Generators of simple Lie superalgebras in characteristic zero

WENDE LIU$^{1,2}$*, AND LIMING TANG$^{1,2}$†

$^1$Department of Mathematics, Harbin Institute of Technology
Harbin 150006, China

$^2$School of Mathematical Sciences, Harbin Normal University
Harbin 150025, China

Abstract: It is shown that any finite dimensional simple Lie superalgebra over an algebraically closed field of characteristic 0 is generated by 2 elements.

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0. Introduction

Our principal aim is to determine the minimal number of generators for a finite-dimensional simple Lie superalgebra over an algebraically closed field of characteristic 0. The present work is dependent on the classification theorem due to Kac [4], which states that a simple Lie superalgebra (excluding simple Lie algebras) is isomorphic to either a classical Lie superalgebra or a Cartan Lie superalgebra (see also [6]). In 2009, Bois [1] proved that a simple Lie algebra in arbitrary characteristic $p \neq 2, 3$ is generated by 2 elements. In 1976, Ionescu [2] proved that a simple Lie algebra $L$ over the field of complex numbers is generated by 1.5 elements, that is, given any nonzero $x$, there exists $y \in L$ such that the pair $(x, y)$ generates $L$. In 1951, Kuranashi [5] proved that a semi-simple Lie algebra in characteristic 0 is generated by 2 elements.

As mentioned above, all the simple Lie superalgebras split into two series: Classical Lie superalgebras and Cartan Lie superalgebras. The Lie algebra (even part) of a classical Lie superalgebra is reductive and meanwhile there exists a similarity in the structure side between the Cartan Lie superalgebras in characteristic 0 and the simple graded Lie algebra of Cartan type in characteristic $p$. Thus, motivated by Bois’s paper [1] and in view of the observation above, we began this work in 2009. In the process we benefit in addition much from the literatures above, especially from [1], which contains a considerable amount of information in characteristic 0 and characteristic $p$. We also use certain information about classical Lie superalgebras from [6].

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†Correspondence: wendeliu@ustc.edu.cn (W. Liu), limingaaa2@sina.com (L. Tang)
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Throughout we work over an algebraically closed field \( F \) of characteristic 0 and all the vector spaces and algebras are finite dimensional. The main result is that any simple Lie superalgebra is generated by 2 elements.

1. Classical Lie superalgebras

1.1. Basics

A classical Lie superalgebra by definition is a simple Lie superalgebra for which the representation of its Lie algebra (its even part) on the odd part is completely reducible. Throughout this section, we always write \( L = L_0 \oplus L_1 \) for a classical Lie superalgebra. Our aim is to determine the minimal number of generators for a classical Lie superalgebra \( L \). The strategy is as follows. First, using the results in Lie algebras \([1, 2]\), we show that the Lie algebra \( L_0 \) is generated by 2 elements. Then, from the structure of semi-simple Lie algebras and their simple modules, we prove that each classical Lie superalgebra is generated by 2 elements.

A classical Lie superalgebra is determined by its Lie algebra in a sense.

**Proposition 1.1.** \([2, p.101, Theorem 1]\) A simple Lie superalgebra is classical if and only if its Lie algebra is reductive.

The following facts including Table 1.1 may be found in \([4, 6]\). The odd part \( L_1 \) as \( L_0 \)-module is completely reducible and \( L_1 \) decomposes into at most two irreducible components. By Proposition 1.1, \( L_0 = C(L_0) \oplus [L_0, L_0] \). If the center \( C(L_0) \) is nonzero, then dim \( C(L_0) = 1 \) and \( L_1 = L_1^1 \oplus L_1^2 \) is a direct sum of two irreducible \( L_0 \)-submodules. For further information the reader is referred to \([4, 6]\).

**Table 1.1**

| \( L \) | \( L_0 \) | \( L_1 \) as \( L_0 \)-module |
|--------|--------|-----------------------------|
| \( A(m,n) \), \( m,n \geq 0, n \neq m \) | \( A_m \oplus A_n + F \) | \( sl_{m+1} \oplus sl_{n+1} \oplus F \oplus (its \ dual) \) |
| \( A(n,n) \), \( n > 0 \) | \( A_n \oplus A_n \) | \( sl_{n+1} \oplus sl_{n+1} \oplus F \oplus (its \ dual) \) |
| \( B(m,n) \), \( m > 0, n > 0 \) | \( B_m \oplus C_n \) | \( so_{2m+1} \oplus sp_{2n} \) |
| \( D(m,n) \), \( m > 2, n > 0 \) | \( D_m \oplus C_n \) | \( sp_{2m} \oplus sp_{2n} \) |
| \( C(n) \), \( n \geq 2 \) | \( C_{n-1} \oplus F \) | \( cap_{2n-2} \oplus (its \ dual) \) |
| \( P(n) \), \( n \geq 2 \) | \( A_n \) | \( \Lambda^2 sl_{n+1} \oplus S^2 sl_{n+1} \) |
| \( Q(n) \), \( n \geq 2 \) | \( A_n \) | \( ad sl_{n+1} \) |
| \( D(2,1;\alpha) \), \( \alpha \in F \setminus \{-1,0\} \) | \( A_1 \oplus A_1 \oplus A_1 \) | \( sl_2 \oplus sl_2 \oplus sl_2 \) |
| \( G(3) \) | \( \mathfrak{so}_2 \oplus \mathfrak{A}_1 \) | \( \mathfrak{so}_2 \oplus sl_2 \) |
| \( F(4) \) | \( \mathfrak{B}_3 \oplus \mathfrak{A}_1 \) | \( spin_7 \oplus sl_2 \) |

1.2. Even parts

Let \( \mathfrak{g} \) be a semi-simple Lie algebra. Consider the root decomposition relative to a Cartan subalgebra \( \mathfrak{h} \): \( \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}^\alpha \). For \( x \in \mathfrak{g} \) we write \( x = x_\mathfrak{h} + \sum_{\alpha \in \Phi} x^\alpha \) for the corresponding root space decomposition. It is well-known that \([2]\)

\[
\dim \mathfrak{g}^\alpha = 1 \quad \text{for all } \alpha \in \Phi, \quad (1.1)
\]

\[
\mathfrak{h} = \sum_{\alpha \in \Phi} [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}], \quad (1.2)
\]

\[
[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \mathfrak{g}^{\alpha + \beta} \quad \text{whenever } \alpha, \beta, \alpha + \beta \in \Phi. \quad (1.3)
\]
Let $V$ be a vector space and $\mathfrak{F} := \{f_1, \ldots, f_n\}$ a finite set of non-zero linear functions on $V$. Write

$$\Omega_\mathfrak{F} := \{v \in V \mid \Pi_{1 \leq i \neq j \leq n}(f_i - f_j)(v) \neq 0\}.$$ 

**Lemma 1.2.** Suppose $\mathfrak{F}$ is a finite set of non-zero functions in $V^*$. Then $\Omega_\mathfrak{F} \neq \emptyset$. If $\mathfrak{F} \subset \mathfrak{F}$ then $\Omega_\mathfrak{F} \subset \Omega_\mathfrak{F}$. 

**Proof.** The first statement is from [1, Lemma 2.2.1] and the second is straightforward. 

This lemma will be usually used in the special situation when $V$ is a Cartan subalgebra of a simple Lie superalgebra.

An element $x$ in a semi-simple Lie algebra $g$ is called *balanced* if it has no zero components with respect to the standard decomposition of simple Lie algebras. If $h$ is a Cartan subalgebra of $g$, $x \in g$ is called $h$-balanced provided that $x^\alpha \neq 0$ for all $\alpha \in \Phi$. 

**Lemma 1.3.** [1] An element of a semi-simple Lie algebra $g$ is balanced if and only if it is $h$-balanced for some Cartan subalgebra $h$. 

**Proof.** One direction is obvious. Suppose $x \in g$ is balanced and let $b'$ be a Cartan subalgebra of $g$. From the proof of [1, Theorem 2.2.3], there exists $\varphi \in g$ such that $\varphi(x)$ is $b'$-balanced. Letting $h = \varphi^{-1}(b')$, one sees that $h$ is a Cartan subalgebra and $x$ is $h$-balanced. 

For an algebra $\mathfrak{A}$ and $x, y \in \mathfrak{A}$, we write $\langle x, y \rangle$ for the subalgebra generated by $x$ and $y$. We should notice that for a Lie superalgebra $\langle x, y \rangle$ is not necessarily a $\mathbb{Z}_2$-graded subalgebra (hence not necessarily a sub-Lie superalgebra). The following technical lemma will be frequently used.

**Lemma 1.4.** Let $\mathfrak{A}$ be an algebra. For $a \in \mathfrak{A}$ write $L_a$ for the left-multiplication operator given by $a$. Suppose $x = x_1 + x_2 + \cdots + x_n$ is a sum of eigenvectors of $L_a$ associated with mutually distinct eigenvalues. Then all $x_i$'s lie in the subalgebra generated $\langle a, x \rangle$. 

**Proof.** Let $\lambda_i$ be the eigenvalues of $L_a$ corresponding to $x_i$. Suppose for a moment that all the $\lambda_i$'s are nonzero. Then

$$(L_a)^k(x) = \lambda_1^k x_1 + \lambda_2^k x_2 + \cdots + \lambda_n^k x_n \quad \text{for } k \geq 1.$$ 

Our conclusion in this case follows from the fact that the Vandermonde determinate given by $\lambda_1, \lambda_2, \ldots, \lambda_n$ is nonzero and thereby the general situation is clear. 

We write down a lemma from [1, Theorem B and Corollary 2.2.5] and the references therein, which is also a consequence of Lemmas 1.2, 1.3 and 1.4.

**Lemma 1.5.** Let $g$ be a semi-simple Lie algebra. If $x \in g$ is balanced then for a suitable Cartan subalgebra $h$ and the corresponding root system $\Phi$ we have $g = \langle x, h \rangle$ for all $h \in \Omega_\Phi$. 

Denote by $\Pi := \{\alpha_1, \ldots, \alpha_n\}$ the system of simple roots of a semi-simple Lie algebra $g$ relative to a Cartan subalgebra $h$. As above, $x \in g$ is referred to as $\Pi$-balanced if $x$ is a sum of all the simple-root vectors, that is, $x = \sum_{\alpha \in \Pi} x^\alpha$, where $x^\alpha$ is a root vector of $\alpha$. Recall that $\Omega_{\Pi} \neq \emptyset$ by Lemma 1.2.
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Corollary 1.6. A semi-simple Lie algebra $g$ is generated by a $\Pi$-balanced element and an element in $\Omega$. 

Proof. This is a consequence of Lemma 1.4 and the facts (1.1), (1.2) and (1.3).

Proposition 1.7. The Lie algebra of a classical Lie superalgebra is generated by 2 elements.

Proof. Let $L = L_0 \oplus L_1$ be a classical Lie superalgebra. By Proposition 1.1, $L_0$ is reductive, that is, $[L_0, L_0]$ is semi-simple and

$$L_0 = C(L_0) \oplus [L_0, L_0].$$

(1.4)

If $C(L_0) = 0$, the conclusion follows immediately from Lemma 1.5. If $C(L_0)$ is nonzero, then $C(L_0) = \mathbb{F} z$ is 1-dimensional. Choose a balanced element $x \in [L_0, L_0]$. By Lemma 1.5 there exists $h \in [L_0, L_0]$ such that $[L_0, L_0] = \langle x, h \rangle$. Claim that $L_0 = \langle x, h + z \rangle$.

Indeed, considering the projection of $L_0$ onto $[L_0, L_0]$ with respect to the decomposition (1.4), denoted by $\pi$, which is a homomorphism of Lie algebras, we have

$$\pi(\langle x, h + z \rangle) = \langle \pi(x), \pi(h + z) \rangle = \langle x, h \rangle = [L_0, L_0].$$

Hence only two possibilities might occur: $\langle x, h + z \rangle = [L_0, L_0]$ or $\dim \langle x, h + z \rangle = \dim [L_0, L_0]$. The first case is the desired. Let us show that the second does not occur. Assume the contrary. Then $\pi$ restricting to $\langle x, h + z \rangle$ is an isomorphism and thereby $\langle x, h + z \rangle$ is semi-simple. Thus

$$\langle x, h + z \rangle = [(x, h + z), (x, h + z)] = \langle x, h \rangle, \langle x, h \rangle = \langle x, h \rangle.$$ 

Hence $h \in \langle x, h + z \rangle$. It follows that

$$z \in \langle x, h + z \rangle = \langle x, h \rangle = [L_0, L_0],$$

contradicting (1.4).

Remark 1.8. By Corollary 1.6 $\mathfrak{sl}(n)$ is generated by a $\Pi$-balanced element $x$ and an element $y$ in $\Omega$. As in the proof of Proposition 1.7 one may prove that $\mathfrak{gl}(n)$ is generated by $h$ and $x + z$, where $z$ is a nonzero central element in $\mathfrak{gl}(n)$.

1.3. Classical Lie superalgebras

Suppose $L$ is a classical Lie superalgebra with the standard Cartan subalgebra $H$. The corresponding weight (root) space decompositions are

$$L_0 = H \oplus \bigoplus_{\alpha \in \Delta_0} L_0^\alpha, \quad L_1 = \bigoplus_{\beta \in \Delta_1} L_1^\beta;$$

$$L = H \oplus \bigoplus_{\alpha \in \Delta_0} L_0^\alpha \oplus \bigoplus_{\beta \in \Delta_1} L_1^\beta.$$ 

(1.5)

Every $x \in L$ has a unique decomposition with respect to (1.5):

$$x = x_H + \sum_{\alpha \in \Delta_0} x_0^\alpha + \sum_{\beta \in \Delta_1} x_1^\beta,$$

(1.6)
where \( x_H \in H, \, x_0^2 \in L_0^2, \, x_1^2 \in L_1^2 \). Write

\[
\Delta := \Delta_0 \cup \Delta_1 \quad \text{and} \quad L^\gamma := L_0^\gamma \oplus L_1^\gamma \quad \text{for} \quad \gamma \in \Delta.
\]

Note that the standard Cartan subalgebra of a classical Lie superalgebra is diagonal:

\[
\text{ad}(h) = \gamma(h)x \quad \text{for all} \quad h \in H, \, x \in L^\gamma, \, \gamma \in \Delta.
\] (1.7)

For \( x \in L \), write

\[
\text{supp}(x) := \{ \gamma \in \Delta \mid x_\gamma \neq 0 \}.
\] (1.8)

For \( x = x_0 + x_1 \in L \),

\[
\text{supp}(x) = \text{supp}(x_0) \cup \text{supp}(x_1).
\]

**Lemma 1.9.**

1. If \( L \neq Q(n) \) then \( 0 \notin \Delta_1 \) and \( \Delta_0 \cap \Delta_1 = \emptyset \).
2. If \( L = Q(n) \) then \( \Delta_1 = \{0\} \cup \Delta_0 \).
3. If \( L \neq A(1,1), \, Q(n) \) or \( P(3) \) then \( \dim L^\gamma = 1 \) for every \( \gamma \in \Delta \).
4. Suppose \( L = \Lambda(m, n), \, A(n, n), \, C(n) \) or \( P(n) \), where \( m \neq n \).
   
   a. \( L_1 = L_1^1 \oplus L_1^2 \) is a direct sum of two irreducible \( L_0 \)-submodules.
   
   b. Let \( \Delta_1^i \) be the weight set of \( L_1^i \) relative to \( H, \, i = 1, 2 \). Then there exist \( \alpha_i^1 \in \Delta_1^i \) such that \( \alpha_i^1 \neq \alpha_i^2 \).

**Proof.** (1), (2) and (3) follow from [6, Proposition 1, p.137]. (4)(a) follows from Table 1.1. Let us consider (4)(b). For \( L = \Lambda(m, n), \, A(n, n) \) or \( C(n) \), it follows from the fact that \( L_0 \)-modules \( L_{-1} \) and \( L_1 \) are contragradient. For \( L = P(n) \), a direct computation shows that \( -e_1 - e_2 \in \Delta_1^1 \) and \( 2e_1 \in \Delta_1^2 \).

**Theorem 1.10.** A classical Lie superalgebra is generated by 2 elements.

**Proof.** Let \( L = L_0 \oplus L_1 \) be a classical Lie superalgebra.

**Case 1.** Suppose \( \dim C(L_0^1) = 1 \). In this case \( L = C(n) \) or \( \Lambda(m, n) \) with \( m \neq n \) (see Table 1.1). Then \( L_1 = L_1^1 \oplus L_1^2 \) is a direct sum of two irreducible \( L_0 \)-submodules and \( [L_0, L_0] \) is simple or a direct sum of two simple Lie algebras. Let \( x_0 \) be a balanced element in \( [L_0, L_0] \). From Lemma 1.3 there exists a Cartan subalgebra \( h \) of \( [L_0, L_0] \) such that \( \text{supp}(x_0) = \Delta_0 \), the latter is viewed as the root system relative to \( h \). By Lemma 1.5 we have \( [L_0, L_0] = \langle x_0, h \rangle \) for all \( h \in \Omega_0 \). Furthermore, from the proof of Proposition 1.7 it follows that \( L_0 = \langle x_0, h + z \rangle \) for \( \emptyset \neq z \in C(L_0^1) \). By Lemma 1.7(1) and (4), there exist \( \alpha_1^1 \in \Delta_1^1 \) and \( \alpha_1^2 \in \Delta_1^2 \) such that \( \alpha_1^1 \neq \alpha_1^2 \) and \( \alpha_1^1, \alpha_1^2 \notin \Delta_0 \). Set \( x := x_0 + x_1^{\alpha_1^1} + x_1^{\alpha_1^2} + z \) for some weight vectors \( x_1^{\alpha_1^i} \in L_1^{\alpha_1^i} \), \( i = 1, 2 \). Then

\[
x = (x_0 + z) + \sum_{\alpha \in \Delta_0} x_0^\alpha + x_1^{\alpha_1^1} + x_1^{\alpha_1^2}.
\]

Write \( \Phi := \Delta_0 \cup \{ \alpha_1^1 \} \cup \{ \alpha_1^2 \} \) and choose an element \( h' \in \Omega_\Phi \). Assert \( \langle x, h' \rangle = L \). To show that, write \( L' := \langle x, h' \rangle \). Lemma 1.4 implies all components \( x_0^\alpha, x_1^{\alpha_1^1}, x_1^{\alpha_1^2} \), and
$x_H + z$ belong to $L'$. Since $x_0^0 \in L'$ for all $\alpha \in \Delta_0$, from (1.2) we have $x_H \in L'$ and then $z \in L'$. As $h' \in \Omega_F \subset \Omega_{\Delta_0}$, we obtain $\langle x_0, h' + z \rangle = L_0 \subset L'$. Since $x_1^{\alpha_1} \in L'$ and $L_1^{\alpha_1}$ is an irreducible $L_0$-module, we have $L_1 \subset L'$, where $i = 1, 2$. Therefore, $L = L'$.

Case 2. Suppose $C(L_0) = 0$. Then $L_0$ is a semi-simple Lie algebra and $L_1$ decomposes into at most two irreducible components (see Table 1.1).

Subcase 2.1. Suppose $L_1$ is an irreducible $L_0$-module. Note that in this subcase, $L$ is of type $B(m, n), D(m, n), D(2; 1; \alpha), Q(n), G(3)$ or $F(4)$. We choose a weight vector $x_1^{\alpha_1} \in L_1^{\alpha_1}$ ($\alpha_1 \neq 0$) and any balanced element $x_0 \in L_0$. By Lemma 1.4 we may assume that $\text{supp}(x_0) = \Delta_0$.

If $L \neq Q(n)$, according to Lemma 1.9(1), $\alpha_1 \notin \Delta_0$. Let $x = x_0 + x_1^{\alpha_1}$. Then

$$x = x_H + \sum_{\alpha \in \Delta_0} x_0^\alpha + x_1^{\alpha_1}$$

is the root-vector decomposition. Let $\Phi = \Delta_0 \cup \{\alpha_1\}$. By Lemmas 1.2 and 1.4 all components $x_H, x_0^\alpha$ and $x_1^{\alpha_1}$ belong to $\langle x, h \rangle$ for $h \in \Omega_F \subset H$. By (1.1) and (1.2), this yields $L_0 = \langle x_0, h \rangle \subset \langle x, h \rangle$. Since $x_1^{\alpha_1} \in \langle x, h \rangle$ and $L_1$ is irreducible as $L_0$-module, we have $L = \langle x, h \rangle$.

Suppose $L = Q(n)$. Denote by $\Pi := \{\delta_1, \delta_2, \ldots, \delta_n\}$ the set of simple roots of $L_0$ relative to the Cartan subalgebra $H$. According to Lemma 1.9(2), without loss of generality we may assume that $\alpha_1 := \delta_1 + \delta_2$. Let $x = x_0 + x_1^{\alpha_1}$. Then

$$x = x_H + \sum_{\alpha \in \Delta_0 \setminus \{\alpha_1\}} x_0^\alpha + (x_0^{\alpha_1} + x_1^{\alpha_1}).$$

By Lemma 1.4 all components $x_H, x_0^\alpha$ ($\alpha \in \Delta_0 \setminus \{\alpha_1\}$), and $x_0^{\alpha_1} + x_1^{\alpha_1}$ belong to $\langle x, h \rangle$, where $h \in \Omega_\Delta_0 \subset H$. From (1.3) and (1.1) we conclude that $x_0^{\alpha_1} \in F[x_0^{\alpha_1}, x_1^{\alpha_1}] \subset \langle x, h \rangle$ and then $x_1^{\alpha_1} \in \langle x, h \rangle$. As above, the irreducibility of $L_1$ yields $L = \langle x, h \rangle$.

Subcase 2.2. Suppose $L_1 = L_1^1 \oplus L_1^2$ is a direct sum of two irreducible $L_0$-submodules. In this case, $L = A(n, n)$ or $P(n)$. Choose any balanced element $x_0 \in L_0$ and weight vectors $x_1^{\alpha_1} \in L_1^{\alpha_1}$, where $\alpha_1^1$ and $\alpha_1^2$ are different nonzero weights and $\alpha_1^1 \notin \Delta_0$ (Lemma 1.2(1) and (4)). Lemma 1.4 allows us to assume that $\text{supp}(x_0) = \Delta_0$.

Let $x := x_0 + x_1^{\alpha_1} + x_1^{\alpha_2}$ and $\Phi := \Delta_0 \cup \{\alpha_1^1\} \cup \{\alpha_1^2\}$. As before, we are able to deduce that $L_0 \subset \langle x, h \rangle$ and $x_1^{\alpha_1}, x_1^{\alpha_2} \in \langle x, h \rangle$ for $h \in \Omega_F \subset \Omega_{\Delta_0} \subset H$. Thanks to the irreducibility of $L_1^1$ and $L_1^2$, we have $L = \langle x, h \rangle$. The proof is complete. □

Remark 1.11. In view of the proof of Theorem 1.10 starting from any balanced element in the semi-simple part of the Lie algebra of a classical Lie superalgebra $L$ we are able to find two elements generating $L$.

By Theorem 1.10 as in the proof of Proposition 1.7 one is able to prove the following

Corollary 1.12. The general linear Lie superalgebra $\text{gl}(m, n)$ is generated by 2 elements.
As a subsidiary result, let us show that a classical Lie superalgebra, except for $A(1,1)$, $Q(n)$ or $P(3)$, is generated by 2 homogeneous elements. By Lemma 1.9 (3), for such a classical Lie superalgebra, all the odd-weight subspaces are 1-dimensional. Here we give a more general description in Remark 1.13. As before, an element $x \in L$ is called $\Delta_1$-balanced if $x$ is a sum of all the odd-weight vectors, namely, $x = \sum_{\gamma \in \Delta_1} x^\gamma_1$, where $x^\gamma_1$ is a weight vector of $\gamma$.

**Remark 1.13.** A finite dimensional simple Lie superalgebra (not necessarily classical) for which all the odd-weight is 1-dimensional center and the semi-simple part:

\[ L = L^0 \oplus L^1\]

where $[A^\gamma_1, e] = \sum_{\gamma \in \Delta_1} \sum_{\gamma^\prime \in \Delta_1} \frac{\epsilon_{\gamma, \gamma^\prime}}{\epsilon_{\gamma^\prime, \gamma}} [A^\gamma_1, x^\gamma_1^\prime]$. By Proposition 1.2(1), p.20, $L^0 = [L^0, L^0]$ and then $\langle x, h \rangle = L$. 

Finally, we give an example to explain how to find the pairs of generators in Theorem 1.10 and Remark 1.13.

**Example 1.14.** Let $A = A(1; 0)$. Find the generators of $A$ as in Theorem 1.10 and Remark 1.13.

Recall that $A = \{x \in gl(2; 1) | \text{str}(x) = 0\}$. Its Lie algebra is a direct sum of the 1-dimensional center and the semi-simple part:

\[ A_0 = \mathbb{F}(e_{11} + e_{22} + 2e_{33}) \oplus [A_0, A_0], \]

where $[A_0, A_0] = \text{span}_F\{e_{11} - e_{22}, e_{12}, e_{21}\}$. The odd part is a direct sum of two irreducible $A_0$-submodules:

\[ A_1 = \bar{A}_1 \oplus A_1' = \text{span}_F\{e_{13}, e_{23}\} \oplus \text{span}_F\{e_{31}, e_{32}\}. \]

The standard Cartan subalgebra is $H = \text{span}_F\{e_{11} - e_{22}, e_{11} + e_{22} + 2e_{33}\}$.

Table 1.2 gives all the roots and the corresponding root vectors.

| roots | $e_1 - e_2$ | $e_2 - e_1$ | $e_1 - 2e_3$ | $e_2 - 2e_3$ | $-e_1 + 2e_3$ | $-e_2 + 2e_3$ |
|-------|-------------|-------------|--------------|--------------|---------------|---------------|
| vectors | $e_{12}$ | $e_{21}$ | $e_{13}$ | $e_{23}$ | $e_{31}$ | $e_{32}$ |

- **Theorem 1.10 Version.** Put $x := (e_{12} + e_{21}) + e_{13} + e_{31} + (e_{11} + e_{22} + 2e_{33})$ and $h := 3e_{11} + e_{22} + 4e_{33}$. From Table 1.2, the weight values corresponding to $e_{12}, e_{21}, e_{13}, e_{31}$ are 2, $-2$, $-5$, 5, respectively. As in the proof of Theorem 1.10 we have

\[ e_{12} + e_{21} + e_{13} + e_{31} + e_{11} + e_{22} + 2e_{33} \in \langle x, h \rangle. \]

Furthermore,\n
\[ (e_{12} + e_{21} + (e_{11} + e_{22} + 2e_{33})) = A_0 \subset \langle x, h \rangle. \]

Since $A_1^i$ is an irreducible $A_0$-module, $A_1^i \subset \langle x, h \rangle$, $i = 1, 2$. Hence $A = \langle x, h \rangle$.

- **Remark 1.13 Version.** Consider the $\Delta_1^-$-balanced element $x := e_{13} + e_{31} + e_{23} + e_{32}$ and write $h := e_{11} + e_{33}$. By Table 1.2, the weight values corresponding to $e_{13}, e_{31}, e_{23}, e_{32}$ are $-1, 1, -2, 2$, respectively. As in the proof of Remark 1.13 we have $e_{13}, e_{31}, e_{23}, e_{32} \in \langle x, h \rangle$. Since $\dim A_1^\lambda = 1$ for $\lambda \in \Delta_1$ and $[A_1^i, A_1^j] = A_0^\lambda$, we obtain $A = \langle x, h \rangle$. 

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**Table 1.2**

| roots | $e_1 - e_2$ | $e_2 - e_1$ | $e_1 - 2e_3$ | $e_2 - 2e_3$ | $-e_1 + 2e_3$ | $-e_2 + 2e_3$ |
|-------|-------------|-------------|--------------|--------------|---------------|---------------|
| vectors | $e_{12}$ | $e_{21}$ | $e_{13}$ | $e_{23}$ | $e_{31}$ | $e_{32}$ |
2. Cartan Lie superalgebras

All the Cartan Lie superalgebras are listed below [4, 6]:

\[ W(n) \ (n \geq 3), \ S(n) \ (n \geq 4), \ S(2m) \ (m \geq 2), \ H(n) \ (n \geq 5). \]

Let \( \Lambda(n) \) be the Grassmann superalgebra with \( n \) generators \( \xi_1, \ldots, \xi_n \). For a \( k \)-shuffle \( u := (i_1, i_2, \ldots, i_k) \), that is, a strictly increasing sequence between 1 and \( n \), we write \( |u| := k \) and \( x^u := \xi_{i_1} \xi_{i_2} \cdots \xi_{i_k} \). Letting \( \deg \xi_i = 1, \ i = 1, \ldots, n \), we obtain the so-called standard \( \mathbb{Z} \)-grading of \( \Lambda(n) \). Let us briefly describe the Cartan Lie superalgebras.

- \( W(n) = \text{der} \Lambda(n) \) is \( \mathbb{Z} \)-graded, \( W(n) = \bigoplus_{k=1}^{n-1} W(n)_k \),
  \[
  W(n)_k = \text{span}_F \{ x^u \partial / \partial \xi_i \mid |u| = k + 1, \ 1 \leq i \leq n \}.
  \]

- \( S(n) = \bigoplus_{k=1}^{n-2} S(n)_k \) is a \( \mathbb{Z} \)-graded subalgebra of \( W(n) \),
  \[
  S(n)_k = \text{span}_F \{ D_{ij}(x^u) \mid |u| = k + 2, \ 1 \leq i, j \leq n \}.
  \]
  Hereafter, \( D_{ij}(f) := \partial(f) / \partial \xi_i \partial / \partial \xi_j + \partial(f) / \partial \xi_j \partial / \partial \xi_i \) for \( f \in \Lambda(n) \).

- \( \tilde{S}(2m) \ (m \geq 2) \) is a subalgebra of \( W(2m) \) and as a \( \mathbb{Z} \)-graded subspace,
  \[
  \tilde{S}(2m) = \bigoplus_{k=-1}^{2m-2} \tilde{S}(2m)_k,
  \]
  where
  \[
  \tilde{S}(2m)_{-1} = \text{span}_F \{ (1 + \xi_1 \cdots \xi_{2m}) \partial / \partial \xi_j \mid 1 \leq j \leq 2m \},
  \]
  \[
  \tilde{S}(2m)_k = S(2m)_k, \ 0 \leq k \leq 2m - 2.
  \]
  Notice that \( \tilde{S}(2m) \) is not a \( \mathbb{Z} \)-graded subalgebra of \( W(2m) \).

- \( H(n) = \bigoplus_{k=0}^{n-3} H(n)_k \) is a \( \mathbb{Z} \)-graded subalgebra of \( W(n) \), where
  \[
  H(n)_k = \text{span}_F \{ D_H(x^u) \mid |u| = k + 2 \}.
  \]
  To explain the linear mapping \( D_H : \Lambda(n) \rightarrow W(n) \), write \( n = 2m \ (m \geq 3) \) or \( 2m + 1 \ (m \geq 2) \). By definition, \( D_H(x^u) := (-1)^{|u|} \sum_{i=1}^{n} \partial(x^u) / \partial \xi_i \partial / \partial \xi_{i'} \) for any shuffle \( u \), where \( i' = i + m \) for \( i \leq m \). For simplicity we usually write \( W, S, \tilde{S}, H \) for \( W(n), S(n), \tilde{S}(2m), H(n) \), respectively. Throughout this section \( L \) denotes one of Cartan Lie superalgebras. Consider its decomposition of subspaces mentioned above:

\[
L = L_{-1} \oplus \cdots \oplus L_s.
\] (2.1)

For \( W, S, \tilde{S} \) and \( H \), the height \( s \) is \( n - 1, n - 2, 2m - 2 \) or \( n - 3 \), respectively. Note that \( S \) and \( H \) are \( \mathbb{Z} \)-graded subalgebras of \( W \) with respect to (2.1), but \( \tilde{S} \) is not. The null \( L_0 \) is isomorphic to \( \mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{sl}(2m), \mathfrak{so}(n) \) for \( L = W, S, \tilde{S}, H \), respectively.
Lemma 2.1.

(1) $L_{-1}$ and $L_s$ are irreducible as $L_0$-modules.

(2) $L_1$ is an irreducible $L_0$-module for $L = S, \tilde{S}$ or $H$, except for $H(6)$. For $L = H(6)$, $L_1$ is a direct sum of two irreducible $L_0$-submodules.

(3) $L$ is generated by the local part $L_{-1} \oplus L_0 \oplus L_1$.

(4) $L$ is generated by $L_{-1}$ and $L_s$ for $L = W, S$ or $H$.

Proof. All the statements are standards (see [4, 6] for example), except for that $\tilde{S}_{-1}$ is irreducible as $\tilde{S}_0$-module. Indeed, a direct verification shows that $\tilde{S}_{-1}$ is an $\tilde{S}_0$-module and the irreducibility follows from the canonical isomorphism of $S_0$-modules $\varphi : S_{-1} \rightarrow \tilde{S}_{-1}$ assigning $\partial/\partial \xi_i$ to $(1 + \xi_1 \cdots \xi_{2m})\partial/\partial \xi_i$ for $1 \leq i \leq 2m$.

The following is a list of bases of the standard Cartan subalgebras $\mathfrak{h}_{L_0}$ of $L_0$.

| $L$       | basis of $\mathfrak{h}_{L_0}$ |
|-----------|--------------------------------|
| $W(n)$    | $\xi_i \partial/\partial \xi_i, 1 \leq i \leq n$ |
| $S(n)$    | $\xi_i \partial/\partial \xi_j - \xi_j \partial/\partial \xi_i, 2 \leq j \leq n$ |
| $S(2m)$   | $\xi_i \partial/\partial \xi_1 - \xi_1 \partial/\partial \xi_i, 2 \leq j \leq 2m$ |
| $H(2m)$   | $\xi_i \partial/\partial \xi_i - \xi_{m+i} \partial/\partial \xi_{m+i}, 1 \leq i \leq m$ |
| $H(2m + 1)$ | $\xi_{i+1} \partial/\partial \xi_{i+1} - \xi_{m+i} \partial/\partial \xi_{m+i}, 1 \leq i \leq m$ |

The weight space decomposition of the component $L_k$ relative to $\mathfrak{h}_{L_0}$ is:

$$L_k = \delta_{k,0} \mathfrak{h}_{L_0} \oplus_{\alpha \in \Delta_k} L_0^\alpha,$$

where $-1 \leq k \leq s$.

By Lemma 2.1(2), $H(6)_1$ is a direct sum of two irreducible $H(6)_0$-modules

$$H(6)_1 = H(6)^1_1 \oplus H(6)^2_1.$$

Let $\Delta_1^i$ be the weight set of $H(6)_1^i$, $i = 1, 2$. Write $\Pi$ for the set of simple roots of $L_0$ relative to the Cartan subalgebra $\mathfrak{h}_{L_0}$. We have

Lemma 2.2.

(1) If $L = W$ or $S$ then $\Pi \cap \Delta_{-1} = \Pi \cap \Delta_s = \Delta_{-1} \cap \Delta_s = \emptyset$.

(2) If $L = \tilde{S}$ then $\Pi \cap \Delta_{-1} = \Pi \cap \Delta_s = \Delta_{-1} \cap \Delta_s = \emptyset$.

(3) If $L = H(2m)$ then $\Pi \cap \Delta_{-1} = \Pi \cap \Delta_1 = \emptyset$ and $\Delta_{-1} \neq \Delta_1$.

(4) If $L = H(2m + 1)$ then $0 \in \Delta_{-1}$, $\Pi \neq \Delta_1$ and $\Delta_{-1} \neq \Delta_1$.

(5) There exist nonzero weights $\alpha_1^i \in \Delta_1^i$ such that $\alpha_1^1 \neq \alpha_1^2$.

Proof. We first compute the weight sets of the desired components and the system of simple roots of $L_0$. For $W(n)$,

$$\Delta_{-1} = \{-\epsilon_j \mid 1 \leq j \leq n\}, \quad \Delta_0 = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\}, \quad \Delta_s = \{\sum_{k=1}^n \epsilon_k - \epsilon_j \mid 1 \leq j \leq n\}.$$

For $S(n)$,

$$\Delta_{-1} = \{-\epsilon_j \mid 1 \leq j \leq n\}, \quad \Delta_0 = \{\epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n\}. \quad \Delta_s = \{\sum_{k=1}^n \epsilon_k - \epsilon_j \mid 1 \leq j \leq n\}.$$
For $S(n)$ and $\tilde{S}(n)$,
\[
\Delta_\pm = \{ \pm \varepsilon_j | 1 \leq j \leq n \}, \quad \Delta_0 = \{ \varepsilon_i - \varepsilon_j | 1 \leq i \neq j \leq n \},
\]
\[
\Pi = \{ \varepsilon_i - \varepsilon_{i+1} | 1 \leq i \leq n - 1 \}, \quad \Delta_1 = \{ \varepsilon_k + \varepsilon_l - \varepsilon_j | 1 \leq k, l, j \leq n \},
\]
\[
\Delta_s = \left\{ \sum_{i=1}^n \varepsilon_i - \varepsilon_j - \varepsilon_k | 1 \leq j, k \leq n \right\}.
\]

For $H(2m)$,
\[
\Delta_\pm = \{ \pm \varepsilon_j | 1 \leq j \leq m \}, \quad \Delta_0 = \{ \pm (\varepsilon_i + \varepsilon_j), \pm (\varepsilon_i - \varepsilon_j) | 1 \leq i < j \leq m \},
\]
\[
\Pi = \{ \varepsilon_i - \varepsilon_{i+1}, \varepsilon_{m-1} + \varepsilon_m | 1 \leq i < m \},
\]
\[
\Delta_1 = \{ \pm (\varepsilon_i + \varepsilon_j) \pm \varepsilon_k \pm (\varepsilon_i - \varepsilon_j) \pm \varepsilon_k | 1 \leq i < j < k \leq m \}
\]
\[
\cup \{ \pm \varepsilon_l | 1 \leq l \leq m \}. \quad (2.2)
\]

For $H(2m + 1)$, write $\varepsilon_i' = \varepsilon_{i+1}$ for $1 \leq i \leq m$. We have
\[
\Delta_\pm = \{ 0 \} \cup \{ \pm \varepsilon_i' | 1 \leq i \leq m \},
\]
\[
\Delta_0 = \{ \pm (\varepsilon_i' + \varepsilon_j'), \pm (\varepsilon_i' - \varepsilon_j') | 1 \leq k, m, 1 \leq i < j \leq m \},
\]
\[
\Pi = \{ \varepsilon_i' - \varepsilon_{i+1}, \varepsilon_m' | 1 \leq i < m \},
\]
\[
\Delta_1 = \{ 0 \} \cup \{ \pm (\varepsilon_i', \varepsilon_i + \varepsilon_j'), \pm (\varepsilon_i' - \varepsilon_j') | 1 \leq l < m, 1 \leq i < j \leq m \}
\]
\[
\cup \{ \pm (\varepsilon_i' + \varepsilon_j') \pm \varepsilon_k \pm (\varepsilon_i' - \varepsilon_j') \pm \varepsilon_k | 1 \leq i < j < k \leq m \}.
\]

All the statements follow directly, except (5) for $L = H(6)$. In this special case, from (2.2) one sees that $0 \notin \Delta_1$ and $|\Delta_1| > 1$. Consequently, (5) holds. \[ \square \]

Recall that an element $x \in \mathfrak{g}$ is referred to as $\Pi$-balanced if $x$ is a sum of all the simple-root vectors.

**Theorem 2.3. A Cartan Lie superalgebra is generated by 2 elements.**

**Proof.** Recall the null $L_0$ is isomorphic to $\mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{sl}(2m)$ or $\mathfrak{so}(n)$. From Remark 1.5 and Corollary 1.6 for a $\Pi$-balanced element $x_0 \in L_0$ and $h_0 \in \Omega_1 \subset \mathfrak{h}_{L_0}$ we have $L_0 = \langle x_0 + \delta_{L,W}z, h_0 \rangle$, where $z$ is a central element in $\mathfrak{gl}(n)$.

For simplicity, write $t := s$ for $L = W$ or $S$ and $t := 1$ for $L = \tilde{S}$ or $H$. Suppose $L \neq H(6)$ and $H(2m + 1)$. According to Lemma 2.2, we are able to choose nonzero weights $\alpha_{-1} \in \Delta_\pm$ and $\alpha_t \in \Delta_t$ such that $\alpha_{-1} \neq \alpha_t$, $\alpha_{-1} \notin \Pi$, and $\alpha_t \notin \Pi$. Put $x := x_{-1} + x_0 + \delta_{L,W}z + x_t$ for some weight vectors $x_{-1} \in L_{\alpha_{-1}}^\alpha$ and $x_t \in L_{\alpha_t}$. Now set $\Phi := \Pi \cup \{ \alpha_{-1} \} \cup \{ \alpha_t \} \subset \mathfrak{h}_{L_0}$ and choose an element $h_0 \in \Omega_1$. Assume $\langle x, h_0 \rangle = L$. Lemma 2.2 implies all components $x_{-1} x_0, \delta_{L,W}z$ and $x_t$ belong to $\langle x, h_0 \rangle$. As $h_0 \in \Omega_1 \subset \Omega_{L_0}$, we obtain $L_0 = \langle x_0 + \delta_{L,W}z, h_0 \rangle \subset \langle x, h_0 \rangle$.

By Lemma 2.1 and (2), since $L_{-1}$ and $L_t$ are irreducible $L_0$-modules, we have $L_{-1} + L_t \subset \langle x, h_0 \rangle$. From Lemma 2.2 (3) and (4) it follows that $L = \langle x, h_0 \rangle$.

If $L = H(6)$, by Lemma 2.2 (3), we are able to choose $\alpha_{-1} \in \Delta_\pm$, $\alpha_1 \in \Delta_1$ and $\alpha_2, \delta_{L,W}z$ such that $\alpha_{-1}, \alpha_1, \alpha_2$ are pairwise distinct and $\alpha_{-1} \notin \Pi$, $\alpha_1 \notin \Pi$ and $\alpha_2 \notin \Pi$. Put $x := x_{-1} + x_0 + x_1 + x_2$ for some weight vectors $x_{-1} \in L_{\alpha_{-1}}^\alpha$ and $x_i \in L_{\alpha_i}^\alpha$, $i = 1, 2$. Write $\Phi := \Pi \cup \{ \delta_{L,W}z \} \cup \{ \alpha_1 \} \cup \{ \alpha_2 \}$. For $h_0 \in \Omega_1 \subset \Omega_{L_0}$, as in the above, one may show that $L = \langle x, h_0 \rangle$.

If $L = H(2m + 1)$, by Lemma 2.2 (4), choose $\alpha_{-1} \in \Delta_\pm$, $\alpha_1 \in \Delta_1$ such that $\alpha_{-1} = 0$, $\alpha_1 \notin \Pi$. Set $x := x_{-1} + x_0 + x_1$ for some weight vectors $x_{-1} \in L_{\alpha_{-1}}^\alpha$ and...
$x_1 \in L_1^{\alpha_1}$. Now put $\Phi := \Pi \cup \{\alpha_1\} \cup \{\alpha_1\} \subset h^*_m$. Let $h_0 \in \Omega_\Phi \subset \Omega_\Pi$ and claim that $L = \langle x, h_0 \rangle$. By Lemma 1.4, $x_0, x_{-1}$ and $x_1 \in \langle x, h_0 \rangle$. Consequently, $L_0 \subset L$. The irreducibility of $L_{-1}$ and $L_1$ ensures $L_{-1} + L_1 \subset \langle x, h_0 \rangle$. By Lemma 2.1(3), the claim holds. The proof is complete.

Theorems 1.10 and 2.3 combine to the main result of this paper:

**Theorem 2.4.** Any simple Lie superalgebra is generated by 2 elements.

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