QUASIMODULAR FORMS AND MODULAR DIFFERENTIAL EQUATIONS WHICH ARE NOT APPARENT AT CUSPS: I

CHANG-SHOU LIN AND YIFAN YANG

ABSTRACT. In this paper, we explore a two-way connection between quasimodular forms of depth 1 and a class of second-order modular differential equations with regular singularities on the upper half-plane and the cusps. Here we consider the cases $\Gamma = \Gamma_0^+(N)$ generated by $\Gamma_0(N)$ and the Atkin-Lehner involutions for $N = 1, 2, 3$ ($\Gamma_0^+(1) = \text{SL}(2, \mathbb{Z})$). Firstly, we note that a quasimodular form of depth 1, after divided by some modular form with the same weight, is a solution of a modular differential equation. Our main results are the converse of the above statement for the groups $\Gamma_0^+(N), N = 1, 2, 3$.

1. INTRODUCTION

Let $\Gamma$ be a discrete subgroup of $\text{SL}(2, \mathbb{R})$ commensurable with $\text{SL}(2, \mathbb{Z})$. The aim of our study is to explore a two-way connection between quasimodular forms with depth $\ell$ on $\Gamma$ and modular differential equations of order $\ell + 1$. In this paper, we shall consider the case $\ell = 1$.

The notion of quasimodular forms was first introduced by Kaneko and Zagier [9]. For example, if $\Gamma = \text{SL}(2, \mathbb{Z})$, a function $f(z)$ on the upper half-plane $\mathbb{H}$ is called a quasimodular form of weight $k$ and depth $\leq 1$ if

$$f(z) = f_1(z)E_2(z) + f_0(z), \quad z \in \mathbb{H},$$

for some modular forms $f_0(z)$ and $f_1(z)$ of weight $k$ and $k - 2$, respectively, on $\text{SL}(2, \mathbb{Z})$, where $E_2(z)$ is the classical Eisenstein series of weight 2 on $\text{SL}(2, \mathbb{Z})$ with $E_2(\infty) = 1$.

For $f(z)$ of (1.1), Pellarin [10] introduced the Wronskian

$$W_f(z) = \det \begin{pmatrix} zf(z) + \frac{6\pi i}{\pi^4} f_1(z) & f(z) \\ (zf(z) + \frac{6\pi i}{\pi^4} f_1(z))^{(1)} & f^{(1)}(z) \end{pmatrix},$$

where $g^{(k)}$ is the $k$-th derivative of a function $g$ with respect to $z$, and a straightforward computation finds that $W_f(z)$ is a modular form of weight $2k$ (see [10] or Section 2 for more details). Let

$$g_1(z) = \frac{zf(z) + 6f_1(z)/\pi i}{\sqrt{W_f(z)}} \quad \text{and} \quad g_2(z) = \frac{f(z)}{\sqrt{W_f(z)}}.$$

Then we have $\det \begin{pmatrix} g_1 & g_2 \\ g_1' & g_2' \end{pmatrix} = 1$, and hence $\det \begin{pmatrix} g_1 & g_2 \\ g_1'' & g_2'' \end{pmatrix} = 0$, by a further differentiation. Thus, both $g_1$ and $g_2$ are solutions of

$$g'' = -4\pi^2 Q(z)g, \quad z \in \mathbb{H},$$

Date: March 9, 2021.
2010 Mathematics Subject Classification. Primary 11F11; secondary 11F25, 11F37, 34M03, 34M35.
where

\begin{equation}
Q(z) = -\frac{1}{4\pi^2} g_1'' - \frac{1}{4\pi^2} g_2''.
\end{equation}

Although generally, \(g_i, i = 1, 2\), are not single-valued functions, it is easy to see that \(Q(z)\) is a single-valued meromorphic function on \(\mathbb{H}\).

Equation (1.3) is a second-order ODE in a complex variable. Such a differential equation is called Fuchsian if the order of pole of \(Q(z)\) is \(\leq 2\). Let \(z_0\) be a singular point and assume that \(Q(z) = A(z - z_0)^{-2} + O\left((z - z_0)^{-1}\right)\). Then the indicial equation at \(z_0\) is

\[\rho(\rho - 1) = -4\pi^2 A,\]

and its roots are \(-\alpha\) and \(\alpha + 1\) for some \(\alpha \in \mathbb{C}\). These two roots are called the local exponents of (1.3) (or of \(Q(z)\)). If \(\alpha \in \frac{1}{2}\mathbb{Z}\), then equation (1.3) might have a solution with a logarithmic singularity near \(z_0\). The (regular) singular point \(z_0\) is called apparent if (a) \(\alpha \in \frac{1}{2}\mathbb{Z}\) and (b) any solution of (1.3) has no logarithmic singularity at \(z_0\). If (a) holds, but (b) does not, we will simply say \(z_0\) is not apparent. In this paper, we always assume that the local exponents of any singularity \(z_0 \in \mathbb{H}\) are integers or half-integers, which will be denoted by \(1/2 \pm \kappa_{z_0}, \kappa_{z_0} \in \frac{1}{2}\mathbb{Z}_{>0}\). It is easy to see that (i) if \(\kappa_{z_0} = 0\), then \(z_0\) is always not apparent, and (ii) if \(\kappa_{z_0} = 1/2\), then \(z_0\) is apparent if and only if \(Q(z)\) is holomorphic at \(z_0\).

We start with the following observation.

**Theorem 1.1.** Let \(Q(z)\) be defined (1.4). Then \(Q(z)\) is a meromorphic modular form of weight 4 on \(\text{SL}(2, \mathbb{Z})\). Moreover, the following hold.

(i) The ODE (1.3) is Fuchsian on \(\mathbb{H} \cup \{\text{cusps}\}\).
(ii) Any pole of \(Q\) on \(\mathbb{H}\) is an apparent singularity for (1.3).
(iii) The cusp \(\infty\) is not apparent.
(iv) Let \(1/2 \pm \kappa_1\) and \(1/2 \pm \kappa_\rho\), \(\kappa_1, \kappa_\rho \in \frac{1}{2}\mathbb{N}\), be the local exponents of (1.3) at \(i = \sqrt{-1}\) and \(\rho = (1 + \sqrt{-3})/2\), respectively. Then \((2\kappa_1, 2) = (2\kappa_\rho, 3) = 1\).

This theorem is a special case of Theorem 3.3 and Proposition 3.6 proved in Section 5.

The unexpected result of Theorem 1.1 is the modularity of \(Q\). In view of Theorem 1.1, a natural question is whether the converse of Theorem 1.1 holds or not. This is the main question studied in this paper, not only for the group \(\text{SL}(2, \mathbb{Z})\). Our methods for proving the main results depend on the structure of \(\Gamma\) and its modular forms. Thus, in this paper, we restrict our attention to \(\Gamma = \Gamma_0(N), N = 1, 2, 3\), where \(\Gamma_0(N)\) is the group generated by \(\Gamma_0(N)\) and all the Atkin-Lehner involutions. (Note that \(\Gamma_0^0(1) = \text{SL}(2, \mathbb{Z})\). In the case \(N = 2, 3, \Gamma_0^+(N)\) is simply the group generated by \(\Gamma_0(N)\) and \(\sqrt{N} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}\).) For such a group \(\Gamma\), there are only one cusp \(\infty\) and two elliptic points \(\rho_i, i = 1, 2\), where the order of \(\rho_1\) is 2, and the order of \(\rho_2\) is \(\{3, 4, 6\}\), depending on \(N\). Throughout the paper, we refer to

\begin{equation}
y''(z) = -4\pi^2 Q(z)y(z)
\end{equation}

as a modular ordinary differential equation (MODE) if \(Q(z)\) is a nonzero meromorphic modular form of weight 4 on \(\Gamma\). Motivated by Theorem 1.1, we consider equations (1.5)
satisfying the following condition \((H_f)\).

(i) The ODE \((1.5)\) is Fuchsian on \(\mathbb{H}\) and all cusps.

(ii) The local exponents of \((1.5)\) at any singular point \(z_0\) on \(\mathbb{H}\) are \(1/2 \pm \kappa_{z_0}, \kappa_{z_0} \in \frac{1}{2}\mathbb{N}\). Moreover, \(z_0\) is apparent for \((1.5)\).

(iii) Let \(e_{\rho_i}, i = 1, 2\), be the order of the elliptic point \(\rho_i\) and \(1/2 \pm \kappa_{\rho_i}, \kappa_{\rho_i} \in \frac{1}{2}\mathbb{N}\), be the local exponents of \((1.5)\) at \(\rho_i\). Then \((2\kappa_{\rho_i}, e_i) = 1\) for \(i = 1, 2\).

In Section \[3\], we will briefly explain the basic notions related to ODE in a complex variable. (We refer to \[4\] for an introduction to elementary theory of complex ODE.) Here we shall give some explanation pertaining to our condition \((H_f)\). Firstly, suppose that the cusp \(\infty\) has width \(N\). Let \(q_N = e^{2\pi i z/N}\). Then \((1.5)\) can be written as

\[
(1.6) \quad \left( q_N \frac{d}{dq_N} \right)^2 y = N^2 Q y
\]

since \(d/dz = 2\pi i N^{-1} q_N d/dq_N\). For other cusps \(s\), we choose \(\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{Z})\) such that \(\gamma \infty = s\). Set

\[
(1.7) \quad \hat{y}(z) = (y|_{-1 \gamma})(z) := (cz + d)y(\gamma z).
\]

Then \(\hat{y}\) satisfies

\[
(1.8) \quad \hat{y}'' = -4\pi^2 Q(z) \hat{y}, \quad \hat{Q}(z) = (Q|_{-1 \gamma})(z).
\]

By \(1.6\) and \(1.8\), we see that \((1.5)\) is Fuchsian at a cusp \(s\) if and only if \(Q\) is holomorphic at \(s\), and the local exponents at \(s\) are \(\pm \kappa_s\), where

\[
(1.9) \quad \kappa_s = N \sqrt{Q(s)}.
\]

Secondly, suppose that \(z_0 \in \mathbb{H}\) is a singular point and \((1.5)\) is a modular ODE. Then the local exponents at \(\gamma z_0, \gamma \in \Gamma\), are the same as those at \(z_0\). Also, if \(z_0\) is apparent, then so is \(\gamma z_0\). See \[4\] for a proof.

For a cusp \(s\), we can define a similar notion of apparentness (or non-apparentness) when \(\kappa_s\) in \(1.9\) is in \(\frac{1}{2} \mathbb{Z}_{\geq 0}\). Suppose that at a cusp \(s\) we have \(\kappa_s \in \frac{1}{2} \mathbb{Z}_{\geq 0}\). Then \((1.5)\) always has a unique solution \(y_+(z)\) of the form

\[
(1.10) \quad y_+(z) = q_N^{\kappa_1} \left( 1 + \sum_{j \geq 1} c_j q_N^j \right).
\]

An elementary theorem in theory of complex ODE says that if \(s\) is not apparent and \(y(z)\) is a solution of \((1.5)\) independent of \(y_+(z)\), then there exists \(\hat{c} \neq 0 \in \mathbb{C}\) and \(m(z)\) such that

\[
(1.11) \quad y(z) = ez y_+(z) + m(z), \quad m(z) = q_N^{-\kappa_1} \sum_{j=0}^{\infty} \hat{c}_j q_N^j, \quad \hat{c}_0 \neq 0.
\]

We now introduce a representation of \(\Gamma\) associated to a modular ODE as follows. It is an elementary fact that any (local) solution \(y(z)\) of \((1.5)\) can be defined on the whole \(\mathbb{H}\) through analytic continuation. Fix a point \(z_0 \in \mathbb{H}\) that is not a singular point of \(Q(z)\). Let \(U\) be a simply connected (small) open set in \(\mathbb{H}\) containing \(z_0\), but not any singularities of \(Q(z)\). For \(\gamma \in \Gamma\), choose a path \(\sigma\) from \(z_0\) to \(\gamma z_0\) and consider the analytic continuation of \(y(z), z \in U\), along the path. Then \(y(\gamma z)\) is well-defined in \(U\). We define \(y|_{-1 \gamma}\) by

\[
(y|_{-1 \gamma})(z) := (cz + d)y(\gamma z), \quad z \in U.
\]
Then \( y|_{-1}\gamma \) is also a solution of \((1.5)\). This can be proved by a direct computation or by using Bol’s identity in literature \([1]\). By choosing a fundamental system \((y_1(z), y_2(z))\) of solutions, it gives a matrix \( \hat{\gamma} \in \text{SL}(2, \mathbb{C}) \) such that
\[
\begin{pmatrix}
y_1|_{-1}\gamma \\
y_2|_{-1}\gamma
\end{pmatrix}(z) = \hat{\gamma} \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}.
\]

(The reason why \( \hat{\gamma} \) has determinant 1 is due to the fact that \( y_1|_{-1}\gamma \) and \( y_2|_{-1}\gamma \) has the same Wronskian as \( y_1 \) and \( y_2 \).) Of course, this matrix \( \hat{\gamma} \) depends on the choice of the path \( \sigma \). However, under the condition \((H_1)\), all local monodromy matrices are \( \pm I \). Hence, different choices of \( \sigma \) will only possibly change \( \hat{\gamma} \) to \( -\hat{\gamma} \). From this, we see that there is a well-defined homomorphism \( \rho \) from \( \Gamma \) to \( \text{PSL}(2, \mathbb{C}) \) such that
\[
(1.12) \quad \begin{pmatrix} y_1|_{-1}\gamma \\
y_2|_{-1}\gamma
\end{pmatrix}(z) = \pm \rho(\gamma) \begin{pmatrix} y_1(z) \\ y_2(z) \end{pmatrix}
\]
for all \( \gamma \in \Gamma \), where \( y_j(\gamma z), j = 1, 2 \), are always understood to take the same path for the analytic continuation. This homomorphism \( \rho \) will be called the Bol representation in this paper. For the group \( \Gamma \) under consideration in this paper, the Bol representation can be lifted to a homomorphism \( \hat{\rho} \) from \( \Gamma \) to \( \text{GL}(2, \mathbb{C}) \). To achieve this, we let \( F(z)^2 \) be a modular form of weight \( 2(\ell + 1) \) with some character for some integer \( \ell \) such that \( \hat{y}(z) = F(z)y(z) \) is well-defined on \( \mathbb{H} \) for any solution \( y(z) \) of \((1.5)\). Note that since \( \hat{y}(z) \) is single-valued, for any \( \gamma \in \Gamma \), there exists \( \hat{\rho}(\gamma) \in \text{GL}(2, \mathbb{C}) \) such that
\[
\begin{pmatrix} \hat{y}_1 \\ \hat{y}_2
\end{pmatrix} = \hat{\rho}(\gamma) \begin{pmatrix} y_1 \\ y_2
\end{pmatrix}.
\]

Obviously, \( \hat{\rho} \) is a lifting of \( \rho \). We remark that there are many possible choices for \( F(z) \). In Sections 4–6, we will choose \( F(z) \) such that \( \hat{y}(z) \) is holomorphic on \( \mathbb{H} \) and at cusps. We emphasize that the integer \( \ell \) will give rise to some important information in this series of papers. For example, we have \( \det \hat{\rho}(\gamma) = 1 \) for all \( \gamma \in \Gamma \) if and only if \( \ell \) is odd.

From now on, we assume that \( \kappa_\infty \in \frac{1}{2}\mathbb{Z}_{\geq 0} \), where \( \pm \kappa_\infty \) are the local exponents of \((1.5)\) at the cusp \( \infty \), and \( \{z_1, \ldots, z_m\} \) is the set of poles of \( Q(z) \) such that \( z_i \) is not an elliptic point and \( z_i \) is not \( \Gamma \)-equivalent to \( z_j \) for \( i \neq j \). Let \( 1/2 \pm \kappa_j \) be the local exponents of \((1.5)\) at \( z_j \).

When \( \Gamma = \text{SL}(2, \mathbb{Z}) \), the integer \( \ell \) in the lift of the Bol representation is defined by
\[
(1.13) \quad \ell = -1 + 12\kappa_\infty + 4 \left( \kappa_\rho - \frac{1}{2} \right) + 6 \left( \kappa_i - \frac{1}{2} \right) + 12 \sum_{j=1}^{m} \left( \kappa_j - \frac{1}{2} \right),
\]
where \( i = \sqrt{-1} \) and \( \rho = (1 + \sqrt{-3})/2 \) are the elliptic points and \( 1/2 \pm \kappa_\rho \) and \( 1/2 \pm \kappa_i \) are the local exponents of \((1.5)\) at \( \rho \) and \( i \), respectively. Since all the \( \kappa \)'s are in \( \frac{1}{2}\mathbb{Z} \), \( \ell \) is always an integer. Furthermore, the integer \( \ell \) is odd if and only if \( \kappa_i \in \frac{1}{2} + \mathbb{Z} \). Thus, \((H_1)\) implies that \( \ell \) is odd.

Let \( F(z) \) be defined by \((4.6)\). Then \( F(z)^2 \) is a modular form of weight \( 2(\ell + 1) \) such that for any solution \( y(z) \) of \((1.5)\), the function \( \hat{y}(z) = F(z)y(z) \) is a single-valued holomorphic function on \( \mathbb{H} \). We use the standard notations \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \), \( S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), and \( R = TS = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \). They satisfy \( S^2 = R^3 = -I \). Our first main result asserts that the converse of Theorem \((1.1)\) holds true.

**Theorem 1.2.** Let \( \Gamma = \text{SL}(2, \mathbb{Z}) \). Suppose that \( Q(z) \), not identically 0, satisfies \((H_1)\) with \( \kappa_\infty \in \frac{1}{2}\mathbb{Z}_{\geq 0} \). Then the following statements hold true.
Theorem 1.4. Let \( y(z) \) be the solution of (1.5) of the form (1.10) and set \( y_1(z) = (\hat{y} + S)(z)/F(z) \) and \( y_2(z) = y_+(z) \). Then \( y_1(z) \) and \( y_2(z) \) form a fundamental system of solutions of (1.5). Moreover, we have
\[
\hat{y}_1(z) = z\hat{y}_2(z) + \hat{m}_1(z)
\]
for some modular form \( \hat{m}_1(z) \) of weight \( \ell - 1 \) on \( \text{SL}(2, \mathbb{Z}) \).

Remark 1.3. Since (1.5) is assumed to be apparent at any pole of \( Q \) on \( \mathbb{H} \), the ratio \( h(z) \) of any two linearly independent solution of (1.5) is a meromorphic (single-valued) function on \( \mathbb{H} \). We say a meromorphic function \( h(z) \) on \( \mathbb{H} \) is equivariant with respect to \( \text{SL}(2, \mathbb{Z}) \) if \( h(z) \) satisfies
\[
h(\gamma z) = \gamma \cdot h(z) := \frac{ah(z) + b}{ch(z) + d}
\]
for all \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{SL}(2, \mathbb{Z}) \) (see (13)).

We note that \( y_1(z) \) is a second solution of (1.5), so \( y_1(z) \) has an expression of the form (1.11) for some \( e \neq 0 \). Then Part (ii) says that \( e = 1 \). It is surprising to see that if the second solution is suitably chosen, then \( m(z) \) of (1.11) is a meromorphic modular form of weight \(-2\).

For the case \( \Gamma = \Gamma_0^+(N), N = 2, 3 \), we define
\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}, \quad R = TS = \frac{1}{\sqrt{N}} \begin{pmatrix} N & -1 \\ 0 & 0 \end{pmatrix}
\]
such that
\[
S^2 = R^{2N} = -I.
\]
The group \( \Gamma_0^+(N), N = 2, 3 \), has one cusp \( \infty \) with width 1 and two elliptic point \( \rho_1 \) of order 2 and \( \rho_2 \) of order \( 2N \), which are the fixed points of \( S \) and \( R \), respectively, i.e.,
\[
\rho_1 = \frac{i}{\sqrt{N}}, \quad \rho_2 = \begin{cases} (1 + i)/2, & \text{if } N = 2, \\
(3 + \sqrt{-3})/6, & \text{if } N = 3.
\end{cases}
\]
The integer \( \ell \) in the lifting in the case \( \Gamma_0^+(2) \) is
\[
\ell = -1 + 8\kappa_\infty + 4 \left( \kappa_{\rho_1} - \frac{1}{2} \right) + 2 \left( \kappa_{\rho_2} - \frac{1}{2} \right) + 8 \sum_{j=1}^{m} \left( \kappa_j - \frac{1}{2} \right).
\]
Note that (H1) implies that both \( \kappa_{\rho_1} - 1/2 \) and \( \kappa_{\rho_2} - 1/2 \) are nonnegative integers. Hence \( \ell \) is an odd integer. Let \( F(z) \) be defined by (5.2) and for a solution \( y(z) \) of (1.5), set \( \hat{y}(z) = F(z)g(z) \). The main theorem for \( \Gamma_0^+(2) \) is as follows.

Theorem 1.4. Let \( \Gamma = \Gamma_0^+(2) \). Suppose that \( Q(z) \) satisfies (H1) with \( \kappa_\infty \in \frac{1}{2}\mathbb{Z}_{\geq 0} \). Then the following statements hold.

(i) The cusp \( \infty \) is not apparent for (1.5).
(ii) Let \( y_1(z) = \left( \hat{y}_1(z) \right)_{1S} \) and \( y_2(z) = y_+(z) \), where \( y_+(z) \) is the unique solution of (1.15) of the form (1.10). Then \( \{y_1(z), y_2(z)\} \) is a fundamental solution of (1.5). Moreover, we have

\[
\hat{y}_1(z) = \left( \frac{2}{\ell} \right) \sqrt{2z} \hat{y}_2(z) + \hat{m}_1(z)
\]

for some modular form \( \hat{m}_1(z) \) in \( \mathcal{M}_{\ell-1}(\Gamma^+_0(2), (\frac{2}{\ell})) \), where \( \left( \frac{2}{\ell} \right) \) is the Legendre symbol whose values are given by

\[
\left( \frac{2}{\ell} \right) = \begin{cases} 1, & \text{if } \ell \equiv 1, 7 \mod 8, \\ -1, & \text{if } \ell \equiv 3, 5 \mod 8. 
\end{cases}
\]

(iii) The ratio \( h(z) = \left( \frac{2}{\ell} \right) y_1(z) / \sqrt{2y_2(z)} \) is equivariant with respect to \( \Gamma^+_0(2) \). That is, for all \( \gamma \in \Gamma^+_0(2) \), we have \( b(\gamma z) = h(z) \).

(iv) Let \( M^*_2(z) = (2E_2(2z) + E_2(z))/3 \) and write \( \hat{y}_+(z) \) as

\[
\hat{y}_+(z) = \left( \frac{2}{\ell} \right) \frac{\pi i}{4\sqrt{2}} \hat{m}_1(z) M^*_2(z) + \hat{m}_2(z).
\]

Then \( \hat{m}_2(z) \) is a modular form in \( \mathcal{M}_{\ell+1}(\Gamma^+_0(2), (\frac{2}{\ell})) \). Consequently, \( \hat{y}_+(z) \) is a quasimodular form in \( \mathcal{M}^{\leq 1}_{\ell+1}(\Gamma^+_0(2), (\frac{2}{\ell})) \).

Note that \( M^*_2(z) \) is a quasimodular form of weight 2 and depth 1 on \( \Gamma^+_0(2) \). Also, \( \mathcal{M}_k(\Gamma^+_0(2), \pm 1) \) denotes the space of modular forms \( f(z) \) of weight \( k \) on \( \Gamma^+_0(2) \) such that \( (f|_kS)(z) = \pm f(z) \). (The reason for the choice of the notation \( \mathcal{M}_k(\Gamma^+_0(2), \pm 1) \) instead of the more common \( \mathcal{M}_k(\Gamma_0(2), \pm 1) \) is that we regard such an \( f \) as a modular form with character on \( \Gamma^+_0(2) \).) For the meaning of the notation \( \mathcal{M}^{\leq 1}_{\ell+1}(\Gamma^+_0(2), (\frac{2}{\ell})) \), see Section 2.2.

Remark 1.5. The identity (1.15) says that the constant \( e \) in (1.11) is \( \left( \frac{2}{\ell} \right) \sqrt{2} \), which depends on \( \ell \). This result is different from Theorem 1.2.

For the case \( \Gamma = \Gamma^+_0(3) \), the group \( \Gamma \) has one cusp \( \infty \) and two elliptic points \( \rho_1 \) and \( \rho_2 \) given by (1.13). In this case, \( \ell \) is defined by

\[
\ell = -1 + 6\kappa_\infty + 3 \left( \kappa_{\rho_1} - \frac{1}{2} \right) + \left( \kappa_{\rho_2} - \frac{1}{2} \right) + 6 \sum_{j=1}^{m} \left( \kappa_j - \frac{1}{2} \right).
\]

Note that by (H_2), both \( \kappa_{\rho_1} - 1/2 \) and \( \kappa_{\rho_2} - 1/2 \) are integers, and \( \ell \) is an integer not divisible by 3. Let \( F(z) \) be defined by (6.3). As before, for a solution \( y(z) \) of (1.3), we let \( \hat{y}(z) = F(z)y(z) \).

Theorem 1.6. Let \( \Gamma = \Gamma^+_0(3) \). Suppose that \( Q(z) \) satisfy (H_1) with \( \kappa_\infty \in \frac{1}{2}\mathbb{Z}_{\geq 0} \). Set

\[
\delta = \begin{cases} \chi^0, & \text{if } \ell \equiv 1, 11 \mod 12, \\ \chi^1, & \text{if } \ell \equiv 2, 4 \mod 12, \\ \chi^2, & \text{if } \ell \equiv 5, 7 \mod 12, \\ \chi^3, & \text{if } \ell \equiv 8, 10 \mod 12, 
\end{cases}
\]

where \( \chi \) is the character of \( \Gamma^+_0(3) \) defined by \( \chi(S) = \chi(R) = -i \). Then the following statements hold.

(i) The cusp \( \infty \) is not apparent.
(ii) Let \( y_1(z) = (\hat{y}_+|S)(z)/F(z) \) and \( y_2(z) = y_+ (z) \). Then \( \{y_1(z), y_2(z)\} \) is a fundamental solution of (1.5). Moreover, we have
\[
\hat{y}_1(z) = \delta(S)^{-1}\sqrt{3z}\hat{y}_2(z) + \hat{n}_1(z)
\]
for some modular form \( \hat{n}_1(z) \) in \( \mathcal{M}_{t-1}(\Gamma_0^+ (3), \delta) \).

(iii) The ratio \( h(z) = \delta(S)^{-1}y_1(z)/\sqrt{3}y_2(z) \) is equivariant with respect to \( \Gamma_0^+ (3) \). That is, for all \( \gamma \in \Gamma_0^+ (3) \), we have \( h(\gamma z) = \gamma \cdot h(z) \).

(iv) Let \( M^*_2(z) = (3E_2(3z) + E_2(z))/4 \) and write \( \hat{y}_+ (z) \) as
\[
\hat{y}_+(z) = \delta(S)^{-1}\pi i/3\sqrt{3}\hat{n}_1(z)M^*_2(z) + \hat{n}_2(z).
\]

Then \( \hat{n}_2(z) \) is a modular form in \( \mathcal{M}_{t+1}(\Gamma_0^+ (3), \delta) \). Consequently, \( \hat{y}_+ (z) \) is a quasimodular form in \( \mathcal{M}_{\leq t}^{\leq 1}(\Gamma_0^+ (3), \delta) \).

Note that \( M^*_2(z) \) is a quasimodular form of weight 2 and depth 1 on \( \Gamma_0^+ (3) \). Also, \( \mathcal{M}_k(\Gamma_0^+ (3), \delta) \) denote the space of modular forms of weight \( k \) with character \( \delta \) on \( \Gamma_0^+ (3) \). See Section 2.1 for a discussion about these spaces and the space \( \mathcal{M}_{\leq t}^{\leq 1}(\Gamma_0^+ (3), \delta) \).

The rest of the paper is organized as follows. In Section 2, we give a quick introduction to quasimodular forms, discuss the Atkin-Lehner decomposition of the space of quasimodular forms on \( \Gamma_0(N) \), and then prove the existence of extremal quasimodular forms in certain spaces, which will used to provide examples of modular differential equations satisfied by quasimodular forms. In Section 3, we will prove that every quasimodular form of depth 1 will give rise to a modular differential equation of the form (1.5) and show that such a differential equation in the case \( \Gamma = \text{SL}(2, \mathbb{Z}), \Gamma_0^+ (2) \), or \( \Gamma_0^+ (3) \) will satisfy the condition (H). In Sections 4 and 5, we consider the converse direction. We will prove that if a modular differential equation (1.5) satisfies (H) with \( \Gamma = \text{SL}(2, \mathbb{Z}), \Gamma_0^+ (2) \), or \( \Gamma_0^+ (3) \), then its solutions must come from quasimodular forms. Finally, in the appendix, we discuss how to determine apparentness of a modular differential equation and prove the existence of meromorphic modular forms \( Q(z) \) satisfying (H).

2. Quasimodular forms

For convenience of non-specialists, we shall give a quick introduction of quasimodular forms in this section. See [2, 13] for more detailed overview and discussion. Note that most of the definitions and properties presented here are valid or have analogues in the setting of general Fuchsian subgroups of the first kind of \( \text{SL}(2, \mathbb{R}) \), but for simplicity, here we restrict our attention to subgroups \( \Gamma \) of \( \text{SL}(2, \mathbb{R}) \) commensurable with \( \text{SL}(2, \mathbb{Z}) \), which we assume throughout the section.

2.1. Basic definitions and properties. The notion of quasimodular forms was first introduced by Kaneko and Zagier [9]. It was defined through the notion of nearly holomorphic modular forms (called almost holomorphic modular forms in [9]).

Definition 2.1. A function \( f : \mathbb{H} \to \mathbb{C} \) is said to be nearly holomorphic if it is of the form
\[
f(z) = \sum_{r=0}^{n} \frac{f_r(z)}{(z - \overline{z})^r}
\]
for some holomorphic functions \( f_r \).
Let $\Gamma$ be a congruence subgroup of $\text{SL}(2, \mathbb{R})$. A function $f : \mathbb{H} \to \mathbb{C}$ is said to be a \textit{nearly holomorphic modular form} on $\Gamma$ if it is nearly holomorphic, satisfies
\begin{equation}
\left( az + b \right) = (cz + d)^k f(z)
\end{equation}
for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and has a finite limit at each cusp of $\Gamma$. The largest integer $r$ such that $f_r \neq 0$ in (2.1) is called the \textit{depth} of $f$. We let $\mathcal{M}_k^*(\Gamma)$ denote the space of nearly holomorphic modular forms of weight $k$ on $\Gamma$.

On the spaces of nearly holomorphic modular forms, we have the so-called Shimura-Maass operator.

\textbf{Definition 2.2.} For a nearly holomorphic function $f : \mathbb{H} \to \mathbb{C}$ and an integer $k$, define the Shimura-Maass operator $\partial_k$ of weight $k$ by
\begin{equation}
(\partial_k f)(z) = \frac{1}{2\pi i} \left( f'(z) + \frac{k f(z)}{z - \bar{z}} \right).
\end{equation}
Also, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}^+(2, \mathbb{R})$, the slash operator of weight $k$ is defined by
\begin{equation}
(f|_k \gamma)(z) = (\det \gamma)^{k/2} (cz + d)^k f\left( az + b \right).
\end{equation}

Then we have the following properties.

\textbf{Lemma 2.3 ([15] Equations (1.5) and (1.8))}. \textit{For any nearly holomorphic functions $f, g : \mathbb{H} \to \mathbb{C}$, any integers $k$ and $\ell$, and any $\gamma \in \text{GL}^+(2, \mathbb{R})$, we have}
\begin{equation}
(\partial_k + \ell)(fg) = (\partial_k f)g + f(\partial_\ell g)
\end{equation}
and
\begin{equation}
(\partial_k (f|_k \gamma)) = (\partial_k f)|_{k + 2}\gamma.
\end{equation}

In particular, the second property in the lemma says that if $f$ is a nearly holomorphic modular form of weight $k$ on $\Gamma$, then $\partial_k f$ is a nearly holomorphic form of weight $k + 2$ on $\Gamma$. That is, $\partial_k$ is a linear transformation from $\mathcal{M}_k^*(\Gamma)$ to $\mathcal{M}_{k+2}^*(\Gamma)$.

\textbf{Definition 2.4.} We say a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a \textit{quasimodular form} of weight $k$ and depth $r$ on $\Gamma$ if it is the holomorphic part of a nearly holomorphic modular form $f$ of weight $k$ and depth $r$ on $\Gamma$. We let $\tilde{\mathcal{M}}_{k,r}^*(\Gamma)$ denote the space of quasimodular forms of weight $k$ and depth $r$ on $\Gamma$. We also let
\begin{equation}
\tilde{\mathcal{M}}^*(\Gamma) = \bigoplus_k \bigcup_r \tilde{\mathcal{M}}_{k,r}^*(\Gamma)
\end{equation}
denote the graded ring of quasimodular forms of all weights and all depths on $\Gamma$.

Using the modular property (2.2), we can show that if $f_0 \in \tilde{\mathcal{M}}_{k,r}^*(\Gamma)$ is the holomorphic part of the nearly holomorphic modular form
\begin{equation}
f(z) = \sum_{r=0}^{\infty} \frac{f_r(z)}{(z - \bar{z})^r},
\end{equation}
then
\begin{equation}
(f_0|_k \gamma)(z) = \sum_{r=0}^{\infty} f_r(z) \left( \frac{cz + d}{cz + d} \right)^r
\end{equation}
for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and vice versa.
The archetypal example of a quasimodular form is $E_2(z)$, which is the holomorphic part of the nearly holomorphic modular form

$$E_2(z) + \frac{6}{\pi i(z - \bar{z})} = E_2(z) - \frac{3}{\pi \text{Im} z}$$

of weight 2 and depth 1 on $\text{SL}(2, \mathbb{Z})$. It can be shown that the graded ring $\widetilde{M}(\text{SL}_2(\mathbb{Z}))$ of quasimodular forms on $\text{SL}(2, \mathbb{Z})$ is generated by $E_2(z), E_4(z),$ and $E_6(z)$. Also, by Ramanujan’s identities

$$D_q E_2 = E_2^2 - E_4, \quad D_q E_4 = E_2 E_4 - E_6,$$

the ring $\widetilde{M}(\text{SL}_2(\mathbb{Z}))$ is closed under the differential operator

$$D_q := q \frac{d}{dq} = \frac{1}{2\pi i} \frac{d}{dz}.$$

More generally, we have the following description of $\widetilde{M}^{\leq r}_k(\Gamma)$.

**Proposition 2.5** ([18, Proposition 20]). Let $\Gamma$ be a subgroup of $\text{SL}(2, \mathbb{R})$ commensurable with $\text{SL}(2, \mathbb{Z})$.

(i) The graded ring $\widetilde{M}(\Gamma)$ of quasimodular forms on $\Gamma$ is closed under differentiation.

More precisely, we have $D_q(\widetilde{M}^{\leq r}_k(\Gamma)) \subseteq \widetilde{M}^{\leq r+1}_{k+2}(\Gamma)$ for all $k, r \geq 0$.

(ii) Let $\phi$ be a quasimodular form of weight 2 and depth 1 on $\Gamma$. Then

$$\widetilde{M}^{\leq r}_k(\Gamma) = \bigoplus_{j=0}^{r} \phi^j \cdot \mathcal{M}_{k-2j}(\Gamma)$$

for all $k, r \geq 0$. In particular, we have

$$\dim \widetilde{M}^{\leq r}_k(\Gamma) = \sum_{j=0}^{r} \dim \mathcal{M}_{k-2j}(\Gamma).$$

The quasimodular form $\phi$ in the proposition can be constructed by taking the logarithmic derivative of a meromorphic modular form of nonzero weight that is nonvanishing on $\mathbb{H}$. In general, the choice of $\phi$ is not unique. For example, when $\Gamma = \Gamma_0(p)$, we can take $\phi$ to be the logarithmic derivative of any $\Delta(z)^m \Delta(pz)^n$, as long as $m + n \neq 0$.

More generally, we can define quasimodular forms on $\Gamma$ with character $\chi$ to be the holomorphic parts of nearly holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying

$$f(\gamma z) = \chi(\gamma)(cz + d)^k f(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

and analytic conditions at cusps. (Thus, if $-I \in \Gamma$ and $\chi(-I) = -1$, we will assume that $k$ is odd.) We similarly let $\widetilde{M}^{\leq r}_k(\Gamma, \chi)$ denote the space of all quasimodular forms on $\Gamma$ of weight $k$ with character $\chi$. In this paper, we will only consider characters of finite order.

The first examples that provides a link between quasimodular forms and modular differential equations were due to Kaneko and Koike [7,8]. Among other things, they proved that if $6 \mid k$, then there is a solution $f(z)$ of the differential equation

$$D_q^2 y(z) - \frac{k}{6} E_2(z) D_q y(z) + \frac{k(k-1)}{12} (D_q E_2(z)) y(z) = 0,$$

with

$$D_q := q \frac{d}{dq} = \frac{1}{2\pi i} \frac{d}{dz}.$$
such that $f$ is an extremal quasimodular form in $\tilde{\mathcal{M}}_{k}^{<1}(\text{SL}(2, \mathbb{Z}))$, defined as an element whose vanishing order at $\infty$ is $\dim \tilde{\mathcal{M}}_{k}^{<1}(\text{SL}(2, \mathbb{Z})) - 1$. Then by a simple transformation of the differential equation, we see that $y(z) = f(z)/\Delta(z)^{k/12}$ is a solution of the differential equation

$$(2.5) \quad D_{z}^{2}y(z) = \left(\frac{k}{12}\right)^{2}E_{4}(z)y(z).$$

Analogues of (2.4) for the groups $\Gamma_{0}^{+}(N)$, $N = 2, 3$, and $\Gamma_{0}(N)$, $N = 2, 3, 4$, were obtained in [11] and [12], respectively. In Section 3 we will give a new proof of these results, stated in the same form as (2.5).

2.2. Atkin-Lehner decomposition of $\tilde{\mathcal{M}}_{k}^{<r}(\Gamma_{0}(N))$. Again, for convenience of non-specialists, we shall give a quick overview of the Atkin-Lehner decomposition of spaces of modular forms.

Let $N$ be a positive integer. It is easy to see that if $\gamma \in \text{SL}(2, \mathbb{R})$ normalizes $\Gamma_{0}(N)$, then $f \mapsto f|_{k}\gamma$ defines an automorphism of $\tilde{\mathcal{M}}_{k}(\Gamma_{0}(N))$ and $\tilde{\mathcal{M}}_{k}^{r}(\Gamma_{0}(N))$, where $f|_{k}\gamma$ is the usual slash operator of weight $k$ acting on $f$. Let $N(\Gamma_{0}(N))$ denote the normalizer of $\Gamma_{0}(N)$ in $\text{SL}(2, \mathbb{R})$. Inside $N(\Gamma_{0}(N))/\Gamma_{0}(N)$, there is a special subgroup, called the Atkin-Lehner subgroup, whose definition is given as follows.

Let $e$ be a positive divisor of $N$ such that $(e, N/e) = 1$. Such a divisor is often called a Hall divisor of $N$. We check that a matrix of the form

$$\frac{1}{\sqrt{e}} \begin{pmatrix} ae & b \\ cN & de \end{pmatrix}, \quad ade^{2} + bcN = e$$

normalizes $\Gamma_{0}(N)$. Let $W_{e}$ denote the set of such matrices. (In particular, $W_{1} = \Gamma_{0}(N)$.) It is easy to see that $W_{e} = w_{e}\Gamma_{0}(N)$ for any element $w_{e}$ of $W_{e}$. Also, if $e$ and $e'$ are two Hall divisors of $N$, $\gamma \in W_{e}$, and $\gamma' \in W_{e'}$, then $\gamma\gamma' \in W_{e''}$, where $e'' = ee'/e, e'$.

Hence, if we let

$$\Gamma_{0}^{+}(N) = \bigcup_{e|N, (e, N/e) = 1} W_{e},$$

then $\Gamma_{0}^{+}(N)$ is a subgroup of $N(\Gamma_{0}(N))$ and the quotient group $\Gamma_{0}^{+}(N)/\Gamma_{0}(N)$ is an elementary abelian 2-group of order $2^{m}$, where $m$ is the number of distinct prime divisors of $N$. This quotient group $\Gamma_{0}^{+}(N)/\Gamma_{0}(N)$ is called the Atkin-Lehner subgroup of $N(\Gamma_{0}(N))/\Gamma_{0}(N)$.

Elements of the Atkin-Lehner groups acts on many things related to $\Gamma_{0}(N)$. For example, they give rise to automorphisms of the modular curve $X_{0}(N)$. They also act on $\tilde{\mathcal{M}}_{k}(\Gamma_{0}(N))$ and $\tilde{\mathcal{M}}_{k}^{r}(\Gamma_{0}(N))$, as mentioned at the beginning of the section. In all case, they are called Atkin-Lehner involutions and denoted by $w_{e}$. Since every $w_{e}$ has order 1 or 2 in $N(\Gamma_{0}(N))/\Gamma_{0}(N)$ and they commute with each other, the space $\tilde{\mathcal{M}}_{k}(\Gamma_{0}(N))$ (respectively, $\tilde{\mathcal{M}}_{k}^{r}(\Gamma_{0}(N))$) decomposes into a direct sum of $2^{m}$ subspaces that are simultaneous eigenspaces with eigenvalues $\pm 1$ for all Atkin-Lehner involutions. For convenience, we introduce the following notation.

**Definition 2.6.** Let $G_{N}$ denote the group of characters of $\Gamma_{0}^{+}(N)/\Gamma_{0}(N)$, that is, let $G_{N}$ be the group of all homomorphisms from $\Gamma_{0}^{+}(N)/\Gamma_{0}(N)$ to $\{\pm 1\}$. We shall let $\epsilon_{0}$ denote the trivial character of $\Gamma_{0}^{+}(N)/\Gamma_{0}(N)$.

For $\epsilon \in G_{N}$, we let

$$\tilde{\mathcal{M}}_{k}(\Gamma_{0}^{+}(N), \epsilon) = \{ f \in \tilde{\mathcal{M}}_{k}(\Gamma_{0}(N)) : f|_{k}w_{e} = \epsilon(w_{e})f \text{ for all } e \}.$$
The notation \( \mathcal{M}_k^* (\Gamma_0^+ (N), \epsilon) \) is similarly defined. (The choice of the notation \( \mathcal{M}_k (\Gamma_0^+ (N), \epsilon) \) instead of \( \mathcal{M}_k (\Gamma_0 (N), \epsilon) \) is to stress that we are considering elements of \( \mathcal{M}_k (\Gamma_0 (N), \epsilon) \) as modular forms on \( \Gamma_0^+ (N) \) with characters.)

With this notation, the Atkin-Lehner decomposition we discussed above can be written as

\[
\mathcal{M}_k (\Gamma_0 (N)) = \bigoplus_{\epsilon \in G_N} \mathcal{M}_k (\Gamma_0^+ (N), \epsilon).
\]

We remark that if \( \Gamma \) is a subgroup of \( \Gamma_0^+ (N) \) containing \( \Gamma_0 (N) \), then

\[
\mathcal{M}_k (\Gamma) = \bigoplus_{\epsilon \in G_N, \Gamma \cap \Gamma_0 (N) \in \ker \epsilon} \mathcal{M}_k (\Gamma_0^+ (N), \epsilon)
\]

Thus, the dimension of each \( \mathcal{M}_k (\Gamma_0^+ (N), \epsilon) \) can be derived from the following general formula and the inclusion-exclusion principle.

**Proposition 2.7** ([16] Theorem 2.23). Let \( \Gamma \) be a subgroup of \( \text{SL}(2, \mathbb{R}) \) commensurable with \( \text{SL}(2, \mathbb{Z}) \). Suppose that the modular curve \( X (\Gamma) \) has genus \( g \), \( c \) cusps, and \( m \) elliptic points of order \( e_1, \ldots, e_m \), respectively. Then for a nonnegative even integer \( k \), we have

\[
\dim \mathcal{M}_k (\Gamma) = (k - 1)(g - 1) + \sum_{j=1}^{m} \left\lfloor \frac{k}{2} \left( 1 - \frac{1}{e_j} \right) \right\rfloor + \frac{ck}{2}.
\]

**Example 2.8.** Consider the case of \( \Gamma_0 (2) \). Since there is only one prime divisor here, we shall use the following more intuitive notations instead.

Let \( \mathcal{M}_k (\Gamma_0^+ (2), \pm) \) denote the Atkin-Lehner eigenspaces with eigenvalue \( \pm 1 \), respectively. We observe that \( \mathcal{M}_k (\Gamma_0^+ (2), +) \) is simply \( \mathcal{M}_k (\Gamma_0^+ (2)) \). To determine its dimension, we need to know how many cusps and elliptic points \( X_0 (2)/w_2 \) has.

The modular curve \( X_0 (2) \) has genus 0, 2 cusps, and an elliptic point \( (1 + i)/2 \) of order 2. Thus,

\[
\dim \mathcal{M}_k (\Gamma_0^+ (2)) = 1 + \left\lfloor \frac{k}{4} \right\rfloor.
\]

To determine how many cusps and elliptic points \( X_0 (2)/w_2 \) has, we note that \( X_0 (2) \to X_0 (2)/w_2 \) is a covering of degree 2. Thus, by the Riemann-Hurwitz formula, the covering ramifies at 2 points and these two points are fixed points of \( w_2 \). Now the matrix \( \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 2 & -1 \\ 2 & 0 \end{array} \right) \) fixes \( (1 + i)/2 \), i.e., the Atkin-Lehner \( w_2 \) fixes \( (1 + i)/2 \). Thus, \( (1 + i)/2 \) becomes an elliptic point of order 4 on \( X_0 (2)/w_2 \). Another fixed point of \( w_2 \) is \( i/\sqrt{2} \), fixed by \( \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 0 & -1 \\ 2 & 0 \end{array} \right) \).

Since we have found two ramified points, all other points are unramified. We conclude that \( X_0 (2)/w_2 \) has one cusp, one elliptic point of order 2, and one elliptic point of order 4. Hence,

\[
\dim \mathcal{M}_k (\Gamma_0^+ (2), +) = \dim \mathcal{M}_k (\Gamma_0^+ (2)) = 1 + \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{3k}{8} \right\rfloor.
\]

It follows that

\[
\dim \mathcal{M}_k (\Gamma_0^+ (2), -) = \dim \mathcal{M}_k (\Gamma_0 (2)) - \dim \mathcal{M}_k (\Gamma_0^+ (2), +) = \frac{k}{2} - \left\lfloor \frac{3k}{8} \right\rfloor.
\]

Now we consider the Atkin-Lehner decomposition of the spaces of nearly holomorphic modular forms and quasimodular forms. Just like \( \mathcal{M}_k (\Gamma_0 (N)) \), we can decompose \( \mathcal{M}_k^* (\Gamma_0 (N)) \) into a direct sum

\[
\mathcal{M}_k^* (\Gamma_0 (N)) = \bigoplus_{\epsilon \in G_N} \mathcal{M}_k^* (\Gamma_0^+ (N), \epsilon).
\]
Definition 2.9. For $\epsilon \in G_N$, we let $\overline{\mathcal{M}}_k(\Gamma_0^+(N), \epsilon)$ denote the space of quasimodular forms that are holomorphic parts of functions in $\mathcal{M}_k^+(\Gamma_0^+(N), \epsilon)$. Also, for a nonnegative integer $r$, we let $\overline{\mathcal{M}}_k^r(\Gamma_0^+(N), \epsilon)$ denote the subspace of $\overline{\mathcal{M}}_k(\Gamma_0^+(N), \epsilon)$ consisting of quasimodular forms of depth $\leq r$. When $\epsilon = \epsilon_0$ is the trivial character, we will often use the notation $\overline{\mathcal{M}}_k^r(\Gamma_0^+(N))$ in place of $\overline{\mathcal{M}}_k^r(\Gamma_0^+(N), \epsilon_0)$.

In addition, when $N = p^n$ is a prime power, we will also use the more intuitive notations $\overline{\mathcal{M}}_k^{\leq r}(\Gamma_0^+(N), +)$ (or simplify $\overline{\mathcal{M}}_k^{\leq r}(\Gamma_0^+(N))$) and $\overline{\mathcal{M}}_k^{\leq r}(\Gamma_0^+(N), -)$ to denote the spaces $\overline{\mathcal{M}}_k^r(\Gamma_0^+(N), \epsilon_0)$ and $\overline{\mathcal{M}}_k^r(\Gamma_0^+(N), \epsilon)$, where $\epsilon$ is the nontrivial element in $G_N$.

Remark 2.10. Let $N$ be a positive integer. Let

$$\phi(z) = \sum_{e|N, (e, N/e) = 1} eE_2(ez),$$

which is a scalar multiple of the logarithmic derivative of $\prod_e \Delta(ez)$ and hence a quasimodular form of weight 2 and depth 1 on $\Gamma_0^+(N)$. Then for $\epsilon \in G_N$, it is easy to see that

$$\overline{\mathcal{M}}_k^{\leq r}(\Gamma_0^+(N), \epsilon) = \bigoplus_{j=0}^r \phi^j \mathcal{M}_{k-2j}(\Gamma_0^+(N), \epsilon).$$

and hence

$$\dim \overline{\mathcal{M}}_k^{\leq r}(\Gamma_0^+(N), \epsilon) = \sum_{j=0}^r \dim \mathcal{M}_{k-2j}(\Gamma_0^+(N), \epsilon).$$

Note that the choice of $\phi$ is not unique in general. For example, when $N = 4$, we can choose $\phi(z) = E_2(2z)$. Then the statement $\overline{\mathcal{M}}_k^{\leq r}(\Gamma_0^+(4), \epsilon) = \bigoplus_{j=0}^r \phi^j \mathcal{M}_{k-2j}(\Gamma_0^+(4), \epsilon)$ still holds.

We will need the following property in the subsequent discussion.

Lemma 2.11. Let $\epsilon \in G_N$. We have

$$D_q \overline{\mathcal{M}}_k^{\leq r}(\Gamma_0^+(N), \epsilon) \subseteq \overline{\mathcal{M}}_k^{\leq r+1}(\Gamma_0^+(N), \epsilon).$$

Proof. This follows from Lemma 2.8. \hfill \square

2.3. Extremal quasimodular forms. Several examples given in the paper are involved with the notion of extremal quasimodular forms, introduced first in [8], so here we shall review their definition and discuss the existence and uniqueness of such quasimodular forms.

Definition 2.12. Given $\epsilon \in G_N$, let $\overline{\mathcal{M}} = \overline{\mathcal{M}}_k^{\leq r}(\Gamma_0^+(N))$ corresponding to $\epsilon$. An element $f \in \overline{\mathcal{M}}$ is said to be extremal if its vanishing order at $\infty$ is equal to $\dim \overline{\mathcal{M}} - 1$. We say $f$ is normalized if its leading Fourier coefficient is 1.

Remark 2.13. Note that in [10], Pellarin called a nonzero quasimodular form $f$ analytically extremal if its vanishing order at $\infty$ is the largest among all nonzero elements of the space $f$ belongs to. Pellarin’s definition of extremality seems to be closer to the literal meaning of the word extremal. However, we will stick to the definition introduced by Kaneko and Koike. It is clear that if $f$ is analytic extremal in a space $\mathcal{M}$ of quasimodular forms and its vanishing order at $\infty$ is $\dim \mathcal{M} - 1$, then it is extremal in the sense of Kaneko and Koike.
Lemma 2.14. Let $\Gamma$ be a subgroup of $SL(2, \mathbb{R})$ commensurable with $SL(2, \mathbb{Z})$ and $\chi$ be a character of $\Gamma$. Let $\phi$ be a quasimodular form of weight 2 and depth 1 on $\Gamma$, i.e.,

$$(cz + d)^{-2}\phi(\gamma z) = \phi(z) + \frac{\alpha c}{cz + d}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ for some nonzero complex number $\alpha$. For $f(z) = f_0(z) + \phi(z)f_1(z) \in \mathcal{M}_k^2(\Gamma, \chi)$, where $k$ is a positive integer and $f_j \in \mathcal{M}_{k-2j}(\Gamma, \chi)$, $j = 1, 2$, let

$$F_f(z) = \begin{pmatrix} zf(z) + \alpha f_1(z) \\ f(z) \end{pmatrix}$$

and define the Wronskian $W_f(z)$ by $W_f(z) = \det(F_f(z), F_f'(z))$. Then $W_f(z)$ is a modular form of weight $2k$ with character $\chi^2$ on $\Gamma$. Moreover, we have

$$v_\infty(W_f) = \begin{cases} 2v_\infty(f), & \text{if } v_\infty(f) \leq v_\infty(f_1), \\ v_\infty(f) + v_\infty(f_1), & \text{if } v_\infty(f) > v_\infty(f_1), \end{cases}$$

where for a modular form or a quasimodular form $g$ we let $v_\infty(g)$ denote the vanishing order of $g$ at $\infty$.

Note that in the case $\Gamma = SL(2, \mathbb{Z})$, a proof of the fact that $W_f(z)$ is a modular form of weight 2 can be found in [3]. (This is also mentioned in [10], although no proof is given.) Since our setting involves quasimodular forms with characters on general $\Gamma$, we shall present a proof here.

Proof of Lemma 2.14. First of all, by the definition of $W_f(z)$, we have

$$W_f(z) = -f(z)^2 + \alpha (f_0(z)f'(z) - f'_1(z)f(z)),$$

from which (2.6) immediately follows. Also, by (an extension of) Proposition 2.5(i), the function $W_f(z)$ belongs to $\mathcal{M}_{2k}^2(\Gamma, \chi^2)$. On the other hand, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we compute that

$$f(\gamma z) = \chi(\gamma)(cz + d)^k \left( f(z) + \frac{\alpha c}{cz + d} f_1(z) \right)$$

and

$$(\gamma z)f(\gamma z) + \alpha f_1(\gamma z) = \chi(\gamma)\frac{az + b}{cz + d}(cz + d)^k \left( f(z) + \frac{\alpha c}{cz + d} f_1(z) \right) + \chi(\gamma)\alpha(az + d)^{k-2} f_1(z).$$

Using the relation $ad - bc = 1$, we may simplify the last expression to

$$\chi(\gamma)(cz + d)^{k-1} ((az + b)f(z) + \alpha f_1(z)).$$

Hence,

$$F_f(\gamma z) = \chi(\gamma)(cz + d)^{k-1} \begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} F_f(z)$$
and
\[ F'_j(\gamma z) = \chi(\gamma) \left( \frac{a}{c} \frac{b}{d} \right) ((k-1)(cz+d)^k F_j(z) + (cz+d)^{k+1} F'_j(z)). \]

From the computation, we see that the Wronskian \( W_f(z) \) satisfies
\[ W_f(\gamma z) = \chi(\gamma)^2 (cz+d)^{2k} W_f(z). \]
That is, \( W_f(z) \) is a quasimodular form of weight \( 2k \) and depth \( \leq 2 \) with character \( \chi^2 \) on \( \Gamma \) that is actually modular. This proves that \( W_f(z) \in \mathcal{M}_{2k}(\Gamma, \chi^2) \). \( \square \)

We now prove an extension of [10] Theorem 1.3]

**Proposition 2.15.** Suppose that \( \widetilde{\mathcal{M}}_k^{\leq 1}(\Gamma_0^1(N), \epsilon) \) is an Atkin-Lehner subspace of \( \widetilde{\mathcal{M}}_k^{\leq 1}(\Gamma_0(N)) \) such that
\[ \dim \mathcal{M}_{2k}(\Gamma_0^+ (N)) = \dim \widetilde{\mathcal{M}}_k^{\leq 1}(\Gamma_0^1(N), \epsilon) \]
and the maximal vanishing order at \( \infty \) of a nonzero element of \( \mathcal{M}_{2k}(\Gamma_0^+ (N)) \) is equal to
\[ \dim \mathcal{M}_{2k}(\Gamma_0^+ (N)) - 1. \]
Then
(i) extremal quasimodular forms exist in \( \widetilde{\mathcal{M}}_k^{\leq 1}(\Gamma_0^1(N), \epsilon) \) and are unique up to scalars, and
(ii) if \( f \) is an extremal quasimodular form in \( \widetilde{\mathcal{M}}_k^{\leq 1}(\Gamma_0^1(N), \epsilon) \), then we have \( v_\infty(f) = v_\infty(W_f) \), where \( W_f \) is the Wronskian associated to \( f \) defined in Lemma 2.14.

**Proof.** The proof follows that of [10] Theorem 1.3. Set \( \widetilde{\mathcal{M}} = \mathcal{M}_k^{\leq 1}(\Gamma_0^1(N), \epsilon) \). Let
\[ \phi(z) = \sum_{e|N, (e, N/\epsilon) = 1} eE_2(\epsilon z), \]
which is a scalar multiple of the logarithmic derivative of \( \prod_e \Delta(\epsilon z) \). It is a holomorphic quasimodular form of weight 2 and depth 1 on \( \Gamma_0^1(N) \) and satisfies
\[ (\phi|z^\alpha)(z) = \phi(z) + \frac{\alpha c}{cz+d} \]
for all \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0^+(N) \) for some nonzero complex number \( \alpha \). Assume that \( f \) is an element of \( \widetilde{\mathcal{M}} \). According to Remark 2.10 we have \( f = f_0 + \phi f_1 \) for some \( f_j \in \mathcal{M}_{k-2j}(\Gamma_0^1(N), \epsilon), j = 1, 2 \). Let \( W_f(z) \) be Wronskian associated to \( f \) defined in Lemma 2.14. By the lemma, \( W_f(z) \) is a modular form in \( \mathcal{M}_{2k}(\Gamma_0^+ (N)) \). (Note that \( \epsilon^2 = \epsilon_0 \) for any \( \epsilon \in G_N \).) Moreover, [2.6] implies that \( W_f \neq 0 \) whenever \( f \neq 0 \). Another consequence of [2.6] is that, by the assumptions that [2.7] holds and that the maximal vanishing order of any nonzero element of \( \mathcal{M}_{2k}(\Gamma_0^+ (N)) \) is \( \dim \mathcal{M}_{2k}(\Gamma_0^+ (N)) - 1 \), we have
\[ \dim \mathcal{M} - 1 = \dim \mathcal{M}_{2k}(\Gamma_0^+ (N)) - 1 \geq v_\infty(W_f) \geq v_\infty(f) \]
for all nonzero elements \( f \) of \( \mathcal{M} \). This implies that the vanishing order at \( \infty \) of an analytically extremal quasimodular form in \( \mathcal{M} \) is \( \dim \mathcal{M} - 1 \) and hence extremal quasimodular forms exists in \( \mathcal{M} = \mathcal{M}_k^{\leq 1}(\Gamma_0^1(N), \epsilon) \) and are unique up to scalars. The argument above also shows that if \( f \) is an extremal quasimodular form in \( \mathcal{M}_k^{\leq 1}(\Gamma_0^1(N), \epsilon) \), then \( v_\infty(f) = v_\infty(W_f) \). \( \square \)

Now we use this result to prove the existence and uniqueness of extremal quasimodular forms in the case \( N = 2 \) or \( N = 3 \). Note that Sakai [11] already established the existence in the case of \( \mathcal{M}_k(\Gamma_0^1(N), +), N = 2, 3 \).
Corollary 2.16. Let $N = 2$ or $N = 3$. Then for all positive even integers $k$, extremal quasimodular forms exist in $\mathcal{M}_k^{<1}(\Gamma_0^+(N))$ and $\mathcal{M}_k^{<1}(\Gamma_0^+(N), -)$ and are unique up to scalars.

Proof. In Example 2.8 we have seen that

$$\dim \mathcal{M}_k(\Gamma_0(2)) = 1 + \left\lfloor \frac{k}{4} \right\rfloor,$$

(2.8)

$$\dim \mathcal{M}_k(\Gamma_0^+(2)) = 1 - \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{3k}{8} \right\rfloor,$$

and

$$\dim \mathcal{M}_k(\Gamma_0^+(2), -) = \frac{k}{2} - \left\lfloor \frac{3k}{8} \right\rfloor.$$

Hence, according to Remark 2.10, we have

$$\dim \mathcal{M}_k^{<1}(\Gamma_0^+(2)) = \begin{cases} 1 + k/4, & \text{if } k \equiv 0 \mod 8, \\ (k + 2)/4, & \text{if } k \equiv 2, 6 \mod 8, \\ k/4, & \text{if } k \equiv 4 \mod 8, \end{cases}$$

(2.9)

and

$$\dim \mathcal{M}_k^{<1}(\Gamma_0^+(2), -) = \begin{cases} k/4, & \text{if } k \equiv 0 \mod 8, \\ (k + 2)/4, & \text{if } k \equiv 2, 6 \mod 8, \\ 1 + k/4, & \text{if } k \equiv 4 \mod 8. \end{cases}$$

(2.10)

It follows that if $k \not\equiv 4 \mod 8$, then

$$\dim \mathcal{M}_{2k}(\Gamma_0^+(2)) = \dim \mathcal{M}_k^{<1}(\Gamma_0^+(2))$$

and if $k \not\equiv 0 \mod 8$, then

$$\dim \mathcal{M}_{2k}(\Gamma_0^+(2)) = \dim \mathcal{M}_k^{<1}(\Gamma_0^+(2), -).$$

The condition (2.7) holds for these spaces.

For the other condition of Proposition 2.15, we observe that the condition is equivalent to there being a basis $\{g_j\}$ for $\mathcal{M}_{2k}(\Gamma_0^+(2))$ such that $v_\infty(g_j) = j - 1$ for each $j = 1, \ldots, \dim \mathcal{M}_{2k}(\Gamma_0^+(2))$. Indeed, we find that $g_j(z) = M_4(z)^{k/2-2j}M_8(z)^j$ for $j = 0, \ldots, \left\lfloor k/4 \right\rfloor$, where

$$M_4(z) = \frac{1}{5}(4E_4(2z) + E_4(z)), \quad M_8(z) = \eta(z)^8 \eta(2z)^8,$$

form such a basis. Therefore, by Proposition 2.15, extremal quasimodular forms exist and are unique up to scalars in the spaces above. For the remaining two cases $\mathcal{M}_k^{<1}(\Gamma_0^+(2))$ for $k \equiv 4 \mod 8$ and $\mathcal{M}_k^{<1}(\Gamma_0^+(2), -)$ for $k \equiv 0 \mod 8$, we observe that when $k \equiv 4 \mod 8$, we have

$$\dim \mathcal{M}_k^{<1}(\Gamma_0^+(2)) = \dim \mathcal{M}_k^{<1}(\Gamma_0^+(2), -).$$

and the map

$$f(z) \mapsto M_4(z)f(z)$$

defines an isomorphism between the two spaces. It is clear that a quasimodular form $f(z)$ is extremal in $\mathcal{M}_k^{<1}(\Gamma_0^+(2))$ if and only if $M_4(z)f(z)$ is extremal in $\mathcal{M}_k^{<1}(\Gamma_0^+(2), -)$. Since we have seen that extremal quasimodular forms exist and are unique up to scalars in the former space, the same thing holds for the latter space. The same argument works for the last case $\mathcal{M}_k^{<1}(\Gamma_0^+(2), -)$, $k \equiv 0 \mod 8$. This proves the case $N = 2$. 


For the case $N = 3$, we note that $\Gamma_0(3)$ has two cusps and one elliptic point of order 3, while $\Gamma_0^+(3)$ has one cusp, one elliptic point of order 2, represented by $i/\sqrt{3}$ with stabilizer subgroup generated by $\frac{1}{\sqrt{3}} \begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix}$, and one elliptic point of order 6, represented by $(3 + \sqrt{-3})/6$ with stabilizer subgroup generated by $\frac{1}{\sqrt{3}} \begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix}$. Thus, by Proposition 2.7 we have

$$\dim \mathcal{M}_k(\Gamma_0(3)) = 1 + \left\lfloor \frac{k}{3} \right\rfloor,$$

(2.11) $$\dim \mathcal{M}_k(\Gamma_0^+(3)) = 1 - \frac{k}{2} + \left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{5k}{12} \right\rfloor,$$

$$\dim \mathcal{M}_k(\Gamma_0^+(3), -) = \frac{k}{2} + \left\lfloor \frac{k}{3} \right\rfloor - \left\lfloor \frac{k}{4} \right\rfloor - \left\lfloor \frac{5k}{12} \right\rfloor.$$  

From these formulas, we deduce that

$$\dim \widetilde{\mathcal{M}}_{\mathcal{L}}^1(\Gamma_0^+(3)) = \begin{cases} 
1 + k/3, & \text{if } k \equiv 0 \mod 12, \\
(k + 1)/3, & \text{if } k \equiv 2, 8 \mod 12, \\
(k - 1)/3, & \text{if } k \equiv 4 \mod 12, \\
k/3, & \text{if } k \equiv 6 \mod 12, \\
(k + 2)/3, & \text{if } k \equiv 10 \mod 12,
\end{cases}$$

and

$$\dim \widetilde{\mathcal{M}}_{\mathcal{L}}^1(\Gamma_0^+(3), -) = \begin{cases} 
k/3, & \text{if } k \equiv 0 \mod 12, \\
(k + 1)/3, & \text{if } k \equiv 2, 8 \mod 12, \\
(k + 2)/3, & \text{if } k \equiv 4 \mod 12, \\
1 + k/3, & \text{if } k \equiv 6 \mod 12, \\
(k - 1)/3, & \text{if } k \equiv 10 \mod 12,
\end{cases}$$

Therefore, the condition (2.7) in Proposition 2.15 is met for $\widetilde{\mathcal{M}}_{\mathcal{L}}^1(\Gamma_0^+(3))$ with $k \equiv 0, 2, 8, 10 \mod 12$ and $\widetilde{\mathcal{M}}_{\mathcal{L}}^1(\Gamma_0^+(3), -)$ with $k \equiv 2, 4, 6, 8 \mod 12$. For the other condition of the proposition, we may use

$$M_4(z) = \frac{1}{10} (9E_4(3z) + E_4(z)),$$

$$M_6(z) = \frac{1}{2} (3E_2(3z) - E_2(z)) \eta(z)^6 \eta(3z)^6,$$

$$M_{12}(z) = \eta(z)^{12} \eta(3z)^{12}$$

to construct a basis $\{g_j\}$ for $\mathcal{M}_{2k}(\Gamma_0^+(3))$ with the property $v_\infty(g_j) = j - 1$ for all $j = 1, \ldots, \dim \mathcal{M}_{2k}(\Gamma_0^+(3))$. Hence, the second condition of Proposition 2.15 also holds and we conclude that extremal quasimodular forms exist in $\widetilde{\mathcal{M}}_{\mathcal{L}}^{\leq 4}(\Gamma_0^+(3))$ for $k \equiv 0, 2, 8, 10 \mod 12$ and $\widetilde{\mathcal{M}}_{\mathcal{L}}^{\leq 4}(\Gamma_0^+(3), -)$ for $k \equiv 2, 4, 6, 8 \mod 12$ and are unique up to scalars.

Finally, similar to the case $N = 2$, for the remaining cases, we observe that

$$\dim \widetilde{\mathcal{M}}_{\mathcal{L}}^{\leq 4}(\Gamma_0^+(3)) = \dim \widetilde{\mathcal{M}}_{\mathcal{L}}^{\leq 4}(\Gamma_0^+(3), -) = \dim \mathcal{M}_{\mathcal{L}}^1(\Gamma_0^+(3), -), \text{ for } k \equiv 10 \mod 12,$$

(2.12) $$\dim \widetilde{\mathcal{M}}_{\mathcal{L}}^{\leq 4}(\Gamma_0^+(3), -) = \dim \mathcal{M}_{\mathcal{L}}^1(\Gamma_0^+(3), -), \text{ for } k \equiv 10 \mod 12.$$
We let \( \varrho_k \) be also defined as modular forms on \( \Gamma \). Forms of weight \( k \) \( R \) and \( M \) will also appear as solutions of modular differential equations when \( \eta \). We conclude that extremal quasimodular forms exist and are unique up to scalars in all \( \varrho_k \) \( \Gamma \). From these observations we define an isomorphism for the pairs of spaces in (2.13), while the map \( f(z) \mapsto (9E_4(3z) + E_4(z))f(z) \) defines an isomorphism for the pairs of spaces in (2.13). From these observations we conclude that extremal quasimodular forms exists and are unique up to scalars in all \( \varrho_k \) \( \Gamma \). Similar to the case of \( \Gamma \), the Atkin-Lehner involutions normalize \( \Gamma \). Hence, we have

\[
\dim \mathcal{M}_k(\Gamma_1(3)) = 1 + \left\lfloor \frac{k}{3} \right\rfloor.
\]

Hence, we have

\[
\sum_{k=0}^{\infty} \dim \mathcal{M}_k(\Gamma_1(3))x^k = \frac{1}{(1-x)(1-x^3)},
\]

and the graded ring \( \bigoplus_{k=0}^{\infty} \mathcal{M}_k(\Gamma_1(3)) \) of modular forms is freely generated by

\[
M_1(z) = \sum_{(m,n) \in \mathbb{Z}^2} q^{m^2 + mn + n^2}, \quad q = e^{2\pi iz},
\]

and

\[
M_3(z) = \frac{9E_4(3z) - E_4(z)}{8M_1(z)} = \frac{\eta(z)^9}{\eta(3z)^3} - \frac{27\eta(3z)^9}{\eta(z)^3}.
\]

2.4. Quasimodular forms of odd weights on \( \Gamma_1(3) \). In this paper, quasimodular forms of odd weights on

\[
\Gamma_1(3) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) : c \equiv 0 \ (mod \ 3), \ a, d \equiv 1 \ (mod \ 3) \right\}
\]

will also appear as solutions of modular differential equations when \( \Gamma = \Gamma_0^+(3) \), so we will review their properties in this section.

For a nonnegative integer \( k \) (even or odd), let \( \mathcal{M}_k(\Gamma_1(3)) \) denote the set of modular forms of weight \( k \) on \( \Gamma_1(3) \). Note that when \( k \) is odd, modular forms of weight \( k \) on \( \Gamma_1(3) \) can also be defined as modular forms on \( \Gamma_0(3) \) with nebentype character \( \chi(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \left( \frac{a}{d} \right) \). By Theorem 2.25 of [16], we have

\[
\dim \mathcal{M}_k(\Gamma_1(3)) = 1 + \left\lfloor \frac{k}{3} \right\rfloor.
\]

Similar to the case of \( \Gamma_0(N) \), the Atkin-Lehner involutions normalize \( \Gamma_1(N) \). Hence, we may decompose \( \mathcal{M}_k(\Gamma_1(3)) \) into eigenspaces of the Atkin-Lehner involution \( w_3 \). Let \( S = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix} \). Because \( S^2 = -I \), the eigenvalues of \( w_3 \) in the case of odd \( k \) can only be \( \pm i \). For convenience, we shall consider the eigenfunctions as modular forms on \( \Gamma_0^+(3) \) with characters. More concretely, we recall that \( \Gamma_0^+(3) \) is generated by

\[
S = \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix}, \quad R = TS = \begin{pmatrix} 3 & -1 \\ 3 & 0 \end{pmatrix},
\]

where \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). They satisfy \( S^2 = R^6 = -I \) and there is no other relation between \( S \) and \( R \). In other words, a presentation of the group \( \Gamma_0^+(3) \) is \( \langle S, R : S^4 = 1, S^2 = R^6 \rangle \).

We let \( \chi \) be the character of \( \Gamma_0^+(3) \) such that

\[
\chi(S) = \chi(R) = -i,
\]
and \( \overline{\chi} \) be its complex conjugate. Then the eigenspace of \( w_3 \) with eigenvalue \(-i\) (respectively, \( i\)) can be identified with \( \mathcal{M}(\Gamma_0^+ (3), \chi) \) (respectively, \( \mathcal{M}(\Gamma_0^+ (3), \overline{\chi}) \)). Using the transformation formula for the Dedekind eta function (see [17, Pages 125–127]), we can show that

\[
M_3(z) \in \mathcal{M}(\Gamma_0^+ (3), \chi)
\]

and hence also

\[
M_1(z) \in \mathcal{M}(\Gamma_0^+ (3), \chi),
\]

where \( M_1(z) \) and \( M_3(z) \) are defined by (2.15) and (2.16), respectively.

**Lemma 2.17.** Let \( k \) be a positive odd integer. Then

\[
\dim \mathcal{M}_k(\Gamma_0^+ (3), \chi) = \begin{cases} 
(k + 5)/6, & \text{if } k \equiv 1 \text{ mod } 12, \\
(k + 3)/6, & \text{if } k \equiv 3, 9 \text{ mod } 12, \\
(k + 1)/6, & \text{if } k \equiv 5, 11 \text{ mod } 12, \\
(k - 1)/6, & \text{if } k \equiv 7 \text{ mod } 12,
\end{cases}
\]

and

\[
\dim \mathcal{M}_k(\Gamma_0^+ (3), \overline{\chi}) = \begin{cases} 
(k - 1)/6, & \text{if } k \equiv 1 \text{ mod } 12, \\
(k + 3)/6, & \text{if } k \equiv 3, 9 \text{ mod } 12, \\
(k + 1)/6, & \text{if } k \equiv 5, 11 \text{ mod } 12, \\
(k + 5)/6, & \text{if } k \equiv 7 \text{ mod } 12.
\end{cases}
\]

**Proof.** Observe that \( \mathcal{M}_{k-1}(\Gamma_0^+ (3)) \cup \mathcal{M}_{k-1}(\Gamma_0^+ (3), -) = \mathcal{M}_k(\Gamma_0^+ (3)) \) and \( \mathcal{M}_k(\Gamma_0^+ (3), \chi) \cup \mathcal{M}_k(\Gamma_0^+ (3), \overline{\chi}) = \mathcal{M}_k(\Gamma_1 (3)) \). Also,

\[
M_1, \mathcal{M}_{k-1}(\Gamma_0^+ (3)) \subseteq \mathcal{M}_k(\Gamma_0^+ (3), \chi), \\
M_1, \mathcal{M}_{k-1}(\Gamma_0^+ (3), -) \subseteq \mathcal{M}_k(\Gamma_0^+ (3), \overline{\chi}).
\]

When \( k \) is not a multiple of 3, by (2.14), we have

\[
\dim \mathcal{M}_k(\Gamma_1 (3)) = \dim \mathcal{M}_{k-1}(\Gamma_0 (3)).
\]

Hence, both inclusions above are equality. When \( 3 | k \), we use

\[
M_1, \mathcal{M}_{k-1}(\Gamma_0^+ (3), \chi) \subseteq \mathcal{M}_{k+1}(\Gamma_0^+ (3), -), \\
M_1, \mathcal{M}_{k-1}(\Gamma_0^+ (3), \overline{\chi}) \subseteq \mathcal{M}_{k+1}(\Gamma_0^+ (3))
\]

instead. Again, by the dimension formula (2.14), both inclusions are equality. This proves the lemma. \( \square \)

Passing to quasimodular forms, we have the following dimension formulas.

**Lemma 2.18.** Let \( k \) be a positive odd integer. We have

\[
\dim \mathcal{M}^{-1}_{k-1}(\Gamma_0^+ (3), \chi) = \begin{cases} 
(k + 2)/3, & \text{if } k \equiv 1 \text{ mod } 12, \\
1 + k/3, & \text{if } k \equiv 3 \text{ mod } 12, \\
(k + 1)/3, & \text{if } k \equiv 5, 11 \text{ mod } 12, \\
(k - 1)/3, & \text{if } k \equiv 7 \text{ mod } 12, \\
k/3, & \text{if } k \equiv 9 \text{ mod } 12,
\end{cases}
\]
we find that $k$ and depth $f$ for some $f$ in Proposition 2.19.

for some complex number $\alpha$ the Wronskian associated to $f$ is unique up to scalars. For the remaining four spaces, we use the facts that $(3.3)$ on $\mathcal{M}(\Gamma_0^+ (3), \chi)$ is the quasimodular form of weight $\frac{k}{6}$, $\tilde{\dim} \mathcal{M}_k(\Gamma_0^+ (3)) = \mathcal{M}_k(\Gamma_0^+ (3), \chi)$ and $\tilde{\dim} \mathcal{M}_k(\Gamma_0^+ (3), \chi)$ is a solution of a modular differential equation. Throughout the section, we assume that $\Gamma$ is a discrete subgroup of $\text{SL}(2, \mathbb{R})$ that is commensurable with $\text{SL}(2, \mathbb{Z})$, and $\phi$ is the quasimodular form of weight 2 and depth 1 appearing in Proposition 2.5, i.e.,

$$(cz + d)^{-2} \phi(\gamma z) = \phi(z) + \frac{\alpha c}{cz + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

for some complex number $\alpha \neq 0$. Suppose that $f(z)$ is a quasimodular form of weight $k$ and depth 1 with character $\chi$ of finite order on $\Gamma$. We have

$$f(z) = f_1(z) \phi(z) + f_0(z)$$

for some $f_0 \in \mathcal{M}(\Gamma, \chi)$ and $f_1 \neq 0 \in \mathcal{M}_{k-1}(\Gamma, \chi)$. Let $W_f(z)$ be the Wronskian associated to $f$ defined in Lemma 2.14. It is a modular form of weight $2k$ with character $\chi^2$ on $\Gamma$. Define

$$g_1(z) = \frac{zf(z) + \alpha f_1(z)}{\sqrt{W_f(z)}} \quad \text{and} \quad g_2(z) = \frac{f(z)}{\sqrt{W_f(z)}}$$

Proposition 2.19. For each positive odd integer $k$, an extremal quasimodular form exists in $\mathcal{M}_k(\Gamma_0^+ (3), \chi)$ or $\mathcal{M}_k(\Gamma_0^+ (3), \overline{\chi})$ and is unique up to scalars.

Proof. For a quasimodular form $f$ in $\mathcal{M}_k^1(\Gamma_0^+ (3), \chi)$ or $\mathcal{M}_k^1(\Gamma_0^+ (3), \overline{\chi})$, let $W_f$ be the Wronskian associated to $f$ defined in Lemma 2.14. By the lemma, we have $W_f \in \mathcal{M}_{2k}(\Gamma_0^+ (3), \chi^2) = \mathcal{M}_{2k}(\Gamma_0^+ (3), -)$. Since

$$\dim \mathcal{M}_{2k}(\Gamma_0^+ (3), -) = \begin{cases} (k + 2)/3, & \text{if } k \equiv 1 \text{ mod } 6, \\
1 + k/3, & \text{if } k \equiv 3 \text{ mod } 6, \\
(k + 1)/3, & \text{if } k \equiv 5 \text{ mod } 6,
\end{cases}$$

we find that

$$\dim \mathcal{M}_k^1(\Gamma_0^+ (3), \chi) = \dim \mathcal{M}_{2k}(\Gamma_0^+ (3), -)$$

for $k \equiv 1, 3, 5, 11 \text{ mod } 12$, and

$$\dim \mathcal{M}_k^1(\Gamma_0^+ (3), \overline{\chi}) = \dim \mathcal{M}_{2k}(\Gamma_0^+ (3), -)$$

for $k \equiv 5, 7, 9, 11 \text{ mod } 12$. Hence, extremal quasimodular forms exist in these spaces and is unique up to scalars. For the remaining four spaces, we use the facts that

$$M_1 \mathcal{M}_{k-1}(\Gamma_0^+ (3)) = \mathcal{M}_k^1(\Gamma_0^+ (3), \chi)$$

for $k \equiv 1, 7, 9, 11 \text{ mod } 12$, and

$$M_1 \mathcal{M}_{k-1}(\Gamma_0^+ (3), -) = \mathcal{M}_k^1(\Gamma_0^+ (3), \overline{\chi})$$

for $k \equiv 1, 3, 5, 7 \text{ mod } 12$ to obtain the same conclusion. □

3. Modular ordinary differential equations

The purpose of this section is to prove that for any quasimodular form $f(z)$ of depth 1, $f(z)/\sqrt{W_f(z)}$ is a solution of a modular differential equation. Throughout the section, we assume that $\Gamma$ is a discrete subgroup of $\text{SL}(2, \mathbb{R})$ that is commensurable with $\text{SL}(2, \mathbb{Z})$, and $\phi$ is the quasimodular form of weight 2 and depth 1 appearing in Proposition 2.5, i.e.,

$$(cz + d)^{-2} \phi(\gamma z) = \phi(z) + \frac{\alpha c}{cz + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$
Then we have \( \det \begin{pmatrix} g_1 & g_2 \\ g_1' & g_2' \end{pmatrix} = 1 \) and hence \( \det \begin{pmatrix} g_1 & g_2 \\ g_1' & g_2' \end{pmatrix} = 0 \). Therefore, both \( g_1 \) and \( g_2 \) are solutions of

\[
y''(z) = -4\pi^2 Q(z) y(z),
\]

where

\[
Q(z) = -\frac{1}{4\pi^2} g_1''(z) = -\frac{1}{4\pi^2} g_2''(z).
\]

Clearly, \( Q(z) \) is a single-valued meromorphic function not identically 0 on \( \mathbb{H} \).

**Proposition 3.1.** Let \( Q(z) \) be defined by (3.5). Then \( Q(z) \) is a meromorphic function on \( \mathbb{H} \) and (3.4) is Fuchsian. Moreover, if \( z_0 \in \mathbb{H} \) is a pole of \( Q(z) \), then \( z_0 \) is an apparent singularity for (3.4).

**Proof.** Clearly, \( g_i(z) = (z - z_0)^{\alpha_i} (c_i + O(z - z_0)) \) near \( z_0 \) for some \( c_i \neq 0 \) and \( \alpha_i \in \frac{1}{2}\mathbb{Z} \). Thus,

\[
Q(z) = \alpha_i(\alpha_i - 1)(z - z_0)^{-2} + O((z - z_0)^{-1})
\]

near \( z_0 \) and we have \( \alpha_1(\alpha_1 - 1) = \alpha_2(\alpha_2 - 1) \), which implies that the local exponents at \( z_i \) are \( \alpha_i \) and \( -\alpha_i + 1 \). (The two sets \( \{\alpha_1, -\alpha_1 + 1\} \) and \( \{\alpha_2, -\alpha_2 + 1\} \) are identical.) Hence (3.4) is Fuchsian and the difference of the local exponents are \( 1 + 2\alpha_i \in \mathbb{Z} \). Since \( g_1 \) and \( g_2 \) are linearly independent and has no logarithmic singularities near \( z_0 \), \( z_0 \) is obviously an apparent singularity. \( \square \)

**Remark 3.2.** Suppose that (3.4) has a solution \( y(z) \) with a logarithmic singularity near \( z_0 \). Then the local exponents \( \rho_1 \) and \( \rho_2 \) at \( z_0 \) satisfy \( \rho_1 - \rho_2 \in \mathbb{Z} \), and \( z_0 \) is not apparent. This remark is used in the proof of Theorem 3.3. See below.

The following result is a generalization of Theorem 1.1

**Theorem 3.3.** Let \( Q(z) \) be defined by (3.5). Then the following hold.

(i) The function \( Q(z) \) is a meromorphic modular form of weight 4 (with trivial character) on \( \Gamma \).

(ii) Moreover, \( Q(z) \) is holomorphic at \( \infty \) with \( Q(\infty) = \kappa_\infty^2 / N^2 \) for some \( \kappa_\infty \in \frac{1}{2}\mathbb{Z}_{\geq 0} \), where \( N \) is the width of the cusp \( \infty \) of \( \Gamma \). Also, the cusp \( \infty \) is not apparent for (3.4).

(iii) For any cusp \( s \), \( Q(z) \) is holomorphic at \( s \) with \( Q(s) \geq 0 \) and \( s \) is not apparent for (3.4).

**Proof.** To prove the modularity of \( Q(z) \), we consider \( y(z) = (g_2|\gamma)(z) := (cz + d)g_2(\gamma z) \) for \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma \). By the definition of \( g_2 \), \( y(z) \) is well-defined up to \( \pm 1 \). A direct computation show that

\[
\left( g_2|\gamma \right)''(z) = \left( g_2''|\gamma \right)(z)
\]

(which is a special case of Bol’s identity (11)), which implies that

\[
y''(z) = \left( Q|\gamma \right)(z)y(z)
\]
On the other hand, we have
\[
y(z) = (cz + d)g_2(\gamma z) = \frac{(cz + d)f(\gamma z)}{\sqrt{W_f(\gamma z)}} \\
= \pm \chi(\gamma)(cz + d)(f(z) + \alpha cf_1(z)/(cz + d))^{\frac{1}{2}} \\
= \pm (cg_1(z) + dg_2(z)).
\]
Thus, \(y(z)\) is also a solution of (3.4), which implies that \((Q|_{a\gamma})(z) = Q(z)\) for all \(\gamma \in \Gamma\). This proves (i).

Let \(N\) be the width of the cusp \(\infty\) and \(q_N = e^{2\pi i z/N}\). By (2.6), the order of \(g_2\) at \(\infty\) is
\[
\kappa_{\infty} = \frac{1}{2} \nu_\infty(f) - \frac{1}{2} \min(\nu_\infty(f), \nu_\infty(f_1)),
\]
which is in \({\frac{1}{2}}Z\geq 0\). (Note that when \(\chi\left(\begin{smallmatrix} N \\ 1 \end{smallmatrix}\right)\) \(\neq 1\), \(\nu_\infty(f)\) and \(\nu_\infty(f_1)\) are not integers, but we still have \(\nu_\infty(f) \equiv \nu_\infty(f_1) \mod 1\) nonetheless.) Since
\[
Q(z) = \frac{1}{4\pi^2} \frac{g_2''(z)}{g_2(z)} = \frac{1}{N^2} \left(\frac{q_N d}{d_{\infty}}\right)^2 \frac{g_2}{g_2},
\]
we find that \(Q(\infty) = \kappa_{\infty}^2/N^2\).

Notice that \(z\) appears in the expression of \(g_1(z)\) and \(g_1(z)\) is another solution of (3.4). This obviously implies that \(\infty\) is not apparent. We now prove (iii).

Let \(s \neq \infty\) be another cusp of \(\Gamma\) and \(\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\) be an element of \(\text{SL}(2, \mathbb{Z})\) such that \(\sigma\infty = s\). Regarding \(f(z)\) as a quasimodular form on \(\Gamma' = \ker \chi \cap \text{SL}(2, \mathbb{Z})\), we have \(f(z) = h_1(z)E_2(z) + h_0(z)\) for some \(h_j \in \mathbb{M}_{k-2j}(\Gamma')\). We have
\[
(g_2|_{-1}\sigma)(z) = \frac{(cz + d)f(\sigma z)}{\sqrt{W_f(\sigma z)}} = \frac{(cz + d)(f|_{k}\sigma)(z)}{\sqrt{W_f|_{2k}\sigma}(z)} \\
= \frac{(cz + d)((h_1|_{k-2}\sigma)E_2(z) + (h_0|_{k}\sigma)) + 6c(h_1|_{k-2}\sigma)/\pi i}{\sqrt{W_f|_{2k}\sigma}(z)}.
\]
Since all terms in the above expression except for \(cz + d\) has a \(q_\infty\)-expansion, where \(M\) is the width of the cusp \(s\), we see that \(z\) appears in the expression of \(g_2|_{-1}\gamma\). (Note that \(c \neq 0\) since \(s\) is assumed to be a cusp different from \(\infty\).) This implies that \(s\) is not apparent and the local exponents of (3.4) at \(s\) are \(\pm \kappa_s\) for some \(\kappa_s \in \frac{1}{2}Z\geq 0\) (see Remark 3.2). As discussed earlier, near the cusp \(s\), there is a solution \(y_+(z)\) of (3.4) of the form
\[
y_+(z) = q_M^{\kappa_s} \left(1 + \sum_{j \geq 1} c_j q_M^j\right).
\]
By the same computation as (3.6), we see that \(Q(s) = M^{-2}\kappa_s^2 \geq 0\). This completes the proof of the theorem.

**Remark 3.4.** We remark that for a given point \(z_0 \in \mathbb{H}\), the local exponents of (3.4) at \(z_0\) and those at \(\gamma z_0\) are the same for any \(\gamma \in \Gamma\). Also, if \(z_0\) is apparent, then any equivalent point \(\gamma z_0\), \(\gamma \in \Gamma\), is also apparent. See [4] for a proof of this fact.

For the special case \(\Gamma\) is one of \(\text{SL}(2, \mathbb{Z})\), \(\Gamma_0^+(2)\), and \(\Gamma_0^+(3)\), we have the following informations about local exponents at elliptic points. We first prove a lemma.

**Lemma 3.5.** Let \(\phi(z)\) be a quasimodular form of weight 2 and depth 1 on \(\Gamma\). Then \(\phi(z)\) does not vanish at any elliptic point of \(\Gamma\).
Proof. Let \( \tilde{\phi}(z) \) be the nearly holomorphic modular form of weight 2 corresponding to \( \phi(z) \). We have

\[
\tilde{\phi}(z) = \phi(z) + \frac{\alpha}{z - \bar{z}}
\]

for some nonzero complex number \( \alpha \). Since \( \tilde{\phi}(z) \) transforms like a modular form of weight 2 on \( \Gamma \), it vanishes at all elliptic points of \( \Gamma \). It follows that \( \phi(z) \) does not vanish at any elliptic point of \( \Gamma \). \( \square \)

**Proposition 3.6.** We have the following properties about local exponents of \( (3.4) \) at an elliptic point.

(i) Let \( f \) be a quasimodular form of depth 1 in \( \tilde{\mathcal{M}}^\leq_1(\text{SL}(2, \mathbb{Z})) \). Let \( 1/2 \pm \kappa_\rho \) be the local exponents of \( (3.4) \) at the elliptic points \( \rho = (1 + \sqrt{-3})/2 \) and \( i = \sqrt{-1} \). Then \((2\kappa_\rho, 3) = (2\kappa_\rho, 2) = 1 \).

(ii) Let \( f \) be a quasimodular form of depth 1 in \( \tilde{\mathcal{M}}^\leq_1(\Gamma_0^+(2), \pm) \). Let \( 1/2 \pm \kappa_{\rho_j} \) be the local exponents of \( (3.4) \) at the elliptic points \( \rho_1 = i/\sqrt{2} \) and \( \rho_2 = (1 + i)/2 \) of \( \Gamma_0^+(2) \). Then \((2\kappa_{\rho_1}, 2) = (2\kappa_{\rho_2}, 4) = 1 \).

(iii) Let \( f \) be a quasimodular form of depth 1 in \( \tilde{\mathcal{M}}^\leq_1(\Gamma_0^+(3), \chi^m) \) with \( m \in \{0, 1, 2, 3\} \) and \( k \equiv m \mod 2 \) that is not a modular form, where \( \chi \) is the character of \( \Gamma_0^+(3) \) defined by \((2.17)\). Let \( 1/2 \pm \kappa_{\rho_j} \) be the local exponents of \( (3.4) \) at the elliptic points \( \rho_1 = i/\sqrt{3} \) and \( \rho_2 = (3 + \sqrt{-3})/6 \) of \( \Gamma_0^+(3) \). Then \((2\kappa_{\rho_1}, 2) = (2\kappa_{\rho_2}, 6) = 1 \).

Proof. We first make a general remark that since the local exponents of \( (3.4) \) at a point \( z_0 \in \mathbb{H} \) are of the form \( 1/2 \pm \kappa_{z_0} \) for some \( \kappa_{z_0} \in \frac{1}{2}\mathbb{N} \), if the order of \( f(z)/\sqrt{W_f(z)} \) at \( z_0 \) is \(-n_{z_0}\) for some \( n_{z_0} \in \frac{1}{2}\mathbb{Z}_{\geq 0} \), then we must have \( 1/2 - \kappa_{z_0} = -n_{z_0} \), i.e., \( \kappa_{z_0} = 1/2 + n_{z_0} \).

Assume that \( f(z) \in \tilde{\mathcal{M}}^\leq_1(\text{SL}(2, \mathbb{Z})) \) that is not a modular form. Write it as \( f(z) = f_1(z)E_2(z) + f_0(z) \) with \( f_j \in \tilde{\mathcal{M}}_{k-2j}(\text{SL}(2, \mathbb{Z})) \). We first observe that if \( f_0(z) \) and \( f_1(z) \) have a common zero at \( i \) (respectively, \( \rho \)), then \( f_j(z)/E_0(z) \) (respectively, \( f_1(z)/E_4(z) \)), \( j = 1, 2 \), are holomorphic modular forms, and \( f(z)/E_0(z) \) (respectively, \( f(z)/E_4(z) \)) is a quasimodular form. Also, it is easy to check that \( W_f(z) = E_0(z)^2W_{f/E_0}(z) \) (respectively, \( W_f(z) = E_4(z)^2W_{f/E_4}(z) \)), so the differential equation \( (3.4) \) from \( f(z) \) and that from \( f(z)/E_0(z) \) (respectively, \( f(z)/E_4(z) \)) are the same. Therefore, without loss of generality, we may assume that \( f_0(z) \) and \( f_1(z) \) have no common zero at \( i \) or \( \rho \). We note that this assumption forces that the weight \( k \) to be congruent to 0 or 2 modulo 6 since any modular form on \( \text{SL}(2, \mathbb{Z}) \) whose weight is not divisible by 6 must have a zero at \( \rho \).

By Lemma \(3.5\), \( E_2(z) \) does not vanish at \( \rho \) and \( i \). Then using

\[
(f|_{k\gamma})(z) = f(z) + \frac{12cf_1(z)}{2\pi i(\gamma z + d)}.
\]

and the assumption that \( f_0(z) \) and \( f_1(z) \) have no common zero at \( i \) and \( \rho \), we see that there exists an element \( \gamma \) in \( \text{SL}(2, \mathbb{Z}) \) such that \( f(\gamma \rho), f(\gamma i) \neq 0 \). Since the local exponents of \( (3.4) \) at \( \rho \) (respectively, \( i \)) and at \( \gamma \rho \) (respectively, \( \gamma i \)) are the same, by the remark we made at the beginning of the proof, we have

\[
2\kappa_\rho = 1 + \text{ord}_\rho W_f(z) = 1 + \text{ord}_\rho W_f(z)
\]

(respectively, \( 2\kappa_i = 1 + \text{ord}_i W_f(z) \)).
Now consider first the case where the weight $k$ is congruent to 0 modulo 6. In this case, the weight of $W_f(z)$ is a multiple of 12 and hence
\[
\frac{\ord_3 W_f(z)}{3} + \frac{\ord_2 W_f(z)}{2}
\]
must be an integer. It follows that $3\mid \ord_3 W_f$ and $2\mid \ord_2 W_f$ and therefore $(2\kappa_{\rho}, 3) = (2\kappa_1, 2) = 1$. Similarly, when the weight $k$ is congruent to 2 modulo 6, we have
\[
\frac{\ord_3 W_f(z)}{3} + \frac{\ord_2 W_f(z)}{2} \equiv \frac{1}{3} \text{ mod } 1.
\]
Then $2\mid \ord_2 W_f$ and $\ord_\rho W_f \equiv 1 \text{ mod } 3$, from which we conclude that $(2\kappa_{\rho}, 3) = (2\kappa_1, 2) = 1$. This proves Part (i).

The proof of Part (ii) is similar. We may assume without loss of generality that $f(z)$ does not vanish at $\gamma_1$ and $\gamma_2$ for some $\gamma \in \Gamma_0^+(2)$. Then
\[
2\kappa_{\rho} = 1 + \ord_{\rho_j} W_f(z), \quad j = 1, 2,
\]
and it suffices to prove that $\ord_{\rho_j} W_f$ is always even. Indeed, if the weight $k$ is a multiple of 4, then the weight of $W_f \in \mathcal{M}_{2k}(\Gamma_0^+(2))$ is a multiple of 8. The orders of such a modular form at any elliptic point must be even. If the weight $k$ is congruent to 2 modulo 4 so that the weight of $W_f$ is congruent to 4 modulo 8, then $W_f$ must be divisible by $M_1(z) = (4E_4(2z) + E_2(z))/5$, which has a zero of order 2 at $\rho_2$ and is nonvanishing at $\rho_1$. Since $W_f/\ M_4$ is again a modular form of a weight congruent to 0 modulo 8, we conclude that the order of $W_f$ at $\rho_j$ must be even for $j = 1, 2$. This proves Part (ii).

The proof of (iii) is slightly more complicated. Write $f(z)$ as $f(z) = f_1(z)M_2^+(z) + f_0(z)$, where $M_2^+(z) = (3E_2(3z) + E_2(z))/4$ and $f_j \in \mathcal{M}_{k-2j}(\Gamma_0^+(3), \chi^m)$. Again, without loss of generality, we assume that $f_0(z)$ and $f_1(z)$ have no common zero at $\rho_1$ and $\rho_2$. This assumption forces the weight $k$ to be congruent to 0 or 2 modulo 3. Then, as above, we have $2\kappa_{\rho_j} = 1 + \ord_{\rho_j} W_f$.

By Lemma 2.14 we have $W_f \in \mathcal{M}_{2k}(\Gamma_0^+(3), \chi^{2m}) = \mathcal{M}_{2k}(\Gamma_0^+(3), (-1)^k)$. Now recall that the ring of modular forms $\oplus_{n=0}^\infty \mathcal{M}_n(\Gamma_1(3))$, where $n$ runs through both even and odd integers, is freely generated by the modular forms $\mathcal{M}(\chi)$, $\mathcal{M}_3(z) \in \mathcal{M}_3(\Gamma_0^+(3), \chi)$ and $\mathcal{M}_3(z) \in \mathcal{M}_3(\Gamma_0^+(3), \chi)$ defined by (2.15) and (2.16). Thus, $W_f$ is a linear combination of products of the form $M_1(z)^a M_3(z)^b$, where $(a, b)$ are pairs of nonnegative integers satisfying
\[
(3.7) \quad a + 3b = 2k, \quad a + b \equiv 2k \text{ mod } 4.
\]
(The latter condition corresponds to the requirement that $M_1(z)^a M_3(z)^b$ must be a modular form on $\Gamma_0^+(3)$ with character $\chi^{2k}$.) The conditions imply that $b$ must be even, while
\[
a \equiv \begin{cases} 
0 \text{ mod } 6, & \text{if } k \equiv 0 \text{ mod } 3, \\
4 \text{ mod } 6, & \text{if } k \equiv 2 \text{ mod } 3,
\end{cases}
\]
Since $M_1(z)$ and $M_3(z)$ have a simple zero at $\rho_2$ and $\rho_1$, respectively, and are nonvanishing elsewhere, the above properties of $(a, b)$ show that $\ord_{\rho_1} W_f \equiv 0 \text{ mod } 2$ and
\[
\ord_{\rho_2} W_f \equiv \begin{cases} 
0 \text{ mod } 6, & \text{if } k \equiv 0 \text{ mod } 3, \\
4 \text{ mod } 6, & \text{if } k \equiv 2 \text{ mod } 3.
\end{cases}
\]
In either case, we find that $(2\kappa_{\rho_1}, 2) = (2\kappa_{\rho_2}, 6) = 1$. The proof of the proposition is completed. \qed
Theorem 3.7. Set
\[ M_2(z) = 2E_2(2z) - E_2(z), \]
\[ M_4(z) = \frac{1}{5}(4E_4(2z) + E_4(z)) = M_2(z)^2, \]
\[ M_8(z) = \eta(z)^8\eta(2z)^8. \]
Assume that \( k \) is a nonnegative integer.

(i) Let \( f(z) \) be an extremal quasimodular form in \( \widetilde{M}^\leq_{4k}(\Gamma_0^+(2), (−1)^k) \). Then \( f(z)/M_8(z)^{k/2} \)
is a solution of
\[ y''(z) = -4\pi^2 \left( \frac{k}{2} \right)^2 M_4(z) y(z). \]

(ii) Let \( f(z) \) be an extremal quasimodular form in \( \widetilde{M}^\leq_{4k+2}(\Gamma_0^+(2), (−1)^k) \). Then \( f(z)/M_8(z)^{k/2}M_2(z) \) is a solution of
\[ y''(z) = -4\pi^2 \left( \frac{k}{2} \right)^2 M_4(z) - 32 M_8(z) M_4(z) y(z). \]

Proof. Let \( W_f(z) \) be the Wronskian associated to \( f \) defined in Lemma 2.14. By the lemma, \( W_f(z) \) is a modular form of weight \( 8k \) on \( \Gamma_0^+(2) \) (with trivial character). We then observe that, by (2.9) and (2.10),
\[ \dim \widetilde{M}^\leq_{4k}(\Gamma_0^+(2), (−1)^k) = 1 + k = \dim \mathcal{M}_{8k}(\Gamma_0^+(2)). \]

It follows that, by Proposition 2.15 if \( f \) is an extremal quasimodular form in \( \widetilde{M}^\leq_{4k}(\Gamma_0^+(2), (−1)^k) \), then \( v_\infty(f) = v_\infty(W_f) = k \). However, up to scalars, \( M_8(z)^k \) is the only modular form in \( \mathcal{M}_{8k}(\Gamma_0^+(2)) \) whose order of vanishing at \( \infty \) is \( k \) (see the proof of Corollary 2.16). In other words, \( W_f(z) = cM_8(z)^k \) for some nonzero complex number \( c \) and the functions \( g_1(z) \) and \( g_2(z) \) in (3.3) are holomorphic throughout \( \mathbb{H} \). It follows that the differential equation (3.4) has no singularities in \( \mathbb{H} \). Then the meromorphic modular form \( Q(z) \) in Theorem 3.3 must be a holomorphic modular form of 4 and hence a multiple of \( M_4(z) \) due to the fact that \( \dim \widetilde{M}_4(\Gamma_0^+(2)) = 1 \). Since \( g_2(z) = c_0 g_1^{k/2} + \cdots \) for some nonzero constant \( c_0 \), we conclude that \( Q(z) = (k/2)^2 M_4(z) \). This proves Part (i).

The proof of Part (ii) is slightly more involved. Let \( f(z) \) be an extremal quasimodular form in \( \widetilde{M}^\leq_{4k+2}(\Gamma_0^+(2), (−1)^k) \). By the same reasoning as in the proof of Part (i), using (2.9), (2.10), and Proposition 2.15 we see that \( W_f(z) \) is a modular form of weight \( 8k + 4 \) such that \( v_\infty(W_f) = k \) and hence must be a multiple of \( M_4(z)M_8(z)^k \). Now \( M_4(z) \) has a zero of order 2 at \( \rho_2 = (1 + i)/2 \) (see Section 5), so the differential equation (3.4) has potentially a singularity at \( \rho_2 \) and we need to determine the local exponents at the point.

Write \( f(z) \) as \( f(z) = M_2^j(z)f_1(z) + f_0(z) \), where \( M_2^j(z) = (2E_2(2z) + E_2(z))/3 \) and \( f_1 \in \mathcal{M}_{4k+2−2j}(\Gamma_0^+(2), (−1)^k), j = 0, 1 \). Since \( \rho_2 = (1 + i)/2 \) is an elliptic point of order 2 of \( \Gamma_0(2) \), we must have \( f_0(\rho_2) = 0 \). We claim that \( f_1(\rho_2) \neq 0 \). Indeed, since \( M_2(z) = 2E_2(2z) - E_2(z) \) has a simple zero at points equivalent to \( \rho_2 \) and is nonvanishing elsewhere, if \( f_1(\rho) = 0 \), then \( f_1(z)/M_2(z) \) and \( f_1(z)/M_2(z) \) are holomorphic modular forms on \( \Gamma_0(2) \) with Atkin-Lehner eigenvalue \( (−1)^{k−1} \). In other words, \( f(z)/M_2(z) \in \mathcal{M}^\leq_{4k}(\Gamma_0^+(2), (−1)^{k−1}) \). However, this is impossible because there is no quasimodular
form in \( \mathcal{M}^{\leq 1}_{2k} (\Gamma^+_0(2), (-1)^{k-1}) \) whose order of vanishing at \( \infty \) is \( k \), by (2.9), (2.10), and Corollary 2.16. We conclude that \( f_1(\rho_2) \neq 0 \).

We next observe that, by Lemma 3.5, \( M_2^\ast(z) \) does not vanish at any elliptic point of \( \Gamma^+_0(2) \). Therefore, \( f(z) = M_2^\ast(z)f_1(z) + f_0(z) \) does not vanish at \( \rho_2 \) and \( f(z)/\sqrt{W_f(z)} \) has a simple zero at \( \rho_2 \). It follows that the local exponents of (3.4) at \( \rho \) must be \(-1\) and \( 2 \), i.e., the indicial equation at \( \rho \) is \( x^2 - x - 2 = 0 \). By (A.15), we must have

\[
Q(z) = \left( \frac{k}{2} \right)^2 M_4(z) - 32 \frac{M_6(z)}{M_2(z)}.
\]

This proves Part (ii). \( \square \)

By the same token, we have the following analogous result for \( \Gamma^+_0(3) \).

**Theorem 3.8.** Set

\[
M_2(z) = \frac{3E_2(3z) - E_2(z)}{2}, \quad (3.8)
\]

\[
M_4(z) = \frac{9E_4(3z) + E_4(z)}{10} = M_2(z)^2,
\]

\[
M_6(z) = \eta(z)^6 \eta(3z)^6.
\]

Let \( \chi \) be the character of \( \Gamma^+_0(3) \) defined by

\[
\chi \left( \begin{array}{cc} 1 & 0 \\ \sqrt{3} & 0 \end{array} \right) = \chi \left( \begin{array}{cc} 0 & -1 \\ 3 & 0 \end{array} \right) = -1.
\]

Let \( k \) be a nonnegative integer.

(i) Let \( f(z) \) be an extremal quasimodular form in \( \mathcal{M}^{\leq 1}_{3k} (\Gamma^+_0(3), \chi^k) \). Then \( f(z)/M_6(z)^{k/2} \) is a solution of the differential equation

\[
y''(z) = -4\pi^2 \left( \frac{k}{2} \right)^2 M_4(z)y(z).
\]

(ii) Let \( f(z) \) be an extremal quasimodular form in \( \mathcal{M}^{\leq 1}_{3k+2} (\Gamma^+_0(3), \chi^k) \). Then \( f(z)/M_6(z)^{k/2} M_2(z) \) is a solution of the differential equation

\[
y''(z) = -4\pi^2 \left( \frac{k}{2} \right)^2 M_4(z) - 18 \frac{M_6(z)}{M_2(z)} y(z).
\]

**Proof.** The proof is similar to that of Theorem 3.7. We omit most of the details and only mention that the Wronskian \( W_f \) associated to \( f \) in Part (ii) is a scalar multiple of \( M_6(z)^k M_4(z) \), where \( M_4(z) \) has a zero of order 4 at \( \rho_2 = (3 + \sqrt{-3})/6 \). Hence, \( f(z)/\sqrt{W_f(z)} \) has a pole of order 2 at \( \rho_2 \), and the indicial equation of (3.4) at \( \rho_2 \) is \( x^2 - x - 6 = 0 \). By (A.18), we therefore have

\[
Q(z) = \left( \frac{k}{2} \right)^2 M_4(z) - 18 \frac{M_6(z)}{M_2(z)},
\]

as claimed. \( \square \)

**Remark 3.9.** The proof of the two theorems also applies to quasimodular forms on \( \text{SL}(2, \mathbb{Z}) \). We can similarly show that if \( f \) is an extremal quasimodular form of weight \( 6k \), then \( f/\Delta^{k/2} \) is a solution of \( y''(z) = -\pi^2 k^2 E_4(z)y(z) \). This gives a new proof of (2.5). Also, if \( f \) is an extremal quasimodular form of weight \( 6k + 2 \) and depth 1 on \( \text{SL}(2, \mathbb{Z}) \), then the Wronskian \( W_f(z) \) is a multiple of \( \Delta(z)^k E_4(z) \). Therefore, the order of \( f(z)/\sqrt{W_f(z)} \) at
\[ \rho = (1 + \sqrt{-3})/2 \text{ is } -1/2 \text{ and the indicial equation at } \rho \text{ is } x^2 - x - 3/4 = 0. \text{ By (A.11),} \]
\[ f(z)/\sqrt{W_f(z)} \text{ is a solution of} \]
\[ y''(z) = -4\pi^2 \left( \left( \frac{k}{2} \right)^2 E_k(z) - 144 \frac{\Delta(z)}{E_k(z)^2} \right) y(z). \]
This corresponds to Theorem 2.1(2) of [8].

We remark that the reason why we do not have a corresponding statement for extremal quasimodular forms of weight \( 6k + 4 \) and depth 1 on \( \text{SL}(2, \mathbb{Z}) \) is that such a quasimodular form \( f(z) \) must be a multiple of \( E_4(z) \), but the differential equation satisfied by \( f(z)/\sqrt{W_f(z)} \) is the same as that for \( f(z)/E_4(z)\sqrt{W_{f/E_4}(z)} \).

4. MODE IN THE CASE OF \( \text{SL}(2, \mathbb{Z}) \)

In this section, we shall consider Fuchsian differential equations of the form
\[ y''(z) = -4\pi^2 Q(z)y(z) \]
on \( \mathbb{H} \) for some meromorphic modular forms \( Q(z) \) of weight 4 on \( \text{SL}(2, \mathbb{Z}) \) satisfying the conditions in (H4). In particular, \( Q(z) \) is holomorphic at \( q = 0, \) \( q = e^{2\pi i z} \), with \( Q(\infty) \geq 0 \). Let \( D_q = q d/dq = (2\pi i)^{-1} d/dz \). At \( q = 0, 4.1 \) becomes
\[ D_q^2 y = Q(z)y. \]
Thus, (4.1) is also Fuchsian at the cusp \( \infty \) and its local exponents at \( \infty \) are \( \pm \kappa_\infty \), where
\[ \kappa_\infty = \sqrt{Q(\infty)}. \]

Note that in [4] Proposition 3.6] we have shown that if (4.1) is apparent at a point \( z_0 \), then it is apparent at all \( \gamma z_0, \gamma \in \text{SL}(2, \mathbb{Z}) \). Let \( z_1, \ldots, z_m \) be the poles of \( Q(z) \) such that \( z_j \) and \( z_k \) are not equivalent under \( \text{SL}(2, \mathbb{Z}) \) when \( j \neq k \), i.e., \( z_k \notin \text{SL}(2, \mathbb{Z})z_j \), and \( z_j \notin \{ \rho = (1 + \sqrt{-3})/2, i = \sqrt{-1} \} \). The apparentness condition implies that the local exponents at \( \rho, i \), and \( z_j, j = 1, \ldots, m \), are \( 1/2 \pm \kappa_\rho, 1/2 \pm \kappa_i \), and \( 1/2 \pm \kappa_j \), \( 1 \leq j \leq m \), where all \( \kappa \)’s are in \( \mathbb{N} \).

To the modular differential equation (4.1), there associates the Bol representation \( \rho \), a homomorphism from \( \text{SL}(2, \mathbb{Z}) \) to \( \text{PSL}(2, \mathbb{C}) \), that is, by choosing a fundamental solution \( Y(z) = (y_1(z), y_2(z))^t \), there is \( \rho(\gamma) \in \text{SL}(2, \mathbb{C}) \) for any \( \gamma \in \text{SL}(2, \mathbb{C}) \) such that
\[ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \pm \rho(\gamma) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \]
For our purpose, we would like to lift \( \rho \) to a homomorphism \( \hat{\rho} : \text{SL}(2, \mathbb{Z}) \to \text{SL}(2, \mathbb{C}) \) such that \( \rho = \pi \circ \hat{\rho} \), where \( \pi \) is the natural projection from \( \text{SL}(2, \mathbb{C}) \) to \( \text{PSL}(2, \mathbb{C}) \). This will be done as follows.

Define \( t_j = E_6(z_j)^2/E_4(z_j)^3 \) such that \( F_j(z_j) = 0 \), where
\[ F_j(z) = E_6(z)^2 - t_j E_4(z)^3. \]

**Lemma 4.1.** If \( t_j \neq 0, 1 \), then \( E_6(z)^2 - t_j E_4(z)^3 \) has zeros only at \( z_j \), and \( z_j \) is a simple zero.

The lemma can be proved by the theorem of counting zeros of modular forms ([14, p. 85, Theorem 3]).

Set
\[ F(z) = \Delta(z)^{\kappa_\infty} E_4(z)^{\kappa_\infty - 1/2} E_6(z)^{-1/2} \prod_{j=1}^m F_j(z)^{\kappa_j - 1/2} \]
It is easy to see that if $\hat{y}_\infty$ is holomorphic throughout $\mathbb{H}$, $\kappa_1 - 1/2$ is an integer. Thus, $\ell$ is always an odd integer.

For any function $f(z)$ on $\mathbb{H}$, we set 
$$\hat{f}(z) = F(z)f(z).$$

It is easy to see that if $y(z)$ is a solution of (4.1), then $\hat{y}(z)$ is single-valued function that is holomorphic throughout $\mathbb{H}$. Furthermore, its order at $z_\infty$ is either 0 or 2$\kappa_j$, and a similar property holds at $\infty$, $\rho$, and $i$. Now choose a fundamental solution $Y(z) = (y_1(z), y_2(z))^t$ of (4.1). Then we have a homomorphism $\hat{\rho} : SL(2, \mathbb{Z}) \to GL(2, \mathbb{C})$ defined by 
$$\hat{Y} \mid_{\ell \gamma} = \hat{\rho}(\gamma) \hat{Y}(z), \quad \text{where} \quad \hat{Y}(z) = F(z)Y(z).$$

The next lemma shows that this homomorphism $\hat{\rho}$ has an image in $SL(2, \mathbb{C})$.

**Lemma 4.2.** We have $\det \hat{\rho}(\gamma) = 1$ for all $\gamma \in SL(2, \mathbb{Z})$. That is, $\hat{\rho}$ is a homomorphism from $SL(2, \mathbb{Z})$ to $SL(2, \mathbb{C})$.

**Proof.** Let 
$$W(z) = \det \begin{pmatrix} y_1 & y'_1 \\ y_2 & y'_2 \end{pmatrix}, \quad \hat{W}(z) = \det \begin{pmatrix} \hat{y}_1 & \hat{y'}_1 \\ \hat{y}_2 & \hat{y'}_2 \end{pmatrix}.$$ 

On the one hand, (4.8) yields that for $\gamma \in SL(2, \mathbb{Z})$, 
$$\hat{W}(z) \mid_{2(\ell+1)\gamma} = \det \hat{\rho}(\gamma) \hat{W}(z).$$

On the other hand, we have $\hat{W}(z) = F(z)^2W(z)$. Hence, 
$$\hat{W}(z) \mid_{2(\ell+1)\gamma} = \left(F(z)^2\right) \hat{W}(z) \mid_{0\gamma} 
= \left(F(z)^2\right) \det \rho(\gamma)W(z) = \frac{F(z)^2 \mid_{2(\ell+1)\gamma}}{F(z)^2} \hat{W}(z).$$

Comparing the two expressions, we find that 
$$\det \hat{\rho}(\gamma) = \frac{F(z)^2 \mid_{2(\ell+1)\gamma}}{F(z)^2} = 1$$
for all $\gamma \in SL(2, \mathbb{Z})$ since $F(z)^2$ is a modular form of weight $2(\ell + 1)$ on $SL(2, \mathbb{Z})$. This proves the lemma. \hfill $\square$

If no confusion arises, we will also call $\hat{\rho}$ the Bol representation.

Throughout the remainder of the section, we will let $y_+(z)$ denote the unique solution of (4.1) of the form 
$$y_+(z) = q^{\kappa_\infty} \left(1 + \sum_{j \geq 1} c_j q^j\right).$$

Also, we will use the standard notations $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $R = TS = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$. They satisfy 
$$S^2 = R^3 = -I.$$
Out main result of this section is the following theorem (stated as Theorem 1.2 in Section 1).

**Theorem 4.3.** Suppose that $Q(z)$ satisfies \((H_1)\) with $\kappa_\infty \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Then the following statements hold true.

1. The differential equation \((4.1)\) is not apparent at $\infty$.
2. Let $y_2(z) = y_\pm(z)$, where $y_\pm(z)$ is the solution of \((4.1)\) of the form \((4.9)\). Let $\hat{y}_1(z) = (\hat{y}_\pm S)(z)$. Then $\hat{y}_1(z) = z\hat{y}_2(z) + \hat{m}_1(z)$ for some modular form $\hat{m}_1(z)$ of weight $\ell - 1$ on $SL(2, \mathbb{Z})$.
3. Using $y_1(z)$ and $y_2(z)$ as the basis, Bol’s representation satisfies
   \[ \hat{\rho}(\gamma) = \gamma, \quad \gamma \in SL(2, \mathbb{Z}). \]
   In particular, the ratio $h(z) = y_1(z)/y_2(z)$ is equivariant, i.e., $h(\gamma z) = \gamma \cdot h(z)$ for all $\gamma \in SL(2, \mathbb{Z})$.
4. Write $\hat{y}_\pm(z)$ as $\hat{y}_\pm(z) = \frac{\hat{m}_1(z)E_2(z)}{\hat{m}_2(z)}$. Then $\hat{m}_2(z)$ is a modular form of weight $\ell + 1$ with respect to $SL(2, \mathbb{Z})$. In other words, $\hat{y}_\pm(z)$ is a quasimodular form of weight $\ell + 1$ and depth 1 on $SL(2, \mathbb{Z})$.

Theorem 4.3 will be proved through a series of lemmas.

**Lemma 4.4.** The ODE \((4.1)\) is not apparent at $\infty$.

*Proof.* Assume that \((4.1)\) is apparent at $\infty$. Then for any fundamental solutions near $\infty$, we have $\hat{\rho}(T) = I$. Now since $\ell$ is odd, we have $\hat{\rho}(-I) = (-1)^{\ell}I = -I$. Thus, by \((4.10)\), $\hat{\rho}(S)^2 = \hat{\rho}(R)^3 = -I$. On the other hand, we have $\hat{\rho}(R)^2 = (\hat{\rho}(T)\hat{\rho}(S))^2 = -I$, which is absurd since $\hat{\rho}(R)^2$ and $\hat{\rho}(R)^3$ cannot be both equal to $-I$. This proves that \((4.1)\) cannot be apparent at $\infty$. \qed

**Remark 4.5.** Observe that the parity of $\ell$ depends only on $\kappa_i$. If $\kappa_i \in \mathbb{N}$, i.e., if $\ell$ is even, then $\hat{\rho}(R^3) = \hat{\rho}(S^2) = \hat{\rho}(-I) = I$, which implies that $\hat{\rho}(S) = \pm I$ and either $\hat{\rho}(R) = I$ or $\hat{\rho}(R)^2 + \hat{\rho}(R) + I = 0$. In any case, there is a matrix $P$ such that both $P\hat{\rho}(S)P^{-1}$ and $P\hat{\rho}(R)P^{-1}$ are diagonal, which implies that $P\hat{\rho}(T)P^{-1}$ is also diagonal, and the apparentness at $\infty$ follows. Thus, the condition $\kappa_i \notin \mathbb{N}$ is necessary in order for \((4.1)\) to be not apparent at $\infty$.

**Lemma 4.6.** Let $y_2(z) = y_\pm(z)$ and $y_1(z) = (\hat{y}_\pm S)(z)/F(z)$. Then the following hold:

1. $y_i, i = 1, 2$, are linearly independent, and
2. the ratio $h(z) = y_1(z)/y_2(z)$ is equivariant.

*Proof.* By Lemma 4.4 \((4.1)\) is not apparent at $\infty$. Hence, we have a solution $y_\mp(z)$ of the form $y_\mp(z) = dz y_\mp(z) + m(z)$ with $m(z) = q^{-\kappa_\infty}(1 + \cdot \cdot \cdot )$ and $d \neq 0$. Then with respect to the choice $\hat{Y}(z) = (\hat{y}_\pm(z), \hat{y}_\mp(z))^t$, we have $\hat{\rho}(T) = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}$.

Now suppose that $y_\pm(z)$ and $(\hat{y}_\pm S)(z)/F(z)$ are not linearly independent, i.e., $(\hat{y}_\pm S)(z) = \lambda \hat{y}_2(z)$ for some $\lambda$. Then $\hat{\rho}(S) = \begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix}$, $c \in \mathbb{C}$. Because $\ell$ is odd, we have $\hat{\rho}(S^2) = \hat{\rho}(-I) = (-1)^{\ell}I = -I$. It follows that $\lambda = \pm i$ and

\[
\hat{\rho}(R) = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ c & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ d\lambda + c & \lambda \end{pmatrix},
\]

which implies $\lambda^3 = -1$, a contradiction because $\lambda = \pm i$. Thus $y_i, i = 1, 2$, are linearly independent.
Since \(4.1\) is not apparent at \(\infty\) and we have shown that \(y_2(z) = y_+(z)\) and \(y_1(z) = (\hat{y}_2|_\ell S) (z) / F(z)\) are linearly independent, by (1.11), there is \(d \neq 0\) such that \(y_1(z) = d \cdot y_2(z) + m_1(z)\) for some function \(m_1(z) = q^{-\rho} (a_0 + \sum_{j \geq 1} a_j q^j)\) with \(a_0 \neq 0\). We claim that

\[
(4.11) \quad d = 1.
\]

Obviously, with respect to the basis \((\hat{y}_1(z), \hat{y}_2(z))^t\), \(\hat{\rho}(T) = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}\). On the other hand, since \(\hat{y}_1(z) = (\hat{y}_2|_\ell S) (z)\), and \(-\hat{y}_2(z) = (\hat{y}_2|_\ell S^2) (z) = (\hat{y}_1|_\ell S) (z)\), we have

\[
(4.12) \quad \hat{\rho}(S) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

and

\[
\hat{\rho}(R) = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} d & -1 \\ 1 & 0 \end{pmatrix}.
\]

By \(\hat{\rho}(R^3) = -I\), i.e. \(\hat{\rho}(R)^2 = -\hat{\rho}(R)^{-1}\), we have

\[
\begin{pmatrix} d & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d^2 & -d \\ d & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & d \end{pmatrix},
\]

which yields \(d = 1\). This proves (4.11).

Using \(h(z) = \hat{y}_1(z) / \hat{y}_2(z)\), we have \(h(z + 1) = (\hat{y}_1(z) + \hat{y}_2(z)) / \hat{y}_2(z) = h(z) + 1\) and

\[
h(-1/z) = \frac{\hat{y}_1(-1/z)}{\hat{y}_2(-1/z)} = (\hat{y}_1|_\ell S)(z) \quad \frac{\hat{y}_2(-1/z)}{\hat{y}_1(z)} = \frac{-\hat{y}_2(z)}{\hat{y}_1(z)} = -1/h(z).
\]

This proves that \(h(z)\) is equivariant. \(\square\)

We recall the identity \(\hat{y}_1(z) = (\hat{y}_2|_\ell S)(z) = z \hat{y}_2(z) + \hat{m}_1(z)\) obtained in the proof of Lemma 4.6.

**Lemma 4.7.** We have

(i) \(\hat{m}_1(z + 1) = \hat{m}_1(z)\), and

(ii) \((\hat{m}_1|_{-1} S)(z) = \hat{m}_1(z)\).

Hence, \(\hat{m}_1(z)\) is a modular form of weight \(\ell - 1\) on \(\text{SL}(2, \mathbb{Z})\).

**Proof.** The identity (i) is obvious. By the definition, \(h(z)\) of Lemma 4.6(ii) can be written as

\[
(4.13) \quad h(z) = z + \frac{m_1(z)}{y_2(z)} = z + \frac{\hat{m}_1(z)}{\hat{y}_2(z)}.
\]

We have

\[
(4.14) \quad -\frac{1}{h(z)} = h \left( -\frac{1}{z} \right) = -\frac{1}{z} + \frac{\hat{m}_1(-1/z)}{\hat{y}_2(-1/z)}.
\]

Thus by \(4.14\),

\[
\frac{\hat{m}_1(-1/z)}{\hat{y}_1(z)} = z^\ell \frac{\hat{m}_1(-1/z)}{\hat{y}_2(-1/z)} = z^\ell \left( \frac{1}{z} - \frac{1}{h(z)} \right) = \frac{(h(z) - z)z^\ell}{zh(z)} \quad \frac{1}{z^1 - h(z)} = \frac{\hat{m}_1(z)}{z^{1-\ell}h(z)}
\]

which implies \(\hat{m}_1(-1/z) = z^{\ell-1} \hat{m}_1(z)\). This proves (ii). \(\square\)
We conclude that $k$ also proved that if $\Delta_z^\ell$ then $z \equiv 2 \mod 6$.

On the other hand, recall from the proof of Lemma 4.6 that the left-hand side is equal to $z \hat{y}_+ (z) + \hat{m}_1 (z)$. Therefore, $\hat{m}_1 (-1/z) = z^{\ell+1} \hat{m}_2 (z)$. That is, $\left( \hat{m}_2 \mid \ell + S \right) (z) = \hat{m}_2 (z)$.

Finally, since $\hat{y}_+ (z+1) = \hat{y}_+ (z)$ and $\hat{m}_1 (z+1) = \hat{m}_1 (z)$, we have $\hat{m}_2 (z+1) = \hat{m}_2 (z)$. We conclude that $\hat{m}_2 (z)$ is a modular form of weight $\ell + 1$ on $SL(2, \mathbb{Z})$.

Having proved Theorem 4.3, we now give some examples.

**Example 4.9.** Consider the differential equation

$$y'' (z) = -4\pi^2 \kappa_\infty^2 E_4 (z) y (z), \quad \kappa_\infty \in \frac{1}{2} \mathbb{N}.$$  

Since $E_4 (z)$ is a holomorphic modular form, this differential equation has no singularities in $\mathbb{H}$. Hence, we have $\kappa_\rho = \kappa_i = 1/2$, $F (z) = \Delta (z)^{\kappa_\infty}$ and the integer $\ell$ in (4.7) is $-1 + 12 \kappa_\infty$. Also, the local exponents at $\infty$ are clearly $\pm \kappa_\infty$. Thus, Theorem 4.3 predicts that $\Delta (z)^{\kappa_\infty} y_+ (z)$ is a quasimodular form of weight $\ell + 1$ and depth 1 on $SL(2, \mathbb{Z})$. Indeed, as mentioned in Section 2.1 by the works of Kaneko and Koike [7, 8] and Pellarin [10], Proposition 3.1, the differential equation (4.15) has $y (z) = f (z) / \Delta (z)^{\kappa_\infty}$ as a solution, where $f (z)$ is an extremal quasimodular form of weight $12 \kappa_\infty = \ell + 1$ and depth 1, agreeing with our Theorem 4.3.

**Example 4.10.** Consider the differential equation

$$y'' (z) = -4\pi^2 \left( \kappa_\infty^2 E_4 (z) - 144 \frac{\Delta (z)}{E_4 (z)^2} \right) y (z), \quad \kappa_\infty \in \frac{1}{2} \mathbb{Z} \geq 0.$$  

The differential equation has only singularities at points equivalent to $\rho$ under $SL(2, \mathbb{Z})$. By (A.11) (or [4] Theorem 1.7 and Corollary 3.10), the indicial equation as $\rho$ is $x^2 - x - 3/4 = 0$. Hence the local exponents at $\rho$ are $-1/2$ and $3/2$, i.e., $\kappa_\rho = 1$, and the differential equation is apparent on $\mathbb{H}$, by Theorem A.3. The integer $\ell$ in (4.7) is $1 + 12 \kappa_\infty$. Theorem 4.3 then asserts that if we let $y_+ (z) = q^{\kappa_\infty} + \cdots$ be the solution with exponent $\kappa_\infty$ at $\infty$, then $\Delta (z)^{\kappa_\infty} E_4 (z)^{1/2} y_+ (z)$ is a quasimodular form of weight $\ell + 1$ and depth 1 on $SL(2, \mathbb{Z})$. To check that this is true, we recall that Kaneko and Koike [8] Theorem 2.1] has also proved that if $k \equiv 2 \mod 6$, then an extremal quasimodular form $f (z)$ of weight $k$ and depth 1 on $SL(2, \mathbb{Z})$ satisfies

$$D_q^2 f - \left( \frac{k}{6} - \frac{1}{3} E_6 \right) D_q f + \left( \frac{k(k-1)}{12} D_q E_2 - \frac{k-1}{18} D_q E_6 \right) f = 0.$$
A straightforward calculation shows that this is equivalent to the assertion that \( f(z) / \Delta(z) = E_4(z)^{1/2} \) is a solution of (4.16) with \( k = 12\kappa_\infty + 2 = \ell + 1 \). This agrees with our result.

**Example 4.11.** Consider the differential equation

\[
y''(z) = -4\pi^2 \left( \frac{1}{4} E_4(z) + 864 \frac{E_4(z) \Delta(z)}{E_6(z)^2} \right) y(z).
\]

We have \( \kappa_\infty = 1/2 \) and by (A.11), \( \kappa_i = 3/2 \). The integer \( \ell \) in this case is 11 and Theorem 4.3 implies that \( \Delta(z)^{1/2} E_6(z) y_+(z) \) is a quasimodular form of weight 12 and depth 1 on \( \text{SL}(2, \mathbb{Z}) \). Indeed, we compute that

\[
y_+(z) = q^{1/2} \left( 1 + 462q + 247494q^2 + 132490928q^3 + \cdots \right)
\]

and find that

\[
11088 \Delta(z)^{1/2} E_6(z) y_+(z) = E_2(z) E_4(z) E_6(z) + 6E_4(z)^3 - 7E_6(z)^2.
\]

### 5. MODE IN THE CASE OF \( \Gamma_0^+ (2) \)

In this section, we will obtain results analogous to Theorem 4.3 for the group \( \Gamma_0^+ (2) \).

First of all, we recall that the graded ring of holomorphic modular forms with respect to \( \Gamma_0^+ (2) \) is generated by three modular forms \( M_4(z) \), \( M_6(z) \) and \( M_8(z) \), whose weights are 4, 6 and 8, respectively. These modular forms can be written explicitly in terms of the Eisenstein series on \( \text{SL}(2, \mathbb{Z}) \) and the Dedekind eta function:

\[
\begin{align*}
M_4(z) &= (4E_4(2z) + E_4(z))/5, \\
M_6(z) &= (8E_6(2z) + E_6(z))/9, \\
M_8(z) &= \eta(z)^6 \eta(2z)^8.
\end{align*}
\]

Noticing that \( \dim \mathcal{M}_{12}(\Gamma_0^+ (2)) = 2 \), there must be a relation among \( M_4(z)^3, M_6(z)^2, \) and \( M_4(z) M_6(z) \). We find that it is \( M_6(z)^2 = M_4(z)(M_4(z)^2 - 256M_8(z)) \). For our purpose, we also need

\[
M_2(z) = 2E_2(2z) - E_2(z),
\]

which is a modular form in \( \mathcal{M}_2(\Gamma_0^+ (2), -) \) and satisfies \( M_2(z)^2 = M_4(z) \). Define

\[
M_2^*(z) = \frac{1}{2\pi i} \frac{M_8(z)}{M_6(z)} = \frac{2E_2(2z) + E_2(z)}{3}.
\]

The holomorphic function \( M_2^*(z) \) is not a modular form. Instead, it satisfies

\[
M_2^*(\gamma z) = (cz + d)^2 M_2^*(z) + \frac{4}{\pi i} c (cz + d), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^+ (2).
\]

We use the notations \( S = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \ T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \ R = TS = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{3}{2} & -1 \\ 1 & 0 \end{pmatrix} \) such that \( S^2 = R^4 = -I \). We collect some facts about \( \Gamma_0^+ (2) \).

(i) There are only two elliptic points \( \rho_1 = i/\sqrt{2} \) and \( \rho_2 = (1 + i)/2 \) of order 2 and 4, respectively. Their stabilizer subgroups are generated by \( S \) and \( R \), respectively. Also, \( \infty \) is the only cusp and has width 1.

(ii) The modular form \( M_2(z) \) has only one zero at \( \rho_2 \). The zero is simple. The modular form \( M_4(z) \) has only one zero (a double zero) at \( \rho_2 \) and \( M_6(z) \) has only zeros at \( \rho_1 \) and \( \rho_2 \). Both zeros are simple.

(iii) \( M_8(z) \neq 0, \forall z \in \mathbb{H}, \) and \( \infty \) is the simple zero of \( M_8(z) \).
Consider the differential equation

\[ y''(z) = -4\pi^2 Q(z)y(z), \]

where \( Q(z) \) is a meromorphic modular form of weight 4 on \( \Gamma_0^+(2) \) such that the conditions \( (H_1) \) are satisfied. Let \( z_1, \ldots, z_m \) be \( \Gamma_0^+(2) \)-inequivalent poles of \( Q(z) \) other than \( \rho_1 \) and \( \rho_2 \), and assume that the local exponents of \( \Gamma \) at \( \infty \), \( \rho_1 \), \( \rho_2 \), and \( z_1, \ldots, z_m \) are \( \pm \kappa_\infty \), \( 1/2 \pm \kappa_1 \), \( 1/2 \pm \kappa_2 \), and \( 1/2 \pm \kappa_3, \ldots, 1/2 \pm \kappa_m \), respectively, where the \( \kappa \)'s satisfy \( (H_2) \).

For \( j = 1, \ldots, m \), set \( t_j = M_4(z_j)^2/M_8(z_j) \) and \( F_j(z) = M_4(z)^2 - t_j M_8(z) \). This modular form \( F_j \) has only one simple zero at \( z_j \). Let

\[ F(z) = M_8(z)^{\kappa_{\infty}} \left( \frac{M_6(z)}{M_2(z)} \right)^{\kappa_{\rho_1}-1/2} M_2(z)^{\kappa_{\rho_2}-1/2} \prod_{j=1}^m F_j(z)^{\kappa_j-1/2}, \]

and

\[ \ell = -1 + 8\kappa_{\infty} + 4 \left( \kappa_{\rho_1} - \frac{1}{2} \right) + 2 \left( \kappa_{\rho_2} - \frac{1}{2} \right) + 8 \sum_{j=1}^m \left( \kappa_j - \frac{1}{2} \right), \]

so that \( F(z)^2 \) is a modular form of weight \( 2(\ell + 1) \) on \( \Gamma_0^+(2) \). We note that by the condition \( (H_1) \), the integer \( \ell \) is always odd.

For any function \( f(z) \) on \( \mathbb{H} \), let \( \hat{f}(z) \) denote the function \( F(z)f(z) \). When \( f(z) = y(z) \) is a solution of \( (5.3) \), by construction, \( \hat{y}(z) \) is a single-valued function holomorphic throughout \( \mathbb{H} \) and its order at \( z_j \) is either 0 or 2\( \kappa_j \). If a fundamental solution \( Y(z) = (y_1(z), y_2(z))^T \) of \( (5.3) \) is chosen, then we have a homomorphism \( \hat{\rho} : \Gamma_0^+(2) \to \text{GL}(2, \mathbb{C}) \) with \( \hat{\rho}(\gamma) \) defined by

\[ \left( \hat{Y} \right|_\gamma \right) (z) = \hat{\rho}(\gamma) \hat{Y}(z). \]

As in the case of \( \text{SL}(2, \mathbb{Z}) \), we can show that the image of \( \hat{\rho} \) actually lies in \( \text{SL}(2, \mathbb{C}) \).

**Lemma 5.1.** We have \( \det \hat{\rho}(\gamma) = 1 \) for all \( \gamma \in \Gamma_0^+(2) \).

**Proof.** The proof of the lemma follows exactly that of Lemma 4.2. We find that for \( \gamma \in \Gamma_0^+(2) \), we have

\[ \det \hat{\rho}(\gamma) = \left( \frac{F(z)^2}{2(\ell + 1)} \right)^{2(\ell + 1)} = 1 \]

because \( F(z)^2 \) is a modular form of weight \( 2(\ell + 1) \) on \( \Gamma_0^+(2) \). \( \square \)

Now let \( y_+(z) \) be the unique solution of \( (5.3) \) of the form

\[ y_+(z) = q^{\kappa_{\infty}} \left( 1 + \sum_{j \geq 1} c_j q^j \right). \]

Then analogous to Theorem 4.3 we have the following result for \( \Gamma_0^+(2) \) (Theorem 1.4 in Section 1).

**Theorem 5.2.** Suppose that \( Q(z) \) satisfies \( (H_1) \) with \( \kappa_{\infty} \in \mathbb{Z}_{\geq 0} \). Then the following statements hold.

(i) The differential equation \( (5.3) \) is not apparent at \( \infty \).
We first prove that

\[ y(z) = y_+(z) + y_1(z) = (\hat{y}_+ | S) (z) \]

for some modular form \( \hat{m}_1(z) \) in \( \mathcal{M}_{\ell - 1}(\Gamma_0^+(2), (\frac{z}{2})) \), where \( (\frac{z}{2}) \) is the Legendre symbol whose values are given by

\[
\left( \frac{2}{\ell} \right) = \begin{cases} 1, & \text{if } \ell \equiv 1, 7 \mod 8, \\ -1, & \text{if } \ell \equiv 3, 5 \mod 8. \end{cases}
\]

(ii) Let \( y_2(z) = y_+(z) \) and \( \hat{y}_1(z) = (\hat{y}_+ | S) (z) \). Then \( \hat{y}_1(z) = z \hat{y}_2(z) + \hat{m}_1(z) \) for some modular form \( \hat{m}_1(z) \) in \( \mathcal{M}_{\ell - 1}(\Gamma_0^+(2), (\frac{z}{2})) \), where \( (\frac{z}{2}) \) is the Legendre symbol whose values are given by

\[
\left( \frac{2}{\ell} \right) = \begin{cases} 1, & \text{if } \ell \equiv 1, 7 \mod 8, \\ -1, & \text{if } \ell \equiv 3, 5 \mod 8. \end{cases}
\]

(iii) The ratio \( h(z) = (\frac{z}{2}) y_1(z)/\sqrt{2} y_2(z) \) is equivariant. That is, for all \( \gamma \in \Gamma_0^+(2) \), we have \( h(\gamma z) = \gamma \cdot h(z) \).

(iv) Write \( \hat{y}_+(z) \) as

\[
\hat{y}_+(z) = \left( \frac{2}{\ell} \right) \frac{\pi i}{\sqrt{2}} \hat{m}_1(z) \Delta_2(z) + \hat{m}_2(z).
\]

Then \( \hat{m}_2(z) \) is a modular form in \( \mathcal{M}_{\ell + 1}(\Gamma_0^+(2), (\frac{z}{2})) \). Hence, \( \hat{y}_+(z) \) is a quasi-modular form in \( \mathcal{M}_{\ell + 1}^{<1}(\Gamma_0^+(2), (\frac{z}{2})) \).

The proof of (i) is the same as that of Lemma 4.2 and is omitted. We remark that if \( \kappa_{\rho_2} \in \mathbb{N} \) then it can be shown that the differential equation (5.3) is apparent at \( \infty \). (C.f. Remark 4.3.) Thus, the condition that \( (2\kappa_{\rho_2}, 4) = 1 \) is necessary in order for (5.3) to be not apparent at \( \infty \). We now prove the other parts of the theorem.

By Part (i) of the theorem, there is another solution \( y_-(z) \) of (5.3) of the form

\[
y_-(z) = zy_+(z) + q^{-\kappa} \left( b_0 + \sum_{j \geq 1} b_j q^j \right), \quad b_0 \neq 0.
\]

Set

\[
y_1(z) = (\hat{y}_+ | S) (z) / F(z), \quad y_2(z) = y_+(z).
\]

We first prove that \( y_1 \) and \( y_2 \) are linearly independent. Hence, there are \( d \neq 0 \) and \( m_1(z) \) such that

\[
y_1(z) = dz y_2(z) + m_1(z),
\]

where \( m_1(z) \) has a \( q \)-expansion of the form \( q^{-\kappa} (c_0 + \sum_{j \geq 1} c_j q^j) \), \( c_0 \neq 0 \).

**Lemma 5.3.** The solutions \( y_i, \ i = 1, 2, \) are linearly independent, and we have \( d = \pm \sqrt{2} \).

**Proof.** The proof of linear independence of \( y_1 \) and \( y_2 \) is similar to that of Part (i) of Lemma 4.6 and is skipped.

To compute \( d \), we use \( \hat{Y}(z) = (\hat{y}_1(z), \hat{y}_2(z))^t \) as the basis. We have \( \hat{\rho}(T) = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \) and \( \hat{\rho}(S) = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \) (see the proof of (4.12)). Hence,

\[
\hat{\rho}(R) = \hat{\rho}(T) \hat{\rho}(S) = \left( \begin{array}{cc} 1 & d \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) = \left( \begin{array}{cc} d & -1 \\ 1 & 0 \end{array} \right).
\]

Then \( \hat{\rho}(R)^2 = -\hat{\rho}(R)^{-2} \) yields

\[
\left( \begin{array}{cc} d^2 - 1 & -d \\ d & -1 \end{array} \right) = \left( \begin{array}{cc} -1 & d \\ -d & d^2 - 1 \end{array} \right).
\]

It follows that \( d = \pm \sqrt{2} \), and the proof of the lemma is completed. \( \Box \)

**Lemma 5.4.** Let \( h(z) = y_1(z)/(dy_2(z)) \). Then \( h(Sz) = S \cdot h(z) \).
Proof. By using the basis \((\tilde{y}_1, \tilde{y}_2)\), we have \(\rho(S) = \left( \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right)\), which implies that
\[
h(Sz) = \frac{-y_2(z)}{dy_1(z)} = \frac{1}{d^2} \frac{-dy_2(z)}{y_1(z)} = -\frac{1}{2h(z)} = \frac{-1/\sqrt{2}}{2h(z)/\sqrt{2}} = S \cdot h(z).
\]
This proves the lemma.

Note that together with Lemma 5.8 below, this lemma yields Part (iii) of the theorem.

Lemma 5.5. Let \(y_1(z)\) and \(m_1(z)\) be defined as in (5.6) and (5.7), respectively. Then \(\hat{m}_1(z)\) satisfies
\[
(i) \quad \hat{m}_1(z + 1) = \hat{m}_1(z),
(ii) \quad \left( \hat{m}_1|_{\ell - 1}S \right)(z) = \epsilon \hat{m}_1(z) \text{ if } d = \epsilon \sqrt{2}, \epsilon \in \{\pm 1\}, \text{ and }
(iii) \quad \hat{m}_1(z) \in \mathcal{M}_{\ell - 1}(\Gamma_0(2), \epsilon).
\]

Proof. Part (i) is obvious. The proof of (ii) is similar to that of Lemma 4.8. For the convenience of readers, we repeat it here. Write
\[
h(z) = \frac{y_1(z)}{dy_2(z)} = z + \frac{m_1(z)}{dy_2(z)}.
\]
By the previous lemma,
\[
-\frac{1}{2h(z)} = h(Sz) = h\left( -\frac{1}{2z} \right) = -\frac{1}{2z} + \frac{m_1(Sz)}{dy_2(Sz)},
\]
and then
\[
\frac{m_1(Sz)}{dy_2(Sz)} = -\frac{1}{2h(z)} + \frac{1}{2} = \frac{h(z) - z}{2h(z)z} = \frac{m_1(z)}{2dh(z)y_2(z)} = \frac{m_3(z)}{2zy_1(z)}.
\]
Since
\[
\hat{y}_1(z) = \left( \hat{y}_2|_\ell S \right)(z) = (\sqrt{2}z)^{-\ell} \hat{y}_2(Sz),
\]
we deduce that
\[
\hat{m}_1(Sz) = \frac{d\hat{m}_1(z)}{2z} \frac{\hat{y}_2(Sz)}{\hat{y}_1(z)} = \epsilon \frac{\hat{m}_1(z)}{\sqrt{2}z} \cdot (\sqrt{2}z)^\ell
\]
\[
= \epsilon (\sqrt{2}z)^{\ell - 1} \hat{m}_1(z) \quad \text{if } d = \epsilon \sqrt{2},
\]
i.e., \(\left( \hat{m}_1|_{\ell - 1}S \right)(z) = \epsilon \hat{m}_1(z)\).

Now we have \(R = TS\). Hence, by (i) and (ii), \(\left( \hat{m}_1|_{\ell - 1}R^2 \right)(z) = \hat{m}_1(z)\). Since \(R^2 = \left( \begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix} \right)\) and \(T\) generate the group \(\Gamma_0(2)\), we find that \(\hat{m}_1(z)\) is a modular form of weight \(\ell - 1\) on \(\Gamma_0(2)\). It is clear from (ii) that it belongs to the Atkin-Lehner subspace with eigenvalue \(\epsilon\).

Define \(m_2(z)\) by
\[
y_4(z) = \frac{\pi i}{4d} m_1(z) M_2^*(z) + m_2(z).
\]
Note that since \(m_1(z) = q^{-\infty}(a_0 + O(q))\), \(a_0 \neq 0\), we have \(m_2(z) = q^{-\infty}(b_0 + O(q))\) for some \(b_0 \neq 0\).

Lemma 5.6. We have \(\hat{m}_2(z) \in \mathcal{M}_{\ell + 1}(\Gamma_0^+(2), \epsilon)\), where \(\epsilon \in \{\pm 1\}\) is determined by \(d = \epsilon \sqrt{2}\).
Proof. As seen in the proof of (5.3), it suffices to prove that
\[ \hat{m}_2(z + 1) = \hat{m}_2(z), \quad \left( \hat{m}_2|_{\ell+1}S \right)(z) = e\hat{m}_2(z). \]
The first identity is obvious. To prove the second identity, by (5.2), (5.8) and Lemma 5.5 we have
\[
\left( \hat{y}_+|_{\ell}S \right)(z) = \sqrt{2}\pi i \frac{\hat{m}_1(z)M_2^*(z) + \hat{m}_1(z)}{2z} + \sqrt{2}\pi i \hat{m}_2(z|_{\ell+1}S)(z)
\]
and
\[
= dz (\hat{y}_+(z) - \hat{m}_2(z)) + \hat{m}_1(z) + \sqrt{2}\pi i \left( \hat{m}_2|_{\ell+1}S \right)(z)
\]
We note that the first sum of the right-hand side is equal to \( \left( \hat{y}_+|_{\ell}S \right)(z) \). Therefore, we have
\[
\left( \hat{m}_2|_{\ell+1}S \right)(z) = e\hat{m}_2(z).
\]
This completes the proof.

It remains to determine \( \epsilon \). We first prove a lemma.

Lemma 5.7. The two modular forms \( \hat{m}_1(z) \) and \( \hat{m}_2(z) \) cannot have a common zero on \( \mathbb{H} \).

Proof. Suppose that \( \hat{m}_1(z) \) and \( \hat{m}_2(z) \) have a common zero at \( z_0 \in \mathbb{H} \). Then \( \hat{y}_1(z_0) = \hat{y}_2(z_0) = 0 \), which implies that both \( y_+ \) and \( y_+|_{-1}S \) have the behavior \( c(z - z_0)^{1/2 + \epsilon} \) for some \( c \neq 0 \) near \( z_0 \). Consequently, \( y_+ \) and \( y_+|_{-1}S \) are linearly dependent, contradicting to Lemma 5.5 We conclude that \( \hat{m}_1(z) \) and \( \hat{m}_2(z) \) have no common zero on \( \mathbb{H} \).

We now use this lemma to determine \( \epsilon \).

Lemma 5.8. We have
\[
\epsilon = \left( \frac{2}{\ell} \right) = \begin{cases} 
1, & \text{if } \ell \equiv 1, 7 \text{ mod } 8, \\
-1, & \text{if } \ell \equiv 3, 5 \text{ mod } 8.
\end{cases}
\]

Proof. We will prove case by case that the assumption \( \epsilon = -\left( \frac{2}{\ell} \right) \) will imply that \( \hat{m}_1(z) \) and \( \hat{m}_2(z) \) have a common zero at \( \rho_2 \). By the lemma above, this is absurd. Hence, \( \epsilon \) must be equal to \( \left( \frac{2}{\ell} \right) \).

In general, if \( z_0 \) is an elliptic point of order \( \epsilon \) of a subgroup \( \Gamma \) of \( \text{SL}(2, \mathbb{R}) \) commensurable with \( \text{SL}(2, \mathbb{Z}) \), then any modular form of even weight \( k \) with \( (2\epsilon) \nmid k \) has a zero at \( z_0 \) (see, for instance, Proposition A.7 in the appendix). From this property, we see that

(i) If \( \ell \equiv 2 \text{ mod } 4 \), then any modular form of weight \( k \) on \( \Gamma_0(2) \) has a zero at \( \rho_2 \) since \( \rho_2 = (1 + i)/2 \) is an elliptic point of order 2 of \( \Gamma_0(2) \).
(ii) If \( \ell \equiv 4 \text{ mod } 8 \), then any modular form of weight \( k \) on \( \Gamma_0^+(2) \) has a zero at \( \rho_2 \) since \( \rho_2 \) is an elliptic point of order 4 of \( \Gamma_0^+(2) \).

Consider first the case \( \ell \equiv 3, 5 \text{ mod } 8 \). Suppose that \( \epsilon = 1 \). Then \( \hat{m}_i(z) \) are modular forms on \( \Gamma_0^+(2) \) of weight \( \ell - 1 \) and \( \ell + 1 \). By (i) and (ii) above, \( \rho_2 \) is a common zero of \( \hat{m}_1 \) and \( \hat{m}_2(z) \), a contradiction to Lemma 5.7. This proves \( \epsilon = -1 \).
For the remaining cases where $\ell + 1$ or $\ell - 1$ is divisible by 8, we claim that if $k$ is divisible by 8, then any modular form in $\mathcal{M}_k(\Gamma_0(2), -)$ has a zero at $\rho_2$, which then implies that the sign $\epsilon$ must be $1 = \left(\frac{2}{2}\right)$. Indeed, if $8|k$ and $f \in \mathcal{M}_k(\Gamma_0(2), -)$, then evaluating the two sides of the identity

$$f \left(\frac{2z - 1}{2z}\right) = f(Rz) = -\sqrt{2z}^k f(z)$$

at $\rho_2$, we see that $f(\rho_2)$ must be 0 because $(\sqrt{2}\rho_2)^8 = 1$. This completes the proof of the lemma.

With this lemma, we conclude the proof of Theorem 5.2. We now give three examples and show they agree with our theorem.

**Example 5.9.** Consider

$$y''(z) = -4\pi^2 \left(\frac{k}{2}\right)^2 M_4(z) y(z),$$

where $k \in \mathbb{N}$. By Theorem 5.2 if $y_+(z)$ is the unique solution of the form $y_+(z) = q^{k/2}(1 + \sum_{j \geq 1} a_j q^j)$, then $M_8(z) k/2 y_+(z)$ is a quasimodular form in $\mathcal{M}_{4k}^{\leq 1}(\Gamma_0(2), (-1)^k)$. (Note that the function $F(z)$ and the integer $f$ in this case are $M_8(z)k/2$ and $-1 + 4k$, respectively.) Moreover, the vanishing order of $M_8(z)k/2 y_+(z)$ at $\infty$ is $k = \dim \mathcal{M}_{4k}^{\leq 1}(\Gamma_0(2), (-1)^k) - 1$. Hence, $M_8(z)k/2 y_+(z)$ is an extremal quasimodular form. This agrees with Part (i) of Theorem 5.7.

**Example 5.10.** Consider

$$y''(z) = -4\pi^2 \left(\frac{k}{2}\right)^2 M_4(z) - 32 \frac{M_8(z)}{M_4(z)} y(z),$$

where $k \in \mathbb{Z}_{\geq 0}$. We have $\kappa_\infty = k/2$, $\kappa_{\rho_1} = 1/2$, $\kappa_{\rho_2} = 3/2$, and $\kappa = 1/2$ elsewhere. Hence, the function $F(z)$ and the integer $f$ in this case are $F(z) = M_8(z)k/2 M_2(z)$ and $f = -1 + 4k + 2 = 4k + 1$. Theorem 5.2 asserts that if $y_+(z)$ is the unique solution of the form $y_+(z) = q^{k/2}(1 + \cdots)$, then $M_8(z)k/2 M_2(z) y_+(z) = M_4(z) y_+(z)$ is a quasimodular form in $\mathcal{M}_{4k+2}^{\leq 1}(\Gamma_0(2), (-1)^k)$. As in the previous example, we find that it is an extremal quasimodular form, agreeing with Part (ii) of Theorem 5.7.

**Example 5.11.** Consider

$$y''(z) = -4\pi^2 \left(\frac{k}{2}\right)^2 M_4(z) + \frac{128 M_4(z) M_8(z)}{M_4(z)^2 - 256 M_8(z)} y(z), \quad k \in \mathbb{Z}_{\geq 0}.$$ 

We have $\kappa_\infty = k/2$ and by (A.15), $\kappa_{\rho_1} = 3/2$. The integer $f$ in Theorem 5.2 is $k + 3$. According to the theorem, $M_8(z)k/2 M_2(z) y_+(z)/M_4(z)$ is a quasimodular form in $\mathcal{M}_{4k+4}^{\leq 1}(\Gamma_0(2), (-1)^k)$. Indeed, when $k = 1$, we have

$$y_+(z) = q^{1/2}(1 + 70q + 5926q^2 + 503696q^3 + 42822181q^4 + \cdots)$$

and

$$1120 M_8(z)^{1/2} \frac{M_8(z)}{M_2(z)} y_+(z) = M_2''(z) M_8(z) - M_4(z)^2 + 1280 M_8(z),$$

which belongs to $\mathcal{M}_8^{\leq 1}(\Gamma_0(2))$, while when $k = 2$, we have

$$y_+(z) = q + \frac{176}{3} q^2 + \frac{13706}{3} q^3 + \frac{1151072}{3} q^4 + \cdots.$$
and
\[ 19008M_8(z) \frac{M_6(z)}{M_2(z)^2} \eta(z) = M_2^*(z)M_2(z) \left( -M_4(z)^2 + 384M_8(z) \right) + M_2(z)M_4(z)M_6(z) - 288M_2^*(z)M_8(z), \]
which lies in \( \mathcal{M}_2^2(\Gamma_0^+(2), -) \), where \( M_4^*(z) = (4E_4(2z) - E_4(z))/3 \in \mathcal{M}_4(\Gamma_0^+(2), -) \).

6. MODE IN THE CASE OF \( \Gamma_0^+(3) \)

We first recall the following information about \( \Gamma_0^+(3) \).

(i) The group \( \Gamma_0^+(3) \) is generated by \( S = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & -1 \\ 3 & 0 \end{pmatrix} \) and \( R = TS = \frac{1}{\sqrt{3}} \begin{pmatrix} 3 & -1 \\ 3 & 0 \end{pmatrix} \), where \( T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). They satisfy \( S^2 = R^6 = -I \).

(ii) \( \Gamma \) has only one cusp at \( \infty \) with width 1, and two elliptic points \( \rho_1 = i/\sqrt{3} \) (order 2) and \( \rho_2 = (3 + \sqrt{3})/6 \) (order 6). Their stabilizer subgroups are generated by \( T, S, \) and \( R \), respectively.

(iii) The ring \( \oplus_{k=0}^\infty \mathcal{M}_k(\Gamma_1(3)) \), where \( k \) runs through both even and odd integers, is freely generated by
\[ M_1(z) = \sum_{m,n \in \mathbb{Z}} q^{m^2+mn+n^2} \]
and
\[ M_3(z) = \frac{9E_4(3z) - E_4(z)}{8M_1(z)} = \frac{\eta(z)^9}{\eta(3z)^3} - \frac{27\eta(3z)^9}{\eta(z)^3} \]
of weight 1 and 3, respectively.

(iv) Let \( \chi \) be the character of \( \Gamma_0^+(3) \) determined by \( \chi(S) = \chi(R) = -i \). Then \( M_1(z) \in \mathcal{M}_1(\Gamma_0^+(3), \chi) \), \( M_3(z) \in \mathcal{M}_1(\Gamma_0^+(3), \chi) \).

For our purpose, we also need the modular forms
\[ M_4^+(z) = \frac{9E_4(3z) + E_4(z)}{10} = M_1(z)^4 \in \mathcal{M}_4(\Gamma_0^+(3)), \]
\[ M_6^- = \eta(z)^6\eta(3z)^6 = \frac{M_1(z)^6 - M_3(z)^2}{108} \in \mathcal{M}_6(\Gamma_0^+(3), -) \]
Up to \( \Gamma_0^+(3) \)-equivalence, \( M_1(z), M_3(z), \) and \( M_6^- \) have simple zeros at \( \rho_2, \rho_1, \) and \( \infty \), respectively. The ratio \( M_1(z)^6/M_6^- \) is a Hauptmodul for \( \Gamma_0^+(3) \). Let also
\[ M_2^*(z) := \frac{1}{2\pi i} \left( \frac{M_6^-}{M_6} \right)'(z) = \frac{3E_2(3z) + E_2(z)}{4}, \]
which is a quasimodular form of weight 2 and depth 1 on \( \Gamma_0^+(3) \) and satisfies
\[ (M_2^*|2\gamma)(z) = M_2^*(z) + \frac{3c}{\pi i(cz+d)}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^+(3). \]

Consider the differential equation
\[ y''(z) = -4\pi^2 Q(z)y(z), \]
where \( Q(z) \) is a meromorphic modular form of weight 4 on \( \Gamma_0^+(3) \), and assume that it satisfies the condition \( (6.2) \). Let \( z_1, \ldots, z_m \) be \( \Gamma_0^+(3) \)-inequivalent singularities of \( (6.2) \) other than \( \rho_1 \) and \( \rho_2 \). Let \( \pm \kappa_\infty, 1/2 \pm \kappa_{\rho_1}, 1/2 \pm \kappa_{\rho_2}, \) and \( 1/2 \pm \kappa_j, j = 1, \ldots, m, \)
be the local exponents of (6.2) at $\infty, \rho_1, \rho_2$, and $z_j$, respectively. For $j = 1, \ldots, m$, let $t_j = M_2^{-}(z_j)^3/M_6^{-}(z_j)$, and set

$$F_j(z) = M_2^{-}(z)^3 - t_j M_6^{-}(z).$$

Define

$$F(z) = M_6^{-}(z)\kappa_\infty M_2(z)\kappa_{\rho_1} - 1/2 M_1(z)\kappa_{\rho_2} - 1/2 \prod_{j=1}^{m} F_j(z)\kappa_j - 1/2,$$

and set

$$\ell = -1 + 6\kappa_\infty + 3 \left(\kappa_{\rho_1} - \frac{1}{2}\right) + \left(\kappa_{\rho_2} - \frac{1}{2}\right) + 6 \sum_{j=1}^{m} \left(\kappa_j - \frac{1}{2}\right).$$

Note that by Condition (H\textsubscript{E}), both $\kappa_{\rho_1} - 1/2$ and $\kappa_{\rho_2} - 1/2$ are integers, and $(2\kappa_{\rho_2}, 3) = 1$. Hence $\ell$ is an integer not divisible by 3. For any function $f(z)$ on $\mathbb{H}$, we let $\hat{f}(z)$ denote

$$\hat{f}(z) = F(z)f(z).$$

Note that $F(z)^2$ is a holomorphic modular form of weight $2(\ell + 1)$ on $\Gamma_0^+(3)$ with some character depending on $\ell$. By construction, for any solution $y(z)$ of (6.2), the function $\hat{y}(z)$ is a single-valued function holomorphic throughout $\mathbb{H}$. Also, its order at a singularity $z_j$ is either 0 or $2\kappa_j$, and a similar property holds for $\infty, \rho_1$, and $\rho_2$. Hence, for any fundamental solution $(y_1(z), y_2(z))$ of (6.2), we have a representation $\hat{\rho} : \Gamma_0^+(3) \to \text{GL}(2, \mathbb{C})$ given by

$$\left(\begin{array}{c} \hat{y}_1 \\gamma \\ \hat{y}_2 \\gamma \end{array}\right) = \hat{\rho}(\gamma) \left(\begin{array}{c} y_1 \\ y_2 \end{array}\right).$$

We now state the main result of this section (Theorem 3.8 of Section II).

**Theorem 6.1.** Suppose that $Q(z)$ satisfies (H\textsubscript{E}) with $\kappa_\infty \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Let $\ell$ be the integer defined by (6.4). Set

$$\delta = \begin{cases} \chi^0, & \text{if } \ell \equiv 1, 11 \text{ mod } 12, \\ \chi^1, & \text{if } \ell \equiv 2, 4 \text{ mod } 12, \\ \chi^2, & \text{if } \ell \equiv 5, 7 \text{ mod } 12, \\ \chi^3, & \text{if } \ell \equiv 8, 10 \text{ mod } 12, \end{cases}$$

where $\chi$ is the character of $\Gamma_0^+(3)$ defined by $\chi(S) = \chi(R) = -i$. The following statements hold.

(i) The differential equation (6.2) is not apparent at $\infty$.

(ii) Let $y_+(z) = y_+(z)$ and $\hat{y}_1(z) = (\hat{y}_+(z))(Z)$. Then $\hat{y}_1(z) = z\hat{y}_2(z) + \hat{m}_1(z)$ for some modular form $\hat{m}_1(z)$ in $\mathcal{M}_{\ell+1}(\Gamma_0^+(3), \delta)$.

(iii) The ratio $h(z) = \delta(S)^{-1}y_1(z)/\sqrt{3}y_2(z)$ is equivariant. That is, for all $\gamma \in \Gamma_0^+(3)$, we have $h(\gamma z) = \gamma \cdot h(z)$.

(iv) Write $\hat{y}_+(z)$ as

$$\hat{y}_+(z) = \delta(S)^{-1} \frac{\pi i}{3\sqrt{3}} \hat{m}_1(z)M_2(z) + \hat{m}_2(z).$$

Then $\hat{m}_2(z)$ is a modular form in $\mathcal{M}_{\ell+1}(\Gamma_0^+(3), \delta)$. Hence, $\hat{y}_+(z)$ is a quasimodular form in $\mathcal{M}_{\ell+1}^{\leq 1}(\Gamma_0^+(3), \delta)$. 

Note that when \( k \) is even, we have \( \mathcal{M}_k(\Gamma_0^+(3), \chi^0) \) and \( \mathcal{M}_k(\Gamma_0^+(3), \chi^2) \) are simply \( \mathcal{M}_k(\Gamma_0^+(3)) \) and \( \mathcal{M}_k(\Gamma_0^+(3), -) \), respectively, so \( \mathcal{M}_{\gamma}^{\pm 1}(\Gamma_0^+(3), \delta) \) in the case of odd \( \ell \) can also be written as \( \mathcal{M}_{\ell+1}^{\pm 1}(\Gamma_0^+(3), (\frac{\ell}{3})) \).

Apart from the difference that the integer \( \ell \) can be even, the proof is very similar to those of Theorems 4.3 and 5.2

**Lemma 6.2.** We have \( \det \hat{\rho}(T) = 1 \) and \( \det \hat{\rho}(S) = (-1)^{\ell-1} \).

**Proof.** Following the proof of Lemma 4.2, we have

\[
\det \hat{\rho}(\gamma) = \left( \frac{F(z)^2}{2(\ell+1)\gamma} \right) \frac{\hat{\rho}(\gamma)}{F(z)}
\]

for \( \gamma \in \Gamma_0^+(3) \). It is clear that when \( \gamma = T \), we have \( \det \hat{\rho}(T) = 1 \). For the case \( \gamma = S \), we note that

\[
\frac{M_6}{M_6^2} = \frac{M_2^2}{M_2^2} = \frac{M_2^2}{M_2^2} = \frac{F_1}{F_1} = -1
\]

It follows that

\[
\det \hat{\rho}(S) = (-1)^{2\kappa_\infty + (\kappa_{\nu_1} - 1/2) + \kappa_{\nu_2} - 1/2 + \sum_j (2\kappa_j - 1)} = (-1)^\ell - 1.
\]

This proves the lemma.

The proofs of the next two lemmas are different from the ones in the case of \( \text{SL}(2, \mathbb{Z}) \) and \( \Gamma_0^+(2) \) because \( F_0 = -I \).

**Lemma 6.3.** The differential equation (6.2) is never apparent at \( \infty \).

**Proof.** If (6.2) is apparent at \( \infty \), then with respect to any basis, we have \( \hat{\rho}(T) = I \). Therefore, there exists a basis \((\hat{y}_1(z), \hat{y}_2(z))\) such that \( \hat{\rho}(T) \), \( \hat{\rho}(S) \), and \( \hat{\rho}(R) \) are all diagonal with the diagonal entries of \( \hat{\rho}(R) \) lying inside \( \{\pm 1, \pm i\} \), which implies that \( \hat{y}_1^{\gamma} \) and \( \hat{y}_2^{\gamma} \) are both modular forms of weight \( 2\ell \) on \( \Gamma_0(3) \). (Note that \( \Gamma_0(3) \) is generated by \( T \) and \( R^2 \).) Hence, the order of \( \hat{y}_1^{\gamma} \) at \( \rho_2 \) is congruent to \( 2\ell \) modulo 3 (see Proposition A.7), which is congruent to \( 2\kappa_\nu_2 \) modulo 3. However, this contradicts to the fact that the order of \( \hat{y}_1^{\gamma} \) at \( \rho_2 \) is either 0 or \( 4\kappa_\nu_2 \). (Recall that by (H.1), \( 3 \nmid 2\kappa_\nu_2 \).) We conclude that (6.2) cannot be apparent at \( \infty \), under the condition (H.1).

**Lemma 6.4.** The solutions \( y_+|_{-1}S \) and \( y_+ \) are linearly independent.

**Proof.** Suppose that \( y_+|_{-1}S \) and \( y_+ \) are linearly dependent. Then \( \hat{y}_+|_{-1}S = \lambda y_+ \) for some \( \lambda \in \mathbb{C} \). Then with respect to the basis \((\hat{y}_-, \hat{y}_+)\), we have

\[
\hat{\rho}(T) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad \hat{\rho}(S) = \begin{pmatrix} (-1)^{\ell-1}/\lambda & c \\ 0 & \lambda \end{pmatrix}
\]

for some complex numbers \( b \neq 0 \) and \( c \). Since \( \hat{\rho}(S)^2 = \hat{\rho}(-I) = (-1)^{\ell}I \), we have \( \lambda \in \{\pm 1\} \) when \( \ell \) is odd and \( \lambda \in \{\pm i\} \) when \( \ell \) is even. In either case, \( \hat{y}_+^2 \) is a modular form of weight \( 2\ell \) on \( \Gamma_0(3) \). By the same reasoning as in the proof of Lemma 6.3, we find that this yields a contradiction. Hence, \( y_+|_{-1}S \) is linearly independent of \( y_+ \).

Since (6.2) is not apparent at \( \infty \) and \( y_+|_{-1}S \) and \( y_+ \) are linearly independent, there is a nonzero constant \( d \) and \( m_1(z) \) with \( m_1(z) = q^{-\kappa}(a_0 + \sum_{j \geq 1} a_j q^j), a_0 \neq 0 \), such that

(6.5)

\[
\left( \hat{y}_+|_{-1}S \right)(z) = dz \hat{y}_+(z) + \hat{m}_1(z)
\]

(and \( dz y_+(z) + m_1(z) \) is a solution of (6.2)).
Lemma 6.5. If \( \ell \) is even, then \( d = \pm \sqrt{-3} \) and if \( \ell \) is odd, then \( d = \pm \sqrt{3} \).

Proof. We use \( \hat{y}_+ |_\ell S \) and \( \hat{y}_+ \) as a basis. By Lemma 6.2,

\[
\hat{\rho}(R) = \hat{\rho}(T) \hat{\rho}(S) = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & (-1)^\ell \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & (-1)^\ell \\ 1 & 0 \end{pmatrix},
\]

which shows that the characteristic polynomial of \( \hat{\rho}(R) \) is \( x^2 - dx + (-1)^\ell \). On the other hand, we have \( \hat{\rho}(R)^6 = \hat{\rho}(-I) = (-1)^\ell I \). When \( \ell \) is even, the eigenvalues of \( \hat{\rho}(R) \) are two sixth roots of unity \( \lambda_1 \) and \( \lambda_2 \) such that \( \lambda_1 \lambda_2 = -1 \) and \( \lambda_1 + \lambda_2 = d \neq 0 \). We find that \( d \) must be one of \( \pm \sqrt{-3} \). Likewise, when \( \ell \) is odd, the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of \( \hat{\rho}(R) \) satisfy \( \lambda_1 \lambda_2 = 1 \) and \( \lambda_1^6 = \lambda_2^6 = -1 \), but \( \lambda_1 + \lambda_2 = d \neq 0 \). We find that \( d \) must be of one \( \pm \sqrt{3} \).

Lemma 6.6. Let \( h(z) = y_1(z)/dy_2(z) \). Then \( h(Sz) = S \cdot h(z) \).

Proof. The proof is almost the same as that of Lemma 5.4. The only difference is that when \( \ell \) is even, we have \( \hat{\rho}(S) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \), but in this case we have \( d^2 = -3 \), by the lemma above, and \( h(Sz) = S \cdot h(z) \) still holds. \( \square \)

Define \( m_2(z) \) by

\[
y_+(z) = \frac{\pi i}{3d} m_1(z) M_2^\ell (z) + m_2(z),
\]

and let \( \epsilon = d/\sqrt{3} \).

Lemma 6.7. The following holds.

(i) \( \hat{m}_1(z + 1) = \hat{m}_1(z) \) and \( \hat{m}_1 |_{\ell-1} S \) (z) = \( \epsilon \hat{m}_1 (z) \).

(ii) \( \hat{m}_2(z + 1) = \hat{m}_2(z) \) and \( \hat{m}_2 |_{\ell+1} S \) (z) = \( \epsilon \hat{m}_2 (z) \).

Proof. The proof follows exactly those of Lemmas 5.5 and 5.6 and is omitted. \( \square \)

It remains to determine \( \epsilon \).

Lemma 6.8. We have \( \epsilon = \delta(S) \).

Proof. We first note that Lemma 5.7 is also valid here. Our strategy is to show that if \( \epsilon \neq \delta(S)(= \delta(R)) \), then \( \hat{m}_1(z) \) and \( \hat{m}_2(z) \) have a common zero at \( \rho_2 \).

When \( \ell \) is odd, the argument is the same as that in the proof of Lemma 5.8. We shall skip the proof in this case.

Now consider the case \( \ell \) is even. Since \( \ell \) is not divisible by 3, exactly one of \( \ell - 1 \) and \( \ell + 1 \) is not a multiple of 3. Let \( k \) be the one in \( \{ \ell - 1, \ell + 1 \} \) that is a multiple of 3, and \( k' \) be the other. Since \( \rho_2 \) is an elliptic point of order 3 on \( X_0(3) \), any modular form of weight \( k' \) has a zero at \( \rho_2 \). On the other hand, if \( f \in \mathcal{M}_k(\Gamma_0^+ (3), \chi) \), then we have

\[
f(Rz) = \chi(R)(\sqrt{3}z)^k f(z).
\]

Observe that \( \sqrt{3} \rho_2 = (\sqrt{3} + i)/2 = e^{2\pi i/12} \). If \( k \equiv 9 \mod 12 \), then \( \chi(R)(\sqrt{3} \rho_2) = (-i)(-i) = -1 \) and hence \( f(\rho_2) = 0 \). Therefore, if \( k \equiv 9 \mod 12 \), i.e., if \( \ell \equiv 8, 10 \mod 12 \), the assumption that \( \epsilon = -i \) leads to a contradiction. Likewise, if \( \ell \equiv 2, 4 \mod 12 \), then \( \epsilon \) cannot be \( i \). This proves the lemma. \( \square \)

Having completed the proof of the theorem, we now give some examples.
Example 6.9. Consider the differential equation
\[ y''(z) = -4\pi^2 \left( \frac{k}{2} \right)^2 M_4^+(z)y(z), \]
where \( k \) is a positive integer. The function \( F(z) \) and the integer \( \ell \) in this case are \( M_6^-(z)k/2 \) and \(-1+3k\). The character \( \delta \) in Theorem 6.1 is \( \chi^k \) and the theorem asserts that \( F(z)y_+(z) \) is a quasimodular form in \( \mathcal{M}_{3k}^{\leq 1}(\Gamma_0^+(3), \chi^k) \). Indeed, according to Part (i) of Theorem 3.8 if \( f(z) \) is an extremal quasimodular form in \( \mathcal{M}_{3k}^{\leq 1}(\Gamma_0^+(3), \chi^k) \), then \( f(z)/M_6^-(z)k/2 \) is a solution of the differential equation above.

Example 6.10. Consider the differential equation
\[ y''(z) = -4\pi^2 \left( \frac{k}{2} \right)^2 M_4^+(z) - 18 \frac{M_6^-(z)}{M_2(z)} y(z), \]
where \( k \) is a positive integer and \( M_2(z) = (3E_2(3z) - E_2(z))/2 = M_1(z)^2 \). The function \( F(z) \) and the integer \( \ell \) in this case are \( M_6^-(z)k/2M_2(z) \) and \( 3k + 1 \), respectively, with \( \kappa_\rho = 5/2 \). The character \( \delta \) in this case is again \( \chi^k \) and the theorem asserts that \( F(z)y_+(z) \) is a quasimodular form in \( \mathcal{M}_{3k+2}^{\leq 1}(\Gamma_0^+(3), \chi^k) \). This agrees with Part (ii) of Theorem 3.8.

Example 6.11. Consider the differential equation
\[ y''(z) = -4\pi^2 \left( \frac{k}{2} \right)^2 M_4^+(z) + 54 \frac{M_4^+(z)M_6^-(z)}{M_3(z)^2} y(z), \quad k \in \mathbb{Z}_{\geq 0}. \]
We have \( \kappa_\infty = k/2 \) and by (A.13), \( \kappa_{\rho_1} = 3/2 \). The integer \( \ell \) in Theorem 6.1 is \( 3k + 2 \) and the character \( \delta \) is \( \chi^{k+1} \). The theorem predicts that \( M_6^-(z)k/2M_3(z)y_+(z) \) belongs to the space \( \mathcal{M}_{3k+3}^{\leq 1}(\Gamma_0^+(3), \chi^{k+1}) \). Indeed, for \( k = 0 \), we compute that
\[ y_+(z) = 1 + 54q + 1944q^2 + 73092q^3 + 2749032q^4 + \cdots \]
and find that
\[ 2M_3(z)y_+(z) = 3M_2(z)M_1(z) - M_3(z) \in \mathcal{M}_{3}^{\leq 1}(\Gamma_0^+(3), \chi). \]
For \( k = 1 \), we have
\[ y_+(z) = q^{1/2} \left( 1 + 30q + 1119q^2 + 42077q^3 + 1582920q^4 + \cdots \right) \]
and
\[ 360M_6(z)^{1/2}M_3(z)y_+(z) = M_2(z)M_1(z)M_3(z) + 3M_1(z)^6 - 4M_3(z)^2, \]
which does lie in \( \mathcal{M}_{6}^{\leq 1}(\Gamma_0^+(3), \chi^2) = \mathcal{M}_{6}^{\leq 1}(\Gamma_0^+(3), -) \). For \( k = 2 \), we find that
\[ y_+(z) = q + 26q^2 + 888q^3 + 32818q^4 + 1231645q^5 + \cdots \]
and
\[ 181446M_6(z)M_3(z)y_+(z) = M_2(z)(3M_1(z)^7 - 9M_1(z)M_3(z)^2) - M_1(z)^6M_3(z) + 7M_3(z)^3, \]
which indeed belongs to \( \mathcal{M}_{6}^{\leq 1}(\Gamma_0^+(3), \chi^3) \). For \( k = 3 \), we have
\[ y_+(z) = q^{3/2} \left( 1 + 27q + \frac{4131}{5}q^2 + \frac{146031}{5}q^3 + \frac{5426163}{5}q^4 + \cdots \right) \]
and

\[2138400M_0(z)^{3/2}M_3(z)y_+(z) = M_3^2(z)\left(M_1(z)^7M_3(z) + 35M_1(z)M_3(z)^3\right) + 3M_1(z)^{12} - 19M_1(z)^6M_3(z)^2 - 20M_3(z)^4,\]

which is indeed lying in \(\mathcal{H}^{-1}_3(\Gamma_0^+(3)).\)

**Appendix A. Existence of \(Q(z)\) satisfying \((H_1)\)**

Let \(\Gamma\) be a subgroup of \(\text{SL}(2, \mathbb{R})\) commensurable with \(\text{SL}(2, \mathbb{Z})\) and \(Q(z)\) be a meromorphic modular form of weight 4 on \(\Gamma\). Consider the modular differential equation

(A.1) \[y''(z) = -4\pi^2Q(z)y(z), \quad z \in \mathbb{H}.\]

We assume that the differential equation is Fuchsian, i.e., that the order of pole at any pole of \(Q(z)\) is \(\leq 2\). Let \(z_0 \in \mathbb{H}\) be a pole of \(Q(z)\). In Section 3 we have discussed how to compute the local exponents and determine whether (A.1) is apparent at \(Q\) properties of \(Q\), here we shall introduce another type of expansions of modular forms.

For an integer \(k\), let \(\partial_k\) be the Shimura-Maass operator defined in Definition 2.2. By Lemma 2.3, when \(f\) is a nearly holomorphic modular form of weight \(k\) on \(\Gamma\), \(\partial_k f\) is a nearly holomorphic modular form of weight \(k + 2\) on \(\Gamma\). For a positive integer \(n\), we let

\[\partial^n_k = \partial_{k+2n-2}\partial_{k+2n-4}\ldots\partial_{k+2}\partial_k.\]

If the weight of \(f\) is clear from the context, we will simply write \(\partial^n_k\) as \(\partial^n\).

**Proposition A.1** ([18 Proposition 17]). Assume that \(f(z)\) is a (holomorphic) modular form of weight \(k\) on \(\Gamma\). Then for a point \(z_0 \in \mathbb{H}\), we have

(A.2) \[f(z) = (1 - w)^k \sum_{n=0}^{\infty} \frac{\partial^n f(z_0)(-4\pi \text{Im} z_0)^n}{n!} w^n, \quad w = \frac{z - z_0}{z - \overline{z_0}}.\]

We will call the expansion of a meromorphic modular form \(f(z)\) of weight \(k\) of the form

\[f(z) = (1 - w)^k \sum_{n=n_0}^{\infty} a_n w^n, \quad w = \frac{z - z_0}{z - \overline{z_0}},\]

the power series expansion of \(f(z)\) at \(z_0\). Note that there is a misprint in [18 Proposition 17], in which a minus sign is missing.

One of the main advantages to use the power series expansion in \(w = (z - z_0)/(z - \overline{z_0})\) instead of the usual expansion in \(z - z_0\) is that the coefficients can be recursively computed. To describe the recursion, let us first recall the definition of Serre’s derivative. In the case of \(\Gamma = \text{SL}(2, \mathbb{Z})\), the well-known Ramanujan’s identities state that

\[D_q E_2(z) = \frac{E_2(z)^2 - E_4(z)}{12},\]

(A.3) \[D_q E_4(z) = \frac{E_2(z)E_4(z) - E_6(z)}{3},\]

\[D_q E_6(z) = \frac{E_2(z)E_6(z) - E_4(z)^2}{2},\]

where \(D_q = qd/dq\). Generalizing these identities, we can easily verify that if \(f(z)\) is a modular form of weight \(k\) on \(\text{SL}(2, \mathbb{Z})\), then \(D_q f(z) - kE_2(z)f(z)/12\) is a modular form.
of weight $k + 2$. Then the Serre’s derivative $\vartheta_k$ of weight $k$ in the case of $\Gamma = \text{SL}(2, \mathbb{Z})$ is defined to

$$\vartheta_k f := D_q f - \frac{k}{12} E_2 f.$$  

For a general group $\Gamma$, we choose a quasimodular form $\phi$ of weight 2 and depth 1 on $\Gamma$ and let $\alpha$ be the nonzero number such that

$$(\phi|_2 \gamma)(z) = \phi(z) + \frac{\alpha c}{2\pi i (cz + d)}$$

for all $\gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma$. Then we can analogously prove that if $f$ is a modular form of weight $k$ on $\Gamma$, then $D_q f - k\phi f/\alpha$ is a modular form of weight $k + 2$ on $\Gamma$. Also,

$$M(z) := D_q \phi(z) - \frac{\phi(z)^2}{\alpha}$$

is a modular form of weight 4 on $\Gamma$.

**Definition A.2.** The differential operator $\vartheta_k : \mathcal{M}_k(\Gamma) \to \mathcal{M}_{k+2}(\Gamma)$ defined by

$$\vartheta_k f := D_q f - \frac{k}{\alpha} \phi f$$

is called Serre’s derivative of weight $k$ (with respect to $\phi$). If the weight of $f$ is clear from the context, we will omit the subscript $k$. We then define recursively $\vartheta^{[n]} f$ by

$$\vartheta^{[0]} f = f, \quad \vartheta^{[1]} f = \vartheta f,$$

and

$$\vartheta^{[n+1]} f = \vartheta(\vartheta^{[n]} f) + \frac{n(k + n - 1)}{\alpha} M \vartheta^{[n-1]} f$$

for $n \geq 1$.

The Shimura-Maass differential operators and Serre’s derivatives are connected through the following combinatorial identity.

**Proposition A.3.** Set

$$\phi^*(z) = \phi(z) + \frac{\alpha}{2\pi i (z - \overline{z})}.$$  

For a modular form $f(z)$ of weight $k$ on $\Gamma$, define two formal power series

$$\tilde{f}_\vartheta(z, X) := \sum_{n=0}^{\infty} \frac{\vartheta^n f(z)}{n!(k)_n} X^n$$

and

$$\tilde{f}_\vartheta(z, X) := \sum_{n=0}^{\infty} \frac{\vartheta^{[n]} f(z)}{n!(k)_n} X^n,$$

where $(k)_n = k(k + 1) \ldots (k + n - 1)$ is the Pochhammer symbol. Then we have

$$\tilde{f}_\vartheta(z, X) = e^{X \phi^*(z)/\alpha} \tilde{f}_\vartheta(z, X)$$

as formal power series in $X$.

**Proof.** See [18 Section 5.2].
Another way to state the proposition is that
\[ \partial^n f(z_0) = \sum_{r=0}^{n} \frac{n!}{r!} (k + n - 1)! \left( \frac{\phi^*(z_0)}{\alpha} \right)^{n-r} \vartheta^{[r]} f(z_0) \]
for all \( n \). This gives a formula for the coefficients of the power series expansion of \( f \). However, while the values \( \vartheta^{[r]} f(z_0) \) can be computed recursively using identities analogous to (A.3), the determination of \( \phi^*(z_0) \) can be problematic in practice, so we will introduce yet another type of expansions using a different local parameter for \( z_0 \).

Letting \( y_0 = \text{Im} z_0 \), by (A.2), we have
\[
\begin{align*}
 f(z) &= (1 - w)^k \sum_{n=0}^{\infty} w^n (-4\pi y_0)^n \\
 &= (1 - w)^k \sum_{r=0}^{\infty} \frac{\vartheta^{[r]} f(z_0)}{r!} (-4\pi y_0 w)^r \\
 &= (1 - w)^k \sum_{r=0}^{\infty} \frac{(k + n + r - 1)!}{n!(k + r - 1)!} \left( -\frac{4\phi^*(z_0)\pi y_0}{\alpha} \right)^r
\end{align*}
\]
Using
\[ (1 + x)^{-k-r} = \sum_{n=0}^{\infty} \frac{(k + n + r - 1)!}{n!(k + r - 1)!} (-x)^n, \]
we find that
\[ f(z) = \left( \frac{1 - w}{1 + Aw} \right)^k \sum_{r=0}^{\infty} \frac{\vartheta^{[r]} f(z_0)}{r!} \left( -\frac{4\pi y_0 w}{1 + Aw} \right)^r, \quad A = \frac{4\phi^*(z_0)\pi y_0}{\alpha}. \]

Let
\[ \tilde{w} = \frac{w}{1 + Aw}, \]
which is also a local parameter at \( z_0 \). Observing that
\[ \frac{1 - w}{1 + Aw} = 1 - (1 + A)\tilde{w}, \]
we obtain the following series expansion of \( f \).

**Proposition A.4.** We have
\[ f(z) = (1 - (1 + A)\tilde{w})^k \sum_{r=0}^{\infty} \frac{\vartheta^{[r]} f(z_0)}{r!} (-4\pi y_0 \tilde{w})^r. \]

The advantage of this expansion in \( \tilde{w} \) over that in \( w \) is that there is no need to compute \( \phi^*(z_0) \).

In order to use Proposition A.4 to compute the local exponents and determine whether (3.4) is apparent at \( z_0 \), we need the following lemma.

**Lemma A.5.** Let \( z_0 \in \mathbb{H} \) be a pole of \( Q(z) \) and assume that \( \tilde{Q}(x) = \sum_{n \geq -2} a_n x^n \) is the power series such that
\[ Q(z) = (1 - (1 + A)\tilde{w})^4 \tilde{Q}(-4\pi \text{Im} z_0 \tilde{w}), \]
where \( \tilde{w} \) is given by (A.4). Then

\[
y(z) = \frac{1}{1 - (1 + A)\tilde{w}} \sum_{n=0}^{\infty} c_n (-4\pi(\text{Im} z_0)\tilde{w})^n a
\]

is a solution of (A.1) if and only if the series \( \tilde{y}(x) = \sum_{n=0}^{\infty} c_n x^{n+\alpha} \) satisfies

\[
\frac{d^2}{dx^2} \tilde{y}(x) = \tilde{Q}(x)\tilde{y}(x),
\]

i.e., if and only if

\[
\alpha^2 - \alpha - a_{-2} = 0
\]

and

\[
((\alpha + n)(\alpha + n - 1) - a_{-2}) c_n = \sum_{j=0}^{n-1} a_{n-j-2} c_j
\]

holds for all \( n \geq 1 \).

**Proof.** Recall that Bol’s identity states that if \( f \) is a twice-differential function and \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{GL}(2, \mathbb{C}) \), then

\[
\left( f \big|_{-1} \gamma \right)''(z) = \left( f'' \big|_{3} \gamma \right)(z).
\]

Let \( x = \gamma z \). We observe that

\[
a - cx = a - c\frac{ax + b - \text{det} \gamma}{cz + d} = \frac{\text{det} \gamma}{cz + d}.
\]

Hence the factor \((\text{det} \gamma)^{1/2}/(cz + d)\) appearing in the slash operator can be written as

\[
\frac{(\text{det} \gamma)^{1/2}}{cz + d} = \frac{a - cx}{(\text{det} \gamma)^{1/2}}.
\]

Therefore, Bol’s identity can be also written as

\[
\frac{d^2}{dz^2} \left( \frac{(\text{det} \gamma)^{1/2}}{a - cx} f(x) \right) = \frac{(a - cx)^3}{(\text{det} \gamma)^{3/2}} \frac{d^2}{dx^2} f(x).
\]

Applying this version of Bol’s identity with

\[
\gamma = \left( \begin{array}{cc} -4\pi\text{Im} z_0 & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ A & 1 \end{array} \right) \left( \begin{array}{cc} 1 & -z_0 \\ 1 & -z_0 \end{array} \right) = \left( \begin{array}{cc} -4\pi\text{Im} z_0 & (4\pi\text{Im} z_0) z_0 \\ 1 + A & -A z_0 - \overline{z_0} \end{array} \right),
\]

\[
\text{det} \gamma = -4\pi(\text{Im} z_0)(z_0 - \overline{z_0}),
\]

\( x = \gamma z = -4\pi(\text{Im} z_0)\tilde{w}, \) and \( f(x) = \tilde{y}(x) \), we obtain

\[
\frac{d^2}{dz^2} \left( \frac{1}{1 - (1 + A)\tilde{w}} \tilde{y}(-4\pi(\text{Im} z_0)\tilde{w}) \right) = -4\pi^2(1 - (1 + A)\tilde{w})^3 \frac{d^2}{dx^2} \tilde{y}(x).
\]

The left-hand side of this identity is simply \( y''(z) \), where \( y(z) \) is the function in (A.5). From this, we see that (A.5) is a solution of (A.1) if and only if \( \tilde{y}(x) \) is a solution of (A.6).

**Remark A.6.** In view of (A.7), we may write the two local exponents at \( z_0 \) at \( 1/2 \pm \kappa \) for some \( \kappa \geq 0 \). When \( \kappa \notin \frac{1}{2} \mathbb{Z} \geq 0 \), the term \((\alpha + n)(\alpha + n - 1) - a_{-2}\) on the left-hand side of (A.8) is never 0 for both \( \alpha = 1/2 \pm \kappa \) and any \( n \geq 1 \). In such a case, we can recursively determine \( c_n \) and get two linearly independent solutions of (A.1). On the other hand, when \( \kappa \in \frac{1}{2} \mathbb{Z} \), we can always get a solution of the form \( \tilde{w}^{1/2 + \kappa}(1 + O(\tilde{w})) \). However, when \( \alpha = 1/2 - \kappa \), we can only solve for \( c_n \) recursively up to \( n = 2\kappa - 1 \) because when \( n = 2\kappa \), the left-hand side of (A.8) becomes 0. In order for (A.1) to have a solution of the...
form $\bar{w}^{1/2-n}(1 + O(\bar{w}))$, i.e., in order for (A.1) to be apparent at $z_0$, we need the numbers $c_0, \ldots, c_{2\kappa - 1}$ to satisfy

$$
(\text{A.9}) \sum_{j=0}^{2\kappa - 1} a_{2\kappa - j} c_j = 0.
$$

Note that this is the only condition required.

We now consider the special case where $z_0$ is an elliptic point of $\Gamma$.

**Proposition A.7.** Assume that the stabilizer subgroup $\Gamma_{z_0}$ of $z_0 \in \mathbb{H}$ in $\Gamma$ has order $N$. Let

$$f(z) = (1 - (1 + A)\bar{w})^k \sum_{n=0}^{\infty} a_n \bar{w}^n$$

be the power series expansion in $\bar{w}$ of a meromorphic modular form $f$ of weight $k$ on $\Gamma$ at $z_0$, where $\bar{w}$ is given by (A.4). Then $a_n = 0$ whenever $k + 2n \not\equiv 0 \pmod{N}$.

**Proof.** The analogous statement for (A.2) is already known in literature (for example, see [6] for the case $k$ is even). For the convenience of the reader, we give a proof here.

Since any meromorphic modular form is the ratio of two holomorphic modular forms, it suffices to prove the proposition under the assumption that $f$ is a holomorphic modular form. By Proposition A.4, $a_n$ is a multiple of $\vartheta^{[n]} f(z_0)$ and the proof reduces to showing that $\vartheta^{[n]} f(z_0) = 0$ whenever $k + 2n \not\equiv 0 \pmod{N}$.

Recall that $\Gamma_{z_0}$ is a finite cyclic group (see [16, Proposition 1.16]), say, $\Gamma_{z_0}$ is generated by $\gamma = (a \ b \ c \ d)$ of order $N$. Since $z_0$ is fixed by $\gamma$, we have

$$z_0 = a - d + \sqrt{(d - a)^2 + 4bc \quad a - d + \sqrt{(a + d)^2 - 4}} \quad \frac{2c}{2c}$$

and

$$cz_0 + d = \frac{a + d + \sqrt{(a + d)^2 - 4}}{2}.$$

Observe that it is a root of the characteristic polynomial $x^2 - (a + d)x + 1$ of $\gamma$, i.e., an eigenvalue of $\gamma$. Hence, $cz_0 + d$ is a primitive $N$th root of unity. Now $\vartheta^{[n]} f$ is a holomorphic modular form of weight $k + 2n$ on $\Gamma$, i.e.,

$$\vartheta^{[n]} f(\gamma z) = (cz + d)^{k + 2n} \vartheta^{[n]} f(z).$$

Evaluating the two sides at $z_0$, we see that $\vartheta^{[n]} f(z_0) = 0$ whenever $N \nmid (k + 2n)$. This proves the proposition. \hfill \Box

Using this proposition, we immediately obtain a sufficient condition for (A.1) to be apparent at an elliptic point.

**Theorem A.8.** Assume that $z_0$ is an elliptic point of order $e$ on $X(\Gamma)$ and the local exponents of (A.1) at $z_0$ is $1/2 \pm \kappa$ for some $\kappa \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ such that $e \nmid (2\kappa z_0)$, then (A.1) is apparent at $z_0$.

**Proof.** Let $Q(z) = (1 - (1 + A)\bar{w})^k \sum_{n=-\infty}^{\infty} a_n \bar{w}^n$ be the expansion of $Q(z)$ in $\bar{w}$. Since $Q(z)$ has weight 4, we may assume that $\Gamma$ contains $-I$. Then the stabilizer subgroup $\Gamma_{z_0}$ of $z_0$ in $\Gamma$ has order $2e$ (see [16, Proposition 1.20]). Hence, by Proposition A.7, $a_n = 0$
whenever $4 + 2n \neq 0 \mod 2e$. Now the differential equation (A.1) is apparent if and only if it has a solution of the form

$$y(z) = \frac{1}{1 - (1 + A)w} \sum_{n=0}^{\infty} c_n w^n, \quad c_0 \neq 0.$$ 

By Lemma [A.5], the coefficients $c_n$ must satisfy (A.8). Since $a_n = 0$ whenever $n \neq -2 \mod e$, we can inductively prove using (A.8) that $c_n = 0$ whenever $e \nmid n$, up to $n = 2\kappa - 1$. Then the apparentness condition (A.9) holds if $2\kappa - j - 2 \equiv -2 \mod e$ and $j \equiv 0 \mod e$ cannot hold simultaneously, by the assumption that $e \nmid (2\kappa, \kappa)$. This proves the theorem.

In the remainder of the section, we will specialize $\Gamma$ to one of the three groups $\text{SL}(2, \mathbb{Z}), \Gamma_0^+(2)$, and $\Gamma_0^+(3)$, and prove the existence of $Q(z)$ satisfying (H$_5$)

Recall that $\rho_1$ and $\rho_2$ are the two elliptic points of $\Gamma$. Given $\{\rho_1, \rho_2, z_1, \ldots, z_m\}$ with $\rho_1, \rho_2, z_1, \ldots, z_m$ mutually $\Gamma$-inequivalent and $\{\kappa_2, \kappa_2, \kappa_1, \ldots, \kappa_m\} \in \frac{1}{2}\mathbb{N}$, where $(2\kappa_1, e_1) = 1$ for $i = 1, 2$, we want to show the existence of a meromorphic modular form $Q(z)$ of weight 4 such that $\{\rho_1, \rho_2, z_1, \ldots, z_m\}$ is the set of singular points of $Q(z)$ and (H$_5$) holds. Note that (H$_5$) allows $\rho_i$ to be smooth if $\kappa_{\rho_i} = 1/2$.

When $\Gamma = \text{SL}(2, \mathbb{Z})$, we let $t_j = E_4(z_j)^3/E_6(z_j)^2$, which is not 0 nor 1 since $z_j$ is not an elliptic point nor a cusp. We observe that if $Q(z)$ satisfies (H$_5$), then $Q(z)$ can be expressed as

$$Q(z) = r E_4(z) + s \frac{\Delta(z)}{E_4(z)^2} + t \frac{E_4(z) \Delta(z)}{E_6(z)^2} + E_4(z) \sum_{j=1}^{m} \left( r_1^{(j)} \frac{\Delta(z)^2}{F_j(z)^2} + r_2^{(j)} \frac{\Delta(z)}{F_j(z)} \right),$$

(A.10)

where $r, s, t, r_1^{(j)},$ and $r_2^{(j)}$ are complex parameters and $F_j(z) := E_4(z)^3 - t_j E_6(z)^2$ are modular forms of weight 12. Note that at any elliptic point $\rho_j$, Proposition [A.7] implies that $Q(z) = A(z - \rho_j)^2 + (\text{a holomorphic function near } \rho_j)$, and the expression (A.10) follows immediately from this fact. The indicial equations at $\rho, i,$ and $z_j$ can be computed using Lemma [A.3] together with Proposition [A.4] and Ramanujan’s identities (A.3). We find they are

$$x^2 - x + \frac{s}{192} = 0, \quad x^2 - x - \frac{t}{432} = 0, \quad x^2 - x - \frac{r_1^{(j)}}{1728 t^2 j} = 0,$$

(A.11)

respectively. Thus, the parameters $s, t,$ and $r_1^{(1)}$ are uniquely determined by the given data $\kappa_{\rho}, \kappa_i,$ and $\kappa_j$. The rest of parameters will be determined by the apparentness condition. See Theorem [A.9] below.

When $\Gamma = \Gamma_0^+(2)$, we define modular forms

$$M_{-}^{+}(z) = 2E_2(2z) - E_2(z), \quad M_{-}^{-}(z) = \frac{4E_4(2z) - E_2(z)}{3},$$

$$M_{+}^{+}(z) = \frac{4E_4(2z) + E_4(z)}{5}, \quad M_{+}^{-}(z) = \frac{8E_4(2z) + E_6(z)}{9},$$

$$M_{-}^{+}(z) = \eta(z)^8 \eta(2z)^8,$$

where the integers in the subscripts denote the weights and the signs in the superscripts denote the eigenvalues under the Atkin-Lehner involution. Among them, $M_{-}^{+}(z)$ and $M_{-}^{-}(z)$ freely generate the ring of modular forms on $\Gamma_0(2)$, while $M_{+}^{+}(z), M_{+}^{-}(z),$ and
generate the ring of modular forms on \( \Gamma_0^+ (2) \) with a single relation \( M_8^+ (z)^2 = M_6^+ (z)(M_4^+ (z)^2 - 256 M_8^+ (z)) \). The expressions for \( M_4^+ \), \( M_6^+ \), and \( M_8^+ \) as products of \( M_2^+ \) and \( M_2^- \) are

\[
M_4^+ = (M_2^-)^2, \quad M_6^+ = M_2^- M_4^- , \quad M_8^+ = \frac{(M_2^-)^4 - (M_2^-)^2}{256}.
\]

Also, set \( M_7^+ (z) = (2E_2(2z) + E_2(z))/3 \), which is a quasimodular form of weight 2 and depth 1 on \( \Gamma_0^+ (2) \). We have

\[
D_q M_2^+(z) = \frac{M_2^+(z)^2 - M_4^+(z)}{8},
\]
\[
D_q M_2^-(z) = \frac{M_2^+(z) M_2^-(z) - M_4^-(z)}{4},
\]
\[
D_q M_4^-(z) = \frac{M_2^+(z) M_4^-(z) - M_4^-(z)^3}{2}.
\]

The modular forms \( M_2^- (z) \) and \( M_4^-(z) \) have a simple zero at \( \rho_2 = (1 + i)/2 \) and \( \rho_1 = i/\sqrt{2} \), respectively, and are nonvanishing elsewhere, up to \( \Gamma_0(2) \)-equivalence.

As in the case of \( \text{SL}(2, \mathbb{Z}) \), we let \( t_j = M_4^- (z_j) M_4^+(z_j)^2 \), which is never 0 nor 1 since \( z_j \) is not a cusp or an elliptic point, and set \( F_j(z) = M_4^- (z) - t_j M_4^+(z)^2 \). When \( j \) observe that the ratios \( M_6^+(z)/M_4^+(z) \) and \( M_8^+(z)/M_4^+(z) \) have double poles at \( \rho_2 \) and \( \rho_1 \), respectively. Also, the modular form \( F_j(z) \) has a simple zero at the point \( z_j \). Hence Proposition\[A.7\] implies that

\[
Q(z) = r M_4^+(z) + s M_2^+(z) + t M_4^+(z) M_4^+(z) - 256 M_8^+(z)
\]
\[
+ M_4^+(z) \sum_{j=1}^m \left( r_1^{(j)} \frac{M_8^+(z)^2}{F_j(z)^2} + r_2^{(j)} \frac{M_8^+(z)}{F_j(z)} \right),
\]

where \( r, s, t, r_1^{(j)}, \) and \( r_2^{(j)}, j = 1, \ldots, m, \) are complex parameters. Among the parameters, \( s, t, \) and \( r_1^{(j)}, \) determined by the data \( \kappa_\rho \) and \( \kappa_j \) since the indicial equations at \( \rho_1, \rho_2, \) and \( z_j \) are

\[
x^2 - x - \frac{t}{64} = 0, \quad x^2 - x + \frac{s}{16} = 0, \quad x^2 - x - \frac{r_1^{(j)}}{256 t_j} = 0,
\]

respectively. (They are computed using Lemma\[A.5\], Proposition\[A.4\] and (A.13).) Again, the remaining parameters will be determined by the apparentness condition.

When \( \Gamma = \Gamma_0^+ (3) \), we define the modular forms

\[
M_4(z) = \sum_{m,n \in \mathbb{Z}} q^{m^2 + mn + n^2}, \quad M_3(z) = \frac{\eta(z)^9}{\eta(3z)^3} - \frac{27}{\eta(3z)^3}
\]

of weight 1 and 3, respectively, on \( \Gamma_1(3) \). They generate the ring of modular forms on \( \Gamma_1(3) \). Set \( M_2^+(z) = (3E_2(3z) + E_2(z))/4 \). We have

\[
D_q M_2^+(z) = \frac{M_2^+(z)^2 - M_4^+(z)^4}{6},
\]
\[
D_q M_1^+(z) = \frac{M_2^+(z) M_1^+(z) - M_3^+(z)}{6},
\]
\[
D_q M_3^+(z) = \frac{M_2^+(z) M_3^+(z) - M_1^+(z)^5}{2}.
\]
Also let
\[ M_2^-(z) = \frac{3E_2(2z) - E_2(z)}{2} = M_1(z)^2 \in \mathcal{M}_2(\Gamma_0^+(3), -), \]
\[ M_4^+(z) = \frac{9E_4(3z) + E_4(z)}{10} = M_1(z)^4 \in \mathcal{M}_4(\Gamma_0^+(3)), \]
\[ M_6^-(z) = \eta(z)^6 \eta(3z)^6 = \frac{M_1(z)^6 - M_3(z)^2}{108} \in \mathcal{M}_6(\Gamma_0^+(3), -). \]

Then meromorphic modular forms \( Q(z) \) of weight 4 on \( \Gamma_0^+(3) \) such that all poles have order \( \leq 2 \) are of the form
\[
Q(z) = r M_4^+(z) + s \frac{M_6^-(z)}{M_2^-(z)} + t \frac{M_7^+(z) M_6^-(z)}{M_3^-(z)^2}
+ M_4^+(z) \sum_{j=1}^m \left( r_1^{(j)} \frac{M_6^-(z)}{F_j(z)^2} + r_2^{(j)} \frac{M_7^+(z)}{F_j(z)} \right),
\]
where \( F_j(z) \) is a modular form in \( \mathcal{M}_6(\Gamma_0^+(3), -) \) of the form \( F_j(z) = M_1(z)^6 - t_j M_3(z)^2 \) with \( t_1, \ldots, t_m \) being distinct complex numbers such that none of them is 0 or 1. The ratio \( M_6^-(z)/M_2^-(z) \) has a double pole at \( \rho_2 = (3 + \sqrt{3})/6 \), while \( M_4^+(z)/M_6^-(z)/M_3^-(z)^2 \) has a double pole at \( \rho_1 = i\sqrt{3} \). The modular form \( F_j(z) \) has a simple zero at the point \( z_j \) such that \( t_j = M_1(z_j)^6/M_3(z_j)^2 \).

Again, \( s, t \), and \( r_1^{(j)} \) are uniquely determined by the indicial equations
\[
x^2 - x - \frac{t}{27} = 0, \quad x^2 - x + \frac{8}{9} = 0, \quad x^2 - x - \frac{r_1^{(j)}}{108 t_j} = 0
\]
at \( \rho_1, \rho_2, \) and \( z_j \), respectively, computed using [A.5] Proposition [A.4] and [A.16]. It remains to determine the other parameters \( r \) and \( r_2^{(j)}, j = 1, \ldots, m \). Note that by Theorem [A.8] the ODE \([A.1] \) is apparent at all the elliptic points.

**Theorem A.9.** Given \( \{\rho_1, \rho_2, z_1, \ldots, z_m\} \) and \( \{\kappa_{\rho_1}, \kappa_{\rho_2}, \kappa_1, \ldots, \kappa_m\} \subset \frac{1}{2} \mathbb{N} \) and \( \kappa_{\infty} \geq 0 \), there are \( \prod_{j=1}^m (2\kappa_j) \) meromorphic modular forms \( Q(z) \) of weight 4 on \( \Gamma \), counted with multiplicity, such that \([H_1]\) holds and the local exponents at \( \infty \) are \( \pm \kappa_{\infty} \).

**Proof.** The theorem was proved in [4] when \( \Gamma = \text{SL}(2, \mathbb{Z}) \). For the other two groups \( \Gamma_0^+(2) \) and \( \Gamma_0^+(3) \), the proofs are basically the same. For the convenience of the reader, we sketch them here.

Using the expansion in Lemma [A.5] we have that \([A.1] \) is apparent at \( z_j \) if and only if there is a solution \( y(z) = (1 - (1 + A) \bar{w})^{-1} \bar{w}^{1/2 - \kappa} \sum_{n=0}^{\infty} c_n \bar{w}^n \), \( c_0 = 1 \). By Remark [A.6] it is equivalent to \([A.9] \) holds for \( c_j, 1 \leq j \leq 2\kappa_i - 1 \), obtained by the recursive relation \([A.8] \). Then the condition \([A.9] \) at \( z_i \) yields a polynomial \( P_j(r, r_2^{(1)}, \ldots, r_2^{(m)}) \) of degree \( 2\kappa_j \) and it is not difficult to see that the polynomial \( P_j \) is of the form
\[
P_j(r, r_2^{(1)}, \ldots, r_2^{(m)}) = B_j(r_2^{(j)})^{2\kappa_j} + \text{(terms of lower order)}
\]
for some constant \( B_j \neq 0 \) independent of \( r \) and \( r_2^{(j)}. \) (For details, we refer to [4] Theorem 1.3.) Suppose that \( \kappa_{\infty} \) is given. Then we have \( r = \kappa_{\infty}^2 \). Thus, \( Q(z) \) is apparent at all \( z_j \) with the given data if and only if \( (r_2^{(1)}, \ldots, r_2^{(m)}) \) is a common root of the \( m \) polynomials \([A.19] \). It is easy to see that there are no common roots at \( \infty \) when all polynomials are homogenized. By Bezout’s theorem, there are \( \prod_{j=1}^m (2\kappa_j) \) solutions, counted with multiplicities. This proves the theorem. \( \square \)
Example A.10. Consider the quasimodular form \( f(z) = E_2(z)E_4(z) + aE_6(z) \) of weight 6 and depth 1 on \( \text{SL}(2, \mathbb{Z}) \), where \( a \) is a complex number. We compute that
\[
W_f(z) = (-1 - 6a)E_4(z)^3 - (a^2 - 4a)E_6(z)^2
\]
and \( f(z)/\sqrt{W_f(z)} \) is a solution of (A.1) with
\[
Q(z) = E_4(z) \left( \frac{3}{4} t_1 - \frac{1728 \Delta(z)^2}{12(1 + 6a) F_1(z)^2} - \frac{1 + 31a}{1728 \Delta(z)} \right),
\]
where \( t_1 = (4a - a^2)/(1 + 6a) \) and \( F_1(z) = E_4(z)^3 - t_1 E_6(z)^2 \). Let \( z_1 \) be a point in \( \mathbb{H} \) such that \( t_1 = E_4(z_1)^3/E_6(z_1)^2 \). The example shows that for a generic point \( z_1 \), there are two meromorphic modular form \( Q(z) \) of weight 4 on \( \text{SL}(2, \mathbb{Z}) \) (corresponding to the two complex numbers \( a \) such that \( t_1 = (4a - a^2)/(1 + 6a) \) such that (A.1) is apparent at \( z_1 \) with local exponents \(-1/2, 3/2\). Theorem A.9 asserts that these are the only two such \( Q(z) \).

REFERENCES

[1] Gerrit Bol. Invarianten linearer differentialgleichungen. Abh. Math. Sem. Univ. Hamburg, 16(3-4):1–28, 1949.
[2] YoungJu Choie and Min Ho Lee. Jacobi-Like Forms, Pseudodifferential Operators, and Quasimodular Forms, volume 11 of Springer Monographs in Mathematics. Springer International Publishing, Princeton, NJ, 2019.
[3] Peter J. Grabner. Quasimodular forms as solutions of modular differential equations. Int. J. Number Theory, 16(10):2233–2274, 2020.
[4] Jia-Wei Guo, Chang-Shou Lin, and Yifan Yang. Metrics with positive constant curvature and modular differential equations I. 2020.
[5] Einar Hille. Ordinary differential equations in the complex domain. Dover Publications, Inc., Mineola, NY, 1997. Reprint of the 1976 original.
[6] Özlem Imamoglu and Cormac O’Sullivan. Parabolic, hyperbolic and elliptic Poincaré series. Acta Arith., 139(3):199–228, 2009.
[7] Masanobu Kaneko and Masao Koike. On modular forms arising from a differential equation of hypergeometric type. Ramanujan J., 7(1-3):145–164, 2003. Rankin memorial issues.
[8] Masanobu Kaneko and Masao Koike. On extremal quasimodular forms. Kyushu J. Math., 60(2):457–470, 2006.
[9] Masanobu Kaneko and Don Zagier. A generalized Jacobi theta function and quasimodular forms. In The moduli space of curves (Texel Island, 1994), volume 129 of Progr. Math., pages 165–172. Birkhäuser Boston, Boston, MA, 1995.
[10] Federico Pellarin. On extremal quasi-modular forms after Kaneko and Koike, 2019. With an appendix by Gabriele Nebe.
[11] Yuichi Sakai. The Atkin orthogonal polynomials for the low-level Fricke groups and their application. Int. J. Number Theory, 7(6):1637–1661, 2011.
[12] Yuichi Sakai and Hiroyuki Tsutsumi. Extremal quasimodular forms for low-level congruence subgroups. J. Number Theory, 132(9):1896–1909, 2012.
[13] Abdellah Sebbar and Ahmed Sebbar. Equivariant functions and integrals of elliptic functions. Geom. Dedicata, 160:373–414, 2012.
[14] Jean-Pierre Serre. A course in arithmetic. Springer-Verlag, New York-Heidelberg, 1973. Translated from the French, Graduate Texts in Mathematics, No. 7.
[15] Goro Shimura. On some arithmetic properties of modular forms of one and several variables. Ann. of Math. (2), 102(3):491–515, 1975.
[16] Goro Shimura. Introduction to the arithmetic theory of automorphic functions, volume 11 of Publications of the Mathematical Society of Japan. Princeton University Press, Princeton, NJ, 1994. Reprint of the 1971 original, Kanō Memorial Lectures, 1.
[17] Heinrich Weber. Lehrbuch der Algebra, Vol. III. Chelsea, New York, 1961.
[18] Don Zagier. Elliptic modular forms and their applications. In The 1-2-3 of modular forms, Universitext, pages 1–103. Springer, Berlin, 2008.
CENTER FOR ADVANCED STUDY IN THEORETICAL SCIENCES (CASTS), NATIONAL TAIWAN UNIVERSITY, TAPEI, TAIWAN 10617.

Email address: csalin@math.ntu.edu.tw

DEPARTMENT OF MATHEMATICS, NATIONAL TAIWAN UNIVERSITY AND NATIONAL CENTER FOR THEORETICAL SCIENCES, TAPEI, TAIWAN 10617.

Email address: yangyifan@ntu.edu.tw