LINE ARRANGEMENTS MODELING CURVES OF HIGH DEGREE: EQUATIONS, SYZYGIES AND SECANTS

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Abstract. We study curves consisting of unions of projective lines whose intersections are given by graphs. These so-called graph curves can be embedded in projective space so their ideals are generated by products of linear forms. We discuss the minimal free resolution of the ideal of a graph curve and are able to produce products of linear forms that generate the ideal under certain hypotheses. We also study the higher-dimensional subspace arrangements obtained by taking the secant varieties of graph curves.

An arrangement of linear subspaces, or subspace arrangement, is the union of a finite collection of linear subspaces in projective space. In this paper we study arrangements of lines called graph curves. Let $G = (V, E)$ be a simple, connected graph with vertex set $V$ and edge set $E$. We assume that $G$ is strictly subtrivalent, meaning that each vertex has degree at most three and there is at least one vertex with degree less than three. The (abstract) graph curve $C_G$ associated to $G$ is constructed by taking the union of $\{L_v \mid v \in V\}$ where each $L_v$ is a copy of $\mathbb{P}^1$ and lines $L_u$ and $L_v$ intersect in a node if and only if there is an edge between $u$ and $v$ in $G$. (Note that if we think of the nodes of $C_G$ as vertices and the lines $L_v$ as edges, then $C_G$ is the graph dual to $G$.) Since we are assuming that each vertex has degree less than or equal to three, $C_G$ is specified by purely combinatorial data; we may assume that on each component of $C_G$ the nodes are at 0, 1 or $\infty$, as the automorphism group of $\mathbb{P}^1$ can take these three points to any other set of three distinct points.

We present an explicit embedding of $C_G$ into projective space (Theorem 1.5) that allows us to use the combinatorics of $G$ to give defining.

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equations for $C_G$ that are products of linear forms (Theorem 1.7). Although a subspace arrangement may always be cut out by products of linear forms set-theoretically, we do not generally expect the ideal of a subspace arrangement to be generated by products of linear forms. For example, a generic set of 7 or more points in $\mathbb{P}^2$ or 3 or more non-intersecting lines in $\mathbb{P}^3$ cannot have an ideal generated by products of linear forms, cf Proposition 5.4 and Proposition 5.7 in [1]. Of course, in both of these cases, the subspaces in the arrangements do not meet each other, so there is little combinatorics to speak of.

The most striking examples of subspace arrangements with ideals generated by products of linear forms occur when the intersections among the subspaces have a rich combinatorial structure. The Boolean, or coordinate, subspace arrangements are defined by monomial ideals and can be studied using the theory of Stanley-Reisner ideals. The “hook” arrangements studied in [13] and the $k$-equal arrangements of [14] also have ideals generated by products of linear forms that are described by a duality between the two [1].

Graph curves associated to graphs in which every vertex is trivalent are canonical curves, and have been studied in several different contexts. For example, Ciliberto, Harris, and Miranda [3] used graph curves to understand the surjectivity of the Wahl map, Ciliberto and Miranda [4] related graph curves to graph colorings, and Bayer and Eisenbud [2] studied graph curves in connection with Green’s conjecture. In fact, Proposition 3.1 in [2] gives an explicit description of the monomial generators of the ideal of a canonical graph curve using the combinatorics of $G$. Note that if each vertex of $G$ is trivalent, then each copy of $\mathbb{P}^1$ in $C_G$ contains three nodes, and $C_G$ is stable. Since our graphs are strictly subtrivalent, the corresponding graph curves $C_G$ are not stable.

In §2 we combine the knowledge that $I_{C_G}$ is generated by quadrics with cohomological techniques to deduce results about minimal free graded resolutions that parallel some of what is known for smooth curves and discuss the effect of the combinatorics of the graphs on syzygies.

We also investigate higher-dimensional subspace arrangements obtained by constructing the secant varieties of these line arrangements.
Definition 0.1. Given a reduced but not necessarily irreducible variety $X$, its $k$th secant variety, $\Sigma_k$, is the closure of the union of all $k$-planes that meet $X$ in a scheme of dimension 0 and length at least $k + 1$.

If $X$ is a subspace arrangement, then each $\Sigma_k$ is again a subspace arrangement. Note in this case that just as the graph curves are not normal, the associated secant varieties will not be normal (except in trivial or degenerate circumstances) as they will have codimension one singularities.

In fact, the motivation for this work comes from a conjectural picture involving a smooth curve and its secant varieties [15], refining and extending conjectures of the last author in [17], in which results of Green [9] for syzygies of curves can be viewed as a base case. Much of the work on secant varieties in [15] uses cohomological methods and is restricted to smooth irreducible curves.

In §3 we show that the girth, or length of the smallest cycle in $G$, is related to the failure of property $N_p$ for $C_G$ and that this may be generalized for secant varieties. We believe that there is much yet to discover about the algebra and combinatorics of line arrangements and their secant varieties, and pose several questions and conjectures in the last section.

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1. Line arrangements generated by products of linear forms

In this section we present an embedding of $C_G$ into projective space over an algebraically closed field $k$ of characteristic zero, and show that this embedding has the property that its ideal may be generated by products of linear forms, which we describe explicitly.

We begin by setting some assumptions and notation. Let $d$ be the number of vertices in $G$ and $m$ the number of edges.
Assumption 1.1. For the remainder of the paper, if \( G \) is a graph we assume that it satisfies the following conditions.

(1) \( G \) is connected.
(2) \( G \) is simple.
(3) \( G \) is strictly subtrivalent.
(4) The shortest path between any two trivalent vertices contains at least 3 edges.
(5) \( G \) does not contain any triangles.

The topology of \( G \) determines the arithmetic genus of \( C_G \) as we may view \( G \) as a 1-dimensional simplicial complex, from which it follows that \( p_a(C_G) = h^1(G, k) \) (see Proposition 1.1 in [2]). We refer to this quantity as the genus \( g \) of \( G \), and \( m = d + g - 1 \) if (1) and (2) of 1.1 hold. Note that \( g \) is not the genus of \( G \) in the usual graph-theoretic sense.

Remark 1.2. The hypothesis that \( G \) does not contain any triangles is necessary in Theorem 1.7. It is also necessary for \( \Sigma_1 \) to be a union of 3-dimensional planes. Indeed, the union of all secant lines between points on intersecting lines spans a 2-plane while the union of the secant lines joining points on disjoint lines spans a 3-plane. If \( L_u \) and \( L_v \) intersect, then the 2-plane that they span is contained in a 3-plane in the secant variety if and only if there exists a line \( L_w \) that intersects exactly one of \( L_u \) and \( L_v \).

We describe the embedding of \( C_G \subset \mathbb{P}^{d-g} \) by giving the ideal of each line in the arrangement. Lemma 1.3 shows that by requiring trivalent vertices to be separated by three edges, \( C_G \) embeds into \( \mathbb{P}^{d-g} \) as a curve of degree at least 2\( g + 2 \). We denote the homogeneous coordinate ring of \( \mathbb{P}^{d-g} \) by \( S = k[x_0, \ldots, x_{d-g}] \).

Lemma 1.3. If \( d \geq 2 \), and \( G \) satisfies Assumption 1.1 then \( d \geq 2g + 2 \).

Proof. We use induction on \( g \). It is easy to check that both of the results hold for \( g \) equal to 0 and and that it also holds for \( g = 1 \) since \( G \) does not contain triangles. If \( g > 1 \), fix any cycle \( \gamma \). Since \( G \) is connected, \( \gamma \) must contain at least one trivalent vertex. Let \( v_0 \) and \( v_1 \) be trivalent vertices in \( \gamma \) connected by a path \( P \) of bivalent vertices, allowing the possibility that \( v_0 = v_1 \). Let \( G' \) be obtained by removing all of the bivalent vertices on \( P \). Then \( G' \) has genus \( g - 1 \) and \( d' \) vertices. By the induction hypothesis \( d' \geq 2g \). Since \( v_0 \) and \( v_1 \) are separated by at least 3 edges, \( d' \leq d - 2 \), and we conclude that \( d \geq 2g + 2 \). \( \square \)
Remark 1.4. The result in Lemma 1.3 is by no means optimal. For example, it is a straightforward combinatorial exercise to show that if $G$ satisfies parts (1)–(3) of Assumption 1.1 and $d$ is even, then $d \geq 2g + 2$ as soon as $G$ has at least 3 non-trivalent vertices, while if $d$ is odd then $d \geq 2g + 3$ as soon as $G$ has at least 4 non-trivalent vertices. As assumptions (4) and (5) are needed for our main results, and a minimum number of non-trivalent vertices is not maintained in some of our inductive proofs, we do not write out the most complete possible statement here.

Theorem 1.5. If $G$ satisfies Assumption 1.4, then $C_G$ can be embedded into $\mathbb{P}^{d-g}$ so that the ideal of each line is generated by variables and binomials of the form $x_j - x_k$.

Proof. Construct a graph $\tilde{G}$ from $G$ by adding a loop to each vertex of degree 1 so that vertices of degree 1 in $G$ are incident to two edges in $\tilde{G}$. The ideal of each line in the embedding of $C_G$ will be specified via a labeling of the edges of $\tilde{G}$.

Let $\Omega = \{x_0, \ldots, x_{d-g}\}$. Label the edges containing a vertex incident to three edges by $e_j, e_k$, and $e_j - e_k$, where $j, k \in \{0, \ldots, d-g\}$. Label the remaining edges with unused elements $e_i$ where $i \in \{0, \ldots, d-g\}$.

If a vertex $v$ is incident to three edges, $e_j, e_k$, and $e_j - e_k$, then the corresponding line has ideal generated by $\Omega \{x_j, x_k\}$. If $v$ is incident to two edges, we obtain generators for the corresponding ideal by modifying $\Omega$ as follows. Delete each variable $x_i$ if $v$ is contained in the edge $e_i$. If $v$ is contained in an edge labeled $e_j - e_k$, we omit the variables $x_j$ and $x_k$ and substitute the binomial $x_j - x_k$.

Example 1.6 shows, the hypothesis that trivalent vertices are separated by three edges is not necessary for Lemma 1.3 or Theorem 1.5.

Example 1.6. Let $G$ be the graph obtained by adding a bivalent vertex to every edge of $K_4$. $C_G$ is a graph curve of degree 10 and genus 3. When $C_G$ is embedded via Theorem 1.5 the result is generated by 18 quadratic products of linear forms. This is despite the fact that each pair of trivalent vertices are separated by exactly two edges.

Theorem 1.7. If $G$ satisfies Assumption 1.4, then an embedding of $C_G$ into $\mathbb{P}^{d-g}$ using Theorem 1.5 has the property that $I_{C_G}$ is generated by elements of the form $x_ix_j, (x_j - x_k)x_i$, and $(x_j - x_k)(x_l - x_m)$, where the variables in each product are distinct.
We prove two technical lemmas before proceeding to the proof of Theorem 1.7. Together, these lemmas help us exploit the fact that many of the lines in the embedding of \( C_G \) are defined by monomial ideals.

**Lemma 1.8.** If \( I \) and \( J \) are ideals that can be decomposed into \( I = L_I + Q_I \) and \( J = L_J + Q_J \), where

1. \( L_I \) and \( L_J \) are generated by disjoint sets of variables
2. \( Q_J \) is generated by quadratic monomials, none of which are divisible by a generator of \( L_J \)
3. \( Q_I \subseteq J \)
4. \( Q_J \subseteq I \),

then, \( I \cap J = L_IL_J + Q_I + Q_J \).

**Proof.** It is clear that \( L_IL_J + Q_I + Q_J \subseteq I \cap J \). We prove the reverse containment. Suppose that \( f \) is an element of \( I \cap J = (L_I + Q_I) \cap (L_J + Q_J) \). As \( f \in L_I + Q_I \), we can find \( g \in L_I \) and \( h \in Q_I \), so that \( f = g + h \). As \( h \in Q_I \subseteq J \), we must have that \( g \in J \) as well. Since \( g \) is in \( L_I \) and \( J \), both monomial ideals, every monomial of \( g \) must be in \( L_I \) and \( J \). So, we may assume that \( g \) is a monomial. If \( g \in Q_J \), we are done, and if not it is in \( L_J \). Therefore, \( g \) must be divisible by a variable in \( L_I \) and a variable in \( L_J \), and hence is in \( L_IL_J \). \( \square \)

**Lemma 1.9.** If \( C_G \) is embedded in \( \mathbb{P}^{d-g} \) via Theorem 1.5 and we interchange the labels \( e_j \) and \( e_j - e_k \) on the edges meeting at a trivalent vertex, the embedding associated to the new labeling may be obtained via a linear automorphism of \( \mathbb{P}^{d-g} \) that is an involution.

**Proof.** Define \( \sigma : \mathbb{P}^{d-g} \to \mathbb{P}^{d-g} \) by \( \sigma(x_k) = x_j - x_k \) and by the identity on all other variables \( x_i \). It is easy to check that \( \sigma \) is an involution and an automorphism. The only ideals in Theorem 1.5 changed by \( \sigma \) are the ones associated to the vertices denoted \( u \) and \( v \) in Figure 11. The ideal corresponding to \( v \) had generators \( x_0, \ldots, \hat{x}_j, \hat{x}_k, x_j - x_k, \hat{x}_i, \ldots, x_{d-g} \), and \( \sigma \) replaces \( x_j - x_k \) with \( x_k \). So now the ideal associated to \( v \) is a monomial ideal. Similarly, in the ideal corresponding to vertex \( u \), the variable \( x_k \) is replaced by \( x_j - x_k \). \( \square \)

**Proof of Theorem 1.7** We proceed with two cases, where \( G \) is a tree and where \( G \) contains cycles.
Case 1: Trees. We argue by induction on $d$. The base case of two lines in $\mathbb{P}^2$ is easy to check. Let $v$ be a vertex of degree 1, and assume by induction that we have the result for $C_{G_v}$. If $v$ is attached to a trivalent vertex, if necessary relabel using Lemma 1.9 so that the ideal $I_{L_v}$ corresponding to $v$ is monomial, and $L_v$ is spanned by a point $p$ in $C_{G_v}$ and the point $[0 : \cdots : 0 : 1]$. Without loss of generality, $I_{C_{G_v}} = Q + \langle x_{d-g} \rangle$, where $Q$ is generated by quadratic products of linear forms in which no term is divisible by $x_{d-g}$.

We argue that $Q \subset I_{L_v}$. Let $q = fh$ be one of the generators of $Q$ fixed above. Since $q$ must vanish at $p$, without loss of generality, we may assume that $f$ vanishes at $p$. Since $x_{d-g}$ does not appear in $f$, then $f$ must also vanish on $[0 : \cdots : 1]$. Thus, $f$ is a linear form vanishing at two points of $L_v$; hence it must vanish on all of $L_v$. Therefore $Q \subset I_{L_v}$.

Applying Lemma 1.8, we see that $I_{C_{G_v}} = Q + \langle x_{d-g} \rangle \cdot I_{L_v}$.

Case 2: $G$ contains cycles. We argue by induction on $g$. If $g = 1$ and $G$ is a cycle, then its vanishing ideal is generated by squarefree quadratic monomials as long as $G$ is not a triangle. Indeed, in this case $C_G$ is the “stick elliptic normal curve” from Example 3.5 in [16]. Otherwise, fix a cycle $\gamma$ so that $C_\gamma$ does not span $\mathbb{P}^{d-g}$. Let $v_0$ and $v_1$ be trivalent vertices on $\gamma$ so that there is a path joining them in $\gamma$ that does not contain any other trivalent vertices. Let $P$ be the set of bivalent vertices in $\gamma$ on this path from $v_0$ to $v_1$ and let $H$ be the subgraph of $G$ induced on $V - P$. See Figure 2 for a diagram. (If there is only one trivalent vertex in $\gamma$, let $P$ be the set of bivalent vertices of $\gamma$.) Since $|P| \geq 2$, $H$ is contained in a proper subspace of $\mathbb{P}^{d-g}$. By induction, $H$ has at least one less cycle than $G$, so we may assume that $I_{C_H}$ is generated by products as specified in the theorem. If $I_{C_\gamma}$ is not a monomial ideal then permute labels on the edges of $\gamma$ using Lemma 1.9 so that the edges labeled $e_j - e_k$ are no longer contained in $\gamma$. 

![Figure 1. A trivalent vertex](image)
Figure 2. A diagram of the induction step

Without loss of generality, let the edges from $v_0$ to $v_1$ be labeled $e_0, \ldots, e_a$. Then we can write $I_{C_H} = L_H + Q_H$ where $L_H = \langle x_1, \ldots, x_{a-1} \rangle$, and $Q_H$ is generated by products of linear forms in which no term is divisible by $x_1, \ldots, x_{a-1}$. Write $I_{C_\gamma} = L_\gamma + Q_\gamma$ where $L_\gamma$ is generated by variables and $Q_\gamma$ is generated by squarefree quadratic monomials that are not in $L_\gamma$. As $C_G = C_H \cup C_\gamma$, it must be that $I_{C_G} = I_{C_H} \cap I_{C_\gamma}$. We will be done if we can show that $I_{C_H} \cap I_{C_\gamma} = L_H L_\gamma + Q_H + Q_\gamma$.

Claim 1: $Q_\gamma \subset I_{C_H}$.

Proof. The ideal $Q_\gamma$ vanishes on all of the lines in $C_H \cap C_\gamma$. The lines in $C_H$ that do not intersect $C_\gamma$ are spanned by coordinates that are orthogonal to the span of $C_\gamma$. Therefore, $Q_\gamma$ vanishes on those lines as well. It remains to show that $Q_\gamma$ vanishes on each line $\ell$ that intersects $C_\gamma$. Each $\ell$ corresponds to a vertex that is connected to a trivalent vertex in $\gamma$ via an edge labeled $e_j - e_k$ for some $j, k$. Suppose that $x_r x_s \in Q_\gamma$ where $r < s$. The line $\ell$ is embedded in $\mathbb{P}^{d-g}$ with coordinates $j$ and $k$ equal and a nonzero coordinate $m$ where $x_m \in L_\gamma$. Note that the fixed monomial generators of $Q$ are not divisible by $x_m$ and $x_j x_k$ is not in $Q_\gamma$ because $C_\gamma$ contains a line embedded with nonzero coordinates in positions $j$ and $k$. Therefore, the monomial $x_r x_s$ vanishes on $\ell$ because it cannot be of the form $x_j x_m, x_j x_k$ or $x_k x_m$. \qed

Claim 2: $Q_H \subset I_{C_\gamma}$.

Proof. There are three cases, generators $x_i x_j$, $x_i (x_j - x_k)$ and $(x_j - x_k)(x_i - x_m)$. As in the previous case, these quadrics vanish on the lines in $Q_H$ that do not intersect $C_\gamma$. There are precisely two lines, $\ell_1$ and $\ell_2$, in $C_\gamma$ that intersect $C_H$ but are not contained in it. The line $\ell_1$ has
nonzero coordinates in positions 0 and 1, and the line $\ell_2$ has nonzero coordinates in positions $a - 1$ and $a$. Recall that $x_1, x_{a-1} \in L_H$. As $Q_H$ does not contain monomials divisible by any variables in $L_H$, the monomials $x_0x_1$ and $x_{a-1}x_a$ are not contained in $Q_H$; so, $x_ix_j$ must vanish on both $\ell_1$ and $\ell_2$.

Suppose now that we have $x_i(x_j - x_k)$. Since $x_i$ cannot be in $L_H$, and $x_1$ and $x_{a-1}$ are in $L_H$, $x_i(x_j - x_k) \neq x_1(x_0 - x_j), x_{a-1}(x_a - x_k)$. Any other choice of quadric of this form would vanish on $\ell_1$ and $\ell_2$. As long as $(x_j - x_k)(x_l - x_m) \neq (x_0 - x_j)^2, (x_a - x_k)^2$, these forms vanish on $\ell_1$ and $\ell_2$ as well. But, by the induction hypothesis, we do not need such forms.

Based on the above, $I_{CG} = I_{CH} + I_{C\gamma} = (L_H + Q_H) \cap (L_\gamma + Q_\gamma)$ satisfies the hypotheses of Lemma 1.8. Therefore, $I_{CG} = L_HL_\gamma + Q_H + Q_\gamma$, which are generated by products of linear forms as stipulated.

It is possible to determine which products in the statement of Theorem 1.5 are in $I_{CG}$ based on the combinatorics of $G$. If $\alpha$ is an edge label then we define its associated linear form

$$\ell(\alpha) = \begin{cases} x_i, & \alpha = e_i \\ x_i - x_j, & \alpha = e_i - e_j. \end{cases}$$

**Definition 1.10.** We will say that an index $i$ appears on an edge if it is labeled with $e_i$ or $\pm(e_i - e_j)$. We will say that indices $i$ and $j$ are adjacent if they appear on adjacent edges.

**Proposition 1.11.** Suppose that $G$ satisfies Assumption 1.1. Let $\alpha$ and $\beta$ be edge labels. The product $\ell(\alpha)\ell(\beta)$ is in $I_{CG}$ if and only if the indices that appear in $\alpha$ and $\beta$ are not adjacent in $G$. Indeed, we can say more.

(1) The product $x_ix_j$ is in $I_{CG}$ unless either subgraph in Figure 3 appears (where we include all permutations of $i, j$ and $k$ for the graph on the right).

**Figure 3.** Subgraphs excluding $x_ix_j$
(2) If $e_j - e_k$ is an edge label $x_i(x_j - x_k)$ is in $I_{C_G}$ unless either subgraph in Figure 4 appears.

Figure 4. Subgraphs excluding $x_i(x_j - x_k)$

(3) If $e_i - e_j$ and $e_k - e_l$ are edge labels then $(x_i - x_j)(x_k - x_l)$ is in $I_{C_G}$.

Proof. A homogeneous polynomial $f$ is in $I_{C_G}$ if and only if it vanishes on $L_v$ for every vertex $v$ in $G$. Fix a vertex $v$ in $G$. Note that $x_i$ vanishes on $L_v$ if and only if $i$ does not appear at $v$. The proof of (1) follows as $x_i(x_j - x_k)$ vanishes on $L_v$ if and only if $i$ and $j$ do not both appear at $v$. The product $x_i(x_j - x_k)$ in (2) vanishes on $L_v$ if $i$ does not appear at $v$. If $e_i - e_m$ appears on an edge incident to $v$, then neither $j$ nor $k$ may appear at $v$ as trivalent vertices are separated by three edges; we conclude that $x_j - x_k$ vanishes on $L_v$. If $i$ appears at $v$ as a single index only, then $x_i(x_j - x_k)$ is in $I(L_v)$ if and only if $x_j - x_k$ is. The two graphs depicted show the only circumstances in which $x_j - x_k$ does not vanish at $v$. For (3), since $e_i - e_j$ and $e_k - e_l$ are both edge labels, if neither $i$ nor $j$ appears at $v$ then $x_i - x_j$ vanishes on $L_v$. If one of $i$ or $j$ appears at $v$, then neither $k$ nor $l$ can, as trivalent vertices are separated by at least three edges, and $x_k - x_l$ vanishes on $L_v$. □

2. Regularity and the Cohen-Macaulay property

In this section we will show that our graph curves $C_G$ are arithmetically Cohen-Macaulay and that $I_{C_G}$ has regularity $\leq 3$. We are interested in statements about the homogeneous ideal of $C_G$, but we sometimes work with ideal sheaves so that we may use cohomological techniques. If $X \subset \mathbb{P}^n$ we will let $I_X$ denote its homogenous ideal and $\mathcal{I}_X$ denote its ideal sheaf. If $v$ is a vertex in $G$ we let $G_{\hat{v}}$ be the subgraph induced on $V(G)\backslash\{v\}$. Note that an embedding via Theorem 1.5 corresponds to an embedding via a complete linear series that restricts to $\mathcal{O}_{L_v}(1)$ for each $v$ in $G$.

Recall that $X \subset \mathbb{P}^n$ is $m$-regular if $H^i(\mathcal{O}_X(m - i)) = 0$ for all $i > 0$.

Theorem 2.1. If $G$ satisfies Assumption 1.1, then the embedding of $C_G \subset \mathbb{P}^n$ via Theorem 1.5 is 3-regular.
Proof. If $G$ is a tree, then the embedding must be 2-regular as $C_G$ is linearly joined as in Theorem 0.4 of [6]. If $G$ is a cycle, then the homogeneous coordinate ring of $C_G$ is Gorenstein and 3-regular as proved in §3 of [7]. Otherwise, let $\mathcal{I}_{C_G}$ be the ideal sheaf of $C_G$. As $C_G$ is embedded via a complete linear series $H^1(\mathcal{O}_X(d)) = 0$ for all $d \geq 0$. Since $\text{dim } C_G = 1$, it remains only to show that $H^1(\mathbb{P}^n, \mathcal{I}_{C_G}(2)) = 0$.

For any vertex $v$ we have

$$0 \to \mathcal{I}_{C_G} \to \mathcal{I}_{C_G^e} \oplus \mathcal{I}_{L_v} \to \mathcal{I}_{C_G^e \cap L_v} \to 0,$$

which gives us

$$H^0(\mathcal{I}_{C_G^e}(2) \oplus \mathcal{I}_{L_v}(2)) \to H^0(\mathcal{I}_{C_G^e \cap L_v}(2)) \to H^1(\mathcal{I}_{C_G}(2)) \to H^1(\mathcal{I}_{C_G^e}(2) \oplus \mathcal{I}_{L_v}(2)).$$

If we can choose $v$ so that we know $H^1(\mathcal{I}_{C_G^e}(2) \oplus \mathcal{I}_{L_v}(2)) = 0$, then it only remains to show that $H^0(\mathcal{I}_{C_G^e}(2) \oplus \mathcal{I}_{L_v}(2)) \to H^0(\mathcal{I}_{C_G^e \cap L_v}(2))$ is surjective.

If $g = 1$ but $G$ not a cycle then $G$ has a vertex $v$ of degree 1 which corresponds to a line $L_v$ that intersects $C_G^e$ in a single point $p$. By Lemma 2.2 then we need to find two linearly independent quadrics that vanish on $p$ but not $L_v$. Every element of $H^0(\mathcal{I}_{C_G^e}(2))$ must vanish at $p$, and there must be a quadratic form containing $p$ but not $L_v$ because $L_v$ is not a component of $C_G^e$. Any such form vanishes at $p$ and an additional point on $q$ on $L_v$. As $\mathcal{I}_{C_G^e}$ is cut out by quadrics by Theorem 1.7 and $q$ is not contained in $C_G^e$ there must be another quadric in $\mathcal{I}_{C_G^e}$ that vanishes at $p$ but not $q$. These two quadrics must be linearly independent.

If $g > 1$, then $G$ has a vertex $v$ of degree 3 whose removal does not disconnect the graph. Then $h^1(G, k) < h^1(G, k)$, so by induction on $g$, $H^1(\mathcal{I}_{C_G^e}(2) \oplus \mathcal{I}_{L_v}(2)) = 0$. Since $C_G \cap L_v$ consists of 3 points then these points are collinear, as they lie on $L_v$. Therefore, any quadratic form that vanishes on the points must also vanish on $L_v$. Hence, $H^0(\mathcal{I}_{L_v}(2))$ surjects onto $H^0(\mathcal{I}_{C_G \cap L_v})$ in this case.

\[\square\]

Lemma 2.2. Let $L \subset \mathbb{P}^n$ be a line and $p$ be a point on $L$. Then $\left(I_p / I_L\right)_2$ is a vector space of dimension 2.

Proof. We may choose coordinates so that $I_L = \langle x_2, \ldots, x_n \rangle$, $I_p = \langle x_1, x_2, \ldots, x_n \rangle$. In these coordinates, $\left(I_p / I_L\right)_2$ is the span of $x_0 x_1, x_1^2$. \[\square\]
Theorem 2.3. If $G$ satisfies Assumption 1.1, then the embedding of $C_G \subset \mathbb{P}^n$ via Theorem 1.5 is arithmetically Cohen-Macaulay.

Proof. Recall that a curve $C$ in $\mathbb{P}^n$ is arithmetically Cohen-Macaulay if the map $H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(k)) \rightarrow H^0(C, O_C(k))$ is surjective for all $k \geq 0$. Surjectivity for $k = 0$ is automatic if $C$ is connected, and the case $k = 1$ follows if $C$ is embedded via a complete linear system. For $k \geq 2$, consider the short exact sequence of sheaves

$$0 \rightarrow \mathcal{I}_{C_G}(k) \rightarrow O_{\mathbb{P}^n}(k) \rightarrow O_C(k) \rightarrow 0.$$ 

Taking cohomology, we have

$$H^0(\mathbb{P}^n, O_{\mathbb{P}^n}(k)) \rightarrow H^0(C, O_C(k)) \rightarrow H^1(\mathbb{P}^n, \mathcal{I}_{C_G}(k)) \rightarrow 0.$$ 

But, $\mathcal{I}_{C_G}$ is 3-regular, so $H^1(\mathbb{P}^n, \mathcal{I}_{C_G}(k)) = 0$ for all $k \geq 2$. \qed

3. Secant varieties and property $N_{k,p}$

In this section we discuss a generalization of the $N_p$ property of Green and Lazarsfeld [11] and what we can say about it in the context of graph curves and their secant varieties.

Definition 3.1 ([6]). A subscheme $X \subseteq \mathbb{P}^n$ satisfies $N_{k,p}$ if $\beta_{i,j}(S_X)$ nonzero implies that $j = i + k - 1$ for $1 \leq i \leq p$. In other words, $X$ satisfies $N_{k,p}$ if $I(X)$ is generated by forms of degree $k$ and all syzygies are linear through the $p$th stage in the resolution.

The authors of [6] state that their Theorem 1.1 has a natural generalization for higher degree forms. We give a precise statement of a special case below.

Theorem 3.2. Suppose that $X$ satisfies $N_{k,p}$. Let $W$ be a linear subspace of dimension $p$ with $Z = X \cap W$. If $\dim Z = 0$, then $Z$ contains at most $\binom{p+k-1}{p}$ points.

Proof. It follows from the proof of Theorem 1.1 from [6] that the ideal of $Z$ in the homogeneous coordinate ring of $W$ is $k$-regular. Via Theorem 4.2 in [5] the degree in which the Hilbert function and Hilbert polynomial of $S_Z$ agree is the regularity of $S_Z$. We know that the Hilbert polynomial of $S_Z$ is constant equal to the number of points in $Z$. If $I(Z)$ is $k$-regular then $S_Z$ is $k - 1$-regular.

If $\dim (S_Z)_{k-1}$ is equal to the size of $Z$, then $\dim S_{k-1}$ must be at least the size of $Z$. Hence, $|Z| \leq \binom{p+k-1}{p}$. \qed
Corollary 3.3. If $C_G$ contains a cycle of $m$ lines, then $N_{2,m-2}$ fails.

Proof. A cycle of $m$ lines spans at most a $\mathbb{P}^{m-1}$. A general hyperplane in the $\mathbb{P}^{m-1}$ containing the lines intersects each line in the cycle in a point. Thus, we have an $(m - 2)$-plane intersecting $X$ in at least $m$ points. □

Corollary 3.4. If $C_G$ contains a cycle of $m$ lines, then $N_{3,m-4}$ fails for the secant variety of $C_G$.

Proof. Since the $m$ lines in the cycle are contained in a $\mathbb{P}^{m-1}$, so is the span of any subset of these lines. Thus, each 3-plane obtained by taking the span of nonadjacent lines in the cycle is contained in this $\mathbb{P}^{m-1}$. There are \( \binom{m}{2} - m = \frac{1}{2}m(m - 3) \) such 3-planes.

A general plane of dimension $m - 4$ intersects a 3-plane in $\mathbb{P}^{m-1}$ in a point. Therefore, a general $(m - 4)$-plane in this $\mathbb{P}^{m-1}$ intersects the secant variety of $X$ in $\frac{1}{2}m(m - 3)$ points. However, \( \binom{m-4+3-1}{m-4} = \binom{m-2}{2} = \frac{1}{2}(m - 2)(m - 3) \). Thus, $N_{3,m-4}$ fails. □

Because $N_{2,m-2}$ fails if $G$ contains a cycle of length $m$, in many situations it will be impossible to reproduce the graded Betti diagrams of smooth curves with the graded Betti diagrams of graph curves. For instance, for genus $g = 2$ and degree $d$, the girth has an upper bound of $\lfloor \frac{2d-3}{3} \rfloor + 1$. For higher genera, minimal girth can be calculated by finding a maximal-girth trivalent graph on $2g - 2$ vertices, and adding bivalent vertices evenly on the paths between trivalent vertices.

4. Questions and conjectures

Computations with Macaulay 2 [10] were essential in our initial explorations of embeddings of graph curves. In addition to the results proved in this article we have several questions and conjectures regarding the defining equations and syzygies of graph curves and their secant varieties motivated by the examples that we have seen.

In §1 we saw that $I_{C_G}$ is generated by products of linear forms that can be described explicitly in terms of the combinatorics of the graph $G$. The $k$th secant variety of $C_G$ is an arrangement of subspaces of dimension $2k + 1$. The combinatorics of such a higher-dimensional subspace arrangement is encoded in an intersection lattice whose elements are constructed by intersecting subsets of the subspaces. From the intersection lattice of an arrangement, we get a partially ordered set ordered by reverse inclusion of subspaces.
**Question 4.1.** Does the partially ordered set associated to the $k$th secant variety have any interesting combinatorial features? We conjecture that $\Sigma_k$ is Cohen-Macaulay, so will the corresponding poset be shellable?

It is also natural to ask if there is an analogue of Theorem 1.7 for secant varieties, perhaps requiring additional hypotheses on the intersection lattice of the secant varieties of $C_G$.

**Question 4.2.** Are the secant varieties of $C_G$ defined by products of linear forms?

Finding generators of $I_{\Sigma_k}$ that are products of linear forms is equivalent to finding an explicit and special basis for the ideal that may have combinatorial interest. Of course, a module does not typically have a unique generating set or a unique minimal free graded resolution. However, the number of minimal generators of degree $j$ of the $i$th syzygy module is invariant under a change of basis. Given a finitely generated graded module $M$, the graded Betti number $\beta_{i,j}$ is the number of minimal generators of degree $j$ required at the $i$th stage of a minimal free graded resolution of $M$. A standard way of displaying the graded Betti numbers of a module is with a graded Betti diagram organized as follows:

```
0 | 0 1 2
---|---
 0 | $\beta_{0,0}$ $\beta_{1,1}$ $\beta_{2,2}$ \cdots
 1 | $\beta_{0,1}$ $\beta_{1,2}$ $\beta_{2,3}$ \cdots
```

Bounds on the number of rows and columns of the graded Betti diagram of a module give a rough sense of how complicated it is. Specifically, recall that the regularity of a finitely generated graded module $M$ is equal to $\sup\{j-i \mid \beta_{i,j} \neq 0 \text{ for some } i\}$, and thus regularity gives a bound on the number of rows of the graded Betti diagram of $M$. Additionally, by the Auslander-Buchsbaum formula, a variety $X \subset \mathbb{P}^n$ is arithmetically Cohen-Macaulay if $\sup\{i \mid \beta_{i,j} \neq 0 \text{ for some } i\} = \operatorname{codim} X$, which bounds the number of columns of the graded Betti diagram of $M$. The following conjecture is the graph curve analogue of Conjecture 1.4 in [15] which refines conjectures from [17].

**Conjecture 4.3.** Under the hypotheses of Theorem 1.5, the $k$th secant variety of $C_G$ has regularity equal to $2k+1$ and is arithmetically Cohen-Macaulay.
Note that as the secant varieties of $C_G$ are not normal, we cannot expect projective normality.

In addition to bounding the length and width of the graded Betti diagram we conjecture that under certain conditions, one particular graded Betti number counts the number of cycles of minimal length in the graph.

**Conjecture 4.4.** Let $G$ be a graph on $d$ vertices, embedded as in Theorem 1.3. Let $n$ denote the girth of $G$. Assume that $d = 2g + 1 + p$ and $n - 2 \leq p$. Then the property $N_{2,p}$ fails and $\beta_{n-2,n}$ is equal to the number of cycles of length $n$ in $G$.

Given the explicit combinatorial nature of the embedding in Theorem 1.5, we found that it is not too hard to guess at representatives for classes of cycles in the Koszul cohomology of $I_{C_G}$ based on computations with Macaulay 2 [10]. However, working explicitly in Koszul cohomology seems to be quite intricate.

**Example 4.5** gives an illustration of the properties discussed in Conjectures 4.3, 4.4.

**Example 4.5** ($g = 2$, $d = 10$). Let $G$ be as given in Figure 4.5.

The ideal of $C_G$ corresponding to this labeling is given below.

\[
I_{C_G} = \langle x_5x_8, x_4x_8, x_3x_8, x_2x_8, x_1x_8, x_6x_7, x_5x_7, x_2x_7, x_1x_7, x_0x_7, x_4x_6, x_3x_6, x_2x_6, x_1x_6, x_3x_5, x_2x_5, x_1x_5, x_0x_5, x_2x_4, x_1x_4, x_0x_4, x_1x_3, x_0x_3, x_0x_2, x_3x_7 - x_4x_7, x_0x_8 - x_6x_8 \rangle
\]

The graded Betti diagram of $S/I_{C_G}$ shows that $N_{2,5}$ fails as $\beta_{5,7} = 2$. As Conjecture 4.4 predicts, the girth of $G$ is 7, and $G$ contains precisely
2 cycles of length 7.

\[
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\text{total:} & 1 & 26 & 98 & 168 & 154 & 72 & 15 & 2 \\
0: & 1 & . & . & . & . & . & . & . \\
1: & . & 26 & 98 & 168 & 154 & 70 & 8 & . \\
2: & . & . & . & . & 2 & 7 & 2 & . \\
\end{array}
\]

We can also compute the ideal of $\Sigma$

\[
I(\Sigma) = (x_3x_5x_8, x_2x_5x_8, x_1x_5x_8, x_0x_4x_8 - x_4x_6x_8, \\
x_2x_4x_8, x_1x_4x_8, x_1x_3x_8, x_0x_3x_8 - x_3x_6x_8, \\
x_2x_6x_7, x_1x_6x_7, x_2x_5x_7, x_0x_2x_8 - x_2x_6x_8, \\
x_1x_5x_7, x_0x_5x_7, x_0x_2x_7, x_3x_6x_7 - x_4x_6x_7, \\
x_2x_4x_6, x_1x_4x_6, x_1x_3x_6, x_1x_3x_7 - x_1x_4x_7, \\
x_1x_3x_5, x_0x_3x_5, x_0x_2x_5, x_0x_3x_7 - x_0x_4x_7, \\
x_0x_2x_4)
\]

and its graded Betti diagram

\[
\begin{array}{ccccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
\text{total:} & 1 & 25 & 58 & 43 & 12 & 3 \\
0: & 1 & . & . & . & . & . \\
1: & . & 25 & 58 & 41 & . & . \\
2: & . & . & . & 7 & . & . \\
3: & . & . & . & 2 & 5 & 3 \\
\end{array}
\]

We see that $N_{3,3}$ fails for $\Sigma$ and that $\beta_{3,7} = 2$, which is the number of cycles of length equal to the girth of $G$. We can also see from the graded Betti diagram that $\Sigma$ is arithmetically Cohen-Macaulay and that $I(\Sigma)$ has regularity 5.

It is natural to ask if combinatorics can be used to compute other values of the $\beta_{i,j}$. One result that gives the flavor of what might be possible is due to Gasharov, Peeva and Welker [8] who used the lcm lattice of a monomial ideal to compute graded Betti numbers of monomial ideals.

**Question 4.6.** Is there an analogue of the lcm lattice for graph curves and their secant varieties embedded via Theorem [3] that would allow us to compute (or estimate) the graded Betti numbers of graph curves?
It is also interesting to consider $C_G$ as a deformation of a smooth curve. In Example 4.5, $C_G$ has a 7-secant $\mathbb{P}^5$ while a smooth curve of genus 2 in $\mathbb{P}^8$ has no such $\mathbb{P}^5$. As any strictly subtrivalent graph curve $C_G \subset \mathbb{P}^n$ is smoothable in $\mathbb{P}^n$ \[12\, 29.9\], it is our expectation that we have a family of seven 6-secant $\mathbb{P}^5$s to smooth curves that collapse to the 7-secant $\mathbb{P}^5$ in the singular limit $C_G$. It also seems reasonable to believe that the secant varieties to embedded curves in a flat family themselves form a flat family, and so the secant varieties to $C_G$ should, in particular, have the same dimension and degree as those to smooth curves. In fact, since each pair of disjoint lines in $C_G$ spans a $\mathbb{P}^3$, we have a 3-dimensional secant plane for each edge in the complement of the graph $G$. If $C_G$ has degree $d$ and genus $g$, then $G$ has $d$ vertices and $d + g - 1$ edges. Thus, the number of edges in the complement of $G$ is $\binom{d}{2} - d - g + 1 = \binom{d-1}{2} - g$, which is the degree of the secant variety of a smooth curve of degree $d$ and genus $g$.

We believe that the type of collapsing described in the previous paragraph cannot happen unless the appropriate secant variety fills up all of $\mathbb{P}^n$. This leads us to propose the following:

**Conjecture 4.7.** If $C_G \subset \mathbb{P}^{g+r}$ is a graph curve of arithmetic genus $p_a$ and degree $d = 2g + r$, then as long as $2k + 1 < g + r$, $C_G$ has no $(k+2)$-secant $\mathbb{P}^k$.

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