BOUNDS FOR CANONICAL GREEN’S FUNCTIONS AT CUSPS

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Abstract. Let $\Gamma$ be a cofinite Fuchsian subgroup. The canonical Green’s function associated with $\Gamma$ arises in Arakelov theory when establishing asymptotics for Arakelov invariants of the modular curve associated with some congruence subgroup of level $N$ with a positive integer $N$. More precisely, in the known cases, canonical Green’s functions at certain cusps contribute to the analytic part of the asymptotics for the self-intersection of the relative dualizing sheaf.

In this article, we prove canonical Green’s function of a cofinite Fuchsian subgroup at cusps bounded by the scattering constants, the Kronecker limit functions, and the Selberg zeta function of the group $\Gamma$. Then as an application, we prove an asymptotic expression of the canonical Green’s function associated with $\Gamma_0(N)$ for any positive integer $N$.

1. Introduction

1.1. Overview. Let $\Gamma$ be a Fuchsian subgroup of $\text{PSL}_2(\mathbb{R})$ with finite hyperbolic volume. Then the quotient space $X = \Gamma \backslash \mathbb{H}$ is conformally equivalent to a non-compact Riemann surface of genus $g_\Gamma$. Let $\mathcal{P}_\Gamma$ denote the set of cusps of $\Gamma$, then $X = X \cup \mathcal{P}_\Gamma$ admits the structure of a compact Riemann surface. On $X$ we have the hyperbolic metric which is defined as

$$
\mu_{\text{hyp}}(z) = \frac{i}{2} \frac{dz \wedge d\bar{z}}{\text{Im}(z)^2}.
$$

On $X$ we also have the canonical metric which is defined as

$$
\mu_{\text{can}}(z) = \frac{i}{2g_\Gamma} \sum_{j=1}^{g_\Gamma} |f_j(z)|^2 \, dz \wedge d\bar{z},
$$

where $\{f_1, \ldots, f_{g_\Gamma}\}$ denote an orthonormal basis of the space of cusp forms of weight 2 with respect to $\Gamma$ equipped with the Petersson inner product.

Let $\mu$ be a smooth metric on $X$. Let $\text{vol}_\mu(X)$ be the volume of $X$ with respect to the metric $\mu$. For $z, w \in X (z \neq w)$, the Green’s function $G_\mu(z, w)$ with respect to metric $\mu$ is defined as the unique solution of a differential equation with a normalization condition

$$
d_z d_z^c G_\mu(z, w) + \delta_w(z) = \frac{\mu(z)}{\text{vol}_\mu(X)}, \quad \int_X G_\mu(z, w) \mu(z) = 0,
$$

where $\delta_w(z)$ is the Dirac delta distribution. Now, if $\mu = \mu_{\text{hyp}}$, we have $\text{vol}_\mu(X) = v_\Gamma$, and in this case we write $G_\mu(z, w) = G_{\text{hyp}}(z, w)$, which is known as the hyperbolic Green’s function. If $\mu = \mu_{\text{can}}$, then we have $\text{vol}_\mu(X) = 1$, and in this case we write $G_\mu(z, w) = G_{\text{can}}(z, w)$, which is known as the canonical Green’s function. The Green’s function $G_\mu(z, w)$ has a logarithmic singularity at the diagonal, meaning,

$$
G_{\text{can}}(z, w) + \log |z - w|^2
$$

is bounded and continuous as $z$ approaches $w$. 

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In this article, we prove a bound for the canonical Green’s function associated with a cofinite Fuchsian subgroup Γ. Following the work of Jorgenson–Kramer [17], we express the difference between the hyperbolic Green’s function and the canonical Green’s function, \( G_{\text{hyp}}(z, w) - G_{\text{can}}(z, w) \), in terms of integrals involving the hyperbolic heat kernel \( K_{\text{hyp}}(t; z, w) \) \((t \in \mathbb{R}_{>0}; z, w \in X)\). Then using the work of Anilatmaja [4], we prove a bound for the canonical Green’s function associated with a cofinite Fuchsian subgroup at two different cusps. This bound depends on scattering constants, Kronecker limit functions, the Selberg zeta constant, and the smallest non-zero eigenvalue of the hyperbolic Laplacian. Then as an example, we take Γ as a congruence subgroup \( \Gamma_0(N) \) with a positive integer \( N \), and we simplify the bound for the canonical Green’s function. In this case, the scattering constants are known from [8], and we simplify the expression for the Kronecker limit function for \( \Gamma_0(N) \) by using the following well-known formula for the Eisenstein series. From Jorgenson–Kramer [20], we use the bound for the Selberg zeta functions for \( \Gamma_0(N) \). Lastly, by using the fact that the smallest non-zero eigenvalue of the hyperbolic Laplacian for congruence subgroups is \( \geq 21/100 \) (see Luo–Rudnick–Sarnak [29]), we prove asymptotics for the canonical Green’s function associated to congruence subgroups \( \Gamma_0(N) \) with a positive integer \( N \).

1.2. Applications. Given a smooth algebraic curve defined over a number field, together with the minimal regular model over the corresponding ring of integers, in [2], Arakelov introduced a real number, called the self-intersection of the dualizing sheaf. The motivation for studying this Arakelov invariant lies in arithmetic geometry. From [2], we know that the Arakelov self-intersection of the dualizing sheaf on a modular curve is defined as the sum of a geometric part and an analytic part, where the analytic part is computed using the canonical Green’s function at cusps.

Considering the congruence subgroup \( \Gamma_0(N) \) with a positive squarefree integer \( N \) with \( 2, 3 \nmid N \), Abbes–Michel–Ullmo [1], [26], proved the following asymptotics.

\[
\bar{\omega}_{\Gamma_0(N)}^2 = 3g_{\Gamma_0(N)} \log N + o(g_{\Gamma_0(N)} \log N) \quad \text{as} \quad N \to \infty,
\]

where \( \bar{\omega}_{\Gamma_0(N)}^2 \) is the Arakelov self-intersection of the dualizing sheaf of the minimal regular model \( \mathcal{X}_0(N) \) over \( \mathbb{Z} \) for the modular curve \( X_0(N) \). In their case, they proved that the leading term, \( 3g_{\Gamma_0(N)} \log N \) is the sum of \( g_{\Gamma_0(N)} \log N \) that comes from the geometric part, and \( 2g_{\Gamma_0(N)} \log N \) come from the analytic part. The analytic part, in this case, is exactly given by

\[
2g_{\Gamma_0(N)}(1 - g_{\Gamma_0(N)}) G_{\text{can}}(0, \infty).
\]

In the present article we remove the squarefree condition on \( N \), and prove that for a positive integer \( N \), the leading term in the asymptotics for \( 2g_{\Gamma_0(N)}(1 - g_{\Gamma_0(N)}) G_{\text{can}}(0, \infty) \) is \( 2g_{\Gamma_0(N)} \log N \).

For a squarefree natural number \( N \) such that \( 2, 3 \nmid N \), Abbes–Michel–Ullmo [1], [26], proved an asymptotic expression for the canonical Green’s function associated with \( \Gamma_0(N) \) by using the following formula.

\[
G_{\text{can}}(p_k, p_l) = 4\pi \mathcal{C}_{p_k p_l} + \frac{4\pi}{v_\Gamma} + \lim_{s \to 1} \left( \int_{X \times X} G_{\text{hyp}, s}(z, w) \mu_{\text{can}}(z) \mu_{\text{can}}(w) - \frac{4\pi}{s(s - 1)v_\Gamma} \right)

- 4\pi \lim_{s \to 1} \left( \int_X E_{p_k}(z, s) \mu_{\text{can}}(z) + \int_{X} E_{p_l}(w, s) \mu_{\text{can}}(w) - \frac{2}{(s - 1)v_\Gamma} \right),
\]

(1.1)

where \( p_k, p_l \) are two different cusps of a cofinite Fuchsian subgroup \( \Gamma \), and \( \mathcal{C}_{p_k p_l} \) is the scattering constant with respect to the cusps \( p_k \) and \( p_l \). By \( G_{\text{hyp}, s}(z, w) \) we denote the automorphic
Green’s function, and by $E_{p_k}(z,s)$ we denote the Eisenstein series corresponding to the cusp $p_k$. Using (1.1), Mayer [24], investigated the case $X_1(N)$ with an odd squarefree integer $N$ which is divisible by at least two relatively prime numbers bigger than or equal to 4. In this case, Mayer proved an asymptotic expression for the Arakelov self-intersection of the dualizing sheaf on $X_1(N)$. Recently, Grados–von Pippich [13] (also see [11]), by following the line of proof in [1], proved an asymptotic expression of this quantity for the modular curve $X(N)$ with an odd squarefree positive integer $N$. In [5], Banerjee–Borah–Chaudhari investigated the case of the modular curve $X_0(p^2)$ with a prime number $p$ by following mostly the lines of proof in [1].

In this article, we do not use the formula (1.1) instead we take a different approach to compute the asymptotics for the canonical Green’s function at cusps. A good reason for that is to avoid lengthy mathematical computations which arise to prove asymptotics for the Rankin-Selberg constant. Furthermore, we prove a bound for the canonical Green’s function for an arbitrary cofinite Fuchsian subgroup of PSL$_2(\mathbb{R})$. So, one can use our result for any congruence subgroup, for example, $\Gamma_0(N), \Gamma_1(N)$ or $\Gamma(N)$ with a positive integer $N$. These bounds will be useful for potential applications to arithmetic algebraic geometry.

1.3. Main results. We now state the main results of the article.

**Theorem 1.1.** Let $\Gamma$ be a cofinite Fuchsian subgroup and $X = \Gamma \backslash \mathbb{H}$. Let $p_k, p_l$ are two cusps of $X$. Then

$$G_{\text{can}}(p_k, p_l) = 4\pi c_{p_k p_l} + \frac{2\pi}{g_\Gamma} \sum_{j=1, j\neq k}^{p_\Gamma} C_{p_k p_j} + \frac{2\pi}{g_\Gamma} \sum_{j=1, j\neq l}^{p_\Gamma} C_{p_l p_j} + \frac{4\pi c_X}{g_\Gamma e_\Gamma}$$

$$+ \frac{2\pi}{g_\Gamma} \sum_{j=1}^{e_\Gamma} (1 - 1/\text{ord}(e_j)) \left( K_{p_k}(e_j) + K_{p_l}(e_j) \right) + \delta_X,$$

where $C_{p_k p_j}$ denotes the scattering constant with respect to cusps $p_k$ and $p_j$, $K_{p_k}(e_j)$ denotes the Kronecker limit function with respect to the cusp $p_k$ evaluated at the elliptic fixed point $e_j$, $c_X$ denotes the Selberg zeta constant on $X$, and the absolute value of $\delta_X$ is bounded by

$$\frac{4\pi}{v_\Gamma g_\Gamma} \sum_{j=1}^{e_\Gamma} (1 + 1/\text{ord}(e_j)) + \frac{4\log 2}{v_\Gamma g_\Gamma} \sum_{j=1}^{e_\Gamma} (\text{ord}(e_j) + 1) + \frac{4\pi (d_X + 1)^2}{\lambda_1 v_\Gamma}$$

$$+ \frac{4\pi}{v_\Gamma} + \frac{2 \log(4\pi)}{g_\Gamma} + \frac{2 p_\Gamma}{g_\Gamma e_\Gamma} \left( \pi + \frac{4\pi^2}{3} + 1 \right).$$

where $p_\Gamma, e_\Gamma$ denote the number of inequivalent cusps and number of inequivalent elliptic fixed points of $\Gamma$ respectively. $\lambda_1$ denotes the smallest non-zero eigenvalue of the hyperbolic Laplacian $\Delta_{\text{hyp,z}}$, and $d_X = \sup_{z \in X} \left| \frac{\mu_{\text{can}}(z)}{\mu_{\text{hyp}}(z)} \right|.$

**Theorem 1.2.** Let $\Gamma = \Gamma_0(N)$ with a positive integer $N$. Then we have the following asymptotic expression.

$$2g_\Gamma (1 - g_\Gamma) G_{\text{can}}(0, \infty) = 2g_\Gamma \log N + o(g_\Gamma \log N) \text{ as } N \to \infty,$$

where $g_\Gamma$ denotes the genus of $X_0(N) = \Gamma \backslash \mathbb{H}$. 

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1.4. Outline of the article. The paper is organized as follows. In section 2, we recall and summarize basic notation and definitions used in this article. In section 3, we recall an expression of the canonical Green’s function in terms of hyperbolic Green’s function and analytic functions derived from the hyperbolic heat kernel. In section 4 by giving some useful estimates, in section 5 we prove Theorem 1.1. Then as an application we consider the congruence subgroup $\Gamma_0(N)$. In section 6 we prove some asymptotic bounds for scattering constants of $\Gamma_0(N)$. In section 7 we prove asymptotic bounds for Kronecker limit functions of $\Gamma_0(N)$. Finally, in section 8 we prove Theorem 1.2.

2. Background material

2.1. Basic notation. As mentioned in the introduction, we let $\Gamma \subset \text{PSL}_2(\mathbb{R})$ denote a cofinite Fuchsian subgroup acting by fractional linear transformations on the hyperbolic upper half-plane $\mathbb{H} := \{ z = x + iy \in \mathbb{C} \mid x, y \in \mathbb{R}; y > 0 \}$. The quotient space $X := \Gamma \backslash \mathbb{H}$ admits the structure of a finite volume hyperbolic Riemann surface. By $\mathcal{P}_\Gamma$ resp. $\mathcal{E}_\Gamma$ we denote a complete set of $\Gamma$-inequivalent cusps and elliptic fixed points of $\Gamma$, respectively, and we set $p_\Gamma := \sharp \mathcal{P}_\Gamma$, $e_\Gamma := \sharp \mathcal{E}_\Gamma$, that is, we assume that $X$ has $e_\Gamma$ elliptic fixed points and $p_\Gamma$ cusps. By $\text{ord}(e_j) \in \mathbb{N}_{\geq 2}$ we denote the order of the elliptic fixed point $e_j \in \mathcal{E}_\Gamma$ ($j = 1, \ldots, e_\Gamma$).

Let $e_j$ be an elliptic fixed point of $\Gamma$, and $\Gamma_{e_j} := \{ \gamma \in \Gamma \mid \gamma e_j = e_j \}$ be the stabilizer of $e_j$. Let $p_j$ be a cusp of $\Gamma$, and $\Gamma_{p_j} := \{ \gamma \in \Gamma \mid \gamma p_j = p_j \}$ be the stabilizer of $p_j$. Then From [25], Theorem 1.5.4, we know that $\Gamma_{e_j}$ is a finite cyclic group, and

$$\Gamma_{p_j}/Z(\Gamma) \cong \mathbb{Z}, \text{ where } Z(\Gamma) = \Gamma \cap \{ \pm 1 \}.$$ 

We identify $X$ locally with its universal cover $\mathbb{H}$. By $d_{\text{hyp}}^2$ resp $\mu_{\text{hyp}}$ we denote the hyperbolic line element and the hyperbolic metric on $X$, which are compatible with the complex structure of $X$, and has constant negative curvature equal to minus one. For $z = x + iy \in \mathbb{H}$, we have

$$d_{\text{hyp}}^2 = \frac{dz \cdot d\overline{z}}{\text{Im}(z)^2} = \frac{dx^2 + dy^2}{y^2}, \text{ resp. } \mu_{\text{hyp}}(z) = \frac{i}{2} \frac{dz \wedge d\overline{z}}{\text{Im}(z)^2} = \frac{dx dy}{y^2}.$$ 

Since $\Gamma$ is cofinite, the hyperbolic volume $v_\Gamma := \int_X \mu_{\text{hyp}}$ is finite. By $\mathcal{F}_\Gamma$ we denote a fundamental domain of $\Gamma$, which is a connected domain of $\mathbb{H}$ which represents the space $X = \Gamma \backslash \mathbb{H}$. Then we can write

$$\int_X \mu_{\text{hyp}} = \int_{\mathcal{F}_\Gamma} \mu_{\text{hyp}}(z).$$

We set the rescaled hyperbolic metric as

$$\mu_{\text{shyp}} := \frac{\mu_{\text{hyp}}}{v_\Gamma}.$$ 

For $z = x + iy \in X$, the hyperbolic Laplacian $\Delta_{\text{hyp},z}$ is given by

$$\Delta_{\text{hyp},z} := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

From [28], p. 10, recall that for any $f$ smooth function on $X$, we have

$$\Delta_{\text{hyp},z} f(z) \mu_{\text{hyp}}(z) = -4\pi d_z d_{\overline{z}} f(z),$$

where $d_z = (\partial_z + \overline{\partial}_z)$, $d_{\overline{z}} = (\partial_z - \overline{\partial}_z) / 4\pi i$ are two differential operators on $X$. 

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By \( d_{hyp}(z,w) \) we denote the hyperbolic distance between \( z, w \in \mathbb{H} \) derived from \( ds^2_{hyp} \). From [7], Theorem 7.2.1, we know the following relation

\[
\cosh(d_{hyp}(z,w)) = 1 + 2u(z,w)
\]

with the point-pair invariant

\[
u(z,w) = \frac{|z-w|^2}{4 \text{Im}(z)\text{Im}(w)}.
\]

By \( g_{\Gamma} \) we denote the genus of the compact Riemann surface obtained from \( X \) by adding the cusps. From [30], Theorem 2.20, we recall the following well-known formula

\[
\frac{v_{\Gamma}}{2\pi} = 2g_{\Gamma} - 2 + p_{\Gamma} + \sum_{j=1}^{e_{\Gamma}} \left( 1 - \frac{1}{\text{ord}(e_j)} \right).
\]

2.2. Scattering constants and Kronecker limit functions. Let \( \Gamma \) be a cofinite Fuchsian subgroup and \( p_j \) be a cusp of \( \Gamma \). Then, for \( z \in X \) and \( s \in \mathbb{C} \) with \( \text{Re}(s) > 1 \), the Eisenstein series \( E_{p_j}(z,s) \) corresponding to the cusp \( p_j \in \mathcal{P}_\Gamma \) is defined by the series

\[
E_{p_j}(z,s) = \sum_{\gamma \in \Gamma /\Gamma \setminus \Gamma_j} \text{Im}(\sigma_{p_j}^{-1}\gamma z)^s,
\]

where \( \sigma_{p_j} \) is a scaling matrix of the cusp \( p_j \), i.e., it satisfies

\[
\sigma_{p_j}\infty = p_j, \quad \text{and} \quad \sigma_{p_j}^{-1}\Gamma_p j \sigma_{p_j} = (\gamma_{\infty}), \quad \text{where} \quad \gamma_{\infty} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

From [16], Chapter 6, we know that the Eisenstein series (2.4) is invariant under the action of \( \Gamma \) and it satisfies the differential equation

\[
(\Delta_{hyp,z} - s(1-s))E_{p_j}(z,s) = 0.
\]

The series (2.4) converges absolutely and uniformly for \( \text{Re}(s) > 1 \), and it admits a meromorphic continuation to all \( s \in \mathbb{C} \) with a simple pole at \( s = 1 \) with residue \( 1/v_{\Gamma} \). The Kronecker limit function associated to the cusp \( p_j \) is defined by

\[
K_{p_j}(z) = \lim_{s \to 1} \left( E_{p_j}(z,s) - \frac{1}{(s-1)v_{\Gamma}} \right).
\]

For example, for \( \Gamma = \text{PSL}_2(\mathbb{Z}) \), the Kronecker limit function is explicitly given by the formula

\[
K_{\infty}(z) = -\frac{1}{4\pi} \log(y^{12}|\Delta(z)|^2) + \frac{3}{4\pi}(\gamma - \log 4\pi),
\]

where \( z = x + iy \), \( \Delta(z) \) denotes the Modular discriminant, and \( \gamma \) denotes the Euler constant.

Let \( p_k, p_l \) be two cusps of \( \Gamma \). From [16], Theorem 3.4, we know the Eisenstein series \( E_{p_k}(\sigma_{p_l}z,s) \) admits the Fourier expansion of the form

\[
E_{p_k}(\sigma_{p_l}z,s) = \delta_{p_k p_l} \text{Im}(z)^s + \varphi_{p_k p_l}(s) \text{Im}(z)^{1-s} + \sum_{n \neq 0} \varphi_{p_k p_l}(n,s) W_s(nz),
\]
where $\delta_{pk}p_i$ denotes the Kronecker delta symbol, $W_s(z)$ denotes the Whittaker function, and
\[
\varphi_{pk}p_i(s) = \sqrt{\pi} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \sum_{c>0} e^{-2s} S_{pk}p_i(0,0;c),
\]
(2.7)\[
\varphi_{pk}p_i(n,s) = \pi^s \Gamma(s)^{-1} |n|^{s-1} \sum_{c>0} e^{-2s} S_{pk}p_i(0,n;c),
\]
where $S_{pk}p_i(m,n;c)$ denotes the Kloosterman sum for $m, n \in \mathbb{Z}$ (see [16] section 2.5).

For example, for $\Gamma = \text{PSL}_2(\mathbb{Z})$, the Fourier coefficients of the Eisenstein series are explicitly given by the formula
\[
\varphi(s) = \sqrt{\pi} \frac{\Gamma(s - 1/2) \zeta(2s - 1)}{\Gamma(s) \zeta(2s)},
\]
\[
\varphi(n,s) = \pi^s \Gamma(s)^{-1} \zeta(2s)^{-1} |n|^{s-1} \sum_{d|n} \sum_{d>0} d^{1-2s}.
\]

Note that, the function $\varphi_{pk}p_i(s)$ which appears in the 0-th Fourier coefficient of the Eisenstein series $E_{pk}p_i(\sigma_{pk}z, s)$ is known as the scattering function. Similar to the Eisenstein series, the scattering function is also holomorphic for $\text{Re}(s) > 1$, and admits a meromorphic continuation to all $s \in \mathbb{C}$ with a simple pole at $s = 1$ with residue $1/v_T$. The scattering constant with respect to the cusps $p_k, p_l$ is defined by
\[
C_{pk}p_l = \lim_{s \to 1} \left( \varphi_{pk}p_l(s) - \frac{1}{(s - 1)v_T} \right).
\]
Here note that $C_{pk}p_l = C_{lp_k}$. In [12], the authors talked about these scattering constants of congruence subgroups.

For example, for $\Gamma = \text{PSL}_2(\mathbb{Z})$, the scattering constant is explicitly given by the formula
\[
C = \frac{6}{\pi} \left( 1 - \log(4\pi) - 12 \zeta'(1) \right).
\]

2.3. Hyperbolic heat kernels. Let $K_{\mathbb{H}}(t; z, w)$ be the hyperbolic heat kernel on $\mathbb{R}_{>0} \times \mathbb{H} \times \mathbb{H}$. From [6] p. 246, we have the following formulas
\[
K_{\mathbb{H}}(t; z, w) = \frac{\sqrt{2} e^{-t/4}}{(4\pi t)^{3/2}} \int_{d_{\text{hyp}}(z,w)}^{\infty} \frac{e^{-r^2/4t}}{\sqrt{\cosh(r) - \cosh(d_{\text{hyp}}(z,w))}} dr,
\]
where $d_{\text{hyp}}(z, w)$ is the hyperbolic distance between $z$ and $w$.

Note that the heat kernel $K_{\mathbb{H}}(t; z, w)$ only depends on the hyperbolic distance $d_{\text{hyp}}(z, w)$. If $z = w$, i.e., if $d_{\text{hyp}}(z, w) = 0$, then we have
\[
K_{\mathbb{H}}(t; z, z) = \frac{1}{2\pi} \int_0^{\infty} e^{-(r^2+1/4)t} r \tanh(\pi r) dr.
\]
The hyperbolic heat kernel $K_{\text{hyp}}(t; z, w)$ on $\mathbb{R}_{>0} \times X \times X$ is defined by averaging over all the elements of $\Gamma$, i.e.,
\[
K_{\text{hyp}}(t; z, w) = \sum_{\gamma \in \Gamma} K_{\mathbb{H}}(t; z, \gamma w).
\]
For notational convenience we write $K_{\text{hyp}}(t; z, z) = K_{\text{hyp}}(t; z)$. 

The hyperbolic heat kernel $K_{\text{hyp}}(t; z, w)$ satisfies the following heat equation

\[
\left( \Delta_{\text{hyp}, z} + \frac{\partial}{\partial t} \right) K_{\text{hyp}}(t; z, w) = 0 \quad \text{for } w \in X.
\]

For any $C^\infty$-function $f$ on $X$, the hyperbolic heat kernel satisfies the relation

\[
\lim_{t \to 0} \int_X K_{\text{hyp}}(t; z, w) f(w) \mu_{\text{hyp}}(w) = f(z) \quad \text{for } z \in X.
\]

Moreover, for $t > 0$ and for all $w \in X$, we have

\[
\int_X K_{\text{hyp}}(t; z, w) \mu_{\text{hyp}}(z) = 1.
\]

From [6] p. 108-112 (see also [21], equation (0.9), p. 349), we have the following spectral expansion of the hyperbolic heat kernel

\[
K_{\text{hyp}}(t; z, w) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \varphi_n(z) \varphi_n(w)
\]

\[
+ \frac{1}{2\pi} \sum_{j=1}^{\infty} \int_0^\infty e^{-(r^2+1/4)t} E_{p_j}(z, 1/2 + ir) E_{p_j}(w, 1/2 - ir) dr,
\]

where $\{\varphi_n(z)\}_{n=0}^{\infty}$ is a system of orthonormal eigenfunctions for the discrete spectrum of $\Delta_{\text{hyp}, z}$ with eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$, and the set of Eisenstein series $\{E_{p_j}(z, 1/2 + ir)\}_{p_j=1}^{\infty}$ is the set of eigenfunctions for the continuous spectrum of $\Delta_{\text{hyp}, z}$ with eigenvalue $1/4 + r^2$.

The hyperbolic heat trace $\text{HTr}K_{\text{hyp}}(t)$ ($t \in \mathbb{R}_{\geq 0}$) is defined by

\[
\text{HTr}K_{\text{hyp}}(t) = \int_X \text{HK}_{\text{hyp}}(t; z) \mu_{\text{hyp}}(z),
\]

where

\[
(2.10) \quad \text{HK}_{\text{hyp}}(t; z) = \sum_{\gamma \in \Gamma} K_{\text{H}}(t; z, \gamma z).
\]

Let $\mathcal{H}_\Gamma$ denote a complete set of representatives of inconjugate, primitive, hyperbolic elements of a cofinite Fuchsian group $\Gamma$. Let $\ell_\gamma$ denote the hyperbolic length of the closed geodesic determined by $\gamma \in \mathcal{H}_\Gamma$. Then, we know that $\text{tr}(\gamma) = 2 \cosh(\ell_\gamma/2)$. For $s \in \mathbb{C}, \text{Re}(s) > 1$, the Selberg zeta function $Z_X(s)$ associated to $X$ is defined via the Euler product expansion

\[
Z_X(s) = \prod_{\gamma \in \mathcal{H}_\Gamma} Z_\gamma(s),
\]

where the local factors $Z_\gamma(s)$ are given by

\[
Z_\gamma(s) = \prod_{n=0}^{\infty} (1 - e^{-(s+n)\ell_\gamma}).
\]

The Selberg zeta function $Z_X(s)$ admits a meromorphic continuation to all $s \in \mathbb{C}$ with zeros and poles characterized by the spectral theory of the hyperbolic Laplacian (see [15], Theorem
For our purpose, it suffices to know that the logarithmic derivative of $Z_X(s)$ has a simple pole at $s = 1$, i.e.,

$$c_X = \lim_{s \to 1} \left( \frac{Z_X'(s)}{Z_X(s)} - \frac{1}{s - 1} \right)$$

is well defined, and we call the constant $c_X$ the Selberg zeta constant. From [25], section 4 (see also [22], p. 656), we know the logarithmic derivative of the Selberg zeta function satisfies the following formula

$$\frac{Z_X'(s)}{Z_X(s)} = (2s - 1) \int_0^\infty e^{-s(s-1)t} \text{HTr} K_{hyp}(t) \, dt,$$

where $\text{HTr} K_{hyp}(t)$ is the hyperbolic heat trace given by (2.9). Then the Selberg zeta constant can be expressed as the following integral

$$c_X - 1 = \int_0^\infty (\text{HTr} K_{hyp}(t) - 1) \, dt.$$

### 2.4. Hyperbolic and canonical Green’s functions.

For $z, w \in X$ and $s \in \mathbb{C}, \text{Re}(s) > 1$, the automorphic Green’s function $G_{hyp,s}(z, w)$ on $X$ is defined by

$$G_{hyp,s}(z, w) = \sum_{\gamma \in \Gamma} G_{\text{H},s}(z, \gamma w),$$

where $G_{\text{H},s}(z, w)$ is defined as

$$G_{\text{H},s}(z, w) = \frac{\Gamma(s)^2}{\Gamma(2s)} u^{-s} F(s, s; 2s, -1/u),$$

where $u = u(z, w)$ is the point-pair invariant given in (2.2), and $F(s, s; 2s, -1/u)$ is the hypergeometric function. We call the function $G_{\text{H},s}(z, w)$ the Green’s function on $\mathbb{H}$. By taking $s = 1$ and using the well-known formula for $F(1, 1; 2, u)$, we get

$$G_{\text{H},1}(z, w) = -\log \left| \frac{z - w}{z - \overline{w}} \right|^2 = \log \left( 1 + \frac{1}{u} \right).$$

For notational convenience we write $G_{\text{H},1}(z, w) = G_{\text{H}}(z, w)$. The hyperbolic Green’s function on $\mathbb{H}$ is related to the hyperbolic heat kernel on $\mathbb{H}$ through the following formula

$$G_{\text{H}}(z, w) = 4\pi \int_0^\infty K_{\text{H}}(t; z, w) \, dt,$$

where $z, w \in \mathbb{H}$.

The automorphic Green’s function $G_{hyp,s}(z, w)$ is holomorphic on $X$, and admits a meromorphic continuation to all $s \in \mathbb{C}$ with a simple pole at $s = 1$ with the residue $4\pi/v_T$. The hyperbolic Green’s function $G_{hyp}(z, w)$ on $X$ is the constant term in the Laurent expansion of $G_{hyp,s}(z, w)$ at $s = 1$, i.e.,

$$G_{hyp}(z, w) = \lim_{s \to 1} \left( G_{hyp,s}(z, w) - \frac{4\pi}{s(s-1)v_T} \right).$$

Away from the diagonal, hyperbolic Green’s function on $X$ is related to the hyperbolic heat kernel on $X$ through the following formula

$$G_{hyp}(z, w) = 4\pi \int_0^\infty \left( K_{hyp}(t; z, w) - \frac{1}{v_T} \right) dt.$$
One can show that the hyperbolic Green’s function $G_{\text{hyp}}(z, w)$ satisfies the differential equation
\[ d_z d_{\overline{z}} G_{\text{hyp}}(z, w) + \delta_w(z) = \mu_{\text{hyp}}(z), \tag{2.16} \]
where $d_z = (\partial_z + \overline{\partial}_z)$, $d_{\overline{z}} = (\partial_z - \overline{\partial}_z) / 4\pi i$, and $d_z d_{\overline{z}} = -\partial_z \overline{\partial}_z / 2\pi i$. The $\delta_w(z)$ is the Dirac delta distribution. The hyperbolic Green’s function $G_{\text{hyp}}(z, w)$ satisfies also satisfies the following normalization condition
\[ \int_X G_{\text{hyp}}(z, w) \mu_{\text{hyp}}(z) = 0. \tag{2.17} \]
Let $S_2(\Gamma)$ denote the $\mathbb{C}$-vector space of cusp forms of weight 2 with respect to a cofinite Fuchsian group $\Gamma$ equipped with the Petersson inner product
\[ \langle f, g \rangle_{\text{pet}} = \int_X f(z) \overline{g(z)} \text{Im}(z)^2 \mu_{\text{hyp}}(z) \text{ with } f, g \in S_2(\Gamma). \]
Let $\{ f_1, \ldots, f_{g_\Gamma} \}$ denote an orthonormal basis of $S_2(\Gamma)$ with respect to the Petersson inner product. Then, the canonical metric on $X = \Gamma \setminus \mathbb{H}$ is defined by
\[ \mu_{\text{can}}(z) = \frac{i}{2g_\Gamma} \sum_{j=1}^{g_\Gamma} |f_j(z)|^2 dz \wedge d\overline{z}. \tag{2.18} \]
The canonical Green’s function $G_{\text{can}}(z, w)$ of $X$ is a function on $X \times X$, which is smooth away from the diagonal and has a logarithmic singularity along the diagonal. Away from the diagonal, it is uniquely characterized by
\[ d_z d_{\overline{z}} G_{\text{can}}(z, w) + \delta_w(z) = \mu_{\text{can}}(z) \text{ where } z, w \in X, \tag{2.19} \]
with the normalization condition
\[ \int_X G_{\text{can}}(z, w) \mu_{\text{can}}(z) = 0 \text{ with } w \in X. \tag{2.20} \]
In [28], p. 26, the author has explicitly given the canonical Green’s function on quotient spaces of genus zero having elliptic fixed points.

2.5. Congruence subgroup $\Gamma_0(N)$. Here we consider $\Gamma = \Gamma_0(N)$, and $X = \Gamma_0(N) \setminus \mathbb{H}$. We define
\[ \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}_2(\mathbb{Z}) \mid c \equiv 0 \mod N \right\} \]
for a positive integer $N$. The hyperbolic volume of $\Gamma$ is given by
\[ v_\Gamma = \frac{\pi N}{3} \prod_{p|N} \left( 1 + \frac{1}{p} \right). \tag{2.21} \]
Every cusp of $\Gamma_0(N)$ is equivalent to one among the following rationals
\[ \frac{m}{n} \text{ with } m, n > 0, \ n|N, \ (m, n) = 1. \]
Two cusps $m/n$ and $m_1/n_1$ of the above type are $\Gamma_0(N)$-equivalent if and only if
\[ n_1 = n \text{ and } m_1 \equiv m \mod \left( n, \frac{N}{n} \right). \]
The number of cusps of \( \Gamma \) is given by

\[
\rho_\Gamma = \sum_{\substack{d \mid N \\
d > 0}} \phi ((d, N/d)),
\]

where \( \phi \) is the Euler function.

From [10], p. 97, we know elliptic fixed points of \( \Gamma \) with \( \text{ord}(e_j) = 2 \) is explicitly given by

\[
e_j = \frac{n + i}{n^2 + 1}
\]
for \( n = 0, \ldots, N - 1 \) such that \( n^2 + 1 \equiv 0 \pmod{N} \)

and elliptic fixed points of \( \Gamma \) with \( \text{ord}(e_j) = 3 \) are explicitly given by

\[
e_j = \frac{n + \frac{1 + i\sqrt{3}}{2}}{n^2 - n + 1}
\]
for \( n = 0, \ldots, N - 1 \) such that \( n^2 - n + 1 \equiv 0 \pmod{N} \).

The number of elliptic fixed points is equal to

\[
\epsilon_\Gamma = \nu_2 + \nu_3,
\]
where

\[
\nu_2 = \begin{cases} 
0 & \text{if } 4 \mid N, \\
\prod_{p \mid N \text{ prime}} \left( 1 + \left( \frac{-1}{p} \right) \right) & \text{otherwise},
\end{cases}
\]

\[
\nu_3 = \begin{cases} 
0 & \text{if } 9 \mid N, \\
\prod_{p \mid N \text{ prime}} \left( 1 + \left( \frac{-3}{p} \right) \right) & \text{otherwise}.
\end{cases}
\]

Remark 2.1. Note that \( \omega(N) = O(\log N / \log \log N) \), where \( \omega(N) \) denote the number of primes which divide \( N \). This implies

\[
\epsilon_\Gamma = O\left(N^\varepsilon\right) \quad \text{for any given } \varepsilon > 0.
\]

Then, from the volume formula (2.3), we have the following asymptotics

\[
\frac{4\pi (g_\Gamma - 1)}{v_\Gamma} = 1 + o(N) \quad \text{as } N \to \infty.
\]

We use these bounds in section 6, section 7 to prove Theorem 1.2.

3. Expressing canonical Green’s functions using hyperbolic data

Here we recall a closed-form expression for the canonical Green’s function in terms of hyperbolic Green’s function and analytic functions derived from the hyperbolic heat kernel. This idea was originally from [17], where the authors proved bounds for the canonical Green’s function for compact Riemann surface associated with a cofinite Fuchsian subgroup having neither cusps nor elliptic fixed points. Then in [3] and [4], the author extended bounds for the canonical Green’s function for non-compact Riemann surface associated with an arbitrary cofinite Fuchsian subgroup.
Lemma 3.1. Let \( \mathcal{G}_{\text{hyp}}(z, w) \) and \( \mathcal{G}_{\text{can}}(z, w) \) be the hyperbolic and the canonical Green's functions associated to \( \Gamma \). Then

\[
\mathcal{G}_{\text{hyp}}(z, w) - \mathcal{G}_{\text{can}}(z, w) = \phi(z) + \phi(w), \quad \text{where}
\]

\[
\phi(z) = \int_X \mathcal{G}_{\text{hyp}}(z, \zeta) \mu_{\text{can}}(\zeta) - \frac{1}{2} \int_X \int_X \mathcal{G}_{\text{hyp}}(\xi, \zeta) \mu_{\text{can}}(\xi) \mu_{\text{can}}(\zeta).
\]

**Proof.** See [3], Proposition 2.6.4. \( \square \)

**Definition 3.2.** We define \( \hat{\mathcal{C}}_{\ell, \ell}(X) \) as the space of all functions \( f : X \to \hat{\mathbb{C}} \) having only finitely many singularities of the following type

1. If the function \( f \in \hat{\mathcal{C}}_{\ell, \ell}(X) \) has a singularity at a cusp \( p_k \in \mathcal{P}_\Gamma \), then there exists a constant \( c_{p_k} \in \mathbb{C} \) such that

\[
f(z) = c_{p_k} \log(\text{Im}(\sigma_{p_k}^{-1} z)) + O(1) \quad \text{as} \quad \text{Im}(\sigma_{p_k}^{-1} z) \to \infty.
\]

2. If the function \( f \in \hat{\mathcal{C}}_{\ell, \ell}(X) \) has a singularity at an elliptic point \( e_j \in \mathcal{E}_\Gamma \), then there exists a constant \( c_{e_j} \in \mathbb{C} \) such that

\[
f(z) = c_{e_j} \text{ord}(e_j) \log\left(\frac{z - e_j}{z - e_j^*}\right) + O(1) \quad \text{as} \quad z \to e_j.
\]

3. If the function \( f \in \hat{\mathcal{C}}_{\ell, \ell}(X) \) has a singularity at \( w \in X \setminus \mathcal{P}_\Gamma \cup \mathcal{E}_\Gamma \), then there exists a constant \( c_w \in \mathbb{C} \) such that

\[
f(z) = c_w \log(z - w) + O(1) \quad \text{as} \quad z \to w.
\]

Lemma 3.3. Let \( f \in \hat{\mathcal{C}}_{\ell, \ell}(X) \), then we have

\[
gr \int_X f(z) \mu_{\text{can}}(z) = \left(\frac{1}{4\pi} + \frac{1}{v_{\Gamma}}\right) \int_X f(z) \mu_{\text{hyp}}(z)
+ \frac{1}{2} \int_X f(z) \left(\int_0^\infty \Delta_{\text{hyp}, z} K_{\text{hyp}}(t; z) dt\right) \mu_{\text{hyp}}(z).
\]

**Proof.** See [3], Corollary 3.2.5. \( \square \)

**Remark 3.4.** Lemma 3.3 plays a crucial role in expressing canonical Green's functions using hyperbolic data. The idea of writing the canonical metric using the heat kernel has already been used in [19], Proposition 4.1, where Jorgenson and Kramer have given the following identity

\[
gr \mu_{\text{can}}(z) = \left(\frac{1}{4\pi} + \frac{1}{v_{\Gamma}}\right) \mu_{\text{hyp}}(z) + \frac{1}{2} \left(\int_0^\infty \Delta_{\text{hyp}, z} K_{\text{hyp}}(t; z) dt\right) \mu_{\text{hyp}}(z),
\]

where \( z \in X \setminus \mathcal{E}_\Gamma \). Then, in [3], Theorem 2.9.5, the author extended the identity (3.1) to elliptic fixed points \( \mathcal{E}_\Gamma \) at the level of currents, meaning

\[
(gr \mu_{\text{can}}(z)) = \left(\frac{1}{4\pi} + \frac{1}{v_{\Gamma}}\right) [\mu_{\text{hyp}}(z)] + \frac{1}{2} \left[\left(\int_0^\infty \Delta_{\text{hyp}, z} K_{\text{hyp}}(t; z) dt\right) \mu_{\text{hyp}}(z)\right],
\]

where \([\mu_{\text{can}}(z)], [\mu_{\text{hyp}}(z)],\) and \([\left(\int_0^\infty \Delta_{\text{hyp}, z} K_{\text{hyp}}(t; z) dt\right) \mu_{\text{hyp}}(z)]\) denote the currents of type \((1, 1)\) on \( X \). For more details see [3], Chapter 2, and Lemma 3.3 is an immediate consequence of the identity (3.2).
Lemma 3.5. Let $G_{\text{hyp}}(z, w)$ be the hyperbolic Green’s function associated to $\Gamma$. Then

$$\int_X G_{\text{hyp}}(z, \zeta) \mu_{\text{can}}(\zeta) = \frac{1}{2g_\Gamma} \int_X G_{\text{hyp}}(z, \zeta) \left( \int_0^\infty \Delta_{\text{hyp}, \zeta} K_{\text{hyp}}(t; \zeta) dt \right) \mu_{\text{hyp}}(\zeta).$$

Proof. From [3], Corollary 1.9.5, we know the following estimate for the automorphic Green’s function. Let $p_j$ and $p_k$ are two cusps, then

$$G_{\text{hyp}, s}(\sigma_{p_j} z, \sigma_{p_k} w) = \frac{4\pi \Im(\sigma_{p_j} z)^{1-s}}{2s-1} E_{p_j}(\sigma_{p_k} w, s) - \delta_{p_j, p_k} \log |1 - e^{2\pi i (z-w)}| + O(e^{-2\pi(\Im(z) - \Im(w)))}.$$

This estimate implies

$$G_{\text{hyp}, s}(z, w) = \frac{4\pi \Im(\sigma_{p_j}^{-1} z)^{1-s}}{2s-1} E_{p_j}(w, s) - \log |1 - e^{2\pi i (\sigma_{p_j}^{-1} z - \sigma_{p_j}^{-1} w)}| + O(e^{-2\pi(\Im(\sigma_{p_j}^{-1} z) - \Im(\sigma_{p_j}^{-1} w)))}.$$

Now using (2.5) and (2.14), we can write

$$(3.3) \quad G_{\text{hyp}}(z, w) = 4\pi K_{p_j}(w) - \frac{4\pi}{\bar{v}_\Gamma} - \frac{4\pi \log (\Im(\sigma_{p_j}^{-1} z))}{\bar{v}_\Gamma} \quad \text{as} \quad z \to p_j.$$

Hence $G_{\text{hyp}}(z, w) \in \hat{\mathcal{C}}_{\ell, \ell}(X)$. Now applying Lemma 3.3, we get

$$\int_X G_{\text{hyp}}(z, \zeta) \mu_{\text{can}}(\zeta) = \frac{1}{g_\Gamma} \left( \frac{1}{4\pi} + \frac{1}{\bar{v}_\Gamma} \right) \int_X G_{\text{hyp}}(z, \zeta) \mu_{\text{hyp}}(\zeta)$$

$$+ \frac{1}{2g_\Gamma} \int_X G_{\text{hyp}}(z, \zeta) \left( \int_0^\infty \Delta_{\text{hyp}, \zeta} K_{\text{hyp}}(t; \zeta) dt \right) \mu_{\text{hyp}}(\zeta).$$

Now, using the characteristic property of the hyperbolic Green’s function, namely (2.17), we get

$$\int_X G_{\text{hyp}}(z, \zeta) \mu_{\text{can}}(\zeta) = \frac{1}{2g_\Gamma} \int_X G_{\text{hyp}}(z, \zeta) \left( \int_0^\infty \Delta_{\text{hyp}, \zeta} K_{\text{hyp}}(t; \zeta) dt \right) \mu_{\text{hyp}}(\zeta).$$

Lemma 3.6. Let $f \in \hat{\mathcal{C}}_{\ell, \ell}(X)$ and let $\text{Sing}(f)$ be the set of all the singularities of the function $f(z)$ on $X$. Then for a fixed $w \in X \setminus \text{Sing}(f)$ we have

$$\int_X G_{\text{hyp}}(z, w) \left( \Delta_{\text{hyp}, z} f(z) \right) \mu_{\text{hyp}}(z) + 4\pi \int_X f(z) \mu_{\text{shyp}}(z)$$

$$= 4\pi f(w) + \sum_{\substack{s \in \text{Sing}(f) \setminus \text{Sing}(\bar{f})}} 2\pi c_s \ G_{\text{hyp}}(s, w),$$

where $c_s \in \mathbb{C}$ is a constant which depends on the singularities of the function $f$.

Proof. See [3], Corollary 3.1.8.

Proposition 3.7. Let $G_{\text{hyp}}(z, w)$ and $G_{\text{can}}(z, w)$ be the hyperbolic and the canonical Green’s functions associated to $\Gamma$. Then

$$G_{\text{hyp}}(z, w) - G_{\text{can}}(z, w) = \phi(z) + \phi(w),$$

where $\phi \in \mathbb{C}$. 

□
where
\[ \phi(z) = \frac{1}{2g_{\Gamma}} \int_{X} G_{\text{hyp}}(z, \zeta) F(\zeta) \mu_{\text{hyp}}(\zeta) \]
\[ - \frac{1}{8g_{\Gamma}^2} \int_{X \times X} G_{\text{hyp}}(\xi, \zeta) F(\xi) F(\zeta) \mu_{\text{hyp}}(\xi) \mu_{\text{hyp}}(\zeta) \]

with
\[ F(\zeta) = \int_{0}^{\infty} \Delta_{\text{hyp}, \zeta} K_{\text{hyp}}(t; \zeta) dt. \]

**Proof.** The proof is clear from (3.5) and Proposition 3.1. \qed

For notational convenience, now we set
\[ E_{\Gamma}(z) = \sum_{\gamma \in \Gamma \setminus \{\text{id}\}} G_{\mathbb{H}}(z, \gamma z), \]
\[ P_{\Gamma}(z) = \sum_{\gamma \in \Gamma \setminus \{\text{id}\}} G_{\mathbb{H}}(z, \gamma z), \]
\[ H_{\Gamma}(z) = 4\pi \int_{0}^{\infty} \left( H_{K_{\text{hyp}}}(t; z) - \frac{1}{v_{\Gamma}} \right) dt, \]

where the function \( G_{\mathbb{H}}(z, w) \) is the Green’s function on \( \mathbb{H} \times \mathbb{H} \), the function \( K_{\text{hyp}}(t; z) \) is the hyperbolic heat kernel. In [19], the authors proved functions \( E_{\Gamma}(z), P_{\Gamma}(z), \) and \( H_{\Gamma}(z) \) are absolutely and locally uniformly convergent.

Also, we set
\[ C_{\text{hyp}} = \int_{X \times X} G_{\text{hyp}}(\xi, \zeta) F(\xi) F(\zeta) \mu_{\text{hyp}}(\xi) \mu_{\text{hyp}}(\zeta), \]
where
\[ F(\zeta) = \int_{0}^{\infty} \Delta_{\text{hyp}, \zeta} K_{\text{hyp}}(t; \zeta) dt. \]

Here \( \Delta_{\text{hyp}, \zeta} \) and \( \Delta_{\text{hyp}, \xi} \) denote the hyperbolic Laplacians with respect to the complex variables \( \zeta \) and \( \xi \), respectively. In [3], section 2.8, the author proved the function \( F(\zeta) \) remains bounded on \( X \).

**Proposition 3.8.** Let \( G_{\text{hyp}}(z, w) \) and \( G_{\text{can}}(z, w) \) be the hyperbolic and the canonical Green’s functions associated \( \Gamma \). Then for \( z, w \in X \setminus \mathcal{E}_{\Gamma} \) we have
\[ G_{\text{hyp}}(z, w) - G_{\text{can}}(z, w) = \phi(z) + \phi(w), \]
where
\[ \phi(z) = \frac{1}{8\pi g_{\Gamma}} \int_{X} G_{\text{hyp}}(z, \zeta) (\Delta_{\text{hyp}, \zeta} P_{\Gamma}(\zeta)) \mu_{\text{hyp}}(\zeta) - \frac{1}{2g_{\Gamma}} \int_{X} E_{\Gamma}(\zeta) \mu_{\text{shyp}}(\zeta) \]
\[ + \frac{H_{\Gamma}(z)}{2g_{\Gamma}} + \frac{E_{\Gamma}(z)}{2g_{\Gamma}} - \frac{C_{\text{hyp}}}{8g_{\Gamma}^2} - \frac{2\pi(c_{X} - 1)}{g_{\Gamma} v_{\Gamma}} - \frac{1}{2g_{\Gamma}} \sum_{j=1}^{e_{\Gamma}} (1 - 1/\text{ord}(e_{j})) G_{\text{hyp}}(z, e_{j}). \]
Here $E_\Gamma(z)$, $P_\Gamma(z)$, $H_\Gamma(z)$ are given by (3.4), $C_{\text{hyp}}$ is given by (3.5), and $c_X$ denotes the Selberg zeta constant on $X$.

**Proof.** From Proposition 3.7, and using the notation (3.5), we have

$$\phi(z) = \frac{1}{2\pi g_\Gamma} \int_X \mathcal{G}_{\text{hyp}}(z, \zeta) \left( \int_0^\infty \Delta_{\text{hyp}, \zeta} K_{\text{hyp}}(t; \zeta) dt \right) \mu_{\text{hyp}}(\zeta) - \frac{C_{\text{hyp}}}{8g_\Gamma^2},$$

Then using the relation (2.13), we get

$$\phi(z) = \frac{1}{8\pi g_\Gamma} \int_X \mathcal{G}_{\text{hyp}}(z, \zeta) \left( \Delta_{\text{hyp}, \zeta} P_\Gamma(\zeta) \right) \mu_{\text{hyp}}(\zeta)$$

$$+ \frac{1}{8\pi g_\Gamma} \int_X \mathcal{G}_{\text{hyp}}(z, \zeta) \left( \Delta_{\text{hyp}, \zeta} E_\Gamma(\zeta) \right) \mu_{\text{hyp}}(\zeta)$$

$$+ \frac{1}{8\pi g_\Gamma} \int_X \mathcal{G}_{\text{hyp}}(z, \zeta) \left( \Delta_{\text{hyp}, \zeta} H_\Gamma(\zeta) \right) \mu_{\text{hyp}}(\zeta) - \frac{C_{\text{hyp}}}{8g_\Gamma^2}.$$  

(3.6)

From [4], Lemma 3.6, we know that the series $E_\Gamma(\zeta)$ is absolutely and locally uniformly convergent for $\zeta \in X \setminus \mathcal{E}_\Gamma$. Furthermore, in [4], Lemma 3.6, the author has proved that

$$E_\Gamma(\zeta) = 2(1 - \text{ord}(e_j)) \log \left( \frac{\zeta - e_j}{\zeta - \bar{e}_j} \right) + O(1) \text{ as } z \to e_j \in \mathcal{E}_\Gamma,$$

(3.7)

and $E_\Gamma(\zeta) = 0$ when $\zeta$ is a cusp of $\Gamma$.

Using Lemma 3.6 by taking the function $f(\zeta) = E_\Gamma(\zeta)$, we get

$$\frac{1}{8\pi g_\Gamma} \int_X \mathcal{G}_{\text{hyp}}(z, \zeta) \left( \Delta_{\text{hyp}, \zeta} E_\Gamma(\zeta) \right) \mu_{\text{hyp}}(\zeta)$$

$$= \frac{E_\Gamma(z)}{2g_\Gamma} - \frac{1}{2g_\Gamma} \int_X E_\Gamma(\zeta) \mu_{\text{shyp}}(\zeta) - \frac{1}{2g_\Gamma} \sum_{j=1}^{e_\Gamma} (1 - 1/\text{ord}(e_j)) \mathcal{G}_{\text{hyp}}(z, e_j).$$

(3.8)

From [3], Proposition 4.3.3, we know that

$$H_\Gamma(\zeta) = 4\pi C_{e_j, p_j} - \frac{8\pi}{v_\Gamma} \log \left( \frac{\text{Im}(\sigma_{e_j}^{-1})}{v_\Gamma} \right) - \frac{4\pi}{v_\Gamma} \text{ as } \zeta \to p_j \in \mathcal{P}_\Gamma,$$

(3.9)

which yields $H_\Gamma(\zeta) \in \mathcal{C}_{\ell, \ell}(X)$ and $\text{Sing}(H_\Gamma(\zeta)) = \mathcal{P}_\Gamma$.

Then, using Lemma 3.6 by taking the function $f(\zeta) = H_\Gamma(\zeta)$, we get

$$\frac{1}{8\pi g_\Gamma} \int_X \mathcal{G}_{\text{hyp}}(z, \zeta) \left( \Delta_{\text{hyp}, \zeta} H_\Gamma(\zeta) \right) \mu_{\text{hyp}}(\zeta)$$

$$= \frac{H_\Gamma(z)}{2g_\Gamma} - \frac{1}{2g_\Gamma} \int_X H_\Gamma(\zeta) \mu_{\text{shyp}}(\zeta) = \frac{H_\Gamma(z)}{2g_\Gamma} - \frac{2\pi(c_X - 1)}{g_\Gamma v_\Gamma}.$$  

(3.10)

Finally, the proof follows directly by using (3.8) and (3.10) in the identity (3.6). 

\[\square\]

4. SOME USEFUL LEMMAS

Here we give bounds for some terms which appear in Proposition 3.8. These bounds help us to simplify Proposition 3.8 further, and they help us to deduce Theorem 1.1.
Lemma 4.1. Let $p_k$ be a cusp of $\Gamma$. Then, as $z$ approaches the cusp $p_k$, we have
\[
\frac{1}{8\pi\Gamma} \int_X G_{\text{hyp}}(z, w) \left( \Delta_{\text{hyp}, w} P_{\Gamma}(w) \right) \mu_{\text{hyp}}(w)
= \frac{2\pi p_{\Gamma} - \nu_{\Gamma}}{g_{\Gamma} v_{\Gamma}} \log \left( \text{Im}(\sigma_{p_k}^{-1} z) \right) - \frac{2\pi}{g_{\Gamma}} \sum_{j=1}^{p_{\Gamma}} c_{\gamma_j p_j} - \frac{\log(4\pi)}{g_{\Gamma}} + \frac{2\pi p_{\Gamma}}{g_{\Gamma} v_{\Gamma}} + R,
\]
where $|R| \leq \left( \frac{4\pi^2}{3} + 1 \right) \frac{p_{\Gamma}}{g_{\Gamma} v_{\Gamma}}$.

Proof. See [23], Proposition 2.4.7 or see [3] Proposition 7.1.12. \qed

Lemma 4.2. Let $E_{\Gamma} = \{e_j | j = 1, \ldots, e_{\Gamma}\}$ be the set of all elliptic fixed points of $\Gamma$. Then
\[
\left| \int_X E_{\Gamma}(\zeta) \mu_{\text{shyp}}(\zeta) \right| \leq \frac{4\pi \log 2}{v_{\Gamma}} \sum_{j=1}^{e_{\Gamma}} (\text{ord}(e_j) - 1),
\]
where $E_{\Gamma}(\zeta)$ is given in (3.4).

Proof. Let $e_j$ be an elliptic fixed point. Then we know that the stabilizer group $\Gamma_{e_j}$ is cyclic, and there exists a scaling matrix $\sigma_{e_j} \in \text{PSL}_2(\mathbb{Z})$ such that
\[
\Gamma_{e_j} = \langle \gamma_{e_j} \rangle, \quad \text{where } \gamma_{e_j} = \sigma_{e_j} \gamma_{i, e_j} \sigma_{e_j}^{-1}
\]
with
\[
\gamma_{i, e_j} = \begin{pmatrix} \cos(\pi/\text{ord}(e_j)) & \sin(\pi/\text{ord}(e_j)) \\ -\sin(\pi/\text{ord}(e_j)) & \cos(\pi/\text{ord}(e_j)) \end{pmatrix}.
\]

Then we have the following disjoint union decomposition
\[
\left\{ \gamma \in \Gamma \setminus \{\text{id}\} \mid \gamma \text{ elliptic} \right\} = \bigcup_{j=1}^{e_{\Gamma}} \bigcup_{\eta \in \Gamma_{e_j} \setminus \Gamma} \left\{ \eta^{-1} \gamma_{e_j} \eta \right\}
= \bigcup_{j=1}^{e_{\Gamma}} \bigcup_{n=1}^{\text{ord}(e_j) - 1} \left\{ \eta^{-1} \gamma_{e_j} \eta \right\} = \bigcup_{j=1}^{e_{\Gamma}} \bigcup_{n=1}^{\text{ord}(e_j) - 1} \left\{ \eta^{-1} \sigma_{e_j} \gamma_{i, e_j} \sigma_{e_j}^{-1} \eta \right\}.
\]

Now using the disjoint union decomposition (4.1), we get
\[
E_{\Gamma}(\zeta) = \sum_{\gamma \in \Gamma \setminus \{\text{id}\}} G_{\text{hyp}}(\zeta, \gamma \zeta) = \sum_{j=1}^{e_{\Gamma}} \sum_{\eta \in \Gamma_{e_j} \setminus \Gamma} \sum_{n=1}^{\text{ord}(e_j) - 1} G_{\text{hyp}}(\zeta, \eta^{-1} \sigma_{e_j} \gamma_{i, e_j} \sigma_{e_j}^{-1} \eta \zeta)
= \sum_{j=1}^{e_{\Gamma}} \sum_{\eta \in \Gamma_{e_j} \setminus \Gamma} \sum_{n=1}^{\text{ord}(e_j) - 1} G_{\text{hyp}}(\sigma_{e_j}^{-1} \eta \zeta, \gamma_{i, e_j} \sigma_{e_j}^{-1} \eta \zeta).
\]

By taking integral on (4.2), we get
\[
\int_X E_{\Gamma}(\zeta) \mu_{\text{shyp}}(\zeta) = \sum_{j=1}^{e_{\Gamma}} \sum_{\eta \in \Gamma_{e_j} \setminus \Gamma} \sum_{n=1}^{\text{ord}(e_j) - 1} \int_{\mathcal{F}_{\Gamma}} G_{\text{hyp}}(\sigma_{e_j}^{-1} \eta \zeta, \gamma_{i, e_j} \sigma_{e_j}^{-1} \eta \zeta) \mu_{\text{shyp}}(\zeta).
\]
Now, making the substitution $\zeta \mapsto \eta^{-1}\sigma_e \zeta$, using the $SL_2(\mathbb{R})$ (or $PSL_2(\mathbb{R})$)-invariance property of $\mu_{hyp}(\zeta)$, we get

$$
\int_X E_\Gamma(\zeta) \mu_{shyp}(\zeta) = \sum_{j=1}^{e_\Gamma} \sum_{\eta \in \Gamma \setminus \Gamma_e} \sum_{n=1}^{\text{ord}(e_j)-1} \int_{\sigma_e^{-1} \eta \Gamma} \mathcal{G}_\mathbb{H}(\zeta, \gamma^n \iota_j \zeta) \mu_{shyp}(\zeta).
$$

From (2.1), we recall

$$
cosh(d_{hyp}(z, w)) = 1 + 2 \sinh^2 \left( \frac{d_{hyp}(z, w)}{2} \right) = 1 + 2u(z, w).
$$

This implies

$$
u = \sinh^2 \left( \frac{d_{hyp}(z, w)}{2} \right).
$$

Hence, from (2.12), we can write

$$
\mathcal{G}_\mathbb{H}(\zeta, \gamma^n \iota_j \zeta) = \log \left( 1 + \frac{1}{\sinh^2 \left( \frac{d_{hyp}(\zeta, \gamma^n \iota_j \zeta)}{2} \right)} \right) = - \log \left( \tanh^2 \left( \frac{d_{hyp}(\zeta, \gamma^n \iota_j \zeta)}{2} \right) \right).
$$

By recalling the triangular inequality of the hyperbolic distance function, we have

$$
\frac{1}{2}d_{hyp}(\zeta, \gamma^n \iota_j \zeta) \leq \frac{1}{2} \left( d_{hyp}(\zeta, \iota) + d_{hyp}(\iota, \gamma^n \iota_j \zeta) \right) = d_{hyp}(\zeta, \iota).
$$

Then, using (4.4) and (4.5), we get

$$
|\mathcal{G}_\mathbb{H}(\zeta, \gamma^n \iota_j \zeta)| \leq |\log \left( \tanh^2 \rho \right)|, \text{ where } \rho = d_{hyp}(\zeta, \iota).
$$

Now, using (4.6) in the R.H.S of (4.3), we get

$$
\left| \int_X E_\Gamma(\zeta) \mu_{shyp}(\zeta) \right| \leq \sum_{j=1}^{e_\Gamma} \sum_{\eta \in \Gamma \setminus \Gamma_e} \sum_{n=1}^{\text{ord}(e_j)-1} \int_{\sigma_e^{-1} \eta \Gamma} \log \left( \tanh^2 \rho \right) \mu_{shyp}(\zeta).
$$

Now we write the hyperbolic metric $\mu_{hyp}(\zeta)$ in polar coordinates, i.e.,

$$
\mu_{hyp}(\zeta) = 2 \sinh(\rho) \, d\rho \, d\theta, \text{ where } \theta \in [0, \pi).
$$

This implies

$$
\left| \int_X E_\Gamma(\zeta) \mu_{shyp}(\zeta) \right| \leq \sum_{j=1}^{e_\Gamma} \frac{(\text{ord}(e_j) - 1)}{v_\Gamma} \int_0^\infty \int_0^\pi \left| \log(\tanh^2 \rho) \right| 2 \sinh(\rho) \, d\theta \, d\rho
$$

$$
= \frac{\pi \log(16)}{v_\Gamma} \sum_{j=1}^{e_\Gamma} (\text{ord}(e_j) - 1).
$$

This completes the proof. □
Lemma 4.3. Let $p_k$ be a cusp of $\Gamma$, and $\sigma_{p_k}$ be a scaling matrix of $p_k$. Then, as $z$ approaches the cusp $p_k$, we have

$$\frac{E_\Gamma(z)}{2g_\Gamma} + \frac{H_\Gamma(z)}{2g_\Gamma} = \frac{2\pi}{g_\Gamma} c_{p_k p_k} - \frac{4\pi \log (\text{Im}(\sigma_{p_k}^{-1} z))}{g_\Gamma v_\Gamma} - \frac{2\pi}{g_\Gamma v_\Gamma},$$

where $E_\Gamma(z)$ and $H_\Gamma(z)$ are given in (3.4).

Proof. From [4], Lemma 3.6, recall that as $z$ approaches the cusp $p_k$, we have

(4.7) $$\frac{E_\Gamma(z)}{2g_\Gamma} = 0.$$

From [3], Proposition 4.3.3, recall that as $z$ approaches the cusp $p_k$, we have

(4.8) $$\frac{H_\Gamma(z)}{2g_\Gamma} = \frac{2\pi}{g_\Gamma} c_{p_k p_k} - \frac{4\pi \log (\text{Im}(\sigma_{p_k}^{-1} z))}{g_\Gamma v_\Gamma} - \frac{2\pi}{g_\Gamma v_\Gamma}.$$

Combining (4.7) and (4.8) we complete the proof. □

Lemma 4.4. The constant $C_{\text{hyp}}$ given in (3.5), satisfies the inequality

$$C_{\text{hyp}} \leq \frac{16\pi g_\Gamma^2 (d_X + 1)^2}{\lambda_1 v_\Gamma},$$

where $d_X = \sup_{z \in X} |\mu_{\text{can}}(z)|$.

Here $g_\Gamma$, $v_\Gamma$ denote the genus and the hyperbolic volume of $\Gamma$ respectively, and $\lambda_1$ is the smallest non-zero eigenvalue of the hyperbolic Laplacian $\Delta_{\text{hyp},z}$ on $X$.

Proof. See [23], Proposition 2.5.2 or see [4], 5.10. □

5. Proof of Theorem 1.1

Finally, to prove Theorem 1.1 we need the following well-known expression of the Kronecker limit function. For the reader’s convenience, we write a short proof of the following proposition.

Proposition 5.1. Let $p_k$, $p_l$ be two cusps of $\Gamma$, and $\sigma_{p_l}$ be a scaling matrix of $p_l$. Then the Kronecker limit function admits a Fourier expansion of the form

$$K_{p_k}(\sigma_{p_l} z) = C_{p_k p_l} - \frac{\log (\text{Im}(z))}{v_\Gamma} + \sum_{n \neq 0} a_{k,l,n}(\text{Im}(z)) e^{2\pi i n \text{Re}(z)},$$

where $C_{p_k p_l}$ is the scattering constant given by (2.8), and non-zero Fourier coefficients $a_{k,l,n}(\text{Im}(z)) = \varphi_{p_k p_l}(n, 1) e^{-2\pi |n| \text{Im}(z)}$ where $\varphi_{p_k p_l}(n, 1)$ is defined in (2.7).

Proof. From (2.5), we have

$$K_{p_k}(\sigma_{p_l} z) = \lim_{s \to 1} \left( E_{p_k}(\sigma_{p_l} z) - \frac{1}{(s-1) v_\Gamma} \right).$$

Now, using the Fourier expansion of the Eisenstein series (2.6), we get

(5.1) $$K_{p_k}(\sigma_{p_l} z) = \lim_{s \to 1} \left( \varphi_{p_k p_l}(s) \text{Im}(z)^{1-s} - \frac{1}{(s-1) v_\Gamma} \right) + \lim_{s \to 1} \sum_{n \neq 0} \varphi_{p_k p_l}(n, s) W_s(nz).$$
Recall that at \( s = 1 \) we have the Laurent expansion
\[
\varphi_{pk,pk}(s) = \frac{1}{v_{\Gamma}(s-1)} + \mathcal{C}_{pk,pk} + O(s-1)
\]
and the Taylor expansion at \( s = 1 \) of \( \text{Im}(z)^{s-1} \) is
\[
\text{Im}(z)^{s-1} = 1 - \log (\text{Im}(z)) (s-1) + O((s-1)^2).
\]
Then from (5.1), we get
\[
K_{pk}(\sigma_{pk}) = \mathcal{C}_{pk,pk} - \frac{\log (\text{Im}(z))}{v_{\Gamma}} + \lim_{s \to 1} \sum_{n \neq 0} \varphi_{pk,pk}(n,s)W_s(nz).
\]
Finally, we have
\[
\lim_{s \to 1} \sum_{n \neq 0} \varphi_{pk,pk}(n,s)W_s(nz) = \sum_{n \neq 0} \varphi_{pk,pk}(n,1)e^{-2\pi |n| \text{Im}(z)} e^{2\pi i \text{Re}(z)},
\]
where the expression of \( \varphi_{pk,pk}(n,1) \) is given in (2.7). This completes the proof. \( \square \)

**Proof of Theorem 1.1.** Recall that, by \( \mathcal{G}_{\text{hyp}}(z,w) \) and \( \mathcal{G}_{\text{can}}(z,w) \) we denote the hyperbolic and the canonical Green’s functions associated to the group \( \Gamma \). Then from Proposition 3.8, we have the identity
\[
\mathcal{G}_{\text{hyp}}(z,w) - \mathcal{G}_{\text{can}}(z,w) = \phi(z) + \phi(w),
\]
where
\[
\phi(z) = \frac{1}{8\pi g_{\Gamma}} \int_X \mathcal{G}_{\text{hyp}}(z,\zeta)(\Delta_{\text{hyp},\zeta} \mathcal{P}_{\Gamma}(\zeta)) \mu_{\text{hyp}}(\zeta) - \frac{1}{2g_{\Gamma}} \int_X \mathcal{E}_{\Gamma}(\zeta) \mu_{\text{shyp}}(\zeta)
\]
\[
+ \frac{H_{\Gamma}(z)}{2g_{\Gamma}} + \frac{E_{\Gamma}(z)}{2g_{\Gamma}} - \frac{C_{\text{hyp}}}{8g_{\Gamma}^2} - \frac{2\pi(c_{X} - 1)}{g_{\Gamma}v_{\Gamma}} - \frac{1}{2g_{\Gamma}} \sum_{j=1}^{c_{\Gamma}} (1 - 1/\text{ord}(e_j)) \mathcal{G}_{\text{hyp}}(z,e_j).
\]
(5.2)

Now, for a fixed \( w \in X \), we set
\[
A(w) = \lim_{z \to \sigma_{pk}} (\mathcal{G}_{\text{hyp}}(z,w) - \phi(z)).
\]
Then
\[
A(w) = \lim_{z \to \sigma_{pk}} (\mathcal{G}_{\text{hyp}}(z,w) - \frac{1}{8\pi g_{\Gamma}} \int_X \mathcal{G}_{\text{hyp}}(z,\zeta)(\Delta_{\text{hyp},\zeta} \mathcal{P}_{\Gamma}(\zeta)) \mu_{\text{hyp}}(\zeta) - \frac{H_{\Gamma}(z)}{2g_{\Gamma}} - \frac{E_{\Gamma}(z)}{2g_{\Gamma}}
\]
\[
+ \frac{1}{2g_{\Gamma}} \sum_{j=1}^{c_{\Gamma}} (1 - 1/\text{ord}(e_j)) \mathcal{G}_{\text{hyp}}(z,e_j)) + \frac{1}{2g_{\Gamma}} \int_X \mathcal{E}_{\Gamma}(\zeta) \mu_{\text{shyp}}(\zeta) + \frac{C_{\text{hyp}}}{8g_{\Gamma}^2} + \frac{2\pi(c_{X} - 1)}{g_{\Gamma}v_{\Gamma}}.
\]
Now, from Lemma 4.3, recall that as \( z \) approaches the cusp \( p_k \), we have
\[
\frac{H_{\Gamma}(z)}{2g_{\Gamma}} + \frac{E_{\Gamma}(z)}{2g_{\Gamma}} = \frac{2\pi}{g_{\Gamma}} C_{pk,pk} - \frac{4\pi \log \left( \text{Im}(\sigma_{pk}^{-1}z) \right)}{g_{\Gamma}v_{\Gamma}} - \frac{2\pi}{g_{\Gamma}v_{\Gamma}}.
\]
From (3.3), recall that as \( z \) approaches the cusp \( p_k \), we have
\[
\mathcal{G}_{\text{hyp}}(z,w) = 4\pi K_{pk}(w) - \frac{4\pi}{v_{\Gamma}} \frac{\log \left( \text{Im}(\sigma_{pk}^{-1}z) \right)}{v_{\Gamma}}.
\]
Then using Lemma 4.1, Lemma 4.2, and Lemma 4.4, we get

\begin{equation}
A(w) = 4\pi K_{p_k}(w) + \frac{2\pi}{gr} \sum_{l=1}^{p_r} C_{p_l p_j} + \frac{2\pi}{gr} \sum_{j=1}^{e_r} \left(1 - 1/\text{ord}(e_j)\right) K_{p_k}(e_j) + \frac{2\pi c_X}{gr v_T} + \delta_X,
\end{equation}

where the absolute value of $\delta_X$ is bounded by

\begin{align*}
&\frac{2\pi}{v_T gr} \sum_{j=1}^{e_r} \left(1 + 1/\text{ord}(e_j)\right) + \frac{2\log 2}{v_T gr} \sum_{j=1}^{e_r} \left(\text{ord}(e_j) + 1\right) + \frac{2\pi(d_X + 1)^2}{\lambda_1 v_T} \\
&+ \frac{2\pi}{v_T} + \log(4\pi) + \frac{p_r}{gr v_T} \left(\pi + \frac{4\pi^2}{3} + 1\right).
\end{align*}

Now, to complete the theorem it remains to compute the following limit

\[ G_{\text{can}}(p_k, p_l) = \lim_{w \to p_l} \left(A(w) - \phi(w)\right). \]

For that we use the formula (5.2) and (5.3). Here note that, from Proposition 5.1, we know that as $w$ approaches the cusp $p_l$, we have

\[ 4\pi K_{p_k}(w) = 4\pi C_{p_k p_l} - \frac{4\pi \log \left(\text{Im}(\sigma^{-1}_{p_l} w)\right)}{v_T}. \]

Then using Lemma 4.1, Lemma 4.2, and Lemma 4.4, we get

\begin{align*}
G_{\text{can}}(p_k, p_l) &= 4\pi C_{p_k p_l} + \frac{2\pi}{gr} \sum_{j=1}^{p_r} C_{p_k p_j} + \frac{2\pi}{gr} \sum_{j=1}^{p_r} C_{p_l p_j} + \frac{4\pi c_X}{gr v_T} \\
&+ \frac{2\pi}{gr} \sum_{j=1}^{e_r} \left(1 - 1/\text{ord}(e_j)\right) \left(K_{p_k}(e_j) + K_{p_l}(e_j)\right) + \delta_X,
\end{align*}

where the absolute value of $\delta_X$ is bounded by

\begin{align*}
&\frac{4\pi}{v_T gr} \sum_{j=1}^{e_r} \left(1 + 1/\text{ord}(e_j)\right) + \frac{4\log 2}{v_T gr} \sum_{j=1}^{e_r} \left(\text{ord}(e_j) + 1\right) + \frac{4\pi(d_X + 1)^2}{\lambda_1 v_T} \\
&+ \frac{4\pi}{v_T} + 2\log(4\pi) + \frac{p_r}{gr v_T} \left(\pi + \frac{4\pi^2}{3} + 1\right).
\end{align*}

This completes the proof. \qed

6. Bounds for scattering constants of $\Gamma_0(N)$

Now, we consider $\Gamma = \Gamma_0(N)$ with a positive integer $N$. Here we give bounds for scattering constants of $\Gamma$.

**Lemma 6.1.** Let $C_{0\infty}$ denote the scattering constant with respect to the cusps 0 and $\infty$ of $\Gamma = \Gamma_0(N)$, where $N$ is a positive integer. Then

\[ 8\pi g_T(1 - g_T) C_{0\infty} = 2g_T \log N + o(g_T \log N) \quad \text{as} \quad N \to \infty. \]
Proof. In the formula [9], p. 247, substituting \( n = 1 \), we get

\[
\varphi_{0\infty}(s) = \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)} \prod_{\substack{p|N \\ \text{p prime}}} \frac{p^{2s} - p}{p^{2s} - 1},
\]

which is the scattering function for \( \Gamma \) with respect to cusps 0 and \( \infty \). Then, from definition (2.8) we can write

\[
C_{0\infty} = \lim_{s \to 1} \left( \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)} \prod_{\substack{p|N \\ \text{p prime}}} \frac{p^{2s} - p}{p^{2s} - 1} - \frac{1}{(s-1)\nu_{\Gamma}} \right).
\]

At \( s = 1 \) we compute the following Taylor expansions

\[
\frac{1}{N^s} = \frac{1}{N} - \frac{\log N}{N} (s - 1) + O\left((s - 1)^2\right),
\]

\[
\prod_{\substack{p|N \\ \text{p prime}}} \frac{p^{2s} - p}{p^{2s} - 1} = \prod_{\substack{p|N \\ \text{p prime}}} \frac{p}{p+1} + \prod_{\substack{p|N \\ \text{p prime}}} \frac{p}{p+1} \sum_{\substack{q|N \\ \text{q prime}}} \frac{2q \log q}{q^2 - 1} (s - 1) + O\left((s - 1)^2\right).
\]

Now note that, \( \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)} \) is the scattering function for \( \text{PSL}_2(\mathbb{Z}) \) and we know that

\[
\sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} \frac{\zeta(2s-1)}{\zeta(2s)} = \frac{3}{\pi(s-1)} + C + O(s - 1) \text{ as } s \to 1,
\]

where \( C = \frac{6}{\pi}(1 - \log(4\pi) - 12\zeta'(-1)) \).

Then, from (6.2), we get

\[
C_{0\infty} = \frac{1}{\nu_{\Gamma}} \left( \frac{\pi}{3} C - \log N + \sum_{\substack{p|N \\ \text{p prime}}} \frac{2p \log p}{p^2 - 1} \right).
\]

Finally, multiplying \( 8\pi g_{\Gamma}(1 - g_{\Gamma}) \) of the both side of (6.4), we get

\[
8\pi g_{\Gamma}(1 - g_{\Gamma}) C_{0\infty} = \frac{8\pi g_{\Gamma}(1 - g_{\Gamma})}{\nu_{\Gamma_0(N)}} \left( \frac{\pi}{3} C - \log N + \sum_{\substack{p|N \\ \text{p prime}}} \frac{2p \log p}{p^2 - 1} \right).
\]

Finally, using (2.26), and taking into account that \( \sum_{p|N} \frac{\log p}{p} = O(\log \log N) \), we have

\[
8\pi g_{\Gamma}(1 - g_{\Gamma}) C_{0\infty} = 2g_{\Gamma} \log N + o(g_{\Gamma} \log N) \text{ as } N \to \infty.
\]

\[\square\]

Lemma 6.2. Let \( a = m/n \) be a cusp of \( \Gamma = \Gamma_0(N) \), where \( N \) is a positive integer. Let \( C_{a\infty} \) denote the scattering constant with respect to the cusps \( a \) and \( \infty \). Then

\[
4\pi (1 - g_{\Gamma}) \sum_{\substack{a \in \mathcal{P}_{\Gamma} \\ a \neq \infty}} C_{a\infty} = o(g_{\Gamma} \log N) \text{ as } N \to \infty.
\]
Proof. From the definition (2.8), we recall
\[ C_{a\infty} = \lim_{s \to 1} \left( \varphi_{a\infty}(s) - \frac{1}{(s-1)v_{\Gamma}} \right), \]
where \( \varphi_{a\infty}(s) \) is given by the formula [9], p. 247. Means, we can write
\[ C_{a\infty} = \frac{\phi(n)}{\phi((n,N/n))} \lim_{s \to 1} \left( \sqrt{\pi} \Gamma(s - \frac{1}{2}) \zeta(2s - 1) \frac{F(s)}{\zeta(2s)} - \frac{1}{(s-1)v_{\Gamma}} \right), \]
where
\[ F(s) = \left( \frac{(n,N/n)}{nN} \right)^s \prod_{p|N \atop p \text{ prime}} \frac{p^{2s}}{p^{2s} - 1} \prod_{p|N \atop p \text{ prime}} \left( 1 - \frac{1}{q^{2s-1}} \right). \]

Now, at \( s = 1 \) we compute the following Taylor expansions
\[ \left( \frac{(n,N/n)}{nN} \right)^s = \left( \frac{(n,N/n)}{nN} \right) + \left( \frac{(n,N/n)}{nN} \right) \log \left( \frac{(n,N/n)}{nN} \right) (s-1) + O((s-1)^2), \]
\[ \prod_{p|N \atop p \text{ prime}} \frac{p^{2s}}{p^{2s} - 1} = \prod_{p|N \atop p \text{ prime}} \frac{p^2}{p^2 - 1} - \prod_{p|N \atop p \text{ prime}} \frac{p^2}{p^2 - 1} \sum_{q|N \atop q \text{ prime}} \frac{2\log q}{q^2-1} (s-1) + O((s-1)^2), \]
\[ \prod_{p|N \atop p \text{ prime}} \frac{p^{2s} - p}{p^{2s}} = \prod_{p|N \atop p \text{ prime}} \frac{p-1}{p} + \prod_{p|N \atop p \text{ prime}} \frac{p-1}{p} \sum_{q|N \atop q \text{ prime}} \frac{2\log q}{q-1} (s-1) + O((s-1)^2). \]

Then, by recalling (6.3), we get
\[ C_{a\infty} = \frac{1}{v_{\Gamma}} \left( \frac{\pi}{3} C + \log \left( \frac{(n,N/n)}{nN} \right) - \sum_{p|N \atop p \text{ prime}} \frac{2\log p}{p^2-1} + \sum_{p|N \atop p \text{ prime}} \frac{2\log p}{p-1} \right). \]

Hence from (2.22), we get
\[ 4\pi(1 - g_{\Gamma}) \sum_{a \in P_{\Gamma} \atop a \neq a_{\infty}} C_{a\infty} = \frac{4\pi(1 - g_{\Gamma})}{v_{\Gamma}} \left( \frac{\pi}{3} C - \sum_{p|N \atop p \text{ prime}} \frac{2\log p}{p^2-1} \sum_{n|N \atop n \neq 1} \phi((n,N/n)) \right) \]
\[ = \frac{4\pi(1 - g_{\Gamma})}{v_{\Gamma}} \sum_{n|N \atop n \neq 1} \phi((n,N/n)) \left( \log \left( \frac{(n,N/n)}{nN} \right) + \sum_{p|N \atop p \text{ prime}} \frac{2\log p}{p-1} \right). \]

Finally, using (2.26), and taking into account that \( \sum_{p|N} \frac{\log p}{p} = O(\log \log N) \), we have
\[ 4\pi(1 - g_{\Gamma}) \sum_{a \in P_{\Gamma} \atop a \neq a_{\infty}} C_{a\infty} = o(g_{\Gamma} \log N) \text{ as } N \to \infty. \]

This completes the proof. \( \Box \)
Lemma 6.3. Let \( a = m/n \) be a cusp of \( \Gamma = \Gamma_0(N) \), where \( N \) is a positive integer. Let \( C_{a0} \) denote the scattering constant with respect to the cusps \( a \) and \( 0 \). Then

\[
4\pi (1 - \gamma_\Gamma) \sum_{a \in \mathcal{P}_\Gamma \atop a \neq 0} C_{a0} = o(g_\Gamma \log N) \text{ as } N \to \infty.
\]

Proof. From the definition (2.8), we recall

\[
C_{a0} = \lim_{s \to 1} \left( \varphi_{a0}(s) - \frac{1}{(s - 1)v_\Gamma} \right),
\]

where \( \varphi_{a0}(s) \) is given by the formula [23], equation (3.6), i.e.,

\[
(6.5) \quad \varphi_{a0}(s) = \sqrt{\frac{\pi}{\Gamma(s - \frac{1}{2})}} \frac{\zeta(2s - 1)}{\zeta(2s)} \phi(N/n) \phi(n, N/n) G(s),
\]

where

\[
G(s) = \frac{(n^2, N)^s}{N^{2s}} \prod_{p \mid N \atop p \text{ prime}} \frac{p^{2s}}{p^{2s} - 1} \prod_{q \mid n \atop q \text{ prime}} \left( 1 - \frac{1}{q^{2s-1}} \right).
\]

Hence we have

\[
C_{a0} = \frac{\phi(N/n)}{\phi(n, N/n)} \lim_{s \to 1} \left( \sqrt{\frac{\pi}{\Gamma(s - \frac{1}{2})}} \frac{\zeta(2s - 1)}{\zeta(2s)} G(s) - \frac{1}{(s - 1)v_\Gamma} \right).
\]

At \( s = 1 \) we compute the following Taylor expansions

\[
\frac{(n^2, N)^s}{N^{2s}} = \frac{(n^2, N)}{N^2} \log \left( \frac{(n^2, N)}{N^2} \right) (s - 1) + O((s - 1)^2),
\]

\[
\prod_{p \mid N \atop p \text{ prime}} \frac{p^{2s}}{p^{2s} - 1} = \prod_{p \mid N} \frac{p^2}{p^2 - 1} \prod_{p \mid N} \frac{p^2}{p^2 - 1} \sum_{q \mid n \atop q \text{ prime}} \frac{2 \log q}{q^2 - 1} (s - 1) + O((s - 1)^2),
\]

\[
\prod_{q \mid n \atop q \text{ prime}} \left( 1 - \frac{1}{q^{2s-1}} \right) = \prod_{q \mid n \atop q \text{ prime}} \left( 1 - \frac{1}{q} \right) + \prod_{q \mid n \atop q \text{ prime}} \left( 1 - \frac{1}{q} \right) \sum_{p \mid N \atop p \text{ prime}} \frac{2 \log p}{p - 1} (s - 1) + O((s - 1)^2).
\]

Then, by recalling (6.3), we get

\[
C_{a0} = \frac{1}{v_\Gamma} \left( \frac{\pi}{3} C + \log \left( \frac{(n^2, N)}{N^2} \right) - \sum_{p \mid N \atop p \text{ prime}} \frac{2 \log p}{p^2 - 1} + \sum_{p \mid n \atop p \text{ prime}} \frac{2 \log p}{p - 1} \right).
\]
Then, from (2.22), we get

\[
4\pi(1 - g_T) \sum_{a \in \mathbb{P}, a \neq 0} C_{a0} = 4\pi(1 - g_T) \left( \frac{\pi}{3} C - \sum_{p|N \text{ prime}} \frac{2 \log p}{p^2 - 1} \sum_{n|N \text{ prime}, n \neq N} \phi((n, N/n)) \right)
\]

Finally, using (2.26), and taking into account that \(\sum_{p|N} \frac{\log p}{p} = O(\log \log N)\), we have

\[
4\pi(1 - g_T) \sum_{a \in \mathbb{P}, a \neq 0} C_{a0} = o(g_T \log N) \quad \text{as} \quad N \to \infty.
\]

This completes the proof.

7. Bounds for Kronecker limit functions of \(\Gamma_0(N)\)

Here we prove a relation between the Kronecker limit functions of \(\Gamma = \Gamma_0(N)\) and \(\text{PSL}_2(\mathbb{Z})\).

For notational convenience, in this section by \(K_{\infty}(z)\) resp. \(E_\infty(z, s)\) we denote the Kronecker limit function resp. the Eisenstein series with respect to the cusp \(\infty\) of the group \(\Gamma(1) = \text{PSL}_2(\mathbb{Z})\). Then we derive bounds for Kronecker limit functions of \(\Gamma_0(N)\).

Lemma 7.1. Let \(K_0(z)\) and \(K_\infty(z)\) denote the Kronecker limit functions with respect to the cusps 0 and \(\infty\) of the group \(\Gamma = \Gamma_0(N)\). Let \(K_\infty(z)_{\Gamma(1)}\) denote the Kronecker limit function with respect to the cusp \(\infty\) of the group \(\Gamma(1) = \text{PSL}_2(\mathbb{Z})\). Then

\[
K_0(z) + K_\infty(z) + \frac{2}{\nu_T} \left( \log N - \sum_{p|N \text{ prime}} \frac{\log p}{p + 1} \right) = \frac{1}{N} \prod_{p|N \text{ prime}} \frac{p^2}{p^2 - 1} \sum_{d|N \text{ prime}, d > 0} \frac{\mu(d)}{d} \left( K_\infty(Nz/d)_{\Gamma(1)} + K_\infty(dz)_{\Gamma(1)} \right),
\]

where \(\mu(d)\) is the Möbius function.

Proof. From [14], p. 240, we know the following formula

\[(7.1) \quad E_\infty(z, s) = N^{-s} \prod_{p|N \text{ prime}} \frac{p^{2s}}{p^{2s} - 1} \sum_{d|N \text{ prime}, d > 0} \frac{\mu(d)}{d^s} E_\infty(Nz/d, s)_{\Gamma(1)},\]

where in the L.H.S. \(E_\infty(z, s)\) denotes the Eisenstein series for \(\Gamma\) with respect to the cusp \(\infty\) and in the R.H.S. \(E_\infty(Nz/d, s)_{\Gamma(1)}\) denotes the Eisenstein series for \(\Gamma(1)\) with respect to the cusp \(\infty\).

Now, using formula (7.1), we can write

\[(7.2) \quad K_\infty(z) = \lim_{s \to 1} \left( N^{-s} \prod_{p|N \text{ prime}} \frac{p^{2s}}{p^{2s} - 1} \sum_{d|N \text{ prime}, d > 0} \frac{\mu(d)}{d^s} E_\infty(Nz/d, s)_{\Gamma(1)} - \frac{1}{(s - 1)\nu_T} \right).\]
To compute this limit (7.2), we use the following expansions.

\[
E_\infty(Nz/d, s)_{\Gamma(1)} = \frac{3}{\pi(s - 1)} + K_\infty(Nz/d)_{\Gamma(1)} + O(s - 1) \quad \text{as} \quad s \to 1,
\]

which is the well-known Laurent expansion of the Eisenstein series.

At \( s = 1 \) we compute the following Taylor series expansions.

\[
\frac{\mu(d)}{d^s} = \frac{\mu(d)}{d} - \frac{\mu(d) \log d}{d} (s - 1) + O((s - 1)^2),
\]

\[
\prod_{\substack{p|N \\text{prime}}} \frac{p^{2s}}{p^{2s} - 1} = \prod_{\substack{p|N \\text{prime}}} \frac{p^2}{p^2 - 1} - \prod_{\substack{p|N \\text{prime}}} \frac{p^2}{p^2 - 1} \sum_{q|N \\text{prime}} \frac{2 \log q}{q^2 - 1} (s - 1) + O((s - 1)^2),
\]

\[
\frac{1}{N^s} = \frac{1}{N} \frac{\log N}{N} (s - 1) + O((s - 1)^2).
\]

Then from (7.2), we get

\[
K_\infty(z) + \frac{1}{v_1} \left( \log N - \sum_{\substack{p|N \\text{prime}}} \frac{\log p}{p + 1} \right)
\]

\[
= \frac{1}{N} \prod_{\substack{p|N \\text{prime}}} \frac{p^2}{p^2 - 1} \sum_{d|N \\text{prime}} \frac{\mu(d)}{d} \left( K_\infty(Nz/d)_{\Gamma(1)} + K_\infty(dz)_{\Gamma(1)} \right),
\]

Now, to compute \( K_0(z) \), we consider a scaling matrix of the cusp 0 as

\[
\sigma_0 = \left( \begin{array}{cc} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{array} \right).
\]

Then using the Fricke involution, we get

\[
E_0(z, s) = E_\infty(\sigma_0^{-1}z, s).
\]

Then we have

\[
K_0(z) = K_\infty(\sigma_0^{-1}z) = K_\infty(-1/Nz)
\]

\[
= \frac{1}{N} \prod_{\substack{p|N \\text{prime}}} \frac{p^2}{p^2 - 1} \sum_{d|N \\text{prime} \\text{prime}} \frac{\mu(d)}{d} \left( K_\infty(Nz/d)_{\Gamma(1)} - \frac{1}{v_1} \left( \log N - \sum_{\substack{p|N \\text{prime} \\text{prime}}} \frac{\log p}{p + 1} \right) \right).
\]

Finally, combining (7.3) and (7.5), we can write

\[
K_0(z) + K_\infty(z) + \frac{2}{v_1} \left( \log N - \sum_{\substack{p|N \\text{prime}}} \frac{\log p}{p + 1} \right)
\]

\[
= \frac{1}{N} \prod_{\substack{p|N \\text{prime}}} \frac{p^2}{p^2 - 1} \sum_{d|N \\text{prime} \\text{prime} \\text{prime}} \frac{\mu(d)}{d} \left( K_\infty(Nz/d)_{\Gamma(1)} + K_\infty(dz)_{\Gamma(1)} \right).
\]

□
Lemma 7.2. Let \( K_\infty(z)_{\Gamma(1)} \) denote the Kronecker limit function with respect to the cusp \( \infty \) of the group \( \Gamma(1) = \text{PSL}_2(\mathbb{Z}) \). Let \( \nu_2 \) denote the number of elliptic fixed points of \( \Gamma \) with \( \text{ord}(e_j) = 2 \). Then

\[
\frac{1 - \gamma N}{N} \sum_{j=1}^{\nu_2} \left( 1 - \frac{1}{\text{ord}(e_j)} \right) \prod_{p \mid N} \frac{p^2}{p^2 - 1} \sum_{d 
mid N, \ d > 0} \frac{\mu(d)}{d} \left( K_\infty(\text{Ne}_j/d)_{\Gamma(1)} + K_\infty(d\text{e}_j)_{\Gamma(1)} \right)
= o(\gamma \log N) \text{ as } N \to \infty,
\]

where \( \mu(d) \) is the M"obius function.

Proof. From (2.23), we recall that if \( e_j \) be an elliptic fixed point of \( \Gamma \) with \( \text{ord}(e_j) = 2 \), then

\[
e_j = \frac{n + i}{n^2 + 1}
\text{ for } n = 0, \ldots, N - 1 \text{ such that } n^2 + 1 \equiv 0 \pmod{N}.
\]

By substituting these values of \( e_j \) in the term \( K_\infty(\text{Ne}_j/d)_{\Gamma(1)} + K_\infty(d\text{e}_j)_{\Gamma(1)} \) we get our desired asymptotics. Recall that

\[
K_\infty(z)_{\Gamma(1)} = -\frac{1}{4\pi} \log(y^{12}\Delta(z)^2) + 3\gamma - \log 4\pi,
\]

where \( z = x + iy \), \( \Delta(z) \) denotes the Modular discriminant, and \( \gamma \) denotes the Euler constant. The Modular discriminant has a Fourier expansion of the form

\[
\Delta(z) = \sum_{n=1}^{\infty} \tau(n)e^{2\pi inz},
\]

where the arithmetic function \( \tau(n) \) is known as Ramanujan’s tau function. Then for any positive integer \( d \), we can easily compute the bound

\[
\log|\Delta(dz)| \leq -2\pi dy + \log \left( 1 + \sum_{n=1}^{\infty} |\tau(n+1)|e^{-2\pi ndy} \right).
\]

When \( e_j \) is an elliptic fixed point of \( \Gamma \) with \( \text{ord}(e_j) = 2 \), then using (2.23) and (7.6), we get

\[
K_\infty(\text{Ne}_j/d)_{\Gamma(1)} + K_\infty(d\text{e}_j)_{\Gamma(1)} = -\frac{3}{\pi} \log \left( \frac{N}{(n^2 + 1)^2} \right) - \frac{1}{2\pi} \log \left| \Delta\left( \frac{N(n+i)}{d(n^2 + 1)} \right) \right|
- \frac{1}{2\pi} \log \left| \Delta\left( \frac{nd + id}{n^2 + 1} \right) \right| + \frac{6}{\pi}(\gamma - \log(4\pi)),
\]

where \( n \) satisfies the conditions given in (2.23), \( d \mid N \) (\( d > 0 \)), and \( \gamma \) is the Euler constant.

When \( n = 0 \) in (7.9), we get

\[
K_\infty(\text{Ne}_j/d)_{\Gamma(1)} + K_\infty(d\text{e}_j)_{\Gamma(1)}
= -\frac{3}{\pi} \log(N) - \frac{1}{2\pi} \log \left( |\Delta(Ni/d)\Delta(di)| \right) + \frac{6}{\pi}(\gamma - \log(4\pi)).
\]

Now, note that using (7.8), we can write

\[
\log \left( |\Delta(Ni/d)| \right) = O(N/d), \quad \log \left( |\Delta(di)| \right) = O(d).
\]

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This implies, there exist some constants $c_1$, independent of $N$, $d$, and $j$, such that
\[
\mathcal{K}_\infty(Ne_j/d)_{\Gamma(1)} + \mathcal{K}_\infty(de_j)_{\Gamma(1)} \leq \frac{c_1 N}{d} + c_2 d + o(N) \quad \text{as} \quad N \to \infty.
\]
When $n = 1, \ldots, N-1$ such that $n^2 + 1 \equiv 0 \pmod{N}$ in (7.9), we get
\[
\mathcal{K}_\infty(Ne_j/d)_{\Gamma(1)} + \mathcal{K}_\infty(de_j)_{\Gamma(1)} \leq \frac{c_3 N}{d(n^2 + 1)} + \frac{c_4 d}{n^2 + 1} + o(N) \quad \text{as} \quad N \to \infty,
\]
where the constants $c_3, c_4$ are independent of $N, d, n, j$.

Now, using the fact that \(\sum_{d|N} \frac{\mu(d)}{d} = \prod_{p|N} \frac{p-1}{p}\), we can write
\[
\frac{1}{N} \sum_{j=1}^{\nu_2} \left(1 - 1/\text{ord}(e_j)\right) \prod_{p|N} \frac{p^2}{p^2 - 1} \sum_{d|N} \frac{\mu(d)}{d} \left(\mathcal{K}_\infty(Ne_j/d)_{\Gamma(1)} + \mathcal{K}_\infty(de_j)_{\Gamma(1)}\right)
\]
\[
= \frac{1}{2N} \prod_{p|N} \frac{p^2}{p^2 - 1} \sum_{d|N} \frac{\mu(d)}{d} \sum_{j=1}^{\nu_2} \left(\mathcal{K}_\infty(Ne_j/d)_{\Gamma(1)} + \mathcal{K}_\infty(de_j)_{\Gamma(1)}\right)
\]
\[
\leq \frac{1}{2N} \prod_{p|N} \frac{p^2}{p^2 - 1} \sum_{d|N} \frac{\mu(d)}{d} \left(\frac{c_1 N}{d} + c_2 d\right)
\]
\[
+ \frac{1}{2N} \prod_{p|N} \frac{p^2}{p^2 - 1} \sum_{d|N} \frac{\mu(d)}{d} \sum_{n=1}^{N-1} \left(\frac{c_3 N}{d(n^2 + 1)} + \frac{c_4 d}{n^2 + 1}\right) + o(N)
\]
(7.10) \quad \leq \frac{c_1}{2} + c_3 + o(N).

Note that, in the last inequality of (7.10) we have used a well-known property of the Möbius function which is \(\sum_{d|N} \frac{\mu(d)}{d^2} = \prod_{p|N} \frac{p^2 - 1}{p^2}\). Then using (7.10), we get
\[
\frac{1 - g_{\Gamma}}{N} \sum_{j=1}^{\nu_2} \left(1 - 1/\text{ord}(e_j)\right) \prod_{p|N} \frac{p^2}{p^2 - 1} \sum_{d|N} \frac{\mu(d)}{d} \left(\mathcal{K}_\infty(Ne_j/d)_{\Gamma(1)} + \mathcal{K}_\infty(de_j)_{\Gamma(1)}\right)
\]
(7.11) \quad = o(g_{\Gamma} \log N) \quad \text{as} \quad N \to \infty.

This completes the proof. □

**Lemma 7.3.** Let \(\mathcal{K}_\infty(z)_{\Gamma(1)}\) denote the Kronecker limit function with respect to the cusp \(\infty\) of the group \(\Gamma(1) = \text{PSL}_2(\mathbb{Z})\). Let \(\nu_3\) denote the number of elliptic fixed points of \(\Gamma\) with \(\text{ord}(e_j) = 3\). Then
\[
\frac{1 - g_{\Gamma}}{N} \sum_{j=1}^{\nu_3} \left(1 - 1/\text{ord}(e_j)\right) \prod_{p|N} \frac{p^2}{p^2 - 1} \sum_{d|N} \frac{\mu(d)}{d} \left(\mathcal{K}_\infty(Ne_j/d)_{\Gamma(1)} + \mathcal{K}_\infty(de_j)_{\Gamma(1)}\right)
\]
\[
= o(g_{\Gamma} \log N) \quad \text{as} \quad N \to \infty,
\]
where \(\mu(d)\) is the Möbius function.
Proof. The proof is similar to the proof of Lemma 7.2. From (2.24), we recall that if \(e_j\) be an elliptic fixed point of \(\Gamma\) with \(\text{ord}(e_j) = 3\), then

\[
e_j = \frac{n + \frac{1 + i\sqrt{3}}{2}}{n^2 - n + 1}\]

for \(n = 0, \ldots, N - 1\) such that \(n^2 - n + 1 \equiv 0 \pmod{N}\).

By substituting these values of \(e_j\) in the term \(K_\infty(Ne_j/d)_{\Gamma(1)} + K_\infty(de_j)_{\Gamma(1)}\), we get

\[
K_\infty(Ne_j/d)_{\Gamma(1)} + K_\infty(de_j)_{\Gamma(1)} = -\frac{3}{\pi} \log \left( \frac{3N}{(2n^2 - 2n + 2)^2} \right) - \frac{1}{2\pi} \log \left| \Delta \left( \frac{N(2n + 1 + i\sqrt{3})}{2d(n^2 - n + 1)} \right) \right|
\]

\[
- \frac{1}{2\pi} \log \left| \Delta \left( \frac{d(2n + 1 + i\sqrt{3})}{2(n^2 - n + 1)} \right) \right| + \frac{6}{\pi} (\gamma - \log(4\pi)),
\]

where \(n\) satisfies the conditions given in (2.24), \(d|N\) and \(\gamma\) is the Euler constant.

When \(n = 0\) in (7.12), we get

\[
K_\infty(Ne_j/d)_{\Gamma(1)} + K_\infty(de_j)_{\Gamma(1)} = -\frac{3}{\pi} \log \left( \frac{3N}{4} \right) - \frac{1}{2\pi} \log \left| \Delta \left( \frac{N\sqrt{3}i}{2d} \right) \Delta \left( \frac{d\sqrt{3}i}{2} \right) \right|
\]

\[
+ \frac{6}{\pi} (\gamma - \log(4\pi)).
\]

Then, using (7.8), we can write

\[
\log \left| \Delta \left( \frac{N\sqrt{3}i}{2d} \right) \right| = O(N/d), \quad \log \left| \Delta \left( \frac{d\sqrt{3}i}{2} \right) \right| = O(d).
\]

This implies, there exist some constants \(c'_1, c'_2\), independent of \(N, d\) and \(j\), such that

\[
K_\infty(Ne_j/d)_{\Gamma(1)} + K_\infty(de_j)_{\Gamma(1)} \leq \frac{c'_1 N}{d} + c'_2 d + o(N) \text{ as } N \to \infty.
\]

When \(n = 1, \ldots, N - 1\) such that \(n^2 - n + 1 \equiv 0 \pmod{N}\) in (7.12), we get

\[
K_\infty(Ne_j/d)_{\Gamma(1)} + K_\infty(de_j)_{\Gamma(1)} \leq \frac{c'_3 N}{d(n^2 - n + 1)} + \frac{c'_4 d}{(n^2 - n + 1)} + o(N) \text{ as } N \to \infty,
\]

where the constants \(c'_3, c'_4\) are independent of \(N, d, n\) and \(j\).
Now, using the fact that \( \sum_{d \mid N} \mu(d) \frac{n(d)}{d} = \prod_{p \mid N} \frac{p-1}{p} \), we can write

\[
\frac{1}{N} \sum_{j=1}^{\nu_3} (1 - 1/\text{ord}(e_j)) \prod_{p \mid N \atop p \text{ prime}} \frac{p^2}{p^2 - 1} \sum_{d \mid N \atop d > 0} \frac{\mu(d)}{d} \left( \mathcal{K}_\infty(N e_j/d)_{\Gamma(1)} + \mathcal{K}_\infty(d e_j)_{\Gamma(1)} \right)
\]

\[
= \frac{2}{3N} \prod_{p \mid N \atop p \text{ prime}} \frac{p^2}{p^2 - 1} \sum_{d \mid N \atop d > 0} \frac{\mu(d)}{d} \sum_{j=1}^{\nu_3} \left( \mathcal{K}_\infty(N e_j/d)_{\Gamma(1)} + \mathcal{K}_\infty(d e_j)_{\Gamma(1)} \right)
\]

\[
= \frac{2}{3N} \prod_{p \mid N \atop p \text{ prime}} \frac{p^2}{p^2 - 1} \sum_{d \mid N \atop d > 0} \frac{\mu(d)}{d} \left( \frac{c'_1 N}{d} + c'_2 d \right)
\]

\[
+ \frac{2}{3N} \prod_{p \mid N \atop p \text{ prime}} \frac{p^2}{p^2 - 1} \sum_{d \mid N \atop d > 0} \frac{\mu(d)}{d} \sum_{n=1}^{N-1} \left( \frac{c'_3 N}{d(n^2 - n + 1)} + \frac{c'_4 d}{n^2 - n + 1} \right) + o(N)
\]

(7.13) \[ \leq \frac{c'_1}{2} + c'_3 + o(N). \]

Note that in the last inequality of (7.13) we have used a well-known property of the Möbius function which is \( \sum_{d \mid N} \mu(d) \frac{n(d)}{d} = \prod_{p \mid N} \frac{p^2-1}{p} \). Then using (7.10), we get

\[
\frac{(1 - g_\Gamma)}{N} \sum_{j=1}^{\nu_3} (1 - 1/\text{ord}(e_j)) \prod_{p \mid N \atop p \text{ prime}} \frac{p^2}{p^2 - 1} \sum_{d \mid N \atop d > 0} \frac{\mu(d)}{d} \left( \mathcal{K}_\infty(N e_j/d)_{\Gamma(1)} + \mathcal{K}_\infty(d e_j)_{\Gamma(1)} \right)
\]

\[
= o(g_\Gamma \log N) \text{ as } N \to \infty.
\]

This completes the proof. \( \square \)

**Proposition 7.4.** Let \( \mathcal{K}_0(z) \) and \( \mathcal{K}_\infty(z) \) denote the Kronecker limit functions for the group \( \Gamma = \Gamma_0(N) \) with respect to the cusps 0 and \( \infty \) respectively. Let \( \{e_j\}_{j=1}^{\nu_3} \) be the set of elliptic fixed points of \( \Gamma \). Then

\[
4\pi(1 - g_\Gamma) \sum_{j=1}^{\nu_3} (1 - 1/\text{ord}(e_j)) (\mathcal{K}_0(e_j) + \mathcal{K}_\infty(e_j)) = o(g_\Gamma \log N) \text{ as } N \to \infty.
\]
Proof. Using Lemma 7.1, we can write

$$4\pi (1-g_{\Gamma}) \sum_{j=1}^{e_{\Gamma}} \left( 1 - \frac{1}{\text{ord}(e_j)} \right) \left( \mathcal{K}_0(e_j) + \mathcal{K}_\infty(e_j) \right)$$

$$= \frac{4\pi (1-g_{\Gamma})}{N} \sum_{j=1}^{e_{\Gamma}} \left( 1 - \frac{1}{\text{ord}(e_j)} \right) \prod_{p \mid N, p \text{ prime}} \frac{p^2}{p^2 - 1} \sum_{d \mid N, d > 0} \frac{\mu(d) \left( \mathcal{K}_\infty(Ne_j/d)_{\Gamma(1)} + \mathcal{K}_\infty(de_j)_{\Gamma(1)} \right)}{d}$$

$$= \frac{4\pi (1-g_{\Gamma})}{N} \sum_{j=1}^{e_{\Gamma}} \left( 1 - \frac{1}{\text{ord}(e_j)} \right) \left( \sum_{p \mid N, p \text{ prime}} \frac{\log p}{p+1} - \log N \right).$$

(7.14)

Now, using the estimates (2.26), (2.25), and taking into account the estimate $\sum_{p \mid N} \frac{\log p}{p} = O(\log \log N)$, we get

$$\frac{8\pi (1-g_{\Gamma})}{v_{\Gamma}} \sum_{j=1}^{e_{\Gamma}} \left( 1 - \frac{1}{\text{ord}(e_j)} \right) \left( \sum_{p \mid N, p \text{ prime}} \frac{\log p}{p+1} - \log N \right) = o(g_{\Gamma} \log N) \text{ as } N \to \infty,$$

which is the second summand of (7.14). So, to complete the proof it suffices to show that

$$\frac{4\pi (1-g_{\Gamma})}{N} \sum_{j=1}^{e_{\Gamma}} \left( 1 - \frac{1}{\text{ord}(e_j)} \right) \prod_{p \mid N, p \text{ prime}} \frac{p^2}{p^2 - 1} \sum_{d \mid N, d > 0} \frac{\mu(d) \left( \mathcal{K}_\infty(Ne_j/d)_{\Gamma(1)} + \mathcal{K}_\infty(de_j)_{\Gamma(1)} \right)}{d}$$

$$= o(g_{\Gamma} \log N) \text{ as } N \to \infty.$$

Now, note that

$$\frac{4\pi (1-g_{\Gamma})}{N} \sum_{j=1}^{e_{\Gamma}} \left( 1 - \frac{1}{\text{ord}(e_j)} \right) \prod_{p \mid N, p \text{ prime}} \frac{p^2}{p^2 - 1} \sum_{d \mid N, d > 0} \frac{\mu(d) \left( \mathcal{K}_\infty(Ne_j/d)_{\Gamma(1)} + \mathcal{K}_\infty(de_j)_{\Gamma(1)} \right)}{d}$$

$$= \frac{(1-g_{\Gamma})}{N} \sum_{j=1}^{e_{\Gamma}} \left( 1 - \frac{1}{\text{ord}(e_j)} \right) \prod_{p \mid N, p \text{ prime}} \frac{p^2}{p^2 - 1} \sum_{d \mid N, d > 0} \frac{\mu(d)}{d} \left( \mathcal{K}_\infty(Ne_j/d)_{\Gamma(1)} + \mathcal{K}_\infty(de_j)_{\Gamma(1)} \right)$$

$$+ \sum_{j=1}^{e_{\Gamma}} \left( 1 - \frac{1}{\text{ord}(e_j)} \right) \prod_{p \mid N, p \text{ prime}} \frac{p^2}{p^2 - 1} \sum_{d \mid N, d > 0} \frac{\mu(d)}{d} \left( \mathcal{K}_\infty(Ne_j/d)_{\Gamma(1)} + \mathcal{K}_\infty(de_j)_{\Gamma(1)} \right).$$

Finally, Lemma 7.2 and Lemma 7.3 completes the proof. □

8. Proof of Theorem 1.2

Finally, we give the proof of Theorem 1.2 using results from section 6 and section 7.
Proof of Theorem 1.2. Recall that $\Gamma = \Gamma_0(N)$ and $X = \Gamma_0(N) \backslash \mathbb{H}$. Then from Theorem 1.1, we have

$$2g_G(1 - g_G) G_{can}(0, \infty) = 8\pi g_G(1 - g_G) C_{0\infty} + 4\pi(1 - g_G) \left( \sum_{a \in \mathcal{P}_G, a \neq \infty} C_{a\infty} + \sum_{a \in \mathcal{P}_G, a \neq 0} C_{a0} \right)$$

$$+ 4\pi(1 - g_G) \sum_{j=1}^{e_G} \left( 1 - 1/\text{ord}(e_j) \right) \left( K_0(e_j) + K_\infty(e_j) \right)$$

$$+ \frac{8\pi(1 - g) e_X}{v_G} + \delta_X,$$

(8.1)

where the absolute value of $\delta_X$ is bounded by

$$\frac{8\pi(g_G - 1)}{v_G} \sum_{j=1}^{e_G} (1 + 1/\text{ord}(e_j)) + \frac{8(g_G - 1)}{v_G} \sum_{j=1}^{e_G} (1 + \text{ord}(e_j)) + \frac{8\pi g_G (g_G - 1)(d_X + 1)^2}{\lambda_1 v_G}$$

$$+ \frac{8\pi g_G (g_G - 1)}{v_G} \sum_{j=1}^{e_G} (1 - 1/\text{ord}(e_j)) \left( K_0(e_j) + K_\infty(e_j) \right)$$

$$+ 4\pi(1 - g_G) \sum_{j=1}^{e_G} \left( 1 - 1/\text{ord}(e_j) \right) \left( K_0(e_j) + K_\infty(e_j) \right) + \frac{8\pi g_G (g_G - 1)(d_X + 1)^2}{\lambda_1 v_G}$$

(8.2)

To show that $\delta_X = o(g_G \log N)$ as $N \to \infty$ we recall the following. From (2.26), (2.25) it is clear that

$$\frac{8\pi(g_G - 1)}{v_G} \sum_{j=1}^{e_G} (1 + 1/\text{ord}(e_j)) + \frac{8(g_G - 1)}{v_G} \sum_{j=1}^{e_G} (1 + \text{ord}(e_j)) = O(N^\varepsilon),$$

which are the first two terms of (8.2). Then from [18], Proposition 5.4, we recall

$$d_X = O(1),$$

where the implied constant is independent of $N$. From [29], Theorem 1.1, we recall

$$\lambda_1 \geq 21/100.$$

Then using (2.26), we get

$$\frac{8\pi g_G (g_G - 1)(d_X + 1)^2}{\lambda_1 v_G} = O(g_G),$$

which is the third term of (8.2). Again using (2.26), (2.25), we have

$$\frac{8\pi g_G (g_G - 1)}{v_G} \sum_{j=1}^{e_G} (1 - 1/\text{ord}(e_j)) \left( K_0(e_j) + K_\infty(e_j) \right) + 4\pi(1 - g_G) \sum_{j=1}^{e_G} \left( 1 - 1/\text{ord}(e_j) \right) \left( K_0(e_j) + K_\infty(e_j) \right) = O(g_G).$$

For the last term of (8.2), from the formula (2.22), we have

$$\frac{4\pi g_G (g_G - 1)}{v_G} \sum_{j=1}^{e_G} (1 - 1/\text{ord}(e_j)) \left( K_0(e_j) + K_\infty(e_j) \right) = O(N),$$

where we use the estimate (2.26), and use a well-known identity $\sum_{d|N} \varphi(d) = N$ for the Euler function $\varphi$.

Then note that, the estimate (2.26) and the formula (2.21) implies that $g_G = O(N \log N)$. Hence we can write

$$\delta_X = o(g_G \log N) \text{ as } N \to \infty.$$
From [20], p. 26-27, we recall that $c_X = O_e(N^e)$, which implies

$$\frac{8\pi(1 - g_\Gamma) c_X}{v_\Gamma} = o(g_\Gamma \log N) \text{ as } N \to \infty.$$  

Then from (8.1), we have

$$2g_\Gamma (1 - g_\Gamma) G_{\text{can}}(0, \infty) = 8\pi g_\Gamma (1 - g_\Gamma) C_{0\infty} + 4\pi(1 - g_\Gamma) \left( \sum_{a \in P_\Gamma \setminus a \neq \epsilon} C_a + \sum_{a \in P_\Gamma \setminus a \neq 0} C_a \right)$$

$$+ 4\pi(1 - g_\Gamma) \sum_{j=1}^{e_\Gamma} \left( 1 - 1/\text{ord}(e_j) \right) (K_0(e_j) + K_{\infty}(e_j)) + o(g_\Gamma \log N) \text{ as } N \to \infty.$$  

From Lemma 6.1, we have

$$8\pi g_\Gamma (1 - g_\Gamma) C_{0\infty} = 2g_\Gamma \log N + o(g_\Gamma \log N) \text{ as } N \to \infty,$$

which is the first term of the R.H.S. of (8.3). Finally, using Lemma 6.2, Lemma 6.3, and Proposition 7.4, it is clear that

$$2g_\Gamma (1 - g_\Gamma) G_{\text{can}}(0, \infty) = 2g_\Gamma \log N + o(g_\Gamma \log N) \text{ as } N \to \infty,$$

which completes the proof. $\square$

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