The choice of dynamic variables in quantum theory

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Abstract. The influence of various choices of dynamic variables in quantum theory on the interpretation of measurement results and experiments are discussed. Dynamic variables in the generally accepted quantum mechanics have no visual sense, which in turn leads to the loss of the image of the studied phenomena and processes of the microworld. Thus, quantum mechanics forces us to abandon the possibility of even mentally imagining the conditions that determine the behavior of an individual system in a quantum region. Also, various representations of the wave function and their use in solving dynamic equations are shown. By now, quantum mechanics is not a complete theory. It resembles classical (Newtonian) mechanics, which explains how movement occurs as a result of the action of certain forces, but does not say what kind of forces it is. Thus, before the advent of basic theories, the application of quantum mechanics to specific physical problems always implied the need to build certain guesses about the nature of these forces.

1. Introduction

Studies have shown that the existing large number of interpretations of quantum mechanics is partly due to the different choice of generalized dynamic variables for describing the same physical system. Very often, by definition, these variables have no visual physical meaning, which makes it difficult to interpret a theory [1]. For example, by now it is already well known that the same elementary particle can be described by both a reducible and an irreducible representation of the Lorentz group. The clearest example of this is the elegant reduction [2] of the Klein-Gordon equation for a scalar particle \((s = 0)\)

\[
(\epsilon + m^2)\varphi = 0 \quad [\varphi \text{ is the d’Alembert operator}] \tag{1}
\]

to the Petiau-Duffin-Kemmer equation [3-5], which is usually written in the form of the Dirac equation (but with matrices \(\beta\) instead of matrices \(\gamma\)). In our work we use the Pauli metric \((+1, +1, +1, +1)\), \(x^\mu = (x_1, x_2, x_3, x_4 = i\epsilon)\), in which "co" and "contr" are variant components of vectors do not differ and all the \(\gamma\) - matrices are Hermitian \((\gamma^\mu_+ = \gamma^\mu)\) [6]:

\[
\gamma_k = -i\beta_k \quad (k = 1, 2, 3), \quad \gamma_4 = \beta, \quad \gamma_5 = -\rho_1 = \gamma_1\gamma_2\gamma_3\gamma_4 = i\alpha_1\alpha_2\alpha_3.
\]

Here, the 5-component wave function realizes the reducible representation of the Lorentz group and consists of a scalar and a 4-vector:

\[
\psi = \left(\begin{array}{c}
\varphi \\
\varphi_\mu
\end{array}\right).
\]

In this case, the matrices \(\beta\) satisfy the Petiau-Duffin-Kemmer algebra (23).
In addition, it is obvious that any linear equation for a complex wave function can be written (separating the real and imaginary parts) as a system of equations only for real functions, but which will be related to each other by the normalization condition. There are such (equivalent to the usual) formulations of quantum mechanics without the explicit use of complexity.

The objective of the current study was to explain that the use of complex wave functions is one of the convenient ways to describe quantum systems, and not their obligatory attribute, as is often considered. The use of complex dynamic variables increases the number of degrees of freedom of a physical system. For example, in quantum field theory, this allows one to simply describe the charges of particles, assuming that real fields do not have charges.

2. Materials and methods of research

It is well known that the wave function of a particle in quantum mechanics is determined up to a unitary transformation. However, there are still "dark" spots.

As is known, the Dirac equation

\[
\left( \gamma^\mu \frac{\partial}{\partial x^\mu} + m_0 \right) \psi_D = 0 ,
\]

(4)
describes particles with left and right helicities, which is evident from the identical representation of the wave function in the form:

\[
\psi_D \equiv \frac{1}{2} (1 + \gamma_5) \psi + \frac{1}{2} (1 - \gamma_5) \psi = \psi_L + \psi_R .
\]

(5)

In the standard model, only the left-handed states of massless leptons with exact \( \gamma_5 \) - invariance (chiral symmetry) enter the seed lepton sector:

\[
\gamma_5 \psi_L = \psi_L .
\]

(6)

In 1952, Corben [6] proposed to add a pseudoscalar mass to the Dirac equation:

\[
\left( \gamma^\mu \frac{\partial}{\partial x^\mu} + m_0 + i m c \gamma_5 \right) \psi_C = 0 .
\]

(7)

When considering specific problems, the Dirac-Corben equation (7) led to the same predictions [7] as the Dirac equation. Later [8] it was shown that there is a unitary transformation

\[
\psi_C = U \psi_D ,
\]

(8)

\[
U = \frac{1}{\sqrt{2}} \left[ 1 + \frac{m_0}{m} \left( 1 + i \frac{m c}{m} \right) \right] ,
\]

(9)

which reduces the Dirac-Corben equation to the Dirac equation, in which the mass term takes the form

\[
m = \sqrt{m_0^2 + m_c^2} .
\]

(10)

That is, the square of the mass can take on different values:

\[
m^2 = m_0^2 + m_c^2 \Rightarrow \begin{cases} 
> 0 \text{ for bradyons } (v < c) , \\
= 0 \text{ for luxons } (v = c) , \\
< 0 \text{ for tachyons } (v > c) .
\end{cases}
\]

All operators are converted by the usual rule:

\[
O_c = U O_D U^{-1} .
\]

(11)

Using the notation \( \frac{m_0}{m} = \cos \theta , \frac{m_c}{m} = \sin \theta \), the unitary operator (9) can be written in a more compact form [9]:
\[ U = \exp \left( -i\gamma_5 \frac{\theta}{2} \right). \]  

Let us make several important observations regarding the Dirac-Corben equation:

1) If the scalar mass is zero \((m_0 = 0)\) from (7), we obtain the equation

\[ \left( \gamma^\mu \frac{\partial}{\partial x^\mu} + im_c \gamma_5 \right) \psi = 0, \]  

in which the real mass \(m_c > 0\) according to (10) in the experiments will be observed as the usual scalar mass \(m_0\) in the Dirac equation:

\[ m_0 = \sqrt{m_c^2}. \]  

2) If \(m_0 = 0\) and \(m_c = i\mu\) (tachyon), then in this case the transition operator to the Dirac equation is no longer unitary and the physical equivalence between equations (4) and (7) no longer exists.

3) When \(m_0 \neq 0\) and \(m_c = i\mu\) for the effective mass, we get

\[ m = \sqrt{m_0^2 - \mu^2}, \]  

and a particle can be a tachyon \((v > c)\) with \(m_0 < \mu\), a luxon \((v = c)\) with \(m_0 = \mu\) and a bradion \((v < c)\) with \(m_0 > \mu\).

4) When \(m_c = im_0\), we obtain from (7) the equation

\[ \left[ \gamma^\mu \frac{\partial}{\partial x^\mu} + m_0(1 - \gamma_5) \right] \psi = 0. \]  

This equation is \(\gamma_5\) - invariant for right-handed neutrinos with an effective zero mass

\[ m = \sqrt{m_0^2 - \mu^2} = 0. \]  

The number \(m_0\) has the meaning of the eigenvalue of the operator of the projection of the spin on the left chiral states (with the right helicity), since

\[ m_c(1 - \gamma_5)\psi_L = 0, \quad m_c(1 + \gamma_5)\psi_R = m_c(1 - \gamma_5)\psi. \]  

Consider the problem of relativistic quantum equations from a more general point of view. It is known [10] that by extracting the square root from the Klein-Gordon operator \((\sqrt{\Box} + m^2)\), one can obtain infinitely many equations of the Dirac type, but with different matrices:

\[ \left( \gamma^\mu \frac{\partial}{\partial x^\mu} + m \right) \psi = 0, \]  

\[ \left( \beta^\mu \frac{\partial}{\partial x^\mu} + m \right) \psi = 0, \]  

\[ \left( u^\mu \frac{\partial}{\partial x^\mu} + m \right) \varphi = 0. \]  

Here the matrices \(\gamma^\mu, \beta^\mu, u^\mu\) satisfy the Dirac-Clifford algebra

\[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\delta^{\mu\nu}, \]  

Petiau-Duffin-Kemmer algebra

\[ \beta^\mu \beta^\nu \beta^\alpha + \beta^\alpha \beta^\nu \beta^\mu = \delta^{\mu\nu} \beta^\alpha + \delta^{\alpha\nu} \beta^\mu \]  

and abelian algebra

\[ u^\mu u^\nu - u^\nu u^\mu = 0. \]  

In the last equation, \(u^\mu\) is the 4-particle velocity vector.
The relativistic covariance of equations (19), (20), (21) is either obvious (equation (21)), or simply proved. We emphasize that there are infinitely many equations of each type (19), (20), (21), if you do not limit the rank of the Lorentz group representations to which the wave functions \( \psi, \varphi \) belong, but for practical purposes most often (but not always \([11], [12]\)) equations only with matrices of lowest rank are used.

But even algebras (22), (23), (24) do not exhaust all possibilities. Consider this on the example of Maxwell’s equation.

Since the time of Majorana\([13]\), dozens of formulations of these equations have been known in the form of (19) and (20), including when mapping the field strengths \( \vec{E} \) and \( \vec{H} \) (electromagnetic field tensor \( F_{\mu\nu} \)) to spaces of spinors and quaternions.

We introduce real non-Hermitian matrices \( \vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) \) of rank 3, which are the generators of the group of three-dimensional rotations \( \text{SO}(3) \)\([14]\)

\[
\begin{align*}
\sigma_3 &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad &\sigma_2 &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad &\sigma_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},
\end{align*}
\]

(25)
satisfying commutation relations

\[
\sigma_1 \sigma_2 - \sigma_2 \sigma_1 = \sigma_3. \tag{26}
\]

Using the obvious identity \((\vec{\sigma} \cdot \vec{\nabla})\vec{a} = \text{rot} \ \vec{a}\) Maxwell’s equation in vacuum we write in the form:

\[
\begin{align*}
\frac{\partial \vec{E}}{\partial t} - (\vec{\sigma} \cdot \vec{\nabla})\vec{H} &= 0, \tag{27. a} \\
\frac{\partial \vec{H}}{\partial t} - (\vec{\sigma} \cdot \vec{\nabla})\vec{E} &= 0. \tag{27. b}
\end{align*}
\]

Multiplying equation (27.b) by “i” and, adding two equations, we get:

\[
\vec{F} = \vec{E} + i\vec{H}, \quad \left[ \frac{\partial}{\partial t} + i(\vec{\sigma} \cdot \vec{\nabla}) \right] \vec{F} = 0. \tag{28}
\]

Here we have introduced, following Majorana\([13]\), a complex vector \( \vec{F} \), satisfying the Cartan isotropy condition:

\[
|\vec{F}|^2 = \vec{F}^* \cdot \vec{F} = \vec{E}^2 + \vec{H}^2 \neq 0, \quad \vec{F}^2 = \vec{F} \cdot \vec{F} = \vec{E}^2 - \vec{H}^2 + 2i(\vec{E} \cdot \vec{H}) = 0. \tag{29}
\]

Let us check whether the quadratic equation satisfies the Dalamber wave equation, as is the case with the Dirac equation for each of the 4 complex (or 8 real) components of the wave function separately. To do this, multiply (28) from the left by the adjoint operator in square brackets:

\[
\left[ \frac{\partial}{\partial t} - i(\vec{\sigma} \cdot \vec{\nabla}) \right] \left[ \frac{\partial}{\partial t} + i(\vec{\sigma} \cdot \vec{\nabla}) \right] \vec{F} = 0, \tag{30}
\]

\[
\left[ \frac{\partial^2}{\partial t^2} + (\vec{\sigma} \cdot \vec{\nabla})(\vec{\sigma} \cdot \vec{\nabla}) \right] \vec{F} = 0. \tag{31}
\]

Considering the relation \((\vec{\sigma} \cdot \vec{\nabla})(\vec{\sigma} \cdot \vec{\nabla}) \vec{F} = \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \Delta \vec{F}\), which in conventional notation is equivalent to the formula \( \text{rot rot} \ \vec{F} = \text{grad div} \ \vec{F} - \Delta \vec{F} \), with the additional condition \( \text{div} \ \vec{F} = 0 \), we obtain the wave equation

\[
\left( \Delta - \frac{\partial^2}{\partial t^2} \right) \vec{F} = 0. \tag{32}
\]
Separating the real and imaginary parts in equation (32), we obtain 6 scalar wave equations for the fields \( \vec{E} \) and \( \vec{H} \):

\[
\varepsilon \vec{H} = 0, \quad \varepsilon \vec{E} = 0. \tag{33}
\]

Thus, Maxwell's equations (27.a) and (27.b) can be written in the Dirac form (19), but using the \textbf{3-row real non-Hermitian} matrices \( \vec{\sigma} \) in (25), which are completely different from the Majorana matrices. Writing Maxwell's equations in the form (28) has certain advantages in the transition to quantum theory, since the complex vector \( \vec{F} = \vec{E} + i\vec{H} \) can be considered as a \textbf{wave function}, since the quantity \( \vec{F}^* \cdot \vec{F} = \vec{E}^2 + \vec{H}^2 \) is the energy density, as

\[
\int \vec{F}^* \cdot \vec{F} \, dV = \int (\vec{E}^2 + \vec{H}^2) \, dV = \mathcal{E}. \tag{34}
\]

After normalization to the total energy \( \mathcal{E} \) (dividing both parts of expression (34) by the constant \( \mathcal{E} \)), the expression under the integral sign can be interpreted as a \textbf{probability density}:

\[
\frac{1}{\sqrt{\mathcal{E}}} \int \psi^* \psi \, dV = 1. \tag{35}
\]

Interestingly, the introduced non-Hermitian matrices \( \vec{\sigma} \) do not satisfy the commutation relations (22), (23), (24).

If we add the complex 3-vector \( \vec{F} = \vec{E} + i\vec{H} \) to the complex 4-vector

\[
\psi = \left\{ \begin{array}{c} \vec{H} + i\vec{E} \\ i(\mathcal{H} + i\mathcal{E}) \end{array} \right\}, \tag{36}
\]

where the additional scalar \( \mathcal{E} \) and pseudoscalar \( \mathcal{H} \) fields are introduced (invariant under the Lorentz transformations), then we arrive at a generalized symmetric system of Maxwell equations with electric and magnetic charges, allowing for the existence of longitudinal electric and magnetic fields in \textbf{vacuum} [15]. In this case, \( \alpha \) - representation of \( 4 \times 4 \) matrices were used, satisfying the algebra (22), but not the \( \gamma \) - matrices by virtue of the additional relation \( \alpha^i \alpha^j = i\alpha^k \). In fact, these are the generators of the rank 4 representation of the SU(2) group, as the matrices \( \rho_i, \sigma^k \), used by Dirac, also satisfying relations (22) and the additional conditions \( \rho^i \rho^j = i\rho^k, \sigma^i \sigma^j = i\sigma^k, (i, j, k = 1, 2, 3; \text{ cyclic permutation of indexes}) \).

3. Results and discussion

The choice of the isotropic vector \( \vec{F} = \vec{E} + i\vec{H} \) as a dynamic variable turns out to be very successful, since it allows describing not only particles with integer spin (photons), but also particles with half-integer spin (for example, neutrinos).

This would seem to contradict the generally accepted opinion that the tensor representations of the Lorentz group (in this particular case, \( F_{\mu\nu} \)) can describe only particles with integer spin. But it turns out that it is not.

Let us demonstrate this using the Cartan method [16], which uses the mapping of 2-component 3-dimensional complex spinors

\[
\xi = \left( \begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right) \in \mathbb{C}^2 \tag{37}
\]

on the space of 3-component 3-dimensional complex isotropic vectors:

\[
\vec{F} = \vec{E} + i\vec{H} \in \mathbb{C}^3. \tag{38}
\]

We emphasize once again that the isotropy condition

\[
\vec{F}^2 = \vec{E}^2 - 2i\vec{E} \cdot \vec{H} + \vec{H}^2 = 0 \tag{39}
\]
means the presence of two scalar non-linear links:
\[ \vec{E}^2 - \vec{H}^2 = 0, \quad \vec{E} \cdot \vec{H} = 0. \]  
(40)

Herewith
\[ \vec{F}^2 \neq |\vec{F}|^2 = \vec{F}^* \cdot \vec{F} = \vec{E}^2 + \vec{H}^2 \neq 0. \]  
(41)

Using mapping [16]
\[ b^0(\xi, \tau) = - (\xi_1 \tau_2 - \xi_2 \tau_1), \]
\[ \tilde{b}(\xi, \tau) = \begin{bmatrix} \xi_1 \tau_1 - \xi_2 \tau_2 \\ i(\xi_1 \tau_1 + \xi_2 \tau_2) \\ -(\xi_1 \tau_2 + \xi_2 \tau_1) \end{bmatrix} \]  
(42)

in the particular case of a single spinor (\( \xi \equiv \tau \)), assuming
\[ b^0(\xi, \xi) = 0, \quad \tilde{b}(\xi, \xi) = -i(\vec{E} + i\vec{H}), \]  
the equation for neutrinos in the spinor form
\[ p^0 \xi = (\dot{\phi} \cdot \vec{\sigma}) \xi \]  
(43)
takes the form of Maxwell-like equations
\[ \text{rot}\vec{E} + \frac{\partial \vec{H}}{\partial t} = \nu_k \vec{v} \vec{H}_k, \]  
(44)
\[ \text{rot}\vec{H} - \frac{\partial \vec{E}}{\partial t} = -\nu_k \vec{v} \vec{E}_k. \]  
(45)

Here, the unit vector \( \vec{v} \) is directed along the rate of energy transfer and is determined by the formula:
\[ \vec{v} = \frac{j}{j_0} = \frac{\vec{E} \times \vec{H}}{\vec{E}^2}. \]  
(46)

Similarly, one can obtain a system of Maxwell-like equations for electrons [17]. In this last case, instead of two already, it will contain 4 equations of the type (44-45) for two waves (\( \vec{E}, \vec{H} \)) and (\( \vec{E}^1, \vec{H}^1 \)), connected by a mass term.

Here are the main properties of the Cartan map:
1. The Cartan map is a non-linear transformation of dynamic variables (from spinors to isotropic vectors)
\[ \psi \leftrightarrow \vec{E} + i\vec{H}, \]  
(47)
similar to nonlinear canonical transformations in classical mechanics.
2. The spinor equation for neutrinos (43) is the Lagrange equation of the second kind without constraints (a total of 4 independent scalar equations).
3. The equations for the neutrino in the Maxwell-like form (44-45) are the Lagrange equations of the first kind with constraints (the number of independent variables does not change: 6-2 constraints = 4).
4. Despite the seemingly noncovariant form of equations (44-45), it can be proved that they satisfy the principle of relativity, like Maxwell’s differential equations written in three-dimensional form.
5. The Cartan map preserves the eigenvalues of the operators of physical quantities.
6. It commutes with the relativistic transformations of the Lorentz group.
7. Despite the choice of the tensor representation of the Lorentz group as the dynamic variables for the neutrino from the equations (42-43), after the second quantization follows a half-integer spin \( \frac{1}{2} \).
8. The isotropic complex vector \( \vec{F} \) plays the role of the wave function.
9. The use of isotropic vectors to calculate the probabilities and cross sections of various processes [18] shows their high efficiency, especially when taking into account the polarizations of particles.

10. Solutions of Maxwell-like nonlinear equations for neutrinos satisfy the Maxwell system of equations (but not vice versa), which makes it possible to treat the neutrino field as a kind of electromagnetic field with half-integer spin quanta \( s = \frac{1}{2} \).

4. Conclusion

Summarizing, we can say that the enormous arbitrariness in the choice of dynamic variables for describing relativistic quantum systems of different spin (and, therefore, a huge amount of interpretation possibilities) allows us to hope that in the end optimal values will be found that will clarify their physical meaning in the most simple way. It may finally be possible to clarify the physical essence of the de Broglie wave, although “the explanation of this meaning will be a difficult task for the extended theory of electromagnetism” [19].

5. References

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