Mean Field Theory of Sandpile Avalanches: 
from the Intermittent to the Continuous Flow Regime.

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We model the dynamics of avalanches in granular assemblies in partly filled rotating cylinders using a mean-field approach. We show that, upon varying the cylinder angular velocity $\omega$, the system undergoes a hysteresis cycle between an intermittent and a continuous flow regimes. In the intermittent flow regime, and approaching the transition, the avalanche duration exhibits critical slowing down with a temporal power-law divergence. Upon adding a white noise term, and close to the transition, the distribution of avalanche durations is also a power-law. The hysteresis, as well as the statistics of avalanche durations, are in good qualitative agreement with recent experiments in partly filled rotating cylinders.
Introduction.— The dynamics of granular materials is not well understood in spite of its widespread scientific and technological interest [1]. Bistability, segregation, arching, hysteresis, instabilities and other properties found in noncohesive granular assemblies make these systems very difficult to analyze. For over two centuries, most studies have focused on the two extreme regimes: the fluidlike continuous flow and the static compact state. The intermediate quasistatic regime, with intermittent transitions between flowing and static behavior has recently generated a flurry of activity including several neat experiments [2, 3] and novel theoretical proposals (see, e.g., Ref. [4]).

Several experiments [2, 3] have studied the dynamics of granular assemblies in partly filled rotating drums. In particular, Rajchenbach [4], monitored the following quantities: (i) the statistics of avalanches as a function of the rotating speed of the cylinder, ω, (ii) the transition from the discrete to the steady regime, including experimental evidence of hysteresis as a function of ω, and (iii) the dependence of the current of particles, j, with the angle θ between the top layer and the horizontal. It is the purpose of this work to study these issues by using a simple mean-field model with two dynamical variables. The recent studies on this subject raise many interesting questions in granular transport, like hysteresis in the angular velocity, ω, the critical slowing down. At a transition from the discrete flow regime to a continuous flow over time but the absence of length and time scales. Our model can produce distributions in agreement with recent experiments [2, 3]. Furthermore, when ω → ω⁺, we predict that the system exhibits a power-law critical slowing down.

Dissipation in Granular Media.— In granular flow, the friction force depends on v² for large values of the velocity. The models of Bagendt [6] and Jenkins and Savage [7] correctly describe this limit. Jaeger et al [8] analyzed how the stress varies as v approaches zero, obtaining the following form for the total friction force: \( \eta v^2/(2\lambda + \alpha m g \cos \theta/\kappa) \), where \( \kappa = 1 + \alpha_1 v^2/gd \cos \theta \); which in dimensionless form can be written as: \( F = \beta v^2 + \alpha \cos \theta/\kappa \), where now \( \kappa = 1 + \alpha_1 v^2/\cos \theta \). Let us consider a stationary \( (v = 0) \) granular assembly with a horizontal \( (\theta = 0) \) top surface contained in a partly filled cylinder. If we slowly rotate the cylinder at an angular velocity \( \dot{\theta} = v \), the average angle of the granular assembly surface, \( \theta \) grows until it reaches the maximum angle of stability, \( \theta_m \), where the system suddenly produces avalanches. During this sudden transport of mass, the average velocity of the sand grains, \( v \), becomes nonzero and \( \theta \) decreases until it reaches the angle of repose, \( \theta_r \), where the system relaxes \( (v=0) \). At this point the cycle repeats itself.

In order to model this dynamics, it is useful to know the time evolution of the mean angle \( \theta \). For this purpose, let us consider the two extreme heights (shown in fig. 2(b)) of granular vertical columns. Since the column heights are proportional to the mass of the columns, the rate of mass transport is proportional to the height difference: \( (0 < x \leq L) \),

\[
\frac{\partial h_1}{\partial t} = -\frac{v}{L} (h_1 - h_2) \quad \frac{\partial h_2}{\partial t} = +\frac{v}{L} (h_1 - h_2) .
\]

The proportionality constant is the inverse of the propagation time. The sum of these equations give the mass conservation condition \( \partial_t (h_1 + h_2) = 0 \), which is valid in a closed rotating drum, while subtracting them gives

\[
\dot{\theta} = -\frac{2v}{L} \tan \theta
\]

since \( h_1 - h_2 = L \sin \theta \).

In this work, we concentrate our efforts in this \( \partial_t \theta = 0 \) case. Elsewhere, we will consider the case with local
steady state (\(\dot{\theta}, \eta\)lates small fluctuations in large values of \(\omega\). We thus model the dynamics of avalanches with the following equations of motion:

\[
\dot{v} = \sin\theta - \beta v^2 - \frac{\alpha \cos\theta}{1 + \alpha_1 v^2 / \cos\theta} = \sin\theta - F(\theta, v) \tag{3}
\]

\[
\dot{\theta} = -\rho v \tan\theta + \omega + \eta(t) \tag{4}
\]

where \(\rho = 2\sqrt{g/L}\), eq.(3) is Newton’s equation of motion, and the very small Gaussian noise term \(\eta(t)\) simulates small fluctuations in \(\omega\) and satisfies: \(<\eta(t) > 0\) and \(<\eta(t)\eta(t') >= D_0 \delta(t-t')\), with \(D_0 \approx \omega\).

In the continuous flow regime, the equation for the steady state (\(\dot{v} = 0, \dot{\theta} = 0\)) is

\[
\sin\theta = \beta \left(\frac{\omega}{\rho \tan\theta}\right)^2 + \alpha \cos^2\theta \left(\cos\theta + \alpha_1 \left(\frac{\omega}{\rho \tan\theta}\right)^2\right)^{-1} \tag{5}
\]

This solution corresponds to any point in the branch \(DC'\) in fig. 2(a) with positive slope in \(F(v)\). Let us denote by \(v_r\) \((v_m)\) the velocity corresponding to \(\theta_r\) \((\theta_m)\) in eq. (3) when \(\dot{v} = 0\). When \(\omega \ll \rho v_r \tan\theta_r\), the system has intermittent avalanches, and in each one of them eq.(4) predicts that \(\theta\) relaxes fast to \(\theta_r\). When \(\omega > \rho v_m \tan\theta_m\), the system is certainly in the continuous flow regime at some point \(C'\) above \(C\) (see fig.2(a)). If \(\rho v_r \tan\theta_r < \omega < \rho v_m \tan\theta_m\), then the system evolves from \(C\) to a point between \(C\) and \(D\) and either remains there (if the stationary solution is stable at this point) or jumps back (i.e., \(\theta \to \theta_r\) and \(v \to 0\)) to the \(AB\) segment and the avalanche stops. If \(\omega < \rho v_r \tan\theta_r\), then the system certainly decays to the \(AB\) segment and has intermittent avalanches evolving through the loop \(ABCD\), and so on, since \(D\) is unstable.

**Numerical Results.**— Figure 1 shows the time dependence of: (a) the flow velocity \(v\), (b) the angle \(\theta\) of the top layer, and (c) the angular velocity \(\omega\) of the rotating drum, obtained by numerically solving eqs.(3-4) with \(D_0 = 0\). For each avalanche, \(v\) has a peak and \(\theta\) has a sudden decrease from \(\theta_m \approx 0.8\) to \(\theta_r \approx 0.65\). The \(v = 0\) regions in between peaks correspond to the loading time in between avalanches, where the angle increases linearly in time from \(\theta_r\) to \(\theta_m\). For clarity, only three avalanches are shown. They are approaching the transition point, \(\omega_+\), to the continuous flow regime, and their periods become longer close to \(\omega_+\), i.e. the system exhibits a critical slowing down. The values of both \(\omega_+\) and \(\omega_-\) depend on \(\omega\), and the constants \(\alpha, \alpha_1, \beta, \) and \(D_0\) in eqs.(3-4). The addition of noise also reduces the hysteresis.

Notice the asymmetric form of the velocity peaks in fig. 1(a). During a loading time, \(\tau_A\), the system moves from \(A\) to \(B\), on the branch \(AB\) in fig.2(a). Right afterwards, an avalanche starts and \(v\) has a sudden increase lasting a time \(\tau_B\) \((B \to C\) in fig.2(a)), a slow decrease with duration \(\tau_C\) \((C \to D)\), and a final decrease lasting a time \(\tau_D\) \((D \to A)\).

Equations (3-4), without a random source term, and for a fixed \(\omega\), produce a periodic sequence of avalanches. However, the real many-body system inevitably has disorder. When \(D_0 \neq 0\), we observe a richer behavior (see figs. 3 and 4(b)) with broad distributions.

The statistics of the avalanche durations are shown in Fig. 3. In order to gain more insight into the several time scales involved in the problem, we show the distributions, \(N\), for (a) \(\tau_A\), (b) \(\tau_B\), (c) \(\tau_C\), and (d) \(\tau_D\), for \(\omega = 3 \times 10^{-4}\) and \(D_0 = 10^{-8}\).

Figure 4(a) shows the evolution of \(\tau_D\) when \(\omega \to \omega_+\). We have considered two different possibilities for the divergence: logarithmic (left vertical axis of the inset) and power-law (right vertical axis). For \(D_0 = 0\), the behavior of \(\tau_D\) at the transition is found to be

\[
\tau_D \sim (\omega_+ - \omega)^{-\beta} \tag{6}
\]

for \(\omega < \omega_+\). Temporal critical slowing down behavior has also been found in bistable optical systems. From eq.(6) and recalling that \(\omega + \eta\) is a Gaussian random variable, one obtains that \(N(\tau_D) \sim \tau_D^{-\beta}\). This is checked numerically in fig. 4(b) where we have obtained the distribution of \(\tau_D\) for fixed \(\omega = 4.6 \times 10^{-3}\), which is slightly below \(\omega_+(D_0 = 0)\), with \(D_0 = 10^{-8}\).

In the intermittent flow regime, our dynamical model exhibits an attractive limit cycle around an unstable fixed point. For \(\theta < \theta_m\), \(v = 0\) is a stable attractor manifold. In the continuum flow regime, the dynamical system is in an attractive fixed point with nonzero velocity. Here, a harmonic approximation to Newton’s equation (eq.(3)) with \(\dot{v} = 0\) gives

\[
v - v_r \approx (\theta - \theta_r)^{1/2} \tag{7}
\]

for the relationship between the mass current flow and the angle, for \(\omega > \omega_-\). This result is consistent with experimental results by Rajchenbach.
when $\omega \to \omega_+$, the avalanche durations exhibit a critical slowing down with a temporal power-law divergence.

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FIG. 1. Shows the time dependence of: (a) the flow velocity $v$, (b) the angle $\theta$ of the top layer, and (c) the angular velocity $\omega$ of the rotating drum. Throughout this work, we use the parameters values: $\alpha = 1$, $\alpha_1 = 2$, $\beta = 1/2$, and $\rho = 10^{-3}$. For these ramping rates, we obtain $\omega_+ = 4.76 \times 10^{-4}$ and $\omega_- = 4.66 \times 10^{-4}$, for the $D_0 = 0$ deterministic case.

FIG. 2. (a) Schematic plot of the friction force versus velocity $v$ for the case $c > 1$. An initially flat ($\theta = 0$) and stationary ($v = 0$) granular surface is tilted with an angular velocity $\omega$, thus having the following sequence of values for the friction force: $O \to A \to B$, from the $F = v = 0$ origin, $O$. The velocity, $v$, is zero in the stick-slip case with infinite slope for the OAB segment. In the AB branch, the loading time, $\tau_A$, satisfies $< \tau_A > = (\theta_{\text{m}} - \theta_{\text{s}})/\omega$. When the maximum angle of stability is reached in $B$, an avalanche ($v \neq 0$) starts and the system jumps to $C$. Different situations, described in the text, can now develop according to the value of $\omega$. (b) Schematic diagram of a rotating drum indicating the extreme heights, the mean downhill velocity $v$, and the mean angle $\theta$.

FIG. 3. Statistics of the avalanche durations. Distributions, $N$, for (a) $\tau_A$, (b) $\tau_B$, (c) $\tau_C$, and (d) $\tau_D$, for $\omega = 3 \times 10^{-4}$ and $D_0 = 10^{-8}$.

FIG. 4. (a) Time evolution of $\tau_D(\omega)$ when $\omega \to \omega_+(D_0 = 0)$. This critical slowing down with a power law divergence is shown in the inset by the dashed line corresponding to the right vertical logarithmic axis. For comparison purposes, the solid line in the inset, corresponding to the left vertical linear axis, has a small curvature because of the small value of the power ($p \simeq 0.34$). (b) Distribution of $\tau_D$ for $\omega = 4.6 \times 10^{-4}$, which is slightly below $\omega_+(D_0 = 0)$. The inset shows the same data in semi-log and log-log scales.