Abstract

Two conjectures of Su and Wang (2008) concerning binomial coefficients are proved. For \( n \geq k \geq 0 \) and \( b > a > 0 \), we show that the finite sequence \( C_j = \binom{n+j}{k+j}a \) is a Pólya frequency sequence. For \( n \geq k \geq 0 \) and \( a > b > 0 \), we show that there exists an integer \( m \geq 0 \) such that the infinite sequence \( \binom{n+j}{k+j}a \), \( j = 0, 1, \ldots, \) is log-concave for \( 0 \leq j \leq m \) and log-convex for \( j \geq m \). The proof of the first result exploits the connection between total positivity and planar networks, while that of the second uses a variation-diminishing property of the Laplace transform.

1 Introduction

A nonnegative sequence \( u_i, i = 0, 1, \ldots, \) is called unimodal if \( u_0 \leq \ldots \leq u_{m-1} \leq u_m \geq u_{m+1} \geq \ldots \) for some \( m \geq 0 \). It is called log-concave (resp. log-convex), if \( u_{i+1}^2 \geq u_i u_{i+2} \) (resp. \( u_{i+1}^2 \leq u_i u_{i+2} \)) for \( i \geq 0 \). As is well-known, a log-concave sequence \( u_i \) with no internal zeros (i.e., there exist no three indices \( j < k < l \) such that \( u_j u_l \neq 0 \) but \( u_k = 0 \)) is unimodal. Moreover, if a polynomial \( \sum_{i=0}^n u_i x^i \) with nonnegative coefficients has only real zeros, then the sequence \( u_i, 0 \leq i \leq n, \) is log-concave with no internal zeros. Unimodal, log-concave and log-convex sequences arise naturally in many problems in combinatorics and elsewhere; see [4, 9] and [13–22], for example.

Unimodality properties of sequences associated with Pascal’s triangle have always been of interest ([17, 18]). Recently, Su and Wang [16] have shown that the sequence of binomial
coefficients located on a ray of Pascal’s triangle is unimodal, as conjectured by Belbachir et al. [3]. At the end of [16], the following new conjectures are proposed.

**Conjecture 1** ([16], Conjecture 2). Let \( n, k, a, b \) be integers such that \( n \geq k \geq 0 \), \( b > a > 0 \), and \( k < b \). Define \( C_j = \binom{n+j a}{k+j b} \), \( j = 0,1,\ldots \). Then the polynomial \( \sum_{j \geq 0} C_j x^j \) has only real zeros.

**Conjecture 2** ([16], Conjecture 3). Let \( n, k, a, b \) be integers such that \( n \geq k \geq 0 \) and \( a > b > 0 \). Then there exists an integer \( m \geq 0 \) such that the sequence \( \binom{n+j a}{k+j b} \), \( j = 0,1,\ldots \), is log-concave for \( 0 \leq j \leq m \) and log-convex for \( j \geq m \).

Note that in Conjecture 1, \( C_j = 0 \) if \( n + j a < k + j b \) by convention. Also, in Conjecture 2, \( m = 0 \) is permitted, in which case the sequence \( \binom{n+j a}{k+j b} \) is log-convex for all \( j \geq 0 \).

In this work we confirm Conjectures 1 and 2. Our proof of Conjecture 1 (in Section 2) follows the combinatorial approach of Gessel and Viennot [7] and Brenti [5]. In contrast, the proof of Conjecture 2 (in Section 3), which uses a variation-diminishing property of the Laplace transform, is analytic. In the process of proving Conjecture 2 we also obtain Theorem 1, which generalizes a result of Su and Wang ([16], Proposition 1) that deals with the case \( n = k = 0 \).

**Theorem 1.** Assume \( n \geq k \geq 0 \) and \( a > b > 0 \). If \( -1 \leq k - (n+1)b/a \leq 0 \) then the sequence \( \binom{n+j a}{k+j b} \), \( j = 0,1,\ldots \), is log-convex.

As usual \( \mathbb{N} \) denotes the set of positive integers and \( \mathbb{Z} \) denotes the set of integers.

## 2 Proof of Conjecture 1

Let us recall some useful terms. An infinite matrix \( W = (w_{ij})_{i,j \in \mathbb{N}} \) is called *totally positive* if every minor of \( W \) is nonnegative. A nonnegative sequence \( u_i, i = 0,1,\ldots \), is called a *Pólya frequency sequence*, or PF sequence, if \( (u_{j-i})_{i,j \in \mathbb{N}} \) \((u_i \equiv 0 \text{ if } i < 0)\) is totally positive. A finite sequence \( u_i, i = 0,\ldots,m \), is a PF sequence if the infinite sequence \( u_0,\ldots,u_m,0,0,\ldots \), is so. The following connection between finite PF sequences and polynomials with real zeros is well-known; see Karlin [10] for further notions and results concerning total positivity.

**Lemma 1.** A nonnegative sequence \( u_i, i = 0,\ldots,n \), is a PF sequence if and only if the polynomial \( \sum_{i=0}^{n} u_i x^i \) has only real zeros.
Showing that a sequence is a PF sequence by definition can be nontrivial. Nevertheless, it is possible to obtain remarkably simple proofs by exploiting the connection between total positivity and planar networks ([5, 6, 7]). In what follows, a planar network is a directed, acyclic, planar graph with no loops or multiple edges. We allow the network to be infinite, but require that it is locally finite, i.e., there exist a finite number of paths between any two vertices. In addition, our network is associated with two sets of distinguished boundary vertices, one set on each side, and numbered from top to bottom as $s_i$, $i = 1, 2, \ldots$ (the sources) and $t_i$, $i = 1, 2, \ldots$ (the sinks) respectively. (See [5] for a more precise formulation.) Define the path matrix $W = (w_{ij})$ of such a network by

$$w_{ij} = \text{the number of paths from } s_i \text{ to } t_j.$$  

The following key lemma dates back to Karlin and McGregor [11] and Lindström [8]. This and related techniques are used by Gessel and Viennot [7], Stembridge [15], Sagan [13] and Brenti [5] to tackle many combinatorial problems.

**Lemma 2.** The path matrix $W$ of a locally-finite planar network is totally positive. Specifically, any $(I, J)$ minor of $W$ is equal to the number of families of vertex-disjoint paths that connect the sources labeled by $I$ with the sinks labeled by $J$.

We now construct a planar network with a particular path matrix suitable for applying Lemmas [11, 12] Fix $n \geq k \geq 0$ and $b > a > 0$. Assume $k < b$ in addition, so that $C_j = \binom{n+a j}{k+b j}$ is indexed starting from $j = 0$. Let us specify the vertex set as

$$V = \{(i, j) : i, j \in \mathbb{Z}, \ 0 \leq i, \ 0 \leq (b-a)i + bj \leq bn - ak\}.$$  

For any two vertices $v_1 = (i_1, j_1)$ and $v_2 = (i_2, j_2)$ in $V$, we place an edge (oriented upwards and to the right) between $v_1$ and $v_2$ if $|i_1 - i_2| + |j_1 - j_2| = 1$, i.e., the edge set is inherited from the square lattice $\mathbb{Z} \times \mathbb{Z}$. Declare the vertices $s_i = (bi, (a-b)i)$, $i = 0, 1, \ldots$, as the sources, and $t_i = (k+bi, n-k+(a-b)i)$, $i = 0, 1, \ldots$, as the sinks. As an illustration, the special case $(n, k, a, b) = (4, 1, 1, 2)$ is displayed in Figure 1.

Evidently, for indices $i$, $j \geq 0$, if $j < i$ or $j > i + (n-k)/(b-a)$, then there are no paths from $s_i$ to $t_j$. If $0 \leq j - i \leq (n-k)/(b-a)$, then there are precisely $C_j-i = \binom{n+a(j-i)}{k+b(j-i)}$ such paths. By Lemma 2 the matrix $(C_{j-i})_{i,j\in\mathbb{N}}$ is totally positive; by Lemma 1 Conjecture 1 is valid.
Figure 1: The planar network corresponding to \((n, k, a, b) = (4, 1, 1, 2)\).

**Remark.** The Delannoy number \((2) D(n, k)\) counts the number of lattice paths from \((0, 0)\) to \((k, n)\) using only east, north and northeast steps; the recursion

\[
D(n, k) = D(n - 1, k) + D(n, k - 1) + D(n - 1, k - 1), \quad n, k \geq 1,
\]

holds with the initial values \(D(n, 0) = D(0, k) = 1, n, k \geq 0\). We have a result analogous to Conjecture 1 for the Delannoy numbers.

**Theorem 2.** Let \(n, k, a, b\) be integers such that \(n \geq k \geq 0, b > a > 0\) and \(k < b\). Define

\[
D_j = D(n - k + (a - b)j, k + bj).
\]

Then the polynomial \(\sum_{j \geq 0} D_j x^j\) has only real zeros.

Indeed, we can modify the planar network in the proof of Conjecture 1 by adding all the edges from \((i, j)\) to \((i + 1, j + 1)\). This new network then has \((D_{j-i})_{i,j \in \mathbb{N}}\) as its path matrix, and Theorem 2 follows from Lemmas 2 and 1 as before.

It would be interesting to know whether results similar to Conjecture 1 and Theorem 2 hold for the Stirling numbers of either kind, the Eulerian numbers, or their \(q\)-analogues. Many matrices associated with these classical numbers are known or conjectured to be totally positive (Brenti [5]).
3 Proof of Conjecture

We shall analyze the quantity of interest as a Laplace transform. A key tool is the following variation-diminishing property of the Laplace transform; see Karlin ([10], Chapter 5) for the precise statements and ramifications.

**Lemma 3.** Let \( f(t) \) be a Borel-measurable function on \((0, \infty)\), and suppose the integral

\[
L(x) = \int_0^\infty f(t)e^{-xt} \, dt
\]

converges absolutely for every \( x \in (0, \infty) \). Then the number of sign changes of \( L(x) \) in \((0, \infty)\) is no more than the number of sign changes of \( f(t) \) in \((0, \infty)\).

Fix \( n \geq k \geq 0 \), \( a > b > 0 \), and define

\[
g(x) = \log \frac{\Gamma(n + ax + 1)}{\Gamma(k + bx + 1)\Gamma(n - k + (a - b)x + 1)}, \quad x \geq 0,
\]

where \( \Gamma \) denotes Euler’s gamma function. Letting \( \psi_1(x) = d^2 \log \Gamma(x)/dx^2 \) as usual, and using the integral representation ([1], p. 260)

\[
\psi_1(x) = \int_0^\infty \frac{te^{-xt}}{1 - e^{-t}} \, dt,
\]

we get

\[
g''(x) = \int_0^\infty \frac{t}{1 - e^{-t}} \left[ a^2 e^{-(n+ax+1)t} - b^2 e^{-(k+bx+1)t} - (a - b)^2 e^{-(n-k+(a-b)x+1)t} \right] dt
\]

\[
= \int_0^\infty a^2te^{-axt} \left[ \frac{e^{-(n+1)t}}{1 - e^{-t}} - \frac{e^{-(k+1)ta/b}}{1 - e^{-ta/b}} - \frac{e^{-(n-k+1)ta/(a-b)}}{1 - e^{-ta/(a-b)}} \right] dt,
\]

where the second step uses two separate changes of variables. For further simplification denote \( u = k - (n + 1)b/a, \ p = a/b, \) and \( q = a/(a - b) \). Note that \( 1/p + 1/q = 1 \). We obtain

\[
g''(x) = \int_0^\infty a^2te^{-axt-(n+1)t}h(t, u) \, dt
\]

with

\[
h(t, u) = \frac{1}{1 - e^{-t}} - \frac{e^{-(u+1)pt}}{1 - e^{-pt}} - \frac{e^{uqt}}{1 - e^{-qt}}.
\]

It is easy to show that \( \lim_{t \to 0} h(t, u) = 1/2 \). Also, \( h(t, u) = O(e^{(n+1)t}), \ t \to \infty, \) for fixed \( u \). By Watson’s Lemma (see [12], for example),

\[
g''(x) = \frac{a^2}{2(ax + n + 1)^2} + o(x^{-2}),
\]

\[5\]
as $x \to \infty$. This shows that $g(x)$ is asymptotically convex. Note that a discrete version of this asymptotic convexity is obtained by Su and Wang [16, Theorem 1, part iii) using a different method.

Next, we examine the number of roots of $h(t,u)$ in $t \in (0, \infty)$ for fixed $u$.

**Lemma 4.** If $-1 \leq u \leq 0$, then $h(t,u) > 0$.

**Proof.** It is easy to see that $h(t,u)$ is concave down in $u$. Thus we only need to show $h(t,u) > 0$ for $u = -1$ and $u = 0$. Let us assume $u = 0$ since the case $u = -1$ can be obtained by switching the roles of $p$ and $q$. We have

$$h(t,0) = \frac{e^{-t}}{1-e^{-t}} - \frac{e^{-pt}}{1-e^{-pt}} - \frac{e^{-qt}}{1-e^{-qt}}.$$  

Consider the function

$$f(s) = \frac{se^{-s}}{1-e^{-s}}, \quad s > 0.$$  

It is easy to show that $f(s)$ strictly decreases in $s$. Using this and $1/p + 1/q = 1$ we get

$$h(t,0) = \frac{f(t) - f(tp)}{tp} + \frac{f(t) - f(tq)}{tq} > 0. \quad \square$$

Note that Theorem 1 follows directly from Lemma 4 and expression (2).

**Lemma 5.** If $u \geq 0$ or $u \leq -1$ then $\partial h(t,u)/\partial t < 0$.

**Proof.** We may assume $u \geq 0$ since, as before, the case $u \leq -1$ can be obtained by switching the roles of $p$ and $q$. We have

$$\frac{\partial h(t,0)}{\partial t} = \frac{-e^{-t}}{(1-e^{-t})^2} + \frac{pe^{-pt}}{(1-e^{-pt})^2} + \frac{qe^{-qt}}{(1-e^{-qt})^2}.$$  

It can be shown (details omitted) that the function

$$l(s) = \frac{s^2e^{-s}}{(1-e^{-s})^2}, \quad s > 0,$$  

strictly decreases in $s$. Thus

$$\frac{\partial h(t,0)}{\partial t} = \frac{l(pt) - l(t)}{pt^2} + \frac{l(qt) - l(t)}{qt^2} < 0. \quad (4)$$

For $u > 0$, it seems hard to determine the sign of $\partial h(t,u)/\partial t$ directly. However, straightforward calculation gives

$$\frac{\partial^2 h(t,u)}{\partial t \partial u} = \frac{p \left[ 1 - pt - e^{-pt} - ptu(1 - e^{-pt}) \right]}{e^{(u+1)pt}(1 - e^{-pt})^2} + \frac{q \left[ (qt + 1)e^{-qt} - 1 - uqt(1 - e^{-qt}) \right]}{e^{-uqt}(1 - e^{-qt})^2}.$$  

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In view of the simple inequalities $1 - pt - e^{-pt} < 0$ and $(qt + 1)e^{-qt} - 1 < 0$, we have

$$\frac{\partial^2 h(t,u)}{\partial t \partial u} < 0, \quad u \geq 0. \quad (5)$$

We obtain $\partial h(t,u)/\partial t < 0$, $u \geq 0$, from (4) and (5).

Lemmas 4 and 5 imply that, for fixed $u$, $h(t,u)$ has at most one root in $t \in (0, \infty)$. Given (2) and Lemma 3 we know that $g''(x)$ changes sign at most once in $(0, \infty)$. By (3), this possible change of sign is from negative to positive as $x$ increases from 0 to $\infty$. Conjecture 2 is then proved when we restrict $x$ to be a non-negative integer.

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