ON THE AVERAGE COMOVING NUMBER DENSITY OF HALOS

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ABSTRACT

I compare the numerical multiplicity function recently given by Yahagi et al. with the theoretical multiplicity function obtained by means of the excursion set model and an improved version of the barrier shape obtained by Del Popolo & Gambera, which implicitly takes into account the total angular momentum acquired by the protostructure during evolution and a nonzero cosmological constant. I show that the multiplicity function obtained here is in better agreement with the recent simulations of Yahagi et al. than other previous models (Sheth & Tormen; Sheth et al.; Jenkins et al.) and that, unlike some previous multiplicity function models (Jenkins et al.; Yahagi et al.), it was obtained from a sound theoretical background.

Subject headings: cosmology: theory — galaxies: formation — large-scale structure of universe

1. INTRODUCTION

Two different kinds of methods are widely used for the study of structure formation. The first are N-body simulations, which are able to follow the evolution of a large number of particles under the influence of mutual gravity from initial conditions to the present epoch. The second are semianalytical methods, among which the Press-Schechter (PS) approach and its extensions are of great interest, since they allow us to compute mass functions (Press & Schechter 1974; Bond et al. 1991), to approximate merging histories (Lacey & Cole 1993, hereafter LC93; Bower 1991; Sheth & Lemson 1999), and to estimate the spatial clustering of dark matter halos (Mo & White 1996; Catelan et al. 1998; Sheth & Lemson 1999a).

Although the analytical framework of the PS model has been greatly refined and extended (Bond et al. 1991; LC93), it is well known that the PS mass function, while qualitatively correct, disagrees with the results of N-body simulations. In particular, the PS formula overestimates the abundance of halos near the characteristic mass $M$, and underestimates the abundance in the high-mass tail (Efstathiou et al. 1988; Lacey & Cole 1994; Tozzi & Governato 1998; Gross et al. 1998; Governato et al. 1999).

A better agreement between the numerical mass function and the analytic mass function can be obtained by incorporating into the PS Ansatz the model with a nonspherical collapse (Del Popolo & Gambera 1998; Sheth & Tormen 1999, hereafter ST99; Sheth et al. 2001; Sheth & Tormen 2002, hereafter ST02; Jenkins et al. 2001, hereafter J01) instead of the spherical model, or by taking into account the spatial correlation of density fluctuations (Nagashima 2001).

More recently, in order to investigate the functional form of the universal multiplicity function, Yahagi et al. performed five runs of N-body simulations with high mass resolution and compared them with different multiplicity functions and with a fit that they proposed.

They showed that discrepancies are observed between some of the quoted analytical multiplicity function with simulations. Here I use an improved version of the barrier shape from Del Popolo & Gambera (1998), obtained from the parameterization of the nonlinear collapse discussed in that paper, taking into account asphericity and tidal interaction between protocahlos and the effects of a nonzero cosmological constant, together with the results of ST99 and ST02 in order to study the “unconditional” multiplicity function.

The motives for this study are several. As previously reported, multiplicity functions such as those in ST99 and J01 only approximately fit high-resolution N-body simulations such as those of YNY04, while the functional form proposed in YNY04 provides a better fit than the ST99 functional form. Unfortunately, the functional form for the multiplicity function proposed in YNY04, and similarly that of J01 (which is a fit to their “Hubble volume” simulations of $\Omega_{\text{CDM}}$ and $\Omega_{\Lambda}$ cosmologies), is not based on any theoretical background. So it is important to find a better analytical form that starts from first principles, is physically motivated, and is able to fit simulations better. I show that the function obtained in the present paper, similar to that in YNY04, provides a better fit than the ST99 or other functional forms used in the literature; moreover, it has been obtained from solid physical and theoretical arguments. In §2, I calculate the “unconditional” multiplicity function; §§3 and 4 are devoted to results and to conclusions, respectively.

2. THE BARRIER MODEL AND THE MULTIPLICITY FUNCTION

According to hierarchical scenarios of structure formation, a region collapses at time $t$ if its overdensity at that time exceeds some threshold. The linear extrapolation of this threshold up to the present time is called a barrier, $B$. A likely form of this barrier is (ST99; ST02)

$$B(\sigma^2, z) = \sqrt{a_S} \left[1 + \beta \left(\frac{S}{a_S}\right)^{\alpha} \right] = \sqrt{a_S} \left[1 + \frac{\beta}{(av)^2}\right].$$

(1)

In the above equation, $a$, $\beta$, and $\alpha$ are constants, $S = \delta_c^2$, where $\delta_c(t)$ is the linear extrapolation up to the present day of the initial overdensity of a spherically symmetric region that collapsed at time $t$, $S = S_c(\sigma/\sigma_c)^2 = S_c/\nu$, $\sigma_c = (S_c)^{1/2}$, and $\nu = |\delta_c(t)/\sigma(M)|^2$, where $\sigma^2(M)$ is the present-day mass dispersion on a comoving scale containing mass $M$. The quantity $S$ depends on the assumed power spectrum. The spherical collapse model has a
barrier that does not depend on the mass (e.g., LC93). For this model the values of the parameters are \( a = 1 \) and \( \beta = 0 \). The ellipsoidal collapse model of ST99 has a barrier that depends on the mass (a moving barrier). The values of the parameters \( \beta = 0.485 \) and \( \gamma = 0.615 \) are adopted from the dynamics of ellipsoidal collapse and the parameter \( \alpha \) is determined by the shape of the mass function at the low-mass end, while \( a = 0.707 \) comes from fits to the results of \( N \)-body simulations. Essentially, the parameter \( a \) is related to the link length. The value of \( a = 0.707 \) is associated with a link length that is 0.2 times the mean interparticle separation or is determined by the number of massive halos in the simulations.

Below I use an improved version of the barrier model used by Del Popolo & Gambera (1998) to get mass functions, which are compared with those obtained by PS, ST99, J01, and YNY04 and with numerical simulations of YNY04. Since the way in which the barrier is obtained is described in previous papers (see Del Popolo & Gambera 1998, 1999, 2000), the reader is referred to those papers for details. If we assume that the barrier is proportional to the threshold for the collapse, similarly to ST99, the barrier can be expressed, in the case of a zero cosmological constant, in the form

\[
B(M) = \delta_c(\nu, z) = \delta_{\text{co}}(1 + \int_0^{r_0} \frac{r_1^2 dr}{G M r^3}) \simeq \delta_{\text{co}}(1 + \frac{\beta_1}{\nu_{\alpha_1}}),
\]

where \( \delta_{\text{co}} \) is the critical threshold for a spherical model, \( r_i \) is the initial radius, \( r_{ta} \) is the turnaround radius, \( I \) is the specific angular momentum, \( \alpha_1 = 0.585 \), and \( \beta_1 = 0.46 \). The specific angular momentum appearing in equation (2) is the specific total angular momentum acquired by the protostructure during evolution. In order to calculate \( I \), I use the same model as described in Del Popolo & Gambera (1998, 1999). (More information on the model and some of the model limits can be found in Del Popolo et al. 2001.)

In this approach, the rms angular momentum acquired through tidal torques is obtained by using the peak formalism and identifying the final angular momentum with that acquired at the maximum expansion of the object (Peebles 1969; Catelan & Theuns 1996a, 1996b). This assumption is justified by the fact that after the maximum expansion time, the angular momentum stops growing, becoming less sensitive to tidal couplings (Peebles 1969).

Similarly to the methods of Ryden (1988) and Eisenstein & Loeb (1995), the protostructure is divided into a series of mass shells, and the torque on each mass shell is computed separately. The density profile of each protostructure is approximated by the superposition of a spherical profile and the same Gaussian density field that is found far from the peak, which provides the quadrupole moment of the protostructure.

The net rms torque on a mass shell depends on the quadrupole moment and tidal field components (see eq. [27] of Ryden 1988). In order to find the total angular momentum imparted to a mass shell by tidal torques, it is necessary to know the time dependence of the torque. This can be found by connecting the quadrupole moment and tidal field components to parameters of the spherical collapse model (eq. [32] of Eisenstein & Loeb 1995; eqs. [32] and [34] of Ryden 1988). The angular momentum acquired during expansion can then be obtained by integrating the torque over time.

An interesting comparison can be made with the result of Catelan & Theuns (1996a), who calculated the angular momentum at the maximum expansion time (see their eqs. [31]–[32]) and compared it with previous theoretical and observational estimates. A comparison of the results of the present paper with those of Catelan & Theuns (1996a, 1996b) shows a good agreement.

If we assume a nonzero cosmological constant, equation (2) is changed as follows (see the Appendix):

\[
B(M) = \delta_c(\nu, z) = \delta_{\text{co}}(1 + \frac{\beta_1}{\nu_{\alpha_1}} + \frac{\Omega_\Lambda \beta_2}{\nu_{\alpha_2}}),
\]

where \( \alpha_2 = 0.4, \beta_2 = 0.02, \) and \( \Omega_\Lambda \) is the contribution to the density parameter from the cosmological constant. In equation (3), I used the functional form of ST99 for the barrier, since in the following calculations of the mass function, this form is simpler. As a consequence, I had to determine the parameters \( \alpha_1, \alpha_2, \beta_1, \) and \( \beta_2 \) in order to ensure that the ST99 barrier would be a good fit to that of the present paper (the first \( \delta_{\text{co}} \) term of eq. [3], which depends on \( I \) and \( \Lambda \)). The values of \( \alpha_1, \alpha_2, \beta_1, \) and \( \beta_2 \) are calculated as follows. I started by getting the values of \( B(M) \) for each mass \( M \). The data obtained were fitted to the functional form of the ST99 barrier (the fit function in the second \( \delta_{\text{co}} \) term of eq. [3]) by means of a nonlinear least-squares fit. If all four parameters are allowed to vary freely, one gets the values previously reported.

Another point to discuss is the profile of the specific angular momentum that appears in the previous equations and its role in the dynamics of the collapse. In the case of the equation of motion, equation (19), which is characterized by \( l = 0 \), the collapse is spherically symmetric. In the case where \( l \neq 0 \), the collapse depends on the profile of the angular momentum. If, for example, the angular momentum on the spin axis is zero, the collapse will be axisymmetrical. As shown in several papers (see Bullock et al. 2001), cylindrical symmetry is the natural symmetry in the spatial distribution of angular momentum as a result of tidal torques or a sequence of mergers.

The spatial distribution of halo angular momentum tends to be more cylindrically symmetric than spherically symmetric. At a fixed distance from the rotation axis, mass near the equatorial plane typically has only about \( \pm 35\% \) more specific angular momentum than corresponding mass near the poles. The mean value of \( l \) in spherical shells encompassing a mass \( M \) is well fitted by a power law, \( l \propto M^s \). The power \( s \) is distributed like a Gaussian, with a mean of \( s = 1.3 \) (see Ryden 1988; Bullock et al. 2001). As a result of this angular momentum distribution, the collapse described in equation (19) will be axisymmetrical and the structure will tend to flatten, similarly to what happens in ellipsoidal models of collapse.

ST02 connected the form of the barrier with the form of the multiplicity function. As shown by ST02, for a given barrier shape \( B(S) \), the first crossing distribution is well approximated by

\[
f(S) dS = |T(S)| \exp \left[ -\frac{B(S)^2}{2S} \right] \frac{dS/S}{\sqrt{2\pi S}},
\]

where \( T(S) \) is the sum of the first few terms in the Taylor expansion of \( B(S) \),

\[
T(S) = \sum_{n=0}^{5} (-S)^n \frac{\partial^n B(S)}{\partial S^n}.
\]
The quantity $S_f(S, t)$ is a function of the variable $\nu$ alone. Since $\delta_c$ and $\sigma$ evolve with time in the same way, the quantity $S_f(S, t)$ is independent of time. Setting $S_f(S, t) = u f(\nu)$, one obtains the so-called multiplicity function $f(\nu)$. The multiplicity function is the distribution of the first crossings of a barrier $B(\nu)$ by independent uncorrelated Brownian random walks (Bond et al. 1991). This is why the shape of the barrier influences the form of the multiplicity function.

In the excursion set approach, the average comoving number density of halos of mass $M$ in the universal or “unconditional” mass function, $n(M, z)$, is given by Bond et al. (1991) as

$$n(M, z) = \frac{\bar{n}}{M^2} \frac{d \log \nu}{d \log M} u f(\nu),$$

where $\bar{n}$ is the background density. In the case of the ellipsoidal barrier shape given in ST99 (eq. [1] of the present paper), equations (4) and (5) give, after truncating the expansion at $n = 5$ (see ST99),

$$u f(\nu) = \sqrt{a \nu} 2 [1 + \beta(\nu) - \alpha(\nu)] \exp \left\{ -0.5a \nu [1 + \beta(\nu) - \alpha(\nu)]^2 \right\},$$

where

$$\alpha(\nu) = \frac{1 - \alpha - \alpha(\alpha - 1)}{2!} - \ldots - \frac{\alpha(\alpha - 1) \cdot \ldots \cdot (\alpha - 4)}{5!}.$$

If the barrier takes into account the cosmological constant, such as in equation (3), and we use the same method that led to equation (7), we find that

$$u f(\nu) = A_1 \left[ 1 + \frac{\beta_1g(\alpha_1)}{(a \nu)^{\alpha_1}} + \frac{\beta_2g(\alpha_2)}{(a \nu)^{\alpha_2}} \right] \sqrt{a \nu} \frac{2 \pi}{a \nu}\exp\left\{-a \nu \left[1 + \frac{\beta_1g(\alpha_1)}{(a \nu)^{\alpha_1}} + \frac{\beta_2g(\alpha_2)}{(a \nu)^{\alpha_2}}\right]^2\right\}. \quad (9)$$

Using the values for $\beta$ and $\alpha$ from ST99 ($a = 0.707, \delta_c(z) = 1.686(1 + z), \beta \simeq 0.485$, and $\alpha \simeq 0.615$) in equation (7), we get (ST92)

$$u f(\nu) \simeq A_2 \left[ 1 + 0.094 \frac{a \nu}{(a \nu)^{\alpha}} \sqrt{a \nu} \frac{2 \pi}{a \nu}\exp\left\{-a \nu \left(1 + \frac{0.5(a \nu)^{0.6}}{2} \right)^2\right\}, \quad (10)$$

with $A_2 \simeq 1$. This last result is in good agreement with the fit of the simulated first crossing distribution (ST99):

$$u f(\nu) d\nu = A_3 \left[ 1 + 0.01 \frac{1}{(a \nu)^{\alpha}} \sqrt{a \nu} \frac{2 \pi}{a \nu}\exp\left(-\frac{a \nu}{2}\right), \quad (11)$$

where $p = 0.3$ and $a = 0.707$.

The normalization factor $A_3$ has to satisfy the constraint

$$\int_0^\infty f(\nu) d\nu = 1,$$

and as a consequence it is not an independent parameter, but is expressed in the form

$$A_3 = \left[ 1 + 2^{-p} \pi^{-1/2} \Gamma(1/2 - p) \right]^{-1} = 0.3222. \quad (13)$$

In the case of the barrier given in equation (2), the unconditional multiplicity function can be approximated by

$$u f(\nu) \simeq A_4 \left[ 1 + \frac{b}{(a \nu)^{0.585}} \sqrt{a \nu} \frac{2 \pi}{a \nu}\exp\left\{-a \nu \left[1 + \frac{d}{(a \nu)^{0.585}}\right]^2\right\}, \quad (14)$$

where $a = 0.707, b = 0.1218, c = 0.4019, d = 0.5526$, and $A_4 \simeq 1.75$ are obtained from the normalization condition. In the case of the barrier with a nonzero cosmological constant, equation (3), a good approximation to the multiplicity function is given by

$$u f(\nu) \simeq A_5 \left[ 1 + 0.1218 \frac{b}{(a \nu)^{0.585}} + 0.0079 \left(\frac{a \nu}{(a \nu)^{0.585}}\right)^{0.4} \sqrt{a \nu} \frac{2 \pi}{a \nu}\exp\left\{-0.4019a \nu \left[1 + \frac{0.5526}{(a \nu)^{0.585}} + \frac{0.02}{(a \nu)^{0.4}}\right]^2\right\}, \quad (15)$$

where $A_5 = 1.75$. For completeness, to the previously reported functions, namely, those in PS, ST99, and equation (15), we have to add that of J01, which produces the equation

$$u f(\nu) = 0.315 \exp\left\{-0.61 + \ln\left[\frac{a \nu}{\sigma^{-1}}\right]\right\} \exp\left\{-0.61 + \ln\sqrt{\nu} - \ln \delta_c\right\}. \quad (16)$$

The above formula is valid for $-1.2 < \ln\sqrt{\nu} - \ln \delta_c < 0.5$. YNY94 (eq. [7]), hereafter YNYeq7 proposes the following function to fit the numerical multiplicity function:

$$u f(\nu) = A_6 \left[ 1 + \left(\frac{B \sqrt{\nu}}{\sqrt{2}}\right)^D \nu^{D/2} \exp\left\{-\left(\frac{B \sqrt{\nu}}{\sqrt{2}}\right)^2\right\}, \quad (17)$$

where $A_6$ is a normalization factor to satisfy the unity constraint,

$$\int_0^\infty f(\nu) d\nu = 1; \text{ therefore}$$

$$A_6 = \left[ B \sqrt{2} \Gamma(D/2) + \Gamma([C + D]/2) \right]^{-1}. \quad (18)$$

The best-fit parameters are given as $B = 0.893, C = 1.39$, and $D = 0.408$, and from these parameters, $A_6$ is constrained so that $A_6 = 0.298$. The CDM spectrum used in the present paper is that of Bardeen et al. (1986; eq. [G3]).

3. RESULTS

In this section, I compare the analytic multiplicity functions of PS, ST99, J01, YNYeq7, and equation (15) of the present paper.
paper with the numerical simulations of YNY04. Those simulations adopt the ΛCDM cosmological parameters of $\Omega_m = 0.3$, $\Omega_b = 0.7$, $h = 0.7$, and $\sigma_8 = 1.0$, using 512$^3$ particles in common (see YNY04 for details).

Figure 1 shows a comparison between numerical multiplicity functions and theoretical ones. In the plot the solid line represents the multiplicity function obtained in the present paper, the short-dashed line represents that of YNY04, the dot-dashed line shows the ST99 multiplicity function, and the long-dashed line represents the J01 multiplicity function. The error bars with open circles represent run 140 of YNY04, error bars with filled squares show case 70a, those with open squares show case 70a, those with filled circles show case 35b, and those with crosses show case 35a.

Note that the comparison of the above curves (except for the PS model) with the results of N-body simulations shows a very good agreement. However, there are some discrepancies between the YNY04 multiplicity function and other model functions (except that in the present paper). First, the multiplicity function of the present paper, like that of YNY04, systematically falls below the ST99 and the J01 functions in the low-$\nu$ region of $\nu \leq 1$. In this region the multiplicity function of the present paper is very close to that of YNY04.

As seen in Figure 1, and in agreement with YNY04, the numerical multiplicity functions reside between the ST99 and J01 functions at $2 \leq \nu \leq 3$ (except in case 35b). In addition, the numerical multiplicity functions have an apparent peak at $\nu \sim 1$ instead of the plateau that is seen in the J01 function.

On the other hand, in the high-$\nu$ region, where $\nu$ is significantly larger than unity, the multiplicity function of the present paper, like that of YNY04, takes values between those of the ST99 and J01 functions. These differences between the numerical multiplicity functions and the analytic ones, such as those of ST99, ST02, and J01, are within 1 $\sigma$ error bars, and they are possibly due to the different box sizes adopted (see YNY04 for a discussion). To be more precise, throughout the peak range of $0.3 \leq \nu \leq 3$, the ST99 multiplicity function is in disagreement with the high mass resolution N-body simulations of YNY04 and that of the present paper. As shown by YNY04, the ST99 functional form provides a good fit to them only by choosing parameter values of $a = 0.664$, $p = 0.321$, and $A_3 = 0.301$. The multiplicity function obtained in the present paper has a peak at $\nu \sim 1$, as in the ST99 function, the YNY04 numerical multiplicity function, and the YNY04$e$, instead of a plateau as in the J01 function.

I want to stress that the functional form proposed in YNY04, namely, that of YNY04$e$, provides a better fit when compared with the ST99 functional form, but it is not based on a theoretical background. The function obtained in the present paper, similarly to YNY04$e$, provides a better fit to simulations than the ST99 functional form, and at the same time it has been obtained from solid physical and theoretical arguments. The better agreement observed between the multiplicity function of the present paper and the YNY04 simulations, when compared with those of ST99, is connected to the shape of the barrier ($h$). By taking into account the effects of asphericity and tidal interaction with neighbors, Del Popolo & Gambera (1998) showed that the threshold is mass dependent, and in particular that of the set of objects that collapse at the same time, the less massive ones must initially have been denser than the more massive ones, since the less massive ones would have had to hold themselves together against stronger tidal forces.

The shape of the barrier given in equation (2) is a direct consequence of the angular momentum acquired by the protostucture during evolution, while equation (3) introduces the effects of the cosmological constant.

Similarly to that of ST99, the barrier increases with $S$ (and decreases with mass, $M$) differently from other models (see Monaco 1997a, 1997b). It is interesting to note that the increase of the barrier with $S$ has several important consequences, and these models have a richer structure than the constant barrier model.

The decrease of the barrier with mass means that, in order to form structure, more massive peaks must cross a lower threshold, $\delta_c(\nu,z)$, than less massive ones. At the same time, since the probability of finding high peaks is larger in more dense regions, this means that, statistically, in order to form structure, peaks in more dense regions may have a lower value of the threshold, $\delta_c(\nu,z)$, with respect to those of underdense regions. This is due to the fact that less massive objects are more influenced by external tides and consequently must be more overdense if they are to collapse by a given time. In fact, the angular momentum acquired by a shell centered on a peak in the CDM density distribution is anticorrelated with density: high-density peaks acquire less angular momentum than low-density peaks (Hoffman 1986; Ryden 1988). A larger amount of angular momentum acquired by low-density peaks (with respect to the high-density ones) implies that these peaks can more easily resist gravitational collapse, and consequently it is more difficult for them to form structure. Therefore, on small scales, where the shear is statistically greater, structures need, on average, a higher density contrast to collapse.

It is evident that the effect of a nonzero cosmological constant adds to that of the angular momentum, $L$. The effect of a nonzero cosmological constant is to slightly change the evolution of the multiplicity function with respect to open models with the
same value of $\Omega_0$. This is caused by the fact that in a flat universe with $\Omega_0 > 0$, the density of the universe remains close to the critical value later in time, promoting perturbation growth at lower redshifts. The evolution is more rapid for larger values (in absolute value) of the spectral index, $n$.

As previously reported, the ST99 model gives a better fit to simulations than the PS model, but it has some discrepancies with simulations. The ST99 model was initially introduced as a fit to the GIF simulations, and the importance of aspherical collapse in the functional form of the mass function was recognized in a subsequent paper (Sheth et al. 2001). The effects of asphericity were taken into account by changing the functional form of the critical overdensity (barrier) by means of a simple intuitive parameterization of elliptical collapse of isolated spheroids. The model proposed in the present paper has several similarities with the models of ST99 and ST02: namely, it uses the excursion set approach as extended by ST02 to calculate the multiplicity function, but at the same time it differs from ST99 and ST02 in the way that the barrier was calculated and in the fact that this model takes into account angular momentum acquisition and a nonzero cosmological constant, aspects that are not taken into account in ST99 and ST02. These differences give rise to a multiplicity function that is in better agreement with simulations. This shows the importance of the form of the barrier. The improvement of the model of the present paper and the ST99 model with respect to that of PS is probably also connected to the fact that incorporating the non-spherical collapse with an increasing barrier into the excursion set approach results in a model in which fragmentation and mergers may occur, effects that are important in structure formation. In the case of non-spherical collapse with an increasing barrier, a small fraction of the mass in the universe remains unbound, while for the spherical dynamics, at the given time, all the mass is bound up in collapsed objects. Moreover, incorporating non-spherical collapse with an increasing barrier in the excursion set approach results in a model in which fragmentation and mergers may occur (ST99). If the barrier decreases with $S$ (Monaco 1997a, 1997b), this implies that all walks are guaranteed to cross it, and so there is no fragmentation associated with this barrier shape.

In other words, the excursion set approach with a barrier that takes into account the effects of the physics of structure formation gives rise to good approximations to the numerical multiplicity function: the goodness of the approximation increases with a more improved form of the barrier (taking into account more and more physical effects: angular momentum acquisition, nonzero cosmological constant, etc.). Another important aspect of this method is its noteworthy versatility: for example, it is very easy to take into account the presence of a nonzero cosmological constant by englobing it in the barrier. The YNY04 numerical multiplicity function assumes a nonzero cosmological constant, while the theoretical models (ST99; ST02; J01) do not take this into account.

Finally, I checked the time dependence of the multiplicity function. Figure 2 shows the multiplicity function from the 35a run of YNY04 for the four redshift ranges of $0 \leq z < 1$ (open circles), $1 \leq z < 3$ (open squares), $3 \leq z < 6$ (open triangles), and $z \geq 6$ (crosses). At high redshifts, high-$\nu$ halos in the exponential part of the YNYeq7 function (solid line) and equation (15) of the present paper (dashed line) are probed. As the redshift decreases, the probe window moves to the lower $\nu$ region. Figure 2 shows that the multiplicity function of this paper, equation (15), and that in YNYeq7 both give a good fit to the numerical simulations. For small values of $\nu$, equation (15) is a slightly better fit to the data, and at large values of $\nu$ the two functions decay in the same way.

4. CONCLUSIONS

In the present paper, I have compared the numerical multiplicity function given in YNY04 with the theoretical multiplicity function obtained by means of the excursion set model and an improved version of the barrier shape obtained in Del Popolo & Gambera (1998), which implicitly takes into account tidal interactions between clusters and a nonzero cosmological constant. I showed that the barrier obtained in Del Popolo & Gambera (1998) gives rise to a better description of the multiplicity functions than other models (e.g., those of ST99 and J01), and the agreement is based on sound theoretical models and not on fitting to simulations.

The main results of this paper can be summarized as follows:

1. The variable barrier of the present paper, combined with the model of ST02, gives “unconditional” multiplicity functions in better agreement with the N-body simulations of YNY04 than other previous models (ST99, ST02, and J01).

2. The comparison of the theoretical multiplicity function of the present paper, in agreement with the YNY04 result, shows some discrepancies with the theoretical multiplicity functions of several authors (ST99, ST02, and J01): for example, the maximum value of the multiplicity function from simulations at $\nu \sim 1$ is smaller, and its low-mass tail is shallower than those of the ST99 multiplicity function.

3. The multiplicity function of the present paper gives a good fit to simulation results as the fit function proposed by YNY04, but it differs from that function in that it was obtained from a sound theoretical background.

4. The excursion set model with a moving barrier is very versatile, since it is very easy to introduce several physical effects in the calculation of the multiplicity function just by modifying the barrier.
The above considerations show that it is possible to get accurate predictions for a number of statistical quantities associated with the formation and clustering of dark matter halos by incorporating a nonspherical collapse that takes into account a nonzero cosmological constant in the excursion set approach. The improvement is probably also connected to the fact that incorporating the nonspherical collapse with an increasing barrier into the excursion set approach results in a model in which fragmentation and mergers may occur, effects that are important in structure formation. Moreover, the effect of a nonzero cosmological constant adds to that of the angular momentum in slightly changing the evolution of the multiplicity function with respect to open models with the same value of the matter density parameter.

APPENDIX

The equation governing the collapse of a density perturbation by taking into account angular momentum acquisition by protostructures can be obtained using a model from Peebles (1993; see also Del Popolo & Gambera 1998, 1999). Let us consider an ensemble of gravitationally growing mass concentrations and suppose that the material in each system collects within the same potential well with inward-pointing acceleration given by \( g(r) \) (see Del Popolo & Gambera 1998). Here \( dP = f(L, r v_\tau, t) dL dv_\tau dr \).\( dv_\tau \) indicates the probability that a particle of mass \( m \) can be found in the proper radius range \( r, r + dr \), in the radial velocity range \( v_\tau, v_\tau + dv_\tau \), where \( v_\tau = \dot{r} \), and with angular momentum \( L = mr\gamma \) in the range \( L, L + dL \), or with specific angular momentum \( l = L/m = r\gamma \). The radial acceleration of the particle is

\[
\frac{dv_\tau}{dt} = \frac{l^2(r)}{r^3} - g(r) = \frac{l^2(r)}{r^3} - \frac{GM}{r^2},
\]

where \( M \) is the mass of the central concentration. Equation (19) can be derived from a potential, and then from Liouville’s theorem it follows that the distribution function, \( f \), satisfies the collisionless Boltzmann equation,

\[
\frac{\partial f}{\partial t} + v_\tau \frac{\partial f}{\partial r} + \frac{\partial f}{\partial v_\tau} \left[ \frac{l^2}{r^3} - g(r) \right] = 0.
\]

If we assume a nonzero cosmological constant, equation (19) becomes (Peebles 1993; Bartlett & Silk 1993; Lahav et al. 1991; Del Popolo & Gambera 1998, 1999)

\[
\frac{dv_\tau}{dt} = -\frac{GM}{r^2} + \frac{l^2(r)}{r^3} + \frac{\Lambda}{3} r.
\]

Integrating equation (21), we have

\[
\frac{1}{2} \left( \frac{dr}{dt} \right)^2 = \frac{GM}{r} + \int_0^l \frac{l^2}{r^3} dr + \frac{\Lambda}{6} r^2 + \epsilon,
\]

where the value of the specific binding energy of the shell, \( \epsilon \), can be obtained using the condition for turnaround, \( dv_\tau dt = 0 \).

In turn, the binding energy of a growing mode solution is uniquely given by the linear overdensity, \( \delta_\gamma \), at a time \( t_i \). From this overdensity, using the linear theory, we can obtain that of the turnaround epoch and then that of the collapse. We find the binding energy of the shell, \( C \), using the relation between \( r \) and \( \delta_\gamma \) for the growing mode (Peebles 1980) in equation (22), and finally the linear overdensity at the time of collapse is given by

\[
\delta_c = \delta_{co} \left( 1 + \int_{0}^{r_\gamma_1} \frac{r_\gamma_1^2 dr}{GMr^3} + \frac{\Lambda r_\gamma_1^2}{6GM} \right) \approx \delta_{co} \left( 1 + \frac{\beta_1}{r^\alpha_1} + \frac{\Omega_\Lambda \beta_2}{r^\alpha_2} \right),
\]

where \( \alpha_1 = 0.585, \beta_1 = 0.46, \alpha_2 = 0.4, \) and \( \beta_2 = 0.02 \).

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