Holomorphic Curves from Matrices

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Abstract

Membranes holomorphically embedded in flat noncompact space are constructed in terms of the degrees of freedom of an infinite collection of 0-branes. To each holomorphic curve we associate infinite-dimensional matrices which are static solutions to the matrix theory equations of motion, and which can be interpreted as the matrix theory representation of the holomorphically embedded membrane. The problem of finding such matrix representations can be phrased as a problem in geometric quantization, where \( \epsilon \propto l_p^3/R \) plays the role of the Planck constant and parametrizes families of solutions. The concept of Bergman projection is used as a basic tool, and a local expansion for the action of the projection in inverse powers of curvature is derived. This expansion is then used to compute the required matrices perturbatively in \( \epsilon \). The first two terms in the expansion correspond to the standard geometric quantization result and to the result obtained using the metaplectic correction to geometric quantization.

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1 Introduction

Matrix theory [1] (see [2, 3, 4, 5] for reviews) has been proposed as a non-perturbative definition of M-theory, and therefore, at low energies, of 11-dimensional supergravity. Matrix theory describes not only fundamental particles like the graviton, but also extended objects like membranes and 5-branes. Matrix theory is very closely related to the light-front formulation of the supermembrane, which was originally studied in the thesis of Hoppe [6] and in the work of de Wit, Hoppe and Nicolai [7]. Using this connection one can describe extended two-dimensional membranes within the context of matrix theory. Compact surfaces with spherical or higher-genus topology were studied in [7, 8]. Compact surfaces, however, are unstable to gravitational collapse, and are expected to radiate gravitons and to eventually disappear. On the other hand, if one focuses attention on static configurations, one is forced to consider noncompact surfaces which are infinite in spatial extent. The planar infinite membrane, in particular, has been extensively discussed in the matrix theory literature [1, 9]. From the point of view of membrane theory, however, the planar brane is nothing but a special case of a larger class of static solutions to the equations of motion, given by holomorphic embeddings of noncompact Riemann surfaces in space. These holomorphic membranes are stable, static configurations corresponding to supergravity solutions which preserve some supersymmetries and are therefore BPS configurations. One can ask if there are matrix theory configurations corresponding to these holomorphic curves in space. This paper is devoted to analyzing this question.

We construct in this paper a set of infinite matrices corresponding to any holomorphic membrane configuration. To motivate the general analysis of the paper, we give here an explicit example of the type of holomorphic matrix membrane in which we are interested. Consider a planar membrane embedded in a pair of holomorphic coordinates \( Z = X_1 + iX_2, W = X_3 + iX_4 \) according to the equation \( W = Z^2 \). A static matrix theory configuration corresponding to this membrane would be a pair of infinite-dimensional complex matrices \( \mathcal{X}_1 = iX_2, \mathcal{W} = X_3 + iX_4 \) satisfying the relation \( \mathcal{W} = \mathcal{X}_1^2 \) and the equations of motion \( [[\mathcal{X}_i, \mathcal{X}_j], \mathcal{X}_j] = 0 \). Such matrices can be constructed by taking \( W = Z^2 \) with

\[
\mathcal{X}_1 = \frac{1}{2} \begin{pmatrix}
0 & \rho_0 & 0 & 0 & \cdots \\
\rho_0 & 0 & \rho_1 & 0 & \cdots \\
0 & \rho_1 & 0 & \rho_2 & \cdots \\
0 & 0 & \rho_2 & 0 & \rho_3 & \cdots \\
\cdots & \cdots & 0 & \rho_3 & 0 & \cdots \\
\end{pmatrix}, \quad \mathcal{X}_2 = \frac{1}{2} \begin{pmatrix}
0 & i\rho_0 & 0 & 0 & \cdots \\
-i\rho_0 & 0 & i\rho_1 & 0 & \cdots \\
0 & -i\rho_1 & 0 & i\rho_2 & \cdots \\
0 & 0 & -i\rho_2 & 0 & i\rho_3 & \cdots \\
\cdots & \cdots & 0 & -i\rho_3 & 0 & \cdots \\
\end{pmatrix}
\]

where the matrix entries \( \rho_n \) are given in terms of \( \rho_0 \approx 0.7502, \rho_{-1} = \rho_{-2} = 0 \) by the recursive
The formula is:

\[ \rho_n = \sqrt{\frac{1 + \rho_{n-2}^2 - \rho_{n-1}^2 + \rho_{n-2}^2\rho_{n-3}^2}{\rho_{n-1}^2}}, \quad n > 0 \]

We will discuss this example in more detail in section 3.

In this paper we rephrase the problem of finding a matrix representation of a general holomorphic curve as a problem in geometric quantization [10]. The constant \( \epsilon \propto l_P^3/R \) (where \( l_P \) is the 11-dimensional Planck length and \( R \) is the light-like compactification radius) plays the role of the Planck constant \( \hbar \), and it parametrizes families of solutions to the matrix theory equations of motion. Given a holomorphic embedding of a Riemann surface \( \Sigma \) in space, we wish to construct matrices which correspond to the embedding coordinates. The matrices must be infinite-dimensional, reflecting the noncompactness of the branes, and are therefore operators acting on a Hilbert space \( \mathcal{H} \). After analyzing some examples, we propose that \( \mathcal{H} \) be taken to be the space of holomorphic functions on the Riemann surface \( \Sigma \), and that the operators corresponding to holomorphic functions act via pointwise multiplication.

The problem of representing holomorphic curves is then reduced to the problem of finding the correct inner product on \( \mathcal{H} \). Different inner products on \( \mathcal{H} \) are naturally given by integrations over \( \Sigma \) with respect to different volume forms \( \Omega \), and therefore the question about the correct choice of inner product can be rephrased in terms of the corresponding volume form \( \Omega \). To make this correspondence we discuss Bergman projections [11, 12] and kernels on \( \Sigma \), and we derive a local expansion of the action of the projection. We then use this expansion to solve for the volume form \( \Omega \), which we do perturbatively in powers of \( \epsilon \). We then devote the last part of the paper to describing the connections between our approach and the theory of geometric quantization. We find that, to first order in \( \epsilon \), our expression for \( \Omega \) reproduces the result expected from geometric quantization. The first correction corresponds to the metaplectic correction to geometric quantization [10, 13]. All higher order terms are needed in order to satisfy the equations of motion, and cannot be determined from standard geometric quantization theory.

The structure of this paper is as follows: in section 2 we review the light-front description of the classical bosonic membrane and its connection with matrix theory. We discuss the static solutions of the membrane equations of motion corresponding to holomorphically embedded membranes. The central problem which we address in this paper, that of finding matrix representations of holomorphic curves, is defined in subsection 2.3. In section 3 we discuss some simple examples of holomorphic curves corresponding to static membranes; subsection 3.4 describes an approach to solving the general problem based on the insights gained from the examples. Section 4 contains a detailed discussion of Bergman integral kernels. We review some basic properties of the projections associated with these kernels, describe a simple example, and derive a general formula for a projection operator which agrees with Bergman projection on a very large class of functions. In section 5 we use the tools developed in section 4 to propose a solution to the general problem of constructing a matrix representation of a holomorphic curve. Section 6 applies the general formalism in
several examples, including the simple examples of section 3 as well as more complicated examples which cannot be solved without the general formalism. Section 7 contains a discussion of the connection between the results in this paper and the theory of geometric quantization. We conclude in section 8 with suggestions for future research.

2 Bosonic Membranes and Matrix Theory

2.1 The light-front bosonic membrane and holomorphic curves

In this subsection we briefly review the theory of a classical bosonic membrane moving in 11 dimensions in light-front coordinates. The derivation of the light-front formalism starting from the Nambu-Goto action has been extensively discussed in the literature, and we refer the reader to the articles [6, 7] for a more detailed explanation. At the end of this subsection we discuss the particular class of static solutions of the equations of motion for the brane that is the central focus of the rest of the paper.

We take the world-volume of the membrane to be the product space $\mathbb{R} \times \Sigma$, where $\Sigma$ is a two dimensional surface (not necessarily compact) with the topology of the brane. Coordinates on the world-volume are $\tau$ and $\sigma^a$, with $a = 1, 2$. The brane propagates in 11-dimensional Minkowski space-time with coordinates $X^\mu$, $\mu = 0 \ldots 10$, and the motion of the surface is described by coordinate functions $X^\mu : \mathbb{R} \times \Sigma \rightarrow \mathbb{R}$ on the world-volume. We use light-front coordinates $X^\pm$ given by

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{10})$$

The light-front formalism is based on a simple observation. The membrane, as it moves in space-time, carries a conserved momentum $p^\mu$, which can be written, as always in field theory, as an integral over $\Sigma$ of densities $P^\mu$

$$p^\mu = \int_\Sigma P^\mu d^2\sigma.$$ 

In general the densities $P^\mu$ will depend on $\tau$. On the other hand, if one imposes the light-front gauge constraints (to be described shortly), one can show that $P^+$ is independent of $\tau$. Therefore $P^+$ singles out, up to a multiplicative constant, a fixed volume form on $\Sigma$ which we call

$$\mu d^2\sigma.$$ 

We may then write the momentum densities $P^\mu$ as

$$P^+ = \Pi^+ \mu$$

$$P^- = \Pi^- \mu$$

$$P^i = \Pi^i \mu, \quad (i = 1 \ldots 9)$$
where $\Pi^+$ is a fixed constant and $\Pi^-, \Pi^i$ are scalar functions on $\Sigma$. The hamiltonian light-front formalism starts then with the choice of a fixed volume form $\mu$ and takes as independent variables the transverse positions $X^i$ together with the corresponding momenta $\Pi^i$, subject to one constraint. The longitudinal coordinates $X^0$ and $X^{10}$ are then derived by solving the gauge-fixing equations.

Let us first describe the hamiltonian which governs the evolution of the independent canonical variables, together with the constraint satisfied by them. To this end we view $\mu$ as a symplectic form and we consider the corresponding Poisson bracket. Specifically, if $A$ and $B$ are functions on $\Sigma$, we define the bracket $\{A,B\}$ by

$$\{A,B\} = \frac{1}{\mu} \epsilon^{ab} \partial_a A \partial_b B,$$

where $\epsilon^{12} = 1$. With this notation we can then write the hamiltonian describing the membrane ($T$ is the brane tension)

$$H = \int_\Sigma \mu \, d^2 \sigma \, \mathcal{H}$$

$$\mathcal{H} = \frac{1}{2\Pi^+} \Pi^i \Pi^i + \frac{T^2}{4\Pi^+} \{X^i, X^j\}^2$$

together with the constraint satisfied by the canonical variables

$$\{X^i, \Pi^i\} = 0. \quad (3)$$

The second term in the hamiltonian can also be rewritten in terms of the induced metric

$$h_{ab} = \partial_a X^i \partial_b X^i$$

on $\Sigma$ by noticing that

$$h = \det h_{ab} = \frac{1}{2} \mu^2 \{X^i, X^j\}^2.$$

It is easy to show, using the canonical commutation relation

$$[X^i(\sigma), \Pi^j(\sigma')]_{P.B.} = \frac{1}{\mu(\sigma)} \delta^2(\sigma - \sigma'),$$

that the equations of motion are given by

$$\Pi^i = \Pi^+ \dot{X}^i$$

$$\ddot{X}^i = \frac{1}{\Pi^+} \ddot{\Pi}^i = \frac{T^2}{\Pi^+} \{\{X^i, X^j\}, X^j\}. \quad (4)$$

A simple application of the Jacobi identity then shows that the constraint (3) is preserved by the hamiltonian evolution (4) and that the hamiltonian system at hand is consistent.

4
We now turn our attention to the gauge fixing equations that determine the constraint coordinates $X^+$ and $X^-$. They read

\begin{align*}
X^+ &= \tau \\
\partial_a X^- &= X^i \partial_a X^i.
\end{align*}

(5)

The first equation simply says that the hamiltonian time $\tau$ measures the light-front coordinate $X^+$. Since the momenta conjugate to $\tau$ and $X^+$ are respectively $H$ and $p^-$, one has that

$$H = \Pi^-.$$

The second equation can be solved for $X^-$, at least locally, if the right hand side of equation (5) is closed. But this is the case since

$$d(\dot{X}^i dX^i) \propto d\Pi^i \wedge dX^i = \epsilon^{ab} \partial_a \Pi^i \partial_b X^i d^2\sigma = \{\Pi^i, X^i\} \mu d^2\sigma = 0.$$ 

Global problems of existence of $X^-$ will not be an issue in this paper and we will not address them.

To end our discussion of the light-front formalism, we derive an equation for the density $\mu$ in terms of the coordinate functions $X^\mu$. We start by taking the time derivative of the gauge constraint (5). This can be rewritten as

$$\partial_a \Delta = 2\ddot{X}^i \partial_a X^i = 2(T^2/\Pi^+)^2 \{\{X^i, X^j\}, X^j\} \partial_a X^i,$$

where

$$\Delta = 2\dot{X}^- - \dot{X}^i \dot{X}^i.$$

A mechanical computation shows that

$$4 \{\{X^i, X^j\}, X^j\} \partial_a X^i = \partial_a \{X^i, X^j\}^2 = 2\partial_a (h/\mu^2).$$

Putting everything together we deduce that

$$\mu = \frac{T}{\Pi^+} \sqrt{\frac{h}{\Delta}}.$$

We conclude this section by focusing our attention on a specific class of static solutions of the equations of motion. We first of all fix, on the space part of Minkowski space-time, a complex structure compatible with the metric. All the possible choices differ only by an $SO(10)$ rotation; we choose the analytic coordinates

$$Z_1 = X^1 + iX^2,$$

$$\ldots$$

$$Z_5 = X^9 + iX^{10}.$$

We then choose a complex structure on the manifold $\Sigma$, denoting the analytic coordinate by

$$z = \sigma^1 + i\sigma^2.$$ 

A class of static solutions of the equations of motion are then given by holomorphic embeddings of $\Sigma$ in $\mathbb{R}^{10} = \mathbb{C}^5$. More specifically we shall take

$$Z_A \in \{\text{analytic functions on } \Sigma \ (\tau \text{ independent})\}$$

$$Z_5 = 0$$

$$X^+ = X^- = \tau.$$ 

(6)
The surface $\Sigma$ cannot be compact. If it were, then the embedding would be trivial since holomorphic functions on a compact Riemann surface are constant, so the brane would degenerate to a point. To prove that the equations of motions are satisfied for a holomorphically embedded membrane, we first note that the gauge fixing equations hold. We then compute the induced metric $h_{ab}$, which is given by

$$
\begin{align*}
h_{zz} &= h_{\bar{z}\bar{z}} = 0 \\
h_{z\bar{z}} &= \frac{1}{2} \partial Z_A \bar{\partial} \bar{Z}_A
\end{align*}
$$

(sum over $A$ will always be implied). Using the fact that

$$
\Delta = 2 \sqrt{\frac{2}{h}} = 2 h_{z\bar{z}}
$$

we can then compute the density $\mu$ given by

$$
\mu = \left( \frac{T}{\sqrt{2\Pi^+}} \right) \partial Z_A \bar{\partial} \bar{Z}_A = \frac{1}{\pi \epsilon} \partial Z_A \bar{\partial} \bar{Z}_A,
$$

where

$$
\epsilon = 2 \sqrt{\frac{\Pi^+}{2\pi T}}.
$$

The symplectic form $\mu d\sigma^1 \wedge d\sigma^2$ can be rewritten as $\frac{i}{2} \mu dz \wedge d\bar{z}$ and correspondingly the bracket $\{ , \}$ can be expressed in terms of holomorphic and antiholomorphic derivatives as

$$
\{ A, B \} = \frac{-2i}{\mu} (\partial A \bar{\partial} B - \bar{\partial} A \partial B).
$$

It is then easy to show that

$$
\{ Z_A, Z_B \} = \{ \bar{Z}_A, \bar{Z}_B \} = 0
$$

and also that

$$
\{ Z_A, Z_A \} = \frac{-2i}{\mu} \partial Z_A \bar{\partial} Z_A = -2\pi i \epsilon. \quad (7)
$$

We will see later the significance of the constant $\epsilon$. For now we can use the above results together with the Jacobi identity to show that

$$
\bar{Z}_A \propto \{ \{ Z_A, Z_B \}, \bar{Z}_B \} + \{ \{ Z_A, \bar{Z}_B \}, Z_B \} = \{ \{ Z_A, \bar{Z}_B \}, Z_B \} = \{ \{ Z_B, \bar{Z}_B \}, Z_A \} + \{ \{ Z_A, Z_B \}, \bar{Z}_B \} = \{-2\pi i \epsilon, Z_A \} = 0. \quad (8)
$$

The equations of motion are therefore satisfied and we indeed have a static solution of the hamiltonian equations (4). The solution (6) can also be boosted in the 10-th direction. If $\omega$ is a boost parameter, then

$$
\begin{align*}
X^+ &\rightarrow \omega X^+ = \omega \tau \\
X^- &\rightarrow \frac{1}{\omega} X^- = \frac{1}{\omega} \tau.
\end{align*}
$$
The above transformation does not preserve the gauge condition, and we have to rescale $\tau \rightarrow \frac{1}{\omega} \tau$. We than have that

$$X^+ = \tau \quad X^- = \frac{1}{\omega^2} \tau.$$ 

The only change in the above discussion is that $\Delta \rightarrow \frac{2}{\omega}$ and therefore $\mu \rightarrow \omega \mu$. This is then reflected in a change

$$\epsilon \rightarrow \frac{1}{\omega} \epsilon$$

in equation (7).

### 2.2 Matrix-membrane correspondence

We have briefly reviewed the theory of classical membranes, with attention to a particular class of static solutions. In this subsection we discuss the relation between light-front membrane theory and matrix theory.

It was pointed out in [6, 7] that the light-front membrane theory discussed in the previous section can be related to a theory of matrices by truncating the space of functions on the brane to a finite number of degrees of freedom. This gives a discrete regularization of the membrane theory which preserves much of the structure of the continuous theory. The matrix theory conjecture of Banks, Fischler, Shenker and Susskind, first proposed in [1] and then further developed in [14, 15, 16] (for reviews see [2, 3, 4, 5]), asserts that the supersymmetric version of this matrix quantum mechanics theory contains all the physics of light-front M-theory. More precisely, the DLCQ version of the conjecture asserts that M-theory compactified on a light-like circle $X^- \sim X^- + 2 \pi R$ is described, within the sector with light-front momentum $p^+ = N/R$, by 10-dimensional $U(N)$ super Yang-Mills theory, dimensionally reduced to $0 + 1$ dimensions. The supermembrane of M-theory is described in matrix theory using precisely the matrix-membrane correspondence worked out by de Wit, Hoppe and Nicolai in [7].

Before discussing the details of the matrix-membrane correspondence, let us fix some conventions. The 11-dimensional Planck length is denoted by $l_P$ and is related to the gravitational constant by $2 \kappa^2 = \frac{2 \pi}{(2 \pi)^2 l_P^3}$. The membrane tension $T$ is

$$T = \frac{1}{(2 \pi)^2 l_P^3}.$$ 

Finally the string scale is given by $\alpha' = l_P^3 / R$.

We now move to an overview of matrix theory. The independent variables are given by the transverse coordinates $X^i$ together with the corresponding canonical momenta $\Pi^i$ ($i = 1 \ldots 9$), where now both $X^i$ and $\Pi^i$ are $N \times N$ hermitian matrices. The canonical variables are not completely independent but satisfy a constraint equation given by

$$[X^i, \Pi^i] = 0.$$  (9)
Time evolution is governed by the Hamiltonian

\[ H = \frac{R}{2} \text{Tr} (\Pi^i \Pi^i) - (2\pi T)^2 \frac{R}{4} \text{Tr} ([X^i, X^j]^2) \tag{10} \]

from which the equations of motion

\[ \Pi^i = \frac{1}{R} \dot{X}^i \]
\[ \ddot{X}^i = R \Pi^i = -(2\pi)^2 T^2 R^2 ([X^i, X^j], X^j) \tag{11} \]

can be derived. The constraint (9) is preserved by (11) and therefore the theory is consistent.

In the 11-dimensional interpretation of matrix theory the conserved momentum \( p^\mu \) is given by

\[ p^+ = \frac{N}{R}, \quad p^- = H, \quad p^i = \text{Tr} (\Pi^i). \]

There is an obvious formal similarity between this matrix quantum mechanics theory and the membrane theory reviewed in the previous section. This connection was made precise in [6, 7]. A configuration of a membrane \( \Sigma \) can be associated with a set of \( N \times N \) matrices by mapping functions on the membrane, like coordinates and momenta, into \( N \times N \) matrices in matrix theory. Through this correspondence, Poisson brackets \( \{ , \} \) on \( \Sigma \) become matrix commutators \( [ , ] \) and integrations \( \int_\Sigma d^2 \sigma \mu \) are replaced by traces \( \text{Tr} \) of matrices. The map from functions on \( \Sigma \) to matrices was described in detail in [7] in the cases where \( \Sigma \) is a Riemann surface of spherical or toroidal topology. We wish to discuss this correspondence in the more general case where the membrane \( \Sigma \) is a noncompact Riemann surface. We conclude this subsection by reviewing the situation when \( \Sigma \) is compact.

To make the matrix-membrane correspondence more precise, let \( A \) be the space of scalar functions on \( \Sigma \) and \( B \) be the space of \( N \times N \) matrices. Both spaces \( A \) and \( B \) carry a similar algebraic structure and we should therefore look for a correspondence \( Q : A \rightarrow B \) which preserves this structure as much as possible. Obviously \( Q \) should be a linear map and should map complex conjugate functions to hermitian conjugate matrices

\[ \bar{X} \rightarrow X^\dagger. \]

For some functions on \( \Sigma \) we wish the product of functions to correspond to a matrix product in \( B \). This is not possible in general, since the matrix product is not commutative. On the other hand, one can at least require that the unit element in \( A \) be mapped to the unit element in \( B \)

\[ 1 \rightarrow 1_{N \times N}. \tag{12} \]
We recall from the last section that the measure \( \mu \) is defined only up to a multiplicative factor (the product \( \Pi^+ \mu = P^+ \) is invariant and we can rescale \( \mu \) as long as we rescale \( \Pi^+ \) accordingly). We use this freedom to fix the correspondence
\[
\int \Sigma d^2 \sigma \mu \rightarrow \text{Tr} (\ ) .
\] (13)
Combining (12) and (13) we see that the normalization of \( \mu \) has been chosen so that
\[
\int \Sigma d^2 \sigma \mu = N.
\]
In the language of matrix theory, the measure \( \mu \) corresponds to the local density of 0-branes on the membrane, and \( N \) is the total number of 0-branes. If we recall that \( p^+ = N/R = \int \Sigma d^2 \sigma \mu \Pi^+ \), we also conclude that
\[
\Pi^+ = \frac{1}{R}.
\]
The last requirement on \( Q \) comes from the fact that both \( A \) and \( B \) are Lie algebras, with bracket \( \{,\} \) and \([,\] \) respectively. In order to match normalizations in the hamiltonians (2) and (10), we require that, under \( Q \),
\[
\{,\} \rightarrow 2\pi i \,[,\] .
\] (14)
This is clearly an example of the classical problem of geometric quantization, if one views \( \Sigma \) as a symplectic manifold with symplectic form \( \mu d^2 \sigma \). We shall see later that we will not be able to satisfy (14) for all elements of \( A \) (this is reminiscent of similar problems in elementary quantum mechanics). On the other hand we will see that, in the cases that we shall study, it is possible and natural to impose (14) on the coordinate functions describing the position of the brane.

This concludes our general discussion of matrix theory and of its correspondence with membrane theory described in the last section. We are now in a position to present clearly the problem that will be analyzed in this paper.

2.3 Holomorphic curves in matrix theory

At the end of section 2.1 we described a family of static solutions of the membrane equations of motion, given by holomorphic embeddings of a Riemann surface \( \Sigma \) in \( \mathbb{C}^4 \) (we had chosen \( Z_5 = 0 \)). As we noted already, \( \Sigma \) cannot be compact and we therefore choose \( \Sigma \) to be a Riemann surface of genus \( g \) with \( n \) points deleted. Moreover we choose the coordinate functions \( Z_A \ (A = 1 \ldots 4) \) to be holomorphic functions on \( \Sigma \), meromorphic at the punctures.

The measure \( \mu \) is given by \( \mu = (1/\pi \epsilon) \partial Z_A \partial \bar{Z}_A \). Since \( N = \int \Sigma d^2 \sigma \mu = \infty \), we have to change our point of view slightly, and let \( \mathcal{B} = \text{End}(\mathcal{H}) \) be the space of operators acting on an infinite-dimensional Hilbert space \( \mathcal{H} \). Our problem will therefore be, given \( \Sigma \) and holomorphic functions \( Z_A \), to find a Hilbert space \( \mathcal{H} \) and a map \( Q : A \rightarrow \text{End}(\mathcal{H}) \) satisfying
the requirements described above. Note that the space $H$ and the map $Q$ depend on the choice of surface $\Sigma$ and the embedding functions $Z_A$.

We now state in detail the properties which must be satisfied by the map $Q$ for a quantization of the holomorphic membrane embedding given by a set of functions $Z_A$. If

$$Z_A = Q(Z_A)$$

we will require that

$$[Z_A, Z_B] = [Z_A^\dagger, Z_B^\dagger] = 0$$

and that

$$[Z_A, Z_A^\dagger] = -\epsilon \text{ Id},$$

where

$$\epsilon = \frac{2\sqrt{2}\Pi^+}{2\pi T} = 2\sqrt{2}\frac{1}{R}(2\pi l^3_p) = 2\sqrt{2}(2\pi \alpha').$$

It is clear that the constant $\epsilon$ plays the role of the Planck constant in the usual classical/quantum correspondence, and that it will parametrize a family of solutions to the quantization problem. For this reason in the rest of the paper we will loosely refer to $\epsilon$ as the Planck constant, to the limit $\epsilon \to 0$ as the classical limit and to the process of going from functions to matrices through $Q$ as quantization. Let us note that one can use equations (15) and (16), together with a manipulation identical to equation (8) to show, starting from the equations of motion, that

$$\ddot{Z}_A = 0.$$

Therefore the matrices $Z_A$ represent a static solution of the matrix theory equations of motion.

Equation (15) allows us to impose one final and crucial constraint on $Q$. Suppose that the embedded surface $\Sigma \hookrightarrow \mathbb{C}^4$ can be described as the locus of points in $\mathbb{C}^4$ satisfying the equations

$$F_i(Z_A) = 0 \quad (i = 1 \ldots 3)$$

for some holomorphic functions $F_i$ (the functions $F_i$ could be polynomial functions of the $Z_A$, but this is not necessary). Since the operators $Z_A$ commute, it makes sense to replace $Z_A$ with $Z_A$ in equation (17) and to require that

$$F_i(Z_A) = 0. \quad (i = 1 \ldots 3)$$

Solutions to equations (15), (16) and (18) will not be unique, since the underlying theory is invariant under $U(N)$ gauge transformations. In particular we are free to transform the operators $Z_A$ to $UZ_AU^\dagger$, where $U$ is a unitary matrix.

As a final remark let us discuss boosted solutions. At the end of the last section we noted that a boost in the 10-th direction with parameter $\omega$ is only reflected in the change

$$\epsilon \to \frac{1}{\omega}\epsilon = \frac{2\sqrt{2}}{\omega R}(2\pi l^3_p).$$
This means that a static solution of equations (15), (16) and (18) with light-like radius $R$ is equivalent to a boosted solution with light-like radius $R/\omega$. This is consistent since, under a boost, $X^- \to X^-/\omega$ and therefore $R\omega$ is invariant.

3 Simple Examples of Holomorphic Membranes

In subsection 2.3 we described the general problem of constructing a Hilbert space $\mathcal{H}$ and a map $Q$ from functions on a membrane $\Sigma$ to matrices which would give a general matrix representation of holomorphic curves. In this section we analyze a few special examples of holomorphic curves where there is a simple and natural choice of $\mathcal{H}$ and $Q$. The purpose in discussing these examples is twofold. First of all, these examples can be solved without resorting to the general machinery developed in the rest of the paper, and are therefore interesting in their own right. Moreover, the analysis of these examples will suggest a solution to the general problem, which is discussed in subsection 3.4 and is the subject of the rest of the paper.

All of the examples that we discuss in this section have a common underlying structure. We take $\Sigma$ to be the full complex plane $\mathbb{C}$, with analytic coordinate $z$, and we look for coordinates $x$ and $y$ on $\Sigma$ such that the symplectic form $\mu d^2 \sigma$ takes the simple form

$$\mu d^2 \sigma = \frac{1}{\pi \epsilon} dx \wedge dy.$$ 

We may then consider $x$ and $y$ to be standard canonical coordinates in a 2-dimensional phase space and we can perform quantization similarly to elementary quantum mechanics. It is convenient to define the complex coordinate

$$s = x + iy$$

even though, as will be clear from the examples, $s$ is not necessarily an analytic coordinate on $\Sigma$ -- i.e. is not necessarily an analytic function of $z$. In terms of $s$ the symplectic form is given by

$$\mu d^2 \sigma = \frac{1}{2\pi i \epsilon} ds \wedge d\bar{s} \quad (19)$$

so that

$$\{s, \bar{s}\} = -2\pi i \epsilon.$$ 

Since the symplectic form is canonical in terms of the coordinate $s$, it is easier to define the quantization map $Q$ on $s$ than directly on the functions $z$ or $Z_A$. If we define

$$a = \frac{1}{\sqrt{\epsilon}} Q(\bar{s})$$

we conclude that

$$[a, a^\dagger] = 1.$$
The above is a canonical creation-annihilation pair and the quantization is standard. The Hilbert space $\mathcal{H}$ on which operators act is spanned by the simple harmonic oscillator states $|n\rangle$ ($n = 0, 1, \ldots$). Moreover functions of $s$ and $\bar{s}$ are quantized using the correspondence

$$
\begin{align*}
  s & \to \sqrt{\epsilon} a^\dagger \\
  \bar{s} & \to \sqrt{\epsilon} a.
\end{align*}
$$

This map is not uniquely defined for a general function of $s$ and $\bar{s}$ due to operator ordering ambiguities. On the other hand these problems can be resolved by imposing equations (15), (16) and (18) as conditions on the operators $Z_A = Q(Z_A)$.

### 3.1 Example: flat membrane

We now move on to the explicit examples. The first one is very simple and well-known. We wish to describe a flat membrane stretched in the $1-2$ plane, defined by

$$
\begin{align*}
  Z_1 & = z \\
  Z_A & = 0. \quad (A = 2, 3, 4)
\end{align*}
$$

Since

$$
\mu d^2\sigma = -\frac{1}{2\pi i \epsilon} dz \wedge d\bar{z},
$$

we may choose $s = z$. Quantization is therefore trivial and is given by

$$
\begin{align*}
  Z_1 & = \sqrt{\epsilon} a^\dagger \\
  Z_A & = 0. \quad (A = 2, 3, 4)
\end{align*}
$$

The operator $a^\dagger$ can be written in terms of hermitian operators $q, p$ satisfying $[q, p] = i$ through $a^\dagger = (q - ip)/\sqrt{2}$. This corresponds to a description of the flat membrane in terms of the hermitian matrices

$$
\begin{align*}
  X_1 & = \frac{\sqrt{\epsilon}}{\sqrt{2}} q \\
  X_2 & = -\frac{\sqrt{\epsilon}}{\sqrt{2}} p
\end{align*}
$$

exactly as discussed in [1]. Note that the space of flat membrane solutions is parameterized by $\epsilon$, which is the inverse of the 0-brane density $\mu$ on the brane.

### 3.2 Example: parabolic membrane

Let us move to a more complicated example, namely that considered in the introduction. In this case we wish to describe a parabolic surface defined by

$$
\begin{align*}
  Z_1^2 & = \beta Z_2 \\
  Z_3 & = Z_4 = 0.
\end{align*}
$$
where $\beta$ is a numerical coefficient which we take to be 1 for simplicity. If we parametrize the surface through

\[
Z_1 = z, \\
Z_2 = z^2,
\]

we can then easily compute

\[
\mu d^2\sigma = -\frac{1}{2\pi i\epsilon}(1 + 4z\bar{z})dz \wedge d\bar{z}.
\]

We now look for a change of coordinates $z \rightarrow s$ so that equation (19) is satisfied. We therefore require that

\[
(1 + 4z\bar{z})dz \wedge d\bar{z} = ds \wedge d\bar{s}.
\]

Both sides of the above equation define rotationally invariant measures on the plane (under rotations $z \rightarrow ze^{i\theta}$ and $s \rightarrow se^{i\theta}$ respectively). We may then assume that $z$ is just a radius-dependent rescaling of $s$, or more precisely that

\[
z = sf(s\bar{s}),
\]

where $f$ is a positive real function. In fact one may use (21) to show that equation (20) is satisfied provided that $\frac{dg}{dx} = 1/(1 + 4g)$, where $g(x) = xf^2(x)$. On the other hand we will not need the precise form of $f$ in the sequel, as will be clear shortly.

Quantization of equation (21) by realizing $Z = Q(z)$ as an operator on $\mathcal{H}$ is ambiguous due to operator-ordering problems. On the other hand, regardless of the ordering prescription chosen, the operator

\[
\rho = Q(z)
\]

must contain one more creation operator then it contains annihilation ones. More formally, if $N = a\dagger a$ is the number operator, then

\[
[N, \rho] = \rho.
\]

Therefore

\[
\rho|n\rangle = \rho_n|n + 1\rangle, \\
\rho^\dagger|n\rangle = \bar{\rho}_{n-1}|n - 1\rangle
\]

for some complex coefficients $\rho_n$ ($n \geq 0$). If we choose

\[
Z_1 = \rho, \\
Z_2 = \rho^2,
\]

then equations (15) and (18) are automatically satisfied. We will just have to impose equation (16), which reads in this case

\[
[\rho, \rho^\dagger] + [\rho^2, \rho^4] = -\epsilon.
\]
The above equation will determine the coefficients $\rho_n$ up to a phase factor. This is expected since phases can be changed with a $U(N)$ gauge transformation $\rho \to U\rho U^\dagger$. Acting with equation (24) on the state $|n\rangle$ and defining the positive coefficients

$$\alpha_n = |\rho_n|^2,$$

we arrive at the recursion relation

$$\alpha_n - \alpha_{n-1} + \alpha_{n+1}\alpha_n - \alpha_{n-2}\alpha_n = \epsilon \tag{25}$$

where we define $\alpha_n = 0$ for $n < 0$. Equation (25) determines $\alpha_{n+1}$ as a rational function of $\alpha_{n-2}, \alpha_{n-1}, \alpha_n$. The only undetermined coefficient is $\alpha_0$ and if we set

$$\alpha_0 = \xi$$

then all of the coefficients $\alpha_n = \alpha_n(\xi)$ are rational functions of $\xi$ alone. It seems as though we have thus constructed, for a fixed $\epsilon$, a family of solutions parametrized by $\xi$. On the other hand we note that the $\alpha_n$ are positive coefficients and, for a generic choice of $\xi$, some of the $\alpha_n(\xi)$ generated by the recursion relation (25) are negative. Let us denote by $\Gamma_N \subset \mathbb{R}$ the subset of the possible values of $\xi$ for which $\alpha_n \geq 0$ for $n \leq N$. Clearly $\Gamma_N \subset \Gamma_M$ for $N \geq M$. Moreover one can show numerically that the $\Gamma_N$’s are nested rectangles which converge rapidly to a unique value for $\xi$, which can be computed to arbitrary accuracy by taking $N$ to be large enough. As a function of $\epsilon$, Figure 1 (at the end of section 3) shows the value of $\alpha_0 = \xi$ determined using this procedure (the curve labeled “exact result” in the figure). The example discussed in the introduction corresponds to taking $\epsilon = 1$. Although it is clearly indicated by the numerics, we do not have a proof that $\alpha_0$ is uniquely determined by this procedure; we will be content here with the numerical result, however, because later we give a more general algebraic procedure which will include this special example.

We have thus found a matrix representation of the holomorphic curve $Z_2 = Z_1^2$ for any value of the parameter $\epsilon$. As in the example of the flat membrane in the previous section, $\epsilon$ is related to the inverse of the 0-brane density on the membrane. A physical picture of how the 0-branes are distributed on the surface of the parabolic membrane can be obtained by considering the matrix

$$\mathcal{X}_1^2 + \mathcal{X}_2^2 = \frac{1}{2}(\rho\rho^\dagger + \rho^\dagger\rho) = \text{Diag}\left(\frac{\alpha_0}{2}, \frac{\alpha_0 + \alpha_1}{2}, \frac{\alpha_1 + \alpha_2}{2}, \frac{\alpha_2 + \alpha_3}{2}, \ldots\right)$$

Because this matrix is diagonal, we can think of the individual 0-branes as having well-defined values of $r^2 = x_1^2 + x_2^2$. Thus, the individual 0-branes are in a sense localized on circular orbits of radii

$$r_n = \sqrt{(\alpha_{n-1} + \alpha_n)/2}. \tag{26}$$

We can compare this picture to the expectation that the 0-branes are uniformly distributed on the membrane surface. For large $N$, $r_N \sim \rho_N$. From (26) we see that as $N$ becomes large,
we expect to have $N$ 0-branes distributed over the portion of the membrane with $r^2 < \alpha_N$. The area of this portion of the membrane is given by

$$A = \frac{1}{\epsilon} \int_0^r (1 + 4t^2)2tdt = \frac{1}{\epsilon}(2r^4 + r^2).$$

Setting $A = N$ gives

$$r^2 = \frac{\sqrt{8\epsilon N + 1} - 1}{4} \sim \frac{\sqrt{\epsilon N}}{\sqrt{2}} - \frac{1}{4} + \mathcal{O}(N^{-1/2}).$$

On the other hand, from the recursion relation (25) we can derive the relation

$$\alpha_{N-1}(\alpha_N + \alpha_{N-2} + 1) = \epsilon N$$

from which we can determine the asymptotic form of $\alpha_N$

$$\alpha_N \sim \frac{\sqrt{\epsilon N}}{\sqrt{2}} - \frac{1}{4} + \mathcal{O}(N^{-1/2}).$$

This shows that the simple physical picture of the 0-branes being localized on circles of radii $\rho_N$ is quite accurate as $N \to \infty$.

3.3 Example: general rotationally invariant curve

We can generalize the previous example with very little effort as follows (the steps are identical and we will be brief). We consider a surface defined by

$$Z_A = c_A \cdot z^{p_A},$$

where the $p_A$ positive integers with no common divisor and the $c_A$ are complex coefficients. The volume form on $\Sigma$ is then

$$\mu d^2\sigma = -\frac{1}{2\pi i\epsilon} \left( \sum_A |c_A|^2 |z^{p_A}| z^{-1} \right) dz \wedge d\bar{z}$$

and is still rotationally invariant. We may then set $z = sf(s\bar{s})$ once again and conclude that $\rho = Q(z)$ satisfies (22) and (23), for some coefficients $\rho_n$ to be determined (up to a phase). We define

$$Z_A = c_A \cdot \rho^{p_A}$$

so that the only equation to be solved is

$$[Z_A, \bar{Z}_A] = \sum_A |c_A|^2 [\rho^{p_A}, \rho^{\bar{p}_A}] = -\epsilon.$$

This can be rewritten in terms of the coefficients $\alpha_n = |\rho_n|^2$ as

$$\sum_A |c_A|^2 (\alpha_n \ldots \alpha_{n+p_A-1} - \alpha_{n-1} \ldots \alpha_{n-p_A}) = \epsilon. \quad (27)$$

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If one calls
\[ q = \max_A \langle p_A \rangle, \]
then equation (27) determines \( \alpha_{n+q-1} \) in terms of \( \alpha_m \), for \( m < n + q - 1 \). On the other hand, the coefficients \( \alpha_0, \ldots, \alpha_{q-2} \) are undetermined by (27). We denote them by
\[ \alpha_j = \xi_j \quad (j = 0, \ldots, q-2) \]
The \( \alpha_n = \alpha_n(\xi) \) are then rational functions of the \( \xi_j \) and, as in the previous example, there should be a unique \( \xi_j \) such that \( \alpha_n(\xi) \geq 0 \) for all \( n \geq 0 \).

### 3.4 General holomorphic curves

The examples that we have discussed so far can all be analyzed by elementary methods. On the other hand, the techniques used in these special cases cannot easily be generalized. First of all, we have used the fact that \( \Sigma = \mathbb{C} \), allowing us to use the canonical quantization of the plane which is well-known from elementary quantum mechanics. More complicated surfaces will have different underlying spaces \( \mathcal{H} \), and the examples given above do not suggest a natural choice of \( \mathcal{H} \) in the general case. Moreover, even if we restrict ourselves to the case \( \Sigma = \mathbb{C} \), it is hard to find a general solution of the problem. The above examples all rely on the rotational symmetry of the volume form \( \mu \), which allows us to conclude that \( \rho \) is an operator satisfying \([N, \rho] = \rho\). This constraint restricts the form of \( \rho \) almost completely. In general the operator \( \rho \) will have matrix elements between eigenstates with arbitrary \( N \) eigenvalues, and the operator-ordering ambiguities will not allow us to explicitly determine the operator \( \rho \), or even to determine its existence.

The idea that allows us to solve the general case comes, on the other hand, from the above examples if one changes point of view. Let us go back to the planar and parabolic examples. In both cases the underlying Hilbert space \( \mathcal{H} \) was the same, but the quantization of the analytic coordinate \( z \) led to different operators, reflecting the difference in the volume form \( \mu \). Let us suppose, on the other hand, that the space \( \mathcal{H} \), considered now just as a vector space (therefore forgetting the inner product), does not change between the two examples, and let us also suppose that the operator \( \rho = Q(z) \) is the same. In both the planar and parabolic case we assume that \( \mathcal{H} \) is spanned by states \(|v_n\rangle, n \geq 0 \) (which are proportional to the states \(|n\rangle\)) and we take
\[ \rho |v_n\rangle = |v_{n+1}\rangle. \]
What changes between the examples, reflecting the change in \( \mu \), is the inner product on \( \mathcal{H} \) and therefore the definition of adjoint operator \( \rho^\dagger \). We will assume that
\[ \langle v_n | v_m \rangle = k_n \delta_{n,m} \]
so that
\[ \rho^\dagger |v_n\rangle = \frac{k_n}{k_{n-1}} |v_{n-1}\rangle. \]
(the absolute normalization of the $k_n$’s is irrelevant, and we fix it by assuming $k_0 = 1$). We now have to solve for the coefficients $k_n$ by imposing equation (16). To make contact with the previous discussion, we define

$$\alpha_n = \frac{k_{n+1}}{k_n}.$$  

Solving for the $k_n$’s is equivalent to solving for the $\alpha_n$’s.

Let us look at the planar case. Equation (16) is simply $[\rho, \rho^\dagger] = -\epsilon$ or, in terms of the $\alpha_n$’s, $\alpha_n - \alpha_{n-1} = \epsilon$. The recursive equation is solved by $\alpha_n = (n+1)\epsilon$, or $k_n = n!\epsilon^n$. The orthonormal states are then given by $|n\rangle = (n!\epsilon^n)^{-\frac{1}{2}}|v_n\rangle$, so that $\rho|n\rangle = \sqrt{\epsilon(n+1)}|n+1\rangle$ as expected (recall $\rho = \sqrt{\epsilon a^\dagger}$ in the planar case).

The parabolic case is solved exactly as before, even though the interpretation is different. Equation (16) reads $[\rho, \rho^\dagger]^2 + [\rho^2, \rho^2] = -\epsilon$ and it implies again the recursive relation (25) for the coefficients $\alpha_n$.

In order to generalize the point of view described above we first have to decide how to choose, given a surface $\Sigma$, a vector space $H_\Sigma$ on which operators act. A natural choice, which is also suggested from coherent state quantization and geometric quantization (more of this in section 7), is to let

$$H_\Sigma = \{ \text{holomorphic functions on } \Sigma \}.$$  

Note that we have not yet chosen an inner product on the space $H$. In the case $\Sigma = \mathbb{C}$, the states $|v_n\rangle$ correspond, up to an overall normalization, to the functions $z^n$.

Quantization of holomorphic functions, as we proposed before, will not depend on the specific inner product chosen, and will be defined as follows. Let $|\phi\rangle$ be a state in $H$ corresponding to the holomorphic function $\phi$, and let $A$ be also holomorphic. The quantized operator $Q(A)$ corresponding to $A$ will then be defined by

$$Q(A)|\phi\rangle = |A\phi\rangle,$$

and it therefore acts by pointwise multiplication of functions. It is clear that operators corresponding to holomorphic functions commute, and that equation (15) is automatically satisfied. Moreover, if the coordinate functions $Z_A$ satisfy (17) then the quantized operators $Z_A$ automatically satisfy (18). In the specific example given above we had $|v_n\rangle = |z^n\rangle$ and $\rho = Q(z)$, so that $\rho|v_n\rangle = |v_{n+1}\rangle$, as we assumed before.

Finally, as in the examples of this section, the crucial constraint comes from equation (16). We need to choose an inner product on $H$ in order to define adjoints of operators. Since states correspond to functions, it is natural to define an inner product via integration over $\Sigma$. In order to do so, we need a volume 2-form on $\Sigma$, which we will denote by $\Omega$ (Note that $\Omega$ is quite different from the measure $\mu$; in particular, $\mu$ does not have a finite integral when $\Sigma$ is noncompact). We can then define the inner product between two elements $\phi, \psi \in H$ by

$$\langle \psi | \phi \rangle = \int_\Sigma \bar{\psi} \phi \Omega.$$  

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In the planar case the correct choice of volume form is given by
\[ \Omega = -\frac{1}{2\pi i\epsilon} e^{-\frac{z\bar{z}}{i\epsilon}} dz \wedge d\bar{z}. \]

The rest of the paper is devoted to the description of how \( \Omega \) can be chosen for any functions \( Z_A \) so that equation (16) is satisfied. In order to find such a description, we need to discuss in more detail how to compute the adjoint of operators, and we will therefore have to make a major mathematical digression devoted to the study of Bergman integral kernels.

4 Bergman Integral Kernels

Let us suppose, in the language of the end of the last section, that \( \phi, \psi \in \mathcal{H} \), and that \( A \) is a holomorphic function on \( \Sigma \). We have proposed that \( Q(A)|\phi\rangle = |A\phi\rangle \), and therefore we have that \( \langle \psi|Q(A)|\phi\rangle = \int_{\Sigma} \bar{\psi} A\phi \Omega \). If we take the adjoint of the previous equation and use the fact that \( Q(A)^\dagger = Q(\bar{A}) \), we then obtain
\[ \langle \phi|Q(\bar{A})|\psi\rangle = \int_{\Sigma} \bar{\phi} \bar{A}\psi \Omega. \]

We might be tempted to deduce from the above equation that \( Q(\bar{A})|\psi\rangle = |\bar{A}\psi\rangle \), but this is wrong since \( \bar{A}\psi \notin \mathcal{H} \). On the other hand, if we consider the larger space \( V \) of functions (not necessarily analytic) on \( \Sigma \) and view \( \mathcal{H} \) as a subspace of \( V \), then the above equation says that the orthogonal projection of the state \( |\bar{A}\psi\rangle \) onto \( \mathcal{H} \) is equal to the state \( Q(\bar{A})|\psi\rangle \). The orthogonal projection is called Bergman projection, and this section is devoted to a detailed study of its properties. The Bergman integral kernel is defined in subsection 4.1. Subsection 4.2 contains a brief review of the geometry of Riemann surfaces. Some basic properties of the Bergman integral kernel are discussed in 4.3. Subsection 4.4 describes a particular example where the Bergman projection operator can be explicitly computed; the form of this projection operator suggests an ansatz for formulating the operator in the general case, which is analyzed in subsection 4.5 and shown to agree with Bergman projection on a general class of functions.

4.1 Definition of Bergman integral kernels

Consider a Riemann surface \( \Sigma \) and fix on the surface, once and for all, a real and non-vanishing 2-form \( \Omega \). We denote by \( V \) the space of complex functions \( f \) on \( \Sigma \) which are square integrable with respect to \( \Omega \) - i.e. such that \( \int_{\Sigma} |f|^2 \Omega < \infty \). The vector space \( V \) has a natural Hilbert space structure, where the inner product between two elements \( \phi, \psi \in V \) is given by
\[ \langle \phi|\psi\rangle = \int_{\Sigma} \bar{\phi}\psi \Omega. \]
The surface $\Sigma$ is endowed with a complex structure, and one is therefore led, following Bergman \cite{11}, to consider the subspace $\mathcal{H} \subset \mathcal{V}$ consisting of holomorphic functions on $\Sigma$. Moreover, since $\mathcal{V}$ is a Hilbert space, one can study the orthogonal projection $\pi : \mathcal{V} \to \mathcal{H}$. (The projection operator does depend on the choice of 2-form $\Omega$. If we want to underline this dependence, we will use the more cumbersome notation $\pi_\Omega$.) In what follows we wish to give an integral representation of the projection operator and to study its properties. To this end we fix an orthonormal basis for $\mathcal{H}$ given by holomorphic functions $f_n$ on $\Sigma$ satisfying $\int_\Sigma \bar{f}_n f_m \Omega = \delta_{n,m}$. (Note that this basis is unrelated to the basis $|v_n\rangle$ discussed in the previous section.) The orthogonal projection of any element $\phi \in \mathcal{V}$ is given by $\pi(\phi) = \sum_n f_n \langle f_n | \phi \rangle$.

We may then introduce a kernel function $K$, called the Bergman integral kernel, defined by

$$K(z, w) = \sum_n f_n(z) \bar{f}_n(w). \quad (28)$$

The function $K$ does not depend on the specific choice of basis $f_n$, since different choices are related by unitary transformations. Moreover it gives an integral representation of the action of the projection operator $\pi$. In fact, if $\phi$ is an element of $\mathcal{V}$, one has that

$$\pi(\phi)(z) = \int_\Sigma K(z, w) \phi(w) \Omega(w).$$

### 4.2 Geometry of Riemann surfaces

In order to gain a deeper understanding of the action of the Bergman projection $\pi$, we need to be able to discuss in more detail the geometry of the underlying Riemann surface $\Sigma$. To this end we could very well use the standard notation of differential and Riemannian geometry. On the other hand, the fact that the manifold $\Sigma$ has a complex structure and is of complex dimension one, greatly simplifies the geometry and the standard notation is very cumbersome in this specific case. Therefore we use this subsection to introduce some specific conventions and notations which will simplify the manipulations and, hopefully, clarify the underlying geometric concepts.

We first focus our attention on tensors, which are classified according to their conformal weight (or dimension). A tensor $T$ has conformal weight $(a, \bar{a})$ if, under coordinate transformations, the expression

$$T \ dz^a d\bar{z}^\bar{a}$$

is invariant. We can use various operations to construct new tensor fields starting from old ones. Some of these manipulations are standard, like tensor addition and multiplication\cite{2}. Some operations, on the other hand, are specific to the case of a complex variety of dimension one, and will be used repeatedly in the rest of the paper. Let us start, for example, with

\footnote{Recall that addition is defined for tensors of the same weight and multiplication is defined for tensors of any weight. Under multiplication, the conformal dimension of the resulting tensor is the sum of the dimensions of the original tensors.}
a non-vanishing tensor field $T$ of conformal weight $(a, \bar{a})$. Since tensor fields are sections of line bundles, we can consider the inverse field

$$1/T$$

of dimension $(-a, -\bar{a})$. As a second example we may start with the same tensor field $T$ and construct a new field

$$\partial \bar{\partial} \ln T$$

of dimension $(1, 1)$. Some explanation is necessary in this case. First of all we must be able to consistently choose a branch of the logarithm. This is possible for example if $a = \bar{a}$ and if $\text{Arg}(T)$ is constant (note that this is a well defined notion because, under change of coordinates $z \to w$, we have that $T \to T|\partial w/\partial z|^{2a}$, so that $\text{Arg}(T)$ is invariant). Secondly we must show that the expression in equation (29) does define a tensor of the correct dimension. Under a coordinate transformation $z \to w$ we have that $\ln(T) \to \ln(T) + a \ln(\partial w/\partial z) + \bar{a} \ln(\partial \bar{w}/\partial \bar{z})$. The second and third term in the transformation law are respectively analytic and antianalytic, and are therefore annihilated by the operator $\partial \bar{\partial}$. Therefore $\partial_z \bar{\partial}_{\bar{z}} \ln(T) \to \partial_w \bar{\partial}_{\bar{w}} \ln(T) |\partial w/\partial z|^2$, as we wanted to show.

Up to this point we have used the 2-form $\Omega$ to define the integration measure on the surface $\Sigma$. It will be very useful later to consider $\Omega$ as the volume form of an underlying riemannian metric $g$. Let us be more specific. If $z = x + iy$ is a local analytic coordinate, we can write

$$\Omega = i \, C \, dz \wedge d\bar{z},$$

where $C$ is a real and positive $(1, 1)$ tensor. We will choose $g$ so that

$$\Omega = \sqrt{\det g_{ab}} \, dx \wedge dy.$$  

We clearly have some freedom in our choice of the metric. If we impose the additional restriction that $g$ be hermitian – i.e. that

$$g_{zz} = g_{\bar{z}\bar{z}} = 0,$$

then the only non-vanishing element of the metric is

$$g_{z\bar{z}} = C.$$ 

The standard riemannian connection is in this case very simple. The only non-vanishing coefficients are given by

$$\Gamma = \Gamma^z_{\bar{z}z} = \partial \ln C$$

$$\bar{\Gamma} = \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = \bar{\partial} \ln C.$$
The covariant derivatives of a tensor $T$ of weight $(a, \bar{a})$ can then be written in terms of the connection as

$$\nabla T = (\partial - a\Gamma)T$$

$$\bar{\nabla} T = (\bar{\partial} - \bar{a}\bar{\Gamma})T.$$ 

We conclude this quick tour of Riemannian geometry by considering the curvature tensor, which measures the lack of commutativity of covariant derivatives. It is a simple computation to check that

$$[\nabla, \bar{\nabla}] T = (a - \bar{a})\mathcal{R}T,$$ 

where $\mathcal{R}$ is the $(1,1)$ curvature tensor, given by

$$\mathcal{R} = \partial\bar{\Gamma} = \bar{\partial}\Gamma = \partial\bar{\partial}\ln C.$$

As a final remark we note that the tensor in equation (29) can be rewritten in terms of covariant derivatives as

$$\partial\bar{\partial}\ln T = \nabla^T \nabla\bar{T} + \bar{a}\mathcal{R}$$

$$= \bar{\nabla}^T \nabla\bar{T} + a\mathcal{R}.$$ 

Note that we will use the convention that covariant derivatives act on everything on the right, unless explicitly indicated.

### 4.3 Properties of Bergman integral kernels

We now have the language to discuss some of the basic properties of the Bergman projection. The first property follows essentially from the definition. If $\phi$ is an element of $\mathcal{V}$, then $\pi(\phi)$ is in $\mathcal{H}$, and is therefore analytic. Moreover, if $\phi$ itself is analytic, then $\pi(\phi) = \phi$. The second property of the projection requires more work. We start by observing that $K(z, w)$ is analytic in $z$ and antianalytic in $w$, as can be readily seen from the defining equation (28). We then let $X$ be a $(-1,0)$ vector field and we consider the scalar field $\nabla X \in \mathcal{V}$. Using the integral representation of $\pi$ we can compute, recalling that $\nabla_w K(z, w) = 0$,

$$\pi(\nabla X)(z) = \int_{\Sigma} K(z, w)\nabla_w X(w) \Omega(w) =$$

$$= -\int_{\Sigma} X(w)\nabla_w K(z, w) \Omega(w) = 0,$$ 

where we assume, as we will from now on, that we can neglect boundary terms when we integrate by parts (this is true in all the cases of interest in this paper). Note that the above
manipulation is possible since $\Omega$ is the volume form of the underlying metric, and therefore integration by parts can be performed, provided we replace partial derivatives with covariant derivatives. We have thus shown that, generically,

$$\pi(\nabla X) = 0.$$  

We may combine the two properties described above as follows. Let $X$ and $\phi$ be, respectively, a holomorphic $(-1,0)$ vector field and a holomorphic function. Then the function $X\nabla\phi$ is itself holomorphic, and therefore $\pi(X\nabla\phi) = X\nabla\phi$. Moreover, as we have shown above, $\pi(\nabla X\phi) = 0$. We can use the fact that $\nabla X\phi = X\nabla\phi + \phi\nabla X$ and the fact that $\pi$ is a linear map to conclude that $\pi(\phi\nabla X) = -X\nabla\phi$. Using the same reasoning inductively we can show that

$$\pi(\phi\nabla X_1\nabla X_2\ldots\nabla X_n) = (-1)^n X_n\nabla X_{n-1}\nabla \ldots X_1\nabla\phi, \quad (31)$$

where the $X_i$ are holomorphic $(-1,0)$ vector fields, and $\phi$ is a holomorphic function.

We have already remarked that the projection operator depends implicitly on the underlying 2-form $\Omega$. Changes in $\Omega$ are reflected non-trivially in changes in $\pi$. In general this relationship is quite complicated. In one specific case, however, we can explicitly relate the projection operators corresponding to different choices of $\Omega$. Let $\chi$ be a holomorphic function and consider the following transformation

$$\Omega \to e^{\chi+\bar{\chi}} \Omega.$$  

Any orthonormal basis $f_n$ of $H$ undergoes the corresponding transformation $f_n \to f_ne^{-\chi}$ and, therefore, the integral kernel is modified as follows

$$K(z, w) \to e^{-\chi(z)}K(z, w)e^{-\bar{\chi}(w)}.$$  

Using the integral representation of the Bergman projection, it is then the work of a minute to show that, for any function $\phi$

$$\pi_{e^{\chi+\bar{\chi}}} (\phi) = e^{-\chi}\pi_\Omega(\phi e^\chi). \quad (32)$$

### 4.4 An example of Bergman projection

We have completed an informal discussion of the Bergman projection and of its basic properties. We now focus our attention once more on the integral representation of the projection $\pi$. Given a function $\phi \in \mathcal{V}$, the value $\pi(\phi)(z)$ will depend generically on the values $\phi(w)$ for arbitrary $w$, and in this sense the operator $\pi$ is non-local. On the other hand, as we will show in detail in this subsection and the following subsection, there is a useful and explicit expansion for $\pi(\phi)(z)$ in terms of the values of $\phi$ and of its covariant derivatives, all evaluated at the same point $z$. The expansion is schematically of the form

$$\pi(\phi)(z) \sim \sum_{n=0}^{\infty} \frac{1}{n!} \nabla^n \nabla^n \phi(z). \quad (33)$$
The interesting feature is that, for surfaces with very large curvature, the first terms dominate the sum and the operator becomes essentially local. We will see that this limit is very natural in the context of geometric quantization of holomorphic surfaces, and corresponds to the limit of zero Planck constant (the classical limit).

In this subsection we describe in detail a simple example where the Bergman projection operator can be computed explicitly. We obtain an expression for the action of this projection operator of the form just described, and this will motivate the ansatz for the general form that is examined in subsection 4.5.

We take $\Sigma$ to be the complex plane $\mathbb{C}$ and

$$\Omega = -\frac{1}{2\pi i\epsilon} e^{-z\bar{z}/\epsilon} dz \wedge d\bar{z},$$

where $\epsilon$ is a positive constant. A natural orthonormal basis for the space $\mathcal{H}$ is given by the functions

$$f_n(z) = c_n z^n,$$

where

$$|c_n|^{-2} = \int_{\Sigma} z^n \bar{z}^n \Omega = \epsilon^n n!.$$

The Bergman integral kernel can be explicitly computed

$$K(z, w) = \sum_{n=0}^{\infty} \frac{1}{\epsilon^n n!} z^n \bar{w}^n = e^{z\bar{w}/\epsilon},$$

and therefore the integral representation of the projection operator is explicitly given by

$$\pi(\phi)(z) = -\frac{1}{\epsilon} \int_{\Sigma} e^{(z-w)\bar{w}/\epsilon} \phi(w) dw \wedge d\bar{w}, \quad (34)$$

where $\phi \in \mathcal{V}$. In order to analyze the above integral we will first of all introduce some auxiliary functions

$$G_n = \left(\frac{\epsilon^n}{(w-z)^n}\right) e^{(z-w)\bar{w}/\epsilon}.$$

for $n \geq 0$. Note that equation (34) can be rewritten as

$$\pi(\phi)(z) = -\frac{1}{\epsilon} \int_{\Sigma} G_0 \phi \ dw \wedge d\bar{w}.$$

Moreover

$$\begin{align*}
G_0 \phi \ dw \wedge d\bar{w} & = d_w (G_1 \phi \ dw) + G_1 \bar{\partial}_w \phi \ dw \wedge d\bar{w} \\
& \vdots \\
G_n \bar{\partial}_w^{(n)} \phi \ dw \wedge d\bar{w} & = d_w (G_{n+1} \bar{\partial}_w^{(n)} \phi \ dw) + G_{n+1} \bar{\partial}_w^{(n+1)} \phi \ dw \wedge d\bar{w}
\end{align*}$$
so that

\[ G_0 \phi \, dw \wedge d\bar{w} = d_w \left( \sum_{n=0}^{\infty} G_{n+1} \bar{\partial}_w^{(n)} \phi \, dw \right). \]

All of the functions \( G_n \), for \( n \geq 1 \), are singular at \( w = z \). This suggests that we should replace the integration region \( \Sigma \) with the region \( \Sigma_\delta \), obtained from the full complex plane by deleting a disk of radius \( \delta \) around \( z \). It will be then understood in the sequel that we are considering the limit \( \delta \to 0 \). We may then use Stokes theorem and write

\[
\pi(\phi)(z) = -\frac{1}{\epsilon} \frac{1}{2\pi i} \int_{\Sigma_\delta} d_w \left( \sum_{n=0}^{\infty} G_{n+1} \bar{\partial}_w^{(n)} \phi \, dw \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{\epsilon} \int_{\Gamma_\delta} dw \frac{G_{n+1} \bar{\partial}_w^{(n)} \phi}{2\pi i},
\]

where \( \Gamma_\delta = -\partial \Sigma_\delta \) is the circle of radius \( \delta \to 0 \) around \( z \), with counterclockwise orientation.

We first recall that for any function, not necessarily analytic,

\[
\lim_{\delta \to 0} \int_{\Gamma_\delta} dw \frac{A(w)}{2\pi i} \frac{1}{(w-z)^{n+1}} = \frac{1}{n!} \partial^{(n)} A(z).
\]

Therefore, using equation (35) for \( G_n \), we obtain

\[
\pi(\phi)(z) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \bar{\partial}_w^{(n)} \left( e^{(z-w)\bar{w}/\epsilon} \bar{\partial}_w^{(n)} \phi(w) \right) \bigg|_{w=z}
\]

\[
= \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \sum_{p=0}^{n} \binom{n}{p} \left( -\frac{\bar{z}}{\epsilon} \right)^p \partial^{n-p} \bar{\partial}^p \phi(z)
\]

and finally

\[
\pi(\phi) = \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \left( \partial - \frac{\bar{z}}{\epsilon} \right)^n \bar{\partial}^n \phi.
\] 

Up to this point we have used a specific coordinate system and therefore the geometric nature of the above expression is not transparent. Let us therefore rewrite equation (36) in a coordinate invariant way. First note that the geometric data for the surface is given by

\[
C = \frac{1}{2\pi \epsilon} e^{-z\bar{z}/\epsilon}
\]

\[
\Gamma = -\frac{\bar{z}}{\epsilon}, \quad \bar{\Gamma} = -\frac{z}{\epsilon}
\]

\[
R = -\frac{1}{\epsilon}
\]

We first of all note that the expansion coefficient \( \epsilon \) is inversely related to curvature, and we start to see the first evidence for the claim (33). Moreover the holomorphic derivatives
\( (\partial - \bar{z}/\epsilon) \) and antiholomorphic derivatives \( \bar{\partial} \) can be replaced with covariant derivatives \( \nabla \) and \( \bar{\nabla} \) as long as they are acting on tensors of holomorphic dimension \(-1\) and antiholomorphic dimension \(0\) respectively. This can be easily done by writing the \(n\)-th term of the sum in equation (36) as

\[
\frac{1}{n!} (-1)^{n} \left( \nabla \frac{1}{R} \right)^{n} R^{n} \left( \frac{1}{R} \bar{\nabla} \right)^{n} \phi \quad (37)
\]

First note that the above expression has the right power of curvature to give a total contribution of \( \epsilon^{n} \) (the minus signs reflect the fact that \( R = -1/\epsilon \)). Starting from the right of equation (37) we can also see that the covariant derivatives act on tensors of the correct dimension. The first antiholomorphic derivative \( \bar{\nabla} \) acts on a \((0,0)\) tensor, thus giving a \((0,1)\) tensor. After dividing by \( R \), we get a \((-1,0)\) tensor. Repeating this process \(n\) times we see that \( \left( \frac{1}{R} \bar{\nabla} \right)^{n} \phi \) has weight \((-n,0)\) and so

\[
R^{n} \left( \frac{1}{R} \nabla \right)^{n} \phi
\]

has conformal dimension \((0,n)\). Division by \( R \) moves the dimension to \((-1,n-1)\) and multiplication by \( \nabla \) moves it to \((0,n-1)\). Continuing this process we see that \( \nabla \) always acts on tensors of holomorphic dimension \(-1\) and \( \bar{\nabla} \) on tensors of antiholomorphic dimension \(0\). Replacing \( \nabla \to \partial + \Gamma \) and \( \bar{\nabla} \to \bar{\partial} \) in equation (37) and using \( R = -1/\epsilon \) we then recover the \(n\)-th term in the sum (36). We thus conclude that

\[
\pi(\phi) = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^{n} \left( \nabla \frac{1}{R} \right)^{n} R^{n} \left( \frac{1}{R} \bar{\nabla} \right)^{n} \phi. \quad (38)
\]

Recall that we are using a convention in which the covariant derivatives act on everything to their right, even when they are inside a parentheses. At this point we could analyze the above expression further and check that it satisfies the properties of the Bergman projection described in the previous section. On the other hand, this will follow as a special case of the general discussion of the next section. We will be therefore content to use equation (38) as a partial motivation for the general ansatz that will be discussed next.

### 4.5 The general Bergman projection operator

In this section we shall argue that the general expansion for the action of the Bergman projection on a generic function \( \phi \in \mathcal{V} \) is given by (again, recall that covariant derivatives act on everything to their right, even when they are inside a parentheses.)

\[
\pi(\phi) = \sum_{n=0}^{\infty} (-1)^{n} (\nabla \frac{1}{P_{1}})(\nabla \frac{1}{P_{2}}) \ldots (\nabla \frac{1}{P_{n}})P_{1} \ldots P_{n}(\frac{1}{P_{n}} \nabla)(\frac{1}{P_{2}} \nabla)(\frac{1}{P_{1}} \nabla)\phi, \quad (39)
\]

where the \( P_{n} \)'s are \((1,1)\) tensors related to the curvature tensor \( R \) and its derivatives. In particular
\[ P_1 = R, \]
and the \( P_n \)'s, for \( n > 1 \), satisfy the recursion relation
\[
P_n = P_1 + P_{n-1} + \sum_{j=1}^{n-1} \partial \bar{\partial} \ln |P_j| = P_1 + P_{n-1} + \sum_{j=1}^{n-1} \partial \bar{\partial} \ln P_j. \tag{40}\]
We will show in particular that the expansion (39) satisfies all of the properties of the projection operator described in section 4.3.

Before giving the formal proof let us show that expression (39) reduces to equation (38) if we are considering the special case of the last section. We recall that, in the canonical coordinate system that we chose in section 4.4, the curvature tensor was \( R = -1/\epsilon \), and was therefore independent of \( z \). Using the recursive equation (40) one can then show inductively that all the \( P_n \)'s are independent of \( z \) and therefore the terms of the form \( \partial \bar{\partial} \ln P_j \) drop from the recursion relation. It is then easy to show that
\[ P_1 = nR, \]
therefore recovering equation (38).

Now back to the main proof. First we note that, if \( \phi \) is holomorphic, all the terms with \( n \geq 1 \) in the expansion (39) vanish, and therefore \( \pi(\phi) = \phi \) as expected. We will now show that, for a generic \( \phi \), \( \pi(\phi) \) is analytic - i.e. that \( \bar{\nabla} \pi(\phi) = 0 \). Let \( \pi_n \) be the \( n \)-th term in the sum (39), so that \( \pi(\phi) = \sum_{n=0}^{\infty} \pi_n \). We write the expression for \( \bar{\nabla} \pi_n \) as a sum
\[
\bar{\nabla} \pi_n = A_n + B_n,
\]
where
\[
A_n = (-1)^n[\bar{\nabla},(\frac{1}{P_1})\ldots(\frac{1}{P_n})P_1\ldots P_n](\frac{1}{P_1})\bar{\nabla}\ldots(\frac{1}{P_1})\bar{\nabla}\phi
\]
\[
B_n = (-1)^n(\frac{1}{P_1})\ldots(\frac{1}{P_n})P_1\ldots P_nP_{n+1}(\frac{1}{P_{n+1}})\bar{\nabla}\ldots(\frac{1}{P_1})\bar{\nabla}\phi.
\]
Clearly \( A_0 = 0 \). Moreover we will show that \( A_n + B_{n-1} = 0 \) for \( n \geq 1 \). This will complete the proof, because \( \bar{\nabla} \pi(\phi) = \sum_{n=0}^{\infty} \bar{\nabla} \pi_n = \sum_{n=0}^{\infty} (A_n + B_n) = A_0 + \sum_{n=1}^{\infty} (A_n + B_{n-1}) = 0 \). We shall use induction to show that \( A_n + B_{n-1} = 0 \). It is convenient to this end to write the expression for \( A_n \) and \( B_n \) as
\[
A_n = (-1)^n \alpha_n(\frac{1}{P_n} \bar{\nabla})\ldots(\frac{1}{P_1} \bar{\nabla})\phi
\]
\[
B_n = (-1)^n \beta_n(\frac{1}{P_{n+1}} \bar{\nabla})\ldots(\frac{1}{P_1} \bar{\nabla})\phi,
\]
where \( \alpha_n \) and \( \beta_n \) are operators acting on tensors of weight \((-n, 0)\) and \((-n-1, 0)\) respectively, and are given by
\[ \alpha_n = \left[ \nabla, (\nabla \frac{1}{P_1}) \ldots (\nabla \frac{1}{P_n}) P_1 \ldots P_n \right] \]

\[ \beta_n = (\nabla \frac{1}{P_1}) \ldots (\nabla \frac{1}{P_n}) P_1 \ldots P_n P_{n+1}. \]

It will then be sufficient to show that \( \alpha_n = \beta_{n-1} \) when acting on tensors of dimension \((-n, 0)\). The case \( n = 1 \) is a simple application of equation (30) for the commutator of covariant derivatives acting on \((-1, 0)\) vector fields

\[ \alpha_1 = \left[ \nabla, \nabla \right] = R = P_1 = \beta_0. \]

The induction step, on the other hand, is proved as follows. Rewrite the expression for \( \alpha_n \) as

\[ \alpha_n = \left[ \nabla, (\nabla \frac{1}{P_1}) \ldots (\nabla \frac{1}{P_{n-1}}) P_1 \ldots P_{n-1} \frac{1}{P_1 \ldots P_{n-1}} (\nabla \frac{1}{P_n}) P_1 \ldots P_n \right] \]

\[ = \alpha_{n-1} \left( \frac{1}{P_1 \ldots P_{n-1}} \nabla P_1 \ldots P_{n-1} \right) + \]

\[ + (\nabla \frac{1}{P_1}) \ldots (\nabla \frac{1}{P_{n-1}}) P_1 \ldots P_{n-1} \left[ \nabla, \frac{1}{P_1 \ldots P_{n-1}} \nabla P_1 \ldots P_{n-1} \right]. \]

Recall that the above operator acts on \((-n, 0)\) tensors. The commutator in the second term of the above expression for \( \alpha_n \), which we will denote by \( O \), can be rewritten as

\[ O = \left[ \nabla, \frac{1}{P_1 \ldots P_{n-1}} \nabla P_1 \ldots P_{n-1} \right] \]

\[ = \left[ \nabla, \frac{1}{P_1 \ldots P_{n-1}} \left[ \nabla, P_1 \ldots P_{n-1} \right] \right] + [\nabla, \nabla] \]

\[ = T + nR, \]

where the tensor \( T \) is given by

\[ T = \nabla \frac{1}{P_1 \ldots P_{n-1}} \nabla P_1 \ldots P_{n-1} = \partial \bar{\partial} \ln P_1 \ldots P_{n-1} - (n - 1)R. \]

Then

\[ O = R + \sum_{j=1}^{n-1} \partial \bar{\partial} \ln P_j = P_n - P_{n-1}. \]

We can then combine the above result with the inductive hypothesis \( \alpha_{n-1} = \beta_{n-2} \) and rewrite equation (41) as
\[ \alpha_n = \beta_{n-2} \frac{1}{P_1 \ldots P_{n-1}} \nabla P_1 \ldots P_{n-1} + (\nabla \frac{1}{P_1}) \ldots (\nabla \frac{1}{P_{n-1}}) P_1 \ldots P_{n-1} O = \]
\[ = (\nabla \frac{1}{P_1}) \ldots (\nabla \frac{1}{P_{n-2}}) \nabla P_1 \ldots P_{n-1} + (\nabla \frac{1}{P_1}) \ldots (\nabla \frac{1}{P_{n-1}}) P_1 \ldots P_{n-1} P_n \]
\[ - (\nabla \frac{1}{P_1}) \ldots (\nabla \frac{1}{P_{n-2}}) \nabla P_1 \ldots P_{n-1} = \beta_{n-1}, \]

thus proving the inductive step.

The next property of the projection operator that we want to prove is that \( \pi(\nabla X) = 0 \)
when \( X \) is a \((-1, 0)\) vector field. The proof is similar to the one just given, and we shall be brief. Suppose that \( \phi = \nabla X \) in equation (98). The expression for \( \pi_n \) can be written, using
the same philosophy, as the sum of two terms
\[ \pi_n = A_n + B_n, \]
where now
\[ A_n = (-1)^{n+1}(\nabla \frac{1}{P_1}) \ldots (\nabla \frac{1}{P_n}) \alpha_n X \]
\[ B_n = (-1)^{n}(\nabla \frac{1}{P_1}) \ldots (\nabla \frac{1}{P_{n+1}}) \beta_n X. \]

The operators \( \alpha_n \) and \( \beta_n \) are now given by
\[ \alpha_n = [\nabla, P_1 \ldots P_n(\frac{1}{P_n} \nabla) \ldots (\frac{1}{P_1} \nabla)] \]
\[ \beta_n = P_1 \ldots P_{n+1}(\frac{1}{P_n} \nabla) \ldots (\frac{1}{P_1} \nabla) \]

and both act on \((-1, 0)\) fields. Again \( A_0 = 0 \) and the proof rests on the fact that \( A_n + B_{n-1} = 0 \). In fact we will prove, like above, that \( \alpha_n + \beta_{n-1} = 0 \). Clearly \( \alpha_1 = [\nabla, \nabla] = -R = -\beta_0 \).

Moreover
\[ \alpha_n = P_1 \ldots P_{n-1} \nabla \frac{1}{P_1 \ldots P_{n-1}} [\nabla, P_1 \ldots P_{n-1}(\frac{1}{P_{n-1}} \nabla) \ldots (\frac{1}{P_1} \nabla)] + \]
\[ + [\nabla, P_1 \ldots P_{n-1} \nabla \frac{1}{P_1 \ldots P_{n-1}}] P_1 \ldots P_{n-1}(\frac{1}{P_{n-1}} \nabla) \ldots (\frac{1}{P_1} \nabla) \]
\[ = P_1 \ldots P_{n-1} \nabla \frac{1}{P_1 \ldots P_{n-1}} \alpha_{n-1} + OP_1 \ldots P_{n-1}(\frac{1}{P_{n-1}} \nabla) \ldots (\frac{1}{P_1} \nabla). \]

As before we can show that, in this case, \( O = P_{n-1} - P_n \). Then
\[ \alpha_n = -P_1 \ldots P_{n-1} \nabla(\frac{1}{P_{n-2}} \nabla) \ldots (\frac{1}{P_1} \nabla) + P_1 \ldots P_{n-1} \nabla(\frac{1}{P_{n-2}} \nabla) \ldots (\frac{1}{P_1} \nabla) \]
\[ -P_1 \ldots P_n(\frac{1}{P_{n-1}} \nabla) \ldots (\frac{1}{P_1} \nabla) = -\beta_{n-1}, \]
as was to be shown.

We can use the fact that \( \alpha_n + \beta_{n-1} = 0 \) to prove an interesting corollary. Suppose that \( X_1, \ldots, X_n \) are holomorphic \((-1,0)\) vector fields and that \( \phi \) is a holomorphic function. We will prove that

\[
\left( \frac{1}{P_n} \nabla \right) \ldots \left( \frac{1}{P_1} \nabla \right) \phi \nabla X_1 \ldots \nabla X_n = \phi X_1 \ldots X_n. \tag{42}
\]

Before we prove this fact, let us note the importance of equation (42). We recall from section 4.3 that \( \pi(\phi \nabla X_1 \ldots \nabla X_n) = (-1)^n X_n \nabla \ldots \nabla X_1 \nabla \phi \). The left-hand side of this equality can be expanded using equation (39), and it is given in general by an infinite sum. On the other hand, in this particular case, equation (42) implies that the sum in (39) stops after \( n \) terms, since the \((n+1)\)-th term will contain an antiholomorphic derivative \( \bar{\nabla} \) acting on the left-hand side of equation (42). Since the right-hand side is manifestly holomorphic, the \((n+1)\)-th term vanishes (and similarly all the \( m \)-th terms, with \( m > n \)).

Now to the inductive proof. First we note that we can take \( \phi = 1 \) with no loss of generality, since \( \phi \) commutes with \( \bar{\nabla} \). We multiply equation (42) by \( P_1 \ldots P_n \), and we write, using the inductive hypothesis,

\[
P_1 \ldots P_n \left( \frac{1}{P_n} \nabla \right) \ldots \left( \frac{1}{P_1} \nabla \right) \nabla X_1 \ldots \nabla X_n =
\]

\[
= \nabla P_1 \ldots P_n \left( \frac{1}{P_n} \nabla \right) \ldots \left( \frac{1}{P_1} \nabla \right) X_1 \nabla \ldots \nabla X_n - \alpha_n X_1 \nabla \ldots \nabla X_n =
\]

\[
= \nabla P_1 \ldots P_{n-1} X_1 \nabla X_2 \ldots X_n + \beta_{n-1} X_1 \nabla \ldots \nabla X_n =
\]

\[
= P_1 \ldots P_n \left( \frac{1}{P_n} \nabla \right) \ldots \left( \frac{1}{P_1} \nabla \right) X_1 \nabla \ldots \nabla X_n =
\]

\[
= P_1 \ldots P_n X_1 \ldots X_n,
\]
as was to be shown.

The last property of \( \pi \) that we would like to check is its behavior under a transformation \( \Omega \to e^{\chi + \bar{\chi}} \Omega \), which is described by equation (32). Under such a transformation we have that \( \ln C \to \chi + \bar{\chi} + \ln C \), and therefore \( R \) is invariant. Moreover, since the recursion relation (40) does not involve the metric, all of the tensors \( P_n \) are invariant. On the other hand the connection does change. In particular \( \Gamma \to \Gamma + \partial \chi \). The expansion in equation (39) does not contain \( \bar{\Gamma} \), since all of the antiholomorphic derivatives act on tensors of antiholomorphic dimension \(-1\), and can therefore be replaced with \( \partial + \Gamma \). If we act with the expansion (39) on the function \( e^{\chi} \), we may consider \( e^{\chi} \) as an operator and move it to the left. It commutes with everything except with the holomorphic covariant derivatives and in this case we pick up a commutator term \( [\nabla, e^{\chi}] = \partial \chi e^{\chi} \). This extra term reflects the change in the connection coefficient \( \Gamma \) given above. Therefore the net effect of moving the operator \( e^{\chi} \) all the way to the left is that \( \nabla_{\Omega} \to \nabla_{\Omega} + \partial \chi = \nabla_{e^{\chi} \bar{\chi}} \Omega \). If we multiply the whole expansion by \( e^{-\chi} \), we then obtain equation (32).

We have thus shown that the projection operator defined in (33) has all the properties we expect of the Bergman projection operator, and that in particular the projection operators
agree on a very general class of functions on $\Sigma$, namely those whose projections are described in (31). This does not give a completely mathematically rigorous proof that the expression (39) correctly describes the action of Bergman projection on an arbitrary function $\phi$. A complete proof might follow, for example, by showing that the set of functions on which the projection operators agree is dense in the requisite function space. For the purposes of this paper, however, we can treat (39) as the definition of Bergman projection on a general function, and we can now use this operator to construct the general matrix representation of any holomorphic curve.

5 Representation of General Holomorphic Curves

We have finally concluded our long digression on the Bergman integral kernel and we can go back to the analysis of the original problem of matrix representations of holomorphic curves.

We first define the quantization operator $Q$ precisely. Let $\phi \in H$ and let $A$ be a generic function on $\Sigma$. The operator $Q(A)$ corresponding to $A$ is defined by

$$Q(A) |\phi\rangle = \pi |A\phi\rangle.$$

In words, we first multiply $\phi$ by the function $A$ and then we extract the holomorphic part using the Bergman projection. We note a few simple properties of the definition. First of all, the above definition is consistent with the one given at the end of section 3 when $A$ is holomorphic, since in this case $\pi |A\phi\rangle = |A\phi\rangle$. We also note that, if $\phi, \psi \in H$, then

$$\langle \phi | Q(A) | \psi \rangle = \langle \phi | A \psi \rangle = \int_\Sigma \bar{\phi} A \psi \Omega,$$

since $\langle \phi | \pi = \langle \phi |$. Moreover, under complex conjugation, we have that

$$\langle \phi | Q^\dagger(A) | \psi \rangle = \langle \psi | Q(A) | \phi \rangle^* = \int_\Sigma \bar{\phi} \bar{A} \psi \Omega$$

so that we have

$$Q(A) = Q^\dagger(A).$$

Finally it is clear from the definition that $Q(1) = \text{Id}_H$.

We are now in a position to analyze equation (16) in its full generality. We will find it convenient to consider the matrix element of equation (16) between two states $\phi, \psi \in H$, and we therefore study the equation

$$\langle \phi | [Z_A, Z_A^\dagger] | \psi \rangle = -\epsilon \langle \phi | \psi \rangle,$$

where we recall that $Z_A = Q(Z_A)$. The right hand side of (16) is simply

$$-\epsilon \langle \phi | \psi \rangle = -\epsilon \int_\Sigma \bar{\phi} \psi \Omega,$$

(44)
The left hand side of (43), on the other hand, requires some manipulations and is given by

\[ \langle \phi | [Z_A, Z_A'] | \psi \rangle = \langle \phi | Q(Z_A)Q(Z_A) - Q(\bar{Z}_A)Q(Z_A) | \psi \rangle \]
\[ = \langle \phi | \pi Z_A \pi \bar{Z}_A | \psi \rangle - \langle \phi | \pi Z_A \pi Z_A | \psi \rangle \]
\[ = \langle \phi | Z_A \pi \bar{Z}_A | \psi \rangle - \langle \phi | \bar{Z}_A Z_A | \psi \rangle , \]

where we have used the properties of \( Q \) described at the beginning of this section. We may now use the expansion (39) for the action of the Bergman projection and write

\[ \langle \phi | [Z_A, Z_A'] | \psi \rangle = \sum_{n=0}^{\infty} (-1)^n \int_{\Sigma} \bar{\phi} Z_A \left( \nabla \frac{1}{P_1} \ldots \nabla \frac{1}{P_n} P_1 \ldots P_n \frac{1}{P_n} \nabla \ldots \frac{1}{P_1} \nabla \bar{Z}_A \right) \Omega - \int_{\Sigma} \bar{\phi} Z_A Z_A \psi \Omega \]
\[ = \sum_{n=1}^{\infty} (-1)^n \int_{\Sigma} \bar{\phi} Z_A \left( \nabla \frac{1}{P_1} \ldots \nabla \frac{1}{P_n} P_1 \ldots P_n \frac{1}{P_n} \nabla \ldots \frac{1}{P_1} \nabla \bar{Z}_A \right) \Omega . \]

We can then integrate by parts to conclude that

\[ \langle \phi | [Z_A, Z_A'] | \psi \rangle = \sum_{n=1}^{\infty} \int_{\Sigma} P_1 \ldots P_n \left( \frac{1}{P_n} \nabla \ldots \frac{1}{P_n} \nabla \bar{\phi} Z_A \right) \left( \frac{1}{P_n} \nabla \ldots \frac{1}{P_n} \nabla \bar{Z}_A \right) \Omega \]
\[ = \sum_{n=1}^{\infty} \int_{\Sigma} \bar{\phi} \psi (P_1 \ldots P_n) \left( \frac{1}{P_n} \nabla \ldots \frac{1}{P_n} \nabla Z_A \right) \left( \frac{1}{P_n} \nabla \ldots \frac{1}{P_n} \nabla \bar{Z}_A \right) \Omega , \]

where, in the last line, we have used the fact that \( \bar{\phi} \) and \( \psi \) are respectively antiholomorphic and holomorphic and therefore commute with the covariant derivatives \( \nabla \) and \( \bar{\nabla} \). Note that in (43) the covariant derivatives \( \nabla \) act only on the terms within the first parentheses; for the remainder of the paper we drop our convention that covariant derivatives act on all terms to their right, and instead take covariant derivatives to act on all terms to their right within a given parenthetical term. Comparing (44) and (43) we conclude that equation (43) is satisfied if

\[ - \epsilon = \sum_{n=1}^{\infty} P_1 \ldots P_n \left( \frac{1}{P_n} \nabla \ldots \frac{1}{P_n} \nabla Z_A \right) \left( \frac{1}{P_n} \nabla \ldots \frac{1}{P_n} \nabla \bar{Z}_A \right) . \tag{45} \]

Note that, in the above equation, all the covariant derivatives can be replaced by partial derivatives (\( \nabla \) acts on tensors of holomorphic dimension 0, \( \bar{\nabla} \) on tensors of antiholomorphic dimension 0). Therefore equation (43) does not contain the connection explicitly and can be considered as an equation for the curvature tensor \( R \).

We have now found a set of equations which in principle could be used to construct a matrix representation of an arbitrary holomorphic curve. Given a set of embedding functions \( Z_A \), we need to solve equations (45) and (40) for the tensors \( P_n \). In principle we can then use \( P_1 = R \) to determine \( \Omega \), which fixes the inner product on \( \mathcal{H} \) and gives a matrix representation of the holomorphic curve defined by the \( Z_A \)'s. Unfortunately, however, we do not have any way of exactly solving these equations in general. Thus, we now describe a perturbative solution of the problem.
We wish to solve (45) perturbatively in $\epsilon$ and, to this end, we recall the planar membrane example of section 3. In that case we noted that the correct integration measure $\Omega$ was given by $\Omega \propto e^{-z\bar{z}/\epsilon}dz \wedge d\bar{z}$, which is the case analyzed in section 4.4. In fact, if one takes $Z_1 = z$, $Z_A = 0 (A = 2, 3, 4)$ and $P_n = -n/\epsilon$, one can quickly verify that equation (45) is satisfied (only the $n = 1$ term is non-zero). The important fact to be learned from this example is that $P_n \sim 1/\epsilon$. In the general case we then define

$$P_n = \frac{1}{\epsilon} Q_n,$$

where, as we will see shortly, the $Q_n$’s are analytic in $\epsilon$ and can therefore be computed perturbatively. Equation (45) can now be written as

$$-1 = \sum_{n=1}^{\infty} \epsilon^{n-1} \pi_n \quad (46)$$

$$\pi_n = Q_1 \ldots Q_n \left( \frac{1}{Q_n} \partial \ldots \frac{1}{Q_1} \partial Z_A \right) \left( \frac{1}{Q_n} \bar{\partial} \ldots \frac{1}{Q_1} \bar{\partial} \bar{Z}_A \right).$$

Let us show that the above equation can be solved for the $Q_n$’s order by order in powers of $\epsilon$. First we note that the tensors $Q_n$ satisfy the recursion relation

$$Q_1 = \epsilon R$$

$$Q_n = Q_{n-1} + Q_1 + \epsilon \sum_{j=1}^{n-1} \partial \bar{\partial} \ln Q_j, \quad (47)$$

which is an immediate consequence of equation (10). Therefore one needs only to solve for $Q_1$ at any given order and the rest of the $Q_n$’s are automatically determined by (17). To solve for $Q_1$ we assume that all the $Q_n$’s are known to order $\epsilon^{N-1}$. From equation (46) we see that also the $\pi_n$’s are known to the same order. We now wish to compute $Q_1$ to order $\epsilon^N$. To this end we rewrite equation (14) as

$$-Q_1 = \partial Z_A \bar{\partial} Z_A + \epsilon \left( \sum_{n=2}^{N+1} \epsilon^{n-2} \pi_n Q_1 \right) + \mathcal{O}(\epsilon^{N+1}) \quad (48)$$

and we note that the term in parentheses multiplies $\epsilon$ and therefore needs only to be computed to order $\epsilon^{N-1}$. But both $\pi_n$ and $Q_1$ are known to order $\epsilon^{N-1}$ by assumption and therefore equation (14) determines $Q_1$ to order $\epsilon^N$. Let us note that this procedure is completely algebraic and that no partial differential equation needs to be solved.

We have solved, at least perturbatively, for $Q_1$. Our real goal, on the other hand, is to determine the integration 2-form $\Omega$. To this end we fix on the surface $\Sigma$ a holomorphic $(1,0)$-form

$$H = h \, dz$$

We then rewrite

$$\Omega = iH \wedge \bar{H} e^{-K/\epsilon}$$

$$= ih \bar{h} e^{-K/\epsilon} \, dz \wedge d\bar{z},$$

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where $K$ is a real scalar function on the surface (we have used the fact that $\Omega$ is real and positive). Recalling that $\Omega = iC\,dz \wedge d\bar{z}$ and that $R = \partial\bar{\partial}\ln C = (1/\epsilon)Q_1$, we conclude that
\[-Q_1 = \partial\bar{\partial}K\] (49)
(we have used the fact that $\partial\bar{\partial}\ln(h\bar{h}) = 0$). This shows that, as long as we succeed in writing $Q_1$ as the laplacian of a scalar function $K$, we have solved the problem completely.

Let us first analyze the solution of equations (46) and (49) in the $\epsilon \to 0$ classical limit. In this case equation (48) reduces to
\[-Q_1 = \partial Z_A \bar{Z}_A + O(\epsilon)\]
and therefore
\[K = Z_A \bar{Z}_A + O(\epsilon).\]
In this limit we then have
\[\Omega \simeq H \wedge \bar{H} e^{-Z_A \bar{Z}_A/\epsilon}. \quad (\epsilon \to 0 \text{ classical limit})\]
(50)
We will see below that the above result is what is expected if one uses geometric quantization techniques. We note that, for $\epsilon \to 0$, $R = -(1/\pi)\mu$. The curvature of the measure $\Omega$ gives, in this limit, the density of 0-branes.

One can go on and compute higher order corrections to $Q_1$ and therefore to $K$. A long but mechanical computation, following the procedure outlined above, shows that, to order $\epsilon^2$
\[K = Z_A \bar{Z}_A - \frac{\epsilon}{2} \ln \left(\frac{\alpha}{h\bar{h}}\right) + \frac{\epsilon^2}{6} \frac{1}{\alpha} \partial\bar{\partial} \ln \alpha + \ldots,\]
(51)
where
\[\alpha = \partial Z_A \bar{\partial} Z_A.\]
Only one point about the derivation of equation (51) is worth mentioning. Using the procedure described previously, one computes $-Q_1 = \ldots - (\epsilon/2) \partial\bar{\partial} \ln \alpha + \ldots$, so that one is tempted to set $K = \ldots -(\epsilon/2) \ln \alpha$. This is wrong, since $\alpha$ is a $(1, 1)$ tensor, and therefore $\ln \alpha$ is not a scalar. On the other hand $\ln(\alpha/h\bar{h})$ is a scalar, and moreover $\partial\bar{\partial} \ln(\alpha/h\bar{h}) = \partial\bar{\partial} \ln \alpha$. Equation (51) implies that the first quantum correction of equation (50) is given by
\[\Omega \simeq dz \wedge d\bar{z} \sqrt{h\bar{h}} \sqrt{\partial Z_A \bar{\partial} Z_A} e^{-Z_A \bar{Z}_A/\epsilon}.\]
(52)
We will see in the next section that this corresponds to the metaplectic correction to geometric quantization.

Finally we note that $K$ is not uniquely defined by equation (49). In particular, if $\chi$ is a holomorphic function, the transformation
\[K \rightarrow K - \epsilon \chi - \epsilon \bar{\chi}\]
leaves $Q_1$ invariant. This transformation corresponds to the one analyzed in section 4.3 since
$\Omega \rightarrow \Omega e^{\chi + \bar{\chi}}$.

We conclude this section with a brief synopsis of our solution to the problem posed in section 2.3. We have found a system of equations (45, 40) whose solution in principle determines a matrix representation of any holomorphic curve. We could not solve this system of equations in full generality, but found a perturbative solution by expanding in the parameter $\epsilon$. To find the perturbative solution to order $\epsilon^n$ it is necessary to solve the system of equations (46, 47) to order $\epsilon^n$. The relation (49) can then be used to construct $K$; the general form of $K$ is given to order $\epsilon^2$ in (51). Once $K$ is known, the corresponding $\Omega$ can be used to explicitly construct the matrices $X_i$ for the holomorphic curve in question.

6 Numerical Interlude

In this section we use the results of the last section and apply them to two numerical examples.

First we analyze once more the parabolic membrane example of section 3. We are going to use the notation of that section. We recall that $\mathcal{H}$ is spanned by the orthogonal states $|v_n\rangle = |z^n\rangle$, with inner product $\langle v_n|v_n\rangle = k_n$. In particular (choosing $H = dz$)

$$k_n = \int dz \wedge d\bar{z} z^n \bar{z}^n e^{-K/\epsilon}.$$ (53)

We also recall that $\alpha_n = k_{n+1}/k_n$ and that $\alpha_0$ could be computed to arbitrary precision. We can now use the successive approximations (51) of $K$ and compute $\alpha_0$ numerically using the integral (53). The results are shown in figure 1, together with the exact numerical result computed following the prescription given in section 3.

![Figure 1](image-url)
As a second example we consider a curve that cannot be analyzed with the techniques of section 3. To be specific we look at the curve defined by
\[ Z_1^2 = Z_2 + 2Z_2^2 + Z_3^3 \]
and parametrized by
\[
\begin{align*}
Z_1 &= z + z^3 \\
Z_2 &= z^2.
\end{align*}
\]
The space \( \mathcal{H} \) is still spanned by the vectors \( |v_n\rangle = |z^n\rangle \), but the basis \( |v_n\rangle \) is not orthonormal (it is not even orthogonal). If we let
\[
I_{nm} = \langle v_n | v_m \rangle = \int dz \wedge d\bar{z} \ z^n \bar{z}^m \ e^{-K/\epsilon}
\]
be the inner-product matrix, we can move to an orthonormal basis by diagonalizing \( I \) with a matrix \( D \) such that \( D^\dagger ID = \text{Id} \). Any operator on \( \mathcal{H} \) which is represented by a matrix \( A \) in the basis \( |v_n\rangle \) is represented by \( D^{-1}AD \) with respect to the orthonormal basis. In particular the matrices \( Z_1 \) and \( Z_2 \) are given by
\[
Z_1 = D^{-1} \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
1 & 0 & 1 & 0 & \cdots \\
\ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix} D, \\
Z_2 = D^{-1} \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots \\
1 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & \cdots \\
& & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix} D
\]
and satisfy (by construction)
\[ Z_1^2 = Z_2 + 2Z_2^2 + Z_3^3. \]
In order to find a numerical approximation to the matrices \( Z_1 \) and \( Z_2 \) we first of all take \( \epsilon \) small (0.001), so that we can approximate \( K \approx Z_A \bar{Z}_A \). Moreover we restrict ourselves to finite \( N \times N \) matrices (more precisely, we restrict to the subspace of \( \mathcal{H} \) spanned by the vectors \( |v_n\rangle, n < N \)). If we take \( N \) large enough (\( N = 10 \)) we expect that the corresponding matrices \( Z_A \) will be accurate, at least in the upper-left corner (physically this corresponds to the region of the brane with \( Z_1, Z_2 \) small). The matrices \( I \) and \( D \) can then be computed numerically and can be used to evaluate the coordinate matrices. The result is
\[
Z_1 \approx \frac{1}{1000} \begin{pmatrix}
0 & -0.1 & 0 & 0 & 0 \\
31 + 1.9i & 0 & -0.3 & 0 & 0 & \cdots \\
0 & 45 - 2.6i & 0 & -0.5 & 0 \\
-0.1 & 0 & 54 - 6.7i & 0 & -0.7 & \cdots \\
0 & 0.15 & 0 & 63 + 7.7i & 0 \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix} + \mathcal{O}(10^{-5})
\]
and

\[ Z_2 \approx \frac{1}{1000} \begin{pmatrix}
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
  1.4 & 0 & 0 & 0 & 0 \\
  0 & 2.4 - 0.4i & 0 & 0 & 0 \\
  0 & 0 & 3.4 & 0 & -0.1 \\
  \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix} + O(10^{-5}) \]

The above matrices satisfy, as they should,

\[ [Z_A, Z_A^\dagger] = -\epsilon \text{Id} + O(10^{-5}). \]

7 Comparison with Geometric Quantization

In this final section we analyze the connection between our results and the theory of geometric quantization \([10, 12, 13]\).

We recall briefly a few key concepts of prequantization. We let \((\Sigma, \omega)\) be a symplectic manifold of dimension \(2n^3\). If \(\omega\) is integral on closed 2-cycles (quantization condition), one may consider \(2\pi \omega\) as the field strength of a \(U(1)\) gauge potential. Locally \(\omega = d\theta\), where \(2\pi \theta\) is the gauge potential. The covariant derivative is then

\[ \nabla = d - 2\pi i\theta. \]

In prequantization one considers the Hilbert space \(\mathcal{V}\) of sections of the above \(U(1)\) bundle with inner product

\[ \langle \eta | \xi \rangle = \int_\Sigma \omega^n \bar{\eta} \xi, \]

and assigns to functions \(A\) on \(\Sigma\) the first order differential operator

\[ Q(A) = -\frac{i}{2\pi} \nabla_{X_A} + A, \]

where \(\nabla_{X_A}\) denotes the covariant derivative in the direction of the hamiltonian flow \(X_A\) of \(A\). Using the fact that \([\nabla_X, \nabla_Y] = \nabla_{[X,Y]} - 2\pi i \omega(X,Y)\), one can show that

\[ Q(\{A, B\}) = 2\pi i [Q(A), Q(B)]. \]

This completes the description of the prequantization. To complete the geometrical quantization process it is necessary to choose a polarization. There is a particularly simple choice of polarization when our symplectic manifold is a Riemann surface. In this case it is a closed non-degenerate 2-form. Given a function \(A\), the corresponding hamiltonian flow vector field \(X_A\) is defined by \(dA(\cdot) + \omega(X_A, \cdot) = 0\). The Poisson bracket of two functions is given by \(\{A, B\} = X_A(B) = \omega(X_A, X_B)\) and it satisfies \([X_A, X_B] = X_{\{A,B\}}\).
natural to choose a polarization given by the condition that the sections $\eta$ be holomorphic.

We now assume that $\Sigma$ is a Riemann surface. Using the Dolbeault lemma (or considering $\omega$ as a Kähler form), we can find a cover $\mathcal{U}_i$ of $\Sigma$ and real functions $\mathcal{K}_i$ on $\mathcal{U}_i$ such that

$$\omega = -\frac{1}{2\pi i} \partial \bar{\partial} \mathcal{K}_i.$$ 

We also have that $\mathcal{K}_j = \mathcal{K}_i + \chi_{ij} + \bar{\chi}_{ij}$, for some $\chi_{ij}$ holomorphic on $\mathcal{U}_i \cap \mathcal{U}_j$. The $U(1)$ potential on $\mathcal{U}_i$ is given by

$$\theta_i = -\frac{1}{4\pi i \epsilon} (\bar{\partial} \mathcal{K}_i - \partial \mathcal{K}_i),$$

and one can check that $\theta_j = \theta_i + d\alpha_{ij}$, where $\alpha_{ij} = \frac{1}{4\pi i \epsilon} (\chi_{ij} - \bar{\chi}_{ij})$. Sections of the $U(1)$ bundle are given by functions $\eta_i$ on $\mathcal{U}_i$ related by $\eta_j = \eta_i e^{2\pi i \alpha_{ij}}$. A holomorphic section $\eta$ satisfies the equation

$$\nabla \eta_i = \bar{\partial} \eta_i + \frac{1}{2\epsilon} \eta_i \partial \mathcal{K}_i = 0$$

which is solved by

$$\eta_i = \phi_i e^{-\mathcal{K}_i/2\epsilon},$$

with $\phi_i$ holomorphic. One can check that $\phi_j = \phi_i e^{\chi_{ij}/\epsilon}$ and therefore the definition is consistent. One may then restrict the attention from $\mathcal{V}$ to the space $\mathcal{H}$ of holomorphic sections of the holomorphic line bundle $L$ with transition functions $\lambda_{ij} = e^{\chi_{ij}/\epsilon}$. The inner product on $\mathcal{H}$ is given by

$$\langle \phi \mid \psi \rangle = \int_{\Sigma} \bar{\phi} \psi e^{-\mathcal{K}/\epsilon} \omega.$$ 

First we note that, if $A$ is a holomorphic function, then $X_A$ is a $(0,-1)$ vector field and therefore $\nabla_{X_A} \propto \bar{\nabla}$. This means that, on holomorphic sections, $Q(A) = A$ as we assumed in the paper. Moreover we note that, in our specific case, $\omega = \mu d^2 \sigma = -\frac{1}{2\pi i \epsilon} \partial \bar{\partial} Z_A \bar{Z}_A$, so that the line bundle $L$ is trivial with globally defined Kähler potential

$$\mathcal{K} = Z_A \bar{Z}_A.$$ 

The space $\mathcal{H}$ can then be identified with the space of holomorphic functions with inner product given by

$$\langle \phi \mid \psi \rangle = \int_{\Sigma} \bar{\phi} \psi \Omega$$

$$\Omega = e^{-Z_A \bar{Z}_A/\epsilon} \omega.$$ 

We see that we recover equation (50), with $\omega$ replacing $H \wedge \bar{H}$. On the other hand this difference is irrelevant in the $\epsilon \to 0$ limit, as can be checked by computing the leading $1/\epsilon$ behavior of $R$ in both cases.

Standard geometric quantization is improved by the metaplectic correction [13], which we now describe (as it reads in our present setting). One assumes that the $\mathcal{U}_i$ are coordinate patches with local coordinate $z_i$ and considers the holomorphic line bundle $K$ with transition
functions given by \( \kappa_{ij} = \sqrt{\frac{\partial \bar{z}_j}{\partial z_i}} \). One then considers, as Hilbert space \( \mathcal{H} \), the holomorphic sections of the tensor bundle \( K \otimes L \), given by analytic functions \( \eta_i \) related by \( \eta_j = \lambda_{ij} \kappa_{ij} \eta_i \). To construct an inner product between two sections \( \eta \) and \( \xi \) one has to build a scalar from \( \bar{\eta} \) and \( \xi_t \). We recall that \( \omega = \frac{i}{2} \mu_i dz_i \wedge d\bar{z}_i \), where \( \mu_j = \kappa^2 \kappa_{ij} \mu_i \). This allows us to construct a scalar, since \( \mu_i^{-1/2} \bar{\eta}_i \xi_t e^{-K/\epsilon} \) is invariant if we change \( i \to j \). If one considers the vector field \( X_i = X_{z_i} = \frac{-2i}{\mu_i} \partial_i \), we see that \( \frac{1}{2\pi i} \omega(\bar{X}_i, X_i) = (\pi \epsilon \mu_i)^{-1} \). Therefore the scalar we where looking for is \( \sqrt{(2\pi i \epsilon)^{-1}} \omega(\bar{X}, X) \bar{\eta} \xi e^{-K/\epsilon} \). The inner product is then given by

\[
\langle \eta | \xi \rangle = \int_{\Sigma} \omega \sqrt{\frac{1}{2\pi i \epsilon}} \omega(\bar{X}, X) \bar{\eta} \xi e^{-K/\epsilon}.
\]

We now specialize to our specific case. First of all we recall that \( \pi \epsilon \mu_i = \partial_i Z_A \partial_i \bar{Z}_A \). We have seen that the bundle \( L \) is trivial. We assume that we can also trivialize the bundle \( K \), and we consider a nowhere zero section \( f_i \) of \( K \). First we note that \( h_i = f_i^2 \) is a section of the canonical line bundle, and therefore \( H = h_i dz_i \) is well defined holomorphic one-form. Moreover any section \( \eta \) of \( K \otimes L \) can be written as \( \eta_i = f_i \phi = \sqrt{h_i} \phi \), where \( \phi \) is a globally defined analytic function. This means that \( \mathcal{H} \) can then be identified with the space of holomorphic functions with inner product given by

\[
\langle \phi | \psi \rangle = \int_{\Sigma} \bar{\phi} \psi \Omega
\]

\[
\Omega = -\frac{1}{2\pi i \epsilon} dz \wedge d\bar{z} \sqrt{h \bar{h} \sqrt{\partial Z_A \partial \bar{Z}_A}} e^{-Z_A \bar{Z}_A / \epsilon}.
\]

We then recover equation (52) of section 5 (up to an irrelevant scale factor).

We conclude this section by analyzing once more the \( \epsilon \to 0 \) limit of the quantization prescription analyzed in section 5. First of all we notice that, given functions \( A \) and \( B \) on \( \Sigma \) and elements \( \phi, \psi \in \mathcal{H} \), one has that \( \langle \phi | Q(A) Q(B) | \psi \rangle = \langle \phi | \pi A \pi B | \psi \rangle = \langle \phi | A \pi B | \psi \rangle \). Following the same steps as in section 5, one can then show that \( \langle \phi | Q(A) Q(B) | \psi \rangle = \langle \phi | Q(A \star B) | \psi \rangle \), where the \( \star \) product is defined by

\[
A \star B = \sum_{n=0}^{\infty} \epsilon^n Q_1 \ldots Q_n \left( \frac{1}{Q_n} \partial \ldots \frac{1}{Q_1} \partial A \right) \left( \frac{1}{Q_n} \partial \ldots \frac{1}{Q_1} \partial B \right).
\]

The product \( \star \) is not commutative. Moreover one has that \( A \star B = AB + (\epsilon / Q_1) \cdot \partial A \partial B + \ldots \), so that one can derive the important equations valid in the \( \epsilon \to 0 \) limit

\[
A \star B = AB + \ldots
\]

\[
A \star B - B \star A = \frac{1}{2\pi i} \{ A, B \} + \ldots,
\]

where we have used that \( Q_1 = -\partial Z_A \partial \bar{Z}_A + \ldots \).

\(^4\)This is the square root of the canonical bundle. Its existence poses problems similar to the ones encountered in choosing a spin structure, since \( \sqrt{\cdot} \) is defined only up to a sign.
8 Conclusion

We have analyzed the problem of constructing matrix representations of holomorphic curves describing static membranes in M-theory. We discussed some simple examples with rotational symmetry for which the corresponding matrices were easy to construct explicitly. We also developed a more general approach which gives the matrix representation of an arbitrary holomorphic curve. We were unable to exactly solve the equations in the general case; we showed, however, that these equations can be solved perturbatively in the parameter $\epsilon$. This parameter is related to the inverse density of 0-branes on the membrane and goes to 0 in the limit of a smooth membrane. The $\epsilon$ expansion is expected to be an asymptotic series for the exact expressions, and should give a good approximation to a matrix representation of any holomorphic curve. The first two terms in the $\epsilon$ expansion correspond to the standard result from geometric quantization and the metaplectic correction to this result.

We conclude this paper by suggesting some future directions for research.

1) First of all, from a practical point of view, it would be very useful to obtain a recursive formula for the higher corrections to the function $\mathcal{K}$. We have not been able to derive a closed form expression for $\mathcal{K}$, but we believe that, with some work, it should be possible to extract it from equation (45).

2) The holomorphic membranes we have considered here are stable BPS configurations which preserve some of the supersymmetries of the theory. It would be interesting to understand the physical properties of these solutions better, either in supergravity or in matrix theory. In particular, to the best of our knowledge the supergravity solutions corresponding to these holomorphic membrane configurations are not known. It would be nice to have an explicit construction of these solutions. From the matrix theory point of view, it would be nice to explicitly demonstrate the supersymmetry of these configurations.

3) Finally, one should analyze the problem of holomorphic curves embedded in compactified space. If the formalism can be generalized, one can apply it, in particular, to the case of compactification on $T^4$. In this case the defining equations (15) and (16) are, after $T$-duality, the equations of self-duality of the dual Yang-Mills field. In particular one can try to understand the relation between holomorphic curves embedded in $T^4$ and Yang-Mills instantons on the dual $T^4$.

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