GENERALIZED STRETCHED LITTLEWOOD-RICHARDSON COEFFICIENTS

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Abstract. In this paper we investigate \( Q(n) = Q^{\lambda,\mu}_{\lambda',\mu'}(n) = \sum_{\nu} c(n\lambda + \lambda'; n\mu + \mu', \nu) \) as a function of \( n \) and show that \( Q(n) \) is bounded above if and only if \( \lambda/\mu \) is a partition or rotated partition. Here \( c(\lambda; \mu, \nu) \) is the Littlewood-Richardson coefficient. So \( Q(n) \) counts the number of LR tableaux of shape \( (n\lambda + \lambda')/(n\mu + \mu') \) or the total number of irreducible characters in the skew character \( [(n\lambda + \lambda')/(n\mu + \mu')] \).

We furthermore investigate the function \( P(n) = P^{\lambda,\mu,\nu}_{\lambda',\mu',\nu'}(n) = c(n\lambda + \lambda'; n\mu + \mu', n\nu + \nu') \) as a function of \( n \).

1. Introduction

Some recent research was concerned with the behaviour of the stretched LR-coefficients \( f(n) = c(n\lambda; n\mu, n\nu) \) (see [KT], [KTW], [Bu1], [DW], [KTT], [Ras]). It is known that this function is a polynomial and therefore is either constant or increases without bound (for example by Lemma 3.1). It is constant if and only if \( c(\lambda; \mu, \nu) = 1 \) (see [KTW]).

In this paper we examine \( Q(n) = Q^{\lambda,\mu}_{\lambda',\mu'}(n) = \sum_{\nu} c(n\lambda + \lambda'; n\mu + \mu', \nu) \) and \( P(n) = P^{\lambda,\mu,\nu}_{\lambda',\mu',\nu'}(n) = c(n\lambda + \lambda'; n\mu + \mu', n\nu + \nu') \) and where \( \lambda + \lambda' = (\lambda_1 + \lambda'_1, \lambda_2 + \lambda'_2, \ldots) \).

So \( f(n) \) is the special case of \( P(n) \) with \( \lambda' = \mu' = \nu' = 0 \).

We show that \( Q^{\lambda,\mu}_{\lambda',\mu'}(n) = \sum_{\nu} c(n\lambda + \lambda'; n\mu + \mu', \nu) \) as a function of \( n \) is bounded above if and only if \( \lambda/\mu \) is a partition or rotated partition. \( Q(n) \) counts the number of LR tableaux of shape \( n\lambda + \lambda'/n\mu + \mu' \) or the total number of irreducible characters in the skew character \( [n\lambda + \lambda'/n\mu + \mu'] \).

2. Notation and Littlewood-Richardson-Symmetries

We mostly follow the standard notation in [Sag] or [Sta]. A partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \) is a weakly decreasing sequence of non-negative integers where only finitely many of the \( \lambda_i \) are positive. We regard two partitions as the same if they differ only by the number of trailing zeros and call the positive \( \lambda_i \) the parts of \( \lambda \). The length is the number of positive parts and we write \( l(\lambda) = l \) for the length and \( |\lambda| = \sum_{i} \lambda_i \) for the sum of the parts. With a partition \( \lambda \) we associate a diagram, which we also denote by \( \lambda \), containing \( \lambda_i \) left-justified boxes in the \( i \)-th row and we use matrix-style coordinates to refer to the boxes.

The conjugate \( \lambda^c \) of \( \lambda \) is the diagram which has \( \lambda_i \) boxes in the \( i \)-th column.

2000 Mathematics Subject Classification. 05E05, 05E10, 14M15, 20C30.
Key words and phrases. Littlewood Richardson, LR-coefficient, LR-tableaux.
The partition $\lambda + (1^n) = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_a + 1, \lambda_{a+1}, \ldots)$ is obtained from $\lambda$ by inserting a column containing $a$ boxes. The partition $\lambda \cup (a)$ is obtained from $\lambda$ by inserting a row containing $a$ boxes. Inserting a row is the conjugate of inserting a column: $(\lambda + (1^n))^c = \lambda^c \cup (a)$. Both operations can be generalized to arbitrary partitions: $\lambda + \lambda'$ and $\lambda \cup \lambda'$.

For $\mu \subseteq \lambda$ we define the skew diagram $\lambda/\mu$ as the difference of the diagrams $\lambda$ and $\mu$ defined as the difference of the set of the boxes. Rotation of $\lambda/\mu$ by $180^\circ$ yields a skew diagram $(\lambda/\mu)^c$ which is well defined up to translation. A skew tableau $T$ is a skew diagram in which positive integers are written into the boxes. A semistandard tableau of shape $\lambda/\mu$ is a filling of $\lambda/\mu$ with positive integers such that the entries weakly increase amongst the rows and strictly increase amongst the columns. The content of a semistandard tableau $T$ is $\nu = (\nu_1, \ldots)$ if the number of occurrences of the entry $i$ in $T$ is $\nu_i$. The reverse row word of a tableau $T$ is the sequence obtained by reading the entries of $T$ from right to left and top to bottom starting at the first row. Such a sequence is said to be a lattice word if for all $i, n \geq 1$ the number of occurrences of $i$ among the first $n$ terms is at least the number of occurrences of $i + 1$ among these terms. The Littlewood-Richardson (LR-) coefficient $c(\lambda; \mu, \nu)$ equals the number of semistandard tableaux of shape $\lambda/\mu$ with content $\nu$ such that the reverse row word is a lattice word. We will call those tableaux LR-tableaux. The LR-coefficients play an important role in different contexts (see [Sag] or [Sta] for further details).

The irreducible characters $[\lambda]$ of the symmetric group $S_n$ are naturally labeled by partitions $\lambda \vdash n$. The skew character $[\lambda/\mu]$ corresponding to a skew diagram $\lambda/\mu$ is defined by the LR-coefficients:

$$[\lambda/\mu] = \sum_\nu c(\lambda; \mu, \nu)[\nu].$$

The translation symmetry gives $[\lambda/\mu] = [\alpha/\beta]$ if the skew diagrams of $\lambda/\mu$ and $\alpha/\beta$ are the same up to translation while rotation symmetry gives $[(\lambda/\mu)^c] = [\lambda/\mu]$. The conjugation symmetry $c(\lambda^c; \mu^c, \nu^c) = c(\lambda; \mu, \nu)$ is also well known and furthermore it is $c(\lambda; \mu, \nu) = c(\lambda; \nu, \mu)$.

We say that a skew diagram $D$ decays into the disconnected skew diagrams $A$ and $B$ if no box of $A$ (viewed as boxes in $D$) is in the same row or column as a box of $B$. We write $D = A \otimes B$ if $D$ decays into $A$ and $B$. A skew diagram is connected if it does not decay. If $D = A \otimes B = C$ then by translation symmetry $[D] = [C]$.

A basic skew diagram $\lambda/\mu$ is a skew diagram which satisfies $\mu_i < \lambda_i$ and $\mu_i \leq \lambda_{i+1}$ for each $1 \leq i \leq l(\lambda)$. This means that we don’t have empty rows or columns in $\lambda/\mu$. Empty rows or columns of a skew diagram don’t influence the filling and so deleting empty rows or columns doesn’t change the skew character or LR-fillings.

A proper skew diagram $\lambda/\mu$ is a skew diagram which is neither a partition nor a rotated partition.

In [Gut] we used the following theorem to classify the multiplicity free skew characters.

**Theorem 2.1 (Theorem 3.1. [Gut]).** Let $\lambda, \mu, \nu$ be partitions and $a, b \geq 0$ integers. Then:

$$c(\lambda; \mu, \nu) \leq c(\lambda + (1^{a+b}); \mu + (1^a), \nu + (1^b)).$$
3. Preliminaries

We can generalize Theorem 2.1 to the following.

**Lemma 3.1.** Let \( \lambda, \mu, \nu, \lambda', \mu', \nu' \) be partitions with \( c(\lambda; \mu, \nu), c(\lambda'; \mu', \nu') \neq 0 \). Then:

\[
c(\lambda; \mu, \nu) \leq c(\lambda + \lambda'; \mu + \mu', \nu + \nu').
\]

*Proof.* Let \( A \) be a fixed LR tableau of shape \( \lambda'/\mu' \) with content \( \nu' \). Let \( A_j \) be the multiset of the entries in the \( j \)th row of \( A \).

For any LR tableau \( C_i \) of shape \( \lambda/\mu \) and content \( \nu \) we let \( C_{ij} \) be the multiset of the entries in the \( j \)th row of \( C_i \).

We can now define for every \( C_i \) a tableau \( D_i \) of shape \( (\lambda + \lambda')/(\mu + \mu') \) with content \( \nu + \nu' \) by placing into row \( j \) the entries of \( A_j \cup C_{ij} \) in weakly increasing order. Because the columns of \( A \) and \( C_i \) are strictly increasing, also the columns of \( D_i \) are strictly increasing. It is also clear that the tableau word is a lattice word because it can be divided into two subsequences (corresponding to the entries in \( D_i \) having their origin either in \( A \) or \( C_i \)) which are both lattice words. So the \( D_i \) are in fact an LR tableaux.

Suppose we have \( D_i = D_i \). Then we know from the construction that the multiset of the entries in the \( j \)th row of \( D_i \) is \( A_j \cup C_{ij} \) while the multiset of the entries in the \( j \)th row of \( D_i \) is \( A_j \cup C_{ij} \). This gives us \( C_{ij} = C_{ij} \) for all \( j \) and since an LR tableau of a given shape is uniquely determined by the content of its row it follows that \( C_i = C_i \). So we have that different LR tableaux of shape \( \lambda/\mu \) with content \( \nu \) give different LR tableaux of shape \( (\lambda + \lambda')/(\mu + \mu') \) with content \( \nu + \nu' \) and so:

\[
c(\lambda; \mu, \nu) \leq c(\lambda + \lambda'; \mu + \mu', \nu + \nu').
\]

\( \square \)

**Remark 3.2.** In the hive model (which we do not use in this paper) the proof is also easy. Choose one LR hive corresponding to the triple \( (\lambda', \mu', \nu') \) and add this hive to all the LR hives corresponding to \( (\lambda, \mu, \nu) \). It is easy to see that all the new hives are different LR hives corresponding to \( (\lambda + \lambda'; \mu + \mu', \nu + \nu') \).

**Remark 3.3.** It is known that \( f(n) = c(n\lambda; n\mu, n\nu) \) is a polynomial which is constant if and only if \( c(\lambda; \mu, \nu) = 1 \) (see [KT1], [KTW]). Suppose \( \lambda, \mu, \nu \) are chosen in such a way that \( f(n) \) is not constant then we know that

\[
F(n) = c(n\lambda + \lambda'; n\mu + \mu', n\nu + \nu')
\]

increases without bound if \( c(\lambda'; \mu', \nu') \neq 0 \).

**Remark 3.4.** It is known (see [Zel]) that the triples of partitions with non-zero LR coefficient form an additive semigroup.

4. Behaviour of \( Q_{\lambda', \mu'}^{\lambda, \mu}(n) \)

For \( \mu \subseteq \lambda, \mu' \subseteq \lambda' \) we define \( Q_{\lambda', \mu'}^{\lambda, \mu}(n) = \sum_{\nu} c(n\lambda + \lambda'; n\mu + \mu', \nu) \) and write simply \( Q(n) \) if \( \lambda, \mu, \lambda', \mu' \) are known from the context.

**Lemma 4.1.** Let \( \lambda/\mu \) be a proper skew diagram. Then \( Q_{\lambda', \mu'}^{\lambda, \mu}(n) \) increases without bound. Furthermore

\[
\sum_{c(n\lambda + \lambda'; n\mu + \mu', \nu) \neq 0} 1 \to \infty \quad \text{for } n \to \infty.
\]
Proof. Since $\lambda/\mu$ is a proper skew diagram it is obtained from the skew diagram $(2,1)/(1)$ by inserting rows and columns and so we have by Lemma 3.1 (in fact the more special Theorem 2.1 is sufficient): $\sum_{\nu} c(\lambda;\mu,\nu) \geq \sum_{\nu} c((2,1);(1),\nu)$. Furthermore we have:

$$\sum_{\nu} c(n\lambda + \lambda';n\mu + \mu',\nu) \geq \sum_{\nu} c(n\lambda;n\mu,\nu) \geq \sum_{\nu} c(n(2,1);n(1),\nu)$$

It is easy to see that $\sum_{\nu} c(n(2,1);n(1),\nu) = n + 1$ because an LR tableau of shape $(2,1)/(1)$ contains $n$ entries in row 1 and $i$ ($0 \leq i \leq n$) entries 1 as well as $n - i$ entries 2 in row 2 and for each such $i$ there is exactly one LR tableau. So $Q_{\lambda'/:\mu'}(n)$ increases without bound.

Furthermore since the number of irreducible characters in $[n(2,1)/(n(1))]$ is $n + 1$ there are also at least $n + 1$ irreducible characters in $[n\lambda + \lambda'/n\mu + \mu']$ and so $\sum c(n\lambda + \lambda',n\mu + \mu',\nu)_{\geq 0} \geq n + 1$. \qed

**Lemma 4.2.** Let $\lambda/\mu$ be a partition or rotated partition.

Then there exists an $m$ with $Q_{\lambda,:\mu}(n) = Q_{\lambda,:\mu'}(m)$ for $n \geq m$.

Furthermore if $\lambda = (\alpha_1^{a_1}, \alpha_2, \alpha_3, \ldots \alpha_k), \alpha_k \neq 0, \mu = (\alpha_1^{a_1-1})$ and $\lambda'/\mu'$ basic we can choose

$$m = \left[ \max_{a_j > a_{j+1}} \left( \frac{\lambda'_i - \lambda'_{a_j} + \lambda'_{a_j + 1} + \mu'_{a_1 - 1} - \mu'_{a_1}}{\alpha_j - \alpha_{j+1}} \right) \right]$$

with $a_j = a_1 - 1 + j, \alpha_{k+1} = 0$ (for $a_1 = 1$ set $\mu'_0 = \lambda'_1$) which then also gives $Q_{\lambda',:\mu'}(m) > Q_{\lambda',:\mu'}(m - 1) > \ldots > Q_{\lambda',:\mu'}(0)$.

These inequalities are also satisfied in the general case if we choose the smallest $m$ satisfying $Q_{\lambda,:\mu}(n) = Q_{\lambda',:\mu'}(m)$ for $n \geq m$.

**Proof.** We look at the skew diagram $A(n) = (n\lambda + \lambda'/(n\mu + \mu')$.

By rotation symmetry we may assume that $\lambda/\mu$ is a partition instead of a rotated partition.

Let $a_1 > a_2 > \ldots > a_k$ be the non-empty rows of $\lambda/\mu$. If we have $\lambda_i = \mu_i > \lambda_{i+1}$ for some $i \neq a_1, \ldots, a_k$ and choose $n$ big enough then $A(n)$ decays into a skew diagram $A^{up}$ containing the upper $i$ rows and a skew diagram $A_{lo}$ containing the rows below row $i$. If we increase $n$ even more then the skew diagrams $A^{up}$ and $A_{lo}$ are translated relative to one another which is irrelevant for the skew character $[A(n)]$. So if there are some $i \neq a_1, \ldots, a_k$ with $\lambda_i = \mu_i > \lambda_{i+1}$ we may choose $n$ large enough so that for each such $i$ $A(n)$ decays into an upper skew diagram and a lower skew diagram. Instead of looking at this situation we may then investigate the case that $\lambda'/\mu' = A(n)$ for an $n$ large enough and have no $i \neq a_1, \ldots, a_k$ with $\lambda_i = \mu_i > \lambda_{i+1}$. So we may assume that $\mu_i = \lambda_i = \lambda_{a_1} = \lambda_{a_2}$ for $i < a_1$ and $\mu_i = \lambda_i = \mu_{a_k}$ for $a_k < i \leq l(\mu)$ (and since $\lambda/\mu$ is a partition we also have $\mu_{a_1} = \mu_{a_k}$). If $\mu_{a_1} > 0$ there is for the same reason as above an $n$ such that $A(n)$ decays into skew diagrams containing the upper $l(\mu)$ rows and the rows below row $l(\mu)$ and increasing $n$ translates these skew diagrams relative to another so we may assume that $\mu_{a_1} = 0$.

So we have without loss of generality $\lambda = (\alpha_1^{a_1}, \alpha_2, \alpha_3, \ldots, \alpha_k), \alpha_k \neq 0$ (not necessarily $a_i \neq a_{i+1}$) and $\mu = (\alpha_1^{a_1-1})$. To prove $Q(n) = Q(m)$ for $n \geq m$ we have to construct an $m$ such that removing in an LR tableau of shape $A(n)$ from the row $a_i$ with $1 \leq i \leq k$ the entry $i(n - m)$ times and translating the upper $a_1 - 1$ rows $(n - m)a_1$ boxes to the left yields an LR tableau of shape $A(m)$. 


By our choice of $\lambda$ and $\mu$ the number $N$ of non-empty columns among the upper $a_1 - 1$ rows of $A(n)$ is independent of $n$ (we have $N \leq \lambda'_1 - \mu'_{a_1-1}$ and may by translation symmetry assume equality, set $\mu'_0 = \lambda'_1$ for $a_1 = 1$). So the number of entries 1 among the upper $a_1$ rows of an LR filling of $A(n)$ is at most $N$. So for $1 \leq i \leq k$ there are at most $N$ entries larger in row $a_i$ of an LR filling of $A(n)$. Furthermore the number of entries smaller $i$ in row $a_i$ is at most $\mu'_{a_i} - \mu'_{a_i}$, also independent of $n$. On the other hand there are in row $a_i$ of $A(n)$ $\lambda'_{a_i} - \mu'_{a_i} + n\alpha_i$ boxes. So the number of entries $i$ in row $a_i$ of an LR filling of $A(n)$ is at least
\[
\lambda'_{a_i} - \mu'_{a_i} + n\alpha_i - N - (\mu'_{a_i} - \mu'_{a_i}) = \lambda'_{a_i} - \mu'_{a_1} - N + n\alpha_i.
\]
Obviously if $\lambda'_{a_k} - \mu'_{a_1} - N + n\alpha_k \geq 0$ then also $\lambda'_{a_i} - \mu'_{a_1} - N + n\alpha_i \geq 0$ for every $1 \leq i \leq k$. So for
\[
n > n' \geq \frac{\mu'_{a_1} + N - \lambda'_{a_k}}{\lambda_{a_k}}
\]
there are at least $(n - n')\alpha_i$ entries $i$ in row $a_i$ of every LR tableau of shape $A(n)$.

We have to investigate the $j$ $(1 \leq j \leq k)$ with $\alpha_j > \alpha_{j+1}$ (for example $j = k$). Removing in an LR tableau $\alpha_i$ times the entry $i$ from row $a_i$ removes more entries $j$ than $j + 1$ so the new tableau can violate the lattice word condition even if there are enough entries $i$ to remove. As calculated above the number of entries $j$ in row $a_j$ of an LR tableau of shape $A(n)$ is at least: $\lambda'_{a_j} - \mu'_{a_1} - N + n\alpha_j$. Furthermore the number of entries $j + 1$ in an LR tableau of shape $A(n)$ below row $j$ is at most $\lambda'_{a_{j+1}} + n\alpha_{j+1}$ since there are only so many columns below row $a_{j+1}$. So for
\[
\lambda'_{a_j} - \mu'_{a_1} - N + n\alpha_j \geq \lambda'_{a_{j+1}} + n\alpha_{j+1}
\]
the number of entries $j$ in row $a_j$ is at least as large as the number of entries $j + 1$ below row $a_{j+1}$ in every LR tableau of shape $A(n)$. We can solve the above inequality and get:
\[
n \geq \frac{\lambda'_{a_{j+1}} - \lambda'_{a_j} + \mu'_{a_1} + N}{\alpha_j - \alpha_{j+1}}.
\]
Since we have $\alpha_k > 0 = \alpha_{k+1}$ setting $j = k$ gives
\[
\frac{\lambda'_{a_{k+1}} - \lambda'_{a_k} + \mu'_{a_1} + N}{\alpha_k} \geq \frac{-\lambda'_{a_k} + \mu'_{a_1} + N}{\alpha_k}
\]
which is the right hand side of inequality [401].

Let us set
\[
m = \left\lceil \max_{1 \leq i \leq k} \left( \frac{\lambda'_{a_i} - \lambda'_{a_i+1} + \mu'_{a_1} + \mu'_j - \mu'_{a_1-1}}{\alpha_j - \alpha_{j+1}} \right) \right\rceil
\]
where $[x]$ denotes as usual the smallest integer larger or equal to $x$.

Then we know for $n \geq m$ from our reasonings above that every LR tableau $C_n$ of shape $A(n)$ contains at least $(n - m)\alpha_i$ entries $i$ in row $a_i$ ($1 \leq i \leq k$) and furthermore removing $(n - m)\alpha_i$ entries $i$ from every row $a_i$ ($1 \leq i \leq k$) and translating the upper $a_1 - 1$ rows $(n - m)\alpha_1$ boxes to the left yields a tableau $C_m$ which contains (for those $j$ with $\alpha_j > \alpha_{j+1}$) in row $a_j$ at least as much entries $j$ as there are entries $j + 1$ below row $a_{j+1}$. So the tableau $C_m$ satisfies the lattice word condition. Furthermore the entries in the rows increase weakly from left to right. We have to check that the entries in the columns are strictly increasing from top to bottom which is not trivial because we remove more entries $j$ from row $a_j$ than
entries $j + 1$ from row $a_j + 1$ if $\alpha_j > \alpha_{j+1}$. But our condition on $m$ ensures that in $C_m$ there is an entry smaller $j + 1$ above every entry in row $a_j + 1$ so there is no problem for the entries weakly larger than $j + 1$ in row $a_j + 1$. But the entries in $C_m$ in row $a_j + 1$ which are smaller than $j + 1$ have an entry smaller than itself in the box directly above itself because $C_n$ is semistandard. So $C_m$ has to be in fact an LR tableau. So every LR tableau of shape $A(n)$ is obtained from an LR tableau of shape $A(m)$ by adding $(n - m)\alpha_i$ entries to row $a_i$ ($1 \leq i \leq k$) and translating the above $a_1 - 1$ rows $(n - m)\alpha_1$ boxes to the right. So for $n \geq m$ we have $Q(n) = Q(m)$.

We now have to prove that $Q_{X_1^{\mu_1}, \mu'}^\lambda(m) > Q_{X_1^{\mu_1}, \mu'}^\lambda(m - 1) > \ldots > Q_{X_1^{\mu_1}, \mu'}^\lambda(0)$ if $\lambda'/\mu'$ is basic.

For $n < \frac{\lambda'_1 - \lambda'_1 + \mu'_1 - \mu'_1 - 1}{\alpha_k}$ we can construct an LR tableau of shape $A(n)$ containing less than $\alpha_k$ entries $k$ in row $a_k$. This gives $Q(n) > Q(n - 1)$.

So now suppose $\frac{\lambda'_1 - \lambda'_1 + \mu'_1 - \mu'_1 - 1}{\alpha_k} \leq n < \frac{\lambda'_1 - \lambda'_1 + \mu'_1 + \mu'_1 - \mu'_1 - 1}{\alpha_j - \alpha_{j+1}}$ for some $1 \leq j \leq k$ with $\alpha_j > \alpha_{j+1}$. We can construct an LR tableau $C_n$ of shape $A(n)$ with the following conditions:

- There are $\lambda'_1 - \mu'_1 - 1$ entries 1 in the upper $a_1 - 1$ rows of $C_n$ (this is possible because $\lambda'/\mu'$ is basic).
- There are $\lambda'_1 - \mu'_1 - 1$ entries $j$ in the upper $a_j - 1$ rows of $C_n$.
- There are $\lambda'_1 - \mu'_1 - 1$ entries $j + 1$ in row $a_j$ (the lower bound on $n$ ensures that there are enough boxes in row $a_j$).
- There are $x \geq \alpha_j$ entries $j$ in row $a_j$ and $x$ entries $j + 1$ below row $a_j$ (the upper bound on $n$ ensures that there are at least $x$ columns below row $a_j$ in which we can write the entry $j + 1$).
- There is no entry $j$ below row $a_j$.

So we have an LR tableau $C_n$ and removing from every row $a_i \alpha_i$ entries $i$ and translating the upper $a_1 - 1$ rows $\alpha_1$ boxes to the right yields a tableau $A_{n-1}$ which contains more entries $j + 1$ than entries $j$ and so is no LR tableau. This gives $Q(n) > Q(n - 1)$.

This proves $Q_{X_1^{\mu_1}, \mu'}^\lambda(m) > Q_{X_1^{\mu_1}, \mu'}^\lambda(m - 1) > \ldots > Q_{X_1^{\mu_1}, \mu'}^\lambda(0)$ in the case $\lambda = (\alpha_1^{a_1}, \alpha_2, \alpha_3, \ldots, \alpha_k), \mu = (\alpha_1^{a_1 - 1})$.

In the more general case there can be $i$ with $\mu_i = \lambda_i > \lambda_{i+1}$ and $\mu'_i < \lambda'_{i+1}$ (so the rows $i$ and $i + 1$ of $A(0) = \lambda'/\mu'$ are connected). We notice that for $n < \frac{\lambda'_{i+1} - \mu'_i}{\mu_i - \lambda_{i+1}}$ we can construct an LR tableau $C_n$ of shape $A(n)$ containing in row $i + 1 \mu'_i - \mu'_i + 1 + n(\mu_i - \mu_{i+1})$ times the entry 1. We notice furthermore that no LR tableau of shape $A(n - 1)$ can contain $\mu'_i - \mu'_i + 1 + n(\mu_i - \mu_{i+1}) - (\lambda_{i+1} - \mu_{i+1})$ entries 1 in row row $i + 1$ because there are not enough boxes in row $i + 1$ without a box directly atop. So we again have $Q(n) > Q(n - 1)$ for these $n$ and for the other $n$ we can specialize to the above case with $\lambda = (\alpha_1^{a_1}, \alpha_2, \alpha_3, \ldots, \alpha_k), \mu = (\alpha_1^{a_1 - 1})$.

Example 4.3. Let $\lambda' = (7^2, 5, 4^3, 3, 2^2), \mu' = (4, 3^3, 2), \lambda = (1^5), \mu = (1^2)$. So
and

\[ A(0) = \frac{\lambda'}{\mu'} = \begin{array}{c} \cdot \cdot \cdot \end{array} \quad A(1) = \begin{array}{c} \cdot \cdot \cdot \end{array} \]

\[ A(2) = \begin{array}{c} \cdot \cdot \cdot \end{array} \quad A(3) = \begin{array}{c} \cdot \cdot \cdot \end{array} \]

and by Lemma 4.2 we have for \( n \geq m = 7 \): \( Q(n) = Q(7) > Q(6) > \ldots > Q(0) \).

And in fact we have:

\[
\begin{array}{cccccccc}
Q(0) & Q(1) & Q(2) & Q(3) & Q(4) & Q(5) & Q(6) & Q(n \geq 7) \\
2184 & 26.421 & 92.030 & 172.795 & 229.660 & 254.420 & 260.761 & 261.512
\end{array}
\]

**Example 4.4.** Let \( \lambda = (6, 5, 3, 2, 1), \mu = (6, 1^4), \lambda' = (8^2, 5, 3^2, 2, 1) \) and \( \mu' = (4, 3, 2, 1^2) \). So

\[
\begin{array}{c}
\lambda/\mu = \end{array}
\]

and

\[ A(0) = \frac{\lambda'}{\mu'} = \begin{array}{c} \cdot \cdot \cdot \end{array} \quad A(1) = \begin{array}{c} \cdot \cdot \cdot \end{array} \]

\[ A(2) = \begin{array}{c} \cdot \cdot \cdot \end{array} \]

By Lemma 4.2 there exists an \( m \) with \( Q(n) = Q(m) \) for \( n \geq m \) but we cannot use the given formula. For \( n = 0 \) the skew diagram \( A(n) \) is connected, for \( 1 \leq n < 4 \) \( A(n) \) decays into 2 skew diagrams, one containing the upper 5 rows and one the rows below row 5. For \( 4 \leq n \) the skew diagram decays into 3 skew diagrams, one containing the topmost row, one containing the rows 2 to 5 and one containing the rows below. Deleting the empty columns in \( A(4) \) and ignoring the parts of \( \lambda/\mu \) which only translate the disconnected skew diagrams we can now use the formula on \( \tilde{A}(4) = (29, 25, 14, 8, 4, 2, 1)/(25, 4, 3, 2, 2) \) and \( \tilde{\lambda}/\mu = (4, 4, 2, 1)/(4) \) which gives
\[ \tilde{m} = 4. \] So in total we have for \( n \geq m = 8 = 4 + \tilde{m} \): \( Q(n) = Q(8) > Q(7) > \ldots > Q(0) \).

And in fact we have:

| \( Q(0) \) | \( Q(1) \) | \( Q(2) \) | \( Q(3) \) | \( Q(4) \) | \( Q(5) \) | \( Q(6) \) | \( Q(7) \) | \( Q(n \geq 8) \) |
|---|---|---|---|---|---|---|---|---|
| 910 | 18.271 | 38.016 | 49.635 | 54.176 | 55.480 | 55.826 | 55.889 | 55.895 |

### 5. Behaviour of \( P_{\lambda',\mu',\nu'}(\tilde{m}) \)

For \( c(\lambda; \mu, \nu), c(\lambda'; \mu', \nu') \neq 0 \) we define \( P_{\lambda',\mu',\nu'}^\lambda,n,\mu,n\nu}(n) = c(n\lambda + \lambda'; n\mu + \mu', n\nu + \nu') \) and write simply \( P(n) \) if \( \lambda, \mu, \nu, \lambda', \mu', \nu' \) are known from the context.

**Lemma 5.1.** Let \( c(\lambda; \mu, \nu) = 1, c(\lambda'; \mu', \nu') > 0 \). Let \( \lambda/\mu, \lambda/\nu \) or \((\lambda_1)^{(\lambda)}/\mu\), \(\nu\) be a partition or a rotated partition.

Then there exists an \( m \) with \( P_{\lambda',\mu',\nu'}^\lambda,n,\mu,n\nu}(n) = P_{\lambda',\mu',\nu'}^\lambda,n,\mu,n\nu}(m) \) for \( n \geq m \).

**Proof.** This follows directly from Lemma 4.2. \( \Box \)

**Remark 5.2.** We can use the formula in Lemma 4.2 to obtain an \( m \) with \( P(n) = P(m) \) for \( n \geq m \) but the \( m \) obtained by the formula in Lemma 4.2 doesn’t have to be minimal.

Many calculations suggest that Lemma 5.1 can be generalized:

**Conjecture 5.3.** Let \( f(n) = c(n\lambda; n\mu, n\nu) \) be a polynomial of degree \( d \). Let \( c(\lambda'; \mu', \nu') \neq 0 \).

Then there exists a polynomial \( g(n) \) of degree \( d \) and an integer \( m \) such that \( P_{\lambda',\mu',\nu'}^\lambda,n,\mu,n\nu}(n) = g(n) \) for \( n \geq m \).

In particular for \( c(\lambda; \mu, \nu) = 1 \) there exists an integer \( m \) with \( P(n) = P(m) \) for \( n \geq m \).

**Example 5.4.** Let \( \lambda = (6, 5, 4, 3^2, 1), \mu = (5, 3, 2, 1), \nu = (5, 3, 2, 1) \) then

\[ c(n\lambda; n\mu, n\nu) = \frac{(n + 1)(n + 2)(n + 3)(n + 4)(n + 5)(2n^2 + 5n + 7)}{840} \]

is of degree 7.

Let \( \lambda' = (9^3, 7, 3^4, 2, 1), \mu' = (7^2, 3, 2^3, 1^2), \nu' = (8, 5, 3^2, 2^2, 1) \). We then have

| \( n \) | 0 | 1 | 2 | 3 | 4 | \( n \geq 5 \) |
|---|---|---|---|---|---|---|
| \( P(n) \) | 39 | 30.920 | 509.202 | 3.101.626 | 12.098.348 | \( g(n) \) |
| \( g(n) \) | 55.407 | 50.333 | 513.782 | 3.102.223 | 12.098.382 | \( g(n) \) |

with

\[ g(n) = \frac{1}{390} (8490n^7 + 214.525n^6 + 1.664.232n^5 + 5.835.910n^4 + 904.140n^3 \\
+ 8.621.725n^2 - 19.075.662n + 19.946.520). \]

(We checked \( P(n) = g(n) \) for \( 5 \leq n \leq 17 \) by computer.)

**Acknowledgement:** John Stembridge’s “SF-package for maple” [Ste] and A. S. Buch’s “Littlewood-Richardson Calculator” [Bn2] were very helpful for computing examples. Furthermore my thanks go to Christine Bessenrodt, Ron King and Martin Rubey for helpful discussions. This paper was inspired by a talk given by Ron King at SLC 60 about stretched LR coefficients.
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