ON QUASILINEAR PARABOLIC SYSTEMS AND FBSDES OF QUADRATIC GROWTH

JOE JACKSON

Abstract. Using probabilistic methods, we establish a-priori estimates for two classes of quasilinear parabolic systems of partial differential equations (PDEs). We treat in particular the case of a nonlinearity which has quadratic growth in the gradient of the unknown. As a result of our estimates, we obtain the existence of classical solutions of the PDE system. From this, we infer the existence of solutions to a corresponding class of forward-backward stochastic differential equations.

1. Introduction

We present a-priori estimates and well-posedness results for two classes of quasi-linear parabolic systems. The first reads

\begin{align*}
\begin{cases}
\partial_t u^i + \text{tr}(a(t, x, u)D^2 u^i) + f^i(t, x, u, Du) = 0, & (t, x) \in (0, T) \times \mathbb{R}^d, \\
u^i(T, x) = g^i(x), & x \in \mathbb{R}^d,
\end{cases}
\end{align*}

(1.1)

for \( i = 1, \ldots, n \). The data consists of functions \( a, f, \) and \( g \), and the unknown is a map \( u = u(t, x) = (u^i(t, x))_{i=1, \ldots, n} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^n \). Precise assumptions will be given below, but we are particularly interested in the case that \( a = \frac{1}{2} \sigma \sigma^T \) is non-degenerate and \( f = f(t, x, u, p) \) exhibits quadratic growth in the variable \( p \). While (1.1) is the main object of the paper, it turns out that roughly the same methods yield estimates and existence results also for the equation

\begin{align*}
\begin{cases}
\partial_t u^i + a(t, x, u, Du)D^2 u^i + f^i(t, x, u, Du) = 0, & (t, x) \in (0, T) \times \mathbb{R}, \\
u^i(T, x) = g^i(x), & x \in \mathbb{R},
\end{cases}
\end{align*}

(1.2)

for \( i = 1, \ldots, n \). The key difference between (1.1) and (1.2) is that the gradient of \( u \) appears as an argument in the function \( a \), which makes the analysis much more difficult. Accordingly, our techniques apply to (1.2) only in one spatial dimension and under the assumption that the driver \( f \) is globally Lipschitz in \( (x, u, p) \).

Systems of the type (1.1) are well-studied, and a classical reference is [LSU68]. For example, Theorem 7.1 of [LSU68] gives the existence of a classical solution to a system similar to (1.1), but on a bounded spatial domain and under the assumption that \( f \) has...
strictly subquadratic growth in $p$. More recently, motivated largely by applications to stochastic differential games, Bensoussan and Frehse undertook an intensive study of elliptic and parabolic systems similar to (1.1). In particular, they focused on systems with quadratic growth. We refer to the book [BF13] for a collection of results in the elliptic setting, as well as the papers [BF95], [BF00], and [BF02] for other relevant contributions. While these results are related to ours in that they treat systems of PDEs with a gradient non-linearity of quadratic growth, we point out that they are all obtained in the semi-linear case $a = a(t, x)$ and in bounded domains. For the system (1.2), it seems that much less is known, and in fact we are not aware of any general solvability result for (1.2) even in one spatial dimension.

One motivation for studying (1.1) comes from the theory of forward-backward stochastic differential equations (FBSDEs), which in turn have diverse applications in mathematical finance, stochastic control, stochastic differential game theory, and even stochastic differential geometry. There is a natural link between the PDE system (1.1) and systems of forward-backward stochastic differential equations (FBSDEs) of the form

\[
\begin{aligned}
    dX_t &= H(t, X_t, Y_t, Z_t)dt + \Sigma(t, X_t, Y_t)dB_t, \\
    dY_t &= -F(t, X_t, Y_t, Z_t)dt + Z_tdB_t, \\
    X_0 &= x_0, \quad Y_T = G(X_T).
\end{aligned}
\] (1.3)

Here $B$ is a Brownian motion, the data consists of appropriate functions $H, \Sigma, F, G$, and the solution is a triple of adapted processes $(X, Y, Z)$. Such FBSDEs have been studied extensively - we refer to [MY07] or [Zha17] for an introduction to the subject, and to [MPY94, Yon99, Yon06, MZZ08a, MZZ08b, MWZZ15] and the references therein for other significant contributions. The connection between (1.3) and (1.1) is that, roughly speaking, regular enough solutions to PDEs of the form (1.1) allow one to deduce existence results for (1.3) - this strategy has been used in many settings since the introduction of the “four-step scheme” by Ma, Protter and Yong in [MPY94]. When the data of (1.3) satisfies standard Lipschitz conditions, this strategy has been used to obtain global existence and uniqueness results for (1.3) in [Del02].

Let us recall in more detail how to solve the FBSDE (1.3) via the PDE (1.1). The idea is to suppose that we find a smooth solution to (1.1) with data $\sigma = \Sigma, g = G,$ and

\[
f^i(t, x, u, p) = F^i(t, x, u, \Sigma(t, x, u)p) + p^i \cdot H(x, u, \Sigma(t, x, u)p),
\] (1.4)

then Itô’s formula shows that we can produce a solution $(X, Y, Z)$ to (1.3) by first solving

\[
X_t = x_0 + \int_0^t H(t, X_t, u(t, X_t), \Sigma(t, X_t, u(t, X_t))Du(t, X_t))dt + \int_0^t \Sigma(t, X_t, u(t, X_t))dB_t,
\]

and then setting $Y_t = u(t, X_t), \quad Z_t = \Sigma(t, X_t, u(t, X_t))Du(t, X_t).

1.1. Related literature and motivation. In terms of the FBSDE (1.3), the present work sits at the intersection of three mathematical challenges:

(1) the quadratic growth of $F$
(2) the fact that $n > 1$, i.e. $Y$ is multidimensional (and hence approaches based on the comparison principle fail)
Each of these issues has received significant attention in the literature, and it would be impossible to give a thorough literature review for all three. Instead, we simply note that while one-dimensional quadratic BSDEs (i.e. decoupled FBSDEs) were given a thorough treatment in the seminal paper of Kobylanski [Kob00], global existence for quadratic BSDE systems has been considered a central open question for several decades, as noted by Peng in [Pen99]. A breakthrough for systems came in the recent paper of Xing and Žitković [XZ18] (see also [HR19], [HT16], and [JŽ21] for related contributions in the non-Markovian setting). When all three of the difficulties listed above are present, we are not aware of any existence results even when $T$ is small - the results of [FI13], [LT17] and [KLT18] do contain results for coupled quadratic FBSDEs with multi-dimensional $Y$, but the results require $\Sigma$ to be independent of $y$ (and even of $x$).

We now highlight three papers which are especially related to the present work, namely [Del03], [XZ18], and [HR19]. In [XZ18], Hölder estimates and existence results are obtained in the semilinear quadratic case, i.e. the case $\sigma = \sigma(t,x)$ does not depend on $u$ but $f$ has quadratic growth in $p$. In particular, it is shown that $L^\infty$ estimates on $u$ lead to Hölder estimates on $u$ as soon as the quadratic driver $f$ admits a “Lyapunov function” - see Theorem 2.5 there. Theorem 2.6 in [HR19] shows how to strengthen the estimates from [XZ18], in particular obtaining a gradient estimate (i.e. an estimate on $\|Du\|_{L^\infty}$) when the data is smooth enough (still in the semi-linear case). We note that in the semi-linear case, an a-priori estimate of $\|Du\|_{L^\infty}$ yields automatically an existence result for classical solutions to the PDE system, thanks to the fact that (1.1) is well-understood when $f$ is Lipschitz (see e.g. Lemma 2.2 of [HY00]), though this argument does not seem to have appeared in the literature until the recent note [Jac23] which studies the FBSDE (4.3) in the semi-linear setting. In the quasi-linear case $\sigma = \sigma(t,x,u)$, Hölder and gradient estimates have been obtained via probabilistic arguments in [Del03] for equations corresponding to FBSDEs with Lipschitz coefficients.

The motivation for understanding quadratic growth comes from the fact that it appears naturally in a variety of applications, for example stochastic differential games, the construction of martingales on Riemannian manifolds, and the existence of equilibria in incomplete financial markets. We refer the reader to Section 3 of [XZ18], where all three of these examples are discussed. In order to treat stochastic differential games (with uncontrolled volatility) in the more natural strong formulation, rather than the weak formulation typically studied through BSDEs, one must solve an FBSDE of the form (4.3) with $F$ having quadratic growth (albeit with $\Sigma$ independent of $Y$). See the recent note [Jac23], where this strategy is executed by relying on a-priori estimates from [XZ18]. The motivation for the present paper is to develop a new approach for quadratic FBSDEs which is flexible enough to allow $\Sigma$ to depend on $y$, or equivalently to allow the corresponding PDE to have a non-linearity in the Hessian term. Even in the case that $\Sigma$ does not depend on $y$, however, the approach we develop here still has merit, since it replaces the analytical arguments of [XZ18] and [Jac23] (which borrow heavily from the strategy of particular proof
strategy of [BF02]) with purely probabilistic (and arguably simpler) methods based on the Krylov-Safonov estimates and the theory of BMO martingales.

The motivation for studying the PDE (1.2), meanwhile, comes largely from the link between PDEs of the form (1.2) and FBSDEs of the form

\[
\begin{align*}
    dX_t &= H(t, X_t, Y_t, Z_t)dt + \Sigma(t, X_t, Y_t, Z_t)dB_t, \\
    dY^i_t &= -F(t, X_t, Y_t, Z_t) + Z_t dB_t, \\
    X_0 &= x_0, \quad Y_T = G(X_T),
\end{align*}
\]  

which appear in particular when the maximum principle is applied to stochastic control problems or stochastic differential games with controlled volatility. FBSDEs of the type (1.5) with \( \Sigma \) depending on \( z \) are notoriously challenging, and they have been successfully treated primarily under a variety of restrictive monotonicity conditions (see e.g. [HP95]). Our results on the PDE (1.2) suggest that it might be possible to obtain positive results for the FBSDE (4.3) using non-degeneracy of \( \Sigma \) instead of monotonicity, but there is an important hurdle still to clear in order to execute this strategy, see Remark 1.1 below.

1.2. Our results. In the case of the equation (1.1), our main results are a Hölder estimate (Theorem 2.6), a gradient estimate (Theorem 2.7) and existence results (Theorems 2.8 and 2.9) for (1.1) under appropriate technical and structural conditions on the data \( f, a = \frac{1}{2} \sigma \sigma^T \) and \( g \). We refer to subsection 2.1 for precise statements of all the hypotheses related to the equation (1.1). For the Hölder estimates, the main structural condition on \( f \) is Hypothesis \( H_{BF} \), which asserts the existence of constants \( C_f > 0, \epsilon \in (0,1) \) such that

\[
|f^i(t, x, u, p)| \leq C_f (1 + |p||p| + \sum_{j<i} |p_j|^2 + |p|^{2-\epsilon}), \quad i = 1, ..., n.
\]

This structural condition is adapted from the conditions appearing in [BF00] and [XZ18], and in that sense our Hölder estimate can be viewed as a generalization of the estimates in [BF00] and [XZ18] to the quasi-linear setting. To prove the Hölder estimate in the quasi-linear case \( \sigma = \sigma(t, x, u) \), it suffices to prove a Hölder estimate for the semi-linear case \( \sigma = \sigma(t, x) \), so long as the estimate depends only on the ellipticity constants of \( \sigma \) (and not the regularity of \( \sigma \)). This is the approach we take. We note that the Hölder estimate in [XZ18] uses the Lipschitz regularity of \( \sigma \) (in particular when Aronson’s estimate is invoked), and so cannot be applied in the quasi-linear setting. Meanwhile the Hölder estimate in [Del03] is independent of the regularity of \( \sigma \), as required, but the argument does not easily adapt to the quadratic case considered here. Our argument for Hölder regularity is similar in spirit to the one in [Del03], in the sense that we combine tools from the theory of BMO-martingales with the Krylov-Safonov estimates, but the execution is different. In particular, to overcome the quadratic growth we use the concept of sliceability together with the structural condition \( H_{BF} \) to execute an inductive argument - first showing \( u^1 \) is Hölder, then showing how this implies that \( u^2 \) is Hölder, and so on.

After obtaining a global Hölder estimate, we show that it can be used to obtain a gradient estimate when we assume some additional regularity on \( f \) (see Hypothesis \( H_Q \)) in
addition to the structural condition $H_{BF}$. The starting point here is to show that the Hölder estimate implies an estimate on the sliceability in bmo of the $Z$-component of the stochastic representation of $u$. This fact has been observed already in Proposition 5.2 in [XZ18], but is used in a novel way here. In particular, we study a BSDE representation of the gradient $Du$, and use results from [JZ21] (see also [HR19] and [DT10]) on linear BSDEs with bmo coefficients to get the desired gradient estimate. As a corollary of our a-priori estimates, we obtain existence results for (1.1) (see Theorems 2.8 and 2.9). Theorem 2.8 gives the existence of classical solutions under sufficient regularity of the data, while 2.9 gives the existence of a “decoupling solution” (defined below) when the data is less regular.

We summarize the results obtained for (1.1) in Table 1 below.

| Hypotheses | Implication | Precise Statement |
|------------|-------------|-------------------|
| $H_{\sigma}$ and $H_{AB}$ | bound on $\|u\|_{L^\infty}$ | Lemma 2.5 |
| $H_{\sigma}$ and $H_{BF}$ | bound on $\|u\|_{L^\infty} \implies \|u\|_{C^{0,\alpha}}$ | Theorem 2.6 |
| $H_{\sigma}$ and $H_{Q}$ | bound on $\|u\|_{C^{0,\alpha}} \implies \|Du\|_{L^\infty}$ | Theorem 2.7 |
| $H_{AB}$, $H_{Reg}$, $g \in C^{2,\alpha}$ | $\exists$ classical solution | Theorem 2.8 |
| $H_{AB}$, $H_{\sigma}$, $H_{Q}$, $g$ is Lipschitz | $\exists$ decoupling solution | Theorem 2.9 |

We note that our existence result for (1.1) allows us to deduce an existence result for the FBSDE (1.3), see Theorem 2.13. In particular, we obtain existence results for (1.3) with $F$ of quadratic growth and satisfying certain structural conditions. This seems to be the first global existence result for a system of the form (1.3) when $n > 1$ and $F$ has quadratic growth. Indeed, as explained above the results so far obtained for coupled FBSDEs of quadratic growth (even for small-time well-posedness results) typically require that $\sigma$ is independent of $y$, or even independent of $x$ (see e.g. [LT17] and [KLT18]). Thus our global existence result is new even in the small-time (meaning $T$ is sufficiently small) regime.

Our results for the (1.2) are similar, but apply only when $f$ is Lipschitz in $(x, u, p)$ and in one spatial dimension. Theorem 2.11 gives a-priori estimates in $C^{1,\alpha}$ and $C^{2,\alpha}$ under appropriate regularity conditions, and Theorem 2.12 gives an existence result for classical solution of (1.1).

Remark 1.1. One might guess that our results for (1.2) should lead to existence results for an FBSDE of the (1.5) where $H$, $\Sigma$ and $F$ are Lipschitz in all arguments and $X, B$ are one-dimensional. Unfortunately this is not the case, because while (1.5) will (under some additional technical conditions) be connected to a PDE of the form (1.2), it will typically not be true that the data $f$, $b$, and $\sigma$ are globally Lipschitz, even if $H$, $\Sigma$, and $F$ are. It would be desirable to extend the existence result for (1.2) to cover the FBSDE (1.5) in some generality, but we leave this interesting question to future work.

1.3. Organization of the paper. In the remainder of the introduction we fix notations and conventions. In section 3 we discuss some preliminaries, mostly related to bmo processes and sliceability. Section 4 states our main assumptions and results. In section 4, we
prove the main a-priori Hölder and gradient estimates for (1.1). Section 5 contains a-priori estimates for (1.2). Finally, Section 6 contains the proofs of the existence results for the PDEs (1.1) and (1.2) and the FBSDE (1.3).

1.4. Notation and conventions.

1.4.1. The probabilistic set-up. Throughout the paper we fix a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) which hosts a \( d \)-dimensional Brownian motion \( B \). We also fix a time horizon \( T \in (0, \infty) \), and \( n \in \mathbb{N} \) which will denote the dimension of the unknown process \( Y \). The augmented filtration of \( B \) is denoted by \( \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T} \).

1.4.2. Conventions regarding multidimensional functions and processes. Given \( u = (u^i)_{i=1,\ldots,n} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^n \), we view the spatial gradient \( Du \) as an element of \((\mathbb{R}^d)^n\), whose \( i \)th element \((Du)^i\) is the gradient \( Du_i \) of \( u^i \). Similarly, we will at times work with stochastic process \( Z \) taking values in \((\mathbb{R}^d)^n\), so the \( i \)th element \( Z^i \) takes values in \( \mathbb{R}^d \). When manipulating elements of \((\mathbb{R}^d)^n\), we interpret multiplication element-wise unless otherwise noted. For example, if \( p \in (\mathbb{R}^d)^n \) and \( Q \in \mathbb{R}^{d \times d} \), \( Qp \) would denote the element of \((\mathbb{R}^d)^n\) whose \( i \)th element is \( Qp^i \in \mathbb{R}^d \). Similarly, if \( p \in (\mathbb{R}^d)^n \) and \( q \in \mathbb{R}^d \), then \( pq \) would denote the element of \( \mathbb{R}^n \) whose \( i \)th element is \( p^i \cdot q \). This philosophy is used in particular when interpreting the symbol \( ZdB \), with \( Z \) a process taking values in \((\mathbb{R}^d)^n\). We note here also that we will use \(| \cdot |\) to denote the Euclidean norm in any finite-dimensional Euclidean space.

1.4.3. Universal constants. We view \( n, d, T \) as fixed universal constants. We will use symbols like \( C \) to denote a generic constant which can change from line to line. Such a constant may always depend implicitly on the universal constants \( n, d, \) and \( T \), but any other dependencies will be made explicit. For example, \( C = C(D) \) would indicate that \( C \) is a constant which depends on the \( D \) as well as possibly on the universal constants \( n, d, \) and \( T \).

1.4.4. Spaces of functions. We will work frequently in parabolic Hölder spaces, so we explain in detail our notations. Fix \( \alpha \in (0,1) \). For a function \( v = v(t,x) : [0,T] \times \mathbb{R}^d \to E \), \( E \) being some Euclidean space with norm \( | \cdot | \) we define the Hölder seminorm

\[
[v]_{C^{0,\alpha}} = [v]_{C^{0,\alpha}([0,T] \times \mathbb{R}^d)} = \sup_{t \neq t', x \neq x'} \frac{|v(t,x) - v(t',x')|}{|t-t'|^{\alpha/2} + |x-x'|^{\alpha}},
\]

and \( C^{0,\alpha} = C^{0,\alpha}([0,T] \times \mathbb{R}^d) \) denotes the functions whose Hölder norm

\[
\|v\|_{C^{0,\alpha}} = \|v\|_{L^\infty} + [v]_{C^{0,\alpha}}
\]

is finite. We define \( C^{1,\alpha} \) to be the set of \( u \in C^{0,\alpha} \) with spatial gradient \( Du \in C^{0,\alpha} \), and \( C^{2,\alpha} \) to be the set of \( u \in C^{0,\alpha} \) with time derivative \( \partial_t u \in C^{0,\alpha} \) and spatial gradient and Hessian \( Du, D^2 u \in C^{0,\alpha} \). We endow \( C^{1,\alpha} \) and \( C^{2,\alpha} \) with the usual norms

\[
\|u\|_{C^{1,\alpha}} = \|u\|_{C^{0,\alpha}} + \|Du\|_{C^{0,\alpha}},
\]

\[
\|u\|_{C^{2,\alpha}} = \|u\|_{L^\infty} + \|Du\|_{L^\infty} + \|\partial_t u\|_{C^{0,\alpha}} + \|D^2 u\|_{C^{0,\alpha}}.
\]
We will at times also use Hölder norms on \([0, t_0] \times \mathbb{R}^d, t_0 < T\), for which we will use obvious notations, e.g. \(u : [0, T] \times \mathbb{R}^d \to \mathbb{R}\),
\[
\|u\|_{C^{0,\alpha}([0, t_0] \times \mathbb{R}^d)} = \|u\|_{L^\infty([0, t_0] \times \mathbb{R}^d)} + \sup_{0 \leq t, t' \leq t_0, t \neq t', x \neq x'} \frac{|v(t, x) - v(t', x')|}{|t - t'|^{\alpha/2} + |x - x'|^\alpha}.
\]
We indicate local versions of these spaces in a natural way using a subscript. In particular, \(C^{2,\alpha}_{loc}([0, T] \times \mathbb{R}^d)\) will denote the space of functions \(u = u(t, x)\) such that for each bounded open set \(U \subset \mathbb{R}^d\), \(\|u\|_{C^{2,\alpha}_{loc}([0, T] \times U)} < \infty\). We will say that \(u^k \to u\) in \(C^{2,\alpha}_{loc}([0, T] \times \mathbb{R}^d)\) if for each bounded open set \(U \subset \mathbb{R}^d\), \(\|u - u^k\|_{C^{2,\alpha}_{loc}([0, T] \times U)} \to 0\).

We define the Hölder spaces of functions defined on \(\mathbb{R}^d\) in the same way, i.e. for \(g : \mathbb{R}^d \to \mathbb{R}\),
\[
\|g\|_{C^{0,\alpha}} = \sup_{x \neq x'} \frac{|g(x) - g(x')|}{|x - x'|^\alpha},
\]
and similarly for \(\|g\|_{C^{k,\alpha}}, k = 1, 2\).

Given an open subset \(U\) of \([0, T] \times \mathbb{R}^d\), we say that \(v \in C^{1,2}(U)\) if \(\partial_t v, Dv, D^2v\) exist and are continuous on \(U\).

1.4.5. Notions of solutions. First, recall that any classical solution to (1.1) is expected to be a “decoupling field” for the FBSDE
\[
\begin{align*}
\left\{ 
\begin{array}{ll}
dX_t = \sigma(t, X_t, Y_t)dB_t, \\
\quad dY_t = -f(t, X_t, Y_t, \sigma^{-1}(t, X_t, Y_t)Z_t)dt + Z_t dB_t,
\end{array}
\right.
\end{align*}
\]
This allows us to define a probabilistic notion of solution to the (1.1) as follows. A bounded and continuous function \(u : [0, T] \times \mathbb{R}^d \to \mathbb{R}^n\) is said to be a decoupling solution of (1.1) if \(Du\) is bounded and continuous on \([0, T] \times \mathbb{R}^d\), for each \(t \in [0, T]\) and \(x \in \mathbb{R}^d\) there is a unique solution \(X^{t,x}\) of the SDE
\[
X^{t,x}_t = x + \int_t^t \sigma(s, X^{t,x}_s, u(s, X^{t,x}_s))dB_s
\]
and with \((Y^{t,x}, Z^{t,x}) := (u(\cdot, X^{t,x}), Du(\cdot, X^{t,x}))\) we have
\[
\begin{align*}
\left\{ 
\begin{array}{ll}
X^{t,x}_t = x + \int_t^t \sigma(s, X^{t,x}_s, Y^{t,x}_s)dB_s, \\
Y^{t,x}_t = g(X^{t,x}_T) + \int_t^T f(s, X^{t,x}_s, Y^{t,x}_s, \sigma^{-1}(s, X^{t,x}_s, Y^{t,x}_s)Z^{t,x}_s)ds - \int_t^T Z^{t,x}_s dB_s
\end{array}
\right.
\end{align*}
\]
on the interval \([t, T]\).

We shall also frequently refer to classical solutions of the PDE (1.1) (or (1.2)). By this, we mean a function \(u = (u^i)_{i=1,...,n} \in C^{1,2}([0, T] \times \mathbb{R}^d; \mathbb{R}^n) \cap C([0, T] \times \mathbb{R}^d; \mathbb{R}^n)\) such that
1. \(u\) and the spatial gradient \(Du\) are bounded on \([0, T] \times \mathbb{R}^d\)
2. the equation (1.1) (or (1.2)) holds pointwise in \([0, T] \times \mathbb{R}^d\)
3. \(u(T, x) = g(x)\), for \(x \in \mathbb{R}^d\).
With this definition in place, it is standard to check via Itô’s formula that if $u$ is a classical solution, then $u$ is a decoupling solution, at least provided some minimal regularity on $\sigma$ (see e.g. $(H_\sigma)$ below).

1.4.6. Spaces of processes. For $1 \leq p \leq \infty$, $L^p$ denotes the space of $p$-integrable $\mathcal{F}_T$-measurable random variables (taking values in some Euclidean space). We indicate measurability with respect to a sub-$\sigma$-algebra when necessary, i.e. for a sub-$\sigma$-algebra $\mathcal{G}$ of $\mathcal{F}_T$, $L^p(\mathcal{G})$ denotes the set of all $\mathcal{G}$-measurable elements of $L^p$. For $1 \leq p \leq \infty$, $S^p$ denotes the space of all continuous processes $Y$ such that $$\|Y\|_{S^p} := \|Y^*\|_{L^p} < \infty \text{ where } Y^* = \sup_{0 \leq t \leq T} |Y_t|.$$ BMO denotes the space of continuous martingales $M$ such that $$\|M\|_{\text{BMO}} := \text{esssup}_\tau \|E_\tau[|M_T - M_\tau|^2]\|_{L^\infty}^{1/2} < \infty,$$ where the supremum is taken over all stopping times $0 \leq \tau \leq T$, while bmo denotes the space of progressive processes $\gamma$ such that $$\|\gamma\|_{\text{bmo}}^2 := \sup_\tau E_\tau \left[ \int_\tau^T |\gamma|^2 \, ds \right] < \infty.$$ Similarly, $\text{bmo}^{1/2}$ denotes the space of progressive processes $\beta$ such that $$\|\gamma\|_{\text{bmo}^{1/2}} := \sup_\tau E_\tau \left[ \int_\tau^T |\gamma| \, ds \right] < \infty,$$ i.e., $\|\gamma\|_{\text{bmo}^{1/2}} = \|\sqrt{\gamma}\|_{\text{bmo}}^2$.

If necessary, we emphasize the co-domain of the space of processes under consideration, e.g. by writing $\text{bmo}(\mathbb{R}^d)$ for the space of bmo processes taking values in $\mathbb{R}^d$. We also note that at times we will work with processes defined only on a subinterval $[t_0, T]$ of $[0, T]$. We can extend all the definitions above to such processes in a natural way. In particular, we highlight that if $Y$ is a continuous process defined on $[t_0, T]$, then $$\|Y\|_{S^p} := \left\| \sup_{t_0 \leq t \leq T} |Y_t| \right\|_{L^p}.$$ If $\gamma$ is defined on $[t_0, T]$, we denote by $\|\gamma\|_{\text{bmo}}$ the quantity $\|\tilde{\gamma}\|_{\text{bmo}}$, where $\tilde{\gamma}$ denotes the extension of $\gamma$ to $[0, T]$ by $0$: $$\tilde{\gamma}_t = \begin{cases} 0 & 0 \leq t < t_0, \\ \gamma_t & t_0 \leq t \leq T. \end{cases}$$ Finally, in an abuse of notation $L^\infty$ denote also the set of progressively measurable processes $Z$ with $\|Z\|_{L^\infty} = \text{esssup}_{t,\omega} |Z_t(\omega)| < \infty.$
2. Assumptions and main results

2.1. Assumptions related to (1.1). The data for (1.1) consists of the three functions \( \sigma, f, \) and \( g \), where

\[
\sigma = \sigma(t, x, u) = (\sigma_{jk}(t, x, u))_{j,k=1,\ldots,d} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}^{d \times d},
\]

\[
f = f(t, x, u, p) = (f_i(t, x, u, p))_{i=1,\ldots,n} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times (\mathbb{R}^d)^n \to \mathbb{R}^n, \quad \text{and}
\]

\[
g = g(x) = (g^i(x))_{i=1,\ldots,n} : \mathbb{R}^d \to \mathbb{R}^n.
\]

We now state the assumptions which will be made at various points on the data. The first assumption concerns the regularity and non-degeneracy of the matrix \( \sigma \).

\[
\left\{
\begin{aligned}
&1) |\sigma(t, x, u) - \sigma(t, x', u')| \leq L_\sigma(|x - x'| + |u - u'|), \\
&2) \frac{1}{2} \sigma_i |z|^2 \leq |\sigma(t, x, u)z|^2 \leq C_\sigma |z|^2, \\
&\text{for all } t \in [0, T], \ x, x' \in \mathbb{R}^d, \ u, u' \in \mathbb{R}^n, \ z \in \mathbb{R}^d.
\end{aligned}
\right. \quad (H_\sigma)
\]

For a general quadratic \( f \), a-priori estimates on \( \|u\|_{L_\infty} \) may not be possible, but there are many structural conditions on \( f \) for which such estimates are known to hold. We give two such conditions here. The first is adapted from [XZ18], and the other one, which is simple to prove, allows us to cover the case studied in [Del02]. We emphasize that the Hölder and Lipschitz estimates proved below do not require the conditions \( H_{AB1} \) or \( H_{AB2} \), given below, which are used only to obtain estimates on \( \|u\|_{L_\infty} \).

The driver \( f \) can be written as

\[
f^i(t, x, u, p) = p^i \cdot b_0(t, x, u, p) + b^i(t, x, u, p),
\]

where \( b_0 \) and \( (b^i)_{i=1,\ldots,n} \) satisfy

\[
\left\{
\begin{aligned}
&1) |b_0(t, x, u, p)| \leq M (1 + \kappa(|u| + |p|)), \\
&2) a_q^T b(t, x, u, p) \leq M + \frac{1}{2} |a_q^T p|^2,
\end{aligned}
\right. \quad (H_{AB1})
\]

for all \((t, x, u, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times (\mathbb{R}^d)^n \) and \( q = 1, \ldots, Q \), and for some constant \( M > 0 \), increasing function \( \kappa : \mathbb{R}^+ \to \mathbb{R}^+ \), and set \( \{a_1, \ldots, a_Q\} \) of vectors positively spanning \( \mathbb{R}^n \).

The driver \( f \) can be written

\[
f^i(t, x, u, p) = p^i \cdot b_0(t, x, u, p) + b^i(t, x, u, p),
\]

where \( b_0 \) and \( (b^i)_{i=1,\ldots,n} \) satisfy

\[
\left\{
\begin{aligned}
&1) |b_0(t, x, u, p)| \leq M (1 + \kappa(|u| + |p|)), \\
&2) |b^i(t, x, u, p)| \leq M (1 + |u| + |p|),
\end{aligned}
\right. \quad (H_{AB2})
\]

for some \( M > 0 \), some increasing function \( \kappa : \mathbb{R}^+ \to \mathbb{R}^+ \) and for all \((t, x, u, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times (\mathbb{R}^d)^n \).

For simplicity, we put these two a-priori boundedness conditions together as follows:

Either \( H_{AB1} \) or \( H_{AB2} \) hold. \quad (H_{AB})
Remark 2.1. We discuss briefly how the conditions \((H_{AB1})\) and \((H_{AB2})\) lead to \(L^\infty\) estimates on \(u\). Firstly, the term \(p^i \cdot b_0(t, x, u, p)\) can typically be safely ignored when searching for \(L^\infty\) bounds. The analytical explanation for this is that it can be viewed as part of the linear operator being applied to each \(u^i\) in the equation (1.1), for example under \((H_{AB1})\) we can rewrite the PDE (1.1) as

\[
\partial_t u^i + \mathcal{L}(u, Du)(u^i) + b^i(t, x, u, Du) = 0,
\]

with \(\mathcal{L}(u, Du)\) denoting the differential operator

\[
\mathcal{L}(u, Du)(v) = \text{tr}(a(t, x, u)D^2 v) + b_0(t, x, u, Du) \cdot Dv.
\]

The corresponding probabilistic explanation is that in the corresponding FBSDE, the term coming from \(p^i \cdot b_0(t, x, u, p)\) can be essentially removed through the Girsanov transformation (see the proof of Lemma 2.5).

Meanwhile, the conditions placed on \(b^i\) in \((H_{AB1})\) are borrowed largely from the “a-priori boundedness condition” in [XZ18], which was in turn inspired by similar conditions in the literature on parabolic systems, see e.g. [BF02]. Roughly speaking, it allows to obtain \(L^\infty\)-estimates for the system by showing that one-dimensional projections of the solution \(u\) (along the directions \(a_q \in \mathbb{R}^n\)) are (approximately) sub-solutions of (scalar) PDEs, and then employing the comparison principle to get \(L^\infty\) estimates on \(u\) in each of the directions \(a_q\).

Finally, the condition on \(b^i\) appearing in \((H_{AB2})\) is fairly easy to explain - it is a linear growth assumption which ensures that (after a Girsanov transformation handles the term coming from \(p^i \cdot b_0(t, x, u, p)\)) the BSDE representing \(u\) can be estimated by standard methods. We again refer to the proof of Lemma 2.5 for more details.

The H"older estimates on \(u\) will be obtained under the following structural condition on the quadratic driver \(f\). We follow [XZ18] in calling it a “Bensoussan-Frehse” condition, because of the resemblance to the structural condition used in [BF00].

\[
\begin{cases}
\text{There are constants } C_f > 0, 0 < \epsilon < 1 \text{ such that } f \text{ satisfies } \\
|f^i(t, x, u, p)| \leq C_f (1 + |p^i| |p| + \sum_{j < i} |p^j|^2 + |p|^{2-\epsilon}), \\
\text{for all } (t, x, u, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times (\mathbb{R}^d)^n, \ i = 1, \ldots, n. 
\end{cases} 
\]

\[(H_{BF})\]

Remark 2.2. We note that following a computation in [BF00], one can show that if \(H_{BF}\) holds, then there are measurable functions \(h^i = h^i(t, x, u, p) : [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times (\mathbb{R}^d)^n \to \mathbb{R}^d, \ k^i = k^i(t, x, u, p) : [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times (\mathbb{R}^d)^n \to \mathbb{R}\) such that

\[
f^i(t, x, u, p) = p^i \cdot h^i(t, x, u, p) + k^i(t, x, u, p) \quad (2.1)
\]

and the estimates

\[
|h^i(t, x, u, p)| \leq C_Q (1 + |p|), \quad |k^i(t, x, u, p)| \leq C_Q (1 + \sum_{j < i} |p^j|^2 + |p|^{2-\epsilon}) \quad (2.2)
\]
hold for all \((t, x, u, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times (\mathbb{R}^d)^n\). Indeed, taking
\[
h^i(t, x, u, p) = \frac{f^i(t, x, u, p)}{(1 + |p^i||p| + \sum_{j<i}|p^j|^2 + |p|^{2-\epsilon})}\]
\[
k^i(t, x, u, p) = \frac{f^i(t, x, u, p)}{(1 + |p^i||p| + \sum_{j<i}|p^j|^2 + |p|^{2-\epsilon})}(1 + \sum_{j<i}|p^j|^2 + |p|^{2-\epsilon}),
\]
it is easy to check that the estimates in \((2.2)\) hold.

To bootstrap from Hölder to gradient estimates, we will need some regularity of the coefficients in addition to the growth condition. The following condition states that \(f\) is locally Lipschitz in \((x, u, p)\), with a Lipschitz constant depending on \(|p|\) in a natural way.

\[
\begin{align*}
\text{In addition to the condition } H_{BF}, \ f \text{ satisfies the estimates } \\
1) \ |f(t, x, u, p) - f(t, x', u', p)| \leq C_f(1 + |p|^2)(|x - x'| + |u - u'|), \\
2) \ |f(t, x, u, p) - f(t, x, u, p')| \leq C_f(1 + |p| + |p'|)|p - p'|.
\end{align*}
\]

Remark 2.3. Suppose that \(f = f(t, x, u, p)\) is continuously differentiable in \((x, u, p)\) for each fixed \(t\). Then \(H_Q\) is equivalent to the estimates
\[
|D_xf(t, x, u, p)| + |D_uf(t, x, u, p)| \leq C_f(1 + |p|^2), \quad |D_pf(t, x, u, p)| \leq C_f(1 + |p|).
\]

Finally, to get a classical solution to \((1.1)\), we will need some Hölder regularity of \(\sigma\) and \(f\) in time:

\[
\begin{align*}
\text{In addition to } H_\sigma \text{ and } H_Q, \text{ we have the estimates } \\
1) \ |\sigma(t, x, u) - \sigma(t', x, u)| \leq L_\sigma|t - t'|^{\alpha_0}, \\
2) \ |f(t, x, u, p) - f(t', x, u, p)| \leq C_f|t - t'|^{\alpha_0},
\end{align*}
\]
for all \(t, t' \in [0, T]\), \((x, u, p) \in \mathbb{R}^d \times \mathbb{R}^n \times (\mathbb{R}^d)^n\), and some \(\alpha_0 \in (0, 1)\).

Remark 2.4. To be clear, we have stated the regularity and structure conditions above in such a way that the implications
\[
H_{Reg} \implies H_\sigma \quad \text{and} \quad H_Q, \quad H_Q \implies H_{BF},
\]
hold.

2.2 Statement of the results for \((1.1)\). We now state our results for the equation \((1.1)\). We begin with an a-priori estimate for \(|u|_{L^\infty}\).

Lemma 2.5. Suppose that \(H_\sigma\) and \(H_{AB}\) hold. Suppose further that \(g\) is bounded. Then for any decoupling solution \(u\) of \((1.1)\), we have
\[
|u|_{L^\infty} \leq C,
\]
for a constant \(C\) depending only on \(|g|_{L^\infty}\), \(C_\sigma\), and either \(\{a_m\}\) and \(\rho\) (if we assume \(H_{AB1}\)) or \(M\) (if we assume \(H_{AB2}\)).

The next result gives an a-priori Hölder estimate for \(u\).
Theorem 2.6. Suppose that $H_{\sigma}$ and $H_{BF}$ hold. Suppose further that $g \in C^{0,\beta}$ for some $\beta \in (0,1)$, and that $u$ is a decoupling solution of (1.1). Then for some $\alpha \in (0,1)$ and $C > 0$ depending on $\beta, \|g\|_{C^{0,\beta}}, C_{\sigma}, C_{Q}, \epsilon, \|u\|_{L^\infty}$, we have

$$\|u\|_{C^{0,\alpha}} \leq C.$$ 

Our next result is a gradient estimate for (1.1).

Theorem 2.7. Assume that $H_{\sigma}$ and $H_{Q}$ hold. Suppose further that $\sigma$ is continuously differentiable in $(x,u)$ and $f$ is continuously differentiable in $(x,u,p)$ for each fixed $t$ and that $g \in C^{1}(\mathbb{R}^d)$ with bounded derivative. Let $u$ be a classical solution of (1.1) with $Du \in C^{1,2}([0,T] \times \mathbb{R}^d)$. Then for any $\alpha \in (0,1)$, we have

$$\|Du\|_{L^\infty} \leq C, \ C = C(\|Dg\|_{L^\infty}, C_{\sigma}, L_{\sigma}, C_{Q}, \alpha, \|u\|_{C^{0,\alpha}}).$$

Finally, we obtain the following existence results as consequences of our a-priori estimates.

Theorem 2.8. Suppose that $H_{AB}$ holds. Suppose also that $H_{Reg}$ holds. Finally, suppose that $g$ is $C^{2,\beta}(\mathbb{R}^d)$ for some $\beta \in (0,1)$. Then, there is a unique classical solution $u$ to (1.1), which satisfies $u \in C^{2,\alpha}$ for some $\alpha \in (0,1)$.

If the terminal condition is only Lipschitz, we can still get decoupling solution to (1.1), and we can also drop the assumption $H_{Reg}$.

Theorem 2.9. Suppose that $H_{AB}$ holds. Suppose also that $H_{\sigma}$ and $H_{Q}$, and that $\sigma, f$ and $g$ are continuous in all arguments. Finally, suppose that $g$ is Lipschitz. Then, there is a unique decoupling solution $u$ to (1.1).

Remark 2.10. The uniqueness statement in Theorem 2.8 is implied by the uniqueness statement in Theorem 2.9, since every classical solution to (1.1) is also a decoupling solution. Moreover, we have defined decoupling solutions to be uniformly Lipschitz in space, so that if $u$ and $\tilde{u}$ were two decoupling solutions, then they would both be decoupling solutions to the PDE (1.1), but with the driver $f$ replaced by the driver $\tilde{f}(t,x,u,p) = f(t,x,u,\pi(p))$ for some smooth cut-off function $\pi$, i.e. $\pi$ is Lipschitz, bounded and $\pi(p) = p$ for $|p| \leq K$ with $K$ sufficiently large. Since $\tilde{f}$ is uniformly Lipschitz in $(x,u,p)$, the uniqueness statement in Theorem 2.8 (hence also in Theorem 2.9) follows easily from the existing literature on Lipschitz FBSDEs (see e.g. [Del02]).

2.3. Assumptions related to (1.2). Now we state the assumptions which we will use when studying (1.2). Recall that in this case the data is

$$\sigma = \sigma(t,x,u,p) : [0,T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},$$

$$f = f(t,x,u,p) = (f^i(t,x,u,p))_{i=1,...,n} : [0,T] \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \text{ and}$$

$$g = g(x) = (g^i(x))_{i=1,...,n} : \mathbb{R} \to \mathbb{R}^n. \quad (2.3)$$
We start with a non-degeneracy and regularity condition for $\sigma$.

\[
\begin{align*}
&\text{There are constants } L_{\sigma}, C_{\sigma} > 0 \text{ such that } \sigma \text{ satisfies the estimates} \\
&\quad 1) \ |\sigma(t, x, u, p) - \sigma(t, x', u', p')| \leq L_{\sigma} (|x - x'| + |u - u'| + |p - p'|), \\
&\quad 2) \ \frac{1}{C_{\sigma}} \leq |\sigma(t, x, u, p)|^2 \leq C_{\sigma}, \\
&\text{hold for all } t \in [0, T], \ x, x' \in \mathbb{R}, \ u, u' \in \mathbb{R}^n, \ p, p' \in \mathbb{R}^d.
\end{align*}
\]

(\(H^1_\sigma\))

Next, we state the appropriate regularity conditions for $f$.

\[
\begin{align*}
&\text{There are constants } C_f \text{ such that the estimates} \\
&\quad 1) \ |f(t, x, u, p) - f(t, x', u', p')| \leq C_f (|x - x'| + |u - u'| + |p - p'|), \\
&\quad 2) \ |f(t, x, u, p)| \leq C_f (1 + |u| + |p|), \\
&\text{hold for all } t \in [0, T], \ x, x' \in \mathbb{R}, \ u, u' \in \mathbb{R}^n, \ p, p' \in (\mathbb{R}^d)^n.
\end{align*}
\]

(\(H^1_{\text{Lip}}\))

In addition to \(H^1_{\text{Lip}}\) and \(H^1_\sigma\), there is a constant $\alpha_0 \in (0, 1)$ such that

\[
\begin{align*}
&\quad 1) \ |\sigma(t, x, u, p) - \sigma(t', x, u, p)| \leq C_0 |t - t'|^{\alpha_0}, \\
&\quad 2) \ |f(t, x, u, p) - f(t', x, u, p)| \leq C_0 |t - t'|^{\alpha_0}, \\
&\text{hold for all } t, t' \in [0, T], \ (x, u, p) \in \mathbb{R}^d \times \mathbb{R}^n \times (\mathbb{R}^d)^n.
\end{align*}
\]

(\(H^1_{\text{Reg}}\))

2.4. Statement of the results for (1.2). We start with an a-priori estimate for (1.2)

**Theorem 2.11.** Suppose that \(H^1_\sigma\) and \(H^1_{\text{Lip}}\) hold. Suppose further that $g \in C^{1,\beta}$ for some $\beta \in (0, 1)$. Finally, suppose that $u$ is a classical solution to (1.2) with $Du \in C^{1,2}$ and $D^2u$ bounded. Then, for some $\alpha = \alpha(\beta, \|g\|_{C^{1,\beta}}, L_{\sigma}, C_{\sigma}, C_f)$, we have

\[
\|Du\|_{C^{0,\alpha}} \leq C, \quad C = C(\beta, \|g\|_{C^{1,\beta}}, L_{\sigma}, C_{\sigma}, C_f).
\]

If in addition \(H^1_{\text{Reg}}\) holds and $g \in C^{2,\beta}$, then for some (potentially different) $\alpha = \alpha(\beta, \|g\|_{C^{1,\beta}}, L_{\sigma}, C_{\sigma}, C_f)$, we have

\[
\|u\|_{C^{2,\alpha}} \leq C, \quad C = C(\beta, \|g\|_{C^{2,\beta}}, L_{\sigma}, C_{\sigma}, C_f, C_0, \alpha_0).
\]

This a-priori estimate can be combined with the method of continuity to give the following existence result.

**Theorem 2.12.** Suppose that \(H^1_\sigma\) and \(H^1_{\text{Reg}}\) hold. Suppose further that for some $\beta \in (0, 1)$, $g \in C^{2,\beta}$. Then, there exists a classical solution $u$ to (1.2).

2.5. Application to the FBSDE (1.3). We now describe the hypothesis on the data

\[
\begin{align*}
H &= H(t, x, y, z) : [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times (\mathbb{R}^d)^n \to \mathbb{R}^d, \\
\Sigma &= \Sigma(t, x, y) : [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times (\mathbb{R}^d)^n \to \mathbb{R}^{d \times d}, \\
F &= F(t, x, y, z) : [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times (\mathbb{R}^d)^n \to \mathbb{R}^n, \\
G &= G(x) : \mathbb{R}^d \to \mathbb{R}^n.
\end{align*}
\]

(2.4)
under which we will obtain existence for (1.3). For Σ and F, we will essentially borrow the conditions we have already defined for σ and f above.

\[ \Sigma \text{ is continuous and satisfies } H_\sigma \quad (H_\Sigma) \]

\[ F \text{ is continuous and satisfies } H_{AB} \text{ and } H_Q. \quad (H_F) \]

\[
\begin{cases}
H \text{ is continuous and there is a constant } C_H > 0 \\
\text{ and an increasing function } \kappa : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that the estimates} \\
1) |H(t, x, y, z) - H(t, x', y', z)| \leq C_H (1 + |z|) (|x - x'| + |y - y'|), \\
2) |H(t, x, y, z) - H(t, x, y, z')| \leq C_H |z - z'| \\
3) |H(t, x, y, z)| \leq C_H (1 + |\kappa(|y|)| + |z|)
\end{cases} 
\quad (H_H)
\]

hold for all \( t, \in [0, T], \ x, x' \in \mathbb{R}^d, \ y, y' \in \mathbb{R}^n, \ z, z' \in (\mathbb{R}^d)^n. \)

Here is the existence result for (1.3).

**Theorem 2.13.** Suppose that \( H_\Sigma, H_F \) and \( H_H \) hold. Suppose further that \( G \) is Lipschitz. Then there is a unique solution \( (X, Y, Z) \in \mathcal{S}^2 \times \mathcal{S}^\infty \times L^\infty \) to (1.3).

**Remark 2.14.** To be clear, Theorem 2.13 asserts uniqueness in the class \( \mathcal{S}^2 \times \mathcal{S}^\infty \times L^\infty \), which follows easily from results on Lipschitz FBSDEs. It seems natural to expect uniqueness also in the slightly larger class \( \mathcal{S}^2 \times \mathcal{S}^\infty \times \text{bmo} \). The standard way to obtain this latter, more general, uniqueness statement would be to first prove existence and uniqueness in \( \mathcal{S}^2 \times \mathcal{S}^\infty \times \text{bmo} \) when \( T \) is sufficiently small, and then bootstrap this local result with the help of the decoupling solution \( u \). Indeed, the arguments introduced in [MPY94] show that as a general rule,

\[
(\exists \text{ smooth solution of PDE } ) + (\text{local well-posedness of FBSDE} ) \implies (\text{global uniqueness of FBSDE}).
\]

But unlike in the Lipschitz case, employing the Banach fixed point theorem to get existence and uniqueness in a space like \( \mathcal{S}^2 \times \mathcal{S}^\infty \times \text{bmo} \) for \( T \) small in the present quadratic case seems relatively challenging - there are some small-time results for quadratic FBSDEs appearing in [LT17] and [KLT18], but none general enough to apply in our setting.

3. Preliminaries

This section is auxiliary in nature, and contains statements and proofs of a number of results which will be necessary for the proof of the main a-priori estimates in the next section.
3.1. The space \( \text{bmo} \). We now recall some basic facts about the space \( \text{bmo} \). The important point is that for algebraically compatible \( a \), \[ \|a\|_{\text{bmo}} = \left\| \int adB \right\|_{\text{BMO}}. \]

The following Lemma can be deduced from Theorem 3.6 of [Kaz94], which explains that a “bmo change of measure” from \( P \) to \( Q \) induces a linear isomorphism from \( \text{BMO}(P) \) to \( \text{BMO}(Q) \).

**Lemma 3.1.** Suppose that \( \|a\|_{\text{bmo}} < \infty \), and define a measure \( Q \) by \[ dQ = E(\int adB)_T dP. \]

Then \( \text{bmo}(P) = \text{bmo}(Q) \), and

\[ \frac{1}{C} \|b\|_{\text{bmo}(Q)} \leq \|b\|_{\text{bmo}(P)} \leq C \|b\|_{\text{bmo}(Q)}, \]

for each \( b \in \text{bmo}(P) \), and some \( C \) depend only on \( \|a\|_{\text{bmo}} \). As a consequence,

\[ \frac{1}{C} \|b\|_{\text{bmo}^{1/2}(Q)} \leq \|b\|_{\text{bmo}^{1/2}(P)} \leq C \|b\|_{\text{bmo}^{1/2}(Q)}, \]

for each \( b \in \text{bmo}^{1/2}(P) \).

This leads to the following Lemma, which will be key in the proof of the Hölder estimate for (1.1).

**Lemma 3.2.** Let \( a \in \text{bmo} \) and \( Q \) be defined by \[ dQ = E(\int adB)_T dP. \]

Then for \( A \in \mathcal{F} \), we have

\[ Q[A] \geq C P[A]^q, \]

for some \( C, q > 0 \) depending only on \( \|a\|_{\text{bmo}} \).

**Proof.** It follows from a computation that

\[ \frac{dP}{dQ} = E(\int -a \cdot dB^a)_T, \quad B^a = B - \int adt. \]

By using Lemma 3.1 together with Theorem 3.1 in [Kaz94], we can find \( p > 1, C > 0 \) depending only on \( \|a\|_{\text{bmo}} \) such that

\[ \left\| E(\int -a \cdot dB^a)_T \right\|_{L^p(Q)} \leq C. \]

Thus by the Hölder inequality

\[ P[A] \leq \int 1_A E(\int -a \cdot dB^a)_T dQ \leq \left\| E(\int a \cdot dB)_T \right\|_{L^p(Q)} \|1_A\|_{L^q(Q)} \leq C Q[A]^{1/q}, \]

where \( q \) is the conjugate exponent of \( p \). This completes the proof.

The next lemma states simply that if \( |a|^{1+\epsilon} \in \text{bmo} \), then \( a \) is sliceable (see subsection 3.2 below for the definition).

---

1Actually, Theorem 3.6 of [Kaz94] implies only the existence of the constant \( C \) appearing in Lemma 3.1, for each \( a \in \text{bmo} \). The fact that \( C \) can be chosen to depend only on \( \|a\|_{\text{bmo}} \) is clear from Kazamaki’s proof.
Lemma 3.3. Suppose that for some $\epsilon > 0$, $\|a^{1+\epsilon}\|_{bmo} < \infty$. Then, for any constants $t, \delta$ such that

$$0 \leq t - \delta \leq t \leq T,$$

we have

$$\|a1_{[t-\delta,t]}\|_{bmo} \leq C\delta^\alpha,$$

where $\alpha = \frac{\epsilon}{1+\epsilon}$, $C = C(\|a^{1+\epsilon}\|_{bmo})$.

Proof. Let $\tau$ be a stopping time. For simplicity, set $\sigma = (\tau \vee (t - \delta)) \wedge t$. Notice that

$$\mathbb{E}_\tau[\int_\tau^T 1_{[t-\delta,t]}|a|^2 ds] = \mathbb{E}_\tau[\int_\sigma^t |a|^2 ds] = \mathbb{E}_\tau[\mathbb{E}_\sigma[\int_\sigma^t |a|^2 ds]],$$

so

$$\left\| \mathbb{E}_\tau[\int_\tau^T 1_{[t-\delta,t]}|a|^2 dt] \right\|_{L^\infty} \leq \left\| \mathbb{E}_\sigma[\int_\sigma^t |a|^2 ds] \right\|_{L^\infty}.$$ 

Since

$$\mathbb{E}_\sigma[\int_\sigma^t |a|^2 ds] \leq (\mathbb{E}_\sigma[\int_\sigma^t |a|^{2+2\epsilon} ds])^{\frac{1}{1+\epsilon}} (\mathbb{E}_\sigma[\int_\sigma^t 1 ds])^{\frac{1}{1+\epsilon}} \leq C\delta^{\frac{1}{1+\epsilon}},$$

we can conclude. \(\square\)

3.2. Sliceability and linear BSDEs with bmo coefficients. We now give some additional preliminaries concerning the concept of sliceability, and linear BSDEs with bmo coefficients. These ideas are taken largely from [\\(JZ21\\)]. First, we define a random partition of $[0, T]$ as a collection $(\tau_k)_{k=0}^m$ of stopping times such that $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_m = T$. The set of all random partitions is denoted by $\mathcal{P}$. For $A \in bmo$, the index of sliceability for $A$ is the function $N_A : (0, \infty) \rightarrow \mathbb{N} \cup \{\infty\}$ defined as follows. For $\delta > 0$, $N_A(\delta)$ is the smallest natural number $m$ such that there exists a random partition $(\tau_k)_{k=0}^m \in \mathcal{P}$ such that

$$\|A1_{[\tau_k-1, \tau_k]}\|_{bmo} \leq \delta$$

for all $1 \leq k \leq m$. \(\text{(3.1)}\)

If no such $m$ exists, we set $N_A(\delta) = \infty$. A bmo-process $A$ is said to be $\delta$-sliceable if $N_A(\delta) < \infty$ and sliceable if it is $\delta$-sliceable for each $\delta > 0$. A family $\mathcal{A} \subseteq bmo$ is said to be uniformly sliceable if

$$\sup_{A \in \mathcal{A}} N_A(\delta) < \infty \text{ for all } \delta > 0.$$ 

Sliceability and the related notions given above are defined for the space $bmo^{1/2}$ in the same way.

Consider now the linear BSDE

$$Y_t = \xi + \int_t^T (\alpha_s Y_s + A_s Z_s + \beta_s) ds - \int_t^T Z_s dB_s, \hspace{1cm} \text{(3.2)}$$
or, unwrapping the conventions on multi-dimensional processes introduced above,

\[
Y^i_t = \xi^i + \int_t^T (\alpha^i \cdot Y_s + \sum_{j=1}^n A^i_{js} \cdot Z^j_s + \beta^i_s) ds - \int_t^T Z^i_s dB_s. \tag{3.3}
\]

The data for this problem is

\[
\alpha = (\alpha^i)_{i=1, \ldots, n} \in \text{bmo}^{1/2}((\mathbb{R}^n)^n), \quad A = (A^i_{ij})_{i,j=1, \ldots, n} \in \text{bmo}(\mathbb{R}^{dn \times n}),
\]

\[
\beta = (\beta^i)_{i=1, \ldots, n} \in \text{bmo}^{1/2}(\mathbb{R}^n), \quad \xi = (\xi^i)_{i=1, \ldots, n} \in L^\infty(\mathbb{R}^n)
\]

and the solution is a pair of processes

\[
Y = (Y^i)_{i=1, \ldots, n} \in \mathcal{S}^\infty(\mathbb{R}^n), \quad Z = (Z^i)_{i=1, \ldots, n} \in \text{bmo}(\mathbb{R}^n)
\]

satisfying (3.2) a.s., for each \( t \in [0, T] \). The following is a consequence of Theorem 2.9 of [JZ21], tailored to our setting.

**Proposition 3.4.** Suppose that \( A \) and \( \alpha \) are sliceable, in the sense that

\[
N_\alpha(\delta) + N_A(\delta) \leq K(\delta),
\]

for some \( K : (0, \infty) \to \mathbb{N} \). Then, for each \( (\beta, \xi) \in \text{bmo}^{1/2} \times L^\infty \), there is a unique solution to (3.2) satisfying

\[
\|Y\|_{L^\infty} + \|Z\|_{\text{bmo}} \leq C(\|\xi\|_{L^\infty} + \|\beta\|_{\text{bmo}^{1/2}}), \quad C = C(K).
\]

### 3.3. Lyapunov functions

**Definition 3.5.** Let \( f \) and \( \sigma \) be as given in (1.1), and \( c \) a constant. A non-negative function \( h \in C^2(\mathbb{R}^n) \) is a \( c \)-Lyapunov function for \( f \) if \( h(0) = 0 \), \( Dh(0) = 0 \), and for some \( k > 0 \) we have

\[
\frac{1}{2} \sum_{i,j=1}^n (D^2 h(y))_{ij} z^i \cdot z^j - Dh(y) \cdot f(t, x, u, \sigma^{-1}(t, x, u)z) \geq |z|^2 - k
\]

for all \( (t, x, u, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \times (\mathbb{R}^d)^n \) with \( |y| \leq c \). In this case, we say that \( (h, k) \in \text{Ly}(f, c) \).

The following is a slight adaptation of Proposition 2.11 in [XZ18].

**Lemma 3.6.** Suppose that \( H_{BF} \) and \( H_{\sigma} \) hold. Then for each \( c > 0 \), there exists a Lyapunov pair \( (h, k) \), depending only on \( C_\sigma \) and \( C_Q \), such that \( (h, k) \in \text{Ly}(f, c) \)

As a consequence, we get the following.

**Lemma 3.7.** Suppose that \( H_{BF} \) and \( H_{\sigma} \) hold, and that \( u \) is a classical solution to (1.1) with \( \|u\|_{L^\infty} < \infty \). Then we have

\[
\sup_{t,x} \|Z^{t,x}\|_{\text{bmo}} \leq C, \quad C = C(C_\sigma, C_Q, \|u\|_{L^\infty}).
\]
Proof. For any \((t, x)\), we have
\[
\begin{align*}
\text{for some martingale } M. \text{ By using the definition of Lyapunov pair, we get}
E_t[\int_\tau^T |Z_s^{t,x}|^2 ds] \leq E_t[h(g(X_T^{t,x})) - h(u(\tau, X_T^{t,x})) + k(T - \tau)] \leq 2\|h \circ u\|_{L^\infty} + kT.
\end{align*}
\]

4. Proofs of the a-priori estimates for \((1.1)\)

Throughout this section, given a decoupling solution \(u = (u^i)_{i=1,\ldots,n} : [0, T] \times \mathbb{R}^d \to \mathbb{R}^n\) to the system \((1.1)\), and a pair \((t_0, x_0) \in [0, T) \times \mathbb{R}^d\), we will denote by \(X^{t_0,x_0}\) the unique strong solution\(^2\) on \([t_0, T]\) to the stochastic differential equation
\[
X^{t_0,x_0}_t = x_0 + \int_{t_0}^t \sigma(s, X^{t_0,x_0}_s, u(s, X^{0,x_0}_s)) dB_s, \quad t \leq s \leq T. \tag{4.1}
\]
We will denote by \(Y^{t_0,x_0}\) and \(Z^{t_0,x_0}\) the processes
\[
Y^{t_0,x_0}_t = u(t, X^{t_0,x_0}_t), \quad Z^{t_0,x_0}_t = \sigma(t, X^{t_0,x_0}_t, Y^{t_0,x_0}_t) Du(t, X^{t_0,x_0}_t). \tag{4.2}
\]
We recall that by hypothesis, the triple \((X^{t_0,x_0}, Y^{t_0,x_0}, Z^{t_0,x_0})\) solves the FBSDE
\[
\begin{cases}
\begin{align*}
&dX^{t_0,x_0}_t = \sigma(t, X^{t_0,x_0}_t, Y^{t_0,x_0}_t) dB_t, \quad t \in [t_0, T)， \\
&dY^{t_0,x_0}_t = -f(X^{t_0,x_0}_t, Y^{t_0,x_0}_t, \sigma^{-1}(t, X^{t_0,x_0}_t)Z^{t_0,x_0}_t) dt + Z^{t_0,x_0}_t dB_t, \quad t \in [t_0, T)， \\
&X^{t_0,x_0}_{t_0} = x_0, \quad Y^{t_0,x_0}_{t_0} = g(X^{t_0,x_0}_t)
\end{align*}
\end{cases} \tag{4.3}
\]
We now proceed with the proof of Lemma 2.5.

Proof of Lemma 2.5. Suppose first that \(H_{AB1}\) holds. For any \((t_0, x_0)\), we set \((X, Y, Z) = (X^{t_0,x_0}, Y^{t_0,x_0}, Z^{t_0,x_0})\), and notice that
\[
\begin{align*}
dY^i_t = -\left[ Z^i_t \cdot \sigma^{-1}(t, X_t, Y_t) b_0(t, X_t, Y_t, \sigma^{-1}(t, X_t, Y_t) Z_t) + b^i(t, X_t, Y_t, \sigma^{-1}(t, X_t, Y_t) Z_t) \right] dt + Z^i_t dB_t \\
= -b^i(t, X_t, Y_t, \sigma^{-1}(t, X_t, Y_t) Z_t) dt + Z^i_t dB_t, \tag{4.4}
\end{align*}
\]
where \(\tilde{B}\) is a Brownian motion under an equivalent probability measure. We can now apply the reasoning from the proof of Proposition 3.8 in [JZ21] to the pair \((Y, Z)\) to get
\[
\|Y\|_{S^\infty} \leq C, \quad C = C(\|Y_T\|_{L^\infty}, \rho, \{a_m\}),
\]
and the result follows. The proof in the case \(H_{AB2}\) holds is essentially the same, but instead of using the reasoning from Proposition 3.8 in [JZ21] to get from the decomposition (4.4)
to the desired estimate, we can instead (because \(b^i\) is Lipschitz) use a standard technique for BSDEs with drivers of linear growth, namely studying the dynamics of \(\exp(\lambda t) |Y_t|^2\) for large enough \(\lambda\). We omit the details. \(\square\)

4.1. The Hölder estimate. This section is devoted to a proof of Theorem 2.6.

4.1.1. Preliminaries on Krylov-Safonov estimates and bmo spaces. The proof of the Hölder estimate is quite technical and relies on a connection between Krylov-Safonov estimates and bmo-spaces which we learned from [Del03]. This sub-section serves two purposes. The first is to introduce notations and lemmas which will be used in the proof of Theorem 2.6. The second is to demonstrate the connection between the Krylov-Safonov estimates and BMO martingales in a simpler setting, for the convenience of the reader. As such, we emphasize that while the lemmas and notations in this sub-section are stated precisely, the rest of this sub-section (e.g. the argument for Hölder regularity of the linear PDE (4.10)) is included to highlight the basic ideas used in the proof of Theorem 2.6, and not meant to be totally rigorous (though it could easily be made so).

For \((t_0, x_0) \in [0, T) \times \mathbb{R}^d\) and \(R \in [0, \sqrt{T - t_0}]\), we define the parabolic cylinder

\[
Q_R(t_0, x_0) = \{(t, x) \in [0, T] \times \mathbb{R}^d : t_0 \leq t \leq t_0 + R^2, \max_i |x_i - x_{0i}| \leq R\}.
\]

Let us recall a basic fact about functions: in order to prove that a function \(v\) is Hölder continuous, it suffices to prove an oscillation decay. In the present parabolic setting, this means that in order to prove that a map \(v : [0, T] \times \mathbb{R}^d \to \mathbb{R}\) is locally Hölder continuous on \([0, T)\), it suffices to prove an estimate of the type

\[
\text{osc}_{Q_{R}}(t_0, x_0)v \leq \beta \text{osc}_{Q_{2R}}(t_0, x_0)v + C_0 R^\gamma
\]

for some \(C_0 > 0, \beta \in (0, 1), \gamma > 0\), and for all \((t_0, x_0)\), \(R\) such that \(t_0 + 4R^2 \leq T\). Here, for any subset \(U \subset [0, T] \times \mathbb{R}^d\),

\[
\text{osc}_U v = \sup_{(t, x) \in U} v(t, x) - \inf_{(t, x) \in U} v(t, x).
\]

If we want global Hölder estimates, we need to complement the oscillation decay (4.5) with a condition which says that oscillation is small over cylinders which are near the terminal time \(T\), i.e. an estimate of the type

\[
\text{osc}_{Q_{\sqrt{T - t_0}}}(t_0, x_0)v \leq C_0 (T - t)^{\alpha_0/2},
\]

for some \(C_0 > 0, \alpha_0 \in (0, 1)\) and all \((t, x) \in [0, T) \times \mathbb{R}^d\).

We formalize this discussion with the following Lemma.

Lemma 4.1. Suppose that \(v : [0, T] \times \mathbb{R}^d \to \mathbb{R}\) is bounded and satisfies (4.5) and (4.6) for some constants \(\alpha_0, \beta \in (0, 1), \gamma > 0, C_0 > 0\). Then, for some \(\alpha = \alpha(\beta, \gamma, \alpha_0)\), we have

\[
\|v\|_{C^{0, \alpha}} \leq C, \quad C = C(C_0, \beta, \gamma, \alpha_0, \|v\|_{L^\infty})
\]
Proof. In this argument, the constant $C$ may change from line to line and depend on any of the constants $C_0$, $\beta$, $\gamma$, $\alpha_0$ and $\|v\|_{L^\infty}$. Let us first record for later use that the estimate (4.6) implies that the function $g(x) = v(T, x)$ satisfies
\[
\text{osc}_{B_R(x_0)} g \leq \text{osc}_{Q_R(t - R^2, x_0)} v \leq C_0 R^{\alpha_0},
\]
which through a standard argument implies
\[
\|g\|_{C^{\alpha, \alpha_0}} \leq C_0.
\]
Now we fix $(t, x)$ apply Lemma 8.23 of [GT77] to the function $\omega(R) = \text{osc}_{Q_R(t, x)} v$, defined on $(0, \sqrt{T - t})$, to conclude that we have
\[
\text{osc}_{Q_R(t, x)} v \leq CR^{\alpha_1} (T - t)^{-\alpha_1/2} \text{osc}_{\sqrt{T - t}}(t, x) v + CR^{\alpha_2} \leq CR^{\alpha_1} (T - t)^{-(\alpha_1 - \alpha_0)/2} + CR^{\alpha_2},
\]
for some $\alpha_1, \alpha_2 \in (0, 1)$ depending only on $\beta, \gamma, \alpha_0$. Now, fix $t \in [0, T)$, $x, y \in \mathbb{R}^d$. Suppose first that $\max_i |x^i - y^i| \leq \sqrt{T - t}$. Then setting $R = \max_i |x^i - y^i|$, we have $(t, y) \in Q_R(t, x)$ By (4.7), we conclude
\[
|v(t, x) - v(t, y)| \leq \text{osc}_{Q_R(t, x)} v \leq C(T - t)^{-(\alpha_1 - \alpha_0)/2} \text{osc}_{\sqrt{T - t}}(t, x) v + C(T - t)^{-(\alpha_1 - \alpha_0)/2} + C|y - x|^{\alpha_2}
\]
\[
\leq C(T - t)^{-(\alpha_1 - \alpha_0)/2} |x - y|^\alpha_1 + C|x - y|^{\alpha_2}.
\]
Now if $\alpha_0 \geq \alpha_1$, clearly we have
\[
|v(t, x) - v(t, y)| \leq C|x - y|^{\alpha}, \quad \alpha = \alpha_0 \land \alpha_1 \land \alpha_2.
\]
If, on the other hand $\alpha_0 < \alpha_1$, then since $\max_i |x^i - y^i| \leq \sqrt{T - t}$, (8.4) gives
\[
|v(t, x) - v(t, y)| \leq C(T - t)^{-(\alpha_1 - \alpha_0)/2} |x - y|^{\alpha_1 - \alpha_0} + C|x - y|^{\alpha_2}
\]
\[
\leq C|x - y|^{\alpha_0} + C|x - y|^{\alpha_2}.
\]
So, at this stage we have established that with $\alpha = \alpha_0 \land \alpha_1 \land \alpha_2$, we have
\[
|v(t, x) - v(t, y)| \leq C|x - y|^{\alpha}
\]
for each $t, x, y$ such that $\max_i |x^i - y^i| \leq \sqrt{T - t}$. If $\max_i |x^i - y^i| > \sqrt{T - t}$, we have
\[
|v(t, x) - v(t, y)| \leq |v(T, x) - v(t, x)| + |v(T, x) - v(T, y)| + |v(T, y) - v(t, y)|
\]
\[
\leq \text{osc}_{\sqrt{T - t}}(t, x) v + \text{osc}_{\sqrt{T - t}}(t, y) v + \|g\|_{C^{\alpha, \alpha_0}} |x - y|^{\alpha_0}
\]
\[
\leq C(T - t)^{\alpha_0/2} + C|x - y|^{\alpha_0} \leq C|x - y|^{\alpha_0}.
\]
So, we have established that (4.9) holds for all $t, x, y$. For time regularity, we fix $t, s, x$ with $t \leq s \leq T$. Then since $(s, x) \in Q_{\sqrt{T - t}}(t, x)$,
\[
|v(t, x) - v(s, x)| \leq \text{osc}_{\sqrt{T - t}}(t, x) v \leq C(s - t)^{\alpha_1/2} (T - t)^{-(\alpha_1 - \alpha_0)/2} + C(s - t)^{\alpha_2}
\]
Once again, we split into cases $\alpha_1 > \alpha_0$ and $\alpha_0 \geq \alpha_1$, and in either case we get the estimate
\[
|v(t, x) - v(s, x)| \leq C(s - t)^{\alpha}.
\]
This completes the proof. \qed

Now, consider a linear, scalar PDE of the type
\[ \partial_t v + \text{tr}(a(t, x)D^2v) + b(t, x) \cdot Dv + f(t, x) = 0, \] (4.10)
with data
\[ a(t, x) = \frac{1}{2} \sigma \sigma^T (t, x) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times d}, \]
\[ b = b(t, x) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d, \quad f = f(t, x) : [0, T] \times \mathbb{R}^d \to \mathbb{R}. \]
When \( a, b, f \) are bounded and \( a \) is uniformly elliptic, i.e.
\[ \frac{1}{C_\sigma} \|z\|^2 \leq |\sigma(t, x)z|^2 \leq C_\sigma \|z\|^2. \]
The Krylov-Safonov estimates show that any bounded solution of (4.10) is locally Hölder continuous on \([0, T] \times \mathbb{R}^d\), with corresponding estimates depending on the \( \|v\|_{L^\infty} \), and the \( L^\infty \) and ellipticity constants of \( b, f \), and \( \sigma \). Now suppose that \( v \) is sufficiently nice and define for \((t_0, x_0) \in [0, T] \times \mathbb{R}^d\) the solution of the SDE
\[ X^{t_0, x_0}_t = x_0 + \int_{t_0}^t \sigma(s, X^{t_0, x_0}_s) dB_s. \]
The key to the probabilistic proof of the Krylov-Safonov estimates is the following Lemma, which can be deduced from the results in the original paper \( [\text{KS79}] \). We use here and in the remainder of the paper the notation \(|A|\) for the Lebesgue measure of a Borel set \( A \).

**Lemma 4.2.** Fix \((t_0, x_0), R\) with \( t_0 + 4R^2 \leq T \), and let \( A \subset Q_{2R}(t_0, x_0) \) with \( |A| \geq \frac{1}{2}|Q_{2R}(t_0, x_0)| \). Then for any \((t, x) \in Q_R(t_0, x_0)\), we have
\[ \mathbb{P} \left[ \tau_A < \tau_{Q_{2R}(t_0, x_0)} \right] \geq \epsilon, \]
where \( \epsilon > 0 \) depends only on the \( C_\sigma \) and
\[ \tau_A = \inf \left\{ s \geq t : (s, X^{t, x}_s) \in A \right\}, \quad \tau_{Q_{2R}(t_0, x_0)} = \inf \left\{ s > t : (s, X^{t, x}_s) \in \partial Q_{2R}(t_0, x_0) \right\}. \]

Let us show how by combining Lemma 4.2 with some facts about the space \( \text{bmo} \), we can obtain an interior Hölder estimate when \( b \) is not necessarily bounded, but satisfies a bound like
\[ \sup_{(t_0, x_0)} \|b(\cdot, X^{t_0, x_0})\|_{\text{bmo}} \leq C_0. \] (4.11)
As explained above, we can focus on checking an oscillation estimate like (4.5). So, we fix \((t_0, x_0) \in [0, T] \times \mathbb{R}^d\), and \( R \) such that \( t_0 + 4R^2 \leq T \). We also fix \((t, x) \in Q_R(t_0, x_0)\), and for simplicity of notation we set \( X = X^{t_0, x_0} \). Set
\[ M^+ = \max_{Q_{2R}(t_0, x_0)} v, \quad M^- = \min_{Q_{2R}(t_0, x_0)} v, \]
\[ A^+ = \{(s, y) \in Q_{2R}(t_0, x_0) : v(s, y) \geq \frac{1}{2}(M^+ + M^-)\}, \]
\[ A^- = \{(s, y) \in Q_{2R}(t_0, x_0) : v(s, y) < \frac{1}{2}(M^+ + M^-)\}. \]
Obviously, we have one of two alternatives:

\[ |A^+| \geq \frac{1}{2} |Q_{2R}(t_0, x_0)|, \quad \text{or} \quad |A^-| \geq \frac{1}{2} |Q_{2R}(t_0, x_0)|. \]

Let us suppose the second of these two possibilities, the first can be handled by a similar argument. Now, set \( \tau = \tau_{A^-} \wedge \tau_{Q_{2R}(t_0, x_0)} \), where \( \tau_{A^-} \) and \( \tau_{Q_{2R}(t_0, x_0)} \) are defined as in the proof of Lemma 4.2. We know from Itô’s formula that

\[
dv(t, X_t) = -[f(t, X_t) - b(t, X_t) \cdot Dv(t, X_t)] dt + Dv(t, X_t) \sigma(t, X_t) dB_t
\]

where \( \tilde{B} = B - \int \sigma^{-1}(s, X_s)b(s, X_s)ds \) is a Brownian motion under \( Q \), with \( \frac{d\tilde{B}}{ds} = f(\sigma^{-1}(s, X_s)b(s, X_s))dB \). Notice that

\[
\|v^{-1}(\cdot, X)\|_{bmo} \leq C_\sigma \|\cdot, X\|_{bmo} \leq C_\sigma C_0.
\]

Thus we find that

\[
v(t, x) = \mathbb{E}^Q[v(\tau, X_\tau)] + \int_{\tau}^T f(s, X_s)ds
\]

\[
\leq \frac{(M^+ + M^-)}{2} Q[\tau_{A^-} < \tau_{Q_{2R}(t_0, x_0)}] + M^+(1 - Q[\tau_{A^-} < \tau_{Q_{2R}(t_0, x_0)}]) + 4R^2 \|f\|_{L^\infty}.
\]

(4.12)

Some arithmetic shows that

\[
v(t, x) - M^- \leq (M^+ - M^-) \left(1 - \frac{1}{2} Q[\tau_{A^-} < \tau_{Q_{2R}(t_0, x_0)}]\right) + CR^2
\]

\[
= \text{osc}_{Q_{2R}(t_0, x_0)} v \left(1 - \frac{1}{2} Q[\tau_{A^-} < \tau_{Q_{2R}(t_0, x_0)}]\right) + CR^2.
\]

Applying Lemma 3.2 (stated below) and then Lemma 4.2 to estimate from below the quantity \( Q[\tau_{A^-} < \tau_{Q_{2R}(t_0, x_0)}] \) lets us conclude that

\[
v(t, x) - M^- \leq \beta \text{osc}_{Q_{2R}(t_0, x_0)} v + CR^2,
\]

for some \( \beta \in (0, 1), C > 0 \) depending only on \( C_\sigma \) and \( \|f\|_{L^\infty} \). Finally, taking a supremum over \( (t, x) \in Q_R(t_0, x_0) \) gives exactly the oscillation decay (4.5).

4.1.2. Proof of Theorem 2.6. Now we give the proof of the Hölder estimate.

**Proof of Theorem 2.6.** To simplify notation, observe that it suffices to assume that \( \sigma = \sigma(t, x) \), but prove an estimate which depends only on the ellipticity constant \( C_\sigma \) of \( \sigma \). That is, we do not assume in this proof that \( H_\sigma \) is satisfied, only that \( \sigma \) is uniformly elliptic with constant \( C_\sigma \). So our equation becomes

\[
\begin{cases}
\partial_t u^i + \text{tr}(a(t, x)D^2 u^i) + f^i(t, x, u, Du) = 0, \\
u^i(T, x) = g^i(x),
\end{cases}
\]

(4.13)
where \( f \) still satisfies (HBF), and \( g \in C^{0,\beta} \). We wish now to prove under these conditions a global Hölder estimate for \( u \). The idea will be to use the preceding three Lemmas to prove by induction that the following statement holds for each \( i \):

\[
\|u^i\|_{C^{0,\alpha}} \leq C \text{ and } \sup_{(t,x)} \sup_{t \leq s \leq s \leq s \leq T} \|Z^t_s, x^i 1_{s=\delta,s} \|_{bmo} \leq C \delta^\alpha,
\]

for some constants \( C \) and \( \alpha \) depending only on \( \beta, \|g\|_{C^{0,\beta}} C_Q, \epsilon, \) and \( \|u\|_{L^\infty} \). (4.14)

Throughout the argument, constants like \( C, \alpha, \) and \( \gamma \) may change freely from line to line but will depend only on \( C, C_Q, \epsilon, \|g\|_{C^{0,\beta}}, \) and \( \|u\|_{L^\infty} \) unless otherwise stated. We start with the base case of our induction argument, namely \( i = 1 \). The idea is to apply (a slightly more sophisticated version of) the argument given above for the linear equation (4.10) to the equation for \( u^1 \). For each \((t,x)\), we recall that \( X^t_s \) denotes the unique solution on \([t,T]\) to

\[
X^t_s = x + \int_t^s \sigma(r, X^t_s) dB_r, \quad t \leq s \leq T,
\]

and that \((Y^t_s,1, Z^t_s,1) = (u^1(\cdot), X^t_s), \sigma(\cdot, X^t_s) Du^1(\cdot, X^t_s))\). We begin by establishing the oscillation decay

\[
\text{osc}_{Q_R(t,x)} u^1 \leq \beta \text{osc}_{Q_{2R}(t,x)} u^1 + C R^\gamma, \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}^d, \quad R > 0 \text{ such that } t + 4R^2 \leq T.
\]

(4.15)

for appropriate constants \( \beta, \gamma, \) and \( C \). We fix \((t_0, x_0) \in [0,T] \times \mathbb{R}^d\) with \( t_0 + 4R^2 \leq T \) and then choose \((t,x) \in Q_R(t_0, x_0)\). Let \( h^i, k^i \) be as discussed in Remark 2.2 (see in particular (2.1)). Thus we have

\[
\partial_t u^1 + \text{tr}(D^2 u^1) + p^1 \cdot h^1(t,x,u,p) + k^1(t,x,u,p),
\]

where

\[
|k^1(t,x,u,p)| \leq C_Q (1 + |p|^{2-\epsilon}), \quad |h^1(t,x,u,p)| \leq C_Q (1 + |p|).
\]

(4.16)

Consequently, we can write

\[
dY^{t,x,1}_s = -(Z^{t,x,1}_s \cdot h_s + k_s) ds + Z^{t,x,1}_s dB_s = -k_s ds + Z^{t,x,1}_s dB_s,
\]

(4.17)

where we have set

\[
h_s = \sigma^{-1}(s, X^{t,x}_s) h^1(s, X^{t,x}_s, \sigma^{-1}(s, X^{t,x}_s) Z^{t,x}_s), \quad k_s = k^1(s, X^{t,x}_s, \sigma^{-1}(s, X^{t,x}_s) Z^{t,x}_s),
\]

and \( \tilde{B} = B - \int h dB \) is a Brownian motion under the measure \( dQ = \mathcal{E}(\int h dB) \). Notice also that by Lemma 3.7 and (4.16) we have \( \|k^{2(2-\epsilon)}\|_{bmo^{1/2}} \leq C \), and in particular we have by Lemma 3.3

\[
\|h\|_{bmo} \leq C, \quad \|k_{[s-\delta,s]}\|_{bmo^{1/2}} \leq C \delta^\gamma.
\]

(4.18)

for each \( t \leq s - \delta \leq s \leq T \) and some \( C, \gamma \). We point out for later use that knowing (4.18) (for each choice of \((t,x)\)) and the bound on \( \|g^1\|_{C^{0,\beta}} \) is the only thing that we will use to
conclude that (4.14) holds for \( i = 1 \). Now if we set

\[
M^+ = \sup_{(s,y)\in Q_{2R}(t_0,x_0)} u^1(s,y), \quad M^- = \inf_{(s,y)\in Q_{2R}(t_0,x_0)} u^1(s,y),
\]

\[
A^+ = \{(s,y) \in Q_{2R}(t_0,x_0) : u^1(s,y) \geq \frac{M^+ + M^-}{2}\},
\]

\[
A^- = \{(s,y) \in Q_{2R}(t_0,x_0) : u^1(s,y) \leq \frac{M^+ + M^-}{2}\},
\]

Then clearly we must either have \(|A^+| \geq \frac{1}{2}|Q_{2R}(t_0,x_0)|\) or \(|A^-| \geq \frac{1}{2}|Q_{2R}(t_0,x_0)|\). We assume the second possibility, and a symmetric argument will take care of the second. Set \( \tau = \tau_{A^-} \land \tau_{Q_{2R}(t_0,x_0)} \), and use (4.17) to write

\[
u^1(t,x) = \mathbb{E}^Q[u^1(\tau, X^{t,x}_\tau) + \int_{t}^{\tau} k_s ds]
\]

\[
\leq \frac{(M^+ + M^-)}{2} Q[\tau_{A^-} \leq \tau_{Q_{2R}(t_0,x_0)}] + M^+(1 - Q[\tau_{A^-} \leq \tau_{Q_{2R}(t_0,x_0)}]) + \|k\|_{bmo^{1/2}(Q_{[t,t_0+4R^2]})}
\]

\[
\leq \frac{(M^+ + M^-)}{2} Q[\tau_{A^-} \leq \tau_{Q_{2R}(t_0,x_0)}] + M^+(1 - Q[\tau_{A^-} \leq \tau_{Q_{2R}(t_0,x_0)}]) + C\|k\|_{bmo^{1/2}(Q_{[t,t_0+4R^2]})}
\]

\[
\leq \frac{(M^+ + M^-)}{2} Q[\tau_{A^-} \leq \tau_{Q_{2R}(t_0,x_0)}] + M^+(1 - Q[\tau_{A^-} \leq \tau_{Q_{2R}(t_0,x_0)}]) + CR^7,
\]

where the second inequality is given by Lemma 3.1, and the third is given by (4.18) together with Lemma 3.3 Some arithmetic then shows that

\[
u^1(t,x) - M^- \leq (1 - \frac{1}{2} Q[\tau_{A^-} \leq \tau_{Q_{2R}(t_0,x_0)}]) \text{osc}_{Q_{2R}(t_0,x_0)} u^1 + CR^7,
\]

and since the estimate holds for all \((t,x) \in Q_R(t_0,x_0)\), we conclude that

\[
\text{osc}_{Q_R(t_0,x_0)} u^1 \leq (1 - \frac{1}{2} Q[\tau_{A^-} \leq \tau_{Q_{2R}(t_0,x_0)}]) \text{osc}_{Q_{2R}(t_0,x_0)} u^1 + CR^7.
\]

We now use Lemma 4.2 to bound from below \( \mathbb{P}[\tau_{A^-} \leq \tau_{Q_{2R}(t_0,x_0)}] \), and then (4.18) together with Lemma 3.2 to translate this to an estimate from below on \( Q[\tau_{A^-} \leq \tau_{Q_{2R}(t_0,x_0)}] \). This allows us to deduce the estimate (4.5). Next, we establish the estimate

\[
\text{osc}_{Q_{\sqrt{T-t_0}}(t_0,x_0)} u^1 \leq C(T - t_0)^{\alpha_0/2}
\]

for some constants \( \alpha_0 \) and \( C \). Note first that because \( g \) is Hölder continuous, it suffices to show that for some \( \gamma \) and \( C \) we have

\[
|u^1(t,x) - g^1(x)| \leq C(T - t)^{\gamma/2}.
\]

For this, we define \( k, h, Q \) as above and notice that

\[
u^1(t,x) = \mathbb{E}^Q[g(X^{t,x}_\tau) + \int_{t}^{T} k_s ds],
\]
so that
\[
|u^1(t, x) - g^1(x)| \leq \mathbb{E}[|g'(X^t_{\tau}) - g'(x)|] + \mathbb{E}[\int_t^\tau |k_s| ds] \\
\leq \mathbb{E}[|X^t_{\tau} - x|^{\beta}] + C(T - t)^{\gamma/2} \leq C(T - t)^{\gamma/2},
\]
where the last inequality follows from the following generalization of Lemma 5.1 in [XZ18]:

**Lemma 4.3.** Let \( \gamma \in \text{bmo} \) and define \( \mathbb{Q} \) by \( d\mathbb{Q} = \mathcal{E}(\int_\gamma dB) \). Then for any stopping times \( \tau \) taking values in \([s - \delta, s]\) and any \((t, x)\) as above, we have
\[
\mathbb{E}_\tau[|X^t_{\tau} - X^t_{s}|^\alpha] \leq C\delta^{\alpha/2},
\]
where \( C \) depends only on \( \|\sigma\|_{L^\infty} \) and the bmo norm of \( \gamma \).

Postponing the proof of Lemma 4.3, we conclude that (4.20) and hence (4.19) holds. From Lemma 4.1, we can now deduce that \( \|u^1\|_{C_{0, \alpha}} \leq C \). Now we show how this implies the second part of the statement (4.14), namely the estimate
\[
\sup_{(t, x)} \sup_{t \leq \tau \leq s} \mathbb{E}_\tau[|Z^{t,x}_{s-h,s}|]_{\text{bmo}} \leq C\delta^{\alpha}.
\]
We fix \((t, x)\) and define \(X^{t,x}, Y^{t,x}, Z^{t,x}, h, q\), etc. as above and compute
\[
d[Y^t_{u}, 1]^2 = (-2Y^t_{u} k_s + |Z^{t,x}_{u}|)^2 ds + dM_s,
\]
where \(M\) is a martingale under \(\mathbb{Q}\). Thus given a stopping time \(\tau\) with \(s - \delta \leq \tau \leq s\), we have
\[
\mathbb{E}_\tau[\int_s^\tau |Z^{t,x}_{u}|^2 du] = \mathbb{E}_\tau[|Y^{t,x}_{s}|^2 - |Y^{t,x}_{\tau}|^2 + 2 \int_s^\tau Y^{t,x}_{u} k_s du] \\
\leq \mathbb{E}_\tau[|u^1(s, X^{t,x}_s)|^2 - |u^1(\tau, X^{t,x}_\tau)|^2 + C\|k\|_{\text{bmo}^{1/2}(\mathbb{Q}, [s - \delta, s])}] \\
\leq \mathbb{E}_\tau[|s - \tau|^{\alpha/2} + |X^{t,x}_s - X^{t,x}_{\tau}|^\alpha + C\|k\|_{\text{bmo}^{1/2}(s - \delta, s)}] \\
\leq C\delta^{\alpha} + \mathbb{E}_\tau[|X^{t,x}_s - X^{t,x}_{\tau}|^\alpha] \leq C\delta^{\alpha},
\]
where we have once again used (4.18) and Lemma 3.1, and the last line follows from Lemma 4.3. We have now shown that \(\mathbb{E}_\tau[\int_s^\tau |Z^{t,x}_{u}|^2 du] \leq C\delta^{\alpha}\) whenever \(t \leq s - \delta \leq \tau \leq s\), from which it follows that
\[
\|Z^{t,x}_{s-h,s}\|_{\text{bmo}} \leq C\|Z^{t,x}_{s-h,s}\|_{\text{bmo}(\mathbb{Q})} \leq C\delta^{\alpha}.
\]
Thus we have established (4.14) in the case \(i = 1\). The induction step is almost exactly the same. Suppose we know that (4.14) holds for all \(i < j\). Then we may again use the decomposition in Remark 2.2 to write
\[
\partial_t u^i + \text{tr}(aD^2 u^i) + k^i(t, x, u, p) + p^i \cdot h^i(t, x, u, p) = 0,
\]
where
\[
|k^i(t, x, u, p)| \leq C_Q(1 + |p|^{2 - \epsilon} + \sum_{j < i} |p|^j), \quad |h^i(t, x, u, p)| \leq C_Q(1 + |p|).
\]
Using the induction hypothesis and Lemma 3.3, we conclude that for any \((t, x)\), if we define

\[ k_s = k^i(s, X^{t,x}_s, Y^{t,x}_s, \sigma^{-1}(s, X^{t,x}_s)Z^{t,x}_s), \quad h_s = h^i(s, X^{t,x}_s, Y^{t,x}_s, \sigma^{-1}(s, X^{t,x}_s)Z^{t,x}_s), \]

then we have

\[ \|h\|_{bmo} \leq C, \quad \|k_{[s-\delta,s]}\|_{bmo} \leq C\delta^\alpha. \]  \(4.21\)

As in the case \(i = 1\), \(4.21\) together with the control of \(\|g^i\|_{C^{0,\alpha}}\) is enough to conclude that \(4.14\) holds for \(i\). This completes the proof. \(\Box\)

**Proof of Lemma 4.3.** For simplicity, we write \(X = X^{t,x}\). Then we have

\[ dX_u = \sigma(u, X_u)dB_u = \sigma(u, X_u)d\tilde{B}_u + \sigma(u, X_u)\gamma_u du, \]

so with \(s\) and \(\tau\) as in the statement of the Lemma,

\[ \mathbb{E}_\tau^Q[|X_s - X_\tau|] \leq \mathbb{E}_\tau^Q[\int_\tau^s \sigma(u, X_u)d\tilde{B}_u] + \mathbb{E}_\tau^Q[\int_\tau^s \sigma(u, X_u)\gamma_u du] \]
\[ \leq C\delta^{1/2} + C\|\gamma_1_{[s-\delta,s]}\|_{bmo^{1/2}(Q)} \leq C\delta^{1/2} + C\|\gamma_1_{[s-\delta,s]}\|_{bmo^{1/2}} \]
\[ \leq C\delta^{1/2} + C\|\gamma_1_{[s-\delta,s]}\|_{bmo^{1/2}} \leq C\delta^{1/2}, \]

where the third inequality uses Lemma 3.1. To complete the proof, note that for \(\alpha \in (0,1)\),

\[ \mathbb{E}_\tau^Q[|X_s - X_\tau|^{\alpha}] \leq \mathbb{E}_\tau^Q[|X_s - X_\tau|^{\alpha}] \leq C\delta^{\alpha/2}. \]

\(\Box\)

### 4.2. The gradient bound.

This section is devoted to a proof of Theorem 2.7. We begin with a Lemma which explains that Hölder estimates always lead to estimates on the sliceability of the processes \(Z^{t,x}\), provided that we have a Lyapunov function.

**Lemma 4.4.** Suppose that \(u\) is a decoupling solution to (1.1), and \(\|u\|_{L^\infty} \leq c\). Suppose further that there exists \((h, k) \in \mathcal{L}(f, c)\). Finally, suppose that we have \(\|u\|_{C^{0,\alpha}} < \infty\), for some \(\alpha \in (0,1)\). Then, for any \((t_0, x_0) \in [0, T) \times \mathbb{R}^d\) and any stopping time \(\tau\) and \(t \in [0, T]\) with \((t_0 \vee (t - \delta)) \leq \tau \leq t\), we have

\[ \mathbb{E}_\tau[\int_\tau^t |Z_s^{t_0,x_0}|^2 ds] \leq k\delta + C\|Dh\|_{L^\infty(B_{\|u\|_{L^\infty}})}\|u\|_{C^{0,\alpha}}(t - s)^{\delta/2}, \]

where \(C = C(\|\sigma\|_{L^\infty})\). In particular, \(Z^{t_0,x_0}\) is sliceable, with an index of sliceability independent of \((t_0, x_0)\), and depending only on \(h, k, \|u\|_{C^{0,\alpha}}\), and \(\|\sigma\|_{L^\infty}\).

**Proof.** The proof is essentially the same as that of Proposition 5.2 in [XZ18]. Namely, fixing any \((t_0, x_0)\) and setting \((X, Y, Z) = (X^{t_0,x_0}, Y^{t_0,x_0}, Z^{t_0,x_0})\) for simplicity, we have

\[ dh(Y_s^{t_0,x_0}) = \left(\frac{1}{2} \sum_{i,j} D_{ij} h(Y_s)Z^i_s \cdot Z^j_s - Dh(Y_s) \cdot f(X_s, Y_s, \sigma^{-1}(s, X_s)Z_s)\right) ds + dM_s, \]
for some martingale $M$. Applying the definition of a Lyapunov pair, we find that
\[
\mathbb{E}_\tau[h(Y_t) - h(Y_\tau)] \geq \mathbb{E}_\tau[\int_\tau^t |Z_s|^2 ds] - k\delta.
\]
We conclude the proof by estimating
\[
\mathbb{E}_\tau[h(Y_t) - h(Y_\tau)] \leq \|Dh\|_{L^\infty(B[|u|]_{L^\infty})}\|u\|_{C^{0,\alpha}}(\delta^{\alpha/2} + \mathbb{E}_\tau[|X_t - X_\tau|^\alpha])
\]
\[
\leq C\|Dh\|_{L^\infty(B[|u|]_{L^\infty})}\|u\|_{C^{0,\alpha}}\delta^{\alpha/2},
\]
where the last inequality comes from Lemma 5.1 of [XZ18].

Now we present the proof of the gradient bound, Theorem 2.7.

**Proof of Theorem 2.7.** For notational simplicity, we give the argument in the case $d = 1$, but the same argument goes through in when $d > 1$. In this case, our equation becomes
\[
\partial_t u^i + a(t, x, u)D^2 u^i + f^i(t, x, u, Du) = 0, \quad (t, x) \in [0, T) \times \mathbb{R},
\]
with the terminal condition $u^i(T, x) = g^i(x)$. We now compute the equations for $v^i = Du^i$, and find
\[
\partial_t v^i + a(t, x, u)D^2 v^i + (D_x a(t, x, u) + D_a a(t, x, u) \cdot Du)Dv^i + D_x f^i(t, x, u, Du)
\]
\[
+ D_a f^i(t, x, u, Du) \cdot v + D_p f^i(t, x, u, Du) \cdot Dv = 0,
\]
with the terminal condition $v^i(T, x) = D g^i(x)$. We fix $(t_0, x_0)$ and set $X = X(t_0, x_0)$, where we continue to define $X(t_0, x_0)$ by (4.1). We then set
\[
Y = v(\cdot, X) = Du(\cdot, X), \quad Z = \sigma(\cdot, X)Dv(\cdot, X) = \sigma(\cdot, X)D^2 u(\cdot, X), \quad U = u(\cdot, X),
\]
so that $(Y, Z)$ solves the linear BSDE
\[
Y_t^i = \xi^i + \int_t^T (\alpha_s^i \cdot Y_s + A_s^i \cdot Z_s + \beta_s^i) ds - \int_t^T Z_s^i dB_s,
\]
where
\[
\xi = D g(X_T), \quad \alpha_t^i = D_a f^i(t, X_t, U_t, Y_t),
\]
\[
A_t^i = \sigma^{-1}(t, X_t, U_t) \left[ D_p f^i(t, X_t, U_t, Y_t) + (D_x a(t, X_t, U_t) + D_a a(t, X_t, U_t) \cdot Y_t) e_t \right],
\]
\[
\beta_t^i = D_x f^i(t, X_t, U_t, Y_t).
\]
Here we use $e_i$ to denote the $i^{th}$ standard basis vector of $\mathbb{R}^n$. Now because of Theorem 2.6, Lemma 4.4, and Lemma 3.6, we can find a $K : (0, \infty) \to \mathbb{N}$ depending on $C_\sigma, L_\sigma, C_Q, \alpha, \|u\|_{C^{0,\alpha}}, \epsilon$ such that
\[
N_\alpha(\delta) + N_A(\delta) \leq K(\delta)
\]
for each $\delta > 0$. Moreover, we have $\|\beta\|_{bmo, 1/2} \leq C, \quad C = C(C_Q, \|Y\|_{bmo})$. The result now follows from Proposition 3.4.
5. Proofs of the a-priori estimates for (1.2)

In this section, given a classical solution $u = (u^i)_{i=1}^n : [0, T] \times \mathbb{R}^d \to \mathbb{R}^n$ to the system (1.2), and a pair $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$, we will denote by $X_{t_0,x_0}$ the unique strong solution on $[t_0, T]$ to the stochastic differential equation

$$X_{t_0,x_0} = x_0 + \int_{t_0}^{t} \sigma(s, X_s, u(s, X_s), Du(s, X_s))dB_s, \quad t \leq s \leq T. \quad (5.1)$$

We will denote by $Y_{t_0,x_0}$ and $Z_{t_0,x_0}$ the processes

$$Y_{t_0,x_0}^t = u(t, X_{t_0,x_0}^t), \quad Z_{t_0,x_0}^t = \sigma(t, X_{t_0,x_0}^t, Y_{t_0,x_0}^t, Du(t, X_{t_0,x_0}^t))Du(t, X_{t_0,x_0}^t). \quad (5.2)$$

We note that $(Y_{t_0,x_0}, Z_{t_0,x_0})$ satisfies

$$\begin{cases}
    dY_{t_0,x_0}^t = -f(X_{t_0,x_0}^t, Y_{t_0,x_0}^t, \sigma^{-1}(t, X_{t_0,x_0}^t, Y_{t_0,x_0}^t, Du(t, X_{t_0,x_0}^t))Z_{t_0,x_0}^t)dt + Z_{t_0,x_0}^t dB_t & t \in [t_0, T], \\
    Y_{t_0,x_0}^T = g(X_{t_0,x_0}^T)
\end{cases} \quad (5.3)$$

**Proof of Theorem 2.11.** In this proof, the constant $C$ will change line to line but only depend on the quantities listed in the statement of Theorem 2.11. First, we note that by using the probabilistic representation (5.3) together with $(H^1_{\text{Lip}})$, it is straightforward to get an estimate of the form

$$\|u\|_{L^\infty} \leq C. \quad (5.4)$$

In fact, once we have this $L^\infty$ estimate, we can also use Theorem 2.6 to obtain an estimate of the form

$$\|u\|_{C^{0,\alpha}} \leq C, \quad (5.5)$$

with $\alpha$ depending only on $C_\sigma$, $\|u\|_{L^\infty}$, and $C_f$. Indeed, notice that under the hypotheses of Theorem 2.11, the function $\tilde{\sigma}(t, x) = \sigma(t, x, u(t, x), Du(t, x))$ satisfies $(H_\sigma)$, with the same ellipticity constant $C_\sigma$ as in $(H^1_\sigma)$. The Lipschitz constant of $\tilde{\sigma}$ will depend, of course, on regularity of $u$, but since the estimate in Theorem 2.6 does not depend on $L_\sigma$, we can infer (5.5) from Theorem 2.6 and (5.4).

Next, we differentiate the equation (1.2). Setting $v^i = Du^i$, we find

$$\partial_t v^i + a(t, x, u, v)D^2 v^i + \left(D_x a(t, x, u, v) + Du a(t, x, u, v) \cdot v + D_p a(t, x, u, v) \cdot Du\right)Dv^i + D_x f^i(t, x, u, v) + Du f^i(t, x, u, v) \cdot v + D_p f^i(t, x, u, v) \cdot Du = 0,$n

with the terminal condition $v^i(T, x) = Dg^i(x)$. We can rewrite this as

$$\partial_t v^i + \tilde{A}(t, x)D^2 v^i + \tilde{f}(t, x, u, Du) = 0,$$

where

$$\tilde{A}(t, x) = \frac{1}{2} \tilde{\sigma} \tilde{\sigma}^T (t, x), \quad \tilde{\sigma}(t, x) = \sigma(t, x, u(t, x), Du(t, x))$$
and
\[ f^i(t, x, v, p) = \left( D_x a(t, x, u(t, x), Dv(t, x)) + D_u a(t, x, u(t, x), Dv(t, x)) \cdot v \right. \\
+ D_p a(t, x, u(t, x), Dv(t, x)) \cdot p \left. \right)^i \\
+ D_x f^i(t, x, u(t, x), Dv(t, x)) + D_u f^i(t, x, u(t, x), Dv(t, x)) \cdot v \\
+ D_p f^i(t, x, u(t, x), Dv(t, x)) \cdot p \]

Using \((H^1_\sigma), (H^1_{\text{Lip}})\), and the bound already established on \(\|u\|_{L^\infty}\), we can check that the data \(\tilde{A}, \tilde{f}\), satisfy the hypotheses \(H_{\text{AB2}}, H_{\text{BF}}, \text{and } H_\sigma\) (and with the relevant constants depending only on \(C_\sigma, L_\sigma, C_f\)). Moreover, clearly the terminal condition \(D g \in C^\beta\). We can thus apply Theorem 2.6 to complete the proof of the estimate on \(\|Du\|_{C^{0,\alpha}}\) (with a smaller \(\alpha\) if necessary).

Now suppose in addition we have \(g \in C^{2,\beta}\) and \(H^1_{\text{Reg}}\) holds. To get the estimate on \(C^{2,\alpha}\), we would like to appeal to Schauder theory. We write the equation for \(u^i\) as
\[ \partial_t u^i + \tilde{A}(t, x) D^2 u^i + \tilde{f}^i(t, x) = 0, \] (5.6)
where
\[ \tilde{A}(t, x) = a(t, x, u(t, x), Dv(t, x)), \quad \tilde{f}^i(t, x) = f^i(t, x, u(t, x), Dv(t, x)). \]

Using the estimates so far obtained on \(u\) and \(Du\), we can check that \(\tilde{A}(t, x) \in C^{0,\alpha}, \tilde{f}^i \in C^{0,\alpha}\) (again, updating \(\alpha\) if necessary, and with corresponding quantitative estimates). Now we can appeal to the classical Schauder estimates to conclude the desired estimate on \(\|u\|_{C^{2,\alpha}}\). \(\square\)

6. Proof of the existence results

Proof of Theorem 2.8. As explained in Remark 2.10, we need only prove existence. The idea is to first truncate and then mollify the data, and then pass to the limit using a compactness argument. For each \(k\), we define \(\pi^{(k)}: (\mathbb{R}^d)^n \to (\mathbb{R}^d)^n\) by
\[ \pi^{(k)}(p) = \begin{cases} p & |p| \leq k, \\ kp & |p| > k. \end{cases} \]

We define for each \(k \in \mathbb{N}\) a driver \(f^{(k)}\) by
\[ f^{(k), i}(t, x, u, p) = f^i(t, x, y, \pi^{(k)}(p)). \] (6.1)

Next, we let \(\rho_c\) be a standard mollifier on \(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^n \times (\mathbb{R}^d)^n\), and we set
\[ f^{(k), \epsilon, i}(t, x, u, p) = \int_{\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^n \times (\mathbb{R}^d)^n} f^{(k), i}(t', x', u', p') \rho_c(t' - t, x - x', u' - u, p' - p) dt' dx' du' dp', \]
where \(f^{(k), i}(t, x, u, p)\) has been extended to \(t \in \mathbb{R}\) by \(f^{(k), i}(t, x, u, p) = f^{(k), i}((t \lor 0) \land T, x, u, p)\). Likewise, define \(\sigma^\epsilon\) through a standard mollification in \((t, x)\). Since \(f^{(k), \epsilon}\) is
bounded with bounded derivatives of all orders, it is standard that for each \( k \in \mathbb{N} \) and \( 0 < \epsilon < 1 \), there is a unique classical solution \( u^{(k),\epsilon} \) to the equation

\[
\begin{align*}
\frac{\partial_t}{\partial x} u^{(k),\epsilon,i} + \text{tr}(\sigma^{\epsilon} D^2 u^{(k),\epsilon,i} + f^{(k),\epsilon,i}(t, x, u^{(k),\epsilon}, Du^{(k),\epsilon}) = 0, \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
u^{(k),\epsilon,i}(T, x) = g^i(x), \quad x \in \mathbb{R}^d,
\end{align*}
\]

(6.2)

see e.g. Proposition 3.3 of [MPY94]. Some tedious but straightforward computations verify that the data \( f^{(k),\epsilon}, \sigma^{\epsilon} \) satisfy the hypotheses \( H_{AB} \) and \( H_{\text{Reg}} \) uniformly in \( k \) and \( \epsilon \). In particular, by Lemma 2.5 and Theorems 2.6 and 2.7, we may conclude that

\[
\sup_{k, \epsilon} \left\| u^{(k),\epsilon} \right\|_L^\infty + \sup_{k, \epsilon} \left\| Du^{(k),\epsilon} \right\|_L^\infty < \infty. \tag{6.3}
\]

But now for a smooth cut-off function \( \kappa : \mathbb{R}^d \to \mathbb{R} \) with \( \kappa(x) = 1 \) for \( |x| \leq 1 \), \( \kappa(x) = 0 \) for \( |x| \geq 2 \), we can compute for each \( x_0 \in \mathbb{R}^d \) the equation satisfied by \( u^{(k),\epsilon,x_0} = u^{(k),\epsilon}(x - x_0) \), and we find that that each component of \( u^{(k),\epsilon,x_0} \) satisfies a linear parabolic equation with uniformly Hölder continuous coefficients with a right-hand side which is bounded uniformly in \( x_0, k, \epsilon \). Then applying the Calderon-Zygmund estimates for this equation (see e.g. Theorem 1 in Chapter 5, Section 2 of [Kry08]) gives

\[
\sup_{k, \epsilon, x_0 \in \mathbb{R}^d} \int_0^T \int_{B_1(x_0)} \left( |\partial_t u^{(k),\epsilon}|^p + |Du^{(k),\epsilon}|^p + |D^2 u^{(k),\epsilon}|^p \right) dx \, dt < \infty, \tag{6.4}
\]

for each \( p < \infty \). By Sobolev embedding (see Appendix E of [FR75] for a nice review of parabolic Sobolev embedding and [LSU68] for the proofs), we conclude that for each \( 0 < \gamma < 1 \), we have

\[
\sup_{k, \epsilon} \left\| u^{(k),\epsilon} \right\|_{C^{1,\gamma}} < \infty.
\]

Now with \( \tilde{f}^{(k),\epsilon,i}(t, x) = f^{(k),\epsilon,i}(t, x, u^{(k),\epsilon}, Du^{(k),\epsilon}) \), we deduce that for some \( \gamma \in (0, 1) \),

\[
\sup_{k, \epsilon} \left\| \tilde{f}^{(k),\epsilon,i}(t, x) \right\|_{C^{0,\gamma}} < \infty,
\]

and so viewing (6.2) as a linear equation and applying the Schauder estimates (see Theorem 9.2.2 in [Kry96]), we get

\[
\sup_{k, \epsilon} \left\| u \right\|_{C^{2,\gamma}} < \infty.
\]

\[\text{To be precise, because of the term involving } \lambda \text{ appearing in the statement of the cited result in [Kry08], we need to use the fact that } \sup_{k, \epsilon} \left\| u^{(k),\epsilon} \right\|_L^\infty < \infty \text{ (which has been noted already in (6.3)) in order to apply the result of [Kry08] and obtain (6.4).}\]
This allows us to find $u \in C^{2, \gamma} \cap k_j \uparrow \infty$, $\epsilon_j \downarrow 0$ such that $u^{(k_j), \epsilon_j} \to u$, $\partial_t u^{(k_j), \epsilon_j} \to \partial_t u$, $Du^{(k_j), \epsilon_j} \to Du$, $D^2 u^{(k_j), \epsilon_j} \to Du$ locally uniformly on $[0, T] \times \mathbb{R}^d$. Then it is clear that $u$ is the desired classical solution to (1.1).

**Proof of Theorem 2.9.** As pointed out in Remark 2.10, we need only prove existence. The proof is very similar to that of Theorem 2.8, so we are brief here. Let $\sigma^\epsilon$, $f^{(k), \epsilon}$ be defined exactly as in the proof of Theorem 2.8, and let $g'$ be a standard mollification of $g$. Then let $u^{(k), \epsilon}$ be the unique classical solution $u^{(k), \epsilon}$ to

$$
\begin{aligned}
\partial_t u^{(k), \epsilon, i} + \text{tr}(\sigma^\epsilon(t, x, u^{(k), \epsilon})D^2 u^{(k), \epsilon, i}) + f^{(k), \epsilon, i}(t, x, u^{(k), \epsilon}, Du^{(k), \epsilon}) = 0, & \quad (t, x) \in (0, T) \times \mathbb{R}^d, \\
u^{(k), \epsilon, i}(T, x) = g'^\epsilon(x), & \quad x \in \mathbb{R}^d.
\end{aligned}
$$

(6.5)

As in the proof of Theorem 2.8, we know that for some $\alpha \in (0, 1)$,

$$
\sup_{k, \epsilon} \left\| Du^{(k), \epsilon} \right\|_{L^\infty} < \infty, \quad \sup_{k, \epsilon} \left\| u^{(k), \epsilon} \right\|_{C^{0, \alpha}} < \infty
$$

This time, there is no way to bootstrap to conclude a uniform bound in $C^{2, \alpha}$. Instead, we can fix a smooth cutoff function $\rho = \rho(x) : \mathbb{R}^d \to \mathbb{R}$ with $0 \leq \rho \leq 1$ and $\rho(x) = 1$ for $|x| \leq 1$, $\rho(x) = 0$ for $|x| \geq 2$ and a smooth cutoff function $\kappa = \kappa(t) : [0, T] \times \mathbb{R}$ with $0 \leq \kappa \leq 1$, $\kappa = 1$ for $t \leq T - \delta$, $\kappa = 0$ for $t > T - \delta/2$. Then for any $x_0$, $k$, $\epsilon$ and $i$ the function $\tilde{u}^{(k), \epsilon, i}(t, x) = u^{k, \epsilon, i}(t, x)\rho(x - x_0)\kappa(t)$ satisfies a linear parabolic of the form

$$
\partial_t \tilde{u}^{(k), \epsilon, i} + \text{tr}(\tilde{a}^{(k), \epsilon, i} D^2 \tilde{u}^{(k), \epsilon, i}) + \tilde{f}^{(k), \epsilon, i} = 0, \quad \tilde{u}^{(k), \epsilon, i}(T, x) = 0,
$$

with $\tilde{a}^{(k), \epsilon, i}$ elliptic uniformly in $k$, $\epsilon$ and the estimates

$$
|\tilde{a}^{(k), \epsilon, i}(t, x) - \tilde{a}^{(k), \epsilon, i}(t, x')| \leq C|x - x'|, \quad \left\| f^{(k), \epsilon, i} \right\|_{L^\infty} \leq C
$$

holding for all $k$, $\epsilon$, $i$, with a constant $C$ independent of $x_0$. By applying Theorem 1 of Chapter 5, Section 2 of [Kry08], we get for any fixed $\delta > 0$ the estimate

$$
\sup_{k, \epsilon} \sup_{x_0 \in \mathbb{R}^d} \int_0^{T - \delta} \int_{B_1(x_0)} \left( |\partial_t u^{(k), \epsilon, i}| + |Du^{(k), \epsilon, i}| + |D^2 u^{(k), \epsilon, i}| \right) dxdt < \infty,
$$

hence by Sobolev embedding

$$
\sup_{k, \epsilon} \left( \left\| Du^{(k), \epsilon} \right\|_{L^\infty} + \left\| u^{(k), \epsilon} \right\|_{C^{0, \alpha}} + \left\| u^{(k), \epsilon} \right\|_{C^{1, \alpha}([0, T - \delta] \times \mathbb{R}^d)} \right) < \infty
$$

(6.6)

for each $\delta > 0$. This lets us find a function

$$
u \in C^{0, \alpha}([0, T] \times \mathbb{R}^d) \cap C^{1, \alpha}_{loc}([0, T] \times \mathbb{R}^d)$$
and sequences \( k_j \uparrow \infty, \epsilon_j \downarrow 0 \) such that
\[
\begin{align*}
u^{(k_j), \epsilon_j} & \to \nu \text{ locally uniformly in } [0, T] \times \mathbb{R}^d, \\
\partial u^{(k_j), \epsilon_j} & \to \partial u \text{ locally uniformly in } [0, T] \times \mathbb{R}^d, \\
 Du^{(k_j), \epsilon_j} & \to Du \text{ locally uniformly in } [0, T] \times \mathbb{R}^d,
\end{align*}
\]
and \( \| Du \|_{L^\infty} < \infty \). For simplicity, let us set
\[
u^{(j)} = \nu^{(k_j), \epsilon_j}, \quad f^{(j)} = f^{(k_j), \epsilon_j}, \quad \sigma^{(j)} = \sigma^{\epsilon_j}, \quad g^{(j)} = g^{\epsilon_j}
\]
Now fix \((t_0, x_0)\), and define processes \((X^{(j)}Y^{(j)}, Z^{(j)})\) and \((X, Y, Z)\) by
\[
X_t^{(j)} = x_0 + \int_{t_0}^t \sigma(s, X_s^{(j)}, u^{(j)}(s, X^{(j)}))dB_s, \quad t_0 \leq t \leq T
\]
\[
X_t = x_0 + \int_{t_0}^t \sigma(s, X_s, u(s, X))dB_s \quad t_0 \leq t \leq T
\]
and
\[
Y_t^{(j)} = \nu^{(k_j)}(t, X_t^{(j)}), \quad Y_t = \nu(t, X_t), \quad \sigma_t^{(j)} = \sigma(t, X_t^{(j)})Du^{(j)(t, X_t^{(j)})}, \quad Z_t = \sigma(t, X_t)Du(t, X_t).
\]
For each \( j \), Itô’s formula gives us the relation
\[
Y_t^{(j)} = g^{\epsilon_j}(X_T^{(j)}) + \int_t^T f^{(j)}(s, X_s^{(j)}, Y_s^{(j)}, Z_s^{(j)})ds - \int_t^T Z_s^{(j)}dB_s.
\]
The fact that \( u \) is Lipschitz and \( u^{(j)} \to u \) uniformly is enough to conclude that \( X^{(j)} \to X \) in \( \mathcal{S}^2 \), and then (6.7) is enough to conclude that \( Y^{(j)} \to Y \) in \( \mathcal{S}^2 \), \( Z^{(j)} \to Z \) in \( L^2 \). This is enough to pass to the limit in (6.8) and conclude that
\[
Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_sdB_s,
\]
which means that \( u \) is a decoupling solution for (1.1).

\[\square\]

**Proof of Theorem 2.12.** First, suppose that \( f, \sigma \) and \( g \) are smooth with bounded derivatives of all orders. In this case, we will establish existence via the method of continuation, and follow closely the presentation in Chapter 17 of [GT77]. Since \( g \) is smooth, we may as well assume that \( g = 0 \) (otherwise we can study the system satisfied by \( \bar{u}'(t, x) = u'(t, x) - g(x) \)). Let \( \alpha \in (0, 1) \) be as given by Theorem 2.11, and define the Banach spaces \( B_1 \) and \( B_2 \) by
\[
B_1 = \{ u \in C^{2, \alpha}([0, T] \times \mathbb{R}^d; \mathbb{R}^n) : u(T, x) = 0 \}, \quad B_2 = C^{0, \alpha}([0, T] \times \mathbb{R}^d; \mathbb{R}^n).
\]
Now fix an arbitrary \( \phi \in B_1 \), and define the functional \( F = F(u, \lambda) : B_1 \times [0, 1] \to B_2 \) by
\[
F^i(u, \lambda) = \partial_i u + (\lambda a(t, x, u, Du) + (1 - \lambda)) Du^i + f^i(t, x, u, Du).
\]
Define \( \Lambda \subset [0, 1] \) by
\[
\Lambda = \{ \lambda \in [0, 1] : F(u, \lambda) = 0 \text{ for some } u \in B_1 \}.
\]
Theorem 2.8 shows that $0 \in \Lambda$. We next claim that the a-priori estimate Theorem 2.11 implies that $\Lambda$ is closed. Indeed, if for $k \in \mathbb{N}$ we have $F(u^k, \lambda^k) = 0$ and $\lambda^k \to \lambda \in [0, T]$, then Theorem 2.11 implies that $\{u^k\}$ is compact in $C^{2,\beta}$ for any $\beta < \alpha$, and this lets us find a $u \in B_1$ such that (up to a subsequence) $u^k \to u$ in $C^{2,\beta}_{\text{loc}}([0, T] \times \mathbb{R}^d; \mathbb{R}^n)$, and so $F(u, \lambda) = 0$, and $\lambda \in \Lambda$.

To see that $\Lambda$ is open, notice that the Frechet derivative $D_u F$ of $F$ in the first argument is given by

$$
(D_u F(u, \lambda)(v))^i = \partial_i v^i + \lambda \text{tr}(a(t, x, u, Du)D^2 v^i) + (1 - \lambda)\Delta v^i + \lambda (D_u a(t, x, u, Du) \cdot v + D_p a(t, x, u, Du)) D^2 u^i + D_u f^i(t, x, u, Du) \cdot v + D_p f^i(t, x, u, Du) \cdot Dv.
$$

It follows from results on solvability of linear parabolic systems in H"older spaces that for each fixed $u \in B_1, \lambda \in [0, 1]$, the map

$$B_1 \to B_2, \quad v \mapsto D_u F(u, \lambda)(v)$$

is invertible (with bounded inverse), and so from the implicit function theorem we see that $\Lambda$ is open. We conclude that $\Lambda = [0, 1]$, and in particular $1 \in \Lambda$, which completes the proof in the case that $f, g$, and $\sigma$ have bounded derivatives of all orders. The general case can now be handled with a mollification procedure and a compactness argument, as in the proofs of Theorem 2.8 and 2.9. \qed

Proof of Theorem 2.13. As explained in Remark 2.14, we need only show existence. It is routine to check that that if $F, H, \Sigma$ and $G$ satisfy the assumptions of Theorem 2.13, then the data $\sigma, f, g$ given by (1.4) satisfy the conditions of Theorem 2.9, so we get functions $(u, v = \sigma Du)$ with the following property: with $\tilde{X}$ defined by

$$
\tilde{X}_t = x_0 + \int_0^t \sigma(s, \tilde{X}_s, u(s, \tilde{X}_s))dB_s, \quad 0 \leq t \leq T,
$$

we have

$$
u^i(t, \tilde{X}_t) = g^i(\tilde{X}_T) + \int_t^T \left( F^i(s, \tilde{X}_s, u(s, \tilde{X}_s), v(s, \tilde{X}_s))
+ v^i(s, \tilde{X}_s) \cdot \sigma^{-1}(s, \tilde{X}_s, u(s, \tilde{X}_s)) H(s, \tilde{X}_s, u(s, \tilde{X}_s), v(s, \tilde{X}_s)) ds
- \int_t^T v^i(s, \tilde{X}_s) \sigma^{-1}(s, \tilde{X}_s, u(s, \tilde{X}_s)) d\tilde{X}_s. \right)
$$

(6.10)

where $\tilde{B} = B - \int \sigma^{-1}(\cdot, \tilde{X}, u(\cdot, \tilde{X})) dt$ is a Brownian motion under the probability measure $Q$ given by $dQ = \mathcal{E}(\int \sigma^{-1}(\cdot, \tilde{X}, u(\cdot, \tilde{X})) dB) d\mathbb{P}$. Now define $X$ by

$$X_t = x_0 + \int_0^t H(s, X_s, u(s, X_s), v(s, X_s)) ds + \int_{t_0}^t \sigma(s, X_s, u(s, X_s)) dB_s, \quad 0 \leq t \leq T.$$
Now by Girsanov there is a probability measure $Q$ under which $X$ has the same law as $\tilde{X}$, so the relation
\[ u^i(t, X_t) = g^i(X_T) + \int_t^T \left( F^i(s, X_s, u(s, X_s), v(s, X_s)) + v^i(s, X_s) \cdot \sigma^{-1}(s, X_s, u(s, X_s)) H(s, X_s, u(s, X_s), v(s, X_s)) \right) ds \]
\[ - \int_t^T v^i(s, X_s) \sigma^{-1}(s, X_s, u(s, X_s)) dX_s. \] (6.11)
holds under $Q$, hence also under $P$. This is equivalent to
\[ u^i(t, X_t) = g^i(X_T) + \int_t^T F^i(s, X_s, u(s, X_s), v(s, X_s)) - \int_t^T v^i(s, X_s) dB_s, \] (6.12)
i.e. this shows that the triple $(X, Y, Z) = (X, u(\cdot, X), \sigma(\cdot, X) Du(\cdot, X))$ solves (4.3). \hfill \Box

References

[BF95] A. Bensoussan and J. Frehse, Ergodic bellman systems for stochastic games in arbitrary dimension, Proceedings: Mathematical and Physical Sciences 449 (1995), no. 1935, 65–77.

[BF00] Alain Bensoussan and Jens Frehse, Stochastic games for n players, Journal of Optimization Theory and Applications 105 (2000), 543—565.

[BF02] , Smooth solutions of systems of quasilinear parabolic equations, ESAIM: Control, Optimisation and Calculus of Variations 8 (2002), 169–193 (en). MR 1932949

[BF13] A. Bensoussan and J. Frehse, Regularity results for nonlinear elliptic systems and applications, Applied Mathematical Sciences, Springer Berlin Heidelberg, 2013.

[Del02] François Delarue, On the existence and uniqueness of solutions to FBSDEs in a non-degenerate case, Stochastic Processes and their Applications 99 (2002), no. 2, 209–286.

[Del03] François Delarue, Estimates of the solutions of a system of quasi-linear PDEs. a probabilistic scheme, pp. 290–332, Springer Berlin Heidelberg, Berlin, Heidelberg, 2003.

[DT10] F. Delbaen and S. Tang, Harmonic analysis of stochastic equations and backward stochastic differential equations, Prob. Theory Relat. Fields 146 (2010), 291 – 336.

[FH77] W. Fleming and R. Rishel, Deterministic and stochastic optimal control, Springer, 1975.

[GT77] David Gilbarg and Neil S. Trudinger, Elliptic differential equations of second order, Springer, 1977.

[HY00] Ying Hu and Jiongmin Yong, Forward–backward stochastic differential equations with nonsmooth coefficients, Stochastic Processes and their Applications 87 (2000), no. 1, 93–106.

[Jac23] Joe Jackson, Global existence for quadratic FBSDE systems and application to stochastic differential games, Electronic Communications in Probability 28 (2023), no. none, 1 – 14.

[JŽ21] Joe Jackson and Gordan Žitković, Existence and uniqueness for non-Markovian triangular quadratic BSDEs, 2021.
[Kaz94] N. Kazamaki, Continuous Exponential Martingales and BMO, Springer, 1994.

[KLT18] Michael Kupper, Peng Luo, and Ludovic Tangpi, Multidimensional Markovian FBSDEs with super-quadratic growth, Stochastic Processes and their Applications 129 (2018), 902–923.

[Kob00] M. Kobylanski, Backward stochastic differential equations and partial differential equations with quadratic growth, Ann. Probab. 28 (2000), no. 2, 558–602.

[Kry96] N.V. Krylov, Lectures on elliptic and parabolic equations in Hölder spaces, American Mathematical Society, 1996.

[Kry08] _______, Lectures on elliptic and parabolic equation in Sobolev spaces, American Mathematical Society, 2008.

[KS79] N.V. Krylov and M.V. Safonov, An estimate for the probability of a diffusion process hitting a set of positive measure, Dokl. Akad. Nauk SSSR 245 (1979), 18–20.

[LSU68] O.A. Ladyženskaja, V.A. Solonnikov, and N.N. Ural’ceva, Linear and Quasi-linear Equations of Parabolic Type, American Mathematical Society, 1968.

[LT17] Peng Luo and Ludovic Tangpi, Solvability of coupled FBSDEs with diagonally quadratic generators, Stochastics and Dynamics 17 (2017), 1750043.

[MPY94] Jin Ma, Philip Protter, and Jiongmin Yong, Solving forward-backward stochastic differential equations explicitly? a four step scheme, Probability Theory and Related Fields 98 (1994), 339–359.

[MWZZ15] Jin Ma, Zhen Wu, Detao Zhang, and Jianfeng Zhang, On well-posedness of forward-backward SDEs—a unified approach, The Annals of Applied Probability 25 (2015), no. 4, 2168–2214.

[MY07] Jin Ma and Jiongmin Yong, Forward-backward stochastic differential equations and their applications, Lecture Notes in Mathematics -Springer-verlag- 1702 (2007), 257–258.

[MZZ08a] Jin Ma, Jianfeng Zhang, and Ziyu Zheng, Weak solutions for forward-backward SDEs: A martingale problem approach, The Annals of Probability 36 (2008), no. 6, 2092–2125.

[MZZ08b] Jin Ma, Jianfeng Zhang, and Ziyu Zheng, Weak solutions for forward–backward SDEs—a martingale problem approach, The Annals of Probability 36 (2008), no. 6, 2092–2125.

[Pen99] S. Peng, Open problems on backward stochastic differential equations, Control of distributed parameter and stochastic systems (Hangzhou, 1998), Kluwer Acad. Publ., Boston, MA, 1999, pp. 265–273.

[XŽ18] Hao Xing and Gordan Žitković, A class of globally solvable Markovian quadratic BSDE systems and applications, Ann. Probab. 46 (2018), no. 1, 491–550.

[Yon99] Jiongmin Yong, Linear forward—backward stochastic differential equations, Applied Mathematics and Optimization 39 (1999), 93–119.

[Yon06] _______, Linear forward—backward stochastic differential equations with random coefficients, Probability Theory and Related Fields 135 (2006), 53–83.

[Zha17] J. Zhang, Backward Stochastic Differential Equations, Springer, 2017.