Enhancement of superconductivity by local inhomogeneity

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We study the effect of inhomogeneity of the paring interaction or the background potential on the superconducting transition temperature, $T_c$. In the weak coupling BCS regime, we find that inhomogeneity which is incommensurate with the Fermi surface nesting vectors enhances $T_c$ relative to its value for the uniform system. For a fixed modulation strength we find that the highest $T_c$ is reached when the characteristic modulation length scale is of the order of the superconducting coherence length.

Many strongly correlated superconductors, and in particular high-temperature superconducting (HTSC) cuprates, exhibit inhomogeneous electronic and/or structural phases at the nanoscale [12, 13]. The coexistence of HTSC and inhomogeneity suggests that the underlying inhomogeneities could be at least partially responsible for the high value of the superconducting transition temperature. Emery and one of us proposed that HTSC is related to frustrated electronic phase separation, commonly expected in strongly correlated systems [14]. These ideas for inhomogeneous superconductivity have been further developed in the context of stripes [15, 16]. It is important to distinguish these and related scenarios for superconductivity creation or enhancement by inhomogeneity from the conventional weak-coupling coexistence of superconductivity and various density waves [17, 18]. In the latter case, the density wave order inevitably suppresses superconductivity due to the competition for the Fermi surface electrons.

It is therefore important to understand the nature of the interplay between superconductivity and inhomogeneities. A complete description of the interplay is clearly impossible. However, in the case in which the characteristic energy scale responsible for the formation of the inhomogeneity is much larger than the superconducting energy scale (the gap $\Delta$), and where the residual interactions are weak, a description based on BCS theory should be reliable. The purpose of this work is to study the effect of such imposed inhomogeneity on superconductivity within the BCS framework. The origin of the inhomogeneity could be either electronic, as in the frustrated phase separation scenario, or structural, that is caused by local lattice distortions or non-uniform carrier concentration due to doping irregularities. We will assume that these structures do not cause Fermi surface nesting either due to the lack of periodicity (e.g. random doping profile) or due to the periodicity being incommensurate with the nested momentum transfers (e.g. frustrated phase separation, or stripes). Under these conditions we generically find that inhomogeneity enhances the global superconducting transition temperature, $T_c$. At the mean-field level, the maximum $T_c$ is achieved when the characteristic length-scale of the inhomogeneities, $L$, is large, in which case the transition temperature is that of the regions with the highest local $T_c$. Upon including the effects of phase fluctuations we find that $T_c$ is maximized when $L$ is comparable to the superconducting coherence length $\xi \sim v_F/T_c$. The increase of the transition temperature occurs at the expense of the superfluid density, which is reduced in inhomogeneous superconductors relative to their homogeneous counterparts.

**Inhomogeneous pairing: Mean-Field treatment.** As a first example we consider a Hubbard model with an inhomogeneous attractive potential $U(r) > 0$,

$$H = H_0 + H_U \tag{1}$$

$$H_0 = \sum_{k\sigma} \xi_k c_k^\dagger c_k c_{\sigma}^\dagger c_{\sigma}$$

$$H_U = -\sum_r U(r)n_\uparrow(r)n_\downarrow(r),$$

where $\xi_k = \epsilon_k - \mu$ and $n_\sigma(r) = c_{\sigma}^\dagger(r)c_\sigma(r)$ is the occupation number of electrons of spin $\sigma$ at position $r$. Within this model, our goal is to understand whether for a fixed average pairing strength $\overline{U}(r)$, a uniform or non-uniform $U(r)$ yields a higher transition temperature, $T_c$. In the weak coupling limit, we can derive the BCS condition for the onset of superconductivity from the Hamiltonian [16],

$$\Delta_q = \int \frac{d^dp}{(2\pi)^d} U(q - p)K(p)\Delta_p, \tag{2}$$

where $\Delta_q = \sum_{k,p} U(k)\langle c_{q/2-k/2+p}^\dagger c_{q/2-k/2-p} \rangle$, $U(k)$ is the Fourier transform of the pairing interaction, and $K(p)$ is the pairing kernel. The pairing kernel depends on temperature $T$ and the mean-field (MF) superconducting transition is defined by the temperature at which the integral equation has a non-trivial solution. The kernel can be calculated from the normal state electron Green functions [16],

$$K(p) \approx N_f \ln \left(\frac{2\gamma\omega_D}{\pi^{1/2} (\omega_D^2 + (v_fp)^2)}\right) \Theta(\omega_D - |v_fp|), \tag{3}$$

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where \( N_f \) is the density of states at the Fermi surface, \( v_f \) is the Fermi velocity, \( T \) is temperature, and \( \ln \gamma \approx 0.577 \) is Euler's constant. Here we also introduced an explicit high-energy cut-off for the attraction, \( \omega_D \). For \( T > v_f p \) this expression reduces to the well known homogeneous result, \( T_c \approx N_f \ln[2\gamma \omega_D/(\pi T)] \).

The modulation of the pairing interaction leads to the mixing between Cooper pairs with different center-of-mass momenta. For simplicity we first assume a harmonic modulation \( (Q \equiv 2\pi/L) \) of the pairing, \( U(r) = \bar{U} + U_Q \cos(Q \cdot r) \). In this case the integral equation \( \Omega n \) reduces to a system of linear equations \( \Delta_n = \bar{U} K_n \Delta_n + (U_Q/2)[K_{n-1}\Delta_{n-1} + K_{n+1}\Delta_{n+1}] \equiv M_{nm}\Delta_m \),

where \( \Delta_n = \delta(nQ + \eta_0) \) and \( K_n \equiv K(nQ + \eta_0) \). The “parent” momentum \( \eta_0 \) defines the minimal momentum of a Cooper pair in the connected family \( \Delta(nQ + \eta_0) \).

The pairing instability occurs at the temperature \( T_c \), such that the largest eigenvalue of matrix \( M \) is equal to 1. In the uniform case, \( U_Q = 0 \), this condition is \( \bar{U} K(0) = 1 \). We will now prove that the mean-field transition temperature is greater than in the uniform case, \( U_Q = 0 \). Consider the \( \eta_0 = 0 \) family and without loss of generality, take \( U_Q > 0 \). Since all of the matrix elements of \( M \) are non-negative, by Perron’s theorem, the maximal eigenvalue is a positive number that is larger than any diagonal matrix element, including \( \bar{U} K(0) \). Thus, generically, the superconducting onset temperature \( T_c \) is increased whenever \( U_Q \neq 0 \).

This result can be understood from an analogy with a quantum mechanical particle in a tight-binding chain. Defining a new variable \( \Lambda_n = \bar{U} K_n \Delta_n \), the BCS condition takes a simple symmetric form, \( \Lambda_n = (1/\bar{U} K_n)\Lambda_n - (U_Q/2\bar{U})(\Lambda_{n+1} + \Lambda_{n-1}) \). The “hopping” term delocalizes the particle and thus reduces the “energy” below its minimal on-site value \((1/\bar{U} K_0)\). Clearly, this leads to a relative increase of \( T_c \).

Large \( Q \) limit. In this limit, the quickly oscillating coupling is ineffective at mixing different modes, so that the off-diagonal terms in \( M \) are rapidly decaying with \( n \). We are then justified in keeping only a small portion of the matrix surrounding the \( n = 0 \) term. The lowest order correction to the homogeneous result is obtained by considering couplings between \( \Delta_0 \) and \( \Delta_{\pm 1} \). The largest eigenvalue in this case is

\[
\lambda_{\text{max}} = \frac{\bar{U}}{2} \left[ K_0 + K_1 + \sqrt{(K_0 - K_1)^2 + \frac{2K_0 K_1 U_Q^2}{\bar{U}^2}} \right],
\]

Given the separation of energy scales, \( T_c \ll v_f Q \ll \omega_D \), we obtain \( T_c \) by solving \( \lambda_{\text{max}} = 1 \),

\[
T_c = \frac{2\gamma}{\pi} \omega_D \exp[-1/N_f(\bar{U} + \eta)],
\]

where \( \eta = U_Q^2 K_1/[2(1 - \bar{U} K_1)] \). While \( \eta \) is positive, and since \( K_1 \sim \log[\omega_D/v_f Q] \) decreases with increasing

![FIG. 1: Critical temperature for the inhomogeneous negative \( U \) Hubbard model with coupling \( U(x) = \bar{U} + U_Q \cos(Qx) \). The thick line denotes the mean-field result, where \( T_{c,a} = (2\gamma/\pi)\omega_D \exp[-1/N_f(\bar{U} + |U_Q|)] \) and \( T_{c,h} = (2\gamma/\pi)\omega_D \exp[-1/N_f(\bar{U} + |U_Q|)] \). The dashed line shows the critical temperature once phase fluctuations of the order parameter are included. For \( Q\xi \ll 1 \), the superconductivity is first established locally in regions where \( U(x) \) is large, but macroscopic phase coherence is achieved at a lower temperature, bounded from below by \( T_{c,i} = (2\gamma/\pi)\omega_D \exp[-1/N_f(\bar{U} + |U_Q|)] \).

\( Q \), so does \( \eta \). For \( v_f Q > \omega_D \), Cooper pairs can no longer scatter off the quickly oscillating coupling landscape, and we recover the critical temperature for the homogeneous case.

Small \( Q \) limit. In the limit \( Q\xi \ll 1 \), the global MF transition temperature is determined by the regions with the strongest pairing interaction, \( T_c \approx (2\gamma/\pi)\omega_D \exp[-1/N_f(\bar{U} + |U_Q|)] \). The deviations from this result due to finite \( Q \) and the effects of the phase fluctuations are discussed below.

Electron density modulation. Before going further, let us in parallel consider the case of homogeneous coupling \( U(r) = \bar{U} \), with inhomogeneity caused by a background potential variation. In the simplest case of the harmonic modulation, the additional contribution to the Hamiltonian is

\[
\mathcal{H}_\rho = \rho \sum_{i \sigma} c_{i \sigma}^\dagger c_{i \sigma} \cos \mathbf{Q} \cdot \mathbf{r}_i.
\]

It is easy to see that for particle-hole symmetric density of states (DOS), the linear in \( \rho \) contributions to the BCS instability condition equations vanish identically. For small values of \( Q \), the modulation acts as a slowly varying shift in the local chemical potential with the amplitude proportional to \( |\rho| \). Thus, we only get a linear in \( \rho \) contribution for asymmetric DOS, \( N(\epsilon) = N_f + N'(\epsilon - \epsilon_F) \). We then find that BCS equations are identical to the case of inhomogeneous pairing interaction with the modulation strength

\[
U_Q^{\text{eff}} = -N'N_f/\bar{N}_f.
\]
Ginzburg-Landau analysis \((Q\xi \ll 1)\). We now consider the general case of slow variation of the pairing strength and/or background potential. The Ginzburg-Landau free energy functional in the presence of inhomogeneity is

\[
F = -\int dr \, dr' K(r-r')\Delta(r)\Delta(r') + \int dr \frac{\Delta(r)^2}{U(r)} + \alpha \int dr \rho(r)\Delta(r)^2 + \frac{\beta}{2} \int dr \Delta(r)^4,
\]

Here we assumed that the order parameter remains real even in the presence of inhomogeneity. We include both the coupling of the superconducting order parameter to a density wave, as well as the inhomogeneity of the pairing interaction. For small amplitude modulation of the pairing interaction, \(U(r) = \bar{U} + \delta U(r)\) with \(|\delta U(r)| < \bar{U}\), the two mechanisms are formally equivalent. For particle-hole asymmetric DOS, from the above considerations, \(\alpha = -\bar{U}N'/N_f\). For simplicity, we only consider the inhomogeneous \(U(r)\) case here. The pair susceptibility kernel is given by Eq. (3). In the long wave length limit,

\[
K(r-r') = \delta(r-r')N_f \left[ \frac{2\omega_D}{\pi T} + \xi^2 \nabla^2 \right],
\]

where \(\xi = v_F/T\). Computing the variation of Eq. (7) with respect to the order parameter, we find the equation for a stationary solution \(\Delta(r)\),

\[
-\xi^2 \nabla^2 \Delta + g(r)\Delta + \frac{\beta}{N_f} \Delta^3 = 0.
\]

This expression is obtained assuming \(g'(x_0) \approx A^2Q\), and therefore valid only for the temperatures sufficiently below \(T_{c0}\). So long as \(T > T_{c0}\), \(T_{c0}\) is the uniform \(T_c\) of a system with pairing strength \(U - |U_Q|\), the distance from the turning point to the minimum point of \(\Delta\) is \(d \approx L\). Notice, that this expression depends on temperature not only explicitly, but also through \(\xi = v_F/T\). Linearization (still with \(Q\xi \ll 1\)). A consequence of the large spatial variations in the mean-field \(\Delta(r)\) is that fluctuation effects are severe where \(\Delta(r)\) is small. Of these, the most important fluctuations are thermal fluctuations in the phase of the order parameter, \(i.e.\) where \(\Delta(r) = |\Delta_{MF}(r)|e^{i\theta}(r)\) is the solution of Eq. (9), \(\theta(r)\) is assumed to be a slowly varying function of \(r\). The free energy cost of such phase fluctuations can then be readily computed

\[
F_\theta = \int dr J(r)(\nabla \theta)^2
\]

where the local superfluid stiffness is \(J(r) = N_f\xi^2|\Delta_{MF}(r)|^2\). In general, the phase ordering temperature estimated using this as the effective Hamiltonian is reduced from the mean-field transition temperature \(T_{c0}\) as the effective Hamiltonian is reduced from the mean-field transition temperature \(T_{c0}\) (i.e. \(J(r)\) vanishes), but by an amount that depends on dimensionality, and on the spatial arrangement of the regions of suppressed stiffness.
For concreteness, we consider the case of a two-dimensional superconductor. At finite temperature, no true long-range order is possible \[13\]. However, at \( T < T_{KT} \), binding of topological excitations into vortex-antivortex pairs leads to a state with quasi-long range order, which has a non-zero superfluid stiffness \[14\]. While for homogeneous BCS superconductors in 2D, the difference between MF and the Kosterlitz-Thouless (KT) transition temperatures is tiny, \((T_{MF}^{KT} - T_{KT})/T_{MF}^{KT} \sim T_{MF}^{KT}/T_F\) (where \( T_F \) is the Fermi temperature), for inhomogeneous superconductors, the suppression of \( T_{KT} \), is generally much larger. For a smooth random distribution of \( J(r) \), an estimate of \( T_{KT} \) can be made based on the effective superfluid density,

\[
T_{KT} \sim \sqrt{\frac{J(r)}{1/J(r)}} = \sqrt{\frac{1}{J_{min} J_{max}}} \tag{14}
\]

This expression has a particularly transparent meaning for the unidirectional “striped-like” variation of \( U(r) \) that we treated explicitly when solving the mean-field equations, above. There, \( J_{max} \) corresponds to the stiffness along the stripes and \( J_{min} \) – perpendicular to the stripes. The corresponding anisotropic XY model directly leads to the result Eq. (14). In this case, we find

\[
T_{KT} \sim \frac{T_f}{T_{KT}} T_{c}^{MF} \Delta_{min}(T_{KT})
\]

Together with Eq. (12) for \( \Delta_{min}(T_{KT}) \sim \Delta(L, T_{KT}) \), this equation implicitly defines \( T_{KT} \). With logarithmic accuracy we find that \( T_{KT} \sim \min(T_c^{MF}, v_f Q/A) \).

In any case, barring certain artificial geometries, it is clear that for a long wavelength modulation, \( \xi \ll 1 \), the Kosterlitz-Thouless temperature \( T_{KT} \) is exponentially lower than the MF transition temperature; on the other hand, for modulations with \( \xi \gg 1 \), the phase fluctuation region is very narrow and \( T_{KT} \approx T_{MF}^{KT} \). In this regime, the mean-field superconducting temperature is still exponentially enhanced relative to its value in the uniform state with the same average pairing interaction strength, \( U \). For even faster modulation, \( \xi Q \gg 1 \), the MF transition temperature drops since the pairings interaction modulation averages out on the length scale of \( \xi \). This trend is presented qualitatively in Fig. 1.

For a “dirty” superconductor with a mean free path shorter than the clean coherence length, \( \ell = v_F \tau < \xi \), the effect of phase fluctuations can be estimated in the same way as in the clean limit, with minor changes (involving prefactors) but with the coherence length redefined as \( \xi_d = \sqrt{\ell} \).

**Summary.** We studied the effect of nanoscale inhomogeneity on the superconducting transition temperature, \( T_c \). We considered two possible kinds of inhomogeneity: the modulations of the pairing strength and of the background potential. In the weak coupling BCS regime, we find that inhomogeneity which is incommensurate with the Fermi surface nesting vectors enhances \( T_c \) relative to its value for the uniform zero center-of-mass momentum pairing. For a fixed modulation depth we find that the highest \( T_c \) is reached when the modulation wavelength is of the order of the superconducting coherence length. For shorter wavelengths, the superconductor cannot take advantage of the locally favorable conditions, while for the longer wavelengths, the global superconductivity is suppressed due to the phase fluctuations on the weak links, where the amplitude of the order parameter is significantly reduced. Although explicitly derived for s-wave superconductors, similar results will also apply to unconventional superconductors in the presence of smooth (on the \( 1/k_f \) length scale) inhomogeneities. Clearly over-simplified, the presented picture bears resemblance to the high-temperature superconducting cuprates, where considerable experimental evidence \[17\] indicates that the maximum \( T_c \) occurs at a crossover between a regime where \( T_c \) is controlled by the pairing scale and where it is a phase ordering transition.

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