IPM strategies to a discrete switching predator-prey model induced by a mate-finding Allee effect

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ABSTRACT
This paper proposes a discrete switching predator-prey model with a mate-finding Allee effect, where also switches are guided by Allee effect. One of the strategies analysed is to use a chemical in order to prevent the pest outbreak when the pest population is free of Allee effect. In this paper, we first study analytically the dynamic behaviors of the two subsystems and the equilibria and their stability of the switched system. Then we provide numerical bifurcation analyses for the switched discrete system. These show that the switched discrete system may have very complex dynamics by 2-parameter bifurcation diagrams which divide the space into regions and study equilibria, and 1-dimensional bifurcation diagrams which reveal that the system has periodic, chaotic solutions, period doubling bifurcations and so on. Furthermore, we try to refer the key parameters and initial densities of both populations associated with pest outbreaks and study their biological implications.

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1. Introduction

The Allee effect [1, 2], which was first described by Allee in the 1930s, has been a relevant topic in ecology but has suffered from widespread confusion since no clear definition had been given before 1999. During this year Stephens et al. [29] defined it as: a positive relationship between any component of individual fitness and either numbers or density of conspecifics, and suggested that it is important to differentiate between component Allee effects and demographic Allee effects.

The formal definition of these two effects has been described by Fauvergue in 2003 [14]. The former one is a decrease in any component of fitness with decreasing population size or density. Especially, a decrease in the probability of a female mating with decreasing male population is therefore a mate-finding Allee effect. The latter one is a decrease in per capita growth rate with decreasing population size, and it can be divided into a weak and strong demographic Allee effect according to the sign of population growth: the weak effect's
Figure 1. EIL: the lowest population density that will cause economic damage. ET: population density at which control measures should be invoked to prevent an increasing pest population from reaching EIL.

growth remains positive even for the smallest populations while the strong effect’s growth may become negative when population size decreases below the Allee threshold. Allee effect plays an important role in the conservation of endangered and exploited species, detailed investigations relating Allee effect may be found in the paper [3–5, 8–10, 13, 14, 19–23, 29, 31, 32, 39].

Meanwhile, it is well known that the outbreak of pests often cause serious ecological and economical problems. In order to control pest more effectively, at the same time, environmental influences and human interventions are taken into account, the Integrated Pest Management (IPM) strategy has been proposed [28, 30, 35]. One of the important concepts in IPM strategy is the Threshold Policy Control (TPC) [25, 26, 33, 34], which maintains the pests’ density below the Economic Injury Level (EIL) [24] by releasing predators and spraying pesticides once the pests’ density reaches the Economic Threshold (ET) [6, 17], see Figure 1. The main purpose of IPM is to maintain the density of the pests below the EIL rather than seeking to eradicate them, and the suitable tactic will only be applied as the density of pests reaches the given ET, when it can minimize the damage of insecticides to non-target pests and to preserve the quality of the environment.

Based on the concept of Allee effect and IPM strategy, we can define the threshold policy control (TPC) related to the IPM and Allee effect in this paper as follows [33, 34]: integrated control (spraying pesticides) is suppressed when the pest population abundance is below a previously chosen threshold density ET, i.e. the pest population is subject to a mate-finding Allee effect; above the threshold, integrated control measures are applied. In order to maintain the density of pest population below the EIL, IPM tactics must be applied once its density reaches and exceeds the ET to avoid delay in response to the control measures. Thus, the ET is chosen as the threshold value to guide the switches of the systems in present work.
By employing the above threshold policy control (TPC), a switching system (or non-smooth Filippov system) has been proposed and studied in present work. Recently, a non-smooth dynamical theory has been developed to study switchings in science and engineering [12, 15], in ecological systems [27, 40], and in epidemiological dynamics [37, 38, 41]. The main goal of this work is to give people a better understanding of a mate-finding Allee effect in a discrete switching predator-prey model. In order to achieve this goal, the effects of switching system on the successful pest control are investigated, as well as the dynamic complexity of proposed switching predator-prey model and their biological implications related to pest control are addressed. The results indicate that a highly Allee effect could decrease the range of the dynamical behaviors of the system, and play a very important role in pest control, initial densities could impact on pest outbreaks. To our knowledge, no work has done for a discrete switching predator-prey model with a mate-finding Allee effect.

The rest of the paper is organized as follows: In the next section, a discrete switching predator-prey model with a mate-finding Allee effect is proposed. The qualitative analysis of two subsystems are given in Section 3. In Section 4, bifurcation analyses including 2-parameter bifurcation and 1-dimensional bifurcation diagrams are studied. In Section 5, we study the important and relevance of key parameters and initial values of both pest and nature enemy populations in pest outbreaks, analyse switching frequencies through numerical simulations and include some final considerations.

2. Model formulation

In 2011, Wang et al. [39] proposed a discrete ecosystem with a mate-finding Allee effect as follows:

\[
\begin{align*}
H_{t+1} &= H_t + rH_t(1 - H_t)(1 - e^{-\theta H_t}) - aH_tP_t, \\
P_{t+1} &= P_t + aP_t(H_t - P_t).
\end{align*}
\] (1)

The assumptions in model (1) are as follows:

\(H_t\) and \(P_t\) are the densities of prey (pest) and predator (natural-enemy) populations in generation \(t\); \(r\) and \(a\) are positive constants; \(r\) is the intrinsic growth rate of pest population; \(a\) is the conversion efficiency of pest to natural-enemy.

The term \(H_t + rH_t(1 - H_t)\) represents the growth rate of pest population in the absence of natural-enemy; \(aH_tP_t\) stands for the rate of decrease of pest due to natural-enemy; \(P_t + aP_t(H_t - P_t)\) represents the variation of natural-enemy density with respect to pest population.

The term \(1 - \exp(-\theta H_t)\) is for mate-finding Allee effect, which means that it is hard to find a mate at low population size or density. \(\theta\) can be defined as the Allee effect constant (see references [10, 31] for details).

The mate-finding Allee effect only takes place when the density of the pest population is below the threshold value \(E_T\) which is the critical point. In case the density is above the threshold value, the mate-finding Allee effect term should be omitted. So model (1) can be
rewritten as
\[ H_{t+1} = H_t + rH_t(1 - H_t) - aH_tP_t, \]
\[ P_{t+1} = P_t + aP_t(H_t - P_t). \]

Model (2) has been researched by Celik and Duman [5] in 2009, he studied the stability of equilibria.

In order to prevent the outbreak of disaster pests, a chemical control strategy is applied with a proportional killing rate \( q \) for pest population only when the density of pest population exceeds the threshold level \( ET \). This yields naturally the following control model with IPM strategy
\[ H_{t+1} = (1 - q)[H_t + rH_t(1 - H_t) - aH_tP_t], \]
\[ P_{t+1} = P_t + aP_t(H_t - P_t). \]

Consequently, combining model (3) with IPM when the density of pest population exceeds \( ET \) and model (1) without control measures when the density falls below \( ET \), the following switching model included by a mate-finding Allee effect is derived

\[
\begin{cases}
H_{t+1} = H_t + rH_t(1 - H_t)(1 - e^{-\theta H_t}) - aH_tP_t, & H_t < ET; \\
P_{t+1} = P_t + aP_t(H_t - P_t), & \\
H_{t+1} = (1 - q)[H_t + rH_t(1 - H_t) - aH_tP_t], & H_t \geq ET; \\
P_{t+1} = P_t + aP_t(H_t - P_t), &
\end{cases}
\]

Model (4) represents a dynamical system subject to a threshold policy, which is referred to as an on-off control or as a special and simple case of variable structure control in the literatures [33, 34].

3. Qualitative analysis for two subsystems

In this section, the existence and stability of positive equilibria of the two subsystems for model (4) are investigated, which will be useful for studying the dynamical behaviors of switching model (4). For convenience, we denote \( H(Z) = H_t - ET \) with vector \( Z = (H_t, P_t)^T \) and
\[
F_{S_1}(Z) = \left[H_t + rH_t(1 - H_t)(1 - e^{-\theta H_t}) - aH_tP_t, \quad P_t + aP_t(H_t - P_t)\right]^T, \\
F_{S_2}(Z) = \left[(1 - q)[H_t + rH_t(1 - H_t) - aH_tP_t], \quad P_t + aP_t(H_t - P_t)\right]^T.
\]

Then model (4) can be rewritten as the following switching system (or filippov system)
\[
\dot{Z}(t) = \begin{cases}
F_{S_1}(Z), & Z \in S_1, \\
F_{S_2}(Z), & Z \in S_2,
\end{cases}
\]

where
\[ S_1 = \{Z \in \mathbb{R}_+^2 | H(Z) < 0, H > 0, P > 0\}, \quad S_2 = \{Z \in \mathbb{R}_+^2 | H(Z) \geq 0, H > 0, P > 0\}. \]

From now on, we call switching system (4) defined in region \( S_1 \) (resp. \( S_2 \)) as system \( F_{S_1} \) (resp. \( F_{S_2} \)).
Figure 2. The two real branches $W(0, z)$ and $W(-1, z)$ of Lambert W function.

The following definitions on regular equilibria of switching system [11, 16, 27, 37, 38, 40], the Lambert W function [7, 36] and the switching frequencies are introduced in this section and will be used throughout this paper.

**Definition 3.1:** A point $Z_*$ is called a real equilibrium of switching system (5) if $F_{S_1}(Z_*) = Z_*, H(Z_*) < 0$, or $F_{S_2}(Z_*) = Z_*, H(Z_*) \geq 0$. $Z_*$ will be denoted by $E_{S_1}^r$ or $E_{S_2}^r$. Analogously, a point $Z_*$ is called a virtual equilibrium if $F_{S_1}(Z_*) = Z_*, H(Z_*) \geq 0$, or $F_{S_2}(Z_*) = Z_*, H(Z_*) < 0$. $Z_*$ will be denoted by $E_{S_1}^v$ or $E_{S_2}^v$. Both real and virtual equilibria are called regular equilibria.

**Definition 3.2:** The Lambert W function is a multivalued inverse of the function $z \mapsto ze^z$ satisfying

$$\text{Lambert } W(z) \cdot e^{\text{Lambert } W(z)} = z. \quad (6)$$

It follows from (6) that

$$\text{Lambert } W'(z) = \frac{\text{Lambert } W(z)}{z[1 + \text{Lambert } W(z)]}.$$ 

For simplicity, the inverse function of $ze^z$ on $(-\infty, -1]$ and $[-1, +\infty)$ will be denoted by $W(-1, z)$ and $W(0, z)$, respectively. Obviously, $W(-1, z)$ and $W(0, z)$ are the real branches of the Lambert W function. In particular, $W(0, z)$ is a monotonically increasing function on $[-\exp(-1), +\infty)$, while $W(-1, z)$ is a monotonically decreasing function on $[-\exp(-1), 0)$, see Figure 2 for detail.

**Definition 3.3:** In switching system (5), if $H_t \geq ET$ and $H(t + 1) < ET$ (or $H_t \leq ET$ and $H_{t+1} > ET$), then system (5) experiences one time switch and $t$ is called as switch-point. The interval generations between two switch-points is defined as switching frequency.
3.1. Stability of equilibria for system $FS_1$

If $H_t < ET$, then the qualitative behaviors of system (5) are determined by system $FS_1$. Obviously, system $FS_1$ has the trivial equilibrium $(0,0)$ and the natural-enemy free equilibrium $(1,0)$, which are not stable. Meanwhile, system $FS_1$ has the unique positive equilibrium $(H^*_S, P^*_S)$ which will be shown in the following theorem.

**Theorem 3.1:** If $a/r < \theta$, then system $FS_1$ presents a unique positive equilibrium $(H^*_S, P^*_S)$, which satisfies

$$H^*_S = P^*_S \quad \text{and} \quad r(1 - H^*_S)[1 - e^{-\theta H^*_S}] - aH^*_S = 0.$$ 

**Proof:** It follows from subsystem $FS_1$ that $(H^*_S, P^*_S)$ should satisfy

$$(1 - H^*_S)[1 - e^{-\theta H^*_S}] - \frac{aH^*_S}{r} = 0.$$ 

For convenience, we denote

$$f(x) \triangleq (1 - x)[1 - e^{-\theta x}] - \frac{ax}{r}.$$ 

Taking derivative of $f(x)$ with respect to $x$, one yields

$$f'(x) = [1 + \theta (1 - x)]e^{-\theta x} - \frac{r + a}{r} = 0.$$ 

Rearranging the above equation we have

$$(-\theta x + 1 + \theta)e^{-\theta x + 1 + \theta} = \frac{r + a}{r}e^{1+\theta}. \quad (7)$$

It is easy to see that

$$\frac{r + a}{r}e^{1+\theta} > e,$$

and according to Definition 3.2, solving Equation (7) with respect to $x$, one yields a unique positive root

$$x^* = \frac{(1 + \theta) - W\left[0, \frac{r + a}{r}e^{1+\theta}\right]}{\theta}.$$ 

On the one hand, by a simple calculation, it gives that (i) if $x < x^*$, then $f'(x) > 0$; (ii) if $x > x^*$, then $f'(x) < 0$. Those show that $x^*$ is the local maximum point of $f(x)$, and since $a/r < \theta$, we can apply the monotonicity property of the Lambert W function which yields

$$W\left[0, \frac{r + a}{r}e^{1+\theta}\right] < W\left[0, (1 + \theta)e^{1+\theta}\right] = (1 + \theta),$$

thus $x^* > 0$. 

On the other hand, according to Definition 3.2, it can be shown that \( W[0, ((r + a)/r)e^{1+\theta}] > 1 \), so we have

\[
-\theta x^* + 1 + \theta = W[0, \frac{r + a}{r}e^{1+\theta}] > 1,
\]

i.e. the equality \( x^* < 1 \) holds true. Thus we can derive that \( 0 < x^* < 1 \). Since \( f(x) \) satisfies the following properties: \( f(x) \in C(-\infty, +\infty), f(0) = 0, f(1) < 0, f(x) \to -\infty \) as \( x \to -\infty \), it is trivial to see that the point \( x^* \) is a unique extreme point of \( f(x) \) and \( f(x^*) > 0 \).

Therefore, combining the following inequalities: \( f(x^*) > 0, f(1) < 0 \) with \( f'(x) < 0, x \in (x^*, +\infty) \), it is easy to show that there is always a unique positive root on the interval \( (x^*, 1) \) provided that \( a/r < \theta \). The proof is completed. \( \square \)

Now we are in a position to investigate the local stability of the unique positive equilibrium \((H^*_S, P^*_S)\) of system \( F_{S_1} \).

**Theorem 3.2:** If the following inequalities

\[
\max \left\{ -aH^*_S, aH^*_S + 2F_1(H^*_S) - 2 \right\} < a^2H^*_S - aH^*_S F_1(H^*_S) < 1 - F_1(H^*_S)
\]

hold, the positive equilibrium point \((H^*_S, P^*_S)\) of system \( F_{S_1} \) is asymptotically stable, and

\[
F_1(H^*_S) = 1 + (r - 2rH^*_S)(1 - e^{-\theta H^*_S}) + rH^*_S(1 - H^*_S) \theta e^{-\theta H^*_S} - aH^*_S.
\]

The proof of Theorem 3.2 is similar to that shown in [39] and we omit it.

**3.2. Stability of equilibria for system \( F_{S_2} \)**

If \( H_t \geq ET \), then the qualitative behaviors of system (5) are determined by system \( F_{S_2} \). Obviously, system \( F_{S_2} \) has a trivial equilibrium \((0, 0)\), which is unstable. In the case in which the following inequality holds:

\[
q < \frac{r}{1 + r},
\]

system \( F_{S_2} \) has a unique positive equilibrium \((H^*_S, P^*_S)\), where

\[
H^*_S = P^*_S = \frac{r - (1 + r)q}{(r + a)(1 - q)}.
\]

In the following theorem, the local stability of the positive equilibrium point \((H^*_S, P^*_S)\) is proved.

**Theorem 3.3:** The unique positive equilibrium point \((H^*_S, P^*_S)\) of system \( F_{S_2} \) is asymptotically stable provided that

\[
\max \left\{ -1 + \text{tr} J, -1 - \text{tr} J \right\} < \det J < 1,
\]

where

\[
J \triangleq \begin{pmatrix}
(1 - q)[1 + r - (2r + a)H^*_S] & a(q - 1)H^*_S \\
\frac{aH^*_S}{1 - aH^*_S} & 1 - aH^*_S
\end{pmatrix}.
\]
**Proof:** Linearizing system $F_{S_2}$ about $(H_{S_2}^*, P_{S_2}^*)$, then we can obtain the characteristic equation about the Jacobian Matrix $\mathcal{J}$ of system $F_{S_2}$

$$\mathcal{P}(\lambda) = \lambda^2 - (\text{tr}\mathcal{J})\lambda + \det\mathcal{J}.$$  \hspace{1cm} (10)

The conditions (8) ensure that all the inequalities

$$\mathcal{P}(1) > 0, \quad \mathcal{P}(-1) > 0, \quad \det\mathcal{J} < 1$$

hold, that is, the modulus of all roots of Equation (10) are less than 1. By employing the Jury Criteria [18], we can derive that the equilibrium point $(H_{S_2}^*, P_{S_2}^*)$ of system $F_{S_2}$ is local asymptotically stable. The proof is completed.

\[\blacksquare\]

**4. Complex dynamical behaviors analysis**

In this section, we investigate the complex dynamical behaviors of switching system (5). Differently from models (1) and (3), the switching system (5) describes the threshold control policy and the mate-finding Allee effect in a much more complex way. The system (5) is a complicated nonlinear dynamical system which is difficult to analyse theoretically so we will focus on the bifurcation phenomena through numerical simulations.

**4.1. Equilibria bifurcation for the switching system (eqn5)**

According to Definition 3.1, the switching system has different types of equilibria [11, 16], and these equilibria play a key role in pest control. Especially, Tang et al. [27] studied pest control with economic threshold. It is difficult to find closed forms for all the interior equilibrium of subsystem $F_{S_1}$ because of the complexity of Allee effect (the nonlinear term), thus the existence of regular equilibria (real and virtual equilibria) and their coexistence will be given by numerical simulation.

We choose $r$ and $ET$ as bifurcation parameters and fix all others as given in Figure 3. In our simulations, ET varies from 0.01 to 0.42 and $r$ varies from 0.2 to 0.85, the parameter space is divided into six regions as shown in Figure 3. In these parameter regions, the existences of regular equilibria depend on the value of $r$ and ET. For example, when the intrinsic growth rate $r$ is relatively small (here $r \in (0.2, 0.28)$), the space can be divided into two regions: I-1 (yellow) and region I-2 (green). In region I-1 only $E_{S_1}^v$ exists, while in region I-2 only $E_{S_1}^r$ exists. However, when the intrinsic growth rate $r$ increases and exceeds a certain threshold value $r = 0.28$, that is, when $r \in (0.28, 0.85)$, the results show that the parameter space has been divided into four regions which are region II-1 (red), II-2 (blue), II-3 (cyan), II-4 (magenta). In region II-1, $E_{S_1}^v$ and $E_{S_2}^v$ coexist; In region II-2, $E_{S_1}^v$ and $E_{S_2}^r$ coexist; In region II-3, $E_{S_1}^r$ and $E_{S_2}^v$ coexist; In region II-4, $E_{S_1}^r$ and $E_{S_2}^r$ coexist.

Obviously, a number of bifurcations occur in Figure 3 as the parameters change. One of the main purposes here is to design optimal control strategies to prevent pest outbreaks or keep the density of the pest population below ET. From a mathematical viewpoint, this can be realized if system can stabilize at the desired level through the integration of different control strategies. To realize this purpose, we can choose the desirable threshold level such that all equilibria of subsystem $F_{S_2}$ become virtual. Thus, in order to prevent the pest outbreak, parameters $r$ and ET should be carefully chosen to keep the interior equilibria of two subsystems in regions I-1, I-2, II-1 and II-2.
4.2. Bifurcation analysis about sensitive parameters

1-dimensional bifurcation analysis is a traditional approach to gain preliminary insight into the properties of a dynamic system, it provides information about the dependence of the dynamics on a certain parameter space. The analysis is expected to reveal the types of attractors and their changes with parameter variations.

To investigate the complex dynamics that system (5) can have, we first choose the intrinsic growth rate $r$ as the bifurcation parameter and fix all other parameters as those in Figure 4. It follows from Figure 4 that system (5) has more complex and interesting dynamic behaviors. That is, when $r$ increases from 2.18 to 2.7, we can find periodic, chaotic solutions, period doubling, multi-stability, crises and so on. As the parameter $r$ further increases from 2.18 to 2.19, system (5) has a stable solution, see Figure 5[A] for detail when $r = 1.9$. When $r$ further increases from 2.19 to 2.2066, system (5) has a periodic solution, see Figure 5[B] for details. However, a chaotic solution emerges abruptly when $r$ becomes larger and reaches 2.4 or 2.65, see Figures 5[C,D] for phase plan.

Furthermore, the complex and interesting dynamic behaviors of system (5) can also be shown respectively through Figures 6 and 7, where we choose the killing rate $q$ and the Allee effect constant $\theta$ as the bifurcation parameter. Comparing Figure 6[A] with Figures 6[B] or 6[C], the chaos may be removed if the pests are subject to a highly effective Allee effect. Meanwhile, the local stability of the equilibrium point increases with a highly effective Allee effect, see Figures 6[B,C] for detail. Furthermore, Figure 7[A] is proof of the fact that dynamical behaviors are very complicated since there may be several hidden factors that can adversely affect our control strategy. However, if we take some control measures to reduce the growth rate of pests, the dynamics of model (5) will become more clear as shown in Figure 7[B], and this will help to prevent pest outbreaks. Therefore, Figures 6 and 7 indicate that a highly effective Allee effect could decrease the complexity of the dynamical behaviors of the system and play a very important role in pest control.
Figure 4. Bifurcation diagram for system (5) with respect to $r$. All other parameters as follows: $a = 2, \theta = 4, q = 0.05, ET = 0.45$ and $(H_0, P_0) = (0.5, 0.4)$.

Figure 5. Phase-plan of system (5) with different $r$. [A] $r = 2.18$; [B] $r = 2.213$; [C] $r = 2.4$; [D] $r = 2.65$. The other parameters are identical to those in Figure 4.
Figure 6. Bifurcation diagram for system (5) with respect to $q$. All other parameters as follows: $a = 1.68$, $ET = 0.72$, $r = 2.58$, $(H_0, P_0) = (0.1, 0.1)$, and [A] $\theta = 9.5$; [B] $\theta = 5$; [C] $\theta = 1$.

Figure 7. Bifurcation diagram for system (5) with respect to $\theta$. All other parameters as follows: $a = 2$, $q = 0.8$, $ET = 0.8$, $r = 2.29$, $(H_0, P_0) = (0.3, 0.2)$, and [A] $r = 2.13$; [B] $r = 2$.

5. Initial sensitivities and switching effects

It is well known that the dynamics can change given different initial densities of both pest and natural-enemy populations. This section focuses on how the initial densities of pest and natural enemy populations affect the control strategies. In addition, we will discuss the effects of key parameters on the switching frequencies of system (5).
Figure 8. Switching effect of system (5) under different initial densities. Parameters are $a = 2, r = 1.9, \theta = 2, q = 0.1, ET = 0.65$.

5.1. Initial sensitivities

Firstly, in order to investigate the interaction between the initial densities of populations and pest control, Figure 8 illustrates the switch effects of initial densities on IPM strategy. In Figure 8[A] the initial densities are $(0.3, 0.4)$ and the simulation result indicates that the pest density never reaches the given $ET = 0.65$, which shows the solution initiating from $(0.3, 0.4)$ is free from IPM measures. If we set the initial densities as $(0.6, 0.1)$ or $(0.5, 0.73)$, the results indicate that the switching system (5) is free from IPM control after 1 or 2 applications of the IPM strategies, see Figures 8[B] and 8[C]. However, when the initial densities are $(0.68, 0.7)$, in order to control the pest population below ET, it is necessary to apply IPM measures several times, see Figure 8[D] for detail.

In addition, we discuss the interaction between initial densities and pest outbreak studying the phase-plane in Figure 9. This plot is separated in five different regions denoted by I (blue), II (magenta), III (green), IV (red), V (yellow). In region I, the pest population never outbreaks and is always stabilized in subsystem $F_{S1}$; In regions II, III and IV, pest density is below the given threshold ET provided that IPM control measures are applied 1, 2 and 3 times, respectively; In region V, the switching system (5) could experience the pest control intervention several times. To sum up, these results indicate that different initial densities may result in different final states of the pest.

Furthermore, the initial densities could also affect the multiple attractors. According to the bifurcation analyses shown in Figures 4, 6 and 7, the multiple attractors can coexist for a wide range of parameters. To confirm this and discuss their biological implications, we fix all parameters as those in Figure 10 and choose different initial densities. In particular, two different pest-outbreak attractors coexist at $r = 2.3$, as shown in Figure 10, which display different amplitudes and frequencies. If we let the initial values be $(H_0, P_0) = (0.6, 0.4)$, then the solution of system (5) approaches the periodic attractor shown in Figure 10[A].
Figure 9. Pest outbreak frequency depends on initial density \((H_0, P_0)\) of system (5). The parameters are fixed as \(a = 1.4, \theta = 6, q = 0.05, ET = 0.65, r = 2.2\).

Figure 10. The coexisting attractors of system (5) with different initial values. Parameters are \(a = 2, \theta = 4, q = 0.05, ET = 0.45, r = 2.3\), and [A] \((H_0, P_0) = (0.6, 0.4)\); [B] \((H_0, P_0) = (0.1, 0.1)\).

On the other hand, we choose the initial value \((H_0, P_0) = (0.1, 0.1)\), the outbreak patterns for pest population become quite complex and a new, unexpected attractor can be seen, as shown in Figure 10[B].

In order to better analyse the initial sensitivities in the examples of Figure 10, in order to illustrate the corresponding initial sensitivities more specifically, basins of attraction with respect to two different pest-outbreak are shown in Figure 11 where the white and black
Figure 11. Basin of attraction of two attractors shown in Fig. 10 with \( H \in [0.3, 0.67] \) and \( P \in [0.2, 0.8] \). The white and black points are attracted to the attractors shown in Figure 10 from left to right.

Figure 12. Attractors’ switch-like behavior of system (5) with \( r_t = r + \sigma u \) has random perturbation as each 90 generations. Parameters are: \( a = 2, \theta = 2, q = 0.3, ET = 0.5, r = 2.5, \sigma = 1 \) and \((H_0, P_0) = (0.5, 0.1)\).

points are attracted to the attractor shown in Figures 10[A,B]. We can also see that the line \( ET = 0.45 \) separates the attraction regions into two parts. The final stable states of pest and natural-enemy depend on their initial densities, and those results are confirmed by
the basins of attraction of initial densities. From a pest control viewpoint, the integrated control strategies may strictly depend on the initial densities of both populations since different attractors have different outbreak amplitudes and frequencies.

Therefore, numerical simulations indicate that the successful biological control depends on the initial densities of both pest and natural-enemy populations. This is because these initial densities can affect the outcome of pest populations and can help us design proper control strategies.

5.2. Switching frequency

In this subsection, the effects of the key parameters on the switching frequencies of system (5) are investigated.

Firstly, in order to understand how the different intrinsic growth rates of pest population affect pest outbreak, the switching system (5) is rewritten as

\[
\begin{align*}
H_{t+1} &= H_t + r_t H_t (1 - H_t) (1 - e^{-\theta H_t}) - a H_t P_t, \\
P_{t+1} &= P_t + a P_t (H_t - P_t), \\
H_{t+1} &= (1 - q_t) [H_t + r_t H_t (1 - H_t) - a H_t P_t], \\
P_{t+1} &= P_t + a P_t (H_t - P_t),
\end{align*}
\]

where \( r_t = r + \sigma u \) is random perturbation of \( q_t \), \( u \) is uniformly distributed variable on \([-1, 1]\) and \( \sigma > 0 \) represents the intensity of noise. What we want to address in this section is how the intensity of noise affects the pest-outbreak amplitudes and whether the stable attractors switch from one attractor to another or not.

To show this, we fix \((H_0, P_0) = (0.5, 0.1)\) and the other parameters as in Figure 12. If we randomly perturb the releasing constant \( r \) every 90 generations with an intensity \( \sigma = 1 \), then the attractors switch-like behavior occurs. In Figure 12[A], within the first 90 generations, the first stable attractor has a really small amplitude and the pests’ densities are below \( ET \). Once the random perturbation occurs at the 90th generation, system (11) quickly switches to a stable pest-outbreak attractor with medium amplitude. At the 180th generation, on the other hand the third pest-outbreak attractor can switch to the chaos attractor with large amplitude and in this case it is much more difficult to control the pest population.

Interesting to notice is also the fact that system (11) can yield completely different dynamics due to the randomness of the perturbation as it can be seen in Figure 12[B]. The amplitudes of of these three stable attractors decrease gradually and below \( ET = 0.5 \), i.e. the pest population is under control as desired.

Analogously, to address the effects of different killing rates on a switch-like behavior, we rewrite switching system (5) as

\[
\begin{align*}
H_{t+1} &= H_t + r_t H_t (1 - H_t) (1 - e^{-\theta H_t}) - a H_t P_t, \\
P_{t+1} &= P_t + a P_t (H_t - P_t), \\
H_{t+1} &= (1 - q_t) [H_t + r_t H_t (1 - H_t) - a H_t P_t], \\
P_{t+1} &= P_t + a P_t (H_t - P_t),
\end{align*}
\]

where \( q_t = q + \eta u \) is a random perturbation of \( q \), \( \eta > 0 \) represents the intensity of noise. By numerical simulations, we can see from Figure 13 that a similar switch-like behavior
Figure 13. Attractors’ switch-like behavior of system (5) with $q_t = q + \eta u$ which random perturbation every 90 generations. Parameters are: $a = 0.8$, $\theta = 5$, $q = 0.5$, $ET = 0.4$, $r = 1$, $\eta = 0.3$ and $(H_0, P_0) = (0.5, 0.4)$.

Figure 14. Switching frequency (S-F) and switching time (S-T) of system (5). Parameters are $a = 2$, $\theta = 2$, $q = 0.01$, $ET = 0.35$, $r = 2.1$. The initial densities from top to bottom are $(0.3, 0.4)$, $(0.2, 0.6)$ and $(0.7, 0.6)$.

can occur once $q_t$ is randomly perturbed at every 90 generations with relatively large intensity (i.e. $\eta > 0.1$). Both Figures 12 and 13 indicate that one solution can switch to another attractor with small amplitude at a random time when a small changes are introduced the parameters. That is, the perturbation of parameter can affect the pest outbreak and bring
some challenges for pest control, it is essential to consider this factor as we design control strategies and make management decisions.

The perturbation of the parameter can bring some challenges for pest control. Furthermore, we investigate the effects of initial densities on switching frequencies. Figure 14[A–C] shows the switching frequencies of system (5) with different initial densities, while Figure 14[D–F] show the relationship between the switching times and frequencies corresponding to Figure 14[A–C], respectively. In particular, the switching frequencies of Figure 14[E,F] are stable at 9, while the switching frequencies of Figure 14[D] fluctuate between 9 and 13. Moreover, it can be seen that the switching frequencies of Figure 14[F] converge three times quicker than Figure 14[E], since the system experiences 3 and 6 times switching in Figure 14[F] and 14[E], respectively. If the switchings occur frequently, then the control measures must be applied more times and this would mean that pesticide applications should be utilized frequently. Unfortunately, this is not cost effective and may lead to adverse effects such as environmental pollution problems. Therefore, the state of the switching frequencies with respect to the switching times depends on the initial values, which can help us to design suitable control measures for pest control.

6. Discussion and biological conclusions

In this contribution, we have deduced a discrete switching predator-prey model with a mate-finding Allee effect where switch strategies and control measures are guided by an economic threshold. The overall objective of this paper is to investigate the long-term dynamic behavior of the discrete switching predator-prey model (5). In order to work towards this goal, many dynamical system elements have been analyzed in this paper including 2-parameter bifurcation diagrams, 1-dimensional bifurcation, initial sensitivity, multiple attractors and switching frequency. Our finding demonstrates that a highly effective Allee effect could decrease the complexity of the dynamical behaviors of the system and plays a very important role in pest control. In particular, the initial densities could greatly impact on pest outbreaks and IPM measures.

For these reasons, the existence and stability of equilibria of subsystems of model (5) have to be investigated. There may be some hidden factors that change dramatically the dynamical behavior of the switching system which increases in complexity making pest control decisions more challenging. To show this, we analyse the possible bifurcation behaviors that model (5) can exhibit, as can be seen in Figures 3, 4, 6 and 7. In Figure 3, the 2-parameter bifurcation diagrams reveal that the regions contain both regular and virtual equilibria. From a pest control viewpoint, it is necessary to design the desirable threshold level ET or appropriate control strategies such that all equilibria of subsystems $F_{S_1}$ and $F_{S_2}$ become virtual. That is, in order to prevent pest outbreak, parameters $r$ and ET should be carefully chosen to keep the internal equilibria of two subsystems $F_{S_1}$ and $F_{S_2}$ in region II-2 in Figure 3.

On the other hand, the 1-dimensional bifurcation diagrams (see Figures 4, 6 and 7 for details) which derive from the switching system (5) presents various dynamical behaviors such as periodic and chaotic solutions, multi-stability, periodic window, crises, period doubling and halving bifurcations etc. According to bifurcation diagrams, we can find that the routes to chaos are very complicated, which indicates that there are several hidden factors
that can adversely affect our control strategy. The increasing number of potential complexities predicted by the theory is a major challenge for pest control in practice. However, the results in Figure 7 indicate that the chaoticity may be removed if the pests are subject to a highly effective Allee effect, and the local stability of the equilibrium increases with a highly effective Allee effect since it could decrease the complexity of the dynamical behaviors of the system, and is a key factor in pest control.

Furthermore, the correlations between initial densities and pest control are investigated. It is shown in this paper that the initial densities of pest and natural-enemy populations can affect the outcome of classical biological control, and the final stable states of both populations depend on their initial densities, see Figures 8, 9, 10 and 11. These results are further confirmed by the switching frequencies since the eventually state of the switching frequencies with respect to the switching times depends on the initial values shown in Figure 14.

In our study, we have investigated the dynamical behavior of a discrete switching predator-prey model with a mate-finding Allee effect. To link the costs of developing and implementing controls to population dynamic modeling, it is necessary to consider resource limitation factors such as limited capacity for pesticides, costs and since such nonlinear factors can affect the outcome of pest control. These topics will be considered in further work in the future.

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Disclosure statement

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References

[1] W.C. Allee, A. Aggregations: A Study in General Sociology, Chicago, 1931.
[2] W.C. Allee, O. Park, A.E. Emerson, Principles of Animal Ecology, WB Saunders, Philadelphia, 1949.
[3] L. Assas, B. Dennis, S. Elaydi, E. Kwessi and G. Livadiotis, Stochastic modified Beverton-Holt model with Allee effect II: the Cushing-Henson conjecture, J. Differ. Equ. Appl. 22 (2016), pp. 164–176.
[4] D.S. Boukal and L. Berec, Single-species models of the Allee effect: extinction boundaries, sex ratios and mate encounters, J. Theor. Biol. 218 (2002), pp. 375–394.
[5] C. Celik and O. Duman, *Allee effect in a discrete-time predator-prey system*, Chaos Solitons Fractals, 40 (2009), pp. 1956–1962.

[6] H.C. Chiang, *General model of the economic threshold level of pest populations*, Plant Protect. Bull (1979).

[7] R.M. Corless, G.H. Gonnet, D.E.G. Hare and D.J. Jeffrey, *On the Lambert W function*, Adv. Comput. Math. 5 (1996), pp. 329–359.

[8] M.I.S. Costa and L. dos Anjos, *Multiple hydra effect in a predator-prey model with Allee effect and mutual interference in the predator*, Ecol. Model. 373 (2018), pp. 22–24.

[9] F. Courchamp, T. Clutton-Brock and B. Grenfell, *Inverse density dependence and the Allee effect*, Trends Ecol. Evol. 14 (1999), pp. 405–410.

[10] B. Dennis, *Allee effects: population growth, critical density, and the chance of extinction*, Nat. Resour. Model. 3 (1989), pp. 481–538.

[11] M. di Bernardo, C.J. Budd, A.R. Champneys, P. Kowalczyk, A.B. Nordmark, G.O. Tost and P.T. Piironen, *Bifurcations in nonsmooth dynamical systems*, SIAM Rev. 50 (2008), pp. 629–701.

[12] S.H. Doole and S.J. Hogan, *A piecewise linear suspension bridge model: nonlinear dynamics and orbit continuation*, Dynam. Stability Syst. 11 (1996), pp. 19–47.

[13] S. Elaydi, E Kwessi and G. Livadiotis, *Hierarchical competition models with the Allee effect III: multispecies*, J. Biol. Dyn. 12 (2018), pp. 271–287.

[14] X. Fauvergue, *A review of mate-finding Allee effects in insects: from individual behavior to population management*, Entomol. Exp. Appl. 146 (2013), pp. 79–92.

[15] M. Guardia, S.J. Hogan and T.M. Seara, *An analytical approach to codimension-2 sliding bifurcations in the dry-friction oscillator*, SIAM J. Appl. Dyn. Syst. 9 (2010), pp. 769–798.

[16] M. Guardia, T.M. Seara and M.A. Teixeira, *Generic bifurcations of low codimension of planar Filippov systems*, J. Differ. Equ. 250 (2011), pp. 1967–2023.

[17] J.C. Headley, *Defining the economic threshold*, in Pest Control Strategies for the Future, National Academy of Sciences, Washington, 1972, pp. 100–108.

[18] E.I. Jury, *Inners and Stability of Dynamic Systems*, Wiley, 1974.

[19] R.R.B. Kaul, A.M. Kramer, F.C. Dobbs and J.M. Drake, *Experimental demonstration of an Allee effect in microbial populations*, Biol. Lett. 12 (2016), pp. 20160070.

[20] H. Kokko and W.J. Sutherland, *Ecological traps in changing environments: ecological and evolutionary consequences of a behaviourally mediated Allee effect*, Evol. Ecol. Res. 3 (2001), pp. 603–610.

[21] M. Kuussaari, I. Saccheri, M. Camara and I. Hanski, *Allee effect and population dynamics in the Glanville fritillary butterfly*, Oikos 82 (1998), pp. 384–392.

[22] M.A. McCarthy, *The Allee effect, finding mates and theoretical models*, Ecol. Modell. 103 (1997), pp. 99–102.

[23] N. Min and M.X. Wang, *Dynamics of a diffusive prey-predator system with strong Allee effect growth rate and a protection zone for the prey*, Discrete Contin. Dyn. Syst. Ser. B 23 (2018), pp. 995–1004.

[24] L.P. Pedigo, S.H. Hutchins and L.G. Higley, *Economic injury levels in theory and practice*, Annu. Rev. Entomol. 31 (1986), pp. 341–368.

[25] W.J. Qin, X.W. Tan, M. Tosato and X.Z. Liu, *Threshold control strategy for a non-smooth Filippov ecosystem with group defense*, Appl. Math. Comput 362 (2019), pp. 124532. doi:10.1016/j.amc.2019.06.046

[26] W.J. Qin, X.W. Tan, X.T. Shi, J.H. Chen and X.Z. Liu Qin, *Dynamics and bifurcation analysis of a Filippov predator-prey ecosystem in a seasonally fluctuating environment*, Internat. J. Bifur. Chaos Appl. Sci. Engrg 29(2) (2019), pp. 1950020. doi:10.1142/S0218127419500202

[27] S.Y. Tang, J.H. Liang, Y.N. Xiao and R.A. Cheke, *Sliding bifurcation of filippov two stage pest control bedels with economic thresholds*, SIAM J. Appl. Math. 72 (2012), pp. 1061–1080.

[28] J.A. Stenberg, *A conceptual framework for integrated pest management*, Trends Plant Sci. 22 (2017), pp. 759–769.

[29] P.A. Stephens, W.J. Sutherland and R.P. Freckleton, *What is the Allee effect?* Oikos 87 (1999), pp. 185–190.
[30] V. Stern, R. Smith, R. Van den Bosch and K. Hagen, *The integrated control concept*, Hilgardia, 29 (1959), pp. 81–101.

[31] A.W. Stoner and M. Ray-Culp, *Evidence for Allee effects in an over-harvested marine gastropod: density-dependent mating and egg production*, Mar. Ecol. Prog. Ser. 202 (2000), pp. 297–302.

[32] G.Q. Sun, *Mathematical modeling of population dynamics with Allee effect*, Nonlinear Dyn. 85 (2016), pp. 1–12.

[33] V.I. Utkin, *Sliding Modes and Their Applications in Variable Structure Systems*, Mir Publishers, 1978.

[34] V.I. Utkin, *Sliding Modes in Control and Optimization*, Springer-Verlag, 1992.

[35] R. Van den Bosch, *The Pesticide Conspiracy*, University of California Press, 1989.

[36] J. Waldvogel, *The period in the Volterra-Lotka predator-prey model*, SIAM J. Numer. Anal. 20 (1983), pp. 1264–1272.

[37] A.L. Wang, Y.N. Xiao and R.A. Cheke, *Global dynamics of a piece-wise epidemic model with switching vaccination strategy*, Discrete Contin. Dyn. Syst. Ser. B 19 (2014), pp. 2915–2940.

[38] A.L. Wang, Y.N. Xiao and H.P. Zhu, *Dynamics of a Filippov epidemic model with limited hospital beds*, Math. Biosci. Eng. 15 (2018), pp. 739–764.

[39] W.X. Wang, Y.B. Zhang and C.Z. Liu, *Analysis of a discrete-time predator-prey system with Allee effect*, Ecol. Complexity. 8 (2011), pp. 81–85.

[40] C.C. Xiang, Z.Y. Xiang, S.Y. Tang and J.H. Wu, *Discrete switching host-parasitoid models with integrated pest control*, Internat. J. Bifur. Chaos Appl. Sci. Eng. 24 (2014), pp. 1450114.

[41] Y.N. Xiao, X.X. Xu and S.Y. Tang, *Sliding mode control of outbreaks of emerging infectious diseases*, Bull. Math. Biol. 74 (2012), pp. 2403–2422.