SPLIT-CM POINTS AND CENTRAL VALUES OF HECKE L-SERIES

KIMBERLY HOPKINS

Abstract. Split-CM points are points of the moduli space $h_2/Sp_4(\mathbb{Z})$ corresponding to products $E \times E'$ of elliptic curves with the same complex multiplication. We prove that the number of split-CM points in a given class of $h_2/Sp_4(\mathbb{Z})$ is related to the coefficients of a weight $3/2$ modular form studied by Eichler. The main application of this result is a formula for the central value $L(\psi, 1)$ of a certain Hecke $L$-series. The Hecke character $\psi$ is a twist of the canonical Hecke character $\psi$ for the elliptic $\mathbb{Q}$-curve $A$ studied by Gross, and formulas for $L(\psi, 1)$ as well as generalizations were proven by Villegas and Zagier. The formulas for $L(\psi, 1)$ are easily computable and numerical examples are given.

1. Introduction

Let $D < 0$, $|D|$ prime be the discriminant of an imaginary quadratic field $K$ with ring of integers $\mathcal{O}_K$. Suppose $N$ is a prime which splits in $\mathcal{O}_K$ and is divisible by an ideal $N$ of norm $N$. We will define Hecke characters $\psi_N$ of $K$ of weight one and conductor $N$ (see Section 3). These are twists of the canonical Hecke characters studied by Rohrlich [Roh80a, Roh80b, Roh82] and Shimura [Shi64, Shi71, Shi73b]. Denote by $L(\psi_N, s)$ the corresponding Hecke $L$-series.

Our main theorem (Theorem 3.6) is a formula in the spirit of Waldspurger’s results [Wal80, Wal81]. It says approximately that

$$L(\psi_N, 1) = \sum_{[R]} \sum_{[a]} \Theta_{[a,R],N} \cdot h^\epsilon_{[a,R]}(-N).$$

Here the first sum is over all conjugacy classes of maximal orders $R$ in the quaternion algebra ramified only at $\infty$ and $|D|$, and the second sum is over the elements $[a]$ of the ideal class group of $\mathcal{O}_K$. We will see that the $h^\epsilon_{[a,R]}(-N)$ are integers related to coefficients of a certain weight $3/2$ modular form, and that the $\Theta_{[a,R],N}$ are algebraic integers equal to the value of a symplectic theta function on ‘split-CM’ points (defined in Section 3) in the Siegel space $h_2/Sp_4(\mathbb{Z})$. We expect the formula (1.1) to be useful for computing the central value $L(\psi_N, 1)$.

Let $A(|D|)$ denote a $\mathbb{Q}$-curve as defined in [Gro80]. This is an elliptic curve defined over the Hilbert class field $H$ of $K$ with complex multiplication by $\mathcal{O}_K$ which is isogenous over $H$ to its Galois conjugates. Its $L$-series is a product of the squares of $L$-series $L(\psi, s)$ over the $h(D)$ Hecke characters of conductor $(\sqrt{|D|})$. A formula for the central value $L(\psi, 1)$ expressed as a square of linear combinations of certain theta functions was proven by Villegas in [RV91]. Extensions of his result to higher weight Hecke characters were given by Villegas in [RV93] and jointly with Zagier in [RVZ93]. The Hecke character $\psi_N$ is a twist of $\psi$ by a quadratic Dirichlet character of conductor $(\sqrt{|D|})N$. Therefore our result (1.1) gives a formula for the central value of the corresponding twist of $A(|D|)$.

Our main theorem can be stated in a particularly nice form when the class number of $\mathcal{O}_K$ is one. Then $[a] = [N] = [\mathcal{O}_K]$ and so in particular $\Theta_{[a,R],N} = \Theta_{[R]}$ and $h^\epsilon_{[a,R]}(-N) = h^\epsilon_{R}(-N)$.
are independent of $[a]$ and $N$. This suggests that formula (1.1) will lead to a generating series for $L(\psi_N, 1)$ as $N$ varies in terms of linear combinations (with scalars in $\{\Theta_R\}$) of half-integer weight modular forms.

We hope to extend these results to higher weight as follows. For certain $k \in \mathbb{Z}_{\geq 1}$ it is well-known that the central value $L(\psi_k^N, k)$ can be written as a trace over the class group of $\mathcal{O}_K$ of a weight $k$ Eisenstein series evaluated at Heegner points of level $N$ and discriminant $D$. It is a general philosophy (see [Zag02], for example) that such traces relate to coefficients of a corresponding modular form of half-integer weight. By the Siegel-Weil formula\footnote{The precise statement of this formula is simplified here for the sake of exposition.} we can write the central value of $L(\psi_k^N, s)$ in terms of a sum of theta-series\footnote{Here $\omega_Q$ is the number of automorphisms of the form $Q$.}

\begin{equation}
L(\psi_k^N, k) = \sum_{[a]} \sum_{[Q]} \frac{1}{\omega_Q} \Theta_Q(\tau_a).
\end{equation}

Here the sum is over $[a]$ in the class group of $\mathcal{O}_K$ and over classes of positive definite quadratic forms $Q : \mathbb{Z}^{2k} \to \mathbb{Z}$ in $2k$ variables and in a given genus. The point $\tau_a \in \mathfrak{h}$ is a Heegner point of level $N$ and discriminant $D$. Analogous to the case of two variables, these quadratic forms correspond to higher rank Hermitian forms (see [Ott71], [HK86], [HK89], [HI80], [HI81], [HI83]). An approach to counting the number of distinct theta values in (1.2) would be to associate the Hermitian forms to isomorphism classes of rank $k$ $R$-modules of $B$, for maximal orders $R$ of $B$. This paper does this for the case $k = 1$. Our intention here is to lay the groundwork for the generalization to arbitrary weight $k$.

This paper is organized as follows. Basic notation is given in Section 2. Background and a statement of results are in Section 3. In Section 4, we analyze the endomorphisms of the principally polarized abelian varieties for the split-CM points, and show they form an explicit maximal order in the quaternion algebra $B$. In Section 5 we identify these orders with explicit right orders in $B$. In Section 6 we prove the main results (Theorems 3.2, 3.3 and 3.6) and provide numerical examples.

2. Notation

Given any imaginary quadratic field $M$ of discriminant $d < 0$, we denote by $\mathcal{O}_M$ its ring of integers, $Cl(\mathcal{O}_M)$ its ideal class group, $h(d)$ its class number, and $Cl(d)$ the isomorphic class group of primitive positive definite binary quadratic forms of discriminant $d$. A nonzero integral ideal of $\mathcal{O}_M$ with no rational integral divisors besides $\pm 1$ is said to be primitive. Any primitive ideal $\mathfrak{a}$ of $\mathcal{O}_M$ can be written uniquely as the $\mathbb{Z}$-module

$$\mathfrak{a} = a\mathbb{Z} + \frac{-b + \sqrt{d}}{2} \mathbb{Z} = [a, -b + \sqrt{d}]$$

with $a := Na$ the norm of $\mathfrak{a}$, and $b$ an integer defined modulo $2a$ which satisfies $b^2 \equiv d \mod 4a$. Conversely any $a, b \in \mathbb{Z}$ which satisfy the conditions above determine a primitive ideal of $\mathcal{O}_M$. The coefficients of the corresponding primitive positive definite binary quadratic form are given by $[a, b, c := \frac{b^2 - d}{4a}]$. The form $[a, -b, c]$ corresponds to the ideal $\bar{\mathfrak{a}}$. We will always assume our forms are primitive positive definite and the same for ideals. The point

$$\tau_\mathfrak{a} := \frac{-b + \sqrt{d}}{2a}$$
3. Statement of Results

We first recall some basic results for Siegel space and symplectic modular forms.

Assume $K$ is an imaginary quadratic field of prime discriminant $D < -4$. Let $L$ be an imaginary quadratic field of discriminant $-N < 0$ where $N$ is a prime which splits in $\mathcal{O}_K$, and is divisible by an ideal $\mathfrak{N}$ of norm $N$. Note $h(D)$ and $h(-N)$ are both odd since $|D|$ and $N$ are prime. Let $\mu : \mathcal{O}_K/\mathfrak{N} \rightarrow \mathbb{Z}/N\mathbb{Z}$ be the natural isomorphism. Composing this with the Jacobi symbol $(\frac{\cdot}{\mathfrak{N}}) : \mathbb{Z}/N\mathbb{Z} \rightarrow \{0, \pm 1\}$ defines a character

$$\chi : (\mathcal{O}_K/\mathfrak{N})^\times \rightarrow \{\pm 1\}.$$ 

This is an odd quadratic Dirichlet character of conductor $N$. Let $I_N$ denote the group of nonzero fractional ideals of $K$ which are coprime to $N$, and let $P_N \subset I_N$ be the subgroup of principal ideals. The map $\psi_N : P_N \rightarrow K^\times$ defined by

$$\psi_N((\alpha)) := \chi(\alpha)\alpha$$

is a homomorphism. There are exactly $h(D)$ extensions of $\psi_N$ to a Hecke character $\psi_N : I_N \rightarrow \mathbb{C}^\times$. This produces $h(D)$ primitive Hecke characters of weight one and conductor $N$. (See [Gro84] [Pac05] and [Roh80a] p.225 for more details). Fix a choice of $\psi_N$. We can extend $\psi_N$ to a multiplicative function on all of $\mathcal{O}_K$ by setting $\psi_N(\mathfrak{a}) := 0$ if $\mathfrak{a}$ is not coprime to $\mathfrak{N}$.

To $\psi_N$ we associate the Hecke $L$-function

$$L(\psi_N, s) := \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\psi_N(\mathfrak{a})}{\mathfrak{N}\mathfrak{a}^s}, \quad \text{Re}(s) > 3/2.$$

We now recall a result due to Hecke which gives the central value $L(\psi_N, 1)$ as a linear combination of certain theta series evaluated at CM points. For each primitive ideal $\mathfrak{Q}$ of $\mathcal{O}_L$, the associated theta series is defined by

$$\Theta_{\mathfrak{Q}}(\tau) := \sum_{\lambda \in \mathfrak{Q}} q^{N(\lambda)/N(\mathfrak{Q})}, \quad q = e^{2\pi i\tau}, \quad \tau \in \mathfrak{h}.$$ 

It is a modular form on $\Gamma_0(N)$ of weight one and character $\text{sgn}(\cdot)(\frac{\tau N}{|\tau|})$ (see [Eic66] p.49, for example).

For each primitive ideal $\mathfrak{a}$ of $\mathcal{O}_K$ with norm prime to $N$, the product ideal $\mathfrak{a}\mathfrak{N}$ is of the form $[a_1N, \frac{-b_1 + \sqrt{D}}{2}]$ for some $a_1, b_1 \in \mathbb{Z}$. The point

$$\tau_{\mathfrak{a}\mathfrak{N}} := \frac{-b_1 + \sqrt{D}}{2a_1N} \in \mathfrak{h}$$

is a Heegner point of level $N$ and discriminant $D$. We will write $\tau_{\mathfrak{a}}$ or just $\tau$ for $\tau_{\mathfrak{a}\mathfrak{N}}$ when the context is clear. Note that as $\mathfrak{a}$ runs over a distinct set of representatives of $\text{Cl}(\mathcal{O}_K)$, so does $\mathfrak{a}\mathfrak{N}$. (The fact that representatives of $\text{Cl}(\mathcal{O}_K)$ can be chosen with norm prime to $N$ is in [Cox89] Lemmas 2.3, 2.25, for example.) By $\mathfrak{a}$ we will always mean a primitive ideal with norm prime to $N$ as above.
Hecke’s formula \cite{Hec59} for the central value of \(L(\psi_N, s)\) states
\[
L(\psi_N, 1) = \frac{2\pi}{\omega_N\sqrt{N}} \sum_{[\alpha] \in Cl(\mathcal{O}_K)} \sum_{[\mathfrak{a}] \in Cl(\mathcal{O}_L)} \frac{\Theta_0(\tau_\mathfrak{a})}{\psi_N(\mathfrak{a})}
\]
where \(\omega_N\) is the number of units in \(\mathcal{O}_L\).

The theta function for \(\mathcal{Q}\) arises from a certain specialization of a symplectic theta function. Let \(\text{Sp}_4(\mathbb{Z})\) denote the Siegel modular group of degree 2. Let \(\Gamma_0\) be the subgroup of \(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{Sp}_4(\mathbb{Z})\) such that both \(\alpha^T \gamma + \beta^T \delta\) have even diagonal entries. The group \(\Gamma_0\) inherits the action of \(\text{Sp}_4(\mathbb{Z})\) on the Siegel upper half plane \(\mathfrak{h}_2 := \{z \in \text{Mat}_2(\mathbb{C}) : z^T z = z, \text{Im}(z) > 0\}\). Define the symplectic theta function by
\[
\theta(z) := \sum_{\vec{x} \in \mathbb{Z}^2} \exp \left[ \pi i \vec{x}^T \vec{z} \vec{x} \right], \quad z \in \mathfrak{h}_2.
\]
The function \(\theta\) satisfies the functional equation
\[
\theta(M \circ z) = \chi(M) [\det(\gamma z + \delta)]^{1/2} \theta(z), \quad M \in \Gamma_0
\]
where \(\chi(M)\) is an eighth root of unity which depends on the chosen square root of \(\det(\gamma z + \delta)\) but is otherwise independent of \(z\). It is a symplectic modular form on \(\Gamma_0\) of dimension \(-1/2\) with multiplier system \(\chi\) (see \cite[p.43]{Eic66} or \cite[p.189]{Mum07}, for example).

Given a primitive ideal \(\mathcal{Q}\) of \(\mathcal{O}_L\), let \(Q := [a, b, c]\) represent the corresponding binary quadratic form of discriminant \(-N\). The product of the matrix of \(Q\) with any Heegner point \(\tau_\mathfrak{a}\) is the Siegel point
\[
Q \tau_\mathfrak{a} := \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \cdot \tau_\mathfrak{a} \in \mathfrak{h}_2.
\]
We will refer to points constructed in this way as \textit{split-CM points} of level \(N\) and discriminant \(D\). This yields the relation
\[
\Theta_0(\tau_\mathfrak{a}) = \theta(Q \tau_\mathfrak{a})
\]
which can be substituted into formula \(\text{(3.1)}\) to get
\[
L(\psi_N, 1) = \frac{2\pi}{\omega_N\sqrt{N}} \sum_{[\alpha] \in Cl(\mathcal{O}_K)} \sum_{[Q] \in Cl(-N)} \frac{\theta(Q \tau_\mathfrak{a})}{\psi_N(\mathfrak{a})}.
\]
If \(Q \sim Q'\) in \(Cl(-N)\), then \(Q \tau_\mathfrak{a} \sim Q' \tau_\mathfrak{a}\) in \(\mathfrak{h}_2/\text{Sp}_4(\mathbb{Z})\), and if \(a \sim a'\) in \(Cl(\mathcal{O}_K)\), then \(Q \tau_\mathfrak{a} \sim Q a'\) in \(\mathfrak{h}_2/\text{Sp}_4(\mathbb{Z})\) (see Remark \textit{[6.2]} and Lemma \textit{[6.12]}). In addition it is shown in \cite[Lemma 53]{Pac05} that these equivalences of Siegel points sustain modulo \(\Gamma_0\). The function \(\theta/\psi_N\) is invariant on such points:

\textbf{Lemma 3.1.} Fix an ideal \(\mathfrak{a} \subset \mathcal{O}_K\) and a prime ideal \(\mathcal{N} \subset \mathcal{O}_K\) of norm \(N\). Let \(Q\) be a binary quadratic form of discriminant \(-N\). Then the value \(\theta(Q \tau_\mathfrak{a})\) depends only on the class \([Q] \in Cl(\mathcal{O}_L)\) and the class \([\mathfrak{a}] \in Cl(\mathcal{O}_K)\).

\textit{Proof.} The value \(\theta(Q \tau_\mathfrak{a})\) is independent of the class representative of \([Q]\) because equivalent forms represent the same values. That \(\text{(3.5)}\) is independent of the representative of \([\mathfrak{a}] \in Cl(\mathcal{O}_K)\) is a short calculation using the functional equation for \(\theta\) in \(\text{(3.2)}\) and is done in \cite[Proposition 22]{Pac05}. \(\square\)

\textsuperscript{3}The symplectic theta function is sometimes defined with extra parameters, \(\theta(z, u, v)\) where \(u, v \in \mathbb{C}^2\), in which case the theta function above is equal to \(\theta(z, 0, 0)\).
Therefore the set of points \([Q]\tau_{[a]N}\) as \([Q]\) runs over \(Cl(-N)\) and \([a]\) runs over \(Cl(O_K)\) are equivalent in \(\mathfrak{h}_2/\Gamma_\theta\) and are identified under \(\theta/\psi_N\). We refer to \([Q]\tau_{[a]N}\) as a split-CM orbit. Thus to determine which values \(\theta(\mathfrak{q}_\tau)\) are equal in (3.4) it is necessary to determine which split-CM orbits \([Q]\tau_{[a]N}\) are equivalent modulo \(\Gamma_\theta\). Since \(\mathfrak{h}_2/Sp_4(\mathbb{Z})\) is a moduli space for the principally polarized abelian varieties of dimension two ([Mum07] or [BL04, Chp. 8]), the classes of split-CM points are determined by the isomorphism classes of the corresponding varieties.

To describe these, we will recall some basic facts about quaternion algebras. Let \(B := (-1, D)_{\mathbb{Q}}\) be the quaternion algebra over \(\mathbb{Q}\) ramified at \(\infty\) and \(|D|\). Recall two maximal orders \(R, R'\) in \(B\) are equivalent if there exists \(x \in B^\times\) such that \(R' = x^{-1}Rx\). Moreover, two optimal embeddings \(\phi: \mathcal{O}_L \hookrightarrow R\) and \(\phi': \mathcal{O}_L \hookrightarrow R'\) are equivalent if there exists \(x \in B^\times\) and \(r \in R^\times\) such that \(R' = x^{-1}Rx\) and \(\phi' = (xr)^{-1}\phi(xr)\). Let \(\mathcal{R}\) denote the set of conjugacy classes of maximal orders in \(B\) and let \(\Phi_\mathcal{R}\) denote the set of classes of optimal embeddings of \(\mathcal{O}_L\) into the maximal orders of \(B\). Let \(\mathcal{R}_N \subset \mathcal{R}\) denote the maximal order classes which admit an optimal embedding of \(\mathcal{O}_L\). Given an optimal embedding \((\phi: \mathcal{O}_L \hookrightarrow R) \in \Phi_\mathcal{R}\), let \((\bar{\phi}: \mathcal{O}_L \hookrightarrow R) \in \Phi_\mathcal{R}\) denote its quaternionic conjugate, so that \(\phi(\sqrt{-N}) = \bar{\phi}(\sqrt{-N})\). The quotient \(\Phi_\mathcal{R}/\sim\) will denote the set \(\Phi_\mathcal{R}\) modulo this conjugation. Let \(h_R(-N)\) denote the number of optimal embeddings of \(\mathcal{O}_L\) into \(R\) modulo conjugation by \(R^\times\). This number is an invariant of the choice of representative of \([R]\) in \(\mathcal{R}\).

Our first theorem says that the classes of split-CM points in Siegel space correspond to classes of maximal orders in \(B\).

**Theorem 3.2.** Fix \([a] \in Cl(O_K)\), \(N \subset O_K\) a prime ideal of norm \(N\), and \(\tau := \tau_{an}\). There is a bijection

\[\Upsilon_1: \{Q\tau: [Q] \in Cl(-N)\} / Sp_4(\mathbb{Z}) \to \mathcal{R}_N.\]

This map is independent of the choice of representative \(a\) of \([a]\).

Let \(\Upsilon_1^{-1}([R])\) for \([R] \in \mathcal{R}_N\) denote the pre-image class in \(\mathfrak{h}_2/Sp_4(\mathbb{Z})\) and set \(\Upsilon_1^{-1}([R]) := \emptyset\) if \([R] \in \mathcal{R} \setminus \mathcal{R}_N\). Our second theorem gives the number of split-CM orbits in a given class.

**Theorem 3.3.** Assume the hypotheses of Theorem 3.2. For any \([R] \in \mathcal{R}\),

\[\# \{[Q]\tau \in \Upsilon_1^{-1}([R]): [Q] \in Cl(-N)\} = h_R(-N)/2.\]

That is, the number of split-CM orbits in the class in \(\mathfrak{h}_2/Sp_4(\mathbb{Z})\) corresponding to \([R]\) under Theorem 3.2 is \(h_R(-N)/2\).

For a maximal order \(R\) of \(B\), define \(S_R := \mathbb{Z} + 2R\) and \(S_R^0 \subset S_R\) to be the suborder of trace zero elements. The suborder \(S_R^0\) is a rank 3 \(\mathbb{Z}\)-submodule of \(R\). Define \(g_R\) to be its theta series

\[g_R(\tau) := \frac{1}{2} \sum_{x \in S_R^0} q^{N(x)} = \frac{1}{2} + \sum_{N > 0} a_R(N)q^N,\]

where \(a_R(N)\) are defined by its \(q\)-expansion. It is well known that \(g_R\) is a weight 3/2 modular form on \(\Gamma_0(4|D)\). Applying [Gro87, Proposition 12.9] to fundamental \(-N\) gives

\[a_R(N) = \frac{\omega_R}{\omega_N} h_R(-N)\]

where \(\omega_R\) is the cardinality of the set \(R^\times / <\pm 1>\).

This gives immediately the following Corollary to Theorem 3.3.
Corollary 3.4. Assume the hypotheses of Theorem 3.3. For any \([R] \in \mathbb{R}\),
\[
\# \{ [Q] \tau \in \mathbb{Y}_1^{-1}([R]) : [Q] \in \text{Cl}(-N) \} = a_R(N) \cdot \frac{2\omega_N}{\omega_R}.
\]
That is, the number of split-CM orbits in the class in \(\mathfrak{h}_2/Sp_4(\mathbb{Z})\) corresponding to \([R]\) under Theorem 3.2 is proportional to the \(N\)-th Fourier coefficient of the weight \(3/2\) modular form \(g_R\).

The application of Theorems 3.2 and 3.3 to a formula for \(L(\psi_N, 1)\) proceeds as follows. Define the following normalization of \(\theta\) given by [Pac05]:
\[
(3.6) \hat{\theta}(Q\tau_{aN}) := \frac{\theta(Q\tau_{aN})}{\eta(N)\eta(\mathcal{O}_K)}
\]
where \(\eta(z) := e_{24}(z) \prod_{n=1}^{\infty} (1 - e^{2\pi iz})\) for \(\text{Im}(z) > 0\) is Dedekind’s eta function and the evaluation of \(\eta\) on ideals is defined in Section 6. It is proven in [Pac05 Proposition 23] (see also [HV97]) that the numbers in \(\hat{\theta}(Q\tau_{aN})/\psi_N(\bar{a})\) are algebraic integers.

Define
\[
\Theta_{[a,Q],N} := \frac{\hat{\theta}(Q\tau_{aN})}{\psi_N(\bar{a})}.
\]
This is well-defined by Lemma 3.1. The following lemma says that the theta-values which correspond to a given class \([R] \in \mathbb{R}\) under Theorem 3.2 are all equal up to \(\pm 1\).

Lemma 3.5. Fix \([a] \in \text{Cl}((\mathcal{O}_K), N \subset \mathcal{O}_K\) a prime ideal of norm \(N\), and \(\tau := \tau_{aN}\). Let \([R] \in \mathbb{R}\). Then the values
\[
(3.7) \{ \Theta_{[a,Q],N} : [Q] \tau \in \mathbb{Y}_1^{-1}([R]) \}
\]
differ by \(\pm 1\).

Assume Lemma 3.5 holds (see Section 5 for the proof). Given \([R] \in \mathbb{R}_N\) and any \([Q] \tau \in \mathbb{Y}_1^{-1}([R])\), define \(\Theta_{[a,R],N}\) to be either \(\Theta_{[a,Q],N}\) or \(-\Theta_{[a,Q],N}\) so that it satisfies \(\text{Re}(\Theta_{[a,R],N}) > 0\). Set \(\Theta_{[a,R],N} := 0\) if \([R] \in \mathbb{R}\setminus\mathbb{R}_N\).

We record the mysterious \(\pm 1\) signs appearing in Lemma 3.5 by defining
\[
(3.8) \varepsilon_{[a,R]} : \{ [Q] \tau \in \mathbb{Y}_1^{-1}([R]) \} \longrightarrow \{ \pm 1 \}
\]
\[
[Q] \tau \mapsto \text{sgn} \left( \text{Re} \left( \Theta_{[a,Q],N} \right) \right).
\]
Note \(\Theta_{[a,Q],N} = \pm \Theta_{[a,R],N}\) by construction. This definition assigns, albeit somewhat arbitrarily, a fixed choice of sign for the theta-values as \([Q]\) varies.

We then define a corresponding twisted variant of \(h_R(-N)\) by
\[
(3.9) h_{[a,R]}^\varepsilon(-N) := \sum_{[Q] \tau \in \mathbb{Y}_1^{-1}([R])} \varepsilon_{[a,R]}([Q] \tau).
\]

The formula for \(L(\psi_N, 1)\) can now be stated as follows.

Theorem 3.6. Let \(N \subset \mathcal{O}_K\) be a prime ideal of norm \(N\). Then
\[
(3.10) L(\psi_N, 1) = \frac{\pi \cdot \eta(N)\eta(\mathcal{O}_K)}{\omega_N\sqrt{N}} \sum_{[R] \in \mathbb{R}} \sum_{[a] \subset \text{Cl}(\mathcal{O}_K)} \Theta_{[a,R],N} \cdot h_{[a,R]}^\varepsilon(-N).
\]
where \(\Theta_{[a,R],N}\) is an algebraic integer and \(h_{[a,R]}^\varepsilon(-N)\) is an integer with \(|h_{[a,R]}^\varepsilon(-N)| \leq h_R(-N)\).
Remark 3.7. The signs in Lemma 3.5 and hence the function \( h_{[\alpha,R]}(-N) \) depend on the character \( \chi \) which appears in the functional equation (6.5) for \( \theta \). In particular, the values of \( \chi \) depend on the entries of the transformation matrices in \( \Gamma_\theta \) which takes one Siegel point to an equivalent one. This value is complicated to compute or even define, and is discussed in detail in [AM73] and [Sta82] and [Eic66, Appendix to Chp 1]. An arithmetic formula for these signs and for \( h_{[\alpha,R]}(-N) \) is yet to be determined. But since the \( h_{[\alpha,R]}(-N) \) are weighted count of optimal embeddings, we expect that, like the \( h_R(-N) \), they will be related to coefficients of a half-integer weight modular form. This will be treated in a subsequent paper.

Theorem 3.6 gives us an upper bound on \( L(\psi_N,1) \) in terms of the computable modular form coefficients \( h_R(-N) \).

Corollary 3.8. Assume the hypotheses of Theorem 3.6. Then

\[
|L(\psi_N,1)| \leq \frac{\pi \cdot |\eta(\bar{N})\eta(\mathcal{O}_K)|}{\omega_N\sqrt{N}} \sum_{[R] \in \mathcal{R}} \sum_{[\alpha] \in \text{Cl}(\mathcal{O}_K)} |\Theta_{[\alpha,R,N]}| \cdot h_R(-N).
\]

If \( h(D)=1 \), then (3.10) has a particularly simple form:

Corollary 3.9. Assume the hypotheses of Theorem 3.6 and suppose \( h(D)=1 \). Then \( \Theta_{[\alpha,R,N]} = \Theta_{[R]} \) and \( h_{[\alpha,R]}(-N) = h_{[R]}(-N) \) are independent of \( \alpha \) and \( N \) and

\[
L(\psi_N,1) = \frac{\pi \cdot |\eta(\mathcal{O}_K)|^2}{\omega_N\sqrt{N}} \sum_{[R] \in \mathcal{R}} \Theta_{[R]} \cdot h_{[R]}(-N).
\]

We conclude this section with a comment regarding varying \( N \). The set

\[
\bigcup_N \{[Q]_{\tau[a]\bar{N}} : [Q] \in \text{Cl}(-N), \; [\alpha] \in \text{Cl}(\mathcal{O}_K), \; N \subset \mathcal{O}_K \text{ of norm } N \}
\]

of split-CM orbits over all prime \( N \) with \( D \equiv \square \mod 4N \) partitions into a finite number of Siegel classes in \( \mathfrak{h}_2/Sp_4(\mathbb{Z}) \). This has a natural explanation from our viewpoint. As a complex torus, \( X_{Q_r} \) is isomorphic to a product \( E \times E' \) of two elliptic curves \( E, E' \) defined over \( \overline{\mathbb{Q}} \) and with complex multiplication by \( \mathcal{O}_K \). (This is the reason the \( Q_r \) are called ‘split-CM.’) It is a general result of [NN81] that there are only finitely many principal polarizations on a given complex abelian variety up to isomorphism. There are also only finitely many isomorphism classes of elliptic curves with CM by \( \mathcal{O}_K \). Together these imply that the number of classes of Siegel points \( (X_{Q_r}, H_{Q_r}) \) for all split-CM points \( Q_r \) of discriminant \( D \) must be finite. See [Pac05, Theorem 58] as well for an alternative interpretation.

4. ENDOMORPHISMS OF \( X_z \) PRESERVING \( H_z \)

In this section we prove that the endomorphisms of the abelian varieties corresponding to split-CM points give maximal orders in the quaternion algebra \( B = (-1,D)_\mathbb{Q} \). Let \( V, V' \) be complex vector spaces of dimension 2 with lattices \( L \subset V, \; L' \subset V' \). The analytic and rational representations are denoted by \( \rho_\alpha : \text{Hom}(X, X') \rightarrow \text{Hom}_\mathbb{C}(V, V') \) and \( \rho_r : \text{Hom}(X, X') \rightarrow \text{Hom}_{\mathbb{Z}}(L, L') \), respectively. Recall the periods matrices \( \Pi, \Pi' \in \text{Mat}_{2 \times 4}(\mathbb{C}) \) of \( X, X' \) commute with \( \rho_\alpha \) and \( \rho_r \) in the following diagram

\[
\begin{array}{ccc}
\mathbb{Z}^{2g} & \overset{\Pi}{\longrightarrow} & \mathbb{C}^g \\
\rho_r(f) \downarrow & & \downarrow \rho_\alpha(f) \\
\mathbb{Z}^{2g'} & \overset{\Pi'}{\longrightarrow} & \mathbb{C}^{g'}
\end{array}
\]
(see \[BL04\], for example).

For any Siegel point \( z \in \mathfrak{h}_2 \), let \( \Pi_2 := [z, 1_2] \in \text{Mat}_{2 \times 4}(\mathbb{C}) \) be its period matrix, \( L_z := \Pi_z \mathbb{Z}^4 \) be its defining lattice, and \( X_z := \mathbb{C}^2/L_z \) be its corresponding complex torus. The Hermitian form \( \mathcal{H}_z : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C} \) defined by \( \mathcal{H}_z(u, v) := T u \text{Im}(z)^{-1} \bar{v} \) determines a principal polarization on \( X_z \). As a point in the moduli space \( \mathfrak{h}_2/\text{Sp}_4(\mathbb{Z}) \), \( z \) corresponds to the principally polarized abelian variety \( (X_z, \mathcal{H}_z) \). Throughout Sections 4 and 5, we fix a representative \( \alpha \in \text{Cl}(\mathfrak{o}_K) \), \( N \subset \mathfrak{o}_K \) a prime ideal of norm \( N \), \( \tau := T_2N := \frac{-b + \sqrt{D}}{2a_1N} \), and a split-CM point \( z = Q \tau \) of level \( N \) and discriminant \( D \) where \( Q := [a, b, c] \) is of discriminant \( -N \). The endomorphisms of \( (X_z, \mathcal{H}_z) \) will be our first main object of study.

We define \( \mathcal{B} \) to be the \( \mathbb{Q} \)-algebra of endomorphisms of \( X_z \) which fix \( \mathcal{H}_z \)
\[
\mathcal{B} := \{ \alpha \in \text{End}_\mathbb{Q}(X_z) : \mathcal{H}_z(\alpha u, v) = \mathcal{H}_z(u, \alpha' v) \quad \forall u, v \in \mathbb{C}^2 \};
\]
here \( \iota \) is the canonical involution inherited from \( \text{Mat}_2(K) \) as defined in \[Shi73a\]. In terms of matrices, let \( H_z := \text{Im}(z)^{-1} \) denote the matrix of \( \mathcal{H}_z \) with respect to the standard basis of \( \mathbb{C}^2 \). Then viewing \( \text{End}_\mathbb{Q}(X_z) \subset \text{Mat}_2(K) \), the set \( \mathcal{B} \) is
\[
\mathcal{B} = \{ M \in \text{End}_\mathbb{Q}(X_z) : \tilde{T} M H_z = H_z M' \}.
\]
The bar denotes complex conjugation restricted to \( K \). The map \( \iota \) sends a matrix \( M \) to its adjoint, or equivalently sends \( M \) to \( \text{Tr}(M) \cdot 1_2 - M \).

We define \( \mathcal{R}_2 \) to be the \( \mathbb{Z} \)-submodule of endomorphisms which fix \( H_z \)
\[
\mathcal{R}_2 := \{ M \in \text{End}(X_z) : \tilde{T} M H_z = H_z M' \}.
\]
The first observation is that \( \mathcal{B} \) is isomorphic to a rational definite quaternion algebra.

**Proposition 4.1.** \( \mathcal{B} \) is isomorphic to \( B \) as \( \mathbb{Q} \)-algebras.

**Remark 4.2.** In \[Shi73a Proposition 2.6\], Shimura proves \( \mathcal{B} \) is a quaternion algebra over \( \mathbb{Q} \) in a much more general setting by showing \( \mathcal{B} \otimes \mathbb{Q} \) is isomorphic to \( \text{Mat}_2(\overline{\mathbb{Q}}) \). Here we give an alternative proof which explicitly gives the primes ramified in \( \mathcal{B} \).

**Proof.** We will need the following elementary lemma.

**Lemma 4.3.** Suppose \( Q_1, Q_2 \in \text{Mat}_2(\mathbb{Z}) \) with determinant \( N \). Set \( H_i := \text{Im}(Q_i \tau)^{-1} \) and \( \mathcal{R}_i := \{ M \in \text{End}(X_{Q_i \tau}) : \tilde{T} M H_i = H_i M' \} \), \( i = 1, 2 \).

Let \( S = \mathbb{Z} \) or \( \mathbb{Q} \) and suppose there exists \( A \in \text{GL}_2(S) \) such that \( Q_2 = (\det A)^{-1} A Q_1^T A \). Then the map
\[
\begin{align*}
\text{End}_S(X_{Q_1 \tau}) &\longrightarrow \text{End}_S(X_{Q_2 \tau}) \\
M &\mapsto A M A^{-1}
\end{align*}
\]
and the induced map
\[
\mathcal{R}_1 \otimes_\mathbb{Z} S \longrightarrow \mathcal{R}_2 \otimes_\mathbb{Z} S
\]
are \( S \)-algebra isomorphisms.

**Proof of Lemma.** Let \( \Pi_i := [Q_i \tau, 1_2] \) be the period matrices for \( Q_i \tau, i = 1, 2 \). Suppose \( M \in \text{End}_S(X_{Q_i \tau}) \). By (4.1), this is if and only if \( M \Pi_i = \Pi_i P \) for some \( P \in \text{Mat}_4(S) \). Set
\[
\tilde{A} := \begin{pmatrix} (\det A)^{-1} A^T & 0 \\ 0 & A^{-1} \end{pmatrix} \in \text{GL}_4(S).
\]
Using the identity \( A \Pi_1 \tilde{A} = \Pi_2 \) gives
\[
(AM A^{-1}) \Pi_2 = \Pi_2 (\tilde{A}^{-1} P \tilde{A}).
\]
Clearly $\tilde{A}^{-1}P\tilde{A} \in \text{Mat}_4(S)$, hence $AMA^{-1} \in \text{End}_S(X_{Q_2\tau})$.

Furthermore the identity $H_1 = (\det A^{-1})^T AH_2 A$ implies $^T(AMA^{-1})^{-1}H_2 = H_2(AMA^{-1})^{-1}$ by a straightforward calculation. \hfill $\square$

Define matrices

\begin{equation}
A := \frac{1}{2a} \begin{pmatrix} 1 & 0 \\ -b & 2a \end{pmatrix} \in GL_2(\mathbb{Q}) \quad \text{and} \quad Q' := \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}.
\end{equation}

By Lemma 4.3, $B$ is isomorphic as a $\mathbb{Q}$-algebra to

$$B' := \{ M \in \text{End}_Q(X_{Q'\tau}) : ^T MH' = H'M' \}$$

where $H' := \text{Im}(Q'\tau)^{-1}$.

We will compute $B'$ explicitly. Let $E_\tau := \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ for any $\tau \in \mathfrak{h}$. Clearly $X_{Q'\tau} \cong E_\tau \times E_{N\tau}$ as complex tori. The endomorphisms of $X_{Q'\tau}$ are characterized as follows.

**Lemma 4.4.**

$$\text{End}(X_{Q'\tau}) = \begin{pmatrix} \mathcal{O}_K & \mathbb{Z} + \mathbb{Z}\omega/\mathcal{O}_K \\ \mathcal{O}_K & \mathbb{Z}\omega \end{pmatrix}$$

where $\omega := a_1 N\tau$.

Assuming this for a moment, we have $\text{End}_Q(X_{Q'\tau}) = \text{Mat}_2(K)$, and a quick calculation shows any $M = (\begin{smallmatrix} \alpha & \beta \\ -N\beta & \alpha \end{smallmatrix}) \in \text{Mat}_2(K)$ satisfies $^T MH = HM'$ if and only if $\delta = \bar{\alpha}$ and $\gamma = -N\beta$. Therefore

$$B' = \left\{ \begin{pmatrix} \alpha & \beta \\ -N\beta & \alpha \end{pmatrix} : \alpha, \beta \in K \right\} \subset \text{Mat}_2(K).$$

The elements

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \sqrt{D} & 0 \\ 0 & -\sqrt{D} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{D} \\ N\sqrt{D} & 0 \end{pmatrix}$$

form a basis of $B'$ and clearly give an isomorphism to $(D,-N)\mathbb{Q}$. We claim $B \cong (D,-N)\mathbb{Q}$. This is a general fact: if $p, q$ are primes with $p \equiv q \equiv 3 \mod 4$ and $-p$ is a square modulo $q$, then $(-p,-q)\mathbb{Q}$ is ramified at $\infty$ and $p$ only, so $(-p,-q)\mathbb{Q} \cong (-1,p)\mathbb{Q}$. Hence $B \cong B' \cong B$ as $\mathbb{Q}$-algebras.

It remains to prove Lemma 4.3.

**Proof of Lemma 4.4.** For any quadratic surds $\tau, \tau' \in K$,

$$\text{Hom}(E_\tau, E_{\tau'}) = \{ \alpha \in K : \alpha(\mathbb{Z} + \mathbb{Z}\tau) \subseteq \mathbb{Z} + \mathbb{Z}\tau' \}.$$ 

Since $X_{Q'\tau} \cong E_\tau \times E_{N\tau}$, we have

$$\text{End}(X_{Q'\tau}) = \begin{pmatrix} \text{End}(E_\tau) & \text{Hom}(E_{N\tau}, E_\tau) \\ \text{Hom}(E_\tau, E_{N\tau}) & \text{End}(E_{N\tau}) \end{pmatrix}.$$

We compute. $\text{End}(E_{N\tau}) = \mathcal{O}_K$ since $\mathbb{Z} + \mathbb{Z}a_1 N\tau = \mathcal{O}_K$ and $[1,N\tau]$ is a (proper) fractional $\mathcal{O}_K$-ideal. Similarly $\text{End}(E_\tau) = \mathcal{O}_K$ since $\mathbb{Z} + \mathbb{Z}\tau$ is a fractional $\mathcal{O}_K$-ideal.

It is straightforward to check $\mathbb{Z} + \mathbb{Z}a_1$ is contained in $\text{Hom}(E_{N\tau}, E_\tau)$. On the other hand, $\text{Hom}(E_{N\tau}, E_\tau) \subseteq \mathbb{Z} + \mathbb{Z}\tau$ by definition, and this is proper containment since otherwise $\mathbb{Z} + \mathbb{Z}N\tau$ would intersect $\mathbb{Z} + \mathbb{Z}\tau$ which is impossible since the former contains $\mathcal{O}_K$. Therefore $\text{Hom}(E_{N\tau}, E_\tau) = \mathbb{Z} + m\mathbb{Z}\tau$ for some integer $m|a_1$ but a quick calculation shows $m = a_1$ else it divides $a_1, b_1$ and $c_1$ whose gcd is assumed to be 1.

It remains to show

$$\text{Hom}(E_\tau, E_{N\tau}) = N\mathbb{Z} + \mathbb{Z}\omega.$$
First observe the ideal \((N)\) in \(\mathcal{O}_K\) is contained in \(\text{Hom}(E_{\tau}, E_{N\tau})\) since
\[
N(Z + \mathbb{Z}a_1 N\tau)(Z + Z\tau) \subseteq N(Z + Z\tau) \subseteq Z + NZ\tau.
\]
Furthermore \((N)\) splits as \((N) = N \cdot \bar{N}\) where \(N = NZ + Z\omega\). Therefore
\[
N \cdot \bar{N} \subseteq \text{Hom}(E_{\tau}, E_{N\tau}) \subseteq \mathcal{O}_K,
\]
where the last containment follows because \(Z + Z\tau\) is a proper fractional \(\mathcal{O}_K\)-ideal which contains \(Z + ZN\tau\). But since \(\mathcal{O}_K\) is Noetherian, there exists a maximal order \(M\) such that
\[
N \cdot \bar{N} \subseteq \text{Hom}(E_{\tau}, E_{N\tau}) \subseteq M \subseteq \mathcal{O}_K.
\]
Therefore either \(N\) or \(\bar{N}\) is in \(M\). Whichever is contained in \(M\) is actually equal to \(M\) since they are both prime and hence maximal. But \(\text{Hom}(E_{\tau}, E_{N\tau})\) is not contained in \(N\). For example, \(\bar{\omega} \in \text{Hom}(E_{\tau}, E_{N\tau})\) but not in \(N\). Thus
\[
\text{Hom}(E_{\tau}, E_{N\tau}) \subseteq \bar{N}.
\]
Finally since the index \([\bar{N} : (N)] = N\) is prime, either \(\text{Hom}(E_{\tau}, E_{N\tau})\) is equal to \(N\) or \(\bar{N}\), but we already showed the former is impossible, hence it is the latter. \(\square\)

This also completes the proof of Proposition \(\PageIndex{1}\). \(\square\)

**Lemma 4.5.** \(\mathcal{R}_z\) is isomorphic to an order in \(B\) as \(\mathbb{Z}\)-algebras, and admits an optimal embedding of \(\mathcal{O}_L\).

**Proof.** The first part is immediate.

The embedding is given in matrix form by \(QS\) where \(S := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\). It is straightforward to check that \((QS)^2 = -N\) and \(1 + QS \notin \mathcal{R}_z\) using definition \(\PageIndex{1}2\). An embedding is optimal if it does not extend to any larger order in the quotient field, but this is immediate since \(\mathcal{O}_L\) is the maximal order in \(L\). (See \cite{Shi73a} for additional discussion of this order.) \(\square\)

The next step is to prove the order \(\mathcal{R}_z\) is maximal.

**Theorem 4.6.** \(\mathcal{R}_z\) is a maximal order.

**Proof.** It suffices to show the local order \((\mathcal{R}_z)_p\) is maximal for all primes \(p\). We do this with the following two lemmas.

**Lemma 4.7.** \((\mathcal{R}_z)_p\) is maximal for all primes \(p \neq 2\).

**Proof of Lemma.** Define \(\mathcal{R}' := \mathcal{R} \cap \text{End}(Q\tau)\) with \(Q'\) defined in \(\PageIndex{1}4\). From Lemma \(\PageIndex{1}3\) and the definition of \(\mathcal{B}'\) above it is clear that \(\mathcal{R}'\) is an order given explicitly by
\[
(4.5) \quad \mathcal{R}' = \left\{ \left( \begin{array}{cc} \alpha & \beta \\ -N\beta & \bar{\alpha} \end{array} \right) : \alpha \in \mathcal{O}_K, \beta \in \mathbb{Z} + \mathbb{Z}\omega/N \right\}.
\]

Its discriminant is \(D^2\), which can be computed using the basis
\[
(4.6) \quad u_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, u_2 := \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix}, u_3 := \begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}, u_4 := \begin{pmatrix} 0 & \omega/N \\ -\bar{\omega} & 0 \end{pmatrix}.
\]

Hence \(\mathcal{R}'\) is maximal. For \(p \nmid a\), the matrix \(A\) from \(\PageIndex{1}4\) is in \(\text{Mat}_2(\mathbb{Z}_p)\) and so gives an isomorphism \(M \rightarrow A M A^{-1}\) from \((\mathcal{R}_z)_p \rightarrow \mathcal{R}'_p\). Hence \((\mathcal{R}_z)_p\) is maximal for \(p \nmid a\).

There exists a form \(\tilde{Q} = \begin{pmatrix} 2a & 0 \\ b & 2\bar{\omega} \end{pmatrix}\) properly equivalent to \(Q\) with \(\gcd(2a, \bar{a}) = 1\) (see \cite{Cox89} p. 25,35], for example). Applying Lemma \(\PageIndex{1}3\) to the pair \(Q\) and \(\tilde{Q}\) gives \(\mathcal{R}_z \cong \mathcal{R}_{\tilde{Q}\tau}\). Hence for \(p\mid a\) we can apply the paragraph above to \(\mathcal{R}_{Q\tau}\) to conclude \((\mathcal{R}_z)_p\) is maximal. \(\square\)

**Lemma 4.8.** \((\mathcal{R}_z)_2\) is maximal.
Proof of Lemma. Note $\gcd(2a, b) = 1$ because $N$ is prime and $b$ is odd. Define $U := (\begin{smallmatrix} 1 & -by \\ -2ax & 1 \end{smallmatrix})$ and $V := (\begin{smallmatrix} y & -b \\ x & 2a \end{smallmatrix})$ where $x, y \in \mathbb{Z}$ such that $2ay + bx = 1$. Then $UQV = Q'$ where $Q'$ was defined in (4.4). Define $H := \mathbb{F}U^{-1}HU^{-1}$, $\mathcal{B} := \{ M \in \text{End}_\mathbb{Q}(X_{Q''}) : T M H = H M T \}$, and $\mathcal{R} := \mathcal{B} \cap \text{End}(X_{Q''})$. The period matrix $\Pi' := [Q'\tau, 12]$ satisfies $\Pi' = U T ^4 \tilde{V}$ where $\tilde{V} := \begin{pmatrix} V_1 \\ U \\ 0 \end{pmatrix} \in \text{Mat}_4(\mathbb{Z})$. Hence the map $M \mapsto U M U^{-1}$ from $\mathcal{B}_z \to \mathcal{R}$ is an isomorphism over $\mathbb{Z}$. Therefore $(\mathcal{R}_p) \cong \mathcal{B}_p$ for all primes $p$. We will show $\mathcal{B}_2$ is maximal.

By Lemma 4.3 the isomorphism $\mathcal{B} \cong \mathcal{B}_2$, a basis for $\mathcal{B}$ by the set $\{ A^{-1} u_i A \}$ with $A$ defined in (4.4) and $u_i$ in (4.10). Hence by above the set $\{ v_i := U A^{-1} u_i A U^{-1} \}$ gives a basis for $\mathcal{B}$ over $\mathbb{Q}$. Replace $v_i$ with $2a v_i$ for $i = 2, 3$ and $v_4$ by $2a N v_4$. Then explicitly,

$$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2a \omega & 0 \\ -N x (b_1 + 2 \omega) & 2a \omega \end{pmatrix},$$

$$v_3 = \begin{pmatrix} 2a N x & 4a^2 \\ -N (N x^2 + 1) & -2a N x \end{pmatrix}, \quad v_4 = \begin{pmatrix} 2a N x \omega & 4a^2 \omega \\ -N (N x^2 \omega + \omega) & -2a N x \omega \end{pmatrix}.$$

By Lemma 4.4 we see $v_i \in \mathcal{R}_2$, $i = 1, \ldots, 4$. To prove $\mathcal{B}_2$ is maximal we will use the elements $\{ v_i \}$ to construct a basis of $\mathcal{B}_2$ whose discriminant is a unit modulo $(\mathbb{Z}_2)^2$.

Associate any matrix $M := (m_{ij} + n_{ij} \omega) \in \text{Mat}_2(\mathbb{Q}(\omega))$ with $m_{ij}, n_{ij} \in \mathbb{Q}$ to the vector

$$\vec{v}_M := (m_{11}, n_{11}, m_{12}, n_{12}, m_{21}, n_{21}, m_{22}, n_{22}) \in \mathbb{Q}^8.$$

Denote the vector $\vec{v}_u$ by $\vec{v}_i$ for simplicity. Let $M_{\text{bas}} \in \text{Mat}_{8 \times 4}(\mathbb{Z})$ be the matrix whose $i$-th column is $\vec{v}_i$ for $i = 1, \ldots, 4$. Given $M \in \text{Mat}_2(K)$, $M \in \mathcal{B}$ if and only if

$$\vec{v}_M = M_{\text{bas}} \cdot \vec{\alpha}_M$$

for some $\vec{\alpha}_M \in \mathbb{Q}^4$. Moreover $M := (m_{ij} + n_{ij} \omega) \in \text{End}(Q')$ if and only if

$$m_{11}, n_{11}, m_{12}, N n_{12}, m_{21} - b_1 n_{21}, n_{21}, m_{22}, n_{22} \in \mathbb{Z}$$

by (4.5). Let $M_{\text{end}} \in \text{Mat}_{8 \times 8}(\mathbb{Q})$ be the matrix which describes the conditions in (4.8) so that $M \in \text{End}(Q')$ if and only if $M_{\text{end}} \cdot \vec{v}_M \in \mathbb{Z}^8$. Therefore the elements $M$ of $\mathcal{B}$ correspond precisely under (4.7) to $\vec{\alpha}_M \in \mathbb{Q}^4$ such that

$$M_{\text{end}} \cdot M_{\text{bas}} \cdot \vec{\alpha}_M \in \mathbb{Z}^8.$$

To show the discriminant of $\mathcal{B}$ is 1 mod $(\mathbb{Z}_2)^2$ amounts to finding solutions $\vec{\beta} \in \mathbb{Z}^4$ such that $M_{\text{end}} \cdot M_{\text{bas}} \cdot \vec{\beta} \equiv 0 \bmod 4$. (Then $\vec{\alpha} := \vec{\beta} / 4$ satisfies (4.9).) Three linearly independent solutions for $\vec{\alpha}$ are given by the vectors

$$\vec{\alpha}_5 := T(0, 0, 1, 0)/2, \quad \vec{\alpha}_6 := T(0, 1, 0, 1)/2, \quad \text{and} \quad \vec{\alpha}_7 := T(2, 0, 1, 0)/4.$$

Therefore $\vec{v}_i := M_{\text{bas}} \cdot \vec{\alpha}_i$ gives an element in $\mathcal{B}$ for $i = 5, 6, 7$. Consider the set $S := \{ \vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4 \}$. Observe the relations

$$\vec{v}_5 = \vec{v}_3/2, \quad \vec{v}_7 = (\vec{v}_1 + \vec{v}_5)/2, \quad \text{and} \quad \vec{v}_6 = (\vec{v}_2 + \vec{v}_4)/2.$$

These imply $\vec{v}_5$ generates $\vec{v}_3$, while $\vec{v}_1$ and $\vec{v}_7$ generate $\vec{v}_5$, and finally $\vec{v}_2$ and $\vec{v}_6$ generate $\vec{v}_1$. Accordingly, replace $\vec{v}_3$ and $\vec{v}_4$ in $S$ with $\vec{v}_6$ and $\vec{v}_7$ so that $S = \{ \vec{v}_1, \vec{v}_2, \vec{v}_6, \vec{v}_7 \}$. Now $S$ is a set of linearly independent vectors over $\mathbb{Z}$ and contained in $\mathcal{B}$, hence a basis. A computation (using PARI/GP (PARI)) of the discriminant of $\mathcal{B}$ with respect to this basis shows it is $D^2 \cdot \mathbb{N} \cdot \mathbb{a}^6$. This is a unit modulo $(\mathbb{Z}_2)^2$ since we may assume $a$ is odd. Hence $\mathcal{B}_2$ is maximal.

□
This concludes the proof that \( \mathcal{R}_z \) is a maximal order. \( \square \)

The next step is to prove \( \mathcal{R}_z \) is the right order of an explicit ideal in \( B \). We first recall a result of Pacetti which constructs Siegel points from certain ideals of \( B \).

5. Split-CM points and right orders in \( B \)

In this section we identify \( \mathcal{R}_z \) with an explicit right order in \( B \). Let \( \mathcal{M} \) be a maximal order of \( B \) such that there exists \( u \in \mathcal{M} \) with \( u^2 = D \). (Such an order must exist by Eichler’s mass formula). Two left \( \mathcal{M} \)-ideals \( I \) and \( I' \) are in the same class if there exists \( b \in B^\times \) such that \( I = Ib \). The number \( n \) of left \( \mathcal{M} \)-ideal classes is finite and independent of the choice of maximal order \( \mathcal{M} \). Let \( J \) be the set of \( n \) left \( \mathcal{M} \)-ideal classes, and recall \( \mathcal{R} \) is the set of conjugacy classes of maximal orders in \( B \). (Equivalently, \( \mathcal{R} \) is the set of conjugacy classes of right orders with respect to \( \mathcal{M} \), taken without repetition.) The cardinality \( t \) of \( \mathcal{R} \) is less than or equal to \( n \) and is called the type number.

Recall \( B \cong (D, -N) \mathbb{Q} \) and let \( 1, u, v, uv \) be a basis for \( B \) where \( u^2 = D \), \( v^2 = -N \), and \( uv = -vu \). Define the \( \mathbb{Z} \)-module

\[
I_z := \left\{ \left( \frac{b_1 - u}{2a_1N} \right) av, \left( \frac{b_1 - u}{2a_1N} \right) \left( \frac{N + bv}{2} \right), \frac{b - v}{2}, -a \right\}_\mathbb{Z}.
\]

It is proven in [Pac05, p. 369-372] that \( I_z \) is a left ideal for a maximal order \( \mathcal{M}_{\mathbb{A}[\mathbb{Q}]} \) which is independent of the class representative of \( \mathcal{N} \) and of the form \( Q \), and contains the element \( u \). Let \( R_z \) denote the right order of \( I_z \). It is maximal because \( \mathcal{M}_{\mathbb{A}[\mathbb{Q}]} \) is maximal.

We will show that the right order \( R_z \) has a natural identification with the maximal order \( \mathcal{R}_z \). To do this, we recall a result of [Pac05] which associates ideals of \( B \) to Siegel points. Namely, let \( (I_R, R) \) be a pair consisting of a left \( \mathcal{M} \)-ideal \( I_R \) with maximal right order \( R \). Define the 4-dimensional real vector space \( V := B \otimes \mathbb{Q} \mathbb{R} \), so that \( V/I_R \) is a real torus. The linear map

\[
J : V \to V
\]

\[
x \mapsto \frac{u}{\sqrt{|D|}} \cdot x
\]

induces a complex structure on \( V \). Hence the data \( (V/I_R, J) \) determines a 2-dimensional complex torus. Define a map \( \mathcal{E}_R : V \times V \to \mathbb{R} \) by

\[
\mathcal{E}_R(x, y) := \text{Tr}(u^{-1}x \overline{y})/\mathcal{N}(I_R),
\]

where \( \mathcal{N}(I_R) \) is the norm of the ideal \( I_R \) and the ‘bar’ denotes conjugation in \( B \). It is straightforward to check that \( \mathcal{E}_R \) is alternating, satisfies \( \mathcal{E}_R(Jx, Jy) = \mathcal{E}_R(x, y) \) for all \( x, y \in V \), is integral on \( I_R \), and that the form \( \mathcal{H}_R : V \times V \to \mathbb{C} \) defined by

\[
\mathcal{H}_R(x, y) := \mathcal{E}_R(Jx, y) + i\mathcal{E}_R(x, y), \quad x, y \in V
\]

is positive definite (see [Pac05] for details). Thus \( \mathcal{E}_R \) is a Riemann form and so there exists a symplectic basis \( \{x_1, x_2, y_1, y_2\} \) of \( I_R \) with respect to \( \mathcal{E}_R \). The matrix \( E_R \) of \( \mathcal{E}_R \) with respect to this basis has determinant

\[
\det(E_R) = \mathcal{N}(I_R)^{-1} \mathcal{N}(u)^{-2} \text{disc}(I_R),
\]

where we have used the fact that \( \text{disc}(I_R) = (\text{det}(u, u_j))_{ij} \) for any basis \( \{u_1, \ldots, u_4\} \) of \( I_R \). But the fact that \( R \) is maximal implies \( \text{disc}(I_R) = D^2 \mathcal{N}(I_R)^4 \) [Piz80], [Pac05, Proposition 32], hence \( \det(E_R) = 1 \). This implies \( \mathcal{E}_R \) is of type 1, its matrix is \( E_R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), and \( \mathcal{H}_R \) is a principal positive definite Hermitian form.
The conclusion is that the data \((I_R, J, E_R)\) determines a Siegel point in \(\mathfrak{h}_2/Sp_4(\mathbb{Z})\). The action of a \(\gamma \in Sp_4(\mathbb{Z})\) on \((I_R, J, E_R)\) is given as a \(\mathbb{Z}\)-linear isomorphism \(I_R \rightarrow \gamma(I_R)\), which sends \(J \rightarrow \gamma^{-1} \circ J \circ \gamma\), and \(E_R \rightarrow \mathcal{E}_R \circ \gamma\).

Left \(\mathcal{M}\)-ideals with the same right order class determine equivalent Siegel points under this construction [Pac05, p. 364]. In other words, there is a well-defined map \(\mathcal{R} \rightarrow \mathfrak{h}_2/Sp_4(\mathbb{Z})\).

This can be seen as follows. Let \(I\) and \(I'\) be two left \(\mathcal{M}\)-ideals with the same right order class \([R]\). Assume first that they are equivalent, that is, \(I = I'b\) for some \(b \in B^\times\). Then multiplication on the right by \(b\) determines a \(\mathbb{Z}\)-linear isomorphism
\[
\gamma : I \rightarrow I',
\]
\[x \mapsto x \cdot b.
\]
Furthermore
\[
E(\gamma(x), \gamma(y)) = \frac{\text{Tr}(u^{-1}x \cdot (y \cdot b))}{\mathcal{N}(I)} = E(x, y) \cdot \frac{\mathcal{N}(b)}{\mathcal{N}(I)} = E'(x, y),
\]
and since \(J\) is a multiplication on the left, and \(b\) on the right, clearly \(\gamma^{-1} \circ J \circ \gamma = J\). Therefore \((I, J, E) \sim (I', J, E')\) for \(I \sim I'\). Now suppose \(I\) and \(I'\) are not equivalent. Then \(uI\) has the same left order and right order class as \(I\) but is not equivalent to \(I\) (see Lemmas 6.7 and 6.9 below). Since there are at most two classes of left \(\mathcal{M}\)-ideals with the same right order class, it must be that \(uI \sim I' \sim uIu^{-1}\). It is straightforward to check that the map from \(I\) to \(uIu^{-1}\) via conjugation by \(u\) gives \((I, J, E) \sim (uIu^{-1}, J, E)\) and so by the above case, \((I, J, E) \sim (I', J, E')\).

The ideal \(I_z\) in (5.1) corresponds to the Siegel point \(z\) under this construction. This is left as an exercise in [Pac05] but can be seen as follows. Let \(\{x_1, x_2, y_1, y_2\}\) denote the basis, taken in order, of \(I_z\) given in (5.1). A straightforward calculation done by Pacetti shows \(\{x_1, x_2, y_1, y_2\}\) is symplectic with respect to \(\mathcal{E}\), and of principal type. Then \(\{y_1, y_2\}\) is a basis for the complex vector space \((V, J)\), and the period matrix for the complex torus \((V/I_z, J)\) is the coefficient matrix of the basis of \(\{x_1, x_2, y_1, y_2\}\) in terms of \(\{y_1, y_2\}\). It suffices to show this period matrix is \(\Pi_z := [z, 1_2]\). Thus one needs to verify
\[
x_1 = 2a\tilde{\tau}y_1 + b\tilde{\tau}y_2
\]
\[
x_2 = b\tilde{\tau}y_1 + 2c\tilde{\tau}y_2,
\]
where \(\tilde{\tau} := \frac{-b_1 + \sqrt{|D|}}{2a_1N}\) is given by the complex multiplication \(J\). This is a simple calculation using the relations \(D = b_1^2 - 4a_1c_1N\) and \(-N = b^2 - 4ac\).

Note this construction determines an isomorphism \(\sigma : I_z \rightarrow L_z\) by
\[
x_1 \mapsto \begin{pmatrix} 2a \\ b \end{pmatrix} \tau, \quad x_2 \mapsto \begin{pmatrix} b \\ 2c \end{pmatrix} \tau, \quad y_1 \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad y_2 \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]
which maps \(J \mapsto i\). In particular \(\mathcal{H}_{\mathfrak{R}_z}(x, y) = \mathcal{H}_{\mathfrak{R}_z}(\sigma(x), \sigma(y))\) for all \(x, y \in I_z\).

The elements of \(R_z\) and \(\mathfrak{R}_z\) can now be related as follows. Any \(b \in R_z\) preserves \(I_z\) (on the right) as well as the complex structure \(J\) and hence defines an endomorphism \(f_b\) of \(X_z\). Likewise, any \(M \in \mathfrak{R}_z\) defines an endomorphism \(f_M\) of the torus \(X_z\) by definition. We claim these rings give the same endomorphisms of \(X_z\):

Proposition 5.1. As endomorphisms, \(R_z\) is identified with \(\mathfrak{R}_z\).
Proof. Suppose \( f_b \in \text{End}(X_z) \) for some \( b \in R_z \). To show \( f_b \) comes from \( \mathcal{R}_z \), it suffices to show \( \rho_\tau(f_b) \) preserves \( H_z \big|_{L_z \times L_z} \). Equivalently by the map \( \sigma \) it suffices to show
\[
\mathcal{H}_{R_z}(x \cdot b, y) = \mathcal{H}_{R_z}(x, y \cdot b').
\]
But this is immediate since \( \text{Tr}(u^{-1}(xb)y) = \text{Tr}(u^{-1}x(\bar{b}y)) = \text{Tr}(u^{-1}x(\bar{yb})) \) and \( \bar{b} = b' \in B \). Therefore as endomorphisms \( R_z \) is contained in \( \mathcal{R}_z \). Conversely any \( f_M \in \text{End}(X_z) \) for \( M \in \mathcal{R}_z \) defines a linear map from \( I_z \) to itself which commutes with the complex structure \( J \), hence corresponds to an element in \( R_z \).

**Corollary 5.2.** \( \mathcal{R}_z \) is isomorphic to the maximal right order \( R_z \) in the quaternion algebra \( \mathcal{B} \), and this map sends \( \frac{1+QS}{2} \mapsto \frac{1+rv}{2} \).

**Proof.** The first part follows immediately from the proposition. Regarding the embedding, the rational representation in \( \text{Mat}_4(\mathbb{Z}) \) of the endomorphism \( \frac{1+QS}{2} \in R_z \) is
\[
\begin{pmatrix}
\frac{b+1}{2} & c & 0 & 0 \\
-a & \frac{1-b}{2} & 0 & 0 \\
0 & 0 & \frac{1-b}{2} & a \\
0 & 0 & -c & \frac{b+1}{2}
\end{pmatrix}.
\]
Its action on the basis \( x_1, x_2, y_1, y_2 \) of \( I_z \) shows immediately that it is the linear transformation given by multiplication on the right by \( \frac{1+rv}{2} \). \( \square \)

### 6. Formula for the Central Value \( L(\psi_N, 1) \)

In this section we prove Theorems 3.2, 3.3 and 3.6.

**Proof of Theorems 3.2 and 3.3.** Fix \( \{a\} \in \text{Cl}(\mathcal{O}_K), N \subset \mathcal{O}_K \) a prime ideal of norm \( N, \tau := \tau_a N \).
Throughout the rest of this section, fix \( z := Q\tau \) and \( z' := Q'\tau \) where \( Q, Q' \) are binary quadratic forms of discriminant \(-N\). Define
\[
\Upsilon_1 : \{Q\tau : [Q] \in \text{Cl}(-N)\} / \text{Sp}_4(\mathbb{Z}) \rightarrow \mathcal{R}_N
\]
\[
[Q] \mapsto [R_{Q\tau}].
\]
Given an \( R_{Q\tau} \), let \( \phi_Q : \mathcal{O}_L \hookrightarrow R_{Q\tau} \) be the optimal embedding defined in Lemma 4.5 and Corollary 5.2. Define a second map
\[
\Upsilon_2 : \text{Cl}(-N) \rightarrow \Phi_8 / -
\]
\[
[Q] \mapsto [\phi_Q : \mathcal{O}_L \hookrightarrow R_{Q\tau}].
\]
We will start by showing that the maps \( \Upsilon_1 \) and \( \Upsilon_2 \) are well-defined. First note \( \Upsilon_1 \) is injective: if \( R_{Q\tau} \sim R_{Q'\tau} \) in \( B \), then we saw in the last section that Pacetti’s map \( \mathcal{R} \rightarrow \mathcal{h}_2 / \text{Sp}_4(\mathbb{Z}) \) sends \( R_{Q\tau} \rightarrow Q\tau \). After proving the maps are well-defined, we will prove \( \Upsilon_2 \) is a bijection and independent of the choice of representative \( a \) of \( \{a\} \). This will simultaneously prove Theorems 3.3 and 3.2.

**Lemma 6.1.** If \( z \sim z' \) in \( \mathcal{h}_2 / \Gamma_\theta \), then \( R_z \sim R_{z'} \) in \( \mathcal{R} \).

**Remark 6.2.** Note that if \( Q \sim Q' \) with \( Q = T AQ'A \) for some \( A \in SL_2(\mathbb{Z}) \), then \( Q\tau \sim Q'\tau \) as Siegel points via the matrix \( \begin{pmatrix} \tau A & 0 \\ 0 & A^{-1} \end{pmatrix} \in \Gamma_\theta \).

**Proof.** Recall \( z \sim z' \) in \( \mathcal{h}_2 / \text{Sp}_4(\mathbb{Z}) \) if and only if the abelian varieties \((X_z, H_z)\) and \((X_{z'}, H_{z'})\) are isomorphic. Write \( X, X', H, H' \) for \( X_z, X_{z'}, H_z, H_{z'} \), respectively. Suppose \( f : X \rightarrow X' \) is
an isomorphism of \((X, H)\) with \((X', H')\), so that \(H'(f(x), f(y)) = H(x, y)\) for all \(x, y \in \mathbb{C}^2\). We claim the isomorphism

\begin{equation}
\text{End}(X) \rightarrow \text{End}(X')
\end{equation}

\(\alpha \mapsto f \circ \alpha \circ f^{-1}\)

induces an isomorphism of \(\mathcal{R}_z\) and \(\mathcal{R}_{z'}\). This follows immediately from the calculation

\[
H'(f \circ \alpha \circ f^{-1}(x), y) = H(\alpha(f^{-1}(x)), f^{-1}(y))
\]

\[
= H(f^{-1}(x), \alpha'(f^{-1}(y)))
\]

\[
= H'(x, f(\alpha'(f^{-1}(y))))
\]

\[
= H'(x, (f \circ \alpha \circ f^{-1})').
\]

The last equality follows because, as a matrix, \(\rho_\alpha(f) = \rho_\alpha(f)^{-1}\det(\rho_\alpha(f))\) and so the determinants in \((f \circ \alpha \circ f^{-1})'\) cancel out. Therefore \(\mathcal{R}_{z'} = f \circ \mathcal{R}_z \circ f^{-1}\) and so by Proposition 5.1 \(R_z \sim R_{z'}\) in \(\mathcal{B}\).

**Lemma 6.3.** If \(Q \sim Q'\) in \(\text{Cl}(-N)\), then the corresponding optimal embeddings \(\frac{v+1}{2} \rightarrow R_z\) and \(\frac{v+1}{2} \rightarrow R_{z'}\) are equivalent.

**Proof.** Suppose \(Q \sim Q'\) with \(Q' = AQ^TA\) for some \(A \in \text{SL}_2(\mathbb{Z})\). Then by Lemma 4.3 the map \(\mathcal{R}_z \rightarrow \mathcal{R}_{z'}\) by \(M \mapsto \text{AMA}^{-1}\) is a \(\mathbb{Z}\)-algebra isomorphism, and extends to a \(\mathbb{Q}\)-algebra isomorphism from \(\mathcal{B} \rightarrow \mathcal{B}'\). In particular it sends \(QS \mapsto A(QS)A^{-1} = Q'S\). By Corollary 5.22 this induces a \(\mathbb{Z}\)-algebra isomorphism of \(R_z \rightarrow R_{z'}\) which sends \(v\) to \(v\), and extends to a \(\mathbb{Q}\)-algebra automorphism of \(B\). Hence by the Skolem-Noether theorem, the map \(R_z \rightarrow R_{z'}\) must be conjugation by some unit of \(B\).

We now turn to proving \(\Upsilon_2\) is a bijection. The following six lemmas will be needed to prove \(\Upsilon_2\) is injective. Let \(\mathcal{O}\) denote the ideal in \(L\) which corresponds to \(Q\).

**Lemma 6.4.**

\(I_z \cong \bar{\mathcal{O}} \oplus \bar{\mathcal{O}}\)

as right \(\mathcal{O}_L\)-modules.

**Proof of Lemma.** Define \(v_1 := x_1, v_2 := x_2, v_3 := y_1, v_4 := -y_2\) where \(x_i, y_j\) is the basis of \(I_z\) defined in Section 4. The \(\{v_i\}\) also form a basis for \(I_z\). The map \(f : I_z \rightarrow \bar{\mathcal{O}} \oplus \bar{\mathcal{O}}\) defined by

\[
v_1 \mapsto (a, 0)
\]

\[
v_2 \mapsto \left(\frac{b - \sqrt{-N}}{2}, 0\right)
\]

\[
v_4 \mapsto (0, a)
\]

\[
v_3 \mapsto \left(0, \frac{b - \sqrt{-N}}{2}\right)
\]

and extended \(\mathbb{Z}\)-linearly is an isomorphism of \(\mathbb{Z}\)-modules. To show it is an \(\mathcal{O}_L\)-module isomorphism, it suffices to show

\[
f\left(v_i \left(\frac{b + v}{2}\right)\right) = f(v_i) \left(\frac{b + \sqrt{-N}}{2}\right)
\]

for all \(i = 1, 2, 3, 4\).

For this, use the identities:

\[
v_1 \left(\frac{b + v}{2}\right) = bv_1 - av_2
\]

\[
v_2 \left(\frac{b + v}{2}\right) = cv_1
\]

\[
v_3 \left(\frac{b + v}{2}\right) = cv_4
\]

\[
v_4 \left(\frac{b + v}{2}\right) = -av_3 + bv_4.
\]
Lemma 6.5. Suppose $S := I_x$ where $x \in B^\times$ commutes with $\frac{p+1}{2}$. Then

$$S \cong \bar{Q} \oplus \bar{Q},$$

as right $\mathcal{O}_L$-modules.

Proof of Lemma. By Lemma 6.4 and the hypotheses on $x$, the composition from $S \to \bar{Q} \oplus \bar{Q}$ given by $g(v, x) := f(v)$ is an isomorphism of $\mathcal{O}_L$-modules.

Lemma 6.6. Suppose $\bar{Q} \oplus \bar{Q} \cong \bar{Q}' \oplus \bar{Q}'$ as right $\mathcal{O}_L$-modules, and $h(-N)$ is odd. Then

$$Q \sim Q'$$

in $\text{Cl}(-N)$.

Proof of Lemma. By a classical theorem of Steinitz [Mil71, Theorem 1.6], $\bar{Q} \oplus \bar{Q} \cong \bar{Q}' \oplus \bar{Q}'$ as right $\mathcal{O}_L$-modules if and only if $[\bar{Q}']^2 = [\bar{Q}]^2$ as classes in the ideal class group of $\mathcal{O}_L$. This is if and only if $[\bar{Q}'/\bar{Q}]^2 = [\text{id}]$ where $\text{id}$ is the identity class. But since the class number $h(-N)$ is odd, this implies $[Q] = [Q']$ in $\text{Cl}(-N)$.

The next three lemmas we need are general results for quaternion algebras. Assume for Lemmas 6.7, 6.8, and 6.9 below that $B$ is a quaternion algebra ramified precisely at $\infty$ and a prime $p$. In addition, assume $M$ and $R$ are maximal orders and there exists $u \in M$ such that $u^2 = -p$.

Lemma 6.7.

$$uMu^{-1} = M.$$

Proof. This is clear locally at primes $q \neq p$ because $u^{-1} = -u/p$. This is also clear locally at $p$ because there is a unique maximal order in the division algebra $B_p$ (see [MR03, Theorem 6.4.1, p.208] or [Vig80] for example).

Lemma 6.8. Suppose $I, I'$ are left $M$-ideals with right order $R$. In addition assume $R$ admits an embedding of a ring of integers $\mathcal{O}$ of some imaginary quadratic field. Set $J := I(I')^{-1}$. Then

$$JI' \cong I'$$

as right $\mathcal{O}$-modules.

Proof. First note $J$ is a bilateral $M$-ideal. Since $u \in M$, $uM = Mu$ by Lemma 6.7 and so is a principal $M$-ideal of norm $p$. Hence it is the unique integral bilateral $M$-ideal of norm $p$, and every bilateral $M$-ideal is equal to $uM \cdot m$ for some $m \in \mathcal{Q}$ [Eic73, Proposition 1, p. 92]. In particular, this implies the bilateral $M$-ideals are principal. Therefore $J = tM = Mt$ for some $t \in B^\times$, and the map,

$$f : I' \longrightarrow JI'$$

$$w \rightarrow tw$$

is a $\mathbb{Z}$-module isomorphism. Since the multiplication by $t$ is on the left, $f$ is an isomorphism of right $\mathcal{O}$ modules.

Lemma 6.9. Suppose $I$ is a left $M$-ideal with right order $R$. Then $uI$ is also a left $M$-ideal with right order $R$. Furthermore, any left $M$-ideal with right order $R$ is equivalent to $I$ or $uI$ (or both).
Proof. The right order of \( uI \) is clearly \( R \). The left order is \( uM u^{-1} = M \) by Lemma 6.7.

Suppose \( J \) is any left \( M \)-ideal with right order \( R \). The ideal \( I^{-1} J \) is \( R \)-bilateral, hence

\[
I^{-1} J = \mathcal{P}^i m, \quad i = 0, 1, m \in \mathbb{Q}
\]

where \( \mathcal{P} \) is the unique bilateral \( R \)-ideal of norm \( p \) [Ec73, Proposition 1, p. 92].

If \( I^{-1} J \) is principal, then \( I \sim J \). Otherwise \( i = 1 \). Then since the ideal \( I^{-1} uI \) is \( R \)-bilateral of norm \( p \), by uniqueness \( I^{-1} uI = \mathcal{P} \) and so

\[
I^{-1} J = I^{-1} uI \cdot m.
\]

Multiplying through by \( I \) we see \( J \sim uI \) as left \( M \)-ideals. \( \Box \)

Now the injectivity of \( \Upsilon_2 \) can be proven.

**Proposition 6.10.** Suppose \(( R_z, \frac{v+1}{2} ) \sim ( R_{z'}, \frac{v+1}{2} ) \). Then \( Q \sim Q' \) in \( Cl(-N) \).

**Proof.** The assumption \(( R_z, \frac{v+1}{2} ) \sim ( R_{z'}, \frac{v+1}{2} ) \) implies there exists \( x \in B^\times \) such that

\[
x^{-1} R_z x = R_{z'}
\]

and \( r \in R_z^\times \) such that

\[
(xr)^{-1} \left( \frac{v+1}{2} \right) xr = \frac{v+1}{2}.
\]

The proof is broken up into two cases.

**Case 1.** Assume \( I_z \sim I_{z'} \). Then \( I_z x \sim I_{z'} \) and they both have right order \( R_{z'} \). Set \( J := I_z x I_{z'}^{-1} \). Then \( J I_{z'} \cong I_{z'} \) as right \( \mathcal{O}_L \)-modules by Lemma 6.8. Combining with Lemma 6.4 applied to \( I_{z'} \) implies

\[
J I_{z'} \cong \mathcal{Q}' + \mathcal{Q}'
\]

as right \( \mathcal{O}_L \)-modules.

On the other hand, \( J I_{z'} = I_z x \). Since \( r \) is a unit, \( I_z x = I_z x r \), so replacing \( x \) by \( xr \) if necessary we may assume \( r = 1 \) and \( x^{-1} (\frac{v+1}{2}) x = \frac{v+1}{2} \). Lemma 6.5 applied to \( I_z x \) gives

\[
J I_{z'} \cong \mathcal{Q} + \mathcal{Q}
\]

as right \( \mathcal{O}_L \)-modules. Hence \( Q \sim Q' \) by Lemma 6.6.

**Case 2.** Assume \( I_z \not\sim I_{z'} \). For each maximal order \( R \), there can be at most two left \( M \)-ideal classes with right orders in the class \([ R ] \). Therefore since \( I_{z'} \) has right order \( R_{z'} \in \mathbb{R} \), but \( I_z \not\sim I_{z'} \), by Lemma 6.9 it must be that

\[
u I_z \sim I_{z'};
\]

note \( u I_z \) is a left \( M \)-ideal by Lemma 6.7. Then \( u I_z x \sim I_{z'} \) and they have the same right order. Let \( J := u I_z x I_{z'}^{-1} \) and use the same argument from Case 1, noting that Lemmas 6.4 and 6.5 hold with \( I_z \) replaced by \( u I_z \) since the multiplication by \( u \) is on the left. This concludes the proof that \( \Upsilon_2 \) is injective. \( \Box \)

It remains to show that \( \Upsilon_2 \) is a surjection. This follows from the fact:

**Lemma 6.11.**

\[
h(-N) = \# \Phi_R \cdot (-N).
\]
Proof of Lemma. For \([R] \in \mathcal{R}\), let \(h_R(-N)\) denote the number of optimal embeddings of \(\mathcal{O}_L\) into \(R\), modulo conjugation by \(R^\times\). Then

\[
\#\Phi_\mathcal{R}/- = \frac{1}{2} \sum_{[R] \in \mathcal{R}} h_R(-N) \text{ by definition,}
\]

\[
= h(-N) \quad \text{by Eichler’s mass formula [Gro84, (1.12)].}
\]

The last task is to prove the maps \(\Upsilon_1\) and \(\Upsilon_2\) are independent of the choice of representative \(a\) of \([a]\). In fact we will prove a slightly stronger result regarding the right orders:

**Lemma 6.12.** If \(a \sim a'\) in \(Cl(\mathcal{O}_K)\) then \(R_{Q_{a;N}} = R_{Q_{a';N}}\).

**Proof.** The hypothesis \(a \sim a'\) implies \(aN \sim a'N\). Suppose \(N\) corresponds to a form \([N, b, c]\). Then we can choose bases so that the products \(aN, a'N\) both correspond to forms with middle coefficient congruent to \(b\mod 2N\) (see [RV91, Lemma 2.3], for example). The CM-points \(\tau_{aN}, \tau_{a'N}\) are Heegner points of level \(N\) and discriminant \(D\) by construction, and by the comment above they have the same ‘root’ \(b\mod 2\) of \(\sqrt{D}\mod 4N\). Hence there exists \(M := \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_0(N)\) such that

\[
M(\tau_{aN}) = \tau_{a'N}.
\]

Set

\[
~\tilde{\tau} := \left(\begin{array}{c} \tilde{\alpha} \\ \tilde{\beta} \\ \tilde{\gamma} \\ \tilde{\delta} \end{array}\right)
\]

where

\[
\tilde{\alpha} := \alpha \cdot 1_2, \quad \tilde{\beta} := \beta \cdot Q, \quad \tilde{\gamma} := \gamma \cdot Q^{-1}, \quad \tilde{\delta} := \delta \cdot 1_2.
\]

It is shown in [AM75, p.233], for example, that \(\tilde{\tau} \in \Gamma_0 \subseteq Sp_4(\mathbb{Z})\). Therefore the relation

\[
M(\tau_{aN}) = \tau_{a'N}
\]

implies \(Q_{\tilde{\tau}_{aN}} \sim Q_{\tilde{\tau}_{a'N}}\) in \(b_2/\Gamma_0\). Let \(\tau := \tau_{aN}\) and \(\tau' := \tau_{a'N}\). An isomorphism \(f_M : X_{Q_{\tau'}} \to X_{Q_{\tau}}\) is given by

\[
T(\tilde{\gamma}Q\tau + \tilde{\delta})[Q\tau', 1_2] = [Q\tau, 1_2]T\left(\begin{array}{c} \tilde{\alpha} \\ \tilde{\beta} \\ \tilde{\gamma} \\ \tilde{\delta} \end{array}\right).
\]

The analytic representation of this isomorphism, which we will also denote by \(f_M\), is

\[
f_M = T(\tilde{\gamma}Q\tau + \tilde{\delta}) = (\gamma\tau + \delta) \cdot 1_2,
\]

where recall \(\gamma, \delta \in \mathbb{Z}\). Therefore the map

\[
\text{End}(X_{Q_{\tau'}}) \to \text{End}(X_{Q_{\tau}})\]

\[
A \mapsto f_M Af_M^{-1} = A
\]

is the identity map, hence \(\text{End}(X_{Q_{\tau'}}) = \text{End}(X_{Q_{\tau}})\). Moreover the equivalence

\[
T\bar{A}H_{Q_{\tau'}} = H_{Q_{\tau}}A^t \iff T\bar{A}Q^t = Q^tA^t
\]

implies the relation on the left hand side is independent of \(\tau\). Hence \(R_{Q_{\tau'}} = R_{Q_{\tau'}}\). \(\square\)

It follows immediately since \(R_{Q_{\tau}} = R_{Q_{\tau'}}\) that the maps \(\Upsilon_1\) and \(\Upsilon_2\) are independent of the choice of representative \(a\) of \([a]\).

This completes the proofs of Theorems 3.2 and 3.3. \(\square\)
Recall the definitions of: the normalized theta values \( \Theta_{[a,b],N} \) in (3.6), the sign function \( \varsigma_{[a,b]} \) on the embeddings in (3.8), and the twisted number of optimal embeddings \( h_{[a,b]}^\varepsilon(-N) \) in (3.9).

The \( \eta \) function in (3.6) is defined on an ideal \( a = [a, -b+\sqrt{D}] \) of \( \mathcal{O}_K \) by

\[
\eta(a) := e_{48}(a(b+3)) \cdot \eta\left(\frac{-b+\sqrt{D}}{2a}\right)
\]

where \( e_n(x) := \exp(2\pi i x/n) \) for \( n \in \mathbb{Z}, x \in \mathbb{C} \), and \( \eta(z) := e_{24}(z) \prod_{n=1}^{\infty} (1-e^{2\pi iz}) \) for \( \text{Im}(z) > 0 \) is Dedekind’s eta function. Using Shimura’s reciprocity law it can be shown that the value \( \Theta_{[a,b],N} \) is an algebraic integer (see [Pac05 Proposition 23, p. 355] and [HV97]).

We now prove Lemma 3.5.

**Proof of Lemma.** Theorem 31 of [Pac05] says that if \( Q \tau_N \sim Q' \tau_N \) in \( \mathfrak{h}_2/\Gamma_\theta \), then

\[
\Theta_{[a,b],N} = \pm \Theta_{[a,b'],N}.
\]

The lemma therefore follows immediately by this fact and Theorem 3.2. \( \square \)

We now prove Theorem 3.6.

**Proof of Theorem.** The remaining step in deriving formula (3.10) for \( L(\psi_N, 1) \) is to determine how \( \theta \) behaves on equivalent split-CM points. The following is a special case of [Pac05 Theorem 31] but we give a slightly simplified proof.

**Lemma 6.13.** Let \( Q \) and \( Q' \) be binary quadratic forms of discriminant \( -N \). If \( Q \tau \sim Q' \tau \) in \( \mathfrak{h}_2/\Gamma_\theta \), then \( \theta(Q \tau) = \pm \theta(Q' \tau) \).

**Proof of Lemma.** Suppose \( Q \tau \sim Q' \tau \) in \( \mathfrak{h}_2/\Gamma_\theta \). Then there exists \( M := \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \in \Gamma_\theta \) such that \( M(Q \tau) = Q' \tau \). Recall the functional equation for \( \theta \) is

\[
\theta(M \circ z) = \chi(M)[\det(\gamma z + \delta)]^{1/2} \theta(z), \quad M \in \Gamma_\theta
\]

where \( \chi(M) \) is a certain 8th root of unity.

Then

\[
\frac{\theta(Q' \tau)}{\theta(Q \tau)} = \chi(M)[\det(\gamma Q \tau + \delta)]^{1/2}.
\]

Applying Smith Normal Form, there exists \( U, V \in SL_2(\mathbb{Z}) \) such that \( UQV = \left( \begin{array}{cc} 1 & 0 \\ 0 & N \end{array} \right) \), and \( U', V' \in SL_2(\mathbb{Z}) \) such that \( U'Q'V' = \left( \begin{array}{cc} 1 & 0 \\ 0 & N \end{array} \right) \). These give isomorphisms \( f_U : X_{Q \tau} \to E_\tau \times E_{N \tau} \) and \( f_{U'} : X_{Q' \tau} \to E_\tau \times E_{N \tau} \) respectively. From the relation \( M(Q \tau) = Q' \tau \), we also get an isomorphism \( f_M : X_{Q' \tau} \to X_{Q \tau} \) given by

\[
T(\gamma Q \tau + \delta)[Q' \tau, \mathbf{1}_2] = [Q \tau, \mathbf{1}_2]^T \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right).
\]

Thus the composition

\[
f_U \circ f_M \circ f_{U'}^{-1} : E_\tau \times E_{N \tau} \longrightarrow E_\tau \times E_{N \tau}
\]

is an automorphism, and the determinant of its analytic representation is a unit and an algebraic integer. This last fact follows from linear algebra or can be deduced directly using Lemma 1.3.

Since \( U \) and \( U' \) are both in \( SL_2(\mathbb{Z}) \), we get \( \det(\gamma Q \tau + \delta) \in \mathcal{O}_K \). Since \( D < -4 \) this implies \( \det(\gamma Q \tau + \delta) = \pm 1 \). Therefore \( [\det(\gamma Q \tau + \delta)]^{1/2} = \pm \sqrt{\pm 1} \).

This proves \( \frac{\theta(Q' \tau)}{\theta(Q \tau)} = \pm \sqrt{\pm 1} \cdot \chi(M) \). But by Theorem 17 of [Pac05], the ratio of theta values on the left is an algebraic integer in the Hilbert class field of \( K \). Hence \( \pm \sqrt{\pm 1} \cdot \chi(M) \) is an 8th root of unity and an algebraic integer in the Hilbert class field of \( K \), which does not contain \( i \).

Therefore

\[
\pm \sqrt{\pm 1} \cdot \chi(M) = \pm 1.
\]
The theorem follows immediately from Lemma 6.13 and Theorems 3.2 and 3.3.

7. Examples

This section provides tables for two class number one examples. All calculations were done in gp/PARI [PAR08]. Given $D$ of class number one, for each admissible $N$ we compute a form $[N, b_1, c_1]$ corresponding to $N$. We set $a_N = N$ since $\text{Cl}(\mathcal{O}_K)$ is trivial, and $\tau_{a_N} := \tau_N := \frac{-b_1 + \sqrt{D}}{2N}$ to be a Heegner point of level $N$ and discriminant $D$. We choose $[1, -\frac{b_1 + \sqrt{D}}{2}]$ for a basis of $\mathcal{O}_K$ so that following definition (6.4),

$$\eta(N)\eta(\mathcal{O}_K) := e_{38}^2(N(b_1 + 3)^2) \cdot \eta\left(\frac{-b_1 + \sqrt{D}}{2N}\right) \cdot \eta\left(\frac{-b + \sqrt{D}}{2}\right).$$

From left to right, the columns of the table are $N$, the absolute values of the integers $\Theta_{[R]}$ for each $[R] \in \mathcal{R}$, the number, denoted $\#\Theta_{[R]}$, of classes $[Q] \in Cl(-N)$ with value $\pm \Theta_{[R]}$ (this equals $h_{R}(-N)$ by Theorem 3.3), and the values $h^{\varepsilon}_{[a,R]}(-N)$.

For $D = -7$, the type number is 1 and so $\#\Theta_{[R]} = \frac{1}{2}h_{R}(-N) = h(-N)$ gives the $N$-th coefficient of the weight $3/2$ level $4D$ form $\frac{1}{2} + \omega_{R} \sum_{N>0} H_D(N) q^N$ defined by the modified Hurwitz invariants $H_D(N)$ (see [Gro84, p. 120] for their definition).

| $N$ | $\Theta_{[R]}$ | $\#\Theta_{[R]}$ | $h^{\varepsilon}_{[a,R]}(-N)$ | $N$ | $\Theta_{[R]}$ | $\#\Theta_{[R]}$ | $h^{\varepsilon}_{[a,R]}(-N)$ |
|-----|----------------|------------------|-------------------------------|-----|----------------|------------------|-------------------------------|
| 11  | 1              | 1                | -1                            | 107 | 1              | 3                | -3                            |
| 23  | 1              | 3                | -1                            | 127 | 1              | 5                | 1                             |
| 43  | 1              | 1                | 1                             | 151 | 1              | 7                | -1                            |
| 67  | 1              | 1                | -1                            | 163 | 1              | 1                | 1                             |
| 71  | 1              | 7                | -3                            | 179 | 1              | 5                | -3                            |
| 79  | 1              | 5                | -1                            | 191 | 1              | 13               | -5                            |

Table 1. $D = -7$, $N \leq 200$, $t = 1.$
| $N$ | $\Theta_{[R]}$ | $\#\Theta_{[R]}$ | $h_{[a,R]}^\gamma(-N)$ | $N$ | $\Theta_{[R]}$ | $\#\Theta_{[R]}$ | $h_{[a,R]}^\gamma(-N)$ |
|-----|----------------|----------------|-----------------|-----|----------------|----------------|-----------------|
| 23  | 0              | 2             | 2               | 103 | 0              | 3             | 3               |
|     | 2              | 1             | 1               | 2   | 2              | 2             | 2               |
| 31  | 0              | 2             | 2               | 163 | 0              | 1             | 1               |
|     | 2              | 1             | -1              | 2   | 0              | 0             | 0               |
| 47  | 0              | 3             | 3               | 179 | 0              | 2             | 2               |
|     | 2              | 2             | 2               | 2   | 3             | 1             | 1               |
| 59  | 0              | 2             | 2               | 191 | 0              | 8             | 8               |
|     | 2              | 1             | -1              | 2   | 5             | 1             | 1               |
| 67  | 0              | 0             | 0               | 199 | 0              | 5             | 5               |
|     | 2              | 1             | -1              | 2   | 4             | 4             | 4               |
| 71  | 0              | 4             | 4               | 223 | 0              | 4             | 4               |
|     | 2              | 3             | -3              | 2   | 3             | 3             | 3               |

Table 2. $D = -11$, $N \leq 250$, $t = 2$.

Acknowledgments

I am deeply grateful to Fernando Rodriguez Villegas for his continuing guidance and support and for sharing his ideas that have enriched this work. I would also like to thank John Voight and Ariel Pacetti for helpful discussions on this subject. Thanks to Jeffrey Stopple for his careful reading of the manuscript. This research was partially funded by the Donald D. Harrington Endowment Fellowship and a Wendell Gordon Endowed Fellowship at the University of Texas at Austin.

References

[AM75] A N Andrianov and G N Maloletkin. Behavior of theta series of degree $n$ under modular substitutions. Mathematics of the USSR-Izvestiya, 9(2):227–241, 1975.

[BL04] Christina Birkenhake and Herbert Lange. Complex abelian varieties, volume 302 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, second edition, 2004.

[Cox89] D. A. Cox. Primes of the form $x^2 + ny^2$. A Wiley-Interscience Publication. John Wiley & Sons Inc., New York, 1989. Fermat, class field theory and complex multiplication.

[Eic66] Martin Eichler. Introduction to the theory of algebraic numbers and functions. Translated from the German by George Striker. Pure and Applied Mathematics, Vol. 23. Academic Press, New York, 1966.

[Eic73] M. Eichler. The basis problem for modular forms and the traces of the Hecke operators. In Modular functions of one variable, I (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pages 75–151. Lecture Notes in Math., Vol. 320. Springer, Berlin, 1973.

[Gro80] Benedict H. Gross. Arithmetic on elliptic curves with complex multiplication, volume 776 of Lecture Notes in Mathematics. Springer, Berlin, 1980. With an appendix by B. Mazur.

[Gro84] B. H. Gross. Heegner points on $X_0(N)$. In Modular forms (Durham, 1983), Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res., pages 87–105. Horwood, Chichester, 1984.
[Gro87] Benedict H. Gross. Heights and the special values of $L$-series. In Number theory (Montreal, Que., 1985), volume 7 of CMS Conf. Proc., pages 115–187. Amer. Math. Soc., Providence, RI, 1987.

[Hec59] Erich Hecke. Mathematische Werke. Herausgegeben im Auftrage der Akademie der Wissenschaften zu Göttingen. Vandenhoeck & Ruprecht, Göttingen, 1959.

[HI80] Ki-ichiro Hashimoto and Tomoyoshi Ibukiyama. On class numbers of positive definite binary quaternion Hermitian forms. J. Fac. Sci. Univ. Tokyo Sect. IA Math., 27(3):549–601, 1980.

[HI81] Ki-ichiro Hashimoto and Tomoyoshi Ibukiyama. On class numbers of positive definite binary quaternion Hermitian forms. II. J. Fac. Sci. Univ. Tokyo Sect. IA Math., 28(3):695–699 (1982), 1981.

[HI83] Ki-ichiro Hashimoto and Tomoyoshi Ibukiyama. On class numbers of positive definite binary quaternion Hermitian forms. III. J. Fac. Sci. Univ. Tokyo Sect. IA Math., 30(2):393–401, 1983.

[HK86] Ki-ichiro Hashimoto and Harutaka Koseki. Class numbers of positive definite binary and ternary unimodular Hermitian forms. Proc. Japan Acad. Ser. A Math. Sci., 62(8):323–326, 1986.

[HK89] Ki-ichiro Hashimoto and Harutaka Koseki. Class numbers of positive definite binary and ternary unimodular Hermitian forms. Tohoku Math. J. (2), 41(2):171–216, 1989.

[HV97] Farshid Hajir and Fernando Rodriguez Villegas. Explicit elliptic units. I. Duke Math. J., 90(3):495–521, 1997.

[Mil71] John Milnor. Introduction to algebraic $K$-theory. Princeton University Press, Princeton, N.J., 1971. Annals of Mathematics Studies, No. 72.

[MR03] Colin Maclachlan and Alan W. Reid. The arithmetic of hyperbolic 3-manifolds, volume 219 of Graduate Texts in Mathematics. Springer-Verlag, New York, 2003.

[Mum07] David Mumford. Tata lectures on theta. I. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, 2007. With the collaboration of C. Musili, M. Nori, E. Previato and M. Stillman, Reprint of the 1983 edition.

[Nar81] M. S. Narasimhan and M. V. Nori. Polarisations on an abelian variety. Proc. Indian Acad. Sci. Math. Sci., 90(2):125–128, 1981.

[Otr71] Gertrud Ottemba. Zur Theorie der hermiteschen Formen in imaginär-quadratischen Zahlkörpern. J. Reine Angew. Math., 249:1–19, 1971.

[Par05] Ariel Pacetti. A formula for the central value of certain Hecke $L$-functions. J. Number Theory, 113(2):339–379, 2005.

[PAR08] PARI Group, Bordeaux. PARI/GP, version 2.3.4, 2008. available from http://pari.math.u-bordeaux.fr.

[Piz80] Arnold Pizer. An algorithm for computing modular forms on $\Gamma_0(N)$. J. Algebra, 64(2):340–390, 1980.

[Roh80a] David E. Rohrlich. The nonvanishing of certain Hecke $L$-functions at the center of the critical strip. Duke Math. J., 47(1):223–232, 1980.

[Roh80b] David E. Rohrlich. On the $L$-functions of canonical Hecke characters of imaginary quadratic fields. Duke Math. J., 47(3):547–557, 1980.

[Roh82] David E. Rohrlich. Root numbers of Hecke $L$-functions of CM fields. Amer. J. Math., 104(3):517–543, 1982.

[RV91] Fernando Rodriguez Villegas. On the square root of special values of certain $L$-series. Invent. Math., 106(3):549–573, 1991.

[RV93] Fernando Rodriguez Villegas. Square root formulas for central values of Hecke $L$-series. II. Duke Math. J., 72(2):431–440, 1993.

[RVZ93] Fernando Rodriguez Villegas and Don Zagier. Square roots of central values of Hecke $L$-series. pages 81–99, 1993.

[Shi64] Goro Shimura. Arithmetic of unitary groups. Ann. of Math. (2), 79:369–409, 1964.

[Shi71] Goro Shimura. On elliptic curves with complex multiplication as factors of the Jacobians of modular function fields. Nagoya Math. J., 43:199–208, 1971.

[Shi73a] G. Shimura. On modular forms of half integral weight. Ann. of Math. (2), 97:440–481, 1973.

[Shi73b] Goro Shimura. On the factors of the jacobian variety of a modular function field. J. Math. Soc. Japan, 25:523–544, 1973.

[Sta82] H. M. Stark. On the transformation formula for the symplectic theta function and applications. J. Fac. Sci. Univ. Tokyo Sect. IA Math., 29(1):1–12, 1982.

[Vig80] Marie-Franco Vignéras. Arithmétique des algèbres de quaternions, volume 800 of Lecture Notes in Mathematics. Springer, Berlin, 1980.

[Wal80] J.-L. Waldspurger. Correspondance de Shimura. J. Math. Pures Appl. (9), 59(1):1–132, 1980.
[Wal81] J.-L. Waldspurger. Sur les coefficients de Fourier des formes modulaires de poids demi-entier. *J. Math. Pures Appls. (9)*, 60(4):375–484, 1981.

[Zag02] Don Zagier. Traces of singular moduli. In *Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998)*, volume 3 of *Int. Press Lect. Ser.*, pages 211–244. Int. Press, Somerville, MA, 2002.