EQUIVARIANT PRINCIPAL BUNDLES ON NONSINGULAR TORIC VARIETIES

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Abstract. We give a classification of the equivariant principal $G$-bundles on a nonsingular toric variety when $G$ is a closed Abelian subgroup of $GL_k(\mathbb{C})$. This is a partial analogue to Klyachko’s classification of equivariant vector bundles on toric varieties.

1. Introduction

Denote the algebraic torus $(\mathbb{C}^*)^n$ by $T$. A $T$-equivariant principal $G$-bundle on a complex manifold $X$ is a locally trivial, principal $G$-bundle $\pi : \mathcal{E} \to X$ such that $\mathcal{E}$ and $X$ are left $T$-spaces, the map $\pi$ is $T$-equivariant and the actions of $T$ and $G$ commute:

$$t(e \cdot g) = (te) \cdot g$$

for all $t \in T$, $g \in G$ and $e \in \mathcal{E}$.

If, in addition, the bundle $\pi : \mathcal{E} \to X$ and the actions of $T$ and $G$ are holomorphic we say that $\mathcal{E}$ is a holomorphic $T$-equivariant principal $G$-bundle.

Let $X_\Xi$ be a complete nonsingular toric variety of dimension $n$ corresponding to a fan $\Xi$. Denote the set of $d$-dimensional cones in $\Xi$ by $\Xi(d)$. For a cone $\sigma$ in $\Xi$, denote the corresponding affine variety by $X_\sigma$ and the corresponding $T$-orbit by $O_\sigma$. Note that each orbit $O_\sigma$ has a natural group structure and the principal orbit $O$ is identified with $T$ (see [6], Proposition 1.6). Let $T_\sigma$ denote the stabilizer of any point in $O_\sigma$. Then $T$ admits a decomposition, $T \cong T_\sigma \times O_\sigma$. Let $\pi_\sigma : T \to T_\sigma$ be the associated projection.

Let $G$ be an Abelian subgroup of $GL(k, \mathbb{C})$ for some positive integer $k$. Note that a classification of such groups is given in [1], Proposition 2.3. Assume further that $G$ is closed in $GL_k(\mathbb{C})$. Then our main theorem is the following (same as Theorem [4,1]).

Theorem 1.1. The isomorphism classes of holomorphic $T$-equivariant principal $G$-bundles on $X_\Xi$ are in one-to-one correspondence with collections of holomorphic group homomorphisms $\{\rho_\sigma : T_\sigma \to G \mid \sigma$ is a maximal cone in $\Xi\}$ which satisfy the extension condition: Each $(\rho_\tau \circ \pi_\tau)(\rho_\sigma \circ \pi_\sigma)^{-1}$ extends to a $G$-valued holomorphic function over $X_\sigma \cap X_\tau$.

A similar classification for algebraic $T$-equivariant bundles over $X_\Xi$ is given in Theorem [5,1]. These theorems provide a partial analogue to Klyachko’s classification of vector bundles on toric varieties in [5].

We also prove that if $G$ is any discrete group, then any holomorphic $T$-equivariant principal $G$-bundle on $X_\Xi$ is trivial with trivial $T$-action (Theorem [4,2]). Our method may be used to prove a similar result for $\Gamma^n$-equivariant principal $G$-bundles over a topological toric manifold [2] where $G$ is discrete and $\Gamma = S^1 \times \mathbb{R}$ acts smoothly.

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2. Local action functions

Let \( X = X_\sigma \) where \( \sigma \in \Xi \). For any sub-cone \( \delta \leq \sigma \) we denote the corresponding \( T \)-orbit by \( O_\delta \). As \( T \) is Abelian, the stabilizer of \( x \) in \( T \) is the same for all \( x \in O_\delta \). Denote this stabilizer subgroup by \( T_\delta \).

Suppose \( \mathcal{E} \) is a \( T \)-equivariant principal \( G \)-bundle over \( X \). Assume that \( \mathcal{E} \) is trivial (we will show in Lemma 3.4 that this holds for suitable \( G \)). Let \( s : X \to \mathcal{E} \) be any holomorphic section. We encode the \( T \)-action on \( \mathcal{E} \) as follows.

**Definition 2.1.** For any \( x \in X \) and \( t \in T \), define \( \rho_s(x,t) \in G \) as follows,

\[
(2.1) \quad ts(x) = s(tx) \cdot \rho_s(x,t)
\]

We say that \( \rho_s : X \times T \to G \) is a local action function.

Since the action of \( G \) on each fiber of \( \mathcal{E} \) is free, it follows that \( \rho_s(x,t) \) is well-defined and holomorphic in \( x \) and \( t \).

It is easy to check that if \( s'(x) = s(x) \cdot \gamma(x) \) is another section, then

\[
(2.2) \quad \rho_{s'}(x,t) = \gamma(tx)^{-1} \rho_s(x,t)\gamma(x)
\]

**Lemma 2.1.** For any \( t_1, t_2 \) in \( T \), \( \rho_s(x,t_1t_2) = \rho_s(t_2x,t_1)\rho_s(x,t_2) \).

**Proof.** Obviously, \( t_1t_2s(x) = s(t_1t_2x) \cdot \rho_s(x,t_1t_2) \). On the other hand,

\[
t_1t_2s(x) = t_1(s(t_2x) \cdot \rho_s(x,t_2)) = (t_1s(t_2x)) \cdot \rho_s(x,t_2) = s(t_1t_2x) \cdot \rho_s(t_2x,t_1)\rho_s(x,t_2).
\]

It follows that if \( x \in O_\delta \) then the restriction

\[
(2.3) \quad \rho_s(x, \cdot) : T_\delta \to G
\]

is a group homomorphism.

Lemma 2.1 implies that the value of \( \rho_s \) on any \( T \)-orbit \( O_\delta \) may be determined from the value of \( \rho_s \) at a point \( x \in O_\delta \). However a stronger statement holds.

**Lemma 2.2.** The map \( \rho_s \) is completely determined by its restriction \( \rho_s(x_0, \cdot) : T \to G \) at any point \( x_0 \) in the principal \( T \)-orbit \( O \).

**Proof.** Let \( \delta \neq \{0\} \) be a sub-cone of \( \sigma \). Let \( x_\delta \) be any point in \( O_\delta \). Then there exist a point \( x_1 \in O \) and a one parameter subgroup \( \lambda'(z) \) of \( T \) such that \( \lim_{z \to 0} \lambda'(z)x_1 = x_\delta \) (see [3], section 2.3). Then using Lemma 2.1 we get \( \rho_s(x_1,t\lambda'(z)) = \rho_s(\lambda'(z)x_1,t)\rho_s(x_1,\lambda'(z)) \).

Hence \( \rho_s(\lambda'(z)x_1,t) = \rho_s(x_1,t\lambda'(z))\rho_s(x_1,\lambda'(z))^{-1} \). Taking limit as \( z \) approaches 0, we get

\[
(2.4) \quad \rho_s(x_\delta,t) = \lim_{z \to 0} \rho_s(x_1,t\lambda'(z))\rho_s(x_1,\lambda'(z))^{-1}.
\]

Since \( \rho_s(x_1, \cdot) \) is determined by \( \rho_s(x_0, \cdot) \), the lemma follows. \( \Box \)

**Lemma 2.3.** Let \( X_1 \) and \( X_2 \) be affine toric varieties. Let \( \alpha : X_1 \to X_2 \) be an isomorphism of \( T \)-spaces up to an automorphism \( a : T \to T \), i.e. \( \alpha \circ t = a(t) \circ \alpha \). Suppose \( \pi_i : \mathcal{E}_i \to X_i \) is a \( T \)-equivariant trivial principal \( G \)-bundle for \( i = 1, 2 \). Let \( \phi : \mathcal{E}_1 \to \mathcal{E}_2 \) be an isomorphism of \( T \)-equivariant principal \( G \)-bundles over \( X \) compatible with \( \alpha \) and \( a \):

\[
\pi_2 \circ \phi = \alpha \circ \pi_1 \quad \text{and} \quad \phi \circ t = a(t) \circ \phi.
\]
Let $s_1$ be any section of $\mathcal{E}_1$ and let $s_2$ be the section of $\mathcal{E}_2$ defined by $s_2(\alpha(x)) = \phi(s_1(x))$ for $x \in X_1$. Then $\rho_{s_1}(x,t) = \rho_{s_2}(\alpha(x),a(t))$ for every $x \in X_1, t \in T$. In particular, if $\alpha$ and $a$ are both identity then $\rho_{s_1} = \rho_{s_2}$.

Proof. The lemma follows from the following calculation.

\[
\begin{align*}
 s_2(a(t)\alpha(x)) \cdot \rho_{s_2}(\alpha(x),a(t)) &= a(t)s_2(\alpha(x)) = a(t)\phi(s_1(x)) \\
 &= \phi(ts_1(x)) = \phi(s_1(tx) \cdot \rho_{s_1}(x,t)) \\
 &= \phi(s_1(tx)) \cdot \rho_{s_1}(x,t) = s_2(\alpha(tx)) \cdot \rho_{s_1}(x,t) \\
 &= s_2(a(t)\alpha(x)) \cdot \rho_{s_1}(x,t).
\end{align*}
\]

\[\square\]

Lemma 2.4. If $\rho_s(x,\cdot)$ is independent of $x$, then it defines a group homomorphism $\rho_s : T \to G$. Conversely if $\rho_s(x_0,\cdot)$ is a group homomorphism for some $x_0 \in O$, then $\rho_s(x,t)$ is independent of $x$.

Proof. The first claim follows immediately from Lemma 2.1. On the other hand if $\rho_s(x_0,\cdot)$ is a group homomorphism, then we have

\[
\rho_s(x_0,t)\rho_s(x_0,u) = \rho_s(x_0,tu) = \rho_s(ux_0,t)\rho_s(x_0,u)
\]

for any $u, t \in T$. Therefore for any $u \in T$

\[
\rho_s(ux_0,\cdot) = \rho_s(x_0,\cdot).
\]

Now the result follows either by using holomorphicity of $\rho_s(x,t)$ in $x$, or equation (2.4).

\[\square\]

Lemma 2.5. Suppose $G$ is a discrete group and $X \cong \mathbb{C}^d$. Let $\mathcal{E}$ be a $T$-equivariant holomorphic principal $G$-bundle over $X$. Then $\mathcal{E}$ is trivial and for any section $s$ of $\mathcal{E}$, $\rho_s(x,\cdot) : T \to G$ is independent of $x$ and is the trivial homomorphism.

Proof. Note that as $X$ is contractible, the bundle $\mathcal{E}$ is topologically trivial. Therefore it admits a continuous section $s$. But as $X$ is connected and $G$ is discrete, $s$ is constant and hence holomorphic.

Fix $t \in T$. Since $X$ is connected and $\rho_s(x,t)$ is continuous is $x$, it is a constant map due to the discreteness of $G$. This implies that $\rho_s(x,\cdot)$ is independent of $x$ and therefore is a group homomorphism $\rho_s(\cdot) : T \to G$ by Lemma 2.4. Continuity in $t$ and connectedness of $T$ then imply that $\rho_s(t)$ is a constant. This completes the proof.

\[\square\]

3. Local action homomorphisms

Suppose $\mathcal{E}$ is a holomorphic principal $G$-bundle over $X \cong \mathbb{C}^n$ where $G$ is a subgroup of $G_k(\mathbb{C})$. Then by Oka-Grauert theory $\mathcal{E}$ is trivial and admits a holomorphic section $s$. We will show that if $G$ is Abelian and closed in $G_k(\mathbb{C})$, then $s$ can be chosen so that the local action function $\rho_s$ is a homomorphism.

Proposition 3.1. Let $G$ be a subgroup of $G_k(\mathbb{C})$ that is closed in $G_k(\mathbb{C})$. Suppose $f : \mathbb{C}^* \to G$ is a holomorphic map such that $\lim_{z \to 0} f(zt)f(z)^{-1} = I$ for every $t \in \mathbb{C}^*$. Then $f$ admits a holomorphic extension $f : \mathbb{C} \to G$. 

Proof. Let

\[ f(z) = \sum_{n=-\infty}^{\infty} C_n z^n \]

be the Laurent series representation of \( f \) for \( z \in \mathbb{C}^* \), where each \( C_n \in M_k(\mathbb{C}) \). Note that \( f(\cdot)^{-1} : \mathbb{C}^* \to G \) is also holomorphic. Let its Laurent series be

\[ f(z)^{-1} = \sum_{n=-\infty}^{\infty} A_n z^n \]

where each \( A_n \in M_k(\mathbb{C}) \). Then we have

\[ f(z) f(z)^{-1} = \sum_{n=-\infty}^{\infty} z^n t^n ( \sum_{m=-\infty}^{\infty} C_{n-m} A_m t^{-m} ). \]

As the limit \( \lim_{z \to 0} f(z) f(z)^{-1} \) exists by assumption, the Riemann extension theorem shows that \( f(z) f(z)^{-1} \) is holomorphic at \( z = 0 \) for every \( t \in \mathbb{C}^* \). Therefore,

\[ \sum_{m=-\infty}^{\infty} C_{n-m} A_m t^{-m} = 0 \]

for every \( n < 0 \) and every \( t \in \mathbb{C}^* \). This implies that

\[ C_{n-m} A_m = 0 \]

for every \( n < 0 \) and every \( m \). Setting \( j = n - m \) and varying \( n \), we obtain

\[ C_j A_m = 0 \]

for every \( j < -m \) for every \( m \).

Let \( S \) be the set of all column vectors of the matrices \( A_m, m \in \mathbb{Z} \). Since every subspace of \( \mathbb{C}^k \) is closed, the span of \( S \) is closed in \( \mathbb{C}^k \). Therefore the columns of \( f(z)^{-1} \) belong to the span of \( S \). Since \( f(z)^{-1} \in GL_k(\mathbb{C}) \), the span of \( S \) must equal \( \mathbb{C}^k \). Therefore there exists a finite set of values of \( m \), say \( m_1, \ldots, m_p \), such that the column spaces of the corresponding \( A_m \)'s span \( \mathbb{C}^k \). Let \( m_0 = \min\{ -m_1, \ldots, -m_p \} \). Consider any \( j < m_0 \). Then \( j < -m_i \) for each \( 1 \leq i \leq p \). Hence by equation (3.10) \( C_j A_{m_i} = 0 \) for each \( 1 \leq i \leq p \). Hence \( C_j = 0 \).

Assume that \( n_0 \) is the minimum value of \( n \) such that \( C_{n_0} \) is nonzero. Then \( f(z) = z^{n_0} \phi(z) \) where \( \phi : \mathbb{C} \to GL_k(\mathbb{C}) \) is holomorphic near zero. Then

\[ I = \lim_{z \to 0} f(z) f(z)^{-1} = \lim_{z \to 0} \phi(z) \phi(z)^{-1} t^{n_0}. \]

Since \( \lim_{z \to 0} \phi(z) = C_{n_0} \) and \( \lim_{z \to 0} \phi(tz) = C_{n_0} t \), multiplying both sides of (3.7) by \( \lim_{z \to 0} \phi(z) \) yields

\[ C_{n_0} = \lim_{z \to 0} \phi(z) = \lim_{z \to 0} \phi(tz) t^{n_0} = C_{n_0} t^{n_0} \]

for every \( t \in \mathbb{C}^* \). Therefore we must have \( n_0 = 0 \). Hence \( f \) is holomorphic at 0 and \( f(0) = C_0 \).

Let \( g(z) = \det(f(z)) \). It follows that \( g : \mathbb{C} \to \mathbb{C} \) is holomorphic and \( \lim_{z \to 0} g(z) g(z)^{-1} = 1 \). If \( g \) has a zero of order \( n \) at zero, we get \( \lim_{z \to 0} g(z) g(z)^{-1} = t^n \). Therefore \( n = 0 \) and \( g(0) \neq 0 \). Hence \( f(0) \) is nonsingular. Thus \( f \) defines a holomorphic map \( f : \mathbb{C} \to GL_k \).

Moreover if \( G \) is closed in \( GL_k(\mathbb{C}) \), it is evident that \( f(0) \in G \) so that \( f \) defines a holomorphic map \( f : \mathbb{C} \to G \). □
Theorem 3.2. Consider the standard action of $T$ on $\mathbb{C}^n$. Suppose $E$ is a holomorphic $T$-equivariant principal $G$-bundle over $\mathbb{C}^n$ where $G$ is a closed Abelian subgroup of $GL_k(\mathbb{C})$. Then there exists a holomorphic section $s'$ of $E$ such that $\rho_s(x, \cdot) : T \to G$ equals $\rho_{s'}(0, \cdot)$ for any $x$ in $\mathbb{C}^n$.

Proof. Let $s$ be a holomorphic section of $E$ over $\mathbb{C}^n$. Fix a point $x_0$ in $O = T$. Note that $\rho_s(0, \cdot) : T \to G$ is a group homomorphism as the origin 0 is fixed by $T$. Define a holomorphic function $F : T \to G$ by

$$F(t) = \rho_s(0, t)^{-1} \rho_s(x_0, t).$$

We may therefore write

$$\rho_s(x_0, t) = \rho_s(0, t) F(t).$$

For any $y, x \in \mathbb{C}^n$, define $yx$ to be coordinate-wise multiplication of $y$ and $x$. Let $z$ denote an element of $T$. For any $y \in \mathbb{C}^n$,

$$\rho_s(yx_0, t) = \lim_{z \to y} \rho_s(zx_0, t) = \lim_{z \to y} \rho_s(x_0, tz) \rho_s(x_0, z)^{-1} = \lim_{z \to y} \rho_s(0, tz) F(tz) F(z)^{-1} \rho_s(0, z)^{-1}.$$

Therefore,

$$\lim_{z \to y} F(tz) F(z)^{-1} = \lim_{z \to y} \rho_s(0, tz)^{-1} \rho_s(yx_0, t) \rho_s(0, z) = \rho_s(yx_0, t) \rho_s(0, t)^{-1},$$

where the last equality follows from the assumption that $G$ is Abelian, and the fact that $\rho_s(0, \cdot)$ is a homomorphism.

Define $P_k = \mathbb{C}^n - \cup_{i=1}^n \{ z_i = 0 \}$ where $1 \leq k \leq n$. Note that $P_1 = (\mathbb{C}^*)^n$ and $P_n = \mathbb{C}^n$. We will now show by induction over $k$ that $F$ admits a holomorphic $G$-valued extension over $\mathbb{C}^n$. Note that $F$ admits an extension over $P_1$ by definition. Now assume that $F$ has a $G$-valued holomorphic extension over $P_k$. Take any point $p = (p_1, \ldots, p_n) \in P_k$. Note that $tp \in P_k$ for any $t \in T$. Setting $y = p$ in (3.11), we get

$$F(tp) F(p)^{-1} = \rho_s(p x_0, t) \rho_s(0, t)^{-1}.$$

Let $\pi_k : \mathbb{C}^n \to \{ z_k = 0 \}$ be the standard projection. Taking limit of (3.12) as $p_k \to 0$, we get

$$\lim_{p_k \to 0} F(tp) F(p)^{-1} = \rho_s(\pi_k(p) x_0, t) \rho_s(0, t)^{-1}.$$

Let $t_k \in \mathbb{C}^*$. Write $\iota(t_k)$ for $(1, \ldots, 1, t_k, 1, \ldots, 1)$ where $t_k$ occupies the $k$-th position.

In the case $p_i \neq 0$ for each $i \neq k$, $t(p) = (p_1, \ldots, p_{k-1}, 1, p_{k+1}, \ldots, p_n)$ defines an element of $T$. Then by Lemma 2.1 and using $G$ is Abelian, we have

$$\rho_s(\pi_k(p) x_0, \iota(t_k)) = \rho_s(\iota(t_k) \pi_k(x_0), \iota(t_k)) \rho_s(\pi_k(x_0), t(p)) \rho_s(\pi_k(x_0), t(p))^{-1} = \rho_s(\pi_k(x_0), \iota(t_k)) \rho_s(\pi_k(x_0), t(p)) \rho_s(\pi_k(x_0), t(p))^{-1} = \rho_s(\pi_k(x_0), \iota(t_k)) \rho_s(\pi_k(x_0), t(p)) \rho_s(\pi_k(x_0), t(p))^{-1} = \rho_s(\pi_k(x_0), \iota(t_k)).$$

The set $\{ \pi_k(p) x_0 | p_i \neq 0 \forall i \neq k \}$ is dense in the hyperplane $z_k = 0$. Hence by holomorphicity of $\rho_s(w, \iota(t_k))$ in $w$, we obtain $\rho_s(w, \iota(t_k)) = \rho_s(\pi_k(x_0), \iota(t_k))$ for every $w$ in
the hyperplane $z_k = 0$. Therefore $\rho_s(w, \iota(t_k))$ is constant for $w \in \{z_k = 0\}$ and equals $\rho_s(0, \iota(t_k))$. In particular,

$$\rho_s(\pi_k(p)x_0, \iota(t_k)) = \rho_s(0, \iota(t_k)) \quad \text{for any } p \text{ in } P_k,$$

since $\pi_k(p)x_0 \in \{z_k = 0\}$.

Fix $p_i$ for each $i \neq k$, and define

$$f(p_k) = F(p_1, \ldots, p_k, \ldots, p_n).$$

Then setting $t = \iota(t_k)$ in (3.13), we get

$$\lim_{p_k \to 0} f(t_k p_k) f(p_k)^{-1} = \rho_s(\pi_k(p)x_0, \iota(t_k)) \rho_s(0, \iota(t_k))^{-1}.$$

Therefore by (3.15),

$$\lim_{p_k \to 0} f(t_k p_k) f(p_k)^{-1} = I$$

for any $t_k \in \mathbb{C}^*$. Therefore Proposition 3.1 applies to $f$. Hence

$$\lim_{p_k \to 0} F(p_1, \ldots, p_n) = \lim_{p_k \to 0} f(p_k)$$

exists and takes value in $G$. As this argument works for every $p \in P_k$, by the Riemann extension theorem, $F$ has a unique $G$-valued extension over $P_{k+1} = \mathbb{C}^n - \cup_{i=k+1}^n \{z_i = 0\}$. Therefore, by induction, $F$ admits a unique $G$-valued holomorphic extension over $\mathbb{C}^n$.

Now define a new section $s'$ of $\mathcal{E}$ by

$$s'(zx_0) = s(x_0) \cdot F(z)$$

for every $z \in \mathbb{C}^n$. Note that $F(I) = I$ as $\rho_s(x_0, I) = I = \rho_s(0, I)$. Then

$$ts'(x_0) = t(s(x_0) \cdot F(I)) = ts(x_0) = s(tx_0) \cdot \rho_s(x_0, t).$$

On the other hand,

$$ts'(x_0) = s'(tx_0) \cdot \rho_{s'}(x_0, t) = s(tx_0) \cdot F(t) \rho_{s'}(x_0, t).$$

Therefore, using (3.10), we have

$$\rho_{s'}(x_0, t) = F(t)^{-1} \rho_s(x_0, t) = F(t)^{-1} \rho_s(0, t) F(t) = \rho_s(0, t).$$

Therefore $\rho_{s'}(x_0, t)$ is a homomorphism and the theorem follows from Lemma 2.4 \qed

**REMARK 3.3.** Note that the homomorphism $\rho_s(0, \cdot)$ is independent of the choice of the section $s$ by (2.2) when $G$ is Abelian.

**Lemma 3.4.** Let $G$ be a closed Abelian subgroup of $GL_k(\mathbb{C})$. Suppose $\sigma$ is any cone of an $n$-dimensional nonsingular fan $\Xi$ and $\mathcal{E}$ a $T$-equivariant holomorphic principal $G$-bundle on the affine toric variety $X_\sigma$. Then $\mathcal{E}$ is trivial and admits a section $s^*$ for which the local action function $\rho_{s^*}$ is a homomorphism. Moreover, there exists a canonical homomorphism $\rho_\sigma : T_\sigma \to G$ and a choice of $s^*$ such that $\rho_{s^*}(t) = \rho_\sigma(\pi_\sigma(t))$, where $\pi_\sigma : T \to T_\sigma$ is the projection associated to the decomposition $T \cong T_\sigma \times O_\sigma$.

**Proof.** Let $d$ denote the dimension of the cone $\sigma$. Note that there exists an isomorphism $\alpha : X_\sigma \to \mathbb{C}^d \times (\mathbb{C}^*)^{n-d}$ where the latter space has the standard $T$-action, such that $\alpha$ is equivariant up to an automorphism $a_\sigma$ of $T$:

$$\alpha(tx) = a_\sigma(t) \alpha(x).$$

Define $H = a_\sigma(T_\sigma)$ and $K = a_\sigma(O_\sigma)$. Note $H \cong (\mathbb{C}^*)^d$ and $K \cong (\mathbb{C}^*)^{n-d}$. 

Let \( F \) be the pull-back of \( E \) with respect to \( \alpha^{-1} \). Let \( \phi : E \to F \) be the natural isomorphism. Note that \( F \) inherits natural actions of \( T \) and \( G \) that satisfy
\[
\phi(te) = a_\sigma(t)\phi(e), \quad \phi(e \cdot g) = \phi(e) \cdot g
\]
for every \( e \in E \). This makes \( F \) a \( T \)-equivariant principal \( G \)-bundle over \( \mathbb{C}^d \times (\mathbb{C}^*)^{n-d} \).

Fix a point \( y_0 \) in \( (\mathbb{C}^*)^{n-d} \). Then by Oka-Grauert theory \( F \) is trivial on \( \mathbb{C}^d \times \{y_0\} \). Let \( s \) be a section of this restricted bundle. We extend \( s \) to a section of \( F \) over \( \mathbb{C}^d \times (\mathbb{C}^*)^{n-d} \) by defining
\[
(3.22) \quad s(x, ky_0) = k s(x, y_0)
\]
for every \( x \in \mathbb{C}^d \) and \( k \in K \). This shows that \( F \), and consequently \( E \), is trivial.

By Theorem 3.2 and Remark 3.3 we may assume that the local action function of the section \( s \) over \( \mathbb{C}^d \times \{y_0\} \) satisfies
\[
(3.23) \quad \rho_s((x, y_0), h) = \rho_s((0, y_0), h)
\]
for all \( h \in H \), and defines a homomorphism \( H \to G \) that is independent of \( s \).

Since
\[
(3.24) \quad h s(x, ky_0) = h k s(x, y_0) = k h s(x, y_0)
\]
we deduce that
\[
(3.25) \quad \rho_s((x, ky_0), h) = \rho_s((x, y_0), h)
\]
for every \( k \in K \). This shows that the homomorphism \( \rho_s : H \to G \) is independent of the choice of \( y_0 \) as well. Recall that \( O_\sigma \cong (\mathbb{C}^*)^{n-d} \) and \( a_\sigma(T_\sigma) = H \). We define \( \rho_\sigma : T_\sigma \to G \) by
\[
(3.26) \quad \rho_\sigma(t) = \rho_s((0, y_0), a_\sigma(t)).
\]

It follows easily from (3.22) that \( k s(x, y) = s(x, ky) \) for any \( y \in (\mathbb{C}^*)^{n-d} \). Then,
\[
(3.27) \quad h s(x, ky) = h s(x, y) = s(h x, ky) \cdot \rho_s((x, ky), h) = s(h x, ky) \cdot \rho_s((0, y_0), h)
\]
for every \( (h, k) \in H \times K \) and \( (x, y) \in \mathbb{C}^d \times (\mathbb{C}^*)^{n-d} \). Therefore,
\[
(3.28) \quad \rho_s((x, y), hk) = \rho_s((0, y_0), h).
\]

Then \( \rho_s((x, y), \cdot) \) is a homomorphism as \( \rho_s((0, y_0), \cdot) \) is so.

We set \( s^* = \phi^{-1}(s) \). Then by Lemma 2.3 and (3.28),
\[
(3.29) \quad \rho_{s^*}(x, t) = \rho_s((\alpha(x), a_\sigma(t)) = \rho_s((0, y_0), pr(a_\sigma(t))'),
\]
where \( pr : T \to H \) is the projection corresponding to the decomposition \( T = H \times K \). Then by definition of \( H \) and \( K \) we have \( pr \circ a_\sigma = a_\sigma \circ \pi_\sigma \). Therefore from (3.29) and (3.26) we have
\[
\rho_{s^*}(x, t) = \rho_s((0, y_0), a_\sigma(\pi_\sigma(t))) = \rho_\sigma(\pi_\sigma(t)).
\]
4. Gluing condition

The main idea of this section is borrowed from [5].

Let $X$ be a nonsingular toric variety of dimension $n$ corresponding to a fan $\Xi$. Suppose $E$ is a $T$-equivariant principal $G$-bundle over $X$ where $G$ is a closed Abelian subgroup of $GL_k(\mathbb{C})$.

Let $\sigma$ be any maximal cone in $\Xi$. Let $\tilde{\rho}_\sigma = \rho_\sigma \circ \pi_\sigma : T \to G$ where $\rho_\sigma : T_\sigma \to G$ is the homomorphism obtained by applying Lemma 3.4 to the bundle $E_\sigma := E|_{X_\sigma}$. Let $s_\sigma$ a section of $E_\sigma$ whose local action homomorphism is $\tilde{\rho}_\sigma$.

Let $\psi_\sigma : E_\sigma \to X_\sigma \times G$ be the trivialization induced by the section $s_\sigma$,

$$\psi_\sigma(s_\sigma(x) \cdot h) = (x, h).$$

Note that

$$\psi_\sigma(ts_\sigma(x) \cdot h) = (tx, \tilde{\rho}_\sigma(t)h).$$

So the $T$ action on the trivialization $X_\sigma \times G$ is defined by

$$t(x, h) = (tx, \tilde{\rho}_\sigma(t)h).$$

Let $\sigma$, $\tau$ be any two maximal cones. Let $\phi_{\tau\sigma} : X_\sigma \cap X_\tau \to G$ denote the transition function defined as follows,

$$\psi_\tau\psi_\sigma^{-1}(x, h) = (x, \phi_{\tau\sigma}(x)h).$$

By equivariance, we have

$$t(\psi_\tau\psi_\sigma^{-1}(x, h)) = \psi_\tau\psi_\sigma^{-1}(t(x, h)).$$

This implies that

$$(tx, \tilde{\rho}_\tau(t)\phi_{\tau\sigma}(x)h) = (tx, \phi_{\tau\sigma}(tx)\tilde{\rho}_\sigma(t)h).$$

Therefore,

$$\phi_{\tau\sigma}(tx) = \tilde{\rho}_\tau(t)\phi_{\tau\sigma}(x)\tilde{\rho}_\sigma(t)^{-1} = \phi_{\tau\sigma}(x)\tilde{\rho}_\tau(t)\tilde{\rho}_\sigma(t)^{-1}.$$ (4.1)

Consider the point $x_0 = (1, \ldots, 1)$ in the principal $T$-orbit $O$ of $X$. For any maximal cones $\tau$ and $\sigma$, we have

$$\tilde{\rho}_\tau(t)\tilde{\rho}_\sigma(t)^{-1} = \phi_{\tau\sigma}(x_0)^{-1}\phi_{\tau\sigma}(tx_0).$$

Therefore the function $\tilde{\rho}_\tau\tilde{\rho}_\sigma^{-1} : T \to G$ admits a holomorphic extension to a function from $X_\tau \cap X_\sigma$ to $G$, namely $\phi_{\tau\sigma}(x_0)^{-1}\phi_{\tau\sigma}(\cdot)$.

**Theorem 4.1.** Let $X$ be an $n$-dimensional nonsingular toric variety with fan $\Xi$. Let $G$ be a closed Abelian subgroup of $GL_k(\mathbb{C})$. Then the isomorphism classes of $T$-equivariant holomorphic principal $G$-bundles on $X$ are in one-to-one correspondence with collections of holomorphic group homomorphisms $\{\rho_\sigma : T_\sigma \to G | \sigma$ is a maximal cone of $\Xi\}$ which satisfy the extension condition: Each $(\rho_\tau \circ \pi_\tau)(\rho_\sigma \circ \pi_\sigma)^{-1}$ extends to a $G$-valued holomorphic function over $X_\sigma \cap X_\tau$.

**Proof.** Given a $T$-equivariant principal $G$-bundle $E$ on $X$, we have a canonical collection of homomorphisms $\{\rho_\sigma : T_\sigma \to G\}$ by Lemma 3.4. We have shown above that this collection satisfies the extension condition. Moreover, the collection of homomorphisms is invariant under an isomorphism of the bundle by Lemma 2.3.

Conversely, given a collection of homomorphisms $\{\rho_\sigma\}$ satisfying the extension condition, define $\tilde{\rho}_\sigma = \rho_\sigma \circ \pi_\sigma$. Let $\phi_{\tau\sigma} : X_\sigma \cap X_\tau \to G$ denote the extension of $\tilde{\rho}_\tau\tilde{\rho}_\sigma^{-1}$. Note that
\{\phi_{\tau\sigma}\} satisfies the cocycle condition. Therefore we may construct a principal $G$-bundle $E$ over $X$ with $\{\phi_{\tau\sigma}\}$ as transition functions,

$$E = (\bigcup_{\sigma} X_{\sigma} \times G)/\sim$$

where $(x, g) \sim (y, h)$ for $(x, g) \in X_{\sigma} \times G$ and $(y, h) \in X_{\tau} \times G$ if and only if

$$x = y, \ x \in X_{\sigma} \cap X_{\tau} \text{ and } h = \phi_{\tau\sigma}(x)g.$$ (4.2)

Define $T$ action on each $X_{\sigma} \times G$ by $t(x, g) = (tx, \tilde{\rho}_\sigma(t)g)$. Then note that if $(y, h) \in X_{\tau} \times G$ is equivalent to $(x, g) \in X_{\sigma} \times G$, then

$$t(y, h) = t(x, \phi_{\tau\sigma}(x)g) = (tx, \tilde{\rho}_\tau(t)\phi_{\tau\sigma}(x)g).$$ (4.3)

Now if $x$ belongs to the open orbit $O = T \subset X_{\sigma} \cap X_{\tau}$, then

$$\phi_{\tau\sigma}(tx)\tilde{\rho}_\sigma(t) = \tilde{\rho}_\tau(tx)\tilde{\rho}_\sigma(tx)^{-1}\tilde{\rho}_\sigma(t) = \tilde{\rho}_\tau(t)\tilde{\rho}_\tau(t)\tilde{\rho}_\sigma(x)\tilde{\rho}_\sigma(x)^{-1} = \tilde{\rho}_\tau(t)\phi_{\tau\sigma}(x).$$ (4.4)

Since both $\phi_{\tau\sigma}(tx)\tilde{\rho}_\sigma(t)$ and $\tilde{\rho}_\tau(t)\phi_{\tau\sigma}(x)$ are continuous in $x$ on $X_{\sigma} \cap X_{\tau}$ and $O$ is dense in $X_{\sigma} \cap X_{\tau}$,

$$\phi_{\tau\sigma}(tx)\tilde{\rho}_\sigma(t) = \tilde{\rho}_\tau(t)\phi_{\tau\sigma}(x) \text{ for all } x \in X_{\sigma} \cap X_{\tau}.$$ (4.5)

From (4.3) and (4.5), we have

$$t(y, h) = (tx, \phi_{\tau\sigma}(tx)\tilde{\rho}_\sigma(t)g)$$

whenever $(x, g) \sim (y, h)$. Since $t(x, g) = (tx, \tilde{\rho}_\sigma(t)g)$, by (4.2) this implies that $t(y, h) \sim t(x, g)$ whenever $(x, g) \sim (y, h)$. In other words, the $T$-actions on the $X_{\sigma} \times G$ are compatible and define an action on $E$. It is obvious that $\{\rho_{\sigma}\}$ are the local homomorphisms associated to $E$. \hfill \Box

**Theorem 4.2.** If $G$ is a discrete group, then any holomorphic $T$-equivariant principal $G$-bundle $E$ over a nonsingular toric variety is trivial with trivial $T$-action.

**Proof.** By using Lemma 2.5 and mimicking the proof of Lemma 3.4, we obtain that $E$ is trivial over any nonsingular affine toric variety $X_{\sigma}$. We also obtain canonical homomorphisms $\rho_{\sigma} : T_{\sigma} \rightarrow G$. Then, we mimic the proof of Theorem 4.1 and obtain an analogous result. Note that a holomorphic homomorphism from any $T_{\sigma}$ to the discrete group $G$ must be trivial. The result follows. \hfill \Box

### 4.1. Examples

Every complete nonsingular toric variety $X$ admits an equivariant principal $G$-bundle: Let $\rho : T \rightarrow G$ be a homomorphism. Set $\rho_{\sigma} = \rho$ for every $\sigma \in \Xi(n)$. Then the extension condition is satisfied as $\rho_{\tau}\rho_{\sigma}^{-1}$ is the identity map.

However $X$ may admit more equivariant principal $G$-bundles. For instance, consider $X = \mathbb{CP}^2$ and let $G \leq GL_2(\mathbb{C})$ be the subgroup of lower triangular matrices with equal diagonal entries,

$$G = \left\{ \begin{pmatrix} \mu & 0 \\ g & \mu \end{pmatrix} : g, \mu \in \mathbb{C} \right\}$$

First we verify that any holomorphic homomorphism $\mathbb{C}^* \rightarrow G$ has the form $\mu(t) = t^a$, $g(t) = 0$, where $a \in \mathbb{Z}$. By the defining property of a homomorphism we have $\mu(t_1t_2) = \mu(t_1)\mu(t_2)$ so that $\mu(t) = t^a$ for some $a \in \mathbb{Z}$. However we also have

$$g(t_1t_2) = g(t_1)\mu(t_2) + \mu(t_1)g(t_2).$$ (4.6)
Differentiating (4.6) with respect to $t_1$ holding $t_2$ constant, we get

\[(4.7)\quad g'(t_1 t_2) t_2 = g'(t_1) \mu(t_2) + \mu'(t_1) g(t_2).\]

Setting $t_2 = 1$ in (4.7) and simplifying, we have

\[(4.8)\quad \mu'(t_1) g(1) = 0.\]

If $\mu'(t_1) = 0$, then $\mu(t) = 1$ for all $t$. Then from (4.6) we have $g(t_1 t_2) = g(t_1) + g(t_2)$. This implies that $g(t) = c \ln(t)$ and by holomorphicity of $g$ on $\mathbb{C}^*$, $c$ must be zero. Therefore in this case $g(t) = 0$.

If $\mu'(t_1) \neq 0$, then $g(1) = 0$. Setting $t_1 = 1, t = t_2$ and using $\mu(t) = t^n$ in (4.7), we get

\[(4.9)\quad g'(t) t = g'(1) t^n + a g(t).\]

Solving this differential equation, and using $g(1) = 0$, we obtain $g(t) = g'(1) t^n \ln(t)$. For $g$ to be holomorphic on $\mathbb{C}^*$, we must have $g'(1) = 0$ and $g(t) = 0$.

It follows that any holomorphic homomorphism $\rho : T = (\mathbb{C}^*)^2 \to G$ satisfies $\mu(t_1, t_2) = t_1^a t_2^b$ and $g(t_1, t_2) = 0$, where $a, b \in \mathbb{Z}$.

Let $e_1 = (1, 0), e_2 = (0, 1)$ and $e_3 = (-1, -1)$. Let $\sigma_i, 1 \leq i \leq 3$, be the top dimensional cone in the fan of $\mathbb{CP}^2$ generated by $\{e_j | j \neq i\}$. Define homomorphisms $\rho_{\sigma_i} : T \to G$ by

$$\mu(t_1, t_2) = t_1^{a_i} t_2^{b_i}, \quad g(t_1, t_2) = 0$$

where $a_i, b_i \in \mathbb{Z}$.

Note that $X_{\sigma_2} \cap X_{\sigma_3} = \text{Spec} \mathbb{C}[t_1, t_2, t_2^{-1}]$. Applying the extension condition to $\rho_{\sigma_2} \rho_{\sigma_3}^{-1}$ and its inverse, we obtain $a_2 - a_3 = 0$. Similarly, considering the extension condition on $X_{\sigma_1} \cap X_{\sigma_3}$ and $X_{\sigma_1} \cap X_{\sigma_2}$, we arrive at the conditions $b_1 - b_3 = 0$ and $a_1 - a_2 + b_1 - b_2 = 0$ respectively. Therefore the set of isomorphism classes of $T$-equivariant principal $G$-bundles on $\mathbb{CP}^2$ is parametrized by $\mathbb{Z}^3$.

5. Algebraic case

Suppose that $E$ is a $T$-equivariant algebraic principal $G$-bundle over $\mathbb{C}^n$ where $G$ is a closed subgroup of $GL_k(\mathbb{C})$. It follows from [7] that $E$ is trivial. Then the local action function corresponding to a section $s$ of $E$ is a regular map $\rho_s : \mathbb{C}^n \times T \to G$. Therefore, the function $F : T \to G$ of (4.9) is regular.

As $G$ is closed in the Zariski topology, it is closed in the analytic topology. So, the proof of Theorem 3.2 shows that $F$ admits a holomorphic extension $F : \mathbb{C}^n \to G$. A priori $F$ is represented by $k \times k$ matrix $A$ with entries in the ring $\mathbb{C}[z_i, z_i^{-1}, 1 \leq i \leq n]$. But as $F$ is holomorphic the entries of $A$ are also convergent power series in $z_1, \ldots, z_n$. Therefore the entries must be polynomials. Hence the extension $F : \mathbb{C}^n \to G$ is regular. This yields an algebraic analogue of Theorem 3.2 which in turn leads to an algebraic analogue of Lemma 3.4. Then the same proof as in the holomorphic case gives the following result.

\[\textbf{Theorem 5.1}. \textit{Let } X \textit{ be an } n \text{-dimensional nonsingular toric variety with fan } \Xi. \textit{Let } G \textit{ be a closed Abelian subgroup of } GL_k(\mathbb{C}). \textit{Then the isomorphism classes of } T \text{-equivariant algebraic principal } G \text{-bundles on } X \text{ are in one-to-one correspondence with collections of algebraic group homomorphisms } \{\rho_\sigma : T_\sigma \to G | \sigma \text{ is a maximal cone of } \Xi\} \text{ which satisfy the extension condition: Each } (\rho_\sigma \circ \pi_\sigma)(\rho_\sigma \circ \pi_\sigma)^{-1} \text{ extends to a } G \text{-valued regular function over } X_\sigma \cap X_\tau.\]
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