On the recurrence and Lyapunov time scales of the motion near the chaos border

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Abstract

Conditions for the emergence of a statistical relationship between $T_r$, the chaotic transport (recurrence) time, and $T_L$, the local Lyapunov time (the inverse of the numerically measured largest Lyapunov characteristic exponent), are considered for the motion inside the chaotic layer around the separatrix of a nonlinear resonance. When numerical values of the Lyapunov exponents are measured on a time interval not greater than $T_r$, the relationship is shown to resemble the quadratic one. This tentatively explains numerical results presented in the literature.

Key words: resonance, chaos, Hamiltonian system, Lyapunov exponents, asteroid.

1 Introduction

On the basis of a lot of numeric experiments in problems dealing with the dynamics of objects of the Solar system, Soper et al. (17), Lecar et al. (9) and Murison et al. (14) argued that the times of “sudden changes” in the chaotic orbital behaviour could be statistically predicted by means of computation of the largest Lyapunov characteristic exponents (LLCE). They established that a simple “universal” statistical dependence existed between the time of a sudden orbital change (designated henceforth $T_r$, the “recurrence time”) and the Lyapunov time (the inverse of the numerically measured LLCE): $T_r \propto T_L^3$, with
a typical value of $\beta = 1.7 \div 1.8$; though a considerable dispersion of the statistical data was usually present. The same kind of dependence, with $\beta \approx 1.9$, was found by Levison and Duncan \cite{10} in simulations of the dynamics of the outer Solar system, namely, the Kuiper asteroidal belt.

Morbidelli and Froeschlé \cite{13} recently considered reasons for the appearance of a statistical relationship between Lyapunov times and “macroscopic diffusion” times in general nearly integrable Hamiltonian systems. In particular, they showed qualitatively that in the regime of multiple resonance overlapping the relationship should be polynomial. However, they did not find any theoretical indication for a universal power law, i.e. a power law with a particular preferable value of the exponent. In their analysis, they considered the strength of perturbation as a main governing parameter, through which both times of interest were expressed. They did not consider any numeric effects; in particular, Lyapunov exponents were treated to be equal to their theoretical values defined on the infinite time scale.

In what follows, a different approach is used. I study a role of the choice of the starting values of trajectories, but not the role of the values of parameters. This implies consideration of selection effects arising in numeric computations due to natural time limitations: of course, the LLCE of a chaotic trajectory computed on the infinite time scale should not depend on starting values, if these starting data belong to one and the same connected chaotic region of phase space. So, a “local” numeric definition is used in what follows for the LLCE. This means that, since the LLCE is computed numerically on a finite time scale, a chaotic trajectory may explore only a finite local domain of all available chaotic region, but not necessarily all this region.

I show that, due to the basic phenomenon of the divided phase space, or, in other words, due to the phenomenon of the presence of chaos border, and the immanent sticking behaviour of chaotic trajectories, there emerges a generic statistical dependence between the times of chaotic transport (recurrence times) and Lyapunov times, when a natural time limitation is used in numeric simulations. This limitation is that the LLCE are computed on time scales equal to or less than the recurrence times. The derived generic relationship explains tentatively the cited results \cite{17; 8; 14; 10}.

Besides, two new numeric examples are given demonstrating the emergence of the generic relationship straightforwardly. One of them concerns the separatrix map describing the motion near the separatrix
of a general nonlinear resonance; the second one deals with the asteroidal motion in the 3/1 mean motion commensurability with Jupiter.

2 Computing Lyapunov exponents

Remember (11) that the LCE of an orbit measures the rate of exponential divergence of trajectories close to this orbit (about the general definition of the LCE of a function see e.g. Ref. (1)). Let \( d(t_0) \ll 1 \) be the initial displacement of a shadow trajectory from the main one, and \( d(t) \) be the displacement at time \( t \). Then the LCE is determined by the formula

\[
\sigma = \lim_{t \to \infty} \frac{1}{t - t_0} \ln \frac{d(t)}{d(t_0)}.
\]  

(1)

Depending on the direction of the initial displacement in phase space, the quantity \( \sigma \) of a trajectory of a Hamiltonian system can have generally different values, where \( N \) is the number of degrees of freedom. However, on an everywhere dense set of starting values of shadow trajectories, it attains the single (maximum) value, the “largest LCE”, shortly, LLCE.

Numerically, the LLCE is found, of course, on a finite time interval (say, of \( M \) time units) by means of the formula

\[
\sigma(M) = \frac{1}{M \Delta t} \sum_{i=1}^{M} \ln r_i.
\]  

(2)

where \( r_i \) is the ratio of the current displacement (at \( t = i \)) to the preceding one (at \( t = i - 1 \)), \( \Delta t \) is the size of the time unit. Current displacements should be periodically renormalized to some small value (preserving direction of the displacement) (11), so that the shadow trajectory is kept in a vicinity of the main one.

The traditional numeric procedure of computation of the LLCE is: build the dependence \( \log \sigma(M) \), given by Eq. (2), versus \( \log M \), and find the value of \( \log \sigma \) at which the dependence is “saturated”, i.e. attains a form of a horizontal plateau (cf. e.g. Ref. (18)). Before saturation, \( \log \sigma \) goes down on average linearly with \( \log M \).

So, on one hand, the value of the LLCE is computed on time intervals not less than the time of saturation; on the other hand, the time of computation cannot be infinite. These limitations from below
and above impose certain selection effects which are necessary to take into account in any numeric statistical study incorporating LLCE.

3 The separatrix map and the generic relationship

The nonlinear pendulum provides a model of a nonlinear resonance under very general conditions (2, 11). The motion in a vicinity of the separatrix of the nonlinear pendulum (the nonlinear resonance) is described by the separatrix map (2, 11), hereafter SM. In what follows the SM in Chirikov’s form (2), or, in identical terms, the “whisker map” is used. It was deduced by Chirikov (2) for the Hamiltonian

\[ H = H_0 + \Lambda \cdot (\xi_1 + \xi_2), \quad (3) \]

where \( H_0 = \frac{Gp^2}{2} - F \cos \kappa \) is the Hamiltonian of the pendulum, and the perturbation is given by the terms \( \xi_1 = \cos(\kappa - \sigma) \), \( \xi_2 = \cos(\kappa + \sigma) \) (where \( \sigma = \Omega t + \sigma_0 \)) and is thus periodic and symmetric. In what follows, the angle \( \kappa \) is referred to as the pendulum’s angle, and \( \sigma \) as the phase angle of perturbation. The quantity \( \Omega \) is the perturbation frequency, and \( \sigma_0 \) is the initial phase; \( p \) is the momentum; \( F, G, \Lambda \) are constants.

The SM is a two-dimensional area-preserving map, with an action-like and phase-like variables, the former one measuring the relative deviation of the energy of the pendulum (with respect to the unperturbed separatrix value), and the latter one measuring the phase of perturbation. The SM in Chirikov’s (2) form is

\[
\begin{align*}
    w_{i+1} &= w_i - W \sin \sigma_i, \\
    \sigma_{i+1} &= \sigma_i + \lambda \ln \frac{32}{|w_{i+1}|} \pmod{2\pi},
\end{align*}
\quad (4)
\]

where \( w \) denotes the relative pendulum’s energy: \( w = \frac{H_0}{p} - 1 \). Constants \( \lambda \) and \( W \) are parameters: \( \lambda \) is the ratio of \( \Omega \), the perturbation frequency, to \( \omega_0 = (FG)^{1/2} \), the frequency of small-amplitude pendulum oscillations; and

\[
W = \frac{\Lambda}{F} \lambda (A_2(\lambda) + A_2(-\lambda)) = \frac{\Lambda}{F} \frac{4\pi\lambda^2}{\sinh \frac{\pi\lambda}{2}}.
\quad (5)
\]
where the function $A_2(x)$ denotes values of the Melnikov–Arnold integral as defined in (2). One iteration of the SM corresponds to one period of pendulum's rotation or a half-period of its libration.

The SM is given by the same Eq. (4) for many other kinds of perturbation terms as well (of course, formulae for the parameter $W$ are then different; see a particular example in Ref. (19)). So, the SM provides a very general description of the motion in a vicinity of the separatrix of a nonlinear resonance.

The SM can be locally linearized in $w$ to give the standard map (2). In other words, the chaotic layer is locally described by the standard map. Consider the chaotic motion in a vicinity of the critical curve separating regular and chaotic domains. Following designations of Refs. (5; 3), let $r_n = p_n/q_n$ be the continued fraction convergents to the winding number of the critical curve. They represent the winding numbers of principal resonances close to the critical curve. The stability of a periodic trajectory $r_n$ can be characterized by the value of the Greene residue (6; 7). Its value for a principal resonance close to the critical curve is given (5) by the formula

$$R_n = R^{(1)} \exp\left(1.20 q_n^{1+\varepsilon} (K - K_G)\right),$$

where $K$ is the stochasticity parameter of the approximating standard map, $K_G \approx 1$ is its critical value, $R^{(1)} \approx 1/4$ is the critical value of the Greene residue; $\varepsilon = 0.013$. Eq. (6) follows from Greene's relation $R \propto K^q$ (7) applied at chaos border, though it is somewhat modified: the coefficient 1.20, put instead of 1, is empirical.

As Chirikov and Shepelyansky (5) noted, Eq. (6) has simple physical meaning: the locally defined Lyapunov exponents

$$l_n \approx 1.20 \Delta K,$$

practically do not depend on $q_n$, i.e. on a particular trajectory, and are equal to the locally defined Kolmogorov–Sinai entropy $h \propto \Delta K \equiv K - K_G$. Namely, one has from Eqs. (6, 7):

$$l_n \approx 1.20 \Delta K.$$
therefore, one can use Eq. (8) to predict local values of the LLCE, \( l \), for all orbits residing in the chaotic layer not far from its border. Hence

\[
l \propto \Delta K. \tag{9}
\]

Thus the value of the LLCE is determined by that of the stochasticity parameter \( K \) of the approximating standard map.

The dependence of the transport time \( T_r \) (equivalently “recurrence” or “sticking” time) near the border on \( \Delta K \) is given by Chirikov’s resonant theory of critical phenomena (3, Section 4.3). This dependence is derived as follows. The recurrence time is of the same order as the time \( \tau_n \) of the transition from a scale \( q_n \) with \( n \) maximal for a given recurrence, to the neighbouring one, since \( \tau_n \) rapidly diminishes with decreasing \( n \) (5). The “mean” relation \( \Delta K \propto \rho \), where \( \rho \) is the detuning \( |r - r_c| \) of the winding number \( r \) with respect to that of the critical curve, \( r_c \), leads to \( \Delta K \propto q_n^{-2} \). Such a dependence is not sufficient to destroy principal critical scales \( q_n \); instead, narrow, \( \sim q_n^{-4} \), chaotic layers are formed between them. From the condition of the flux balance in statistical equilibrium, one has \( \tau_n \propto q_n^4 \) (5; 3). Since the recurrence time \( T_r \propto \tau_n \), and the dependence of \( \Delta K \) on the winding number detuning is set to be linear, one has

\[
T_r \propto \Delta K^{-2}. \tag{10}
\]

Via Eqs. (9, 10), one can then express the recurrence time \( T_r \) through the Lyapunov time \( T_L \), which is the inverse of the locally defined LLCE. One has finally

\[
T_r \propto T_L^2. \tag{11}
\]

Eq. (11) is valid if, during the time of measurement of the LLCE, the chaotic motion mostly takes place not far from chaos border, i.e. in the “sticking regime”. The “sudden orbital change” means escape from the border.

In the opposite limit of \( K \gg 1 \), i.e. for short recurrences, the motion is completely diffusive, and the “\( T_L - T_r \)” relationship is not so simple. The relationship in the case of diffusion was studied by Morbidelli and Froeschlé (13). They showed that it did not follow a uniform pattern; though power laws with values of exponents in a wide range could be indeed observed and explained.
The short-recurrence time (diffusive) part of the relationship seems to be normally not observed in computations dealing with asteroidal dynamics, due to a selection effect (the time of the measurement of LCE cannot be too short). Besides, the diffusive stage can be simply absent, e.g. as in the case of the SM with $\lambda \sim 1$.

The analysis above has been performed for the perturbed nonlinear pendulum, which represents a system with one and a half degrees of freedom (one degree of freedom plus dependence on time). Whether the resulting formula has any relevance to systems with many degrees of freedom? In such systems, the Arnol’d diffusion takes place \[3\]. This universal instability is not possible in systems with the number of degrees of freedom less than or equal to two. The situation in higher dimensions is therefore more complicated. However, different resonances in multi-dimensional systems generically have different strengths. According to Chirikov’s classification \[3\], the “guiding” resonance may be chosen arbitrarily; this depends on the region of phase space where the motion is considered. Remaining resonances are considered as driving. The strongest one among the driving resonances is the so-called “layer” resonance. It drives transport across the chaotic layer of the guiding resonance. This chaotic transport is faster than the Arnol’d diffusion driven by remaining resonances, and can be described by the usual separatrix map (see Ref. \[2\], p. 355), retaining all statistical properties of the latter. Therefore, one may expect that the generic relationship \[T_L - T_r\] derived above in the framework of the universal description of a perturbed one-degree of freedom nonlinear resonance is generically valid for multi-dimensional systems as well.

Another important problem concerns conditions for the emergence of the generic relationship in numeric simulations. There are at least two natural numeric selection effects, mentioned already in Section \[2\] making the presence of the quadratic relationship \[11\] ubiquitous. These effects both concern the procedure of measurement of the LLCE. First, the LLCE is measured on a long enough time interval in order that its value were “saturated” (see Section \[2\]). This eliminates small recurrence times from consideration. Second, when relationships of the kind \[T_L - T_r\] are constructed, the LLCE is not measured normally on time intervals greater than \[T_r\]. It is usually measured on some limited time interval, though long enough for the LLCE to be saturated (e.g. \[9\]). This limitation is usually justified by that the variation of the LLCE after its saturation is slow.
Imposing a lower limit on the time of the measurement leads to a situation that the impact of chaos border (or, equivalently, the role of the sticking regime) may become prominent; imposing the upper limit leads to that the LLCE corresponds to a particular local domain of the chaotic region. These factors put in action the generic relationship.

4 Numeric examples

I consider two examples of the emergence of the “$T_L - T_r$” relationship, one in a simpler and one in a computationally more sophisticated problem. The statistical behaviour of a single trajectory and that of a set of trajectories on a grid of starting values are accordingly analyzed.

The first example straightforwardly concerns the SM Eq. (4). I build the dependence “$\log T_L - \log T_r$”, where $T_L$ is the inverse of the LLCE computed on a time interval during which the trajectory stays at one side of the chaotic layer; $T_r$ is the duration of this time interval. Time is measured in iterations of the map. “Staying at one side” of the layer means that the variable $w$ in Eq. (4) has a particular sign; when a trajectory crosses the central line of the layer, the sign alternates; an orbit segment between crossings of the central line forms a recurrence.

In computations, the SM Eq. (4) was used in its equivalent form (4)

\[
\begin{align*}
y_{i+1} &= y_i + \sin x_i, \\
x_{i+1} &= x_i - \lambda \ln |y_{i+1}| + c \pmod{2\pi},
\end{align*}
\]  

(12)

where $y = w/W$, $x = \sigma + \pi$; the parameter $c = \lambda \ln 32/|W|$.

In Fig. 1, the dependence is shown for the values of the SM parameters $\lambda = 3.22$, $c = 0$. These values are the same as used in Ref. [4]. They correspond to a case of the critical curve with the “golden” winding number, i.e. the winding number equal to the golden mean $(\sqrt{5} - 1)/2$. The latter is the irrational number furthest away from neighbouring rationals (see e.g. Ref. [11]). The effect of marginal resonances is thus reduced to a minimum, and the quadratic nature of the “$T_L - T_r$” relationship manifests itself most clearly.

In captions to the figures, $n_{it}$ is the total number of iterations in the computation run, $n_{points}$ is the number of points in the plot. Logarithmic scales in all figures are decimal. Note that, in Fig. 1, the recurrences with duration $T_r < 10$ are eliminated in order that
the LLCE were saturated. One can see that the dependence in Fig. 1 has a major pattern that with large scatter but follows the straight line of the generic relationship, as expected. I have built “$T_L - T_r$” dependences for various values of the SM parameters, not only for the “golden” case. They always have the major pattern close to the quadratic law. When marginal resonances are present, there are disturbances, as expected.

Consider now an analytically more complicated problem, which nevertheless has an established theoretical link to the SM paradigm. The problem concerns the asteroidal motion in the 3/1 mean motion commensurability with Jupiter. The following analysis is performed in the planar-elliptic restricted three-body problem and is limited to asteroidal orbits with eccentricities less than 0.4. Trajectories exhibiting the mode of jumps to very high eccentricities $e \approx 1$ (cf. Ref. [8]) are not considered.

This asteroidal problem was shown [15] to be reducible, after averaging on the orbital time scale and in certain regimes, to the SM with $\lambda \approx 1.4$. Since the value of $\lambda$ is low, the diffusive stage is absent. A manifestation of the behaviour well-known already in the case of the SM was observed by Shevchenko and Scholl [16] in the appearance of statistical distributions of duration of intervals between eccentricity bursts of intermittent asteroidal orbits in the 3/1 Jovian resonance. The distributions in the tails followed the power-law decay. This kind of dependence, according to Chirikov [3], is immanent to trajectories’ sticking to chaos border.

Let us see how the relationship “$\log T_L - \log T_r$” looks like in the considered asteroidal problem. The computations are performed with Wisdom’s map [18]. The following notations are adopted henceforth: $l$ and $l_J$ are mean longitudes of an asteroid and Jupiter; $\varpi$ is the longitude of perihelion of the asteroid; $a$ and $e$ are its semimajor axis and eccentricity. The ratio of the mass of Jupiter to that of the Sun is set to be equal to $1/1047.355$. Jupiter’s perihelion is at the origin of longitudes, i.e. $\varpi_J = 0$.

Consider orbits with starting values on the rectangular grid $0.48025 \leq a_0 \leq 0.48200$, $0.005 \leq e_0 \leq 0.050$, with the step in $a_0$ equal to 0.00005 and that in $e_0$ equal to 0.005. For Jupiter, set the eccentricity $e_J = 0.048$; the initial value of its mean longitude $l_J$ set to be zero. For an asteroid, set $l_0 = \pi$, $\varpi_0 = 0$. This choice of $l_0$, $\varpi_0$ forms a representative plane of starting values of the asteroidal motion [18]; almost every orbit in the phase space of the 3/1 Jovian resonance in-
tersects this plane. The chosen rectangle covers the domain of the chaotic motion at \(e_0 \leq 0.05\) completely, as well as parts of the neighbouring space of regular motion. With such a grid, one has 350 orbits in all; 166 among them, those with \(\log T_L < 5.3\), are chaotic. The latter threshold value follows from an analysis of the bimodal structure of the differential distribution of computed values of the LLCE. One or two orbits with \(\log T_L\) close to this value may be controversial. In what follows, regular orbits are excluded.

Each trajectory, together with its LLCE, was computed on the time interval \(n_{it} = 10^7\) iterations of Wisdom’s map (18) (one iteration equals to one Jupiter period), or less if a burst of eccentricity was encountered. The burst was considered to take place if the value 0.2 of eccentricity was surmounted. This provides a good empirical criterion in the given range of starting eccentricities.

The resulting “\(\log T_L - \log T_r\)” relationship is shown in Fig. 2. As in the case of the SM, one can see again that the statistical dependence tentatively follows the generic relationship expected for the motion near the separatrix of a nonlinear resonance.

An important feature of the observed dependence is that there exists a group of chaotic orbits for which the recurrence time is “infinite”, i.e. these orbits do not ever exhibit eccentricity bursts, at least during the adopted time interval of computation. They are displayed in Fig. 2 as points with \(\log T_r\) at the limiting value of 7. Such orbits form a group located mostly at \(\log T_L = 4.0 \div 4.2\); thus they are definitely chaotic. A closer analysis shows that these orbits have a very narrow spectrum of winding numbers: the ratio \(Q\) of frequencies of rotation of angles used in the SM approximation of the relevant motion (15), \(\sigma \equiv 2\omega + l - 3l_J\) and \(\kappa \equiv \omega + l - 3l_J\), lies within limits \(4/3\) and \(3/2\) for these orbits; i.e. they are associated with the chaotic layers around separatrices of the minor resonances \(Q = 4/3\) and \(3/2\). The absence of eccentricity bursts simply means that these resonances do not overlap with the integer one, \(Q = 1\), which is responsible for the eccentricity bursts.

The existence of a chaotic asteroidal orbit without bursts was already encountered by Milani and Nobili (12) in a study of the asteroid Helga. Such phenomenon seems to be in an apparent contradiction to the statistical law “\(T_L - T_r\)” as found by (17; 9; 14; 10). The example of the asteroidal problem considered above shows that there is no contradiction; the matter is in the definition of a “sudden orbital change”. A single definition should not be used when trajectories in
a statistical set belong to disconnected chaotic domains.

Concluding on the numerics, it is necessary to note the following.

(1) Exact values of the exponent $\beta$ of the power law fitting the observed relationships in the examples considered are not presented here, since the data cannot be straightforwardly corrected for selection effects. The most important selection effect modifying the value of the exponent calculated by the mean square fit is due to concentration of the points to the lower part of the dependence, since long recurrences are rare. Therefore the exponent value somewhat depends on the mode of the data cutting at the lower time edge. Evidently, longer recurrences should be taken with a greater weight, presumably directly proportional to their duration. Neglecting the weights would lead to a bias in computed values of the exponent; author’s numeric experiments with the SM show that the value of $\beta$ somewhat diminishes as a rule. Maybe the small deviation of the reported values of $\beta = 1.7 \pm 1.9$ in Refs. (17, 9, 14, 10) from the theoretical quadratic result is due to this selection effect.

(2) Whether the lower time limit of applicability of the theoretical law (11) is low enough for the examples considered? Empirically, it seems to be the case. Indeed, the distribution of duration of recurrences for the SM with $\lambda \sim 1$ transforms into the algebraic law with the exponent $\approx -1.5$ (for the integral distribution) characteristic of the sticking regime when recurrences are just several iterations long. According to (3), the threshold value of $T_r$ for this transformation is $\approx 0.3\lambda^2$. Concerning the case of asteroidal trajectories in the 3/1 Jovian resonance, the transformation of the distribution of the inter-burst interval duration to the algebraic decay with the exponent $\approx -1.5$ for integral distributions is often observed already at $T_r \approx 10^5$ Jupiter periods (16), i.e. again the lower time limit for the theory’s applicability is reasonably small. These considerations are mostly empirical, of course; a universal theoretical estimate for the lower time limit is still to be found.

5 Conclusions

In this paper, conditions for the emergence of a statistical relationship between $T_r$, the recurrence time, and $T_L$, the local Lyapunov time (the inverse of the locally defined largest Lyapunov characteristic exponent, LLCE), were investigated for the motion inside the
The generic relationship is shown to resemble the quadratic one. The reasons leading to its emergence are very general. There are two main factors. The first one is immanent to any Hamiltonian system; this is the effect of trajectories’ sticking to chaos border. The second one is a natural selection effect arising in the procedure of computation of LCE: though they are measured on time intervals long enough for their values to be saturated, the computation is normally stopped before or when sudden orbital changes, which signal the end of the “recurrence time”, take place.

In order to check the validity of this theoretical relationship straightforwardly, a statistical dependence between the duration of a recurrence (the time which a chaotic trajectory stays at one side of the chaotic layer) and the Lyapunov time (the inverse of the numeric value of the LLCE measured on the time interval of the recurrence) was constructed by means of computation with the separatrix map (14). Besides, as a particular applied and numerically more complicated example, a statistical dependence of the time of a sudden orbital change (namely, the time of a burst of eccentricity) on the Lyapunov time was constructed for chaotic asteroidal orbits in the 3/1 Jovian resonance in the planar-elliptic three-body problem by means of Wisdom’s map (18). In both cases, the LLCE were measured on a time interval equal to the recurrence time (i.e. until a sudden orbital change), and the observed dependences follow the generic relationship (11), as expected.

The existence of the generic relationship (11) tentatively provides a theoretical explanation of the statistical dependences of times of “sudden orbital changes” on Lyapunov times, found by (17; 9; 14; 10) in numeric experiments in a number of problems of celestial mechanics.

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Figure 1: The statistical dependence “log $T_L$ — log $T_r$” for the SM with parameters $\lambda = 3.22$, $c = 0$. The winding number of the critical curve is “golden”. $n_{it} = 10^5$, $n_{points} = 1682$. Time is in iterations of the map. The straight (dotted) line of the generic relationship is shown for reference.
Figure 2: The statistical dependence “log $T_L$ — log $T_r$” for chaotic asteroidal trajectories in the 3/1 Jovian resonance, as described in the text. $n_{\text{points}} = 166$. Time is in Jupiter periods. The straight (dotted) line of the generic relationship is shown for reference.