Nonuniqueness of Solutions of a Class of $\ell_0$-minimization Problems

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ABSTRACT
Recently, finding the sparsest solution of an underdetermined linear system has become an important request in many areas such as compressed sensing, image processing, statistical learning, and data sparse approximation. In this paper, we study some theoretical properties of the solutions to a general class of $\ell_0$-minimization problems, which can be used to deal with many practical applications. We establish some necessary conditions for a point being the sparsest solution to this class of problems, and we also characterize the conditions for the multiplicity of the sparsest solutions to the problem. Finally, we discuss certain conditions for the boundedness of the solution set of this class of problems.

KEYWORDS
$\ell_0$-minimization; Sparsity; Nonuniqueness; Boundedness.

1. Introduction

Let $\|x\|_0$ denote the number of nonzero components of the vector $x$ in this paper. We consider the following $\ell_0$-minimization problem:

$$
(P_0) \quad \min_{x \in R^n} \|x\|_0 \quad \text{s.t.} \quad \|y - Ax\|_2 \leq \epsilon, \quad Bx \leq b,
$$

where $A \in R^{m \times n}$ and $B \in R^{l \times n}$ are two matrices with $m \ll n$ and $l \leq n$, $y \in R^m$ and $b \in R^l$ are two given vectors, $\epsilon \geq 0$ is a given parameter, and $\|x\|_2 = (\sum_{i=1}^{n} |x_i|^2)^{1/2}$ is the $\ell_2$-norm of $x$. In compressed sensing (CS), the parameter $\epsilon$ is often used to estimate the level of the measurement error $e = y - Ax$. Clearly, the purpose of (1) is to find the sparsest point in the convex set $T$ defined by

$$
T = \{x : \|y - Ax\|_2 \leq \epsilon, Bx \leq b\}.
$$

The constraint $Bx \leq b$ is motivated by some practical applications which lets the model (1) be general enough to cover several sparsity models including a few models widely used in compressed sensing [1, 2, 5, 6], 1-bit compressed sensing [10, 12, 21], and statistical regression [11, 13, 14]. For example, some structured sparsity models,
including the nonnegative sparsity model \cite{2, 3, 9, 17} and the monotonic sparsity model (isotonic regression) \cite{15}, are the special cases of the model \eqref{model1}. Clearly, the following commonly used $\ell_0$-minimization models are also the special cases of \eqref{model1}:

\[
\begin{align*}
(C1) \quad & \min_x \{\|x\|_0 : \ y = Ax\}; \\
(C2) \quad & \min_x \{\|x\|_0 : \ \|y - Ax\|_2 \leq \varepsilon\}.
\end{align*}
\]

The problems (C1) and (C2) can be called the standard $\ell_0$-minimization problems \cite{3, 9, 17}.

From theory to computation methods, an intensive study of (C1) has been carried out over the past decade. Some sufficient criteria have been developed for the problem (C1) to have a unique sparsest solution, for example, the criteria based on the spark \cite{7}, mutual coherence \cite{8}, null space property (NSP) \cite{4}, restricted isotonic property (RIP) \cite{3}, exact recovery condition \cite{16}, and the range space property (RSP) \cite{17, 19, 20}. Zhao also \cite{18} developed several other sufficient conditions for the uniqueness of the solution to the problem (C1), such as sub-mutual coherence, scaled mutual coherence, coherence rank and sub-Babel function.

However, the above existing sufficient conditions are still very restrictive from a practical viewpoint. In practical signal recovery scenarios, the measured data is always inaccurate, in which case we use the sparsity model (C2) instead of (C1) or more complex ones such as the model \eqref{model1}. Different from (C1), the model \eqref{model1} involves a perturbation parameter $\varepsilon$. As a result, the uniqueness of the sparse solutions of \eqref{model1} might not be guaranteed, and hence it also makes sense to understand the conditions under which the model has multiple sparsest solutions. It is known that an $\ell_1$-minimization problem may solve (C1) under the NSP \cite{4} and RIP assumptions \cite{3} which ensures that the problem (C1) has a unique sparsest solution. However, Zhao \cite{19} has shown that even if an underdetermined linear system admits multiple sparsest solutions, the $\ell_1$-minimization problem is still able to solve (C1) under a mild RSP assumption which does not necessarily require the uniqueness of the sparsest solution of the problem. Therefore, in order to broadly understand the property of $\ell_0$-problems, it is meaningful to identify some conditions under which the $\ell_0$-problem has multiple solutions. To this goal, we characterize the necessary conditions for a vector to be the sparsest solution of the problem, and sufficient conditions for the multiplicity of the solutions of \eqref{model1}, and the condition for the solution set of \eqref{model1} to be bounded.

This paper is organized as follows. In Section \ref{sec2} we show some theoretical properties of the problem \eqref{model1} such as the necessary conditions for a point being the sparsest solution to the problem \eqref{model1}. Section \ref{sec3} gives some sufficient conditions for the nonuniqueness of the sparsest solutions of the problem \eqref{model1}. In Section \ref{sec4} we develop some sufficient conditions for the boundedness of the solution set of the problem \eqref{model1}.

**Notation**: The $\ell_p$-norm on $\mathbb{R}^n$ is defined as $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$, where $p \geq 1$. The field of real numbers is denoted by $\mathbb{R}$ and the $n$-dimensional Euclidean space is denoted by $\mathbb{R}^n$. The complementary set of $S \subseteq \{1, \ldots, n\}$ with respect to $\{1, \ldots, n\}$ is denoted by $\overline{S}$, i.e., $\overline{S} = \{1, \ldots, n\} \setminus S$. For a given vector $x \in \mathbb{R}^n$, $x_S$ and $|x|$ denotes the vector supported on $S$ and the vector with components $|x|_j = |x_j|$, $j = 1, \ldots, n$, respectively. Given a matrix $A$, $a_{i,j}$ denotes the entry of $A$ in row $i$ and column $j$. $A_S$ denotes the submatrix of $A \in \mathbb{R}^{m \times n}$ obtained by deleting the columns indexed by $\overline{S}$, and $A_{I,S}$ denotes the submatrix of $A$ with components $a_{i,j}$ for $i \in I$, $j \in S$. 

2. Necessary conditions for the solutions of \((P_0)\)

We first develop some necessary conditions for a point to be the solution of \((P_0)\), which are summarized in the following Theorem 2.1 and Theorem 2.2.

**Theorem 2.1.** If \(x^*\) is the sparsest solution to \((1)\) where \(A \in \mathbb{R}^{m \times n}\) and \(B \in \mathbb{R}^{l \times n}\) are two matrices with columns \(a_i (i = 1, 2, \ldots, n)\) and \(b_i (i = 1, 2, \ldots, n)\) respectively, then

\[
\text{Null}(A_S) \cap \text{Null}(B_S) = \{0\}, \tag{2}
\]

where \(S \subseteq \{1, 2, \ldots, n\}\) is the support set of \(x^*\).

**Proof.** Let \(x^*\) be the sparsest solution of \((1)\) and \(k\) be the optimal value of \((1)\). We prove this result by contradiction. If \(\text{Null}(A_S) \cap \text{Null}(B_S) \neq \{0\}\), there exists a nonzero vector \(\Delta x \in \mathbb{R}^n\) with \((\Delta x)_S \neq 0\) such that

\[
A_S (\Delta x)_S = 0 \quad \text{and} \quad B_S (\Delta x)_S = 0.
\]

Since \((\Delta x)_S \neq 0\), there is a nonzero component \((\Delta x)_j, j \in S\), such that the corresponding \(a_j\) and \(b_j\) can be represented as the linear combination of the other columns, that is,

\[
a_j = - \sum_{i \in S, i \neq j} a_i (\Delta x)_i, \quad b_j = - \sum_{i \in S, i \neq j} b_i (\Delta x)_i. \tag{3}
\]

Since \(x^*\) is feasible to the problem \((1)\), we have

\[
\left\| y - \left( \sum_{i \in S, i \neq j} a_i x^*_i \right) - \Delta x_j \right\|_2 \leq \epsilon, \quad \left( \sum_{i \in S, i \neq j} b_i x^*_i \right) + b_j x^*_j \leq b.
\]

Substituting \(a_j\) and \(b_j\) in \((3)\) into the above system yields

\[
\left\| y - \sum_{i \in S, i \neq j} \left( x^*_i - \frac{(\Delta x)_i}{(\Delta x)_j} x^*_j \right) a_i \right\|_2 \leq \epsilon, \quad \sum_{i \in S, i \neq j} \left( x^*_i - \frac{(\Delta x)_i}{(\Delta x)_j} x^*_j \right) b_i \leq b. \tag{4}
\]

The inequalities in \((4)\) imply that the vector \(\bar{x}\) with \(\|\bar{x}\|_0 \leq k - 1\) defined as

\[
\bar{x}_i = \begin{cases} 
  x^*_i - \frac{(\Delta x)_i}{(\Delta x)_j} x^*_j, & i \in S, i \neq j, \\
  0, & i = j, \\
  0, & i \notin S.
\end{cases}
\]

is a feasible solution of \((1)\). This means that \(\bar{x}\) is a solution of \((1)\) sparser than \(x^*\). This is a contradiction. The desired result follows. \(\square\)
Note that \( \text{Null}(A_S) \cap \text{Null}(B_S) = \{0\} \) means \[
\begin{bmatrix}
A \\
B_S
\end{bmatrix}
\] has full column rank. We make the following comments for the condition \( \text{Null}(A_S) \cap \text{Null}(B_S) = \{0\} \).

**Remark 1.** It can be seen that \( \text{Null}(A_S) \cap \text{Null}(B_S) = \{0\} \) has some equivalent forms. Since \( B_S x_S^* \leq b \) can be decomposed by active and inactive constraints, the following conditions can be regarded as the equivalent conditions for (2):

(i) \( \text{Null}(A_{B_I,S}) \cap \text{Null}(B_{I,S}) = \{0\}; \)

(ii) \( \text{Null}(A_{B_I,S}) \cap \text{Null}(B_{I,S}) = \{0\}; \)

(iii) \( \text{Null}(A_S) \cap \text{Null}(B_{I,S}) \cap \text{Null}(B_{I,S}) = \{0\}. \)

Here \( I \subseteq \{1,2,\ldots,m\} \) is the index set of active constraints in \( B_S x_S^* \leq b \) and \( I = \{1,2,\ldots,m\} \setminus I \) is the index set of inactive constraints in \( B_S x_S^* \leq b \).

Let \( |I(x)| \) be the cardinality of active constraints in \( Bx \leq b \) with respect to \( x \). Denote the sparsest solution set by

\[
\Lambda = \{x \in \mathbb{R}^n : \|x\|_0 = k, \ x \in T\}, \tag{5}
\]

where \( k \) is the optimal value of (1). From the above remark, we see that the condition (2) is equivalent to (ii) above. We may develop more specific necessary conditions than these conditions. For instance, in terms of maximum cardinality of \( I(x) \), we can prove the following result.

**Theorem 2.2.** Let \( x^* \) be a solution to (1) and \( S \) be the support of \( x^* \). If \( x^* \) admits the maximum cardinality of \( I(x), \ x \in \Lambda \), i.e., \( |I(x^*)| = \max\{|I(x)| : x \in \Lambda\} \), then

\[
M^* = \begin{bmatrix}
A_S \\
B_{I,S}
\end{bmatrix}
\tag{6}
\]

has full column rank where \( I = I(x^*) \).

**Proof.** Let \( x^* \) be a sparsest solution of (1) which satisfies the assumption in Theorem 2.2. We prove the result by contradiction. Assume that \( \text{Null}(M^*) \neq \{0\} \). Then there exists a nonzero vector \( \Delta x \) with \( (\Delta x)_S = 0 \) and \( (\Delta x)_S \neq 0 \) such that

\[
A_S(\Delta x)_S = 0 \quad \text{and} \quad B_{I,S}(\Delta x)_S = 0. \tag{7}
\]

Then we construct a new vector \( \tilde{x}(\lambda) \) such that

\[
\tilde{x}(\lambda) = x^* + \lambda(\Delta x)
\]

where \( \lambda \) is a parameter. Clearly, \( \tilde{x}(\lambda) \) continuously changes with \( \lambda \) and

\[
\text{supp}(\tilde{x}(\lambda)) \subseteq \text{supp}(x^*) \quad \text{and} \quad \|\tilde{x}(\lambda)\|_0 \leq \|x^*\|_0 \tag{8}
\]

for all \( \lambda \). If \( \tilde{x}(\lambda) \) satisfies the following system:

\[
\|y - A_S z_S\|_2 \leq \epsilon, \ B_{I,S} z_S \leq b_I, \ B_{I,S} z_S \leq b_I, \tag{9}
\]

then \( \tilde{x}(\lambda) \) is a feasible solution to (1), and hence \( \tilde{x}(\lambda) \) is a sparsest solution to (1) which follows from (8) and the fact that \( x^* \) is a sparsest solution. We now prove that
there exists a nonzero \( \lambda \) such that \( \bar{x}(\lambda) \) satisfies the system (9). Based on (7), the following two constraints are satisfied for all \( \lambda \):

\[
\|y - A_S \bar{x}_S(\lambda)\|_2 \leq \epsilon, \quad B_{I,S} \bar{x}_S(\lambda) = b_I.
\]  

(10)

We only need to check if \( \bar{x}(\lambda) \) satisfies the third inequality in (9). First we denote three disjoint sets \( J_+ , J_- , J_0 \) as follows,

\[
J_+ = \{ j : (B_{I,S}(\Delta x)_S)_j > 0 \}, \quad J_- = \{ j : (B_{I,S}(\Delta x)_S)_j < 0 \}, \quad J_0 = \{ j : (B_{I,S}(\Delta x)_S)_j = 0 \}.
\]  

(11)

Consider the following cases:

(M1) \( J_+ \cup J_- = \emptyset \). In this case \( B_{I,S}(\Delta x)_S = 0 \). Combining with (7) yields \( (\Delta x)_S \in \text{Null}(M^*) \cap \text{Null}(B_{I,S}) \). This contradicts to Theorem 2.1. Thus we have only the next case.

(M2) \( J_+ \cup J_- \neq \emptyset \). In this case \( B_{I,S}(\Delta x)_S \neq 0 \). Let \( \lambda \in [\lambda_{\min}, \lambda_{\max}] \) be continuously increased from \( \lambda_{\min} \) to \( \lambda_{\max} \) where

\[
\lambda_{\max} = \min_{j \in J_+} \left\{ \frac{(b_j - B_{I,S}x^*_S)_j}{(B_{I,S}(\Delta x)_S)_j} \right\}, \quad \lambda_{\min} = \max_{j \in J_-} \left\{ \frac{(b_j - B_{I,S}x^*_S)_j}{(B_{I,S}(\Delta x)_S)_j} \right\}.
\]

Clearly, due to (11), \( \lambda_{\min} < 0 \) and \( \lambda_{\max} > 0 \). For \( \lambda \in (0, \lambda_{\max}] \), we have that:

\[
(B_{I,S} \bar{x}_S(\lambda))_i \begin{cases} 
\leq (b_f)_i, & i \in J_+, \\
< (b_f)_i + \lambda * 0 = (b_f)_i, & i \in J_-,
\end{cases} \quad (b_f)_i < (b_f)_i,
\]

\[
i \in J_0.
\]

The above second and third inequalities are obvious, and the first inequality follows from the fact that for \( i \in J_+ , \)

\[
(B_{I,S} \bar{x}_S(\lambda))_i = (B_{I,S}x^*_S)_i + \lambda (B_{I,S}(\Delta x)_S)_i \\
\leq (B_{I,S}x^*_S)_i + \lambda_{\max} (B_{I,S}(\Delta x)_S)_i \\
\leq (B_{I,S}x^*_S)_i + \frac{(b_j - B_{I,S}x^*_S)_j}{(B_{I,S}(\Delta x)_S)_j}, (B_{I,S}(\Delta x)_S)_i = (b_f)_i.
\]

For \( \lambda \in [\lambda_{\min}, 0) \), we have that

\[
(B_{I,S} \bar{x}_S(\lambda))_i \begin{cases} 
< (b_f)_i + \lambda * 0 = (b_f)_i, & i \in J_+, \\
\leq (b_f)_i, & i \in J_-,
\end{cases} \quad (b_f)_i < (b_f)_i,
\]

\[
i \in J_0.
\]

where the second inequality follows from the fact that for \( i \in J_- , \)

\[
(B_{I,S} \bar{x}_S(\lambda))_i \leq (B_{I,S}x^*_S)_i + \lambda_{\min} (B_{I,S}(\Delta x)_S)_i \\
\leq (B_{I,S}x^*_S)_i + \frac{(b_j - B_{I,S}x^*_S)_j}{(B_{I,S}(\Delta x)_S)_j}, (B_{I,S}(\Delta x)_S)_i = (b_f)_i.
\]

Note that \( \bar{x}(\lambda) = x^* \) when \( \lambda = 0 \). Thus we have

\[
B_{I,S} \bar{x}_S(\lambda) \leq b_f
\]

for all \( \lambda \in [\lambda_{\min}, \lambda_{\max}] \). Combining this with (10), we see that \( \bar{x}(\lambda) \neq x^* \) for all \( \lambda \in [\lambda_{\min}, \lambda_{\max}] \) satisfying (9) and hence \( \bar{x}(\lambda) \) is a feasible solution to (11). Now starting
from $\lambda = 0$, we continuously increase the value $|\lambda|$. Thus, without loss of generality, we assume $\text{supp}(\tilde{x}(\lambda)) = \text{supp}(x^*)$ when $|\lambda|$ is increased continuously. Note that there exists a $\lambda^* \in [\lambda_{\text{min}}, \lambda_{\text{max}}]$ such that at least one index of inactive constraints in $B_Sx^*_S \leq b$ will be added to the index set of active constraints in $B_S\tilde{x}_S(\lambda^*) \leq b$. That is, the index set of active constraints in $B_S\tilde{x}_S(\lambda^*) \leq b$ includes $I$ and $D$:

$$I(\tilde{x}(\lambda^*)) = I \cup D,$$

where $D = \{ j : (B_{I,S} \tilde{x}_S(\lambda^*))_j = (b_j)_j \}, \ D \neq \emptyset.$

This means $|I(\tilde{x}(\lambda^*))| > |I(x^*)|$ which contradicts the fact that $I(x^*)$ has the maximum cardinality of $I(x)$ amongst all sparsest solutions of (11). This contradiction shows that $M^*$ given in (6) has full column rank.

3. Multiplicity of sparsest solutions of ($P_0$)

The sparsest solutions of (11) might not be unique when the null space of $(A^T, B^T)^T$ is not reduced to the zero vector. In fact, any slight perturbation of the problem data $(A, B, b, y, \epsilon)$ may lead to the nonuniqueness of the solutions to the modified problem. This means that in most cases, the sparsest solutions for the problem (11) are non-unique. In this section, we show that (11) has infinitely many solutions under some mild conditions. Let $x^*$ be a sparsest solution to (11). From Theorem 2.1 we know that

$$\text{Null}(A_S) \cap \text{Null}(B_S) = \{0\}, \quad (12)$$

which can be separated into four cases:

$$\begin{cases}
\text{Null}(A_S) \neq \{0\}, \text{ Null}(B_{I,S}) = \{0\}, \\
\text{Null}(A_S) \neq \{0\}, \text{ Null}(B_{I,S}) \neq \{0\}, \text{ Null}(B_S) = \{0\}, \\
\text{Null}(A_S) = \{0\}, \text{ Null}(B_{I,S}) \neq \{0\}, \\
\text{Null}(A_S) = \{0\}, \text{ Null}(B_{I,S}) = \{0\},
\end{cases} \quad (13)$$

where $I$ and $\bar{I}$ are the index sets of active and inactive constraints in $B_Sx^*_S \leq b$ respectively. Under some conditions, it can be shown that for each case in (13), (11) has infinite sparsest solutions admitting the same support as that of the sparsest solution $x^*$, as indicated by the following Theorems 3.1 and 3.2. Theorem 3.1 covers the first three cases and Theorem 3.2 covers the last case in (13) respectively.

**Theorem 3.1.** Let $x^*$ be an arbitrary sparsest solution to (11) and $S$ be the support of $x^*$. The problem (11) has infinitely many optimal solutions which have the same support as $x^*$ if the following condition (C1) holds:

- (C1) $\text{Null}(A_S) \cap \text{Null}(B_S) = \{0\}$ and $x^*$ does not admit the maximum cardinality, i.e., $|I(x^*)| \neq \max\{|I(z)| : z \in \Lambda\}$ where $\Lambda$ is given in (5).

If the corresponding error vector $e^*$, i.e., $e^* = y - Ax^*$, satisfies $\|e^*\|_2 < \epsilon$, then (11) has infinitely many optimal solutions which have the same support as $x^*$ if one of the following conditions (C2), (C3) and (C4) holds:

- (C2) $\text{Null}(M^*) = \{0\}$ and $\text{Null}(B_S) \neq \{0\}$. 

$$\text{Null}(A_S) \cap \text{Null}(B_S) = \{0\}, \quad (12)$$

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(C3) \( \text{Null}(M^*) = \{0\} \) and \( \{d : B_{I,S}d > 0\} \cap \text{Null}(B_{I,S}) \neq \emptyset \).

(C4) \( \text{Null}(M^*) = \{0\} \) and \( \{d : B_{I,S}d < 0\} \cap \text{Null}(B_{I,S}) \neq \emptyset \).

**Proof.** (C1) Consider the case (C1) in Theorem 3.1 We can find a nonzero \( d \) such that \( d_S \in \text{Null}(M^*) \) and \( d_S = 0 \), leading to

\[
A_Sd_S = 0 \text{ and } B_{I,S}d_S = 0.
\]

Due to (12), we know that \( B_{I,S}d_S \neq 0 \). Let \( z(\lambda) \) be a vector which is constructed as

\[
z(\lambda) = x^* + \lambda d
\]

where \( \lambda \) is a parameter. It is easy to check that \( z_S(\lambda) \) satisfies

\[
\|y - A_Sz_S(\lambda)\|_2 \leq \epsilon, \quad B_{I,S}z_S(\lambda) = b_I.
\]

Let the sets \( J_+, J_- \) and \( J_0 \) be still defined as the corresponding sets in (11) by replacing \( (\Delta x)_S \) with \( d_S \). Let \( \lambda \) be restricted in \( [\lambda_{\min}, \lambda_{\max}] \) where

\[
\lambda_{\max} = \min_{j \in J_+} \left\{ \frac{(b_I - B_{I,S}x^*_S)_j}{(B_{I,S}d_S)_j} \right\}, \quad \lambda_{\min} = \max_{j \in J_-} \left\{ \frac{(b_I - B_{I,S}x^*_S)_j}{(B_{I,S}d_S)_j} \right\}.
\]

Similar to the case (M2) in the proof of Theorem 2.2, it can be proven that for all \( \lambda \in [\lambda_{\min}, \lambda_{\max}] \), we have \( B_{I,S}z_S(\lambda) \leq b_I \). Then \( z(\lambda) \) is a feasible solution to (1) when \( \lambda \in [\lambda_{\min}, \lambda_{\max}] \), which together with the fact that \( x^* \) is a sparsest solution and \( \text{supp}(z(\lambda)) \subseteq \text{supp}(x^*) \), implies for all \( \lambda \in [\lambda_{\min}, \lambda_{\max}] \) \( z(\lambda) \) is a sparsest solution of (1) and hence

\[
\text{supp}(x^*) = \text{supp}(z(\lambda)).
\]

Since \( z(\lambda) \) varies when \( \lambda \) is changed continuously in the interval \( [\lambda_{\min}, \lambda_{\max}] \), it implies that (1) has infinitely many sparsest solutions with the same support as \( x^* \).

(C2) Consider the case (C2) in Theorem 3.1. We choose a nonzero vector \( \mu \) from the set \( \text{Null}(B_S) \). Due to (12), we have \( A_S\mu \neq 0 \). Let \( t(\lambda) \) be a vector with components

\[
t_S(\lambda) = x^*_S + \lambda \mu, \quad t_S(\lambda) = 0.
\]

Then we have \( B_{I,S}t_S(\lambda) < b_I \) and \( B_{I,S}t_S(\lambda) = b_I \) for all \( \lambda \) which imply \( B_S t_S(\lambda) \leq b \). Let \( |\lambda| \) be restricted in \((0, \lambda_{\max}')\) with

\[
\lambda_{\max}' = \frac{\epsilon - \|e^*\|_2}{\|A_S\mu\|_\infty \sqrt{m}}.
\]

and \( e^* = y - A_S x^*_S \). We have

\[
\|y - A_S(x^*_S + \lambda \mu)\|_2 = \|e^* - \lambda A_S \mu\|_2,
\]

\[
\leq \|e^*\|_2 + |\lambda| \|A_S \mu\|_2 \leq \|e^*\|_2 + \lambda_{\max}' \|A_S \mu\|_2,
\]

\[
= \|e^*\|_2 + \frac{\epsilon - \|e^*\|_2}{\sqrt{m}} \|A_S \mu\|_\infty \|A_S \mu\|_2,
\]

\[
\leq \|e^*\|_2 + \frac{\epsilon - \|e^*\|_2}{\sqrt{m}} \|A_S \mu\|_\infty = \epsilon,
\]

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where the first inequality follows from the triangle inequality and $e^m$ is the vector of ones with $m$ dimension. Combining this with the fact $B_S t_S(\lambda) \leq b$ implies that $t(\lambda)$ is a feasible solution of \((\text{I})\) when $\lambda \in [0, \lambda'_{\text{max}}]$. Same as the proof in \((C1)\), it implies that $t(\lambda)$ is the sparsest solution of \((\text{I})\) when $\lambda \in [0, \lambda'_{\text{max}}]$, and hence we obtain the desired result. Moreover, the active and inactive indices in $B t(\lambda) \leq b$ are the same as that in $B x^* \leq b$.

\((\text{C3})\) Consider the case \((\text{C3})\) in Theorem \(3.1\). We can find a nonzero vector $\xi$ from the set \(\{ d : B_{I,S} d > 0 \} \cap \text{Null}(B_{I,S})\) satisfying

$$B_{I,S} \xi = 0 \text{ and } B_{I,S} \xi > 0.$$ 

Since the two cases $A_S \xi = 0$ and $A_S \xi \neq 0$ do not contradict $\text{Null}(M^*) = \{0\}$, we consider both of them. Let $v(\lambda)$ be a vector with components

$$v_S(\lambda) = x^*_S + \lambda \xi \text{ and } v(\lambda) = 0,$$

where $\lambda$ is a parameter. Clearly, $\text{supp}(v(\lambda)) \subseteq \text{supp}(x^*)$ for $\lambda$. Now we claim that $v(\lambda)$ is a sparsest solution to \((\text{I})\) in both cases of $A_S \xi = 0$ and $A_S \xi \neq 0$ when $\lambda$ is restricted in certain interval.

1) $A_S \xi \neq 0$. When $\lambda \in [-\lambda''_{\text{max}}, 0]$ with $\lambda''_{\text{max}} = \frac{\epsilon - \| e^* \|_\infty}{\| \lambda \|_\infty \sqrt{m}}$, by the same proof as in \((C2)\), we have

$$\| y - A_S v_S(\lambda) \|_2 \leq \epsilon.$$ 

It is easy to check that

$$B_{I,S} v_S(\lambda) < b_I \text{ and } B_{I,S} v_S(\lambda) < b_f.$$ 

Thus $v(\lambda)$ is a feasible point in $T$ for all $\lambda \in [-\lambda''_{\text{max}}, 0]$, $\text{supp}(v(\lambda)) \subseteq \text{supp}(x^*)$ and the fact that $x^*$ is a sparsest point in $T$ imply that $v(\lambda)$ is a sparsest point in $T$ when $\lambda \in [-\lambda''_{\text{max}}, 0]$.

2) $A_S \xi = 0$. Here $\lambda$ can be any negative number so that $v(\lambda)$ is a feasible point in $T$. Similarly, $v(\lambda)$ is a sparsest solution to \((\text{I})\) when $\lambda \leq 0$. Combining 1) and 2) implies the desired result.

\((\text{C4})\) This proof is omitted. Note that \(\{ d : B_{I,S} d > 0 \} \cap \text{Null}(B_{I,S}) \neq \emptyset\) is equivalent to \(\{ d : B_{I,S} d < 0 \} \cap \text{Null}(B_{I,S}) \neq \emptyset\). Thus we can directly get the desired result. \(\square\)

It follows from Theorem \(2.2\) that the linear dependence of the columns of $M^*$ implies that $I(x^*)$ does not have the maximum cardinality amongst $I(x), x \in \Lambda$. Therefore the condition in \((C1)\) is mild. Note that the case \((C1)\) corresponds to the first two cases in \((\text{I3})\), and the cases \((C2) - (C4)\) correspond to the third case in \((\text{I3})\). Now we consider the last case in \((\text{I3})\) and have the following theorem.

**Theorem 3.2.** Let $x^*$ be an arbitrary sparsest solution of \((\text{I})\). $S$ be the support of $x^*$. Assume that $\text{Null}(M^*) = \{0\}$ and $\text{Null}(B_{I,S}) = \{0\}$. Then \((\text{I})\) has infinitely many optimal solutions with the same support as $x^*$ if one of the following conditions holds:

- \((D1)\) \(\{ d : B_{I,S} d > 0 \} \cap \{ d : A_S d = 0 \} \neq \emptyset\).
- \((D2)\) \(\{ d : B_{I,S} d < 0 \} \cap \{ d : A_S d = 0 \} \neq \emptyset\).
If the corresponding error vector $e^*$, i.e., $e^* = y - Ax^*$, satisfies $\|e^*\|_2 < \epsilon$, then (1) has infinitely many optimal solutions which have the same support as $x^*$ if one of the following conditions holds:

- (D3) $\text{Null}(B_{I,S}) \neq \{0\}$.
- (D4) $\{d : B_{I,S}d > 0\} \cap \{d : A_Sd \neq 0\} \neq \emptyset$.
- (D5) $\{d : B_{I,S}d < 0\} \cap \{d : A_Sd \neq 0\} \neq \emptyset$.

**Proof.** We start from (D3).

**(D3)** Since $\text{Null}(M^*) = \{0\}$ and $\text{Null}(B_{I,S}) \neq \{0\}$, for $\forall \tilde{d} \in \text{Null}(B_{I,S})$, we have

$$B_{I,S}\tilde{d} = 0 \text{ and } A_S\tilde{d} \neq 0.$$ 

Since $\text{Null}(B_{I,S}) = \{0\}$, we have $B_{I,S}\tilde{d} \neq 0$. Denote

$$G_0 = \{j : (B_{I,S}\tilde{d})_j = 0\}, \quad G_- = \{j : (B_{I,S}\tilde{d})_j < 0\}, \quad G_+ = \{j : (B_{I,S}\tilde{d})_j > 0\}.$$ 

Clearly, $G_+ \cup G_- \neq \emptyset$. Let $\bar{z}(\lambda)$ be a vector with components

$$\bar{z}_S(\lambda) = x^* + \lambda \tilde{d} \text{ and } \bar{z}_S(\lambda) = 0.$$ 

Clearly, $\text{supp}(\bar{z}(\lambda)) \subseteq \text{supp}(x^*)$ for all $\lambda$. Let $|\lambda|$ be restricted in $(0, \min(\lambda_1, \lambda_2)]$ where

$$\lambda_1 = \min_{j \in G_+ \cup G_-} \frac{(b_j - B_{I,S}x^*_S)_j}{|(B_{I,S}\tilde{d})|_j}, \quad \lambda_2 = \frac{\epsilon - \|e^*\|_2}{\|A_Sd\|_\infty \sqrt{m}}.$$ 

For $i \in G_+ \cup G_-,$

$$(B_{I,S}\bar{z}_S(\lambda))_i = (B_{I,S}x^*_S)_i + \lambda (B_{I,S}\tilde{d})_i \leq (B_{I,S}x^*_S)_i + |\lambda|(B_{I,S}\tilde{d})_i \leq (B_{I,S}x^*_S)_i + \lambda_1(B_{I,S}\tilde{d})_i \leq (B_{I,S}x^*_S)_i + \frac{(b_j - B_{I,S}x^*_S)_j}{|(B_{I,S}\tilde{d})|_i}|(B_{I,S}\tilde{d})_i| = (b_j)_i.$$ 

The above fact, combined with $(B_{I,S}\bar{z}_S(\lambda))_i < (b_j)_i, i \in G_0,$ implies that $B_{I,S}\bar{z}_S(\lambda) \leq b_I.$ We also have $\|y - A_S\bar{z}_S(\lambda)\|_2 \leq \epsilon$ which has been proven for many times in Theorem 3.1. These, combined with the fact that $B_{I,S}\bar{z}_S(\lambda) = b_I,$ implies that $\bar{z}(\lambda)$ is a sparsest point in $T$ with the same support as $x^*$ when $\lambda \in [0, \min(\lambda_1, \lambda_2)].$

**(D4)** Clearly, there exists a nonzero vector $d'$ such that

$$B_{I,S}d' > 0, \quad A_Sd' \neq 0.$$ 

Since $\text{Null}(B_{I,S}) = \{0\},$ we have $B_{I,S}d' \neq 0.$ Denote

$$J_0' = \{j : (B_{I,S}d')_j = 0\}, \quad J_- = \{j : (B_{I,S}d')_j < 0\}, \quad J_+ = \{j : (B_{I,S}d')_j > 0\}.$$ 

Clearly, $J_+ \cup J_- \neq \emptyset$. Let $z'(\lambda)$ be a vector with components $z'_S(\lambda) = x^*_S + \lambda d'$ and
Due to $J_z$ is equivalent to (D4). Thus the desired results can be obtained immediately.

Let $\lambda$ be restricted in $\max(\lambda_1', \lambda_2')$, we have

$$\lambda_1' = \max_{j \in J_1'} \frac{(b_I - B_{I,S}x^*_S)_j}{(B_{I,S}d''_j)_j}, \quad \lambda_2' = -\frac{(\varepsilon - \|e^*\|_2)}{\|Ad''\|_{\infty} \sqrt{m}}.$$ 

For $i \in J_1'$, we have

$$(B_{I,S}z'_S(\lambda))_i = (B_{I,S}x^*_S)_i + \lambda(B_{I,S}d''_i) \leq (B_{I,S}x^*_S)_i + \lambda_i'(B_{I,S}d''_i),$$

$$\leq (B_{I,S}x^*_S)_i + \frac{(b_I - B_{I,S}x^*_S)_i}{(B_{I,S}d''_i)_i}(B_{I,S}d''_i)_i = (b_I)_i.$$ 

For $i \in J_+ \cup J_0'$, we have $(B_{I,S}z'_S(\lambda))_i < (b_I)_i$. It can be proven that $\|y - A_Sz'_S(\lambda)\|_2 \leq \varepsilon$ for $\lambda \in [\max(\lambda_1', \lambda_2'), 0)$, which combined with the fact $B_{I,S}z'_S(\lambda) < b_I$ implies that $z'(\lambda)$ is a sparsest point in $T$ with the same support as $x^*$ when $\lambda \in [\max(\lambda_1', \lambda_2'), 0]$, i.e., $\text{supp}(x^*) = \text{supp}(z'(\lambda))$.

(D1) Clearly, there exists a nonzero vector $d''$ such that

$$B_{I,S}d'' > 0, \quad A_Sd'' = 0.$$ 

Since $\text{Null}(B_{I,S}) = \{0\}$, we have $B_{I,S}d'' \neq 0$. Denote

$$J''_0 = \{j : (B_{I,S}d'')_j = 0\}, \quad J''_+ = \{j : (B_{I,S}d'')_j < 0\}, \quad J''_+ = \{j : (B_{I,S}d'')_j > 0\}.$$ 

Clearly, $J''_+ \cup J''_+ \neq \emptyset$. Let $z''(\lambda)$ be a vector with components

$$z''_S(\lambda) = x^*_S + \lambda d''$$

and $z''_S(\lambda) = 0$.

Due to $A_Sd'' = 0$, $\|y - A_Sz''_S(\lambda)\|_2 \leq \varepsilon$ is satisfied. Let $\lambda$ be restricted in $[\lambda''_1, 0)$ where

$$\lambda''_1 = \max_{j \in J''_0} \frac{(b_I - B_{I,S}x^*_S)_j}{(B_{I,S}d''_j)_j}.$$ 

Similar to the proof of $B_{I,S}z''_S(\lambda) < b_I$ in (D4), we have $B_{I,S}z''_S(\lambda) < b_I$. The fact $B_{I,S}z''_S(\lambda) < b_I$ and $\|y - A_Sz''_S(\lambda)\|_2 \leq \varepsilon$ implies that $z''(\lambda)$ is a sparsest point in $T$ with the same support as $x^*$ when $\lambda \in [\lambda''_1, 0]$, i.e., $\text{supp}(x^*) = \text{supp}(z''(\lambda))$.

(D2) The proof is omitted. Note that (D2) is equivalent to (D1) and that (D5) is equivalent to (D4). Thus the desired results can be obtained immediately. 

Through the above theoretical analysis, we know that (1) may have infinitely many sparsest solutions. We also want to know whether the sparsest solution set $\Lambda$ given in (5) is bounded or not. This question will be explored in Section 3. The example below is given to illustrate the results of Theorems 3.1 and 3.2.

Example 3.3. Consider the system $\|y - Ax\|_2 \leq \varepsilon$, $Bx \leq b$ with $\varepsilon = 10^{-1}$, where

$$A = \begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 4 & -9 \\ 1 & 0 & -2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -0.5 & 0 & 1 & -2.5 \\ 0.5 & -0.5 & -1 & 2 \\ -3 & -3 & -2 & 3 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} -0.5 \\ 1 \\ 1 \end{bmatrix}.$$ 

It can be seen that $(0, 0, 2, 1)^T$ and $(0, 1, -1/2, 0)^T$ are the sparsest solutions to the above convex system. Next, we show that the above two sparsest solutions satisfy some assumptions in Theorems 3.1 and 3.2.
(i) \( x = (0, 0, 2, 1)^T \): We have \( A_S = \begin{bmatrix} -2 & 5 \\ 4 & -9 \\ -2 & 5 \end{bmatrix}, B_{I,S} = \begin{bmatrix} 1 & -2.5 \\ -2 & 3 \end{bmatrix} \) and \( B_{I,S} = \begin{bmatrix} -1 & 2 \end{bmatrix} \). We can see that
\[
\text{Null}(A_S) = \{0\}, \text{Null}(B_{I,S}) = \{0\}, \text{Null}(B_{I,S}) \neq \{0\},
\]
and
\[
(2, 1)^T \in \{d : B_{I,S}d < 0\} \cap \text{Null}(B_{I,S}), \ (-2, -1)^T \in \{d : B_{I,S}d > 0\} \cap \text{Null}(B_{I,S})
\]
which satisfy (C4) and (C3) in Theorem 3.1. The value of \( \lambda \) in the proof of (C4) or (C3) can be determined, i.e.,
\[
\lambda \in (0, 1/10\sqrt{3}] \text{ for } (2, 1)^T, \lambda \in [-1/10\sqrt{3}, 0) \text{ for } (-2, -1)^T.
\]
Then another sparsest solution can be formed as
\[
(0, 0, 2, 1)^T + \lambda (0, 0, 2, 1)^T, \lambda \in (0, 1/10\sqrt{3}]
\]
and hence the system \( T \) in this example has infinitely many sparsest solutions.

(ii) \( x = (0, 1, -1/2, 0)^T \): We have \( A_S = \begin{bmatrix} 0 & -2 \\ 1 & 4 \\ 0 & -2 \end{bmatrix}, B_{I,S} = (0, 1) \) and \( B_{I,S} = \begin{bmatrix} -0.5 & -1 \\ -3 & -2 \end{bmatrix} \). It is easy to check
\[
\text{Null}(A_S) = \text{Null}(B_{I,S}) = \{0\} \quad \text{and} \quad \text{Null}(B_{I,S}) \neq \{0\}
\]
so that this example satisfies \( \text{Null}(M^*) = \{0\} \) and \( \text{Null}(B_{I,S}) = \{0\} \). We can find two vectors which meet (D5) and (D4) in Theorem 3.2, i.e.,
\[
(4, -1)^T \in \{d : B_{I,S}d < 0\} \cap \{d : A_Sd \neq 0\}, \ (-4, 1)^T \in \{d : B_{I,S}d > 0\} \cap \{d : A_Sd \neq 0\}.
\]
Then the value of \( \lambda \) in the proof of (D5) or (D4) can be determined. Analogously, for all \( \lambda \in [\max(-1/10, -1/20\sqrt{3}), 0] \), the vector \((0, 1, -1/2, 0)^T + \lambda (0, -4, 1, 0)^T\) is a sparsest point in \( T \). Note that \( \text{Null}(B_{I,S}) \neq \{0\} \), which also meets (D3) in Theorem 3.2. We can find \((1, 0)^T \in \text{Null}(B_{I,S})\), and therefore \( \lambda_1 \) and \( \lambda_2 \) in the proof of (D3) can be determined. Consequently, for all \( \lambda \) such that \(|\lambda| \in [0, 1/10\sqrt{3}] \), the vector \((0, 1, -1/2, 0)^T + \lambda (0, 1, 0, 0)^T\) is a sparsest point in \( T \).

4. Boundedness of the solution set of \((P_0)\)

In this section, some sufficient conditions for the boundedness of the solution set \( \Lambda \) of \((P_0)\) are also identified. We start to discuss the lower bound on the absolute value of nonzero components of vectors in \( \Lambda \) given in \([5]\). We only consider the case that \( \Lambda \) is bounded.
Lemma 4.1. Let $k$ be the optimal value of (1). If the solution set $\Lambda$ is bounded, then there exists a positive lower bound $\gamma^*$ for the nonzero component $|x_i|$ of any vector $|x|$, $x \in \Lambda$, i.e.,

$$|x_i| \geq \gamma^*, \ i \in \text{supp}(x). \quad (14)$$

Proof. We prove this result by considering only two situations: $\Lambda$ is finite or infinite.

(i) Let the set $\Lambda$ be finite and bounded. Denote the cardinality of $\Lambda$ as $L$ and the sparsest solutions of (1) as $\{x^p\}$, where $1 \leq p \leq L$. Obviously, we can find the minimum value among the nonzero absolute entries of all vectors in $\Lambda$ and set such a minimal value as $\gamma^*$, which is expressed as

$$\gamma^* = \min_{1 \leq p \leq L} \min_{i \in \text{supp}(x^p)} |x^p_i|. \quad (15)$$

This implies that the absolute values of the nonzero components of vectors in $\Lambda$ have a positive lower bound $\gamma^*$.

(ii) Let the set $\Lambda$ be infinite and bounded. In this case, $L$ is an infinite number. Since $\Lambda$ is bounded, there exists a positive number $U$ such that the absolute value of all entries of vectors in $\Lambda$ is less or equal than $U$. We assume that (14) does not hold for $x \in \Lambda$. This means there exists a sequence $\{x^p\} \in \Lambda$, such that the minimum nonzero absolute entries of $x^p$ approach to 0, i.e.,

$$\min_{i \in \text{supp}(x^p)} |x^p_i| \rightarrow 0 \ \text{as} \ p \rightarrow \infty.$$ 

Since $\Lambda$ is bounded, this implies that

$$|x^p_i| \leq U, \ i \in \text{supp}(x^p).$$

Following by Bolzano-Weierstrass Theorem, the sequence $\{x^p\}$ has at least one convergent subsequence, denoted still by $\{x^p\}$, with a limit point $x^* \in \Lambda$ satisfying $\|x^*\|_0 \leq k - 1$. This is a contradiction, and hence the lower bound is ensured when $\Lambda$ is infinite and bounded. Combining (i) and (ii) obtains the desired result.

The above lemma ensures the existence of a positive lower bound for the absolute value of the nonzero components of the vectors in $\Lambda$ when $\Lambda$ is bounded. In the following lemma, some sufficient conditions are developed to guarantee the boundedness of $\Lambda$.

Lemma 4.2. Let $k$ be the optimal value of (1). The sparse solution set $\Lambda$ is bounded if one of the following conditions holds:

- (E1) For any $\Pi \subseteq \{1, \ldots, n\}$ and $|\Pi| = k$, we have

  $$\{\eta : A_{\Pi}\eta = 0\} \cap \{\eta : B_{\Pi}\eta \leq 0\} = \{0\}. \quad (15)$$

- (E2) Any $k$ columns in $A$ are linearly independent.

- (E3) $k < \text{spark}(A)$, where $\text{spark}(A)$ denote the minimum number of linearly dependent columns in $A$.

Proof. First of all, we suppose that the set $\Lambda$ is unbounded. There exists a sequence
of the sparsest solutions of (1), denoted by \( \{x^p\} \), satisfying the following properties:

\[ \|x^p\|_\infty \to \infty \quad \text{as} \quad p \to \infty \]

and there is a fixed index set \( S_1 (|S_1| \leq k) \) such that

\[ |x^p_i| \to \infty \quad \text{for all} \quad i \in S_1, \quad \text{as} \quad p \to \infty \]

and the remaining components \( x^p_i, \quad i \in S_2 = \text{supp}(x^p) \setminus S_1 \) are bounded. Based on the fact that \( x^p \) satisfies the constraints in (1), we have

\[ \|A_{S_2}x^p_{S_2} + A_{S_1}x^p_{S_1} - y\|_2 \leq \epsilon, \quad B_{S_2}x^p_{S_2} + B_{S_1}x^p_{S_1} \leq b. \]

We divide the above two inequalities by \( \|x^p_{S_1}\|_2 \) to obtain

\[ \frac{\|A_{S_2}x^p_{S_2} + A_{S_1}x^p_{S_1} - y\|_2}{\|x^p_{S_1}\|_2} \leq \frac{\epsilon}{\|x^p_{S_1}\|_2}, \quad \frac{B_{S_2}x^p_{S_2} + B_{S_1}x^p_{S_1}}{\|x^p_{S_1}\|_2} \leq \frac{b}{\|x^p_{S_1}\|_2}. \]

Then we have

\[ \frac{\|A_{S_2}x^p_{S_2}\|_{S_1} + A_{S_1}y - \|x^p_{S_1}\|_2}{\|x^p_{S_1}\|_2} \leq \frac{\epsilon}{\|x^p_{S_1}\|_2}, \quad \frac{B_{S_2}x^p_{S_2} + B_{S_1}y}{\|x^p_{S_1}\|_2} \leq \frac{b}{\|x^p_{S_1}\|_2}, \]

where \( \bar{\eta} \) is a unit vector in \( \mathbb{R}^{\|S_1\|} \). Note that

\[ \lim_{p \to \infty} \frac{x^p_{S_2}}{\|x^p_{S_1}\|_2} = 0, \quad \lim_{p \to \infty} \frac{y}{\|x^p_{S_1}\|_2} = 0, \quad \lim_{p \to \infty} \frac{b}{\|x^p_{S_1}\|_2} = 0, \quad \lim_{p \to \infty} \frac{\epsilon}{\|x^p_{S_1}\|_2} = 0. \]

Thus there exists a unit vector \( \bar{\eta} \in \mathbb{R}^{\|S_1\|} \) satisfying

\[ A_{S_1} \bar{\eta} = 0, \quad B_{S_1} \bar{\eta} \leq 0. \]

This means

\[ \{ \eta : A_{S_1} \eta = 0 \} \cap \{ \eta : B_{S_1} \eta \leq 0 \} \neq \{0\}. \]

which contradicts to the assumption (15). Thus under (15), \( \Lambda \) is bounded. It is clear that if any \( k \) columns of \( A \) are linearly independent or \( k < \text{spark}(A) \), then the set \( \{ \eta : A_{\Pi} \eta = 0 \} = \{0\} \) and thus (15) holds. Hence the second and third conditions in Lemma 4.2 can also ensure \( \Lambda \) to be bounded.

5. Conclusion

In this paper, some basic properties of the solutions of (1) are developed such as the necessary conditions for a point being the sparsest point in the feasible set of (1). Some sufficient conditions for the nonuniqueness of the sparsest solutions of (1) are also
developed. We also discussed the boundedness of the solution set of (1) under certain conditions. Based on this, a positive lower bound for the absolute nonzero entries of the solutions to (1) can be guaranteed when the solution set of (1) is bounded. These results can be applied to a class of \( \ell_0 \)-problems such as the standard \( \ell_0 \)-minimization problems (C1) and (C2), and even some structured sparsity models.

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