ is defined everywhere and has dense range).
We obtain
\[
\langle (A - I)^2 y, (A - I)^2 y \rangle - 2 \langle (A^2 - I)y, (A^2 - I)y \rangle + \langle (A + I)^2 y, (A + I)^2 y \rangle = 4 \langle A^2 y, y \rangle + 8 \langle Ay, Ay \rangle + 4 \langle y, A^2 y \rangle \geq 0.
\]

Thus condition (7) holds if and only if the cogenerator \(T\) is 2-hypercontractive.

Thus every normal contraction semigroup is 2-hypercontractive. Moreover, even every hyponormal contraction semigroup is 2-hypercontractive. Note, that a semigroup is hyponormal if the generator \(A\) satisfies \(D(A) \subset D(A^*)\) and \(\|A^*x\| \leq \|Ax\|\) for all \(x \in D(A)\), see [14, 18]. Clearly, a \(C_0\)-semigroup \((T(t))_{t \geq 0}\) is contractive if and only if the adjoint semigroup \((T^*(t))_{t \geq 0}\) is contractive. Unfortunately, a similar statement does not hold for 2-hypercontractions: The right shift semigroup on \(L^2(0, \infty)\) is 2-hypercontractive, but the adjoint semigroup, the left shift semigroup on \(L^2(0, \infty)\) is not.

3 Discrete-time \(\beta\)-admissibility

Let \(\mathcal{H}, \mathcal{U}, \mathcal{Y}\) be Hilbert spaces, \(T \in \mathcal{L}(\mathcal{H}), E \in \mathcal{L}(\mathcal{U}, \mathcal{H})\) and \(F \in \mathcal{L}(\mathcal{H}, \mathcal{Y})\). Consider the discrete time linear system:

\[
x_{n+1} = Tx_n + Eu_{n+1}, \quad y_n = Fx_n \quad \text{with} \quad x_0 \in \mathcal{H}
\]

and \(u_n \in \mathcal{U}, n \in \mathbb{N}\).

Following [8] and [23], we say that \(F\) is a \(\beta\)-admissible observation operator for \(T\), if there exists a constant \(M > 0\) such that

\[
\sum_{n=0}^{\infty} (1 + n)^\beta \|FT^n x\|^2 \leq M \|x\|^2
\]

for every \(x \in \mathcal{H}\). Moreover, we say that \(E\) is a \(\beta\)-admissible control operator for \(T\), if there exists a constant \(M > 0\) such that

\[
\left\| \sum_{n=0}^{\infty} T^n E u_n \right\|_{\mathcal{H}} \leq M \|\{u_n\}_n\|_\beta
\]

for every \(\{u_n\}_n \in \ell_\beta^2(\mathcal{U})\).

**Remark 3.1.** Let \(x \in \mathcal{H}\) and \(\{y_n\}_n \in \ell_\beta^2(\mathcal{Y})\). Then the calculation

\[
|\langle \{FT^n x\}_n, \{y_n\}_n \rangle_{\beta \times \beta^-} | = \sum_{n=0}^{\infty} \langle FT^n x, y_n \rangle_{\mathcal{Y}} = \langle x, \sum_{n=0}^{\infty} (T^*)^n F^* y_n \rangle_{\mathcal{H}}
\]

implies that \(F\) is a \(\beta\)-admissible observation operator for \(T\) if and only if \(F^*\) is a \((-\beta)\)-admissible control operator for \(T^*\).
A characterisation of \( \beta \)-admissibility with respect to \( \gamma \)-hypercontractions (\( \gamma > 1 \)) may be obtained by characterising \( \beta \)-admissibility with respect to the shift operator on vector-valued weighted Bergman spaces, as defined in Definition 2.3.

It is shown in [10] that in the case of a scalar-valued Bergman spaces, \( \beta \)-admissibility with respect to \( S_\alpha \) can be characterised by the resolvent growth bound (3). This result was obtained by noting that \( \beta \)-admissibility is equivalent to boundedness of an appropriate little Hankel operator, while (3) is equivalent to boundedness of the same Hankel operator on a set of reproducing kernels. That such Hankel operators satisfy a Reproducing Kernel Thesis (boundness on the reproducing kernels is equivalent to operator boundedness) is equivalent to the characterisation of \( \beta \)-admissibility by the growth condition (3).

To extend this analysis to the vector-valued setting, let \( \mathcal{K}, \mathcal{Y} \) be Hilbert spaces and consider an analytic function \( C : \mathbb{D} \to L(\mathcal{K}, \mathcal{Y}) \) given by

\[
C(z) = \sum_{n=0}^{\infty} C_n z^n, \quad z \in \mathbb{D},
\]

where \( C_n \in L(\mathcal{K}, \mathcal{Y}) \), for each \( n \). We write \( L^2_\alpha(\mathbb{D}, \mathcal{K}) \) for the space of measurable functions \( f : \mathbb{D} \to \mathcal{K} \) satisfying (5). We also write

\[
A^2_\alpha(\mathbb{D}, \mathcal{K}) = \{ g(\mathcal{K}) : g \in A^2_\alpha(\mathbb{D}, \mathcal{K}) \}.
\]

The little Hankel operator \( h_C : A^2_{\beta-1}(\mathbb{D}, \mathcal{K}) \to \overline{A^2_\alpha(\mathbb{D}, \mathcal{Y})} \) acting between weighted Bergman spaces is defined by

\[
h_C(f) := \overline{\mathcal{P}}_\alpha(C(\mathcal{T}) f(\mathbb{T})), \quad f \in A^2_{\beta-1}(\mathbb{D}, \mathcal{K}),
\]

where \( \overline{\mathcal{P}}_\alpha : L^2_\alpha(\mathbb{D}, \mathcal{K}) \to \overline{A^2_\alpha(\mathbb{D}, \mathcal{K})} \) is the orthogonal projection onto the anti-analytic functions and \( \mathbb{T}(z) = z, z \in \mathbb{D} \). The following result links \( \beta \)-admissibility with little Hankel operators of the form (10).

**Proposition 3.2.** Let \( \alpha > -1 \) and \( \beta > 0 \). Let \( \mathcal{K}, \mathcal{Y} \) be Hilbert spaces. Given \( F \in L(A^2_\alpha(\mathbb{D}, \mathcal{K}), \mathcal{Y}) \), define bounded linear operators \( F_\alpha \in L(\mathcal{K}, \mathcal{Y}) \) by

\[
F_\alpha x = F(x t^n), \quad x \in \mathcal{K}, n \in \mathbb{N},
\]

and symbols \( C : \mathbb{D} \to L(\mathcal{K}, \mathcal{Y}), \tilde{C} : \mathbb{D} \to L(\mathcal{Y}, \mathcal{K}) \) by

\[
C(z) = \sum_{n=0}^{\infty} (1 + n)^\alpha F_\alpha z^n, \quad \tilde{C}(z) = \sum_{n=0}^{\infty} (1 + n)^\alpha F_\alpha^* z^n.
\]

The following conditions are equivalent:

(i) The resolvent condition (3) holds with \( T = S_\alpha \) and \( \mathcal{H} = A^2_\alpha(\mathbb{D}, \mathcal{K}) \);

(ii) The Hankel operator \( h_{\tilde{C}} : A^2_{\beta-1}(\mathbb{D}, \mathcal{Y}) \to \overline{A^2_\alpha(\mathbb{D}, \mathcal{K})} \) satisfies

\[
\sup_{\omega \in \mathbb{D}, \|y\| = 1} \| h_{\tilde{C}} k_{\omega, y}^{\beta-1} \|_{A^2_\alpha(\mathbb{D}, \mathcal{K})} < \infty,
\]

where

\[
k_{\omega, y}^{\beta-1}(z) := y (1 - |\omega|^2) \frac{1 + \beta}{(1 - \overline{\omega} z)^{1+\beta}}, \quad z, \omega \in \mathbb{D}, \ y \in \mathcal{Y},
\]

are the normalized reproducing kernels for \( A^2_{\beta-1}(\mathbb{D}, \mathcal{Y}) \);
(iii) The Hankel operator \( h_C : A^2_{\beta-1}(\mathbb{D}, \mathcal{K}) \to A^2_{\alpha}(\mathbb{D}, \mathcal{Y}) \) satisfies
\[
h_C \in \mathcal{L}(A^2_{\beta-1}(\mathbb{D}, \mathcal{K}), A^2_{\alpha}(\mathbb{D}, \mathcal{Y}));
\]

(iv) \( F \) is \( \beta \)-admissible for \( S_\alpha \) on \( A^2_{\alpha}(\mathbb{D}, \mathcal{K}) \).

**Proof** \((i) \iff (ii)\) follows directly from a vectorial analogue of [10] Proposition 2.3 (ii).

\((ii) \Rightarrow (iii)\) Note first that [10] Theorem 2.7 extends to the vector-valued setting to imply that \( h_C : A^2_{\beta-1}(\mathbb{D}, \mathcal{Y}) \to A^2_{\alpha}(\mathbb{D}, \mathcal{K}) \) is bounded. An alternative characterisation of boundedness of little Hankel operators can be given in terms of generalized Hankel matrices of the form
\[
\Gamma^{a,b} := \left( (1 + m)^{a}(1 + n)^{b}\Phi_{n+m} \right)_{m,n \geq 0}
\]
where \( a, b > 0 \) and \( \Phi : \mathbb{D} \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \) is given by \( \Phi(z) = \sum_{n \geq 0} \Phi_n z^n \), for some Hilbert spaces \( \mathcal{H}_1, \mathcal{H}_2 \). In particular, the vectorial analogue of [10] Proposition 2.3 (i)] implies that
\[
h_C \in \mathcal{L}\left( A^2_{\beta-1}(\mathbb{D}, \mathcal{Y}), A^2_{\alpha}(\mathbb{D}, \mathcal{K}) \right) \iff \Gamma^{\frac{1}{\beta},\frac{1}{\alpha}} \in \mathcal{L}(\ell^2(\mathcal{Y}), \ell^2(\mathcal{K})). \tag{11}
\]
Now, it is shown in [10] Theorem 9.1 that
\[
\Gamma^{\frac{1}{\beta},\frac{1}{\alpha}} \in \mathcal{L}(\ell^2(\mathcal{Y}), \ell^2(\mathcal{K})) \iff \tilde{C} \in \Lambda^{1+\alpha,1+\beta}_{\frac{1}{\beta}}(\mathcal{L}(\mathcal{Y}, \mathcal{K})). \tag{12}
\]
Here, for \( s > 0 \) and a Banach space \( X \), \( \Lambda_s(X) \) is the Besov space containing functions \( f \in L^\infty(\mathbb{D}, X) \) for which
\[
\sup_{\tau \in \mathbb{T}, \tau \neq 1} \frac{\|\Delta^n f\|_{L^\infty(\mathbb{D}, X)}}{|1 - \tau|^s} < \infty,
\]
for some integer \( n > s \). It follows immediately that \( C \in \Lambda^{1+\alpha,1+\beta}_{\frac{1}{\beta}}(\mathcal{L}(\mathcal{K}, \mathcal{Y})) \) and hence, by (11) and (12), that
\[
h_C \in \mathcal{L}(A^2_{\beta-1}(\mathbb{D}, \mathcal{K}), A^2_{\alpha}(\mathbb{D}, \mathcal{Y})).
\]

\((iii) \iff (iv)\): The vectorial analogue of [10] Proposition 2.1] implies that \((iv)\) holds if and only if \( \Gamma^{\frac{1}{\beta},\frac{1}{\alpha}} \in \mathcal{L}(\ell^2(\mathcal{K}), \ell^2(\mathcal{Y})) \). By (11), boundedness \((iii)\) of the little Hankel operator \( h_C \) is equivalent to \( \Gamma^{\frac{1}{\beta},\frac{1}{\alpha}} \in \mathcal{L}(\ell^2(\mathcal{K}), \ell^2(\mathcal{Y})) \). That \((iii)\) and \((iv)\) are equivalent then follows from [19] Theorem 9.1 and the fact that \( \alpha > -1, \beta > 0 \).

\((iv) \Rightarrow (i)\) is well known. See, for example, [25]. \( \square \)

**Theorem 3.3.** Let \( \beta > 0 \). Let \( \mathcal{H}, \mathcal{Y} \) be Hilbert spaces and let \( T^* \in \mathcal{L}(\mathcal{H}) \) be a \( \gamma \)-hypercontraction for some \( \gamma > 1 \). Let \( F \in \mathcal{L}(\mathcal{H}, \mathcal{Y}) \). Then the following are equivalent:

1. \( F \) is a \( \beta \)-admissible observation operator for \( T \).

2. \[
\sup_{z \in \overline{\mathbb{D}}}(1 - |z|^2)^{\frac{1+\beta}{2}} \|F(1 - \bar{z}T)^{-\beta-1}\|_{\mathcal{L}(\mathcal{H}, \mathcal{Y})} < \infty.
\]
Proof The implication \( (1) \Rightarrow (2) \) follows as usual from the testing on fractional derivatives of reproducing kernels. For \( (2) \Rightarrow (1) \), write \( K = \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\frac{1+\alpha}{2}} \|F(1 - zT)^{-\beta-1}\|_{L(H,Y)} \) and let us first replace \( T \) by \( rT \) for some \( 0 < r < 1 \). Write \( \gamma = 2 + \alpha \). By Theorem 2.4, \( (rT)^* \) is the restriction of \( S_\alpha^* \) to the invariant subspace \( H \subset A^2(\mathbb{D},K) \). Extend \( F \) trivially to \( A^2(\mathbb{D},K) \) by letting \( F = 0 \) on \( H^+ \subset A^2(\mathbb{D},K) \). Then \( F^* y \in H \) for all \( y \in Y \). Then for each \( z \in \mathbb{D} \) we obtain

\[
\|F(1 - zS_\alpha)^{-\beta-1}\|_{L(A^2(\mathbb{D},K),Y)} = \sup_{h \in A^2(\mathbb{D},K), \|h\| = 1} \|F(1 - zS_\alpha)^{-\beta-1}h\|_{Y} \\
= \sup_{h \in A^2(\mathbb{D},K), \|h\| = 1} \sup_{y \in Y, \|y\| = 1} |\langle (1 - zS_\alpha)^{-\beta-1}h, F^* y \rangle| \\
= \sup_{h \in A^2(\mathbb{D},K), \|h\| = 1} \sup_{y \in Y, \|y\| = 1} |\langle h, (1 - zS_\alpha^*)^{-\beta-1}F^* y \rangle| \\
= \sup_{h \in A^2(\mathbb{D},K), \|h\| = 1} \sup_{y \in Y, \|y\| = 1} |\langle h, (1 - z(rT)^*)^{-\beta-1}F^* y \rangle| \\
= \sup_{h \in H, \|h\| = 1} \sup_{y \in Y, \|y\| = 1} |\langle h, (1 - z(rT)^*)^{-1}F^* y \rangle| \\
\leq K \frac{1}{(1 - |rz|^2)^{\frac{1+\alpha}{2}}} \\
\leq K \frac{1}{(1 - |z|^2)^{\frac{1+\alpha}{2}}}.
\]

Hence, by Proposition 3.2, \( F \) is an \( \beta \)-admissible observation operator for \( S_\alpha \).

Thus there exists a constant \( M \) such that for each \( x \in H \),

\[
\sum_{n=0}^{\infty} (1 + n)^{\beta} \|F(rT)^nx\|_Y^2 = \sum_{n=0}^{\infty} (1 + n)^{\beta} \sup_{y \in Y, \|y\| = 1} |\langle (rT)^nx, F^* y \rangle|^2 \\
= \sum_{n=0}^{\infty} (1 + n)^{\beta} \sup_{y \in Y, \|y\| = 1} |\langle x, ((rT)^n)^* F^* y \rangle|^2 \\
= \sum_{n=0}^{\infty} (1 + n)^{\beta} \sup_{y \in Y, \|y\| = 1} |\langle x, (S_\alpha^n)^* F^* y \rangle|^2 \\
= \sum_{n=0}^{\infty} (1 + n)^{\beta} \sup_{y \in Y, \|y\| = 1} |\langle S_\alpha^n x, F^* y \rangle|^2 \\
= \sum_{n=0}^{\infty} (1 + n)^{\beta} \|FS_\alpha^n x\|_Y^2 \leq M \|x\|^2
\]

Here, the constant \( M \) depends only on \( K, \alpha \) and \( \beta \), but not on \( r \). It therefore follows easily from the Monotone Convergence Theorem that

\[
\sum_{n=0}^{\infty} (1 + n)^{\beta} \|F^n x\|_Y^2 \leq M \|x\|^2 \quad (x \in H)
\]
and $F$ is a $\beta$-admissible observation operator for $T$.

By duality we obtain the following result.

**Theorem 3.4.** Let $\beta \in (-1,0)$. Let $\mathcal{H}$, $\mathcal{U}$ be Hilbert spaces and let $T \in \mathcal{L}(\mathcal{H})$ be a $\gamma$-hypercontraction for some $\gamma > 1$. Let $E \in \mathcal{L}(\mathcal{U}, \mathcal{H})$. Then the following are equivalent:

1. $E$ is a $\beta$-admissible control operator for $T$.
2. $$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{\frac{1+\beta}{2}} \| (1 - zT)^{-\beta-1} E \|_{\mathcal{L}(\mathcal{H}, \mathcal{Y})} < \infty.$$  

**Remark 3.5.** Theorem 3.3 in particular shows Wynn’s result [23] for $\beta$-admissibility of normal discrete contractive semigroups, also for infinite-dimensional output space.

## 4 Continuous-time $\beta$-admissibility

We consider a continuous-time control system of the form

$$
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \geq 0, \\
y(t) &= Cx(t), \quad t \geq 0.
\end{align*}
$$

Here $A$ is the generator of a $C_0$-semigroup $(T(t))_{t \geq 0}$ on a Hilbert space $\mathcal{H}$. Writing $\mathcal{H}_1 = D(A)$ and $\mathcal{H}_{-1} = D(A^*)^*$, we suppose that $B \in \mathcal{L}(\mathcal{U}, \mathcal{H}_1)$ and $C \in \mathcal{L}(\mathcal{H}_1, \mathcal{Y})$, where $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces as well.

**Definition 4.1.** Let $\beta > -1$.

1. $B$ is called a $\beta$-admissible control operator for $(T(t))_{t \geq 0}$, if there exists a constant $M > 0$ such that

$$
\left\| \int_0^\infty T(t)Bu(t) \, dt \right\| \leq M \| u \|_{L^2_\beta(0,\infty; \mathcal{U})}
$$

for every $u \in L^2_\beta(0,\infty; \mathcal{U})$.

2. $C$ is called a $\beta$-admissible observation operator for $(T(t))_{t \geq 0}$, if there exists a constant $M > 0$ such that

$$
\int_0^\infty t^\beta \| CT(t)x \|^2 \, dt \leq M \| x \|^2_{\mathcal{H}_1}
$$

for every $x \in \mathcal{H}_1$.

**Remark 4.2.** Similarly as for discrete-time systems it can be shown for $\beta \in (-1,1)$ that $B$ is a $\beta$-admissible control operator for $(T(t))_{t \geq 0}$ if and only if $B^*$ is a $(-\beta)$-admissible observation operator for $(T^*(t))_{t \geq 0}$.

The following result is proven in [25, Propositions 2.1 and 2.2] for $\beta \in (0,1)$. The trivial extension to the case $\beta > 0$ is given for completeness. For $\alpha > -1$ we write $A^\alpha_\beta(\mathbb{C}_+)$ for the Bergman space on the right half-plane corresponding to the measure $x^\alpha \, dx \, dy$.

**Proposition 4.3.** Let $\beta > 0$. Suppose that $A$ generates a contraction semigroup on $\mathcal{H}$ and that $C \in \mathcal{L}(D(A), \mathcal{Y})$. Define the cogenerator $T \in \mathcal{L}(\mathcal{H})$ by $T := (I + A)(I - A)^{-1}$ and $F := C(I - A)^{-(1+\beta)} \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$. Then the following statements hold.
1. $C$ is a $\beta$-admissible observation operator for $(T(t))_{t \geq 0}$ if and only if $F$ is a $\beta$-admissible observation operator for $T$.

2. The resolvent condition (3) for $(F, T)$ holds if and only if

$$\sup_{\lambda \in \mathbb{C}_+} (\text{Re} \lambda)^{\frac{1+\beta}{2}} \| C(\lambda - A)^{-(1+\beta)} \| < \infty.$$ 

Proof 1. $F$ is $\beta$-admissible for $T$ if and only if $\Lambda: A^2_{2-\beta}(\mathbb{D}) \to \mathcal{L}(\mathcal{H}, \mathcal{Y})$ defined initially on reproducing kernels by $\Lambda f = F f(T)$ extends to a bounded linear operator. On the other hand, $C$ is $\beta$-admissible for $A$ if and only if $\tilde{\Lambda}: A^2_{2-\beta}(\mathbb{C}_+) \to \mathcal{L}(\mathcal{H}, \mathcal{Y})$ defined initially on reproducing kernels by $\tilde{\Lambda}(g) = Cg(-A)$ extends to a bounded linear operator. That the two conditions are equivalent follows from the fact that for any $\beta > 0$ there is an isomorphism $J_\beta: A^2_{2-\beta}(\mathbb{D}) \to A^2_{2-\beta}(\mathbb{C}_+)$ for which $\Lambda = \tilde{\Lambda} \circ J_\beta$ holds on each reproducing kernel.

2. Follows directly from the identities

$$D(I - zT)^{-(1+\beta)} = \frac{CR\left(\frac{1-z}{1+z}, A\right)^{1+\beta}}{(1+z)^{1+\beta}}, \quad z \in \mathbb{D}$$

and

$$\text{Re} \left(\frac{1-z}{1+z}\right)|1+z|^2 = (1-|z|^2), \quad z \in \mathbb{D}.$$ 

Our main theorems concerning continuous-time systems are as follows.

**Theorem 4.4.** Let $\beta > 0$. Let $(T(t))_{t \geq 0}$ be a contraction semigroup on $\mathcal{H}$ such that the adjoint of the cogenerator $T^*$ is $\gamma$-hypercontractive for some $\gamma > 1$. Then the following are equivalent:

1. $C$ is $\beta$-admissible observation operator for $(T(t))_{t \geq 0}$.

2. $$\sup_{\lambda \in \mathbb{C}_+} (\text{Re} \lambda)^{\frac{1+\beta}{2}} \| C(\lambda - A)^{-(1+\beta)} \| < \infty.$$ 

Proof The statement of the theorem follows from Proposition 4.3 together with Theorem 3.3.

**Remark 4.5.** $T^*$ is $\gamma$-hypercontractive if every operator $T^*(t), t \geq 0$, is $\gamma$-hypercontractive. If $A$ generates a contraction semigroup of normal operators, then the adjoint of the cogenerator of $(T(t))_{t \geq 0}$ is $\gamma$-hypercontractive for each $\gamma \geq 1$, see Section 2.

By duality we obtain the following result.

**Theorem 4.6.** Let $\beta \in (-1, 0)$. Let $(T(t))_{t \geq 0}$ be a contraction semigroup on $\mathcal{H}$ such that the cogenerator $T$ is $\gamma$-hypercontractive for some $\gamma > 1$. Then the following are equivalent:

1. $B$ is $\beta$-admissible control operator for $(T(t))_{t \geq 0}$.
2. 

$$\sup_{\lambda \in \mathbb{C}_+} (Re\lambda)^{\frac{1+\beta}{2}} \| (\lambda - A)^{-(1+\beta)} B \| < \infty.$$ 

Theorems 4.4 and 4.6 give positive results for $\beta > 0$ and adjoints of $\gamma$-hypercontractions in the case of observation operators, and for $\beta < 0$ and $\gamma$-hypercontractions in the case of control operators. The remaining possibilities for $\beta \in (-1,0) \cup (0,1)$ can be shown not to hold by means of various counterexamples. For $\beta \in (-1,0)$ the counterexample for normal semigroups given in [24] shows that there is no positive result for observation operators in either the $\gamma$-hypercontractive or adjoint $\gamma$-hypercontractive case. For $\beta \in (0,1)$, there is a counterexample in [24] based on the unilateral shift, which is 2-hypercontractive, see Figure 3. By Remark 4.2 these provide appropriate counterexamples for control operators as well.

| $\beta \in (-1,0)$ | $\beta \in (0,1)$ |
|-------------------|-------------------|
| Counterexample [24] | Counterexample [24] |
| $T \gamma$-hypercontr. for some $\gamma > 1$ | $T^* \gamma$-hypercontr. for some $\gamma > 1$ |

Figure 3: Weighted Weiss conjecture

References

[1] J. Agler, The Arveson extension theorem and coanalytic models. Integral Equations Operator Theory 5 (1982), no. 5, 608–631.

[2] J. Agler, Hypercontractions and subnormality. J. Operator Theory 13 (1985), no. 2, 203–217.

[3] A. Aleman and O. Constantin, Hankel operators on Bergman spaces and similarity to contractions. Int. Math. Res. Not. 2004, no. 35, 1785–1801.

[4] C.-G. Ambrozie, M. Engliš and V. Müller, Operator tuples and analytic models over general domains in $\mathbb{C}^n$. J. Operator Theory 47 (2002), no. 2, 287–302.

[5] B. Aupetit, A primer on spectral theory. Universitext. Springer-Verlag, New York, 1991.

[6] W.S. Cohn and I. E. Verbitsky, Factorization of tent spaces and Hankel operators. J. Funct. Anal. 175 (2000), no. 2, 308–329.

[7] O. Constantin, Weak product decompositions and Hankel operators on vector-valued Bergman spaces. J. Operator Theory 59 (2008), no. 1, 157–178.

[8] B. Haak and C. Le Merdy, $\alpha$-admissibility of observation and control operators. Houston J. Math. 31 (2005), no. 4, 1153–1167.
[9] Z. Harper, Applications of the discrete Weiss conjecture in operator theory. *Integral Equations Operator Theory* 54 (2006), no. 1, 69–88.

[10] B. Jacob, E. Rydhe and A. Wynn, The weighted Weiss conjecture and reproducing kernel theses for generalized Hankel operators. *J. Evol. Equ.* 14 (2014), no. 1, 85–120.

[11] B. Jacob and J.R. Partington, The Weiss conjecture on admissibility of observation operators for contraction semigroups. *Integral Equations Operator Theory* 40 (2001), no. 2, 231–243.

[12] B. Jacob, J.R. Partington and S. Pott, Admissible and weakly admissible observation operators for the right shift semigroup. *Proc. Edinburgh Math. Soc.* 45 (2002), 353–362.

[13] B. Jacob and H. Zwart, Counterexamples concerning observation operators for $C_0$-semigroups. *SIAM J. Control Optim.* 43 (1) (2004), 137–153.

[14] J. Janas, On unbounded hyponormal operators. *Ark. Mat.* 27 (1989), no. 2, 273–281.

[15] J.R. Partington and G. Weiss, Admissible observation operators for the right-shift semigroup. *Math. Control Signals Systems* 13 (2000), 179–192.

[16] C. Le Merdy, The Weiss conjecture for bounded analytic semigroups. *J. Lond. Math. Soc.*, 67 (3) (2003), 715–738.

[17] C. Le Merdy, $\alpha$-admissibility for Ritt operators. *Complex Anal. Oper. Theory*, 8 (2014), 665–681.

[18] S. Öta and K. Schmüdgen, On some classes of unbounded operators. *Integral Equations Operator Theory* 12 (1989), no. 2, 211–226.

[19] V.V. Peller, *Hankel Operators and Their Applications*. Springer-Verlag New York, Inc. (2003).

[20] E. Rydhe, An Agler model theorem for $C_0$-semigroups of Hilbert space contractions. Submitted, 2015.

[21] B. Sz.-Nagy, C. Foias, H. Bercovici and L. Kérchy, *Harmonic analysis of operators on Hilbert space*. Second edition. Revised and enlarged edition. Universitext. Springer, New York, 2010.

[22] G. Weiss, Two conjectures on the admissibility of control operators. *Estimation and control of distributed parameter systems* (Vorau, 1990), 367–378, Internat. Ser. Numer. Math., 100, Birkhäuser, Basel, 1991.

[23] A. Wynn, $\alpha$-admissibility of observation operators in discrete and continuous time. *Complex Anal. Oper. Theory* 4 (2010), no. 1, 109–131.

[24] A. Wynn, Counterexamples to the discrete and continuous weighted Weiss conjectures, *SIAM J. Control Optim.*, 48 (4) (2009), 2620–2635.

[25] A. Wynn, $\alpha$-admissibility of the right-shift semigroup on $L^2(R_+)$. *Systems Control Lett.* 58 (2009), no. 9, 677–681.
[26] K. Zhu, *Spaces of holomorphic functions in the unit ball*. Graduate Texts in Mathematics, 226. Springer-Verlag, New York, 2005.