Modified Newtonian Dynamics and Induced gravity

W. F. Kao
Institute of Physics, Chiao Tung University, Hsinchu, Taiwan

Modified Newtonian dynamics, a successful alternative to the cosmic dark matter model, proposes that gravitational field deviates from the Newtonian law when the field strength $g$ is weaker than a critical value $g_0$. We will show that the dynamics of MOND can be derived from an induced gravity model. New dynamics is shown to be compatible with the spatial deformation of scalar fields coupled to the system. Approximate solutions are shown explicitly for a simple toy model.

PACS numbers: PACS numbers: 98.80.-k, 04.50.+h

I. INTRODUCTION

The rotation curve (RC) observations indicates that less than 10% of the gravitational mass can be measured from the luminous part of spiral galaxies. This is the first evidence calling for the existence of un-known dark matter and dark energy. In the meantime, an alternative approach, Modified Newtonian dynamics (MOND) proposed by Milgrom [1], has been shown to agree with many rotation curve observations [1, 2].

Milgrom argues that dark matter is redundant in the approach of MOND. The missing part was, instead, proposed to be derived from the conjecture that gravitational field deviates from the Newtonian $1/r^2$ form when the field strength $g$ is weaker than a critical value $g_0 \sim 0.9 \times 10^{-8}$ cm/s$^2$ [2].

The phenomenological foundations for MOND are based on two observations: (1) flat asymptotic rotation curve (RC) is a common feature for many spiral galaxies, (2) the Tully-Fisher (TF) law, $M \sim V^\alpha$ [3] is very successful explaining the relation between rotation velocity and luminosity in many spiral galaxies. $\alpha$ is shown to be close to 4. Note that first fact indicates that gravitational field $g$ goes like $1/r$ asymptotically in the flat RC region.

There have been effort trying to connect non-relativistic MOND theory with a relativistic version [4]. We will show that the dynamics of MOND can be due to the effect of scalar fields in an induced gravity model. The associated effective potential is also a source of dark energy. In fact, one of the original effect of the scalar field is to deform the definition of scale and distance in the Weyl invariant model.

Indeed, Weyl [5, 6] proposed that the invariant length scale should be defined as $ds^2 = \phi g_{\mu\nu}dx^\mu dx^\nu$. It is easy to find that $ds^2$ is invariant under the local scale transformation $\phi(x) \rightarrow \Lambda^2(x)\phi(x)$ and $g_{\mu\nu} \rightarrow \Lambda^{-2}(x)g_{\mu\nu}$. The local length scale $d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu$ measured by any observers will then be deformed due to the scale chosen by the dimension two scalar field $\phi$.

Therefore one of the effect of the scalar field is to deform the effective distance between two distant matters. One will show explicitly in this paper a possible resolution derived from a simple model with a spherically symmetric mass-distribution directed by similar effect.

Einstein theory works with the latest CMB anisotropic survey in a manageable way. One would like to study if there is any way to induce the MOND under the Einstein-Hilbert framework with minimum modifications. Indeed, one is able to show that the dynamics of MOND could be due to the large-scale spatial inhomogeneity of the coupled scalar field. A specific example will be presented in this paper to show its effect in the large $r$ limit where the effective potential $V(r)$ goes like $\ln r$ in the absence of any dark matter.

In section II, the idea of MOND will be briefly reviewed. The force law of MOND will also be rewritten in term of dimensionless field variables for convenience in this section. The effective potential for MOND will be shown in section III. In section IV, one shows that induced gravity model could provide a possible resolution to the force law of MOND with the help of two scalar fields $\psi$ and $\phi$. Two different approximate solutions to the field equations associated with a simple toy model are presented in section V. Some conclusions and remarks are drawn in section VI.

II. MOND

It was proposed that there exists a critical acceleration parameter $g_0 = 0.9 \times 10^{-8}$ cms$^{-2}$ [2] (Sanders 2002) characterizing the turning point of the effective power law associated with the gravitational field in MOND. Gravitational field of the following form was suggested

$$g \cdot \mu \left( \frac{g}{g_0} \right) = g_N$$

(1)
with a function $\mu(x)$ considered as modified inertial. Here $g_N$ is the Newtonian gravitational field produced by any sort of mass distribution. Milgrom suggests that the following function-form

$$\mu(x) = \frac{x}{\sqrt{1 + x^2}}$$

agrees with the rotational curve of many spiral galaxies.

It is interesting to find that there exists a length scale factor $r_0$ associated with our Milky Way via the following equation

$$g_0 \equiv \frac{GM_0}{r_0^2}$$

with $M_0 \equiv 2 \times 10^{11} M_s$ roughly the order of total mass of the Milky Way. $M_s$ denotes the solar mass. As a result one can show that $r_0 \sim 49700 ly$ which is roughly the size of the luminous part our Milky Way.

Note that $M_0$ is the typical mass of any galaxy with a size similar to our Milky Way. In fact, $r_0$ chosen here will only served as a convenient unit of length scale. Selecting different value of $M_0$ and hence $r_0$ will not affect the physics presented in this paper. It turns out that writing all physical variables in dimensionless form will make it easier for us to focus on important physics.

One will try to write the modified field strength $g$ in a dimensionless form with the help of the parameters $M_0$, $r_0$ and $g_0$. The idea is to write $M$, $r$ and $g$ in units of $M_0$, $r_0$ and $g_0$ respectively.

Milgrom argues that one can take the effective inertial $\mu$ as $\mu(x) = x/\sqrt{1 + x^2}$. Therefore, one obtains

$$\frac{(g/g_0)^2}{\sqrt{1 + (g/g_0)^2}} = g_N/g_0$$

by dividing both sides of the equation [1] by the critical parameter $g_0$. One can therefore write above equation as

$$\frac{g^2}{\sqrt{1 + g^2}} = g_N$$

with $g$ and $g_N$ written in unit of $g_0$. This makes $g$ and $g_N$ dimensionless from now on.

Throughout this paper, we will focus on the study of the system with a Newtonian attraction of the form $g_N = Gm/r^2$ for simplicity. This is the field strength at a radial distance $r$ from a spherically distributed matter with total mass $m$. System with different mass distribution can be obtained straightforwardly. To summarize, one has chosen dimensionless scale according to following redefinitions:

$$m' = \frac{m}{M_0},$$

$$g' = \frac{g}{g_0},$$

$$r' = \frac{r}{r_0}.$$  

(6)

For example, $g'_N = m'/r'^2$. One will suppress the superscript $'$ for convenience. Therefore, the modified field strength $g$ becomes

$$\frac{g^2}{\sqrt{1 + g^2}} = \frac{m}{r^2}$$

(7)

One can further remove the parameter $m$ by defining $r_c \equiv r_0\sqrt{m}$ and write $r$ as a dimensionless coordinate variable in unit of $r_c$ instead of $r_0$.

$$r'' = \frac{r}{r_c} = \frac{r}{\sqrt{mr_0}}$$

(8)

As a result, one can write above equation in a very compact form

$$\frac{g^2}{\sqrt{1 + g^2}} = \frac{1}{r''^2}$$

(9)
in terms of the dimensionless parameters \( g \), \( g_N \), and \( r'' \). Note that it is straightforward to restore all dimension parameters to evaluate any corresponding physical values. Suppressing the superscript \( '' \) again for convenience, one can write it as

\[
\frac{g^2}{\sqrt{1 + g^2}} = \frac{1}{r^2}
\]

(Note that the dimensionless physical coordinate \( r'' \equiv r/r_c = r/(\sqrt{m(r)}r_0) \) is in fact a function of \( r \) for a system with total mass distribution given by \( m(r) \). Gauss law implies that, for a spherically symmetric system, exterior mass throughout \( r > r_1 \) will not affect the gravitational field \( g(r < r_1) \). From now on, one will assume \( m \) is a constant independent of the coordinate \( r \) for simplicity keeping in mind that the result is valid only for the exterior region of a spherical system.

Once \( m(r) \) is not a constant, the generalization is still straightforward. One can simply take \( r'' \) as co-moving coordinate. All physics can be re-derived by a proper coordinate transformation.

Note that Eq. (10) can be solved directly to write \( g(r) \) as a function of \( r \):

\[
g(r) = \frac{\sqrt{1 + \sqrt{1 + 4r^4}}}{\sqrt{2r^2}}.
\]

It is apparent that \( g(r) \) goes like \( 1/r^2 \) at short distance scale where \( r \ll 1 \). On the other hand, \( g(r) \) goes like \( 1/r \) at large distance scale where \( r \gg 1 \).

### III. EFFECTIVE POTENTIAL OF MOND

In order to take a close look at the changing pattern of \( g \), one can split it into two different parts:

\[
g = g_l + g_s = \frac{2\sqrt{2}r^2}{\sqrt{1 + 4r^4}\sqrt{1 + \sqrt{1 + 4r^4}}} + \frac{\sqrt{1 + \sqrt{1 + 4r^4}}}{\sqrt{2r^2}\sqrt{1 + 4r^4}}.
\]

Note that the first term is \( g_l \) that goes like \( 1/r \) when \( r \gg 1 \), while the second term is \( g_s \) that goes like \( 1/r^2 \) when \( r \ll 1 \). Therefore it is easy to see that \( g_l \) and \( g_s \) represents the long distance and short distance field strength of the \( g \) respectively. One is hoping that successful separation of \( g \) may help shedding light to the search of the underlying theory.

One can integrate \( g \) for the effective potential. After some algebra, one can show that the effective potential \( V_l \equiv -\int g_l \cdot dr = \int g_l dr \) and \( V_s \equiv \int g_s dr \) can be evaluated directly to give

\[
V_l = \ln \frac{\sqrt{1 + \sqrt{1 + 4r^4}}}{\sqrt{2}} - \sum_{n=1}^{\infty} \frac{\pi^2}{2n \cdot n!} \frac{1}{(1 + \sqrt{1 + 4r^4})^n},
\]

and

\[
V_s = -\frac{\sqrt{1 + \sqrt{1 + 4r^4}}}{\sqrt{2r}}.
\]

One can verify directly that \( V'_l = g_l \) and \( V'_s = g_s \) and prove that above equations are indeed correct up to an irrelevant integration constant.

In fact, it is difficult to specify this constant of integration in the conventional approach which take \( V(r \to \infty) \to 0 \). This is because the effective potential in fact diverges at spatial infinity due to the logarithm behavior of the dominating 2D-like potential.

One remarks that the potential \( V(r) \) derived here remain valid in the co-moving coordinate chosen as \( r/r_c \) which depends on the mass content of the spherically symmetric system.

### IV. INDUCED GRAVITY THEORY AND MOND

One is looking for a theory that goes to Newtonian theory in the small \( r \) region and reproduce the \( g \sim 1/r \) effect in the large \( r \) region. The dynamics proposed by MOND gives a specific force law shown earlier in section II. In
particular, the induced gravity (or equivalently the Brans-Dicke theory) provides a nice framework for our purpose. Note that induced gravity model, similar to the Brans-Dicke theory, proposes that the gravitational constant is a dynamical variable given by the vacuum expectation value of a scalar field $\phi$ derivable from the Lagrangian $\mathcal{L}_\phi$.

$$\mathcal{L}_\phi = -\frac{1}{8\omega}\phi R - \frac{1}{8\phi} (\partial \phi)^2 - W(\phi).$$

(15)

Here $R$ is the scalar curvature and $\phi$ is a scalar field producing a space-time dependent gravitational constant. In addition, $W(\phi)$ is a spontaneously symmetry broken effective potential coupled to the system. In addition to the Lagrangian of scalar field $\mathcal{L}_\phi$, there is another matter Lagrangian

$$\mathcal{L}_M = \mathcal{L}_M(m, \psi)$$

(16)

with a modified function $\psi(x)$ introduced here to induce the MOND effect. One will assume that the variation of $\mathcal{L}_M$ with respect to the metric $g_{\mu\nu}$ will contribute a term $\psi T_{\mu\nu}^M$ in the field equation accounting for the matter effect. Here $\psi$ is treated as an auxiliary field without any dynamics. The main purpose of this auxiliary field is to introduce a distortion of Newtonian potential in the Newtonian limit.

The physical interpretation of $\psi$ field:

Note that mass of a fermion is induced from a Yukawa coupling term, e.g. $b\bar{u}_b b$ for a baryon $b$. The mass of the baryon $b$ is thus induced by the vacuum expectation value of the coupled scalar field $<\psi'> \rightarrow m_b$. Assuming that $m_b(x) = \langle \psi'(x) \rangle$ is spatial dependent, one can in principle induce a very different mass effect for the system in coherent to the spatial dependent scalar field $\phi(x)$ introduced earlier. To be more specific, the inhomogeneous $m_b(x)$ represents the local mass deformation in our approach.

The overall effect can be integrated in order to produce global deformation on the Newtonian potential $V(x, m)$. In our approach, one assumes that the deformation function $\psi(r)$ coupled to the matter energy-momentum tensor represents collectively the total effect of the inhomogeneity distortion of MOND potential $V(x)$. In other words, $\psi$ is assumed to provide the collective deformation of the entire system accommodating the MOND potential as a physical resolution. In addition, one assumes that the mass generating scalar field $\psi'$ is an auxiliary field without a kinetic term. Moreover, one would like to impose the equation

$$\psi V_0 = \phi V$$

(19)

as the auxiliary constraint for $\psi$. Note that what one imposes here is a relation between $\psi$ and $V$ instead of relating $\psi$ and $\phi$. Indeed, one can see from the structure of the field equation [14] that only the ratio $\psi/\phi$ coupled to $T_{\mu\nu}^M$ matters. Indeed the field equation reads $G_{\mu\nu} = [\psi/\phi]T_{\mu\nu}^M + \cdots$ with the term $[\psi/\phi]T_{\mu\nu}^M$ serves as the generalized energy momentum tensor. Therefore, one can parameterize $\psi$ differently, but the final result will remain the same. By all means, one finds that the constraint [15] is the best way to relate $\psi$ and $V$. This constraint introduced here is similar to the effect of modified inertial introduced by the scalar-tensor theory [3]. By all means, it requires a constraint to be imposed by hand in order to reproduce the dynamics of MOND in an exact form. Hopefully, the approach shown here will shed light to the finding of a more realistic approach.

In fact, the constraint one introduced here is equivalent to setting $\psi/\phi = V/V_0$ such that the system will become Newtonian in the limit $V \rightarrow V_0$. To be more specific, the constraint is in fact introduced via the ration of $\phi/\psi$. Bring $\psi/\phi$ to the right hand side of above equations, one can write $G_{\mu\nu} = [\psi/\phi]T_{\mu\nu}^M + \cdots$ as $\phi' G_{\mu\nu} = [\psi/\phi]G_{\mu\nu} = T_{\mu\nu}^M + \cdots$. This is equivalent to the induced gravity theory without a $\psi$ field. One can introduce the constraint via a re-scaling of the scalar field $\phi' = \phi/\psi$ in the same induced gravity theory. In this approach, one can as well say that the constraint is imposed on the choice of scalar field $\phi' = \phi/\psi$ by choosing a proper gauge of scale transformation [3]. And hopefully, the constraint $\phi' = V_0/V$ (equivalent to the constraint [15]) governing the deformation information of
the Newtonian force law will also lead one from Newtonian theory to the dynamics proposed by MOND. In short, one can either view the constraint Eq. (14) as the collective mass function $\psi$ or the deformation of the Newtonian constant $G$ prescribed by $\phi^\prime$.

Note however that the field equations for these two different approaches will be slightly different due to the kinetic term of the scaled $\phi$ field. In a moment we will try to solve this model in the limit where kinetic term of $\phi$ is omitted. The field equations of the scalar field will be the same in this limit. Therefore, it does not matter which viewpoints one decide to take in the toy model we will study later. The view of the collective mass deformation is however a quite promising idea, therefore, one will stick to Eqs. (17,18) throughout this paper. In short, the auxiliary field introduced here is not a dynamical field. In addition, the effect of $\psi$ has already been absorbed into the collective mass term included in $\psi T_{\mu\nu}$.

The classical Newtonian field equation can be obtained from the time-time component of the Einstein equation in the Newtonian limit with the metric identification

$$g_{00} = -1 - 2V. \quad (20)$$

Indeed, the geodesic equation

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \quad (21)$$

becomes

$$\ddot{r} = -\frac{\partial V}{\partial r} \quad (22)$$

in the Newtonian limit. Note that one can write $V = \varphi(r)V_0$ for convenience with $\nabla^2 V_0 = 4\pi\rho$ the Newtonian potential for a spherically symmetric system. Therefore $\varphi = V/V_0$ represents the deformation factor signifying the deviation of the gravitational potential $V$ as compared to the Newtonian potential $V_0$. One would like to study if there exists a consistent solution in an induced gravity model that will accommodate the gravitational potential of the form $V(r) = (V_l + V_c)$ given by Eq. (19).

Note that the time-time component of the Einstein equation can be shown to be

$$(\frac{1}{2} + V)[\phi'' + \frac{2}{r}\phi'] - [V_0(\varphi'' + \frac{2}{r}\varphi') + 2V_0^\prime\varphi']\phi - \phi\varphi \nabla^2 V_0 + 4\pi\psi\rho = -4\omega(\frac{1}{2} + V)W - \omega(\frac{1}{2} + V)\frac{\phi^2}{2\phi} \quad (23)$$

with $\omega$ a coupling constant.

Since one has $\nabla^2 V_0 = 4\pi\rho$, the constraint $\psi V_0 = \phi V$ implying $\psi = \phi \varphi$ can be used to write Eq. (28) as

$$(\frac{1}{2} + V)[\phi'' + \frac{2}{r}\phi'] - [V_0(\varphi'' + \frac{2}{r}\varphi') + 2V_0^\prime\varphi']\phi = -4\omega(\frac{1}{2} + V)W - \omega(\frac{1}{2} + V)\frac{\phi^2}{2\phi} \quad (24)$$

In addition, one has

$$\phi'' + (\frac{2}{r} + V^\prime)\phi' = \frac{8\pi}{3 + 2\omega}\psi\rho + \frac{8\omega}{3 + 2\omega}[\phi\partial_\phi W - 2W] \quad (25)$$

from Eq. (19).

Therefore one has a set of equation with two correlated ODEs and an unknown scalar potential $W(\phi)$ to be dealt with. In principle, one can choose some appropriate, possibly exotic, combination of scalar potential $W$ such that the equations (23-24) do accommodate consistent solution for $\phi$. One should first solve the second Eq. (24) to find the solution of $\phi(r)$. Substituting this solution back to Eq. (23), one should be able to find $\varphi(r)$.

In practice, what one actually is doing is: (i) given any prescribed form of Newtonian potential $V(r)$ and insert this function to equations (24), (ii) one can in principle solve for a consistent set of solution $(\phi(r), W(\phi))$. Finally, one can reconstruct $W(r)$ as a function of $\phi(r)$. This in principle will give us a solution for $W(\phi)$. One may however end up with a very complicated expression for the scalar potential $W$.

To be more specifically, one can write the scalar potential as

$$W = \left[ -\left(\frac{1}{2} + V\right)[\phi'' + \frac{2}{r}\phi'] + [V_0(\varphi'' + \frac{2}{r}\varphi') + 2V_0^\prime\varphi']\phi \right] / \left[ 4\omega(\frac{1}{2} + V) \right] - \frac{\phi^2}{8\phi} \quad (26)$$
from Eq. 24. In addition, one can put Eq. 25 as
\[
\phi' \left[ \phi'' + \left( \frac{2}{r} + V' \right) \phi' \right] = \frac{8\pi}{3 + 2\omega} \psi\phi' + \frac{8\omega}{3 + 2\omega} \left[ \phi W' - 2\phi' W \right]
\] (27)
by multiplying it with \( \phi' \). Eliminating \( W' \) and \( W \), with the help of Eq. 26, one can write Eq. 24 as a differential equation of \( \phi \) independent of \( W \) with a given \( V(x) \) and \( \rho(x) \). This equation can be solved to give a formal expression of \( \phi(r) \). Insert this solution of \( \phi(r) \) back to Eq. 24, one can hence write the scalar potential \( W(r) \) as a function of \( r \). Since we know the form of \( \phi(r) \), one can invert the function to find \( r = r(\phi) \). Therefore, one should be able to write \( W(\phi(r)) \) as a functional of \( \phi(r) \) straightforwardly. One will show later as an example that the process does work in the large \( r \) limit where the gravitational force become dramatically deformed to the MOND limit \( g \sim 1/r \).

For our purpose, one needs to know if the set of field equations accommodates arbitrarily specified \( \varphi(r) \) as a consistent solution with a properly chosen \( W \). One has two variables \( \phi \) and \( W(\phi) \) at our disposal. Therefore, the answer turns out to be yes. Indeed, one can always solve \( \phi(\varphi) \) as a function of \( \varphi \) with any given \( \varphi(r) \) and \( W(\phi) \) from Eq. (23). The additional Eq. (24) would then require a very special form of \( W \) in order to have the needed function \( \phi(\varphi(r)) \) as a consistent solution.

Since consistent solution with any given \( \varphi(r) \) can be in principle made possible with certain properly chosen potential coupled to the system. Given a carefully chosen potential \( W \), one is naturally lead to the desired deformation factor \( \varphi(r) \) that induce MOND as an alternative theory via the identity \( V(r) = \varphi(r)V_0(r) \).

Note that one assumes that the scalar potential \( W \) could be of very complicated origin. It might have to do with some complicated gravitational field interaction similar to the temperature-dependent effective potential. One is unable to offer a solution here. Hopefully the approach shown here may shed a light to a better understanding of the physical origin of MOND.

For a simple demonstration, one will show explicitly in next section how to obtain a desired asymptotic solution with a toy model.

## V. A TOY MODEL

One will try to show that the system of equations 24, 25 does accommodate consistent set of solution for \( \phi \) and \( W(\phi) \) in the large \( r \) region in this section with the help of a simple toy model. For simplicity, one will consider the limit \( \omega \to 0 \). In order to keep the effective potential term \( W_1 \) in our toy model, one also has to assume that \( W_1 = W/\omega \). Equivalently, one is trying to remove the effect of the kinetic term from the field equations such that the scalar field \( \phi \) becomes an auxiliary field without dynamical term. To be more specific, one is considering the scalar Lagrangian given by
\[
\mathcal{L}_\phi = -\frac{1}{8} \phi R - W(\phi).
\] (28)
We will try to find models which enable a MOND solution in large \( r \) region where no matter is present. In this case, the Newtonian potential \( V_0 \to 1/r \) according to the Gauss law.

In constructing the model, what one actually did is that: (i) let \( V \to \ln(r/r_1) \) be an approximate solution in this limit, (ii) insert this solution to equation 24, 25 and solve for a consistent set of solution \( (\phi(r), W(r)) \). Finally, one can reconstruct \( W(r) \) as a function of \( \phi(r) \).

### A. weakly coupled \( V \)

In the region \( V \ll 1 \), \( W \ll \phi'' \) and \( \phi W' \ll \phi'' \), the field equations become:
\[
\phi'' + \frac{2}{r} \phi' \sim 2V_0 \phi'' \phi
\] (29)
\[
\phi'' + \left( \frac{2}{r} + V' \right) \phi' \sim 0
\] (30)
with a potential \( W \) is either 0 or close nothing in this region. Note that one has used the identification \( V = \varphi V_0 \) and \( V_0 = -1/r \) in this limit.
It is straightforward to show that the following solutions
\[ \phi \to \frac{\phi_0}{r^2} \]  
\[ V \to \ln \frac{r}{r_1} \]
solve Eq. (30) in a consistent way. Here the parameter \( r_1 \) represents a local re-scaling of the zero point of the gravitational potential. The constant \(-\ln r_1\) will not affect the force law Eq. (24).

In addition, the constraint \( V \ll 1 \) implies that this solution is good for \( r \) close to \( r_1 \). For \( r \approx r_0 \), the solution does reproduce the \( 1/r \) force law beyond the luminous region of spiral galaxies. It was shown in Ref. [7] that \( V \rightarrow \ln r \) quickly when \( r > r_c \). Hence region close to \( r_c \) is a perfect domain for our assumption.

One knows that the scalar potential \( W \) may be affected by the detailed dynamics of the system. For example, it will depend on the temperature of the system [9]. Therefore, it is also natural to expect a potential correlated to the scale length \( r \) in a large-scale astrophysical system. Once the effective scalar potential becomes negligible in a domain near and beyond our galactic size, the toy model discussed here may provide a good resolution for the new dynamics in that region.

### B. large \( r \) region

One will show that if the scalar potential is given by
\[ W \sim -\frac{3\phi_0^2}{8\phi} \exp[-2\phi/\phi_0] \]  
in the large \( r \) region, one will be able to reproduce a gravitational potential \( V \to \ln r \) asymptotically in the large \( r \) region.

Note that we are assuming that the weak field approximation \( g_{00} \to -1 - 2V \) and the resulting field equations (23-24) still hold in the large \( r \) limit even the assumption \( V \ll 1 \) is not valid in the large \( r \) limit.

There are two main reasons for this approach. Firstly, MOND approach also have similar problem with the asymptotic divergent potential \( V \) that goes like \( \ln r \) in the large \( r \) region. One has to deal with the potential difficult region in any case. If the induced gravity model considered here has anything to with the real physics, there must, hopefully, exist certain sort of normalization process, similar to the re-normalization theory, holding Eq. (24) valid in the asymptotic region. Secondly, the purpose dealing with this toy model is simply to demonstrate how it is possible to derive the desired large scale (or weak field) gravitational potential \( V \) consistently, at least in the large \( r \) region.

The field equations become
\[ V\phi'' + \frac{2}{r} V \phi' \sim [V_0(\phi'' + \frac{2}{r} \phi')] + 2V_0' \phi' - 2VW \]
\[ \phi'' + (\frac{2}{r} + V') \phi' \sim \frac{8}{3} [\phi \partial_{\phi} W - 2W] \]
\[ \sim 2\phi_0 \exp[-2\phi/\phi_0] - 8W. \]

It is easy to show that Eq. (35) does have an asymptotic approximate-solution \( \phi \to \phi_0 \ln r \) by ignoring the negligible terms proportional to \( W \). One can readily show that the \( W \) term is indeed negligible when \( \phi \to \phi_0 \ln r \) in Eq. (35). Substituting this asymptotic solution \( \phi \) into Eq. (34), one can derive the following equation
\[ \ln r \phi'' \sim \frac{1}{r^2} \phi \]

Note that one has also ignored the negligible \( W \)-dependent term in Eq. (34). One can hence easily show that the solution to above equation is
\[ \phi \to -r \ln r \]
which implies the asymptotic solution \( V \to \ln r \) in the large \( r \) region.

Therefore, one shows that it is indeed possible to derive the asymptotic solution with a toy model given by the action (28). In principle, it is possible to reproduce any physical deformation of the gravitational potential one desires with the help of some properly chosen scalar potential.

Note that one in fact expects that \( V \to \ln r \) and hence \( \varphi \to -r \ln r \) as given by Eq. (37). Therefore, one inserts this solution back to the field equation and try to find potential \( W \) which works along with the field equations. As a result, one finds that \( W \sim -3\phi_0^2 \exp[-2\phi/\phi_0]/8 \) is the effective potential one is looking for.

VI. CONCLUSION

The force law of MOND is reviewed and rewritten in term of dimensionless field variables for a spherically symmetric system in this paper.

One of the effect of the scalar field is to deform the effective distance between two distant matters. The scalar field \( \psi \) coupled to the matter energy momentum tensor is proposed to represent the spatial inhomogeneous effect of the distance deformation in a collective way. One has also shown explicitly in this paper a possible resolution derived from a simple model incorporated with a spherically symmetric mass-distribution generated by similar effect.

Indeed, one shows that the dynamics of MOND could be due to the large-scale spatial inhomogeneity of the coupled scalar field. A specific example with a toy model is presented in this paper to demonstrate its effect in two different regions: (i) one presents an approximate solution in the limit \( V \sim \ln(r/r_1) \ll 1 \) which holds in the region near \( r = r_1 \).
(ii) approximate solution is obtained in the large \( r \) limit. Note that the effective potential \( V(r) \) goes like \( \ln r \) in the absence of any dark matter in this toy model. Indeed,

Eqs (21-25) are in principle solvable with some properly chosen potential even one is unable to obtain an analytical solution at the moment. The method shown in this paper also applies to different force law, with a different \( \varphi(r) \), that may properly describe the RC with or without any sort of dark matter. The constraint (19) is introduced as an auxiliary constraint. One does not know the physical origin of this constraint other than the consistency in dimension. It deserves more attention studying the physical origin of this constraint.

Acknowledgments This work is supported in part by the National Science Council of Taiwan.

References

[1] Milgrom, M., a modification of the newtonian dynamics as a possible alternative to the hidden mass hypothesis, 1983, ApJ, 270, 365; Milgrom, M., a modification of the newtonian dynamics: Implications for galaxies, 1983, ApJ, 270, 371; Milgrom, M., mond as modified inertia, astro-ph/0510117
[2] Sanders, R.H., Verheijen M.A.W., Rotation curves of UMa galaxies in the context of modified Newtonian dynamics, 1998 ApJ, 503, 97; Sanders, R.H., The formation of cosmic structure with modified Newtonian dynamics, 2001, ApJ (in press), astro-ph/0011439; R. H. Sanders and S. S. McGaugh, Modified Newtonian Dynamics as an Alternative to Dark Matter, Ann.Rev.Astron.Astrophys. 40 (2002) 263-317, astro-ph/0204521
[3] Tully, R.B. & Fisher, J.R. 1977, A&A, 54, 661
[4] Bekenstein, J.D., Relativistic gravitation theory for the modified Newtonian dynamics paradigm, 2004, Phys.Rev.D. 70, 083509; Sanders, R. H.; A tensor-vector-scalar framework for modified dynamics and cosmic dark matter, astro-ph/0502222
[5] H. Weyl, electron and gravitation, Z.Phys.56:330-352,1929, Surveys High Energ.Phys.5:261-267,1986;
[6] H. Cheng, the possible existence of weyl’s vector meson, Phys.Rev.Lett. 61, 2182 (1988);
[7] W.F. Kao, Modified Newtonian Dynamics In Dimensionless Form, astro-ph/0504009
[8] Weinberg, S., Gravitation and Cosmology (Wiley, New York, 1972)
[9] Dolan, L. and Jackiw, R., Phys. Rev. D9, 3320 (1974); Kapusta, J.I., Finite-temperature field theory, (Cambridge University Press, Cambridge, 1989);