The timelike half-supersymmetric backgrounds of $\mathcal{N} = 2, D = 4$ supergravity with Fayet-Iliopoulos gauging

Dietmar Klemm and Emanuele Zorzan

Dipartimento di Fisica dell’Università di Milano,
Via Celoria 16, I-20133 Milano and
INFN, Sezione di Milano, Via Celoria 16, I-20133 Milano.

Abstract: Subject to some relatively mild assumptions, we derive the complete form of all timelike half-supersymmetric solutions to $\mathcal{N} = 2, D = 4$ gauged supergravity coupled to an arbitrary number of abelian vector multiplets. This is done using spinorial geometry techniques. Explicit examples are given for a simple prepotential. Among the solutions, there are near-horizon geometries of extremal rotating BPS black holes still to be discovered, with a nontrivial dependence of the scalar fields on one of the horizon coordinates.

Keywords: Superstring Vacua, Black Holes, Supergravity Models.
1. Introduction

BPS solutions to supergravity theories have played, and continue to play, an important role in string theory developments. Supersymmetric black holes represent perhaps one of the most notable examples of this: In presence of a sufficient amount of supersymmetry, non-renormalization theorems allow to extrapolate an entropy computation at weak string coupling (when the system is generically described by a configuration of strings and branes) to the strong-coupling regime, where a description in terms of a black hole is valid [1]. These entropy calculations have been essential for our current understanding of black hole microstates. It is therefore important to dispose of a systematic classification of BPS solutions, that allows to construct such backgrounds.
without the necessity to guess suitable ansaetze. Of particular interest in this context are gauged supergravities, which are related to supersymmetric field theories by the AdS/CFT correspondence. While we know by now a broad landscape of BPS solutions to ungauged supergravities, including many different types of black holes and black rings [2], only a few of their analogues in gauged supergravity have been constructed\(^1\). For instance, in four dimensions, there should exist rotating black holes in gauged \(\mathcal{N} = 8\) supergravity (that admits a truncation to \(\mathcal{N} = 2\) gauged supergravity coupled to three abelian vector multiplets [4]) with four independent electromagnetic charges. Until now, the only known solutions of this type are the Kerr-Newman AdS black holes, which correspond to setting the four charges equal, and the black holes in SO(4) gauged \(\mathcal{N} = 4\) supergravity with two pairwise equal charges [5].

In this paper, we consider the theory of \(\mathcal{N} = 2, D = 4\) gauged supergravity coupled to an arbitrary number of abelian vector multiplets, but with no hypermultiplets (so-called Fayet-Iliopoulos gauging). The constraints obeyed by backgrounds admitting at least one timelike Killing spinor were given in [6], generalizing the results for minimal gauged supergravity [7]. Although the equations determining the BPS geometries are rather involved, some explicit solutions of them describing static black holes with nontrivial scalars turned on have been obtained in [8]. These black holes provide a new ground to test the AdS/CFT correspondence: In principle it should be possible to compute their microscopic entropy using the recently discovered Chern-Simons-matter theories [9], and to compare it then with the macroscopic Bekenstein-Hawking result.

Here we go one step further with respect to [6] and impose the existence of at least two Killing spinors, so we want to determine the most general half-supersymmetric configurations\(^2\). There are several reasons motivating this:

First of all, it is of special interest to address cases of the \(\text{AdS}_4/\text{CFT}_3\) correspondence with less than maximal supersymmetry. For instance, supergravity vacua with lower supersymmetry may have an interpretation on the CFT side as vacua with non-zero expectation values of certain operators (spontaneous symmetry breaking), or as deformations of the CFT (explicit symmetry breaking).

The second point is the attractor mechanism [13–17]. While the BPS attractor flow has been studied extensively for asymptotically flat black holes, the AdS case was considered only recently [8]\(^3\). In order to explore the BPS attractor flow in AdS, one

\(^1\)Note that some of these analogues might not exist [3].

\(^2\)In five dimensions, this was done in [10] and [11] for the timelike and null cases respectively. Maximally supersymmetric solutions to four-dimensional \(\mathcal{N} = 2\) gauged supergravity were classified in [12].

\(^3\)For an analysis of the attractor mechanism in \(\mathcal{N} = 2, D = 4\) supergravity with SU(2) gauging cf. [18].
needs the near-horizon geometry of (possibly rotating) AdS black holes with scalar fields turned on. In the asymptotically flat case, such near-horizon geometries are typically fully supersymmetric, whereas, as we shall see below, in AdS they generically break one half of the supersymmetries.

Furthermore, in gauged supergravity, interesting mathematical structures appear in the base manifolds of reduced holonomy, over which supersymmetric spacetimes are fibered. For instance, one can have U(1) holonomy with torsion [6] (the torsion coming from the gauging), Einstein-Weyl spaces [19] or hyper-Kähler torsion manifolds [20], and one might ask how these structures are modified if one imposes the existence of more than one Killing spinor.

Finally, in minimal \( \mathcal{N} = 2, D = 4 \) gauged supergravity, the equations determining the BPS solutions reduce, under some assumptions, to the equations of motion following from the gravitational Chern-Simons action [21]. While the deeper reason for this remains obscure, it indicates that the full set of equations actually might be integrable, i.e., it should be possible to construct a Lax pair for them. Requiring additional supersymmetries can help to better understand the integrability structure of this system.

The remainder of this paper is organized as follows: In section 2, we briefly review the theory of \( \mathcal{N} = 2, D = 4 \) supergravity with Fayet-Iliopoulos gauging. After that, in 3, we impose the existence of a second Killing spinor, obtain the linear system into which the Killing spinor equations turn, and derive the time-dependence of this second covariantly constant spinor. Subsequently, the linear system is solved under some relatively mild assumptions, and the spacetime geometry, the fluxes as well as a scalar flow equation are obtained. The reader who is interested only in the final results can skip the technical details and immediately jump to the summaries in sections (3.3.1), (3.4.1), (3.4.2) and (3.4.3).

2. \( \mathcal{N} = 2, D = 4 \) supergravity with Fayet-Iliopoulos gauging

We consider \( \mathcal{N} = 2, D = 4 \) gauged supergravity coupled to \( n_V \) abelian vector multiplets \[22]\4. Apart from the vierbein \( e_\mu^a \), the bosonic field content includes the vectors \( A_I^\mu \) enumerated by \( I = 0, \ldots, n_V \), and the complex scalars \( z^\alpha \) where \( \alpha = 1, \ldots, n_V \). These scalars parametrize a special Kähler manifold, i.e., an \( n_V \)-dimensional Hodge-Kähler manifold that is the base of a symplectic bundle, with the covariantly holomorphic sections

\[
\mathcal{V} = \begin{pmatrix} X^I \\ F_I \end{pmatrix}, \quad \mathcal{D}_\alpha \mathcal{V} = \partial_\alpha \mathcal{V} - \frac{1}{2}(\partial_\alpha \mathcal{K})\mathcal{V} = 0 ,
\]

\[\text{(2.1)}\]

\[\text{4} \] Throughout this paper, we use the notations and conventions of [23].
where $K$ is the Kähler potential and $\mathcal{D}$ denotes the Kähler-covariant derivative. $\mathcal{V}$ obeys the symplectic constraint
\[
\langle \mathcal{V}, \bar{\mathcal{V}} \rangle = X_I \bar{F}_I - F_I \bar{X}^I = i.
\] (2.2)

To solve this condition, one defines
\[
\mathcal{V} = e^{K(z, \bar{z})/2} v(z),
\] (2.3)
where $v(z)$ is a holomorphic symplectic vector,
\[
v(z) = \left( \frac{Z^I(z)}{\partial Z^I} F(Z) \right).
\] (2.4)

$F$ is a homogeneous function of degree two, called the prepotential, whose existence is assumed to obtain the last expression. The Kähler potential is then
\[
e^{-K(z, \bar{z})} = -i \langle v, \bar{v} \rangle.
\] (2.5)

The matrix $\mathcal{N}_{IJ}$ determining the coupling between the scalars $z^\alpha$ and the vectors $A_\mu^I$ is defined by the relations
\[
F_I = \mathcal{N}_{IJ} X^J, \quad \mathcal{D}_\alpha \bar{F}_I = \mathcal{N}_{IJ} \mathcal{D}_\alpha \bar{X}^J.
\] (2.6)

The bosonic action reads
\[
e^{-1} \mathcal{L}_{\text{bos}} = \frac{1}{16\pi G} R + \frac{1}{4} (\text{Im} \mathcal{N})_{IJ} F^I_{\mu\nu} F^{J\mu\nu} - \frac{1}{8} (\text{Re} \mathcal{N})_{IJ} e^{-1} \epsilon^{\mu\rho\sigma} F^I_{\mu\nu} F^{J}_{\rho\sigma}
\]
\[-g_{\alpha\beta} \partial_\mu z^\alpha \partial^\mu \bar{z}^\beta - V,
\] (2.7)

with the scalar potential
\[
V = -2g^2 \xi_I \xi_J [(\text{Im} \mathcal{N})^{-1}]^{IJ} + 8 \bar{X}^I X^J,
\] (2.8)

that results from U(1) Fayet-Iliopoulos gauging. Here, $g$ denotes the gauge coupling and the $\xi_I$ are constants. In what follows, we define $g_I = g \xi_I$.

The supersymmetry transformations of the gravitini $\psi^I_\mu$ ($i = 1, 2$) and gaugini $\lambda^I_\alpha$ are\(^5\)
\[
\delta \psi^I_\mu = D_\mu (\omega) e^i + ig_I X^I \gamma_\mu \sigma_3^{ij} \epsilon_j + \frac{1}{4} \gamma_{ab} F^{-I}_{ab} \epsilon^{ij} \gamma_\mu \epsilon_j (\text{Im} \mathcal{N})_{IJ} X^J,
\] (2.9)

\(^5\)They result from the expressions given in [23] by taking $\vec{P}_I = \vec{e} \xi_I$ for the moment maps (FI gauging), where $\vec{e}$ denotes a unit vector that can be chosen to point in the 3-direction without loss of generality. The antiselfdual parts $F^{-I}$ of the fluxes as well as the $\sigma$-matrices and the Kähler-covariant derivatives $\mathcal{D}$ are also given in [23].
\[ \delta \lambda^i = -\frac{1}{2} g^{\alpha\beta} D_{\beta} \bar{X}^I (\text{Im} \mathcal{N})_{IJ} F^{-J}_{\mu\nu} \gamma^{\mu\nu} \epsilon_i \epsilon^j + \gamma^\mu \partial_\mu z^\alpha \epsilon_i - 2ig_1 \sigma_{3ij} g^{\alpha\beta} D_{\beta} \bar{X}^I \epsilon^j, \] (2.10)

where

\[ D_\mu (\omega) \epsilon^i = (\partial_\mu + \frac{1}{4} \omega^{ab}_\mu \gamma_{ab}) \epsilon^i + \frac{i}{2} A_\mu \epsilon^i + ig_1 A^I_\mu \sigma_{3j} \epsilon^j. \] (2.11)

Here, \( A_\mu \) is the gauge field of the Kähler U(1),

\[ A_\mu = -\frac{i}{2} (\partial_\alpha \mathcal{K} \partial_\mu z^\alpha - \partial_\alpha \mathcal{K} \partial_\mu z^\alpha). \] (2.12)

The most general timelike supersymmetric background of the theory described above was constructed in [6], and is given by

\[ ds^2 = -4|b|^2 (dt + \sigma)^2 + |b|^{-2} (dz^2 + e^{2\Phi} dw d\bar{w}), \] (2.13)

where the complex function \( b(z, w, \bar{w}) \), the real function \( \Phi(z, w, \bar{w}) \) and the one-form \( \sigma = \sigma_w dw + \sigma_{\bar{w}} d\bar{w} \), together with the symplectic section \((2.1)\) are determined by the equations

\[ \partial_z \Phi = 2ig_1 \left( \frac{\bar{X}^I}{b} - \frac{X^I}{b} \right), \] (2.14)

\[ 4\partial\bar{\partial} \left( \frac{X^I}{b} - \frac{\bar{X}^I}{b} \right) + \partial_z \left[ e^{2\Phi} \partial_z \left( \frac{X^I}{b} - \frac{\bar{X}^I}{b} \right) \right] \] (2.15)

\[ -2ig_J \partial_z \left\{ e^{2\Phi} \left[ |b|^{-2} (\text{Im} \mathcal{N})^{-1|IJ} + 2 \left( \frac{X^I}{b} + \frac{\bar{X}^I}{b} \right) \left( \frac{X^J}{b} + \frac{\bar{X}^J}{b} \right) \right] \right\} = 0, \]

\[ 4\partial\bar{\partial} \left( \frac{F^I}{b} - \frac{\bar{F}^I}{b} \right) + \partial_z \left[ e^{2\Phi} \partial_z \left( \frac{F^I}{b} - \frac{\bar{F}^I}{b} \right) \right] \] (2.16)

\[ -2ig_J \partial_z \left\{ e^{2\Phi} \left[ |b|^{-2} \text{Re} \mathcal{N}_{III} (\text{Im} \mathcal{N})^{-1|IJ} + 2 \left( \frac{F^I}{b} + \frac{\bar{F}^I}{b} \right) \left( \frac{X^J}{b} + \frac{\bar{X}^J}{b} \right) \right] \right\} \] (2.17)

\[ 2\partial\bar{\partial} \bar{\Phi} = e^{2\Phi} \left[ ig_J \partial_z \left( \frac{X^I}{b} - \frac{\bar{X}^I}{b} \right) + 2 \frac{|b|^2}{|b|^2} g_{IJ} g_J (\text{Im} \mathcal{N})^{-1|IJ} + 4 \left( \frac{g_I X^I}{b} + \frac{\bar{g}_I X^I}{b} \right)^2 \right] \] (2.18)

\[ d\sigma + 2 *^{(3)} \langle \mathcal{I}, d\mathcal{I} \rangle - \frac{i}{|b|^2} g_I \left( \frac{\bar{X}^I}{b} + \frac{X^I}{b} \right) e^{2\Phi} dw \wedge d\bar{w} = 0. \] (2.18)

\[ ^6 \text{Note that also } \sigma \text{ and } \mathcal{V} \text{ are independent of } t. \]
Here \( *^{(3)} \) is the Hodge star on the three-dimensional base with metric\(^7\)

\[
ds^2_3 = dz^2 + e^{2\Phi} dw \, \bar{dw} , \tag{2.19}
\]

and we defined \( \partial = \partial_w, \, \bar{\partial} = \partial_{\bar{w}}, \) as well as

\[
\mathcal{I} = \text{Im} \left( \mathcal{V}/b \right) . \tag{2.20}
\]

Given \( b, \Phi, \sigma \) and \( \mathcal{V}, \) the fluxes read

\[
F^I = 2(dt + \sigma) \wedge d \left[ b X^I + \bar{b} X^I \right] + |b|^{-2} dz \wedge dw \left[ X^I (\bar{\partial}b + i A_w \bar{b}) \right] + (D_\alpha X^I) \bar{b} \partial z^\alpha - X^I (\partial b - i A_w b) + (D_\alpha X^I) \bar{b} \partial z^\alpha - \frac{1}{2} |b|^{-2} e^{-2\Phi} dw \wedge dw \left[ X^I (\bar{\partial} \bar{b} + i A_{\bar{w}} \bar{b}) \right] + (D_\alpha X^I) (\bar{b} \partial z^\alpha - X^I (\partial b - i A_b b) - (D_\alpha X^I) (\bar{b} \partial z^\alpha) - 2ig_I (\text{Im} N)^{-1/|f|} . \tag{2.21}
\]

If the constraints (2.14)-(2.18) are satisfied, the solution admits the Killing spinor \( (\epsilon^1, \epsilon^2) = (1, be_2) \) (cf. appendix A for a summary of the essential information needed to realize spinors in terms of forms).

Before we continue, a short comment on Kähler-covariance is in order. Under a Kähler transformation

\[
\mathcal{K} \mapsto \mathcal{K} + f(z^\alpha) + \bar{f}(\bar{z}^\alpha) , \tag{2.22}
\]

the Killing spinors transform as

\[
\epsilon^1 \mapsto e^{(\bar{f}-f)/4} \epsilon^1 , \quad \epsilon_i \mapsto e^{-(f-f)/4} \epsilon_i . \tag{2.23}
\]

On the other hand, under a U(1) gauge transformation

\[
A_\mu^I \mapsto A_\mu^I + \partial_\mu \chi^I , \tag{2.24}
\]

we have

\[
\epsilon^1 \mapsto e^{-ig_I \chi^I} \epsilon^1 , \quad \epsilon_2 \mapsto e^{-ig_I \chi^I} \epsilon_2 . \tag{2.25}
\]

Under a combined Kähler/U(1) transformation with \( ig_I \chi^I = (\bar{f} - f)/4, \) the Killing spinor representative \( (\epsilon^1, \epsilon_2) = (1, be_2) \) is forminvariant; it goes over into \((1, b'e_2), \) with \( b' = e^{-(f-f)/2b}. \) One easily checks that the eqns. (2.14)-(2.18) are covariant under Kähler transformations if \( b \) is replaced by \( b'. \) In what follows we sometimes use the Kähler-covariant derivatives of \( b \) defined by

\[
D_\mu b = (\partial_\mu - i A_\mu) b , \quad D_\mu \bar{b} = (\partial_\mu + i A_\mu) \bar{b} , \tag{2.26}
\]

as well as \( D \equiv D_w, \, \bar{D} \equiv D_{\bar{w}}. \) These satisfy \( D'_\mu b' = e^{-(f-f)/2} D_\mu b. \)

\(^7\) Whereas in the ungauged case, this base space is flat and thus has trivial holonomy, here we have U(1) holonomy with torsion [6].
3. Half-supersymmetric backgrounds

Let us now investigate the additional conditions satisfied by half-supersymmetric vacua in the timelike class. As the stability subgroup of the first Killing spinor was already used in [6] to obtain the eqns. (2.14)-(2.18), the second one cannot be simplified anymore, and is thus of the general form

$$\epsilon^1 = \alpha + \beta e_{12} , \quad \epsilon^2 = \gamma + \delta e_{12} , \quad \epsilon_1 = \bar{\alpha} e_1 - \bar{\beta} e_2 , \quad \epsilon_2 = \bar{\gamma} e_1 - \bar{\delta} e_2 , \quad (3.1)$$

where $\alpha, \beta, \gamma, \delta$ are complex-valued functions.

The conditions coming from an additional Killing spinor are easily obtained by plugging (3.1) into (2.9) and (2.10) (with $\delta \psi_\mu = \delta \lambda^\mu = 0$), and taking into account the constraints on the bosonic fields implied by the first Killing spinor $(\epsilon^1, \epsilon_2) = (1, be_2)$, given in [6]. This will be done in the following subsection.

3.1 The linear system

From the vanishing of the gaugini supersymmetry transformations (2.10) we get

$$\begin{align*}
(\bar{\beta} - b\gamma)\partial_z z^\alpha + 2e^{-\Phi}\sqrt{b}\left(\bar{b}\alpha + \delta\right)\partial z^\alpha &= 4ig^{\alpha\beta}D_\beta X^t g_{t\gamma} , \\
(\bar{b}\alpha + \delta)\partial_z z^\alpha - 2e^{-\Phi}\sqrt{b}\left(\bar{\beta} - b\gamma\right)\partial z^\alpha &= 0 , \\
(b\alpha + \bar{\delta})\partial_z z^\alpha - 2e^{-\Phi}\sqrt{b}\left(\beta - \bar{b}\gamma\right)\partial z^\alpha &= 0 , \\
(\beta - \bar{b}\gamma)\partial_z z^\alpha + 2e^{-\Phi}\sqrt{b}(b\alpha + \bar{\delta})\partial z^\alpha &= -\frac{4i}{b}g^{\alpha\beta}D_\beta X^t g_{t\beta} ,
\end{align*}$$

while the gravitini variations (2.9) yield

$$\begin{align*}
\partial_t \alpha &= -ib\Omega_z(b\alpha + \bar{\delta}) + 2ie^{-\Phi}|b|\Omega_w(\beta - \bar{b}\gamma) , \\
\partial_t \beta &= 2ie^{-\Phi}b|b|\Omega_w(b\alpha + \bar{\delta}) + ib\bar{b}\Omega_z(\beta - \bar{b}\gamma) + 4i(bX\cdot g + \bar{bX}\cdot g)\beta - 4ibbX\cdot g\gamma , \\
\partial_t \gamma &= 2ib|e^{-\Phi}\Omega_w(b\alpha + \bar{\delta}) + i\bar{b}\Omega_z(\beta - \bar{b}\gamma) + 4iX\cdot g\beta - 4i(bX\cdot g + \bar{bX}\cdot g)\gamma , \\
\partial_t \delta &= ib\bar{b}\Omega_z(b\alpha + \bar{\delta}) - 2ie^{-\Phi}b|b|\Omega_w(\beta - \bar{b}\gamma) ,
\end{align*}$$

$$\begin{align*}
\partial_z \alpha &= -\frac{i\Omega_z}{2b}(b\alpha + \bar{\delta}) - \frac{ie^{-\Phi}}{|b|}\Omega_w(\beta - \bar{b}\gamma) , \\
\partial_z \beta &= i\sqrt{b}e^{-\Phi}\Omega_w(b\alpha + \bar{\delta}) - \frac{i}{2}\Omega_z(\beta - \bar{b}\gamma) + \beta\partial_z \ln|b| + 2iX\cdot g\gamma ,
\end{align*}$$

- 7 -
\[
\partial_\gamma = -\frac{ie^{-\phi}}{|b|} \Omega_w(b\alpha + \delta) + \frac{i}{2b} \Omega_z(\beta - \gamma) + \frac{2iX \cdot g}{bb} \beta - \frac{\gamma}{2} \partial_z \ln \frac{b}{\beta}, \\
\partial_\delta = -ie^{-\phi} \sqrt{\frac{b}{b}} \Omega_w(\beta - \gamma) - \frac{i}{2} \Omega_z(b\alpha + \delta) + \delta \partial_z \ln \frac{b}{\beta},
\]

(3.7)

\[
\partial_\alpha = -\frac{i}{b}(\Omega_w + b\Omega_z \sigma_w)(b\alpha + \delta) + 2ie^{-\phi}e^{-2\Phi} \Omega_w \sigma_w(\beta - \gamma), \\
\partial_\beta = -\frac{ie^\phi}{2} \sqrt{\frac{b}{b}} \left( \Omega_z - 4e^{-2\Phi} bb \Omega_w \sigma_w + \frac{4X \cdot g}{b} \right) (b\alpha + \delta) - \beta \partial(\Phi - \ln |b|) \\
+ i\Omega_z \sigma_w(\beta - \bar{\gamma}) + 4i(bX \cdot g + \bar{b}X \cdot g)\sigma_w \beta - 4i\Omega_z \sigma_w \gamma, \\
\partial_\gamma = \frac{i}{b}(\Omega_w + b\Omega_z \sigma_w)(\beta - \gamma) + \gamma \partial \left( \frac{1}{2} \ln \frac{b}{\beta} \right) \\
+ 2i|b|e^{-\Phi} \Omega_w \sigma_w(b\alpha + \delta) + 4iX \cdot g \sigma_w \beta - 4i(bX \cdot g + \bar{b}X \cdot g)\sigma_w \gamma, \\
\partial_\delta = i\Omega_z \sigma_w(b\bar{\alpha} + \delta) + \frac{ie^\phi}{2} \sqrt{\frac{b}{b}} \left( \Omega_z - 4e^{-2\Phi} bb \Omega_w \sigma_w \right) (\beta - \gamma) \\
- 2iX \cdot g e^{\Phi} \sqrt{\frac{b}{b}} \gamma + \delta \partial \ln \frac{b}{\beta},
\]

(3.8)

\[
\bar{\partial}_\alpha = -\bar{b} \Omega_z \sigma_w(b\alpha + \delta) + \frac{2iX \cdot g e^\phi}{bb} \beta + \frac{ie^\phi}{2|b|} \left( \Omega_z + 4bb e^{-2\Phi} \Omega_w \sigma_w \right)(\beta - \bar{\gamma}), \\
\bar{\partial}_\beta = -i(\Omega_w - \bar{b} \Omega_z \sigma_w)(\beta - \bar{\gamma}) + \beta \bar{\partial}(\Phi + \ln |b|) \\
+ 2ie^{-\Phi} \bar{b} |b| \Omega_w \sigma_w (b\alpha + \delta) + 4i(bX \cdot g + \bar{b}X \cdot g)\sigma_w \beta - 4i\Omega_z \sigma_w \gamma, \\
\bar{\partial}_\gamma = \frac{ie^\phi}{2|b|} \left( \Omega_z + 4bb e^{-2\Phi} \Omega_w \sigma_w + \frac{4X \cdot g}{b} \right) (b\bar{\alpha} + \delta) - \gamma \bar{\partial} \left( \frac{1}{2} \ln \frac{b}{\beta} \right) \\
+ i\Omega_z \sigma_w(\beta - \gamma) + 4iX \cdot g \sigma_w \beta - 4i(bX \cdot g + \bar{b}X \cdot g)\sigma_w \gamma, \\
\bar{\partial}_\delta = -i \left( \Omega_w - \bar{b} \Omega_z \sigma_w \right) (b\bar{\alpha} + \delta) - 2i e^{-\Phi} \bar{b} |b| \Omega_w \sigma_w (\beta - \gamma) + \delta \bar{\partial} \ln \frac{b}{\beta},
\]

(3.9)

where \(X \cdot g = X^I g_I\) and \(\Omega_\mu = A_\mu - i \partial_\mu \ln \bar{b}\).

To proceed it is convenient to set \(b = re^{i\phi}\) and to introduce the new basis\(^8\)

\[
\bar{\psi} = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_{12} \end{pmatrix} = \begin{pmatrix} \alpha \\ -r^2 \alpha - \bar{b}\delta \\ re^{-\Phi} \bar{b}\gamma \\ re^{-\Phi} \beta \end{pmatrix}, \tag{3.10}
\]

\(^8\)Note that the first Killing spinor has components \((1, 0, 0, 0)\) in this basis.
in which the gaugini conditions (3.2)-(3.5) become

\[
\bar{\psi}_- \partial_z z^\alpha + 2e^{-2\Phi} \bar{\psi}_1 \partial_z z^\alpha = -\frac{4i}{\beta} g^{\alpha \beta} D_\beta \bar{X}_1 g_1 \psi_2 ,
\]

(3.11)

\[
\bar{\psi}_1 \partial_z z^\alpha - 2\bar{\psi}_- \partial_z z^\alpha = 0 ,
\]

(3.12)

\[
\psi_1 \partial_z z^\alpha - 2\psi_- \partial_z z^\alpha = 0 ,
\]

(3.13)

\[
\psi_- \partial_z z^\alpha + 2e^{-2\Phi} \psi_1 \partial_z z^\alpha = \frac{4i}{\beta} g^{\alpha \beta} D_\beta \bar{X}_1 g_1 \psi_2 ,
\]

(3.14)

with \(\psi_\pm = \psi_2 \pm \psi_1\). In general the Killing spinor equations do not readily provide information and one has to resort to their integrability conditions. Rewriting the linear system (3.6)-(3.9) in the basis (3.10), and defining \(Q = e^{-2\Phi} \bar{b} D \bar{b}\), \(P = e^{-2\Phi} b D b\), one finds that the \(t\)-\(w\) integrability condition implies

\[
-\frac{1}{2} (D_z Q - ie^{-2\Phi} \bar{b} D F_{zw}) \psi_1 + (DQ) \psi_- = 0 ,
\]

(3.15)

\[
-\frac{1}{2} (D_z P + ie^{-2\Phi} b D F_{zw}) \psi_1 + (DP) \psi_- = 0 ,
\]

(3.16)

\[
f_A \psi_1 + f_B \psi_- - 2i \partial(\bar{b}X.g) \psi_2 = 0 ,
\]

(3.17)

\[
f_C \psi_1 + f_D \psi_- + 2i \partial(b \bar{X}.g) \psi_{12} = 0 ,
\]

(3.18)

where \(F_{\mu \nu}\) denotes the field strength of the Kähler U(1) (2.12), and

\[
f_A = \frac{\bar{b}}{2b} \left[ -2e^{-2\Phi} D b \bar{D} b + 2e^{-2\Phi} b D \bar{D} b - (D_z b)^2 + 6i \bar{X}.g D_z b + 8(\bar{X}.g)^2 \right] ,
\]

\[
f_B = \frac{\bar{b}}{2b} e^{2\Phi} (D_z P + ie^{-2\Phi} b^2 F_{zw}) - 2i [X.g D b + \bar{b} D \bar{X}.g] ,
\]

\[
f_C = -\frac{b}{2b} \left[ -2e^{-2\Phi} D b \bar{D} b + 2e^{-2\Phi} b D \bar{D} b - (D_z b)^2 - 6i X.g D_z b + 8(X.g)^2 \right] ,
\]

\[
f_D = -\frac{b}{2b} e^{2\Phi} (D_z Q - ie^{-2\Phi} \bar{b}^2 F_{zw}) - 2i [\bar{X}.g D \bar{b} + b D X.g] .
\]

### 3.2 Time-dependence of second Killing spinor

In this subsection we will make use of the Killing spinor equations (3.10)-(3.14) and the integrability conditions (3.15)-(3.18) to derive the time-dependence of the second Killing spinor. Let us define \(g(t, z, w, \bar{w})\) by

\[
\psi_- = \frac{1}{2} g(t, z, w, \bar{w})(D_z P + ie^{-2\Phi} b^2 F_{zw}) .
\]

Plugging this into (3.10), one gets under the assumption \(D_z P + ie^{-2\Phi} b^2 F_{zw} \neq 0\)

\[
\psi_1 = g D P .
\]

\footnote{The case \(D_z P + ie^{-2\Phi} b^2 F_{zw} = 0\) will be considered in appendix 3.}
Using this form of $\psi_-$ and $\psi_1$, the integrability condition (3.17) becomes

$$f_A gDP + \int_B \frac{g}{2} (D_z P + ie^{-2\Phi} b^2 F_{zw}) - 2i \psi_2 \partial(bX \cdot g) = 0 .$$  \hspace{1cm} (3.19)

Now, if $g = 0$ the gravitini equations (3.6)-(3.9) imply that $X \cdot g = 0$. If we exclude for the time being this degenerate subcase, we have $g \neq 0$ and thus $g =: e^G$. Dividing (3.19) by $g$ and deriving with respect to $t$ yields $\partial_t (\psi_2/g) = 0$ (if $\partial(bX \cdot g) \neq 0$) and hence

$$\psi_2 = e^{\frac{G}{g}} \psi_2(z, w, \bar{w}) .$$

It is then clear that $\partial_t \psi_1 = \psi_1 \partial_t G$, $i = 1, 2, 12$. The Killing spinor equations are of the form $\partial_\mu \psi_1 = M_{\mu ij} \psi_j$, for some time-independent matrices $M_\mu$. Taking the derivative of this with respect to $t$, one gets $\partial_\mu \partial_t G = 0$, and therefore

$$G = G_0 t + \tilde{G}(z, w, \bar{w}) ,$$

with $G_0 \in \mathbb{C}$ constant. We have thus

$$\partial_t \psi_1 = G_0 \psi_1$$  \hspace{1cm} (3.20)

Furthermore the time-dependence of $\psi_0$ can be easily deduced from the Killing spinor equations for $\psi_0$,

$$\partial_t \psi_0 = i \Omega_z \psi_1 - 2i \Omega_w \psi_- ,$$  \hspace{1cm} (3.21)

$$\partial_z \psi_0 = \frac{i}{2r^2} \Omega_z \psi_1 + \frac{i}{r^2} \Omega_w \psi_- ,$$  \hspace{1cm} (3.22)

$$\partial \psi_0 = \left( \frac{i}{r^2} \Omega_w + i \Omega_z \sigma_w \right) \psi_1 - 2i \Omega_w \sigma_w \psi_- ,$$  \hspace{1cm} (3.23)

$$\partial \psi_0 = i \Omega_z \sigma_w \psi_1 - \left( \frac{ie^{2\Phi}}{2r^2} \Omega_z + 2i \Omega_w \sigma_w \right) \psi_- + \frac{2i X \cdot g e^{2\Phi}}{br^2} \psi_{12} .$$  \hspace{1cm} (3.24)

Deriving (3.21)-(3.24) with respect to $t$ and taking into account (3.20), one obtains $\partial_\mu \partial_\mu \psi_0 = G_0 \partial_\mu \psi_0$. Hence $\partial_t \psi_0 = G_0 \psi_0 + \lambda$ where $\lambda$ is an arbitrary constant. If $G_0 \neq 0$, this implies

$$\psi_0 = -\frac{\lambda}{G_0} + \tilde{\psi}_0(z, w, \bar{w}) e^{G_0 t} .$$  \hspace{1cm} (3.25)

In that case one can set $\lambda = 0$ without loss of generality, because a nonvanishing $\lambda$ simply corresponds to adding a multiple of the first Killing spinor to the second. The time-dependence of $\psi_0$ is thus of the same exponential form as that of the other components of the second Killing spinor,

$$\psi_0 = \tilde{\psi}_0(z, w, \bar{w}) e^{G_0 t} , \quad \psi_1 = \tilde{\psi}_1(z, w, \bar{w}) e^{G_0 t} .$$

– 10 –
If $G_0$ vanishes we have

$$\psi_0 = \lambda t + \tilde{\psi}_0(z, w, \bar{w}) \ , \ \ \ \psi_1 = \tilde{\psi}_1(z, w, \bar{w}) \quad (3.26)$$

(so that one cannot choose $\lambda = 0$ in this case).

Plugging this time-dependence into the subsystem of the Killing spinor equations not containing $\psi_0$ one obtains the following reduced system for $\psi_1$:

$$\partial_z \psi_1 + \left( \frac{G_0}{2bb} - \frac{\partial_z b}{b} + iA_z \right) \psi_1 + 2 \left( \frac{\partial b}{b} - iA_w \right) \psi_\perp = 0 \ , \quad (3.27)$$

$$\partial_z \psi_2 + \left( \frac{G_0}{2bb} - \frac{\partial_z \bar{b}}{b} - 4i \frac{X\cdot g}{b} - iA_z \right) \psi_2 - \left( \frac{\partial_z b}{b} - 4i \frac{\bar{X}\cdot g}{b} - iA_z \right) \psi_1 = 0 \ , \quad (3.28)$$

$$\partial_z \psi_{12} + 2e^{-2\Phi} \left( \frac{\partial \bar{b}}{b} + iA_w \right) \psi_1 + \left( \frac{G_0}{2bb} - \frac{\partial_z b}{b} - \frac{\partial \bar{b}}{b} - 4i \frac{X\cdot g}{b} \right) \psi_{12} = 0 \ , \quad (3.29)$$

$$\partial_z \psi_1 - \left( \frac{G_0}{2bb} + \frac{\partial \bar{b}}{b} + iA_z \right) \psi_1 + 2 \left( \frac{\partial \bar{b}}{b} + iA_w \right) \psi_\perp = 0 \ , \quad (3.30)$$

$$\partial_z \psi_2 - 2e^{-2\Phi} \left( \frac{\partial \bar{b}}{b} - iA_w \right) \psi_1 - \left( \frac{G_0}{2bb} + \frac{\partial_z b}{b} + \frac{\partial \bar{b}}{b} - 4i \frac{\bar{X}\cdot g}{b} \right) \psi_2 = 0 \ , \quad (3.31)$$

$$\partial_z \psi_{12} - \left( \frac{\partial \bar{b}}{b} + 4i \frac{X\cdot g}{b} + iA_z \right) \psi_2 - \left( \frac{G_0}{2bb} + \frac{\partial_z b}{b} - 4i \frac{\bar{X}\cdot g}{b} - iA_z \right) \psi_{12} = 0 \ , \quad (3.32)$$

$$\partial \psi_1 - G_0 \sigma_w \psi_1 = 0 \ , \quad (3.33)$$

$$\partial \psi_2 + \left( \frac{\partial_z b}{2b} - 2i \frac{\bar{X}\cdot g}{b} - i \frac{3}{2} A_z \right) \psi_1 - \left( G_0 \sigma_w + \frac{\partial \bar{b}}{b} + \frac{\partial \bar{b}}{b} - 2\partial \Phi \right) \psi_2 = 0 \ , \quad (3.34)$$

$$\partial \psi_{12} - \left( \frac{\partial \bar{b}}{2b} + 2i \frac{X\cdot g}{b} + i \frac{3}{2} A_z \right) \psi_1 - \left( G_0 \sigma_w + \frac{\partial \bar{b}}{b} + \frac{\partial \bar{b}}{b} - 2\partial \Phi \right) \psi_{12} = 0 \ , \quad (3.35)$$

$$\bar{\partial} \psi_1 - \left( G_0 \sigma_{\bar{w}} + \frac{\partial b}{b} + \frac{\partial b}{b} \right) \psi_1 - e^{2\Phi} \left[ \left( \frac{\partial_z b}{2b} + \frac{\partial \bar{b}}{2b} \right) \psi_\perp - 2i \left( \frac{\bar{X}\cdot g}{b} \psi_2 + \frac{X\cdot g}{b} \psi_{12} \right) \psi_{12} \right] = 0 \ , \quad (3.36)$$

$$\bar{\partial} \psi_2 - \left( G_0 \sigma_{\bar{w}} + \frac{\partial b}{b} + iA_{\bar{w}} \right) \psi_2 - \left( \frac{\partial b}{b} - iA_w \right) \psi_{12} = 0 \ , \quad (3.37)$$

$$\bar{\partial} \psi_{12} - \left( \frac{\partial b}{b} + iA_{\bar{w}} \right) \psi_2 - \left( G_0 \sigma_{\bar{w}} + \frac{\partial b}{b} - iA_w \right) \psi_{12} = 0 \ . \quad (3.38)$$
From the difference of eqns. (3.28)-(3.32) and (3.37)-(3.38) one gets respectively

\[
\partial_z \psi_+ = \frac{G_0}{2bb} \psi_+ , \quad \bar{\partial}_z \psi_- = G_0 \sigma \bar{w} \psi_- . \tag{3.39}
\]

Furthermore, [(3.31) − (3.29) − 2e^{−2Φ} (3.36)] yields

\[
\bar{\partial}_z \psi_1 = \frac{e^{2Φ}}{2} \partial_z \psi_- - G_0 \left( \frac{e^{2Φ}}{4bb} \psi_+ - \sigma \bar{w} \psi_1 \right) . \tag{3.40}
\]

Obviously for \( G_0 = 0 \), the equations (3.27)-(3.38) simplify significantly. Let us now study this particular case under the additional assumption \( \psi_- \neq 0 \) and \( \psi_1 \neq 0 \).

### 3.3 Case \( G_0 = 0 \), \( \psi_- \neq 0 \) and \( \psi_1 \neq 0 \)

For \( G_0 = 0 \) one gets from (3.33), (3.39) and (3.40)

\[
\psi_1 = \psi_1(z) , \quad \psi_- = \psi_-(w) .
\]

Assuming \( \psi_- \neq 0 \), the gaugini equations (3.11)-(3.14) imply

\[
\partial z^\alpha = -\frac{4i}{b} g^{\alpha\beta} D_\beta \bar{X}^I g_I \frac{\psi_- \bar{\psi}_2}{\psi_- \psi_- + e^{-2Φ} \psi_1}, \tag{3.41}
\]

\[
\partial z^\alpha = \frac{\psi_1}{2\psi_-} \partial z^\alpha , \tag{3.42}
\]

\[
\bar{\partial} z^\alpha = \frac{\bar{\psi}_1}{2\psi_-} \partial z^\alpha , \tag{3.43}
\]

\[
0 = g^{\alpha\beta} D_\beta \bar{X}^I g_I \left( \psi_2 \bar{\psi}_2 - \psi_1 \bar{\psi}_1 \right) . \tag{3.44}
\]

From eqns. (3.42) and (3.43) we obtain

\[
A_z \psi_1 - 2A_w \psi_- = 0 . \tag{3.45}
\]

(3.27)+(3.30) and (3.29)-(3.31) yield respectively

\[
\partial_z \psi_1 = \psi_1 \partial_z \ln |b| - 2 \psi_- \partial_z \ln |b| , \tag{3.46}
\]

\[
0 = \psi_- \partial_z \ln |b| + 2e^{-2Φ} \psi_1 \bar{\partial} \ln |b| - 2i \left( \frac{\bar{X} \cdot g}{b} \psi_2 + \frac{X \cdot g}{b} \psi_1 \right) . \tag{3.47}
\]

Using (3.46) and (3.47) it is easy to shew that

\[
\bar{\psi}_1 \partial_z \psi_1 - \psi_1 \partial_z \bar{\psi}_1 = 2ie^{2Φ} \left( \frac{X \cdot g}{b} + \frac{\bar{X} \cdot g}{b} \right) \left( \psi_2 \bar{\psi}_2 - \psi_1 \bar{\psi}_1 \right) . \tag{3.48}
\]
Because we are interested only in the case in which $g^{\alpha\bar{\beta}}D_{\bar{\beta}}X^I g_I \neq 0^{10}$, (3.44) implies $|\psi_2| = |\psi_{12}|$ and thus from (3.48) one gets

$$\tilde{\psi}_1 \partial_z \psi_1 - \psi_1 \partial_z \tilde{\psi}_1 = 0.$$  \hfill (3.49)

Hence $\psi_1 = \zeta(z)e^{i\theta_0}$ where $\theta_0$ is a constant and $\zeta(z)$ is a real function. By rescaling $\psi_1 \rightarrow e^{-i\theta_0}\psi_1$ we can take $\psi_1$ real and positive without loss of generality. By assumption both $\psi_1$ and $\psi_-$ are non-vanishing, which allows to introduce new coordinates $Z$, $W$ and $\bar{W}$ such that

$$dZ = -\frac{2dz}{\psi_1(z)} , \quad dW = \frac{dw}{\psi_-(w)} , \quad d\bar{W} = \frac{d\bar{w}}{\psi_-(\bar{w})} .$$

Note that one can set $\psi_- = 1$ using the residual gauge invariance $w \mapsto W(w)$, $\Phi \mapsto \Phi - \frac{1}{2}\ln(dW/dw) - \frac{1}{2}\ln(d\bar{W}/d\bar{w})$ leaving invariant the metric $e^{2\Phi}dwd\bar{w}$. We can thus take $W = w$ in the following. (3.27) and (3.30) are then equivalent to

$$(\partial_Z + \partial)\varphi = 0 , \quad \partial_Z \ln \psi_1 - (\partial_Z + \partial) \ln r = 0 .$$

From the real part of the first equation one has

$$\varphi = \varphi(Z - w - \bar{w}) .$$

Using $\psi_1 = \psi_1(Z)$, the second equation implies

$$ (\partial_Z + \partial) \frac{r}{\psi_1} = 0 , $$  \hfill (3.50)

and therefore

$$ \frac{r}{\psi_1} = \rho(Z - w - \bar{w}) .$$

The function $b$ must thus have the form

$$ b(Z, w, \bar{w}) = \psi_1(Z)B(Z - w - \bar{w}) ,$$

where $B(Z - w - \bar{w}) = \rho(Z - w - \bar{w})e^{i\varphi(Z - w - \bar{w})}$. Taking into account (2.14) and (3.50), the difference between (3.34) and (3.35) yields

$$(\partial_Z + \partial)(\ln \psi_1 - \Phi) = 0 ,$$

---

$^{10}$One readily shows that $g^{\alpha\bar{\beta}}D_{\bar{\beta}}X^I g_I = 0$ leads to $\partial_{\beta}V = 0$, where $V$ is the scalar potential (2.8). Unless there are flat directions in the potential, these equations completely fix the moduli which are thus constant.
so that \( \ln \psi_1 - \Phi = -H(Z - w - \bar{w}) \) with \( H \) real. This gives

\[
e^{2\Phi} = \psi_1^2 e^{2H}
\]

for the conformal factor. The conditions (3.41)-(3.44) coming from the gaugino variations boil down to

\[
\partial_Z z^\alpha = \frac{i}{B} g^{\alpha \beta} D_\beta \bar{X} I g_I \frac{1 - \psi_+}{1 + e^{-2H}} , \quad (3.51)
\]

\[
\partial z^\alpha = \bar{\partial} z^\alpha = -\partial_Z z^\alpha , \quad (3.52)
\]

\[
\bar{\psi}_+ = -\psi_+ . \quad (3.53)
\]

From equation (3.52) we obtain that \( z^\alpha = z^\alpha (Z - w - \bar{w}) \). In terms of the new coordinate \( Z, \ (2.14) \) reads

\[
\partial_Z \Phi + i \left( \frac{\bar{X} \cdot g B - X \cdot g \bar{B}}{B} \right) = 0 .
\]

Using the definition of \( H \) we get

\[
\partial_Z \ln \psi_1 = -\dot{H} - i \left( \frac{\bar{X} \cdot g B - X \cdot g \bar{B}}{B} \right) , \quad (3.54)
\]

where a dot denotes a derivative w.r.t. \( Z - w - \bar{w} \). As the lhs depends only on \( Z \) and the rhs depends only on \( Z - w - \bar{w} \), we can conclude that \( \partial_Z \ln \psi_1 = \kappa \) with some real constant \( \kappa \), i.e., \( \psi_1 (Z) = \psi_1^{(0)} e^{\kappa Z} \). By shifting \( Z \) one can set \( \psi_1^{(0)} = 1 \). The only remaining nontrivial equations in the system (3.27)-(3.38) read

\[
\partial_Z \psi_+ - 2 \left( \frac{\dot{\rho}}{\rho} - \dot{H} \right) \psi_+ + 2i (\dot{\varphi} - AZ) + 2i \left( \frac{\bar{X} \cdot g B}{B} + \frac{X \cdot g}{B} \right) = 0 , \quad (3.55)
\]

\[
\partial_Z \psi_+ - \left( 2 \frac{\dot{\rho}}{\rho} - \dot{H} + \kappa \right) \psi_+ - 2i e^{-2H} (\dot{\varphi} - AZ) - i \left( \frac{\bar{X} \cdot g B}{B} + \frac{X \cdot g}{B} \right) = 0 , \quad (3.56)
\]

\[
\partial \psi_+ + 2 \left( \frac{\dot{\rho}}{\rho} - \dot{H} \right) \psi_+ - 2i (\dot{\varphi} - AZ) - 2i \left( \frac{\bar{X} \cdot g B}{B} + \frac{X \cdot g}{B} \right) = 0 , \quad (3.57)
\]

\[
\frac{i}{\dot{\rho}} \left( \frac{\bar{X} \cdot g B}{B} + \frac{X \cdot g}{B} \right) \psi_+ + 2 (1 + e^{-2H}) \frac{\dot{\rho}}{\rho} - \dot{H} + \kappa = 0 . \quad (3.58)
\]

From (3.55)+(3.57) and (3.53)+(3.58) we obtain respectively

\[
(\partial_Z + \bar{\partial}) \psi_+ = 0 , \quad (3.60)
\]

\[
(\partial_Z + \bar{\partial}) \psi_+ = -2\dot{H} \psi_+ - 2i \left( \frac{\bar{X} \cdot g B}{B} + \frac{X \cdot g}{B} \right) . \quad (3.61)
\]
Since $\psi_+$ is imaginary (cf. (3.53)), (3.60) implies $\psi_+ = \psi_+(Z - w - \bar{w})$ so that (3.61) yields

$$\dot{H}\psi_+ + i\left(\frac{\dot{X}\cdot g}{B} + \frac{X\cdot g}{B}\right) = 0 .$$

(3.62)

Using these informations, eqns. (3.55)-(3.59) reduce further to

$$\begin{align*}
\left[(1 + e^{2H})\frac{\psi_+}{\rho^2}\right]' - \kappa e^{2H}\frac{\psi_+}{\rho^2} &= 0 , \\
\left(\frac{\psi_+}{\rho^2}\right)' + 2i\frac{\dot{\phi} - A_Z}{\rho^2} &= 0 , \\
\dot{H}(1 + \psi_+^2) - 2\frac{\dot{H}}{\rho}(1 + e^{-2H}) &= \kappa .
\end{align*}$$

(3.63)

(3.64)

(3.65)

Eliminating $\dot{\rho}/\rho$ from (3.63) and (3.65) leads to

$$\dot{H}\psi_+(1 - \psi_+^2) + (1 + e^{-2H})\dot{H}\psi_+ = 0 ,$$

(3.66)

that can be integrated to give

$$\psi_+ = \frac{ia}{\sqrt{1 + e^{2H} - a^2}} ,$$

(3.67)

where $a$ is real integration constant. To proceed we observe that from (3.54) and (3.62) one obtains for the function $B$,

$$B = -\frac{2i\dot{X}\cdot g}{\dot{H}(1 + \psi_+) + \kappa} ,$$

(3.68)

and thus for its absolute value $\rho$ and phase $\varphi$

$$\rho^{-2} = \frac{(\kappa + \dot{H})^2 - \dot{H}^2\psi_+^2}{4X\cdot g\dot{X}\cdot g} ,$$

(3.69)

$$\tan \varphi = i\frac{(X\cdot g + \dot{X}\cdot g)(\kappa + \dot{H}) + (X\cdot g - \dot{X}\cdot g)(\dot{H}\psi_+)}{(X\cdot g - \dot{X}\cdot g)(\kappa + \dot{H}) + (X\cdot g + \dot{X}\cdot g)(\dot{H}\psi_+)} .$$

(3.70)

Using (3.69), (3.63) yields a relation between $H$ and $X\cdot g$,

$$0 = 2(1 + e^{-2H})\dot{H} + \dot{H}^2(1 + 3\psi_+^2) - \kappa^2$$

$$-\frac{(\dot{H} + \kappa)^2 - \dot{H}^2\psi_+^2}{\dot{H}(1 - \psi_+^2) + \kappa} (1 + e^{-2H}) \left(\frac{\dot{X}\cdot g}{X\cdot g} + \frac{\dot{X}\cdot g}{X\cdot g}\right) ,$$

(3.71)
while (3.64) gives $A_Z$,

$$A_Z = \frac{i}{2} \left\{ (1 + \psi_+) \frac{\dot{X} \cdot g}{X \cdot g} - (1 - \psi_+) \frac{\dot{X} \cdot g}{X \cdot g} \right\} \quad (3.72)$$

$$- \frac{\dot{H} \psi_+ (1 - \psi_+^2) (1 + e^{-2H})^{-1}}{(\dot{H} + \kappa)^2 - \dot{H}^2 \psi_+^2} \left[ 2 \left(1 + e^{-2H}\right) \ddot{H} + (1 + 3 \psi_+^2) - \kappa^2 \right].$$

Making use of (3.71), this boils down to

$$A_Z = - \left[ \dot{H} (1 - \psi_+^2) + \kappa \right]^{-1} \text{Im} \left\{ \left[ \dot{H} (1 - \psi_+) + \kappa \right] (1 + \psi_+) \frac{\dot{X} \cdot g}{X \cdot g} \right\}. \quad (3.73)$$

The condition (2.17) is then automatically satisfied: Plugging the relation

$$\dot{X} \cdot g + iA_Z X \cdot g = \dot{\mathcal{N}} B^\alpha D_\alpha X \cdot g + \frac{i}{B} g^{\alpha \beta} D_\alpha X \cdot g D_\beta \dot{X} \cdot g \frac{1 - \psi_+}{1 + e^{-2H}},$$

(where we used (3.51) in the second step) into

$$- \frac{1}{2} (\text{Im} \mathcal{N})^{-1/2} g_I g_J = X \cdot g \dot{\mathcal{N}} B^\alpha D_\alpha X \cdot g + g^{\alpha \beta} D_\alpha X \cdot g D_\beta \dot{X} \cdot g,$$

that follows from special geometry [23], one gets

$$(\text{Im} \mathcal{N})^{-1/2} g_I g_J = -2X \cdot g \dot{\mathcal{N}} B^\alpha D_\alpha X \cdot g + \frac{4 \dot{X} \cdot g}{\dot{H}(1 + \psi_+) + \kappa} \frac{1 + e^{-2H}}{1 - \psi_+} \left( \dot{X} \cdot g + iA_Z X \cdot g \right).$$

Inserting this into (2.17), the latter becomes

$$0 = 2 \left(1 + e^{-2H}\right) \ddot{H} + (1 + 3 \psi_+^2) - \kappa^2$$

$$-2 \left[ \dot{H} (1 - \psi_+) + \kappa \right] \frac{1 + e^{-2H}}{1 - \psi_+} \left( \dot{X} \cdot g + iA_Z X \cdot g \right), \quad (3.74)$$

which coincides with (3.71) once we substitute in it the expression (3.73) for $A_Z$.

The Bianchi identities (2.15) and Maxwell equations (2.16) can be integrated once, with the result

$$\left(1 + e^{2H}\right) \left( \frac{X^I}{B} - \frac{\dot{X}^I}{B} \right) = - \kappa e^{2H} \left( \frac{X^I}{B} - \frac{\dot{X}^I}{B} \right) + i e^{2H} \left[ (\text{Im} \mathcal{N})^{-1/2} g_I \right] + 2i \dot{H} \psi_+ \left( \frac{X^I}{B} + \frac{\dot{X}^I}{B} \right) = i p^I, \quad (3.75)$$

\[\text{Page 16}\]
\begin{align}
(1 + e^{2H}) \left( \frac{F_I}{B} - \frac{\bar{F}_I}{B} \right) - \kappa e^{2H} \left( \frac{F_I}{B} - \frac{\bar{F}_I}{B} \right) - g_I e^{2H} \frac{\psi_+}{\rho^2} + i e^{2H} \left[ \frac{\text{Re} \, \mathcal{N}_{II} \left( \text{Im} \, \mathcal{N} \right)^{-1} J_I}{BB} g_I + 2i \bar{H} \psi_+ \left( \frac{F_I}{B} + \frac{\bar{F}_I}{B} \right) \right] = iq_I , \quad (3.76)
\end{align}

where \( p^I, q_I \) are integration constants. It is straightforward to show that (3.75) and (3.76) are implied by (3.51), (3.64)-(3.66) and (3.68) iff \( p^I = q_I = 0 \). \footnotemark

Finally, the shift vector \( \sigma \) follows from (2.18) that simplifies to
\begin{align}
\partial_Z \sigma_w = \frac{e^{-\kappa Z}}{4} \left( \frac{\psi_+}{\rho^2} \right) , \quad \partial \sigma_w - \bar{\partial} \sigma_w = -\frac{e^{-\kappa Z}}{2} \left( \frac{e^{2H} \psi_+}{\rho^2} \right) , \quad (3.77)
\end{align}
whose solution is
\begin{align}
\sigma = -\frac{e^{-\kappa Z}}{4} e^{2H} \frac{\psi_+}{\rho^2} (dw - d\bar{w}) . \quad (3.78)
\end{align}

Note that in the case \( \kappa \neq 0 \) one can always set \( \kappa = 1 \) by rescaling the coordinates.

The missing component \( \psi_0 \) of the second Killing spinor is determined by the system (3.21)-(3.24) that can be integrated straightforwardly. This yields (after going back to the original basis)
\begin{align}
\alpha = \hat{\alpha} - 2\kappa t - \frac{\psi_1}{2b} - \frac{e^{2\Phi} \psi_+}{2\psi_1 b} , \quad \beta = -\frac{e^{\Phi}}{2|b|} (1 - \psi_+ ) , \\
\gamma = -\beta b , \quad \delta = -\bar{b}\hat{\alpha} - \frac{\psi_1}{b} \quad (3.79)
\end{align}
for the second Killing spinor. Here, \( \hat{\alpha} \) denotes an integration constant. As is clear from (2.14) and (2.10), \( C(\epsilon^1, \epsilon_2) \), with \( C \in \mathbb{C} \) an arbitrary constant, is again Killing if \( (\epsilon^1, \epsilon_2) \) is. This means that multiplication of \( \alpha \) and \( \beta \) by \( C \) and of \( \gamma \) and \( \delta \) by \( C \) gives again a solution of the Killing spinor equations. Choosing \( \hat{\alpha} = 1/C \), in order to obtain the first Killing spinor when \( C \to 0 \), the norm squared of the associated Killing vector \( V_\mu = A(\epsilon^i, \gamma_i \epsilon_i) \) (with \( A \) given in (A.4)) turns out to be
\begin{align}
V^2 = -4|b|^2 \left\{ |1 - 2\kappa Ct|^2 - \left[ \frac{|C| \psi_1 (1 + e^{2H})}{2|b|} \right]^2 \frac{1 - a^2}{1 + e^{2H} - a^2} \right. \\
+ \left. \frac{\psi_1 e^{2H}}{|b|^2} \frac{a \text{Im} C}{\sqrt{1 + e^{2H} - a^2}} \right\}^2 - \left( \frac{2\psi_1 \text{Im} C}{|b|} \right)^2 . \quad (3.80)
\end{align}

\footnotetext{This does not mean that all the fluxes vanish.}
For $V^2 = 0$ the solution belongs also to the null class considered in [24]. This happens for $\text{Im} C = 0$, $\kappa = 0$, $a^2 < 1$ and

$$\dot{H} = \sqrt{\frac{8X \cdot \bar{X} \cdot g}{|C|(1 - a^2)^{1/2}} \frac{(1 + e^{2H} - a^2)^{3/4}}{1 + e^{2H}} } .$$

(3.81) is actually the general form of $\dot{H}$ in the case $\kappa = 0$. To see this, observe that (3.63) implies

$$(1 + e^{2H}) \psi + \rho^2 = i h_0 ,$$

(3.82) if $\kappa = 0$, where $h_0$ is a real integration constant. Using the expressions (3.67) and (3.69) for $\psi$ and $\rho^2$ we obtain exactly (3.81), with $h_0 |C|(1 - a^2)^{1/2} = 2a$. Plugging the expression for $\dot{H}$ into (3.51) we find that the scalars have to satisfy the flow equation

$$z^\alpha = - \left( h_0 X \cdot g \over a X \cdot g \right)^{1/2} \frac{g^{\alpha \beta} D_\beta \bar{X} \cdot g}{(1 + e^{2H})(1 + e^{-2H} - a^2)^{1/4}}.$$  

(3.83)

Using $w = x + iy$ and $dZ = dH \dot{H} + 2 dx$, the metric reads

$$ds^2 = -4 \rho^2 \left[ dt - e^{2H} \frac{i \psi_+}{2 \rho^2} dy \right]^2 + \frac{1}{4 \rho^2} \left( \frac{dH}{H} + 2 dx \right)^2 + \frac{e^{2H}}{\rho^2} (dx^2 + dy^2) ,$$

(3.84)

where $\psi_+$, $\rho^2$ and $\dot{H}$ are given by (3.67), (3.69) and (3.81) respectively. As a check, let us show that this solution does indeed coincide with one of the 1/2 BPS lightlike case classified in [24]. To this end, consider the coordinate transformation

$$u = \frac{2a}{h_0} (1 - a^2)^{-1/2} t + x + \mu(\chi) , \quad v = \frac{t}{\sqrt{2}} - \frac{h_0}{2 \sqrt{2} a} (1 - a^2)^{1/2} x + \nu(\chi) ,$$

$$\Psi = 4a \left( \frac{a}{h_0} \right)^{1/2} (1 - a^2)^{-1/4} t - 2 \left( \frac{h_0}{a} \right)^{1/2} (1 - a^2)^{3/4} y ,$$

$$\coth \chi = (1 - a^2)^{-1/2} (1 + e^{2H} - a^2)^{1/2} ,$$

with

$$\frac{d\nu}{d\chi} = \frac{(\tanh \chi)^{1/2}}{8 \sqrt{2} (X \cdot g \bar{X} \cdot g)^{1/2} (1 - a^2)^{1/4}} \left( \frac{h_0}{a} \right)^{1/2} , \quad \frac{d\mu}{d\chi} = - \frac{2 \sqrt{2} a}{h_0} (1 - a^2)^{-1/2} \frac{d\nu}{d\chi} .$$

Then, the metric (3.81), the fluxes (2.21) and the flow equation (3.83) become

$$ds^2 = -2 \sqrt{2} \coth \chi dudv + \frac{d\chi^2}{16 \sinh^2 \chi X \cdot g \bar{X} \cdot g} + \frac{d\Psi^2}{2 \sinh 2\chi} ,$$

(3.85)
\[
F^I = \frac{(\text{Im } \mathcal{N})^{-1/2} g_J}{4 \cosh^2 \chi (X \cdot g \overline{X} \cdot g \tanh \chi)^{1/2}} d\Psi \wedge d\chi, \quad \frac{dz^\alpha}{d\chi} = \frac{g^{\alpha\beta} D_\beta \overline{X} \cdot g}{\overline{X} \cdot g \sinh 2\chi},
\]

which are exactly the eqns. (5.33), (5.34) and (5.24) of [24]. We also see that in this case, \( a \) can be eliminated by a diffeomorphism, and thus is not really a parameter of the solution.

### 3.3.1 Summary

In the case \( D_z P + ie^{-2\Phi} b^2 \mathcal{F}_{zw} \neq 0 \) and \( \mathcal{G}_0 = 0 \) and under the additional assumptions \( \psi_- \neq 0 \) and \( \psi_1 \neq 0 \), the fields are given in terms of the solutions of the system

\[
\dot{z}^\alpha = - \left[ \dot{H} (1 + \psi_+) + \kappa \right] \frac{1 - \psi_+}{1 + e^{-2H}} \frac{g^{\alpha\beta} D_\beta \overline{X} \cdot g}{2 \overline{X} \cdot g},
\]

and (3.71), where \( \kappa = 0, 1 \), the scalars \( z^\alpha \) and the real function \( H \) depend only on the combination \( Z - w - \overline{w} \), and \( \psi_+ \) is given by (3.67), with \( a \in \mathbb{R} \) an arbitrary constant. Furthermore, a dot denotes a derivative w.r.t. \( Z - w - \overline{w} \). Once a solution \((z^\alpha, H)\) is determined, one defines \( \rho \) by (3.69). Then, the metric and the fluxes read respectively

\[
ds^2 = -4\rho^2 e^{2\kappa Z} \left[ dt - e^{2H - \kappa Z} \frac{\psi_+}{4\rho^2} (dw - d\overline{w}) \right]^2 + \frac{1}{\rho^2} \left( \frac{dZ^2}{4} + e^{2H} dwd\overline{w} \right),
\]

\[
F^I = 8\kappa e^{\kappa Z} \text{Im} \left[ \frac{\overline{X} \cdot gX^I}{\dot{H}(1 + \psi_+) + \kappa} \right] dt \wedge dZ
\]

\[
+ \frac{2ie^{\kappa Z}}{1 + e^{-2H}} \left\{ \psi_+ \left( \text{Im } \mathcal{N} \right)^{-1/2} g_J \right\} dt \wedge d(Z - w - \overline{w})
\]

\[
+ 4i\kappa \text{Im} \left[ \frac{(1 + \psi_+) \overline{X} \cdot gX^I}{\dot{H}(1 + \psi_+) + \kappa} \right] dt \wedge d(Z - w - \overline{w})
\]

\[
+ \frac{i}{4X \cdot g} \left( \dot{H} + \kappa \right)^2 \overline{X} \cdot g \left( 1 + e^{2H} \psi_+^2 \right) \left( \text{Im } \mathcal{N} \right)^{-1/2} g_J dt \wedge d(Z - w - \overline{w})
\]

\[
+ 4\kappa \text{Re} \left[ \frac{\overline{X} \cdot gX^I}{\dot{H}(1 + \psi_+) + \kappa} \right] \left[ \frac{dZ}{2} \wedge (dw - d\overline{w}) + dw \wedge d\overline{w} \right].
\]

### 3.3.2 Explicit solutions

We shall now give some explicit solutions for the simple model determined by the prepotential \( F = -i Z^0 Z^1 \) that has \( n_V = 1 \) (one vector multiplet), and thus just one
complex scalar $\tau$. Choosing $Z^0 = 1$, $Z^1 = \tau$ (cf. [23]), the symplectic vector $v$ reads

$$v = \begin{pmatrix} 1 \\ \tau \\ -i\tau \\ -i \end{pmatrix}.$$  

The Kähler potential, metric and kinetic matrix for the vectors are given respectively by

$$e^{-K} = 2(\tau + \bar{\tau}), \quad g_{\tau\bar{\tau}} = \partial_\tau \partial_{\bar{\tau}} K = (\tau + \bar{\tau})^{-2},$$

$$N = \begin{pmatrix} -i\tau & 0 \\ 0 & -i/\tau \end{pmatrix}.$$  

Note that positivity of the kinetic terms in the action requires $\text{Re} \tau > 0$. For the scalar potential one obtains

$$V = \frac{-4}{\tau + \bar{\tau}}(g_0^2 + 2g_0g_1\tau + 2g_0g_1\bar{\tau} + g_1^2\tau\bar{\tau}),$$

which has an extremum at $\tau = \bar{\tau} = |g_0/g_1|$. In what follows we assume $g_1 > 0$. The Kähler $U(1)$ is

$$A_\mu = \frac{i}{2(\tau + \bar{\tau})}\partial_\mu(\tau - \bar{\tau}).$$

In order to proceed we shall take $\tau = \bar{\tau}$ (this includes the extremum of the potential and thus the AdS vacuum). Then $A = 0$ and equation (3.73) imposes $\kappa \psi_+ = 0$ if $\dot{X}g \neq 0$. The case $\kappa = 0$ was considered in generality above, and an explicit solution of the flow equation (3.86) for the prepotential of this paragraph can be found in section 4.5 of [24] (put $G = 0$ there). Thus, we shall focus on the case $\psi_+ = 0$ in the following. Then, eqns. (3.71) and (3.87) boil down to

$$2(1 + e^{-2H})\ddot{H} + \dot{H}^2 - \kappa^2 + (1 + e^{-2H}) (\dot{H} + \kappa) \frac{g_0 - g_1\tau \dot{\tau}}{g_0 + g_1\tau \dot{\tau}} = 0,$$

$$\frac{\dot{\tau}}{\tau} = \frac{\dot{H} + \kappa \frac{g_0 - g_1\tau}{1 + e^{-2H} g_0 + g_1\tau}}{g_0 + g_1\tau}.$$  

Plugging (3.96) into (3.95) yields an expression for $\tau$ in terms of $H$ and its derivatives. Reinserting this into (3.96) gives a third order differential equation for $H$ only,

$$(1 + e^{-2H})^2 \dddot{H} + \left[(3 - 2e^{-2H}) (1 + e^{-2H}) \ddot{H} + \dot{H}^2 - \kappa^2\right] \dot{H} = 0.$$  

\[\text{– 20 –}\]
that can be integrated twice, with the result
\[
\dot{H} = \frac{1}{(1 + e^{2H})^{1/4}} \sqrt{2E_1 + \frac{E_2}{2(1 + e^{2H})^{1/2}} + \kappa^2 (1 + e^{2H})^{1/2}}, \tag{3.98}
\]
where \(E_1\) and \(E_2\) are two integration constants. If \(\dot{H} \neq 0\), we can use the function \(H\) in place of \(w + \bar{w}\) as a new coordinate. Using \(w = x + iy\), in the coordinate system \(\{t, H, y, Z\}\) the solution is given by
\[
ds^2 = -\left[\frac{2(g_0 + g_1 \tau)}{\sqrt{T}(\dot{H} + \kappa)}\right]^2 e^{2\kappa Z} dt^2
+ \left[\frac{2(g_0 + g_1 \tau)}{\sqrt{T}(\dot{H} + \kappa)}\right]^{-2} \left[dZ^2 + e^{2H} \left(dZ - \frac{dH}{H}\right)^2 + 4e^{2H} dy^2\right], \tag{3.99}
\]
\[
F^0 = -\frac{(\dot{H} + \kappa) (\kappa g_1 \tau - g_0 \dot{H})}{\dot{H} (g_0 + g_1 \tau)^2 (1 + e^{-2H})} dH \wedge dy,
\]
\[
F^1 = -\frac{\tau (\dot{H} + \kappa) (\kappa g_0 - g_1 \dot{H} \tau)}{\dot{H} (g_0 + g_1 \tau)^2 (1 + e^{-2H})} dH \wedge dy, \tag{3.100}
\]
\[
\tau = \frac{g_0 \sqrt{2(\dot{H} + \kappa)} (1 + e^{2H})^{1/2} - \sqrt{E_2}}{g_1 \sqrt{2(\dot{H} + \kappa)} (1 + e^{2H})^{1/2} + \sqrt{E_2}}. \tag{3.101}
\]
Asymptotically for \(H \to \infty\) the scalar field goes to its critical value, \(\tau \to g_0/g_1\), and the metric approaches AdS_4. A more detailed analysis of the geometry (3.99) will be presented elsewhere.

3.4 \(G_0 = \psi_- = 0\)

For \(G_0 = \psi_- = 0\) one has \(\psi_1 = \psi_1(z)\) by virtue of (3.33) and (3.40). Moreover, the sum of (3.27) and (3.30) yields
\[
\psi_1 = r \chi(w, \bar{w}), \tag{3.102}
\]
with \(\chi(w, \bar{w})\) an arbitrary function, while the difference of (3.27) and (3.30) implies \(A_z = \partial_z \varphi\). Subtracting (3.33) from (3.34) leads to
\[
\partial_z \ln r + 2i \left(\frac{X \cdot g}{b} - \frac{X^* \cdot g}{b}\right) = 0. \tag{3.103}
\]
Plugging this into (3.28), one gets $\partial_z \psi_2 = 0$. Using equ. (2.14) in (3.103), we obtain

$$e^\Phi = r \Lambda(w, \bar{w}) ,$$

(3.104)

where $\Lambda$ is again an arbitrary function. (3.38), together with $\partial_z \psi_2 = 0$, gives

$$\psi_2 = \frac{r^2}{\psi_1} \nu(w) ,$$

(3.105)

with $\nu(w)$ holomorphic. Note that (3.102), combined with $\psi_1 = \psi_1(z)$, forces the phase $\theta$ of $\psi_1$ to be constant. By rescaling all the $\psi_i$’s with $e^{-i\theta}$ we can thus choose $\psi_1$ real without loss of generality. From the gaugino equations (3.11)-(3.14) one has

$$A_w = \partial \varphi , \quad A_{\bar{w}} = \bar{\partial} \varphi ,$$

(3.107)

whereas their difference leads to

$$\psi_2^{-1} e^{-2\Phi} \bar{\partial} \ln r = i \left( \frac{X \cdot g}{b} + \bar{X} \cdot g \right) .$$

(3.108)

Taking the sum of (3.108) and its complex conjugate, and using (3.105), one obtains

$$(\bar{\nu}(\bar{w}) \bar{\partial} + \nu(w) \partial) r = 0 .$$

(3.109)

Let us first consider the subcase $\psi_2 \neq 0$, i.e., $\nu(w) \neq 0$. (The case $\psi_2 = 0$ will be dealt with in section 3.4.3.) This allows to introduce new coordinates $W, \bar{W}$ such that $\nu \partial = \partial_W, \nu \bar{\partial} = \partial_{\bar{W}}$. Using the residual gauge invariance $w \mapsto W(w), \Phi \mapsto \Phi - \frac{1}{2} \ln(dW/dw) - \frac{1}{2} \ln(d\bar{W}/d\bar{w})$ leaving invariant the metric $e^{2\Phi} dwd\bar{w}$, one can set $\nu(w) = 1$ and hence $w = W$ without loss of generality. Then, eqns. (3.106) and (3.109) boil down to

$$\partial_z z^\alpha = \partial_x z^\alpha = \partial_x r = 0 ,$$

(3.110)
where $x$ is defined by $w = x + iy$. Thus, $r = r(y)$, $z^\alpha = z^\alpha(y)$, $A_x = 0$, and from (3.107) also $\partial_x \varphi = 0$ so that $\varphi = \varphi(y)$. (3.108) simplifies to
\[
e^{-2\Phi} \partial_y r - 2r^2 (X \cdot g e^{i\varphi} + \bar{X} \cdot g e^{-i\varphi}) = 0.
\]
(3.111)

Plugging this into the sum of (3.34) and (3.35) yields
\[
\partial e^{2\Phi} = \frac{i}{2r^3} \partial_y r,
\]
(3.112)
which implies $(\partial + \bar{\partial}) \Phi = 0$, and thus $\Phi = \Phi(y)$. Integration of (3.112) gives then
\[
e^{2\Phi} = \frac{1}{4r^4} + L,
\]
(3.113)
with $L$ a real constant. In what follows, we shall use $r$ as a new coordinate in place of $y^{12}$. The only nontrivial gaugino equation of the system (3.11)-(3.14) becomes
\[
\frac{d}{dr} \frac{dz^\alpha}{r} = \frac{g^{\alpha\beta} D_\beta \bar{X} \cdot g}{X \cdot g}.
\]
(3.114)

One also has to check whether the equations (2.15)-(2.17) for the first Killing spinor are satisfied. The Bianchi identities (2.13) and Maxwell equations (2.16) can be integrated once, with the result
\[
\partial_y \left( \frac{X^I}{b} - \frac{\bar{X}^I}{b} \right) = ip^I, \quad \partial_y \left( \frac{F_I}{b} - \frac{\bar{F}_I}{b} \right) - \frac{ig_I}{r^4} = iq_I,
\]
(3.115)
where $p^I, q_I$ are integration constants. Using the flow equation (3.114) together with the special geometry relation [23]
\[
-\frac{1}{2} (\text{Im} \mathcal{N})^{-1/2} = \bar{X}^I X^J + g^{\alpha\beta} D_\alpha X^I D_\beta \bar{X}^J,
\]
(3.116)
one finds that (3.113), as well as (2.17), indeed hold, if $p^I = 0, q_I = 4Lg_I$.

Finally, the shift vector $\sigma$ follows from (2.18), which implies
\[
\sigma = \frac{dx}{4r^4}.
\]
Then the metric and the fluxes read respectively
\[
ds^2 = -4r^2 \left( dt + \frac{dx}{4r^4} \right)^2 + \frac{dz^2}{r^2} + \left( \frac{1}{4r^4} + L \right) \frac{dx^2}{r^2} + \frac{dr^2}{16r^6 X \cdot g \bar{X} \cdot g \left( \frac{1}{4r^4} + L \right)}.
\]
(3.117)

\footnote{This is possible as long as $X \cdot g \neq 0$, cf. (3.111).}
\[ F^I = -\frac{2}{\sqrt{X \cdot g X \cdot g}} (\text{Im} N)^{-1} g_J g_J dt \wedge dr . \]  

(3.118)

Actually the solutions with \( L \neq 0 \) can be cast into a simpler form by the coordinate transformation

\[ Lx = t - \psi , \quad \zeta = |L|^{1/2} z , \quad \rho^2 = \frac{1}{|L|^2} . \]

Defining also \( q^2 \equiv 4/|L| \), we get for \( L > 0 \)

\[ ds^2 = -\left( \rho^2 + \frac{q^2}{\rho^2} \right) dt^2 + \frac{d\rho^2}{4X \cdot g X \cdot g \left( \rho^2 + q^2/\rho^2 \right)} + \rho^2 (d\zeta^2 + d\psi^2) , \]

(3.119)

and for \( L < 0 \)

\[ ds^2 = \left( \rho^2 - \frac{q^2}{\rho^2} \right) dt^2 + \frac{d\rho^2}{4X \cdot g X \cdot g \left( \rho^2 - q^2/\rho^2 \right)} + \rho^2 (d\zeta^2 - d\psi^2) . \]

(3.120)

In both cases, the fluxes and the flow equation (3.114) become

\[ F^I = \frac{q}{\rho^2 \sqrt{X \cdot g X \cdot g}} (\text{Im} N)^{-1} g_J g_J dt \wedge d\rho , \quad -\rho \frac{d z^\alpha}{d\rho} = \frac{g^{\alpha \beta} D_{\beta} \bar{X} \cdot g X \cdot g}{X \cdot g} . \]

(3.121)

(3.119) represents a generalization of the naked singularity solution to minimal gauged supergravity found in [25] with nontrivial scalars turned on. Its double analytic continuation \( t \mapsto it, \psi \mapsto i\psi, q \mapsto -iq \) yields (3.120), which has the interpretation of a bubble of nothing [26]: In order to avoid the conical singularity at \( \rho^2 = q \equiv \rho_s^2 \) in the \((t, \rho)\)-hypersurface, we must compactify \( t \) such that\(^{13}\)

\[ t \sim t + \frac{\pi}{2 \rho_s |X_s|} . \]

Note that the limit \( L \to 0 \) is naively singular in the coordinates \( t, \rho, \zeta, \psi \), because the charge \( q \) diverges, but it can be taken if we perform a Penrose limit [27]: Start for instance from the \( L > 0 \) solution and set

\[ \psi - t = -\epsilon^2 X^+ , \quad \psi + t = 2X^- , \quad \rho = \frac{1}{\epsilon R} , \quad \zeta = \epsilon Z , \quad q = \frac{2}{\epsilon} . \]

Then, the limit \( \epsilon \to 0 \) leads to the regular solution

\[ ds^2 = -4R^2 dX^-^2 - \frac{2}{R^2} dX^- dX^+ + \frac{dR^2}{4R^2 X \cdot g X \cdot g} + \frac{dZ^2}{R^2} . \]

\(^{13}\)We assumed that \( \lim_{\rho \to \rho_s} g_I X^I(\rho) \equiv X_s \neq 0 . \)
\[ F^I = -\frac{2}{\sqrt{X \cdot g X \cdot g}} (\text{Im} \mathcal{N})^{-1/2} g^I_J dX^J \wedge dR , \]

which is nothing else than (3.117) and (3.118) for \( L = 0 \).

Integration of the system (3.21)-(3.24) yields

\[ \psi_0 = \hat{\psi}_0 - \frac{1}{2r^2} , \]

with \( \hat{\psi}_0 \) a complex constant. The second Killing spinor is thus

\[ \epsilon^1 = \left( \hat{\psi}_0 - \frac{1}{2r^2} \right) 1 + re^{\Phi} \epsilon_{12} , \quad \epsilon^2 = e^{\Phi - i\varphi} 1 - \left( \frac{1}{2b} + \bar{b} \bar{\psi}_0 \right) \epsilon_{12} . \]  

(3.122)

For \( \hat{\psi}_0 = 0 \), the norm squared of the associated Killing vector \( V_\mu = A(\epsilon^i, \gamma_\mu \epsilon^i) \) (with \( A \) given in (A.4)) reads

\[ V^2 = -4r^2 L^2 , \]  

(3.123)

which vanishes for \( L = 0 \), so that in this case the solution belongs to the null class as well. To understand what happens for \( L \neq 0 \), we have to consider a general linear combination of the two Killing spinors. As was explained earlier, the rescaling \( (\epsilon^1, \epsilon^2) \mapsto (C \epsilon^1, C \epsilon^2) \), with \( C \in \mathbb{C} \) an arbitrary constant, gives again a Killing spinor. If we apply this to (3.122) and choose \( \hat{\psi}_0 = 1/C \) (in order to recover the first covariantly constant spinor for \( C \to 0 \)), the associated Killing vector has norm squared

\[ V^2 = -4r^2 \left[ (1 + L|C|^2)^2 + \frac{\text{Im}^2 C}{r^4} \right] . \]  

(3.124)

This is zero iff \( \text{Im} C = 0 \), \( L = -1/|C|^2 \), i.e. \( L < 0 \). In conclusion, the half-BPS solutions of this subsection belong also to the lightlike class for \( L \leq 0 \). They must therefore correspond to some of the geometries of [24], where the half-supersymmetric null case was classified. This is indeed the case: Take the 1/2-BPS solutions with \( d\chi = 0 \) in section 5.2 of [24]. Consider there the subcase \( d = \bar{b} X \cdot g / \bar{X} \cdot g \), equ. (5.49). In order to solve the equations for half-supersymmetry, make the additional assumption that the function \( H \), the scalars \( z^a \) and the wave profile \( \mathcal{G} \) depend on \( w - \bar{w} \) only. Moreover, choose \( m_J = g_J \) and \( l^I = 0 \) in the expression (5.67) that determines the fluxes. As a solution of the eqns. (5.59), (5.62) for the wave profile take \( \mathcal{G} = -1/(4\rho^4) \). Finally, set \( u = -2\sqrt{2t} \), \( v = -x/8 \), \( w + \bar{w} = \sqrt{2}z \) and \( \rho = 1/r \). This yields the solution (3.114), (3.117), (3.118) with \( L = 0 \). Note that for constant scalars, the \( L = 0 \) solution reduces to a subclass of the charged generalization of the Kaigorodov spacetime found in [28].
If one starts instead from the half-BPS null case with \(d\chi \neq 0\), eqns. (5.24), (5.33), (5.34) in [24], and sets
\[
\begin{align*}
u &= A(t - Lx) + \frac{z}{\sqrt{2}A}, \\
v &= A(t - Lx) - \frac{z}{\sqrt{2}A},
\end{align*}
\]
where \(A = (2|L|)^{-1/4}\), one obtains the \(L < 0\) solution. Notice that the geometry described by eqns. (5.24), (5.33) and (5.34) of [24] appeared also in subsection 3.3.

### 3.4.2 \(X\cdot ge^{i\varphi} - \bar{X}\cdot ge^{-i\varphi} \neq 0\)

For \(X\cdot ge^{i\varphi} - \bar{X}\cdot ge^{-i\varphi} \neq 0\), taking into account that the scalar fields \(z^\alpha\) and the phase \(\varphi\) are independent of \(z\), integration of (3.103) yields
\[
r = 2iz(\bar{X}\cdot ge^{-i\varphi} - X\cdot ge^{i\varphi}) ,
\]
(3.125)
where a possible integration constant has been eliminated by shifting \(z\). Using this in (3.102) and keeping in mind that \(\psi_1\) depends on \(z\) only, one gets \(\psi_1 = cz\), with \(c\) a real integration constant that we can set equal to one without loss of generality by rescaling the \(\psi_1\)'s. Plugging (3.125) into (3.104), we have \(e^\Phi = z e^H\), with the real function \(H(w, \bar{w})\) given by
\[
e^H = 2i(\bar{X}\cdot ge^{-i\varphi} - X\cdot ge^{i\varphi})\Lambda(w, \bar{w}).
\]
From (3.105) one obtains
\[
\psi_2 = -4\nu (\bar{X}\cdot ge^{-i\varphi} - X\cdot ge^{i\varphi})^2.
\]

In what follows, it is convenient to introduce the real function \(Y = Y(w, \bar{w})\),
\[
Y = -\frac{e^{i\varphi}X\cdot g + e^{-i\varphi}\bar{X}\cdot g}{e^{i\varphi}X\cdot g - e^{-i\varphi}\bar{X}\cdot g},
\]
(3.126)
which is related to the phase \(\varphi\) of \(b\) by
\[
e^{2i\varphi} = -\frac{1 + iY \bar{X}\cdot g}{1 - iY \bar{X}\cdot g}.
\]

In terms of \(Y\), the expressions for \(\psi_2\) and \(b\) simplify to
\[
\psi_2 = \frac{16X\cdot g\bar{X}\cdot g}{1 + Y^2}\nu, \quad b = \frac{4i\bar{X}\cdot g}{1 - iY}z.
\]
(3.127)
The system (3.27)-(3.38) boils down to

\[
e^{2H} \nu = -\frac{i}{8X \cdot gX \cdot g} \left[ \partial Y - \frac{1 + Y^2}{2Y} \bar{\partial} \ln (X \cdot g \bar{X} \cdot g) \right],
\]

(3.128)

\[
\partial \left( e^{2H} \nu \right) = -\frac{i e^{2H} Y (1 + Y^2)}{32 X \cdot g X \cdot g},
\]

(3.129)

together with

\[
A_w = \frac{1}{2Y} \left[ (1 + iY) \partial \ln (X \cdot g) + (1 - iY) \bar{\partial} \ln (\bar{X} \cdot g) \right].
\]

Equ. (2.17) becomes

\[
2 \partial \bar{\partial} H = e^{2H} \left[ \frac{1}{2} + Y^2 + \frac{1 + Y^2}{8X \cdot gX \cdot g} (\text{Im} \mathcal{N})^{-1} g_{IJ} g_{IJ} \right].
\]

(3.130)

Using

\[
(\text{Im} \mathcal{N})^{-1} g_{IJ} g_{IJ} = -2X \cdot g \bar{X} \cdot g + \frac{i (1 + Y^2)}{8e^{2H}Y \nu} \partial \ln (X \cdot g \bar{X} \cdot g),
\]

that follows from (3.116), it is easy to shew that (3.130) is automatically satisfied if (3.128) and (3.129) hold.

The case \( \nu = 0 \) (and thus \( \psi_2 = \psi_{12} = 0 \)) will be considered in 3.4.3. In the remaining part of this subsection we shall assume \( \nu \neq 0 \), which allows to define new coordinates \( W, \bar{W} \) such that

\[
\partial W = \nu \partial, \quad \partial \bar{W} = \bar{\nu} \bar{\partial}.
\]

Making use of the residual gauge invariance \( w \mapsto W(w), \Phi \mapsto \Phi - \frac{1}{2} \ln (dW/dw) - \frac{1}{2} \ln (d\bar{W}/d\bar{w}) \) leaving invariant the metric \( e^{2\Phi} dw d\bar{w} \), one can set \( \nu(w) = 1 \) and hence \( w = W \) without loss of generality. The gaugino eqns. (3.11) and (3.14) reduce to

\[
(\partial + \bar{\partial}) z^\alpha = 0, \quad \partial z^\alpha = -\frac{8e^{2H} X \cdot g}{1 + i Y} g^{\alpha \beta} D_\beta \bar{X} \cdot g,
\]

(3.131)

which imply that \( z^\alpha = z^\alpha (w - \bar{w}) \). Note also that from (3.129) it follows that the functions \( H, Y \) depend on \( w - \bar{w} \) only.

The Bianchi identities (2.13) and Maxwell equations (2.16) are automatically satisfied. Finally, integration of (2.18) gives the shift vector

\[
\sigma = \frac{e^{2H}}{2z} (dw + d\bar{w}).
\]

(3.132)
Denoting with a dot the derivative w.r.t. \( i(w - \bar{w}) \), (3.131), (3.128) and (3.129) become

\[
\dot{z}^\alpha = \frac{8ie^{2iH}X \cdot g^{\alpha \beta}D_\beta \bar{X} \cdot g}{1 + iY} , \quad (3.133)
\]

\[
e^{2iH} = -\frac{1}{8X \cdot gX \cdot g} \left\{ \dot{Y} - \frac{1 + Y^2}{2Y} \left[ \ln (X \cdot g \bar{X} \cdot g) \right]' \right\} , \quad (3.134)
\]

\[
\dot{H} = -\frac{Y(1 + Y^2)}{64X \cdot gX \cdot g} . \quad (3.135)
\]

Combining (3.134) and (3.135) yields

\[
\begin{bmatrix}
\dot{Y} \\
X \cdot gX \cdot g
\end{bmatrix}' = -\frac{Y(1 + Y^2)}{32(X \cdot gX \cdot g)^2} \left\{ \dot{Y} - \frac{1 + Y^2}{2Y} \left[ \ln (X \cdot g \bar{X} \cdot g) \right]' \right\} + \left\{ \frac{1 + Y^2}{2Y} \left[ \ln (X \cdot g \bar{X} \cdot g) \right]' \right\} , \quad (3.136)
\]

which, integrated once, gives

\[
\left( \ln \frac{X \cdot g \bar{X} \cdot g}{1 + Y^2} \right)' = \frac{Y(1 + Y^2)}{64X \cdot gX \cdot g} - \frac{64YLX \cdot g \bar{X} \cdot g}{1 + Y^2} , \quad (3.137)
\]

where \( L \) is a real integration constant. Let us define

\[
e^\xi = \frac{64X \cdot g \bar{X} \cdot g}{1 + Y^2} ,
\]

and use \( \xi \) as a new coordinate instead of \( w - \bar{w} \). Then, the flow equation (3.133) becomes

\[
\frac{dz^\alpha}{d\xi} = \frac{i}{2X \cdot g}(1 - iY)g^{\alpha \beta}D_\beta X \cdot g , \quad (3.138)
\]

with \( Y \) given by \( Y^2 = 64e^{-\xi}X \cdot g \bar{X} \cdot g - 1 \). Setting \( x = (w + \bar{w})/2 \), the metric and the fluxes read respectively

\[
ds^2 = -z^2e^\xi \left[ dt + 4(e^{-2\xi} - L)dx \frac{dz}{z} + 4e^{-\xi} \frac{dz^2}{z^2} \right] + 16e^{-\xi}(e^{-2\xi} - L)dx^2 + \frac{4e^{-2\xi}d\xi^2}{Y^2(e^{-\xi} - Le^\xi)} , \quad (3.139)
\]

\[
F^I = 8i \left( \frac{X \cdot g X^I}{1 - iY} - \frac{X \cdot g \bar{X}^I}{1 + iY} \right) dt \wedge dz + \frac{4}{Y} \left[ \frac{2X \cdot g X^I}{1 - iY} + \frac{2X \cdot g \bar{X}^I}{1 + iY} + (\text{Im}N)^{-1}g_{IJ} \right] (zdt - 4Ldx) \wedge d\xi . \quad (3.140)
\]
For $L > 0$, the line element (3.139) can be cast into the simple form

$$ds^2 = 4e^{-\xi} \left( -z^2 d\hat{t}^2 + \frac{dz^2}{z^2} \right) + 16L(e^{-\xi} - Le^\xi) \left( dx - \frac{z}{2\sqrt{L}} d\hat{t} \right)^2$$

$$+ \frac{4e^{-2\xi} d\xi^2}{Y^2(e^{-\xi} - Le^\xi)},$$

(3.141)

where $\hat{t} \equiv t/(2\sqrt{L})$. (3.141) is of the form (3.3) of [29], and describes the near-horizon geometry of extremal rotating black holes. From (3.138) it is clear that the scalar fields have a nontrivial dependence on the horizon coordinate $\xi$ unless $D_\alpha X \cdot g = 0$.

While the generic hairy black holes with the near-horizon geometry (3.141) are still to be discovered, the solution with constant scalars is actually known: Start from the rotating generalization of the hyperbolic black hole solution to minimal gauged supergravity, given by [25]

$$ds^2 = -\frac{\Delta r}{\rho^2} \left[ dt + \frac{a}{\Xi} \sinh^2 \theta d\phi \right]^2 + \frac{\rho^2}{\Delta r} dr^2 + \frac{\rho^2}{\Delta \theta} d\theta^2 + \frac{\Delta \theta \sinh^2 \theta}{\rho^2} \left[ d\theta - \frac{r^2 + a^2}{\Xi} d\phi \right]^2,$$

$$A = -\frac{q_e r}{\rho^2} \left[ dt + \frac{a}{\Xi} \sinh^2 \theta d\phi \right] - \frac{q_m \cosh \theta}{\rho^2} \left[ d\theta - \frac{r^2 + a^2}{\Xi} d\phi \right],$$

with

$$\Delta_r = (r^2 + a^2) \left( -1 + \frac{r^2}{\ell^2} \right) - 2mr + q_e^2 + q_m^2,$$

$$\Delta_\theta = 1 + \frac{a^2}{\ell^2} \cosh^2 \theta,$$

$$\rho^2 = r^2 + a^2 \cosh^2 \theta,$$

$$\Xi = 1 + \frac{a^2}{\ell^2}.$$

Here, $a$, $m$, $q_e$ and $q_m$ denote the rotation parameter, mass parameter, electric and magnetic charge respectively, and $\ell$ is related to the cosmological constant by $\Lambda = -3/\ell^2$. This black hole is both extremal and supersymmetric iff [25]

$$m = q_e = 0, \quad q_m = \pm \frac{\ell}{2} \Xi,$$

(3.142)

which leaves a one-parameter family of solutions, with horizon at $r^2 = r_h^2 = (\ell^2 - a^2)/2$.

In order to obtain the near-horizon family of solutions, with horizon at $r^2 = r_h^2 = (\ell^2 - a^2)/2$. In order to obtain the near-horizon limit, we introduce new coordinates $z, \hat{t}, \hat{\phi}$ according to

$$r = r_h + \epsilon r_0 z, \quad t = \frac{\hat{t}r_0}{\epsilon}, \quad \phi = \hat{\phi} + \Omega \frac{\hat{t}r_0}{\epsilon},$$

(3.143)

where $\Omega = a\Xi/(r_h^2 + a^2)$ is the angular velocity of the horizon, and $r_0$ is defined by

$$r_0^2 = \frac{\ell^2(r_h^2 + a^2)}{4r_h^2}. $$

– 29 –
After taking the limit $\epsilon \to 0$, the metric becomes

$$ds^2 = \frac{\ell^2 \rho_h^2}{4r_h^2} \left[ -z^2 dt^2 + \frac{dz^2}{z^2} \right] + \frac{\rho_h^2}{\Delta_\phi} d\theta^2 + \frac{\Delta_\theta \sinh^2 \theta}{\rho_h^2 z^2} (r_h^2 + a^2)^2 (d\phi + k z dt)^2 ,$$

(3.144)

with

$$\rho_h^2 = r_h^2 + a^2 \cosh^2 \theta , \quad k = \frac{2r_h r_0^2 \Omega}{r^2 + a^2} .$$

If we set

$$e^{-\xi} = \frac{\ell^2 \rho_h^2}{16 r_h^2} , \quad x = \frac{32r_h^3 (r_h^2 + a^2)}{\ell^6 \Xi^2 a} \hat{\phi} , \quad L = \frac{\ell^8 \Xi^2}{1024 r_h^4} , \quad X \cdot g \bar{X} \cdot g = \frac{1}{4\ell^2} , \quad (\text{3.141})$$

reduces precisely to the near-horizon geometry (3.144).

Let us now come back to the case of arbitrary $L$. The missing component $\psi_0$ of the second Killing spinor is determined by the system (3.21)-(3.24), that simplifies to

$$\partial_t \psi_0 = 1 , \quad \partial_z \psi_0 = \frac{1 + Y^2}{32 z^2 X \cdot g \bar{X} \cdot g} , \quad \partial \psi_0 = -\bar{\partial} \psi_0 = \frac{ie^{2H}Y}{2z} .$$

(3.145)

Integration of (3.145) yields (after going back to the original basis)

$$\alpha = \hat{\alpha} + t - \frac{1 + Y^2}{32 z X \cdot g \bar{X} \cdot g} , \quad \beta = -\frac{4iX \cdot g e^{H}}{1 + iY} e^{i\phi} ,$$

$$\gamma = \frac{e^{H}}{z} e^{-i\phi} , \quad \delta = \frac{4iX \cdot g}{1 + iY} z (\bar{\alpha} + t) - \frac{1 - iY}{8iX \cdot g} ,$$

(3.146)

where $\hat{\alpha} \in \mathbb{C}$ denotes an integration constant. As before, we rescale $\alpha, \beta$ by $C$ and $\gamma, \delta$ by $\bar{C}$, with $C \in \mathbb{C}$ constant, and choose $\hat{\alpha} = 1/C$ in order to obtain the first Killing spinor for $C \to 0$. Then, the norm squared of the associated Killing vector turns out to be

$$V^2 = -4|b|^2 \left[ 1 + Ct \right]^2 + \left( \frac{e^{2H}}{z^2} - \frac{z^2}{4|b|^4} \right) |C|^2 \right]^2 - \left( \frac{2z \text{Im} C}{|b|} \right)^2 ,$$

(3.147)

which is always negative, so that the solutions considered here do not belong to the null class.\(^{14}\)

Notice that in minimal supergravity, the analogue of eqns. (3.134), (3.135) follow from the dimensionally reduced gravitational Chern-Simons action [30]. It would be interesting to see if something similar happens here. For instance, (3.133)-(3.135) might be related to the gravitational Chern-Simons system coupled to scalar fields. We hope to come back to these points in a future publication.

\(^{14}\)Of course, the choice $\hat{\alpha} = 1/C$ does not cover the case $\hat{\alpha} = 0$, which has to be treated separately. It is easy to show that the result is again a timelike vector.
3.4.3 $\psi_2 = 0$

In 3.4.1 and 3.4.2 we assumed $\nu \neq 0$, that is $\psi_2 \neq 0$. Let us now consider the case $G_0 = 0$ and $\psi_2 = \psi_{12} = 0$. The gaugino equations (3.11)-(3.14) imply that the scalars $z^\alpha$ are constant, while the system (3.21)-(3.24) and (3.27)-(3.38) reduces to

$$
\begin{align*}
\partial_t \psi_0 &= -\frac{4iX \cdot g}{b} \psi_1 , \\
\partial_z \psi_0 &= \frac{\partial_t \psi_0}{2r^2} , \\
\partial \psi_0 &= \sigma_w \partial_t \psi_0 , \\
\partial_z \psi_0 &= \frac{4iX \cdot g}{b} \psi_1 ,
\end{align*}
$$

(3.148)

together with

$$
\begin{align*}
\partial_z r &= -4iX \cdot ge^{i\varphi} , \\
\partial r &= \partial \varphi = \partial_z \varphi = 0 , \\
e^{i\varphi} X \cdot g + e^{-i\varphi} \bar{X} \cdot g &= 0 .
\end{align*}
$$

(3.150)

From (2.18) one gets $\sigma = 0$, and (3.148)-(3.150) give

$$
\begin{align*}
\psi_0 &= \hat{\alpha} + t - \frac{1}{32zX \cdot g \bar{X} \cdot g} , \\
\psi_1 &= z , \\
b &= 4iX \cdot gz ,
\end{align*}
$$

(3.151)

where $\hat{\alpha} \in \mathbb{C}$ is an integration constant. It is straightforward to show that the Killing vector associated to a general linear combination of the two Killing spinors is always timelike. Integration of (2.14) yields $e^\Phi = ze^H$, with $H = H(w, \bar{w})$ a real function satisfying

$$
8\partial \bar{\partial} H = e^{2H} 
$$

(3.152)

due to (2.17). (3.152) is the Liouville equation and implies that the two-dimensional metric $e^{2H} dw d\bar{w}$ has constant negative curvature. Note that the Bianchi identities (2.13) and Maxwell equations (2.16) are automatically satisfied. The metric and fluxes read respectively

$$
\begin{align*}
ds^2 &= -64X \cdot g \bar{X} \cdot g z^2 dt^2 + \frac{dz^2}{16X \cdot gX \cdot g} + \frac{e^{2H} dw d\bar{w}}{16X \cdot gX \cdot g} ,
\end{align*}
$$

(3.153)

$$
F^I = -16\text{Im}(\bar{X} \cdot g X^I) dt \wedge dz + \frac{ie^{2H}}{16X \cdot gX \cdot g} \left[ 4\text{Re}(\bar{X} \cdot g X^I) + g_J (\text{Im}N)^{-1/2} \right] dw \wedge d\bar{w} .
$$

We have thus a product spacetime $\text{AdS}_2 \times \mathbb{H}^2$, with constant electric flux on $\text{AdS}_2$ and magnetic flux on $\mathbb{H}^2$. This is the near-horizon geometry of static supersymmetric black holes, like the ones discovered in [8].

3.5 Case $G_0 \neq 0$

For $G_0 \neq 0$, the gaugino eqns. (3.11)-(3.14) suggest to define new coordinates $Z, W, \bar{W}$ according to

$$
\begin{align*}
z &= z(Z, W, \bar{W}) , \\
w &= W , \\
\bar{w} &= \bar{W} ,
\end{align*}
$$

(3.154)
where
\[ \frac{\partial z}{\partial W} = -\frac{\psi_1}{2\psi_-}. \tag{3.155} \]
Then, (3.12) and (3.13) simplify to
\[ \partial W z^\alpha = \partial W z^\alpha = 0, \tag{3.156} \]
so that the scalars depend on \( Z \) only. The integrability conditions
\[ \frac{\partial^2 z}{\partial W \partial W} = \frac{\partial^2 z}{\partial W \partial \bar{W}}, \]
of (3.155) and its complex conjugate read
\[ \partial W \frac{\psi_1}{\psi_-} = \partial W \frac{\bar{\psi}_1}{\bar{\psi}_-}. \tag{3.157} \]
Remarkably, it can be shown that (3.157) is implied by the Killing spinor eqns. (3.27)-(3.38). Unfortunately, the system (3.27)-(3.38) does not seem to simplify much after the introduction of the coordinates \( Z, W, \bar{W} \), at least not in an obvious way, so that we were unable to solve it in general in the case \( G_0 \neq 0 \). For minimal \( \mathcal{N} = 2 \) gauged supergravity, all known 1/2 BPS solutions have either \( G_0 = 0 \), or are related to the case \( G_0 = 0 \) by a diffeomorphism [30]. This might be a general feature, and hold in the matter-coupled case as well, but we know of no way to show this in general.

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**A. Spinors and forms**

In this appendix, we summarize the essential information needed to realize spinors of \( \text{Spin}(3,1) \) in terms of forms (cf. also [31] and references therein).

Let \( V = \mathbb{R}^{3,1} \) be a real vector space equipped with the Lorentzian inner product \( \langle \cdot, \cdot \rangle \). Introduce an orthonormal basis \( e_1, e_2, e_3, e_0 \), where \( e_0 \) is along the time direction, and consider the subspace \( U \) spanned by the first two basis vectors \( e_1, e_2 \). The space of Dirac spinors is \( \Delta_c = \Lambda^*(U \otimes \mathbb{C}) \), with basis \( 1, e_1, e_2, e_{12} = e_1 \wedge e_2 \). The gamma matrices are represented on \( \Delta_c \) as
\[ \gamma_0 \eta = -e_2 \wedge \eta + e_2 \rbracket \eta, \quad \gamma_1 \eta = e_1 \wedge \eta + e_1 \lbracket \eta, \]
\[ \gamma_2 \eta = e_2 \wedge \eta + e_2 \rbracket \eta, \quad \gamma_3 \eta = ie_1 \wedge \eta - ie_1 \lbracket \eta, \tag{A.1} \]
where
\[ \eta = \frac{1}{k!} \eta_{j_1\ldots j_k} e_{j_1} \wedge \ldots \wedge e_{j_k} \]
is a \( k \)-form and
\[ e_i | \eta = \frac{1}{(k - 1)!} \eta_{j_1\ldots j_{k-1}} e_{j_1} \wedge \ldots \wedge e_{j_{k-1}}. \]

One easily checks that this representation of the gamma matrices satisfies the Clifford algebra relations \( \{ \gamma_a, \gamma_b \} = 2 \eta_{ab} \). The parity matrix is defined by \( \gamma_5 = i \gamma_0 \gamma_1 \gamma_2 \gamma_3 \), and one finds that the even forms \( 1, e_{12} \) have positive chirality, \( \gamma_5 \eta = \eta \), while the odd forms \( e_1, e_2 \) have negative chirality, \( \gamma_5 \eta = -\eta \), so that \( \Delta_c \) decomposes into two complex chiral Weyl representations \( \Delta^+_c = \Lambda^{\text{even}}(U \otimes \mathbb{C}) \) and \( \Delta^-_c = \Lambda^{\text{odd}}(U \otimes \mathbb{C}) \).

Let us define the auxiliary inner product
\[
\langle \sum_{i=1}^2 \alpha_i e_i, \sum_{j=1}^2 \beta_j e_j \rangle = \sum_{i=1}^2 \alpha_i^* \beta_i \tag{A.2}
\]
on \( U \otimes \mathbb{C} \), and then extend it to \( \Delta_c \). The Spin(3,1) invariant Dirac inner product is then given by
\[
D(\eta, \theta) = \langle \gamma_0 \eta, \theta \rangle. \tag{A.3}
\]
The Majorana inner product that we use is
\[
A(\eta, \theta) = \langle C\eta^*, \theta \rangle, \tag{A.4}
\]
with the charge conjugation matrix \( C = \gamma_{12} \). It is easy to show [6] that (A.4) is Spin(3,1) invariant as well.

A Killing spinor can be viewed as an SU(2) doublet \( (\epsilon^1, \epsilon^2) \), where an upper index means that a spinor has positive chirality. \( \epsilon^i \) is related to the negative chirality spinor \( \epsilon_i \) by charge conjugation, \( \epsilon_i^C = \epsilon^i \), with
\[
\epsilon_i^C = \gamma_0 C^{-1} \epsilon_i^* . \tag{A.5}
\]

As was shown in [6], there are three orbits of spinors under Spin(3,1), two of them with corresponding null bilinear \( V_\mu = A(\epsilon^i, \gamma_\mu \epsilon_i) \), and one with timelike \( V_\mu \). In the latter case, one can choose \( (\epsilon^1, \epsilon^2) = (1, be_2) \) as representative [6], with \( b \) a complex-valued function.

**B. The case** \( D_z P + i e^{-2\Phi} b^2 F_{zw} = 0 \)

In section 3.2, we simplified the equations for the second Killing spinor under the assumption \( D_z P + i e^{-2\Phi} b^2 F_{zw} \neq 0 \). Here we consider the case \( D_z P + i e^{-2\Phi} b^2 F_{zw} = 0 \).

\[ -33 - \]
From (3.16), one obtains then $DP = 0$ or $\psi_\perp = 0$. Let us first assume the latter, i.e., $\psi_2 = \psi_{12}$. Then, the $\partial_\beta, \partial_{\tilde{\gamma}}, \partial_\gamma, \partial_{\tilde{\gamma}}$ and $\partial_\gamma$ eqns. of (3.6)-(3.9) imply
\[
\partial_z \psi_2 = 2 \left[ \partial_z \ln r + 2i \left( \frac{X \cdot g}{b} - \frac{\bar{X} \cdot g}{b}\right) \right] \psi_2, \quad \partial_t \psi_2 = 0, \quad (B.1)
\]
\[
\left[ \partial_z \ln r + 2i \left( \frac{X \cdot g}{b} - \frac{\bar{X} \cdot g}{b}\right) \right] \psi_1 = 0, \quad (B.2)
\]
\[
e^{-2\Phi} (\bar{r} \ln r) \psi_1 - i \left( \frac{X \cdot g}{b} + \frac{\bar{X} \cdot g}{b}\right) \psi_2 = 0, \quad (B.3)
\]
\[
e^{-2\Phi} (A_{\bar{w}} - \bar{\partial} \varphi) \psi_1 + \left( \frac{X \cdot g}{b} - \frac{\bar{X} \cdot g}{b}\right) \psi_2 = 0. \quad (B.4)
\]
We have to suppose $\psi_1 \neq 0$ because otherwise (B.3) and (B.4) lead to $\psi_2 = 0$\textsuperscript{15} and thus there exists no further Killing spinor. Hence, (B.1) and (B.2) yield $\partial_z \psi_2 = 0$. Deriving (B.3) and (B.4) with respect to $t$ we get
\[
0 = \bar{\partial} r \partial_t \psi_1, \quad 0 = (A_{\bar{w}} - \bar{\partial} \varphi) \partial_t \psi_1.
\]
If $\partial_t \psi_1 \neq 0$ then $\bar{\partial} r = 0, \bar{\partial} \varphi = A_{\bar{w}}$ and (B.3), (B.4) give $\psi_2 = 0$. The gaugini equations (3.11)-(3.14) imply then that the scalar fields $z^\alpha$ must be constant. Moreover, since in this case $A_\mu = 0$, one has also $\partial \varphi = \bar{\partial} \varphi = 0$, which, together with $\partial r = \bar{\partial} r = 0$ leads to $b = b(z)$.

If instead $\partial_t \psi_1 = 0$, all the $\psi_i, i = 1, 2, 12$, are independent of $t$, and the Killing spinor equations reduce to the system (3.27)-(3.38) with $G_0 = 0$ and $\psi_\perp = 0$, which is solved in section 3.4.

In the case $DP = 0$, consider the integrability condition (3.15). As long as $D_z Q - ie^{-2\Phi} \bar{b}^2 F_{zw} \neq 0$ one could proceed exactly in the same way as in section 3.2. If $D_z Q - ie^{-2\Phi} \bar{b}^2 F_{zw} = 0$, (3.15) implies $\psi_\perp = 0$ or $DQ = 0$. The case $\psi_\perp = 0$ was already considered above, so the only remaining case is
\[
D_z P + ie^{-2\Phi} \bar{b}^2 F_{zw} = DP = D_z Q - ie^{-2\Phi} \bar{b}^2 F_{zw} = DQ = 0.
\]
For minimal gauged supergravity, one can show [30] that this brings us back again to the case $\psi_\perp = 0$. Perhaps an analogous reasoning can be applied here as well, although we shall not attempt to do this.

\textsuperscript{15}This is true if $X \cdot g \neq 0$. 

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- 34 –
References

[1] A. Strominger and C. Vafa, “Microscopic origin of the Bekenstein-Hawking entropy,” Phys. Lett. B 379 (1996) 99 [arXiv:hep-th/9601029].

[2] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, “A supersymmetric black ring,” Phys. Rev. Lett. 93 (2004) 211302 [arXiv:hep-th/0407065].

[3] H. K. Kunduri, J. Lucietti and H. S. Reall, “Do supersymmetric anti-de Sitter black rings exist?,” JHEP 0702 (2007) 026 [arXiv:hep-th/0611351].

[4] M. Cvetič et al., “Embedding AdS black holes in ten and eleven dimensions,” Nucl. Phys. B 558 (1999) 96 [arXiv:hep-th/9903214].

[5] Z. W. Chong, M. Cvetič, H. Lü and C. N. Pope, “Charged rotating black holes in four-dimensional gauged and ungauged supergravities,” Nucl. Phys. B 717 (2005) 246 [arXiv:hep-th/0411045].

[6] S. L. Cacciatori, D. Klemm, D. S. Mansi and E. Zorzan, “All timelike supersymmetric solutions of $\mathcal{N} = 2$, $D = 4$ gauged supergravity coupled to abelian vector multiplets,” JHEP 0805 (2008) 097 [arXiv:0804.0009 [hep-th]].

[7] M. M. Caldarelli and D. Klemm, “All supersymmetric solutions of $\mathcal{N} = 2$, $D = 4$ gauged supergravity,” JHEP 0309 (2003) 019 [arXiv:hep-th/0307022].

[8] S. L. Cacciatori and D. Klemm, “Supersymmetric AdS$^4$ black holes and attractors,” arXiv:0911.4926 [hep-th].

[9] O. Aharony, O. Bergman, D. L. Jafferis and J. Maldacena, “$\mathcal{N} = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals,” JHEP 0810 (2008) 091 [arXiv:0806.1218 [hep-th]].

[10] J. B. Gutowski and W. A. Sabra, “Half-supersymmetric solutions in five-dimensional supergravity,” JHEP 0712 (2007) 025 [arXiv:0706.3147 [hep-th]].

[11] J. Grover, J. B. Gutowski and W. Sabra, “Null half-supersymmetric solutions in five-dimensional supergravity,” JHEP 0810 (2008) 103 [arXiv:0802.0231 [hep-th]].

[12] K. Hristov, H. Looyestijn and S. Vandoren, “Maximally supersymmetric solutions of $D = 4$, $\mathcal{N} = 2$ gauged supergravity,” JHEP 0911 (2009) 115 [arXiv:0909.1743 [hep-th]].

[13] S. Ferrara, R. Kallosh and A. Strominger, “$\mathcal{N} = 2$ extremal black holes,” Phys. Rev. D 52 (1995) 5412 [arXiv:hep-th/9508072].

[14] A. Strominger, “Macroscopic entropy of $\mathcal{N} = 2$ extremal black holes,” Phys. Lett. B 383, 39 (1996) [arXiv:hep-th/9602111].
[15] S. Ferrara and R. Kallosh, “Supersymmetry and attractors,” Phys. Rev. D 54 (1996) 1514 [arXiv:hep-th/9602136].

[16] S. Ferrara and R. Kallosh, “Universality of supersymmetric attractors,” Phys. Rev. D 54 (1996) 1525 [arXiv:hep-th/9603090].

[17] S. Ferrara, G. W. Gibbons and R. Kallosh, “Black holes and critical points in moduli space,” Nucl. Phys. B 500 (1997) 75 [arXiv:hep-th/9702103].

[18] M. Huebscher, P. Meessen, T. Ortín and S. Vaulà, “Supersymmetric $\mathcal{N} = 2$ Einstein-Yang-Mills monopoles and covariant attractors,” Phys. Rev. D 78 (2008) 065031 [arXiv:0712.1530 [hep-th]].

[19] J. Grover, J. B. Gutowski, C. A. R. Herdeiro, P. Meessen, A. Palomo-Lozano and W. A. Sabra, “Gauduchon-Tod structures, Sim holonomy and de Sitter supergravity,” JHEP 0907 (2009) 069 [arXiv:0905.3047 [hep-th]].

[20] J. Grover, J. B. Gutowski, C. A. R. Herdeiro and W. Sabra, “HKT Geometry and de Sitter Supergravity,” Nucl. Phys. B 809 (2009) 406 [arXiv:0806.2626 [hep-th]].

[21] S. L. Cacciatori, M. M. Caldarelli, D. Klemm and D. S. Mansi, “More on BPS solutions of $\mathcal{N} = 2$, $D = 4$ gauged supergravity,” JHEP 0407 (2004) 061 [arXiv:hep-th/0406238].

[22] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D’Auria, S. Ferrara, P. Fré and T. Magri, “$\mathcal{N} = 2$ supergravity and $\mathcal{N} = 2$ super Yang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map,” J. Geom. Phys. 23 (1997) 111 [arXiv:hep-th/9605032].

[23] A. Van Proeyen, “$\mathcal{N} = 2$ supergravity in $d = 4, 5, 6$ and its matter couplings,” extended version of lectures given during the semester “Supergravity, superstrings and M-theory” at Institut Henri Poincaré, Paris, november 2000; http://itf.fys.kuleuven.ac.be/~toine/home.htm#B

[24] D. Klemm and E. Zorzan, “All null supersymmetric backgrounds of $\mathcal{N} = 2$, $D = 4$ gauged supergravity coupled to abelian vector multiplets,” Class. Quant. Grav. 26 (2009) 145018 [arXiv:0902.4186 [hep-th]].

[25] M. M. Caldarelli and D. Klemm, “Supersymmetry of anti-de Sitter black holes,” Nucl. Phys. B 545 (1999) 434 [arXiv:hep-th/9808097].

[26] E. Witten, “Instability of the Kaluza-Klein vacuum,” Nucl. Phys. B 195 (1982) 481.

[27] R. Penrose, “Any space-time has a plane wave as a limit,” in *Differential geometry and relativity*, Reidel, Dordrecht (1976), pp. 271-275.
[28] R. G. Cai, “Boosted domain wall and charged Kaigorodov space,” Phys. Lett. B 572 (2003) 75 [arXiv:hep-th/0306140].

[29] D. Astefanesei, K. Goldstein, R. P. Jena, A. Sen and S. P. Trivedi, “Rotating attractors,” JHEP 0610 (2006) 058 [arXiv:hep-th/0606244].

[30] S. L. Cacciatori, M. M. Caldarelli, D. Klemm, D. S. Mansi and D. Roest, “Geometry of four-dimensional Killing spinors,” JHEP 0707 (2007) 046 [arXiv:0704.0247 [hep-th]].

[31] J. Gillard, U. Gran and G. Papadopoulos, “The spinorial geometry of supersymmetric backgrounds,” Class. Quant. Grav. 22 (2005) 1033 [arXiv:hep-th/0410155].