EQUIVARIANT HOMOLOGY FOR PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. We compute the cyclic homology for the cross-product algebra \( \mathcal{A}(M) \rtimes \Gamma \) of the algebra of complete symbols on a compact manifold \( M \) with action of a finite group \( \Gamma \). A spectral sequence argument shows that these groups can be identified using deRham cohomology of the fixed point manifolds \( S^*M^\rho \). In the process we obtain new results about the homologies of general cross-product algebras and provide explicit identification of the homologies for \( C^\infty(M) \rtimes \Gamma \).

1. INTRODUCTION

On a closed manifold \( M \) the (classical) pseudodifferential operators form an algebra \( \Psi^\infty(M) \). The space of smoothing operators \( \Psi^{-\infty}(M) \) is then an ideal and the quotient \( \mathcal{A}(M) := \Psi^\infty(M)/\Psi^{-\infty}(M) \) is called the algebra of complete symbols. Let \( \Gamma \) be a finite group acting on \( M \) by diffeomorphisms. Then by push-forward of operators \( \Gamma \) acts on \( \Psi^\infty(M) \) and on \( \mathcal{A}(M) \), namely if \( D \) be a pseudodifferential operator and \( g \in \Gamma \), then

\[
g.D(f) := gD(g^{-1}f) \quad \forall f \in C^\infty(M).
\]

In this paper we compute the Hochschild and cyclic homology groups of the cross-product algebra \( \mathcal{A}(M) \rtimes \Gamma \).

Results on the cyclic homology for algebras of complete symbols over compact manifolds were obtained in \([7, 27]\). In particular, these homology calculations recover the noncommutative residue of Guillemin\([16]\) and Wodzicki\([27]\). In a similar way among other things our calculations of these homology groups tells us exactly how many linearly independent equivariant traces to expect on the algebra \( \mathcal{A}(M) \rtimes \Gamma \). These traces are computed in \([13]\) where they are considered as equivariant generalization of the noncommutative residue Wodzicki\([27]\) and Guillemin\([16]\).

We shall begin with the motivation for our calculations.

As noticed in \([13]\) that certain traces on the crossed product algebra \( \mathcal{A}(M) \rtimes \Gamma \) can be considered as equivariant versions of the noncommutative residue. To a generator \( Ag \) in \( \mathcal{A}(M) \rtimes \Gamma \) and an equivariant order 1 positive elliptic operator \( D \) one associates the zeta function

\[
\zeta_{Ag,D}(z) := \text{Tr}(D^{-z}Ag).
\]

By means of stationary phase analysis near the fixed points of the diffeomorphism \( g \), a meromorphic extension of these zeta functions to whole of \( \mathbb{C} \)

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with some simple poles can be shown. Then for a fixed conjugacy class $\langle \gamma \rangle$ a trace on $\mathcal{A}(M) \rtimes \Gamma$ is obtained by

$$\text{Tr}(\langle \gamma \rangle) \left( \sum_{g \in \gamma} A_g g \right) := \text{res}_{z=0} \left( \sum_{g \in \langle \gamma \rangle} \zeta_{A_g,D}(z) \right).$$

Analogous to Guillemin [16] the equivariant traces can be used to obtain an equivariant Weyl’s formula [], namely if $\Gamma$ acts faithfully and $\pi$ is an irreducible representation of $\Gamma$ then for an invariant operator $D$

$$N_{\pi,D}(\lambda) := \sum_{\lambda_i < \lambda} \text{“multiplicity of } \pi \text{ in } V_i,” \approx \frac{C}{\dim \pi} \lambda^{\dim M - \text{order}(D)}.$$  

Other equivariant results such as an equivariant Connes trace formula as well as extensions of the logarithmic symbols based on a 2 cocycle in $H^2(S_{\log}(M) \rtimes \Gamma)$ as in [?] can also be obtained from the above mentioned traces on $\mathcal{A}(M) \rtimes \Gamma$.

Here we are interested in knowing the higher versions of these equivariant noncommutative residues.

Our result is as follows. Let $\Gamma_{\gamma} := \{g \in \Gamma, g\gamma = \gamma g\}$ be the centralizer of $g$ in $\Gamma$. Let $k_\gamma = \dim(T^* M^\gamma)$. Then

$$HH_k(\mathcal{A}(M) \rtimes \Gamma) = \sum_{\langle \gamma \rangle} H^{k_\gamma - k}(S^* M^\gamma \times S^1)^{\Gamma_{\gamma}}.$$  

Here the sum is taken over a set of representatives of the conjugacy classes. Also

$$HC_k(\mathcal{A}(M) \rtimes \Gamma) = \sum_{j \geq 0} HH_{k-2j}(\mathcal{A}(M) \rtimes \Gamma).$$

Our determination of these homology groups extend the results of [4], using also techniques from [8, 11]. Interestingly, there are some qualitatively new phenomena arising at the nontrivial conjugacy classes that are not expected from the non-equivariant case.

The cross-product algebra $\mathcal{B} := \mathcal{A}(M) \rtimes \Gamma$ has a natural filtration that comes from the order of the operators on $\mathcal{A}(M)$. We use the spectral sequence associated to this filtration in our homological computation. The first hurdle here is that the associated graded algebra $Gr(\mathcal{B}) = Gr(\mathcal{A}(M)) \rtimes \Gamma$ is noncommutative, unlike $Gr(\mathcal{A}(M))$ which is commutative. Nevertheless, this algebra is the cross-product of a commutative algebra by a finite group, and as such it preserves many features of commutativity. In particular, its Hochschild homology has a description using differential forms on the fixed point sets of the elements of the group [3]. The differentials in the spectral sequence turn out to preserve this structure. The action of the first relevant differential, $d_2$, is similar to the one in the case without group action [7], albeit technically different. Here one can exploit the structure of certain symplectic submanifolds of the cotangent bundle. Moreover, the residue trace associated to each conjugacy class of $\Gamma$ (provided that that conjugacy
class has a nonempty, connected fixed point set) will no longer have the property of being localized to a singly homogeneous component of the symbol in any coordinate neighborhood. This is in contrast with the usual case when there is no group action, the residue trace is localized on the component of homogeneity $-n$ of the complete symbol. Moreover, the study of the equivariant residue traces requires a nontrivial use of the stationary action principle.

As mentioned above, we need to know as explicitly as possible the Hochschild homology groups of $\mathcal{C}^\infty(S^*M) \rtimes \Gamma$. In the process of these computations, several new results on these groups are obtained. These results fit into the general philosophy of noncommutative geometry that Hochschild homology is the analogue of smooth forms on a compact manifold and (periodic) cyclic homology is the analogue of deRham cohomology for manifolds. These results are consistent with a famous theorem of Connes that computes the Hochschild and cyclic homology groups of $\mathcal{C}^\infty(M)$, the algebra of smooth functions on $M$. They are also consistent with the results of Baum and Connes [3] on the homology of cross-products by proper actions.

2. HOCHSCHILD AND CYCLIC HOMOLOGY FOR CROSS-PRODUCTS

We recall the definitions for Hochschild and cyclic homology of algebras. We describe the properties needed, and set up our notation for the subsequent sections. A reference for most of the results cited here is Loday [20]. Unless otherwise stated all algebras in this section are over the complex numbers and shall be unital.

The Hochschild homology of an unital algebra $\mathcal{A}$, denoted by $HH_*(\mathcal{A})$ is the homology of the complex $\mathcal{H}_*(\mathcal{A}) := (\mathcal{A}^{\otimes n+1}, b)$ where the differential $b$ is given by

$$b(a_0 \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_n) := \sum_{i=0}^{n-1} (-1)^i a_0 \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_i a_{i+1} \ldots a_n + (-1)^n a_n a_0 \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_{n-1}.$$

Let $\mathcal{A}$ be an algebra and $\Gamma$ be a finite group acting on it by $\pi : \Gamma \to Auto(\mathcal{A})$. For most purposes, $\mathcal{A}$ will be a locally convex topological algebra with jointly continuous product. We define the cross-product algebra $\mathcal{B} = \mathcal{A} \rtimes \Gamma$ as the algebra generated by elements of the form $\{a_g g | a_g \in \mathcal{A}, g \in \Gamma\}$ with the product given by

$$a_g g \cdot b_h h = a_g (b_h) g h.$$

The Hochschild homology of $\mathcal{B}$ admits a natural decomposition. For every conjugacy class $\langle \gamma \rangle$ of the group $\Gamma$, there is a subcomplex of $\mathcal{H}_*(\mathcal{B})$ given by

$$\mathcal{H}_*(\mathcal{A})_\gamma = \{ (a_{g_0} g_0 \otimes a_{g_1} g_1 \otimes \ldots \otimes a_{g_n} g_n) | g_1 g_2 \ldots g_n g_0 \in \langle \gamma \rangle \},$$
which yields the decomposition:

$$\mathcal{H}_*(B) = \bigoplus_{\langle \gamma \rangle} \mathcal{H}_*(A)_\gamma.$$ 

We shall also use the notation $L_* (A, \Gamma, \gamma)$ for $H_* (A)_\gamma$ if we need to specify the group $\Gamma$ explicitly.

Our aim is to first identify each of the homology of a conjugacy components $H_* (A)_\gamma$ with that of a certain twisted Hochschild complex.

2.1. Twisted Hochschild Complex for Commutative Algebras. We consider a commutative algebra $A$. Let $h$ be an automorphism of $A$. Let as usual $A^e = A \otimes A$. Consider the $A^e$ module $A_h$ with the same linear structure as $A$, but the module structure given by

$$(a \otimes b) \cdot c = ac \cdot h(b).$$

Furthermore let us consider the complex $C_* (A)_h = \bigoplus A^{\otimes i+1}, b_h$, where the twisted Hoschild differential $b_h$ is defined as

$$(1)$$

$$b_h(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = (a_0 h(a_1) \otimes \ldots \otimes a_n) + \sum_{i=1}^n (-1)^i(a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n).$$

In case of a group action on a manifold $M$ we shall be able to very easily identify the homology of twisted complex $C_* (C^\infty (M))_h$ with the differential forms on the fixed point manifold $M^h$. At the same time the result in this section show that twisted homology gives the same homology as the conjugacy component $H_* (C^\infty (M) \rtimes \Gamma)_{(h)}$.

Then we have the following result.

**Lemma 2.1.** We have $C_* (A)_h \simeq Tor^*_b (A, A_h)$.

**Proof.** Let $H'_* (A)$ denote the bar resolution of $A$. That is, $H'_* (A) = (A^{\otimes i+1}, b')$ is the standard projective resolution of $A$ by $A^e$ modules. Consider the map

$$\psi : H_n (A) \otimes A^e A_h \to C_n (A)_h$$

$$\psi(a_0 \otimes a_1 \otimes \ldots \otimes a_n) \otimes a : \to (a_n ah(a_0) \otimes a_1 \otimes \ldots \otimes a_{n-1}).$$

Then we check that $b_h \circ \psi = \psi \circ b' \otimes 1$, which means that $\psi$ is a morphism of complexes. The result follows from the fact that $\psi$ is an isomorphism and the definition of the $Tor$ groups. \hfill \Box

Abstractly the above result can tell us that the twisted Hochschild complex $C_* (A)_h$ has the same homology as the conjugacy component $H_* (A)_{(h)}$ but we are in the lookout for a concrete quasisomorphism.

To this end we start with another well known acyclic model. For any finite group $G$, let $\beta G$, $\beta G)_n = \mathbb{C}[G \times G \times \cdots \times G]$, be the complex endowed
with the differential
\[ d(g_0, g_1, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, g_1, \ldots, \hat{g}_i, \ldots, g_n), \]
where \( \hat{g}_i \) as usual means that the entry is omitted. This differential also comes from a simplicial object structure on \( \beta G \). It is well known that
\[ H_q(\beta G) = \begin{cases} \mathbb{C} & \text{if } q = 0 \\ 0, & \text{if } q > 0 \end{cases} \]
and hence \( \beta G \) is a free resolution for the trivial \( G \) module \( \mathbb{C} \).

For any subgroup \( G \subseteq \Gamma \), we define the complex
\[ \tilde{L}_n(A, G, h) := (C_\ast(A)_h)_n \otimes \beta G_n \]
with induced simplicial structure and differential given by
\[ b_h \otimes d(a_0g_0, a_1g_1, \ldots, a_ng_n) = (a_0h(a_1)g_1, a_2g_2, \ldots, a_ng_n) + \sum_{i=1}^{n-1} (-1)^i (a_0g_0, \ldots, a_i a_{i+1} g_{i+1}, \ldots, a_ng_n) + (-1)^n (a_n a_0g_0, a_1g_1, \ldots, a_{n-1}g_{n-1}). \]
As one would expect this \( \tilde{L}_\ast(A, G, h) \) complex to be a simply connected cover of \( H_\ast(A)_{(h)} \). (See [23] for more detailed presentation.)

An application of the Eilenburg-Zibler isomorphism gives us the following result.

**Lemma 2.2.** We have \( \tilde{L}(A, G, h) \) is quasi-isomorphic to \( C_\ast(A)_h \).

**Proof.** By definition, \( \tilde{L}(A, G, h)_n \simeq \beta G_n \otimes (C_\ast(A)_h)_n \). Thus by the Künneth formula and the Eilenberg-Zibler theorem, we have the following diagram:

\[ \begin{diagram}
\node{\tilde{L}(A, G, h)} 
\arrow{e, \pi_G} 
\node{C_\ast(A)_h} 
\arrow{s, \pi_G} 
\node{\beta G \otimes C_\ast(A)_h} 
\arrow{n, g} 
\node{\beta G \otimes C_\ast(A)_h} 
\end{diagram} \]

where we have denoted
\[ \pi_G(a_0g_0, a_1g_1, \ldots, a_ng_n) := (a_0, a_1, \ldots, a_n) \]
and similarly,
\[ \pi^G(a_0, a_1, \ldots, a_n) := (a_0e, a_1e, \ldots, a_ne), \]
and also the maps \( f \) and \( g \) are the maps for the Eilenberg-Zibler quasi-isomorphism, and \( \pi \) is the projection on the first component. (The Eilenberg-Zibler isomorphism applies because the complex \( \tilde{L}(A, G, h) \) is obtained from the product of two simplicial objects.) That is,
\[ \pi((\beta G)_l \otimes C_k(A)) = \begin{cases} 0 & \text{if } l \neq 0 \\ C_k(A) & \text{if } l = 0. \end{cases} \]
Since $H_q(\beta G) = 0$ for $q > 0$, $\pi$ turns out to be a quasi-isomorphism. Thus $\pi_G$ and $\pi^G$ are quasi-isomorphisms.

Let $A$ be a commutative unital algebra with an action of a group $\Gamma$. Let $h \in \langle \gamma \rangle$ be an element of the conjugacy class $\langle \gamma \rangle$ and $C_*(A)_h$ be the corresponding twisted complex under the action of $h$, then we can make the following identifications:

**Proposition 2.3.** The conjugacy component of the Hochschild homology $H_*(A)_h$ is quasi-isomorphic to $C_*(A)_h\Gamma_h$.

And the chain map $G : H_*(A)_{\langle \gamma \rangle} \rightarrow C_*(A)_h\Gamma_h$ is given by the explicit formula

$$G(b_0h_0, b_1h_1, \ldots, b_nh_n) = \frac{1}{|\Gamma_h|} \sum_{g_0 \in \Gamma_h} (hg_0h_0^{-1}(b_0), g_0(b_1), \ldots, g_0h_1 \ldots h_{n-1}(b_n)).$$

**Proof.** There is a covering map $\alpha : \tilde{L}(A, \Gamma, h) \rightarrow H_*(A)_h$ given by

$$\alpha(a_0g_0, a_1g_1, \ldots, a_ng_n) = (g_0^{-1}a_0h_0g_0, g_0^{-1}a_1g_1, \ldots, g_0^{-1}a_ng_n) = (g_0^{-1}(a_0)g_0^{-1}h_0g_0, g_0^{-1}(a_1)g_0^{-1}g_1, \ldots, g_0^{-1}(a_n)g_0^{-1}g_n),$$

which is a chain map. In fact, $\alpha$ is a morphism of simplicial objects and $\Gamma_h$ equivariant. We would like to lift this map $\alpha$ from $H_*(A)_h$ to $\tilde{L}(A, \Gamma, h)\Gamma_h$.

The map $\alpha$ above also restricts to a chain map on the quasi-isomorphic complex $\tilde{L}(A, \Gamma_h, h)$ which we denote by $\alpha_{\Gamma_h}$. An explicit lifting is easy to construct for $\alpha_{\Gamma_h}$ as follows:

Define, for any $g_0 \in \Gamma_h$ a linear map $T_{g_0} : L(A, \Gamma_h, h) \rightarrow \tilde{L}(A, \Gamma_h, h)$ by the formula

$$T_{g_0}(b_0h_0, b_1h_1, \ldots, b_nh_n) = (hg_0h_0^{-1}(b_0)g_0, g_0(b_1)g_0h_1, \ldots, g_0h_1 \ldots h_{i-1}(b_{i-1})g_0h_i \ldots h_n).$$

Then verify directly that

$$T_{g_0}h - bT_{g_0} = 0.$$

Let us define the map $T = \frac{1}{|\Gamma_h|} \sum_{g_0 \in \Gamma_h} T_{g_0}$. Clearly $T$ maps $L(A, \Gamma_h, h)$ to $\tilde{L}(A, \Gamma_h, h)\Gamma_h$. We next observe that $\alpha \circ T = Id_{H_*(A)_h}$ and $T \circ \alpha = Id_{C_*(A)_{\Gamma_h}}$ and that $T = \alpha^{-1}$ and therefore must be a chain map. Thus we have established the following commutative diagram of quasi-isomorphisms.

\[
\begin{array}{ccc}
C_*(A)_{\Gamma_h} & \xrightarrow{T} & L(A, \Gamma_h, h)_{\Gamma_h} \\
\pi_T & \Downarrow & \alpha_{\Gamma_h} \\
\tilde{L}(A, \Gamma, h)_{\Gamma_h} & \xrightarrow{T} & H_*(A)_{\langle \gamma \rangle} \\
\pi_{\Gamma_h} & \Downarrow & \\
L(A, \Gamma, h)_{\Gamma_h} & \xrightarrow{T} & L(A, \Gamma_h, h).
\end{array}
\]
Here $F := \alpha \circ \pi : C_*(A)^{\Gamma_h}_h \to \mathcal{H}_*(A)_\gamma$ is of the form
\begin{equation}
F(a_0, a_1, \ldots, a_n) = (a_0 h, a_1 e, \ldots a_n e).
\end{equation}

At the same time the inverse quasi-isomorphism $G := \pi \circ T : \mathcal{H}_*(A)_\gamma \to C_*(A)^{\Gamma_h}_h$ is given by the formula (2).

Thus as a consequence we immediately have that $\mathcal{H}_*(A)_\gamma \simeq C_*(A)^{\Gamma_h}_h \simeq \text{Tor}^*_A e(A, A^h)^{\Gamma_h}$.

As already mentioned earlier the computation shall be of interest in case of $A = C^\infty(M)$. We now begin our efforts to show that in this case homology of $C_*(C^\infty(M))^h$ can be described by differential forms on the fixed point sets $N^h$.

3. Local computations

Let us now specialize to the case when the algebra $A = C^\infty(V)$ is the algebra of smooth functions on a vector space $V$, and $\gamma : V \to V$ is a linear transformation. In this section we shall identify our twisted Hochschild homology that is the homology of the complex $C_*(A)_\gamma$ with the differential forms on the fixed point subspace $V^\gamma$. To this end it is is most convenient to introduce the language of Koszul complexes.

3.1. Koszul Complex. Let $R$ be a commutative ring. Let $f_1, f_2, \ldots, f_q \in R$. Let $\{v_j\}$ be a basis for $C^q$. We define the Koszul complex of $R$ generated by $f_1, f_2, \ldots, f_q$ by
$$K_\ell(R : f_1, f_2, \ldots, f_q) = R \otimes \wedge^\ell C^q$$
$$\delta(r \otimes \wedge v_1 \wedge v_2 \wedge \ldots \wedge v_1) = \sum_{j=1}^{j=q} (-1)^j (r f_j \otimes \wedge v_1 \wedge v_2 \ldots \wedge \hat{v}_j \wedge \ldots \wedge v_1).$$

This differential arises naturally from a simplicial module structure. We observe the following properties:

**Lemma 3.1.**

1. Let $R$, $R'$ be two algebras over $\mathbb{C}$. with $S \subset R$ and $S' \subset R'$ be subsets. Denote by $S \coprod S' = S \otimes 1 \cup 1 \otimes S' \subset R \otimes R'$. Then
$$K_s(R \otimes R' : S \coprod S') = K_s(R : S) \otimes K_s(R' : S').$$

2. Let $V = \mathbb{R}^n$. Let $A = C^\infty(V)$, and let $X_i$ be the coordinate functions on $V$. Then
$$\mathcal{H}_q(K_\ell(A : \{X_i\}_{i=1}^n) = \begin{cases} 0 & q > 0 \\ \mathbb{C} & q = 0. \end{cases}$$

**Proof.** Since the differential comes from a simplicial object structure the first fact is a consequence of the Eilenberg-Zibler theorem.

The second fact follows from Poincare lemma and properties of Fourier transform. \qed
3.2. **Linear Action on** $\mathbb{R}^n$. Given a linear transformation $\gamma$ of a real vector space $V$ we decompose $V$ into fixed point subspace $V^\gamma$ and an invariant complement $(1 - \gamma)V$, 

$$V = V^\gamma \oplus (1 - \gamma)V.$$  

We come now to the main result of this section which is the local version of our desired result.

**Theorem 3.2.** Let $\gamma$ be a linear automorphism of the algebra $\mathcal{A} = C^\infty(V)$. The homology of the twisted complex $(C_*(A), \delta_r)$ is then given by the space of forms on the fixed point $V^\gamma$

$$H_q(C_*(A), \gamma) \simeq \Omega^q(V^\gamma),$$

and the identification is $\Gamma$ equivariant.

**Proof.** Let $e_i$ be a basis of $V$ such that $e_i \in V^\gamma$ for $1 \leq i \leq m$ and $e_i \in (1 - \gamma)V$ for $m + 1 \leq i \leq n$. Let $X_i$ denote the corresponding coordinate functions.

Consider the Koszul complex,

$$K_*(A^e : \{X_i \otimes 1 - 1 \otimes X_i\}_{i=1}^n)$$

which is a projective resolution of $\mathcal{A}$ over $A^e$. And hence by Lemma 2.1,

$$C_*(A) \simeq_q K_*(A^e : \{X_i \otimes 1 - 1 \otimes X_i\}_{i=1}^n) \otimes_{\mathcal{A}} \mathcal{A}.$$

Observe that $K_*(A^e : \{X_i \otimes 1 - 1 \otimes X_i\}_{i=1}^n) \otimes_{\mathcal{A}} \mathcal{A}_\gamma$ can be identified with another Koszul complex namely, $K_*(A : \{X_i - \gamma(X_i)\})$ via the map

$$\psi(a \otimes b \otimes v_{i_1} \wedge \ldots \wedge v_{i_k} \otimes c) = a\gamma(b)c \otimes v_{i_1} \wedge \ldots \wedge v_{i_k}.$$  

This results in the following diagram.

$$
\begin{array}{c}
\cdots \longrightarrow (A^e \otimes \Lambda^l V) \otimes_{A^e} A_\gamma \xrightarrow{\delta \otimes 1} (A^e \otimes \Lambda^{l-1} V) \otimes_{A^e} A_\gamma \longrightarrow \cdots \\
\downarrow \hspace{1cm} \downarrow \\
\cdots \longrightarrow (A \otimes \Lambda^l V) \longrightarrow (A \otimes \Lambda^{l-1}) \longrightarrow \cdots \\
\end{array}
$$

Let $R = C^\infty(V^\gamma)$ be smooth functions on the fixed point manifold $V^\gamma$ and let $R' = C^\infty((1 - \gamma)V)$ be smooth functions on the invariant compliment of $V^\gamma$.

Since $\mathcal{A} = C^\infty(V) = C^\infty(V^\gamma) \otimes C^\infty((1 - \gamma)V) = R \otimes R'$, by Lemma 3.1\(1\),

$$K(\mathcal{A} : \{X_i - \gamma(X_i)\}_{i=1}^{n+q}) = K(R : \{X_i - \gamma(X_i)\}_{i=1}^m) \otimes K(R' : \{X_i - \gamma(X_i)\}_{i=m+1}^{m+q})$$

$$\simeq K(R : \{0\}) \otimes K(R' : \{X_i\}_{i=m+1}^{m+q}).$$

Now by applying Lemma 3.1\(2\) to $K(R' : \{X_i\}_{i=m+1}^{m+q})$, we have

$$K(\mathcal{A} : \{X_i - \gamma(X_i)\}_{i=1}^{n+q}) = K(R : \{0\}) \simeq \Omega(V^\gamma).$$


Here the last identification is due to the following theorem, which we formulate only in the \(\sigma\)-compact case, for simplicity.

**Theorem 3.3** (Connes’ HKR Theorem). Let \(X\) be a smooth, \(\sigma\)-compact manifold. Then the Hochschild homology of the algebra \(A = \mathcal{C}^\infty(X)\) is given by the differential forms on \(X\). The map

\[
\chi_k(a_0 \otimes a_1 \otimes \ldots \otimes a_k) \mapsto a_0 da_1 da_2 \ldots da_k
\]

induces an isomorphism

\[
HH_k(\mathcal{C}^\infty(X)) = \Omega_k(X)
\]

**Proof.** Let us consider \(X = \mathbb{R}^n\), which is sufficient to prove the equality in \(\mathbb{R}\). It is easily seen that the map \(\chi\) is a chain map. We define the inverse map (which is only well define on the homology)

\[
E_k : \Omega^k(\mathbb{R}^n) \rightarrow \mathcal{H}_*(\mathcal{C}^\infty(\mathbb{R}^n))
\]

Then the following diagram commutes.

\[
\begin{array}{ccc}
\Omega^k(\mathbb{R}^n) & \xrightarrow{\chi_k} & \mathcal{H}_k(A) \\
\downarrow 0 & & \downarrow b \\
\Omega^{k-1}(\mathbb{R}^n) & \xrightarrow{\chi_{k-1}} & \mathcal{H}_{k-1}(A).
\end{array}
\]

Then the map \(\chi_k\) induces an isomorphism

\[
HH_k(\mathcal{C}^\infty(\mathbb{R}^n)) \simeq \Omega^k(\mathbb{R}^n).
\]

The case of a closed manifold \(X\) will be proved in the next section. \(\square\)

Thus combining all this information we get the following diagram.

\[
\begin{array}{ccc}
\mathcal{C}_*(A)_{\gamma} & \xrightarrow{\sigma} & \mathcal{H}_*(A) \otimes \mathcal{A}_\gamma \\
\downarrow \mathcal{K}(A^c : \{X_i \otimes 1 - 1 \otimes X_i\}) & & \downarrow Tor^\mathcal{A}(A, A_{\gamma}) \\
\Omega^*(V^\gamma) & \xrightarrow{\gamma} & \mathcal{H}_*(A) \otimes \mathcal{A}_\gamma \\
\end{array}
\]

We note that all the maps involved are \(\Gamma\) equivariant. \(\square\)

**Corollary 3.4.** Let \(\Gamma\) be a finite group acting linearly on \(A = \mathcal{C}^\infty(V)\) and

\[B = A \rtimes \Gamma.\]

And let \(\gamma \in \Gamma\). Then the \(\langle \gamma \rangle\) component of Hochschild homology \(HH(B)_{\gamma}\) is given by

\[
HH_{\gamma}(B) = \Omega(V^\gamma)^{\Gamma_{\gamma}}.
\]

The following result now follows immediately by Proposition 2.3.
4. Localization with Groupoids and Sheaf

We return to the situation where $\mathcal{A} = \mathcal{C}^\infty(M)$ is the algebra of smooth functions over a closed manifold $M$ and $\Gamma$ acts by diffeomorphisms. In this section we shall obtain the “normalized complex” for the Hochschild homology of our cross-product algebra $\mathcal{C}^\infty(M) \rtimes \Gamma$. To this end we identify the cross-product algebra as usual to the convolution algebra over the transformation groupoid $M \rtimes \Gamma$. The normalized complex thus provides a complex of sheaves of germs of smooth functions that reduces the calculation of Hochschild homology to the previous example of linear action on euclidian space.

Recall that the transformation groupoid as a space is just $G = M \times \Gamma := \{(x, g) | x \in M, g \in \Gamma\}$ with units $M$ and the source, the range and the composition maps given by

- $s(x, g) = g^{-1}x$ and $r(x, g) = x$
- $(x, g).(y, h) = (x, gh)$ when $x = g.y$

and the inverse defined by $(x, g)^{-1} := (g^{-1}x, g^{-1})$. The convolution algebra of $G$ is the space $\mathcal{C}^\infty(G)$ with the product given by

$$f_1 \ast f_2(x, g) := \sum_{h \in \Gamma} f_1(x, h).f_2((x, h)^{-1}.(x, g))$$

We can identify the cross-product algebra $\mathcal{B} = \mathcal{C}^\infty(M) \rtimes \Gamma$ with the convolution algebra $\mathcal{C}^\infty(G)$, by the map

$$\mathcal{C}^\infty(G) \ni f \mapsto \sum_{g \in \Gamma} f_g g \in \mathcal{C}^\infty(M) \rtimes \Gamma$$

with $f_g(x) = f(x, g)$.

This leads to another description of the Hochschild and cyclic complexes for $\mathcal{B}$. First define $\Phi : \mathcal{B}^\otimes n \simeq \mathcal{C}^\infty(G)^{\otimes n} \to \mathcal{C}^\infty(G^n)$ as a map of nuclear Fréchet algebras by

$$\Phi((f_0 \otimes f_1 \otimes \ldots \otimes f_{n-1})(a_0, a_1, \ldots a_{n-1}) = f_0(a_0).f_1(a_1)\ldots f_{n-1}(a_{n-1})$$

where $f_i \in \mathcal{C}^\infty(G)$ and $a_i := (x_i, g_i) \in G$. Then $\Phi$ is an isomorphism [15]. With this in mind, the Hochschild differential takes the form

$$b(F)(a_0, a_1, \ldots, a_{n-1})$$

$$= \sum_{i=0}^{n-2} (-1)^i \sum_{\gamma \in \Gamma} F(a_0, a_1, \ldots, a_{i-1}, (x_i, \gamma), (\gamma^{-1}x_i, \gamma^{-1}g_i), a_{i+1}, \ldots, a_{n-1})$$

$$+ (-1)^{n-1} \sum_{\gamma \in \Gamma} F((\gamma^{-1}x_i, \gamma^{-1}g_i), a_1, \ldots, a_{n-1}, (x_o, \gamma)).$$

It turns out (see [8]) that the normalized complex for the convolution algebra $\mathcal{C}^\infty(G)$ provides a complex of sheaves of germs of smooth functions over certain “loop spaces”. To observe this let us denote by $B^{(n)} \subset G^{n+1} := \{(a_0, a_1, \ldots, a_n) | s(a_i) = r(a_{i+1}) \forall 0 \leq i \leq n\}$ the space of loops in $G^{n+1}$ Here
and through this section we shall use the convention $n + 1 = 0$ while talking about the loops space $B^{(n)}$. For example, $B^{(0)} = \{ (x, g) \in G | g^{-1}x = x \}$ is the disjoint union of the fixed point manifolds.

Let's denote by $I_n$ the submodule (in fact an ideal) of $C^\infty(G^{n+1})$ defined by

$$I_n = \{ f \in C^\infty(G^{n+1}) | \text{ supp}(f) \cap B^{(n)} = \emptyset \}.$$  

Then we have the following result.

**Proposition 4.1.** The complex $(I_n, b)$ is an acyclic subcomplex of $\mathcal{H}_n(B)$ and hence the quotient map $B^{\otimes n+1} \rightarrow B^{\otimes n+1}/I_n$ is a quasi-isomorphism.

**Proof.** Let $U_j^{n+1} \subset G^{n+1} = \{(a_0, a_1, \ldots, a_n) | s(a_i) \neq r(a_{i+1}) \}$ be a sequence of open sets. This yields an filtration of the complex $B^{\otimes n+1}$ as

$$F_i = F_i B^{\otimes n+1} = \sum_{j=0}^{i} \{ f \mid \text{ supp}(f) \subseteq U_j^{n+1} \}.$$  

Also, let $F_i B^{\otimes n+1} := F_n B^{\otimes n+1}$ for $i > n$. Then $F_i$ is a filtration on the complex $(I_n, b)$. We can check that in the corresponding spectral sequence $E^0_n = (\oplus F_{i+1}/F_i, b)$ is acyclic. \hfill $\Box$

Thus the Hochschild chains are reduced from functions on $G^{n+1}$ to germs of function near $B^{(n)}$. Denote by $i_n : B^{(0)} \rightarrow G^{n+1}$ the loop inclusion map $(x, g) \rightarrow ((g^{-1}x, e), (g^{-1}x, e), \ldots, (g^{-1}x, e))$. Let

$$\mathcal{F}_n = i_n^{-1} C^\infty(G^{n+1})$$

be the pull-back of the sheaf of smooth functions on $G^{n+1}$ to $B^{(n)}$.  

To make a more geometrical identification let us denote by the projection $\pi : B^{(n)} \rightarrow B^{(0)}$ defined by

$$\pi((x_0, g_0), (x_1, g_1), \ldots, (x_n, g_n)) = (x_0, g_0), (x_1, g_1) \cdots (x_n, g_n) = (x_0, g_0, g_1 \cdots g_n)$$

is a local homeomorphism. It is in fact a diffeomorphism on when restricted to each connected component. For a fixed $u = (g_0, g_1, \ldots, g_n) \in \Gamma^{n+1}$, let $\gamma = g_0 g_1 \ldots g_n$ and let $B^{(n)}_u$ denote the component of $B^{(n)}$ consisting of $\{(x_0, g_0), (x_1, g_1), \ldots, (x_n, g_n) \in B^{(n)} \}$. Then $\pi$ maps $B^{(n)}_u$ diffeomorphically onto $B^{(0)}_u \cong M^n$. Noting again that $B^{(0)}$ is just the disjoint union of the fixed point manifolds, we have the following.

**Proposition 4.2.** The pull back of the sheaf $\mathcal{F}_n$ under the local diffeomorphism $\pi : B^{(n)} \rightarrow B^{(0)}$ gives an isomorphism

$$\Gamma(B^{(n)}_u, \pi^{-1} \mathcal{F}_n) = B^{\otimes n+1}/I_n,$$

as vector spaces.
Proof. Every point \( u \in B^{(n)} \) has a neighborhood \( W_u \) in \( G^{n+1} \) which is diffeomorphic to \( V^{n+1} \) for some small enough neighborhood \( V \) of \( s(a_u) = g_0^{-1} a_u \) in \( G \). One possible identification can be \( W_u = (V, g_0) \times (g_0 V, g_1) \times \ldots \times (g_0 g_1 \ldots g_{n-1} V, g_n) \). Then this diffeomorphism gives a map \( \psi : C^\infty(V^{n+1}) \to C^\infty(W_u) \). Thus after covering \( B^{(n)} \) with small enough neighborhoods like \( W_u \), would match the global sections in \( \Gamma(B^{(n)}, \pi^{-1} F_n) \) to functions smoothly supported in some neighborhood of \( B^{(n)} \) in \( G^{n+1} \). □

Using the proposition above, one may obtain a sheafified version of Proposition 2.3 with an induced differential on \( F_n \).

Remark 4.3. At each \( a = (x, \gamma) \in B^{(0)} \), the map \( \phi : C^\infty(V^{n+1}) \to C^\infty(W) \) defined above induces an isomorphism \( \phi_a \) on the stalks \( (F_n)_a = C^\infty(M)_{\gamma^{-1} x} \) to the stalk \( (C^\infty(G^{n+1}))_{(a,a,a)} \) of the sheaf of smooth functions which are germs of smooth functions on \( G^{n+1} \) at \( (a,a,a,\ldots,a) \). And this isomorphism on the stalk is given by the formula

\[
\phi_a(f_0 \otimes f_1 \otimes \ldots \otimes f_n) \to (f_0 \gamma \otimes f_1 e \otimes \ldots \otimes f_n e).
\]

Under \( \phi \) at every point \((x, \gamma) \in B^{(0)}\) the stack \((F_n)_{(x,\gamma)}\) have a simplicial structure given by the differential \( b_\gamma \) from Equation (1) and the map \( \phi \) is a chain map. Using Corollary 3.4, we can compute the homology of each stalk. In the following section, we study the properties of sheaves of complexes for which a quasi-isomorphism on stalks give global quasi-isomorphisms on complexes of global sections.

4.1. Sheaves. Let \( X \) be a compact Hausdorff space. A sheaf \( S \) on \( X \) is called flabby if for any open set \( U \subseteq X \), the restriction map

\[
\text{res}_{XU} : \Gamma(X, S) \to \Gamma(U, S)
\]

is surjective.

Let \( K \) be a sheaf of unital algebras on \( X \). In particular, we assume that the restriction maps are unital \((1 \in \Gamma(X, K)\) and the image under restriction \( \text{res}_{XU}(1) \) is a unit in \( \Gamma(U, K) \)). We say that the sheaf \( K \) has a partition of unity, if for any locally finite cover \( U_\alpha \) of \( X \) there exists \( \Phi_\alpha \in \Gamma(X, K) \) such that

1. \( \text{supp}(\Phi_\alpha) \subseteq U_\alpha \).
2. \( \sum_\alpha \Phi_\alpha = 1 \).

Let \( F_i \) be flabby sheaves of \( K \) modules. By this we mean that for any open set \( U \subseteq X \), the space of sections \( \Gamma(U, F_i) \) is a module over \( \Gamma(U, K) \). Let

\[
F = F_0 \xleftarrow{d_0} F_1 \xleftarrow{d_1} F_2 \xleftarrow{d_2} \ldots
\]

be a sheaf of complexes such that each \( d_i \) is \( K \)-linear.

Lemma 4.4. : Let \( F \) be a flabby sheaf of complexes \( F \) over a sheaf of unital algebra \( K \). If \( K \) has partition of unity then

\[
\Gamma(X, H_*(F)) \cong H_*(\Gamma(X, F)).
\]
Proof. Let \( b \in \Gamma(X, \mathcal{F}_*) \) be such that \( d^X_s(b) = 0 \). So \([b] \in \mathcal{H}_s(\Gamma(X, \mathcal{F})) \).

For any open set \( U \subseteq X \), define

\[
b^U = \text{res}_{XU}(b) \quad \text{and} \quad [b^U] \in \Gamma(U, \mathcal{H}_s(\mathcal{F})).
\]

Then for any open cover \( \{U_\alpha\} \) of \( X \),

\[
\text{res}_{U_\alpha \cap U_\beta}([b^U_\alpha]) = [b^U_\alpha \cap U_\beta] = \text{res}_{U_\beta \cap U_\alpha}([b^U_\beta]).
\]

Hence there is a global section \( \Phi(b) \in \Gamma(X, \mathcal{H}_s(\mathcal{F})) \). It follows from the definition that \( \Phi(b) \) depends only on the class of \([b] \in \mathcal{H}_s(\Gamma(X, \mathcal{F})) \).

The map \( \Phi \) above \( \mathcal{H}_s(\Gamma(X, \mathcal{F})) \rightarrow \Gamma(X, \mathcal{H}_s(\mathcal{F})) \) is always defined. We now prove that under our assumptions on \( k \in \mathcal{F} \), it is in fact an isomorphism.

We construct an inverse as follows. Choose an open cover \( U_\alpha \) and a subordinate partition of unity \( \Phi_\alpha \). Let \( a \in \Gamma(X, \mathcal{H}_s(\mathcal{F})) \) and \( a_\alpha = \text{res}_{XU_\alpha}(a) \in \Gamma(U_\alpha, \mathcal{H}_s(\mathcal{F})) \). Pick representatives \( b_\alpha \in \Gamma(U_\alpha, \mathcal{F}_*) \) with \([b_\alpha] = a_\alpha \). Since \( \mathcal{F}_* \) is flabby, there exists a \( \tilde{b}_\alpha \in \Gamma(X, \mathcal{F}_*) \) so that \( \text{res}_{XU_\alpha}(\tilde{b}_\alpha) = b_\alpha \).

Let \( b_1 = \sum \Phi_\alpha \tilde{b}_\alpha \in \Gamma(X, \mathcal{F}_*) \). Then \( d_s(b_1)|_{U_\alpha} = 0 \) and hence \( d_s(b) = 0 \).

For any other choice of representatives and partition of unity or cover, we can see that the class of \( b_1, [b_1] \in \mathcal{H}_s(X, \mathcal{F}) \), is independently defined. We say that \( \sigma(a) = b_1 \), and from definition of \( \sigma \) and \( \Phi \), they are inverses of each other. \( \square \)

5. Smooth Manifolds

Let \( M \) be a closed manifold. We are now ready to find the Hochschild and cyclic homology of \( \mathcal{C}^\infty(M) \times \Gamma \) using localization argument and the results for \( \mathbb{R}^n \) of the previous sections.

Recall that for every conjugacy class \( \langle \gamma \rangle \) of the group \( \Gamma \), there is a subcomplex of \( \mathcal{H}_s(\mathcal{B}) \) given by \( \mathcal{H}_s(\mathcal{B})_{\gamma} = \{(a_{g_0}g_0 \otimes a_{g_1}g_1 \otimes \cdots \otimes a_{g_n}g_n)|g_1g_2 \cdots g_n \in \langle \gamma \rangle \} \) and which yields the decomposition : \( \mathcal{H}_s(\mathcal{B}) = \bigoplus_{\langle \gamma \rangle} \mathcal{H}_s(\mathcal{B})_{\gamma} \).

Fix a conjugacy class by choosing a representative \( \gamma \). We denote by \( \Gamma_{\gamma} \) the stabilizer of \( \gamma \) in \( \Gamma \). Also if \( \gamma' \in \langle \gamma \rangle \), then we denote by \( S_{\gamma'\gamma} = \{k \in \Gamma | k\gamma = k\gamma'k^{-1} \} \). Define a map \( \chi_s^\gamma : \mathcal{H}_s(\mathcal{B})_{\gamma} \rightarrow \Omega(M\gamma) \) from the Hochschild complex to the forms on the fixed point manifold, by

\[
\chi^\gamma_m(a_{g_0}g_0 \otimes a_{g_1}g_1 \otimes \cdots \otimes a_{g_n}g_n) := \\
\left( \frac{1}{|\Gamma_{\gamma}|} \sum_{h \in S_{\gamma'\gamma}} \gamma hg_0^{-1}(a_0)dh(a_1)dg_1(a_2) \cdots dhg_1g_2 \cdots g_n(a_n) \right)_{M\gamma},
\]

where \( \gamma' = g_1g_2 \cdots g_n g_0 \). Then \( \chi^\gamma_m \circ b = 0 \) and \( \chi^\gamma_m \circ B = d_{M\gamma} \) is the de Rham differential on \( M\gamma \).

**Theorem 5.1.** The Hochschild homology of the complex \( \mathcal{H}_s(\mathcal{B})_{\gamma} \) is isomorphic to the \( \Gamma_{\gamma} \) invariant forms on the fixed point manifold \( M\gamma \). More precisely, the map

\[
\chi_s^\gamma : \mathcal{H}_s(\mathcal{B})_{\gamma} \rightarrow \Omega^*(M\gamma)^{\Gamma_{\gamma}}.
\]
is a quasi-isomorphism between \((\mathcal{H}_n(B)_\gamma, b)\) and \((\Omega^n(M^\gamma)^\Gamma, 0)\)

Proof. By Proposition 4.1, the map \(\Phi : \mathcal{H}_\ast(B) \to \Gamma(B(0)^\gamma, F_n^\gamma), b_\gamma)\) is an isomorphism. The sheaves \(F_n\) are flabby and are modules over the sheaf of invariant functions \(k = C^\infty(M)^\Gamma\). This is the sheaf associated to the presheaf \(U \to C^\infty(U)^\gamma\). Also the differentials \(b_\gamma\) are \(C^\infty(M)^\Gamma\)-linear. Similarly, the sheaf of differential \(n\)-forms \(\Omega_n\) too is a flabby sheaf of \(k\) modules. Then by Theorem 3.3, the chain map \(\widetilde{\chi}_n(a_0, a_1, \ldots a_n) \to a_0da_1 \ldots da_n\) is in fact an quasi-isomorphism on each stalk and inverse (defined only on homology) given by

\[
E_k(a_0da_1da_2 \ldots da_k) = \frac{1}{k!} \sum_{\pi \in S_k} \text{sign}(\pi)(\tilde{a}_0 \gamma \otimes \tilde{a}_{\pi(1)}e \otimes \tilde{a}_{\pi(2)}e \otimes \ldots \otimes \tilde{a}_{\pi(k)}e).
\]

By Lemma 4.4, \(\tilde{\chi}\) is an isomorphism from \(\Gamma(B(0)^\gamma, F_n) \to \Omega(M^\gamma)\).

Thus the theorem follows by observing that \(\chi_0 = \tilde{\chi} \circ \Phi\). □

Theorem 5.2.

\[
HC_k(B)_\gamma = \Omega^k(M^\gamma)^\Gamma, \bigoplus_{j>0} H_{de Rham}^{k-2j}(M^\gamma)^\Gamma.
\]

Proof. Since \(\chi : (B_\ast(B)_\gamma, b, B) \to (\Omega^\ast(M^\gamma)^\Gamma, 0, d)\) is a map of mixed complexes which is an isomorphism on the columns by Theorem 5.1 above, \(\chi^\gamma\) must be an isomorphism on the total complexes of these mixed complexes. □

Corollary 5.3.

\[
HP_k(B)_\gamma = \sum_{j \in \mathbb{Z}} H_{de Rham}^{k-2j}(M^\gamma)^\Gamma.
\]

Proof. Since \(HH_n(B)_\gamma = 0\) for \(n > \dim(M)\), the periodicity map \(S : HC_{n+2}(B)_\gamma \to HC_n(B)_\gamma\) is an isomorphism by the SBI-exact sequence. In particular, \(\lim_{\leftarrow} HC_\ast(B)_\gamma = \sum_{j \in \mathbb{Z}} H_{de Rham}^{k-2j}(M^\gamma)^\Gamma\). Also \(\lim_{\leftarrow} CC_\ast(B)_\gamma = 0\). □

6. Topologically Filtered Algebra

The algebra of complete symbols with the topology that is described below, is not a topological algebra in the above sense because the multiplication is not jointly continuous. Hence we need a larger category, namely that of topologically filtered algebra defined in [4]. Let \(A\) be a algebra with filtration \(A = \bigcup_{p \in \mathbb{Z}} F_pA\). That is to say, each \(F_pA\) is a subspace \(F_pA \subset F_{p+1}A\) and the multiplication map takes \(F_pA \times F_qA \to F_{p+q}A\). We would say that \(A\) is a filtered algebra for short.

Since, by definition, \(F_0A\) is a subalgebra of \(A\) and \(F_{-1}A\) is an ideal of \(F_0A\), \(M_0 := F_0A/F_{-1}A\) is naturally an algebra and each \(M_j := F_pA/F_{p-j}A\) is
a module over \( \mathcal{M}_0 \). Similarly, \( \mathcal{I} := \bigcap_p \mathcal{F}_p A \) is an ideal in \( A \). We call our algebra a symbol algebra if \( \mathcal{I} = 0 \). We only consider symbol algebra for now. That is, if \( A' \) is any filtered algebra, we consider the algebra \( A = A'/\mathcal{I} \).

**Definition 6.1.** We say that a filter algebra is **topologically filtered** if

1. \( A \) is a symbol algebra, that is \( \mathcal{I} := \bigcap_p \mathcal{F}_p A = 0 \).
2. Each \( \mathcal{M}^j_p = \mathcal{F}_p A/\mathcal{F}_{p-j} A \) is a nuclear Frechet space for all \( p \) and each \( j \).
3. Each module map \( \mathcal{M}^j_p \otimes \mathcal{M}^j_q \rightarrow \mathcal{M}^j_{p+q} \) induced by the multiplication in \( A \) is continuous.
   
   We call an element \( P \in \mathcal{F}_1 A \) **elliptic** if it is invertible and \( P^n \in \mathcal{F}_n A \) for all integers \( n \).
4. There exists an elliptic element such that the map \( \mathcal{F}_n A/\mathcal{F}_{n-1} A \ni [P^n] \rightarrow P^n \in \mathcal{F}_n A \) gives a linear splitting of \( \mathcal{F}_{n-1} A \hookrightarrow \mathcal{F}_n A \rightarrow \mathcal{F}_n A/\mathcal{F}_{n-1} A \).

If \( A \) is a topologically filtered algebra then using the existence of an elliptic element, we have

\[
\mathcal{F}_p A = \mathcal{F}_{p-1} \oplus \mathcal{F}_p A/\mathcal{F}_{p-1} A \\
= \mathcal{F}_{p-2} \oplus \mathcal{F}_p A/\mathcal{F}_{p-1} A \oplus \mathcal{F}_{p-1} A/\mathcal{F}_{p-2} A \\
= \mathcal{F}_{p-2} \oplus \mathcal{F}_p A/\mathcal{F}_{p-2} A.
\]

Since \( \mathcal{I} = 0 \), by repeating the iterations, we obtain

\[
\mathcal{F}_p A = \lim_{\rightarrow p} \mathcal{F}_p A/\mathcal{F}_{p-k} A = \prod_{j \leq p} \mathcal{F}_j A/\mathcal{F}_{j-1} A.
\]

We use this description to endow \( \mathcal{F}_p A \) with the projective limit topology from the Frechet topologies on \( \mathcal{F}_p A/\mathcal{F}_{p-k} A \). \( A \) is then endowed with the inductive limit topology from \( A = \lim_{\rightarrow} \mathcal{F}_p A = \prod_p \mathcal{F}_p A/\mathcal{F}_{p-1} A \). Since strong inductive limits of nuclear spaces is nuclear, the topology on \( A \) is nuclear. We call the resultant topology the weak topology on \( A \). (The inductive limit topologies are ‘strong’ topologies. The nomenclature here is to distinguish the inductive limit topology on \( A \) with yet another topology which is stronger.) Since \( A \) is a filtered algebra \( A_0 := \mathcal{F}_0 A \) is a subalgebra and further \( A_{-1} := \mathcal{F}_{-1} A \) is an ideal in \( A_0 \). By Axiom \( 4 \) each \( A_i = \mathcal{F}_i A/\mathcal{F}_{i-1} A \) is a module over \( A_0 = A_0/A_{-1} \) generated by a single element \([P^i]\).

**Proposition 6.2.** The multiplication in \( A \) is separately continuous with respect to the weak topology. The multiplication is jointly continuous on the subalgebra \( A_0 = \mathcal{F}_0 A \).

**Proof.** First we prove that multiplication is jointly continuous on \( A_0 \). This is an immediate consequence of universal property of projective limits. Since the multiplication map on \( \mathcal{F}_0 A/\mathcal{F}_{-j} A \otimes \mathcal{F}_0 A/\mathcal{F}_{-j} A \rightarrow \mathcal{F}_0 A/\mathcal{F}_{-j} A \) is continuous for each \( j \), by composition there is a continuous map

\[
\phi_j : \lim_{\rightarrow} \mathcal{F}_0 A/\mathcal{F}_{-j} A \otimes \lim_{\rightarrow} \mathcal{F}_0 A/\mathcal{F}_{-j} A \rightarrow \mathcal{F}_0 A/\mathcal{F}_{-j} A.
\]
Using the universal property of \( \lim \), there is thus a unique continuous map on \( \phi : \lim F_0A/F_{-j}A \otimes \lim F_0A/F_{-j}A \to \lim F_0A/F_{-j}A = A_0 \). To complete the proof, we must verify that the map \( \phi \) is indeed the multiplication map. But multiplication on \( A_0 \) composed with the projection \( A_0 \to F_0A/F_{-1}A \) is just the map \( \phi_j \) above and hence the multiplication must be \( \phi \) by uniqueness (as algebraic maps) on the projective limit.

The separate continuity of the multiplication on \( A \) can be proved similarly.

Although a topologically filtered algebra \( A \) is a nuclear space and the completed projective tensor product \( A_{\hat{\otimes}n} \) is again nuclear by Proposition 6.2, the Hochschild boundary map \( b \) may not be continuous on \( A_{\hat{\otimes}n} \). We could define a Hochschild complex for \( A \) in the following fashion.

Let

\[
F_p := \sum_{\sum p_i \leq p} F_{p_0}A \hat{\otimes} F_{p_1}A \hat{\otimes} \ldots \hat{\otimes} F_{p_n}A.
\]

Let \( F_pC_n(A) := \lim F'_p/F'_{p-f} \). Then the differentials \( b \) and \( b' \) are continuous on each filtration. And if \( 1 \in F_0A \) then the operators \( s, t, N \) and \( B \) induce continuous maps on \( F_pC_n(A) \). Denote by \( F_pHH_s(A) \) the homology of the complex \( F_pC_s(A) \). We define the Hochschild homology of \( A \) to be \( HH_s(A) = \lim F_pHH_s(A) \).

Since it is often useful to use spectral sequences to compute homologies for such filtered algebras, it is useful to describe the topology on the associated graded algebra of \( A \) by using the identification

\[
G_\tau(A) = \lim \bigoplus_{p=-N}^N F_pA/F_{p-1}A.
\]

And the topology on the Hochschild complex for \( G_\tau(A) \) is defined by

\[
H_n(G_\tau(A)) := \lim \left( \bigoplus_{p=-N}^N F_pA/F_{p-1}A \right)^{\otimes n+1}.
\]
Again the Hochschild boundary map would be continuous with respect to this topology. The homogeneous components of $\mathcal{G}(\mathcal{A})$ can be defined by

$$\mathcal{H}_n(\mathcal{G}(\mathcal{A}))_p := \lim_{\rightarrow} \bigoplus_{p = -N}^{N} \otimes k_j \mathcal{F}_{k_j} \mathcal{A}/\mathcal{F}_{k_j-1} \mathcal{A},$$

where $-N \leq k_j \leq N$ and $\sum k_i \leq p$.

The following examples are of interest. The algebra of complete symbols $\mathcal{A} = \Psi^\infty(M)/\Psi^{-\infty}(M)$ on a closed manifold $M$ is a topologically filtered algebra with the filtration given by the order of an operator $\mathcal{A} = \bigcup_m \Psi^m(M)/\Psi^{-\infty}(M)$. If $P$ be an elliptic operator in $\Psi^1(M)$ then the principal symbol map $\rho := \sigma(P) \rightarrow P$ defines a splitting of $\Psi^m(M) \rightarrow C^\infty(S^*M)_{\rho_m}$. Furthermore each $\Psi^m(M)/\Psi^{m-j}(M)$ is isomorphic to $\Gamma^\infty(S^*M \times \mathbb{C}^j : S^*M)$ and hence has a Frechet topology. As described before we would topologize $\mathcal{A}$ by

$$\mathcal{A} = \lim_{\rightarrow} \lim_{\leftarrow} \Psi^m(M)/\Psi^{m-j}(M).$$

It's important here to note that the multiplication on the algebra of complete symbols is not jointly continuous, but only separately continuous.

From above it also follows that

$$\mathcal{G}(\mathcal{A}) \simeq \bigoplus_{m \in \mathbb{Z}} C^\infty(S^*M)_{\rho^m} = C^\infty(S^*M) \otimes \mathbb{C}[\rho, \rho^{-1}].$$

7. Homology of Complete Symbols

Let $\mathcal{A}$ be the algebra of complete symbols on a closed manifold $M$, that is,

$$\mathcal{A}(M) = \Psi^\infty(M)/\Psi^{-\infty}(M) = \bigcup_m \Psi^m(M)/\Psi^{-\infty}(M) = \bigcup_m \mathcal{F}_m \mathcal{A}(M).$$

A group $\Gamma$ acting smoothly on $M$ induces an action on the algebra $\mathcal{A}(M)$ by $(\gamma, A) \rightarrow \gamma A \gamma^{-1}$. This action preserves the filtration on the algebra $\mathcal{A}(M)$. Thus the algebraic cross-product algebra $\mathcal{B}(M) = \mathcal{A}(M) \times \Gamma$ is again a filtered algebra with induced filtration

$$\mathcal{F}_m \mathcal{B}(M) = \mathcal{F}_m \mathcal{A}(M) \times \Gamma$$

. We would compute the Hochschild and cyclic homology of $\mathcal{B}(M)$ using the spectral sequence associated with the above filtration. But first we prepare some background in the symplectic structure of the cotangent bundle $T^*M$.

8. Symplectic Poisson Structure

**Definition 8.1.** Let $(X, \omega)$ be a symplectic manifold. A submanifold $i : Y \hookrightarrow X$ is called a symplectic submanifold if the pull-back $i^*\omega$ is a symplectic form on $Y$. Since $i^*\omega$ is certainly a closed form, it is only required that it is non-degenerate on $Y$ to be a symplectic form.
A symplectic form on $X$ gives rise to a Poisson structure on $C^\infty(X)$. For $f \in C^\infty(X, \mathbb{R})$, the Hamiltonian vector field generate by $f$ is the unique vector field $X_f$ such that $df = i(X_f)\omega$. Poisson structure on $X$ can now be defined by

$$\{f, g\}_X = \omega(X_f, X_g) \quad \forall f, g \in C^\infty(X, \mathbb{R}).$$

9. Normal Darboux Coordinates

Let $X$ be a symplectic manifold and $Y$ be a symmetric submanifold at a point $x \in Y$. The tangent space $T_x X$ decomposes to

$$T_x X = T_x Y \oplus T_x Y^\perp,$$

where $T_x X^\perp$ is the symplectic orthogonal to $T_x Y$. Let $TY^\perp$ be the subbundle on $Y$ with fiber $T_x Y^\perp$ at each $x$. We call $TY^\perp$ as the symplectic normal bundle.

For a submanifold $Y$ of $X$, we define the inhalator $TY^\perp$ of $Y$ is the subbundle of the pull-back $TX|_Y$ defined by

$$TY^\perp = \{ Y \in \Gamma^\infty(TX)|_Y \text{ such that } \omega|_Y(X, Y) = 0, \forall X \in \Gamma^\infty(TY) \}.$$

The following lemma states that symplectic Poisson structure on $M$ naturally restricts to the symplectic Poisson structure on a symplectic submanifold $N$.

**Lemma 9.1.** A submanifold $Y$ of the symplectic manifold $(X, \omega)$ is symplectic if and only if $TY^\perp \cap TY = 0$

We assume for the rest of the section that $X$ is a symplectic manifold and $Y$ a symplectic submanifold of $X$.

Let $U(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ be a Darboux coordinate chart at $x$ in $X$.

We say $U$ is normalDarboux chart if

$$U \cap Y = \{ x_{k+1} = \cdots = x_n = 0 = \xi_{k+1} = \cdots = \xi_n \}$$

and

$U \cap Y$ is a trivializing neighborhood of $TY^\perp$ such that $(x_{k+1}, \ldots, x_n, \xi_{k+1}, \ldots, \xi_n)$ give a trivialization. This is to say

$$U \cong T(U \cap Y)^\perp.$$

We use the short hand $U(x_1, x_2, \xi_1, \xi_2)$ with $x_1 = (x_1, \ldots, x_k)$ and $x_2 = (x_{k+1}, \ldots, x_n)$ and so on.

We consider $TY^\perp$ as the default normal bundle to $Y$ in $X$. Any local chart at a point $x \in Y$ would trivialize $TY^\perp$ on $Y$. If $\phi : U(x_1, x_2, \xi_1, \xi_2) \rightarrow U(y_1, y_2, \eta_{s1}, \eta_{s2})$ be a change of normal coordinates then it is a bundle map

$$\phi : T(U \cap Y)^\perp \rightarrow T(U \cap Y)^\perp$$

(and hance is linear in $x_2$ and $\xi_2$).
Proposition 9.2. Let $Y$ be a symplectic submanifold of $X$. Then there exists an extension $C^\infty(Y) \rightarrow C^\infty(X)$ denoted by $f \rightarrow \hat{f}$ such that
\[
\{\hat{f}, \hat{g}\}_{X|Y} = \{f, g\}_Y \text{ for all } f, g \in C^\infty(Y).
\]

Proof. Let $\{U_\alpha\}$ be a cover of $Y$ by normal Darboux coordinates. Assume that $\cup \alpha U_\alpha$ is a tubular neighbor of $Y$ and $\varphi_\alpha \prec U_\alpha \cap Y$ is a partition of unity subordinate to this cover.

In each coordinate chart, let $\pi_\alpha : U_\alpha \rightarrow U_\alpha \cap Y$ be the projection along the symplectic orthogonal. For a function $f$, define $f_1 = \sum_\alpha \pi_\alpha \ast (\phi_\alpha f)$ and extend $f_1$ to $\hat{f}$ on $X$. [Choose an open set $U$ that separates $Y$ and $X - \cup \alpha U_\alpha \subset U$, and further use partition of unity.]

The process of extension is independent of the function $f$, and Poisson bracket can be compared in local charts to check the property required from the extension as Poisson bracket in any Darboux coordinates is given by
\[
\{f, g\} = \sum_i \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i} - \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i}.
\]

We call an extension $C^\infty(Y) \rightarrow C^\infty(X)$ a symplectic extension if for all $f, g \in C^\infty(Y)$ the identity $\{\hat{f}, \hat{g}\}_{X|Y} = \{f, g\}_Y$ holds.

Let $\Gamma$ be a finite group acting on $X$ by diffeomorphisms, and let $Y = X^\gamma$ be the fixed point manifold for some $\gamma \in \Gamma$. Then $\Gamma$ acts on $T^*X$ by symplectomorphisms. We would identify $T^*Y = T^*(X^\gamma)$ with $(T^*X)^\gamma$.

Corollary 9.3. For the inclusion of the symplectic submanifold $T^*X^\gamma \hookrightarrow T^*X$, there exists a symplectic extension which is $\Gamma_\gamma$ equivariant.

10. Canonical Homology of Symplectic Manifolds

Let $X$ be a $2n$-dimensional symplectic manifold. Let $I : T^*X \rightarrow TX$ be the isomorphism induced by symplectic form $\omega$, namely, if $\alpha \in T^*X$ and $\xi \in TX$ then $\langle \alpha, \xi \rangle = \omega(\xi, I(\alpha))$. Then Poisson tensor on $X$ is defined as $G = -\wedge^2 I(\omega)$. Let $i(G)$ be an interior product by $G$, that is to consider a bilinear pairing $G : T^*X \times T^*X \rightarrow C^\infty(X)$. For all $k \geq 0$, denote by $\wedge^k G$ the associated bilinear pairing $\wedge^k G : \wedge^k T^*X \times \wedge^k T^*X \rightarrow C^\infty(X)$, which is $(-1)^k$ symmetric. Then $G$ can be used to define the Poisson structure on $X$ instead as $\{f, g\} := i(G)(df \wedge dg)$. Set $\nu_X := \omega^n/n!$ as the volume form on $X$. The symplectic * operator is the map $* : \Omega^k(X) \rightarrow \Omega^{2n-k}(X)$ which is defined by
\[
\beta \wedge * \alpha = \wedge^k G(\beta, \alpha) \nu_X.
\]

Definition 10.1 (Koszul complex). Let $\delta = i(G) \circ d - d \circ i(G)$. Then
\[
\delta : \Omega^k(M) \rightarrow \Omega^{k-1}(M),
\]
and is defined in local expression as
\[ \delta(f_0 df_1 \wedge \ldots \wedge df_k) = \sum_{1 \leq i \leq k} (-1)^{i-1} \{f_0, f_i\}_M df_1 \wedge \ldots \wedge \widehat{df_i} \ldots \wedge df_k \]

\[ + \sum_{1 \leq i < j \leq k} (-1)^{i+j-1} f_0 d\{f_i, f_j\}_M \wedge df_1 \wedge \ldots \widehat{df_i} \wedge \ldots \wedge \widehat{df_j} \wedge \ldots \wedge df_k. \]

Then \( \delta^2 = 0 \) is in fact a differential and we call the complex \((\Omega^\ast(M), \delta)\) as the Koszul complex of the Poisson manifold \((M < \emptyset)\), and its homology as Poisson homology and denote it by \( HK_\ast(M) \).

Although \( \delta \) can be defined for any Poisson manifold, we would restrict to the symplectic case. The following results were proved in [6].

**Lemma 10.2.** \( X \) be a symplectic manifold. Then \( \delta, d \) and \( * \) satisfy the following relations,

1. If \( \alpha \in \Omega^k(X) \) then \( ** \alpha = \alpha \)
2. \( \delta = (-1)^{k+1} * d * \) on \( \Omega^k(X) \).
3. \( d\delta + \delta d = 0 \) aa a map on \( \Omega^k(X) \).

A direct consequence of the lemma above is the following.

**Proposition 10.3** (The Poisson Homology of a symplectic Poisson manifold). Let \((M, \omega)\) be a compact symplectic manifold of dimension \( 2n \). Then \((-1)^{k+1} * \) is a chain map between \((\Omega^k(X), \delta)\) and \((\Omega^{2n-k}(X), d)\), and the symplectic Poisson homology is given by

\[ HK_k(M) = H^{2n-k}_{dRham}(M). \]

Now let \( X = T^*M/\{0\} \) be the cotangent bundle minus the zero section. Then there is a \( \mathbb{R}^+ \) action on \( X \). Let \( \Xi \) be the Euler vector field on \( X = T^*M/\{0\} \) generated by the cation of \( \mathbb{R}^+ \). The canonical one form on \( X \) is the form \( \alpha = i(\Xi)\omega \). Let \( \varepsilon(\alpha) : \Omega^k(X) \to \Omega^{k+1}(X) \) be the exterior multiplication by \( \alpha \). We would find the following result useful.

**Lemma 10.4.** Let \( \mathcal{L}_Y \) denote the Lie derivative with respect to a vector field \( Y \). On \( k \)-forms \( \Omega^k \), the operator

\[ \delta \varepsilon(\alpha) + \varepsilon(\alpha) \delta = \mathcal{L}_\Xi + n - k \]

In particular, if \( \beta \) is a \( k \)-form which is homogeneous of degree \( l \), then \( \mathcal{L}_\Xi \beta = l\beta \).

11. **Spectral Sequence for the Cross-Product**

We come back to \( \mathcal{B}(M) = \mathcal{A}(M) \rtimes \Gamma = \Psi^\infty(M)/\Psi^{-\infty}(M) \rtimes \Gamma \) the cross-product algebra. Since the group action on \( \mathcal{A}(M) \) preserves the filtration, \( \mathcal{B}(M) \) would again be a filtered algebra \( \mathcal{F}_i \mathcal{B}(M) = \mathcal{F}_i \mathcal{A}(M) \rtimes \Gamma \). Let \( P \) be a positive elliptic operator invariant under the \( \Gamma \) action. Let \( \rho \) be the symbol for \( P \). Then for any \( Q \in \mathcal{F}_m \mathcal{A}(M) \) an order \( m \) operator, the map
Let $M$ be the fixed point manifold $\xi$. Then the $k$-form $\gamma$ gives a class in $HH(\xi)$. More importantly, $\xi$ generates a class in $HH(\xi)$. To evaluate $\iota$ through symplectomorphisms, the extensions $\tilde{\iota}$ of $\iota$ to whole of $T^*M$ are in fact invariant extensions of $\iota$ to whole of $T^*M$. We only have to check that $\delta = \chi \circ d^1 \circ E$. To evaluate $d^1$ on $HH_s(\mathcal{G}(\mathcal{B}(M)))$, we must lift the form $\iota$ to a tensor. We
now choose operators $A_0, A_1, \ldots, A_k \in \mathcal{F}_{p_i}^\Gamma(A(M) = \Psi^\Gamma_p(M)/\Psi^{-\infty}(M)$, which are $\Gamma_\gamma$ invariant and such that $\sigma(A_j) = \hat{f}_j$. (This can be done by averaging on a $\Gamma$ equivariant splitting of $\sigma$.) Using the above choice of lifting, 

$$\sigma^{-1}(E_k(\xi)) = \sum_{\pi \in S_k} \epsilon(\pi) A_0 \gamma \otimes A_{\pi(1)} e \otimes A_{\pi(2)} e \ldots \otimes A_{\pi(n)} e.$$ 

On applying $b$ the Hochschild differential, there are three different kinds of expressions for each $\pi \in S_k$. So the resultant expression can be broken up as the sum of 

$$A = \sum_{\pi \in S_k} \epsilon(\pi) A_0 A_{\pi(1)} \gamma \otimes A_{\pi(2)} e \otimes \ldots \otimes A_{\pi(k)} e,$$

$$B = \sum_{\pi \in S_k} \sum_{i=1}^{k-1} (-1)^{-1} \epsilon(\pi) A_0 \gamma \otimes \ldots \otimes A_{\pi(i)} A_{\pi(i+1)} e \otimes \ldots \otimes A_{\pi(k)} e,$$

$$C = \sum_{\pi \in S_k} (-1)^k \epsilon(\pi) A_{\pi(k)} A_0 \gamma \otimes A_{\pi(1)} e \otimes \ldots \otimes A_{\pi(k-1)} e.$$ 

(Remember that $A_i$'s have been chosen to be invariant under $\gamma$.) By replacing $\pi$ in $A$ by $\pi \tau$ where $\tau$ is a cyclic permutation, we could rewrite it as 

$$A = \sum_{\pi \in S_k} (-1)^{k+1} \epsilon(\pi) A_0 A_{\pi(k)} \gamma \otimes A_{\pi(1)} e \otimes \ldots \otimes A_{\pi(k-1)} e.$$ 

Thus 

$$A + C = \sum_{\pi \in S_k} (-1)^{k+1} \epsilon(\pi) \{A_0, A_{\pi(k)}\} \gamma \otimes A_{\pi(1)} e \otimes \ldots \otimes A_{\pi(k-1)} e$$ 

or in $HH_*(\text{Gr}(\mathcal{B}(M))$, it is represented by 

$$\sum_{\pi \in S_k} (-1)^{k+1} \epsilon(\pi) \{f_0, f_{\pi(k)}\} \gamma \otimes f_{\pi(1)} e \otimes \ldots \otimes f_{\pi(k-1)} e.$$ 

For any permutation $\pi$ for which $\pi(n) = i$ is fixed, the image of the tensor 

$$(-1)^{k+1} \epsilon(\pi) \{f_0, f_{\pi(k)}\} \gamma \otimes f_{\pi(1)} e \otimes \ldots \otimes f_{\pi(k-1)} e$$ 

under $\chi$ of Theorem 11 is the same namely 

$$\frac{1}{k!} (-1)^i \{f_0, f_i\} df_1 df_2 \ldots df_i \ldots df_n.$$ 

(Again remember that all the $f_j$'s are $\Gamma_\gamma$ invariant.) There are $(k-1)!$ permutations such that for a fixed $i$, $\pi(n) = i$, and therefore the parts corresponding to $A$ and $C$ in $d^1$ become 

$$\sum_{1 \leq i \leq k} (-1)^{i-1} \{f_0, f_i\} df_1 \wedge \ldots \wedge df_i \ldots \wedge df_k.$$
The summand $B$ pairs each $\pi$ with the transpositions $\pi(i\,i+1)$

$$B = \frac{1}{2} \sum_{\pi \in S_k} \sum_{i=1}^{k-1} (-1)^{i-1} \epsilon(\pi) A_{0 \gamma} \otimes \ldots \otimes \{A_{\pi(i)}, A_{\pi(i+1)}\} e \otimes \ldots \otimes A_{\pi(k)} e$$

and so $\sigma(B) = \frac{1}{2} \sum_{\pi \in S_k} \sum_{i=1}^{k-1} (-1)^{i-1} \epsilon(\pi) f_0 \gamma \otimes \ldots \otimes \{f_{\pi(i)}, f_{\pi(i+1)}\} e \otimes \ldots \otimes f_{\pi(k)} e$. All pairs $(\pi, i)$ such that the set $\{\pi(i), \pi(i-1)\}$ is the same as the set $\{m, n\}$, $m < n$ have the same image under $\chi$, namely the form

$$\frac{(-1)^{m+n}}{k-1} f_0 d\{f_m, f_n\} \wedge df_1 \wedge \ldots \wedge df_m \wedge \ldots \wedge df_n \wedge \ldots \wedge df_k.$$ 

As there would be $2(k-1)!$ such pairs for each $\{m, n\}$, the terms in $B$ would map to the remainder of $\delta$

$$\sum_{1 \leq i < j \leq k} (-1)^{i+j-1} f_0 d\{f_i, f_j\} \wedge df_1 \wedge \ldots \wedge df_i \wedge \ldots \wedge df_j \wedge \ldots \wedge df_k.$$

\[\square\]

**Corollary 11.2.** We have

$$HH_*(\Psi^\infty(M)/\Psi^{-\infty}(M) \rtimes \Gamma) = \sum_{(\gamma)} H^{2k_\gamma-\ast}(S^* M^\gamma \times S^1)^{\Gamma_\gamma},$$

where $k_\gamma = \dim(M^\gamma)$.

**Proof.** Since the $\Gamma_\gamma$ action on $T^* M^\gamma/\{0\}$ is by symplectomorphisms, all $\gamma \in \Gamma$ commute with the symplectic $*$ operator and therefore with the operator $\delta$. Hence by Theorem 3.3 and the previous proposition, $E^2_{k\gamma} \simeq H^{2k_\gamma-\ast}(S^* M^\gamma \times S^1)^{\Gamma_\gamma}$.

We now observe that the differential $d^2$ for this spectral sequence vanishes. The $E^1$ can be given a $Z$ grading with $E^1_{k,l}$ be $k$-forms of homogeneity $l$ on $T^*(M)\{0\}$. Then the differential $d^1 = \delta$ maps $E^1_{lk} \rightarrow E^1_{k-l-1}$. This is because the Poisson bracket decreases the homogeneity by 1. But on $E^1_{kl}$ by 10.3 we have

$$\delta \varepsilon(\alpha) + \varepsilon(\alpha) \delta = \mathcal{L}_\Xi + n - k = l + n - k.$$ 

Thus unless $l = k - n$ there is a contraction for $\delta$ on $E^1_{pq}$. Therefore the only nonzero terms on $E^2_{kl}$ correspond to $l = n - k$. The differential $d^2$ must either start or end in a 0 term. \[\square\]

Since the Hochschild homology groups are finite dimensional, the Hochschild cohomology groups are the dual of the Hochschild homology groups. We still write down explicitly $HH^0(B(M))$, the space of traces on $B(M)$

**Corollary 11.3.** We have $HH^0(B(M)) = \sum_{(\gamma)} H_0(S^* M^\gamma \times S^1)$. hence its rank is the number of path components of $S^* M^\gamma$. 

Proof. We have $HH_0(\mathcal{B}(M))_\gamma = H^{2k_\gamma}(S^*M^\gamma \times S)$, because the action of $\Gamma$ on $S^*M$ is orientation preserving. By Poincare duality, $H^{2k_\gamma}(S^*M^\gamma \times S^1) = H^0(S^*M^\gamma \times S^1)$. 

To compute the cyclic homology, we use Connes’ SBI exact sequence:

$$\ldots \xrightarrow{B} HH_n(\mathcal{B}(M)) \xrightarrow{I} HC_n(\mathcal{B}(M)) \xrightarrow{S} HC_{n-2}(\mathcal{B}(M)) \xrightarrow{B} HH_{n-1}(\mathcal{B}(M)) \xrightarrow{} \ldots$$

Here the connecting morphism $B : \text{Tot}(B_*(\mathcal{B}(M))) \to \mathcal{H}_*(\mathcal{B}(M))[−1]$ is the cyclic boundary map $B$.

**Proposition 11.4.** The connecting morphism $B$ in the SBI exact sequence for $\mathcal{B}(M)$ vanishes and hence,

$$HC_K(\mathcal{B}(M)(M)) = \sum_{j \geq 0} HH_{k-2j}(\mathcal{B}(M)(M)).$$

Proof. Since $\mathcal{B}(M)$ is unital, $B$ is induced from a chain map

$$\text{Tot}(B_*(\mathcal{B}(M))) \to \mathcal{H}_*(\mathcal{B}(M))[−1]$$

that respects the filtration on both the source and the range complex. Thus $B$ induces a map on the spectral sequence $EC^*$ of $\text{Tot}(B_*(\mathcal{B}))$ into spectral sequence $EH^*$ of $\mathcal{H}_*(\mathcal{B}(M))[−1]$. The induced map on $EC^1 \to EH^1$ when identified with the space of differential forms is the deRham differential $d$ which vanishes on $E^2$.

$$d : EC^1_{k,l} \oplus_j \Omega^{k-2j}(S^*M^\gamma \times S^1)^{g_p_{\gamma}} \to EH^1_{k,l} \Omega^{k+1}(S^*M^\gamma \times S^1)^{\Gamma_{\gamma}}.$$

For the $EC$ term to contribute to nonzero homology, $l = 2k_{\gamma} - k$, where as for the $EH$ term $l = 2k_{\gamma} - k - 1$.

The formula for $HC_K(\mathcal{B})$ then follows from the SBI sequence. 

We can note that the map induced by $B$ on $EH^1$ is a differential implies in particular that

**Corollary 11.5.** $HP_j(\mathcal{B}(M)) = \sum_{k \in \mathbb{Z}} HH_{j-2k}(\mathcal{B}(M))$.

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