GLOBAL EXISTENCE OF A STRONG SOLUTION TO A FOURTH-ORDER EXPONENTIAL PDE MODELING CRYSTAL SURFACE GROWTH WITH METROPOLIS-TYPE RATES

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ABSTRACT. In this article we prove the global existence of a unique strong solution to the initial boundary-value problem for a fourth-order exponential PDE modeling crystal surface growth. The model we study is derived as the limit of a microscopic Markov jump process with Metropolis-type transition rates. Our investigation reveals that, in opposition to the models with Arrhenius rates, where the exponent may contain a singular part, the exponent in our model is a $W^{1,2}$ function.

1. INTRODUCTION

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial \Omega$ and $T > 0$. We consider the initial boundary-value problem,

\begin{align}
\frac{\partial u}{\partial t} - \Delta \sinh(-\Delta u) &= 0 \quad \text{in } \Omega_T \equiv \Omega \times (0, T), \\
\nabla u \cdot \nu &= \nabla \sinh(-\Delta u) \cdot \nu = 0 \quad \text{on } \Sigma_T \equiv \partial \Omega \times (0, T), \\
u(x, 0) &= u_0(x) \quad \text{on } \Omega,
\end{align}

where $\nu$ is the unit outward normal vector to the boundary. Equation (1.1) can be used to describe the evolution of a crystal surface. It arises as the limit of a microscopic Markov jump process, with Metropolis-type transition rates. For a deeper discussion on the origins of the equation, along with the 1-dimensional existence assertion and numerical simulations see [2]. The analogous exponential PDE derived using Arrhenius rates is given by,

\begin{equation}
\frac{\partial u}{\partial t} = \Delta e^{-\Delta u}
\end{equation}

For this equation it was observed in [6] that one had to allow the possibility that the exponent could be a measure-valued function. This is due to the lack of estimates for the exponent term. To remove the singularity in the exponent, one must impose a smallness condition on the initial data [5, 7]. For the exponential PDE (1.1) derived using Metropolis-type rates, we are able to improve this, and find that the exponent is a $W^{1,2}(\Omega)$ function without any smallness assumptions on the given data, and, in particular, is not a measure.

One approach when studying nonlinear problems is to take a linear approximation, in this case $1 + x$ for the natural exponential $e^x$. This approach has an unfortunate drawback however, as the linear approximation treats local maxima and minima symmetrically, whereas the original exponential causes local maxima to form expanding facets, while local minima remain stationary [1].

Before we state our main theorem, we give the following definition of a strong solution,
Definition 1.1. We say a pair of functions \((u, w)\) is a strong solution of 1.1 - 1.3 if the following conditions hold:

\((D1)\) \(u \in W^{1,2}(\mathbb{R}^N) \cap L^2(0, T; W^{2,2} (\Omega))\) with \(\partial_t u, |\nabla u| \in L^\infty(0, T; L^2(\Omega))\), and \(w \in W^{1,2}(\mathbb{R}^N)\) is such that \(\sinh(w) \in L^2(0, T; W^{2,2}(\Omega))\),

\((D2)\) we have,

\[
\begin{align*}
\partial_t u - \Delta \sinh(w) &= 0 \text{ a.e. on } \Omega_T, \\
-\Delta u &= w \text{ a.e. on } \Omega_T, \\
\nabla u \cdot \nu &= \nabla \sinh(w) \cdot \nu = 0 \text{ a.e. on } \Sigma_T, \\
u(x, 0) &= u_0(x),
\end{align*}
\]

where the initial condition (1.8) is satisfied in the space \(C \left([0, T]; L^2(\Omega)\right)\).

Our result is the following,

Theorem 1.2 (Main Theorem). Assume:

\((H1)\) \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) which is either convex or has \(C^2\) boundary;

\((H2)\) \(N \geq 2\);

\((H3)\) \(u_0 \in W^{2,2}(\Omega)\) is such that \(\cosh(-\Delta u_0) \in L^1(\Omega), \ \sinh(-\Delta u_0) \in W^{2,2}(\Omega), \ \text{and} \ \nabla u_0 \cdot \nu = 0,\)

\(\nabla \sinh(-\Delta u_0) \cdot \nu = 0 \text{ on } \partial \Omega\).

Then there is a unique global strong solution to equation 1.1 - 1.3 in the sense of Definition 1.1.

By a global strong solution, we mean that for each \(T > 0\) there is a strong solution \(u\) to (1.1)-(1.3) on \(\Omega_T\).

Our result here indicates that the two principal nonlinear terms in equation (1.1) have a balancing effect. Indeed, according to [6], the term \(e^{-\Delta u}\) can behave well even if \(-\Delta u\) is a Radon measure with the support of its singular part contained in the set where the absolutely continuous part is negative infinity. This suggests that good estimates on \(e^{-\Delta u}\) do not control what may happen on the set \(\{\Delta u = \infty\}\). Naturally, one would expect that estimates for the second nonlinear term \(e^{\Delta u}\) in our equation could make up for this. Our analysis here shows that this is exactly what has happened. Note that measure exponents do not arise in the one-dimensional case. See [3, 4].

The uniqueness assertion in Theorem 1.2 is simple. We will prove it now. Let \(v\) be a second solution of 1.5 - 1.8. Then we have,

\[
\frac{\partial}{\partial t} (u - v) = \Delta \sinh(-\Delta u) - \Delta \sinh(-\Delta v).
\]

Multiply through this equation by \(u - v\) and then integrate the resulting equation over \(\Omega\) to get,

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (u - v)^2 \, dx &= \int_{\Omega} (\Delta \sinh(-\Delta u) - \Delta \sinh(-\Delta v)) (u - v) \, dx \\
&= - \int_{\Omega} (\sinh(-\Delta u) - \sinh(-\Delta v)) (-\Delta u - (-\Delta v)) \, dx \\
&\leq 0.
\end{align*}
\]

Here we have used the fact that \(\sinh\) is an increasing function. Integrate the above inequality with respect to \(t\) to complete the proof.

1.1. A-Priori Estimates for smooth solutions. The core of our approach to this problem lies in the following a-priori estimates, which resemble those in [6].

Towards our first estimate we square both sides of 1.5 and integrate over \(\Omega\) to obtain,

\[
\int_{\Omega} (\partial_t u)^2 \, dx - 2 \int_{\Omega} \partial_t u \Delta \sinh(w) \, dx + \int_{\Omega} (\Delta \sinh(w))^2 \, dx = 0.
\]
For the second integral in the above equation we have,

\[ -2 \int_{\Omega} \partial_t u \Delta \sinh (w) \, dx = -2 \int_{\Omega} \partial_t \Delta u \sinh (w) \, dx \]

\[ = 2 \frac{d}{dt} \int_{\Omega} \cosh (w) \, dx. \]  

(1.12)

Putting this back into the previous equation we get,

\[ \int_{\Omega} (\partial_t u)^2 \, dx + 2 \frac{d}{dt} \int_{\Omega} \cosh (w) \, dx + \int_{\Omega} (\Delta \sinh (w))^2 \, dx = 0. \]

(1.13)

Upon integrating with respect to time, we obtain our first estimate,

\[ \int_{\Omega_T} (\partial_t u)^2 \, dx + 2 \sup_{0 \leq t \leq T} \int_{\Omega} \cosh (w) \, dx + \int_{\Omega_T} (\Delta \sinh (w))^2 \, dx \leq 2 \int_{\Omega} \cosh (-\Delta u_0) \, dx \]

(1.14)

Next, we take the gradient of both sides of equation 1.5, and then we take the dot product of the resulting equations with \( \nabla \) and finally integrate over \( \Omega \) to get,

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} \nabla (\Delta \sinh (w)) \cdot \nabla u \, dx = 0. \]

(1.15)

For the second integral in this equation we have,

\[ - \int_{\Omega} \nabla (\Delta \sinh (w)) \cdot \nabla u \, dx = - \int_{\Omega} \nabla \sinh (w) \cdot \nabla \Delta u \, dx \]

\[ = \int_{\Omega} \cosh (w) |\nabla w|^2 \, dx \]

\[ \geq \int_{\Omega} |\nabla w|^2 \, dx. \]

Putting this back into 1.15, and then integrating with respect to \( t \) gives us our second estimate,

\[ \frac{1}{2} \sup_{0 \leq t \leq T} \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega_T} |\nabla w|^2 \, dx \leq \int_{\Omega} |\nabla u_0|^2 \, dx. \]

(1.16)

Towards our third estimate we first take the derivative of 1.5 with respect to \( t \). Then we multiply through the resulting equation by \( \partial_t u \) and integrate over \( \Omega \) to obtain,

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} (\partial_t u)^2 \, dx - \int_{\Omega} \partial_t \Delta \sinh (w) \partial_t u \, dx = 0. \]

(1.17)

For the second integral in this equation we calculate,

\[ - \int_{\Omega} \partial_t \Delta \sinh (w) \partial_t u \, dx = - \int_{\Omega} \partial_t \sinh (w) \partial_t \Delta u \, dx \]

\[ = \int_{\Omega} \partial_t \sinh (w) \partial_t w \, dx \]

\[ = \int_{\Omega} \cosh (w) (\partial_t w)^2 \, dx \]

\[ \geq \int_{\Omega} (\partial_t w)^2 \, dx. \]
Putting this into the original equation and then integrating with respect to \( t \) gives our third estimate,

\[
\frac{1}{2} \sup_{0 \leq t \leq T} \int_{\Omega} (\partial_t u)^2 \, dx + \int_{\Omega} (\partial_t w)^2 \, dx \leq \int_{\Omega} (\Delta \sinh (-\Delta u_0))^2 \, dx.
\]

Finally, we integrate (1.1) with respect to \( x \) over \( \Omega \) to get

\[
\frac{d}{dt} \int_{\Omega} u(x, t) \, dx = 0,
\]

from whence follows

\[
\int_{\Omega} u(x, t) \, dx = \int_{\Omega} u_0(x) \, dx.
\]

Similarly, we can deduce from (1.6) that

\[
\int_{\Omega} w(x, t) \, dx = 0.
\]

We shall see that the preceding a-priori estimates combined with relevant interpolation inequalities for Sobolev spaces and Lemma 2.1 below imply (D1) in the definition 1.1.

A solution to 1.1 - 1.3 will be constructed as the limit of a sequence of approximate solutions. In section 2 we will present our approximate problems, and we establish the existence of a classical solution for these problems. We then form a sequence of approximate solutions based upon implicit discretization in the time variable. Section 3 is then devoted to the proof of the discrete versions of the estimates obtained in Subsection 1.2. These estimates are then shown to be enough to justify passing to the limit.

## 2. Approximate Problems

Before we present our approximate problems, we state a few preparatory lemmas.

**Lemma 2.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \). If \( \Omega \) is convex, then,

\[
\int_{\Omega} (\Delta u)^2 \, dx \geq \int_{\Omega} |\nabla^2 u|^2 \, dx
\]

for all \( u \in W^{2,2}(\Omega) \) with \( \nabla u \cdot \nu = 0 \) on \( \partial \Omega \). If \( \partial \Omega \) is \( C^2 \), then there is a positive constant \( c \) depending only on \( N, \Omega \), and the smoothness of the boundary such that

\[
\int_{\Omega} (\Delta u)^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx \geq c \int_{\Omega} |\nabla^2 u|^2 \, dx
\]

for all \( u \in W^{2,2}(\Omega) \) with \( \nabla u \cdot \nu = 0 \) on \( \partial \Omega \).

For more information on this lemma we refer the reader to [8].

Next, we present some relevant interpolation inequalities for Sobolev spaces,

**Lemma 2.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \). Then,

1. \( \|f\|_{q,\Omega} \leq \epsilon \|f\|_{r,\Omega} + \epsilon^\sigma \|f\|_{p,\Omega} \), where \( \epsilon > 0 \), \( p \leq q < r \), and \( \sigma = \left( \frac{1}{r} - \frac{1}{q} \right) / \left( \frac{1}{q} - \frac{1}{r} \right) \);

2. If \( \partial \Omega \) is \( C^2 \), for each \( \epsilon > 0 \) and each \( p \in [2,2^*) \), where \( 2^* = \frac{2N}{N-2} \) if \( N > 2 \) and any number bigger than 2 if \( N = 2 \), there is a positive number \( c = c(\epsilon, p) \) such that

\[
\|f\|_p \leq \epsilon \|\nabla f\|_2 + c \|f\|_1,
\]

\[
\|\nabla g\|_p \leq \epsilon \|\nabla^2 g\|_2 + c \|g\|_1,
\]

for all \( f \in W^{1,2}(\Omega) \) and \( g \in W^{2,2}(\Omega) \).

Our existence theorem is based on the following fixed point theorem, which is often called the Leray-Schauder Theorem.
Lemma 2.3. Let $B$ be a map from a Banach space $X$ into itself. Assume:
(1) $B$ is continuous,
(2) the images of bounded sets under $B$ are precompact,
(3) there exists a constant $c$ so that,
\begin{equation}
\|z\|_X \leq c
\end{equation}
for all $z \in X$, and $\sigma \in (0, 1)$ satisfying,
\begin{equation}
z = \sigma B(z).
\end{equation}
Then $B$ has a fixed point.

Next we collect a few useful elementary inequalities.

Lemma 2.4. (1) If $f$ is an increasing function and $F$ an anti-derivative of $f$, then,
\begin{equation}
f(s)(s - t) \geq F(s) - F(t).
\end{equation}
(2) for $a, b \in [0, \infty)$ there hold
\begin{equation}
(a + b)^\alpha \leq a^\alpha + b^\alpha \text{ if } 0 < \alpha \leq 1,
\end{equation}
\begin{equation}
(a + b)^\alpha \leq 2^{\alpha - 1}(a^\alpha + b^\alpha) \text{ if } \alpha > 1.
\end{equation}

Lemma 2.5. Let $w \in W^{1,2}(\Omega)$ be a weak solution of the boundary value problem,
\begin{equation}
-\Delta w = f \quad \text{in } \Omega,
\end{equation}
\begin{equation}
\nabla w \cdot \nu = 0 \quad \text{on } \partial \Omega.
\end{equation}
Then for each $p > \frac{N}{2}$ there is a positive number $c = c(N, p, \Omega)$ so that,
\begin{equation}
\|w\|_{\infty, \Omega} \leq c \|w\|_{1, \Omega} + c \|f\|_{p, \Omega}.
\end{equation}
This result is well known.

For our approximate problems, let $\tau > 0$, and $v \in L^\infty(\Omega)$ be given. We consider the following boundary-value problem,
\begin{equation}
\frac{u - v}{\tau} - \Delta \sinh(w) + \tau w = 0 \quad \text{on } \Omega,
\end{equation}
\begin{equation}
-\Delta u + \tau u = w \quad \text{on } \Omega,
\end{equation}
\begin{equation}
\nabla w \cdot \nu = \nabla u \cdot \nu = 0 \quad \text{on } \partial \Omega.
\end{equation}

Proposition 2.6. There exists a weak solution $(u, w)$ to equation (2.10)-(2.12) in the space $(W^{2,2}(\Omega) \cap C^\alpha(\overline{\Omega}))^2$ for some $\alpha \in (0, 1)$.

Proof. To prove the existence of a solution, we will employ the Leray - Schauder Theorem. To this end we define a mapping from $L^\infty(\Omega)$ into itself as follows: for $\varphi \in L^\infty(\Omega)$ we first define $u$ to be the weak solution to the problem,
\begin{equation}
-\Delta u + \tau u = \varphi \quad \text{in } \Omega,
\end{equation}
\begin{equation}
\nabla u \cdot \nu = 0 \quad \text{on } \partial \Omega.
\end{equation}
From the classical theory for linear elliptic equation there is a unique weak solution $u$ in the space $W^{1,2}(\Omega)$. Furthermore, $u$ is Hölder continuous in $\overline{\Omega}$. Using this $u$ we then form the problem,
\begin{equation}
-\text{div}(\cosh(\varphi)\nabla w) + \tau w = \frac{u - v}{\tau} \quad \text{on } \Omega,
\end{equation}
\begin{equation}
\nabla w \cdot \nu = 0 \quad \text{on } \partial \Omega.
\end{equation}
Since $\varphi \in L^\infty(\Omega)$ and $\cosh \varphi \geq 1$, equation (2.15) is uniformly elliptic, and so from the classical existence theory there is a unique weak solution $w \in W^{1,2}(\Omega) \cap C^\beta(\overline{\Omega})$ for some $\beta \in (0, 1)$. We set
Clearly, we can conclude that $B$ is well defined, continuous, and maps bounded sets into precompact ones. We still need to show that there is a positive number $c$ so that

$$\|w\|_{\infty, \Omega} \leq c,$$

for all $w \in L^\infty(\Omega)$ and $\sigma \in (0, 1)$ satisfying,

$$w = \sigma B(w).$$

This is equivalent to the boundary value problem,

$$-\text{div} (\cosh(w) \nabla w) + \tau w = -\frac{\sigma}{\tau} u - v \text{ in } \Omega,$$

$$-\Delta u + \tau u = w \text{ in } \Omega,$$

$$\nabla u \cdot \nu = \nabla w \cdot \nu = 0 \text{ on } \partial\Omega.$$

First use $w$ as a test function in (2.19) and use the fact that $\cosh(w) \geq 1$ to get,

$$\int_{\Omega} |\nabla w|^2 dx + \tau \int_{\Omega} w^2 dx \leq \frac{1}{\tau} \int_{\Omega} (u - v) wx dx$$

$$= \frac{1}{\tau} \int_{\Omega} uw dx + \frac{1}{\tau} \int_{\Omega} vwdx.$$

We now use $u$ as a test function in (2.20), yielding,

$$\int_{\Omega} wu dx = \int_{\Omega} |\nabla u|^2 dx + \tau \int_{\Omega} u^2 dx \geq 0.$$

Using the above equation in (2.22), we are able to derive,

$$\int_{\Omega} |\nabla w|^2 dx + \tau \int_{\Omega} w^2 dx \leq \frac{1}{\tau} \int_{\Omega} vwdx.$$

Using the above equation and (2.23) we then have,

$$\int_{\Omega} |\nabla w|^2 dx + \tau \int_{\Omega} w^2 dx \leq c(\tau) \int_{\Omega} v^2 dx$$

$$\int_{\Omega} |\nabla u|^2 dx + \tau \int_{\Omega} u^2 dx \leq c(\tau) \int_{\Omega} w^2 dx.$$

Then for each $p > 2$, we use the function $|w|^{p-2} w$ as a test function in 2.19 to obtain,

$$(p - 1) \int_{\Omega} |w|^{p-2} \cosh(w) |\nabla w|^2 dx + \tau \int_{\Omega} |w|^p dx \leq \int_{\Omega} \frac{u - v}{\tau} |w|^{p-1} dx$$

$$\leq \left\| \frac{u - v}{\tau} \right\|_{p,\Omega} \|w\|_{p,\Omega}^{p-1}. $$

After dropping the first integral, we obtain,

$$\tau \left\| w \right\|_{p,\Omega} \leq \left\| \frac{u - v}{\tau} \right\|_{p,\Omega}. $$

Letting $p \to \infty$ then gives,

$$\tau \left\| w \right\|_{\infty,\Omega} \leq \left\| \frac{u - v}{\tau} \right\|_{\infty,\Omega}. $$

From Lemma 2.5 we then get that for each $q > \max \left\{ \frac{N}{2}, 2 \right\}$ there is a positive number $c = c(N, \Omega, \tau)$ so that,

$$\|u\|_{\infty,\Omega} \leq c\|u\|_{1,\Omega} + c\|w\|_{q,\Omega} \leq c\|w\|_{q,\Omega}.$$
The last step is due to (2.25). Using this in conjunction with 2.28 we have,
\[ \|w\|_{\infty, \Omega} \leq c\|u\|_{\infty, \Omega} + c\|v\|_{\infty, \Omega} \]
(2.30)
\[ \leq c\|w\|_{\infty, \Omega} + c\|v\|_{\infty, \Omega} \]
\[ \leq c\|w\|_{\infty, \Omega} + c(e)\|w\|_{1, \Omega} + c\|v\|_{\infty, \Omega} \]
Taking \( \epsilon \) suitably small we finally get,
\[ \|w\|_{\infty, \Omega} \leq c\|v\|_{\infty, \Omega} + c\|w\|_{1, \Omega} \leq c. \]
(2.31)

Here we have used (2.25).

The fact that both \( u \) and \( \sinh w \) lie in \( W^{2,2}(\Omega) \) is a consequence of Lemma 2.1. Then we can deduce \( w \in W^{2,2}(\Omega) \) from the boundedness of \( w \). The proof is complete. \( \Box \)

3. PROOF OF THE MAIN THEOREM

The proof of our main Theorem will be accomplished in several stages. First we present the time discretized problem. The existence of a solution to this problem is dependent on our approximate problem from section 3. We then derive estimates similar to our apriori estimates, and show that this is enough to justify in passing to the limit.

Let \( T > 0 \) be given. For each \( j \in \{1, 2, 3, \ldots\} \) we divide the time interval \([0, T]\) into \( j \) equal sub-intervals. Set
\[ \tau = \frac{T}{j}, \quad t_k = k\tau, \quad k = 0, 1, \cdots, j. \]
(3.1)

Let \( u_0 \) be given, satisfying (H3). It is not difficult to see from (H3) that \( u_0 \in L^\infty(\Omega) \). Thus by Proposition 2.6 we can recursively solve the system,
\[ \frac{u_k - u_{k-1}}{\tau} - \Delta \sinh (w_k) + \tau w_k = 0 \quad \text{in} \quad \Omega, \]
(3.2)
\[ -\Delta u_k + \tau u_k = w_k \quad \text{in} \quad \Omega, \]
(3.3)
\[ \nabla u_k \cdot \nu = \nabla w_k \cdot \nu = 0 \quad \text{on} \quad \partial \Omega. \]
(3.4)
We can pick \( w_0 \) and \( u_{-1} \) so that the equations
\[ \frac{u_0 - u_{-1}}{\tau} - \Delta \sinh (w_0) + \tau w_0 = 0 \quad \text{in} \quad \Omega, \]
\[ -\Delta u_0 + \tau u_0 = w_0 \quad \text{in} \quad \Omega \]
are satisfied. This together with (H3) implies that (3.2)-(3.4) still hold for \( k = 0 \).

Next, we define the functions \( \tilde{u}_j(x,t), \tilde{u}_j(x,t), \tilde{w}_j(x,t), \tilde{w}_j(x,t) \) on \( \Omega_T \) as follows: For each \( (x,t) \in \Omega_T \) there is \( k \) such that \( t \in [t_{k-1}, t_k] \). Subsequently, set
\[ \tilde{u}_j(x,t) = \frac{t - t_{k-1}}{\tau} u_k(x) + \left( 1 - \frac{t - t_{k-1}}{\tau} \right) u_{k-1}(x), \]
(3.6)
\[ \tilde{u}_j(x,t) = u_k(x), \]
(3.7)
\[ \tilde{w}_j(x,t) = \frac{t - t_{k-1}}{\tau} w_k(x) + \left( 1 - \frac{t - t_{k-1}}{\tau} \right) w_{k-1}(x), \]
(3.8)
\[ \tilde{w}_j(x,t) = w_k(x). \]
(3.9)
Using these functions we may then write our discretized system as,
\[ \partial_t \tilde{u}_j - \Delta \sinh (\tilde{w}_j) + \tau \tilde{w}_j = 0 \quad \text{on} \quad \Omega_T, \]
(3.10)
\[ -\Delta \tilde{u}_j + \tau \tilde{u}_j = \tilde{w}_j \quad \text{on} \quad \Omega_T. \]
(3.11)
We now proceed to derive the discrete analogues of our A-priori estimates. First we have the discrete version of 1.14.
Proposition 3.1. We have the estimate,

\[
\int_{\Omega_T} \left( \frac{\partial \tilde{u}_j}{\partial t} \right)^2 dx dt + \int_{\Omega_T} (\Delta \sinh (\tilde{w}_j))^2 dx dt + \tau^2 \int_{\Omega_T} \tilde{w}_j^2 dx dt
+ 2\tau \int_{\Omega_T} \left| \nabla \sinh (\tilde{w}_j) \right|^2 dx dt + 2\tau^2 \int_{\Omega_T} \tilde{w}_j \sinh (\tilde{w}_j) dx dt
\]

\[
+ 2 \max_{0 \leq t \leq T} \int_{\Omega} \cosh (\tilde{w}_j) dx + 2 \max_{0 \leq t \leq T} \left( \tau \int_{\Omega} \left| \nabla \tilde{u}_j \right|^2 dx + \tau^2 \int_{\Omega} \tilde{u}_j^2 dx \right)
+ 2 \tau \int_{\Omega_T} \tilde{w}_j^2 dx dt
\]

\leq 2 \int_{\Omega} \cosh (w_0) dx + 2 \tau \int_{\Omega} \left| \nabla u_0 \right|^2 dx + 2\tau^2 \int_{\Omega} u_0^2 dx

(3.12)

Proof. First we square both sides of equation (3.2), and the integrate the resulting equation over \( \Omega \) to obtain,

\[
\int_{\Omega} \left( \frac{u_k - u_{k-1}}{\tau} \right)^2 dx + \int_{\Omega} (\Delta \sinh (w_k))^2 dx + \tau^2 \int_{\Omega} w_k^2 dx
- \frac{2}{\tau} \int_{\Omega} (u_k - u_{k-1}) \Delta \sinh (w_k) dx + 2 \int_{\Omega} (u_k - u_{k-1}) w_k dx
- 2\tau \int_{\Omega} w_k \Delta \sinh (w_k) dx = 0.

(3.13)

The first three integrals in this equation are good, we need only worry about the final three. For the fourth integral in the above equation we calculate,

\[
\frac{2}{\tau} \int_{\Omega} (u_k - u_{k-1}) \Delta \sinh (w_k) dx = -\frac{2}{\tau} \int_{\Omega} \Delta (u_k - u_{k-1}) \sinh (w_k) dx

= \frac{2}{\tau} \int_{\Omega} \left[ -\tau (u_k - u_{k-1}) + (w_k - w_{k-1}) \right] \sinh (w_k) dx

= -2\tau \int_{\Omega} \left( \frac{u_k - u_{k-1}}{\tau} \right) \sinh (w_k) dx + \frac{2}{\tau} \int_{\Omega} (w_k - w_{k-1}) \sinh (w_k) dx

= -2\tau \int_{\Omega} \Delta \sinh (w_k) \sinh (w_k) dx + 2\tau^2 \int_{\Omega} w_k \sinh (w_k) dx

+ \frac{2}{\tau} \int_{\Omega} (w_k - w_{k-1}) \sinh (w_k) dx
\]

\[
= 2\tau \int_{\Omega} \left| \nabla \sinh (w_k) \right|^2 dx + 2\tau^2 \int_{\Omega} w_k \sinh (w_k) dx + \frac{2}{\tau} \int_{\Omega} (w_k - w_{k-1}) \sinh (w_k) dx
\]

\[
\geq 2\tau \int_{\Omega} \left| \nabla \sinh (w_k) \right|^2 dx + 2\tau^2 \int_{\Omega} w_k \sinh (w_k) dx

+ \frac{2}{\tau} \int_{\Omega} (\cosh (w_k) - \cosh (w_{k-1})) dx
\]

(3.14)
Then for the fifth integral in equation (3.13) we calculate,

\[
2 \int_{\Omega} (u_k - u_{k-1}) w_k dx = 2 \int_{\Omega} (u_k - u_{k-1})(-\Delta u_k + \tau u_k) dx
\]

\[
= -2 \int_{\Omega} (u_k - u_{k-1}) \Delta u_k dx + 2\tau \int_{\Omega} (u_k - u_{k-1}) u_k dx
\]

(3.15)

\[
= 2 \int_{\Omega} (\nabla u_k - \nabla u_{k-1}) \nabla u_k dx + 2\tau \int_{\Omega} (u_k - u_{k-1}) u_k dx
\]

\[
\geq 2 \int_{\Omega} (|\nabla u_k|^2 - |\nabla u_{k-1}|^2) dx + 2\tau \int_{\Omega} (u_k^2 - u_{k-1}^2) dx.
\]

Finally for the last integral in equation (3.13) we calculate,

\[
-2\tau \int_{\Omega} w_k \Delta \sinh (w_k) dx = 2\tau \int_{\Omega} \nabla w_k \cdot \nabla \sinh (w_k) dx
\]

\[
= 2\tau \int_{\Omega} \nabla w_k \cdot (\cosh (w_k) \nabla w_k) dx
\]

(3.16)

\[
= 2\tau \int_{\Omega} \cosh (w_k) |\nabla w_k|^2 dx
\]

\[
\geq 2\tau \int_{\Omega} |\nabla w_k|^2 dx.
\]

We plug each of the integrals (3.14) - (3.16) back into equation (3.13) to then get the inequality,

\[
\int_{\Omega} \left( \frac{u_k - u_{k-1}}{\tau} \right)^2 dx + \int_{\Omega} (\Delta \sinh (w_k))^2 dx + \tau^2 \int_{\Omega} w_k^2 dx + 2\tau \int_{\Omega} |\nabla \sinh (w_k)|^2 dx
\]

(3.17)

\[
+ 2\tau^2 \int_{\Omega} w_k \sinh (w_k) dx + \frac{2}{\tau} \int_{\Omega} (\cosh (w_k) - \cosh (w_{k-1})) dx + 2\tau \int_{\Omega} |\nabla w_k|^2 dx
\]

\[
+ 2 \int_{\Omega} (|\nabla u_k|^2 - |\nabla u_{k-1}|^2) dx + 2\tau \int_{\Omega} (u_k^2 - u_{k-1}^2) dx = 0.
\]

Finally we multiply through this inequality by \(\tau\) and sum up over \(k\) to get the result. \(\square\)

Next, we have the discrete version of 1.16.

**Proposition 3.2.** We have the estimate,

\[
\max_{0 \leq \ell \leq T} \left( \int_{\Omega} |\nabla \bar{u}_j|^2 dx + \tau \int_{\Omega} \bar{u}_j^2 dx \right) + \tau^3 \int_{\Omega_T} \bar{u}_j^2 dx dt
\]

\[
+ \int_{\Omega_T} |\nabla \bar{w}_j|^2 dx dt + \tau \int_{\Omega_T} (\Delta \bar{u}_j)^2 dx dt + 2\tau^2 \int_{\Omega_T} |\nabla \bar{u}_j|^2 dx dt
\]

(3.18)

\[
\leq \int_{\Omega} |\nabla u_0|^2 dx + \tau \int_{\Omega} u_0^2 dx.
\]

**Proof.** We take the gradient of both side of equation (3.2), and then dot the resulting equation with \(\nabla u_k\) and integrate over \(\Omega\) to get,

\[
\frac{1}{\tau} \int_{\Omega} \nabla (u_k - u_{k-1}) \cdot \nabla u_k dx - \int_{\Omega} \nabla (\Delta \sinh (w_k)) \cdot \nabla u_k dx + \tau \int_{\Omega} \nabla u_k \cdot \nabla u_k dx = 0
\]

(3.19)

For the first integral in the above equation we calculate,

\[
\frac{1}{\tau} \int_{\Omega} \nabla (u_k - u_{k-1}) \cdot \nabla u_k dx \geq \frac{1}{\tau} \int_{\Omega} \left( |\nabla u_k|^2 - |\nabla u_{k-1}|^2 \right) dx.
\]
For the second integral in equation (3.19) we calculate,

\[- \int_{\Omega} \nabla (\Delta \sinh(w_k)) \cdot \nabla u_k dx = - \int_{\Omega} \nabla \sinh(w_k) \cdot \nabla \Delta u_k dx\]

\[= \int_{\Omega} \nabla \sinh(w_k) \cdot \nabla (-\tau u_k + w_k) dx\]

\[= -\tau \int_{\Omega} \nabla \sinh(w_k) \cdot \nabla u_k dx + \int_{\Omega} \nabla \sinh(w_k) \cdot \nabla w_k dx\]

\[= \tau \int_{\Omega} \Delta \sinh(w_k) u_k dx + \int_{\Omega} \cosh(w_k) |\nabla w_k|^2 dx\]

\[= \tau \int_{\Omega} \left[ \frac{u_k - u_{k-1}}{\tau} + \tau w_k \right] u_k dx + \int_{\Omega} \cosh(w_k) |\nabla w_k|^2 dx\]

(3.21)

\[= \int_{\Omega} (u_k - u_{k-1}) u_k dx + \tau^2 \int_{\Omega} w_k u_k dx + \int_{\Omega} \cosh(w_k) |\nabla w_k|^2 dx\]

\[= \int_{\Omega} (u_k - u_{k-1}) u_k dx + \tau^2 \int_{\Omega} (-\Delta u_k + \tau u_k) u_k dx + \int_{\Omega} \cosh(w_k) |\nabla w_k|^2 dx\]

\[= \int_{\Omega} (u_k - u_{k-1}) u_k dx - \tau^2 \int_{\Omega} \Delta u_k dx + \tau^3 \int_{\Omega} dx + \int_{\Omega} \cosh(w_k) |\nabla w_k|^2 dx\]

\[= \int_{\Omega} (u_k - u_{k-1}) u_k dx + \tau^2 \int_{\Omega} |\nabla u_k|^2 dx + \tau^3 \int_{\Omega} u_k^2 dx + \int_{\Omega} \cosh(w_k) |\nabla w_k|^2 dx\]

\[\geq \int_{\Omega} (u_k^2 - u_{k-1}^2) dx + \tau^2 \int_{\Omega} |\nabla u_k|^2 dx + \tau^3 \int_{\Omega} u_k^2 dx + \int_{\Omega} |\nabla w_k|^2 dx.\]

Finally for the last integral in equation (3.19) we calculate,

\[\tau \int_{\Omega} \nabla w_k \cdot \nabla u_k dx = -\tau \int_{\Omega} w_k \Delta u_k dx\]

\[= -\tau \int_{\Omega} (-\Delta u_k + \tau u_k) \Delta u_k dx\]

(3.22)

\[= \tau \int_{\Omega} (\Delta u_k)^2 dx - \tau^2 \int_{\Omega} u_k \Delta u_k dx\]

\[= \tau \int_{\Omega} (\Delta u_k)^2 dx + \tau^2 \int_{\Omega} |\nabla u_k|^2 dx.\]

Upon putting (3.20) - (3.22) back into equation (3.19) we obtain,

\[\frac{1}{\tau} \int_{\Omega} \left( |\nabla u_k|^2 - |\nabla u_{k-1}|^2 \right) dx + \int_{\Omega} (u_k^2 - u_{k-1}^2) dx\]

\[+ \tau^3 \int_{\Omega} u_k^2 dx + \int_{\Omega} |\nabla w_k|^2 dx\]

\[+ \tau \int_{\Omega} (\Delta u_k)^2 dx + 2\tau^2 \int_{\Omega} |\nabla u_k|^2 dx = 0.\]

(3.23)

Multiply through this inequality by \(\tau\) and sum up over \(k\) to obtain the result. \(\square\)

Now we can obtain the discrete version of (1.18).
Proposition 3.3. We have the estimate,

\[
\int_{\Omega_T} \left( \frac{\partial \tilde{w}_j}{\partial t} \right)^2 dx dt + \frac{1}{\tau} \max_{0 \leq t \leq T} \int_{\Omega} \left( \frac{\partial \tilde{u}_j}{\partial t} \right)^2 dx + \tau \max_{0 \leq t \leq T} \int_{\Omega} \tilde{w}_j \sinh (\tilde{w}_j) dx \\
+ \frac{1}{2} \max_{0 \leq t \leq T} \int_{\Omega} \left| \nabla \sinh (\tilde{w}_j) \right|^2 dx + \tau \int_{\Omega} \left| \nabla \tilde{u}_j \right|^2 dx dt + \tau^2 \int_{\Omega_T} \left( \frac{\partial \tilde{u}_j}{\partial t} \right)^2 dx dt \\
\leq \int_{\Omega} (\Delta \sinh (w_0) - \tau w_0)^2 dx + \tau \int_{\Omega} \left| \nabla \sinh (w_0) \right|^2 dx + 2\tau^2 \int_{\Omega} w_0 \sinh (w_0) dx.
\]

Proof. From equation (3.2) we derive the equation, for \( k = 1, 2, \ldots \)

\[
\frac{1}{\tau} \left( \frac{u_k - u_{k-1}}{\tau} - \frac{u_{k-1} - u_{k-2}}{\tau} \right) - \Delta \left( \frac{\sinh (u_k) - \sinh (u_{k-1})}{\tau} \right) + \tau \frac{w_k - w_{k-1}}{\tau} = 0 \text{ in } \Omega.
\]

We multiply through this equation by the function \( \frac{u_k - u_{k-1}}{\tau} \), and integrate over \( \Omega \) to obtain,

\[
\frac{1}{\tau} \int_{\Omega} \left( \frac{u_k - u_{k-1}}{\tau} - \frac{u_{k-1} - u_{k-2}}{\tau} \right) \left( \frac{u_k - u_{k-1}}{\tau} \right) dx \\
- \int_{\Omega} \Delta \left( \frac{\sinh (u_k) - \sinh (u_{k-1})}{\tau} \right) \left( \frac{u_k - u_{k-1}}{\tau} \right) dx \\
+ \tau \int_{\Omega} \left( \frac{w_k - w_{k-1}}{\tau} \right) \left( \frac{u_k - u_{k-1}}{\tau} \right) dx = 0.
\]

For the first integral in the above equation we then calculate,

\[
\frac{1}{\tau} \int_{\Omega} \left( \frac{u_k - u_{k-1}}{\tau} - \frac{u_{k-1} - u_{k-2}}{\tau} \right) \left( \frac{u_k - u_{k-1}}{\tau} \right) dx \\
\geq \frac{1}{2\tau} \int_{\Omega} \left[ \left( \frac{u_k - u_{k-1}}{\tau} \right)^2 - \left( \frac{u_{k-1} - u_{k-2}}{\tau} \right)^2 \right] dx.
\]

For the second integral in (3.26) we first calculate,

\[
- \int_{\Omega} \Delta \left( \frac{\sinh (u_k) - \sinh (u_{k-1})}{\tau} \right) \left( \frac{u_k - u_{k-1}}{\tau} \right) dx \\
= - \int_{\Omega} \left( \frac{\sinh (u_k) - \sinh (u_{k-1})}{\tau} \right) \Delta \left( \frac{u_k - u_{k-1}}{\tau} \right) dx \\
= \int_{\Omega} \left( \frac{\sinh (u_k) - \sinh (u_{k-1})}{\tau} \right) \left( -\tau \left( \frac{u_k - u_{k-1}}{\tau} \right) + \left( \frac{w_k - w_{k-1}}{\tau} \right) \right) dx \\
= -\tau \int_{\Omega} \left( \frac{\sinh (u_k) - \sinh (u_{k-1})}{\tau} \right) \left( \frac{u_k - u_{k-1}}{\tau} \right) dx \\
+ \int_{\Omega} \left( \frac{\sinh (u_k) - \sinh (u_{k-1})}{\tau} \right) \left( \frac{w_k - w_{k-1}}{\tau} \right) dx \\
= - \int (\sinh (u_k) - \sinh (u_{k-1})) (-\Delta \sinh (u_k) + \tau w_k) dx \\
+ \int \left( \frac{\sinh (u_k) - \sinh (u_{k-1})}{\tau} \right) \left( \frac{w_k - w_{k-1}}{\tau} \right) dx \\
= - \int (\sinh (u_k) - \sinh (u_{k-1})) \Delta \sinh (u_k) dx + \tau \int (\sinh (u_k) - \sinh (u_{k-1})) w_k dx \\
+ \int (\sinh (u_k) - \sinh (u_{k-1})) \left( \frac{w_k - w_{k-1}}{\tau} \right) dx.
\]
We now have three integrals to worry about here. For the first integral on the right most side of this equation we have,

\[- \int_{\Omega} (\sinh (w_k) - \sinh (w_{k-1})) \Delta \sinh (w_k) \, dx\]

(3.29)

\[= \int_{\Omega} (\nabla \sinh (w_k) - \nabla \sinh (w_{k-1})) \nabla \sinh (w_k) \, dx\]

\[\geq \frac{1}{2} \int_{\Omega} (|\nabla \sinh (w_k)|^2 - |\nabla \sinh (w_{k-1})|^2) \, dx\]

Towards estimating the second integral we first calculate,

\[(\sinh (w_k) - \sinh (w_{k-1})) w_k = \left[ \frac{1}{2} (e^{w_k} - e^{-w_k}) - \frac{1}{2} (e^{w_{k-1}} - e^{-w_{k-1}}) \right] w_k\]

\[= \left[ \frac{1}{2} (e^{w_k} - e^{w_{k-1}}) + \frac{1}{2} (e^{-w_k} - e^{-w_{k-1}}) \right] w_k\]

\[= \frac{1}{2} (e^{w_k} - e^{w_{k-1}}) w_k + \frac{1}{2} (e^{-w_k} - e^{-w_{k-1}}) (-w_k)\]

(3.30)

\[\geq \frac{1}{2} (w_k e^{w_k} - w_{k-1} e^{w_{k-1}}) - (w_k - w_{k-1})\]

\[+ \frac{1}{2} (-w_k e^{-w_k} + w_{k-1} e^{-w_{k-1}}) - (-w_k + w_{k-1})\]

\[= w_k \frac{1}{2} (e^{w_k} - e^{-w_k}) - w_{k-1} \frac{1}{2} (e^{w_{k-1}} - e^{-w_{k-1}})\]

\[= w_k \sinh (w_k) - w_{k-1} \sinh (w_{k-1})\]

Consequently we then have,

(3.31)

\[\tau \int_{\Omega} (\sinh (w_k) - \sinh (w_{k-1})) w_k \, dx \geq \tau \int_{\Omega} (w_k \sinh (w_k) - w_{k-1} \sinh (w_{k-1})) \, dx.\]

For the last integral in equation (3.28) we then calculate using the mean value theorem.

\[\int_{\Omega} \left( \frac{\sinh (w_k) - \sinh (w_{k-1})}{\tau} \right) \left( \frac{w_k - w_{k-1}}{\tau} \right) \, dx = \int_{\Omega} \cosh (\xi) \left( \frac{w_k - w_{k-1}}{\tau} \right)^2 \, dx\]

(3.32)

\[\geq \int_{\Omega} \left( \frac{w_k - w_{k-1}}{\tau} \right)^2 \, dx\]

Using the above inequality and (3.29) - (3.31) in equation (3.28) we are then able to obtain,

\[- \int_{\Omega} \Delta \left( \frac{\sinh (w_k) - \sinh (w_{k-1})}{\tau} \right) \left( \frac{w_k - w_{k-1}}{\tau} \right) \, dx\]

(3.33)

\[\geq \frac{1}{2} \int_{\Omega} (|\nabla \sinh (w_k)|^2 - |\nabla \sinh (w_{k-1})|^2) \, dx\]

\[+ \tau \int_{\Omega} (w_k \sinh (w_k) - w_{k-1} \sinh (w_{k-1})) \, dx + \int_{\Omega} \left( \frac{w_k - w_{k-1}}{\tau} \right)^2 \, dx\]
Finally for the last integral in equation (3.26) we have,
\[
\tau \int_{\Omega} \left( \frac{w_k - w_{k-1}}{\tau} \right) \left( \frac{u_k - u_{k-1}}{\tau} \right) dx = -\tau \int_{\Omega} \Delta \left( \frac{u_k - u_{k-1}}{\tau} \right) \left( \frac{u_k - u_{k-1}}{\tau} \right) dx
\]
\[
+ \tau^2 \int_{\Omega} \left( \frac{u_k - u_{k-1}}{\tau} \right)^2 dx
\]
\[
= \tau \int_{\Omega} \left| \nabla \left( \frac{u_k - u_{k-1}}{\tau} \right) \right|^2 dx + \tau^2 \int_{\Omega} \left( \frac{u_k - u_{k-1}}{\tau} \right)^2 dx
\]
Then using (3.27), (3.33), and (3.34), in equation (3.26), we obtain the inequality,
\[
\frac{1}{2\tau} \int_{\Omega} \left[ \left( \frac{u_k - u_{k-1}}{\tau} \right)^2 - \left( \frac{u_{k-1} - u_{k-2}}{\tau} \right)^2 \right] dx + \int_{\Omega} \left( \frac{w_k - w_{k-1}}{\tau} \right)^2 dx
\]
\[
+ \frac{1}{2} \int_{\Omega} \left| \nabla \sinh (w_k) \right|^2 - \left| \nabla \sinh (w_{k-1}) \right|^2 dx + \tau \int_{\Omega} \left| \nabla \left( \frac{u_k - u_{k-1}}{\tau} \right) \right|^2 dx
\]
\[
+ \tau \int_{\Omega} (w_k \sinh (w_k) - w_{k-1} \sinh (w_{k-1})) dx + \tau^2 \int_{\Omega} \left( \frac{u_k - u_{k-1}}{\tau} \right)^2 dx \leq 0.
\]
Multiply through this inequality by \( \tau \), sum up over \( k \), and take into account (3.5) to get the result. 

We are now ready to prove the necessary compactness to justify taking the limit in our equations. First we have

**Proposition 3.4.** The sequences \( \{\tilde{u}_j\} \) and \( \{\tilde{w}_j\} \) are bounded in \( W^{1,2}(\Omega_T) \), and hence precompact in \( L^2(\Omega_T) \).

**Proof.** To demonstrate this we first notice that by Proposition 3.1 the sequence \( \{\frac{\partial \tilde{u}_j}{\partial t}\} \) is bounded in \( L^2(\Omega_T) \). For each \( t \in (0,T] \) there is a \( k \) such that \( t \in (t_{k-1}, t_k] \). Then we have
\[
\int_{\Omega} \left| \nabla \tilde{u}_j (x,t) \right|^2 dx dt = \int_{\Omega} \left| \frac{t-t_{k-1}}{\tau} \nabla u_k + \left( 1 - \frac{t-t_{k-1}}{\tau} \right) \nabla u_{k-1} \right|^2 dx dt
\]
\[
\leq \frac{t-t_{k-1}}{\tau} \int_{\Omega} \left| \nabla u_k \right|^2 dx + \left( 1 - \frac{t-t_{k-1}}{\tau} \right) \int_{\Omega} \left| \nabla u_{k-1} \right|^2 dx
\]
\[
\leq \sup_{0 \leq \tau \leq T} \int_{\Omega} \left| \nabla \tilde{u}_j \right|^2 dx \leq c.
\]
The last step is due to Proposition 3.2. Now we integrate (3.10) over \( \Omega \) to get
\[
\frac{d}{dt} \int_{\Omega} \tilde{u}_j (x,t) dx + \tau \int_{\Omega} \tilde{w}_j (x,t) dx = 0.
\]
Integrate with respect to \( t \) and keep (3.12) in mind to deduce
\[
\int_{\Omega} \tilde{u}_j (x,t) dx \leq c.
\]
By Poincaré’s inequality, we have
\[
\int_{\Omega} \tilde{u}_j^2 (x,t) dx \leq 2 \int_{\Omega} \left( \tilde{u}_j (x,t) - \frac{1}{|\Omega|} \int_{\Omega} \tilde{u}_j (x,t) dx \right)^2 dx + \frac{2}{|\Omega|} \left( \int_{\Omega} \tilde{u}_j (x,t) dx \right)^2
\]
\[
\leq c \int_{\Omega} \left| \nabla \tilde{u}_j (x,t) \right|^2 dx + c \leq c.
\]
Now we can conclude that the sequence \( \{ \tilde{u}_j \} \) is bounded in \( W^{1,2}(\Omega_T) \). As such we may conclude that the sequence is precompact in \( L^2(\Omega_T) \). In the same manner, only using Propositions 3.2 and 3.3, we also have that the sequence \( \{ \tilde{w}_j \} \) is bounded in \( W^{1,2}(\Omega_T) \). The main difference is that instead of (3.36) we use the estimate

\[
\int_{\Omega_T} |\nabla \tilde{w}_j (x,t)|^2 \, dx \, dt = \sum_{k=1}^{j} \int_{t_k}^{t_{k-1}} \int_{\Omega} \left( \frac{t - t_{k-1}}{\tau} \nabla w_k + \left( 1 - \frac{t - t_{k-1}}{\tau} \right) \nabla w_{k-1} \right)^2 \, dx \, dt
\]

\[
\leq \sum_{k=1}^{j} \int_{t_k}^{t_{k-1}} \left[ \frac{t - t_{k-1}}{\tau} \int_{\Omega} |\nabla w_k|^2 \, dx + \left( 1 - \frac{t - t_{k-1}}{\tau} \right) \int_{\Omega} |\nabla w_{k-1}|^2 \, dx \right] \, dt
\]

\[
\leq \sum_{k=1}^{j} \tau \left[ \int_{\Omega} |\nabla w_k|^2 \, dx + \int_{\Omega} |\nabla w_{k-1}|^2 \, dx \right]
\]

\[
\leq c \int_{\Omega_T} |\nabla \tilde{w}_j|^2 \, dx \, dt + \tau c \int_{\Omega} |\nabla w_0|^2 \, dx \leq c
\]

and

\[
\left| \int_{\Omega_T} \tilde{w}_j (x,t) \, dx \right| \leq c
\]

is a consequence of (3.11). The proof is complete. \( \square \)

**Proposition 3.5.** The sequences \( \{ \tilde{u}_j \} \) and \( \{ \tilde{w}_j \} \) are precompact in \( L^2(\Omega_T) \).

**Proof.** To go about proving this proposition we calculate, for \( t \in (t_{k-1}, t_k) \),

\[
\tilde{u}_j (x,t) - \bar{u}_j (x,t) = \frac{t - t_k}{\tau} (u_k - u_{k-1})
\]

\[
= (t - t_k) \frac{\partial \tilde{u}_j}{\partial t}.
\]

Consequently,

\[
\int_{\Omega_T} (\tilde{u}_j - \bar{u}_j)^2 \, dx \, dt = \sum_{k=1}^{j} \int_{t_{k-1}}^{t_k} \left( t - t_k \right) \left( \frac{\partial \tilde{u}_j}{\partial t} \right)^2 \, dt \, dx
\]

\[
\leq \frac{\tau^2}{3} \int_{\Omega_T} \left( \frac{\partial \tilde{u}_j}{\partial t} \right)^2 \, dx \, dt \leq c \tau^2.
\]

This implies that the sequence \( \{ \tilde{u}_j \} \) is precompact in \( L^2(\Omega_T) \).

The estimate (3.40) also holds for \( (\tilde{w}_j - \bar{w}_j) \). Subsequently we may conclude that the sequence \( \{ \tilde{w}_j \} \) is precompact in \( L^2(\Omega_T) \). This completes the proof. \( \square \)

**Proposition 3.6.** The sequences \( \{ \sinh (\tilde{w}_j) \} \) and \( \{ \bar{w}_j \} \) are bounded in \( L^2(0,T;W^{2,2}(\Omega)) \).

**Proof.** We can easily infer from (3.11) and Lemma 2.1 that \( \{ \tilde{w}_j \} \) is bounded in \( L^2(0,T;W^{2,2}(\Omega)) \). To see the rest, we clearly have for \( s \geq 0 \) the following inequality,

\[
\sinh(s) \leq \cosh(s).
\]

Then since \( \sinh \) is odd and \( \cosh \) is even we are able to conclude that,

\[
|\sinh(s)| \leq \cosh(s).
\]

Then using Proposition 3.1, we get the estimate,

\[
\int_{\Omega} |\sinh(\tilde{w}_j)| \, dx \leq \int_{\Omega} \cosh(\tilde{w}_j) \, dx \leq c.
\]
We calculate from Lemmas 2.2 and 2.1 that
\[
\| \nabla \sinh (\bar{w}_j) \|_{2,\Omega} \leq \epsilon \| \nabla^2 \sinh (\bar{w}_j) \|_{2,\Omega} + c \| \sinh (\bar{w}_j) \|_{1,\Omega} \\
\leq \epsilon \| \Delta \sinh (\bar{w}_j) \|_{2,\Omega} + c \| \nabla \sinh (\bar{w}_j) \|_{2,\Omega} + c.
\]
Choose \( \epsilon \) suitably small to get
\[
\| \nabla \sinh (\bar{w}_j) \|_{2,\Omega} \leq c \| \Delta \sinh (\bar{w}_j) \|_{2,\Omega} + c.
\]
Square the above inequality and then integrate to get
\[
\int_{\Omega_T} |\nabla \sinh (\bar{w}_j)|^2 \, dx \, dt \leq c \int_{\Omega_T} |\Delta \sinh (\bar{w}_j)|^2 \, dx \, dt \leq c.
\]
The last step is due to Proposition 3.1. Use Lemma 2.2 again to get
\[
\int_{\Omega_T} |\sinh (\bar{w}_j)|^2 \, dx \, dt \leq c.
\]
Invoking Proposition 3.1 and Lemma 2.1 one more time, we can derive
\[
\int_{\Omega_T} |\nabla^2 \sinh (\bar{w}_j)|^2 \, dx \, dt \leq c \int_{\Omega_T} |\Delta \sinh (\bar{w}_j)|^2 \, dx \, dt + c \int_{\Omega_T} |\nabla \sinh (\bar{w}_j)|^2 \, dx \, dt \leq c.
\]
This finishes the proof. \( \square \)

We are now ready to prove our main theorem.

**Proof of Main Theorem.** Passing to subsequences if need be we may assume,
\[
\bar{u}_j \to u \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \quad \text{strongly in } L^2(\Omega_T), \quad \text{and a.e. on } \Omega_T, \\
\bar{w}_j \to w \quad \text{weakly in } L^2(0,T;W^{1,2}(\Omega)), \quad \text{strongly in } L^2(\Omega_T), \quad \text{and a.e. on } \Omega_T.
\]
By (3.40), we also have \( \tilde{u}_j \to u \) weakly in \( W^{1,2}(\Omega_T) \).

On account of Proposition 3.6 we obtain
\[
\sinh (\bar{w}_j) \to \sinh (w) \quad \text{weakly in } L^2(0,T;W^{2,2}(\Omega)).
\]
Thus we may pass to the limit in (3.10) and (3.11). The proof is complete. \( \square \)

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