Dynamical symmetries of the Kepler problem

Marco Cariglia and Eduardo Silva Araújo

Universidade Federal de Ouro Preto, ICEB, Departamento de Física, Campus Morro do Cruzeiro, Morro do Cruzeiro, 35400-000 Ouro Preto, MG, Brazil

E-mail: marco@iceb.ufop.br and duduktang@hotmail.com

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Abstract
This paper comes from a first-year undergraduate research project on hidden symmetries of the dynamics for classical Hamiltonian systems. For pedagogical reasons the main subject chosen was Kepler’s problem of motion under a central potential, since it is a completely solved system. It is well known that for this problem the group of dynamical symmetries is strictly larger than the isometry group $O(3)$, the extra symmetries corresponding to hidden symmetries of the dynamics. By taking the point of view of examining the group action of the dynamical symmetries on the allowed trajectories, it is possible to teach the basic elements of many important physics subjects in the same project, including the Hamiltonian formalism, hidden symmetries, integrable systems, group theory and the use of manifolds.

(Some figures may appear in colour only in the online journal)

1. Introduction
Keplerian motion is a well-understood subject and suitable for introducing physics students to concrete, real-world problems. Depending on the level at which it is taught it may involve a direct study of the properties of the allowed trajectories, ellipses, parabolae and hyperbolae; a study of Hamiltonian mechanics; and a discussion of the role of dynamical symmetries generated by the angular momentum, associated with isometries, and by the Runge–Lenz vector, associated with hidden symmetries. The role of dynamical symmetries tends to be left to more advanced treatments, since it is usually expressed in terms of the abstract Poisson brackets algebra of the conserved quantities of motion.

This paper takes its motivation from a first-year undergraduate research project developed under the ‘Jovens talentos para a Ciência’ program, sponsored by the Brazilian funding agency, Capes [1]. The aim is to introduce the student to the concept of dynamical and hidden symmetries using a concrete, well-understood example, while employing a point of view on the subject that does not require mastery of symplectic space methods. The project is at the same time designed to expose the student to symplectic methods as well as other important topics in physics, such as integrable systems, group theory and the use of manifolds. It should
be noted, however, that the point of view taken, that the dynamical symmetry group of the Kepler problem induces a group action on allowed trajectories seen in a single object, provides a simple way to explain what the dynamical symmetry group is and to geometrically visualize the action of hidden symmetries.

The paper is structured as follows. In section 2.1 we formulate Kepler’s problem using Hamiltonian dynamics and basic symplectic geometry, introducing the Poisson brackets. In section 2.2 we discuss the conserved quantities of Keplerian motion: the angular momentum and the Runge–Lenz vector, and the concept of dynamical symmetry group that they generate. We comment on the system’s specific property of being maximally super-integrable and discuss the general concept of integrability and super-integrability. In section 2.3 we discuss the allowed trajectories, ellipses, parabolae and hyperbolae, and the fact that there is a group action of the dynamical symmetry group that transforms trajectories to have the same energy. Then we calculate explicitly the non-trivial hidden symmetry action originated by the Runge–Lenz vector in both infinitesimal and finite forms. The result is a smooth change in the eccentricity of the conics performed at fixed energy, which runs through all the allowed values of the eccentricity. The action is thus transitive on trajectories of a given energy. Finally, in section 2.4 we discuss the global properties of the manifold of allowed trajectories, and how the action of the dynamical symmetry group can be used to classify them. We finish in section 3 with a summary and concluding remarks.

2. Kepler’s problem

2.1. Equations of motion, Hamiltonian dynamics and symplectic geometry

Kepler’s problem consists of the classical motion of a point particle with mass $m$ in flat three-dimensional space $\mathbb{R}^3$, under a central potential $V = -\frac{k}{r}$. Here $k$ is a positive constant and $r = \sqrt{x^2 + y^2 + z^2}$ is the distance from the origin, $x$, $y$, $z$ being Cartesian coordinates. This finds applications in, for example, study of the gravitational interaction of planets and comets with the Sun in the solar system, or in the attractive interactions of two electric charges of different sign.

We use the following notation: $\{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$ is an orthonormal Cartesian base for vectors, $\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$ is the radial vector and $\vec{e}_t = \hat{r}/r$, $t$ is the absolute time coordinate and we write time derivatives of functions $f(t)$ as $\frac{df}{dt} = \dot{f}$, $\frac{d^2f}{dt^2} = \ddot{f}$. The modulus of a vector $\vec{v}$ is indicated with $v$. The motion under a central potential $V = -\frac{k}{r}$ is a well-understood dynamical system and there are many excellent texts, both older and more recent, discussing its main properties, along with the properties of Hamiltonian systems and symplectic spaces in general. We refer the reader to, for example, [2, 3], with these not representing an exhaustive list.

Newton’s equations of motion are

$$\ddot{m}\vec{r} = -\frac{k}{r^2} \vec{e}_r,$$  \hspace{1cm} (2.1)

where the right-hand side of the equation corresponds to the attractive central force.

Equation (2.1) can be conveniently reformulated in symplectic space, that is the space defined by six coordinates $y^a$, $a = 1, \ldots, 6$, where $y^1 = x, y, z$ for $i = 1, 2, 3$, and $y^{i+3} = p_x, p_y, p_z$. $\vec{p}$ has the interpretation of a physical momentum vector. In symplectic space we can exchange equation (2.1), that is a system of three second order equations, with the following system of six first order equations:

$$\dot{\vec{r}} = \frac{\vec{p}}{m},$$  \hspace{1cm} (2.2)
\[ \dot{\vec{p}} = -\frac{k}{r^3} \hat{r}, \] (2.3)

These equations can in turn be given a geometrical origin. First define the Hamiltonian function in symplectic space as
\[ H(\vec{r}, \vec{p}) = \frac{p^2}{2m} - \frac{k}{r}, \] (2.4)
physically representing the total energy written as kinetic plus potential energy. Then define the following antisymmetric matrix:
\[ \omega^{ab} = \sum_{i=1}^{3} (\delta^a_i \delta^b_{i+3} - \delta^a_{i+3} \delta^b_i). \] (2.5)

This is the so-called inverse symplectic matrix. With this matrix we can define two operations. The first is the symplectic gradient of a function \( f(\vec{r}, \vec{p}) \) as a six-dimensional vector \( X_f \) with components
\[ X^a_f = \sum_{b=1}^{6} \omega^{ab} \frac{\partial f}{\partial y^b}. \] (2.6)

Using the symplectic gradient we can rewrite equations (2.2), (2.3) as
\[ \dot{y}^a = X^a_f. \] (2.7)

which has the interpretation of defining a trajectory \( t \mapsto y^a(t) \) in symplectic space that is tangent to \( X_H \). This trajectory is the Hamiltonian flux passing through an appropriate initial point.

The second we can define is a bracket, or multiplication, acting on the space of functions \( f(y), g(y) \) defined on symplectic space. We define it as
\[ \{f, g\} = \sum_{a,b=1}^{6} \omega^{ab} \frac{\partial f}{\partial y^a} \frac{\partial g}{\partial y^b} \in \mathbb{R}. \] (2.8)

This is called the Poisson bracket. The Poisson bracket is antisymmetric, \( \{f, g\} = -\{g, f\} \), and it can be shown that is satisfies the Jacobi identity
\[ \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0. \] (2.9)

Then the equations of motion equations (2.2), (2.3) can also be rewritten as
\[ \dot{y}^a = \{y^a, H\}. \] (2.10)

From this equation we can interpret \( H \) as being the generator of time translations, \( \frac{dy^a}{dt} = \{y^a, H\} \). The time derivative of any function \( f(\vec{r}, \vec{p}) \) along an allowed trajectory \( t \mapsto (\vec{r}(t), \vec{p}(t)) \) obeying (2.7) can then be written as
\[ \dot{f} = \sum_{a=1}^{6} \frac{\partial f}{\partial y^a} \dot{y}^a = \sum_{a,b=1}^{6} \frac{\partial f}{\partial y^a} \omega^{ab} \frac{\partial H}{\partial y^b} = \{f, H\}. \] (2.11)

As a consequence, \( H \) is always conserved on trajectories.

2.2. Conserved quantities, dynamical symmetries and integrable systems

An important quantity in Kepler’s problem is the angular momentum of the particle with respect to the origin of coordinates:
\[ \vec{L} = \vec{r} \times \vec{p}. \] (2.12)
It can easily be checked that the angular momentum is conserved, \( \dot{\mathbf{L}} = 0 \), by applying the equations of motion (2.2), (2.3).

In general, given any regular function \( f(\mathbf{r}, \mathbf{p}) \) on symplectic space it is possible to generate an infinitesimal transformation of the coordinates

\[
\begin{align*}
\delta x^i &= \eta [x^i, f], \\
\delta p_i &= \eta [p_i, f],
\end{align*}
\]

(2.13)

where \( \eta \) is an infinitesimal parameter. It is possible to show that the inverse symplectic matrix is preserved under the transformation, \( \mathcal{L}(\omega^{ab}) = 0 \), where \( \mathcal{L} \) is the Lie derivative, and therefore this is an infinitesimal canonical transformation. In particular, when \( f \) is a conserved quantity, \( \{ f, H \} = 0 \), this means the Hamiltonian function is conserved along the transformation, \( \delta H = 0 \).

The angular momentum, defined above, provides three such independent conserved quantities and therefore three independent infinitesimal transformations, namely \( \forall i = 1, 2, 3 \):

\[
\begin{align*}
\delta x^i &= \eta [x^i, L^j], \\
\delta p_j &= \eta [p_j, L^i],
\end{align*}
\]

(2.14)

where \( \epsilon_{ijk} \) is the totally antisymmetric Levi-Civita symbol, with \( \epsilon^{123} = 1 \). One can immediately recognize in equation (2.14) the form of an infinitesimal rotation. In fact, the \( L^i \) quantities satisfy the Poisson algebra

\[
\{ L^i, L^j \} = \sum_{k=1}^{3} \epsilon_{ijk} L^k.
\]

(2.15)

This is the Lie algebra of the group \( O(3) \) of rotations in three dimensions, associated with infinitesimal transformations of the group. So for every pair \( (\mathbf{r}, \mathbf{p}) \) in symplectic space equation (2.14) represents an infinitesimal transformation of the group \( O(3) \). Moving from infinitesimal to finite transformations one obtains a group action of the group \( O(3) \) on symplectic space.

The angular momentum \( \mathbf{L} \) and the energy \( H \) are not the only conserved quantities for Kepler’s problem. Another important vectorial conserved quantity is the Runge–Lenz vector

\[
\mathbf{A} = \mathbf{p} \times \mathbf{L} - m \mathbf{e}_r.
\]

(2.16)

This is quadratic in the momenta, as opposed to \( \mathbf{L} \), which is linear, and a derivation that \( \dot{\mathbf{A}} = 0 \) can be found in \([2, 3]\). \( \mathbf{H}, \mathbf{L}, \mathbf{A} \) can be seen to form a closed algebra under Poisson brackets, given by (2.15) and

\[
\begin{align*}
\{ L^i, A^j \} &= \sum_{k=1}^{3} \epsilon_{ijk} A^k, \\
\{ A^i, A^j \} &= -2mH \sum_{k=1}^{3} \epsilon_{ijk} L^k.
\end{align*}
\]

(2.17)

(2.18)

Equations (2.15) and (2.17) have the interpretation that, under the infinitesimal \( O(3) \) transformation (2.14), \( \mathbf{L} \) and \( \mathbf{A} \) transform as vectors in \( \mathbb{R}^3 \). Equation (2.18) shows that the algebra is closed. If we restrict ourselves to solutions with zero energy, \( H = 0 \), the right-hand side of (2.18) is zero and the algebra of \( \mathbf{L} \) and \( \mathbf{A} \) is that of \( O(3) \times \mathbb{R}^3 \). For solutions with \( H = E \neq 0 \) we can introduce the new quantity \( \mathbf{B} = \frac{\mathbf{A}}{\sqrt{2m|E|}} \), which satisfies

\[
\{ L^i, B^j \} = \sum_{k=1}^{3} \epsilon_{ijk} B^k.
\]

(2.19)
\[ \{B', B''\} = -\text{sgn}(E) \sum_{k=1}^{3} \epsilon^{ijk} L^j, \tag{2.20} \]

where \(\text{sgn}(x) = x/|x|\) for \(x \neq 0\), \(\text{sgn}(0) = 0\) is the signum function. The algebra is that of \(O(4)\) for \(E < 0\), and \(O(1, 3)\) for \(E > 0\). Both groups admit two Casimir operators, and it can be shown that they are given by

\[ C_1 = B^2 - \text{sgn}(H)L^2, \tag{2.21} \]
\[ C_2 = \vec{L} \cdot \vec{B}. \tag{2.22} \]

One can check directly that \(\{C_{1,2}, L^j\} = 0 = \{C_{1,2}, B^j\}\) using the Poisson brackets. Then, for the specific dynamics of Kepler’s problem one can substitute equations (2.12), (2.16) into (2.21), (2.22) and obtain \(C_1 = \frac{m}{\sqrt{2E}}, C_2 = 0\). For the case \(H = 0\) it is possible to use \(C_1 = A^2 = m^2 k^2\) and \(C_2 = \vec{L} \cdot \vec{A}\).

The set \(\{H, C_1, C_2\}\) represents a maximal set of independent mutually commuting functions on phase space. In general, such a set can have at most \(n\) elements, where \(n\) is the dimension of the position space, three in our case. A Hamiltonian system admitting a maximal set of independent mutually commuting functions on phase space is called Liouville integrable; this means that there is a set of natural ‘action-angle’ variables, using which the system becomes trivially solvable. In fact, the Kepler system is maximally super-integrable, because it admits five independent conserved quantities of the dynamics, meaning that motion in the six-dimensional symplectic space occurs on a one-dimensional curve once the five independent conserved quantities are fixed. In general, in \(n\) dimension the maximum number admissible is \(2n - 1\), given that one of the \(2n\) variables must always remain free in order to describe the time evolution. There are five such independent quantities because the seven quantities \(\{H, \vec{L}, \vec{A}\}\) are all conserved, but they are not all independent, the functions \(C_1\) and \(C_2\) providing two constraints.

The functions \(\vec{L}, \vec{B}\) generate, for a fixed energy \(H = E\), a group of transformations according to the rule (2.13). Each of the transformations preserves the energy, \(\delta H = 0\), and this is called the dynamical symmetry group. The \(O(3)\) part of the dynamical symmetry group represents isometries, i.e. rigid rotations of the three-dimensional configuration space. It is possible to show that the remaining dynamical symmetries cannot be written as isometries. They are genuine transformations in symplectic space, not in configuration space (see, for example, [4] for a discussion of such symmetries for generic Hamiltonian systems) and are called hidden symmetries of the dynamics.

It is common practice to introduce the dynamical symmetry group algebraically, as in this section, i.e. presenting the generators of the group and their Poisson algebra. While correct and useful, this approach is abstract and tends to make it difficult to interpret the action of the hidden symmetry transformations. Some authors instead display explicitly the action of the dynamical symmetry group by extending the symplectic space to a larger space, for Kepler’s problem see, for example, [5]. This approach is interesting, although it seems it may be necessary to search for the correct set of extended coordinates on an individual, problem by problem, basis. In the next sections we show how for the Kepler problem the dynamical symmetry group acts transforming trajectories into trajectories of the same energy. There is therefore an explicit group action on the space of trajectories, and no extended space needs to be invoked. There are already works in the literature showing compatible results, for example that hidden symmetries in the Kepler motion can change negative energy trajectories, which are ellipses, into ellipses of a different eccentricity [6]. In this paper we only analyse the Kepler problem explicitly (for all types of trajectories, ellipses, hyperbolae and parabolae) because of the
focused nature of the current research project. However, even if we do not discuss the general case, the underlying reasoning used here applies to generic systems. The generic case will be discussed in a separate paper. The hidden symmetry transformations become transformations that alter the shape of the trajectories, while isometries keep the shape unchanged. The action of the symmetry group is well defined on the whole manifold of the possible trajectories, and by taking the quotient with respect to the group action we can classify the different trajectories.

2.3. Allowed trajectories and group actions

It is well known that the allowed trajectories that do not pass through the origin in Kepler’s problem are ellipses, hyperbolae and parabolae, according to whether the total energy \( E \) is negative, positive or zero. There are also arbitrary energy straight line solutions that correspond to a particle running into, or leaving from, the centre of coordinates.

We focus on the conical trajectories, the straight line solutions will be recovered as a limiting case of the conics for \( L \to 0 \). The conical trajectories lie in a plane perpendicular to the vector \( \vec{L} \). In this plane we can introduce Cartesian coordinates \( x, y \) and two-dimensional radial coordinates \( \rho, \theta \), with \( \rho = \sqrt{x^2 + y^2} \) the distance from the centre and \( \theta \) the angle with the \( x \) axis. A generic conic with one focus at the origin of the coordinates can be parameterized as

\[
\frac{1}{\rho} = C[1 + e \cos(\theta - \theta^*)], \tag{2.23}
\]

where \( e \) is the eccentricity and \( \theta^* \) the angle of the perihelion, that is the angle between the point of minimum \( \rho \) and the \( x \) axis. For \( e < 1 \) this is an ellipse, in particular a circle when \( e = 0 \), for \( e = 1 \) a parabola and for \( e > 1 \) a hyperbola.

In the specific Kepler problem one has \( C = \frac{mk}{2} \), and

\[
e = \sqrt{1 + \frac{2EL^2}{mk^2}} = \frac{A}{mk}. \tag{2.24}
\]

In particular, circles have \( E = -\frac{mk^2}{2L} \). We are excluding for now the cases with \( L = 0 \), which are the straight lines. The direction of the perihelion is given by that of \( \vec{A} \).

We can parameterize a possible trajectory using the following five variables. The components of \( \vec{L} \) and \( \vec{A} \), subject to the constraints that \( C_1 \) and \( C_2 \) in equations (2.21), (2.22) are constant. The reason five coordinates are required is the following: one can think that for each point \((\vec{r}, \vec{p})\) in symplectic space there passes a unique trajectory satisfying equations (2.2), (2.3). This gives six coordinates, but one of these is redundant: the time coordinate along the trajectory. Similarly, trajectories of a given fixed energy \( E \) are described by four coordinates. We can also choose another set of local coordinates. First, for each given orientation of \( \vec{L} \) we define smoothly coordinates \( x, y \) in the perpendicular plane. This can only be done locally, since it is equivalent to choosing a vector \( \hat{e}_x \), perpendicular to \( \hat{n} = \frac{\vec{L}}{L} \in S^2 \), and by the hairy ball theorem the choice cannot be done at the same time globally and smoothly. However, as we will see in section 2.4, we can limit ourselves to consider only half of the sphere \( S^2 \) and therefore this limitation does not apply. So we can define a \( \theta^* \) variable as the angle between \( \vec{A} \) and the chosen \( x \) axis. Then we can use as variables the energy \( E \), the angular momentum \( \vec{L} \) which can be decomposed into its modulus \( L \) and the associated unit norm vector \( \hat{n} = \frac{\vec{L}}{L} \), and the perihelion angle \( \theta^* \).

We now show that the dynamical symmetry group induces an action on the space of trajectories. We introduce the following notation. For a given function \( f(\vec{r}, \vec{p}) \) let \( \Phi_{(f,s)} \) be a map of symplectic space into itself, such that \( \Phi_{(f,s)} : (\vec{r}, \vec{p}) \mapsto \Phi_{(f,s)}(\vec{r}, \vec{p}) \), which is the image of \((\vec{r}, \vec{p})\) under the finite transformation generated by (2.13), for a finite parameter \( s \).
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If \( f \) is a constant of motion, \( [f, H] = 0 \), then \( \Phi_{(f,t)} \) is a finite canonical transformation and one element of the dynamical symmetry group. Since \( f \) and \( H \) Poisson commute then \( \Phi_{(f,t)} \) commutes with the Hamiltonian flow \( \Phi_{(H,t)} \) for any value of \( t \) and \( s \). Consider now a trajectory passing through \((\vec{r}_0, \vec{p}_0)\): this is given by all points of the kind \( \Phi_{(H,t)}(\vec{r}_0, \vec{p}_0) \), for some allowed \( t \). If we act with \( \Phi_{(f,s)} \) on any of these points we get

\[
\Phi_{(f,s)}(\Phi_{(H,t)}(\vec{r}_0, \vec{p}_0)) = \Phi_{(H,t)}(\Phi_{(f,s)}(\vec{r}_0, \vec{p}_0)).
\] (2.25)

showing that this is the trajectory associated to the initial point \( \Phi_{(f,s)}(\vec{r}_0, \vec{p}_0) \). Then this means that \( \Phi_{(f,t)} \) transforms a trajectory into another trajectory, with the same energy.

For the Kepler problem, the \( O(3) \) transformations are straightforward, since they correspond to rigid rotations of trajectories where \( \vec{L} \) and \( \vec{A} \) rotate accordingly without changing modulus. We now examine the action of the hidden symmetries, the transformations generated by \( \vec{A} \). We do this in the following way. We start with a reference system where \( \vec{L} \) lies in the \( z \) direction and \( \vec{A} \) in the \( x \) direction. Then we apply an infinitesimal transformation according to (2.13) and infer the result of the corresponding finite transformation. For a transformation generated by \( A_z \), using the algebra (2.17), (2.18), we find that the only non-zero transformation is

\[
\delta A_z = \eta [A_x, A_z] = 2mEL,
\] (2.26)

with \( \delta L^2 = 0 = \delta A^2 \). This is then the infinitesimal version of a rotation around the \( \vec{L} \) axis, and belongs to the \( O(3) \) isometry subgroup. Similarly, the transformation generated by \( A_x \) has

\[
\delta L_x = \eta [L_y, A_x] = -2mE, \quad \delta A_x = \eta [L_z, A_x] = -\eta A_x.
\] (2.27)

Thus the amplitude of both \( \vec{L} \) and \( \vec{A} \) change, respecting \( \delta C_1 = 0 = \delta C_2 \) as can be explicitly checked. A direct calculation shows that under this transformation

\[
\delta \epsilon = -\eta \frac{2EL}{k} = -\eta \text{sgn}(E) \sqrt{2mE} \sqrt{|e^2 - 1|}.
\] (2.29)

Then the corresponding finite transformation will be a change in eccentricity at fixed energy. For parabola, which have \( E = 0 \), the eccentricity stays constant and equal to 1, and only the amplitude of \( \vec{L} \) changes, which is consistent with the general condition \( A^2 = m^2k^2 + 2mEL^2 \). Then changing \( \vec{L} \) amounts to parabola with different distances between the perihelion and the origin of the coordinates.

It is easy to find the finite form of transformations (2.28), (2.29). It is convenient to use the identities \( A = mke \) and \( A_z = mE + 2mEL^2 \), and \( e = \sqrt{1 + \text{sgn}(E)L^2} \), where we defined the variable \( \vec{L} = \sqrt{\frac{2mE}{mke}}L \). In terms of these equations (2.28), (2.29) are integrated as follows. If \( E < 0 \) then \( 0 < \vec{L} \leq 1 \) and

\[
\vec{L}(s) = \cos(\sqrt{2mE}s + \cos^{-1} \hat{L}_0),
\] (2.30)

\[
e(s) = \sin(\sqrt{2mE}s + \sin^{-1} \hat{e}_0),
\] (2.31)

where \( s \) is the transformation finite parameter, \( 0 \leq e \leq 1 \) and \( e^2 + \vec{L}^2 = 1 \). It can be seen that this transformation deforms a circle, \( e = 0, \vec{L} = 1 \), into ellipses of higher and higher eccentricity, with limit \( e = 1, L = 0 \), which is a segment of straight line and not, to clarify possible doubts, a parabola.
For $E = 0$ the transformation is just

$$L(s) = -mk s + L_0,$$  \hspace{1cm} (2.32)

$$e(s) = e_0,$$  \hspace{1cm} (2.33)

with no change in eccentricity—parabolas are mapped into parabolas, since the energy is unchanged—but with a varying $L$, which means that the shape of the parabola changes. For $L \to 0$ the conic degenerates into a straight line.

Finally, for $E > 0$ the variable $\tilde{L}$ satisfies $\tilde{L} \geq 0$ the transformations are given by

$$\tilde{L} = \sinh(-\sqrt{2m|E|s + \sinh^{-1} \tilde{L}_0}),$$  \hspace{1cm} (2.34)

$$e = \cosh(-\sqrt{2m|E|s + \cosh^{-1} e_0}),$$  \hspace{1cm} (2.35)

with $e^2 - \tilde{L}^2 = 1$, and the hyperbola degenerating into a straight line for $\tilde{L} \to 0$.

Figures 1–3 display a number of trajectories related by finite hidden symmetry transformations in the case of negative, zero and positive energy, respectively.

### 2.4. The space of allowed trajectories as a manifold with global properties

One of the advantages of thinking of the dynamical symmetry group as a group acting on the space of allowed trajectories is that it allows us to consider the latter space as a global object on its own, and to classify it according to the group action.

We can start by studying the sections of constant $E$ separately. For $E < 0$ we have ellipses with $0 \leq L \leq 1$. $L = 1$ represents the circles, or equivalently $E = -m_0^2$. To describe a general ellipse we need to specify three coordinates for $\tilde{L}$, which amounts to specifying the orbital plane and the value of $L$, and a fourth coordinate corresponding locally to $\theta^*$ or in general to $\tilde{A}$ subject to the constraints $C_1$ and $C_2$. This last freedom can be parameterized by a variable on the circle $S^1$. Given that $L \leq \sqrt{\frac{m_0^2}{2E}}$, one might think that the variable $\tilde{L}$ should lie within a sphere of radius $\sqrt{\frac{m_0^2}{2E}}$, which increases to infinity when the energy increases to zero. However, the operation $\tilde{L} \to -\tilde{L}$ is redundant since it yields the same orbital plane...
Figure 2. The finite transformation (2.32) is displayed for the following values of the parameter $L$: 0.025, 0.25, 0.5, 0.75, 1 and 1.25. The parabolae lay in the $x, y$ plane, with focus on the $x$ axis. To generate the image we have set $mk = 1$.

Figure 3. The finite transformation (2.35) is displayed for the following values of the eccentricity: 1.0005, 1.05, 1.15, 1.25, 1.5 and 1.75. The hyperbolae lay in the $x, y$ plane, with foci on the $x$ axis. To generate the image we have set $\frac{k}{|E|} = 1$. 

and the same value of $L$, and should be quotiented out. So $\vec{L}$ lies in a closed ball of radius $\sqrt{\frac{mk^2}{2|E|}}$ modulo the inversion operation, $\vec{L} \in S(\sqrt{\frac{mk^2}{2|E|}})/\mathbb{Z}_2$. For all values of $\vec{L}$ inside the ball the eccentricity is different from zero and there is an $S^1$ freedom to rotate the ellipse around the $\vec{L}$ axis. However, for $\vec{L}$ on the surface of the ball the trajectories are circles and the $S^1$ freedom disappears: rotations around the $\vec{L}$ axis no longer generate new trajectories, so the $S^1$ collapses into a point. We can therefore picture the space of trajectories of a given negative energy $E$ as a cone with base given by $S(\sqrt{\frac{mk^2}{2|E|}})/\mathbb{Z}_2$, the tip of the cone lying on the border of the sphere. Since using the dynamical symmetries we can change the direction of $\vec{L}$ and $\vec{A}$ (isometries), and their moduli (the non-trivial hidden symmetry), then the action of $O(4)$ is transitive on trajectories of negative energy. In other words, if we use dynamical symmetry transformations as an equivalence relation on the space of trajectories of a given negative energy $E < 0$, then there is only one representative for each value of the energy.

For $E \geq 0$ the analysis is similar, but the $S^1$ circle does not collapse at any point, and moreover $\vec{L}$ is unbounded from above, so the fixed energy manifold is given by $\mathbb{R}^3/\mathbb{Z}_2 \times S^1$.

In both cases, the action of $O(4)$ is transitive and there is only one representative per given value of the energy.

3. Conclusions

We have discussed the classical Kepler problem from the point of view of its dynamical symmetries. While it is well known that the group of symmetries of the dynamics is strictly larger than $O(3)$, this is normally discussed in terms of Poisson brackets algebra and infinitesimal canonical transformations. We have taken the point of view here that the group of symmetries of dynamics acts on the space of allowed trajectories as a whole, and found the explicit form of the finite group transformations. We have shown that the dynamical symmetry group acts transitively on the space of trajectories of fixed energy $E$, for any allowed value of the energy, and we discussed the global structure of the manifold of allowed trajectories. Thus in a single project it has been possible to touch on several important subjects in physics.

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