T-Duality in 2-D Integrable Models

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ABSTRACT

The non-conformal analog of abelian T-duality transformations relating pairs of axial and vector integrable models from the non abelian affine Toda family is constructed and studied in detail.
1 Introduction

Abelian T-duality in $U(1)^{\otimes s}$ invariant 2-D conformal field theories (CFT’s) and in string theory represents a set of specific canonical transformations that relate pairs of equivalent models sharing the same spectrum, but with different $\sigma$-model-like Lagrangians [1], [2]. The axial and vector gauged $G/H$-WZW models provide a vast variety of examples of such pairs of T-dual models [3], [4]. From the other side, the integrable perturbations of these $G/H$-WZW models have been identified with the family of the so called Non-Abelian Affine Toda theories [5], [6], [7]. An important feature of these integrable models (IM’s) is their $U(1)^{\otimes k}$, $k \leq s$ global symmetry and the fact that they admit both topological and/or non-topological soliton solutions carrying $U(1)^{\otimes k}$ charges as well [7], [8]. Hence, an interesting problem to be addressed is about the T-duality of pairs of axial and vector IM’s within this family. More precisely: whether the perturbation breaks a part (or all) of the isometries (i.e., $U(1)^{\otimes s_{CFT}}$ to $U(1)^{\otimes s_{Im}}$, $s_{Im} \leq s_{CFT}$ ) and whether certain non conformal analog of the abelian T-duality transformations takes place. The simplest example of a pair of T-dual IM’s with only one isometry (i.e., $s_{Im} = 1$) have been studied in detail in our recent paper [7],[9]. As one expects the mass spectrum of the solitons is indeed invariant under the corresponding non-critical T-duality, but the $U(1)$-charges of the solitons of the axial model are mapped into the topological charges of the solitons of the vector IM and vice-versa. An interesting example of T-self-dual IM’s is given by the complex sine-Gordon [6] and the the Fateev’s IM’s [10].

The present paper is devoted to the investigation of T-duality properties of family of IM’s representing relativistic IM belonging to the same hierarchy as the Fordy-Kulish (multi-component) nonlinear Schroedinger model (NLS)[11], [12]. They can be considered as a specific Hamiltonian reduction of the $A_{n}^{(1)}$- Homogeneous sine-Gordon models [13]. Their main property is the large global symmetry group $SL(N) \otimes U(1)$, i.e., they admit $N$-isometries, as in the $SL(N + 1)/SL(N) \otimes U(1)$-WZW model. As a consequence the T-duality transformations relating the corresponding axial and vector IM’s of this family are indeed more involved.

The paper is organized as follows. Sect. 2 contains a brief summary of the general formalism for the construction of the effective action of a large class of NA-affine Toda theories. In Sect. 3 we apply these methods for the derivation of the Lagrangians of axial and vector IM’s of reduced homogeneous SG-type. Sect. 4 is devoted to the symmetries of such models while in Sect. 5 we explicitly construct the corresponding T-duality transformations.

2 NA affine Toda models as gauged two-loop WZW models

The basic ingredient in constructing massive Toda models is the decomposition of an affine Lie algebra $\mathcal{G}$ in terms of graded subspaces defined according to a grading operator $Q$,

$$\left[Q, \mathcal{G}_l\right] = l\mathcal{G}_l, \quad \mathcal{G} = \bigoplus \mathcal{G}_l, \quad \left[\mathcal{G}_l, \mathcal{G}_k\right] \subset \mathcal{G}_{l+k}, \quad l, k = 0, \pm 1, \cdots$$

(2.1)
In particular, the zero grade subspace $G_0$ plays an important role since it is parametrized by the Toda fields. The grading operator $Q$ induces the notion of negative ($G_<$) and positive ($G_>$) grade subalgebras and henceforth the decomposition of a group element in the Gauss form,

$$g = NBM$$  \hspace{1cm} (2.2)

where $N = \exp(G_<)$, $B = \exp(G_0)$ and $M = \exp(G_>)$.

The action of the corresponding affine Toda models can be derived from the gauged two-loop $^1$ Wess-Zumino-Witten (WZW) action [15], [7],

$$S_{G/H}(g, A, \bar{A}) = S_{WZW}(g) - \frac{k}{2\pi} \int d^2x Tr \left( A(\partial g g^{-1} - \epsilon_+) + \bar{A}(g^{-1} \partial g - \epsilon_-) + Ag\bar{A}g^{-1} \right)$$  \hspace{1cm} (2.3)

where $A = A_\in G_<, \bar{A} = \bar{A}_+ \in G_>$ and $\epsilon_\pm$ are constant elements of grade $\pm 1$. The action (2.3) is invariant under

$$g' = \alpha_- g \alpha_+, \quad A' = \alpha_- A \alpha_-^1 + \alpha_- \partial \alpha_-^1, \quad \bar{A}' = \alpha_+^1 \bar{A} \alpha_+ + \bar{\alpha}_+ \alpha_+^1$$  \hspace{1cm} (2.4)

where $\alpha_- \in G_<, \alpha_+ \in G_>$. It therefore follows that $S_{G/H}(g, A, \bar{A}) = S_{G/H}(B, A', \bar{A}')$.

Integrating over the auxiliary fields $A$ and $\bar{A}$ in the partition function

$$Z = \int DAD\bar{A}DBe^{-S}$$  \hspace{1cm} (2.5)

we find the effective action for an integrable model defined on the group $G_0$,

$$S_{\text{eff}}(B) = S_{WZW}(B) - \frac{k}{2\pi} \int Tr \left( \epsilon_+ B \epsilon_- B^{-1} \right) d^2x$$  \hspace{1cm} (2.6)

The corresponding equations of motion have the following compact form [16]

$$\partial(B^{-1} \partial B) + [\epsilon_-, B^{-1} \epsilon_+ B] = 0, \quad \partial(\partial B B^{-1}) - [\epsilon_+, B \epsilon_- B^{-1}] = 0$$  \hspace{1cm} (2.7)

It is straightforward to derive from the eqns. (2.7) the chiral conserved currents associated to the subalgebra $G_0^0 \subset G_0$ defined as $G_0^0 = \{ X \in G_0, \text{ such that } [X, \epsilon_\pm] = 0 \}$, i.e.,

$$J_X = Tr \left( X B^{-1} \partial B \right), \quad \bar{J}_X = Tr \left( X \partial B B^{-1} \right), \quad \partial J_X = \partial \bar{J}_X = 0$$  \hspace{1cm} (2.8)

The conservation of such currents is consequence of the invariance of the action (2.6) under the $G_0^0 \otimes G_0^0$ chiral transformation,

$$B' = \Omega(z) B \Omega(z)$$  \hspace{1cm} (2.9)

where $\Omega(z), \Omega(z) \in G_0^0$.

$^1$the $\hat{G}$-WZW model in the case where $\hat{G}$ is an affine Kac-Moody algebra is called two-loop WZW model [14]
The fact that the currents $J_X$ and $\tilde{J}_X$ in (2.8) are chiral, allows further reduction of the IM (2.6) by imposing a set of subsidiary constraints,

$$
J_X = Tr \left( X B^{-1} \partial B \right) = 0, \quad \tilde{J}_X = Tr \left( X \tilde{\partial} B B^{-1} \right) = 0, \quad X \in G_0^0
$$

which reduces the model defined on the group $G_0$ to the one on coset $G_0/G_0^0$. Such constraints are incorporated into the action by repeating the gauged WZW action argument for the subgroup $G_0$. For a general non abelian $G_0^0$, we define a second grading structure $Q'$ which decomposes $G_0^0$ into positive, zero and negative graded subspaces, i.e., $G_0^0 = G_0^0> \oplus G_0^0= \oplus G_0^0<$. Following the same principle as in [15], [7] and [8] we seek for an action invariant under

$$
B'' = \gamma_0(\bar{z}, z) \gamma_{-}(\bar{z}, z) B \gamma_{+}(\bar{z}, z) \gamma_{0}'(\bar{z}, z), \quad \gamma_0, \gamma_0' \in G_0^0_0, \quad \gamma_{-} \in G_0^0<, \quad \gamma_{+} \in G_0^0>
$$

and choose $\gamma_0(\bar{z}, z), \gamma_0'(\bar{z}, z), \gamma_{-}(\bar{z}, z), \gamma_{+}(\bar{z}, z) \in G_0^0$ such that $B'' = \gamma_0 \gamma_{-} B \gamma_{+} \gamma_0' = g_0^f \in G_0/G_0^0$ since $B$ can also be decomposed into the Gauss form according to the second grading structure $Q'$. Denote $\Gamma_{-} = \gamma_0 \gamma_{-}$ and $\Gamma_{+} = \gamma_{+} \gamma_0'$. Then the action

$$
S(B, A^{(0)}, \tilde{A}^{(0)}) = S_{WZW}(B) - \frac{k}{2\pi} \int Tr \left( \epsilon_+ B \epsilon_- B^{-1} \right) d^2x
$$

$$
- \frac{k}{2\pi} \int Tr \left( \eta A^{(0)} \tilde{\partial} B B^{-1} + \bar{A}^{(0)} B^{-1} \partial B + \eta A^{(0)} B \bar{A}^{(0)} B^{-1} + A^{(0)} \tilde{A}^{(0)} \right) d^2x
$$

(2.12)

(with $\eta = +1, -1$ correspond to $\gamma_0' = \gamma_0$ for axial or $\gamma_0' = \gamma_0^{-1}$ for vector gaugings\(^2\) respectively, $A^{(0)} = A_0^{(0)} + A_{-}^{(0)}$ and $\bar{A}^{(0)} = \bar{A}_0^{(0)} + \bar{A}_{+}^{(0)}$, is invariant under $\Gamma_{\pm}$ transformations

$$
B' = \Gamma_{-} B \Gamma_{+}, \quad A_0^{(0)} = A_0^{(0)} - \eta \gamma_{-}^{-1} \partial \gamma_0, \quad \bar{A}_0^{(0)} = \bar{A}_0^{(0)} - \gamma_{-}^{-1} \bar{\partial} \gamma_0,
$$

$$
A^{(0)} = \Gamma_{-} A^{(0)} \Gamma_{+} - \eta \partial \Gamma_{-} \Gamma_{+}, \quad \bar{A}'^{(0)} = \Gamma_{-} \bar{A}^{(0)} \Gamma_{+} - \Gamma_{-} \bar{\partial} \Gamma_{+},
$$

(2.13)

where $A_0^{(0)}, \bar{A}_0^{(0)} \in G_0^0, A_{-}^{(0)} \in G_0^0<, \bar{A}_{+}^{(0)} \in G_0^0>$. Hence we have

$$
S(B, A^{(0)}, \bar{A}^{(0)}) = S(g_0^f, A^{(0)}, \bar{A}^{(0)}),
$$

(2.14)

The general construction above provides a systematic classification of relativistic integrable models in terms of its algebraic structure, i.e. $\{G, Q, \epsilon_{\pm}, G_0^0\}$. For example, within the affine $G = SL(N + 1)$ algebra we have the following families of integrable models:

1. $G_0^0 = \emptyset$ characterizes the choices of

$$
Q = (N + 1) d + \sum_{l=1}^{N} \lambda_{l} \cdot H, \quad G_0 = U(1)^{N} = \{h_1, \ldots, h_N\},
$$

$$
\epsilon_{\pm} = \mu (\sum_{l=1}^{N} E_{\pm}^{(0)} + E_{\pm}^{(\pm 1, \ldots, \pm N)})
$$

which gives rise to the well known abelian affine Toda model (see for instance [17], [16]).

\(^2\)Notice that for non abelian $G_0^0$ the invariance of the vector action in (2.12) is consequence of the Borel structure of the subgroup elements $\Gamma_{\pm}$, i.e., we consider the left-right coset $\Gamma_{-} \setminus G/\Gamma_{+}$. 

2. (a) $G_0^0 = U(1) = \{\lambda_1 \cdot H\}$

\[
Q = N d + \sum_{l=2}^N \lambda_l \cdot H, \quad G_0 = SL(2) \otimes U(1)^{N-1} = \{E_{\pm \alpha_1}, h_1, \cdots, h_N\},
\]

\[
\epsilon_\pm = \mu \left( \sum_{l=2}^N E_{\pm \alpha_l}^{(0)} + E_{\pm \alpha_1 + \cdots + \alpha_N}^{(\pm 1)} \right)
\]

corresponds to the simplest non abelian affine Toda model of dyonic type, admitting electrically charged topological solitons (see for instance [15], [7]).

(b) $G_0^0 = U(1) \otimes U(1) = \{\lambda_1 \cdot H, \lambda_N \cdot H\}$

\[
Q = (n - 1) d + \sum_{l=2}^{N-1} \lambda_l \cdot H, \quad \epsilon_\pm = \mu \left( \sum_{l=2}^{N-1} E_{\pm \alpha_l}^{(0)} + E_{\pm \alpha_2 + \cdots + \alpha_{N-1}}^{(\pm 1)} \right)
\]

\[
G_0 = SL(2) \otimes SL(2) \otimes U(1)^{N-2} = \{E_{\pm \alpha_1}, E_{\pm \alpha_N}, h_1, \cdots, h_N\},
\]

is of the same class of $U(1)^k$ dyonic type IM's, but now yielding multicharged solitons ([8]).

3. $G_0^0 = SL(2) \otimes U(1) = \{E_{\pm \alpha_1}, \lambda_1 \cdot H, \lambda_2 \cdot H\}$

\[
Q = (N - 1) d + \sum_{l=3}^N \lambda_l \cdot H, \quad \epsilon_\pm = \mu \left( \sum_{l=3}^N E_{\pm \alpha_l}^{(0)} + E_{\pm \alpha_3 + \cdots + \alpha_{N-1}}^{(\pm 1)} \right)
\]

\[
G_0 = SL(3) \otimes U(1)^{N-2} = \{E_{\pm \alpha_1}, E_{\pm \alpha_2}, E_{\pm \alpha_1 + \alpha_2}, h_1, \cdots, h_N\},
\]

and $Q' = \lambda_1 \cdot H$, such that $G_0^{0,=} = \{E_{-\alpha_1}\}$, $G_0^{0,>} = \{E_{\alpha_1}\}$, $G_0^{0,0} = \{\lambda_1 \cdot H, \lambda_2 \cdot H\}$ leads to dyonic models with non abelian global symmetries (see Sect. 6 of [8]).

The classical integrability of all these models follows from their zero curvature (Lax) representation:

\[
\partial \bar{\mathcal{A}} - \bar{\partial} \mathcal{A} - [\mathcal{A}, \bar{\mathcal{A}}] = 0, \quad \mathcal{A}, \bar{\mathcal{A}} \in \oplus_{i=0, \pm 1} G_i
\]  

\[
(2.15)
\]

with

\[
\mathcal{A} = -B \epsilon_- B^{-1}, \quad \bar{\mathcal{A}} = \epsilon_+ + \bar{\partial} B B^{-1}
\]  

\[
(2.16)
\]

where the constraints (2.10) are imposed. It can be easily verified that substituting (2.16) into (2.15) taking into account (2.10), one reproduces the equations of motion (2.7). Then the existence of an infinite set (of commuting) conserved charges $P_m$, $m = 0, 1, \cdots$ is a simple consequence of eqn. (2.15), namely,

\[
P_m(t) = Tr \left( T(t) \right)^m, \quad \partial_t P_m = 0, \quad T(t) = \lim_{L \to \infty} \mathcal{P} \text{exp} \int_{-L}^{L} \mathcal{A}_x(t, x) dx
\]

Hence the above described procedure for derivation of the abelian and NA affine Toda models as gauged $G/H$ two loop WZW models leads to integrable models by construction.
3 Homogeneous Gradation and the Lund-Regge Type Models

An interesting class of integrable models, that generalizes the Lund-Regge model [18], can be constructed from the affine Kac-Moody algebra \( \mathcal{G} = SL(N + 1) \) endowed with homogeneous gradation \( Q = d \) and the specific choice of \( \epsilon_\pm = \mu \lambda_N \cdot H(\pm 1) \), where \( \lambda_N \) is the \( N^{th} \) fundamental weight of \( SL(N + 1) \). The zero grade subalgebra \( \mathcal{G}_0 \) corresponds to the finite dimensional Lie algebra \( G_0 = SL(N + 1) \) and \( G_0^0 = SL(N) \otimes U(1) \). Let us parametrize the auxiliary gauge fields as follows

\[
\begin{align*}
A_0^{(0)} &= \sum_{i=1}^{N} a_i(\lambda_i - \lambda_{i-1}) \cdot H^{(0)}, \\
\bar{A}_0^{(0)} &= \sum_{i=1}^{N} \bar{a}_i(\lambda_i - \lambda_{i-1}) \cdot H^{(0)}, \quad \lambda_0 = 0
\end{align*}
\]

where \( a_{ij}(x,t), a_i(x,t), \bar{a}_{ij}(x,t), \bar{a}_i(x,t) \) are arbitrary functions of space-time variables. We next consider two different gauge fixings of \( G_0^0 \), the vector and the axial, in order to derive the effective Lagrangians for the pair of T-dual IM’s.

3.1 Axial Gauging

According to the axial gauging (2.11), \( \eta = 1, \gamma_0^f = \gamma_0 \), the factor group element \( g^f_0 \in G_0/G_0^0 \) is parametrized as follows

\[
g^f_0 = g^f_{0,v} = nm, \quad n = e^{\sum_{i=1}^{N} \xi_i E_{-(\alpha_i + \cdots + \alpha_N)}}, \quad m = e^{\sum_{i=1}^{N} \psi_i E_{\alpha_i + \cdots + \alpha_N}}
\]

After a tedious but straightforward calculation we find

\[
\begin{align*}
&Tr \left( A_0^{(0)} \bar{A}_0^{(0)} + A^{(0)} g^f_0 \bar{A}^{(0)} g_0^{f-1} + A^{(0)} \bar{g}_0^{f-1} g_0^{f} + A^{(0)} g_0^{f-1} \bar{g}_0^{f-1} \right) \\
&= \bar{a}_i M_{ij} a_j + \bar{a}_i N_i + \bar{N}_i a_i + \sum_{j=1}^{N-1} \sum_{i=j}^{N-1} \bar{a}_{j,i+1} \alpha_{k+1,j} (\delta_{i,k} + \psi_{i+1} \chi_{k+1}) \\
&- \sum_{j=1}^{N-1} \sum_{i=j}^{N-1} \bar{a}_{j,i+1} \psi_{i+1} \partial \chi_j - \sum_{j=1}^{N-1} \sum_{i=j}^{N-1} a_{i+1,j} \chi_{i+1} \partial \psi_j
\end{align*}
\]

where we have introduced \( M_{ij} \) and \( N_j, \bar{N}_j \) as

\[
\begin{align*}
M_{ij} &= 2(\lambda_i - \lambda_{i-1}) \cdot (\lambda_j - \lambda_{j-1}) + \psi_i \chi_i \delta_{i,j}, \quad i, j = 1, \cdots N, \quad \lambda_0 = 0 \\
N_j &= \left( \sum_{i=j}^{N-1} a_{i+1,j} \chi_{i+1} - \partial \chi_j \right) \psi_j, \quad \bar{N}_j = \left( \sum_{i=j}^{N-1} \bar{a}_{j,i+1} \psi_{i+1} - \partial \psi_j \right) \chi_j
\end{align*}
\]

In order to derive the effective Lagrangian of the axial model we have to integrate the auxiliary fields \( a_1, \bar{a}_i, a_{j,i+1} \) and \( \bar{a}_{i+1,j} \). We shall consider the particular case \( N = 2 \), i.e. \( G = \)
\( \hat{SL}(3) \), where the Gaussian matrix integration is quite simple. Then, in the parametrization (3.18)

\[
B = e^{\hat{\chi}_1 E_{-a_1} + \tilde{\chi}_2 E_{-a_2} + \tilde{\chi}_3 E_{-a_3}} e^{\phi_1 h_1 + \phi_2 h_2} e^{\psi_1 E_{a_1} + \psi_2 E_{a_2}} e^{\tilde{\psi}_1 E_{a_1}} = e^{\hat{\chi}_1 E_{-a_1} \frac{1}{2} \left( \lambda_1 H R_1 + \lambda_2 H R_2 \right)} \left( g_{0,ax}^f \right) e^{\frac{1}{2} \left( \lambda_1 H R_1 + \lambda_2 H R_2 \right)} e^{\tilde{\psi}_1 E_{a_1}}
\]

\[
g_{0,ax}^f = e^{\chi_1 E_{-a_1} + \chi_2 E_{-a_2} + \chi_3 E_{-a_3}} e^{\psi_1 E_{a_1} + \psi_2 E_{a_2}} e^{\phi_1 h_1 + \phi_2 h_2} e^{\psi_1 E_{a_1} + \psi_2 E_{a_2}} \quad \phi_1 h_1 + \phi_2 h_2 = \lambda_1 \cdot H R_1 + \lambda_2 \cdot H R_2 \quad (3.21)
\]

we have \( M_{ij}, N_j \tilde{N}_j, \ i, j = 1, 2 \) in the form

\[
M = \begin{pmatrix}
\frac{4}{3} + \psi_1 \chi_1 & -\frac{2}{3} \\
-\frac{2}{3} & \frac{4}{3} + \psi_2 \chi_2
\end{pmatrix}
\]

and

\[
\tilde{N} = \begin{pmatrix}
-(\partial \psi_1 - \tilde{a}_1 \psi_2) \chi_1 \\
-(\chi_2 \partial \psi_2)
\end{pmatrix}, \quad N = \begin{pmatrix}
-(\partial \chi_1 - a_2 \chi_2 \psi_1) \\
-\psi_1 \partial \chi_1
\end{pmatrix}
\]

(3.23)

Integrating first over the \( a_i \) and \( \tilde{a}_i \) and next on the \( a_{12}, \tilde{a}_{21} \) we derive the effective action of the \( SL(3) \) axial model

\[
S_{ax} = -\frac{k}{2\pi} \int dz \tilde{z} \left( \frac{1}{\Delta} (\partial \psi_2 \partial \chi_2 (1 + \psi_1 \chi_1 + \psi_2 \chi_2) + \partial \psi_1 \partial \chi_1 (1 + \psi_2 \chi_2) - \frac{1}{2} (\psi_2 \chi_1 \partial \psi_2 \partial \chi_2 + \chi_2 \psi_1 \partial \psi_2 \partial \chi_1) - V) \right)
\]

(3.24)

where \( V = \mu^2 \left( \frac{\psi_1 \chi_1 + \psi_2 \chi_2}{2} + \chi_1 \partial \chi_1 \right) \) and \( \Delta = (1 + \psi_2 \chi_2)^2 + \psi_1 \chi_1 (1 + \frac{3}{4} \psi_2 \chi_2) \).

### 3.2 Vector Gauging

For the explicit \( SL(3) \) case, the zero grade group element \( B \) is written according to the vector gauging (\( \eta = -1, \gamma_0^* = \gamma_0^{-1} \)) as

\[
B = e^{\hat{\chi}_1 E_{-a_1} + \tilde{\chi}_2 E_{-a_2} + \tilde{\chi}_3 E_{-a_3}} e^{\phi_1 h_1 + \phi_2 h_2} e^{\psi_1 E_{a_1} + \psi_2 E_{a_2}} e^{\tilde{\psi}_1 E_{a_1}} = e^{\hat{\chi}_1 E_{-a_1} \frac{1}{2} \left( \lambda_1 H u_1 + \lambda_2 H u_2 \lambda_3 \right)} \left( g_{0,vec}^f \right) e^{\frac{1}{2} \left( \lambda_1 H u_1 + \lambda_2 H u_2 \lambda_3 \right)} e^{\tilde{\psi}_1 E_{a_1}}
\]

(3.25)

where \( g_{0,vec}^f = e^{-t_2 E_{-a_2} - t_1 E_{-a_3}} e^{\phi_1 h_1 + \phi_2 h_2} e^{t_2 E_{a_2} + t_1 E_{a_3}} \). We next choose \( u_1, u_2 \) such that

\[
\tilde{\chi}_2 e^{-\frac{1}{2} u_2} = -t_2, \quad \tilde{\psi}_2 e^{\frac{1}{2} u_2} = t_2, \quad \tilde{\chi}_3 e^{-\frac{1}{2} (u_1 + u_2)} = -t_1, \quad \tilde{\psi}_3 e^{\frac{1}{2} (u_1 + u_2)} = t_1
\]

Taking into account the parametrization (3.17) for \( SL(3) \) we find

\[
Tr \left( A^{(0)}_0 \tilde{A}^{(0)}_0 \right) - A^{(0)}_0 g_{0,vec}^f \tilde{A}^{(0)}_0 g_{0,vec}^f + \tilde{A}^{(0)}_0 g_{0,vec}^{-1} g_{0,vec} f g_{0,vec}^{-1} \quad - A^{(0)}_0 \tilde{g}_{0,vec}^f g_{0,vec}^{-1} \quad A^{(0)}_0 \tilde{g}_{0,vec}^{-1} g_{0,vec}^{-1} = a_1 \tilde{a}_1 \Delta + a_1 (a_0 t_2) \partial \phi_1 \phi_1 + \phi_2 + a_1 (a_0 t_2 - t_2 \tilde{a}_1) e^{\phi_1 + \phi_2}
\]

(3.26)
where $\tilde{\Delta} = t_2^2 e^{\phi_1 + \phi_2} - e^{2\phi_1 - \phi_2}$. We first take the integral over $a_1$ and $\bar{a}_1$ in the partition function (2.5) with the action given by (2.12). As a result we get

$$\mathcal{L}_{int} = \bar{a}_{0i} M_{ij} a_{0j} + \bar{a}_{0i} N_i + \bar{N}_i a_{0i} + \frac{t_1^2 \partial t_1 \bar{\partial} t_1}{\Delta} e^{2(\phi_1 + \phi_2)}$$

(3.27)

where

$$M_{11} = -\frac{t_1^2}{\Delta} e^{3\phi_1}, \quad M_{22} = t_2^2 e^{2\phi_2 - \phi_1}, \quad M_{12} = M_{21} = 0$$

(3.28)

and

$$N_1 = \partial \phi_1 + t_1 \partial t_1 e^{\phi_1 + \phi_2} - \frac{t_1 t_2^2 \partial t_1}{\Delta} e^{2(\phi_1 + \phi_2)}, \quad N_2 = \partial \phi_2 - \partial \phi_1 + t_2 \partial t_2 e^{-\phi_1 + 2\phi_2},$$

$$\bar{N}_1 = -\tilde{\partial} \phi_1 - t_1 \bar{\partial} t_1 e^{\phi_1 + \phi_2} + \frac{t_1 t_2^2 \bar{\partial} t_1}{\Delta} e^{2(\phi_1 + \phi_2)}, \quad \bar{N}_2 = -\tilde{\partial} \phi_2 + \tilde{\partial} \phi_1 - t_2 \bar{\partial} t_2 e^{-\phi_1 + 2\phi_2}$$

(3.29)

We next integrate the fields $\bar{a}_{0i}$ and $a_{0i}$, $i = 1, 2$ in eqn. (3.27). Together with the standard form of WZW action $S_{WZW}(g_{0,vec}^f)$ we arrive at the following effective Lagrangian for the vector IM,

$$\mathcal{L}_{vec} = \frac{1}{2} \sum_{i=1}^{2} \eta_{ij} \partial \phi_i \bar{\partial} \phi_j + \frac{\partial \phi_1 \bar{\partial} \phi_1}{t_1} e^{-\phi_1 - \phi_2} + \bar{\partial} \phi_1 \partial t_1 + \partial \phi_1 \bar{\partial} ln(t_1) + \partial \phi_1 \partial t_1 - \partial \phi_1 \bar{\partial} \phi_1 \left(\frac{t_2}{t_1}\right)^2 e^{-2\phi_1 + \phi_2}$$

$$+ \frac{\bar{\partial}(\phi_2 - \phi_1) \partial(\phi_2 - \phi_1)}{t_2} e^{\phi_1 - 2\phi_2} + \tilde{\partial}(\phi_2 - \phi_1) \partial t_2 + \partial(\phi_2 - \phi_1) \bar{\partial} ln(t_2) - V$$

(3.30)

where $V = \mu^2(\frac{2}{3} - t_2^2 e^{-\phi_1 + 2\phi_2} - t_1^2 e^{\phi_1 + \phi_2})$ and $\eta_{ij} = 2\delta_{ij} - \delta_{i,j-1} - \delta_{i,j+1}$. The integrability of the axial (3.24) and vector (3.30) models is a consequence of the Lax representation (2.15) and (2.16) valid for both models.

4 Local and Global Symmetries

Before imposing the subsidiary constraints (2.10) the model on the group $G_0$ described by (2.6) is invariant under chiral transformation (2.9) generated by $G_0^0 \otimes G_0^0$. For the explicit $SL(3)$ case, the associated Noether currents are given in terms of the axial variables defined in (3.21) as:

$$J_{a_1} = \partial \tilde{\psi}_1 - \tilde{\psi}_1^2 \partial \tilde{\chi}_1 e^{R_1} + \partial \tilde{\chi}_2 (\tilde{\psi}_1 \tilde{\psi}_2 - \tilde{\psi}_3) e^{R_2}$$

$$+ (\partial \tilde{\chi}_3 - \tilde{\chi}_2 \partial \tilde{\chi}_1) (\tilde{\psi}_1 \tilde{\psi}_2 - \tilde{\psi}_3) \tilde{\psi}_1 e^{R_1 + R_2} + \tilde{\psi}_1 \partial R_1,$$

$$J_{a_1} = \partial \tilde{\chi}_1 e^{R_1} - \tilde{\psi}_2 (\partial \tilde{\chi}_3 - \tilde{\chi}_2 \partial \tilde{\chi}_1) e^{R_1 + R_2},$$

$$J_{\lambda_1, H} = \frac{1}{3} (2 \partial R_1 + \partial R_2) - \tilde{\psi}_1 \partial \tilde{\chi}_1 e^{R_1} + (\tilde{\psi}_1 \tilde{\psi}_2 - \tilde{\psi}_3) (\partial \tilde{\chi}_3 - \tilde{\chi}_2 \partial \tilde{\chi}_1) e^{R_1 + R_2},$$

$$J_{\lambda_2, H} = \frac{1}{3} (\partial R_1 + 2 \partial R_2) - \tilde{\psi}_2 \partial \tilde{\chi}_2 e^{R_2} - \tilde{\psi}_3 (\partial \tilde{\chi}_3 - \tilde{\chi}_2 \partial \tilde{\chi}_1) e^{R_1 + R_2}$$

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\[ \tilde{J}_{\alpha_1} = \partial \tilde{\chi}_1 - \tilde{\chi}^2 \partial \tilde{\psi}_1 e^{R_1} + \partial \tilde{\psi}_2 (\tilde{\chi}_1 \tilde{\chi}_2 - \tilde{\chi}_3) e^{R_2}, \]
\[ + (\partial \tilde{\psi}_3 - \tilde{\psi}_2 \partial \tilde{\psi}_1) (\tilde{\chi}_1 \tilde{\chi}_2 - \tilde{\chi}_3) \tilde{\chi}_1 e^{R_1+R_2} + \tilde{\chi}_1 \partial R_1, \]
\[ \tilde{J}_{\lambda_2-H} = \frac{1}{3} (2\partial R_1 + \partial R_2) - \tilde{\chi}_1 \tilde{\psi}_1 e^{R_1} + (\tilde{\chi}_1 \tilde{\chi}_2 - \tilde{\chi}_3) (\partial \tilde{\psi}_3 - \tilde{\psi}_2 \partial \tilde{\psi}_1) e^{R_1+R_2}, \]
\[ \tilde{J}_{\lambda_2-H} = \frac{1}{3} (\partial R_1 + 2\partial R_2) - \tilde{\chi}_2 \tilde{\psi}_2 e^{R_2} - \tilde{\chi}_3 (\partial \tilde{\psi}_3 - \tilde{\psi}_2 \partial \tilde{\psi}_1) e^{R_1+R_2} \]

(4.31)

where \( \tilde{\partial} J = \partial \tilde{J} = 0 \) and \( J = J_{\lambda_1-H} h_1 + J_{\lambda_2-H} h_2 + \sum_{\alpha} J_{\alpha} E_{-\alpha} + J_{-\alpha} E_{\alpha} \), \( \alpha = \alpha_1, \alpha_2, \alpha_1 + \alpha_2 \).

Apart from those Noether currents (4.31) notice the existence of topological currents
\[ j_{\varphi, \mu} = \epsilon_{\mu \nu} \partial \varphi, \quad \varphi = \{ R_i, i = 1, 2, \tilde{\chi}_j, \tilde{\psi}_j, j = 1, 2, 3 \}. \]

(4.32)

The reduction from the group \( G_0 \) to the coset \( G_0/G_0^0 \) implies the vanishing of currents (4.31), which defines the unphysical nonlocal fields \( R_i \) in terms of \( \psi_i, \chi_i \):
\[ \partial R_1 = \frac{\psi_1 \partial \chi_1}{\Delta} (1 + \frac{3}{2} \psi_2 \chi_2) - \frac{\psi_2 \partial \chi_2}{\Delta} \frac{3}{2} \psi_1 \chi_1, \]
\[ \partial R_2 = \frac{\psi_1 \partial \chi_1}{\Delta} + \frac{\psi_2 \partial \chi_2}{\Delta} (2\Delta + \frac{3}{2} \psi_1 \chi_1), \]
\[ \partial \tilde{R}_1 = \frac{\chi_1 \partial \tilde{\psi}_1}{\Delta} (1 + \frac{3}{2} \psi_2 \chi_2) - \frac{\chi_2 \partial \tilde{\psi}_2}{\Delta} (2\Delta + \frac{3}{2} \psi_1 \chi_1), \]
\[ \partial \tilde{R}_2 = \frac{\chi_1 \partial \tilde{\psi}_1}{\Delta} + \frac{\chi_2 \partial \tilde{\psi}_2}{\Delta} (2\Delta + \frac{3}{2} \psi_1 \chi_1) \]

(4.33)

where \( \Delta = (1 + \psi_2 \chi_2)^2 + \psi_1 \chi_1 (1 + \frac{3}{4} \psi_2 \chi_2), \quad \Delta_2 = 1 + \psi_2 \chi_2 \) and
\[ \tilde{\chi}_1 = \chi_3 e^{-\frac{1}{2} R_1}, \quad \tilde{\psi}_1 = \psi_3 e^{-\frac{1}{2} R_1}, \]
\[ \tilde{\chi}_2 = \chi_2 e^{-\frac{1}{2} R_1}, \quad \tilde{\psi}_2 = \psi_2 e^{-\frac{1}{2} R_1}, \]
\[ \tilde{\chi}_3 = \chi_1 e^{-\frac{1}{2} (R_1 + R_2)}, \quad \tilde{\psi}_3 = \psi_1 e^{-\frac{1}{2} (R_1 + R_2)} \]

(4.34)

In addition we find
\[ \partial \tilde{\chi}_1 = \frac{\psi_2}{\Delta} \left( \partial \chi_1 \Delta_2 - \frac{1}{2} \chi_1 \psi_2 \partial \chi_2 \right) e^{-\frac{1}{2} R_1}, \]
\[ \partial \tilde{\psi}_1 = \frac{\psi_1}{\Delta} \left( \partial \chi_2 (1 + \psi_1 \chi_1 + \psi_2 \chi_2) - \frac{1}{2} \chi_2 \psi_1 \partial \chi_1 \right) e^{-\frac{1}{2} R_1}, \]
\[ \partial \tilde{\psi}_2 = \frac{\chi_2}{\Delta} \left( \partial \psi_1 \Delta_2 - \frac{1}{2} \chi_2 \psi_1 \partial \psi_2 \right) e^{-\frac{1}{2} R_1}, \]
\[ \partial \tilde{\chi}_1 = \frac{\chi_1}{\Delta} \left( \partial \psi_2 (1 + \psi_1 \chi_1 + \psi_2 \chi_2) - \frac{1}{2} \chi_1 \psi_2 \partial \psi_1 \right) e^{-\frac{1}{2} R_1} \]

(4.35)

Using the equations of motion derived from (3.24), we prove the following conservation laws\(^3\)
\[ \tilde{\partial} j = \partial \tilde{J}, \quad j = j_{\psi}, \quad j_{\tilde{\chi}_i}, \quad j = j_{R_i}, i = 1, 2, \]

(4.36)

\(^3\)Notice that (4.35) denotes non local fields \( R_1, R_2, \tilde{\psi}, \tilde{\chi} \) in terms of the physical fields \( \psi, \chi_1 \) and \( \chi_2 \) and hence conservation of (4.37) is non trivial.
where \( j = \frac{1}{2}(j_0 + j_1), \quad \bar{j} = \frac{1}{2}(j_0 - j_1), \) and

\[
j_{R_i,\mu} = \epsilon_{\mu\nu} \partial_\nu R_i, \quad i = 1, 2, \quad j_{\psi_i,\mu} = \epsilon_{\mu\nu} \partial_\nu \psi_i, \quad j_{\bar{\psi}_i,\mu} = \epsilon_{\mu\nu} \partial_\nu \bar{\psi}_i, \quad (4.37)
\]

Under the reduction (2.10), the topological currents (4.32) in the group \( G_0 \) become Noether currents (4.37) in the coset \( G_0/G_0^0 \) and their conservation is consequence of the invariance of action (3.24) under the following nonlocal global transformations

\[
\delta \psi_1 = \frac{1}{2}(-\epsilon_1 - \epsilon_2 + \bar{\epsilon}_1 + \bar{\epsilon}_2) \psi_1 - \frac{1}{2}\epsilon_- \psi_1 \bar{\psi}_1 + \epsilon_+ \left( \psi_2 e^{-\frac{1}{2}R_1} + \frac{1}{2} \psi_1 \bar{\psi}_1 \right),
\]
\[
\delta \chi_1 = \frac{1}{2}(\epsilon_1 + \epsilon_2 - \bar{\epsilon}_1 - \bar{\epsilon}_2) \chi_1 - \frac{1}{2}\bar{\epsilon}_+ \chi_1 \bar{\chi}_1 + \epsilon_- \left( \chi_2 e^{-\frac{1}{2}R_1} + \frac{1}{2} \chi_1 \bar{\chi}_1 \right),
\]
\[
\delta \psi_2 = \epsilon_- \left( \frac{1}{2} \psi_2 \bar{\psi}_1 - \psi_1 e^{-\frac{1}{2}R_1} \right) - \frac{1}{2}\bar{\epsilon}_+ \psi_2 \bar{\chi}_1 + \frac{1}{2}(-\epsilon_2 + \bar{\epsilon}_2) \psi_2,
\]
\[
\delta \chi_2 = \epsilon_+ \left( \frac{1}{2} \chi_2 \bar{\chi}_1 - \chi_1 e^{-\frac{1}{2}R_1} \right) - \frac{1}{2}\epsilon_- \chi_2 \bar{\psi}_1 + \frac{1}{2}(\epsilon_2 - \bar{\epsilon}_2) \chi_2.
\]

where \( \epsilon_1 - \bar{\epsilon}_1, \epsilon_2 - \bar{\epsilon}_2, \epsilon_- \) and \( \epsilon_+ \) are arbitrary constants. The algebra of such transformations can be shown to be the \( q \)-deformed Poisson bracket algebra \( SL(2)_q \otimes U(1) \) [19], with \( q = \exp(-2\pi \tau) \). The global symmetries of the vector model generate the same algebra.

### 5 Non-Conformal T-Duality

T-duality in the context of the conformal \( \sigma \)-models

\[
S_{\sigma}^{conf} = \frac{1}{4\pi \alpha'} \int d^2 z \left( (g_{MN}(X)\eta^{\mu\nu} + \epsilon^{\mu\nu} b_{MN}(X)) \partial_\mu X^M \partial_\nu X^N + \frac{\alpha'}{2} R^{(2)} \varphi(X) \right)
\]

(\( \mu, \nu = 0, 1, \quad M, N = 1, 2, \cdots D \) and \( R^{(2)} \) is the worldsheet curvature), represents specific canonical transformations (CT): \( (\Pi_{X_M}, X^M) \to (\Pi_{\tilde{X}_M}, \tilde{X}^M) \) that map (5.39) into its dual \( \sigma \)-model \( S_{\sigma}^{conf} \left( G_{M,N}(\tilde{X}), B_{M,N}(\tilde{X}), \phi(\tilde{X}) \right) \). In the case of curved backgrounds with \( d \)-isometric directions (i.e., the metric \( g_{MN}(X^m) \), the antisymmetric tensor \( b_{MN}(X^m) \) and the dilaton \( \varphi(X^m) \) are independent of the \( d \leq D \) fields \( X_\alpha(z, \bar{z}), \alpha = 1, 2, \cdots d \) the corresponding CT has the form:

\[
\Pi_{\tilde{X}_\alpha} = -2\partial_x X_\alpha, \quad \Pi_{\bar{X}_\alpha} = -2\partial_{\bar{x}} \bar{X}_\alpha
\]

and the others \( \Pi_{X_m} \) and \( X_m, \quad m = d + 1, \cdots D \) remain unchanged. Then T-duality manifests as (matrix) transformations of the target-space geometry data of (5.39): \( e_{MN}(X) = b_{MN}(X) + g_{MN}(X) \) and \( \varphi(X) \) to its T-dual \( E_{MN}(\tilde{X}) = B_{MN}(\tilde{X}) + G_{MN}(\tilde{X}) \) and \( \phi(\tilde{X}) \) [20]:

\[
E_{\alpha\beta} = (e^{-1})_{\alpha\beta}, \quad E_{\alpha\mu} = e_{\alpha\mu} - e_{\alpha a}(e^{-1})^{a\beta} e_{\beta\mu},
\]
\[
E_{\alpha\mu} = (e^{-1})^{a\beta} e_{\beta\alpha}, \quad E_{\alpha a} = -e_{\alpha a}(e^{-1})^{a\beta} e_{\beta\mu}, \quad \phi = \varphi - \ln(\text{det}e_{\alpha\beta})
\]

By construction the dual pair of \( \sigma \)-models \( S_{\sigma}^{conf}(e, \varphi) \) and \( \tilde{S}_{\sigma}^{conf}(E, \phi) \) share the same spectra and partition functions. Their Lagrangians are related by the generating function \( \mathcal{F} [1] \)

\[
\mathcal{L}(e, \varphi) = \mathcal{L}(E, \phi) + \frac{dF}{dt}, \quad \mathcal{F} = \frac{1}{8\pi \alpha'} \int dx \left( X \cdot \partial_x \tilde{X} - \partial_x X \cdot \tilde{X} \right)
\]
An important feature of the abelian T-duality (5.40) and (5.41) is that it maps the $U(1)^{\otimes d}$ Noether charges $Q^\alpha = \int_{-\infty}^{\infty} J_\alpha^\alpha dx$ of $S_{\sigma}^{\text{conf}}(e, \phi)$ into the topological charges $\tilde{Q}_{\text{top}}^\alpha = \int_{-\infty}^{\infty} \partial_\alpha \tilde{X}^\alpha dx$ of its T-dual model $\tilde{S}_{\sigma}^{\text{conf}}(E, \phi)$ and vice-versa, i.e., we have

\[
J^\alpha_\mu = e^{\alpha \beta}(X_n) \partial_\mu X_\beta + e^{\gamma \mu m}(X_n) \partial_\mu X_m = \epsilon_{\mu \nu} \partial^\nu \tilde{X}^\alpha, \\
\tilde{J}^\alpha_\mu = E^{\alpha \beta}(\tilde{X}_n) \partial_\mu \tilde{X}_\beta + E^{\gamma \mu n}(\tilde{X}_n) \partial_\mu \tilde{X}_n = \epsilon_{\mu \nu} \partial^\nu X^\alpha
\]

and therefore

\[T : (Q^\alpha, Q_{\text{top}}^\alpha) \to (\tilde{Q}_{\text{top}}^\alpha, \tilde{Q}^\alpha)\]

Different examples of such T-dual pairs of conformal $\sigma$-models have been constructed in terms of axial and vector gauged $G/H$-WZW models (see [4] and references therein).

From the other side the IM’s considered in Sect. 2 and 3 have as their conformal limits ($\mu = 0$, i.e. $V = 0$ in (3.24) and (3.30)) the corresponding axial and vector gauged $SL(3, R)/SL(2, R) \otimes U(1)$-WZW models which are T-dual by construction. They have $d = 2$ isometric directions, i.e., $e_{MN}(\psi, \chi)$ are independent of $\Theta_i = ln(\frac{\psi_i}{\chi_i})$. T-duality group in this case is known to be $O(2, 2|Z)$ (see for instance [2]). The problem we address in this section is about T-duality of the IM’s (3.24) and (3.30). We first note the important property of these IM’s, namely adding the potentials $V = Tr(\epsilon_+ g_0^\dagger \epsilon_-(g_0^{-1})^{-1})$ breaks the conformal symmetry, but one still keep two isometries, i.e., $U(1) \otimes U(1)$ invariance, say $\Theta_i \to \Theta_i + \alpha_i$ in the axial case. This suggests that the T-duality of the conformal $G/H$-WZW models can be extended to T-duality for their integrable perturbations (3.24) and (3.30). In order to prove it we extend the Buscher procedure [20] of deriving T-dual of a given conformal $\sigma$-model (with $d$ isometries) to the case of IM’s, i.e., in the presence of the potential $V(X_n)$.

### 5.1 Isometries and T-Dual Actions

Let us consider the Lagrangian density of the form

\[\mathcal{L}_{IM}^{ax} = \mathcal{L}_{\text{conf}}^{\sigma}(\Theta_\alpha, X_m) - V(X_m)\]  

(5.44)

where $\mathcal{L}_{\text{conf}}^{\sigma}$ is the Lagrangian (5.39) with $X_\alpha = \Theta_\alpha$ and the potential $V(X_m)$ is independent of $\Theta_\alpha$. We next rewrite (5.39) in a symbolic form separating the isometric fields $\Theta_\alpha$, $\alpha = 1, 2, \cdots d$ from the remaining ones $X_m$, $m = d + 1, \cdots D$:

\[\mathcal{L}_{IM}^{ax} = \partial_\alpha e^{\alpha \beta}(X_m) \partial_\beta \Theta_\alpha + \partial_\alpha N_\alpha + \tilde{N}_\alpha \partial_\alpha + \mathcal{L}'(X_m)\]

(5.45)

In order to derive $\mathcal{L}_{IM}^{vec}(\tilde{\Theta}_\alpha, \tilde{X}_m)$ of the T-dual IM we apply eq. (5.42), i.e.,

\[\mathcal{L}_{IM}^{vec}(\tilde{\Theta}_\alpha, \tilde{X}_m) = \mathcal{L}_{IM}^{ax}(\Theta_\alpha, X_m) - \tilde{\Theta}_\alpha \left( \partial \tilde{P}_\alpha - \tilde{\partial} P_\alpha \right)\]

(5.46)

where we denote $P_\alpha = \partial \Theta_\alpha$, $\tilde{P}_\alpha = \tilde{\partial} \Theta_\alpha$ and the second term is nothing but the contribution of the generating function $\mathcal{F}(\Theta_\alpha, \tilde{\Theta}_\alpha) \sim \epsilon^{\mu \nu} \partial_\mu \Theta_\alpha \partial_\nu \tilde{\Theta}_\alpha$. We first integrate (5.46) by parts

\[\mathcal{L}_{IM}^{vec} = \tilde{P}_\alpha e^{\alpha \beta} P_\beta + \tilde{P}_\alpha \left( N_\alpha + \partial \tilde{\Theta}_\alpha \right) + \left( \tilde{N}_\alpha - \tilde{\partial} \Theta_\alpha \right) P_\alpha + \mathcal{L}'(X_m)
\]

(5.47)
and next we can take the Gaussian integral in $\bar{P}_\alpha$ and $P_\alpha$ in the corresponding path integral. Therefore the effective action for the T-dual model has the form

$$L_{IM}^{vec}(\tilde{\Theta}_\alpha, X_m) = -\left(\bar{N}_\alpha - \tilde{\partial}\tilde{\Theta}_\alpha\right) e^{-1}_{\alpha\beta} \left(N_\beta + \tilde{\partial}\tilde{\Theta}_\beta\right) + L'(X_m) - 4\pi(\alpha')^2 \ln(\text{det}_{\alpha\beta}) R^{(2)}$$

(5.48)

in accordance with eqs. (5.41).

The second question to addressed is whether the Lagrangians (5.44) and (5.48) are related by canonical transformations (5.40). In order to answer it, we shall compare their Hamiltonians:

$$H^{ax} = \dot{\Theta}_\alpha \Pi_{\Theta_\alpha} + \dot{X}_m \Pi_{X_m} - L^{ax}, \quad H^{vec} = \dot{\tilde{\Theta}}_\alpha \Pi_{\tilde{\Theta}_\alpha} + \dot{X}_m \Pi_{X_m} - L^{vec}$$

since by definition

$$\Pi_{\Theta_\alpha} = \frac{\delta L^{ax}}{\delta \dot{\Theta}_\alpha} = 2\dot{\Theta}_\beta e_{\alpha\beta} N_\alpha + \bar{N}_\alpha,$$
$$\Pi_{\tilde{\Theta}_\alpha} = \frac{\delta L^{vec}}{\delta \dot{\tilde{\Theta}}_\alpha} = e^{-1}_{\alpha\beta} (2\dot{\tilde{\Theta}}_\beta + N_\beta - \bar{N}_\beta),$$

(5.49)

we find that

$$H^{ax} = \frac{1}{4} \Pi_{\Theta_\alpha} e^{-1}_{\alpha\beta} \Pi_{\Theta_\beta} - \frac{1}{2} \Pi_{\Theta_\alpha} e^{-1}_{\alpha\beta} (N_\beta + \bar{N}_\beta) + \partial_x \Theta_\alpha e_{\alpha\beta} \partial_x \Theta_\beta + \partial_x \Theta_\alpha (N_\alpha - \bar{N}_\alpha)$$
$$+ \frac{1}{4} (N_i + \bar{N}_i) e^{-1}_{ij} (N_j + \bar{N}_j) + \mathcal{H}(X_m, \Pi_{X_m})$$

(5.50)

and

$$H^{vec} = \frac{1}{4} \Pi_{\tilde{\Theta}_\alpha} e_{\alpha\beta} \Pi_{\tilde{\Theta}_\beta} - \frac{1}{2} \Pi_{\tilde{\Theta}_\alpha} (N_\alpha - \bar{N}_\alpha) + \partial_x \tilde{\Theta}_\alpha e^{-1}_{\alpha\beta} \partial_x \tilde{\Theta}_\beta + \partial_x \tilde{\Theta}_\alpha (N_\beta + \bar{N}_\beta)$$
$$+ \frac{1}{4} (N_i + \bar{N}_i) e^{-1}_{ij} (N_j + \bar{N}_j) + \mathcal{H}(X_m, \Pi_{X_m})$$

(5.51)

where $\mathcal{H}(X_m, \Pi_{X_m}) = \dot{X}_m \Pi_{X_m} - L'(X_m)$. Finally we observe that $H^{ax} = H^{vec}$, i.e. integrable models (5.44) and (5.48) have coinciding Hamiltonians if the transformation

$$\Pi_{\Theta_\alpha} = -2\partial_x \tilde{\Theta}_\alpha, \quad \Pi_{\tilde{\Theta}_\alpha} = -2\partial_x \Theta_\alpha$$

(5.52)

takes place. This is precisely the canonical transformation (5.40) relating the T-dual pair of $\sigma$-models.

### 5.2 Axial-Vector Duality for Homogeneous Grading Models

In order to prove that the axial (3.24) and vector (3.30) IM’s are T-dual to each other, we apply the procedure explained in Sect. 5.1. Starting from eq. (3.24) we recognize the two isometric “coordinates” to be $\Theta_\alpha = \ln(\psi_{\alpha})$, $\alpha = 1, 2$. By changing variables

$$\psi_{\alpha}, \chi_{\alpha} \rightarrow \Theta_{\alpha}, \quad a_m = \psi_{m} \chi_{m}, \quad m = 1, 2$$

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one can rewrite $\mathcal{L}^{ax}$ in (3.24) in the form (5.45) with

$$\mathcal{L}'(X_m) = \frac{\bar{\partial}a_1\partial a_1}{4\Delta a_1}(1 + a_2) + \frac{\bar{\partial}a_2\partial a_2}{4\Delta a_2}(1 + a_1 + a_2) - \frac{\bar{\partial}a_1\partial a_2}{8\Delta} - \frac{\bar{\partial}a_2\partial a_1}{8\Delta} - \mu^2(\frac{2}{3} + a_1 + a_2)$$

and

$$e_{11} = -\frac{1}{4\Delta}(1 + a_2)a_1, \quad e_{22} = -\frac{1}{4\Delta}(1 + a_1 + a_2)a_2, \quad e_{12} = e_{21} = \frac{1}{8\Delta}a_1a_2$$

$$N_1 = \frac{1}{4\Delta}((1 + a_2)\partial a_1 - \frac{1}{2}a_1\partial a_2), \quad N_2 = \frac{1}{4\Delta}((1 + a_1 + a_2)\partial a_2 - \frac{1}{2}a_2\partial a_1),$$

$$N_1 = \frac{1}{4\Delta}(-(1 + a_2)\partial a_1 + \frac{1}{2}a_1\partial a_2), \quad N_2 = -\frac{1}{4\Delta}((1 + a_1 + a_2)\partial a_2 - \frac{1}{2}a_2\partial a_1).$$

Therefore, according to eqns. (5.46) and (5.47) the axial and vector IM’s are related by canonical transformation (5.52). The identification of $\mathcal{L}_{IM}^{vec}$ in (5.48) with the vector model Lagrangian (3.30) becomes evident by observing the relations among the fields,

$$a_1 = -t_2^2e^{\phi_1 + \phi_2}, \quad a_2 = -t_2^2e^{-\phi_1 + 2\phi_2}, \quad \Theta_1 = -\frac{1}{2}\phi_1, \quad \Theta_2 = -\frac{1}{2}(\phi_2 - \phi_1).$$

Another important feature of the axial-vector T-duality is the simple relation between the isometric fields $\tilde{\Theta}_a$ of the vector model (3.30) and the nonlocal fields $R_i$ (see (4.33) ) of the axial model,

$$R_1 = 2(\Theta_2 - \tilde{\Theta}_1), \quad R_2 = -2(\tilde{\Theta}_1 + 2\tilde{\Theta}_2)$$

The above identification can be established by solving the constraints (2.10) (or in the explicit form (4.33) for the $SL(3)$ case) in favour of the non local field of the vector model $\Theta_i$:

$$\partial \Theta_1 = \partial \ln a_1 - \partial (R_1 + R_2) - \frac{2}{3}\frac{a_2 + 1}{a_1}\partial (2R_1 + R_2),$$

$$\partial \Theta_2 = \partial \ln a_2 + \frac{2}{3}\frac{a_2 + 1}{a_2}\partial (R_1 - R_2) - \frac{1}{3}\partial (2R_1 + R_2),$$

$$\bar{\partial} \Theta_1 = -\bar{\partial} \ln a_1 + \bar{\partial} (R_1 + R_2) + \frac{2}{3}\frac{a_2 + 1}{a_2}\bar{\partial} (2R_1 + R_2),$$

$$\bar{\partial} \Theta_2 = -\bar{\partial} \ln a_2 - \frac{2}{3}\frac{a_2 + 1}{a_2}\bar{\partial} (R_1 - R_2) - \frac{1}{3}\bar{\partial} (2R_1 + R_2)$$

and next comparing the RHS of eqn. (5.56) with the $U(1) \otimes U(1)$ conserved currents of the vector model Lagrangian (3.30). We can further write eqns. (5.56) and (4.33) in the compact form

$$J_{top}^{i,ax} = \epsilon_{\mu\nu}\partial^\nu \Theta_1 = \tilde{J}_{top}^{i,vec}, \quad \tilde{J}_{top}^{i,vec} = \epsilon_{\mu\nu}\partial^\nu R_i = J_{top}^{i,ax},$$

or equivalently

$$\tilde{J}_{top}^{1,vec} = \epsilon_{\mu\nu}\partial^\nu \tilde{\Theta}_1 = -\frac{1}{6}(J^{2,ax}_\mu + 2J^{1,ax}_\mu),$$

$$\tilde{J}_{top}^{2,vec} = \epsilon_{\mu\nu}\partial^\nu \tilde{\Theta}_2 = \frac{1}{6}(J^{1,ax}_\mu - J^{2,ax}_\mu).$$

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These equations exemplify for the $SL(3)$-case in consideration the main property (5.43) of the T-dual pairs of models

$$Q_{\alpha,ax}^{\alpha,ax} = Q_{\alpha,vec}^{\alpha,vec}, \quad Q_{\alpha,ax}^{\alpha,vec} = Q_{\alpha,vec}^{\alpha,ax} \quad (5.59)$$

namely that T-duality relates the topological charges $Q_{\alpha,vec}^{\alpha,vec} = \int dx \partial_x \tilde{\Theta}_\alpha$ to the $U(1) \otimes U(1)$-charges $Q_{\alpha,ax}^{\alpha,ax}$ of the axial IM and vice-versa.

An explicit realization of the above exchange of topological and $U(1)$-Noether charges (similar to the momentum-winding numbers exchange in string theory) have been observed in ref. [7], analyzing the 1-soliton structure spectrum of the corresponding dyonic IM. The masses of the solitons of axial and vector models remains equal, but the $U(1)$ charge of the axial non-topological solitons is transformed into the topological charge of the vector model solitons. Similar relations take place in the pair of T-dual nonabelian dyonic models (3.24) and (3.30) in consideration [19].

6 Conclusions

We have demonstrated how one can extend the abelian T-duality of the conformal gauged $G/H$-WZW models to their integrable perturbations, that appears to be identical to specific homogeneous gradation NA affine Toda models. More general considerations (presented in Sect. 5) of generic (relativistic) IM’s (as well as for non integrable models) admitting isometric directions (i.e., with few global $U(1)$ symmetries) make evident that one can construct their T-dual partners by approprietly chosen cannonical transformations. The most important new feature of the T-duality in the context of 2-D integrable models consists in its action on the spectrum of the solitons of the corresponding pair of dual IM’s. As one can expect it maps the $U(1)^{\otimes d}$-charges of the solitons of the axial model (with $d$-isometries) to the topological charges of the solitons of its T-dual counterpart, leaving the soliton masses unchanged.

The quantization of the NA affine Toda models usualy require nontrivial counterterms [7], [10], [21] together with the renormalization of the couplings and masses. Hence, an interesting open problem is whether the quantum vector and axial IM’s continue to be T-dual to each other.

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