Abstract. We describe how turbulence distributes tracers away from a localized source of injection, and analyze how the spatial inhomogeneities of the concentration field depend on the amount of randomness in the injection mechanism. For that purpose, we contrast the mass correlations induced by purely random injections with those induced by continuous injections in the environment. Using the Kraichnan model of turbulent advection, whereby the underlying velocity field is assumed to be shortly correlated in time, we explicitly identify scaling regions for the statistics of the mass contained within a shell of radius $r$ and located at a distance $\rho$ away from the source. The two key parameters are found to be (i) the ratio $s^2$ between the absolute and the relative timescales of dispersion and (ii) the ratio $\Lambda$ between the size of the cloud and its distance away from the source. When the injection is random, only the former is relevant, as previously shown by Celani et al (2007 J. Fluid Mech. 583 189–98) in the case of an incompressible fluid. It is argued that the space partition in terms of $s^2$ and $\Lambda$ is a robust feature of the injection mechanism itself, which should remain relevant beyond the Kraichnan model. This is for instance the case in a generalized version of the model, where the absolute dispersion is prescribed to be ballistic rather than diffusive.

Keywords: diffusion in random media, exact results, transport properties, computational fluid dynamics
1. Introduction

Modeling how tracer particles released from a source distribute in a turbulent environment is a long-standing problem with obvious practical implications, from hazard control to biology [1–4]. In such a scenario, the statistics of the concentration field are non-homogeneous, and obviously depend on the space-time statistics of both the
underlying turbulent velocity field and of the injection mechanism. In the case of a nearly-punctual source, the latter reduces to the specific features of the time statistics of the injection. In the engineering community, those matters have led to the development and systematic use of Lagrangian stochastic models, whose versatility to accommodate both various turbulent statistics and various types of source make them appropriate for commercial use [5–7]. From a more fundamental perspective, quantitative estimates are however seldom, that study how the spatial distribution of the concentration depends upon the properties of the flow field, and its interplay with the injection statistics. Among exceptions are the papers [8] and [9], where the advecting velocity field is assumed to have a vanishing correlation time and is prescribed by the ‘Kraichnan ensemble’ (later defined). More specifically, the authors in [8] consider a memoryless point-source that emits randomly in time, and show that the two-point correlation of the concentration field exhibit non-trivial intermittent scaling regimes, that are essentially determined by the ratio between the absolute and relative dispersion timescales.

In practice, this kind of a random ‘source’ only modulates the number of particles, but does not on average inject any matter in the environment. To model a genuine source, one therefore needs to superimpose over the modulation a continuous contribution, that determines the net injection rate. The presence of a continuous contribution might however fundamentally alter the spatial patterns of the concentration field away from the source, which are determined by an intricate interplay between the turbulent environment and the injection statistics. The Lagrangian point of view ties the statistics of order \( n \) of the concentration field to Lagrangian averages over \( n \) distinct trajectories (see e.g. [10]), with the statistical ensemble being determined by the injection mechanism itself.

For the white-in-time point-source considered in [8]: the \( n \)-point spatial correlations are in principle determined by the Lagrangian statistics for puffs of \( k \leq n \) tracers that transit simultaneously through the source [4]. While the statistics of such turbulent puffs may prove intricate [11], the spatial two-point correlations patterns induced by white-in-time injections remain simple in the sense that they relate to standard two-time Lagrangian statistics.

On the other hand, when the injection is continuous, every Lagrangian trajectory transiting through the source (regardless when) contributes to the statistics. In the point-source setting the Lagrangian statistics involved to determine the \( n \)-point statistics are therefore non-standard \( N + 1 \) Lagrangian statistics—involving the \( N \) different times of injections and the measurement time. In this heuristic picture, a continuously emitting point-source therefore mixes many Lagrangian time-scales, and it is not clear whether scaling should be expected at all when it comes to the statistics of the concentration field (see for instance [12, section 3]).

One purpose of the present paper is to clarify this issue, and to contrast in a quantitative fashion how the concentration statistics depend on the injection mechanism, and in particular on the amount of randomness in the injection statistics. To do so, we contrast the spatial correlations of the concentration fields induced by a localized white-in-time modulation (later WIT) to those induced by a localized continuous-in-time injection (later CIT). The concentration depends linearly upon the source statistics, and the full correlation induced by a source with both continuous and white components.
is recovered by adding together the WIT and the CIT contributions. Another purpose of the paper is to highlight the non-trivial dynamical interplay between absolute and relative dispersion in determining the statistics of the concentration field.

The present work builds on the paper by [8], but discusses two types of prototypical velocity fields. The first is naturally the Kraichnan ensemble, whose dynamical interplay between a diffusive absolute and a super-diffusive (Richardson-like) relative dispersion yet turns out to be highly unrepresentative of a genuine turbulent flow. The second is obtained by slightly altering the Kraichnan velocity field in the spirit of the so-called ‘puff-particles models’ of [13], in order to incorporate some large-scale sweeping, as typically found in Navier–Stokes turbulence. In both cases, it is found that the statistics of the mass contained in a cloud of size \( r \) located at a distance \( \rho \) away from the source are determined by two key parameters, namely (i) the ratio \( s^2 \) between the absolute and the relative timescales of dispersion and (ii) the ratio \( \Lambda \) between the size of the cloud and its distance away from the source. This partition is robust and is independent on the specificity of the velocity field. When they exist, the specific scaling behaviours are however non-trivial and intrinsically depend upon the velocity statistics.

The paper is organized as follows. The next section gives a qualitative account on the differences between CIT and WIT injections, and provides some background definitions on the Kraichnan ensemble. The concentration statistics for the Kraichnan ensemble are derived in section 3. The effects of the large-scale sweeping are discussed in section 4. While sections 3 and 4 are rather technical, the reader can refer to figure 6 and tables 1 and 2 in the conclusion to find the main analytical results of the paper readily summarized.

2. Statistics of the concentration

This section defines the averages and the correlations of the concentration field, and provides some insights on the physical picture behind both random and continuous injections. Some useful background material related to Kraichnan velocity ensembles is also recalled.

2.1. The concentration field

Let us consider a scenario where the source only operates from an ‘initial time’, say \( t = 0 \), but is before-hands non-active. For negative times, the physical domain \( \mathcal{D} = \mathbb{R}^d \) contains a large number of massless particles (‘tracers’) that are advected by a prescribed turbulent velocity field \( \mathbf{v} \) and subject to a small thermal noise. Mathematically, this means that the trajectory \( \mathbf{X}^{\varpi}(t|\mathbf{x}_0, t_0) \) of a tracer \( \varpi \) that is at \( \mathbf{x}_0 \) at time \( t_0 < t \) is obtained as a specific realisation of the stochastic system:

\[
d\mathbf{X}^{\varpi}_i = v_i(\mathbf{X}^{\varpi}, t)\,dt + \sqrt{2\kappa}\,dW^{\varpi}_i,
\]

where \( W \) is a Wiener process. The statistics of \( \mathbf{v}(\cdot, t) \) are prescribed to be homogeneous and isotropic in space, but need not yet be fully specified. Note that the subscripts \( i \in [1,d] \) relate to spatial coordinates, while the superscript \( \varpi \) denotes a realisation of the noise, different for each tracer.
Turbulent mass inhomogeneities induced by a point-source

At time $t = 0$, the particles distribute over the domain according to an equilibrium distribution $n_0$, obtained after averaging the stochastic trajectories (1) over the noise. The number of particles contained within the infinitesimal volume $dx$ is then $n_0(x)dx$. For positive times $t \geq 0$, the source $S$ becomes active and locally injects or removes particles: the initial equilibrium concentration field is then altered into

$$n(x, t) := n_0(x) + \int_0^t dt_0 \int \mathcal{D}x_0 S(x_0, t_0) p_\kappa(x, t|x_0, t_0),$$

where $p_\kappa(x, t|x_0, t_0) := \langle \delta(x - X_\kappa(t|x_0, t_0)) \rangle_\kappa$ represents the transition probability from $x_0$ to $x$, when averaging the tracer trajectories over the noise. To see that formula (2) indeed defines a quantity that is transported as a density field [14], one needs to use the invariance property of the equilibrium distribution: $n_0(x) = \int \mathcal{D}x_0 p_\kappa(x, t|x_0, 0)n_0(x_0)$, and observe that the $p_\kappa$’s represent forward transition probabilities.

In this work, we focus on the case of ‘point sources’, whose spatial extensions $\epsilon$ are taken to be smaller than the integral scale $\lambda$ yet larger than the smallest turbulent scale $\eta$ (namely, the dissipation scale in 3D). In order to investigate how the turbulence propagates small-scale inhomogeneities on scales $\epsilon \ll \delta \ll \lambda$, we will eventually take the three subsequent limits $\eta_K \to 0$, $\epsilon \to 0$ and then $\delta \to 0$. In order to incorporate some fluctuations in the injection rate, a general source-term could be modeled as:

$$S_\epsilon(x, t) := \phi_0 \delta_\epsilon(x)(1 + \sigma\eta(t)),$$

where $\delta_\epsilon$ denotes a compact-support regularization of the Dirac $\delta$ function (see also appendix A.1 for further details). The $\epsilon$ subscripts will later be dropped, and the dependence on the source extension will be made explicit only when necessary. $\eta$ is a Gaussian white noise with vanishing mean, that is $\langle \eta \rangle = 0$ and $\langle \eta(t)\eta(t') \rangle = \delta(t - t')$. The injection rate $\phi_0$ is a positive quantity with dimension (time)$^{-1}$. The relative modulation rate $\sigma$ has dimension (time)$^{1/2}$. 

The distinction between WIT and CIT injections is obtained by isolating in the previous formula the random contribution from the continuous one. We will therefore study the following simplified emission schemes:

$$S(x, t) := \phi_\sigma \delta(x)\eta(t) \quad \text{(WIT)},$$

or

$$S(x, t) := \phi_0 \delta(x) \quad \text{(CIT)},$$

where we write $\phi_\sigma := \phi_0 \sigma$ the WIT modulation rate, with dimension (time)$^{-1/2}$. Let us here again observe, that in this approach, only the CIT source term genuinely acts as a source of particles. By contrast, the WIT ‘source term’ is vanishing on average: it therefore both acts as a source and a sink of particles, and only generates fluctuations in the distribution in the total number of particles. Naturally, the physical meaning of the WIT mechanism is tied to the the concentration $n(x, t)$ not becoming negative. This constrains the value of $\phi_\sigma$ to be sufficiently small, so as to guarantee that the fluctuations remain small with respect to the average equilibrium profile.

In the remainder of the paper, we analyse the steady properties of the concentration field. The CIT statistics do not depend on the specific equilibrium distribution $n_0$ and the latter can be safely chosen to be vanishing. While this is not so in the WIT case,
the notation \( n(x, t) \) will be slightly abused to denote the fluctuation \( n(x, t) - n_{\text{ref}}(x) \), with respect to the underlying reference distribution, be the latter the equilibrium or the CIT one.

### 2.2. Averages and correlations of the concentration field

In practice, one wants to describe the mass statistics away from the source, that is the distribution of the mass \( m(\rho, r, t) \) contained in a ball of diameter \( r \) centered at a position \( \rho \) (see the sketch in figure 1):

\[
m(\rho, r, t) = \int_{|x-\rho|<r/2} \text{d}x \, n(x, t).
\]

The statistics of the mass is tied to the spatial inhomogeneities of the concentration field: the average mass \( \langle m \rangle \) obviously relates to the average concentration \( C_1(x, t) := \langle n(x, t) \rangle \), while the mass fluctuation \( \langle m^2 \rangle - \langle m \rangle^2 \) relates to the correlation function \( C_2(x, x', t) := \langle n(x, t) n(x', t) \rangle \). Please note, that the averages \( \langle \cdot \rangle \) are to be understood in terms of ensemble averages: over the possible realizations of the turbulent velocity field for the CIT case, and over both the turbulent field and the source statistics for the WIT case.

The correlation functions \( C_1 \) and \( C_2 \) are the lowest-order non trivial statistics related to the concentration field, and are the statistical objects we now essentially focus on.

In the Lagrangian framework, the quantities \( C_1 \) and \( C_2 \) can be conveniently written in terms of single-point and two-point forward transition probabilities from the source, obtained by averaging the stochastic trajectories (1) over both the noise and the realization of the velocity field [15]:

\[
p_1(x, t|t_0) := \langle p_\kappa(x, t|0, t_0) \rangle, \quad \text{and} \quad p_2(x, x', t|t_0, t'_{0}) := \langle p_\kappa(x, t|0, t_0) p_\kappa(x', t|0, t'_{0}) \rangle.
\]

The WIT statistics then read

\[
C_1(x, t) = 0, \quad \text{and} \quad C_2(x, x', t) = \phi_0^2 \int_0^t dt_0 p_2(x, x', t|t_0, t_0),
\]

while the CIT statistics are obtained as

\[
C_1(x, t) = \phi_0 \int_0^t dt_0 p_1(x, t|t_0), \quad \text{and} \quad C_2(x, x', t) = \phi_0^2 \int_0^t dt_0 \int_0^t dt'_{0} p_2(x, x', t|t_0, t'_{0}).
\]

Due to the linear dependence of the concentration field with respect to the source term, and the white-in-time nature of the WIT source, one obtains the average and correlation field induced by the full source (3) as the sum of the WIT and CIT statistics. The average concentration is solely prescribed by the CIT contribution, but the
Turbulent mass inhomogeneities induced by a point-source

https://doi.org/10.1088/1742-5468/aab509

J. Stat. Mech. (2018) 033210

2.3. Isotropic correlation and quasi-Lagrangian mass

To characterize the correlation field $C_2(x, x', t)$, it proves convenient to introduce the midpoint $\rho = \frac{x + x'}{2}$ and the separation $r = x - x'$. In the steady state, the isotropic nature of the advecting velocity field makes $C_2$ only depend on three parameters: (i) the absolute distance $\rho = |\rho|$, (ii) the relative distance $r = |r|$ and (iii) the angle $\theta = (r, \rho)$, so that $C_2 = C_2(r, \rho, \theta)$.

To simplify the discussion, we focus on the isotropic contribution $c(r, \rho)$ to the correlation field, obtained by averaging $C_2$ over the azimuthal solid-angle. Physically, the quantity $c(r, \rho)$ relates to the concept of quasi-Lagrangian mass $d_{\text{QL}}$, that describes the probability of a puff having some mass at a distance $r$, knowing that its center is located at a distance $\rho$ away from the source:

$$d_{\text{QL}}(r, \rho) = r^{d-1} d\rho c(r, \rho) / \mathcal{Z}(\rho) \quad \text{with} \quad \mathcal{Z}(\rho) = \int_{\epsilon}^{R} \int_{0}^{2\pi} \int_{0}^{\pi} r^{d-1} c(r, \rho) \sin \theta \, d\theta \, dr,$$

Figure 1. Qualitative distinction between WIT and CIT injections for the Lagrangian contribution to the mass $d_{\text{QL}}$. In the WIT case, correlations between tracers that transit simultaneously through the source provide the only contribution. In the CIT case, correlations between non-coincident trajectories do also contribute. (a) WIT injection. (b) CIT injection.

correlation has both non-trivial CIT and WIT contributions. Let us point out, that higher order statistics would involve additional correlations between the WIT and the CIT terms.
and where $R$ is a regularizing cut-off, that we can take to be $\infty$ if the defining integral for $\mathcal{Z}(\rho)$ is convergent. From equations (7) and (8), one qualitatively expects the statistics of the correlation $c(r, \rho)$ to be more intricate in the CIT than in the WIT scenario. In the latter case, the correlations of the concentration field are due to the correlations between pairs of trajectories that pass simultaneously through the source. In the former case, correlations between particles released from the source at different times do also contribute (see figure 1). To go beyond this very qualitative remark, the space-time statistics of the turbulent velocity field that intervenes in equation (1) now need to be specified further.

2.4. Kraichnan velocity ensemble

In order to treat a soluble model of turbulent transport, the velocity field is now prescribed to have a vanishing correlation time and Gaussian spatial statistics, hereby yielding statistics in the so-called ‘Kraichnan velocity ensemble’ (see [15–17], and references therein):

$$\langle v_i(\mathbf{x}, t) \rangle = 0, \text{ and}$$
$$\langle v_i(\mathbf{x}, t)v_j(\mathbf{x}', t') \rangle = R_{ij}(|\mathbf{x} - \mathbf{x}'|)\delta(t - t').$$

(10)

The covariance matrix $R_{ij}$ is chosen so as to mimic the spatial correlations of a $d$-dimensional rough turbulent velocity field in the inertial range, taken to be homogeneous, isotropic and compressible [15, 18]:

$$R_{ij}(r) = D_0 (\delta_{ij} - d_{ij}(r)), \text{ where}$$

$$d_{ij}(r) = \left( \frac{r}{\lambda} \right)^\xi \left( \frac{\gamma\delta_{ij} - \beta \frac{r_i r_j}{r^2}}{r^2} \right), \text{ and using}$$

$$\gamma = \frac{d - 1 + \xi(1 - \varphi)}{(d - 1)(1 + \varphi\xi)}, \text{ and } \beta = \frac{\xi(1 - d\varphi)}{(d - 1)(1 + \varphi\xi)}.$$

(11)

The compressibility degree

$$\varphi = \lim_{r \to 0} \frac{\partial^2}{\partial^2 r} R_{ij}(r)/\partial^2_{jj} R_{ii}(r)$$

ranges from 0 to 1. The roughness of the velocity field is given by the coefficient $\xi$, which ranges from 0 to 2. The coefficient $\lambda$ represents the integral length scale, and the inertial scales hence correspond to $r \ll \lambda$. The parameters relevant to describe the 3D direct cascade or the 2D inverse cascade of homogeneous isotropic incompressible turbulence are $\varphi = 0$ and $\xi = 4/3$.

It is well known that the Kraichnan model is not realistic, in the sense that the statistics of a genuine turbulent velocity field are usually both non-Gaussian in space and non trivially correlated in time [19, 20]. While those features may fundamentally alter the phenomenology of tracer dispersion [21], the Kraichnan model can however be expected to provide a qualitative understanding of pure Lagrangian phenomena. For example, the anomalous features of passive transport have been tied in the Kraichnan model to the existence of so-called ‘zero-modes’ (see for instance [15, 17, 22], and references therein). The concept of zero-mode has proven fruitful for DNS, where it is reflected in terms of statistical conservation laws [10, 23–26].
2.5. Steady states of the concentration field

Combining equations (10)–(11) to (1), and using standard Itô calculus [27], it is easy to show that the transition probabilities (6) propagate as \( \partial_t p_i = -M_i p_i \), with

\[
M_1[x] = -\left( \kappa + \frac{D_0}{2} \right) \partial^2_{x_ix_i} \quad \text{and} \\
M_2[x, x'] = M_1[x] + M_1[x'] - \partial^2_{x_ix_j} R_{ij}(r).
\]

(12)

It then follows from the definitions (7) and (8), that the steady statistics of the WIT concentration field satisfy:

\[
C_1(x) = 0, \quad \text{and} \quad M_2 C_2(x, x') = \phi^2 \delta(x) \delta(x'),
\]

(13)

while the CIT statistics are determined by:

\[
M_1 C_1(x) = \phi_0 \delta(x), \quad \text{and} \\
M_2 C_2(x, x') = \phi_0 (\delta(x) C_1(x') + \delta(x') C_1(x)).
\]

(14)

To proceed further, it is useful to write the propagator \( M_2[x, x'] \) in terms of the absolute separation vector \( \rho \) and the relative separation \( r \) as

\[
M_2[x, x'] = M_1[\rho] + M_\xi[r], \quad \text{with} \\
M_\xi[r] = -D_0 \partial^2_{r_ir_j} d_{ij}(r).
\]

(15)

Upon inspection of the previous equations, one qualitatively expects the behaviour of the fluctuations to depend crucially on the features of the one-point motion. This is trivial in the CIT case, where \( C_1 \) appears explicitly on the right-hand side of the steady-state equation (14). For both the WIT and the CIT cases though, the propagator \( M_2 \) involves an interplay between the absolute dispersion propagator \( M_1 \) and the relative dispersion propagator \( M_\xi \), a feature that might affect the mass statistics in a less immediate manner. This intuition will be substantiated in the next two sections.

3. Fluctuation statistics in the Kraichnan ensemble

This section discusses the statistics of the concentration field in the Kraichnan model, and contrast WIT and CIT statistics. While the effective computation is only described in outline, technical details can be found in appendix. For the purpose of brevity, the isotropic contribution \( c(r, \rho) \) of the correlation field is later referred to as being itself the ‘correlation field’.
3.1. Computing the correlation field

Both the WIT and the CIT correlation fields \( c(r, \rho) \) are obtained by taking the Hankel transforms of equations (13) and (14) with respect to \( \rho \). More explicitly, we look for a solution in the form:

\[
c(r, \rho) = \int_0^\infty \frac{dk}{k^{d-1}} g_0(k \rho) c(r, k), \quad \text{so that}
\]

\[
c(r, k) := \left( \frac{2}{\pi} \right)^{d-2} \int_0^\infty \frac{d\rho}{\rho} \rho^{d-1} g_0(k \rho) c(r, \rho),
\]

and then solve for \( c(r, k) \). \( g_0 \) is shorthand for the standard Bessel function of the first kind \( J_0 \) when \( d = 2 \), and the spherical Bessel function of the first kind \( j_0 \) when \( d = 3 \).

The Hankel transforms of the steady equations (13) and (14) read:

\[
\left( \frac{M_\xi}{\rho} + \frac{D_0}{2} k^2 \right) c(r, k) = \text{rhs}(r, k), \quad \text{where}
\]

\[
\text{rhs}(r, k) = \left( \frac{\varphi^2}{2\pi^{d-1}} \delta_\epsilon(r) \right) (\text{WIT case}),
\]

or \( \text{rhs}(r, k) = \left( \frac{\varphi_0}{\pi^{d-1}} j_0 \left( \frac{kr}{2} \right) c_1(r) \right) (\text{CIT case}). \) (16)

To deal carefully with the \( \delta \) function involved in the WIT right-hand side, the small source extension \( \epsilon \) is here again made explicit. The coefficients \( c(r, k) \) are then found explicitly after some long but straightforward algebra. Reconstructing the correlation field there from yields the final result:

\[
c(r, \rho) = 2^{1-d/2} (2 - \xi)^{d-1} D_0^{-1} \lambda^{(1-d/2)} \rho^{(d/2-1)(\xi-2)} (c_-(r, \rho) + c_+(r, \rho)), \quad \text{with}
\]

\[
c_-(r, \rho) = \int_0^\infty du u^{d-1} g_0(\rho u) I_\omega(u) \int_0^1 dv v^{1+m-\xi} \text{rhs} \left( \frac{rv}{\rho} \right) K_\omega \left( \frac{uv^{1-\xi}}{2} \right),
\]

\[
c_+(r, \rho) = \int_0^\infty du u^{d-1} g_0(\rho u) I_\omega(u) \int_0^\infty dv v^{1+m-\xi} \text{rhs} \left( \frac{rv}{\rho} \right) K_\omega \left( \frac{uv^{1-\xi}}{2} \right).
\] (17)

\( I_\omega \) and \( K_\omega \) are the modified Bessel functions of the first and second kind order \( \omega \). The previous formula involves a crucial dimensionless parameter, that will be later commented on:

\[
s^2 := \frac{(2 - \xi)^2}{2} \left( \frac{\rho}{\lambda} \right)^2 \left( \frac{r}{\lambda} \right)^\xi \, ,
\]

along with the explicit coefficients

\[
\omega := \frac{f}{2 - \xi}, \quad \text{where} \quad f = ((m - 1)^2 - n)^{1/2},
\]

\[
m := (d + \xi - 1) \left( 1 + \frac{\varphi \xi}{1 + \varphi \xi} \right), \quad \text{and}
\]

\[
n := (d + \xi - 2) (d + \xi) \frac{\varphi \xi}{1 + \varphi \xi}. \quad \text{(19)}
\]
Plugging the expressions (16) into the general expression (17) yields the final result. Quite remarkably, the integrals (17) can be computed explicitly in the WIT case. While this is not so in the CIT case, asymptotic scaling regimes can still be identified.

### 3.2. WIT statistics

#### 3.2.1. WIT fluctuations

The explicit expression for the WIT correlations is:

\[
c(r, \rho) = c_d \left( \frac{\epsilon}{r} \right)^g \left( \frac{r}{\lambda} \right)^{\xi(d/2-1)} \frac{r^{2-2d}}{(1 + s^2)^{\omega+d/2}}, \quad \text{with} \quad c_d = \frac{2^{d/2} \Gamma \left( \omega + \frac{d}{2} \right)}{\pi^{d/2} V_d \Gamma(\omega + 1)} \frac{(2 - \xi)^{d-1} \varphi_d^2}{g + d} D_0, \tag{20}
\]

and where \( g := \frac{3+m+f}{2} - \xi - d, \) \( V_d \) is the \( d \)-dimensional volume of the unit-sphere, and the other coefficients given by (19). The scaling behaviors of the correlation depend on the large-scale \( \lambda \) and are therefore anomalous. As in the incompressible case discussed by [8], the specific scaling properties are controlled by the value of the dimensionless coefficient \( s^2 \). This coefficient is essentially a ratio between two timescales, namely \( s^2 \sim \frac{\tau_0(\rho)}{\tau_1(r)} \), where \( \tau_0(\rho) \sim \frac{\rho^2}{D_0} \) and \( \tau_1(r) \sim \frac{r^2}{D_0(r/\lambda)^3} \) respectively represent the Lagrangian time-scales for the absolute and relative separations. In those asymptotics, and without keeping track of the constants, the correlation behaves as:

\[
c(r, \rho) \sim \begin{cases} 
\left( \frac{\epsilon}{\xi} \right)^{-g} \left( \frac{r}{\lambda} \right)^{\xi(d/2-1)} r^{2-2d} & \text{for } s \ll 1 \\
\left( \frac{\epsilon}{\xi} \right)^{-g} \left( \frac{r}{\lambda} \right)^{-\xi(1+\omega)} \left( \frac{\xi}{\rho} \right)^{2\omega+d} r^{2-2d} & \text{for } s \gg 1.
\end{cases} \tag{21}
\]

Let us here emphasize that while the ratio \( s^2 \) is a ratio between the two Lagrangian quantities \( \tau_0(\rho) \) and \( \tau_1(r) \), it here intervenes as a parameter for the stationary Eulerian field \( c_2(r, \rho) \). Upon suitable normalization, the field \( c_2 \) can be thought of as the Eulerian probability that in the stationary state, the mass present in the domain lies in some infinitesimal shell volume \( r^d \text{d}r \) centered around a position at a distance \( \rho \) away from the source (see figure 6). All values of \( s \) are therefore allowed. Shells characterized by \( s^2 \sim 1 \) are those for which the Eulerian probability field is determined by the typical Lagrangian events, namely the bulk of Richardson’s distribution. Similarly, shells with a small spatial extension are characterized by small values of \( s^2 \), and the Eulerian probability is then determined by those particles that separate faster than average. Conversely, large values of \( s^2 \) relate to untypical trajectories that do not separate. In the Lagrangian framework, those would correspond to the left-end tail of Richardson’s distribution.

#### 3.2.2. Effects of compressibility

Let us first observe that while a non-vanishing compressibility seemingly only mildly affects the scaling exponents (see figure 2), it makes the point-source problem become degenerate in the limit of an infinitesimal source extension \( \epsilon \to 0 \). Because the coefficient \( g \) is strictly positive unless the underlying flow is incompressible (see the left panel of figure 2), the correlation should vanish in that limit. One way to circumvent the problem and define a non-trivial limit \( \epsilon \to 0 \) is to
focus on the properties of the quasi-Lagrangian mass $dm_{QL}(r, \rho) = \frac{1}{Z(\rho)} r^d c(r, \rho)dr$ for $\rho$ strictly positive, with the normalization $Z(\rho) = \int_0^\infty r^{d-1} c(r, \rho)dr$. With this choice of normalization, the quasi-Lagrangian mass becomes independent of the source extension $\epsilon$, that can safely be taken to 0. This definition naturally relies upon the quasi-Lagrangian mass being indeed integrable as $r \to 0$ and $r \to \infty$, and hence on the exponents $\gamma_0 := \lim_{r \to 0} \frac{\log dm_{QL}/dr}{\log(r)} > -1$ and $\gamma_\infty := \lim_{r \to \infty} \frac{\log dm_{QL}/dr}{\log(r)} < -1$. Figure 3 shows that this is indeed the case unless $d = 2$ and the flow is incompressible. Only in that specific case, does one need to introduce a large-scale cut-off $R$ in the definition of $Z(\rho)$.

It is known from previous work, that Lagrangian trajectories advected by a compressible Kraichnan velocity field are essentially explosive for small values of the compressibility and become ‘sticky’ when the compressibility increases above the critical value $d/\xi^2$ [28]. The phenomenon can be qualitatively related to the presence of shocks. In our point-source setting, compressibility shapes the scaling behavior of the quasi-Lagrangian mass (9). A phase transition can be identified by looking at the statistics of the large shells, which are characterized by the exponent $\gamma_\infty$. The top panel of figure 3 shows an apparent transition at the critical value $\phi^*$ defined by $\partial_\xi \gamma_\infty = 0$, whereby the exponent $\gamma_\infty$ become independent of the roughness of the velocity field. We identify $\phi^* = 0$ for $d = 2$ and $\phi^* \simeq 0.25$ for $d = 3$. For $\phi > \phi^*$, $\gamma_\infty$ decreases with $\xi$: The mass distribution becomes steeper as smoothness increases ($\xi \to 2$). In other words, the sticky behavior due to compressibility prevents the mass to spread broadly, unless the flow is very rough ($\xi \to 0$). The critical value of $\phi^*$ can be interpreted as a case where the compressive ‘stickiness’ compensate the roughness-induced explosive behavior.

Let us finally note that formulas (3.2) and (4.3) of [8] are recovered as special cases of (20) and (21), in the incompressible limit $\phi \to 0$.

### 3.3. CIT statistics

#### 3.3.1. CIT average concentrations

Computing the CIT fluctuation field requires to know the CIT average concentration $C_1(|x|) = c_1(|x|)$. The latter is obtained by direct integration of (14), with the prescription that $c_1$ vanishes at $\infty$. For $d = 2$, this is only possible provided a large scale cut-off $L$ is introduced$^2$, so that

$$c_1(|x|) = \begin{cases} -\frac{\phi_0}{\pi D_0} \log \frac{|x|}{L} H(L - |x|) & \text{for } d = 2, \\ \frac{\phi_0}{2\pi D_0|\xi|} & \text{for } d = 3, \end{cases}$$

where $H$ here denotes the Heaviside function, that takes value 1 for positive arguments and vanishes otherwise.

#### 3.3.2. CIT correlations

While the CIT correlation field does not seem to have a fully explicit expression beyond (17), scaling behaviours can still be analysed. From (16), one computes rhs $(rv, uv/s) = \phi_0 \pi^{1-d} c_1(rv) J_0(\pi vr/2\rho)$, and observes that the integrand quantities of (17) now depend not only on the coefficient $s$ but also on the

$^2$ This feature is due to the recurring nature of the Brownian motion for $d = 2$, and its transiting nature for $d = 3$. 

https://doi.org/10.1088/1742-5468/aab509

J. Stat. Mech. (2018) 033210
value $\Lambda := r/\rho$. Using the asymptotic properties of the modified Bessel functions $I_\omega$ and $K_\omega$ [29, section 9], three different asymptotic regimes can be explicitly determined: (i) $s \gg 1$, (ii) $s \ll 1$ and $\Lambda \gg 1$, (iii) $s \ll 1$ and $\Lambda \ll 1$, as explained in details in appendix A.3. For each of those three regions, one can identify the following behaviors (the constants are here documented):

**Figure 2.** Dependence of the exponents $g$ and $f = (2 - \xi)\omega$ with respect to the compressibility degree $\nu$ for various values of the roughness coefficient $\xi$, and $d = 2$ (solid) or $d = 3$ (dashed).

**Figure 3.** Scaling exponents $\gamma_\infty$ (top) and $\gamma_0$ (bottom) defining the large-$r$ and small-$r$ dependence of the quasi-Lagrangian mass $dm_{QL} \sim r^\gamma dr$, for $d = 2$ (left side) and $d = 3$ (right side) for the WIT Kraichnan case.
(i) \( s \gg 1 \):
\[
c(r, \rho) \sim \frac{2^{2-d}}{\omega \pi (2 - \xi)} \left( \frac{r}{\lambda} \right)^{- \xi} \left( \frac{r}{2 \rho} \right)^{\xi \frac{m+1}{2}} c_1(2 \rho) \frac{\phi_0}{D_0} r^2 \rho^{-d},
\]
with \( \pm = \text{sign}(r - 2 \rho) \).

(ii) \( s \ll 1 \) and \( \Lambda \gg 1 \):
\[
c(r, \rho) \sim \frac{2^{5(\bar{d}-2)/2}}{\pi^{\bar{d}-2}} \left( \frac{r}{\lambda} \right)^{\xi} c_1(r) \frac{\phi_0}{D_0} \rho^{2-d} \chi_d(s).
\]

(iii) \( s \ll 1 \) and \( \Lambda \ll 1 \):
\[
c(r, \rho) \sim \frac{2^{3(\bar{d}-2)/2}}{\pi^{\bar{d}-2}} \left( \frac{r}{\lambda} \right)^{\xi} c_1(r) \frac{\phi_0}{D_0} \rho^{2-d} \chi_d(s),
\]
where in the last two formulas \( \chi_d(s) = -\log s \) for \( \bar{d} = 2 \) and constant \( \frac{\pi}{4} \) for \( \bar{d} = 3 \).

Let us first observe, that while in 2D, logarithmic corrections are present far from the source (\( s \ll 1 \)), scaling regions can be identified in all three asymptotic regimes, in spite of the strong Lagrangian mixing due to the continuous nature of the injection. As in the WIT case the scaling is intermittent, in the sense that it is affected by the integral scale \( \lambda \). The second observation is that the asymptotic behaviors here depend on both the timescale ratio and the aspect ratio \( \Lambda = r/\rho \) between the relative and absolute dispersion. The dependence on \( \Lambda \) is not surprising: averages over shells that incorporate the source correspond to \( \Lambda \geq 2 \), and incorporate the constant contribution of the injection rate, a feature that can naturally be expected to alter the statistics. The dependence on the Lagrangian time-scales via the coefficient \( s^2 \) is however more surprising, as one could have expected that this ratio was tied to the fact that the contributing Lagrangian trajectories would pass simultaneously through the source. The present results show that this is however not the case.

Note that the compressibility degree does not here affect the statistics of the fluctuation field in a spurious way: unlike in the WIT case, taking \( \epsilon \to 0 \) does not make the correlation field become infinite. To analyze further the effect of compressibility, it however remains instructive to comment on the scaling properties of the CIT Lagrangian mass \( dm \sim r^{d-1} c(r, \rho) \, dr \), through the small and large \( r \) scaling exponent \( \gamma_0 \), \( \gamma_\infty \), defined such that \( dm \sim r^{\gamma_0} \) for small \( r \) and \( dm \sim r^{\gamma_\infty} \) for large \( r \). Computing \( \gamma_\infty = 4 - d + \xi \), it is apparent that the large-\( r \) behavior is independent from the compressibility degree. This results probably owes to the fact that the continuous contribution from the source there dominates the statistics. Besides, it also shows that a large-scale cut-off \( R \) needs to be prescribed for the mass to be accurately normalized. Figure 4 shows the iso-lines of the exponent \( \gamma_0 = d + (f - m - 1)/2 \) for the small \( r \) behavior. As in the WIT case, compressibility only weakly alters the small-\( r \) scaling.
4. Concentration statistics in the presence of a large-scale sweeping

4.1. Modeling the large-scale sweeping

As mentioned in the introduction, the statistics of the correlation depend on the interplay between the absolute and the relative dispersion. This is particularly obvious in the CIT case, where the average $c_1$ appears explicitly in the expression for the correlation given by equations (23)–(25). However, the specific interplay that appears in the Kraichnan ensemble, between an absolute diffusive dispersion and a relative explosive separation can look paradoxical with respect to the Lagrangian phenomenology of time-correlated turbulence à la Kolmogorov. In DNS and experiments, both the absolute and relative separation are known to become diffusive only at times greater than the Lagrangian integral time-scale $\tau_L$. Below $\tau_L$, the phenomenologies of absolute and relative dispersion differ. On the one hand, it is known from state-of-the art numerics that after an initial transient ballistic regime the bulk statistics of relative separation are reasonably well-described by Richardson diffusion both in two and three dimensions, (see [30–34]): this therefore justifies the use of the relative dispersion operator $M_\xi$. On the other hand, the Lagrangian velocity measured along a single trajectory is typically correlated over the integral time-scale [35, 36]. Unlike the Kraichnan model, the absolute dispersion is therefore not diffusive, except for timescales far greater than the integral time scale [37]. Refined treatments of single-particle dispersion have motivated in the past the development of Lagrangian stochastic models in terms of Langevin process (see for instance [6, 38]) but go beyond the point of this paper. For the present purpose, it is probably reasonable to consider that the absolute dispersion is essentially ballistic for times below $\tau_L$.

In order to investigate quantitatively how a ‘Ballistic/Explosive’ interplay differs from the ‘Diffusive/Explosive’ interplay studied in the previous section, the Kraichnan velocity ensemble (10) is now altered into:

$$v_i(x, t) = U_0 \frac{x_i}{|x|} + u_i(x, t),$$

(26)

where $U_0$ is a constant velocity, and $u$ is a fluctuating turbulent field with Kraichnan statistics, as prescribed by equation (10). The ratio $D_0/U_0$ defines a length scale $r_0$, which is here assumed to be small compared to the integral length scale. Below $r_0$, the diffusive nature of the Kraichnan model dominates the one point motions, and those therefore diffuse. Only for $r > r_0$ does the motion become ballistic. For our present purpose, we therefore wish to to consider statistics on scales $r \gg r_0$. In the spirit of the so-called ‘puff-particles models’ described in [13] in the context of atmospheric dispersion modeling, the idea of model (26) is to prescribe the barycenter of puffs of tracers to have a dynamics independent from the fluctuating turbulent field. In the present case, the barycenter is essentially prescribed by the large-scale velocity $U_0$ and is insensitive to Kraichnan diffusion when it varies on scales greater than $r_0$.

In the limit of vanishing diffusivity $\kappa$, one can check that the single point motion for the inertial scales is essentially ballistic, that is $\langle |x| \rangle = U_0 t$ for $r_0 \ll |x| \ll \lambda$. The relative motion is left unaltered and given by Kraichnan statistics. In other words, puffs of tracers move ballistically on average but spread explosively. The velocity ensemble

https://doi.org/10.1088/1742-5468/aab509
Taking $c(r, k)$ instead of $c(r, k)$ makes the connection with the calculation of section 3 particularly apparent.

(26) is therefore later referred to as the ‘Ballistic/Explosive’ (B/E) model. It is easily checked that the steady states equations (12)–(15) for the concentration statistics carry through, with the only difference that the single-point propagator is now given by:

$$m_{1}^{\text{bal}}[x] := U_{0} \partial_{x_{i}} \left( \frac{x_{i}}{|x|} \right).$$

(27)

4.2. B/E correlation field

The (isotropic) correlation field of the B/E ensemble (26) is computed along the same lines as in the previous section, except that Laplace rather than Hankel transforms are used. More specifically, the correlation is solved as:

$$c(r, \rho) = \frac{1}{\rho^{d-1}} \mathcal{L}^{-1} \left[ c(r, k) \right] \left[ \rho \right], \text{ with}$$

$$c(r, k) := \int_{0}^{+\infty} d\rho \rho^{d-1} e^{-k\rho} c(r, \rho),$$

(28)

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform with respect to the pair of variables $\rho, k$. From the steady equations (13) and (14), the equation on $c(r, k^2)^{3}$ is now obtained as:

$$\left( M_{2} r + U_{0} k^2 \right) c(r, k^2) = \text{rhs}(r, k^2)$$

where $\text{rhs}(r, k^2) = \frac{\phi_{2}^{2}}{2^{d-1} \pi} \delta_{1}(r)$ (WIT case),

or $\text{rhs}(r, k^2) = \frac{\phi_{0}}{2^{d-2} \pi} c_{1}(r)e^{-k^{2}r/2}$ (CIT case).

(29)

Solving the previous equation and using (28) to reconstruct the correlation field yields after some routine algebra:

\footnote{Taking $c(r, k^2)$ instead of $c(r, k)$ makes the connection with the calculation of section 3 particularly apparent.}
Turbulent mass inhomogeneities induced by a point-source

\[ \rho(r, \rho) = (1 - \xi/2) U_0^{-1} \rho^{1-d} \left( c_-(r, \rho) + c_+(r, \rho) \right), \]

with

\[ c_-(r, \rho) = \mathcal{L}^{-1} \left[ K_\omega \left( u^{1/2} \right) \int_0^1 dv \, v^{1+\omega} - \xi \text{rhs} \left( rv, \frac{us^2}{\rho} \right) I_\omega \left( u^{1/2} v^{1-\xi/2} \right) \right] [s^2], \]

and

\[ c_+(r, \rho) = \mathcal{L}^{-1} \left[ I_\omega \left( u^{1/2} \right) \int_1^{+\infty} dv \, v^{1+\omega} - \xi \text{rhs} \left( rv, \frac{us^2}{\rho} \right) K_\omega \left( u^{1/2} v^{1-\xi/2} \right) \right] [s^2], \]

where \( m \) and \( \omega \) are the coefficients already referenced in equation (19) and the dimensionless parameter \( s^2 \) is now given by:

\[ s^2 = (1 - \xi/2)^2 \frac{\rho D_0}{r^2 U_0} \left( \frac{r}{\lambda} \right)^\xi. \]  

(31)

Once again, \( s^2 \approx \tau_{bal}(\rho)/\tau_{\parallel}(r) \) is essentially the ratio between the absolute and relative separation Lagrangian time scales, with the former being now the sweeping time-scale, that is \( \tau_{\text{sweep}}(\rho) \sim \rho/U_0 \).

4.3. B/E WIT statistics

The B/E WIT correlation field can be computed explicitly, by combining equations (29) and (30). The final result is:

\[ c(r, \rho) = \tilde{c}_d \left( \frac{r}{\epsilon} \right)^{-g} \rho^{1-d} \pi^{d-1} s^{2-2\omega} \exp \left( -\frac{1}{4s^2} \right), \]

where \( \tilde{c}_d = \frac{1 - \xi/2}{2^{\omega+d} \pi^{d+g}} \frac{\rho D_0}{r^2 U_0} \phi_\sigma^2. \)

(32)

Similarly to the Kraichnan case, the control parameter \( s^2 \) determines the scaling regions. Contrarily to the Kraichnan case, pure scaling is here only present for \( s^2 \gg 1 \). In that case, the correlation behaves as:

\[ c(r, \rho) \sim \left( \frac{r}{\epsilon} \right)^{-g} \left( \frac{r}{\lambda} \right)^{(1-\omega)} \rho^{2-d-\omega \pi^{2\omega-2-d}}, \]  

(33)

and is obviously very different from (21).

Figure 5. The small \( r \) scaling exponent \( \gamma_0 = d + (f - m - 1)/2 \) of the quasi-Lagrangian mass in the case of a CIT injection, for \( d = 2 \) (left) and \( d = 3 \) (right).
Turbulent mass inhomogeneities induced by a point-source

Figure 6. Scaling regions of the WIT and CIT correlation field (here denoted $c_{\text{wit}}$ and $c_{\text{cit}}$) in the case $\xi = 4/3$, relevant to the 2D inverse cascade and 3D direct cascade, for both the Kraichnan model (top) and the B/E model (bottom). $r$ and $\rho$ are scaled by the large scale $\lambda$.

The spurious effect of compressibility found in the limit $\epsilon \to 0$ in the Kraichnan case is here still present. The large scale sweeping here translates into the correlation being exponentially damped for large values of $r$. The competing effects between $\xi$ and $\varphi$ that showed up in the Kraichnan case for large $r$ is therefore being obliterated. The small-scale exponent for the quasi-Lagrangian mass is found to be $\gamma_0 = -1 - g + (1 - \omega)(\xi - 2)$, and its behavior is shown on figure 5. Not surprisingly, the exponents show little dependence with compressibility. Note that the exponents are negative for small values of $\xi$. However the scaling are *stricto sensu* only valid for $r \gg r_0$, so that $r_0$ should be taken as a small scale cut-off to make the quasi-Lagrangian mass well defined.

4.4. B/E CIT statistics

4.4.1. Average concentration. Combining equations (27) and (14), and prescribing vanishing boundary condition at $\infty$, the average B/E concentration is found to be:

$$c_1(|x|) = \frac{\phi_0}{2^{d-1} \pi U_0 |x|^{d-1}}. \quad (34)$$

4.4.2. Correlations. The B/E correlation is again obtained from (30), observing that $\text{rhs}(rv, us^2/\rho) = \phi_0 2^{2-d} \pi^{-1} c_1(rv) \exp(-uv\Lambda s^2/2)$ with $\Lambda = r/\rho$. The three asymptotic regions previously determined for the Kraichnan case can also be worked out. As for the B/E WIT case, the statistics in the regions $s \ll 1$ are damped by a factor $\exp(-\frac{1}{4s^2})$, and therefore do not display scaling. Only for the case $s \gg 1$, a scaling regime can be identified, namely:

$$c(r, \rho) \sim \frac{4}{2^d \omega \pi (2 - \xi)} \left( \frac{r}{\lambda} \right)^{-\frac{1}{2}} \left( \frac{r}{2\rho} \right)^{\xi - \frac{m_f + 3}{2}} \phi_0^2 \rho^{2-\xi} r^{-\frac{2}{\xi}}, \quad (35)$$

for $s \gg 1$,

where $\pm = \text{sign}(r - 2\rho)$. Up to a constant factor, this expression exactly matches the expression (23) found in the Kraichnan case, as does the small-scale scaling exponent.
Turbulent mass inhomogeneities induced by a point-source

19https://doi.org/10.1088/1742-5468/aab509

J. Stat. Mech. (2018) 033210

of the quasi-Lagrangian mass. The only difference comes from the scaling of the one-point motion, namely $c_1(\rho) \sim \rho^{1-d}$. This is a surprising result, as it suggests that averages over small clouds sizes $r$'s are insensitive to the nature of the absolute/relative interplay.

Let us finally remark that the ballistic behavior of the center of mass destroys the scaling for large $r$'s (region $s \ll 1$), as was already the case in the WIT scenario.

5. Conclusion

In order to get an overview of the results, some specific scaling behaviors are summarized in tables 1 and 2, that correspond to the incompressible case ($\rho = 0$). Figure 6 provides a sketch of the different scaling regions, which are determined by the two parameters $\Lambda = r/\rho$ and $s^2 \sim \tau(\rho)/\tau(r)$. Please recall that $\tau(r) = r^{2-d} \Lambda^2 / D_0$ is essentially the time-scale of relative separation, while $\tau(\rho)$ is the time-scale for the one-point motion, which can be identified to $\tau_0(\rho) = \rho^2 / D_0$ (diffusive time-scale) in the Kraichnan ensemble and $\tau_{\text{sweep}} = \rho / U_0$ (sweeping time-scale) in the B/E ensemble. The salient features are the following:

- The CIT statistics differ from the WIT statistics in that they depend on both the Lagrangian timescales ratio $s^2$ and on the aspect ratio $\Lambda$, while only $s^2$ is relevant for the WIT statistics. This observation does not depend on the statistics of the advecting velocity field: It carries through whether the flow is compressible or not, whether $d = 2$ or $d = 3$, and whether sweeping effects are or not included. This is therefore a robust signature of the injection mechanism itself.

Table 1. Scaling of the correlation field for the WIT scenario, in the incompressible case $\rho = 0$. Please recall that $s^2 \sim \tau_0(\rho)/\tau(r)$ for the Kraichnan model and $s^2 \sim \tau_{\text{sweep}}(\rho)/\tau(r)$ for the B/E model.

| $s^2$ | Kraichnan | B/E |
|-------|-----------|-----|
| $s^2 \gg 1$: | $\rho^{2-d-\xi}^{1-\frac{\xi}{2}}$ | $\rho^{3-d-\xi}^{1-\frac{\xi}{2}}$ |
| $s^2 \ll 1$: | $r^{2-\xi+\frac{\xi}{2}(1-4)}$ | No scaling |

Table 2. Same as table 1, but for the CIT scenario.

| $r \ll 2\rho$ | Kraichnan | B/E |
|----------------|-----------|-----|
| $s^2 \gg 1$: | $\rho^{1-2d-\xi}$ | $\rho^{1-2d-\xi}$ |
| $s^2 \ll 1$: | $r^{2-\xi+\xi(1-4)}$ | No scaling |

| $r \gg 2\rho$ | Kraichnan | B/E |
|----------------|-----------|-----|
| $s^2 \gg 1$: | $r^{2-d-\xi}^{1-\frac{\xi}{2}}\rho^{1-\xi}$ | $r^{2-d-\xi}^{1-\frac{\xi}{2}}\rho^{1-\xi}$ |
| $s^2 \ll 1$: | $r^{4-2d+\xi}$ | No scaling |

https://doi.org/10.1088/1742-5468/aab509
For small values of $s$, scaling exists. In both ensemble it is intermittent, in the sense that the inertial scaling of the concentration depends on the large scale $\lambda$. Note the $\lambda$ dependence is not shown explicitly on tables 1 and 2. The one-point motion affects the correlation when averaged over large shells ($s \ll 1$) in a drastic manner, and is likely to destroy pure scaling behaviors. Physically, this is consistent with the idea that the ballistic motion is faster than the explosive motion for small times. Large clouds correspond to pairs of particles that have essentially spread symmetrically with respect to the source. In the presence of a ballistic one-point motion, the rapid sweeping by the large-scale velocity field make those events extremely unlikely.

Compressibility essentially affects the large $r$ behavior of the correlation, a feature that can be seen from the properties of the quasi-Lagrangian mass. It is here only explicitly apparent for the WIT statistics in the Kraichnan ensemble. In all the other cases, the effect of compressibility is obliterated by the continuous injection of mass in the system and by the large-scale sweeping.

In both ensembles, a source with both a WIT and a CIT contributions induce different scaling behaviors depending on the level of noise in the injection. Averages over shells with a small extension $r$ are described by the limit $s \gg 1$, and the correlation field is there a function of the distance from the source only. The specific scaling exponents however depend on the type of injection. For a genuine point source, this means that they will depend on levels of fluctuations in the injection.

As emphasized throughout this work, the analytical predictions that are here documented rely heavily on the white-in-time nature of the underlying prototype turbulent statistics. While this feature is highly unrealistic, it can be hoped that the aforementioned conclusions still hold true in the presence of non trivial Lagrangian correlations, at least at a qualitative level. While the specific values of the scaling regimes that were here found in this work are hardly likely to be seen in a real flow, it could be expected that some robust features might carry trough. For instance in the incompressible, figure 6 shows that for a velocity field with Kolmogorov-like scaling, one should be able to distinguish between regions where the statistics of the correlations are dominated by one point motion, and where no dependence on $r$ is shown ($s \gg 1$) from regions dominated by relative separation where on the contrary no dependence on $\rho$ is shown ($s \ll 1$).

In finite-Reynolds-number turbulent flows, both relative and absolute dispersion however have multiple stages. Whether scaling regimes for the concentration field are indeed to be found is not granted. This is an open question that would benefit from being investigated using either Direct Numerical Simulations or laboratory experiment in the light of the present framework. For realistic injection mechanisms, it is to be tested whether scaling regimes of the fluctuation field depend on the level of noise in the injection mechanism, as implied for instance by figure 6. It is to be seen whether the statistics have connections with either the B/E or the Kraichnan statistics.

In practice, one might also wish to consider positive-definite random sources, that emits puff of particles randomly in time, but unlike our WIT injections do not remove any. This situation can in fact be checked to be ‘intermediate’ between the WIT and
the CIT case: the average concentration $c_1$ is non-zero and prescribed by the average CIT concentration, while the correlation field is prescribed by the WIT correlation field.

The approach described in this paper could also naturally be extended to study non ideal turbulent transport, either involving inertial or active particles as initiated by [9], or involving non isotropic turbulent statistics.

Acknowledgments

I acknowledge insightful discussions with Jérémie Bec, Giorgio Krstulovic and François Laenen. I also thank ICTS-TIFR for their hospitality that led to final completion of this work: In particular the ICTS programs ICTS/taly2018/01 and ICTS/iscpm2018/02 as well as the support from the DST (India) project ECR/2015/00036.

Appendix. Computation of the correlation field

This appendix contains some details about the algebra involved in the computation of the correlation fields, namely (1) the derivation of the mode-to-mode equation (16), (2) the general solution (17), (3) the asymptotics of the CIT correlation.

A.1. Hankel transforms of the steady states equation

Equation (16) is obtained from the steady states equations (13) and (14):

$$M_2[x, x'] c(r, \rho) = \text{rhs}(x, x') \quad \text{with}$$

$$\text{rhs}(x, x') = \begin{cases} \phi_\rho \delta(x) \delta(x') & (\text{WIT}) \\ \phi_0 (c_1(x') \delta(x) + c_1(x) \delta(x')) & (\text{CIT}) \end{cases},$$

where we recall that $r(x, x') := |x - x'|$ and $\rho(x, x') := |x + x'|/2$, and that $M_2 = M_1[\rho] + M_2[r]$ from equation (15).

Hankel transforming both sides of the previous equation yields:

- for the left-hand side:
  $$\text{lhs}(r, k) = \left(\frac{2}{\pi}\right)^{d-2} \int_0^\infty d\rho \rho^{d-1} g_0(k\rho) (M_1[\rho] + M_2(x)) c(r, \rho)$$
  $$= \left(\rho M_\xi(x) + \frac{D_0}{2} k^2\right) c(r, k),$$

  where the second line comes from a double integration by parts with respect to $\rho$, and from the $g_0$'s being the isotropic Eigen-functions of the diffusion operator $M_1$;

- for the right-hand side:
  $$\text{rhs}(r, k) = \left(\frac{2}{\pi}\right)^{d-2} \int_0^\infty d\rho \rho^{d-1} g_0(k\rho) \text{rhs}(x, x')$$
  $$= \lim_{\epsilon \to 0} \left(\frac{2}{\pi}\right)^{d-2} \frac{1}{2^{d-1-\pi}} \int_0^\infty dxdx' \delta_\epsilon (r - r(x, x')) \rho^{d-1}$$
  $$g_0(k\rho(x, x')) \text{rhs}(x, x'),$$

https://doi.org/10.1088/1742-5468/aab509
where the notation \( \delta_\epsilon(r) \) essentially denotes a compact-support approximation to a radial Dirac distribution, namely a step function that vanishes for \( r > \epsilon \) and otherwise takes the constant value \( \epsilon^{-d}V_d^{-1} \), where \( V_d \) is the volume of the unit sphere in dimension \( d \).

Equation (16) follows. For the WIT statistics, the final result involves a Dirac distribution for the right-hand-side. To avoid any confusion, we find it safer to keep track of the ‘source extension’ \( \epsilon \ll 1 \), and not take directly the limit \( \epsilon \to 0 \), hence the \( \epsilon \) subscript in equation (16).

A.2. General form of the fluctuation field

The general solution (17) is obtained by solving equation (16) explicitly and transforming \( c(r, k) \) back into \( c(r, \rho) \). To solve for \( c(r, k) \), one first works out the isotropic contribution to the operator \( M_\xi[r] \) as

\[
M_\xi[r] = -D_0 \left( \frac{\xi}{\lambda} \right)^\xi \left( \partial_{rr} + \frac{m(\varphi, \xi)}{r} \partial_r + \frac{n(\varphi, \xi)}{r^2} \right), \text{ where}
\]

\[
m(\varphi, \xi) = (d + \xi - 1) \left( 1 + \frac{\varphi \xi}{1 + \varphi \xi} \right) \quad \text{and}
\]

\[
n(\varphi, \xi) = (d + \xi - 2) (d + \xi) \frac{\varphi \xi}{1 + \varphi \xi}.
\]

One may observe that a pair of independent homogeneous solutions of (17) is

\[
\phi_I(r, k) = r^{(1-m)/2} I_\omega \left( \eta kr^{1-\xi/2} \right),
\]

and \( \phi_K(r, k) = r^{(1-m)/2} K_\omega \left( \eta kr^{1-\xi/2} \right), \)

where \( \omega = \frac{(m - 1)^2 - n}{2 - \xi} \) and \( \eta = \sqrt{2 - \xi} \lambda^{\xi/2}, \)

and where \( I_\omega \) and \( K_\omega \) are the modified Bessel of the first and second kind. The solution \( c(r, k) \) is then obtained by a brute-force use of the ‘variation of the constant’ method, which yields

\[
c(r, k) = \frac{\lambda^\xi}{D_0(1 - \xi/2)} \left( \phi_I(r, k) \int_r^\infty dr' r'^{m-\xi} \phi_K(r', k) \text{rhs}(r', k) \right.
\]

\[
+ \left. \phi_K(r, k) \int_0^r dr' r'^{m-\xi} \phi_I(r', k) \text{rhs}(r', k) \right).
\]

Reconstructing the fluctuation field as \( c(r, \rho) = \int_0^\infty dk \, k^{d-1} c(r, k) g_0(k\rho) \), and performing the change of variables \( 'v = r'/r \) and \( u = \eta r^{1-\xi/2} k' \) yields the looked-for general expression (17).

A.3. CIT asymptotics

The CIT asymptotics (23)–(25) are obtained from (17) by (i) approximating the modified Bessel functions \( I_\omega, K_\omega \) with their asymptotic behaviour, and (ii) integrating over the
Turbulent mass inhomogeneities induced by a point-source

\( u \) before the \( v \)-variable. After changing \( u \to 2u/(\Lambda s) \), one gets from equation (17), in the limit \( s \gg 1 \):

\[
\begin{align*}
\phi_0 & \approx \frac{2^{1-s}}{2^{1-d}} \int_0^1 dv v^{1-s/2-\xi/4} \int_0^{+\infty} du u^{d-1} J_0 \left( u \frac{2}{\Lambda} \right) \phi_0 (uv), \\
\phi_0 & \approx \frac{2^{1-s}}{2^{1-d}} \int_0^{+\infty} du u^{d-2} \frac{\phi_0 (uv)}{uv} \left( \frac{2}{2-d} \right) e^{-\frac{u}{2} (1-v^{1-\xi/2})}, \\
\phi_0 & \approx \frac{2^{1-s}}{2^{1-d}} \int_0^{+\infty} du u^{d-2} \frac{\phi_0 (uv)}{uv} \left( \frac{2}{2-d} \right) e^{-\frac{u}{2} (1-v^{1-\xi/2}-1)}.
\end{align*}
\]

Similarly for \( \Lambda \gg 1 \) and \( \Lambda \ll 1 \). Noticing that \( \int_0^{+\infty} du \frac{\phi_0 (uv)}{uv} \exp(-|x|/u) = (x^2 + v^2)^{(1-d)/2} \), we obtain the asymptotic behaviour for \( \Lambda \ll 1 \):

\[
\begin{align*}
\phi_0 & \approx \frac{2^{1-s}}{2^{1-d}} \int_0^{+\infty} dv v^{1-s/2-\xi/4} \int_0^{+\infty} du u^{d-1} J_0 \left( u \frac{2}{\Lambda} \right) \frac{u^{m/2-3\xi/4}}{\left( 1 + \frac{X(v)}{4} \right)^{(d-1)/2}}, \\
\phi_0 & \approx \frac{2^{1-s}}{2^{1-d}} \int_0^{+\infty} dv v^{m/2-3\xi/4} \left( \frac{2}{2-d} \right) \left( \frac{v^2 + \frac{4}{\Lambda^2} X(v)}{4} \right)^{(d-1)/2},
\end{align*}
\]

Similarly for \( \Lambda \gg 1 \), the asymptotic behaviour is:

\[
\begin{align*}
\phi_0 & \approx \frac{2^{1-s}}{2^{1-d}} \int_0^{+\infty} dv v^{m/2-3\xi/4} \left( \frac{2}{2-d} \right) \left( \frac{v^2 + \frac{4}{\Lambda^2} X(v)}{4} \right)^{(d-1)/2},
\end{align*}
\]

where \( X(v) := |1 - v^{1-\xi/2}|. \)

Since \( s \ll 1 \), the behaviours of those integrals are dominated by the behaviours near \( v = 1 \), for instance:

\[
\begin{align*}
\int_0^{+\infty} dv \frac{c_1 (rv) v^{m/2-3\xi/4}}{\left( 1 + \frac{X(v)}{4} \right)^{(d-1)/2}} & \sim 2c_1 (r) \int_0^{1-\epsilon} dv \left( \frac{v^2 + \frac{4}{\Lambda^2} X(v)}{4} \right)^{(d-1)/2}, \\
& \sim 2c_1 (r) \left( \frac{s}{1 - \xi/2} \right) \chi_d(s),
\end{align*}
\]

where \( \chi_d(s) = \text{asinh}((1 - \xi/2)\epsilon/s) \sim -\log s \) for \( d = 2 \) and \( \chi_d(s) = \text{atan}((1 - \xi/2)\epsilon/s) \sim \Pi/2 \) for \( d = 3 \). The behaviours (25) and (24) follow. The asymptotic behaviour (35) valid for the B/E ensemble is obtained along the same line.
Turbulent mass inhomogeneities induced by a point-source

References

[1] Canadé G T 2012 Turbulent Diffusion in the Environment vol 3 (Berlin: Springer)
[2] Nasr S, Sugiyama G, Baskett R L, Larsen S C and Bardley M M 2007 Int. J. Emerg. Manage. 4 524–50
[3] Devenish B J, Francis P N, Johnson B T, Sparks R S J and Thomson D J 2012 J. Geophys. Res.: Atmos. 117 (https://doi.org/10.1029/2011JD016782)
[4] Celani A, Villermaux E and Vergassola M 2014 Phys. Rev. X 4 041015
[5] Flesch T K, Wilson J D and Yee E 1995 J. Appl. Meteorol. 34 1320–32
[6] Wilson J D and Sawford B L 1996 Bound.-Layer Meteorol. 78 191–210
[7] Jones A R, Thomson D J, Hort M and Devenish B 2004 Air Pollution Modeling and its Application XVII, Proc. of the 27th NATO/CCMS Int. Technical Meeting on Air Pollution Modelling and its Application pp 24–9
[8] Celani A, Martins Afonso M and Mazzino A 2007 J. Fluid Mech. 583 189–98
[9] Martins Afonso M and Mazzino A 2011 Geophys. Astrophys. Fluid Dyn. 105 553–65
[10] Celani A and Vergassola M 2001 Phys. Rev. Lett. 86 424
[11] Bianchi S, Biferale L, Celani A and Cencini M 2016 Eur. J. Mech. B 55 324–9
[12] Laenen F 2017 Mixing, transport and turbulence modulation in solid suspensions: study and modelling PhD Thesis Côte d’Azur
[13] Haan P D and Rotach M W 1995 Int. J. Environ. Pollut. 5 350–9
[14] Cardy J, Falkovich G and Gawedzki K 2008 Non-Equilibrium Statistical Mechanics and Turbulence vol 355 (Cambridge: Cambridge University Press)
[15] Gawedzki K 2008 arXiv:0806.1949
[16] Kraichnan R H 1970 Phys. Fluids (1958–1988) 13 22–31
[17] Gawedzki K 2001 Intermittency in Turbulent Flows ed J C Vassilicos (Cambridge: Cambridge University Press) pp 86–104
[18] Orszag S A 1974 Lectures on the Statistical Theory of Turbulence (Cambridge, MA: MIT)
[19] Frisch U 1995 Turbulence: the Legacy of AN Kolmogorov (Cambridge: Cambridge University Press)
[20] Canet J, Rossetto V, V schebor N and Balac G 2017 Phys. Rev. E 95 023107
[21] Chaves M, Gawedzki K, Horvai P, Kupiainen A and Vergassola M 2003 J. Stat. Phys. 113 643–92
[22] Falkovich G, Gawedzki K and Vergassola M 2001 Rev. Mod. Phys. 73 913–75
[23] Pumir A, Shraiman B I and Chertkov M 2000 Phys. Rev. Lett. 85 5324
[24] Arad I, Biferale L, Celani A, Procaccia I and Vergassola M 2001 Phys. Rev. Lett. 87 164502
[25] Sreenivasan K R and Schumacher J 2010 Phil. Trans. R. Soc. A 368 1561–77
[26] Falkovich G E 2008 Lecture Notes on Turbulence and Coherent Structures in Fluids (Plasmas and Nonlinear Media vol 4) (Singapore: World Scientific)
[27] Risken H 1984 The Fokker–Planck Equation (Berlin: Springer) pp 63–95
[28] Gawedzki K and Horvai P 2004 J. Stat. Phys. 116 1247–300
[29] Abramowitz M and Stegun I A 1964 Handbook of Mathematical Functions: with Formulas, Graphs and Mathematical Tables (New York: Dover)
[30] Boffetta G and Celani A 2000 Physica A 280 1
[31] Boffetta G and Sokolov I M 2002 Phys. Rev. Lett. 88 094501
[32] Bitane R, Homann H and Bec J 2012 Phys. Rev. E 86 045302
[33] Thalabard S, Krstulovic G and Bec J 2014 J. Fluid Mech. 755 R4
[34] Bourgoin M 2015 J. Fluid Mech. 772 678–704
[35] Yeung P K and Pope S B 1989 J. Fluid Mech. 207 531–86
[36] Mordant N, Metz P, Michel O and Pinton J F 2001 Phys. Rev. Lett. 87 214501
[37] Taylor G I 1922 Proc. London Math. Soc. 20 196–212
[38] LaCasce J H 2008 Progr. Oceanogr. 77 1–29

https://doi.org/10.1088/1742-5468/aab509