REGULARITY OF LIMIT SETS OF ANOSOV REPRESENTATIONS

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Abstract. In this paper we establish necessary and sufficient conditions for the limit set of a projective Anosov representation to be a $C^\alpha$-submanifold of projective space for some $\alpha \in (1, 2)$. We also calculate the optimal value of $\alpha$ in terms of the eigenvalue data of the Anosov representation.

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1. INTRODUCTION

Suppose that $\mathbb{H}^d_\mathbb{R}$ is real hyperbolic $d$-space. Let $\partial_\infty \mathbb{H}^d_\mathbb{R}$ denote the geodesic boundary of $\mathbb{H}^d_\mathbb{R}$ and let $\text{Isom}(\mathbb{H}^d_\mathbb{R})$ denote the isometry group of $\mathbb{H}^d_\mathbb{R}$. Given a representation $\rho : \Gamma \to \text{Isom}(\mathbb{H}^d_\mathbb{R})$, the limit set of $\rho$ is defined to be

$$L_\rho = \overline{\rho(\Gamma) \cdot x_0 \cap \partial_\infty \mathbb{H}^d_\mathbb{R}}$$

where $x_0 \in \mathbb{H}^d_\mathbb{R}$ is any point. If we further assume that $\Gamma$ is a hyperbolic group and $\rho$ is a convex co-compact representation, then there is a $\rho$-equivariant, continuous map from $\partial_\infty \Gamma$, the Gromov boundary of $\Gamma$, to the limit set $L_\rho$. The limit set in this

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setting is generically very irregular, for instance when $\partial_\infty \Gamma$ is a topological manifold Yue [Yue96] proved: unless $\rho$ is a co-compact action on a totally geodesic subspace of $\mathbb{H}_g^d$, its limit set is fractal like, and in particular, has Hausdorff dimension strictly greater than its topological dimension.

The group $\text{Isom}(\mathbb{H}_g^d)$ is a semisimple Lie group. For a general semisimple Lie group $G$, there is a rich class of representations from a hyperbolic group $\Gamma$ to $G$ called \textit{Anosov representations}, which generalize the convex co-compact representations from $\Gamma$ to $\text{Isom}(\mathbb{H}_g^d)$. Anosov representations were introduced by Labourie [Lab06] and extended by Guichard-Wienhard [GW12]. Since then, they have been heavily studied, [KLP18a, KLP14, KLP18b, GGKW17, BPS16]. One reason for their popularity is that they are rigid enough to retain many of the good geometric properties that convex co-compact representations have, while at the same time are flexible enough to admit many new and interesting examples.

In this paper, we investigate the regularity of the limit sets of Anosov representations from $\Gamma$ into $\text{PGL}_d(\mathbb{R})$. We will give precise definitions in Section 2 but informally: if $\Gamma$ is a word hyperbolic group with Gromov boundary $\partial_\infty \Gamma$, a representation $\rho: \Gamma \to \text{PGL}_d(\mathbb{R})$ is said to be $k$-Anosov if there exist continuous $\rho$-equivariant maps $\xi^{(k)}: \partial_\infty \Gamma \to \text{Gr}_k(\mathbb{R}^d)$ and $\xi^{(d-k)}: \partial_\infty \Gamma \to \text{Gr}_{d-k}(\mathbb{R}^d)$ which satisfy certain dynamical properties. For a $k$-Anosov representation, it is reasonable to call the image of $\xi^{(k)}$ in $\text{Gr}_k(\mathbb{R}^d)$ the "$k$-limit set of $\rho$ in $\text{Gr}_k(\mathbb{R}^d)$.”

We will largely focus our attention on 1-Anosov representations; by a result of Guichard-Wienhard [GW12, Proposition 4.3], for any Anosov representation $\rho: \Gamma \to \text{PGL}_d(\mathbb{R})$ into a semisimple Lie group $G$, there exists $d > 0$ and an irreducible representation $\phi: G \to \text{PGL}_d(\mathbb{R})$ such that $\phi \circ \rho$ is 1-Anosov. Thus, up to post composition with irreducible representations, the class of 1-Anosov representations contains all other types of Anosov representations. Further, the flag maps induced by $\phi$ are smooth. Thus, another result of Guichard-Wienhard [GW12, Proposition 4.4] implies that all regularity properties of the limit set can be investigated by reducing to the case of 1-Anosov representations.

Our first main result gives a sufficient condition for the 1-limit set of a 1-Anosov representation to be a $C^\alpha$-submanifold of $\mathbb{P}(\mathbb{R}^d)$ for some $\alpha > 1$.

**Theorem 1.1.** (Theorem 4.2) Suppose $\Gamma$ is a hyperbolic group, $\partial_\infty \Gamma$ is a topological $(m-1)$-manifold, and $\rho: \Gamma \to \text{PGL}_d(\mathbb{R})$ is a 1-Anosov representation. If

1. $\rho$ is $m$-Anosov, and $\xi^{(1)}(x) + \xi^{(1)}(z) + \xi^{(d-2m)}(y)$ is a direct sum for all pairwise distinct $x, y, z \in \partial_\infty \Gamma$,

then

2. $M := \xi^{(1)}(\partial_\infty \Gamma)$ is a $C^\alpha$-submanifold of $\mathbb{P}(\mathbb{R}^d)$ for some $\alpha > 1$.

Moreover, $T_{\xi^{(1)}(x)}M = \xi^{(m)}(x)$ for any $x \in \partial_\infty \Gamma$.

**Remark 1.2.**

1. A weaker version of this result, only deducing $C^1$ regularity was independently proven by Pozzetti-Sambarino-Wienhard [PSW18].
2. Property (i) and $k$-Anosovness in Theorem 1.1 are open conditions in $\text{Hom}(\Gamma, \text{PGL}_d(\mathbb{R}))$, see Section 6.1.
Theorem 1.1 is a generalization of the following theorem due to Benoist in the setting of divisible, properly convex domains in $\mathbb{P}(\mathbb{R}^d)$. A group of projective transformations $\Gamma \subset \text{PGL}_d(\mathbb{R})$ divides a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ if $\Gamma$ acts properly discontinuously and co-compactly on $\Omega$.

**Theorem 1.3** ([Ben04]). Let $\Gamma \subset \text{PGL}_d(\mathbb{R})$ be a hyperbolic group that divides a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$. Then then $\text{id} : \Gamma \to \text{PGL}_d(\mathbb{R})$ is a 1-Anosov representation whose 1-limit set is $\partial \Omega$. Furthermore, $\partial \Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a $C^\alpha$-submanifold for some $\alpha > 1$.

Theorem 1.1 also generalizes a result due to Labourie in the setting of Hitchin representations. Let $S$ be a closed orientable hyperbolizable surface and fix a Fuchsian representation $\rho_0 : \pi_1(S) \to \text{PGL}_2(\mathbb{R})$. Then let $\tau_d : \text{PGL}_2(\mathbb{R}) \to \text{PGL}_d(\mathbb{R})$ be the standard irreducible representation (see Section 8.1). A representation $\rho : \pi_1(S) \to \text{PGL}_d(\mathbb{R})$ is Hitchin if it is conjugate to a representation in the connected component of $\text{Hom}(\pi_1(S), \text{PGL}_d(\mathbb{R}))$ that contains $\tau_d \circ \rho_0$.

**Theorem 1.4** ([Lab06]). If $\rho : \pi_1(S) \to \text{PGL}_d(\mathbb{R})$ is a Hitchin representation, then $\rho$ is $k$-Anosov for every $k \in \{1, \ldots, d-1\}$, and the 1-limit set of $\rho$ is a $C^\alpha$-submanifold in $\mathbb{P}(\mathbb{R}^d)$ for some $\alpha > 1$.

Using Theorem 1.1 we can find more examples of representations that preserve $C^\alpha$-submanifolds in $\mathbb{P}(\mathbb{R}^d)$.

**Example 1.5.** (See Section 10) Suppose $\tau : \text{PO}(m, 1) \to \text{PGL}_d(\mathbb{R})$ is an irreducible representation, $\Gamma \leq \text{PO}(m, 1)$ is a co-compact lattice, and $\rho := \tau|_\Gamma : \Gamma \to \text{PGL}_d(\mathbb{R})$. If $\rho$ is 1-Anosov, then there exists a neighborhood $\mathcal{O}$ of $\rho$ in $\text{Hom}(\Gamma, \text{PGL}_d(\mathbb{R}))$ such that every representation in $\mathcal{O}$ is a 1-Anosov representation whose 1-limit set is a $C^\alpha$-submanifold of $\mathbb{P}(\mathbb{R}^d)$ for some $\alpha > 1$.

**Example 1.6.** (See Section 9) If $\rho : \pi_1(S) \to \text{PGL}_d(\mathbb{R})$ is a Hitchin representation, then for all $k = 1, \ldots, d-1$, there is an open set $\mathcal{O}$ of $\bigwedge^k \rho$ in $\text{Hom}(\Gamma, \text{PGL}(\bigwedge^k \mathbb{R}^d))$ so that every representation in $\mathcal{O}$ is a 1-Anosov representation whose 1-limit set is a $C^\alpha$-submanifold of $\mathbb{P}

\left(\bigwedge^k \mathbb{R}^d\right)$ for some $\alpha > 1$. See Section 7.1 for the definition of $\bigwedge^k \rho$. In particular, by applying [GW12, Proposition 4.4], the $k$-limit set of $\rho$ is a $C^\alpha$-submanifold of $\text{Gr}_k(\mathbb{R}^d)$ for some $\alpha > 1$.

**Remark 1.7.** Example 1.6 was independently observed by Pozzetti-Sambarino-Wienhard [PSW18].

In fact, Theorem 1.1 is a consequence of a more general theorem, see Theorem 4.2, that is stated using $\rho$-controlled subsets $M \subset \mathbb{P}(\mathbb{R}^d)$, of which the 1-limit set of $\rho$ is an example, see Definition 3.1. In the main body of our paper, all our results will be stated for $\rho$-controlled subsets. These statements are stronger than the results we mention in this introduction, but are more technical to state.

We also investigate the extent to which the converse of Theorem 1.1 holds. In general, there are 1-Anosov representations $\rho$ whose 1-limit set are $C^\infty$-submanifolds of $\mathbb{P}(\mathbb{R}^d)$, but for which (1) in Theorem 1.1 does not hold, see Example 8.2. However, we prove that when $\Gamma$ is a surface group and $\rho$ is irreducible, the conditions in Theorem 1.1 are both necessary and sufficient.
Theorem 1.8. (Theorem 8.1) Suppose $\Gamma$ is a hyperbolic group, $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ is an irreducible $1$-Anosov representation, and $\partial_\infty \Gamma$ is homeomorphic to a circle. Then the following are equivalent:

(\dagger) $\rho$ is a $2$-Anosov representation and $\xi^{(1)}(x) + \xi^{(1)}(y) + \xi^{(d-2)}(z)$ is a direct sum for all $x, y, z \in \partial \Gamma$ distinct,

(\ddagger) $\xi^{(1)}(\partial_\infty \Gamma)$ is a $C^\alpha$-submanifold of $\mathbb{P}(\mathbb{R}^d)$ for some $\alpha > 1$.

From Theorem 1.8 and (3) of Remark 1.2 we have the following corollary.

Corollary 1.9. Suppose $\Gamma$ is a hyperbolic group with $\partial_\infty \Gamma$ homeomorphic to a circle. Let $\mathcal{C} \subset \text{Hom}(\Gamma, \text{PGL}_d(\mathbb{R}))$ denote the set of representations that are irreducible, $1$-Anosov, and whose $1$-limit set is a $C^\alpha$-submanifold of $\mathbb{P}(\mathbb{R}^d)$ for some $\alpha > 1$ (which may depend on $\rho$). Then $\mathcal{C}$ is an open set in $\text{Hom}(\Gamma, \text{PGL}_d(\mathbb{R}))$.

For non-surface groups the situation is more complicated; there exist irreducible $1$-Anosov representations $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ whose $1$-limit set is a $C^\alpha$-submanifold of $\mathbb{P}(\mathbb{R}^d)$, but $\rho$ does not satisfy the condition (\dagger) in Theorem 1.1; see Example 7.3. However, if one assumes a stronger irreducibility condition on $\rho$, then the conditions in Theorem 1.1 are both necessary and sufficient.

Theorem 1.10. (Theorem 7.1) Suppose $\Gamma$ is a hyperbolic group, $\partial_\infty \Gamma$ is a $(m-1)$-dimensional topological manifold, and $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ is an irreducible $1$-Anosov representation such that $\Lambda^m \rho : \Gamma \to \text{PGL}(\Lambda^m \mathbb{R}^d)$ is also irreducible. Then the following are equivalent:

(\dagger) $\rho$ is a $m$-Anosov representation and $\xi^{(1)}(x) + \xi^{(1)}(y) + \xi^{(d-m)}(x)$ is a direct sum for all pairwise distinct $x, y, z \in \partial_\infty \Gamma$,

(\ddagger) $\xi^{(1)}(\partial_\infty \Gamma)$ is a $C^\alpha$-submanifold of $\mathbb{P}(\mathbb{R}^d)$ for some $\alpha > 1$.

Recall that if $\rho : \Gamma \to H$ is a Zariski-dense representation and $\tau : H \to \text{PGL}_d(\mathbb{R})$ is an irreducible representation, then $\tau \circ \rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ is irreducible. Thus, (3) of Remark 1.2 and Theorem 1.10 give the following corollary.

Corollary 1.11. Suppose $\Gamma$ is a hyperbolic group. Let $\mathcal{C} \subset \text{Hom}(\Gamma, \text{PGL}_d(\mathbb{R}))$ denote the set of representations $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ where $\rho$ is $1$-Anosov, has Zariski dense image, and whose $1$-limit set is a $C^\alpha$-submanifold of $\mathbb{P}(\mathbb{R}^d)$ for some $\alpha > 1$ (which may depend on $\rho$). Then $\mathcal{C}$ is an open set in $\text{Hom}(\Gamma, \text{PGL}_d(\mathbb{R}))$.

Finally, for representations satisfying certain irreducibility conditions, we also determine the optimal regularity of the $1$-limit set in terms of the spectral data of $\rho(\Gamma)$. More precisely, given $g \in \text{PGL}_d(\mathbb{R})$, let $\vec{g} \in \text{GL}_d(\mathbb{R})$ be a lift of $g$, and let $\lambda_1(\vec{g}) \geq \lambda_2(\vec{g}) \geq \cdots \geq \lambda_d(\vec{g})$ denote the absolute values of the eigenvalues of $\vec{g}$. Note that for all $i, j$, the ratio

$$\frac{\lambda_i}{\lambda_j}(g) := \frac{\lambda_i(\vec{g})}{\lambda_j(\vec{g})}$$

does not depend on the choice of lift $\vec{g}$ of $g$. Then given a representation $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ and $2 \leq m \leq d-1$ define

$$\alpha_m(\rho) = \inf_{\gamma \in \Gamma} \left\{ \frac{\log \lambda_1}{\log \lambda_{m+1}}(\rho(\gamma)) \Bigg/ \frac{\log \lambda_1}{\log \lambda_m}(\rho(\gamma)) : \frac{\lambda_1}{\lambda_m}(\rho(\gamma)) \neq 1 \right\}.$$

If $\rho$ is $(1, m)$-Anosov, it follows from definition that $\alpha_m(\rho) > 1$ (see Section 2).
Theorem 1.12. (Theorem 6.1) Suppose $\Gamma$ is a hyperbolic group and $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ is an irreducible $(1,m)$-Anosov representation so that $\xi^{(1)}(x) + \xi^{(1)}(y) + \xi^{(d-m)}(z)$ is a direct sum for all $x, y, z \in \partial_\infty \Gamma$ distinct. Then

$$\alpha_m(\rho) \leq \sup \left\{ \alpha \in (1, 2) : \xi^{(1)}(\partial_\infty \Gamma) \text{ is a } C^\alpha\text{-submanifold} \right\}$$

with equality if $\xi^{(1)}(\partial_\infty \Gamma) \cap \left( \xi^{(1)}(x) + \xi^{(1)}(y) + \xi^{(d-m)}(z) \right)$ spans $\xi^{(1)}(x) + \xi^{(1)}(y) + \xi^{(d-m)}(z)$ for all $x, y, z \in \partial_\infty \Gamma$ distinct.

Remark 1.13.

(1) In Theorem 1.12 when $\xi^{(1)}(\partial_\infty \Gamma)$ has either dimension one or co-dimension one, the extra hypothesis for equality is automatically satisfied. If the dimension is one (i.e. $m = 2$), then

$$\xi^{(1)}(x) + \xi^{(1)}(y) + \xi^{(d-m)}(z) = \xi^{(1)}(x) + \xi^{(1)}(y) + \xi^{(d-2)}(z) = \mathbb{R}^d.$$ 

So the extra hypothesis follows from the irreducibility of $\rho$. If the co-dimension is one (i.e. $m = d - 1$), then

$$\xi^{(1)}(x) + \xi^{(1)}(y) + \xi^{(d-m)}(z) = \xi^{(1)}(x) + \xi^{(1)}(y) + \xi^{(1)}(z)$$

is spanned by $\xi^{(1)}(x), \xi^{(1)}(y), \xi^{(1)}(z)$. Hence the extra hypothesis always holds in this case.

(2) In general, the extra hypothesis for equality is an open condition, see Section 6.1.

(3) The irreducibility of $\rho$ is necessary in Theorem 1.12. For instance, if $\tau_d : \text{PGL}_2(\mathbb{R}) \to \text{PGL}_d(\mathbb{R})$ is the standard irreducible representation, see Section 6.1 and $\Gamma \leq \text{PGL}_2(\mathbb{R})$ is a co-compact lattice, then $\rho = (\tau_2 \otimes \tau_2)|\Gamma$ is 1-Anosov and $\xi^{(1)}(\partial_\infty \Gamma)$ is a 1-dimensional $C^\infty$-submanifold of $\text{P}(\mathbb{R}^7)$. At the same time, for any infinite order $\gamma \in \Gamma$,

$$\log \frac{\lambda_1}{\lambda_3}(\rho(\gamma)) / \log \frac{\lambda_1}{\lambda_2}(\rho(\gamma)) = 3/2.$$ 

(4) Notice that the quantity $\alpha_m(\rho)$ is invariant under passing to finite index subgroups. In particular, if $\Gamma_0 \leq \Gamma$ is a finite index subgroup and $\gamma \in \Gamma$, then there exists some $k \in \mathbb{N}$ such that $\gamma^k \in \Gamma_0$. Further,

$$\log \frac{\lambda_1}{\lambda_{m+1}}(\rho(\gamma^k)) / \log \frac{\lambda_1}{\lambda_m}(\rho(\gamma^k)) = \log \frac{\lambda_1}{\lambda_{m+1}}(\rho(\gamma)) / \log \frac{\lambda_1}{\lambda_m}(\rho(\gamma)).$$

Hence $\alpha_m(\rho|\gamma^k) = \alpha_m(\rho)$.

In Section 6.1 we establish a generalization of Theorem 1.12 which holds for $\rho$-controlled subsets. One example of such a subset is the boundary of a properly convex domain $\Omega \subset \text{P}(\mathbb{R}^d)$ that admits a $\Gamma$-action induced by a 1-Anosov representation $\rho$. In this case, Theorem 6.1 implies the following.

Theorem 1.14. Suppose $\Gamma$ is a hyperbolic group and $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ is an irreducible 1-Anosov representation. Also, suppose $\Omega \subset \text{P}(\mathbb{R}^d)$ is a $\rho(\Gamma)$-invariant properly convex domain so that $\xi^{(d-1)}(x) \cap \partial \Omega = \xi^{(1)}(x)$ for all $x \in \partial \Omega$. If

* $p_1 + p_2 + \xi^{(1)}(y)$ is a direct sum for all pairwise distinct $p_1, p_2, \xi^{(1)}(y) \in \partial \Omega$,
then
\[(\ast\ast) \partial \Omega \text{ is } C^\alpha \text{ along } \xi^{(1)}(\partial_\infty \Gamma) \text{ for some } \alpha > 1.\]
Moreover, \(T_{\xi^{(1)}(x)} \partial \Omega = \xi^{(d-1)}(x)\) for any \(x \in \partial_\infty \Gamma\), and
\[
\alpha_{d-1}(\rho) = \sup \left\{ \alpha \in (1, 2) : \partial \Omega \text{ is } C^\alpha \text{ along } \xi^{(1)}(\partial_\infty \Gamma) \right\}.
\]

In the case when the \(\Gamma\)-action on \(\Omega\) is co-compact, Theorem 1.14 was previously proven by Guichard \[Gui05\] using different techniques. Also, a weaker version of Theorem 1.13 (without the optimal bound for \(\alpha\)) was previously proven independently by Danciger-Gueritaud-Kassel \[DGK17\] and the second author \[Zim17\].

1.1. Terminology. Through out the paper we will use the following terminology:

1. ||·||_2 will always denote the standard \(\ell^2\)-norm on \(\mathbb{R}^d\).
2. A \((m-1)\)-dimensional topological manifold \(M \subset P(\mathbb{R}^d)\) is \(C^\alpha\) for some \(\alpha \in (1, 2)\) if for every \(p \in M\) there exists local coordinates around \(p\) and a differentiable map \(f : \mathbb{R}^{m-1} \to \mathbb{R}^{d-m}\) such that \(M\) coincides with the graph of \(f\) near \(p\) and
\[
f(u + h) = f(u) + df_u(h) + O(\|h\|_2^\alpha)
\]
for all \(u, h \in \mathbb{R}^{m-1}\).
3. A \((m-1)\)-dimensional topological manifold \(M \subset P(\mathbb{R}^d)\) is \(C^\alpha\) along a subset \(N \subset M\) for some \(\alpha \in (1, 2)\) if for every \(p \in N\) there exists local coordinates around \(p\) and a continuous map \(f : \mathbb{R}^{m-1} \to \mathbb{R}^{d-m}\) such that \(M\) coincides with the graph of \(f\) near \(p\) and if \((u, f(u)) \in N\), then \(f\) is differentiable at \(u\) and satisfies
\[
f(u + h) = f(u) + df_u(h) + O(\|h\|_2^\alpha)
\]
for all \(h \in \mathbb{R}^{m-1}\).

2. Anosov representations

For the rest of this article, \(\Gamma\) will denote a hyperbolic group, and \(\partial_\infty \Gamma\) will be its Gromov boundary. In this section, we define Anosov representations from \(\Gamma\) to \(\text{PGL}_d(\mathbb{R})\), and mention some of their properties.

2.1. A definition of Anosov representations. Since they were introduced, several other characterizations of Anosov representations have been given by Kapovich et al. \[KLP18a, KLP14, KLP18b\], Guéritaud et al. \[GGKW17\], and Bochi et al. \[BPS16\]. The definition we give below comes from \[GGKW17\] Theorem 1.7.

First, let \(S\) be a finite symmetric generating set of \(\Gamma\), and \(d_S\) the induced word metric on \(\Gamma\). For \(\gamma \in \Gamma\), let \(\ell_S(\gamma)\) denote the minimal translation distance of \(\gamma\) acting on \(\Gamma\), that is
\[
\ell_S(\gamma) := \inf_{x \in \Gamma} d_S(\gamma \cdot x, x).
\]
Also, recall that for any \(g \in \text{PGL}_d(\mathbb{R})\) and any \(i, j \in \{1, \ldots, d\}\), we have defined
\[
\frac{\lambda_i}{\lambda_j}(g) := \frac{\lambda_i(g)}{\lambda_j(g)},
\]
where \(\lambda_1(\mathbf{g}) \geq \cdots \geq \lambda_d(\mathbf{g})\) are the absolute values of the (generalized) eigenvalues of a representative \(\mathbf{g} \in \text{GL}_d(\mathbb{R})\) of \(g\).
A third ingredient we need to define Anosov representations are appropriate definitions of “well-behaved” flag maps. More precisely, we have the following.

**Definition 2.1.** Let \( \rho : \Gamma \to \text{PGL}_d(\mathbb{R}) \) be a representation. If \( 1 \leq k \leq d-1 \), then a pair of maps \( \xi^{(k)} : \partial_\infty \Gamma \to \text{Gr}_k(\mathbb{R}^d) \) and \( \xi^{(d-k)} : \partial_\infty \Gamma \to \text{Gr}_{d-k}(\mathbb{R}^d) \) are called:

- **\( \rho \)-equivariant** if \( \xi^{(k)}(\gamma \cdot x) = \rho(\gamma) \cdot \xi^{(k)}(x) \) and \( \xi^{(d-k)}(\gamma \cdot x) = \rho(\gamma) \cdot \xi^{(d-k)}(x) \) for all \( x \in \partial_\infty \Gamma \) and \( \gamma \in \Gamma \),
- **dynamics-preserving** if for every \( \gamma \in \Gamma \) of infinite order with attracting fixed point \( \gamma^+ \in \partial_\infty \Gamma \), the points \( \xi^{(k)}(\gamma^+) \in \text{Gr}_k(\mathbb{R}^d) \) and \( \xi^{(d-k)}(\gamma^+) \in \text{Gr}_{d-k}(\mathbb{R}^d) \) are attracting fixed points of the action of \( \rho(\gamma) \) on \( \text{Gr}_k(\mathbb{R}^d) \) and \( \text{Gr}_{d-k}(\mathbb{R}^d) \), and
- **transverse** if for every distinct pair \( x, y \in \partial_\infty \Gamma \) we have
  \[ \xi^{(k)}(x) + \xi^{(d-k)}(y) = \mathbb{R}^d. \]

With these definitions, we can now define Anosov representations.

**Definition 2.2.** A representation \( \rho : \Gamma \to \text{PGL}_d(\mathbb{R}) \) is \( k \)-Anosov if

- there exist continuous, \( \rho \)-equivariant, dynamics preserving, and transverse maps \( \xi^{(k)} : \partial \Gamma \to \text{Gr}_k(\mathbb{R}^d) \), \( \xi^{(d-k)} : \partial \Gamma \to \text{Gr}_{d-k}(\mathbb{R}^d) \), and
- for any sequence \( \{ \gamma_i \}_{i=1}^\infty \subset \Gamma \) so that \( \lim_{i \to \infty} \ell_S(\gamma_i) = \infty \), we have
  \[ \lim_{i \to \infty} \log \frac{\lambda_k}{\lambda_{k+1}} (\rho(\gamma_i)) = \infty. \]

If \( \rho \) is \( k \)-Anosov for all \( k \in \{k_1, \ldots, k_j\} \) we say that \( \rho \) is \( (k_1, \ldots, k_j) \)-Anosov.

If \( S' \) is another finite symmetric generating set of \( \Gamma \), then \( \text{id} : (\Gamma, d_S) \to (\Gamma, d_{S'}) \) is a quasi-isometry. In particular, the notion of an Anosov representation does not depend on the choice of \( S \). Also, it follows from the definition that a representation \( \rho : \Gamma \to \text{PGL}_d(\mathbb{R}) \) is \( k \)-Anosov if and only if it is \( (d-k) \)-Anosov. We refer to \( \xi^{(k)} \) as the \( k \)-flag map of \( \rho \), and \( \xi^{(k)}(\partial \Gamma) \subset \text{Gr}_k(\mathbb{R}^d) \) as the \( k \)-limit set of \( \rho \).

Given a subspace \( V \subset \mathbb{R}^N \) define

\[ [V] = \{ [v] \in \mathbb{P}(\mathbb{R}^N) : v \in V \}. \]

Often, we will view \( \xi^{(k)}(x) \) as the projective subspace \( [\xi^{(k)}(x)] \subset \mathbb{P}(\mathbb{R}^d) \). However, to simplify notation, we will denote \( [\xi^{(k)}(x)] \) simply by \( \xi^{(k)}(x) \) in those settings.

**Remark 2.3.** In many other places in the literature, what we call a \( k \)-Anosov representation is usually known as a \( P_k \)-Anosov representation, where \( P_k \) is the stabilizer in \( \text{PGL}_d(\mathbb{R}) \) of a point in \( \text{Gr}_k(\mathbb{R}^d) \). This notation is an artifact of a more general definition of Anosov representations to an arbitrary non-compact semisimple Lie group. Since we do not use that generality here, we will use \( k \) in place of \( P_k \) to simplify the notation.

### 2.2. Singular values and Anosov representations

We will now briefly discuss singular values, which we use to give an alternate description of Anosov representations. This description was initially due to Kapovich et al. [KLP14, KLP18b], but was also later proven by Bochi et al. [BPS16] using different techniques.

**Definition 2.4.** Let \( | \cdot | \) and \( \| \cdot \| \) be norms on \( \mathbb{R}^d \), and let \( L : (\mathbb{R}^d, | \cdot |) \to (\mathbb{R}^d, \| \cdot \|) \) be a linear map.
• For any $X \in (\mathbb{R}^d, |\cdot|)$, the stretch factor of $X$ under $L$ is the quantity
  $$\sigma_X(L) := \frac{||L(X)||}{|X|}.$$ 

• For $i = 1, \ldots, n$, the $i$-th singular value of $L$ is the quantity
  $$\sigma_i(L) := \max_{W \subseteq \mathbb{R}^d, \dim W = i} \min_{X \in W} \sigma_X(L) = \min_{W \subseteq \mathbb{R}^d, \dim W = d-i+1} \max_{X \in W} \sigma_X(L).$$

Observe that for all $i = 1, \ldots, d-1$, $\sigma_i(L) \geq \sigma_{i+1}(L)$, and if $L$ is invertible, then $\sigma_1(L) = \min_{1 \leq i \leq d} L_{ii}$.

When $L = \overline{g} \in \text{GL}_d(\mathbb{R})$ and $||\cdot|| = |\cdot|$ is the standard norm $||\cdot||_2$ on $\mathbb{R}^d$, we denote $\sigma_i(L)$ by $\mu_i(\overline{g})$. In that case, if $A$ is a $d \times d$ real-valued matrix representing $\overline{g}$ in an orthonormal basis for the standard inner product on $\mathbb{R}^d$, then the singular values $\mu_1(\overline{g}) \geq \cdots \geq \mu_d(\overline{g}) > 0$ are the square roots of the eigenvalues of $A^T A$. Using this, we may define, for any $g \in \text{PGL}_d(\mathbb{R})$ and all $i, j \in \{1, \ldots, d\}$, the quantity
  $$\frac{\mu_i}{\mu_j}(g) := \frac{\mu_i(\overline{g})}{\mu_j(\overline{g})},$$

where $\overline{g} \in \text{GL}_d(\mathbb{R})$ is a lift of $g$.

We can now state the following theorem due to Kapovich et al. [KLP14, KLP18b], (see Bochi et al. [BPS16, Proposition 4.9]).

**Theorem 2.5.** Suppose $\Lambda$ is a finitely generated group and $S$ is a finite symmetric generating set. A representation $\rho : \Lambda \to \text{PGL}_d(\mathbb{R})$ is $k$-Anosov if and only if there are constants $C, c > 0$ such that

$$\log \frac{\mu_k}{\mu_{k+1}}(\rho(\gamma)) \geq C d_S(\gamma, \text{id}) - c$$

for all $\gamma \in \Lambda$.

**Remark 2.6.** In Theorem 2.5 it is implied, not assumed, that $\Lambda$ is a hyperbolic group.

### 2.3. Properties of Anosov representations

Next, we recall some important properties of Anosov representations.

Define respectively the Cartan and Jordan projection $\mu, \lambda : \text{GL}_d(\mathbb{R}) \to \mathbb{R}^d$ by

$$\mu(\overline{g}) := (\log \mu_1(\overline{g}), \ldots, \log \mu_d(\overline{g})) \quad \text{and} \quad \lambda(\overline{g}) := (\log \lambda_1(\overline{g}), \ldots, \log \lambda_d(\overline{g})).$$

Observe that while the Jordan projection is invariant under conjugation in $\text{GL}_d(\mathbb{R})$, the Cartan projection is not. These two projections can be interpreted geometrically in the following way.

Associated to the Lie group $\text{PGL}_d(\mathbb{R})$ is the Riemannian symmetric space $X$, on which $\text{PGL}_d(\mathbb{R})$ acts transitively and by isometries. As a $\text{PGL}_d(\mathbb{R})$-space, $X = \text{PGL}_d(\mathbb{R})/\text{PO}(d)$. Furthermore, the distance $d_X$ on $X$ induced by its Riemannian metric can be computed from the Cartan projection by the formula

$$d_X(g_1 \cdot \text{PO}(d), g_2 \cdot \text{PO}(d)) = \left\| \mu \left( g_1^{-1} g_2 \right) \right\|_2,$$

where $g_1^{-1} g_2 \in \text{SL}_d(\mathbb{R}) := \{ g \in \text{GL}_d(\mathbb{R}) : \det(g) = \pm 1 \}$ is a lift of $g_1^{-1} g_2$, and $||\cdot||_2$ is the $L^2$-norm. On the other hand, if $g \in \text{PGL}_d(\mathbb{R})$ and $\overline{g} \in \text{SL}_d(\mathbb{R})$ is a representative of $g$, then

$$\inf_{p \in \Lambda} d_X(p, g \cdot p) = ||\lambda(\overline{g})||_2.$$
As such, if \( g \in \text{PGL}_d(\mathbb{R}) \) and \( \overline{g} \in \text{SL}^+_{d/2}(\mathbb{R}) \) is a lift of \( g \), then \( \mu(\overline{g}) \) is a refinement of the distance by which \( g \) translates the identity coset in \( X \), and \( \lambda(\overline{g}) \) is a refinement of the minimal translation distance of \( g \) in \( M \).

As an immediate consequence of Theorem \ref{thm:metric_preservation}, an Anosov representations coarsely preserve the metric \( d_S \) on \( \Gamma \).

**Corollary 2.7.** Let \( \rho : \Gamma \to \text{PGL}_d(\mathbb{R}) \) be \( k \)-Anosov for any \( k \). Then the map \( \Gamma \to M \) defined by \( \gamma \mapsto \rho(\gamma) : \text{PSO}(d) \) is a quasi-isometric embedding. In other words, there are constants \( C \geq 1 \) and \( c \geq 0 \) such that for all \( \gamma_1, \gamma_2 \in \Gamma \),

\[
\frac{1}{C} \| \mu(\rho(\gamma_1^{-1}\gamma_2)) \|_2 - c \leq d_S(\gamma_1, \gamma_2) \leq C \| \mu(\rho(\gamma_1^{-1}\gamma_2)) \|_2 + c,
\]

where \( \rho(\gamma_1^{-1}\gamma_2) \in \text{SL}^+_{d/2}(\mathbb{R}) \) is a lift of \( \rho(\gamma_1^{-1}\gamma_2) \).

We also have the following proposition due to Quint (see \cite[Lemma 2.19]{BCLS15} for a proof), which restricts the possible Zariski closures of Anosov representations to \( \text{PGL}_d(\mathbb{R}) \).

**Proposition 2.8.** Let \( \rho : \Gamma \to \text{PGL}_d(\mathbb{R}) \) be a 1-Anosov representation. If \( \rho \) is irreducible, then the Zariski closure of \( \rho(\Gamma) \) is a semisimple Lie group without compact factors.

We will also use the following observation of Guichard-Wienhard.

**Proposition 2.9.** \cite[Lemma 5.12]{GW12} Let \( \rho : \Gamma \to \text{PGL}_d(\mathbb{R}) \) be an irreducible 1-Anosov representation. If \( \Gamma_0 \leq \Gamma \) is a finite index subgroup, then \( \rho|_{\Gamma_0} \) is also irreducible.

In many places in our argument, it will be more convenient to work with representations into \( \text{SL}_d(\mathbb{R}) \) instead of \( \text{PGL}_d(\mathbb{R}) \). Then next observation allows us to make this reduction. Let \( \pi : \text{GL}_d(\mathbb{R}) \to \text{PGL}_d(\mathbb{R}) \) denote the obvious projection.

**Observation 2.10.** For any representation \( \rho : \Gamma \to \text{PGL}_d(\mathbb{R}) \), there exists a subgroup \( \Lambda_{\rho} \leq \text{SL}_d(\mathbb{R}) \) so that \( \pi|_{\Lambda_{\rho}} : \Lambda_{\rho} \to \text{PGL}_d(\mathbb{R}) \) is a representation whose kernel is a subgroup of \( \mathbb{Z}_2 \), and whose image is a subgroup of \( \rho(\Gamma) \) with index at most two.

**Proof.** Define \( \Lambda_0 := \{ g \in \text{SL}^+_{d/2}(\mathbb{R}) : \{ g \} \in \rho(\Gamma) \} \), and let \( \Lambda_{\rho} := \Lambda_0 \cap \text{SL}_d(\mathbb{R}) \). Then \( \pi(\Lambda_0) \subset \text{PGL}_d(\mathbb{R}) \) coincides with \( \rho(\Gamma) \), and \( \Lambda_{\rho} \) has index at most two in \( \Lambda_0 \). \( \square \)

In particular, if \( \Gamma \) is a hyperbolic group, then so is \( \Lambda_{\rho} \), and there are canonical identifications \( \partial_{\infty}\Gamma = \partial_{\infty}\rho(\Gamma) = \partial_{\infty}\pi(\Lambda_{\rho}) = \partial_{\infty}\Lambda_{\rho} \). Furthermore, the following proposition is an immediate consequence of \cite[Corollary 1.3]{GW12}.

**Proposition 2.11.** Let \( \rho : \Gamma \to \text{PGL}_d(\mathbb{R}) \) be a representation. The representation \( \rho' := \pi|_{\Lambda_{\rho}} : \Lambda_{\rho} \to \text{PGL}_d(\mathbb{R}) \) is \( k \)-Anosov if and only if \( \rho \) is \( k \)-Anosov. If so, the \( k \)-flag maps of \( \rho \) and \( \rho' \) agree.

**Remark 2.12.** To prove any properties about the \( k \)-limit sets of \( \rho \), it is now sufficient to show those properties hold for the \( k \)-limit sets of \( \rho' \). The advantage of working with \( \rho' \) in place of \( \rho \) is that \( \rho' : \Lambda_{\rho} \to \text{PGL}(d, \mathbb{R}) \) admits a lift to a representation from \( \Lambda_{\rho} \) to \( \text{SL}_d(\mathbb{R}) \). With this, we can henceforth assume that \( \rho : \Gamma \to \text{PGL}(d, \mathbb{R}) \) admits a lift to a representation \( \overline{\rho} : \Gamma \to \text{SL}(d, \mathbb{R}) \).
2.4. Gromov geodesic flow space. In their proof of Theorem 2.5 Bochi et al. [BPS16] gave a description of Anosov representations using dominated splittings. Our next goal is to give this description. To do so, we recall the definition of the flow space of a hyperbolic group, and state some of their well-known properties. For more details, see for instance [Gro87], [Cha94], or [Mat91].

As a topological space, the flow space for $\Gamma$, denoted $\hat{UT}$, is homeomorphic to $\partial_\infty \Gamma(2) \times \mathbb{R}$, where $\partial_\infty \Gamma(2) := \{(x, y) \in \partial_\infty \Gamma^2 : x \neq y\}$. This flow space admits a natural $\mathbb{R}$-action by translation in the $\mathbb{R}$-factor called the geodesic flow on $\hat{UT}$. We will use the notation $v = (v^+, v^-, v_0) \in \hat{UT}$, and denote the geodesic flow on $\hat{UT}$ by $\phi_t$, i.e.

$$\phi_t(v) = (v^+, v^-, v_0 + t) = (\phi_t(v^+), \phi_t(v^-), \phi_t(v_0)).$$

There is a proper, co-compact $\Gamma$-action on $\hat{UT}$ that commutes with $\phi_t$, and satisfies $\gamma \cdot (v^+, v^-, \mathbb{R}) = (\gamma \cdot v^+, \gamma \cdot v^-, \mathbb{R})$. There is also a natural $\mathbb{Z}/2\mathbb{Z}$ action on $\hat{UT}$ which satisfies

$$(1 + 2 \mathbb{Z}) \cdot (x, y, \mathbb{R}) = (y, x, \mathbb{R}).$$

This action commutes with the $\Gamma$ action, but not the $\phi_t$ action. Instead:

$$\alpha \phi_t \alpha = \phi_{-t}$$

where $\alpha = (1 + 2 \mathbb{Z})$. So the actions of $\Gamma$, $\phi_t$, and $\mathbb{Z}/2\mathbb{Z}$ combine to yield an action of $\Gamma \times (\mathbb{R} \times \psi \mathbb{Z}/2\mathbb{Z})$ on $\hat{UT}$ where $\psi : \mathbb{Z}/2\mathbb{Z} \to \text{Aut}(\mathbb{R})$ is given by $\psi(\alpha)(t) = -t$.

Since the $\Gamma$ action commutes with $\phi_t$, the geodesic flow on $\hat{UT}$ descends to a flow on the compact space $UT := \hat{UT}/\Gamma$, which we refer to as the geodesic flow on $UT$, and denote by $\hat{\phi}_t$. This also implies that if $v^+ = \gamma^+$ and $v^- = \gamma^-$ are the attracting and repelling fixed points of some infinite order $\gamma \in \Gamma$, then the orbit $(\gamma^+, \gamma^-, \mathbb{R}) \subset \hat{UT}$ of $\hat{\phi}_t$ descends to a closed orbit of $\hat{\phi}_t$ in $UT$. We will denote the period of this closed orbit by $T_\gamma \in \mathbb{R}$, and refer to $T_\gamma$ as the period of $\gamma$. In other words, for all $v_0 \in \mathbb{R}$, $\gamma \cdot (\gamma^+, \gamma^-, v_0) = (\gamma^+, \gamma^-, v_0 + T_\gamma)$.

Furthermore, $\hat{UT}$ admits a $\Gamma \times \mathbb{Z}/2\mathbb{Z}$-invariant metric so that every orbit $(v^+, v^-, \mathbb{R})$ of $\hat{\phi}_t$ is a continuous quasi-geodesic. Since the $\Gamma$-action on $\hat{UT}$ is also co-compact, any $\Gamma$-orbit is a quasi-isometry. As a consequence, there is a canonical $\Gamma$-invariant homeomorphism $\partial_\infty \hat{UT} \simeq \partial_\infty \hat{\Gamma} \cup \partial_\infty \hat{\Gamma}$ between the Gromov boundaries of $\hat{UT}$ and $\hat{\Gamma}$, and $v^+$ and $v^-$ in $\partial_\infty \hat{UT}$ are the forward and backward endpoints of $(v^+, v^-, \mathbb{R}) \subset \hat{UT}$ respectively.

Remark 2.13. In the case when $\Gamma$ is the fundamental group of a compact Riemannian manifold $M$ with negative sectional curvature, this geodesic flow space is what one would expect. In particular, let $T^1 M$ denote the unit tangent bundle of $M$, let $\hat{M}$ denote the universal cover of $M$, and let $T^1 \hat{M}$ denote the unit tangent bundle of $\hat{M}$. Then we may take $\hat{UT}$ to be $T^1 \hat{M}$ and $UT$ to be $T^1 M$. The geodesic flow on both $T^1 M$ and $T^1 \hat{M}$ is the usual geodesic flow associated to the Riemannian metrics on $\hat{M}$ and $M$, and the $\Gamma$-invariant metric $d_{\hat{UT}}$ is the lift of the Riemannian metric on $T^1 M$ that is locally given by the product of the Riemannian metric on $M$ and the spherical metric on the fibers. Further, the $\mathbb{Z}/2\mathbb{Z}$ action is given by $v \mapsto -v$.

Gromov proved that the geodesic flow space $\hat{UT}$ is unique up to homeomorphism.
Theorem 2.14. [Gro87, Theorem 8.3.C] Suppose that $\mathcal{G}$ is a proper Gromov hyperbolic metric space such that

1. $\Gamma \times (\mathbb{R} \times \mathbb{Z}/2\mathbb{Z})$ acts on $\mathcal{G}$,
2. the actions of $\Gamma$ and $\mathbb{Z}/2\mathbb{Z}$ are isometric,
3. for every $v \in \mathcal{G}$, the map $\gamma \in \Gamma \to \gamma \cdot v \in \mathcal{G}$ is a quasi-isometry,
4. the action is free and every $\mathbb{R}$-orbit is a quasi-geodesic in $\mathcal{G}$. Further, the induced map $\mathcal{G}/\mathbb{R} \to \partial_{\infty} \mathcal{G}(2)$ is a homeomorphism.

Then there exists a $\Gamma \times \mathbb{Z}/2\mathbb{Z}$-equivariant homeomorphism $T : \mathcal{G} \to \tilde{\mathcal{G}}$ that maps $\mathbb{R}$-orbits to $\mathbb{R}$-orbits.

2.5. Dominated Splittings. Next, we describe an alternate characterization of Anosov representations in $\text{GL}_d(\mathbb{R})$ using dominated splittings due to Bochi et al. [BPS16].

Let $\rho : \Gamma \to \text{GL}_d(\mathbb{R})$ be a representation. Let $E := \tilde{\mathcal{G}} \times \mathbb{R}^d$ be the product bundle over $\mathcal{G}$, and define the flat vector bundle $E_{\rho} := E/\Gamma$ over $\mathcal{G}$, where the $\Gamma$ action on $E$ is given by $\gamma \cdot (v, X) = (\gamma \cdot v, \rho(\gamma) \cdot X)$. Since $E_{\rho}$ is naturally a flat vector bundle over $\mathcal{G}$, it admits a continuous norm, and the compactness of $\mathcal{G}$ ensures that any two such norms are bi-Lipschitz. For any continuous norm on $E_{\rho}$, choose a lift of this norm to a $\Gamma$-invariant, continuous norm $\|\cdot\|$ on $E$. With this, we can state the following theorem due to Bochi et al (see Theorem 2.2, Proposition 4.5 and Proposition 4.9 in [BPS16]).

Theorem 2.15. A representation $\rho : \Gamma \to \text{GL}_d(\mathbb{R})$ is $k$-Anosov if and only if there exist

- continuous, $\phi_t$-invariant, $\rho$-equivariant maps
  
  
  
  
  

  so that $F_1(v) + F_2(v) = \mathbb{R}^d$ for all $v \in \tilde{\mathcal{G}}$, and

- constants $C > 0$, $\beta > 0$ such that
  
  
  
  

  for all $v \in \tilde{\mathcal{G}}$, $X_i \in F_i(v)$ non-zero, and $t \geq 0$.

Here, we may think of $F_1$ and $F_2$ as $\Gamma$-invariant sub-bundles of $E$. The maps $F_1$ and $F_2$ are related to the flag maps $\xi^{(k)}$ and $\xi^{(d-k)}$ by

$F_1(v) = \xi^{(k)}(v^+) \quad \text{and} \quad F_2(v) = \xi^{(d-k)}(v^-)$

for all $v = (v^+, v^-, v_0) \in \tilde{\mathcal{G}}$.

3. $\rho$-controlled sets

In this section we introduce $\rho$-controlled sets and construct a useful family of projections.

Definition 3.1. Suppose that $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ is a 1-Anosov representation. A closed $\rho(\Gamma)$-invariant subset $M \subset \mathbb{P}(\mathbb{R}^d)$ is $\rho$-controlled if

(i) $\xi^{(1)}(\partial_{\infty} \Gamma) \subset M$ and
(ii) $M \cap \xi^{(d-1)}(x) = \xi^{(1)}(x)$ for every $x \in \partial_{\infty} \Gamma$. 
If $\rho$ also happens to be $m$-Anosov for some $m = 2, \ldots, d - 1$, then a $\rho$-controlled subset $M \subset \mathbb{P}(\mathbb{R}^d)$ is $m$-hyperconvex if

$$p_1 + p_2 + \xi^{(d-m)}(y)$$

is a direct sum for all $p_1, p_2 \in M$ and $y \in \partial_\infty \Gamma$ with $p_1, p_2, \xi^{(1)}(y)$ pairwise distinct.

**Remark 3.2.** We will typically consider the case when $M$ is a topological $(m-1)$-dimensional manifold and then require that $M$ is $m$-hyperconvex.

The three main examples of $\rho$-controlled subsets $M \subset \mathbb{R}^d$ that we will be concerned with are the following.

**Example 3.3.** When $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ is a 1-Anosov representation, then the 1-limit set $\xi^{(1)}(\partial_\infty \Gamma)$ for $\rho$ is obviously $\rho$-controlled. Furthermore, if $\rho$ is $m$-Anosov for some $m = 1, \ldots, d - 1$, then $\xi^{(1)}(\partial_\infty \Gamma)$ is $m$-hyperconvex if and only if

$$\xi^{(1)}(x) + \xi^{(1)}(z) + \xi^{(d-m)}(y)$$

is a direct sum for all pairwise distinct $x, y, z \in \partial_\infty \Gamma$.

**Example 3.4.** Suppose that $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ is a 1-Anosov representation and $\rho(\Gamma)$ preserves a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ so that $\xi^{(d-1)}(x) \cap \partial \Omega = \xi^{(1)}(x)$ for all $x \in \partial_\infty \Gamma$, see Section 3.1. Then $M := \partial \Omega$ is obviously $\rho$-controlled. Notice that in this case, the requirement that $M$ is $(d-1)$-hyperconvex is simply that

$$p_1 + p_2 + \xi^{(1)}(y)$$

is a direct sum for all pairwise distinct $p_1, p_2, \xi^{(1)}(y) \in \partial \Omega$. This is satisfied if and only if $\xi^{(1)}(\partial_\infty \Gamma)$ does not intersect any proper line segments in $\partial \Omega$.

**Example 3.5.** Suppose $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ is a 1-Anosov representation and $\Gamma_1 \leq \Gamma$ is a quasi-convex subgroup. Then $\rho_1 := \rho|_{\Gamma_1} : \Gamma_1 \to \text{PGL}_d(\mathbb{R})$ is also 1-Anosov. In this case, $\rho_1$ is $1$-Anosov, and the 1-limit set $\xi^{(1)}(\partial_\infty \Gamma)$ for $\rho$ is obviously $\rho_1$-controlled. Furthermore, if $\rho_1$ is $m$-Anosov for some $m = 1, \ldots, d - 1$, then $\xi^{(1)}(\partial_\infty \Gamma)$ is $m$-hyperconvex if and only if

$$\xi^{(1)}(x) + \xi^{(1)}(z) + \xi^{(d-m)}(y)$$

is a direct sum for all pairwise distinct $x, y, z \in \partial_\infty \Gamma$ with $y \in \partial_\infty \Gamma_1$.

Recall that $\partial_\infty \Gamma^{(2)}$ is the set of all pairs $(x, y) \in \partial_\infty \Gamma^{(2)}$ with $x \neq y$. Then for any $(x, y) \in \partial_\infty \Gamma^{(2)}$, let $L_{x,y}$ denote the orbit $(x, y, \mathbb{R}) \subset \cup \mathcal{T}$ of $\phi_2$. The following proposition is one of the key tools we use to investigate regularity properties of $\rho$-controlled subsets.

**Proposition 3.6.** Suppose that $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ is a 1-Anosov representation and $M \subset \mathbb{P}(\mathbb{R}^d)$ is $\rho$-controlled. Then there exists a continuous family of continuous maps

$$\pi_{x,y} : M \setminus \left\{ \xi^{(1)}(x), \xi^{(1)}(y) \right\} \to L_{x,y}$$

indexed by $(x, y) \in \partial_\infty \Gamma^{(2)}$ such that

$$\pi_{x,y} = \pi_{\gamma \cdot x, \gamma \cdot y} \circ \rho(\gamma),$$

$$x = \lim_{p \to \xi^{(1)}(x)} \pi_{x,y}(p), \text{ and}$$

$$y = \lim_{p \to \xi^{(1)}(y)} \pi_{x,y}(p).$$
for all \((x, y) \in \partial_\infty \Gamma(2)\) and \(\gamma \in \Gamma\).

The proof of Proposition \ref{prop:main-application} is given in Section \ref{sec:proof-of-main-application}. Delaying the proof of Proposition \ref{prop:main-application} for a moment, we describe the main application. Suppose that \(\rho : \Gamma \to \text{PGL}_d(\mathbb{R})\) is a 1-Anosov representation and \(M \subset \mathbb{P}^d\) is \(\rho\)-controlled. Let \(\{\pi_{x,y} : (x, y) \in \partial_\infty \Gamma(2)\}\) be a family of maps satisfying Proposition \ref{prop:main-application}. Then define the following space

\[
P(M) := \left\{ (v, p) \in \tilde{U}\Gamma \times M : p \in \xi_1(v^\pm) \text{ and } v = \pi_{v^+,v^-}(p) \right\}.
\]

Notice that there is a natural \(\Gamma\) action on \(P(M)\) given by

\[
\gamma \cdot (v, p) = (\gamma \cdot v, \rho(\gamma)p).
\]

This space has the following properties.

**Observation 3.7.** With the notation above,

1. \(\Gamma\) acts co-compactly on \(P(M)\),
2. for any \(v \in \tilde{U}\Gamma\) and \(z \in M\), there exists \(t \in \mathbb{R}\) such that \((\phi_t(v), z) \in P(M)\),
3. for any compact set \(K \subset \tilde{U}\Gamma\) there exists \(\delta > 0\) such that: If \(v \in K\) and \(p \in M\) satisfies \(d_\rho(\xi_1(v^+), p) \leq \delta\), then \((\phi_t(v), p) \in P(M)\) for some \(t > 0\).

**Proof.** (1): Since the \(\Gamma\)-action on \(\tilde{U}\Gamma\) is co-compact, there exists a compact set \(K \subset \tilde{U}\Gamma\) such that \(\Gamma \cdot K = \tilde{U}\Gamma\). Since the \(\pi_{x,y}\) is a continuous family of maps, the set

\[
\hat{K} := \{(v, z) \in K \times M : v = \pi_{v^+,v^-}(z)\}
\]

is compact. Further, by definition, \(\Gamma \cdot \hat{K} = P(M)\).

(2) Follows directly from the definition.

(3) Fix a compact set \(K \subset \tilde{U}\Gamma\). If such a \(\delta > 0\) does not exist, then there exists \(v_n \in K\), \(p_n \in M\), and \(t_n \leq 0\) such that

\[
d_\rho(\xi_1(v^+_n), p_n) \leq 1/n
\]

and \(\phi_{t_n}(v_n) = \pi_{v_n^+,v_n^-}(p_n)\). By passing to a subsequence we can suppose that \(v_n \to v \in K\). But then \(p_n \to \xi_1(v^+)\) as \(n \to \infty\), so

\[
v^+ = \lim_{n \to \infty} \pi_{v_n^+,v_n^-}(p_n) = \lim_{n \to \infty} \phi_{t_n}(v_n) \in L_{v^+,v^-} \cup \{v^-\}
\]

which is a contradiction. \(\Box\)

**Remark 3.8.** The set \(P(M)\) is designed to be a generalization of the following construction: Suppose \(\Gamma\) is the fundamental group of \(M\) a compact negatively curved Riemannian manifold, \(\tilde{M}\) is the universal cover of \(M\), \(T^1\tilde{M}\) is the unit tangent bundle of \(\tilde{M}\), and \(\phi_t\) is the geodesic flow on \(T^1\tilde{M}\). Then we define

\[
\text{Perp} \subset T^1\tilde{M} \times \partial_\infty \Gamma
\]

to be the set of pairs \((v, z)\) such that there exists \(w \in T^1_{\pi(v)}\tilde{M}\) with \(w \perp v\) and \(\lim_{t \to \infty} \pi(\phi_tw) = z\).
3.1. Properly convex domains. We now describe properly convex domains and some of their relevant properties. These will be used to prove Proposition 3.6.

**Definition 3.9.**

1. An open set $\Omega \subset \mathbb{P}(\mathbb{R}^d)$ is a **properly convex domain** if its closure lies in an affine chart in $\mathbb{P}(\mathbb{R}^d)$, and it is convex, i.e. for every pair of distinct points $x, y \in \Omega$, there is a projective line segment in $\Omega$ whose endpoints are $x$ and $y$.
2. Given a subset $X \subset \mathbb{P}(\mathbb{R}^d)$ the **projective automorphism group** of $X$ is defined to be $$\text{Aut}(X) = \{g \in \text{PGL}_d(\mathbb{R}) : gX = X\}.$$ Given a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$, there is a canonical distance on $\Omega$ which is defined as follows. For any pair of points $x, y \in \Omega$, let $l$ be a projective line through $x$ and $y$, and let $a$ and $b$ be the two points of intersection of $l$ with $\partial \Omega$, ordered so that $a < x \leq y < b$ lie along $l$. Then define $$H_\Omega(x, y) := \log C(a, x, y, b).$$ Here, $C$ is the cross ratio along the projective line $l$, i.e. $$C(a, x, y, b) := \frac{\|a - y\|_2 \|b - x\|_2}{\|a - x\|_2 \|b - y\|_2},$$ where $\|\cdot\|_2$ is the standard norm on some (equiv.) any affine chart of $\mathbb{P}(\mathbb{R}^d)$ containing the closure of $\Omega$. One can verify from properties of the cross ratio that the map $H_\Omega : \Omega \times \Omega \to \mathbb{R}_+ \cup \{0\}$ is a well-defined, continuous, distance function. This is commonly known as the **Hilbert metric** on $\Omega$.

Let $T\Omega$ denote the tangent bundle of $\Omega$ and $\pi : T\Omega \to \Omega$ the natural projection. Also, for any $v \in T\Omega$, let $l_v$ denote the oriented projective line segment in $\Omega$ through $\pi(v)$ in the direction given by $v$, and with endpoints in $\partial \Omega$. Then let $v^+$ and $v^-$ be the forward and backward endpoints of $l_v$ respectively. The Hilbert metric $H_\Omega$ is infinitesimally given by the norm $$h_\Omega : T\Omega \to \mathbb{R} \cup \{0\},$$ $$v \mapsto \|v\|_2 \left(\frac{1}{\|\pi(v) - v^+\|_2} + \frac{1}{\|\pi(v) - v^-\|_2}\right),$$ where $\|\cdot\|_2$ is the standard norm on any affine chart of $\mathbb{P}(\mathbb{R}^d)$ containing the closure of $\Omega$. With this, define the **unit tangent bundle** of $\Omega$ to be $$T^1\Omega = \{v \in T\Omega : h_\Omega(v) = 1\}.$$ We recall the definition of a convex co-compact action on $\Omega$.

**Definition 3.10.** A discrete subgroup $\Lambda \leq \text{PGL}_d(\mathbb{R})$ acts convex co-compactly on a properly convex domain $\Omega$ if $\Lambda \leq \text{Aut}(\Omega)$ and there exists a closed non-empty convex subset $C \subset \Omega$ such that $\Lambda \leq \text{Aut}(C)$ and the quotient $\Lambda \backslash C$ is compact.

**Remark 3.11.** This is not the definition of convex co-compactness used in [DGK17], instead they say groups satisfying Definition 3.10 act **naive convex co-compactly**.

We now use work of Danciger-Gueritaud-Kassel [DGK17] and the second author [Zim17] to construct a convex co-compact action.
Theorem 3.12. [DGK17, Theorem 1.4], [Zim17, Theorem 1.27] Suppose that $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ is a 1-Anosov representation and there exists a properly convex domain $\Omega_0 \subset \mathbb{P}(\mathbb{R}^d)$ with $\rho(\Gamma) \leq \text{Aut}(\Omega_0)$. Then $\rho(\Gamma)$ acts convex co-compactly on a properly convex domain $\Omega \subset \mathbb{P}(\mathbb{R}^d)$. Moreover,

1. $\xi^{(1)}(\partial_\infty \Gamma) \subset \partial \Omega$,
2. for every $x, y \in \partial_\infty \Gamma$ distinct, $\Omega_0$ and $\Omega$ are contained in the same connected component of $\mathbb{P}(\mathbb{R}^d) \setminus \left( \xi^{(d-1)}(x) \cup \xi^{(d-1)}(y) \right)$,
3. we can assume $C = \Omega \cap \text{ConvHull} \left\{ \xi^{(1)}(\partial_\infty \Gamma) \right\}$ and $\overline{C} \cap \partial \Omega = \xi^{(1)}(\partial_\infty \Gamma)$.

Remark 3.13. In [Zim17, Theorem 1.27] it is assumed that $\rho$ is irreducible.

3.2. The proof of Proposition 3.6. We will prove Proposition 3.6 by constructing a projective model of the geodesic flow space $\tilde{U}_\Gamma$. This construction has several steps: first we post compose to obtain a new 1-Anosov representation that preserves a properly convex domain. Theorem 3.12 then gives us a convex co-compact action, which we then use to construct a projective model of the geodesic flow space. Finally we use this projective model to construct maps to $\tilde{U}_\Gamma$.

3.2.1. Constructing an invariant properly convex domain. In general, a 1-Anosov representation will not preserve a properly convex domain. In Section 4 we will study the regularity of these sets.

Example 3.14. [DGK17, Proposition 1.7] If $d$ is even and $\rho : \pi_1(S) \to \text{PGL}_d(\mathbb{R})$ is Hitchin (see Definition 9.1), then $\rho(\pi_1(S))$ does not preserve a properly convex domain.

However, we will show that after post composing with another representation we can always find an invariant properly convex domain.

Denote the vector space of symmetric 2-tensors by $\text{Sym}_2(\mathbb{R}^d)$ and let $D := \text{dim} \text{Sym}_2(\mathbb{R}^d)$. Then let $S : \text{GL}_d(\mathbb{R}) \to \text{GL}(\text{Sym}_2(\mathbb{R}^d))$ be the representation $S(g)(v \otimes v) = gv \otimes gv$.

Then given a representation $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$, let $S(\rho) : \Gamma \to \text{PGL}_d(\mathbb{R})$ be the representation $S(\rho) = S \circ \rho$.

Associated to $S$ are smooth embeddings $\Phi : \mathbb{P}(\mathbb{R}^d) \to \mathbb{P}(\text{Sym}_2(\mathbb{R}^d))$ and $\Phi^* : \text{Gr}_{d-1}(\mathbb{R}^d) \to \text{Gr}_{D-1}(\text{Sym}_2(\mathbb{R}^d))$ defined by $\Phi(v) = [v \otimes v]$ and $\Phi^*(W) = \text{Span}\{v \otimes w + w \otimes v : w \in W, v \in \mathbb{R}^d\}$.

Notice that $\Phi$ and $\Phi^*$ are both $S$-equivariant.

Proposition 3.15. If $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ is 1-Anosov with boundary maps $\xi^{(1)}$ and $\xi^{(d-1)}$, then $S(\rho)$ is 1-Anosov with boundary maps $\Phi \circ \xi^{(1)}$ and $\Phi^* \circ \xi^{(d-1)}$. 

Proof. The maps $\Phi \circ \xi^{(1)}$ and $\Phi^* \circ \xi^{(d-1)}$ are clearly $S(\rho)$-equivariant, dynamics-preserving, and transverse.

Suppose that $\gamma \in \Gamma$ and $\overline{\gamma}$ is a lift of $\rho(\gamma)$. If $\lambda_1 \geq \cdots \geq \lambda_d$ are the absolute values of the eigenvalues of $\overline{\gamma}$, then

$$\lambda_1(S(\overline{\gamma})) = \lambda_1^2,$$

and

$$\lambda_2(S(\overline{\gamma})) = \lambda_1 \lambda_2.$$

So

$$\frac{\lambda_1}{\lambda_2}(S(\rho)(\gamma)) = \frac{\lambda_1}{\lambda_2}(\rho(\gamma)).$$

Then since $\rho$ is 1-Anosov, we see that $S(\rho)$ is also 1-Anosov. □

Now we construct a properly convex domain in $\mathbb{P}(\text{Sym}_2(\mathbb{R}^d))$ which is invariant under the action of $S(\text{PGL}_d(\mathbb{R}))$. Given $X \in \text{Sym}_2(\mathbb{R}^d)$ we say that $X$ is positive definite, and write $X > 0$, if $(f \otimes f)(X) > 0$ for every $f \in \mathbb{R}^d$. Also, we say that $X$ is positive semidefinite, and write $X \geq 0$, if $(f \otimes f)(X) \geq 0$ for every $f \in \mathbb{R}^d$.

Then define

$$\mathcal{P}^+ := \{ [X] : X \in \text{Sym}_2(\mathbb{R}^d), X > 0 \}.$$

Observation 3.16.

1. $\mathcal{P}^+$ is a properly convex domain in $\mathbb{P}(\text{Sym}_2(\mathbb{R}^d))$,
2. $S(\text{PGL}_d(\mathbb{R})) \leq \text{Aut}(\mathcal{P}^+)$,
3. $\Phi(\mathbb{P}(\mathbb{R}^d)) \subset \overline{\mathcal{P}^+}$.

Proof. (1): Clearly $C := \{ X : X \in \text{Sym}_2(\mathbb{R}^d), X > 0 \}$ is a convex open cone in $\text{Sym}_2(\mathbb{R}^d)$. Since

$$(f \otimes f)(X + tY) = (f \otimes f)(X) + t(f \otimes f)(Y)$$

it is clear that $C$ does not contain any real affine lines. Thus $C$ is properly convex. Since $C$ projects to $\mathcal{P}^+$ we see that $\mathcal{P}^+$ is a properly convex domain.

(2): Notice that

$$(f \otimes f)(S(g)X) = ((f \circ g) \otimes (f \circ g))(X)$$

when $f \in \mathbb{R}^d$, $g \in \text{GL}_d(\mathbb{R})$, and $X \in \text{Sym}_2(\mathbb{R}^d)$. So $S(\text{PGL}_d(\mathbb{R})) \leq \text{Aut}(\mathcal{P}^+)$. (3): Suppose that $[v] \in \mathbb{P}(\mathbb{R}^d)$. Then $\Phi([v]) = [v \otimes v]$ and

$$(f \otimes f)(v \otimes v) = f(v)f(v) \geq 0$$

when $f \in \mathbb{R}^d$. Thus $\Phi([v]) \subset \overline{\mathcal{P}^+}$. □

3.2.2. Constructing a projective geodesic flow. In this step we construct a “projective” geodesic flow for 1-Anosov representations that act convex co-compactly on a properly convex domain. For the rest of this subsection, suppose that $\rho : \Gamma \rightarrow \text{PGL}_d(\mathbb{R})$ is a 1-Anosov representation and $\rho(\Gamma)$ acts convex co-compactly on a properly convex domain $\Omega$. We can further assume that

$$\mathcal{C} = \Omega \cap \text{ConvHull} \left\{ \xi^{(1)}(\partial_\infty \Gamma) \right\}.$$
and
\[ \overline{\mathcal{C}} \cap \partial \Omega = \xi^{(1)}(\partial_{\infty} \Gamma). \]

Every projective line segment in \( \Omega \) can be parametrized to be a geodesic in \( H_{\Omega} \) and thus \( T^1 \Omega \) has a natural geodesic flow, which we denote by \( \psi_t \), obtained by flowing along the projective line segments. Using this flow we can construct a model of the flow space \( \tilde{U}_{\Gamma} \).

For \( x, y \in \partial_{\infty} \Gamma \) distinct, let \( \ell_{x,y} \subset T^1 \Omega \) be the unit tangent vectors whose based points are contained in the line segment joining \( \xi^{(1)}(x) \) to \( \xi^{(1)}(y) \) and who point in the direction of \( \xi^{(1)}(y) \). Then the set
\[ G := \bigcup_{(x,y) \in \partial_{\infty} \Gamma} \ell_{x,y} \]
is invariant under the action of \( \rho(\Gamma) \), the flow \( \psi_t \), and the natural \( \mathbb{Z}/2\mathbb{Z} \) action on \( T^1 \Omega \) given by \( v \to -v \). Using Theorem 2.14 we will construct a homeomorphism \( G \to \tilde{U}_{\Gamma} \).

**Corollary 3.17.** With the notation above, there exists a homeomorphism \( T : G \to \tilde{U}_{\Gamma} \) with the following properties:

1. \( T \) is equivariant relative to the \( \Gamma \) and \( \mathbb{Z}/2\mathbb{Z} \) actions
2. for every \( (x,y) \in \partial_{\infty} \Gamma \), \( T \) maps the flow line \( \ell_{x,y} \) to the flow line \( L_{x,y} \).

**Proof.** By construction \( \Gamma \times (\mathbb{R} \rtimes \psi_{\mathbb{Z}/2\mathbb{Z}}) \) acts on \( G \) and the \( \mathbb{R} \) action is free. Further, \( G \) is homeomorphic to \( \partial_{\infty} \Gamma \times \mathbb{R} \) and so we have a homeomorphism
\[ G / \mathbb{R} \to \partial_{\infty} \Gamma \times \mathbb{R}. \]

Hence to apply Gromov’s theorem we just have to verify that \( G \) has a complete metric \( d \) with the following properties:

1. the actions of \( \Gamma \) and \( \mathbb{Z}/2\mathbb{Z} \) are isometric,
2. for every \( v \in G \), the map \( \gamma \in \Gamma \to \gamma \cdot v \in G \) is a quasi-isometry.
3. every \( \mathbb{R} \)-orbit is a quasi-geodesic in \( G \).

To do this, define \( d : G \times G \to \mathbb{R} \) by
\[ d(v,w) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} H_{\Omega}(\gamma_v(t),\gamma_w(t)) e^{-t^2} \, dt \]
where \( \gamma_v \) and \( \gamma_w \) are the unit speed geodesics with \( \gamma'_v(0) = v \) and \( \gamma'_w(0) = w \). Then conditions (1) and (3) are easy to check. To verify condition (2), notice that
\[ |H_{\Omega}(\gamma_v(t),\gamma_w(t)) - H_{\Omega}(\pi(v),\pi(w))| \leq 2 |t| \]
and
\[ \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} 2 |t| e^{-t^2} \, dt = \frac{2}{\sqrt{\pi}}. \]
Hence
\[ H_{\Omega}(\pi(v),\pi(w)) - \frac{2}{\sqrt{\pi}} \leq d(v,w) \leq H_{\Omega}(\pi(v),\pi(w)) + \frac{2}{\sqrt{\pi}}. \]
And so \( \pi : (G,d) \to (C,H_{\Omega}) \) is a quasi-isometry. Since \( (C,H_{\Omega}) \) is a geodesic metric space and \( \Gamma \) acts co-compactly on \( C \), the fundamental lemma of geometric group theory states that for every \( c \in C \), the map \( \gamma \in \Gamma \to \gamma \cdot c \in C \) is a quasi-isometry. So the \( \Gamma \) orbits in \( G \) are also quasi-isometries. □
3.2.3. **Finishing the proof of Proposition 3.6.** Suppose that $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ is a 1-Anosov representation and $M \subset \text{P}(\mathbb{R}^d)$ is $\rho$-controlled.

By Proposition 3.15, the representation $S(\rho) : \Gamma \to \text{PGL}(\text{Sym}_2(\mathbb{R}^d))$ is 1-Anosov with boundary maps $\xi_S^{(1)} := \Phi \circ \xi^{(1)}$ and $\xi_S^{(d-1)} := \Phi^* \circ \xi^{(d-1)}$. By Observation 3.16, $S(\rho)(\Gamma) \leq \text{Aut}(\mathcal{P}^+)$. Then by Theorem 3.12 there exists a properly convex domain $\Omega \subset \text{P}(\text{Sym}_2(\mathbb{R}^d))$ where $S(\rho)(\Gamma)$ acts convex co-compactly on $\Omega$. We can assume that

$$\mathcal{C} = \Omega \cap \text{ConvHull} \left\{ \xi^{(1)}(\partial_{\infty}\Gamma) \right\}$$

and

$$\mathcal{C} \cap \partial \Omega = \xi^{(1)}(\partial_{\infty}\Gamma).$$

Now let $\mathcal{G} \subset T^1\Omega$ be the projective model of the geodesic flow constructed in Section 3.2.2 and let $T : \mathcal{G} \to \overline{UT}$ denote the homeomorphism in Corollary 3.17.

Next, for $x, y \in \partial_{\infty}\Gamma$ distinct we define a projection

$$p_{x,y} : \text{P}\left(\text{Sym}_2(\mathbb{R}^d)\right) \setminus \left(\xi^{(d-1)}_S(x) \cap \xi^{(d-1)}_S(y)\right) \to \xi^{(1)}_S(x) + \xi^{(1)}_S(y)$$

by

$$\{p_{x,y}(v)\} = \left(\xi^{(1)}_S(x) + \xi^{(1)}_S(y)\right) \cap \left(v + \xi^{(d-1)}_S(x) \cap \xi^{(d-1)}_S(y)\right).$$

**Observation 3.18.** If $m \in M \setminus \{\xi^{(1)}(x), \xi^{(1)}(y)\}$, then $p_{x,y}(\Phi(m))$ is contained in the line segment joining $\xi^{(1)}_S(x)$ to $\xi^{(1)}_S(y)$ in $\Omega$.

**Proof.** Since $M$ is $\rho$-controlled,

$$m \notin \xi^{(d-1)}(x) \cup \xi^{(d-1)}(y).$$

Hence

$$\Phi(m) \notin \xi^{(d-1)}_S(x) \cup \xi^{(d-1)}_S(y).$$

Observation 3.16 implies that $\Phi(m) \in \mathcal{P}^+$ and (2) of Theorem 3.12 says that $\mathcal{P}^+$ and $\Omega$ are in the same connected component of

$$\text{P}\left(\text{Sym}_2(\mathbb{R}^d)\right) \setminus \left(\xi^{(d-1)}_S(x) \cup \xi^{(d-1)}_S(y)\right).$$

Hence $p_{x,y}(\Phi(m))$ is contained in the line segment joining $\xi^{(1)}_S(x)$ to $\xi^{(1)}_S(y)$ in $\Omega$. \qed

Next, for $x, y \in \partial_{\infty}\Gamma$ distinct we define a map

$$\hat{p}_{x,y} : M \setminus \{\xi^{(1)}(x), \xi^{(1)}(y)\} \to \mathcal{G}$$

by letting $\hat{p}_{x,y}(m)$ be the unit vector above $\pi(p_{x,y}(\Phi(m)))$ pointing towards $y$.

Finally, we define

$$\pi_{x,y} : M \setminus \{\xi^{(1)}(x), \xi^{(1)}(y)\} \to L_{x,y}$$

by $\pi_{x,y} = T \circ \hat{p}_{x,y}$.

By construction we have

$$\pi_{x,y} = \pi_{\gamma \cdot x, \gamma \cdot y} \circ \rho(\gamma)$$
for all \((x,y) \in \partial \infty \Gamma\) and \(\gamma \in \Gamma\). Further, by (2) of Corollary 3.17
\[
\lim_{p \to \xi(t)(x)} \pi_{x,y}(p) = x \quad \text{and} \quad \lim_{p \to \xi(t)(y)} \pi_{x,y}(p) = y
\]
for all \((x,y) \in \partial \infty \Gamma\).

3.2.4. The construction for non-surface groups: It is worth noting that for many word hyperbolic groups, post composing with the representation \(S : \text{GL}_d(\mathbb{R}) \to \text{GL}(\text{Sym}_d(\mathbb{R}^d))\) is not necessary to construct a convex co-compact action.

**Theorem 3.19.** [Zim17, Theorem 1.25] Suppose \(\Gamma\) is a non-elementary word hyperbolic group which is not commensurable to a non-trivial free product or the fundamental group of a closed hyperbolic surface. Then any irreducible 1-Anosov representation \(\rho : \Gamma \to \text{PGL}_d(\mathbb{R})\) acts convex co-compactly on a properly convex domain \(\Omega \subset \mathbb{P}(\mathbb{R}^d)\).

4. **Sufficient conditions for differentiability of \(\rho\)-controlled subsets**

The goal of this section is to prove Theorem 4.2, which is generalization of Theorem 4.1 in terms of \(\rho\)-controlled subsets of \(\mathbb{P}(\mathbb{R}^d)\) instead of the 1-limit set.

4.1. **The quantity \(\alpha^m(\rho)\).** Suppose that \(\rho\) is \((1,m)\)-Anosov for some \(m = 2, \ldots, d-1\). To state Theorem 4.2, we first define a quantity \(\alpha^m(\rho)\) as follows. Let \(\overline{\rho} : \Gamma \to \text{SL}_d(\mathbb{R})\) be a lift of \(\rho\) (see Remark 2.12), and let \(\|\cdot\|\) be a \(\Gamma\)-invariant norm on \(E\), i.e. \(v \mapsto \|\cdot\|_v\) is a continuous family of norms on \(\mathbb{R}^d\) parameterized by \(\widehat{\Gamma}\), so that \(\|\overline{\rho}(\gamma) \cdot X\|_{\gamma,v} = \|X\|_v\) for all \(\gamma \in \Gamma\), \(v \in \widehat{\Gamma}\) and \(X \in \mathbb{R}^d\). For any \(v = (v^+, v^-, v_0) \in \widehat{\Gamma}\), let
\[
E_1(v) = \xi^{(1)}(v^+),
E_2(v) = \xi^{(d-1)}(v^-) \cap \xi^{(m)}(v^+),
E_3(v) = \xi^{(d-m)}(v^-),
\]
and define \(f : \widehat{\Gamma} \times \mathbb{R} \to \mathbb{R}\) by
\[
f(v, t) := \inf_{X_i \in S_i(v)} \left\{ \log \frac{\|X_3\|_{\overline{\rho}(v)}}{\|X_1\|_{\overline{\rho}(v)}} / \log \frac{\|X_2\|_{\overline{\rho}(v)}}{\|X_1\|_{\overline{\rho}(v)}} \right\},
\]
where \(S_i(v) := \{ X \in E_i(v) : \|X\|_v = 1 \} \) for \(i = 1, 2, 3\). Then define
\[
\alpha^m(\rho) := \liminf_{t \to \infty} \inf_{v \in \widehat{\Gamma}} f(v, t).
\]
To see that \(\alpha^m(\rho)\) is well-defined and strictly larger than 1, we need the following observation.

**Observation 4.1.** There exists \(C_1 \geq 1\) and \(\beta_1 \geq 0\) such that
\[
\frac{1}{C_1} e^{-\beta_1 t} \|X\|_v \leq \|X\|_{\overline{\rho}(v)} \leq C_1 e^{\beta_1 t} \|X\|_v
\]
for all \(v \in \widehat{\Gamma}\), \(t > 0\), and \(X \in \mathbb{R}^d\).
Proof. Since $\Gamma$ acts co-compactly on $\hat{U}\Gamma$ there exists $\beta_1 \geq 0$ such that 
\[ e^{-\beta_1} \|X\|_v \leq \|X\|_{\phi_1(v)} \leq e^{\beta_1} \|X\|_v \]
for all $v \in \hat{U}\Gamma$, $t \in [0, 1]$, and $X \in \mathbb{R}^d$. Then for any $t > 0$, let $k \in \mathbb{Z}^+$ so that $t \in [k^{-1}, k)$, and note that 
\[ e^{-k\beta_1} \|X\|_v \leq \|X\|_{\phi_1(v)} \leq e^{k\beta_1} \|X\|_v . \]

Thus, if we let $C_1 := e^{\beta_1}$, then 
\[ \frac{1}{C_1} e^{-t\beta_1} \|X\|_v \leq \frac{1}{C_1} e^{-(k-1)\beta_1} \|X\|_v \leq \|X\|_{\phi_1(v)} \leq C_1 e^{(k-1)\beta_1} \|X\|_v \leq C_1 e^{t\beta_1} \|X\|_v . \]

By Theorem 2.15, the assumption that $\rho$ is $(1, m)$-Anosov ensures that there are constants $C_2, C_3 \geq 1$ and $\beta_2, \beta_3 \geq 0$ so that for all $v \in \hat{U}\Gamma$, $X_i \in F_i(v)$ non-zero, and $t \geq 0$, we have 
\[ \frac{\|X_2\|_{\phi_1(v)}}{\|X_1\|_{\phi_1(v)}} \geq \frac{1}{C_2} e^{\beta_2 t} \frac{\|X_2\|_v}{\|X_1\|_v} \quad \text{and} \quad \frac{\|X_3\|_{\phi_1(v)}}{\|X_2\|_{\phi_1(v)}} \geq \frac{1}{C_3} e^{\beta_3 t} \frac{\|X_3\|_v}{\|X_2\|_v} . \]

This, together with Observation 4.1, then implies that 
\[ \log \frac{\|X_3\|_{\phi_1(v)}}{\|X_1\|_{\phi_1(v)}} \left/ \log \frac{\|X_2\|_{\phi_1(v)}}{\|X_1\|_{\phi_1(v)}} \right. = 1 + \log \left( \frac{\|X_3\|_{\phi_1(v)}}{\|X_2\|_{\phi_1(v)}} \right) = 1 + \frac{\beta_3 t - \log C_3 + \log \|X_3\|_v}{2\beta_1 t + 2\log C_1 + \log \|X_2\|_v} \]
and 
\[ \log \frac{\|X_2\|_{\phi_1(v)}}{\|X_1\|_{\phi_1(v)}} \left/ \log \frac{\|X_2\|_{\phi_1(v)}}{\|X_1\|_{\phi_1(v)}} \right. \leq 1 + \frac{2\beta_1 t + 2\log C_1 + \log \|X_2\|_v}{\beta_2 t - \log C_2 + \log \|X_2\|_v} \]

In particular, $\alpha^m(\rho)$ is a well-defined real number that is strictly larger than 1. Also, observe that $\alpha^m(\rho)$ does not depend on the choice of $\|\cdot\|$, nor on the choice of lift $\overline{\rho}$ of $\rho$.

With this, we can state the main theorem of this section.

**Theorem 4.2.** Let $m = 2, \ldots, d-1$, and let $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ be $(1, m)$-Anosov. Suppose that $M \subset \mathbb{P}(\mathbb{R}^d)$ is a $\rho$-controlled subset that is also a topological $(m-1)$-dimensional manifold. If

1. $\rho$ is $m$-Anosov and $M$ is $m$-hyperconvex,

then

2. $M$ is $C^\alpha$ along $\xi^{(1)}(\partial_{\infty}\Gamma)$ for all $\alpha$ so that $1 < \alpha < \alpha^m(\rho)$.

Moreover, for all $x \in \partial_{\infty}\Gamma$, the tangent space to $M$ at $\xi^{(1)}(x)$ is $\xi^{(m)}(x)$.

**Remark 4.3.**

1. See Section 1.1 for the definition of “$C^\alpha$ along.”

2. As mentioned in the introduction, in the special case when $M = \xi^{(1)}(\partial_{\infty}\Gamma)$ this theorem was independently proven by Pozzetti-Sambarino-Wienhard [PSW18] without the estimate on $\alpha$. 

It is clear from Example 3.3 and 3.4 that Theorem 1.1 and the first part of Theorem 1.14 follow immediately from Theorem 4.2.

4.2. The key inequality. Suppose that $\rho$ is $(1, m)$-Anosov for some $m = 2, \ldots, d - 1$. Fix a distance $d_\rho$ on $\mathbb{P}(\mathbb{R}^d)$ induced by a Riemannian metric. The following lemma is the key inequality needed to prove Theorem 4.2.

Lemma 4.4. Suppose that $M \subset \mathbb{P}(\mathbb{R}^d)$ be $\rho$-controlled and $m$-hyperconvex. Then for all $\alpha$ satisfying $0 < \alpha < \alpha^m(\rho)$, there exists $D \geq 1$ with the following property: for every $x \in \partial_\infty \Gamma$ and $p \in M$, we have

$$d_\rho \left( p, \xi^{(m)}(x) \right) \leq D d_\rho \left( p, \xi^{(1)}(x) \right)^\alpha.$$  

We prove Lemma 4.4 via a series of small observations. First, from the definition of $\alpha^m(\rho)$, one observes the following.

Observation 4.5. If $0 < \alpha < \alpha^m(\rho)$, then there is a constant $B \geq 1$ so that

$$\frac{\|X_1\|_{\phi_t(v)}}{\|X_3\|_{\phi_t(v)}} \leq B \left( \frac{\|X_1\|_{\phi_t(v)}}{\|X_2\|_{\phi_t(v)}} \right)^\alpha$$

for all $v \in \tilde{\Gamma} T$, $t \geq 0$, and $X_i \in S_i(v)$.

Proof. Since $0 < \alpha < \alpha^m(\rho)$, there exists $T > 0$ such that $\alpha < \alpha(f(v, t))$ for all $t \geq T$ and $v \in \tilde{\Gamma} T$, so

$$\frac{\|X_1\|_{\phi_t(v)}}{\|X_3\|_{\phi_t(v)}} \leq \left( \frac{\|X_1\|_{\phi_t(v)}}{\|X_2\|_{\phi_t(v)}} \right)^\alpha$$

for all $t \geq T$, $v \in \tilde{\Gamma} T$, and $X_i \in S_i(v)$. On the other hand, if $0 \leq t \leq T$, then the $\Gamma$-invariance of $\|\| \|$ implies that both sides of the inequality (8) are continuous positive functions on $[0, T] \times S_\pi$, where $S_\pi \subset E_\pi$ is a compact fiber bundle over $UT$ whose fiber over $[v] \in UT$ is $S_1(v) \times S_2(v) \times S_3(v)$. Thus, there exists some $B \geq 1$ so that (7) holds.

Next, for $i = 1, 2, 3$ and $v \in \tilde{\Gamma} T$ define $P_{i,v} : \mathbb{R}^d \to E_i(v)$ to be the projection with kernel $E_{i-1}(v) + E_{i+1}(v)$, where the arithmetic in the subscripts are done modulo 3.

The following observation is an immediate consequence of the fact that $M$ is $\rho$-controlled and $m$-hyperconvex.

Observation 4.6. If $v \in \tilde{\Gamma} T$ and $p \in M \setminus \{\xi^{(1)}(v^+), \xi^{(1)}(v^-)\}$ then $P_{i,v}(X) \neq 0$ for all non-zero $X \in p$ and $i = 1, 2, 3$.

Choose a compact set $K \subset \tilde{\Gamma} T$ so that $\Gamma \cdot K = \tilde{\Gamma} T$. By enlarging $K$ if necessary, we can ensure that $\{v^+ : v \in K\} = \partial_\infty \Gamma$.

Next let

$$\pi_{x,y} : M \setminus \{\xi^{(1)}(x), \xi^{(1)}(y)\} \to L_{x,y},$$

be a family of maps which satisfy Proposition 3.6. Then, as in Section 3, define

$$P(M) := \{(v, z) \in \tilde{\Gamma} T \times M : p \in \xi^{(1)}(v^+) \text{ and } v = \pi_{v^+, v^-}(p)\}.$$
Using the fact that $\Gamma \backslash P(M)$ is compact (see (1) of Observation 3.7) and Observation 4.6, we deduce the next three observations, which we use to prove Theorem 4.4.

**Observation 4.7.** There is a constant $C \geq 1$ so that

\[ \frac{1}{C} \leq \frac{\| P_{i,v} (X) \|_v}{\| P_{j,v} (X) \|_v} \leq C \]

for all $(v, p) \in P(M)$, all non-zero $X \in p$, and all $i, j \in \{1, 2, 3\}$, and

\[ \frac{1}{C} \leq \frac{\| X \|_v}{\| X \|_2} \leq C \]

for all $v \in K$ and non-zero $X \in \mathbb{R}^d$.

**Proof.** Since $\| \cdot \|$ is $\Gamma$-invariant, Observation 4.6 implies that the map $P(M)/\Gamma \to \mathbb{R}$ defined by

\[ [v, p] \mapsto \frac{\| P_{i,v} (X) \|_v}{\| P_{j,v} (X) \|_v} \]

where $X \in p$ is a non-zero vector, is a well-defined, continuous, positive function on $P(M)/\Gamma$. Hence, (1) of Observation 3.7 implies that there exists $C \geq 1$, so that (9) holds. Also, since the function $K \times P(\mathbb{R}^d) \to \mathbb{R}$ defined by

\[ (v, [X]) \mapsto \frac{\| X \|_v}{\| X \|_2} \]

is also well-defined, continuous, and positive, by further enlarging $C$ if necessary, we may assume that (10) holds. \qed

Using Observations 4.7 and 4.8, we will now prove Lemma 4.4.

**Proof of Lemma 4.4.** Let $\delta > 0$ be sufficiently small so that Observation 4.8 holds. Using (3) of Observation 3.7 and possibly decreasing $\delta > 0$ we may also assume that for all $v \in K$ and $p \in M$ satisfying $d_\varphi (\xi^{(1)}(v^+), p) \leq \delta$, there is some $t > 0$ so that $(\phi_t(v), p) \in P(M)$.

Elementary considerations imply that it is sufficient to prove Lemma 4.4 for all $x \in \partial_\infty \Gamma$ and $p \in M$ so that $d_\varphi (\xi^{(1)}(x), p) \leq \delta$. By the assumptions on $K$, there exist some $v \in K$ such that $v^+ = x$. Further, by our choice of $\delta$, there exists $t > 0$ such that $(\phi_t(v), p) \in P(M)$.
For any non-zero $X \in p$ and for $i = 1, 2, 3$, let
\[ X_i := \frac{P_{i,v}(X)}{\|P_{i,v}(X)\|_v} \in S_i(v). \]

By \((9), (10),\) and \((11),\)
\[ d_p \left( p, \xi^{(m)}(x) \right) \leq A \cdot \frac{\|P_3,v(X)\|_v}{\|P_1,v(X)\|_v} \]
\[ \leq AC^3 \cdot \frac{\|P_3,v(X)\|_v \cdot \|P_1,v(X)\|_{\phi(v)}}{\|P_3,v(X)\|_{\phi(v)}} \]
\[ = AC^3 \cdot \frac{\|X_1\|_{\phi(v)}}{\|X_3\|_{\phi(v)}}. \]

Repeating a similar argument, but with \((12)\) in place of \((11),\) proves
\[ d_p \left( p, \xi^{(1)}(x) \right) \geq \frac{1}{AC^3} \frac{\|X_1\|_{\phi(v)}}{\|X_2\|_{\phi(v)}}. \]

Finally, since $0 < \alpha < \alpha^m(\rho)$, Observation \((4.5)\) and \((13)\) gives
\[ d_p \left( p, \xi^{(m)}(x) \right) \leq ABC^3 \cdot \left( \frac{\|X_1\|_{\phi(v)}}{\|X_2\|_{\phi(v)}} \right)^\alpha. \]

Combining this with \((14)\) yields
\[ d_p \left( p, \xi^{(m)}(x) \right) \leq Dd_p \left( p, \xi^{(1)}(x) \right)^\alpha \]
where $D := A^{1+\alpha}BC^{3+3\alpha}$.

**4.3. Proof of Theorem 4.2.** We now use Lemma \((14)\) to prove Theorem \((4.2)\).

Again, suppose that $\rho$ is $(1,m)$-Anosov for some $m = 2, \ldots, d - 1$.

We begin by making the following simple observation. Fix a hyperplane $\mathcal{H} \subset \mathbb{R}^d$ and a $(d - m)$-dimensional subspace $\mathcal{V} \subset \mathcal{H}$. Then consider the affine chart $\mathcal{A}_\mathcal{H} := \mathbb{P}(\mathbb{R}^d) \setminus [\mathcal{H}]$ of $\mathbb{P}(\mathbb{R}^d)$. Recall that $[\mathcal{H}]$ denotes the projectivization of $\mathcal{H}$, see \((1)\).

For any $m$-dimensional subspace $\mathcal{U} \subset \mathbb{R}^d$ that is transverse to $\mathcal{V}$, let
\[ \Pi_{\mathcal{U},\mathcal{V}} : \mathcal{A}_\mathcal{H} \to [\mathcal{U}] \cap \mathcal{A}_\mathcal{H} \]
be the projection given by $[X] \mapsto [U_X]$, where $X = U_X + V_X$ with $U_X \setminus \{0\} \in \mathcal{U}$ and $V_X \in \mathcal{V}$. Observe that the fibers of $\Pi_{\mathcal{U},\mathcal{V}}$ are of the form $[\mathbb{R} \cdot X + \mathcal{V}] \cap \mathcal{A}_\mathcal{H}$ for some $X \in \mathbb{R} \setminus \mathcal{H}$. In particular, the fibers of $\Pi_{\mathcal{U},\mathcal{V}}$ do not depend on $\mathcal{U}$, i.e. if $\mathcal{U}' \subset \mathbb{R}^d$ is another $m$-dimensional subspace of $\mathbb{R}^d$ that is transverse to $\mathcal{V}$, then the fibers of $\Pi_{\mathcal{U}',\mathcal{V}}$ and the fibers of $\Pi_{\mathcal{U},\mathcal{V}}$ agree.

Now, fix $y \in \partial_\infty \Gamma$. We will specialize the observation in the previous paragraph to the case where $\mathcal{H} = \xi^{(d-1)}(y)$ and $\mathcal{V} = \xi^{(d-m)}(y)$. This yields the following statement, which we record as an observation.

**Observation 4.9.** Let $\mathcal{A}_y := \mathcal{A}_{\xi^{(d-1)}(y)}$. If $x \in \partial_\infty \Gamma \setminus \{y\}$, then
\[ \Pi_{x,y} := \Pi_{\xi^{(m)}(x),\xi^{(d-m)}(y)} : \mathcal{A}_y \to \xi^{(m)}(z) \cap \mathcal{A}_y \]
is a projection whose fibers do not depend on $x$. 

\[ \square \]
Lemma 4.10. If \( x \in \partial_\infty \Gamma \setminus \{ y \} \), then the map
\[
F_{x,y} : M \setminus \{ \xi^{(1)}(y) \} \to \xi^{(m)}(x) \cap A_y.
\]
is a homeomorphism.

Remark 4.11. Lemma 4.10 implies that \( M \setminus \{ \xi^{(1)}(y) \} \) can be viewed as the graph of a map from
\[
\xi^{(m)}(x) \cap A_y \quad \text{to} \quad \Pi^{-1}_{x,y}(\xi^{(1)}(x)) = \left( \xi^{(1)}(x) + \xi^{(d-m)}(y) \right) \cap A_y.
\]
In particular, \( M \setminus \{ \xi^{(1)}(y) \} \) is diffeomorphic to \( \mathbb{R}^{m-1} \).

The proof of Lemma 4.10 requires a basic result from topology.

Theorem 4.12 (The Invariance of Domain Theorem). If \( U \subset \mathbb{R}^d \) is open and \( f : U \to \mathbb{R}^d \) is continuous injective map, then \( f(U) \) is open and \( f \) induces a homeomorphism \( U \to f(U) \).

Proof of Lemma 4.10. We first observe that the map \( F_{x,y} \) is injective. If \( p_1, p_2 \in M \setminus \{ \xi^{(1)}(y) \} \) and \( F_{x,y}(p_1) = F_{x,y}(p_2) \), then
\[
p_1 + \xi^{(d-m)}(y) = p_2 + \xi^{(d-m)}(y),
\]
so \( p_1 + p_2 + \xi^{(d-m)}(y) \) is not direct. The assumption that \( M \) is \( m \)-hyperconvex implies that \( p_1 = p_2 \).

Since \( F_{x,y} \) is continuous and injective, we can now apply the invariance of domain theorem to deduce that \( F_{x,y} \) is a homeomorphism onto an open set in \( I(x,y) \) in \( \xi^{(m)}(x) \cap A_y \). To finish the proof, we now need to show that
\[
I(x,y) = \xi^{(m)}(x) \cap A_y.
\]

Suppose \( \gamma \in \Gamma \) has infinite order, and denote its attracting and repelling fixed points in \( \partial_\infty \Gamma \) by \( \gamma^+ \) and \( \gamma^- \) respectively. Note that \( \rho(\gamma) : I(\gamma^+, \gamma^-) = I(\gamma^+, \gamma^-) \) and \( \xi^{(1)}(\gamma^+) \in I(\gamma^+, \gamma^-) \). Since \( \rho \) is 1-Anosov,
\[
\xi^{(m)}(\gamma^+) \cap A_{\gamma^-} = \bigcup_{n \in \mathbb{N}} \rho(\gamma)^{-n} \cdot O
\]
for any open set \( O \subset \xi^{(m)}(\gamma^+) \cap A_{\gamma^-} \) containing \( \xi^{(1)}(\gamma^+) \). Hence
\[
I(\gamma^+, \gamma^-) = \bigcup_{n \in \mathbb{N}} \rho(\gamma)^{-n} \cdot I(\gamma^+, \gamma^-) = \xi^{(m)}(\gamma^+) \cap A_{\gamma^-}.
\]
The density of \( \{ (\gamma^+, \gamma^-) : \gamma \in \Gamma \) has infinite order \} \) in \( \partial_\infty \Gamma \times \partial_\infty \Gamma \) proves (15). \( \square \)

With Lemma 4.10, we can now proceed to the proof of Theorem 4.12.

Proof of Theorem 4.12. Fix \( y \in \partial_\infty \Gamma \), and as before, consider the affine chart
\[
A_y := \mathbb{P}(\mathbb{R}^d) \setminus \xi^{(d-1)}(y).
\]
By working in some particular affine coordinates in the affine chart \( A_y \), we will show that Theorem 4.12 holds for all \( x \in \partial_\infty \Gamma \setminus \{ y \} \). Since \( y \) was chosen arbitrarily, this suffices to prove the theorem.

Let \( x \in \partial_\infty \Gamma \setminus \{ y \} \) and choose affine coordinates \( A_y \simeq \mathbb{R}^{d-1} \) so that in these coordinates,
\( \xi^{(1)}(x) = 0 \),
\( \xi^{(m)}(x) \cap A_y = \mathbb{R}^{m-1} \times \{0\} \),
\( (\xi^{(1)}(x) + \xi^{(d-m)}(y)) \cap A_y = \{0\} \times \mathbb{R}^{d-m} \).

For any \( z \in \partial_\infty \Gamma \) sufficiently close to \( x \), there exists a unique affine map
\[
A_z : \mathbb{R}^{m-1} \times \{0\} \to \{0\} \times \mathbb{R}^{d-m}
\]
whose graph is \( H_z := \xi^{(m)}(z) \cap A_y \), i.e.
\[
H_z = \{ u + A_z(u) : u \in \mathbb{R}^{m-1} \times \{0\} \},
\]
see Figure 1. Let \( L_z : \mathbb{R}^{m-1} \times \{0\} \to \{0\} \times \mathbb{R}^{d-m} \) denote the linear part of \( A_z \) (in our choice of affine coordinates). Note that the maps \( z \mapsto A_z \) and \( z \mapsto L_z \) are continuous.

For any \( z \in \partial_\infty \Gamma \setminus \{y\} \), Observation 4.9 implies that \( \Pi^{-1}_{x,y}(\xi^{(1)}(z)) \) is parallel to \( \Pi^{-1}_{x,y}(\xi^{(1)}(x)) = \{0\} \times \mathbb{R}^{d-m} \) in \( A_y \). Thus, as a consequence of Lemma 4.10, there exists a map
\[
f_z : H_z \to \{0\} \times \mathbb{R}^{d-m}
\]
whose graph is \( \xi^{(1)}(\partial_\infty \Gamma \setminus \{y\}) \), i.e.
\[
\xi^{(1)}(\partial_\infty \Gamma \setminus \{y\}) = \{u + f_z(u) : u \in H_z\}.
\]
Further, Theorem 4.4 implies that for all \( \alpha \) satisfying \( 1 \leq \alpha < \alpha^m(\rho) \) and all \( \xi^{(1)}(z) + h \in H_z \), we have
\[
f_z (\xi^{(1)}(z) + h) = o(\|h\|^\alpha).
\]
Now, for any \( u \in \mathbb{R}^{m-1} \times \{0\} \) and \( z \in \partial_\infty \Gamma \setminus \{y\} \),
\[
u + f_z(u) = \left( u + A_z(u) \right) + f_z \left( u + A_z(u) \right).
\]
Also, if \( u_z := \Pi_{x,y}(\xi^{(1)}(z)) \) then \( u_z + A_z(u_z) = \xi^{(1)}(z) \), which means that \( f_x(u_z) = A_z(u_z) \). Thus, for all \( h \in \mathbb{R}^{m-1} \times \{0\} \),

\[
\begin{align*}
f_x(u_z + h) &= A_z(u_z + h) + f_x(u_z + h + A_z(u_z + h)) \\
&= A_z(u_z) + L_z(h) + f_x(u_z + h + A_z(u_z) + L_z(h)) \\
&= f_x(u_z) + L_z(h) + f_x(\xi^{(1)}(z) + h + L_z(h)) \\
&= f_x(u_z) + L_z(h) + o(\|h + L_z(h)\|^\alpha). \\
&= f_x(u_z) + L_z(h) + o(\|h\|^\alpha).
\end{align*}
\]

This proves the theorem. \( \square \)

5. Eigenvalue description of \( \alpha^m(\rho) \)

For the rest of this section, let \( \rho : \Gamma \to \text{PGL}(d, \mathbb{R}) \) be a \((1, m)\)-Anosov representation. Recall that in the introduction, we defined

\[
\alpha_m(\rho) := \inf_{\gamma \in \Gamma} \left\{ \log \frac{\lambda_1}{\lambda_{m+1}}(\rho(\gamma)) \middle/ \log \frac{\lambda_1}{\lambda_m}(\rho(\gamma)) : \frac{\lambda_1}{\lambda_m}(\rho(\gamma)) \neq 1 \right\}.
\]

The main result of this section is the following theorem.

**Theorem 5.1.** If \( \rho \) is irreducible, then

\[
\alpha_m(\rho) = \alpha^m(\rho),
\]

where \( \alpha^m(\rho) \) is the quantity defined by (1).

The proof of Theorem 5.1 will be given in the following two subsections. In the first, we will use general properties of singular values to relate the quantity \( f(v, t) \) (the function \( f \) was defined by (1)) to the ratios of eigenvalues of \( \rho(\gamma) \) when \( v^\pm = \gamma^\pm \). In the second, we will use a deep result due to Benoist to finish the proof.

Before starting the proof we make several reductions. First, \( \alpha_m(\rho) \) and \( \alpha^m(\rho) \) are invariant under passing to a finite index subgroup (see Remark 2.12). So by Remark 2.12, we can assume that \( \rho \) admits a lift \( \tilde{\rho} : \Gamma \to \text{SL}_d(\mathbb{R}) \). Then, by passing to another finite index subgroup, we may also assume that the Zariski closure of \( \rho(\Gamma) \) is connected. By Proposition 2.9, this representation is still irreducible.

### 5.1. Singular values along closed orbits.

Let \( E := \widetilde{U} \times \mathbb{R}^{d} \), and for \( i = 1, 2, 3 \), let \( E_i \) be the \( \Gamma \)-invariant sub-bundle of \( E \) defined by (3). (Recall that the \( \Gamma \)-action on \( E \) is given by \( \gamma \cdot (v, X) = (\gamma \cdot v, \rho(\gamma) \cdot X) \).) Also, choose a \( \Gamma \)-invariant inner product \( \langle \cdot, \cdot \rangle \) on \( E \) so that \( E = E_1 \oplus E_2 \oplus E_3 \) is an orthogonal splitting. We may assume that the norm \( \|\cdot\| \) used in the definition of \( \alpha^m(\rho) \) is given by \( \|\cdot\|_v = \sqrt{\langle \cdot, \cdot \rangle_v} \) for all \( v \in \widetilde{U} \).

For any \( v, w \in \widetilde{U} \), let \( \sigma_i(v, w) \) denote the \( i \)-th singular value of

\[
\text{id} = \text{id}_{v,w} : (\mathbb{R}^d, \|\cdot\|_v) \to (\mathbb{R}^d, \|\cdot\|_w),
\]

where \( \text{id} = \text{id}_{v,w} \) is the identity map.
The functions \( h \) and \( f \) (recall that \( f \) is defined by (4)) are related by the following lemma.

**Lemma 5.2.** For all \( v \in \widehat{\Gamma} \) and for sufficiently large \( t \), we have

\[
 f(v, t) = h(v, t).
\]

In particular, \( \alpha^m(\rho) = \liminf_{t \to \infty} \inf_{v \in \widehat{\Gamma}} h(v, t) \).

**Proof.** Since \( E = E_1 \oplus E_2 \oplus E_3 \) is an orthogonal splitting, Theorem 2.15 implies that for all \( v \in \widehat{\Gamma} \) and for sufficiently large \( t \),

- \( \sigma_d(v, t) = \| X \|_{\phi_t(v)} \) for all \( X \in S_1(v) \),
- \( \sigma_{d-m+1}(v, t) = \sup_{X \in S_2(v)} \| X \|_{\phi_t(v)} \),
- \( \sigma_{d-m}(v, t) = \inf_{X \in S_3(v)} \| X \|_{\phi_t(v)} \).

Thus,

\[
 h(v, t) = \log \frac{\sigma_{d-m}(v, t)}{\sigma_d(v, t)} / \log \frac{\sigma_{d-m+1}(v, t)}{\sigma_d(v, t)} = \log \frac{\inf_{X \in S_3(v)} \| X \|_{\phi_t(v)}}{\sup_{X \in S_1(v)} \| X \|_{\phi_t(v)}} / \log \frac{\sup_{X \in S_2(v)} \| X \|_{\phi_t(v)}}{\inf_{X \in S_1(v)} \| X \|_{\phi_t(v)}} = \inf_{X_i \in S_i(v)} \left\{ \log \frac{\| X_i \|_{\phi_t(v)}}{\| X_1 \|_{\phi_t(v)}} / \log \frac{\| X_2 \|_{\phi_t(v)}}{\| X_1 \|_{\phi_t(v)}} \right\} = f(v, t)
\]

Notice that in the third equality we used the fact that \( \dim E_1(v) = 1 \).

The following observation gives a simple but important bound for ratios of singular values. The proof is a straightforward calculation which we omit.

**Observation 5.3.** Suppose that for \( i = 1, \ldots, 4 \), \( \| \cdot \|_{(i)} \) are norms on \( \mathbb{R}^d \) so that for all \( X \in \mathbb{R}^d \), \( \frac{1}{A} \leq \frac{\| X \|_{(1)}}{\| X \|_{(2)}} \leq A \) and \( \frac{1}{A'} \leq \frac{\| X \|_{(3)}}{\| X \|_{(4)}} \leq A' \) for some \( A, A' > 1 \). Let \( L : \left( \mathbb{R}^d, \| \cdot \|_{(1)} \right) \to \left( \mathbb{R}^d, \| \cdot \|_{(3)} \right) \) and \( L' : \left( \mathbb{R}^d, \| \cdot \|_{(2)} \right) \to \left( \mathbb{R}^d, \| \cdot \|_{(4)} \right) \) denote the identity maps. Then

\[
 \frac{1}{AA'} \leq \frac{\sigma_1(L)}{\sigma_1(L')} \leq AA'.
\]

The next lemma relates the function \( h \) to the eigenvalues of \( \rho(\gamma) \).

**Lemma 5.4.** Let \( \gamma \in \Gamma \setminus \{ \text{id} \} \) be an infinite order element, and let \( v = (v^+, v^-, v_0) \in \widehat{\Gamma} \) so that \( v^\pm = \gamma^\pm \). Then

\[
 \lim_{t \to \infty} h(v, t) = \log \frac{\lambda_1}{\lambda_{m+1}}(\rho(\gamma)) / \log \frac{\lambda_1}{\lambda_m}(\rho(\gamma)).
\]
Proof. Let $T$ denote the period of $\gamma$ (see Section 2.4). For all $k \in \mathbb{Z}_+^+$ and $X \in \mathbb{R}^d$

$$\|X\|_{\phi_{kT}(v)} = \|X\|_{\gamma^k \cdot v} = \|\bar{\rho}(\gamma^{-k}) \cdot X\|_v.$$ 

Hence, the singular values of the two linear maps

$$\text{id}: (\mathbb{R}^d, \|\cdot\|_v) \to (\mathbb{R}^d, \|\cdot\|_{\phi_{kT}(v)}) \quad \text{and} \quad \bar{\rho}(\gamma^{-k}) : (\mathbb{R}^d, \|\cdot\|_v) \to (\mathbb{R}^d, \|\cdot\|_v)$$

agree.

It is a straightforward calculation to show that for any inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^d$ and any invertible linear map $\bar{\gamma} : (\mathbb{R}^d, \langle \cdot, \cdot \rangle) \to (\mathbb{R}^d, \langle \cdot, \cdot \rangle)$,

$$\lim_{k \to \infty} \frac{1}{k} \log \sigma_i(\bar{\gamma}^k) = \log \lambda_i(\bar{\gamma}).$$

Thus, we can deduce that

$$\lim_{k \to \infty} \frac{1}{k} \log \sigma_i(v, kT)^{\frac{1}{i}} = \lim_{k \to \infty} \sigma_i(\bar{\rho}(\gamma^{-k}))^{\frac{1}{i}} = \frac{1}{|\lambda_1(\bar{\rho}(\gamma))|},$$

which implies that

$$\lim_{k \to \infty} h(v, kT) = \lim_{k \to \infty} \left( \log \frac{\sigma_{d-m}(v, kT)}{\sigma_d(v, kT)} \right) = \log \frac{\lambda_1}{\lambda_{m+1}}(\rho(\gamma)) / \log \frac{1}{\lambda_m}(\rho(\gamma)).$$

For any $t > 0$, let $k \in \mathbb{Z}_+$ so that $t \in [kT, (k + 1)T)$. Then Lemma 4.4 implies that there are constants $C \geq 1$ and $\beta \geq 0$ so that

$$\frac{1}{C} e^{-\beta T} \leq \frac{\|X\|_{\phi_{kT}v}}{\|X\|_{\phi_{kT}v}} \leq C e^{\beta T}$$

for all $t \in \mathbb{R}$ and $X \in \mathbb{R}^d$. This, together with Observation 5.3 implies that for all $i = 1, \ldots, d$,

$$\frac{1}{C} e^{-\beta T} \leq \frac{\sigma_i(v, kT)}{\sigma_i(v, t)} \leq C e^{\beta T}.$$ 

Also, since $\rho$ is $(1, m)$-Anosov, we know that

$$\lim_{k \to \infty} \log \frac{\sigma_{d-m}(v, kT)}{\sigma_d(v, kT)} = \infty = \lim_{k \to \infty} \log \frac{\sigma_{d-m+1}(v, kT)}{\sigma_d(v, kT)}.$$ 

Hence,

$$\limsup_{t \to \infty} h(v, t) = \limsup_{t \to \infty} \log \frac{\sigma_{d-m}(v, t)}{\sigma_d(v, t)} \geq \log \frac{\sigma_{d-m+1}(v, t)}{\sigma_d(v, t)} \leq \limsup_{k \to \infty} \frac{2 \log C + 2 \beta T + \log \sigma_{d-m}(v, kT)}{-2 \log C - 2 \beta T + \log \sigma_{d-m+1}(v, kT)}$$

$$= \lim_{k \to \infty} h(v, kT).$$

By a similar argument, $\liminf_{t \to \infty} h(v, t) \geq \lim_{k \to \infty} h(v, kT)$, so $\lim_{t \to \infty} h(v, t) = \lim_{k \to \infty} h(v, kT)$. This, together with (18) implies the lemma.

$\square$
5.2. Asymptotic cones and eigenvalues. Recall that $\lambda, \mu : GL_d(\mathbb{R}) \to \mathbb{R}^d$ respectively denote the Jordan and Cartan projections defined in Section 2.3. For any subgroup $G \leq SL_d(\mathbb{R})$, let $C_\lambda(G) \subset \mathbb{R}^d$ denote the smallest closed cone containing $\lambda(G)$, that is

$$C_\lambda(G) := \bigcup_{\gamma \in G} \mathbb{R}_{>0} \cdot \lambda(\gamma).$$

Also, let $C_\mu(G)$ denote the asymptotic cone of $\mu(G)$, that is

$$C_\mu(G) := \{ x \in \mathbb{R}^d : \exists \bar{y}_n \in G, \exists t_n \to 0, \text{ with } \lim_{n \to \infty} t_n \mu(\bar{y}_n) = x \}.$$

A deep result of Benoist [Ben97] implies the following.

**Theorem 5.5.** If $G \leq SL_d(\mathbb{R})$ is a connected semisimple real algebraic subgroup which acts irreducibly on $\mathbb{R}^d$ and $\Lambda \leq G$ is a Zariski dense subgroup, then

$$C_\mu(\Lambda) = C_\lambda(\Lambda).$$

**Remark 5.6.** Notice that for any subgroup $\Lambda \subset SL_d(\mathbb{R})$, the fact that $C_\lambda(\Lambda) \subset C_\mu(\Lambda)$ is a consequence of [16].

A proof of Theorem 5.5 is given in the appendix. Theorem 5.5 can be used to prove the following lemma.

**Lemma 5.7.** For any $\epsilon > 0$ there exists $R > 0$ such that

$$\alpha_m(\rho) - \epsilon < \log \frac{\mu_1}{\mu_{m+1}}(\rho(\gamma)) \left/ \log \frac{\mu_1}{\mu_m}(\rho(\gamma)) \right.$$

for all $\gamma \in \Gamma$ with $||\mu(\rho(\gamma))||_2 \geq R$.

**Proof.** By Proposition 2.8 and Theorem 5.5, $C_\mu(\bar{\rho}(\Gamma)) = C_\lambda(\bar{\rho}(\Gamma))$. Fix $\epsilon > 0$ and suppose for contradiction that there exists a sequence $\{\gamma_n\}_{n=1}^\infty \subset \Gamma$ such that for all $n$, $||\mu(\rho(\gamma_n))||_2 \geq n$ and

$$\alpha_m(\rho) - \epsilon \geq \log \frac{\mu_1}{\mu_{m+1}}(\rho(\gamma_n)) \left/ \log \frac{\mu_1}{\mu_m}(\rho(\gamma_n)) \right.$$

By passing to a subsequence we can suppose that

$$\frac{1}{||\mu(\rho(\gamma_n))||_2} \mu(\rho(\gamma_n)) \to x = (x_1, \ldots, x_d) \in C_\mu(\bar{\rho}(\Gamma)) = C_\lambda(\bar{\rho}(\Gamma)).$$

It follows that $\alpha_m(\rho) - \epsilon \geq \frac{x_1 - x_{m+1}}{x_1 - x_m}$. On the other hand, the definition of $\alpha_m(\rho)$ and $C_\lambda(\bar{\rho}(\Gamma))$, implies that

$$\alpha_m(\rho) \leq \frac{x_1 - x_{m+1}}{x_1 - x_m},$$

which is a contradiction. \qed

**Proof of Theorem 5.1.** It is clear from Lemma 5.2 and Lemma 5.4 that $\alpha_m(\rho) \leq \alpha_m(\rho)$. We will now prove $\alpha_m(\rho) \geq \alpha_m(\rho)$. Let $K \subset \tilde{U}$ be a compact fundamental domain for the $\Gamma$-action on $\tilde{U}$. Since $h(v, t) = h(\gamma \cdot v, t)$ for all $\gamma \in \Gamma$ and all $v \in \tilde{U}$, by Lemma 5.2 it is enough to show that

$$\alpha_m(\rho) \leq \liminf_{t \to \infty} \inf_{v \in K} h(v, t)$$
Fix $C > 1$ such that $\frac{1}{C'} \|X\|_2 \leq \|X\|_v \leq C \|X\|_2$ for all $v \in K$ and $X \in \mathbb{R}^d$. By Lemma 5.7 there exists, for every $\epsilon > 0$, a positive number $R' > 0$ such that
\[
\alpha_m(\rho) - \epsilon < \log \frac{\mu_1}{\mu_{m+1}}(\rho(\gamma)) \log \frac{\mu_1}{\mu_{m+1}}(\rho(\gamma))
\]
for all $\gamma \in \Gamma$ with $\|\mu(\varphi(\gamma))\|_2 \geq R'$. Since $\rho$ is 1-Anosov and
\[
\log \frac{\mu_1}{\mu_2}(\rho(\gamma)) \leq \log \frac{\mu_1}{\mu_k}(\rho(\gamma))
\]
for $k > 1$, Theorem 2.5 and Corollary 2.7 together imply that
\[
\log \frac{\mu_1}{\mu_{m+1}}(\rho(\gamma)) \log \frac{\mu_1}{\mu_{m+1}}(\rho(\gamma)) \geq \frac{1}{A''} \|\mu(\varphi(\gamma))\|_2 - B''
\]
for some $A'' \geq 1$ and $B'' \geq 0$. Hence, there exists $R \geq R'$ such that
\[
\alpha_m(\rho) - 2\epsilon < \left( \log \frac{\mu_1}{\mu_{m+1}}(\rho(\gamma)) - 4\log C \right) \left( \log \frac{\mu_1}{\mu_{m+1}}(\rho(\gamma)) + 4\log C \right)
\]
for all $\gamma \in \Gamma$ with $\|\mu(\varphi(\gamma))\|_2 \geq R$.

Let $d = d_{\tilde{U} \Gamma}$ denote the $\Gamma$-invariant metric on $\tilde{U} \Gamma$ specified in Section 2.4 and let $D$ be the diameter of $K$. By Corollary 2.7 and the fact that any $\Gamma$-orbit in $\tilde{U} \Gamma$ is a quasi-isometry, there exists $A \geq 1$ and $B \geq 0$ such that
\[
\frac{1}{A} \|\mu(\varphi(\gamma))\|_2 - B \leq d(v, \gamma \cdot v) \leq A \|\mu(\varphi(\gamma))\|_2 + B
\]
for all $v \in \tilde{U} \Gamma$. Also, since every $\phi_t$-orbit in $\tilde{U} \Gamma$ is a quasi-isometric embedding, there exists $A' \geq 1$ and $B' \geq 0$ so that,
\[
\frac{1}{A'}|t| - B' \leq d(v, \phi_t(v)) \leq A'|t| + B'
\]
for all $t \in \mathbb{R}$ and $v \in \tilde{U} \Gamma$.

Fix $t > A'(B' + D + AR + B)$ and $v \in K$. Let $\gamma \in \Gamma$ such that $\gamma^{-1} \cdot \phi_t(v) \in K$. By the definition of $C$, we see that for any $X \in \mathbb{R}^d,$
\[
\frac{1}{C} \leq \frac{\|X\|_v}{\|X\|_2} \frac{\|\mu(\gamma)^{-1} \cdot X\|_{\gamma^{-1}, \varphi_t(v)}}{\|\mu(\gamma)^{-1} \cdot X\|_2} \leq C.
\]

Since $\|\mu(\gamma)^{-1} \cdot X\|_{\gamma^{-1}, \varphi_t(v)} = \|X\|_{\varphi_t(v)}$ and $X \mapsto \|\mu(\gamma)^{-1} \cdot X\|_2$ are both norms on $\mathbb{R}^d$, it follows from Proposition 5.3 that
\[
\frac{1}{C^2} \frac{1}{\mu_{d+1-l}(\mu(\gamma))} = \frac{1}{C^2} \mu_l(\varphi(\gamma)^{-1}) \leq \sigma_l(v, t) \leq C^2 \mu_l(\varphi(\gamma)^{-1}) = C^2 \frac{1}{\mu_{d+1-l}(\varphi(\gamma))}.
\]

Also, $d(\gamma \cdot v, v) \geq d(v, \phi_t(v)) - d(\phi_t(v), \gamma \cdot v) \geq \frac{1}{A}t - B' - D$, which means
\[
\|\mu(\varphi(\gamma))\|_2 \geq \frac{1}{A} (d(\gamma \cdot v, v) - B) \geq \frac{\frac{1}{A}t - B' - D - B}{A} \geq R.
\]
Hence,
\[ h(v,t) = \log \frac{\sigma_{d-m}(v,t)}{\sigma_d(v,t)} \geq \left( \log \frac{\mu_1}{\mu_{m+1}}(\rho(\gamma)) - 4 \log C \right) \geq \left( \log \frac{\mu_1}{\mu_m}(\rho(\gamma)) + 4 \log C \right) \]
\[ > \alpha_m(\rho) - 2\epsilon. \]
Since \( v \in K \) and \( t > A'(B' + D + AR + B) \) was arbitrary,
\[ \alpha_m(\rho) - 2\epsilon \leq \lim\inf_{t \to \infty} \inf_{v \in K} h(v,t). \]
Then since \( \epsilon > 0 \) was also arbitrary we see that
\[ \alpha_m(\rho) \leq \lim\inf_{t \to \infty} \inf_{v \in K} h(v,t). \]

6. Optimal regularity

In this section we prove Theorem 1.12 and the second part of Theorem 1.14. By Example 3.3 and 3.4, it is sufficient to prove the following theorem.

**Theorem 6.1.** Suppose that \( \rho: \Gamma \to \text{PGL}_d(\mathbb{R}) \) is an irreducible, \((1,m)\)-Anosov representation for some \( m = 2, \ldots, d-1 \), and suppose that \( M \subset \mathbb{P}(\mathbb{R}^d) \) is a \( \rho \)-controlled, \( m \)-hyperconvex, topological \((m-1)\)-dimensional submanifold. Then
\[ \alpha_m(\rho) \leq \sup \left\{ \alpha \in (1,2) : M \text{ is } C^\alpha \text{ along } \xi^{(1)}(\partial_\infty \Gamma) \right\} \]
with equality if
\[ (*) \ M \cap (p_1 + p_2 + \xi^{(d-m)}(y)) \text{ spans } p_1 + p_2 + \xi^{(d-m)}(y) \text{ for all pairwise distinct } p_1, p_2, \xi^{(1)}(y) \in M. \]

As mentioned in the introduction (see (2) of Remark 1.13), the condition (*) is trivial when \( m = 2 \) and \( m = d-1 \). In Section 6.1, we show that when \( M = \xi^{(1)}(\partial_\infty \Gamma) \), (*) is an open condition in \( \text{Hom}(\Gamma, \text{PSL}_d(\mathbb{R})) \). Then, in Section 6.2, we prove Theorem 6.1.

6.1. Stability of hypotheses. To show that (*) is an open condition when \( M = \xi^{(1)}(\partial_\infty \Gamma) \), we use the following two statements. The first is a standard fact about hyperbolic groups.

**Proposition 6.2.** The \( \Gamma \)-action on \( \partial_\infty \Gamma^{(3)} := \{(x, y, z) \in \partial_\infty \Gamma^3 : x, y, z \text{ distinct}\} \) is co-compact.

The second is a well-known result about Anosov representations due to Guichard-Wienhard. In the case when \( \Gamma \) is the fundamental group of a negatively curved Riemannian manifold, Theorem 6.3 was established by Labourie [Lab06, Proposition 2.1].

**Theorem 6.3.** [GW12, Theorem 5.13] Let
\[ \mathcal{O}_k := \{ \rho \in \text{Hom}(\Gamma, \text{PGL}_d(\mathbb{R})) : \rho \text{ is } k\text{-Anosov}\}. \]
Then \( \mathcal{O}_k \) is open, and the map
\[ \rho \in \mathcal{O}_k \to \xi^{(k)}_\rho \in C \left( \partial_\infty \Gamma, \text{Gr}_k(\mathbb{R}^d) \right) \]
is continuous.
If $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ is a $(1,m)$-Anosov representation for some $m = 2, \ldots, d-1$, let $\xi_p^{(k)} : \partial_{\infty} \Gamma \to \text{Gr}_k(\mathbb{R}^d)$ denote the $k$-flag map of $\rho$ for $k = 1, m, d-m, d-1$.

**Corollary 6.4.** Suppose $\partial_{\infty} \Gamma$ is a topological $(m-1)$-manifold, and $\rho_0 : \Gamma \to \text{PGL}_d(\mathbb{R})$ is a $(1,m)$-Anosov representation. If $\xi_p^{(1)}(x) + \xi_p^{(1)}(z) + \xi_p^{(d-m)}(y)$ is a direct sum and

$$\xi_p^{(1)}(\partial_{\infty} \Gamma) \cap \left( \xi_p^{(1)}(x) + \xi_p^{(1)}(z) + \xi_p^{(d-m)}(y) \right)$$

spans $\xi_p^{(1)}(x), \xi_p^{(1)}(z), \xi_p^{(d-m)}(y)$ for all $x, y, z \in \partial_{\infty} \Gamma$ distinct, then any sufficiently small deformation of $\rho_0$ also has these properties.

**Proof.** It follows easily from Theorem 6.3 and Proposition 6.2 that there exists a neighborhood $O \subset \text{Hom}(\Gamma, \text{PGL}_d(\mathbb{R}))$ of $\rho_0$ with the following property: if $\rho \in O$, then $\rho$ is a $(1,m)$-Anosov representation and $\xi_p^{(1)}(x) + \xi_p^{(1)}(z) + \xi_p^{(d-m)}(y)$ is a direct sum for all $x, y, z \in \partial_{\infty} \Gamma$ distinct.

By Proposition 6.2, it is enough to fix $(x_0, y_0, z_0) \in \partial_{\infty} \Gamma(3)$ and prove that there exists a neighborhood $U$ of $(x_0, y_0, z_0)$ in $\partial_{\infty} \Gamma(3)$ such that

$$\xi_p^{(1)}(\partial_{\infty} \Gamma) \cap \left( \xi_p^{(1)}(x) + \xi_p^{(1)}(z) + \xi_p^{(d-m)}(y) \right)$$

spans $\xi_p^{(1)}(x), \xi_p^{(1)}(z), \xi_p^{(d-m)}(y)$ for all $(x, y, z) \in U$ and any $\rho$ that is a sufficiently small deformation of $\rho_0$.

Let $e_1, \ldots, e_d$ be the standard basis of $\mathbb{R}^d$. By changing coordinates we can assume that

$$\xi_p^{(1)}(x_0) = \mathbb{R} \cdot e_1,$$

$$\xi_p^{(m)}(x_0) = \text{Span}\{e_1, \ldots, e_m\},$$

$$\xi_p^{(d-m)}(y_0) = \text{Span}\{e_{m+1}, \ldots, e_d\},$$

$$\xi_p^{(d-1)}(y_0) = \text{Span}\{e_2, \ldots, e_d\},$$

$$\xi_p^{(1)}(z_0) = \mathbb{R} \cdot (e_1 + e_2 + e_d).$$

Using Theorem 6.3 and possibly shrinking $O$, we can find a neighborhood $U_0$ of $(x_0, y_0, z_0)$ such that there exists a continuous map

$$(\rho, (x, y, z)) \in O \times U_0 \to g_{\rho, (x, y, z)} \in \text{PGL}_d(\mathbb{R})$$

such that $g_{\rho_0, (x_0, y_0, z_0)} = \text{id}$,

$$g_{\rho, (x, y, z)} \cdot \xi_p^{(1)}(x) = \mathbb{R} \cdot e_1,$$

$$g_{\rho, (x, y, z)} \cdot \xi_p^{(m)}(x) = \text{Span}\{e_1, \ldots, e_m\},$$

$$g_{\rho, (x, y, z)} \cdot \xi_p^{(d-m)}(y) = \text{Span}\{e_{m+1}, \ldots, e_d\},$$

$$g_{\rho, (x, y, z)} \cdot \xi_p^{(d-1)}(y) = \text{Span}\{e_2, \ldots, e_d\},$$

$$g_{\rho, (x, y, z)} \cdot \xi_p^{(1)}(z) = \mathbb{R} \cdot (e_1 + e_2 + e_d).$$

By Theorem 1.1 for each $\rho, (x, y, z)) \in O \times U_0$, there exists a unique $C^1$ function $f_{\rho, (x, y, z)} : \mathbb{R}^{m-1} \to \mathbb{R}^{d-m}$ such that

$$g_{\rho, (x, y, z)} : \xi_p^{(1)}(\partial_{\infty} \Gamma \setminus \{y\}) = \{1 : v \cdot f_{\rho, (x, y, z)}(v) : v \in \mathbb{R}^{m-1}\}.$$
Then by Theorem 6.3, the map $\mathcal{O} \times U_0 \to C \left( \mathbb{R}^{m-1}, \mathbb{R}^{d-m} \right)$ given by

$$(\rho, (x, y, z)) \mapsto f_{\rho,(x,y,z)}$$

is continuous. Notice that

$$\xi^{(1)}_\rho(\partial_\infty \Gamma \setminus \{y\}) \cap \left( \xi^{(1)}_\rho(x) + \xi^{(1)}_\rho(z) + \xi^{(d-m)}_\rho(y) \right) = g_{\rho,(x,y,z)}^{-1} \cdot \{ [1 : t e_2 : f_{\rho,(x,y,z)}(t e_2)] : t \in \mathbb{R} \}.$$

So if

$$[1 : t_1 e_2 : f_{\rho_0,(x_0,y_0,z_0)}(t_1 e_2)], \ldots, [1 : t_{d-m+1} e_2 : f_{\rho_0,(x_0,y_0,z_0)}(t_{d-m+1} e_2)]$$

spans $\xi^{(1)}_{\rho_0}(x_0) + \xi^{(1)}_{\rho_0}(y_0) + \xi^{(d-m)}_{\rho_0}(z_0)$, then

$$g_{\rho,(x,y,z)}^{-1} [1 : t_1 e_2 : f_{\rho,(x,y,z)}(t_1 e_2)], \ldots, g_{\rho,(x,y,z)}^{-1} [1 : t_{d-m+1} e_2 : f_{\rho,(x,y,z)}(t_{d-m+1} e_2)]$$

spans $\xi^{(1)}_{\rho}(x) + \xi^{(1)}_{\rho}(y) + \xi^{(d-m)}_{\rho}(z)$ when $(\rho, (x, y, z))$ is sufficiently close to $(\rho_0, (x_0, y_0, z_0))$. \hfill \Box

### 6.2. Proof of Theorem 6.1

We begin with the following observation. Let $e_1, \ldots, e_d$ denote the standard basis of $\mathbb{R}^d$, and let $\gamma \in \text{GL}_d(\mathbb{R})$ be a proximal element so that

- $e_1$ spans the eigenspace corresponding to $\lambda_1(\gamma)$,
- $e_m$ lies in the generalized eigenspace corresponding to $\lambda_m(\gamma)$,
- $e_{m+1}$ lies in the generalized eigenspace corresponding to $\lambda_{m+1}(\gamma)$.

Then observe that

$$\log \lambda_1(\gamma) = \lim_{n \to \infty} \frac{1}{n} \log \| \gamma^n \cdot e_1 \|,$$

$$\log \lambda_m(\gamma) = \lim_{n \to \infty} \frac{1}{n} \log \| \gamma^n \cdot e_m \|,$$

and

$$\log \lambda_{m+1}(\gamma) = \lim_{n \to \infty} \frac{1}{n} \log \left\| \gamma^n \cdot \sum_{j=m+1}^d v_j e_j \right\| \text{ when } v_{m+1} \neq 0.$$

**Proof of Theorem 6.1** From Theorem 4.2 and Theorem 5.1, we see that

$$\alpha_m(\rho) \leq \sup \left\{ \alpha \in (1, 2) : M \text{ is } C^\alpha \text{ along } \xi^{(1)}(\partial_\infty \Gamma) \right\}.$$

To prove the equality case, fix some $\gamma \in \Gamma$ with infinite order and let $\gamma^\pm \in \partial_\infty \Gamma$ denote the attracting and repelling fixed points of $\gamma$. We can make a change of basis and assume that $\xi^{(1)}(\gamma^+) = \mathbb{R} \cdot e_1$, $\xi^{(m)}(\gamma^+) = \text{Span}\{e_1, \ldots, e_m\}$, $\xi^{(d-m)}(\gamma^-) = \text{Span}\{e_{m+1}, \ldots, e_d\}$, and $\xi^{(d-1)}(\gamma^-) = \text{Span}\{e_2, \ldots, e_d\}$. Now fix a lift $\tilde{\gamma} \in \text{GL}_d(\mathbb{R})$ of $\rho(\gamma) \in \text{PGL}_d(\mathbb{R})$. Then

$$\tilde{\gamma} = \begin{pmatrix} \lambda & U \\ V & \lambda \end{pmatrix}$$

where $\lambda \in \mathbb{R}$, $U \in \text{GL}_{m-1}(\mathbb{R})$, and $V \in \text{GL}_{d-m}(\mathbb{R})$. By a further change of basis, we can assume that $e_m$ lies in the generalized eigenspace corresponding to $\lambda_m(\gamma)$, and $e_{m+1}$ lies in the generalized eigenspace corresponding to $\lambda_{m+1}(\gamma)$. 
By Theorem 4.2, $M$ is $C^1$ along $\xi^{(1)}(\partial_\infty \Gamma)$, and the tangent space to $M$ at $\xi^{(1)}(\gamma^+) = \xi^{(m)}(\gamma^+)$. Thus, for any $\epsilon > 0$ sufficiently small there exists some $p \in M$ such that

$$p = \left[ e_1 + \epsilon e_m + \sum_{j=m+1}^d y_j e_j \right].$$

Then

$$\xi^{(1)}(\gamma^+) + p + \xi^{(d-m)}(\gamma^-) = \text{Span}\{e_1, e_m, e_{m+1}, \ldots, e_d\}$$
and by hypothesis there exists some $q \in M$ such that

$$q = \left[ z_1 e_1 + z_m e_m + \sum_{j=m+1}^d z_j e_j \right]$$
and $z_{m+1} \neq 0$. The sums $q + \xi^{(d-1)}(\gamma^-)$ and $\xi^{(1)}(\gamma^+) + q + \xi^{(d-m)}(\gamma^-)$ are both direct, so $z_1 \neq 0 \neq z_m$.

Next fix a distance $d_p$ on $\mathbb{P}(\mathbb{R}^d)$ induced by a Riemannian metric. Since

$$\lim_{n \to \infty} \rho(\gamma^n) \cdot q = \xi^{(1)}(\gamma^+),$$
Observation 4.8 implies that if

$$X_n := \overline{\rho(\gamma^n) \cdot \left( z_1 e_1 + z_m e_m + \sum_{j=m+1}^d z_j e_j \right)},$$
then there is some $A \geq 1$ so that for sufficiently large $n$,

$$\frac{1}{A} \frac{\|P_{3,p}(X_n)\|_2}{\|P_{1,p}(X_n)\|_2} \leq d_p \left( \rho(\gamma^n) \cdot q, \xi^{(m)}(\gamma^+) \right) \leq A \frac{\|P_{3,p}(X_n)\|_2}{\|P_{1,p}(X_n)\|_2}$$
and

$$\frac{1}{A} \frac{\|P_{2,p}(X_n)\|_2}{\|P_{1,p}(X_n)\|_2} \leq d_p \left( \rho(\gamma^n) \cdot q, \xi^{(1)}(\gamma^+) \right) \leq A \frac{\|P_{2,p}(X_n)\|_2 + \|P_{3,p}(X_n)\|_2}{\|P_{1,p}(X_n)\|_2}.$$

It then follows from (19) that

$$\lim_{n \to \infty} \frac{1}{n} \log d_p \left( \rho(\gamma^n) \cdot q, \xi^{(m)}(\gamma^+) \right) = \log \frac{\lambda_{m+1}}{\lambda_1}.$$ 

and

$$\lim_{n \to \infty} \frac{1}{n} \log d_p \left( \rho(\gamma^n) \cdot q, \xi^{(1)}(\gamma^+) \right) = \log \frac{\lambda_m}{\lambda_1}.$$

Finally, if $M$ is $C^\alpha$ along $\xi^{(1)}(\partial_\infty \Gamma)$, then there exists $C > 0$ such that

$$d_p \left( \xi^{(m)}(\gamma^+), \rho(\gamma^n) \cdot q \right) \leq C d_p \left( \xi^{(1)}(\gamma^+), \rho(\gamma^n) \cdot q \right)^\alpha$$
for all sufficiently large $n$. By taking the logarithm to both sides, dividing by $n$, and then taking the limit, we see that

$$\alpha \leq \frac{\log \frac{\lambda_1}{\lambda_{m+1}}}{\log \frac{\lambda_m}{\lambda_1}}.$$

Since $\gamma \in \Gamma$ was arbitrary, we see that $\alpha \leq \alpha_m(\rho)$.

$\square$
7. Necessary conditions for differentiability of $\rho$-controlled subsets

In this section, we establish Theorem 1.10. By Example 3.3, it is sufficient to prove the following theorem.

**Theorem 7.1.** Suppose $\Gamma$ is a hyperbolic group and $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ is an irreducible $1$-Anosov representation such that $\bigwedge^m \rho : \Gamma \to \text{PGL}(\bigwedge^m \mathbb{R}^d)$ is also irreducible. Also, suppose that $M$ is a $\rho$-controlled, $(m - 1)$-dimensional topological manifold. If

(‡) $M$ is $C^\alpha$ along $\xi(1)(\partial_\infty \Gamma)$ for some $\alpha > 1$,

then

(†') $\rho$ is $m$-Anosov and $\xi(1)(x) + p + \xi(d-m)(y)$ is a direct sum for all pairwise distinct $\xi(1)(x), p, \xi(1)(y) \in M$.

**Remark 7.2.** Note that (†’) in Theorem 7.1 is a weaker condition than (†) in Theorem 4.2. However, when $M = \xi(1)(\partial_\infty \Gamma)$, then the two conditions are identical.

First, in Section 7.1, we define, for any $1 \leq m \leq d$ and any representation $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$, the representation

$$\bigwedge^m \rho : \Gamma \to \text{PGL}\left(\bigwedge^m \mathbb{R}^d\right),$$

whose irreducibility appears as a hypothesis in the statements of Theorem 7.1. Then, in Section 7.2, we give an example to demonstrate the necessity of the irreducibility of $\bigwedge^m \rho$ as a hypothesis of Theorem 7.1 (and also in Theorem 1.10). Next, we prove Theorem 7.1, whose proof can be broken down into two main steps. In Section 7.3, we use the fact that $M$ is an $(m - 1)$-dimensional topological manifold that is $C^\alpha$ along the 1-limit set of $\rho$ for some $\alpha > 1$, to deduce that $\log \lambda_{m+1}(\rho(\gamma))$ grows linearly with respect to the word length of $\gamma$. Then, in Section 7.4, we use this to deduce that $\rho$ is $m$-Anosov, and obtain the required transversality condition.

**7.1. The wedge representation.** Observe that for any $m \leq d - 1$, there is a natural linear $\text{GL}_d(\mathbb{R})$-action on $\bigwedge^m \mathbb{R}^d$ given by

$$g \cdot (u_1 \wedge \cdots \wedge u_m) := (g \cdot u_1) \wedge \cdots \wedge (g \cdot u_m),$$

where $u_i \in \mathbb{R}^d$ for all $i$. This defines a representation

$$\iota_{d,m} : \text{GL}_d(\mathbb{R}) \to \text{GL}\left(\bigwedge^m \mathbb{R}^d\right),$$

which in turn defines a representation

$$\widehat{\iota}_{d,m} : \text{PGL}_d(\mathbb{R}) \to \text{PGL}\left(\bigwedge^m \mathbb{R}^d\right).$$

Using this, we may define the $m$-wedge representation of $\overline{\rho} : \Gamma \to \text{GL}_d(\mathbb{R})$ (resp. $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$) to be

$$\bigwedge^m \overline{\rho} := \iota_{d,m} \circ \overline{\rho} : \Gamma \to \text{GL}\left(\bigwedge^m \mathbb{R}^d\right) \quad \text{(resp.} \quad \bigwedge^m \rho := \widehat{\iota}_{d,m} \circ \rho : \Gamma \to \text{PGL}\left(\bigwedge^m \mathbb{R}^d\right)\text{)}. $$
Also, if \( \langle \cdot, \cdot \rangle_{\mathbb{R}^d} \) denotes the standard inner product on \( \mathbb{R}^d \) with orthonormal basis \( e_1, \ldots, e_d \), then we may define a bilinear pairing \( \langle \cdot, \cdot \rangle_{\Lambda^m \mathbb{R}^d} \) on \( \Lambda^m \mathbb{R}^d \) by first defining
\[
\langle u_{i_1} \wedge \cdots \wedge u_{i_m}, v_{j_1} \wedge \cdots \wedge v_{j_m} \rangle_{\Lambda^m \mathbb{R}^d} := \prod_{\sigma \in S_m} \prod_{k=1}^m \text{sgn}(\sigma)\langle u_{i_k}, v_{j_{\sigma(k)}} \rangle
\]
for all \( u_{i_k}, v_{j_k} \in \mathbb{R}^d \), and then extending it linearly to all of \( \Lambda^m \mathbb{R}^d \). Observe that
\[
\{ e_{i_1} \wedge \cdots \wedge e_{i_m} : 1 \leq i_1 < \cdots < i_k \leq d \}
\]
is an orthonormal basis of \( \Lambda \mathbb{R}^d \), so \( \langle \cdot, \cdot \rangle_{\Lambda^m \mathbb{R}^d} \) is an inner product. Using this, we may define the norm \( \| \cdot \|_{\Lambda^m \mathbb{R}^d} \) on \( \Lambda^m \mathbb{R}^d \) associated to \( \langle \cdot, \cdot \rangle_{\Lambda^m \mathbb{R}^d} \).

Next, let \( \mathbf{g} \in \text{GL}_d(\mathbb{R}) \). For all \( i \), let \( \mu_i(\iota_{d,m}(\mathbf{g})) \) denote the \( i \)-th singular value of \( \iota_{d,m}(\mathbf{g}) \) with respect to the norm \( \| \cdot \|_{\Lambda^m \mathbb{R}^d} \) on \( \Lambda^m \mathbb{R}^d \). One can verify from the definition of the \( \text{GL}_d(\mathbb{R}) \)-action on \( \Lambda^m \mathbb{R}^d \) that for all \( i \), there exists \( i_1 < i_2 < \cdots < i_m \) such that
\[
\lambda_i(\iota_{d,m}(\mathbf{g})) = \lambda_{i_1}(\mathbf{g}) \cdots \lambda_{i_m}(\mathbf{g}) \quad \text{and} \quad \mu_i(\iota_{d,m}(\mathbf{g})) = \mu_{i_1}(\mathbf{g}) \cdots \mu_{i_m}(\mathbf{g}).
\]
This implies that
\[
\lambda_1(\iota_{d,m}(\mathbf{g})) = \prod_{i=1}^m \lambda_i(\mathbf{g}) \quad \text{and} \quad \lambda_2(\iota_{d,m}(\mathbf{g})) = \lambda_{m+1}(\mathbf{g}) \prod_{i=1}^{m-1} \lambda_i(\mathbf{g}) \tag{20}
\]
and
\[
\mu_1(\iota_{d,m}(\mathbf{g})) = \prod_{i=1}^m \mu_i(\mathbf{g}) \quad \text{and} \quad \mu_2(\iota_{d,m}(\mathbf{g})) = \mu_{m+1}(\mathbf{g}) \prod_{i=1}^{m-1} \mu_i(\mathbf{g}). \tag{21}
\]
Hence, for any \( \gamma \in \Gamma \),
\[
\lambda_1 \left( \bigwedge^m \rho(\gamma) \right) \lambda_2 \left( \bigwedge^m \rho(\gamma) \right) = \frac{\lambda_1 \left( \bigwedge^m \rho(\gamma) \right)}{\lambda_{m+1} \left( \bigwedge^m \rho(\gamma) \right)} \tag{22}
\]
and
\[
\mu_1 \left( \bigwedge^m \rho(\gamma) \right) \mu_2 \left( \bigwedge^m \rho(\gamma) \right) = \frac{\mu_1 \left( \bigwedge^m \rho(\gamma) \right)}{\mu_{m+1} \left( \bigwedge^m \rho(\gamma) \right)} \tag{23}
\]

### 7.2. Irreducibility of \( \Lambda^m \rho \)

Now, we will discuss an example to demonstrate that the irreducibility of \( \Lambda^m \rho \) is a necessary hypothesis for Theorem 7.1 to hold.

The identification of \( \mathbb{C}^3 \) with \( \mathbb{R}^6 \) given by
\[
(z_1, z_2, z_3) \mapsto (\Re(z_1), \Im(z_1), \Re(z_2), \Im(z_2), \Re(z_3), \Im(z_3))
\]
defines an inclusion \( j : \text{SL}_3(\mathbb{C}) \to \text{SL}_4(\mathbb{R}) \). The image of \( j \) can be characterized as the subgroup of \( \text{SL}_4(\mathbb{R}) \) that commutes with the linear endomorphism \( J \) on \( \mathbb{R}^6 \) defined by \( J(x_1, y_1, x_2, y_2, x_3, y_3) := (-y_1, x_1, -y_2, x_2, -y_3, x_3) \). Let \( \text{SU}(2,1) \subset \text{SL}_3(\mathbb{C}) \) be the subgroup that leaves invariant the bilinear pairing that is represented in the standard basis of \( \mathbb{C}^3 \) by the matrix
\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix},
\]
and define
\[ \tau_0 := (\iota_{d,2} \circ j)|_{\text{SU}(2,1)} : \text{SU}(2,1) \to \text{SL}\left(\bigwedge^2 \mathbb{R}^6\right). \]

Recall that \( \iota_{d,m} \) was defined in Section 7.1. Let \( \bigwedge^2 J \) be the linear endomorphism on \( \bigwedge^2 \mathbb{R}^6 \) given by
\[
\left(\bigwedge^2 J\right)(u_1 \wedge u_2) = J(u_1) \wedge J(u_2).
\]
Consider the \( \tau_0 \)-invariant subspace
\[ E = \left\{ v \in \bigwedge^2 \mathbb{R}^6 : \left(\bigwedge^2 J\right)(v) = v \right\}, \]
and let \( \tau : \text{SU}(2,1) \to \text{GL}(E) \) be the representation defined by the \( \tau_0 \) action on \( E \). Observe that if \( e_1, \ldots, e_6 \) is the standard basis for \( \mathbb{R}^6 \), then
\[
\begin{align*}
f_1 &:= e_1 \wedge e_2, & f_4 &:= e_3 \wedge e_4, & f_7 &:= e_3 \wedge e_5 + e_4 \wedge e_6, \\
f_2 &:= e_2 \wedge e_3 - e_1 \wedge e_4, & f_5 &:= e_2 \wedge e_5 - e_1 \wedge e_6, & f_8 &:= e_4 \wedge e_5 - e_3 \wedge e_6, \\
f_3 &:= e_1 \wedge e_3 + e_2 \wedge e_4, & f_6 &:= e_1 \wedge e_5 + e_2 \wedge e_6, & f_9 &:= e_5 \wedge e_6
\end{align*}
\]
is a basis of \( E \). One can then explicitly verify that \( \tau \) is irreducible.

If \( g \in \text{SU}(2,1) \), then there exists some \( \lambda \geq 1 \) such that the (complex) eigenvalues of \( g \in \text{SU}(2,1) \) have absolute values \( \lambda, 1, \lambda^{-1} \). By conjugating \( g \) by an appropriate element in \( h \in \text{SU}(2,1) \), we may also assume that the eigenvectors of \( hgh^{-1} \) corresponding to \( \lambda, 1, \lambda^{-1} \) are \( (1,0,0)^T \), \( (0,1,0)^T \) and \( (0,0,1)^T \) respectively. This implies that the (generalized) eigenvalues of \( j(hgh^{-1}) \) have absolute values \( \lambda, 1, \lambda^{-1} \) (each with multiplicity 2), and the corresponding (generalized) eigenspaces are \( \text{Span}_\mathbb{R}\{e_1, e_2\} \), \( \text{Span}_\mathbb{R}\{e_3, e_4\} \) and \( \text{Span}_\mathbb{R}\{e_5, e_6\} \) respectively. Using the basis of \( E \) described above, one can then compute that the eigenvalues of \( \tau(hgh^{-1}) \), and hence \( \tau(g) \), have absolute values
\[ \lambda^2, \lambda, 1, 1, 1, \lambda^{-1}, \lambda^{-1}, \lambda^{-2}. \]
In particular, the image of \( \tau \) lies in \( \text{SL}(E) \).

With this set up, we can now give our example. It describes a 1-Anosov, irreducible representation of a co-compact lattice \( \Gamma \subset \text{SU}(2,1) \) to \( \text{SL}(E) \), where \( E \) is a 9-dimensional vector space. We show that the 1-limit set of this representation is a 3-dimensional, \( C^\infty \)-submanifold of \( \mathbb{P}(E) \), but \( \rho \) is not 4-Anosov.

**Example 7.3.** Fix a co-compact lattice \( \Gamma \subset \text{SU}(2,1) \). Since \( \text{SU}(2,1) \) is a rank-one Lie group, it acts transitively and by isometries on a negatively curved Riemannian symmetric space \( \mathbb{H}_C^2 \) (the 2-dimensional complex hyperbolic space), whose visual boundary \( \partial \mathbb{H}_C^2 \) has the structure of a 3-dimensional smooth sphere. Thus, the inclusion of \( \Gamma \) into \( \text{SU}(2,1) \) specifies an identification of \( \partial_\infty \Gamma = \partial \mathbb{H}_C^2 \).

As \( \text{SU}(2,1) \)-spaces, \( \mathbb{H}_C^2 \simeq \text{SU}(2,1)/B \), where \( B \subset \text{SU}(2,1) \) is the subgroup of upper triangular matrices. It is straightforward to check that \( \tau(B) \subset \text{SL}(E) \) lies in \( P \cap Q \), where \( P \subset \text{SL}(E) \) is the subgroup that preserves the line spanned by \( f_1 \), and \( Q \subset \text{SL}(E) \) is the subgroup that preserves \( \text{Span}_\mathbb{R}\{f_1, \ldots, f_9\} \). In particular, there are smooth, \( \tau \)-equivariant maps
\[ \xi^{(1)} : \partial_\infty \Gamma = \text{SU}(2,1)/B \to \text{SL}(E)/P = \mathbb{P}(E) \]
and
\[ \xi^{(8)} : \partial_\infty \Gamma = \text{SU}(2, 1)/B \to \text{SL}(E)/Q = \mathbb{P}^*(E) = \text{Gr}_s(E). \]

Furthermore, a result of Guichard-Wienhard [GW12, Proposition 4.4] imply that \( \tau_{\Gamma} : \Gamma \to \text{SL}(E) \) is a 1-Anosov representation whose 1-flag map and 8-flag map are \( \xi^{(1)} \) and \( \xi^{(8)} \) respectively. The eigenvalue calculation above then implies that \( \rho \) is not 4-Anosov. However, the 1-limit set \( \xi^{(1)}(\partial_\infty \Gamma) \) is a 3-dimensional \( C^\infty \)-submanifold of \( \mathbb{P}(\mathbb{R}^3) \).

7.3. Eigenvalue gaps from the \( C^\alpha \) property along the 1-limit set. Our goal now will be to prove the following proposition.

**Proposition 7.4.** Suppose that \( \rho : \Gamma \to \text{PGL}_d(\mathbb{R}) \) is a 1-Anosov representation. Also, suppose that \( \bigwedge^m \rho : \Gamma \to \text{PGL} \left( \bigwedge^m \mathbb{R}^d \right) \) is irreducible and \( M \) is \( \rho \)-controlled, \( (m-1) \)-dimensional topological manifold that is \( C^\alpha \) along the 1-limit set of \( \rho \) for some \( \alpha > 1 \). If \( \gamma \in \Gamma \), then
\[ \frac{\lambda_{m+1}}{\lambda_m}(\rho(\gamma)) \leq \left( \frac{\lambda_2}{\lambda_1}(\rho(\gamma)) \right)^{\alpha - 1}. \]

In particular, \( \log \frac{\lambda_{m+1}}{\lambda_m}(\rho(\gamma)) \) grows linearly with the word-length of \( \gamma \).

The proof of Proposition 7.4 requires two observations and a lemma.

**Observation 7.5.** Let \( g \in \text{PGL}_d(\mathbb{R}) \) be proximal, let \( \bar{g} \in \text{GL}_d(\mathbb{R}) \) be a lift of \( g \), and let \( g^+ \in \mathbb{P}(\mathbb{R}^d) \) and \( g^- \in \text{Gr}_{d-1}(\mathbb{R}^d) \) be the attracting fixed point and repelling fixed hyperplane of \( g \) respectively. Also, let \( d_\rho \) be a distance on \( \mathbb{P}(\mathbb{R}^d) \) induced by a Riemannian metric. If \( p \in \mathbb{P}(\mathbb{R}^d) \) satisfies \( p \not\equiv g^+ \) and \( p \not\equiv g^- \), then
\[ \log \frac{\lambda_2}{\lambda_1}(g) \geq \limsup_{n \to \infty} \frac{1}{n} \log d_\rho \left( g^n \cdot p, g^+ \right). \]

Moreover, there is a proper subspace \( V \subset \mathbb{R}^d \) so that if \( p \not\in [V] \), then the above inequality holds as equality.

**Remark 7.6.** In the above observation, we identify \( g^- \in \text{Gr}_{d-1}(\mathbb{R}^d) \) with a hyperplane of \( \mathbb{P}(\mathbb{R}^d) \), which we also denote by \( g^- \).

**Proof.** Note that the affine chart \( A_{\bar{g}^-} \) contains both \( p \) and \( g^+ \). Equip \( A_{\bar{g}^-} \) with an Euclidean metric \( d_\mathbb{A} \), and let \( B \) be the unit ball in \( A_{\bar{g}^-} \) centered at \( g^+ \). Since \( p \not\in g^- \), \( g^n \cdot p \in B \) for sufficiently large \( n \). On \( B \), \( d_\mathbb{A} \) and \( d_\mathbb{R} \) are bi-Lipschitz, so there is a constant \( A \) so that for sufficiently large \( n \),
\[ \frac{1}{A} \frac{\| P_2(X) \|_2}{\| P_1(X) \|_2} \leq d_\rho(g^n \cdot p, g^+) \leq A \frac{\| P_2(X) \|_2}{\| P_1(X) \|_2}, \]
where \( X \in \mathbb{R}^d \) is a non-zero vector in \( g^n \cdot p \), \( P_1 : \mathbb{R}^d \to g^+ \) is the projection with kernel \( g^- \), and \( P_2 : \mathbb{R}^d \to g^- \) is the projection with kernel \( g^+ \). On the other hand, it is straightforward that
\[ \log \frac{\lambda_2}{\lambda_1}(g) \geq \limsup_{n \to \infty} \frac{1}{n} \log \frac{\| P_2(g^n \cdot X) \|_2}{\| P_1(g^n \cdot X) \|_2}, \]
thus giving the desired inequality.

To determine \( V \), choose a basis \( \{ e_1, \ldots, e_d \} \) for \( \mathbb{R}^d \) so that \( g \) is in real Jordan normal form in this basis. We may assume that \( e_1 \) is an eigenvector of \( g \) corresponding
to \(\lambda_1\), and there is some \(l\) so that \(e_2, \ldots, e_l\) spans the generalized eigenspace corresponding to \(\lambda_2\). Let \(V\) be the span of \(e_1, e_{l+1}, \ldots, e_d\), and it is easy to see that the inequality (25) holds with equality when \(v \notin [V]\). This proves the observation.  

**Observation 7.7.** Let \(g \in \text{PGL}_d(\mathbb{R})\) so that \(\lambda_1(g) > 1\) for some \(i = 2, \ldots, d\), and let \(\mathbb{R} = V_1 + V_2\) be a direct sum decomposition so that \(V_1\) is the sum of all the generalized eigenspaces of \(g\) corresponding to \(\lambda_1(g)\) and \(V_2\) is the sum of all the other generalized eigenspaces. Suppose that \(\dim(V_1) > 1\) and \(g\) has an invariant line \(l \in [V_1]\). Then for all \(p \in \mathbb{P}(\mathbb{R}^d) \setminus [l + V_2]\),

\[
0 = \lim_{n \to \infty} \frac{1}{n} \log d_p\left(g^n \cdot p, l\right),
\]

where \(d_p\) is a distance on \(\mathbb{P}(\mathbb{R}^d)\) induced by a Riemannian metric.

**Proof.** First, note that since \(A\) implies that there exists \(\lambda_1\) and \(V\),

\[
0 = \lim_{n \to \infty} \frac{1}{n} \log d_p\left(g^n \cdot p, l\right),
\]

Now assume for a contradiction that Equation (26) does not hold for some \(p \in \mathbb{P}(\mathbb{R}^d) \setminus [l + V_2]\). Then by taking a subsequence, we may assume that

\[
\lim_{k \to \infty} \frac{1}{n_k} \log d_p\left(g^n \cdot p, l\right) < 0.
\]

Notice that this implies that \(g^n \cdot p \to l\) as \(k \to \infty\).

Let \(\overline{g} \in \text{GL}_d(\mathbb{R})\) be a lift of \(g\). Using the real Jordan normal form of \(\overline{g}\), we can decompose \(V_1 = R_{j=1} V_{1,j}\) where

1. \(V_{1,1} = l\),
2. for \(2 \leq j \leq r\)
   a. \(V_{1,j}\) is either one or two dimensional,
   b. there exists a linear transformation \(L_j : V_{1,j} \to V_{1,j}\) such that
      \[
      \overline{g} \cdot Y \in L_j \cdot Y + V_{1,j-1}
      \]
   for all \(Y \in V_{1,j}\),
3. \(\|L_j \cdot Y\|_2 = \lambda_1(\overline{g}) \|Y\|_2\) for all \(Y \in V_{1,j}\).

Also, let \(P_{1,j} : \mathbb{R}^d \to V_{1,j}\) and \(P_2 : \mathbb{R}^d \to V_2\) be the projections relative to the decomposition \(\mathbb{R}^d = V_{1,1} \oplus \cdots \oplus V_{1,r} \oplus V_2\).

Since \(g^n \cdot p\) converges to \(l\), (24) in the first part of the proof of Observation 7.5 implies that there exists \(A \geq 1\) such that

\[
1/A \left( \sum_{j=1}^l \|P_{1,j}(\overline{g}^n \cdot X)\|_2^2 + \|P_2(\overline{g}^n \cdot X)\|_2^2 \right) \leq d_p\left(g^n \cdot p, \Phi(\gamma)\right)
\]

for all non-zero \(X \in p\) and all sufficiently large \(k\). Since \(X \notin [l + V_2]\), there exists \(2 \leq j_0 \leq r\) such that \(P_{1,j_0}(X) \neq 0\). By increasing \(j_0\) if necessary, we can also assume that \(P_{1,j}(X) = 0\) for \(j_0 < j \leq r\). This implies that

\[
\|P_{1,j_0}(\overline{g}^n \cdot X)\|_2 = \lambda_1(\overline{g}) \|P_{1,j_0}(X)\|_2.
\]

Further, by increasing \(A \geq 1\) if necessary, we can assume that

\[
\|P_{1,1}(\overline{g}^n \cdot X)\|_2 \leq A \|\overline{g}^n \cdot X\|_2 \leq A \|\overline{g}^n\|_{\text{op}} \|X\|_2
\]

for all \(n \geq 0\).
Then by Equations (28), (29), and (30),
\[
\lim_{k \to \infty} \frac{1}{n_k} \log d_2 \left( g^{n_k} \cdot p, l \right) \geq \limsup_{k \to \infty} \frac{1}{n_k} \log \frac{\|P_{1,j_0}(\overline{g}^{n_k} \cdot X)\|_2}{A \|P_{1,1}(\overline{g}^{n_k} \cdot X)\|_2} \\
\geq \log(\lambda_1(\overline{g})) + \limsup_{k \to \infty} \frac{1}{n_k} \log \frac{\|P_{1,j_0}(X)\|_2}{A^2 \|\overline{g}^{n_k}\|_{op} \|X\|_2} \\
\geq \log(\lambda_1(\overline{g})) - \liminf_{k \to \infty} \frac{1}{n_k} \log \|\overline{g}^{n_k}\|_{op}.
\]

But Gelfand’s formula states that
\[
\lim_{n \to \infty} \frac{1}{n} \log \|\overline{g}^{n}\|_{op} = \log(\lambda_1(\overline{g})),
\]
so by (27),
\[
0 > \lim_{k \to \infty} \frac{1}{n_k} \log d_2 \left( g^{n_k} \cdot p, \Phi(\xi(1)(\gamma^+)) \right) \geq 0.
\]
and we have a contradiction. \hfill \Box

Next, suppose \( \rho : \Gamma \to \text{PGL}_d(\mathbb{R}) \) and \( M \subset \mathbb{P}(\mathbb{R}^d) \) satisfy the hypothesis of Proposition 7.4. Define the map

\[
F_{d,m} : \text{Gr}_m(\mathbb{R}^d) \to \mathbb{P} \left( \bigwedge^m \mathbb{R}^d \right) \text{ by } F_{d,m} : V \mapsto \left[ \bigwedge_{i=1}^m v_i \right],
\]
where \( v_1, \ldots, v_m \) is a basis of \( V \). Note that \( F_{d,m} \) is well defined, smooth, and \( \iota_{d,m} \)-equivariant. Since \( M \) is differentiable along the 1-limit set \( \xi(1)(\partial_\infty \Gamma) \) of \( \rho \), we can define \( \Phi : \xi(1)(\partial_\infty \Gamma) \to \text{Gr}_m(\mathbb{R}^d) \) to be the map that associates to every point in \( \xi(1)(\partial_\infty \Gamma) \) its tangent space. Then define

\[
\Phi := F_{d,m} \circ \overline{\Phi} : \xi(1)(\partial_\infty \Gamma) \to \mathbb{P} \left( \bigwedge^m \mathbb{R}^d \right).
\]

Remark 7.8. Fix distances \( d_1 \) on \( \mathbb{P}(\mathbb{R}^d) \) and \( d_2 \) on \( \mathbb{P} \left( \bigwedge^m \mathbb{R}^d \right) \) which are induced by Riemannian metrics. Since \( M \) is \( C^\alpha \) along \( \xi(1)(\partial_\infty \Gamma) \) and \( F_{d,m} \) is smooth, a calculation shows that there is some \( C \geq 1 \) so that

\[
d_2(\Phi(q_1), \Phi(q_2)) \leq C d_1(q_1, q_2) \alpha^{-1}
\]
for all \( q_1, q_2 \in \xi(1)(\partial_\infty \Gamma) \).

Lemma 7.9. Suppose that \( \rho : \Gamma \to \text{PGL}_d(\mathbb{R}) \) is a 1-Anosov representation. Also, suppose that \( \bigwedge^m \rho : \Gamma \to \text{PGL} \left( \bigwedge^m \mathbb{R}^d \right) \) is irreducible and \( M \) is \( \rho \)-controlled, \( (m-1) \)-dimensional topological manifold that is \( C^\alpha \) along the 1-limit set of \( \rho \) for some \( \alpha > 1 \). If \( \gamma \in \Gamma \) has infinite order, then \( g := (\bigwedge^m \rho)(\gamma) \) is proximal and \( \Phi(\xi(1)(\gamma^+)) \in \mathbb{P} \left( \bigwedge^m \mathbb{R}^d \right) \) is the attracting fixed point of \( g \).

Proof. Let \( \overline{\gamma} \in \text{GL}_d(\mathbb{R}) \) be a lift of \( \rho(\gamma) \), \( \overline{g} := \bigwedge^m \overline{\gamma} \), and \( \lambda_i = \lambda_i(\overline{g}) \) for \( i = 1, \ldots, d \). Then by (20), \( \lambda_1(\overline{g}) = \lambda_1 \cdots \lambda_m \). Thus it is equivalent to prove that \( \overline{g} \) is proximal and \( \Phi(\xi(1)(\gamma^+)) \) is the eigenline of \( \overline{g} \) whose eigenvalue has absolute value \( \lambda_1 \cdots \lambda_m \).
We first show that \( \Phi(\xi^{(1)}(\gamma^+)) \) is an eigenline of \( \mathcal{G} \) whose eigenvalue has absolute value \( \lambda_1 \cdots \lambda_m \). Let \( \{n_k\}_{k=1}^\infty \) be an increasing sequence of integers such that

\[
\frac{1}{\|g^{n_k}\|} g^{n_k}
\]

converges to some \( T \in \text{End}\left(\bigwedge^m \mathbb{R}^d\right) \). Also, let \( \bigwedge^m \mathbb{R}^d = V_1 \oplus V_2 \) be a \( \mathcal{G} \)-invariant decomposition of \( \bigwedge^m \mathbb{R}^d \), where every eigenvalue of \( \mathcal{G}|_{V_1} \) has absolute value \( \lambda_1 \cdots \lambda_m \) and every eigenvalue of \( \mathcal{G}|_{V_2} \) has absolute value strictly less than \( \lambda_1 \cdots \lambda_m \). Observe that the image of \( T \) is contained in \( V_1 \). Since

\[
\mathcal{G} \cdot \Phi(\xi^{(1)}(\gamma^+)) = \Phi(\xi^{(1)}(\gamma \cdot \gamma^+)) = \Phi(\xi^{(1)}(\gamma^+)),
\]

\( \Phi(\xi^{(1)}(\gamma^+)) \) is an eigenline of \( \mathcal{G} \). Thus, we only need to show that \( \Phi(\xi^{(1)}(\gamma^+)) \) is contained in the image of \( T \).

We claim that the image of \( T \) is exactly \( \Phi(\xi^{(1)}(\gamma^+)) \). Notice that if \( p = [v] \in \mathbb{P}(\bigwedge^m \mathbb{R}^d) \) and \( v \notin \text{ker} T \) then

\[
[T(v)] = \lim_{k \to \infty} g^{n_k} \cdot p
\]

(recall that \([v]\) denotes the projective line containing \( v \)). Further, since \( \bigwedge^m \rho : \Gamma \to \text{PGL}\left(\bigwedge^m \mathbb{R}^d\right) \) is irreducible, the set \( \{\Phi(x) : x \in \xi^{(1)}(\partial_\infty \Gamma)\} \) spans \( \bigwedge^m \mathbb{R}^d \). Thus there exists \( x_1, \ldots, x_N \in \partial_\infty \Gamma \) such that

\[
\Phi(\xi^{(1)}(x_1)), \ldots, \Phi(\xi^{(1)}(x_N))
\]

span \( \bigwedge^m \mathbb{R}^d \). By perturbing and relabelling the \( x_i \) (if necessary) we can also assume that \( \gamma^- \notin \{x_1, \ldots, x_N\} \), and that there exists \( 1 \leq \ell \leq N \) such that

\[
\Phi(\xi^{(1)}(x_1)) + \cdots + \Phi(\xi^{(1)}(x_\ell)) + \text{ker} T = \bigwedge^m \mathbb{R}^d
\]

is a direct sum. For \( 1 \leq i \leq \ell \),

\[
T(\Phi(\xi^{(1)}(x_i))) = \lim_{k \to \infty} g^{n_k} \Phi(\xi^{(1)}(x_i)) = \lim_{k \to \infty} \Phi(\xi^{(1)}(\gamma \cdot g^{n_k} \cdot x_i)) = \Phi(\xi^{(1)}(\gamma^+)),
\]

so the image of \( T \) is \( \Phi(\xi^{(1)}(\gamma^+)) \). Thus, \( \Phi(\xi^{(1)}(\gamma^+)) \) is an eigenline of \( \mathcal{G} \) whose eigenvalue has absolute value \( \lambda_1 \cdots \lambda_m \).

We next argue that \( \mathcal{G} \) is proximal, or equivalently that \( \text{dim} V_1 = 1 \). Let

\[
W := V_2 + \Phi(\xi^{(1)}(\gamma^+)),
\]

and suppose for contradiction that \( \text{dim} V_1 > 1 \). This implies that \( W \subset \mathbb{R}^d \) is a proper subspace. By Lemma \ref{lem:67}

\[
0 = \lim_{n \to \infty} \frac{1}{n} \log d_2\left(g^n \cdot p, \Phi(\xi^{(1)}(\gamma^+))\right)
\]

when \( p \in \mathbb{P}\left(\bigwedge^m \mathbb{R}^d\right) \setminus [W] \).

Since \( \{\Phi(x) : x \in \xi^{(1)}(\partial_\infty \Gamma)\} \) spans \( \bigwedge^m \mathbb{R}^d \), there exists \( x \in \partial_\infty \Gamma \) such that \( \Phi(\xi^{(1)}(x)) \notin [W] \). By perturbing \( x \) (if necessary) we can assume that \( x \neq \gamma^- \).

Then

\[
\lim_{n \to \infty} \rho(\gamma)^n \cdot \xi^{(1)}(x) = \xi^{(1)}(\gamma^+) \quad \text{and} \quad \lim_{n \to \infty} g^n \cdot \Phi(\xi^{(1)}(x)) = \Phi(\xi^{(1)}(\gamma^+)).
\]
So, by Observation 7.5

\[ 0 > \log \frac{\lambda_2}{\lambda_1} \geq \limsup_{n \to \infty} \frac{1}{n} \log d_1 \left( (\rho(\gamma))^n \cdot \xi(1)(x), \xi(1)(\gamma^+) \right) \]

\[ \geq \limsup_{n \to \infty} \frac{1}{(\alpha - 1)n} \log d_2 \left( \Phi(\xi(1)(\gamma^n \cdot x)), \Phi(\xi(1)(\gamma^+)) \right) \]

\[ = \limsup_{n \to \infty} \frac{1}{(\alpha - 1)n} \log d_2 \left( g^n \cdot \Phi(\xi(1)(x)), \Phi(\xi(1)(\gamma^+)) \right) = 0, \]

where the last inequality is Remark 7.8. This is a contradiction, so \( \overline{g} \) is proximal.

\( \square \)

**Proof of Proposition 7.4.** Fix some \( \gamma \in \Gamma \). If \( \gamma \) has finite order, then

\[ \frac{\lambda_i}{\lambda_j}(\rho(\gamma)) = 1 \]

for all \( 1 \leq i, j \leq d \) and there is nothing to prove. So suppose that \( \gamma \) has infinite order and let \( \gamma^+ \in \partial_\infty \Gamma \) be the attracting fixed point of \( \gamma \). By Lemma 7.9 \( g := \bigwedge^m \rho(\gamma) \) is proximal and \( \Phi(\xi(1)(\gamma^+)) = g^+ \).

By 22 and Observation 7.5 there exists a proper subspace \( V \subset \bigwedge^m \mathbb{R}^d \) such that: if \( p \in \mathbb{P}(\bigwedge^m \mathbb{R}^d) \setminus [V] \) and \( p \) is not in the repelling hyperplane of \( g \), then

\[ \log \frac{\lambda_{m+1}}{\lambda_m}(\rho(\gamma)) = \lim_{n \to \infty} \frac{1}{n} \log d_2 \left( g^n \cdot p, \Phi(\xi(1)(\gamma^+)) \right). \]

Since \( \{\Phi(g) : q \in \xi(1)(\partial_\infty \Gamma)\} \) spans \( \bigwedge^m \mathbb{R}^d \) we can find \( x \in \partial_\infty \Gamma \) such that \( \Phi(\xi(1)(x)) \notin [V] \). By perturbing \( x \) if necessary, we can also assume that \( x \neq \gamma^- \).

Then

\[ \lim_{n \to \infty} g^n \cdot \Phi(\xi(1)(x)) = \Phi(\xi(1)(\gamma^+)), \]

so \( \Phi(\xi(1)(x)) \) does not lie in the repelling hyperplane of \( g \). Thus, by Observation 7.5 and Remark 7.8

\[ \log \frac{\lambda_{m+1}}{\lambda_m}(\rho(\gamma)) \leq (\alpha - 1) \lim_{n \to \infty} \frac{1}{n} \log d_1 \left( (\rho(\gamma))^n \cdot \xi(1)(x), \xi(1)(\gamma^+) \right) \]

\[ \leq (\alpha - 1) \log \frac{\lambda_2}{\lambda_1}(\rho(\gamma)). \]

\( \square \)

### 7.4. Anosovness from eigenvalue gaps

To prove Theorem 7.1 we will use the following proposition.

**Proposition 7.10.** Suppose that \( \rho : \Gamma \to \text{PGL}_d(\mathbb{R}) \) is an irreducible 1-Anosov representation and \( M \) is a \( \rho \)-controlled subset which is \( C^1 \) along \( \xi(1)(\partial_\infty \Gamma) \). Suppose also that for all \( \gamma \in \Gamma \) with infinite order,

\[ \frac{\lambda_{m+1}}{\lambda_m}(\rho(\gamma)) \leq \left( \frac{\lambda_2}{\lambda_1}(\rho(\gamma)) \right)^{\alpha - 1}, \]

\( g := (\bigwedge^m \rho)(\gamma) \) is proximal, and \( \Phi(\xi(1)(\gamma^+)) \in \mathbb{P}(\bigwedge^m \mathbb{R}^d) \) is the attracting fixed point of \( g \). Then \( \rho \) is \( m \)-Anosov, and \( \xi(1)(x) + p + \xi^{(d-m)}(y) \) is a direct sum for all pairwise distinct \( \xi(1)(x), p, \xi(1)(y) \in M \).

Assuming Proposition 7.10 we can prove Theorem 7.1.
Proof of Theorem 7.1. If Condition (‡) holds, then Proposition 7.4, Lemma 7.9 and Proposition 7.10 imply Condition (†′). □

We start the proof of Proposition 7.10 by making some initial reductions. First, notice that the reduction made in Remark 2.12 does not impact the hypothesis or conclusion of the Proposition. So we may assume that there exists a lift \( \rho : \Gamma \to \mathrm{SL}_d(\mathbb{R}) \) of \( \rho \). Second, notice that passing to a finite index subgroup also does not impact the hypotheses or conclusion of the Proposition (see Proposition 2.9). Hence we may also assume that the Zariski closure of \( \overline{\rho(\Gamma)} \) is connected.

The proof of Proposition 7.10 requires the following lemma.

Lemma 7.11. If \( 1 < \beta < \alpha \), then there exists \( C > 0 \) such that
\[
\log \frac{\mu_{m+1}}{\mu_m}(\rho(\gamma)) \leq (\beta - 1) \log \frac{\mu_2}{\mu_1}(\rho(\gamma)) + C
\]
for all \( \gamma \in \Gamma \).

Proof. Let \( C_\mu = C_\mu(\overline{\rho(\Gamma)}) \) and \( C_\lambda = C_\lambda(\overline{\rho(\Gamma)}) \) be the cones defined in Section 5.2. Then \( C_\mu = C_\lambda \) by Proposition 2.8 and Theorem 5.5. By hypothesis, if \( x = (x_1, \ldots, x_d) \in C_\lambda \), then
\[
x_{m+1} - x_m \geq (\beta - 1)(x_2 - x_1).
\]
Further, since \( \rho \) is 1-\( \alpha \)-Anosov, \( x_2 - x_1 < 0 \) for all \( x = (x_1, \ldots, x_d) \in C_\lambda \).

Next, we will prove that there exists \( R > 0 \) with the following property: if \( \|\mu(\overline{\rho(\gamma)})\|_2 \geq R \), then
\[
\log \frac{\mu_{m+1}}{\mu_m}(\rho(\gamma)) \leq (\beta - 1) \log \frac{\mu_2}{\mu_1}(\rho(\gamma)).
\]
Suppose for contradiction that there exists \( \{\gamma_n\}_{n=1}^\infty \subset \Gamma \) with \( \|\mu(\overline{\rho(\gamma_n)})\|_2 \to \infty \) and
\[
\log \frac{\mu_{m+1}}{\mu_m}(\rho(\gamma_n)) > (\beta - 1) \log \frac{\mu_2}{\mu_1}(\rho(\gamma_n)).
\]
By passing to a subsequence, we can assume that
\[
\frac{1}{\|\mu(\overline{\rho(\gamma_n)})\|_2} \mu(\overline{\rho(\gamma_n)}) \to x = (x_1, \ldots, x_d).
\]
Then \( x \in C_\mu = C_\lambda \) and
\[
x_{m+1} - x_m \geq (\beta - 1)(x_2 - x_1) > (\alpha - 1)(x_2 - x_1)
\]
so we have a contradiction.

The lemma then follows from the observation that since \( \rho \) is 1-\( \alpha \)-Anosov, the set \( \{\gamma \in \Gamma : \|\mu(\overline{\rho(\gamma)})\|_2 \leq R\} \) is finite. □

Proof of Proposition 7.10. Since \( \rho \) is 1-\( \alpha \)-Anosov, Theorem 2.5 implies that there exists \( C_0, c_0 > 0 \) such that
\[
\frac{\mu_2}{\mu_1}(\rho(\gamma)) \leq C_0 e^{-c_0 d_2(\gamma, \text{id})}
\]
for all \( \gamma \in \Gamma \). Then by Lemma 7.11, there exists \( C, c > 0 \) such that
\[
\frac{\mu_{m+1}}{\mu_m}(\rho(\gamma)) \leq C e^{-c d_2(\gamma, \text{id})}
\]
for all \( \gamma \in \Gamma \). Thus, Theorem 2.5 implies that \( \rho \) is \( m \)-Anosov.
To finish the proof, we will now show that $\xi^{(1)}(x) + p + \xi^{(d-m)}(y)$ is a direct sum for all pairwise distinct $\xi^{(1)}(x), p, \xi^{(1)}(y) \in M$. Let
\[ \hat{M} := \{(\xi^{(1)}(x), p, \xi^{(1)}(y)) \in M^3 : \xi^{(1)}(x), p, \xi^{(1)}(y) \text{ are pairwise distinct}\} \]
and let
\[ \mathcal{O} := \left\{ (\xi^{(1)}(x), p, \xi^{(1)}(y)) \in \hat{M} : \xi^{(1)}(x) + p + \xi^{(d-m)}(y) \text{ is a direct sum} \right\}. \]
Notice that $\mathcal{O}$ is open and $\Gamma$-invariant. Also, recall that $\Gamma$ acts co-compactly on $\hat{U}$, the flow space associated to $\Gamma$ described in Section 2.4. Hence, there exists a compact set $K \subset \hat{U}$ such that $\Gamma \cdot K = \hat{U}$. Then define
\[ 0 < \epsilon := \min\{d(p, (v^+, v^-)) : v \in K\}, \]
where $d_p$ is a distance on $P(\mathbb{R}^d)$ induced by a Riemannian metric.

Given two proper subspaces $V, W \subset \mathbb{R}^d$ define
\[ d(V, W) := \min\{|v - w|_2 : v \in V, w \in W, |v|_2 = |w|_2 = 1\}. \]

Note that $\{(x, y) \in \partial_{\infty} \Gamma^2 : d_p(\xi^{(1)}(x), \xi^{(1)}(y)) \geq \epsilon\}$ is compact. Since $\xi^{(m)}(x) + \xi^{(d-m)}(y) = \mathbb{R}^d$ when $x \neq y$, this implies that there exists $\theta_0 > 0$ with the following property: if $x, y \in \partial_{\infty} \Gamma$ and $d_p(\xi^{(1)}(x), \xi^{(1)}(y)) \geq \epsilon$, then
\[ d(\xi^{(m)}(x), \xi^{(d-m)}(y)) \geq \theta_0. \]

Also, by hypothesis, if $\gamma \in \Gamma$ has infinite order and $\gamma^+ \in \partial_{\infty} \Gamma$ is the attracting fixed point of $\gamma$, then
\[ \xi^{(m)}(\gamma^+) = T_{\xi^{(1)}(x)} M. \]

So by the continuity of $\xi^{(m)}$ and the density of $\{\gamma^+ : \gamma \in \Gamma \text{ has infinite order}\}$ in $\partial_{\infty} \Gamma$ we see that $\xi^{(m)}(x) = T_{\xi^{(1)}(x)} M$ for all $x \in \partial_{\infty} \Gamma$. Thus, the compactness of $M$ implies that there exists $\delta > 0$ with the following property: if $\xi^{(1)}(x), p \in M$ and $d_p(\xi^{(1)}(x), p) \leq \delta$, then
\[ d\left(\xi^{(1)}(x) + p, \xi^{(m)}(x)\right) < \theta_0/2. \]

Using this, define
\[ \mathcal{U} := \left\{ (\xi^{(1)}(x), p, \xi^{(1)}(y)) \in \hat{M} : d_p(\xi^{(1)}(x), \xi^{(1)}(y)) \geq \epsilon \text{ and } d_p(\xi^{(1)}(x), p) \leq \delta \right\}. \]

We claim that $\mathcal{U} \subset \mathcal{O}$. Indeed, by the definition of $\theta_0$ and $\delta$, if $(\xi^{(1)}(x), p, \xi^{(1)}(y)) \in \mathcal{U}$ then
\[ d\left(\xi^{(1)}(x) + p, \xi^{(d-m)}(x)\right) > \theta_0/2. \]

This implies that $\xi^{(1)}(x) + p + \xi^{(d-m)}(y)$ is direct, so $(\xi^{(1)}(x), p, \xi^{(1)}(y)) \in \mathcal{O}$.

Next, let $P(M) \subset \hat{U} T \times M$ be the set defined by [2], and recall that $\phi_t$ denotes the geodesic flow on $\hat{U} T$. Note that there exists $T \geq 0$ such that if $v \in K$, $t \geq T$, and $(\phi_t(v), p) \in P(M)$, then $(\xi^{(1)}(v^+), p, \xi^{(1)}(v^-)) \in \mathcal{U} \subset \mathcal{O}$. Now, choose any $(\xi^{(1)}(x), p, \xi^{(1)}(y)) \in \hat{M}$. From the definition of $P(M)$, there exists $v \in \hat{U} T$ such that $v^+ = x$, $v^- = y$, and $(v, p) \in P(M)$. Further, there exists $\gamma \in \Gamma$ such that $w := \gamma \cdot \phi_{-T}(v) \in K$. Since the $\Gamma$-action on $\hat{U} T$ commutes with the geodesic flow,
\[ (\phi_T(w), \rho(\gamma) \cdot p) = \gamma \cdot (v, p) \in P(M) \]
and so \( \gamma \cdot (\xi^{(1)}(x), p, \xi^{(1)}(y)) = (\xi^{(1)}(w^+), \rho(\gamma) \cdot p, \xi^{(1)}(w^-)) \in \mathcal{O} \), which means 
\( (\xi^{(1)}(x), p, \xi^{(1)}(y)) \in \mathcal{O} \). Thus, \( \mathcal{O} = \mathcal{M} \).

\[ \square \]

8. Necessary conditions for differentiability of 1-dimensional \( \rho \)-controlled subsets

In this section we prove Theorem 1.8. Again by Example 3.3 it is sufficient to prove the following theorem.

**Theorem 8.1.** Suppose \( \Gamma \) is a hyperbolic group and \( \rho : \Gamma \to \text{PGL}_d(\mathbb{R}) \) is an irreducible 1-Anosov representation. Also, suppose that \( M \) is a \( \rho \)-controlled, topological circle. If

\( (\dagger) \) \( M \) is \( C^\alpha \) along \( \xi^{(1)}(\partial_\infty \Gamma) \) for some \( \alpha > 1 \),

then

\( (\ddagger) \) \( \rho \) is \( m \)-Anosov and \( \xi^{(1)}(x) + p + \xi^{(d-m)}(y) \) is a direct sum for all pairwise distinct \( \xi^{(1)}(x) \), \( p \), \( \xi^{(1)}(y) \in \mathcal{M} \).

Before proving Theorem 8.1 we give an example to demonstrate that the irreducibility of \( \rho \) is a necessary hypothesis in Theorem 8.1 (and also in Theorem 1.8) to hold.

8.1. Irreducibility of \( \rho \). For \( d \in \mathbb{N} \), let \( \tau_d : \text{GL}_2(\mathbb{R}) \to \text{GL}_d(\mathbb{R}) \) be the standard irreducible representation, which is constructed as follows. First, identify \( \mathbb{R}^d \) with the space of homogeneous degree \( d - 1 \) polynomials in two variables with real coefficients by

\[
(a_1, \ldots, a_d) \mapsto \sum_{i=1}^{d} a_i X^{d-i} Y^{i-1}.
\]

Using this, we may define an \( \text{GL}_2(\mathbb{R}) \)-action on \( \mathbb{R}^d \) by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P(X, Y) = P \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \cdot (X, Y).
\]

It is easy to check that this \( \text{GL}_2(\mathbb{R}) \)-action is linear. Thus, it has an associated linear representation \( \tau_d : \text{GL}_2(\mathbb{R}) \to \text{GL}_d(\mathbb{R}) \), which descends to a representation \( \tau_d : \text{PGL}_2(\mathbb{R}) \to \text{PGL}_d(\mathbb{R}) \).

One can verify that if \( \lambda, \lambda^{-1} \) are the absolute value of the eigenvalues of \( \overline{\gamma} \in \text{SL}_2^+(\mathbb{R}) \), then

\[
\lambda^{d-1}, \lambda^{d-3}, \ldots, \lambda^{-(d-1)}
\]

are the absolute values of the eigenvalues of \( \tau_d(\overline{\gamma}) \). Further, if \( B_d \subset \text{GL}_d(\mathbb{R}) \) denotes the subgroup of upper triangular matrices, then \( \tau_d(B_2) \subset B_d \). In particular, \( \tau_d \) induces a smooth map

\[
\Psi_d : \mathbb{P}(\mathbb{R}^2) \simeq \text{PGL}_2(\mathbb{R})/B_2 \to \text{PGL}_d(\mathbb{R})/B_d.
\]

Since \( \text{GL}_d(\mathbb{R})/B_d \) is the space of complete flags in \( \mathbb{R}^d \), there is an obvious smooth projection \( p_m : \text{GL}_d(\mathbb{R})/B_d \to \text{Gr}_m(\mathbb{R}^d) \) for each \( m = 1, \ldots, d - 1 \). Using this, define \( \Psi_{d,m} := p_m \circ \Psi_d : \mathbb{P}(\mathbb{R}^2) \to \text{Gr}_m(\mathbb{R}^d) \). It is clear that \( \Psi_{d,m} \) is \( \tau_d \)-equivariant.

Next, observe that the subgroup \( \mathbb{P}(\text{GL}_d(\mathbb{R}) \times \text{GL}_{d+2}(\mathbb{R})) \subset \text{PGL}_{2d+2}(\mathbb{R}) \) preserves both the subspaces \( \mathbb{P}(\mathbb{R}^d) \) and \( \mathbb{P}(\mathbb{R}^{d+2}) \) of \( \mathbb{P}(\mathbb{R}^{2d+2}) \) induced respectively by
the obvious inclusions of $\mathbb{R}^d \simeq \mathbb{R}^d \oplus \{0\}$ and $\mathbb{R}^{d+2} \simeq \{0\} \oplus \mathbb{R}^{d+2}$ into $\mathbb{R}^d \oplus \mathbb{R}^{d+2} \simeq \mathbb{R}^{2d+2}$. Similarly, the subspaces $\text{Gr}_{d-1}(\mathbb{R}^d)$ and $\text{Gr}_{d+1}(\mathbb{R}^{d+2})$ of $\text{Gr}_{2d+1}(\mathbb{R}^{2d+2})$ that are respectively defined by the inclusions $V \mapsto V \oplus \mathbb{R}^{d+2}$ and $U \mapsto \mathbb{R}^d \oplus U$ are $\mathbb{P}(\text{GL}_d(\mathbb{R}) \times \text{GL}_{d+2}(\mathbb{R}))$-invariant. In particular, the representation

$$\tau_d \oplus \tau_{d+2} : \text{GL}_2(\mathbb{R}) \to \text{GL}_d(\mathbb{R}) \times \text{GL}_{d+2}(\mathbb{R}) \subset \text{GL}_{2d+2}(\mathbb{R})$$

defines the representation $\tau_d \oplus \tau_{d+2} : \text{PGL}_2(\mathbb{R}) \to \text{PGL}_{2d+2}(\mathbb{R})$ by projectivizing, and the maps

$$\Psi_{d,1} : \mathbb{P}(\mathbb{R}^2) \to \mathbb{P}(\mathbb{R}^d) \subset \mathbb{P}(\mathbb{R}^{2d+2}),$$

$$\Psi_{d+1,1} : \mathbb{P}(\mathbb{R}^2) \to \mathbb{P}(\mathbb{R}^{d+2}) \subset \mathbb{P}(\mathbb{R}^{2d+2}),$$

$$\Psi_{d,d-1} : \mathbb{P}(\mathbb{R}^2) \to \text{Gr}_{d-1}(\mathbb{R}^d) \subset \text{Gr}_{d+1}(\mathbb{R}^{2d+2}),$$

$$\Psi_{d+2,d+1} : \mathbb{P}(\mathbb{R}^2) \to \text{Gr}_{d+1}(\mathbb{R}^{d+2}) \subset \text{Gr}_{d+1}(\mathbb{R}^{2d+2}).$$

are smooth and $\tau_d \oplus \tau_{d+2}$-equivariant.

One can check that $(\Psi_{d,1}, \Psi_{d,d-1})$ and $(\Psi_{d+1,1}, \Psi_{d+2,d+1})$ are transverse pairs of maps. It also follows from (33) that for any $g \in \text{PGL}_2(\mathbb{R})$, $\tau_d \oplus \tau_{d+2}(g)$ is proximal. However, the attracting eigenline and repelling hyperplane of $\tau_d \oplus \tau_{d+2}(g)$ lies in the image of $\Psi_{d+2,1}$ and $\Psi_{d+2,d+1}$ respectively, so only the pair of maps $(\Psi_{d+2,1}, \Psi_{d+2,d+1})$ is dynamics preserving.

Example 8.2. Fix a co-compact lattice $\Gamma \leq \text{PGL}_2(\mathbb{R})$. The inclusion of $\Gamma$ into $\text{PGL}_2(\mathbb{R})$ induces an identification $\partial_\infty \Gamma \simeq \mathbb{P}(\mathbb{R}^2)$, and thus equips $\partial_\infty \Gamma$ with the structure of a smooth manifold. Consider the representation

$$\rho := \tau_d \oplus \tau_{d+2} | \Gamma : \Gamma \to \text{PGL}(\mathbb{R}^d \oplus \mathbb{R}^{d+2}).$$

By the discussion above,

$$\Psi_{d+2,1} : \partial_\infty \Gamma \to \mathbb{P}(\mathbb{R}^{2d+2})$$

and

$$\Psi_{d+2,d+1} : \partial_\infty \Gamma \to \text{Gr}_{d+1}(\mathbb{R}^{2d+2})$$

is a pair of smooth, dynamics preserving, $\rho$-equivariant, transverse maps. Thus, one deduces from (33) that $\rho$ is 1-Anosov, but it is not 2-Anosov because

$$\frac{\lambda_2}{\lambda_3}(\rho(\gamma)) = 1$$

for any $\gamma \in \Gamma$. However, since $\Psi_{d+2,1}$ is a smooth map, the 1-limit set of $\rho$ is a 1-dimensional, $C^\infty$-submanifold of $\mathbb{P}(\mathbb{R}^{2d+2})$. This shows that the conclusion of Theorem 8.1 does not hold if we do not assume the irreducibility hypothesis of Theorem 8.1.

8.2. Proof of Theorem 8.1 Lemma 8.4 and Lemma 8.6 stated below are respectively the analogs of Lemma 7.9 and Proposition 7.4 in the case when $M$ is a 1-dimensional topological manifold. With these two lemmas, we can replicate the proof of Theorem 7.1 to prove Theorem 8.1.

Remark 8.3. In Lemma 7.9 and Proposition 7.4 we assumed that $\Lambda^2 \rho$ is irreducible, but in Lemma 8.4 and Lemma 8.6 we assume that $\rho$ is irreducible.

Lemma 8.4. Suppose that $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ is a 1-Anosov representation. Also, suppose that $M$ is $\rho$-controlled, topological circle that is $C^\alpha$ along the 1-limit set of $\rho$ for some $\alpha > 1$, and let $\Phi : M \to \mathbb{P}\left(\Lambda^2 \mathbb{R}^d\right)$ be as defined in (52). If $\gamma \in \Gamma$
has infinite order, then \( \bigwedge^2 \rho(\gamma) \) is proximal and \( \Phi(\xi^{(1)}(\gamma^+)) \in \mathbb{P}\left(\bigwedge^2 \mathbb{R}^d\right) \) is the attracting fixed point of \( \bigwedge^2 \rho(\gamma) \).

**Proof.** Define \( \overline{\Psi} : \xi^{(1)}(\partial_{\infty} \Gamma) \times \xi^{(1)}(\partial_{\infty} \Gamma) \to \text{Gr}_2(\mathbb{R}^d) \) by letting \( \overline{\Psi}(p, q) \) be the projective line containing \( p, q \) when \( p \neq q \) and letting \( \overline{\Psi}(p, p) \) be the projective line tangent to \( M \) at \( p \). Then define
\[
\Psi := F_{d,2} \circ \overline{\Psi} : \xi^{(1)}(\partial_{\infty} \Gamma) \times \xi^{(1)}(\partial_{\infty} \Gamma) \to \mathbb{P}\left(\bigwedge^2 \mathbb{R}^d\right),
\]
where \( F_{d,2} \) is defined by \([31]\). Observe that \( \Psi \) is continuous and \( \Phi(p) = \Psi(p, p) \) for all \( p \in \xi^{(1)}(\partial_{\infty} \Gamma) \).

Fix distances \( d_1 \) on \( \mathbb{P}(\mathbb{R}^d) \) and \( d_2 \) on \( \mathbb{P}\left(\bigwedge^2 \mathbb{R}^d\right) \) that are induced by Riemannian metrics. Since \( M \) is \( C^\alpha \) along the 1-limit set of \( \rho \) for some \( \alpha > 1 \), there exists \( C > 0 \) such that
\[
d_2\left(\Psi(p, p), \Psi(p, q)\right) \leq C d_1(p, q)^{\alpha-1}
\]
for all \( p, q \in \xi^{(1)}(\partial_{\infty} \Gamma) \). Also, since \( \rho \) is irreducible, the elements of \( \xi^{(1)}(\partial_{\infty} \Gamma) \) span \( \mathbb{R}^d \), so
\[
\Psi \left(\xi^{(1)}(\partial_{\infty} \Gamma) \times \xi^{(1)}(\partial_{\infty} \Gamma)\right)
\]
spans \( \bigwedge^2 \mathbb{R}^d \). Now the rest of the proof closely follows the proof of Lemma 7.9 but we use \( \Psi(\xi^{(1)}(\gamma^+), \xi^{(1)}(\gamma^+)) \) in place of \( \Phi(\xi^{(1)}(\gamma^+)) \) and \( \Psi(\xi^{(1)}(x), \xi^{(1)}(\gamma^+)) \) in place of \( \Phi(\xi^{(1)}(x)) \). \( \square \)

**Remark 8.5.** In the case when \( M \) is a topological \((m - 1)\)-dimensional manifold with \( m > 2 \), it is not true that \( \xi^{(1)}(x_1) + \cdots + \xi^{(1)}(x_m) \) converges to \( \xi^{(m)}(x) \) as \( x_i \to x \), so the direct analog of (34) \( \square \) cannot hold. As such, we need the additional assumption that \( \bigwedge^m \rho(\gamma) \) is irreducible in Theorem 7.1.

**Lemma 8.6.** Suppose that \( \rho : \Gamma \to \text{PGL}_d(\mathbb{R}) \) is a 1-Anosov representation. Also, suppose that \( M \) is \( \rho \)-controlled, topological circle that is \( C^\alpha \) along the 1-limit set of \( \rho \) for some \( \alpha > 1 \). If \( \gamma \in \Gamma \), then
\[
\frac{\lambda_{m+1}}{\lambda_m} (\rho(\gamma)) \leq \left(\frac{\lambda_2}{\lambda_1}(\rho(\gamma))\right)^{\alpha-1}.
\]

**Proof.** Use the same argument as we did in the proof of Proposition 7.4 but with \( \Psi(\xi^{(1)}(\gamma^+), \xi^{(1)}(\gamma^+)) \) and \( \Psi(\xi^{(1)}(x), \xi^{(1)}(\gamma^+)) \) in place of \( \Phi(\xi^{(1)}(\gamma^+)) \) and \( \Phi(\xi^{(1)}(x)) \) respectively, and Lemma 8.4 in place of Lemma 7.9. \( \square \)

**Proof of Theorem 1.8.** Use the same proof as Theorem 1.10 but replace Lemma 7.9 and Proposition 7.4 by Lemma 8.4 and Lemma 8.6 respectively. \( \square \)

9. \( \text{PGL}_d(\mathbb{R}) \)-Hitchin representations

In this section, let \( \Gamma := \pi_1(\Sigma) \), where \( \Sigma \) is a closed, orientable, connected hyperbolic surface of genus at least 2.

**Definition 9.1.** A \( \text{PGL}_d(\mathbb{R}) \)-Hitchin representation is a continuous deformation (in \( \text{Hom}(\Gamma, \text{PGL}_d(\mathbb{R})) \)) of \( \tau_d \circ j \), where \( j : \Gamma \to \text{PGL}_2(\mathbb{R}) \) is a Fuchsian representation, and \( \tau_d : \text{PGL}_2(\mathbb{R}) \to \text{PGL}_d(\mathbb{R}) \) is the representation defined in Section 8.1.
The goal of this section is to show that if $\rho$ is a $\text{PGL}_d(\mathbb{R})$-Hitchin representation, then for all $k = 1, \ldots, d - 1$, $\bigwedge^k \rho : \Gamma \to \text{PGL}(\bigwedge^k \mathbb{R}^d)$ satisfies the hypothesis of Theorem 1.1 (see Example 1.6). The following proposition is a straightforward consequence of Labourie’s deep work on the Hitchin component \cite{Lab06} and has also been observed by Pozzetti-Sambarino-Wienhard \cite{PSW18}.

**Proposition 9.2.** Let $\rho$ be a $\text{PGL}_d(\mathbb{R})$-Hitchin representation and $D := \dim(\bigwedge^k \mathbb{R}^d)$.

If $k \in \{1, \ldots, d - 1\}$, then $\bigwedge^k \rho : \Gamma \to \text{PGL}(\bigwedge^k \mathbb{R}^d)$ is $(1, 2)$-Anosov, and its $1$-flag map $\zeta^{(1)}(x) + \zeta^{(1)}(y) + \zeta^{(D-2)}(z)$, is a direct sum for all $x, y, z \in \partial_\infty \Gamma$ distinct.

For the rest of the section fix some $\text{PGL}_d(\mathbb{R})$-Hitchin representation $\rho$ and some finite generating set $S$ of $\Gamma$.

9.1. **Preliminaries.** Before proving the proposition, we recall some results of Labourie. By Theorem 4.1 and Proposition 3.2 in \cite{Lab06},

1. $\rho$ is $k$-Anosov for every $1 \leq k \leq d$. Denote the $k$-flag map of $\rho$ by $\xi^{(k)}$.
2. If $x, y, z \in \partial_\infty \Gamma$ are distinct, $k_1, k_2, k_3 \geq 0$, and $k_1 + k_2 + k_3 = d$, then
   \[
   \xi^{(k_1)}(x) + \xi^{(k_2)}(y) + \xi^{(k_3)}(z) = \mathbb{R}^d
   \]
   is a direct sum.
3. If $x, y, z \in \partial_\infty \Gamma$ are distinct and $0 \leq k < d - 2$, then
   \[
   \xi^{(k+1)}(y) + \xi^{(d-k-2)}(x) + \left(\xi^{(k+1)}(z) \cap \xi^{(d-k)}(x)\right) = \mathbb{R}^d
   \]
   is a direct sum.
4. $\rho$ admits a lift $\overline{\rho} : \Gamma \to \text{GL}_d(\mathbb{R})$ whose image lies in $\text{SL}_d(\mathbb{R})$.
5. If $\gamma \in \Gamma \setminus \{1\}$, then the absolute values of the eigenvalues of $\overline{\rho}(\gamma)$ satisfy
   \[
   \lambda_1(\overline{\rho}(\gamma)) > \cdots > \lambda_d(\overline{\rho}(\gamma)).
   \]
6. If $\gamma \in \Gamma \setminus \{1\}$, then $\xi^{(k)}(\gamma^+)$ is the span of the eigenspaces of $\rho(\gamma)$ corresponding to the eigenvalues
   \[
   \lambda_1(\overline{\rho}(\gamma)), \ldots, \lambda_k(\overline{\rho}(\gamma)).
   \]

9.2. **Proof of Proposition 9.2.** Since $\rho$ is $k$-Anosov, Theorem 2.5 implies that there exists $C, c > 0$ such that
\[
\log \frac{\mu_k}{\mu_{k+1}}(\rho(\gamma)) \geq C d S(1, \gamma) - c
\]
for all $\gamma \in \Gamma$ and $1 \leq k \leq d$.

**Lemma 9.3.** $\bigwedge^k \rho$ is $(1, 2)$-Anosov.

**Proof.** By Theorem 2.5 it is enough to prove that there exists $A, a > 0$ such that
\[
\log \frac{\mu_1}{\mu_2} \left(\bigwedge^k \rho(\gamma)\right) \geq A d S(1, \gamma) - a
\]
and
\[
\log \frac{\mu_2}{\mu_3} \left( \bigwedge^k \rho(\gamma) \right) \geq Ad_S(1, \gamma) - a
\]
for all \( \gamma \in \Gamma \).

Fix \( \gamma \in \Gamma \) and let \( g \in \text{SL}_d(\mathbb{R}) \) be a lift of \( \rho(\gamma) \). Then let
\[
\sigma_1 \geq \cdots \geq \sigma_d
\]
denote the singular values of \( g \) (in the Euclidean norm on \( \mathbb{R}^d \)), and let
\[
\chi_1 \geq \cdots \geq \chi_D
\]
denote the singular values of \( \bigwedge^k g \) (in the induced norm on \( \bigwedge^k \mathbb{R}^d \)).

Recall, that Equation (21) says that
\[
\chi_1 = \sigma_1 \cdots \sigma_k \quad \text{and} \quad \chi_2 = \sigma_1 \cdots \sigma_{k-1} \sigma_{k+1}.\]

Hence
\[
\log \frac{\mu_1}{\mu_2} \left( \bigwedge^k \rho(\gamma) \right) = \log \frac{\chi_1}{\chi_2} = \log \frac{\sigma_k}{\sigma_{k+1}} \geq C\ell_S(\gamma) - c.
\]

To verify the other inequality, pick \( 1 \leq i_1 < \cdots < i_k \leq d \) such that
\[
\chi_3 = \sigma_{i_1} \cdots \sigma_{i_k}.
\]

We consider two cases based on the value of \( i_{k-1} \).

**Case 1:** Suppose \( i_{k-1} = k - 1 \). Then \( i_j = j \) for \( j \leq k - 1 \) and \( i_k \geq k \). Since
\[
(i_1, \ldots, i_k) \notin \{(1, \ldots, k), (1, \ldots, k-1, k+1)\}
\]
we must have \( i_k \geq k + 2 \). So
\[
\log \frac{\chi_2}{\chi_3} = \log \left( \frac{\sigma_1}{\sigma_{i_1}} \cdots \frac{\sigma_{k-1}}{\sigma_{i_{k-1}}} \frac{\sigma_{k+1}}{\sigma_{i_k}} \right) = \log \frac{\sigma_{k+1}}{\sigma_{i_k}} \geq \log \frac{\sigma_{k+1}}{\sigma_{k+2}} \geq C\ell_S(\gamma) - c.
\]

**Case 2:** Suppose \( i_{k-1} \geq k \). Then \( i_k \geq k + 1 \) and \( i_j \geq j \) for all \( j \) so
\[
\log \frac{\chi_2}{\chi_3} = \log \left( \frac{\sigma_1}{\sigma_{i_1}} \cdots \frac{\sigma_{k-1}}{\sigma_{i_{k-1}}} \frac{\sigma_{k+1}}{\sigma_{i_k}} \right) \geq \log \frac{\sigma_{k-1}}{\sigma_{i_{k-1}}} \geq \log \frac{\sigma_{k-1}}{\sigma_k} \geq C\ell_S(\gamma) - c.
\]

In either case
\[
\log \frac{\mu_2}{\mu_3} \left( \bigwedge^k \rho(\gamma) \right) = \log \frac{\chi_2}{\chi_3} \geq C\ell_S(\gamma) - c.
\]

Then since \( \gamma \in \Gamma \) was arbitrary, we see that \( \bigwedge^k \rho \) is \((1, 2)\)-Anosov.

Given subspaces \( V_1, \ldots, V_k \subset \mathbb{R}^d \), we will let \( V_1 \wedge \cdots \wedge V_k \) denote the subspace of \( \bigwedge^k \mathbb{R}^d \) that is spanned by \( \{X_1 \wedge \cdots \wedge X_k : X_i \in V_i\} \). For \( \ell \in \{1, 2, D - 2, D - 1\} \) define maps
\[
\zeta^{(\ell)} : \partial_\infty \Gamma \to \text{Gr}_\ell \left( \bigwedge^k \mathbb{R}^d \right)
\]
by
\[
\zeta^{(1)}(x) = \bigwedge^k \zeta^{(k)}(x),
\]
These maps are clearly continuous and $\wedge^k \rho$-equivariant.

**Lemma 9.4.** $\zeta^{(1)}$, $\zeta^{(2)}$, $\zeta^{(D-2)}$, $\zeta^{(D-1)}$ are the flag maps of $\wedge^k \rho$.

**Proof.** By the density of attracting fixed points in $\partial_\infty \Gamma$ and the continuity of the maps, it is enough to verify that $\zeta^{(j)}(\gamma^+)$ is the attracting fixed point of $\wedge^k \rho(\gamma)$ in $\text{Gr}_j(\wedge^k \mathbb{R}^d)$ when $\gamma^+ \in \partial_\infty \Gamma$ is the attracting fixed point of $\gamma \in \Gamma$.

By Property (6) in Section 9.1, there exists a basis $v_1, \ldots, v_d$ of $\mathbb{R}^d$ of eigenvectors of $\rho(\gamma)$ such that

$$\zeta^{(j)}(\gamma^+) = \text{Span}\{v_{j+1}, \ldots, v_d\} \text{ for } j = 1, \ldots, d.$$ 

Let $I_1 = \{d-k+1, d-k+2, \ldots, d\}$ and $I_2 = \{d-k, d-k+2, d-k+3, \ldots, d\}$. Then a calculation shows that

$$\zeta^{(1)}(\gamma^+) = [v_1 \wedge \cdots \wedge v_k],$$

$$\zeta^{(2)}(\gamma^+) = \{v_1 \wedge \cdots \wedge v_{k-1} \wedge (av_k + bv_{k+1}) : a, b \in \mathbb{R}\},$$

$$\zeta^{(D-2)}(\gamma^+) = \text{Span}\{v_i \wedge \cdots \wedge v_k : i \in I_1 \cup I_2\},$$

$$\zeta^{(D-1)}(\gamma^+) = \text{Span}\{v_i \wedge \cdots \wedge v_k : i \in I_1 \cup I_2\} \neq I_1 \setminus I_2.$$ 

So $\zeta^{(j)}(\gamma^+)$ is the attracting fixed point of $\wedge^k \rho(\gamma)$ in $\text{Gr}_j(\wedge^k \mathbb{R}^d)$. $\square$

**Lemma 9.5.** $\zeta^{(1)}(x) + \zeta^{(1)}(y) + \zeta^{(D-2)}(z)$ is a direct sum for all $x, y, z \in \partial_\infty \Gamma$ distinct.

**Proof.** Fix $x, y, z \in \partial_\infty \Gamma$ distinct, and choose a basis $v_1, \ldots, v_d \in \mathbb{R}^d$ such that

$$[v_{i_1}] = \xi^{(i_1)}(x) \cap \xi^{(d-\ell+1)}(y)$$

for $1 \leq \ell \leq d$. Next pick $u_1, \ldots, u_k \in \mathbb{R}^d$ such that

$$\xi^{(k)}(z) = \text{Span}\{u_1, \ldots, u_k\}.$$ 

Then $\zeta^{(1)}(z) = [u_1 \wedge \cdots \wedge u_k]$.

If $I = \{1, \ldots, k-1, k+1\}$, then a computation shows that

$$\zeta^{(1)}(x) + \zeta^{(D-2)}(y) = \text{Span}\{v_{i_1} \wedge \cdots \wedge v_k : i_1, \ldots, i_k \neq I\}.$$ 

Since

$$\xi^{(k)}(z) + \left(\xi^{(k)}(x) \cap \xi^{(d-\ell+1)}(y)\right) + \xi^{(d-\ell+1)}(y) = \mathbb{R}^d$$

for $1 \leq \ell \leq d$, we have

$$\zeta^{(1)}(x) + \zeta^{(1)}(y) + \zeta^{(D-2)}(z) = \mathbb{R}^d.$$ 

$\square$
is a direct sum and
\[
(\xi^{(k)}(x) \cap \xi^{(d-k+1)}(y)) + \xi^{(d-k-1)}(y) = \text{Span}\{v_k, v_{k+2}, \ldots, v_d\}
\]
we see that
\[
(u_1 \wedge \cdots \wedge u_k) \wedge (v_k \wedge v_{k+2} \wedge \cdots \wedge v_d) \neq 0.
\]
This implies that
\[
\zeta^{(1)}(x) + \zeta^{(1)}(y) + \zeta^{(d-m)}(z) = k \bigwedge \mathbb{R}^d.
\]

10. Real hyperbolic lattices

The goal of this section is to justify Example 1.5. More precisely, we need to prove the following proposition.

**Proposition 10.1.** Suppose \(\tau : \text{PO}(m,1) \to \text{PGL}_d(\mathbb{R})\) is a representation, \(\Gamma \leq \text{PO}(m,1)\) is a co-compact lattice, and \(\rho = \tau|:\Gamma \to \text{PGL}_d(\mathbb{R})\) is the representation obtained by restricting \(\tau\) to \(\Gamma\). If \(\rho\) is irreducible and 1-Anosov, then \(\rho\) is also \(m\)-Anosov and
\[
(\xi^{(1)}(x) + \xi^{(1)}(y) + \xi^{(d-m)}(z))
\]
is a direct sum for all \(x, y, z \in \partial_\infty \Gamma\) distinct. Thus, the same is true for any small deformation of \(\rho\).

Let \(\text{PO}(m,1) \subset \text{PGL}_{m+1}(\mathbb{C})\) be the subgroup that leaves invariant the bilinear pairing that is represented in the standard basis of \(\mathbb{R}^{m+1}\) by the matrix.

\[
\begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & -1
\end{pmatrix}
\]

10.1. Preliminaries. Consider the unit ball \(\mathbb{B}_m \subset \mathbb{R}^m\) endowed with the metric
\[
d(x,y) = \frac{1}{2} \log \frac{\|y-b\|_2}{\|x-b\|_2}
\]
where \(a,b \in \partial \mathbb{B}_m \cap (x + \mathbb{R}(y-x))\) ordered \(a,x,y,b\), and \(\|\cdot\|_2\) is the standard Euclidean norm on \(\mathbb{R}^m\). The metric space \((\mathbb{B}_m, d)\) is usually known as the *Klein-Beltrami model* of real hyperbolic \(m\)-space. Further, \(\text{PO}(m,1)\) acts transitively and by isometries on \((\mathbb{B}_m, d)\) via fractional linear transformations, that is
\[
\begin{bmatrix}
A & u \\
\ell v & a
\end{bmatrix} : x = \frac{Ax + u}{\ell vx + a}.
\]

Using the formula for the distance, one can compute that \(d(e^{sH} \cdot 0,0) = s\), where
\[
H := \begin{bmatrix}
0 & e_1 \\
e_1 & 0
\end{bmatrix}.
\]
In fact, one can verify that the map \(\gamma_0 : \mathbb{R} \to \mathbb{B}_m\) given by
\[
\gamma_0 : s \mapsto \tanh(s)e_1 = e^{sH} \cdot 0
\]
is a unit-speed geodesic in \(\mathbb{B}_m\) with \(-e_1\) and \(e_1\) as its backward and forward endpoints respectively.
Lemma 10.5. If $g \in \text{PO}(m, 1)$, then there exists $k_1, k_2 \in K$ such that
\[
g = k_1 e^{d(g \cdot 0) H} k_2.
\]
Recall that an element $g \in \text{PO}(m, 1)$ is called hyperbolic if there exists some geodesic $\gamma : \mathbb{R} \to \mathbb{B}_m$ and some $\ell(g) > 0$ such that
\[
g(\gamma(t)) = \gamma(t + \ell(g))
\]
for all $t \in \mathbb{R}$. The number $\ell(g)$ is called the translation length of $g$. For co-compact lattices in $\text{PO}(m, 1)$, we have the following proposition.

**Proposition 10.3.** If $\Gamma \leq \text{PO}(m, 1)$ is a co-compact lattice and $\gamma \in \Gamma$ has infinite order, then $\gamma$ is a hyperbolic element.

**Proof.** See for instance [4C92, Chapter 12, Proposition 2.6]. $\square$

Let $\gamma_0$ be the geodesic as defined above, and let $M$ be the subgroup of $\text{PO}(m, 1)$ that fixes the image of the geodesic $\gamma_0$ pointwise, i.e.
\[
M := \{k \in K : k \cdot e_1 = e_1\}.
\]

**Proposition 10.4.** If $h \in \text{PO}(m, 1)$ is hyperbolic, then $h = ge^{\ell(h)H}g^{-1}$ for some $k \in M$ that commutes with $e^{\ell(h)H}$.

**Proof.** Since $h$ is hyperbolic, there exists some geodesic $\gamma : \mathbb{R} \to \mathbb{B}_m$ such that $h\gamma(t) = \gamma(t + \ell(h))$ for all $t \in \mathbb{R}$. Also, $\text{PO}(m, 1)$ acts transitively on the unit tangent bundle of $\mathbb{B}_m$, there exists $g \in \text{PO}(m, 1)$ so that $g \circ \gamma = \gamma_0$. Since $h$ translates along $\gamma$ by $\ell(h)$ and $e^{-\ell(h)H}$ translates along $\gamma_0$ by $-\ell(h)$, we see that $e^{-\ell(h)H}ghg^{-1} = ghg^{-1}e^{-\ell(h)H}$ fixes the image of $\gamma_0$ pointwise, and therefore lies in $M$. Hence, there is some $k \in M$ so that
\[
ghg^{-1} = e^{\ell(h)H}k = ke^{\ell(h)H}
\]
for some $k \in M$. $\square$

10.2. **Proof of Proposition 10.1.** Let $\tau$, $\rho$, and $\Gamma$ satisfy the hypothesis of Proposition 10.1. To prove Proposition 10.1, we use the following two lemmas.

Let $\tau_0 = \tau|_{e^H}$ and let $\tau_0 : e^{\mathbb{R} \cdot H} \to \text{SL}_d(\mathbb{R})$ be the lift of $\tau_0$ (since $\mathbb{R}$ is simply connected, such a lift exists).

**Lemma 10.5.** $\tau_0(e^H)$ is proximal and the eigenvalue with maximal modulus is a positive real number.
Proof. The group $\tau(M)$ is a compact subgroup of $\text{PGL}_d(\mathbb{R})$, so every element in $\tau(M)$ is elliptic. Now suppose that $\gamma \in \Gamma$ has infinite order. Since $\rho$ is 1-Anosov, $\tau(\gamma)$ has a representative in $\text{SL}_d^+(\mathbb{R})$ whose eigenvalue of maximal absolute value has multiplicity 1. On the other hand, by Proposition 10.4 $\gamma$ is conjugate to $ke^{sH}$ for some $s > 0$ and $k \in M$. Then since $\tau(k)$ is elliptic and commutes with $\tau(e^{sH})$, the eigenvalues of $\tau(e^{sH})$ and $\tau(k)\tau(e^{sH}) = \tau(ke^{sH})$ have the same absolute values. So $\tau(e^{sH})$ also has a unique eigenvalue with maximal absolute value. This implies that $\tau(e^{tH})$ is proximal for every $t \geq 0$.

Since $\tau_0(id) = \text{id}$ has all positive eigenvalues, we see that the eigenvalue with maximal modulus of $\tau_0(e^{tH})$ is positive for all $t \geq 0$. □

Lemma 10.6. Let $e^{\lambda}$ denote the eigenvalue of $\tau_0(e^H)$ with maximal modulus. There is some integer $k$ so that the set of eigenvalues of $\tau_0(e^H)$ is

$$\{e^{\lambda-n} : 0 \leq n \leq k\}.$$

Furthermore, the eigenspace corresponding to $e^{\lambda-1}$ has dimension $m-1$.

The proof of Lemma 10.6 is a standard argument from the theory of weight spaces. We give this argument in Appendix B. With Lemma 10.5 and 10.6, we can prove Proposition 10.1.

Proof of Proposition 10.1. By Lemma 10.5 and 10.6, the eigenvalues of $\tau_0(e^{sH})$ are

$$e^{\lambda s}, e^{(\lambda-1)s}, \ldots, e^{(\lambda-1)s}, e^{(\lambda-2)s}, \ldots,$$

and the multiplicity of $e^{(\lambda-1)s}$ is $m - 1$. In particular,

$$\frac{\mu_m}{\mu_{m+1}}(\tau(e^{sH})) = e^s.$$

Also, the group $\tau(K) \subset \text{PGL}_d(\mathbb{R})$ is compact, so it lifts to a compact subgroup $\tilde{K} \subset \text{SL}_d^+(\mathbb{R})$. Hence, there exists some $C > 1$ such that

$$\frac{1}{C} \mu_i(T) \leq \mu_i(k_1k_2) \leq C \mu_i(T)$$

for all $1 \leq i \leq n$, all $k_1, k_2 \in \tilde{K}$, and all $T \in \text{End}(\mathbb{R}^n)$. By Observation 10.2

$$\log \frac{\mu_m}{\mu_{m+1}}(\rho(\gamma)) \geq \log \frac{\mu_m}{\mu_{m+1}} \left(\tau(e^{d(\gamma \cdot 0)H})\right) - \log(C^2) = \lambda d(\gamma \cdot 0) - \log(C^2),$$

which implies that $\rho$ is $m$-Anosov.

Since $\rho$ is the restriction of $\tau$ to $\Gamma$, the $\rho$-equivariant flag maps

$$\xi^{(i)} : \partial \Gamma \simeq \partial B_m \to \text{Gr}_i(\mathbb{R}^d)$$

are $\tau$-equivariant for $i = 1, d - 1, m, d - m$. Further, by the description of $K$ given by (55), we see that $\text{PO}(m, 1)$ acts transitively on triples of distinct points $x, y, z \in \partial B_m$. Thus it is enough to show that

$$\xi^{(1)}(x) + \xi^{(1)}(y) + \xi^{(d-m)}(z)$$

is direct for some $x, y, z \in \partial B_m$ distinct. Fix $y, z \in \partial B_m$ distinct. Then since $\tau$ is irreducible we must have

$$\mathbb{R}^d = \text{Span}\{\xi^{(1)}(x) : x \in \partial B_m\}.$$
and so there exists some \( x \in \partial \mathbb{B}_m \) such that
\[
\xi^{(1)}(x) + \xi^{(1)}(y) + \xi^{(d-m)}(z)
\]
is direct.

\[\square\]

**APPENDIX A. THEOREM 5.5**

**Proof.** First notice that \( C_\lambda(G) \) is invariant under conjugation in \( \text{SL}_d(\mathbb{R}) \), i.e. \( C_\lambda(G) = C_\lambda(gGg^{-1}) \) for all \( g \in \text{SL}_d(\mathbb{R}) \). Further, if \( h \in G \), then from the geometric description of the Cartan projection given in [2.3] there exists some \( C > 0 \) such that
\[
\left\| \mu(g) - \mu(ghg^{-1}) \right\|_2 \leq C
\]
for all \( g \in \text{SL}_d(\mathbb{R}) \). Hence \( C_\mu(G) \) is also invariant under conjugation in \( \text{SL}_d(\mathbb{R}) \).

Let \( \mathfrak{sl}_d(\mathbb{R}) = \mathfrak{t} + \mathfrak{p} \) denote the standard Cartan decomposition of \( \mathfrak{sl}_d(\mathbb{R}) \), that is
\[
\mathfrak{t} = \{ X \in \mathfrak{sl}_d(\mathbb{R}) : \imath X = -X \} \quad \text{and} \quad \mathfrak{p} = \{ X \in \mathfrak{sl}_d(\mathbb{R}) : \imath X = X \}.
\]
Let \( \mathfrak{g} \) denote the Lie algebra of \( G \). Using Theorem 7 in [Mos55] and conjugating \( G \) we may assume that
\[
\mathfrak{g} = \mathfrak{t} \cap \mathfrak{g} + \mathfrak{p} \cap \mathfrak{g}
\]
is a Cartan decomposition of \( \mathfrak{g} \). Fix a maximal abelian subspace \( \mathfrak{a} \subseteq \mathfrak{p} \cap \mathfrak{g} \).

By [Hel01, Chapter V, Lemma 6.3], there exists some \( k \in \text{SO}(d) \) such that \( \text{Ad}(k) \mathfrak{a} \) is a subspace of the diagonal matrices in \( \mathfrak{sl}_d(\mathbb{R}) \). Since \( \text{Ad}(k) \mathfrak{p} = \mathfrak{p} \), by replacing \( G \) with \( kGk^{-1} \) we can assume that \( \mathfrak{a} \) is itself a subspace of the diagonal matrices. Finally fix a Weyl chamber \( \mathfrak{a}^+ \) of \( \mathfrak{a} \).

Next let \( \mathfrak{K} \subseteq G \) denote the subgroup corresponding to \( \mathfrak{t} \cap \mathfrak{g} \), let \( A = \exp(\mathfrak{a}) \), and let \( A^+ = \exp(\mathfrak{a}^+) \). By [Hel01, Chapter IX, Theorem 1.1], each \( g \in G \) can be written as
\[
g = k_1 \exp(\mu_G(g))k_2
\]
where \( k_1, k_2 \in K \) and \( \mu_G(g) \in \mathfrak{a}^+ \) is unique. The map \( \mu_G : G \rightarrow \mathfrak{a}^+ \) is called the Cartan projection of \( G \) relative to the decomposition \( G = K\mathfrak{a}^+K \). Since \( K \subseteq \text{SO}(d) \) and \( \mathfrak{a} \) is a subspace of the diagonal matrices, the diagonal entries of \( \mu_G(g) \) coincide with the entries of \( \mu(g) \) up to permuting indices.

Every \( g \in G \) can be written as a product \( g = g_1g_2g_3 \) of commuting elements, where \( g_1 \) is elliptic, \( g_2 \) is hyperbolic, and \( g_3 \) is unipotent. This is called the Jordan decomposition of \( g \) in \( G \). The element \( g_2 \) is conjugate to a unique element \( \exp(\lambda_G(g)) \in \overline{\mathfrak{a}^+} \) and the map \( \lambda_G : G \rightarrow \overline{\mathfrak{a}^+} \) is called the Jordan projection. Since \( G \) is an irreducible real algebraic subgroup of \( \text{SL}_d(\mathbb{R}) \), the Jordan decomposition in \( G \) coincides with the standard Jordan decomposition in \( \text{SL}_d(\mathbb{R}) \). Then, since \( \mathfrak{a} \) is a subspace of the diagonal matrices, the diagonal entries of \( \lambda_G(g) \) coincide with the entries of \( \lambda(g) \) up to permuting indices.

Next define cones \( C_1, C_2 \subseteq \mathfrak{a}^+ \) as follows:
\[
C_1 := \bigcup_{\gamma \in \Gamma} \mathbb{R}_{>0} \cdot \lambda_G(\gamma)
\]
and
\[
C_2 := \{ x \in \mathbb{R}^d : \exists \gamma_n \in \Gamma, \exists t_n \downarrow 0, \text{ with } \lim_{n \rightarrow \infty} t_n \mu_G(\gamma_n) = x \}.
\]
Then the main result in [Ben97] says that $C_1 = C_2$. Since $\mu_G(g)$ and $\mu(g)$ (respectively $\lambda_G(g)$ and $\lambda(g)$) coincide up to permuting indices, this implies that $C_\mu(\Lambda) = C_\lambda(\Lambda)$. \qed

**Appendix B. Proof of Lemma 10.6**

Let $\mathfrak{so}(m, 1)$ denote the Lie algebra of PO($m, 1$), and let $e_1, \ldots, e_{m+1}$ be the standard basis of $\mathbb{R}^{m+1}$. By fixing the signature $(m, 1)$-form on $\mathbb{R}^{m+1}$ that is represented in this basis by the matrix

$$
\begin{bmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & -1
\end{bmatrix},
$$

one can compute that $\mathfrak{so}((m, 1)) = \begin{bmatrix} A & u \\ t^u & 0 \end{bmatrix} : tA = -A$.

Define vector following subspaces of $\mathfrak{so}(m, 1)$:

- $\mathfrak{a} = \left\{ \begin{bmatrix} 0 & \lambda e_1 \\ \lambda^t e_1 & 0 \end{bmatrix} : \lambda \in \mathbb{R} \right\}$,
- $\mathfrak{g}_0 = \left\{ \begin{bmatrix} A & \lambda e_1 \\ \lambda^t e_1 & 0 \end{bmatrix} : tA = -A, \ Ae_1 = 0, \text{ and } \lambda \in \mathbb{R} \right\}$,
- $\mathfrak{g}_{-1} = \left\{ \begin{bmatrix} -u^t e_1 + e_1 t^u & u \\ t^u & 0 \end{bmatrix} : \langle u, e_1 \rangle = 0 \right\}$, and
- $\mathfrak{g}_1 = \left\{ \begin{bmatrix} u^t e_1 - e_1 t^u & u \\ t^u & 0 \end{bmatrix} : \langle u, e_1 \rangle = 0 \right\}$.

Then $\mathfrak{a} \subset \mathfrak{g}_0$ is a maximal abelian subalgebra, and the decomposition $\mathfrak{so}(1, m) = \mathfrak{g}_0 + \mathfrak{g}_{-1} + \mathfrak{g}_1$ is the associated (restricted) root space decomposition of $\mathfrak{so}(1, m)$.

Recall that $H := \begin{bmatrix} 0 & e_1 \\ t e_1 & 0 \end{bmatrix} \in \text{PO}(m, 1)$.

The following lemma states some basic properties of the root space decomposition [Kna02, Chapter II.1], and can be verified explicitly in this special case.

**Lemma B.1.**

1. Let $\sigma \in \{0, 1, -1\}$, and $Y \in \mathfrak{g}_\sigma$. Then

$$[H, Y] = \sigma Y \quad \text{and} \quad \text{Ad}(e^{sH}) Y = e^{\sigma s} Y.$$

2. Let $\alpha, \beta \in \{0, -1, 1\}$. Then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, where $\mathfrak{g}_{-2} := \{0\} =: \mathfrak{g}_2$.

Next, suppose that $\tau : \text{PO}(m, 1) \to \text{PGL}_d(\mathbb{R})$ is an irreducible representation so that $\tau(e^H)$ is proximal and $\tau_0 : e^{\mathbb{R} : H} \to \text{SL}_d(\mathbb{R})$ is the lift of $\tau_0 := \tau\big|_{e^{\mathbb{R} : H}}$.

Let $\mathfrak{sl}_d(\mathbb{R})$ denote the Lie algebra of $\text{PGL}_d(\mathbb{R})$ and let $d\tau : \mathfrak{so}(m, 1) \to \mathfrak{sl}_d(\mathbb{R})$ be the derivative at the identity of the homomorphism $\tau : \text{PO}(m, 1) \to \text{PGL}_d(\mathbb{R})$. The next lemma gives a description of the eigenvalues and eigenspaces of $\tau_0(e^H)$. 


Lemma B.2. Let $e^\lambda$ denote the largest eigenvalue of $\tau_H(e^H)$ and let $V_0 \subset \mathbb{R}^d$ denote the eigenspace of $\tau_0(e^H)$ corresponding to $e^\lambda$. For $n \in \mathbb{N}$, define

$$V_{n+1} := d\tau(g_{-1})V_n,$$

(1) If $v \in V_n$, then $\tau_0(e^H)v = e^{\lambda-n}v$.
(2) If $Z \in g_0$, then $d\tau(Z)V_n \subset V_n$.
(3) If $Z \in g_1$, then $d\tau(Z)V_0 = \{0\}$ and $d\tau(Z)V_n \subset V_{n-1}$ when $m > 0$.
(4) $\sum_{n \geq 0} V_n = \mathbb{R}^d$.

Proof. (1): By definition $v = d\tau(Y)w$ for some $Y \in g_{-1}$ and $w \in V_{n-1}$. Then by induction

$$\tau_0(e^H) d\tau(Y)w = d\tau(Ad(e^H)Y)\tau_0(e^H)w$$

$$= d\tau(e^{-1}Y) (e^{\lambda-(n-1)}w)$$

$$= e^{\lambda-n}d\tau(Y)w,$$

where the second equality is a consequence of (1) of Lemma B.1.

(2): Fix some $v \in V_n$. Then by definition $v = d\tau(Y)w$ for some $Y \in g_{-1}$ and $w \in V_{n-1}$. Then $[Z,Y] \in g_0$ by (2) of Lemma B.1 so

$$d\tau(Z)d\tau(Y)w = d\tau([Z,Y])w - d\tau(Y)d\tau(Z)w \in V_n$$

by induction.

(3): If $v_0 \in V_0$, then

$$\tau_0(e^H) d\tau(Z)v_0 = d\tau(Ad(e^H)Z)\tau_0(e^H)v_0$$

$$= e^{\lambda+1}d\tau(Z)v_0.$$

Since $e^\lambda$ is the largest eigenvalue of $\tau_0(e^H)$ we must have $d\tau(Z)v_0 = 0$. Since $v_0 \in V_0$ was arbitrary, we then have $d\tau(Z)V_0 = \{0\}$.

Next fix some $v \in V_n$. Then by definition $v = d\tau(Y)w$ for some $Y \in g_{-1}$ and $w \in V_{n-1}$. Then $[Z,Y] \in g_0$ by (2) of Lemma B.1 so

$$d\tau(Z)d\tau(Y)w = d\tau([Z,Y])w - d\tau(Y)d\tau(Z)w \in V_{n-1}$$

by (2) and induction.

(4): The previous parts show that $\sum_{n \geq 0} V_n$ is an $d\tau$ and hence $\tau$ invariant subspace. Since $\tau$ is irreducible, we then have $\sum_{n \geq 0} V_n = \mathbb{R}^d$. □

Proof of Lemma 10.6. The first statement of the lemma is an immediate consequence of Lemma 10.5 and B.2. To prove the second statement, fix some non-zero $v_0 \in V_0$, and consider the linear map $T : g_{-1} \to V_1$ given by

$$T(Y) = d\tau(Y)v_0.$$ 

Since $T$ is onto and $\dim_{\mathbb{R}} g_{-1} = m - 1$, we see that $\dim_{\mathbb{R}} V_1 \leq m - 1$. It is now sufficient to prove that $\ker T = \{0\}$. To see this, again let

$$M := \{ k \in K : k \cdot e_1 = e_1 \}.$$ 

Then a calculation shows that $Ad(M)$ preserves and acts irreducibly on $g_{-1}$. Notice that $\tau(M)v_0 \subset V_0$ is a compact connected set and so $\tau(M)v_0 = v_0$. Further, if $Y \in g_{-1}$ and $k \in M$, then

$$T(Ad(k)Y) = d\tau(Ad(k)Y)v_0 = \tau(k)d\tau(Y)\tau(k^{-1})v_0 = \tau(k)T(Y).$$
So \( \ker T \) is an \( \text{Ad}(M) \)-invariant subspace. So either \( \ker T = \{0\} \) or \( \ker T = \mathfrak{g}_{-1} \). If \( \ker T = \mathfrak{g}_{-1} \), then \( V_0 = \mathbb{R}^d \) and since \( d > 1 \) this is impossible. So \( \ker T = \{0\} \) and hence \( \dim V_1 \geq m - 1 \).

\[ \square \]

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