The proportional UAP characterizes weak Hilbert spaces*

by W. B. Johnson and G. Pisier

Abstract: We prove that a Banach space has the uniform approximation property with proportional growth of the uniformity function iff it is a weak Hilbert space.

* Both authors were supported in part by NSF DMS 87-03815
Introduction

The “weak Hilbert spaces” were introduced and studied in [P 2]. Among the many equivalent characterizations in [P 2] perhaps the simplest definition is the following. A Banach space is a weak Hilbert space if there is a constant $C$ such that for all $n$, for all $n$-tuples $(x_1, \ldots, x_n)$ and $(x_1^*, \ldots, x_n^*)$ in the unit balls of $X$ and $X^*$ respectively, we have

$$|\det(< x_i^*, x_j>)|^\frac{1}{n} \leq C.$$ 

The first example of a non Hilbertian weak Hilbert space was obtained by the first author (cf. [FLM], Example 5.3 and [J]).

Recall that a Banach space $X$ has the uniform approximation property (in short UAP) if there is a constant $K$ and a function $n \to f(n)$ such that for all $n$ and all $n$-dimensional subspaces $E \subset X$, there is an operator $T : X \to X$ with $\text{rk}(T) \leq f(n)$ such that $\|T\| \leq K$ and $T|_E = I|_E$.

For later use, given $K > 1$ we introduce

$$k_X(K,n) = \sup_{E \subset X, \dim E = n} \inf \{\text{rk}(T)\}$$

where the infimum runs over all $T : X \to X$ such that $\|T\| \leq K$ and $T|_E = I|_E$.

Note that $X$ has the UAP iff there is a constant $K$ such that $k_X(K,n)$ is finite for all $n$; we then say that $X$ has the $K$-UAP. The asymptotic growth of the function $n \to k_X(K,n)$ provides a quantitative measure of the UAP of the space $X$.

For instance, if $X$ is a Hilbert space we have clearly $k(1,n) = n$, hence if $X$ is isomorphic to a Hilbert space there is a constant $K$ such that

$$k_X(K,n) = n \quad \text{for all } n.$$ 

The converse is also true by the complemented subspace theorem of Lindenstrauss-Tzafriri [LT 1].

The main result in this paper can be viewed as an analogous statement for weak Hilbert spaces, as follows.
**Main Theorem.** A Banach space $X$ is a weak Hilbert space iff there are constants $K$ and $C$ such that

$$k_X(K, n) \leq Cn \quad \text{for all } n.$$  

That is, proportional asymptotic behavior of the uniformity function in the definition of the UAP characterizes weak Hilbert spaces.

It was proved in [P 2] that weak Hilbert spaces have the UAP but no estimate of the function $n \to k_X(K, n)$ was obtained.

For the purposes of this paper we will say that $X$ has the proportional UAP if there are constants $K$ and $C$ such that (0.1) holds.

The authors thank V. Mascioni and G. Schechtman for several discussions concerning the material in this paper.

§1. **Weak Hilbert spaces have proportional UAP**

We first recall a characterization of weak Hilbert spaces in terms of nuclear operators. Recall that an operator $u : X \to X$ is called nuclear if it can be written

$$u(x) = \sum_{n=1}^{\infty} x^*_n(x)x_n$$

with $x^*_n \in X^*$, $x_n \in X$ such that $\sum \|x^*_n\| \|x_n\| < \infty$. Moreover the nuclear norm $N(u)$ is defined as

$$N(u) = \inf \left\{ \sum \|x^*_n\| \|x_n\| \right\}$$

where the infimum runs over all possible representations. We also recall the notation for the approximation numbers

$$\forall k \geq 1 \quad a_k(u) = \inf \{\|u - v\| \mid v : X \to X, \ \rk(v) < k\}.$$
By [P 2], a Banach space \( X \) is a weak Hilbert space iff there is a constant \( C \) such that for all nuclear operators \( u : X \to X \) we have

\[
(1.1) \quad \sup_{k \geq 1} ka_k(u) \leq CN(u)
\]

The following observation is identical to reasoning already used by V. Mascioni [Ma 2].

**Proposition 1.1.** Let \( X \) be a weak Hilbert space. Assume that there is a constant \( K' \) such that for all \( n \) and all \( n \)-dimensional subspaces \( E \subset X \) there is an operator \( u : X \to X \) such that \( u|_E = I|_E \), \( \|u\| \leq K' \) and \( N(u) \leq K'n \). Then \( X \) has the proportional UAP. (Recall that if \( u \) has finite rank then \( N(u) \leq \text{rk} (u) \|u\| \), hence the converse to the preceding implication is obvious.)

**Proof:** Let \( u \) be as in the preceding statement. We use (1.1) with \( k = [2CK'n] + 1 \), so that

\[
a_k(u) \leq CN(u)k^{-1} \leq CK'nk^{-1} \leq \frac{1}{2}.
\]

This means that there is an operator \( v : X \to X \) with \( \text{rk} (v) \leq 2CK'n \) such that \( \|u - v\| \leq \frac{1}{2} \). By perturbation, it follows that the operator

\[
V = v - u + I
\]

is invertible on \( X \) with \( \|V^{-1}\| \leq 2 \). Moreover we have

\[
(1.2) \quad V|_E = v|_E.
\]

It follows that if we let \( T = V^{-1}v \), then we have

\[
\|T\| \leq \|V^{-1}\| \|v\| \leq 2(\|u\| + \|u - v\|) \leq 2K' + 1,
\]

also \( \text{rk} (T) \leq \text{rk} (v) \leq 2CK'n \) and \( T|_E = I|_E \) by (1.2).

We conclude that \( X \) has the UAP with \( k_X (K, n) \leq 2CK'n \), where \( K = 2K' + 1 \).  

We will use duality via the following proposition (a similar kind of criterion was used by Szankowski [S] to prove that certain spaces fail the UAP):
Proposition 1.2. Let $X$ be a reflexive Banach space with the approximation property; in short, AP; let $\alpha, \beta$ be positive constants; and let $n \geq 1$ be an integer. The following are equivalent.

(i) For all nuclear operators $T_1, T_2$ on $X$ such that $T_1 + T_2$ has rank $\leq n$, we have

$$|\text{tr} (T_1 + T_2)| \leq \alpha N(T_1) + \beta n \|T_2\|.$$ 

(ii) Same as (i) with $T_1, T_2$ of finite rank.

(iii) For any subspace $E \subset X$ with dimension $\leq n$ there is an operator $u : X \to X$ such that $u|_E = I_E$, $\|u\| \leq \alpha$ and $N(u) \leq \beta n$.

Proof: (i) $\Rightarrow$ (ii) is trivial.

Assume that (ii) holds.

We equip $X^* \otimes X$ with the norm $|w| = \inf \{\alpha N(T_1) + \beta n \|T_2\|\}$ where the infimum runs over all decompositions

$$u = T_1 + T_2$$

with $T_1$ and $T_2$ in $X^* \otimes X$ (identified with the set of finite rank operators on $X$). On $X^* \otimes X$ this norm is clearly equivalent to the operator norm.

Now let $E \subset X$ be a fixed subspace with $\dim (E) \leq n$. Let $S \subset X^* \otimes X$ be the subspace $X^* \otimes E$ of all the operators on $X$ with range in $E$. On this linear subspace the linear form $\xi$ defined by $\xi(w) = \text{tr} (w)$ has norm $\leq 1$ relative to $|\cdot|$ by our assumption (ii).

Therefore there is a Hahn-Banach extension $\tilde{\xi}$ defined on the whole of $X^* \otimes X$ which extends $\xi$ and satisfies

$$|\tilde{\xi}(w)| \leq |w| \quad \forall w \in X^* \otimes X.$$ 

(1.3)

By classical results, $\tilde{\xi}$ can be identified with an integral operator $u : X \to X^{**}$. Since $X$ is reflexive, $u$ is actually a nuclear operator on $X$, and we have $\tilde{\xi}(w) = \text{tr} (wu)$ for all $w$ in $X^* \otimes X$.

Since $\tilde{\xi}$ extends $\xi$, we must have
∀x∗ ∈ X∗ ∀e ∈ E < ˜ξ, x∗ ⊗ e >= tr (x∗ ⊗ e) = x∗(e) hence x∗(ue) = x∗(e).

Equivalently

\[ u|_E = I|_E. \]

On the other hand, by (1.3) we have

\[ |\text{tr} (uT_1)| \leq \alpha N(T_1) \quad \text{and} \quad |\text{tr} (uT_2)| \leq \beta n \| T_2 \| \]

for all finite rank operators \( T_1 \) and \( T_2 \) on \( X \).

This implies \( \| u \| \leq \alpha \) and (again using the reflexivity of \( X \)) \( N(u) \leq \beta n \).

This shows that (ii)⇒(iii).

Finally we show (iii)⇒(i). Assume (iii). Let \( T_1, T_2 \) be as in (i), let \( E \) be the range of \( T_1 + T_2 \) and let \( u \) be as in (iii). Then we have \( T_1 + T_2 = u(T_1 + T_2) \) hence since \( X \) has the AP (which ensures that \( |\text{tr} (T)| \leq N(T) \) for all nuclear operator \( T : X \rightarrow X \)) we have

\[ |\text{tr} (T_1 + T_2)| = |\text{tr} (uT_1) + \text{tr} (uT_2)| \]

\[ \leq \| u \| N(T_1) + N(u)\| T_2 \| \]

\[ \leq \alpha(T_1) + \beta n \| T_2 \|. \]

**Remark:** Note that (i) is also equivalent to (i'):

(i') For all \( T_1, T_2 \) on \( X \) such that \( (T_1 + T_2) \) has rank \( \leq n \), we have

\[ N(T_1 + T_2) \leq \alpha N(T_1) + \beta n \| T_2 \|. \]

Indeed; (assuming the AP and reflexivity) we have

\[ N(T_1 + T_2) = \sup \{ \text{tr} (S(T_1 + T_2)) ; S : X \rightarrow X, \| S \| \leq 1 \}. \]

This shows that (i)⇒(i'). Since \( X \) has the AP the converse is obvious.

Of course, a similar remark holds for (ii).

**Remark:** If \( X \) is not assumed to have the AP a variant of Proposition 1.2 will still hold provided we use the projective tensor norm on \( X^* \otimes X \) instead of the nuclear norm.
We will use the following result already exploited in [P 2] to prove that weak Hilbert spaces have the AP. Whenever \( u : X \to X \) is a finite rank operator, we denote by \( \det (I + u) \) the quantity

\[
\Pi(1 + \lambda_j(u))
\]

where \( \{ \lambda_j(u) \} \) are the eigenvalues of \( u \) repeated according to their algebraic multiplicity. Equivalently, \( \det (I + u) \) is equal to the ordinary determinant of the operator \( (I + u)|_E \) restricted to any finite dimensional subspace \( E \subset X \) containing the range of \( u \).

**Lemma 1.3.** Let \( u, v \) be finite rank operators on a weak Hilbert space \( X \) with \( \text{rk}(u) \leq n \). Then we have

\[
|\det (I + u + v)| \leq \left( \sum_{j=0}^{n} \frac{C^j}{j!} N(u)^j \right) \exp CN(v)
\]

where \( C \) is the “weak Hilbert space constant” of \( X \); that is to say,

\[
C = \sup_{x_i \in B_X} |\det \langle x_i^*, x_j \rangle|^{1/n}.
\]

For the proof we refer to [P 3] p. 229. Note that if \( N(u) \geq 1 \) then (1.4) implies for all complex numbers \( z \),

\[
\det (I + z(u + v)) \leq N(u)^n \exp \{ C|z| + C|z|N(v) \}.
\]

Let \( f(z) = \det (I + z(u + v)) \). Then \( f \) is a polynomial function of \( z \in \mathbb{C} \) such that

\[
f(0) = 1 \text{ and } f'(0) = \text{tr}(u + v).
\]

In [G], Grothendieck showed that the function \( u \to \det (I + u) \) is uniformly continuous on \( X^* \otimes X \) equipped with the projective norm, and therefore extends to the completion \( X^* \hat{\otimes} X \). This shows that if \( X \) has the AP, the determinant \( \det (I + v) \) can be defined unambiguously for any nuclear operator \( v \) on \( X \). As shown in [G], the function \( z \to \det (I + z(u + v)) \) is an entire function satisfying (1.4) if \( u \) is of rank \( \leq n \) and \( v \) possibly of infinite rank. We use
this extension in Theorem 1.5 below, but in the proof of our main result the special case of \( v \) of finite rank in Theorem 1.5 is sufficient. This makes our proof more elementary.

We will make crucial use of the following classical inequality of Carathéodory; we include the proof for the convenience of the reader.

**Lemma 1.4.** Let \( h \) be an analytic function in a disc \( D_R = \{ z \in \mathbb{C} ; \ |z| < R \} \) such that \( h(0) = 0 \). Then for any \( 0 < r < R \) we have

\[
|h'(0)| \leq \frac{2}{r} \sup_{|z|=r} \text{Re} \ (h(z)).
\]

**Proof.** Let \( M = \sup \{ \text{Re} \ (h(z)), |z| < r \} \). Note that \( M \geq 0 \).

Let \( g(z) = \frac{h(z)}{2M - h(z)} \). Then \( |g(z)| \leq 1 \) if \( |z| \leq r \), \( g \) is analytic in \( D_r \) and \( g(0) = 0 \). By the Schwarz lemma we have

\[
|g(z)| \leq \frac{|z|}{r} \quad \text{for all } z \text{ in } D_r \quad \text{and} \quad |g'(0)| \leq 1/r.
\]

Since \( h(z) = \frac{2Mg(z)}{1+g(z)} \) we have \( h'(0) = 2Mg'(0) \) hence \( |h'(0)| \leq 2M/r \). \( \blacksquare \)

We now prove the main result of this section, namely that any weak Hilbert space has the proportional UAP. Let \( X \) be a weak Hilbert space. We will show that \( X \) satisfies (ii) in Proposition 1.2. Actually, we obtain the following result of independent interest.

**Theorem 1.5.** Let \( X \) be a weak Hilbert space with constant \( C \) as in (1.5). Let \( u, v \) be nuclear operators on \( X \) and let \( \rho \) be the spectral radius of \( u + v \). Then if \( \text{rk}(u) \leq n \) and \( N(u) > 1 \)

\[
|\text{tr} \ (u + v)| \leq 2n \rho \log N(u) + 2C + 2CN(v)
\]

hence also

\[
N(u + v) \leq 2n \|u + v\| \log N(u) + 2C + 2CN(v).
\]

**Proof:** Let \( R = 1/\rho \). The function \( f(z) = \det (I + z(u + v)) \) is entire and does not vanish in \( D_R \). Therefore there is an analytic function \( h \) on \( D_R \) such that \( f = \exp(h) \) and since \( f(0) = 1 \) we can assume \( h(0) = 0 \).
Note that $f'(0) = h'(0) = \text{tr} (u + v)$. By (1.6) we have if $N(u) \geq 1$ and $r < R$

$$\sup_{|z|=r} \text{Re} \ h(z) \leq n \log N(u) + Cr + CN(v)$$

hence by Lemma 1.4

$$|\text{tr} (u + v)| = |h'(0)| \leq \frac{2n}{r} \log N(u) + 2C + 2CN(v)$$

Letting $r$ tend to $R = \frac{1}{\rho}$, we obtain (1.7).

For (1.8) we simply observe that if $N(u) > 1$ we have

\begin{equation}
N(u) = \sup\{|\text{tr} (uS)|; \quad S : X \to X, \quad \|S\| \leq 1, \quad N(uS) > 1\}.
\end{equation}

Therefore (1.8) follows from (1.7) since $\rho \leq \|u + v\|$ and we can take the supremum of (1.7) over all $S$ as in (1.9). ■

Finally we prove the “only if” part of our main theorem. Let $X$ be a weak Hilbert space. The first and second authors proved, respectively, that $X$ is reflexive (cf. [P 3], chapter 14) and that $X$ has the AP ([P 3], chapter 15). We will show that (ii) in Proposition 1.2 holds for suitable constants. Let $T_1, T_2$ be as in Proposition 1.2. Let $u = T_1 + T_2$ and $v = -T_1$.

By homogeneity we may assume $n\|T_2\| + N(T_1) = 1$.

Then if $N(T_1 + T_2) > 1$ we have by (1.8)

$$N(T_1 + T_2) \leq N(u + v) + N(v) \leq 2n\|T_2\| \log N(T_1 + T_2) + 2C + (2C + 1)N(T_1) \leq 2\log N(T_1 + T_2) + 4C + 1,$$

and (since $2\log x \leq (x/2) + 2$ if $x > 1$) this implies that if $N(T_1 + T_2) > 1$ then

$$N(T_1 + T_2) \leq 8C + 6.$$

Since in the case $N(T_1 + T_2) \leq 1$ this bound is trivial, we conclude by homogeneity that (if $T_1 + T_2$ has rank $\leq n$)
\[ N(T_1 + T_2) \leq (8C + 6)n\|T_2\| + N(T_1). \]

By proposition 1.2 and 1.1 we conclude that \( X \) has the proportional UAP. ■

**Remark:** Replacing \((u+v)\) by \(\epsilon(u+v)\) in (1.7) yields that if \(\epsilon \geq N(u)^{-1}\), then \(|\text{tr} (u+v)| \leq 2n\rho\log (\epsilon N(u)) + 2C\epsilon^{-1} + 2CN(v)\) hence after minimization over \(\epsilon \geq N(u)^{-1}\) we find that if \(N(u) \geq n\rho/C\), then

\[ |\text{tr} (u+v)| \leq 2n\rho\log \left(\frac{CN(u)}{n\rho}\right) + 1) + 2CN(v). \]

On the other hand if \(N(u) < n\rho/C\) we have trivially since \(C \geq 1\)

\[ |\text{tr} (u+v)| \leq N(u+v) \]
\[ \leq n\rho/C + N(v) \]
\[ \leq 2n\rho + 2CN(v) \]

hence we conclude that without any restriction on \(N(u)\) we have if \(\text{rk} (u) \leq n\)

\[ |\text{tr} (u+v)| \leq 2n\rho\log \left(\frac{CN(u)}{n\rho}\right) + 1) + 2CN(v). \]

Even in the case of a Hilbert space we do not see a direct proof of this inequality.
§2. The converse

Recall that $X$ is a weak cotype 2 space if there are constants $C$ and $0 < \delta < 1$ such that every finite dimensional subspace $E \subset X$ contains a subspace $F \subset E$ with $\dim F \geq \delta \dim E$ such that $d_F \equiv d(F, l_2^{\dim F}) \leq C$ (cf. [MP]).

We begin with a slightly modified presentation of Mascioni’s [Ma 1] proof that a Banach space $X$ which has proportional UAP must have weak cotype 2. Suppose that $k_X(n, K) \leq L n$ for all $n = 1, 2, \ldots$. Take any $(1 + \delta)n$-dimensional subspace $G_0$ of $X$, and, using Milman’s subspace of quotient theorem [M] (or see [P 3], chapters 7 & 8), choose an $n$–dimensional subspace $G$ of $G_0$ for which there exists a subspace $H$ of $G$ such that $\dim H \leq \delta n$ and $d = d_{G/H}$ is bounded by a constant which is independent of $n$, where $\delta$ is chosen so that $\delta L \leq \frac{1}{2}$.

Take $T: X \to X$ with $T|_H = I_H$, $\|T\| \leq K$, and $\text{rk}(T) \leq \delta Ln$. Let $Q: G \to G/H$ be the quotient map and set $E = \ker(T) \cap G$. If $x$ is in $E$, then

$$\|Qx\| = \inf_{h \in H} \|x - h\|$$

$$\geq \inf_{h \in H} \frac{\|(I - T)(x - h)\|}{\|I - T\|}$$

$$= \frac{\|x\|}{\|I - T\|};$$

that is, $Q|_E$ is an isomorphism and $\|(Q|_E)^{-1}\| \leq \|I - T\|$. Thus $d_E \leq \|I - T\|d_{G/H}$, which finishes the proof since $\dim E \geq n - \delta Ln \geq \frac{n}{2}$.

Since we do not know a priori that the proportional UAP dualizes, we need to prove Mascioni’s theorem under a weaker hypothesis.

**Theorem 2.1.** Let $X$ be a Banach space. Assume that there are constants $K$ and $L$ such that for all finite dimensional subspaces $E \subset X$ there is an operator $T: X \to X$ satisfying $T|_E = I_E$ and such that $\|T\| \leq K$ and $\pi_2(T) \leq L(\dim E)^{1/2}$. Then $X$ is a weak cotype 2 space.

**Proof:** Recall that if $T: X \to Y$ is an operator, then

$$\pi_2(T)^2 = \sup \left\{ \sum \|T u(e_i)\|^2; u: l_2 \to X, \|u\| \leq 1 \right\}$$
where $\{e_i\}_{i=1}^\infty$ is the unit vector basis for $l_2$.

Take any $(1 + \delta)n$-dimensional subspace $G_0$ of $X$, where $\delta$ is chosen so that $L\sqrt{\delta} \leq \frac{1}{8}$.

By Milman’s subspace of quotient theorem ([M] or [P 3], chapters 7 & 8), we can choose an $n$-dimensional subspace $G$ of $G_0$ for which there exists a subspace $H$ of $G$ such that $\dim(H) \leq \delta n$ and $d = d_{G/H}$ is bounded by a constant which is independent of $n$. Using the ellipsoid of maximal volume, we get from the Dvoretzky-Rogers lemma (cf. [P 3], lemma 4.13) a norm one operator $J: l_2^n \to G$ such that $\|x_i\| \geq \frac{1}{2}$ for all $i = 1, 2, \ldots, \frac{n}{2}$, where $x_i = Je_i$. By [FLM], all we need to check is that there is a constant $\tau$ so that

$$\left(\text{Average}_\pm \| \sum_{i=1}^n \pm x_i \|^2 \right)^{\frac{1}{2}} \geq \tau \sqrt{n}.$$

Let $Q: G \to G/H$ be the quotient map. Then

$$\tau_0 \sqrt{n/4} \equiv \left(\text{Average}_\pm \| \sum_{i=1}^{n/2} \pm Qx_i \|^2 \right)^{\frac{1}{2}} \geq \frac{1}{d} \left(\sum_{i=1}^{n/2} \|Qx_i\|^2 \right)^{\frac{1}{2}}.$$

So we can assume without loss of generality that $\|Qx_i\| \leq d\tau_0$ for $1 \leq i \leq \frac{n}{4}$. Now take $T: X \to X$ with $T|_H = I_H$ and $\pi_2(T) \leq L\sqrt{\delta n}$. Thus also

$$\left(\sum_{i=1}^{n/4} \|Tx_i\|^2 \right)^{\frac{1}{2}} \leq L\sqrt{\delta n}$$

hence without loss of generality $\|Tx_1\| \leq 2L\sqrt{\delta}$. But then

$$d\tau_0 \geq \|Qx_1\| = \inf_{h \in H} \|x_1 - h\|$$

$$\geq \frac{\| (I - T)x_1 \|}{\| I - T \|} \geq \frac{\frac{1}{2} - \|Tx_1\|}{\| I - T \|} \geq \frac{\frac{1}{2} - 2L\sqrt{\delta}}{\| I - T \|} \geq \frac{1}{4\| I - T \|};$$

that is, $\tau_0 \geq (4d\|I - T\|)^{-1}$. 

12
Proof of converse of Main Theorem: By [LT 2], $X^{**}$ also has proportional UAP; in fact, $k_{X^{**}}(K,n) = k_X(K,n)$ for all $K$ and $n$. Then, just as in the proof of Theorem 4 in [Ma 2], Lemma 1 in [Ma 2] (or its unpublished predecessor proved by Bourgain and mentioned in [Ma 2]) yields that $X^*$ satisfies the hypothesis of Theorem 2.1 and hence $X^*$ as well as $X$ has weak cotype 2. By the results of [P 1], it only remains to check that $X$ has non-trivial type; this is done as in Theorem 3.3 of [Ma 1]: since $X$ and $X^*$ have weak cotype 2, they both have cotype $2+\epsilon$ for all $\epsilon > 0$. Since $X$ has the bounded approximation property, the main result of [P 1] yields that $X$ has non-trivial type. ■

Remark: With a bit more work, the converse of the Main Theorem can be improved. Following Mascioni [Ma 2], given an ideal norm $\alpha$, a normed space $X$, and $K > 1$, we write

$$\alpha-k_X(K,n) = \sup_{E \subset X, \dim E=n} \inf \{\alpha(T)\}$$

where the infimum runs over all finite rank operators $T : X \to X$ such that $\|T\| < K$ and $T|_E = I|_E$. (We use “$< K$” instead of “$\leq K$” in order to avoid in the sequel statements involving awkward “$K + \epsilon$ for all $\epsilon > 0$”.) We say that $X$ has the $\alpha$-UAP if there is a $K > 1$ such that for all $n$, $\alpha-k_X(K,n) < \infty$; when the value of $K$ is important, we say that $X$ has the $K$-$\alpha$-UAP. Notice that the “finite rank” can be ignored if the space $X$ has the metric approximation property or (by adjusting $K$) if $X$ has the bounded approximation property. Here we are interested in $\alpha = \pi_2$ and $\alpha = \pi_2^d$, where $\pi_2^d(T) \equiv \pi_2(T^*)$. Since for either of these $\alpha$’s, $\alpha(T) < \infty$ implies that $T^2$ is uniformly approximable by finite rank operators ($T^2$ factors through a Hilbert-Schmidt operator), for these two $\alpha$’s the $K$-$\alpha$-UAP implies the bounded approximation property. In fact, by passing to ultraproducts and using [H], it follows that for either of these $\alpha$’s the $K$-$\alpha$-UAP implies the $(K^2+\epsilon)$-UAP; in particular, $X^{**}$ has the bounded approximation property. (This is really a sloppy version of Mascioni’s reasoning [Ma 2]; Mascioni gives a better estimate for $k_X(K',n)$ in terms of $\alpha-k_X(K,n)$.)

We now state an improvement of the converse in the Main Theorem:
**Theorem 2.2.** Suppose that there are constants $K$ and $L$ so that the Banach space $X$ satisfies for all $n$ \( \pi_2 k_X(K, n) \leq L \sqrt{n} \) and \( \pi_2^d k_X(K, n) \leq L \sqrt{n} \). Then $X$ is a weak Hilbert space.

**Proof:** In view of the discussion above, we can ignore the “finite rank” condition in the definition of $\alpha$-UAP. It is then easy to see for $\alpha = \pi_2$ or $\alpha = \pi_2^d$ that for all $n$ and $K$ $\alpha k_{X^{**}}(K, n) = \alpha k_X(K, n)$, hence by Lemma 1 of [Ma 2] and Theorem 2.1 we conclude that $X$ and $X^*$ have weak cotype 2. The argument used in the proof of the converse in the Main Theorem shows that $X$ has non-trivial type, so $X$ is weak Hilbert. ■
§3. Related results and concluding remarks

In [Ma 1], Mascioni proved (but stated in slightly weaker form) that for \(2 < p < \infty\) and all \(K\), there exists \(\delta = \delta(p, K) > 0\) so that for all \(n\), \(k_p(K, n) \geq \delta n^{p/2}\). (We write \(k_p(K, n)\) for \(k_{L_p}(K, n)\) and \(\alpha-k_p(K, n)\) for \(\alpha-k_{L_p}(K, n)\). See [FJS], [JS], and [Ma 1] for results about the UAP in \(L_p\)-spaces, \(1 \leq p \leq \infty\).) We prove here the corresponding result for \(1 < p < 2\).

**Theorem 3.1.** For each \(2 < p < \infty\) and \(K > 1\), there exists \(\epsilon = \epsilon(p, K) > 0\) so that for all \(n\), \(\pi_2-k_p(K, n) \geq \epsilon n^{p/2}\). Consequently, for \(1 < q < 2\), \(k_q(K, n) \geq \pi_2^{d} - k_q(K, n)^2 \geq \epsilon^2 n^{p/2}\) where \(\frac{1}{p} + \frac{1}{q} = 1\).

**Proof:** The proof is basically the same as the proof of Theorem 2.1 once we substitute a result of Gluskin [Gl] for Milman’s subspace of quotient theorem, so we use notation similar to that used in Theorem 2.1. Fix \(n\), set \(G = l^n_p\), let \(J\) denote the formal identity from \(l^n_2\) into \(l^n_p\), and let \(x_i = J e_i\) be the unit vector basis of \(l^n_p\). By Gluskin’s theorem [Gl], given any \(\gamma > 0\) there is \(M = M(p, \gamma)\) independent of \(n\) and a subspace \(H\) of \(G\) with \(\dim H \leq M n^{p/2}\) so that \(\delta = d_{G/H} \leq \gamma n^{p/2} - \frac{1}{2}\). Let \(Q: G \to G/H\) be the quotient map. Define \(\epsilon_0\) by the formula \(\pi_2-k_p(K, \dim H) = \epsilon_0 \left( \frac{1}{2} \right)^{p/2} \left( \sum_{i=1}^{n} \left\| x_i \right\| \right)^{1/2}\), and choose \(T: X \to X\) with \(T|_H = I_H\), \(\|T\| \leq K\), and \(\pi_2(T) \leq \epsilon_0 M^2 \sqrt{n}\). We need to show that \(\epsilon_0\) is bounded away from 0 independently of \(n\). Now

\[
\frac{1}{2} \left( \sum_{i=1}^{n} \| x_i \| \right)^{1/2} \leq \epsilon_0 \left( \frac{1}{2} \right)^{p/2} \left( \sum_{i=1}^{n} \| x_i \| \right)^{1/2} \leq \frac{1}{d} \left( \sum_{i=1}^{n} \| Q x_i \| \right)^{1/2},
\]

So we can assume without loss of generality that for \(i = 1, \ldots, n/2\),

\[
\| Q x_i \| \leq \sqrt{2} d \left( \frac{1}{2} \right)^{p/2} \cdot \frac{1}{2}.
\]

Since \(\left( \sum_{i=1}^{n/2} \| T x_i \| \right)^{1/2} \leq \pi_2(T) \leq \epsilon_0 M^2 \sqrt{n}\), we can also assume without loss of generality that \(\| T x_1 \| \leq \sqrt{2} \epsilon_0 M^2 \). But then

\[
\sqrt{2} d \left( \frac{1}{2} \right)^{p/2} \cdot \frac{1}{2} \geq \| Q x_1 \| = \inf_{h \in H} \| x_1 - h \| \geq \frac{\| (I - T) x_1 \|}{\| I - T \|} \geq \frac{1 - \| T x_1 \|}{\| I - T \|} \geq \frac{1 - \sqrt{2} \epsilon_0 M^2}{\| I - T \|}.
\]
that is,
\[ d \geq \frac{1 - \sqrt{2} \epsilon_0 M^\frac{k}{2}}{\sqrt{2\|I - T\|} n^{\frac{1}{2}}}. \]
For sufficiently small \( \gamma \) (e.g., \( \gamma \leq \frac{1}{2} (K + 1)^{-1} \)), this gives a lower bound on \( \epsilon_0 \) since \( d \leq \gamma n^{\frac{1}{2}} \).

The “consequently” statement follows by duality from Lemma 1 in [Ma 2].

\[ \bullet \quad \bullet \quad \bullet \]

The trick of Mascioni’s [Ma 1] mentioned at the beginning of section 2 can be used to answer a question Pełczyński asked the authors twelve years ago; namely, whether every \( n \)-dimensional subspace of \( l^{2n}_\infty \) well-embeds into \( l^{(1+\epsilon)n}_\infty \) for each \( \epsilon > 0 \). Since \( l^{2n}_\infty \) has an \( n \)-dimensional quotient \( F \) with \( d_F \) bounded independently of \( n \) by Kašin’s theorem ([K] or [P 3], corollary 6.4), Proposition 3.2 gives a strong negative answer to Pełczyński’s question.

**Proposition 3.2.** Set \( G = l^n_\infty \), let \( H \) be a subspace of \( G \), set \( d = d_{G/H} \), and assume that \( H \) is \( K \)-isomorphic to a subspace of \( l^k_\infty \). Then \( d \geq e^{-2} (K + 1)^{-1} \left( \frac{n - k}{\log k} \right)^\frac{1}{2} \).

**Proof:** Let \( u: H \to l^k_\infty \) satisfy \( \|u\| = 1 \), \( \|u^{-1}\| \leq K \), let \( U \) be a norm one extension of \( u \) to an operator from \( G \) to \( l^k_\infty \), let \( S \) be an extension of \( u^{-1} \) to an operator from \( l^k_\infty \) to \( G \) with \( \|S\| \leq K \), and set \( T = SU \). So \( T|_H = I_H \) and \( \|T\| \leq K \). Let \( Q: G \to G/H \) be the quotient map and set \( E = \ker(T) \), so that \( \dim E \geq n - k \). Thus (see the argument at the beginning of section 2)
\[ d_E \leq \|I - T\| d \leq (K + 1)d. \]

But by [BDGJN], p. 182 (let \( s = \log k \) there),
\[ d_E \geq e^{-2} \left( \frac{n - k}{\log k} \right)^\frac{1}{2}. \]

\[ \bullet \quad \bullet \quad \bullet \]

We conclude with two open problems related to the material in section 1.

**Problem 3.3.** If \( X \) is a weak Hilbert space, then is \( k_X(K, n) \) proportional to \( n \) for all \( K > 1 \)?
Since weak Hilbert spaces are superreflexive, for all \( K > 1 \) \( k_X(K, n) < \infty \) for every weak Hilbert space \( X \) by a result of Lindenstrauss and Tzafriri [LT 2], but their argument does not give a good estimate of \( k_X(K, n) \) for \( K \) close to one when one has a good estimate for large \( K \).

For the known weak Hilbert spaces \( X \), the growth rate of \( k_X(K, n) - n \) is very slow (cf. [J], [CJT]), at least for sufficiently large \( K \). It follows from recent work of Nielsen and Tomczak-Jaegermann that \( k_X(K, n) - n \) has the same kind of slow growth for any weak Hilbert space \( X \) which has an unconditional basis. On the other hand, we do not know any improvement of the result presented in section 2 for general weak Hilbert spaces. This suggests:

**Problem 3.4.** *If \( X \) is a weak Hilbert space, does there exist \( K \) so that \( k_X(K, n) - n = o(n) \)?*
References

[BDGJN] G. Bennett, L. E. Dor, V. Goodman, W. B. Johnson, and C. M. Newman, On uncomplemented subspaces of $L_p$, $1 < p < 2$, Israel J. Math. 26 (1977), 178–187.

[CJT] P. G. Casazza, W. B. Johnson, and L. Tzafriri, On Tsirelson’s space, Israel J. Math. 47 (1984), 81–98.

[FJS] T. Figiel, W. B. Johnson, and G. Schechtman, Factorizations of natural embeddings of $l^n_p$ into $L_r$, I, Studia Math. 89 (1988), 79–103.

[FLM] T. Figiel, J. Lindenstrauss, and V. Milman, The dimension of almost spherical sections of convex bodies, Acta Math. 139 (1977), 53–94.

[Gl] E. D. Gluskin, Norms of random matrices and diameters of finite dimensional sets, Mat. Sbornik 120 (1983), 180–189.

[G] A. Grothendieck, La Théorie de Fredholm, Bull. Soc. Math. France 84 (1956), 319–384.

[H] S. Heinrich, Ultraproducts in Banach space theory, J. Reine Angew. Math. 313 (1980), 72–104.

[J] W. B. Johnson, Banach spaces all of whose subspaces have the approximation property, Special Topics of Applied Mathematics North-Holland (1980), 15–26.

[JS] W. B. Johnson and G. Schechtman, Sums of independent random variables in rearrangement invariant function spaces, Ann. Prob. 17 (1989), 789–808.

[K] B. S. Kašin, Sections of some finite-dimensional sets and classes of smooth functions, Izv. Akad. Nauk SSSR 41 (1977), 334–351 (Russian).

[LT 1] J. Lindenstrauss and L. Tzafriri, On the complemented subspaces problem, Israel J. Math. 9 (1971), 263–269.

[LT 2] J. Lindenstrauss and L. Tzafriri, The uniform approximation property in Orlicz spaces, Israel J. Math. 23 (1976), 142–155.
[Ma 1] V. Mascioni, *Some remarks on the uniform approximation property in Banach spaces*, Studia Math. (to appear).

[Ma 2] V. Mascioni, *On the duality of the uniform approximation property in Banach spaces*, Illinois J. Math. (to appear)

[M] V. D. Milman, *Almost Euclidean quotient spaces of subspaces of finite dimensional normed spaces*, Proc. AMS 94 (1985), 445–449.

[MP] V. Milman and G. Pisier, *Banach spaces with a weak cotype 2 property*, Israel J. Math. 54 (1986), 139–158.

[PT] A. Pajor and N. Tomczak-Jaegermann, *Subspaces of small codimension of finite dimensional Banach spaces*, Proc. A.M.S. 97 (1986), 637–642.

[P 1] G. Pisier, *On the duality between type and cotype*, Springer Lecture Notes 939 (1982), 131–144.

[P 2] G. Pisier, *Weak Hilbert spaces*, Proc. London Math. Soc. 56 (1988), 547–579.

[P 3] G. Pisier, *The volume of convex bodies and Banach space Geometry*, Cambridge Univ. Press (1989).

[S] A. Szankowski, *On the uniform approximation property in Banach spaces*, Israel J. Math. 49 (1984), 343–359.

Texas A&M University, College Station, TX 77843, U.S.A.
Texas A&M University, College Station, TX 77843, U.S.A., and Equipe d’Analyse, Université Paris VI, 75230 Paris, FRANCE