MARCINKIEWICZ SPACES, GARSIA-RODEMICH SPACES AND THE SCALE OF JOHN-NIRENBERG SELF IMPROVING INEQUALITIES

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Abstract. We extend to n-dimensions a characterization of the Marcinkiewicz $L(p, \infty)$ spaces first obtained by Garsia-Rodemich in the one dimensional case. This leads to a new proof of the John-Nirenberg self-improving inequalities. We also show a related result that provides still a new characterization of the $L(p, \infty)$ spaces in terms of distribution functions, reflects the self-improving inequalities directly, and also characterizes $L(\infty, \infty)$, the rearrangement invariant hull of $BMO$. We show an application to the study of tensor products with $L(\infty, \infty)$ spaces, which complements the classical work of O’Neil [19] and the more recent work of Astashkin [2].

1. Introduction

In their seminal paper [11], John-Nirenberg introduced the space $BMO$ and proved the celebrated John-Nirenberg inequality for functions in $BMO$. It is also well known, although perhaps somewhat less so, that in the same paper, John-Nirenberg showed that the $BMO$ self improvement inequality can be refined and framed as a scale of inequalities. These inequalities (or embeddings) are associated with what we nowadays call “John-Nirenberg spaces”. This result of John-Nirenberg, which we now describe, is the starting point of our development in this paper.

Let $Q_0 \subset \mathbb{R}^n$, be a fixed cube, $1 \leq p < \infty$. Let $P(Q_0) = \{\{Q_i\}_{i \in N} : \text{countable families of subcubes } Q_i \subset Q_0, \text{with pairwise disjoint interiors}\}$. The John-Nirenberg spaces are defined by

$$JN_p(Q_0) = \{f \in L^1(Q_0) : JN_p(f, Q_0) < \infty\},$$

where

$$JN_p(f, Q_0) = \sup_{(Q_i) \in P(Q_0)} \left\{ \left\{ \sum_i |Q_i| \left( \frac{1}{|Q_i|} \int_{Q_i} |f - f_{Q_i}| \, dx \right)^p \right\}^{1/p} \right\}.$$
Let us also recall that, for a given measure space, the Marcinkiewicz $L(p, \infty)$ spaces, $1 \leq p < \infty$, are defined by demanding\(^3\) that \(\|f\|_{L(p, \infty)} < \infty\), where

\[
(1.1) \quad \|f\|_{L(p, \infty)} = \sup_{t>0} \{f^+(t)^{1/p}\} = \sup_{t>0} \{t (\lambda_f(t))^{1/p}\};
\]

while for $p = \infty$, the space $L(\infty, \infty)$ (cf. \cite{11}) is defined\(^4\) by the condition \(\|f\|_{L(\infty, \infty)} < \infty\), where

\[
(1.2) \quad \|f\|_{L(\infty, \infty)} = \sup_{t>0} \{f^*(t) - f^+(t)\},
\]

and

\[
f^*(t) = \frac{1}{t} \int_0^t f^*(s) ds.
\]

Then (cf. \cite{11} Lemma 3), and also \cite{21} Theorem 4.1, pag 209 for a more detailed proof),

**Theorem 1.** Let \(1 < p < \infty\). Suppose that \(f \in JN_p(Q_0)\), then \(f - f_{Q_0} \in L(p, \infty)(Q_0)\), and there exists a constant \(A(p, Q_0, \nu)\) such that

\[
\|f - f_{Q_0}\|_{L(p, \infty)(Q_0)} \leq A(p, Q_0, \nu) JN_p(f, Q_0).
\]

In particular,

\[
f - f_{Q_0} \in \bigcap_{r < p} L^r(Q_0).
\]

The limiting condition defining \(JN_p(Q_0)\) when \(p = \infty\) corresponds\(^5\) to \(BMO\), and in this case Theorem \(1\) corresponds to a version of the well known John-Nirenberg inequality \cite{11}.

In the one dimensional case, Garsia and Rodemich \cite{10} improved on Theorem \(1\). To formulate the Garsia and Rodemich result it will be convenient to introduce a different scale of spaces which we shall term Garsia-Rodemich spaces. It will be useful for later use to give the relevant definitions in the \(n\)-dimensional case.

Let \(Q_0 \subset \mathbb{R}^n\) be a fixed cube, let \(1 \leq p < \infty\), and let \(p'\) be defined by \(\frac{1}{p} + \frac{1}{p'} = 1\).

The Garsia-Rodemich spaces \(GaRo_p(Q_0)\) are defined as follows. We shall say that \(f \in GaRo_p(Q_0)\), if and only if \(f \in L^1(Q_0)\), and \(\exists C > 0\) such that for all \(\{Q_i\}_{i \in \mathbb{N}} \in P(Q_0)\) we have

\[
(1.2) \quad \sum_i \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx \, dy \leq C \left( \sum_i |Q_i| \right)^{1/p'}.
\]

We let

\[
GaRo_p(f, Q_0) = \inf \{C > 0 : \text{such that (1.2) holds}\}.
\]

Then we have (cf. \cite{10})

\(^3\)Here \(f^+\) denotes the non-increasing rearrangement of \(f\) and \(\lambda_f\) its distribution function (cf. \cite{3}).

\(^4\)Some authors (including sometimes the author of this paper) use a different notation and let \(W\) denote what we call \(L(\infty, \infty)\), at the same time that they use the notation \(L(\infty, \infty) = L^\infty\).

\(^5\)The \(JN_\infty(Q_0)\) condition would read

\[
\sup_{\{Q_i\} \in P(Q_0)} \frac{1}{|Q_i|} \int_{Q_i} |f - f_{Q_i}| \, dx < \infty.
\]
Theorem 2. Let \( 1 < p < \infty \), and let \( Q_0 = I = [0, 1] \). Then, as sets
\[
GaRo_p(I) = L(p, \infty)(I).
\]

Remark 1. The elementary proof of the embedding \( L(p, \infty) \subset GaRo_p \) outlined in \cite{10} works in \( n \)-dimensions and actually shows that (cf. Theorem 2 part (ii), below)
\[
(1.3) \quad GaRo_p(f, I) \leq \frac{p}{p-1} 2 \| f \|^*_{L(p, \infty)(I)}.
\]

By Theorem 11 we have
\[
JN_p(I) \subset L(p, \infty)(I),
\]
therefore by Theorem 2 (cf. Section 2 below for a direct proof of the \( n \)-dimensional case) it follows that
\[
(1.4) \quad JN_p(I) \subset GaRo_p(I).
\]

In conclusion, Theorem 2 not only improves on Theorem 1 in the one dimensional case, but also gives us an interesting characterization of the Marcinkiewicz \( L(p, \infty)(I) \) spaces, \( 1 < p < \infty \). Unfortunately, one part of the proof of Theorem 2 uses a non-trivial rearrangement inequality, also due to Garsia-Rodemich \cite{10}, which is only proved there in the one dimensional case.

In \cite{10}, the authors briefly suggest a possible different method to prove Theorem 2 in \( n \)-dimensions, and without dimensional constants, but no details are provided. In this note we give a new proof Theorem 2 that is valid in \( n \) dimensions (cf. Theorem 5 below) . Our approach is different from the one given in \cite{10}, and does not use martingale techniques. Instead, our method is ultimately based on Calderón-Zygmund type decompositions, following classical ideas in \cite{3}.

As we shall see (cf. Section 2) the verification that the John-Nirenberg conditions are stronger than the Garsia-Rodemich conditions (e.g. \( 1.4 \)) is immediate. Therefore, the crucial aspect of this approach to the John-Nirenberg theorem is the fact that the Garsia-Rodemich spaces are the same as the Marcinkiewicz \( L(p, \infty) \) spaces! This clarifies the self improvement results of John-Nirenberg. Moreover, these ideas could potentially be useful in the investigation of related issues, e.g. the dimensional constants involved in the John-Nirenberg embeddings (cf. \cite{7}).

Now the classical definitions of the \( L(p, \infty) \) spaces are given in terms of growth conditions on rearrangements or distribution functions (cf. \cite{4}, \cite{5}, \cite{18}, \cite{6}, \cite{20}, etc.). The case \( p = \infty \), which corresponds to \( L(\infty, \infty) \) ("the rearrangement invariant hull of BMO", cf. \cite{3}), also admits a similar characterization through the use
of the oscillation operator \( f^{**} - f^* \), and indeed one can find a characterization of all the \( L(p, \infty) \) spaces, \( p \in (1, \infty] \), in the same fashion, namely
\[
\|f\|_{L(p, \infty)}^\# = \sup_{s} \{(f^{**}(s) - f^*(s))s^{1/p}\} < \infty.
\]

This characterization, while extremely useful in many problems (cf. [4], [16]) is not always easy to implement, and does not reflect immediately the self improvement of the Garsia-Rodemich construction. In this direction, we found a different characterization of \( L(\infty, \infty) \), which gives an implicit differential inequality reflecting the exponential decay of the distribution function of elements of \( L(\infty, \infty) \), via the use of distribution functions (cf. Section 3 below).

**Theorem 3.** Let \((\Omega, \mu)\) be a measure space. Then, \( f \in L(\infty, \infty) := L(\infty, \infty)(\Omega) \) if and only if there exists \( C > 0 \) such that for all \( t > 0 \),
\[
(1.5) \quad \int_{t}^{\infty} \lambda_f(s)ds \leq C \lambda_f(t),
\]
and
\[
\|f\|_{L(\infty, \infty)}^\#: = \inf \{C : \text{such that (1.5) holds} \} = \|f\|_{L(\infty, \infty)}.
\]

This characterization gives immediately the exponential integrability of functions \((1.7)\) which in turn coincides with the set of all \( \phi \) such that \( \lambda(\phi) = 0 \), and
\[
\|f\|_{L(\infty, \infty)}^\#: = \sup_{s>0} \{f^{**}(s) - f^*(s)\} < \infty,
\]
which in turn coincides with the set of all \( f \) such that \( f^{**}(\infty) = 0 \), and
\[
(1.7) \quad \|f\|_{L(\infty, \infty)}^\#: = \sup_{t>0} \frac{1}{\lambda_f(t)^{1-1/p}} \int_{t}^{\infty} \lambda_f(s)ds < \infty.
\]

If one combines (1.7) with the usual definition of the spaces \( L(p, \infty) \) (cf. (1.6)), one readily obtains a known characterization of the \( L(p, \infty) \) spaces which was apparently first given by O’Neil [19].

**Corollary 1.** Let \( 1 < p < \infty \), then
\[
(1.8) \quad \|f\|_{L(p, \infty)}^* \sim \inf \{C^{1/p} : \int_{t}^{\infty} \lambda_f(s)ds \leq C \lambda_f^p(t)\}.
\]

**Remark 2.** One difference between (1.7) and (1.8) is given by the fact that the former also works in the case \( p = \infty \). Both formulations can be extended to more general Marcinkiewicz spaces, \( M_\phi \), where \( \phi \) is a concave function. In particular, we refer to [19] for the corresponding theory of generalized Marcinkiewicz spaces \( M_\phi \) defined via (1.8).

\[\text{Note however that } (f^{**}(t) - f^*(t)) = t^{\#}(f^{**}(t)).\]
In his expansive work [19], O’Neil used the formulae (1.8) to study tensor products of $L(p, q)$ spaces (cf. also [2] and [17]). The space $L(\infty, \infty)$ was introduced later (cf. [3]), and consequently was not considered in [19]. In the last section of this paper we give an application of (1.5) to show that (cf. Theorem 6 in Section 4 below)

(1.9) \[ L(\infty, \infty)(\Omega_1) \otimes L^\infty(\Omega_2) \subset L(\infty, \infty)(\Omega_1 \times \Omega_2). \]

While we think that (1.9) could be useful in establishing other embeddings of tensor products involving $L(\infty, \infty)$, such an undertaking falls outside the scope of this note.

In conclusion, we should mention that this paper is part of series of papers by the author on $BMO$, self improvement and interpolation, that go back at least to [13], [14], [15], with the most recent opus being [16], to which we refer for background information and further references.

2. John-Nirenberg Spaces and Garsia-Rodemich Spaces

It is easy to see the connection of the John-Nirenberg spaces with $BMO$. Fix a cube $Q_0 \subset \mathbb{R}^n$ and let

\[ \|f\|_{BMO(Q_0)} = \sup \left\{ \frac{1}{|Q|} \int_Q |f - f_Q| \, dx : Q \text{ subcube of } Q_0 \right\}. \]

Then, for $1 \leq p < \infty$

\[ JN_p(f, Q_0) \leq \|f\|_{BMO(Q_0)} |Q_0|^{1/p}. \]

Indeed, if $\{Q_i\}_{i \in N} \in P(Q_0)$, then we clearly have

\[ \left\{ \sum_i |Q_i| \left( \frac{1}{|Q_i|} \int_{Q_i} |f - f_{Q_i}| \, dx \right)^p \right\}^{1/p} \leq \left\{ \sum_i |Q_i| \left( \|f\|_{BMO(Q_0)} \right)^p \right\}^{1/p}
\]

\[ \leq \|f\|_{BMO(Q_0)} |Q_0|^{1/p}. \]

The purpose of this section is to prove the following

**Theorem 5.** Let $1 < p < \infty$, and let $Q_0 \subset \mathbb{R}^n$ be a fixed cube. Then

(i) $JN_p(Q_0) \subset GaRo_p(Q_0)$, in fact

(2.1) \[ GaRo_p(f, Q_0) \leq 2JN_p(f, Q_0). \]

(ii) $GaRo_p(Q_0) = L(p, \infty)(Q_0)$, in fact we have

\[ GaRo_p(f, Q_0) \leq \frac{2p}{p-1} \|f\|_{L(p, \infty)}^p, \]

\[ \sup_t t^{1/p} (f^{**}(t) - f^*(t)) \leq 2^{n/p'+1} GaRo_p(f, Q_0) + \left( \frac{4}{|Q_0|} \right)^{1/p'} \|f\|_{L^1}. \]

**Proof.** (i). Suppose that $\{Q_i\}_{i \in N} \in P(Q_0)$. Then for all $Q_i$, $i \in N$, we have,

\[ \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx \, dy \leq \int_{Q_i} \int_{Q_i} |f(x) - f_{Q_i}| \, dx \, dy + \int_{Q_i} \int_{Q_i} |f_{Q_i} - f(y)| \, dx \, dy \]

\[ = 2|Q_i| \int_{Q_i} |f - f_{Q_i}| \, dx. \]
Therefore,

\[ \sum_{i} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx \, dy \leq 2 \sum_{i} \frac{1}{|Q_i|} \int_{Q_i} |f - f_{Q_i}| \, dx \]

\[ = 2 \sum_{i} |Q_i|^{1/p'} |Q_i|^{1/p} \frac{1}{|Q_i|} \int_{Q_i} |f - f_{Q_i}| \, dx \]

\[ \leq 2 \left( \sum_{i} |Q_i| \right)^{1/p'} \left\{ \sum_{i} \left( \frac{1}{|Q_i|} \int_{Q_i} |f - f_{Q_i}| \, dx \right)^p \right\}^{1/p}, \]

and \((2.1)\) follows.

(ii). We show first that \(L(p, \infty)(Q_0) \subset GaRo_p(Q_0)\). Let \(\{Q_i\}_{i \in N} \in P(Q_0)\), then

\[ \sum_{i} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx \, dy \leq \sum_{i} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} (|f(x)| + |f(y)|) \, dx \, dy \]

\[ \leq 2 \int_{Q_0} |f(x)| \, dx \]

\[ \leq 2 \int_{0}^{\sum_{i} |Q_i|} f^*(t) \, dt \]

\[ \leq 2 \|f\|_{L(p, \infty)}^{*} \int_{0}^{\sum_{i} |Q_i|} t^{-1/p} \, dt \]

\[ = \frac{2p}{p-1} \|f\|_{L(p, \infty)}^{*} \left( \sum_{i} |Q_i| \right)^{1/p'}. \]

Consequently,

\[ GaRo_p(f, Q_0) \leq \frac{2p}{p-1} \|f\|_{L(p, \infty)}^{*}. \]

To show the remaining inclusion, \(GaRo_p(Q_0) \subset L(p, \infty)(Q_0)\), we argue as in [4 Chapter 5]. We provide all the details for the sake of completeness.

To show that a function \(f\) belongs to \(L(p, \infty)(Q_0)\) it is equivalent to show that \(|f| \in L(p, \infty)(Q_0)\), therefore, since

\[ GaRo_p(|f|, Q_0) \leq GaRo_p(f, Q_0), \]

to show that \(f \in GaRo_p(Q_0)\) belongs to \(L(p, \infty)(Q_0)\), we can assume without loss that \(f \geq 0\). Let \(f \in GaRo_p(Q_0)\), \(f \geq 0\). Fix \(t > 0\), such that \(t < |Q_0| / 4\), and let \(E = \{x \in Q_0 : f(x) > f^*(t)\}\). By definition, \(|E| \leq t < |Q_0| / 4\), consequently, we can find a relatively open subset of \(Q_0, \Omega\), say, such that \(E \subset \Omega\) and \(|\Omega| \leq 2t \leq |Q_0| / 2\). By [4 Lemma 7.2, page 377] we can find a sequence of cubes \(\{Q_i\}_{i \in N}\), with pairwise disjoint interiors, such that:

\[ (i) \quad |\Omega \cap Q_i| \leq \frac{1}{2} |Q_i| \leq |\Omega^c \cap Q_i|, \quad i = 1, 2... \]

\[ (ii) \quad \Omega \subset \bigcup_{i \in N} Q_i \subset Q_0 \]

\[ (iii) \quad |\Omega| \leq \sum_{i \in N} |Q_i| \leq 2^{n+1} |\Omega|. \]
Now, to estimate $t^{1/p} (f^{**}(t) - f^*(t))$, it will be more convenient, by homogeneity, to consider $t (f^{**}(t) - f^*(t))$ first. Then, we have

$$t (f^{**}(t) - f^*(t)) = \int_E \{f(x) - f^*(t)\} dx$$

$$\leq \sum_{i \in N} \int_{E \cap Q_i} \{f(x) - f^*(t)\} dx$$

$$= \sum_{i \in N} \left( \int_{E \cap Q_i} \{f(x) - f_{Q_i}\} dx + |E \cap Q_i| \{f_{Q_i} - f^*(t)\} \right)$$

$$\leq \sum_{i \in N} \left( \int_{Q_i} \{f(x) - f_{Q_i}\} dx + |E \cap Q_i| \{f_{Q_i} - f^*(t)\} \right)$$

$$= (I) + (II).$$

Let $J = \{i : f_{Q_i} > f^*(t)\}$, then

$$(II) = \sum_{i \in J} |E \cap Q_i| \{f_{Q_i} - f^*(t)\}$$

$$\leq \sum_{i \in J} |E \cap Q_i| \{f_{Q_i} - f^*(t)\}$$

$$\leq \sum_{i \in J} |\Omega \cap Q_i| \{f_{Q_i} - f^*(t)\}$$

$$\leq \sum_{i \in J} |\Omega^c \cap Q_i| \{f_{Q_i} - f^*(t)\}$$

$$= \sum_{i \in J} \int_{\Omega^c \cap Q_i} \{f_{Q_i} - f^*(t)\} dx$$

$$\leq \sum_{i \in J} \int_{\Omega^c \cap Q_i} \{f_{Q_i} - f(x)\} dx (\text{since } \Omega^c \subset E^c)$$

$$\leq \sum_{i \in J} \int_{Q_i} |f_{Q_i} - f(x)| dx$$

$$\leq \sum_{i \in J} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(y) - f(x)| dy dx$$

$$\leq GaRop(f, Q_0) \left( \sum_{i \in N} |Q_i| \right)^{1/p'}.$$
Likewise,

\[
(I) = \sum_{i \in N} \int_{Q_i} \{ f(x) - f_{Q_i} \} \, dx
\]

\[
= \sum_{i \in N} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} (f(x) - f(y)) \, dx \, dy
\]

\[
\leq \sum_{i \in N} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx \, dy
\]

\[
\leq GaRo_p(f, Q_0) \left( \sum_{i \in N} |Q_i| \right)^{1/p'}.
\]

Combining the inequalities we have obtained,

\[
t (f^{**}(t) - f^*(t)) \leq 2GaRo_p(f, Q_0) \left( \sum_{i \in N} |Q_i| \right)^{1/p'}
\]

\[
\leq 2GaRo_p(f, Q_0)(2^{n+1})^{1/p'} 2^{-1/p'} t^{1/p'}.
\]

Therefore,

\[
\sup_{t \leq |Q_0|/4} t^{1/p} (f^{**}(t) - f^*(t)) \leq 2^{n/p'+1} GaRo_p(f, Q_0).
\]

To deal with \( t > |Q_0|/4 \), we note that \( t (f^{**}(t) - f^*(t)) = \int_{f^*(t)}^{\infty} \lambda_f(s) \, ds \leq \int_0^{\infty} \lambda_f(s) \, ds = \|f\|_{L^1} \); therefore,

\[
t^{1/p} (f^{**}(t) - f^*(t)) \leq t^{-1/p'} \|f\|_{L^1}
\]

\[
\leq \left( \frac{4}{|Q_0|} \right)^{1/p'} \|f\|_{L^1}.
\]

Thus,

\[
\sup_t t^{1/p} (f^{**}(t) - f^*(t)) \leq 2^{n/p'+1} GaRo_p(f, Q_0) + \left( \frac{4}{|Q_0|} \right)^{1/p'} \|f\|_{L^1},
\]

and the desired result follows by Theorem 4.

\[\square\]

### 3. Another characterization of the \( L(p, \infty) \) spaces, \( 1 < p \leq \infty \)

The purpose of this section is to give a proof of Theorem 2 and Theorem 4.

We start with the former.

**Proof.** Let \( f \) be such that there exists \( C > 0 \) such that (1.5) holds for all \( t > 0 \). Then, we have

\[
(f^{**}(t) - f^*(t)) t = \int_{f^*(t)}^{\infty} \lambda_f(s) \, ds
\]

\[
\leq C\lambda_f(f^*(t))
\]

\[
\leq Ct.
\]
Thus,
\[
\| f \|_{L(\infty, \infty)} \leq \inf \{ C : (1.5) \text{ holds} \} = \| f \|^{#}_{L(\infty, \infty)}.
\]
Conversely, suppose that \( f \in L(\infty, \infty) \). Then, for all \( t > 0 \), we have,
\[
\int_{f^*(t)}^{\infty} \lambda_f(s)ds = (f^{**}(t) - f^*(t)) t \leq t \| f \|_{L(\infty, \infty)}.
\]
Therefore,
\[
\int_{f^*(\lambda_f(t))}^{\infty} \lambda_f(s)ds \leq \lambda_f(t) \| f \|_{L(\infty, \infty)}.
\]
Now, since \( f^*(\lambda_f(t)) \leq t \), we have
\[
\int_{t}^{\infty} \lambda_f(s)ds \leq \lambda_f(t) \| f \|_{L(\infty, \infty)}.
\]
Consequently,
\[
\| f \|^{#}_{L(\infty, \infty)} \leq \| f \|_{L(\infty, \infty)} ,
\]
concluding the proof.

We proceed with the proof of Theorem 4.

Proof. We trivially have \( \| f \|^{#}_{L(p, \infty)} \leq \| f \|_{L(p, \infty)} \). Moreover, if \( f^{**}(\infty) = 0 \), then
\[
f^{**}(t)t^{1/p} = t^{1/p} \int_{t}^{\infty} \frac{f^{**}(s) - f^*(s)}{s} ds
= t^{1/p} \int_{t}^{\infty} (f^{**}(s) - f^*(s)) s^{1/p} s^{-1/p} ds
\leq t^{1/p} \| f \|^{#}_{L(p, \infty)} \int_{t}^{\infty} s^{-1/p} ds
= p \| f \|^{#}_{L(p, \infty)}.
\]
Consequently,
\[
\| f \|_{L(p, \infty)} \leq p \| f \|^{#}_{L(p, \infty)}.
\]
The last part of the result follows exactly as the proof of Theorem 3 (the case \( p = \infty \)). For example, from
\[
\int_{t}^{\infty} \lambda_f(s)ds \leq \| f \|^{#}_{L(p, \infty)} (\lambda_f(t))^{1-1/p}
\]
we get
\[
(f^{**}(t) - f^*(t)) t = \int_{f^*(t)}^{\infty} \lambda_f(s)ds \leq \| f \|^{#}_{L(p, \infty)} t^{1-1/p}
\]
and therefore
\[
\| f \|^{#}_{L(p, \infty)} \leq \| f \|^{#}_{L(p, \infty)}.
\]
Conversely, for all \( t > 0 \),
\[
t \| f \|^{#}_{L(p, \infty)} \geq t t^{1/p} (f^{**}(t) - f^*(t)) \geq t^{1/p} \int_{f^*(t)}^{\infty} \lambda_f(s)ds.
\]

Thus,
\[ \lambda_f(t) \|f\|_{L(p, \infty)}^\# \geq (\lambda_f(t))^{1/p} \int_{f^{*}(\lambda_f(t))}^\infty \lambda_f(s) ds \]
\[ \geq (\lambda_f(t))^{1/p} \int_t^\infty \lambda_f(s) ds, \]
and the desired result follows. \[ \square \]

**Remark 3.** Observe that when \( p = 1 \), the previous considerations provide a characterization of \( L^1 \), not of \( L(1, \infty) \). Indeed, the corresponding result for \( p = 1 \) is
\[ \sup_t f^{**}(t) t = \sup_t \int_0^t f^*(s) ds = \|f\|_{L^1} = \sup_t \int_t^\infty \lambda_f(s) ds. \]
In other words, \( L^1 \) is characterized by the condition
\[ \sup_{t > 0} \int_t^\infty \lambda_f(s) ds \leq C. \]
Note that in this case \( (\lambda_f(t))^{1-1/1} = 1 \).

To conclude this section we prove Corollary 1.

**Lemma 1.** Let \( 1 < p < \infty \). Then
\[ \|f\|_{L(p, \infty)}^\# \approx \inf \{ C^{1/p} : \int_t^\infty \lambda_f(s) ds \leq C t^{1-p} \}. \]

**Proof.** Note that
\[ (\lambda_f(t))^{1/p} \leq \|f\|_{L(p, \infty)} t^{-1} \Leftrightarrow \lambda_f(t) \leq \|f\|_{L(p, \infty)}^p t^{-p}. \]
Suppose that \( f \in L(p, \infty) \). By the previous Theorem,
\[ \int_t^\infty \lambda_f(s) ds \leq \|f\|_{L(p, \infty)}^\# (\lambda_f(t))^{1-1/p} \]
\[ \leq \|f\|_{L(p, \infty)}^\# \|f\|_{L(p, \infty)}^p t^{-(1-1/p)} (\text{by (3.1)}) \]
\[ \leq C \|f\|_{L(p, \infty)}^p t^{1-p}. \]
Conversely, suppose that
\[ \int_t^\infty \lambda_f(s) ds \leq C t^{1-p}. \]
Then, since \( \lambda_f \) decreases,
\[ \lambda_f(t)^{1/p} \leq \int_{t/2}^t \lambda_f(s)^{1/p} ds = \int_{t/2}^t \lambda_f(s) \lambda_f(s)^{1/p-1} ds \]
\[ \leq \lambda_f(t/2)^{1/p-1} \int_{t/2}^t \lambda_f(s) ds \]
\[ \leq \lambda_f(t/2)^{1/p-1} \int_{t/2}^\infty \lambda_f(s) ds \]
\[ \leq \lambda_f(t/2)^{1/p-1} C 2^{p-1} t^{1-p}. \]
Therefore,

\[ \lambda_f(t/2)^{-1/p+1}\lambda_f(t)^{1/p} \leq \tilde{C}t^{-p} \]

and, consequently,

\[ \lambda_f(t)^{-1/p+1}\lambda_f(t)^{1/p} \leq \tilde{C}t^{-p}. \]

The desired result follows. □

4. Final Remarks

4.1. Products and tensor products with \( L(\infty, \infty) \). The new formulae we presented for the computation of the “norm” \( |||L(\infty, \infty)||| \) has several applications. Here, following O’Neil [19] (cf. also [2], [17]), we shall briefly consider tensor products with \( L(\infty, \infty) \). It is not our purpose to develop the most general results, but to give a flavor of the ideas involved.

**Theorem 6.** Let \((\Omega_1, \mu_1), (\Omega_2, \mu_2)\) be measure spaces. Then,

\[ L(\infty, \infty)(\Omega_1) \otimes L(\infty, \infty)(\Omega_2) \subset L(\infty, \infty)(\Omega_1 \times \Omega_2), \]

with

\[ \| f \otimes g \|_{L(\infty, \infty)(\Omega_1 \times \Omega_2)} \leq \| f \|_{L(\infty, \infty)(\Omega_1)} \| g \|_{L(\infty, \infty)(\Omega_2)}. \]

**Proof.** Let \( f \in L(\infty, \infty)(\Omega_1) \), \( g \in L(\infty, \infty)(\Omega_2) \). The distribution function of \( f \otimes g \) is computed in [19, Lemma 7.1 (2), page 97]

\[ \lambda_{f \otimes g}(z) = \int_0^\infty \lambda_f(z u) d(-\lambda g(u)), z > 0. \]

Therefore, on account that \( g \in L(\infty, \infty) \), we have

\[ \lambda_{f \otimes g}(z) = \int_0^\| g \|_{L(\infty, \infty)(\Omega_2)} \lambda_f(z u) d(-\lambda g(u)). \]

Then, by Tonnelli’s theorem, for all \( t > 0 \),

\[
\int_t^\infty \lambda_{f \otimes g}(z) dz = \int_t^\infty \int_0^{\| g \|_{L(\infty, \infty)(\Omega_2)}} \lambda_f(z u) d(-\lambda g(u)) dz \\
= \int_0^{\| g \|_{L(\infty, \infty)(\Omega_2)}} \int_t^\infty \lambda_f(z u) d(-\lambda g(u)) dz \\
= \int_0^{\| g \|_{L(\infty, \infty)(\Omega_2)}} \int_t^\infty \lambda_f(z u) d(-\lambda g(u)) dr \\
= \int_0^{\| g \|_{L(\infty, \infty)(\Omega_2)}} \| f \|_{L(\infty, \infty)(\Omega_1)} \lambda_f(z u) d(-\lambda g(u)) \\
= \| g \|_{L(\infty, \infty)(\Omega_2)} \| f \|_{L(\infty, \infty)(\Omega_1)} \lambda_{f \otimes g}(t) \text{ (by (4.3)).}
\]

Hence, by Theorem [3]

\[ \| f \otimes g \|_{L(\infty, \infty)(\Omega_1 \times \Omega_2)} \leq \| f \|_{L(\infty, \infty)(\Omega_1)} \| g \|_{L(\infty, \infty)(\Omega_2)}. \]

□
4.2. More Problems. 1. It seems to us that our approach to prove the second part of Theorem 5 can be modified to study the rearrangement inequality of Garsia-Rodemich [10, Theorem 7.3] in the $n-$dimensional case.

2. It would be of interest to follow up on the suggestion of Garsia-Rodemich and prove a version of Theorem 5 using the methods of [9].

3. It would be of interest to complete the study of tensor products with $L(\infty, \infty)$.

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