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Fractional Quantum Hall Effect in Topological Flat Bands with Chern Number Two

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Recent theoretical works have demonstrated various robust Abelian and non-Abelian fractional topological phases in lattice models with topological flat bands carrying Chern number $C = 1$. Here we study hard-core bosons and interacting fermions in a three-band triangular-lattice model with the lowest topological flat band of Chern number $C = 2$. We find convincing numerical evidence of bosonic fractional quantum Hall effect at the $\nu = 1/3$ filling characterized by three-fold quasi-degeneracy of ground states on a torus, a fractional Chern number for each ground state, a robust spectrum gap, and a gap in quasi-hole excitation spectrum. We also observe numerical evidence of a robust fermionic fractional quantum Hall effect for spinless fermions at the $\nu = 1/5$ filling with short-range interactions.

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Introduction.—Topological states of matter have been the focus of intensive studies since the discovery of the integer quantum Hall effect (QHE) [1] and the fractional QHE (FQHE) [2]. The latter, occurring at fractional filling of Landau levels (LLs), provides the first example of fractionalization in two dimensions. The precise quantization of Hall conductance was found to be directly connected to a topological invariant Chern number [3, 4] soon after its experimental discovery. The FQHE is further characterized by quasi-particles with fractional charge [5] and fractional statistics [6, 7] as well as topological ground-state degeneracy [8], which are manifestations of its topological order [9]. The idea of flux attachment and composite-particle theory has provided a simple but profound picture of the FQHE [10, 11].

Recently, a series of numerical works have demonstrated convincing evidence of the Abelian [12–14] and non-Abelian FQHEs [15–17] in topological flat band (TFB) models [18] without an external magnetic field. These TFB models, belonging to the topological class of the well-known Haldane model [19], have at least one topologically nontrivial nearly flat band with a Chern number $C = 1$, which is separated from the other bands by large gaps [18, 20–22]. This intriguing fractionalization effect in TFBs without LLs, defines a new class of fractional topological phases (also known as fractional Chern insulators), and has stimulated a lot of recent research activities [23–32].

In contrast to the continuum model in a magnetic field where the Chern number of a LL is always one, higher Chern numbers are possible for nearly flat bands in lattice models [33]. The Abelian and non-Abelian FQHE states found in $C = 1$ TFBs [12–17] generally have analogy with ones in the continuum LL, while FQHE in TFBs with high Chern numbers might do not have such simple analogy; thus new exotic topological states of matter might occur in these TFBs [34]. Nonetheless, exotic FQHE in TFBs with high Chern numbers has not been studied in microscopic models. In this Letter, we aim to fill in the gap by demonstrating that exotic FQHE for both hard-core bosons and interacting fermions can indeed be realized in TFBs with $C = 2$.

We first introduce a three-band triangular lattice tight-binding model whose lowest TFB has Chern number $C = 2$. Through extensive exact diagonalization (ED) studies of hard-core bosons in the TFB, we find convincing numerical evidence of the bosonic FQHE at the filling $\nu = 1/3$ in the $C = 2$ TFB. This 1/3 bosonic FQHE is characterized by three-fold quasi-degeneracy ($d = 3$) of ground states on a torus, a fractional quantized Chern number for each ground state, a robust spectrum gap, and a gap in quasi-hole excitation spectrum. For the 1/5 filling of spinless fermions in the $C = 2$ TFB, clear FQHE features are also observed. We discuss the plausibility of understanding the exotic $\nu = 1/3$ bosonic and $\nu = 1/5$ fermionic FQHEs in the $C = 2$ TFB in terms of effective two-component (or bilayer) FQHEs.

Formulation.—We introduce a three-band triangular-lattice model of interacting hard-core bosons:

$$H = \pm t \sum_{\langle rr' \rangle} [b^\dagger_r b_{r'} \exp(i\phi_{r-r'}) + \text{H.c.}]$$
$$\pm t' \sum_{\langle\langle rr'\rangle\rangle} [b^\dagger_r b_{r} \exp(i\phi_{r-r}) + \text{H.c.}]$$
$$+ V_1 \sum_{\langle rr' \rangle} n_r n_{r'} + V_2 \sum_{\langle\langle rr'\rangle\rangle} n_r n_{r'}$$  \hspace{1cm} (1)

where $b^\dagger_r$ creates a hard-core boson at site $r$, $(\langle\ldots\rangle)$ and $(\langle\langle\ldots\rangle\rangle)$ denote the nearest-neighbor (NN) and the next-nearest-neighbor (NNN) pairs of sites, respectively.
The NN and NNN hopping amplitudes are positive (negative) along the solid (dashed) lines; The arrows represent the phases ±2φ (signs are represented by arrow directions) in the NN hoppings and ±φ in the NNN hoppings. (b) Edge states of the triangular-lattice model in (a), and the lower TFB owns the Chern number $C = +2$.

FIG. 2: (color online). The 1/3 bosonic FQHE. (a) Low energy spectrum $E_i - E_1$ versus the momentum $k_1N_2 + k_2$ of the 1/3 bosonic FQHE phase for three lattice sizes $N_s = 45$, 54 and 63 at $\nu = 1/3$ filling with $V_1 = V_2 = 0.0$. (b) Spectrum gaps versus $1/N_s$ for four lattice sizes.

The triangular-lattice model has a unit cell of three sites, and therefore has three single-particle bands. Here, we adopt the parameters $t = 1$, $t' = 1/4$ [the signs of hoppings are described in Fig. 1(a)] and $\phi/2\pi = 1/6$, such that a lowest TFB of $C = 2$ is formed with a flatness ratio (of the band gap over bandwidth) of about 15 [Fig. 1(b)]. In our ED study, we consider a finite system of $N_1 \times N_2$ unit cells (total number of sites $N_s = 3 \times N_1 \times N_2$ and total number of single-particle orbitals $N_{\text{orb}} = N_1N_2$ in each band) with basis vectors shown in Fig. 1(a) and we use periodic boundary conditions. We denote the boson numbers as $N_b$, and the filling factor of the flat band is $\nu = N_b/N_{\text{orb}}$. The momentum vector $\mathbf{q} = (2\pi k_1/N_1, 2\pi k_2/N_2)$ will be denoted by a pair of integer quantum numbers $(k_1, k_2)$. The amplitude of the NN hopping $t$ is set as the unit of energy.

The 1/3 bosonic FQHE. (a) Low energy spectrum.—We first look at the low-energy spectrum for a finite lattice with $N_s = 45$ (3 x 3 x 5) sites at filling $\nu = 1/3$ with $V_1 = V_2 = 0.0$ as shown by Fig. 2(a). We denote $E_1$ as the energy of the i-th lowest many-body eigenstate. The ground state manifold (GSM) is defined as a set of lowest states with close energies well separated from other excited states by a finite spectrum gap. For the $\nu = 1/3$ bosonic FQHE phase, two necessary conditions are satisfied: a GSM with three quasi-degenerate ($d = 3$) lowest eigenstates ($E_3 - E_1 \sim 0$); and the $d = 3$ GSM being separated from the higher eigenstates by a finite spectrum gap $E_3 - E_4 \gg E_3 - E_1$.

We have also obtained numerical results from other lattice sizes of $N_s = 36$ (3 x 3 x 4), 54 (3 x 3 x 6) and 63 (3 x 3 x 7) around $V_1 = V_2 = 0.0$. Similar to the FQHE in the C = 1 TFBs [12–14], if $(k_1, k_2)$ is the momentum sector for one of the states in the GSM, we find that other state in the GSM can be found in the sector $(k_1 + n_b, k_2 + n_b)$ [module $(N_1, N_2)$] demonstrating the momentum space translation invariant as an emerging symmetry of the system. Indeed, for $N_s = 36, 45, 63$, the three GSs are in the (0,0), (1,0), and (2,0) sectors, respectively; while for $N_s = 54$, both $N_b/N_1$ and $N_b/N_2$ are integers, and all three GSs are in the same (0,0) sector [with very close energies as shown in Fig. 2(a)]. Therefore, for each system size, there is an obvious GSM with three-fold quasi-degenerate states, which is well separated from the higher energy spectrum by a large spectrum gap. We have also attempted a scaling plot of spectrum gaps for the four lattice sizes [as shown in Fig. 2(b)], which indicates that the spectrum gap of the 1/3 bosonic FQHE phase should survive in the thermodynamic limit. We further find that this $\nu = 1/3$ FQHE is stable (with large spectrum gap and well-defined $d = 3$ GSM) in the presence of relatively weak repulsions ($V_1 < 0.5$ and $V_2 < 0.5$).

(b) Berry curvature and Chern number.—The Chern number [3] (which is the Berry phase in units of $2\pi$) of a many-body state is an integral invariant in the boundary phase space [4, 35]: $C = \frac{1}{2\pi} \int d\theta_1 d\theta_2 F(\theta_1, \theta_2)$, where two boundary phases $\theta_1$ and $\theta_2$ are introduced as the generalized boundary conditions in both direc-
of 1 phase (Fig. 3). For each GSM of the 1/3 bosonic FQHE, the three GSs maintain their quasi-degeneracy and are well separated from the other low-energy quasihole states is consistent with the counting rule of splitting one hole into three quasiholes (each with fractional charge 1/3), i.e. the quasi-hole-counting in Laughlin's 1/3 fermionic FQHE state, based upon the generalised Pauli principle [14, 16]. Similarly, for the $N_s = 54$ and $N_b = 5$ case [Fig. 5(b)], and the $N_s = 63$ and $N_b = 6$ case [Fig. 5(c)], there are also distinguishable spectrum gaps and well-separated lower-energy quasihole manifolds.

The 1/5 fermionic FQHE. —For the case of interacting spinless fermions [where the bosonic operators in Eq. (1) are replaced by the fermionic ones] in the $C = 2$ TFB, we have also observed FQHE features at the $\nu = 1/5$ filling [36] for four different lattice sizes of $N_s = 45$ (3 $\times$ 5 $\times$ 3), 60 (3 $\times$ 5 $\times$ 4), 75 (3 $\times$ 5 $\times$ 5) and 90 (3 $\times$ 5 $\times$ 6). We denote the fermion numbers as $N_f$, and the filling factor of the TFB is $\nu = N_f/N_{orb}$. In contrast to the $\nu = 1/3$ bosonic FQHE around $V_1 = V_2 = 0$, the onset of fermionic FQHE features at $\nu = 1/5$ needs finite values of short-range repulsions ($V_1$, $V_2$ and a third-neighbor interaction $V_3$), similar to the fermionic 1/5 FQHE [12] or the bosonic 1/4 FQHE [13] in the $C = 1$ TFBs. At $\nu = 1/5$, the topological ground-state degeneracy is $d = 5$: for $N_s = 45$, 60, 90, the five GSs are in five different sectors, e.g. the (0,0), (1,0), (2,0), (3,0) and (4,0) sectors for moving one boson from the $\nu = 1/3$ filling. As shown in Fig. 5(a), for the case of $N_s = 45$ and $N_b = 4$, the quasihole spectrum exhibits a distinguishable gap which separates 5 lowest states in each momentum sector from the other higher-energy states, and there are 75 low-energy quasihole states in total. This number of low-energy quasihole states is consistent with the counting rule of splitting one hole into three quasiholes (each with fractional charge 1/3), i.e. the quasi-hole-counting in Laughlin’s 1/3 fermionic FQHE state, based upon the generalised Pauli principle [14, 16]. Similarly, for the $N_s = 54$ and $N_b = 5$ case [Fig. 5(b)], and the $N_s = 63$ and $N_b = 6$ case [Fig. 5(c)], there are also distinguishable spectrum gaps and well-separated lower-energy quasihole manifolds.
At the 2/3 total filling. For the 1/5 filling of interacting spinless fermions in the $C = 2$ TFB, clear FQHE features have also been observed with a five-fold degenerate GSM and a fractional quantized Hall conductance of $2e^2/5h$ per GS, which could also be consistent with the Halperin 332 state. Nonetheless, the two “components” in a $C = 2$ TFB are mutually entangled, one may speculate that our states may be different from the conventional bilayer $mnm$ states where the two separated layers are only coupled by interaction. We believe that more direct evidence is needed to verify the nature of the obtained FQHE states, which calls for further studies in the future.

Other topological phases are also possible in other fractional fillings. For the $\nu = 1/4$ hard-core bosons, we have observed a bosonic state with some topological features. However we can not determine the nature of this 1/4 bosonic state as its ground-state degeneracy and the total Chern number varies with the particle numbers which will be presented in the Supplementary Material [37].

Although our results are obtained in a specific three-band triangular-lattice model, one may speculate that such results, especially the very robust 1/3 bosonic state, might be universal for generic $C = 2$ TFB models [38]. In the future, it would be also highly interesting to demonstrate more exotic fractional topological phases in microscopic TFB models with high Chern numbers.

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Note added.—After the submission of the present Letter, a few related works appeared very recently [39–43]. Two works have shown that flat bands with arbitrary high Chern numbers can be systematically constructed using multi-layer lattice models [39, 40]. Another two works reported some numerical evidence for a series of fermionic and bosonic fractional incompressible states in a multi-layer kagome-lattice model with high Chern numbers [41, 43]. These results together with ours suggest some universality of FQHE in TFBs with high Chern numbers.

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