Mertens’ theorem and prime number theorem for Selberg class

Yoshikatsu Yashiro
Graduate School of Mathematics, Nagoya University,
464-8602 Chikusa-ku, Nagoya, Japan
E-mail: m09050b@math.nagoya-u.ac.jp

Abstract

In 1874, Mertens proved the approximate formula for partial Euler product for Riemann zeta function at $s = 1$, which is called Mertens’ theorem. In this paper, we shall generalize Mertens’ theorem for Selberg class and show the prime number theorem for Selberg class.

1 Introduction

In 1874, Mertens [6] proved the following theorem:

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} = e^\gamma \log x + O(1),$$

where $x \in \mathbb{R}_{\geq 2}$ and $\gamma$ is Euler’s constant. The above formula is the approximate formula of the finite Euler product for the Riemann $\zeta$-function $\zeta(s)$ at $s = 1$, which is called Mertens’ (3rd) theorem. Later, in 1999 Rosen [9] generalized Mertens’ theorem for Dedekind $\zeta$-function $\zeta_K(s)$:

$$\prod_{Np \leq x} \left(1 - \frac{1}{Np}\right)^{-1} = a_K e^\gamma \log x + O(1),$$

2010 Mathematics Subject Classification: Primary 11M41; Secondary 11N05.
Key words and phrases: Selberg class, Mertens’ theorem, prime number theorem.
where $K$ is an algebraic number field, and $\alpha_K$ is the residue of $\zeta_K(s)$ at $s = 1$. This theorem is obtained by using the following approximate formula:

$$\sum_{Np \leq x} \log(Np) = x + O(xe^{-c_K\sqrt{\log x}}) \quad (1.1)$$

which was proved by Landau [4]. Note that (1.1) is equivalent to the prime number theorem for the algebraic number field $K$, where $c_K$ is a positive constant depending on $K$.

In this paper, we consider Mertens’ theorem for Dirichlet series introduced by Selberg [10]. Selberg class $S$ is defined by the class of Dirichlet series satisfying the following conditions:

(a) (Absolute convergence) The series $F(s) = \sum_{n=1}^{\infty} a_F(n)n^{-s}$ is absolutely convergent for $\text{Re } s > 1$.

(b) (Analytic continuation) There exists $m \in \mathbb{Z}_{\geq 0}$ such that $(s - 1)^m F(s)$ is an entire function of finite order.

(c) (Functional equation) The function $F(s)$ satisfies $\Phi(s) = Q \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s)$, $r \in \mathbb{Z}_{\geq 1}$, $Q \in \mathbb{R}_{>0}$, $\lambda_j \in \mathbb{R}_{>0}$, $\text{Re } \mu_j \in \mathbb{R}_{>0}$, and $\omega \in \mathbb{C}$ satisfying $|\omega| = 1$.

(d) (Ramanujan conjecture) For any fixed $\varepsilon \in \mathbb{R}_{>0}$, $a_F(n) = O(n^\varepsilon)$.

(e) The logarithmic function of $F(s)$ is given by $\log F(s) = \sum_{n=1}^{\infty} b_F(n)n^{-s}$, where $b_F(n) = 0$ when $n \neq p^r$ ($r \in \mathbb{Z}_{\geq 1}$), and there exists $\theta \in \mathbb{R}_{<1/2}$ such that $b_F(n) = O(n^\theta)$.

Moreover, the extended Selberg class $S^\#$ is defined by a class of Dirichlet series satisfying only the conditions (e)–(c). Clearly $S \subset S^\#$. For example $\zeta(s)$ belongs to $S$ and $\zeta_K(s)$ belong to $S^\#$. The function $\zeta(s)$ and $\zeta_K(s)$ have the Euler products and zero-free regions. It is expected that $F \in S^\#$ satisfies the following conditions:

(I) (Euler product) There exists a positive integer $k$ (depending on $F$) such that

$$F(s) = \prod_p \sum_{r=0}^{\infty} \frac{a_F(p^r)}{p^{rs}} = \prod_p \prod_{j=1}^{k} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1} \quad (1.2)$$

with $|\alpha_j(p)| \leq 1$ for $\text{Re } s > 1$. 

---

Y. Yashiro
(II) (Zero-free region) There exists a positive constant $c_F$ (depending on $F$) such that $F(s)$ has no zeros in the region

$$\text{Re } s \geq 1 - \frac{c_F}{\log(|\text{Im } s| + 2)};$$

except $s = 1$ (if $F(s)$ has zero at $s = 1$) and the Siegel zeros of $F(s)$.

**Remark 1.1.** If $F \in S^\#$ satisfies (I), then (e) are satisfied with $b_F(p^r) = (\alpha_1(p) + \cdots + \alpha_k(p))r$ and $\theta = 0$, where the constant of $O$-term depends on $k$.

By the same discussion as in the proof of the zero-free region of $\zeta(s)$ (see Montgomery and Vaughan [7, Lemma 6.5 and Theorem 6.6]), the following fact is obtained:

If $F \in S^\#$ satisfies (e) with $\text{Re } b_F(n) \geq 0$ and has a zero or a simple pole at $s = 1$, then $F$ satisfies (II).

Applying Motohashi’s method [8, Chapter 1.5], we can extend Mertens’ theorem for Selberg class by using Perron’s formula and complex analysis.

**Theorem 1.2** (Mertens’ 3rd theorem for Selberg class). Let $F \in S^\#$ and suppose the condition (I) and (II). Then we have

$$\prod_{p \leq x} \prod_{j=1}^k \left(1 - \frac{\alpha_j(p)}{p}\right)^{-1} = c_{-m} e^{\gamma m} (\log x)^m (1 + O(e^{-\sqrt{x}})), \quad (1.3)$$

where $m$ denotes the order $m$ of pole for $F(s)$ at $s = 1$ when $m \in \mathbb{Z}_{>0}$, and the order $-m$ of zero for $F(s)$ at $s = 1$ when $m \in \mathbb{Z}_{<0}$. Moreover $c_{-m}$ is given by $c_{-m} = \lim_{s \to 1} (s - 1)^m F(s)$, and $C_F$ is a positive constant smaller than $c_F$ of the condition (II).

Indeed instead of the condition (II), we can prove the following weaker formula than (1.3):

$$\prod_{p \leq x} \prod_{j=1}^k \left(1 - \frac{\alpha_j(p)}{p}\right)^{-1} = c_{-m} e^{\gamma m} (\log x)^m \left(1 + O\left(\frac{1}{\log x}\right)\right), \quad (1.4)$$

under the assumption of the prime number theorem for Selberg class. In order to improve the error term in (1.4), it is necessary to assume (II).
It is well-known that the prime number theorem is equivalence to $\zeta(1 + it) \neq 0 (t \in \mathbb{R})$. Kaczorowski and Perelli [3] proved the equivalence of prime number theorem in Selberg class:

$$F(1 + it) \neq 0 (t \in \mathbb{R}) \iff \sum_{n \leq x} b_F(n) \log n = mx + o(x) \quad (1.5)$$

where $F \in \mathcal{S}$. If we apply (1.3), we can improve the above $o(x)$:

**Theorem 1.3** (Prime number theorem for Selberg class). Let $F \in \mathcal{S}^\#$ and suppose the conditions (I) and (II). Then we have

$$\sum_{n \leq x} b_F(n) \log n = mx + O(xe^{-C'_F \sqrt{\log x}}) \quad (1.6)$$

where $C'_F$ is a positive constant smaller than $C_F$ in Theorem 1.2.

We shall give an example of Theorems 1.2 and 1.3. In the case of $\zeta_K \in \mathcal{S}^\#$, we know that $\zeta_K$ satisfies (I), (II), and it is known that $\zeta_K(s)$ has a simple pole as $s = 1$. Therefore the following fact is obtained:

**Corollary 1.4.** We obtain the Metens' theorem for $\zeta_K(s)$:

$$\prod_{Np \leq x} \left(1 - \frac{1}{Np}\right)^{-1} = \alpha_K e^\gamma (\log x)(1 + O(e^{-C_K \sqrt{\log x}}))$$

and the prime number theorem for $\zeta_K(s)$:

$$\sum_{Np^r \leq x} \log(Np^r) = x + O(xe^{-c_K \sqrt{\log x}}),$$

where $\alpha_K$ is the residue for $\zeta_K(s)$ at $s = 1$, and $C_K, c_K$ are positive constants such that $c_K < C_K$ depending on $K$.

In the case of automorphic $L$-function, we see that the Rankin-Selberg $L$-function $L_{f \times g}(s)$ belongs to the Selberg class, and it is known that if $L_{f \times g}(s)$ has a simple pole at $s = 1$ when $f = g$ and no pole in the whole $s$-plane when $f \neq g$, where $f$ and $g$ are cusp forms of weight $k$ for $SL_2(\mathbb{Z})$. Assume that $f, g$ are normalized Hecke eigenforms, and the Fourier expansion of $X = f, g$ are given by $X(z) = \sum_{n=1}^{\infty} \lambda_X(n)n^{(k-1)/2}e^{2\pi inz}$. Then we obtain the following corollary:
Corollary 1.5. We have the Mertens’ theorem for $L_{f \times g}(s)$:

$$\prod_{p \leq x} \left(1 - \frac{(\alpha_f \alpha_g)(p)}{p}\right)^{-1} \left(1 - \frac{(\alpha_f \beta_g)(p)}{p}\right)^{-1} \left(1 - \frac{(\beta_f \alpha_g)(p)}{p}\right)^{-1} \times \left(1 - \frac{(\beta_f \beta_g)(p)}{p}\right)^{-1} = \begin{cases} A_{f \times f} e^{\gamma} (\log x) (1 + O(e^{-c_{f,g} \sqrt{\log x}})), & f = g, \\ L_{f \times g}(1) + O(e^{-c_{f,g} \sqrt{\log x}}), & f \neq g, \end{cases}$$

and the prime number theorem for $L_{f \times g}(s)$:

$$\sum_{n \leq x} b_{f \times g}(n) \log n = \begin{cases} x + O(x e^{-c_{f,g} \sqrt{\log x}}), & f = g, \\ O(x e^{-c_{f,g} \sqrt{\log x}}), & f \neq g. \end{cases}$$

Where $\alpha_j, \beta_j$ satisfy $(\alpha_j + \beta_j)(p) = \lambda_j(p)$, $(\alpha_j \beta_j)(p) = 1$, $A_{f \times f}$ is the residue for $L_{f \times f}(s)$ at $s = 1$, $c_{f,g}, C_{f,g}$ are positive constants such that $c_{f,g} < C_{f,g}$ depending on $f, g$, and $b_{f \times g}(n)$ are given by

$$b_{f \times g}(n) = \begin{cases} (\alpha_f^r + \beta_f^r + \alpha_g^r + \beta_g^r)(p)/r, & n = p^r, \\ 0, & n \neq p^r. \end{cases}$$

In this corollary, we used the fact $b_{f \times g}(n) \geq 0$ when $f = g$, and the result that if $f \neq g$ then the condition (II) are satisfied (see Ichihara [2]). In this paper, we shall show Theorem 1.2 in Section 2 and Theorem 1.3 in Section 3.

2 Proof of Theorem 1.2

Let $F \in S^\#$ and put the left hand of (1.3) by $F_x(1)$. We shall give the approximate formula of $\log F_x(1)$. By using Remarks 1.1 and (1.2), we can write

$$\log F_x(1) = \sum_{p \leq x} \sum_{r=1}^{\infty} \frac{b_F(p^r)}{p^r} = \sum_{n \leq x} \frac{b_F(n)}{n} + \sum_{\sqrt{x} \leq p \leq x} \sum_{p^r > x} \frac{b_F(p^r)}{p^r} + \sum_{p \leq \sqrt{x}} \sum_{p^r > x} \frac{b_F(p^r)}{p^r}. \quad (2.1)$$
It is clear that the second and third terms of (2.1) are estimated as

\[
\sum_{\sqrt{x} < p, x' > x} \frac{b_F(p^r)}{p^r} \ll \sum_{\sqrt{x} < p, x' > x} \frac{1}{p^r} \ll \sum_{\sqrt{x} < p, x' > x} \frac{1}{p^2} \ll \frac{1}{\sqrt{x}},
\]

(2.2)

\[
\sum_{p \leq \sqrt{x}, x' > x} \frac{b_F(p^r)}{p^r} \ll \sum_{p \leq \sqrt{x}, x' > x} \frac{1}{p^r} \ll \sum_{p \leq \sqrt{x}} \frac{1}{x' x} \ll \frac{1}{\sqrt{x}}.
\]

(2.3)

Applying Perron’s formula to the first term of (2.1) (see Liu and Ye [5, Corollary 2.2] or [7, Chapter 5.1]), we get

\[
\sum_{n \leq x} \frac{b_F(n)}{n} = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \frac{x^s}{s} \log F(1+s) ds + O(e^{-c\sqrt{\log x}}),
\]

(2.4)

where we put \( b = 1/\log x \) and \( T = e^{\sqrt{\log x}} \), and the following fact were used:

\[
\log F(\sigma) \ll \zeta(\sigma) \ll \frac{1}{\sigma-1} \quad (\sigma \in \mathbb{R} > 1), \quad \frac{\sqrt{\log x}}{2} + \log \log x \leq -c\sqrt{\log x}
\]

where \( c \in (0, 1/2) \) is a constant.

Now we consider the integral of (2.4). If we put \( b' = c_F/\log T = c_F/\sqrt{\log x} \) and take large \( x \), from (11) we see that

\[
F(\sigma + it) \neq 0; \quad t \in [-T, T], \quad \sigma \geq 1 - b'
\]

(2.5)

on condition that \( s = 1 \) is excluded when \( F(s) \) has zero on \( s = 1 \). Note that \( F(s) \) has no Siegel zeros in the region \( \text{Re } s \leq 1 - b' \) because \( x \) is taken large. Define the contour

\[
L_{+1} = \{-b' + it \mid t \in [0, T]\}, \quad L_{-1} = \{-b' + it \mid t \in [-T, 0]\},
\]

\[
L_{-2} = \{-\sigma - iT \mid \sigma \in [-b', b]\}, \quad L_{+2} = \{\sigma + iT \mid \sigma \in [-b', b]\},
\]

\[
C = \{b'e^{i\theta} \mid \theta \in [-\pi, \pi]\}.
\]

Using (2.5) and Cauchy’s residue theorem, we have

\[
\frac{1}{2\pi i} \int_{b-iT}^{b+iT} \frac{x^s}{s} \log F(s+1) ds = \frac{1}{2\pi i} \int_C \frac{x^s}{s} \log F(s+1) ds + O\left( \sum_{j=\pm 1, \pm 2} I_j \right),
\]

(2.6)
where

$$I_j = \frac{1}{2\pi i} \int_{L_j} \frac{x^s}{s} \log F(s + 1) ds.$$ 

First we shall calculate estimates of $I_j$. In the case of $j = \pm 2$, Phragmén-Lindelöf theorem and Stirling’s formula give that $\log F(1 - \sigma \pm iT) \ll (\log T)^2$ for $\sigma \in [-b', b]$. Then $I_{\pm 2}$ are estimated as

$$I_{\pm 2} \ll \frac{(\log T)^2}{T} \int_{-b'}^b x^\sigma d\sigma \ll (\sqrt{\log x})^2 e^{-\sqrt{\log x}} \ll e^{-CF\sqrt{\log x}}. \quad (2.7)$$

In the case of $j = \pm 1$, we use the following result (see [7, Lemma 6.3]):

**Lemma 2.1.** Let $f(z)$ be an analytic function in the region containing the disc $|z| \leq 1$, supposing $|f(z)| \leq M$ for $|z| \leq 1$ and $f(0) \neq 0$. Fix $r$ and $R$ such that $0 < r < R < 1$. Then, for $|z| \leq r$ we have

$$\frac{f'}{f}(z) = \sum_{|\rho| \leq R} \frac{1}{z - \rho} + O\left(\frac{\log M}{|f(0)|}\right)$$

where $\rho$ is a zero of $f(s)$.

Put $f(z) = (z + 1/2 + it)^m F(1 + z + (1/2 + it))$, $R = 5/6$, $r = 2/3$ in Lemma 2.1 and use the assuming the condition (11). Then the following estimates are obtained by the same discussion of the proof of [7, Theorem 6.7]:

$$\log s^m F(s + 1) \ll \begin{cases} \log(|t| + 4), & |t| \geq 7/8 \text{ and } \sigma \geq -b', \\ 1, & |t| \leq 7/8 \text{ and } \sigma \geq -b', \end{cases}$$

and $I_{\pm 1}$ are estimated as

$$I_{\pm 1} \ll \left(\int_0^{\gamma/8} + \int_{\gamma/8}^T\right) \frac{x^{-b'}}{|s|} \left(\log s^m + |\log s^m F(s + 1)|\right) dt$$

$$\ll \int_0^{\gamma/8} \frac{x^{-b'}}{|s|} (\log b' + 1) dt + \int_{\gamma/8}^T \frac{x^{-b'}}{t} (\log (t + b') + \log(t + 4)) dt$$

$$\ll e^{-CF\sqrt{\log x}} \sqrt{\log x} \log \log x + e^{-CF\sqrt{\log x}} \log x \ll e^{-CF\sqrt{\log x}}. \quad (2.8)$$
Secondly we consider the integral term of (2.6). From (1), we see that $F(s)$ has a pole of order $m$ on $s = 1$ where $m \in \mathbb{Z}_{\geq 1}$, or has a zero of order $-m$ in $s = 1$ where $m \in \mathbb{Z}_{\leq 0}$. Considering the Laurent expansion of $F(s)$ in $s = 1$, we get $c_{-m} = \lim_{s \to 1} (s - 1)^m F(s) \neq 0$ for $m \in \mathbb{Z}$. Therefore, the following formula is obtained by Cauchy’s residue theorem:

$$\frac{1}{2\pi i} \int_C \frac{x^s \log F(s + 1)}{s} ds = -\frac{m}{2\pi i} \int_C \frac{x^s \log s ds}{s} + \log c_{-m}$$  \hspace{1cm} (2.9)

Here, the first term of (2.9) is written as

$$\int_C \frac{x^s \log s ds}{s} = i(b') \int_{-\pi}^{\pi} e^{b' \log x} d\theta - \int_{-\pi}^{\pi} \theta e^{b' \log x} d\theta.$$  \hspace{1cm} (2.10)

Using termwise integration, the first and second terms on the right hand side of (2.10) are calculated as

$$\int_{-\pi}^{\pi} e^{b' \log x} d\theta = \int_{-\pi}^{\pi} d\theta + \sum_{r=1}^{\infty} \frac{(b' \log x)^r}{r!} \int_{-\pi}^{\pi} e^{ir\theta} d\theta = 2\pi,$$  \hspace{1cm} (2.11)

$$\int_{-\pi}^{\pi} \theta e^{b' \log x} d\theta = \sum_{r=1}^{\infty} \frac{(b' \log x)^r}{r!} \int_{-\pi}^{\pi} \theta e^{ir\theta} d\theta = \frac{2\pi i}{i} \sum_{r=1}^{\infty} \frac{(-1)^r}{r!} \int_0^{b' \log x} u^{r-1} du$$

$$= \frac{2\pi}{i} \int_0^{b' \log x} \frac{e^{-u} - 1}{u} du.$$  \hspace{1cm} (2.12)

Moreover, (2.12) is calculated as

$$\int_0^{b' \log x} \frac{e^{-u} - 1}{u} du = \gamma + \int_1^{b' \log x} \frac{du}{u} - \int_{b' \log x}^{\infty} \frac{e^{-u}}{u} du = \gamma + \log \log x + \log b' + O(e^{-C_F \sqrt{\log x}}),$$  \hspace{1cm} (2.13)

where the following result was used:

$$\int_0^1 \frac{1 - e^{-u}}{u} du - \int_1^{\infty} \frac{e^{-u}}{u} du = \gamma.$$  \hspace{1cm} (2.14)

Finally combining (2.11)–(2.13), we get

$$\log F_x(1) = \log c_{-m} + m\gamma + m \log \log x + O(e^{-C_F \sqrt{\log x}}).$$  \hspace{1cm} (2.15)

Taking exponentials in both sides of (2.14) and using the fact $e^y = 1 + O(y)$ for $y \ll 1$, we complete the proof of Theorem 1.2.
3 Proof of Theorem 1.3

First we shall show the following result from Mertens’ 3rd theorem:

**Proposition 3.1** (Mertens’ 2nd theorem). Let \( F \in S^\# \) and assume (I) and (II). Then we have

\[
\sum_{p \leq x} \frac{b_F(p)}{p} = m \log \log x + M + O(e^{-C_F \sqrt{\log x}}),
\]

where \( M \) is a constant given by

\[
M = \log c_m + m\gamma - \sum_{p \text{ prime}} \sum_{r=2}^{\infty} \frac{b_F(p^r)}{p^r}.
\]

Namely we may call the constant \( M \) generalized Mertens’ constant.

**Proof.** By (2.1) the sum of statement of Proposition 3.1 is written as

\[
\sum_{p \leq x} \frac{b_F(p)}{p} = \log F_x(1) - \sum_{p \leq x} \sum_{r=2}^{\infty} \frac{b_F(p^r)}{p^r} = \log F_x(1) - \sum_{p \text{ prime}} \sum_{r=2}^{\infty} \frac{b_F(p^r)}{p^r} + \sum_{p>x} \sum_{r=2}^{\infty} \frac{b_F(p^r)}{p^r}. \tag{3.1}
\]

By the trivial estimate, the third term of the right-hand side of (3.1) is estimated as

\[
\sum_{p>x} \sum_{r=2}^{\infty} \frac{b_F(p^r)}{p^r} \ll \sum_{p>x} \frac{1}{p^2} \ll \frac{1}{x}. \tag{3.2}
\]

Therefore from (2.14)–(3.2), Proposition 3.1 is obtained. \(\square\)

Secondly we shall show the following formula from Proposition 3.1:

**Proposition 3.2** (Mertens’ 1st theorem). Let \( F \in S^\# \) and assume (I) and (II). Then we have

\[
\sum_{p \leq x} \frac{b_F(p) \log p}{p} = m \log x + M_1 + O(e^{-C'_F \sqrt{\log x}}),
\]

where \( M_1 \) is a constant given by

\[
M_1 = \log c_m + m\gamma - \sum_{p \text{ prime}} \sum_{r=2}^{\infty} \frac{b_F(p^r) \log p}{p^r}.
\]
where $C'_F$ is a positive constant smaller than $C_F$ on Theorem 1.2. $M_1$ is a constant given by

$$
M_1 = -\int_{2}^{\infty} \frac{\Delta_2F(u)}{u} du + M \log 2 + m(\log 2)(\log \log 2 - 1),
$$

and $\Delta_2F(u)$ is given by

$$
\Delta_2F(u) = \sum_{p \leq u} \frac{b_F(p)}{p} - m \log u - M,
$$

which is estimated as $\Delta_2F(u) = O(e^{-C'_F \sqrt{\log u}})$.  

Proof. Using partial summation formula, we have

$$
\sum_{p \leq x} \frac{b_F(p) \log p}{p} = (\log x) \sum_{p \leq x} \frac{b_F(p)}{p} - \int_{2}^{x} \frac{1}{u} \sum_{p \leq u} \frac{b_F(p)}{p} du
$$

$$
= (\log x) \sum_{p \leq x} \frac{b_F(p)}{p} - \int_{2}^{x} \frac{m \log \log u + M}{u} du - \int_{2}^{x} \frac{\Delta_2F(u)}{u} du
$$

$$
= S_1 + S_2 + S_3. \tag{3.3}
$$

From Proposition 3.1 we see that $\Delta_2F(u) = O(e^{-C'_F \sqrt{\log x}})$ and

$$
S_1 = m(\log x) \log \log x + M \log x + O(e^{-C'_F \sqrt{\log x}}), \tag{3.4}
$$

$$
S_2 = -m(\log x) \log \log x - M \log x + m \log x + m(\log 2) \log 2 - m \log 2 + M \log 2, \tag{3.5}
$$

$$
S_3 = -\int_{2}^{\infty} \frac{\Delta_2F(u)}{u} du + O(e^{-C'_F \sqrt{\log x}}). \tag{3.6}
$$

Combining (3.3)–(3.6), we complete the proof of Proposition 3.2.  

Finally we shall show Theorem 1.3 from Proposition 3.2. The left hand side of Theorem 1.3 is written as follows:

$$
\sum_{n \leq x} b_F(n) \log n = \sum_{p \leq x} b_F(p) \log p + \sum_{p^r \leq x, r \geq 2} b_F(p^r) \log p^r. \tag{3.7}
$$
The second term on right-hand side of (3.7) is estimated as
\[
\sum_{p^r \leq x, \ r \geq 2} b_F(p^r) \log p^r \ll \sum_{p \leq \sqrt{x}} \sum_{r \leq \frac{\log x}{\log p}} \log p^r \ll \sqrt{x} (\log x)^2. \tag{3.8}
\]

Applying partial summation to the first term of the right-hand of (3.7), we have
\[
\sum_{p \leq x} b_F(p) \log p = x \sum_{p \leq x} \frac{b_F(p) \log p}{p} - \int_2^x \sum_{p \leq u} \frac{b_F(p) \log p}{p} \, du
= x \sum_{p \leq x} \frac{b_F(p) \log p}{p} - \int_2^x (m \log u + M_1) du - \int_2^x \Delta_{1F}(u) \, du
=: T_1 + T_2 + T_3 \tag{3.9}
\]
where \(\Delta_{1F}(u)\) is given by
\[
\Delta_{1F}(u) = \sum_{p \leq u} \frac{b_F(p) \log p}{p} - m \log u - M_1.
\]

Proposition 3.2 gives that \(\Delta_{1F}(u) = O(ue^{-C_p' \sqrt{\log u}})\) and
\[
\begin{align*}
T_1 &= mx \log x + M_1 x + O(xe^{-C_p' \sqrt{\log x}}), \\
T_2 &= -mx \log x + mx - M_1 x + 2(m \log 2 + M_1), \\
T_3 &\ll \left( \int_2^{\sqrt{x}} + \int_2^{x} \right) e^{-C_p' \sqrt{\log u}} du \ll \sqrt{x} + xe^{-C_p' \sqrt{\log x}} \ll xe^{-C_p'' \sqrt{\log u}}. \tag{3.12}
\end{align*}
\]

Therefore from (3.7)–(3.12), the proof of Theorem 1.3 is completed.

References

[1] P. Deligne, *La conjecture de Weil. I*, Publ. Math. Inst. Hautes Études Sci. 43 (1974), 273–307.

[2] Y. Ichihara, *The Siegel-Walfisz theorem for Rankin-Selberg L-functions associated with two cusp forms*, Acta Arith. (3) 92 (2000), 215–227.
[3] J. Kaczorowski and A. Perelli, *On the prime number theorem for the Selberg class*, Arch. Math. 80 (2003) 255–263.

[4] E. Landau, *Einführung in die Elementare und Analytische Theorie der Algebraischen Zahlen und der Ideale*, second edition, Chelsea Publishing Co., New York, 1949.

[5] J. Liu and Y. Ye, *Perron's formula and the prime number theorem for automorphic L-functions*, Pure Appl. Math. Quart 3 (2007), 481–497.

[6] F. Mertens, *Ein Beitrag zur analytischen Zahlentheorie*, J. Reine Angew. Math. 78 (1874), 46–62.

[7] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory. I. Classical Theory*, Cambridge University Press, 2007.

[8] Y. Motohashi, *Analytic Number Theory I*, Asakura Publishing, Tokyo, 2009, (in Japanese).

[9] M. Rosen, *A generalization of Mertens' theorem*, J. Ramanujan Math. Soc. (1) 14 (1999), 1–19

[10] A. Selberg, *Old and new conjectures and results about a class of Dirichlet series*, In Proc. Amalfi Conf. Analytic Number Theory, E. Bombieri et al. eds., 367–385, Universitàd Salerno 1992; Collected Papers, Vol. II, 47–63, Berlin-Heidelberg-New York 1991.