AN ALGEBRAIC SATO-TATE GROUP AND SATO-TATE CONJECTURE

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Abstract. We make explicit a construction of Serre giving a definition of an algebraic Sato-Tate group associated to an abelian variety over a number field, which is conjecturally linked to the distribution of normalized $L$-factors as in the usual Sato-Tate conjecture for elliptic curves. The connected part of the algebraic Sato-Tate group is closely related to the Mumford-Tate group, but the group of components carries additional arithmetic information. We then check that in many cases where the Mumford-Tate group is completely determined by the endomorphisms of the abelian variety, the algebraic Sato-Tate group can also be described explicitly in terms of endomorphisms. In particular, we cover all abelian varieties (not necessarily absolutely simple) of dimension at most 3; this result figures prominently in the analysis of Sato-Tate groups for abelian surfaces given recently by Fité, Kedlaya, Rotger, and Sutherland.

1. Introduction

Let $F$ be a number field with absolute Galois group $G_F$. Let $A$ be an abelian variety of dimension $g \geq 1$ over $F$. Pick a prime number $l$ and consider the action of $G_F$ on the $l$-adic Tate module of $A$. A theorem of Weil implies that for each prime ideal $p$ of $F$ at which $A$ has good reduction, the characteristic polynomial $L(A/F,T)$ of any geometric Frobenius element of $G_F$ corresponding to $p$ is a monic polynomial with integer coefficients, whose roots in the complex numbers all have absolute value $q^{1/2}$ for $q$ the absolute norm of $p$.

One can ask how the renormalized characteristic polynomials $\overline{L}(A/F,T) = q^{-g}L(A/F,q^{1/2}T)$ are distributed in the limit as $q \to \infty$. For $A$ an elliptic curve without complex multiplication, the Sato-Tate conjecture predicts equidistribution with respect to the image of the Haar measure on SU(2); this result is now known when $F$ is a totally real field. For $A$ an elliptic curve with complex multiplication, the situation is easier to analyze: for all $F$, one can prove equidistribution for the image of the Haar measure on a certain subgroup of SU(2). This group may be taken to be SO(2) when $F$ contains the field of complex multiplication; otherwise, one must instead take the normalizer of SO(2) in SU(2).

With this behavior in mind, it is reasonable to formulate an analogous Sato-Tate conjecture for general $A$, in which the equidistribution is with respect to the image of the Haar measure on a suitable compact Lie group. Such a conjecture seems to
have been first formulated by Serre \cite{Se4} in the language of motives. More recently, Serre has given an alternate formulation in terms of \(l\)-adic Galois representations \cite[§8]{Se5}.

The purpose of this paper is to make explicit one aspect of Serre’s \(l\)-adic construction of the putative Sato-Tate group, namely the role of an algebraic group closely related to (but distinct from) the classical Mumford-Tate group associated to \(A\). This algebraic Sato-Tate group has the property that its connected part is conjecturally determined by the Mumford-Tate group; however, whereas the Mumford-Tate group is by construction connected, the algebraic Sato-Tate group has a component group with Galois-theoretic meaning. For example, in those cases where the Mumford-Tate group is determined entirely by the endomorphisms of \(A\) (e.g., the cases studied by the first author jointly with W. Gajda and P. Krason in \cite{BGK1, BGK2}), we can determine the algebraic Sato-Tate group and interpret its component group as the Galois group of the minimal extension of \(F\) over which all geometric endomorphisms are defined.

One key motivation for this work is to establish the properties of the algebraic Sato-Tate group for all abelian varieties of dimension at most 3. (Note that includes only cases where the Mumford-Tate group is determined by endomorphisms, as the first counterexamples are Mumford’s famous examples in dimension 4.) This result is applied by the second author in \cite{FKRS} in order to classify the possible Sato-Tate groups arising in dimension 2, and makes the corresponding classification in dimension 3 feasible as well.

2. The algebraic Sato-Tate group and conjecture

We begin with the definition of the algebraic Sato-Tate group and the formulation of an associated conjecture refining the standard Mumford-Tate conjecture.

**Definition 2.1.** Let \(A/F\) be an abelian variety over a number field \(F\). Choose an embedding \(F \subset \mathbb{C}\), then put \(V := V(A) := H_1(A(\mathbb{C}), \mathbb{Q})\) and \(V^* := \text{Hom}_\mathbb{Q}(V, \mathbb{Q})\). Put \(D := D(A) := \text{End}(A)^0 := \text{End}_F(A) \otimes \mathbb{Q}\).

We obtain a rational Hodge structure of weight 1 on \(V^*\) from the Hodge decomposition
\[
V_C^* := V^* \otimes_{\mathbb{Q}} \mathbb{C} = H^{1,0} \oplus H^{0,1},
\]
where \(H^{p,q} = H^p(A; \Omega^q_A/\mathbb{C})\) and \(\overline{H^{p,q}} = H^{q,p}\). Observe that \(H^{p,q}\) are invariant subspaces with respect to the action of \(D\) on \(V^* \otimes_{\mathbb{Q}} \mathbb{C}\), so the \(H^{p,q}\) are right \(D\)-modules. Put \(H^{-p,-q} := \text{Hom}_\mathbb{C}(H^{p,q}, \mathbb{C})\). The left action of \(D\) on \(H^{-p,-q}\) is defined via the right action of \(D\) on \(H^{p,q}\). Let
\[
\psi = \psi_V : V \times V \to \mathbb{Q}
\]
be the \(\mathbb{Q}\)-bilinear, nondegenerate, alternating form coming from the Riemann form of \(A\).

**Definition 2.2.** Put \(T_i := T_i(A)\) and \(V_i := V_i(A) := T_i(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l\). It is well known \cite[§24, p. 237]{Md} (cf. \cite[§16, p. 133]{Mi}) that \(V_i \cong V \otimes_{\mathbb{Q}} \mathbb{Q}_l\). The Galois group \(G_F\) acts naturally on \(V_i\) and because of the Weil pairing \(\psi_i : V_i \times V_i \to \mathbb{Q}_l\) we have the \(l\)-adic representation
\[
\rho_l : G_F \to GSp(V_i)
\]
induced by the natural representation \(\rho_l : G_F \to GSp(T_i)\).
Let $K/F$ be a finite extension. Let $G_{l,K}^\text{alg} \subseteq GSp_{(V_l,\psi_l)}$ be the Zariski closure of $\rho_l(G_K)$ in $GSp_{(V_l,\psi_l)}$. Define

$$\rho_l(G_K)_1 := \rho_l(G_K) \cap Sp_{(V_l,\psi_l)}.$$ 

Let $G_{l,K,1}^\text{alg} \subseteq Sp_{(V_l,\psi_l)}$ be the Zariski closure of $\rho_l(G_K)_1$ in $Sp_{(V_l,\psi_l)}$. The algebraic group $G_{l,K}^\text{alg}$ is reductive by [F, Theorem 3]. If $K = F$, we will put $G_{l,F}^\text{alg} := G_{l,K}^\text{alg}$ and $G_{l,1,F} := G_{l,K,1}^\text{alg}$.

On the basis of the work of Mumford, Serre, and Tate, it has been conjectured for about 50 years (see Conjecture 4.7) that there is a connected reductive group scheme $G \subseteq GSp_{(V,\psi)}$ such that for all $l \gg 0$,

$$(2.2) \quad G_{\mathbb{Q}l} = (G_{l,K}^\text{alg})^\circ,$$

where $(G_{l,K}^\text{alg})^\circ$ denotes the connected component of the identity. This implies that $G$ does not depend on $K$ (see §3). We propose the following refinement of this conjecture which does detect the field $K$.

**Conjecture 2.3. (Algebraic Sato-Tate conjecture)** There is an algebraic group $\text{AST}_{K}(A) \subseteq Sp_{(V,\psi)}$ over $\mathbb{Q}$ such that $\text{AST}_{K}(A)^\circ$ is reductive and for each $l$,

$$(2.3) \quad \text{AST}_{K}(A)_{\mathbb{Q}l} = G_{l,K,1}^\text{alg}.$$ 

**Definition 2.4.** The algebraic Sato-Tate group plays a key role in Serre’s definition of the putative Sato-Tate group $ST_{K}(A)$. If Conjecture 2.3 holds, then base extension of $\text{AST}_{K}(A)_{\mathbb{Q}l}$ along an embedding $\mathbb{Q}l \subset \mathbb{C}$ gives a group $\text{AST}_{K}(A)_{\mathbb{C}}$ that does not depend on $l$ or the embedding. Taking a maximal compact subgroup of $\text{AST}_{K}(A)_{\mathbb{C}}$, we obtain the Sato-Tate group $ST_{K}(A) \subseteq USp(V_{\mathbb{C}})$.

**Remark 2.5.** Besides Conjecture 2.3 and the usual Sato-Tate conjecture, we expect $\text{AST}_{K}(A)$ to have the following properties:

$$(2.4) \quad \text{AST}_{K}(A) \subseteq DL_{K}(A)$$

$$(2.5) \quad \text{AST}_{K}(A)^\circ \subseteq H(A).$$

Here $H(A)$ is the Hodge group and $DL_{K}(A)$ is the twisted decomposable Lefschetz group, to be defined in §5. If the Mumford-Tate conjecture also holds for $A$, we may further expect that

$$(2.6) \quad \text{AST}_{K}(A)^\circ = H(A)$$

$$(2.7) \quad \text{AST}_{K}(A)/\text{AST}_{K}(A)^\circ = ST_{K}(A)/ST_{K}(A)^\circ.$$ 

In §5, we exhibit a wide class of abelian varieties such that Conjecture 2.3 and conditions (2.4)–(2.7) hold (see Theorem 6.1).

**3. Some remarks on Zariski closure. A theorem of Serre.**

Before proceeding further, we collect some observations about the groups we have just defined.
Remark 3.1. We have the following commutative diagram with exact columns:

\[
\begin{array}{ccc}
\rho_l(G_K)_1 & \rightarrow & \mathfrak{G}_{l,K,1}^{\text{alg}} \\
\downarrow & & \downarrow \\
\rho_l(G_K) & \rightarrow & \mathfrak{G}_{l,K}^{\text{alg}} \\
\downarrow & & \downarrow \\
\mathbb{Q}_l^\times & \rightarrow & \mathbb{G}_m \\
\end{array}
\]

By the theorem of Bogomolov [Bo] on homotheties, \(\rho_l(G_K)\) is an open subgroup of \(\mathfrak{G}_{l,K}^{\text{alg}}\). Since \(\mathfrak{G}_{l,K}(V_{i,\psi_i})\) is closed in \(GSp(V_{i,\psi_i})\), \(\rho_l(G_K)_1 = \rho_l(G_K) \cap \mathfrak{G}_{l,K}(V_{i,\psi_i})\) is an open subgroup of \(\mathfrak{G}_{l,K}^{\text{alg}} \cap \mathfrak{G}_{l,K}(V_{i,\psi_i})\). It follows that

\[(3.1) \quad \mathfrak{G}_{l,K,1}^{\text{alg}} = \mathfrak{G}_{l,K}^{\text{alg}} \cap \mathfrak{G}_{l,K}(V_{i,\psi_i})\]

is closed in \(\mathfrak{G}_{l,K}^{\text{alg}}\), and that there is an exact sequence

\[(3.2) \quad 1 \rightarrow \mathfrak{G}_{l,K,1}^{\text{alg}} \rightarrow \mathfrak{G}_{l,K}^{\text{alg}} \rightarrow \mathbb{G}_m \rightarrow 1.\]

Since \(\mathfrak{G}_{l,K,1}^{\text{alg}}\) is the kernel of a homomorphism from a reductive group to a torus, it is also reductive.

Remark 3.2. Let \(F \subseteq K \subseteq L \subseteq \overline{F}\) be a chain of extensions such that \(K/F\) and \(L/K\) are finite. Let \(L' \subseteq \overline{F}\) be the Galois closure of \(L\) over \(K\). It is clear that \(\mathfrak{G}_{l,L'}^{\text{alg}} \subseteq \mathfrak{G}_{l,L}^{\text{alg}} \subseteq \mathfrak{G}_{l,K}^{\text{alg}}\) and \(\mathfrak{G}_{l,L',1}^{\text{alg}} \subseteq \mathfrak{G}_{l,L,1}^{\text{alg}} \subseteq \mathfrak{G}_{l,K,1}^{\text{alg}}\). On the other hand, there is a surjective homomorphism \(\rho_l(G_K)/\rho_l(G_{L'}) \rightarrow \mathfrak{G}_{l,K}^{\text{alg}}/\mathfrak{G}_{l,L'}^{\text{alg}}\), so \(\mathfrak{G}_{l,L'}^{\text{alg}}\) is a subgroup of \(\mathfrak{G}_{l,K}^{\text{alg}}\) of finite index, as then is \(\mathfrak{G}_{l,L}^{\text{alg}}\). In particular, \((\mathfrak{G}_{l,L}^{\text{alg}})^\circ = (\mathfrak{G}_{l,K}^{\text{alg}})^\circ\). By a similar argument, \((\mathfrak{G}_{l,L,1}^{\text{alg}})^\circ = (\mathfrak{G}_{l,K,1}^{\text{alg}})^\circ\).

Proposition 3.3. There is a finite Galois extension \(L_0/K\) such that \(\mathfrak{G}_{l,L_0}^{\text{alg}} = (\mathfrak{G}_{l,K}^{\text{alg}})^\circ\) and \(\mathfrak{G}_{l,L_0,1}^{\text{alg}} = (\mathfrak{G}_{l,K,1}^{\text{alg}})^\circ\).

Proof. Let \(Z\) be the set of \(\mathbb{Q}_l\)-points of the closed subscheme \((\mathfrak{G}_{l,K}^{\text{alg}} - (\mathfrak{G}_{l,K,1}^{\text{alg}})^\circ) \cup (\mathfrak{G}_{l,L,1}^{\text{alg}} - (\mathfrak{G}_{l,L,1}^{\text{alg}})^\circ)\) of \(GSp(V_{i,\psi_i})\). Since \(\rho_l\) is continuous, we can find a finite Galois extension \(L_0/K\) such that \(\rho_l(G_{L_0}) \cap Z = \emptyset\). Such an extension satisfies \(\mathfrak{G}_{l,L_0}^{\text{alg}} \subseteq (\mathfrak{G}_{l,K}^{\text{alg}})^\circ\) and \(\mathfrak{G}_{l,L_0,1}^{\text{alg}} \subseteq (\mathfrak{G}_{l,K,1}^{\text{alg}})^\circ\). Since we already have the reverse inclusions, we obtain the desired equalities. \(\square\)

The following theorem is a special case of a result of Serre [Se5] §8.3.4].

Theorem 3.4. The following map is an isomorphism:

\[
i_CC : \mathfrak{G}_{l,K,1}/(\mathfrak{G}_{l,K,1}^{\text{alg}})^\circ \cong \mathfrak{G}_{l,K}^{\text{alg}}/(\mathfrak{G}_{l,K}^{\text{alg}})^\circ.\]
Proof. Choose $L_0$ as in Proposition 3.3. Then the following commutative diagram has exact rows:

$$
\begin{array}{cccc}
1 & \rightarrow & G_{l,L_0}^{\text{alg}} & \rightarrow & G_{l,K}^{\text{alg}} & \rightarrow & G_{l,K,1}^{\text{alg}} & \rightarrow & (G_{l,K,1}^{\text{alg}})^\circ & \rightarrow & 1 \\
1 & \rightarrow & G_{l,L_0}^{\text{alg}} & \rightarrow & G_{l,K}^{\text{alg}} & \rightarrow & G_{l,K}^{\text{alg}} & \rightarrow & (G_{l,K}^{\text{alg}})^\circ & \rightarrow & 1 \\
1 & \rightarrow & \mathbb{G}_m & \rightarrow & \mathbb{G}_m & \rightarrow & 1 \\
1 & \rightarrow & 1 & \rightarrow & 1
\end{array}
$$

The first two columns are also exact by (3.2), so a diagram chase (as in the snake lemma) shows that the third column is also exact. □

Remark 3.5. Observe that the natural action by left translation

$$(3.3) \quad G_K \times G_{l,K}^{\text{alg}} / (G_{l,K}^{\text{alg}})^\circ \rightarrow G_{l,K}^{\text{alg}} / (G_{l,K}^{\text{alg}})^\circ$$

is continuous. By Theorem 3.4 we also have a continuous action

$$(3.4) \quad G_K \times G_{l,K,1}^{\text{alg}} / (G_{l,K,1}^{\text{alg}})^\circ \rightarrow G_{l,K,1}^{\text{alg}} / (G_{l,K,1}^{\text{alg}})^\circ.$$ 

Choose a suitable field embedding $\mathbb{Q}_l \rightarrow \mathbb{C}$ and put $G_{l,K,1 \mathbb{C}}^{\text{alg}} := G_{l,K,1}^{\text{alg}} \otimes \mathbb{Q}_l \mathbb{C}$. By Theorem 3.4 and [FKRS] Lemma 2.8, we have the following.

Proposition 3.6. Let $K/F$ be any finite extension such that $F \subseteq K \subset \overline{F}$. Then there are natural isomorphisms

$$(3.5) \quad G_{l,K,1}^{\text{alg}} / (G_{l,K,1}^{\text{alg}})^\circ \cong G_{l,K,1 \mathbb{C}}^{\text{alg}} / (G_{l,K,1 \mathbb{C}}^{\text{alg}})^\circ \cong \text{ST}_A / \text{ST}^\circ_A.$$ 

If the algebraic Sato-Tate conjecture (Conjecture 2.3) holds, then all of the groups in (3.5) are isomorphic to $\text{AST}_K(A) / \text{AST}_K(A)^\circ$.

Remark 3.7. Consider the continuous homomorphism

$$\epsilon_{l,K} : G_K \rightarrow G_{l,K}^{\text{alg}}(\mathbb{Q}_l).$$

Serre [Se3, Se2] proved that $\epsilon_{l,K}^{-1}((G_{l,K}^{\text{alg}})^\circ(\mathbb{Q}_l))$ is independent of $l$. Since $(G_{l,K}^{\text{alg}})^\circ$ is open in $G_{l,K}^{\text{alg}}$, this preimage equals $G_{L_0} = G(\overline{K}/L_0)$ for some finite Galois extension $L_0/K$. This theorem of Serre can also be seen from Propositions 3.3 and 3.6 and Theorem 3.4. In fact, $L_0/K$ is the minimal extension such that $G_{l,L_0}^{\text{alg}}$ and $G_{l,L_0,1}^{\text{alg}}$ are connected for all $l$. Let $H_{l,K,1} := H_{l,K}^{-1}(\rho_l(G_K))$. Put $K_1 := \overline{K}^{H_{l,K,1}}$. Observe that

$$\epsilon_{l,K}^{-1}((G_{l,K,1}^{\text{alg}})^\circ(\mathbb{Q}_l)) = \epsilon_{l,K}^{-1}(G_{l,L_0,1}^{\text{alg}}(\mathbb{Q}_l)) = \epsilon_{l,K}^{-1}((G_{l,L_0}^{\text{alg}} \cap \text{Sp}(V_{l,\psi})) / (\mathbb{Q}_l)) =$$

$$= \epsilon_{l,K}^{-1}(G_{l,L_0}^{\text{alg}}(\mathbb{Q}_l)) \cap \epsilon_{l,K}^{-1}(\text{Sp}(V_{l,\psi})) / (\mathbb{Q}_l)) = G_{L_0} \cap G_{K_1} = G_{L_0,1}.$$
4. Mumford-Tate group and Mumford-Tate conjecture

We next recall some standard definitions and results concerning the Mumford-Tate group. We continue to assume that $A$ is an abelian variety over a number field $F$ equipped with a fixed embedding into $\mathbb{C}$.

**Definition 4.1.** For $V = H_1(A, \mathbb{Q})$, define the cocharacter

$$\mu_{\infty,A} : \mathbb{G}_m(\mathbb{C}) \to GL(V_{\mathbb{C}})$$

such that for any $z \in \mathbb{C}^\times$, the automorphism $\mu_{\infty,A}(z)$ acts as multiplication by $z$ on $H^{-1,0}$ and as the identity on $H^{0,-1}$. Observe that by definition we have $\mu_{\infty,A}(\mathbb{C}^\times) \subset GSp(V,\psi)(\mathbb{C})$.

**Definition 4.2.** (Mumford-Tate and Hodge groups)

1. The **Mumford-Tate group** of $A/F$ is the smallest algebraic subgroup $MT(A) \subseteq GL_V$ over $\mathbb{Q}$ such that $MT(A)(\mathbb{C})$ contains $\mu_{\infty}(\mathbb{C})$.
2. The **decomposable Hodge group** is $DH(A) := MT(A) \cap SL_V$.
3. The **Hodge group** $H(A) := DH(A)^\circ$ is the connected component of the identity in $DH(A)$.

By definition, if $K/F$ is a field extension such that $K \subset F$, then $MT(A)$ and $H(A)$ for $A/F$ and for $A/K$ are the same. That is, we do not need to include $K$ in the notation.

**Remark 4.3.** Note that $MT(A)$ is a reductive subgroup of $GSp(V,\psi)$ [D, Prop. 3.6], $H(A) \subseteq Sp(V,\psi)$, and $MT(A) \subseteq CD(GSp(V,\psi))$. Hence

$$H(A) \subseteq CD(Sp(V,\psi)).$$

**Definition 4.4.** The algebraic group $L(A) = C_D^0(Sp(V,\psi))$ is called the **Lefschetz group** of $A$. By (4.1) and the connectedness of $H(A)$, we have $H(A) \subseteq L(A)$.

**Definition 4.5.** For a field extension $L/\mathbb{Q}$ put

$$MT(A)_L := MT(A) \otimes_{\mathbb{Q}} L, \quad DH(A)_L := DH(A) \otimes_{\mathbb{Q}} L,$$

$$H(A)_L := H(A) \otimes_{\mathbb{Q}} L, \quad L(A)_L := L(A) \otimes_{\mathbb{Q}} L.$$

**Theorem 4.6.** (Deligne [D, I, Prop. 6.2]) For any prime number $l$,

$$(G^{alg}_l)^\circ \subseteq MT(A)_{\mathbb{Q}_l}.$$  

**Conjecture 4.7.** (Mumford-Tate) For any prime number $l$,

$$(G^{alg}_l)^\circ = MT(A)_{\mathbb{Q}_l}.$$  

**Remark 4.8.** Observe that (4.2) is equivalent to the inclusion

$$(G^{alg}_l)^\circ \subseteq MT(A)_{\mathbb{Q}_l},$$

while the Mumford-Tate conjecture is equivalent to the equality

$$(G^{alg}_{l,K,1})^\circ = H(A)_{\mathbb{Q}_l}.$$  

This follows immediately from the following commutative diagram in which every column is exact and every horizontal arrow is a containment of corresponding group
schemes.

\[
\begin{array}{ccc}
1 & & 1 \\
\downarrow & & \downarrow \\
G_{l,K,1}^{\text{alg}} & \rightarrow & DH(A)_{\mathbb{Q}_l} & \rightarrow & S_{pV_1,\psi_1} \\
\downarrow & & \downarrow & & \downarrow \\
G_{l,K}^{\text{alg}} & \rightarrow & MT(A)_{\mathbb{Q}_l} & \rightarrow & GSp_{V_1,\psi_1} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{G}_m & \sim & \mathbb{G}_m & \sim & \mathbb{G}_m \\
1 & & 1 & & 1
\end{array}
\]

It is known that the Mumford-Tate and Hodge groups do not behave well in general with respect to products of abelian varieties [3] p. 316]. However one can prove the following theorem.

**Theorem 4.9.** The Mumford-Tate group of abelian varieties have the following properties.

1. An isogeny \( \phi : A_1 \rightarrow A_2 \) of abelian varieties induces isomorphisms \( MT(A_1) \cong MT(A_2) \) and \( H(A_1) \cong H(A_2) \).
2. If \( A \) is an abelian variety and \( A^* := \prod_{i=1}^s A \), then \( MT(A^*) \cong MT(A) \) and \( H(A^*) \cong H(A) \).

**Proof.** 1. This follows because an isogeny induces an isomorphism of associated polarised rational Hodge structures.

2. Observe that there is a natural isomorphism \( H^1(A^*(\mathbb{C}), \mathbb{Q}) \cong \bigoplus_{i=1}^s H^1(A(\mathbb{C}), \mathbb{Q}) \) or simply \( V(A^*) \cong V(A)^s \). So in this case, the Hodge structure on \( H^1(A(\mathbb{C}), \mathbb{Q}) \) is isomorphic to the direct sum of the Hodge structures on \( H^1(A(\mathbb{C}), \mathbb{Q}) \). Via this identification, we have \( \mu_{\infty,A^*}^s = \Delta(\mu_{\infty,A}) \) where \( \Delta : GSp(V_\mathbb{C}) \rightarrow GSp(V_\mathbb{C})^s \subseteq GL(V_\mathbb{C})^s \) is the diagonal homomorphism. So we can consider \( MT(A^*) \) as a subgroup of \( MT(A)^s := \prod_{i=1}^s MT(A) \). Consider the diagonal homomorphism \( \Delta : MT(A) \rightarrow MT(A)^s \). Since \( \mu_{\infty,A}^s(\mathbb{G}_m(\mathbb{C})) = \Delta(\mu_{\infty,A})((\mathbb{G}_m(\mathbb{C})) \subseteq \Delta(MT(A))(\mathbb{C})) \), we have \( MT(A^*) \subseteq \Delta(MT(A)) \cong MT(A) \). Since \( MT(A^*)(\mathbb{C}) \) contains \( \Delta(\mu_{\infty,A}^s(\mathbb{G}_m(\mathbb{C}))) \) then the image of \( MT(A^*) \) via projection \( \pi : MT(A)^s \rightarrow MT(A) \) on every factor contains the image of \( \mu_{\infty,A}(\mathbb{G}_m(\mathbb{C})) \). By the definition of \( MT(A) \), this means that the projection \( \pi \) must surject onto \( MT(A) \). On the other hand, since \( MT(A^*) \subseteq \Delta(MT(A)) \), it is clear that for \( \pi \) to be onto, we must have \( MT(A^*) = \Delta(MT(A)) \). To get the isomorphism of Hodge groups, we note that the isomorphism \( V(A^*) \cong V(A)^s \) gives an isomorphism \( SL_{V(A^*)} \cong SL_{V(A)^s} \). We thus have the following isomorphisms of algebraic groups over \( \mathbb{Q} \):

\[
MT(A^*) \cap SL_{V(A^*)} \cong \Delta(MT(A)) \cap SL_{V(A)^s} \cong \\
\cong \Delta(MT(A)) \cap (SL_{V(A)})^s \cong MT(A) \cap SL_{V(A)}.
\]

Taking connected components, we get the desired isomorphism of Hodge groups. \( \square \)
One can make a corresponding calculation also on the Galois side.

**Theorem 4.10.** We have the following results.

1. An isogeny \( \phi : A_1 \rightarrow A_2 \) of abelian varieties over a number field \( K \) induces isomorphisms \( G_{l,K}^{\text{alg}}(A_1) \cong G_{l,K}^{\text{alg}}(A_2) \) and \( G_{l,K,1}^{\text{alg}}(A_1) \cong G_{l,K,1}^{\text{alg}}(A_2) \).

2. If \( A \) is an abelian variety over a number field \( K \), then for any positive integer \( s \), \( G_{l,K}^{\text{alg}}(A^s) \cong G_{l,K}^{\text{alg}}(A) \) and \( G_{l,K,1}^{\text{alg}}(A^s) = G_{l,K,1}^{\text{alg}}(A) \).

**Proof.**

1. Observe that an isogeny \( \phi \) induces an isomorphism of Galois modules \( V_l(A_1) \cong V_l(A_2) \).

2. The natural isomorphism \( V_l(A^s) \cong V_l(A)^s \) of \( G_K \)-modules shows that \( \rho_l,A \cong \Delta \rho_l,A \), where \( \Delta \rho_l,A : G_K \rightarrow GSp(V_l(A))^s \) is the natural diagonal representation \( \Delta \rho_l,A = \text{diag}(\rho_l,A, \ldots, \rho_l,A) \).

**Corollary 4.11.** Let \( A/K \) be an abelian variety such that the Mumford-Tate conjecture holds for \( A \). Then the Mumford-Tate conjecture holds for \( A^s \) for any positive integer \( s \).

**Proof.** It follows from Theorems 4.9 and 4.10. □

**Remark 4.12.** Observe that if the Mumford-Tate conjecture holds for \( A \) and \( K \) is a finite extension of \( F \) for which \( G_{l,K}^{\text{alg}} \) is connected, then

\[
G_{l,K,1}^{\text{alg}}(A^s) = H(A^s)_{Q_l}.
\]

for any \( s \geq 1 \). Hence the algebraic Sato-Tate conjecture holds for \( A^s \) for any \( s \geq 1 \) with

\[
AST_K(A^s) = H(A^s).
\]

5. **Twisted Lefschetz groups and the algebraic Sato-Tate conjecture**

We next use the Lefschetz group to define an upper bound on the algebraic Sato-Tate group. In those cases where the Mumford-Tate group is determined entirely by endomorphisms, this allows us to deduce the algebraic Sato-Tate conjecture from the Mumford-Tate conjecture. We continue to take \( A \) to be an abelian variety over a number field \( F \) embedded in \( C \) and \( K \) to be a finite extension of \( F \).

**Definition 5.1.** Note that we have a continuous representation

\[
\rho_e : G_K \rightarrow Aut_Q(D).
\]

Put \( G_{L_e} := \text{Ker} \rho_e \), so that \( L_e/K \) is a finite Galois extension. For \( \tau \in G(L_e/K) \), define

\[
DL^K_L(A) := \{ g \in Sp_V : g\beta g^{-1} = \rho_e(\tau)(\beta) \ \forall \ \beta \in D \}.
\]

It is enough to impose the condition for \( \beta \) running over a \( Q \)-basis of \( D \); therefore, \( DL^K_L(A) \) is a closed subscheme of \( Sp_V \) for each \( \tau \).

**Definition 5.2.** Define the **twisted decomposable algebraic Lefschetz group** of \( A \) over \( K \) to be the closed algebraic subgroup of \( Sp_{V,\rho} \) given by

\[
DL^K(A) = \bigcup_{\tau \in G(L_e/K)} DL^K_L(A).
\]
For any subextension \( L/K \) of \( \overline{F}/K \) we have \( DL_L(A) \subseteq DL_K(A) \) and \( DL_L^{id}(A) = DL_L^{id}(A)K \). Hence the classical Lefschetz group of \( A \) can be written as:

\[
(5.4) \quad L(A) = DL_L^{id}(A)K = DL_K(A)K.
\]

and \( L(A) = DL_L^{id}(A)K = DL_L(A)K \), for any subextension \( L/K \) of \( \overline{F}/K \). In particular \( DL_L^{id}(A) = DL_L(A) = DL_F(A) = DL_F^{id}(A) \) and \( L(A) = DL_L(A)K = DL_F(A)K \).

**Theorem 5.3.** The twisted decomposable Lefschetz groups of abelian varieties have the following properties.

1. An isogeny \( \phi : A_1 \to A_2 \) of abelian varieties over \( K \) induces an isomorphism \( DL_K(A_1) \cong DL_K(A_2) \) for every \( \tau \in G(L_c/K) \), and consequently an isomorphism \( DL_K(A_1) \cong DL_K(A_2) \).
2. If \( A/K \) is an abelian variety, then \( DL_K(A) = DL_K(A)K \) for every \( \tau \in G(L_c/K) \).
3. If \( A = \prod_{i=1}^s A_i \) with \( A_i/K \) simple for all \( i \) and \( Hom_{\overline{F}}(A_i, A_j) = 0 \) for all \( i \neq j \), then \( DL_K(A) = \prod_{i=1}^s DL_K(A_i) \) for every \( \tau \in G(L_c/K) \).
4. If \( A = \prod_{i=1}^s A_i^{\tau_i} \) with \( A_i/K \) for all \( i \neq j \), then \( DL_K(A) = \prod_{i=1}^s DL_K(A_i) \) for every \( \tau \in G(L_c/K) \).
5. If \( A = \prod_{i=1}^s A_i^{\tau_i} \) with all \( A_i/K \) simple, pairwise nonisogenous then \( DL_K(A) = \prod_{i=1}^s DL_K(A_i) \) for every \( \tau \in G(L_c/K) \).

**Proof.**

1. An isogeny induces the isomorphism \( (V(A_1), \psi_{V(A_1)}) \cong (V(A_2), \psi_{V(A_2)}) \) of \( \mathbb{Q} \)-bilinear spaces. It also induces an isomorphism \( D(A_1) = \text{End}_{\overline{F}}(A_1) \otimes \mathbb{Z} \mathbb{Q} \cong D(A_2) = \text{End}_{\overline{F}}(A_2) \otimes \mathbb{Q} \) of \( \mathbb{Q}[G_K] \)-modules, and so the first claim follows.
2. There is a natural isomorphism:

\[
(V(A^s), \psi_{V(A^s)}) \cong (V(A), \psi_{V(A)}^s) := \bigoplus_{i=1}^s (V(A), \psi_{V(A_i)})
\]

and an isomorphism \( D(A^s) \cong M_s(D(A)) \) of \( \mathbb{Q} \)-algebras. Let \( \Delta \) be the homomorphism that maps \( Sp_{V(A), \psi_{V(A)}} \) onto \( \text{diag}(Sp_{V(A), \psi_{V(A_1)}}, \ldots, Sp_{V(A), \psi_{V(A_s})}) \subseteq Sp_{\bigotimes_{i=1}^s V(A_i)} \psi_{V(A_i)}) \), and \( \Delta \) is defined from the definition of the twisted decomposable Lefschetz group, we get \( DL_K(A^s) \cong \Delta(DL_K(A)) \cong DL_K(A) \).
3. The proof is similar to the proof of 2. under the observation that \( D(A) \cong \prod_{i=1}^s D(A_i) \) and \( (V(A), \psi_{V(A)}) \cong \bigoplus_{i=1}^s (V(A_i), \psi_{V(A_i)}) \).
4. This follows immediately from 2. and 3.
5. Immediate corollary from 4. \( \square \)

**Remark 5.4.** Theorem 5.3 remains true if we replace \( DL_K(B) \) with \( DL_K^o(B) \) for all abelian varieties \( B \) that appear in the theorem. Since \( L(B) = DL_K^o(B) \), then the Lefschetz group satisfies properties 1-5 of Theorem 5.3. The property 5. of this theorem in the case of the Lefschetz group was obtained previously by K. Murty [My Lemma 2.1], cf. [G Lemma B.70].

**Remark 5.5.** Observe that we have

\[
DL_K(A) := \{ g \in Sp_{V, \psi} : \exists \tau \in G_K \forall \beta \in \mathbb{D} \quad g \beta g^{-1} = \rho_c(\tau)(\beta) \}
\]
Changing quantifiers we get another group scheme

\[ \widetilde{DL}_K(A) := \{ g \in Sp(V, \psi) : \forall \beta \in D \exists \tau \in G_K \ g\beta g^{-1} = \rho_\tau(\beta) \} \]

Observe that \( DL_K(A) \subseteq \widetilde{DL}_K(A) \).

We now use the twisted Lefschetz group to obtain an upper bound on the algebraic Sato-Tate group.

**Remark 5.6.** Observe that (4.1) implies that

\[ DH(A) \subseteq DL_{id}^K(A) \subseteq DL_K(A). \]

On the other hand, for any \( P \in A(\overline{F}) \), any \( \beta \in End_F(A) \), and any \( \sigma \in G_K \), we have \( \sigma \beta \sigma^{-1}(P) = \sigma(\beta)(P) \). Hence \( \rho_\sigma(G_K)^1 \subseteq DL_K(A)(\overline{Q}) \). In particular,

\[ G_{t,K,1}^{alg} \subseteq DL_K(A)_{\overline{Q}}. \]

We will be particularly interested in cases where \( G_{t,K,1}^{alg} = DL_K(A)_{\overline{Q}} \). We next check that this condition does not depend on the field \( K \).

**Remark 5.7.** Let \( \tilde{\tau} \in G_K \) be a lift of \( \tau \in G(L_c/K) \). The coset \( \tilde{\tau} G_{L_c} \) does not depend on the lift. The Zariski closure of \( \rho_\tilde{\tau}(G_{L_c}) = \rho_\tau(G_{L_c}) \) in \( Sp_{V_l} \) is

\[ \rho_\tau(G_{L_c}) G_{t,L_c,1}^{alg} \subseteq DL_K^T(A)_{\overline{Q}}. \]

We observe that

\[ G_{t,K,1}^{alg} = \bigsqcup_{\tau \in G(L_c/K)} \rho_\tau G_{t,L_c,1}^{alg}. \]

Since \( DL_{id}^K(A) = DL_{L_c}(A) = DL_{\overline{A}}(A) \), we get

\[ G_{t,L_c,1}^{alg} \subseteq DL_{L_c}(A)_{\overline{Q}}. \]

Hence the following two equalities are equivalent:

\[ G_{t,L_c,1}^{alg} = DL_{L_c}(A)_{\overline{Q}}. \]

\[ G_{t,K,1}^{alg} = DL_K(A)_{\overline{Q}}. \]

6. **Some cases of the algebraic Sato-Tate conjecture**

In general, the containment \( H(A) \subseteq L(A) \) can be strict, which makes the Mumford-Tate conjecture a somewhat subtle problem; for instance, Mumford exhibited examples of simple abelian fourfolds for which \( H(A) \neq L(A) \). Fortunately, when \( A \) has a large endomorphism algebra as compared to its dimension, such pathologies do not occur: one has \( H(A) = L(A) \) and one can often show that the Mumford-Tate conjecture holds. When this happens, we will say that the Mumford-Tate conjecture for \( A \) is *explained by endomorphisms*.

The following theorem asserts that in cases where the Mumford-Tate conjecture is explained by endomorphisms and the twisted decomposable Lefschetz group over \( \overline{F} \) is connected, the algebraic Sato-Tate conjecture is in a sense also explained by endomorphisms.

**Theorem 6.1.** Assume that the following conditions hold.

1. We have \( H(A) = L(A) \).
2. We have \( (G_{t,K}^{alg})^o = MT(A)_{\overline{Q}} \).
3. The group $DL_{\mathbb{F}}(A)$ is connected.

Then (6.12) holds for every $l$. Consequently, the algebraic Sato-Tate conjecture (Conjecture 2.3) holds for $A/K$ with

$$AST_K(A) = DL_K(A).$$

**Proof.** It is enough to prove (6.11). By our assumptions and Remark 4.8 we get

$$(G^a_{l,L,n,1})^o = H(A)_{Q_l} = L(A)_{Q_l} = DL_{L_n}(A)_{Q_l}.\text{ Hence }DL_{L_n}(A)_{Q_l}\text{ is also connected for every }l\text{ and by (5.10) we obtain }\begin{cases} G^a_{l,L,n,1})^o = G^a_{l,L,n,1} \text{ for every }l. \end{cases}$$

**Remark 6.2.** Under assumptions of Theorem 6.1 the results of Theorems 4.9, 4.10 and 5.3 show that the algebraic Sato-Tate conjecture holds for $A/L$. Hence by Remark 4.8, we obtain (6.4), (6.3) gives

$$H(A) = L(A).$$

as desired.

**Remark 6.4.** Recall that $L(A) = DL_K(A)^o \triangleleft DL^{id}_K(A) \triangleleft DL_K(A)$. Consider the following epimorphism of groups:

$$(6.5) \quad DL_K(A)/L(A) \rightarrow DL_K(A)/DL^{id}_K(A) \cong G(L_c/K).$$

If $A$ satisfies the assumptions of Theorem 6.1 then the epimorphism (6.5) is an isomorphism. This gives an identification $G(L_c/K) \cong AST_K(A)/AST_K(A)^o$.

We now assemble some cases where the Mumford-Tate and algebraic Sato-Tate conjecture are explained by endomorphisms. We start with some cases with $A$ simple.

**Definition 6.5.** Put $g := \dim(A)$. We say $A$ is of CM type if $D$ contains a commutative $\mathbb{Q}$-subalgebra of dimension $2g$. If $A$ is simple, then $DL_{\mathbb{F}}(A)$ is a torus of dimension at most $g$; equality holds if $g \leq 3$ [R Example 3.7]. In this case, $L(A) = DL_{\mathbb{F}}(A)$.

**Theorem 6.6.** Let $A/F$ be an abelian variety of CM type. Then $A$ satisfies the conditions of Theorem 6.1, so the algebraic Sato-Tate conjecture holds with $AST_K(A) = DL_K(A)$. 

Lemma 6.8. Let $A/F$ be an absolutely simple abelian variety for which the endomorphism algebra $D$ is of type $I$, $II$, or $III$ in the Albert classification. Then $C_DSp(V,\psi)$ is connected.

Proof. Recall that $MT(A), (G_{l,K}^{alg})^\circ, (G_{l,F,1}^{alg})^\circ$ are independent of the field extension $K/F$. We may thus take $K$ large enough that $G_{l,K}^{alg}$ is connected. In this case, conditions 1 and 2 of Theorem 6.1 are established in [BGK1] (in the type I and II case) and [BGK2] (in the type III case). Condition 3 holds by Lemma 6.7.

One can also show that the Mumford-Tate conjecture and the algebraic Sato-Tate conjecture are explained by endomorphisms for all abelian varieties of dimension at most $3$. This result is the starting point of the analysis of the Sato-Tate conjecture for abelian surfaces given in [FKRS]; the corresponding analysis for threefolds has not yet been carried out.

Lemma 6.9. For $i = 1, 2$, let $A_i/F$ be an abelian variety satisfying the Mumford-Tate conjecture. Suppose that over $\overline{F}$, $A_1$ has no factors of type IV while $A_2$ either is of CM type or has no factors of type IV. Suppose in addition over $\overline{F}$, there is no nontrivial homomorphism from $A_1$ to $A_2$. Then $A = A_1 \times A_2$ also satisfies the Mumford-Tate conjecture and $H(A) = H(A_1) \times H(A_2)$.

Proof. Let $G, G_1, G_2$ be the groups $(G_{l,F,1}^{alg})^\circ$ associated to $A_1, A_2$. By hypothesis, we have $G_1 = H(A_1)_{Q_l}$ and $G_2 = H(A_2)_{Q_l}$. Since $A_1$ has no factors of type IV, $G_1$ is semisimple [11, Lemma 1.4]. Similarly, if $A_2$ has no factors of type IV, then $G_2$ is semisimple; if instead $A_2$ is of CM type, then $G_2$ is a torus.

There is a natural inclusion $G \subseteq G_1 \times G_2$ with the property that the induced projection maps $\pi_1 : G \to G_1$ and $\pi_2 : G \to G_2$ are surjective. Since $G$ is reductive, we can write $G$ as an almost direct product $G' \cdot T$ with $G'$ semisimple and $T$ a torus.

In case $A_2$ is of CM type, we argue following Imai [11]. In this case, $p(G') = p([G, G]) = [p(G), p(G)] = G_1$ because $G_1$ is semisimple, while $q(T) = q(G) = G_2$ because $G'$ has no nontrivial character. Hence $\dim(G) \geq \dim(G_1 \times G_2)$, forcing $G = G_1 \times G_2$. By the same reasoning, $H(A_1) = H(A_1) \times H(A_2)$.

In case $A_2$ has no factors of type IV, then an argument of Hazama [15, Proposition 1.8] shows that $H(A) = H(A_1) \times H(A_2)$. The same argument implies that if $G \neq G_1 \times G_2$, there is a nonzero $G$-equivariant homomorphism $V_1(A_1) \to V_1(A_2)$. However, the latter would imply the existence of a nonzero homomorphism $A_1 \to A_2$ of abelian varieties over $\overline{F}$, by virtue of Faltings’s proof of the Tate conjecture for abelian varieties over number fields [13].

In both cases, we conclude $G = H(A)_{Q_l} = H(A_1)_{Q_l} \times H(A_2)_{Q_l}$, as desired. □
Theorem 6.10. Let $A/F$ be an abelian variety of dimension at most 3. Then $A$ satisfies the conditions of Theorem 6.7, so the Mumford-Tate conjecture holds with $H(A) = L(A)$, and the algebraic Sato-Tate conjecture holds with $\text{AST}_{K}(A) = DL_{K}(A)$.

Proof. Since we are free to enlarge $F$, we may assume that all simple factors of $A$ are absolutely simple. We may invoke [MZ, Theorem 0.1] to establish condition 1 of Theorem 6.1 in all cases we are considering, so we focus on conditions 2 and 3.

We start with the case where $A$ is simple. If $A$ is a CM elliptic curve, it is straightforward to check condition 2 (or one can apply Theorem 6.6) and condition 3 holds because $D\ell_{\mathbb{Q}}(A)$ is a torus. If $A$ is a non-CM elliptic curve, condition 2 follows from Serre’s open image theorem or from Theorem 6.8 and condition 3 holds because $D\ell_{\mathbb{Q}}(A) = SL_{2}$. In the other cases, the dimension is prime, so we may deduce condition 2 from [Ch, Concluding Remark] and condition 3 from a theorem of Tankeev [MZ, Theorem 2.7].

We next consider the case where $A$ is not simple. By Theorem 6.8, condition 3 for $A$ reduces to the corresponding condition for each of the simple factors of $A$. We thus need only check condition 2. In case $A$ is isogenous to a product of elliptic curves, this is a result of Imai [I]. The only other possibility is that $A$ is isogenous to the product of an elliptic curve $E$ with a simple abelian surface $B$; this case may be treated using Lemma 6.9. □

Remark 6.11. As noted earlier, Theorem 6.10 cannot be extended to dimension 4, due in part to Mumford’s examples of absolutely simple abelian fourfolds for which the Hodge conjecture is not explained by endomorphisms. However, there are also some nonsimple examples with the same property. For a complete classification of Hodge groups for abelian varieties of dimension at most 5, see [MZ].

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