An Asymptotically Good Tower of Curves Over the Field with Eight Elements

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Introduction

In this note we construct an explicit asymptotically good tower of curves over the field $F_8$. For a curve $C$ defined over a finite field $F_q$ of cardinality $q$ we denote by $\#C(F_q)$ the number of $F_q$-rational points on $C$. If furthermore we denote, as usual, by $N_q(g)$ the function

$$N_q(g) = \max\{\#C(F_q) : C/F_q, g(C) = g\},$$

where $C$ runs through the set of smooth absolutely irreducible projective curves of genus $g$ defined over $F_q$ then Drinfeld and Vladuts showed in [D-V] the inequality

$$\limsup_{g \to \infty} \frac{N_q(g)}{g} \leq \sqrt{q} - 1,$$

(1)

and Ihara constructed in [I] for $q$ a square a sequence of modular curves which attains the upper bound in (1).

It then came as a surprise when in 1995 Garcia and Stichtenoth constructed in [G-S1] for $q$ a square a tower of Artin-Schreier covers

$$\ldots \to C_i \to C_{i-1} \to \ldots \to C_1 \to C_0$$

which is defined over $F_q$ and given by a simple recursive equation such that

$$\lim_{i \to \infty} g(C_i) = \infty \quad \text{and} \quad \lim_{i \to \infty} \frac{\#C_i(F_q)}{g(C_i)} = \sqrt{q} - 1.$$

An infinite tower $C_\bullet$ of covers of curves over $F_q$

$$\ldots \to C_i \to C_{i-1} \to \ldots \to C_1 \to C_0$$

with $g(C_i) > 1$ for some $i \geq 0$ is called an asymptotically good tower if

$$\ell(C_\bullet) = \lim_{i \to \infty} \frac{\#C_i(F_q)}{g(C_i)} > 0.$$
Note that in [G-S2] it is shown that this limit exists for towers having at least one index \( i \) with \( g(C_i) > 1 \).

Apart from having an evident charm of their own, asymptotically good towers are important for coding theory, since such towers enable the construction of long error correcting codes over \( \mathbf{F}_q \) which can correct a fixed percentage of errors per codeword and have a positive transmission rate. However, for this application it is essential that the curves are in explicit form.

For \( q \) is not a square the results are much less complete. It is not known how good the Drinfeld-Vladuts upper bound (1) is for that case. For \( q \) not a square asymptotically good towers of curves are mainly obtained by class field theory, see for example [N-X]. These constructions are not explicit. In 1985 Zink using certain Shimura surfaces constructed in [Z] a tower \( C_\bullet \) of curves defined over \( \mathbf{F}_{p^3} \) for \( p \) a prime with limit

\[
\ell(C_\bullet) \geq \frac{2(p^2 - 1)}{p + 2},
\]

but that construction is far from explicit. In [G-S-T] there is an explicit asymptotically good tower of Kummer covers over \( \mathbf{F}_{q = p^m} \) for \( m \geq 2 \) with limit

\[
\ell(C_\bullet) \geq \frac{2}{q - 2}.
\]

Here we present an explicit tower \( C_\bullet \) of Artin-Schreier curves defined over \( \mathbf{F}_8 \) given by a simple recursive equation with limit

\[
\ell(C_\bullet) = 3/2.
\]

One should compare this with (2). It remains an interesting problem to see whether our explicit tower is related to that of Zink, cf. the remarks made by Elkies at the end of [E]. Another interesting problem is to extend our construction to other fields of odd degree over the prime field.

We give explicit formulas for the genus and number of rational points for the curves \( C_i \) in our tower. The ramification behavior turns out to be rather subtle with alternating ramification and non-ramification.

\section*{1 The basic equation}

In our search for curves over finite fields with many points we came across a curve defined over \( \mathbf{F}_8 \) with a remarkable property. The curve of genus 1 given by the affine equation

\[x^2_1 + x_1 = x_0 + 1 + 1/x_0\]

has 14 \( \mathbf{F}_8 \)-rational points and attains the Hasse-Weil bound for \( \mathbf{F}_8 \). To each \( x_0 \in \mathbf{F}_8 - \mathbf{F}_2 \) there correspond 2 solutions \( x_1 \in \mathbf{F}_8 - \mathbf{F}_2 \) and if \( x_0 \) runs through \( \mathbf{F}_8 - \mathbf{F}_2 \) then so does \( x_1 \). This implies that the system of equations

\[x^2_{i+1} + x_{i+1} = x_i + 1 + 1/x_i \quad i = 0, 1, 2, \ldots\]
has sequences of solutions \((x_0, x_1, x_2, \ldots)\) for every \(x_0 \in \mathbb{F}_8 - \mathbb{F}_2\).

Consider in \(\mathbb{P}^1 \times \mathbb{P}^1\) over the field \(\mathbb{F}_2\) the closure of the affine curve given by the equation
\[
x_1^2 + x_1 = x_0 + 1 + 1/x_0.
\]
This defines a smooth projective curve \(C\) of genus 1 together with two morphisms \(b_1 : C \to \mathbb{P}^1, (x_0, x_1) \mapsto x_0,\) and \(e_1 : C \to \mathbb{P}^1, (x_0, x_1) \mapsto x_1\) of degree 2. The curve \(C\) possesses 2 points rational over \(\mathbb{F}_2\), 8 points rational over \(\mathbb{F}_4\) and 14 points rational over \(\mathbb{F}_8\). The correspondence \(C\) in \(\mathbb{P}^1 \times \mathbb{P}^1\) preserves the points of \(\mathbb{P}^1(\mathbb{F}_8) - \mathbb{P}^1(\mathbb{F}_2)\) and surprisingly also those of \(\mathbb{P}^1(\mathbb{F}_4)\).

We consider the following infinite tower \(C_i\) of smooth projective curves defined over \(\mathbb{F}_2\)
\[
\rightarrow C_i \xrightarrow{\pi_i} C_{i-1} \xrightarrow{\pi_{i-1}} \cdots \xrightarrow{\pi_2} C_1 \xrightarrow{\pi_1} C_0 = \mathbb{P}^1,
\]
where we take an affine coordinate \(x_0\) on \(C_0\) and where the cover \(C_i \to C_{i-1}\) is given by the affine equation
\[
x_i^2 + x_i = x_{i-1} + 1 + \frac{1}{x_{i-1}} \quad \text{for } i \geq 1.
\]
Equivalently, we can describe the curve \(C_i\) as the normalization of the curve \(D_i\) defined by
\[
D_i = \{(p_0, p_1, \ldots, p_i) \in \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 : (p_j, p_{j+1}) \in C \text{ for } j = 0, \ldots, i-1\}.
\]
This shows that \(C_i\) for \(i \geq 1\) admits two maps \(b_i : C_i \to C_{i-1}\) (resp. \(e_i : C_i \to C_{i-1}\)) given (on the model \(D_i\)) by \((p_0, \ldots, p_i) \mapsto (p_0, \ldots, p_{i-1})\) (resp. \(\mapsto (p_1, \ldots, p_{i})\)). The curve \(C_i\) is then the normalization of the fibre product
\[
C_{i-1} \times_{C_{i-2}} C_{i-1}
\]
via the maps \(b_{i-1}\) and \(e_{i-1}\).

We now work over the algebraic closure \(\mathbb{F}\) of \(\mathbb{F}_2\) and consider geometric points of \(C_i \otimes \mathbb{F}\). It will turn out that ramification in \(C_i/C_{i-1}\) can occur only in points \(P\) that map to a point of \(D_i\) with coordinates in \(\mathbb{P}^1(\mathbb{F}_4)\). We therefore introduce the following notation for such points. By \(P = P(a_0, a_1, \ldots, a_i)\) with \(a_j \in \mathbb{P}^1(\mathbb{F}_4)\) we denote a point on \(C_i\) such that \(x_j(P) = a_j\) for \(0 \leq j \leq i\). That is, the point \((a_0, \ldots, a_i)\) is the image point in \(D_i\) and will be called the index sequence of the point. Note that because of the normalization, a point \(P(a_0, \ldots, a_i)\) of \(C_i\) is not necessarily uniquely determined by its index sequence \((a_0, \ldots, a_i)\) on \(D_i\).

We shall write \(\rho\) for a primitive element of \(\mathbb{F}_4\). Note that in an index sequence \((a_0, a_1, \ldots, a_i)\) of a point \(P\) we have

\[
\begin{align*}
\infty & \quad \infty \\
0 & \quad \infty \\
1 \quad \text{is followed by} & \quad \rho \text{ or } \rho^2 \\
\rho & \quad 0 \text{ or } 1 \\
\rho^2 & \quad 0 \text{ or } 1
\end{align*}
\]
Sometimes we shall write \((a_0, \ldots, a_i, \infty^j)\) for a point \((a_0, \ldots, a_i, \infty, \ldots, \infty)\).

§2 The principal part of the \(x_i\)

The main problem to find the limit \(\ell(C_i)\) of our tower lies in the determination of the genus \(g(C_i)\). In order to compute it we have to find the ramification divisor of \(C_{i+1}\) over \(C_i\). Since we are dealing with Artin-Schreier equations we can restrict ourselves to the points which are poles of the function \(f_i = x_i + 1 + 1/x_i\). The contribution to the ramification is determined by the orders \(\text{ord}_P(f_i^*)\) for the poles \(P\) on \(C_i\) of the Artin-Schreier reduction \(f_i^*\) of the function \(f_i\).

(2.1) Lemma. The zeros of \(x_i\) on \(C_i\) are of the form \(P(a_0, a_1, \ldots, a_i)\) with \(a_i = 0\), \(a_i-j \in \mathbb{F}_4 - \mathbb{F}_2\) for \(j \geq 1\) odd and \(a_i-j = 1\) for \(j \geq 2\) even. The poles of \(x_i\) are of the form \(P(b_0, b_1, \ldots, b_j, \infty^{i-j})\) with \(0 \leq j \leq i-1\) and \((b_0, \ldots, b_j)\) an index sequence of a zero of \(x_j\) or of the form \(P(\infty^{i+1})\).

Proof. By induction on \(i\). The lemma is true for \(x_0\). From the equation

\[
x_i^2 + x_i = \frac{x_{i-1}^2 + x_{i-1} + 1}{x_{i-1}} = f_{i-1}
\]

it follows that we have the equality of divisors on \(C_i\)

\[(f_{i-1}) = (x_i) + (x_i + 1) = (x_i)_0 + (x_i)_1 - 2(x_i)_\infty.\]

So poles of \(x_i\) lie above poles of \(f_{i-1}\) and the points \(P\) on \(C_i\) with \(x_i(P) \in \mathbb{F}_2\) lie above the zeros of \(f_{i-1}\). Moreover, we have

\[(f_{i-1}) = (x_{i-1} + \rho) + (x_{i-1} + \rho^2) - (x_{i-1}) = (x_{i-1})_\rho + (x_{i-1})_\rho^2 - (x_{i-1})_0 - (x_{i-1})_\infty.\]

which implies that the poles of \(f_{i-1}\) are the zeros and poles of \(x_{i-1}\), while the zeros of \(f_{i-1}\) are the points \(P\) on \(C_{i-1}\) with \(x_{i-1}(P) \in \mathbb{F}_4 - \mathbb{F}_2\). Hence the poles of \(x_i\) are the points on \(C_i\) above the zeros and the poles of \(x_{i-1}\) on \(C_{i-1}\), whereas the zeros of \(x_i\) lie above points \(P\) on \(C_{i-1}\) with \(x_{i-1}(P) \in \mathbb{F}_4 - \mathbb{F}_2\). So we obtain the index sequence of a pole of \(x_i\) by adding \(\infty\) to a zero or pole of \(x_{i-1}\) and we obtain the index sequence of a zero of \(x_i\) by adding a zero to an index sequence which ends with an element of \(\mathbb{F}_4 - \mathbb{F}_2\) and in which 1 and elements of \(\{\rho, \rho^2\}\) alternate. \(\square\)

In the following we shall develop rational functions on \(C_i\) as a power series in a local parameter at a given point \(P\), that is, we consider the function as an element of the quotient field of the completion of the local ring of \(P\). Often we are only interested in the principal part and neglect elements that are regular, i.e. elements of (the completion of) the local ring. By the notation

\[f = g + O(P)\]
we mean that \( f - g \) is regular in \( P \), that is, is an element of \( O_P \) or of \( \hat{O}_P \).

Consider now a sequence of points \( P_0 \in C_0, P_1 \in C_1, \ldots, P_i \in C_i \) with \( \pi_\ell(P_\ell) = P_{\ell-1} \) for \( \ell = 1, \ldots, i \) and with the property that 1 and \( \rho \) or \( \rho^2 \) alternate in the index sequence \( (a_0, \ldots, a_i) \) of \( P_i \).

We shall first assume that \( a_0 = 1 \). Then the function \( t = x_0 + 1 \) provides a local parameter at \( P_0 \) on \( C_0 \) and the pull back (under the maps \( \pi_\ell \)) of this function (again denoted by \( t \)) is still a local parameter at the points \( P_j \) on \( C_j \) for \( j \leq i \).

In the completion of the local ring \( \hat{O}_P \cong \mathbb{F}[t] \) the function \( x_j \) can be written as a power series in \( t \)

\[
x_j = a_j + m_j(t),
\]

where \( m_j(t) \in \mathbb{F}[t] \) has ord\(_t\)(\( m_j \)) \( \geq 1 \).

**Lemma.** In the quotient field \( \mathbb{F}((t)) \) of the formal power series ring \( \hat{O}_P \cong \mathbb{F}[t] \) the function \( m_j(t) \) satisfies for \( 0 \leq j \leq i \) the relations

\[
\frac{1}{m_j} = \frac{a_{j-1}}{m_{j-1}} + O(P_j) \quad \text{for } j \geq 2 \text{ even},
\]

\[
\frac{1}{m_j} = \frac{1}{m_{j-1}^2} + \frac{1}{m_{j-1}} + O(P_j) \quad \text{for } j \text{ odd}.
\]

**Proof.** We start with \( m_0(t) = t \). For even \( j \geq 2 \) we have \( a_{j-1} \in \{\rho, \rho^2\} \) and \( a_j = 1 \) since we assumed that \( a_0 = 1 \). From the relation

\[
x_j^2 + x_j = x_{j-1} + 1 + 1/x_{j-1}
\]

we obtain

\[
m_j^2 + m_j = a_{j-1} + m_{j-1} + 1 + 1/(a_{j-1} + m_{j-1})
\]

\[
= a_{j-1} + m_{j-1} + 1 + (1/a_{j-1}) \sum_{n=0}^{\infty} (m_{j-1}/a_{j-1})^n
\]

\[
= a_{j-1}^2 m_{j-1} + m_{j-1}^2 + \text{higher powers of } m_{j-1}.
\]

This implies that \( m_j \) is the product of \( a_{j-1}^2 m_{j-1} \) with a 1-unit \( u \) in \( m_{j-1} \), i.e. a unit of the form \( u = 1 + r \) with \( r \in (m_{j-1}) \). So we get

\[
\frac{1}{m_j} = \frac{a_{j-1}}{m_{j-1}} \cdot u = \frac{a_{j-1}}{m_{j-1}} + O(P_j).
\]

For \( j \) odd we have \( a_j \in \{\rho, \rho^2\} \) and \( a_{j-1} = 1 \). In the same way as for \( j \) even we obtain

\[
a_j^2 + m_j^2 + a_j + m_j = 1 + m_{j-1} + 1 + \frac{1}{1+m_{j-1}} = 1 + \sum_{n=2}^{\infty} m_{j-1}^n,
\]
that is, \[ m_j^2 + m_j = \sum_{n=2}^{\infty} m_j^{-n}, \] so that we have
\[ m_j = m_j^{2} + m_j^{3} + \text{higher powers of } m_j^{-1}. \]

This means that
\[
\frac{1}{m_j} = \frac{1}{m_j^{2}} + \frac{1}{m_j^{3}} + \text{higher powers of } m_j^{-1}
= \frac{1}{m_j^{2}} + \frac{1}{m_j^{3}} + O(P_j).
\]
This completes the proof of the lemma. □

We denote the principal part of \(1/m_j\) by \(F_j\). We now can deduce the following corollary.

(2.3) Corollary. The principal part \(F_j\) of \(1/m_j\) satisfies:
\[
F_j = \begin{cases} 
F_j^{2} + F_j^{-1} & \text{for } j \text{ odd} \\
(a_j-1) \cdot F_j^{-1} & \text{for } j \geq 2 \text{ even}.
\end{cases}
\]

Furthermore, \(F_j\) is a 2-linearized polynomial in \(1/t\) of the form
\[
F_j = \frac{b_k}{t^{2k}} + \frac{b_{k-1}}{t^{2k-1}} + \ldots + \frac{b_0}{t},
\]
where \(k = [(j + 1)/2]\), the coefficients \(b_\ell\) are in \(F_4\) and \(b_k \neq 0\).

Proof. The relations for \(F_j\) follow at once from Lemma.(2.2). We have \(F_0 = 1/t\) and \(F_1 = 1/t^2 + 1/t\) from which the formula for \(F_j\) follows by induction. □

For an index sequence \((a_0, a_1, \ldots, a_i)\) where \(a_0 \in \{\rho, \rho^2\}\) we have a similar result.

§3 The ramification behavior

Now we study the ramification behavior in a point \(P_i = P(a_0, \ldots, a_i-1, a_i = 0)\) which is a zero of \(x_i\) on \(C_i\) for \(i \geq 2\). We assume \(a_0 = 1\). Then \(a_{\text{odd}} \in \{\rho, \rho^2\}\) and \(i\) is even. In a point \(P_i\) where \(a_0 \in \{\rho, \rho^2\}\) the ramification behavior is similar.

Since we are working with Artin-Schreier covers here we introduce the standard notation \(\varphi(f) = f^2 + f\) for an element \(f\) in one of our function fields.

(3.1) Lemma. A linear combination \(\sum_{j=2, \text{even}}^{i} B_{j,i} F_j\) with coefficients \(B_{j,i} \in F_4\) can be written as
\[
\sum_{j=2, \text{even}}^{i} B_{j,i} F_j = \varphi(\sum_{j=0, \text{even}}^{i-2} B_{j,i-2} F_j) + B_i^* F_0
\]
with
\[
B_i^* = \varphi(\sum_{j=2, \text{even}}^{i} B_{j,i} a_{j-1})
\]
and
\[ B_{j,i-2} = (B_i^* + \varphi(\sum_{k=2,\text{even}}^j B_{k,i}a_{k-1}))a_{j+1}^2 + \varphi(B_{j+2,i}). \]  

(6)

**Proof.** Using Corollary (2.3) we find for even \( j \geq 2 \)
\[ B_{j,i}F_j = B_{j,i}a_{j-1}F_{j-1} = B_{j,i}a_{j-1}\varphi(F_{j-2}) = \varphi(B_{j,i}^2a_{j-1}^2F_{j-2}) + \varphi(B_{j,i}a_{j-1})F_{j-2}, \]
that is
\[ B_{j,i}F_j = \varphi(B_{j,i}^2a_{j-1}^2F_{j-2}) + \varphi(B_{j,i}a_{j-1})F_{j-2}. \]  

(7)

Applying (7) to the second term in the RHS of (7) we obtain
\[ \varphi(B_{j,i}a_{j-1})F_{j-2} = \varphi(\varphi(B_{j,i}a_{j-1})^2a_{j-3}^2F_{j-4}) + \varphi(B_{j,i}a_{j-1})F_{j-4}, \]
where we use \( a_{\text{odd}}^2 + a_{\text{odd}} = 1 \). Continuing this way we find an expression for \( B_{j,i}F_j \) as
\[ \varphi(\text{linear combination of } F_{j-2}, \ldots, F_0) + \varphi(B_{j,i}a_{j-1})F_0. \]
Adding these relations for all terms in \( \sum_{j=2,\text{even}}^i B_{j,i}F_j \) we find (4) with coefficients satisfying the equations (5) and (6). \( \square \)

Note that all coefficients are in \( \mathbf{F}_4 \) and that \( B_i^* \) is in \( \mathbf{F}_2 \).

The cover \( C_{i+1}/C_i \) is given by the equation
\[ x_{i+1}^2 + x_{i+1} = x_i + 1 + 1/x_i. \]
In a point \( P_i \) with index sequence \( (1, a_1, \ldots, a_{i-1}, 0) \) we have the relation
\[ x_{i+1}^2 + x_{i+1} = \frac{1}{m_i} + O(P_i) = F_i + O(P_i). \]  

(8)

Therefore the principal part of \( x_{i+1}^2 + x_{i+1} \) in \( P_i \) is \( F_i \). According to Lemma (3.1) we have
\[ F_i = \varphi(\sum_{j=0,\text{even}}^{i-2} B_{j,i-2}F_j) + B_i^*F_0 \]
with \( B_i^* = a_{i-1}^2 + a_{i-1} = 1 \) and \( B_{j,i-2} = a_{j+1}^2 \) for even \( j = 0, 2, \ldots, i-2 \).

By the substitution
\[ X_{i+1} = \sum_{j=0}^{i-2} B_{j,i-2}F_j + x_{i+1} \]
we can reduce the equation (8) to
\[ X_{i+1}^2 + X_{i+1} = B_i^*F_0 + O(P_i) = F_0 + O(P_i) \]  

(9)

with \( F_0 = 1/t \).
(3.2) Corollary. The point \( P_i = P(a_0 = 1, a_1, \ldots, a_i = 0) \) is totally ramified in \( C_{i+1}/C_i \) and the contribution of \( P_i \) to the ramification divisor of \( C_{i+1}/C_i \) is 2. In the pole \( P_{i+1} = P_i(\infty) \) of \( x_{i+1} \) we have
\[
\text{ord}_{P_{i+1}}(x_{i+1}) = -2^{[(i+1)/2]}.
\]

As the next step we consider the behavior of the pole \( P_{i+1} = P_i(\infty) \) of the function \( f_{i+1} = x_{i+1} + 1 + (1/x_{i+1}) \) in the cover \( C_{i+2}/C_{i+1} \).

In \( P_{i+1} \) the equation of \( C_{i+2}/C_{i+1} \) is
\[
x_{i+2}^2 + x_{i+2} = x_{i+1} + O(P_{i+1})
\]
\[
= \sum_{j=0, \text{even}}^{i-2} B_{j,i-2} F_j + X_{i+1} + O(P_{i+1}). \tag{10}
\]

If we apply Lemma (3.1) to the linear combination \( \sum_{j=2, \text{even}}^{i-2} B_{j,i-2} F_j \) the RHS of (10) becomes
\[
\varphi(\sum_{j=0, \text{even}}^{i-4} B_{j,i-4} F_j) + B_{i-2}^* F_0 + B_{0,i-2} F_0 + X_{i+1} + O(P_{i+1}). \tag{11}
\]

Then by using (9), i.e., by substituting \( F_0 = X_{i+1}^2 + X_{i+1} + O(P_{i+1}) \), the expression (11) is converted to
\[
\varphi(\sum_{j=0, \text{even}}^{i-4} B_{j,i-4} F_j) + \varphi(B_{i-2}^* X_{i+1}) + \varphi(B_{0,i-2}^2 X_{i+1}) + (B_{0,i-2}^2 + B_{0,i-2} + 1) X_{i+1} + O(P_{i+1})
\]
with \( B_{0,i-2}^2 + B_{0,i-2} + 1 = a_1^2 + a_1 + 1 = 0 \). Hence the equation of \( C_{i+2}/C_{i+1} \) in \( P_{i+1} \) is of the form
\[
x_{i+2}^2 + x_{i+2} = \varphi(\sum_{j=0}^{i-4} B_{j,i-4} F_j) + \varphi((B_{i-2}^* + B_{0,i-2}^2) X_{i+1}) + O(P_{i+1}). \tag{12}
\]

(3.3) Corollary. The pole \( P_{i+1} \) of \( f_{i+1} \) is unramified in the cover \( C_{i+2}/C_{i+1} \) and in a point \( P_{i+2} = P_{i+1}(\infty) \) above \( P_{i+1} \) we have
\[
\text{ord}_{P_{i+2}}(x_{i+2}) = -2^{[(i+1)/2]-1}.
\]

Note that (12) implies that
\[
x_{i+2} + (\sum_{j=0, \text{even}}^{i-4} B_{j,i-4} F_j) + (B_{i-2}^* + B_{0,i-2}^2) X_{i+1}
\]
is integral in the point \( P_{i+2} \).
To analyze the situation in $P_{i+2}$ we start with the equation of $C_{i+3}/C_{i+2}$ in this point:

$$x_{i+3}^2 + x_{i+3} = x_{i+2} + O(P_{i+2})$$

$$= \left( \sum_{j=0, \text{even}}^{i-4} B_{j,i-4}F_j + (B^*_{i-2} + B^2_{0,i-2})X_{i+1} + O(P_{i+2}) \right).$$

(13)

Using Lemma (3.1) and (9) the RHS of (13) is of the form

$$\varphi\left( \sum_{j=0, \text{even}}^{i-6} B_{j,i-6}F_j + \varphi((B^*_{i-4} + B^2_{0,i-4})X_{i+1}) + \varphi(B_{0,i-4}) + B^*_{i-2} + B^2_{0,i-2})X_{i+1} + O(P_{i+2}) \right).$$

(14)

Since (6) implies $\varphi(B_{0,i-4}) = B^*_{i-2}$ the coefficient of $X_{i+1}$ in (14) is $B^2_{0,i-2} = a_1^2$. So the right hand side of (13) has the form

$$\varphi(\gamma) + B^2_{0,i-2}X_{i+1} + O(P_{i+2})$$

for some $\gamma \in F(C_{i+2})$. If we set

$$X_{i+3} = \sum_{j=0}^{i-6} B_{j,i-6}F_j + (B^*_{i-4} + B^2_{0,i-4})X_{i+1}$$

the equation of $C_{i+3}/C_{i+2}$ becomes

$$x_{i+3}^2 + x_{i+3} = B^2_{0,i-2}X_{i+1} + O(P_{i+2}).$$

(15)

(3.4) Corollary. The point $P_{i+2}$ is totally ramified in the cover $C_{i+3}/C_{i+2}$ and the contribution to the ramification divisor is 2. In $P_{i+3} = P_{i+2}(\infty)$ above $P_{i+2}$ we have

$$\text{ord}_{P_{i+3}}(x_{i+3}) = -2^{[(i+1)/2]-1}.$$

If we continue along these lines we obtain the following formulas:

(3.5) Formula. For $t$ even and $2 \leq t \leq i$ the equation of $C_{i+t}/C_{i+t-1}$ in a point $P_{i+t-1}$ is

$$x_{i+t}^2 + x_{i+t} = \varphi\left( \sum_{j=0, \text{even}}^{i-2t} B_{j,i-2t}F_j + \varphi((B^*_{i-2t+2} + B^2_{0,i-2t+2})X_{i+1}) + \sum_{k=1}^{t/2-1} \varphi((B^2_{2k-2,i-2t+4k}B^2_{2k-2,i-2})X_{i+2k+1}) + O(P_{i+t-1}) \right).$$
(3.6) Formula. For \( t \) odd and \( 3 \leq t \leq i - 1 \) the equation of \( C_{i+t}/C_{i+t-1} \) in a point \( P_{i+t-1} \) is

\[
x_{i+t}^2 + x_{i+t} = \varphi \left( \sum_{j=0, \text{even}}^{i-2t} B_j, i-2tF_j \right) + \varphi \left( (B_{i-2t+2}^* + B_0, i-2t+2)X_{i+1} \right) + \]
\[
\sum_{k=1}^{(t-3)/2} \varphi \left( (B_{2k-2, i-2t+4k}^* B_{2k-2, i-2}^2 X_{i+2k+1}) + B_{t-3, i-2}^2 X_{i+t-2} + O(P_{i+t-1}) \right).
\]

(3.7) Remark. The function \( X_{i+2k+1} \) with \( k \geq 1 \) satisfies an equation of the form

\[
X_{i+2k+1}^2 + X_{i+2k+1} = B_{2k-2, i-2}^2 X_{i+2k-1} + O(P_{i+2k}).
\]

We also find

\[
\text{ord}_{P_{i+t}}(x_{i+t}) = -2\lceil (i+2)/2 \rceil - \lfloor t/2 \rfloor \quad \text{for} \quad 2 \leq t \leq i.
\]

Hence \( \text{ord}_{P_{2i}}(x_{2i}) = -1 \) and this implies that from \( P_{2i} \) on the extensions in the tower are totally ramified above \( P_{2i} \) and their contribution to the ramification divisor is 2.

We summarize the preceding results in the following theorem.

(3.8) Theorem. A pole \( P_{i+j} = P(a_0, \ldots, a_{i-1}, 0, \infty^j) \) of \( x_{i+j} \) on \( C_{i+j} \) for \( j \geq 1 \) (resp. of \( 1/x_i \) for \( j = 0 \)) with \( a_0 = 1 \) is

1) totally ramified in \( C_{i+j+1}/C_{i+j} \) for \( j \) with \( j = 0, 2, 4, \ldots, i-2 \) or with \( j \geq i \) and each of these contributes 2 to the ramification divisor,

2) unramified in \( C_{i+j+1}/C_{i+j} \) for \( j = 1, 3, 5, \ldots, i-1 \).

For a pole whose index sequence starts with \( a_0 \in \{ \rho, \rho^2 \} \) there is the following similar result.

(3.9) Theorem. A pole \( P_{i+j} = P(a_0, \ldots, a_{i-1}, 0, \infty^j) \) of the function \( x_{i+j} \) on \( C_{i+j} \) with \( a_0 \in \{ \rho, \rho^2 \} \) is

1) totally ramified in \( C_{i+j+1}/C_{i+j} \) for all \( j \) with \( j = 0, 2, 4, \ldots, i-3 \) and for all \( j \) with \( j \geq i-1 \) with contribution 2 to the ramification divisor,

2) unramified in \( C_{i+j+1}/C_{i+j} \) for \( j = 1, 3, 5, \ldots, i-2 \).

(3.10) Remark. We always have the totally ramified points \( P(\infty, \infty, \ldots, \infty) \) and also \( P(0, \infty, \infty, \ldots, \infty) \) which contribute 2 to the ramification divisor.

§4 The genus and the number of points in the tower

In order to compute the genus of our curve \( C_i \) we have to count the number of points on the curve \( C_i \) which contribute to the ramification divisor. We find using (3.8)–(3.10)
Theorem. Let $n_i$ be the number of points on $C_i$ which are totally ramified in $C_{i+1}/C_i$. Then

$$n_i = \begin{cases} 
((i+2)/4) + 2 \cdot 2^{i/2} & \text{for } i \text{ even}, \\
[i/4] + 2 \cdot 2^{(i+1)/2} & \text{for } i \text{ odd}.
\end{cases}$$

Now it is not difficult to determine the genus $g(C_i)$ of $C_i$. From the Hurwitz formula it follows that

$$g(C_i) = 1 + \sum_{j=1}^{i-1} 2^{i-1-n_j}. \quad (16)$$

If we combine (16) with theorem (4.1) we get:

Theorem. The genus $g(C_i)$ of $C_i$ satisfies

$$g(C_i) = 2^{i+2} + 1 - \begin{cases} 
(i+10)2^{(i/2)-1} & \text{for } i \text{ even}, \\
(i+2[i/4]+15)2^{(i-3)/2} & \text{for } i \text{ odd}.
\end{cases}$$

Now we count the number $\#C_i(F_8)$ of $F_8$-rational points on $C_i$.

Theorem. We have $\#C_i(F_8) = 6 \cdot 2^i + 2$.

Proof. Let $\alpha$ be a primitive element of $F_8$ which satisfies $\alpha^3 + \alpha + 1 = 0$. For $x \in F_8 - F_2$ we find

$$\{x + \frac{1}{x} + 1 : x \in F_8 - F_2\} = \{\alpha, \alpha^2, \alpha^4\},$$

but also

$$\{y^2 + y : y \in F_8 - F_2\} = \{\alpha, \alpha^2, \alpha^4\}.$$

This means that a point $x \in \mathbb{P}^1(F_8)$ with $x \not\in \mathbb{P}^1(F_2)$ splits completely in the tower. This yields $6 \cdot 2^i$ rational points over $F_8$ on $C_i$. Besides these we have two totally ramified points $P(0, \infty, \ldots, \infty)$ and $P(\infty, \ldots, \infty)$ defined over $F_2$. □

Combining this with the formula for the genus we obtain the following Theorem.

The tower of curves $C_i$ over $F_8$ is asymptotically good with limit

$$\lim_{i \to \infty} \frac{\#C_i(F_8)}{g(C_i)} = \frac{3}{2}.$$ 

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