A THERMODYNAMIC STUDY OF THE TWO-DIMENSIONAL PRESSURE-DRIVEN CHANNEL FLOW

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Dedicated to Professor Peter Lax on the occasion of his 90th birthday.

Abstract. The instability of the two-dimensional Poiseuille flow in a long channel and the subsequent transition is studied using a thermodynamic approach. The idea is to view the transition process as an initial value problem with the initial condition being Poiseuille flow plus noise, which is considered as our ensemble. Using the mean energy of the velocity fluctuation and the skin friction coefficient as the macrostate variable, we analyze the transition process triggered by the initial noises with different amplitudes. A first order transition is observed at the critical Reynolds number $Re^* \sim 5772$ in the limit of zero noise. An action function, which relates the mean energy with the noise amplitude, is defined and computed. The action function depends only on the Reynolds number, and represents the cost for the noise to trigger a transition from the laminar flow. The correlation function of the spatial structure is analyzed.

1. Introduction. One of the most studied problems in fluid mechanics is the instability of shear flows and the subsequent transition to turbulence. Beginning with the pioneering work of Reynolds in 1883 [26], a huge amount of efforts, both theoretical and experimental, have been devoted to this topic. From the theoretical viewpoint, past work has been concentrated in two directions:

1. Linear stability analysis. Classical normal mode analysis started with the work of Rayleigh, Orr and Sommerfeld and has been associated with such names as Heisenberg and C. C. Lin. By now this has become an important chapter both in theoretical fluid dynamics and asymptotic methods. The most important result is the identification of a critical Reynolds number, $Re^* \sim 5772$, at which the plane Poiseuille flow becomes linearly unstable. However, this is

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also the most disturbing piece of information, for two reasons. One is that experimentally, instability and transition to turbulence can be triggered at much lower Reynolds number. A variety of secondary flows in the form of traveling waves have been found numerically for Reynolds numbers $2700 < Re < Re^*$. More importantly, it is widely believed that a similar normal mode analysis for the 3D pipe flow would yield a critical Reynolds number to be infinity [31, 32], although there is only indirect and partial evidence at this point. This is in contrast to the experimental results of Reynolds which clearly showed instability and transition to turbulence.

The most plausible explanation of this discrepancy is that the experimentally observed instability is a nonlinear phenomenon. The question is then: How do we analyze such a nonlinear instability? Up until now, this question has been addressed mainly from the viewpoint of dynamical systems.

2. Dynamical systems approach. One of the most important objects in dynamical systems theory is the invariant sets. They are the building blocks for the long time behavior of general solutions. For channel flow, the first nontrivial invariant solution was found by Nagata for plane Couette flow using homotopy methods starting from the rotating plane Couette flow [21]. Similar unstable 3D traveling waves were discovered in plane Poiseuille flow [35], and in pipe flow [9, 37]. The traveling wave solutions exhibit large scale features of vortices and streaks that resemble coherent structures observed in turbulence. In addition, various bifurcations involving more complicated solutions such as periodic orbits were identified, which form the hierarchy of invariant solutions in state space [15, 19, 38].

However, all these invariant solutions are unstable and can only appear as transient behavior in 3D flows. Indeed it was revealed in a variety of experiments and numerical simulations that the life time of turbulence is exponentially distributed in pipe flow [12] and plane Couette flow [16, 30]. This can be qualitatively explained by viewing the turbulent state as a chaotic saddle or a strange saddle in state space (for instance, see a review article by Eckhardt et al. [8]). In this regard, the most important advance has been accomplished in the recent experimental works by Avila et al. [2]: By tracking a single puff, the decay and splitting times for the puffs were measured. A critical Reynolds number of about 2040 was identified as the balance point between the two opposite processes, namely the decay and splitting of puffs.

In this series of papers, we propose a thermodynamic formalism for dealing with problems of this type, namely, problems having nonlinear instabilities. The main ingredients of the this formalism are:

1. We take the infinite volume limit and view the problem as one of phase transition.
2. We define coarse-grained or macroscopic variables and the associated action or free energy.
3. The statistical ensemble is defined by the probability distribution for the random initial condition. The dynamics stays deterministic. Consequently, the dynamics is far from being ergodic. In fact, the different phases are represented by the different domain of attraction for different invariant sets of the dynamics.
A natural question is whether a thermodynamic formalism exists for such a setting, e.g. whether the thermodynamic limit exists. While there are no general theorems about this, there are indeed examples of the Ising type, which suggest that the limit is well-defined and the various thermodynamic quantities do converge in the limit [34].

When applying the thermodynamic formalism to study the channel flow driven by the pressure gradient, we note that there are at least three phases: The Poiseuille flow, the localized complex states (localized wave packets) with Poiseuille flow as the background and the fully developed states. We will refer the localized complex states as puffs in analogy with the localized turbulent states in pipe flow. In a companion paper [34], we focused on the transition from the Poiseuille flow to the puff states. In this paper, we focus on the second transition from the puff states to fully developed states.

We note that studying subcritical transitions of fluid dynamics in a thermodynamic setting is not a new idea [1, 18]. Back in 1986, Pomeau suggested an analogy between laminar-turbulent transition in the shear flow and the nucleation phenomena in first order phase transitions [25]. This was pursued in some detail by Manneville for the plane Couette flow using a simplified model [17]. Subsequent work by Schneider et al. and Shi et al. seems to point to a contradictory picture: On one hand, direct numerical simulation by Schneider et al. further confirms that the growing turbulent spots in the plane Couette flow does act like the critical nuclei in spatially extended domains [28]. On the other hand, it was pointed out recently that the transition to turbulence in the plane Couette flow behaves more like a second order phase transition, analogues to directed percolation [30]. For pipe flow, Barkley proposed a simplified one-dimensional model, and argued that the phase transition in pipe flow closely resembles the second-order phase transition in directed percolation [4, 5]. Our contribution is to develop a full thermodynamic formalism as well as the quantitative tools needed.

This paper is organized as follows. In the next section, we briefly review the existing literature. In section 3, we present our setting. In sections 4 and 5, we describe results on the thermodynamic quantities, namely the average energy, the skin friction coefficient and the action function. In section 6, we present the results on the statistics and structures of the traveling waves and turbulent structure. Some conclusions are drawn in section 7.

2. Review of the current literature. As mentioned above, some interesting secondary flows have been discovered for the 2D channel flow. Specifically, there is a branch of stable periodic traveling wave solutions from a Hopf bifurcation of a finite amplitude traveling wave at \( Re \approx 2939 \) and channel length \( L = 2\pi/1.32 \) [6] (assuming the width of the channel to be 2). In a longer channel with length \( L = 2\pi/0.15 \), this nontrivial solution was observed at a smaller Reynolds number of \( Re \approx 2750 \) [27]. Direct numerical simulations revealed that the wave train becomes unstable at \( Re \approx 5600 \), and undergoes bifurcations into more disordered states at larger \( Re \) [13, 14]. In addition to traveling wave solutions, stability analysis on several groups of quasi-periodic orbits that bifurcate from traveling waves has been carried out [6].

More recently, stochastic Navier-Stokes equations in a short channel has been studied [33]. A small Gaussian noise was introduced into the forcing term and the long time stochastic dynamics has been considered. The most probable transition
path between the Poiseuille flow and the non-trivial solutions has been calculated using an extension of the Freidlin-Wentzell large deviation theory. As a consequence, one can define a critical Reynolds number for the relative instability of two locally stable solutions for this stochastic system.

For three-dimensional channel flows, a secondary two-dimensional traveling wave solution was discovered for Reynolds number $Re \approx 2900$, and was used to explain the experimentally observed transition to turbulence at much smaller Reynolds number ($Re \approx 1000$) [11, 23]. Unstable 3D traveling wave solutions that arises from saddle-node bifurcations were discovered at $Re < 1000$ [22, 35, 36]. Additionally, there are localised coherent structures that bifurcate from extended traveling waves [39], though these coherent structures are usually linearly unstable. In another direction, edge states located at the boundary of the laminar-turbulent basins have been investigated [40]. This work has revealed some common features shared by general shear flow regardless of specific details of flow type.

For the 3D pipe flow, several families of nontrivial traveling wave solutions with symmetric arrangement of vortices were found in short pipes when $Re \approx 1250$ [9, 37]. Later, traveling waves with asymmetric states were found at $Re \approx 770$ [24]. More recently, the streamwise-localised invariant solutions have been discovered and were found to be responsible for inducing some chaotic motion [3]. In addition, the connection between the spatially localised solutions and the many known streamwise-periodic solutions was addressed, and it was shown that the localized solutions were originated from a Hopf bifurcation from some global traveling wave solutions [7].

In a long pipe, Moxey and Barkley observed that spatially localized puffs dominate the transitional pipe flow for $Re < 2300$, while for $Re > 2300$ a spatio-temporal intermittent flow structure develops as a result of the spatial growth process caused by puff-splitting [20]. The fraction of turbulence grows as the Reynolds number is increased, until a uniformly turbulent state is established for $Re > 2800$. It is widely accepted now that both puff decaying and splitting processes are approximately memoryless, and their time scales have approximate exponential distributions [2]. Based on this information, Barkley proposed a one-dimensional model that seems to capture many observed flow patterns in the pipe flow, including metastable localized puffs, puff splitting and slugs [4, 5]. Numerical simulation reveals that phase transition in this model closely resembles the second-order phase transition in directed percolation.

3. Setting and the numerical method. We consider the 2D shear flow in a long channel driven by a pressure gradient with random initial condition. The governing equations of the flow, in dimensionless form, are given by:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + f,
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right),
\]

where, $x$ and $y$ are streamwise and transverse coordinates respectively. $u$ is the streamwise velocity component, $v$ is the transverse velocity component, $p$ stands for the pressure, and $Re$ denotes the Reynolds number. The boundary conditions are: $u = 0$ and $v = 0$ at the two parallel walls $y = \pm h$, $h = 1$, and periodic in the
streamwise direction, namely, \( u(x, y, t) = u(x + L, y, t) \), \( v(x, y, t) = v(x + L, y, t) \), and \( p(x, y, t) = p(x + L, y, t) \), where \( L \) is the length of the channel.

We use a constant driving force \( f = 2/Re \) to represent the pressure gradient. Note that some researchers have used a constant mass flux \( Q = 4/3 \) to control the flow, where \( Q = \int_0^L u \, dy \). At “statistical” steady states, pressure gradient and mass flux are nearly constant in time. Hence, both conditions are equivalent. We can also define two different Reynolds numbers. For the constant flux condition, \( Re_Q = 3Q/4\nu \); for the constant pressure gradient condition, \( Re_p = -h^3(\partial P/\partial x)/2\nu^2 \), where \( \partial P/\partial x \) is the mean pressure gradient and \( \nu \) is the kinematic viscosity coefficient of the fluid [14, 27]. In our numerical simulations, \( Re = Re_p \).

The laminar Poiseuille flow is given by: \( u_0(y) = 1 - y^2 \), and \( v_0 = 0 \). In this situation, the two Reynolds numbers defined above are identical: \( Re = Re_Q = Re_p \).

In the general situation, we decompose the velocity field into:

\[
\begin{align*}
\alpha &= \text{kinematic viscosity coefficient of the fluid} \\
\beta &= \text{constant flux condition, where} \beta \text{ and } \nu \text{ are constant} \text{,} \\
\gamma &= \text{the mean pressure gradient and } \nu \text{ is the kinematic viscosity} \text{ coefficient of the fluid} \\
\theta &= \text{kinematic viscosity coefficient of the fluid} \\
\phi &= \text{constant flux condition, where} \phi \text{ and } \nu \text{ are constant} \text{.}
\end{align*}
\]

For each parameter pair of \((Re, \epsilon)\), 200 initial conditions are simulated. Four different noise amplitudes \( \epsilon = 0.04, 0.08, 0.1, 0.14 \) have been used.

The field \((u^G, v^G)\) is numerically constructed by the following steps: Denote by \( \{x_i, y_j\} \) the set of collocation points. (1) Generate statistically independent random variables with normal distribution \( p_N(z) = \exp(-z^2/2)/\sqrt{2\pi} \) at the collocation
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Figure 1. 2D profiles of velocity and streamlines at $Re = 3000$.

points $\{(x_i, y_j)\}$, denoted these random variables by $\{(u_{i,j}^{G0}, v_{i,j}^{G0})\}$. (2) Project the field $(u_{i,j}^{G0}, v_{i,j}^{G0})$ into the space of divergence vector fields. The field after projection is denoted as $(u_{i,j}^{G}, v_{i,j}^{G})$. We note that this kind of noise is quite specific and the results will quantitatively depend on the choice of noise. For example, adding a small spatial correlation to the noise will lead to a larger deviation of transitional flow from the laminar state for the same noise amplitude. Though initially there is a high energy level at high wavenumbers, viscosity effect will damp out small scale noise rapidly. We expect that qualitative features of transitional flow are not affected by the choice of noise as was pointed out in a previous work [34].

Direct numerical simulation reveals the following general scenario: With perturbed random initial condition, the fluctuations in the background Poiseuille flow grow and a new statistical steady state is reached after several thousand dimensionless time units. We plot 2D profiles of velocity and streamlines in a realization at $Re = 3000$ in figure 1. We observe three puffs (localized wave packets) with characteristic length scale of 20 in the interval $[0,150]$.

4. Behavior of the order parameters. We define two macrostate variables. The first is the skin friction coefficient [27], defined by

$$C_f = -\frac{\langle \partial_x p \rangle_{xyt} h}{\frac{1}{2} \rho_0 U^2},$$

(14)

where $U$ is the mean streamwise velocity. $\langle \cdot \rangle_{xyt}$ denotes average over space and time. The second is the average energy of the velocity fluctuation, $\langle q \rangle = \langle q(x, t) \rangle_{xt}$, where
$q(x,t)$ is the local average energy: $q(x,t) = \frac{1}{2} \int_{-1}^{1} \frac{1}{2} (u^2 + v^2) \, dy$, $u_1$ is the deviation of the streamwise velocity from the Poiseuille flow, and $\langle \rangle_x$ denotes average over $x$ and $t$.

The skin friction coefficient at different Reynolds numbers is depicted in figure 2. The dashed line denotes the result for the Poiseuille flow, $C_f = 9/Re$. If the Reynolds number is large enough, there is a threshold of noise amplitude $\epsilon_t(Re)$, above which the transitional flow is saturated, i.e. the statistical properties of the flow do not depend on $\epsilon$ as long as $\epsilon > \epsilon_t(Re)$. The dash-dotted line in figure 2 is the result for the “saturated flow”. Numerically, we obtain this state by tracking the solution of $(Re, \epsilon) = (4800, 0.14)$ continuously while decreasing $Re$. We will report about $\epsilon_t(Re)$ below. For a given $\epsilon$, the discrepancy relative to the laminar flow increases with the Reynolds number. All $C_f - Re$ curves collapse to the same line at large Reynolds numbers. As the noise amplitude decreases, the $C_f - Re$ curve changes rather abruptly from the laminar state to the saturated transition state, at large Reynolds numbers. We further suspect that, as $\epsilon$ decreases to zero, there is a jump of the $C_f - Re$ curve at $Re_\ast = 5772$ where $\epsilon_t(Re_\ast) = 0$, signaling a first order transition. The reason is as follows: for any $Re$ below $Re_\ast$, the laminar flow is linearly stable, and the probability for the noise-perturbed flow to fall in the basin of attraction of the laminar flow increases to 1 as the noise amplitude decreases to zero; for $Re$ above $Re_\ast$, the laminar flow is linearly unstable, and the probability of transition to the saturated state is always 1 no matter how small the noise amplitude is. Similar features are observed for the $\langle q \rangle - Re$ curves, as shown in figure 3. All the curves lie between the laminar state of $\langle q \rangle = 0$, and the saturated state.

In a previous study, we identified a critical Reynolds number $Re_c = 2332$ above which the transition occurs in the thermodynamic limit [34]. For the value of $L (= 400)$ considered here, the critical Reynolds number $Re_{c,L} = 2341$ is slightly larger than $Re_c = 2332$ [34]. As long as $Re > Re_{c,L}$, the noise-induced transitional flow deviates from the Poiseuille flow, even though the difference of the order parameter between the transitional flow and the Poiseuille flow is negligibly small for $Re_{c,L} < Re \leq 2800$ and $\epsilon \leq 0.14$.

To define $\epsilon_t(Re)$ more precisely, we look at the normalized average energy by $q_n = \langle q \rangle / q_\ast$, where $q_\ast$ is the value of order parameter $\langle q \rangle$ in the saturated flow. For each $Re$, we can find a particular noise amplitude for which $q_n = 0.95$. This will be defined as the threshold of noise amplitude $\epsilon_t(Re)$. The numerically estimated noise threshold is displayed in figure 4 (a), for several Reynolds numbers between 4000 and 5700. In addition, we display the relation between normalized average energy $q_n$ and the normalized noise amplitude $\epsilon / \epsilon_t$ in figure 4 (b). All curves nearly overlap, revealing a uniform dependence of the order parameter on the noise amplitude in the range $4400 \leq Re \leq 5700$.

5. The action function. To further explore the behavior of the order parameter as the noise amplitude decreases to zero, we show in figure 5 (a)(b)(c) the behavior of $-\log(\langle q \rangle)$ as a function of $1/\epsilon^2$ for several Reynolds numbers. At a given $Re$, the quantity $-\log(\langle q \rangle)$ is nearly constant at small values of $1/\epsilon^2$ corresponding to the saturated transition state. As $1/\epsilon^2$ increases, a linear profile develops, suggesting the scaling relation:

$$\langle q \rangle \sim \exp \left[ -\frac{S(Re)}{\epsilon^2} \right],$$  

(15)
for some function $S(Re)$ that depends only on $Re$. Following the practice in large deviation theory, we call this the action function. According to the large deviation principle (LDP) in the Freidlin-Wentzell theory, the action function $S(Re)$ measures the smallest kinetic energy needed to for the initial noise to trigger a transition [10, 33]. The action function $S(Re)$ can be calculated using the least square estimation, as suggested by the straight lines in the figure. The estimated action $S$ is shown in figure 5 (d) as a function of normalized Reynolds number $Re_n = (Re_\ast - Re)/Re_\ast$ using a log-log scale plot. For Reynolds number close to $Re_\ast = 5772$, numerical results suggest a power-law scaling relation:

$$S \sim Re_n^{0.4}. \quad (16)$$
Figure 4. (a) The threshold of noise amplitude $\epsilon_t$ for the saturated flow as a function of Reynolds number $Re$. (b) Normalized average energy $q_n$ as a function of the normalized noise amplitude $\epsilon/\epsilon_t$.

Figure 5. (a)(b)(c) The logarithm of average energy $-\log(\langle q \rangle)$ as a function of $1/\epsilon^2$. Straight lines represent the least square fit in the region of large $1/\epsilon^2$. (d) The action $S$ as a function of normalized Reynolds number $Re_n = (Re - Re_\ast)/Re_\ast$. The dashed line is a power law fit with exponent 0.4, i.e., $S \sim Re_n^{0.4}$.

Figure 6 shows the entire action function in the subcritical transition region of $Re_\ast < Re < Re_\ast$. Here the results denoted by the dashed line with plus were quoted...
Figure 6. The action $S$ as a function of Reynolds number $Re$.
The three different phases can be defined as: (1) the Poiseuille flow
phase, $S = +\infty$, (2) the intermediate phase, the puff states, $S$ is
finite, and (3) the fully developed states, $S = 0$.

from [34], where a longer channel with $L = 1600$ was used to report results at lower
Reynolds numbers. It can be seen that $S$ is a monotonically decreasing function
of $Re$. In particular, $S = 0$ at the critical Reynolds number for linear instability
$Re_\ast = 5772$, since above $Re_\ast$, white noise with any finite amplitude can trigger the
transition from the Poiseuille flow.

6. The spatial structure of the transitional flow. To gain more insight into
the subcritical transition originated by the initial disturbance of the laminar base
flow, we carried out a study of the microscopic structures of the transition flow.
A good indicator is the profile of local average energy $q$. As shown in figure 7, at
small Reynolds numbers, $Re \leq 3800$, the transitional flow consists of a collection
of ‘puff’ structures, namely, a wave train composed of localized wave packets. Each
puff has a well-defined profile with a characteristic length scale around 20. At the
noise amplitude $\epsilon = 0.14$, the density of puff increases gradually as $Re$ increases. In
saturated flows, the distance between neighboring puffs is always about $d_{\text{min}} = 30$.

The profile of $q$ at larger Reynolds numbers are displayed in figure 8. For $Re$
around 4000, some irregular yet coherent structures appear as shown by a particular
example at $r = 380$ in figure 8 (a). On the left of the irregular structure, the interval
$[0, 330]$ covers a very orderly wave train composed by 11 puffs. Profiles of local
average energy at $Re = 4400$, 5000 and 6000 are depicted in figure 8 (b), (c) and
(d). Obviously, the flow becomes more chaotic as $Re$ increases from 4000. Yet the
coherent structures are all in the form of a vortex trains and they all exhibit the
following pattern: The leading vortex is quite energetic and causes a sharp rise in
the turbulent energy. The following vorticies then weaken gradually.

Generally speaking, there are different types of stable solutions that can par-
ticipate in the channel flow. The volume fraction of each solution depends on the
initial noise. If the noise is weak, the Poiseuille profile will dominate the overall flow
field. As the noise becomes stronger, more puffs and turbulent (coherent) structures
are generated from the background laminar flow. In the saturated flow, puffs and
coherent structures will occupy the entire channel. Figure 9 (a) and (b) visualizes two particular realizations for $Re = 4800$, at two different noise amplitude $\epsilon = 0.06$ and 0.08. At the smaller noise amplitude $\epsilon = 0.06$, a few puffs are generated in the channel. At the larger noise amplitude $\epsilon = 0.08$, puffs and coherent structures appear everywhere in the channel. From figure 9 (c) and (d), we see that for $Re = 5600$, with noise amplitude $\epsilon = 0.04$, both laminar (Poiseuille) and turbulent regions make a substantial contribution to the intermittency of the system. At a larger noise amplitude $\epsilon = 0.06$, it gives rise to something that can be regarded as homogenous turbulence, for lack of a better term.
In order to provide a statistical description of the above observation, we introduce the streamwise correlation function of the local averaged energy $q$, defined by

$$C_q(r) = \frac{\langle q(x,t)q(x+r,t) \rangle_{xt}}{\langle q(x,t)q(x,t) \rangle_{xt}},$$

(17)

where, $\langle \rangle_{xt}$ denotes the average over $x$ and $t$. $C_q$ is a periodic even function of $r$, due to the periodic boundary condition of the velocity field and the symmetry of the definition with respect to the origin of $r$. Moreover, we notice that the correlation function $C_q(r)$ nearly vanishes for $r > 150$, indicating that the channel length $L = 400$ reported here is long enough to cover the correlated coherent regions.

The correlation functions at $Re = 3000$, $3400$, and $3800$ for noise amplitude $\epsilon = 0.14$ are presented in figure 10 (a). Oscillations of the correlation function are observed. For the fixed Reynolds number, the period of oscillation is nearly constant, and the magnitude of oscillation decays gradually with the increase of distance $r$. The correlation of the local energy is still significant for the separation $r = 100$, revealing a pronounced nonlocal feature of the streamwise structures in the transitional flows. As $Re$ becomes larger, the correlation becomes stronger and the period of oscillation decreases. In figure 10 (b), we plot the correlation functions at $Re = 3000$, $3400$, and $3800$ for the saturated transition flow. All the correlation functions collapses to the same curve, in consistent with previous discussions on saturated flows.

Figure 10 (c) shows the correlation functions at $Re = 4000$, $4400$, and $4800$ for noise amplitude $\epsilon = 0.14$. For $Re$ between $4000$ and $4800$, the oscillation in the correlation function weakens as $Re$ is increased, but the change of the period of the oscillation is negligible. This suggests that the density of coherent structure is saturated, and the spatial distribution of the coherent structure becomes more irregular as $Re$ becomes larger. As shown in figure 10 (d), for $Re \geq 5000$, the oscillation in the correlation function disappears and the correlation at distance $r > 50$ is negligible, which is attributed to the irregularity in the distribution of the coherent structures in the fully developed transitional flow at high Reynolds numbers.
To provide a more clear picture for the dependence of spatial correlation on the Reynolds numbers, we calculated the minimum value $C_{\min}$ of the correlation function and the location $r_{\min}$ for the minimum correlation in the saturated flow. The results are shown in figure 11. Both $C_{\min}$ and $r_{\min}$ are nearly constant for $3000 \leq Re \leq 4000$ which is in agreement with the collapse of correlation functions shown in figure 10 (b). The separation $r_{\min}$ of largest negative correlation is about half of the distance between two neighboring puffs, i.e., $r_{\min} = d_{\min}/2 = 15$. As $Re$ increases from 4000, the value $C_{\min}$ increases, due to the weakening of the correlation caused by the irregularities in the flow. For $Re > 5000$, the correlation becomes quite small ($r_{\min}$ is around $-0.1$). For $Re \leq 5000$, the location $r_{\min}$ is insensitive to the change of Reynolds number since the puffs and turbulent structures are already saturated, with a constant characteristic length scale. We do not show $r_{\min}$ at $Re > 5200$ since the minimum value of the correlation function is close to zero, and the problem of finding a minimum location is ill-defined.

Finally, we show profiles of the vorticity $\omega$ defined by:

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u_1}{\partial y}.$$  \hspace{1cm} (18)

Figure 12 displays the profile of $\omega$ in the entire channel at several Reynolds numbers for noise amplitude $\epsilon = 0.14$. For $Re \leq 4000$, puff structures can be clearly identified by high levels of vorticity intensity. The number of puffs increases as $Re$ becomes larger. For $Re = 4000$, a more complex pattern of vorticity profile appears in the region of $350 \leq x \leq 400$. At $Re = 6000$, the overall pattern becomes rather irregular along the streamwise direction. The detailed flow pattern in the puffs or coherent structures is shown in figure 13, where we display 2D profiles of $u_1$, $v$, $\omega$ and the streamlines at $Re = 3000$ and $Re = 6000$. The subregion shown here has a length scale of 20, which is roughly the length scale of a puff. Interestingly, the flow patterns at $Re = 6000$ looks quite similar to those at $Re = 3000$.

To explore the transversal features of the flow, we introduce a normalized mean velocity and a normalized mean vorticity:

$$u_n(y) = u_m(y)/u_m(0),$$  \hspace{1cm} (19)
Figure 11. Minimum value of the correlation function $C_{\min}$ and the corresponding location $r_{\min}$ at different Reynolds numbers in the saturated flow.

Figure 12. 2D profiles of vorticity at $Re = 3000, 3400, 4000$ and 6000 for $\epsilon = 0.14$. 
and,

\[ \omega_n(y) = \omega_m(y) / \omega_m^{\text{rms}}, \quad \omega_m^{\text{rms}} = \sqrt{\frac{1}{2} \int_{-1}^{1} \omega_m^2(y) dy}, \]

where, the mean velocity \( u_m \) and mean vorticity \( \omega_m \) are defined by

\[ u_m(y) = \langle u_1(x, y, t) \rangle_{xt}, \]

and,

\[ \omega_m(y) = \langle \omega(x, y, t) \rangle_{xt}. \]

We display the normalized profiles of mean velocity \( u_n \) and mean vorticity \( \omega_n \) in figure 14. One striking feature is that the profiles of both mean velocity \( u_n \) and mean vorticity \( \omega_n \) almost all collapse to the same lines for four different Reynolds numbers \( Re = 3000, 4000, 5000, \) and 6000, revealing a similar averaged transversal structure of the flow over a wide range of Reynolds numbers. One can also see that there is a nearly constant mean streamwise velocity \( u_n \) in the central region of the channel for \( y \) between \(-0.3 \) and \( 0.3 \), where the mean vorticity \( \omega_n \) is negligibly small.

7. Summary and conclusions. Let us summarize the picture suggested by the numerical results of this and the companion paper \cite{34}. At finite noise amplitudes, there are three phases: The Poiseuille flow, localized states (puffs) with Poiseuille flow background, and fully developed states. In the second phase, the average distance between puffs is a function of the noise amplitude and goes to infinity as the noise amplitude \( \epsilon \) goes to 0 or as the Reynolds number approaches the transition point from above. As a result, at finite values of \( \epsilon \), we see a continuous transition at the low critical Reynolds number \( Re_c \sim 2332 \). In the zero noise limit, we see a discontinuous transition at a higher critical Reynolds number \( Re_* \sim 5772 \). However, in contrast to discontinuous transitions that we see in other thermodynamic systems, for \( Re > Re_* \), the Poiseuille flow phase is not linearly stable. Yet it still exhibits...
some kind of metastability in the sense that, for small $\epsilon$ and Reynolds number close to $Re_*$, the time scale for instability is very long.

From the viewpoint of dynamical systems, the transition at the lower critical Reynolds number is a finite amplitude or nonlinear instability, the transition at the upper critical Reynolds number is a zero amplitude or linear instability. So naively, if we draw an analogy with statistical physics, we might have expected a discontinuous transition at the lower critical Reynolds number and a continuous transition at the upper critical Reynolds number. This is opposite to what we saw from our numerical results. The discrepancy at the second transition is quite easy to understand: The suggestion of a continuous transition is from the analogy between supercritical bifurcation and second order phase transition, in particular, the analogy between weakly nonlinear stability theory and Landau’s theory of second order transition. This simply does not apply here. The first discrepancy is harder to appreciate and could be a special feature of this particular problem. But it is also a manifestation of the fact that the system under consideration is far from the ones usually considered in equilibrium statistical physics.

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