Abstract—To account for joint tracking and classification (JTC) of multiple targets from observation sets in presence of detection uncertainty, noise, and clutter, this paper develops a new trajectory probability hypothesis density (TPHD) filter, which is referred to as the JTC-TPHD filter. The JTC-TPHD filter classifies different targets based on their motion models and each target is assigned with multiple class hypotheses. By using this strategy, we can not only obtain the category information of the targets, but also a more accurate trajectory estimation than the traditional TPHD filter. The JTC-TPHD filter is derived by finding the best Poisson posterior approximation over trajectories on an augmented state space using the Kullback-Leibler divergence (KLD) minimization. The Gaussian mixture is adopted for the implementation, which is referred to as the GM-JTC-TPHD filter. The L-scan approximation is also presented for the GM-JTC-TPHD filter, which possesses lower computational burden. Simulation results show that the GM-JTC-TPHD filter can classify targets correctly and obtain accurate trajectory estimation.

Index Terms—Multi-target tracking, trajectory RFS, joint tracking and classification.

I. INTRODUCTION

The probability hypothesis density (PHD) filter [1]–[3] is a widely used approach in the multi-target tracking, which aims to model the appearance and disappearance of targets, false detections and misdetections of measurements based on the random finite set (RFS) [1]. The PHD filter is known for its low computational burden, which considers a Poisson multi-target filtering density. If the prior or posterior density is not Poisson, the PHD filter finds the best Poisson approximation to enjoy a conjugate closure by minimizing the Kullback-Leibler divergence (KLD) [4]. There are several common implementations for the PHD filter, such as sequential Monte Carlo (SMC) [5] or Gaussian mixture (GM) [3]. Following the same routine, the cardinalized PHD (CPHD) filter [6], the multi-Bernoulli (MB) filter [7], the Poisson multi-Bernoulli mixture (PMBM) filter [8], the generalized labeled multi-Bernoulli (GLMB) filter [9], [10], and the labeled multi-Bernoulli (LMB) filter [11] are also devised.

In the multi-target tracking, we also need to obtain more accurate trajectory estimation and eliminate the trajectory fragmentation [12]. Recently, using a set of trajectories as the posterior density [13]–[16] or a (labeled) multi-target state sequence posterior [12] provides an efficient approach to the requirements above. Among these approaches, the trajectory PHD (TPHD) filter [13] establishes trajectories from first principles using trajectory RFSs. The TPHD filter propagates the best Poisson multi-trajectory approximation using the KLD minimization [4]. The Gaussian mixture is proposed to obtain a closed-form solution of the TPHD filter, which is given as the GM-TPHD filter. Meanwhile, the L-scan approximation strategy is suggested to achieve the fast implementation of the TPHD filter by only updating the multi-trajectory density of the last $L$ and keeps the rest unchanged.

The joint tracking and classification (JTC) [17]–[23] of targets is also a critical problem in radar detection fields. For example, in the battlefield surveillance, we need to identify incoming aircrafts and missiles instead of only considering tracking them. Generally, the motion models [17] or other characteristics of targets like the extended target model [19] are related to their categories. In principle, correctly classifying targets and assigning a class-matched filter can also improve the accuracy of both detection and track estimation. In [17], the JTC model is extended to the PHD filter with the particle implementation, which considers each target class is assigned with a class-matched PHD-like filter. This filter propagates particles based on their class-dependent motion models in the prediction step and exchanges the mutual information between these PHD-like filters by updating the particle weights. Later, in [18], the Gaussian implementation of the jump Markov system [24]–[27] PHD (JMS-PHD) filter combined with the JTC model is discussed. Both above JTC methods are based on the estimation of the mixed density probability, and the classification is significantly dependent on the estimation. It is worth noting that, in [22], the GLMB filter is combined with a novel joint decision and estimation algorithm [28], [29] based on the generalized Bayes risk to solve the JTC problem. Such risk defined in the GLMB filter involves both the estimation costs of cardinality and states, and the classification cost.

In this paper, the concept of JTC is developed into the trajectory RFS. Combined with the JTC model, a new TPHD filter is proposed to obtain the trajectory information and classify different kinds of targets, which is referred to as the JTC-TPHD filter. In this filter, we assign possible category hypotheses for each trajectory and the corresponding multiple motion models for each category. After the prediction and update step, the category hypothesis with the maximum posterior probability is extracted to serve as the classification result of the target at this moment, and then the trajectory estimation of the target is extracted from this category hypothesis. Besides, the Gaussian mixture is adopted to obtain an efficient implementation of the
Bayesian Filtering Recursion

\[ k \]

Kronecker delta function is expressed as,

\[ b \]

of \( T \)

\[ \beta \]

step \( k \) at time \( n \) dimension of target state. Supposing there are union, \( \{ \beta, \xi \} \) \( 1 \leq \beta \leq k \) and \( 1 \leq \xi \leq k - \beta + 1 \). Therefore, a single trajectory up to time step \( k \) belongs to the space \( \mathcal{T}_k = \mathcal{U}_{(\beta, \xi) \in \mathcal{U}_k} \{ \beta \} \times \mathbb{R}^{x_{n \times n}}, \) where \( \mathcal{U} \) denotes the disjoint union, \( \times \) denotes a Cartesian product, and \( n_x \) represents the dimension of target state. Supposing there are \( N_k \) trajectories at time \( k \), the set of trajectories is denoted as

\[ X_k = \{ X_1, \ldots, X_{N_k} \} \subset \mathcal{F}(\mathbb{T}_k), \]

where \( \mathcal{F}(\mathbb{T}_k) \) is the respective collections of all finite subsets of \( \mathbb{T}_k \). The inner product between two real valued functions is expressed as,

\[ \langle \cdot \rangle \]

surements of trajectories. As the measurements come from the ground, \( \mathcal{G} \) contains the multi-target state at \( \tau \) where \( \lambda_k \) denotes the corresponding multi-target state at \( \tau \) and \( \lambda_k \) is the density of measurements \( z_k \) at time \( k \).

B. Bayesian Filtering Recursion

Given the posterior multi-trajectory density \( \pi_{k-1} (\cdot) \) at time \( k-1 \) and the set of measurements \( z_k \) at time \( k \), the posterior density \( \pi_k (\cdot) \) can be obtained by using the Bayes recursion

\[ \pi_{k|k-1} (X_k) = \int f (X_k|X_{k-1}) \pi_{k-1} (X_{k-1}) \delta X_{k-1}, \]

\[ \pi_k (X) = \frac{\ell_k (z_k|X_k) \pi_{k|k-1} (X_k)}{\ell_k (z_k|X_k) \pi_{k|k-1} (X_k)}, \]

where \( f (\cdot) \) denotes the transition density, \( \pi_{k|k-1} (\cdot) \) denotes the predicted density, \( \ell_k (z_k|X_k) \) denotes the density of measurements of trajectories. As the measurements come from the target states of a single frame time, \( \ell_k (z_k|X_k) \) can be also written as

\[ \ell_k (z_k|X_k) = \ell_k (z_k|\tau_k (X_k)), \]

where \( \tau_k (X) \) denotes the corresponding multi-target state at the time \( k \). Similarly, the measurement likelihood \( l_k (z|X) \) of the single trajectory \( X = (\beta, x^{1:1}) \) equals to \( l_k (z|x^\beta) \).

C. The TPHD Filter

The TPHD filter considers a Poisson multi-trajectory density \( p_k (\cdot) \) at time \( k \), which is expressed as

\[ p_k (\{ X_1, \ldots, X_{N_k} \}) = e^{-\lambda_k} \lambda_k^{N_k} \prod_{j=1}^{N_k} \tilde{p}_k (X_j), \]

where \( \tilde{p}_k (\cdot) \) denotes the single trajectory density and \( \lambda_k \geq 0 \). A Poisson PDF is characterized by its PHD \[ \gamma_k (\cdot) \]

\[ D_k (X) = \lambda_k \tilde{p}_k (X). \]

The clutter RFS is also Poisson with intensity \( \kappa (z) \). The TPHD filter follows the assumptions \[ 13] ;

- The trajectories at time \( k \) consist of surviving trajectories at time \( k - 1 \) with surviving probability \( p_{S,k} (\cdot) \) and the trajectories born at time \( k \) with the PHD \( \gamma_k (\cdot) \) of a Poisson density. The birth and the surviving RFSs are independent of each other.

- The trajectory RFS at time \( k - 1 \) is Poisson and the clutter RFS is also Poisson and independent of measurement RFS.

Given the posterior PHD \( D_{k-1} (\beta, x^{1:1}) \) at time \( k - 1 \) and the transition density \( f (x^\gamma|x^{\beta-1}) \), the prediction step of the TPHD filter is obtained by

\[ D_k (X) = \gamma_k (X) + D_k^S (X), \]

where

\[ D_k^S (X) = p_{S,k} (x^\gamma) f (x^\gamma|x^{\beta-1}) D_{k-1} (\beta, x^{\beta-1}). \]

As only alive trajectories are considered in \[ 13] , if \( \beta \neq k - \beta + 1 \), \( D_k^S (X) \) equals to zero. Besides, in (11), it is required \( \beta \in \{ 1, 2, \ldots, k - 1 \} \) to represents the trajectories born before time \( k \). The update step of the TPHD filter is obtained by

\[ D_k (X) = D_{k|k-1} (X) \]

\[ + D_{k|k-1} (X) p_{D,k} (x^\gamma) \]

\[ \times \sum_{z \in Z_k} \kappa (z) \left\{ \gamma_k (z) + \left\langle p_{D,k} \cdot \gamma_k (z) \right\rangle, D_{k|k-1} \right\}, \]

where

\[ D_{k|k-1} (x^\gamma) = \sum_{\beta=1}^{k-1} D_{k|k-1} (t, x^{1:k-\beta+1} d x^{1:k-\beta}), \]

with \( \zeta = k - \beta + 1 \), which denotes the PHD of the prior target density at time \( k \).

III. THE JTC-TPHD FILTER

In this section, the recursion of the JTC-TPHD filter is elaborated. The JTC-TPHD filter can classify the target when tracking and each target is assigned with multiple class hypotheses. Using this strategy, we can obtain both the category information of the targets and a more accurate trajectory estimation than the traditional TPHD filter. In Section III-A, the JTC model is presented. In Section III-B, the JTC-TPHD filter is derived in detail.
A. The Joint Tracking and Classification Model

Assume that a considered target possesses the class hypothesis \( c \in C_k \) at time \( k \), where \( C_k \) represents the state space of classes. Each target class is then characterized by a finite number of possible motion models denoted by \( r \in M(c) \), where \( r \) denotes the possible motion model and \( M(c) \) denotes the corresponding space of motion models with class \( c \). The class state is augmented to the trajectory state, and the state space of the augmented trajectories is defined as

\[
Y = \{ \beta \in \beta_k \times \mathbb{R}^{d_x \times n_x} \times C \}, \tag{14}
\]

where \( Y \) represents the space of the augmented trajectories \( X_c \), which consists of trajectories \( X \) and the corresponding classes \( c \), i.e., \( X_c = (X, c) \). For a single trajectory \( X \) at time \( k \), the classification result is determined by the category hypothesis with the maximum posterior probability density at this moment. Given the sets of measurements \( z_{1:k} \) from time 1 to \( k \), the prior density \( p(z_{1:k} | c) \) and the likelihood function \( p(z_k | c) \) of each class \( c \), the posterior probability density is computed by the Bayes rule

\[
P_r(c | z_{1:k}) = \frac{p(z_k | z_{1:k-1})Pr(c | z_{1:k-1})}{\sum_{c' \in C} p(z_k | c')Pr(c' | z_{1:k-1})}. \tag{15}
\]

Besides, we define the density of augmented trajectory sets as the augmented multi-trajectory density. The augmented trajectory RFS is denoted as

\[
X_c = \{ X_c = (\beta, x^{1:c}, c') \in Y \}. \tag{16}
\]

More specifically, considering an augmented trajectory possesses a motion model \( r' \), then the state is expressed as

\[
X_{c,r} = (\beta, x^{1:c}, c', r') \in Y \times M(c'), \tag{17}
\]

For an augmented trajectory \( X_{c,r} = (\beta, x^{1:c}, c', r') \) with a motion model at time \( k \), its transition density is given as \( f(X_{c,r} | X_{c,r}) \), where \( X_{c,r} = (\beta, x^{1:c}, c', r') \) represents the state at time \( k - 1 \). In some applications, the switch of motion models is independent of the trajectory states, which is given as \( f_r(r' | r^{-1}) \). However, for a certain target, its class does not change with time. In other words, there is no transition between different target classes without considering spawning targets. Besides, the transition \( f(X_{c,r} | X_{c,r}) \) is considered as the first-order Markov process, thus the following equation is established

\[
f(X_{c,r} | X_{c,r}) = f(x^{1:c} | x^{c-1}, r^{c-1}) f_r(r' | r^{c-1}) \delta_{\beta} \delta_{x^{c} | x^{c-1} | c}, \tag{18}
\]

In this paper, the measurements are only concerned about target states of a single frame time, thus the measurement likelihood \( l_k(z | X_{c,r}) \) can be simplified to

\[
l_k(z | X_{c,r}) = l_k(z | x^c). \tag{19}
\]

B. The JTC-TPHD Filter

Following the work of the JTC-PHD filter [17], the recursion of the JTC-TPHD filter not only extends the JTC model into trajectory RFS, but also presents the prediction and update process of the categories and multiple motion models of targets in detail. The JTC-TPHD filter follows the same assumptions as the TPHD filter [13], which is not elaborated here. Combined with JTC model, the PHD of the multi-trajectory density (8) at time \( k \) is expressed as

\[
D_c(k) = \sum_{c \in C} D_c(k | X_c) = \sum_{c \in C} \sum_{r \in M(c)} D_{r,k}(X_{c,r}), \tag{20}
\]

where \( D_{c,k}(X_c) \) and \( D_{r,k}(X_{c,r}) \) represent the PHD of the augmented multi-trajectory density concerning class and the PHD of the augmented multi-trajectory density with motion model \( r \), respectively. By propagating the PHD \( D_c(X_c) \) of the augmented trajectory \( X_c = (\beta, x^{1:c}, c') \), the recursion of the JTC-TPHD filter is given as follows.

**Proposition 1.** Given the posterior PHD \( D_{c,k-1}(X_c) \) at time \( k - 1 \), which equals to

\[
D_{c,k-1}(\beta, x^{1:c-1}, c') = \sum_{r^{-1} \in M(c')} D_{r,k-1}(\beta, x^{1:c-1}, c', r^{-1}) \tag{21}
\]

the predicted PHD \( D_{c,k}(X_c) \) of the augmented multi-trajectory density is obtained by

\[
D_{c,k}(X_c) = \gamma_{c,k}(X_c) + D_{c,k}^S(X_c), \tag{22}
\]

where

\[
\gamma_{c,k}(X_c) = \delta_{c} \sum_{r^{-1} \in M(c')} f_r(r' | r^{-1}) \times \sum_{r \in M(c')} p_{S,k}(x^{r-1}, c') f(x^c | x^{r-1}, r^{-1}) \times D_{r,k-1}(\beta, x^{1:c-1}, c', r^{-1}) \tag{23}
\]

and

\[
D_{c,k}^S(X_c) = \delta_{c} \sum_{r \in M(c')} f_r(r' | r^{-1}) \times \sum_{r \in M(c')} p_{S,k}(x^{r-1}, c') f(x^c | x^{r-1}, r^{-1}) \tag{24}
\]

As indicated by (24), for surviving trajectories, the prediction step includes the switches of different target motion models of the same class and the prediction of trajectory states. Note that, different from [17], the previous states of trajectories are retained in (24).

**Proposition 2.** Given the predicted PHD \( D_{c,k|k-1}(X_c) \) at time \( k \), the posterior PHD \( D_{c,k}(X_c) \) of the multi-trajectory density is obtained by

\[
D_{c,k}(\beta, x^{1:c}, c') = D_{c,k|k-1}(\beta, x^{1:c}, c')(1 - p_D(x^c, c')) + D_{c,k|k-1}(\beta, x^{1:c}, c') p_D(x^c, c') \\
\times \sum_{z \in \mathbb{Z}_k} l_k(z | x^c) \tag{25}
\]

+ \Theta_k [z, X_c],
where
\[
\Theta_k [z, X_c] = \sum_{c^\ell \in \mathcal{C}} \int p_D(x^\ell, c^\ell) \cdot l_k(z|x^\ell) \, dx^\ell \quad (26)
\]
\[
\times D^\tau_{c,k-1} (x^\ell, c^\ell) \, dx^\ell,
\]
\[
D^\tau_{c,k-1} (x^\ell, c^\ell) = \sum_{\beta=1}^k \int D_{c,k-1}(\beta, x^{1:k-\beta+1}, c^{k-\beta+1}) \times dx^{1:k-\beta}. \quad (27)
\]

In Proposition 2, \( \zeta = k-\beta+1 \), and \( D^\tau_{c,k-1} (x^\ell, c^\ell) \) denotes the PHD of the prior density of augmented targets at time \( k \). The updated PHD contains information about the trajectories, corresponding classes and motion models.

IV. THE GAUSSIAN MIXTURE IMPLEMENTATION

In this section, the Gaussian mixture is presented to obtain a closed-form implementation for the JTC-TPHD filter, which is referred to as the GM-JTC-TPHD filter. There are some notations given as follows. At time \( k \), the Gaussian density of a trajectory which is born at time \( i \) of length \( n \) is given as
\[
\mathcal{N}(X_{c,r}; \beta, \hat{m}^k(r), \hat{P}^k(r)) = \mathcal{N}(x^{1:n}; \hat{m}^k(r), \hat{P}^k(r)) \quad (28)
\]
\[
\times I_{M(c)}(r) 1_{U_k(\beta, \zeta)}1_{C_k(c)},
\]
with the mean \( \hat{m}^k \in \mathbb{R}^{n_x} \) and the covariance \( \hat{P}^k \in \mathbb{R}^{n_x \times n_x} \). For a matrix \( V \), the notation \( V_{[n:m,s:t]} \) represents the submatrix of \( V \) for rows from \( n \) to \( m \) and columns from \( s \) to \( t \). The notation \( V_{[n:m]} \) is used to present the submatrix of \( V \) for rows from time steps \( n \) to \( m \). Besides, the notations \( V_{[n:s:t]} \) and \( V_{[n:n]} \) represent \( V_{[n:n:s:t]} \) and \( V_{[n:n]} \), respectively. The recursion of the GM-JTC-TPHD filter follows the assumptions of

Assumption 1: The transition of target state and observation function are the linear Gaussian model
\[
f(x^\ell|x^{k-1}, r^{k-1}) = \mathcal{N}(x^\ell; F(x^{k-1}, Q(r^{k-1}))), \quad (29)
\]
\[
l(z|x^\ell) = \mathcal{N}(z; Hx^\ell, R), \quad (30)
\]
where \( F \in \mathbb{R}^{n_x \times n_x} \) denotes the state transition matrix and \( Q \in \mathbb{R}^{n_x \times n_x} \) is the process noise covariance, \( H \in \mathbb{R}^{n_z \times n_x} \) is the observation matrix and \( R \in \mathbb{R}^{n_z \times n_z} \) denotes the observation noise covariance.

Assumption 2: The values of surviving probability and detection probability are considered as class dependent in the implementation, which are expressed as, \( p_S(c) \) and \( p_D(c) \), respectively.

Assumption 3: The PHD of the birth density \( \gamma_{c,k} \) at time \( k \) is a Gaussian mixture
\[
\gamma_{c,k}(X_c) = \sum_{j=1}^{J^k(c)} \omega^k_{c,j} \sum_{r^k \in \mathbb{M}(c^k)} \omega^k_{r^k,j}(r^k) \quad (31)
\]
\[
\times \mathcal{N}(X_{c,r}; \hat{m}^k_{r^k,j}(r^k), \hat{P}_{r^k,j}(r^k)),
\]
where \( J^k \in \mathbb{N} \) represents the number of Gaussian components. For the \( j \)-th birth component at time \( k \), \( \omega^k_{c,j} \), and \( \omega^k_{r^k,j}(r^k) \) represents the weight of components and motion models, respectively. The mean and covariance of the Gaussian density are expressed as \( \hat{m}^k_{r^k,j} \in \mathbb{R}^{n_x} \) and \( \hat{P}^k_{r^k,j} \in \mathbb{R}^{n_x \times n_x} \), respectively.

Proposition 3. The posterior PHD \( D_{c,k-1}(X_c) \) at time \( k-1 \) can be given as the Gaussian mixture as follows
\[
D_{c,k-1}(X_c) = \sum_{j=1}^{J^k(c-1)} \omega^k_{c,j}(c-1) \sum_{r^k-1 \in \mathbb{M}(c-1)} \omega^k_{r^k,j}(r^k-1) \times \mathcal{N}(X_{c,r}; \beta, \hat{m}^k_{r^k,j}(r^k-1), \hat{P}_{r^k,j}(r^k-1)), \quad (32)
\]
where, at time \( k-1 \), the length \( c^k-1 \) of the \( j \)-th augmented trajectory \( X_{c,r} \) is given as \( k-\beta \). The mean and covariance of the Gaussian density are given as \( \hat{m}^k_{r^k,j}(r^k-1) \) and \( \hat{P}^k_{r^k,j}(r^k-1) \), respectively. Then the prior PHD \( D_{c,k|k-1}(X_c) \) is given as
\[
D_{c,k|k-1}(X_c) = \gamma_{c,k}(X_c) + D^S_{c,k}(X_c), \quad (33)
\]
\[
D^S_{c,k}(X_c) = \sum_{r^k \in \mathbb{M}(c^k)} \sum_{j=1}^{J^k(c-1)} p_S(c-1) \delta_{x^k-1}(c^k) \omega^k_{c,j}(c^k-1) \times \sum_{r^k-1 \in \mathbb{M}(c-1)} \omega^k_{r^k,j}(r^k-1) f_r(r^k|y^k-1) \times \mathcal{N}(X_{c,r}; \beta, \hat{m}^k_{r^k,j}(r^k), \hat{P}_{r^k,j}(r^k-1)), \quad (34)
\]
where
\[
\hat{m}^k_{r^k,j}(r^k) = \begin{bmatrix} \hat{m}^k_{r^k,j}(r^k-1) \\ F(x^k) \hat{m}^k_{r^k,j}(r^k-1) \end{bmatrix}, \quad (35)
\]
\[
\hat{P}^k_{r^k,j}(r^k) = \begin{bmatrix} \hat{P}^k_{r^k,j-1}(r^k-1) & P_1 \\ P_1^\top & P_2 \end{bmatrix}, \quad (36)
\]
\[
P_1 = \hat{P}^k_{r^k,j-1}(r^k-1) F(x^k) \hat{P}^k_{r^k,j-1}(r^k-1), \quad (37)
\]
\[
P_2 = F(x^k) \hat{P}^k_{r^k,j-1}(r^k-1) F(x^k)^\top + Q(r^k). \quad (38)
\]

Proposition 3 is the consequence of Proposition 1. The prediction step trajectory in the GM-JTC-TPHD filter roots in the change of the target state and the prediction of motion models depends on the transition density function \( f_r(\cdot|\cdot) \).

Proposition 4. If at time \( k \), the prior PHD \( D_{c,k|k-1}(X_c) \) is given as the Gaussian mixture of the form
\[
D_{c,k|k-1}(X_c) = \sum_{j=1}^{J^k(c-1)} \omega^k_{c,j}(c-1) \sum_{r^k \in \mathbb{M}(c^k)} \omega^k_{r^k,j}(r^k) \times \mathcal{N}(X_{c,r}; \beta, \hat{m}^k_{r^k,j}(r^k), \hat{P}_{r^k,j}(r^k)), \quad (39)
\]
then, given a measurement set \( Z_k \), the posterior PHD \( D_{c,k}(X_c) \) is given as
where
\[ \omega_{r,j}(r^k, z, \tau^k) = \frac{\omega_{D,J}(r^k, z, \tau^k)}{\phi_j(r^k, z)}, \]
\[ \omega_{c,j}(c^k, z, \tau^k) = \frac{\omega_{D,J}(c^k, z, \tau^k) \phi_j(c^k, z)}{\phi_j(c^k, z)}, \]
\[ \phi_j(c^k, z) = \sum_{r^k \in M(c^k)} \omega_{r,j}(r^k, z, \tau^k) g_j(r^k, z), \]
\[ \tilde{z}_j(r^k) = H \tilde{m}_{[j,k]}(r^k), \]
\[ S_j(r^k) = H \tilde{P}_{[j,k]}(r^k) H^T + R, \]
\[ q_j(r^k, z) = N(z; \tilde{z}_j(r^k), S_j(r^k)), \]
\[ \tilde{P}_j(r^k) = \tilde{P}_{[j,k]}(r^k) - K_j H \tilde{P}_{[j,k]}(r^k), \]
\[ K_j = \frac{\tilde{P}_{[j,k]}(r^k)}{\phi_j(c^k, z)} H^{-1}(S_j(r^k)), \]

Different from the JTC-PHDF [17], the GM-JTC-PHDF filter aims at the whole trajectory in the update step. It not only updates the estimation of the target state at current time, but also smooths the estimation of previous states. When there is only one case of the target class c, the GM-JTC-PHDF filter degrades into the GM-JMS-PHDF filter [30], which only considers tracking maneuvering targets without classification.

Similar to the traditional GM-PHDF [13] filter, the number of Gaussian components for the GM-JTC-PHDF filter increases as time progresses. Hence, to limit unbounded Gaussian components, the pruning and absorption techniques are needed, which can be found in [13]. In addition, the L-scan approximation [13] is also needed to deal with the increasing length of trajectories, which only updates the multi-trajectory density of the last L time and leaves the rest unaltered. At the end of each recursion, the classification result of the j-th target is determined by
\[ \omega_{c,j}^k(c^k), \]
then the estimation of the number of alive trajectories of all possible classes at time k is given as
\[ N^k = \text{round} \left( \sum_{j=1}^{N^k} \sum_{c^k \in C_k} \omega_{c,j}^k(c^k) \right). \]

The state estimations of \( N^k \) trajectories are given as \( \{ (\beta_1, \zeta_1, \tilde{m}_1^k), ..., (\beta_{N^k}, \zeta_{N^k}, \tilde{m}_{N^k}^k) \} \).

V. Simulations

This section presents numerical studies for the GM-JTC-PHDF filter. A six targets simulation is set inside of a three-dimensional space with the size of \([-8000, 12000] \times [-12000, 6000] \times [0, 10000] \) m for 100 seconds. The target state matrix is given as \( x_k = [p_x, p_y, p_z, \dot{p}_x, \dot{p}_y, \dot{p}_z] \), where \( p_x, p_y, p_z \) denote the position information and \( \dot{p}_x, \dot{p}_y, \dot{p}_z \) represent the velocity information. Consider an approximate linear observation process
\[ z_k = h(x_k, v_k) = \left[ \tan^{-1} \left( \frac{x_k}{y_k} \right) \right] + v_k, \]
where \( v_k \sim N(0; 0, R_k) \) and \( R_k = \text{diag}([\pi/180]^2, [\pi/180]^2, 10^2]^2 \). By using the extended Kalman (EK) [31] filter to local linearizations of the (nonlinear) mapping \( h_k \), we can obtain an approximate linear observation matrix (30)
\[ H = \frac{\partial h(x_k, 0)}{\partial x_k}, \]
where \( r_1 = p_{x_k}^2 + p_{y_k}^2 \) and \( r_2 = p_{x_{k+1}}^2 + p_{y_k}^2 + p_{z_{k+2}}^2 \).

The targets are divided into two classes: the plane \((c_p)\) and unmanned aerial vehicle (UAV) \((c_u)\). In other words, the space of class is given as \( C = \{ c_p, c_u \} \). In this scenario, the plane goes into the nosedive from the high, belonging to the weak maneuvering target and the UAV flies at a low altitude, belonging to the high maneuvering target. The maneuvering motion model consists of the CT and CV modes. The linear state transition matrices for the CV and CT models are given as follows
\[ F_{CV} = I_3 \otimes \begin{bmatrix} 1 & \delta t \\ 0 & 1 \end{bmatrix}, \]
\[ F_{CT} = \begin{bmatrix} 0 & 0 & \frac{\sin \zeta \delta t}{\zeta} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\cos \zeta \delta t}{\zeta} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sin \zeta \delta t}{\zeta} & 0 \\ 0 & 0 & 0 & \cos \zeta \delta t & 0 & \delta t^2 \\ 0 & 0 & 0 & 0 & 0 & \delta t \end{bmatrix}, \]
\[ Q_{CV} = Q_{CT} = \sigma^2 \begin{bmatrix} \delta t^2 \\ \delta t \end{bmatrix}, \]
where \( I_3 \) represents the \( 3 \times 3 \) unit matrix, \( \sigma^2 = 5 \text{ms}^{-2} \), \( \otimes \) represents the Kronecker product and \( \delta t = 1 \text{s} \) denotes the sampling interval. The plane consists of three motion modes i.e. \( M(c_p) = 3 \) with the linear motion, a counterclockwise turn rate of \(-4^\circ/s\) and a clockwise turn rate of \(4^\circ/s\). For
the plane, the notation $\zeta$ equals to $-4\pi/180$ or $4\pi/180$. The UAV consists of three motion modes i.e. $M(c_u) = 3$ with the linear motion, a counterclockwise turn rate of $-15^\circ/s$ and a clockwise turn rate of $15^\circ/s$. For the UAV, the notation $\zeta$ equals to $-15\pi/180$ or $15\pi/180$. The surviving probabilities of the plane and UAV are given as $p_S(c_p) = p_S(c_u) = 0.99$. The detection probabilities of the plane and UAV are given as $p_D(c_p) = 0.99$, $p_D(c_u) = 0.95$, respectively. The number of clutter per scan is Poisson distributed with mean $\lambda_c = 30$. We assume that surviving probabilities, detection probabilities and clutter rate are given as a prior knowledge. The initial models of targets are shown in Table I and the death time here refers to the last time a target existing.

Besides, the birth process is Poisson with parameters $J^k = 4$, $\omega^k = 0.03 \times [0.5, 0.5]$, $\omega^k_{r_\gamma} = [0.3, 0.35, 0.35]$ and $P^k_{\gamma} = \text{diag}([50, 50, 50, 50, 50])^2$. For each $j \in \{1, 2, 3, 4\}$, $\tilde{m}^k_{\gamma, 1} = [0, 0, 0, 0, 8000, 0]^T$, $\tilde{m}^k_{\gamma, 2} = [-1000, 0, 1000, 0, 9000, 0]^T$, $\tilde{m}^k_{\gamma, 3} = [0, 0, 100, 0, 7500, 0]^T$, $\tilde{m}^k_{\gamma, 4} = [50, 0, 50, 0, 0, 0]^T$ The value of the $L$-scan approximation is set as $L = 5$. The switching between different motion modes in both classes is taken as the same, which is given by the following Markovian model transition probability matrices

$$f_r(r^i|r^{i-1}) = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}, \quad (55)$$

In pruning and absorption, the threshold of weight is $\Gamma_p = 10^{-5}$, the threshold of absorption is given as $\Gamma_a = 4$ and the maximum is limited as $J_{\text{MAX}} = 50$. The trajectory metric error (TM) [32] with parameters $p = 2, c = 100, \gamma = 1$ is used to characterize the error between the estimated and truth for the GM-JTC-TPHD filter. By running 1000 Monte Carlo experiments, the performance of the GM-JTC-TPHD filter is given as follows.

It can be seen from Fig. 2 that the GM-JTC-TPHD filter can achieve excellent performance in the estimation of the trajectory states and classification results. Then, it can be seen from Figs.3 and 4 that the GM-JTC-TPHD filter can correctly classify the plane and UAV with time progressing, but performs fluctuation with the changes of the number of targets. Finally, the influence of different values of the $L$-scan approximation is shown in Fig. 5. As expect, an increasing $L$ can improve estimation performance and reduce the errors, while the improvement becomes less and less with increasing $L$. In addition, The averaged times to run one Monte Carlo iteration with a 2.8 GHz Intel i7 laptop are given as : 8.69
VI. CONCLUSION

In this paper, the recursion of the JTC-TPHD filter is derived, which can not only estimate trajectories, but also classify different kinds of targets. The Gaussian mixture implementation is presented for the JTC-TPHD filter, which is referred to as the GM-JTC-TPHD filter. The L-scan approximation is also applied to the GM-JTC-TPHD filter to achieve a fast implementation. Simulation results demonstrate that the GM-JTC-TPHD filter can achieve excellent performance in both tracking and classification. However, this paper only considers the classification based on the target state and motion models of a single frame time. Future works will research the special association between trajectories and classes to obtain a better classification results.
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