More on the initial singularity problem in gravity’s rainbow cosmology

M. Khodadi†, K. Nozari† and H. R. Sepangi‡

1 Department of Physics, Faculty of Basic Sciences, University of Mazandaran, P. O. Box 47416-95447, Babolsar, Iran
2 Department of Physics, Shahid Beheshti University, G.C., Evin, Tehran 19839, Iran

November 8, 2016

Abstract

Using a one-dimensional minisuperspace model with a dimensionless ratio \( \frac{E}{E_{Pl}} \), we study the initial singularity problem at the quantum level for the closed rainbow cosmology with a homogeneous, isotropic classical space-time background. We derive the classical Hamiltonian within the framework of Schutz’s formalism for an ideal fluid with a cosmological constant. We characterize the behavior of the system at the early stages of the universe evolution through analyzing the relevant shapes for the potential sector of the classical Hamiltonian for various matter sources, each separately modified by two rainbow functions. We show that for both rainbow universe models presented here, there is the possibility of eliminating the initial singularity by forming a potential barrier and static universe for a non-zero value of the scale factor. We investigate their quantum stability and show that for an energy-dependent space-time geometry with energies comparable with the Planck energy, the non-zero value of the scale factor may be stable. It is shown that under certain constraints the rainbow universe model filled with an exotic matter as a domain wall fluid plus a cosmological constant can result in a non-singular harmonic universe. In addition, we demonstrate that the harmonically oscillating universe with respect to the scale factor is sensitive to \( \frac{E}{E_{Pl}} \) and that at high energies it may become stable quantum mechanically. Through a Schrödinger-Wheeler-De Witt (SWD) equation obtained from the quantization of the classical Hamiltonian, we also extract the wave packet of the universe with a focus on the early stages of the evolution. The resulting wave packet supports the existence of a bouncing non-singular universe within the context of gravity’s rainbow proposal.

Keywords: Rainbow Cosmology; Potential Barrier; Static Universe; Quantum Stability; Wave Packet

PACS numbers: 04.60.-m; 98.80.Qc; 04.20.Dw

1 Introduction

One of the serious problems from which standard cosmology suffers is the so-called “initial singularity problem.” At first, Einstein and many others believed that the origin of this issue goes back to the simplifying idea of the “cosmological principle.” However, this was challenged by Lemaitre in [1]. He argued that in a certain class of anisotropic universe models, the tendency towards the appearance of singularities is even greater than in isotropic ones and concluded that this problem cannot be linked to the cosmological principle. Later on, Penrose and Hawking presented some theorems on the existence of singularities in the solutions of Einstein’s field equations [2, 3, 4, 5]. The Penrose
theorem considers space-like singularities which are characteristic of non-rotating uncharged black-holes, whereas Hawking’s singularity theorem covers the whole universe. For a comprehensive review of the concept of singularity see [6, 7].

To study the early universe for which quantum gravitational effects are to be considered, General Relativity (GR) alone is insufficient and quantum gravity (QG) should come to the fore. More precisely, QG proposal could be used to avoid the initial singularity through a potential barrier [8]-[13]. Theories like Loop Quantum Gravity (LQG) [14,15] and Super Strings (SS) [16] are attempts in this direction [17,18,19,20]. Despite the absence of a fully self-consistent theory for QG, semi-classical approaches have been very much in the spotlight in recent years. Generally, semi-classical QG approaches contain two eminent characteristics: the existence of a natural cutoff in the order of the Planck length $l_{Pl}$ which represents the minimal accessible distance (see, e.g., [21]-[33]) and deviation from the standard relativistic dispersion relation. This implies that at regions dominated by QG effects, the relativistic dispersion relation should be modified. One of these approaches in which both the attributes noted above are addressed was presented by Amelino-Camelia [34,35] and is knows as “Doubly Special Relativity” (DSR). Specifically, DSR is an advanced version of special relativity in the presence of an excess invariant object named Planck energy $E_{Pl}$ [36,37,38]. Following this proposal, Magueijo and Smolin [39] generalized DSR to “Doubly General Relativity” (DGR) through involvement of gravity. The core idea of the DGR is that at high energy regimes we will not meet a single geometry of classical space-time, rather it is a “running geometry”. Indeed, the geometry of space-time is detected by the energy dependent quantum particle(s) known as “probe particle(s)” which are traveling in it. As in quantum mechanics (QM) where the system under measurement has interaction with measuring device, the classical background geometry can be affected by the movements of these probe particle(s) with various energies and interactions between them which would lead to changes in the standard relativistic dispersion relation, written as

$$E^2 f_1 \left( \frac{E}{E_{Pl}} \right) - p^2 f_2 \left( \frac{E}{E_{Pl}} \right) = m^2 .$$

(1)

Depending on the energy level with which the space-time is explored, i.e. the value of dimensionless ratio $\frac{E}{E_{Pl}}$, probe particles record the various pictures of the space-time background. Inspired by such a feature, the DGR scenario is known as the “gravity’s rainbow.” Accordingly, $f_{1,2} \left( \frac{E}{E_{Pl}} \right)$ in the modified dispersion relation (MDR) [11], are called the “rainbow functions” and have a two-facet significance; they lead to a MDR which in one hand must be consistent with some outcomes of other QG approaches and on the other hand they should help in resolving the paradoxes created in justifying some of the cosmological phenomenon. It should be noted that to respect the usual formula, the rainbow functions should obey

$$\lim_{\frac{E}{E_{Pl}} \to 0} f_{1,2} \left( \frac{E}{E_{Pl}} \right) = 1 .$$

(2)

In this work, we focus attention on one of the cosmological applications of this semi-classical QG approach, namely the status of the initial singularity problem by taking the quantum corrections allowed by the DGR proposal. Numerous works have been carried out in recent years regarding gravity’s rainbow proposal. For instance in [40], the authors study a rainbow FRW cosmology which is fixed by two rainbow functions and derive some non-singular analytical solutions. In [41], a quantum cosmological perfect fluid model is considered in the context of rainbow gravity and the possibility of avoiding the initial singularity is studied, leading to a solution predicting the existence of a bouncing non-singular universe. In addition to the mechanism of establishing the potential barrier to remove the initial singularity, there is another idea known as the static universe (SU) which is extensively discussed in recent years. Based on this scenario, the present universe could have been commenced from an initial frozen state known as the SU at the asymptotic earliest times [42,43] in the absence of

1Semi-classical approach here means a theoretical framework in which one treats matter fields as being quantum and the gravitational field as being classical. Indeed, the matter fields are propagating on the classical space-time background, as described in GR.
the Big Bang. In other words, the heart of this scenario is that there is no time origin for the beginning of the universe. In another scenario known as the emergent cosmology (EC) [44, 45, 46], one considers a closed universe having a positive curvature constant with a primary origin from which the SU begins where there are no issues such as initial singularity and horizon [17, 18]. Note that, while the positive spatial curvature idea has no consequence at late time cosmology, it can address some fundamental problems of GR in the early universe. Of course, this idea is endorsed implicitly by observational data since observations have revealed that we do not live in an exactly flat universe [49, 50]. The classical stability of the initial SU with respect to perturbations of the scalar and tensor modes within the framework of the DGR proposal have been studied by fixing two distinct rainbow functions [51]. Depending on the type of rainbow function, one meets the different stability conditions. It would therefore be of interest to investigate the quantum stability of rainbow cosmology. More precisely, in this paper we endeavor to find an answer to the question of whether or not the closed rainbow cosmology remains stable from the QM viewpoint. To this end, we have limited our analysis to a minisuperspace model containing one degree of freedom, i.e. scale factor of the universe.

The plan of the paper is as follows: in Section 2 we derive the classical Hamiltonian by means of the Schutz’s formalism for an ideal fluid plus cosmological constant. Section 3 deals with a closed rainbow universe model with two components of matter sources; dust and the cosmological constant. By analyzing the potential part of classical Hamiltonian and WKB approximation, we examine the quantum stability of non-singular solutions. We then move on to consider a closed rainbow FRW universe model including an exotic matter field as a domain wall fluid plus the cosmological constant. Our analysis on the initial singularity of the model in the context of gravity’s rainbow cosmology finishes in section 4 by the quantization of the classical Hamiltonian and consequently derivation of the wave function of the universe. Section 5 is devoted to conclusions.

2 The Hamiltonian

We start by considering the action rose up from ADM formalism for gravity in the presence of a cosmological constant \( \Lambda \) and perfect fluid as

\[
S = \int_M d^4x \sqrt{-g} \, R + 2 \int_{\partial M} d^3\sqrt{h} \, h_{ab} \, K^{ab} + \int_M d^4x \sqrt{-g} \, (P + \Lambda) , \quad (16\pi G = \hbar = 1) . \tag{3}
\]

Here \( h_{ab} \) represents the induced metric over three dimensional spatial hypersurfaces, \( K^{ab} \) is the extrinsic curvature and \( P \) is the pressure defined via the usual equation of state (EOS) \( P = \omega \rho \) with \( \rho \) being the energy density. So one see in the above action included the perfect fluid energy density inside the Lagrangian. In [44] has been presented an action known as Hawking-Ellis as \( S_M = - \int d^4x \sqrt{-g} \zeta (1 + v) \) corresponding to the action of perfect fluid so that in which \( \zeta \) and \( v \) represent the density of fluid’s and internal energy, respectively. In the same reference it is proven that in the case of defining \( \rho = \zeta (1 + v) \) and \( P = \zeta^2 \frac{d\zeta}{dv} \), by varying the mentioned action in terms of metric obtain the same standard expression for the energy-momentum tensor. Surprisingly, in [52, 53] shown that this result also can be derivable in the same way by considering the perfect fluid energy density inside the Lagrangian. Note that the above action suggested according to ADM formalism so that the seconded term - as a boundary term- will be remove via the variation of \( \int_M d^4x \sqrt{-g} R \), see [54] for more details. For an isotropic and homogeneous universe the general form of the rainbow FRW metric reads

\[
d s^2 = \frac{N^2(t)}{f_1^2 \left( \frac{E}{\epsilon_{pl}} \right) dt^2 - \frac{a^2(t)}{f_2^2 \left( \frac{E}{\epsilon_{pl}} \right)} \left[ \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2 \right]} . \tag{4}
\]

Here \( N \) represents the lapse function and \( k \) takes one of the three values \(-1, 0, +1\) corresponding to an open, flat or closed universe respectively. In the modified FRW metric [44], the quantum corrections are embedded in the rainbow functions \( f_{1,2} \left( \frac{E}{\epsilon_{pl}} \right) \). In other words, these functions indicate how the standard FRW metric can be deformed as a consequence of the motion of quantum probe particles
in the early classical space-time geometry. One my urge that $f_{1,2} \left( \frac{E}{E_{Pl}} \right)$ can be absorbed in the lapse function and scale factor respectively just by redefinition of these quantities and therefore there is no trace of energy-dependence in this metric in essence. While this seems to be the case for $f_1$, we note however that energy dependence of the background metric in gravity’s rainbow needs a reconsideration of the measurement process and therefore one can not say that this is just a re-parametrization of the lapse function and scale factor. In order to calculate the Hamiltonian for the action $\mathcal{L}$ we start from the fluid part. Although the fluid’s four velocity $U_\mu$ in Schutz’s formalism $[52, 53]$ is defined in terms of six potentials, it can be rewritten in terms of four independent potentials $h, S, \epsilon$ and $\theta$ as follows:

$$U_\mu = \frac{1}{\hbar}(\epsilon, \mu + \theta S, \mu) .$$

The fluid’s four velocity $U_\mu$ obeys $U_\mu U^\mu = 1$. To make contact with thermodynamic quantities, we can interpret $h$ and $S$ as the specific enthalpy and specific entropy, respectively. Therefore, the fluid part of the action $\mathcal{L}$ can be rewritten using the following relevant thermodynamic equations $[? ]$

$$\rho = \rho_0 (1 + \Pi) ,$$
$$h = 1 + \Pi + \frac{E}{\rho_0} ,$$
$$\tau dS = d\Pi + P d\left( \frac{1}{\rho_0} \right).$$

where $\tau, \rho, \rho_0$ and $\Pi$ denote temperature, total mass energy density, rest mass density and specific internal energy, respectively. By combining the thermodynamic relations given in $[6]$ one can easily prove that the EoS reads as

$$P = \frac{\omega}{(1 + \omega)\omega^\omega^\omega^\omega} h \frac{e^{-S}}{e^{-S}} .$$

In a comoving system with a perfect fluid four vector velocity $U_\mu = \left( N f_1^{-1}(E/E_{Pl}), 0, 0, 0 \right)$ on can deduce the following Lagrangian

$$L_{f-A} = \frac{f_1^{\frac{1}{2}}(E/E_{Pl})}{f_2^{\frac{3}{2}}(E/E_{Pl})} a^3 N \frac{1}{2} \omega (\dot{\epsilon} + \theta \dot{S}) e^{-S} \omega + \frac{Na^3}{f_1(E/E_{Pl})f_2(E/E_{Pl})} \Lambda .$$

so that $h > 0$ and $(\dot{\epsilon} + \theta \dot{S}) > 0$. Also, the Hamiltonian takes the following form

$$H_{f-A} = \epsilon P_\epsilon + \dot{S} P_S - L_{f-A} = \frac{N f_2^{3\omega}(E/E_{Pl})}{f_1(E/E_{Pl})} \frac{P_T}{a^6} - \frac{Na^3}{f_1(E/E_{Pl})f_2(E/E_{Pl})} \Lambda .$$

As is clearly seen, the Hamiltonian is a linear function of $P_T$ such that $P_\epsilon = \frac{\partial}{\partial h} L_{f-A}, P_S = \frac{\partial}{\partial S} L_{f-A}$. To obtain the Hamiltonian in terms of $P_T$, we have employed the following canonical transformations introduced in $[54, 55]$

$$T = -P_S e^{-S} P_\epsilon^{-(\omega + 1)} ,$$
$$P_T = P_\epsilon^{\omega + 1} e^{S} .$$

The Lagrangian and consequently the Hamiltonian corresponding to the gravity part of the action $\mathcal{L}$ can be written as

$$L_g = -6 \frac{f_1(E/E_{Pl})}{f_2(E/E_{Pl})} a \dot{a}^2 + 6 \frac{N k a}{f_1(E/E_{Pl})f_2(E/E_{Pl})} .$$

\footnote{Among the six potentials, two are linked to rotation. The FRW type models, on the other hand, admit time-like geodesics which are hyper-surface normal i.e. the vorticity tensor $\omega_{\mu\nu}$ is zero. Note that $\omega_{\mu\nu}$ refers to the rotation of time-like geodesics.}

\footnote{Let us recall that by way of such canonical transformations, we may pursue the status of a dynamical system with more variables $[56, 57]$.
and
\[ H_g = \dot{a} P_a - L_g = -\frac{N f_3^3(E_{\text{Ep}})}{24 f_1^3(E_{\text{Ep}})} \frac{P_a^2}{a} - \frac{6 N k a}{f_1(E_{\text{Ep}}) f_2(E_{\text{Ep}})^2}, \]

(12)

respectively with \( p_a = \frac{\partial L_a}{\partial a} \) which is the momentum canonically conjugated to the scale factor \( a \). The super Hamiltonian for the minisuperspace of this model can now be written as

\[ H = H_g + H_{f-A} = \frac{N f_1(E_{\text{Ep}})}{} \left( -\frac{f_3^3(E_{\text{Ep}})}{24} \frac{P_a^2}{a} - \frac{6 k a f_2(E_{\text{Ep}})}{} + f_3^2(E_{\text{Ep}}) \frac{P_T}{a^3} - \frac{\Lambda a^3}{f_2^3(E_{\text{Ep}})} \right). \]

(13)

In Eq. (13), \( N \) is called the Lagrange multiplier preserving the classical constraint equation \( H = 0 \).

Given that \( \dot{T} = \{T, H\} = \frac{N f_3^3(E_{\text{Ep}})}{f_1(E_{\text{Ep}}) a^{3\omega}} \), \( T \) in Eq. (13) might serve the role of the time (i.e. \( T = t \)) provided that

\[ N = \frac{f_1(E_{\text{Ep}})}{} a^{3\omega}. \]

(14)

Thereupon, we can write the final form of the super Hamiltonian (13) as

\[ H = f_2^2 f_3^{3-3\omega} \left( \frac{E}{E_{\text{Ep}}} \right) a^{3\omega - 1} \left[ \frac{P_a^2}{24} + \frac{6 k a^2 f_2^4}{f_2(E_{\text{Ep}})} \left( \frac{E}{E_{\text{Ep}}} \right) - a^{1-3\omega} f_2^{3\omega - 3} \left( \frac{E}{E_{\text{Ep}}} \right) p_T + a^4 f_2^4 \left( \frac{E}{E_{\text{Ep}}} \right) \Lambda \right]. \]

(15)

At this point an important issue should be noted. As the above relation indicates, the rainbow function \( f_1(E_{\text{Ep}}) \) does not contribute to the final form of the super Hamiltonian in the same way as the lapse function \( N \) does. More precisely, since the lapse function is arbitrary, the rainbow function \( f_1(E_{\text{Ep}}) \) can always be absorbed by re-scaling \( N \). In fact, based on some symmetry properties of the spacetime in cosmological scales, one can define the well known standard (or global) time coordinate, the cosmic time \( \eta \equiv \int N dt \). Here, by rescaling the lapse function, definition of the cosmic time is modified as \( \eta = \int_{f_1(E_{\text{Ep}})}^N dt \) which, based on the correspondence principle, recovers the standard definition of the cosmic time in low energy scales. Using \( N \) in (14), one finds that the energy dependent function \( f_1(E_{\text{Ep}}) \) disappears and has no affect on the physical properties or measurements expressed by the cosmic time coordinate \( \eta \). In effect, it is equivalent to fixing the value of \( f_1(E_{\text{Ep}}) \) to unity in temporal part of the line element. One may argue that the results included in the present analysis, given by two energy-dependent functions \( f_1 \) and \( f_2 \) in the line element (4), appear to be a coordinate effect. While it is indeed the case for \( f_1 \) due to definition of time as we have explained, the situation is different for \( f_2 \). In gravity’s rainbow proposal the unknown energy-dependent function \( f_2(E_{\text{Ep}}) \) has physical implications and changes measurement process in essence. The physical roles of this function cannot be ignored by a simple rescaling of the scale factor. For instance, the scale factor rescaled by \( f_2(E_{\text{Ep}}) \) in the spatial part of the line element (4) has the potential to remove the initial singularity in the cosmic history, see [57] for details. Indeed, Magueijo and Smolin in their seminal work [38], by using rescaling of the time coordinate \( \tau(E) = \frac{b(E)}{f(E)} \) via introducing some unknown energy dependent functions \( h \) and \( f \), have opened novel avenues for the solution of the horizon problem. It is noteworthy that here the solution of the horizon problem is subject to an appropriate choice of the above mentioned rainbow functions. In summary, the role of the rainbow function \( f_2 \) cannot be reduced to merely a coordinate transformation and has significant effects on the cosmological scenario under consideration. In what follows, one of the its cosmological application will be discussed in details.

3 Quantum stability of closed rainbow cosmology

Based on classical theory, rainbow universe can be considered as a constrained dynamical system resulting from Hamiltonian (15). The constraint equation \( H = 0 \) allows us to rewrite Hamiltonian
in terms of the kinetic and potential energies as follows

\[ p_a^2 + V \left( a, \frac{E}{E_{Pl}} \right) = 0, \quad (16) \]

so that

\[ p_a = -12 a^{1-3\omega} \hat{a} f_2^{3\omega-3} \left( \frac{E}{E_{Pl}} \right), \quad (17) \]

and

\[ V \left( a, \frac{E}{E_{Pl}} \right) = 144 k f_2^{-4} \left( \frac{E}{E_{Pl}} \right)^2 - 24 f_2^{3\omega-3} \left( \frac{E}{E_{Pl}} \right) p_T a^{1-3\omega} + 24 f_2^{-6} \left( \frac{E}{E_{Pl}} \right) \Lambda a^4. \quad (18) \]

The benefit of the above decomposition is that we can imagine the universe as a non-relativistic particle which is under the influence of the one-dimensional potential. We note that from a classical viewpoint, regions \( V \leq 0 \) are accessible to the traveling particle, here the universe. In what follows, by taking two common rainbow functions for two universe models filled with matter sources mentioned in Section 2, we survey the stability of a closed rainbow universe from a QM viewpoint.

### 3.1 Non-relativistic dust matter plus cosmological constant \( \Lambda \)

For a closed rainbow universe model with a source of attraction as non-relativistic dust, i.e. \( \omega = 0 \), along with cosmological constant (attraction or repulsion), the potential function becomes

\[ V \left( a, \frac{E}{E_{Pl}} \right) = 24 f_2^{-6} \left( \frac{E}{E_{Pl}} \right) \Lambda a^4 + 144 f_2^{-4} \left( \frac{E}{E_{Pl}} \right) a^2 - 24 f_2^{-3} \left( \frac{E}{E_{Pl}} \right) p_T a . \quad (19) \]

As previously mentioned, finding a relevant and useful rainbow function from both theoretical and phenomenological considerations is one of the open issues in rainbow gravity proposal. Here, we restrict ourselves to two most widely used rainbow functions

\[ f_2 \left( \frac{E}{E_{Pl}} \right) = \left( 1 - \frac{E}{E_{Pl}} \right)^{-1}, \quad (20) \]

and

\[ f_2 \left( \frac{E}{E_{Pl}} \right) = \sqrt{1 - \left( \frac{E}{E_{Pl}} \right)^n}, \quad n = 1, 2, 3, ... \quad (21) \]

which have been suggested in [37] and [58], respectively. Let us emphasize that either choice of the above functions does not mean that they are problem free. However, their phenomenological aspects have particular importance. For instance, the MDR obtained using the rainbow function with \( f_1 \left( \frac{E}{E_{Pl}} \right) = 1 \) to describe many of the phenomenon of interest seems to work well [59]. Now, starting from the rainbow function [20], we arrive at

\[ V \left( a, \frac{E}{E_{Pl}} \right) = 24 \left( 1 - \frac{6E}{E_{Pl}} \right) \Lambda a^4 + 144 \left( 1 - \frac{4E}{E_{Pl}} \right) a^2 - 24 \left( 1 - \frac{3E}{E_{Pl}} \right) p_T a . \quad (22) \]

Through the minimization of the above potential function we find that under the conditions listed in Eq. [23], we are dealing with an early static universe (SU) \( V_{min} = 0 \) which is fluctuating in the

---

\[^4\text{While the sign of the cosmological constant } \Lambda \text{ is not yet clear, an analysis of the potential part of the Hamiltonian in the presence of the two known rainbow functions leads to its determination.}\]
vicinity of non-zero values of the scale factor $a_0$

\[
p_T = 0, \quad \frac{1}{6} < \frac{E}{E_{pl}} \leq \frac{1}{4}, \quad \Lambda \leq 0,
\]

\[
p_T = 0, \quad 0 \leq \frac{E}{E_{pl}} < \frac{1}{6}, \quad \Lambda \geq 0,
\]

\[
p_T > 0, \quad \frac{1}{3} < \frac{E}{E_{pl}} \leq 1, \quad \Lambda \leq F(\frac{E}{E_{pl}}),
\]

\[
p_T \leq 0, \quad 0 \leq \frac{E}{E_{pl}} < \frac{1}{6}, \quad \Lambda \geq 0,
\]

\[
p_T < 0, \quad \frac{1}{6} < \frac{E}{E_{pl}} \leq \frac{1}{4}, \quad \Lambda \leq 0.
\]

(23)

where $F(\frac{E}{E_{pl}})$ reads as

\[
F(\frac{E}{E_{pl}}) = \frac{-2048(\frac{E}{E_{pl}})^3 + 1536(\frac{E}{E_{pl}})^2 - 384(\frac{E}{E_{pl}}) + 32}{(54(\frac{E}{E_{pl}})^3 - 45(\frac{E}{E_{pl}})^2 + 12(\frac{E}{E_{pl}}) - 1) p_T^2}.
\]

(24)

We see from Figure 1 (left panel) that around $a_0 \sim 2 - 2.5$, a SU takes shape. While from the classical viewpoint, $a_0$ is a half stable point, this figure reflects the fact that from the QM viewpoint, there is the chance of a tunneling via the potential barrier to a zero value of the scale factor. Surprisingly, Figure 1 (left panel) shows qualitatively that as the dimensionless ratio $\frac{E}{E_{pl}}$ grows, the barrier height also increases and the chance of tunneling becomes tiny. We know from ordinary QM that the probability of tunneling is given in the WKB-approximation by [60, 61]

\[
P \sim e^{-2I},
\]

(25)

where $I$ is called WKB tunneling action and is defined as

\[
I = \int_0^{a_0} \sqrt{V} \, da.
\]

(26)

The third constraints in Eq. (23) is very suitable in the sense that, unlike other circumstances, it allows us to explore the geometry of space-time by high energy particle probes. Substituting the potential function (22) in (26) and using parameter values $p_T = 15$, $a_0 \sim 2$ with $\Lambda = F(\frac{E}{E_{pl}})$ allows us to perform a numerical analysis based on Eq. (25), see Table 1. Indeed, this analysis represents the possibility of the transition of SU through the barrier to a zero value of the scale factor. In agreement with Figure 1 (left panel), values reported in Table 1 suggest that from QM viewpoint, increasing the energy levels of the probe particle(s) makes the possibility of the SU collapse via quantum tunneling small. Overall, it can be said that for a rainbow universe model filled with dust plus cosmological constant, by satisfying the third constraint in (23), we are dealing with an early SU in the vicinity of a non-zero scale factor $a_0 \neq 0$ which avoids the singularity. The same results obtained from Figure 1 (left panel) and Table 1, can be summarized as follows: the point $a_0 \neq 0$ may be stable quantum mechanically. What should be noted is that each of the numerical values presented in the above Table and also next Tables are not particularly illuminating. In fact, their up or down trends in terms of dimensionless ratio $\frac{E}{E_{pl}}$ point to a physical interpretation. Recall that the SU fluctuating in the vicinity of the point $a_0$ does not have exact classical stability, rather it is a half stable point.

The conditions below imply that for the potential function (22) there is a non zero value in the classically allowed region $V < 0$.

\[
p_T = 0, \quad \frac{1}{4} < \frac{E}{E_{pl}} \leq 1, \quad \Lambda < 0,
\]

\[
p_T > 0, \quad \frac{1}{6} < \frac{E}{E_{pl}} < \frac{1}{4}, \quad \Lambda = 0,
\]

\[
p_T > 0, \quad \frac{E}{E_{pl}} = \frac{1}{3}, \quad \Lambda < 0.
\]

(27)
the rainbow function (21). This time the potential function (19) can be rewritten as

\[ V(a, \frac{E}{E_{Pl}}) = 24 \left( 1 + 3 \left( \frac{E}{E_{Pl}} \right)^n \right) \Lambda a^4 + 144 \left( 1 + 2 \left( \frac{E}{E_{Pl}} \right)^n \right) a^2 - 24 \left( 1 + \frac{3}{2} \left( \frac{E}{E_{Pl}} \right)^n \right) p_T a. \]

To satisfy the third constraint in (23), use has been made of \( \Lambda = \frac{E}{E_{Pl}} \) and the parameter value \( p_T = 15 \).

| \( n \) | \( \frac{E}{E_{Pl}} \) | \( \bar{t} \) | \( P \) |
|---|---|---|---|
| 2 | 0.4 | 31.5648 | \( e^{-63.1290} \) |
| 3 | 0.45 | 36.1046 | \( e^{-72.2092} \) |
| 4 | 0.5 | 63.3813 | \( e^{-126.763} \) |
| 5 | 0.55 | 96.5461 | \( e^{-193.092} \) |
| 6 | 0.6 | 132.062 | \( e^{-264.124} \) |
| 7 | 0.65 | 168.745 | \( e^{-337.49} \) |
| 8 | 0.7 | 206.091 | \( e^{-412.182} \) |
| 9 | 0.75 | 243.848 | \( e^{-487.696} \) |
| 10 | 0.8 | 281.878 | \( e^{-564.796} \) |

The first constraint given in Eq. (27) is the most suitable constraint for exploring the geometry of classical space-time background at energies comparable to Planck energy. It suggests a harmonic universe in the presence of an initial singularity which is oscillating between two classical turning points \( a_1 \) and \( a_2 \), so that

\[ a_1 = 0, \quad a_2 = \sqrt[3]{\frac{6 \left( \frac{4E}{E_{Pl}} - 1 \right)}{\Lambda \left( 1 - \frac{6E}{E_{Pl}} \right)}}. \]  

We see from relation obtained above and as well as Figure 1 (right panel) that by increasing the dimensionless ratio \( \frac{E}{E_{Pl}} \) the point \( a_1 \) also shifts towards greater values. Now, let us determine the status of the classical stability of the points \( a_1 = 0 \) and \( a_2 \). The classical turning point \( a_1 = 0 \) is an unstable saddle point since \( \frac{d^2}{da^2} V \bigg|_{a=a_1} = 0 = \frac{d^2}{da^2} V \bigg|_{a=a_2} \). The point \( a_2 \) also is an unstable point since \( \frac{d}{da} V \bigg|_{a=a_2} < 0 \).

Another set of constraints derived from the analysis of the potential function (22) together with the rainbow function (20) can be written as

\[
\begin{cases}
    p_T = 0, & \frac{1}{6} < \frac{E}{E_{Pl}} \leq 1, \quad \Lambda > 0, \\
    p_T \geq 0, & \frac{1}{6} < \frac{E}{E_{Pl}} < \frac{1}{4}, \quad \Lambda > 0, \\
    p_T > 0, & \frac{1}{4} \leq \frac{E}{E_{Pl}} \leq 1, \quad \Lambda > 0, \\
    p_T < 0, & \frac{1}{6} < \frac{E}{E_{Pl}} \leq \frac{1}{4}, \quad \Lambda > 0.
\end{cases}
\]  

Constraints presented in (29) indicate that there is no minimum for the potential function (22). We also note that the potential is commenced from a zero value for the scale factor and grows eventually to \( |V| = \infty \). As mentioned earlier, in the context of the semi-classical QG approaches, it is believed that the introduction of a non-zero minimal length would render the initial singularity avoidable. Therefore, under constraints given in Eqs. (27) and (29), the rainbow cosmology filled with matter sources considered above is not devoid of the initial singularity. Let us now pursue our analysis using the rainbow function (21). This time the potential function (19) can be rewritten as

\[ V \left( a, \frac{E}{E_{Pl}} \right) = 24 \left( 1 + 3 \left( \frac{E}{E_{Pl}} \right)^n \right) \Lambda a^4 + 144 \left( 1 + 2 \left( \frac{E}{E_{Pl}} \right)^n \right) a^2 - 24 \left( 1 + \frac{3}{2} \left( \frac{E}{E_{Pl}} \right)^n \right) p_T a. \]
By the same method, we go through minimizing the potential function \( E_{\text{Pl}} \) and examine the initial singularity of the rainbow cosmology. While we know that the value of the dimensionless ratio \( \frac{E}{E_{\text{Pl}}} \) lies in the interval \( 0 \leq \frac{E}{E_{\text{Pl}}} \leq 1 \), we find that for the odd values of \( n \) minimization of the potential \( (30) \) leads to the constraint \( \frac{E}{E_{\text{Pl}}} < 0 \), which is not allowed. Also, for the even values of \( n \) under no circumstances will we encounter a SU \( (V_{\text{min}} = 0) \) in the early universe. This is while for the following condition there is a non-zero minimum value in the classically allowed region \( V < 0 \)

\[
p_T > 0 \ , \quad \Lambda = 0 \ ,
\]

so that \( V_{\text{min}} < 0 \). Akin to Figure 1 (right panel), this condition also leads to an early singular harmonic universe which is oscillating between \( a_1 \) and \( a_2 \), so that

\[
a_1 = 0 , \quad a_2 = \frac{p_T}{6} \left( \frac{1 + \frac{1}{2}(\frac{E}{E_{\text{Pl}}})^n}{1 + 2(\frac{E}{E_{\text{Pl}}})^n} \right) , \quad n = 2, 4, \ldots
\]

However, for the following constraint

\[
p_T \leq 0 \ , \quad \Lambda < 0 \ ,
\]

there is an early potential barrier for non-zero values of the scale factor \( a_2 \neq 0 \) which creates a repulsive force preventing the formation of the initial singularity, see Figure 2 (left and right panels). We see from this figure that by increasing the dimensionless ratio \( \frac{E}{E_{\text{Pl}}} \) (left panel) and also the canonically conjugate momenta to \( T \) (right panel), the chance of quantum mechanical penetration via the potential barrier from \( a_2 \neq 0 \) to \( a_1 = 0 \), becomes small. Interestingly, in Figure 2 (right panel) we see that for a fixed value of \( \frac{E}{E_{\text{Pl}}} \), an increase in \( |p_T| \) causes a repulsive force to arise from the potential barrier which appears at a greater minimum scale factor. Therefore, unlike the rainbow function \( (20) \), here the initial singularity gets eliminated through a pure repulsion mechanism and causes the formation of a potential barrier for non-zero values of the scale factor. Of course, from a classical dynamics approach, it is easily recognizable that the point \( a_2 \) is stable.

### 3.2 Exotic matter as domain wall fluid plus cosmological constant \( \Lambda \)

Our goal here is to examine the initial singularity of rainbow cosmology through analyzing the potential part of a one-dimensional minisuperspace model with a dimensionless ratio \( \frac{E}{E_{\text{Pl}}} \). This time however it is done in the context of a closed rainbow FRW universe model filled with a repulsive (exotic) source.
suggest a slight diversion of ΛCDM models towards ω > 0 from the domain walls form at the interface their between. It is interesting to note that present cosmological data highly support spontaneous symmetry breaking [63]. Technically, the phase transition happens due to losing the temperatures below the threshold associated with a certain Higgs field with non zero vacuum expectation value. Given that the universe is so big, it is a reasonable expectation that in separated regions vacuum expectation value not be the same. Therefore, these regions take arbitrarily different expectation vacuum values along the phase transition so that the domain walls form at the interface their between. It is interesting to note that present cosmological data highly suggest a slight diversion of ΛCDM models towards ω > −1.

Figure 2: Left: The potential V, Eq. (30) as a function of the scale factor a for various values of dimensionless ratio \( \frac{E}{E_{pl}} = 0.4, 0.6, 0.8, 1 \), from bottom to top respectively. To satisfy constraint Eq. (33) we have used parameters values \( P_T = -35, \Lambda = -1 \) and \( n = 2 \). Right: Various values of \( P_T = -15, -25, -35, -45 \) from bottom to top respectively. We have used parameter values \( \frac{E}{E_{pl}} = 0.8, \Lambda = -1 \) and \( n = 2 \).

as a domain wall fluid with \( \omega = -\frac{2}{3} \) plus a cosmological constant \( \Lambda \). Therefore, Eq. (18) reads

\[
V \left( \frac{a}{E_{pl}}, \frac{E}{E_{pl}} \right) = 24 \int_2^6 \left( \frac{E}{E_{pl}} \right) \Lambda a^4 - 24 \int_2^5 \left( \frac{E}{E_{pl}} \right) p_T a^3 + 144 \int_2^4 \left( \frac{E}{E_{pl}} \right) a^2 ,
\]

where by substituting the rainbow functions [20] and [21] we arrive at

\[
V \left( \frac{a}{E_{pl}}, \frac{E}{E_{pl}} \right) = 24(1 - \frac{6E}{E_{pl}}) \Lambda a^4 - 24(1 - \frac{5E}{E_{pl}}) p_T a^3 + 144(1 - \frac{4E}{E_{pl}}) a^2 ,
\]

and

\[
V \left( \frac{a}{E_{pl}}, \frac{E}{E_{pl}} \right) = 24 \left( 1 + 3 \left( \frac{E}{E_{pl}} \right) \right) \Lambda a^4 - 24 \left( 1 + \frac{3}{2} \left( \frac{E}{E_{pl}} \right) \right) p_T a^3 + 144 \left( 1 + 2 \left( \frac{E}{E_{pl}} \right) \right) a^2 .
\]

respectively. As before, by means of minimizing the potential function [35], we note that for the underlying constraints

\[
\begin{align*}
& p_T = 0, \quad \frac{1}{6} < \frac{E}{E_{pl}} \leq \frac{1}{4}, \quad \Lambda \leq 0 , \\
& p_T > 0, \quad 0 \leq \frac{E}{E_{pl}} < \frac{1}{6}, \quad \Lambda \geq \frac{25 \left( \frac{E}{E_{pl}} \right)^2 - 10 \left( \frac{E}{E_{pl}} \right) + 1}{\left( 576 \left( \frac{E}{E_{pl}} \right)^2 - 240 \left( \frac{E}{E_{pl}} \right) + 24 \right) p_T^2} .
\end{align*}
\]

there is a zero minimum value which points to the presence of an early SU fluctuating in the vicinity of a non-zero value of the scale factor \( a_0 \neq 0 \). Due to the two constraints listed above, the space-time background is just restricted to low energy levels. Now, using the parameter value \( P_T = 10 \) with

\[
\Lambda = \frac{25 \left( \frac{E}{E_{pl}} \right)^2 - 10 \left( \frac{E}{E_{pl}} \right) + 1}{\left( 576 \left( \frac{E}{E_{pl}} \right)^2 - 240 \left( \frac{E}{E_{pl}} \right) + 24 \right) p_T^2} ,
\]

We know that a perfect fluid with negative pressure results in instabilities on the very short scales. Nevertheless if we assume the dark matter is a solid, with an elastic resistance to pure shear deformations, then such short wavelength instabilities are avoidable by an EOS parameter \( \omega < 0 \) [62]. One of possible candidates for a solid dark matter component is a frustrated network of domain walls with fixed EOS parameter \( \omega = -\frac{2}{3} \). In fact, domain walls are topological defects which are expected to be formed along the phase transitions in the initial moments of universe as it expands. Historically, the idea was raised for the first time in 1974 that a domain walls structure may be occur in the framework of theories with spontaneous symmetry breaking [63]. Technically, the phase transition happens due to losing the temperatures of universe below the threshold associated with a certain Higgs field with non zero vacuum expectation value. Given that the universe is so big, it is a reasonable expectation that in separated regions vacuum expectation value not be the same. Therefore, these regions take arbitrarily different expectation vacuum values along the phase transition so that the domain walls form at the interface their between. It is interesting to note that present cosmological data highly suggest a slight diversion of ΛCDM models towards ω > −1.
To be elaborated
Figure 3: Left: the potential $V$, Eq. (35) as a function of scale factor $a$ for various values of dimensionless ratio $\frac{E}{E_{\text{Pl}}} = 0.160, 0.155, 0.150, 0.145$, from top to bottom respectively. In plotting this figure, the satisfaction of the forth constraint in (37) is taken into account. Right: The potential $V$, Eq. (36) as a function of scale factor $a$ for various values of dimensionless ratio $\frac{E}{E_{\text{Pl}}} = 0.7, 0.8, 0.9, 1$ from bottom to top respectively. To satisfy constraint (42) we have used parameter values corresponding to $p_T = 10$ and $n = 4$.

Table 3: Numerical analysis of the probability of the SU collapse using WKB-approximation for potential $V$ (36). To satisfy constraint (42), we have set parameter values at $p_T = 10$ and $n = 4$.

| $\frac{E}{E_{\text{Pl}}}$ | $\mathcal{I}$     | $P$                |
|--------------------------|---------------------|--------------------|
| 2                        | 0.6                 | $63.4409$          |
| 3                        | 0.65                | $73.204$           |
| 4                        | 0.7                 | $86.7097$          |
| 5                        | 0.75                | $105.072$          |
| 6                        | 0.8                 | $129.74$           |
| 7                        | 0.85                | $162.598$          |
| 8                        | 0.9                 | $206.146$          |
| 9                        | 0.95                | $264.009$          |
| 10                       | 1                   | $343.985$          |
for the case $n = 4$ and some values of $\frac{E}{E_{Pl}}$, we are dealing with a harmonic universe which oscillates between the minimum and maximum values of the scale factor $a_1$ and $a_2$, for which $V(a_{1,2}) = 0$

$$a_{1,2} = \frac{24 \left( 18 \left( \frac{E}{E_{Pl}} \right)^8 + 15 \left( \frac{E}{E_{Pl}} \right)^6 + 15 \left( \frac{E}{E_{Pl}} \right)^4 + 10 \left( \frac{E}{E_{Pl}} \right)^2 + 2 \mp A \right)}{(27 \left( \frac{E}{E_{Pl}} \right)^8 + 36 \left( \frac{E}{E_{Pl}} \right)^6 + 21 \left( \frac{4E}{E_{Pl}} \right)^4 + 12 \left( \frac{4E}{E_{Pl}} \right)^2 + 2)^{PT}}$$

so that

$$A = \sqrt{-162 \left( \frac{E}{E_{Pl}} \right)^{14} - 63 \left( \frac{E}{E_{Pl}} \right)^{12} + 33 \left( \frac{E}{E_{Pl}} \right)^{10} + 54 \left( \frac{E}{E_{Pl}} \right)^8 + 83 \left( \frac{E}{E_{Pl}} \right)^6 + 47 \left( \frac{E}{E_{Pl}} \right)^4 + 8 \left( \frac{E}{E_{Pl}} \right)^2}.$$ (44)

The above solution shows that no matter how the energy levels of probe particle(s) increase the interval between $a_1$ and $a_2$ becomes smaller up to where the oscillating universe turns into a SU which is fluctuating around a non-zero value of the scale factor, see Figure 3 (right panel). Theoretically, this recent case is rather dramatic for two reasons. First, both of the turning points correspond to non-zero values of the scale factor, in contrast to the initial singular harmonic universe discussed in previous cases. Secondly, constraints (42) address the existence of a positive cosmological constant which entails no violation of positive energy requirement. At first sight, one may get the illusion that under conditions (42) there is an expanding and oscillating universe in the absence of the initial singularity.

Expansion in rainbow universe model in our study is due to the existence of a two component source of view, the minimum turning point is perfectly stable, Figure 3 (right panel) explicitly shows that in the QM view there is a chance of the minimum classical turning point collapsing by tunneling the potential barrier to zero scale factor $a = 0$. To be more specific, in Table 3 the outcome of a numerical analysis by means of WKB-approximation is shown. These results can be interpreted as the probability of collapse via quantum tunneling as the rainbow universe bounces at the scale factor $a_1$. Overall, this numerical analysis reflects the fact that as the dimensionless ratio $\frac{E}{E_{Pl}}$ is getting close to unity, the chance of the minimum scale factor collapsing becomes tiny and ignorable. Concretely speaking, the closed rainbow universe model, when satisfying the constraints (12), can results in a non-singular oscillating cosmology which at a high energy phase may become quantum mechanically stable.

### 4 Quantization, wave function and initial singularity

Let us now address the issue of initial singularity from the perspective of the solution of the Schrödinger-Wheeler-DeWitt (SWD) equation for the wave function of a closed rainbow universe. to this end, we start with the classical super Hamiltonian (13) and try to quantize it. The existence of Lapse function $N$ signals the classical constraint equation $\mathcal{H} = 0$ because it explicitly refers to a Lagrange multiplier. So, the operator version of this constraint acting on the wave function $\Psi(a,T)$ for a closed, $k = +1$, rainbow modified FRW model is written as

$$\mathcal{H} \Psi(a,T) = \left( -\frac{P_a^2}{2a} - \frac{6a}{f_2^3 \left( \frac{E}{E_{Pl}} \right)} + f_2^{3\omega - 3} \left( \frac{E}{E_{Pl}} \right)^2 \frac{P_T}{a}\Lambda^3 - \frac{\Lambda a^3}{f_2^3 \left( \frac{E}{E_{Pl}} \right)} \right) \Psi(a,T) = 0.$$ (45)

Here, $\Psi(a,T)$ is the wave function of the universe. We choose the ordering $a^{-1}P_a^2 = P_a a^{-1}P_a$ to make the Hamiltonian Hermitian and use the usual operator representations ($P_a, P_T \rightarrow -i\left( \frac{\partial}{\partial a}, \frac{\partial}{\partial T} \right)$) to find

$$\left( \frac{1}{a} \frac{\partial^2 \Psi(a,T)}{\partial a^2} - \frac{1}{a^2} \frac{\partial \Psi(a,T)}{\partial a} - \frac{144a}{f_2^6 \left( \frac{E}{E_{Pl}} \right)} - \frac{24\Lambda a^3}{f_2^6 \left( \frac{E}{E_{Pl}} \right)} - \frac{24iE}{f_2^6 \left( \frac{E}{E_{Pl}} \right)} \frac{f_2^{3\omega - 3} \left( \frac{E}{E_{Pl}} \right)}{\partial T} \right) \Psi(a,T) = 0.$$ (46)

\footnote{For the case $n = 2$, constraints (12) result in having the SU around a non-zero value of the scale factor akin to shapes displayed in Figures 1 and 3 (left panels).}
Introducing the following separation of variables

\[ \Psi(a, T) = e^{-iET} \chi(a), \]  

(47)

Eq. (46) reduces to

\[ a \frac{\partial^2 \chi(a)}{\partial a^2} - \frac{\partial \chi(a)}{\partial a} - A_1 a^3 - A_2 a^5 + A_3 a^{2-3\omega} \chi(a) = 0 \]

(48)

where

\[ A_1 = 144 f_2^{-4} \left( \frac{E}{E_{Pl}} \right), \quad A_2 = 24 \Lambda f_2^{-6} \left( \frac{E}{E_{Pl}} \right), \quad A_3 = 24 E f_2^{3\omega - 3} . \]

(49)

The differential equation [48] in this form is not manageable. Given that we are interested in the high energy (or short distance) phase of the universe, we expect to drop terms of order \( a^2 \) or higher. Therefore, as can be seen, the term including \( \omega = -2/3 \) in Eq. (48) does not contribute to the rainbow function \( f_2 \). Thus, we look for the solution and analysis of the differential equation relevant to the case of \( \omega = 0 \), i.e.

\[ a \frac{\partial^2 \chi(a)}{\partial a^2} - \frac{\partial \chi(a)}{\partial a} + A_3 a^2 \chi(a) = 0 . \]

(50)

Elimination of the terms \( a^3 \) and \( a^5 \) in equation [48] means that non flat spatial curvature and also non zero cosmological constant corrections should not affect the wave function in the short distance regimes (around the Planck scale). The general solution of equation (50) can be written in terms of Bessel functions \( J_\nu \) and \( Y_\nu \).

\[ \chi(a) = a \left[ C_1 J_{2/3} \left( \sqrt{\frac{32E}{3} f_2^{\frac{3-\omega}{2}} \left( \frac{E}{E_{Pl}} \right) a^{3/2}} \right) + C_2 Y_\nu \left( \sqrt{\frac{32E}{3} f_2^{\frac{3-\omega}{2}} \left( \frac{E}{E_{Pl}} \right) a^{3/2}} \right) \right] . \]

(51)

Here, \( C_{1,2} \) can be interpreted as integration constants. However, we should set \( C_2 = 0 \) in the above solution since \( Y_\nu \) goes to infinity at the origin. Thereupon, the final form of the eigenfunction of SWD equation reads

\[ \psi(a, T) = e^{-iET} a \left[ J_{2/3} \left( \sqrt{\frac{32E}{3} f_2^{\frac{3-\omega}{2}} \left( \frac{E}{E_{Pl}} \right) a^{3/2}} \right) \right] . \]

(52)

Let us now introduce a weight function \( A(E) \) and write the wave packet solution to the SWD equation as

\[ \Psi(x, T) = \int_{E=0}^{\infty} A(E) \Psi_E(x, T) dE . \]

(53)

In order to come to an analytical expressions for the integral relation [53] we introduce the following quasi-Gaussian weight factor for the function \( A(E) \)

\[ A(E) = \left( \frac{32E}{3} \right)^{1/3} \exp \left( -\frac{32\gamma E}{3} \right) , \]

(54)

where \( \gamma \) is a positive numerical factor. Now, Eq. [53] reads

\[ \frac{16a}{3} \int_0^{\infty} \left( \frac{32E}{3} \right)^{1/3} \exp \left( -\frac{32\gamma E}{3} \right) e^{-iET} \left[ J_{2/3} \left( \sqrt{\frac{32E}{3} f_2^{\frac{3-\omega}{2}} \left( \frac{E}{E_{Pl}} \right) a^{3/2}} \right) \right] dE . \]

(55)

Setting \( \eta = \sqrt{\frac{32E}{3}} \), the above integral becomes

\[ a \int_0^{\infty} \exp \left( -\alpha \eta^2 \right) \eta^{5/2} J_{2/3} \left( \eta f_2^{\frac{3}{2}} \left( \frac{E}{E_{Pl}} \right) a^{3/2} \right) d\eta , \]

(56)
with $\alpha = (\gamma + \frac{3i}{16}T)$. The final step before solving is to noted that due to the existence of an explicit cutoff in the energy scale at which the minisuperspace is probed by test photons, the measure of the integral under consideration over $E$ is deformed as $dE \to f_E(E) dE$. Therefore, by including the rainbow functions (20) and (21), the integral (56) takes the form

$$a \int_0^\infty \exp \left(-\alpha \eta^2\right) J_{2/3} \left(\eta a^{3/2} \left(1 - \frac{9\eta^2}{64E_{Pl}^2}\right)\right) \left(1 + \frac{3\eta^2}{32E_{Pl}}\right) d\eta$$

and

$$a \int_0^\infty \exp \left(-\alpha \eta^2\right) J_{2/3} \left(\eta a^{3/2} \left(1 + \frac{3\eta^{2n}}{128(E_{Pl})^n}\right)\right) \left(1 - \frac{3\eta^{2n}}{64(E_{Pl})^n}\right) d\eta$$

respectively. Finally, we get the following expressions

$$\psi(a, T) = \mathcal{N} \left[a^2 \left(2\gamma + \frac{3i}{16}T\right)^{-5/3} + \frac{3}{32E_{Pl}} \left(2\gamma + \frac{3i}{16}T\right)^{-11/3}\right] \times \left[a^5 + \left(\frac{20\gamma}{3} + \frac{5i}{8}T\right) a^2\right] \exp \left(-\frac{a^3}{(4\gamma + \frac{3i}{8}T)}\right)$$

for the wave function. Note that we have used (see 61)

$$\int_0^\infty e^{-ax^2} x^{\nu+1} J_{\nu}(bx) dx = \frac{b^\nu}{(2a)^{\nu+1}} e^{-\frac{b^2}{4a}}$$

and

$$\int_0^\infty e^{-ax^2} x^{\nu+5} J_{\nu}(bx) dx = \frac{b^\nu}{(2a)^{\nu+5}} e^{-\frac{b^2}{4a}} (b^4 - 8(\nu + 2)ab^2 + 16(\nu + 1)(\nu + 2)a^2)$$

We have also applied the approximations $\left(1 - \frac{9\eta^2}{64E_{Pl}^2}\right) \approx 1$ and $\left(1 + \frac{3\eta^{2n}}{128(E_{Pl})^n}\right) \approx 1$ in the argument of the Bessel functions. Here, $\mathcal{N}$ is a numerical factor used for normalization purposes. Our analysis in the previous section has shown that the values of $n$ are limited to even numbers since to obtain the wave function (59) we have set $n = 2$. At first glance, one notices that these wave functions (59) and (60) go to zero as one approaches the origin (i.e. $a \to 0$). This means that non zero bouncing rainbow universe addressed in the previous section is free of quantum collapse, recalling that the boundary condition $\psi(a \to 0, T \to 0) = 0$ was first suggested in 65. Figure 4 shows the behavior of probability density $|\Psi(a, T)|^2$ using (59). We see that the wave packet is peaked at the non-zero values of $a$ at $T = 0$ which supports the idea of non zero bouncing universe. We also find that increasing the value of $\gamma$, the peak of $|\Psi(a, T)|^2$ emerges in higher non zero values of $a$ at $T = 0$. Needless to say that generally all the results for the wave packet of the rainbow universe (60) are retained here as well. As final word in the section, we recommended reference 65 in which displayed one of other benefits of the WDW equation within gravity’s rainbow proposal. Briefly, in the mentioned reference using a WDW equation (within a FRW and spherically symmetric background, respectively) authors can show the equivalence of gravity’s rainbow with Hořava-Lifshitz model of gravity as theories which incorporate a correction in the high energy regime.
Figure 4: The probability density function $|\Psi(a,T)|^2$ derived from the wave packet (59). To plot this figure we have set the numerical values $\gamma = 1 = E_{Pl}$ and $N = 8$. Here, the peak of $|\Psi(a,T)|^2$ emerges around $a = 1$ at $T = 0$.

5 Conclusions

Following the previous studies, in this work we use “rainbow metric” (4) as one of many possible hypotheses to provide an effective description of some QG effects in the early stages of the formation of the universe. Our principal goal in this paper was dedicated to the investigation of the initial singularity issue at the quantum level in a closed rainbow cosmology with a homogeneous isotropic space-time background. We used a one-dimensional minisuperspace model including the dimensionless ratio $\frac{E}{E_{Pl}}$, for an ideal fluid with a cosmological constant and constructed the Hamiltonian, in the framework of the Schutz’s formalism. It should be stressed that the dimensionless ratio $\frac{E}{E_{Pl}}$ directly arises from the rainbow gravity proposal. For the matter sources we considered the following cases:

1. Non-relativistic dust matter with $\omega = 0$ plus the cosmological constant $\Lambda$,

2. Exotic matter as domain wall fluid with $\omega = -\frac{2}{3}$ plus the cosmological constant $\Lambda$.

For each of these two cases, our closed FRW universe model was modified by rainbow functions (20) and (21). By analyzing the potential sector of the relevant Hamiltonian, we demonstrated that for both options above, when certain conditions are satisfied, we get either a potential barrier or a static universe for the non-zero values of the scale factor. These two situations provide the possibility of removing the initial singularity. However, the main point is that if the non-zero values of the scale factor become unstable, any discussion of the initial singularity becomes meaningless and misleading. As we saw, from a classical dynamical system viewpoint, they can be stable. This does not rule out the non-zero probability from a QM perspective of a return to a zero value scale factor via tunneling.

By considering the shape of the potential as well as using an approximation procedure such as WKB, we showed that the higher the energy levels of probe particle(s) (i.e. $\frac{E}{E_{Pl}}$), the higher the barrier height. That is, the higher the probing energy, the chance of tunneling through the barrier to a zero value of the scale factor diminishing to zero. Therefore, despite the probability of minimal scale factor crumbling in both closed rainbow universe models, for the energy-dependent space-time geometry with energies around the Planck energy, there is the possibility of quantum stability.

We also noted that the rainbow function (21) modification to FRW universe with a domain wall fluid plus cosmological constant provided the constraints (12), which in turn resulted in a non-singular harmonic universe. The behavior of this harmonic universe is interesting in the sense that it is oscillating between minimum and maximum values of the scale factor and is sensitive to the dimensionless ratio $\frac{E}{E_{Pl}}$ (Eq. (13)). Figure 3 (right panel) displays this behavior in that by increasing the value of $\frac{E}{E_{Pl}}$, the oscillating interval becomes smaller and smaller in such a way that at the high energy phase...
the harmonic universe reduces to a SU. We also demonstrated that such a harmonic universe could be stable quantum mechanically at high energies.

Finally, we quantized the classical Hamiltonian (15) within the rainbow framework and used separation of variable method to obtain analytical solutions relevant to the SWD equation. Introduction of a suitable superposition of the eigenfunctions was then used to derive the wave packet of the universe modified by two rainbow functions $f_2$, (20) and $f_1$, (21), respectively. We showed that both wave functions (59) and (60) satisfy the boundary condition $\psi(a \to 0, T \to 0)$. This would mean that the non zero bouncing rainbow universe will not lead to a quantum collapse. The peak of the probability density function $|\Psi(a, T)|^2$, shown in Figure 4, coincides with the non zero value of $a$ at $T = 0$ which supports a bouncing non-singular universe.

References

[1] G. Lemaitre, Gen. Rel. Grav. 29 (1997) 641
[2] R. Penrose. Phys. Rev. Lett. 14(3) (1965) 57
[3] S. W. Hawking, P. Roy. Soc. A-Math. Phy. 294(1439) (1966) 511
[4] S. W. Hawking. P. Roy. Soc. A-Math. Phy. 295(1443) (1966) 490
[5] S. W. Hawking. P. Roy. Soc. A-Math. Phy. 300(1461) (1967) 187
[6] S. W. Hawking and G. F. R. Ellis, “The Large Scale Structure of Space-Time”, Cambridge University Press, (1973)
[7] J. Earman, “Bangs, Crunches, Whimpers and Shrieks: Singularities and A causalities in Relativistic Spacetimes”, Oxford University Press, USA (1995)
[8] D. Battefeld and P. Peter, Phys. Rept. 571 (2015) 1
[9] L. J. Garay, M. Martin-Benito and E. Martin-Martinez, Phys. Rev. D 89 (2014) 043510
[10] N. Pinto-Neto and J. C. Fabris, Class. Quant. Grav. 30 (2013) 143001
[11] A. Ashtekar and P. Singh, Class. Quant. Grav. 28 (2011) 213001
[12] R. Brandenberger, Phys. Rev. D 80 (2009) 043516
[13] G. Calcagni, JHEP 0909 (2009) 1112
[14] R. Gambini, and J. Pullin, Phys. Rev. D 59 (1999) 124021
[15] A. Ashtekar, J. Lewandowski, Class. Quant. Grav. 21, R53-R152 (2004)
[16] V. A. Kostelecky, and S. Samuel, Phys. Rev. D 39 (1989) 683
[17] M. Bojowald. Phys. Rev. Lett. 86 (2001) 5227
[18] A. Ashtekar, T. Pawlowski, P. Singh, Phys. Rev. Lett. 96 (2006) 141301
[19] A. Ashtekar, T. Pawlowski, P. Singh, Phys. Rev. D 74 (2006) 084003
[20] B. Craps, T. Hertog, and N. Turok. Phys. Rev. D 80 (2009) 086007
[21] A. Kempf, G. Mangano and R. B. Mann, Phys. Rev. D 52 (1995) 1108
[22] R. J. Adler, P. Chen and D. I. Santiago, Gen. Rel. Grav. 33 (2001) 2101
[23] A. J. M. Medved and E. C. Vagenas, Phys. Rev. D 70 (2004) 124021
[24] Y. Ling, B. Hu and X. Li, Phys. Rev. D 73 (2006) 087702
[25] B. Vakili, Phys. Rev. D 77 (2008) 044023
[26] A. F. Ali, S. Das and E. C. Vagenas, Phys. Lett. B 678 (2009) 497
[27] S. Das, E. C. Vagenas and A. F. Ali, Phys. Lett. B 690 (2010) 407
[28] K. Nozari and S. Saghafi, JHEP 1211 (2012) 005
[29] K. Nozari and A. Etemadi, Phys. Rev. D 85 (2012) 104029
[30] S. Jalalzadeh, M. A. Gorji and K. Nozari, Gen. Rel. Grav. 46 (2014) 1632
[31] P. Pedram, K. Nozari and S. H. Taheri, JHEP 03 (2011) 093
[32] K. Nozari, M. Khodadi, M. A. Gorji, Europhys. Lett. 112 (2015) 60003
[33] K. Nozari, M. A. Gorji, V. Hosseinzadeh and B. Vakili, Class. Quantum Grav. 33 (2016) 025009
[34] G. Amelino-Camelia, Phys. Lett. B 510 (2001) 255
[35] G. Amelino-Camelia, Int. J. Mod. Phys. D 11 (2002) 35
[36] G. Amelino-Camelia, J. Kowalski-Glikman, G. Mandanici and A. Procaccini, Int. J. Mod. Phys. A 20 (2005) 6007
[37] J. Magueijo and L. Smolin, Phys. Rev. D 67 (2003) 044017
[38] J. Magueijo and L. Smolin, Phys. Rev. Lett. 88 (2002) 190403
[39] J. Magueijo and L. Smolin, Class. Quant. Grav. 21 (2004) 1725
[40] A. Awad, A. F. Ali, B. Majumder, JCAP 10 (2013) 052
[41] B. Majumder, Int. J. Mod. Phys. D 22 (2013) 1350079
[42] A. T. Mithani, A. Vilenkin, JCAP 1201 (2012) 028
[43] A. Borde, A. H. Guth and A. Vilenkin, Phys. Rev. Lett. 90 (2003) 151301
[44] S. del Campo, E. Guendelman, A. B. Kaganovich, R. Herrera and P. Labrana, Phys. Lett. B 699 (2011) 211
[45] P. Wu and H. Yu, Phys. Rev. D 81 (2010) 103522
[46] D. J. Mulryne, R. Tavakol, J. E. Lidsey and G. F. R. Ellis, Phys. Rev. D 71 (2005) 123512
[47] G. F. R. Ellis and R. Maartens, Class. Quant. Grav. 21 (2004) 223
[48] G. F. R. Ellis, J. Murugan and C. G. Tsagas, Class. Quant. Grav. 21 (2004) 233
[49] C. L. Bennet, et al., Astrophys. J. Suppl. 148 (2003) 1
[50] D. N. Spergel, et al., Astrophys. J. Suppl. 148 (2003) 175
[51] M. Khodadi, Y. Heydarzade, K. Nozari, F. Darabi, Eur. Phys. J. C 75 (2015) 590
[52] B. F. Schutz, Phys. Rev. D 2 (1970) 2762
[53] B. F. Schutz, Phys. Rev. D 4 (1971) 3559
[54] V. G. Lapchinskii and V. A. Rubakov, Theor. Math. Phys. 33 (1977) 1076
[55] B. Vakili, Phys. Lett. B 688 (2010) 129
[56] M. Khodadi, K. Nozari, B. Vakili, Gen. Rel. Grav. 48 (2016) 64
[57] Yi Ling and Qingzhang Wu, Phys. Lett. B 687 (2010) 103
[58] G. Amelino-Camelia, J. R. Eliss, N. Mavromatos and D. V. Nanopoulos, Int. J. Mod. Phy. A. 12 (1997) 607
[59] G. Amelino-Camelia, Living Rev. Rel. 16 (2013) 5
[60] M. P. Dabrowski and A. L. Larsen, Phys. Rev. D 52 (1995) 3424
[61] D. Atkatz, Am. J. Phys. 62 (1994) 619
[62] M. Bucher and D. N. Spergal, Phys. Rev. D 60 (1999) 043505
[63] Y. B. Zeldovich, I. Y. Kobzarev, and L. B. Okun, Zh. Eksp. Teor. Fiz. 67 (1974) 3 (also in Sov. Phys. JETP, 40 (1974) 1)
[64] M. Abramowitz and I. A. Stegun, “Handbook of Mathematical Functions”, New York: Dover (1972)
[65] B. S. Dewitt, Phys. Rev. 160 (1967) 1113
[66] R. Garattini and E. N. Saridakisc, Eur. Phys. J. C 75 (2015) 343