ON WEAKLY ALMOST PERIODIC MEASURES

DANIEL LENZ AND NICOLAE STRUNGAU

Abstract. We study the diffraction and dynamical properties of translation bounded weakly almost periodic measures. We prove that the dynamical hull of a weakly almost periodic measure is a weakly almost periodic dynamical system with unique minimal component given by the hull of the strongly almost periodic component of the measure. In particular the hull is minimal if and only if the measure is strongly almost periodic and the hull is always measurably conjugate to a torus and has pure point spectrum with continuous eigenfunctions. As an application we show the stability of the class of weighted Dirac combs with Meyer set or FLC support and deduce that such measures have either trivial or large pure point respectively continuous spectrum. We complement these results by investigating the Eberlein convolution of two weakly almost periodic measures. Here, we show that it is unique and a strongly almost periodic measure. We conclude by studying the Fourier-Bohr coefficients of weakly almost periodic measures.

Contents

Introduction 2
1. General background and notation 4
2. Measure dynamical systems and weakly almost periodic measures 7
2.1. Measure dynamical systems 7
2.2. Weakly almost periodic functions and measures 8
2.3. Product and weak topology on $M^G(G)$ and $WAP(G)$ 12
3. The hull of a weakly almost periodic measure 13
3.1. The hull is a weakly almost periodic system 13
3.2. Stability of almost periodicity along the hull 17
3.3. The unique minimal component and unique ergodicity 19
3.4. The structure of the hull 22
4. Spectral and diffraction theory of weakly almost periodic measures 25
5. On the hull of the autocorrelation 28
6. Application to weighted Dirac combs 30
6.1. On almost periodic Delone sets 30
6.2. The support of the Eberlein decomposition 31
7. Eberlein convolution of weakly almost periodic measures 33
References 41
INTRODUCTION

Over the past 100 and more years physical diffraction gave us valuable insight into the atomic structure of solids. Periodic crystals produce a clear diffraction pattern, consisting exclusively of bright spots (Bragg peaks) positioned on a lattice (the dual lattice of the periods lattice of the crystal).

While often the atomic structure of the crystal can be reconstructed from the diffraction pattern, this is not always the case: there exists different periodic crystals with the same diffraction pattern. The problem becomes even more complex in the case of quasicrystals, or more generally, systems with mixed spectrum (see e.g. [24] for a recent overview of the inverse problem).

Mathematically, physical diffraction is described as follows: starting from the underlying structure, which is modeled by a discrete set or more generally by a measure, we construct a measure $\gamma$, called the autocorrelation measure, which encodes the frequencies of vectors between atoms in the structure. This measure is positive definite, therefore Fourier transformable with a positive Fourier transform $\hat{\gamma}$ [2, 12, 40]. The positive measure $\hat{\gamma}$ is called the diffraction measure or diffraction pattern and models the physical diffraction of the original solid. The Lebesgue decomposition $\hat{\gamma} = \hat{\gamma}_d + \hat{\gamma}_c$ gives us a decomposition of the diffraction into the discrete or pure point part $\hat{\gamma}_d$ and the continuous part $\hat{\gamma}_c$. (The continuous part can then be further split into the absolutely continuous part and the singularly continuous part.) The pure point component $\hat{\gamma}_d$ can be written as $\hat{\gamma}_d = \sum_{\chi \in B} \hat{\gamma}(\{\chi\}) \delta_{\chi}$, where $B$ is the set of Bragg peaks.

As any positive definite translation bounded measure, the autocorrelation is a weakly almost periodic measure [28, 40]. The Eberlein decomposition $\gamma = \gamma_s + \gamma_0$, into the strong and null weakly almost periodic part is the Fourier dual of the above mentioned Lebesgue decomposition of $\hat{\gamma}$ into the discrete and continuous part [40, 28] or [45, Thm. 2.8]. This implies in particular that $\gamma$ is strongly almost periodic if and only if $\hat{\gamma}$ is a pure point measure.

Recently there is also increased interest in diffraction of systems which are described by more general structures than measures [11, 32, 49, 50]. In particular, [11, 49, 50] deal with tempered distributions. Also, in these cases we end up with weakly almost periodic distributions and there exists also an Eberlein decomposition of tempered distributions which is dual to the Lebesgue decomposition [48].

Almost periodicity has long been known to play a role in the study of pure point diffraction, see e.g. the paper [43] of Solomyak for an early explicit investigation. The possible connections between almost periodicity properties of the original point set or measure and pure point diffraction were then the topic of a survey article of
Lagarias [27], which contains many questions and conjectures on this topic. The actual statement on the equivalence of pure point diffraction and strong almost periodicity of the autocorrelation measure discussed above was first given explicitly in the work of Baake-Moody [10]. The main thrust of their work concerns the case of measures with Meyer set support. In that case, strong almost periodicity of the autocorrelation can even be replaced by the stronger notion of norm almost periodicity [10]. Connections of strong almost periodicity of measures and pure point diffraction were then also studied by Moody-Strungaru [36] and Gouéré [23] (see also Favarov [20] and Kellendonk-Lenz [25] for related material discussing strongly almost periodic point sets). Also, the connection between almost periodicity and the cut and project formalism thoroughly investigated by Baake-Moody [10], Richard [38], Lenz-Richard [33] and Strungaru [47]. As shown recently in Lenz [31] it is even possible to extend parts of the above theory from dynamical systems based on measures to general dynamical systems allowing for a metric.

All these results emphasize that the class of (weakly) almost periodic measures is important for diffraction theory. It is our goal in this paper to systematically study the long range order properties of this class via their dynamical system and the diffraction measure.

As far as methods and tools go the paper builds up on two related theories. On the one hand it can be seen as a continuation of the work of Eberlein [16, 17, 18] on weakly almost periodic functions and measures in the context of pure point diffraction. On the other hand it can be seen as an application of the theory of weakly almost periodic systems to a specific class of measure dynamical systems. This theory was started by Ellis and Nerurkar [19] and has later been extended in various works, see e.g. the recent work of Akin-Glasner [1], where various further references can be found.

The paper is organized as follows: we start by reviewing basic concepts in harmonic analysis on locally compact abelian groups in Section 1. There, we also introduce one of the main players of our investigation viz the autocorrelation of a measure. We will be interested in the autocorrelation of weakly almost periodic measures and much of our study is built on dynamical systems. The corresponding background dynamical systems, measure dynamical systems and almost periodic measures and corresponding topologies on the set of all translation bounded measures is discussed in Section 2. This material is essentially all known. We discuss it at some length in order to keep the present paper accessible to a wide audience.

Our actual investigation starts in the subsequent Section 3. There, we present a thorough analysis of the hull of a weakly almost periodic measure. In particular, we show in Section 3.1 that the hull is a weakly almost periodic dynamical system. In the subsequent Section 3.2 we discuss how the classes of weakly, null-weakly and strongly almost periodic measures are stable under taking the hull. We show that the hull of a weakly almost periodic measure has a unique minimal component in Section 3.3. In this context, a main result of the paper, Theorem 3.15, is proven. This result shows that for every weakly almost periodic measure \( \mu \), its strongly almost periodic component \( \mu_s \) belongs to the hull \( X(\mu) \). Most of the subsequent
considerations rely on this result. One implication is that the hull generated by \( \mu_s \) is the unique minimal component. In fact, based on this result we can discuss the structure of the hull of a weakly almost periodic system in quite some detail in the subsequent Section 3.3. There, we show that the projections \( P_s, P_0 \) onto the strongly and the null-weakly almost periodic part are continuous \( G \)-mappings with range contained inside \( X(\mu_s) \) respectively \( X(\mu_0) \).

In Section 4 we show that the dynamical system of a weakly almost periodic measure \( \mu \) has pure point dynamical and diffraction spectra. Here, we also provide a closed formula for the eigenfunctions of the system, which are shown to be continuous, and for the diffraction measure. When combining this with Theorem 3.12 from Section 3.2 we infer as a remarkable feature of our setting that the eigenfunctions relevant for diffraction are just given by the Fourier coefficients of the measure in question.

The preceding results on weakly almost periodic measures have an application to arbitrary measure dynamical systems. The reason is that each measure dynamical system gives rise to an autocorrelation which is weakly almost periodic. It turns out that the diffraction of the dynamical system arising as the hull of the autocorrelation is just the pure point part of the diffraction of the original system. This is discussed in Section 5.

Our results have various application to Delone sets. This is discussed in Section 6. There, we first relate in Section 6.1 for weakly almost periodic weighted Dirac combs \( \mu \) with FLC support, the support of \( \mu_s \) and of \( \mu_0 \) to the support of \( \mu \). We then use our results to give rather direct alternative proofs to recent results on the support of the Eberlein decomposition for weighted Dirac combs. This is discussed in Section 6.2.

We complete the paper by looking at the Eberlein convolution of weakly almost periodic measures in Section 7. We show that given any two weakly almost periodic measures they have an unique Eberlein convolution, which exists with respect to any van Hove sequence and is strongly almost periodic. As a consequence we get (an alternative proof) that any weakly almost periodic measure has pure point diffraction and hence pure point dynamical spectrum.

By combining weakly almost periodic dynamical systems and aperiodic order in our paper we emphasize that there is a deep connection between these concepts, which could be of interest to mathematicians from both areas. In order to keep the paper accessible for both communities we have taken special care to discuss basic concepts at sufficient length.

1. **General background and notation**

For this entire paper \( G \) denotes a locally compact Abelian group (LCAG). We will denote by \( C_u(G) \) the space of uniformly continuous and bounded functions on \( G \), and by \( C_c(G) \) the subspace of \( C_u(G) \) consisting of functions with compact support.

For a compact set \( K \subset G \) we define
\[
C(G : K) := \{ f \in C_c(G) | \text{supp}(f) \subset K \}.
\]
The space \((C(G : K), \| \cdot \|_x)\) is a Banach space. On \(C_c(G)\) we define a locally convex topology as the inductive limit topology defined by the embeddings \(C(G : K) \hookrightarrow C_c(G)\), for \(K \subset G\) compact.

For a function \(f\) on \(G\) we define the functions \(f^\dagger\) and \(\tilde{f}\) on \(G\) by

\[
f^\dagger(x) = f(-x) \quad \text{and} \quad \tilde{f}(x) = \overline{f(-x)}.
\]

Similarly, for a measure \(\mu\) we define a new measure \(\tilde{\mu}\) by

\[
\tilde{\mu}(f) := \mu(\tilde{f}).
\]

By Riesz-Markov theorem, a measure \(\mu\) on \(G\) can be viewed as a linear functional on \(C_c(G)\), which is continuous with respect to the inductive topology on \(C_c(G)\). We will refer to the weak-* topology of this duality as the vague topology for measures.

Any measure \(\mu\) gives rise to an unique positive measure \(|\mu|\), called the variation measure of \(\mu\), satisfying

\[
|\mu| (f) = \sup\{|\mu(g)| : g \in C_c(G : \mathbb{R}), \ |g| \leq f\},
\]
for any non-negative \(f \in C_c(G)\). We can define the convolution between \(c \in C_c(G)\) and \(g \in C_c(G)\) as

\[
c * g(x) = \int_G c(x-t)g(t)dt.
\]

Then \(c * g \in C_c(G)\). Similarly, the convolution between a function \(c \in C_c(G)\) and a measure \(\mu\) is the function

\[
c * \mu(x) = \int_G c(x-t)d\mu(t).
\]

We will restrict attention to a class of measures which are “equi-bounded” on \(G\):

**Definition 1.1.** A measure \(\mu\) on \(G\) is called translation bounded if for each compact \(K \subset G\) we have

\[
\|\mu\|_K := \sup_{x \in G}|\mu|(x + K) < \infty
\]

where \(|\mu|\) denotes the variation of \(\mu\). The space of all translation bounded measures on \(G\) is denoted by \(\mathcal{M}^e(G)\).

To check that a measure is translation bounded, it is sufficient to check that (1) holds for one fixed compact set \(K\) with non-empty interior [10]. Also, the translation boundedness property of measures can be understood via convolutions with compactly supported continuous functions. Indeed, by [2 Thm.1.1] a measure \(\mu\) is translation bounded if and only if for all \(c \in C_c(G)\) we have \(c * \mu \in C_c(G)\). The convolution of function can be extended to measures the following way: We say that the measures \(\mu\) and \(\nu\) are convolvable, if for all \(f \in C_c(G)\) the function \((x, y) \to |f(x + y)|\) is in \(L^1(|\mu| \times |\nu|)\). In this case, we define the convolution \(\mu * \nu\) by

\[
\mu * \nu(f) = \int_G \int_G f(x+y)d\mu(x)d\nu(y).
\]
In general for translation bounded measures we can try to also define an averaged convolution (Eberlein convolution) of the measures. This point will be taken up and discussed extensively later on in the last section of the article. Here, we introduce next what makes a good averaging set.

**Definition 1.2.** A sequence \((A_n)\) of compact subsets of \(G\) called a **van Hove sequence** if for all compact sets \(K \subset G\) we have
\[
\lim_{n \to \infty} \frac{\theta_G(\partial^K(A_n))}{\theta_G(A_n)} = 0,
\]
where the **K-boundary** is defined by:
\[
\partial^K(A_n) = ((A_n + K) \setminus A_n) \cup ((G \setminus A_n) \cap B_n).
\]

A sequence \((A_n)\) of compact subsets of \(G\) is called a **Fölner sequence** if for all \(x \in G\) we have
\[
\lim_{n \to \infty} \frac{\theta_G(A_n \Delta (x + A_n))}{\theta_G(A_n)} = 0.
\]

It is easy to see that any van Hove sequence is a Fölner sequence.

We are now ready to introduce the notion of autocorrelation measure.

**Definition 1.3.** Let \(\mu \in \mathcal{M}(G)\) be a measure and let \((A_n)\) be a van Hove sequence. We say that \(\mu\) has the autocorrelation \(\gamma\) with respect to \((A_n)\) if the following vague limit exists:
\[
\gamma := \lim_{n \to \infty} \frac{1}{\theta_G(A_n)} \mu|_{A_n} * \widetilde{\mu|_{A_n}},
\]
where \(\mu|_{A_n}\) denotes the restriction of the measure \(\mu\) to the set \(A_n\). (Here, the convolution makes sense as the measures \(\mu|_{A_n}\) and \(\widetilde{\mu|_{A_n}}\) are finite due to the compactness of the \(A_n\).

Note that existence of the limit in the definition is rather a matter of convention. Indeed, the limit will always exist for a suitable subsequence of \((A_n)\). If the limit exists for all van Hove sequences \((A_n)\), it must be the same for all. In this case we will say that \(\mu\) has an **unique autocorrelation**.

The autocorrelation has an extra positivity property which we introduce next:

**Definition 1.4.** A function \(f : G \to \mathbb{C}\) is called **positive definite** if for all \(N \in \mathbb{N}\) and \(x_1, \ldots, x_N \in G\), the matrix \((f(x_k - x_l))_{k,l=1,\ldots,N}\) is positive Hermitian.

A measure \(\mu\) on \(G\) is called **positive definite** if for all \(f \in C_c(G)\) we have
\[
\mu(f * \tilde{f}) \geq 0.
\]

It is well known (see e.g. [12]) that a measure \(\mu\) is positive definite if and only if for all \(f \in C_c(G)\) the function \(\mu * (f * \tilde{f})\) is positive definite.

It is also easy to see that for any finite measure \(\nu\), the measure \(\nu * \tilde{\nu}\) is positive definite, and that vague limits of positive definite measures are positive definite. Thus, we immediately obtain the following corollary.

**Corollary 1.5.** Let \(\gamma\) be the autocorrelation of \(\mu\) with respect to the van Hove sequence \((A_n)\). Then, \(\gamma\) is positive definite.
2. Measure dynamical systems and weakly almost periodic measures

The use of dynamical systems in the study of long range order has a long history, see e.g. the survey \[42\] for an early account. As in statistical mechanics, the basic idea is to look at ensembles rather than at individual manifestations. More specifically, the idea is to construct out of a given point set \( \Lambda \subset G \), the collection \( \mathcal{X}(\Lambda) \) of all point sets which locally look like \( \Lambda \). This collection becomes compact with respect to the local rubber topology, and comes with a natural \( G \)-action, therefore it becomes a topological dynamical system. The diffraction of \( \Lambda \) can then be connected to the dynamical spectrum of \( \mathcal{X}(\Lambda) \) via the Dworkin argument \[7, 23, 30, 15\], and all these ideas can be generalized to measures \[7, 34\]. We recommend \[8\] for a recent review of this.

2.1. Measure dynamical systems. We will be interested in dynamical systems coming from translation bounded measures.

Whenever \( X \) is a compact space equipped with a continuous action \( T : G \times X \longrightarrow X \) of the group \( G \), we call \((X, T)\) a dynamical system. Such a system is called minimal if the orbit \( \{ T^t x : t \in G \} \) of \( x \in X \) is dense in \( X \) for every \( x \in X \). If there exists only one dense orbit \( \{ T^t x : t \in G \} \) then the system is called transitive. A system is called uniquely ergodic if there exists only one \( G \)-invariant probability measure on \( X \). A special focus of ours will be on continuous eigenfunctions. Here, an \( f \in C(X) \) with \( f \neq 0 \) is called an eigenfunction to \( \chi \in \hat{G} \) if

\[
f(T^t x) = \chi(t) f(x)
\]

holds for all \( t \in G \) and \( x \in X \). Such a \( \chi \) is called a continuous eigenvalue.

We will be interested in special transitive dynamical systems coming from measures and more specifically coming from weakly almost periodic measures. In this context we define for a translation bounded measure \( \mu \) on \( G \) the measure \( T^t \mu \) by

\[
T^t \mu := \delta_t \ast \mu,
\]

where \( \delta_t \) denotes the unique point mass at \( t \in G \).

It is not hard to see that the translation action

\[
G \times \mathcal{M}^c(G) \longrightarrow \mathcal{M}^c(G), (t, \mu) \mapsto T^t \mu,
\]

is continuous when restricted to compact \( G \)-invariant subsets of \( \mathcal{M}^c(G) \). So any such compact \( G \)-invariant \( X \) is a dynamical system \((X, T)\). We refer to such a dynamical system as measure dynamical system.

In particular, any \( \mu \in \mathcal{M}^c(G) \) gives rise to a dynamical system \( \mathcal{X}(\mu) \) defined as

\[
\mathcal{X}(\mu) := \{ T^t \mu \mid t \in G \},
\]

where the closure is taken in the vague topology. This dynamical system is called the hull of \( \mu \). We recommend \[7\] for more details.

A crucial feature of any measure dynamical system \((X, T)\) is that there is a linear map \( \phi \) from \( C_c(G) \) to the set \( C_c(X) \) of continuous functions on \( X \). More specifically,
each \( c \in C_c(G) \) defines a function \( \phi_c : M^\infty \rightarrow \mathbb{C} \) via
\[
\phi_c(\nu) = c * \nu(0) = \int_G c(-x) d\nu(x).
\]
These functions are continuous \([7]\), and in fact the vague topology on \( M^\infty(G) \) is the weakest topology which makes these functions continuous \([7]\). When dealing with a given measure dynamical system we can (and will tacitly) restrict these functions to the dynamical system. Accordingly, these functions will appear often in our computations. We will also make often use of
\[
\phi_c(T^t \nu) = c * \nu(t).
\]

2.2. Weakly almost periodic functions and measures. In this section we review the basic definitions and properties of weakly almost periodic functions and measures.

We start by recalling the notions of almost periodicity we will use. Here, for a function \( f \) on \( G \) and \( t \in P(G) \) we define the translate \( T^t f \) of \( f \) by \( t \) to be the function with
\[
T^t f(x) = f(-t + x)
\]
for all \( x \in G \).

**Definition 2.1.** A function \( f \in C_u(G) \) is called **Bohr-almost periodic** if the set \( \{ T^t f | t \in G \} \) is precompact in \( (C_u(G), \| \|_\infty) \). The set of all Bohr-almost periodic function on \( G \) is denoted by \( SAP(G) \).

A function \( f \in C_u(G) \) is called **weakly almost periodic** if the set \( \{ T^t f | x \in G \} \) is precompact in the weak topology of the Banach space \( (C_u(G), \| \|_\infty) \). The space of all weakly almost periodic functions is denoted by \( WAP(G) \).

We summarize now the basic structural stability properties of the spaces \( WAP(G) \) as proven in \([16]\) and of \( SAP(G) \) as discussed recently in \([40]\).

**Theorem 2.2.** \([16]\) Thm. 11.2, Thm. 12.1, \([40]\).

(a) \( SAP(G) \) and \( WAP(G) \) are closed subspaces of \( (C_u(G), \| \|_\infty) \).
(b) \( SAP(G) \subset WAP(G) \).
(c) \( SAP(G) \) and \( WAP(G) \) are closed under translation, reflection and complex conjugation.
(d) \( SAP(G) \) and \( WAP(G) \) are closed under multiplication.
(e) If \( f \in SAP(G) \) and \( p \geq 0 \) then \( |f|^p \in SAP(G) \).
(f) If \( f \in WAP(G) \) and \( p \geq 0 \) then \( |f|^p \in WAP(G) \).

For our further investigation we will also need the following fundamental fact.

**Theorem 2.3.** \([16]\) If \( f \in C_u(G) \) is a positive definite function on \( G \) then \( f \) belongs to \( WAP(G) \).

Next we recall the notion of amenability for continuous functions.
Definition 2.4. Let \((A_n)\) be a Föllner sequence. A function \(f \in C_u(G)\) is called **amenable** if there exists a number \(M(f)\), called the **mean** of \(f\) such that
\[
\lim_{n} \frac{1}{\theta_G(A_n)} \int_{x+A_n} f(t) dt = M(f),
\]
uniformly in \(x \in G\).

**Remark.** [16] The definition of amenability and the mean are independent of the choice of the Föllner sequence.

The basic properties of the mean \(M : WAP(G) \to \mathbb{C}\) are summarised below.

**Theorem 2.5.** [16, Thm. 14.1] Any weakly almost periodic function \(f\) is amenable.

If \(f, g \in WAP(G), C_1, C_2 \in \mathbb{C}\) and \(t \in G\) then we have:

(a) \(M(C_1 f + C_2 g) = C_1 M(f) + C_2 M(g)\).
(b) \(M(1) = 1\).
(c) If \(f \geq 0\) then \(M(f) \geq 0\).
(d) \(|M(f)| \leq M(|f|) \leq \|f\|_x\).
(e) \(M(T^t f) = M(f)\).
(f) \(M(f^1) = M(f)\).
(h) \(|M(fg)|^2 \leq (M(|f|^2)) (M(|g|^2))\).

Moreover, if \(L : WAP(G) \to \mathbb{C}\) is a function which satisfies (a), (b), (c), (d) and (f) then \(L(f) = M(f)\) for all \(f \in WAP(G)\).

As a consequence of Theorem 2.4 and Theorem 2.5 we note the following.

**Corollary 2.6.** [16] Let \(f \in WAP(G)\). Then \(M(|f|) = 0\) if and only if \(M(|f|^2) = 0\).

We are now ready to introduce the notion of null-weakly almost periodicity.

**Definition 2.7.** A function \(f\) is called **null weakly almost periodic** if \(f \in WAP(G)\) and \(M(|f|) = 0\). We will denote the space of null weakly almost periodic functions by \(WAP_0(G)\).

The relevance of the space of null-weakly almost periodic functions comes from the following result.

**Theorem 2.8.** [15] \(WAP(G) = SAP(G) \oplus WAP_0(G)\).

The theorem says that any weakly almost periodic function \(f\) can uniquely be written as the sum
\[
f = f_s + f_0,
\]
of a strongly almost periodic function \(f_s\) and a weakly almost periodic function \(f_0\). We will refer to this decomposition as the **Eberlein decomposition** of weakly almost periodic functions.

Next we see the Eberlein convolution of functions:
Definition 2.9. If \( f, g \in WAP(G) \) we can define a new function \( f \circledast g \) via
\[
 f \circledast g(x) = M_t(f(x-t)g(t)),
\]
where \( M_t \) denotes the mean with respect to the variable \( t \). The function \( f \circledast g \)
is called the **Eberlein convolution** of \( f \) and \( g \). (For a fixed \( x \in G \) we have \( f(x-\cdot)g(\cdot) \in WAP(G) \) and therefore the mean exists.)

Theorem 2.10. [16, Thm. 15.1] Given \( f, g \in WAP(G) \) we have \( f \circledast g \in SAP(G) \) and
\[
 f \circledast g(x) = M_t(f(t)g(x-t)).
\]

The above concepts can be extended to measures via convolutions. This has been carried out by deLamadrid and Argabright [28] (see also [40]):

Definition 2.11. A translation bounded measure \( \mu \) is called **strongly, weakly** respectively **null weakly almost periodic** if for all \( c \in C_c(G) \) the function \( c \ast \mu \) is Bohr, weakly respectively null weakly almost periodic. We denote the spaces of weakly, strongly and null weakly almost periodic measures by \( WAP(G) \), \( SAP(G) \) respectively \( WAP_0(G) \).

Exactly as for functions, we can define in a simple way the mean of a weakly almost periodic measure.

Proposition 2.12. [28, 40] Let \( \mu \) be a weakly almost periodic measure. Then, there exists a constant \( M(\mu) \) such that, for all \( c \in C_c(G) \) we have
\[
 M(c \ast \mu) = M(\mu) \int_G c(t) dt.
\]

Definition 2.13. Given a measure \( \mu \in WAP(G) \), we call the number \( M(\mu) \) from Proposition 2.12 the **mean** of \( \mu \).

The mean of a measure can be calculated by a formula which is similar to the formula for functions.

Lemma 2.14. [40] Let \( \mu \in WAP(G) \) and \( (A_n) \) a van-Hove sequence on \( G \). Then
\[
 M(\mu) = \lim_{n} \frac{\mu(A_n)}{\theta_G(A_n)}.
\]

The Eberlein decomposition can then be extended to measures:

Theorem 2.15. [28, 40] \( WAP(G) = SAP(G) \oplus WAP_0(G) \).

As for functions, we will refer to this decomposition as the **Eberlein decomposition**.

Let us emphasize here that the space of null weakly almost periodic measures is more enigmatic than the space of null weakly almost periodic functions: Consider a weakly almost periodic measure \( \mu \). Then, by definition, it is null weakly almost periodic if and only if for all \( c \in C_c(G) \) we have \( M(|c \ast \mu|) = 0 \). However, this is not equivalent to \( M(|\mu|) = 0 \) (even if the variation measure \( |\mu| \) is weakly almost periodic). Indeed, as proven in [44, 47], for weakly almost periodic measures with
uniformly discrete support we have the equivalence $\mu \in WAP_0(G)$ if and only if $M(|\mu|) = 0$, but in general we only have the implication $M(|\mu|) = 0 \Rightarrow \mu \in WAP_0(G)$. The measure

$$\mu = \sum_{n \in \mathbb{Z} \setminus \{0\}} \delta_{m+\frac{1}{m}} - \delta_m,$$

is a null weakly almost periodic measure, but $M(|\mu|) = 2$, see [44].

Next we introduce the Fourier Bohr coefficients of a weakly almost periodic measure. Whenever $f$ is strongly almost periodic function and $\mu$ is a weakly almost periodic measure we have $f \mu \in WAP(G)$ [28 Thm. 6.2]. We can thus define.

**Definition 2.16.** The **Fourier Bohr** coefficients of a weakly almost periodic measure are defined for each character $\chi \in \hat{G}$ as

$$c_\chi(\mu) := M(\bar{\chi}\mu).$$

Similarly, if $f \in WAP(G)$ we define the **Fourier Bohr** coefficients of $f$ as

$$c_\chi(f) := M(\bar{\chi}f).$$

It is easy to see that $c_\chi(f) = c_\chi(f \theta_G)$.

As proven in [28 Thm. 8.1], the null weakly almost periodic measures/functions are exactly the weakly almost periodic measures/functions with vanishing Fourier-Bohr coefficients. It follows from the uniqueness of the Eberlein decomposition that a strongly almost periodic measure/function is uniquely determined by its Fourier Bohr coefficients.

The Fourier Bohr coefficients of the Eberlein convolution are exactly the products of the Fourier Bohr coefficients of the two functions. This result is proven in [16 Lemma 1.15] in the particular case $g = \tilde{f}$ and in [28] in the case of a convolution between a strong almost periodic function and a weakly almost periodic measure. For completeness reasons we include a proof (which is the standard one) for the general case.

**Theorem 2.17.** Let $f, g \in WAP(G)$ and $\chi \in \hat{G}$. Then

$$c_\chi(f \circ g) = c_\chi(f)c_\chi(g).$$

**Proof.** A direct computation gives

$$c_\chi(f \circ g) = M_x(\bar{\chi}(x)f \circ g(x)) = M_x(\bar{\chi}(x)M_t(f(x-t)g(t)))
= M_x(M_t(\bar{\chi}(x)f(x-t)g(t))).$$

Now, by the Fubini Theorem for the mean [16 Thm.14.2], we have

$$c_\chi(f \circ g) = M_t(M_x(\bar{\chi}(x)f(x-t)g(t))) = M_t(\bar{\chi}(t)g(t)M_x(\bar{\chi}(x-t)f(x-t)))
= M_t(\bar{\chi}(t)g(t)\tilde{a}_\chi(f)).$$

As $a_\chi(f)$ is a constant, the claim follows. □
2.3. **Product and weak topology on** \( \mathcal{M}^\infty(G) \) and \( \mathcal{WAP}(G) \). In this section we provide another perspective on the weak and strong almost periodic measures by considering two further topologies on \( \mathcal{M}^\infty(G) \).

Consider the natural embedding

\[
\mathcal{M}^\infty(G) \hookrightarrow [C_0(G)]^{C_0(G)} : \mu \to \{c \ast \mu\}_{c \in C_0(G)}.
\]

Via this embedding the product topology on \([C_0(G)]^{C_0(G)}\) induces a topology on \(\mathcal{M}^\infty(G)\). This topology is called the **product topology for measures**. Let us note that the product topology is a locally convex topology defined by the family of seminorms \(\|c\|_{C_0(G)}\) given by

\[
\|\mu\|_c := \|c \ast \mu\|_\infty.
\]

Thus, it also allows for a corresponding weak topology. We will refer to this weak topology as the **weak topology for measures**.

**Proposition 2.18.** *The vague topology is weaker than the weak topology for measures*.

**Proof.** This follows since for each \(c \in C_0(G)\) the mapping

\[
\mu \to c \ast \mu,
\]

is a linear functional which is continuous with respect to the product topology. □

**Remark.** It is not known if the image of \(\mathcal{M}^\infty(G)\) under the above embedding is closed in \([C_0(G)]^{C_0(G)}\), but it is bounded closed, and hence quasi-complete (See [28, Thm. 2.4, Cor. 2.1]).

Here comes a characterization of (weak) almost periodicity of measures via the previously introduced topologies.

**Theorem 2.19.** [28] Let \(\mu \in \mathcal{M}^\infty(G)\). Then

(a) A measure \(\mu\) belongs to \(\mathcal{SAP}(G)\) if and only if \(\{T_t \mu| t \in G\}\) is precompact in the product topology.

(b) A measure \(\mu\) belongs to \(\mathcal{WAP}(G)\) if and only if \(\{T_t \mu| t \in G\}\) is precompact in the weak topology of the product topology.

We complete the section by showing the continuity of the functions \(c_\chi\) on \(\mathcal{WAP}(G)\). We start with a standard lemma from topology.

**Lemma 2.20.** [29] Let \(\tau_1\) and \(\tau_2\) be topologies on \(X\) such that \(\tau_1\) is stronger than \(\tau_2\) and \(\tau_2\) is a Hausdorff topology. If \(A \subset X\) is a compact set with respect to \(\tau_1\) then \(A\) is compact with respect to \(\tau_2\) and the two topologies coincide on \(A\).

**Theorem 2.21.** Let \(\chi \in \hat{G}\). Then, the Fourier coefficient \(c_\chi : \mathcal{WAP}(G) \to \mathbb{C}\) is weakly continuous on \(\mathcal{WAP}(G)\). In particular, if \(K\) is any weakly compact subset of \(\mathcal{WAP}(G)\), then \(c_\chi\) is vaguely continuous on \(K\).
Proof. First, let us observe that we have
\[ |c_\chi(f)| \leq \|f\|_\infty, \]
This shows that \( c_\chi \) is continuous on \( (\text{WAP}(G), \|\cdot\|_\infty) \). Then, by the Hahn-Banach Theorem, we can extend this to a continuous functional \( f_\chi \) on \( (C_u(G), \|\cdot\|_\infty) \). Furthermore, by the definition of the product topology, for each \( c \in C_c(G) \) the mapping
\[ (\mathcal{M}_c(G), \text{product topology}) \to (C_u(G), \|\cdot\|_\infty), \mu \to c * \mu \]
is continuous from \( \mathcal{M}_c \). It follows that this mapping is weakly continuous, see e.g. [40]. Therefore, the composition \( \mu \to f_\chi(c * \mu) \) is a weakly continuous mapping on \( \mathcal{M}_c \), and hence its restriction \( \mu \to c_\chi(c * \mu) \) is weakly continuous on \( \text{WAP}(G) \). Finally, let us pick some \( c \in C_c(G) \) with \( \tilde{c}(\chi) \neq 0 \). Then, as
\[ c_\chi(c * \mu) = \tilde{c}(\chi)c_\chi(\mu), \]
we get the weak continuity of \( c_\chi \) on \( \text{WAP}(G) \).

The last statement follows as by Lemma 2.20 and Proposition 2.18 the vague and weak topologies coincide on \( K \). □

Remark. When we extended \( c_\chi \) to a continuous linear functional on \( C_u(G) \) in order to get the weak continuity of \( c_\chi \) we have actually seen a very particular case of the following general phenomena, which is an immediate consequence of the Hahn-Banach Theorem: If \( (E, \|\|) \) is a Banach space, and \( F \) is a closed subspace of \( E \), then the weak topology of \( (F, \|\|) \) is the same as the inherited weak topology on \( F \) as a subspace of \( E \).

3. The hull of a weakly almost periodic measure

In this section we present a rather thorough study of the hull of a weakly almost periodic measure.

3.1. The hull is a weakly almost periodic system. In this section we provide an important structural insight concerning the hull of a weakly almost periodic measure: This hull is a weakly almost periodic dynamical system. This makes a wealth of results available to the study of weakly almost periodic measures.

Whenever \( (\mathcal{X}, T) \) is a dynamical system a function \( f \in C(\mathcal{X}) \) is called weakly almost periodic if the set \( \{f \circ T^t : t \in G\} \) is relatively compact in the weak topology of \( (C(\mathcal{X}), \|\cdot\|_\infty) \). We denote by \( \text{WAP}(\mathcal{X}) \) the subset of \( C(\mathcal{X}) \) consisting of weakly almost periodic functions. A dynamical system \( (\mathcal{X}, T) \) is called weakly almost periodic if every \( f \in C(\mathcal{X}) \) is an weakly almost periodic function, or equivalently if \( C(\mathcal{X}) = \text{WAP}(\mathcal{X}) \).

In order to prove the main result of this section we next carry out a study of \( \text{WAP}(\mathcal{X}) \) for general dynamical systems. As is will appear in many proofs, we define weak the hull of \( f \) by
\[ \mathbb{W}(f) := \{f \circ T^t : t \in G\} \]
the closure being in the weak topology of \( C(\mathcal{X}) \).
Lemma 3.1. ([16] Thm. 1.3, [40] Prop. 1.4.8) If $\Omega$ is a compact space, and $f_n, f \in C(\Omega)$, then $f_n \to f$ weakly if and only if $\|f_n\|_\infty$ is bounded and $f_n \to f$ pointwise.

Next, we will often make use of the following classical result, which allows us to use sequences instead of nets to prove weakly almost periodicity.

Lemma 3.2 (Grothendieck), [22] Thm. 1.43 (1)) Let $f \in C(\mathcal{X})$. Then $f \in WAP(\mathcal{X})$ if and only if for each sequence $(t_n)$ in $G$ we can find a subsequence $(t_{k_n})$ such that $f \circ T^{t_{k_n}}$ is weakly convergent to some $g \in C(\mathcal{X})$.

We are now ready to prove a few useful results about the space $WAP(\mathcal{X})$. Note that the proofs are almost identical to the ones from [16, 40], but since those in the cited papers dealt only with $WAP(G)$, we are in a completely different situation. Nevertheless the basic ideas and tools remain the same.

Proposition 3.3. $WAP(\mathcal{X})$ is a subalgebra of $C(\mathcal{X})$ which contains the constant function 1.

Proof. Let $f, g \in WAP(\mathcal{X})$, and let $(t_n)$ be any sequence in $G$. By $f \in WAP(\mathcal{X})$ there exists by Corollary 3.2 a subsequence $(t_{k_n})$ such that $f \circ T^{t_{k_n}}$ is weakly convergent to some $f_1 \in C(\mathcal{X})$. Similarly, by $g \in WAP(\mathcal{X})$, there exists again by Corollary 3.2 a subsequence $(t_{l_n})$ of $(t_{k_n})$ such that $g \circ T^{t_{l_n}}$ is weakly convergent to some $g_1 \in C(\mathcal{X})$.

As $f \circ T^{t_{l_n}}$ respectively $g \circ T^{t_{l_n}}$ converge weakly to $f_1$ and $g_1$, respectively, by Lemma 3.1 $f \circ T^{t_{l_n}}$ respectively $g \circ T^{t_{l_n}}$ are equi-bounded and converge pointwise to $f_1$ and $g_1$, respectively. Then, it is immediate to see that $(f + g) \circ T^{t_{l_n}}$ respectively $(f \cdot g) \circ T^{t_{l_n}}$ are equi-bounded and converge pointwise to $f_1 + g_1$ and $f_1 \cdot g_1$, respectively. It now follows from Corollary 3.2 that $f + g, f \cdot g \in WAP(\mathcal{X})$.

It is trivial to check that $WAP(\mathcal{X})$ is closed under scalar multiplication, and it contains the constant function 1. \qed

Proposition 3.4. $WAP(\mathcal{X})$ is closed subspace of $(C(\mathcal{X}), \| \cdot \|_\infty)$.

Proof. Since $(C(\mathcal{X}), \| \cdot \|_\infty)$ is a Banach space, it suffices to work with sequences.

Let $(f_n)$ be a sequence in $WAP(\mathcal{X})$ such that, in the sup norm, $f_n \to f \in C(\mathcal{X})$. We need to show that $f \in WAP(\mathcal{X})$. We prove this by using Corollary 3.2. Let $(t_m)$ in $G$ be arbitrary. Then, we can pick some subsequence $(t_{m_n})$ of $(t_m)$ such that $f_1 \circ T^{t_{m_n}}$ is weakly convergent to some $g_1 \in C(\mathcal{X})$. By induction, for each $n \geq 2$ we can pick some subsequence $(t_{m_n})$ of $(t_{m-1})$ such that $f_n \circ T^{t_{m_n}}$ is weakly convergent to some $g_n \in C(\mathcal{X})$. Consider now the diagonal $s_m := t_{m \cdot m}, m \in \mathbb{N}$. Then, for all $n$ we have

$$w - \lim_{m} f_n \circ T^{s_m} = g_n.$$
We show next that \( g_n \) is Cauchy in \((C(X), \| \cdot \|_\infty)\). Indeed
\[
\|g_n - g_p\|_\infty = \sup_{\psi \in C(X) : \|\psi\|_1 = 1} |\psi(g_n) - \psi(g_p)| = \sup_{\psi \in C(X) : \|\psi\|_1 = 1} \lim_m \|\psi(f_n \circ T^{s_m} - f_p \circ T^{s_m})\| = \sup_{\psi \in C(X) : \|\psi\|_1 = 1} \lim_m \|\psi||f_n - f_p\| \circ T^{s_m} = \|f_n - f_p\|_\infty
\]

Therefore, as \((C(X), \| \cdot \|_\infty)\) is complete, the sequence \((g_n)\) converges to some \( g \in C(X) \).

We claim now that \( f \circ T^{s_m} \) converges weakly to \( g \). This claim, once proven, completes the proof by Corollary 3.2.

Let \( \epsilon > 0 \) and \( \psi \in C(X)' \). Since \( g_n \to g \) and \( f_n \to f \), there exists some \( N \) such that for all \( n > N \) we have
\[
\|g_n - g\|_\infty < \frac{\epsilon}{\|\phi\| + 1} \quad \text{and} \quad \|f_n - f\|_\infty < \frac{\epsilon}{\|\phi\| + 1}
\]

Pick some \( n_0 > N \). As \( w - \lim_m f_{n_0} \circ T^{s_m} = g_{n_0} \) there exists some \( M \) such that for all \( m > M \) we have
\[
|\psi(f_{n_0} \circ T^{s_m} - g_{n_0})| < \epsilon.
\]

Then, for all \( m > M \) we have
\[
|\psi(g - f \circ T^{s_m})| \leq |\psi(g - g_{n_0})| + |\psi(g_{n_0} - f_{n_0} \circ T^{s_m})| + |\psi(f_{n_0} \circ T^{s_m} - f \circ T^{s_m})| \\
\leq \|\psi\|\|g - g_{n_0}\|_\infty + \epsilon + \|\psi\|\|f - f_{n_0}\|_\infty < 3\epsilon.
\]

This proves the claim.

□

We complete our considerations on general dynamical systems by proving the following result. Part (a) can also be found in [22, Thm. 1.43 (4)]

**Theorem 3.5.** Let \((X, G)\) be a dynamical system. Let \( f \in C(X) \) and \( \omega \in \mathcal{X} \) be given and define \( g : G \to \mathbb{C} \) via \( g(t) = f(T^t \omega) \). Then, the following holds:

(a) If \( f \in WAP(X) \) then \( g \in WAP(G) \).

(b) If \( g \in WAP(G) \) and \( \{T^t \omega \mid t \in G\} \) is dense in \( X \) then \( f \in WAP(X) \).

**Proof.** Before proving (a) and (b) we need a little preparation. Define
\[
F : C(X) \to C_u(G); \ F(h)(s) = h(T^s(\omega)) .
\]

It is easy to see that \( F \) is well defined. Indeed, as \( X \) is compact, \( h \) is bounded and hence so is \( F(h) \). Moreover, \( h \) is uniformly continuous, and by the continuity of the group action on \( X \) so is \( F(h) \). Moreover,
\[
(2) \quad \|F(h)\|_\infty = \sup_{s \in G} |h(T^s(\omega))| \leq \|h\|_\infty .
\]

Let us also observe that if \( \{T^t \omega t \in G\} \) is dense in \( X \) then we have equality in \((2)\).

Now we proceed to prove (a) and (b).

(a) \( F \) is a continuous operator between the Banach spaces \((C(X), \| \cdot \|_\infty)\) and \((C_u(G), \| \cdot \|_\infty)\) and hence it is clearly weakly continuous (see also [40] for a recent discussion of this type of reasoning). Since \( \mathbb{W}(f) \) is weakly compact, and \( F \) is
weakly continuous, it follows that $F(\mathcal{W}(f))$ is weakly compact in $C_u(G)$. Now, by construction we have $T^tF(f) = F(T^{-t}f)$. Therefore
\[
\{T^tF(f) | t \in G\} \subset F(\mathcal{W}(f))
\]
and hence $g = F(f) \in WAP(G)$.

(b) Since $\{T^t\omega t | G\}$ is dense in $\mathcal{X}$, we have equality in (2). Therefore, $F$ gives an isometric embedding of $C(\mathcal{X}) \to C_u(G)$. Denote by
\[
\mathbb{Y} := \{F(h) | h \in C(\mathcal{X})\} \subset C_u(G).
\]
Then, $F$ is an isometry between $C(\mathcal{X})$ and $\mathbb{Y}$. Since $C(\mathcal{X})$ is a Banach space, so is $\mathbb{Y}$ and therefore $\mathbb{Y}$ is closed in $C_u(G)$. As $F$ is linear, $\mathbb{Y}$ is a subspace of $C_u(G)$. As a closed subspace $\mathbb{Y}$ is also weakly closed in $C_u(G)$.

Consider now the isometry $F^{-1} : \mathbb{Y} \to C(\mathcal{X})$. Since
\[
T^tF(f) = F(T^{-t}f),
\]
we have $T^t g \in \mathbb{Y}$. As $\mathbb{Y}$ is weakly closed, and $g \in WAP(G)$ it follows that weak closure $\{T^t g | t \in G\}$ of the orbit of $g$ is compact and contained $\mathbb{Y}$. Then, as $F^{-1}$ is continuous, it is weakly continuous, and therefore $F^{-1}(J)$ is weakly compact. As this set contains the orbit of $f$, we get that $f \in WAP(\mathcal{X})$. \hfill $\square$

We can now come to the main result of this section.

**Theorem 3.6.** Let $\mu \in M^\infty(G)$ be given. Then, $\mu$ is weakly almost periodic if and only if $(\mathcal{X}(\mu), T)$ is a weakly almost periodic dynamical system.

**Proof.** This follows from Theorem 3.3 applied with the functions $\phi_c : M^\infty(G) \to \mathbb{C}, \phi_c(\mu) = c \ast \mu(0)$ for $c \in C_c(G)$. Note that $\phi_c(T^t \mu) = c \ast \mu(t)$. Here are the details:

Assume that $(\mathcal{X}(\mu), G)$ is weakly almost periodic. Let $c \in C_c(G)$. Since $\phi_c \in WAP(\mathcal{X}(\mu))$ it follows from Theorem 3.3 (a) that $c \ast \mu \in WAP(G)$. As this is true for all $c \in C_c(G)$ we get $\mu \in WAP(G)$.

Assume conversely that $\mu$ is a weakly almost periodic function. If $c \in C_c(G)$ then it follows from Theorem 3.3 (b) that $\phi_c \in WAP(\mathcal{X}(\mu))$. This shows that $\phi_c \in WAP(\mathcal{X}(\mu))$ for all $c \in C_c(G)$. Then, by Proposition 3.3 $WAP(\mathcal{X}(\mu))$ contains the algebra generated by $\{\phi_c | c \in C_c(G)\}$. Since this algebra is dense in $C(\mathcal{X}(\mu))$ and since $WAP(\mathcal{X}(\mu))$ is a closed subspace of $(C(\mathcal{X}(\mu)), \| \cdot \|_\infty)$, it follows that
\[
C(\mathcal{X}(\mu)) = WAP(\mathcal{X}(\mu)).
\]
This finishes the proof. \hfill $\square$

We now derive a few corollaries from this concerning strongly almost periodic systems and measures. Here, a dynamical system $(\mathcal{X}, T)$ is called **strongly almost periodic** if for any $f \in C(\mathcal{X})$ the set $\{f \circ T^t : t \in G\}$ is relatively compact in $(C(\mathcal{X}), \| \cdot \|_\infty)$. We will need the following well-known statements on strongly almost periodic systems.

**Proposition 3.7.** (a) Any minimal component of a weakly almost periodic dynamical system is strongly almost periodic.
For a transitive dynamical system \((X, G)\) with dense orbit of \(\omega \in X\) the following statements are equivalent:

(i) \((X, G)\) is strongly almost periodic.

(ii) \((X, G)\) admits a structure of a locally compact group such that \(G \to X, t \mapsto T^t \omega\) becomes a continuous group homomorphism.

(iii) \((X, G)\) is weakly almost periodic and minimal.

Proof. (a) By [19, Prop. 2.8] any minimal weakly almost periodic dynamical system is strongly almost periodic. Now, (a) follows as any minimal component of a weakly-almost periodic system is also weakly almost periodic. (Indeed, if \(Y\) is an invariant compact subset of \(X\) then any \(f \in C(Y)\) can be extended by compactness to some \(g \in C(X)\). Define now \(R : C(X) \to C(Y)\) to be the restriction of the domain to \(Y\). Then we have \(|R| \leq 1\), which shows that \(R\) is continuous, hence weakly continuous. Therefore, as \(g \in C(X) = \text{WAP}(X)\) we have \(f = R(g) \in \text{WAP}(Y)\).

(b) The equivalence of (i) and (ii) can be found e.g. in the book of Auslander [3]. The implication (iii) \(\Rightarrow\) (i) is immediate from (a). On the other hand (i) clearly implies weak almost periodicity and (ii) clearly implies minimality of \((X, G)\) and so (iii) follows from (i) and (ii).

Here are the corollaries.

Corollary 3.8. A translation bounded measure \(\mu\) belongs to \(\text{SAP}(G)\) if and only if \((X(\mu), G)\) is strongly almost periodic.

Proof. As is well-known, see e.g. [33] for a recent discussion, \((X(\mu), G)\) admits a group structure such that \(t \mapsto T^t \mu\) is a continuous group homomorphism if and only if \(\mu\) is strongly almost periodic. Now, the statement follows from the equivalence of (i) and (ii) in (b) of Proposition 3.7.

Corollary 3.9. Let \(\mu \in \text{WAP}(G)\). Then, \(X(\mu)\) is minimal if and only if \(\mu \in \text{SAP}(G)\).

Proof. By Theorem 3.6 \((X(\mu), G)\) is weakly almost periodic. Hence, by the equivalence between (i) and (iii) in (b) of Proposition 3.7 it is strongly almost periodic if and only if it is minimal. Now, the desired statement follows from the preceding corollary.

3.2. Stability of almost periodicity along the hull. In this section we show that all elements in the hull of a strongly, weakly and null-weakly almost periodic measure are strongly, weakly and null-weakly almost periodic respectively.

We begin with the following well-known consequence of Lemma 2.20.

Lemma 3.10. Let \(\tau_1\) and \(\tau_2\) be topologies on \(X\) such that \(\tau_1\) is stronger than \(\tau_2\) and \(\tau_2\) is Hausdorff. If \(A \subset X\) is precompact with respect to \(\tau_1\), then the closures of \(A\) with respect to \(\tau_1\) and \(\tau_2\) coincide, and the two topologies coincide on this closure.

Here is the first stability result of this section.

Proposition 3.11. Let \(\mu \in \mathcal{M}^\infty(G)\) be given.

(a) The following assertions are equivalent:
(i) \( \mu \in \mathcal{WAP}(G) \).

(ii) The hull \( \mathcal{X}(\mu) \) agrees with the closure of \( \{T^t \mu \mid t \in G\} \) in the weak topology and is, in particular, compact in the weak topology.

(iii) The vague and the weak topologies agree on \( \mathcal{X}(\mu) \).

(iv) \( \mathcal{X}(\mu) \subset \mathcal{WAP}(G) \).

(b) The following assertions are equivalent:

(i) \( \mu \in \mathcal{SAP}(G) \).

(ii) The hull \( \mathcal{X}(\mu) \) agrees with the closure of \( \{T^t \mu \mid t \in G\} \) in the product topology and is, in particular, compact in the product topology.

(iii) The vague and the product topology agree on \( \mathcal{X}(\mu) \).

(iv) \( \mathcal{X}(\mu) \subset \mathcal{SAP}(G) \).

Proof. (a) (i) \( \implies \) (ii): As \( \mu \) is weakly almost periodic, the closure of its orbit in the weak topology is compact. Now, (ii) follows from the previous Lemma as the weak topology is stronger than the vague topology.

(ii) \( \implies \) (iii): This follows from Lemma 2.21.

(iii) \( \implies \) (iv): Let \( \nu \in \mathcal{X}(\mu) \) be arbitrary. We then have \( \mathcal{X}(\nu) \subset \mathcal{X}(\mu) \). As the vague and the weak topology agree on \( \mathcal{X}(\mu) \) we infer that \( \mathcal{X}(\nu) \) is compact in the weak topology. Hence, \( \nu \) belongs to \( \mathcal{WAP}(G) \). This gives (iv).

(iv) \( \implies \) (i): This is clear.

(b) This can be shown by analogous arguments. \( \square \)

We next show that for weakly almost periodic measures, the Fourier Bohr coefficients depend continuously on the measure.

Theorem 3.12. Let \( \mu \in \mathcal{WAP}(G) \) and \( \chi \in \hat{G} \). Then \( c_\chi : \mathcal{X}(\mu) \to \mathbb{C} \) is continuous and satisfies

\[
c_\chi(T^t \nu) = \chi(t) c_\chi(\nu).
\]

In particular, if \( c_\chi(\mu) \neq 0 \), then \( c_\chi \) is a continuous eigenfunction of \( \mathcal{X}(\mu) \).

Proof. Since \( \mu \in \mathcal{WAP}(G) \) the hull \( \mathcal{X}(\mu) \) is weak compact. The continuity of \( c_\chi \) follows now from Theorem 2.21.

Moreover, for all \( t \in G \) we have

\[
c_\chi(T^t \nu) = M(\chi T^t \nu) = M(T^t (\chi T^{-t} \nu)) = M(\chi(t) \nu) = \chi(t) c_\chi(\nu)
\]

and the proof is finished. \( \square \)

Remark. As we will see later, the intensity of the Bragg peak at \( \chi \in \hat{G} \) in the diffraction of \( \mu \) is given by \( |c_\chi(\mu)|^2 \). So Theorem 3.12 gives that each Bragg peak \( \chi \), comes from a continuous eigenfunction on \( \mathcal{X}(\mu) \).

Proposition 3.13. If \( \mu \in \mathcal{WAP}_0(G) \) then \( \mathcal{X}(\mu) \subset \mathcal{WAP}_0(G) \).

Proof. This follows easily from the previous theorem and the fact that a weakly almost periodic measure is null weakly almost periodic if and only if all of its Fourier coefficients are zero [23, Thm. 8.1]. Specifically, as \( \mu \) is null weakly almost periodic all of its Fourier coefficients are zero. By the formula given in the previous theorem
this then holds as well for all elements in the orbit of \( \mu \) and by the continuity statement of the previous theorem, this then holds for the whole hull.

3.3. The unique minimal component and unique ergodicity. In this section we show that the hull of a weakly almost periodic measure has a unique minimal component and that this component is just the hull of the strongly almost periodic part of the measure. This has strong consequences. For example a weakly almost periodic dynamical system is uniquely ergodic if and only if it has a unique minimal component.

A first step (and main result in this section) is that the strongly almost periodic part \( \mu_s \) of \( \mu \) belongs to the hull. As a consequence we can then derive a wealth of results on weakly almost periodic measures in this section and the next one.

As a preparation of the proof of the main result of this section we show that for a uniformly continuous function \( f \), whose average absolute value integral is 0, we can find arbitrarily large regions in \( G \) where \( |f| \) is arbitrarily small.

Lemma 3.14. Let \( (A_n) \) be a van Hove sequence. Let \( f \) be an uniformly continuous bounded function on \( G \) such that

\[
\lim_{n} \int_{A_n} \frac{|f(x + t)| \, dx}{\theta_G(A_n)} = 0.
\]

uniformly in \( t \in G \). Then for all compact sets \( K \subset G \) and all \( \varepsilon > 0 \), there exists some \( t \in G \) so that

\[
|f(x)| < \varepsilon
\]

for all \( x \in t + K \).

Remark. Note that the lemma applies to any null weakly almost periodic function.

Proof. Assume the contrary. Then, there exists a compact \( K \) and an \( \varepsilon > 0 \) such that for any \( t \in G \) there exists an \( x_t \in K \) with \( |f(t + x_t)| \geq \varepsilon \). As \( f \) is uniformly continuous, there exists then an open relatively compact set \( U \) with

\[
|f(t + y)| \geq \varepsilon / 2
\]

for all \( t \in G \) and all \( y \in x_t + U \). This implies, in particular,

\[
\int_{K + U} |f(t + x)| \, dx \geq \varepsilon / 2 \theta_G(U) =: c > 0
\]

for all \( t \in G \) and, hence,

\[
\liminf_{n} \frac{\int_{A_n} \int_{K + U} |f(t + x)| \, dx \, dt}{\theta_G(A_n)} \geq c > 0.
\]

(3)

On the other hand, from the assumption on \( f \) and as \( (A_n) \) is van Hove we have

\[
\int_{A_n} |f(t + x)| \, dx / \theta_G(A_n) \to 0, n \to \infty
\]

uniformly in \( t \in G \).
From Fubini-theorem, we then obtain
\[
\frac{\int_{A_n} f(t,x) \, dx}{\theta_G(A_n)} = \frac{\int_{K+U} \left( \frac{\int_{A_n} f(t,x) \, dx}{\theta_G(A_n)} \right) \, dt}{\theta_G(K+U)}
\]

Now, by the uniform convergence of \( \frac{\int_{A_n} f(t,x) \, dx}{\theta_G(A_n)} \), there exists some \( N \) such that, for all \( n > N \) we have
\[
\frac{\int_{A_n} f(t,x) \, dx}{\theta_G(A_n)} < \frac{c}{2 \theta_G(K+U)}.
\]
Then, for all \( n > N \) we have
\[
\frac{\int_{A_n} \int_{K+U} f(t,x) \, dx \, dt}{\theta_G(A_n)} = \frac{\int_{K+U} \left( \frac{\int_{A_n} f(t,x) \, dx}{\theta_G(A_n)} \right) \, dt}{\theta_G(K+U)} \leq \frac{c}{2 \theta_G(K+U)} \leq \frac{c}{2},
\]
which contradicts (3). This completes the proof. \( \square \)

**Remark.** Let us shortly discuss the assumptions in the lemma: The uniform continuity of \( f \) is essential in Lemma 3.14, as can be seen by considering the following example: For each \( n \in \mathbb{Z} \) pick some \( f_n \in C_c(\mathbb{R}) \) such that \( f_n \geq 0 \), \( f_n(n) = 1 \), \( \text{supp}(f_n) \subset (n - \frac{1}{2}, n + \frac{1}{2}) \) and \( \int f_n = \frac{1}{2^n} \). Let
\[
f := \sum_{n \in \mathbb{Z}} f_n.
\]
Then \( f \) is non-negative and has finite integral. Hence, it has zero average integral. However, clearly any translate of \( [-1,1] \) will meet an element where \( f \) takes the value 1.

The assumptions of boundedness of \( f \) and of uniform existence of the limit can be removed if vanishing of the limit is known for any van Hove sequence.

We are now ready to prove the main result of this section viz that the hull of a weakly almost periodic function contains its strong almost periodic part.

**Theorem 3.15.** Let \( \mu \) be a weakly almost periodic measure. Then
\[
\mu_s \in \mathbb{X}(\mu).
\]

In particular, \( \mathbb{X}(\mu_s) \subset \mathbb{X}(\mu) \).

**Proof.** We need to show that for every natural number \( n \) and all \( c_1, \ldots, c_n \in C_c(G) \) and \( \epsilon > 0 \) there exists some \( r \in G \) so that for all \( 1 \leq i \leq n \) we have
\[
\|(T^r \mu)(c_i) - \mu_s(c_i)\| < \epsilon.
\]

The basic idea of the proof is now to choose a compact \( K \) so that any translate of \( K \) contains a vector \( r \) such that \( T^r \mu_s \) is very close to \( \mu_s \) (which is possible by almost periodicity of \( \mu_s \)) and then to shift this \( K \) so that \( T^r \mu_0(c_i) \) is very small for all \( r \) in the shifted \( K \) (which is possible by the previous lemma).

Here, are the details: Let \( c_1, \ldots, c_n \in C_c(G) \) and \( \epsilon > 0 \) be given. Since the functions \( \mu_s \circ c_i \) are almost periodic, the set
\[
P := \{ t \in G | \max_{1 \leq i \leq n} \left\| \mu_s \circ c_i - T^t \mu_s \circ c_i \right\|_\infty < \frac{\epsilon}{2} \},
\]
is relatively dense. Fix a compact $K$ such that $0 \in K$ and $P + K = G$. Define 
\[ g_0 = \sum_{i=1}^{n} \mu_0 * c_i. \]
Then $g_0$ is a null weakly almost periodic function. By Lemma 3.14 there exists then some $t \in G$ so that 
\[ |g_0(x)| < \epsilon \]
for all $x \in t - K$. Since $t \in G = P + K$, there exists some $r \in P$ and $k \in K$ such that $t = r + k$. Therefore 
\[ |g_0(r)| = |g_0(t - k)| < \epsilon. \]
and hence, by the definition of $g_0$, for all $1 \leq i \leq n$ we have 
\[ \left| \mu_0 * c_i \right|(r) < \frac{\epsilon}{2}. \]
Moreover, as $r \in P$ we also have 
\[ \|\mu_s * c_i - T^r \mu_s * c_i\| < \frac{\epsilon}{2}. \]
Combining these two relations we get: 
\[ \left| \mu_s * c_i(0) - [T^r \mu_s + T^r \mu_0] * c_i(0) \right| < \epsilon. \]
Therefore 
\[ |\mu_s(c_i) - [T^r \mu](c_i)| < \epsilon. \]
which completes the proof. \qed

**Corollary 3.16.** Let $\mu$ be a weakly almost periodic measure. Then, $\mathcal{X}(\mu_s)$ is the unique minimal component of $\mathcal{X}(\mu)$.

**Proof.** By definition $\mu_s$ is strongly almost periodic. Hence, its hull $\mathcal{X}(\mu_s)$ is minimal. Moreover, by the previous theorem, this hull is part of $\mathcal{X}(\mu)$. It remains to show the uniqueness part of the statement: Let $\nu$ be any element of $\mathcal{X}(\mu)$ belonging to a minimal component. Then, it is strongly almost periodic by abstract principles (see above). Hence, $\nu = \nu_s$. Now, by assumption we have $T^{t_\beta} \mu \to \nu$ for a net $(t_\beta)$. Without loss of generality we can assume that $T^{t_\beta} \mu_0$ converges to, say, the measure $\alpha$ and $T^{t_\beta} \mu_s$ converges to, say, the measure $\rho$ say. Then, we have 
\[ \alpha + \rho = \nu = \nu_s. \]
By Proposition 3.13 we have that $\alpha \in \mathcal{WAP}_0(G)$ and clearly $\rho \in \mathcal{SAP}(G)$. Thus, by uniqueness of the decomposition, we have $\alpha = 0$. Hence, we have that $\nu = \nu_s = \rho$ belongs to the hull of $\mu_s$. This finishes the proof. \qed

An immediate consequence is the following simple characterisation of null almost periodicity. To state it we will use the notation $\underline{0}$ to denote the zero measure on $G$.

**Corollary 3.17.** Let $\mu \in \mathcal{WAP}(G)$. Then $\mu$ is null weakly almost periodic if and only if the zero measure $\underline{0} \in \mathcal{K}(\mu)$. In this case $\delta_{\underline{0}}$ is the uniquely ergodic measure on $\mathcal{K}(\mu)$.
As is well-known, for weakly almost periodic systems unique ergodicity is equivalent to uniqueness of the minimal component. Indeed, each minimal component certainly carries ergodic measure and conversely uniqueness of such a component implies unique ergodicity [19, Prop. 2.10]. So from the previous corollary and Theorem 3.6 we directly infer the following result.

**Corollary 3.18.** Let $\mu$ be a weakly almost periodic measure. Then, $(\mathcal{X}(\mu), G)$ is uniquely ergodic.

**Remark.** We note that unique ergodicity can also be derived from existence of means of weakly almost periodic functions: For any $c_1, \ldots, c_n \in C_c(G)$ and the function $t \to \prod_{j=1}^n \phi_{c_j}(T^t \mu)$ is weakly almost periodic. Hence, its mean exists in a rather uniform manner. Since such linear combinations form a dense subset of $C(\mathcal{X}(\mu))$. one obtains uniform existence of means for all functions in $C(\mathcal{X}(\mu))$. This, in turn, implies unique ergodicity, see [35].

### 3.4. The structure of the hull

In this section we look a bit closer at the relation between $X(\mu)$ and $X(\mu_s)$. This will allow us to get a rather precise description of the hull $\mathcal{X}(\mu)$.

The dynamical system $(\mathcal{X}(\mu_s), G)$ will play a central role in subsequent considerations. Hence, we will give it a special name.

**Definition 3.19.** Let $\mu \in \mathcal{WAP}$. We define $S(\mu) := \mathcal{X}(\mu_s)$.

As $\mu_s$ is strongly almost periodic, we infer from Corollary 3.18 that $S(\mu)$ is a compact abelian group. We denote the Haar measure on this group by $\theta_{S(\mu)}$. Moreover, by Theorem 3.15 we have that $S(\mu)$ is a subset of $\mathcal{X}(\mu)$. Clearly, it is a closed $G$-invariant subset. Hence, from the unique ergodicity of $(\mathcal{X}(\mu), G)$ and the fact that $\theta_{S(\mu)}$ is an invariant measure, we obtain the following corollary.

**Corollary 3.20.** Let $\mu \in \mathcal{WAP}(G)$. The Haar measure $\theta_{S(\mu)}$ is the uniquely ergodic measure of $\mathcal{X}(\mu)$.

**Remark.** Of course, this is just a special case of the general fact that a weakly almost periodic system with a unique minimal component has the Haar measure on this component as its unique translation invariant measure.

We next show that $S(\mu)$ consists exactly of the strong almost periodic measures in $\mathcal{X}(\mu)$. First we present a more general result.

**Lemma 3.21.** Let $\mathcal{X}$ be a weakly almost periodic dynamical system which has an unique minimal component $S$. Then, for an element $\omega \in \mathcal{X}$ we have $\omega \in S$ if and only if $\mathcal{X}(\omega)$ is an almost periodic dynamical system.

**Proof.** Assume $\omega \in S$. Since $S$ is a minimal component, we have $\mathcal{X}(\omega) = S$. By the minimality of $S$ we get that $S$ is an almost periodic dynamical system.

Assume now that $\mathcal{X}(\omega)$ is an almost periodic dynamical system. Hence, it must be minimal. Hence, by the uniqueness of the minimal component we get $\mathcal{X}(\omega) = S$ and hence $\omega \in S$. \qed
As a corollary we obtain the following.

**Lemma 3.22.** Let $\mu \in \mathcal{WAP}(G)$. Then $\mathcal{X}(\mu) \cap \mathcal{SAP}(G) = \mathcal{S}(\mu)$.

Next we show that the mapping $P_0 : \mathcal{WAP}(G) \to \mathcal{WAP}_0(G)$ takes $\mathcal{X}(\mu)$ into $\mathcal{X}(\mu_0)$.

**Lemma 3.23.** Let $\mu \in \mathcal{WAP}(G)$. If $\nu \in \mathcal{X}(\mu)$ then $\nu_0 \in \mathcal{X}(\mu_0)$.

**Proof.** Let $\nu \in \mathcal{X}(\mu)$. Then there exists some net $(t_\alpha)$ in $G$ such that

$$\nu = \lim_{\alpha} T_{t_\alpha} \mu.$$  

Next, let us look at $T_{t_\alpha} \mu_\alpha$. As $T_{t_\alpha} \mu_\alpha \in \mathcal{X}(\mu_\alpha)$ and $\mathcal{X}(\mu_\alpha)$ is compact, the net has cluster points. Thus, we can find some subnet $(\beta)$ of $(\alpha)$ and some $\omega \in \mathcal{X}(\mu_\alpha)$ such that

$$T_{t_{\beta}} \nu_\alpha \to \omega.$$  

As $T_{t_{\beta}} \nu_\alpha \in \mathcal{X}(\mu_\alpha)$ we get $\omega \in \mathcal{X}(\mu_\alpha)$. Therefore, by Proposition 3.14 we get that $\omega \in \mathcal{SAP}(G)$.

Now, as $T_{t_\alpha} \mu_0 \in \mathcal{X}(\mu_0)$ and $\mathcal{X}(\mu_0)$ is compact, we can find a subnet $\gamma$ of $\beta$ such that

$$T_{t_{\gamma}} \mu_0 \to v \in \mathcal{X}(\mu_0).$$  

Then, by Proposition 3.13 we have $v \in \mathcal{WAP}_0(G)$. Now, we have

$$\nu = \lim_{n} T_{t_{n}} \mu = \lim_{n} T_{t_{n}} \mu_{\alpha} = \lim_{n} T_{t_{n}} (\mu_{\alpha} + \mu_0) = \lim_{n} T_{t_{n}} \mu_{\alpha} + \lim_{n} T_{t_{n}} \mu_0 = \omega + \nu.$$  

The uniqueness of the decomposition gives us that $\nu = \nu_0$. Since $\nu \in \mathcal{X}(\mu_0)$ we get the claim. \hfill $\Box$

Now, we can prove that the projections $P_\xi$ and $P_0$ are continuous when restricted to the hull $\mathcal{X}(\mu)$.

**Lemma 3.24.** Let $\mu \in \mathcal{WAP}(G)$. Then the maps

$$P_\xi : \mathcal{X}(\mu) \to \mathcal{S}(\mu), \nu \mapsto \nu_\xi$$  

and $P_0 : \mathcal{X}(\mu) \to \mathcal{X}(\mu_0), \nu \mapsto \nu_0$,

are continuous.

**Proof.** First, let us observe that $P_\xi$ is well defined i.e. maps into $\mathcal{S}(\mu)$. Indeed, if $\nu \in \mathcal{X}(\mu)$ we know that $\nu \in \mathcal{WAP}(G)$ and hence

$$P_\xi(\nu) \in \mathcal{SAP}(G) \cap \mathcal{X}(\nu) \subset \mathcal{SAP}(G) \cap \mathcal{X}(\nu) = \mathcal{S}(\mu).$$  

In the same way $P_0$ is well defined.

We next prove the continuity of $P_\xi$. To do this we have to show that whenever $\nu_\alpha \to \nu$ in $\mathcal{X}(\mu)$ we have

$$(\nu_\alpha)_\xi \to \nu_\xi.$$  

As $\mathcal{X}(\mu_\alpha) = \mathcal{S}(\mu)$ is compact, and $(\nu_\alpha)_\xi, \nu_\xi \in \mathcal{X}(\mu_\alpha)$, the net $(\nu_\alpha)_\xi$ as well as any subnet has cluster points. To prove that $(\nu_\alpha)_\xi \to \nu_\xi$, it thus suffices to show that every cluster point of $(\nu_\alpha)_\xi$ is equal to $\nu_\xi$. 


Let $\omega$ be a cluster point of $(\nu_\alpha)_\alpha$. Then, there exists a subnet $\beta$ of $\alpha$ such that

$$\omega = \lim_{\beta} (\nu_\beta)_\beta .$$

As $(\nu_\beta)_\beta \in \mathcal{SAP}(G) \cap \mathcal{X}(\mu) = \mathcal{S}(\mu)$ which is compact, hence complete in the vague topology, we get $\omega \in \mathcal{S}(\mu)$ and hence, $\omega \in \mathcal{SAP}(G)$.

Moreover, by Lemma 3.23, as $\nu_\beta \in \mathcal{X}(\mu)$ we have $(\nu_\beta)_0 \in \mathcal{X}(\mu_0)$. As before, by compactness of $\mathcal{X}(\mu_0)$ we can also pick a subnet $(\nu_\gamma)_\gamma$ such that $(\nu_\gamma)_0$ is convergent in $\mathcal{X}(\mu_0)$. Let $v$ be the limit of this net, then $v \in \mathcal{WAP}_0(G)$.

Then

$$\nu = \lim_{\alpha} \nu_\alpha = \lim_{\gamma} \nu_\gamma = \lim_{\gamma} (\nu_\gamma)_0 + (\nu_\gamma)_0 = \omega + v .$$

As $\omega \in \mathcal{SAP}(G)$ and $v \in \mathcal{WAP}_0(G)$, the uniqueness of the decomposition shows that

$$\nu_\beta = \omega .$$

This shows that $P_s$ is continuous.

Now, consider $P_0 : \mathcal{X}(\mu) \to \mathcal{WAP}(G)$. Then, we have

$$P_0(\nu) = \nu - P_s(\nu) ,$$

is the difference of two functions which are continuous on $\mathcal{X}(\mu)$, and hence $P_0$ is continuous on $\mathcal{X}(\mu)$. Restricting now the codomain to $\mathcal{X}(\mu_0)$ will not change the continuity, and prove the claim. \hfill $\square$

**Remark.** For the preceding results the restriction to the hull of one element is crucial. On the whole space $\mathcal{WAP}(G)$ the maps $P_s$ and $P_0$ are not continuous with respect to the vague topology. Indeed, for example for $G = \mathbb{R}^N$ it suffices to consider the sequence $\mu_n = \delta_{n\mathbb{Z}^N}$. Then, each $\mu_n$ is of fully periodic and hence belongs to $\mathcal{SAP}(G)$. However, the vague limit of the $\mu_n$ is given by $\mu = \delta_0$. So, in this case we then have

$$P_s(\mu_n) \to \mu \neq 0 = P_s(\mu) \text{ and } P_0(\mu_n) = 0 \to 0 \neq \mu = P_0(\mu) .$$

We summarize part of the preceding considerations as follows.

**Theorem 3.25.** Let $\mu \in \mathcal{WAP}(G)$. Then

(a) $\mathcal{S}(\mu)$ is a compact abelian group with Haar measure $\theta_{\mathcal{S}(\mu)}$.

(b) $\mathcal{S}(\mu_0)$ and $\delta_0$ is the uniquely ergodic measure on $\mathcal{X}(\mu_0)$.

(c) The mapping

$$S : \mathcal{X}(\mu) \to \mathcal{S}(\mu) \times \mathcal{X}(\mu_0) ; \quad S(\mu) = (\mu_s, \mu_0)$$

is a continuous $G$-mapping, one to one, and has full measure range.

**Proof.** The only thing which was not proven is (c). The continuity of $S$ is an immediate consequence of Theorem 3.24. The $G$-invariance of $S$ follows immediately from the uniqueness of the Eberlein decomposition:

$$T_l \mu = T_l (\mu_s + \mu_0) = T_l (\mu_s) + T_l (\mu_0) .$$
is the Eberlein decomposition of $T_t \mu$ and therefore
\[ T_t(\mu_s) = (T_t \mu)_s \quad ; \quad T_t(\mu_0) = (T_t \mu)_0. \]
The fact that the $S$ is one to one is obvious by the Eberlein decomposition. Finally, we have that
\[ S(\mu) \times \{0\} = S(S(\mu)) \subset S(\mathcal{X}(\mu)), \]
has full measure in $S(\mu) \times \mathcal{X}(\mu_0)$. □

From these considerations we also easily infer the following result.

**Theorem 3.26.** Let $\mu \in \mathcal{WAP}(G)$. Then the mappings $i : S(\mu) \leftrightarrow \mathcal{X}(\mu), i(\nu) = \nu$ and $P_\lambda : \mathcal{X}(\mu) \to S(\mu)$ are well defined, continuous $G$-mappings, and
\[ P_\lambda \circ i(\nu) = \nu \quad \text{for all } \nu \in S(\mu) \quad \text{and} \]
\[ i \circ P_\lambda(\nu) = \nu \quad \text{for } \theta_{S(\mu)} - \text{almost all } \nu \in \mathcal{X}(\mu). \]

**Proof.** The mappings are well defined and continuous by Lemma 3.24.

Now, if $\nu \in S(\mu)$ then $\nu \in S(\mathcal{X}(\mu))$. Therefore
\[ P_\lambda \circ i(\nu) = P_\lambda(\nu) = \nu = \nu. \]

Which proves the first relation. Also, for all $\nu \in \mathcal{X}(\mu)$ we have
\[ i \circ P_\lambda(\nu) = \nu. \]

Therefore $i \circ P_\lambda(\nu) = \nu$ for all $\nu \in S(\mu)$. As $\theta_{S(\mu)}(S(\mu)) = 1$, we get that
\[ i \circ P_\lambda(\nu) = \nu \quad \text{for } \theta_{S(\mu)} - \text{almost all } \nu \in \mathcal{X}(\mu). \]

□

4. Spectral and diffraction theory of weakly almost periodic measures

Based on the discussion in the preceding section we can rather directly set up the diffraction theory for a weakly almost periodic measures. In fact, the preceding section has immediate consequence for the spectral theory of the hull of the measure and this, in turn, gives results on the diffraction.

We start with the following rather direct consequence of Theorem 3.26.

**Proposition 4.1.** Let $\mu \in \mathcal{WAP}(G)$. Then,
\[ U : L^2(S(\mu), \theta_{S(\mu)}) \to L^2(\mathcal{X}(\mu), \theta_\mu), g \mapsto g \circ P_\lambda \]
is a $G$-invariant unitary map mapping $C(S(\mu))$ into $C(\mathcal{X}(\mu))$.

From this proposition we can infer that the spectral theory of the two dynamical systems $(S(\mu), G)$ and $(\mathcal{X}(\mu), G)$ is the same. As the spectral theory of $(S(\mu), G)$ is well-known we then obtain a precise description of the spectral theory of $(\mathcal{X}(\mu), G)$. To make this explicit we need a bit more notation.
Let \((X, G)\) be a dynamical system and \(\mu\) an \(G\)-invariant probability measure on \(G\). Then, an \(\chi \in \hat{G}\) is called an **eigenvalue** if there exists an \(f \in L^2(X, \mu)\) with \(f \neq 0\) and
\[
f \circ T^t = \chi(t)f
\]
for all \(t \in G\). Such an \(f\) is then called an **eigenfunction**. A dynamical system is said to have **pure point dynamical spectrum** if there exists an orthonormal basis of \(L^2(X, \mu)\) consisting of eigenfunctions.

**Theorem 4.2.** Let \(\mu \in WAP(G)\). Then the dynamical system \((X(\mu), G)\) has pure point dynamical spectrum with an orthonormal basis of continuous eigenfunctions given by
\[
f_\lambda = \lambda \circ P_s
\]
for \(\lambda \in \hat{S}(\mu)\).

**Remark.**
(a) The group homomorphism \(G \longrightarrow S(\mu), t \mapsto T^t\mu\), has dense range. Hence, its dual map \(j : \hat{S}(\mu) \longrightarrow \hat{G}\) is injective and in this way the elements of \(\hat{S}(\mu)\) can be seen as elements of \(\hat{G}\). A \(\lambda \in \hat{S}(\mu)\) then gives rise to the eigenvalue \(j(\lambda)\).

(b) The result implies that \((S(\mu), G)\) is what is called the maximal equicontinuous factor of \((X(\mu), G)\). Indeed, one way to define this factor is as the dual to the group of all eigenvalues with continuous eigenfunctions equipped with the discrete topology, see e.g. [5] for a recent discussion in the context of diffraction. Now, in our case this clearly leads to \(S(\mu)\).

(c) Uniquely ergodic systems with pure point spectrum with continuous eigenfunctions are called **isomorphic extensions of their maximal equicontinuous factor**. By the previous corollary the hull of a weakly almost periodic measure is an example of such a system. As shown recently by Downarowicz and Glasner [14] isomorphy of the extension is equivalent to mean equicontinuity in the minimal case.

(d) We have already met some eigenfunctions in Theorem 3.12. Let us emphasize that the eigenfunctions given in that theorem are completely canonical. This is rather remarkable as usually eigenfunctions are only determined up to some overall phase factor. Note also that Theorem 3.12 does not necessarily provide an eigenfunction for each eigenvalue (as the Fourier coefficients may vanish for some group elements).

**Proof.** As \(S(\mu)\) is a compact group, it is known to have pure point dynamical spectrum with an orthonormal basis of continuous eigenfunctions given by the elements \(\lambda \in \hat{S}(\mu)\). Now, the statement follows from the previous proposition and the continuity of \(P_s\). \(\square\)

As a consequence we also obtain the following application to diffraction theory. To express the corollary we will need the map
\[
j : \hat{S}(\mu) \longrightarrow \hat{G}
\]
with \(j(\lambda)(t) := \lambda(T^t\mu)\) (compare (a) of the preceding remark).
Theorem 4.3. Let $\mu \in WAP(G)$ be given. Then, the measures $\mu$ and $\mu_\ast$ have the same autocorrelation $\gamma$, and pure point diffraction. In fact, for any $\lambda \in S(\mu)$ the functions $c^n_\lambda : X(\mu) \to \mathbb{C}$ defined by

$$c^n_\lambda(\omega) := \int_{A_n} j(\lambda)(t)d\omega(t)$$

converge uniformly on $X(\mu)$ to the Fourier Bohr coefficient function

$$c_{j(\lambda)} : X(\mu) \to \mathbb{C}, \omega \mapsto c_{j(\lambda)}(\omega);$$

$A_\lambda = |c_{j(\lambda)}(\omega)|^2$ does not depend on $\omega \in X(\mu)$ and the diffraction is given by

$$\hat{\gamma} = \sum_{\lambda \in S(\mu)} A_\lambda \delta_{j(\lambda)}.$$ 

Remark. Validity of uniform convergence of the $|c^n_\lambda|^2$ to the point parts of $\hat{\gamma}$ is sometimes discussed under the name of Bombieri-Taylor conjecture, see [31] for details. Thus, our result gives in particular validity of Bombieri/Taylor conjecture for the dynamical systems coming from weakly almost periodic measures.

Proof. We first show that the autocorrelations agree: As $X(\mu), X(\mu_\ast)$ are uniquely ergodic, each of $\mu$ and $\mu_\ast$ have unique autocorrelation. Let us denote by $\gamma_1, \gamma_2$ the two autocorrelations. Then, as $X(\mu)$ is uniquely ergodic, we have, see e.g. [7],

$$c * \tilde{c} * \gamma_1(t) = \langle \phi_c, T_t \phi_c \rangle$$

with the inner product calculated in $L^2(X(\mu), \theta_\mu)$, where $\phi_c$ is defined via $\phi_c(\nu) = \nu * c(0)$ see above.

Then, as $X(\mu)$ is uniquely ergodic, we have

$$c * \tilde{c} * \gamma_2(t) = \langle \phi_c, T_t \phi_c \rangle$$

with the inner product calculated in $L^2(S(\mu), \theta_\mu)$. By Theorem 3.20 those are equal, and hence

$$c * \tilde{c} * \gamma_1 = c * \tilde{c} * \gamma_2$$

for all $c \in C_c(G)$. This shows that

$$\gamma_1 = \gamma_2.$$ 

We now turn to proving pure point diffraction: This follows directly as the systems have pure point dynamical spectra. In fact, it is known that pure point diffraction and pure point dynamical spectrum are equivalent for translation bounded measure dynamical systems [7] (and in fact even more general situations [34, 32]).

We finally discuss the formula for $\hat{\gamma}$: By the preceding corollary, the dynamical system $(X(\mu), G)$ has pure point spectrum with continuous eigenfunctions to the eigenvalues $j(\lambda), \lambda \in S(\mu)$. Thus, the uniform convergence of the $c^n_\lambda$ to a function $\tilde{c}_\lambda$ follows from Corollary 2 of [31]. Theorem 4 of [31] then gives that $A_\lambda := |\tilde{c}_\lambda(\omega)|^2$ is independent of $\omega \in X(\mu)$ and satisfies $\hat{\gamma}(\{j(\lambda)\}) = A_\lambda$. Now, the equality

$$\hat{\gamma} = \sum_{\lambda \in S(\mu)} A_\lambda \delta_{j(\lambda)}$$
follows from Corollary 2 of [31]. It remains to show that \( \tilde{\gamma}_\lambda(\omega) \) equals the Fourier coefficient \( c_{\lambda}(\phi_\omega) \). This, however, is clear from the definition of the Fourier coefficients. □

5. ON THE HULL OF THE AUTOCORRELATION

In the preceding considerations we essentially always started with a weakly almost periodic measure. Here, we discuss that any measure dynamical system gives rise to the hull of a weakly almost periodic measure viz its autocorrelation and how this hull carries important information on the pure point diffraction spectrum of the original system. In this sense hulls of weakly almost periodic measures are somehow unavoidable when dealing with diffraction.

Let \((X, T)\) be a measure dynamical system and \(m\) an invariant probability measure on \(X\). Then, there exists a unique positive definite measure \(\gamma\) with

\[
\gamma \ast c \ast \delta(0) = \langle \phi_c, \phi_d \rangle
\]

for all \(c, d \in C_c(G)\). This measure is called the autocorrelation of \(m\).

Collecting the results from this paper we get:

**Theorem 5.1.** Let \((X, G, m)\) be any ergodic system of translation bounded measures, and let \(\gamma\) be the autocorrelation of this system. Then

(a) \(\gamma\) is a weakly almost periodic system, therefore, uniquely ergodic.

(b) \(\gamma\) has pure point dynamical spectrum.

(c) The diffraction spectrum of \((X, T, m)\) agrees with the diffraction spectrum of \((\gamma, T)\) and generates the whole dynamical spectrum of \((\gamma, T)\) as a group.

(d) If \(X = X(\mu) \subset WAP(G)\) then

1. \(\gamma\) is a compact group;

2. For \(m\)-almost all \(\omega \in X\), \(X(\omega)\) is isomorphic as compact group with \(X(\omega)\).

3. Composing \(P_\chi : X \to SAP(G)\) with the isomorphism from (2) gives us for \(m\)-almost all \(\omega \in X\) a continuous factor \(G\)-mapping \(X(\omega) \to X(\gamma)\).

**Proof.** (a) Since \(\gamma\) is positive definite and translation bounded, it is weakly almost periodic [40].

(b) Follows from (a) and Theorem [41,2]

(c) By Theorem [43,3] the intensity of the diffraction of \((\gamma, G)\) is given by

\[
A_\chi = |\hat{\gamma}(\{\chi\})|^2.
\]

It follows that \(A_\chi \neq 0 \iff \hat{\gamma}(\{\chi\}) \neq 0\), that is the systems \((\gamma, G)\) and \((X, m, G)\) have the same Bragg peaks.

Now, the system \((X(\gamma), G)\) is weakly almost periodic, and hence it has pure point dynamical spectrum, which is the same as the group generated by the diffraction spectrum, [7].

(d): (1) Since \(X(\mu)\) is a weakly almost periodic dynamical system, and hence it has pure point diffraction. Then \(\gamma \in SAP(G)\). [28, 40].
(2) Since $\gamma \in \mathcal{SAP}(G)$ the hull $X(\gamma)$ is a compact group whose dual is exactly the pure point dynamical spectrum of $X(\gamma)$.

By [7] for almost all $\omega \in X$, $\gamma$ is the autocorrelation of $\omega$. Therefore, by (c), the diffraction and dynamical spectra of $\omega$ are exactly the dual of $X(\omega)$.

As $X \subset \mathcal{WAP}(G)$, it follows that for $m$-almost all $\omega$, the system $X(\omega)$ has also pure point spectrum, and the spectrum is the dual of the compact group $X(\omega)$.

Therefore $X(\omega)$ and $X(\gamma)$ are compact groups with isomorphic duals.

(3) Follows immediately from (2) and Lemma 3.24. □

Next, consider the set $E$ of all dynamical systems of translation bounded measures on $G$ equipped with invariant probability measures, that is

$$E := \{(X, m, G) | X \subset \mathcal{M}_\mathbb{F}(G), m \text{ invariant probability measure on } X\}.$$

We can define a function $F: E \to E$ via

$$F(X, m, G) = (X(\gamma), m', G),$$

where $\gamma$ is the autocorrelation of $(X, m, G)$ and $m'$ is the unique ergodic measure on $X(\gamma)$. Then, the previous theorem yields the following consequence.

**Proposition 5.2.** Let $(X, m, G) \in E$, let $S$ denote the pure point spectrum of $(X, m, G)$ and let $T = \hat{\mathbb{S}}$.

(a) $F(X, m, G)$ is a weakly almost periodic system with spectrum $S$.

(b) For each $n \geq 2$, $F^n((X, m, G))$ is an almost periodic system, which is isomorphic as an abelian group to $T$.

**Proof.** (a) Follows from Theorem 5.1 (a).

(b) Follows by induction from Theorem 5.1 (c) and (d). □

**Remark.** Note that while they are topologically isomorphic as dynamical systems, the systems $F^n((X, m, G))$ have typically different autocorrelation measures, and hence are in general supported on different subsets of $\mathcal{M}_\mathbb{F}(G)$.

Finally, if we denote by $\gamma_{(0)}$ the autocorrelation of $(X, m, G)$ and by $\gamma_{(n)}$ the autocorrelation of $F^n(X, m, G)$ we get

**Proposition 5.3.**

(a) $\gamma_{(0)} \in \mathcal{WAP}(G)$.

(b) $\gamma_{(n)} \in \mathcal{SAP}(G)$ for all $n \geq 1$.

(c) For each $n \geq 1$ and all $\chi \in \hat{G}$ we have

$$c_\chi(\gamma_{(n+1)}) = |c_\chi(\gamma_n)|^2 = |c_\chi(\gamma_{(0)})|^{2n+1}.$$

In particular, all $\gamma_{(n)}$ have the same set of characters with non-trivial Fourier-Bohr coefficient.

**Proof.** (a) and (b) are immediate consequences of Proposition 5.2.

(c) Since $\gamma_{n+1}$ is the autocorrelation of $X(\gamma_n)$, by Theorem 5.3 we have

$$\widehat{\gamma_{n+1}}(\chi) = A_\chi = |c_\chi(\gamma_n)|^2.$$

Since $\gamma_{n+1}$ is Fourier transformable, we also have
$$\hat{\gamma_{n+1}}(\{\chi\}) = c_\chi(\gamma_{n+1}),$$
which completes the proof. \qed

**Remark.** The strong almost periodicity of $\gamma_1$ is the key for the proof of [47, Thm. 7.1].

6. Application to weighted Dirac combs

In this Section we apply part of the results of the previous sections to study the support of the Eberlein decomposition for weighted Dirac combs.

Recall that at subset $A$ of $G$ is called **uniformly discrete** if there exists an open set $U$ containing the neutral element of $G$ such that $(x+U) \cap (y+U) = \emptyset$ for all $x, y \in A$ with $x \neq y$. Such a subset is called **relatively dense** if there exists a compact set $C$ such that $A + C = G$. A set which is both relatively dense and uniformly discrete is called a **Delone set**.

A set $A \subset G$ is called a **Meyer set** if $A$ is relatively dense and if $A - A - A$ is uniformly discrete. If $G$ is compactly generated, then the second condition can be replaced by the weaker $A - A$ is uniformly discrete [9, 47].

To connect subsets to the measures we note that any uniformly discrete set (and hence any Delone set) $A$ gives rise to a measure
$$\delta_A := \sum_{x \in A} \delta_x,$$
where $\delta_x$ denotes the unique point mass at $x$. This measure is known as the **Dirac comb of $A$**. The map
$$\delta : \text{Uniform discrete subset of } G \rightarrow \mathcal{M}^\sigma$$
is injective. In this way, the set of uniform discrete subsets of $G$ inherits a topology and we can form also the **hull** $\mathcal{X}(A)$ of $A$. Clearly, this hull is homeomorphic to $\mathcal{X}(\delta_A)$ via $\delta$.

**6.1. On almost periodic Delone sets.** Here, we consider the case that all weights are equal to one i.e. we look at some consequences of Theorem 3.15 for Delone sets.

**Corollary 6.1.** Let $\Lambda$ be a Delone set such that $\delta_\Lambda$ is a weakly almost periodic measure. Then there exists some Delone set $\Lambda'$ so that $(\delta_\Lambda)_\sigma = \delta_{\Lambda'}$.

**Remark.** It doesn’t follow from Proposition 6.1 that $\Lambda$ itself is strongly almost periodic (in the sense that $\delta_\Lambda$ is strongly almost periodic). For example, if $\Lambda = \mathbb{Z} \setminus \{0\}$ then $\delta_\Lambda$ is not strongly almost periodic. The problem is that while $\mathbb{Z} \in \mathcal{X}(\Lambda)$, we don’t have $\Lambda \in \mathcal{X}(\mathbb{Z})$.

An interesting question is what happens if we also ask for $\Lambda$ to be minimal. We address this question next.

Let us recall first a Theorem of Favarov [20] and Kellendonk-Lenz [25]:
Theorem 6.2. Let $\Lambda$ be a Delone set with FLC. If $\delta_{\Lambda}$ is strongly almost periodic, then $\Lambda$ is a fully periodic crystal.

Corollary 6.3. Let $\Lambda$ be a Delone set with FLC. If $\delta_{\Lambda}$ is weakly almost periodic and $\mathbb{X}(\Lambda)$ is minimal, then $\Lambda$ is a fully periodic crystal.

Combining Theorem 6.2 with Theorem 3.15 we get:

Theorem 6.4. Let $\Lambda$ be a Delone set with FLC such that $\delta_{\Lambda}$ is weakly almost periodic. Then, there exists a lattice $L$ and a finite set $F$ so that $F \sim L \sim P_{X} \equiv \delta_{L}$ and $\delta_{L} = \delta_{\Lambda} \ast \delta_{F}.$

Let us look next at two interesting examples of weakly almost periodic measures. Note that these examples are coming from Delone sets without FLC, and emphasize the importance of FLC in the previous results.

Example 6.5. Let $\Lambda := \{n + \frac{1}{n} | n \in \mathbb{Z} \backslash \{0\} \} \cup \{0\}.$ Let $\nu := \delta_{\Lambda} - \delta_{\mathbb{Z}}.$ Then $\nu \ast g$ is a function vanishing at infinity for all $g \in C_{c}(\mathbb{R})$, and thus null-weakly almost periodic [10]. This proves that $\nu$ is a null weakly almost periodic measure. Hence, using the uniqueness of the almost periodic decomposition, we have:

$$(\delta_{\Lambda})_{s} = \delta_{\mathbb{Z}}; \ (\delta_{\Lambda})_{0} = \nu.$$ 

Example 6.6. Let $\Lambda := \{(n + \frac{1}{n}, 2m) | m \in \mathbb{Z} \} \cup \{(n \sqrt{2}, 2m + 1) | m, n \in \mathbb{Z} \} \subset \mathbb{R} \times \mathbb{R}.$ A similar computation as in the previous example shows that, with $\Lambda' = [Z \times 2Z] \cup [Z \sqrt{2} \times (2Z + 1)]$ we have

$$(\delta_{\Lambda})_{s} = \delta_{\Lambda'}.$$ 

Note that $\Lambda'$ doesn’t have FLC, and is the union of translates of lattices.

6.2. The support of the Eberlein decomposition. Given a weakly almost periodic measure $\mu$, we know that $\mu_{s} \in \mathbb{X}(\mu)$. We use this result to show that for weighted dirac combs the support of $\mu_{s}$ and $\mu_{0}$ cannot be much larger than the support of $\mu$. We then use this result to rederive some recent results about Eberlein decomposition of weighted Dirac combs with Meyer set support [46].

Let us start by recalling a result of [4].

Proposition 6.7. [4 Prop. 5.2] Let $S$ be a set with finite local complexity and

$$\mu = \sum_{x \in S} \omega_{x} \delta_{x},$$

be a translation bounded measure supported inside $S$. If $\nu \in \mathbb{X}(\mu)$ then there exists some $S' \in \mathbb{X}(S)$ such that $\nu$ is supported inside $S'$.

Combining this result with Theorem 3.15 we get
**Theorem 6.8.** Let $\mu$ be a weakly almost periodic measure supported on a set with FLC. Then $\mu_s$ and $\mu_0$ are pure point measures and $\sup(\mu_s)$ has FLC.

**Proof.** It follows from Proposition 6.7 that $\sup(\mu_s)$ has FLC. In particular $\mu_s$ is a pure point measure. Since both $\mu$ and $\mu_s$ are pure point measures, so is their difference $\mu_0$. □

As a consequence we get the following result about diffraction.

**Theorem 6.9.** Let $\mu$ be a translation bounded measure and let $\gamma$ be an autocorrelation of $\omega$. If $\gamma$ is supported on an FLC set then the following statements hold:

(a) $\hat{\gamma}_d \in SAP(\hat{G})$ and $\hat{\gamma}_c \in SAP(\hat{G})$.

(b) Each of the pure point and continuous spectra is either empty or relatively dense.

**Proof.** (a) By Theorem 6.8, each of $\mu_s$ and $\mu_0$ is a pure point measure. Moreover, since $\gamma$ is positive definite, it is Fourier transformable, and hence so are $\mu_s$ and $\mu_0 [40]$. Now since the Fourier transform of a pure point Fourier transformable measure is a strongly almost periodic measure [28, 40], (a) follows.

(b) follows now immediately by combining (a) with [44, Prop. 4.5]. □

In particular we get new proofs of some of the results in [46].

**Theorem 6.10.** [46, Thm. 5.3] Let $\mu$ be a translation bounded measure supported inside a Meyer set and let $\gamma$ be an autocorrelation of $\omega$. Then

(a) $\hat{\gamma}_d \in SAP(\hat{G})$ and $\hat{\gamma}_c \in SAP(\hat{G})$.

(b) Each of the pure point and continuous spectra is either empty or relatively dense.

We complete the section by showing that in this case, if we consider a cut and project scheme which gives our Meyer set, the decomposition doesn’t go outside a translate of the closure of the window. This gives a result similar to [46, Thm. 4.6].

Let us start by recalling that a **Cut and project scheme** is a triple $(G, H, \mathcal{L})$ consisting of two LCAG’s $G$ respectively $H$ and a lattice $\mathcal{L} \subset G \times H$ such that

- the restriction of the first projection $\pi_G : G \times H \to H$ to $\mathcal{L}$ is one to one.
- the second projection $\pi_H : G \times H \to H$ satisfies $\pi_H(\mathcal{L})$ is dense in $H$.

Given any cut and project scheme $(G, H, \mathcal{L})$, each pre-compact set $W \subset H$ defines an uniformly discrete set $\wedge(W) \subset G$ via

\[ \wedge(W) := \{ x \in G | \text{there exists } y \in W \text{ such that } (x, y) \in \mathcal{L} \}. \]

Next we need to recall the following lemma.

**Lemma 6.11.** [4] Thm. 5.9(i)] Let $(G, H, L)$ be a cut and project scheme, $W \subset G$ be compact and

\[ \mu := \sum_{x \in \wedge(W)} \omega_x \delta_x, \]

be a translation bounded measure. If $\nu \in X(\mu)$ then there exists some $(s, t) \in G \times H$ such that

\[ \sup(\nu) \subset s + \wedge(t + W). \]
Combining this result with Theorem 3.15 we get (compare with [46])

**Theorem 6.12.** Let $(G, H, \mathcal{L})$ be a cut and project scheme, $W \subset G$ be compact and

$$
\mu := \sum_{x \in \Lambda \setminus (W)} \omega_x \delta_x ,
$$

be a weakly almost periodic measure. Then there exists some $p$ and $t$

Remark. As the difference $\mu - \mu_s$ is a null weakly almost periodic measure with Meyer set support, it is essentially a small measure, and hence their supports cannot differ "too much". This is most likely the reason why in [46] the author proved that $(s, t)$ in Theorem 6.12 can be chosen to be $(0, 0)$.

7. **Eberlein Convolution of Weakly Almost Periodic Measures**

In this section we have a thorough look at the Eberlein convolution of weakly almost periodic measures. In particular, we show that this convolution always exists and is independent of the choice of van Hove sequence. We then go on and prove that it defines a strongly almost periodic measure. As an application we obtain an alternative proof for the pure pointedness of the autocorrelation of a weakly almost periodic measure (already shown above in Section 4).

**Lemma 7.1.** Let $\mu, \nu \in \mathcal{M}^c(G)$. If the limit

$$
\varpi = \lim_n \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) \ast (\nu|_{A_n}) ,
$$

exists then for all $f, g \in C_c(G)$ we have

$$
\varpi \ast f \ast g(t) = \lim_n \frac{1}{\theta_G(A_n)} \int_{A_n} (\mu \ast f)(s)(\nu \ast g)(t-s)ds .
$$

**Proof.** Fix $t \in G$.

$$
(4) \left| \frac{\int_G (f \ast \mu)(s)(g \ast \nu)(t-s)1_{A_n}(s)ds - \int_G (f \ast \mu_{A_n})(s)(g \ast \nu_{A_n})(t-s)ds}{\theta_G(A_n)} \right| \\
\leq \frac{\int_G |(f \ast \mu)(s)1_{A_n}(s)(g \ast \nu)(t-s) - (f \ast \mu_{A_n})(s)(g \ast \nu_{A_n})(t-s)|ds}{\theta_G(A_n)} .
$$

Let $K_0$ be any compact set containing $\pm \text{supp}(f)$ and let $K = K_0 \cup (t + K_0)$.

We claim that

$$(f \ast \mu)(s)1_{A_n}(s)(g \ast \nu)(t-s) - (f \ast \mu_{A_n})(s)(g \ast \nu_{A_n})(t-s) \neq 0 \Rightarrow s \in \partial^K(A_n) .$$

**Case 1:** $s \in A_n$. Then

$$(f \ast \mu)(s)(g \ast \nu)(t-s) \neq (f \ast \mu_{A_n})(s)(g \ast \nu_{A_n})(t-s) .$$

This implies $f \ast \mu (s) \neq f \ast \mu_{A_n} (s)$ or $g \ast \nu (t-s) \neq g \ast \nu_{A_n} (t-s)$.
Subcase 1.a: If \( f * \mu(s) \neq f * \mu_{A_n}(s) \) then
\[
\int_K f(s - r)(1 - 1_{A_n}(r))d\mu(r) \neq 0.
\]

Therefore, there exists some \( r \) such that \( f(s - r)(1 - 1_{A_n}(r)) \neq 0 \). Then \( r \notin A_n \) and \( s - r \in \text{supp}(f) \subset K \).
Thus \( s \in (G \setminus A_n) + K \). Since \( s \in A_n \) we get \( s \in \partial^K(A_n) \).

Subcase 1.b: \( g * \nu(t - s) \neq g * \nu_{A_n}(t - s) \) then
\[
\int_K g(t - s - r)(1 - 1_{A_n}(r))d\mu(r) \neq 0.
\]

Therefore, there exists some \( r \) such that \( g(t - s - r)(1 - 1_{A_n}(r)) \neq 0 \). Then \( r \notin A_n \) and \( t - s - r \in \text{supp}(f) \subset K_0 \). Therefore
\[
s \in t - r - K_0 \subset t - K_0 - r \subset K - r.
\]

As \( r \notin A_n = -A_n \) it follows that \( s \in (G \setminus A_n) + K \). This completes Case 1.

Case 2: \( s \notin A_n \). Then, exactly as before
\[
(f * \mu)(s)1_{A_n}(s)(g * \nu)(t - s) \neq (f * \mu_{A_n})(s)(g * \nu_{A_n})(t - s)\]

implies
\[
(f * \mu)(s)1_{A_n}(s) \neq (f * \mu_{A_n})(s) \quad \text{or} \quad 1_{A_n}(s)(g * \nu)(t - s) \neq (g * \nu_{A_n})(t - s)\]

As \( 1_{A_n}(s) = 0 \) we get
\[
0 \neq (f * \mu_{A_n})(s)(g * \nu_{A_n})(t - s)
\]
This implies that \( \int_K f(s - t)d\mu_{A_n}(t) \neq 0 \), and therefore, there exists some \( t \in A_n \) so that \( f(s - t) \neq 0 \). Hence \( s - t \in \text{supp}(f) \) and therefore, \( s \in A_n + K \).

It follows that \( s \in (A_n + K) \setminus A_n \). This completes Case 2.

Now, since \( (f * \mu)(s)1_{A_n}(s)(g * \nu)(t - s) - (f * \mu_{A_n})(s)(g * \nu_{A_n})(t - s) = 0 \) for \( s \) outside \( \partial^K(A_n) \), we get in (1):
\[
\begin{align*}
&\left| \int_G (f * \mu)(s)(g * \nu)(t - s)1_{A_n}(s)ds - \int_G (f * \mu_{A_n})(s)(g * \nu_{A_n})(t - s)ds \right| \\
\leq & \int_G \left| (f * \mu)(s)1_{A_n}(s)(g * \nu)(t - s) - (f * \mu_{A_n})(s)(g * \nu_{A_n})(t - s)ds \right| \\
= & \frac{\int_{\partial^K(A_n)} \left| (f * \mu)(s)1_{A_n}(s)(g * \nu)(t - s) - (f * \mu_{A_n})(s)(g * \nu_{A_n})(t - s)ds \right|}{\theta_G(A_n)} \\
\leq & \frac{\int_{\partial^K(A_n)} 2\|f\| \|\mu\| \|g\| \|\nu\| \|x\|}{\theta_G(A_n)} \\
= & 2\|f\| \|\mu\| \|g\| \|\nu\| \|x\| \frac{\theta_G(\partial^K(A_n))}{\theta_G(A_n)}.
\end{align*}
\]

Therefore, by the van Hove sequence property, we get that
\[
\lim_n \left| \int_G (f * \mu)(s)(g * \nu)(t - s)1_{A_n}(s)ds - \int_G (f * \mu_{A_n})(s)(g * \nu_{A_n})(t - s)ds \right| = 0.
\]
As
\[
\lim_n \frac{1}{\theta_G(A_n)} \int_{A_n} (f * \mu_{A_n})(s)(g * \nu_{A_n})(t-s) \, ds = \varpi \ast f(g(t)),
\]

it follows that
\[
\lim_n \frac{1}{\theta_G(A_n)} \int_{A_n} (f * \mu)(s)(g * \nu)(t-s) \, ds = \lim_n \frac{1}{\theta_G(A_n)} \int_{A_n} (f * \mu)(s)(g * \nu)(t-s)1_{A_n}(s) \, ds = \varpi \ast f(g(t)).
\]

\[\square\]

Using this result we can now prove the existence of the Eberlein convolution of weakly almost periodic measures.

**Theorem 7.2.** Let \( \mu, \nu \in \mathcal{WAP}(G) \). Then the limit
\[
\mu \ast \ast \nu := \lim_n \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) \ast (\nu|_{A_n}),
\]
exists, is translation bounded and is independent of the choice of the van Hove sequence.

Moreover, for all \( f, g \in C_c(G) \) we have \( (\mu \ast \ast \nu) \ast f \ast g = (f * \mu) \ast (g * \nu) \).

**Proof.** By [7], there exists a constant \( C \) and a compact set \( K \subset G \) so that for all \( n \) we have
\[
\frac{1}{\theta_G(A_n)} (\mu|_{A_n}) \ast (\nu|_{A_n}) \in \mathcal{M}_K^C(G) := \{ \varpi \in \mathcal{M}^{\infty}(G) | ||\mu||_K \leq C \}.
\]
Moreover, the set \( \mathcal{M}_K^C(G) \) is compact in the vague topology.

Thus, to prove that the limit exists, it suffices to show that any two cluster points of \( \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) \ast (\nu|_{A_n}) \) are equal. Let \( \gamma_1, \gamma_2 \) be any two cluster points of \( \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) \ast (\nu|_{A_n}) \).

Let \( f, g \in C_c(G) \). Then, the functions \( f * \mu \) and \( g * \nu \) are weakly almost periodic, therefore their Eberlein \( (f * \mu) \ast (g * \nu) \) convolution exists, and it can be calculated with respect to any van Hove sequence.

Let \( A_{k_n} \) and \( A_{l_n} \) be the subsequences which define these cluster points:

\[
\gamma_1 = \lim_n \frac{1}{\theta_G(A_{k_n})} (\mu|_{A_{k_n}}) \ast (\nu|_{A_{k_n}}),
\]
and
\[
\gamma_2 = \lim_n \frac{1}{\theta_G(A_{l_n})} (\mu|_{A_{l_n}}) \ast (\nu|_{A_{l_n}}),
\]

By applying Lemma 7.1 to (5) we get:
\[
\gamma_1 \ast f \ast g(s) = \lim_n \frac{1}{\theta_G(A_{k_n})} \int_{A_{k_n}} (f * \mu)(s-t)(g * \nu) \, dt = (f * \mu) \ast (g * \nu)(s),
\]

while by applying Lemma 7.1 to (6) we get:
\[
\gamma_2 \ast f \ast g = \lim_n \frac{1}{\theta_G(A_{l_n})} \int_{A_{l_n}} (f * \mu)(s-t)(g * \nu) \, dt = (f * \mu) \ast (g * \nu)(s).
\]
This shows that \( \gamma_1 \ast f \ast g = \gamma_2 \ast f \ast g \) for all \( f, g \in C_c(G) \). In particular, evaluating at 0 we get \( \gamma_1(f \ast g^1) = \gamma_2(f \ast g^1) \), for all \( f, g \in C_c(G) \). It follows that \( \gamma_1, \gamma_2 \) are two continuous linear functionals on \( C_c(G) \), which are equal on the dense subset \( \{ f \ast g | f, g \in C_c(G) \} \), therefore \( \gamma_1 = \gamma_2 \).

This shows the existence of the limit. The independence of the van Hove sequence is done exactly the same way: If \( A_n, B_n \) are van Hove sequences, and

\[
\gamma_1 = \lim_{n} \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) \ast (\nu|_{A_n}),
\]

and

\[
\gamma_2 = \lim_{n} \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) \ast (\nu|_{A_n}),
\]

then redoing the above computation, we get \( \gamma_1 \ast f \ast g = (f \ast \mu) \circ (g \ast \nu) = \gamma_2 \ast f \ast g \) for all \( f, g \in C_c(G) \). Therefore, exactly as before \( \gamma_1 = \gamma_2 \).

The fact that \( \mu \circ \nu \) is translation bounded follows immediately from the fact that it is a vague cluster point of a net from \( M^c_{\mathcal{K}}(G) \), which is compact in the vague topology.

The last claim is obvious. \( \square \)

**Definition 7.3.** Let \( \mu, \nu \in \mathcal{WAP}(G) \). We call the limit

\[
\mu \circ \nu := \lim_{n} \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) \ast (\nu|_{A_n}),
\]

the **Eberlein convolution** of \( \mu \) and \( \nu \).

Exactly as in the case of autocorrelation, the Eberlein convolution can be calculated by truncating only one term:

**Lemma 7.4.** Let \( \mu, \nu \in \mathcal{WAP}(G) \). Then \( \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) \ast \nu \) and \( \frac{1}{\theta_G(A_n)} \mu \ast (\nu|_{A_n}) \) converge in the vague topology to \( \mu \circ \nu \).

**Proof.** Let \( f \in C_c(G) \). Then

\[
\left( \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) \ast \nu - \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) \ast (\nu|_{A_n}) \right)(f) =
\]

\[
= \frac{1}{\theta_G(A_n)} \int_G \int_G f(s + t) d\mu|_{A_n}(s) d(\nu - \nu|_{A_n})(t)
\]

\[
= \frac{1}{\theta_G(A_n)} \int_G \int_G f(s + t) 1_{A_n}(s)(1 - 1_{A_n})(t) d\mu(s) d\nu(t).
\]

Then,

\[
\left| \left( \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) \ast \nu - \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) \ast (\nu|_{A_n}) \right)(f) \right| \leq
\]

\[
\leq \frac{1}{\theta_G(A_n)} \int_G \int_G |f(s + t)| 1_{A_n}(s) |(1 - 1_{A_n})| (t) d\mu(s) d\nu(t).
\]
Let us note that $|f(s + t)| 1_{A_n}(s) |(1 - 1_{A_n})|(t) \neq 0$ when $s \in A_n, t \in G \setminus A_n$ and $s + t \in \text{supp}(f)$. Denoting $K = \pm \text{supp}(f)$, and using $A_n = -A_n$ we get that

$$|f(s + t)| 1_{A_n}(s) |(1 - 1_{A_n})|(t) \neq 0 \Rightarrow t \in \partial^K(A_n).$$

Therefore

$$\left| \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) * \nu - \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) * (\nu|_{A_n}) \right| \leq \frac{1}{\theta_G(A_n)} \int_{\partial^K(A_n)} \int_G |f(s + t)| 1_{A_n}(s) d|\mu|(s) d|\nu|(t)$$

$$\leq \frac{1}{\theta_G(A_n)} \int_{\partial^K(A_n)} \int_G |f(s + t)| d|\mu|(s) d|\nu|(t)$$

$$\leq \frac{1}{\theta_G(A_n)} \int_{\partial^K(A_n)} |f| \cdot |\mu|(t) d|\nu|(t) \leq \|f\| \cdot |\mu| \cdot \frac{1}{\theta_G(A_n)} \cdot |\partial^K(A_n)| .$$

Now, by using the translation boundedness of $\mu, \nu$ and the van Hove property, we get that

$$\lim_n \left( \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) * \nu - \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) * (\nu|_{A_n}) \right) (f) = 0 .$$

This shows that in the vague topology we have

$$\lim_n \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) * \nu - \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) * (\nu|_{A_n}) = 0 .$$

Since

$$\lim_n \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) * (\nu|_{A_n}) = \mu \oplus \nu. $$

we get

$$\lim_n \frac{1}{\theta_G(A_n)} (\mu|_{A_n}) * \nu = \mu \oplus \nu. $$

This completes the first claim. The second claim follows by interchanging $\mu$ and $\nu$. \hfill \Box

Note that Theorem 7.2 tells us that the limit always exists, and the definition doesn’t depend on the choice of the van Hove sequence.

**Theorem 7.5.** Let $\mu, \nu \in WAP(G)$ and let $A_n$ be a van Hove sequence. Then

(a) $\mu \oplus \nu \in SAP(G).$

(b) $c_\chi(\mu \oplus \nu) = c_\chi(\mu)c_\chi(\nu)$

(c) $(\mu \oplus \nu)_b = (\mu)_b * (\nu)_b$
Proof. (a) We know that for all $f, g \in C_c(G)$ we have $(\mu \ast \nu) \ast f \ast g = (f \ast \mu) \ast (g \ast \nu)$. As the Eberlein convolution of two weakly almost periodic functions is a strong almost periodic function \[16, 40\], it follows that $(\mu \ast \nu) \ast f \ast g \in SAP(G)$ for all $f, g \in C_c(G)$.

But this implies \[40\] that $\mu \ast \nu \in SAP(G)$.

(b) For all $f, g \in C_c(G)$ we have

$$c_\chi((\mu \ast \nu) \ast f \ast g) = c_\chi((f \ast \mu) \ast (g \ast \nu)).$$

Basic properties of Fourier Bohr coefficients yield

$$c_\chi((\mu \ast \nu) \ast f \ast g) = c_\chi((f \ast \mu) \ast (g \ast \nu)) = c_\chi(f \ast \mu) c_\chi(g \ast \nu)$$

Therefore

$$c_\chi((\mu \ast \nu) \ast f \ast g) = c_\chi(f \ast \mu) c_\chi(g \ast \nu) = c_\chi(f \ast \mu) \hat{c}_\chi(g \ast \nu) = c_\chi(f \ast \mu) \hat{c}_\chi(g \ast \nu) = c_\chi(f \ast \mu) \hat{c}_\chi(g \ast \nu).$$

As this is true for all $f, g \in C_c(G)$, the claim follows now by picking some $f, g$ whose Fourier transform doesn’t vanish at $\chi$.

(c) As both $(\mu \ast \nu)_b$ and $(\mu \ast \nu)_b$ are finite measures on the compact group $G_b$, it suffices to prove that they have the same Fourier transform. Indeed

$$(\mu \ast \nu)_b(\chi) = c_\chi((\mu \ast \nu)_b) = c_\chi((\mu \ast \nu)_b) = (\mu \ast \nu)_b(\chi).$$

□

An immediate consequence of this is the following.

**Theorem 7.6.** Let $\mu \in WAP(G)$. Then

(a) $\mu$ has unique autocorrelation $\gamma = \mu \ast \tilde{\mu}$.

(b) $\mu$ is pure point diffractive.

(c) The intensity of the Bragg peaks is given by

$$\tilde{\gamma}(|\chi|) = |c_\chi(\mu)|^2.$$

**Remark.** If $\chi$ is a Bragg peak, then by Theorem 8.12 $c_\chi$ is a continuous eigenfunction of the system.

We also know that $X(\mu)$ is uniquely ergodic and has pure point dynamical spectrum. Then, the spectral group is generated by the set of Bragg peaks \[7, 30\]. Therefore, each eigenvalue $\lambda$ can be written in the form $\lambda = \chi_1 + \ldots + \chi_k - \chi_{k+1} - \ldots - \chi_n$ for some Bragg peaks $\chi_1, \ldots, \chi_n$.

Then, it follows that

$$f_\lambda = c_{\chi_1} \cdot \ldots \cdot c_{\chi_k} \cdot c_{\chi_{k+1}} \cdot \ldots \cdot c_{\chi_n},$$

is a continuous eigenfunction for the eigenvalue $\lambda$.

This yields and alternate proof of Corollary 4.2.

As immediate consequences of Theorem 7.6 and Corollary 3.18 we get:
**Corollary 7.7.** Let $\mu \in \mathcal{M}_+(G)$ be a positive definite measure. Then $\mu$ is pure point diffractive and uniquely ergodic.

**Corollary 7.8.** Let $\mu \in \mathcal{WAP}(G)$. Then $\mu$ has pure point dynamical spectra.

**Corollary 7.9.** Let $\mu \in \mathcal{WAP}(G)$. Then $\mu$ and $\mu_{\pi}$ have the same diffraction.

We complete the paper by stating and proving a result which is hidden behind the proof of uniqueness of autocorrelation.

**Lemma 7.10.** Let $\mu, \nu$ be two translation bounded measures. If $\mu - \nu$ is null weakly almost periodic, then $\mu$ and $\nu$ have the same autocorrelation(s).

**Proof.** Let $\phi, \psi \in C_c(G)$, and let $A_n$ be any van hove Sequence. Then

\[
\begin{aligned}
&\left| \phi * (\mu|A_n) * (\nu|A_n) * \psi - \phi * (\nu|A_n) * (\nu|A_n) * \psi \right| \\
&\leq \left| \phi * (\mu|A_n) * (\nu|A_n) * \psi - \phi * (\nu|A_n) * (\nu|A_n) * \psi \right| \\
&\quad + \left| \phi * (\mu|A_n) * (\mu|A_n) * \psi \right| \\
&\quad + \left| \phi * (\mu|A_n - \nu|A_n) * (\nu|A_n) * \psi \right|.
\end{aligned}
\]

(7)

Let $\epsilon > 0$.

Since $\mu - \nu$ is null weakly almost periodic, we have

\[
\lim_{n} \frac{\int_{A_n} \left| (\mu|A_n - (\nu|A_n)) * \psi \right| (t) dt}{\theta_G(A_n)} = 0,
\]

and

\[
\lim_{n} \frac{\int_{A_n} \left| \phi * (\mu|A_n - \nu|A_n) \right| (t) dt}{\theta_G(A_n)} = 0.
\]

Let

\[
C := \max\{\|\phi\| \mu, \|\phi\| \nu, \|\phi\| \psi, \|\mu\| \psi, \|\mu\| \mu, \|\psi\| \psi, \|\psi\| \nu, \|\nu\| \nu\}.
\]

Let $K$ be any compact set which contains $\pm \text{supp}(\phi)$ and $\pm \text{supp}(\psi)$.

An easy computation shows that

\[
\left| \phi * (\mu|A_n) * (\mu|A_n - (\nu|A_n)) * \psi \right| \leq C \cdot 1_{A_n + K} * \left| (\mu|A_n - (\nu|A_n)) * \psi \right|,
\]

and

\[
\left| \phi * (\mu|A_n - \nu|A_n) * (\nu|A_n) * \psi \right| \leq C \cdot 1_{A_n + K} * \left| \phi * ((\mu|A_n - (\nu|A_n)) \right|.
\]
Then for all \( t \in G \) we have

\[
\begin{align*}
\left| \phi \ast (\mu|A_n) \ast \left[ (\mu|A_n) \overline{-(\nu|A_n)} \right] \ast \psi \right| (t) & \\
\leq C \frac{1_{A_n} + 1_{K} \ast \left[ (\mu|A_n) \overline{-(\nu|A_n)} \right] \ast \psi}{\theta_G(A_n)} (t) \\
& = C \frac{\int_{A_n} \left[ (\mu|A_n) \overline{-(\nu|A_n)} \right] \ast \psi (t-s) ds}{\theta_G(A_n)} \\
& \leq C \frac{\int_{A_n} \left[ (\mu|A_n) \overline{-(\nu|A_n)} \right] \ast \psi (t-s) ds}{\theta_G(A_n)} \\
& + C \frac{\int_{K} \left[ (\mu|A_n) \overline{-(\nu|A_n)} \right] \ast \psi (t-s) ds}{\theta_G(A_n)}.
\end{align*}
\]

Since \( \mu - \nu \) is null weakly almost periodic, the function \( \left[ (\mu) \overline{-(\nu)} \right] \ast \psi \) is null weakly almost periodic and thus

\[
\lim_n \frac{\int_{A_n} \left[ (\mu) \overline{-(\nu)} \right] \ast \psi (t-s) ds}{\theta_G(A_n)} = 0,
\]

uniformly in \( t \).

A simple computation shows that this implies that

\[
\lim_n \frac{\int_{A_n} \left[ (\mu|A_n) \overline{-(\nu|A_n)} \right] \ast \psi (t-s) ds}{\theta_G(A_n)} = 0.
\]

Also, since \( \left[ (\mu|A_n) \overline{-(\nu|A_n)} \right] \ast \psi \) is bounded, by the van Hove property we have

\[
\lim_n C \frac{\int_{K} \left[ (\mu|A_n) \overline{-(\nu|A_n)} \right] \ast \psi (t-s) ds}{\theta_G(A_n)} = 0.
\]

This shows that

\[
\lim_n \frac{\phi \ast (\mu|A_n) \ast \left[ (\mu|A_n) \overline{-(\nu|A_n)} \right] \ast \psi (t)}{\theta_G(A_n)} = 0,
\]

pointwise for \( t \in G \).

Similarly we can show that

\[
\lim_n \frac{\phi \ast \left[ (\mu|A_n) - \nu|A_n \right] \ast \psi (t)}{\theta_G(A_n)} = 0,
\]

pointwise for \( t \in G \).

Using this in (17) we get that for all \( t \in G \) and all van Hove sequences \( \{A_n\} \) we have

\[
\lim_n \frac{\phi \ast (\mu|A_n) \ast \left[ (\mu|A_n) \overline{-(\nu|A_n)} \right] \ast \psi (t)}{\theta_G(A_n)} = 0.
\]
In particular, in the vague topology, we have
\[
\lim_n \left( \mu_{\mathcal{A}_n} * \mu_{\mathcal{A}_n} - \nu_{\mathcal{A}_n} * \nu_{\mathcal{A}_n} \right) = 0.
\]
Thus the limit \( \lim_n \frac{\mu_{\mathcal{A}_n} * \mu_{\mathcal{A}_n}}{\theta_G(\mathcal{A}_n)} \) exists if and only if the limit \( \lim_n \frac{\nu_{\mathcal{A}_n} * \nu_{\mathcal{A}_n}}{\theta_G(\mathcal{A}_n)} \) exists, and in this case they are the same. □

**Remark.** In Lemma 7.10 the measures \( \mu, \nu \) need not be weakly almost periodic.

**Acknowledgment:** Part of this work is inspired by stimulating talks given at the 2016 meeting ‘Dynamical systems for aperiodicity’ in Lyon. DL would like to thank the organizers for inviting him there. NS was supported by NSERC, under grant 2014-03762, and would like to thank NSERC for their support. Part of this work was done when NS visited DL at Jena University, and NS would like to thank the mathematics department for hospitality.

**References**

[1] E. Akin, E. Glasner, WAP Systems and Labeled Subshifts, preprint arXiv:1410.4753.
[2] L. N Argabright, J. Gil de Lamadrid, Fourier analysis of unbounded measures on locally compact abelian groups, Memoirs of the Amer. Math. Soc., Vol 145, 1974.
[3] J. Auslander, Minimal Flows and their Extensions, North-Holland Mathematical Studies 153, Elsevier 1988.
[4] J.-B. Aujogue, Pure Point/Continuous Decomposition of Translation-Bounded Measures and Diffraction, preprint arXiv:1510.06381
[5] J.-B. Aujogue, M. Barge, J. Kellendonk, D. Lenz, Equicontinuous factors, Proximality and Ellis semigroup for Delone sets in: [26].
[6] M. Baake, U. Grimm, Aperiodic Order. Vol. I: A Mathematical Invitation (Cambridge University Press, Cambridge), 2013.
[7] M. Baake, D. Lenz, Dynamical systems on translation bounded measures: Pure point dynamical and diffraction spectra, Ergod. Th. & Dynam. Syst. 24, 1867–1893, 2004. arXiv:math.DS/0302231.
[8] M. Baake, D. Lenz, Spectral notions of aperiodic order, preprint, 2016. arXiv:1601.06629.
[9] M. Baake, D. Lenz, R.V. Moody, A characterisation of Model sets via Dynamical systems, Ergod. Th. & Dynam. Syst. 27, 341–382, 2007. arXiv:math.DS/0511648.
[10] M. Baake, R.V. Moody, Weighted Dirac combs with pure point diffraction, J. reine angew. Math. (Crelle) 573, 61–94, 2004. arXiv:math.MG/0203030.
[11] S. Beckus, D. Lenz, F. Pogorzelski, M. Schmidt, Diffraction theory for processes of tempered distributions, in preparation.
[12] C. Berg, G. Forst, Potential Theory on Locally Compact Abelian Groups, Springer Berlin, New York, 1975.
[13] X. Deng, R.V. Moody, Dworkin’s argument revisited: point processes, dynamics, diffraction, and correlations, J. Geom. Phys. 58 506–541, 2008. arXiv:0712.3287.
[14] T. Downarowicz, E. Glasner, Isomorphic extensions and applications, preprint, arXiv:1502.06999.
[15] S. Dworkin, Spectral theory and X-ray diffraction, J. Math. Phys. 34 2965–2967, 1993.
[16] W.F. Eberlein, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc., 67, 217-24, 1949.
[17] W.F. Eberlein, *A Note on Fourier-Stieltjes Transforms*, Proc. Amer. Math. Soc., Vol 6, No. 2, 310-312, 1955.
[18] W.F. Eberlein, *The Point Spectrum of Weakly Almost Periodic Functions*, Michigan Math J. 3, 137-139, 1955-1956.
[19] R. Ellis, M. Nerurkar *Weakly Almost Periodic Flows*, Trans. Amer. Math. Soc., 313, vol 1, 103-119, 1989.
[20] S. Favorov, *Bohr and Besicovitch almost periodic discrete sets and quasicrystals*, Proc. Amer. Math. Soc. 140, 1761-1767, 2012. arXiv:math.MG/1011.4036.
[21] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton Univ. Press, Princeton, NJ, 1981.
[22] E. Glasner, *Ergodic Theory via Joinings*, Mathematical Surveys and Monographs, Vol. 101: 2003.
[23] J.-B. Gouéré, *Quasicrystals and almost periodicity*, Comm. Math. Phys. 255,655–681, 2005. arXiv:math-ph/0212012.
[24] U. Grimm, M. Baake *Homometric Point Sets and Inverse Problems* Z. Krist. 223, 777-781, 2008. arXiv:0808.0094.
[25] J. Kellendonk, D. Lenz, *Equscontinuous delone dynamical systems*, Canadian Journal of Mathematics 65, 149–170, 2013. math.DS/1105.3855.
[26] J. Kellendonk, D. Lenz, J. Savinien (eds), *Mathematics of aperiodic order*, Progress in Mathematics 309, Birkhaeuser 2016.
[27] J. C. Lagarias, *Mathematical quasicrystals and the problem of diffraction*. In: Directions in Mathematical Quasicrystals (eds. M. Baake and R.V Moody ), CRM Monograph Series, Vol 13, Amer. Math. Soc., Providence, RI, 61-93, 2000.
[28] J . Gil. de Lamadrid, L. A. R. Argabright, *Almost Periodic Measures*, Memoirs of the Amer. Math. Soc., Vol 85, No. 428, 1990.
[29] L. H. Loomis, *An introduction to abstract harmonic analysis*, D. Van Nostrand Company, Toronto-New York-London, 1953.
[30] J.-Y. Lee, R. V. Moody, B. Solomyak, *Pure point dynamical and diffraction spectra*, Annales H. Poincaré 3 1003–1018; mp.arxiv/02-39, 2002.
[31] D. Lenz, *An autocorrelation and discrete spectrum for dynamical systems on metric spaces*, preprint 2016.
[32] D. Lenz, R.V. Moody, *Stationary processes with pure point diffraction*, to appear in: Ergodic Theory & Dynam. Sys., arXiv:1111.3617v1.
[33] D. Lenz, C. Richard, *Pure Point Diffraction and Cut and Project Schemes for Measures: the Smooth Case*, Mathematische Zeitschrift 256, 347–378, 2007. math.DS/0603453.
[34] D. Lenz, N. Strungaru, *Pure point spectrum for measurable dynamical systems on locally compact Abelian groups*, J. Math. Pures Appl. 92 , 323-341, 2009. arXiv:0704.2498.
[35] D. Lenz, N. Strungaru, *On uniquely ergodicity of systems of measures*, in preparation.
[36] R. V. Moody, N. Strungaru, *Point sets and dynamical systems in the autocorrelation topology*, Canad. Math. Bull. Vol. 47, 82-99, 2004.
[37] H. Reiter, J. D. Stegeman *Classical Harmonic Analysis and Locally Compact Groups*, London Mathematical Society Monographs, Clarendon Press, 2000.
[38] C. Richard, *Dense Dirac combs in Euclidean space with pure point diffraction*, J. Math. Phys. 44, 4436–4449, 2003. arXiv:math-ph/0302049.
[39] C. Richard, N. Strungaru, *A short guide to pure point diffraction in cut-and-project sets*, preprint. arXiv:1606.08831.

[40] R.V. Moody, N. Strungaru, *Almost Periodic Measures and their Fourier Transforms: Results from Harmonic Analysis*, to appear in: M. Baake and U. Grimm (eds.), Aperiodic Order. Vol. 2: Crystallography and Almost Periodicity, Cambridge Univ. Press, in preparation.

[41] G. K. Pedersen, *Analysis Now*, Springer, New York (1989); Revised printing (1995).

[42] C. Radin, *Miles of tiles*, Ergodic theory of $\mathbb{Z}^d$-actions (Warwick, 19931994), 237–258, London Math. Soc. Lecture Note Ser., 228, Cambridge Univ. Press, Cambridge, 1996.

[43] B. Solomyak, *Spectrum of dynamical systems arising from Delone sets*, in: J. Patera (ed.), Quasicrystals and Discrete Geometry, Fields Institute Monographs, 10, AMS, Providence, RI (1998), pp. 265–275.

[44] N. Strungaru, *Almost periodic measures and long-range order in Meyer sets*, Discrete and Computational Geometry vol. 33(3), 483-505, 2005.

[45] N. Strungaru, *Almost periodic Measures and Bragg Diffraction*, J. Phys. A: Math. Theor. 46., 125205, 2013. arXiv:1209.2168.

[46] N. Strungaru, *On Weighted Dirac Combs Supported Inside Model Sets*, J. Phys. A: Math. Theor. 47, 2014. arXiv:1309.7947.

[47] N. Strungaru, *Almost periodic measures and Meyer sets*, to appear in: M. Baake and U. Grimm (eds.), Aperiodic Order. Vol. 2: Crystallography and Almost Periodicity, Cambridge Univ. Press, in preparation. arXiv:1501.00945.

[48] N. Strungaru, V. Terauds, *Diffraction theory and almost periodic distributions*, J. Stat. Phys. 164, no. 5, 1183–1216, 2016. arXiv:1603.04796.

[49] V. Terauds, *The inverse problem for pure point diffraction – examples and open questions*, J. Stat. Phys. 152, no. 5, 954–968, 2013. arXiv:1303.3260.

[50] V. Terauds, M. Baake, *Some comments on the inverse problem of pure point diffraction*, in: Aperiodic Crystals, eds. S. Schmid, R.L. Withers and R. Lifshitz, Springer, Dordrecht, 35–41, 2013. arXiv:1210.3460.

[51] P. Walters, *An introduction to ergodic theory*, Springer Graduate Texts in Mathematics, Springer-Verlag, New York, 1982.