Existence of Whittaker models
related to
four dimensional symplectic Galois representations

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Let \( \mathbb{A} = \mathbb{A}_{\text{fin}} \otimes \mathbb{R} \) be the ring of rational adeles and \( GSp(4) \) be the group of symplectic similitudes in four variables. Suppose \( \Pi \cong \Pi_{\text{fin}} \otimes \Pi_\infty \) is a cuspidal irreducible automorphic representation of the group \( GSp(4, \mathbb{A}) \), where \( \Pi_\infty \) belongs to a discrete series representation of the group \( GSp(4, \mathbb{R}) \). The discrete series representations of the group \( GSp(4, \mathbb{R}) \) are grouped into local \( L \)-packets \( [W1][W2] \), which have cardinality two and consist of the class of a holomorphic and the class of a nonholomorphic discrete series representation. Two irreducible automorphic representations \( \Pi = \otimes_v \Pi_v \) and \( \Pi' = \otimes_v \Pi'_v \) are said to be weakly equivalent, if \( \Pi_v \cong \Pi'_v \) holds for almost all places \( v \).

The aim of this article is to prove the following

**Theorem 1.** Let \( \Pi \) be a cuspidal irreducible automorphic representation of the group \( GSp(4, \mathbb{A}) \). Suppose \( \Pi \) is not CAP and suppose \( \Pi_\infty \) belongs to the discrete series of the group \( GSp(4, \mathbb{R}) \). Then \( \Pi \) is weakly equivalent to an irreducible globally generic cuspidal automorphic representation \( \Pi_{\text{gen}} \) of the group \( GSp(4, \mathbb{A}) \), whose archimedean component \( \Pi_{\text{gen}, \infty} \) is the nonholomorphic discrete series representation contained in the local archimedean \( L \)-packet of \( \Pi_\infty \).

For an automorphic representation \( \Pi \) as in theorem 1 by [W2] there exists an associated Galois representation \( \rho_{\Pi, \lambda} \). From [W2] and theorem 1 we obtain, that these Galois representations \( \rho_{\Pi, \lambda} \) are symplectic in the following sense

**Theorem 2.** Suppose \( \Pi \) is as in theorem 1. Then the associated Galois representation \( \rho_{\Pi, \lambda} \) preserves a nondegenerate symplectic \( \mathbb{Q}_l \)-bilinear form \( \langle \cdot, \cdot \rangle \), such that the Galois group acts with the multiplier \( \omega_{\Pi, \mu_l}^{-w} \)

\[
\langle \rho_{\Pi, \lambda}(g)v, \rho_{\Pi, \lambda}(g)w \rangle = \omega_{\Pi}(g)\mu_l^{-w}(g) \cdot \langle v, w \rangle, \quad g \in \text{Gal}(\mathbb{Q}/\mathbb{Q})
\]

where \( \mu_l \) is the cyclotomic character.
A variant of this construction also yields certain orthogonal four dimensional Galois representations. See the remark at the end of this article.

For an irreducible cuspidal automorphic representation $\Pi$ of $GSp(4, \mathbb{A})$, which is not CAP and whose archimedean component belongs to the discrete series, we want to show that $\Pi$ is weakly equivalent to a globally generic representation $\Pi_{gen}$, whose archimedean component again belongs to the discrete series. $\Pi$ not being CAP implies, that $\Pi_{gen}$ again is cuspidal. Hence [W2], theorem III now also holds unconditionally, since the multiplicity one theorem is known for the generic representation $\Pi_{gen}$.

A careful analysis of the proof shows, that the arguments imply more. For this we refer to the forthcoming work of U.Weselmann [Wes].

Proof of theorem 1: The proof will be based on the hypotheses A,B of [W2] proved in [W1], and theorem 3 and theorem 4, which will be formulated further below during the proof of theorem 1. Theorem 3 is a consequence of results of [GRS]. Theorem 4 is proved in [Wes]. For the following it is important, that under the assumptions made in theorem 1 Ramanujan’s conjecture holds for the representation $\Pi$ at almost all places, as explained in [W2] section 1.

Restriction to $Sp(4)$. The restriction of $\Pi$ to $Sp(4, \mathbb{A})$ contains an irreducible constituent, say $\tilde{\Pi}$. In the notation of [W2], section 3 consider the degree five standard $L$-series

$$\zeta^S(\Pi, \chi, s)$$

of $\tilde{\Pi}$ for $Sp(4)$. For our purposes it suffices to consider this $L$-series for the primes outside a sufficiently large finite set of place $S$ containing all ramified places. So this partial $L$-series depends only on $\Pi$, and does not depend on the chosen $\tilde{\Pi}$.

Euler characteristics. $\Pi$ is cohomological in the sense of [W2],[W1], i.e. $\Pi$ occurs in the cohomology of the Shimura variety $M$ of principally polarized abelian varieties of genus two for a suitable chosen $\mathbb{Q}_l$-adic coefficient system $V_\mu(\mathbb{Q}_l)$. Since $\Pi$ is of cohomological type and since we excluded CAP-representations $\Pi$, the representations $\Pi_{fin}$ and also $\Pi^S$ (i.e. $\Pi^S = \otimes_{\nu \in S} \Pi_{\nu}$, outside a finite set $S$ of bad primes) only contribute to the cohomology $H^i(M, V_\mu(\mathbb{Q}_l))$ for the coefficient system $V_\mu$ [W2] in the middle degree $i = 3$ and not for the other degrees (see [W2] hypothesis B(1),(2) and [W1]). This cohomological property is inherited to the subgroup $Sp(4, \mathbb{A})$ by restriction and induction using the following easy observation: Given two irreducible automorphic representations $\Pi_1, \Pi_2$ of $GSp(4, \mathbb{A})$.
having a common irreducible constituent after restriction to $Sp(4, \mathbb{A})$. Then, if $\Pi_1$ is cuspidal but not CAP, then also $\Pi_2$ is cuspidal and not CAP.

Therefore, if we consider the generalized $\Pi^S$-isotypic subspaces for either of the groups $GSp(4, \mathbb{A}^S)$ or $Sp(4, \mathbb{A}^S)$ in the middle cohomology group of degree 3, we may as well replace the middle cohomology group by the virtual representation $H^*(M, \mathcal{V}_\mu(\mathbb{Q}_i))$. Up to a minus sign this does not change the traces of Hecke operators , which are later considered in theorem 4. Here $S$ may be any finite set containing the archimedean place.

*First temporary assumption.* For the moment suppose $\tilde{\Pi}$ admits a weak lift $\tilde{\Pi}'$ to an irreducible automorphic representation of the group $PGl(5, \mathbb{A})$ in the sense below (we later show using theorem 4 that this in fact always holds). A representation $\tilde{\Pi}'$ of $PGl(5, \mathbb{A})$ can be considered to be a representation of $Gl(5, \mathbb{A})$ with trivial central character. In this sense the lifting property just means, that there exists an irreducible automorphic representation $\tilde{\Pi}'$ of $PGl(5, \mathbb{A})$, for which

$$L^S(\tilde{\Pi}' \otimes \chi, s) = \zeta^S(\Pi, \chi, s)$$

holds for the standard $L$-series of $Gl(5)$ and all idele class characters $\chi$ and certain sufficiently large finite sets $S = S(\chi, \tilde{\Pi}')$ of exceptional places.

Considered as an irreducible automorphic representation of $Gl(5, \mathbb{A})$ the representation $\tilde{\Pi}'$ need not be cuspidal (by the way this is essentially used in the additional remark on orthogonal representations made at the end after the proof of theorem 1). There exist irreducible cuspidal automorphic representations $\sigma_i$ of groups $Gl(n_i, \mathbb{A})$ where $\sum_i n_i = 5$, such that $\tilde{\Pi}'$ is a constituent of the representation induced from a parabolic subgroup with Levi subgroup $\prod_i Gl(n_i, \mathbb{A})$. See [L], prop.2. Each of the cuspidal representations $\sigma_i$ can be written in the form $\sigma_i = \chi_i \otimes \sigma_i^0$ for some unitary cuspidal representations $\sigma_i^0$ and certain one dimensional characters $\chi_i$.

Let $\omega_{\sigma_i}$ denote the central character of $\sigma_i$. The identity $L^S(\tilde{\Pi}', s) = \zeta^S(\Pi, 1, s)$ implies, that the characters $\omega_{\sigma_i} = \chi_i^{n_i} \omega_{\sigma_i}$ are unitary. In fact, since $\Pi^S$ satisfies the Ramanujan conjecture by [W2], they have absolute value one at all places outside $S$. Therefore by the approximation theorem all $\chi_i$ are unitary. Hence the $\sigma_i$ itself have been cuspidal unitary representations.
Now, since the $\sigma_i$ are cuspidal unitary, the well known theorems of Jaquet-Shalika and Shahidi on $L$-series for the general linear group [JS] imply the non-vanishing $L^S(\tilde{\Pi}' \otimes \chi, 1) = \prod_i L^S(\sigma_i \otimes \chi, 1) \neq 0$ for arbitrary unitary idele class characters $\chi$. By the (temporary) assumption, that $\tilde{\Pi}'$ is a lift of $\tilde{\Pi}$, this implies
\[
\zeta^S(\Pi, \chi, 1) \neq 0
\]
for all unitary idele class characters $\chi$.

Second temporary assumption. Now in addition we suppose, that $\Pi$ can be weakly lifted to an irreducible automorphic representation $(\Pi', \omega)$ of the group $Gl(4, A) \times A^*$. By this we mean, that there exists an irreducible automorphic representation $\Pi'$ of $Gl(4, A)$ and an idele class character $\omega$, such that for the central characters of $\Pi$ and $\Pi'$ we have
\[
\omega_{\Pi'} = \omega^2 \quad \text{and} \quad \omega_{\Pi} = \omega,
\]
and that furthermore
\[
L^S(\Pi \otimes \chi, s) = L^S(\Pi' \otimes \chi, s)
\]
holds for sufficiently large finite sets of places $S$ containing all ramified places. Here, following the notation of [W2] section three, $L^S(\Pi, s)$ denotes the standard degree four $L$-series of $\Pi$.

These conditions imposed at almost all unramified places of course completely determines the automorphic representation $\Pi'$ by the strong multiplicity one theorem for $Gl(n)$. In particular this implies, that the global lift $\Pi \mapsto (\Pi', \omega)$ commutes with character twists $\Pi \otimes \chi \mapsto (\Pi' \otimes \chi, \omega \chi^2)$. Furthermore it implies $(\Pi')^\vee \cong \Pi' \otimes \omega^{-1}$. In particular this finally holds locally at all places including the archimedean place.

The archimedean place. From the last observation we obtain $\Pi'_\infty \otimes sign \cong \Pi'_\infty$ from the corresponding $\Pi_\infty \otimes sign \cong \Pi_\infty$, which is known for discrete series representations $\Pi_\infty$ of $GSp(4, \mathbb{R})$. Hence it is easy to see, that $\Pi'_\infty$, respectively $\Pi_\infty$ are determined by their central characters and their restriction to $Sl(4, \mathbb{R})$ respectively $Sp(4, \mathbb{R})$. In fact we will later show, that the lift $\Pi'$ can be assumed to have a certain explicitly prescribed behavior at the archimedean place. What this means will become clear later.

Properties of $\Pi'$. Again the irreducible automorphic representation $\Pi'$ of $Gl(4, A)$ need not be cuspidal. $\Pi'$ is a constituent of a representation induced from some
parabolic subgroup with Levi subgroup $\prod_j \text{Gl}(m_j)$ for $\sum_j m_j = 4$, with respect to some irreducible cuspidal representations $\tau_j$ of $\text{Gl}(m_j, \mathbb{A})$.

The same argument, as already used for the first temporary assumption, implies the $\tau_j$ to be unitary cuspidal. This excludes $m_j = 1$ for some $j$, since otherwise this would force the existence of a pole of $L^S(\Pi \otimes \chi, s)$ for $\chi = \tau_j^{-1}$ at $s = 1$ again by the results [JS] of Jaquet-Shalika and Shahidi on the analytic behavior of $L$-series for $\text{Gl}(n)$ on the line $\text{Re}(s) = 1$. Notice, a pole at $s = 1$ would imply $\Pi$ to be a CAP representation (see [P]). This however contradicts the assumptions of theorem 1, by which $\Pi$ is not a CAP representation.

Let us return to $\Pi$. Either $\Pi'$ is cuspidal; or $\Pi'$ comes by induction from a pair $(\tau_1, \tau_2)$ of irreducible cuspidal representations $\tau_i$ of $\text{Gl}(2, \mathbb{A})$, for which $\omega_{\tau_1} \omega_{\tau_2} = \omega^2$ holds. In this second case we do have

$$L^S(\Pi', s) = L^S(\tau_1, s)L^S(\tau_2, s).$$

By $L^S(\Pi, s) = L^S(\Pi', s)$ therefore $\Pi$ is a weak endoscopic lift, provided the central characters $\omega_{\tau_1} = \omega_{\tau_2}$ coincide. For weak endoscopic lifts theorem 1 obviously holds. See [W2], hypothesis A part (2) and (6) combined with [W1]. To complete the discussion of this case we establish the required identity $\omega_{\tau_1} = \omega_{\tau_2}$. For this we need some further argument and therefore we make a digression on theta lifts first.

The theta lift. $\text{Gl}(1)$ acts on $\text{Gl}(4) \times \text{Gl}(1)$ such that $t$ maps $(h, x)$ to $(ht^{-1}, xt^2)$. By our temporary assumptions made, the central character of $\Pi'$ is completely determined so that the representation $(\Pi', \omega)$ of $\text{Gl}(4) \times \text{Gl}(1)$ descends to a representation on the quotient group $G(\mathbb{A}) = (\text{Gl}(4, \mathbb{A}) \times \text{Gl}(1, \mathbb{A}))/\text{Gl}(1, \mathbb{A})$. This quotient group $G(\mathbb{A})$ is isomorphic to the special orthogonal group of similitudes $GSO(3, 3)(\mathbb{A})$ attached to the split 6 dimensional Grassmann space $\Lambda^2(\mathbb{Q}^4)$ with the underlying quadratic form given by the cup-product.

The (generalized) theta correspondence of the pair

$$\langle \text{GSp}(4), \text{GO}(3, 3) \rangle$$

preserves central characters. If we apply the corresponding theta lift to the representation $\Pi$ of $\text{GSp}(4, \mathbb{A})$, then according to [AG] p.40 the theta lift of $\Pi$ to $\text{GO}(3, 3)(\mathbb{A})$ is nontrivial if and only if $\Pi$ is a globally generic representation. In this case it is easy to see, that the lift is globally generic. For the converse we need
the following result announced by Jaquet, Piatetski-Shapiro and Shalika [JPS]. See also [AG] and [S2], [PSS].

**Theorem 3.** An irreducible cuspidal automorphic representation $\Pi'$ of $\text{Gl}(4, \mathbb{A})$ lifts nontrivially to $\text{GSp}(4, \mathbb{A})$ under the generalized theta correspondence of the pair $(\text{GSp}(4), \text{GO}(3, 3))$ if and only if the alternating square degree six $L$-series $L(\Pi', \chi, s, \Lambda^2)$ or equivalently some partial $L$-series $L^S(\Pi', \chi, s, \Lambda^2)$ (for a suitably large finite set of places $S$ containing all bad places) has a pole at $s = 1$ for some unitary idele class character $\chi$. In this case the lift of $\Pi'$ is globally generic, and also $\Pi$ is generic.

**Remark on the degree six $L$-series.** To apply this it is enough to observe, that under our second temporary assumptions we have enough control on $L^S(\Pi', \chi, s, \Lambda^2)$ to apply theorem 3 for the representation $\Pi'$ of $\text{Gl}(4, \mathbb{A})$, attached to the representation $\Pi$ of $\text{GSp}(4, \mathbb{A})$ subject to our second temporary assumption. Indeed by an elementary computation the second temporary assumption implies the identity

$$L^S(\Pi', \chi, s, \Lambda^2) = L^S(\omega \chi, s) \zeta^S(\Pi, \omega \chi, s),$$

where $\omega = \omega_\Pi$ denotes the central character of $\Pi$. Now compare $L^S(\Pi', \chi, s, \Lambda^2)$ with the standard $L$-series of the special orthogonal group $\text{SO}(3, 3)$, which is used in the proof of theorem 3:

We have exact sequences

$$1 \longrightarrow \text{SO}(3, 3) \longrightarrow \text{GSO}(3, 3) \xrightarrow{\lambda} \text{Gl}(1) \longrightarrow 1,$$

$$1 \longrightarrow \text{Gl}(1) \xrightarrow{i} \text{Gl}(4) \times \text{Gl}(1) \longrightarrow \text{GSO}(3, 3) \longrightarrow 1,$$

where $\lambda$ is the similitude homomorphism and where $i(t) = (t^{-1} \cdot id, t^2)$. Hence for the Langlands dual groups, which at the unramified places describe the restriction of spherical representations of $\text{GSO}(3, 3)(F)$ to $\text{SO}(3, 3)(F)$, we get

$$1 \longrightarrow \mathbb{C}^\times \xrightarrow{\lambda} \text{GSO}(3, 3) \longrightarrow \text{SO}(3, 3) \longrightarrow 1.$$

Notice $\text{GSO}(3, 3) \subseteq \hat{\text{Gl}}(4) \times \hat{\text{Gl}}(1)$. The 6-dimensional complex representation of $\text{Gl}(4, \mathbb{C}) \times \text{Gl}(1, \mathbb{C})$ on $\Lambda^2(\mathbb{C}^4)$ defined by

$$(A, t) \cdot X = t^{-1} \cdot \Lambda^2(A)(X)$$
is trivial on the subgroup $\hat{\lambda}(\mathbb{C}^*)$, hence defines a 6-dimensional representation of the $L$-group $SO(3, 3)$. The $L$-series of this representation defines the degree 6 standard $L$-series $L^S(\sigma, s)$ of an irreducible automorphic representation $\sigma$ of the group $SO(3, 3)(\mathbb{A})$. Apparently, for $\sigma$ spherical outside $S$ in the restriction of $(\Pi', \chi)$ so that $\omega_{\Pi'} = \chi^2$, we therefore get

$$L^S(\sigma, s) = L^S(\Pi', \chi^{-1}, s, \Lambda^2).$$

**Proof of theorem 3:** Using the remark on the degree six $L$-series we now can invoke [GRS], theorem 3.4 to deduce hypothesis C. The condition on genericity made in loc. cit. automatically holds for a cuspidal irreducible automorphic representation $\sigma$ of $SO(n, n)(\mathbb{A})$ for $n = 3$, since this condition is true for $Gl(4, \mathbb{A})$. Therefore by [GRS] a pole of $L^S(\sigma, s)$ at $s = 1$ implies, that $\sigma$ has a nontrivial, cuspidal generic theta lift to the group $Sp(2n-2)(\mathbb{A}) = Sp(4, \mathbb{A})$. This in turn easily implies the same for the extended theta lift from $GSO(3, 3)(\mathbb{A})$ to $GSp(4, \mathbb{A})$. This proves theorem 3.

**Additional remark.** If in addition the spherical representation $\Pi'^S$ is a constituent of an induced representation attached to a pair of unitary cuspidal irreducible automorphic representations $\tau_1, \tau_2$ of $Gl(2, \mathbb{A})$, then furthermore

$$L^S(\Pi', \chi, s, \Lambda^2) = L^S(\tau_1 \times (\tau_2 \otimes \chi), s) L^S(\omega_{\tau_1} \chi, s) L^S(\omega_{\tau_2} \chi, s).$$

This, as well as $\omega^2 = \omega_{\Pi'} = \omega_{\tau_1} \omega_{\tau_2}$, are rather obvious. But by the wellknown analytic properties of $L$-series [JS] it implies, that the right side now has poles for $\chi = \omega_{\tau_i}^{-1}$ and $i = 1, 2$. Notice the $\tau_i$ are unitary cuspidal.

This being said we now complete the discussion of the case, where $\Pi'$ is not cuspidal.

**Reduction to the case $\Pi'$ cuspidal.** If $\Pi'$ is not cuspidal, then as already shown $\Pi'$ is obtained from a pair of irreducible unitary cuspidal representations $(\tau_1, \tau_2)$ by induction. To cover theorem 1 in this case we already remarked, that it suffices to show $\omega_{\tau_1} = \omega_{\tau_2}$. If these two characters were different, then $\chi_i = \omega_{\omega_{\tau_i}^{-1}} = 1$ for $i = 1$ or $i = 2$. Therefore $L^S(\Pi', \chi, s, \Lambda^2)$ would have a pole at $s = 1$ for $\chi = \omega_{\tau_i}^{-1}$ by the previous ‘additional remark’. Hence the identity $L^S(\Pi', \omega_{\tau_i}^{-1}, s, \Lambda^2) = L^S(\chi_i, s) \zeta^S(\Pi, \chi_i, s)$ would imply the existence of poles at $s = 1$ for the $L$-series $\zeta^S(\Pi, \chi_i, s)$. Thm.4.2 of [W2], section 4 then would imply
\((\chi_i)^2 = 1\). Since \(\chi_1\chi_2 = 1\) holds by definition of \(\Pi'\), therefore \(\chi_1 = \chi_2\). Thus 
\(\omega_{\tau_1} = \omega_{\tau_2}\). So we are in the case already considered in [W2]: \(\Pi\) is a weak lift. In this case the statement of theorem 1 follows from the multiplicity formula for weak endoscopic lifts [W2], hypothesis A (6). Thus we may suppose from now on, that \(\Pi'\) is cuspidal.

**Applying theorem 3.** Both our temporary assumptions on the existence of the lifts \(\tilde{\Pi}'\) and \(\Pi'\) imply, that from now on we can assume without restriction of generality, that \(\Pi'\) is a unitary cuspidal representation. Since we deduced \(\zeta^S(\Pi, \chi, 1) \neq 0\) for all unitary characters \(\chi\) from our first temporary assumption, the crucial identity 
\(L^S(\Pi', \chi, s, \Lambda^2) = L^S(\omega \chi, s)\zeta^S(\Pi, \omega \chi, s)\) forces the existence of a pole for 
\(L^S(\Pi', \omega^{-1}, s, \Lambda^2)\) at \(s = 1\). Therefore we are in a situation where we can apply theorem 3: Since \(\Pi'\) is cuspidal, the pair \((\Pi', \omega)\) defines a cuspidal irreducible automorphic representation of \(GSO(3,3)(\mathbb{A})\). It nontrivially gives a backward lift from \(GSO(3,3)\) to a globally generic automorphic representation \(\Pi_{gen}\) of \(GSp(4,\mathbb{A})\) using theorem 3. Comparing both lifts at the unramified places gives

\[L^S(\Pi, s) = L^S(\Pi', s) = L^S(\Pi_{gen}, s)\,.

The first equality holds by assumption. The second equality follows from the behavior of spherical representations under the Howe correspondence [R]. See also [PSS], p.416 for this particular case. Hence \(\Pi\) and the generic representation \(\Pi_{gen}\) are weakly equivalent.

In other words, using two temporary assumptions, we now have almost deduced theorem 1. In fact the generic representation \(\Pi_{gen}\), that has been constructed above, is weakly equivalent to \(\Pi\). Hence it is cuspidal, since \(\Pi\) is not CAP. However, for the full statement of theorem 1 one also needs control over the archimedean component of \(\Pi_{gen}\). We postpone this archimedean considerations for the moment and rather explain first, how to establish the two temporary assumptions to hold unconditionally. This will be deduced from the topological trace formula.

**Construction of the weak lifts \(\Pi'\) and \(\tilde{\Pi}'\) of \(\Pi\).** The existence of these lifts will follow from a comparison of the twisted topological trace formula of a group \((G, \sigma)\) with the ordinary topological trace formula for a group \(H\) for the pairs \(H = GSp(4,\mathbb{A})\) and \((G, \sigma) = (GSO(3,3)(\mathbb{A}), \sigma)\) respectively \(\tilde{H} = Sp(4,\mathbb{A})\) and \(\tilde{G} = (PGL(5,\mathbb{A}), \tilde{\sigma})\). Here \(\sigma\) respectively \(\tilde{\sigma}\) denote automorphisms of order two of \(G\) resp. \(\tilde{G}\). In both cases the group of fixed points in the center under the automorphism \(\sigma, \tilde{\sigma}\) will be a Zariski connected group, a condition imposed in
Notice G is isomorphic to the quotient of $Gl(4) \times Gl(1)$ divided by the subgroup $S$ of all zentral elements of the form $(z \cdot id, z^{-2})$, hence is isomorphic to $Gl(4)/\{\pm 1\}$. Hence for a local field $F$

$$GSO(3, 3)(F) \cong \left( Gl(4, F) \times F^* \right)/F^*,$$

where $t \in F^*$ acts on $(h, x) \in Gl(4, F) \times F^*$ via $(h, x) \mapsto (ht^{-1}, xt^2)$. The group $GSO(3, 3)(F)$ can be realized to act on the six dimensional Grassmann space $\Lambda^2(F^4)$. See [AG], p.39ff and [Wa], p.44f. This identifies the quotient group $G(F)$ with the special orthogonal group of similitudes $GSO(3, 3)(F)$ attached to the splitt 6 dimensional quadratic space $\Lambda^2(F^4)$ defined by the cup-product. The similitude character is $\lambda(h, x) = det(h)x^2$. The automorphism $\sigma$ of $G$ is induced by the map $(h, x) \mapsto (\omega(h')^{-1}\omega^{-1}, det(h)x)$ for a suitable matrix $\omega \in Gl(4, F)$ chosen in such a way, that $\sigma$ stabilizes a fixed splitting [BW1]. Then $\sigma$ is the identity on the center $Z(G) \cong Gl(1)$ of $G$.

Let us start with some

**Notations.** Let $G$ be a split reductive $\mathbb{Q}$-group with a fixed splitting $(B, T, \{x_\alpha\})$ over $\mathbb{Q}$ and center $Z(G)$. Let $\sigma$ be a $\mathbb{Q}$-automorphism of $G$, which stabilizes the splitting, such that the group $Z(G)^\sigma$ of fixed points is connected for the Zariski topology. Let $K^\pm_\infty$ be the topological connected component of a maximal compact subgroup of $G(\mathbb{R})$, similarly let $Z^\pm_\infty$ be the connected component of the $\mathbb{R}$-valued points of the maximal $\mathbb{R}$-split subtorus of the center of $G$. Let $X_G = G(\mathbb{R})/K^\pm_\infty Z^\pm_\infty$ be the associated symmetric domain as in [Wes] (5.22). Let $V$ be an irreducible finite dimensional complex representation of $G$ with highest weight $\chi \in X^*(T)$, which is invariant under $\sigma$. It defines a bundle $V_G = G(\mathbb{Q}) \backslash [G(\mathcal{A}_{fin}) \times X_G \times V]$ over $M_G = G(\mathbb{Q}) \backslash [G(\mathcal{A}_{fin}) \times X_G]$. See also [Wes] (3.4). Let $V_\chi$ denote the associated sheaf. For the natural right action of $G(\mathcal{A}_{fin})$ on $V_G$ the cohomology groups $H^i(M_G, V_\chi)$ become admissible $G(\mathcal{A}_{fin})$-modules, on which $\sigma$ acts. Let $H^\bullet(M_G, V_\chi) = \sum_k (-1)^k H^k(M_G, V_\chi)$ be the corresponding virtual modules. Then for $f \in C^\infty_c(G(\mathcal{A}_{fin}))$ the traces

$$T(f, \sigma, G, \chi) = \text{Trace}(f \cdot \sigma, H^\bullet(M_G, V_\chi))$$

are well defined.

Let $G_1$ be the maximal or stable endoscopic group for $(G, \sigma, 1)$ in the sense of $\sigma$-twisted endoscopy (see [KS](2.1) and [Wes], section 5), where we assume that
the character $\omega = \omega_v$ is trivial. The corresponding endoscopic datum $(G_1, \mathcal{H}, s, \xi)$ has the property $\mathcal{H} = ^L G_1$. The dual group $\hat{G}_1$ is the group of fixed points of $\hat{G}$ under $\hat{\sigma}$ and is again split, defining $\xi : ^L G_1 \to ^L G$. Let $T_v$ be a maximal $\mathbb{Q}$-torus of $G_1$, then we can identify the character group $X^*(T_v)$ with the fixed group $X^*(T)^\sigma$. Hence $\chi$ defines a coefficient system $V_{\chi_1}$ on $M_{G_1}$ and we can similarly define $T(f_1, id, G_1, \chi_1)$. Functions $f = \prod_{v \neq \infty} f_v \in C^\infty_c(G(\mathbb{A}_{\text{fin}}))$ and $f_1 = \prod_{v \neq \infty} f_{1,v} \in C^\infty_c(G_1(\mathbb{A}_{\text{fin}}))$ are called matching functions, if each of the local pairs are matching in the sense of [KS] (5.5.1) up to $z$-extensions, so that in particular $f_v$ and $f_{v,1}$ are characteristic functions of suitable hyperspecial maximal compact subgroups $K_v \subseteq G_v = G(\mathbb{Q}_v)$ resp. $K_{v,1} \subseteq G_{1,v} = G_1(\mathbb{Q}_v)$ for almost all $v \notin S$ ($S$ a suitable finite set of places which may be chosen arbitrarily large). By simplicity here we tacitly neglect, that $f_1$ and $f$ have to be chosen as in [KS] p. 24 or 70 up to an integration over the central group denoted $Z_1(F)$ in loc. cit. The functions $f_1 = \prod_v f_{1,v}$ and $f = \prod_v f_v$ are said to be globally matching functions, if the analog of formula [KS] (5.5.1) holds for the global stable orbital integrals defined over the finite adeles all global elements $\delta, \delta_H$. This slightly weaker global condition suffices for the comparison of trace formulas.

The homorphism $b_\xi$. For $v \notin S$ the classes of irreducible $K_{1,v}$-spherical representations $\Pi_{v,1}$ of $G_1(\mathbb{Q}_v)$ are parameterized by their ‘Satake parameter’ $\alpha_1 \in Y_1(\mathbb{C}) = \hat{T}_1/W(\hat{G}_1)$. Since $Y_1(\mathbb{C})$ can be identified with $(\hat{G}_1)_{ss}/\text{int}(\hat{G}_1)$ (see [Bo], lemma 6.5) and similar for $G$, the endoscopic map $\xi$ induces an algebraic morphism $(\hat{G}_1)_{ss}/\text{int}(\hat{G}_1) \to (\hat{G})_{ss}/\text{int}(\hat{G})$, hence a map $\xi_Y : Y_1(\mathbb{C}) \to Y(\mathbb{C})$. Since the spherical Hecke algebra of $(G_{1,v}, K_{1,v})$ resp. $(G_v, K_v)$ can be identified with the ring of regular functions $\mathbb{C}[Y_1]$ resp. $\mathbb{C}[Y]$, we obtain an induced algebra homomorphism $b_\xi = \xi_Y^*$ from the spherical Hecke algebra $C^*_c(G_v/K_v)$ to spherical Hecke algebra $C^*_c(G_{1,v}/K_{1,v})$. The Satake parameter $\alpha = \xi_Y(\alpha_1)$ defines a $K_v$-spherical representation of $G_{1,v}$, also denoted $\Pi_v = r_\xi(\Pi_{v,1})$. By construction of the map it satisfies $\Pi_v \simeq \Pi_v$. Hence the action of $G_v$ on $\Pi_v$ can uniquely be extended to an action of the semidirect product $G_v \rtimes < \sigma >$ assuming, that $\sigma$ fixes the spherical vector.

The topological trace formula. We give a review of the $\sigma$-twisted topological trace formula of Weselmann [Wes] theorem 3.21, which generalizes the topological trace formula of Goresky and MacPherson [GMP], [GMP2] from the untwisted to the twisted case. Weselmann shows that these trace formulas themselves are ‘stable’ trace formulas [Wes] theorem 4.8 and remark 4.5, i.e. can be written entirely in terms of stable twisted orbital integrals $SO^{G;\sigma}_\gamma(f)$ for the maximal elliptic...
endoscopic group of \((G, \sigma)\) in the sense of twisted endoscopy

\[
T(f, \sigma, G, \chi) = \sum_{I \subseteq \Delta, \sigma(I) = I} (-1)^{|I \setminus \sigma|} \cdot T_I(f, \sigma, G, \chi)
\]

\[
T_I(f, \sigma, G, \chi) = \sum_{\gamma \in P_I(Q)/\sim} \alpha_\infty(\gamma, 1) \cdot SO_{\gamma}^{G, \sigma}(f) \cdot \text{Trace}(\gamma \cdot \sigma, V)
\]

where the summation runs over all stable conjugacy classes \(\gamma\) in \(P_I(Q)\) of \(I\)-contractive elements, whose norm is in \(L^1_{\infty}\) (see loc. cit theorem 4.8). In principle these trace formulas can be compared without simplifying assumptions (see [Wes], section 5) except for a different notion of matching functions due to the factors \(\alpha_\infty(\gamma, 1)\). See [Wes] (5.15).

**Strongly matching functions.** Globally matching functions \((f, f_1)\) are said to be **strongly matching**, if there exists a universal constant \(c = c(G, \sigma) \neq 0\) such that

\[
T_I(f, \sigma, G, \chi) = c \cdot T_I(f_1, id, G_1, \chi)
\]

holds for all \(I\). Then

\[
T(f, \sigma, G, \chi) = c \cdot T(f_1, id, G_1, \chi)
\]

holds by definition. In the lemma below we will show for constants \(c_I = c(G, \sigma, I) \neq 0\)

\[
|\alpha_\infty(\gamma_0, 1)| = c_I \cdot |\alpha_\infty(\gamma_1, 1)|
\]

for all summands of the sum defining \(T_I(f, \sigma, G, \chi)\) respectively \(T_I(f_1, id, G_1, \chi)\) and sufficiently regular \(\gamma_0\) respectively \(\gamma_1\). Notice \((\gamma_0, \gamma_1) = (\delta, \delta_H)\) are the global \(Q\)-rational elements to be compared in the notions above. For \(\gamma = \gamma_0\) (or \(\gamma = \gamma_1\)) and the respective group \(G\) (or \(G_1\)) by [Wes] theorem 4.8

\[
O_{\sigma}^\infty(\gamma, 1) \cdot \#H^1(\mathbb{R}, T)
\]

\[
d_{\gamma, \sigma}^{\infty} \cdot \text{vol}_{db_{\infty}}((G_{1, \sigma}^I) (\mathbb{R})/\zeta)
\]

This is the factor for the terms in the sum defining \(T_I(f, \sigma, G, \chi)\), i.e. \(\gamma \in P_I(Q)\) for the \(\sigma\)-stable \(Q\)-rational parabolic subgroup \(P_I = M_IU_I\). We now discuss the ingredients of the formula defining \(\alpha_\infty(\gamma, 1)\).

**Regularity.** We will later choose a pair of matching functions \((f, f_1)\), whose stable orbital integrals have strongly \(\sigma\)-regular semisimple support. For the notion of strongly \(\sigma\)-regular see [KS] p.28. Taking this for granted, we therefore analyze the
above terms under the assumption, that $\text{Int}(\gamma) \circ \sigma$ is strongly $\sigma$-regular semisimple. For simplicity of notation we consider the case $(G, \theta)$ in the following. The case of its maximal elliptic $\sigma$-endoscopic group $(G_1, id)$ of course is analogous. We will also assume $Z(G)^\theta$ to be Zariski connected, since this suffices for our applications. Although the following discussion holds more generally, at the end we restrict to our two relevant cases for convenience.

The groups $G^I_{\gamma,\sigma}$. Fix a $\sigma$-stable rational parabolic subgroup $P_I$ of $G$ (or similar $G_1$). For a strongly $\sigma$-regular semisimple elements $\theta = \text{Int}(\gamma) \circ \sigma$, where $\gamma \in P_I(\mathbb{Q})$, the twisted centralizer $G^I_{\gamma,\sigma} = (P_I)^\theta$ is abelian. In fact, $G^\theta$ is abelian by [KS] p.28 and the centralizer $G^\theta$ in $G$ is a maximal torus $T \subseteq G$. $T$ is $\theta$-stable and $T := G_{\gamma,\sigma}$ is the group of $\theta$-fixed points in $T$.

$G^\theta = T^\theta$.

$\theta$ is strongly $\sigma$-regular, hence there exists a pair $(T, B)$ ($B$ a Borel containing $T$ defined over the algebraic closure), which is $\theta$-stable. Since $T$ acts transitively on splittings, there exists a $t$ in $T$ (again over the algebraic closure), such that $\theta^* = \text{int}(t) \theta$ respects a fixed splitting $(T, B, \{x_\alpha\})$. Then

$T^\theta = T^{\text{int}(t)\theta} = T^{\theta^*}$.

By [KS] p.14 $G^{\theta^*}$ is Zariski connected if and only if $T^{\theta^*}$ is Zariski connected. Therefore $T^\theta$ is Zariski connected, if $G^{\theta^*}$ is Zariski connected. Now $G^{\theta^*} = G^1 \cdot Z(G)^{\theta^*}$ by [KS], p.14, for the Zariski connected component $G^1 = (G^{\theta^*})^0$.

By our assumption $Z(G)^{\theta^*} = Z(G)^\sigma$ is Zariski connected. Hence $G^{\theta^*}$ and therefore $T^\theta$ are both Zariski connected. In other words $T^\theta$ is a subtorus of $T$. Notice $\sigma^2 = 1$ implies $(\theta^*)^2 = 1$, since an inner automorphism fixing a splitting is trivial. Therefore to describe $(T, \theta^*)$ over the algebraic closure, we can replace $\theta^*$ by our original automorphism $\sigma$, which also fixes (some other but conjugate) splitting of $G$. Over the algebraic closure $(T, \theta^*)$ is isomorphic to a direct product $\prod_i(T_i, \theta^*_i)$, where the factors are either $(Gl(1), id)$ or $(Gl(1), \text{inv})$ or $(Gl(1)^2, \theta^*)$, where $\theta^*$ is the flip automorphism of the two factors $\theta^*(x, y) = (y, x)$. Which types arise does not depend on $I$ as shown below, and can be directly read of from the way in which $\sigma$ acts on the $\sigma$-stable diagonal reference torus [BWW]. Two cases are relevant: [BWW] example 1.8 where $G = PGL_{2n+1}$ with $\sigma(g) = J(t^4g^{-1})J^{-1}$ and $J$ as in loc. cit., and [BWW] example 1.9 where $G = Gl(2n) \times Gl(1)$ and $\theta(g, a) = (J(t^4g^{-1})J^{-1}, \text{det}(g)a)$. For $n = 2$ these specialize to the cases considered in theorem 4, except that in example 1.9 one has to divide by $Gl(1)$ to obtain $GSO(3, 3)$ as explained already.
Absolute independence from $I$. $T$ is the unique maximal torus of $G$, which contains $G^\theta$ as subgroup and which is $\theta$-stable. Furthermore $G^I_{\gamma,\sigma} = T^\theta \cap P_I$. Since $U_I \cap T^\theta$ is trivial, the projection $P_I \to P_I/U_I \cong M_I$ induces an isomorphism of $G^I_{\gamma,\sigma}$ with its image in $M_I$. We can find a $\theta$-stable maximal torus in $M_I$ containing the image. By dimension reason, this maximal torus coincides with the image of $G^I_{\gamma,\sigma}$. By the uniqueness of $T$ this determines $T$ and $T = T^\theta$ within the subgroup $(M_I, \theta)$ of $(G, \theta)$. For strongly regular $\gamma$ this implies

$$G^I_{\gamma,\sigma} \cong T^\theta = T^\theta = T,$$

which is independent from $I$. In our situation $\sigma = \eta_1$ holds for $\eta_1$ as defined in [Wes] (2.1). By the Gauss-Bonnet formula [Wes] (3.9) the torus $T$ is $\mathbb{R}$-anisotropic modulo the center of $M_I$. By a global approximation argument therefore the groups $\zeta$ in the formula above are trivial. Similarly, since in our case the centers of $G$ and $G_1$, hence also the centers of their respective $\mathbb{Q}$-Levi subgroups $M_I$, are split $\mathbb{Q}$-tori, the groups $\zeta$ can be assumed to be trivial. Hence $d^I_{\zeta,\gamma} = 1$ by [Wes] (2.23).

The other factors. By [Wes] Lemma (2.17) and (2.15) and the isomorphism $\pi_0(F(g, \gamma)^{\eta_1, h_{\infty}}) \cong \pi_0(F(g, \gamma)^{\eta_1, h_{\infty}})$ the factors $O^\infty_\sigma(\gamma, 1) = \# R_{\gamma,\sigma}$ are

$$O^\infty_\sigma(\gamma, 1) = \frac{\#(K^I_{\infty, m} / K^I_{\infty})^{\eta_2}}{\# \pi_0(G^I_{\eta_1, \sigma}(\mathbb{R}) / (G_{\eta_1, \sigma}(\mathbb{R}) \cap p_1 K_{\infty}^Z Z_{\infty}^+ A_1 p_1^{-1}))}.$$

$G^I_{\sigma,\gamma}(\mathbb{R})$ is topologically connected, since $G^I_{\gamma,\sigma} = T$ is a torus. Therefore the denominator is trivial, and the factor $O^\infty_\sigma(\gamma, 1)$ becomes $(K^I_{\infty, m} / K^I_{\infty})^{\eta_2}$ with notations from [Wes], (2.2). It only depends on $I$, but does not depend on $\gamma$. So we need compare the factors

$$|\alpha^\infty_\infty(\gamma, 1)| = \#(K^I_{\infty, m} / K^I_{\infty})^{\eta_2} \cdot \frac{\# H^1(\mathbb{R}, T)}{vol_{db_\infty} (T'(\mathbb{R}))}.$$

The $\sigma$-stable torus $T$ contains $G^I_{\gamma,\sigma} = T = T^\sigma$ ($\sigma$-invariant subtorus). Its $\mathbb{R}$-structure might a priori depend on $\gamma$, but in fact does not. $T' = (T^\sigma)'$ is the maximal $\mathbb{R}$-anisotropic subtorus of $T$, as follows from the description of [Wes] (3.9). Now, since

$$G^I_{\gamma_0,\sigma} = T \cong T^\sigma \longrightarrow T_\sigma \cong T_1 = (G_1)^{I_1}_{\gamma_1, \iota}$$
are isogenous tori, the definition of measures $db^\gamma_\infty$ ([Wes], (3.9)) and the measures used the definition of matching of Kottwitz-Shelstad ([KS], p. 71) shows, that both factors $\rho\text{vol}_{db^\gamma_\infty}((G_{\gamma_0,\sigma})'(\mathbb{R}))$ for $G$ and $\gamma_0$ and $\rho\text{vol}_{db^\gamma_\infty}((G_1)^I_{\gamma_1,\sigma,\sigma'}'(\mathbb{R}))$ for $G_1$ and $\gamma_1$ differ by a constant independent of $(\gamma_0, \gamma_1)$. Recall that $\gamma = \gamma_0 \in G(\mathbb{Q})$ and $\gamma = \gamma_1 \in G_1(\mathbb{Q})$ is a pair of points related by the Kottwitz-Shelstad norm. For such a pair we have the tori $T \subseteq G$ and $T_1 \subseteq G_1$ and $T_1 = T_\sigma$ ($\sigma$-coinvariant quotient torus of $T \subseteq G$) by the definition of the Kottwitz-Shelstad norm. See [KS], chapter 3 and [Wes], chapter 5. The relative factor $\rho\text{vol}_{db^\gamma_\infty}((G_{\gamma_0,\sigma})'(\mathbb{R}))/\rho\text{vol}_{db^\gamma_\infty}((G_1)^I_{\gamma_1,\sigma,\sigma'}'(\mathbb{R}))$ turns out to be the degree of the isogeny $T_{\sigma^*} \to T_\sigma$, independently from $I$. Notice, up to conjugacy over $\mathbb{R}$ (hence up to isomorphism over $\mathbb{R}$) the tori $T$ respectively $T_1$ only depend on $I$ and not on $\gamma_0, \gamma_1$.

**Example.** For $G = (\text{PGl}(2n+1), \sigma)$ and $G_1 = (\text{Sp}(2n), \text{id})$ as in [Wes] (5.18) the quotient $|\alpha_{\infty}(\gamma_0, 1)|/|\alpha_{\infty}(\gamma_1, 1)|$ is equal to the relative measure factor $2^n$. In fact $K_{\infty}^{I, m} = K_{\infty}^{I, +}$ both for $(G, \sigma)$ and $(G_1, \text{id})$, since $O(2n + 1, \mathbb{R})/\{\pm \text{id}\} \cong S O(2n + 1, \mathbb{R})$ and since $U(n)$ is connected. If it holds for $I = \Delta$, then for all $I$. In all cases $H^1(\mathbb{R}, T) \cong H^1(\mathbb{R}, T_1)$. For $M_I \cong \text{PGl}(2n+1) \times \prod_i \text{Gl}(r_i) \subseteq G = \text{PGl}(2n+1)$ and the corresponding $(M_1)_I \cong \text{Sp}(2m) \times \prod_i \text{Gl}(r_i) \subseteq G_1 = \text{Sp}(2n)$ only the cases $r_i = 0, 1, 2$ give nonvanishing contributions to the trace formula, using that $T$ and $T_1$ are anisotropic modulo the center of $M_I$ resp. $(M_1)_I$.

The corresponding decomposition $T = T_m \times \prod_i T_{r_i}$ and $T_1 = T_{1, m} \times \prod_i T_{1, r_i}$ easily implies $H^1(\mathbb{R}, T_m) \cong H^1(\mathbb{R}, T_{1, m})$ (being $\mathbb{R}$-anisotropic of the same rank) and $H^1(\mathbb{R}, T_{r_i}) \cong H^1(\mathbb{R}, T_{1, r_i})$ by direct inspection, again using that $T$ and $T_1$ are anisotropic modulo the center of $M_I$ resp. $(M_1)_I$.

We have shown

**Lemma.** For matching elements strongly $\sigma$-regular elements $\gamma_0$ and matching $(\gamma_0, \gamma_1)$ the quotient $|\alpha_{\infty}(\gamma_0)|/|\alpha_{\infty}(\gamma_1)| = c_I$ only depends on $I$ and $(G, \sigma, G_1)$, but not on the stable conjugacy class of $\gamma \in P_I(\mathbb{Q})$. For $I = \Delta$ this quotient defines a universal constant $c = c(G, \sigma, G_1) \neq 0$.

In the other relevant case $(G, \sigma) = (\text{GSO}(3, 3), \sigma)$ and $G_1 = \text{GSp}(4)$ the constants $c_I$ depend on $I$. 

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Theorem 4. For $G = \text{GSO}(3, 3)$ resp. $G = \text{PGL}(5)$ and $\sigma$ as above, so that $G_1$ is $\text{GSp}(4)$ respectively $G_1$ is $\text{Sp}(4)$, the following holds

1. Suppose $\Pi_1$ is a cuspidal automorphic representation of $\text{GSp}(4, \mathbb{A})$ (respectively an irreducible component of its restriction to $\text{Sp}(4, \mathbb{A})$), which is not CAP (nor a weak endoscopic lift). Suppose $(\Pi_1)_{\text{fin}}$ contributes to $H^\bullet(M_{G_1}, V_1)$. Suppose $S$ is a sufficiently large finite set of places for which $\Pi_1^S$ defined by $\Pi_1 = \Pi_{1, S} \Pi_1^S$ is nonarchimedean and unramified. Let $\Psi$ be the virtual representation of $G_1(\mathbb{A}_{\text{fin}}) = G_1(\mathbb{A}_S) \times G_1(\mathbb{A}^S)$ on the generalized (cuspidal) eigenspaces contained in $H^\bullet(M_{G_1}, V_1)$, on which $G_1(\mathbb{A}_S)$ acts by $\Pi_1^S$. Then there exists a pair of globally and strongly matching functions $f, f_1$ with strongly regular support, so that $\text{Trace}((\Pi_1)_{\text{fin}}(f_1)) \neq 0$ and

$$\text{Trace}(\Psi(f_1)) \neq 0.$$ 

2. The fundamental lemma holds: In the situation of 1) there exists a finite set of places $S = S(\Pi_1, G_1, G)$ such that for $v \notin S$ the representation $\Pi_{1, v}$ is unramified, the spherical Hecke algebra $C^\infty_c(G_v/\mathbb{A}_v)$ is defined, so that for all $f_v \in C^\infty_c(G_v/\mathbb{A}_v)$ the functions $f_v$ and $b_\xi(f_v)$ are locally matching functions for the endoscopic datum $(G_1, G_1, \xi, s)$ for $(G, \sigma, 1)$ in the sense of [KS], (5.5.1) up to an appropriate $z$-extension. So in particular the following generalized Shintani identities hold for $v \notin S$

$$\text{Trace}(b_\xi(f_v); \Pi_{1, v}) = \text{Trace}(f_v \cdot \sigma; r_\xi(\Pi_{1, v})).$$

3. For globally and strongly matching functions $(f, f_1)$ there exists a universal constant $c = c(G, G_1) \neq 0$ such that the trace identity

$$T(f, \sigma, G, \chi) = c \cdot T(f_1, \text{id}, G_1, \chi_1)$$

holds.

Proof of theorem 4. Assertion (4.3) has already been explained.

Assertion (4.2). The comparison (4.3) of trace formulas would be pointless unless there exist matching functions at least of the form (4.2). Assertion (4.2) is a statement, which is enough to prove for almost all places $v \notin S$ for a suitable large finite set $S$. For an arbitrary spherical Hecke operator at $v \notin S$ it can be reduced
to the case, where \( f_v = 1_{K_v} \) is the unit element of the spherical Hecke algebra. In the untwisted case this was shown by Hales [Ha] in full generality. In [W4] this is done for a large class of twisted cases, including those considered above, by extending a method of Clozel and Labesse. We remark, that in the situation of theorem 4, the arguments of [W4] can be simplified by the use of the topological trace identity (4.3) explained above. Concerning unit elements: The case of the unit element \( f_v = 1_{K_v} \) was established by Flicker [Fl] for the first kind of trace comparison involving \( GSO(3, 3) \) or more precisely \( GL(4) \times GL(1) \), and was deduced from that result for the second kind of trace comparison involving \( PGL(5) \) in [BWW] theorem 7.9 and corollary 7.10. In both cases this is obtained for the unit elements of the spherical Hecke algebras for large enough primes and for sufficiently regular \((\gamma_0, \gamma_1)\). That these regularity assumptions do no harm is shown below.

Assertion (4.1). We claim that the assertions (4.2) and (4.3) of theorem 4 imply assertion (4.1). Recall that \((\Pi_1)_{\text{fin}}\) and the coefficient system \(\mathcal{V}_\chi\) are fixed. Since \((\Pi_1)_{\text{fin}}\) is cuspidal but not CAP, it only contributes to cohomology in degree 3, if it contributes nontrivially to the Euler characteristic of \(\mathcal{V}_\chi\) (in our case this amounts to the assumption that the archimedean component belongs to the discrete series). All constituents of \(\Psi\), being weakly equivalent to \(\Pi_1\) and isomorphic to \(\Pi_1\) outside \(S\), are not CAP. Hence the same applies for them.

To construct \(f_1\) such that \(T(f_1, id, G_1, \chi) \neq 0\) and \(\text{Trace}(\Psi(f_1)) \neq 0\) the easiest candidate to come into mind is the following: Let \(N\) be some principal congruence level, i.e. assume \((\Pi_1)_{\text{fin}}^{K(N)} \neq 0\) where \(K(N) \subseteq GSp(4, \mathbb{Z}_{\text{fin}})\) is defined by the congruence condition \(k \equiv id \mod N\). Choose a sufficiently large finite set of places \(S\) containing the finite set of divisors of \(N\). Then put \(f_1 = \prod_{v \neq \infty} f_{1,v}\), where \(f_{1,v}\) is chosen to be the function which is zero for all \(g \neq k \cdot z\) for \(k\) is in the principal congruence subgroup of level \(N\) and chosen equal to \(f_{1,v} = \omega(z)^{-1}\) else. Then \(f_{1,v}\) is the unit element of the Hecke algebra at almost all places not in \(S\) and the required condition \(\text{Trace}((\Psi)_{\text{fin}}(f_1)) \geq \text{Trace}((\Pi_1)_{\text{fin}}(f_1)) > 0\) holds by definition. To start to prove (4.1) we first have to find a matching function \(f\) on \(G(A_{\text{fin}})\). This involves the fundamental lemma, which is known for elements \((\gamma_0, \gamma_1)\) sufficiently regular. Therefore we modify our first naive choice \(f\) slightly. For this we choose an auxiliary place \(w \notin S\), where \(\Pi_1\) is unramified. At this auxiliary place \(w\) our previous function \(f_{1,w}\) will be replaced by an elementary function in the sense of [La], [W4]. We will see, that this allows us to restrict ourselves to consider matching conditions at sufficiently regular elements \((\gamma_0, \gamma_1)\).
Once these data in $S$ and $w$ are fixed, we finally allow for some additional modification at other unramified place $w' \neq w$ in order to construct a good $\Pi_1^S$-projector in the sense of [W1].

At the auxiliary place $w$ we choose $f_{1,w}$ to be an elementary function as in [La] §2 attached to some sufficiently regular element $t_0$ in the split diagonal torus, to be specified later. Then there exists a matching $\sigma$-twisted elementary function on $G(\mathbb{Q}_w)$ [W4]. Notice, the stable orbital integrals $SO_{G_1}^G(f_{1,w})$ vanish, unless $t$ is conjugate in $G_1(F)$ to some element in the torus of diagonal matrices, which differs from $t_0$ by a unimodular diagonal matrix and a central factor. In other words, the stable orbital integral of $f_{1,w}$ then locally has regular semisimple support for suitable sufficiently regular element $t_0$. For the global topological trace formulas of $G_1$ and $G$ this has the effect, that only regular semisimple global elements give a nonzero contribution on the geometric side. Hence for the construction a global matching functions $(f_1, f)$ suffices to show local matching for $(f_v, f_{1,v})$ at $v \neq w$ by considering stable orbital integrals at regular elements only! Since in our situation the Kottwitz-Shelstad transfer factors are identically one for all regular elements, this means we have to show local matching of stable orbital integrals at all semisimple regular elements $\gamma_1$ at the places $v \neq w$. In addition we have to find a function $f_w$ matching with $f_{1,w}$ at the place $w$ supported in strongly $\sigma$-regular elements $\gamma_0$ of $G(\mathbb{Q}_w)$, and we have to guarantee $\text{Trace}(\Psi_{1,w})(f_{1,w}) \neq 0$ by a suitable choice of $t_0$. By taking finite linear combination of elementary functions of this type one can also achieve the vanishing of all representation $\text{Trace}(\pi_{1,w})(f_{1,w})$ for the finitely many representations $\pi = \Pi_i$ weakly equivalent to $\Pi_1$ with level $N \cdot p_w$, which contribute to cohomology of the fixed coefficient system $\mathcal{V}_\chi$. Hence the trace coincides with the trace of our first naive choice of $f_1$. In fact, as already mentioned, one can then also find a corresponding finite linear combination $f_w$ of $\sigma$-twisted elementary functions matching with $f_{1,w}$, and for this choice of $f_{1,w}$ then $f_w$ has support in strongly $\sigma$-regular elements (see [W4]).

This modification using $t_0$ at the place $w$, which we further discuss below, is useful also for other purposes. At the moment, taking the situation at the place $w$ for granted, let us look at the other places first. For the nonarchimedean places $v \neq w$ not in $S$ we now can use assertion (4.2) to obtain the matching functions $f_v$ to be unit elements at all these places. Recall, the fundamental lemma for the unit elements at sufficiently regular suﬃces for this. Concerning the place $v \in S$, the existence of local matching function can be reduced to existence of matching
germs. This can be achieved by the argument of [LS], §2.2 or the similar argument of [V]. So one is reduced to the matching of germs of stable orbital integrals. Since we deal with a purely local question now, of course one has to allow singular elements now. Fortunately for the pairs \((G, G_1)\) under consideration the matching of germs of stable orbital integrals has been proven by Hales [Ha2]. This completes our construction of the functions \(f_v\) at the places \(v \neq w\) defining the globally matching pair \((f_1, f)\). So let us come back to the auxiliary unramified place \(w\).

Concerning the choice of \(t_0\): The trace \(\text{Trace}(f_1, \Pi_1) = 0\) of any irreducible cuspidal automorphic representation \(\Pi_1\) vanishes unless \(\Pi_1\) admits a nontrivial fixed vector for the group \(K(N, w) = K_w \prod_{v \neq \infty, w} K(N)_v\), where \(K_w\) is chosen to be a Iwahori subgroup. Hence only finitely many irreducible automorphic representations with fixed archimedean component and \(\text{Trace}(f_1, \Pi_1) \neq 0\) exist. \(\text{Trace}(\Pi_{\text{fin}}(f_1))\) considered in assertion (4.1) is a linear combination of the type \(\sum_i m(\Pi_i) \text{Trace}(f_{1, w}, \Pi_i)\) as a function of \(t_0\), involving a finite number of local irreducible admissible representations \(\pi = \Pi_i\) of \(G_1(\mathbb{Q}_w)\) of Iwahori level, with certain multiplicities \(m(\Pi_i) > 0\). Since the unramified representation \(\Pi_{1, w}\) is one of these, this sum is nonempty. By the argument of [La] (section 5) \(t_0\), which defines the elementary function \(f_{1, w}\), can be chosen to be strongly regular so that this sum is nonzero, provided \(G_1\) is of adjoint type. This adjointness assumption is true for \(G_1 = \text{GSp}(4)\), but not for \(G_1 = \text{Sp}(4)\). However, since we consider irreducible components \(\Pi_1\) of the restriction of the global representation in the case of the group \(\text{Sp}(4)\), this kind of argument carries over also for the case \(G_1 = \text{Sp}(4)\). Of course this remark proves the required nonvanishing \(\text{Trace}(\Pi(f_1)) \neq 0\), since it allows reconstruct the spherical trace of the spherical function of \(\Pi_{1, w}\) at the place \(w\) by a linear combination \(f_{1, w}\) of elementary functions attached to finitely many strongly regular elements \(t_0\).

We finally make some further adjustment, but only using spherical functions. This is necessary to obtain strongly matching functions, and this is only necessary in the case \((G, G_1) = (\text{GSO}(3, 3), \text{GSp}(4))\). For this we consider the unramified representation \(\Pi^S = r_\xi(\Pi^S)\). Using the method of CAP-localization [W1] (with levels and coefficient system fixed) it is clear that we can modify our \(f\) at unramified places not in \(S\) using a good spherical \(\Pi^S\)-projector \(f^S\). Similarly modify \(f_1\) by the corresponding good spherical \(\Pi^S_1\)-projector \(f_1^S = b_\xi(f^S)\). Then all contributions \(T_I(f_1, id, G_1, \chi)\) in the trace formula for \(P_I \neq G_1\) in the trace formula vanish. This is possible, since we assumed \(\Pi^S\) not to be a CAP-representation. Then, since \(f\) is globally matching, we get \(T_I(f, G, \sigma, \chi) = c_I \cdot T_I(f_1, id, G_1, \chi)\)

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for constants $c_I$ by the last lemma. Hence also $T_I(f, G, \sigma, \chi) = 0$ for $P_I \neq G$. Therefore the modified pair $(f \tilde{f}, f_1 \tilde{f})$ is not only globally matching, but also a strongly matching pair of functions with the properties required by assertion 4.1. This completes the proof of theorem 4.

Applying theorem 4. For an irreducible cuspidal representation $\Pi$, which is not CAP, suppose $\Pi_{\infty}$ belongs to the discrete series. For $G_1(A) = GSp(4, A)$ the representation $\Pi_{fin}$ contributes to $H^3(M_1, V_{\chi_1})$ for some character $\chi_1$ ([W2], hypothesis $A$ and $B$) and we can therefore assume that $Trace(\Pi_{fin}(f_1); H^*(M_1, V_{\chi_1}))$ does not vanish for all $f_1$. Now choose $f_1$ and a globally and strongly matching function $f$ on the group $G(A_{fin})$ for $G = GSO(3,3)$ (or its z-extension $Gl(4) \times Gl(1)$) as in theorem (4.1). Then by theorem (4.2) and (4.3) there must exist a $\sigma$-stable irreducible $G(A_{fin})$-constituent $(\Pi', \omega)$ of $H^*(M_G, V_{\chi_1})$, for which $Trace(\Pi', \omega)(f \cdot \sigma) \neq 0$ holds. By a theorem of Franke [Fr] this representation is automorphic. Any modification of the pair $(f_1, f)$ to $(f_1, \prod_{w \neq v} f_1, f_v \prod_{w \neq v} f_w)$ at a place $v \notin S$ for $f_1, v = b_\xi(f_w)$ and spherical Hecke operators $f_v$ again gives a pair of globally strongly matching functions. Then (4.2) and (4.3) and the linear independence of characters of the group $G(A_{fin}) \rtimes < \sigma >$ imply, that the representation identity $(\Pi'_v, \omega_v) = r_\xi(\Pi_v)$ holds for all $v \notin S$. Since the partial $L$-series outside $S$ can be easily computed from the Satake parameters, we have constructed a weak lift: The $L$-identity

$$L^S(\Pi' \otimes \chi, s) = L^S(\Pi \otimes \chi, s)$$

holds for all idele class characters $\chi$ for a sufficiently large finite set $S$. This is just a transcription of the fundamental lemma, i.e. follows from part (4.2) and (4.3) of theorem 4, and defines the weak lift $\Pi'$. The weak lift $\Pi'$, now constructed, is uniquely determined by $\Pi$, since by the strong multiplicity 1 theorem for $Gl(n)$ it is determined by the partial $L$-series $L^S(\Pi' \otimes \chi, s)$ (at the places $v \notin S$), once it exists. This shows, that our second temporary assumption is satisfied. The required identity for the central characters $\omega^2_{\Pi} = \omega_{\Pi'}$ holds, since it locally holds at all nonarchimedean places outside $S$ by the $L$-identity above.

The case $G_1 = Sp(4), G = PGL(5)$ is completely analogous and the corresponding considerations now also allows us to get rid of our first temporary assumption.

Concerning the archimedean place. The topological trace formulas compute the traces of the Hecke correspondences on the virtual cohomology of a given coefficient system in terms of orbital integrals. Underlying the trace comparison (4.3)
is a corresponding fixed lift of the coefficient system \( \mathcal{V}_{\mu_1} \). The coefficient system being fixed, only finitely many archimedean representations contribute to the topological trace formula for this coefficient system. Also notice, the representation \( \Pi'_{\infty} \) has to be a representation of \( Gl(4, \mathbb{R}) \), which has nontrivial cohomology for the given lift of the coefficient system. In fact this determines \( \Pi'_\infty \) in terms of \( \mathcal{V}_\mu \) and the cohomology degree. Furthermore in our relevant case, i.e. for the lift to \( Gl(4) \times Gl(1) \), well known vanishing theorems for Lie algebra cohomology and duality completely determine \( \Pi'_{\infty} \) in terms of the coefficient system. We leave this as an exercise. This uniquely determines \( \Pi'_{\infty} \) in terms of \( \Pi_{\infty} \). This yields matching archimedean representations, which have nontrivial cohomology in degree 3 for the group \( H \) with \( \sigma \)-equivariant representations, that have nontrivial cohomology for \( G \).

Consider the theta lift from \( GO(3, 3) \) to \( GSp(4) \), which maps the class of \((\Pi'_\infty, \omega)\) to the class of \( \Pi_{gen, \infty} \). At the archimedean place we claim, that for this lift \((\Pi'_{\infty, \omega_\infty})\) locally (!) uniquely determines \( \Pi_{gen, \infty} \). Since theta lifts behave well with respect to central characters, it is enough to prove this for the dual pair \((O(3, 3), Sp(4))\) by our earlier remarks on \( \Pi'_{\infty} \). In fact by Mackey’s theory the restriction of \( \Pi'_{\infty} \) to \( Sl(4, \mathbb{R}) \cdot \mathbb{R}^* \) decomposes into two nonisomorphic irreducible constituents, from which (each of them) in turn \( \Pi'_{\infty} \) is obtained by induction. The same holds for the restriction of discrete series representations of \( GSp(4, \mathbb{R}) \) to \( Sp(4, \mathbb{R}) \cdot \mathbb{R}^* \). It is therefore enough to determine the restriction of \( \Pi_{gen, \infty} \) to \( Sp(4, \mathbb{R}) \cdot \mathbb{R}^* \).

Once we have shown, that this restriction contains an irreducible constituent in the discrete series, we have therefore as a consequence, that the representation \( \Pi_{gen, \infty} \) of \( GSp(4, \mathbb{R}) \) is uniquely determined by \( \Pi'_{\infty} \) and that \( \Pi_{gen, \infty} \) belongs to the discrete series. We now still have to understand, why \( \Pi_{gen, \infty} \) should be in the archimedean \( L \)-packet of \( \Pi_{\infty} \).

This being said, we first replace \( Gl(4, \mathbb{R}) \) by \( Sl(4, \mathbb{R}) \) or the quotient \( SO(3, 3)(\mathbb{R}) = Sl(4, \mathbb{R})/\mathbb{Z}_2 \) (recall the central character was \( \omega^2 \)) and replace \( \Pi'_{\infty} \) by a suitable irreducible constituent of the restriction. For simplicity of notation still denote it \( \Pi'_{\infty} \). If the representation on \( O(3, 3)(\mathbb{R}) \) induced from the irreducible representation \( \Pi'_{\infty} \) of \( SO(3, 3)(\mathbb{R}) \) would be irreducible, we can immediately apply \([H]\) to conclude, that its theta lift on \( Sp(4, \mathbb{R}) \) is uniquely determined by \( \Pi'_{\infty} \). Otherwise there exist two different extensions of \( \Pi'_{\infty} \) to \( O(3, 3)(\mathbb{R}) \), by simplicity denoted \( \Pi'_{\infty} \) and \( \Pi'_{\infty} \otimes \epsilon \), where \( \epsilon \) is the quadratic character of \( O(3, 3)(\mathbb{R}) \) defined by the quotient \( O(3, 3)(\mathbb{R})/SO(3, 3)(\mathbb{R}) \). However if this happens, we now claim only one of the two possibilities contributes to the theta correspondence, so that again
we can apply [H]. The claim made follows from an archimedean version of part c) of the Proposition of [V], p.483. For the convenience of the reader we prove this in the archimedean exercise below.

So passing from \( \Pi_\infty \) to \((\Pi'_\infty, \omega_\infty)\) and then back to \(\Pi_{\text{gen}, \infty}\) turns out to be a well defined local assignment at the archimedean place; to be accurate, in the first instance only up to twist by the sign-character. That it is well defined then follows a posteriori, once we know the assigned image is contained in the discrete series. For this see the archimedean remarks made following our second temporary assumption. To compute the local assignment - first only up to a possible character twist - it is enough to do this for a single suitably chosen global automorphic cuspidal representation \(\Pi\). We have to show, that passing forth back with these two lifts locally at the archimedean place, we do not leave the local archimedean \(L\)-packet. Since for every archimedean discrete series representation \(\Pi_\infty\) of weight \((k_1, k_2)\) there exists a global weak endoscopic lift \(\Pi\) with this given archimedean component \(\Pi_\infty\) (see [W2]), it is now enough to do this calculation globally for this global weak endoscopic lift \(\Pi\). For global weak endoscopic lifts there exists a unique globally generic cuspidal representation \(\Pi_{\text{gen}}\) in the weak equivalence class of \(\Pi\) by [W2], hypothesis A. So \(\Pi_{\text{gen}}\) is uniquely determined by the description of this generic component given in [W2], hypothesis A. Therefore looking at the archimedean place the archimedean component \(\Pi_{\text{gen}, \infty}\), which is uniquely determined by \(\Pi_\infty\) as we already have seen, has to be this unique generic nonholomorphic member of the discrete series \(L\)-packet of \(GSp(4, \mathbb{R})\) of weight \((k_1, k_2)\). Since it depends only on the archimedean local representation, and since this holds true in the special case, this holds in general. This proves theorem 1 modulo the following

**Archimedean exercise.** We now prove the archimedean analog of proposition [V], p.483, which we used above. See also [W5]. Let \(V\) be a nondegenerate real quadratic space of dimension \(m = p + q\), let \(G = O = O(p, q)\) be its orthogonal group with maximal compact subgroup \(K = O(p) \times O(q)\). Let \(O^\vee\) be the set of equivalence classes of irreducible \((\text{Lie}(O), K)\)-modules. The metaplectic cover \(Mp(2N)\) of \(Sp(2N)\) for \(N = m \cdot n\) naturally acts on the Fock space. Its associated Harish-Chandra module \(P_V(n) \cong \text{Sym}^n(V \otimes \mathbb{C}^n)\) defines the oscillator representation \(\omega_{\text{Fock}}\). For the pair \((G, G') = (O, Mp(2n, \mathbb{R}))\) the restriction of \(\omega_{\text{Fock}}\) to \(G \times G' \subseteq Mp(2N, \mathbb{R})\) induces the theta correspondence. The action of \(G\) by \(\omega_{\text{Fock}}\) does not coincide with the natural action on \(\text{Sym}^n(V \otimes \mathbb{C}^n)\), except when \(V\) is anisotropic. Nevertheless the action of \(K\) does. Let \(R(n) \subseteq O^\vee\)
[111x214]is isomorphic to $i$ in the restriction of $\pi$ in the sense of [H]. The restriction of $\deg([H], p.541)$, $\tau'$ to $\pi'$ in $R(n')$ the irreducible quotients of $\pi \otimes \tau'$ contribute to $R(n + n')$. For $\pi \in O'$ let $\pi'$ denote the contragredient representation. Since the quadratic character $\varepsilon$ is an irreducible quotient of the representation $\pi \otimes (\pi' \otimes \varepsilon)$, we obtain $n(\pi) + n(\pi' \otimes \varepsilon) \geq n(\varepsilon)$. We claim $n(\varepsilon) \geq \dim_{\mathbb{R}}(V)$, which implies as desired

$$n(\pi) + n(\pi' \otimes \varepsilon) \geq \dim_{\mathbb{R}}(V).$$

For the proof of $n(\varepsilon) \geq m$ put $n = n(\varepsilon)$. The restriction of $\varepsilon$ to $K = O(p) \times O(q)$ is $\sigma = \varepsilon \boxtimes \varepsilon$. $M' = Mp(2n, \mathbb{R}) \times Mp(2n, \mathbb{R})$ covers the centralizer of $K$ in $Sp(2N)$ and $M'_0 = \tilde{U}(n, \mathbb{C}) \times \tilde{U}(n, \mathbb{C})$ its maximal compact subgroup. By [KV] and [H], lemma 3.3 a unique irreducible representation $\tau'$ of $M'_0$ is attached to $\sigma$, which is the external tensor product of the irreducible highest weight representations with highest weights $(\frac{p}{2} + 1, \ldots, \frac{p}{2} + 1)$ respectively $(\frac{q}{2} + 1, \ldots, \frac{q}{2} + 1)$ of $\tilde{U}(n, \mathbb{C})$, which is a twofold covering group of $U(n, \mathbb{C})$. Here $\frac{p}{2} + 1$ occurs $p$ times respectively $\frac{q}{2} + 1$ occurs $q$ times. So in particular $n \geq \max(p, q)$. The two components are represented by pluriharmonic polynomials in the Fock space of degree $p$ respectively $q$. As a consequence, $\tau'$ is realized in $Sym^d(V \otimes_{\mathbb{R}} \mathbb{C}^n)$ for $d = p + q = m$. In other words $\deg(\tau') = m$ for the degree $\deg$ in the sense of [H]. The restriction of $\tau'$ to the maximal compact subgroup $K' = \tilde{U}(N, \mathbb{C}) \cap G'$ is the tensor representation $\det\frac{n-2}{2} \wedge^p (\mathbb{C}^n) \otimes \wedge^q (\mathbb{C}^n)^\vee$ of $\tilde{U}(n, \mathbb{C})$, since the negative definite part gives an antiholomorphic embedding ([H], p.541). The highest weights of all irreducible constituents of the representation $\wedge^p (\mathbb{C}^n) \otimes \wedge^q (\mathbb{C}^n)^\vee$ of $\tilde{U}(n, \mathbb{C})$ are of the form $(1, 1, 0, 0, \ldots, 0, -1, \ldots, -1)$ with $i \leq p$ digits 1, $j \leq q$ digits -1 and $n - i - j$ digits 0. According to [H], lemma 3.3 there must be a unique irreducible representation $\sigma'$ of $\tilde{U}(n, \mathbb{C})$ in the restriction of $\tau'$ such that $\deg(\sigma') = \deg(\tau') = m$. The representation of $M'_0 \cong \tilde{U}(n, \mathbb{C}) \times \tilde{U}(n, \mathbb{C})$ on the polynomials of degree $k$ in the Fock space $P_V(n)$ is isomorphic to $\bigoplus_{a+b=k} \det \frac{n-2}{2} \cdot Sym^a(\mathbb{R}^p \otimes_{\mathbb{R}} \mathbb{C}^n) \otimes \det \frac{n-2}{2} \cdot Sym^b(\mathbb{R}^q \otimes_{\mathbb{R}} \mathbb{C}^n)$. Its restriction to $K' \cong \tilde{U}(n, \mathbb{C})$ in $G'$ therefore is isomorphic to the representation $\det \frac{n-2}{2} \bigoplus_{a+b=k} Sym^a(\mathbb{R}^p \otimes_{\mathbb{R}} \mathbb{C}^n) \otimes Sym^b(\mathbb{R}^q \otimes_{\mathbb{R}} (\mathbb{C}^n)^\vee)$ induced by the natural action of $\tilde{U}(n, \mathbb{C})$ on $\mathbb{C}^n$. Hence for $i \leq p$ and $j \leq q$ (the number of digits 1 resp. -1 of the highest weight) we get $\deg(\det \frac{n-2}{2} \otimes (1, 1, 0, 0, \ldots, 1, \ldots, 1)) = i + j$ by considering the representation generated by the product of some $i \times i$-minor in $Sym^i(\mathbb{R}^i \otimes_{\mathbb{R}} \mathbb{C}^n)$ and some $j \times j$-minor in $Sym^j(\mathbb{R}^j \otimes_{\mathbb{R}} \mathbb{C}^n)$. Thus $\deg(\sigma') = m$
implies, that there exists \( i \leq p \) and \( j \leq q \) such that \( \deg(\sigma') = i + j = m \). Therefore \( i = p \) and \( j = q \), hence \( m = i + j \leq n \). So we have shown \( n(\varepsilon) \geq m \). In fact it is not hard to see \( n(\varepsilon) = m \).

**The special case considered.** In the situation of theorem 1 and its proof this gives

\[
n(\pi) + n(\pi \otimes \varepsilon) \geq 6.
\]

The underlying representation \( \Pi'_{\infty} \) of \( GO(3,3)(\mathbb{R}) \) decomposes into two nonisomorphic representations \( \pi_1, \pi_2 \) of \( O(3,3)(\mathbb{R}) \) with \( n(\pi_1) = n(\pi_2) = 2 \). Therefore \( n(\pi_1 \otimes \varepsilon) \geq 4 \) and \( n(\pi_2 \otimes \varepsilon) \geq 4 \) by the above inequality, since in our case \( \pi^\vee \cong \pi \) follows from \( (\Pi'_\infty)^\vee \cong \Pi'_\infty \otimes \omega_{\infty}^{-1} \) (which was a consequence of the second temporary assumption).

This finally completes the proof of theorem 1.

**Remark on orthogonal representations.** For a weak endoscopic lift \( \Pi \) the statement of theorem 1 is known by [W2], hypothesis A. So there was no need not go to the trace formula arguments required otherwise. Nevertheless the above arguments applied for a weak endoscopic lift nevertheless yield something interesting. Let \( \Pi \) be cuspidal irreducible representation \( \Pi \) of \( GSp(4,\mathbb{A}) \), which is a weak endoscopic lift attached to a pair of cuspidal representations \( (\sigma_1, \sigma_2) \) of \( Gl(2,\mathbb{A}) \) with central character \( \omega \), but which is not CAP. Then we can still construct the irreducible automorphic representation \( \tilde{\Pi}' \) of \( Gl(5,\mathbb{A}) \) from \( \Pi \) as above. However the representation \( \tilde{\Pi}' \) will not be cuspidal any more, since its \( L \)-series

\[
L^S(\tilde{\Pi}', s) = L^S(\sigma_1 \times \sigma_2 \otimes \omega^{-1}, s)\zeta^S(s)
\]

has a pole at \( s = 1 \). Hence the automorphic representation \( \tilde{\Pi}' \) is Eisenstein, in fact induced from an automorphic irreducible representation \( (\pi_1, \pi_1) \) of the Levi subgroup \( Gl(4,\mathbb{A}) \times Gl(1,\mathbb{A}) \). This gives rise to an irreducible automorphic representation \( \pi_4 \) of \( Gl(4,\mathbb{A}) \) attached to \( (\sigma_1, \sigma_2) \), which after a character twist by \( \omega \) will be denoted

\[
\sigma_1 \times \sigma_2.
\]

The automorphic representation \( \sigma_1 \times \sigma_2 \) is uniquely determined by its partial \( L \)-series \( L^S(\sigma_1 \times \sigma_2, s) \), which coincides with the partial \( L \)-series attached to the orthogonal four dimensional \( E_\lambda \)-valued \( \lambda \)-adic Galois representation \( \rho_{\sigma_1,\lambda} \otimes E_\lambda \rho_{\sigma_2,\lambda} \), the tensor product of the two dimensional Galois representations \( \rho_{\sigma_i,\lambda} \) attached to \( \sigma_1 \) and \( \sigma_2 \). This four dimensional tensor product is an orthogonal Galois representation. It should not be confused with the symplectic four dimensional Galois
representation obtained in [W2], theorem I, which for weak endoscopic lifts is the direct sum of the Galois representations $\rho_{\sigma_1, \lambda}$. D.Ramakrishnan obtained a completely different and more general construction of the automorphic representation $\sigma_1 \times \sigma_2$ using converse theorems.

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