FILTER REGULAR SEQUENCES AND GENERALIZED LOCAL COHOMOLOGY MODULES

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Abstract. Let \( a, b \) be ideals of a commutative Noetherian ring \( R \) and let \( M, N \) be finite \( R \)-modules. The concept of an \( a \)-filter grade of \( b \) on \( M \) is introduced and several characterizations and properties of this notion are given. Then, using the above characterizations, we obtain some results on generalized local cohomology modules \( H^i_a(M, N) \). In particular, first we determine the least integer \( i \) for which \( H^i_a(M, N) \) is not Artinian. Then we prove that \( H^i_a(M, N) \) is Artinian for all \( i \in \mathbb{N}_0 \) if and only if \( \dim R/(a + \operatorname{Ann} M + \operatorname{Ann} N) = 0 \). Also, we establish the Nagel-Schenzel formula for generalized local cohomology modules. Finally, in a certain case, the set of attached primes of \( H^i_a(M, N) \) is determined and a comparison between this set and the set of attached primes of \( H^i_a(N) \) is given.

1. Introduction

Throughout this paper, \( R \) is a commutative Noetherian ring with nonzero identity, \( a, b \) are ideals of \( R \) and \( M, N, L \) are finite \( R \)-modules. We will use \( \mathbb{N} \) (respectively \( \mathbb{N}_0 \)) to denote the set of positive (respectively non-negative) integers. The generalized local cohomology functor was first introduced, in the local case, by Herzog [10] and, in the general case, by Bijan-Zadeh [3]. The \( i \)-th generalized local cohomology functor \( H^i_a(\cdot, \cdot) \) is defined by

\[
H^i_a(X, Y) = \lim_{n \to \infty} \operatorname{Ext}^i_R(X/a^n X, Y)
\]

for all \( R \)-modules \( X, Y \) and \( i \in \mathbb{N}_0 \). Clearly, this notion is a natural generalization of the ordinary local cohomology functor [5].

There is a lot of current interest in the theory of filter regular sequences in commutative algebra; and, in recent years, there have appeared many papers concerned with the role of these sequences in the theory of local cohomology. In particular case, when one works on a local ring, the concept of a filter regular sequence has been studied in [23, 26] and has led to some interesting results. We will denote the supremum of all numbers \( n \in \mathbb{N}_0 \) for which there exists an \( a \)-filter regular \( M \)-sequence of length \( n \) in \( b \) by \( \operatorname{f-grad}(a, b, M) \). In a local ring \( (R, m) \), \( \operatorname{f-grad}(m, a, M) \) is known as \( \operatorname{f-depth}(a, M) \). Lü and Tang [13] proved that

\[
\operatorname{f-depth}(a, M) = \inf \{ i \in \mathbb{N}_0 | \dim \operatorname{Ext}^i_R(R/a, M) > 0 \}
\]

and that \( \operatorname{f-depth}(a, M) \) is the least integer \( i \) such that \( H^i_a(M) \) is not Artinian. As a theorem, we generalize their results and characterize \( \operatorname{f-grad}(a, b, M) \) to non local
cases as follows.

\[
f\text{-grad}(a, b, M) = \inf\{i \in \mathbb{N}_0 | \text{Supp} \text{Ext}_R^i(R/b, M) \not\subseteq V(a)\}
\]

\[
= \inf\{i \in \mathbb{N}_0 | \text{Supp} H^i_a(M) \not\subseteq V(a)\},
\]

\[
f\text{-grad}(a, b + \text{Ann} N, M) = \inf\{i \in \mathbb{N}_0 | \text{Supp} H^i_a(N, M) \not\subseteq V(a)\},
\]

and

\[
sup_{A \in \mathcal{M}} f\text{-grad}(\bigcap_{m \in A} m, a + \text{Ann} M, N)
\]

\[
= \inf\{i \in \mathbb{N}_0 | H^i_a(M, N) \text{ is not Artinian}\}
\]

\[
= \inf\{i \in \mathbb{N}_0 | \text{Supp} H^i_a(M, N) \not\subseteq \max(R)\}
\]

\[
= \inf\{i \in \mathbb{N}_0 | \text{Supp} H^i_a(M, N) \not\subseteq A \text{ for all } A \in \mathcal{M}\}
\]

\[
= \inf\{i \in \mathbb{N}_0 | \dim \text{Ext}_R^i(M/aM, N) > 0\},
\]

where \(\mathcal{M}\) is the set of all finite subsets of \(\max(R)\).

As an application of this theorem, we show that, if \(n \in \mathbb{N}\), then \(H^i_a(M, N)\) is Artinian for all \(i < n\) if and only if \(H^i_{aR}(M_p, N_p)\) is Artinian for all \(i < n\) and all prime ideals \(p\). Also, we prove that \(H^i_a(M, N)\) is an Artinian \(R\)-module for all \(i \in \mathbb{N}_0\) if and only if \(\dim R/(a + \text{Ann} M + \text{Ann} N) = 0\). In particular, \(\text{Ext}_R^i(M, N)\) has finite length for all \(i \in \mathbb{N}_0\) if and only if \(\dim R/(\text{Ann} M + \text{Ann} N) = 0\).

Let \(x_1, \ldots, x_n\) be an \(a\)-filter regular \(N\)-sequence in \(a\). Then the formula

\[
H^i_a(N) = \begin{cases} H^{i}_{\{x_1, \ldots, x_n\}}(N) & \text{if } i < n, \\ H^{i-n}_{a}(H^{n}_{\{x_1, \ldots, x_n\}}(N)) & \text{if } i \geq n, \end{cases}
\]

is known as Nagel-Schenzel formula (see [20] and [11]). We generalize the above formula for the generalized local cohomology modules. Indeed, we prove that:

(i) \(H^i_a(M, N) \cong H^{i}_{\{x_1, \ldots, x_n\}}(M, N)\) for all \(i < n\);

(ii) if proj \(\dim M = d\) and \(L\) is projective, then

\[
H^{i+n}_{a}(M \otimes_R L, N) \cong H^i_a(M, H^{n}_{\{x_1, \ldots, x_n\}}(L, N))
\]

for all \(i \geq d\).

Assume that \(\bar{R} = R/(a + \text{Ann} M + \text{Ann} N)\) and that the ideal \(r\) is the inverse image of the Jacobson radical of \(\bar{R}\) in \(R\). If \(\bar{R}\) is semi local, then, by using the isomorphisms described in (i) and Theorem 4.2, we prove that

\[
f\text{-grad}(r, a + \text{Ann} M, N) = \inf\{i \in \mathbb{N}_0 | H^i_a(M, N) \text{ is not Artinian}\}
\]

\[
= \inf\{i \in \mathbb{N}_0 | H^i_a(M, N) \not\cong H^i_{\bar{R}}(M, N)\}.
\]

Let \((R, \mathfrak{m})\) be a local ring and \(\dim N = n\). Macdonald and Sharp [15] Theorem 2.2] show that

\[
\text{Att} H^i_a(N) = \{p \in \text{Ass } N | \dim R/p = n\}.
\]

As an extension of this result, Dibaei and Yassemi [8] Theorem A] proved

\[
\text{Att} H^i_a(N) = \{p \in \text{Ass } N | \text{cd}_a(R/p) = n\},
\]
Theorem 2.2. In the case where $M$ is a weak $R,$ in addition, if $x$ is a sequence, if $x$ is a sequence, if $x$ is a sequence, if $x$ is a sequence, then $\dim M < \infty,$ then Gu and Chu [9, Theorem 2.3] proved that $H^a_{n+d}(M, N)$ is Artinian and

$$\text{Att} H^a_{n+d}(M, N) = \{p \in \text{Ass } N | cd_a(M, R/p) = n + d\},$$

where, for an $R$-module $Y,$ $cd_a(M, Y)$ is the greatest integer $i$ such that $H^i_a(M, Y) \neq 0.$ Notice that $cd_a(M, N) \leq d + n$ [3, Lemma 5.1]. We prove the above result in general case where $R$ is not necessarily local. As a corollary we deduce that $\text{Att} H^a_{n+d}(M, N) \subseteq \text{Att} H^a_0(N).$ Also, we give an example to show that this inclusion may be strict. Indeed, our example not only show that the Theorem 2.1 of [17] is not true, but it also rejects all of the following conclusions in [17].

Finally, Let $\dim M = d < \infty$ and $\dim N = n < \infty$ and $b = \text{Ann } H^a_0(N).$ We prove that, if $R/b$ is a complete semilocal ring, then

$$\text{Att} H^a_{n+d}(M, N) = \text{Supp} \text{Ext}_R^d(M, R) \cap \text{Att} H^0_0(N).$$

In particular, if in addition, $\dim_{R_p} M_p = \dim M$ for all $p \in \text{Supp } M,$ then

$$\text{Att} H^a_{n+d}(M, N) = \text{Supp } M \cap \text{Att} H^0_0(N).$$

2. Filter regular sequences

We say that a sequence $x_1, \ldots, x_n$ of elements of $R$ is an $a$-filter regular $M$-sequence, if $x_i \notin p$ for all $p \in \text{Ass } M/(x_1, \ldots, x_{i-1})M \setminus V(a)$ and for all $i = 1, \ldots, n.$ In addition, if $x_1, \ldots, x_n$ belong to $b,$ then we say that $x_1, \ldots, x_n$ is an $a$-filter regular $M$-sequence in $b.$ Note that $x_1, \ldots, x_n$ is an $R$-filter regular $M$-sequence if and only if it is a weak $M$-sequence in the sense of [6, Definition 1.1.1].

Some parts of the next elementary proposition are included in [20, Proposition 2.2] in the case where $(R, m)$ is local and $a = m.$

Proposition 2.1. Let $x_1, \ldots, x_n$ be a sequence of elements of $R$ and $n \in \mathbb{N}.$ Then the following statements are equivalent.

(i) $x_1, \ldots, x_n$ is an $a$-filter regular $M$-sequence.
(ii) $\text{Supp}((x_1, \ldots, x_{i-1})M :_M x_i)/(x_1, \ldots, x_{i-1})M \subseteq V(a)$ for all $i = 1, \ldots, n.$
(iii) $x_1/1, \ldots, x_n/1$ is a weak $M_p$-sequence for all $p \in \text{Supp } M \setminus V(a).$
(iv) $x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}$ is an $a$-filter regular $M$-sequence for all positive integers $\alpha_1, \ldots, \alpha_n.$
(v) $x_i$ is a weak $(M/(x_1, \ldots, x_{i-1})M)/\Gamma_n(M/(x_1, \ldots, x_{i-1})M)$-sequence for all $i = 1, \ldots, n.$
(vi) $(x_1, \ldots, x_{i-1})M :_M x_i \subseteq (x_1, \ldots, x_{i-1})M :_M (a)$ for all $i = 1, \ldots, n,$ where $N :_M (a) = \{x \in M | a^n x \subseteq N \text{ for some } n \in \mathbb{N}\}$ for any submodule $N$ of $M.$

It is clear from definition, that, for a given $n \in \mathbb{N},$ one can find an $a$-filter regular $M$-sequence of length $n.$ The following theorem characterizes the existence of an $a$-filter regular $M$-sequence of length $n$ in $b.$

Theorem 2.2. Let $n \in \mathbb{N}.$ Then the following statements are equivalent.

(i) $b$ contains an $a$-filter regular $M$-sequence of length $n.$
(ii) Any $a$-filter regular $M$-sequence in $b$ of length less than $n$ can be extended to an $a$-filter regular $M$-sequence of length $n$ in $b.$
(iii) $\text{Supp} \text{Ext}_R^i(R/b, M) \subseteq V(a)$ for all $i < n.$
(iv) If $\text{Supp } N = V(b),$ then $\text{Supp} \text{Ext}_R^i(N, M) \subseteq V(a)$ for all $i < n.$
(v) $\text{Supp } H^i_b(M) \subseteq V(a)$ for all $i < n.$
(vi) If $\text{Ann} N \subseteq b$, then $\text{Supp} H^i_b(N, M) \subseteq V(a)$ for all $i < n$.

**Proof.** The implications (ii)⇒ (i), (iv)⇒ (iii) and (vi)⇒ (v) are clear.

(i)⇒ (ii). Assume the contrary that $x_1, \ldots, x_t$ is an $a$-filter regular $M$-sequence in $b$ such that $t < n$ and that it can not be extended to an $a$-filter regular $M$-sequence of length $n$ in $b$. Then $b \subseteq p$ for some $p \in \text{Ass} M/(x_1, \ldots, x_t)M \setminus V(a)$. So that $bR_p \subseteq pR_p \in \text{Ass}_{R_p} M_p/(x_1/1, \ldots, x_t/1)M_p$. It follows that $x_1/1, \ldots, x_t/1$ is a maximal $M_p$-sequence in $bR_p$, which is a contradiction in view of the hypothesis, Proposition 2.1 and [6, Theorem 1.2.5].

(i)⇒ (iv) Suppose that $x_1, \ldots, x_n$ is an $a$-filter regular $M$-sequence in $b$. Let $t \in \mathbb{N}$ be such that $x_i^t \in \text{Ann} N$ for all $i = 1, \ldots, n$. By Proposition 2.1 for any $p \in \text{Supp} M \setminus V(a)$, $x_i^t/1, \ldots, x_n^t/1$ is a weak $M_p$-sequence in $\text{Ann}_{R_p} N_p$. So that, for all $i < n$, we have $\text{Ext}^1_{R_p}(N_p, M_p) = 0$. Therefore (iv) holds.

(i)⇒(vi) Suppose that $x_1, \ldots, x_n$ is an $a$-filter regular $M$-sequence in $b$. For any $p \in \text{Supp} M \setminus V(a)$, $x_1/1, \ldots, x_n/1$ is a weak $M_p$-sequence in $bR_p$. So that, by [3, Proposition 5.5], $H^i_{bR_p}(N_p, M_p) = 0$ for all $i < n$. This proves the implication (i)⇒(vi).

Next we prove the implications (iii)⇒(i) and (v)⇒(i) by induction on $n$. Let $n = 1$. In either cases $\text{Supp} \text{Hom}_R(R/b, M) \subseteq V(a)$. Therefore (i) holds. Suppose that, for all $i \in \mathbb{N}_0$, $T^i(\cdot)$ is either $\text{Ext}^i_{R}(R/b, \cdot)$ or $H^i_{b}(\cdot)$. Assume that $n > 1$ and $\text{Supp} T^i(M) \subseteq V(a)$ for all $i < n$. Then $b$ contains an $a$-filter regular $M$-sequence, say $x_1$. The exact sequences
\[
0 \rightarrow 0 :_M x_1 \rightarrow M \xrightarrow{x_1} x_1M \rightarrow 0
\]
and
\[
0 \rightarrow x_1M \rightarrow M \rightarrow M/x_1M \rightarrow 0
\]
induce the long exact sequences
\[
\cdots \rightarrow T^i(0 :_M x_1) \rightarrow T^i(M) \rightarrow T^i(x_1M) \rightarrow T^{i+1}(0 :_M x_1) \rightarrow \cdots
\]
and
\[
\cdots \rightarrow T^i(x_1M) \rightarrow T^i(M) \rightarrow T^i(M/x_1M) \rightarrow T^{i+1}(x_1M) \rightarrow \cdots.
\]
Since $\text{Supp} 0 :_M x_1 \subseteq V(a)$, by Proposition 2.1 it follows that $\text{Supp} T^i(0 :_M x_1) \subseteq V(a)$ for all $i \in \mathbb{N}_0$. Therefore, using the above long exact sequences, we have $\text{Supp} T^i(M/x_1M) \subseteq V(a)$ for all $i < n - 1$. Hence, by inductive hypothesis, $b$ contains an $a$-filter regular $M/x_1M$-sequence of length $n - 1$ such as $x_2, \ldots, x_n$. This completes the inductive step, since $x_1, \ldots, x_n$ is an $a$-filter regular $M$-sequence in $b$. \hfill $\square$

**Remark 2.3.** One may use Theorem 2.2 (iii)⇒(ii) and Proposition 2.1 to see that $\text{Supp} M/bM \subseteq V(a)$ if and only if, for each $n \in \mathbb{N}$, there exists an $a$-filter regular $M$-sequence of length $n$ in $b$. Moreover, if $\text{Supp} M/bM \not\subseteq V(a)$, then it follows from Theorem 2.2 that any two maximal $a$-filter regular $M$-sequences in $b$ have the same length. Therefore, we may give the following.

**Definition 1.** Let $\text{Supp} M/bM \not\subseteq V(a)$. Then the common length of all maximal $a$-filter regular $M$-sequences in $b$ is denoted by $f\text{-grad}(a, b, M)$ and is called the $a$-filter grade of $b$ on $M$. We set $f\text{-grad}(a, b, M) = \infty$ whenever $\text{Supp} M/bM \subseteq V(a)$. 

Let \((R, \mathfrak{m})\) be a local ring. Then the \(\mathfrak{m}\)-filter grade of \(\mathfrak{a}\) on \(M\) is called the filter depth of \(\mathfrak{a}\) on \(M\) and is denoted by \(\text{f-depth}(\mathfrak{a}, M)\). Notice that, by Remark 2.3, \(\text{f-depth}(\mathfrak{a}, M) < \infty\) if and only if \(M/\mathfrak{a}M\) has finite length.

**Remark 2.4.** The following equalities follows immediately from Theorem 2.2
\[
\text{f-grad}(\mathfrak{a}, \text{Ann} N, M) = \inf \{ i \in \mathbb{N}_0 | \text{Supp} \text{Ext}^i_R(\mathfrak{a}, N, M) \not\subseteq V(\mathfrak{a}) \},
\]
\[
\text{f-grad}(\mathfrak{a}, \mathfrak{b} + \text{Ann} N, M) = \inf \{ i \in \mathbb{N}_0 | \text{Supp} \text{H}^i_{\mathfrak{b}}(N, M) \not\subseteq V(\mathfrak{a}) \}.
\]
In particular,
\[
\text{f-grad}(\mathfrak{a}, \mathfrak{b}, M) = \inf \{ i \in \mathbb{N}_0 | \text{Supp} \text{Ext}^i_R(\mathfrak{a}, M) \not\subseteq V(\mathfrak{a}) \}
= \inf \{ i \in \mathbb{N}_0 | \text{Supp} \text{H}^i_{\mathfrak{b}}(M) \not\subseteq V(\mathfrak{a}) \}.
\]
Suppose in addition that \((R, \mathfrak{m})\) is local. Then
\[
\text{f-depth}(\mathfrak{a}, M) = \inf \{ i \in \mathbb{N}_0 | \dim \text{Ext}^i_R(\mathfrak{a}, M) > 0 \}
= \inf \{ i \in \mathbb{N}_0 | \text{Supp} \text{H}^i_{\mathfrak{b}}(M) \not\subseteq \{ \mathfrak{m} \} \}.
\]

3. **A generalization of Nagel-Schenzel formula**

Let \(x_1, \ldots, x_n\) be an \(\mathfrak{a}\)-filter regular \(M\)-sequence in \(\mathfrak{a}\). Then, by [11 Proposition 1.2],
\[
\text{H}^i_{\mathfrak{a}}(M) = \begin{cases} 
\text{H}^i_{\mathfrak{a},(x_1,\ldots,x_n)}(M) & \text{if } i < n, \\
\text{H}^{i-n}_{\mathfrak{a}}(\text{H}^n_{\mathfrak{a},(x_1,\ldots,x_n)}(M)) & \text{if } i \geq n.
\end{cases}
\]
This formula was first obtained by Nagel and Schenzel, in [20 Lemma 3.4], in the case where \((R, \mathfrak{m})\) is a local ring and \(\mathfrak{a} = \mathfrak{m}\). Afterwards Khashyarmanesh, Yassi and Abbasi [12 Theorem 3.2] and Mafi [16 Lemma 2.8] generalized the second part of this formula for the generalized local cohomology modules as follows.

Suppose that \(M\) has finite projective dimension \(d\) and that \(x_1, \ldots, x_n\) is an \(\mathfrak{a}\)-filter regular \(N\)-sequence in \(\mathfrak{a}\). Then
\[
\text{H}^{i+n}_{\mathfrak{a}}(M, N) \cong \text{H}^{i}_{\mathfrak{a}}(M, \text{H}^n_{\mathfrak{a},(x_1,\ldots,x_n)}(N))
\]
for all \(i \geq d\).

The following Theorem establishes the Nagel-Schenzel formula for the generalized local cohomology modules. The first part of the following theorem is needed in the proof of the Corollary 4.5.

**Theorem 3.1.** Let \(x_1, \ldots, x_n\) be an \(\mathfrak{a}\)-filter regular \(N\)-sequence in \(\mathfrak{a}\). Then the following statements hold.

(i) \(\text{H}^i_{\mathfrak{a}}(M, N) \cong \text{H}^i_{\mathfrak{a},(x_1,\ldots,x_n)}(M, N)\) for all \(i < n\).

(ii) If \(\text{proj dim } M = d < \infty\) and \(L\) is projective, then
\[
\text{H}^{i+n}_{\mathfrak{a}}(M \otimes_R L, N) \cong \text{H}^{i}_{\mathfrak{a}}(M, \text{H}^n_{\mathfrak{a},(x_1,\ldots,x_n)}(L, N))
\]
for all \(i \geq d\).

**Proof.** (i) Set \(x = x_1, \ldots, x_n\). Since \(\Gamma_{\mathfrak{a}}(N) \subseteq \Gamma_{(x)}(N)\) we have a natural monomorphism \(\varphi_{M,N} : \text{H}^0_{\mathfrak{a}}(M, N) \to \text{H}^0_{\mathfrak{x}}(M, N)\). Now, let \(\mu_i(p, N)\) be the \(i\)-th Bass number of \(N\) with respect to a prime ideal \(p\) and let \(0 \to E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \to \cdots\) be
the minimal injective resolution of $N$. Then, by Proposition 2.1, $\mu_i(p, N) = 0$ for all $p \in \text{Supp } N \cap V(x) \setminus V(a)$ and all $i < n$. So

$$\Gamma_a(E^i) = \bigoplus_{p \in \text{Supp } N \cap V(a)} E(R/p)^{\mu_i(p, N)}$$

$$= \bigoplus_{p \in \text{Supp } N \cap V(a)} E(R/p)^{\mu_i(p, N)} = \Gamma_a(E^i)$$

for all $i < n$. Therefore $\varphi_{M, E^i}$ is an isomorphism for all $i < n$. Now let $i < n$. Since $\varphi_{M, E^{i-1}}$ and $\varphi_{M, E^i}$ are isomorphisms and $\varphi_{M, E^{i+1}}$ is a monomorphism, one can use the following commutative diagram

$$\begin{array}{ccc}
H^0_\alpha(M, E^{i-1}) & \xrightarrow{\varphi_{M, E^{i-1}}} & H^0_\alpha(M, E^i) \\
\downarrow{\varphi_{M, E^i}} & & \downarrow{\varphi_{M, E^i}} \\
H^0_{(x)}(M, E^{i-1}) & \xrightarrow{\varphi_{M, E^i}} & H^0_{(x)}(M, E^i)
\end{array}$$

$$\begin{array}{ccc}
H^0_\alpha(M, E^i) & \xrightarrow{\varphi_{M, E^i}} & H^0_\alpha(M, E^{i+1}) \\
\downarrow{\varphi_{M, E^{i+1}}} & & \downarrow{\varphi_{M, E^{i+1}}} \\
H^0_{(x)}(M, E^i) & \xrightarrow{\varphi_{M, E^{i+1}}} & H^0_{(x)}(M, E^{i+1})
\end{array}$$

to see that the induced homomorphism

$$\tilde{\varphi}_{M, E^i} : H^i_a(M, N) = \frac{\ker H^0_\alpha(M, d^i)}{\text{im } H^0_\alpha(M, d^{i-1})} \rightarrow \frac{\ker H^0_{(x)}(M, d^i)}{\text{im } H^0_{(x)}(M, d^{i-1})} = H^i_{(x)}(M, N),$$

is an isomorphism.

(ii) Set $F(\cdot) = H^0_a(M, \cdot)$ and $G(\cdot) = H^0_{(x)}(L, \cdot)$. Then $F$ and $G$ are left exact functors and $FG(\cdot) \cong H^0_a(M \otimes_R L, \cdot)$. Furthermore if $E$ is an injective $R$-module and $R^pF(p \in \mathbb{N}_0)$ is the $p$-th right derived functor of $F$, then it follows from [27, Lemma 1.1] and (i) that

$$R^pF(G(E)) = H^0_a(M, H^0_{(x)}(L, E)) \cong H^0_a(M, H^0_a(L, E))$$

$$\cong \text{Ext}^p_M(M, \text{Hom}_R(L, \Gamma_a(E))) = 0$$

for all $p \geq 1$. This yields the following spectral sequence

$$E_2^{p,q} = H^p_a(M, H^q_{(x)}(L, N)) \Rightarrow H^{p+q}_a(M \otimes_R L, N)$$

(see for example [21, Theorem 11.38]). Let $t = p + q \geq d + n$. If $q > n$, then $H^q_{(x)}(N) = 0$ by [31, Corollary 3.3.3]. Since $L$ is projective, it therefore follows that $H^q_{(x)}(L, N) = 0$. On the other hand if $q < n$, then $p > d = \text{proj dim } M$. Hence

$$E_2^{p,q} = H^p_a(M, H^q_{(x)}(L, N)) \cong H^p_a(M, H^q_a(L, N)) \cong \text{Ext}^p_M(M, H^q_a(L, N)) = 0.$$ 

Therefore, for $t \geq n + d$, there is a collapsing on the line $q = n$. Thus, there are isomorphisms

$$H^{t-n}_a(M, H^q_a(L, N)) \cong H^t_a(M \otimes_R L, N)$$

for all $t \geq n + d$. 

4. Artinianness of Generalized Local Cohomology Modules

Let $(R, \mathfrak{m})$ be a Noetherian local ring. In view of [19, Theorem 3.1] and [13, Theorem 3.10], one can see that $f\text{-depth}(a, M)$ is the least integer $i$ for which $H^i_a(M)$ is not Artinian. Also, as a main result, it was proved in [7, Theorem 2.2] that $f\text{-depth}(a + \text{Ann } M, N)$ is the least integer $i$ such that $H^i_a(M, N)$ is not Artinian.
We use rather a short argument to generalize this to the case in which $R$ is not necessarily a local ring. The following lemma is elementary.

**Lemma 4.1** ([22] Exercise 8.49). Let $X$ be an Artinian $R$-module, then $\text{Ass } X = \text{Supp } X$ is a finite subset of $\text{max}(R)$.

**Theorem 4.2.** Let $\mathcal{M}$ be the set of all finite subsets of $\text{max}(R)$. Then

\[
\sup_{A \in \mathcal{M}} \text{f-grad}((\cap_{m \in A} m, a + \text{Ann } M, N)) = \inf \{ i \in \mathbb{N}_0 \mid H^i_a(M, N) \text{ is not Artinian} \}
\]

\[
= \inf \{ i \in \mathbb{N}_0 \mid \text{Supp } H^i_a(M, N) \not\subseteq \text{max}(R) \}
\]

\[
= \inf \{ i \in \mathbb{N}_0 \mid \text{Supp } H^i_a(M, N) \not\subseteq A \text{ for all } A \in \mathcal{M} \}
\]

**Proof.** Since $H^i_a(M, N) \cong H^i_{a + \text{Ann } M}(M, N)$, we can assume that $\text{Ann } M \subseteq a$. It is clear that

\[
\sup_{A \in \mathcal{M}} \text{f-grad}((\cap_{m \in A} m, a, N)) = \inf \{ i \in \mathbb{N}_0 \mid \text{Supp } H^i_a(M, N) \not\subseteq A \text{ for all } A \in \mathcal{M} \}.
\]

Let $\mathcal{S}$ be either $\{ X \in \mathcal{C}_R \mid \text{Supp } X \subseteq \text{max}(R) \}$ or $\{ X \in \mathcal{C}_R \mid \text{Supp } X \subseteq A \text{ for some } A \in \mathcal{M} \}$, where $\mathcal{C}_R$ is the category of $R$-modules. Set $r = \inf \{ i \in \mathbb{N}_0 \mid H^i_a(M, N) \text{ is not Artinian} \}$ and $s = \inf \{ i \in \mathbb{N}_0 \mid H^i_a(M, N) \not\subseteq \mathcal{S} \}$. By Lemma [4.1] $r \leq s$. If $r = \infty$, there is nothing to prove. Assume that $r < \infty$. We show by induction on $r$, that $H^r_a(M, N) \not\subseteq \mathcal{S}$.

If $r = 0$, then $H^0_a(M, N) \not\subseteq \mathcal{S}$. Now suppose, inductively, that $r > 0$ and that the result has been proved for smaller values of $r$. In view of [27] Lemma 1.1 the exact sequence

\[
0 \longrightarrow \Gamma_a(N) \longrightarrow N \longrightarrow N/\Gamma_a(N) \longrightarrow 0
\]

induces the following long exact sequence

\[
\cdots \rightarrow \text{Ext}^i_R(M, \Gamma_a(N)) \rightarrow H^i_a(M, N) \rightarrow H^i_a(M, N/\Gamma_a(N)) \rightarrow \cdots.
\]

Since $H^0_a(M, N)$ has finite length, we have

\[
\text{Supp } H^0_a(M, N) = \text{Ass } \text{Hom}_R(M, \Gamma_a(N)) = \text{Ass } \Gamma_a(N);
\]

so that $\Gamma_a(N) \in \mathcal{S}$. Thus $\text{Ext}^i_R(M, \Gamma_a(N)) \in \mathcal{S}$ for all $i \in \mathbb{N}_0$. It follows that for each $i \in \mathbb{N}_0$, $H^i_a(M, N) \in \mathcal{S}$ if and only if $H^i_a(M, N/\Gamma_a(N)) \in \mathcal{S}$. Also we have $H^i_a(M, N)$ is Artinian if and only if $H^i_a(M, N/\Gamma_a(N))$ is Artinian. Hence we can replace $N$ by $N/\Gamma_a(N)$ and assume that $N$ is an $a$-torsion free $R$-module. Thus there exists an element $x \in a$ which is a non-zero divisor on $N$. The exact sequence

\[
0 \longrightarrow N \overset{x}{\longrightarrow} N \longrightarrow N/xN \longrightarrow 0
\]

induces the long exact sequence

\[
\cdots \rightarrow H^i_a(M, N) \rightarrow H^i_a(M, N) \rightarrow H^i_a(M, N/xN) \rightarrow H^{i+1}_a(M, N) \rightarrow \cdots.
\]

Since $H^i_a(M, N)$ is Artinian for all $i < r$, we may use the above sequence to see that $H^i_a(M, N/xN)$ is Artinian for all $i < r - 1$. On the other hand, $H^i_a(M, N)$ is not Artinian. Hence, using the above exact sequence and [27] Theorem 7.1.2, we see that $0 : H^{r-1}_a(M, N/xN) \not\subseteq \text{im } f_{r-1}$ is not Artinian. Thus $H^{r-1}_a(M, N/xN)$ is not Artinian; and hence, by inductive hypothesis, $H^{r-1}_a(M, N/xN) \not\subseteq \mathcal{S}$. So, again
by using the above sequence, we get $H^i(M, N) \notin \mathcal{S}$. This completes the inductive step. □

**Corollary 4.3.** Suppose that $\text{Supp} \ L = \text{Supp} \ M/aM$. Then

$$\inf \{i \in \mathbb{N}_0 | H^i_d(M, N) \text{ is not Artinian} \} = \inf \{i \in \mathbb{N}_0 | \dim \text{Ext}^i_R(L, N) > 0 \}.$$  

**Proof.** Let $n \in \mathbb{N}_0$. Then, by the Theorem 4.2, $H^i_d(M, N)$ is an Artinian $R$-module for all $i \leq n$ if and only if $n < f\text{-grad}(m_1 \cap \ldots \cap m_i, a + \text{Ann} M, N)$ for some maximal ideals $m_1, \ldots, m_i$ of $R$. By the Remarks 2.3(i), it is equivalent to $\text{Supp} \text{Ext}^i_R(L, N) \subseteq \{m_1, \ldots, m_i\}$ for some maximal ideals $m_1, \ldots, m_i$ of $R$ and for all $i \leq n$. This proves the assertion. □

The following corollary extend the main result of [25] to the generalized local cohomology modules.

**Corollary 4.4.** Let $n \in \mathbb{N}$. Then $H^i_b(M, N)$ is Artinian for all $i < n$ if and only if $H^i_{aR_p}(M_p, N_p)$ is Artinian for all $i < n$ and all prime ideal $p$.

**Proof.** This is immediate by the Corollary 1.3 □

**Corollary 4.5.** Let $\overline{R} = R/(a + \text{Ann} M + \text{Ann} N)$ be a semi local ring and let $\mathfrak{t}$ be the inverse image of the Jacobson radical of $R$ in $R$. Then we have

$$f\text{-grad}(\mathfrak{t}, a + \text{Ann} M, N) = \inf \{i \in \mathbb{N}_0 | H^i_d(M, N) \text{ is not Artinian} \} = \inf \{i \in \mathbb{N}_0 | H^i_d(M, N) \not\cong H^i_d(M, N) \}.$$  

**Proof.** The first equality is immediate by Theorem 4.2. To prove the second equality, let $n \leq f\text{-grad}(\mathfrak{t}, a + \text{Ann} M, N)$ and let $x_1, \ldots, x_n$ be an $\mathfrak{t}$-filter regular $N$-sequence in $a + \text{Ann} M$. Then $x_1, \ldots, x_n$ is an $a + \text{Ann} M$-filter regular $N$-sequence. So by Theorem 8.1(i),

$$H^i_a(M, N) \cong H^i_{aR_p}(M_p, N_p) \cong H^i_{(x_1, \ldots, x_n)} (M, N) \cong H^i_{\mathfrak{t}}(M, N)$$  

for all $i < n$. If $f\text{-grad}(\mathfrak{t}, a + \text{Ann} M, N) = \infty$, then the above argument shows that, $\inf \{i \in \mathbb{N}_0 | H^i_a(M, N) \not\cong H^i_d(M, N) \} = \infty$ and therefore the required equality holds. Therefore, we may assume that $f\text{-grad}(\mathfrak{t}, a + \text{Ann} M, N) = n < \infty$. By the first equality, $H^a(M, N)$ is not Artinian while $H^a_{\mathfrak{t}}(M, N)$ is Artinian. Hence the second equality holds. □

It was shown in [25] Theorem 2.2 that if $\dim R/a = 0$, then $H^i_a(M, N)$ is Artinian for all $i \in \mathbb{N}_0$. The following corollary is a generalization of this.

**Corollary 4.6.** Let $\overline{R} = R/(a + \text{Ann} M + \text{Ann} N)$. Then $H^i_a(M, N)$ is an Artinian $R$-module for all $i \in \mathbb{N}_0$ if and only if $\dim \overline{R} = 0$. In particular, $\text{Ext}^i_R(M, N)$ has finite length for all $i \in \mathbb{N}_0$ if and only if $\dim R/(\text{Ann} M + \text{Ann} N) = 0$.

**Proof.** Assume that $p$ is a prime ideal of $R$. By the Corollary 4.5, $H^i_{aR_p}(M_p, N_p)$ is Artinian for all $i < n$ if and only if $f\text{-depth}((a + \text{Ann} M)R_p, N_p) = \infty$ or equivalently $\dim R_p N_p/(aR_p + (\text{Ann} M)R_p)N_p = 0$ (Remark 2.3). Now, the result follows by corollary 4.4. □
5. Attached primes of the top generalized local cohomology modules

Let $X \neq 0$ be an $R$-module. If, for every $x \in R$, the endomorphism on $X$ given by multiplication by $x$ is either nilpotent or surjective, then $p = \sqrt{\text{Ann} X}$ is prime and $X$ is called a $p$-secondary $R$-module. If for some secondary submodules $X_1, \ldots, X_n$ of $X$ we have $X = X_1 + \ldots + X_n$, then we say that $X$ has a secondary representation.

One may assume that the prime ideals $p_i = \sqrt{\text{Ann} X_i}$, $i = 1, \ldots, n$, are distinct and, by omitting redundant summands, that the representation is minimal. Then the set $\text{Att} X = \{p_1, \ldots, p_n\}$ does not depend on the choice of a minimal secondary representation of $X$. Every element of $\text{Att} X$ is called an attached prime ideal of $X$. It is well known that an Artinian $R$-module has a secondary representation. The reader is referred to [14] for more information about the theory of secondary representation.

Let $(R, m)$ be a local ring and $n = \dim N < \infty$ and $d = \text{proj dim } M < \infty$. It was proved in [3] Theorem 2.3 that $H_{a}^{n+d}(M, N)$ is Artinian and that

$$\text{Att } H_{a}^{n+d}(M, N) = \{p \in \text{Ass } N| \text{cd}_{a}(M, R/p) = n + d\},$$

where, for an $R$-module $Y$, $\text{cd}_{a}(M, Y)$ is the greatest integer $i$ such that $H_{a}^{i}(M, Y) \neq 0$. Notice that $\text{cd}_{a}(M, N) \leq d + n$ [3] Lemma 5.1]. Next, we prove the above result without the local assumption on $R$. The following lemmas are needed.

**Lemma 5.1** ([3] Theorem A and B ). Let $\text{proj dim } M < \infty$. Then

(i) $\text{cd}_{a}(M, N) \leq \text{cd}_{a}(M, L)$ whenever $\text{Supp } N \subseteq \text{Supp } L$.

(ii) $\text{cd}_{a}(M, L) = \max\{\text{cd}_{a}(M, N), \text{cd}_{a}(M, K)\}$ whenever $0 \to N \to L \to K \to 0$ is an exact sequence.

**Lemma 5.2.** Let $\text{proj dim } M < \infty$, $\dim N < \infty$, $t = \text{cd}_{a}(M, N) \geq 0$ and

$$\Sigma = \{L \subsetneq N| \text{cd}_{a}(M, L) < t\}.$$

Then $\Sigma$ has the largest element with respect to inclusion, $L$ say, and the following statements hold.

(i) If $K$ is a non-zero submodule of $N/L$, then $\text{cd}_{a}(M, K) = t$.

(ii) $H_{a}^{1}(M, N) \cong H_{a}^{t}(M, N/L)$.

(iii) If $t = \text{proj dim } M + \dim N$, then

$$\text{Ass } N/L = \{p \in \text{Ass } N| \text{cd}_{a}(M, R/p) = t\}.$$ 

**Proof.** Since $N$ is Noetherian, $\Sigma$ has a maximal element, say $L$. Now assume that $L_1, L_2$ are elements of $\Sigma$. Using the exact sequence

$$0 \to L_1 \cap L_2 \to L_1 \oplus L_2 \to L_1 + L_2 \to 0$$

and Lemma 5.1 we see that $t > \text{cd}_{a}(M, L_1 + L_2)$. Hence the sum of any two elements of $\Sigma$ is again in $\Sigma$. It follows that $L$ contains every element of $\Sigma$; and so it is the largest one.

(i) Let $K = K'/L$ be a non-zero submodule of $N/L$. Since $L$ is the largest element of $\Sigma$, by applying Lemma 5.1 to the exact sequence

$$0 \to L \to K' \to K \to 0$$

we see that $t = \text{cd}_{a}(M, K)$.

(ii) The exact sequence $0 \to L \to N \to N/L \to 0$ induces the exact sequence

$$0 = H_{a}^{t}(M, L) \to H_{a}^{t}(M, N) \to H_{a}^{t}(M, N/L) \to H_{a}^{t+1}(M, L) = 0.$$
This completes the proof. □

(iii) Assume that $\text{cd}_\alpha(M, N) = \text{proj dim } M + \text{dim } N$. For each $p \in \text{Ass } L$, we have $\text{cd}_\alpha(M, R/p) < t$; so that

$$\{ p \in \text{Ass } N | \text{cd}_\alpha(M, R/p) = t \} \subseteq \text{Ass } N/L.$$ 

To establish the reverse inclusion, let $p \in \text{Ass } N/L$. Then by (i) and [3 Lemma 5.1] $t = \text{proj dim } M + \text{dim } R/p$. Therefore $p \in \text{Ass } N$ and equality holds.

**Theorem 5.3.** Let $d = \text{proj dim } M < \infty$ and $n = \dim N < \infty$. Then the $R$-module $H_n^{n+d}(M, N)$ is Artinian and

$$\text{Att } H_n^{n+d}(M, N) = \{ p \in \text{Ass } N | \text{cd}_\alpha(M, R/p) = n + d \}.$$ 

**Proof.** Let $x = x_1, \ldots, x_n$ be an $a$-filter regular $N$-sequence in $a$ and let $E^\bullet$ be the minimal injective resolution of $H_n^{(x)}(N)$. Since, by [3 Exercise 7.1.7], $H_n^{(x)}(N)$ is Artinian, every component of $E^\bullet$ is Artinian. On the other hand by 3.1

$$H_n^{n+d}(M, N) \cong H_n^d(M, H_n^n(x)(N)) \cong H^d(\text{Hom}_R(M, \Gamma_a(E^\bullet))).$$ 

It follows that $H_n^{n+d}(M, N)$ is Artinian.

Now we prove that $\text{Att } H_n^{n+d}(M, N) = \{ p \in \text{Ass } N | \text{cd}_\alpha(M, R/p) = n + d \}$. If $\text{cd}_\alpha(M, N) < n + d$, then $\text{Att } H_n^{n+d}(M, N) = \emptyset = \{ p \in \text{Ass } N | \text{cd}_\alpha(M, R/p) = n + d \}$. So one can assume that $t = \text{cd}_\alpha(M, N) = n + d$. Let $L$ be the largest submodule of $N$ such that $\text{cd}_\alpha(M, L) < t$. By Lemma 5.2 there is no non-zero submodule $K$ of $N/L$ such that $\text{cd}_\alpha(M, K) < t$. Also we have $H_n^d(M, N) \cong H_n^d(M, N/L)$ and $\text{Ass } N/L = \{ p \in \text{Ass } N | \text{cd}_\alpha(M, R/p) = t \}.$ Moreover $t = \text{cd}_\alpha(M, N/L) = \text{proj dim } M + \text{dim } N/L$. Thus we may replace $N$ by $N/L$ and prove that $\text{Att } H_n^d(M, N) = \text{Ass } N$. Now, for any non-zero submodule $K$ of $N$, $\text{cd}_\alpha(M, K) = t$ and $\text{dim } K = n$.

Assume that $p \in \text{Att } H_n^d(M, N)$. We have $p \supseteq \text{Ann } H_n^d(M, N) \supseteq \text{Ann } N$. Hence $p \in \text{Supp } N$. Now let $x \in \text{Reg } \setminus \bigcup_{p \in \text{Ass } N} p$. The exact sequence

$$0 \rightarrow N \xrightarrow{x} N \rightarrow N/xN \rightarrow 0$$ 

induces the exact sequence

$$H_n^d(M, N) \xrightarrow{x} H_n^d(M, N) \rightarrow H_n^d(M, N/xN) = 0.$$ 

Therefore $x \notin \bigcup_{p \in \text{Att } H_n^d(M, N)} p$. So $\bigcup_{p \in \text{Att } H_n^d(M, N)} p \subseteq \bigcup_{p \in \text{Ass } N} p$. Thus $p \subseteq q$ for some $q \in \text{Ass } N$. Hence $p = q$ and $\text{Att } H_n^d(M, N) \subseteq \text{Ass } N$. Next we prove the reverse inclusion. Let $p \in \text{Ass } N$ and let $T$ be a $p$-primary submodule of $N$. We have $t = \text{cd}_\alpha(M, R/p) = \text{cd}_\alpha(M, N/T)$. Moreover $N/T$ has no non-zero submodule $K$ such that $\text{cd}_\alpha(M, K) < t$. Hence, using the above argument, one can show that $\text{Att } H_n^d(M, N/T) \subseteq \text{Ass } N/T = \{ p \}$. It follows that

$$\{ p \} = \text{Att } H_n^d(M, N/T) \subseteq \text{Att } H_n^d(M, N).$$ 

This completes the proof. □

**Corollary 5.4.** Let $d = \text{proj dim } M < \infty$ and $n = \dim N < \infty$. Then

$$\text{Att } H_n^{n+d}(M, N) \subseteq \text{Supp } M \cap \text{Att } H_n^d(N).$$
Proof: If \( \text{Att } H_a^{n+d}(M, N) = \emptyset \), there is nothing to prove. Assume that \( \mathfrak{p} \in \text{Att } H_a^{n+d}(M, N) \). Then, by Lemma 19.1(iii) \( \mathfrak{p} \in \text{Ass } N \) and \( H_a^{n+d}(M, R/\mathfrak{p}) \neq 0 \). Next one can use the spectral sequence

\[
E_2^{p,q} = \text{Ext}_{R/\mathfrak{p}}^p(M, H_a^q(R/\mathfrak{p})) \Rightarrow H_a^{p+q}(M, R/\mathfrak{p})
\]

to see that \( H_a^{n+d}(M, R/\mathfrak{p}) \cong \text{Ext}_{R/\mathfrak{p}}^d(M, H_a^0(R/\mathfrak{p})) \). Therefore \( H_a^0(R/\mathfrak{p}) \neq 0 \); and hence \( \text{cd}_a(R/\mathfrak{p}) = n \). Thus, again by Lemma 5.5 \( \mathfrak{p} \in \text{Att } H_a^0(N) \). Also, we have \( \mathfrak{p} \supseteq \text{Ann } \text{Ext}_{R}^d(M, H_a^0(N)) \supseteq \text{Ann } M \), which completes the proof. 

Let \( X \) be an \( R \)-module. Set \( E = \bigoplus_{m \in \max R} E(R/m) \) (minimal injective cogenerator of \( R \)) and \( D = \text{Hom}_R(\cdot, E) \). We note that the canonical map \( X \to DDX \) is an injection. If this map is an isomorphism we say that \( X \) is (Matlis) reflexive. The following lemma yields a classification of modules which are reflexive with respect to \( E \).

Lemma 5.5 (Theorem 12). An \( R \)-module \( X \) is reflexive if and only if it has a finite submodule \( S \) such that \( X/S \) is artinian and that \( R/\text{Ann } X \) is a complete semilocal ring.

Assume that \( a \subseteq b \) and \( R/a \) is a complete semilocal ring. By above lemma \( R/a \) is reflexive as an \( R \)-module. On the other hand, the category of reflexive \( R \)-modules is a Serre subcategory of the category of \( R \)-modules. Therefore \( R/b \) is reflexive as an \( R \)-module and hence, by the above lemma, \( R/b \) is a complete semilocal ring. We shall use the conclusion of this discussion in the proof of the next theorem.

Theorem 5.6. Let \( M, N \) be two finite \( R \)-modules with \( \text{proj dim } M = d < \infty \) and \( \text{dim } N = n < \infty \). Let \( b = \text{Ann } H_a^0(N) \). If \( R/b \) is a complete semilocal ring, then

\[
\text{Att } H_a^{n+d}(M, N) = \text{Supp } \text{Ext}_{R}^d(M, R) \cap \text{Att } H_a^0(N).
\]

In particular, if in addition, \( \text{proj dim }_{R/\mathfrak{p}} M/\mathfrak{p} \) = \( \text{proj dim } M \) for all \( \mathfrak{p} \in \text{Supp } M \), then

\[
\text{Att } H_a^{n+d}(M, N) = \text{Supp } M \cap \text{Att } H_a^0(N).
\]

Proof. Since \( \text{Ext}_{R}^d(M, \cdot) \) is a right exact \( R \)-linear covariant functor, we have

\[
H_a^{n+d}(M, N) \cong \text{Ext}_{R}^d(M, H_a^0(N)) \cong \text{Ext}_{R}^d(M, R) \otimes_R H_a^0(N).
\]

Set \( \mathfrak{c} = \text{Ann } H_a^{n+d}(M, N) \). It is clear that \( b \subseteq c \). Therefore \( R/c \) is a complete semilocal ring. Now, by Lemma and [5, Exercise 7.2.10] and [4, VI.1.4 Proposition 10] we have

\[
\text{Att } H_a^{n+d}(M, N) = \text{Ass } D D H_a^{n+d}(M, N)
\]

\[
= \text{Ass } D H_a^{n+d}(M, N)
\]

\[
= \text{Ass } D(\text{Ext}_{R}^d(M, R) \otimes_R H_a^0(N))
\]

\[
= \text{Ass } \text{Hom}_R(\text{Ext}_{R}^d(M, R), D H_a^0(N))
\]

\[
= \text{Supp } \text{Ext}_{R}^d(M, R) \cap \text{Ass } D H_a^0(N)
\]

\[
= \text{Supp } \text{Ext}_{R}^d(M, R) \cap \text{Att } DD H_a^0(N)
\]

\[
= \text{Supp } \text{Ext}_{R}^d(M, R) \cap \text{Att } H_a^0(N)
\]

The final assertion follows immediately from the first equality, [18, Lemma 19.1(iii)] and the fact that \( \text{Supp } \text{Ext}_{R}^d(M, R) \subseteq \text{Supp } M \).
By Corollary 5.3 $\text{Att} H_{m}^{n+d}(M, N) \subseteq \text{Att} H_{m}^{n}(N)$. Next, we give an example to show that this inclusion may be strict even if $(R, m)$ is a complete regular local ring and $a = m$. Also, this example shows that the following theorem of Mafi is not true.

**[17, Theorem 2.1]**: Let $(R, m)$ be a commutative Noetherian local ring and $n = \dim N, d = \text{proj dim} M < \infty$. If $H_{m}^{n+d}(M, N) \neq 0$, then

\[ \text{Att} H_{m}^{n+d}(M, N) = \text{Att} H_{m}^{n}(N). \]

**Example 5.7.** Let $(R, m)$ be a complete regular local ring of a dimension $n \geq 2$ and assume that $R$ has two distinct prime ideals $p, q$ such that $\dim R/p = \dim R/q = 1$. Set $M = R/p$ and $N = R/p \oplus R/q$. Then, by Theorem 5.3

\[ \text{Att} H_{m}^{1}(N) = \{p, q\}. \]

On the other hand, $\text{proj dim} M = \dim R - \depth M = n - 1$ and $\dim N = 1$. Now, by Theorem 5.6

\[ \text{Att} H_{m}^{n}(M, N) = \text{Supp} M \cap \text{Att} H_{m}^{1}(N) = \{p\}. \]

Therefore [17, Theorem 2.1] is not true. Also, by [5, Proposition 7.2.11],

\[ \sqrt{(\text{Ann} H_{m}^{n}(M, N))} = \bigcap_{p \in \text{Att} H_{m}^{n}(M, N)} p = p \]

and

\[ \sqrt{(\text{Ann} H_{m}^{1}(N))} = \bigcap_{p \in \text{Att} H_{m}^{1}(N)} p = p \cap q. \]

Hence, again, Corollary 2.2 and Corollary 2.3 of [17] are not true. We note that, the other results of [17] are concluded from [17, Theorem 2.1, Corollary 2.2 and Corollary 2.3].

It is known that if $(R, m)$ is a local ring and $\dim M = n > 0$, then $H_{m}^{n}(M)$ is not finite [5, Corollary 7.3.3]. It was proved in [9, Proposition 2.6] that if $d = \text{proj dim} M < \infty$ and $0 < n = \dim N$, then $H_{m}^{n+d}(M, N)$ is not finite whenever it is non-zero. Next, we provide a generalization of this result. The following lemma, which is needed in the proof of the next proposition, is elementary.

**Lemma 5.8.** Let $X$ be an $R$-module. Then $X$ has finite length if and only if $X$ is Artinian and $\text{Att} X \subseteq \text{max} R$. Moreover if $X$ has finite length, then $\text{Att} X = \text{Supp} X = \text{Ass} X$.

**Proposition 5.9.** Let $d = \text{proj dim} M < \infty, 0 < n = \dim N < \infty$. If $H_{a}^{n+d}(M, N) \neq 0$, then it is not finite.

**Proof.** Assume that $p \in \text{Att} H_{a}^{n+d}(M, N)$. By [5.3] $H_{a}^{n+d}(M, N)$ is an Artinian $R$-module and $n + d = \text{cd}_{a}(M, R/p) = \text{proj dim} M + \dim R/p$. Therefore $\dim R/p = n > 0$; So that $\text{Att} H_{a}^{n+d}(M, N) \not\subseteq \text{max} R$. It follows that, in view of [5.8] $H_{a}^{n+d}(M, N)$ is not finite.

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