PRODUCT SYSTEMS OVER ORE MONOIDS

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Abstract. We interpret the Cuntz–Pimsner covariance condition as a non-degeneracy condition for representations of product systems. We show that Cuntz–Pimsner algebras over Ore monoids are constructed through inductive limits and section algebras of Fell bundles over groups. We construct a groupoid model for the Cuntz–Pimsner algebra coming from an action of an Ore monoid on a space by topological correspondences. We characterize when this groupoid is effective or locally contracting and describe its invariant subsets and invariant measures.

1. Introduction

Let $A$ and $B$ be $C^*$-algebras. A correspondence from $A$ to $B$ is a Hilbert $B$-module $E$ with a nondegenerate $^*$-homomorphism from $A$ to the $C^*$-algebra of adjointable operators on $E$. It is called proper if the left $A$-action is by compact operators, $A \to \mathbb{K}(E)$. If $E_{AB}$ and $E_{BC}$ are correspondences from $A$ to $B$ and from $B$ to $C$, respectively, then $E_{AB} \otimes_B E_{BC}$ is a correspondence from $A$ to $C$.

A triangle of correspondences consists of three $C^*$-algebras $A$, $B$, $C$, correspondences $E_{AB}$, $E_{AC}$ and $E_{BC}$ between them, and an isomorphism of correspondences $u: E_{AB} \otimes_B E_{BC} \to E_{AC}$; that is, $u$ is a unitary operator of Hilbert $C$-modules that also intertwines the left $A$-module structures. Such triangles appear naturally if we study the correspondence bicategory of $C^*$-algebras introduced in [9].

This article started with the observation that a correspondence triangle with $A = B$ and $E_{BC} = E_{AC}$ is the same as a Cuntz–Pimsner covariant representation of the correspondence $E := E_{AB}$ by adjointable operators on $F := E_{BC} = E_{AC}$, provided $E_{AB}$ is proper. Thus we get to the Cuntz–Pimsner algebra directly, without going through the Cuntz–Toeplitz algebra.

This is limited, however, to proper correspondences and the absolute Cuntz–Pimsner algebra; that is, we cannot treat the relative Cuntz–Pimsner algebras introduced by Muhly and Solel [34] and Katsura [26]. The relative versions are most relevant if the left action map $A \to \mathbb{K}(E)$ is not faithful. Then the map from $A$ to the Cuntz–Pimsner algebra is not faithful, and the latter may even be zero.

Our observation about the Cuntz–Pimsner algebra of a single proper correspondence has great conceptual value because it exhibits these (absolute) Cuntz–Pimsner algebras as a special case of a general construction, namely, colimit in the correspondence bicategory, see [2]. Other examples of such colimits are crossed products for group and crossed module actions, inductive limit for chains of $^*$-homomorphisms, and Cuntz–Pimsner algebras for proper essential product systems.

In this article, we apply our observation on the Cuntz–Pimsner covariance condition to the case of Cuntz–Pimsner algebras for proper essential product systems over monoids. Much less is known about their structure. Following Fowler [21], they are always defined and treated through the corresponding Cuntz–Toeplitz algebra. Many articles never get farther than the Nica–Toeplitz algebra. We shall...
prove strong results about the structure of Cuntz–Pimsner algebras of proper essential product systems over Ore monoids. As for a single correspondence, we must assume the product system to consist of proper correspondences, which includes nondegeneracy of the left actions, and we can only treat absolute Cuntz–Pimsner algebras.

Let $P$ be a cancellative Ore monoid and let $G$ be its group completion. Let $A$ be a $C^*$-algebra and let $(\mathcal{E}_p)_{p \in P}$ be a proper, essential product system over $P$ with unit fibre $\mathcal{E}_1 = A$; that is, the left $A$-action on $\mathcal{E}_p$ for $p \in P$ is by a nondegenerate $\ast$-homomorphism $A \to \mathbb{K}(\mathcal{E}_p)$. The Ore conditions for $P$ ensure that the diagram formed by the $C^*$-algebras $\mathbb{K}(\mathcal{E}_p)$ for $p \in P$ with the maps $\mathbb{K}(\mathcal{E}_p) \to \mathbb{K}(\mathcal{E}_q \otimes_A \mathcal{E}_q) \cong \mathbb{K}(\mathcal{E}_{pq})$ for $q, p \in P$ is indexed by a directed set. Hence the colimit for this diagram behaves like an inductive limit; it may indeed be rewritten as an inductive limit of a chain of maps $\mathbb{K}(\mathcal{E}_p) \to \mathbb{K}(\mathcal{E}_{p+1})$ for a suitable set. Thus the colimit for this diagram is an inductive limit of maps $\mathbb{K}(\mathcal{E}_p) \to \mathbb{K}(\mathcal{E}_{p+1})$. Let $\mathcal{O}_p$ be the inductive limit of this diagram of $C^*$-algebras. We construct a Fell bundle $(\mathcal{O}_p)_{p \in G}$ over $G$ with $\mathcal{O}_1$ as its unit fibre, such that its section algebra is the Cuntz–Pimsner algebra $\mathcal{O}$ of the given product system. Thus the construction of the Cuntz–Pimsner algebra of a product system over $P$ has two steps: inductive limits and Fell bundle section algebras.

Assume now that the correspondences $\mathcal{E}_p$ are full as Hilbert $A$-modules. Then the $C^*$-algebras $\mathbb{K}(\mathcal{E}_p)$ for $p \in P$ are all Morita–Rieffel equivalent to $A$ and the Fell bundle $(\mathcal{O}_p)_{p \in G}$ is saturated. Let $\mathbb{K}$ be the $C^*$-algebra of compact operators. By the Brown–Green–Rieffel Theorem, $\mathcal{E}_p \otimes \mathbb{K} \cong A \otimes \mathbb{K}$ as a Hilbert $A \otimes \mathbb{K}$-module, so we may replace the property $\mathcal{E}_p$ by an endomorphism $\varphi_p: A \otimes \mathbb{K} \to A \otimes \mathbb{K}$. Choose a cofinal sequence $(p_i)_{i \geq 0}$ in $P$ as above with $p_0 := 1$, and let $q_i \in P$ be such that $p_i = p_{i-1} q_i$. Then $\mathcal{O}_1 \otimes \mathbb{K}$ is the inductive limit of the inductive system

$$A \otimes \mathbb{K} \xrightarrow{\varphi_{q_1}} A \otimes \mathbb{K} \xrightarrow{\varphi_{q_2}} A \otimes \mathbb{K} \xrightarrow{\varphi_{q_3}} A \otimes \mathbb{K} \to \cdots ;$$

this inductive limit carries a natural $G$-action with $G \ltimes \mathcal{O}_1 \cong \mathcal{O} \otimes \mathbb{K}$.

Thus the $K$-theory of $\mathcal{O}_1$ is an inductive limit of copies of the $K$-theory of $A$; the maps are induced by the proper correspondences $\mathcal{E}_p$ or, equivalently, the endomorphisms $\varphi_q$ of $A \otimes \mathbb{K}$. Roughly speaking, we have reduced the problem of computing the $K$-theory for Cuntz–Pimsner algebras of proper product systems over an Ore monoid $P$ to the problem of computing the $K$-theory for crossed products with the group $G$. This latter problem may be difficult, but is much studied. We cannot hope for more because crossed products for $G$-actions are special cases of Cuntz–Pimsner algebras over $P$.

Many Cuntz–Pimsner algebras are constructed from generalised dynamical systems, such as higher-rank, topological graphs. The appropriate topological analogue of a product system over $P$ is given by locally compact spaces $X$ and $M_p$ for $p \in P$ with continuous maps $r_p, s_p: M_p \to X$ and $\sigma_p: M_{pq} \xrightarrow{\sim} M_p \times_{s_p \times r_p} M_q$. We assume $r_p$ to be proper and $s_p$ to be local homeomorphisms to turn $(M_p, s_p, r_p)$ into a proper correspondence over $C_0(X)$. These form a product system over $P$ with unit fibre $C_0(X)$. The data above may be called a topological higher-rank graph over $P$; we prefer to call it an action of $P$ on $X$ by topological correspondences.

In the above situation, we construct a groupoid model for the Cuntz–Pimsner algebra of our product system. This model is a Hausdorff, locally compact, étale groupoid. We translate what it means for this groupoid to be effective, locally contracting, or minimal into the original data $(X, M_p, s_p, r_p, \sigma_{p,q})$. We also describe invariant subsets and invariant measures for the object space of our groupoid model. This gives criteria when the Cuntz–Pimsner algebra of an action by topological correspondences is simple or purely infinite and often describes its set of traces.
Our results are interesting already for the commutative Ore monoids \((\mathbb{N}^k, +)\). Several authors have considered examples of products systems over these and other commutative cancellative monoids \((\mathbb{N}^k, +)\). Commutativity seems to be a red herring: what is relevant are Ore conditions. Commutativity is hidden also in Exel’s idea in [18] to extend a semigroup action to an “interaction semigroup.” Examples in [19] show that interaction semigroups for product systems over \((\mathbb{N}^2, +)\) only exist under some commutativity assumptions about certain conditional expectations. Our approach shows that a deep study of these examples is possible without such technical commutativity assumptions.

2. The Cuntz–Pimsner covariance condition

We first reinterpret the Cuntz–Pimsner covariance condition for a single correspondence as a nondegeneracy condition.

**Definition 2.1.** A correspondence from \(A\) to \(B\) is a Hilbert \(B\)-module \(F\) with a nondegenerate left action of \(A\) by adjointable operators. A correspondence is proper if \(A\) acts by compact operators. We often write the left action multiplicatively as \(a \cdot \xi\) for \(a \in A, \xi \in F\). An isomorphism between two correspondences from \(A\) to \(B\) is a unitary operator on the underlying Hilbert \(B\)-modules that intertwines the left actions of \(A\).

**Definition 2.2.** Let \(A\) and \(B\) be \(C^*\)-algebras and let \(E\) be a correspondence from \(A\) to itself. A transformation from \((A, E)\) to \(B\) is a correspondence \(F\) from \(A\) to \(B\) with an isomorphism of correspondences \(u: E \otimes_B F \xrightarrow{\sim} F\).

This definition is a special case of standard definitions in the realm of bicategories, from which the name “transformation” is taken. It leads to a nice interpretation of Cuntz–Pimsner algebras of proper correspondences as colimits in the correspondence bicategory in [2], based on the following results. First we relate transformations to certain Toeplitz representations of correspondences. The next proposition is already implicit in [33, §5], and has also been used by other authors before.

**Proposition 2.3.** Let \(A, B_1\) and \(B_2\) be \(C^*\)-algebras. Let \(E: A \rightarrow B_1, \mathcal{F}_1: B_1 \rightarrow B_2\) and \(\mathcal{F}_2: A \rightarrow B_2\) be correspondences. Correspondence isomorphisms \(E \otimes_B \mathcal{F}_1 \rightarrow \mathcal{F}_2\) are in natural bijection with linear maps \(S: E \rightarrow B(\mathcal{F}_1, \mathcal{F}_2)\) that satisfy

1. \(S(a\xi) = aS(\xi)\) for all \(a \in A, \xi \in E\);
2. \(S(\xi_1) \cdot S(\xi_2) = (\xi_1, \xi_2)_{\eta_1}\) for all \(\xi_1, \xi_2 \in E\);
3. \(S(E) \cdot \mathcal{F}_1\) spans a dense subspace of \(\mathcal{F}_2\).

Furthermore, (2) implies

4. \(S(\xi b) = S(\xi)b\) for all \(b \in B_1\).

**Proof.** First let \(u: E \otimes_B \mathcal{F}_1 \rightarrow \mathcal{F}_2\) be an isomorphism of correspondences. Define \(S(\xi)(\eta) := u(\xi \otimes \eta)\) for \(\xi \in E, \eta \in \mathcal{F}_1\). For fixed \(\xi\), this is an adjointable operator \(S(\xi): \mathcal{F}_1 \rightarrow \mathcal{F}_2\) because \(u\) and the operator \(\mathcal{F}_1 \rightarrow E \otimes_B \mathcal{F}_1, \eta \mapsto \xi \otimes \eta\) are adjointable. This map \(S\) clearly satisfies (1) since \(u\) is isometric,

\[
\langle \eta_1, S(\xi_1) \cdot S(\xi_2) \eta_2 \rangle = \langle S(\xi_1) \eta_1, S(\xi_2) \eta_2 \rangle = \langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, (\xi_1, \xi_2)_{\eta_1} \rangle
\]

for all \(\xi_1, \xi_2 \in E, \eta_1, \eta_2 \in \mathcal{F}_1\). This is equivalent to (2). Since \(u\) is unitary, it has dense range, which gives (3).

Conversely, let \(S: E \rightarrow B(\mathcal{F}_1, \mathcal{F}_2)\) be given. Define \(u\) on the algebraic tensor product of \(E\) and \(\mathcal{F}_1\) by linear extension of \(u(\xi \otimes \eta) := S(\xi)(\eta)\). Condition (2) ensures that this is an isometry and hence extends to the completion \(E \otimes_B \mathcal{F}_1\). Hence \(S\) satisfies \(S(\xi b)(\eta) = S(\xi)(b\eta)\) for all \(\xi \in E, \eta \in \mathcal{F}_1\), which is equivalent to (4). Condition (1) says that \(u\) is \(A\)-linear, and (3) says that it has dense range.
We now relate nondegeneracy to the Cuntz–Pimsner covariance condition: 

Definition 2.4. If \( F_1 = F_2 \), then a map \( S \) satisfying [1] and [2] is called a representation of \( E \). It is called nondegenerate if it also satisfies [3].

By Proposition 2.3, a transformation from \((A, E)\) to \( B \) is equivalent to a correspondence \( F : A \to B \) with a nondegenerate representation of \( E \) by operators on \( F \). We now relate nondegeneracy to the Cuntz–Pimsner covariance condition:

Proposition 2.5. Nondegenerate representations of a correspondence \( E \) are Cuntz–Pimsner covariant. The converse holds if \( E \) is proper.

Proof. Let \( A, B_1 \) and \( B_2 \) be \( C^* \)-algebras and let \( E : A \to B_1, F_1 : B_1 \to B_2 \) and \( F_2 : A \to B_2 \) be correspondences as in Proposition 2.3. The left actions of \( A \) in our correspondences are nondegenerate \( * \)-homomorphisms

\[
\varphi_E : A \to B(E), \quad \varphi_{F_2} : A \to B(F_2).
\]

Let \( u : E \otimes B_1 F_1 \to F_2 \) be an isomorphism of correspondences. The map

\[
\vartheta : B(E) \to B(F_2), \quad T \mapsto u(T \otimes 1)u^*,
\]

is a strictly continuous, unital \( * \)-homomorphism. It satisfies \( \vartheta \circ \varphi_E = \varphi_{F_2} \) because \( u \) intertwines the left actions of \( A \). If \( \xi_1, \xi_2 \in E \) and \( [\xi_1] \langle \xi_2 \rangle \) is the corresponding rank-one compact operator on \( E \), then

\[
\vartheta([\xi_1] \langle \xi_2 \rangle) = S(\xi_1)S(\xi_2)^*.
\]

This formula still defines a (possibly degenerate) \( * \)-homomorphism \( \vartheta : \mathbb{K}(E) \to B(F_2) \) for any representation \( S : E \to B(F_1, F_2) \), see [36, p. 202].

Definition 2.6. A representation \( S \) is Cuntz–Pimsner covariant if \( \vartheta(\varphi_E(a)) = \varphi_{F_2}(a) \) for all \( a \in A \) with \( \varphi_E(a) \in \mathbb{K}(E) \).

By Proposition 2.3, a nondegenerate representation comes from an isomorphism of correspondences. We have already seen \( \vartheta(\varphi_E(a)) = \varphi_{F_2}(a) \) for all \( a \in A \) in that case. So nondegenerate representations are Cuntz–Pimsner covariant.

Conversely, let \( S \) be Cuntz–Pimsner covariant and assume that \( E \) is proper, that is, \( \varphi_E(A) \subseteq \mathbb{K}(E) \). Let \( (X) \) denote the closed linear span of \( X \). We have

\[
\langle S(E)F_1 \rangle \supseteq \langle S(E)S(E)^*F_2 \rangle = \langle \vartheta(\mathbb{K}(E))F_2 \rangle \supseteq \langle \vartheta(\varphi_E(A))F_2 \rangle = \langle \varphi_{F_2}(A)F_2 \rangle = \langle F_2 \rangle
\]

because \( \varphi_{F_2} \) is nondegenerate. Thus \( S \) is nondegenerate.

For a proper correspondence \( E \), we may now reformulate the universal property that defines its Cuntz–Pimsner algebra \( O_E \): it is the universal \( C^* \)-algebra for nondegenerate representations of \( E \). Equivalently, \( O_E \) is the universal target for transformations from \((A, E)\) to \( C^* \)-algebras. The Cuntz–Pimsner algebra comes with a nondegenerate \( * \)-homomorphism \( \varphi_0 : A \to O_E \) and a representation \( S_0 : E \to O_E \), which is Cuntz–Pimsner covariant and thus nondegenerate. This is equivalent to a transformation from \((A, E)\) to \( O_E \); the underlying correspondence is \( O_E \) itself as a Hilbert \( O_E \)-module, with \( A \) acting via \( \varphi_0 \). The isomorphism \( u_0 : E \otimes_A O_E \cong O_E \) is the unitary that corresponds to \( S_0 \) by Proposition 2.3.

The transformation \((O_E, u_0)\) has the following universal property: if \((F, u)\) is another transformation from \((A, E)\) to a \( C^* \)-algebra \( B \), then there is a unique representation \( \psi : O_E \to B(F) \) for which \( u = u_0 \otimes \psi \text{id}_F \). Conversely, a representation \( \psi : O_E \to B(F) \) provides a unitary \( u = u_0 \otimes \psi \text{id}_F \) from \( E \otimes_A F \cong E \otimes A O_E \otimes_{O_E} F \) to \( F \cong O_E \otimes_{O_E} F \). The pair \((F, \psi)\) is the same as a correspondence from \( O_E \) to \( B \). Thus transformations from \((A, E)\) to \( B \) are equivalent to correspondences from \( O_E \) to \( B \).
What happens for a representation $S \colon \mathcal{E} \to \mathbb{B}(\mathcal{F})$ that does not satisfy the Cuntz–Pimsner covariance condition? The construction in the proof of Proposition 2.8 still gives a map $u \colon \mathcal{E} \otimes_A \mathcal{F} \to \mathcal{F}$, which is an $A, \mathcal{B}$-bimodule map and isometric for the $B$-valued inner product. But this isometry $u$ need not be unitary, even not adjointable. Thus allowing all Toeplitz representations replaces the unitary in the definition of a transformation by a possibly non-adjointable isometry.

**Example 2.7.** What goes wrong for a non-proper correspondence? This is shown by the simplest case, $A = \mathbb{C}$ and $\mathcal{E} = \ell^2(\mathbb{N})$. In this case, no non-zero element of $A$ acts by a compact operator, so there is no difference between the Cuntz–Pimsner and the Cuntz–Toeplitz algebras. A correspondence from $A$ to $B$ is the same as a Hilbert $B$-module. The Cuntz–Pimsner algebra $\mathcal{O}_E$ is the famous Cuntz algebra $\mathcal{O}_\infty$.

The identity map on $\mathcal{O}_\infty$ corresponds to a Cuntz–Pimsner covariant representation $S_0 \colon \ell^2(\mathbb{N}) \to \mathcal{O}_\infty$, which maps the basis vector $\delta_i$ to the generating isometry $S_i$. The induced $^\ast$-homomorphism $\mathbb{K}(\ell^2(\mathbb{N})) \to \mathcal{O}_\infty$ is degenerate, however, because $\mathcal{O}_\infty$ is unital. It corresponds to the isometry of Hilbert $\mathcal{O}_\infty$-modules $\ell^2(\mathbb{N}) \otimes \mathcal{O}_\infty \hookrightarrow \mathcal{O}_\infty$, $E_{ij} \otimes x \mapsto S_i x S_j^\ast$. If this were adjointable, its range would be of the form $p \mathcal{O}_\infty$ for a projection $p \in \mathcal{O}_\infty$ because $\mathcal{O}_\infty$ is unital. Then $[1] + p = p$ in $K_0(\mathcal{O}_\infty)$ because $\mathcal{O}_\infty \otimes \ell^2(\mathbb{N}) \otimes \mathcal{O}_\infty \cong \ell^2(\mathbb{N}) \otimes \mathcal{O}_\infty$, giving $[1] = 0$ in $K_0(\mathcal{O}_\infty)$, which is false.

For a relative Cuntz–Pimsner algebra, we only ask the Cuntz–Pimsner covariance condition on a certain ideal $K \triangleleft A$ that acts on $\mathcal{E}$ by compact operators (see Katsura’s [26, Definition 3.4]). We may reformulate this as a partial nondegeneracy condition:

**Proposition 2.8.** Let $A$ and $B$ be $C^\ast$-algebras, let $\mathcal{E}$ and $\mathcal{F}$ be correspondences from $A$ to $A$ and from $A$ to $B$, respectively. Let $K$ be an ideal in $A$ that acts on $\mathcal{E}$ by compact operators. A representation $S \colon \mathcal{E} \to \mathbb{B}(\mathcal{F})$ satisfies the Cuntz–Pimsner covariance condition on $K$ if and only if $K \cdot S(\mathcal{E}) \mathcal{F} = K \cdot \mathcal{F}$. Equivalently, the isometry $\mathcal{E} \otimes_A \mathcal{F} \to \mathcal{F}$ induced by $S$ restricts to an isomorphism of correspondences $K \mathcal{E} \otimes_A \mathcal{F} \to K \mathcal{F}$.

**Proof.** Proposition 2.3 says that an isomorphism $\mathcal{K} \mathcal{E} \otimes_A \mathcal{F} \to K \mathcal{F}$ is equivalent to a nondegenerate representation $\mathcal{K} \mathcal{E} \to \mathbb{B}(K \mathcal{F})$. Now apply Proposition 2.3 to the correspondences $\mathcal{K} \mathcal{E} : K \to A$, $\mathcal{F} : A \to B$, and $K \mathcal{F} : K \to B$, so substitute $K, A, B, \mathcal{E}, \mathcal{F}, K \mathcal{F}$ for $A, B_1, B_2, \mathcal{E}, \mathcal{F}_1, \mathcal{F}_2$. Since we assume $K$ to act by compact operators on $\mathcal{E}$, the correspondence $\mathcal{K} \mathcal{E} : K \to A$ is always proper. So the nondegeneracy condition $\mathcal{K} \mathcal{E} \cdot \mathcal{F} = K \mathcal{F}$ is equivalent to the Cuntz–Pimsner covariance condition for the restriction of the left action of $K$ to $K \mathcal{F}$. That is, $\vartheta(\varphi \mathcal{E}(k)) \mathcal{F} = \varphi \mathcal{F}(k) \mathcal{E}$ for all $k \in K$ and $\mathcal{E} \in K \mathcal{F}$, with $\vartheta : \mathcal{K} \mathcal{E} \to \mathbb{B}(K \mathcal{F})$ as in the proof of Proposition 2.3. It remains to show that this equality for all $\mathcal{E} \in K \mathcal{F}$ implies the same equality for all $\mathcal{E} \in \mathcal{F}$: the latter is the usual coisometry condition for the ideal $K$. Let $T_k := \vartheta(\varphi \mathcal{E}(k)) \mathcal{F} = \varphi \mathcal{F}(k)$ for $k \in K$. Both $T_k$ and $T_k^\ast = T_k$, map $\mathcal{F}$ to $K \mathcal{F} = K \mathcal{S}(\mathcal{E}) \mathcal{F}$, and they vanish on $K \mathcal{F}$ by the above computation. Therefore, $\langle T_k \mathcal{E}, T_k \mathcal{F} \rangle = \langle T_k \mathcal{E}, T_k \mathcal{F} \rangle = 0$ for all $\mathcal{E} \in \mathcal{F}$. \hfill $\square$

### 3. Cuntz–Pimsner Algebras of Product Systems over Ore Monoids

Let $P$ be a monoid. Product systems over $P$ were introduced by Fowler [21]. The following data is equivalent to a product system in Fowler’s sense with the mild extra condition that each fibre be an essential left module over the unit fibre:

- a $C^\ast$-algebra $A$;
- correspondences $\mathcal{E}_p$ from $A$ to itself for all $p \in P \setminus \{1\}$;
- isomorphisms of correspondences $\mu_{p,q} : \mathcal{E}_p \otimes_A \mathcal{E}_q \to \mathcal{E}_{pq}$ for all $p, q \in P \setminus \{1\}$, which are required to be associative, that is,

$$\mu_{p,q} \circ (\text{id} \otimes_A \mu_{q,r}) = \mu_{pq,r} \circ (\mu_{p,q} \otimes_A \text{id}) : \mathcal{E}_p \otimes_A \mathcal{E}_q \otimes_A \mathcal{E}_r \to \mathcal{E}_{pqr}.$$
here we let \( E_1 = A \), and we let \( \mu_{1,q} \) and \( \mu_{p,1} \) be the isomorphisms \( A \otimes_A E_q \cong E_q \) and \( E_p \otimes_A A \cong E_p \) from the left and right \( A \)-module structures, respectively; this is needed to write down \( \mu_{p,q} \) if \( p \cdot q = 1 \) and to formulate the associativity condition for \( E_p \otimes_A E_q \otimes_A E_r \rightarrow E_{pqr} \) if \( p \cdot q = 1 \) or \( q \cdot r = 1 \).

Our main theorems will only hold if all correspondences \( E_p \) are proper. Then we speak of a proper product system over \( P \).

**Definition 3.1.** Let \((A, E_p, \mu_{p,q})\) be a product system over \( P \). A transformation from it to a \( C^* \)-algebra \( B \) consists of a correspondence \( \mathcal{F} \) from \( A \) to \( B \) and isomorphisms of correspondences \( V_p : E_p \otimes_A \mathcal{F} \rightarrow \mathcal{F} \) for \( p \in P \setminus \{1\} \), such that for all \( p,q \in P \setminus \{1\} \), the following diagram of isomorphisms commutes:

\[
\begin{array}{c}
E_p \otimes_A E_q \otimes_A \mathcal{F} \\
\mu_{p,q} \otimes_A \text{id}_\mathcal{F}
\end{array}
\begin{array}{c}
\Rightarrow \mathcal{F} \\
V_{pq}
\end{array}
\]

We let \( V_1 \) be the canonical isomorphism \( A \otimes_A \mathcal{F} \cong \mathcal{F} \) and use this in (3.2) if \( p \cdot q = 1 \).

By Proposition 2.3 each isomorphism \( V_p \) corresponds to a nondegenerate representation \( S_p : E_p \rightarrow \mathcal{B}(\mathcal{F}) \) of the correspondence \( E_p \). By convention, \( S_1 : \mathcal{E}_1 = A \rightarrow \mathcal{B}(\mathcal{F}) \) is the representation of \( A \) that is part of the correspondence \( \mathcal{F} \). Equation (3.2) means that both maps around the square agree on all monomials \( \xi_p \otimes \xi_q \otimes \eta \in E_p \otimes_A E_q \otimes_A \mathcal{F} \). This amounts to the condition \( S_p(\xi_p) \cdot S_q(\xi_q) = S_{pq}(\mu_{p,q}(\xi_p \otimes \xi_q)) \) for all \( \xi_p \in E_p, \xi_q \in E_q \), which is standard for representations of product systems over \( \mathbb{P}^P \).

Example 2.7 shows that we cannot expect enough transformations to exist unless our product system is proper. We assume this from now on. By Proposition 2.5 the nondegeneracy of the representations \( S_p \) is equivalent to the Cuntz–Pimsner covariance condition for all of them. Hence the universal property that defines the Cuntz–Pimsner algebras gives a natural bijection between correspondences from it to a \( C^* \)-algebra \( B \) and transformations from the product system to \( B \); this bijection leaves the underlying Hilbert module \( \mathcal{F} \) unchanged.

A transformation \((\mathcal{F}, V_p)\) gives unital, strictly continuous \( * \)-homomorphisms

\[ \vartheta_p : \mathcal{B}(E_p) \rightarrow \mathcal{B}(E_p \otimes_A \mathcal{F}) \xrightarrow{\cong} \mathcal{B}(\mathcal{F}), \quad T \mapsto V_p(T \otimes_A \text{id}_\mathcal{F})V_p^*, \]

for all \( p \in P \). Similarly, the isomorphism \( \mu_{p,q} : E_p \otimes_A E_q \rightarrow E_{pq} \) induces a nondegenerate \( * \)-homomorphism

\[ \varphi_{p,q} : \mathcal{K}(E_p) \rightarrow \mathcal{K}(E_{pq}), \quad T \mapsto \mu_{p,q}(T \otimes_A \text{id}_{E_q})\mu_{p,q}^*, \]

Since \( E_q \) is proper, \( \varphi_{p,q}(\mathcal{K}(E_p)) \) is contained in \( \mathcal{K}(E_{pq}) \). The commuting diagram (3.2) gives \( \vartheta_{pq} \circ \varphi_{p,q} = \vartheta_p \) for all \( p, q \in P \).

This situation invites us to take a colimit (or inductive limit) of the \( C^* \)-algebras \( \mathcal{K}(E_p) \) along the maps \( \varphi_{p,q} \). More precisely, let \( \mathcal{C}_P \) be the category with object set \( P \) and arrow set \( P \times P \), where \((p,q)\) is an arrow from \( p \) to \( pq \), and where \((pq,r) \cdot (p,q) := (pq,r)\) for all \( p,q,r \in P \).

**Lemma 3.4.** The maps \( p \mapsto \mathcal{K}(E_p) \) and \((p,q) \mapsto \varphi_{p,q}\) form a functor from \( \mathcal{C}_P \) to the category of \( C^* \)-algebras and nondegenerate \( * \)-homomorphisms.

**Proof.** Functoriality means that \( \varphi_{pq,r} \circ \varphi_{p,q} = \varphi_{p,qr} \) for all \( p,q,r \in P \). This is equivalent to the associativity of \( \mu \).

□

Any such diagram has a colimit in the category of \( C^* \)-algebras and nondegenerate \( * \)-homomorphisms. This colimit will act nondegenerately on \( \mathcal{F} \) by its universal
property. Therefore, it is part of the Cuntz–Pimsner algebra of the product system. In general, the colimit involves amalgamated free products, which make it rather intractable. To get a well-behaved Cuntz–Pimsner algebra, we assume that \( C_P \) is a filtered category in the following sense:

**Definition 3.5** ([22] Section IX.1). A category \( C \) is filtered if it is nonempty and

(F1) for any two objects \( x, y \in C_0 \), there are an object \( z \in C_0 \) and arrows \( g \in C(x, z) \) and \( h \in C(y, z) \);

(F2) for any two parallel arrows \( g, h \in C(x, y) \), there are \( z \in C_0 \) and \( k \in C(y, z) \) with \( kg = kh \).

These conditions for \( C_P \) are equivalent to the following Ore conditions for \( P \):

(O1) for all \( x_1, x_2 \in P \), there are \( y_1, y_2 \in P \) with \( x_1y_1 = x_2y_2 \);

(O2) if \( x_1 = x_2y_2 \) for \( y_1, y_2 \in P \), then there is \( z \in P \) with \( y_1z = y_2z \).

**Definition 3.6.** We call \( P \) a right Ore monoid if it has these two properties or, equivalently, \( C_P \) is filtered. We call \( P \) a left Ore monoid if \( P^{op} \) is a right Ore monoid.

Condition [O2] follows if \( P \) has cancellation. Both hold if \( P \subseteq G \) for a group \( G \) with \( PP^{-1} = G \). Cancellative Ore monoids have already been considered by C*-algebraists; see, for instance, [29]. We know no interesting non-cancellative monoids satisfying the weaker assumption [O2] but it costs little extra effort to work in this greater generality.

**Remark 3.7.** A nondegenerate \(*\)-homomorphism \( f: A \to B \) gives a proper correspondence \( \mathcal{E}_f \) from \( A \) to \( B \): take \( \mathcal{E}_f = B \) with \( A \) acting through \( f \). For two composable nondegenerate \(*\)-homomorphisms, we have a natural isomorphism \( \mathcal{E}_f \otimes_B \mathcal{E}_g \cong \mathcal{E}_{gf} \). This change in the order of products means that product systems over \( P \) should be viewed as actions of \( P^{op} \) by correspondences. Thus the left Ore condition is needed in [29] to study actions of \( P \) by endomorphisms, while we need the right Ore conditions to study product systems over \( P \).

**Example 3.8.** The cancellative, commutative monoid \( (\mathbb{N}^k, +) \) is clearly Ore. We may also leave out finitely many elements of \( \mathbb{N}^k \); if the result is still a monoid, then this monoid is Ore because it contains all “sufficiently large” elements of \( \mathbb{N}^k \), and in condition (O1) we may arrange for \( x_1, x_2 \) to be as large as we need.

**Example 3.9.** The matrices of the form
\[
h(a, b, c) := \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}
\]
for \( a, b, c \in \mathbb{N} \) form a noncommutative, cancellative monoid \( H_\mathbb{N} \) under matrix multiplication:

\[
h(a_1, b_1, c_1) \cdot h(a_2, b_2, c_2) = h(a_1 + a_2, b_1 + b_2, c_1 + c_2 + a_1b_2).
\]

This is an Ore monoid. To check the Ore condition [O1] pick \( h(a_1, b_1, c_1) \) and \( h(a_2, b_2, c_2) \) in \( H_\mathbb{N} \). Let
\[
a := \max(a_1, a_2), \quad b := \max(b_1, b_2), \quad c := \max(c_1 + a_1(b_1 - b_1), c_2 + a_2(b - b_2))
\]
For \( i = 1, 2 \), let \( a_i^+ := a - a_i \), \( b_i^+ := b - b_i \), and \( c_i^+ := c - c_i + a_i b_i^+ \); then
\[
h(a_1, b_1, c_1) \cdot h(a_i^+, b_i^+, c_i^+) = h(a, b, c) \text{ for } i = 1, 2,
\]
so we have found the desired common multiple. A similar formula works for the opposite monoid, so \( H_\mathbb{N} \) is both left and right Ore.
Let \( P \) be a right Ore monoid. We may construct a group out of \( P \) by taking equivalence classes of formal quotients \( pq^{-1} := (p, q) \) for \( p, q \in P \), where \(( p_1, q_1 ) \sim ( p_2, q_2 )\) if there are \( r_1, r_2 \in P \) with \(( p_1 r_1, q_1 r_1 ) = ( p_2 r_2, q_2 r_2 )\) (see also \( \text{(10)} \)). Condition \( \text{(O1)} \) implies that this relation is transitive and that products \( p_1 q_1^{-1} p_2 q_2^{-1} \) may be rewritten as \( pq^{-1} \) by finding a common multiple of \( q_1 \) and \( p_2 \): if \( q_1 r_1 = p_2 r_2 \), then

\[
p_1 q_1^{-1} p_2 q_2^{-1} = (p_1 r_1)(q_1 r_1)^{-1} \cdot (p_2 r_2)(q_2 r_2)^{-1} = (p_1 r_1)(q_2 r_2)^{-1}.
\]

Hence we define the multiplication by \([p_1, q_1] \cdot [p_2, q_2] := [p_1 r_1, q_2 r_2]\) for \( r_1, r_2 \in P \) with \( q_1 r_1 = p_2 r_2 \). The conditions \( \text{(O1)} \) and \( \text{(O2)} \) imply that this is a well-defined group structure on \( G := P/\sim \).

An inductive limit in the usual sense is the same as a colimit over the category associated to the poset \(( \mathbb{N}, \leq )\), which is easily seen to be filtered. Colimits over general filtered categories behave very much like inductive limits. This is well-known to category theorists. For the operator algebraists, we now assume that \( P \) is countable, so that \( C_P \) is countable. We may replace any countable filtered colimit by an inductive limit over \(( \mathbb{N}, \leq )\):

**Lemma 3.10.** Let \( C \) be a countable filtered category. Then there is a sequence of objects \((x_n)_{n \in \mathbb{N}}\) and maps \( f_n \in C(x_{n-1}, x_n) \) such that for any object \( y \) of \( C \) there is \( n \in \mathbb{N} \) and an arrow \( y \to x_n \). Furthermore, if \( y \to x_n \) and \( y \to x_m \) are two such arrows, they become equal by composing with \( f_{N-1} \circ \cdots \circ f_n : x_n \to x_{n+1} \to \cdots \to x_N \) and \( f_{N-1} \circ \cdots \circ f_m : x_m \to x_{m+1} \to \cdots \to x_N \) for sufficiently large \( N \).

Such a sequence of objects and maps is called cofinal or final. More precisely, the functor \(( \mathbb{N}, \subseteq ) \to C \) given by the objects \( x_n \) and the maps \( f_n \) is called final in \( \text{[32]} \).

**Proof.** It is shown in \( \text{[5]} \) that any filtered category receives a cofinal functor from a directed (partially ordered) set. A partially ordered set is viewed as a category by putting a unique arrow \( x \to y \) if \( x \leq y \), and no arrow otherwise. A category is of this form if and only if for any two objects there is at most one arrow between them. To simplify the proof, we first use \( \text{[3]} \) to reduce to a countable, directed set.

The category \( C_P \) comes from a directed set if and only if \( P \) has cancellation.

Let \((y_n)_{n \in \mathbb{N}}\) be an enumeration of the objects of \( C \). We construct \( x_n \) for \( n \in \mathbb{N} \) inductively so that it receives maps from \( y_1, \ldots, y_n \). We start with \( x_0 = y_0 \). Assume \( x_i \) and \( f_i \) have been constructed for \( i < n \). Since \( C \) is filtered, there is an object \( x_n \) that receives maps from \( y_n \) and \( x_{n-1} \). Let \( f_n \) be the arrow \( x_{n-1} \to x_n \). Since already \( x_{n-1} \) receives maps from \( y_i \) for \( i < n \), so does \( x_n \) by composing with \( f_n \). Thus every object \( y \) has a map to some \( x_n \). Our simplifying assumption makes the second part of the lemma trivial. \( \square \)

We now describe the colimit of the inductive system on \( C_P \) given by the \( \text{C}^*\)-algebras \( \mathbb{K}(E_p) \) for \( p \in P \) and the maps \( \varphi_{p, q} \) for \( p, q \in P \) defined in \( \text{(3.3)} \).

We first do this quickly in the countable case. Then Lemma 3.10 allows us to choose a cofinal functor \(( \mathbb{N}, \subseteq ) \to C_P \), that is, we get sequences \((p_n)_{n \in \mathbb{N}}\) and \((q_n)_{n \in \mathbb{N}}\) in \( P \) with \( p_{n+1} = p_n q_n \) for all \( n \in \mathbb{N} \) that is “cofinal” in \( C_P \). The \( \text{C}^*\)-algebras \( \mathbb{K}(E_{p_n}) \) and the nondegenerate *-homomorphisms \( \varphi_{p_n, q_n} : \mathbb{K}(E_{p_n}) \to \mathbb{K}(E_{p_n q_n}) = \mathbb{K}(E_{p_{n+1}}) \) form an inductive system in the usual sense. Let \( \mathcal{O}_1 \) be its inductive limit \( \text{C}^*\)-algebra. Cofinality implies that this inductive limit is also a colimit of the whole diagram on \( C_P \).

Now we give the more complicated construction without using Lemma 3.10 which also works in the uncountable case. Let

\[
\mathcal{O}_\downarrow := \bigsqcup_{p \in P} \mathbb{K}(E_p).
\]
Let $O_\sim$ be the set of equivalence classes for the equivalence relation on $O_\cup$ generated by the relations $(x, p) \sim (\varphi_{p, q}(x), pq)$ for all $p, q \in P$, $x \in \mathbb{K}(E_p)$.

**Lemma 3.11.** There are a unique $\ast$-algebra structure and $C^\ast$-seminorm on $O_\sim$ for which all maps $\mathbb{K}(E_p) \to O_\sim$ are contractive $\ast$-homo\-morphism\s. Let $O_1$ be its $C^\ast$-com\-pletion. If $(F, V_p)$ is a transformation from $(A, E_p, \mu_{p,q})$ to a $C^\ast$-algebra $B$, then the resulting maps $\vartheta_p : \mathbb{K}(E_p) \to B(F)$ factor through a unique nondegenerate $\ast$-homo\-morphism $\Theta : O_1 \to B(F)$.

**Proof.** Let $x \in \mathbb{K}(E_p)$, $y \in \mathbb{K}(E_q)$. There are $r_1, r_2 \in P$ with $pr_1 = qr_2$. Then $(x, p) \sim (\varphi_{p,r_1}(x), pr_1)$ and $(y, q) \sim (\varphi_{q,r_2}(y), qr_2)$ both belong to the $C^\ast$-algebra $\mathbb{K}(E_{pr_1}) = \mathbb{K}(E_{qr_2})$; this dictates what their sum or product should be in $O_\sim$. If we choose $r_1', r_2' \in P$ with $pr_1' = qr_2'$ instead, then we may find $m_1, m_2 \in P$ with $pr_1'm_1 = pr_2'm_2$ and hence $qr_2'm_1 = qr_2'm_2$. If not yet $r_2'm_1 = r_2m_2$, then we find $n \in P$ with $r_2'm_1n = r_2m_2n$ and replace $m_1, m_2$ by $n_1m_1, m_2n$. Similarly, we achieve $r_1'm_1 = r_2m_2$. Then multiplication with $m_2$ and $m_1$ will map our two choices of the sum or product to the same sum or product in $\mathbb{K}(E_{pr_1m_2})$, respectively. Thus the $\ast$-algebra structure on $O_\sim$ is well-defined.

A similar argument shows that any finite subset of $O_\sim$ belongs to the image of $\mathbb{K}(E_p)$ in $O_\sim$ for some $p \in P$. Since the algebraic operations are defined using those in $\mathbb{K}(E_p)$, $O_\sim$ is a $\ast$-algebra.

The kernel of the map $\mathbb{K}(E_p) \to O_\sim$ is the union of the kernels of the $\ast$-homo\-morphism\s $\varphi_{p,q} : \mathbb{K}(E_p) \to \mathbb{K}(E_q)$. Thus the image of $\mathbb{K}(E_p)$ in $O_\sim$ is the quotient by a union of closed $\ast$-ideals. We equip it with the quotient seminorm, which is a $C^\ast$-seminorm (there may be a nullspace because the union of ideals need not be closed).

All these $C^\ast$-sem\-norms on subalgebras of $O_\sim$ together are compatible with each other and thus define a $C^\ast$-seminorm on $O_\sim$. Let $O_1$ be the (Hausdorff) completion of $O_\sim$ for this $C^\ast$-seminorm. This is a $C^\ast$-algebra with $\ast$-homo\-morphism\s $\vartheta_p : \mathbb{K}(E_p) \to O_1$ for all $p \in P$ that satisfy $\vartheta_0 = \vartheta_p \circ \varphi_{p,q} = \vartheta_q$ for all $p, q \in P$.

Now take a transformation to $B$ as above. The resulting maps $\vartheta_p : \mathbb{K}(E_p) \to B(F)$ satisfy $\vartheta_0 = \varphi_{p,q} = \vartheta_p$. Hence the map $\bigcup_{p \in P} \vartheta_p : O_\sim \to B(F)$ descends to a map $f : O_1 \to B(F)$. Since all $\vartheta_p$ are $\ast$-homo\-morphism\s, so is $f$. The composite map $f \circ \vartheta_0 : \mathbb{K}(E_p) \to B(F)$ is a $\ast$-homo\-morphism, hence contractive, and it vanishes on the kernel of the map $\mathbb{K}(E_p) \to O_\sim$. Thus it vanishes on the closure of the kernel as well, so it is still contractive for the quotient norm defined by dividing out this closure. Hence the restriction of $f$ to the image of $\mathbb{K}(E_p)$ in $O_\sim$ is contractive for the canonical $C^\ast$-seminorm on $O_\sim$. This means that $f$ itself is contractive, so it extends to the $C^\ast$-com\-pletion. This gives a unique $\ast$-homo\-morphism $\Theta : O_1 \to B(F)$ with $\Theta \circ \vartheta_0 = \vartheta_p$ for all $p \in P$. \hfill $\Box$

Any functor $(p_\alpha, q_\alpha) : (N, \leq) \to C_P$ induces a $\ast$-homo\-morphism from the inductive limit $C^\ast$-algebra of the inductive system $(\mathbb{K}(E_{p_\alpha}), \varphi_{p_\alpha, q_\alpha})$ described above Lemma 3.11 to $O_1$. If the functor is cofinal, then this map is an isomorphism. Hence the simplified construction for countable $P$ gives the same $C^\ast$-algebra $O_1$.

So far, we have described only a part of the Cuntz–Pimsner algebra of the product system. For a single endomorphism, this is the fixed-point subalgebra of the canonical gauge action. We now describe the whole Cuntz–Pimsner algebra through a Fell bundle over the group completion $G$ of $P$.

Elements of $G$ are equivalence classes of formal fractions $p_1p_2^{-1}$ for $p_1, p_2 \in P$, with $p_1p_2^{-1} \sim (p_1) (p_2)^{-1}$. The fibre of the desired Fell bundle over $G$ at $1 \in G$ is the $C^\ast$-algebra $O_1$ described above. Fix $g \in G$ and let

$$R_g = \{(p_1, p_2) \in P \times P \mid p_1p_2^{-1} = g \in G\}$$
be its set of representatives. If \( p = (p_1, p_2) \in R_g \), \( q = (q_1, q_2) \in R_g \), let \( C^0_P(p, q) \) be the set of all \( h \in P \) with \( p_1 h = q_1 \) and \( p_2 h = q_2 \). These are the arrows of a category, with multiplication given by that in \( P \).

**Lemma 3.12.** If \( P \) is an Ore monoid, then the categories \( C^0_P \) for \( q \in G \) are filtered.

**Proof.** Let \( p = (p_1, p_2) \) and \( q = (q_1, q_2) \) be elements of \( R_g \). We must prove two statements. First, there should be arrows \( h: p \to r \) and \( k: q \to r \) with the same target \( r \in R_g \). Secondly, if \( h, k: p \to q \) are two parallel arrows, there is an arrow \( l: q \to r \) for some object \( r \) such that \( l \circ h = l \circ k \). Since \( p \) and \( q \) both represent \( g \in G \), there are \( h, k \in P \) with \( p_1 h = q_1 k \) and \( p_2 h = q_2 k \). Hence \( h: (p_1, p_2) \to (p_1 h, p_2 h) \) and \( k: (q_1, q_2) \to (q_1 k, q_2 k) \) have the same target, as desired. The second statement is immediate from \((O2)\) we may simply forget \( p_2 \) and \( q_2 \).

For \( (p_1, p_2) \in R_g \), let \( O_{p_1, p_2} := \mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}) \); for now, we view this as a Banach space. For \( h \in P, (p_1, p_2) \in R_g \), we define a contraction

\[ \mathcal{E}_{p_1, p_2, h}: \mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}) \to \mathbb{K}(\mathcal{E}_{p_2, h}, \mathcal{E}_{p_1 h}), \quad T \mapsto \mu_{p_1, h}(T \otimes A \text{id}_{\mathcal{E}_{p_2, h}}) \mu_{p_2, h}. \]

These maps form a functor from \( C^0_P \) to the category of Banach spaces with linear contractions. Since \( C^0_P \) is filtered by Lemma 3.12, the colimit \( O_g \) of this diagram may be constructed as in Lemma 3.11 first take the disjoint union of the Banach spaces \( O_{p_1, p_2} \) for all \( (p_1, p_2) \in R_g \); then divide out the relations given by the maps \( \mathcal{E}_{p_1, p_2, h} \); this gives a vector space, and it inherits a canonical seminorm by taking the quotient norms on the images of \( \mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}) \); finally, take the completion to get \( O_g \). If \( P \) is countable, then we may also use a cofinal sequence in \( C^0_P \) to describe the colimit as an inductive limit over \((\mathbb{N}, \leq)\).

If \( g = 1 \) is the unit element of \( G \), then there is a fully faithful functor \( C_P \to C^1_P \) mapping \( p \mapsto (p, p) \). If \( (p_1, p_2) \in R_1 \), then there are \( q, h \in P \) with \( (p_1 h, p_2 h) = (q, q) \). Thus the image of \( C_P \) in \( C^1_P \) is cofinal and the colimit of a diagram over \( C^1_P \) is the same as the colimit of its restriction to \( C_P \). Hence the construction of \( O_g \) for \( g = 1 \) gives the same \( C^* \)-algebra \( O_1 \) as our previous construction, as suggested by our notation.

If \( g_1, g_2 \in G, (p_1, p_2) \in R_{g_1}, (p_2, p_3) \in R_{g_2} \), then \( p_1 p_2^{-1} p_3^{-1} = p_1^{-1} p_2^{-1} p_3^{-1} = g_1 \cdot g_2 \), that is, \( (p_1, p_3) \in R_{g_1 g_2} \). The composition of compact operator gives a bounded bilinear map \( O_{p_1, p_2} \times O_{p_2, p_3} \to O_{p_1, p_3} \). These maps define a bounded bilinear map

\[ O_{g_1} \times O_{g_2} \to O_{g_1 g_2} \]

because for any \((p_1', p_2'), (p_2', p_3') \in R_{g_1} \times R_{g_2} \) there are \( h, k \in P \) with \( p_2 h = p_2' k \), so that the composition is defined on \( O_{p_1' h, p_2' k} \times O_{p_2' h, p_3' k} \), and these composition maps are compatible with the structure maps of the inductive limits. Similarly, taking adjoints gives maps \( O_{p_1, p_2} \to O_{p_1, p_2}, T \mapsto T^* \), for all \( p_1, p_2 \in R_g \); these maps induce an involution \( O_g \to O^{*-1}_g \). These multiplication maps and involutions on \((O_g)_{g \in G}\) give a Fell bundle over the group \( G \). The resulting \( C^* \)-algebra structure on its unit fibre \( O_1 \) is the one already described in Lemma 3.11.

**Theorem 3.13.** Let \( P \) be an Ore monoid and let \((A, \mathcal{E}_P, \mu_{p,q})\) be a proper, nondegenerate product system over \( P \). Its Cuntz–Pimsner algebra is isomorphic to the full sectional \( C^* \)-algebra of the Fell bundle \((O_g)_{g \in G}\) described above.

**Proof.** Let \( C \) denote the Cuntz–Pimsner algebra of our product system. By construction, a nondegenerate \(^*\)-homomorphism \( C \to M(B) \) for a \( C^* \)-algebra \( B \) is the same as a Cuntz–Pimsner covariant representation of our product system on \( B \) that is nondegenerate on the unit fibre \( A \). The Cuntz–Pimsner covariance condition is equivalent to the nondegeneracy condition \( \mathcal{E}_P \cdot B = B \) for all \( p \in P \) by Proposition 3.24 because we assume all \( \mathcal{E}_p \) to be proper and nondegenerate left \( A \)-modules, and the left \( A \)-action on \( B \) is nondegenerate as well.
We are going to find a natural bijection between representations of the product system with \( \mathcal{E}_p \cdot B = B \) for all \( p \in P \) and representations of the Fell bundle \((O_g)_{g \in G}\) in \( \mathcal{M}(B) \). By the universal property of the sectional \( C^*\)-algebra of a Fell bundle, this gives a natural bijection between nondegenerate \(^*\)-homomorphisms \( C \to \mathcal{M}(B) \) and \( C^*((O_g)_{g \in G}) \to \mathcal{M}(B) \), and this implies \( C \cong C^*((O_g)_{g \in G}) \).

By Proposition 3.15 a representation of the product system that is nondegenerate in the above sense is equivalent to a transformation from \((A, \mathcal{E}_p, \mu_{p,q})\) to \( B \) with underlying Hilbert \( B \)-module \( B \). We write \( \mathcal{F} = B \) to be consistent with our previous notation. We already constructed \(^*\)-homomorphisms \( \vartheta_p : \mathcal{K}(\mathcal{E}_p) \to B(\mathcal{F}) \) with \( \vartheta_{pq} \circ \varphi_{p,q} = \vartheta_p \) for all \( p, q \in P \). The same recipe gives linear contractions

\[
\vartheta_{p_1, p_2} : \mathcal{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}) \to B(\mathcal{F}), \quad T \mapsto V_{p_1}(T \otimes \text{id}_\mathcal{F})V_{p_2}^*.
\]

These satisfy \( \vartheta_{p_1 h, p_2 h} \circ \varphi_{p_1, p_2, h} = \vartheta_{p_1, p_2} \) for all \( p_1, p_2, h \in P \). Hence they induce maps \( \Theta_g : O_g \to B(\mathcal{F}) \) on the Banach space inductive limits. Routine shows that

\[
(3.14) \quad \vartheta_{p_2, p_1}(T^*) = \vartheta_{p_1, p_2}(T^*), \quad \vartheta_{p_1, p_2}(T) \circ \vartheta_{p_2, p_3}(T_2) = \vartheta_{p_1, p_3}(T \circ T_2)
\]

for all \( p_1, p_2, p_3 \in P, T \in \mathcal{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}), T_2 \in \mathcal{K}(\mathcal{E}_{p_3}, \mathcal{E}_{p_2}) \). Hence the maps \( \Theta_g \) form a representation of the Fell bundle \((O_g)_{g \in G}\).

Conversely, a representation of the Fell bundle \((O_g)_{g \in G}\) gives maps \( \mathcal{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}) \to B(\mathcal{F}) \) that satisfy (3.14). For \( p_2 = 1 \), there is a canonical isomorphism \( \mathcal{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}) \cong \mathcal{E}_{p_1} \) because \( \mathcal{E}_1 = A \). Hence the Fell bundle representation gives maps \( S_p : \mathcal{E}_p \to B(\mathcal{F}) \). Since \( A = \mathcal{K}(\mathcal{E}_1) \subseteq O_1 \), the conditions of a Fell bundle representation imply that the maps \( S_p \) form a representation of the product system. Since the maps \( \mathcal{K}(\mathcal{E}_g) \to O_1 \to B(\mathcal{F}) \) are nondegenerate, we have \( S_p(\mathcal{E}_p)F \supseteq \mathcal{K}(\mathcal{E}_p)F = F \). This gives the desired bijection between Fell bundle representations and Cuntz–Pimsner covariant representations of the product system and finishes the proof.

**Proposition 3.15.** The Fell bundle \((O_g)_{g \in G}\) is saturated if \( \mathcal{E}_p \) is a full Hilbert \( A \)-module for each \( p \in P \).

**Proof.** Let \( g \in G \) and let \( p \in P \). We want to show that the image of \( \mathcal{K}(\mathcal{E}_p) \) in \( O_1 \) is contained in the space of right inner products from \( O_g \). There is \( (p_1, p_2) \in \mathcal{R}_g \) and \( q \in P \) with \( pq = p_1 \). The image of \( \mathcal{K}(\mathcal{E}_q) \) in \( O_1 \) is contained in the image of \( \mathcal{K}(\mathcal{E}_{pq}) = \mathcal{K}(\mathcal{E}_{p_1}) \).

Since \( \mathcal{E}_{p_1} \) and \( \mathcal{E}_{p_2} \) are full, both \( \mathcal{K}(\mathcal{E}_{p_1}) \) and \( \mathcal{K}(\mathcal{E}_{p_2}) \) are Morita–Rieffel equivalent to \( A \) and hence equivalent to each other. The equivalence between them is \( \mathcal{K}(A, \mathcal{E}_{p_2}) \otimes_A \mathcal{K}(\mathcal{E}_{p_2}, A) \cong \mathcal{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}) \). Hence the latter is a full Hilbert bimodule over \( \mathcal{K}(\mathcal{E}_{p_2}) \) and \( \mathcal{K}(\mathcal{E}_{p_1}) \). Since \( \mathcal{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}) \subseteq O_g \), it follows that the right inner products from \( O_g \) give a dense subspace of \( \mathcal{K}(\mathcal{E}_{p_1}) \) in \( O_1 \). Thus the Fell bundle \((O_g)_{g \in G}\) is saturated.

**Remark 3.16.** The criterion in Proposition 3.15 is not necessary for rather trivial reasons. If the left \( A \)-actions on \( \mathcal{E}_p \) are not faithful, then it may happen that \( O_1 = 0 \). Since this has nothing to do with \( \mathcal{E}_p \) being full as a right Hilbert module, the Fell bundle \((O_g)_{g \in G}\) may be saturated although not all \( \mathcal{E}_p \) are full.

Saturated Fell bundles over a group \( G \) are interpreted as actions of \( G \) by correspondences in \([9]\). Long before, it was known that one may replace a saturated Fell bundle \((O_g)_{g \in G}\) with unit fibre \( O_1 \) by an action of \( G \) by automorphisms on a \( C^* \)-algebra \( O_1 \) that is Morita–Rieffel equivalent to \( O_1 \): this is the Packer–Raeburn Stabilisation Trick. Non-saturated Fell bundles over \( G \) are interpreted in \([8]\) as actions of \( G \) by Hilbert bimodules, that is, partial Morita–Rieffel equivalences. The analogue of the Packer–Raeburn Stabilisation Trick says that any Fell bundle, saturated or not, is equivalent to an action of \( G \) by partial \(^*\)-isomorphisms.
A saturated Fell bundle over $G$ may, of course, be restricted to a product system over $P$.

**Proposition 3.17.** A proper product system over $P$ is the restriction of a saturated Fell bundle over $G$ if and only if each $\mathcal{E}_p$ is an $A$-$A$-imprimitivity bimodule, that is, each $\mathcal{E}_p$ is a full right Hilbert $A$-module and the left action is by an isomorphism $A \cong \mathbb{K}(\mathcal{E}_p)$. The saturated Fell bundle over $G$ is unique up to isomorphism.

**Proof.** In a saturated Fell bundle over $G$, each $\mathcal{E}_p$ is an imprimitivity bimodule. Conversely, assume that $\mathcal{E}_p$ is an imprimitivity bimodule for each $p \in P$. Then the maps $\mathbb{K}(\mathcal{E}_p) \to \mathbb{K}(\mathcal{E}_{pq})$ in our inductive system are all isomorphisms, so that the inductive limit $\mathcal{O}_1$ is isomorphic to $A = \mathbb{K}(\mathcal{E}_1)$. Similarly, $\mathcal{O}_p \cong \mathcal{E}_p$ for all $p \in P$. Thus our product system is the restriction to $P$ of a Fell bundle over $G$. Since all $\mathcal{E}_p$ are assumed to be full, this Fell bundle is saturated by Proposition 3.15.

Now start with a saturated Fell bundle $(\mathcal{O}_p)_{p \in G}$, restrict it to $P$, and then go back to a Fell bundle over $G$. The maps $\mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}) \to \mathbb{K}(\mathcal{E}_{pq}, \mathcal{E}_{pq})$ are isomorphisms for all $p_1, p_2, q \in P$, so the inductive systems that give the fibres of the new Fell bundle are also constant. Thus the colimit $\mathcal{O}_p$ is canonically isomorphic to $\mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1})$ for any $(p_1, p_2) \in R_p$, and our construction of a Fell bundle from $(\mathcal{E}_p)_{p \in P}$ reproduces the original Fell bundle up to isomorphism. Hence the product system on $P$ determines the saturated Fell bundle over $G$ uniquely up to isomorphism. □

A non-saturated Fell bundle over $G$ need not give a proper product system on $P$: this requires $\mathcal{E}_p$ to be full as a left Hilbert $A$-module for each $p \in P$.

**Theorem 3.18.** If $A$ is nuclear or exact, then so is $\mathcal{O}_1$. If $A$ is nuclear and the group $G$ generated by $P$ is amenable, then the Cuntz–Pimsner algebra $\mathcal{O}$ is nuclear. If $A$ is exact and $G$ is amenable, then $\mathcal{O}$ is exact.

**Proof.** The first claim follows because $\mathcal{O}_1$ is an inductive limit of $C^*$-algebras Morita–Rieffel equivalent to $A$ and because nuclearity and exactness are hereditary under Morita–Rieffel equivalence and filtered inductive limits.

The second statement for exact groups follows from Theorem 3.13 and general results about nuclearity and exactness of Fell bundle $C^*$-algebras. First, if the group is amenable, then any Fell bundle over it has the approximation property, which implies that the full and reduced sectional $C^*$-algebras coincide (see [14]). The exactness of the reduced sectional $C^*$-algebra is proved in [13], assuming exact unit fibre and an exact group. The nuclearity of the full sectional $C^*$-algebra is proved in [14], assuming nuclear unit fibre and an amenable group. □

Next we describe the K-theory of the unit fibre $\mathcal{O}_1$ of our Fell bundle. Since $\mathcal{E}_p$ is a proper correspondence from $A$ to $A$, it gives an element $[\mathcal{E}_p] \in KK_0(A, A)$ with zero operator $F$. This gives a map

$$(\mathcal{E}_p)_*: \text{K}_*(A) \to \text{K}_*(A);$$

here $\text{K}_*(A)$ denotes the $\mathbb{Z}/2$-graded K-theory of $A$ comprising both $K_0$ and $K_1$. The Kasparov product of $[\mathcal{E}_p]$ and $[\mathcal{E}_q]$ is $[\mathcal{E}_p \otimes A \mathcal{E}_q]$ with zero operator; since the Fredholm operator is irrelevant, this is a very easy special case of the Kasparov product. The isomorphisms $\mu_{p,q}$ now show that $[\mathcal{E}_p] \otimes A [\mathcal{E}_q] = [\mathcal{E}_{pq}]$ and hence $(\mathcal{E}_q)_* \circ (\mathcal{E}_p)_* = (\mathcal{E}_{pq})_*$ for all $p, q \in P$. The order of $p$ and $q$ is changed here because $\otimes A$ is the composition product in KK in reverse order. Hence our product system over $P$ gives an action of $P$ on $\text{K}_*(A)$. We view this as a right module structure over the monoid ring $\mathbb{Z}[P]$. The group ring $\mathbb{Z}[G]$ is a left module over $\mathbb{Z}[P]$.

**Theorem 3.19.** Let $P$ be an Ore monoid and let $(A, \mathcal{E}_p, \mu_{p,q})$ give a proper, non-degenerate product system over $P$. Assume also that all $\mathcal{E}_p$ are full right Hilbert
A-modules. Then the K-theory of $\mathcal{O}_1$ is $K_*(A) \otimes_{\mathbb{Z}[P]} \mathbb{Z}[G]$. This is an isomorphism of right $\mathbb{Z}[G]$-modules for the canonical right module structure on $K_*(A) \otimes_{\mathbb{Z}[P]} \mathbb{Z}[G]$.

Lemma 3.10

We leave it to the reader interested in uncountable monoids to check that the result remains true for arbitrary filtered colimits. Hence $K_*(\mathcal{O}_1)$ is the colimit of the diagram over $\mathcal{C}_P$ that maps $p \in P$ to $K_*([\mathcal{E}_p])$ and $pq \in \mathcal{C}_P$ induces the map $K_*([\mathcal{E}_p]) \to K_*([\mathcal{E}_{pq}])$.

Since $\mathcal{E}_p$ is a full Hilbert bimodule, it gives a Morita–Rieffel equivalence from $K_*([\mathcal{E}_p])$ to $A$. This correspondence with zero operator $F$ is a cycle for $K\mathcal{K}_0([\mathcal{E}_p])$, $A$. This is a KK-equivalence: the inverse is the inverse imprimitivity bimodule $\mathcal{K}(A)$ with zero operator $F$. We use this KK-equivalence to identify $K_*(\mathcal{E}_p) \cong K_*(A)$ for all $p \in P$. Composing the maps $K_*(A) \to K_*([\mathcal{E}_p]) \to K_*([\mathcal{E}_{pq}]) \to K_*(A)$ requires composing three KK$_0$-cycles with zero operator $F$, which amounts to tensoring the underlying correspondences. Identifying $\mathcal{E}_{pq} \cong \mathcal{E}_p \otimes_A \mathcal{E}_q$ as in the definition of $\varphi_{p,q}$, we see that this composite is $[\mathcal{E}_q]$. Thus the inductive system with colimit $K_*(\mathcal{O}_1)$ is isomorphic to the inductive system with entries $K_*(A)$ at all $p \in P$, where the arrow $(p,q) : p \to pq$ in $\mathcal{C}_P$ induces the map $[\mathcal{E}_q] : K_*(A) \to K_*(A)$.

Define a diagram of left $\mathcal{Z}[P]$-modules over $\mathcal{C}_P$ by taking the free module $\mathcal{Z}[P]$ at all objects and letting $h : p \to ph$ act by $\delta_x \mapsto \delta_{xh}$ for all $x, p, h \in P$. The colimit of this diagram of modules is isomorphic to $\mathcal{Z}[G]$ by mapping $\mathcal{Z}[P] \ni \delta_x$ to $h : p \to ph$ in $\mathcal{C}_P$ for objects $p \in \mathcal{C}_P$ to $\delta_{xh} \in \mathcal{Z}[G]$. Hence $M \otimes_{\mathcal{Z}[P]} \mathcal{Z}[G]$ for a right $\mathcal{Z}[P]$-module $M$ is the colimit of the diagram over $\mathcal{C}_P$ with entries $M \otimes_{\mathcal{Z}[P]} \mathcal{Z}[P] \cong M$ and with $h : p \to ph$ acting by $m \mapsto m \cdot h$ for all $m \in M$, $p, h \in P$. Now compare this with our description of the inductive system that computes $K_*(\mathcal{O}_1)$ to get $K_*(\mathcal{O}_1) \cong K_*(A) \otimes_{\mathcal{Z}[P]} \mathcal{Z}[G]$.

Since the Hilbert modules $\mathcal{E}_p$ are full, the Fell bundle $(\mathcal{O}_g)_{g \in G}$ is saturated by Proposition 3.15. Then each $\mathcal{O}_g$ is a proper correspondence from $\mathcal{O}_1$ to itself and hence gives a class $[\mathcal{O}_g]$ in $K\mathcal{K}_0(\mathcal{O}_1, \mathcal{O}_1)$. This induces maps $[\mathcal{O}_g] : K_*(\mathcal{O}_1) \to K_*(\mathcal{O}_1)$. Since we have a saturated Fell bundle, we have $\mathcal{O}_g \otimes_{\mathcal{O}_1} \mathcal{O}_h \cong \mathcal{O}_{gh}$. Therefore, $g \mapsto [\mathcal{O}_g]$ defines a representation of $G^{\text{op}}$ on the Abelian group $K_*(\mathcal{O}_1)$; we view this as a right $\mathcal{Z}[G]$-module structure.

To describe this action, it suffices to compute, for $p \in P$, how $[\mathcal{O}_g]$ acts on the image of $K_*(\mathcal{E}_p)$ in $K_*(\mathcal{O}_1)$ under the map $K_*(\mathcal{E}_p) \to K_*(\mathcal{O}_1)$.

First choose $(p_1, p_2) \in R_g$ and then $q \in P$ with $pq = p_1$. Then $K_*(\mathcal{E}_{pq}) = K_*(\mathcal{E}_{p_1}) \otimes K_*(\mathcal{E}_{p_2}, \mathcal{E}_{p_1})$, so it suffices to describe how $[\mathcal{O}_g]$ acts on the image of $K_*(\mathcal{E}_{pq})$. The map $K_*(\mathcal{E}_{pq}) \to \mathcal{O}_g$ shows that $(\mathcal{E}_{pq})^* \cong \mathcal{E}_{pq} \otimes_{\mathcal{O}_{p_2}} \mathcal{O}_1$ as correspondences from $\mathcal{E}_{pq}$ to $\mathcal{O}_1$. Now compose these correspondences with the KK-equivalences between $\mathcal{E}_{pq}$ and $A$. Then we see that $(\mathcal{O}_g)^*$ acts on the entry $K_*(A)$ at $p_1$ in the inductive system describing $K_*(\mathcal{O}_1)$ by sending it to the same entry at $p_2$. Right multiplication by $g = p_1 p_2^{-1}$ in $K_*(A) \otimes_{\mathcal{Z}[P]} \mathcal{Z}[G]$ has the same effect. Thus the action of $G$ on $K_*(\mathcal{O}_1)$ induced by the Fell bundle corresponds to the one by right multiplication on $K_*(A) \otimes_{\mathcal{Z}[P]} \mathcal{Z}[G]$.

By the Packer–Raeburn Stabilisation Trick, there is a $G$-action by automorphisms on the stabilisation $\hat{\mathcal{O}}_1 := \mathcal{O}_1 \otimes K(L^2G)$ such that the (full) crossed product $G \times \hat{\mathcal{O}}_1$ is Morita–Rieffel equivalent to the Cuntz-Pimsner algebra $\mathcal{O}$ (this follows...
from [13 Corollary 5.5]. Thus computing the K-theory of the Cuntz–Pimsner algebra becomes a matter for the (full) Baum–Connes conjecture for \( G \) with certain coefficients.

For \( \alpha \)-T-nilpotent groups, the Baum–Connes assembly map is known to be an isomorphism for all coefficients, also for the full crossed product (see [24]). The meaning of the Baum–Connes conjecture here is that we may compute \( K_*(\mathcal{O}_H) \) by topological means from \( K_*(\mathcal{O}|_H) \), the section algebras for restrictions of \( (\mathcal{O}_g)_{g \in G} \) to all finite subgroups \( H \). These topological means may be expressed as a spectral sequence, and it can be quite hard to perform this computation in practice. At least, the results above show that the computation for a Cuntz–Pimsner algebra over \( P \) is not more difficult than in the special case of an action of \( G \) by automorphisms.

For instance, let \( P = (\mathbb{N}^k, +) \) for \( k \in \mathbb{N} \). Then \( G = \mathbb{Z}^k \), and the computation of \( K_*(\mathcal{O}_H) \) is a matter of iterating the Pimsner–Voiculescu sequence \( k \) times. Already two iterations are often impossible to do explicitly because the relevant boundary maps for the second iteration are not determined by the original data.

By the way, the iteration of the Pimsner–Voiculescu sequence that we get is different from Deaconu in [12]. This is because the Pimsner–Voiculescu maps for the second iteration are not determined by the original data.

3.1. Making left actions faithful. Let \((A, \mathcal{E}_p, \mu_{p,q})\) be a proper product system over an Ore monoid \( P \). Taking suitable quotients of \( A \) and \( \mathcal{E}_p \), we are going to construct another product system \((A', \mathcal{E}'_p, \mu'_{p,q})\) with the same nondegenerate representations and hence the same Cuntz–Pimsner algebra, such that the left actions \( \varphi'_{p,q} : A' \to \mathbb{B}(\mathcal{E}'_p) \) are injective for all \( p \in P \).

For \( p \in P \), let \( \varphi_p : A \to \mathbb{K}(\mathcal{E}_p) \) denote the left action map and let \( I_p = \text{ker} \varphi_p \). These ideals are ideals in \( A \) with \( I = \bigcup_{p \in P} I_p \). Recall the maps \( \varphi_{p,q} : \mathbb{K}(\mathcal{E}_p) \to \mathbb{K}(\mathcal{E}_{pq}) \) for \( p, q \in P \). Since \( \varphi_{p,q} \circ \varphi_p = \varphi_{pq} \), we have \( I_p \subseteq I_{pq} \) for all \( p, q \in P \). Since \( \mathcal{E}_p \) is filtered, this implies that the ideals \( I_p \) form a directed set of ideals in \( A \). Thus \( I := \bigcup_{p \in P} I_p \) is another ideal in \( A \). We let \( A' := A/I \) and \( \mathcal{E}'_p := \mathcal{E}_p \otimes_A A' \), which is canonically isomorphic to the quotient \( \mathcal{E}_p / (\mathcal{E}_p : I) \).

Lemma 3.20. The induced left action \( A \to \mathbb{K}(\mathcal{E}'_p) \) factors through \( A' \), and the isomorphism \( \mu_{p,q} : \mathcal{E}_p \otimes A \mathcal{E}_q \xrightarrow{\sim} \mathcal{E}_{pq} \) descends to an isomorphism \( \mu'_{p,q} : \mathcal{E}'_p \otimes A \mathcal{E}'_q \xrightarrow{\sim} \mathcal{E}'_{pq} \). This gives a product system \((A', \mathcal{E}'_p, \mu'_{p,q})\).  

Proof. Let \( p \in P \). To prove that the induced left \( A' \)-module structure on \( \mathcal{E}'_p \) descends to \( A' \), we must show that \( I \mathcal{E}_p \subseteq \mathcal{E}_p I \). Since \( P \) is an Ore monoid, the subset \( pP \) is cofinal in \( P \), so \( \bigcup_{q \in P} I_{pq} \) is still dense in \( I \). Thus it suffices to prove \( I_{pq} \mathcal{E}_p \subseteq \mathcal{E}_p I \) for all \( p, q \in P \). We will prove the following more precise result:

\[
I_{pq} = \{ a \in A \mid a\mathcal{E}_p \subseteq \mathcal{E}_p I_q \}.
\]

Let \( \xi \in \mathcal{E}_p \). We have \( \xi \otimes_A \eta = 0 \in \mathcal{E}_p \otimes_A \mathcal{E}_q \) for all \( \eta \in \mathcal{E}_q \) if and only if

\[
0 = \langle \xi \otimes \eta_1, \xi \otimes \eta_2 \rangle = \langle \eta_1, \varphi_{p,q}(\langle \xi, \xi \rangle_A)\eta_2 \rangle
\]

for all \( \eta_1, \eta_2 \in \mathcal{E}_q \), if and only if \( \varphi_{p,q}(\langle \xi, \xi \rangle_A) = 0 \), if and only if \( \langle \xi, \xi \rangle_A \in I_q \). We claim that this is equivalent to \( \xi \in \mathcal{E}_p : I_q \). Since \( I_q \) is an ideal, we have \( \langle \xi, \xi \rangle_A \in I_q \) for \( \xi \in \mathcal{E}_p : I_q \). Conversely, if \( \langle \xi, \xi \rangle_A \in I_q \), then the closure of \( \xi \) in \( \mathcal{E}_p \) is a Hilbert \( I_q \)-module, and thus it is nondegenerate as a right \( I_q \)-module, so that \( \xi \in \mathcal{E}_p : I_q \). Hence \( \xi \otimes_A \eta = 0 \in \mathcal{E}_p \otimes_A \mathcal{E}_q \) for all \( \eta \in \mathcal{E}_q \) if and only if \( \xi \in \mathcal{E}_p : I_q \).

Now let \( a \in A \). Then \( a\xi \in \mathcal{E}_p : I_q \) for all \( \xi \in \mathcal{E}_p \) if and only if \( a\xi \otimes_A \eta = 0 \) for all \( \xi \in \mathcal{E}_p \), \( \eta \in \mathcal{E}_q \), if and only if the left action by \( a \) vanishes on \( \mathcal{E}_p \otimes_A \mathcal{E}_q \). This
Theorem 3.22. The product system $(A', \mathcal{E}'_p, \mu'_p,q)$ has faithful left action maps $A' \to \mathbb{K}(\mathcal{E}'_p)$, and it has the same nondegenerate representations as the original system. Hence it also has the same Cuntz–Pimsner algebra.

Proof. Fix $p \in P$. An operator on $\mathcal{E}_p$ induces the zero operator on $\mathcal{E}_p / \mathcal{E}_p I_q \cong \mathcal{E}_p \otimes_A (A/I_q)$ if and only if it maps $\mathcal{E}_p$ into $\mathcal{E}_p I_q$. Thus (3.21) shows that the map $\varphi_p: A \to \mathbb{K}(\mathcal{E}_p)$ descends to an injective $^*$-homomorphism $A/I_{pq} \hookrightarrow \mathbb{K}(\mathcal{E}_p / \mathcal{E}_p I_q)$. The $C^*$-algebras $A/I_{pq}$ and $\mathbb{K}(\mathcal{E}_p / \mathcal{E}_p I_q)$ for $q \in P$ form inductive systems indexed by the filtered category $\mathcal{E}_P$, and the maps $A/I_{pq} \hookrightarrow \mathbb{K}(\mathcal{E}_p / \mathcal{E}_p I_q)$ form a morphism of inductive systems, consisting of injective maps. It follows that the induced map between the inductive limits $\lim_p A/I_{pq} = A / \bigcup_{q \in P} I_{pq} = A'$ and $\lim_p \mathbb{K}(\mathcal{E}_p / \mathcal{E}_p I_q) = \mathbb{K}(\mathcal{E}_p')$ is injective as well. That is, the left action $A' \to \mathbb{K}(\mathcal{E}_p')$ is faithful.

Now let $\vartheta': A' \to \mathbb{B}(\mathcal{F})$ and $S_p': \mathcal{E}'_p \to \mathbb{B}(\mathcal{F})$ for $p \in P$ give a nondegenerate representation of the product system $(A', \mathcal{E}'_p, \mu'_p,q)$. Composing with the quotient maps $A \to A'$ and $\mathcal{E}_p \to \mathcal{E}'_p$, then gives a nondegenerate representation of $(A, \mathcal{E}_p, \mu_p,q)$. We claim that any nondegenerate representation $(\vartheta, S_p)$ of $(A, \mathcal{E}_p, \mu_p,q)$ factors through the quotient maps $A \to A'$ and $\mathcal{E}_p \to \mathcal{E}'_p$ and thus comes from a unique representation $(\vartheta', S'_p)$. This gives a bijection on the level of nondegenerate representations and thus an isomorphism of Cuntz–Pimsner algebras because they are universal for nondegenerate representations by Proposition 2.5.

Recall the maps $\vartheta_p: \mathbb{K}(\mathcal{E}_p) \to \mathbb{B}(\mathcal{F})$ with $\vartheta_{pq} \circ \varphi_{p,q} = \vartheta_p$ for all $p,q \in P$. In particular, $\vartheta = \vartheta_p \circ \varphi_p: A \to \mathbb{B}(\mathcal{F})$, so $\vartheta$ must vanish on $I_p$. Since this holds for all $p \in P$, we get $\vartheta_{I} = 0$, so $\vartheta$ factors through the quotient map $A \to A'$. Since $S_p(\xi) = \vartheta(\xi, \xi) I$ for $\xi \in \mathcal{E}_p$ and $(\xi, \xi) \in I$ for $\xi \in \mathcal{E}_p I$, we also get $S_p(\xi) = 0$ for $\xi \in \mathcal{E}_p \cdot I$. Thus $S_p$ factors through $\mathcal{E}'_p$. □

Proposition 3.23. If the maps $A \to \mathbb{K}(\mathcal{E}_p)$ are injective for all $p \in P$, then so are the induced maps $\varphi_{p,q,t}: \mathbb{K}(\mathcal{E}_q, \mathcal{E}_p) \to \mathbb{K}(\mathcal{E}_{qt}, \mathcal{E}_{pt})$ for $p,q,t \in P$ and the maps $\mathbb{K}(\mathcal{E}_q, \mathcal{E}_p) \to \mathcal{O}$ to the Cuntz–Pimsner algebra.

Proof. We assume that $I_p = \{0\}$ for all $p \in P$. The proof of 3.21 shows that $\xi \in \mathcal{E}_p$ satisfies $\xi \otimes_A \eta = 0$ in $\mathcal{E}_p \otimes_A \mathcal{E}_q$ for all $\eta \in \mathcal{E}_q$ if and only if $\xi = 0$. Hence the maps $\varphi_{p,q,t}$ are injective. Since $\mathcal{O}_g \subseteq \mathcal{O}$ is the filtered colimit of the spaces $\mathbb{K}(\mathcal{E}_q, \mathcal{E}_p)$, this implies the same for the maps $\mathbb{K}(\mathcal{E}_q, \mathcal{E}_p) \to \mathcal{O}_g \subseteq \mathcal{O}$. □

3.2. What happens without the Ore conditions? We now consider an example of a monoid without the Ore conditions where we can, nevertheless, describe the Cuntz–Pimsner algebra by hand. Let $F_n^+$ be the free monoid on $n$ generators, $n \geq 2$. Elements in $F_n^+$ are finite words in the letters $a_1, \ldots, a_n$, including the empty word. This monoid violates the Ore conditions: there are no words $w_1, w_2 \in F_n^+$ with $a_1 w_1 = a_2 w_2$. A proper product system over $F_n^+$ is equivalent to a $C^*$-algebra $A$ with proper correspondences $E_i$ from $A$ to itself for $i = 1, \ldots, n$, without any further data or conditions: given this data, we may define $E_w$ for a word $w$ by composing the correspondences for the letters in $w$, and we use the canonical multiplication maps between them.
Proposition 3.24. Let $A$ be a C$^*$-algebra and let $E_i$ for $1 \leq i \leq n$ be proper correspondences from $A$ to $A$. Let $O_i$ be the Cuntz–Pimsner algebra of $E_i$ for $1 \leq i \leq n$. The Cuntz–Pimsner algebra of the resulting product system over $F_n^+$ is the amalgamated free product of the Cuntz–Pimsner algebras $O_i$ over $A$.

Proof. Let $D$ be another C$^*$-algebra and let $G$ be a Hilbert module over $D$. A nondegenerate representation of our product system over $F_n^+$ on $G$ is already determined by what it does on the correspondences $E_i$, and $E_i$ may act by arbitrary nondegenerate representations because $F_n^+$ is a free monoid. A nondegenerate representation of $E_i$ is equivalent to a representation of the Cuntz–Pimsner algebra $O_i$ by Proposition 2.5. Since all these representations give the same representation when we compose with the canonical map $A \to O_i$, we get a representation of the amalgamated free product of the C$^*$-algebras $O_i$ over $A$. Conversely, a representation of this free product gives nondegenerate representations of the correspondences $E_i$ and thus of $A$, and it gives the same representation on $A$ for each $i$. This data may be extended to a nondegenerate representation of the product system over $F_n^+$. □

Free products with amalgamation are, unfortunately, rather large and complicated. In particular, they are almost never nuclear or exact. Thus we view Proposition 3.24 as a negative result: it tells us that we should not expect Cuntz–Pimsner algebras for proper product systems over $F_n^+$ to have a nice structure. Standard assumptions in the theory of Cuntz–Toeplitz and Cuntz–Pimsner algebras are that the underlying semigroup be “quasi-lattice ordered” and the product system “compactly aligned,” see [21]. Both assumptions are satisfied in our case. First, if two elements in $F_n^+$ have an upper bound, they have a least upper bound. This is because two elements in $F_n^+$ only have an upper bound if one of them is a subword of the other, and then the longer of the two is a least upper bound. Secondly, any proper product system is compactly aligned (see [21, Definition 5.7]). Hence the usual assumptions do not guarantee that the Cuntz–Pimsner algebra or the Cuntz–Toeplitz algebra is a tractable object.

4. Actions of Ore monoids on spaces

Now let $X$ be a locally compact, Hausdorff space and let $A = C_0(X)$. Since any automorphism of $A$ comes from a homeomorphism on $X$, we may turn an action of a group $G$ on $A$ by automorphisms into an action of $G$ on the space $X$ and form a transformation groupoid $G \ltimes X$. The crossed product $G \ltimes C_0(X)$ is canonically isomorphic to the groupoid C$^*$-algebra of $G \ltimes X$. When is there such a groupoid model for a self-correspondence on $A$?

As a counterexample, consider a Hermitian vector bundle over $X$. It gives a proper self-correspondence from $A$ to itself by taking the Hilbert module of sections with its usual inner product and the left action by pointwise multiplication. The resulting Cuntz–Pimsner algebra is a locally trivial field of C$^*$-algebras over $X$ with Cuntz algebras as fibres. Such C$^*$-algebras are classified by Dădărlat in [11] in terms of certain cohomology groups. Unless the field of C$^*$-algebras over $X$ is particularly simple, there seems to be no natural groupoid model for it.

Therefore, we restrict attention to self-correspondences of $C_0(X)$ that are induced by topological correspondences (see [27]). We define product systems of such topological correspondences in the obvious fashion, so that they induce a product system of C$^*$-correspondences. We will build a “transformation groupoid” for a proper product system of topological correspondences and show that its groupoid C$^*$-algebra is isomorphic to the Cuntz–Pimsner algebra of the product system. Our transformation groupoid construction is similar in spirit to the boundary path groupoid of Yeend [46] for a higher-rank topological graph, that is, for the case.
We can treat all Ore monoids, and our construction is simpler because we only consider the Cuntz–Pimsner algebra and assume product systems to be proper.

A topological correspondence between two spaces $X$ and $Y$ is given by a third space $M$ with two maps $r: M \to X$ and $s: M \to Y$. We want to turn this into a $C^*$-correspondence from $C_0(X)$ to $C_0(Y)$. There are two ways to do this. First, we may assume that $s$ is a local homeomorphism; this is Katsura’s definition of a topological correspondence in [27]. Secondly, we may add extra data, namely, a family of measures $(\lambda_x)_{x \in X}$ on the fibres of $s$; this is what Muhly and Tomforde call a topological quiver in [33]. The family of measures $(\lambda_x)$ is equivalent to a transfer operator for $s$ in the notation of Exel [16]. A topological correspondence gives a topological quiver when combined with the family of counting measures on the (discrete) fibres of $s$.

A topological quiver $(M, r, s, \lambda_2)$ gives a $C^*$-correspondence $\mathcal{E}_{r,M,s}$ over $C_0(X)$: complete $C_c(M)$ with respect to the $C_0(X)$-valued inner product

$$\langle \xi_1, \xi_2 \rangle(x) := \int_{s^{-1}(x)} (\overline{\xi_1}(y) \lambda_1(y)) \, d\lambda_2(y)$$

for $\xi_1, \xi_2 \in C_c(M)$, $x \in X$; the left and right module structures are $(f \xi)(m) := f(r(m)) \xi(m)$ and $(\xi f)(m) := \xi(m) f(s(m))$ for all $m \in M$, $f \in C_0(X)$, $\xi \in C_c(M)$. In particular, this construction applies to topological correspondences, where we always take the family of counting measures.

**Proposition 4.1.** The $C^*$-correspondence $\mathcal{E}_{r,M,s}$ is proper if and only if $r$ is proper and $s$ is a local homeomorphism. In that case, the isomorphism class of $\mathcal{E}_{r,M,s}$ does not depend on $(\lambda_x)$, so we may always use the family of counting measures. The $C^*$-correspondence $\mathcal{E}_{r,M,s}$ is full if and only if $s$ is surjective.

**Proof.** The $C^*$-correspondence $\mathcal{E}_{r,M,s}$ is proper if and only if $\varphi^{-1}(\mathbb{K}(\mathcal{E}_{r,M,s})) = C_0(X)$. In the notation of [35] Definition 3.14, all vertices are finite emitters. Corollary 3.12] shows that this happens if and only if $r$ is proper and $s$ is a local homeomorphism.

Let $(\lambda_2)$ and $(\lambda'_2)$ be two families of measures that make $(M, r, s)$ into a topological quiver. Since both $\lambda_2$ and $\lambda'_2$ have the same discrete subset $s^{-1}(x)$ as support, they are equivalent, say, $\lambda'_2 = f_x \cdot \lambda_2$ for a unique function $f_x: s^{-1}(x) \to (0, \infty)$. The functions $f_x$ may be pieced together to a function $f: M \to (0, \infty)$. The continuity of $(\lambda_x)$ and $(\lambda'_x)$ implies that $f$ is a continuous function. Hence multiplication with $\sqrt{f}$ is a unitary operator between the Hilbert modules over $C_0(X)$ associated to the two families of measures. This unitary also intertwines the left actions, which are by multiplication operators.

It is routine to check that $\mathcal{E}_{r,M,s}$ is full if and only if $s$ is surjective. \qed

**Definition 4.2.** A topological correspondence is called proper if $r$ is proper and $s$ is a local homeomorphism.

**Lemma 4.3.** Consider two topological correspondences

$$X \xleftarrow{\varepsilon_1} M_1 \xrightarrow{s_1} X \xleftarrow{\varepsilon_2} M_2 \xrightarrow{s_2} X.$$

Define $M := \times_{s_1} X, r: M \to X, (m_1, m_2) \mapsto r_1(m_1), s: M \to X, (m_1, m_2) \mapsto s_2(m_2)$. Then

$$\mathcal{E}_{r_1, M_1, s_1} \otimes_{C_0(X)} \mathcal{E}_{r_2, M_2, s_2} \cong \mathcal{E}_{r,M,s}.$$

If $r_1$ and $r_2$ are proper, so is $r$. If $s_1$ and $s_2$ are local homeomorphisms, so is $s$. If $s_1$ and $s_2$ are surjective, so is $s$. 

$P = \mathbb{N}^k$ for some $k \geq 1$. Yeend’s construction depends on special features of $\mathbb{N}^k$.
Proof. The first part is routine to prove and holds even for topological quivers, see [55, Lemmas 6.1–4]. The statements in the last paragraph are also routine to check. When we combine the first two properties, they amount to the statement that tensor products of proper or full \( C^* \)-correspondences are again proper or full, respectively. \(\square\)

Proposition 4.1 says that the \( C^* \)-correspondence associated to a topological quiver is proper if and only if we are dealing with a proper topological correspondence; the family of measures does not matter. We restrict attention to proper topological correspondences from now on.

The notion of a “topological graph algebra” interprets a topological correspondence as a “topological graph,” where vertices and (oriented) edges form topological spaces. This interpretation, however, fails to elucidate the lack of symmetry to commute for all \( r, M, s \) in the construction of the \( C^* \)-correspondence. Another interpretation is that a topological correspondence \( (r, M, s) \) is a multi-valued map from \( Y \) to \( X \), where \( r(m) \in X \) for \( m \in s^{-1}(y) \) are the possible values at \( y \in Y \). If \( s \) is a local homeomorphism and \( r \) is proper, then the subset of values \( r(s^{-1}(y)) \) of \( y \) is discrete. The interpretation as a multi-valued map breaks down, however, if there are different \( m, m' \in M \) with \( s(m) = s(m') \) and \( r(m) = r(m') \). We suggest the following more dynamical interpretation of a (proper) topological correspondence.

We consider points in \( M \) as possible developments or, briefly, stories. Each story \( m \in M \) assumes a certain initial situation \( s(m) \in Y \) and leads to a certain ending \( r(m) \in X \). Several stories may have the same initial situation and ending.

How does this interpretation account for the assumptions that \( s \) be a local homeomorphism and \( r \) be proper? That \( s \) is a local homeomorphism means the following: if we modify the initial situation \( s(m) \) of a story \( m \) a little bit, then there is a unique story \( m_\sigma \) close to \( m \) with initial situation \( \sigma \). Roughly speaking, \( m_\sigma \) describes how “the same” story would go in a slightly different initial situation, and fits our intuition of story-telling. That \( r \) is proper means that, given a compact set of possible endings, the set of stories with such an ending is also compact. This is a rather technical finiteness condition on the space of possible stories. It ensures that the space of complete histories defined below is locally compact.

**Definition 4.4.** Let \( P \) be a monoid. An action of \( P \) on \( X \) by proper topological correspondences consists of the following data:

- proper topological correspondences \( (M_p, r_p, s_p) \) from \( X \) to \( X \) for \( p \in P \setminus \{1\} \);
- homeomorphisms \( \sigma_{p,q}: M_{pq} \to M_p \times_{s_p X, r_q} M_q \) for \( p, q \in P \setminus \{1\} \).

Let \( M_1 = X \) and \( r_1 = s_1 = \text{id}_X \), and let \( \sigma_{p,1} \) and \( \sigma_{1,q} \) be the canonical homeomorphisms \( M_p \cong M_p \times_{s_p X, \text{id}_X} X \) and \( M_q \cong X \times_{\text{id}_X X, r_q} M_q \) for \( p, q \in P \). For an action of \( P \), we require the diagram

\[
\begin{array}{c}
M_p \times_X M_q \times_X M_t \\
\downarrow \text{id}_{M_p} \times_X \sigma_{q,t} \\
M_p \times_X M_{qt} \\
\downarrow \sigma_{p,qt} \\
M_{pq} \\
\end{array}
\]

(4.5) to commute for all \( p, q, t \in P \setminus \{1\} \) (since \( pq = 1 \) or \( qt = 1 \) is possible, we have to define \( (M_1, s_1, r_1), \sigma_{1,q} \) and \( \sigma_{p,1} \) for this condition to make sense). This diagram commutes automatically if \( p = 1, q = 1 \) or \( t = 1 \), so our assumption implies that it commutes for all \( p, q, t \in P \).

**Example 4.6.** An action of \( \mathbb{N}^k \) on a countable discrete set \( X \) by proper topological correspondences is equivalent to a row-finite rank-\( k \) graph by [22]. The Cuntz–Pimsner algebra that we shall attach to this data is not always the higher-rank
We fix an action of $P$ on $X$ by proper topological correspondences as above. The proper topological correspondences $(M_p, r_p, s_p)$ induce proper $C^*$-correspondences $\mathcal{E}_q$ from $C_0(X)$ to itself for $p \in P \setminus \{1\}$, and we let $\mathcal{E}_1 := C_0(X)$. The homeomorphisms $\sigma_{p,q}$ induce isomorphisms of $C^*$-correspondences

$$
\mu_{p,q} : \mathcal{E}_p \otimes_{C_0(X)} \mathcal{E}_q \to \mathcal{E}_{pq}
$$

for $p, q \in P \setminus \{1\}$ by Lemma 4.3 and we let $\mu_{1,q}$ and $\mu_{p,1}$ be the canonical isomorphisms. The diagram (4.5) ensures the associativity of these multiplication maps $\mu_{p,q}$ for all $p, q, t \in P \setminus \{1\}$ (even if $pq = 1$ or $qt = 1$); associativity is automatic if $p = 1$, $q = 1$ or $t = 1$. So an action of $P$ on $X$ by topological correspondences induces a proper product system over $P$ with unit fibre $C_0(X)$, as expected.

The defining property of the fibre product means that $\sigma_{p,q} = (r_{p,q}, s_{p,q})$ for two continuous maps

$$
r_{p,q} : M_{pq} \to M_p, \quad s_{p,q} : M_{pq} \to M_q
$$

with $s_p \circ r_{p,q} = r_q \circ s_{p,q}$. Since $\sigma_{p,1}$ and $\sigma_{1,q}$ are the canonical maps,

$$
s_{p,1} = s_p, \quad r_{p,1} = \text{id}_{M_p}, \quad s_{1,q} = \text{id}_{M_q}, \quad r_{1,q} = r_q
$$

for all $p, q \in P$. The associativity condition (4.7) is equivalent to

$$
(4.7) \quad r_{p,q} \circ r_{pq,t} = r_{p,qt}, \quad s_{p,q} \circ r_{pq,t} = r_{qt} \circ s_{pq,t}, \quad s_{q,t} \circ s_{pq,t} = s_{pq,t}.
$$

**Lemma 4.8.** The maps $r_{p,q}$ are proper and the maps $s_{p,q}$ are local homeomorphisms. If all $s_p$ are surjective, then so are the maps $s_{p,q}$.

**Proof.** The map $r_{p,q}$ is the composite of the homeomorphism $\sigma_{p,q}$ and the coordinate projection $M_p \times_{s_p, X, r_q} M_q \to M_p$. This coordinate projection is proper if $r_q$ is proper because properness is hereditary under this type of fibre-products. Similarly, the map $s_{p,q}$ is the composite of the homeomorphism $\sigma_{p,q}$ and the coordinate projection $M_p \times_{s_p, X, r_q} M_q \to M_q$; the latter inherits the property of being surjective or a local homeomorphism from $s_p$. □

We interpret elements of $P$ as a (multi-dimensional) kind of *time*, and elements of $M_p$ as *stories of length* $p \in P$; a story $m \in M_p$ starts in the situation $s_p(m)$ and ends in $r_q(m)$. The maps $r_{p,q} : M_{pq} \to M_p$ and $s_{p,q} : M_{pq} \to M_q$ cut a story $m$ of length $pq$ into two stories of length $p$ and $q$: its *ending* $m_1 = r_{p,q}(m) \in M_p$ and its *beginning* $m_2 = s_{p,q}(m) \in M_q$. These satisfy $s_p(m_1) = r_q(m_2)$, that is, the story $m_1$ starts when $m_2$ ends. The inverse of $\sigma_{p,q}$ combines a pair $m_1 \in M_p$, $m_2 \in M_q$ of stories of lengths $p$ and $q$ to a story $m_1 \circ m_2$ of length $pq$, provided $m_1$ starts when $m_2$ ends. The assumption that $\sigma_{p,q}$ be a homeomorphism says that $m \in M_{pq}$ and $(m_1, m_2) \in M_p \times M_q$ with $s_p(m_1) = r_q(m_2)$ determine each other uniquely and continuously.

The length $1 \in P$ is the neutral element, so nothing can happen in time 1, and adding a story of length $1$ before or after another story does nothing. This means that $M_1 = X$ and that $\sigma_{p,1}$ and $\sigma_{1,q}$ are the canonical maps. The associativity conditions (4.7) say that the two ways of cutting a story of length $pq$ into three pieces of length $p, q$ and $t$ give the same results.

If $P$ is a free monoid on $n$ generators (which, however, is not Ore), then the situation above may be interpreted as describing a game where the players may do $n$ different things in each time interval. If, say, the player has the three options $a, b, c$, then $p = baac$ means a time interval of length 4 in which the player first does $c$, then twice $a$, then $b$. If the game was in situation $x \in X$ initially, then the points in $s_p^{-1}(x) \subseteq M_p$ are the possible game developments in this length-4 time period,
provided the player’s actions are \( baac \). And \( r_p(m) \) for \( m \in s_p^{-1}(x) \) is the situation after this time period. If \( s_p^{-1}(x) \) has more than one point, then the game contains randomness. It makes sense to quantify this randomness by a transfer operator with \( \sum s_p(m) = \mu_p(m) = 1 \) for all \( x \in X \), where \( \mu_p(m) \) is the probability that the game develops as in story \( m \), given the initial situation \( y \). Since these probabilities are irrelevant for us, we refrain from adding them and use the standard transfer operator, see Proposition \ref{prop:transfer}.

A relation in the monoid \( P \) means that certain actions of the player always and automatically have the same effect on the game. For instance, if \( P \) is the free Abelian monoid \( \mathbb{N}^n \) on \( n \) generators, then the order in which the player does various things does not matter. I know no game with this property; so the interpretation through games works best for free monoids.

There are three simple special cases of actions by proper topological correspondences:

1. \( M_p = X \) and \( s_p = \text{id}_X \) for all \( x \in X \), \( p \in P \); that is, a situation \( x \in X \) determines its future uniquely;
2. \( M_p = X \) and \( r_p = \text{id}_X \) for all \( x \in X \), \( p \in P \); that is, a situation \( x \in X \) determines its past uniquely;
3. \( M_p \) is arbitrary, but \( s_p = r_p \) for all \( p \in P \); that is, the situation never changes; then \( s_p = r_p \) must be both proper and a local homeomorphism; equivalently, it is a finite covering map.

Now we assume that \( P \) is a right Ore monoid. In this case, the Cuntz–Pimsner algebra of the product system \( (\mathcal{E}_p, \mu_{p,q}) \) over \( P \) is described more concretely in Section \ref{sec:cpalg}. We are going to identify this Cuntz–Pimsner algebra with the groupoid \( C^* \)-algebra of an étale, locally compact groupoid \( H \).

We first describe the object space \( H^0 \) of this groupoid. The first associativity condition in \ref{eq:assoc} says that the spaces \( M_p \) for \( p \in P \) and the continuous maps \( r_{p,q} \) for \( p,q \in P \) form a projective system of locally compact spaces indexed by the directed category \( \mathcal{C}_P \). We let

\[
H^0 := \lim_{\mathcal{C}_P} (M_p, r_{p,q}).
\]

We call a point in \( H^0 \) a complete history. It consists of stories \( m_p \in M_p \) for each \( p \in P \) that describe what happened in the past length-\( p \) time period. To be a history, \( m_p \) must be the ending of \( m_{pq} \) for all \( p,q \in P \); that is, \( r_{p,q}(m_{pq}) = m_p \) for all \( p,q \in P \).

**Lemma 4.9.** The space \( H^0 \) is locally compact and Hausdorff. The maps \( \pi_q : H^0 \to X, (m_p)_{p \in P} \mapsto m_q \), are proper for all \( q \in P \).

**Proof.** Fix \( (m_p)_{p \in P} \in H^0 \) and let \( K \subset X = M_1 \) be a compact neighbourhood of \( m_1 \). The preimage of \( K \) in \( H^0 \) is the subset of all \( (m'_p)_{p \in P} \) with \( m'_1 \in K \) and hence \( r_p(m'_p) \in K \) for all \( p \in P \). Since all the maps \( r_p \) are proper, the subsets \( r^{-1}_p(K) \subseteq M_p \) are compact. Hence so is the product \( L := \prod_{p \in P} r^{-1}_p(K) \) by Tychonov’s Theorem. Thus the map \( \pi_1 \) is proper; the same argument shows that all the maps \( \pi_q \) are proper. Since \( L \) is also a compact neighbourhood of \( (m_p)_{p \in P} \) in \( H^0 \), the space \( H^0 \) is locally compact. If \( (m'_p) \neq (m_p) \), then there is \( p \in P \) with \( m'_p \neq m_p \). There are open neighbourhoods in \( M_p \) that separate \( m'_p \) and \( m_p \). These yield open neighbourhoods in \( H^0 \) that separate \( (m'_p) \) and \( (m_p) \), so \( H^0 \) is Hausdorff. \[ \square \]

Given a complete history \( (m_p)_{p \in P} \) and \( t \in P \), we may forget what happened in the last time period of length \( t \); this gives another complete history, defined formally by \( m'_p := s_{t,p}(m_{tp}) \) for \( p \in P \); the second condition in \ref{eq:assoc} implies \( r_{p,q}(m'_{pq}) = m'_p \).
for all $p,t \in P$, that is, $(m'_p)_p \in P$ is again a complete history as expected. Hence we get well-defined maps $\tilde{s}_t : H^0 \to H^0$ for $t \in P$. These are clearly continuous.

**Lemma 4.10.** The maps $\tilde{s}_t$ are local homeomorphisms. If the maps $s_t$ are surjective, then so are the maps $\tilde{s}_t$.

**Proof.** Let $(m'_p) \in H^0$ be a complete history and let $m_t \in M_t$ be a length-$t$ story with $m'_1 = s_t(m_t) \in X$. We claim that there is a unique complete history $(m_p)$ with given $m_t$ and with $\tilde{s}_t((m_p)) = (m'_p)$, which depends continuously on $(m'_p)$ and $m_t$. Since $s_t$ is a local homeomorphism, this shows that $\tilde{s}_t$ is a local homeomorphism as well. Roughly speaking, $(m_p)_p \in P$ is the complete history that continues the length-$t$ story $m_t$. The assumption $m'_1 = s_t(m_t) \in X$ ensures that the two may be combined.

First assume that $(m_p)_p \in P$ as above has been found. Let $q \in P$. Then we may write $qu = tp$ for some $u,p \in P$ because $P$ is a right Ore monoid. The story $m_{tp}$ is the concatenation $m_o m'_p$ of $r_{tp}(m_p) = m_t$ and $s_t(m_p) = m'_p$, which exists because $r_m(m'_p) = m'_1 = s_t(m_t)$. Thus $m_q = r_{q,u}(m_{qu}) = r_{q,u}(m_t \circ m'_p)$, so there is at most one possible solution $(m_p)_p \in P$, and it depends continuously on $(m'_p)$ and $m_t$. It remains to show that the length-$q$ stories $m_q := r_{q,u}(m_t \circ m'_p)$ with $u,p \in P$ chosen as above form a complete history, that is, $r_{q,v}(m_{qv}) = m_q$ for all $q,v \in P$.

We have $m_{qv} = r_{qv,u_2}(m_t \circ m'_p)$ for some $u_2,P \in P$ with $qvu_2 = tp_2$. Since $P$ is a right Ore monoid, there are $u_3,u_4 \in P$ with $vu_2u_3 = uu_4$. Then $tpu_3 = quv_2u_3 = quu_4 = tp_4$. Since $P$ is a right Ore monoid, there is $u_5 \in P$ with $p_{u_3u_5} = p_{u_4u_5}$. To simplify notation, we replace $(u_3,u_4)$ by $(u_3u_5,u_4u_5)$; thus $p_{u_3u_5} = p_{u_4u_5}$.

Since $quu_4 = tp_4$, we could also use $(uu_4,p_{u_4})$ instead of $(u,p)$ to define $m_q$.

The associativity conditions in (4.7) show that this gives the same result:

$$r_{q,u_4}(m_t \circ m'_p) = r_{q,u}r_{qu,u_4}(m_t \circ m'_p)$$

Similarly, we get the same result for $m_{qv}$ if we use $(u_2u_3,p_{u_3u_5})$ instead of $(u_2,p_2)$. Thus we may assume that $p_2 = p$ and $u = vu_2$. Then

$$r_{q,v}(m_{qv}) = r_{q,v}(r_{qv,u_2}(m_t \circ m'_p)) = r_{q,v}(m_t \circ m'_p) = r_{q,v}(m_t \circ m'_p) = m_q.$$

This computation finishes the proof. \qed

First forgetting the last length-$t$ time period and then the last length-$u$ time period gives the same result as directly forgetting the last time period of length $tu$. That is, $\tilde{s}_u \circ \tilde{s}_t = \tilde{s}_{tu}$ for all $t,u \in P$. Formally, this follows from the third condition in (4.7). Thus the monoid $P_{\operatorname{pop}}$ acts on $H^0$ by local homeomorphisms.

Why do we get the opposite monoid here? The maps $\tilde{s}_t : H^0 \to H^0$ and $\tilde{r}_t := \operatorname{id}_{H^0}$ form an action of $P$ by topological correspondences with the extra property that any situation determines its past uniquely: a “situation” in $H^0$ is a complete history, which simply contains its past. Thus we still have an action by topological correspondences, but one where the maps $\tilde{r}_t$ are all identity maps, so that we may forget about them. This gives an action of the opposite monoid $P_{\operatorname{pop}}$ by local homeomorphisms because of the direction of the maps $\tilde{s}_t$.

From now on, we write the action of $P_{\operatorname{pop}}$ on $H^0$ multiplicatively as $p^{-1}.x := \tilde{s}_t(x)$ for $x \in H^0$, $p \in P$. We are going to define the groupoid $H$ as the “transformation groupoid” of this action:
Definition 4.11. The transformation groupoid \( H := P^{op} \times H^0 \) associated to the \( P^{op} \)-action on \( H^0 \) by local homeomorphisms has object space \( H^0 \), arrow space

\[ \text{H}^1 := \{(x, g, y) \in H^0 \times G \times H^0 \mid \exists p_1, p_2 \in P, \ g = p_1 p_2^{-1}, \ p_1^{-1} x = p_2^{-1} y\}, \]

range and source maps \( r(x, g, y) := x \), \( s(x, g, y) := y \), and multiplication

\[ (x_1, g_1, y_1) \cdot (x_2, g_2, y_2) := (x_1 g_1 g_2, y_2) \]

if \( y_1 = x_2 \). The unit on \( x \in H^0 \) is \((x, 1, x)\), the inverse of \((x, g, y)\) is \((y, g^{-1}, x)\).

We describe the topology on \( H^1 \). For \( p_1, p_2 \in P \), let

\[ R_{p_1, p_2} := H^0 \times S_{p_1} H^0, S_{p_2} H^0 := \{(x, y) \in H^0 \times H^0 \mid p_1^{-1} x = p_2^{-1} y\}, \]

the fibre product of the diagram \( H^0 \xrightarrow{S_{p_1}} H^0 \xleftarrow{S_{p_2}} H^0 \). We give each \( R_{p_1, p_2} \) the subspace topology from the product \( H^0 \times H^0 \), and \( \bigsqcup_{p_1, p_2 \in P} R_{p_1, p_2} \) the disjoint union topology. We map \( R_{p_1, p_2} \to H^1 \) by \((x, y) \mapsto (x, p_1 p_2^{-1} y) \in H^1 \). This gives a surjection \( \bigsqcup R_{p_1, p_2} \to H^1 \). We give \( H^1 \) the quotient topology from this map.

To verify that this construction has desirable properties, we rewrite \( H \) using filtered colimits. Let \( H^1_g := \{(x, g, y) \in H^1 \} \) for \( g \in G \); so \( H^1 = \bigsqcup_{g \in G} H^1_g \). We describe \( H^1_g \) for fixed \( g \in G \) as a filtered colimit over \( C^1_g \); the objects of this category are \((p_1, p_2) \in P^2 \) with \( p_1 p_2^{-1} = g \), the arrows \((p_1, p_2) \to (p_3, p_4)\) are \( q \in P \) with \( p_1 q = p_3, \ p_2 q = p_4 \); the composition is the one from \( P \). This category is filtered by Lemma 3.12. If \( p_1 q = p_3, \ p_2 q = p_4 \), then \( p_1^{-1} x = p_2^{-1} y \) implies \( p_3^{-1} x = p_4^{-1} y \), so \( R_{p_1, p_2} \subseteq R_{p_3, p_4} \subseteq H^0 \times H^0 \). Since right multiplication with \( g \) is locally injective, any \((x, y) \in R_{p_1, p_2}\) has a neighbourhood in \( H^0 \times H^0 \) so that for \((x', y')\) in this neighbourhood, \( p_1^{-1} x' \neq p_2^{-1} y' \) implies \((p_1 q)^{-1} x' \neq (p_2 q)^{-1} y' \). Thus the subset \( R_{p_1, p_2} \) is relatively open in \( R_{p_3, p_4} \), so the spaces \( R_{p_1, p_2} \) for \((p_1, p_2) \in P^2 \) and the inclusion maps \( R_{p_1, p_2} \to R_{p_3, p_4} \) form a diagram of subsets of \( H^0 \times H^0 \) with open inclusion maps. By definition, \( H^1_g \) is the inductive limit (union) of this diagram.

Lemma 4.12. If \( p_1 p_2^{-1} = p_3 p_4^{-1} \), then there are \( p_5, p_6 \in P \) with \( p_5 p_6^{-1} = p_1 p_2^{-1} \) and so that both \( R_{p_1, p_2} \) and \( R_{p_3, p_4} \) are open subsets of \( R_{p_5, p_6} \); hence the subspace topologies from \( R_{p_1, p_2} \) and \( R_{p_3, p_4} \) coincide on \( R_{p_1, p_2} \cap R_{p_3, p_4} \), and this subset is open both in \( R_{p_1, p_2} \) and in \( R_{p_3, p_4} \). Each \( R_{p_1, p_2} \) is open in the inductive limit topology on \( H^1_g \), and the inductive limit topology restricts to the given topology on each \( R_{p_1, p_2} \).

Proof. Since \( C^1_g \) is filtered, there is an object \((p_5, p_6)\) that dominates both \((p_1, p_2)\) and \((p_3, p_4)\). This has all required properties. The last statement now follows from the definition of the inductive limit topology and because all the embeddings \( R_{p_1, p_2} \to R_{p_3, p_4} \) are open.

Thus the subsets \( R_{p_1, p_2} \) for \( p_1, p_2 \in P \) form an open covering of \( H^1 \), and the topology on \( H^1 \) restricts to the usual topology on each \( R_{p_1, p_2} \). In the following, we identify \( R_{p_1, p_2} \) with its image in \( H^1 \), which is an open subset.

Proposition 4.13. The groupoid \( H \) is étale, locally compact and Hausdorff. The decomposition \( H^1 = \bigsqcup_{g \in G} H^1_g \) satisfies \( H^1_g \cdot H^1_h \subseteq H^1_{gh} \cdot (H^1_g)^{-1} = H^1_{g^{-1}} \). If the maps \( s_p \) for \( p \in P \) are surjective, then \( H^1_g \cdot H^1_h = H^1_{gh} \) for all \( g, h \in G \).

The groupoid \( H^1 \subseteq H^1 \) is an increasing union of open subgroupoids that are proper and étale and describe equivalence relations on \( H^0 \).

Proof. The space \( H^0 \) is locally compact and Hausdorff by Lemma 4.12. The coordinate projections \( R_{p_1, p_2} \to H^0 \) are étale because \( P \) acts by local homeomorphisms. Since the subsets \( R_{p_1, p_2} \) for \( p_1, p_2 \in P \) form an open covering of \( H^1 \), the coordinate projections \( H^1 \to H^0 \) are étale. Any two points of \( H^1 \) are contained in the same Hausdorff, locally compact, open subset \( R_{p_1, p_2} \) for suitable \( p_1, p_2 \), so they may be
separated by open subsets of $H^1_y$; since the subsets $H^1_g$ are open, it is also possible to separate points in $H^1_g$ and $H^1_h$ for $g \neq h$. Thus $H^1$ is Hausdorff.

If the maps $s_p$ for $p \in P$ are surjective, then so are the maps $\tilde{s}_p$ for $p \in P$ by Lemma 4.10. Let $(x, gh, y) \in H^1_{gh}$. Hence there are $p_1, p_2 \in P$ with $gh = ps_p^{-1}$ and $p_1^{-1}x = p_2^{-1}y$. Write $g = ps_p^{-1} = ps_p^{-1}$, $h = ps_p^{-1}$. Then also $g = ps_q(pq)^{-1}$ and $h = (ps_t)(ps_t)^{-1}$ for all $q, t \in P$. The Ore condition (O1) allows us to choose $q$ and $t$ such that $ps_q = ps_t$. Hence we may assume without loss of generality that $p_4 = p_5$. Then $p_3p_4^{-1} = gh = ps_p^{-1}p_3p_6^{-1} = ps_p^{-1}$. If $p_1^{-1}x = p_2^{-1}y$, then also $(p_1t)^{-1}x = (ps_t)^{-1}y$ for any $t \in P$, so we may rewrite $gh = (p_1t)(ps_t)^{-1}$. We may also replace $(p_3, p_4, p_5, p_6)$ by $(p_3q, ps_q, ps_q, ps_q)$ for any $q \in P$. Choosing $q$ and $t$ by condition (O1) we may achieve $ps_q = ps_t$, $ps_q = ps_t$, by the definition of the group $G$. Hence we find $p_1, p_2, p_3 \in P$ with $gh = ps_p^{-1}, g = ps_p^{-1}, h = ps_p^{-1}$ and $p_1^{-1}x = p_2^{-1}y$. Since $\tilde{s}_p$ is surjective, we may choose $z \in H^0$ with $ps_p^{-1}z = p_1^{-1}x = p_2^{-1}y$. Then $(x, g, z) \in H^1_y$ and $(z, h, y) \in H^1_h$ satisfy $(x, g, z) \cdot (z, h, y) = (x, gh, y)$.

The open subgroupoid $H^1_y$ is defined as the union of $R_{P, P_{1y}}$ with $p_1p_2^{-1} = 1$ in $G$; that is, $p_1q = p_1q$ for some $q \in P$. We have already seen in the previous section that the subcategory of pairs $(p, p)$ is cofinal in $C^1_p$. Hence $H^1_y = \bigcup_{p \in P} R_{p, p}$. Here $R_{p, p}$ is the set of all $(x, y) \in H^0 \times H^0$ with $p_1^{-1}x = p_1^{-1}y$, and it carries the subspace topology from $H^0 \times H^0$. So $R_{p, p}$ is a proper equivalence relation on $H^0$, and $H^1_y$ is the union of these open subgroupoids. □

If $P$ is countable, then we may choose a cofinal sequence in $C^1_p$ and write $H^1_y$ as an increasing union of a sequence of proper étale equivalence relations. Hence $H^1_y$ is an approximately proper equivalence relation in the notation of [21]. These are called hyperfinite relations in [28]. We allow ourselves to call $H^1_y$ approximately proper also if $P$ is uncountable, replacing a sequence of proper (finite) open subrelations by a directed set of such subrelations.

If the maps $s_p$ for $p \in P$ are surjective, then the subsets $(H^1_y)$ form a $G$-grading in the notation of [8]. This is equivalent to an action of $G$ on the groupoid $H^1_y$ by partial equivalences with transformation groupoid $H$. Thus we may think of $H$ as the transformation groupoid associated to an action of $G$ on the noncommutative orbit space $H^1_y/H^1_y$. Points in this orbit space are equivalence classes of complete histories, where two complete histories are identified if they coincide in the distant past, that is, $p_1^{-1}x = p_1^{-1}y$ for some $p \in P$. The group $G$ acts on this by “time translations.”

If the maps $s_p$ are not surjective, then the $G$-action on $H^1_y$ is only a partial action because time translations $x \mapsto px$ into the future are not everywhere defined. A partial $G$-action is the same as an action of a certain inverse semigroup associated to $G$, see [13].

**Definition 4.14.** A situation $x \in X$ is (historically) possible if $x \in r_{p}(M'_p)$ for all $p \in P$.

Let $X' \subseteq X$ be the subset of possible situations. We have $X' = X$ if and only if all the maps $r_p$ are surjective. Let $M'_p = s_p^{-1}(X')$ and let $r'_p$ and $s'_p$ be the restrictions of $r_p$ and $s_p$ to $M'_p$. Any situation that occurs at some point in a complete history is possible, so we have $m_p \in M'_p$ for any $(m_p)_{p \in P} \in H^0$. Conversely, a situation that is historically possible is the endpoint $m_1$ of some complete history $(m_p)_{p \in P}$ because the maps $r_p$ are proper (Lemma 4.10) and a projective limit of non-empty compact spaces is non-empty. Thus $\pi'_p(H^0) = M'_p$, $r'_p(M'_p) = X'$ for all $p \in P$, and $s'_p(M'_p) \subseteq X'$ by associativity: if a situation has a possible past, then it is itself possible because we may concatenate stories. We still have isomorphisms $M'_p \times X' M'_q \cong M'_{pq}$; so restricting to the historically possible
situations gives a new action by topological correspondences. By construction, both systems \((X, M, s_p, r_p, \sigma_{p,q})\) and \((X', M'_p, s'_p, r'_p, \sigma'_{p,q})\) have the same complete histories and thus the same groupoid model \(H\).

**Lemma 4.15.** Let \(q, a_1, a_2 \in P\) satisfy \(qa_1 = qa_2\). Then \(r_{q,a_1} |_{M_{qa_1}} = r_{q,a_2} |_{M_{qa_2}}\).

**Proof.** Condition \((O2)\) gives us \(b \in P\) with \(a_1 b = a_2 b\). The associativity property \((L7)\) of the maps \(r_{p,q}\) gives \(r_{q,a_1} \circ r_{qa_1, b} = r_{q,a_2, b}\). Hence \(r_{q,a_1}\) and \(r_{q,a_2}\) coincide on the range of \(r_{q,a_1b} = r_{q,a_2b}\). The subspace \(M_{qa_1}\) is contained in that range. \(\square\)

Write \(p \geq q\) for \(p, q \in P\) if there is \(a \in P\) with \(p = qa\). Lemma 4.15 shows that after restricting to the possible situations, the truncation map \(r_{q,a}: M'_p \to M'_q\) for \(p \geq q\) does not depend on the choice of \(a\).

**Theorem 4.16.** The groupoid \(C^\ast\)-algebra \(C^\ast(H)\) is canonically isomorphic to the Cuntz–Pimsner algebra of the product system \((E_p)_{p \in P}\) over \(P\) described above.

**Proof.** Here \(E_p\) is the Hilbert \(C_0(X)\)-module completion of \(C_c(M_p)\). A function \(f \in C_0(X)\) acts trivially on \(E_p\) if and only if \(f \circ r_p = 0\), that is, \(f\) vanishes on the range of \(r_p\). Since \(r_p\) is a proper map, this range is a closed subspace of \(X\).

As a consequence, the quotient \(A' = C_0(X)\) in Theorem 3.22 is \(C_0(X')\) with the space of possible situations \(X'\) as defined above. In fact, the modified product system in Theorem 3.22 is simply the product system associated to the restricted action by topological correspondences \((X', M'_p, \sigma'_{p,q})\). We have just observed that restricting our action to the closed subspace \(X'\) also gives the same groupoid \(H\). Since this modification changes neither the Cuntz–Pimsner algebra nor the groupoid model, we may assume \(X = X'\), that is, the range maps \(r_p: M_p \to X\) and hence the maps \(r_{p,q}: M_p \to M_q\) and \(\pi_p: H^0 \to M_p\) are surjective. Dually, the maps \(C_0(X) \to \mathbb{K}(E_p)\) are faithful for all \(p \in P\). We assume this from now on.

The following proof is rather technical. We begin by explaining the main point. Let \(p, q \in P\). We want to relate the space \(\mathbb{K}(E_q, E_p)\) that appears in the construction of the Cuntz–Pimsner algebra to the space \(C_c(M_p \times_{s_p, X, r_q} M_q)\); we will see below how the latter relates to the groupoid algebra of \(H\). Roughly speaking, compact operators may be described by kernel functions.

First, given a function \(k \in C_c(M_p \times_{s_p, X, s_q} M_q)\), we define

\[
T_k: E_q \to E_p, \quad (T_k \xi)(m_1) := \sum_{s_p(m_1) = s_p(m_2)} k(m_1, m_2) \xi(m_2);
\]

these sums are uniformly finite for \(m_1\) in a compact subset because \(s_p\) and \(s_q\) are local homeomorphisms and the support of \(k\) is compact. The operator \(T_k\) is a rank-one operator if \(k(m_1, m_2) = k_1(m_1) \cdot k_2(m_2)\) with \(k_1 \in C_c(M_p)\), \(k_2 \in C_c(M_q)\).

Functions \(k\) of this form are dense in \(C_c(M_p \times_{s_p, X, s_q} M_q)\) and the map \(k \mapsto T_k\) is continuous, we have \(T_k \in \mathbb{K}(E_q, E_p)\) for all \(k \in C_c(M_p \times_{s_p, X, s_q} M_q)\).

Secondly, given \(T \in \mathbb{K}(E_q, E_p)\), we claim that there is a unique function \(k \in C_0(M_p \times_{s_p, X, s_q} M_q)\) with

\[
(\xi, T \eta)(x) = \sum_{s_p(m_1) = s_q(m_2) = x} k(m_1, m_2) \xi(m_1) \overline{\eta(m_2)}
\]

for all \(\xi \in E_q\), \(\eta \in E_p\). If \(T\) is a rank-one operator \([\zeta_1]|_{[\zeta_2]}\), this this holds with \(k(m_1, m_2) = \zeta_1(m_1) \overline{\zeta_2(m_2)}\); here we also use that \(E_q \subseteq C_0(M_p)\). The claim extends from rank-one operators to all compact operators by linearity and continuity. Thus \(\mathbb{K}(E_q, E_p)\) lies between the spaces of \(C_c\)- and \(C_0\)-functions on \(M_p \times_{s_p, X, s_q} M_q\).

The product and involution on compact operators are also easily described in terms of kernel functions: \(T_k \circ T_l\) for \(k \in C_c(M_p \times_{s_p, X, s_q} M_q)\) and \(l \in C_c(M_q \times_{s_q, X, s_q} M_l)\) has the kernel \((m_1, m_2) \mapsto \sum_{s_q(m) = s_p(m_1) = s_l(m_2)} k(m_1, m) l(m, m_2)\), and the
adjoint of \( T_0 \) has the kernel \((m_1, m_2) \mapsto k(m_2, m_1)\). The remaining technical work is to relate the groupoid \( H \) to the spaces \( M_p \times_{s_p \times s_q} M_q \) appearing above.

First we recall the definition of \( C^*(H) \). It is the \( C^* \)-completion of the dense \( * \)-subalgebra \( C_c(H^1) \) of compactly supported, continuous functions on \( H^1 \), equipped with the usual convolution and involution

\[
f_1 \ast f_2(h) := \sum_{h_1, h_2 = h} f_1(h_1) f_2(h_2), \quad f^*(h) := \overline{f((h^{-1})},
\]

for \( f_1, f_2, f \in C_c(H^1), \ h \in H^1 \) (see [17, Section 3]). Here \( \text{"\( C^* \)-completion" means that we complete in the largest \( C^* \)-seminorm on \( C_c(H^1) \). There is no need to assume boundedness for the \( I \)-norm. First, the argument in [17] shows that every Hilbert space representation and hence every \( C^* \)-seminorm is continuous for the inductive limit topology; secondly, [39, Corollaire 4.8] shows that such representations and \( C^* \)-seminorms are bounded for the \( I \)-norm.

The disjoint union decomposition \( H^1 = \bigsqcup_{g \in G} H^1_g \) gives \( C_c(H^1) = \bigoplus_{g \in G} C_c(H^1_g) \). This is a \( G \)-grading, that is, \( C_c(H^1_g) \ast C_c(H^1_h) \subseteq C_c(H^1_{gh}) \) and \( C_c(H^1_g) \ast = C_c(H^1_{g^{-1}}) \). This \( G \)-grading turns \( C^*(H) \) into the section algebra of a Fell bundle over \( G \). Of course, our proof will show that this Fell bundle structure corresponds to the same structure on the Cuntz–Pimsner algebra.

The space \( H^1_g \) is an increasing union of the open subsets \( R_{p_1, p_2} \). Thus any function in \( C_c(H^1_g) \) already belongs to \( C_c(R_{p_1, p_2}) \) for some \( p_1, p_2 \in P \) with \( p_1 p_2^{-1} = g \):

\[
C_c(H^1_g) = \bigcup_{p_1 p_2^{-1} = g} C_c(R_{p_1, p_2}).
\]

Let \( p_1, p_2 \in P \) satisfy \( p_1 p_2^{-1} = g \). The space \( R_{p_1, p_2} \) is defined as a fibre product. Thus it is the limit of the diagram

\[
H^0 \xrightarrow{s_{p_1}} H^0 \xrightarrow{s_{p_2}} H^0.
\]

The space \( H^0 \) is defined as the limit of the diagram of spaces \( (M_q)_{q \in P} \) and maps \( r_{p, q} : M_{p q} \to M_p \). Taking limits twice again gives a limit, so \( R_{p_1, p_2} \) is a limit of some diagram. We now describe this diagram. Its entries are the fibre products \( M_{p_1 q_1} \times_{s_{p_1} \times s_{q_1}} M_{q_2} \) for \( q \in P \). If \( q_1, q_2 \in P \), there is a continuous map

\[
M_{p_1 q_1, q_2} \times_{s_{p_1, q_1}, s_{q_2}} M_{q_1, q_2} \xrightarrow{\beta_{p_1, p_2, q_1, q_2}} M_{p_1 q_1} \times_{s_{p_2, q_1}, s_{q_1}} M_{q_2},
\]

because of [4.17], and these maps satisfy \( \beta_{p_1, p_2, q_1, q_2} \circ \beta_{p_1, p_2, q_2, q_3} = \beta_{p_1, p_2, q_1, q_3} \) for all \( q_1, q_2, q_3 \in P \). We have a projective system of topological spaces over \( C^*_P \).

The maps \( \beta_{p_1, p_2, q_1, q_2} \) are proper because the maps \( r_{p, q} \) are proper.

There are canonical continuous maps

\[
\alpha_{p_1, p_2, q} : R_{p_1, p_2} \to M_{p_1 q} \times_{s_{p_1} \times s_{q_1}} M_{q}, \quad ((x_q)_{q \in P}, (y_q)_{q \in P}) \mapsto (x_{p_1 q}, y_{p_2 q}),
\]

because \( p_1^{-1}(x_q) = p_2^{-1}(y_q) \) implies \( s_{p_1}(x_{p_1 q}) = s_{p_2}(y_{p_2 q}) \) by evaluating at \( q \in P \). Equation [4.17] implies \( \beta_{p_1, p_2, q_1, q_2} \circ \alpha_{p_1, p_2, q_1 q_2} = \alpha_{p_1, p_2, q_1} \) for all \( q_1, q_2 \in P \), so the maps \( \alpha_{p_1, p_2, q} \) induce a map

\[
\alpha_{p_1, p_2} : R_{p_1, p_2} \to \lim \ (M_{p_1 q} \times_{s_{p_1} \times s_{q_1}} M_{q}),
\]

Lemma 4.17. The map \( \alpha_{p_1, p_2} \) is a homeomorphism.

Proof. We want to verify that \( R_{p_1, p_2} \) has the universal property of the projective limit. So let \( Z \) be an auxiliary space. A continuous map \( Z \to R_{p_1, p_2} \) is equivalent to a pair of continuous maps \( h_1, h_2 : Z \to H^0 \) with \( \tilde{s}_{p_1} \circ h_1 = \tilde{s}_{p_2} \circ h_2 \). Since \( H^0 \) is a projective limit, the maps \( h_i \) for \( i = 1, 2 \) are equivalent to families of maps.
The convolution on $C$ where the vertical maps extend functions given on open subspaces by 0 outside. Thus $\alpha h c$ of compact operators both restrict to the same map $K$ dense vector subspace of $C$.

Similarly, the involution on $C$ the image of $\alpha h c$ are cofinal; that is, taking only the maps $h'$ extend to the projective limit. Thus a continuous map $Z \to R_{q, \alpha} p$ is equivalent to a family of continuous maps $h_{1, q}: Z \to M_{p1, q}$ and $h'_{2, q}: Z \to M_{p2, q}$ for $q \in P$ with $q, \alpha h_{1, q} = h'_{1, q}$ for all $q, \alpha p1, q2 \in P$; the condition $\delta_{p1, q} h_{1, q} = \delta_{p2, q} h_{2, q}$ means that $\alpha p1, q h_{1, p1, q} = \alpha p2, q h_{2, p2, q}$ for all $p1, p2, q \in P$.

For $q \in P$, the right Ore conditions give that the functors $C, p \to C, p, q \mapsto p q$ are cofinal; that is, taking only the maps $h'$ extend to the projective limit. Thus a continuous map $Z \to R_{q, \alpha} p$ is equivalent to a family of continuous maps $h_{1, q}: Z \to M_{p1, q}$ and $h'_{2, q}: Z \to M_{p2, q}$ for $q \in P$ with $q, \alpha h_{1, q} = h'_{1, q}$ for all $q, \alpha p1, q2 \in P$; the condition $\delta_{p1, q} h_{1, q} = \delta_{p2, q} h_{2, q}$ means that $\alpha p1, q h_{1, p1, q} = \alpha p2, q h_{2, p2, q}$ for all $p1, p2, q \in P$.

We have related compact operators $E, p \mapsto E, p, q \mapsto p q$ to kernel functions above, getting continuous compactly supported functions:

$$\beta_{p1, p2, q1, q2} : C_{C}(M_{p1, q1} \times s_{p1, q1}, s_{p2, q1} M_{p2, q1}) \to C_{C}(M_{p1, q1} \times s_{p1, q1}, s_{p2, q1}, s_{p2, q1}, s_{p2, q1} M_{p2, q1}).$$

The maps $\alpha_{p1, p2, q}$ induce a canonical map

$$\alpha_{p1, p2, q}^* : \lim_{\to} C_{C}(M_{p1, q} \times s_{p1, q}, s_{p2, q} M_{p2, q}) \to C_{C}(R_{p1, p2}) \to C_{C}(H^1_{q}).$$

Lemma \[137\] implies that the image of $\alpha_{p1, p2, q}^*$ is dense in the inductive limit topology: any $C, p$-function on a projective limit with proper maps may be approximated uniformly by a function on one of the spaces in the system. We have seen above that we may assume that all the maps $r_p$ and $r_p, q$ are surjective. Hence so are $\alpha_{p1, p2, q}$ and $\beta_{p1, p2, q1, q2}$.

For $p1, p2, q \in P$, $M_{p1, q} \times s_{p1, q}, s_{p2, q} M_{p2, q} \subseteq M_{p1, q} \times s_{p1, q}, s_{p2, q} M_{p2, q}$ because $s_q s_{p1, q} = s_{p1, q}$ for $q, \alpha p1, q2 \in P$, and this subset is open because the maps $s_q$ are local homeomorphisms. Hence we have a commuting diagram

$$\begin{array}{ccc}
C_{C}(M_{p1, q} \times s_{p1, q}, s_{p2, q} M_{p2, q}) & \xrightarrow{\alpha_{p1, p2, q}^*} & C_{C}(R_{p1, p2}) \\
\downarrow & & \downarrow \\
C_{C}(M_{p1, q} \times s_{p2, q}, s_{p2, q} M_{p2, q}) & \xrightarrow{\alpha_{p1, p2, q1}^*} & C_{C}(R_{p1, p2, q2})
\end{array}$$

where the vertical maps extend functions given on open subsets by 0 outside. Thus the image of $\alpha_{p1, p2, q}^*$ is contained in the image of $\alpha_{p1, p2, q1}^*$. The subspaces

$$K_{p1, p2} := \alpha_{p1, p2, q1}^*(C_{C}(M_{p1, q} \times s_{p1, q}, s_{p2, q} M_{p2, q}))$$

satisfy $K_{p1, p2} \subseteq K_{p1, p2, q2}$. Since $C, p$ is a directed category, $\bigcup_{p1, p2} K_{p1, p2}$ is a dense vector subspace of $C_{C}(H^1_{q})$.

We have related compact operators $E, q \mapsto E, p$ to kernel functions above, getting continuous linear maps

$$K_{p1, p2} \cong C_{C}(M_{p1, q} \times s_{p1, q}, s_{p2, q} M_{p2, q}) \subseteq K(E_{p1}, E_{p2}) \subseteq C_{C}(M_{p1, q} \times s_{p1, q}, s_{p2, q} M_{p2, q}).$$

The convolution on $C_{C}(H^1_{q})$ restricted to $K_{p2, p2, p3} \otimes K_{p1, p2}$ for $p1, p2, p3 \in P$ is a map to $K_{p1, p2}$; the description of the composition of compact operators $E_{p1, q} \mapsto E_{p2, q} \mapsto E_{p3}$ through kernel functions shows that the convolution on $C_{C}(H^1_{q})$ and the composition of compact operators both restrict to the same map $K_{p2, p2} \otimes K_{p1, p2} \mapsto K_{p1, p2}$. Similarly, the involution on $C_{C}(H^1_{q})$ and the involution $\overline{K(E_{p2}, E_{p1})} \mapsto K(E_{p1, q}, E_{p2})$ restrict to the same map $K_{p1, p2} \mapsto K_{p2, p2}$. 
Thus $K_{p,p}$ is a dense *-subalgebra both in $C_c(H^1)$ and in $\mathbb{K}(\mathcal{E}_p, \mathcal{E}_p)$. The *-algebra structure on $C_c(K_{p,p})$ also comes from a groupoid, namely, the groupoid with object space $M_p$ and arrow space $M_p \times s_p X s_p M_p$. This is an equivalence relation, and the map $M_p \to M_p/\sim = s_p(M_p) \subseteq X$ is proper. So the $C^*$-completion $C^*(K_{p,p})$ of $K_{p,p}$ is Morita–Rieffel equivalent to $C_0(s_p(M_p))$. Thus the inclusion map $K_{p,p} \to \mathbb{K}(\mathcal{E}_p, \mathcal{E}_p)$ extends to an isomorphism $C^*(K_{p,p}) \to \mathbb{K}(\mathcal{E}_p, \mathcal{E}_p)$. Thus the inclusion $K_{p,p} \to C_c(H^1) \subseteq C^*(H)$ extends to a *-homomorphism $\mathbb{K}(\mathcal{E}_p, \mathcal{E}_p) \to C^*(H)$.

Let $f \in K_{p_1, p_2}$. Then the norm of $f$ in $\mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1})$ is equal to the norm of $f^* f$ in $\mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_2})$. Therefore, the inclusion $K_{p_1, p_2} \to C_c(H)$ extends to a bounded linear map $\mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}) \to C^*(H)$.

We claim that any $C^*$-seminorm on the dense subspace of $C_c(H^1)$ generated by the subspaces $K_{p_1, p_2}$ extends to $C_c(H^1)$ and hence to $C^*(H)$. First restrict attention to $C_c(H^0) \subseteq C_c(H^1)$; the intersection $C_c(H^0) \cap K_{p,p}$ consists of those functions on $H^0$ that have compact support and that factor through the projection $H^0 \to M_p$. Hence $C_c(H^0) \cap K_{p,p}$ is a $C^*$-algebra, and so carries a unique $C^*$-norm. Thus any $C^*$-seminorm on $C_c(H^0) \cap \bigcup_{p \in P} K_{p,p}$ is dominated by the supremum norm and extends continuously to $C_c(H^0)$. Next if $f \in K_{p_1, p_2}$ is supported in a single bisection $u \subseteq H^1$, then $f^* f \in C_c(H^0) \cap K_{p_2, p_2}$, so any $C^*$-norm has $\|f\|^2 \leq \|f^* f\|_\infty$ and hence extends to $C_0(u)$. We may write elements of $K_{p_1, p_2}$ as finite linear combinations of functions supported in bisections using a partition of unity on $H^1$. Thus any $C^*$-seminorm on $\bigcup K_{p_1, p_2}$ extends continuously first to $C_c(H^1)$ and then to $C^*(H)$.

Thus $C^*(H^1)$ is the $C^*$-completion of the dense *-subalgebra

$$\bigoplus_{g \in G_{p_1, p_2}} \bigcup K_{p_1, p_2}. $$

Since we have bounded linear maps $K_{p_1, p_2} \subseteq \mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1}) \to C^*(H)$, we may replace $K_{p_1, p_2}$ by $\mathbb{K}(\mathcal{E}_{p_2}, \mathcal{E}_{p_1})$ here. This gives exactly the description of the Cuntz–Pimsner algebra of the product system $(\mathcal{E}_p)_{p \in P}$ in the proof of Theorem 3.13.

How does our groupoid model compare to that of Yeend [40]? A topological rank-$k$ graph is the same as an action of the Ore monoid $\mathbb{N}^k$ by topological correspondences. Yeend requires the source maps to be local homeomorphisms, but does not require the range maps to be proper; instead, he assumes a weaker condition called “compact alignment.” Under this assumption, he constructs groupoid models for the Toeplitz $C^*$-algebra and the relative Cuntz–Pimsner algebra of the product system. The relative and absolute Cuntz–Pimsner algebras agree if and only if all range maps $\sigma_p$ are surjective. If these maps are both surjective and proper, then the groupoid model constructed by Yeend [40] is the same one that we have constructed above (see also [14, Lemma 6.6]).

What makes Yeend’s construction difficult, in general, is that he needs certain finite and partially infinite paths to treat non-proper product systems as well. The correct definition of the appropriate set of paths is subtle and specific to the monoids $\mathbb{N}^k$. An aspect that complicates our construction is that we treat more general Ore monoids.

5. Properties of the Groupoid Model

Let an Ore monoid $P$ act on a locally compact space $X$ by topological correspondences $(M_p, \sigma_{p,q})$. The Cuntz–Pimsner algebra of the resulting product system over $P$ is identified with a groupoid $C^*$-algebra $C^*(H)$ in Theorem 4.16. Many properties of $C^*(H)$ are equivalent or closely related to properties of the underlying groupoid $H$. We harvest some known results of this type regarding nuclearity, simplicity or ideal structure, traces and KMS states, and pure infiniteness. We
reformulate some of the relevant dynamical properties of $H$ in terms of the original action of $P$ on $X$.

First, we remark without proof that a groupoid $C^*$-algebra of an étale locally compact groupoid is separable if and only if the underlying groupoid is second countable. This happens if and only if the closed subspace $X' \subseteq X$ is second countable and $G$ is countable. This follows if $X$ is second countable and $P$ is countable; in the latter case, it can be seen directly that the Cuntz–Pimsner algebra is separable, without using the groupoid model.

Secondly, $C^*(H)$ is unital if and only if $H^0$ is compact. This happens if and only if the closed subspace $X' \subseteq X$ is compact because the projection map $r: H^0 \to X'$ is continuous, proper and surjective.

**Theorem 5.1.** The full groupoid $C^*$-algebra $C^*(H)$ is nuclear if and only if the groupoid $H$ is topologically amenable. In that case, $C^*(H)$ belongs to the bootstrap class. The groupoid $H_1 \subseteq H$ is always topologically amenable. If $G$ is amenable, then the groupoid $H$ is also amenable.

**Proof.** An étale, Hausdorff, locally compact groupoid is (topologically) amenable if and only if its reduced $C^*$-algebra is nuclear by [44, Corollary 6.2.14]. Furthermore, if $H$ is amenable, then its reduced and full $C^*$-algebras coincide, so the full one is also nuclear. Conversely, if the full groupoid $C^*$-algebra is nuclear, then so is the reduced one because nuclearity is hereditary for quotients. Hence nuclearity of the full groupoid $C^*$-algebra is also equivalent to amenability of the groupoid.

Any amenable groupoid is “a-T-menable” by [44, Lemma 3.5]; that is, it acts properly and isometrically on a continuous field of affine Euclidean spaces. The proof of the Baum–Connes conjecture for a-T-menable groupoids also shows that their groupoid $C^*$-algebras belong to the bootstrap class, see [44, Proposition 10.7].

Since $C_0(X)$ is nuclear, Theorem 3.18 shows that the unit fibre $O_1$ in the associated Cuntz–Pimsner algebra is always nuclear; then $O$ itself is nuclear if $G$ is amenable. Theorem 4.10 and its proof identify $O_1$ and $O$ with $C^*(H_1)$ and $C^*(H)$, respectively. So the statements about amenability of $H_1$ and $H$ follow from the first sentence in the theorem.

It is elementary to prove the topological amenability of $H_1$ directly. The open subgroupoids $R_{p,p}$ for $p \in P$ are proper equivalence relations. So we may normalise the counting measure on the fibres of $R_{p,p}$ to give an invariant mean on $R_{p,p}$. When we view these invariant means on $R_{p,p}$ as means on $H_1$ for $p$ in the filtered category $C_P$, we get an approximately invariant mean on $H_1$. □

We have not tried to characterise amenability of $H$ in terms of the original action by topological correspondences.

### 5.1. Open invariant subsets and minimality.

**Definition 5.2.** A topological groupoid $H$ is minimal if $H^0$ has no non-trivial open, invariant subsets.

Being minimal is a necessary condition for $C^*(H)$ to be simple because open invariant subsets of $H^0$ generate ideals in $C^*(H)$. We are going to describe the open, invariant subsets of $H^0$ in terms of the original data $(M_p, \sigma_{p,q})$.

**Lemma 5.3.** For $p \in P$ and an open subset $U \subseteq M_p$, let

$$\pi_p^*(U) := \{(m_q)_{q \in P} \in H^0 \mid m_p \in U\} \subseteq H^0.$$ 

The family $B$ of subsets of this form is a base for the topology on $H^0$ and, for each $x \in H^0$, the subsets $\pi_p^*(U)$ with $x \in U$ form a neighbourhood base at $x$. This
base for the topology is closed under finite unions, finite intersections, and under applying \( s_t^{-1} \) for all \( t \in P \); and

\[
(5.4) \quad \tilde{s}_p(\pi^*_p(U)) = \pi^*_t(s_p(U)).
\]

**Proof.** By definition of the product topology, intersections \( \bigcap_{p \in F} \pi^*_p(U_p) \) for finite subsets \( F \subseteq P \) and open subsets \( U_p \subseteq M_p \) for \( p \in F \) form a base of the topology on \( H^0 \), and intersections of such sets with \( x \in \pi^*_p(U_p) \) form a neighbourhood basis for any \( x \in H^0 \). If \( B \) is closed under finite intersections, then \( B \) itself is this canonical base, and similarly for neighbourhoods of \( x \).

Let \( p, q \in P \). Then \( r_{p,q}(m_{pq}) = m_p \) for all \( (m_{t})_{t \in P} \in H^0 \). Thus

\[
(5.5) \quad \pi^*_p(r_{p,q}^{-1}(U)) = \pi^*_p(U)
\]

for each open subset \( U \subseteq M_p \). Since the maps \( r_{p,q} \) are continuous, \( r_{p,q}^{-1}(U) \) is again open. Now let \( F = \{ p_1, \ldots, p_n \} \) and write \( U_i = U_{p_i} \). Since \( P \) is a right Ore monoid, there are \( p \in P \) and \( q_i \in P \) with \( p_i q_i = p \) for \( i = 1, \ldots, n \). Then \( \pi^*_p(U_i) = \pi^*_p(r_{p_i,q_i}^{-1}(U_i)) \). Thus

\[
\begin{align*}
\bigcap_{i=1}^n \pi^*_p(U_i) &= \pi^*_p \left( \bigcap_{i=1}^n r_{p_i,q_i}^{-1}(U_i) \right), \\
\bigcup_{i=1}^n \pi^*_p(U_i) &= \pi^*_p \left( \bigcup_{i=1}^n r_{p_i,q_i}^{-1}(U_i) \right).
\end{align*}
\]

Thus \( B \) is closed under finite intersections and finite unions. We have

\[
(5.6) \quad \tilde{s}_t^{-1}(\pi^*_p(U)) = \pi^*_p(s_{t,p}(U))
\]

because \( \tilde{s}_t((m_{pq})) = (s_{t,p}(m_{pq}))_{p \in P} \), so \( B \) is closed under \( \tilde{s}_t^{-1} \) for all \( t \in P \).

We check (5.4). Let \( (m_{pq})_{q \in P} \in H^0 \) and let \( m'_q := s_{p,q}(m_{pq}) \), so that \( (m'_q) = \tilde{s}_p(m_p) \). If \( m_p \in U \), then \( m'_q \in s_{p,1}(m_p) = s_p(m_p) \in s_p(U) \). Since the map \( s_p \) is open, \( s_p(U) \) is open and invariant of \( X \). We have \( \tilde{s}_p(\pi^*_p(U)) \subseteq \pi^*_t(s_p(U)) \). Conversely, that \( (m'_q)_{q \in P} \in H^0 \) with \( m'_q \in s_p(U) \). We must show that there is \( (m_q)_{q \in P} \in H^0 \) with \( \tilde{s}_p((m_q)) = (m'_q) \) and \( m_p \in U \). By assumption, there is \( m_p \in U \) with \( s_p(m_p) = m'_1 \). The proof of Lemma 4.10 gives a unique complete history \( (m_q)_{q \in P} \) containing \( m_p \) with \( \tilde{s}_p((m_q)) = (m'_q) \). Since \( m_p \in U \), \( (m_q) \in \pi^*_p(U) \), so \( m'_q \in \tilde{s}_p(\pi^*_p(U)) \) as asserted. \( \square \)

**Definition 5.7.** Call an open subset \( U \subseteq X \) (past) **invariant** if \( s_p(r_{p}^{-1}(U)) \subseteq U \) for all \( p \in P \).

The subset \( s_p(r_{p}^{-1}(U)) \subseteq X \) consists of all situations \( x \) for which there is a length-\( p \) story that begins with \( x \) and ends in \( U \). Thus being invariant means that \( U \) contains any possible past of a situation in \( U \).

**Theorem 5.8.** The complete lattice of open \( H \)-invariant subsets of \( H^0 \) is isomorphic to the complete lattice of open invariant subsets of the subset \( X' \subseteq X \) of possible situations. The isomorphism maps an open invariant subset \( V \subseteq X' \) to the open \( H \)-invariant subset \( \bigcup_{p \in P} \pi^*_p(s_{p}^{-1}(V)) \).

A complete history \( (m_q)_{q \in P} \) belongs to \( \pi^*_p(s_{p}^{-1}(V)) \) if and only if \( s_p(m_p) \in V \). Here \( s_p(m_p) \) is the situation at the (past) time \( p \). Thus the open \( H \)-invariant subset of \( H^0 \) associated to an open invariant subset \( V \subseteq X' \) is the set of all complete histories that involve a situation in \( V \) at some time \( p \in P \).

**Proof.** A subset \( W \) of \( H^0 \) is \( H \)-invariant if and only if \( \tilde{s}_t(W) \subseteq W \) and \( \tilde{s}_t^{-1}(W) \subseteq W \) for all \( t \in P \). First we check that subsets of \( H^0 \) of the form \( W := \bigcup_{p \in P} \pi^*_p(s_{p}^{-1}(V)) \) for \( V \subseteq X \) open and invariant are \( H \)-invariant; they are clearly open.
We have \( \pi^*_p(s^{-1}_p(V)) = \tilde{s}^{-1}_p(\pi^*_1(V)) \) by (5.10), so our definition ensures that \( W \) is invariant under \( \tilde{s}_t^{-1} \) for all \( t \in P \). Let \( t \in P \) and let \( x \in W \); we must show that \( \tilde{s}_t(x) \in W \). We have \( x \in \tilde{s}_q^{-1}(\pi^*_1(V)) \) for some \( p \in P \), so \( \tilde{s}_p(x) \in \pi^*_1(V) \). Since \( P \) is a right Ore monoid, there are \( q, u \in P \) with \( pu = tq \). Thus \( \tilde{s}_u \tilde{s}_p = \tilde{s}_q \tilde{s}_t \) and

\[
\tilde{s}_q \tilde{s}_t(x) = \tilde{s}_u(\tilde{s}_p(x)) \in \tilde{s}_u(\pi^*_1(V)) = \tilde{s}_u(\pi^*_u(r^{-1}_u(V))) = \pi^*_1(s_u r^{-1}_u(V)) \subseteq \pi^*_1(V)
\]

by (5.5) and (5.4). That is, \( \tilde{s}_t(x) \in \pi^*_1(V) \). This shows that subsets of \( H^0 \) of the form \( \bigcup_{p \in P} \pi^*_p(s^{-1}_p(V)) \) for \( V \subseteq X \) open and invariant are \( H \)-invariant.

Now let \( W \subseteq H^0 \) be any \( H \)-invariant subset. Write \( W = \bigcup_{i \in I} \pi^*_p(U_i) \) for some index set \( I \) and \( p_i \in P \), \( U_i \subseteq m \), open, using the base \( B \) from Lemma 5.3.

Let \( p = p_i, U = U_i \). Then \( W \) also contains \( \tilde{s}_p(\pi^*_p(U)) = \pi^*_1(V_0) \) with \( V_0 := s_p(U) \); here we used (5.4), and \( V_0 \subseteq X \) is open because the map \( s_p \) is open. Next, for \( q \in P \),

\[
W \supseteq \pi^*_1(V_0) = \pi^*_q(r^{-1}_q(V_0)).
\]

By the argument above, this implies \( W \supseteq \pi^*_1(s_q(r^{-1}_q(V_0))) \). Thus \( W \) contains \( \pi^*_1((V_0)) \), where

\[
\langle V_0 \rangle := \bigcup_{q \in P} s_q(r^{-1}_q(V_0))
\]

denotes the smallest invariant subset of \( X \) containing \( V_0 \). Then \( W \) also contains \( \bigcup_{q \in P} \pi^*_q(s^{-1}_q((V_0))) \). The latter is an open and \( H \)-invariant subset of \( H^0 \), and it contains \( \pi^*_p(s^{-1}_p(V_0)) = \pi^*_q(s^{-1}_q(V_0)) \supseteq \pi^*_q(U) \). Hence

\[
W = \bigcup_{i \in I} \pi^*_p(U_i) = \bigcup_{i \in I, q \in P} \pi^*_q(s^{-1}_q(V_i)) = \bigcup_{q \in P} \pi^*_q(s^{-1}_q(V)),
\]

where the subsets \( V_i := \langle s_p(U_i) \rangle \) and \( V := \bigcup_{i \in I} V_i \) of \( X \) are open and invariant. Thus any open \( H \)-invariant subset of \( H^0 \) comes from an open invariant subset of \( X \).

We may replace \( X \) by \( X' \) everywhere above because restricting to this subspace gives the same groupoid model: impossible situations cannot be the endpoints of complete histories. Thus any open \( H \)-invariant subset of \( H^0 \) comes from an open invariant subset of \( X' \). This subset of \( X' \) is uniquely determined by the projection \( H^0 \rightarrow X \), \( (m_p)_{p \in P} \mapsto m_1 = r_q(m_q) \), maps

\[
\bigcup_{q \in P} \pi^*_q(s^{-1}_q(V)) \mapsto \bigcup_{q \in P} r_q(s^{-1}_q(V)) = V
\]

if \( V \subseteq X' \) is invariant; it is crucial here that for any \( m_q \in s^{-1}_q(V) \), there is a complete history \( (m_p)_{p \in P} \) with given \( m_q \) because all situations in \( V \subseteq X' \) are possible, so endpoints of some complete history.

**Corollary 5.9.** The groupoid \( H \) is minimal if and only if any non-empty, closed invariant subset of \( X \) contains \( X' \).

**Proof.** Taking complements, Theorem 5.8 shows that closed invariant subsets of \( H^0 \) are in bijection with closed invariant subsets of \( X' \). Since \( X' \) is closed in \( X \), the result follows.

**5.2. Effectivity.**

**Definition 5.10.** An étale topological groupoid \( H \) is essentially free if the subset of objects with trivial isotropy is dense in \( H^0 \). It is effective if every open subset of \( H^1 \) of \( H^0 \) contains an arrow \( x \) with \( s(x) \neq r(x) \). Equivalently, the interior of the set \( \{ h \in H^1 \mid r(h) = s(h) \} \) is empty.
For a second countable, locally compact, étale groupoid, being effective or essentially free are equivalent properties by [41, Proposition 3.6] or [6, Lemma 3.1] (these articles use the names “topologically principal” for “essentially free”).

Being essentially free is a variant of the aperiodicity condition that is used to characterise when topological higher-rank graph $C^*$-algebras are (see, for example, [46, Definition 5.2]). We cannot, however check whether $H$ is essentially free without looking at points in $H^0$, that is, infinite paths. In contrast, we will translate the property of $H$ being effective into a property of the action by topological correspondences. Similar criteria for boundary path groupoids of higher-rank topological graphs being effective have been found by Wright [45]: for higher-rank graphs without topology, such criteria are also given in [25, 31, 43].

The following technical definition describes when the groupoid $H$ is effective in terms of the original data of an action by topological correspondences:

**Definition 5.11.** An action $(M_p, \sigma_{p,q})$ of an Ore monoid $P$ on a locally compact Hausdorff space $X$ by proper topological correspondences is effective if for all $p, q \in P$ with $pq^{-1} \neq 1$ in $G$ and for all non-empty, open subsets $U \subseteq X'$, there are $a, f, g \in P$ with $pa = qag$ and $y \in M_{paf} = M_qag$ with $r_{paf}(y) \in U$ and $mid_{p,a,f}(y) \neq mid_{q,a,g}(y)$ in $M_a$. Here $X' \subseteq X$ denotes the closed subset of possible situations; $mid_{p,a,f}(y)$ denotes the component in the middle factor $M_a$ after identifying $M_{paf} \cong M_p \times X M_a \times X M_f$, and similarly for $mid_{q,a,g}(y)$.

**Theorem 5.12.** The groupoid $H$ is effective if and only if the action $(M_p, \sigma_{p,q})$ is effective.

*Proof.* We may assume without loss of generality that $X = X'$.

First we assume that the action $(M_p, \sigma_{p,q})$ is not effective. This means that there are $p, q \in P$ with $pq^{-1} \neq 1$ in $G$ and a non-empty open subset $U \subseteq X'$ such that $mid_{p,a,f}(y) = mid_{q,a,g}(y)$ in $M_a$ for all $a, f, g \in P$ with $pa = qag$ and all $y \in M_{paf}$ with $r_{paf}(y) \in U$. This means that $(p^{-1}x)_a = (q^{-1}x)_a$ for all $x \in \pi_1(U)$ and all $a \in P$. Hence $p^{-1}x = q^{-1}x$ in $H^0$ for all $x \in \pi_1(U)$. Thus the elements of the form $(x, pq^{-1}, x)$ for $x \in \pi_1(U)$ form a bisection $B$ in $H^1 \setminus H^0$ with $r_B = s_B$, which means that $H$ is not effective.

Now assume that the action $(M_p, \sigma_{p,q})$ is effective. Let $U \subseteq H^1 \setminus H^0$ be a non-empty open subset. We need to find $(x, g, y) \in U$ with $x \neq y$.

First, $U \cap R_{p,q}$ is non-empty for some $p, q \in P$. Replacing $U$ by $U \cap R_{p,q}$, we may arrange that $U \subseteq R_{p,q}$. The subgroupoid $H_1$ is an increasing union of equivalence relations, so it is certainly effective. Hence we are done if $pq^{-1} = 1$ in $G$, and may assume from now on that $pq^{-1} \neq 1$ in $G$.

Secondly, we may shrink $U$ to be a bisection because $H$ is étale. We may then shrink further so that $r(U) = \pi_1^*(U_t)$ for some $t \in P$ and some non-empty open subset $U_t \subseteq M_t$, because subsets of the form $\pi_1^*(U_t)$ form a base for the topology on $H^0$ by Lemma [5,3]. Since the map $s_t : M_t \to X$ is a local homeomorphism, we may shrink $U_t$ even further, so that $s_t$ restricts to a homeomorphism from $U_t \subseteq M_t$ onto $s_t(U_t) \subseteq X$.

We are going to show that there is $x \in \pi_1^*(U_t)$ with $p^{-1}x \neq q^{-1}x$. Thus $(x, g, x)$ is not an arrow in $H$; since $r(U) = \pi_1^*(U_t)$ and $U \subseteq R_{p,q}$, there must be $y \in H^0$ with $(x, g, y) \in U \subseteq R_{p,q}$. Since $p^{-1}x = q^{-1}y \neq q^{-1}x$, we have $x \neq y$, as desired.

Since $P$ is Ore, we may find $h, i \in P$ with $ph = ti$. Then we may find $h', i' \in P$ with $qhh' = ti'$. Thus $phh' = tih'$ and $qhh' = ti'$. Since $R_{p,q} \subseteq R_{phh', qhh'}$, we may replace $(p, q)$ by $(phh', qhh')$. Thus we may assume without loss of generality that there are $p, q \in P$ with $p = tp'$ and $q = tq'$.

Recall that $s_t$ restricts to a homeomorphism $U_t \xrightarrow{s_t} s_t(U_t)$. Hence $x \mapsto t^{-1}x$ is a homeomorphism from $\pi_1^*(U_t)$ onto $V := \pi_1^*(s_t(U_t))$. By assumption, there are
$a, f, g \in P$ with $p'af = q'ag$ and $y \in M_p'af$ with $r(y) \in V$ and $\mid f, a, f(y) \neq \mid f, a, g(y)$ in $M_a$. By construction, there is $z \in M_t$ with $s(z) = y$. Then $(z, y) \in M_t \times X M_p'af \cong M_p'af$. We have $p^{-1}(z, y) = (p')^{-1}(y)$ and $q^{-1}(z, y) = (q')^{-1}(y)$ because $p = tp'$ and $q = tq'$. The $M_p$-component of $(p')^{-1}(y)$ is $\mid f, a, f(y)$, and that of $(q')^{-1}(y)$ is $\mid f, a, g(y)$. Since these are different, $p^{-1}x \neq q^{-1}x$ for any $x \in H^0$ with $tp'af$-component $(z, y)$. Such $x$ exist because we have restricted to possible histories throughout, making the maps $\pi_p : H^0 \to M_p$ surjective for all $p \in P$. 

**Theorem 5.13.** Assume that $P$ is countable and $X$ is second countable or, more generally, that $H$ is second countable. The Cuntz–Pimsner algebra $\mathcal{O}$ or, equivalently, the $C^*$-algebra $C^*(H)$, is simple if and only if the following three conditions are satisfied:

1. $C^*(H) = C^*_\tau(H)$;
2. the action $(M_p, \sigma_{p,q})$ is effective;
3. any non-empty closed invariant subset of $X$ contains $X'$.

The first condition above follows if $H$ is amenable and, in particular, if $G$ is amenable.

**Proof.** We use [6, Theorem 5.1], which characterises when the groupoid $C^*$-algebra of a second-countable, locally compact, Hausdorff, étale groupoid is simple. The groupoid $H$ is always locally compact, Hausdorff, and étale. The third condition is equivalent to the minimality of $H$ by Corollary [5.29]. Since $H$ is second countable, it is essentially free if and only if it is effective by [6, Lemma 3.1]. This is equivalent to the effectivity of the action $(M_p, \sigma_{p,q})$ by Theorem [5.12]. Thus our conditions are equivalent to the three conditions in [6, Theorem 5.1].

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### 5.3 Invariant measures

In the following, a “measure” on a locally compact space $X$ means a Radon measure or, equivalently, a positive linear functional on $C_c(X)$. We assume $X$ to be second countable and $P$ to be countable so that $H$ is second countable. Let $c : P \to (0, \infty)$ be a homomorphism to the multiplicative group of positive real numbers. This extends to the group completion $G$ and then to $H$, by letting $c|_{M_p} = c(p)/c(q)$. This cocycle on $H$ generates a 1-parameter group of automorphisms of $C^*(H)$, see [38, Section 5].

**Definition 5.14.** A measure $\mu$ on $H^0$ is $c$-invariant if $\mu(r(B)) = c(g)\mu(s(B))$ for any bisection $B \subseteq H^0_g$. If $c \equiv 1$, we speak simply of invariant measures.

A $c$-invariant measure on $H^0$ gives a KMS-weight on $C^*(H)$ for the corresponding automorphism group (with temperature 1); conversely, if $H$ has trivial isotropy groups, then any KMS-weight on $C^*(H)$ for this 1-parameter group of automorphisms is of this form for a unique $c$-invariant measure on $H^0$ (see [38, Proposition 5.4]). In particular, invariant measures on $H^0$ give traces on $C^*(H)$, and if $H$ has trivial isotropy, then any trace on $C^*(H)$ comes from an invariant measure on $H^0$.

We are going to describe invariant measures on $H^0$ in terms of measures on $X$. This requires two operations on measures: push-forwards along continuous maps and pull-backs along local homeomorphisms. The first is standard; if $f : X \to Y$ is a continuous map and $\mu$ is a measure on $X$, then $f_\mu$ is the measure on $Y$ defined by $f_\mu(B) = \mu(f^{-1}(B))$ for Borel subsets $B \subseteq Y$. If $f : X \to Y$ is a local homeomorphism and $\lambda$ is a measure on $Y$, then $f^*\lambda$ is the measure on $X$ defined by

$$f^*\lambda(B) = \int_Y |f^{-1}(y) \cap B| \, d\lambda(y).$$
If \( h : X \to \mathbb{C} \) is Borel measurable with compact support, then
\[
\int_X h(x) \, d(f^* \lambda)(x) = \int_Y \sum_{\{x \in f^{-1}(y)\}} h(x) \, d\lambda(y)
\]
because this holds for characteristic functions of Borel subsets.

**Definition 5.15.** A measure \( \lambda \) on \( X \) is \( c \)-invariant if \( \lambda = c(p)(r_p)_* s_p^*(\lambda) \) for all \( p \in P \).

**Theorem 5.16.** The map \( (\pi_1)_* \) induced by the projection \( \pi_1 : H^0 \to X \) is a bijection between \( c \)-invariant measures on \( H^0 \) and \( c \)-invariant measures on \( X \).

**Proof.** A measure \( \mu \) on \( H^0 \) gives measures \( \mu_p := (\pi_p)_*(\mu) \) on \( M_p \) for all \( p \in P \). These are linked by \( (r_{p,q})_* \mu_{pq} = \mu_p \) for all \( p, q \in P \) because \( r_{p,q} \circ \pi_{pq} = \pi_p \). Conversely, we claim that any family of measures \( (\mu_p)_{p \in P} \) with \( (r_{p,q})_* \mu_{pq} = \mu_p \) for all \( p, q \in P \) comes from a unique measure \( \mu \) on \( H^0 \). This is because \( \bigcup_{p \in P} \pi_p^*(C_c(M_p)) \) is a dense subspace in \( C_c(H^0) \). The consistency condition \( r_{p,q} \circ \pi_{pq} = \pi_p \) implies that the positive linear maps \( C_c(M_p) \to \mathbb{C} \), \( f \mapsto \int_{M_p} f(x) \, d\mu_p(x) \), well-define a positive linear map on \( \bigcup_{p \in P} \pi_p^*(C_c(M_p)) \). Any such positive linear map extends uniquely to a positive linear map on \( C_c(H^0) \). Thus we may replace a measure \( \mu \) on \( H^0 \) by a family of measures \( (\mu_p)_{p \in P} \) on the spaces \( M_p \) whenever this is convenient.

Next we claim that \( \mu \) is \( c \)-invariant if and only if \( \mu_p = c(p)s_p^*(\mu) \) for all \( p \in P \).

Assume first that \( \mu \) is \( c \)-invariant. Let \( p \in P \) and let \( U \subseteq M_p \) be an open subset such that \( s_p|U : U \to X \) is injective. Then \( \{(x, p^{-1}x) \mid x \in \pi_p^*(U)\} \subseteq R_{p,1} \) is a bisection with range \( \pi_p^*(U) \) and source \( \pi_1^*(s_p(U)) \). Since \( \mu \) is \( c \)-invariant,
\[
\mu_p(U) = \mu(\pi_p^*(U)) = c(p)\mu(\pi_1^*(s_p(U))) = c(p)\mu_1(s_p(U)).
\]
This equality holds for all open subsets of \( U \) because \( s_p \) is still injective on them. This implies \( \mu_p(B) = c(p)\mu_1(s_p(B)) \) for all Borel subsets \( B \) of \( U \).

If \( B \subseteq M_p \) is an arbitrary Borel subset, then we may cover \( B \) by open subsets on which \( s_p \) is injective because \( s_p \) is a local homeomorphism. Then we may decompose \( B \) as a countable disjoint union \( B = \bigsqcup_{i \in \mathbb{N}} B_i \) of Borel subsets with \( B_i \subseteq U_i \) for open subsets \( U_i \) such that \( s_p|U_i \) is injective for all \( i \). Applying the formula above for each \( i \) gives
\[
\mu_p(B) = c(p)\int_X |s_p^{-1}(x) \cap B| \, d\mu_1(x) = c(p)s_p^*\mu_1(B).
\]
Thus \( \mu_p = c(p)s_p^*\mu_1 \) if \( \mu \) is \( c \)-invariant.

Conversely, assume \( \mu_p = c(p)s_p^*\mu_1 \). Let \( g = pq^{-1} \in G \) and let \( V \subseteq H^0_1 \) be a bisection. We may assume without loss of generality that \( V \subseteq R_{p,q} \), replacing \( (p, q) \) by \( (ph, qh) \) for some \( h \in P \) if necessary. Decomposing \( V \) into disjoint Borel subsets, we may further reduce to the case where \( s(V) \subseteq H^0 \) is one of the base neighbourhoods in Lemma 5.13. Say, \( s(V) = \pi_v^*(U) \) for an open subset \( U \subseteq M_t \). Replacing \( (p, q, t) \) by \( (pa, qa, tb) \) for suitable \( a, b \in P \), we may arrange \( t = p \) by the Ore condition. We assume this from now on. Decomposing \( V \) even further, we may arrange that \( s_p|U \) is injective. Then \( V \) is the product of two bisections, one of the form \( \{(x, p^{-1}x) \mid x \in \pi_p^*(U)\} \), the other of the form \( \{y, q^{-1}y \mid y \in r(V)\} \); here \( q^{-1}r(V) = p^{-1}\pi_p^*(U) = \pi_1^*(s_p(U)) \). So \( r(V) \) must be of the form \( \pi_q^*(W) \) for some \( W \subseteq M_q \) for which \( s_p|W \) is a homeomorphism onto \( s_p(U) \).

The upshot of these reductions is that \( \mu \) is \( c \)-invariant once the \( c \)-invariance condition holds for bisections of the form \( \{(x, p^{-1}x) \mid x \in \pi_p^*(U)\} \) with \( p \in P \) and open \( U \subseteq M_p \). But this is exactly the condition \( \mu_p = c(p)s_p^*\mu_1 \). This finishes the proof of the claim.
The claim shows that the family of measures \((\mu_p)_{p \in P}\) and hence the measure \(\mu\) is determined uniquely by the measure \(\mu_1\) on \(M_1 = X\) provided \(\mu\) is \(c\)-invariant. If we are given a measure \(\lambda\) on \(X\), then \(\mu_1 := c(p) s_p^* \lambda\) is the only possibility for a \(c\)-invariant measure on \(H^0\) with \(\mu_1 = \lambda\). This family of measures gives a measure on \(H^0\) if and only if \((r_{p,q})_* \mu_{pq} = \mu_p\) for all \(p, q \in P\). In particular, for \(p = 1\) and \(q \in P\), this gives the condition \(c(q)(r_q)_* s_q^* \lambda = (r_q)_* \mu_q = \lambda\), that is, \(\lambda\) has to be \(c\)-invariant.

The proof of the theorem will be finished by checking that
\[
c(p) s_p^* \lambda = c(pq)(r_{p,q})_* s_{pq}^* \lambda
\]
holds for all \(p, q \in P\) if \(\lambda\) is a \(c\)-invariant measure on \(X\). Since \(c(pq) = c(p)c(q)\), we have to prove \(c(q)(r_{p,q})_* s_{pq}^* \lambda = s_q^* \lambda\). Let \(U \subseteq M_p\) be an open subset. On the one hand, \((s_q^* \lambda)(U) := \int_X |s_q^{-1}(x) \cap U| d\lambda(x)\). Substituting \(c(q)(r_q)_* s_q^* \lambda\) for \(\lambda\), this becomes
\[
(5.17) \quad c(q) \int_X |s_q^{-1}(x) \cap U| d(r_q)_* s_q^* \lambda(x) = c(q) \int_{M_q} |s_q^{-1}(r_q(z)) \cap U| d\lambda(z)
\]
On the other hand,
\[
(5.18) \quad c(q)(r_{p,q})_* s_{pq}^* \lambda(U) = c(q) s_{pq}^* \lambda(r_{p,q}(U)) = c(q) \int_{X} |s_{pq}^{-1}(x) \cap r_{p,q}(U)| d\lambda(x).
\]
We may identify \(M_{pq} \cong M_p \times_{s_p, X, r_q} M_q\), and \(r_{p,q}\) is the projection to the first factor. Hence \(s_{pq}^{-1}(x) \cap r_{p,q}(U)\) is the set of pairs \((y, z)\) with \(y \in U \subseteq M_p, \ z \in M_q, \ s_p(y) = r_q(z)\) and \(s_q(z) = x\). The cardinality of this set is the sum over all \(z \in s_q^{-1}(x)\) of the cardinalities of \(s_p^{-1}(r_q(z)) \cap U\). Hence the right hand sides in (5.17) and (5.18) are equal, as desired. □

Remark 5.19. It is shown in [23] that any stably finite, exact, unital C*-algebra has a tracial state. Conversely, a C*-algebra with a tracial state is stably finite. Thus a unital, exact C*-algebra has a tracial state if and only if it is stably finite.

We have already remarked that \(C^*(H)\) is unital if and only if \(X^f\) is compact, and exactness follows if \(H\) is amenable, compare Theorem 5.1. Hence Theorem 5.1 gives a necessary and sufficient criterion for \(C^*(H)\) to be stably finite in the case where \(X^f\) is compact metrisable, \(P\) is countable, and \(H\) is amenable.

5.4. Local contractivity.

Definition 5.20 ([3]). A locally compact, Hausdorff, étale groupoid \(\mathcal{G}\) is locally contracting if, for every non-empty open subset \(U \subseteq \mathcal{G}^0\), there are an open subset \(V \subseteq U\) and a bisection \(B\) of \(\mathcal{G}\) such that \(\nabla \subseteq s(B)\) and \(r(B\nabla) \subseteq V\).

Since \(H\) is an étale, locally compact groupoid, [3, Proposition 2.4] shows that \(C^*_r(H)\) is purely infinite (that is, every hereditary C*-subalgebra contains an infinite projection) if \(H\) is essentially free and locally contracting. Actually, we only need \(H\) to be effective here by [3, Lemma 3.1.(4)]: this is exactly the condition in [3, Lemma 2.3] that is used in the proof of [3, Proposition 2.4]. We would, however, not expect local contractivity to be necessary for \(C^*_r(H)\) to be purely infinite.

The following definition characterises when the groupoid \(H\) is locally contracting in terms of the original action by topological correspondences:

Definition 5.21. An action \((M_p, \sigma_{p,q})\) of an Ore monoid \(P\) on a locally compact Hausdorff space \(X\) by proper topological correspondences is locally contracting if for any open subset \(S \subseteq X^f\), there are \(n \in \mathbb{N}\) and \(p_1, q_i, a_i, b_i \in P\) and \(W_i \subseteq M_{p_1} \times_{s_{p_1}, X, s_{P}} M_{q_i}^0\) for \(1 \leq i \leq n\) such that...
(LC1) \( p_1a_1 = p_2a_2 = \cdots = p_na_n = q_1b_1 = \cdots = q_nb_n; \)

(LC2) the restricted coordinate projections \( \text{pr}_1: W_i \to M'_{p_i}, \) and \( \text{pr}_2: W_i \to M'_{q_i}, \) are injective and open;

(LC3) the subsets
\[
\text{pr}_1(W_i)M'_{a_i} := \{ \sigma_{p_i,a_i}(x_1, x_2) \mid x_1 \in \text{pr}_1(W_i), \ x_2 \in M'_{a_i}, \ s_{p_i}(x_1) = r_{a_i}(x_2) \}
\]
of \( M'_{p_i,a_i} \) are disjoint, and so are the subsets \( \text{pr}_2(W_i)M'_{b_i}; \)

(LC4) \( \bigcup_{i=1}^n \text{pr}_1(W_i)M'_{a_i} \subseteq \bigcup_{i=1}^n \text{pr}_2(W_i)M'_{b_i}; \)

(LC5) \( r_b, \text{pr}_2(W_i) \subseteq \mathcal{S}; \)

(LC6) \( p_iq_i^{-1} \neq p_jq_j^{-1} \) in \( G \) for \( i \neq j. \)

Here \( X' \subseteq X \) is the closed invariant subspace of possible situations, which is different from \( X \) if some of the range maps are not surjective.

The choice of \( a_i, b_i \) does not really matter: if the conditions hold for one choice satisfying (LC1) then also for all others. This follows from Lemma 4.15 and the surjectivity of the maps \( r_{pq} \) on the \( \mathcal{M}_{pq}. \)

Giving up some symmetry, we may use the Ore conditions to simplify the data above slightly: we may assume either \( p_1 = p_2 = \cdots = p_n \) and \( a_1 = a_2 = \cdots = a_n \) or \( q_1 = q_2 = \cdots = q_n, \) and \( b_1 = b_2 = \cdots = b_n. \)

Condition (LC6) says, roughly speaking, that we cannot make \( n \) smaller by combining the data for any \( i \neq j. \) This is its only role, and it could be left out.

The case \( n = 1 \) in this criterion is less complicated and gives a sufficient condition for local contractivity. It says that for any subset \( S \subseteq X' \), there are \( p, q, a, b \in \mathcal{P} \) with \( p\mathcal{A} = q\mathcal{B} \) and a subset \( W \subseteq M_p \times s_p, x, s_q, M_q \) such that the projections \( \text{pr}_1: W \to M_p \) and \( \text{pr}_2: W \to M_q \) are injective and open, and \( \text{pr}_1(W) \cdot M_p \subseteq \text{pr}_2(W) \cdot M_q \) as subsets of \( \mathcal{M}_{pq} = \mathcal{M}_{q,p}. \) We could not prove that this condition is necessary for \( H \) to be locally contracting.

[42] Proposition 5.8] gives a sufficient condition for the boundary path groupoid of a higher-rank topological graph to be locally contracting. [42] Lemma 5.9 shows that the condition in [42] Proposition 5.8] implies that the action by topological correspondences is locally contracting, with \( n = 1 \) in the above definition.

**Theorem 5.22.** The groupoid \( H \) is locally contracting if and only if the action \( (M_p, \sigma_{pq}) \) is locally contracting.

**Proof.** We replace \( X \) by \( X' \) throughout, so that the maps \( r_{pq} \) are all surjective.

Call a subset \( U \) of \( H^0 \) good if there are \( V, B, \) as in Definition 5.20. If \( U_1 \subseteq U_2 \) and \( U_1 \) is good, then so is \( U_2. \) And if \( T \subseteq H^1 \) is a bisection, then \( r(T) \) is good if and only if \( s(T) \) is good: given \( V, B \) for \( s(T), \) then \( TV \) and \( TBT^{-1} \) will work for \( r(T). \)

If \( U \subseteq H^0 \) is an arbitrary open subset, then there are \( p \in \mathcal{P} \) and \( U_p \subseteq M_p \) such that \( \pi_p(U_p) \subseteq U, \) \( s_p|_{U_p} \) is injective, and \( U_p \) is relatively compact and open; here we use that subsets of the form \( \pi_p(U_p) \) for \( U_p \subseteq M_p \) open form a base (Lemma 5.3), that \( M_p \) is locally compact, and that \( s_p \) is a local homeomorphism for each \( p \) (Lemma 4.10). Hence there is a bisection with range \( \pi_p(U_p) \) and source \( \pi_1(s_p(U_p)). \) Thus \( U \) is good when \( \pi_1(s_p(U_p)) \) is good. Summing up, all non-empty open subsets of \( H^0 \) are good once all non-empty subsets of the form \( \pi_1(S) \) with \( S \subseteq X \) relatively compact and open are good.

Assume now that the action \( (M_p, \sigma_{pq}) \) is locally contracting. Let \( S \subseteq X' \) be a non-empty, open, relatively compact set. Pick \( n \in \mathbb{N} \) and \( p_i, q_i, a_i, b_i \in \mathcal{P} \) and subsets \( W_i \subseteq M_{p_i} \times s_{p_i}, x, s_{q_i}, M_{q_i} \) as in Definition 5.21. Let

\[
B_i := \{(x_1y, p_iq_i^{-1}, x_2y) \mid (x_1, x_2) \in W_i, \ y \in H^0, \ s_{pi}(x_1) = \pi_1(y)\};
\]
here \(x_1y\) and \(x_2y\) are well-defined because \(s_{p_i}(x_1) = s_{q_i}(x_2) = \pi_1(y)\), which uses that \(s_{p_i}(x_1) = s_{q_i}(x_2)\) for \((x_1, x_2) \in W_i\). Since \(p_i^{-1}(x_1y) = y = q_i^{-1}(x_2y)\), we have \(B_i \subseteq \mathcal{R}_{p_i,q_i}\). Condition \([\text{LC2}]\) ensures that the range and source maps are open and injective on each \(B_i\), so these are bisections. For different \(i\), they have disjoint ranges and sources and so \([\text{LC3}]\). Hence \(B := B_1 \cup B_2 \cup \cdots \cup B_n\) is a bisection as well. Condition \([\text{LC4}]\) gives \(r(B) \subseteq s(B)\), and we have \(s(B) \subseteq \pi_1^*(S)\) because of \([\text{LC5}]\). Since \(S\) is a relatively compact, the closed subset \(r(B)\) is compact. So there is an open subset \(V\) with \(r(V) \subseteq V \subseteq V' \subseteq s(B)\). That is, \(\pi_1^*(S)\) is good. This implies that \(H\) is locally contracting.

Conversely, assume that \(H\) is locally contracting. Let \(S \subseteq X'\) be a relatively non-empty, open subset and let \(U := \pi_1^*(S)\). Then \(U\) is relatively compact, non-empty and open because \(\pi_1: H^0 \to X'\) is surjective, continuous, and proper. Since \(H\) is locally contracting, there is a bisection \(B \subseteq H^1\) with \(r(B) \subseteq s(B) \subseteq U\). Next we have to analyse this bisection \(B\) locally. This does not yet use the special feature of \(B\) and gives slightly more, namely, a “base” for the inverse semigroup of bisections of \(H\). By this we mean an inverse subsemigroup closed under finite intersections that covers \(H^1\). This can be used to study actions of \(H\) on \(C^\star\)-algebras as in \([8]\).

**Lemma 5.23.** Let \(B \subseteq H^1\) be a bisection and \(\eta \in B\). Then there is an open bisection \(B_\eta\) with \(\eta \in B_\eta \subseteq B\) of the following form:

\[
B_\eta := \{(x_1y, pq^{-1}, x_2y) \mid (x_1, x_2) \in W, y \in H^0, s_p(x_1) = \pi_1(y)\}
\]

for some \(p, q \in P\) and a subset \(W \subseteq M_p \times s_p, x, s_q M_q\) such that \(\text{pr}_1: W \to M_p\) and \(\text{pr}_2: W \to M_q\) are injective and open.

If \(pl = p_1, q_1 = q_1\) and

\[
W_1 := \{(x_1y, x_2y) \mid (x_1, x_2) \in W, y \in W, s_p(x_1) = \pi_1(y)\}
\]

for some \(t \in P\), then the data \((p, q, W)\) and \((p_1, q_1, W_1)\) define the same bisection \(B_\eta\).

**Proof.** Let \(\xi := s(\eta)\) and \(B\xi := r(\eta)\); since \(\eta \in B\) is uniquely determined by \(\xi\), this defines a homeomorphism \(\xi \mapsto B\xi\) from \(s(B)\) to \(r(B)\).

There are \(p, q \in P\) with \(\eta \in R_{p,q}\) because these subsets cover \(H^1\). Since the subsets \(B\) and \(R_{p,q}\) and the map \(s: H^1 \to H^0\) are open, \(s(B \cap R_{p,q})\) is an open neighbourhood of \(\xi\). Let \(h \in B \cap R_{p,q}\). Then we may write \(s(h)\) uniquely as \(s(h) = x_2y\) for \(x_2 \in M_q\) and \(y \in H^0\) with \(s_q(x_2) = \pi_1(y)\); namely, \(x_2 = \pi_q(s(h))\) and \(y = q^{-1}(s(h))\). Similarly, \(r(h) = x_1y'\) for \(x_1 \in M_p\) and \(y' \in H^1\) with \(s_p(x_1) = \pi_1(y')\); namely, \(x_1 = \pi_p(r(h))\) and \(y' = p^{-1}(r(h))\). The assumption \(h \in R_{p,q}\) means exactly that \(y = y'\). Thus \(h = (x_1y, pq^{-1}, x_2y)\) for \(x_1 \in M_p, x_2 \in M_q, y \in H^0\) with \(s_p(x_1) = s_q(x_2) = \pi_1(y)\).

Since \(s_p\) is a local homeomorphism, there is an open neighbourhood \(V\) around \(\pi_p(B\xi) \in M_p\) so that \(s_p|_V: V \to X\) is a homeomorphism onto an open subset of \(X\). The subset

\[
V' := \{x \in s(B \cap R_{p,q}) \mid \pi_p(Bx) \in V\}
\]

is still an open neighbourhood of \(\xi\). It contains a neighbourhood of \(\xi\) that belongs to the base in Lemma \([5.3]\). This gives us \(q_2 \in P\) and \(V'' \subseteq M_{q_2}\) with \(\xi \in \pi_{q_2}^*(V'') \subseteq V'\). Since \(s_{q_2}\) is a local homeomorphism as well, we may further shrink \(V''\) so that \(s_{q_2}|_{V''}\) becomes injective; we assume this. The first Ore condition gives us \(a, b \in P\) with \(qa = qb\). Let \(p' := pa\) and \(q' := qa = qb\). Let

\[
W := \{((\pi_{p'}(r(h))), \pi_{q'}(s(h))) \mid h \in B, \pi_{q_2}(s(h)) \in V''\}.
\]

We claim that \(p', q', W\) have the asserted properties.
Let \( h \in B \) satisfy \( \pi_{q_2}(s(h)) \in V'' \). Write \( s(h) = x_{2y}, r(h) = x_1y \) with \( x_1 \in M_p, x_2 \in M_q, y \in H^0 \) and \( s_p(x_1) = s_q(x_2) = \pi_1(y) \) as above. Then \( x_1 \in V \), so \( x_1 \) is the unique point in \( V \) with \( s_p(x_1) = s_q(x_2) \). Now write \( y = y_1y_2 \) with \( y_1 \in M_p, y_2 \in H^0 \), \( s_a(y_1) = \pi_1(y_2) \). Then \( s(h) = x_1y_1y_2 \) and \( r(h) = x_2y_1y_2 \). The point \( x_1y_1 \in M_{pq} = M_{p'} \) is the unique one in \( r_{p,o}(V) \) with \( s_{p,o}(x_1y_1) = y_1 \). This shows that \( \pi_{p'}(r(h)) \) is determined by \( \pi_p(s(h)) \) and that the map that takes \( \pi_{p'}(s(h)) \) to \( \pi_{q'}(r(h)) \) is continuous. Hence the second coordinate projection \( pr_2 : W \to M_{p'} \) is injective and open. Since we assumed \( s_{q_2} \) to be injective on \( V'' \), the same holds for \( pr_1 : W \to M_{p'} \).

If \( (x_{1y}, pq^{-1}, x_2y) \in B_\eta \), then \( p^{-1}(x_{1y}) = y = q^{-1}(x_2y) \), so \( (x_{1y}, pq^{-1}, x_2y) \in R_{p,q} \) and \( B_\eta \subseteq H^1 \). Since \( pr_1 : W \to M_{p'} \) and \( pr_2 : W \to M_{q'} \) are injective and open, so are the maps \( s, r : B_\eta \to H^0 \) because \( y_2 \) is common to both source and range and \( x_1y_1 \) and \( x_2y_1 \) determine each other uniquely and continuously. Thus \( B_\eta \) is a bisection. It is clear from the construction that \( \eta \in B_\eta \subseteq B \). The last statement about different data giving the same \( B_\eta \) is implicitly shown above. 

Fix \( \xi_0 \in s(B) \setminus \bigcup \eta \). The subset \( \bigcup \{ \xi_0 \} \subseteq s(B) \subseteq U = \pi_1(S) \) is closed and contained in the relatively compact subset \( \pi_1(S) \), so it is compact. For each \( x \in \bigcup \{ \xi_0 \} \subseteq s(B) \), there is a unique \( \eta \in B \) with \( s(\eta) = x \). Let \( B_\eta \subseteq B \) be some bisection as in Lemma \( 5.23 \). The open subsets \( s(B_{\xi_0}) \) for these chosen bisections cover \( \bigcup \{ \xi_0 \} \). By compactness, this only needs finitely many of them, say, \( B_1, \ldots, B_t \). Let \( B' = B_1 \cup B_2 \cup \cdots \cup B_t \subseteq B \). This is still an open bisection, and it satisfies \( \bigcup \{ \xi_0 \} \subseteq s(B') \subseteq s(B) \subseteq U \). Since \( \xi_0 \in s(B') \setminus \bigcup \{ \xi_0 \} \), we get \( \bigcup \{ \xi_0 \} \subseteq s(B'). \) Hence we may replace \( B \) by \( B' \).

Each \( B_\eta \) is constructed from certain \( p_i, q_i \in P \) and \( W_i \subseteq M_{p_i} \times M_{q_i} \), as in Lemma \( 5.23 \). The Ore conditions also provide \( e, a_i, b_i \in P \) with \( p_i a_i = \pi(c_i) \) for all \( i \). Our construction so far already achieves the most crucial properties \( \{ \text{LC1}, \text{LC2}, \text{LC4} \} \) and \( \{ \text{LC5} \} \) the last one follows from \( s(B') \subseteq s(B) \subseteq \pi_1(S) \). So far, however, the subsets \( s(B_\eta) \) and \( r(B_\eta) \) may still overlap, and among the group elements \( p_i q_i^{-1} \) may well be equal. We now rectify this.

Let \( p_i q_i^{-1} = p_j q_j^{-1} \) for some \( i \neq j \). We may replace \( (p_i, q_i) \) by \( (p_i c_i, q_i c_i) \) for \( c_i \in P \) using the last statement in Lemma \( 5.23 \). By this, we can arrange that \( p_i = p_j \) and \( q_i = q_j \), which we now assume. Then

\[
B_i \cup B_j = \{(x_{1y}, pq^{-1}, x_2y) \mid (x_1, x_2) \in W_i \cup W_j, \ y \in H^0, s_p(x_1) = \pi_1(y)\}.
\]

We know that \( B_i \cup B_j \) is a bisection because it is contained in the bisection \( B \). Since \( X' = X' \), this implies that the coordinate projections remain injective on \( W_i \cup W_j \). They are open because this holds locally on \( W_i \) and \( W_j \). Hence we may simply merge the data \( (p_i, q_i, W_i) \) and \( (p_j, q_j, W_j) \) into one piece, without changing the bisection \( B' \).

We go on merging part of our data until all group elements \( g_i := p_i q_i^{-1} \) are different. This achieves \( \{ \text{LC6} \} \). By construction, \( B_i \subseteq R_{p_i, q_i} \subseteq H_{p_i}^0 \), and these subsets are disjoint for different \( g_i \). Hence the \( B_i \) are disjoint bisections. This implies that their range subsets and their source subsets are disjoint, which is \( \{ \text{LC3} \} \). Hence all the conditions in Definition \( 5.21 \) hold.

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