REAL REGULATORS FOR PRODUCTS OF ELLIPTIC CURVES

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Abstract. Assuming the Künneth decomposition of the Chow groups of products of very general Kummer surfaces, we prove that the Hodge-$D$-conjecture fails for the real regulator $r_{k,1}$ on a product of $n$ very general elliptic curves for $2n \geq 3k - 1 \geq 8$.

1. Introduction

Let $X$ be a projective variety and let $(k, m)$ be a pair of integers. The higher Chow groups $\text{CH}^k(X, m)$ were introduced by S. Bloch [Blo86]. For the purpose of this paper, let us give a quick definition of $\text{CH}^k(X, 1)$ as follows

$$Z^k(X, 1) = \left\{ \sum_j (f_j, Z_j) : \text{cd}_X Z_j = k - 1, \ f_j \in \mathcal{C}(Z_j)^\times \right\}$$

$$\text{CH}^k(X, 1) = \left\{ \xi = \sum_j (f_j, Z_j) \in Z^k(X, 1) : \text{div}(\xi) = \sum \text{div}(f_j) = 0 \right\}.$$  

where $Z_j$ are irreducible subvarieties of $X$ of codimension $k - 1$, $\mathcal{C}(Z_j)$ is the space of rational functions on $Z_j$ and $\text{div}(f_j)$ is the divisor on $Z_j$ defined by $f_j$. We will not explain the Tame symbol since it is not needed in this paper.

If $X$ is smooth, then similar to the cycle maps $\text{CH}^k(X) \rightarrow H^{2k}(X)$ on Chow groups, there are maps, called regulators, from the higher Chow groups of $X$ to its Deligne cohomologies (see, for example, [KLMS06, KL07]). Again, for our purpose, we just need the real regulator map

$$\text{CH}^k(X, 1) \otimes \mathbb{R} \xrightarrow{r_{k,1}} H^{n-k+1,n-k+1}(X, \mathbb{R})$$

Date: October 17, 2023.

2020 Mathematics Subject Classification. Primary 14C25; Secondary 14F43, 14J28, 14K30.

Key words and phrases. Higher Chow group, real regulator, algebraic cycle.

Both authors partially supported by grants from the Natural Sciences and Engineering Research Council of Canada.
defined on $\text{CH}^{k}(X,1)$ for a smooth projective variety $X$ of dimension $n$, which is explicitly given by

$$r_{k,1}(\xi)(\omega) = \sum_{j} \int_{Z_j} \log |f_j| \omega$$

for $\xi \in \text{CH}^{k}(X,1)$ represented by $\xi = \sum (f_j, Z_i) \in Z^{k}(X,1)$ satisfying $\text{div}(\xi) = 0$.

The Hodge-\(D\)-conjecture states that this map is surjective. It is expected to be true for varieties over $\overline{\mathbb{Q}}$. For surfaces over $\mathbb{C}$, it is known to be true for rational surfaces and general Abelian and K3 surfaces [CL05]. It fails for very general surfaces in $\mathbb{P}^3$ of degree $\geq 5$ [MS97].

Let us consider the real regulator for a product of elliptic curves.

**Conjecture 1.1.** For $n$ very general complex elliptic curves $E_{1}, E_{2}, \ldots, E_{n}$ and $X = E_{1} \times E_{2} \times \ldots \times E_{n}$, the real regulator map $r_{k,1}$ on $X$ is surjective for $k = 2$ and “trivial” (explained below) for all $2n \geq 3k - 1 \geq 8$. The triviality of $r_{k,1}$ is measured by whether its image is orthogonal to one of the subspaces

$$(1.1) \quad T_{m}(H^{2n-2k+2}(X)) = \sum_{|I|=m} \bigotimes_{i \in I} H^{1}(E_{i}) \otimes H^{2n-2k+2-m}(\prod_{j \notin I} E_{j})$$

of $H^{2n-2k+2}(X)$ for some $1 \leq m \leq n$. We expect that

$$(1.2) \quad r_{k,1}(\text{CH}_{\mathbb{R}}^{k}(X,1)) \subset T_{2r+2}(H^{2n-2k+2}(X))^\perp$$

for all $3 \leq k \leq 2r + 1$. For example, when $(k, r, n) = (3, 1, 4)$, we expect that $r_{3,1}(\text{CH}_{\mathbb{R}}^{3}(X,1)) \subset T_{4}(H^{4}(X))^\perp = (H^{1}(E_{1}) \otimes H^{1}(E_{2}) \otimes H^{1}(E_{3}) \otimes H^{1}(E_{4}))^\perp$.

It is easy to show that $r_{k,1}$ is surjective if and only if

$$(1.3) \quad \bigotimes_{i=1}^{2l} H^{1}(E_{a_{i}}) \cap H^{l+l}(E_{a_{1}} \times E_{a_{2}} \times \ldots \times E_{a_{2l}}, \mathbb{R}) \subset r_{l+1,1}(\text{CH}_{\mathbb{R}}^{l+1}(E_{a_{1}} \times E_{a_{2}} \times \ldots \times E_{a_{2l}}, 1))$$

for all $1 \leq l \leq k - 1$ and $1 \leq a_{1} < a_{2} < \ldots < a_{2l} \leq n$. This holds for $k = 2$ [CL05]. So the question here is whether $r_{k,1}$ is trivial for $3 \leq k \leq n - 1$.

In this paper, we will reduce the question regarding regulators on products of elliptic curves to those on products of Kummer surfaces. Here a *Kummer surface* is the minimal resolution of $E_{1} \times E_{2}/ \mathbb{Z} \pm 1$ for a product of elliptic curves $E_{1}$ and $E_{2}$. Roughly, if we assume a K"unneth decomposition of some Chow group of products of very general Kummer surfaces, then we have the triviality of $r_{k,1}$.
Theorem 1.2. Let $k, r$ and $m$ be integers satisfying $3 \leq k \leq 2r+1 \leq 2m-1$. Suppose that the natural map

$$
\bigoplus_{\sum d_i = k-1} \bigotimes_{j=1}^{r} \mathrm{CH}^d_Q(Y_j) \otimes \mathrm{CH}^{d_{r+1}}_{Q} \left( \prod_{j=r+1}^{m} Y_j \right) \rightarrow \mathrm{CH}^{k-1}_{Q} \left( \prod_{j=1}^{m} Y_j \right)
$$

is surjective for very general Kummer surfaces $Y_1, \ldots, Y_r, Y_{r+2}, \ldots, Y_m$ and all Kummer surfaces $Y_{r+1}$ with rank$_\mathbb{Z} \text{Pic}(Y_{r+1}) \leq 19$. Then (1.2) holds on a product $X$ of $n \leq 2m$ very general elliptic curves.

In the above theorem, for example, if $n = 2m = 4$, $r = 1$ and $k = 3$, then

$$\sum_{i=1}^{2} d_i = 2.$$

And (1.4) becomes

$$\sum_{d_1 + d_2 = 2} \mathrm{CH}^{d_1}_Q(Y_1) \otimes \mathrm{CH}^{d_2}_Q(Y_2) \rightarrow \mathrm{CH}^{2}_Q(Y_1 \times Y_2),$$

which is equivalent to the Künneth decomposition of $\mathrm{CH}^{2}_Q(Y_1 \times Y_2)$.

Keep in mind the difference between $Y_{r+1}$ and the rest of $Y_j$.

Note that if (1.2) holds for $n = n_0$, then we see that it holds for all $k + 1 \leq n \leq n_0$ by projecting $E_1 \times E_2 \times \ldots \times E_{n_0}$ to $E_1 \times E_2 \times \ldots \times E_n$. So we just have to prove the above theorem for $n = 2m$.

We will prove Theorem 1.2 in sections 2 and 3. In section 4, we will show that the Künneth decomposition of $\mathrm{CH}^{2}_Q(Y_1 \times Y_2)$ is a consequence of the Bloch-Beilinson conjecture on Abel-Jacobi maps. Hence (1.2) holds for $(k, r, n) = (3, 1, 4)$ if we assume the Bloch-Beilinson conjecture. Consequently, either the Hodge-$\mathcal{D}$-conjecture fails for $r_{3,1}$ on a product of four very general elliptic curves or the Bloch-Beilinson conjecture fails.

We work exclusively over $\mathbb{C}$ unless otherwise stated.

We are grateful to the referee for doing a splendid job.

2. Completion of Higher Chow Cycles

Roughly speaking, we will follow the same argument in [CL06]. That is, we will construct a family of products of Kummer surfaces, extend a higher Chow cycle to the whole family and use a standard monodromy argument to show that it has trivial regulator on a general fiber. First we need a generalization of [CL06, Theorem 0.1], which allows us to extend a higher Chow cycle over the family after some modification.

Theorem 2.1. Let $f : W \rightarrow \Gamma$ be a dominant morphism with connected fibers from a smooth projective variety $W$ to a smooth projective curve $\Gamma$, let
$Y_1, Y_2, \ldots, Y_r$ be smooth projective surfaces with $H^1(Y_j) = 0$ and let $k \leq 2r+1$ be a positive integer such that the natural map

\[
\bigoplus_{d_i=k-1}^{r} \bigotimes_{j=1}^{r} \text{CH}^d_j(Y_j) \otimes \text{CH}_Q^{d_{i+1}}(V) \longrightarrow \text{CH}_Q^{k-1}(\prod_{j=1}^{r} Y_j \times V)
\]

is surjective for all irreducible components $V$ of $W_t$ and all $t \in \Gamma$, where $W_t$ is the fiber of $W/\Gamma$ over $t$. Let $\xi \in \text{CH}_Q^k(\prod Y_j \times W_U, 1)$ be a higher Chow cycle defined on $\prod Y_j \times W_U = \prod Y_j \times (W \times U)$ for an open set $U \subset \Gamma$. Then there exist $\eta \in \text{CH}_Q^k(\prod Y_j \times W, 1)$ and pre-higher Chow cycles $\alpha_0, \alpha_1, \ldots, \alpha_r$ on $\prod Y_j \times W$ such that

\[
\begin{align}
\text{(2.2)} & \quad \alpha_0 \in f^*Z^k_Q(\prod Y_j \times \Gamma, 1), \\
\text{(2.3)} & \quad \alpha_i \in Z^0_Q(Y_i) \otimes Z^k_Q(\prod_{j \neq i} Y_j \times W, 1) \oplus Z^1_Q(Y_i) \otimes Z^{k-1}_Q(\prod_{j \neq i} Y_j \times W, 1)
\end{align}
\]

for $i = 1, 2, \ldots, r$, and

\[
\eta + \sum_{i=0}^{r} \alpha_i = \xi
\]

on $\prod Y_j \times W_U$, where $Z^m(X)$ is the free abelian group of Chow cycles of codimension $m$ on $X$ and $Z^m(X, 1)$ is the free abelian group of pre-higher Chow cycles defined at the beginning.

**Proof.** We can extend $\xi$ to a pre-higher Chow cycle $\tilde{\xi}$ on $\prod Y_j \times W$ with $\text{div}(\tilde{\xi})$ supported on $\prod Y_j \times W_B$ for $B = \Gamma \setminus U$. By the surjection (2.1), we may choose the completion $\tilde{\xi}$ of $\xi$, after some modification by a cycle in $Z^{k-1}_Q(\prod Y_j \times W_B, 1)$, such that

\[
\text{div}(\tilde{\xi}) = \sum_{i=1}^{r} R_i + R_0
\]

where

\[
R_i \in \bigotimes_{a=1}^{i-1} Z^2_Q(Y_a) \otimes \left( Z^0_Q(Y_i) \otimes Z^{k-2i+1}_Q(\prod_{j=i+1}^{r} Y_j \times W_B) \right. \\
\left. \oplus Z^1_Q(Y_i) \otimes Z^{k-2i}_Q(\prod_{j=i+1}^{r} Y_j \times W_B) \right)
\]

for $i = 1, 2, \ldots, r$ and

\[
R_0 \in \bigotimes_{a=1}^{r} Z^2_Q(Y_a) \otimes Z^{k-2r-1}_Q(W_B).
\]

Note that

\[
\sum_{i=0}^{r} R_i \sim_{\text{rat}} 0
\]
in $\text{CH}^k_\mathbb{Q}((\prod Y_j \times W))$.

Let us prove by induction that $R_i = \text{div}(\alpha_i)$ for $i = 1, 2, ..., r$ and some $\alpha_i$ in the space (2.3).

Starting with $R_1$, we write

$$R_1 = R_{1,0} + R_{1,1}$$

for $R_{1,0} \in \text{Z}_\mathbb{Q}^0(Y_1) \otimes \text{Z}_\mathbb{Q}^{k-1}(\prod_{j=2}^r Y_j \times W_B)$

$$R_{1,1} \in \text{Z}_\mathbb{Q}^1(Y_1) \otimes \text{Z}_\mathbb{Q}^{k-2}(\prod_{j=2}^r Y_j \times W_B)$$

Clearly, $R_{1,0}$ can be written as

$$R_{1,0} = Y_1 \otimes S$$

for some $S \in \text{Z}_\mathbb{Q}^{k-1}(\prod_{j=2}^r Y_j \times W_B)$

By intersecting $\sum R_i$ with $p \times Y_2 \times ... \times Y_r \times W$ for a point $p \in Y_1$, we see that $S \sim_{\text{rat}} 0$ in $\text{CH}^k_\mathbb{Q}((\prod_{j=2}^r Y_j \times W))$. Therefore,

$$R_{1,0} = \text{div}(\alpha_{1,0})$$

for some $\alpha_{1,0} \in \text{Z}_\mathbb{Q}^0(Y_1) \otimes \text{Z}_\mathbb{Q}^1(Y_1) \otimes \text{Z}_\mathbb{Q}^{k-2}(\prod_{j=2}^r Y_j \times W, 1)$

Hence $R_{1,0} \sim_{\text{rat}} 0$ and

$$R_{1,1} + \sum_{i=2}^r R_i + R_0 \sim_{\text{rat}} 0$$

in $\text{CH}^k_\mathbb{Q}((\prod Y_j \times W))$.

We can write

$$R_{1,1} = \sum L_a \otimes S_a$$

for some $L_a \in \text{Z}_\mathbb{Q}^1(Y_1)$ and $S_a \in \text{Z}_\mathbb{Q}^{k-2}(\prod_{j=2}^r Y_j \times W_B)$

We may assume that $L_a$ are linearly independent in $\text{CH}^1_\mathbb{Q}(Y_1)$, after further modifying $\xi$ by some cycle supported on $\prod Y_j \times W_B$.

Since $H^1(Y_1) = 0$, the intersection pairing

$$\text{CH}^1_\mathbb{Q}(Y_1) \otimes \text{CH}^1_\mathbb{Q}(Y_1) \longrightarrow H^2(Y_1, \mathbb{Q}) \cong \mathbb{Q}$$

is nondegenerate. Therefore, by intersecting $R_{1,1} + R_2 + ... + R_r + R_0$ with cycles in

$$\text{Z}_\mathbb{Q}^1(Y_1) \otimes \text{Z}_\mathbb{Q}^0(\prod_{j=2}^r Y_j \times W),$$

we obtain $S_a \sim_{\text{rat}} 0$ in $\text{CH}^{k-1}(\prod_{j=2}^r Y_j \times W)$ for all $a$. Consequently,

$$R_{1,1} = \text{div}(\alpha_{1,1})$$

for some $\alpha_{1,1} \in \text{Z}_\mathbb{Q}^1(Y_1) \otimes \text{Z}_\mathbb{Q}^{k-1}(\prod_{j=2}^r Y_j \times W, 1)$.
Then

\[ R_1 = \text{div}(\alpha_{1,0} + \alpha_{1,1}) = \text{div}(\alpha_1) \]

and hence

\[ \sum_{i=2}^{r} R_i + R_0 \sim_{\text{rat}} 0 \]

in \( \text{CH}^k_{\mathbb{Q}}(\prod Y_j \times W) \).

In this way, we can inductively show that

\[ R_i = \text{div}(\alpha_i) \]

for \( i = 1, 2, ..., r \) by intersecting \( R_i + ... + R_r + R_0 \) with cycles in

\[ Z^0_{\mathbb{Q}}(Y_i) \otimes Z^0_{\mathbb{Q}}(\prod_{j \neq i} Y_j \times W) \]

and

\[ Z^1_{\mathbb{Q}}(Y_i) \otimes Z^0_{\mathbb{Q}}(\prod_{j \neq i} Y_j \times W) \].

It follows that

\[ R_0 \sim_{\text{rat}} 0 \]

in \( \text{CH}^k_{\mathbb{Q}}(\prod Y_j \times W) \). It remains to find \( \alpha_0 \) in the space (2.2) such that \( R_0 = \text{div}(\alpha_0) \).

If \( k < 2r + 1 \), then \( R_0 = 0 \) and there is nothing to prove. Suppose that \( k = 2r + 1 \). In this case,

\[ R_0 \in Z^{2r}_{\mathbb{Q}}(Y) \otimes Z^0_{\mathbb{Q}}(W_B) \quad \text{for} \quad Y = \prod_{j=1}^{r} Y_j. \]

Let us write

\[ R_0 = \sum L_a \otimes S_a \]

where \( S_a \) are irreducible components of \( W_B \) and \( L_a \in Z^{2r}_{\mathbb{Q}}(Y) \). Let \( \mu_a \) be the multiplicity of \( S_a \) in \( W_B \). We claim that for every pair \( S_a \) and \( S_b \) with \( f(S_a) = f(S_b) \), i.e., for any two components \( S_a \) and \( S_b \) of \( W_p \) and all \( p \in B \),

\[ \mu_b L_a \sim_{\text{rat}} \mu_a L_b \]

over \( \mathbb{Q} \) on \( Y \).

Since \( W \) is smooth, the components of \( W_B \) are Cartier divisors of \( W \). We take a sufficiently ample divisor \( A \) on \( W \) and cut \( W \) by \( n-2 \) general members \( A_1, A_2, ..., A_{n-2} \in |A| \) for \( n = \dim W \). The resulting \( D = A_1 \cap A_2 \cap ... \cap A_{n-2} \) is a smooth projective surface and a flat family of curves over \( \Gamma \). The basic intersection theory on surfaces tells us that for every \( p \in B \), the intersection matrix of any \( m-1 \) irreducible components of \( D_p \) is negative definite, where \( m \) is the number of irreducible components of \( W_p \). Therefore, for any two components \( S_a \) and \( S_b \) of \( W_p \), there exists \( \Lambda \in Z^1(W) \), supported on \( W_p \), such that

\[ \Lambda.D.S \equiv 0 \quad \text{for all components} \quad S \neq S_a, S_b \quad \text{of} \quad W_p, \]

\[ \Lambda.D.S_a \neq 0, \quad \text{and} \quad \Lambda.D.(\mu_a S_a + \mu_b S_b) = 0 \]

where “\( \equiv \)” is numerical equivalence. And since \( \Lambda \) is supported on \( W_p \), we actually have \( \Lambda.D.S_i \equiv 0 \) for all \( i \neq a, b \). For simplicity, by choosing the cycle \( \Lambda \in Z^1_{\mathbb{Q}}(W) \) over \( \mathbb{Q} \), we may assume that \( \Lambda.D.S_a \equiv \mu_b \). In summary, by
letting $C = \Lambda.D$, we conclude that for every $p \in B$ and any two components $S_a$ and $S_b$ of $W_p$, we can find a 1-cycle $C \in Z^u_{Q}(W)$ such that

$$C.S_i \equiv 0 \text{ for } i \neq a, b, \ C.S_a \equiv \mu_b, \text{ and } C.S_b \equiv -\mu_a.$$

Then

$$f_*(Y \otimes C.R_0) = (C.S_a)L_a \otimes p + (C.S_b)L_b \otimes p$$

$$= ((C.S_a)L_a + (C.S_b)L_b) \otimes p$$

for $f : Y \times W \to Y \times \Gamma$. Thus

$$(C.S_a)L_a + (C.S_b)L_b \sim_{rat} 0 \Rightarrow \mu_b L_a - \mu_a L_b \sim_{rat} 0.$$ 

Therefore, $\mu_b L_a \sim_{rat} \mu_a L_b$ for all pairs of components $S_a$ and $S_b$ of $W_p$. This implies that after replacing $\xi$ by $\xi + \beta$ for some $\beta \in Z^0_{Q}(W_B) \otimes Z^{k-1}_{Q}(Y, 1)$, we may write $R_0$ as

$$R_0 = \sum_{p \in B} M_p \otimes W_p$$

where $M_p = (1/\mu_a)L_a$ for a component $S_a$ of $W_p$. Namely, $R_0 = f^*G$ for some $G \in Z^k_{Q}(Y) \otimes Z^1_{Q}(\Gamma)$. Since $R_0 \sim_{rat} 0$ on $Y \times W, G \sim_{rat} 0$ on $Y \times \Gamma$. So there exists $\alpha_0 \in f^*Z^k_{Q}(Y \times \Gamma, 1)$ such that $R_0 = \text{div}(\alpha_0)$.

In conclusion,

$$\eta = \xi - \sum_{i=0}^r \alpha_i$$

is a higher Chow cycle in $CH^k_{Q}(Y \times W, 1)$ with the required property. □

3. Products of Kummer Surfaces

We will reduce the triviality of $r_{k,1}$ on products of elliptic curves to that on products of Kummer surfaces.

For a product $E_1 \times E_2$ of two elliptic curves, we fix two involutions $\sigma_1$ and $\sigma_2$ on $E_i$ and let $E_1 \times E_2/\sigma_1 \times \sigma_2$ be the quotient of $E_1 \times E_2$ by the action $\sigma_1 \times \sigma_2$. Usually, we simply write it as $E_1 \times E_2/_{\pm 1}$. Note that the action of $\sigma_1 \times \sigma_2$ is invariant on $H^2(E_1 \times E_2)$, i.e.,

$$\ (\sigma_1 \times \sigma_2)\omega = \omega \quad \text{for all } \omega \in H^2(E_1 \times E_2)$$

The resulting surface $E_1 \times E_2/_{\pm 1}$ has 16 ordinary double points, corresponding to 16 fixed points of $\sigma_1 \times \sigma_2$. Blowing up at the 16 double points, we obtain a Kummer K3 surface $Y$. Indeed, we have a diagram

$$\begin{array}{ccc}
Z & \xrightarrow{f} & Y \\
\downarrow g & & \downarrow \\
X & \longrightarrow & X/_{\pm 1}
\end{array}$$

where $X = E_1 \times E_2$, $Z$ is the blowup of $X$ at the 16 fixed points of $\sigma_1 \times \sigma_2$ and $f$ is a finite morphism of degree 2 ramified at the 16 exceptional divisors of $Z \to X$. The action $\sigma_1 \times \sigma_2$ on $X$ extends to the Galois action $\sigma$ on
Clearly, $\sigma$ preserves the 16 exceptional divisors of $g$. Combining this with (3.1), we see that

$$\sigma(\omega) = \omega \quad \text{for all } \omega \in H^2(Z).$$

Thus, we have

$$f^*f_*\omega = \omega + \sigma(\omega) = 2\omega \quad \text{for all } \omega \in H^2(Z).$$

Combined with the projection formula $f_*f^*\omega = 2\omega$, (3.3) implies that $f^*$ and $f_*$ are isomorphisms between $H^2(Y)$ and $H^2(Z)$ satisfying (with $\deg f = 2$)

$$H^2(Y) \xrightarrow{f^*} H^2(Z) \cong H^2(Y) \xrightarrow{f_*} H^2(Z).$$

It follows that $(1/\deg f)f_* = (f^*)^{-1}$ preserves the intersection pairing and hence

$$\langle f_*\alpha, f_*\beta \rangle = (\deg f)f_*\langle \alpha, \beta \rangle$$

for all $\alpha, \beta \in H^2(Z)$.

Furthermore, $f^*$ and $f_*$ induce isomorphisms between the $\mathbb{Q}$-Hodge structures on $H^2(Y, \mathbb{Q})$ and $H^2(Z, \mathbb{Q})$. Thus they induce isomorphisms between the algebraic/transcendental parts of $H^2(Y)$ and $H^2(Z)$.

We define

$$H^2_{tr}(Y) = f_*g^*(H^1(E_1) \otimes H^1(E_2)).$$

Strictly speaking, this is not exactly the transcendental part of $H^2(Y)$. It is the subspace orthogonal to the 18 algebraic classes of $H^2(Y)$ corresponding to the two fibers of $E_1 \times E_2$ over $E_i$ and 16 exceptional divisors of $g$. For very general $E_1$ and $E_2$, this is the transcendental part of $H^2(Y)$. For arbitrary $E_1$ and $E_2$, it contains the transcendental part of $H^2(Y)$ as a subspace.

Based on the above observations, we have

**Proposition 3.1.** Let $E_1, E_2, \ldots, E_{2m}$ be $n = 2m$ elliptic curves and let $Y_{ij}$ be the Kummer surface birational to $E_i \times E_j/\pm 1$. Then (1.2) holds if the real regulator $r_{k,1}$ on

$$Y = Y_{a_1a_2} \times Y_{a_3a_4} \times \ldots \times Y_{a_{2m-1}a_{2m}}$$

satisfies

$$r_{k,1}(\text{CH}^k_{\mathbb{R}}(Y, 1)) \subset \left( \bigotimes_{i=1}^{r+1} H^2_{tr}(Y_{a_2i-1a_2i}) \otimes H^{4m-2k-2r} \left( \prod_{i=r+2}^m Y_{a_{2i-1}a_{2i}} \right) \right)^\perp$$

for all $\{a_1, a_2, \ldots, a_{2m}\} = \{1, 2, \ldots, 2m\}$, where $H^2_{tr}(Y_{ij})$ is the subspace of $H^2(Y_{ij})$ defined by (3.6).
Proof. Clearly, $T_{2r+2}(H^{4m-2k+2}(X))$ is spanned by the forms

$$\omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_{2r+2} \otimes \eta$$

for some $\omega_i \in H^1(E_{a_i})$ and

$$\eta \in H^{4m-2k-2r}(\prod_{i \neq a_1, \ldots, a_{2r+2}} E_i).$$

It suffices to prove that

$$r_{k,1}(\text{CH}^k_a(X, 1)) \subset (\omega \otimes \eta)^\perp$$

(3.7)

For simplicity, we may assume that $(a_1, a_2, \ldots, a_{2r+2}) = (1, 2, \ldots, 2r + 2)$.

Let $Z_{ij}$ be the blowup of $E_i \times E_j$ at the 16 fixed points and let

$$Y = Y_{12} \times Y_{34} \times \ldots \times Y_{2m-1,2m}$$

and

$$Z = Z_{12} \times Z_{34} \times \ldots \times Z_{2m-1,2m}.$$

We have the commutative diagram (3.2).

By (3.4) and (3.5),

$$\langle r_{k,1}(\xi), \omega \otimes \eta \rangle = g^* \langle r_{k,1}(\xi), \omega \otimes \eta \rangle = (g^* (r_{k,1}(\xi)), g^* (\omega \otimes \eta))$$

$$= f_*(r_{k,1}(g^* \xi), g^* (\omega \otimes \eta)) = \frac{1}{\text{deg } f} (f_*(r_{k,1}(g^* \xi)), f_*(g^* (\omega \otimes \eta)))$$

$$= \frac{1}{\text{deg } f} \langle r_{k,1}(f_*(g^* \xi)), f_*(g^* (\omega \otimes \eta)) \rangle = 0$$

for all $\xi \in \text{CH}^k_a(X, 1)$ and (3.7) follows. \hfill \square

By the above proposition, to prove the triviality of $r_{k,1}$ on a product of $2m$ very general elliptic curves in our main Theorem 1.2, it suffices to prove

$$r_{k,1}(\text{CH}^k_a(\prod_{i=1}^m Y_i, 1)) \subset \left(\bigotimes_{i=1}^{r+1} H^2_{tr}(Y_i) \otimes H^{4m-2k-2r}(\prod_{i=r+2}^m Y_i)\right)^\perp$$

(3.8)

for a product of $m$ very general Kummer surfaces $Y_1, Y_2, \ldots, Y_m$.

Now let us try to use the argument in [CL06] to prove (3.8) and thus Theorem 1.2. As in [CL06], we first construct a one-parameter family of Kummer surfaces with “nice” singularities.

We start with the construction of two flat projective families $S/B$ and $T/B$ of curves over a smooth projective curve $B$ satisfying

- $S$ and $T$ are smooth,
- there is a nonempty finite set $\Sigma \subset B$ such that $S_b$ and $T_b$ are rational curves with a node for $b \in \Sigma$ and they are smooth elliptic curves for $b \not\in \Sigma$,
- $S_b \times T_b$ is a general product of two elliptic curves for $b \in B$ general,
- $h^{1,1}(S_b \times T_b, \mathbb{Q}) := \dim H^{1,1}(S_b \times T_b, \mathbb{Q}) \leq 3$ for all $b \in B \setminus \Sigma$,
- and both $S/B$ and $T/B$ have sections.
By (3.9), the one-parameter family of Kummer surfaces constructed from $S \times_B T$ is generically of Picard rank 18 and of Picard rank 19 (but never 20) at finitely many points of $B \setminus \Sigma$.

It is not hard to construct such $S/B$ and $T/B$ individually. The difficulty is that we have to make sure that $S/B$ and $T/B$ are singular over the same points $b \in B$. Here is one construction.

We let $G \subset \mathbb{P}^2 \times \mathbb{P}^1$ be a general pencil of cubic curves. It is well known that $G/\mathbb{P}^1$ has exactly 12 nodal fibers over $p_1, p_2, \ldots, p_{12} \in \mathbb{P}^1$. We choose two different morphisms $g_i : \mathbb{P}^1 \to \mathbb{P}^1$ of degree 12 that map all $p_1, p_2, \ldots, p_{12}$ to the same point $q \in \mathbb{P}^1$, i.e.,

$$g_i^*(q) = p_1 + p_2 + \ldots + p_{12} \quad \text{for } i = 1, 2.$$ 

Let $B$ be the normalization of the fiber product of $g_1 : \mathbb{P}^1 \to \mathbb{P}^1$ and $g_2 : \mathbb{P}^1 \to \mathbb{P}^1$ with the diagram

$$
\begin{array}{c}
B \\
\pi_1 \\
\pi_2 \\
\pi \\
\mathbb{P}^1 \\
g_1 \\
g_2 \\
\pi_1 \\
\pi_2 \\
\pi \\
\mathbb{P}^1 \\
\mathbb{P}^1
\end{array}
$$

Then $$\Sigma = \pi^{-1}(q) = \pi_1^{-1}\{p_1, p_2, \ldots, p_{12}\} = \pi_2^{-1}\{p_1, p_2, \ldots, p_{12}\}.$$ 

Indeed, for general choices of $g_1$ and $g_2$, $B$ is irreducible and $\pi : B \to \mathbb{P}^1$ has degree 144.

Let $S = G \times_{\mathbb{P}^1} B$ be the fiber product of $G \to \mathbb{P}^1$ and $\pi_1 : B \to \mathbb{P}^1$ and let $T = G \times_{\mathbb{P}^1} B$ be the fiber product of $G \to \mathbb{P}^1$ and $\pi_2 : B \to \mathbb{P}^1$. It is not hard to see that $S/B$ and $T/B$ have the required properties for very general choices of $g_1$ and $g_2$: (3.9) holds since there are only countably many products of elliptic curves $E \times F$ with $h^{1,1}(E \times F, \mathbb{Q}) = 4$; it is easy to choose $g_i$ such that $S_b \times T_b$ is not one of them for all $b \in B \setminus \Sigma$; the pencil $G/\mathbb{P}^1$ has infinitely many sections so the same holds for both $S/B$ and $T/B$. This shows the existence of such $S/B$ and $T/B$.

Since $S/B$ has a section, we have an involution $\sigma_S : S/B \to S/B$ defined on smooth fibers of $S/B$. This involution extends to singular fibers of $S/B$ as well: for a nodal fiber $S_b$, it extends to an automorphism $S_b \to S_b$ fixing three points including the node. Indeed, this is the Galois action induced by a degree 2 map $S_b \to \mathbb{P}^1$. So we have an automorphism $\sigma_S : S \to S$ preserving the base $B$ of order 2. The fixed locus of $\sigma_S$ consists of a multisection of $S/B$ which meets each smooth fiber transversely at 4 points and each singular fiber at 3 points including the node. Of course, the same holds for $T/B$ and we have an involution $\sigma_T : T/B \to T/B$.

Let $R = S \times_B T/\sigma$ for $\sigma = \sigma_S \times \sigma_T$. After resolving the singularities of $R$, we obtain a family of Kummer surfaces over $B$. Let $Y_1, \ldots, Y_r, Y_{r+2}, \ldots, Y_m$
be $m - 1$ very general Kummer surfaces and let us try to prove (3.8) where $Y_{r+1}$ is the Kummer surface birational to $R_t$ for $t \in B$ general. If (3.8) fails, then there exist a finite base change $\phi : \Gamma \to B$, a desingularization $Z \to R \times_B \Gamma$ and a higher Chow cycle

$$\xi \in \text{CH}^k_{\mathbb{Q}}(\prod_{i=1}^r Y_i \times Z_U \times \prod_{i=r+2}^m Y_i, 1)$$

over a nonempty open set $U \subset \Gamma$ such that

- $Z_t$ is a Kummer surface birational to $R_{\phi(t)}$ for $t \notin \phi^{-1}(\Sigma)$, and
- for every $t \in U$,

$$r_{k,1}(\xi_t) \notin (\bigotimes_{i=1}^r H^2_{tr}(Y_i) \otimes H^2_{tr}(Z_t) \otimes H^4m-2k-2r(\prod_{i=r+2}^m Y_i))^\perp.$$  

We claim that we can choose $Z_t$ after a further finite base change $\Gamma' \to \Gamma$, such that every irreducible component of $Z_t$ is a smooth rational surface for all $t \in \phi^{-1}(\Sigma)$. Namely, we claim

**Proposition 3.2.** Let $R = R_{S \times_B T/\sigma}$ be constructed as above and let $\phi : \Gamma \to B$ be a finite morphism from a smooth projective curve $\Gamma$ to $B$ such that the ramification index of $\phi$ at each point of $\phi^{-1}(\Sigma)$ is even. Then there exists a desingularization $Z \to R \times_B \Gamma$ such that every irreducible component of $Z_t$ is a smooth rational surface for all $t \in \phi^{-1}(\Sigma)$.

The hypothesis on the ramification index in the above proposition can be easily met by a further finite base change $\Gamma' \to \Gamma$. Assuming Proposition 3.2, let us finish the proof of (3.8).

For $t \notin \phi^{-1}(\Sigma)$, by (3.9), we have $\text{rank}_{\mathbb{Z}} \text{Pic}(Z_t) \leq 19$. Therefore, by our hypothesis (1.4), the map

(3.10) $$\bigotimes_{i=1}^r \text{CH}^*_\mathbb{Q}(Y_i) \otimes \text{CH}^*_\mathbb{Q}(Z_t \times \prod_{i=r+2}^m Y_i) \longrightarrow \text{CH}^{k-1}_\mathbb{Q}(\prod_{i=1}^r Y_i \times Z_t \times \prod_{i=r+2}^m Y_i)$$

is surjective for all $t \notin \phi^{-1}(\Sigma)$.

For $t \in \phi^{-1}(\Sigma)$, every irreducible component $P \subset Z_t$ is a smooth rational surface by Proposition 3.2. The Chow groups of $P \times X$ have Künneth decomposition

(3.11) $$\text{CH}^*(P \times X) = \text{CH}^*(P) \otimes \text{CH}^*(X)$$

for every smooth rational projective surface $P$ and every smooth projective variety $X$. 
Let \( Y_{r+1} \) be a general Kummer surface. Choosing a finite morphism \( g : Y_{r+1} \to \mathbb{P}^2 \), we have the diagram
\[
\bigotimes_{i=1}^r \text{CH}^*_{Q}(Y_i) \otimes \text{CH}^*_{Q}(\prod_{i=r+1}^m Y_i) \longrightarrow \text{CH}^{k-1}_{Q}(\prod_{i=1}^m Y_i)
\]
\[
\downarrow g_* \quad \downarrow g_*
\]
\[
\bigotimes_{i=1}^r \text{CH}^*_{Q}(Y_i) \otimes \text{CH}^*_{Q}(\mathbb{P}^2 \times \prod_{i=r+1}^m Y_i) \longrightarrow \text{CH}^{k-1}_{Q}(\prod_{i=1}^r Y_i \times \mathbb{P}^2 \times \prod_{i=r+1}^m Y_i)
\]

Clearly, we see from the above diagram that its bottom row is also surjective. Combining this with the K"unneth decomposition (3.11), we have surjections
\[
(3.12) \quad \bigotimes_{i=1}^r \text{CH}^*_{Q}(Y_i) \otimes \text{CH}^*_{Q}(\prod_{i=r+1}^m Y_i) \longrightarrow \text{CH}^d_{Q}(\prod_{i=1}^r Y_i \times \prod_{i=r+1}^m Y_i)
\]
for \( d = k - 1, k - 2, k - 3 \). Then we obtain the surjection
\[
(3.13) \quad \bigotimes_{i=1}^r \text{CH}^*_{Q}(Y_i) \otimes \text{CH}^*_{Q}(P \times \prod_{i=r+1}^m Y_i) \longrightarrow \text{CH}^{k-1}_{Q}(\prod_{i=1}^r Y_i \times P \times \prod_{i=r+1}^m Y_i)
\]
from (3.11) and (3.12) for every irreducible component \( P \subset Z_t \) and all \( t \in \phi^{-1}(\Sigma) \).

Combining (3.10) and (3.13), we see that the map (2.1) is surjective in Theorem 2.1 for every irreducible component \( V \) of \( W_t \) and all \( t \in \Gamma \) with
\[
W = Z \times \prod_{i=r+2}^m Y_i.
\]

So we can apply the theorem and obtain a higher Chow class
\[
\eta \in \text{CH}^k_{Q}(\prod_{i=1}^r Y_i \times Z \times \prod_{i=r+2}^m Y_i, 1)
\]
and pre-higher Chow cycles \( \alpha_0, \alpha_1, ..., \alpha_r \) as in the theorem such that
\[
\eta - \sum_{i=0}^r \alpha_i = \xi
\]
on \( Y_1 \times \ldots \times Y_r \times Z_{U} \times Y_{r+2} \times \ldots \times Y_m \). For \( \alpha_0, \alpha_1, ..., \alpha_r \) given in Theorem 2.1, it follows from the explicit regulator formula applied to the precycles that
\[
r_{k,1}(\eta_t) - r_{k,1}(\xi_t) \in \left( \bigotimes_{i=1}^r H^2_{tr}(Y_i) \otimes H^2_{tr}(Z_t) \otimes H^{4m-2k-2r}(\prod_{i=r+2}^m Y_i) \right)^{1}
\]
for all \( t \in U \). Then a standard monodromy argument shows that \( r_{k,1}(\xi_t) \) is trivial for \( t \in U \) general (see, for example, [CL06]). We will sketch this argument at the end of this section.

It remains to prove Proposition 3.2. This is achieved by finding an explicit resolution of the singularities of \( R \times_B \Gamma \).

**Proof of Proposition 3.2.** The problem is local at every point \( \phi^{-1}(\Sigma) \). Let us replace \( \Gamma \) by a disk centered at a point \( 0 \in \phi^{-1}(\Sigma) \). So \( S \times_B \Gamma \) and \( T \times_B \Gamma \) have singularities of type \( xy = t^{2m} \) at the nodes of \( S_{\phi(0)} \) and \( T_{\phi(0)} \), respectively, where \( 2m \) is the ramification index of \( \phi \) at 0. Let \( \hat{S} \) and \( \hat{T} \) be the minimal resolution of \( S \times_B \Gamma \) and \( T \times_B \Gamma \), respectively.

The central fiber
\[
\hat{S}_0 = C_0 \cup C_1 \cup C_2 \cup \ldots \cup C_{2m-1}
\]
of \( \hat{S}/\Gamma \) is a union of \( 2m \) smooth rational curves of simple normal crossings whose dual graph is a circle, where \( C_0 \) is the proper transform of \( S_{\phi(0)} \) and \( C_i \cap C_{i+1} \neq \emptyset \) for \( i = 0, 1, \ldots, 2m - 1 \) with \( C_{2m} = C_0 \).

It is easy to see that the involution \( \sigma_S : S \to S \) lifts to an involution \( \hat{\sigma}_S : \hat{S} \to \hat{S} \) whose action on \( \hat{S}_0 \) is given by
\[
\hat{\sigma}_S(C_0 \cap C_1) = C_{2m-1} \cap C_0, \quad \hat{\sigma}_S(C_1 \cap C_2) = C_{2m-2} \cap C_{2m-1}, \\
\hat{\sigma}_S(C_2 \cap C_3) = C_{2m-3} \cap C_{2m-2}, \ldots, \quad \hat{\sigma}_S(C_{m-1} \cap C_m) = C_m \cap C_{m+1}
\]
In the case of \( m = 1 \), \( \hat{\sigma}_S \) switches the two intersections of \( C_0 \) and \( C_1 \). The fixed locus \( \hat{\sigma}_S \) consists of four disjoint sections \( P_1, P_2, P_3, P_4 \) meeting \( C_0 \) and \( P_1 \) and \( P_2 \) meeting \( C_0 \) and \( P_3 \) and \( P_4 \) meeting \( C_m \).

The exact same holds for \( \hat{T} \):
\[
\hat{T}_0 = D_0 \cup D_1 \cup D_2 \cup \ldots \cup D_{2m-1}
\]
is a union of \( 2m \) smooth rational curves of simple normal crossings whose dual graph is a circle and the involution \( \sigma_T : T \to T \) lifts to an involution \( \hat{\sigma}_T : \hat{T} \to \hat{T} \) whose fixed locus consists of four disjoint sections \( Q_1, Q_2, Q_3, Q_4 \) of \( \hat{T}/\Gamma \).

Let \( \hat{\sigma} = \hat{\sigma}_S \times \hat{\sigma}_T \) be the involution on \( \hat{S} \times \hat{T} \). Then the singular locus of \( \hat{S} \times \hat{T}/\hat{\sigma} \) consists of the images of the 16 sections \( P_i \times Q_j \) and \( 4m^2 \) isolated points \( (C_a \cap C_{a+1}) \times (D_b \cap D_{b+1}) \). At each point among
\[
(C_a \cap C_{a+1}) \times (D_b \cap D_{b+1}),
\]
\( \hat{\sigma}_S \times \hat{\sigma}_T \) has a 3-fold rational double point \( xy = zw = t \); the same is true for \( \hat{S} \times \hat{T}/\hat{\sigma} \) at the images of \( (C_a \cap C_{a+1}) \times (D_b \cap D_{b+1}) \). So we can easily resolve the singularities of \( \hat{S} \times \hat{T}/\hat{\sigma} \) by blowing it up along its singular locus. Let \( Z \) be the resulting blowup. Clearly, all components of \( Z_0 \) are smooth rational surfaces. So we have obtained a resolution of \( R \times_B \Gamma \) with
the required property via the diagram
\[
\begin{array}{ccc}
Z & \longrightarrow & \hat{S} \times_{\Gamma} \hat{T}/\hat{\sigma} \\
\downarrow & & \downarrow \\
(S \times_B T/\sigma) \times_B \Gamma & = & R \times_B \Gamma
\end{array}
\]

\[\Box\]

We will outline the monodromy argument. To set this up, suppose that we have a smooth projective family \(Z/U\) of Kummer surfaces of maximal moduli over a smooth quasi-projective surface \(U\) and a higher Chow class
\[
\xi \in \text{CH}^k_{Q}(\prod_{i=1}^r Y_i \times Z \times \prod_{i=r+2}^m Y_i, 1).
\]
We want to show that \(r_{k,1}(\xi_t)\) is trivial for \(b \in U\) general.

Given our construction of the one parameter family \(S \times_B T\), after a base change, we can find a morphism \(C \rightarrow U\) from a smooth quasi-projective curve \(C\) to \(U\) whose image passing through a general point of \(U\) with the following property. The one-parameter family \(Z_C = Z \times_U C\) over \(C\) can be extended to a family \(Z_{\overline{C}}\) of Kummer surfaces over the completion \(\overline{C}\) of \(C\) such that \(Z_{\overline{C}}\) is smooth and the pullback \(\xi_C\) of \(\xi\) to \(Z_C\) can be extended to a higher Chow class \(\eta_{\overline{C}} \in \text{CH}^k_{Q}(Z_{\overline{C}}, 1)\) satisfying
\[
(3.14)\ r_{k,1}(\eta_t) - r_{k,1}(\xi_t) \in \left(\bigotimes_{i=1}^r H^2_{\text{tr}}(Y_i) \otimes H^2_{\text{tr}}(Z_t) \otimes H^{4m-2k-2r}(\prod_{i=r+2}^m Y_i)\right)_{\perp}
\]
for all \(t \in C\). Actually, (3.14) holds for the full regulator \(\text{cl}_{k,1}\). That is,
\[
(3.15)\ \tilde{\text{cl}}_{k,1}(\eta_t) - \tilde{\text{cl}}_{k,1}(\xi_t) \in \left(\bigotimes_{i=1}^r H^2_{\text{tr}}(Y_i) \otimes H^2_{\text{tr}}(Z_t) \otimes H^{4m-2k-2r}(\prod_{i=r+2}^m Y_i)\right)_{\perp}
\]
where \(\tilde{\text{cl}}_{k,1}(\eta_t)\) and \(\tilde{\text{cl}}_{k,1}(\xi_t)\) are local lifts of \(\text{cl}_{k,1}(\eta_t)\) and \(\text{cl}_{k,1}(\xi_t)\), respectively. The Gauss-Manin connection \(\nabla\) on \(Y_1 \times \ldots \times Y_r \times Z_C \times Y_{r+2} \times \ldots \times Y_m/C\) acts on \(\tilde{\text{cl}}_{k,1}(\eta_t)\) and \(\tilde{\text{cl}}_{k,1}(\xi_t)\) (see, for example, [CDKL16]).

Let us fix a class
\[
(3.16)\ \omega \in \bigotimes_{i=1}^r H^2_{\text{tr}}(Y_i) \otimes H^{4m-2k-2r}(\prod_{i=r+2}^m Y_i).
\]
Then by (3.15),
\[
(3.17)\ \pi_*(\tilde{\text{cl}}_{k,1}(\eta_t) \wedge \omega - \tilde{\text{cl}}_{k,1}(\xi_t) \wedge \omega) \in H^2_{\text{tr}}(Z_t)_{\perp}
\]
for all \(t \in C\), where \(\pi\) is the projection \(Y_1 \times \ldots \times Y_r \times Z \times Y_{r+2} \times \ldots \times Y_m \rightarrow Z\).

It follows from (3.17) that
\[
(3.18)\ \nabla((\pi_*(\tilde{\text{cl}}_{k,1}(\eta_t) \wedge \omega - \tilde{\text{cl}}_{k,1}(\xi_t) \wedge \omega)) = 0
\]
for the Gauss-Manin connection $\nabla$ on $Z_C/C$. Since $\tilde{c}_{k,1}(\eta)$ is the restriction of $c_{k,1}(\eta)$ defined on the smooth projective variety $Z_C$, we have

$$\nabla \left( \pi_* (\tilde{c}_{k,1}(\eta) \wedge \omega) \right) = 0.$$  

(3.19)

Combining (3.18) and (3.19), we obtain

$$\nabla \left( \pi_* (\tilde{c}_{k,1}(\xi_t) \wedge \omega) \right) = 0$$

on $Z_C/C$.

By our construction of $S \times_B T$, we can choose two such curves $C_i$ with two points $p_i \in C_i$ and maps $f_i : C_i \to U$ for $i = 1, 2$ satisfying that $f_1(p_1) = f_2(p_2) = b$ and the differential maps $df_i$ of $f_i$ on the tangent spaces of $C_i$ at $p_i$ satisfy that

$$\nabla \left( \pi_* (\tilde{c}_{k,1}(\xi_b) \wedge \omega) \right) = 0$$

on $Z/C$.

Then by (3.21), we see that (3.20) actually holds on $Z/U$. Namely,

$$\nabla \left( \pi_* (\tilde{c}_{k,1}(\xi_b) \wedge \omega) \right) = 0$$

on $Z/U$ for the Gauss-Manin connection $\nabla$ on $Z/U$. And since $Z/U$ is a complete family of Kummer surfaces, (3.22) implies that

$$\pi_* (\tilde{c}_{k,1}(\xi_b) \wedge \omega) \in H^2_{tr}(Z_b)^\perp$$

for all $b \in U$. And since (3.23) holds for all $\omega$ in the space (3.16), we conclude that

$$r_{k,1}(\xi_b) \in \left( \bigotimes_{i=1}^{r} H^2_{tr}(Y_i) \otimes H^2_{tr}(Z_b) \otimes H^{4m-2k-2r}( \prod_{i=r+2}^{m} Y_i) \right)^\perp$$

for all $b \in U$.

4. Bloch-Beilinson Conjecture on Abel-Jacobi Maps

The following conjecture stated in [Lew01], can be thought of as a variant of the Bloch-Beilinson conjecture:

**Conjecture 4.1.** Let $V/\overline{Q}$ be a smooth quasiprojective variety. Then the Abel-Jacobi map $\Phi_{k, Q} : CH^k_{hom}(V/\overline{Q}; Q) \to J^k(V(\mathbb{C})) \otimes Q$ is injective.

Here the definition of the Abel-Jacobi map for smooth quasiprojective varieties, which is an extension of Griffiths’ prescription, involves Carlson’s extension class interpretation of intermediate jacobians ([Car80]). A detailed description of this map for example can be found in [Jan90, §9]. We now make use of the following result:

**Theorem 4.2.** ([Lew01]) Assume given a smooth projective variety $X/C$. Then for all $k$, there is a filtration

$$CH^k(X; \mathbb{Q}) = F^0 \supset F^1 \supset \cdots \supset F^\ell \supset F^{\ell+1} \supset \cdots \supset F^k \supset F^{k+1} = F^{k+2} = \cdots,$$
which satisfies the following

(i) $F^1 = \text{CH}^k_{\text{hom}}(X; \mathbb{Q})$

(ii) $F^2 \subset \ker \Phi_{k,Q} : \text{CH}^k_{\text{hom}}(X; \mathbb{Q}) \to J^k(X) \otimes \mathbb{Q}$.

(iii) $F^\ell \bullet F^r \subset F^{\ell+r}$, where $\bullet$ is the intersection product.

(iv) $F^\ell$ is preserved under push-forwards $f_*$ and pull-backs $f^*$, where $f : X \to Y$ is a morphism of smooth projective varieties. [In short, $F^\ell$ is preserved under the action of correspondences between smooth projective varieties.]

(v) $\text{Gr}_F := F^\ell/F^\ell+1$ factors through the Grothendieck motive. More specifically, let us assume that the Künneth components of the diagonal class $[\Delta] = \bigoplus_{p+q=2n}[\Delta(p,q)] \in H^{2n}(X \times X, \mathbb{Q})$ are algebraic. Then

$$\Delta(2n - 2k + r, 2k - r)_* \big|_{\text{Gr}_F \text{CH}^k(X; \mathbb{Q})} = \begin{cases} \text{Identity} & \text{if } r = \ell \\ 0 & \text{otherwise} \end{cases}$$

(vi) Let $D^k(X) := \bigcap \ell F^\ell$. If Conjecture 4.1 above holds, then $D^k(X) = 0$.

Using Theorem 4.2, it was proved in [CL06, Lemma 3.2] that if Conjecture 4.1 holds, $\text{CH}^2(X \times Y)$ has Künneth decomposition for a product $X \times Y$ of two smooth projective surfaces satisfying $H^1(X) = H^1(Y) = 0$ and

$$H^2(X, \mathbb{Q}) \otimes H^2(Y, \mathbb{Q})) \cap H^{2,2}(X \times Y) = H^{1,1}(X, \mathbb{Q}) \otimes H^{1,1}(Y, \mathbb{Q}).$$

(4.1)

Let us verify (4.1) for a very general Kummer surface $X$ and a Kummer surface $Y$ with $\text{rank}_{\mathbb{Q}} \text{Pic}(Y) \leq 19$. Actually, we have

**Proposition 4.3.** Let $\pi : X \to B$ be a non-isotrivial smooth family of K3 surfaces over a smooth variety $B$ and let $Y$ be a smooth K3 surface. Then

$$H^2(X_b, \mathbb{Q}) \otimes H^2(Y, \mathbb{Q})) \cap H^{2,2}(X_b \times Y) = H^{1,1}(X_b, \mathbb{Q}) \otimes H^{1,1}(Y, \mathbb{Q})$$

for $b \in B$ very general. In particular, the identity (4.1) holds for the product of a very general Kummer surface and an arbitrary smooth K3 surface.

**Proof.** It suffices to prove

$$H^{1,1}(X_b, \mathbb{Q})^\perp \otimes H^{1,1}(Y, \mathbb{Q})^\perp \cap H^{2,2}(X_b \times Y) = 0$$

(4.2)

for $b \in B$ very general, where $H^{1,1}(X_b, \mathbb{Q})^\perp$ and $H^{1,1}(Y, \mathbb{Q})^\perp$ are the orthogonal complements of $H^{1,1}(X_b, \mathbb{Q})$ and $H^{1,1}(Y, \mathbb{Q})$ in $H^2(X_b, \mathbb{Q})$ and $H^2(Y, \mathbb{Q})$, respectively.

We may take $B$ to be a polydisk and assume that the Kodaira-Spencer map

$$T_{B,b} \longrightarrow H^1(T_{X_b})$$

is nonzero at all $b \in B$.

If (4.2) fails, after shrinking $B$, there exists

$$\eta \in H^0(B, (R^2\pi_*\mathcal{Q})_{\text{tr}}) \otimes H^{1,1}(Y, \mathbb{Q})^\perp$$
such that
\[ \eta_b \neq 0 \in H^{2,2}(X_b \times Y) \]
for all \( b \in B \), where \((R^2\pi_*\mathbb{Q})_{\text{tr}}\) is the subsheaf of \( R^2\pi_*\mathbb{Q} \) orthogonal to the relative algebraic cycles of \( X/B \).

Since \( \eta_b \) is orthogonal to \( F_1^r \), we have
\[ \langle \eta, \gamma \otimes \omega \rangle = 0 \]
for all \( \gamma \in H^0(B, F_1^r R^2\pi_*\mathbb{C}) \), where \( \omega \) is a nonvanishing holomorphic 2-form on \( Y \). Applying the Gauss-Manin connection, we obtain
\[ \langle \eta, \nabla \gamma \otimes \omega \rangle = 0 \]
where we observe that \( \nabla \eta = 0 \). Since the Kodaira-Spencer map of \( \pi \) is nonzero, we have
\[ \nabla(F^1 R^2\pi_*\mathbb{C}) \not\subset F_1^r R^2\pi_*\mathbb{C} \otimes \Omega_B \]
due to the fact that the pairing \( H^{1,1}(X_b) \otimes H^1(T_{X_b}) \rightarrow H^{0,2}(X_b) \) is nondegenerate. Thus, we conclude
\[ \langle \eta_b, \xi_b \otimes \omega \rangle = 0 \]
for all \( \xi_b \in H^2(X_b) \) and \( b \in B \). That is,
\[ \eta_b \in (H^2(X_b) \otimes H^{2,0}(Y))^\perp. \]

But we know that
\[
\begin{align*}
(H^{1,1}(X_b, \mathbb{Q})^\perp \otimes H^{1,1}(Y, \mathbb{Q})^\perp) & \cap (H^2(X_b) \otimes H^{2,0}(Y))^\perp \\
= H^{1,1}(X_b, \mathbb{Q})^\perp & \otimes (H^{1,1}(Y, \mathbb{Q})^\perp \cap H^{1,1}(Y)) \\
= H^{1,1}(X_b, \mathbb{Q})^\perp & \otimes (H^{1,1}(Y, \mathbb{Q})^\perp \cap H^{1,1}(Y, \mathbb{Q})) = 0.
\end{align*}
\]
This leads to \( \eta_b = 0 \), which is a contradiction. \( \square \)

Combining the above proposition and [CL06, Lemma 3.2], we are able to apply Theorem 1.2 to the case \( (k, r, m, n) = (3, 1, 2, 4) \) and conclude that the Hodge-\( \mathcal{D} \)-conjecture fails for the real regulator \( r_{3,1} \) on a product of four very general elliptic curves, if the Bloch-Beilinson Conjecture 4.1 holds.

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