NONEXISTENCE AND SYMMETRY OF SOLUTIONS FOR SCHröDINGER SYSTEMS INVOLVING FRACTIONAL LAPLACIAN

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Abstract. In this paper, we consider the following Schrödinger systems involving pseudo-differential operator in $\mathbb{R}^n$
\[
\begin{align*}
(-\Delta)_{\alpha}^\frac{\beta_1}{2} u(x) &= u^{\beta_1}(x)v^{\tau_1}(x), \quad \text{in } \mathbb{R}^n, \\
(-\Delta)_{\gamma}^\frac{\gamma_2}{2} v(x) &= u^{\beta_2}(x)v^{\tau_2}(x), \quad \text{in } \mathbb{R}^n,
\end{align*}
\]
where $\alpha$ and $\gamma$ are any number between 0 and 2, $\alpha$ does not identically equal to $\gamma$.

We employ a direct method of moving planes to partial differential equations (PDEs) (1). Instead of using the Caffarelli-Silvestre’s extension method and the method of moving planes in integral forms, we directly apply the method of moving planes to the nonlocal fractional order pseudo-differential system. We obtained radial symmetry in the critical case and non-existence in the subcritical case for positive solutions.

In the proof, combining a new approach and the integral definition of the fractional Laplacian, we derive the key tools, which are needed in the method of moving planes, such as, narrow region principle, decay at infinity. The new idea may hopefully be applied to many other problems.

1. Introduction. In recent years, there has been a great deal of interest in using the fractional Laplacian to model diverse physical phenomena, such as anomalous diffusion and quasi-geostrophic flows, turbulence and water waves, molecular dynamics, and relativistic quantum mechanics of stars (see [3],[4],[9],[23]). In particular, the fractional order Laplacian can be understood as the infinitesimal generator of a stable Lévy process (see [2]).
The fractional Laplacian in $\mathbb{R}^n$ is a nonlocal pseudo-differential operator, taking the form
\[
(-\Delta)^{\alpha/2}u(x) = C_{n,\alpha} \text{PV} \int_{\mathbb{R}^n} \frac{u(x) - u(z)}{|x - z|^{n+\alpha}} dz,
\]
where $\alpha$ is any real number between 0 and 2 and PV stands for the Cauchy principal value. This operator is well defined in $\mathcal{S}$, the Schwartz space of rapidly decreasing $C^\infty$ functions in $\mathbb{R}^n$. In this space, it can also be defined equivalently in terms of the Fourier transform
\[
(-\Delta)^{\alpha/2}u(\xi) = |\xi|^\alpha \hat{u}(\xi),
\]
where $\hat{u}$ is the Fourier transform of $u$. One can extend this operator to a wider space of distributions as the following.

Let
\[
L_\alpha = \{ u : \mathbb{R}^n \to \mathbb{R} | \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} dx < \infty \} \text{ (see [21]).}
\]
For $u \in L_\alpha$, we define $(-\Delta)^{\alpha/2}u$ as a distribution:
\[
< (-\Delta)^{\alpha/2}u(x), \phi > = u(-\Delta)^{\alpha/2}\phi, \quad \forall \phi \in \mathcal{S}.
\]
Throughout the paper, we consider the solutions in this distributional sense. One can verify that, when $u$ is in $\mathcal{S}$, all the above definitions coincide.

In this paper, we study the Schrödinger systems (see [5], [18], [20]) involving fractional Laplacian in $\mathbb{R}^n$
\[
\left\{
\begin{array}{ll}
(-\Delta)^{\frac{\alpha}{2}}u(x) = u^{\beta_1}(x)v^{\gamma_1}(x), & \text{in } \mathbb{R}^n, \\
(-\Delta)^{\frac{\alpha}{2}}v(x) = u^{\beta_2}(x)v^{\gamma_2}(x), & \text{in } \mathbb{R}^n,
\end{array}
\right.
\]
where $\alpha$ and $\gamma$ are any real numbers between 0 and 2. We also assume that $(n + \alpha)\beta_i - (n - \alpha)\beta_i - (n - \gamma)\tau_i \geq 0$, $(n + \gamma) - (n - \alpha)\beta_i - (n - \gamma)\tau_i \geq 0$, $\beta_i \neq \beta_2$, $\tau_i \neq \tau_2$, and for $i = 1, 2$, $\beta_i, \tau_i \geq 1$.

When $\alpha = \gamma = 2$, Li and Ma [17] studied a similar system in the whole space $\mathbb{R}^n$:
\[
\left\{
\begin{array}{ll}
-\Delta u(x) = u^\beta(x)v^\tau(x), & \text{in } \mathbb{R}^n, \\
-\Delta v(x) = v^\beta(x)u^\tau(x), & \text{in } \mathbb{R}^n,
\end{array}
\right.
\]
for $n \geq 3$, $1 \leq \beta, \tau \leq \frac{n+2}{n-2}$ and $\beta + \tau = \frac{n+2}{n-2}$. When $n = 3$ and $\beta = 2$, $\tau = 3$, Eq. (4) is the stationary Schrödinger system with critical exponents for Bose-Einstein condensate. There they proved

**Proposition 1.** (See Li and Ma [17]) Assume that $1 \leq \beta < \tau \leq \frac{n+2}{n-2}$. Then any $L^{\frac{2n}{n+2}}(\mathbb{R}^n) \times L^{\frac{2n}{n+2}}(\mathbb{R}^n)$ radially symmetric solution pair $(u, v)$ to system (4) with critical exponents are unique such that $u = v$.

In [24] and [25], the authors considered the same Schrödinger system with high order Laplacian on a upper half space $\mathbb{R}^n_+$ with Navier and Dirichlet boundary conditions:
\[
\left\{
\begin{array}{ll}
(-\Delta)^{\frac{\alpha}{2}}u(x) = u^{\beta_1}(x)v^{\gamma_1}(x), & \text{in } \mathbb{R}^n_+, \\
(-\Delta)^{\frac{\alpha}{2}}v(x) = u^{\beta_2}(x)v^{\gamma_2}(x), & \text{in } \mathbb{R}^n_+, \\
u(x) = -\Delta u(x) = \cdots = (-\Delta)^{\frac{\alpha}{2}-1}u(x) = 0, & \text{on } \partial\mathbb{R}^n_+, \\
v(x) = -\Delta v(x) = \cdots = (-\Delta)^{\frac{\alpha}{2}-1}v(x) = 0, & \text{on } \partial\mathbb{R}^n_+.
\end{array}
\right.
\]
and
\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u(x) = u^{\beta_1}(x)v^{\gamma_1}(x), & \text{in } R^n_+,

(-\Delta)^{\frac{\alpha}{2}} v(x) = u^{\beta_2}(x)v^{\gamma_2}(x), & \text{in } R^n_+,

u = \frac{\partial u}{\partial x_n} = \cdots = \frac{\partial^{n-\gamma_1+1} u}{\partial x_n^{n-\gamma_1+1}} = 0, & \text{on } \partial R^n_+,

v = \frac{\partial v}{\partial x_n} = \cdots = \frac{\partial^{n-\gamma_2+1} v}{\partial x_n^{n-\gamma_2+1}} = 0, & \text{on } \partial R^n_+,
\end{cases}
\]

where \( R^n_+ \) is the \( n \)-dimensional upper half Euclidean space, \( R^n_+ = \{ x = (x_1, x_2, \cdots, x_n) \in R^n | x_n > 0 \} \), and \( \beta_1, \gamma_1, \beta_2, \text{ and } \gamma_2 \) satisfy the condition \((f_1)\):

\[ 0 \leq \beta_1, \gamma_1, \beta_2, \gamma_2 \leq \frac{n+\alpha}{n-\alpha} \text{ with } \frac{n}{n-\alpha} < \beta_1 + \gamma_1 = \beta_2 + \gamma_2 \leq \frac{n+\alpha}{n-\alpha}, \beta_1 \neq \beta_2, \gamma_1 \neq \gamma_2. \]

They considered the corresponding integral systems
\[
\begin{cases}
u(x) = \int_{R^n_+} G_N(x,y) u^{\beta_1}(y)v^{\gamma_1}(y)dy, \\
v(x) = \int_{R^n_+} G_N(x,y) u^{\beta_2}(y)v^{\gamma_2}(y)dy,
\end{cases}
\]
and
\[
\begin{cases}
u(x) = \int_{R^n_+} G_D(x,y) u^{\beta_1}(y)v^{\gamma_1}(y)dy, \\
v(x) = \int_{R^n_+} G_D(x,y) u^{\beta_2}(y)v^{\gamma_2}(y)dy,
\end{cases}
\]
where
\[
G_N(x,y) = c_n\left( \frac{1}{|x-y|^{n-\alpha}} - \frac{1}{|x-z|^{n-\alpha}} \right)
\]
is the Green’s function with Navier boundary conditions, and
\[
G_D(x,y) = \frac{C_n}{|x-y|^{n-\alpha}} \int_0^{\frac{\alpha}{n-\alpha}} z^{\frac{\alpha}{2}-1} \left( z + 1 \right)^{-\frac{\alpha}{2}} dz,
\]
is the Green’s function with Dirichlet boundary conditions.

Due to the non-locality of the fractional Laplacian with \( 0 < \alpha < 2 \), they technically required that \( \alpha \) is any even number between \( 0 \) and \( n \) in PDEs. They proved that the solutions of the corresponding integral equations must satisfy PDEs. Because of technical limitations, they only conjectured that the converse is also true. By using the method of moving planes in integral forms, they verified

**Proposition 2.** (See Zhuo, Li and Lv \cite{25}) For \( \beta_1, \gamma_1, \beta_2, \text{ and } \gamma_2 \) satisfying \((f_1)\), if \((u,v)\) is a pair of non-negative solution of integral systems \((7)\), then \( u \equiv 0 \) and \( v \equiv 0 \), where \( \alpha \) is any real number between \( 0 \) and \( n \) if \( n > 3 \), and \( 1 < \alpha < n \) if \( n = 3 \).

**Proposition 3.** (See Zhuo and Li \cite{24}) For \( \beta_1, \gamma_1, \beta_2, \text{ and } \gamma_2 \) satisfying \((f_1)\), if \((u(x), v(x))\) is a pair of non-negative solutions of \((8)\), with \( u, v \in L^p_{\text{loc}}(R^n_+) \), and \( p = \frac{n(\beta_1+\gamma_1-1)}{\alpha} \). Then \( u(x) \equiv 0 \) and \( v(x) \equiv 0 \).

Using the equivalence between PDEs and the corresponding integral equations, they partially proved the nonexistence of solutions for PDEs \((5)\) and \((6)\). For more information of this method, please see \cite{1},\cite{12},\cite{13},\cite{15},\cite{16}. In this paper, we directly work on the nonlocal operator to circumvent the difficulty of equivalence between partial differential equations and integral equations.

For fractional Laplacian problems, a useful method is the extension method introduced by Caffarelli and Silvestre (see \cite{6}). By extending the fractional operator to one more dimension, the nonlocal problem becomes a local one taking the form
of a second order elliptic equation. Specifically, let \( u(x) \) be a function: \( R^n \rightarrow R \), and \( U(x,y): R^n \times [0, +\infty) \rightarrow R \). Here \( U(x,y) \) satisfies the following equation

\[
\begin{aligned}
\begin{cases}
\text{div}(y^{1-\alpha}\nabla U(x,y)) = 0, & \text{in } R^{n+1}_+, \\
U(x,0) = u(x), & \text{on } \partial R^{n+1}_+,
\end{cases}
\end{aligned}
\]  

(9)

where \( R^{n+1}_+ = R^n \times [0, +\infty) \). One can prove

\[
(-\Delta)^\frac{n}{2} u(x) = -C \lim_{y \to 0^+} y^{1-\alpha} \frac{\partial U}{\partial y} \text{ in } R^n.
\]

In particular, when \( \alpha = 1 \), PDE (9) is reduced to the following problem

\[
\begin{aligned}
\begin{cases}
\text{div}(y^{1-\alpha}\nabla U(x,y)) = \Delta U(x,y) = 0, & \text{in } R^{n+1}_+, \\
U(x,0) = u(x), & \text{on } \partial R^{n+1}_+.
\end{cases}
\end{aligned}
\]

(10)

For (10), one can express the solution \( U(x,y) \) by the Poisson kernel \( P(x,y) \):

\[
\begin{aligned}
\begin{cases}
\Delta P(x,y) = 0, & \text{in } R^{n+1}_+, \\
P(x,0) = \delta_0(x), & \text{on } \partial R^{n+1}_+.
\end{cases}
\end{aligned}
\]

(11)

here \( \delta_0(x) \) is \( \delta \)-function centered at the origin.

Then,

\[
U(x,y) = \int_{R^n} P(x-\xi,y)u(\xi)d\xi.
\]

However, the extension method can not be applied to uniformly elliptic nonlocal operators and fully nonlinear nonlocal operators. Moreover, it's hard to apply this method to system involving fractional Laplacian. In this paper, we employ the method of moving planes directly to the fractional Laplacian, and derive the symmetry and non-existence of solutions for system involving the fractional operator. For more information of the method of moving planes, please see \([10],[11],[8]\) and \([19]\). Furthermore, this method can be generalized to study the uniformly elliptic nonlocal problem, such as (22)

\[
A_\alpha u(x) = f(x,u),
\]

where

\[
A_\alpha u(x) = C_{n,\alpha} \lim_{\epsilon \to 0} \int_{R^n \setminus B_\epsilon(x)} \frac{a(x-y)(u(x)-u(y))}{|x-y|^{n+\alpha}} dy,
\]

\[0 < c_0 < a(y) < c_1,
\]

and hopefully can be applied to equations involving fully nonlinear nonlocal operators.

In the paper, we need some key tools, such as the narrow region principle and decay at infinity to carry out the method of the moving planes. In Section 2, we will accomplish this. In Section 3, we apply the key technical results in the method of moving planes together with a new idea to show that

**Theorem 1.** Assume that \( u \in L_\alpha \cap C^{1,1}_{\text{loc}}, v \in L_\gamma \cap C^{1,1}_{\text{loc}}, \) and \( \tau_i, \beta_i \geq 1, i=1,2. \) If \( (u,v) \) is a pair of positive solutions for (3), then

(i) in subcritical case, that is, \( (n+\alpha)-(n-\alpha)\beta_1 -(n-\gamma)\tau_1 > 0 \) and \( (n+\gamma)-(n-\alpha)\beta_2 -(n-\gamma)\tau_2 > 0, \) (3) has no positive solution;

(ii) in critical case, that is, \( (n+\alpha)-(n-\alpha)\beta_1 -(n-\gamma)\tau_1 = 0 \) and \( (n+\gamma)-(n-\alpha)\beta_2 -(n-\gamma)\tau_2 = 0, \) \( u \) and \( v \) must be radially symmetric with the same center.
In above theorem, we prove the results under the weak condition that \( u \in L^\alpha \cap C^{1,1}_{\text{loc}}, v \in L^\gamma \cap C^{1,1}_{\text{loc}} \). Because there is no degeneracy assumption on the solution \((u(x), v(x))\), we need to apply the Kelvin transform. Let

\[
\tilde{u}(x) = \frac{1}{|x - x^0|^{n - \alpha}} u\left(\frac{x - x^0}{|x - x^0|} + x^0\right),
\]

\[
\tilde{v}(x) = \frac{1}{|x - x^0|^{n - \gamma}} v\left(\frac{x - x^0}{|x - x^0|} + x^0\right)
\]

be the Kelvin transform centered at any given point \( x^0 \).

Then \( \tilde{u}, \tilde{v} \) satisfy

\[
(-\Delta)^{\frac{\gamma}{2}} \tilde{u} = \frac{1}{|x - x^0|^{(n + \alpha)(n - \alpha)\beta_1 - (n - \alpha)\gamma_1}} \tilde{u}^{\beta_1} \tilde{v}^{\gamma_1},
\]

\[
(-\Delta)^{\frac{\gamma}{2}} \tilde{v} = \frac{1}{|x - x^0|^{(n + \gamma)(n - \alpha)\beta_2 - (n - \gamma)\gamma_2}} \tilde{u}^{\beta_2} \tilde{v}^{\gamma_2}.
\]

When \((n + \alpha) - (n - \alpha)\beta_1 - (n - \gamma)\gamma_1 > 0\) and \((n + \gamma) - (n - \alpha)\beta_2 - (n - \gamma)\gamma_2 > 0\), that is subcritical case, then due to the presence of the singular terms \(\frac{1}{|x - x^0|^{(n + \alpha)(n - \alpha)\beta_1 - (n - \alpha)\gamma_1}} \) and \(\frac{1}{|x - x^0|^{(n + \gamma)(n - \alpha)\beta_2 - (n - \gamma)\gamma_2}}\), we are able to use the method of moving planes to show that \( \tilde{u} \) and \( \tilde{v} \) must be radially symmetric about the point \( x^0 \). Since \( x^0 \) is an arbitrary point in \( R^n \), we conclude that \( u, v \) are constants. This contradicts system (3). This establishes the non-existence of positive solutions.

In critical case, \((n + \alpha) - (n - \alpha)\beta_1 - (n - \gamma)\gamma_1 = 0\) and \((n + \gamma) - (n - \alpha)\beta_2 - (n - \gamma)\gamma_2 = 0\), we can still utilize the method of moving planes to derive that \( \tilde{u} \) and \( \tilde{v} \) must be radially symmetric about the point \( x^0 \). Hence \( u, v \) are symmetry about some point in \( R^n \).

Based on the proof of above theorem, we will investigate the same system in the half space in our next paper. We firmly believe that we can get similar results in the half space. Together with the results in the whole space and in the half space, we can establish a priori estimates of solutions for a family of nonlocal operators on bounded domains of Euclidean space. In general, let

\[
Lu = (-\Delta)^{\frac{\gamma}{2}} u + \sum_{i=1}^n \frac{\partial u}{\partial x_i} + c(x) u,
\]

where \((-\Delta)^{\frac{\gamma}{2}}\) was introduced in [7] as the uniformly elliptic nonlocal operator,

\[
(-\Delta)^{\frac{\gamma}{2}} u = \frac{C_{n, \alpha}}{2} \int_{R^n} \frac{2u(x) - u(x + y) - u(x - y)}{|y|^{n + \alpha}} a(y) dy,
\]

and

\[0 < c_0 < a(x) < C_0.\]

One can use the Liouville type theorems to obtain a priori estimate for solutions of \( Lu = f(x, u) \) with the corresponding boundary conditions.

2. Key tools in the method of moving planes. In the section, we will show the key ingredients in the method of moving planes, such as narrow region principle and decay at infinity. First we introduce some basic notation needed in the method of moving planes. For a given real number \( \lambda \), denote

\[
\Sigma_\lambda = \{ x = (x_1, x_2, \ldots, x_n) \in R^n | x_1 \leq \lambda \},
\]
there exist some points $x$. Actually, by (12), one can further deduce that

$$
\Sigma_\lambda = \{ x^\lambda | x \in \Sigma_\lambda \},
$$

and let

$$
x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n)
$$

be the reflection of the point $x = (x_1, x_2, \cdots, x_n)$ about the plane $T_\lambda$.

**Theorem 2.1.** (Narrow Region Principle) Let $\Omega \subseteq \{ x | -\lambda < x_1 < \lambda \}$ be a bounded narrow region in $\Sigma_\lambda$ for $\lambda > 0$ small. Assume that $U \in L_\alpha \cap C^{1,1}_\text{loc}(\Omega)$ and $V \in L_\gamma \cap C^{1,1}_\text{loc}(\Omega)$ are lower semi-continuous on $\Omega$. If $b_i(x)$ and $c_i(x)$ are positive and bounded from below in $\Omega$, $i = 1, 2$,

$$
\begin{align*}
(-\Delta)^\alpha U(x) &\geq b_1(x)U(x) + c_1(x)V(x), \quad \text{in } \Omega, \\
(-\Delta)^\alpha V(x) &\geq b_2(x)U(x) + c_2(x)V(x), \quad \text{in } \Omega, \\
U(x) &\geq 0, \quad V(x) \geq 0, \quad \text{in } \Sigma_\lambda \setminus \Omega, \\
U(x^\lambda) = -U(x), \quad V(x^\lambda) = -V(x) \quad \text{in } \Sigma_\lambda,
\end{align*}
$$

then for sufficiently small $\lambda$, we get

$$
U(x) \geq 0, \quad V(x) \geq 0, \quad \forall x \in \Omega. \tag{13}
$$

Furthermore, if $U(x) = 0$ and $V(x) = 0$ at some point in $\Omega$, then

$$
U(x) = 0, \quad V(x) = 0, \quad \text{a.e. } x \in \mathbb{R}^n.
$$

For an unbounded narrow region $\Omega$, if we suppose

$$
\lim_{|x| \to \infty} U(x) \geq 0, \quad \lim_{|x| \to \infty} V(x) \geq 0,
$$

the above conclusions also hold.

**Proof of Theorem 2.1.** If (13) does not hold, by the lower semi-continuity of $U(x)$ and $V(x)$ on $\Omega$, there exist some points $x^0$, $x^1 \in \Omega$, such that

$$
U(x^0) = \min_{x \in \Omega} U(x) < 0, \quad V(x^1) = \min_{x \in \Omega} V(x) < 0.
$$

Actually, by (12), one can further deduce that $x^0$ and $x^1$ are in the interior of $\Omega$.

By the elementary calculation, we derive

$$
(-\Delta)^\alpha U(x^0) = C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{U(x^0) - U(y)}{|x^0 - y|^{n+\alpha}} dy
$$

$$
= C_{n,\alpha} PV \int_{\Sigma_\lambda} U(x^0) - U(y) \frac{dy}{|x^0 - y|^{n+\alpha}} + \int_{\Sigma_\lambda} U(x^0) - U(y) \frac{dy}{|x^0 - y|^{n+\alpha}}
$$

$$
= C_{n,\alpha} PV \int_{\Sigma_\lambda} U(x^0) - U(y) \frac{dy}{|x^0 - y|^{n+\alpha}} + \int_{\Sigma_\lambda} U(x^0) - U(y^\lambda) \frac{dy}{|x^0 - y^\lambda|^{n+\alpha}}
$$

$$
= C_{n,\alpha} PV \int_{\Sigma_\lambda} U(x^0) - U(y) \frac{dy}{|x^0 - y|^{n+\alpha}} + \int_{\Sigma_\lambda} U(x^0) + U(y) \frac{dy}{|x^0 - y|^{n+\alpha}}
$$

$$
\leq C_{n,\alpha} PV \int_{\Sigma_\lambda} U(x^0) - U(y) + U(x^0) + U(y) \frac{dy}{|x^0 - y|^{n+\alpha}}
$$

$$
\leq C_{n,\alpha} PV \int_{\Sigma_\lambda} U(x^0) - U(y) \frac{dy}{|x^0 - y^\lambda|^{n+\alpha}}. \tag{14}
$$

$$
\leq C_{n,\alpha} PV \int_{\Sigma_\lambda} 2U(x^0) \frac{dy}{|x^0 - y^\lambda|^{n+\alpha}}.
$$
Set $H = \{ y = (y_1, y') \in \mathbb{R}^n | |y_1 - x_1| < 1, |y' - (x_0')| < 1 \}$, and let $\rho = y_1 - x_1$, $t = |y' - (x_0')|$, 
\[
\int_{\Sigma_\lambda} \frac{1}{|x^0 - y|^n + \alpha} dy \geq \int_H \frac{1}{|x^0 - y|^n + \alpha} dy \geq \int_1^1 \int_0^1 \frac{\omega_{n-1} \rho s^{n-2} dt}{(\rho^2 + s^2)^{\frac{n+\alpha}{2}}} d\rho.
\]

Let $t = \rho s$, it follows from the above inequality that
\[
\int_{\Sigma_\lambda} \frac{1}{|x^0 - y|^n + \alpha} dy \geq \int_1^1 \int_0^1 \frac{\omega_{n-1}(\rho s)^{n-2} \rho ds}{\rho^{n+\alpha}(1 + s^2)^{\frac{n+\alpha}{2}}} d\rho \geq \int_1^1 \frac{1}{\rho^{n+\alpha}} d\rho \int_0^1 \frac{\omega_{n-1}s^{n-2}}{(1 + s^2)^{\frac{n+\alpha}{2}}} ds \geq \frac{C}{\lambda^\alpha},
\]
where $\omega_{n-1}$ is the area of $(n - 1)$-dimensional unit sphere, and $C$ denotes some constant.

By (14) and (15), we derive
\[
(-\Delta)^\frac{\alpha}{2} U(x^0) \leq \frac{C}{\lambda^\alpha} U(x^0).
\]

Similarly, we get
\[
(-\Delta)^\frac{\alpha}{2} V(x^1) \leq \frac{C}{\lambda^\alpha} V(x^1).
\]

By the first inequality of (12) and (16), one can see that there exists some constant $a_1 > 0$, such that for sufficiently small $l$ we have
\[
a_1 U(x^0) \frac{1}{l^\alpha} \geq c_1(x^0) V(x^0).
\]

Similarly, there exists some constant $a_2 > 0$ such that for sufficiently small $l$, we have
\[
a_2 V(x^1) \frac{1}{l^\gamma} \geq b_2(x^1) U(x^1).
\]

Combining (18) with (19), it gives
\[
a_1 U(x^0) \frac{1}{l^\alpha} \geq c_1(x^0) V(x^1) \geq a_2 c_1(x^0) b_2(x^1) U(x^0),
\]
that is
\[
\frac{1}{a_2 \alpha c_1(x^0) b_2(x^1)} \leq 1.
\]

It’s trivial that the inequality does not hold for sufficiently small $l$. Hence (13) must be true.

\begin{theorem}
(Decay at Infinity) Let $\Omega$ be an unbounded region in $\Sigma_\lambda$. Assume that $U \in L^\alpha_{\text{loc}} \cap C^{1,1}_{\text{loc}}(\Omega)$, $V \in L^\gamma_{\text{loc}} \cap C^{1,1}_{\text{loc}}(\Omega)$, $(U, V)$ satisfy the following equations
\[
\begin{cases}
(-\Delta)^\frac{\alpha}{2} U(x) \geq b_1(x) U(x) + c_1(x) V(x), & \text{in } \Omega, \\
(-\Delta)^\frac{\alpha}{2} V(x) \geq b_2(x) U(x) + c_2(x) V(x), & \text{in } \Omega, \\
U(x) \geq 0, V(x) \geq 0, & \text{in } \Sigma_\lambda \setminus \Omega, \\
U(x^\lambda) = -U(x), V(x^\lambda) = -V(x) & \text{in } \Sigma_\lambda,
\end{cases}
\]
\end{theorem}
Proof of Theorem 2.2. Similar to (14), we have

\[
(-\Delta)^{\frac{\gamma}{2}} U(x^0) \leq C_{n, \alpha} PV \int_{\Sigma_{\lambda}} \frac{2U(x^0)}{|x^0 - y|^{n+\alpha}}dy.
\]  

(24)

For fixed \( \lambda \), when \( |x^0| \geq \lambda \) and \( |x^1| \geq \lambda \), it’s easy to derive

\[
\int_{\Sigma_{\lambda}} \frac{1}{|x^0 - y|^{n+\alpha}}dy \geq \frac{C}{|x^0|^{n+\alpha}}.
\]  

(25)

here \( C \) denotes some constant.

Combining (24) with (25),

\[
(-\Delta)^{\frac{\gamma}{2}} U(x^0) \leq CU(x^0)\frac{1}{|x^0|^{\alpha}}.
\]  

(26)

Similarly, we get

\[
(-\Delta)^{\frac{\gamma}{2}} V(x^1) \leq CV(x^1)\frac{1}{|x^1|^{\gamma}}.
\]  

(27)

It follows from the first inequality of (21) and (26) that

\[
CU(x^0)\frac{1}{|x^0|^{\alpha}} \geq c_1(x^0)V(x^0) + b_1(x^0)U(x^0).
\]

Combining this with degenerate assumption of \( b_1(x) \), it’s easy to see that there exists some constant \( C_1 > 0 \) such that

\[
\frac{C_1}{|x^0|^{\alpha}}U(x^0) \geq c_1(x^0)V(x^0).
\]  

(28)

Similarly, there exists some constant \( C_2 > 0 \) such that

\[
\frac{C_2}{|x^1|^{\gamma}}V(x^1) \geq b_2(x^1)U(x^1).
\]  

(29)

From (28) and (29), we get

\[
\frac{C_1}{|x^0|^{\alpha}|x^1|^{\gamma}}U(x^0) \geq c_1(x^0)V(x^1) \geq c_1(x^0)|x^1|^{\gamma}b_2(x^1)U(x^1)
\]

\[
\geq c_1(x^0)|x^1|^{\gamma}b_2(x^1)U(x^0).
\]

That is

\[
\frac{C_1}{|x^0|^{\alpha}|x^1|^{\gamma}c_1(x^0)b_2(x^1)} \leq 1.
\]
However, for $|x^0|$ and $|x^1|$ sufficiently large, the inequality above is not true. Therefore, there exists $R_0 > 0$ such that

$$|x^0| \leq R_0, \text{ or } |x^1| \leq R_0.$$  

This completes the proof.  

3. **Rotational symmetry of solutions for Schrödinger system.** In this section, we give the proof of Theorem 1.  

Because there is no decay conditions on $u$ and $v$ in infinity, we apply the Kelvin transform. For any $z^0 \in \mathbb{R}^n$, consider the Kelvin transform centered at $z^0$

$$\bar{u}(x) = \frac{1}{|x - z^0|^{n-\alpha}} u(\frac{x - z^0}{|x - z^0|^2} + z^0),$$  

(30)

$$\bar{v}(x) = \frac{1}{|x - z^0|^{n-\gamma}} v(\frac{x - z^0}{|x - z^0|^2} + z^0).$$  

(31)

For simplicity of arguments, we will only show the case when $z^0$ is the origin, while the proof for a general $z^0$ is entirely similar.

Let

$$\bar{u}(x) = \frac{1}{|x|^{n-\alpha}} u(\frac{x}{|x|^2}),$$  

(32)

$$\bar{v}(x) = \frac{1}{|x|^{n-\gamma}} v(\frac{x}{|x|^2})$$  

(33)

be the Kelvin transform of $u$ and $v$ centered at the origin.

It’s easy to see

$$\bar{u}(x) \sim \frac{1}{|x|^{n-\alpha}}, \quad \bar{v}(x) \sim \frac{1}{|x|^{n-\gamma}}, \text{ for large } |x|. $$  

(34)

It is well known that

$$(-\Delta)^{\frac{\alpha}{2}} \bar{u}(x) = \frac{1}{|x|^{n+\alpha}} ((-\Delta)^{\frac{\alpha}{2}} u)(\frac{x}{|x|^2})$$

$$= \frac{1}{|x|^{n+\alpha}} u^{\beta_1}(\frac{x}{|x|^2}) v^{\tau_1}(\frac{x}{|x|^2})$$

$$= \frac{1}{|x|^{n+\alpha}} |x|^{(n-\alpha)\beta_1} |x|^{(n-\gamma)\tau_1} \bar{u}^{\beta_1}(x) \bar{v}^{\tau_1}(x)$$

$$= \frac{1}{|x|^{n+\alpha}} u^{\beta_1}(x) \bar{v}^{\tau_1}(x),$$

(35)

where $a_1 = (n + \alpha) - (n - \alpha)\beta_1 - (n - \gamma)\tau_1$.

Similarly, we have

$$(-\Delta)^{\frac{\gamma}{2}} \bar{v}(x) = \frac{1}{|x|^{n+\gamma}} \bar{u}^{\beta_2}(x) \bar{v}^{\tau_2}(x),$$

(36)

here $a_2 = (n + \gamma) - (n - \alpha)\beta_2 - (n - \gamma)\tau_2$.

Let

$$x^\lambda = (2\lambda - x_1, x_2, \cdots, x_n)$$

be the reflection of the point $x = (x_1, x_2, \cdots, x_n)$ about the plane $T_\lambda$.

From (35) and (36), it’s easy to derive

$$(-\Delta)^{\frac{\alpha}{2}} \bar{u}(x^\lambda) = \frac{1}{|x^{\lambda}|^{n+\alpha}} \bar{u}^{\beta_1}(x^\lambda) \bar{v}^{\tau_1}(x^\lambda),$$

(37)
\[-(\Delta)^{\frac{\gamma}{2}} \bar{v}(x^\lambda) = \frac{1}{|x|^\alpha_1} \bar{u}^{\beta_1}(x^\lambda)\bar{v}^{\gamma_2}(x^\lambda), \tag{38}\]

where \(a_1 = (n + \alpha) - (n - \alpha) \beta_1 - (n - \gamma) \tau_1\), \(a_2 = (n + \gamma) - (n - \alpha) \beta_2 - (n - \gamma) \tau_2\).

### 3.1. System in subcritical case.

In the subcritical case, \(a_1 = (n + \alpha) - (n - \alpha) \beta_1 - (n - \gamma) \tau_1 > 0\) and \(a_2 = (n + \gamma) - (n - \alpha) \beta_2 - (n - \gamma) \tau_2 > 0\), we show that (3) has no positive solution.

**Proof.** Let

\[U_\lambda(x) = \bar{u}(x^\lambda) - \bar{u}(x), \quad V_\lambda(x) = \bar{v}(x^\lambda) - \bar{v}(x).\]

By the definition of \(U_\lambda\) and \(V_\lambda\), we have

\[\lim_{|x| \to \infty} U_\lambda(x) = 0, \quad \lim_{|x| \to \infty} V_\lambda(x) = 0.\]

This implies that \(U_\lambda\) and \(V_\lambda\) attain negative minimum in the interior of \(\Sigma_\lambda\).

Define

\[\Sigma^\alpha_\lambda = \{x \in \Sigma_\lambda | U_\lambda(x) < 0\}, \quad \Sigma^\beta_\lambda = \{x \in \Sigma_\lambda | V_\lambda(x) < 0\}.\]

The proof consists of two steps.

**Step 1.** We will show that, for \(\lambda\) sufficiently negative,

\[U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda \setminus \{0^\lambda\}. \tag{39}\]

By an elementary calculation, we derive that, for \(x \in \Sigma^\alpha_\lambda \cap \Sigma^\beta_\lambda\),

\[-(\Delta)^{\frac{\gamma}{2}} U_\lambda(x) = \frac{1}{|x|^\alpha_1} \bar{u}^{\beta_1}(x^\lambda)\bar{v}^{\gamma_2}(x^\lambda) - \frac{1}{|x|^\alpha_1} \bar{u}^{\beta_1}(x)\bar{v}^{\gamma_2}(x)\]

\[= \frac{1}{|x|^\alpha_1} \bar{u}^{\beta_1}(x^\lambda)\bar{v}^{\gamma_2}(x^\lambda) - \frac{1}{|x|^\alpha_1} \bar{u}^{\beta_1}(x)\bar{v}^{\gamma_2}(x^\lambda)\]

\[+ \frac{1}{|x|^\alpha_1} \bar{u}^{\beta_1}(x)\bar{v}^{\gamma_2}(x^\lambda) - \frac{1}{|x|^\alpha_1} \bar{u}^{\beta_1}(x)\bar{v}^{\gamma_2}(x)\]

\[= \left( \frac{1}{|x|^\alpha_1} - \frac{1}{|x|^\alpha_1} \right) \bar{u}^{\beta_1}(x^\lambda)\bar{v}^{\gamma_2}(x^\lambda) + \frac{1}{|x|^\alpha_1} \bar{u}^{\beta_1}(x^\lambda)\bar{v}^{\gamma_2}(x^\lambda) - \bar{u}^{\beta_1}(x)\bar{v}^{\gamma_2}(x)\]

\[\geq \frac{1}{|x|^\alpha_1} \left( \bar{u}^{\beta_1}(x^\lambda)\bar{v}^{\gamma_2}(x^\lambda) - \bar{u}^{\beta_1}(x)\bar{v}^{\gamma_2}(x) \right)\]

\[\geq \frac{1}{|x|^\alpha_1} \left( \bar{u}^{\beta_1}(x^\lambda)\bar{v}^{\gamma_2}(x^\lambda) - \bar{u}^{\beta_1}(x^\lambda)\bar{v}^{\gamma_2}(x) + \bar{u}^{\beta_1}(x^\lambda)\bar{v}^{\gamma_2}(x) - \bar{u}^{\beta_1}(x)\bar{v}^{\gamma_2}(x) \right)\]

\[\geq \frac{1}{|x|^\alpha_1} \left[ \bar{u}^{\beta_1}(x^\lambda)\bar{v}^{\gamma_2}(x^\lambda) - \bar{u}^{\beta_1}(x)\bar{v}^{\gamma_2}(x) + \bar{u}^{\beta_1}(x^\lambda)\bar{v}^{\gamma_2}(x) - \bar{u}^{\beta_1}(x)\bar{v}^{\gamma_2}(x) \right]\]

For the above inequality, applying the Mean Value Theorem,

\[-(\Delta)^{\frac{\gamma}{2}} U_\lambda(x) \geq \frac{c}{|x|^\alpha_1} \left[ \bar{u}^{\beta_1}(x^\lambda)\bar{v}^{\gamma_2-1}(\xi)(\bar{v}(x^\lambda) - \bar{v}(x)) + \bar{v}^{\gamma_2-1}(\eta)(\bar{u}(x^\lambda) - \bar{u}(x)) \right]\]

\[\geq \frac{c}{|x|^\alpha_1} \left[ \bar{u}^{\beta_1}(x^\lambda)\bar{v}^{\gamma_2-1}(x)(\bar{v}(x^\lambda) - \bar{v}(x)) + \bar{v}^{\gamma_2-1}(x)(\bar{u}(x^\lambda) - \bar{u}(x)) \right]\]

\[\geq \frac{c}{|x|^\alpha_1} \bar{u}^{\beta_1}(x^\lambda)\bar{v}^{\gamma_2-1}(x)V_\lambda(x) + \frac{c}{|x|^\alpha_1} \bar{v}^{\gamma_2-1}(x)\bar{u}^{\beta_1-1}(x)U_\lambda(x)\]

\[\geq c_1(x)V_\lambda(x) + b_1(x)U_\lambda(x),\]

where \(\xi\) and \(\eta\) are valued between \(x^\lambda\) and \(x\), \(c_1(x) = \frac{c}{|x|^\alpha_1} \bar{u}^{\beta_1}(x^\lambda)\bar{v}^{\gamma_2-1}(x)\), \(b_1(x) = \frac{c}{|x|^\alpha_1} \bar{v}^{\gamma_2-1}(x)\bar{u}^{\beta_1-1}(x)\).
That is,

\[ (-\Delta)^{\gamma} U_\lambda(x) \geq c_1(x)V_\lambda(x) + b_1(x)U_\lambda(x). \]

By (34), it is easy to derive that

\[ c_1(x) \sim \frac{1}{|x|^{\alpha+\gamma}}, \quad b_1(x) \sim \frac{1}{|x|^{2\alpha}}, \text{ for sufficiently large } |x|. \]

Similarly,

\[ (-\Delta)^{\gamma} V_\lambda(x) \geq c_2(x)V_\lambda(x) + b_2(x)U_\lambda(x), \]

where \( c_2(x) = \frac{c}{|x|^{2\gamma}} \tilde{a}^2(x)\tilde{\tilde{a}}^{\beta_2}(x)\tilde{\tilde{a}}^{\gamma_2}(x), \)

\[ b_2(x) = \frac{c}{|x|^{2\gamma}} \tilde{a}^{\beta_2}(x)\tilde{\tilde{a}}^{\gamma_2}(x). \]

By (34), we have

\[ c_2(x) \sim \frac{1}{|x|^{2\gamma}}, \quad b_2(x) \sim \frac{1}{|x|^{\alpha+\gamma}}, \text{ for sufficiently large } |x|. \]

Suppose there exists some point \( x^0 \) such that

\[ U_\lambda(x^0) = \min_{x \in \Sigma_\lambda} U_\lambda(x) < 0. \]

Similarly to (16), for \( |x^0| > \lambda \), we get

\[ (-\Delta)^{\gamma} U_\lambda(x^0) \leq cU_\lambda(x^0) \frac{1}{|x^0|^{\alpha}}. \]

Combining this with (40), we deduce

\[ cU_\lambda(x^0) \frac{1}{|x^0|^{\alpha}} \geq c_1(x^0)V_\lambda(x^0) + b_1(x^0)U_\lambda(x^0). \]

By the degeneracy of \( b_1(x) \) at infinity and (45), for sufficiently negative \( \lambda \),

\[ \frac{c}{|x^0|^{\alpha}} U_\lambda(x^0) \geq c_1(x^0)V_\lambda(x^0). \]

Next we suppose that there is some point \( x^1 \) such that

\[ V_\lambda(x^1) = \min_{x \in \Sigma_\lambda} V_\lambda(x) < 0. \]

Similar to (44), we obtain

\[ (-\Delta)^{\gamma} V_\lambda(x^1) \leq cV_\lambda(x^1) \frac{1}{|x^1|^{\gamma}}. \]

Combining (42) and (47),

\[ cV_\lambda(x^1) \frac{1}{|x^1|^{\gamma}} \geq c_2(x^1)V_\lambda(x^1) + b_2(x^1)U_\lambda(x^1). \]

From the degeneracy of \( c_2(x) \) at infinity and (48), for sufficiently negative \( \lambda \), we have

\[ \frac{c}{|x^1|^{\gamma}} V_\lambda(x^1) \geq b_2(x^1)U_\lambda(x^1). \]

Combining (46) with (49), we derive that

\[ \frac{c}{|x^1|^{\gamma}} V_\lambda(x^1) \geq b_2(x^1)U_\lambda(x^0) \geq b_2(x^1)|x^0|^{\alpha} c_1(x^0)V_\lambda(x^0) \]

\[ \geq b_2(x^1)|x^0|^{\alpha} c_1(x^0)V_\lambda(x^1). \]

Using the degeneracy of \( b_2(x) \) and \( c_1(x) \) at infinity, we arrive at

\[ \frac{c}{|x^1|^{\gamma}} V_\lambda(x^1) \geq \frac{1}{|x^1|^{\alpha+\gamma}} \frac{1}{|x^0|^{\gamma}} V_\lambda(x^1). \]
That is,
\[ c \frac{1}{|x|^\alpha|x^0|^\gamma} \geq 1. \]
For sufficiently negative \( \lambda \), the inequality does not hold. From Theorem 2.2 (Decay at Infinity), for sufficiently negative \( \lambda \) (or \( |\lambda| < R_0 \) in Theorem 2.2), at least one of \( U_\lambda \) and \( V_\lambda \) is greater than or equal to 0. Without loss of generality, we assume that
\[ U_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda \setminus \{0^\lambda\}. \] (50)

To prove (50) also holds for \( V_\lambda \), we argue by contradiction. If \( V_\lambda \) is negative somewhere in \( \Sigma_\lambda \setminus \{0^\lambda\} \), then there must exist some \( \bar{x} \in \Sigma_\lambda \) such that
\[ V_\lambda(\bar{x}) = \min_{x \in \Sigma_\lambda} V_\lambda(x) < 0. \]
From previous arguments of (42) and (47), we know that
\[ 0 > cV_\lambda(\bar{x}) \frac{1}{|\bar{x}|^\gamma} \geq (\Delta)^\frac{2}{\gamma} V_\lambda(\bar{x}) \]
\[ \geq c_2(\bar{x}) V_\lambda(\bar{x}) + b_2(\bar{x}) U_\lambda(\bar{x}), \]
here \( c_2(x) = \frac{c}{|x|^{\alpha_2}} \bar{u}^{\beta_2}(x^\lambda) \bar{v}^{\tau_2}(x) \bar{u}^{\beta_2-1}(x) \), \( b_2(x) = \frac{c}{|x|^{\alpha_2}} \bar{u}^{\tau_2}(x) \bar{v}^{\beta_2-1}(x) \).

For the above inequality, combining with (49), we derive that
\[ 0 > (\Delta)^\frac{2}{\gamma} V_\lambda(\bar{x}) \]
\[ \geq c_2(\bar{x}) V_\lambda(\bar{x}) + b_2(\bar{x}) U_\lambda(\bar{x}) \]
\[ \geq 0 \]
This is a contradiction. And we complete step 1.

**Step 2.** Step 1 provides a starting point for us to move the plane \( T_\lambda \) to the right along \( x_1 \) direction as long as inequality (39) holds.
Define
\[ \lambda_0 = \sup \{ \lambda < 0 | U_\mu(x) \geq 0, \forall x \in \Sigma_\mu \setminus \{0^\mu\}, \mu \leq \lambda \}. \]
In the step, we will prove that
\[ \lambda_0 = 0, \] (51)
and
\[ U_{\lambda_0}(x) \equiv 0, \quad V_{\lambda_0}(x) \equiv 0, \quad \forall x \in \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\}. \] (52)
Suppose that
\[ \lambda_0 < 0, \]
we will show that the plane \( T_\lambda \) can be moved further more. That is, there exists some small \( \epsilon > 0 \), such that for any \( \lambda \in (\lambda_0, \lambda_0 + \epsilon) \), we have
\[ U_\lambda(x) \equiv 0, \quad V_\lambda(x) \equiv 0, \quad \forall x \in \Sigma_\lambda \setminus \{0^\lambda\}. \] (53)
This is a contradiction with the definition of \( \lambda_0 \). Therefore, we derive that
\[ \lambda_0 = 0. \]
Actually, for \( \lambda_0 < 0 \),
\[ U_{\lambda_0}(x) > 0, \quad V_{\lambda_0}(x) > 0, \quad \forall x \in \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\}. \] (54)
Otherwise, at least one of \( U_{\lambda_0}(x) \) and \( V_{\lambda_0}(x) \) is greater than or equal to zero. Without loss generality, we may assume that \( U_{\lambda_0}(x) \geq 0 \). That is, there exists some point \( \hat{x} \) such that
\[
U_{\lambda_0}(\hat{x}) = \min_{\Sigma_{\lambda_0}} U_{\lambda_0}(x) = 0.
\]

It follows that
\[
(-\Delta)^{\frac{1}{2}} U_{\lambda_0}(\hat{x}) = C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{-U_{\lambda_0}(y)}{|x-y|^{n+\alpha}} dy
\]
\[
= C_{n,\alpha} PV \int_{\Sigma_{\lambda_0}} \frac{-U_{\lambda_0}(y)}{|x-y|^{n+\alpha}} dy + \int_{\mathbb{R}^n \setminus \Sigma_{\lambda_0}} \frac{-U_{\lambda_0}(y)}{|x-y|^{n+\alpha}} dy
\]
\[
= C_{n,\alpha} PV \int_{\Sigma_{\lambda_0}} \left( \frac{1}{|x-y_{\lambda_0}|^{n+\alpha}} - \frac{1}{|x-y|^{n+\alpha}} \right) U_{\lambda_0}(y) dy
\]
\[
\leq 0. \tag{55}
\]

On the other hand,
\[
(-\Delta)^{\frac{1}{2}} U_{\lambda_0}(\hat{x}) = \frac{\tilde{u}^{\beta_1}(\hat{x}) \tilde{v}^{\tau_1}(\hat{x})}{|x_{\lambda_0}|_{a_1}} - \frac{\tilde{u}^{\beta_1}(\hat{x}) \tilde{v}^{\tau_1}(\hat{x})}{|x|_{a_1}}
\]
\[
= \frac{\tilde{u}^{\beta_1}(\hat{x}) \tilde{v}^{\tau_1}(\hat{x})}{|x_{\lambda_0}|_{a_1}} - \frac{\tilde{u}^{\beta_1}(\hat{x}) \tilde{v}^{\tau_1}(\hat{x})}{|x|_{a_1}}
\]
\[
= \frac{\tilde{u}^{\beta_1}(\hat{x}) (\tilde{v}^{\tau_1}(\hat{x}) \tilde{v}^{\tau_1}(\hat{x}) - \tilde{v}^{\tau_1}(\hat{x}))}{|x|_{a_1}}
\]
\[
+ \left( \frac{1}{|x_{\lambda_0}|_{a_1}} - \frac{1}{|x|_{a_1}} \right) \tilde{v}^{\tau_1}(\hat{x})
\]
\[
> 0.
\]

This is a contradiction with (55). Hence (54) holds.

From [26], we have the integral expressions of \( U_{\lambda_0} \) and \( V_{\lambda_0} \). Combining it with the proof of Appendix A in [14], we can show that there exists \( c_0 > 0 \) such that, for sufficiently small \( \epsilon \),
\[
U_{\lambda_0}(x), V_{\lambda_0} \geq c_0, \quad \forall x \in B_\epsilon(0^{\lambda_0}) \setminus \{0^{\lambda_0}\}. \tag{56}
\]

Together with the above bounded-away-from-0 result, we derive that for \( \delta > 0 \), there exists some constant \( c_0 > 0 \) such that
\[
U_{\lambda_0}(x), V_{\lambda_0} \geq c_0, \quad \forall x \in (\Sigma_{\lambda_0} - \delta \setminus \{0^{\lambda_0}\}) \cap B_{R_0}(0). \tag{57}
\]

For \( \epsilon, \delta \ll |\lambda_0| \),
\[
0^{\lambda_0} \in (\Sigma_{\lambda_0} - \delta \setminus \{0^{\lambda_0}\}) \cap B_{R_0}(0).
\]

Since \( U_{\lambda} \) and \( V_{\lambda} \) depend on \( \lambda \) continuously, we have
\[
U_{\lambda_0}(x), V_{\lambda_0} \geq 0, \quad \forall x \in (\Sigma_{\lambda_0} - \delta \setminus \{0^{\lambda_0}\}) \cap B_{R_0}(0). \tag{58}
\]

By Theorem 2.2 (Decay at infinity), we know that if
\[
U_\lambda(\hat{x}) = \min_{\Sigma_\lambda} U_\lambda < 0,
\]
then there exists a large \( R_0 \) such that
\[
|\hat{x}| \leq R_0.
\]
Hence \( \hat{x} \in (\Sigma_\lambda \setminus \Sigma_{\lambda_0 - \delta}) \cap B_{R_0}(0) \).

For sufficiently large \( R_0 \), similar to (46), we obtain
\[
V_\lambda(\hat{x}) < 0.
\]

Therefore, there exists some point \( \bar{x} \) such that
\[
V_\lambda(\bar{x}) = \min_{\Sigma} V_\lambda < 0.
\]

If \( \bar{x} \in B_{R_0}^C \cap \Sigma_\lambda \), similar to (49),
\[
0 > \frac{cV_\lambda(\bar{x})}{|\bar{x}|^\gamma} \geq b_2(\bar{x})U_\lambda(\bar{x}).
\]

Meanwhile, for \( U_\lambda \) at \( \hat{x} \), similar to (46), we get
\[
\frac{cU_\lambda(\hat{x})}{|\epsilon + \delta|^{\gamma}} \geq c_1(\hat{x})V_\lambda(\hat{x}).
\]

By (59) and (60), we have
\[
b_2(\bar{x})|\bar{x}|^\gamma c_1(\hat{x})|\epsilon + \delta|^{\gamma} \geq c.
\]

Using the degeneracy of \( c_1 \), we know that \( c_1(\hat{x}) \) is bounded. Notice that \( b_2(\bar{x})|\bar{x}|^\gamma \) is also bounded for \( |\bar{x}| > R_0 \). Hence for \( \delta \) sufficiently small, (61) does not hold. This implies \( \bar{x} \in B_{R_0}^C \cap \Sigma_\lambda \) will not happen.

Employing Theorem 2.1 (Narrow region principle), let the narrow region \( \Omega = (\Sigma_\lambda \setminus \Sigma_{\lambda_0 - \delta}) \cap B_{R_0}(0) \), while \( U_\lambda \) and \( V_\lambda \) satisfy system (12), we obtain
\[
U_\lambda(x) \geq 0, \quad V_\lambda(x) \geq 0, \quad \forall x \in (\Sigma_\lambda \setminus \Sigma_{\lambda_0 - \delta}) \cap B_{R_0}(0).
\]

Now we conclude that neither \( U_\lambda(x) \) nor \( V_\lambda(x) \) has negative minimum in \( \Sigma_\lambda \setminus \{0^\lambda\} \).

Hence,
\[
U_\lambda(x), \quad V_\lambda(x) \geq 0, \quad \forall x \in \Sigma_\lambda \setminus \{0^\lambda\}.
\]

This completes the proof of (53). Thus we obtain
\[
\lambda_0 = 0.
\]

Similarly, we can move the plane from \( x_1 = +\infty \) near to the left, and we can show that
\[
U_\lambda(x), \quad V_\lambda(x) \leq 0, \quad \forall x \in \Sigma_\lambda \setminus \{0^\lambda\}.
\]

Therefore we conclude that
\[
\lambda_0 = 0, \quad U_{\lambda_0}(x) \equiv 0, \quad V_{\lambda_0}(x) \equiv 0, \quad \forall x \in \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\}.
\]

Since the direction of \( x_1 \)-axis is arbitrary, we derive that \( \bar{u} \) and \( \bar{v} \) are radially symmetric about the origin.

For any point \( z^0 \in R^n \), applying the Kelvin transform centered at \( z^0 \) (see (30),(31)), and by an entirely similar argument, one can show that \( \bar{u} \) and \( \bar{v} \) are radially symmetric about \( z^0 \).

Let \( z^1 \) and \( z^2 \) be any two points in \( R^n \), and we choose the coordinate system so that the midpoint \( z^0 = \frac{z^1 + z^2}{2} \) is the origin. Since \( \bar{u} \) and \( \bar{v} \) are radially symmetric about \( z^0 \), we have \( u(z^1) = u(z^2) \) and \( v(z^1) = v(z^2) \). This implies that \( u \) and \( v \) must be constants. Positive constant solutions do not satisfy system (3). That is, in subcritical case, there is no positive solution for system (3). \( \square \)
3.2. **System in critical case.** In critical case, \( a_1 = (n + \alpha) - (n - \alpha)\beta_1 - (n - \gamma)\tau_1 = 0 \) and \( a_2 = (n + \gamma) - (n - \alpha)\beta_2 - (n - \gamma)\tau_2 = 0 \). In this section, we still utilize the Kelvin transform of \( u \) and \( v \) centered at the origin (see (32),(33)). We will show that either \( \tilde{u} \) and \( \tilde{v} \) are symmetric about the origin or \( \bar{u} \) and \( \bar{v} \) are symmetric about some point.

We still use the notations introduced in the subcritical case. The argument is quite similar to, but not entirely the same as that in the subcritical case. Hence we still present some details here.

**Proof.** In critical case, similar to (40), we get

\[
(-\Delta)\tilde{u}_\lambda(x) \geq \bar{c}_1(x)V_\lambda(x) + \bar{b}_1(x)\bar{U}_\lambda(x),
\]

where \( \bar{b}_1 = \tilde{v}_{\tau_2}(x)\tilde{u}_{\beta_1}^{-1}(x), \bar{c}_1(x) = \tilde{u}_\lambda(x)\tilde{u}_{\beta_1}^{-1}(x) \).

Similar to (42), we have

\[
(-\Delta)\tilde{u}_\lambda(x) \geq \bar{c}_2(x)V_\lambda(x) + \bar{b}_2(\lambda)\bar{U}_\lambda(x),
\]

where \( \bar{b}_2(x) = \tilde{v}_{\tau_2}(x)\tilde{u}_{\beta_2}^{-1}(x), \bar{c}_2(x) = \tilde{u}_\lambda(x)\tilde{u}_{\beta_2}^{-1}(x) \).

By (34), it is easy to derive that

\[
c_1(x) \sim \frac{1}{|x|^\alpha + \gamma}, \quad b_1(x) \sim \frac{1}{|x|^{2\alpha}}, \quad \text{for large } |x|.
\]

\[
c_2(x) \sim \frac{1}{|x|^{2\gamma}}, \quad b_2(x) \sim \frac{1}{|x|^{\alpha + \gamma}}, \quad \text{for lager } |x|.
\]

The remaining proof is the same as that in the subcritical case. We can show that, for \( \lambda \) sufficiently negative,

\[
\bar{U}_\lambda, V_\lambda \geq 0, \quad x \in \Sigma_\lambda \setminus \{0^\lambda\}.
\]

Define

\[
\lambda_0 = \sup\{\lambda \leq 0 | U_\mu(x) \geq 0, \forall x \in \Sigma_\mu \setminus \{0^\mu\}, \lambda \leq \lambda \}.
\]

We consider two possible cases.

**Case (i).** \( \lambda_0 < 0 \).

For \( \lambda_0 < 0 \), either

\[
U_{\lambda_0}(x) = V_{\lambda_0}(x) \equiv 0, \quad x \in \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\},
\]

or

\[
U_{\lambda_0}(x), V_{\lambda_0}(x) > 0, \quad x \in \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\}.
\]

We suppose that there exists some point \( \hat{x} \in \Sigma_{\lambda_0} \) such that

\[
U_{\lambda_0}(\hat{x}) = \min_{x \in \Sigma_{\lambda_0}} U_{\lambda_0} = 0,
\]

then

\[
U_{\lambda_0}(x) \equiv 0, \quad x \in \Sigma_{\lambda_0}.
\]

Otherwise,

\[
(-\Delta)^{\frac{\alpha}{2}} U_{\lambda_0}(\hat{x}) = C_{n, \alpha} \text{PV} \int_{\mathbb{R}^n} \frac{-U_{\lambda_0}(y)}{|\hat{x} - y|^{n + \alpha}} dy < 0.
\]

On the other hand,

\[
(-\Delta)^{\frac{\alpha}{2}} U_{\lambda_0}(\hat{x}) = \tilde{u}_{\lambda_0}^{\beta_1}(\hat{x})\tilde{v}_{\tau_2}(\hat{x}) - \tilde{u}_{\lambda_0}^{\beta_1}(\hat{x})\tilde{v}_{\tau_2}(\hat{x})
\]

\[
= \bar{u}_{\lambda_0}^{\beta_1}(\hat{x}) (\tilde{v}_{\tau_2}(\hat{x}) - \tilde{v}_{\tau_2}(\hat{x}))
\]

\[
\geq 0.
\]
This is a contradiction. Hence (68) holds. When \( U_{\lambda_0} \equiv 0 \), by the anti-symmetry of \( U_{\lambda} \), that is,
\[
U_{\lambda_0}(x) = -U_{\lambda_0}(x^{\lambda_0}),
\]
we derive that
\[
U_{\lambda_0}(x) \equiv 0, \ x \in \mathbb{R}^n.
\]
Obviously,
\[
(-\Delta)^{\frac{a}{2}} U_{\lambda_0}(x) = 0.
\]
Since
\[
(-\Delta)^{\frac{a}{2}} U_{\lambda_0}(x) = \bar{u}_{\beta_1}(x^{\lambda_0})(\bar{v}_{\tau_1}(x^{\lambda_0}) - \bar{v}_{\tau_1}(x)) = 0,
\]
it must be true that
\[
\bar{v}(x^{\lambda_0}) = \bar{v}(x), \ x \in \mathbb{R}^n,
\]
That is
\[
V_{\lambda_0}(x) \equiv 0, \ x \in \mathbb{R}^n.
\]
Similarly, if \( V_{\lambda_0}(x) = 0 \) somewhere, then we can prove that
\[
U_{\lambda_0}(x) \equiv 0, \ x \in \mathbb{R}^n.
\]
When
\[
U_{\lambda_0}(x), V_{\lambda_0}(x) > 0, \ x \in \Sigma_{\lambda_0} \setminus \{0^{\lambda_0}\},
\]
using an entirely similar argument of Step 2 in subcritical case, one can keep moving the plane \( T_\lambda \). That is, there exists some small \( \epsilon > 0 \), such that for any \( \lambda \in (\lambda_0, \lambda_0 + \epsilon) \), we have
\[
U_{\lambda}(x) \equiv 0, \ V_{\lambda}(x) \equiv 0, \ \forall x \in \Sigma_\lambda \setminus \{0^{\lambda}\}. \tag{69}
\]
This is a contradiction with the definition of \( \lambda_0 \). Therefore (67) must not be true. We conclude that
\[
U_{\lambda_0}(x) = V_{\lambda_0}(x) \equiv 0, \ x \in \mathbb{R}^n.
\]
This implies \( u \) and \( v \) are symmetric about some point in \( \mathbb{R}^n \).

**Case (ii).** \( \lambda_0 = 0 \).

In this case, we can move the plane from near \( x_1 = +\infty \) to the left, and derive that
\[
U_0(x), V_0(x) > 0, \ x \in \Sigma_0.
\]
Hence
\[
U_0(x), V_0(x) \equiv 0, \ x \in \Sigma_0.
\]
This proves that \( \bar{u} \) and \( \bar{v} \) are symmetric about the origin. So are \( u \) and \( v \). In any case, \( u \) and \( v \) are symmetric about some point in \( \mathbb{R}^n \).

This completes the proof of Theorem 1. \( \square \)
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