Linear versus lattice embeddings between Banach lattices

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\textbf{ABSTRACT}

A well-known classical result states that $c_0$ is linearly embeddable in a Banach lattice if and only if it is lattice embeddable. Improving results of H.P. Lotz, H.P. Rosenthal and N. Ghoussoub, we prove that $C[0,1]$ shares this property with $c_0$. Furthermore, we show that any infinite-dimensional closed sublattice of $C[0,1]$ is either lattice isomorphic to $c_0$ or contains a closed sublattice isomorphic to $C[0,1]$. As a consequence, it is proved that for a separable Banach lattice $X$ the following conditions are equivalent:

1. $X$ is linearly embeddable in a Banach lattice if and only if it is lattice embeddable;
2. $X$ is lattice embeddable into $C[0,1]$.

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1. Introduction

A Banach lattice is a Banach space equipped with compatible lattice operations. Understanding the relation between the linear and lattice structures of Banach lattices has been the driving force behind a large part of research in the topic. However, many fundamental questions concerning this relation still remain open.

It is well-known that given a Banach lattice $X$, the class of Banach lattices which can arise as a sublattice of $X$ can be very different from those which can arise as a subspace of $X$. Nevertheless, a number of results relating the containment of a Banach lattice as a subspace to the containment as a sublattice can be found in the literature (see [2, §4.3]). For instance, a well-known fact, attributed to P. Meyer-Nieberg [18,19], states that the Banach lattice $c_0$ is linearly embeddable in a Banach lattice if and only if it is lattice embeddable (cf. [2, Theorem 4.61]).

Under additional assumptions, similar results on this line are known: $\ell_1$ embeds as a closed sublattice of a Banach lattice $X$ if and only if it embeds as a complemented subspace (cf. [2, Theorem 4.69]); if $X$ is a Dedekind $\sigma$-complete Banach lattice, then $\ell_\infty$ embeds as a closed sublattice of $X$ precisely when it embeds as a subspace (cf. [2, Theorem 4.56]). In connection with these facts, a classical result attributed to G. Ja. Lozanovski [16] states that neither $c_0$ nor $\ell_1$ are lattice embeddable in a Banach lattice $X$ if, and only if, neither of them linearly embeds in $X$ (which actually provides a characterization of reflexivity, cf. [2, Theorem 4.71]). Finally, let us also mention the deep result due to N. Kalton that if $X$ is an order-continuous rearrangement-invariant Banach lattice on $[0,1]$ different from $L_2$, then $X$ embeds as a complemented subspace of an order-continuous Banach lattice $Y$ if and only if $X$ embeds as a complemented closed sublattice [13].

The problem we address in the present paper is to characterize those Banach lattices which embed as a closed sublattice whenever they embed as a subspace in an arbitrary Banach lattice. To our knowledge, no example other than $c_0$ can be found in the literature. The main result of the article is the following:

**Theorem A.** The Banach lattice $C[0,1]$ is linearly embeddable in a Banach lattice if and only if it is lattice embeddable.

The proof of Theorem A relies on some classical and recent results in the theory of Banach lattices. On the one hand, H. P. Lotz and H. P. Rosenthal proved in [17] that for a Banach lattice $X$ it is equivalent that:

1. $L_1$ is lattice embeddable in $X^*$;
2. $C(\Delta)^*$ is lattice embeddable in $X^*$;
(3) there exists a positive embedding \( T : C(\Delta) \rightarrow X \);

where \( \Delta \) denotes the Cantor space. Furthermore, they asked whether these conditions are in turn equivalent to the fact that \( C(\Delta) \) linearly embeds into \( X \). Notice that one cannot expect that the linear embeddability of \( C(\Delta) \) into \( X \) implies the lattice embeddability of \( C(\Delta) \). Indeed, \( C(\Delta) \) linearly embeds in \( C[0, 1] \) (they are in fact linearly isomorphic by Miljutin’s Theorem [25, Theorem 21.5.10]) but \( C(\Delta) \) is not lattice isomorphic to a closed sublattice of \( C[0, 1] \) (see, for example, Lemma 3.1). The previous question of Lotz and Rosenthal was answered in the affirmative by N. Ghoussoub in [9]. Although the proof of Theorem A relies on the aforementioned results of Lotz, Rosenthal and Ghoussoub, note that our theorem provides a strengthening of Ghoussoub’s answer, since the linear embeddability of \( C(\Delta) \) is equivalent to the linear embeddability of \( C[0, 1] \), whereas the lattice embeddability of \( C[0, 1] \) into \( X \) implies condition (3), since there are positive linear embeddings from \( C(\Delta) \) into \( C[0, 1] \) (for instance, because of the above mentioned result of N. Ghoussoub).

On the other hand, D.H. Leung, L. Li, T. Oikhberg and M.A. Tursi recently proved that every separable Banach lattice embeds lattice isometrically into \( C(\Delta, L_1) \) [14, Theorem 1.1], where \( C(\Delta, L_1) \) denotes the Banach lattice of continuous functions \( f: \Delta \rightarrow L_1 \), endowed with the norm \( \|f\| = \sup_{t \in \Delta} \|f(t)\|_{L_1} \). This result will also play a fundamental role in the proof of Theorem A.

The last key ingredient of Theorem A is the recent concept of projectivity for Banach lattices introduced by B. de Pagter and A.W. Wickstead in [21] (see Definition 2.1 below). The study of projectivity for \( C(K) \)-spaces was initiated by these authors and a complete characterization was given in [4]. We will make use of the fact that \( C[0, 1] \) is projective [21, Corollary 11.5] and, more precisely, of Lemma 2.2, which is a direct consequence of this fact.

Whereas Section 2 is devoted to the proof of Theorem A, in Section 3, we study what other Banach lattices share the aforementioned property with \( c_0 \) and \( C[0, 1] \), providing a characterization in the separable case:

**Theorem B.** For a separable Banach lattice \( X \) the following conditions are equivalent:

1. \( X \) is linearly embeddable in a Banach lattice if and only if it is lattice embeddable;
2. \( X \) is lattice embeddable into \( C[0, 1] \).

**Theorem C.** An infinite-dimensional closed sublattice of \( C[0, 1] \) is either lattice isomorphic to \( c_0 \) or it contains a closed sublattice isomorphic to \( C[0, 1] \).
We finish Section 3 with some consequences of Theorem B, a simple characterization of those \( C(K) \)-spaces which lattice embed into \( C[0,1] \), and a discussion on the possible existence of nonseparable Banach lattices for which linear embeddability implies lattice embeddability, leaving several open questions.

**Notation.** Our terminology is standard; any unexplained notation can be found in [2], [15] and [20]. By an operator we shall mean a bounded linear operator. An operator \( T: X \to Y \) between Banach lattices is said to be a lattice homomorphism if \( Tx \vee Ty = T(x \vee y) \) and \( Tx \wedge Ty = T(x \wedge y) \). We say that \( T \) is a lattice isomorphism if it is a lattice homomorphism and a Banach space isomorphism (notice that this notation differs from the one used in [17], where a lattice isomorphism might not have closed range). We say that a Banach space \( X \) is *linearly embeddable* into another Banach space \( Y \) whenever there exists an operator \( T: X \to Y \) which is an isomorphism onto its range, i.e. \( T \) is a linear embedding. If, in addition, \( X \) and \( Y \) are Banach lattices and \( T \) is also a lattice homomorphism then \( X \) is said to be *lattice embeddable* into \( Y \). For any Banach lattice \( X \) and any compact space \( K \), we denote by \( C(K,X) \) the Banach lattice consisting of all continuous functions \( f: K \to X \) endowed with the norm \( \|f\| = \sup_{t \in K} \|f(t)\|_X \). If we do not require the functions to be continuous then we get the Banach lattice \( \ell_\infty(K,X) \). We see \( C(K,X) \) as a closed sublattice of \( \ell_\infty(K,X) \) in the canonical way. Furthermore, for any \( f \in \ell_\infty(K,X) \), by \( \text{supp}(f) \) we denote its support, i.e. \( \text{supp}(f) := \{ t \in K : f(t) \neq 0 \} \).

2. Lattice embeddability of \( C[0,1] \)

In this section we prove Theorem A. First, we need a simple criterion to guarantee that \( C[0,1] \) is lattice embeddable in a Banach lattice, which uses the concept of projectivity.

**Definition 2.1.** [21, Definition 10.1] A Banach lattice \( P \) is said to be *projective* if, whenever \( X \) is a Banach lattice, \( J \) is a closed ideal in \( X \) and \( Q: X \to X/J \) is the quotient map, for every lattice homomorphism \( T: P \to X/J \) and every \( \varepsilon > 0 \) there exists a lattice homomorphism \( \hat{T}: P \to X \) such that \( T = Q \circ \hat{T} \) and \( \|\hat{T}\| \leq (1 + \varepsilon)\|T\| \).

The fact that \( C[0,1] \) is projective [21, Corollary 11.5], yields the following result:

**Lemma 2.2.** Let \( X \) be a Banach lattice and \( J \) be a closed ideal in \( X \). If \( C[0,1] \) is lattice embeddable into \( X/J \) then \( C[0,1] \) is lattice embeddable into \( X \).

**Proof.** Let \( Q: X \to X/J \) be the quotient map and let \( T: C[0,1] \to X/J \) be a lattice embedding such that \( \alpha \|f\| \leq \|Tf\| \leq \|f\| \) for every \( f \in C[0,1] \). Since \( C[0,1] \) is projective, for every \( \varepsilon > 0 \) there exists a lattice homomorphism \( \hat{T}: C[0,1] \to X \) such that \( T = Q \circ \hat{T} \) and \( \|\hat{T}\| \leq (1 + \varepsilon)\|T\| \). Note that for \( f \in C[0,1] \) we have

\[
(1 + \varepsilon)\|f\| \geq \|\hat{T}f\| \geq \|Q\|^{-1}||Q\hat{T}f|| = \|Tf\| \geq \alpha\|f\|.
\]
Therefore, $\hat{T}: C[0, 1] \to X$ defines a lattice embedding (with embedding constant as close as we want to that of $T$). \(\Box\)

To prove Theorem A we need to show that if $X$ is a Banach lattice and $C[0, 1]$ linearly embeds into $X$ then $C[0, 1]$ is lattice embeddable into $X$. By Miljutin’s Theorem [25, Theorem 21.5.10], the linear embeddability of $C[0, 1]$ is equivalent to the linear embeddability of $C(\Delta)$. Now, by [9, Theorem (A)], there exists a positive linear embedding $T: C(\Delta) \to X$. By the equivalence between (c) and (d) in [17, Theorem 2], there is no loss of generality in assuming that there is $0 < \varepsilon < \frac{1}{4}$ such that $(1 - \varepsilon)\|f\| \leq \|Tf\| \leq \|f\|$ for every $f \in C(\Delta)$. Thus, Theorem A will be a consequence of the following theorem which we prove below.

**Theorem 2.3.** Let $X$ be a Banach lattice, $0 < \varepsilon < \frac{1}{4}$ and $T: C(\Delta) \to X$ be a positive linear embedding with $(1 - \varepsilon)\|f\| \leq \|Tf\| \leq \|f\|$ for every $f \in C(\Delta)$. Then $C[0, 1]$ is lattice embeddable into $X$.

**Proof.** Without loss of generality we assume that $X$ is the closed sublattice generated by $T(C(\Delta))$. In particular, it is separable, so by [14, Theorem 1.1] we can suppose that $X$ is a closed sublattice of $C(\Delta, L_1)$. Let us give a brief overview of the proof. Instead of working directly with $C[0, 1]$ we are going to construct a closed sublattice $Z \subseteq X$, a suitable compact space $K \subseteq \Delta$ and lattice homomorphisms $Q_1: C(\Delta, L_1) \to C(K, L_1)$ and $Q_2: \ell_\infty(K, L_1) \to \ell_\infty(K, L_1)$ such that $Q_2(Q_1(Z))$ is a Banach lattice isomorphic to $C(\Delta)$. Since $C[0, 1]$ is lattice embeddable into $C(\Delta)$, the conclusion will follow from Lemma 2.2. We shall split the proof into five steps for simplicity. First, we obtain a family $\{f_\sigma: \sigma \in 2^{<\omega}\}$ of norm-one positive functions in $C(\Delta)$ such that $C(\Delta) = \mathfrak{sp}\{f_\sigma: \sigma \in 2^{<\omega}\}$ and such that $f_\sigma = f_{\sigma_0} + f_{\sigma_1}$ and $f_{\sigma_0} = f_{\sigma_1}$ for every $\sigma$ (see Step 1 for further details and unexplained notation). If we set $g_\sigma = T f_\sigma$ for every $\sigma \in 2^{<\omega}$, then $g_{\sigma_0}, g_{\sigma_1}$ might not be pairwise disjoint for some $\sigma \in 2^{<\omega}$. In Steps 2 and 3 the lattice homomorphisms $Q_1$ and $Q_2$ are constructed in such a way that the images of $g_{\sigma_0}, g_{\sigma_1}$ through $Q_2 \circ Q_1$ are in some sense close to be disjoint. Then, in Step 4 a suitable perturbation of these images yields a seminormalized system of positive vectors $\{y_\sigma\}_{\sigma \in 2^{<\omega}}$ which not only fulfills that $y_\sigma = y_{\sigma_0} + y_{\sigma_1}$ for every $\sigma \in 2^{<\omega}$ but also that $y_{\sigma_0}$ and $y_{\sigma_1}$ are disjoint. This in turn implies that $\hat{Y} = \mathfrak{sp}\{y_\sigma: \sigma \in 2^{<\omega}\}$ is a Banach lattice lattice isomorphic to $C(\Delta)$. Finally, in Step 5 the closed sublattice $Z \subseteq X$ is constructed using similar perturbations in such a way that $Q_2(Q_1(Z)) = \hat{Y}$ as desired.

**Step 1.** Writing $C(\Delta)$ as the closure of an increasing sequence of finite dimensional closed sublattices.

We identify $\Delta$ with $2^\omega$ with the product topology. By $2^{<\omega}$ we denote the family of all functions $\sigma: \{1, 2, \ldots, n\} \to \{0, 1\}$ for some $n < \omega$. For any such function $\sigma$ we write $\text{supp}(\sigma) = \{1, 2, \ldots, n\}$ and $|\sigma| = n$. Moreover, for any $i \in \{0, 1\}$ we denote by
\( \sigma \mapsto i \) the map defined on \( \{1, \ldots, n, n + 1\} \) which coincides with \( \sigma \) on \( \text{supp}(\sigma) \) and takes the value \( i \) on \( n + 1 \). For each \( \sigma, \sigma' \in 2^{<\omega} \) we write \( \sigma \preceq \sigma' \) if \( \sigma' \) extends \( \sigma \) (that is, if \( \text{supp}(\sigma) \subseteq \text{supp}(\sigma') \) and \( \sigma'|_{\text{supp}(\sigma)} = \sigma) \). Thus, \( 2^{<\omega} \) with this order is a partially ordered set with a least element denoted by \( \emptyset \), which is the only function whose domain is the empty set.

Set \( \Delta_\sigma = \{ t \in \Delta : t|_{\text{supp}(\sigma)} = \sigma \} \) and \( f_\sigma = \chi_{\Delta_\sigma} \) its characteristic function for every \( \sigma \in 2^{<\omega} \). Since each \( \Delta_\sigma \) is clopen we have that each \( f_\sigma \) is a positive norm-one continuous function in \( C(\Delta) \). Moreover, \( f_\sigma = f_{\sigma - 0} + f_{\sigma - 1} \) and \( f_{\sigma - 0}, f_{\sigma - 1} \) are pairwise disjoint for every \( \sigma \). It follows that \( \{ f_\sigma : |\sigma| = n \} \) are pairwise disjoint functions and that \( f_\beta \in \text{span}\{ f_\sigma : |\sigma| = n \} \) whenever \( |\beta| \leq n \). Thus, the closed sublattice generated by \( \{ f_\sigma : |\sigma| = n \} \) is just the Banach lattice generated by \( \{ f_\sigma : |\sigma| \leq n \} \). Since the Banach lattice generated by finitely many pairwise disjoint vectors is just the linear span of these vectors, we conclude that the closed sublattice generated by \( \{ f_\sigma : |\sigma| \leq n \} \) is just \( Y_n := \text{span}\{ f_\sigma : |\sigma| = n \} \). \( Y_n \) is a Banach lattice of dimension \( 2^n \), lattice isometric to \( \ell_\infty^n \). If we take \( Y = \bigcup_n Y_n \), then the lattice version of the Stone-Weierstrass Theorem (see, for instance, [20, Theorem 2.1.1]) gives that \( \overline{Y} = C(\Delta) \).

**Step 2. Construction of a lattice homomorphism \( Q_1 : C(\Delta, L_1) \to C(K, L_1) \) induced by the restriction map to a closed subset \( K \) in \( \Delta \).**

Since \( X \subseteq C(\Delta, L_1) \) and \( T \) is a positive linear embedding, each \( g_\sigma := Tf_\sigma \) is a positive continuous function \( g_\sigma : \Delta \to L_1 \). By hypothesis, \( \| g_\sigma \| \geq 1 - \varepsilon \) for each \( \sigma \in 2^{<\omega} \). Let \( K_\sigma := \{ t \in \Delta : \| g_\sigma(t) \| \geq 1 - \varepsilon \} \). Since the function \( t \mapsto \| g_\sigma(t) \| \) is continuous and \( \| g_\sigma \| \geq 1 - \varepsilon \), we get that \( K_\sigma \) is a nonempty closed subset of \( \Delta \) for every \( \sigma \in 2^{<\omega} \).

Furthermore, notice that \( K_{\sigma - 0}, K_{\sigma - 1} \subseteq K_\sigma \) for every \( \sigma \). Indeed, if \( t \in K_{\sigma - 0} \), then \( \| g_{\sigma - 0}(t) \| \geq 1 - \varepsilon \) and since \( g_{\sigma - 0} \leq g_\sigma \) and both are positive, we have that \( \| g_\sigma(t) \| \geq 1 - \varepsilon \), so \( t \in K_\sigma \). Moreover, \( K_{\sigma - 0} \cap K_{\sigma - 1} = \emptyset \) since if \( t \in K_{\sigma - 0} \cap K_{\sigma - 1} \) then, bearing in mind that the norm of \( L_1 \) is additive on the positive cone, \( 1 \geq \| g_\sigma(t) \| = \| g_{\sigma - 0}(t) + g_{\sigma - 1}(t) \| = \| g_{\sigma - 0}(t) \| + \| g_{\sigma - 1}(t) \| \geq 2(1 - \varepsilon) \geq 2(1 - \varepsilon)^\frac{3}{2} \), which yields a contradiction. Thus, we can take \( K := \bigcap_{k \in \mathbb{N}} (\bigcup_{|\sigma| = k} K_\sigma) \), which is a nonempty closed subset of \( \Delta \). We set \( Q_1 : C(\Delta, L_1) \to C(K, L_1) \) the lattice homomorphism given by the restriction map \( Q_1 f = f|_K \). Set \( K_\sigma' := K_\sigma \cap K \) and \( h_\sigma := Q_1 g_\sigma \) for every \( \sigma \in 2^{<\omega} \). We point out the following property, which will be used in the following step:

\[
\int (h_\emptyset(s)(t) - h_\sigma(s)(t)) dt \leq \varepsilon \quad \text{for every } \sigma \in 2^{<\omega} \quad \text{and every } s \in K_\sigma'.
\] (2.1)

Indeed, \( \int (h_\emptyset(s)(t) - h_\sigma(s)(t)) dt = \| h_\emptyset(s) \| - \| h_\sigma(s) \| \) and (2.1) follows from the fact that \( 1 - \varepsilon \leq \| h_\sigma(s) \| \) for every \( s \in K_\sigma' \).

**Step 3. Construction of the lattice homomorphism \( Q_2 : \ell_\infty(K, L_1) \to \ell_\infty(K, L_1) \).**

Notice that \( h_\sigma(s) \in L_1 \) is an equivalence class of functions for every \( \sigma \in 2^{<\omega} \) and every \( s \in K \). For simplicity, we identify each equivalence class \( h_\sigma(s) \) with a representative, still denoted by \( h_\sigma(s) \). We know that if \( \sigma \preceq \sigma' \) then \( h_\sigma(s) \geq h_{\sigma'}(s) \), which means
that $h_{\sigma}(s)(t) \geq h_{\sigma'}(s)(t)$ for almost every $t \in [0,1]$. There is no loss of generality in assuming that the representatives have been taken in such a way that the inequality $h_{\sigma}(s)(t) \geq h_{\sigma'}(s)(t)$ holds for every $t \in [0,1]$.

For each $\sigma \in 2^{<\omega}$ and $s \in K'_{\sigma}$ define

$$L_{\sigma}(s) := \{t \in [0,1] : h_{\sigma}(s)(t) \geq 2(h_{\emptyset}(s)(t) - h_{\sigma}(s)(t))\}.$$ 

We prove the following inequality

$$\int_{L_{\sigma}(s)} h_{\emptyset}(s)(t)dt \geq 1 - 4\varepsilon \text{ for every } s \in K'_{\sigma}, \sigma \in 2^{<\omega}. \tag{2.2}$$

Indeed,

$$\int_{L_{\sigma}(s)} h_{\emptyset}(s)(t)dt = \int_{[0,1]} h_{\emptyset}(s)(t)dt - \int_{L_{\sigma}(s)^{c}} h_{\emptyset}(s)(t)dt$$

$$= \|h_{\emptyset}(s)\| - \int_{L_{\sigma}(s)^{c}} (h_{\sigma}(s)(t) + (h_{\emptyset}(s)(t) - h_{\sigma}(s)(t)))dt$$

$$\geq \|h_{\emptyset}(s)\| - \int_{L_{\sigma}(s)^{c}} 3(h_{\emptyset}(s)(t) - h_{\sigma}(s)(t))dt \overset{(\ast)}{=} 1 - 4\varepsilon,$$

where in $(\ast)$ we have used the definition of $L_{\sigma}(s)$ and in $(\ast\ast)$ we have used (2.1) and the fact that $\|h_{\emptyset}(s)\| \geq 1 - \varepsilon$.

Notice that if $\sigma \preceq \sigma'$ and $s \in K'_{\sigma'}$, then $L_{\sigma'}(s) \subseteq L_{\sigma}(s)$, since if $t \in L_{\sigma'}(s)$ then

$$h_{\sigma}(s)(t) \geq h_{\sigma'}(s)(t) \geq 2(h_{\emptyset}(s)(t) - h_{\sigma'}(s)(t)) \geq 2(h_{\emptyset}(s)(t) - h_{\sigma}(s)(t)).$$

We define for every $s \in K$ the set

$$L(s) := \bigcap_{\sigma: s \in K'_{\sigma}} L_{\sigma}(s).$$

Notice that for every $s \in K$ and every $n$ there exists a unique $\sigma_{n}$ with $|\sigma_{n}| = n$ such that $s \in K'_{\sigma_{n}}$. Thus, $L(s) = \bigcap_{n} L_{\sigma_{n}}(s)$ and since this is a decreasing sequence of sets, we have that

$$\int_{L(s)} h_{\emptyset}(s)(t)dt = \int_{L(s)} h_{\emptyset}(s)(t)\chi_{L(s)}(t)dt = \lim_{|\sigma| \to \infty, s \in K'_{\sigma}} \int_{L(s)} h_{\emptyset}(s)(t)\chi_{L_{\sigma}(s)}(t)dt \geq 1 - 4\varepsilon \tag{2.3}$$

for every $s \in K$, by the Monotone Convergence Theorem, where in the last inequality we have used (2.2).
We consider now the map $Q_2: \ell_\infty(K, L_1) \rightarrow \ell_\infty(K, L_1)$ given by the formula $(Q_2f)(s) = f(s)\chi_{L(s)}$ for every $s \in K$. It is easily checked that $Q_2$ is a lattice homomorphism of norm 1 with closed range. The range consists of the functions whose value at each $s$ is supported on $L(s)$. Since $C(K, L_1)$ is a closed sublattice of $\ell_\infty(K, L_1)$, we can define $x_\sigma = Q_2h_\sigma$ for every $\sigma \in 2^{<\omega}$.

**Step 4. Construction of a copy of $C(\Delta)$ in $Q_2(Q_1(X))$.
**
We consider the sequence $(y_\sigma)_{\sigma \in 2^{<\omega}}$ in $Q_2(Q_1(X))$ defined recursively as follows:

- $y_\emptyset = x_\emptyset$;
- $y_{\sigma \cup 0} = (2x_{\sigma \cup 0} - x_{\sigma \cup 1})^+ \land y_\sigma$;
- $y_{\sigma \cup 1} = (2x_{\sigma \cup 1} - x_{\sigma \cup 0})^+ \land y_\sigma$.

It is immediate that every $y_\sigma$ is positive. We prove the following claims:

**Claim 1.** $\text{supp}(y_\sigma) \subseteq K'_\sigma$ and therefore $y_{\sigma \cup 0} \land y_{\sigma \cup 1} = 0$.

The claim is trivial for $\sigma = \emptyset$. Suppose the claim holds for $\sigma$ and we show that it is true for $\sigma \cup 0$ (the case $\sigma \cup 1$ is analogous). By the induction hypothesis, $\text{supp}(y_{\sigma \cup 0}) \subseteq \text{supp}(y_\sigma) \subseteq K'_\sigma = K'_{\sigma \cup 0} \cup K'_{\sigma \cup 1}$, so we only need to show that if $s \in K'_{\sigma \cup 1}$ then $y_{\sigma \cup 0}(s)(t) = 0$ for every $t \in [0, 1]$. By the definition of $Q_2$, it is immediate that if $t \notin L(s)$ then $y_{\sigma \cup 0}(s)(t) = 0$. Nevertheless, since $s \in K'_{\sigma \cup 1}$, if $t \in L(s) \subseteq L_{\sigma \cup 1}(s)$ then

$$h_{\sigma \cup 1}(s)(t) \geq 2(h_\emptyset(s)(t) - h_{\sigma \cup 0}(s)(t)) \geq 2h_{\sigma \cup 0}(s)(t).$$

But then $x_{\sigma \cup 1}(s)(t) \geq 2x_{\sigma \cup 0}(s)(t)$ and therefore $y_{\sigma \cup 0}(s)(t) = (2x_{\sigma \cup 0}(s)(t) - x_{\sigma \cup 1}(s)(t))^+ \land y_\sigma = 0 \land (y_\sigma(s)(t) = 0$ as desired.

**Claim 2.** $y_\sigma(s) = x_\emptyset(s)$ for every $\sigma \in 2^{<\omega}$, $s \in K'_\sigma$. The result is trivial for $\sigma = \emptyset$. Suppose the claim holds for a fixed $\sigma$. We show that it holds for $\sigma \cup 0$ (the case $\sigma \cup 1$ is analogous). It is clear that $y_{\sigma \cup 0} \leq x_\emptyset$, hence we prove the converse inequality. Notice that if $s \in K'_{\sigma \cup 0}$ and $t \in L_{\sigma \cup 0}(s)$ then

$$h_{\sigma \cup 0}(s)(t) \geq 2(h_\emptyset(s)(t) - h_{\sigma \cup 0}(s)(t)) = (h_\emptyset(s)(t) - h_{\sigma \cup 0}(s)(t)) + (h_\emptyset(s)(t) - h_{\sigma \cup 0}(s)(t)) \geq (h_\emptyset(s)(t) - h_{\sigma \cup 0}(s)(t)) + h_{\sigma \cup 1}(s)(t),$$

where in the last inequality we have used that

$$(h_\emptyset - h_{\sigma \cup 0}) = \sum_{\sigma' \neq \sigma \cup 0, |\sigma'| = |\sigma| + 1} h_{\sigma'} \geq h_{\sigma \cup 1}.$$

Thus, we obtain that

$$2h_{\sigma \cup 0}(s)(t) - h_{\sigma \cup 1}(s)(t) \geq h_\emptyset(s)(t),$$

for $s \in K'_{\sigma \cup 0}$ and $t \in L_{\sigma \cup 0}(s)$. By the definition of $Q_2$ and $y_{\sigma \cup 0}$ we get that
\[ y_{\sigma^{-0}}(t) = (2h_{\sigma^{-0}}(t) - h_{\sigma^{-1}}(t)) \wedge y_{\sigma}(s)(t) \geq h_0(s)(t) \wedge y_{\sigma}(s)(t) \]

and the conclusion follows from the induction hypothesis.

The following claim is a direct consequence of Claims 1 and 2, where \( \chi_{K_{\sigma}} : K \rightarrow L_1 \) is the function which is null on \( K \setminus K'_{\sigma} \) and is the constant function one on \( K'_{\sigma} \).

Claim 3. \( y_{\sigma} = \chi_{K_{\sigma}} x_{\emptyset} \) and therefore \( y_{\sigma} = y_{\sigma^{-0}} + y_{\sigma^{-1}} \) for every \( \sigma \in 2^{<\omega} \).

Claim 4. \( \| y_{\sigma} \| \geq 1 - 4\varepsilon \) for every \( \sigma \). The inequality \( \| y_{\sigma} \| \) is immediate, whereas the inequality \( \| y_{\sigma} \| \geq 1 - 4\varepsilon \) follows from Claim 3 and (2.3).

Finally, by Claims 1 and 3 we get that the closed sublattice generated by \( \widehat{Y}_n := \{ y_{\sigma} : |\sigma| \leq n \} \) is just the closed lattice generated by \( \{ y_{\sigma} : |\sigma| = n \} \). As happened at Step 1, this Banach lattice is just the linear span of these vectors, i.e. \( \widehat{Y}_n = \text{span}\{ y_{\sigma} : |\sigma| = n \} \).

In fact, if we denote by \( \widehat{Y} \) the closed sublattice generated by \( \{ y_{\sigma} : \sigma \in 2^{<\omega} \} \), then it follows from Claim 4 that the unique operator \( \widehat{T} : C(\Delta) \rightarrow \widehat{Y} \) satisfying \( \widehat{T}(f_{\sigma}) = y_{\sigma} \) is a well-defined lattice isomorphism onto \( \widehat{Y} \).

**Step 5. Embedding \( C[0,1] \) into \( X \).**

Consider the sequence \( (z_{\sigma})_{\sigma \in 2^{<\omega}} \) in \( X \) defined recursively as follows:

- \( z_{\emptyset} = g_{\emptyset} \);
- \( z_{\sigma^{-0}} = (2g_{\sigma^{-0}} - g_{\sigma^{-1}})^+ \wedge z_{\sigma} \);
- \( z_{\sigma^{-1}} = (2g_{\sigma^{-1}} - g_{\sigma^{-0}})^+ \wedge z_{\sigma} \).

Since \( x_{\sigma} = Q_2 h_{\sigma} = Q_2 (Q_1 g_{\sigma}) \) for every \( \sigma \in 2^{<\omega} \), it is immediate that \( Q_2 (Q_1 z_{\sigma}) = y_{\sigma} \) for every \( \sigma \in 2^{<\omega} \). Let \( Z \) be the closed sublattice of \( X \) generated by \( (z_{\sigma})_{\sigma \in 2^{<\omega}} \). Then \( Q_2 (Q_1 Z) \subseteq \widehat{Y} \). We are going to show that \( Q_2 (Q_1 Z) = \widehat{Y} \). Let \( n \geq 1 \) and \( a_{\sigma} \) be a family of scalars with \( |\sigma| = n \). On one hand,

\[
\sum_{|\sigma| = n} a_{\sigma} z_{\sigma} \leq \sum_{|\sigma| = n} |a_{\sigma}| z_{\sigma} \leq \sup_{|\sigma| = n} |a_{\sigma}| \sum_{|\sigma| = n} z_{\sigma} = \sup_{|\sigma| = n} |a_{\sigma}| \sum_{|\sigma'| = n-1} (z_{\sigma^{-0}} + z_{\sigma^{-1}}) \leq \\
\leq \sup_{|\sigma| = n} |a_{\sigma}| \sum_{|\sigma'| = n-1} ((2g_{\sigma^{-0}} - g_{\sigma^{-1}})^+ + (2g_{\sigma^{-1}} - g_{\sigma^{-0}})^+) \leq \\
\leq \sup_{|\sigma| = n} |a_{\sigma}| \sum_{|\sigma'| = n-1} (2g_{\sigma^{-0}} + 2g_{\sigma^{-1}}) = \\
= 2 \sup_{|\sigma| = n} |a_{\sigma}| \sum_{|\sigma| = n} g_{\sigma} = 2 \sup_{|\sigma| = n} |a_{\sigma}| g_{\emptyset}.
\]

Thus,

\[
\left\| \sum_{|\sigma| = n} a_{\sigma} z_{\sigma} \right\| \leq 2 \sup_{|\sigma| = n} |a_{\sigma}| \|g_{\emptyset}\| \leq 2 \sup_{|\sigma| = n} |a_{\sigma}|.
\]
On the other hand, by Claims 3 and 4 we have that
\[
(1 - 4\varepsilon) \sup_{|\sigma|=n} |a_\sigma| \leq \left\| \sum_{|\sigma|=n} a_\sigma y_\sigma \right\| \leq \sup_{|\sigma|=n} |a_\sigma|.
\]

Hence, it follows from the previous inequalities that \( B_\hat{Y} \subseteq \frac{2}{1 - 4\varepsilon} Q_2(Q_1(B_Z)) \). Then, by the classical Banach Open Mapping Theorem (see, for instance, [7, Lemma 2.23]), we conclude that \( Q_2(Q_1(Z)) = \hat{Y} \) as desired.

Let \( J = \ker((Q_2 \circ Q_1)|_Z) \), which is a closed ideal in \( Z \). Then, \( Z/J \) is lattice isomorphic to \( \hat{Y} \), which is in turn lattice isomorphic to \( C(\Delta) \) by Step 4. Any continuous surjection \( \pi: \Delta \rightarrow [0, 1] \) induces a lattice embedding \( S: C[0, 1] \rightarrow C(\Delta) \) through the formula \( Sf = f \circ \pi \) for every \( f \in C[0, 1] \). Thus, \( C[0, 1] \) is lattice embeddable into \( C(\Delta) \) and therefore it is also lattice embeddable into \( Z/J \). By Lemma 2.2, \( C[0, 1] \) is lattice embeddable into \( Z \subseteq X \), which concludes the proof. \( \square \)

As a consequence of Theorem A, [17, Theorem 2] and [9, Theorem (A)], we get the following corollary:

**Corollary 2.4.** Let \( X \) be a Banach lattice. Then the following assertions are equivalent:

1. \( C[0, 1] \) is linearly embeddable into \( X \);
2. \( C[0, 1] \) is lattice embeddable into \( X \);
3. \( L_1 \) is lattice embeddable into \( X^* \);
4. \( C(\Delta)^* \) is lattice embeddable into \( X^* \);
5. There is an order bounded sequence in \( X \) with no weak Cauchy subsequence.

The following corollary is a simple consequence of Theorem A, [10, Theorem II.2] and the fact that \( \ell_1 \) embeds complementably in a Banach lattice if and only if it is lattice embeddable [2, Theorem 4.69].

**Corollary 2.5.** Let \( X \) be a Banach lattice such that \( \ell_1 \) is linearly embeddable in \( X \). Then \( \ell_1 \) is lattice embeddable in \( X \) or \( C[0, 1] \) is lattice embeddable in \( X \).

3. **Lattice embeddability of other Banach lattices**

We wonder in this section what other Banach lattices, different from \( c_0 \) and \( C[0, 1] \), satisfy the property exhibited in Theorem A. Before giving a complete characterization in the separable case, we prove Theorem C, which will be the key ingredient in the proof of Theorem B.

**Proof of Theorem C.** Let \( X \) be an infinite-dimensional closed sublattice of \( C[0, 1] \). Suppose that \( X \) is not isomorphic to \( c_0 \). By [20, Lemma 2.7.12], \( X \) does not have order
continuous norm. By [20, Theorem 2.4.2], there exists a disjoint sequence \((g_n)_{n \in \mathbb{N}}\) and \(g\) in \(X^+\) with \(g\) an upper bound for the sequence \(g_n\) and such that \((g_n)_{n \in \mathbb{N}}\) is not norm-convergent to zero. Without loss of generality, we can suppose \(\|g_n\| = 1\) for every \(n \in \mathbb{N}\). Let \(I_n\) be a connected component of the open set \(\{ t \in [0, 1] : g_n(t) > 0 \}\) where \(g_n\) attains its maximum. Then, \(I_n = (a_n, b_n)\) is an open interval and there is \(c_n \in (a_n, b_n)\) such that \(g_n(c_n) = 1\). Since \((g_n)_{n \in \mathbb{N}}\) is a disjoint sequence, the intervals \(I_n\) are disjoint. Thus, the length of \(I_n\) converges to zero. By the uniform continuity of \(g\), there exists \(n \in \mathbb{N}\) such that the length of the interval \(g(I_n)\) is smaller than \(\frac{1}{2}\). In particular, since \(g \geq g_n\) and \(g_n(c_n) = 1\), we have that \(g(t) \geq \frac{1}{2}\) for every \(t \in I_n\). We claim that \([0, 1]\) is lattice embeddable into the closed sublattice \(Y\) generated by \(g\) and \(g_n\).

Let \(\{(t_\alpha, t_\alpha', \lambda_\alpha)\}_{\alpha \in \Gamma}\) be the family of all elements \(t_\alpha, t_\alpha', \lambda_\alpha \in [0, 1]\) such that \(g(t_\alpha) = \lambda_\alpha g(t_\alpha')\) and \(g_n(t_\alpha) = \lambda_\alpha g_n(t_\alpha')\) for every \(\alpha \in \Gamma\). By [12, Theorem 3],

\[
Y = \{ f \in [0, 1] : f(t_\alpha) = \lambda_\alpha f(t_\alpha') \text{ for every } \alpha \in \Gamma \}.
\]

Define \(T : [0, 1] \to Y\) by the formula

\[
Tf(t) = f \left( \frac{g_n(t)}{g(t)} \right) g(t) \text{ whenever } t \in [0, 1] \text{ and } g(t) \neq 0,
\]

and \(Tf(t) = 0\) otherwise. It is immediate that \(Tf \in [0, 1]\) for every \(f \in [0, 1]\). A routine computation shows that \(Tf \in Y\) for every \(f \in [0, 1]\). Thus, \(T\) is a lattice homomorphism with \(\|T\| \leq \|g\|\). Let \(M := \sup_{t \in I_n} \frac{g_n(t)}{g(t)}\) (notice that \(0 < M \leq 1\) and define

\[
Z = \{ f \in [0, 1] : f \text{ is constant on the interval } [M, 1] \},
\]

which is clearly lattice isomorphic to \([0, 1]\). We claim that \(T|_Z\) is an isomorphism. Indeed, if \(f \in Z\) then

\[
\|Tf\| = \sup_{t \in [0, 1]} |Tf(t)| = \sup_{t \in [0, 1] \setminus g^{-1}(0)} \left| f \left( \frac{g_n(t)}{g(t)} \right) g(t) \right| \leq \sup_{t \in I_n} f \left( \frac{g_n(t)}{g(t)} \right) g(t) \geq \frac{1}{2} \sup_{t \in I_n} \left| \frac{g_n(t)}{g(t)} \right| = \frac{1}{2} \|f\|,
\]

where the last equality follows from the definition of \(Z\) and \(M\) and the fact that \([0, M] = \{ \frac{g_n(t)}{g(t)} : t \in I_n\}\), by definition of \(I_n\). Thus, \(T|_Z\) yields a lattice embedding from \(Z\) into \(Y \subseteq X\), which concludes the proof.

**Proof of Theorem B.** Suppose first that \(X\) satisfies (1). Then, since every separable Banach space is linearly embeddable into \([0, 1]\), it follows that \(X\) is lattice embeddable into \([0, 1]\).
We show now that (2) implies (1). We do know that (1) holds if \( X \) is lattice isomorphic to \( c_0 \) or finite-dimensional. We suppose then that \( X \) is an infinite-dimensional closed sublattice of \( C[0, 1] \) not lattice isomorphic to \( c_0 \). By Theorem C, \( C[0, 1] \) is lattice embeddable into \( X \). Let \( Y \) be an arbitrary Banach lattice containing an isomorphic copy of \( X \). Then, it contains an isomorphic copy of \( C[0, 1] \), so by Theorem A, \( C[0, 1] \) is lattice embeddable into \( Y \). Nevertheless, since \( X \) is lattice embeddable into \( C[0, 1] \) we conclude that \( X \) is lattice embeddable into \( Y \) as desired. \( \square \)

After Theorem B, one may wonder what are the closed sublattices of \( C[0, 1] \). For \( C(K) \) spaces we can give a characterization. Recall that a compact space is said to be a Peano compactum if it is a continuous image of the interval \([0, 1]\).

**Proposition 3.1.** For a compact space \( K \), \( C(K) \) is lattice embeddable into \( C[0, 1] \) if and only if \( K \) is a disjoint finite union of Peano compacta.

**Proof.** We only prove that if \( C(K) \) is lattice embeddable into \( C[0, 1] \) then \( K \) is a finite union of Peano compacta, which is the nontrivial implication. Let \( T : C(K) \rightarrow C[0, 1] \) be a lattice embedding. By [20, Theorem 3.2.10], we have continuous functions \( u : [0, 1] \rightarrow [0, \infty) \) and \( h : [0, 1] \setminus u^{-1}(0) \rightarrow K \) so that the embedding \( T : C(K) \rightarrow C[0, 1] \) is given by the formula

\[
T(f)(t) = u(t)f(h(t)) \text{ whenever } t \in [0, 1] \setminus u^{-1}(0)
\]

and \( T(f)(t) = 0 \) otherwise. We must have a constant \( C > 0 \) such that \( \|f\| \leq C\|Tf\| \) for every \( f \in C(K) \). This implies that \( h(\{t : u(t) \geq \frac{1}{2C} \}) = K \), because otherwise, taking a function \( f \) of norm 1 that vanishes on \( h(\{t : u(t) \geq \frac{1}{2C} \}) \) we get a contradiction.

Since \( \{t : u(t) \geq \frac{1}{2C} \} \) is closed and \( u^{-1}( (0, \infty) ) \) is open, we can find a cover of \( \{t : u(t) \geq \frac{1}{2C} \} = \bigcup_{i \in I} U_i \), where each \( U_i \) is an open interval such that \( \overline{U_i} \subseteq u^{-1}( (0, \infty) ) \). Since \( \{t : u(t) \geq \frac{1}{2C} \} \) is compact, there is a finite union \( L \) of closed intervals such that \( \{t : u(t) \geq \frac{1}{2C} \} \subseteq L \subseteq u^{-1}( (0, \infty) ) \). Therefore \( K \) is a finite union of Peano compacta. Consider the maximal unions of these finitely many Peano compacta that are path connected. They constitute a partition of \( K \) into compact sets. Each of them is a Peano compactum, because it is easy to see that a path connected finite union of Peano compacta is a Peano compactum. \( \square \)

We finish with a few comments concerning how a nonseparable Banach lattice satisfying condition (1) in Theorem B should look like. Suppose that \( X \) is a Banach lattice such that whenever it is linearly embeddable into a Banach lattice then it is lattice embeddable. There are two distinguished Banach lattices where \( X \) linearly embeds in. The first one is \( C(B_X^*) \), with \( B_X^* \) being the unit ball of \( X^* \) endowed with the weak*-topology. Thus, \( X \) is a closed sublattice of a \( C(K) \)-space or, equivalently, \( X \) is an AM-space by the classical Kakutani-Bohnenblust-Krein Theorem (see, for instance, [1, Theorem 3.6]). On
the other hand, $X$ linearly embeds into the free Banach lattice generated by $X$, denoted by $FBL[X]$. This notion was introduced in [6] and we recall its definition next:

**Definition 3.2.** Let $E$ be a Banach space. The free Banach lattice $FBL[E]$ generated by $E$ is a Banach lattice for which there is an isometry $\phi_E : E \rightarrow FBL[E]$ such that for every Banach lattice $Y$ and every operator $T : E \rightarrow Y$ there exists a lattice homomorphism $\hat{T} : FBL[E] \rightarrow Y$ such that $T = \hat{T} \circ \phi_E$ and $\|T\| = \|\hat{T}\|$.

Thus, if $X$ is lattice embeddable into a Banach lattice whenever it is linearly embeddable then $X$ must be lattice embeddable into $FBL[X]$. One consequence of this fact is that $X$ must have the $\sigma$-bounded chain condition (see Theorem 1.2 and Definition 1.1 in [5]). In particular, $X$ cannot be $c_0(\Gamma)$ for any uncountable $\Gamma$.

These remarks and the results in this article motivate the following questions, for which we do not know the answer.

**Problem 3.3.** Is there a nonseparable Banach lattice $X$ such that $X$ is lattice embeddable in a Banach lattice whenever it is linearly embeddable? Does $C([0, 1]^\Gamma)$ satisfy this property for some uncountable set $\Gamma$?

By Theorem B and Proposition 3.1, $C([0, 1]^\Gamma)$ does have this property when $\Gamma$ is finite or countable, because the finite or countable powers of the interval are Peano compacta. If we restrict our attention to embeddings inside $C(K)$-spaces, we may ask the following.

**Problem 3.4.** If $C([0, 1]^\Gamma)$ linearly embeds into $C(K)$, does $C([0, 1]^\Gamma)$ lattice embed into $C(K)$?

It follows from a result of Plebanek [24, Corollary 4.6] that there is a lattice embedding from $C([0, 1]^\Gamma)$ into $C(K)$ if and only if there is a positive linear embedding, if and only if $K$ maps continuously onto $[0, 1]^\Gamma$. Haydon [11], Plebanek [22] and Fremlin [8] found various conditions on the cardinality of $\Gamma$ that imply that if $\ell_1(\Gamma)$ linearly embeds into $C(K)$, then $K$ maps continuously onto $[0, 1]^\Gamma$, which in turn ensures a positive answer to Problem 3.4, cf. [23]. The question for arbitrary uncountable $\Gamma$ seems however open.

On the other hand, $C([0, 1]^\Gamma)$ does lattice embed into $FBL[C([0, 1]^\Gamma)]$, in fact as a complemented closed sublattice, cf. [3, Proposition 2.2 and Corollary 2.8] and [4, Theorem 1.4].

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