Morse–Sard theorem and Luzin $N$-property: a new synthesis result for Sobolev spaces

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Abstract

For a regular (in a sense) mapping $v : \mathbb{R}^n \to \mathbb{R}^d$ we study the following problem: let $S$ be a subset of $m$-critical set $\tilde{Z}_{v,m} = \{ \text{rank} \nabla v \leq m \}$ and the equality $\mathcal{H}^\tau(S) = 0$ (or the inequality $\mathcal{H}^\tau(S) < \infty$) holds for some $\tau > 0$. Does it imply that $\mathcal{H}^\sigma(v(S)) = 0$ for some $\sigma = \sigma(\tau, m)$? (Here $\mathcal{H}^\tau$ means the $\tau$-dimensional Hausdorff measure.)

For the classical classes $C^k$-smooth and $C^{k+\alpha}$-Holder mappings this problem was solved in the papers by Bates and Moreira. We solve the problem for Sobolev $W^k_p$ and fractional Sobolev $W^{k+\alpha}_p$ classes as well. Note that we study the Sobolev case under minimal integrability assumptions $p = \max(1, n/k)$, i.e., it guarantees in general only the continuity (not everywhere differentiability) of a mapping.

In particular, there is an interesting and unexpected analytical phenomena here: if $\tau = n$ (i.e., in the case of Morse–Sard theorem), then the value $\sigma(\tau)$ is the same for the Sobolev $W^k_p$ and for the classical $C^k$-smooth case. But if $\tau < n$, then the value $\sigma$ depends on $p$ also; the value $\sigma$ for $C^k$ case could be obtained as the limit when $p \to \infty$. The similar phenomena holds for Holder continuous $C^{k+\alpha}$ and for the fractional Sobolev $W^{k+\alpha}_p$ classes.

The proofs of the most results are based on our previous joint papers with J. Bourgain and J. Kristensen (2013, 2015). We also crucially use very deep Y. Yomdin’s entropy estimates of near critical values for polynomials (based on algebraic geometry tools).

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1 Introduction

The Morse–Sard theorem in its classical form states that the image of the set of critical points of a $C^{n-d+1}$ smooth mapping $v : \mathbb{R}^n \to \mathbb{R}^d$ has zero Lebesgue measure in $\mathbb{R}^d$. 

More precisely, assuming that \( n \geq d \), the set of critical points for \( v \) is \( Z_v = \{ x \in \mathbb{R}^n : \text{rank} \nabla v(x) < d \} \) and the conclusion is that
\[
\mathcal{L}^d(v(Z_v)) = 0
\] (1.1)
whenever \( v \in C^k \) with \( k \geq \max(1, d - m + 1) \). The theorem was proved by Morse [47] in 1939 for the case \( d = 1 \) and subsequently by Sard [52] in 1942 for the general vector-valued case. The celebrated results of Whitney [57] show that the \( C^{n-d+1} \) smoothness assumption on the mapping \( v \) is sharp.

Another important item of the real analysis, \( N \)-property, means that the image \( v(E) \) has zero measure whenever \( E \) has zero measure (see the recent paper [26], where we discuss the history of the topic).

We need some usual notation. Fix a pair of positive parameters \( \tau \) and \( \sigma \). A continuous mapping \( v : \mathbb{R}^n \to \mathbb{R}^d \) is said to satisfy \((\tau, \sigma)-N\)-property, if
\[
\mathcal{H}^\sigma(v(E)) = 0 \text{ whenever } \mathcal{H}^\tau(E) = 0,
\]
where \( \mathcal{H}^\tau \) means the Hausdorff measure.

For a \( C^1 \)-smooth mapping \( v : \mathbb{R}^n \to \mathbb{R}^d \) and for an integer number \( m \in \mathbb{Z}_+ \) denote
\[
\tilde{Z}_{v,m} = \{ x \in \mathbb{R}^n : \text{rank} \nabla v(x) \leq m \}.
\]
Then for parameters \( \tau, \sigma > 0 \) we say that that a mapping \( v : \mathbb{R}^n \to \mathbb{R}^d \) satisfies \((\tau, \sigma, m)-N\)-property, if
\[
\mathcal{H}^\sigma(v(E)) = 0 \text{ whenever } E \subset \tilde{Z}_{v,m} \text{ with } \mathcal{H}^\tau(E) = 0.
\]

Further, we say that that a mapping \( v : \mathbb{R}^n \to \mathbb{R}^d \) satisfies strict \((\tau, \sigma, m)-N\)-property, if
\[
\mathcal{H}^\sigma(v(E)) = 0 \text{ whenever } E \subset \tilde{Z}_{v,m} \text{ with } \mathcal{H}^\tau(E) < \infty.
\]

Using this notation, the above classical Morse–Sard theorem means, that every \( C^k \)-mapping \( v : \mathbb{R}^n \to \mathbb{R}^d \) has strict \((n, d, d-1)-N\)-property if \( k \geq n - d + 1 \).

The starting point for our research is the following recent result for classically smooth case.

**Theorem 1.1** (Bates S.M. and Moreira C., 2002 [10, 46]). Let \( m \in \{0, \ldots, n-1\} \), \( k \geq 1 \), \( d \geq m \), \( 0 \leq \alpha \leq 1 \), and \( v \in C^{k,\alpha}(\mathbb{R}^n, \mathbb{R}^d) \). Then for any \( \tau \in [m, n] \) the mapping \( v \) has \((\tau, \sigma, m)-N\)-property with
\[
\sigma = m + \frac{\tau - m}{k + \alpha}.
\]

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1We use the symbol \( \tilde{Z} \), since in our previous papers we denoted \( Z_{v,m} = \{ x \in \mathbb{R}^n : \text{rank} \nabla v(x) < m \} \). So in the present notation \( \tilde{Z}_{v,m} = Z_{v,m+1} \).
Moreover, this \(N\)-property is strict if at least one of the following additional assumptions is fulfilled:

1) \(\tau = n\) (in particular, it includes the case of the classical Morse–Sard theorem);
2) \(\tau > m\) and \(\alpha = 0\) (that means \(v \in C^k\));
3) \(\tau > m\) and \(v \in C^{k,\alpha+}(\mathbb{R}^n, \mathbb{R}^d)\).

Here we say that a mapping \(v : \mathbb{R}^n \to \mathbb{R}^d\) belongs to the class \(C^{k,\alpha}\) for some positive integer \(k\) and \(0 < \alpha \leq 1\) if \(v \in C^k\) and there exists a constant \(L \geq 0\) such that

\[
|\nabla^k v(x) - \nabla^k v(y)| \leq L|x - y|\alpha \quad \text{for all } x, y \in \mathbb{R}^n.
\]

To simplify the notation, let us make the following agreement: for \(\alpha = 0\) we identify \(C^{k,\alpha}\) with usual spaces of \(C^k\)-smooth mappings.

Analogously, we say that a mapping \(v : \mathbb{R}^n \to \mathbb{R}^d\) belongs to the class \(C^{k,\alpha+}\) for some positive integer \(k\) and \(0 < \alpha \leq 1\), if there exists a function \(\omega : \mathbb{R}_+ \to \mathbb{R}_+\) such that \(\omega(r) \to 0\) as \(r \to 0\) and

\[
|\nabla^k v(x) - \nabla^k v(y)| \leq \omega(r) \cdot |x - y|\alpha \quad \text{whenever } |x - y| < r.
\] (1.3)

Note that the assertion of Theorem 1.1 is rather sharp: for example, if its conditions 1)–3) are not satisfied, then the corresponding \((\tau, \sigma, m)\)-\(N\)-property is not strict in general, it follows from Whitney’s counterexamples [57], see also [48] for commentaries.

Of course, the assertion of Theorem 1.1 includes Morse–Sard theorem and many other results on this topic as partial cases; for convenience, we made some historical references below in Subsection 1.2. The purpose of our paper is to extend this result to the mappings of Sobolev spaces.

### 1.1 Morse–Sard–Luzin type theorem for the case of Sobolev spaces

In this subsection \(W^k_p(\mathbb{R}^n, \mathbb{R}^d)\) means the space of Sobolev mappings with all derivatives of order \(j \leq k\) belonging to the Lebesgue space \(L_p\).

Let \(k \in \mathbb{N}, 1 < p < \infty\) and \(0 \leq \alpha < 1\). One of the most natural type of fractional Sobolev spaces is (Bessel) potential spaces \(L^{k+\alpha}_p\).

Recall, that a function \(v : \mathbb{R}^n \to \mathbb{R}^d\) belongs to the space \(L^{k+\alpha}_p\), if it is a convolution of a function \(g \in L_p(\mathbb{R}^n)\) with the Bessel kernel \(G_{k+\alpha}\), where \(\hat{G}_{k+\alpha}(\xi) = (1 + 4\pi^2\xi^2)^{-(k+\alpha)/2}\). It is well known that for the integer exponents (i.e., when \(\alpha = 0\)) one has the identity

\[
L^k_p(\mathbb{R}^n) = W^k_p(\mathbb{R}^n) \quad \text{if} \quad 1 < p < \infty.
\]

As well-known, if \((k+\alpha)p > n\), then functions from the potential space \(L^{k+\alpha}_p(\mathbb{R}^n)\) are continuous by Sobolev Imbedding theorem, but in general the gradient \(\nabla v\) is not well-defined everywhere. Thus now for the Sobolev case the \(m\)-critical set is defined as

\[
\tilde{Z}_{v,m} = \{ x \in \mathbb{R}^n : x \in A_v \text{ or } x \in \mathbb{R}^n \setminus A_v \text{ with rank } \nabla v(x) \leq m \}.
\]
Here $A_v$ means the set of ‘bad’ points at which either the function $v$ is not differentiable or which are not the Lebesgue points for $\nabla v$. So in the paper\(^2\) we consider these ‘bad’ nonregular points automatically as $m$-critical for any $m$ (such assumption, of course, makes the corresponding $(\tau, \sigma, m)$-$N$-properties more stronger).

**Theorem 1.2.** Let $m \in \{0, \ldots, n - 1\}$, $k \geq 1$, $d \geq m$, $0 \leq \alpha < 1$, $p > 1$, $(k + \alpha)p > n$, and let $v \in \mathcal{L}_p^{k+\alpha}(\mathbb{R}^n, \mathbb{R}^d)$. Denote $\tau_* = n - (k + \alpha - 1)p$. Suppose in addition that
\[
\tau > m \quad \text{and} \quad \tau > \tau_*,
\]
then the mapping $v$ has strict $(\tau, \sigma, m)$-$N$-property with
\[
\sigma = m + \frac{p(\tau - m)}{\tau + (k + \alpha)p - n}.
\]

Further, if $\tau = m > \tau_*$, then $v$ has nonstrict $(\tau, m, m)$-$N$-property.

We need to make several remarks here.

- First of all, let us note, that the value $\sigma$ in Theorems 1.1–1.2 coincide for the boundary cases $\tau = m$ or $\tau = n$, but they are different for $m < \tau < n$ (of course, then $\sigma$ for Sobolev case is larger). Nevertheless, $\sigma$ in Theorem 1.1 could be obtained by taking a limit in (1.4) as $p \to \infty$;

- Recall, that by approximation results (see, e.g., [54] and [36]) the set of ‘bad’ points $A_v$ is rather small, i.e., it has the Hausdorff dimension $\tau_*:
\[
\mathcal{H}^\tau(A_v) = 0 \quad \forall \tau > \tau_* := n - (k + \alpha - 1)p \quad \text{if} \quad v \in \mathcal{L}_p^{k+\alpha}(\mathbb{R}^n).
\]
In particular, $A_v = \emptyset$ if $(k + \alpha - 1)p > n$.

- The condition $\tau > \tau_*$ in Theorem 1.2 is essential and sharp: namely, in the paper [26] we constructed a counterexample of a mapping from $\mathcal{L}_p^{k+\alpha}(\mathbb{R}^n)$ not satisfying the $(\tau, \sigma, m)$-$N$-property with $\tau = \tau_* = m = \sigma = 1$.

- The usual $(\tau, \sigma)$-$N$-properties (without constraints on the gradient, i.e., when $m = n$) were studied in our previous paper [26], see also subsection 1.2, Theorems 1.4–1.5. (One has to use these usual $N$-properties also if the assumptions $\tau > m$ and $\tau > \tau_*$ of Theorem 1.2 are not satisfied.)

Thus above Theorem 1.2 omits the limiting cases $(k + \alpha)p = n$ and $\tau = \tau_*$. However, it is possible to cover these cases as well using the Lorentz norms. Namely, denote by $\mathcal{L}_{p,1}^{k+\alpha}(\mathbb{R}^n, \mathbb{R}^d)$ the space of functions which could be represented as a convolution of the Bessel potential $G_{k+\alpha}$ with a function $g$ from the Lorentz space $L_{p,1}$ (see the definition of these spaces in the section 2).

\(^2\)In our previous papers we consider the $m$-critical points and ‘bad’ points $A_v$ separately.
Theorem 1.3. Let $m \in \{0, \ldots, n-1\}$, $k \geq 1$, $d \geq m$, $0 \leq \alpha < 1$, $p \geq 1$ and let $v : \mathbb{R}^n \to \mathbb{R}^d$ be a mapping for which one of the following cases holds:

(i) $\alpha = 0$, $k \geq n$, and $v \in W^k_1(\mathbb{R}^n, \mathbb{R}^d)$;

(ii) $0 \leq \alpha < 1$, $p > 1$, $(k + \alpha)p \geq n$, and $v \in \mathcal{L}^{k+\alpha}_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$.

Denote $\tau_* = n - (k + \alpha - 1)p$. Suppose in addition that

$\tau > m \quad \text{and} \quad \tau \geq \tau_*$,

then the mapping $v$ has strict $(\tau, \sigma, m)$-$N$-property with the same $\sigma$ defined by (1.4). Further, if $\tau = m \geq \tau_*$, then $v$ has the corresponding nonstrict $(\tau, m, m)$-$N$-property.

So here the limiting case $\tau = \tau_*$ is included. Some other commentaries:

- Recall, that by approximation results (see, e.g., [54] and [36]) the set of ‘bad’ points $A_v$ for this Sobolev–Lorentz case has the same Hausdorff dimension $\tau_* = n - (k + \alpha - 1)p$, but it is smaller in a sense, namely:

\[ \mathcal{H}^{\tau_*}(A_v) = 0 \quad \text{if} \quad v \text{ is from Theorem 1.3}. \]  

(compare with (1.5)). In particular, $A_v = \emptyset$ if $(k + \alpha - 1)p \geq n$.

- For the integer exponents (i.e., when $\alpha = 0$) the Sobolev–Lorentz potential space has a more simple and natural description:

\[ \mathcal{L}^k_{p,1}(\mathbb{R}^n) = W^k_{p,1}(\mathbb{R}^n) \quad \text{if} \quad 1 < p < \infty, \]

there by $W^k_{p,1}$ we denote the subspace of $W^k_p$ consisting of functions whose derivatives of order $k$ belongs to the Lorentz space $L^p_{p,1}$ (see, e.g., [26]).

1.2 Some historical remarks

There are a lot of papers devoted to the Morse–Sard theorem, and the above formulated results includes many previous theorems as partial cases. For example, for smooth case if $\alpha = 0$, $\tau = n$, then we have

\[ \sigma = m + \frac{n - m}{k}, \]

and the assertion of Theorem 1.1 coincides with the classical Federer–Dubovitskiǐ theorem, obtained almost simultaneously by Dubovitskiǐ [22] in 1967 and Federer [25, Theorem 3.4.3] in 1969. Of course, it includes the original Morse–Sard theorem as partial case (when $k = n - m, \sigma = m + 1$).

Note also, that Theorem 1.1 was formulated as a Conjecture by A.Norton in [48, page 369] and it includes as partial cases some relative results of other mathematicians: Norton himself (who proved the assertion for the case $\sigma = d$, $\tau = (k + \alpha)(d - m) + m$),
Y. Yomdin [58] (case $\tau = n$, $v \in C^{k,\alpha+}$, see also [13]), M. Kucera [38] (case $\tau = n$, $m = 1$, i.e., when the gradient totally vanishes on the critical set), etc.

Concerning the Sobolev case, in the pioneering paper by De Pascale [18] the assertion of the initial Morse–Sard theorem (1.1) (i.e., when $k = n - d + 1$, $m = d - 1$, $\sigma = d$) was obtained for the Sobolev classes $W^k_p(\mathbb{R}^n, \mathbb{R}^m)$ under additional assumption $p > n$ (in this case the classical embedding $W^k_p(\mathbb{R}^n, \mathbb{R}^m) \hookrightarrow C^{k-1}$ holds, so there are no problems with nondifferentiability points).

Some other Morse–Sard type theorems for Sobolev cases were obtained in [13] and [29], these papers mainly concern the Dubovitskii–Fubini type properties for the Morse–Sard theorem, which will be discussed in the next subsection.

In addition to the above mentioned papers there is a growing number of papers on the topic, including [6, 7, 8, 9, 17, 28, 49, 55, 56].

Finally, Theorems 1.2 and 1.3 for the most important case $\tau = n$ were obtained in our previous paper [27] (see also our preceding articles [15, 16, 30, 35, 36] of the first author with J. Bourgain, J. Kristensen, and P. Hajlasz on this topic).

The usual $(\tau, \sigma)$-$N$-properties (without constraints on the gradient, i.e., when $m = n$) were studied in our previous paper [26], where we proved the following two theorems:

**Theorem 1.4** ([26]). Let $\alpha > 0$, $1 < p < \infty$, $\alpha p > n$, and $v \in \mathcal{L}^\alpha_p(\mathbb{R}^n, \mathbb{R}^d)$. Suppose that $0 < \tau \leq n$. Then the following assertions hold:

1. If $\tau \neq \tau_* = n - (\alpha - 1)p$, then $v$ has the $(\tau, \sigma)$-$N$-property, where the value $\sigma = \sigma(\tau)$ is defined as

$$\sigma(\tau) := \begin{cases} \tau, & \text{if } \tau \geq \tau_* := n - (\alpha - 1)p; \\ \frac{p}{\alpha p - n + \tau}, & \text{if } 0 < \tau < \tau*. \end{cases} \tag{1.7}$$

2. If $\alpha > 1$ and $\tau = \tau_* > 0$ then $\sigma(\tau) = \tau_*$ and the mapping $v$ in general has no $(\tau_*, \tau_*)$-$N$-property, i.e., it could be $\mathcal{H}^{\tau_*}(v(E)) > 0$ for some $E \subset \mathbb{R}^n$ with $\mathcal{H}^{\tau_*}(E) = 0$.

The similar results were announced in [5], see [26] for our commentaries and other historical remarks on this important case.

The above Theorem 1.4 omits the limiting cases $\alpha p = n$ and $\tau = \tau_*$. As above, it is possible to cover these cases as well using the Lorentz norms.

**Theorem 1.5** ([30, 26]). Let $v : \mathbb{R}^n \to \mathbb{R}^d$ be a mapping for which one of the following cases holds:

1. $v \in W^k_1(\mathbb{R}^n, \mathbb{R}^d)$ for some $k \in \mathbb{N}$, $k \geq n$;
2. $v \in \mathcal{L}^\alpha_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$ for some $\alpha > 0$, $p \in (1, \infty)$ with $\alpha p \geq n$.

Suppose that $0 < \tau \leq n$. Then $v$ is a continuous function satisfying the $(\tau, \sigma)$-$N$-property, where again the value $\sigma = \sigma(\tau)$ is defined in (1.7) (with $\alpha = k$ and $p = 1$ for the (i) case).

So, in the last theorem the critical case $\tau = \tau_*$ is included.
1.3 The Dubovitskiĭ–Fubini type properties for the Morse–Sard theorem

As it was mentioned by A.Norton [48, page 369], the absence of a Fubini theorem for Hausdorff measure makes an obstacle for proofs of some new Morse–Sard type theorems. Nevertheless, in 1957 Dubovitskiĭ proved, that surprisingly some Fubini type properties always hold for the Morse–Sard topic.

Theorem A (Dubovitskiĭ 1957 [21]). Let \( n, d, k \in \mathbb{N} \), and let \( v : \mathbb{R}^n \rightarrow \mathbb{R}^d \) be a \( C^k \)-smooth mapping. Then

\[
\mathcal{H}^\mu (Z_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{L}^d \text{-a.a. } y \in \mathbb{R}^d,
\]

where \( \mu = n - d + 1 - k \) and \( Z_v = \{ x \in \mathbb{R}^n : \text{rank } \nabla v(x) < d \} \).

Here and in the following we interpret \( \mathcal{H}^\beta \) as the counting measure when \( \beta \leq 0 \). Thus for \( k \geq n - d + 1 \) we have \( \nu \leq 0 \), and \( \mathcal{H}^\mu \) in (1.8) becomes simply the counting measure, so the Dubovitskiĭ theorem contains the Morse–Sard theorem as particular case.

It turns out that the similar Fubini type extensions hold for the Theorems 1.1–1.3 stated above.

Remark 1.1. The following language below may seem too technical and cumbersome. So, a disinterested reader can omit them; anyway the main results of the article are the above theorems 1.2–1.3. Nevertheless, authors consider the following theorems as important strengthens of theorems 1.1–1.3, as they allow to realise the idea of Dubovitsky’s approach in general situation, and include all the theorems given in this article as a particular case; moreover, they are new even for the classical smooth cases \( C^k \) and \( C^{k,\alpha} \).

We need some notation. For parameters \( \mu \geq 0, q \geq m, \tau > 0 \) we say that that a mapping \( v : \mathbb{R}^n \rightarrow \mathbb{R}^d \) satisfies \((\tau, \mu, q, m)\)-\( N \)-property, if

\[
\mathcal{H}^\mu (E \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q \text{-almost all } y \in v(E) \quad \text{whenever } E \subset \tilde{Z}_{v,m} \text{ with } \mathcal{H}^\tau (E) = 0.
\]

Recall, that here as above \( \tilde{Z}_{v,m} = \{ x \in \mathbb{R}^n : \text{rank } \nabla v(x) \leq m \} \). Obviously,

\[
\text{if } \mu \leq 0, \text{ then the } (\tau, \mu, q, m)\)-\( N \)-property is equivalent to the \( (\tau, q, m)\)-\( N \)-property.
\]

Further, we say that that a mapping \( v : \mathbb{R}^n \rightarrow \mathbb{R}^d \) satisfies strict \((\tau, \mu, q, m)\)-\( N \)-property, if

\[
\mathcal{H}^\mu (E \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q \text{-almost all } y \in v(E) \quad \text{whenever } E \subset \tilde{Z}_{v,m} \text{ with } \mathcal{H}^\tau (E) < \infty.
\]

Theorem 1.6 (Smooth case \( v \in C^{k,\alpha}(\mathbb{R}^n, \mathbb{R}^d) \)). Under assumptions of Theorem 1.1 one can replace the assertion about \((\tau, \sigma, m)\)-\( N \)-properties by the more strong assertion about \((\tau, q, \mu, m)\)-\( N \)-property for any \( \tau \in [m, n] \) and \( q \in [m, \sigma] \) with

\[
\mu = \tau - m - (k + \alpha)(q - m).
\]
Further, if \( q > m \) and at least one of the corresponding conditions 1)–3) of Theorem 1.1 is fulfilled, then this \((\tau, q, \mu, m)\)-N-property is strict.

The similar assertions hold for Sobolev and Sobolev–Lorentz cases (we use the definition from subsection 1.1 for the \( m \)-critical set \( \tilde{Z}_{v,m} \) of Sobolev functions).

**Theorem 1.7** (Sobolev case \( v \in \mathcal{L}^{k,\alpha}_p(\mathbb{R}^n, \mathbb{R}^d), (k + \alpha)p > n \)). Under assumptions of Theorem 1.2 one can replace the assertion about strict \((\tau, \sigma, m)\)-N-properties by the more strong assertion about strict \((\tau, q, \mu, m)\)-N-property for any \( \tau > \max(\tau_*, m) \), \( q \in (m, \sigma] \) with

\[
\mu = \tau - m - \left( k + \alpha - \frac{n}{p} + \frac{\tau}{p} \right)(q - m).
\]  

(1.12)

Further, if \( q = m \), \( \tau > \tau_* \), and \( \tau \geq m \), then \( v \) has nonstrict \((\tau, m, \mu, m)\)-N-property with \( \mu = \tau - m \).

**Theorem 1.8** (Sobolev–Lorentz case \( v \in \mathcal{L}^{k,\alpha}_{p,1}(\mathbb{R}^n, \mathbb{R}^d), kp \geq n \)). Under assumptions of Theorem 1.3 one can replace the assertion about strict \((\tau, \sigma, m)\)-N-properties by the more strong assertion about strict \((\tau, q, \mu, m)\)-N-property for any \( \tau \geq \tau_* \), \( \tau > m \), \( q \in (m, \sigma] \), and with the same \( \mu \) as in (1.12). Further, if \( q = m \) and \( \tau \geq \max(m, \tau_*) \), then \( v \) has nonstrict \((\tau, m, \mu, m)\)-N-property with \( \mu = \tau - m \).

It is easy to see, that in formulation of Theorems 1.6–1.8 if we take \( q = \sigma \), then \( \mu = 0 \), where \( \sigma \) is defined in formulation of the corresponding Theorems 1.1–1.3. It means (see (1.10)), that Theorems 1.6–1.8 include the previous Theorems 1.1–1.3 as particular case.

**Remark 1.2** (Some historical remarks). It is interesting to note that this Dubovitskiĭ Theorem A remained almost unnoticed by West mathematicians for a long time; another proof was given in the recent paper Bojarski B. et al. [13], where they proved also a version of this theorem for Holder classes \( C^{k,\alpha+} \) with vanishing condition (1.3). Further, in [29] Hajlasz and Zimmerman replaced the assumption \( v \in C^k(\mathbb{R}^n, \mathbb{R}^d) \) of Theorem A by the assumption of Sobolev regularity \( v \in W^k_p(\mathbb{R}^n, \mathbb{R}^d) \) with \( p > n \) (this is an analog of DePascale extension for the Morse-Sard, see subsection 1.2, cf. with our assumptions \( kp > n \) or \( kp \geq n \) in theorems 1.2–1.8).

It is easy to see that Dubovitskiĭ Theorem A is a partial case of Theorem 1.6 of the present paper with parameters \( \tau = n \), \( \alpha = 0 \) and \( q = m + 1 = d \). Note, that the last assumption (which also used in [13], [29]) simplifies the proofs very essentially, because automatically one has that the image \( v(E) \) is \( \mathcal{H}^{\alpha-}\sigma \)-finite. But in general in theorems 1.6–1.8 the image \( v(E) \) may have Hausdorff dimension much larger than \( q \) for \( E \subset \tilde{Z}_{v,m} \) with \( \mathcal{H}^\tau(E) = 0 \). Nevertheless, the equality \( \mathcal{H}^{\alpha}(v^{-1}(y) \cap Z_{v,m}) = 0 \) is fulfilled for \( q \)-almost all \( y \in v(E) \) as required in definition (1.9).

Finally, let us note that the assertions of Theorems 1.6–1.8 for the case \( \tau = n \) were proved in our previous paper [27] and in the papers of [30] by Hajlasz, Korobkov, Kristensen.
Without the gradient constraints, the Dubovitskiĭ–Fubini analogs of Theorems 1.4–1.5 were obtained in our previous paper [26].

**Theorem 1.9** ([26], Sobolev case). Let $\alpha > 0$, $1 < p < \infty$, $\alpha p > n$, and $v \in \mathcal{L}^\alpha_p(\mathbb{R}^n, \mathbb{R}^d)$. Suppose that $0 < \tau \leq n$ and $\tau \neq \tau_*= n - (\alpha - 1)p$. Then for every $q \in [0, \sigma]$ and for any set $E \subset \mathbb{R}^n$ with $H^\tau(E) = 0$ the equality

\[
\mathcal{H}^\mu(E \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d
\]  

holds, where $\mu = \tau(1 - \frac{q}{\sigma})$ and the value $\sigma = \sigma(\tau, \alpha, p)$ is defined in (1.7).

The above Theorem 1.9 omits the limiting cases $\alpha p = n$ and $\tau = \tau_*$. As above, it is possible to cover these cases as well using the Lorentz norms.

**Theorem 1.10** ([26], Sobolev–Lorentz case). Let $v : \mathbb{R}^n \rightarrow \mathbb{R}^d$ be a mapping for which one of the following cases holds:

(i) $v \in W^k_1(\mathbb{R}^n, \mathbb{R}^d)$ for some $k \in \mathbb{N}, k \geq n$;

(ii) $v \in \mathcal{L}^\alpha_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$ for some $\alpha > 0$, $p \in (1, \infty)$ with $\alpha p \geq n$.

Suppose that $0 < \tau \leq n$. Then for every $q \in [0, \sigma]$ and for any set $E \subset \mathbb{R}^n$ with $H^\tau(E) = 0$ the equality (1.13) holds with the same $\mu$ and $\sigma$ defined in (1.7) (with $\alpha = k$ and $p = 1$ for the case (i)).

Taking $\tau \geq \tau_*$, we obtain, in particular,

**Corollary 1.1.** Let $\alpha > 0$, $1 < p < \infty$, $\alpha p > n$, and $v \in \mathcal{L}^\alpha_p(\mathbb{R}^n, \mathbb{R}^d)$. Suppose that $0 < \tau \leq n$ and $\tau > \tau_* = n - (\alpha - 1)p$. Then for every $q \in [0, \tau]$ and for any set $E \subset \mathbb{R}^n$ with $H^\tau(E) = 0$ the equality

\[
\mathcal{H}^{\tau-q}(E \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-a.a. } y \in \mathbb{R}^d
\]  

holds. Further, if $v \in \mathcal{L}^\alpha_{p,1}(\mathbb{R}^n, \mathbb{R}^d)$ or if $v \in W^k_1(\mathbb{R}^n, \mathbb{R}^d)$, then the same assertion holds under weaker assumptions $\alpha p \geq n$ (respectively, $k \geq n$) and $\tau \geq \tau_*$.

## 2 Preliminaries

By an $n$–dimensional interval we mean a closed cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. If $Q$ is an $n$–dimensional cubic interval then we write $\ell(Q)$ for its sidelength.

For a subset $S$ of $\mathbb{R}^n$ we write $\mathcal{L}^n(S)$ for its outer Lebesgue measure (sometimes we use the symbol $\lambda\,m\,$ for the same purpose). The $m$–dimensional Hausdorff measure is denoted by $\mathcal{H}^m$ and the $m$–dimensional Hausdorff content by $\mathcal{H}^m_\infty$. Recall that for any subset $S$ of $\mathbb{R}^n$ we have by definition

\[
\mathcal{H}^m(S) = \lim_{\epsilon \searrow 0} \mathcal{H}^m_\epsilon(S) = \sup_{t > 0} \mathcal{H}^m_t(S),
\]
where for each $0 < t \leq \infty$,

\[ \mathcal{H}^n_t(S) = \inf \left\{ \sum_{i=1}^{\infty} (\text{diam } S_i)^n \mid \text{diam } S_i \leq t, \ S \subset \bigcup_{i=1}^{\infty} S_i \right\}. \]

It is well known that $\mathcal{H}^n(S) = H^\infty(S) \sim L^n(S)$ for sets $S \subset \mathbb{R}^n$.

To simplify the notation, we write $\| f \|_{L^p}$ instead of $\| f \|_{L^p(\mathbb{R}^n)}$, etc.

The Sobolev space $W^k_p(\mathbb{R}^n, \mathbb{R}^d)$ is as usual defined as consisting of those $\mathbb{R}^d$-valued functions $f \in L^p_p(\mathbb{R}^n)$ whose distributional partial derivatives of orders $l \leq k$ belong to $L^p(\mathbb{R}^n)$ (for detailed definitions and differentiability properties of such functions see, e.g., [23], [45], [59], [19]). We use the norm

\[ \| f \|_{W^k_p} = \| f \|_{L^p} + \| \nabla f \|_{L^p} + \cdots + \| \nabla^k f \|_{L^p}, \]

and unless otherwise specified all norms on the spaces $\mathbb{R}^s$ ($s \in \mathbb{N}$) will be the usual euclidean norms.

Working with locally integrable functions, we always assume that the precise representatives are chosen. If $w \in L_{1,\text{loc}}(\Omega)$, then the precise representative $w^*$ is defined for all $x \in \Omega$ by

\[ w^*(x) = \begin{cases} \lim_{r \to 0} \int_{0}^{r} w(z) \, dz, & \text{if the limit exists and is finite}, \\ 0, & \text{otherwise}, \end{cases} \]

where the dashed integral as usual denotes the integral mean,

\[ \int_{B(x,r)} w(z) \, dz = \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} w(z) \, dz, \]

and $B(x, r) = \{ y : |y - x| < r \}$ is the open ball of radius $r$ centered at $x$. Henceforth we omit special notation for the precise representative writing simply $w^* = w$.

If $k < n$, then it is well-known that functions from Sobolev spaces $W^k_p(\mathbb{R}^n)$ are continuous for $p > \frac{n}{k}$ and could be discontinuous for $p \leq p_0 = \frac{n}{k}$ (see, e.g., [45, 59]). The Sobolev–Lorentz space $W^{k,p_0,1}(\mathbb{R}^n) \subset W^k_{p_0}(\mathbb{R}^n)$ is a refinement of the corresponding Sobolev space. Among other things functions that are locally in $W^{k,p_0,1}$ on $\mathbb{R}^n$ are in particular continuous (see, e.g., [35]).

Here we only mentioned the Lorentz space $L^p_{p,1}$, $p \geq 1$, and in this case one may rewrite the norm as (see for instance [42, Proposition 3.6])

\[ \| f \|_{L^p_{p,1}} = \int_{0}^{+\infty} \left[ \mathcal{L}^n(\{ x \in \mathbb{R}^n : |f(x)| > t \}) \right]^\frac{1}{p} \, dt. \]

Of course, we have the inequality

\[ \| f \|_{L^p} \leq \| f \|_{L^p_{p,1}}. \] (2.1)
Denote by $W^k_{p,1}(\mathbb{R}^n)$ the space of all functions $v \in W^k_p(\mathbb{R}^n)$ such that in addition the Lorentz norm $\|\nabla^k v\|_{L^p,1}$ is finite.

By definition put $\|g\|_{L^p,1}(E) := \|1_E \cdot g\|_{L^p,1}$, where $1_E$ is the indicator function of $E$.

We need the following analog of the additivity property for the Lorentz norms:

$$\sum_i \|f_i\|_{L^p,1}(Q_i) \leq \|f\|_{L^p,1}(\bigcup_i Q_i)$$

(see, e.g., [42, Lemma 3.10] or [51]).

For a function $f \in L_{1,\text{loc}}(\mathbb{R}^n)$ we often use the classical Hardy–Littlewood maximal function:

$$Mf(x) = \sup_{r>0}\int_{B(x,r)} |f(y)| \, dy.$$  

2.1 On Fubini type theorems for graphs of continuous functions

Recall that by usual Fubini theorem, if a set $E \subset \mathbb{R}^2$ has a zero plane measure, then for $\mathcal{H}^1$-almost all straight lines $L$ parallel to coordinate axes we have $\mathcal{H}^1(L \cap E) = 0$. The next result could be considered as functional Fubini type theorem.

**Theorem 2.1** (see Theorem 5.3 in [30]). Let $\mu \geq 0$, $q > 0$, and $v : \mathbb{R}^n \to \mathbb{R}^d$ be a continuous function. For a set $E \subset \mathbb{R}^n$ define the set function

$$\Phi(E) = \inf_{E \subset \bigcup_j D_j} \sum_j (\text{diam } D_j)^\mu [\text{diam } v(D_j)]^q,$$

where the infimum is taken over all countable families of compact sets $\{D_j\}_{j \in \mathbb{N}}$ such that $E \subset \bigcup_j D_j$. Then $\Phi(\cdot)$ is a countably subadditive and the implication

$$\Phi(E) = 0 \Rightarrow \left[ \mathcal{H}^\mu \big( E \cap v^{-1}(y) \big) = 0 \right. \text{ for } \mathcal{H}^q\text{-almost all } y \in \mathbb{R}^d$$

holds.

3 Estimates of the critical values on cubes

In this section we formulate estimates of the above defined set function $\Phi$ obtained in [27, Appendix] for subsets of critical set in cubes for different classes of mappings\(^\text{3}\).

For all the following four subsections fix $m \in \{0, \ldots, n-1\}$ and $d \geq m$. Take also a positive parameter $q \geq m$ and nonnegative $\mu \geq 0$ required in the definition of the set–function $\Phi$ in (2.3).

\(^3\text{The only technical difference is that in [27] we used the notation } Z'_v = \{ x \in \mathbb{R}^n \setminus A_v : \text{rank } \nabla v(x) < m \}, \text{ i.e., there } m-1 \text{ plays the role of the parameter } m \text{ of the present article.}\)
For a regular (in a sense) mapping \( v : \mathbb{R}^n \to \mathbb{R}^d \) denote
\[
Z_v = \{ x \in \mathbb{R}^n \setminus A_v : \text{rank} \nabla v(x) \leq m \}.
\]
Here \( A_v \) means the set of ‘bad’ points, where \( v \) is not differentiable or or which are not Lebesgue points for \( \nabla v \) (of course, \( A_v = \emptyset \) if the gradient \( \nabla v \) is a continuous function).

It is convenient (and sufficient for our purposes) to restrict our attention on the following subset of critical points
\[
Z'_v = \{ x \in Z_v : |\nabla v(x)| \leq 1 \}.
\] (3.1)

### 3.1 Estimates on cubes for Holder classes of mappings.
Fix \( k \geq 1, \ 0 \leq \alpha \leq 1, \) and \( v \in C^{k,\alpha}(\mathbb{R}^n, \mathbb{R}^d) \). By definition of the space \( C^{k,\alpha} \), there exists a constant \( A \in \mathbb{R}_+ \) such that
\[
|\nabla^k v(x) - \nabla^k v(y)| \leq A \cdot |x - y|^\alpha \text{ for all } x, y \in \mathbb{R}^n.
\] (3.2)

**Theorem 3.1** ([27]). Under above assumptions, for any sufficiently small \( n \)-dimensional interval \( Q \subset \mathbb{R}^n \) the estimate
\[
\Phi(Q \cap Z'_v) \leq C \Lambda^{q-m} \ell(Q)^{q+\mu+(k+\alpha-1)(q-m)}
\] (3.3)
holds, where the constant \( C \) depends on \( n, m, k, \alpha, d \) only.

### 3.2 Estimates on cubes for Sobolev classes of mappings.
Fix \( k \geq 1, \ 0 \leq \alpha < 1, \ 1 < p < \infty, \) and \( v \in \mathcal{L}^{k+\alpha,p}_p(\mathbb{R}^n, \mathbb{R}^d) \). In this subsection we consider the case, when \( k + \alpha > 1 \) and
\[
(k + \alpha)p > n,
\] (3.4)
i.e., when \( v \) is a continuous function (see, e.g., [35]), but the gradient \( \nabla v \) could be discontinuous in general (if \( (k + \alpha - 1)p < n \)).

**Theorem 3.2** ([27]). Under above assumptions, there exists a function \( h \in L^p(\mathbb{R}^n) \) (depending on \( v \)) such that the following statements are fulfilled:

(i) if \( (k + \alpha - 1)p > n \), then the gradient \( \nabla v \) is continuous and uniformly bounded function, and for any sufficiently small \( n \)-dimensional interval \( Q \subset \mathbb{R}^n \) the estimate
\[
\Phi(Z'_v \cap Q) \leq C \sigma^{q-m} \ell(q+\mu+(k+\alpha-1-\frac{q}{p})(q-m)}
\] (3.5)
holds, where
\[
r = \ell(Q), \quad \sigma = \|h\|_{L^p(Q)}.
\] (3.6)
and the constant \( C \) depends on \( n, m, k, \alpha, d, p \) only.
(ii) if \((k + \alpha - 1)p < n\), then under additional assumption
\[
q + \mu > \tau_* := n - (k + \alpha - 1)p
\]  
for any \(n\)-dimensional interval \(Q \subset \mathbb{R}^n\) the estimate
\[
\Phi(Z'_v \cap Q) \leq C \left( \sigma^q r^{(k+\alpha-\frac{n}{p})q+\mu} + \sigma^{q-m} r^{q+\mu+(k+\alpha-1-\frac{n}{p})(q-m)} \right)
\]  
holds with the same \(\sigma, r\).

### 3.3 Estimates on cubes for Sobolev–Lorentz classes of mappings.

Fix \(k \geq 1\), \(0 \leq \alpha < 1\), \(1 < p < \infty\), and \(v \in \mathcal{L}^{k+\alpha}_{p,1}(\mathbb{R}^n, \mathbb{R}^d)\). In this subsection we consider the case, when \(k + \alpha > 1\) and
\[
(k + \alpha)p \geq n,
\]  
i.e., when \(v\) is a continuous function (see, e.g., [35]), but the gradient \(\nabla v\) could be discontinuous in general (if \((k + \alpha - 1)p < n\).

**Theorem 3.3** ([27]). Under above assumptions, there exists a function \(h \in L_{p,1}(\mathbb{R}^n)\) (depending on \(v\)) such that the following statements are fulfilled:

1. **if \((k + \alpha - 1)p \geq n\), then** the gradient \(\nabla v\) is continuous and uniformly bounded function, and for any sufficiently small \(n\)-dimensional interval \(Q \subset \mathbb{R}^n\) the estimate (3.5) holds with
\[
r = \ell(Q), \quad \sigma = \|h\|_{L_{p,1}(Q)}.
\]  

2. **if \((k + \alpha - 1)p < n\), then** under additional assumption
\[
q + \mu \geq \tau_* := n - (k + \alpha - 1)p
\]  
for any \(n\)-dimensional interval \(Q \subset \mathbb{R}^n\) the estimate (3.8) holds with the same \(\sigma, r\) as in (3.10).

**Remark 3.1.** Formally estimates in Theorem 3.3 are the same as in Theorems 3.2, the only difference is in the definition of \(\sigma\) (using the Lorentz norm instead of Lebesgue one). However, Theorem 3.3 is ‘stronger’ in a sense than the previous Theorems 3.2. Namely, there are some important (limiting) cases, which are not covered by Theorem 3.2, but one could still apply the Theorem 3.3 for these cases. It happens for the following values of the parameters:
\[
(k + \alpha)p = n,
\]
or

\[(k + \alpha - 1)p = n,\]

or

\[q + \mu = \tau.\]

It means, that the Lorentz norm is a sharper and more accurate tool here than the Lebesgue norm.

### 3.4 Estimates on cubes for Sobolev classes of mappings \(W^k_1(\mathbb{R}^n), \ k \geq n.\)

In this subsection we consider the limiting case \(p = 1\) for Sobolev spaces \(W^k_1\). It is well known that functions from the Sobolev space \(W^k_1(\mathbb{R}^n, \mathbb{R}^d)\) are continuous if

\[k \geq n,\]  

so we assume this condition below. Fix \(k \geq n\) and \(v \in W^k_1(\mathbb{R}^n, \mathbb{R}^d)\).

**Theorem 3.4 ([27]).** Under above assumptions, the following statements hold:

(i) if \(k - 1 \geq n\), then the gradient \(\nabla v\) is continuous and uniformly bounded function, and for any sufficiently small \(n\)-dimensional interval \(Q \subset \mathbb{R}^n\) the estimate

\[\Phi(Z'_v \cap Q) \leq C \sigma^{q-m} r^{q+\mu+(k-1-n)(q-m)}\]  

holds, where again

\[r = \ell(Q), \quad \sigma = \|\nabla^k v\|_{L^1(Q)}\]  

and the constant \(C\) depends on \(n, m, k, d\) only.

(ii) if \(k = n\), then under additional assumption

\[q + \mu \geq 1\]  

for any \(n\)-dimensional interval \(Q \subset \mathbb{R}^n\) the estimate

\[\Phi(Z'_v \cap Q) \leq C \left(\sigma^{q-\mu} + \sigma^{q-m} r^{\mu+m}\right),\]  

holds with the same \(r, \sigma\), and with \(C\) depending on \(n, m, k, d\) only.
4 Proofs of the main results

We have to prove three theorems 1.6–1.8 (because other two theorems 1.2–1.3 are the partial cases of Theorems 1.7–1.8 when $q = \sigma$ and $\mu = 0$).

For the extremal case $\tau = n$ all these three theorems were proved in [30] and [27], so below we always assume that

$$0 < \tau < n. \quad (4.1)$$

Let us first check the assertions about strict $N$-properties. Fixed the corresponding parameters $m \in \{0, 1, \ldots, n-1\}$, $\mu \geq 0$, $q \in (m, \sigma]$, and a mapping $v : \mathbb{R}^n \to \mathbb{R}^d$ satisfying assumptions of one of the Theorems 1.6–1.8. We have to prove that

$$\mathcal{H}^\mu(E \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-almost all } y \in \mathbb{R}^d \text{ whenever } E \subset \tilde{Z}_{v,m} \text{ with } \mathcal{H}^\tau(E) < \infty. \quad (4.2)$$

First of all, we will simplify the situation and eliminate some technical difficulties associated with irregular points of mappings from Sobolev classes. Recall, that for the Sobolev case the $m$-critical set is defined as

$$\tilde{Z}_{v,m} = \{x \in \mathbb{R}^n : x \in A_v \text{ or } x \in \mathbb{R}^n \setminus A_v \text{ with } \text{rank } \nabla v(x) \leq m\}.$$

Here $A_v$ means the set of ‘bad’ points at which either the function $v$ is not differentiable or which are not the Lebesgue points for $\nabla v$. Recall that the set $A_v$ is relatively small:

$$\mathcal{H}^t(A_v) = 0 \quad \forall t > \tau_* := n - (k+\alpha-1)p \quad \text{if } v \in \mathscr{L}_p^{k+\alpha}(\mathbb{R}^n) \quad \text{(case of Theorem 1.7)}; \quad (4.3)$$

$$\mathcal{H}^{\tau_*}(A_v) = 0 \quad \text{if } v \text{ is from Theorem 1.8}. \quad (4.4)$$

In particular, $A_v = \emptyset$ if $(k + \alpha - 1)p > n$ (respectively, if $(k + \alpha - 1)p \geq n$).

For the case of Theorem 1.7, take $t \in (\tau_*, \tau)$. Then by Corollary 1.1 we have

$$\mathcal{H}^{t-q}(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-almost all } y \in \mathbb{R}^d. \quad (4.5)$$

By elementary direct calculation, if $\tau_* > 0$, then

$$\tau_* - q < \mu = \tau - m - (k + \alpha - \frac{n}{p} + \frac{\tau}{p})(q - m). \quad (4.6)$$

Indeed, by definition of $\tau_* = n - (k + \alpha - 1)p$, the last inequality is equivalent to

$$(\tau - \tau_*)(1 - \frac{q - m}{p}) > 0. \quad (4.7)$$

But really by our assumptions

$$\frac{q - m}{p} \leq \frac{\sigma - m}{p} = \frac{\tau - m}{\tau + (k + \alpha)p - n} < \frac{\tau}{\tau} = 1,$$
so (4.6)–(4.7) is fulfilled. From inequality (4.6) it follows that for \( t \in (\tau_*, \tau) \) sufficiently close to \( \tau_* \) we have
\[
t - q < \mu.
\]
From this inequality and (4.5) we obtain
\[
\mathcal{H}^\mu(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^\mu\text{-almost all } y \in \mathbb{R}^d,
\]
so indeed \( A_v \) is negligible in property (4.2).

If \( v \) is from Theorem 1.8, then again Corollary 1.1 implies
\[
\mathcal{H}^{\tau_*-q}(A_v \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^q\text{-almost all } y \in \mathbb{R}^d.
\]
And by the same calculations we obtain
\[
\tau_* - q \leq \mu,
\]
therefore, the identity (4.8) is fulfilled as well and in any case the 'bad' set \( A_v \) is negligible in property (4.2).

It means, that in the required property (4.2) we could replace the set \( \tilde{Z}_{v,m} \) by smaller (regular) set
\[
Z_v = \{ x \in \mathbb{R}^n \setminus A_v : \text{rank } \nabla v(x) \leq m \}.
\]
Moreover, since the countable union of the sets of \( \mathcal{H}^\mu\)-measure zero has again \( \mathcal{H}^\mu\)-measure zero, we could replace the set \( Z_v \) by the smaller set
\[
Z'_v = \{ x \in \mathbb{R}^n \setminus A_v : |\nabla v(x)| \leq 1 \text{ and rank } \nabla v(x) \leq m \},
\]
i.e., instead of (4.2) we need to check only
\[
\mathcal{H}^\mu(E \cap v^{-1}(y)) = 0 \quad \text{for } \mathcal{H}^\mu\text{-almost all } y \in \mathbb{R}^d \text{ whenever } E \subset Z'_v \text{ with } \mathcal{H}^r(E) < \infty.
\]
Because of Theorem 2.1, for the proof of the last assertion it is sufficient to check, that
\[
\Phi(E) = 0 \quad \text{whenever } E \subset Z'_v \text{ with } \mathcal{H}^r(E) < \infty,
\]
where the set function \( \Phi \) was defined in Theorem 2.1.

In our previous paper [27, Appendix] we obtained the general estimates for \( \Phi(Z'_v \cap Q) \), here \( Q \) is an arbitrary \( n \)-dimensional cube, for all considered cases: Holder, Sobolev (including fractional Sobolev), and Sobolev–Lorentz (see their formulation in Section 3 of the present paper). From these estimates and from the Holder inequality the required property (4.12) follows easily\(^4\).

\(^4\)Really, the present paper and [27] were written in the same time, so we had in mind the purposes of the present paper when we formulated and proved the estimates for \( \Phi(Z'_v \cap Q) \) in [27, Appendix].
The detailed description of application of these estimates and Holder inequalities was given in [27] for the case \( \tau = n \). The present case \( \tau < n \) is even simpler: indeed, the most difficult and subtle part in [27] was to prove the strict \((\tau, q, \mu, m)\)-N-property for Holder case when \( \tau = n \), — it requires the application of some generalised Coarea formula, etc. We do not need to touch these difficulties here. The strictness of considered N-properties for the present case \( 0 < \tau < n \) follows from the following three simple facts:

\[
|\nabla^k v(x) - \nabla^k v(y)| \leq \omega(r) \cdot |x - y|^\alpha \quad \text{whenever} \quad |x - y| < r
\]

with \( \omega(r) \to 0 \) as \( r \to 0 \) for \( v \in C^{k,\alpha+} \) or \( v \in C^k \) (i.e., \( \alpha = 0 \));

\[
\sum_i \|h\|_{L_p(Q_i)}^p \to 0 \quad \text{as} \quad \sum_i \ell(Q_i)^\tau \leq C, \quad \sup_i \ell(Q_i) \to 0,
\]

for any (fixed) function \( h \in L_p(\mathbb{R}^n) \), where \( Q_i \) is a family of nonoverlapping \( n \)-dimensional cubes;

\[
\sum_i \|h\|_{L_p,1(Q_i)}^p \to 0 \quad \text{as} \quad \sum_i \ell(Q_i)^\tau \leq C, \quad \sup_i \ell(Q_i) \to 0,
\]

for any (fixed) function \( h \in L_{p,1}(\mathbb{R}^n) \), where again \( Q_i \) is a family of nonoverlapping \( n \)-dimensional cubes (see (2.2)). Since there are no any difficulties in realisation of these arguments, we omit the details.

The proof of the nonstrict N-properties in Theorems 1.6–1.8 is based on the same estimates with evident simplifications in calculations.

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