ON SOME FAMILIES OF MODULES FOR THE CURRENT ALGEBRA

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Abstract. Given a finite-dimensional module, $V$, for a finite-dimensional, complex, semi-simple Lie algebra $\mathfrak{g}$ and a positive integer $m$, we construct a family of graded modules for the current algebra $\mathfrak{g}[t]$ indexed by simple $\mathbb{C}\mathfrak{S}_m$-modules. These modules have the additional structure of being free modules of finite rank for the ring of symmetric polynomials and so can be localized to give finite-dimensional graded $\mathfrak{g}[t]$-modules. We determine the graded characters of these modules and show that if $\mathfrak{g}$ is of type $A$ and $V$ the natural representation, these graded characters admit a curious duality.

1. Introduction

The current algebra associated to a simple Lie algebra $\mathfrak{g}$ is the Lie algebra $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ with the obvious bracket (see Section 2.3). The study of graded representations for the current algebra $\mathfrak{g}[t]$ has been of significant interest for several decades. One of the reasons for this is that the current algebra is a maximal parabolic subalgebra of the associated affine Kac-Moody Lie algebra and many interesting representations for the affine algebra specialize to graded modules for the current algebra. One can also get modules by taking the so called graded-limit of modules for the quantum affine algebras (see for example [4] and [14]).

Of particular interest is the category $I$, comprising graded modules $M$ for the current algebra with the condition that the graded components are finite dimensional. The simple modules in $I$ are indexed by pairs, $(\lambda, r) \in P^+ \times \mathbb{Z}$, where $P^+$ is the set of dominant integral weights for the Lie algebra $\mathfrak{g}$. The category $I$ is not a semi-simple category; there are many indecomposable, yet reducible, modules in $I$. The subcategory of $I$ consisting of objects with only finitely many non-zero graded components of negative grade is known to admit a BGG-style reciprocity (see [6] for the most general case) and a tilting theory (see [1]), while the subcategory consisting of finitely-generated modules is a motivating example of an affine highest weight category (see [12] and [13]).

Our work can be thought of as a generalization of the following classical construction. Because we will work exclusively over $\mathbb{C}$, the representation theory of the symmetric group $\mathfrak{S}_m$ is semi-simple, with the simple modules indexed by partitions, $\gamma \in \mathcal{P}(m)$, of $m$. Given a finite-dimensional $\mathfrak{g}$-module $V$, the module $N = V^\otimes m$ is naturally a bimodule for $\mathfrak{g}$ and $\mathfrak{S}_m$ (both with left actions) and the actions commute. If we decompose $N \cong \bigoplus_\gamma S(\gamma) \otimes \text{Hom}_{\mathfrak{S}_m}(S(\gamma), N)$ as a $\mathbb{C}\mathfrak{S}_m$-module, we have an action of $\mathfrak{g}$ on the multiplicity spaces $\text{Hom}_{\mathfrak{S}_m}(S(\gamma), N)$. As special cases, the multiplicity space associated to the trivial module is the space of invariants $N^\mathfrak{S}_m$, while that associated to the sign representation is the exterior power $\wedge^m V$.

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In our situation, we start with a finite-dimensional $\mathfrak{g}$-module $V$, and let $V \otimes \mathbb{C}[t]$ be a $\mathfrak{g}[t]$-module with the natural action. The tensor product $M = (V \otimes \mathbb{C}[t])^\otimes m$ is naturally a $\mathfrak{g}[t]$-$\mathbb{C}[\mathfrak{s}_m]$-bimodule and again, the actions commute. The grading on $\mathbb{C}[t]$ by powers of $t$ induces a grading on $M$. Clearly, $\mathbb{C}_m$ preserves the finite-dimensional graded components of $M$ and so we can decompose $M$ as follows.

$$M \cong \bigoplus_{k \in \mathbb{Z}} \bigoplus_{\gamma \in \mathbb{P}(m)} S(\gamma) \otimes \text{Hom}_{\mathbb{C}_m}(S(\gamma), M[k]).$$

These multiplicity spaces are naturally $\mathfrak{g}[t]$-modules; we denote them

$$B(\gamma, V) := \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{\mathbb{C}_m}(S(\gamma), M[k]).$$

Note that these modules lie in $I$. The purpose of this paper is to study the modules $B(\gamma, V)$.

We note here that the modules $B(\gamma, V)$ have additional structure. Using the natural isomorphism $M \cong V^\otimes m \otimes \mathbb{C}[t_1, \ldots, t_m]$, we see that $M$ admits a right action by the polynomial ring $\mathbb{C}[t_1, \ldots, t_m]$, which is simply right multiplication. This action commutes with the action of $\mathfrak{g}[t]$, but not with the action of the symmetric group. However, if we restrict this right action to the ring of symmetric polynomials $\mathbb{A}_m$, then all three actions commute. In fact, $M$ is a free $\mathbb{A}_m$-module of finite rank. It follows from the celebrated result of Quillen and Suslin that $B(\gamma, V)$ is a free right $\mathbb{A}_m$-module. We let $I_m$ be the unique graded maximal ideal in $\mathbb{A}_m$ and define the graded localization $B_{\text{loc}}(\gamma, V) := B(\gamma, V) \otimes_{\mathbb{A}_m} \mathbb{A}_m/I_m$. This is a finite-dimensional graded $\mathfrak{g}[t]$-module. The main result of our paper is the following theorem.

**Theorem.** The multiplicity space $B(\gamma, V)$ is a graded module for the current algebra and so is $B_{\text{loc}}(\gamma, V)$. The graded characters of these modules behave in the following way.

(i) For each $\gamma \in \mathbb{P}(m)$, the graded character of $B_{\text{loc}}(\gamma, V)$ is given by

$$\text{ch}_{\text{gr}} B_{\text{loc}}(\gamma, V) = \sum_{\mu \in P^+} \sum_{\tau, \sigma \in \mathbb{P}(m)} s_{\mu}(\tau, V) c_{\tau,\sigma}^{\gamma} f_{\sigma}(u)e(O(\mu)).$$

The notation here is defined in Section 2.

(ii) Up to a fixed graded shift, the graded characters of localizations corresponding to conjugate partitions and dual $\mathfrak{g}$-modules are dual to one another in the following way

$$\text{ch}_{\text{gr}} B_{\text{loc}}(\gamma, V) = u(\gamma) \text{ch}_{\text{gr}} B_{\text{loc}}(\gamma^\vee, V^\vee)^*.$$
result that \([B(\tau, V(I)) : \tau \in P(m), m \in \mathbb{Z}_{>0}]\) is a two parameter family of modules admitting filtrations by global Weyl modules. The main theorem also gives a new formula for the graded character of the global Weyl modules \(W(m_\omega)\) and the tilting modules \(T(m)\). We also note that the character equality implied by [2] Proposition 3.7 can be thought of as a special case of part (ii) of the Theorem.

We also note the following generalizations of this construction. Given a commutative associative algebra \(A\), \(g \otimes A\) is naturally a Lie algebra. Given a \(g\)-module \(V\), the space \(V \otimes A\) is naturally a \(g \otimes A\)-module, and \((V \otimes A)^{\otimes m}\) is a bimodule for \(g \otimes A\) and \(\Xi_m\) such that the actions commute. Therefore the multiplicity spaces associated to simple modules \(S(\tau)\) will be \(g \otimes A\)-modules. A key component to understanding the multiplicity spaces when \(g\) is a Lie algebra \(A\) is naturally a Lie algebra. Given a \(g\)-module \(V\) and \(m\) a fixed Cartan subalgebra of rank \(n\). We set \(I = \{1, \ldots, n\}\) and let \(\{\alpha_i : i \in I\}\) be a set of simple roots for \(h\) with respect to \(h\). We let \(R\) denote the set of roots, \(R^+\) the positive roots, \(Q\) the root lattice and \(Q^+\) the positive root lattice. We also let \(P\) be the set of weights and \(P^+\) the dominant integral weights, with \(\{\alpha_i : i \in I\}\) the fundamental weights. We put a partial order on \(h^*\) by letting \(\lambda \geq \mu\) if \(\lambda - \mu \in Q^+\).

Let \(W \subset \text{Aut}(h^*)\) be the Weyl group and \(w_0\) the longest word. For \(\alpha \in R\) let \(g_\alpha\) be the corresponding root space. We have a decomposition \(g = (\oplus_{\alpha \in R} g_\alpha) \oplus h\) and set \(n^+ = \oplus_{\alpha \in R^+} g_{\pm \alpha}\). Let \(\{x_{i\alpha}, h_i : \alpha \in R^+, i \in I\}\) be a Chevalley basis for \(g\). We set \(x_i^+ = x_{i\alpha_i}\).

Given a \(g\)-module \(M\) and \(\lambda \in h^*\), denote by \(M_\lambda\) the \(\lambda\) weight space of \(M\); that is \(M_\lambda := \{m \in M : h \cdot m = \lambda(h)m, \text{ for all } h \in h\}\). Finite-dimensional \(g\)-modules decompose as a direct sum of their weight spaces. Any module with such a decomposition is referred to as a weight module. The finite-dimensional simple \(g\)-modules each correspond to a dominant integral weight \(\lambda \in P^+\), and we call this module \(V(\lambda)\); it is generated by a vector \(v_\lambda\) of weight \(\lambda\) such that \(n^+ \cdot v_\lambda = 0\). The finite dimensional representation theory of \(g\) is semi-simple, meaning that every finite dimensional \(g\)-module is isomorphic to a direct sum of simple modules.

Given a finite dimensional \(g\)-module \(V\), the dual space \(V^*\) is also a finite dimensional \(g\)-module, and hence a weight module. Note that \((V_\lambda)^* \cong (V^*)_{-\lambda}\). (2.1)

It is well known that if \(\lambda \in P^+\) then \(\lambda^\vee := -\omega_0 \cdot \lambda \in P^+\). On simple modules, this duality satisfies \(V(\lambda)^* \cong V(\lambda^\vee)\).
The **character** of a finite-dimensional \( g \)-module, \( M \), is the formal sum in the group ring of the weight lattice,
\[
\text{ch}_g M := \sum_{\lambda \in \mathcal{P}} \dim M_{\lambda} e(\lambda) \in \mathbb{Z}[P].
\]
A standard result says that if \( w \in \mathcal{W} \) and \( \lambda \in \mathfrak{h}^* \), then \( \dim M_{\lambda} = \dim M_{w \cdot \lambda} \) for any finite-dimensional module \( M \). If we let \( O(\lambda) = \{ w \cdot \lambda : w \in \mathcal{W} \} \) and \( e(O(\lambda)) = \sum_{\mu \in O(\lambda)} e(\mu) \) then we can also write the character of \( M \) as
\[
\text{ch}_g M = \sum_{\lambda \in \mathcal{P}} \dim M_{\lambda} e(O(\lambda)).
\]

Given a finitely generated, graded algebra \( A = \bigoplus_{i \in \mathbb{Z}} A[i] \), we call an \( A \)-module \( M \) a **graded module** if it decomposes as a vector space into graded pieces \( M = \bigoplus_{k \in \mathbb{Z}} M[k] \) such that \( A[i]M[j] \subset M[i + j] \). If each graded component is finite dimensional we can associate to \( M \) its **Hilbert series**, \( H(M) := \sum_{k \in \mathbb{Z}} \dim M[k] u^k \in \mathbb{Z}[\left[ u, u^{-1}\right]] \)
where \( u \) is an indeterminate.

If \( M \) is also a \( g \)-module, we define its **graded character** to be the following power series with coefficients in the group ring \( \mathbb{Z}[P] \):
\[
\text{ch}_{g^r} M := \sum_{r \in \mathbb{Z}} \text{ch}_g M[r] u^r \in \mathbb{Z}[\left[ u, u^{-1}\right]].
\]

### 2.2. The coinvariant ring for \( \mathbb{Z}_m \)

Let \( \mathcal{P}(m) = \{ \tau = (\tau_1 \geq \tau_2 \geq \cdots \geq \tau_m) : \sum \tau_i = m \} \) be the set of partitions of \( m \). Given \( \tau \in \mathcal{P}(m) \) we let \( \tau^\vee \) denote the conjugate partition. The complex irreducible representations of \( \mathbb{Z}_m \) are indexed by partitions of \( m \). Given such a partition, \( \tau \), we will denote the corresponding simple complex \( \mathbb{C} \mathbb{Z}_m \)-module \( S(\tau) \). We will denote the one-dimensional modules corresponding to the trivial and sign representations by \( \text{triv} \) and \( \text{sgn} \) respectively. The group algebra \( \mathbb{C} \mathbb{Z}_m \) has a comultiplication induced by the assignment \( g \mapsto g \otimes g \); in this way, we define the tensor product of \( \mathbb{C} \mathbb{Z}_m \)-modules. We can also define the dual representation \( S^* = \text{Hom}_{\mathbb{Z}_m}(S, \mathbb{C}) \), which on simple modules satisfies \( S(\gamma)^* \cong S(\gamma) \).

Define non-negative integers, \( c_{\tau,\gamma}^\vee \), by
\[
S(\tau) \otimes S(\sigma) \cong \bigoplus_{\gamma \in \mathcal{P}(m)} c_{\tau,\gamma}^\vee S(\gamma);
\]  
(2.2)
the \( c_{\tau,\gamma}^\vee \) are known as Kronecker coefficients. Note that the decomposition above depends on a choice of basis for each isotypic component of \( S(\tau) \otimes S(\sigma) \) and hence is not canonical. However, for any finite-dimensional module, \( N \), the following decomposition is canonical.
\[
N \cong \bigoplus_{\tau \in \mathcal{P}(m)} S(\tau) \otimes \text{Hom}_{\mathbb{Z}_m}(S(\tau), N)
\]  
(2.3)
where the dimension of \( \text{Hom}_{\mathbb{Z}_m}(S(\tau), N) \) records the multiplicity of the isotypic component corresponding to \( \tau \).

Given a finite dimensional \( g \) module \( V \), the module \( N = V^\otimes m \) is a left \( \mathbb{Z}_m \)-module via permutation of tensorands, and this action commutes with the left action of \( g \). In particular
\( \mathfrak{S}_m \) preserves the weight spaces of \( N \). By the discussion above, for each \( \mu \in P \), the weight space \( N_{\mu} \) decomposes into a direct sum of simple \( \mathfrak{S}_m \)-modules. We define non-negative integers \( s_\mu(t, V) \) to satisfy
\[
N_{\mu} \cong \mathfrak{S}_m \bigoplus_{\gamma} s_\mu(\gamma, V)S(\gamma). \tag{2.4}
\]

Let \( A_m := \mathbb{C}[t_1, \ldots, t_m] \), the polynomial ring in \( m \) indeterminates, which we consider to be \( \mathbb{Z}_{\geq 0} \) graded in the natural way. Then \( \mathfrak{S}_m \) acts on \( A_m \) by permuting the \( t_i \). Clearly, \( \mathfrak{S}_m \) preserves the graded components of \( A_m \), so \( A_m \) is a graded \( \mathbb{C}\mathfrak{S}_m \)-module. Let \( A_m := A_{\mathfrak{S}_m} \) be the ring of polynomials invariant under the action of \( \mathfrak{S}_m \) and \( (A_m)_+ = \oplus_{k \geq 1} A_m[k] \). It is know that \( A_m \) is itself a polynomial ring, and that \( A_m \) is a free module over \( A_m \). The **coinvariant ring**\(^1\) is the quotient
\[
A_m^{\text{coin}} := A_m/(A_m)_+.
\]

It is well known (for example, [5]) that the module \( A_m^{\text{coin}} \) is isomorphic to the regular representation of \( \mathfrak{S}_m \); however, in the category of graded representations, these two are not isomorphic: the coinvariant has a non-trivial grading which reflects some of the more subtle features of the symmetric group, such as the ordering of simple \( \mathfrak{S}_m \)-modules induced by the natural ordering of partitions. More precisely, given a partition \( \gamma \in \mathcal{P}(m) \), we can decompose the \( S(\gamma) \)-isotypic component of \( A_m^{\text{coin}} \) into graded pieces. We denote the Hilbert series of these components by \( f_\gamma(u) \); that is to say,
\[
H(A_m^{\text{coin}}) = \sum_{\gamma \in \mathcal{P}(m)} f_\gamma(u) \dim S(\gamma). \tag{2.5}
\]

Note that the \( f_\gamma(u) \) are actually polynomials.

The proof of following result on the Hilbert series of free \( A_m \) modules is straight forward

**Lemma.** Let \( M \) be a free, finitely generated, graded \( A_m \) module with graded basis \( S \) such that there are \( n_r \) elements of \( S \) of grade \( r \). Then \( H(M) = H(A_m) \sum n_r u^r \).

### 2.3. The Current Algebra

For a Lie algebra, \( \mathfrak{a} \), the **current algebra of** \( \mathfrak{a} \) is the vector space \( \mathfrak{a}[t] = \mathfrak{a} \otimes \mathbb{C}[t] \) with the Lie bracket determined by the rule \([a \otimes t', a' \otimes t^s] := [a, a'] \otimes t^{r+s} \) for \( a, a' \in \mathfrak{a} \). Let \( \mathbb{U}(\mathfrak{a}) \) denote the universal enveloping algebra of \( \mathfrak{a} \). The current algebra is graded by putting \( t \) in degree one, as is \( \mathbb{U}(\mathfrak{a}[t]) \). Note that \( \mathfrak{g} \otimes 1 \subset \mathfrak{g}[t] \) is isomorphic to \( \mathfrak{g} \) and we will simply write \( \mathfrak{g} \subset \mathfrak{g}[t] \). The universal enveloping algebra \( \mathbb{U}(\mathfrak{a}) \) is an associative Hopf algebra with co-multiplication induced by the assignment \( \Delta(x) = 1 \otimes x + x \otimes 1 \) for \( x \in \mathfrak{a} \). This is a map of graded algebras for the case \( \mathbb{U}(\mathfrak{a}[t]) \).

We let \( I \) be the category of graded \( \mathfrak{g}[t] \)-modules, \( M \), such that each graded component satisfies \( \dim M[k] < \infty \) with graded morphisms (that is, they preserve degrees). Note that, because the graded components are finite-dimensional \( \mathfrak{g} \)-modules, the objects in \( I \) are necessarily weight modules. The simple modules in \( I \) are indexed by \((\lambda, r) \in \mathbb{P}^+ \times \mathbb{Z} \); the simple module \( V(\lambda, r) \) is isomorphic as a \( \mathfrak{g} \)-module to \( V(\lambda) \), and is concentrated in the \( r \)th graded component.

\(^1\)also known as the co(in)variant algebra
The category $\mathcal{I}$ is not strictly a tensor category: given two objects $M$ and $N$, the tensor product $M \otimes N$ satisfies $(M \otimes N)[k] = \sum_{i \in \mathbb{Z}} M[k-i] \otimes N[i]$. This will be an object of $\mathcal{I}$ if and only if this sum is finite for all $k$. A sufficient condition is that both $M$ and $N$ have a lower bound on their grades.

Given an object, $N \in \mathcal{I}$, its graded dual is the module $N^* = \oplus_{k \in \mathbb{Z}} \text{Hom}(N[k], \mathbb{C})$, which has graded components

$$(N^*)[k] = (N[-k])^*.$$  

The following lemma explains the connection between the graded characters of a module $N \in \mathcal{I}$ and that of its dual, $N^*$.

**Lemma.** Let $\text{ch}_{gr} N = \sum_{\lambda \in P^+} g_\lambda(u) e(O(\lambda))$, where $g_\lambda(u) \in \mathbb{Z}[u^{\pm 1}]$. Then the graded character of $N^*$ is given by

$$\text{ch}_{gr} N^* = \sum_{\lambda \in P^+} g_\lambda(u^{-1}) e(O(\lambda^\vee)).$$

**Proof.** It is enough to show that the coefficient of $e(O(\lambda^\vee))$ is $g_\lambda(u^{-1})$, for which it is enough to show that the coefficient of $(N^*)_{\lambda^\vee}$ is $g_\lambda(u^{-1})$. For this, it is enough to show that the dimension of $(N^*)_{\lambda^\vee}[-k]$ is equal to the dimension of $N_\lambda[k]$.

If $N_\lambda \neq 0$ then $(N_\lambda)^* = (N^*)_{\lambda^\vee} \neq 0$. By the invariance of dimensions of weight spaces under the action of the Weyl group and equation 3 it follows that

$$\dim N[k]_\lambda = \dim N[k]_{\omega_0 \lambda} = \dim (N^*)[-k]_{\lambda^\vee}.$$  

The result follows. \hfill \Box

The subcategory of $\mathcal{I}$ where objects have a lowest graded component admits a tilting theory([1]); for each $(\lambda, r) \in P^+ \times \mathbb{Z}$ there is an indecomposable tilting module $T(\lambda, r)$ which admits a filtration by standard modules and a filtration by costandard modules. The standard modules are the global Weyl modules $W(\lambda, r)$; these are universal highest weight modules, and can be defined using generators and relations. If $\lambda \neq 0$ the global Weyl module is infinite dimensional. The costandard objects are the graded duals of so called local Weyl modules $(W_{\text{loc}}(\lambda, r))^*$. The local Weyl module is a finite dimensional quotient of the global Weyl module, and can be defined using generators and relations. Global and local Weyl modules were introduced in [8] and defined in broad generality in [7]. The modules $W(\lambda, r)$ and $W(\lambda, s)$ are grade shifts of each other.

If $V$ is a finite-dimensional $\mathfrak{g}$-module then $V[t] := V \otimes \mathbb{C}[t]$ is an object in $\mathcal{I}$, with an action defined by letting $x \otimes t^r \cdot v := (x \cdot v) \otimes t^{r+s}$, for $x \in \mathfrak{g}$, $v \in V$ and $r, s \in \mathbb{Z}$. We have $\text{ch}_{gr} V[t] = \sum_{t \geq 0} \text{ch}_\lambda V_{rt}$. In [3] it is shown that the fundamental global Weyl module $W(\omega_1, 0) \cong V(\omega_1) \otimes \mathbb{C}[t]$.

3. A natural construction

For $V$ a finite-dimensional $\mathfrak{g}$-module, the module $M = (V[t])^{\otimes m} \in \mathcal{I}$ admits a natural right action of $\mathfrak{s}_m$, permuting tensorands; therefore, $M$ is a $\mathfrak{g}[t] \mathbb{C} \mathfrak{s}_m$-bimodule and the two
actions commute. Because \( E_m \) preserves the finite-dimensional graded components, the decomposition of \( M \) as a representation of \( E_m \) follows Equation 2.3:

\[
M \cong \bigoplus_{\gamma \in P(m)} \bigoplus_{k \in \mathbb{Z}} S(\gamma) \otimes \text{Hom}_{E_m}(S(\gamma), M[k]).
\]

For each \( k \in \mathbb{Z} \), we set \( B(\gamma, V)[k] = \text{Hom}_{E_m}(S(\gamma), M[k]) \) and we define

\[
B(\gamma, V) = \bigoplus_{k \in \mathbb{Z}} \text{Hom}_{E_m}(S(\gamma), M[k]).
\]

We will show at the end of the section that \( B(\gamma, V) \) is a \( g[t] \)-module.

Consider the vector space isomorphism

\[
M = (V \otimes \mathbb{C}[t])^\otimes m \cong V^\otimes m \otimes A_m. \tag{3.1}
\]

Using equation 3.1 we see that \( M \) admits the structure of a right \( A_m \)-module. This action does not commute with the action of \( E_m \); however, if we restrict this right action to \( A_m \), the actions commutes with \( E_m \) and \( g[t] \). Because the polynomial ring \( A_m \) is a free graded \( A_m \)-module of rank \( = \dim A_m^{\text{con}} \), we see that \( M \) is a free \( A_m \)-module of rank \( = (\dim V)^m \times \dim A_m^{\text{con}} \). The \( B(\gamma, V) \) are then projective \( A_m \) modules, and the Quillen-Suslin Theorem tells us that the \( B(\gamma, V) \) are in fact free (graded) \( A_m \)-module. Given \( J \) a maximal ideal of \( A_m \), we define the localization of \( B(\gamma, V) \) at \( J \) to be

\[
B_J(\gamma, V) := B(\gamma, V) \otimes_{A_m} A_m/J
\]

and analogously define \( M_J \). These will always be a finite dimensional vector space, but will only be a graded vector space if \( J \) is a graded ideal. We let \( I_m \) be the unique graded maximal ideal in \( A_m \), and denote \( B_{I_m}(\gamma, V) \) by \( B_{I_m}(\gamma, V) \) and \( M_{I_m} \) by \( M_{I_m} \).

The main result of this paper is the following theorem. See equations 2.2, 2.4 and 2.5 for the definitions of the coefficients.

**Theorem.** The multiplicity space \( B(\gamma, V) \) is a graded module for the current algebra and so is \( B_{I_m}(\gamma, V) \). The graded characters of these modules behave in the following way.

(i) For each \( \gamma \in P(m) \), the graded character of \( B(\gamma, V) \) is given by

\[
\text{ch}_{gr} B_{I_m}(\gamma, V) = \sum_{\mu \in P^+} \sum_{\sigma, \tau \in P(m)} s_{\mu}(\tau, V)c_{\tau, \sigma}^{\gamma} f_\sigma(u)e(O(\mu)).
\]

(ii) Up to a fixed graded shift, the graded characters of localizations corresponding to conjugate partitions and dual \( g \)-modules are dual to one another as \( g[t] \)-modules in the following sense

\[
\text{ch}_{gr} B_{I_m}(\gamma, V) = u^{(x)} \text{ch}_{gr} B(\gamma', V)^*,
\]

This theorem is proved in the next section. We end this section by stating the following, which is an immediate corollary of Theorem 3(i) and Lemma 2.2

**Corollary.** The graded character of \( B(\gamma, V) \) is given by

\[
\text{ch}_{gr} B(\gamma, V) = H(A_m) \sum_{\mu \in P^+} \sum_{\sigma, \tau \in P(m)} s_{\mu}(\tau, V)c_{\tau, \sigma}^{\gamma} f_\sigma(u)e(O(\mu)).
\]
We now explain why the (graded) multiplicity space $B(\gamma, V) = \text{Hom}_{\mathbb{C}^{\infty}}(S(\gamma), M)$ is a $g[t]$ module. Recall that the actions of $g[t]$ and $\mathbb{C}^{\infty}$ on $M$ commute. This means that, as elements of $\text{End}(M)$, we can view $g[t] \subset \text{End}_{\mathbb{C}^{\infty}}(M)$. It is now sufficient to show that $B(\gamma, V)$ is a module for $\text{End}_{\mathbb{C}^{\infty}}(M)$. To the end, let $f \in B(\gamma, V)$ and $\phi \in \text{End}_{\mathbb{C}^{\infty}}(M)$. Let $\phi \cdot f$ denote the composition $f$ followed by $\phi$. Then

$$S(\gamma) \xrightarrow{f} M \xrightarrow{\phi} M$$

is clearly an element of $B(\gamma, V)$.

4. Proof of Theorem 3

In this section we will look more closely at several $\mathbb{C}^{\infty}$-modules.

4.1. Hilbert polynomials for the coinvariant ring. Recall that we defined the polynomials $f_\alpha(u)$ by the decomposition $H(A^m_{\text{coin}}) = \sum_\alpha f_\alpha(u)S(\alpha)$. In this section we prove the following proposition

**Proposition.** Let $\alpha \in P(m)$ and $\alpha^\vee$ denote its conjugate. The polynomial $f_\alpha(u)$ satisfies

$$f_\alpha(u) = u^{(\frac{m}{2})}f_{\alpha^\vee}(u^{-1}).$$

The first step is to understand the polynomials $f_\alpha(u)$ better. Let $\text{Tab}(\alpha)$ (respectively $\text{STab}(\alpha)$) denote the set of tableau (respectively the set of standard tableau) of shape $\alpha$. Given $T \in \text{Tab}(\alpha)$ we define its descent by $\text{Dec}(T) := \{a : a + 1 \text{ is in a row strictly below the row of } a\}$. Define the major index of $T$ to be the integer $\text{Maj}_T = \sum_{a \in \text{Dec}(T)} a$. For each $\alpha$ we define a polynomial

$$\text{Maj}_\alpha(u) = \sum_{T \in \text{STab}(\alpha)} u^{\text{Maj}_T}.$$

The graded multiplicity of $S(\alpha)$ in $A^m_{\text{coin}}$ is given by the major index,

$$f_\alpha(u) = \text{Maj}_\alpha(u); \quad (4.1)$$

see, for example, [10].

**Lemma.** The major indexes of the tableau $T$ and $T^\vee$ are related by $\text{Maj}_T + \text{Maj}_{T^\vee} = \binom{m}{2}$.

**Proof.** It is easy to see that the conjugate map induces a bijection of sets $\vee : \text{STab}(\alpha) \to \text{STab}(\alpha^\vee)$ sending $T \mapsto T^\vee$.

If $b \in \{1, \ldots, m-1\}$ is such that $b \notin \text{Dec}(T)$, then $b + 1$ must be in the same row as $b$, and to the right. Then $b + 1$ must be below $b$ in $T^\vee$, and so $b \in \text{Dec}(T^\vee)$. It follows that $\text{Dec}(T) \cap \text{Dec}(T^\vee) = \emptyset$ and that $\text{Dec}(T) \cup \text{Dec}(T^\vee) = \{1, \ldots, m-1\}$. Hence $\text{Maj}_T + \text{Maj}_{T^\vee} = \sum_{i=1}^{m-1} i = \binom{m}{2}$. \qed

Proposition 4.1 is an immediate corollary.
4.2. The simple modules. Recall the Kronecker coefficients are non-negative integers defined by $S(\tau) \otimes S(\sigma) = \oplus c_{\tau,\sigma}^{\gamma} S(\gamma)$. We will show that

**Lemma.** The Kronecker coefficients satisfy $c_{\tau,\sigma}^{\gamma} = c_{\tau,\sigma}^{\gamma'}$.

The key tool to proving this is the fact [9, Exercise 4.51] that $S(\tau) \otimes \text{sgn} = S(\tau^\vee)$.

**Proof.** We start with $S(\tau) \otimes S(\sigma) = \oplus c_{\tau,\sigma}^{\gamma} S(\gamma)$ and then tensor by $\text{sgn}$ on the right. This shows us that

$$S(\tau) \otimes S(\sigma) = \bigoplus_{\gamma} c_{\tau,\sigma}^{\gamma} S(\gamma).$$

However, by definition we also have

$$S(\tau) \otimes S(\sigma) = \bigoplus_{\gamma} c_{\tau,\sigma}^{\gamma} S(\gamma).$$

Thus we must have $c_{\tau,\sigma}^{\gamma} = c_{\tau,\sigma}^{\gamma'}$. □

4.3. Weight modules.

**Proposition.** Let $V$ be a $\mathfrak{g}[t] - \mathfrak{g}$ bimodule and $\mu \in h^*$. Then we have an isomorphism of $\mathfrak{g}$-modules $V_\mu \cong \odot_{\mathfrak{g}} (V^*)_{-\mu}$.

**Proof.** This follows from the fact that the $\mathfrak{g}[t]$ duality satisfies $(V_\mu)^* \cong (V^*)_{-\mu}$ together with the fact that the $\mathfrak{g}$, dual of a simple module is itself. □

4.4. Proof of the main result. We can now prove Theorem 3. We have already established the opening remark. We will start by proving the first equality. We freely use the notation of the previous sections. Recall that for all $\mu \in P^+$, the weight space $(M_{\text{loc}})_\mu$ is a $\mathfrak{g}$-module, and

$$(M_{\text{loc}})_\mu = (V^m \otimes A_m^{\text{coin}})_\mu = (V^m)_\mu \otimes A_m^{\text{coin}}.$$  

Using the decompositions of $(V^m)_\mu$ and $A_m^{\text{coin}}$ as $\mathfrak{g}$-modules, we see that as a $\mathfrak{g}$-module, $(M_{\text{loc}})_\mu$ is isomorphic to

$$\left( \sum_\tau s_\mu(\tau, V) S(\tau) \right) \otimes \left( \sum_\sigma f_\sigma(u) S(\sigma) \right) = \sum_\tau \sum_\sigma s_\mu(\tau, V) f_\sigma(u) S(\sigma) \otimes S(\sigma) \quad (4.2)$$

$$= \sum_\tau \sum_\sigma \sum_\gamma s_\mu(\tau, V) f_\sigma(u) c_{\tau,\sigma}^\gamma S(\gamma).$$

Another way to state this is that $H((M_{\text{loc}})_\mu) = \sum_\tau \sum_\sigma \sum_\gamma s_\mu(\tau, V) f_\sigma(u) c_{\tau,\sigma}^\gamma S(\gamma)$.

The Hilbert series of $B_{\text{loc}}(\gamma, V)_\mu$ is obtained by collecting the coefficient of $S(\gamma)$ in the decomposition of $(M_{\text{loc}})_\mu$. We know that $B_{\text{loc}}(\gamma, V)$ is a finite-dimensional module for $\mathfrak{g}$, and so is a direct sum of its weight spaces. Therefore the graded character of $B_{\text{loc}}(\gamma, V)$ is given by summing over the Hilbert series for its weight spaces:

$$\text{ch}_{\text{gr}} B_{\text{loc}}(\gamma, V) = \sum_{\mu \in P^+} \left( \sum_\tau \sum_\sigma \sum_\gamma s_\mu(\tau, V) c_{\tau,\sigma}^\gamma f_\sigma(u) \text{sgn}(\mu) \right).$$
To prove the second equality in Theorem 3, first note that by using the equality proved above, and Lemma 2.3, we have

\[ \text{ch}_{f_0} B_{loc}(\gamma^\vee, V^\vee)^* = \sum_{\mu \in P^+} \left( \sum_{\tau} \sum_{\sigma} s_\mu(\tau, V^\vee)c_{\tau, \sigma}^{\gamma^\vee} f_\sigma(u^{-1}) \right) e(O(\mu^\vee)). \]

It is enough to show that, for a fixed \( \mu \in P^+ \) and \( \tau \in P(m) \)

\[ \sum_{\sigma} s_\mu(\tau, V)c_{\tau, \sigma}^{\gamma^\vee} f_\sigma(u) = \sum_{\sigma} s_\mu(\tau, V)c_{\tau, \sigma}^{\gamma^\vee} f_\sigma(u^{-1}). \]

Starting on the left, we apply Proposition 4.1 and Lemma 4.2

\[ u^{(g)} \sum_{\sigma} s_\mu(\tau, V)c_{\tau, \sigma}^{\gamma^\vee} f_\sigma(u^{-1}). \]

Using Proposition 4.3 we get

\[ u^{(m)} \sum_{\sigma} s_\mu(\tau, V)c_{\tau, \sigma}^{\gamma^\vee} f_\sigma(u^{-1}) \]

and reindexing gives the result.

5. The Case \( V = V(\omega_1) \)

5.1. Weights and Characters. A special case of our construction is when we take \( g = s_{l+1} \) and \( V \) to be the natural representation \( V(\omega_1) \). In this case we can identify those dominant weights, \( \mu \in P^+ \), such that \( (V^\otimes m)_\mu \neq 0 \) and we can explicitly describe the \( \Xi_m \)-structure of these weight spaces.

The following results are well known.

**Proposition.** The module corresponding to the natural representation, \( V(\omega_1) \), satisfies the following properties.

1. It has a basis \( \{v_0, \ldots, v_n\} \), defined by \( v_i := x_i^{-1}x_{i-1}^{-1} \cdots x_1^{-1}v_0 \) for \( i \geq 1 \), where the \( x_i \) are Chevalley basis elements.
2. Each basis vector, \( v_i \), has weight \( \text{wt}(v_i) = -\omega_i + \omega_{i+1} \), where we use the convention that \( \omega_0 = \omega_{n+1} = 0 \).

Let \( N = V^\otimes n \). Let \( \underline{a} = (a_0, \ldots, a_n) \in Z_{\geq 0}^{n+1} \) be such that \( \sum a_i = m \) and define \( \underline{a^\prime} := \otimes_{i=0}^{n} a_i^\prime \); this element of \( N \) has weight \( \text{wt}(\underline{a^\prime}) = \sum_{i=1}^{n}(a_i - a_i^\prime)\omega_i \). It follows that \( \text{wt}(\underline{a^\prime}) \in P^+ \) if and only if \( a_{i+1} \leq a_i \) for all \( i \); that is to say, precisely when \( \underline{a^\prime} \) is a partition. Given such a partition \( \underline{a^\prime} \in P(m) \), define an \( n \)-tuple, \( \underline{a^\prime} \in Z_{\geq 0}^n \), by \( a_i^\prime = \sum_{j=i}^{n} a_j \). We can now write the weight of \( \underline{a^\prime} \) as

\[ \text{wt}(\underline{a^\prime}) = m\omega_1 - \sum a_i^\prime \omega_i. \]

The following properties follow immediately from the definition.

**Lemma.** The weight spaces of \( N \) can be described as follows

1. Suppose \( N_{\mu} \neq 0 \), then there exists some \( \underline{a} \) such that \( \mu = m\omega_1 - \sum a_i^\prime \omega_i \), where \( a_i^\prime \) is associated to \( \underline{a} \) as above. Furthermore, \( \mu \in P^+ \) if and only if \( a_i \geq a_{i+1} \) for all \( i \).
2. The space \( N_{\mu} \) is spanned by permutations of the tensorands of the vector \( \underline{a^\prime} \).
Proposition. Fix $a \in P(m)$ as above and let $\mu$ be an arbitrary weight. The decomposition of the module $N_{\mu}$ is described by Kostka numbers; that is,

$$ N_{\mu} \cong \bigoplus_{\tau \geq \underline{a}} K_{\tau, \underline{a}} S(\tau). $$

Proof. Define $Y(\underline{a}) = \Xi_{a_1} \times \cdots \times \Xi_{a_n} \leq \Xi_{m}$, the Young subgroup associated to $\underline{a}$. Clearly the action of $Y(\underline{a})$ fixes $v_{\underline{a}}$. Lemma 5.1 shows that $v_{\underline{a}}$ generates $N_{\mu}$ as a left $C \Xi_n$-module, that is

$$ N_{\mu} \cong C \Xi_n \otimes_{Y(\underline{a})} \tau_{\underline{a}} = \text{Ind}_{\Xi_n}^{\Xi_{m}} \text{triv}. $$

The proposition follows by applying Young’s Rule (see, for example, [9, Corollary 4.39]) in the special case that the representation is trivial. $\square$

Corollary. The character of $B_{\text{loc}}(\gamma, V(\omega_1))$ can be described explicitly:

$$ \text{ch}_{gr} B_{\text{loc}}(\gamma) = \sum_{a, \tau \in P(m) | \tau \geq \underline{a}} K_{a, \tau} f_\gamma(u) c_{\tau, \underline{a}} e(O(\mu_{\underline{a}})). $$

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