Invited Comment

A wave equation interpolating between classical and quantum mechanics

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Abstract
We derive a ‘master’ wave equation for a family of complex-valued waves $\Phi \equiv R \exp[iS^{(cl)}/\hbar]$ whose phase dynamics is dictated by the Hamilton–Jacobi equation for the classical action $S^{(cl)}$. For a special choice of the dynamics of the amplitude $R$ which eliminates all remnants of classical mechanics associated with $S^{(cl)}$ our wave equation reduces to the Schrödinger equation. In this case the amplitude satisfies a Schrödinger equation analogous to that of a charged particle in an electromagnetic field where the roles of the scalar and the vector potentials are played by the classical energy and the momentum, respectively. In general this amplitude is complex and thereby creates in addition to the classical phase $S^{(cl)}/\hbar$ a quantum phase. Classical statistical mechanics, as described by a classical matter wave, follows from our wave equation when we choose the dynamics of the amplitude such that it remains real for all times. Our analysis shows that classical and quantum matter waves are distinguished by two different choices of the dynamics of their amplitudes rather than two values of Planck’s constant.

Keywords: Schrödinger equation, Hamilton–Jacobi equation, classical-quantum border

1. Introduction

The emergence of the macroscopic world from the microscopic one is a central question [1] in quantum theory. In this transition the phenomenon of decoherence which leads to a rapid decay of macroscopic superpositions [2] plays a crucial role. In the present paper we pursue this path in the opposite direction and ask how to reach quantum mechanics starting from classical mechanics. We identify ‘more freedom in phase’ as the essential ingredient of this journey from the classical to the quantum world.

1.1. From Hamilton–Jacobi to ‘master’ wave equation

Quantum mechanics rests [3] on three principles: (i) the motion of a particle is described by interfering waves, the phases of which are governed by the action. (ii) In order to ensure Born’s probability interpretation [4] only waves which can be normalized are permissible. (iii) There is a democracy
of all space-time trajectories rather than a single one, leading to the propagator as expressed by the Feynman path integral formulation [5] anticipated [6] already by Gregor Wentzel in 1924. Since each trajectory contributes a phase the propagator is a result of many interfering waves only one of which gives the classical motion. It is this increased freedom in the phase of the quantum wave compared to its classical counterpart which we illuminate in this article using a ‘master’ wave equation for a complex-valued wave whose phase is governed in its dynamics by the Hamilton–Jacobi equation [7].

The Hamilton–Jacobi equation [7] plays an important role in building a bridge [8] between classical mechanics and quantum theory [3]. Three examples may suffice to illustrate this statement: (i) Erwin Schrödinger [9] was guided by it in establishing his wave equation. (ii) It has not only been instrumental in Wentzel’s approach [10] of obtaining approximate solutions of the time-independent Schrödinger equation, but has also been used [11] to improve this approximation, and (iii) Erwin Madelung and David Bohm [12] have shown that the Schrödinger equation implies the Hamilton–Jacobi equation with an additional potential frequently referred to as the quantum potential.

In a recent article [13] we have demonstrated that the Schrödinger equation follows from a mathematical identity of complex vector analysis when we require the linearity of the wave equation. The approach pursued in the present article is different [14] since we start from the fact that the wave \( \Phi \equiv R \exp[iS^{(cl)}(\hat{r}, t)/\hbar] \) composed of an amplitude, \( R = R(\hat{r}, t) \) and the classical action \( S^{(cl)} = S^{(cl)}(\hat{r}, t) \) determined in its dynamics by the Hamilton–Jacobi equation [7], obeys a wave equation of the form of a Schrödinger equation with additional terms. They are constructed from derivatives in time and space of \( R \) as well as \( S^{(cl)} \).

1.2. The role of the amplitude

When we now specify the dynamics of \( R \) a specific wave equation arises. Since in our approach \( R \) is arbitrary we obtain a set of wave equations that includes the linear Schrödinger equation of quantum mechanics as well as the non-linear one describing [15] classical statistical mechanics. For this reason we refer to our wave equation for arbitrary \( R \) as ‘master’ wave equation.

1.2.1. Linear Schrödinger equation: quantum mechanics. It is interesting to note that the amplitude \( R^{(q)} \) of the wave which gives rise to the emergence of the Schrödinger equation, follows again from a Schrödinger equation which is reminiscent of that of a particle in an electromagnetic field represented by a scalar and a vector potential. However, in the present discussion these potentials are in terms of derivatives of the classical action and therefore carry the memory of classical mechanics.

Since the dynamics of \( R^{(q)} \) is dictated by a Schrödinger equation \( R^{(q)} \) cannot remain real but must assume complex values. As a consequence, we obtain a quantum phase in addition to the classical one determined by the action \( S^{(cl)} \).

It is instructive to compare and contrast our ‘derivation’ of the Schrödinger equation to the one proposed in [16]. There momentum fluctuations have been added to the Hamilton–Jacobi equation which when subjected to the Heisenberg uncertainty relation lead to the Schrödinger equation.

The connection between [16] and our work comes to light when we recall that the momentum is determined by the gradient of the phase. Hence, additional contributions to the momentum imply additional phases. Indeed, our approach eliminates the remnants of classical physics by allowing the phase of the wave function to go beyond the one given by the classical action, that is by allowing \( R^{(q)} \) to assume complex values. Since this quantum phase is determined by a Schrödinger equation, the uncertainty principle is satisfied automatically.

1.2.2. Non-linear Schrödinger equation: classical statistical mechanics. In order to make the transition from the ‘master’ wave equation to the linear Schrödinger equation we have to allow the amplitude \( R \) to assume complex values. However, when we restrict \( R \) to real values and still eliminate the remnants of classical mechanics represented by \( S^{(cl)} \) and its derivatives we arrive at a non-linear Schrödinger equation [15] which provides us with a wave description of classical statistical mechanics.

This unusual representation is an effective way of combining two real-valued equations such as the Hamilton–Jacobi equation for the dynamics of the phase and the continuity equation for the square of the amplitude \( R^{(cl)} \) into a single complex-valued one. Although the underlying equations are independent of Planck’s constant the non-linear Schrödinger equation contains this signature of quantum mechanics. This on first sight surprising feature is a consequence of the condition that the phase of the wave which is governed by the classical action must be dimensionless. We emphasize that despite the appearance of Planck’s constant, the non-linear Schrödinger equation represents only classical physics.

1.2.3. More phase versus symmetric coupling. The two extreme cases of the linear Schrödinger equation representing quantum mechanics and the non-linear Schrödinger equation describing classical statistical mechanics suggest that it is the dynamics of the amplitude that determines the quantumness of the wave. This observation is in accordance with [13] where we have shown that these two theories differ in their couplings between the phase and the amplitude of the respective waves.

Indeed, in classical statistical mechanics the dynamics of the amplitude is determined by the phase, but the dynamics of the phase is independent of the amplitude. In contrast, quantum dynamics acts symmetrically. Here the phase governs the amplitude and vice versa.

The connection between the present approach and [13] comes to light when we recall that there we have not specified the dynamics of the phase. However, throughout this article we confine ourselves to the one given by the Hamilton–Jacobi equation. In this framework the difference between classical and quantum dynamics contained in [13] in different couplings manifests itself in the fact that the amplitude takes on complex values and thereby creates a phase in addition to
the classical action. This feature demonstrates that the quantum world allows more freedom in phase.

1.3. Linearity from wave functions in phase space

It is remarkable that this distinct difference between the time evolutions of quantum mechanics and classical statistical mechanics, that is a linear versus a non-linear equation of motion, only manifests itself when we describe the system of interest by a wave function, that is a probability amplitude. A description based on probabilities rather than probability amplitudes leads to a linear time evolution in both theories. Indeed, the von Neumann equation for the density operator of a quantum system emerges [3] from the corresponding Liouville equation of the classical phase space density by replacing appropriately the Poisson brackets by commutators. Since the Liouville equation is linear, the same must hold true for the von Neumann equation.

The similarity of the two theories is even more apparent in the formulation of classical statistical mechanics by Bernard Koopman [17] and John von Neumann [18]. They interpret the classical phase space density as a wave function which not only depends on position but on position and momentum and enjoys a time evolution governed by the linear Liouville equation. Together with an appropriate definition of a scalar product between two such wave functions both develop a Hilbert space description of classical statistical mechanics. Obviously in this framework, classical mechanics and its quantum counterpart only differ in the fact that position and momentum represent either commuting, or non-commuting conjugate variables [19].

Since in classical mechanics the probability density is real and positive, the wave function in the Koopman–von Neumann theory satisfies the same properties. In the same spirit it is possible [20] to consider the Wigner function [21] to be a wave function in quantum phase space rather than position space. We note that this interpretation of the Wigner function being a probability amplitude has the additional benefit that it is natural for the Wigner function to assume negative values.

Unfortunately, a more detailed discussion of these intriguing similarities to and differences between our approach based on wave functions in position space rather than phase space and the Koopman–von Neumann Schrödinger equation has to be postponed to a future publication.

1.4. Outline

Our article is organized as follows. In section 2 we use the Hamilton–Jacobi equation for the classical action \( S^{(c)} \) together with three mathematical identities of complex analysis to obtain a ‘master’ wave equation for the wave \( \Phi \equiv R e^{i S^{(c)}/\hbar} \) constructed out of the amplitude \( R \) and the phase factor \( \exp(i S^{(c)}/\hbar) \). Since at this point \( R \) is not specified yet we deal with a wave equation for a whole family of waves. Indeed, we show in section 3 that for specific choices of \( R \) we either arrive at the linear Schrödinger equation governing the microscopic world, or at a non-linear wave equation summarizing classical statistical mechanics. In section 4 we then turn to a discussion of the dynamics of the quantum amplitude \( R^{(q)} \). In particular, we show that \( R^{(q)} \) has to assume complex values and thereby accumulates, apart from the classical phase \( S^{(cl)}/\hbar \), a quantum phase. The latter leads to an additional contribution to the classical current. Moreover, we demonstrate that the classical dynamics dictates through the classical momentum and energy the dynamics of the quantum amplitude. We conclude in section 5 by providing a brief summary of our results and an outlook.

2. A master wave equation

In classical mechanics we are interested in determining the trajectory \( r = r(t) \) of a particle of mass \( m \) moving in a position- and time-dependent potential \( V = V(r, t) \). Many ways of treating this problem offer themselves ranging from Newton’s equation to Hamilton’s equations [7]. Most relevant for the present discussion is the formulation of classical mechanics as a field theory expressed by the action \( S^{(cl)} = S^{(cl)}(r, t) \) which satisfies the Hamilton–Jacobi equation [7]

\[
-\frac{\partial S^{(cl)}}{\partial t} = \frac{1}{2m} \left( \nabla S^{(cl)} \right)^2 + V. \tag{1}
\]

Indeed, we find the trajectory of the particle from the action \( S^{(cl)} \) with the help of the relation \( m\, dr/dt \equiv p = \nabla S^{(cl)} \).

Since \( S^{(cl)} \) represents the phase of a wave, it is suggestive to define the complex-valued wave \( \Phi = \Phi(r, t) \) as the product

\[
\Phi \equiv R e^{i S^{(cl)}/\hbar} \tag{2}
\]

of an amplitude \( R = R(r, t) \) and a phase determined by the classical real-valued action \( S^{(cl)} \).

At this point we have also introduced a constant \( \hbar \) with the dimension of action in order to ensure that the phase of the wave is dimensionless. Needless, to say there is no justification to identify this constant with the reduced Planck constant \( \hbar \). Indeed, we could have chosen any quantity with the units of an action to make the phase dimensionless.

Next we recall that an appropriately differentiable complex-valued function

\[
Z \equiv A e^{i \alpha} \nonumber
\]

represented by the amplitude \( A = A(r, t) \) and the phase \( \alpha \equiv \alpha(r, t) \) satisfies the identities

\[
\frac{\partial}{\partial t} Z = \frac{\partial A}{\partial r} e^{i \alpha} + i \frac{\partial \alpha}{\partial r} Z \tag{3}
\]

and

\[
\nabla Z = \nabla A e^{i \alpha} + i \nabla \alpha A e^{i \alpha} \tag{4}
\]

together with

\[
\Delta Z = \Delta A e^{i \alpha} + 2i \nabla A \cdot \nabla \alpha e^{i \alpha} - (\nabla \alpha)^2 Z + i \Delta \alpha Z. \tag{5}
\]

We emphasize that in these relations \( A \) does not have to be identical to \( |Z| \).
We now use equations (3) and (5) for $A \equiv R$ and $\alpha \equiv S^{(c)}/\hbar$ as well as the Hamilton–Jacobi equation, equation (1), to connect the derivatives $\partial \Phi/\partial t$ and $\Delta \Phi$. For this purpose we express $(\nabla S^{(c)})^2$ appearing for our choice of $\alpha$ in equation (5) by the appropriate combination of the potential $V$ and $\partial S^{(c)}/\partial t$, which with the help of equation (3) leads us to

$$\ih \frac{\partial}{\partial t} \Phi = \hat{H} \Phi + C. \quad (6)$$

Here we have introduced the abbreviation

$$\hat{H} \equiv -\frac{\hbar^2}{2m} \Delta + V, \quad (7)$$

where

$$C \equiv C[R, S^{(c)}] \equiv \left[ \frac{\hbar^2}{2m} \Delta R + \ih \int \nabla S^{(c)} \right] \exp i S^{(c)}/\hbar \quad (8)$$

and

$$\mathcal{F}[R, S^{(c)}] \equiv \frac{\partial R}{\partial t} + \nabla R \cdot \left( \frac{\nabla S^{(c)}}{m} \right) + \frac{1}{2} m \nabla \cdot \left( \frac{\nabla S^{(c)}}{m} \right) \quad (9)$$

are functionals of $R$ and $S^{(c)}$ as indicated by square brackets.

Hence, we have obtained a wave equation for $\Phi$ which, due to the appearance of the first order derivative with respect to time and the familiar Hamiltonian $\hat{H}$, equation (7), is reminiscent of the Schrödinger equation. However, equation (6) also displays an additional term $C$ which according to equation (8) contains three ingredients: (i) the Laplacian of $R$, (ii) the functional $\mathcal{F}$ with derivatives of $R$ and $S^{(c)}$, and (iii), the phase factor $\exp(x S^{(c)}/\hbar)$. It is the quantity $C$ which allows us to interpolate between classical and quantum mechanics.

Indeed, so far we have only specified the dynamics of $S^{(c)}$ as given by the classical Hamilton–Jacobi equation, equation (1). Since the time evolution of the amplitude $R$ is still open, equation (6) represents a wave equation for a family of waves, that is a ‘master’ wave equation.

### 3. Special choices for the amplitude

In this section we discuss two examples of $R$ giving rise to two fundamentally different wave equations: the first one is the linear Schrödinger equation governing the quantum world, and the second one the non-linear Schrödinger equation of classical statistical mechanics. It is in this sense that the quantumness of a wave is determined by the dynamics of its amplitude rather than Planck’s constant. In both cases we first present our choices of $R$ together with the resulting wave equations and then motivate them.

#### 3.1. Quantum matter wave

The form of the master wave equation, equation (6), implies that it would be prudent to connect $R$ with $S^{(c)}$ to simplify equation (6). Many choices for such a connection offer themselves. However, one stands out in the sense that it erases all remnants of classical mechanics contained in equation (6), that is it eliminates all terms involving the classical action as well as its derivatives and leads to a linear wave equation.

Indeed, when we define the amplitude $R^{(q)}$ by the condition

$$\frac{\hbar^2}{2m} \Delta R^{(q)} + \ih \int \left( R^{(q)}, S^{(c)} \right) \equiv 0, \quad (10)$$

the additional term $C$, given by equation (8) vanishes and the corresponding wave

$$\psi^{(q)} \equiv \Phi \left[ R^{(q)} \right] \equiv R^{(q)} e^{i S^{(c)}/\hbar} \quad (11)$$

resulting from this choice satisfies the familiar Schrödinger equation

$$\ih \frac{\partial}{\partial t} \psi^{(q)} = \hat{H} \psi^{(q)}, \quad (12)$$

which contains $R^{(q)}$ and $S^{(c)}$ only through the combination, equation (11), defining the quantum matter wave $\psi^{(q)}$. Since the condition, equation (10), allows us to make the transition from the master wave equation, equation (6), to the Schrödinger equation, equation (12), governing the quantum world, we refer to equation (10) as the quantum condition.

We emphasize that the requirement of the elimination of the classical action and its derivatives is not sufficient to obtain the Schrödinger equation. Indeed, the condition

$$C \equiv \hat{F} \psi \quad (13)$$

where $\hat{F}$ is a non-linear integro-differential operator acting on $\psi$, together with equation (6) leads us to the wave equation

$$\ih \frac{\partial}{\partial t} \psi = \hat{H} \psi + \hat{F} \psi \quad (14)$$

that solely depends on $\psi$.

However, since $\hat{F}$ is non-linear in $\psi$, equation (14) represents a non-linear wave equation. In contrast, the quantum condition equation (10), which by virtue of the definition, equation (8), of $C$ implies $C \equiv \hat{F} \equiv 0$, ensures not only the elimination of the classical action and its space derivative from equation (6), but also the linearity of the wave equation.

In [22] we study this aspect in more detail and show that a ‘derivation’ of the Schrödinger equation starting from equation (6) requires three ingredients: (i) the elimination of classical concepts, (ii) the linearity of the wave equation, and (iii) the continuity equation for the probability density. The combination of these three ideas enforces a non-vanishing form of $\hat{F}$ which introduces in a natural way gauge transformations and thus the principle of minimal coupling [23]. The latter will be discussed from a different point of view in the next section.

#### 3.2. Classical matter wave

Another choice of $R$ in terms of $S^{(c)}$ yields waves which provide us with a description of classical statistical mechanics. In order to bring this fact out most clearly we first note
that the condition
\[ i \left[ R^{(c)} \right] S^{(c)} = 0 \] \tag{15}
reduces the wave equation, equation (6), for the wave
\[ \psi^{(c)} \equiv \Phi \left[ R^{(c)} \right] \equiv R^{(c)} e^{iS^{(c)}/\hbar} \] \tag{16}
to
\[ i \hbar \frac{\partial}{\partial t} \psi^{(c)} = \hat{H} \psi^{(c)} + \frac{\hbar^2}{2m} \frac{\Delta R^{(c)}}{R^{(c)}} \psi^{(c)}. \] \tag{17}
Next we recognize that the equation of motion
\[ \frac{\partial R^{(c)}}{\partial t} + \nabla R^{(c)} \cdot \left( \frac{\nabla S^{(c)}}{m} \right) + \frac{1}{2} \nabla \cdot \left( \nabla S^{(c)} \right) = 0 \] \tag{18}
for the amplitude \( R^{(c)} \) following from the condition equation (15) does not involve the imaginary unit. Since \( S^{(c)} \) is real, \( R^{(c)} \) has to remain real for all times provided it was real at time \( t = 0 \). As a result, we obtain the connection
\[ |\psi^{(c)}| = R^{(c)} \] \tag{19}
between the absolute value of the complex-valued wave \( \psi^{(c)} \) and its amplitude \( R^{(c)} \).

With the help of this identity the wave equation equation (17) reduces to the non-linear Schrödinger equation [15]
\[ i \hbar \frac{\partial}{\partial t} \psi^{(c)} = \hat{H} \psi^{(c)} + \frac{\hbar^2}{2m} \frac{\Delta |\psi^{(c)}|}{|\psi^{(c)}|} \psi^{(c)}, \] \tag{20}
which involves \( \psi^{(c)} \) as well as \( |\psi^{(c)}| \).

This equation contains \( \hbar \) explicitly and one might wonder in which sense \( \psi^{(c)} \) can be considered to be classical. The answer to this question springs from the fact that the Hamilton–Jacobi equation, equation (1), as well as the equation of motion, equation (18), for \( R^{(c)} \) are both independent of \( \hbar \). However, when we combine them into a single complex equation, that is the non-linear Schrödinger equation, equation (20), by defining the wave \( \psi^{(c)} \) according to equation (16), Planck’s constant emerges since for dimensional reasons it must appear in the phase of \( \psi^{(c)} \).

We gain more insight into the condition, equation (15), when we multiply equation (18) by \( 2\frac{\partial R^{(c)}}{\partial t} \) which yields the classical continuity equation
\[ \frac{\partial}{\partial t} \left( R^{(c)} \right)^2 + \nabla \cdot \left( \left( R^{(c)} \right)^2 \frac{\nabla S^{(c)}}{m} \right) = 0 . \] \tag{21}

Indeed,
\[ \left( R^{(c)} \right)^2 = |\psi^{(c)}|^2 \equiv \rho^{(c)} \] and
\[ \frac{\partial}{\partial t} \left( R^{(c)} \right)^2 = \frac{\partial |\psi^{(c)}|^2}{\partial t} \equiv \rho^{(c)} \] which are both real, play the role of a density and a current.

With the choice, equation (18), of the time evolution of the amplitude \( R^{(c)} \) we have arrived at a probabilistic description of classical statistical mechanics governed by the classical Hamilton–Jacobi equation, equation (1), and the classical continuity equation, equation (21). For this reason we refer to equation (15) as the classicality condition.

In this approach we have started from the classicality condition, equation (15), which implies that \( R^{(c)} \) is real and have derived from it the continuity equation, equation (21), for the probability density. However, we can also reverse [13] the logic, that is we begin with the conservation law, equation (21), and obtain from it the classicality condition. However, there is even no need to assume the existence of the continuity equation, equation (21). Indeed, the Hamilton–Jacobi equation already implies [13, 15, 24, 25] this equation provided the amplitude of the classical wave is given by the inverse of the square root of the Van Vleck determinant [24]. Since in this case the amplitude \( R^{(c)} \) is real, we can reverse our steps and reach from the continuity equation, equation (21), the equation of motion, equation (18), for \( R^{(c)} \). Hence, for this choice of \( R^{(c)} \) given by the Van Vleck determinant we obtain [13] the classicality condition, equation (15), directly from the Hamilton–Jacobi equation, equation (1).

### 3.3. Amplitude determines quantumness

The two extreme cases of \( \psi^{(c)} \) and \( \psi^{(q)} \) show that we can make the transition between the classical and the quantum world, that is between classical statistical mechanics as described by the non-linear Schrödinger equation, equation (20), and the linear one, equation (12), by a change of the dynamics of \( R \) rather than Planck’s constant. In this sense the quantumness of the wave \( \Phi \) is determined by the choice of the amplitude \( R \).

### 4. Consequences of quantum condition

In the present section we analyze the consequences of the quantum condition, equation (10), from three different perspectives: (i) from the point of view of the quantum wave \( \psi^{(q)} \) giving rise to an additional phase, (ii) from the continuity equation for the probability density creating an additional contribution to the classical current, and (iii) from the equation of motion for \( R^{(q)} \). The latter leads us to an interpretation in which the fields of classical momentum and energy given by the appropriate derivatives of the classical action serve as potentials for the amplitude \( R^{(q)} \) of the wave and thereby guide it.

#### 4.1. Quantum phase

Whereas the classicality condition, equation (15), is independent of \( \hbar \), the quantum condition, equation (10), involves \( \hbar \) with two different powers: In the first term we have \( \hbar^2 \) but in front of \( \hat{\mathbf{f}} \) we find \( \hbar \). Moreover, in contrast to the classicality condition, equation (15), which is purely real, the imaginary unit \( \hat{\mathbf{i}} \) appears explicitly in equation (10). Both properties have important consequences.

Indeed, in the course of time the quantum amplitude \( R^{(q)} \) has to assume complex values and the quantum wave \( \psi^{(q)} \) must develop a phase \( \beta^{(q)} \) in addition to the one given by the classical action, that contains \( \hbar \). As a result, we find
\[ \psi^{(q)} \equiv |R^{(q)}| e^{i\beta^{(q)}} e^{iS^{(q)}/\hbar} \equiv |R^{(q)}| e^{iS^{(q)}/\hbar} , \] \tag{22}
that is the quantum condition, equation (10), creates the total action

\[ S \equiv \hbar \beta(q) + S^{(cl)} + S^{(q)} \]  

consisting of the sum of the quantum action \( S^{(q)} \equiv \hbar \beta(q) \) and the classical one \( S^{(cl)} \).

Nowhere do we see the difference between classical and quantum dynamics clearer than in a comparison of equations (16) and (22): The amplitude \( \psi(q) \) of a quantum wave \( \psi^{(q)} \) is not identical to \( R(q) \) but to \( |R(q)| \).

4.2. Quantum current

We now analyze the quantum condition, equation (10), in more detail. With the help of the definition, equation (9), we obtain the equation of motion \(^7\)

\[
\text{i} \hbar \frac{\partial R^{(q)}}{\partial t} = -\frac{\hbar^2}{2m} \Delta R^{(q)} + \frac{\hbar}{i} \nabla \cdot \left( \frac{\nabla S^{(cl)}}{m} \right) R^{(q)} 
+ \frac{\hbar}{2} \nabla \cdot \left( \frac{\nabla S^{(cl)}}{m} \right) R^{(q)}
\]

(24)

for the amplitude \( R^{(q)} \) of the quantum wave \( \psi^{(q)} \).

When we identify \( Z \equiv R^{(q)} \equiv |R^{(q)}| \exp(i\beta^{(q)}) \), that is \( \mathcal{A} \equiv |R^{(q)}| \) and \( \alpha \equiv \beta^{(q)} \) we find from the mathematical identities, equations (3)–(5), the representation

\[
0 = R^{(q)} \left\{ i \frac{\hbar}{2 |R^{(q)}|^2} \left[ \frac{\partial}{\partial t} |R^{(q)}|^2 + \nabla \cdot \left( |R^{(q)}|^2 \nabla S \right) \right] 
\right.

\[ - \left[ \frac{\partial S^{(q)}}{\partial t} + \frac{1}{2} \nabla S^{(q)}^2 + \frac{\nabla S^{(cl)}}{m} \cdot \nabla S^{(cl)} 
\right.

\[ + \frac{\hbar}{2m} \frac{\Delta |R^{(q)}|^2}{|R^{(q)}|} \left. \right\}. \]

(25)

Here we have also recalled the definitions, equation (23), of the quantum and the total actions, \( S^{(q)} \) and \( S \), respectively.

Next we multiply the Hamilton–Jacobi equation equation (1) by \( R^{(q)} \) and subtract the resulting expression

\[ R^{(q)} \left[ \frac{\partial S^{(cl)}}{\partial t} + \frac{1}{2m} (\nabla S^{(cl)})^2 + V \right] = 0 \]

(26)

from equation (25) which allows us to complete the square, that is combine the squares of the gradients of the actions \( S^{(q)} \) and \( S^{(cl)} \) with the product of the gradients of the two actions into the square of the gradient of the total action \( S \). Moreover, we obtain the time derivative of \( S \).

When we now divide by \( R^{(q)} \) and take real and imaginary parts we arrive at the quantum Hamilton–Jacobi equation

\[ \frac{\partial S}{\partial t} = \frac{1}{2m} (\nabla S)^2 + V - \frac{\hbar^2}{2m} \frac{\Delta |R^{(q)}|^2}{|R^{(q)}|^2} \]

(27)

\(^7\) Needless to say, we are not aware of a solution of equation (24) for \( R^{(q)} \) for an arbitrary \( S^{(cl)} \). Such an expression would allow us, via the definition equation (11) of \( \psi^{(q)} \) in terms of \( R^{(q)} \) and \( S^{(cl)} \), to solve the Schrödinger equation in the presence of an arbitrary potential \( V = V(r, t) \).

and the quantum continuity equation

\[ \frac{\partial}{\partial t} |R^{(q)}|^2 + \nabla \cdot \left( |R^{(q)}|^2 \nabla S \right) = 0, \]

familiar from the Madelung–Bohm theory \(^{12}\).

Here in contrast to the classical Hamilton–Jacobi equation, equation (1), the dynamics of the total action \( S \) consisting of the sum of the quantum and the classical actions is determined by the amplitude \( R^{(q)} \) of the wave through the quantum potential

\[ V_q = -\frac{\hbar^2 \Delta |R^{(q)}|^2}{2m |R^{(q)}|}. \]

(28)

Moreover, the quantum current \( j^{(q)} \equiv |R^{(q)}| \nabla (\hbar \beta^{(q)} + S^{(cl)})/m \) contains the sum of the gradients of the phase \( \beta^{(q)} \) of the complex-valued amplitude \( R^{(q)} \) and the classical action \( S^{(cl)} \).

4.3. Classical dynamics guides quantum dynamics

More insight into the equation of motion, equation (24), for \( R^{(q)} \), emerges when we add equation (26) to the right-hand side of equation (24) and complete the square, which gives rise to the equation of motion

\[ \text{i} \hbar \frac{\partial R^{(q)}}{\partial t} = \frac{1}{2m} \left( \frac{\hbar}{i} \nabla + \nabla S^{(cl)} \right)^2 R^{(q)} \]

\[ \quad + \left( V + \frac{\partial S^{(cl)}}{\partial t} \right) R^{(q)}. \]

(29)

Hence, the amplitude \( R^{(q)} \) of the quantum wave \( \psi^{(q)} \) satisfies a Schrödinger equation in which the operator \( \hat{p} \equiv -\text{i} \hbar \nabla \) is replaced by \( \hat{p} + \hat{V} S^{(cl)} \) and the potential \( V \) by \( V + \partial S^{(cl)}/\partial t \). These replacements are reminiscent of the minimal substitution \(^{23}\)

\[ \hat{p} \rightarrow \hat{p} - eA \equiv \frac{\hbar}{i} \nabla - eA \]

(30)

and

\[ V \rightarrow V + e\phi \]

(31)

providing the coupling between a particle of charge \( e \) and an electromagnetic field represented by scalar and vector potentials \( \phi \) and \( A \), respectively.

In the present case the role of the vector potential is played by \( \nabla S^{(cl)} \) and that of the scalar potential by \( \partial S^{(cl)}/\partial t \). As a result, \( R^{(q)} \) evolves as if it would be the Schrödinger wave function of a particle moving in a vector and a scalar potential given solely in terms of derivatives of the classical action.

Since \( \nabla S^{(cl)} \) is the classical momentum and \( \partial S^{(cl)}/\partial t \) is the energy we can argue that classical dynamics guides the quantum wave—an interpretation which is reminiscent of, but also opposite to the de Broglie
pilot wave theory [8]. Indeed, in the latter it is the wave which guides\(^8\) the particle and not the other way around, as suggested by equation (29).

We emphasize that this picture emerges from the equation of motion for \(R^{(0)}\). However, the Schrödinger equation for the quantum wave \(\psi^{(0)}\) does not contain terms reminiscent of classical mechanics. Hence, we find two sides of the same coin: the wave equation for \(\psi^{(0)}\), that is the Schrödinger equation, is free of any remnants of classical physics whereas the amplitude of \(\psi^{(0)}\) is guided by it. The definition, equation (22), of \(\psi^{(0)}\) connects the two sides \(^9\).

5. Summary and outlook

We conclude by summarizing our main results. In particular, we compare and contrast the decisive role of the quantum potential, equation (28), in making the resulting equations of motion linear which is central to the discussion of [13], to the analysis of the present article based on the master wave equation, equation (6). Moreover, we provide an outlook by briefly addressing issues such as the derivation of the Schrödinger equation from gauge transformations and the effect of the multi-valuedness of the classical action.

5.1. The role of the quantum potential

We emphasize that the quantum condition, equation (10), eliminates all remnants of classical mechanics in the master wave equation, equation (6), in the sense that neither \(S^{\text{cl}}\) nor its derivatives enter. Moreover, it ensures the linearity of the wave equation. Nevertheless, the Schrödinger-type equation, equation (29), for \(R^{(0)}\) remembers its roots in the classical world.

It is interesting to note that in [13] we have arrived at the linear, or the non-linear Schrödinger equation corresponding to quantum mechanics or classical statistical mechanics using a slightly different argument. In this approach the quantum potential has played a prominent role and we have either included it in the equation of motion, equation (27), for the quantum action \(S\) arriving at the linear Schrödinger equation, or in the wave function creating a non-linear Schrödinger equation, equation (20).

In the present treatment the quantum potential is not as visible, yet equally important. On first sight it is the special choice of the time evolution of the amplitude of the wave governed by the quantum condition, equation (10), which decides the emergence of the non-linear rather than the linear equation. Indeed, we do not recognize in this approach the quantum potential. Nevertheless, it hides in the Laplacian in the quantum condition, equation (10), which transforms into a Schrödinger-type equation for the amplitude \(R^{(0)}\) of the quantum wave. Hence, the quantum potential manifests itself in the kinetic energy operator for the amplitude. On the other hand, the classicality condition, equation (15), creates in the Schrödinger equation the non-linearity very much in the spirit of [13].

5.2. Schrödinger equation from gauge invariance

At this point we must recall that in deriving the wave equation, equation (29), for \(R^{(0)}\) we have twice taken advantage of the classical Hamilton–Jacobi equation, equation (1): the first time when we connected the time derivative of the wave with its space derivative to arrive at the master wave equation, equation (6), and the second time when we added it on the right-hand side of equation (24).

This observation suggests that our derivation might not even rely on the use of the phase of the wave \(\Phi\) being given by the classical action but might allow an arbitrary phase, pointing to phase invariance as the deeper origin of the Schrödinger equation. Unfortunately, this idea of the emergence of the Schrödinger equation from gauge invariance inspired by the derivation [23] of the Weyl equation goes beyond the scope of the present article and will be the subject of a future publication [25].

5.3. Consequences of multi-valuedness of classical action

Likewise, we cannot elaborate on the consequences of the classical action being multi-valued. Indeed, in the most elementary case of a particle in a one-dimensional binding potential we find two solutions for \(S^{\text{cl}}\) corresponding to the motion to the right and to the left and being associated with positive and negative momentum, respectively. This feature is a consequence of the fact that the Hamilton–Jacobi equation, equation (1), involves the square of the gradient of \(S^{\text{cl}}\).

This multi-valuedness of the classical action is crucial for the formation of standing waves, that is of bound states and stands out most clearly in the semiclassical limit à la Wentzel, Kramers and Brillouin [21], where the wave is the product of two contributions: (i) a classical amplitude given by the inverse of the square root of the classical momentum, and (ii) a phase governed by the classical action and expressed in units of Planck’s constant. The latter which is already apparent in the ansatz, equation (2), represents [21] the analogue of the dynamical phase in the discussion of the Berry phase.

However, the classical contribution to the amplitude of the wave emerges from the equation of motion, equation (29), for \(R^{(0)}\), and in particular, from the kinetic energy contribution. However, it not only provides us with the classical part but also contains quantum corrections such as the geometric phase [21], that is the Berry phase. Since the latter is closely related to the Aharonov–Bohm effect we expect a new perspective from the analysis of equation (29). However, this question as well as the derivation of the Bohr–Sommerfeld–Kramers quantization rule from equation (29) go beyond the topic of the present paper.

\(^8\) In this context it is interesting to recall experiments [26] studying the motion of single oil drops due to their associated surface waves. They seem to guide the drops which can even display the familiar interference pattern of Young’s double-slit experiment.

\(^9\) We emphasize that both \(\psi^{(0)}\) and \(R^{(0)}\) must be normalizable. Moreover, according to equation (2) their initial values \(\psi^{(0)}(r, 0)\) and \(R^{(0)}(r, 0)\) are connected by the identity \(\psi^{(0)}(r, 0) = R^{(0)}(r, 0)\exp\{1S^{\text{cl}}(r, 0)/\hbar\}\) to the initial action \(S^{\text{cl}}(r, 0)\).
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References

[1] Zurek W H 1991 Phys. Today 44 36
[2] Deléglise S, Dotsenko I, Sayrin C, Bernu J, Brune M, Raimond J M and Haroche S 2008 Nature 455 510
[3] See, for example Bohm D 1989 Quantum Theory (New York: Dover Publications)
[4] Zurek W H 2009 Nat. Phys. 5 181
[5] Feynman R P 1948 Rev. Mod. Phys. 20 367
[6] Wentzel G 1924 Z. Phys. 22 193
[7] See, for example Goldstein H 1980 Classical Mechanics (Addison-Wesley, Reading)
Rund H 1966 Hamilton–Jacobi Theory in the Calculus of Variations: Its Role in Mathematics and Physics (New York: Van Nostrand)
[8] See, for example Jammer M 1974 The Philosophy of Quantum Mechanics (New York: Wiley)
[9] Schrödinger E 1982 Collected Papers on Wave Mechanics (Providence, RI: AMS Chelsea Publisher)
[10] Wentzel G 1926 Z. Phys. 38 518
[11] Eleuch H, Rostovtsev Y V and Scully M O 2010 EPL 89 50004
[12] See, for example Holland P R 1993 The Quantum Theory of Motion: An Account of the de Broglie–Bohm Causal Interpretation of Quantum Mechanics (Cambridge: Cambridge University Press)
[13] Schleich W P, Greenberger D M, Kobe D H and Scully M O 2013 Proc. Natl. Acad. Sci. USA 110 5374
[14] For a derivation of the Schrödinger equation using quantum field theory, see for example Scully M O 2008 J. Phys. Conf. Series 99 012019
[15] Schiller R 1962 Phys. Rev. 125 1100
[16] Hall M J W and Reginatto M 2002 J. Phys. A 35 3289
[17] Koopman B O 1931 Proc. Natl. Acad. Sci. USA 17 315
[18] Neumann J v 1932 Ann. Math. 33 587
[19] Jaurslin H R and Sagny D 2010 Mathematical Horizons for Quantum Physics ed H Araki, B-G Englert, L-C Kwek and J Suzuki (Singapore: World Scientific)
[20] Bondar D I, Cabrera R, Zhdanov D V and Rabitz H A 2013 Phys. Rev. A 88 052108
[21] Schleich W P 2001 Quantum Optics in Phase Space (Weinheim: Wiley-VCH)
[22] Schleich W P, Greenberger D M, Kobe D H and Scully M O 2014 Phys. Rev. A (submitted)
[23] See for example Weyl H 1929 Z. Phys. 56 330
Weyl H 1931 The Theory of Groups and Quantum Mechanics (New York: E.P. Dutton)
[24] Van Vleck J H 1928 Proc. Natl. Acad. Sci. USA 14 178
[25] Schleich W P, Greenberger D M, Kobe D H and Scully M O Phys. Rep. (to be published)
[26] See for example Couder Y and Fort E 2006 Phys. Rev. Lett. 97 154101
or the review by Cartwright J 2013 Phys. World 26(11) 33