FINITE – DIMENSIONAL REPRESENTATIONS
OF $U_q[gl(2/1)]$ IN A BASIS OF $U_q[gl(2) \oplus gl(1)]$

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Abstract

The quantum superalgebra $U_q[gl(2/1)]$ is given as both a Drinfel’d–Jimbo
deformation of $U[gl(2/1)]$ and a Hopf superalgebra. Finite–dimensional repre-
sentations of this quantum superalgebra are constructed and investigated in a
basis of its even subalgebra $U_q[gl(2) \oplus gl(1)]$. The present method for construct-
ing representations of a quantum superalgebra combines previously suggested
ones for the cases of superalgebras and quantum superalgebras, and, therefore,
has an advantage in comparison with the latter.

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I. INTRODUCTION

Emerged about twenty years ago $^{[1] - [6]}$ from the study on the quantum inverse
scattering method and Yang–Baxter equations $^7$, quantum groups (QG’s) readily
became one of the most interesting concepts in physics and mathematics in the last
two decades. For a short time QG’s and their representations have been investigated
in details in both the physical and the mathematical aspects and have found various
applications in physics $^8 - [14]$.

One of the approaches to QG’s is the Drinfel’d–Jimbo (DJ), or "quantum", de-
formation of universal enveloping algebras $^2 [4]$. This kind of deformation depends on
one or more parameters which could be generic complex numbers or roots of unity. The defined in this way QG’s appear to be Hopf algebras which are typically noncommutative and noncocommutative \cite{2}. The latter, in turn, can be used for introducing and studying QG’s. Hopf algebra structures of QG’s are shown to be an efficient tool for investigating QG’s as the whole and their representations in particular. Moreover, these investigations can be extended to quantum supergroups (QSG’s), a notion combining QG’s with supersymmetry \cite{15} – \cite{18}. For their generality and importance QSG’s, e.g., $U_q[gl(m/n)]$, which are deformations of universal enveloping algebras $U[gl(m/n)]$ of superalgebras $gl(m/n)$ are a subject of research interest \cite{16} – \cite{28}. Representations of these QSG’s called also quantum superalgebras (QSA’s) are explicitly known in a number of cases but their constructions are sometimes complicated with heavy calculations, especially for higher rank cases. We suggested in \cite{22} a method (procedure) for constructing and investigating finite–dimensional representations of a QSA. This method is very efficient for one-parametric deformations $U_q[gl(m/n)]$, at least with $m$ and $n$ not very high \cite{22} – \cite{25}, and it can be also applicable to the two-parametric case \cite{24, 27, 28}. In general, the method proposed is good, however, as in the classical, i.e., non-deformed, case \cite{29} – \cite{31}, its practical realisation is not always convenient because the calculation process based on using (deformed) commutation relations between generators (without using their Hopf algebra structures) is cumbersome in some stages. Besides that, the latter method does not give us an easy way to get an explicit description of representations of a QSA in a basis of its even subalgebra (for example, it does not express the so-called induced basis of a QSA in terms of a basis of the even subalgebra and, therefore, we do not have matrix elements in the induced basis as we could do in the classical case \cite{29, 30}). Such a description may be physically necessary as in it both the origin and the structure of multiplets can be seen explicitly.

Exploiting the Hopf algebra structure of quantum superalgebras $U_q[gl(m/n)]$ we can investigate in a transparent and consistent way their module structure and representations. Taking a demonstration on $U_q[gl(2/1)]$ (which can be applied to physical problems such as those of strongly correlated electron systems \cite{32} – \cite{34}) we construct its induced module and find all its finite–dimensional irreducible representations in a basis of the even subalgebra $U_q[gl(2/1)_0] \equiv U_q[gl(2) \oplus gl(1)]$. The results obtained are hopefully useful for the above–mentioned applications. The present method combines the advantage of the previously suggested methods for the classical case \cite{29, 30} and the quantum deformation case \cite{22, 25} and may be more convenient in some practical (calculation and application) aspects. This paper is organized as follows.
The quantum superalgebra $U_q[gl(2/1)]$ as a DJ deformation of $U[gl(2/1)]$ and as a Hopf superalgebra is given in Section II where its induced representations are also considered. Finite-dimensional representations of this quantum superalgebra in a basis of its even subalgebra $U_q[gl(2) \oplus gl(1)]$ (or simply, a $U_q[gl(2) \oplus gl(1)]$ – basis) are constructed in Section III and classified in Section IV. Finally, section V is devoted to some discussions and the conclusion.

II. $U_q[gl(2/1)]$ AND ITS INDUCED REPRESENTATIONS

The quantum superalgebra $U_q[gl(2/1)]$ can be completely defined through the Weyl–Chevalley generators $E_{12}, E_{21}, E_{23}, E_{32}$ and $E_{ii}, i = 1, 2, 3$, which satisfy the following defining relations $[22, 25]$

\begin{align*}
\text{a) the super-commutation relations } (1 \leq i, i+1, j, j+1 \leq 3): \\
[E_{ii}, E_{jj}] &= 0, \quad (2.1a) \\
[E_{ii}, E_{j,j+1}] &= (\delta_{ij} - \delta_{i,j+1})E_{j,j+1}, \quad (2.1b) \\
[E_{ii}, E_{j+1,j}] &= (\delta_{i,j+1} - \delta_{ij})E_{j+1,j}, \quad (2.1c) \\
[E_{12}, E_{21}] &= [H_1], \quad (2.1d) \\
\{E_{23}, E_{32}\} &= [H_2], \quad (2.1e) \\
H_i &= (E_{ii} - \frac{d_{i+1}}{d_i}E_{i+1,i+1}), \quad (2.1g)
\end{align*}

where $d_1 = d_2 = -d_3 = 1$, and

\begin{align*}
\text{b) the Serre relations:} \\
E_{23}^2 &= E_{32}^2 = 0, \quad (2.2a) \\
[E_{12}, E_{13}]_q &= [E_{21}, E_{31}]_q = 0, \quad (2.2b)
\end{align*}

with $E_{13}$ and $E_{31}$,

\begin{align*}
E_{13} := [E_{12}, E_{23}]_q^{-1}, \quad E_{31} := -[E_{21}, E_{32}]_q^{-1}, \quad (2.3)
\end{align*}

defined as new generators, where the notation

$$[A, B]_r := AB - rBA$$

is used. The newly defined generators are odd and have vanishing squares. The generators $E_{ij}, i, j = 1, 2, 3$, including the new ones, are quantum deformation analogues ($q$–analogues) of the Weyl generators $e_{ij}$ of the classical superalgebra $gl(2/1)$ whose
universal enveloping algebra $U[gl(2/1)]$ is a classical limit of $U_q[gl(2/1)]$ at $q \to 1.$ The defined in this way quantum superalgebra $U_q[gl(2/1)] \equiv U_q$ is a Hopf superalgebra endowed with the following additional maps:

1) \textit{coproduct} $\Delta$: \quad $U_q \to U_q \otimes U_q,$

$$\Delta(1) = 1 \otimes 1,$$

$$\Delta(E_{ii}) = E_{ii} \otimes 1 + 1 \otimes E_{ii},$$

$$\Delta(E_{12}) = E_{12} \otimes q^{H_1} + 1 \otimes E_{12},$$

$$\Delta(E_{21}) = E_{21} \otimes 1 + q^{-H_1} \otimes E_{21},$$

$$\Delta(E_{23}) = E_{23} \otimes q^{H_2} + 1 \otimes E_{23},$$

$$\Delta(E_{32}) = E_{32} \otimes 1 + q^{-H_2} \otimes E_{32}.$$ \quad (2.4)

2) \textit{antipode} $S$: \quad $U_q \to U_q,$

$$S(1) = 1,$$

$$S(E_{ii}) = -E_{ii},$$

$$S(E_{12}) = -E_{12}q^{-H_1},$$

$$S(E_{21}) = -q^{H_1}E_{21},$$

$$S(E_{23}) = -E_{23}q^{-H_2},$$

$$S(E_{32}) = -q^{H_2}E_{32}.$$ \quad (2.5)

3) \textit{counit} $\varepsilon$: \quad $U_q \to \mathbb{C},$

$$\varepsilon(1) = 1,$$

$$\varepsilon(E_{ii}) = \varepsilon(E_{12}) = \varepsilon(E_{21}) = \varepsilon(E_{23}) = \varepsilon(E_{32}) = 0.$$ \quad (2.6)

These maps are either homomorphisms (\textit{coproduct} and \textit{counit}) or an anti-homomorphism (\textit{antipode}) and are consistent with the defining relations (2.1) – (2.3). These relations are quantum deformations, or $q$-deformations, of the ordinary (super-) commutation relations and they can be obtained from the latter by replacing the classical adjoint operation $ad$ with the quantum deformation one $ad_q,$

$$ad_q = (\mu_L \otimes \mu_R)(id \otimes S)\Delta,$$ \quad (2.7)

where $\mu_L$ (respectively, $\mu_R$) is the left (respectively, right) multiplication: $\mu_L(x)y = xy$ (respectively, $\mu_R(x)y = (-1)^{deg_x \cdot deg_y}yx$). Then, the generators $E_{13}$ and $E_{31}$ defined in (2.3) can be written in an adjoint form,

$$E_{13} = ad_q(E_{12})(E_{23}q^{H_2})q^{H_1-H_2}, \quad E_{31} = -ad_q(E_{21})(E_{32}),$$ \quad (2.8)
being a $q$–analogue of the classical one,

$$e_{13} = ad(e_{12})e_{23} \equiv [e_{12}, e_{23}], \quad e_{31} = -ad(e_{21})e_{32} \equiv -[e_{21}, e_{32}] .$$

Of course, one can rescale the generators $E_{12}$ and $E_{23}$ to make $E_{13}$ in (2.8) to resemble more its classical counterpart $e_{13} = [e_{12}, e_{23}]$.

We see from the relations (2.1) – (2.3) that each of the odd spaces $A_{\pm}$,

$$A_+ = \text{lin. env.} \{ E_{13}, E_{23} \}, \quad (2.9)$$

$$A_- = \text{lin. env.} \{ E_{31}, E_{32} \}, \quad (2.10)$$

is a representation space of the even subalgebra $U_q[gl(2/1)] \equiv U_q[gl(2) \oplus gl(1)]$, which, generated by generators $E_{12}$, $E_{21}$, and $E_{ii}$, $i = 1, 2, 3$, is a stability subalgebra of $U_q[gl(2/1)]$. Therefore, we can construct a representation of $U_q[gl(2/1)]$ induced from some (finite–dimensional irreducible, for example) representation of $U_q[gl(2/1)]_0$ which is realized in a representation space (module) $V^q$ being a tensor product of a $U_q[gl(2)]$-module $V^q(gl_2)$ and a $gl(1)$-module ($gl(1)$-factors) $V^q(gl_1)$. Let us take throughout this paper $V^q$ to be an irreducible (later also finite–dimensional) $U_q[gl(2/1)]_0$-module.

If we demand

$$E_{23}V^q = 0 \quad (2.11)$$

hence

$$U_q(A_+)V^q = 0, \quad (2.12)$$

we turn the $U_q[gl(2/1)]_0$-module $V^q$ into a $U_q(B)$-module with

$$B = A_+ \oplus gl(2) \oplus gl(1). \quad (2.13)$$

The $U_q[gl(2/1)]$-module $W^q$ induced from the $U_q[gl(2/1)]_0$- module $V^q$ is the factor space

$$W^q = [U_q \otimes V^q]/I^q \quad (2.14)$$

where

$$U_q \equiv U_q[gl(2/1)], \quad (2.15)$$

and $I^q$ is the subspace

$$I^q = \text{lin. env.} \{ ub \otimes v - u \otimes bv | u \in U_q, b \in U_q(B), v \in V^q \}. \quad (2.16)$$

By construction, any vector $w \in W^q$ can be represented as

$$w = u \otimes v, \quad u \in U_q, \quad v \in V^q. \quad (2.17)$$
Then $W^q$ is a $U_q[gl(2/1)]$-module in the sense
\[ gw \equiv g(u \otimes v) = gu \otimes v \in W^q \] (2.18)
for $g, u \in U_q, w \in W^q$ and $v \in V^q$.

Using the commutation relations (2.1) – (2.2) and the definitions (2.3) we can prove an analogue of the Poincaré–Birkhoff–Witt theorem.

**Proposition 1:** The quantum deformation $U_q := U_q[gl(2/1)]$ is spanned on all possible linear combinations of the elements
\[ g = (E_{23})^{\eta_1}(E_{13})^{\eta_2}(E_{31})^{\theta_1}(E_{32})^{\theta_2}g_0, \] (2.19)
where $\eta_i, \theta_i = 0, 1$ and $g_0 \in U_q[gl(2/1)] \equiv U_q[gl(2)] \oplus gl(1)$.

Then the following proposition can be also proved:

**Proposition 2:** The induced $U_q[gl(2/1)]$-module $W^q$ is the linear span
\[ W^q = \text{lin. env.} \{(E_{31})^{\theta_1}(E_{32})^{\theta_2} \otimes v || v \in V^q, \, \theta_1, \theta_2 = 0, 1\}, \] (2.20)
and, consequently, the set of all the vectors
\[ |\theta_1, \theta_2; (m)\rangle := (E_{31})^{\theta_1}(E_{32})^{\theta_2} \otimes (m), \, \theta_1, \theta_2 = 0, 1, \] (2.21)
constitutes a basis of $W^q$, with $(m)$ being a basis of $V^q$.

Thus, we can write the $U_q[gl(2/1)]$-module $W^q$ in the form
\[ W^q([m]) = T^q \otimes V^q([m]), \] (2.22)
where
\[ T^q = \text{lin. env.} \{(E_{31})^{\theta_1}(E_{32})^{\theta_2}, \theta_i = 0, 1\} \] (2.23)
and $[m]$ is a signature (an highest weight, in the case of finite–dimensional representations) characterizing the module $V^q$ and, therefore, also the module $W^q$. The basis (2.21) referred to as the induced basis of $W^q$ is a tensor product
\[ |\theta_1, \theta_2; (m)\rangle = |\theta_1, \theta_2\rangle \otimes (m) \] (2.21’)
between a basis
\[ |\theta_1, \theta_2\rangle := (E_{31})^{\theta_1}(E_{32})^{\theta_2}, \, \theta_i = 0, 1, \] (2.24)
of $T^q$ and a basis $(m)$ of $V^q$.

Taking the fact that

$$ad_q(U_q[gl(2/1)_0]) T^q \subset T^q$$

(2.25)

we can consider $T^q$ as a module of the even subalgebra $U_q[gl(2/1)_0]$. This module is completely reducible since it represents a direct sum of three irreducible submodules

$$T^q = T^q_0 \oplus T^q_1 \oplus T^q_2,$$

(2.26)

where

$$T^q_0 = \text{lin. env. } \{(E_{31})^0(E_{32})^0 \equiv \mathbf{1}\} \equiv \mathbf{C},$$

(2.27a)

$$T^q_1 = \text{lin. env. } \{E_{31}, E_{32}\},$$

(2.27b)

$$T^q_2 = \text{lin. env. } \{E_{31}E_{32}\}.$$  

(2.27c)

Every subspace $T^q_i$, $i = 0, 1, 2$, as an irreducible $U_q[gl(2/1)_0]$-module, is characterized by a signature, say $[\mu]_i$, which is always fixed and will be determined in the next section (see (3.11)): 

$$T^q_i = T^q_i([\mu]_i).$$

(2.28)

So, the module $W^q$ being a tensor product of two $U_q[gl(2/1)_0]$-modules, $T^q$ and $V^q_0$, is also a $U_q[gl(2/1)_0]$-module which, in general, is reducible and can be written now in the form

$$W^q([m]) = D_0 \oplus D_1 \oplus D_2,$$

(2.29)

where

$$D_i = T^q_i \otimes V^q([m]), \quad i = 0, 1, 2.$$  

(2.30)

Here, as seen later, $D_0$ and $D_2$ are irreducible $U_q[gl(2/1)_0]$-modules, but $D_1$ is a reducible one (see, (3.23)).

**Proposition 3:** The $U_q[gl(2/1)]$-module $W^q$ is decomposed into (four or less) finite-dimensional irreducible modules $V^q_k$ of the even subalgebra $U_q[gl(2/1)_0]$,

$$W^q([m]) = \bigoplus_{0 \leq k \leq 3} V^q_k([m]_k),$$

(2.31)

where $[m]$ and $[m]_k$ are some signatures (highest-weights) characterizing the module $W^q \equiv W^q([m])$ and the modules $V^q_k \equiv V^q_k([m]_k)$, respectively.

Now we are ready to construct finite-dimensional representations of $U_q[gl(2/1)]$ in a basis of its even subalgebra $U_q[gl(2/1)_0]$. These representations are induced from
finite–dimensional irreducible representations of the even subalgebra $U_q[gl(2/1)_0]$. For a basis of the latter we can chose a Gel’fand–Zetlin (GZ) one.

III. FINITE – DIMENSIONAL REPRESENTATIONS OF $U_q[gl(2/1)]$

A finite–dimensional representation of $U_q[gl(2/1)_0]$ is realized in some space (module) which could be one of the above $V^q_k$ whose basis, a $U_q[gl(2/1)_0]$-basis, can be chosen as a tensor product

$$\begin{bmatrix} m_{12} & m_{22} & m_{32} \\ m_{11} & m_{31} \end{bmatrix} \equiv (m)_{gl(2)} \otimes m_{31} \equiv (m)_k \quad (3.1a)$$

between a (GZ) basis $(m)_{gl(2)}$ of $U_q[gl(2)]$ and $gl(1)$-factors $m_{31}$, where $m_{ij}$ are complex numbers such that

$$m_{12} - m_{11}, \quad m_{11} - m_{22} \in \mathbb{Z}_+, \quad (3.1b)$$
$$m_{32} = m_{31}. \quad (3.1c)$$

Indeed, finite–dimensional representations of $U_q[gl(2)]$ are highest weight and the generators $E_{ij}, \; i, j = 1, 2$, and $E_{33}$ (called the even generators of $U_q[gl(2/1)]$) really satisfy the commutation relations (2.1a) – (2.1d) for $U_q[gl(2/1)_0]$ if they are defined on (3.1) as follows

$$E_{11}(m)_k = (l_{11} + 1)(m)_k, \quad (3.2a)$$
$$E_{22}(m)_k = (l_{12} + l_{22} - l_{11} + 2)(m)_k, \quad (3.2b)$$
$$E_{12}(m)_k = (l_{12} - l_{11})[l_{11} - l_{22}]^{1/2}(m)_k^{11}, \quad (3.2c)$$
$$E_{21}(m)_k = (l_{12} - l_{11} + 1)[l_{11} - l_{22} - 1]^{1/2}(m)_k^{-11}, \quad (3.2d)$$
$$E_{33}(m)_k = (l_{31} + 1)(m)_k, \quad (3.2e)$$

where

$$l_{ij} = m_{ij} - (i - 2\delta_{i,3}), \quad (3.3)$$

and $(m)_k^{\pm i j}$ is a vector obtained from $(m)$ by replacing $m_{ij}$ with $m_{ij} \pm 1$. The signature of a basis vector $(m)_k$ now is the highest weight described by the first (top) row of the patterns (3.1)

$$[m]_k = [m_{12}, m_{22}, m_{32}] \quad (3.4)$$

remaining unchanged under the action of the even generators is nothing but an ordered set of eigen-values of the Cartan generators $E_{ii}, \; i = 1, 2, 3$, on the highest weight vector $(M)_k$ defined as follows

$$E_{12}(M)_k = 0, \quad (3.5)$$
\[ E_{ii}(M)_k = m_{i2}(M)_k. \] (3.6)

The highest weight vector \((M)_k\) is a vector \((m)_k\) with \(m_{11} = m_{12}\),

\[
(M)_k = \begin{bmatrix} m_{12} & m_{22} ; m_{32} = m_{31} \\ m_{11} & m_{31} \end{bmatrix},
\] (3.7)

and vice versa a (lower weight) vector \((m)_k\) can be derived from \((M)_k\) via the formula

\[
(m)_k = \left( \frac{[m_{11} - m_{22}]!}{[m_{12} - m_{22}]![m_{12} - m_{11}]!} \right)^{1/2} (E_{21})^{m_{12} - m_{11}}(M)_k. \] (3.8)

The subscript \(k\) in the l.h.s of (3.4) can be omitted when there is no degeneration among signatures of basis vectors. Additionally, for the case \(k = 0\), as will be seen \(V_0^q \equiv V^q\), we can always skip the subscript 0,

\[
(m)_0 \equiv (m), \quad [m]_0 \equiv [m], \quad (M)_0 \equiv (M). \] (3.9)

In a GZ basis (3.1), the highest weights \([\mu]_i\) of the subspaces \(T_i^q\) have the form (3.4), \([\mu]_i \equiv [\mu_{12}, \mu_{22}, \mu_{32}]\), that is

\[
T_i^q = T^q([\mu]_i) \equiv T^q([\mu_{12}, \mu_{22}, \mu_{32}]). \]

Let us denote the GZ basis vectors of \(T_i^q([\mu]_i)\) by

\[
\begin{bmatrix} \mu_{12} & \mu_{22} ; \mu_{32} = \mu_{31} \\ \mu_{11} & \mu_{31} \end{bmatrix} \equiv (\mu)_{gl(2)} \otimes \mu_{31} \equiv (\mu). \] (3.10)

Using the action of \(U_q[gl(2/1)_0]\) on \(T_i^q\) we identify the basis vectors (2.24) as follows:

\[
|0, 0\rangle \equiv 1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in T^q([0, 0; 0]) = T_0^q, \] (3.11a)

\[
|1, 0\rangle \equiv E_{31} = -\begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \in T^q([0, -1; 1]) = T_1^q, \] (3.11b)

\[
|0, 1\rangle \equiv E_{32} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \in T^q([-1, -1; 2]) = T_2^q, \] (3.11c)
In the latter formulae the subscripts $i$ of signatures of the subspaces $T^q_i$ can be omitted since there is no degeneration among these signatures.

We can combine all the basis vectors (3.11) in a common formula:

$$|\theta_1, \theta_2\rangle = (-1)^{\theta_1 + (\theta_2 + 1)}(\mu),$$  \hspace{1cm} (3.12)

where $\theta_1, \theta_2 = 0, 1$, and

$$(\mu) = \left[ \begin{array}{ccc} \mu_{12} & \mu_{22} & \mu_{32} \\ \mu_{11} & \mu_{31} \\ \end{array} \right];$$

$$= \left[ \begin{array}{ccc} -\theta_1 \theta_2 & -\frac{1}{2} \{1 + (-1)^{(1-\theta_1)(1-\theta_2)}\} \theta_1 + \theta_2 \\ -\theta_1 & \theta_1 + \theta_2 \\ \end{array} \right].$$  \hspace{1cm} (3.13)

The action of the even subalgebra $U_q[gl(2/1)]$ on the basis (3.11) of $T^q_i$ is the following

$$E_{ij} |\theta_1, \theta_2\rangle = -\theta_i (1 - \theta_j). |1 - \theta_1, 1 - \theta_2\rangle, \text{ for } i, j = 1, 2, \ i \neq j,$$

$$E_{ii} |\theta_1, \theta_2\rangle = -\theta_i. |\theta_1, \theta_2\rangle, \ i = 1, 2,$$

$$E_{33} |\theta_1, \theta_2\rangle = (\theta_1 + \theta_2). |\theta_1, \theta_2\rangle.$$  \hspace{1cm} (3.14)

Now the induced basis (2.21) can be written in the form

$$|\theta_1, \theta_2; (m)\rangle = (-1)^{\theta_1 + (\theta_2 + 1)}(\mu) \otimes (m).$$  \hspace{1cm} (3.15)

To find the transformation of the latter basis under $U_q[gl(2/1)]$, it is sufficient to find transformations of this basis under the Weyl–Chevalley generators, which are those $E_{ij}$ with $|i - j| \leq 1, \ i, j = 1, 2, 3$. The actions of the even generators follow from their co-product structure and their actions (3.2) on $|\mu\rangle$ and $(m)$, while those of the odd generators follow from

$$E_{32}(E_{31})^{\theta_1}(E_{32})^{\theta_2} = (-q)^{\theta_1}(E_{31})^{\theta_1}(E_{32})^{\theta_2 + 1},$$

$$E_{23}(E_{31})^{\theta_1}(E_{32})^{\theta_2} = (-1)^{\theta_1 + \theta_2}(E_{31})^{\theta_1}(E_{32})^{\theta_2}E_{23} + (-1)^{\theta_1 \theta_2}(E_{31})^{\theta_1}[H_2]$$

$$-\theta_1 \theta_2 E_{31} q^{-H_2 - 1} + \theta_1 q^{-\theta_2} (E_{32})^{\theta_2} E_{21} q^{-H_2 - 1}.  \hspace{1cm} (3.16)$$

The latter in turn can follow from a more general (deformed) commutation relation

$$E_{ij}(E_{31})^{\theta_1}(E_{32})^{\theta_2} = q^{(\delta_{i3}\delta_{j2} + \delta_{i2}\delta_{j3})\theta_1 - \delta_{i2}\delta_{j1}\theta_2}(-1)^{\delta_{i3}\delta_{j3}(\theta_1 + \theta_2) + \delta_{i3}\delta_{j2}\theta_1}(E_{31})^{\theta_1}(E_{32})^{\theta_2} E_{ij}. $$
\[- \{\delta_{i1}\delta_{j1}\theta_1 + (\delta_{i2}\delta_{j2} - \delta_{i3}\delta_{j3})\theta_2\} (E_{31})^{\theta_1}(E_{32})^{\theta_2} \]
\[-\theta_i(1 - \theta_j)(E_{31})^{1-\theta_1}(E_{32})^{1-\theta_2}q^{\delta_{i1}\delta_{j2}H_1} \]
\[+\delta_{i2}\delta_{j3}\theta_1 \left\{ q^{-\theta_2}(E_{32})^{\theta_2}E_{21} - \theta_2 E_{31} \right\} q^{-1-H_2} \]
\[+\delta_{i2}\delta_{j3}\theta_2 (-E_{31})^{\theta_1}[H_2], \]
(3.17)

where \(i, j = 1, 2, 3\), \(|i - j| \leq 1\), \(\theta_1, \theta_2 = 0, 1\), \(\theta_3 = \theta_2\). This commutation relation is, of course, consistent with (3.14).

Taking into account (3.11) – (3.17) we get representations of \(U_q[gl(2/1)]\) in the induced basis (2.21)

\[E_{ij} |\theta_1, \theta_2; (m)\rangle = (1 - \delta_{i2}\delta_{j3})q^{(\delta_{i3}\delta_{j2} + \delta_{i2}\delta_{j1})\theta_1 - \delta_{i2}\delta_{j1}\theta_2 (-1)^{\delta_{i2}\delta_{j3}(\theta_1 + \theta_2) + \delta_{i3}\delta_{j2}\theta_1}} \]
\[\times |\theta_1, \theta_2 + \delta_{i3}\delta_{j2}; (m)\rangle \]
\[- \{\delta_{i1}\delta_{j1}\theta_1 + (\delta_{i2}\delta_{j2} - \delta_{i3}\delta_{j3})\theta_2\} \cdot |\theta_1, \theta_2; (m)\rangle \]
\[-\theta_i(1 - \theta_j)q^{\delta_{i1}\delta_{j2}h_1} \cdot |1 - \theta_1, 1 - \theta_2; (m)\rangle \]
\[+\delta_{i2}\delta_{j3}\theta_1 q^{-1-h_2} \left\{ q^{-\theta_2} \cdot |1 - \theta_1, \theta_2; (m)\rangle_{21} - \theta_2 \cdot |\theta_1, 1 - \theta_2; (m)\rangle \right\} \]
\[+\delta_{i2}\delta_{j3}\theta_2 (-1)^{\theta_1}[h_2] \cdot |\theta_1, 1 - \theta_2; (m)\rangle, \]
(3.18a)

where \(l\) and \(h_i\) are respectively eigenvalues of \(L\) and \(H_i\) on \((m)\), while

\[(m)_{ij} = \begin{cases} E_{ij}(m), \text{given by (3.2), if } i, j = 1, 2 \text{ or } i = j = 3, \\ (m), \text{otherwise.} \end{cases} \]
(3.18b)

These transformations give different representations for different \([m]\). The representations of \(U_q[gl(2/1)]\) constructed are in general reducible. However, the induced basis is not convenient for investigating the representation structure. Let us go to another, more appropriate to this goal, basis.

The module \(V^q\) is a tensor product of a \(U_q[gl(2)]\)-module with a \(U_q[gl(1)]\)-module (in fact, a \(gl(1)\)-factor),

\[V^q([m_{12}, m_{22}, m_{32}]) = V^q([m_{12}, m_{22}]) \otimes V^q([m_{32}]), \]
(3.19a)

and so is the module \(T^q\),

\[T^q([\mu_{12}, \mu_{22}, \mu_{32}]) = T^q([\mu_{12}, \mu_{22}]) \otimes T^q([\mu_{32}]). \]
(3.19b)

Then the module \(W^q\) in (2.22) can be written as follows

\[W^q([m]) = \{T^q([\mu_{12}, \mu_{22}]) \otimes V^q([m_{12}, m_{22}])\} \otimes \{T^q([\mu_{32}]) \otimes V^q([m_{32}])\}. \]
(3.20)
Here the notation $\odot$ is used for a tensor product between two modules of one and
the same (quantum) algebra, whereas $\otimes$ is a more general notation used for a ten-
sor product of two arbitrary spaces or modules. In general, the $U_q[gl(2)]$-module
$T^q([\mu_{12}, \mu_{22}]) \odot V([m_{12}, m_{22}])$ in (3.20) is reducible and can be decomposed into a
direct sum of irreducible modules

$$T^q([\mu_{12}, \mu_{22}]) \odot V^q([m_{12}, m_{22}]) = \bigoplus_{i=0}^{n} V^q([\mu_{12} + m_{12} - i, \mu_{22} + m_{22} + i]), \quad (3.21)$$

where

$$n = \min (\mu_{12} - \mu_{22}, m_{12} - m_{22}),$$

while the $gl(1)$-factor $T^q([\mu_{32}]) \odot V^q([m_{32}])$ is just

$$T^q([\mu_{32}]) \odot V^q([m_{32}]) = V^q([\mu_{32} + m_{32}]). \quad (3.22)$$

Taking into account (2.30) and (3.19) – (3.22) we get

$$D_0 \equiv T^q([0, 0, 0]) \odot V^q([m_{12}, m_{22}, m_{32}]) \equiv V^q([m_{12}, m_{22}, m_{32}]),$$

$$D_1 \equiv T^q([-1, 1, 1]) \otimes V^q([m_{12}, m_{22}, m_{32}])
= \bigoplus_{i=0}^{1} V^q([m_{12} - i, m_{22} + i - 1, m_{32} + i]),$$

$$D_2 \equiv T^q([-1, -1, 2]) \odot V^q([m_{12}, m_{22}, m_{32}])
= V^q([m_{12} - 1, m_{22} - 1, m_{32} + 2]). \quad (3.23)$$

Inserting (3.23) in (2.28) we prove (2.31) with $V^q_k$ identified as follows

$$V^q_0 \equiv V^q([m_{12}, m_{22}, m_{32}]) = V^q,$$

$$V^q_1 \equiv V^q([m_{12}, m_{22} - 1, m_{32} + 1]),$$

$$V^q_2 \equiv V^q([m_{12} - 1, m_{22}, m_{32} + 1]),$$

$$V^q_3 \equiv V^q([m_{12} - 1, m_{22} - 1, m_{32} + 2]). \quad (3.24)$$

Instead of the induced basis (2.21) for a basis of $W^q$ we can chose the union of
the bases, the GZ bases (3.1) in the case, of all its subspaces $V^q_k$. This new basis of
$W^q$ is referred to as its reduced basis which is related to the induced one (2.21) via the
Clebsch–Gordan (CG) decomposition. In order to derive such a relation between the
two bases for the whole $W^q$ we should have it first for each of the subspaces (3.21) and
(3.22). Within the subspace (3.21), which is a $U_q[gl(2)]$-module, the relation between the induced basis

$$(\mu)_{gl(2)} \otimes (m)_{gl(2)} \equiv \begin{bmatrix} \mu_{12} & \mu_{22} \\ \mu_{11} & \mu_{12} \end{bmatrix} \otimes \begin{bmatrix} m_{12} & m_{22} \\ m_{11} & m_{12} \end{bmatrix} \in T^q([\mu_{12}, \mu_{22}]) \otimes V^q([m_{12}, m_{22}])$$

and the reduced basis

$$(m'_{gl(2)}) \equiv \begin{bmatrix} m'_{12} & m'_{22} \\ m'_{11} & m'_{11} \end{bmatrix} \in V^q([m'_{12}, m'_{22}]),$$

$m'_{12} = \mu_{12} + m_{12} - i, \ m'_{22} = \mu_{22} + m_{22} + i,$

can be written in the form

$$(m'_{gl(2)}) = \sum_{\mu_{11}, m_{11}} \begin{bmatrix} m'_{12} & m_{22} \\ m'_{11} & m_{11} \end{bmatrix} \cdot \begin{bmatrix} \mu_{12} & \mu_{22} \\ \mu_{11} & \mu_{11} \end{bmatrix} \cdot \begin{bmatrix} m_{12} & m_{22} \\ m_{11} & m_{11} \end{bmatrix} \cdot (\mu)_{gl(2)} \otimes (m)_{gl(2)}, \quad (3.25)$$

where

$$\begin{bmatrix} m'_{12} & m_{22} \\ m'_{11} & m_{11} \end{bmatrix} \cdot \begin{bmatrix} \mu_{12} & \mu_{22} \\ \mu_{11} & \mu_{11} \end{bmatrix} \cdot \begin{bmatrix} m_{12} & m_{22} \\ m_{11} & m_{11} \end{bmatrix}$$

are the Clebsch–Gordan coefficients of $U_q[gl(2)]$. The relation between the two bases within the subspace (3.22) is simply

$$m'_{31} = \mu_{31} + m_{31}. \quad (3.27)$$

Now taking into account (3.25) – (3.27) we can express the reduced basis of $W^q$ in terms of the induced one (2.21):

$$(m)_0 \equiv \begin{bmatrix} m_{12} & m_{22} \\ m_{11} & m_{32} \end{bmatrix} \equiv (m) \in V^q_0 \equiv V^q,$$

$$(m)_1 \equiv \begin{bmatrix} m_{12} & m_{22} - 1 \\ m_{11} & m_{32} + 1 \end{bmatrix},$$

$$= \begin{bmatrix} m_{12} & m_{22} - 1 \\ m_{11} & m_{32} + 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} m_{12} & m_{22} \\ m_{11} & m_{11} \end{bmatrix} \cdot |0, 1; (m)\rangle$$

$$- \begin{bmatrix} m_{12} & m_{22} - 1 \\ m_{11} & m_{11} + 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} m_{12} & m_{22} \\ m_{11} & m_{11} + 1 \end{bmatrix} \cdot |1, 0; (m)^{+11}\rangle \in V^q_1,$$
By construction, $W^q$ is characterized by the highest weight of $V^q$, the signature $[m]$ in (3.9). In order to describe $W^q$ as the whole we unify the basis vectors (3.28) in a single notation,

\[
(m)_2 \equiv \left[ \begin{array}{ccc}
  m_{12} - 1 & m_{22} & m_{32} + 1 \\
  m_{11} & m_{32} + 1 & m_{32} + 1
\end{array} \right],
\]

\[
= \left[ \begin{array}{ccc}
  m_{12} - 1 & m_{22} & 0 - 1 \\
  m_{11} & m_{11} & m_{11}
\end{array} \right]. \|0, 1; (m)\rangle
\]

\[
- \left[ \begin{array}{ccc}
  m_{12} - 1 & m_{22} & 0 - 1 \\
  m_{11} & m_{11} & m_{11} + 1
\end{array} \right]. \|1, 0; (m)^{+11}\rangle \in V^q_2,
\]

\[
(m)_3 \equiv \left[ \begin{array}{ccc}
  m_{12} - 1 & m_{22} - 1 & m_{32} + 2 \\
  m_{11} & m_{32} + 2 & m_{32} + 2
\end{array} \right]
\]

\[
= \left[ \begin{array}{ccc}
  m_{12} - 1 & m_{22} - 1 & -1 - 1 \\
  m_{11} & m_{11} + 1 & m_{11} + 1
\end{array} \right]. \|1, 1; (m)^{+11}\rangle \in V^q_4. \quad (3.28)
\]

This row of (3.29) remains unchanged throughout the whole $W^q$, while the second row depending on $k$ represents the first row of one of the patterns (3.28) and tells us which subspace $V^q_k$ the considered basis vector (3.29) of $W^q$ belongs to. The basis (3.29) reflects the branching rule $U_q[gl(2/1)] \supset U_q[gl(2) \otimes gl(1)]$ and it can be called a quasi-GZ basis. The subspaces (3.24) in this new notation is

\[
V^q_0 \equiv V^q([m_{13}, m_{23}, m_{33}]) \equiv V^q,
\]

\[
V^q_1 \equiv V^q([m_{13}, m_{23} - 1, m_{33} + 1]),
\]

\[
V^q_2 \equiv V^q([m_{13} - 1, m_{23}, m_{33} + 1]),
\]

\[
V^q_3 \equiv V^q([m_{13} - 1, m_{23} - 1, m_{33} + 2]). \quad (3.31)
\]
Let us now determine the CG coefficients in (3.28). To do that we use the Hopf algebra structure which is again helpful. We start with the subspace $V_1^q$. The highest vector here is

$$(M)_1 = a_1 (E_{32} \otimes (M)),$$  \hfill (3.32)

where $a_1$ is an arbitrary complex coefficient which may depend on $q$. Formula (3.8) now becomes

$$(m)_1 = \left( \frac{[l_{11} - l_{23}]!}{[2l + 1]!l_{13} - l_{11}]!} \right)^{1/2} (E_{21})^{l_{13} - l_{11}} (M)_1,$$  \hfill (3.33a)

where

$$l = (m_{13} - m_{23})/2.$$  \hfill (3.33b)

Replacing

$$(E_{21})^{l_{13} - l_{11}} (M)_1 \equiv a_1 \{\Delta(E_{21})\}^{l_{13} - l_{11}} (E_{32} \otimes (M))$$

$$= -a_1[l_{13} - l_{11}] \left( \frac{[2l]![l_{13} - l_{11} - 1]!}{[l_{11} - l_{23}]!} \right)^{1/2} (E_{31} \otimes (m)^{+11})$$

$$+ a_1 q^{l_{13} - l_{11}} \left( \frac{[2l]![l_{13} - l_{11}]!}{[l_{11} - l_{23} - 1]!} \right)^{1/2} (E_{32} \otimes (m))$$  \hfill (3.34)

in (3.33) we obtain

$$(m)_1 = a_1 \left\{ - \left( \frac{[l_{13} - l_{11}]}{[2l + 1]} \right)^{1/2} \cdot |1, 0; (m)^{+11}\rangle + q^{l_{13} - l_{11}} \left( \frac{[l_{11} - l_{23}]}{[2l + 1]} \right)^{1/2} \cdot |0, 1; (m)\rangle \right\}.$$  \hfill \(\blacksquare\)

So with the help of the Hopf algebra structure the necessary CG coefficients can be found directly and easily without knowing in advance a general formula for them. In the same way, from the highest weight vectors $(M)_2 \in V_2^q$ and $(M)_3 \in V_3^q$,

$$(M)_2 = a_2 \{E_{31} \otimes (M) + q^{2l}[2l]^{-1/2}E_{32} \otimes (M)^{-11}\},$$

$$(M)_3 = a_3 E_{31} E_{31} \otimes (M),$$

we can find explicit expressions for $(m)_2$ and $(m)_3$, respectively. Thus, we have the following relation between the reduced and the induced basis

$$(m)_0 \equiv \begin{bmatrix} m_{13} & m_{23} & m_{33} \\ m_{13} & m_{23} & m_{33} \\ m_{11} & 0 & m_{33} \end{bmatrix} = |0, 0; (m)\rangle \equiv (m),$$

$$(m)_1 \equiv \begin{bmatrix} m_{13} & m_{23} & m_{33} \\ m_{13} & m_{23} - 1 & m_{33} + 1 \\ m_{11} & 0 & m_{33} + 1 \end{bmatrix}.$$
\[
\begin{aligned}
&= a_1 \left\{ -\left( \frac{[l_{13} - l_{11}]}{[2l + 1]} \right)^{1/2} \cdot |1, 0; (m)^{+11} \rangle + q^{l_{11} - l_{13}} \left( \frac{[l_{11} - l_{23}]}{[2l + 1]} \right)^{1/2} \cdot |0, 1; (m) \rangle \right\}, \\
(m)_2 &\equiv \begin{bmatrix}
m_{13} & m_{23} & m_{33} \\
m_{13} - 1 & m_{23} & m_{33} + 1 \\
m_{11} & 0 & m_{33} + 1
\end{bmatrix} \\
&= a_2 \left\{ \left( \frac{[l_{11} - l_{23}]}{[2l]} \right)^{1/2} \cdot |1, 0; (m)^{+11} \rangle \\
+ q^{l_{11} - l_{23}} \left( \frac{[l_{13} - l_{11}]}{[2l]} \right)^{1/2} \cdot |0, 1; (m) \rangle \right\}, \\
(m)_3 &\equiv \begin{bmatrix}
m_{13} & m_{23} & m_{33} \\
m_{13} - 1 & m_{23} - 1 & m_{33} + 2 \\
m_{11} & 0 & m_{33} + 2
\end{bmatrix} = a_3 \cdot |1, 1; (m)^{+11} \rangle & (3.35)
\end{aligned}
\]

and, equivalently, the inverse relation
\[
|0, 0; (m) \rangle = (m) \\
|1, 0; (m) \rangle = -\frac{1}{a_1} q^{l_{11} - l_{23} - 1} \left( \frac{[l_{13} - l_{11} + 1]}{[2l + 1]} \right)^{1/2} (m)_1^{-11} \\
+ \frac{1}{a_2} q^{l_{11} - l_{13} - 1} \left( \frac{[l_{11} - l_{23} - 1][2l]}{[2l + 1]} \right)^{1/2} (m)_2^{-11}, \\
|0, 1; (m) \rangle = \frac{1}{a_1} \left( \frac{[l_{11} - l_{23}]}{[2l + 1]} \right)^{1/2} (m)_1 \\
+ \frac{1}{a_2} \left( \frac{[l_{13} - l_{11}][2l]}{[2l + 1]} \right)^{1/2} (m)_2, \\
|1, 1; (m) \rangle = \frac{1}{a_3} (m)_3^{-11}. & (3.36)
\]

Now we are ready to compute all the matrix elements of the generators in the basis (3.35) which allows a clear description of the structure of the module \( W_q \). Since the finite–dimensional representations of the \( U_q[gl(2/1)] \) in some basis are completely defined by the actions of the even generators and the odd Weyl–Chevalley ones \( E_{23} \) and \( E_{32} \) in the same basis, it is sufficient to write down the matrix elements of these generators only. For the even generators the matrix elements have already been given
in (3.2), while for $E_{23}$ and $E_{32}$, using the relations (2.1)–(2.3), (3.35) and (3.36) we get

$$E_{23}(m) = 0,$$

$$E_{23}(m)_1 = a_1 \left( \frac{l_{11} - l_{23}}{2l + 1} \right)^{1/2} [l_{23} + l_{33} + 3](m),$$

$$E_{23}(m)_2 = a_2 \left( \frac{l_{13} - l_{11}}{2l} \right)^{1/2} [l_{13} + l_{33} + 3](m),$$

$$E_{23}(m)_3 = a_3 \left( \frac{1}{a_1 q} \left( \frac{l_{13} - l_{11}}{2l + 1} \right) \right)^{1/2} [l_{13} + l_{33} + 3](m)_1$$

$$- \frac{1}{a_2 q} ([l_{11} - l_{23}][2l])^{1/2} [l_{23} + l_{33} + 3] (m)_2 \right),$$

$$E_{32}(m) = \frac{1}{a_1} \left( \frac{l_{11} - l_{23}}{2l + 1} \right)^{1/2} (m)_1 + \frac{1}{a_2} \left( \frac{l_{13} - l_{11}}{2l + 1} \right)^{1/2} (m)_2,$$

$$E_{32}(m)_1 = \frac{a_1 q}{a_3} \left( \frac{l_{13} - l_{11}}{2l + 1} \right)^{1/2} (m)_3,$$

$$E_{32}(m)_2 = -\frac{a_2 q}{a_3} \left( \frac{l_{11} - l_{23}}{2l} \right)^{1/2} (m)_3,$$

$$E_{32}(m)_3 = 0. \quad (3.37)$$

All the matrix elements of the Chevalley generators obtained here coincide, of course, with the ones obtained previously by another (but longer) way [24, 25]. Besides that, we can easily find matrix elements for non–Chevalley generators too:

$$E_{31}(m) = -\frac{1}{a_1} q^{l_{11} - l_{23} - 1} \left( \frac{l_{13} - l_{11} + 1}{2l + 1} \right)^{1/2} (m)_1^{-11}$$

$$+ \frac{1}{a_2} q^{l_{11} - l_{13} - 1} \left( \frac{l_{11} - l_{23} - 1}{2l + 1} \right)^{1/2} \frac{[l_{23} + l_{33} + 3]}{[2l + 1]} (m)_2^{-11},$$

$$E_{31}(m)_1 = \frac{a_1}{a_3} q^{l_{11} - l_{13}} \left( \frac{l_{11} - l_{23}}{2l + 1} \right)^{1/2} (m)_3^{-11},$$

$$E_{31}(m)_2 = \frac{a_2}{a_3} q^{l_{11} - l_{23}} \left( \frac{l_{13} - l_{11}}{2l} \right)^{1/2} (m)_3^{-11},$$

$$E_{31}(m)_3 = 0,$$
\[ E_{13}(m) = 0, \]
\[ E_{13}(m)_1 = -a_1 q^{l_{23} - l_{11} - l} \left( \frac{[l_{13} - l_{11}]}{2l + 1} \right)^{1/2} [l_{23} + l_{33} + 3] (m)^{+11}, \]
\[ E_{13}(m)_2 = a_2 q^{l_{13} - l_{11} - l} \left( \frac{[l_{11} - l_{23}]}{2l} \right)^{1/2} [l_{13} + l_{33} + 3] (m)^{+11} \]
\[ E_{13}(m)_3 = a_3 \left\{ \frac{q^{l_{13} - l_{11} - l}}{a_1} \left( \frac{[l_{11} - l_{23} + 1]}{2l + 1} \right)^{1/2} [l_{13} + l_{33} + 3] (m)^{+11} \right. \]
\[ \left. + \frac{q^{l_{23} - l_{11} - l}}{a_2} ([l_{13} - l_{11} - 1][2l])^{1/2} \frac{[l_{23} + l_{33} + 3]}{[2l + 1]} (m)^{+11} \right\}. \quad (3.38) \]

A question arising here is when the representations constructed are irreducible and how they are classified. It will be dealt with in the next section.

**IV. TYPICAL AND NONTYPICAL REPRESENTATIONS OF \( U_q[gl(2/1)] \)**

The finite–dimensional representations constructed above are either irreducible or indecomposable. We can prove the following proposition.

**Proposition 4:** The finite–dimensional representations of \( U_q[gl(2/1)] \) given in (3.37) and (3.38) are irreducible and called typical if and only if the condition

\[ [l_{13} + l_{23} + 3][l_{23} + l_{33} + 3] \neq 0 \quad (4.1) \]

holds.

When this condition (4.1) is violated, i.e. one of the following pairs of conditions

\[ [l_{13} + l_{33} + 3] = 0 \quad \text{and} \quad [l_{23} + l_{33} + 3] \neq 0 \quad (4.2) \]

or

\[ [l_{13} + l_{33} + 3] \neq 0 \quad \text{and} \quad [l_{23} + l_{33} + 3] = 0 \quad (4.3) \]

(but not both of them simultaneously) holds, the module \( W^q \) is no longer irreducible but indecomposable. In this case, however, there exists an invariant subspace, say \( I_k^q \), of \( W^q \) such that the factor representation in the factor module

\[ W_k^q := W_q/I_k^q \quad (4.4) \]

is irreducible. We call this irreducible representation non-typical in the non-typical module \( W_k^q \). Then, as in [25], it is not difficult for us to prove the following assertions.
Proposition 5:

\[ V_3^q \subset I_k^q \]  \hspace{1cm} (4.5)

and

\[ V_0^q \cap I_k^q = 0. \]  \hspace{1cm} (4.6)

From (3.37) – (4.3) we can easily find all non-typical representations of \( U_q[gl(2/1)] \) which are classified into two classes.

IV.1. Non-typical representations of class 1

This class is characterized by the conditions (4.2) which, for generic \( q \), take the forms

\[ l_{13} + l_{33} + 3 = 0 \]

(4.2a)

and

\[ l_{23} + l_{33} + 3 \neq 0. \]

(4.2b)

In the other words, we have to replace everywhere all \( m_{33} \) by \(-m_{13} - 1\), keeping (4.2b) valid. Thus we have the following proposition.

Proposition 6:

\[ I_1^q = V_3^q \oplus V_2^q. \]  \hspace{1cm} (4.7)

Then the class 1 non-typical representations in

\[ W_1^q = W_1^q([m_{13}, m_{23}, -m_{13} - 1]) \]

(4.8)

are given through (3.31) by keeping the conditions (4.2a) and (4.2b) and replacing all vectors belonging to \( I_1^q \) with 0:

\[ E_{23}(m) = 0, \]

\[ E_{23}(m)_1 = a_1 \left( \frac{l_{11} - l_{23}}{2l + 1} \right)^{1/2} [l_{23} - l_{13}](m), \]

\[ E_{32}(m) = \frac{1}{a_1} \left( \frac{l_{11} - l_{23}}{2l + 1} \right)^{1/2} (m)_1, \]

\[ E_{32}(m)_1 = 0. \]  \hspace{1cm} (4.9)

IV.2. Non-typical representations of class 2
For this class non-typical representations we must keep the conditions

\[ l_{13} + l_{33} + 3 \neq 0 \quad (4.3a) \]

and

\[ l_{23} + l_{33} + 3 = 0, \quad (4.3b) \]

derived from (4.3) when the deformation parameter \( q \) is generic. Equivalently, we have to replace everywhere all \( m_{33} \) by \( -m_{23} \) and keep (4.3a) valid. Now the invariant subspace \( I^q_2 \) is determined as follows.

**Proposition 7:**

\[ I^q_2 = V^q_3 \oplus V^q_1. \quad (4.10) \]

The class 2 non-typical representations in

\[ W^q_2([m_{13}, m_{23}, -m_{23}]) \quad (4.11) \]

are also given through (3.31) but by keeping the conditions (4.3a) and (4.3b) valid and replacing all vectors belonging to the invariant subspace \( I^q_2 \) by 0:

\[
\begin{align*}
E_{23}(m) &= 0, \\
E_{23}(m)_2 &= a_1 \left( \frac{|l_{13} - l_{11}|}{[2l]} \right)^{1/2} [2l + 1](m), \\
E_{32}(m) &= \frac{1}{a_2} \left( \frac{|l_{13} - l_{11}|[2l]^{1/2}}{[2l + 1]} \right)(m)_2, \\
E_{32}(m)_2 &= 0. \quad (4.12)
\end{align*}
\]

We complete this section with the following statement.

**Proposition 8:** The class of the finite–dimensional representations determined above contains all finite–dimensional irreducible representations of the quantum superalgebra \( U_q[gl(2/1)] \).

V. CONCLUSION

The quantum superalgebra \( U_q[gl(2/1)] \) is given as both a Drinfel’d–Jimbo deformation of the universal enveloping \( U[gl(2/1)] \) and a Hopf superalgebra. Using the Hopf algebra structure of \( U_q[gl(2/1)] \) we have constructed all its finite (irreducible) dimensional representations in a basis of the even subalgebra \( U_q[gl(2/1)_0] \). This method combines the advantage of previously suggested methods for constructing representations of a classical superalgebra \[29] – \[31\] and a quantum superalgebra \[22, 23\] and
shows that the method used in the classical case can be extended to the quantum deformation case. It proves once again the usefulness of knowing a Hopf algebra structure of a quantum group. In particular, using a Hopf algebra structure of a quantum superalgebra, \( U_q[gl(2/1)] \) in the case, one can calculate in an easier way matrix elements in the induced basis and express the latter in terms of a basis of the even subalgebra. All that could not be done via the previously suggested procedure \[22 \] – \[24\]. Such a description of an induced basis may be physically necessary as in it both the origin and the structure of multiplets can be seen clearer. Certainly, the method of the present paper can be applied to a bigger quantum superalgebra and may be also applicable to the case of multi-parameter deformations, for example, the two-parametric \( U_{pq}[gl(2/1)] \). We hope this method and the results obtained here could be useful for physics applications.

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