THE DYNAMICS OF THREE-PLANET SYSTEMS: AN APPROACH FROM A DYNAMICAL SYSTEM

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ABSTRACT

We study in detail the motions of three planets interacting with each other under the influence of a central star. It is known that the system with more than two planets becomes unstable after remaining quasi-stable for long times, leading to highly eccentric orbital motions or ejections of some of the planets. In this paper, we are concerned with the underlying physics for this quasi-stability as well as the subsequent instability and advocate the so-called stagnant motion in the phase space, which has been explored in the field of a dynamical system. We employ the Lyapunov exponent, the power spectra of orbital elements, and the distribution of the durations of quasi-stable motions to analyze the phase-space structure of the three-planet system, the simplest and hopefully representative one that shows the instability. We find from the Lyapunov exponent that the system is almost non-chaotic in the initial quasi-stable state whereas it becomes intermittently chaotic thereafter. The non-chaotic motions produce the horizontal dense band in the action–angle plot whereas the voids correspond to the chaotic motions. We obtain power laws for the power spectra of orbital eccentricities. Power-law distributions are also found for the durations of quasi-stable states. With all these results combined together, we may reach the following picture: the phase space consists of the so-called KAM tori surrounded by satellite tori and imbedded in the chaotic sea. The satellite tori have a self-similar distribution and are responsible for the scale-free power-law distributions of the duration times. The system is trapped around one of the KAM torus and the satellites for a long time (the stagnant motion) and moves to another KAM torus with its own satellites from time to time, corresponding to the intermittent chaotic behaviors.

Key words: celestial mechanics – chaos – planets and satellites: dynamical evolution and stability

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1. INTRODUCTION

More than 300 exoplanets have been discovered so far and, interestingly, some of them have a quite different appearance from that in our solar system. The existence of so-called eccentric planets, that is, planets with high orbital eccentricities, for example, attracts attentions of many researchers (Butler et al., 2006). Stimulated by these observations, planetary formation theory including the origin of the eccentric planets has made substantial progress over the years (Kokubo et al., 2006). In the standard theory, terrestrial planets are thought to be formed through giant impacts of protoplanets (planets with sub-Earth masses) in crossing orbits (Chambers & Wetherill, 1998). Since N-body simulations suggest that protoplanets are formed in nearly circular orbits separated by several Hill radii (Kokubo & Ida, 1995), some destabilizing processes are expected to operate and make protoplanets originally in the well-separated circular orbits collide with each other and grow up to terrestrial planets.

The stability of the system with two planets around a central star has been thoroughly investigated in celestial mechanics and it is known that there exists a critical orbital separation between the planets, beyond which the planets never experience close encounters and the system remains stable forever (Marchal & Bozis, 1982; Gladman, 1993). The situation changes drastically, however, if another planet is added to the system. Using numerical simulations, Chambers et al. (1996) demonstrated that the systems with more than two planets become unstable even for large orbital separations. Although the planetary motions remain regular for some time at first, one of the planets eventually comes close enough to another (that is, within the Hill radius of the latter), leading to subsequent orbital crossings.

This is good news for the terrestrial formation theory and may also account for the formation of eccentric planets. In fact, Marzari & Weidenschilling (2002) numerically integrated the motions of three Jupiter-mass planets and found in most of their simulations that one of the planets is ejected from the system and the others are left in the system with high eccentricities. Several attempts (Juric & Tremaine, 2007; Ford & Rasio, 2008) have been made to reproduce the observed eccentricity distribution by orbital instability. Their results seem to be consistent with the observations although the latter itself may be somewhat biased (Shen & Turner, 2008).

The planetary motions in these systems are interesting in their own right. As mentioned above, we commonly observe a long period of quasi-regular motions that look like independent Keplerian motions before the eventual orbital crossings. The switch is sudden and quick. The duration of the quasi-regular motions is sensitive to the initial orbital separations and eccentricities (Chambers et al., 1996; Yoshinaga et al., 1999). Using numerical simulations and simplified analytical models, Zhou et al. (2007) claimed that the gradual deviation from the Keplerian motions can be regarded as a random walk process. These efforts notwithstanding, the underlying physics behind the phenomena such as the long period of quasi-regular motions followed by the sudden transition to chaotic states remains to be revealed and is the main concern of this paper.

We attempt to understand this phenomenon as the so-called stagnant motion in the phase space, which will be described below. We pay attention to a similar phenomenon known in the field of the dynamical system, a...
sudden transition from a regular motion sustained for a long time to chaotic motions is often observed. In the nonlinear lattice problems, for example, Hirooka & Saito (1969) found that an initially imposed normal mode experiences sudden energy exchanges among several other modes after long regular oscillations. They called this “the induction phenomenon” and referred to the duration of the regular motion as “the induction period” (Saito et al. 1970).

Aizawa et al. (1989) constructed a so-called stagnant motion model for this induction phenomenon. According to the KAM theorem (Kolmogorov 1979; Moser 1958; Arnold’ 1963), the phase space of a nearly integrable system retains tori, which exist in the integrable system, if perturbations to the integrable system are sufficiently weak. It is generally expected that the so-called KAM tori will survive even for not so small perturbations. In the stagnant motion model, it is assumed that the KAM torus exists in “the chaotic sea,” the region corresponding to chaotic motions of the system, being surrounded by a thin layer called “the stagnant layer,” in which smaller tori are distributed in a self-similar manner (see Figure 1). The system shows nearly regular behavior when the phase-space orbit is trapped in the stagnant layer whereas the sudden transition to chaotic motions occurs when the orbit escapes out of the layer. This model is successful in reproducing the scale-free power spectra and the distribution of the induction period. In this paper, we show some evidence to support the interpretation of the motions of the three-planet system as one of the induction phenomena and attempt to understand them in the framework of the stagnant motion model.

The organization of the paper is as follows. We summarize the numerical models in Section 2. In Section 3, we describe the methods of analysis. The results are presented in Section 4 and the summary and discussions are given in Section 5.

2. MODELS

In this paper, we restrict the investigation to the simplest multi-planet system, which shows the behavior mentioned above: the system consists of a central star with $M_\ast = 1 M_\odot$ and three planets with an identical mass in coplanar orbits.

We consider two cases for the planetary mass, $m_{pl}$: (1) $m_{pl} = 10^{-7} M_\odot$ (the protoplanet system) and (2) $m_{pl} = 10^{-3} M_\odot$ (the Jupiter system). The initial semimajor axis of the innermost planet is set to be $a_1 = 1$ AU for the protoplanet system and $a_1 = 5$ AU for the Jupiter system. Following Chambers et al. (1996), we give the initial radial locations of the outer planets as

$$a_{i+1} = a_i + \Delta r_{h(i,i+1)},$$

where $a_i$ is the semimajor axis of the $i$th planet counted from the innermost one, and $r_{h(i,i+1)}$ is the Hill radius for the pair of $i$th and $(i+1)$th planets defined as

$$r_{h(i,j)} = \left( \frac{m_i + m_j}{3 M_\ast} \right)^{1/3} \frac{a_i + a_j}{2}.$$

In addition, we also consider the case with $m_{pl} = 10^{-3} M_\odot$, $a_1 = 5$ AU, $a_2 = 7.25$ AU, $a_3 = 9.5$ AU, that is, the same parameter set as that used in Marzari & Weidenschilling (2002), except for no inclination in our model.

The Hamiltonian in the barycentric coordinates consists of three parts and is given as

$$H = H_{Kep} + H_\ast + H_{int}.$$  

$$H_{Kep} = \sum_{i=1}^{n} \left( \frac{p_{i}^2}{2m_i} - \frac{G m_0 m_i}{r_i} \right),$$  

$$H_\ast = \frac{p_0^2}{2m_0} - \sum_{i=1}^{n} \left( \frac{G m_0 m_i}{r_{i0}} - \frac{G m_0 m_i}{r_i} \right),$$  

$$H_{int} = - \sum_{i,j=1, i \neq j} \left( \frac{G m_i m_j}{r_{ij}} \right).$$

In the above equations, $G$ is the gravitational constant and $m_i$, $p_i$, and $r_i$ denote the mass, momentum, and radial position of the $i$th object, where $i = 0$ corresponds to the central star. $r_{ij}$ stands for the distance between the $i$th and $j$th objects. $H_{Kep}$ is an integrable Hamiltonian corresponding to independent Keplerian motions of three planets with respect to the barycenter. $H_\ast$ is a correction originated from the orbital motion of the central star itself by the attraction of the planets. $H_{int}$ expresses the interactions between the planets.

For each model listed in Table 1, we generate 10,000 different initial conditions, which have the same integrals of motion, that is, the same linear and angular momenta and total energy. This constraint is important in exploring the structure in the phase space. The orbital phases of planets are given randomly. Since the radial location of the innermost planet, $a_1$, is fixed, the remaining parameters, the radial locations of the outer planets, $a_2$, $a_3$, and the velocity of the central star, $V_{sx}$, $V_{sy}$, are determined so that the system should have the same values of the integrals of motion. As a result of this constraint, the resultant initial radial locations of the planets differ only slightly among 10,000 realizations. The planets have slight eccentricity ($e \sim 10^{-7}$ for $m_{pl} = 10^{-7}$ and $e \sim 10^{-3}$ for $m_{pl} = 10^{-3}$) initially because of the non-zero velocity of the central star relative to the barycenter. Using the ensemble obtained in this way, we obtain various distributions and take statistics thereof. We summarize in Table 1 the input parameters ($a_1$ and the integrals of motion) as well as the orbital separations and eccentricities averaged over the 10,000 realizations for each model.
Numerical integrations are performed with the MERCURY6 package, which was developed by Chambers (2000). For the protoplanet system, we integrate the orbital motions until the first close encounter occurs. For arbitrarily chosen three models among 10,000 realizations, we continue the integration after the close encounter up to 10^5 yr in order to compute the Lyapunov exponent and power spectra in the post-encounter phase. For the Jupiter system, the integration is terminated when one of the planets is ejected from the system.

Before closing this section, we discuss the relative magnitude of each part of the Hamiltonian given in Equation (3) and the existence of KAM tori in our system. The relative magnitude of $H_e$ to $H_{Kep}$ is always of the order of $m_pl/M_*$ and so is the ratio of $H_{int}$ to $H_{Kep}$ unless $r_{ij}/r_{i0}$ becomes as small as $\sim m_pl/M_*$. In our models, the value of $m_pl/M_*$ is $10^{-7}$ for the protoplanet system or $10^{-3}$ for the Jupiter system and the minimum value of $r_{ij}/r_{i0}$, which is achieved when two planets have the same orbital phase, is $\sim (m_pl/M_*)^{1/3} \gg m_pl/M_*$. Thus, $H_e$ and $H_{int}$ are always small in our models.

As mentioned in Section 1, the KAM tori exist in the phase space of the nearly integrable system, whose Hamiltonian is expressed as the sum of the integrable part $H_0(I)$ and the perturbation $\epsilon H(I, \theta)$,

$$H(I, \theta) = H_0(I) + \epsilon H(I, \theta) \quad (\epsilon \ll 1),$$

provided the perturbation is sufficiently small. Here $I$ and $\theta$ are the action and angle variable, respectively. It should be noted that our Hamiltonian does not meet this condition. Hence, we look for evidence that this is really the case, employing the Lyapunov exponent, power spectra of orbital elements, and induction periods, in addition to the trajectories in the $I-\theta$ plane, which will be described in the following section.

### 3. ANALYSIS METHODS

In order to get some insight into the phase-space structure of our models, we employ three tools: the Lyapunov exponent, the power spectra of orbital elements, and the distribution of the duration of the quasi-regular motions. The first two are useful to see the degree of chaos of the system. If there indeed remain KAM tori in the phase space, the system is expected to show both non-chaotic and chaotic features, which will then be captured by these measures. The last quantity will tell us if the planetary motions of our models can be interpreted as the induction phenomenon. In fact, the distribution is expected to have a power law if it is really the case.

#### 3.1. Lyapunov Exponent

We compute the so-called maximum Lyapunov exponent. The maximum global Lyapunov exponent, $\lambda_{global}$, is the local growth rate of the distance, $||\delta x(t)||$, between adjacent orbits in the phase space and is defined more precisely as

$$\lambda_{global} \equiv \lim_{T \to \infty} \frac{1}{T} \log \frac{||\delta x(T)||}{||\delta x(0)||}. \quad (5)$$

It is, of course, impossible in practice to compute the growth of the distance over the infinite time. We calculate instead the following quantity as a function of time and look into their behavior:

$$\lambda_{global}(t) = \frac{1}{t} \log \frac{||\delta x(t)||}{||\delta x(0)||}. \quad (6)$$

The asymptotic limit of this function at $t \to \infty$ gives the original Lyapunov exponent.

The integrable system has a null Lyapunov exponent while chaotic systems have a positive Lyapunov exponent. If the
system is nearly integrable in particular, the Lyapunov exponent
defined by Equation (6) oscillates around a small but finite positive
value and does not converge as $t \to \infty$.

In analyzing the system that shows both quasi-regular and
chaotic behaviors alternatively, it is useful to look also into the
local Lyapunov exponent, $\lambda_{\text{local}}$, which is the same local
growth rate of the orbital separation in much shorter times and is
expressed as

$$ \lambda_{\text{local}}(n, \tau) = \frac{1}{\tau} \log \frac{\|\delta x(n, \tau)\|}{\|\delta x((n-1), \tau)\|}, $$

(7)

where $n$ specifies an interval with a period of $\tau$. The choice
of $\tau$ is rather arbitrary. It should be longer than the typical
orbital period but shorter than the timescale of the quasi-regular
or chaotic motions of interest. If chosen appropriately, it will
indicate the local degree of chaos.

In the following, the interval $\tau$ for the local Lyapunov exponent
is chosen to be 100 yr, which corresponds to 10–100 times the orbital periods. Both $\lambda_{\text{global}}$ and $\lambda_{\text{local}}$ are obtained by numerically integrating the linearized equations of motion along the phase-space orbit given by the integration of the equations of motion (Wolf et al. 1985).

3.2. Power Spectra of Orbital Elements

The power spectrum, $S(f)$, of an orbital element denoted by $z(t)$ is defined as

$$ S(f) = \left| \int_0^\infty z(t)e^{ij\beta t} dt \right|^2 $$

(8)

and is also useful to characterize the chaotic system. If one defines the autocorrelation function of $z(t)$ as

$$ \Phi_z(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} z(t)z(t+\tau)dt, $$

(9)

its Fourier transform is equal to the power spectrum according to the Wiener–Khintchin theorem (e.g., Leichl 1980):

$$ S(f) = \int_{-\infty}^{\infty} \Phi_z(\tau)e^{-ij\beta t} d\tau. $$

(10)

Hence by investigating the power spectrum, we can acquire the knowledge of the temporal autocorrelation of the orbital element.

The integrable systems have a discrete spectrum with peaks at the orbital periods provided appropriate variables are chosen. On the other hand, the chaotic systems have a rather featureless continuum spectrum. In particular, it is known that nearly integrable systems show in general a power-law spectrum in their low frequency regime as

$$ S(f) \propto f^{-\nu}, $$

(11)

indicating a long-time correlation for the variables.

In this paper, we discuss the power spectra of the orbital eccentricity, $e_i$, for each planet in the system. We confirmed, however, that other orbital elements such as the semimajor axis give a similar result.

3.3. Distribution of Induction Periods

For the induction phenomena, it is known that the distribution of the induction periods, $T$, or the duration of quasi-regular motions generally obeys a power law,

$$ P(T) \propto T^{-\beta} \quad (\beta > 0), $$

(12)

for large $T$ (Baouer & Bertsch 1990). This implies that there is no characteristic timescale for the induction period. This type of distribution can be obtained in the stagnant motion model by evoking a collection of tori, which have a self-similar distribution, in the so-called stagnant layer around a KAM torus in the phase space (Aizawa et al. 1989). Figure 1 illustrates schematically the phase-space structure assumed in the stagnant motion model. The stagnant layer is filled with self-similarly distributed tori, which trap the system around them for a long time. Since the trapping time is scale-free thanks to the self-similar distributions of the tori, the power law is obtained for the induction period, which is nothing but the trapping time.

We will use this feature to see if the planetary motions of our concern are indeed the induction phenomenon. We expect that the duration of quasi-regular motions corresponds to the induction period. More precisely, we define the duration of the regular motion as the interval from the start of the integration of motion until the first close encounter between two planets. Close encounter here means that the distance between the pair of planets is smaller than their Hill radius. It is known from previous papers and confirmed in this paper that the transition from the regular motion to the chaotic one occurs in general after the first close encounter (Chambers et al. 1996). Incidentally, since the ejection of one planet occurs rather soon after the encounter for the Jupiter system, we also study the distribution of the time from the encounter to the ejection.

4. RESULTS

4.1. Lyapunov Exponents

We first show in Figure 2 the evolution of the orbital semimaj-
or axis and eccentricity of each planet for model 11, which is
representative of the protoplanet system. In this model, the first
close encounter between two planets occurs at $t = 16,400$ yr.
Before the encounter, the planetary motions are almost regular
and the orbital elements remain unchanged essentially. After
the encounter, on the other hand, they start to vary on short
timescales. Then comes a period ($t \approx 23,500–38,500$ yr) when
the semimajor axes do not change very much and the eccen-
tricities of the two planets vary rather monotonically in this
period. Thereafter the orbital elements change in time violently
again. Note, however, that the quasi-regular phases, although
with much shorter periods, emerge intermittently.

In Figure 3, both the global and local Lyapunov exponents,
$\lambda_{\text{global}}(t)$ and $\lambda_{\text{local}}(n, \tau)$, are displayed as a function of time for the same model as in Figure 2. As mentioned earlier, the time interval $\tau$ for the integration of the local Lyapunov exponent is set to be 100 yr, that is roughly 100 times the orbital period for the protoplanet system, whose innermost planet is initially
located at 1 AU. It is clear that $\lambda_{\text{global}}(t)$ monotonically decreases toward zero before the close encounter ($\lambda \approx 0.0005$ just prior to the encounter), indicating that the motions are non-chaotic (or very weakly chaotic) in this phase. After the encounter, on
the other hand, $\lambda_{\text{global}}(t)$ increases drastically by about 2 orders of magnitude and fluctuates slowly around a constant value ($\sim 0.02$) thereafter. For comparison, we also employ MEGNO, another indicator of chaos suggested by Cincotta & Simó (2000), to estimate the global Lyapunov exponent. The values derived from MEGNO agree with the ones obtained above within 1% typically.

The local Lyapunov exponent also shows that the orbital motions are almost non-chaotic before the encounter. In fact, the almost constant small value during this period is consistent with the evolution of the global Lyapunov exponent for the first 100 yr. It is also clear that the local Lyapunov exponent shows remarkable peaks rather intermittently after the close encounter. Moreover, it is found by comparison between Figures 2 and 3 that $\lambda_{\text{local}}$ takes a small constant value when the semimajor axis of each planet remains nearly constant in time. It is interesting to point out again that the eccentricities of planetary orbits are not zero and change in time during this period. This suggests that the planetary system has settled to a quasi-stable configuration that is different from the initial condition and has substantial orbital eccentricities.

In Figure 4, we show the orbital evolution of the Jupiter system (model 17). In this model, the first close encounter happens at $t = 162,416$ yr and one of the planets is ejected from the system at $t = 239,279$ yr. Just as in the protoplanet system shown in Figure 2, the semimajor axes remain almost unchanged in time and the eccentricities oscillate around the initial value with small amplitudes before the close encounter but they start to vary rapidly in time after the encounter. It is noted that the amplitudes of the variations are much larger for the Jupiter system than those for the protoplanet system, the fact responsible for the ejection of a planet in short times in the Jupiter system.

In Figure 5, the global and local Lyapunov exponents are shown for the same Jupiter system. Again, we see the monotonic decrease of the exponent for the first $\sim 20,000$ yr. In this case, however, the exponent is then saturated and stays at a small but finite level. This reflects the fact that the system is close to integrable but still non-integrable. The global Lyapunov exponent increases quickly after the close encounter as in the protoplanet system. The local Lyapunov exponent obtained for every 100 yr, which is about 10 times the orbital period for the Jupiter system, shows some intermittent spikes after the close encounter although the interval from the close encounter to the ejection of a planet is rather short. It is also confirmed that the violent variations of the orbital elements occur in the spikes of the local Lyapunov exponent and that when $\lambda_{\text{local}}$ has a small, nearly constant value, the semimajor axes are not changed very much whereas the eccentricities are non-zero and fluctuate rather slowly. The Lyapunov exponent increases toward the ejection of the planet in this case.

The above-mentioned features in the orbital evolutions as well as in the Lyapunov exponents are common to all the models. (The behavior of the Lyapunov exponent close to the
Figure 4. Time evolutions of the semimajor axis (upper panel) and eccentricity (lower panel) of each planet for model 17.

Figure 5. Time evolutions of the global (upper panel) and local (lower panel) Lyapunov exponents for the same model as in Figure 4.

Figure 6. Action \( I_{0,i} \) and angle \( \theta_{0,i} \) variables for the same model as in Figure 4. See Equation (13) for the definitions of these variables. The horizontal line indicated by an arrow corresponds to the initial regular motion.

ejection of a planet is an exception and no clear trend can be seen.) This suggests that the underlying structure in the phase space is not very much different from each other. In particular, the quasi-regular motions before the close encounter, which are very weakly chaotic at most, strongly suggest the existence of the KAM torus. This may be true even of the periods that occur intermittently after the close encounter, in which the local Lyapunov exponent returns to a small value and the planetary motions become quasi-regular again. Then, the following picture is inferred: the phase space consists of a chaotic sea and KAM tori surrounded by a stagnant layer that consists of satellite tori. The phase-space orbits go from one system of KAM torus and stagnant layer to another through the chaotic sea. When the phase-space orbit is moving around one of the KAM tori, the local Lyapunov exponent takes a small constant value whereas it becomes spiky once the phase-space orbit moves into the chaotic sea.

This picture is also supported by the plot in Figure 6 of the action \( I_{0} \) and angle \( \theta_{0} \) variables of the non-perturbed Hamiltonian \( H_{\text{Kep}} \) in Equation (3). These variables are given as

\[
I_{0,i} = m_i \sqrt{G m_0 a_i}, \tag{13}
\]

\[
\theta_{0,i} = u_i - e_i \sin u_i. \tag{14}
\]

In the above equations, the angle variable \( \theta_{0,i} \) is called the mean anomaly and the so-called eccentric anomaly, \( u_i \), is defined as

\[
\tan \frac{u_i}{2} = \sqrt{\frac{1 - e_i}{1 + e_i}} \tan \frac{\phi_i}{2}, \tag{15}
\]

where \( e_i \) denotes the eccentricity and \( \phi_i \) the orbital phase of the \( i \)th planet measured in the barycentric coordinates. In Figure 6, we plot the action and angle variables for the innermost planet.
every ten steps. The behavior of the variables for other planets is essentially the same.

One can recognize some horizontal bands and voids in the figure. These bands are regions where the phase-space orbit lingers, whereas the voids are passed through quickly. It is also seen that some of the bands undulate. The densest band pointed by an arrow at \( t_0,1 = 1.98 \times 10^9 \, \text{g cm}^2 \, \text{s}^{-1} \) in the figure corresponds to the KAM torus of the initial regular motion and its satellite tori, whereas other bands represent other KAM tori and their satellites, which are visited by the phase-space orbit during the evolution. The bands with undulation correspond to the motion approaching the ejection of a planet. Since the \( I_{0,1}-\theta_{0,1} \) plane is filled by horizontal lines (or tori) everywhere uniformly if the perturbations are absent, the voids can be interpreted as chaotic regions produced by the perturbations to \( H_{\text{Kep}} \).

Looking more closely, one finds that all the lines composing a band that corresponds to a quasi-regular motion are oscillating with finite amplitudes. These oscillations are studied by Fourier analysis and the power spectrum for the initial quasi-regular motion is plotted in Figure 7. It is found that the power spectrum obeys a power law over a wide frequency range. This means that there is no characteristic frequency scale, a fact which seems to be consistent with the stagnant motion model, in which satellite tori are supposed to exist around a KAM torus with a fractal size distribution.

In the following, we will further look for evidence that the dynamics in these periods, that is, relatively long quasi-regular motions followed by an abrupt transition to chaotic motions, can indeed be interpreted as an induction phenomenon.

### 4.2. Power Spectra of Orbital Eccentricities

In Figure 8, we show the power spectra of the orbital eccentricity of the innermost planet before and after the close encounter for model 11. This is a representative model for the protoplanet system and the one we employed to demonstrate the behavior of the Lyapunov exponents in Figure 3. This model also has a merit in that it has a relatively long duration up to the first close encounter. Although the data are quite noisy, they can be roughly fit by the power law, \( \propto 1/\nu^\nu \). It is apparent that the power-law indices are different between the two phases. We obtain \( \nu = 1.01 \) before the encounter whereas the spectrum becomes steeper with \( \nu = 1.85 \) thereafter.

The same trend can also be seen for the Jupiter system, whose typical results are displayed in Figure 9 for model 17. Again the spectra are fit by the power law approximately both before and after the encounter and the spectral indices are \( \nu = 0.88 \) before the encounter and \( \nu = 1.85 \) thereafter. The spectrum given in the upper panel of Figure 9 has finer features than that in the upper panel of Figure 8 because the pre-encounter phase is longer for the model in Figure 8 than for the model in Figure 9. Note also that the post-encounter phase was computed up to the ejection of a planet in this case.

In Tables 2 and 3, we summarize the power-law spectral indices of the arbitrarily chosen three realizations for each model both before the encounter (first phase in the table) and after the encounter (second phase). Although we again employ the orbital eccentricity of the innermost planet, the orbits of outer planets behave similarly. Table 2 gives the results for the protoplanet system while Table 3 corresponds to the Jupiter system. For some of the models, the duration of the phase is too short to obtain the spectral index and ‘-’ is put instead of the spectral index for them. Although the spectral indices vary substantially even among different realizations for the same model, it is clear that phase 2 has spectral indices clustered around \( \nu = 2 \) whereas phase 1 has smaller indices, \( 0 \lesssim \nu \lesssim 1.69 \), in general. The
mean square \( (e^2)^{1/2} \) grows more slowly than for the ordinary Brownian motion (Mandelbrot & Van Ness 1968).

### 4.3. Distribution of Induction Periods

#### 4.3.1. Protoplanet System

As mentioned in Section 3.3, the distributions of the durations of various phases should be one of the key ingredients if the dynamics were to be interpreted as an induction phenomenon and described by the stagnant motion model. For the protoplanet systems, we study the statistics of the duration, \( T_{ce} \), of the pre-encounter phase, where the orbits are nearly circular. We show in Figure 10 the distribution of \( T_{ce} \) for 10,000 realizations of each model for the protoplanet systems listed in Table 1. For the plots we employed 100 equal bins between the maximum and minimum values of \( T_{ce} \). For some models with small initial orbital separations, these two values are quite different and, as a result, there are some bins with very small populations near the longest \( T_{ce} \).

Except for the model with the smallest \( \Delta \), the distribution has a peak, which shifts to longer times as \( \Delta \) becomes larger. It is also clear from the log–log plot that the dispersion around the peak gets larger, too, as \( \Delta \) becomes greater. We are particularly interested in the long-time regime, where the stagnant motion model predicts power laws, which implies that there is no characteristic timescale for the trapping in the stagnant layer and is supposed to be a consequence of the self-similar distribution of tori in the stagnant layer. In the figure, the straight lines are the power-law fit to the long-time part of the distributions. We employ 30 data points down from the one with the longest \( T_{ce} \). The bins with \( T_{ce} \) shorter than that at the peak or those with population of less than 2% are discarded. If there are less than 30 data points that satisfy the criteria, all of them are used. The obtained power-law indices lie between \(-5.68 \) and \(-2.27 \) as shown in each panel, which are expected to reflect the difference in the phase-space structures.

It is obvious that the distributions deviate from power laws both at the short and long durations. The initial conditions prepared so that planets are initially in regular motions may lead to the underestimation of the duration time, since it might have cut short the earlier portion of the pre-encounter phase. Note, however, that there is no reason to expect power laws for short timescales. As for longer timescales, on the other hand, power laws are expected if the stagnant motion model can be applied. We suspect that the main reason for the deviation from the power law for the very long durations is that the number of realizations, that is, 10,000 for each model, is not large enough. In fact, only a small number of realizations are contributing to the longest time portion of the distributions, in which the deviation from the power law is remarkable.

In Figure 11, we present a histogram in the \( \Delta-T_{ce} \) plane expressed in color for the number of cases in our model calculations. Connecting the peak, \( T_{ce,\text{peak}} \), for each \( \Delta \), we find the following relation:

\[
\log T_{ce,\text{peak}} \approx 1.008\Delta - 1.307, \tag{16}
\]

which is very similar to what Chambers et al. (1996) found \((a = 1.176, b = -1.663 \) in their results\) with a much smaller number (5) of realizations. It is clear from our results that the relation holds only for the durations corresponding to the peaks and, in fact, the durations for each \( \Delta \) have a distribution as demonstrated above. Incidentally, the integrals of motion, the linear and angular momenta, and total energy, are fixed in average spectral indices over all the models in Table 2 are 0.84 and 1.85 for the phases 1 and 2, respectively. The counterparts for the Jupiter system given in Table 3 are 0.91 before the encounter and 1.77 after the encounter.

The power-law spectrum is a characteristic feature of the stagnant motions although the power-law indices are not specified by the theory. The time variations of the orbital elements are induced by the energy exchange among the planets. The power law \( \propto 1/f^2 \) observed for the post-encounter phase indicates that the time evolution of the orbital eccentricity is a Brownian motion with the root mean square being \( \propto t^{1/2} \). On the other hand, the pre-encounter phase has a smaller spectral index in general. Although the numbers of the planets in the system are different, our results on the growth rate of eccentricity in the post-encounter phase are consistent with Zhou et al. (2007), who studied the system with 50 planets and claimed that the eccentricity of planetary orbits roughly evolves as \( (e^2)^{1/2} \propto t^{1/2} \) both before and after the encounter. As mentioned above, however, we found for our system with three planets different power-law indices before the encounter. The smaller power-law indices of \( e(t) \) in the pre-encounter phase in our models might correspond to the so-called fractional Brownian motion, for which the root
producing different realizations in this paper, which was not the case for Chambers et al. (1996).

### 4.3.2. Jupiter System

One of the characteristics of the dynamics of the Jupiter systems is that one of the planets is ejected from the system eventually. In addition to the durations of the pre-encounter phase, $T_{ce}$, we also take the statistics of the time from the first close encounter until the ejection of a planet, $T_{ej} - T_{ce}$, for the Jupiter systems.

Figure 12 shows the distributions of $T_{ce}$ for models 17 (left panel) and 18 (right panel) for the Jupiter system. Note that model 18 is meant to mimic the models in Marzari & Weidenschilling (2002) and the initial condition is prepared differently from the other models. The straight lines in the figure are the power-law fit to the long-duration part of the distributions and obtained just in the same manner as for the protoplanet systems (see Figure 10). It is seen again that the power-law distribution is a good approximation in this regime, which suggests that the quasi-regular motions before the close encounter in the Jupiter system can be also understood as the stagnant motion just as for the protoplanet system. The obtained power-law indices are $-5.57$ and $-1.56$ for models 17 and 18, respectively.

Now we turn our attention to the distributions of the time from the encounter to the ejection, $T_{ej} - T_{ce}$, which are given in Figure 13 for models 17 (left panel) and 18 (right panel). It is clear from the figure that the long-time portions of both distributions are again approximated by power laws with the indices of $-2.65$ and $-2.67$ for models 17 and 18, respectively. This implies that there is no characteristic timescale for the duration, during which the phase-space orbit of the system remains in the part of the phase space corresponding to the bound states of three planets.

If the power-law distribution of $T_{ce}$ reflects the self-similar distribution of smaller tori in the stagnant layer of a KAM torus (see Figure 1) as claimed in the stagnant motion model, the power-law distribution of $T_{ej} - T_{ce}$ might suggest a self-similar distribution of these KAM tori in the part of the phase space for the bound motions of three planets (see Figure 14 for a schematic picture of the phase space).

### 5. SUMMARY AND DISCUSSIONS

In this paper, we have numerically investigated the dynamics of the three-planet system and inferred its phase-space structure from the obtained Lyapunov exponents, power spectra of orbital elements, and distributions of induction periods based on the stagnant motion model. What we have found are the following.

1. The global and local Lyapunov exponents show that the system is almost non-chaotic until the first close encounter between two planets and it then turns into chaotic motions.
with intermittent non-chaotic periods. This suggests that the phase space consists of KAM tori surrounded by the stagnant layer and immersed in the chaotic sea. In fact, the dense bands are formed in the action–angle plot, corresponding to the quasi-regular motions. The lines composing a band are undulating with frequencies that obey a power law and this may represent the motions around the satellite tori in the stagnant layer, which have a self-similar distribution. The phase-space orbit goes from one system of KAM torus and stagnant layer to another through the chaotic sea.

2. The power spectra of the orbital eccentricities of planets can be approximated by the power law, $\propto 1/f^p$, in general. Such power-law spectra are known to be one of the characteristic...
features of the stagnant motions although the power-law index is not predicted by the theory. In our models, the power-law index is \( \nu \approx 1 \) for the pre-encounter phase whereas it becomes \( \nu \approx 2 \) after the encounter. The spectrum in the post-encounter phase is similar to that of Brownian motions or the random walks. On the other hand, the spectrum in the pre-encounter phase might have originated from fractional Brownian motions.

3. The distributions of the duration of the pre-encounter phase that was referred to as the induction periods obey the power law in the long-duration part. The power-law indices are substantially different between models. It is stressed that the stagnant motion model predicts the power law for the distribution of the induction periods as a consequence of the self-similar distribution of smaller tori in the stagnant layer around a KAM torus. The distributions deviate from
the power law in the short-duration part and has a peak in between. Connecting the peaks for various models with different initial orbital separations, we have obtained the relation similar to what Chambers et al. (1996) found. It is also shown that the duration of the pre-encounter phase has actually a considerably broad distribution.

4. For the Jupiter system, the distribution of the time from the first encounter to the ejection of a planet from the system also obeys a power law, which was not expected initially. From the analogy to the stagnant motion model, we might be able to infer the phase-space structure as shown schematically in Figure 14: many KAM tori with its own stagnant layer and satellite tori in it are distributed self-similarly in the chaotic sea.

Although we expect that the phase-space structure depicted in Figure 14 is true for both the protoplanet system and the Jupiter system, the difference between them should also be mentioned. In general, the number of KAM tori in the phase space becomes smaller and the stagnant layers around them get thinner as the perturbation to the integrable system is greater. In the system of our concern, the perturbation is the interactions among the planets and hence it is larger for more massive planets. The pre-encounter phase is an exception, though. In this phase, the planets have nearly circular orbits separated by several Hill radii. Then the strength of the interactions between the planets depends only weakly on the planetary mass thanks to the definition of the Hill radius given in Equation (2). These facts suggest that the KAM torus corresponding to the initial regular motion and its stagnant layer are robust and similar for
Figure 12. Normalized distributions of the duration of the pre-encounter phase, $T_{ce}$, for models 17 (left panel) and 18 (right panel). The straight lines are the fit by power law to the data in the long-duration regime.

Figure 13. Normalized distributions of the time from the first close encounter up to the ejection of a planet, $T_{ej} - T_{ce}$, for models 17 (left panel) and 18 (right panel). The straight lines are the fit by power law to the data in the long-time regime.

Figure 14. Schematic illustration of the phase space suggested by the power-law distributions of the time from the encounter to the ejection of a planet for the Jupiter system. KAM tori of various sizes with a stagnant layer and smaller tori in it are distributed self-similarly in the chaotic sea. An exemplary phase-space orbit is given by a curve with an arrow.

The results obtained in this paper appear to be consistent with our interpretation that the dynamics of the three-planet system is a stagnant motion at least in the pre-encounter phase. The results for the Jupiter system also suggest that even the post-encounter phase may be described by some extension of the stagnant motion model. It is true, however, that a more direct capture of satellite tori in the phase space is certainly desirable. We have attempted to do this with the so-called Poincaré mapping, but our attempts have been in vain so far. We are afraid that the degree of freedom of our system is just too large to find an appropriate two-dimensional section in the 12 dimensional phase space. Maybe other approaches should be pursued in future work. In so doing, the number of realizations should be increased and other initial settings should be tried.
Furthermore, we are also interested in how the results will change as the number of planets are varied.

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