Off-Criticality and the Massive Brownian Loop Soup

Federico Camia *

Abstract

The Brownian loop soup is a conformally invariant Poissonian ensemble of loops in the plane recently introduced by Lawler and Werner. It has attracted significant attention for its connection to the Schramm-Loewner Evolution and Conformal Loop Ensembles and its consequent ability to describe the critical scaling limit of two-dimensional statistical mechanical models. In this paper, we introduce a natural “massive” (non-scale-invariant) version of the Brownian loop soup as a candidate to describe near-critical scaling limits, and study some of its properties, such as conformal covariance, exponential decay of correlations, and Hausdorff dimension. We also show that the massive Brownian loop soup arises as the near-critical scaling limit of a “gas” of random walk loops which is closely related to the discrete Gaussian free field.

Key words and phrases: Brownian loop soup, off-critical regime, near-critical scaling limit, conformal covariance, random walk loop soup, Gaussian free field, Conformal Loop Ensembles.

AMS subject classification: 60J65, 82B41, 60K35, 81T40.

*VU University Amsterdam and NYU Abu Dhabi. E-mail: f.camia@vu.nl
†Partially supported by NWO grant VIDI 639.032.916.
1 Introduction

Several interesting models of statistical mechanics, such as percolation and the Ising and Potts models, can be described in terms of clusters. In two dimensions and at the critical point, the scaling limit geometry of the boundaries of such clusters is known (see [30, 7, 8, 10]) or conjectured (see [15]) to be described by some member of the one-parameter family of Schramm-Loewner Evolutions (SLE$_\kappa$ with $\kappa > 0$) and related Conformal Loop Ensembles (CLE$_\kappa$ with $8/3 < \kappa < 8$). What makes SLEs and CLEs natural candidates is their conformal invariance, a property expected of the scaling limit of two-dimensional statistical mechanical models at the critical point. SLEs can be used to describe the scaling limit of single interfaces; CLEs are collections of loops and are therefore suitable to describe the scaling limit of the collection of all macroscopic boundaries at once. For example, the scaling limit of the critical percolation exploration path is SLE$_6$ [30, 8], and the scaling limit of the collection of all critical percolation interfaces in a bounded domain is CLE$_6$ [7, 9]. For $8/3 < \kappa \leq 4$, CLE$_\kappa$ can be obtained [29] from the Brownian loop soup introduced by Lawler and Werner [20].

A meaningful continuum scaling limit that differs from the critical one can usually be obtained by considering a system “near” the critical point.
This is done by adjusting some parameter of the model and sending it to the critical value at a specific rate while taking the scaling limit, in such a way that the correlation length, in macroscopic units, stays bounded away from 0 and $\infty$. We call that a near-critical scaling limit. In the Ising model at the critical temperature, for instance, one can introduce an external magnetic field $h$. Sending $h$ to zero at the appropriate rate while taking the scaling limit yields a one-parameter ($h$) family of nontrivial magnetization fields \[6\], with $h = 0$ corresponding to the critical case. The fields corresponding to $h \neq 0$ are called off-critical, and are not expected to be conformally (or even scale) invariant. Near-critical scaling limits are expected to possess a weaker property, which we will call conformal covariance, a sort of “local” conformal invariance that takes into account the fact that the continuum model has a finite correlation length.

Scaling limits of two-dimensional statistical mechanical models at the critical point are known to correspond to massless field theories, and are believed (and sometimes proved) to be conformally invariant, while near-critical scaling limits correspond to massive field theories. (Here, as in Euclidean field theory, we use the term massless to describe a scale-invariant system, while the term massive refers to a system with exponential decay of correlations.) The lack of conformal invariance implies that the geometry of near-critical scaling limits cannot be described by an SLE or CLE. Indeed, much less has been proved about the geometry of off-critical models than about that of critical ones.

In this paper we introduce a massive variant of the Brownian loop soup that possesses the properties of conformal covariance and exponential decay of correlations that are expected to characterize off-critical continuum models obtained via a near-critical scaling limit. This massive Brownian loop soup is perhaps the simplest modification of the massless Brownian loop soup of Lawler and Werner combining those properties; the mechanism by which it arises from the Brownian loop soup of Lawler and Werner is analogous to that appearing in the standard Brownian motion representation of the continuum Gaussian free field when a mass term is present. Moreover, it can be seen as the only possible near-critical scaling limit of a random walk loop soup with killing whose critical version (with no killing) has the massless Brownian loop soup as its scaling limit (see \[19\]).

As mentioned above, CLE$_\kappa$ for $8/3 < \kappa \leq 4$ can be obtained \[29\] from the massless Brownian loop soup. By applying the same procedure to the massive Brownian loop soup introduced in this paper, one obtains a new, conformally
covariant loop ensemble with the property that the diameter of individual loops has an exponential tail. Because of the previous considerations, it seems reasonable to conjecture that such massive, conformally covariant versions of CLE$_\kappa$ for $8/3 < \kappa \leq 4$ may describe the scaling limit of some off-critical systems, including perhaps one of the near-critical versions of the Ising model.

We arrive at the definition of our massive Brownian loop soup in two different ways, as we explain below.

The Brownian loop soup of [20] is a Poisson process of loops in the plane with intensity measure $\lambda \mu$, obtained by multiplying by a positive constant $\lambda$ the Brownian loop measure $\mu$ introduced in [35]. In Section 3.1 we define the massive Brownian loop soup as a Poisson process of loops with a new intensity, $\lambda \mu^m$, such that $d\mu^m(\gamma) = \eta_m(\gamma)d\mu(\gamma)$ and $\eta_m(\gamma) = \exp(-\int_0^{t_\gamma} m^2(\gamma(t))dt)$, where $m : \mathbb{C} \to \mathbb{R}$ is a nonnegative function and $t_\gamma$ is the duration of the loop $\gamma$. This particular form of the density, $\eta_m$, of $\mu^m$ with respect to $\mu$ (the Radon-Nikodym derivative) ensures that large Brownian loops are exponentially suppressed while the loop soup as a whole is conformally covariant.

In Sect. 3.2 we give precise definitions of the Brownian loop soup and its massive version, and study percolation in the massive Brownian loop soup, establishing a sharp connectivity phase transition. We also show that the Hausdorff dimension of the massive Brownian loop soup carpet in a bounded domain is the same as that of its massless counterpart (see Sect. 3.2 for the definition of carpet).

The second way to arrive at the massive Brownian loop soup is presented in Sect. 4 where we show that it can be obtained as the near-critical scaling limit of a “gas” (a Poissonian ensemble) of lattice loops of the type studied in [21, 22], which in turn is a generalization of the random walk loop soup introduced in [19]. The latter is a discrete analog of the Brownian loop soup.

In the final section, Sect. 5, we discuss some interesting relations between the gas of loops introduced in Sect. 4 and the discrete Gaussian free field in a bounded domain with Dirichlet boundary condition. In particular, we show that the gas of loops provides a measure of how much the free field at a point inside the domain feels a change in the shape of the domain. The results of this section hold for both the massless and massive Gaussian free field.
2 Brownian Loop Soups and the Geometry of Critical and Off-Critical Models

Symanzik, in his seminal work on Euclidean quantum field theory [31], introduced a representation of the $\phi^4$ Euclidean field as a “gas” of weakly interacting random paths. The use of random paths in the analysis of Euclidean field theories and statistical mechanical models was subsequently developed by various authors, most notably Brydges, Fröhlich, Spencer and Sokal [5, 4], and Aizenman [1], proving extremely useful (see [13] for a comprehensive account). The probabilistic analysis of Brownian and random walk paths and associated local times was carried out by Dynkin [11, 12]. More recently, “gases” or “soups” (i.e., Poissonian ensembles) of Brownian and random walk loops have been extensively studied in connection with the Schramm-Loewner Evolution and the Gaussian free field (see, e.g., [33, 20, 34, 21, 22, 32]).

As already mentioned, the Brownian loop soup with intensity $\lambda > 0$ [20] is a Poisson point process with intensity measure $\lambda$ times the Brownian loop measure $\mu$ introduced in [35]. If $D$ is a domain of $\mathbb{C}$, $\mu_D$ is $\mu$ restricted to those loops that stay in $D$: it is a measure on (equivalence classes of) loops $\gamma$ in $D$ of duration $t_\gamma$, where $\gamma : [0, t_\gamma] \to D$ is a continuous function (see Sect. 3.2 for precise definitions).

Given a conformal map $f : D \to D'$, let $f \circ \gamma(s)$ denote the loop $f(\gamma(t))$ in $D'$ withparametrization

$$s = s(t) = \int_0^t |f'(\gamma(u))|^2 \, du.$$  \hfill (1)

Given a subset $A$ of the space of loops in $D$, let $f \circ A = \{ \hat{\gamma} = f \circ \gamma \text{ with } \gamma \in A \}$. Up to a multiplicative constant, $\mu_D$ is the unique measure satisfying the following two properties, collectively known as conformal restriction property.

- For any conformal map $f : D \to D'$,
  $$\mu_{D'}(f \circ A) = \mu_D(A).$$  \hfill (2)

- If $D' \subset D$, $\mu_{D'}$ is $\mu_D$ restricted to loops that stay in $D'$.

A sample of the Brownian loop soup in $D$ with intensity $\lambda$ is the collection of loops, contained in $D$, from a Poisson realization of $\lambda \mu_D$. When $\lambda \leq 1$, the loop soup is composed of disjoint clusters of loops [33, 34, 29] (where a cluster
is a maximal collection of loops that intersect each other). When $\lambda > 1$, there is a unique cluster and the set of points not surrounded by a loop is totally disconnected (see [3]). Furthermore, when $\lambda \leq 1$, the outer boundaries of the loop soup clusters are distributed like Conformal Loop Ensembles (CLE) with $8/3 < \kappa \leq 4$. The latter are conjectured to describe the scaling limit of cluster boundaries in various critical models of statistical mechanics, such as the critical Potts models for $q \in (1, 4)$.

More precisely, if $8/3 < \kappa \leq 4$, then $0 < \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa} \leq 1$ and the collection of all outer boundaries of the clusters of the Brownian loop soup with intensity $\lambda = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}$ is distributed like CLE. For example, the continuum scaling limit of the collection of all macroscopic boundaries of critical Ising spin clusters is conjectured to correspond to CLE and to a Brownian loop soup with $\lambda = 1/2$. (A proof of this conjecture is due to appear in [10].)

The intensity $\lambda$ of the Brownian loop soup is related to the central charge $c$ of the corresponding statistical mechanical model. In some sense, $\lambda$ determines how much the system feels a change in the shape of the domain. To understand this, suppose that $\mathcal{A}(\lambda, D)$ is a realization of the Brownian loop soup in $D$ with intensity $\lambda$, and consider a subdomain $D' \subset D$. By removing from $\mathcal{A}(\lambda, D)$ all loops that are not contained in $D'$, one obtains the loop soup $\mathcal{A}(\lambda, D')$ in $D'$ with the same intensity $\lambda$. (This property of the Brownian loop soup follows from its Poissonian nature and the second bullet of the conformal restriction property of the Brownian loop measure.) The number of loops removed is stochastically increasing in $\lambda$, so that larger values of $\lambda$ imply that the system is more sensitive to changes in the shape of the domain. In some sense, the loops can be seen as mediators of correlations from the boundary of the domain. (A precise formulation of this observation is presented in Sect. 5 in the context of the discrete Gaussian free field.) We note that the change from $\mathcal{A}(\lambda, D)$ to $\mathcal{A}(\lambda, D')$ has a nonlocal effect, since the loops that are removed are extended objects, and that even a small local change to the shape of the domain can have an effect very far away, due to the scale invariance of the Brownian loop soup. This is a manifestation of the criticality of the system.

In view of the results and observations presented above, it seems natural to ask the following question:

(⋆) Can a near-critical scaling limit be described in terms of a loop soup?

Based on the previous discussion, in case of a positive answer, in the putative loop soup, loops of diameter larger than the correlation length should
be rare, since then the effect of a local change to the shape of the domain would typically only extend at most to a distance of the order of the correlation length away from the location where the change is made. At the same time one would expect the system of loops to closely resemble the Brownian loop soup at all scales much smaller than the correlation length.

In the next section we introduce a loop soup with these properties by modifying appropriately the Brownian loop soup of Lawler and Werner. The new loop soup is not scale invariant, but it is conformally covariant (see Eq. (4) below). In Sect. 4 we show that this loop soup arises as the near-critical scaling limit of a “gas” of loops which is a generalization of the random walk loop soup studied in [19].

3 The Massive Brownian Loop Soup

3.1 Heuristic Derivation

Suppose that the answer to (⋆) is positive and that the relevant loop soup is defined just like the Brownian loop soup, but with a different intensity measure, \( \mu^m \). Assume moreover that \( \mu^m \) is absolutely continuous with respect to \( \mu \), i.e.,

\[
d\mu^m(\gamma) = \eta_m(\gamma)d\mu(\gamma),
\]

where \( \eta_m \) denotes the Radon-Nikodym derivative; \( \eta_m \) should depend on the correlation length of the off-critical system.

If one is interested in loop soups restricted to bounded domains \( D \) and in their transformation properties under conformal maps, \( f : D \to D' \), it is natural to consider inhomogeneous systems with a correlation length that is a function of space. (Such a situation would in any case arise when conformally mapping a homogeneous system, unless \( |f'| \) is constant.) We describe this situation by introducing a positive function \( m \) that represents the inverse correlation length. We call \( m \) the mass function of the system.

Since \( |f'(z)| \) gives the amount of dilation at \( z \) under the conformal map \( f \), \( m \) should change locally as follows:

\[
m(z) \mapsto \tilde{m}(w) = |f'(f^{-1}(w))|^{-1} m(f^{-1}(w)),
\]

where \( w = f(z) \). As a consequence, conformal invariance (Eq. (2)), which in critical systems originates from the absence of a characteristic length due to
the infinite correlation length, should be replaced by conformal covariance:
\[ \mu_D^m(f \circ A) = \mu_D^m(A), \tag{4} \]
where \(A\) and \(f \circ A\) have the same meaning as in Eq. (2).

Now let \(\gamma(t)\) denote a loop of the Brownian loop soup of duration \(t_\gamma\), with a given time parametrization. Conformal invariance (Eq. (2)) requires that \(\gamma\) is mapped to the loop \(\tilde{\gamma}(s) = f \circ \gamma(t) = f(\gamma(t))\) with time parametrization
\[ s = s(t) = \int_0^t |f'(\gamma(u))|^2 du. \tag{5} \]
This is a consequence of Brownian scaling and implies, together with Eq. (3), that \(m^2 dt = \tilde{m}^2 ds\). Therefore, if \(\eta_m(\gamma)\) depends on \(m\) via some functional of
\[ R_m(\gamma) := \int_0^{t_\gamma} m^2(\gamma(t)) dt, \tag{6} \]
the measure \(\mu^m\) automatically satisfies conformal covariance (Eq. (4)). With this in mind, a simple choice for \(\eta_m(\gamma)\) is \(e^{-R_m(\gamma)}\), which leads to
\[ d\mu^m(\gamma) = \exp \left( -\int_0^{t_\gamma} m^2(\gamma(t)) dt \right) d\mu(\gamma). \tag{7} \]

We define the massive Brownian loop soup in \(D\) with intensity \(\lambda > 0\) and mass function \(m\) to be the Poisson process with intensity measure \(\lambda \mu_D^m\), where \(\mu_D^m\) denotes the restriction of \(\mu^m\) to \(D\).

Note that, for a homogeneous system, loops are exponentially suppressed at a rate proportional to their time duration. We will sometimes call the conformally invariant Brownian loop soup introduced by Lawler and Werner critical, to distinguish it from the massive Brownian loop soup defined above.

The above definition has a nice interpretation in terms of “killed” Brownian motion, which makes it appear very natural. For a given function \(f\) on the space of loops, one can write
\[ \int f(\gamma) e^{-R_m(\gamma)} d\mu(\gamma) = \int \mathbb{E}_{T_\gamma} f(\gamma) 1_{\{R_m(\gamma) < T_\gamma\}} d\mu(\gamma), \]
where \(\mathbb{E}_{T_\gamma}\) denotes expectation with respect to the law of the mean-one, exponential random variable \(T_\gamma\), and \(1_{\{\cdot\}}\) denotes the indicator function. In view of this, one can think of the Brownian loop \(\gamma\) under the measure \(\mu^m\) defined in (11) as being “killed” at rate \(m^2(\gamma(t))\). More precisely, one has the following alternative and useful characterization.
**Proposition 3.1** A massive Brownian loop soup in $D$ with intensity $\lambda$ and mass function $m$ can be realized in the following way.

1. Take a realization of the critical Brownian loop soup in $D$ with intensity $\lambda$.

2. Assign to each loop $\gamma$ of duration $t_{\gamma}$ an independent, mean-one, exponential random variable, $T_{\gamma}$.

3. Remove from the soup all loops $\gamma$ such that
   \[ \int_0^{t_{\gamma}} m^2(\gamma(t))dt > T_{\gamma}. \] (8)

**Remark 3.2** Note that Eq. (8) requires choosing a time parametrization for the loop $\gamma$ but is independent of the choice.

**Proof.** Let $\mathcal{L}^D$ denote the set of loops contained in $D$ and define $\mathcal{L}_{>\varepsilon}^D := \{ \gamma \in \mathcal{L}^D : \text{diam}(\gamma) > \varepsilon \}$ and $\mathcal{L}_{>\varepsilon,r}^D := \{ \gamma \in \mathcal{L}_{>\varepsilon}^D : R_m(\gamma) = r \}$. For a subset $A$ of $\mathcal{L}_{>\varepsilon}^D$, let $A_r = A \cap \mathcal{L}_{>\varepsilon,r}^D$. For every $\varepsilon > 0$, the restriction to loops of diameter larger than $\varepsilon$ of the massive Brownian loop soup in $D$ with mass function $m$ is a Poisson point process on $\mathcal{L}_{>\varepsilon}^D$ such that the expected number of loops in $A \subset \mathcal{L}_{>\varepsilon}^D$ at level $\lambda > 0$ is

\[
\lambda \mu^m(A) = \lambda \int_A e^{-R_m(\gamma)}d\mu(\gamma) = \lambda \int_0^\infty \int_{A_r} e^{-r}d\mu(\gamma)dr = \lambda \int_0^\infty e^{-r}\mu(A_r)dr.
\]

We will now show that, when attention is restricted to loops of diameter larger than $\varepsilon$, the construction of Proposition 3.1 produces a Poisson point process on $\mathcal{L}_{>\varepsilon}^D$ with the same expected number of loops at level $\lambda > 0$.

Let $N_\lambda(A)$ denote the number of loops in $A$ obtained from the construction of Proposition 3.1. Because the Brownian loop soup is a Poisson point process and loops are removed independently, for every $A \subset \mathcal{L}_{>\varepsilon}^D$ we have that

(i) $N_0(A) = 0,$
(ii) $\forall \lambda, \delta > 0$ and $0 \leq \ell \leq \lambda$, $N_{\lambda+\delta}(A) - N_{\lambda}(A)$ is independent of $N_{\ell}(A),$

(iii) $\forall \lambda, \delta > 0$, $\Pr(N_{\lambda+\delta}(A) - N_{\lambda}(A) \geq 2) = o(\delta),$

(iii) $\forall \lambda, \delta > 0$, $\Pr(N_{\lambda+\delta}(A) - N_{\lambda}(A) = 1) = \lambda(A)\delta + o(\delta),$

where

$$\tilde{\lambda}(A) := \lambda \int_0^\infty \frac{e^{-r}\mu(A_r)}{\mu(A)} dr$$

and (iii) follows from the fact that, conditioned on the event $N_{\lambda+\delta}(A) - N_{\lambda}(A) = 1$, the additional point (i.e., loop) that appears going from $\lambda$ to $\lambda + \delta$ is distributed according to the density $\frac{\mu(A_r)dr}{\mu(A)}$ on $A$. Conditions (i)-(iii) ensure that the point process is Poisson.

In order to identify the Poisson point process generated by the construction of Proposition 3.1 with the massive Brownian loop soup, it remains to compute the expected number of loops in $A$ at level $\lambda$. For every $\varepsilon > 0$ and $A \subset \mathcal{L}^D_{>\varepsilon}$, this is given by

$$\int_0^\infty e^{-r}\lambda\mu(A_r)dr = \lambda\mu^m(A),$$

which concludes the proof. □

### 3.2 Precise Definitions and Some Properties

In the previous section we gave a heuristic derivation of the massive Brownian loop soup and a characterization (Proposition 3.1) that provides a useful probabilistic coupling with the critical (i.e., conformally invariant) Brownian loop soup. In this section we first give more precise definitions of the critical and massive Brownian loop soups and then state and prove some properties of the massive Brownian loop soup.

A **rooted loop** $\gamma : [0, t] \to \mathbb{C}$ is a continuous function with $\gamma(0) = \gamma(t)$. We will consider only loops with $t_\gamma \in (0, \infty)$. The **Brownian bridge measure** $\mu^{br}$ is the probability measure on rooted loops of duration 1 with $\gamma(0) = 0$ induced by the Brownian bridge $B_t := W_t - tW_1$, $t \in [0, 1]$, where $W_t$ is standard, two-dimensional Brownian motion. A measure $\mu^{br}_{z,t}$ on loops rooted at $z \in \mathbb{C}$ (i.e., with $\gamma(0) = z$) of duration $t$ is obtained from $\mu^{br}$ by Brownian scaling, using the map

$$(\gamma, z, t) \mapsto z + t^{1/2}\gamma(s/t), \ s \in [0, t].$$
More precisely, we let
\[ \mu_{br}^{z,t}(\cdot) := \mu^{br}(\Phi_{z,t}^{-1}(\cdot)), \]
where
\[ \Phi_{z,t} : \gamma(s), s \in [0,1] \mapsto z + t^{1/2}\gamma(s/t), s \in [0,t]. \]

The rooted Brownian loop measure is defined as
\[ \mu_r := \int_{\mathbb{C}} \int_0^\infty \frac{1}{2\pi t^2} \mu_{br}^{z,t} dt dA(z), \]
where \( A \) denotes area.

The (unrooted) Brownian loop measure \( \mu \) is obtained from the rooted one by “forgetting the root.” More precisely, if \( \gamma \) is a rooted loop, \( \theta_u \gamma : t \mapsto \gamma(u+t \text{ mod } t) \) is again a rooted loop. This defines an equivalence relation between rooted loops, whose equivalence classes we refer to as (unrooted) loops; \( \mu(\gamma) \) is the \( \mu_r \)-measure of the equivalence class \( \gamma \). With a slight abuse of notation, in the rest of the paper we will use \( \gamma \) to denote an unrooted loop and \( \gamma(\cdot) \) to denote any representative of the equivalence class \( \gamma \).

The massive (unrooted) Brownian loop measure \( \mu^m \) is defined by the relation
\[ d\mu^m(\gamma) = \exp(-R_m(\gamma)) d\mu(\gamma), \]
where \( m : \mathbb{C} \to \mathbb{R} \) is a nonnegative mass function and
\[ R_m(\gamma) := \int_0^{t_\gamma} m^2(\gamma(t)) dt \]
for any rooted loop \( \gamma(t) \) in the equivalence class of the unrooted loop \( \gamma \). (Analogously, one can also define a massive rooted Brownian loop measure: \( d\mu^m_r(\gamma) := \exp(-R_m(\gamma)) d\mu_r(\gamma). \))

If \( D \) is a subset of \( \mathbb{C} \), we let \( \mu_D \) (respectively, \( \mu^m_D \)) denote \( \mu \) (resp., \( \mu^m \)) restricted to loops that lie in \( D \). The family of measures \( \{\mu_D\}_D \) (resp., \( \{\mu^m_D\}_D \)), indexed by \( D \subset \mathbb{C} \), satisfies the restriction property, i.e., if \( D' \subset D \), then \( \mu_{D'} \) (resp., \( \mu^m_{D'} \)) is \( \mu_D \) (resp., \( \mu^m_D \)) restricted to loops lying in \( D' \). Moreover, it is shown in [20] that the family \( \{\mu_D\}_D \) satisfies conformal invariance (see Eq. (2)). As a consequence, it is straightforward to check that the family \( \{\mu^m_D\}_D \) satisfies conformal covariance (see Eq. (4)).

**Definition 3.3** A Brownian loop soup in \( D \) with intensity \( \lambda \) is a Poissonian realization from \( \lambda \mu_D \).

A massive Brownian loop soup in \( D \) with intensity \( \lambda \) and mass function \( m \) is a Poissonian realization from \( \lambda \mu^m_D \).
Let $A(\lambda, m, D)$ denote a massive Brownian loop soup in $D \subseteq \mathbb{C}$ with mass function $m$ and intensity $\lambda$. We say that two loops are adjacent if they intersect; this adjacency relation defines clusters of loops, denoted by $\mathcal{C}$. (Note that clusters can be nested.) For each cluster $\mathcal{C}$, we write $\overline{\mathcal{C}}$ for the closure of the union of all the loops in $\mathcal{C}$; furthermore, we write $\hat{\mathcal{C}}$ for the filling of $\mathcal{C}$, i.e., the complement of the unbounded connected component of $\mathcal{C} \setminus \overline{\mathcal{C}}$. With a slight abuse of notation, we call $\hat{\mathcal{C}}$ a cluster and denote by $\hat{\mathcal{C}}_z$ the cluster containing $z$. We set $\hat{\mathcal{C}}_z = \emptyset$ if $z$ is not contained in any cluster $\hat{\mathcal{C}}$, and call the set $\{ z \in D : \hat{\mathcal{C}}_z = \emptyset \}$ the carpet (or gasket). (Informally, the carpet is the complement of the “filled-in” clusters.)

It is shown in [29] that, in the critical case ($m = 0$), if $D$ is bounded, the set of outer boundaries of the clusters $\hat{\mathcal{C}}$ that are not surrounded by other outer boundaries are distributed like a Conformal Loop Ensemble in $D$, as explained in Section 1.

**Theorem 3.4** Let $A(\lambda, m, D)$ be a massive Brownian loop soup in $D$ with intensity $\lambda$ and mass function $m$, and denote by $\mathbb{P}_{\lambda,m}$ the distribution of $A(\lambda, m, \mathbb{C})$.

- If $\lambda > 1$, $m$ is bounded and $D$ is bounded, with probability one the vacant set of $A(\lambda, m, D)$ is totally disconnected.
- If $\lambda \leq 1$ and $m$ is bounded away from zero, the vacant set of $A(\lambda, m, \mathbb{C})$ contains a unique infinite connected component. Moreover, there is a $\xi < \infty$ such that, for any $z \in \mathbb{C}$ and all $L > 0$,

$$
\mathbb{P}_{\lambda,m}(\text{diam}(\hat{\mathcal{C}}_z) \geq L) \leq e^{-L/\xi}.
$$

(13)

Note that, although in a massive loop soup individual large loops are exponentially suppressed, Eq. (13) is far from obvious, and in fact false when $\lambda > 1$, since in that equation the exponential decay refers to clusters of loops.

**Theorem 3.5** For any bounded domain $D \subseteq \mathbb{C}$ and any $m : D \to \mathbb{R}$ non-negative and bounded, the carpet of the massive Brownian loop soup in $D$ with mass function $m$ and intensity $\lambda$ has the same Hausdorff dimension as the carpet of the critical Brownian loop soup in $D$ with the same intensity.

It is expected that certain features of a near-critical scaling limit be the same as for the critical scaling limit. One of these features is the Hausdorff
dimension of certain geometric objects. For instance, it is proved in [25] that
the almost sure Hausdorff dimension of near-critical percolation interfaces in
the scaling limit is 7/4, exactly as in the critical case. In view of the results
in Sect. 4, Theorem 3.5 can be interpreted in the same spirit.

We conclude this section with the proofs of the two theorems. To prove
Theorem 3.4 we will use the following lemma, where, according to the nota-
tion of the theorem, \( P_{\lambda,m} \) denotes the distribution of the massive Brownian
loop soup in \( \mathbb{C} \) with intensity \( \lambda \) and mass function \( m \).

**Lemma 3.6** Let \( D \subset \mathbb{C} \) be a bounded domain with \( \text{diam}(D) > 1 \) and \( m \) a
positive function bounded away from zero. There exist constants \( c < \infty \) and
\( m_0 > 0 \), independent of \( D \), such that, for every \( \lambda > 0 \) and \( \ell_0 > 1 \),
\[
P_{\lambda,m}(\exists \gamma \text{ with diam}(\gamma) > \ell_0 \text{ and } \gamma \cap D \neq \emptyset) \geq 1 - \lambda c (\text{diam}(D) + 2\ell_0)^2 e^{-m_0 \ell_0}.
\]

**Proof.** Let \( D_\ell := \bigcup_{z \in D} B_\ell(z) \), where \( B_\ell(z) \) denotes the disc of radius \( \ell \)
centered at \( z \). We define several sets of loops, namely,
\[
\mathcal{L}_\ell := \{ \text{loops } \gamma \text{ with } \text{diam}(\gamma) = \ell \}
\]
\[
\mathcal{L}_\ell' := \{ \text{loops } \gamma \text{ with } \text{diam}(\gamma) = \ell \text{ and } t_\gamma \geq \ell \}
\]
\[
\mathcal{L}_\ell'' := \{ \text{loops } \gamma \text{ with } \text{diam}(\gamma) = \ell \text{ and } t_\gamma < \ell \}
\]
\[
\mathcal{L}_D := \{ \text{loops } \gamma \text{ with } \text{diam}(\gamma) = \ell \text{ and } \gamma \cap D \neq \emptyset \}
\]
\[
\mathcal{L}_{> \ell_0} := \bigcup_{\ell > \ell_0} \mathcal{L}_\ell = \{ \text{loops } \gamma \text{ with } \text{diam}(\gamma) > \ell_0 \text{ and } \gamma \cap D \neq \emptyset \}.
\]

We note that
\[
P_{\lambda,m}(\exists \gamma \text{ with diam}(\gamma) > \ell_0 \text{ and } \gamma \cap D \neq \emptyset) = \exp[-\lambda \mu^m(\mathcal{L}_{> \ell_0}^D)]
\]
\[
\geq 1 - \lambda \mu^m(\mathcal{L}_{> \ell_0}^D), \quad (14)
\]
where \( \mu^m \) denotes the massive Brownian loop measure with mass function \( m \).
Thus, in order to prove the lemma, we look for an upper bound for \( \mu^m(\mathcal{L}_{> \ell_0}^D) \).

Denoting by \( \mu^m_{D_\ell} \) the restriction of \( \mu^m \) to \( D_\ell \), we can write
\[
\mu^m(\mathcal{L}_{> \ell_0}^D) \leq \int_{\ell > \ell_0} d\mu^m_{D_\ell}(\mathcal{L}_\ell) = \int_{\ell > \ell_0} d\mu^m_{D_\ell}(\mathcal{L}_\ell') + \int_{\ell > \ell_0} d\mu^m_{D_\ell}(\mathcal{L}_\ell''). \quad (15)
\]

From the definition of the massive Brownian loop measure (see Eqs. (12)
and (11)), we have that
\[
\int_{\ell > \ell_0} d\mu^m_{D_\ell}(\mathcal{L}_\ell') \leq \int_{\ell_0}^\infty \frac{\pi}{4} \frac{\text{diam}^2(D_\ell)}{2\pi \ell^2} \exp\left(-\lambda \inf_{z \in D_\ell} m(z)\right) d\ell
\]
\[
< \frac{1}{8} \int_{\ell_0}^\infty \text{diam}^2(D_\ell) \exp\left(-\lambda \inf_{z \in D_\ell} m(z)\right) d\ell. \quad (16)
\]
To bound the second term of the RHS of (15), we observe that, if a loop $\gamma$ of duration $t_\gamma$ has $\text{diam}(\gamma) \geq \ell$, for any time parametrization of the loop, there exists a $t_0 \in (0, t_\gamma)$ such that $|\gamma(t_0) - \gamma(0)| \geq \ell/2$. The image $\hat{\gamma}$ of $\gamma$ under $\Phi^{-1}$ must then satisfy $|\hat{\gamma}(t_0/t_\gamma)| \geq \ell/(2\sqrt{t_\gamma})$ (see Eq. (10)).

Let $W_s$ and $B_s := W_s - sW_1$ denote standard two-dimensional Brownian motion and Brownian bridge, respectively, with $s \in [0, 1]$. Let $T(a)$ be the first time a standard one-dimensional Brownian motion hits level $a$. Noting that $|B_s| \leq |W_s| + |W_1| \leq 2|W_s| + |W_1 - W_s|$, the above observation gives the bound

\[
\mu_{\text{br},z,t}^\ell(\gamma : \text{diam}(\gamma) \geq \ell) \leq \mu_{\text{br},z,t}^\ell(\exists s \in (0, t) : |\gamma(s) - z| \geq \ell/2) = \mu_{\text{br}}^\ell\left(\exists s \in (0, 1) : |\hat{\gamma}(s)| \geq \frac{\ell}{2\sqrt{t}}\right) \leq \Pr\left(\exists s \in (0, 1) : |W_s| + |W_1 - W_s| \geq \frac{\ell}{4\sqrt{t}}\right) \leq \Pr\left(\exists s \in (0, 1) : |W_s| \geq \frac{\ell}{12\sqrt{t}}\right) \leq 4\Pr\left(T\left(\frac{\ell}{12\sqrt{2t}}\right) < 1\right) \leq \frac{96\sqrt{t}}{\sqrt{\pi}\ell} \exp\left(-\frac{\ell^2}{576t}\right),
\]

where we have used the fact that $T(a)$ has probability density $\frac{a}{\sqrt{2\pi}s^3} \exp(-a^2/2s)$ (see, e.g., Theorem 2.35 on p. 51 of [23]) and that

\[
\Pr(T(a) < 1) = \int_0^1 \frac{a}{\sqrt{2\pi}s^3} \exp\left(-\frac{a^2}{2s}\right) ds \leq \frac{a}{\sqrt{2\pi}} \int_0^1 \frac{1}{s^2} \exp\left(-\frac{a^2}{2s}\right) ds = \frac{\sqrt{2}}{\pi} \frac{e^{-a^2/2}}{a}.
\]

Using the above upper bound for $\mu_{\text{br},z,t}^\ell(\gamma : \text{diam}(\gamma) \geq \ell)$, we have that, for
\[ \ell_0 > 1, \]
\[
\int_{\ell > \ell_0} d\mu^m_{D,\ell}(\mathcal{L}^D) \leq \int_{\ell_0}^{\infty} \frac{\pi}{4} \frac{\text{diam}^2(D)}{t} \int_{0}^{\ell} \frac{1}{2\pi t^2} \mu^m_{0,t}(\gamma : \text{diam}(\gamma) \geq \ell) \, dt \, d\ell
\]
\[
\leq \int_{\ell_0}^{\infty} \frac{\pi}{4} \frac{\text{diam}^2(D)}{\ell} \int_{0}^{\ell} \frac{1}{\ell^2} \exp \left( -\frac{\ell^2}{576t} \right) \, dt \, d\ell
\]
\[
\leq \frac{12}{\sqrt{\pi}} \int_{\ell_0}^{\infty} \frac{\text{diam}^2(D)}{\ell} \int_{0}^{1} \frac{1}{\ell^2} \exp \left( -\frac{\ell^2}{576t} \right) \, dt \, d\ell
\]
\[
+ \frac{12}{\sqrt{\pi}} \int_{\ell_0}^{\infty} \frac{\text{diam}^2(D)}{\ell} \int_{1}^{\ell} \exp \left( -\frac{\ell^2}{576t} \right) \, dt \, d\ell
\]
\[
\leq \tilde{c} \int_{\ell_0}^{\infty} \text{diam}^2(D) e^{-\ell/576} \, d\ell
\]

where \( \tilde{c} \) is a suitably chosen constant that does not depend on \( D, \ell_0 \) or \( m \).

Combining this bound with the bound \((16)\), we can write

\[ \mu^m(\mathcal{L}^D_{> \ell_0}) \leq \tilde{c} \int_{\ell_0}^{\infty} \text{diam}^2(D) e^{-m_0 \ell} \, d\ell, \]

where \( \tilde{c} < \infty \) is a suitably chosen constant independent of \( D, \ell_0 \) and \( m \), and \( m_0 > 0 \) is any positive number smaller than \( \min(1/576, \inf_{z \in \mathbb{C}} m(z)) \). From this, a simple calculation leads to

\[ \mu^m(\mathcal{L}^D_{> \ell_0}) \leq \tilde{c} \int_{\ell_0}^{\infty} (\text{diam}(D) + 2\ell)^2 e^{-m_0 \ell} \, d\ell \leq c (\text{diam}(D) + 2\ell_0)^2 e^{-m_0 \ell_0}, \]

where \( c < \infty \) is a constant that does not depend on \( D \) and \( \ell_0 \). The proof is concluded using inequality \((14)\). \( \square \)

**Proof of Theorem 3.4.** We first prove the statement in the first bullet. Let \( \overline{m} := \sup_{z \in \mathbb{C}} m(z) \); since \( m \) is bounded, \( \overline{m} < \infty \). Let \( \tau > 0 \) be so small that \( \lambda' := e^{-\overline{m}\tau} \lambda > 1 \) and denote by \( A(\lambda, 0, D) \) a critical loop soup in \( D \) with intensity \( \lambda \) obtained from the full-plane loop soup \( A(\lambda, 0, \mathbb{C}) \). Let \( A(\lambda, m, D) \) be a massive soup obtained from \( A(\lambda, 0, D) \) via the construction of Prop. 3.1, and let \( \overline{A}(\lambda', \tau, D) \) denote the collection of loops obtained from \( A(\lambda, 0, D) \) by removing all loops of duration \( > \tau \) with probability one, and other loops with probability \( 1 - e^{-\overline{m}\tau} \), so that \( \overline{A}(\lambda', \tau, D) \) is a loop soup with intensity \( \lambda' = e^{-\overline{m}\tau} \lambda \), restricted to loops of duration \( \leq \tau \). If we use the same exponential random variables to generate \( A(\lambda, m, D) \) and \( \overline{A}(\lambda', \tau, D) \)
from $\mathcal{A}(\lambda, 0, D)$, the resulting loop soups are coupled in such a way that $\mathcal{A}(\lambda, m, D)$ contains all the loops that are contained in $\overline{\mathcal{A}}(\lambda', \tau, D)$, and the vacant set of $\mathcal{A}(\lambda, m, D)$ is a subset of the vacant set of $\overline{\mathcal{A}}(\lambda', \tau, D)$.

We will now show that, for every $S \subset D$ at positive distance from the boundary of $D$, the intersection with $S$ of the vacant set of $\overline{\mathcal{A}}(\lambda', \tau, D)$ is totally disconnected. The same statement is then true for the intersection with $S$ of the vacant set of $\mathcal{A}(\lambda, m, D)$, which concludes the proof of the first bullet, since the presence of a connected component larger than one point in the vacant set of $\mathcal{A}(\lambda, m, D)$ would lead to a contradiction.

For a given $S \subset D$ and any $\varepsilon > 0$, take $\tau = \tau(S, \varepsilon)$ so small that $e^{-m^2 \tau} \lambda > 1$ and the probability that a loop from $\mathcal{A}(\lambda, 0, \mathbb{C})$ of duration $\leq \tau$ intersects both $S$ and the complement of $D$ is less than $\varepsilon$. (This is possible because the $\mu$-measure of the set of loops that intersect both $S$ and the complement of $D$ is finite.) If that event does not happen, the intersection between $S$ and the vacant set of $\overline{\mathcal{A}}(\lambda', \tau, D)$ coincides with the intersection between $S$ and the vacant set of the full-plane loop soup $\overline{\mathcal{A}}(\lambda', \tau, \mathbb{C})$ with cutoff $\tau$ on the duration of loops. The latter intersection is a totally disconnected set with probability one by an application of Theorem 2.5 of [3]. (Note that the result does not follow directly from Theorem 2.5 of [3], which deals with full-space soups with a cutoff on the diameter of loops, but can be easily obtained from it, for example with a coupling between $\overline{\mathcal{A}}(\lambda', \tau, \mathbb{C})$ and a full-plane soup with a cutoff on the diameter of loops chosen to be much smaller than $\sqrt{\tau}$, and using arguments along the lines of those in the proof of Lemma 3.6. We leave the details to the interested reader.) Since $\tau$ can be chosen arbitrarily small, this shows that the intersection with any $S$ of the vacant set of $\overline{\mathcal{A}}(\lambda', \tau, D)$ is totally disconnected with probability one.

To prove the statement in the second bullet we need some definitions. Let $R_l := [0, 3l] \times [0, l]$ and denote by $A_l$ the event that the vacant set of a loop soup contains a crossing of $R_l$ in the long direction, i.e., that it contains a connected component which stays in $R_l$ and intersects both $\{0\} \times [0, l]$ and $\{3l\} \times [0, l]$. Furthermore, let $E^\ell_l := \{ \not\exists \gamma \text{ with diam}(\gamma) > \ell \text{ and } \gamma \cap R_l \neq \emptyset \}$ and denote by $\overline{\mathbb{P}}_{\lambda, \ell}$ the distribution of the critical Brownian loop soup in $\mathbb{C}$ with intensity $\lambda$ and cutoff $\ell$ on the diameter of the loops (i.e., with all the loops of diameter $> \ell$ removed).
We have that, for any $\ell_0 > 0$ and $n \in \mathbb{N}$,
\[
\mathbb{P}_{\lambda,m}(A_{3^n}) \geq \mathbb{P}_{\lambda,m}(A_{3^n} \cap E_{3^n}^{\ell_0}) = \mathbb{P}_{\lambda,m}(A_{3^n} \mid E_{3^n}^{\ell_0}) \mathbb{P}_{\lambda,m}(E_{3^n}^{\ell_0}) \geq \mathbb{P}_{\lambda,n\ell_0}(A_{3^n}) \mathbb{P}_{\lambda,m}(E_{3^n}^{\ell_0}) = \mathbb{P}_{\lambda,3^{-n}n\ell_0}(A_1) \mathbb{P}_{\lambda,m}(E_{3^n}^{\ell_0})
\]
where we have used the Poissonian nature of the loop soup in the second inequality, and the last equality follows from scale invariance.

Now consider a sequence $\{A(\lambda, 3^{-n}n\ell_0, \mathbb{C})\}_{n \geq 1}$ of full-plane soups with cutoffs $\{3^{-n}n\ell_0\}_{n \geq 1}$, obtained from the same critical Brownian loop soup $\mathcal{A}(\lambda, 0, \mathbb{C})$ by removing all loops of diameter larger than the cutoff. The soups then couple in such a way that their vacant sets form an increasing (in the sense of inclusion of sets) sequence of sets. Therefore, by Kolmogorov’s zero-one law, $\lim_{n \to \infty} \mathbb{P}_{\lambda,3^{-n}n\ell_0}(A_1)$ is either 0 or 1. (Note that this limit can be seen as the probability of the union over $n \geq 1$ of the events that the rectangle $R_1$ is crossed in the long direction by the vacant set of the soup with cutoff $3^{-n}n\ell_0$.) Since $\mathbb{P}_{\lambda,3^{-n}n\ell_0}(A_1)$ is strictly positive for $n = 1$ (see Section 3 of [3]) and clearly increasing in $n$, we conclude that $\lim_{n \to \infty} \mathbb{P}_{\lambda,3^{-n}n\ell_0}(A_1) = 1$.

Moreover, it follows from Lemma 3.6 that
\[
\mathbb{P}_{\lambda,m}(E_{3^n}^{\ell_0}) \geq 1 - \lambda c (\sqrt{10} + 2\ell_0)^2 e^{(2\log 3 - m_0\ell_0)n},
\]
for some constants $c < \infty$ and $m_0 > 0$ independent of $\ell_0$ and $n$. Choosing $\ell_0 > (2\log 3)/m_0$, we have that $\mathbb{P}_{\lambda,m}(E_{3^n}^{\ell_0}) \to 1$ as $n \to \infty$, which implies that $\lim_{n \to \infty} \mathbb{P}_{\lambda,m}(A_{3^n}) = 1$. Note that the result does not depend on the position and orientation of the rectangles $R_1$ chosen to define the event $A_1$.

Crossing events for the vacant set are decreasing in $\lambda$ and are therefore positively correlated (see, e.g., Lemma 2.2 of [17]). (An event $A$ is decreasing if $A \not\subset A'$ implies $A' \not\subset A$ whenever $A$ and $A'$ are two soup realizations such that $A'$ contains all the loops contained in $A$.) Let $A_n := [-3^{n+1}/2, 3^n + 1/2] \times [-3^{n+1}/2, 3^n + 1/2] \setminus [-3^n/2, 3^n/2] \times [-3^n/2, 3^n/2]$ and $C_n$ denote the event that a connected component of the vacant set makes a circuit inside $A_n$ surrounding $[-3^n/2, 3^n/2] \times [-3^n/2, 3^n/2]$. Using the positive correlation of crossing events, and the fact that the circuit described above can be obtained by “pasting” together four crossings of rectangles, we conclude that $\lim_{n \to \infty} \mathbb{P}_{\lambda,m}(C_n) = 1$.

The existence of a unique unbounded component in the vacant set now follows from standard arguments (see, e.g., the proof of Theorem 3.2 of [3]).
The exponential decay of loop soup clusters also follows immediately, since the occurrence of $C_n$ prevents the cluster of the origin from extending beyond the square $[-3^{n+1}/2, 3^{n+1}/2] \times [-3^{n+1}/2, 3^{n+1}/2]$.

**Proof of Theorem 3.5.** Let $m_D := \sup_{z \in D} m(z)$; since $m$ is bounded, $m_D < \infty$. Fix $\tau \in (0, \infty)$ and define $\lambda' := e^{-m_D \tau} \lambda < \lambda$. Denote by $\mathcal{A}(\lambda, 0, D)$ a critical loop soup in $D$ with intensity $\lambda$. Let $\mathcal{A}(\lambda, m, D)$ be a massive loop soup obtained from $\mathcal{A}(\lambda, 0, D)$ via the construction of Prop. 3.1, and let $\mathcal{A}(\lambda', \tau, D)$ denote the set of loops obtained from $\mathcal{A}(\lambda, 0, D)$ by removing all loops of duration $> \tau$ with probability one, and other loops with probability $1 - e^{-m_D \tau}$, so that $\mathcal{A}(\lambda', \tau, D)$ is a loop soup with intensity $\lambda' = e^{-m_D \tau} \lambda$, restricted to loops of duration $\leq \tau$. If we use the same exponential random variables to generate $\mathcal{A}(\lambda, m, D)$ and $\mathcal{A}(\lambda', \tau, D)$ from $\mathcal{A}(\lambda, 0, D)$, the resulting loop soups are coupled in such a way that $\mathcal{A}(\lambda, m, D)$ contains all the loops that are contained in $\mathcal{A}(\lambda', \tau, D)$, and the vacant set of $\mathcal{A}(\lambda, m, D)$ is a subset of the vacant set of $\mathcal{A}(\lambda', \tau, D)$. Let $\mathbb{P}$ denote the probability distribution corresponding to the coupling between soups described above.

Note that, if we denote carpets by $G(\cdot)$, we have that

$$G[\mathcal{A}(\lambda', \tau, D)] \supset G[\mathcal{A}(\lambda, m, D)] \supset G[\mathcal{A}(\lambda, 0, D)].$$

(17)

Moreover, letting

$$h(\ell) := \frac{187 - 7\ell + \sqrt{25 + \ell^2 - 26\ell}}{96},$$

denoting the Hausdorff dimension of a set $S$ by $H(S)$, and combining the computation of the expectation dimension for carpets/gaskets of Conformal Loop Ensembles [27] with the results of [24] (see, in particular, Section 4.5 of [24]), we have that, for any $\ell \in [0, 1]$, with probability one,

$$H(G(\mathcal{A}(\ell, 0, D))) = h(\ell).$$

We will now show that the almost sure Hausdorff dimension of $\mathcal{A}(\lambda', \tau, D)$ equals $h(\lambda')$. To do this, consider the event $E$ that $\mathcal{A}(\lambda, 0, D)$ contains no loop of duration $> \tau$. Note that $\mathbb{P}(E) > 0$, since the set of loops of duration $> \tau$ that stay in $D$ has finite mass for the Brownian loop measure $\mu$. Note also that, on the event $E$, $\mathcal{A}(\lambda', \tau, D)$ coincides with $\mathcal{A}(\lambda', 0, D)$. Thus, since the sets of loops of duration $> \tau$ and $\leq \tau$ are disjoint, the Poissonian nature of the loop soups implies that

$$\mathbb{P}(H(G[\mathcal{A}(\lambda', \tau, D)]) = h(\lambda')) = \mathbb{P}(H(G[\mathcal{A}(\lambda', 0, D)]) = h(\lambda') | E) = 1.$$
From this and (17), it follows that, with probability one,
\[ h(\lambda') = H(G[\mathcal{A}(\lambda', \tau, D)]) \geq H(G[\mathcal{A}(\lambda, m, D)]) \geq H(G[\mathcal{A}(\lambda, 0, D)]) = h(\lambda). \]
Since \( h \) is continuous, letting \( \tau \to 0 \) (so that \( \lambda' \to \lambda \)) concludes the proof. \( \square \)

4 Random Walk Loop Soups and Scaling Limits

In this section we show that the massive Brownian loop soup emerges as the near-critical scaling limit of the random walk loop soup with killing. In what follows, we will consider \( \mathbb{Z}^2 \) as a subset of \( \mathbb{C} \).

Let \( k_x \geq 0 \) for every \( x \in \mathbb{Z}^2 \) and define \( p_{x,y} = 1/(k_x + 4) \) if \( |x - y| = 1 \) and \( p_{x,y} = 0 \) otherwise. If \( k_x = 0 \) for all \( x \), \( \{p_{x,y}\}_{y \in \mathbb{Z}^2} \) is the collection of transition probabilities for the simple symmetric random walk on \( \mathbb{Z}^2 \). If \( k_x \neq 0 \), then \( p_{x,y} = \frac{1}{4}(1 + \frac{k_x}{4})^{-1} < \frac{1}{4} \) and one can interpret \( \{p_{x,y}\}_{y \in \mathbb{Z}^2} \) as the collection of transition probabilities for a random walker “killed” at \( x \) with probability \( 1 - (1 + \frac{k_x}{4})^{-1} = \frac{k_x}{k_x + 4} \). (Equivalently, one can introduce a “cemetery” state \( \Delta \) not in \( \mathbb{Z}^2 \) to which the random walker jumps from \( x \in \mathbb{Z}^2 \) with probability \( \frac{k_x}{k_x + 4} \), and where it stays forever once it reaches it.) Because of this interpretation, we will refer to the collection \( k = \{k_x\}_{x \in \mathbb{Z}^2} \) as killing rates.

Given a \((2n + 1)\)-tuple \((x_0, x_1, \ldots, x_{2n})\) with \( x_0 = x_{2n} \) and \( |x_i - x_{i-1}| = 1 \) for \( i = 1, \ldots, 2n \), we call rooted lattice loop the continuous path \( \tilde{\gamma} : [0, 2n] \to \mathbb{C} \) with \( \tilde{\gamma}(i) = x_i \) for integer \( i = 0, \ldots, 2n \) and \( \tilde{\gamma}(t) \) obtained by linear interpolation for other \( t \). We call \( x_0 \) the root of the loop and denote by \( |\tilde{\gamma}| = 2n \) the length or duration of the loop.

Now let \( D \) denote either \( \mathbb{C} \) or a connected subset of \( \mathbb{C} \). Following Lawler and Trujillo Ferreras [19], but within the more general framework of the previous paragraph, we introduce the rooted random walk loop measure \( \nu_{D}^{r,k} \) which assigns the loop \( \tilde{\gamma} \) of length \( |\tilde{\gamma}| \), with root \( x \), weight \( |\tilde{\gamma}|^{-1}p_{x,x_1}p_{x_1,x_2} \cdots p_{x_{|\tilde{\gamma}|-1},x} \) if \( x, \ldots, x_{|\tilde{\gamma}|-1} \in D \) and 0 otherwise.

The unrooted random walk loop measure \( \nu_{D}^{u,k} \) is obtained from the rooted one by “forgetting the root.” More precisely, if \( \tilde{\gamma} \) is a rooted lattice loop and \( j \) a positive integer, \( \theta_j \tilde{\gamma} : t \mapsto \tilde{\gamma}(j + t \mod |\tilde{\gamma}|) \) is again a rooted loop. This defines an equivalence relation between rooted loops; an unrooted lattice loop
is an equivalence class of rooted lattice loops under that relation. By a slight abuse of notation, in the rest of the paper we will use ˜γ to denote unrooted lattice loops and ˜γ(·) to denote any rooted lattice loop in the equivalence class of ˜γ. The νD u,k-measure of the unrooted loop ˜γ is the sum of the νD r,k-measures of the rooted loops in the equivalence class of ˜γ. The length or duration, |˜γ|, of an unrooted loop ˜γ is the length of any one of the rooted loops in the equivalence class ˜γ.

A random walk loop soup in D with intensity λ is a Poissonian realization from λνu,kD. A realization of the random walk loop soup in D is a multiset (i.e., a set whose elements can occur multiple times) of unrooted loops. If we denote by Nγ the multiplicity of ˜γ in a loop soup with intensity λ, {Nγ} is a collection of independent Poisson random variables with parameters λνD u,k(˜γ). Therefore, the probability that a realization of the random walk loop soup in D with intensity λ contains each loop ˜γ in D with multiplicity nγ ≥ 0 is equal to

\[ \prod_{\tilde{\gamma}} \exp \left( -\lambda \nu_{D}^{u,k}(\tilde{\gamma}) \right) \frac{1}{n_\gamma!} \left( \lambda \nu_{D}^{u,k}(\tilde{\gamma}) \right)^{n_\gamma} = \frac{1}{Z_{\lambda,k}} \prod_{\tilde{\gamma}} \frac{1}{n_\gamma!} \left( \lambda \nu_{D}^{u,k}(\tilde{\gamma}) \right)^{n_\gamma}, \]

(18)

where the product \( \prod_{\tilde{\gamma}} \) is over all unrooted lattice loops in D and

\[ Z_{\lambda,k} := \exp \left( \lambda \sum_{\tilde{\gamma}} \nu_{D}^{u,k}(\tilde{\gamma}) \right) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n)} \prod_{i=1}^{n} \lambda \nu_{D}^{u,k}(\tilde{\gamma}_i), \]

(19)

where the sum over (\( \tilde{\gamma}_1, \ldots, \tilde{\gamma}_n \)) is over all ordered configurations of n loops, not necessarily distinct. From a statistical mechanical viewpoint, \( Z_{\lambda,k} \) can be interpreted as the grand canonical partition function of a “gas” of loops, and one can think of the random walk loop soup as describing a grand canonical ensemble of noninteracting loops (an “ideal gas”) with the killing rates \{k_x\} and the intensity \( \lambda \) as free “parameters.” (For more on the statistical mechanical interpretation of the model, see Sect. 6.4 of [2].)

When \( k_x = 0 \) \( \forall x \in D \cap \mathbb{Z}^2 \), we use \( \nu_{D}^{u} \) to denote the unrooted random walk loop measure in D; for reasons that will be clear when we talk about scaling limits, later in this section, a random walk loop soup obtained using such a measure will be called critical.

Now let \( m : \mathbb{C} \rightarrow \mathbb{R} \) be a nonnegative function; we say that a random walk loop soup has mass (function) \( m \) if \( k_x = 4(e^{m^2(x)} - 1) \) for all \( x \in D \cap \mathbb{Z}^2 \), and call massive a random walk loop soup with mass \( m \) that is not identically
zero on $D \cap \mathbb{Z}^2$. For a massive random walk loop soup in $D$ with intensity $\lambda$ and mass $m$ we use the notation $\tilde{A}(\lambda, m, D)$.

The next proposition gives a construction for generating a massive random walk loop soup from a critical one, establishing a useful probabilistic coupling between the two (i.e., a way to construct the two loop soups on the same probability space).

**Proposition 4.1** A random walk loop soup in $D$ with intensity $\lambda$ and mass function $m$ can be realized in the following way.

1. Take a realization of the critical random walk loop soup in $D$ with intensity $\lambda$.
2. Assign to each loop $\tilde{\gamma}$ an independent, mean-one, exponential random variable $T_{\tilde{\gamma}}$.
3. Remove from the soup the loop $\tilde{\gamma}$ of length $|\tilde{\gamma}|$ if
   \[
   \sum_{i=0}^{|\tilde{\gamma}|-1} m^2(\tilde{\gamma}(i)) > T_{\tilde{\gamma}}. \tag{20}
   \]

**Remark 4.2** Note that Eq. \(20\) requires choosing a rooted loop from the equivalence class $\tilde{\gamma}$ but is independent of the choice.

**Proof.** The proof is analogous to that of Proposition 3.1, so we leave the details to the reader. Below we just compare the expected number of loops in $D$ generated by the construction of Proposition 4.1 with that of the massive random walk loop soup, to verify the relation between killing rates $\{k_x\}$ and mass function $m$.

Writing $p_{x,y} = \frac{1}{k_x + 1} = \frac{1}{4} \frac{4}{k_x + 4}$ when $|x - y| = 1$, for the massive random walk soup we have

\[
\lambda \sum_{\tilde{\gamma}} \nu_D^{nu}(\tilde{\gamma}) = \lambda \sum_{\tilde{\gamma}} \nu_D^{nu}(\tilde{\gamma}) \prod_{i=0}^{|\tilde{\gamma}|-1} \frac{4}{k_{x_i} + 4}, \tag{21}
\]

where the sum $\sum_{\tilde{\gamma}}$ is over all unrooted loops in $D$ and the right hand side implies that we have chosen a representative for $\tilde{\gamma}$ such that $\tilde{\gamma}(i) = x_i$, but is independent of the choice.
The expected number of loops resulting from the construction of Prop. 4.1 is

$$
\lambda \sum_{\tilde{\gamma}} \nu_{D}^{\mu}(\tilde{\gamma}) \int_{0}^{\infty} e^{-t} \mathbb{1}_{\{\sum_{i=0}^{|\tilde{\gamma}|} m^{2}(\tilde{\gamma}(i)) < t\}} \, dt
$$

$$
= \lambda \sum_{\tilde{\gamma}} \nu_{D}^{\mu}(\tilde{\gamma}) \prod_{i=0}^{|\tilde{\gamma}| - 1} e^{-m^{2}(x_{i})}, \quad (22)
$$

where we have chosen the same representative with $\tilde{\gamma}(i) = x_{i}$ as before and the result is again independent of the choice.

Comparing Eqs. (21) and (22) gives

$$
e^{-m^{2}(x)} = \frac{4}{k_{x} + 4} = (1 + k_{x}/4)^{-1} \quad \text{or} \quad k_{x} = 4(e^{m^{2}(x)} - 1).
$$

We are now going to consider scaling limits for the random walk loop soup defined above. Consider first a critical, full-plane, random walk loop soup $\tilde{A}_{\lambda} \equiv \tilde{A}(\lambda, 0, \mathbb{Z}^{2})$. Following [19], for each integer $N \geq 2$, we define the rescaled loop soup

$$
\tilde{A}_{\lambda}^{N} = \{ \tilde{\Phi}_{N}\tilde{\gamma} : \tilde{\gamma} \in \tilde{A}_{\lambda} \} \quad \text{with} \quad \tilde{\Phi}_{N}\tilde{\gamma}(t) = N^{-1}\tilde{\gamma}(2N^{2}t).
$$

$\tilde{\Phi}_{N}\tilde{\gamma}$ is a lattice loop of duration $t_{\tilde{\gamma}} := |\tilde{\gamma}|/(2N^{2})$ on the rescaled lattice $\frac{1}{N}\mathbb{Z}^{2}$ and so $\tilde{A}_{\lambda}^{N}$ is a random walk loop soup on $\frac{1}{N}\mathbb{Z}^{2}$, with rescaled time. It is shown in [19] that, as $N \to \infty$, $\tilde{A}_{\lambda}^{N}$ converges to the Brownian loop soup of [20] in an appropriate sense. This means that the critical random walk loop soup has a conformally invariant scaling limit (the Brownian loop soup), which explains our use of the term critical.

If we rescale in the same way a massive random walk loop soup with constant mass function $m > 0$, the resulting scaling limit is trivial, in the sense that it does not contain any loops larger than one point. This is so because, under the random walk loop measure, only loops of duration of order at least $N^{2}$ have diameter of order at least $N$ with non-negligible probability as $N \to \infty$, and are therefore “macroscopic” in the scaling limit. It is then clear that, in order to obtain a nontrivial scaling limit, the mass function needs to be rescaled while taking the scaling limit.

Suppose, for simplicity, that the mass function $m$ is constant, and let $m_{N}$ denote the rescaled mass function. When $m_{N}$ tends to zero, $k_{x} \approx 4m_{N}^{2}$ and one has the following dichotomy.
• If \( \lim_{N \to \infty} Nm_N = 0 \), loops with a number of steps of the order of \( N^2 \) or smaller are not affected by the killing in the scaling limit and one recovers the critical Brownian loop soup.

• If \( \lim_{N \to \infty} Nm_N = \infty \), all loops with a number of steps of the order of \( N^2 \) or more are removed from the soup in the scaling limit and no “macroscopic” loop (larger than one point) is left.

In view of this observation, a near-critical scaling limit, that is, a nontrivial scaling limit that differs from the critical one, can only be obtained if the mass function \( m \) is rescaled by \( O(1/N) \). This leads us to considering the loop soup \( \tilde{A}_N^\lambda m \) defined as a random walk loop soup on the rescaled lattice \( \frac{1}{\sqrt{2}} \mathbb{Z}^2 \) with mass function \( m/(\sqrt{2}N) \) and rescaled time as in (23). Such a soup can be obtained from \( \tilde{A}_N^\lambda \) using the construction in Prop. 4.1, replacing \( m^2(\tilde{\gamma}(i)) \) with \( \frac{1}{2N^2} m^2(\tilde{\gamma}(i))/N \) in Eq. (20).

**Theorem 4.3** Let \( m \) be a nonnegative function such that \( m^2 \) is Lipschitz continuous. There exist two sequences \( \{A_{\lambda,m}^N\}_{N \geq 2} \) and \( \{\tilde{A}_{\lambda,m}^N\}_{N \geq 2} \) of loop soups defined on the same probability space and such that the following holds.

• For each \( \lambda > 0 \), \( A_{\lambda,m}^N \) is a massive Brownian loop soup in \( \mathbb{C} \) with intensity \( \lambda \) and mass \( m \); the realizations of the loop soup are increasing in \( \lambda \).

• For each \( \lambda > 0 \), \( \tilde{A}_{\lambda,m}^N \) is a massive random walk loop soup on \( \frac{1}{\sqrt{2}} \mathbb{Z}^2 \) with intensity \( \lambda \), mass \( m/(\sqrt{2}N) \) and time scaled as in (23); the realizations of the loop soup are increasing in \( \lambda \).

• For every bounded \( D \subset \mathbb{C} \), with probability going to one as \( N \to \infty \), loops from \( A_{\lambda,m}^N \) and \( \tilde{A}_{\lambda,m}^N \) that are contained in \( D \) and have duration at least \( N^{-1/6} \) can be put in a one-to-one correspondence with the following property. If \( \gamma \in A_{\lambda,m}^N \) and \( \tilde{\gamma} \in \tilde{A}_{\lambda,m}^N \) are paired in that correspondence and \( t_\gamma \) and \( t_{\tilde{\gamma}} \) denote their respective durations, then

\[
|t_\gamma - t_{\tilde{\gamma}}| \leq \frac{5}{8}N^{-2}
\]

\[
\sup_{0 \leq s \leq 1} |\gamma(st_\gamma) - \tilde{\gamma}(st_{\tilde{\gamma}})| \leq c_1 N^{-1} \log N
\]

for some constant \( c_1 \).
Proof. Let $A_\lambda$ be a critical Brownian loop soup in $\mathbb{C}$ with intensity $\lambda$ and $\tilde{A}_\lambda$ a critical random walk loop soup on $\mathbb{Z}^2$ with intensity $\lambda$, coupled as in Theorem 1.1 of [19]. Consider the scaled loop soups $A^N_\lambda$ and $\tilde{A}^N_\lambda$, where $\tilde{A}^N_\lambda$ is defined in (23) and $A^N_\lambda := \{\Phi^N_N : \gamma \in A_\lambda\}$ with $\Phi^N_N \gamma(t) = N^{-1}\gamma(N^2 t)$ for $0 \leq t \leq t_\gamma/N^2$. Note that, because of scale invariance, $A^N_\lambda$ is a critical Brownian loop soup in $\mathbb{C}$ with parameter $\lambda$.

It readily follows from Theorem 1.1 of [19] that, if one considers only loops of duration greater than $N^{-1/6}$, loops from $A^N_\lambda$ and $\tilde{A}^N_\lambda$ contained in $D$ can be put in a one-to-one correspondence with the properties described in Theorem 4.3, except perhaps on an event of probability going to zero as $N \to \infty$. For simplicity, in the rest of the proof we will call macroscopic the loops of duration greater than $N^{-1/6}$.

On the event that such a one-to-one correspondence between macroscopic loops in $D$ exists, we construct the massive loop soups $A^N_{\lambda,m}$ and $\tilde{A}^N_{\lambda,m}$ in the following way. To each pair of macroscopic loops $\gamma \in A^N_\lambda$ and $\tilde{\gamma} \in \tilde{A}^N_\lambda$, paired in the correspondence of Theorem 1.1 of [19], we assign an independent, mean-one, exponential random variable $T_\gamma$. We let $t_\gamma$ denote the (rescaled) duration of $\gamma$ and $t_{\tilde{\gamma}}$ the (rescaled) duration of $\tilde{\gamma}$, and let $M = 2N^2 t_{\tilde{\gamma}}$ denote the number of steps of the lattice loop $\tilde{\gamma}$. As in the constructions described in Props. 3.1 and 1.1 we remove $\gamma$ from the Brownian loop soup if $\int_0^{t_\gamma} m^2(\gamma(s))ds > T_\gamma$ and remove $\tilde{\gamma}$ from the random walk loop soup if $\frac{1}{2N^2} \sum_{k=0}^{M-1} m^2(\tilde{\gamma}(\frac{k}{2N^2})) > T_\gamma$. The resulting loop soups, $A^N_{\lambda,m}$ and $\tilde{A}^N_{\lambda,m}$, are defined on the same probability space and are distributed like a massive Brownian loop soup with mass function $m$ and a random walk loop soup with mass function $m/(\sqrt{2N})$, respectively. We use $\mathbb{P}$ to denote the joint distribution of $A^N_{\lambda,m}$, $\tilde{A}^N_{\lambda,m}$ and the collection $\{T_\gamma\}$.

For loops that are not macroscopic, the removal of loops is done independently for the Brownian loop soup and the random walk loop soup. If there is no one-to-one correspondence between macroscopic loops in $D$, the removal is done independently for all loops, including the macroscopic ones.

We want to show that, on the event that there is a one-to-one correspondence between macroscopic loops in $D$, the one-to-one correspondence survives the removal of loops described above with probability going to one as $N \to \infty$. For that purpose, we need to compare $\int_0^{t_\gamma} m^2(\gamma(s))ds$ and $\frac{1}{2N^2} \sum_{k=0}^{M-1} m^2(\tilde{\gamma}(\frac{k}{2N^2}))$ for loops $\gamma$ and $\tilde{\gamma}$ paired in the above correspondence.
In order to do that, we write

\[
\int_0^{t_\gamma} m^2(\gamma(s))ds = t_\gamma \int_0^1 m^2(\gamma(t_\gamma u))du = \lim_{n \to \infty} t_\gamma \sum_{i=0}^{\lfloor nt_\gamma \rfloor} m^2 \left( \gamma \left( \frac{t_\gamma i}{nt_\gamma} \right) \right)
\]

\[
= \lim_{q \to \infty} \frac{t_\gamma}{4qN^2} \sum_{i=0}^{2qM-1} m^2 \left( \gamma \left( \frac{i}{2qM} \right) \gamma \right),
\]

where \(t_\gamma = \frac{N}{2N^2}\) and the last expression is obtained by letting \(n = 4qN^2\), with \(q \in \mathbb{N}\). Thus, for fixed \(N\) and \(\gamma\), the quantity

\[
\Omega(N, q; \gamma) := \left| \int_0^{t_\gamma} m^2(\gamma(s))ds - \frac{t_\gamma}{4qN^2} \sum_{i=0}^{2qM-1} m^2 \left( \gamma \left( \frac{i}{2qM} \right) \right) \right|
\]

can be made arbitrarily small by choosing \(q\) sufficiently large.

Define the sets of indices

\[
I_0 = \{ i : 0 \leq i < q \} \cup \{ i : (2M-1)q \leq i < 2qM \}
\]

and \(I_k = \{ i : (2k-1)q \leq i < (2k+1)q \}\) for \(1 \leq k \leq M - 1\). For \(i \in I_k\), \(0 \leq k \leq M - 1\), we have that

\[
\left| \gamma \left( \frac{i}{2qM} t_\gamma \right) - \tilde{\gamma} \left( \frac{k}{M} t_\gamma \right) \right|
\]

\[
\leq \left| \gamma \left( \frac{i}{2qM} t_\gamma \right) - \tilde{\gamma} \left( \frac{i}{2qM} t_\gamma \right) \right| + \left| \tilde{\gamma} \left( \frac{i}{2qM} t_\gamma \right) - \tilde{\gamma} \left( \frac{k}{M} t_\gamma \right) \right|
\]

\[
\leq c_1 \log \frac{N}{N} + \frac{\sqrt{2}}{N}
\]

for some constant \(c_1\), where the first term in the last line comes from Theorem 1.1 of [19] and the second term comes from the fact that \(\tilde{\gamma}(s)\) is defined by interpolation and that

- if \(i \in I_0\), either \(0 \leq \frac{i}{2qM} t_\gamma < \frac{1}{2M} t_\gamma\) so that \(\tilde{\gamma}(\frac{i}{2qM} t_\gamma)\) falls on the edge of \(\frac{1}{N} \mathbb{Z}^2\) between \(\tilde{\gamma}(0)\) and \(\tilde{\gamma}(\frac{1}{2M} t_\gamma)\), or \((1 - \frac{i}{2M}) t_\gamma \leq \frac{i}{2qM} t_\gamma < t_\gamma\) so that \(\tilde{\gamma}(\frac{i}{2qM} t_\gamma)\) falls on the edge between \(\tilde{\gamma}(t_\gamma - \frac{i}{2M}) = \tilde{\gamma}(t_\gamma - \frac{1}{2N^2})\) and \(\tilde{\gamma}(t_\gamma) = \tilde{\gamma}(0)\),
\[ \text{if } i \in I_k \text{ with } 1 \leq k \leq M - 1, \quad \frac{k}{M} t_{\tilde{\gamma}} - \frac{h_i}{2M} \leq \frac{k}{M} t_{\tilde{\gamma}} + \frac{t_i}{2M} \leq \frac{k}{M} t_{\tilde{\gamma}} \quad \text{so that} \quad \tilde{\gamma}(\frac{i}{2qM} t_{\tilde{\gamma}}) \text{ falls either on the edge of } \frac{1}{N} \mathbb{Z}^2 \text{ between } \tilde{\gamma}(\frac{k-1}{M} t_{\tilde{\gamma}}) \quad \text{and} \quad \tilde{\gamma}(\frac{k}{2N^2}), \text{ or on the edge between } \tilde{\gamma}(\frac{k}{2N^2}) \quad \text{and} \quad \tilde{\gamma}(\frac{k+1}{M} t_{\tilde{\gamma}}) = \tilde{\gamma}(\frac{k}{2N^2}). \]

Since \( m^2 \) is Lipschitz continuous, for each \( i \in I_k, 0 \leq k \leq M - 1 \), we have that
\[
\left| m^2 \left( \gamma \left( \frac{i}{2qM} t_{\tilde{\gamma}} \right) \right) - m^2 \left( \tilde{\gamma} \left( \frac{k}{2N^2} \right) \right) \right| \leq \frac{c_2 \log N}{N}
\]
for some constant \( c_2 < \infty \) and all \( N \geq 2 \). We let \( \bar{m}_D^2 := \sup_{x \in D} m^2(x) \) and observe that, since \( t_{\tilde{\gamma}} \geq N^{-1/6} \), the inequality \( |t_{\gamma} - t_{\tilde{\gamma}}| < \frac{5}{8} N^{-2} \) from Theorem 1 of [19] implies that \( t_{\gamma}/t_{\tilde{\gamma}} < 1 + \frac{5}{8} N^{-11/6} \) and that \( M = 2N^2 t_{\tilde{\gamma}} < 2N^2 t_{\gamma} + \frac{5}{4} \). It then follows that
\[
\left| \frac{1}{2N^2} \sum_{k=0}^{M-1} m^2 \left( \gamma \left( \frac{k}{2N^2} \right) \right) - \frac{t_{\gamma}/t_{\tilde{\gamma}}}{4qN^2} \sum_{i=0}^{2qM-1} m^2 \left( \gamma \left( \frac{i}{2qM} t_{\tilde{\gamma}} \right) \right) \right| \leq \frac{1}{2N^2} \sum_{k=0}^{M-1} \left| m^2 \left( \gamma \left( \frac{k}{2N^2} \right) \right) - \frac{t_{\gamma}/t_{\tilde{\gamma}}}{2q} \sum_{i \in I_k} m^2 \left( \gamma \left( \frac{i}{2qM} t_{\tilde{\gamma}} \right) \right) \right|
\]
\[
\leq \frac{1}{2N^2} \sum_{k=0}^{M-1} \frac{1}{2q} \sum_{i \in I_k} \left| m^2 \left( \gamma \left( \frac{k}{2N^2} \right) \right) - \frac{t_{\gamma}/t_{\tilde{\gamma}}}{2q} m^2 \left( \gamma \left( \frac{i}{2qM} t_{\tilde{\gamma}} \right) \right) \right|
\]
\[
\leq \frac{|1 - t_{\gamma}/t_{\tilde{\gamma}}|}{2N^2} \sum_{k=0}^{M-1} m^2 \left( \gamma \left( \frac{k}{2N^2} \right) \right)
\]
\[
+ \frac{t_{\gamma}/t_{\tilde{\gamma}}}{4qN^2} \sum_{k=0}^{M-1} \sum_{i \in I_k} \left| m^2 \left( \gamma \left( \frac{k}{2N^2} \right) \right) - m^2 \left( \gamma \left( \frac{i}{2qM} t_{\tilde{\gamma}} \right) \right) \right|
\]
\[
\leq \frac{|1 - t_{\gamma}/t_{\tilde{\gamma}}|}{2N^2} \sum_{k=0}^{M-1} m^2 \left( \gamma \left( \frac{k}{2N^2} \right) \right) + t_{\gamma} \frac{c_2 \log N}{N}
\]
\[
\leq \frac{|1 - t_{\gamma}/t_{\tilde{\gamma}}|}{2N^2} \bar{m}_D^2 + t_{\gamma} \frac{c_2 \log N}{N} < t_{\gamma} \frac{c_3 \log N}{N},
\]
for some positive constant \( c'_3 = c'_3(D, m) < \infty \) independent of \( \gamma \) and \( \tilde{\gamma} \).

Therefore, for fixed \( N \) and pair of macroscopic loops, \( \gamma \) and \( \tilde{\gamma} \), and for any \( q \in \mathbb{N} \),
\[
\left| \int_0^{t_{\gamma}} m^2(\gamma(s))ds - \frac{1}{2N^2} \sum_{k=0}^{M-1} m^2 \left( \gamma \left( \frac{k}{2N^2} \right) \right) \right| < t_{\gamma} \frac{c'_3 \log N}{N} + \Omega(N, q; \gamma).
\]

26
For fixed $N$ and $\gamma$, one can choose $q^*$ so large that

$$\Omega(N, q^*; \gamma) < t_\gamma \frac{c_3 \log N}{N}.$$ 

Hence, there is a positive constant $c_3 = 2c'_3$ such that, for every $N \geq 2$ and every pair of macroscopic loops, $\gamma$ and $\tilde{\gamma}$, paired in the correspondence of Theorem 1.1 of [19],

$$\left| \int_0^{t_*} m^2(\gamma(s)) ds - \frac{1}{2N^2} \sum_{k=0}^{M-1} m^2 \left( \tilde{\gamma} \left( \frac{k}{2N^2} \right) \right) \right| < t_\gamma \frac{c_3 \log N}{N}.$$ 

We now need to estimate the number of macroscopic loops contained in $D$. For that purpose, we note that, using the rooted Brownian loop measure (11), the mean number, $M$, of macroscopic loops contained in $D$ can be bounded above by

$$M = \lambda \int_D \int_{\mathbb{R}^2} \mu_{z,t}^{br}(\gamma : \gamma \subset D) \, dt \, dA(z) \leq \frac{\lambda \text{diam}^2(D)}{8} N^{1/6}. \quad (24)$$ 

Let $A_N^\lambda(\lambda, m; D)$ (respectively, $\bar{A}_N^\lambda(\lambda, m; D)$) denote the massive Brownian (resp., random walk) loop soup in $D$, i.e., the set of loops from $A_N^\lambda_{\lambda,m}$ (respectively, $\bar{A}_N^\lambda_{\lambda,m}$) contained in $D$. For the critical soups, we use the same notation omitting the $m$.

Let $A_N$ denote the event that there is a one-to-one correspondence between macroscopic loops from $A_N^\lambda(\lambda; D)$ and $\bar{A}_N^\lambda(\lambda; D)$, and let $A_N^{\text{op}}$ denote the event that there is a one-to-one correspondence between macroscopic loops from $A_N^\lambda(\lambda, m; D)$ and $\bar{A}_N^\lambda(\lambda, m; D)$. Furthermore, we denote by $X$ the number of macroscopic loops in $A_N^\lambda(\lambda; D)$, and by $T$ a mean-one exponential random variable. We have that, for any $c_4, \theta > 0$ and for all $N$ sufficiently large,
\[ \mathbb{P}(A_N^m) \geq \mathbb{P}(A_N \cap X \leq c_4 N^{1/6}) \cap \{ \# \gamma \in \mathcal{A}^N(\lambda; D) : t_\gamma \geq \theta \} \]

\[ = \mathbb{P}(A_N^m | A_N \cap X \leq c_4 N^{1/6}) \cap \{ \# \gamma \in \mathcal{A}^N(\lambda; D) : t_\gamma \geq \theta \} \]

\[ \geq \left[ 1 - \sup_{x \geq 0} \Pr \left( x \leq T \leq x + \frac{c_3 \theta \log N}{N} \right) \right]^{c_4 N^{1/6}} \]

\[ \mathbb{P}(A_N \cap X \leq c_4 N^{1/6}) \cap \{ \# \gamma \in \mathcal{A}^N(\lambda; D) : t_\gamma \geq \theta \} \]

\[ = \exp \left( - \frac{c_5 \theta \log N}{N^{5/6}} \right) \]

\[ \mathbb{P}(A_N \cap X \leq c_4 N^{1/6}) \cap \{ \# \gamma \in \mathcal{A}^N(\lambda; D) : t_\gamma \geq \theta \}, \]

where \( c_5 = c_3 c_4 \).

Since \( \exp \left( - \frac{c_5 \theta \log N}{N^{5/6}} \right) \to 1 \) as \( N \to \infty \) for any \( c_5, \theta > 0 \), in order to conclude the proof, it suffices to show that \( \mathbb{P}(A_N \cap X \leq c_4 N^{1/6}) \cap \{ \# \gamma \in \mathcal{A}^N(\lambda; D) : t_\gamma \geq \theta \} \) can be made arbitrarily close to one for some choice of \( c_4 \) and \( \theta \), and \( N \) sufficiently large. But by Theorem 1 of [19], \( \mathbb{P}(A_N) \geq 1 - c(\lambda + 1)\text{diam}^2(D)N^{-7/2} \to 1 \) as \( N \to \infty \); moreover, if \( c_4 > \frac{\lambda \text{diam}^2(D)}{8} \), by Eq. (24), \( c_4 N^{1/6} \) is larger than the mean number of macroscopic loops in \( D \). Since \( X \) is a Poisson random variable with parameter equal to the mean number \( \mathcal{M} \) of macroscopic loops in \( D \), the latter fact (together with a Chernoff bound argument) implies that

\[ \mathbb{P}(X > c_4 N^{1/6}) \leq \frac{e^{-\mathcal{M}}(e\mathcal{M})^{c_4 N^{1/6}}}{(c_4 N^{1/6})^{c_4 N^{1/6}}} \leq \left( \frac{e \lambda \text{diam}^2(D)}{8c_4} \right)^{c_4 N^{1/6}}. \]

This shows that, if \( c_4 > e \lambda \text{diam}^2(D)/8 \), \( \mathbb{P}(X \leq c_4 N^{1/6}) \to 1 \) as \( N \to \infty \).

To find a lower bound for \( \mathbb{P}(\# \gamma \in \mathcal{A}^N(\lambda; D) : t_\gamma \geq \theta) \), we define

\[ \mathcal{L}_{\theta, D} := \{ \text{loops } \gamma \text{ with } t_\gamma \geq \theta \text{ that stay in } D \}. \]

We then have

\[ \mathbb{P}(\# \gamma \in \mathcal{A}^N(\lambda; D) : t_\gamma \geq \theta) = \exp \left[ -\lambda_{\mathcal{L}_D}(\mathcal{L}_{\theta, D}) \right] \]

\[ \geq 1 - \lambda_{\mathcal{L}_D}(\mathcal{L}_{\theta, D}) \]

\[ \geq 1 - \frac{\lambda \text{diam}^2(D)}{\theta}. \quad (25) \]
where the last line follows from the bound

$$\mu_D(L_{\theta,D}) = \int_D \int_0^\infty \frac{1}{2\pi t^2} \mu_{t,x}^{br}(\gamma : \gamma \text{ stays in } D) \, dt \, dA(z) \leq \frac{\text{diam}^2(D)}{\theta}.$$ 

The lower bound (25), together with the previous observations, shows that \(\mathbb{P}(A_N)\) can be made arbitrarily close to one by choosing \(c_4 > e \lambda \text{diam}^2(D)/8\), \(\theta\) sufficiently large, depending on \(D\), and then \(N\) sufficiently large, depending on the values of \(c_4\) and \(\theta\). \(\square\)

5 Random Walk Loop Soups and the Discrete Gaussian Free Field

In this section we discuss some interesting relations between the random walk loop soups of the previous section and the discrete Gaussian free field. We will use the setup of the previous section, but we need some additional notation and definitions.

Let \(D\) be a bounded subset of \(\mathbb{C}\), define \(D^\# := D \cap \mathbb{Z}^2\) and let \(\Phi_D^k = \{\phi_x\}_{x \in D^\#}\) denote a collection of mean-zero Gaussian random variables with covariance \(\mathbb{E}_D^k(\phi_x \phi_y) = G_D^k(x,y)\), where \(G_D^k(x,y)\) denotes the Green function of the random walk introduced at the beginning of Sect. 4 with killing rates \(k = \{k_x\}_{x \in D^\#}\) and killed upon exiting the domain \(D\) (i.e., if the random walker attempts to leave \(D\), it is sent to the cemetery \(\Delta\), where it stays forever). The lattice field \(\Phi_D^k\) is the discrete Gaussian free field in \(D\) with zero (Dirichlet) boundary condition. If \(k_x = 0 \forall x \in D^\#\), the field is called massless, otherwise we will call it massive. (If the nature of the field is not specified, it means that it can be either massless or massive.)

The distribution of \(\Phi_D^k\) has density with respect to the Lebesgue measure on \(\mathbb{R}^{D^\#}\) given by

$$\frac{1}{Z_D^k} \exp \left( - H_D^k(\varphi) \right),$$

where

$$Z_D^k = \int_{\mathbb{R}^{D^\#}} \exp \left( - H_D^k(\varphi) \right) \prod_{x \in D^\#} d\varphi_x$$

is a normalizing constant (the partition function of the model) and the Hamiltonian

$$H_D^k(\varphi) := \frac{1}{2} \sum_{x \in D^\#} \sum_{y \in D^\#} \kappa_{x,y} \phi_x \phi_y$$

with

$$\kappa_{x,y} := \frac{k_x}{d(x,y)}$$

for \(d(x,y)\) the Euclidean distance between \(x\) and \(y\) in \(\mathbb{Z}^2\).
tonian $H^k_D$ is defined as follows:

\[
H^k_D(\varphi) := \frac{1}{4} \sum_{x,y \in D^\#: x \sim y} (\varphi_y - \varphi_x)^2 + \frac{1}{2} \sum_{x \in D^\#} k_x \varphi_x^2 + \frac{1}{2} \sum_{x \in \partial D^\#: y \notin D^\#: x \sim y} \varphi_x^2
\]

where the first sum is over all ordered pairs $x, y \in D^\#$ such that $|x - y| = 1$ (denoted by $x \sim y$), and $\partial D^\#$ is the set $\{x \in D^\#: \exists y \notin D^\# \text{ such that } |x - y| = 1\}$.

In the first expression for $H^k_D$, the second sum accounts for the massive nature of the field, while the third sum accounts for the Dirichlet boundary condition. (To understand the third sum, note that one can extend the field $\Phi_k$ on $D^\#$ to a field $\Phi_k = \{\varphi_x\}_{x \in \mathbb{Z}^2}$ on $\mathbb{Z}^2$ by setting $\varphi_x = 0 \forall x \notin D^\#$.)

The next theorem shows that the probability that the value of the field at a point inside the domain is affected by a change in the shape of the domain can be computed using the random walk loop soup with intensity $\lambda = 1/2$.

**Theorem 5.1** Let $m$ be a nonnegative function (possibly identically zero), $D$ and $D'$ be bounded subsets of $\mathbb{C}$ containing $x_0 \in \mathbb{Z}^2$, with $D' \subset D$, and $k$ denote the collection $\{4(e^{m^2(x)} - 1)\}_{x \in D \cap \mathbb{Z}^2}$. There exist versions of $\Phi^k_D = \{\varphi_x\}_{x \in D \cap \mathbb{Z}^2}$ and $\Phi^k_{D'} = \{\varphi'_x\}_{x \in D' \cap \mathbb{Z}^2}$, defined on the same probability space, such that

\[
P(\varphi_{x_0} \neq \varphi'_{x_0}) = P_{1/2, m}(\text{there is a loop through } x_0 \text{ that intersects } D \setminus D'),
\]

where $P$ denotes the joint probability distribution of $\Phi^k_D$ and $\Phi^k_{D'}$, and $P_{1/2, m}$ is the law of the random walk loop soup in $D$ with intensity $\lambda = 1/2$ and mass function $m$.

**Remark 5.2** In the near-critical scaling limit of the random walk loop soup discussed in the previous section, the mass squared is rescaled by $1/(2N^2)$: $m^2(x) \mapsto m^2(x)/(2N^2)$. When $N$ is large, this corresponds to killing rates $k^N_x = 4(e^{m^2(x)/(2N^2)} - 1) \approx 2m^2(x)/N^2$. It is a standard result that the scaling limit of the discrete Gaussian free field with masses rescaled by $O(1/N^2)$, obtained by scaling the field itself by $1/N^2$, yields a continuum massive Gaussian free field. This observation provides an indirect link between the massive Brownian loop soup with intensity 1/2 and the continuum massive Gaussian free field. The relation between these objects is being investigated in work in progress by the author with T. van de Brug and M. Lis.
The proof of Theorem 5.1 will follow from a probabilistic coupling that allows us to define the random walk loop soup in $D$ with intensity $1/2$ and the Gaussian free field in $D$ with Dirichlet boundary condition on the same probability space. The coupling is given in Proposition 5.3 below, but first we need some additional notation.

We say that $x \in \mathbb{Z}^2$ is touched by the unrooted loop $\tilde{\gamma}$, and write $x \in \tilde{\gamma}$, if $\tilde{\gamma}(i) = x$ for some $i \in \{0, \ldots, |\tilde{\gamma}| - 1\}$ and some representative $\tilde{\gamma}(\cdot)$ of $\tilde{\gamma}$. If $\tilde{\gamma}(\cdot)$ is any rooted version of $\tilde{\gamma}$, the number of indices in $\{0, \ldots, |\tilde{\gamma}| - 1\}$ such that $\tilde{\gamma}(i) = x$ is denoted by $n(x, \tilde{\gamma})$ (note that the notation makes sense because $n(x, \tilde{\gamma})$ is independent of the choice of representative $\tilde{\gamma}(\cdot)$). To each $x \in \mathbb{Z}^2$ touched by $\tilde{\gamma}$, we associate $n(x, \tilde{\gamma})$ independent, exponentially distributed random variables with mean one, denoted by $\{\tau_{i,x}^{\tilde{\gamma}}\}_{i=1}^{n(x, \tilde{\gamma})}$. We call the quantity

$$T_{x}(\tilde{\gamma}) := \left\{ \begin{array}{ll}
\sum_{i=1}^{n(x, \tilde{\gamma})} \frac{\tau_{i,x}^{\tilde{\gamma}}}{k + 4} & \text{if } x \in \tilde{\gamma} \\
0 & \text{if } x \notin \tilde{\gamma}
\end{array} \right.$$  

the occupation time at $x$ associated to $\tilde{\gamma}$.

If $\tilde{A}_{\lambda,m}$ is a realization of the random walk loop soup, we define the occupation field at $x$ associated to $\tilde{A}_{\lambda,m}$ as

$$L_{x}(\tilde{A}_{\lambda,m}) := \sum_{\tilde{\gamma} \in \tilde{A}_{\lambda,m}} T_{x}(\tilde{\gamma}) + \frac{\tau_{0,x}^{\tilde{\gamma}}/2}{k_x + 4},$$

where $\{\tau_{0,x}^{\tilde{\gamma}}\}_{x \in \mathbb{Z}^2}$ is an additional collection of independent, exponential random variables with mean one. We denote by $L^k_D$ the collection $\{L_{x}\}_{x \in \mathbb{D}^#}$.

The next result provides the coupling needed to prove Theorem 5.1. It says that, if $\{S_{x}\}$ are random variables with the Ising-type distribution (26) below, where the $\{L_{x}\}$ are distributed like the components of the occupation field of the random walk loop soup in $D$ with intensity $1/2$ and mass function $m$, then the random variables $\psi_{x} = \sqrt{2L_{x}}S_{x}$ are equidistributed with the components of the discrete Gaussian free field in $D$ with $k_x = 4(e^{m^2(x)} - 1)$ and Dirichlet boundary condition. More formally, one has the following proposition, where $P_{1/2,m}$ denotes the joint distribution of the random walk loop soup in $D$ with intensity $1/2$ and mass function $m$, and the collection of all exponential random variables needed to define the occupation field. The proof of the proposition follows easily from Theorem A.1 in the appendix, which is a version of a recent result of Le Jan [21] (see also [22] and Theorem 4.5 of [32]).
Proposition 5.3. Let \( L^D_k = \{L_x\}_{x \in D^\#} \), denote the occupation field of the random walk loop soup in \( D \) with intensity \( 1/2 \) and mass function \( m \). Let \( S = \{S_x\}_{x \in D^\#} \) be \((\pm 1)\)-valued random variables with \((\text{random})\) distribution \( P^D_L(S_x = \sigma_x \forall x \in D^\#) \)

\[
P^D_L(S_x = \sigma_x \forall x \in D^\#) = \frac{1}{Z} \exp \left( \sum_{x,y \in D^\#: x \sim y} \sqrt{L_x L_y} \sigma_x \sigma_y \right),
\]

(26)

where \( \sigma_x = 1 \) or \(-1\) and \( Z \) is a normalization constant. Let \( \psi_x = \sqrt{2L_x S_x} \) for all \( x \in D^\# \); then under \( \hat{P}_{1/2,m} \otimes P^D_L, \{\psi_x\}_{x \in D^\#} \) is distributed like the discrete Gaussian free field \( \Phi^D_k = \{\phi_x\}_{x \in D^\#} \) in \( D \) with \( k_x = 4(e^{m^2(x)} - 1) \) and Dirichlet boundary condition.

Proof. According to Theorem A.1, \( \phi^2_x \) and \( \psi^2_x = 2L_x \) have the same distribution for every \( x \in D^\# \). The proposition follows immediately from this fact and the observation that, letting \( \sigma_x = \text{sgn}(\phi_x) \), one can write

\[
H^k_D(\varphi) = -\frac{1}{2} \sum_{x,y \in D^\#: x \sim y} \sqrt{\varphi^2_x \varphi^2_y} \sigma_x \sigma_y + \frac{1}{2} \sum_{x \in D^\#} (k_x + 4) \varphi^2_x.
\]

Proof of Theorem 5.1. Let \( \tilde{A}_{1/2,m} \) be a realization of the random walk loop soup in \( D \) with intensity \( \lambda = 1/2 \) and mass function \( m \), and let \( \tilde{A}'_{1/2,m} \) denote the collection of loops obtained by removing from \( \tilde{A}_{1/2,m} \) all loops that intersect \( D \setminus D' \). \( \tilde{A}'_{1/2,m} \) is a random walk loop soup in \( D' \) with intensity \( 1/2 \) and mass function \( m \).

If \( x_0 \) is touched by a loop from \( \tilde{A}_{1/2,m} \) that intersects \( D \setminus D' \), use \( \tilde{A}_{1/2,m} \) and Proposition 5.3 to generate a collection \( \{\psi_x\}_{x \in D \cap \mathbb{Z}^2} \) distributed like the discrete Gaussian free field in \( D \) with Dirichlet boundary condition, and \( \tilde{A}'_{1/2,m} \) and Proposition 5.3 to generate a collection \( \{\psi'_x\}_{x \in D' \cap \mathbb{Z}^2} \) distributed like the free field in \( D' \). Then, with probability one,

\[
\psi^2_{x_0} = 2L_{x_0}(\tilde{A}_{1/2,m}) > 2L_{x_0}(\tilde{A}'_{1/2,m}) = (\psi'_{x_0})^2.
\]

If \( x_0 \) is not touched by any loop from \( \tilde{A}_{1/2,m} \) that intersects \( D \setminus D' \), use \( \tilde{A}_{1/2,m} \) and Proposition 5.3 to generate a \( \{\psi_x\}_{x \in D \cap \mathbb{Z}^2} \) distributed like the free field in \( D \). Because of symmetry, \( \psi_{x_0} > 0 \) with probability \( 1/2 \).
If $\psi x_0 > 0$, use $\tilde{A}'_{1/2,m}$ and Proposition 5.3 to generate a collection 
$\{\psi' x\}_{x \in D' \cap \mathbb{Z}^2}$, conditioned on $\psi' x_0 > 0$.

If $\psi x_0 < 0$, use $\tilde{A}'_{1/2,m}$ and Proposition 5.3 to generate a collection 
$\{\psi' x\}_{x \in D' \cap \mathbb{Z}^2}$, conditioned on $\psi' x_0 < 0$.

Because of the $\pm$ symmetry of the Gaussian free field, it follows immediately 
that $\{\psi' x\}_{x \in D' \cap \mathbb{Z}^2}$ is distributed like a Gaussian free field in $D'$, and that $\psi' x_0 = \psi x_0$. \hfill $\square$

A Appendix: Occupation Field and Gaussian Free Field

The theorem used in the proof of Proposition 5.3, stated below, is a version 
of a recent result of Le Jan [21] (see also [22] and Theorem 4.5 of [32]). In this appendix we give a self-contained proof, both for completeness and because 
the occupation field $\{L_x\}$ in Theorem A.1 below is not exactly the same as 
the occupation field of [21] and [22] and Theorem 4.5 of [32]. Indeed, the loop soup that appears 
in [21] and [32] is not the same as the loop soup studied in the present paper, although the two are closely related. In this appendix we use the notation of Sects. 4 and 5.

**Theorem A.1** Let $m$ be a nonnegative function and $k$ denote the collection 
$\{4(\frac{e^{m(x)} - 1)}{2}\}_{x \in D^\#}$. Then the occupation field $\{L_x(\tilde{A}_{1/2,m})\}_{x \in D^\#}$ has the same distribution as the field $\{\frac{1}{2}\phi^2 x\}_{x \in D^\#}$, where $\{\phi x\}_{x \in D^\#}$ is the Gaussian free field with covariance $E(\phi x \phi y) = G_D(x, y)$.

**Proof.** We will show that the Laplace transform of the occupation field is the same as that of the field $\{\frac{1}{2}\phi^2 x\}_{x \in D'}$. For that purpose, it is crucial to notice that $L_x(\tilde{A}_{m})$ has the gamma distribution with parameters $\sum_{\tilde{\gamma}} N_{\tilde{\gamma}} n(x, \tilde{\gamma}) + 1/2$ and $\frac{1}{k_x + 4}$, and consequently, for any real number $v$ and any collection $\{n_{\tilde{\gamma}}\}$ of nonnegative numbers,

$$E[\exp(-v L_x) | \{N_{\tilde{\gamma}}\} = \{n_{\tilde{\gamma}}\}] = \left(1 + \frac{v}{k_x + 4}\right)^{-1/2} \prod_{\tilde{\gamma}} \left(1 + \frac{v}{k_x + 4}\right)^{-n_{\tilde{\gamma}} n(x, \tilde{\gamma})}$$,

where the product if over all unrooted lattice loops in $D$. 33
Let \( \{v_x\}_{x \in D^\#} \) be a collection of real numbers, the Laplace transform of the occupation field is given by the following expectation, where the sum \( \sum_{\{n_\gamma\}} \) is over all collections of possible multiplicities for the lattice loops \( \tilde{\gamma} \) in \( D \),

\[
\mathbb{E} \left[ \exp \left( - \sum_{x \in D^\#} v_x L_x \right) \right] = \sum_{\{n_\gamma\}} \mathbb{E} \left[ \exp \left( - \sum_{x \in D^\#} v_x L_x \right) \mid \{N_\gamma\} = \{n_\gamma\} \right] \mathbb{P}(\{N_\gamma\} = \{n_\gamma\}) .
\]

Recalling (18), we can write

\[
\mathbb{E} \left[ \exp \left( - \sum_{x \in D^\#} v_x L_x \right) \right] = \sum_{\{n_\gamma\}} \left[ \prod_{x \in D^\#} \left( 1 + \frac{v_x}{k_x + 4} \right) \right]^{-1/2} \prod_{\tilde{\gamma}} \left( 1 + \frac{v_x}{k_x + 4} \right)^{-n_\gamma n(x, \tilde{\gamma})} \prod_{\tilde{\gamma}} \exp \left( - \lambda n(x, \tilde{\gamma}) \nu_D^{u,k}(\tilde{\gamma}) \frac{1}{n_\gamma} \right) \nu_D^{u,k}(\tilde{\gamma})^{n_\gamma} \prod_{x \in D^\#} \left( \frac{v_x + k_x + 4}{k_x + 4} \right)^{-n(x, \tilde{\gamma})} .
\]

Now let \( \rho_\gamma \) denote the number of rooted loops in the equivalence class \( \tilde{\gamma} \), then we can write

\[
\nu_D^{u,k}(\tilde{\gamma}) = \frac{\rho_\gamma}{|\tilde{\gamma}|} \prod_{x \in D^\#} (k_x + 4)^{-n(x, \tilde{\gamma})}
\]

and, letting \( k + v \) denote the collection \( \{k_x + v_x\}_{x \in D^\#} \), it is easy to see that

\[
\nu_D^{u,k}(\tilde{\gamma}) \prod_{x \in D^\#} \left( \frac{v_x + k_x + 4}{k_x + 4} \right)^{-n(x, \tilde{\gamma})} = \nu_D^{u,k+v}(\tilde{\gamma}) .
\]
Using this fact, the fact that (18) defines a probability distribution, and Eq. (19), one has that

\[
\sum \prod_{n_i} \frac{1}{n_i!} \left[ \lambda \nu_{D}^{u,k}(\tilde{\gamma}) \prod_{x \in D^\#} \left( \frac{v_x + k_x + 4}{k_x + 4} \right)^{-n(x,\tilde{\gamma})} \right]^{n_i} \\
= \sum \prod_{n_i} \frac{1}{n_i!} \left( \lambda \nu_{D}^{u,k+v}(\tilde{\gamma}) \right)^{n_i} \\
= Z_{\lambda,k+v} = \exp \left( \lambda \sum_{\tilde{\gamma}} \nu_{D}^{u,k+v}(\tilde{\gamma}) \right).
\]

Then the Laplace transform of the occupation field can be written as

\[
E \left[ \exp \left( - \sum_{x \in D^\#} v_x L_x \right) \right] \\
= \left( \prod_{x \in D^\#} (k_x + 4) \right)^{1/2} \exp \left( -\lambda \sum_{\tilde{\gamma}} \nu_{D}^{u,k}(\tilde{\gamma}) \right) \left( \prod_{x \in D^\#} (v_x + k_x + 4) \right)^{-1/2} \exp \left( \lambda \sum_{\tilde{\gamma}} \nu_{D}^{u,k+v}(\tilde{\gamma}) \right) \\
= \left( \prod_{x \in D^\#} (k_x + 4) \right)^{1/2} \exp \left( -\lambda \sum_{\tilde{\gamma}} \frac{\rho_{\tilde{\gamma}}}{|\tilde{\gamma}|} \prod_{x \in D^\#} (k_x + 4)^{-n(x,\tilde{\gamma})} \right) \left( \prod_{x \in D^\#} (v_x + k_x + 4) \right)^{-1/2} \exp \left( \lambda \sum_{\tilde{\gamma}} \frac{\rho_{\tilde{\gamma}}}{|\tilde{\gamma}|} \prod_{x \in D^\#} (v_x + k_x + 4)^{-n(x,\tilde{\gamma})} \right).
\]

If \( \lambda = 1/2 \), using Lemma 1.2 of [5] and a standard Gaussian integration formula, the last expression can be written as

\[
\left( \int_{R^D^\#} e^{-H_{D}^{k}(\varphi)} \prod_{x \in D^\#} d\varphi_x \right)^{-1} \int_{R^D^\#} \exp \left( -H_{D}^{k}(\varphi) - \frac{1}{2} \sum_{x \in D^\#} v_x \varphi_x^2 \right) \prod_{x \in D^\#} d\varphi_x \\
= \mathbb{E}_{D}^k \left[ \exp \left( - \sum_{x \in D^\#} v_x \frac{\varphi_x^2}{2} \right) \right],
\]
which concludes the proof. □

Acknowledgements. The author thanks Erik Broman for a useful conversation and Tim van de Brug and Marcin Lis for comments on earlier drafts of the paper and for numerous discussions.

References

[1] M. Aizenman, Geometric Analysis of $\phi^4$ Fields and Ising Models. Parts I and II, Commun. Math. Phys. 86, 1-48 (1982).

[2] M. Bauer and D. Bernard, 2D growth processes: SLE and Loewner chains, Physics Reports 432, 115-221 (2006).

[3] E.I. Broman, F. Camia, Universal behavior of connectivity properties in fractal percolation models, Electron. J. Probab. 15, 1394-1414 (2010).

[4] D.C. Brydges, J. Fröhlich, A.D. Sokal, The random-walk representation of classical spin systems and correlation inequalities. II. The skeleton inequalities, Commun. Math. Phys. 91, 117-139 (1983).

[5] D.C. Brydges, J. Fröhlich, T. Spencer, The random walk representation of classical spin systems and correlation inequalities, Commun. Math. Phys., 83, 123-150 (1982).

[6] F. Camia, C. Garban, C.M. Newman, Planar Ising magnetization field II. Properties of the critical and near-critical scaling limits, arXiv:1307.3926 (2013).

[7] F. Camia, C. M. Newman, Two-dimensional critical percolation: the full scaling limit, Comm. Math. Phys. 268, 1-38 (2006).

[8] F. Camia, C. M. Newman, Critical Percolation Exploration Path and $\text{SLE}_6$: a Proof of Convergence, Probab. Theory Related Fields 139, 473-519 (2007).

[9] F. Camia, C. M. Newman, SLE(6) and CLE(6) from critical percolation, in Probability, geometry and integrable systems, 103-130, Math. Sci. Res. Inst. Publ., 55, Cambridge Univ. Press, Cambridge, 2008.
[10] D. Chelkak, H. Duminil-Copin, C. Hongler, A. Kemppainen, S. Smirnov, Convergence of Ising interfaces to Schramm’s SLEs, preprint.

[11] E.B. Dynkin, Markov processes as a tool in field theory, *J. of Funct. Anal.* 50, 167-187 (1983).

[12] E.B. Dynkin. Gaussian and non-Gaussian random fields associated with Markov processes, *J. of Funct. Anal.* 55, 344-376 (1984).

[13] R. Fernández, J. Fröhlich, A.D. Sokal, *Random Walks, Critical Phenomena, and Triviality in Quantum Field Theory*, Springer-Verlag (1992).

[14] G. Grimmett, *Percolation* (2nd ed.), Springer (1999).

[15] W. Kager and B. Nienhuis, A Guide to Stochastic Löwner Evolution and Its Applications, *J. Phys. A* 115, 1149-1229 (2004).

[16] H. Kesten, Scaling relations for 2D-percolation, *Comm. Math. Phys.* 105, 109-156 (1987).

[17] S. Janson, Bounds of the distribution of extremal values of a scanning process, *Stochastic Processes Applications* 18, 313-328 (1984).

[18] G.F. Lawler, *Conformally Invariant Processes in the Plane*, Mathematical Surveys and Monographs 114, American Mathematical Society, Providence, RI, 2005.

[19] G.F. Lawler, J.A. Trujillo Ferreras, *Random Walk Loop Soup*, Transactions of the American Mathematical Society 359, 7676-787 (2006).

[20] G.F. Lawler and W. Werner, The Brownian loop soup, *Probab. Theory Relat. Fields* 128, 565-588 (2004).

[21] Y. Le Jan, Markov loops and renormalization, *Ann. Probab.* 38, 1280-1319 (2010).

[22] Y. Le Jan, *Markov paths, loops and fields*, in Lecture Notes in Mathematics, volume 2026, Ecole d’Eté de Probabilité de St. Flour, Berlin (2012).

[23] P. Mörters, Y. Peres, *Brownian Motion*, Cambridge University Press, Cambridge (2010).
[24] S. Nacu, W. Werner, Random soups, carpets and fractal dimensions, *J. London Math. Soc.* **83** (3), 789-809 (2011).

[25] P. Nolin, W. Werner, *Asymmetry of near-critical percolation interfaces*, Journal of the American Mathematical Society **22**, 797-819 (2009).

[26] O. Schramm, Scaling limits of loop-erased random walks and uniform spanning trees, *Israel Journal of Mathematics* **118**, 221-288 (2000).

[27] O. Schramm, S. Sheffield, D.B. Wilson, Conformal Radii for Conformal Loop Ensembles, *Comm. Math. Phys.* **288**, 43-53 (2009).

[28] S. Sheffield, Exploration trees and conformal loop ensembles, *Duke Math. J. Volume* **147**, 79-129 (2009).

[29] S. Sheffield and W. Werner, Conformal Loop Ensembles: the Markovian characterization and the loop-soup construction, *Ann. Math.* **176**, 1827-1917, 2012.

[30] S. Smirnov, Critical percolation in the plane: Conformal invariance, Cardy’s formula, scaling limits, *C. R. Acad. Sci. Paris* **333**, 239-244 (2001).

[31] K. Symanzik, *Euclidean quantum field theory*. In: Local quantum theory, Proceedings of the International School of Physics “Enrico Fermi,” course 45 (R. Jost editor), pp. 152-223, Academic Press (1969).

[32] A.-S. Sznitman, *Topics in Occupation Times and Gaussian Free Field*, Notes of the course “Special topics in probability” ETH Zurich (Spring term 2011).

[33] W. Werner, SLEs as boundaries of clusters of Brownian loops, *C. R. Math. Acad. Sci. Paris* **337**, 481-486 (2003).

[34] W. Werner, Some recent aspects of random conformally invariant systems, in *Les Houches School Proceedings: Session LXXXII, Mathematical Statistical Physics* (A. Bovier, F. Dunlop, A. van Enter, J. Dalibard editors), pp. 57-98, Elsevier (2006).

[35] W. Werner, The conformally invariant measure on self-avoiding loops, *J. Amer. Math. Soc.* **21**, 137-169 (2008).