Weakly robust periodic solutions of one-dimensional cellular automata with random rules

JANKO GRAVNER and XIAOCHEM LIU
Department of Mathematics
University of California
Davis, CA 95616
gravner@math.ucdavis.edu, xchliu@math.ucdavis.edu

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Abstract

We study 2-neighbor one-dimensional cellular automata with a large number \( n \) of states and randomly selected rules. We focus on the rules with weakly robust periodic solutions (WRPS). WRPS are global configurations that exhibit spatial and temporal periodicity and advance into any environment with at least a fixed strictly positive velocity. Our main result quantifies how unlikely WRPS are: the probability of existence of a WRPS within a finite range of periods is asymptotically proportional to \( 1/n \), provided that a divisibility condition is satisfied. Our main tools come from random graph theory and the Chen-Stein method for Poisson approximation.

1 Introduction

We continue our study of one-dimensional cellular automata (CA) with random rules, initiated in [6]. As in that paper, we investigate rules with \( n \) states and 2 neighbors, with a rule chosen uniformly from all \( n^{n^2} \) rules. In [6], we provided the asymptotic probability, as \( n \) goes to infinity, that such a rule has a periodic solution (PS) with a given spatial and temporal period. In this paper, we demand a certain additional stability property of a PS, and explore the analogous probability of existence of a PS in this special class.

To be precise, we consider one-dimensional cellular automata with \( n \)-state space, encoded by \( \mathbb{Z}_n = \{0, \ldots, n-1\} \), and 2-neighbor rules \( f : \mathbb{Z}_n^2 \to \mathbb{Z}_n \). Assume that a CA given by the rule \( f \) starts from a periodic global configuration \( \xi_0 : \mathbb{Z} \to \mathbb{Z}_n \) that satisfies \( \xi_0(x) = \xi_0(x+\sigma) \), for all \( x \in \mathbb{Z} \). If we also have \( \xi_\tau = \xi_0 \), and \( \tau \) and \( \sigma \) are both minimal, then we have found a periodic solution (PS) under rule \( f \), with spatial period \( \sigma \) and temporal period \( \tau \). We will not distinguish between spatial and temporal shifts of a PS. Therefore, each configuration \( \xi_t \in \mathbb{Z}_n^2 \), \( t \geq 0 \), characterizes the PS and is called a PS configuration. We call the map \( (x,t) \mapsto \xi_t(x) \) from \( \mathbb{Z} \times \mathbb{Z}_+ \) to \( \mathbb{Z} \) the space-time configuration; within it, any rectangle with \( \tau \) rows and \( \sigma \) columns also characterizes

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the PS, and we call any such rectangle the tile of the PS. Thus we do not distinguish between tiles which are spatial or temporal rotations of each other.

In the present paper, we are interested in PS with an expansion property, which we first illustrate by an example and provide some motivation, and then give a formal definition. Figure 1 demonstrates two pieces of the space-time configurations under the 3-state rule $102222210$. (As in [6], we name a rule by listing its values for all pairs in reverse alphabetical order from $(n-1, n-1)$ to $(0,0)$.) The tile

$$
\begin{array}{cccccc}
0 & 2 & 2 & 2 & 1 & 1 \\
2 & 2 & 1 & 1 & 0 & 2 \\
1 & 1 & 0 & 2 & 2 & 2
\end{array}
$$

characterizes a PS under this rule and for such PS, even if the spatially periodic configuration is replaced by an arbitrary configuration to the right of some site in $\mathbb{Z}$, the periodic configuration will “repair” itself, that is, it will advance to the right with a minimal velocity $v > 0$ as time increases, uniformly over the perturbed environment.

PS with such property are of particular interest and importance, as they are related to stable limit cycles in continuous dynamical systems. Limit cycles, also known as isolated closed trajectories, are such that neighboring trajectories either spiral toward or away from them. In the former case, when a perturbation of a limit cycle converges back, the limit cycle is called stable [10]. Thus we consider an analogous stability property for CA: after a one-sided perturbation of a periodic configuration, the dynamics make the configuration converge back. In this paper, we keep the terminology from [3] and refer to such stability as robustness. We remark that the minimal velocity $v$ gives the minimal exponential rate of convergence to the PS in the standard metric, by which the distance between $\xi, \eta \in \mathbb{Z}_n^Z$ is $m(\xi, \eta) = 2^{-n}$, where $n = \inf\{|x| : \xi(x) \neq \eta(x)\}$.

Figure 1: Two pieces of the space-time configuration of the 3-state rule $102222210$. The underlying PS exhibits weak robustness: the periodicity expands if it is terminated and continued by an arbitrary configuration, for example a random configuration (left) or all 0s (right).

Proceeding to the formal definition, let $\xi_0$ be a PS configuration under rule $f$ and $\eta_0$ be any initial configuration that agrees with $\xi_0$ on all $x \leq y$, for some $y \in \mathbb{Z}$. Adapting the definition from [3], we call such initial configurations proper for the PS $\xi_0$. Let $\xi_t$ and $\eta_t$ be the configurations obtained by running $f$ starting with $\xi_0$ and $\eta_0$, respectively. Let

$$s_t(\eta_0) = \sup\{x \in \mathbb{Z} : \eta_t(x) = \xi_t(x)\}$$

be the rightmost location at which $\eta_t$ agrees with $\xi_t$ at time $t$. Then the expansion velocity in the initial environment $\eta_0$ is

$$v(\eta_0) = \lim_{t \to \infty} \inf \frac{s_t}{t}.$$
which describes the rate at which spatial periodicity expands. The **expansion velocity**

\[ v = \inf \{ v(\eta_0) : \eta_0 \text{ is proper for } \xi_0 \} \]

then measures uniformity over all environments. If \( v > 0 \), then the PS \( \xi_i \) is **weakly robust**. With this terminology, we distinguish this property from the more restrictive robustness from [3].

As in [6], we are interested in the existence of WRPS of a randomly selected \( n \)-state 2-neighbor rule \( f \). To this end, fix two sets \( T, \Sigma \subset \mathbb{N} = \{1, 2, \ldots \} \) and let \( R_{T,\Sigma} \) be the (random) set of WRPS of a randomly selected \( n \)-state rule \( f \), with temporal period \( \tau \) and spatial period \( \sigma \) satisfying \( (\tau, \sigma) \in T \times \Sigma \). While our results for existence of PS [6] are valid for arbitrary finite \( T \times \Sigma \subset \mathbb{N} \times \mathbb{N} \), we impose a divisibility restriction for our result on WRPS.

**Theorem 1.1.** Let \( T \times \Sigma \subset \mathbb{N} \times \mathbb{N} \) be fixed and finite. If there exists \((\tau, \sigma) \in T \times \Sigma \) such that \( \sigma \mid \tau \), then \( \Pr(R_{T,\Sigma} \neq \emptyset) = c(T, \Sigma)/n + o(1/n) \), where \( c(T, \Sigma) \) is a constant depending only on \( T \) and \( \Sigma \).

In addition to [6], we have investigated periodic solutions for cellular automata in [5, 4], where the emphasis is on maximal temporal periods; some further results and conjectures on robustness are in [7]. The initial motivation for the present paper comes from the investigation of robust periodic solutions (RPS) in [3], in which all the 64 one-dimensional binary 3-neighbor edge CA rules and their RPS are studied. To our knowledge, robustness of PS is first addressed for the *Exactly 1* rule, i.e., the elementary CA Rule 22, in [2].

This paper is organized as follows. In the next section, we recall some preliminary results from [6]. While we summarize major definitions and tools, we omit the proofs and refer the reader to [6] for a more detailed discussion. In Section 3 we introduce the property of a tile that distinguishes a WRPS from a PS, i.e., the decidability of labels in a tile. We establish the probability that a label exhibits such property for a randomly selected rule in Section 4 and give the proof of Theorem 1.1 in Section 5. In the final section, we discuss the possible directions and methods to extend and generalize our results.

## 2 Preliminaries

The main purpose of this section is to gather the relevant definitions and results from [6]. All lemmas are restatements of results in [6], where the proofs are provided.

### 2.1 Tiles of PS

We may express a tile with periods \( \tau \) and \( \sigma \) as \( T = (a_{i,j})_{i=0, \ldots, \tau-1, j=0, \ldots, \sigma-1} \), once we fix an element in \( T \) to be placed at the position \((0,0)\). We use the notation \( \text{row}_i \) and \( \text{col}_j \) to denote the \( i \)th row and \( j \)th column of a tile \( T \) and use \( a_{i,j} \) to denote the element at the \( i \)th row and \( j \)th column of \( T \), where we always interpret the two subscripts modulo \( \tau \) and \( \sigma \), respectively.

Let \( T_1 \) and \( T_2 \) be two tiles and \( a_{i,j}, b_{k,m} \) be the corresponding elements. If \( (a_{i,j}, a_{i,j+1}) \neq (b_{k,m}, b_{k,m+1}) \) for \( i, j, k, m \in \mathbb{Z}_+ \), then \( T_1 \) and \( T_2 \) are called **orthogonal**, denoted by \( T_1 \perp T_2 \). In this case, we observe that two assignments \((a_{i,j}, a_{i,j+1}) \mapsto a_{i+1,j+1} \) and \((b_{k,m}, b_{k,m+1}) \mapsto b_{k+1,m+1} \) occur independently. We say that \( T_1 \) and \( T_2 \) are **disjoint**, and denote this property by \( T_1 \cap T_2 = \emptyset \), if \( a_{i,j} \neq b_{k,m} \), for \( i, j, k, m \in \mathbb{Z}_+ \). Clearly, every pair of disjoint tiles is orthogonal, but not vice versa.
The following quantities associated with a tile play a important role in the sequel. We define the assignment number of $T$ to be $p(T) = \# \{(a_{i,j}, a_{i,j+1}) : (a_{i,j}, a_{i,j+1}) \in T\}$, i.e., the number of values of the rule $f$ specified by $T$. Also, let $s(T) = \# \{a_{i,j} : a_{i,j} \in T\}$ be the number of different states in the tile. Clearly, $p(T) \geq s(T)$, so we define $\ell = \ell(T) = p(T) - s(T)$ to be the lag of $T$.

The following lemma from [6] lists two immediate properties of the tile of a PS.

**Lemma 2.1.** Let $T = (a_{i,j}) = 0, \ldots, \tau - 1, j = 0, \ldots, \sigma - 1$ be the tile of a PS with periods $\tau$ and $\sigma$. Then $T$ satisfies the following properties:

1. Uniqueness of assignment: if $(a_{i,j}, a_{i,j+1}) = (a_{k,m}, a_{k,m+1})$, then $a_{i+1,j+1} = a_{k+1,m+1}$.
2. Aperiodicity of rows: each row of $T$ cannot be divided into smaller identical pieces.

We remark that for a tile of a PS that is not weakly robust, there may exist periodic columns. However, in Section 3 we will show that, if $T$ is a tile of a WRPS, its columns are necessarily aperiodic.

### 2.2 Circular Shifts

We also recall the concept of circular shifts operation on $Z_n^\sigma$ (or $Z_n^\tau$), the set of words of length $\sigma$ (or $\tau$) from the alphabet $Z_n$, which will be used in Section 2.5.

**Definition 2.2.** Let $Z_n^\sigma$ consist of all length-$\sigma$ words. A circular shift is a map $\pi : Z_n^\sigma \to Z_n^\sigma$, given by an $i \in Z_+$ as follows: $\pi(a_0a_1 \ldots a_{\sigma-1}) = a_{i}a_{i+1} \ldots a_{i+\sigma-1}$, where the subscripts are modulo $\sigma$. The order of a circular shift $\pi$ is the smallest $k$ such that $\pi^k(A) = A$ for all $A \in Z_n^\sigma$, and is denoted by $\text{ord}(\pi)$. Circular shifts on $Z_n^\sigma$ will also appear in the sequel and are defined in the same way.

**Lemma 2.3.** Let $\pi$ be a circular shift on $Z_n^\sigma$ and let $A \in Z_n^\sigma$ be an aperiodic length-$\sigma$ word from alphabet $Z_n$. Then: (1) $\text{ord}(\pi) | \sigma$; and (2) for any $d | \sigma$,

$$\# \{B \in Z_n^\sigma : A = \pi(B) \text{ for some } \pi \text{ with } \text{ord}(\pi) = d\} = \varphi(d).$$

Two words $A$ and $B$ of length $\sigma$ are **equal up to a circular shift** if $B = \pi(A)$ for some circular shift $\pi$.

### 2.3 Directed Graph on Labels

In our study of PS [6], we extended the notion of label trees from [3] to define the **label digraph**. As this object is also of relevance to WRPS, we recall its definition in this subsection.

**Definition 2.4.** Let $A = a_0 \ldots a_{\tau-1}$ and $B = b_0 \ldots b_{\tau-1}$ be two words from alphabet $Z_n$, which we call **labels** of length $\tau$. (While it is best to view them as vertical columns, we write them horizontally for reasons of space, as in [3].) We say that $A$ **right-extends to** $B$ if $f(\pi_i, b_i) = b_{i+1}$, for all $i \in Z_+$, where (as usual) the indices are modulo $\tau$, and we write $A \rightarrow B$. We form the **label digraph** associated with a given $\tau$ by forming an arc from a label $A$ to a label $B$ if $A$ right-extends to $B$.

The right extension relation is the basis for the Algorithm 2.5 below for finding all the PS with temporal period $\tau$.
Algorithm 2.5.

\begin{verbatim}
input : Label digraph $D_{\tau,f}$ of $f$ with temporal period $\tau$

Find all the directed cycles in $D_{\tau,f}$
for each cycle $A_0 \to A_1 \to \cdots \to A_{\sigma-1} \to A_0$ do
    form the tile $T$ by placing labels $A_0, A_1, \ldots, A_{\sigma-1}$ on successive columns.
    if both spatial and temporal periods of $T$ are minimal then
        print $T$ as a PS
end
\end{verbatim}

Proposition 2.6. All PS of temporal period $\tau$ of $f$ can be obtained by the Algorithm 2.5.

2.4 Chen-Stein Method for Poisson Approximation

The most useful tool in proving Poisson convergence is the Chen-Stein method [1]. The local version stated below (Theorem 4.7 from [8]) was instrumental in [6] and continues to play a similar role in the present paper.

Let $\text{Poisson}(\lambda)$ be a Poisson random variable with expectation $\lambda$, and let $d_{TV}$ be the total variation distance between measures on $\mathbb{Z}_+$. Assume that $I_i$, $i \in \Gamma$, are indicators of a finite family of events, $p_i = \mathbb{E}(I_i)$, $W = \sum_{i \in \Gamma} I_i$, $\lambda = \sum_{i \in \Gamma} p_i = \mathbb{E}W$, and $\Gamma_i = \{j \in \Gamma : j \neq i, I_i$ and $I_j$ are not independent\}.

Lemma 2.7. We have

$$d_{TV}(W, \text{Poisson}(\lambda)) \leq \min\left(1, \lambda^{-1}\right) \left[ \sum_{i \in \Gamma} p_i^2 + \sum_{i \in \Gamma, j \in \Gamma_i} (p_i p_j + \mathbb{E}(I_i I_j)) \right].$$

2.5 Simple Tiles

If a tile $T$ has zero lag, we call $T$ simple. In [6], we show that the probability of existence of PS with simple tiles provides the dominant terms of the existence of PS. In Section 5, we show that it is also the dominant term for WRPS.

Lemma 2.8. Assume $T = (a_{i,j})_{i=0,\ldots,\tau-1,j=0,\ldots,\sigma-1}$ is a simple tile. Then

1. the states on each row of $T$ are distinct;
2. if two rows of $T$ share a state, then they are circular shifts of each other;
3. the states on each column of $T$ are distinct; and
4. if two columns of $T$ share a state, then they are circular shifts of each other.
Let $T = (a_{i,j})_{i=0,...,\tau-1,j=0,...,\sigma-1}$ be a simple tile. Let

$$i = \min\{k = 1,2,\ldots,\tau - 1 : \text{row}_k = \pi(\text{row}_0), \text{ for some circular shift } \pi : \mathbb{Z}^\sigma \to \mathbb{Z}^\sigma\}$$

be the smallest $i$ such that $\text{row}_i$ is a circular shift of $\text{row}_0$, and let $i = 0$ if and only if $T$ does not have circular shifts of $\text{row}_0$ other than this row itself. Then this circular shift satisfies $\text{row}_{(j+i) \mod \tau} = \pi(\text{row}_j)$, for all $j = 0,\ldots,\tau - 1$ and $i$ is determined by the tile $T$; we denote this circular shift by $\pi_T^{i \sigma}$. We denote by $\pi_T^{i \sigma}$ the analogous circular shift for columns.

**Lemma 2.9.** Let $T$ be a simple tile of a PS, and let $d_1 = \text{ord}(\pi_T^1)$ and $d_2 = \text{ord}(\pi_T^2)$. Then $d_1$ and $d_2$ are equal and divide $\gcd(\tau,\sigma)$.

**Lemma 2.10.** An integer $s \leq n$ is the number of states in a simple tile $T$ of PS if and only if there exists $d \mid \gcd(n,\sigma)$, such that $s = \tau\sigma/d$.

The above lemma gives the possible values of $s(T)$ for a simple tile $T$ and the next one enumerates the number of simple tiles of PS containing $s$ different states.

**Lemma 2.11.** The number of simple tiles of PS with temporal periods $\tau$ and spatial period $\sigma$ containing $s$ states is $\varphi(d)\binom{n}{s}(s-1)!$, where $d = \tau\sigma/s$.

Consider two different simple tiles $T_1$ and $T_2$ under the rule. The following lemma provides a lower bound on the combined number of values of the rule $f$ assigned by $T_1$ and $T_2$, in terms of the number of states.

**Lemma 2.12.** Let $T_1$ and $T_2$ be two different simple tiles for the same rule. If $T_1$ and $T_2$ have at least one state in common, then there exist $a_{i,j} \in T_1$ and $b_{k,m} \in T_2$ such that $a_{i,j} = b_{k,m}$ and $a_{i,j+1} \neq b_{k,m+1}$.

As a result, if $s(T_1) = s_1$, then $p(T_1) \geq s_1$, i.e., there are at least $s_1$ values assigned by $T_1$. If there are $s_2'$ states in $T_2$ that are not in $T_1$, then there are at least $s_2'$ additional values to assign. With the above lemma, a lower bound of the number of values to be assigned in $T_1$ and $T_2$ is $s_1 + s_2' + 1$.

### 3 Decidability and WRPS

In order for a PS to be weakly robust, we need one more condition on the directed cycle in the label digraph, which requires that each label decides its unique child. To be more accurate, let $A$ and $B$ be two labels. Assume that at a site $k \in \mathbb{Z}$ the temporal evolution of the states, arranged vertically, is the repeated label $A$: $a_0 \ldots a_{\tau-1}a_0 \ldots a_{\tau-1} \ldots$. Suppose that the states at site $k + 1$ eventually “converge” to repetition of $B$: $b_0 \ldots b_{\tau-1}b_0 \ldots b_{\tau-1} \ldots$, regardless of the initial state at site $k + 1$. In this case, we say that $A$ decides $B$, and then it is clear that $A$ does not decide $C$ for any other length-$\tau$ label $C$ that is not equal to $B$ up to a circular shift. We now provide a more formal definition.

**Definition 3.1.** Let $A = a_0 \ldots a_{\tau-1}$ and $B = b_0 \ldots b_{\tau-1}$ be two length-$\tau$ labels. We call that label $A$ decides $B$, denoted as $A \Rightarrow B$, if the following two conditions are satisfied:

1. label $A$ right-extends to $B$, i.e., $A \Rightarrow B$;
2. for an arbitrary \( c_0 \in \mathbb{Z}_n \), recursively define
\[ c_{j+1} = f(a_{j \mod \tau}, c_j) \] such that
\[ c_j \mod \tau = b_j \mod \tau. \]

The following proposition, analogous to Proposition 2.2 in [3], provides an algorithm to verify whether a PS is weakly robust.

**Proposition 3.2.** A tile is a WRPS if and only if each column decides the column to its right.

**Proof.** Assume that a tile \( T = (a_{i,j}) \) is a WRPS with columns \( A_j, j = 0, \ldots, \sigma - 1 \). Let \( \eta \) be the initial configuration formed by doubly infinite repetition of \( a_{0,0} \ldots a_{0,\sigma-1} \). If \( A_j = a_{0,j} \ldots a_{\tau-1,j} \) does not decide \( A_{j+1} = a_{0,j+1} \ldots a_{\tau-1,j+1} \), for some \( j = 0, \ldots, \tau - 1 \), then there exists a \( c_0 \in \mathbb{Z}_n \) such that in the position to the right of \( A_j \), the states do not converge to a repetition of \( A_{j+1} \).

Now, construct an initial configuration \( \eta' \) by replacing one \( a_{0,j+1} \) by \( c_0 \) in \( \eta \). Then \( \eta' \) is proper for \( \eta \), but the advance of the spatial period is stopped, thus \( v(\eta') = 0 \) and \( T \) cannot be weakly robust.

Conversely, note that if label \( A_j \) decides \( A_{j+1} \), then for any \( c_0 \in \mathbb{Z}_n \) to the right of \( a_{0,j} \), the label converges to \( A_{j+1} \) within \( n\tau \) iterations. Thus the expansion velocity must be at least \( 1/(\tau n) \).

Recall that by Lemma 2.1 a tile of a PS does not have periodic rows. The following lemma concludes that a periodic label cannot be a part of WRPS tile, since otherwise the temporal period of the WRPS is reduced.

**Lemma 3.3.** If \( T \) is a tile of WRPS of period \( \tau \), then every column has minimal period \( \tau \).

**Proof.** Assume that \( A \) is a label of length \( \tau \) that is formed by concatenating shorter label \( A' \) that has length \( \tau' \). It is clear that if \( A \Rightarrow B = b_0 \ldots b_{\tau-1} \), \( A \) also decides the circular shift \( b_{\tau} b_{\tau-1} \ldots b_2 b_1 b_0 \ldots b_{\tau-1} \). This implies that \( b_0 = b_{\tau'} \), \( b_1 = b_{\tau'+1} \), etc. That is, \( B \) is also periodic with period \( \tau' \). By induction, every label in \( T \) is periodic with period \( \tau' \), thus \( T \) is temporally reducible.

In a label digraph \( D_{\tau,f} \), we call an arc \( A \rightarrow B \) deciding arc if \( A \Rightarrow B \) and a directed cycle deciding cycle if all the arcs contained in this cycle are deciding arcs. The following algorithm finds all WRPS of temporal period \( \tau \) for rule \( f \).

**Algorithm 3.4.**

\[
\begin{align*}
\text{input :} & \quad \text{Label digraph } D_{\tau,f} \text{ of } f \text{ with temporal period } \tau \\
\text{Find all deciding cycles in } D_{\tau,f} \\
\quad \text{for each deciding cycle } A_0 \Rightarrow A_1 \Rightarrow \cdots \Rightarrow A_{\sigma-1} \Rightarrow A_0 \text{ do} \\
\quad \quad \quad & \text{form the tile } T \text{ by placing labels } A_0, A_1, \ldots, A_{\sigma-1} \text{ on successive columns.} \\
\quad \quad \quad \quad \text{if both spatial and temporal periods of } T \text{ are minimal then} \\
\quad \quad \quad \quad \quad \quad & \text{print } T \text{ as a WRPS} \\
\quad \quad \quad \text{end} \\
\text{end} \\
\end{align*}
\]
4 Decidability Probability

We call a label $A = a_0 \ldots a_{\tau-1}$ simple if $a_i \neq a_j$ for $i \neq j$. We next prove the main result regarding the probability of the decidability of simple labels.

**Theorem 4.1.** Fix a number of states $n$ and a $\tau \leq n$. Let $A = a_0 \ldots a_{\tau-1}$ be a simple label with length $\tau$ and $B = b_0 \ldots b_{\tau-1}$ be any other label (not necessarily simple) of length $\tau$. Then

$$\Pr(A \Rightarrow B) = \frac{n^\tau - (n-1)^\tau}{n^\tau} \cdot \frac{1}{n^\tau}.$$

The theorem is proved in four lemmas below. The key idea reduces to calculating the probability that a random $\tau$-partite graph is a directed pseudo-tree, i.e., a weakly connected directed graph that has at most one directed cycle. To be precise, we construct label assignment digraph (LAD) $G_{\tau,n}(f, A)$ of a label $A$ under a rule $f$ in the following manner.

We consider $\tau$-partite digraphs with a $i$th part denoted by $(i, \ast) = \{(i, j): j = 0, \ldots, n-1\}$, $i = 0, \ldots, \tau - 1$. The arcs of the digraph $G_{\tau,n}(f, A)$ are determined as follows: for all $i = 0, \ldots, \tau - 1$ and $j = 0, \ldots, n-1$, there is an arc $(i, j) \rightarrow (i+1, j')$ if $f(a_i, j) = j'$. As usual, we identify $i = \tau$ with $i = 0$, $i = \tau + 1$ with $i = 1$, etc. We next state the conditions for $G_{\tau,n}(f, A)$ that characterize when $A \Rightarrow B$ and when $A \Rightarrow B$.

**Definition 4.2.** Let $A = a_0 \ldots a_{\tau-1}$ and $B = b_0 \ldots b_{\tau-1}$ be two labels. Consider the following conditions on a $\tau$-partite graph $G$:

1. $G$ contains the cycle $(0, b_0) \rightarrow (1, b_1) \rightarrow \cdots \rightarrow (\tau - 1, b_{\tau-1}) \rightarrow (0, b_0)$;
2. there is a directed path in $G$ from $(i, j)$ to $(0, b_0)$ for all $i = 0, \ldots, \tau - 1$ and $j = 0, \ldots, n - 1$.

The set $E(A, B)$ is the set of all $\tau$-partite digraphs $G$, which satisfy condition (1) and the set $D(A, B)$ is the set of all such digraphs $G$ that satisfy both conditions (1) and (2).

**Lemma 4.3.** Let $A = a_0 \ldots a_{\tau-1}$ and $B = b_0 \ldots b_{\tau-1}$ be any two labels. Then $A \Rightarrow B$ if and only if $G_{\tau,n}(f, A) \in E(A, B)$ and $A \Rightarrow B$ if and only if $G_{\tau,n}(f, A) \in D(A, B)$.

We skip the proof as it follows immediately from the definitions, and instead give two examples for different rules by Figure 2. For the reader’s convenience, we denote a node $(i[a_i], j)$ instead of $(i, j)$ as in the definition. The two labels are $A = 12$ and $B = 00$ in both cases. Under the rule that generates the left LAD, $A \Rightarrow B$, but $A \nsucc B$, i.e., $G_{\tau,n}(f, A) \in E(A, B) \setminus D(A, B)$; under the rule that generates the right LAD, $A \Rightarrow B$, i.e., $G_{\tau,n}(f, A) \in D(A, B)$.

![Figure 2: Two LADs of label $A = 12$ under two different rules. We use $(i[a_i], j)$ to represent a node for the reader’s convenience. In the left one, $A \rightarrow 00$ but $A \nsucc 00$; in the right one, $A \Rightarrow 00$.](image-url)
Fix a label $A = a_0 \ldots a_{\tau-1}$. The LAD $G_{\tau,n}(f, A)$ becomes a random graph if the rule $f$ is selected randomly and we are interested in $\mathbb{P}(G_{\tau,n}(f, A) \in \mathcal{E}(A, B))$ and $\mathbb{P}(G_{\tau,n}(f, A) \in \mathcal{D}(A, B))$. The case that $A$ is simple is easier as we can take advantage of independence of assignments of $f$. To be precise, let $A$ be a simple label with length $\tau$ and $B$ be an arbitrary label with the same length. We clearly have that $\mathbb{P}(G_{\tau,n}(f, A) \in \mathcal{E}(A, B)) = 1/n^\tau$, as the assignments on $(a_j, b_j)$’s are independent.

Next, we find $\mathbb{P}(G_{\tau,n}(f, A) \in \mathcal{D}(A, B))$ for simple label $A$ thus complete the proof of Theorem L1. We start by the following observation.

**Lemma 4.4.** If $A$ and $A'$ are simple labels with the same length, $\mathbb{P}(A \Rightarrow B) = \mathbb{P}(A' \Rightarrow B)$ for any label $B$; if $B$ and $B'$ are labels with the same length, $\mathbb{P}(A \Rightarrow B) = \mathbb{P}(A \Rightarrow B')$ for any simple label $A$.

To find $\mathbb{P}(G_{\tau,n}(f, A) \in \mathcal{D}(A, B))$, we adapt the counting techniques in [9] to enumerate $\mathcal{D}(A, B)$. We start by proving the following combinatorial result.

**Lemma 4.5.** Let $A_{k,\ell} = \binom{n-1}{k}(\ell+1)^k(n-1-\ell)^{n-1-k}$, and assume that $k_{m+1}$ is a non-negative integer. Then

$$S_m := \sum_{k_m=0}^{n-1} A_{k_m,k_{m+1}} \cdots \left[ \sum_{k_2=0}^{n-1} A_{k_2,k_3} \left[ \sum_{k_1=0}^{n-1} A_{k_1,k_2}(k_1+1)n^{n-2} \right] \right]$$

$$= n^{(m+1)(n-2)} \left[ P_{m+1} + k_{m+1}(n-1)^m \right],$$

where $P_m = n^{m} - (n-1)^m$.

**Proof.** We use induction on $m$. Assume $m = 1$. Observe that

$$A_{k,\ell} = n^{1-1} \mathbb{P} \left( \text{Binomial} \left( n-1, \frac{\ell+1}{n} \right) = k \right).$$

Therefore,

$$\sum_{k_1=0}^{n-1} A_{k_1,k_2}(k_1+1)n^{n-2} = n^{n-2} \cdot n^{n-1} \cdot \left[ 1 + (n-1) \frac{k_2+1}{n} \right]$$

$$= n^{2(n-2)} \left[ P_2 + k_2(n-1) \right].$$

Now, by the induction hypothesis

$$S_m = \sum_{k_m=0}^{n-1} A_{k_m,k_{m+1}}S_{m-1}$$

$$= n^{m(n-2)} \sum_{k_m=0}^{n-1} \binom{n-1}{k_m} (k_{m+1}+1)^{k_m}(n-1-k_{m+1})^{n-1-k_m} \left[ P_m + k_m(n-1)^{m-1} \right]$$

$$= n^{m(n-2)} \left[ n^{n-1} P_m + (n-1)^m(k_{m+1}+1)n^{n-2} \right]$$

$$= n^{(m+1)(n-2)} \left[ nP_m + k_{m+1}(n-1)^m + (n-1)^m \right]$$

$$= n^{(m+1)(n-2)} \left[ P_{m+1} + k_{m+1}(n-1)^m \right],$$

which is the desired result. \(\square\)
Now, we are ready to prove the key combinatorial result.

**Lemma 4.6.** Let $A$ and $B$ be labels with length $\tau$ and let $A$ be simple. Then $\#D(A, B) = n^{\tau(n-2)}(n^\tau - (n-1)^\tau)$.

**Proof.** The argument we give partly follows the proof of Theorem 1 in [9]. Applying Lemma 4.4 we may assume that $B = 0 \ldots 0$, without loss of generality.

First, choose a $k_{\tau-1} \in \{0, \ldots, n-1\}$, pick $k_{\tau-1}$ nodes in $(\tau - 1, *) \setminus \{(\tau - 1, 0)\}$, and form $k_{\tau-1}$ arcs from those nodes to the node $(0, 0)$. There are $\binom{n-1}{k_{\tau-1}}$ choices for a fixed $k_{\tau-1}$. Denote this subset of $(\tau - 1, *)$ together with $(\tau - 1, 0)$ as $(\tau - 1, *)'$; thus, $(\tau - 1, *)' \subset (\tau - 1, *)$ are the nodes in $(\tau - 1, *)$ that are mapped to $(0, 0)$. Assign the images of the nodes in $(\tau - 1, *) \setminus (\tau - 1, *)'$ to $(0, *) \setminus \{(0, 0)\}$, for which there are $(n-1)^{n-1-k_{\tau-1}}$ choices. So, for a fixed $k_{\tau-1}$ to assign the image of nodes in $(\tau - 1, *)$, there are

$$\binom{n-1}{k_{\tau-1}}(n-1)^{n-1-k_{\tau-1}}$$

choices.

Second, we need to assign the image of the nodes in $(\tau - 2, *)$ to $(\tau - 1, *)$. Choose a $k_{\tau-2} \in \{0, \ldots, n-1\}$, pick $k_{\tau-2}$ nodes in $(\tau - 2, *) \setminus (\tau - 2, 0)$, and form $k_{\tau-2}$ arcs from those nodes to the nodes in $(\tau - 1, 0)'$. There are $\binom{n-1}{k_{\tau-2}}$ choices to choose those nodes for a fixed $k_{\tau-2}$ and $(k_{\tau-1} + 1)^{k_{\tau-2}}$ choices to assign the images. Denote this subset of $(\tau - 2, *)$ together with $(\tau - 2, 0)$ as $(\tau - 2, *)'$. Now, the images of the nodes in $(\tau - 2, *) \setminus (\tau - 2, *)'$ should be in $(\tau - 1, *) \setminus (\tau - 1, *)'$, for which there are $(n-1-k_{\tau-1})^{n-1-k_{\tau-2}}$ choices. Hence, for fixed $k_{\tau-1}$ and $k_{\tau-2}$, to assign the image of the nodes in $(\tau - 2, *)$ to $(\tau - 1, *)$, there are

$$\binom{n-1}{k_{\tau-1}}(k_{\tau-1} + 1)^{k_{\tau-2}}(n-1-k_{\tau-1})^{n-1-k_{\tau-2}}$$

choices.

Repeat the above steps for $(\tau - 3, *)$, ..., $(1, *)$. To complete the construction, we assign the images of the nodes in $(0, *) \setminus \{(0, 0)\}$. We choose a $t \in \{0, \ldots, n-2\}$, and add $t$ arcs from $(0, *) \setminus \{(0, 0)\}$ to $(1, *) \setminus (1, *)'$ consecutively as specified below, making sure to avoid creating a cycle that does not include $(0, 0)$.

In the evolving digraph, a **component** is a weakly connected component, obtained by ignoring the orientation of edges. First note that there are $n$ components in the current digraph; more precisely, each node of $(0, *)$ belongs to a different component (possibly consisting of a single node).

To select the first arc, pick a $b \in (1, *) \setminus (1, *)'$ $(n-1-k_1$ choices). There is one component that contains $(0, 0)$ and one other component containing $b$. As a result, there are $n-2$ components and among each of them, there is a node in $(0, *) \setminus \{(0, 0)\}$ with zero out-degree. Among these $n-2$ nodes, we select one and connect it to $b$. Therefore, there are $(n-2)(n-1-k_1)$ choices for the first arc. The addition of this arc decreases the number of components by one.

To assign the second arc, again pick a $b \in (1, *) \setminus (1, *)'$ (again $n-1-k_1$ choices). Now there are exactly $n-3$ components, among which there is a node in $(0, *) \setminus \{(0, 0)\}$ with zero out-degree. We again select one and connect it with this $b$, leading to $(n-3)(n-1-k_1)$ choices.

In subsequent steps, we add an arc from $a$ to $b$, where $b \in (1, *) \setminus (1, *)'$ is arbitrary, while $a \in (0, *) \setminus \{(0, 0)\}$ is a unique node with zero out-degree in any component not containing $b$ in the
Proof of Theorem 4.1. It is clear that the number of components decreases by one after each arc is added, i.e., that a cycle not including \((0,0)\) is never created.

In the above steps we add \(t\) arcs, with the number of choices, in order: \((n-2)(n-1-k_1), (n-3)(n-1-k_1), \ldots, (n-t-1)(n-1-k_1)\). As any order in which they are assigned produces the same digraph, there are

\[
\frac{(n-2)(n-1-k_1)(n-3)(n-1-k_1) \cdots (n-t-1)(n-1-k_1)}{t!} = \binom{n-2}{t}(n-1-k_1)^t
\]

cases. Finally, we assign the remaining \(n-1-t\) arcs to \((1,*)',\) for which we have \((k_1+1)^{n-1-t}\) choices. Hence, for a fixed \(k_1,\) to assign the arcs originating from \((0,*) \setminus \{(0,0)\},\) there are

\[
\sum_{t=0}^{n-2} \binom{n-2}{t}(n-1-k_1)^t(n_1+1)^{n-1-t} = (k_1+1)n^{n-2}
\]

choices, in total. Lastly, we use Lemma 4.5 to get

\[
\#D(A, B) = \sum_{k_r=1}^{n-1} \binom{n-1}{k_r-1}(n-1)^{n-1-k_r-1} \\
\cdot \left[ \sum_{k_{r-2}=0}^{n-1} A_{k_{r-2},k_{r-1}} \cdots \sum_{k_2=0}^{n-1} A_{k_2,k_3} \left[ \sum_{k_1=0}^{n-1} A_{k_1,k_2}(k_1+1)n^{n-2} \right] \right] \cdots \\
= n^{(\tau-1)(n-2)} \sum_{k_r=0}^{n-1} \binom{n-1}{k_r-1}(n-1)^{n-1-k_r-1}[P_{\tau-1} + k_{\tau-1}(n-1)^{\tau-2}] \\
= n^{(\tau-1)(n-2)} [n^{n-1}P_{\tau-1} + (n-1)^{\tau-1}n^{n-2}] \\
= n^{\tau(n-2)}P_{\tau},
\]

as claimed. \(\square\)

Now, proof of Theorem 4.1 is straightforward.

**Proof of Theorem 4.1.** It is clear that the number of LAD \(G_{\tau,n}(f,A)\) is \(n^\tau n\). Then, by Lemma 4.6

\[
P(A \Rightarrow B) = \mathbb{P}(G_{\tau,n}(f,A) \in D(A, B)) = \frac{n^{\tau(n-2)}[n^\tau - (n-1)^\tau]}{n^{\tau n}} = \frac{n^\tau - (n-1)^\tau}{n^\tau} \cdot \frac{1}{n^\tau},
\]

as claimed. \(\square\)

By Theorem 4.1 assuming that \(A\) is simple and \(B\) is any label of the same length \(\tau\), we have

\[
P(A \Rightarrow B \mid A \Rightarrow B) = \frac{n^\tau - (n-1)^\tau}{n^\tau} = \frac{\tau}{n} + o\left(\frac{1}{n}\right).
\]

The case when \(A\) is not simple is much harder, since the parts of \(G_{\tau,n}(f,A)\) are no longer independent from each other for a random rule \(f\). While it is possible to obtain the deciding probability for a specific label using a similar method as in Theorem 4.1, it is hard to find a general formula or even to prove this probability is always \(O(1/n)\). We are, however, able to obtain the following weaker result.
Theorem 4.7. Let $A = a_0 \ldots a_{\tau-1}$ and $B = b_0 \ldots b_{\tau-1}$ be two fixed labels (not necessarily simple) with length $\tau$. Then

$$\mathbb{P}(G_{\tau,n}(f, A) \in \mathcal{D}(A, B) \mid G_{\tau,n}(f, A) \in \mathcal{E}(A, B)) = o(1).$$

Equivalently, we have

$$\mathbb{P}(A \Rightarrow B \mid A \rightarrow B) = o(1).$$

Proof. Again, we assume that $B = 0 \ldots 0$. We remark that, unlike Theorem 4.1, label $B$ here does not affect the deciding probability. However, the case of general $B$ does not significantly alter the proof but it makes it transparent, so we choose this $B$ for readability.

Let $a'_0, \ldots, a'_{\ell-1}$ be the different states in $A$ and $m_i$ be the repetition numbers of $a_i$'s, for $i = 0, \ldots, \ell - 1$. Clearly, $\sum_{i=0}^{\ell-1} m_i = \tau$. Let $\zeta$ be the cycle $(0, 0) \rightarrow (1, 0) \rightarrow \cdots \rightarrow (\tau-1, 0) \rightarrow (0, 0)$.

It suffices to show that

$$\mathbb{P}(\text{there are no other cycles in } G_{\tau,n}(f, A) \mid \zeta \in G_{\tau,n}(f, A)) = o(1).$$

To accommodate the conditional probability, our probability space will be a uniform choice of a digraph from $\mathcal{E}(A, B)$ for the remainder of the proof.

Fix an integer $K \geq 1$. Call a cycle $\zeta' = (0, j_0) \rightarrow (1, j_1) \rightarrow \cdots \rightarrow (0, j_0)$ simple with respect to $\zeta$ if:

1. $\zeta'$ contains no parallel arcs, i.e., if $(i, j)$ and $(i', j)$ are nodes in $\zeta'$, then $a_i \neq a_{i'}$; and
2. if $(i, j)$ is on $\zeta$ and $(i', j')$ on $\zeta'$, then $(a_{i'}, b_{j'}) \neq (a_i, b_j)$.

Let $Y_k$ be the random number of simple cycles with respect to $\zeta$ with length exactly $\tau_k$ and $Z_K = \sum_{k=1}^K Y_k$ be the random variable that counts the number of such cycles with length less than or equal to $\tau K$. We will show that, for any $K$, $\lim_{n \to \infty} \mathbb{P}(Z_K \geq 1) = 1 - \exp\left(-\sum_{k=1}^K 1/k\right)$, converging to 1 as $K \to \infty$. As a consequence, the LAD has another simple cycle asymptotically almost surely (in $n$), and this will conclude the proof.

We first compute the expectation of $Y_k$:

$$\mathbb{E}Y_k = \frac{(n-1)m_{1k} \cdots (n-1)m_{\ell k}}{k} \cdot \frac{1}{n^{\tau_k}} \to \frac{1}{k}, \quad \text{as } n \to \infty.$$ 

Here and in the sequel, we use the falling factorial notation $(x)_n = x(x-1) \cdots (x-n+1)$. The first factor counts the number of simple cycles with respect to $\zeta$ and the second factor is the probability that a fixed simple cycle with length $\tau k$ is formed.

Now, let $\lambda_K = \mathbb{E}Z_K = \sum_{k=1}^K \mathbb{E}Y_k$. We use the notation $\Gamma^k$ to denote the set of all possible simple cycles with length $\tau k$ and define $\Gamma = \bigcup_{1 \leq k \leq K} \Gamma^k$ as set of such cycles with length less than or equal
to $\tau K$. The set $\Gamma_i$ consists of cycles in $\Gamma$ that has at least one node in common with the cycle $i$. The random variable $I_i$ is the indicator that the cycle $i \in \Gamma$ is formed and $p_i = \mathbb{E} I_i$.

We use Lemma 2.7 to find an upper bound for $d_{TV}(Z_K, \text{Poisson}(\lambda_K))$. For the first term $\sum_{i \in \Gamma} p_i^{2}$, we have

$$\sum_{i \in \Gamma} p_i^{2} = \sum_{k=1}^{K} \frac{(n-1)_{m_{1k}} \cdots (n-1)_{m_{rk}}}{k} \frac{1}{n^{2\tau r}} = O \left( \frac{1}{n^{r}} \right).$$

To obtain an upper bound for $\sum_{i \in \Gamma} \sum_{j \in \Gamma_i} p_i p_j$, we note that if $i$ is the index of a simple cycle of length $\tau r$, then we may count the number of length-$\tau k$ simple cycles that have no common vertex with the cycle $i$, that is

$$\# \left( \Gamma^k \setminus \Gamma_i \right) = \frac{(n-1-r)_{m_{1k}} \cdots (n-1-r)_{m_{rk}}}{k}.$$

It immediately follows that,

$$\# \left( \Gamma^k \cap \Gamma_i \right) = \frac{(n-1-r)_{m_{1k}} \cdots (n-1-r)_{m_{rk}}}{k} - \frac{(n-1-r)_{m_{1k}} \cdots (n-1-r)_{m_{rk}}}{k} = O \left( n^{r^2-1} \right),$$

as the highest powers of $n$ in the numerator cancel. Hence, for a fixed $r$ and $k$, we have

$$\sum_{i \in \Gamma^r} \sum_{k \in \Gamma_i} \sum_{\Gamma^k} p_i p_j = \frac{(n-1-r)_{m_{1r}} \cdots (n-1-r)_{m_{rk}}}{r} \cdot \frac{1}{n^{r^2}} \cdot \frac{1}{n^{r^2}} = O \left( \frac{1}{n} \right).$$

Therefore, the total sum

$$\sum_{i \in \Gamma} \sum_{j \in \Gamma_i} p_i p_j = O \left( \frac{K^2}{n} \right).$$

For the last term in the upper bound in Lemma 2.7, we observe that $\mathbb{E} I_i I_j = 0$ if two cycles have shared vertices.

Now, by Lemma 2.7

$$\mathbb{P} (Z_K = 0) \leq e^{-\lambda_K} + O \left( \frac{K^2}{n} \right) \leq \frac{1}{K+1} + O \left( \frac{K^2}{n} \right).$$

Sending $n \to \infty$ and noting that $K$ is arbitrary conclude the proof.

5 Proof of Theorem 1.1

Let $T$ be a tile with $\tau$ rows and $\sigma$ columns. Define the rank of $T$ to be the largest $x$ such that there exist $x$ columns of $T$ with distinct $x \tau$ states. We denote the rank of a tile as rank$(T)$. For
example, the tiles

\[
T_1 = \begin{bmatrix}
0 & 1 & 2 & 3 \\
2 & 3 & 0 & 1
\end{bmatrix}, \quad T_2 = \begin{bmatrix}
0 & 1 & 2 & 1 \\
2 & 1 & 0 & 1
\end{bmatrix}
\]

have \( \text{rank}(T_1) = 2 \) and \( \text{rank}(T_2) = 1 \).

As in [6], we denote by \( R(\ell)_{\tau,\sigma,n} \) the set of tile of WRPS that has lag \( \ell \). Thus the set of simple WRPS is \( R(0)_{\tau,\sigma,n} \). We also use the notation \( R(0,y)_{\tau,\sigma,n} \subset R(0)_{\tau,\sigma,n} \) to denote the set of WRPS whose tile is simple and has rank \( y \). We use \( T_{\tau,\sigma,n} \) to denote the set of all PS tiles; to be more precise, this is the set of all \( \tau \times \sigma \) arrays \( T \) with state space \( \mathbb{Z}_n \) that satisfy properties 1 and 2 in Lemma 2.1, so that there exists a CA rule with a PS given by \( T \). We also use \( T_{\tau,\sigma,n}^{(0)} \) and \( T_{\tau,\sigma,n}^{(0,y)} \) to denote the tiles in \( T_{\tau,\sigma,n} \) that are simple, and that are simple with rank \( y \), respectively.

Our first step is to study the probability that \( R(0,x)_{\tau,\sigma,n} \) is not empty, where \( x = \sigma / \text{gcd}(\tau,\sigma) \).

Before we advance, we state two lemmas on simple tiles.

**Lemma 5.1.** Let \( T \) be a simple tile. Then

1. \( \text{rank}(T) \geq \sigma / \text{gcd}(\sigma,\tau) \);
2. \( \text{rank}(T) = y \) if and only if \( s(T) = \tau y \). In particular, \( \text{rank}(T) = \sigma / \text{gcd}(\sigma,\tau) \) if and only if \( s(T) = \tau \sigma / \text{gcd}(\sigma,\tau) = \text{lcm}(\sigma,\tau) \).

**Proof.** By Lemma 2.8, the states on each column of \( T \) are distinct and two columns either share no common states or are circular shifts of each other. As a result, \( \text{rank}(T) \geq s(T)/\tau \). Together with Lemma 2.10, this proves (1) and implication (\( \Rightarrow \)) of (2). The reverse implication in (2) follows from \( s(T) \geq \tau \cdot \text{rank}(T) \).

In the sequel, we write \( d = \text{gcd}(\tau,\sigma) \), \( k = \text{lcm}(\sigma,\tau) \). By Lemma 5.1, \( k \) is the number of distinct states in a simple tile with rank \( x = \sigma/d \). As before, \( \varphi \) is the Euler totient function. We index the tiles in \( T_{\tau,\sigma,n}^{(0,x)} \) in an arbitrary way. Let

\[
\mathcal{I}_m = \left\{(T_i, T_j) \in T_{\tau,\sigma,n}^{(0,x)} \times T_{\tau,\sigma,n}^{(0,x)} : i < j \text{ and } T_i, T_j \text{ have } m \text{ states in common}\right\}.
\]

The following lemma gives the cardinality of these sets.

**Lemma 5.2.** The following enumeration results hold:

1. the set \( T_{\tau,\sigma,n}^{(0,x)} \) has cardinality \( \varphi(d) \binom{n}{k} (k-1)! \);
2. if \( m < k \), the set \( \mathcal{I}_m \) has cardinality

\[
\frac{1}{2} \varphi(d) \binom{n}{k} (k-1)! \varphi(d) \binom{k}{m} \binom{n-k}{k-m} (k-1)! = \mathcal{O} \left( n^{2k-m} \right);
\]

3. if \( m = k \), the set \( \mathcal{I}_m \) has cardinality

\[
\frac{1}{2} \varphi(d) \binom{n}{k} (k-1)! (\varphi(d)(k-1)! - 1) = \mathcal{O} \left( n^k \right).
\]
Proof. Part (1) follows directly from Lemma 2.11. Then, part (2) follows from (1). Part (3) also follows from (1), after we note that once we select \( T_i \), we have all \( k \) colors fixed and we are not allowed to select \( T_j \) equal to \( T_i \).

We will also need the following consequence of Theorem 4.1.

**Lemma 5.3.** Let \( T \) be a simple tile and \( \text{rank}(T) = y \). Let \( A_0, \ldots, A_{\sigma-1} \) be the labels in \( T \). Then we have

\[
P(A_i \Rightarrow A_{i+1}, \text{ for } i = 0, \ldots, \sigma - 1 \mid A_i \rightarrow A_{i+1}, \text{ for } i = 0, \ldots, \sigma - 1) = \left( \frac{T}{n} + o \left( \frac{1}{n} \right) \right)^y.
\]

**Proof.** Assume that the \( y \) columns with \( yT \) states have indices in \( I \subset \{0, \ldots, \sigma - 1\} \) and let those columns have labels \( A_i, i \in I \). As \( A_i \)'s do not share any states, the events \( \{A_i \Rightarrow A_{i+1}\}, i \in I \) are independent, and so are \( \{A_i \rightarrow A_{i+1}\}, i \in I \). We use Lemma 2.8 and Theorem 4.1 to get

\[
P(A_i \Rightarrow A_{i+1}, \text{ for } i = 0, \ldots, \sigma - 1 \mid A_i \rightarrow A_{i+1}, \text{ for } i = 0, \ldots, \sigma - 1) = \frac{P(A_i \Rightarrow A_{i+1}, \text{ for } i \in I)}{P(A_i \rightarrow A_{i+1}, \text{ for } i \in I)}
= \frac{\prod_{i \in I} P(A_i \Rightarrow A_{i+1})}{\prod_{i \in I} P(A_i \rightarrow A_{i+1})}
= \left( \frac{n^y - (n-1)^y}{n^y} \cdot \frac{1}{n^y} \right)^y / \left( \frac{1}{n^y} \right)^y
= \left( \frac{T}{n} + o \left( \frac{1}{n} \right) \right)^y,
\]
as desired. \( \square \)

Theorem 1.1 will now be established through next three propositions, the first one of which deals with existence of WRPS with zero lag and minimal rank \( x = \sigma/d \).

**Proposition 5.4.** We have

\[
P\left( \mathcal{R}^{(0,x)}_{\tau,\sigma,n} \neq \emptyset \right) = c(\tau, \sigma) \frac{n^x}{n^x} + o \left( \frac{1}{n^x} \right),
\]

for some constant \( c(\tau, \sigma) \).

**Proof.** We first find an upper bound by Markov inequality.

By Lemma 5.2 we have that \#\( \mathcal{T}^{(0,x)}_{\tau,\sigma,n} \) is \( \varphi(d) \binom{n}{k} (k-1)! \). The probability that a tile in \( \mathcal{T}^{(0,x)}_{\tau,\sigma,n} \) forms a PS is \( 1/n^k \) and the probability that the desired decidability, thus weak robustness, holds is \( (T/n + o(1/n))^x \) by Lemma 5.3. As a result, we have

\[
E(\#\mathcal{R}^{(0,x)}_{\tau,\sigma,n}) = \varphi(d) \binom{n}{k} (k-1)! \frac{1}{n^x} \left( \frac{T}{n} + o \left( \frac{1}{n} \right) \right)^x = \frac{c(\tau, \sigma)}{n^x} + o \left( \frac{1}{n^x} \right),
\]
as an upper bound.
To find an asymptotically matching lower bound, we use the Bonferroni’s inequality
\[ \mathbb{P} \left( \bigcup_i A_i \right) \geq \sum_i \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j). \]

Here, \( A_i \) is the event that \( T_i \in T^{(0,x)}_{\tau,\sigma,n} \) is formed as a simple WRPS, for \( i = 1, \ldots, \varphi(d) \binom{n}{k}(k-1)! \).

Clearly, \( \sum_i \mathbb{P}(A_i) = \mathbb{E} \left( \#R^{(0,x)}_{\tau,\sigma,n} \right) \). Then it suffices to show that \( \sum_{i<j} \mathbb{P}(A_i \cap A_j) = o(1/n^x) \).

For a pair of tiles \( (T_i, T_j) \in \mathcal{T}_m \), there are \( 2k-m \) different colors in \( T_i \cup T_j \). By Lemma 2.12, there is at least one additional restriction on the number of maps. Using this lemma, the enumeration result Lemma 5.2 and Lemma 5.3 we have
\[
\begin{align*}
\sum_{i<j} \mathbb{P}(A_i \cap A_j) &= \sum_{m=0}^{k} \sum_{i<j} \mathbb{P}(A_i \cap A_j \cap \{(T_i, T_j) \in \mathcal{T}_m\}) \\
&= \sum_{m=0}^{k} \mathcal{O} \left( n^{2k-m} \right) \frac{1}{n^{2k-m+1}} \left( \frac{\tau}{n} + o \left( \frac{1}{n} \right) \right)^x \\
&= \mathcal{O} \left( \frac{1}{n^{x+1}} \right).
\end{align*}
\]

Next, we consider all simple tiles and show that among simple tiles, the WRPS with rank \( x \) provide the dominant probability.

**Proposition 5.5.** We have
\[ \mathbb{P} \left( R^{(0)}_{\tau,\sigma,n} \neq \emptyset \right) = \frac{c(\tau, \sigma)}{n^x} + o \left( \frac{1}{n^x} \right), \]
for the same constant \( c(\tau, \sigma) \) as in Proposition 5.4.

**Proof.** First, we note the following bounds for \( \mathbb{P}(R^{(0)}_{\tau,\sigma,n} \neq \emptyset) \),
\[ \mathbb{P} \left( R^{(0)}_{\tau,\sigma} \neq \emptyset \right) \leq \mathbb{P} \left( R^{(0)}_{\tau,\sigma,n} \neq \emptyset \right) \leq \mathbb{P} \left( R^{(0,0)}_{\tau,\sigma,n} \neq \emptyset \right) + \sum_y \mathbb{P} \left( R^{(0,y)}_{\tau,\sigma,n} \neq \emptyset \right), \]
where the last sum is over \( y = \sigma/d' \) for \( d' \mid \gcd(\tau, \sigma) \) and \( d < \gcd(\tau, \sigma) \). As \( x < y \), we have from Lemmas 5.1 and 5.3
\[
\begin{align*}
\mathbb{P} \left( R^{(0,y)}_{\tau,\sigma,n} \neq \emptyset \right) &\leq \mathbb{E} \left( \#R^{(0,y)}_{\tau,\sigma,n} \right) \\
&= \varphi(d_y) \binom{n}{k_y} (k_y - 1)! \frac{1}{n^{k_y}} \left( \frac{\tau}{n} + o \left( \frac{1}{n} \right) \right)^y \\
&= o \left( \frac{1}{n^x} \right),
\end{align*}
\]
where, \( k_y = \tau y \) is the number of states in a tile in \( R^{(0,y)}_{\tau,\sigma,n} \) and \( d_y = \sigma/y \). The conclusion now follows from Proposition 5.4.
Lemma 5.6. If $\ell > 0$, then
\[
\Pr \left( R_{\tau,\sigma,n}^{(\ell)} \neq \emptyset \right) = o \left( \frac{1}{n} \right).
\]

Proof. For a fixed $\ell$, let $g_{\tau,\sigma}(s)$ count the number of tiles with periods $\tau$ and $\sigma$, and $s$ different fixed states. By Theorem 4.7,
\[
\Pr \left( R_{\tau,\sigma,n}^{(\ell)} \neq \emptyset \right) \leq E \left( \# R_{\tau,\sigma,n}^{(\ell)} \right) = \sum_{s=1}^{\tau \sigma} \binom{n}{s} g_{\tau,\sigma,\ell}(s) \frac{1}{n^{s+\ell}} \cdot o(1)
\]
\[
= o \left( \frac{1}{n^{\ell}} \right) = o \left( \frac{1}{n} \right).
\]
\[\square\]

Next, we extend Proposition 5.5 to cover non-simple tiles. It is here that we impose the condition that $\sigma \mid \tau$.

Proposition 5.7. If $\sigma \mid \tau$, then
\[
\Pr \left( R_{\tau,\sigma,n} \neq \emptyset \right) = c(\tau,\sigma) \left( \frac{x}{n} \right) + o \left( \frac{1}{n} \right).
\]

Proof. First, note that $\sigma \mid \tau$ implies that $x = \sigma / \gcd(\tau,\sigma) = 1$ and as a result of Proposition 5.5, we have
\[
\Pr \left( R_{\tau,\sigma,n}^{(0)} \neq \emptyset \right) = \frac{c(\tau,\sigma)}{n} + o \left( \frac{1}{n} \right).
\]

The desired result now follows from the bounds
\[
\Pr \left( R_{\tau,\sigma,n}^{(0)} \neq \emptyset \right) \leq \Pr \left( R_{\tau,\sigma,n} \neq \emptyset \right) \leq \sum_{\ell=0}^{\tau \sigma} \Pr \left( R_{\tau,\sigma,n}^{(\ell)} \neq \emptyset \right)
\]
and Lemma 5.6. \[\square\]

Proof of Theorem 1.1. If $\sigma \nmid \tau$, then $x = \sigma / \gcd(\tau,\sigma) > 1$, and by Proposition 5.5 and Lemma 5.6,
\[
\Pr \left( R_{\tau,\sigma,n} \neq \emptyset \right) \leq \Pr \left( R_{\tau,\sigma,n}^{(0)} \neq \emptyset \right) + \sum_{\ell=1}^{\tau \sigma} \Pr \left( R_{\tau,\sigma,n}^{(\ell)} \neq \emptyset \right) = c(\tau,\sigma) \left( \frac{1}{n^{x}} \right) + o \left( \frac{1}{n} \right) = o \left( \frac{1}{n} \right).
\]

These bounds, together with Proposition 5.7, now give the desired result:
\[
\frac{c(T,\Sigma)}{n} + o \left( \frac{1}{n} \right) = \sum_{\sigma \mid \tau} \Pr \left( R_{\tau,\sigma,n} \neq \emptyset \right)
\]
\[
\leq \Pr \left( R_{T,\Sigma,n} \neq \emptyset \right)
\]
\[
\leq \sum_{\sigma \mid \tau} \Pr \left( R_{\tau,\sigma,n} \neq \emptyset \right) + \sum_{\sigma \nmid \tau} \Pr \left( R_{\tau,\sigma,n} \neq \emptyset \right) \leq \frac{c(T,\Sigma)}{n} + o \left( \frac{1}{n} \right).
\]
\[\square\]

17
6 Discussion

Inspired by [3], we prove that the probability that a randomly chosen CA has a weakly robust periodic solution with periods in the finite set \( T \times \Sigma \) is asymptotically \( c(T, \Sigma) / n \), provided that \( T \times \Sigma \) contains a pair \((\tau, \sigma)\) with \( \sigma | \tau \). A natural first question is whether the divisibility condition may be removed.

**Question 6.1.** Let \( R_{\tau,\sigma,n} \) be the set of WRPS with periods \( \tau \) and \( \sigma \) from a random rule \( f \). Do we have

\[
\mathbb{P}(R_{\tau,\sigma,n} \neq \emptyset) = c(\tau, \sigma) n^x + o\left(\frac{1}{n^x}\right),
\]

where \( x = \sigma / \gcd(\tau, \sigma) \)?

A possible strategy to answer Question 6.1 affirmatively is through proving the following two conjectures, the first of which provides a lower bound of the rank of a tile. Recall that \( x = \sigma / \gcd(\tau, \sigma) \).

**Conjecture 6.2.** Let \( T \) be a tile of a WRPS of period \( \tau \) and \( \sigma \) and \( \ell = p(T) - s(T) \). Then

\[
\text{rank}(T) \geq x - \ell.
\]

We recall that a tile of a WRPS satisfies the properties stated in Lemmas 2.1 and 3.3. The next conjecture presents an asymptotic property similar to the one in Theorem 4.7. In its formulation, we assume validity of Conjecture 6.2: for a tile \( T \) of a WRPS, we let \( I = I(T) \subset \{0, \ldots, \sigma - 1\} \) be the index set with \( \#I = x - \ell \), such that the labels indexed by \( I \) are the leftmost \( x - \ell \) labels without a repeated state.

**Conjecture 6.3.** Assume that \( T \) is a tile of a WRPS. Then there exists a label \( A_j \) with index \( j \notin I \) so that

\[
\mathbb{P}(A_j \Rightarrow A_{j+1} \mid \{A_i \Rightarrow A_{i+1} \text{ for all } i \in I\}) = o(1).
\]

If there exists a label \( j \) that does not share any state with \( A_i \), for any \( i \in I \), the conjecture can be proved in the same way as Theorem 4.7. To see how Question 6.1 is settled in the case that both of the conjectures are satisfied, use again the bounds

\[
\mathbb{P}(R_{\tau,\sigma,n}^{(0)} \neq \emptyset) \leq \mathbb{P}(R_{\tau,\sigma,n} \neq \emptyset) \leq \mathbb{P}(R_{\tau,\sigma,n}^{(0)} \neq \emptyset) + \sum_\ell \mathbb{E}(\#R_{\tau,\sigma,n}^{(\ell)}),
\]

and then, with \( g_{\tau,\sigma}(s) \) as in the proof of Lemma 5.6 and using Lemma 5.3

\[
\mathbb{E}(\#R_{\tau,\sigma,n}^{(\ell)}) = \sum_{s=1}^{\tau \sigma} \binom{n}{s} g_{\tau,\sigma}(s) \frac{1}{n^m} \cdot O\left(\frac{1}{n^{x-\ell}}\right) \cdot o(1) = o\left(\frac{1}{n^x}\right).
\]

To provide some modest evidence for the validity of Conjecture 6.2 we prove that it holds when \( \sigma = 2 \) or \( \tau = 2 \). Conjecture 6.3 remains open even in these cases. We begin by the following lemma.

**Lemma 6.4.** Let \( T \) be a tile of a WRPS with \( \sigma = 2 \) and odd \( \tau \). Fix an arbitrary row as the 0th row. Let \( M_t = \{ \text{maps up to } t \text{ th row} \} \), \( S_t = \{ \text{states up to } t \text{ th row} \} \) and \( \ell_t = \#M_t - \#S_t \), for \( t = 0, 1, \ldots, \tau - 1 \). Assume the \((t + 1)\text{th row of the tile is ab. Then:}\)

1. if \( a \in S_t \) and \( b \in S_t \), \( \ell_{t+1} - \ell_t = 2 \);
2. if exactly one of $a$ and $b$ is in $S_t$, then $\ell_{t+1} - \ell_t = 1$; and

3. if $a \notin S_t$ and $b \notin S_t$, $\ell_{t+1} - \ell_t = 0$.

Proof. Write $\ell_{t+1} - \ell_t = (\#M_{t+1} - \#M_t) - (\#S_{t+1} - \#S_t)$. Observe that $a \neq b$, as otherwise the spatial period of the tile is reducible. In addition, $(a, b) \notin M_t$, as otherwise $T$ is temporally reducible, and $(b, a) \notin M_t$, as otherwise $\tau$ is even. Hence, $\#M_{t+1} - \#M_t = 2$, which implies the claim.

Proof of Conjecture 6.2 when $\sigma = 2$. If $\tau$ is even, we need to show that $\text{rank}(T) \geq 1 - \ell$. This is trivial if $\ell \geq 1$, and follows from Lemma 2.8 when $\ell = 0$.

If $\tau$ is odd, we must show that $\text{rank}(T) \geq 2 - \ell$. We may assume $\ell = 1$ as otherwise this is immediate (as above). Then there exists exactly one $t \in \{0, \ldots, \tau - 1\}$ at which Case 2 of Lemma 6.4 happens, and otherwise Case 3 happens. If $a \in S_t$, then column with $b$ has no repeated state, and vice versa.

Proof of Conjecture 6.2 when $\tau = 2$. We will prove this for any tile that satisfies the properties stated in Lemmas 2.1 and 3.3. We assume that no two different labels of $T$ are rotations of each other; otherwise the argument is similar.

We use induction on the lag. If $\ell(T) = 0$, $T$ is simple and Lemma 2.8 applies. Suppose now the statement is true for any tile $T$ with $\ell(T) = \ell \geq 0$. Now, consider a tile $T$ with $\ell(T) = \ell + 1$. As $\ell(T) \geq 1$, there is at least one repeated state, say $a$. Consider two appearance of $a$ and its neighbors: $bac$ and $b'ac'$.

As $\tau = 2$ and $T$ has no rotated columns, $b \neq b'$ and $c \neq c'$. Now replace the $a$ in $bac$ by an arbitrary state not represented in $T$, say $z$, and denote the new tile by $T'$. Note that $T'$ also satisfies the properties in Lemmas 2.1 and 2.8. Moreover, $p(T') = p(T)$ and $s(T') = s(T) + 1$ imply that $\ell(T') = \ell$. By inductive hypothesis, $\text{rank}(T') \geq \sigma / \gcd(\sigma, \tau) - \ell$. Among rank($T'$) labels of $T'$ without a repeated state, at most one has the state $z$. Excluding this label, if necessary, we conclude that $\text{rank}(T) \geq \sigma / \gcd(\sigma, \tau) - (\ell + 1)$.

Besides the above two special cases, we are also able to prove Conjecture 6.2 for a special class of tiles, which may give a hint about the general case. Within $T$, fix an arbitrary row as the 0th row and find the smallest $\tilde{\tau}$ such that row is a cyclic permutation of row$\cdot$ It is likely that such $\tilde{\tau}$ does not exist, in which case define $\tilde{\tau} = \tau$. We call $T$ semi-simple if $p(T) = \tilde{\tau}\sigma$; i.e., within the first $\tilde{\tau}$ rows in $T$, there are no repeated states. We omit the proof of our last lemma, as it is very similar to the argument above.

Lemma 6.5. A semi-simple tile $T$ has rank at least $\sigma / \gcd(\tau, \sigma) - \ell$.

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