How Fabulous Is Fab 5 Cosmology?

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Extended gravity origins for cosmic acceleration can solve some fine tuning issues and have useful characteristics, but generally have little to say regarding the cosmological constant problem. Fab 5 gravity can be ghost free and stable, have attractor solutions in the past and future, and possess self tuning that solves the original cosmological constant problem. Here we show however it does not possess all these qualities at the same time. We also demonstrate that the self tuning is so powerful that it not only cancels the cosmological constant but also all other energy density, and we derive the scalings of its approach to a renormalized de Sitter cosmology. While this strong cancellation is bad for the late universe, it greatly eases early universe inflation.

I. INTRODUCTION

Accelerated expansion is a property of our universe of immense importance yet lacking a clear physical explanation. Adding an energy density with accompanying negative pressure involves a seemingly arbitrary scalar field potential (flat in the case of a cosmological constant) that is moreover unnatural, not just in initial conditions but in its reaction under quantum radiative corrections. This last aspect can be partly ameliorated by imposing, e.g., a shift symmetry on the field but much arbitrariness remains. One approach is to remove the potential entirely and deal with only the kinetic behavior of the field, as in k-essence [1–4].

Altering the gravitational theory is another approach. Within scalar-tensor theories one can again employ shift symmetric fields, and considerable work has been done on these Galileon theories [5–7]. None of these address why the vacuum energy does not generate a cosmological constant characteristic of the Planck energy scale or other early universe scale. One interesting development has been the identification of four unique self tuning terms in the action, called the Fab Four [8–10], that can dynamically cancel a high energy cosmological constant.

An early approach to combining several of these characteristics was the purely kinetic coupled gravity of [11], using what later became called the $L_{\text{John}}$ term of Fab Four, but this version was later found to be unhealthy due to having a ghost (negative kinetic energy state). By generalizing the coefficient of the coupling to a free constant, the ghost could be exorcised and this becomes the derivatively coupled Galileon model, studied in detail in [12]. Promoting the constant coefficient to a potential depending on the field value leads to the full $L_{\text{John}}$ term, and allowing it to depend further on the field kinetic term gives the $L_5$ term of Horndeski scalar-tensor gravity [6, 13, 14]. However, these terms by themselves can (and without a potential do) have a gradient instability in the early universe, rapidly violating homogeneity.

By taking the nonlinear generalization of purely kinetic coupled gravity, [15] showed that one could go beyond Fab Four for obtaining self tuning, introducing the name Fab 5 gravity, while avoiding a potential. Interestingly, despite the nonlinearity this theory added no new propagating degrees of freedom (on a homogeneous, isotropic spacetime) and hence avoided a ubiquitous ghost. Fab 5 gravity was shift symmetric and so avoided many naturalness problems with radiative corrections, could have an attractor behavior in the early universe and so not only removed an arbitrary potential but also fine tuning of initial conditions, could have an attractor behavior in the late universe giving cosmic acceleration and a de Sitter state, and could dynamically cancel a cosmological constant. It is also related to multifield Galileons [16].

While Fab 5 gravity possesses many interesting and desirable characteristics, not the least being cancellation of the high energy cosmological constant, [15] did not show that these characteristics existed simultaneously. In fact, we will find that some are exclusive. Section II investigates the health and naturalness of Fab 5 gravity. We present explicit demonstrations of self tuning, including in the presence of other energy-momentum components, in Sec. III. Conclusions about the overall state of the theory are discussed in Sec. IV.

II. SOUNDNESS AND NATURALNESS

Fab 5 gravity uses a derivative coupling of a scalar field to the Einstein tensor, turning the kinetic term into a disformal theory. (For more on disformal theories, see e.g. [17–23].) The field has no potential term so the theory is manifestly shift symmetric. Matter is minimally coupled. The derivatively coupled term is promoted to a nonlinear function, in a manner similar to how the Ricci scalar is promoted to a nonlinear function in $f(R)$ gravity. This can be viewed as creating a new, auxiliary scalar field, but due to the symmetries of the Einstein tensor in a homogeneous, isotropic spacetime the new field has no propagating degrees of freedom and preserves the second order nature of the field equations.
Explicitly, the action is

\[ S = \int d^4x \sqrt{-g} \left[ \frac{M_{\text{Pl}}^2}{2} R + c_1 X + f(C_2 X + \frac{G_{\mu\nu} \phi_\mu \phi_\nu}{M^2}) \right] + S_m(g_{\mu\nu}), \]

where \( X = (-1/2)g^{\mu\nu} \phi_\mu \phi_\nu \), \( G_{\mu\nu} \) is the Einstein tensor, and \( S_m \) is the action for the matter fields. The mass scales are the Planck scale \( M_{\text{Pl}} \) and the scalar field scale \( M \), which we normalize to the Hubble constant \( H_0 \) when using Fab 5 to give late universe cosmic acceleration.

We build on the characteristics of Fab 5 gravity discussed in [15]. A key ingredient is the nonlinear function \( f(\chi) \), where

\[ \chi = c_2 X + \frac{c_G}{M^2} G_{\mu\nu} \phi_\mu \phi_\nu, \]

\[ = \frac{c_2}{2} \tilde{H}^2 x^2 + 3 c_G \tilde{H}^4 x^2. \]

We define dimensionless variables \( \tilde{H} = H/H_0 \) and \( x = d(\phi/M_\text{Pl})/d \ln a \), and assume a Friedmann-Robertson-Walker cosmology. The major constraints on the health of the theory will arise at high redshift, when the \( c_1 \) term dominates over either the \( c_1 \) or \( c_2 \) terms (since it has an extra \( H^2 \) factor). Therefore our main conclusions are independent of whether we include the canonical kinetic term inside the nonlinear function or not, i.e. whether we set \( c_1 = 0 \) or \( c_2 = 0 \); for definiteness we set \( c_1 = 0 \), hence including the usual kinetic term in the function. This gives a disformal structure

\[ \chi = \left[ -\frac{c_2}{2} g^{\mu\nu} + \frac{c_G}{M^2} G^{\mu\nu} \right] \phi_\mu \phi_\nu. \]

A. General criteria

Although the theory is guaranteed by its structure to have second order field equations, and that the auxiliary scalar field \( \chi \) does not propagate and so has no ghost, we still must check that the main scalar field \( \phi \) is well behaved. In order to have a healthy theory we must require that it is free from ghosts and is stable against instabilities. General expressions for these conditions were given in [15]; in the early universe limit, where the effective energy density contribution \( \Omega_\chi \ll 1 \), these simplified to

\[ 3c_G \tilde{H}^2 (f_\chi + 2\chi f_{\chi \chi}) > 0 \]

\[ \frac{2\tilde{H} + 3H^2}{H^2} \frac{f_\chi}{f_\chi + 2\chi f_{\chi \chi}} > 0, \]

respectively.

Note that in the linear theory, where \( f(\chi) = \chi \) hence \( f_{\chi \chi} = 0 \), the stability condition is violated during the radiation domination epoch where \( \tilde{H} = -2H^2 \), as pointed out by [12]. This gradient instability (also known as Laplace instability) rules out the purely kinetic coupled gravity of [11]. Furthermore, it constrains derivatively coupled Galileons and many Fab Four gravity models to have the derivative coupling and \( L_{\text{John}} \) respectively to be unimportant aspects of the theory in the early universe, considerably weakening their utility.

Thus, the nonlinear promotion at the heart of Fab 5 is a key element guarding against instability and turning the theory healthy. The form of \( f(\chi) \) must be chosen to satisfy stability.

In addition to the absolute requirements of Eqs. (5) and (6) to have a healthy theory, there are some desiderata that allow it to thrive. We might ask that the model be reasonably natural, i.e. not fine tuned in its initial conditions. (Note that the theory has technical naturalness due to shift symmetry, i.e. once the initial conditions are imposed they will not obtain large quantum corrections. We do, however, have to choose \( M = H_0 \) in order for \( c_2, c_G \sim O(1) \) to give current cosmic acceleration, as for any theory in the literature.) Since one of the most interesting properties of Fab 5 gravity is its self tuning, we might ask that the model succeed in tuning a large early universe cosmological constant to zero. Finally we might ask that it accords with observations, having the standard progression of radiation dominated epoch, matter dominated epoch, and late time cosmic acceleration with a matter density contribution today \( \Omega_m \approx 0.3 \).

Note that a model that achieves all of these would be counted as wildly successful. Most cosmic acceleration models are fine tuned and few are technically natural (the cosmological constant itself being a notable failure). None can dynamically cancel a large cosmological constant. For example, if we only succeed in the two requirements (stable and no ghosts) and the last desideratum (agreement with observations) then we have done as well as \( f(R) \) theories, with the bonus of adding stability to quantum corrections and having an innate de Sitter attractor in the future.

B. Power law models

In the case of power law models, \( f(\chi) = A\chi^n \), the stability condition in the radiation dominated era requires

\[ n < 1/2, \]

as noted by [15]. However, in this era the effective dark energy density is on an attractor trajectory, scaling with expansion factor as

\[ \rho_\phi \sim a^{-2n/(2n-1)}, \]

and so it is phantom, growing with time. (We require \( n > 0 \), otherwise in the Minkowski vacuum \( f(0) \neq 0 \) and we have included an explicit cosmological constant.) In order for the dark energy not to overwhelm the standard radiation or matter eras this requires a fine tuning, one more extreme than for the cosmological constant (which corresponds to \( n = 0 \)).
Furthermore, the no ghost condition implies
\[
A (2n - 1) > 0,
\]
requiring \( A < 0 \). (We keep \( c_G > 0 \) otherwise at early times \( \chi < 0 \), causing problems for the necessarily noninteger \( n \).) The magnitude of the dark energy density at early times is
\[
\rho_\phi \approx (4\chi f_\chi - f) = A (4n - 1),
\]
and hence for \( 1/4 < n < 1/2 \) we have \( \rho_\phi < 0 \). This is potentially a good thing, since if \( \rho_\phi > 0 \) and phantom (as for \( 0 < n < 1/4 \)) the fine tuning is strong. Moreover, for self tuning one requires \( \rho_\phi < 0 \) in order to cancel a positive early universe cosmological constant. Finally, although its magnitude initially becomes more negative (being phantom), at later times it can turn around and become positive giving the usual cosmic acceleration.

Thus for power law models we will always have some fine tuning but may have the usual cosmic acceleration or may have self tuning. First consider the least promising case of \( 0 < n < 1/4 \). Here the density is positive and increases so we must start at a (fine tuned) low level in order not to violate radiation domination. Eventually the density is so strong that it dominates, but continues growing and does not approach a de Sitter attractor. Figure 1 shows a broken power law model that does go to a de Sitter state.

![Figure 1](image)

**FIG. 1.** Healthy models are forced to be phantom models at early times, so there is a fine tuning in the initial energy density of the scalar field. If we keep the energy density positive, then we require \( 0 < n < 1/4 \) for the early time behavior. Here we show a case with \( n = 0.2 \) at high redshift, which is fine tuned, but not much more than a cosmological constant.

Next, take \( 1/4 < n < 1/2 \). Here the negative dark energy density can self tune, canceling a large cosmological constant. The expansion approaches a de Sitter attractor given by \( H_{dS} = \sqrt{-c_2/(6c_G)} \). At this fixed point \( x \to \text{const} \), i.e. the field keeps rolling to dynamically cancel the other, positive energy densities, leaving behind a net small positive energy density. That is, Fab 5 turns a large early universe cosmological constant into a small late universe cosmological constant, but as discussed in Sec. III does not generally give complete radiation and matter dominated eras.

Within the restrictions of the power law form, we end up with no fully satisfactory cosmology, even without self tuning. It is a general property of the power law models that in order for them to be Laplace stable, they must also be phantom and hence more fine tuned than a cosmological constant. Therefore we now consider non-power law models.

### C. Non-power law models

Fab 5 gravity has early time attractor solutions during both radiation and matter domination, one of its positive aspects in ameliorating fine tuning. For radiation domination this gives [15]
\[
\chi \sim a^{-2/(1+2b)},
\]
where \( b = \chi f_{\chi\chi}/f_{\chi} \). The no ghost and Laplace stability conditions can be written as \( f_{\chi}(1+2b) > 0 \) and \( 1+2b < 0 \) respectively, so we see that we must have \( b < -1/2 \) and \( f_{\chi} < 0 \). For power law models this gave us \( n < 1/2 \) and \( A < 0 \), but now we have more freedom.

The stability condition assures that \( \chi \) grows during radiation domination, therefore it starts small, as does the function \( f(\chi) \) since we require it adds no explicit cosmological constant, so \( f(0) = 0 \). Since that is the minimum, we expect that \( f \) should look like a power law in the early universe. That is, if the function is analytic near 0 then we could expand in a Maclaurin series and have \( f(\chi) \approx f(0) + \chi f_{\chi}(0) + \ldots \) or if \( f \) is a noninteger power law then we reduce to the previous case. One exception is where all the derivatives of \( f \) vanish at the origin. Another possibility is to allow \( f \) to approximate a power law but with a constant term so that \( \rho \approx (4\chi f_{\chi} - f) \) is not proportional to \( \chi^n \) and hence does not start so small that it is fine tuned. Finally, we could break the attractor behavior (for example by choosing \( c_2 \) huge compared to \( c_G \), so that the derivative coupling is unimportant) but this requires fine tuning.

Considering the first two cases, examples are
\[
\text{Case 1:} \quad f = A e^{-\chi a/\chi} \quad \text{(12)}
\]
\[
\text{Case 2:} \quad f = A (e^{-\chi a/\chi} - 1). \quad \text{(13)}
\]
Case 1 has \( b = \chi a/\chi - 2 \); in the early universe when \( \chi \) should be small then \( b \) violates the stability condition. Case 2 has \( b = -\chi a/\chi \) and again is unstable sufficiently early. We have not been able to find any form that both
preserves stability and is not fine tuned in the early universe (again, recall we are holding ourselves to a high standard – neither the cosmological constant nor \( f(R) \) gravity for example satisfy the criteria).

In summary, we have not been able to resolve the tension that avoiding instability enhances fine tuning, and avoiding fine tuning leads to instability.

### III. CANCELING THE COSMOLOGICAL CONSTANT

Let us return to perhaps the most interesting feature of Fab 5 (and Fab Four) gravity, the possibility of self tuning to cancel a high energy scale cosmological constant. As mentioned in the previous section, and illustrated in Fig. 2, this does indeed work, but works too well. Because the field is coupled to the Einstein tensor (\( G^{\mu\nu} \sim H^2 \sim \rho \)), the field dynamically cancels all energy density, leaving behind only a small cosmological constant giving de Sitter expansion with \( H_{\Lambda S} = \sqrt{-c_2 / (6c_G)} \). It makes no distinction between a large initial cosmological constant and other forms of energy density such as radiation and matter.

Figure 2 exhibits the behavior of the cosmological expansion \( H^2 \) and the dark energy density \( \rho_\Lambda \) for several cases. We find that the self tuning indeed cancels an arbitrarily large cosmological constant – in contrast to [15] where the (Laplace unstable) \( n = 1.5 \) case had a limited range of self tuning. However, the self tuning is in fact too powerful and does not deliver an observationally viable cosmology.

We see that the resulting expansion history deviates at early times from the standard radiation domination and matter domination. While it follows this initially, it begins to deviate once the Fab 5 density has grown in amplitude (due to its phantom nature) to become comparable to the background energy density \( \rho_5 \). Recall that the field evolves along an attractor as \( \rho \sim a^{-3(1-w_5)n/(2n-1)} \) as long as its energy density is small. For the \( n = 0.4 \) case plotted, this corresponds to \( a^{47} \) when \( \rho_5 \) is dominated by radiation \( (w_5 = 1/3) \) and \( a^{12} \) when \( \Lambda \) dominates (as in the top, blue case).

Once \( \rho_5 \) has grown sufficiently, the field “eats” the background energy density and \( \rho_5 \approx -\rho_\Lambda \). It almost exactly cancels it, even through the transition when the background density changes from being dominated by radiation and matter to being dominated by the bare \( \Lambda \), at which point \( \rho_5 \approx -\rho_\Lambda \) from then on. It leaves behind a small, positive, “renormalized” cosmological constant \( \Lambda/(H_{\Lambda S}^2 M_{Pl}^2) = -c_2 / (6c_G) \). During the time when \( \rho_5 \approx -\rho_\Lambda \), the expansion rapidly evolves toward the de Sitter attractor \( H_{\Lambda S}^2 \) as a highly negative power of \( a \) (a \( a^{-8} \) in the case shown in Fig. 2), switching over to \( a^{-6} \) when \( \rho_5 \approx -\rho_\Lambda \).

From the coupled evolution equations for the field \( \phi \) and expansion rate \( H \), given in [15], we can find attractor solutions. When Fab 5 is self tuning in a background otherwise dominated by an energy density with equation of state \( w_5 \), the scalings are

\[
\begin{align*}
x &\sim a^{\frac{3(1+w_5)(1-4n)}{n}} \\
H &\sim a^{\frac{3(1+w_5)(1-2n)}{n}}.
\end{align*}
\]

In the presence of a large cosmological constant \( (w_5 = -1) \), this yields \( H^2 \sim a^{-6} \) (as noted by [10] for the linear case). This should not be viewed as kination of a stiff fluid \( (w_5 = 1) \), however, as the field velocity \( \dot{\phi} \sim a^3 \) is increasing not decreasing due to the nonlinear kinetic terms, and the dark energy density is constant. Note that in this case the results are independent of the power law index \( n \); indeed this attractor is independent of the form of \( f \) (as long as self tuning operates to achieve the attractor dynamics). In the presence of background radiation or matter dominating over a cosmological constant, the scaling during self tuning does depend on \( n \).

Previous works on self tuning with Fab Four [8–10] and Fab 5 [15] investigated the case with solely the scalar field and a large cosmological constant, and so did not emphasize the fact that the standard radiation or matter dominated eras are eaten. As \( G^{\mu\nu} \sim H^2 \) is independent...
of the form of $f$, there does not seem to be a way around this. Indeed, it does not even depend on nonlinearity of $f$, so $L_{\text{John}}$ in Fab Four, or its Horndeski theory equivalent, also appear in danger if they are dominant at late times. It is not clear whether adding a potential (breaking shift symmetry) or a function of $\phi$ and $X$ can ameliorate this (though [10] used a potential to give “fake” radiation and matter eras).

While this is bad news for self tuning to explain late universe cosmic acceleration, it can be very useful for early universe inflation. The self tuning dramatically increases the ease of inflation starting, regardless of the other energy density components. Even for potentials that would not normally allow slow roll, the kinetically coupled gravity converts them into reasonable inflation models. This was discussed for the linear function $f$ in [24]. The derivative coupling to the Einstein tensor has also been used for inflation in [25–27]. Indeed the self tuning means that inflation can even occur without any potential, since Fab 5 has none. The expansion rate during inflation will be $H_{\text{DS}} = M \sqrt{-c_2/(6c_G)}$ and by choosing the mass scale $M$ to be a high energy scale rather than $H_0$ then the values of $c_2$ and $c_G$ remain of order unity.

IV. CONCLUSIONS

The bright hopes for Fab 5 gravity being simultaneously healthy, natural, observationally viable, and solving the cosmological constant constant problem have been shown to be fabulous in the sense of being fictional. The theory can deliver these characteristics individually but not simultaneously. We emphasize this merely puts the theory on the same level as many others considered for dark energy, no worse but with equally regrettable as-

fields nonminimally, in particular with regard to instabilities, and the strong constraints this engenders on the structure of the model. We find that stable forms of the theory require fine tuning of the initial conditions to allow radiation and matter domination, while viable cosmological solutions with respect to the expansion history tend to have instabilities in the field perturbations. We also note that the derivative coupling affects not only the scalar sound speed but also shifts the speed of gravitational waves from being the speed of light [15]; this may lead to gravitational Cherenkov radiation, which is highly constrained [28]. On solar system scales further considerations arise (though Vainshtein screening should remain), as discussed for Fab Four by [29]; for Fab 5, research is in progress to evaluate whether the nonpropagating degree of freedom becomes dynamical there.

Self tuning, or dynamical cancellation, of a high energy cosmological constant is one of the freshest and most attractive ideas for solving the cosmological constant problem. Unfortunately it is less than even a Pyrrhic victory, but rather suicidal, as the kinetic gravity coupling considered here (also known as $L_{\text{John}}$ in the Fab Four, $L_5$ in Horndeski theory, and even appearing in some massive gravity theories [30]) cancels all other energy density including radiation and matter. On the plus side this can be used in the early universe to greatly ease the onset of inflation.

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[19] N. Kaloper, Phys. Lett. B 583, 1 (2004) [arXiv:hep-th/0312002]
[20] J. Noller, JCAP 1207, 013 (2012) [arXiv:1203.6639]
[21] T.S. Koivisto, D.F. Mota, M. Zumalacarregui, Phys. Rev. Lett. 109, 241102 (2012) [arXiv:1205.3167]
[22] P. Brax, C. Burrage, A.-C. Davis, JCAP 1210, 016 (2012) [arXiv:1206.1809]
[23] M. Zumalacarregui, T.S. Koivisto, D.F. Mota, Phys. Rev. D 87, 083010 (2013) [arXiv:1210.8016]
[24] S. Tsujikawa, Phys. Rev. D 85, 083518 (2012) [arXiv:1201.5926]
[25] S. Sushkov, Phys. Rev. D 80, 103505 (2009) [arXiv:0910.0980]
[26] C. Germani, A. Kehagias, Phys. Rev. Lett. 105, 011302 (2010) [arXiv:1003.2635]
[27] J.-P. Bruneton, M. Rinaldi, A. Kanfon, A. Hees, S. Schlögel, A. Füzfa, Adv. Astron. 430694 (2012) [arXiv:1203.4446]
[28] R. Kimura, K. Yamamoto, JCAP 1207, 050 (2012) [arXiv:1112.4284]
[29] N. Kaloper, M. Sandora, arXiv:1310.5058
[30] C. de Rham, L. Heisenberg, Phys. Rev. D 84, 043503 (2011) [arXiv:1106.3312]