The Loewy length of the descent algebra of $D_{2m+1}$.

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Abstract. In this article the Loewy length of the descent algebra of $D_{2m+1}$ is shown to be $m + 2$, for $m \geq 2$, by providing an upper bound that agrees with the lower bound in [Bonnafé and Pfeiffer, 2006]. The bound is obtained by showing that the length of the longest path in the quiver of the descent algebra of $D_{2m+1}$ is at most $m + 1$. To achieve this bound, the geometric approach to the descent algebra is used, in which the descent algebra of a finite Coxeter group $W$ is identified with an algebra associated to the reflection arrangement of $W$.

1. Introduction

Cédric Bonnafé and Götz Pfeiffer determined the Loewy length of the descent algebra $\Sigma_k(W)$ for irreducible finite Coxeter groups $W$ of all types except $D_{2m+1}$ [Bonnafé and Pfeiffer, 2006]. (The case of the symmetric group was established earlier; see [Garsia and Reutenauer, 1989] Theorems 5.6 and 5.7 and [Atkinson, 1992, Theorem 3.4].) For type $D_{2m+1}$, Bonnafé and Pfeiffer prove that the Loewy length of $\Sigma_k(D_{2m+1})$ is at least $m + 2$, and state that they suspect this is an equality. In this paper, we show that $m + 2$ is also an upper bound—by showing that the length of the longest path in the quiver of $\Sigma_k(D_{2m+1})$ is at most $m + 1$—confirming their suspicion.

We briefly outline the argument, and the structure of the paper. The plan is to use the geometric approach to the descent algebra; the descent algebra $\Sigma_k(W)$ can be identified with a subalgebra of an algebra $kF$ associated to the reflection arrangement of $W$. This is explained in Section 2. Specifically, there is an action of $W$ on $kF$ and an anti-isomorphism between $\Sigma_k(W)$ and the $W$-invariant subalgebra $(kF)_W$. After defining quivers and path algebras in Section 3, Section 4 describes the quiver $Q$ of $kF$ and the construction of a $W$-equivariant surjection $\varphi : kQ \to kF$. This results in a surjection $(kQ)_W \to (kF)_W$ that we use to gain information about the quiver $Q_W$ of $(kF)_W$ in Section 5. We then specialize in Section 6 to the irreducible finite Coxeter group of type $D$ and bound the length of the longest path in the quiver $Q_{D_{2m+1}}$. The reader familiar with the above theory may decide to begin in Section 6.

Throughout this paper $k$ denotes a field whose characteristic is zero or does not divide the order of the Coxeter group.
2. The Geometric Approach to the Descent Algebra

We begin by recalling the definition of Coxeter systems and the descent algebra. We then explain the connection between the descent algebra and the face semigroup algebra of the reflection arrangement of the Coxeter group.

2.1. Coxeter systems and reflection arrangements. Let \( V \) be a finite dimensional real vector space. A \emph{finite Coxeter group} \( W \) is a finite group generated by a set of reflections of \( V \). The \emph{reflection arrangement} of \( W \) is the hyperplane arrangement \( \mathcal{A} \) consisting of the hyperplanes of \( V \) fixed by some reflection in \( W \).

Let \( W \) denote a finite Coxeter group with reflection arrangement \( \mathcal{A} \) and let \( c \) denote a connected component of the complement of \( \bigcup_{H \in \mathcal{A}} H \) in \( V \). A \emph{wall} of \( c \) is a hyperplane \( H \in \mathcal{A} \) such that \( H \cap \overline{c} \) spans \( H \), where \( \overline{c} \) is the closure of \( c \) in \( V \). Let \( S \subset W \) denote the set of reflections in the walls of \( c \). Then \( S \) is a generating set of \( W \) [Brown, 1989 §1.5A] and the pair \((W,S)\) is called a \emph{Coxeter system} with \emph{fundamental chamber} \( c \).

2.2. The descent algebra. Fix a Coxeter system \((W,S)\). For \( J \subseteq S \), let \( W_J = \langle J \rangle \) denote the subgroup of \( W \) generated by the elements in \( J \). Each coset of \( W_J \) in \( W \) contains a unique element of minimal length, where the \emph{length} \( \ell(w) \) of an element \( w \) of \( W \) is the smallest number of generators \( s_1, \ldots, s_t \in S \) such that \( w = s_1 \cdots s_t \) [Humphreys, 1990 Proposition 1.10(c)]. Let \( X_J \) denote the set of \emph{minimal length coset representatives} of \( W_J \) and let \( x_J = \sum_{w \in X_J} w \) denote the formal sum of the elements of \( X_J \). Then \( x_J \) is an element of the group algebra \( kW \) of \( W \) with coefficients in a field \( k \). Louis Solomon proved that the elements \( x_J \), one for each \( J \subseteq S \), form a basis of a subalgebra of \( kW \) [Solomon, 1976 Theorem 1]. This subalgebra is denoted by \( \Sigma_k(W) \) and is called the \emph{descent algebra} of \( W \).

2.3. The faces of \( \mathcal{A} \). For each hyperplane \( H \in \mathcal{A} \), let \( H^+ \) and \( H^- \) denote the two open half spaces of \( V \) determined by \( H \). The choice of labels \( H^+ \) and \( H^- \) is arbitrary, but fixed throughout. For convenience, let \( H^0 = H \). A \emph{face} of the arrangement \( \mathcal{A} \) is a non-empty intersection of the form \( x = \bigcap_{H \in \mathcal{A}} H^{\sigma_H(x)} \), where \( \sigma_H(x) \in \{ +, 0, - \} \) for each hyperplane \( H \in \mathcal{A} \). The sequence \( \sigma(x) = (\sigma_H(x))_{H \in \mathcal{A}} \) is called the \emph{sign sequence} of \( x \). The set \( \mathcal{F} \) of all faces of \( \mathcal{A} \) is a partially ordered set with partial order given by \( x \leq y \) if \( x \subseteq \overline{y} \), where \( \overline{y} \) denotes the closure of the set \( y \). A \emph{chamber} of \( \mathcal{A} \) is a face that is maximal with respect to this order.

2.4. The intersection lattice. For each face \( x \in \mathcal{F} \), the \emph{support} \( \text{supp}(x) \) of \( x \) is the intersection of all hyperplanes in \( \mathcal{A} \) that contain \( x \). Equivalently, \( \text{supp}(x) \) is the subspace of \( V \) spanned by \( x \). The \emph{dimension} of \( x \) is the dimension of the subspace \( \text{supp}(x) \). The \emph{intersection lattice} \( \mathcal{L} \) of \( \mathcal{A} \) is the image of \( \text{supp} \); that is, \( \mathcal{L} = \text{supp}(\mathcal{F}) \). The elements of \( \mathcal{L} \) are subspaces of \( V \) and are ordered by inclusion. With this partial order, \( \mathcal{L} \) is a finite lattice, where the meet of two subspaces is their intersection, and the join of two subspaces is the smallest subspace that contains both. It follows that \( \text{supp} : \mathcal{F} \to \mathcal{L} \) is an order-preserving surjection of posets. (N.B. Some authors order \( \mathcal{L} \) by reverse inclusion rather than inclusion.)

2.5. The face semigroup algebra. Define the \emph{product of two faces} \( x, y \in \mathcal{F} \) to be the face \( xy \) with sign sequence \( (\sigma_H(xy))_{H \in \mathcal{A}} \) given by

\[
\sigma_H(xy) = \begin{cases} 
\sigma_H(x), & \text{if } \sigma_H(x) \neq 0, \\
\sigma_H(y), & \text{if } \sigma_H(x) = 0,
\end{cases}
\]
where $\sigma(x)$ and $\sigma(y)$ are the sign sequences of $x$ and $y$. This product has a geometric interpretation: the product $xy$ of two faces $x$ and $y$ is the face obtained by moving a small positive distance along a straight line from a point in $x$ to a point in $y$. It is straightforward to verify that this product gives $\mathcal{F}$ the structure of an associative semigroup with identity, and that $x^2 = x$ and $xyx = xy$ for all $x, y \in \mathcal{F}$. A semigroup satisfying these identities is called a left regular band.

The semigroup algebra $k\mathcal{F}$ is called the face semigroup algebra of $A$ over the field $k$. It consists of finite $k$-linear combinations of elements of $\mathcal{F}$ with multiplication induced by the product defined on elements of $\mathcal{F}$.

### 2.6. The invariant subalgebra.

Since $W$ is a group of orthogonal transformations of the vector space $V$, there is an action of $W$ on $V$ defined by setting $w(\vec{v})$ to be the image of $\vec{v} \in V$ under the transformation $w$. Under this action the set $\mathcal{A}$ is permuted [Humphreys, 1990 Proposition 1.2], so there is an induced action of $W$ on $\mathcal{L}$ and on $\mathcal{F}$. The action preserves the semigroup structure of $\mathcal{F}$, so it extends linearly to an action on $k\mathcal{F}$. Let $(k\mathcal{F})^W$ denote the subalgebra of $k\mathcal{F}$ consisting of the elements of $k\mathcal{F}$ fixed by all elements of $W$:

$$(k\mathcal{F})^W = \left\{ a \in k\mathcal{F} : w(a) = a \text{ for all } w \in W \right\}.$$

The following was first proved by T. P. Bidigare [Bidigare, 1997]. Another proof was given by K. S. Brown and can be found in [Brown, 2000, Theorem 7] or [Saliola, 2007, Theorem 2.7].

**Theorem 2.1.** Let $W$ be a finite reflection group and let $k\mathcal{F}$ denote the face semigroup algebra of the reflection arrangement of $W$. The $W$-invariant subalgebra $(k\mathcal{F})^W$ is anti-isomorphic to the descent algebra $\Sigma_k(W)$ of $W$.

We briefly describe an anti-isomorphism. The faces of the fundamental chamber $c$ are parametrized by the subsets of $S$; if $J \subseteq S$, then there is a unique face $\iota_J$ of the fundamental chamber $c$ that is fixed by all elements of $J$ [Brown, 1989 §4.5F]. Moreover, every face of $\mathcal{A}$ is in the $W$-orbit of a unique face of $c$ [Brown, 1989 §4.5F]. So if $O_J$ denotes the $W$-orbit of $\iota_J$, then the elements $x_J = \sum_{y \in O_J} y$, one for each $J \subseteq S$, form a basis of $(k\mathcal{F})^W$. The function $(k\mathcal{F})^W \rightarrow \Sigma_k(W)$ defined by mapping $x_J$ to $x_J$ is an anti-isomorphism.

### 3. Quivers and Path Algebras

Let $k$ be a field and $A$ a finite dimensional $k$-algebra.

#### 3.1. Complete system of primitive orthogonal idempotents.

An element $a \in A$ is an idempotent if $a^2 = a$. Two idempotents $e, f \in A$ are orthogonal if $ef = 0 = fe$. An idempotent $e \in A$ is primitive if it cannot be written as $e = f + g$ with $f$ and $g$ non-zero orthogonal idempotents of $A$. A complete system of primitive orthogonal idempotents of $A$ is a set $\{e_1, e_2, \ldots, e_n\}$ of primitive idempotents of $A$ that are pairwise orthogonal and that sum to $1_A$.

#### 3.2. The quiver of a split basic algebra.

The Jacobson radical $\text{rad} A$ of $A$ is the smallest ideal of $A$ such that $A/\text{rad} A$ is semisimple. If $A/\text{rad} A$ is a direct product of copies of $k$, then $A$ is a split basic $k$-algebra. Equivalently, $A$ is a split basic algebra if and only if all the simple $A$-modules are of dimension one.

The quiver $Q$ of a split basic finite dimensional $k$-algebra $A$ is the finite directed graph constructed as follows. Let $\{e_v : v \in I\}$ be a complete system of primitive
orthogonal idempotents of $A$, where $\mathcal{I}$ is some index set. The vertex set of $Q$ is the
index set $\mathcal{I}$, so there is one vertex $v$ in $Q$ for each idempotent $e_v$. If $x, y \in \mathcal{I}$, then
the number of arrows $x \to y$ is $\dim_k e_y (\text{rad}(A)/\text{rad}^2(A)) e_x$. This construction does
not depend on the complete system of primitive orthogonal idempotents, so $Q$ is
canonical defined.

3.3. The path algebra. The path algebra $kQ$ of a quiver $Q$ is the $k$-algebra with
basis the set of paths in $Q$ and with multiplication defined on paths by

$$(w_0 \to \cdots \to w_s) \cdot (v_0 \to \cdots \to v_r) = \begin{cases} (v_0 \to \cdots \to v_r \to w_1 \to \cdots \to w_s), & \text{if } w_0 = v_r, \\ 0, & \text{if } w_0 \neq v_r, \end{cases}$$

where $(w_0 \to \cdots \to w_s)$ and $(v_0 \to \cdots \to v_r)$ are paths in $Q$. If $F$ denotes the ideal of $kQ$
gerated by the arrows of $Q$, then an ideal $I \subseteq kQ$ is said to be admissible if $F^m \subseteq I \subseteq F^2$ for some $m \in \mathbb{N}$.

If $Q$ is the quiver of $A$, then there is a surjection $\varphi : kQ \to A$ defined by
mapping each vertex $x$ to the idempotent $e_x$ and by mapping the arrows from $x$ to
$y$ to elements in $e_y (\text{rad}(A)/\text{rad}^2(A)) e_x$ whose image in $e_y (\text{rad}(A)/\text{rad}^2(A)) e_x$ forms a basis
of the quotient space. Moreover, ker $\varphi$ is an admissible ideal of $kQ$.

3.4. Loewy length. The Loewy length $\text{LL}(A)$ of a finite dimensional $k$-algebra $A$ is the smallest $l \in \mathbb{N}$ such $(\text{rad}A)^l = 0$. The following observation is pertinent.

Lemma 3.1. Suppose $Q$ is a finite acyclic quiver. If $A \cong kQ/I$ for some quiver $Q$
and some admissible ideal $I$ of $kQ$, then $\text{LL}(A) \leq l + 1$, where $l$ is the length of the
longest path in $Q$.

Proof. Let $\varphi : kQ \to A$ denote a surjection with kernel $I$. If $F \subseteq kQ$ denotes
the ideal generated by the arrows of $Q$, then $\varphi(F^l) = (\text{rad}A)^l$ for all $l \geq 1$
[Assem et al., 2006 Corollary 2.11]. So if $l \in \mathbb{N}$ is the length of the longest path in
$Q$, then $F^{l+1} = 0$. Hence, $(\text{rad}A)^{l+1} = 0$. Thus, $\text{LL}(A) \leq l + 1$. \qed

4. A $W$-Equivariant Surjection

Notation. Throughout this section: $(W, S)$ is a Coxeter system with fundamental
domain $c$, $A$ is the reflection arrangement of $W$; $\mathcal{L}$ is the intersection lattice of $\mathcal{A}$;
and $k\mathcal{F}$ is the face semigroup algebra of $\mathcal{A}$, where $k$ is a field whose characteristic
does not divide the order of $W$.

In this section we recall the construction a complete system of primitive ortho-
ogonal idempotents for $(k\mathcal{F})^W$ and the construction of a $W$-equivariant surjection
$\varphi : kQ \to k\mathcal{F}$, where $Q$ is the quiver of $k\mathcal{F}$.

4.1. The orbit poset. For each $x \in \mathcal{F}$ let $\mathcal{O}_x = \{w(x) : w \in W\}$ denote the
$W$-orbit of $x$, and for each $X \in \mathcal{L}$ let $\mathcal{O}_X = \{w(X) : w \in W\}$ denote the $W$-orbit
of $X$. The $W$-orbits of elements of $\mathcal{L}$ form a poset $\mathcal{L}/W = \{\mathcal{O}_X : X \in \mathcal{L}\}$ with
partial order given by $\mathcal{O}_X \leq \mathcal{O}_Y$ if and only if there exists $w \in W$ with $w(X) \leq Y$.

Remark 4.1. The poset $\mathcal{L}/W$ is isomorphic to a poset of equivalence classes of
subsets of $S$. Indeed, define a relation on subsets $J, K \subseteq S$ by setting $J \sim K$ if and
only if supp$(c_J)$ and supp$(c_K)$ belong to the same orbit. Equivalently, $J \sim K$ if and
only if $W_J$ and $W_K$ are conjugate subgroups of $W$. The poset $S/\sim$, with partial
order induced by reverse inclusion of subsets of $S$, is isomorphic to $\mathcal{L}/W$. 

4.2. **Complete system of primitive orthogonal idempotents.** The construction requires, for each \( X \in \mathcal{L} \), a linear combination \( \ell(X) \) of faces of support \( X \) with coefficients summing to 1. Moreover, the elements \( \ell(X) \) need to satisfy \( w(\ell(X)) = \ell(w(X)) \) for all \( w \in W \).

We provide one example of such elements; see §3.4 of [Saliola, 2007] for other examples. For every orbit \( O \in \mathcal{L}/W \), fix a face \( f_O \) such that \( \text{supp}(f_O) \in O \). For each \( X \in \mathcal{L} \), let \( f_X = f_{O_X} \) and define

\[
\ell(X) = \frac{1}{\lambda_X} \sum_{z \in \mathcal{O}_{f_X} : \text{supp}(z) = X} z, \quad \text{where } \lambda_X = |\{z \in \mathcal{O}_{f_X} : \text{supp}(z) = X\}|.
\]

Note that \( \lambda_X \) is the index of the stabilizer subgroup \( W_z = \{w \in W : w(x) = x\} \) of \( x \) in the stabilizer subgroup \( W_X = \{w \in W : w(X) = X\} \) of \( X \), where \( x \) is any face with support \( X \). Hence, \( \lambda_X \) depends only on the orbit of \( X \) and so the elements \( \ell(X) \) satisfy \( w(\ell(X)) = \ell(w(X)) \) for all \( w \in W \).

Define elements \( e_X \in k\mathcal{F} \), one for each \( X \in \mathcal{L} \), recursively by the formula

\[
e_X = \ell(X) - \ell(X) \sum_{Y > X} e_Y.
\]

These elements form a complete system of primitive orthogonal idempotents for \( k\mathcal{F} \) [Saliola, 2006] Theorem 5.2]. Moreover, they satisfy \( w(e_X) = e_{w(X)} \) for all \( w \in W \) and all \( X \in \mathcal{L} \), so the elements

\[
\varepsilon_O = \sum_{X \in O} e_X,
\]

one for each \( O \in \mathcal{L}/W \), form a complete system of primitive orthogonal idempotents for \( (k\mathcal{F})^W \) [Saliola, 2007] Theorem 3.7].

**Remark 4.2.** The above leads to a construction of a complete system of primitive orthogonal idempotents directly within the descent algebra \( \Sigma_k(W) \). Let \( S/\sim \) denote the poset defined in Remark 4.1. For each \( O \in S/\sim \), fix a subset \( J_O \subseteq S \) with \( J_O \in \mathcal{O} \) and define elements \( \varepsilon_O \), one for each \( O \in S/\sim \), recursively by the formula

\[
\varepsilon_O = \frac{1}{\lambda_O} x_{J_O} - \sum_{O' > O} \varepsilon_{O'} \left( \frac{1}{\lambda_O} x_{J_O} \right),
\]

where \( \lambda_O \) is the index of \( W_J \) in the normalizer of \( W_J \). These elements correspond, under the anti-isomorphism of [22], to the elements defined in Equation 4.3 for a suitable choice of \( f_O \). Therefore, they form a complete system of primitive orthogonal idempotents for \( \Sigma_k(W) \). See [Saliola, 2007] Proposition 3.9 for details.

4.3. **The quiver of** \( k\mathcal{F} \). The quiver of \( k\mathcal{F} \) is the directed graph \( \mathcal{Q} \) constructed as follows. The vertex set of \( \mathcal{Q} \) is \( \mathcal{L} \), and there is exactly one arrow \( X \to Y \), for \( X, Y \in \mathcal{L} \), if and only if \( Y \prec X \). There exists a surjection \( \varphi : k\mathcal{Q} \to k\mathcal{F} \) with kernel generated by the sum of all the paths in \( \mathcal{Q} \) of length two. A proof of this for any central hyperplane arrangement can be found in [Saliola, 2006]. Below we recall the construction of a \( W \)-equivariant \( \varphi : k\mathcal{Q} \to k\mathcal{F} \). See [Saliola, 2007] §4 for details.

Define an action of \( W \) on the path algebra \( k\mathcal{Q} \) as follows. Fix an orientation \( \epsilon_X \) on each subspace \( X \in \mathcal{L} \). Thus, \( \epsilon_X \) is a map that assigns 1 or \(-1\) to a basis of \( X \) depending on whether the basis is positively or negatively oriented. For \( w \in W \) and
Lemma 5.1. If for every path $O \in L$ ends at a vertex in the structure of $Q_k$, $O$ is no arrow from $\nu$ if there is an arrow $\nu$.

Proof. (see Theorem 4.3). Therefore, $\nu$ is generated by the sum of all the paths of length two in $Q$.

We briefly recall that construction. Let $\epsilon_X$ denote the orientations chosen above. Define $\varphi$ on each vertex $X$ and arrow $X \to Y$ of $Q$ by

$$\varphi(X) = \epsilon_X \quad \text{and} \quad \varphi(X \to Y) = \lambda_Y \epsilon_Y \left( \sum_{x \in X} [y : x] \right) \epsilon_X,$$

where $\lambda_Y$ is defined in Equation (4.1), where $y$ is any face of support $Y$ and

$$[y : x] = \epsilon_{\text{supp}(y)}(\bar{y}_1, \ldots, \bar{y}_t) \epsilon_{\text{supp}(x)}(\bar{y}_1, \ldots, \bar{y}_t, \bar{x}_1),$$

where $\bar{y}_1, \ldots, \bar{y}_t$ is a basis of $\text{supp}(y)$ and $\bar{x}_1$ is a vector in $x$. Then $\varphi$ extends linearly and multiplicatively to a $W$-equivariant $k$-algebra surjection $\varphi : kQ \to kF$.

5. On the Quiver of $(k\mathcal{F})^W$

We continue with the notation of the previous section and let $Q_W$ denote the quiver of $(k\mathcal{F})^W$. The quiver $Q_W$ is not known for arbitrary $W$, but the idempotents of Equation (4.3) and the surjection of Theorem 4.3 provide some information about the structure of $Q_W$. In the next section we specialize to $W$ of type $D$.

The following result is our main tool.

Lemma 5.1. If for every path $P$ in $Q$ that begins at a vertex in $O' \in L/W$ and ends at a vertex in $O \in L/W$ there exists $w \in W$ such that $w(P) = -P$, then there is no arrow from $O'$ to $O$ in $Q_W$.

Proof. If there is an arrow $O' \to O$, then the vector space $\epsilon_O(k\mathcal{F})^W \epsilon_{O'}$ is nonzero (see [3,2]). We'll show that this vector space is zero if the hypothesis holds.

For each $O \in L/W$, let $\nu_O = \sum_{X \in O} X \in kQ$. Let $\varphi : kQ \to k\mathcal{F}$ denote the $W$-equivariant surjection of Theorem 4.3. Then $\varphi$ restricts to a surjection $\nu_O(kQ)^W \nu_{O'} \to \epsilon_O(k\mathcal{F})^W \epsilon_{O'}$.

We'll show that $\nu_O(kQ)^W \nu_{O'} = 0$. This subspace is spanned by elements of the form $\sum_{P' \in O_P} P'$, where $P$ is a path of $Q$ that begins at a vertex in $O'$ and ends at a vertex in $O$, and where $O_P$ is the $W$-orbit of $P$. The hypothesis implies $w(P) = -P$ for some $w \in W$, so

$$\sum_{P' \in O_P} P' = \sum_{P' \in O_{-P}} w(P') = \sum_{P' \in O_{-P}} P' = -\sum_{P' \in O_P} P'.$$

Therefore, $\sum_{P' \in O_P} P' = 0$. So $\nu_O(kQ)^W \nu_{O'} = 0$. \hfill $\Box$
Our first result on the structure of $Q_W$ shows that it contains no oriented cycles.

**Proposition 5.2.** There is exactly one vertex in $Q_W$ for each element of $L/W$. If $O' \rightarrow O$ is an arrow in $Q_W$, then $O \leq O'$ in $L/W$. In particular, $Q_W$ does not contain any oriented cycles.

**Proof.** Since the elements in Equation [3.3] form a complete system of primitive orthogonal idempotents for $(kF)^W$, the vertex set of $Q_W$ is the poset $L/W$.

If $(X_0 \rightarrow \cdots \rightarrow X_l)$ is a path in $Q$, then $X_l \leq X_0$. In particular, $O_X \leq O_{X_0}$. So if $O \not\leq O'$, then the condition of Lemma 5.1 is satisfied. Therefore, there is no arrow from $O'$ to $O$ in $Q_W$. It follows that $Q_W$ cannot contain an oriented cycle. 

Our next result shows that the quiver $Q_W$ contains at least one isolated vertex.

**Proposition 5.3.** There are no arrows in $Q_W$ beginning at $\{V\}$.

**Proof.** Let $(X_0 \rightarrow \cdots \rightarrow X_l)$ be a path in $Q$ with $X_0 = V$. Let $w \in W$ denote the reflection in the hyperplane $X_l$. Then

$$w (X_0 \rightarrow \cdots \rightarrow X_l) = \sigma_X(w)\sigma_X(l)(w(X_0) \rightarrow \cdots \rightarrow w(X_l)) = -(X_0 \rightarrow \cdots \rightarrow X_l).$$

By Lemma 5.1, there is no arrow in $Q_W$ beginning at $\{V\}$. 

6. **The Loewy Length of $\Sigma_k(D_{2m+1})$**

**Notation.** Throughout this section: $D_n$ is a Coxeter group of type $D$, $A$ is the reflection arrangement of $D_n$; $\mathcal{L}$ is the intersection lattice of $A$; and $kF$ is the face semigroup algebra of $A$, where $k$ is a field whose characteristic does not divide the order of $D_n$.

6.1. **Coxeter groups of type $D$.** For $n \in \mathbb{N}$, let $[n] = \{1, 2, \ldots, n\}$ and let $\pm [n] = \{1, 2, \ldots, n\} \cup \{-1, -2, \ldots, -n\}$. A **signed permutation** of $\pm [n]$ is a permutation $w$ of $\pm [n]$ satisfying $w(-i) = -w(i)$ for all $i \in [n]$. A signed permutation $w$ of $\pm [n]$ acts on $\mathbb{R}^n$ by permuting and negating coordinates:

$$w(v_1, v_2, \ldots, v_n) = (v_{w^{-1}(1)}, v_{w^{-1}(2)}, \ldots, v_{w^{-1}(n)}),$$

where $v_{-i} = -v_i$ for $i \in [n]$.

The Coxeter group $D_n$ is the subgroup of the group of signed permutations of $\pm [n]$ that negate an even number of elements of $[n]$. The reflection arrangement $A$ of $D_n$ consists of the hyperplanes

$$H_{ij} = \{\bar{v} \in \mathbb{R}^n : v_i = v_j\},$$

where $i \neq j$, $-i \in [\pm n]$.

6.2. **Intersection lattice.** A set partition of $\pm [n]$ is a collection of nonempty subsets $B = \{B_1, \ldots, B_r\}$ of $\pm [n]$ such that $\bigcup B_i = [\pm n]$ and $B_i \cap B_j = \emptyset$ for $i \neq j$. The sets $B_i$ in $B$ are called the **blocks** of $B$. The collection of set partitions of a finite set form a finite lattice, with partial order given by: $B \leq C$ if and only if every block of $C$ is contained in a block of $B$. If $A \subseteq [\pm n]$, then $\overline{A} = \{-a : a \in A\}$.

The intersection lattice $\mathcal{L}$ of $A$ is isomorphic to the sublattice of set partitions of $\pm [n]$ of the form $\{B_1, \ldots, B_r, \overline{C}, \overline{B_1}, \ldots, \overline{B_1}\}$, where $C$ satisfies: $C = \emptyset$ or $|C| \geq 4$; and $C = \overline{C}$ [Barcelo and Thrig, 1999, Theorem 4.1]. The isomorphism is given by

$$P = \{P_1, \ldots, P_r\} \mapsto \{\bar{v} \in V : v_i = v_j \text{ if } i, j \in P_h \text{ for some } h \in [r]\} = \bigcap_{h=1}^{r} \bigcap_{i, j \in P_h} H_{ij},$$
where \( P \) is a set partition of \([±n]\) and \( v_{-i} = -v_i \) for \( i \in \mathbb{N} \).

To simplify notation, we let \( \pi(X) \) denote the set partition of \([±n]\) induced by \( X \in \mathcal{L} \), and we let \( \{B_1, \ldots, B_r; C\} \) denote the set partition \( \{B_1, \ldots, B_r, C, \overline{B}_1, \ldots, \overline{B}_r\} \). The block \( C \) is called the central block. Under this isomorphism the action of \( D_n \) on \( X \in \mathcal{L} \) is given by permuting the elements of \( \pi(X) \). That is, \( \pi(w(X)) = w(\pi(X)) \) for all \( w \in D_n \) and \( X \in \mathcal{L} \).

6.3. **Canonical basis.** The set partition \( \pi(X) = \{B_1, \ldots, B_r; C\} \) describes a basis of \( X \). For each \( i \in [r] \), let

\[
\beta_i = \sum_{j \in B_i} \tilde{e}_j,
\]

where \( \tilde{e}_1, \ldots, \tilde{e}_n \) is the standard basis of \( \mathbb{R}^n \) and \( \tilde{e}_{-j} = -\tilde{e}_j \) for \( j \in [n] \). The vectors \( \beta_1, \ldots, \beta_r \) form a basis of the subspace \( X \) called the canonical basis of \( X \).

6.4. **The length of the longest path in \( Q_{D_{2m}} \).** This serves as a quick example to illustrate the approach we take in the following section.

The Coxeter group \( D_{2m} \) contains an element \( w_0 \) that acts on \( V \) by central reflection. That is, \( w_0(\vec{v}) = -\vec{v} \) for all \( \vec{v} \in V \). Therefore, \( \sigma_X(w_0) = (-1)^{\dim(X)} \). So if \( A \) is an arrow in \( Q \), then \( w_0(A) = -A \). It follows from Lemma 5.1 that there is no arrow \( \mathcal{O} \to \mathcal{O}' \) in \( Q_{D_{2m}} \), if \( \mathcal{O} < \mathcal{O}' \). Combined with Proposition 5.2, this implies that the length of the longest path in \( Q_{D_{2m}} \) is at most \( 2m-1 \), since \( 2m \) is the length of the longest path in \( Q \). This establishes the following.

**Proposition 6.1.** The length of the longest path in \( Q_{D_{2m}} \) is at most \( m - 1 \).

This implies that the Loewy length of the descent algebra \( \Sigma_k(D_{2m}) \), for \( m \geq 2 \), is at most \( m \) (see the proof Theorem 6.5). Also, the same argument also gives an upper bound of \( \left\lceil \frac{m}{2} \right\rceil \) for the Loewy length of the descent algebra \( \Sigma_k(B_n) \). These are both equalities [Bonnafé and Pfeiffer, 2006, §5E].

6.5. **The Loewy length of \( \Sigma_k(D_{2m+1}) \).** In this section we develop necessary conditions on \( O_X \) and \( O_Y \) for there to be an arrow from \( O_X \) to \( O_Y \) in \( Q_{D_{2m+1}} \).

For \( X \in \mathcal{L} \), let \( \pi(X) = \{B_1, \ldots, B_r; C\} \) denote the set partition induced by \( X \) (see §6.2). Let \( \text{Even}(X) \) denote the number of \( i \in [r] \) with \( |B_i| \) even, and let \( \text{Odd}(X) \) denote the number of \( j \in [r] \) with \( |B_j| \) odd.

**Lemma 6.2.** If there is an arrow \( \mathcal{O}' \to \mathcal{O} \) in \( Q_{D_{2m+1}} \), then \( \text{Even}(Y) \leq \text{Even}(X) \) for all \( X \in \mathcal{O}' \) and \( Y \in \mathcal{O} \).

**Proof.** We prove that if \( \text{Even}(X) < \text{Even}(Y) \), then there is no arrow \( O_X \to O_Y \).

Suppose \( P \) is a path in \( Q \) beginning at a vertex \( X' \) in \( O_X \) and ending at a vertex \( Y' \) in \( O_Y \). Since \( \pi(X') \) and \( \pi(X) \) are in the same orbit, \( \text{Even}(X') = \text{Even}(X) \). Similarly, \( \text{Even}(Y') = \text{Even}(Y) \). So \( \text{Even}(X') < \text{Even}(Y') \).

If every even-sized non-central block \( B_i \) in \( \pi(X') = \{B_1, \ldots, B_r; C\} \) contains an even-sized non-central block of \( \pi(X') \), then \( \text{Even}(X') \geq \text{Even}(Y') \), contrary to our assumption. Therefore, for some \( i \in [r] \) the block \( B_i \in \pi(X') \) is even-sized and is a union of an even number of odd-sized blocks of \( \pi(X') \).

Let \( w \) be the signed permutation that negates all elements of \( B_i \). Then \( w \in D_{2m+1} \) since \( |B_i| \) is even and \( w(Z) = Z \) for all vertices of \( P \). Since \( w \) negates an even number of blocks of \( \pi(X') \) and fixes the others, \( w \) negates an even number of vectors (and fixes the others) in a canonical basis of \( X' \). Hence \( \sigma_{X'}(w) = 1 \). Similarly, \( w \)
negates exactly 1 block of $\pi(Y)$ while fixing the others, so $\sigma_Y(w) = -1$. Hence, $w(P) = -P$. By Lemma 5.1, there is no arrow $\mathcal{O}_X \rightarrow \mathcal{O}_Y$ in $\mathcal{Q}_{2m+1}$.

**Lemma 6.3.** If $X' \rightarrow Y'$ is an arrow in $\mathcal{Q}$ and if $\text{Odd}(X') \neq 1$, then there is no arrow $\mathcal{O}_{X'} \rightarrow \mathcal{O}_{Y'}$ in $\mathcal{Q}_{2m+1}$.

**Proof.** Suppose $X \rightarrow Y$ is an arrow in $\mathcal{Q}$ with $X \in \mathcal{O}_{X'}$ and $Y \in \mathcal{O}_{Y'}$. Since $\pi(X')$ is in the orbit of $\pi(X)$, $\text{Odd}(X) = \text{Odd}(X') \neq 1$. Since $Y \prec X$, $\pi(Y')$ is obtained from $\pi(X) = \{B_1, \ldots, B_r; C\}$ by merging two blocks. Let $\beta_1, \ldots, \beta_r$ be the canonical basis for $X$ (see [6.3]).

**Case 1.** $\pi(Y)$ is obtained from $\pi(X)$ by merging $B_i$ and $B_j$, where $i \neq j$.

If $|B_i \cup B_j|$ is even, then let $w$ be the signed permutation that negates the elements of $B_i$ and $B_j$. Then $w$ negates two elements of the canonical basis of $X$ and fixes the other basis elements, so $w(X) = X$ and $\sigma_X(w) = 1$. Since $\{\beta_i + \beta_j\} \cup \{\beta_h : h \neq i, j\}$ is a basis of $Y$, and since $w$ negates $\beta_i + \beta_j$ and fixes the others, it follows that $w(Y) = Y$ and $\sigma_Y(w) = 1$. Therefore, $w(X \rightarrow Y) = -(X \rightarrow Y)$.

If $|B_i \cup B_j|$ is odd, then let $w$ be the signed permutation that negates the elements of $B_i$, $B_j$ and the elements of $B_h$, where $h \neq i, j$ and $|B_h|$ is odd (such a block exists since $\text{Odd}(X) \neq 1$). Then $w$ negates three elements of the canonical basis of $X$ and two elements of the basis $\{\beta_i + \beta_j\} \cup \{\beta_a : a \neq i, j\}$ of $Y$. Therefore, $\sigma_X(w) = 1$ and $\sigma_Y(w) = 1$. It follows that $w(X \rightarrow Y) = -(X \rightarrow Y)$.

**Case 2.** $\pi(Y)$ is obtained from $\pi(X)$ by merging $B_i$ and $\overline{B}_j$, where $i \neq j$.

This is argued as is Case 1.

**Case 3.** $\pi(Y)$ is obtained from $\pi(X)$ by merging the blocks $B_i, \overline{B}_i$ and $C$.

If $|B_i|$ is even, then let $w$ be the signed permutation that negates the elements of $B_i$. Then $w$ negates one element of the canonical basis of $X$ and no elements of the basis $\{\beta_a : a \neq i\}$ of $Y$. Thus, $w(X \rightarrow Y) = -(X \rightarrow Y)$.

If $|B_i|$ is odd, then let $w$ be the signed permutation that negates the elements of $B_i$ and the elements of $B_h$, where $h \neq i$ and $|B_h|$ is odd. Then $w$ negates two elements of the canonical basis of $X$ and one element of the basis $\{\beta_a : a \neq i\}$ of $Y$. It follows that $w(X \rightarrow Y) = -(X \rightarrow Y)$.

It follows from Lemma 5.1 that there is no arrow $\mathcal{O}_{X'} \rightarrow \mathcal{O}_{Y'}$ in $\mathcal{Q}_{2m+1}$.

**Proposition 6.4.** The length of the longest path in $\mathcal{Q}_{2m+1}$ is at most $m + 1$.

**Proof.** Suppose $(\mathcal{O}_0 \rightarrow \mathcal{O}_1 \rightarrow \cdots \rightarrow \mathcal{O}_l)$ is a path in $\mathcal{Q}_{2m+1}$. Then for $0 \leq i \leq l$, there is $X_i \in \mathcal{O}_i$ such that $X_0 \leq \cdots \leq X_i \leq X_l$. Note that $X_0 \neq V$ by Proposition 5.3.

For each $j \in [l]$, let $d_j = \dim(X_{j-1}) - \dim(X_j)$. If $d_i \geq 2$ for all $i \in [l]$, then

$$2l \leq \sum_{i=1}^{l} d_i = \dim(X_0) - \dim(X_l) \leq \dim(X_0) \leq 2m,$$

so $l \leq m$. Suppose instead that $d_j = 1$ for some $j \in [l]$, and let $i$ be the smallest such $j$. By the choice of $i$, $X_{i-1} \rightarrow X_i$ is an arrow in $\mathcal{Q}$ with $X_{i-1} \in \mathcal{O}_{i-1}$ and $X_i \in \mathcal{O}_i$. Then $\text{Odd}(X_{i-1}) = 1$ by Lemma 5.3 and $\text{Even}(X_{i-1}) \leq \text{Even}(X_0)$ by Lemma 6.2.

Recall that for each $X \in \mathcal{L}$, if $\pi(X) = \{B_1, \ldots, B_r; C\}$, then $\dim(X) = r$ (6.3). In particular, $\dim(X) = \text{Even}(X) + \text{Odd}(X)$ and $\dim(X) \leq (2m + 1) - \text{Even}(X)$. Therefore, since $\text{Even}(X_{i-1}) \leq \text{Even}(X_0)$,

$$\dim(X_0) \leq (2m + 1) - \text{Even}(X_0) \leq (2m + 1) - \text{Even}(X_{i-1}).$$
By the choice of $i$, $d_j \geq 2$ for all $j \in [i-1]$, so
\[
2(i-1) \leq \dim(X_0) - \dim(X_{i-1})
\leq \left( (2m + 1) - \text{Even}(X_{i-1}) \right) - \left( \text{Even}(X_{i-1}) + \text{Odd}(X_{i-1}) \right)
\leq 2(m - \text{Even}(X_{i-1})).
\]
Since the length of $(O_{i-1} \rightarrow \cdots \rightarrow O_l)$ is bounded by $\dim(X_{i-1})$,
\[
l = (l - (i-1)) + (i-1)
\leq \dim(X_{i-1}) + (m - \text{Even}(X_{i-1}))
\leq (\text{Even}(X_{i-1}) + \text{Odd}(X_{i-1})) + (m - \text{Even}(X_{i-1}))
\leq m + 1.
\]

**Theorem 6.5.** For all $m \geq 2$, the Loewy length of $\Sigma_k(D_{2m+1})$ is $m + 2$.

**Proof.** By Theorem 2.1, the Loewy length of $\Sigma_k(D_{2m+1})$ is the Loewy length of $(kF)^{D_{2m+1}}$. By Lemma 3.1, and the previous Proposition, the Loewy length of $(kF)^{D_{2m+1}}$ is bounded by $m+2$. By Corollary 5.9(b) of [Bonnafé and Pfeiffer, 2006], this is also a lower bound. □

**References**

[Assem et al., 2006] Assem, I., Skowronski, A., and Simson, D. (2006). *Elements of the Representation Theory of Associative Algebras: Techniques of Representation Theory*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge.

[Atkinson, 1992] Atkinson, M. D. (1992). Solomon’s descent algebra revisited. *Bull. London Math. Soc.*, 24(6):545–551.

[Barcelo and Ihrig, 1999] Barcelo, H. and Ihrig, E. (1999). Lattices of parabolic subgroups in connection with hyperplane arrangements. *J. Algebraic Combin.*, 9(1):5–24.

[Bidigare, 1997] Bidigare, T. P. (1997). *Hyperplane Arrangement Face Algebras and Their Associated Markov Chains*. PhD thesis, University of Michigan.

[Bonnafé and Pfeiffer, 2006] Bonnafé, C. and Pfeiffer, G. (2006). Around Solomon’s descent algebra. arXiv:math/0601317v1 [math.RT].

[Brown, 1989] Brown, K. S. (1989). *Buildings*. Springer-Verlag, New York.

[Brown, 2000] Brown, K. S. (2000). Semigroups, rings, and Markov chains. *J. Theoret. Probab.*, 13(3):871–938.

[Garsia and Reutenauer, 1989] Garsia, A. M. and Reutenauer, C. (1989). A decomposition of Solomon’s descent algebra. *Adv. Math.*, 77(2):189–262.

[Humphreys, 1990] Humphreys, J. E. (1990). *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge.

[Saliola, 2006] Saliola, F. V. (2006). The face semigroup algebra of a hyperplane arrangement. arXiv:math/0511717v2 [math.RA].

[Saliola, 2007] Saliola, F. V. (2007). On the quiver of the descent algebra. arXiv:0708.4213v1 [math.RT].

[Solomon, 1976] Solomon, L. (1976). A Mackey formula in the group ring of a Coxeter group. *J. Algebra*, 41(2):255–264.

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