MARKOFF–ROSENBERGER TRIPLES IN GEOMETRIC PROGRESSION

ENRIQUE GONZÁLEZ–JIMÉNEZ

Abstract. We study solutions of the Markoff–Rosenberger equation $ax^2 + by^2 + cz^2 = dxyz$ whose coordinates belong to the ring of integers of a number field and form a geometric progression.

1. Introduction

The study of sequences of points on algebraic varieties is nowadays a topic in number theory studied by many authors. A special interest has been shown for the case of arithmetic progressions on plane curves. Let $C : F(x,y) = 0$ be a plane curve defined over a field $K$. An arithmetic progression (a.p. for short) of length $n$ on $C$ is a sequence of points $(x_1,y_1),\ldots,(x_n,y_n) \in C(K)$ such that $x_1,\ldots,x_n$ form an arithmetic progression. If $C$ is an elliptic curve, many researchers have studied this problem depending on how $C$ is described: Weierstrass form [15, 18, 20, 21, 6, 33], Mordell [27, 23], congruent [16, 35], quartic form [36, 24, 5], Edwards [28], Huff [29]. In genus greater than one the case of $C$ being a hyperelliptic curve has been treated in [37, 4, 38]. As for the case of genus zero is concerned: Pellian equations [19, 30, 11], conic section [7].

Recently, some authors have considered similar problems but replacing arithmetic progressions by geometric progressions (g.p. for short): Bérczes and Ziegler [13] and Bremner and Ulas [17] on Pell type equations.

One can take another point of view and consider a hypersurface $S : F(x_1,\ldots,x_n) = 0$ in $\mathbb{A}^n$. One may study if the coordinates of a point in $S$, considered as a sequence, satisfy some property. For example, if they form an a.p. The case of solutions of norm equation such that their coordinates form an a.p. has been deeply studied in [10, 11, 12, 9]. Recently, the author and J. M. Tornero [22] have studied the case of triples in a.p. on the Markoff–Rosenberger equation over number fields. That is, if $\mathcal{O}_K$ denotes the ring of integers of a number field $K$, we studied triples $x,y,z \in \mathcal{O}_K$ in a.p. such that $ax^2 + by^2 + cz^2 = dxyz$ for some $a,b,c,d \in \mathcal{O}_K$. In the current paper, we fix our attention on the case of g.p. instead of a.p. to prove results for g.p. analogous to those obtained in [22]. That is, our main objective is to study the set

$$\mathcal{GP}_{(a,b,c,d)}(K) := \{ \mathcal{O}_K–\text{non-trivial triples in g.p. to } ax^2 + by^2 + cz^2 = dxyz \}.$$

Date: December 12, 2013.
2010 Mathematics Subject Classification. Primary: 11D25; Secondary: 11D45, 14G05.
Key words and phrases. Markoff equation, geometric progression.
The author was partially supported by the grant MTM2012–35849.
In section 2, we will see that $\mathcal{GP}_{(a,b,c,d)}(K)$ is in bijection with a subset of the affine solutions over $\mathcal{O}_K$ of a genus zero curve, denoted by $\mathcal{G}_{(a,b,c,d)}$. Then the description of $\mathcal{GP}_{(a,b,c,d)}(K)$ is translated to the computation of integer points of a genus zero curve. For this task we will use mainly and heavily the research of Poulakis on solving genus zero Diophantine equations, developed in successive papers with Alvanos, Bilu and Voskos [2, 3, 31]. In section 3, we present an algorithm based on Alvanos and Poulakis’ work [3] that allows us to compute $\mathcal{G}_{(a,b,c,d)}(\mathcal{O}_K)$. We show how the algorithm described in this section works on some examples not covered by the theoretical results from section 4.

All the main theoretical results appear at section 4. There, we show that infinitely many integral Markoff–Rosenberger triples in g.p. over a number field $K$ can exist only if $K$ is neither the rational field nor a quadratic imaginary field. When $K$ is the rational field we give an explicit description of this finite set. Finally we fix our attention to our original goal, the study of the generalized Markoff equation:

$$x^2 + y^2 + z^2 = dxyz.$$  

2. A genus zero curve

Our starting point is the Markoff equation

$$x^2 + y^2 + z^2 = 3xyz.$$  

The integer solutions (so-called Markoff triples) of this Diophantine equation were deeply studied by Markoff in [25, 26] obtaining, among other results, that infinitely many Markoff triples exist. Later on, the Markoff equation has been generalized by several authors. We are going to focus our attention on the one studied by Rosenberger [32]:

$$ax^2 + by^2 + cz^2 = dxyz.$$  

We will call this equation the Markoff–Rosenberger equation and its solutions will be called Markoff–Rosenberger triples. Notice that Rosenberger imposed some extra conditions on the coefficients $a, b, c, d \in \mathbb{N}$: $a|d$, $b|d$, $c|d$ and $(a, b) = (a, c) = (b, c) = 1$. With these requirements he proved that non–trivial integral Markoff–Rosenberger triples exist only if $(a, b, c, d)$ belongs to the set \{(1, 1, 1, 1), (1, 1, 1, 3), (1, 1, 2, 2), (1, 1, 2, 4), (1, 2, 3, 6), (1, 1, 5, 5)\}. We will not assume these extra conditions for the rest of the paper.

This paper is devoted to the study of Markoff–Rosenberger triples that form a g.p. over $\mathcal{O}_K$. Let $a, b, c, d \in \mathcal{O}_K$ and $x, y, z \in \mathcal{O}_K$ be a Markoff–Rosenberger triple in g.p. Then there exist $\alpha, \beta \in \mathcal{O}_K$ such that

$$x = \alpha, \quad y = \alpha \beta, \quad z = \alpha \beta^2,$$

satisfying

$$\alpha^2 (c\beta^4 - d\alpha \beta^3 + b\beta^2 + a) = 0.$$  

Therefore, if we exclude the solutions with $\alpha = 0$ that correspond to the trivial solution $(x, y, z) = (0, 0, 0)$, we obtain that non–trivial Markoff–Rosenberger triples in g.p. over $\mathcal{O}_K$ are in bijection with the affine solutions over $\mathcal{O}_K$ of the curve

$$\mathcal{G} = \mathcal{G}_{(a,b,c,d)} : F(X,Y) = cY^4 - dXY^3 + bY^2 + a = 0.$$  

2. A genus zero curve
with $X \neq 0$. Namely:

$$\mathcal{G}(a,b,c,d)(\mathcal{O}_K) \rightarrow \mathcal{GP}(a,b,c,d)(K).$$

(3)

$$\begin{align*}
(X, Y) & \mapsto (X, XY, XY^2) \\
(x, z/y) & \mapsto (x, y, z)
\end{align*}$$

Let $\tilde{F}(X, Y, Z)$ be the homogenization of $F(X, Y)$ and denote by $\tilde{G}$ the projective curve defined by $\tilde{G} : \tilde{F}(X, Y, Z) = 0$. Now, the curve $\tilde{G}$ has two points at infinity: $[c : d : 0]$ and the singular point $[1 : 0 : 0]$. In particular, $\tilde{G}$ has genus $0$ and the birational map

$$\tilde{\psi} : \mathbb{P}^1 \rightarrow \tilde{G}$$

$$[U : V] \mapsto \left( cU^4 + bV^2U^2 + aV^4 : dU^3V \right)$$

(4)

gives a parametrization of the curve $\tilde{G}$ over $K$.

Denote by $\tilde{G}_\infty$ the set of infinity places of the field $\overline{K}(C)$. Then by [31, Lemma 2.2] we have that

$$|\tilde{G}_\infty| = |\{(U : V) \in \mathbb{P}^1 : dU^3V = 0\}| = |\{[1 : 0], [0 : 1]\}| = 2.$$ 

Note that both points at infinity are defined over $\mathbb{Q}$.

3. The general algorithm over number fields

Let $K$ be a number field of degree $n = [K : \mathbb{Q}]$ and $a, b, c, d \in \mathcal{O}_K$. Our objective in this section is to describe the set

$$\mathcal{G}(\mathcal{O}_K) = \{(X, Y) \in \mathcal{O}_K^2 | F(X, Y) = 0\}.$$ 

For this purpose we use the affine part of the parametrization $\tilde{\psi}$:

$$\psi : \mathbb{P}^1 \rightarrow \mathcal{G}$$

$$[U : V] \mapsto \left( \frac{cU^4 + bV^2U^2 + aV^4}{dU^3V}, \frac{U}{V} \right)$$

and we are going to develop an algorithm heavily based on the Alvanos and Poulakis’ algorithm INTEGRAL-POINTS2A from the paper [3]. In fact, the algorithm presented here is just the application of INTEGRAL-POINTS2A to the curve $\mathcal{G}$.

**Step 1:** Let $\alpha \in \mathcal{O}_K$ and denote by $\tilde{\alpha} = \alpha^s$ where $s = 0$ or $s = 1$ depending on whether $\alpha \in \mathbb{Z}$ or $\alpha \notin \mathbb{Z}$. Let $E = \mathbb{Q}(\tilde{\alpha})$ and $L = \mathbb{Q}(\tilde{d})$. Compute

$$\delta_0 = \gcd\left( \frac{c}{\tilde{c}}, \frac{d}{\tilde{d}}N_{K}(\tilde{\alpha}), N_{L}(\tilde{d}) \right) \in \mathbb{Z}.$$ 

We denote by $N_K$ the absolute norm map for a number field $K$.

**Step 2:** Compute

$$M = \{ t \in \mathcal{O}_K | N_{K}(t) \text{ divides } N_{K}(\alpha)\delta_0^{4n} \}/\sim,$$

where $\sim$ denotes equivalence class (modulo units of $\mathcal{O}_K$).

**Step 3:** Compute a basis for the unit group $\mathcal{U}(\mathcal{O}_K)$. By Dirichlet’s Unit Theorem we have that $\mathcal{U}(\mathcal{O}_K)$ is a finitely generated abelian group and therefore:

$$\mathcal{U}(\mathcal{O}_K) = \langle \zeta_k \rangle \oplus \langle \varepsilon_1, \ldots, \varepsilon_r \rangle$$

where $\zeta_k$ is a $k$-th root of unity ($k$ is the torsion order) and $r$ is the rank. Moreover, $r = r_1 + r_2 - 1$ where $r_1$ denotes the number of real embeddings of $K$ and $r_2$ the
number of complex pairs embeddings of $K$. In particular, $\mathcal{U}(\mathcal{O}_K)$ is finite if and only if $K = \mathbb{Q}$ or $K$ is an imaginary quadratic field.

**Step 4:** For any $t \in M$ compute $\tau(i, t)$ the order of the class of $\varepsilon_i$ in $\mathcal{U}(\mathcal{O}_K/\delta_0dt^3)$. Note that if $\delta_0dt^3 \in \mathcal{U}(\mathcal{O}_K)$ then we define $\tau(i, t) = 1$.

**Step 5:** Now, for every $t \in M$ compute the set $H(t)$ of units $\eta = \zeta_k^{\ell_1} \cdots \zeta_r^{\ell_r}$ with $0 \leq l < k$, $0 \leq l_i < \tau(i, t)$ for $i = 1, \ldots, r$ such that $\psi(t\eta, \delta_0) \in \mathcal{O}_K^*$. From this condition we obtain that $d(t\eta)^3\delta_0$ divides $c(t\eta)^4 + b\delta_0^2(t\eta)^2 + a\delta_0^3$ and $\delta_0$ divides $t\eta$. Then

$$H(t) = \left\{ \eta = \zeta_k^{\ell_1} \cdots \zeta_r^{\ell_r} \mid 0 \leq l < k, \quad 0 \leq l_i < \tau(i, t), \quad i = 1, \ldots, r \right\}.$$ 

Notice that in the case that $\delta_0 \neq \pm 1$ we have $\delta_0$ divides $t$.

**OUTPUT:** Denote by $\Theta(t) = \prod_{i=1}^{r} \varepsilon_i^{\tau(i, t)\cdot z_i}$, $z_i \in \mathbb{Z}, i = 1, \ldots, r$. Then

$$\mathcal{G}(\mathcal{O}_K) = \bigcup_{t \in M} \{ \psi(t\eta\varepsilon, \delta_0) \mid \eta \in H(t) \text{ and } \varepsilon \in \Theta(t) \}.$$ 

Note that in order to apply the algorithm for some fixed values $a, b, c, d$ in the ring of integers of some number field $K$ we should be able to solve some problems. All of them are sorted out in Magma [14]. In the following table we show the main problems to be solved and the Magma functions that may be used:

| Step | Problems | Magma functions |
|------|----------|-----------------|
| 2    | $M$      | NormEquation    |
| 3    | $\mathcal{U}(\mathcal{O}_K)$ | UnitGroup |
| 4    | $\tau(i, t)$ | quo, MultiplicativeGroup.Order |
| 5    | $\psi(t\eta, \delta_0) \in \mathcal{O}_K^*$ | IsIntegral |

3.1. **The algorithm at work.** We recall that Rosenberger [32] proved that integral Markoff–Rosenberger triples exist if and only if

$$(a, b, c, d) \in \{(1, 1, 1, 1), (1, 1, 1, 3), (1, 1, 2, 2), (1, 1, 2, 4), (1, 2, 3, 6), (1, 1, 5, 5)\},$$

with the extra requirements: $a, b, c, d \in \mathbb{N}$: $a|d, b|d, c|d$ and $(a, b) = (a, c) = (b, c) = 1$. Note that Theorem [1,3] applied to the previous cases tells us that

| $(a, b, c, d)$ | $\mathcal{G}(\mathcal{O}_K)(\mathbb{Q})$ |
|---------------|---------------------------------|
| $(1, 1, 1, 1)$ | $(\pm 3, 3, \pm 3)$ |
| $(1, 1, 1, 3)$ | $(\pm 1, 1, \pm 1)$ |
| $(1, 1, 2, 2)$ | $(\pm 2, 2, \pm 2)$ |
| $(1, 1, 2, 4)$ | $(\pm 1, 1, \pm 1)$ |
| $(1, 2, 3, 6)$ | $(\pm 1, 1, \pm 1)$ |
| $(1, 1, 5, 5)$ | $\emptyset$ |

That is, integral Markoff–Rosenberger triples in g.p. exist for all these cases except for the last one. In this section we are going to study the case $(1, 1, 5, 5)$ over the first two quadratic real fields. Thanks to Theorem [1,1] we know that if $D$ is a squarefree positive integer then a Markoff–Rosenberger triple in g.p. over $\mathbb{Q}(\sqrt{D})$ exists if and only if infinitely many exist. We show one case of each of these possibilities. In particular, in $\mathbb{Q}(\sqrt{2})$ we describe the infinitely many triples in g.p.; meanwhile in $\mathbb{Q}(\sqrt{3})$ we will show that triples in g.p. do not exist. Let us apply our algorithm for this purpose.
Let $K$ be a real quadratic field. We have that $\delta_0 = 5$, in particular $M = \{ t \in \mathcal{O}_K \mid N_K(t) = \pm 5^k, \; k = 2, \ldots, 8 \}$. Moreover, a fundamental unit $\varepsilon_D$ exists, such that $\mathcal{U}(\mathcal{O}_K) = \{ \pm \varepsilon_D^k \mid k \in \mathbb{Z} \}$. Now we work out the cases $K = \mathbb{Q}(\sqrt{2})$ and $K = \mathbb{Q}(\sqrt{3})$:

- $K = \mathbb{Q}(\sqrt{2})$. The fundamental unit is $\varepsilon_2 = 1 + \sqrt{2}$ which has order 12 on $\mathcal{U}(\mathcal{O}_K)$. We have $M = \bigcup_{k=1}^4 \{ 5^k, 5^k \varepsilon_2 \}$. Then, for any $t \in M$ we obtain that $H(t) = \emptyset$, except in the following two cases:
  
  $H(5\varepsilon_2) = \{ \pm \varepsilon_2^2, \pm \varepsilon_2^3 \}$ and $H(5) = \{ \pm \varepsilon_2^3, \pm \varepsilon_2^4 \}$.

Now, we have $\Theta(t) = \{ \varepsilon_2^{12z} \mid z \in \mathbb{Z} \}$ for any $t \in M$. Therefore

$$\mathcal{P}_{(1,1,5,5)}(\mathcal{O}_\mathbb{Q}(\sqrt{2})) = \bigcup_{k \in \{1,2\}} \bigcup_{z \in \mathbb{Z}} \{ \psi(\pm 5\varepsilon_2^{3k+12z},5) \}.$$  

Then we have obtained infinitely many triples in g.p., described by

$$\mathcal{P}_{(1,1,5,5)}(\mathcal{O}_\mathbb{Q}(\sqrt{2})) = \bigcup_{k=1}^2 \bigcup_{z \in \mathbb{Z}} \left\{ (\pm \beta, \beta \gamma, \pm \beta \gamma^2) \mid \gamma = \varepsilon_2^{3k+12z}, \beta = \gamma + 5^{-1}(\gamma^{-1} + \gamma^{-3}) \right\}.$$

- $K = \mathbb{Q}(\sqrt{3})$. The fundamental unit is $\varepsilon_3 = 2 + \sqrt{3}$ which has order 3 on $\mathcal{U}(\mathcal{O}_K)$.

In this case, we have $M = \{ 5^k \mid k = 1, 2, 3, 4 \}$. Finally, we obtain that $H(t) = \emptyset$ for all $t \in M$. That is, $\mathcal{G}_{(1,1,5,5)}(\mathcal{O}_\mathbb{Q}(\sqrt{3})) = \emptyset$, and therefore

$$\mathcal{P}_{(1,1,1,5)}(\mathcal{O}_\mathbb{Q}(\sqrt{3})) = \emptyset.$$

4. THEORETICAL RESULTS

This section is dedicated to show the theoretical results obtained on Markoff–Rosenberger triples in g.p. over number fields.

**Theorem 4.1.** Let $K$ be a number field and $a, b, c, d \in \mathcal{O}_K$. Assume that $\mathcal{G}(\mathcal{O}_K)$ contains a non–singular point, then the set $\mathcal{P}_{(a,b,c,d)}(K)$ is finite if and only if $K = \mathbb{Q}$ or $K$ is an imaginary quadratic field.

**Proof.** Let $C$ be an affine algebraic curve of genus $g$ defined over a number field $K$ and denote by $C_\infty$ the set of infinity places of the field $\overline{K}(C)$. Siegel [34] proved that if $g > 0$ or $|C_\infty| > 2$ then $C(\mathcal{O}_K)$ is finite. However, $C(\mathcal{O}_K)$ may be finite if $g = 0$ and $|C_\infty| \leq 2$. These last cases were treated by Alvanos, Bilu and Poulakis [2], obtaining a complete characterization of the cases in which $C(\mathcal{O}_K)$ is finite. In particular, our genus zero curve $\mathcal{G}$ satisfies $|\mathcal{G}_\infty| = 2$ and both points at infinity are defined over $\mathbb{Q}$, therefore [2] Theorem 1.2] asserts that $\mathcal{G}(\mathcal{O}_K)$ is finite if and only if $K = \mathbb{Q}$ or $K$ is an imaginary quadratic field. 

**Remark 4.2.** Assume that $d$ divides $a + b + c$ on $\mathcal{O}_K$ where $K$ neither is $\mathbb{Q}$ nor an imaginary quadratic field, then $\#\mathcal{P}_{(a,b,c,d)}(K) = \infty$.

Now we fix our attention on the case of the rational field or an imaginary quadratic field, since in these cases we have proved that only a finite number of triples in g.p. exist. In the rational case we have the following result:

---

*Baer and Rosenberger [3] studied the set of solutions over the ring of integers of an imaginary quadratic field of the equation $ax^2 + by^2 + cz^2 = dxyz$ with the extra requirements $a, b, c | d$.}*
Theorem 4.3. Let \( a, b, c, d \in \mathbb{Z} \). For any \( u \in \mathbb{Z} \) denote by \( \gamma_u = \frac{c u^2 + b u^2 + a}{du^2} \) then
\[
\mathcal{GP}_{(a,b,c,d)}(\mathbb{Q}) = \left\{ (\pm \gamma_u, u \gamma_u, \pm u^2 \gamma_u) \mid u^2 | a, u > 0, du^2 | (cu^2 + bu^2 + a) \right\}.
\]
Furthermore, if \( a \) is squarefree and \( \gamma = \frac{a + b + c}{d} \), then
\[
\mathcal{GP}_{(a,b,c,d)}(\mathbb{Q}) = \left\{ \begin{aligned}
(\pm \gamma, \gamma, \pm \gamma) & \quad \text{if } d | (a + b + c), \\
\emptyset & \quad \text{otherwise}.
\end{aligned} \right.
\]

Proof. Let \( x, y, z \in \mathbb{Z} \) be a triple in g.p. to the equation \( ax^2 + by^2 + cz^2 = dxyz \). That is, there exist \( u, v \in \mathbb{Z} \) such that \( u \neq 0 \) and \( x = v, y = vu, z = vu^2 \). Then \( au^2 + bu^2 v^2 + cu^2 v^4 = dv^3 u^3 \). Dividing by \( u^2 \) we obtain the simpler equation: \( a + bu^2 + cu^4 - dvu^3 = 0 \). This implies \( u^2 | a \) and \( v = \gamma_u \). In particular, since \( v \in \mathbb{Z} \) we have \( du^3 | (cu^4 + bu^2 + a) \). Finally, we may assume \( u > 0 \) since \( \gamma_{-u} = -\gamma_u \) and therefore if \( (\gamma_u, u \gamma_u, u^2 \gamma_u) \in \mathcal{GP}_{(a,b,c,d)}(\mathbb{Q}) \) then \( (-\gamma_u, u \gamma_u, -u^2 \gamma_u) \in \mathcal{GP}_{(a,b,c,d)}(\mathbb{Q}) \).

Now, from the condition \( u^2 | a \), it is clear that if \( a \) is squarefree then \( u = \pm 1 \) must hold. \( \square \)

Remark 4.4. Notice that the condition on the squarefreeness of \( a \) is necessary. For example, \( \mathcal{GP}_{(4,1,1,1)}(\mathbb{Q}) = \{ (\pm 6, 6, \pm 6), (\pm 3, 6, \pm 12) \} \).

The next two results give complete descriptions on the case of generalized Markoff triples in g.p. over number fields of very low degree, that is one or two. Notice that \( (x, y, z) \) is a triple in g.p. to the equation \( x^2 + y^2 + z^2 = dxyz \) if and only if \( (x, -y, z) \) is a triple in g.p. to the equation \( x^2 + y^2 + z^2 = -dxyz \). Then we will assume that \( d \) is positive. Now, the first result describes precisely the cases in which only finitely many triples in g.p. exist.

Theorem 4.5. Let \( d, D \in \mathbb{Z}_{>0} \), with \( D \) squarefree. If \( (d, D) \neq (1, 1) \), then
\[
\mathcal{GP}_{(1,1,1,d)}(\mathbb{Q}(\sqrt{-D})) = \mathcal{GP}_{(1,1,1,1)}(\mathbb{Q}) = \left\{ \begin{aligned}
(\pm 3, 3, \pm 3) & \quad \text{if } d = 1, \\
(\pm 1, 1, \pm 1) & \quad \text{if } d = 3, \\
\emptyset & \quad \text{if } d \neq 1, 3,
\end{aligned} \right.
\]
and \( \mathcal{GP}_{(1,1,1,1)}(\mathbb{Q}(i)) = \{ (\pm 3, 3, \pm 3), (\pm i, -1, \mp i) \} \).

Proof. First notice that the second equality, for \( (d, D) \neq (1, 1) \), is a consequence of Theorem 4.3 with \( a = b = c = 1 \). To prove the first equality we use the algorithm of section 4. We have that \( \delta_0 = 1 \). Let \( K = \mathbb{Q}(\sqrt{-D}) \) then \( M = \{ 1 \} \), since on an imaginary quadratic field all elements have positive norm. Now we should compute the unit group of \( \mathcal{O}_K \). But it is well-known that \( \mathcal{U}(\mathcal{O}_K) = \{ \zeta_k \} \), when \( k = 2 \) (\( \zeta_2 = -1 \)) if \( D \neq 1, 3 \), \( k = 4 \) (\( \zeta_4 = i \)) if \( D = 1 \) and \( k = 6 \) \((\zeta_6 = (1 + \sqrt{-3})/2)\) if \( D = 3 \). Next step is to compute the set \( H(1) \): in this case its elements are \( \zeta_k^l \) for \( 0 \leq l < k \) such that \( d | (\zeta_k^l + \zeta_k^\ell) + 1 \). It is a straightforward computation to determine \( H(1) \) depending on \( D \) and \( d \):

\[
H(1) = \left\{ \begin{aligned}
\emptyset & \quad \text{if } D \neq 3 \text{ and } d \neq 1, 3, \\
\{ \pm 1 \} & \quad \text{if } D \neq 1, 3 \text{ and } d = 1, 3, \\
\{ \zeta_k^l \mid l = 1, 2, 4, 5 \} & \quad \text{if } D = 3 \text{ and } d \neq 1, 3, \\
\{ \zeta_k^l \mid 0 \leq l < 6 \} & \quad \text{if } D = 3 \text{ and } d = 1, 3, \\
\{ \zeta_4^l \mid 0 \leq l < 4 \} & \quad \text{if } D = 1 \text{ and } d = 1, \\
\{ \pm 1 \} & \quad \text{if } D = 1 \text{ and } d = 3.
\end{aligned} \right.
\]
Then the algorithm outputs:

$$
\mathcal{G}(O_K) = \left\{ \left( \frac{\eta^4 + \eta^2 + 1}{d\eta^3}, \eta \right) \mid \eta \in H(1) \right\}.
$$

That is

$$
\mathcal{G}(O_K) = \left\{
\begin{array}{ll}
\emptyset & \text{if } D \neq 3 \text{ and } d \neq 1, 3, \\
\{ \pm(3, 1) \} & \text{if } D \neq 1, 3 \text{ and } d = 1, \\
\{ \pm(1, 1) \} & \text{if } D \neq 3 \text{ and } d = 3, \\
\{(0, \zeta_d^l) \mid l = 1, 2, 4, 5\} \cup \{ \pm(3, 1) \} & \text{if } D = 3 \text{ and } d \neq 1, 3, \\
\{(0, \zeta_d^l) \mid l = 1, 2, 4, 5\} \cup \{ \pm(1, 1) \} & \text{if } D = 3 \text{ and } d = 3, \\
\{ \pm(3, 1), \pm(i, i) \} & \text{if } D = 1 \text{ and } d = 1.
\end{array}
\right\}
$$

Therefore, the bijection given by (3) gives:

$$
\mathcal{GP}_{(1,1,1,d)}(\mathbb{Q}(\sqrt{-D})) = \left\{
\begin{array}{ll}
\emptyset & \text{if } d \neq 1, 3, \\
\{ \pm(1, 1), \pm(3, 1) \} & \text{if } d = 3, \\
\{ \pm(3, 3), \pm(3, 3) \} & \text{if } D \neq 1 \text{ and } d = 1, \\
\{ \pm(3, 3, 3), (\pm i, -1, \mp i) \} & \text{if } D = 1 \text{ and } d = 1.
\end{array}
\right\}
$$

\[ \square \]

**Remark 4.6.** In particular the previous result proves:

$$
\bigcup_{d, D \in \mathbb{Z}_{>0}} \mathcal{GP}_{(1,1,1,d)}(\mathbb{Q}(\sqrt{-D})) = \{ \pm(1, 1), \pm(3, 3, 3), (\pm i, -1, \mp i) \}
$$

**Remark 4.7.** Note that a similar study for fixed $a, b, c \in \mathbb{Z}$, with $a$ squarefree, could be done. That is, to compute explicitly $\mathcal{GP}_{(a,b,c,d)}(\mathbb{Q}(\sqrt{-D}))$ and $\mathcal{GP}_{(a,b,c,d)}(\mathbb{Q})$ for any $d, D \in \mathbb{Z}_{>0}$, with $D$ squarefree.

Once we have treated the cases where there are only a finite number of generalized Markoff triples in g.p., we are going to give a complete description of the set of generalized Markoff triples in g.p. over a real quadratic field. In this case, we will have infinitely many such triples or none at all.

**Theorem 4.8.** Let $D$ be a squarefree positive integer and $\varepsilon_D$ be the fundamental unit of the real quadratic field $K = \mathbb{Q}(\sqrt{D})$. Denote by $n$ the order of the class of $\varepsilon_D$ in $\mathcal{U}(O_K/d)$ and define the set

$$
\mathcal{H} = \left\{ k \in \{0, \ldots, n\} \mid \varepsilon_D^{4k} + \varepsilon_D^{2k} + 1 \equiv 0 \pmod{d(O_K)} \right\}.
$$

Then

$$
\mathcal{GP}_{(1,1,1,d)}(\mathbb{Q}(\sqrt{D})) = \bigcup_{k \in \mathcal{H}, z \in \mathbb{Z}} \left\{ (\pm \beta, \beta \gamma, \pm \beta \gamma^2) \mid \gamma = \varepsilon^{nz+k}_D, \beta = \frac{1}{d} \sum_{j=0}^{2} \gamma^{1-2j} \right\}.
$$

**Proof.** To prove this result we are going to use the algorithm of section [3]. First of all, let $\varepsilon_D$ be the fundamental unit of the real quadratic field $K = \mathbb{Q}(\sqrt{D})$. That is, $\varepsilon_D$ satisfies $\mathcal{U}(O_K) = \{ \pm \varepsilon_D^k \mid k \in \mathbb{Z} \}$. To compute $M$ we have only to observe that since $\delta_0 = 1$ the set $M$ consists on 1 and $\varepsilon_D$ in the case that $\mathcal{N}_K(\varepsilon_D) = -1$. Note that this is the so-called negative Pell equation. While the Pell equation $\mathcal{N}_K(\varepsilon_D) = 1$ has always a solution for any positive integer $D$, the negative Pell equation does not always have one, although there exists an effective algorithm to determine, for a fixed $D$, which is the case.
Now, denote by \( n \) the order of the class of \( \varepsilon_D \) in \( \mathcal{U}(\mathcal{O}_K/d) \). Then at step 4 we must compute \( H(t) \) for \( t \in M \). Note that 1 always belongs to \( M \), then we have

\[
H(1) = \left\{ \pm \varepsilon_D^n, \varepsilon_D^{2k} + 1 \equiv 0 \pmod{d\mathcal{O}_K}, k = 0, \ldots, n - 1 \right\}.
\]

In the case that \( \mathcal{N}_K(\varepsilon_D) = -1 \) we have that \( \varepsilon_D \in M \) and

\[
H(\varepsilon_D) = \left\{ \pm \varepsilon_D^{k+1} + \varepsilon_D^{2(k+1) + 1} \equiv 0 \pmod{d\mathcal{O}_K}, k = 0, \ldots, n - 1 \right\}.
\]

Finally, we have \( \Theta(t) = \left\{ \varepsilon_D^n \mid z \in \mathbb{Z} \right\} \) for any \( t \in M \). Then the algorithm outputs

\[
\mathcal{G}_{(1,1,1,d)}(\mathcal{O}_K) = \{ \psi(\pm \varepsilon_D^{k+nz}, 1) \mid k \in \mathcal{H}, z \in \mathbb{Z} \}.
\]

Then the bijection given by (3) together with the parametrization \( \psi \) given in (5) give the result. \( \square \)

**Remark 4.9.** The triples in g.p. over the ring of integer of a real quadratic field of the generalized Markoff equation \( x^2 + y^2 + z^2 = dxyz \) have a very special shape. That is, let \( (\beta, \gamma, \beta^2 \gamma) \) be such a triple. Then \( \gamma = \varepsilon_D^{n+2k} \) and \( \beta = \frac{1}{\beta} \sum_{j=0}^{2} \gamma^{j-2j} \) for some \( n, k, z \in \mathbb{Z} \), where \( \varepsilon_D \) is the fundamental unit on the real quadratic field \( \mathbb{Q}(\sqrt{D}) \). Moreover, \( \gamma \in \mathbb{Z} \) since \( \gamma \beta = \frac{1}{\beta}(1 + \text{Trace}_K(\gamma^2)) \in \mathcal{O}_K \) and the trace of any \( \alpha \in \mathcal{O}_K \) belongs to \( \mathbb{Z} \). But there is something more. The third term of the g.p. satisfies \( \gamma^{2\beta} = \mathcal{N}_K(\gamma) \beta^2 \gamma = \pm \beta \) where \( \sigma \) is the nontrivial automorphism of the Galois group of \( \mathbb{Q}(\sqrt{D}) \) and the sign depends on \( \mathcal{N}_K(\varepsilon_D) \) and on \( n, k, z \in \mathbb{Z} \).

**Example 4.10.** We are looking for all Markoff triples in g.p. over \( \mathbb{Q}(\sqrt{5}) \). First, the fundamental unit is \( \varepsilon_5 = (1 + \sqrt{5})/2 \) whose class in \( \mathcal{U}(\mathcal{O}_K) \) has order \( n = 8 \). Next, we obtain \( \mathcal{H} = \{0, 4, 8\} \). Therefore, we conclude:

\[
\mathcal{G}_{(1,1,1,3)}(\mathbb{Q}(\sqrt{5})) = \bigcup_{z \in \mathbb{Z}} \left\{ (\pm \beta, \beta \gamma, \pm \beta \gamma^2) \mid \gamma = \varepsilon_5^{4z}, \beta = \frac{1}{3} \sum_{j=0}^{2} \gamma^{1-2j} \right\}.
\]

The following table shows some examples of Markoff triples in g.p. over \( \mathbb{Q}(\sqrt{5}) \):

| \( z \) | Markoff triples in g.p. over \( \mathbb{Q}(\sqrt{5}) \) |
|---|---|
| 0 | \( (1,1,1) \) |
| 1 | \( (56 - 24\sqrt{5}, 16, 56 + 24\sqrt{5}) \) |
| 2 | \( (17296 - 7728\sqrt{5}, 736, 17296 + 7728\sqrt{5}) \) |
| 3 | \( (5564321 - 2488392\sqrt{5}, 34561, 5564321 + 2488392\sqrt{5}) \) |
| 4 | \( (1791660256 - 801254496\sqrt{5}, 1623616, 1791660256 + 801254496\sqrt{5}) \) |

Finally, we present a partial result valid for any number field.

**Theorem 4.11.** Let \( K \) be a number field and \( a, b, c, d \in \mathcal{O}_K \) such that \( a, d \in \mathcal{U}(\mathcal{O}_K) \). For any \( u \in \mathcal{U}(\mathcal{O}_K) \) denote by \( \gamma_u = d^{-1}(cu + bu^{-1} + au^{-3}) \). Then

\[
\mathcal{G}P(a, b, c, d, K) = \{ (\gamma_u, u\gamma_u, u^2\gamma_u) \mid u \in \mathcal{U}(\mathcal{O}_K) \}.
\]

**Proof.** Let us apply the algorithm of section 3. Now, since \( a, d \in \mathcal{U}(\mathcal{O}_K) \) we have \( \delta_0 = 1, M = \{ t \in \mathcal{O}_K \mid \mathcal{N}_K(t) = \pm 1 \} \) and \( \tau(i, t) = 1 \) for \( i = 1, \ldots, r = \text{rank}_E \mathcal{U}(\mathcal{O}_K) \) and any \( t \in M \). Therefore, if \( t \in M \) we have that \( H(t) = \mathcal{U}(\mathcal{O}_K)_{\text{tors}} \)
and \( \Theta(t) = U(O_K)_{\text{free}} \), the torsion and free part of the unit group \( U(O_K) \) respectively. Then the algorithm outputs

\[
G(O_K) = \bigcup_{t \in M} \{ \psi(t \eta \varepsilon, 1) \mid \eta \in U(O_K)_{\text{tors}}, \varepsilon \in U(O_K)_{\text{free}} \} = \bigcup_{u \in U(O_K)} \{ \psi(u, 1) \}.
\]

Then we conclude the proof using (3) and (5).

\[\Box\]

Acknowledgements. We would like to thank Paraskevas Alvanos and Dimitrios Poulakis for several useful discussions on their papers; and José M. Torner and David Torres Teigell for their careful readings of a preliminary version of this manuscript.

We wish to thank the anonymous referee whose comments and suggestions have been very useful; in particular for giving us a simplified proof of Theorem 4.3.

References

[1] J. Aguirre, A. Dujella and J. C. Peral. Arithmetic progressions and pellian equations. Preprint, 2012.
[2] P. Alvanos, Y. Bilu and D. Poulakis. Characterizing algebraic curves with infinitely many integral points. *Int. J. Number Theory*, 5 (2009), 585–590.
[3] P. Alvanos and D. Poulakis. Solving genus zero Diophantine equations over number fields. *J. Symbolic Comput.*, 46 (2011), 54–69.
[4] A. Alvarado. An arithmetic progression on quintic curves. *J. Integer Seq.*, 12 (2009), Article 09.7.3 (electronic).
[5] A. Alvarado. Arithmetic progressions on quartic elliptic curves. *Ann. Math. Inform.*, 37 (2010), 3–6.
[6] A. Alvarado. Arithmetic progressions in the \( y \)-coordinates on certain elliptic curves. In F. Luca and P. Stanica, editors, *Aportaciones Matemáticas, Investigación 20: Proceedings of the Fourteenth International Conference on Fibonacci Numbers*, pages 1–9. Sociedad Matemática Mexicana, 2011.
[7] A. Alvarado and E. H. Goins. Arithmetic progressions on conic sections. Preprint arXiv:1210.6612, 2012.
[8] C. Baer and G. Rosenberger. The equation \( ax^2 + by^2 + cz^2 = dxyz \) over quadratic imaginary fields. *Results Math.*, 33 (1998), 30–39.
[9] A. Bazsó. Further computational experiences on norm form equations with solutions forming arithmetic progressions. *Publ. Math. Debrecen*, 71 (2007), 489–497.
[10] A. Bérczes, L. Hajdu and A. Pethő. Arithmetic progressions in the solution sets of norm form equations. *Rocky Mountain J. Math.*, 40 (2010), 383–395.
[11] A. Bérczes and A. Pethő. On norm form equations with solutions forming arithmetic progressions. *Publ. Math. Debrecen*, 65 (2004), 281–290.
[12] A. Bérczes and A. Pethő. Computational experiences on norm form equations with solutions forming arithmetic progressions. *Glas. Mat. Ser. III*, 41 (2006), 1–8.
[13] A. Bérczes and V. Ziegler. On geometric progressions on Pell equations and Lucas sequences. To appear in *Glasnik Matematicki*.
[14] W. Bosma, J. Cannon, C. Fieker and A. Steel, editors. *Handbook of Magma functions, Edition 2.19-2* [http://magma.maths.usyd.edu.au/magma], 2012.
[15] A. Brenner. On arithmetic progressions on elliptic curves. *Experiment. Math.*, 8 (1999), 409–413.
[16] A. Brenner, J. H. Silverman and N. Tzanakis. Integral points in arithmetic progression on \( y^2 = x(x^2 - n^2) \). *J. Number Theory*, 80 (2000), 187–208.
[17] A. Brenner and M. Ulas. Rational points in geometric progressions on certain hyperelliptic curves. To appear in *Publ. Math. Debrecen*.
[18] G. Campbell. A note on arithmetic progressions on elliptic curves. *J. Integer Seq.*, 6 (2003), Article 03.1.3 (electronic).
[19] A. Dujella, A. Pethő and P. Tadić. On arithmetic progressions on Pellian equations. *Acta Math. Hungar.*, 120 (2008), 29–38.
[20] I. García-Selfa and J. M. Tornero. Searching for simultaneous arithmetic progressions on elliptic curves. Bull. Austral. Math. Soc., 71 (2005), 417–424.
[21] I. García-Selfa and J. M. Tornero. On simultaneous arithmetic progressions on elliptic curves. Experiment. Math., 15 (2006), 471–478.
[22] E. González-Jiménez and J. M. Tornero. Markoff–Rosenberger triples in arithmetic progression. J. Symbolic Comput., 53 (2013), 53–63.
[23] J.-B. Lee and W. Y. Vélez. Integral solutions in arithmetic progression for $y^2 = x^3 + k$. Period. Math. Hungar., 25 (1992), 31–49.
[24] A. J. MacLeod. 14-term arithmetic progressions on quartic elliptic curves. J. Integer Seq., 9 (2006), Article 06.1.2 (electronic).
[25] A. Markoff. Sur les formes quadratiques binaires indéfinies. Math. Ann., 15 (1879), 381–406.
[26] A. Markoff. Sur les formes quadratiques binaires indéfinies. Math. Ann., 17 (1880), 379–399.
[27] S. P. Mohanty. On consecutive integer solutions for $y^2 = k$. Proc. Amer. Math. Soc., 48 (1975), 281–285.
[28] D. Moody. Arithmetic progressions on Edwards curves. J. Integer Seq., 14 (2011), Article 11.1.7, 4 (electronic).
[29] D. Moody. Arithmetic progressions on Huff curves. Ann. Math. Inform., 38 (2011), 111–116.
[30] A. Pethő and V. Ziegler. Arithmetic progressions on Pell equations. J. Number Theory, 128 (2008), 1389–1409.
[31] D. Poulakis and E. Voskos. On the practical solution of genus zero Diophantine equations. J. Symbolic Comput., 30 (2000), 573–582.
[32] G. Rosenberger. Über die diophantische Gleichung $ax^2 + by^2 + cz^2 = dxyz$. J. Reine Angew. Math., 305 (1979), 122–125.
[33] R. Schwartz, J. Solymosi and F. de Zeeuw. Simultaneous arithmetic progressions on algebraic curves. Int. J. Number Theory, 7 (2011), 921–931.
[34] C. L. Siegel. Über einige anwendungen diophantischer approximationen. Abh. Preuss Akad. Wiss. Phys.-Math. Kl., 1 (1929), 41–69.
[35] B. K. Spearman. Arithmetic progressions on congruent number elliptic curves. Rocky Mountain J. Math., 41 (2011), 2033–2044.
[36] M. Ulas. A note on arithmetic progressions on quartic elliptic curves. J. Integer Seq., 8 (2005), Article 05.3.1 (electronic).
[37] M. Ulas. On arithmetic progressions on genus two curves. Rocky Mountain J. Math., 39 (2009), 971–980.
[38] M. Ulas. Rational points in arithmetic progressions on $y^2 = x^n + k$. Canad. Math. Bull., 55 (2012), 193–207.

Universidad Autónoma de Madrid, Departamento de Matemáticas and Instituto de Ciencias Matemáticas (ICMAT), Madrid, Spain
E-mail address: enrique.gonzalez.jimenez@uam.es
URL: http://www.uam.es/enrique.gonzalez.jimenez