THE $\bar{\partial}_b$ EQUATION ON WEAKLY PSEUDOCONVEX CR MANIFOLDS OF DIMENSION 3

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1. Introduction

The starting point of this work is the second author’s thesis [N] accomplished under the direction of the first author. In [N] the main result is that on a smooth, compact, orientable, and weakly pseudoconvex CR manifold of real dimension 5 or higher, embedded in $\mathbb{C}^N$ and of codimension one or higher, the tangential CR operator $\bar{\partial}_b$ has closed range in $L^2$ and every Sobolev space $H^s$ with $s > 0$. This is proven using microlocalization with a specially constructed variant of a strongly plurisubharmonic function as weight, this function being the way in which the embedding of the manifold is exploited. Such a microlocalization, however, does not hold for a CR manifold of real dimension three, whose Levi form is merely a function, because some positive semi-definite quadratic forms involving the Levi form that appear in the proof are identically zero in this case. The aim of this paper is to extend the results of [N] to the case of a CR manifold of dimension three.

We will employ the same setup as in [N], which is described in detail in Section 2. We will use the same type of microlocalization, namely the division of the cotangent space into two truncated cones, a positive and a negative one, plus a region called the zero portion that overlaps somewhat these two and on which one obtains the best possible estimates, elliptic ones. On the positive cone, the estimates for $(0, 1)$ forms are the same as the ones in [N], but as explained above, one cannot expect any such estimates to hold on the negative cone. There, one can only prove estimates for functions. On the zero region, good estimates hold for both functions and $(0, 1)$ forms. Putting all these estimates together, we are able to prove the following theorem:

Theorem 1.1. Let $M$ be a smooth, compact, orientable, and weakly pseudoconvex manifold of real dimension 3 embedded in a complex space $\mathbb{C}^N$ and endowed with the induced CR structure.
If the range of $\partial_b$ is closed in $L^2(M)$, then the range of $\partial_b$ is also closed in $H^s(M)$ for each $s > 0$. Moreover, for every $\partial_b$-closed $(0,1)$ form $\alpha \in C^\infty$ such that $\alpha \perp \mathcal{N}(\partial_b^*)$, there exists some function $u \in C^\infty$ such that $\partial_b u = \alpha$.

Compared to the main result in [N], this theorem contains the extra hypothesis that the range of the tangential CR operator $\partial_b$ is closed in $L^2(M)$. We do not believe this hypothesis to be necessary; on the contrary, we conjecture that the embeddability assumption combined with weak pseudoconvexity should suffice for the range for $\partial_b$ to be closed in $L^2(M)$. None of the current methods, however, is strong enough to allow us to prove this conjecture. Until such a method is developed, we shall justify somewhat this additional hypothesis by pointing out that Dan Burns proved in [B] that the range of $\bar{\partial}$ method is developed, we shall justify somewhat this additional hypothesis by pointing out that Dan Burns proved in [B] that the range of $\bar{\partial}$ fails to be closed in $L^2$ on the well-known Rossi example of a non-embeddable CR structure on the three dimensional sphere discussed in [R].

2. Definitions and Notation

Definition 2.1. Let $M$ be a smooth manifold of real dimension 3 embedded in a complex space $\mathbb{C}^N$. An induced CR structure on $M$ is a complex line bundle of the complexified tangent bundle $\mathbb{T}(M)$ denoted by $T^{1,0}(M)$ satisfying $T^{1,0}(M) = \mathbb{T}(M) \cap T^{1,0}(\mathbb{C}^N)$. $M$ endowed with such a CR structure is called a CR manifold.

Let $B^q(M)$ be the bundle of $(0,q)$ forms which consists of skew-symmetric multi-linear maps of $(T^{0,1}(M))^q$ into $\mathbb{C}$. $B^0(M)$ is then the set of functions on $M$. Since $M$ is endowed with the induced CR structure coming from $\mathbb{C}^N$, there exists a natural restriction of the de Rham exterior derivative to $B^q(M)$, which we will denote by $\partial_b$. Thus, $\partial_b : B^q(M) \to B^{q+1}(M)$.

Clearly, the target space contains non-zero elements only for $q = 0$.

Next, it is only natural to choose as a metric the restriction on $\mathbb{T}(M)$ of the usual Hermitian inner product on $\mathbb{C}^N$, since this Riemannian metric is compatible with the CR structure on $M$, namely the spaces $T^{1,0}_p(M)$ and $T^{0,1}_p(M)$ are orthogonal under it because $\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \rangle = 0$ for all $1 \leq i, j \leq N$. We then define a Hermitian inner product on $B^q(M)$ by

$$\langle \phi, \psi \rangle = \int \langle \phi, \psi \rangle_z dV,$$

where $dV$ denotes the volume element and $\langle \phi, \psi \rangle_z$ the inner product induced on $B^q(M)$ by the metric on $\mathbb{T}(M)$ at each $z \in M$. Let $\| \cdot \|$ be the corresponding norm and $L^2_b(M)$ the Hilbert space obtained by completing $B^q(M)$ under this norm. We will only consider the $L^2$ closure of $\partial_b$, which we will once again denote it by $\partial_b$. Its domain is defined as follows:

Definition 2.2. $\mathrm{Dom}(\partial_b)$ is the subset of $L^2_b(M)$ composed of all forms $\phi$ for which there exists a sequence of $\{\phi_\nu\}_\nu$ in $B^q(M)$ satisfying:

(i) $\phi = \lim_{\nu \to \infty} \phi_\nu$ in $L^2$, where $\phi_\nu$ is smooth and

(ii) $\{\partial_b \phi_\nu\}_\nu$ is a Cauchy sequence in $L^2_b(M)$. 

For all $\phi \in \text{Dom}(\bar{\partial}_b)$, let $\lim_{\nu \to \infty} \bar{\partial}_b \phi_\nu = \bar{\partial}_b \phi$ which is thus well-defined. We need to define next $\bar{\partial}_b^*$, the $L^2$ adjoint of $\bar{\partial}_b$. Again, we first define its domain:

**Definition 2.3.** $\text{Dom}(\bar{\partial}_b^*)$ is the subset of $L^2_b(M)$ composed of all forms $\phi$ for which there exists a constant $C > 0$ such that

$$|\langle \phi, \bar{\partial}_b \psi \rangle| \leq C ||\psi||$$

for all $\psi \in \text{Dom}(\bar{\partial}_b)$.

For all $\phi \in \text{Dom}(\bar{\partial}_b^*)$, we let $\bar{\partial}_b^* \phi$ be the unique form in $L^2_b(M)$ satisfying

$$\langle \bar{\partial}_b^* \phi, \psi \rangle = \langle \phi, \bar{\partial}_b \psi \rangle,$$

for all $\psi \in \text{Dom}(\bar{\partial}_b)$.

The tangent space to $M$ in the neighborhood $U$, $T(U)$ is spanned by some $(1,0)$ vector $L$, its conjugate $\overline{L}$, and one more vector $T$ taken to be purely imaginary, i.e. $\overline{T} = -T$. Since we only consider orientable CR manifolds, the Levi form can be defined globally. We let $\gamma$ be a purely imaginary global 1-form on $M$ which annihilates $T^{1,0}(M) \ominus T^{0,1}(M)$. $\gamma$ is not unique, so we normalize by choosing it in such a way that $\langle \gamma, T \rangle = -1$. Note this implies $\gamma$ is nowhere vanishing.

**Definition 2.4.** The Levi form at a point $z \in M$ is the Hermitian form given by $\langle d\gamma_z, L \wedge \overline{L} \rangle$, where $L$ is a vector field in $T_z^{1,0}(U)$, $U$ a neighborhood of $z$ in $M$. In dimension 3, the Levi form has only one coefficient, so we call $M$ weakly pseudo-convex if there exists a form $\gamma$ such that this coefficient is non-negative at all $z \in M$ and strongly pseudo-convex if there exists a form $\gamma$ such that it is positive at all $z \in M$.

If $L$ is the vector field that spans $T^{1,0}(U)$ for some neighborhood $U$ of $M$, let $\omega$ be the $(1,0)$ form dual to it. In this setup, $\bar{\partial}_b$ is given by the following: If $u$ is a smooth function on $U$, then

$$\bar{\partial}_b(u) = \overline{L}(u) \overline{\omega}.$$

If $\varphi = v \overline{\omega}$ is a $(0,1)$ form on $U$ automatically $\bar{\partial}_b \varphi = 0$. As for the $L^2$ adjoint, $\bar{\partial}_b^*$,

$$\bar{\partial}_b^* \varphi = \overline{L}^*(v) = -L(v),$$

where $\overline{L}^*$ is the $L^2$ adjoint of $\overline{L}$.

We will work with the same inner product on $L^2_b(M)$ as in $\mathbb{N}$, which will be the sum of three inner products: the inner product without any weight defined above, which we will denote from now on by $(\cdot, \cdot)_0$; the inner product with weight $e^{-\lambda}$, $(\cdot, \cdot)_t = (e^{-t\lambda} \cdot, \cdot)_0$; and the inner product with weight $e^{\lambda}$, $(\cdot, \cdot)_{-t} = (e^{t\lambda} \cdot, \cdot)_0$. Notice that each of these three inner products determines an $L^2$ adjoint, so we denote by $\bar{\partial}_b^{*,+}$ the $L^2$ adjoint on $(\cdot, \cdot)_t$, by $\bar{\partial}_b^{*,-}$ the $L^2$ adjoint on $(\cdot, \cdot)_{-t}$, and by $\bar{\partial}_b^{*,0}$ the $L^2$ adjoint on $(\cdot, \cdot)_0$, although since there is no weight function in this case, $\bar{\partial}_b^{*,0}$ equals precisely $\bar{\partial}_b^*$. Now let us compute $\bar{\partial}_b^{*,+}$ and $\bar{\partial}_b^{*,-}$.

Let $u$ be a smooth function and $\varphi = v \overline{\omega}$ a smooth $(0,1)$ form, then

$$\langle \bar{\partial}_b u, \varphi \rangle_t = \langle \bar{\partial}_b u, e^{-t\lambda} \varphi \rangle_0 = \langle u, \bar{\partial}_b^* (e^{-t\lambda} \varphi) \rangle_0 = (u, e^{-t\lambda} (\bar{\partial}_b^* - t[\bar{\partial}_b^* \lambda]) \varphi)_0 = (u, (\bar{\partial}_b^* - t[\bar{\partial}_b^* \lambda]) \varphi)_t.$$
Therefore, $\tilde{\partial}_b^* \varphi = (\tilde{\partial}_b^* - t[\tilde{\partial}_b^*, \lambda])\varphi = \overline{\mathcal{T}}^{*,t}(\varphi) = -L(v) + tL(\lambda)v$, where $\overline{\mathcal{T}}^{*,t}$ is the $L^2$ adjoint of $\overline{\mathcal{T}}$ with respect to $\langle \cdot, \cdot \rangle_t$. Similarly,

$$(\tilde{\partial}_b u, \varphi)_{-t} = (\tilde{\partial}_b u, e^{tA}\varphi)_0 = (u, \tilde{\partial}_b^*(e^{tA}\varphi))_0 = (u, e^{tA}(\tilde{\partial}_b^* + t[\tilde{\partial}_b^*, \lambda])\varphi)_0 = (u, (\tilde{\partial}_b^* + t[\tilde{\partial}_b^*, \lambda])\varphi)_{-t}.$$ 

So then $\tilde{\partial}_b^* \varphi = (\tilde{\partial}_b^* + t[\tilde{\partial}_b^*, \lambda])\varphi = \overline{\mathcal{T}}^{*-t}(v) = -L(v) - tL(\lambda)v$, where $\overline{\mathcal{T}}^{*-t}$ is the $L^2$ adjoint of $\overline{\mathcal{T}}$ with respect to $\langle \cdot, \cdot \rangle_{-t}$. Just as in $\mathbb{N}$, $\lambda$ is chosen to be CR plurisubharmonic, which is defined as follows:

**Definition 2.5.** Let $M$ be a CR manifold. A $C^\infty$ real-valued function $\lambda$ defined in the neighborhood of $M$ is called strongly CR plurisubharmonic if $\exists A_0 > 0$ such that $\langle \frac{1}{2} (\tilde{\partial}_b\tilde{\partial}_b\lambda - \tilde{\partial}_b\tilde{\partial}_b\lambda) + A_0 d\gamma, L \wedge \overline{\mathcal{T}} \rangle$ is strictly positive $\forall L \in T^{1,0}(M)$, where $\langle d\gamma, L \wedge \overline{\mathcal{T}} \rangle$ is the invariant expression of the Levi form. $\lambda$ is called weakly CR plurisubharmonic if $\langle \frac{1}{2} (\tilde{\partial}_b\tilde{\partial}_b\lambda - \tilde{\partial}_b\tilde{\partial}_b\lambda) + A_0 d\gamma, L \wedge \overline{\mathcal{T}} \rangle$ is just non-negative.

**Remark.** Note that the previous definition is trivially satisfied for any $\lambda$ if $M$ is strongly pseudoconvex. This reflects the fact that in such a case the microlocal argument can be carried out in absence of a weight function.

The following properties of CR plurisubharmonic functions stated here for 3-dimensional CR manifolds were proven for $(2n - 1)$-dimensional CR manifolds in $\mathbb{N}$ and are relevant for the upcoming argument:

**Proposition 2.6.** Let $M$ be a compact, smooth, orientable, weakly pseudoconvex CR manifold of real dimension 3 embedded in a complex space $\mathbb{C}^N$ and endowed with an induced CR structure.

(i) Around each point $P \in M$, there exists a small enough neighborhood $U$ and a local orthonormal basis $L, L', \overline{\mathcal{T}}, \overline{\mathcal{T}}'$ of the 2 dimensional complex bundle containing $TM$ when restricted to $U$, satisfying $[L, \overline{\mathcal{T}}]|_P = cT$, where $T = L' - \overline{\mathcal{T}}$ and $c$ is the coefficient of the Levi form in the local basis $L, \overline{\mathcal{T}}, T$ of $TM$.

(ii) If $\lambda$ defined on $M$ is strongly CR plurisubharmonic, then $\exists A_0 > 0$ such that $\langle \frac{1}{2} (\overline{\mathcal{T}}\lambda + LL(\lambda)) + A_0 c \rangle$ is positive in a neighborhood $U'$ around $P$ which is smaller than $U$. $A_0$ is of course independent of $P$ or $U$, and the size of $U'$ depends on it. If $\lambda$ is weakly CR plurisubharmonic, then $\frac{1}{2} (\overline{\mathcal{T}}L(\lambda) + LL(\lambda)) + A_0 c$ is just non-negative.

(iii) If $\lambda$ is strongly plurisubharmonic on $\mathbb{C}^N$, then $\lambda$ is also strongly CR plurisubharmonic on $M$.

(iv) Let

$$\overline{\mathcal{T}}^{*,t} = -L \pm tL(\lambda).$$

For each small positive number $\epsilon_G$, there exists a covering $\{V_\mu\}_\mu$, a local basis $L, \overline{\mathcal{T}}, T$ of $TV_\mu$ for each $\mu$, and $C^\infty$ functions $a, b, g, \epsilon$ so that the bracket $[L, \overline{\mathcal{T}}]$ has the following form:

$$[L, \overline{\mathcal{T}}] = cT + a\overline{\mathcal{T}} + b\overline{\mathcal{T}}^{*,t} \pm t \epsilon,$$

(2.1)
where $|e|$ is bounded independently of $t$, namely

$$|e| \leq \epsilon_G.$$  

Just as in [N], we will use microlocalization to prove the main estimates, so the following definitions are necessary:

**Definition 2.7.** Let $P$ be a pseudodifferential operator of order zero, then another pseudodifferential operator of order zero, $\tilde{P}$ is said to dominate $P$, if the symbol of $\tilde{P}$ is identically equal to 1 on a neighborhood of the support of the symbol of $P$ and the support of the symbol of $\tilde{P}$ is slightly larger than the support of the symbol of $P$.

In particular, the previous definition also applies to cutoff functions, all of which are pseudodifferential operators of order zero.

**Definition 2.8.** Let $P(t)$ be a family of pseudodifferential operators depending upon a parameter $t$. Such a family is called zero order $t$ dependent if it can be written as $P_1 + tP_2$, where $P_1$ and $P_2$ are pseudodifferential operators independent of $t$ and $P_2$ has order zero.

**Definition 2.9.** Let $P(t)$ be a family of pseudodifferential operators depending upon a parameter $t$ such that $P(t)$ has order zero. Such a family $P(t)$ is called inverse zero order $t$ dependent if its symbol $\sigma(P)$ satisfies $D_\xi \sigma(P) = D_\xi P(x, \xi) = \frac{1}{\epsilon_0} q(x, \xi)$ for $|\alpha| \geq 0$, where $q(x, \xi)$ is bounded independently of $t$.

Notice that the two adjoint operators $\bar{\partial}_b^{\ast, +}$ and $\bar{\partial}_b^{\ast, -}$ are zero order $t$ dependent, according to the definition given above.

The microlocalization consists in dividing the Fourier transform space into three conveniently chosen regions, two truncated cones $C^+$ and $C^-$ and another region $C^0$, with some overlap. Let the coordinates on the Fourier transform space be $\xi = (\xi_1, \xi_2, \xi_3)$. Write $\xi' = (\xi_1, \xi_2)$, so then $\xi = (\xi', \xi_3)$. The work is done in coordinate patches on $M$, each of which has defined on it local coordinates such that $\xi'$ is dual to the holomorphic part of the tangent bundle $T^{1,0}(M) \oplus T^{0,1}(M)$ and $\xi_3$ is dual to the totally real part of the tangent bundle of $M$ spanned by the "bad direction," $T$. Duality in this context merely means that the pairing between the specified cotangent and tangent sub-bundles is nondegenerate at each point of the coordinate patch. Define $C^+ = \{\xi | \xi_3 \geq \frac{1}{2}|\xi'| and |\xi| \geq 1\}$. Then $C^- = \{\xi | -\xi \in C^+\}$, and finally $C^0 = \{\xi | -\frac{3}{4}|\xi'| \leq \xi_3 \leq \frac{3}{4}|\xi'|\} \cup \{\xi | |\xi| \leq 1\}$. Notice that by definition, $C^+$ and $C^0$ overlap on two smaller cones and part of the sphere of radius 1 and similarly $C^-$ and $C^0$, whereas $C^+$ and $C^-$ do not intersect.

Let us now define three functions on $\{|\xi'|^2 + |\xi_3|^2 = 1\}$, which is the unit sphere in $\xi$ space. $\psi^+$, $\psi^-$, and $\psi^0$ are smooth, take values in $[0, 1]$, and satisfy the condition of symbols of pseudodifferential operators of order zero. Moreover, $\psi^+$ is supported in $\{\xi | \xi_3 \geq \frac{1}{2}|\xi'|\}$ and $\psi^+ \equiv 1$ on the subset $\{\xi | \xi_3 \geq \frac{3}{4}|\xi'|\}$. Then let $\psi^-(x, \xi) = \psi^+(x, -\xi)$ which means that $\psi^-$ is supported in $\{\xi | \xi_3 \leq -\frac{1}{2}|\xi'|\}$ and $\psi^- \equiv 1$ on the subset $\{\xi | \xi_3 \leq -\frac{3}{4}|\xi'|\}$. Finally, let $\psi^0(\xi)$
satisfy 
\[(\psi^0(\xi))^2 = 1 - (\psi^+(\xi))^2 - (\psi^-(\xi))^2\]
which means that \(\psi^0\) is supported in \(C^0\) and \(\psi^0 \equiv 1\) on the subset \(\{\xi | -\frac{1}{2} |\xi'| \leq \xi_3 \leq \frac{1}{2} |\xi'|\}\). Next extend \(\psi^+, \psi^-,\) and \(\psi^0\) homogeneously by setting 
\(\psi^+(\xi) = \psi^+(\frac{\xi}{|\xi|})\), for \(\xi \in C^+\) outside of the unit sphere. Similarly, let \(\psi^-(\xi) = \psi^-(\frac{\xi}{|\xi|})\), for \(\xi \in C^-\) outside of the unit sphere. Extend \(\psi^+, \psi^-,\) and \(\psi^0\) inside the unit sphere in some smooth way so that \((\psi^+)^2 + (\psi^-)^2 + (\psi^0)^2 = 1\) still holds. Now we have the functions \(\psi^+, \psi^-\), and \(\psi^0\) defined everywhere on \(\xi\) space. We then define \(\tilde{\psi}^+(\xi) = \psi^+(\frac{\xi}{|\xi|})\), \(\tilde{\psi}^0(\xi) = \psi^0(\frac{\xi}{|\xi|})\), and \(\tilde{\psi}^-(\xi) = \psi^-(\frac{\xi}{|\xi|})\) for some positive constant \(A\) to be chosen later. Let \(\tilde{\Psi}^+, \tilde{\Psi}^0\), and \(\tilde{\Psi}^-\) be pseudodifferential operators of order zero with symbols \(\tilde{\psi}^+, \tilde{\psi}^-,\) and \(\tilde{\psi}^0\) respectively. By construction, the following is true:
\[(\tilde{\Psi}^+_t)^*\tilde{\Psi}^+_t + (\tilde{\Psi}^0_t)^*\tilde{\Psi}^0_t + (\tilde{\Psi}^-_t)^*\tilde{\Psi}^-_t = Id,\]
modulo a smoothing operator. The global norm is defined using a very special covering of \(M\), \(\{U_\nu\}_\nu\), satisfying a number of technical conditions proven in [N] and summarized in the next lemma:

**Lemma 2.10.** Let \(M\) be a compact, smooth, orientable, weakly pseudoconvex CR manifold of real dimension 3 embedded in a complex space \(\mathbb{C}^N\) and endowed with an induced CR structure. Let \(\tilde{\Psi}^+_t\), \(\tilde{\Psi}^0_t\), and \(\tilde{\Psi}^-_t\) be the pseudodifferential operators of order zero defined on \(U_\nu\) for each \(\nu\) and \(C^+\), \(C^-\), and \(C^n\) be the three regions of the \(\xi\) space dual to \(U_\nu\) on which the symbol of each of those three pseudodifferential operators is supported. Moreover, define \(\tilde{\Psi}^+_t\) and \(\tilde{\Psi}^-_t\) so that they dominate \(\tilde{\Psi}^+_t\) and \(\tilde{\Psi}^-_t\) respectively and are also inverse zero order \(t\) dependent. We denote by \(\tilde{\mathcal{C}}^+\) and \(\tilde{\mathcal{C}}^-\) the supports of the symbols of \(\tilde{\Psi}^+_t\) and \(\tilde{\Psi}^-_t\) respectively.

(i) If \(\{V_\mu\}_\mu\) is the covering of \(M\) given by part (iv) of Proposition 2.6, then \(\{U_\nu\}_\nu\) is such that for each \(\nu, \exists \mu(\nu)\) with the property that \(U_\nu\) and all \(U_\eta\) satisfying \(U_\nu \cap U_\eta \neq \emptyset\) are contained in the neighborhood \(V_{\mu(\nu)}\) of \(\{V_\mu\}_\mu\);

(ii) Let \(U_0\) be two neighborhoods such that \(U_\nu \cap U_\mu \neq \emptyset\). There exists a diffeomorphism \(\vartheta\) between \(U_\nu\) and \(U_\mu\) with Jacobian \(\mathcal{J}_\vartheta\) satisfying \(\mathcal{J}_\vartheta(C^+_{\mu}) \cap C^-_{\nu} = \emptyset\) and \(C^+_{\nu} \cap \mathcal{J}_\vartheta(C^-_{\mu}) = \emptyset\), where \(\mathcal{J}_\vartheta\) is the inverse of the transpose of the Jacobian of \(\vartheta\);

(iii) Let \(\vartheta \Psi^+_t, \vartheta \Psi^0_t, \vartheta \Psi^-_t\) be the transfers of \(\tilde{\Psi}^+_t, \tilde{\Psi}^0_t, \tilde{\Psi}^-_t\) respectively via \(\vartheta\), then on \(\{\xi | \xi_{2n-1} \geq \frac{1}{3} |\xi'| \text{and} |\xi| \geq (1+\epsilon) \tau A\}\), the principal symbol of \(\vartheta \Psi^+_t\) is identically equal to 1, on \(\{\xi | \xi_{2n-1} \leq -\frac{1}{3} |\xi'| \text{and} |\xi| \geq (1+\epsilon) \tau A\}\), the principal symbol of \(\vartheta \Psi^0_t\) is identically equal to 1, and on \(\{\xi | -\frac{1}{3} |\xi'| \leq \xi_{2n-1} \leq \frac{1}{3} |\xi'| \text{and} |\xi| \geq (1+\epsilon) \tau A\}\), the principal symbol of \(\vartheta \Psi^0_t\) is identically equal to 1, where \(\epsilon > 0\) and can be very small;

(iv) \(\mathcal{J}_\vartheta(\tilde{\mathcal{C}}^+) \cap \tilde{\mathcal{C}}^- = \emptyset\) and \(\mathcal{J}_\vartheta(\tilde{\mathcal{C}}^-) \cap \tilde{\mathcal{C}}^+ = \emptyset\);

(v) Let \(\vartheta \tilde{\Psi}^+_t, \vartheta \tilde{\Psi}^-_t\) be the transfers via \(\vartheta\) of \(\tilde{\Psi}^+_t\) and \(\tilde{\Psi}^-_t\) respectively. Then the principal symbol of \(\vartheta \tilde{\Psi}^+_t\) is identically 1 on \(C^+_\mu\) and the principal symbol of \(\vartheta \tilde{\Psi}^-_t\) is identically 1 on \(C^-\mu\);

(vi) \(\tilde{\mathcal{C}}^+ \cap \tilde{\mathcal{C}}^- = \emptyset\).
Henceforth, the left superscript $\vartheta$ indicating the transfer of a pseudodifferential operator into another local coordinate system will be suppressed to simplify our notation. Let now $\{\zeta_\nu\}$, be a partition of unity subordinate to the covering $\{U_\nu\}$ satisfying $\sum_\nu \zeta_\nu^2 = 1$, and for each $\nu$ let $\tilde{\zeta}_\nu$ be a cutoff function that dominates $\zeta_\nu$ such that $\text{supp}(\tilde{\zeta}_\nu) \subset U_\nu$. Then we define the global norm as follows:

$$\langle|\varphi|\rangle_t^2 = \sum_\nu (||\tilde{\zeta}_\nu \Psi_{t,\nu}^+ v^\nu||_t^2 + ||\tilde{\zeta}_\nu \Psi_{t,\nu}^0 v^\nu||_0^2 + ||\tilde{\zeta}_\nu \Psi_{t,\nu}^0 v^\nu||_{-t}^2)$$

where $\varphi = v\overline{\omega}$ is a $(0,1)$ form in $L^2_1(M)$ and $v^\nu$ is the coefficient of the form expressed in the local coordinates on $U_\nu$. We define the norm exactly the same way for functions. The following two facts are proven in [N]:

**Lemma 2.11.** Let $M$ be a compact, smooth, orientable, weakly pseudoconvex CR manifold of real dimension 3 embedded in a complex space $\mathbb{C}^N$ and endowed with an induced CR structure.

(i) For any $t$, there exist two positive constants depending on $t$, $C_t$ and $C'_t$ such that

$$C_t ||\varphi||_{L^1}^2 \leq \langle |\varphi| \rangle_t^2 \leq C'_t ||\varphi||_{L^1}^2,$$

where $\varphi$ is a form in $L^1_2(M)$;

(ii) There exists a self-adjoint operator $G_t$ such that

$$(\varphi, \phi)_0 = \langle |\varphi|, G_t \phi \rangle_t,$$

for any two forms $\varphi$ and $\phi$ in $L^1_2(M)$. $G_t$ is the inverse of the operator $F_t$ given by

$$\sum_\nu \left( \zeta_\nu (\Psi_{t,\nu}^+) \ast \tilde{\zeta}_\nu e^{-t\lambda} \tilde{\zeta}_\nu \Psi_{t,\nu}^+ \zeta_\nu + \zeta_\nu (\Psi_{t,\nu}^0) \ast \tilde{\zeta}_\nu e^{-t\lambda} \tilde{\zeta}_\nu \Psi_{t,\nu}^0 \zeta_\nu + \zeta_\nu (\Psi_{t,\nu}^-) \ast \tilde{\zeta}_\nu e^{t\lambda} \tilde{\zeta}_\nu \Psi_{t,\nu}^- \zeta_\nu \right).$$

Finally, here is the definition of the Sobolev norm for forms in this context. Clearly, we make the same definition for functions.

**Definition 2.12.** Let the Sobolev norm of order $s$ for a form $\varphi = v\overline{\omega}$ supported on $M$ be given by:

$$||\varphi||_s^2 = \sum_\eta ||\tilde{\zeta}_\eta \Lambda^s \zeta_\eta v^\eta||_0^2,$$

where as usual $\Lambda$ is defined to be the pseudodifferential operator with symbol $(1 + |\xi|^2)^{\frac{s}{2}}$. Then

$$H^s = \{\varphi \in B^1(M) \mid ||\varphi||_s < +\infty\}.$$

A few more definitions are needed to lay out the terminology used in the statement of Theorem 1.1.

**Definition 2.13.** An operator $P$ has closed range if $\forall \alpha \in \overline{\mathcal{R}(P)}$, where $\overline{\mathcal{R}(P)}$ is the closure of the range of $P$, $\alpha \in \mathcal{R}(P)$.
In particular, for $\bar{\partial}_b$ to have closed range in $L^2(M)$ means that if we denote by $L^1_2(M)$ the set of $(0,1)$ forms in $L^2$ of $M$, the set of closed, $(0,1)$ forms in $L^2$ decomposes as follows:

$$L^1_2(M) \cap \mathcal{N}(\bar{\partial}_b) = \mathcal{R}(\bar{\partial}_b) \oplus \mathcal{H}_t,$$

where for each $t$ the corresponding harmonic space is

$$\mathcal{H}_t = \{ \varphi \in \text{Dom}(\bar{\partial}_b) \cap \text{Dom}(\bar{\partial}^*_{b,t}) \mid \bar{\partial}_b \varphi = 0 \text{ and } \bar{\partial}_{b,t} \varphi = 0 \}.$$

In other words, given some $\bar{\partial}_b$-closed $(0,1)$ form $\alpha$ that is in $L^2$ of $M$ and such that $\alpha \perp \mathcal{H}_t$, where orthogonality is defined with respect to the $\langle | \cdot | \rangle_t$ norm, there exists some function $u_t$ in $L^2$ of $M$ such that

$$\bar{\partial}_b u_t = \alpha.$$

Note that we index the function by $t$ since the norm depends on $t$, hence such a function exists for each $t$, and functions corresponding to different values of $t$ are not necessarily equal.

The closed range property is equivalent to three other properties because $\bar{\partial}_b$ is a linear, closed, densely defined operator on a Hilbert space. See for example [H]:

**Theorem 2.14.** The following four conditions are equivalent:

(i) $\mathcal{R}(\bar{\partial}_b)$ is closed in $H^s$;

(ii) There exists a $t$-dependent constant $C_t$ such that

$$\langle | \Lambda^s u_t | \rangle_t \leq C_t \langle | \Lambda^s \alpha | \rangle_t,$$

where $u_t \in \text{Dom}(\bar{\partial}_b) \cap \overline{\mathcal{R}(\bar{\partial}_{b,t})} \subset \text{Dom}(\bar{\partial}_b) \cap \perp \mathcal{N}(\bar{\partial}_b) \text{ and } \bar{\partial}_b u_t = \alpha$;

(iii) $\mathcal{R}(\bar{\partial}^*_{b,t})$ is closed in $H^s$;

(iv) There exists a $t$-dependent constant $C_t$ such that

$$\langle | \Lambda^s \varphi_t | \rangle_t \leq C_t \langle | \Lambda^s u_t | \rangle_t,$$

where $\varphi_t \in \text{Dom}(\bar{\partial}^*_{b,t}) \cap \overline{\mathcal{R}(\bar{\partial}_b)} \subset \text{Dom}(\bar{\partial}^*_{b,t}) \cap \perp \mathcal{N}(\bar{\partial}^*_{b,t}) \text{ and } \bar{\partial}_{b,t} \varphi_t = u_t$.

The best constants in (2.2) and in (2.3) are the same.

In particular, it follows that the $\bar{\partial}^*_{b,t}$ problem can be solved, i.e. for every function $u_t \perp \mathcal{H}_t$, there exists a $(0,1)$ form $\varphi_t$ in $L^2$, $\varphi_t \perp \mathcal{N}(\bar{\partial}^*_{b,t})$, such that

$$\bar{\partial}_{b,t} \varphi_t = u_t.$$

On $\perp \mathcal{H}_t$ we define the Kohn Laplacian

$$\Box_{b,t} = \bar{\partial}_b \bar{\partial}^*_{b,t} + \bar{\partial}^*_{b,t} \bar{\partial}_b.$$

$\Box_{b,t}$ is self-adjoint, and on $(0,1)$ forms on a manifold $M$ of dimension $3$, $\Box_{b,t} = \bar{\partial}_b \bar{\partial}^*_{b,t}$. Moreover, by the considerations above, for every $(0,1)$ form $\alpha \perp \mathcal{H}_t$, there exists a $(0,1)$ form $\varphi_t \perp \mathcal{H}_t$ such that

$$\Box_{b,t} \varphi_t = \alpha.$$
Let $N_t$ be the solution operator which takes $\alpha$ to $\varphi_t$. Define $N_t$ to be identically zero on $\mathcal{H}_t$. Equations 2.2 and 2.3 together imply that
\[ \langle \psi \rangle_t = \langle \varphi_t \rangle_t \leq C_t' \langle \alpha \rangle_t, \]
for some $t$-dependent constant $C_t'$, i.e. the solution operator $N_t$ is bounded on $L^2$. Furthermore, notice that $u_t = \bar{\partial}_{b,t} N_t \alpha$ solves the $\bar{\partial}_b$ problem
\[ \bar{\partial}_b u_t = \alpha. \]
Let us now relate $\perp \mathcal{N}(\bar{\partial}^*_{b,t})$ to $\perp \mathcal{N}(\bar{\partial}^*_{b})$ in order to simplify the hypotheses of our main result, Theorems 1.1.

Lemma 2.15. Let $\perp$ denote perpendicularity with respect to the unweighted norm $\| \cdot \|_0$, and let $\perp_t$ denote perpendicularity with respect to the weighted norm $\langle \cdot \rangle_t$. $\alpha \perp \mathcal{N}(\bar{\partial}^*_{b,t})$ implies $\alpha \perp_t N(\bar{\partial}^*_{b,t})$ for each $t$ and each form or function $\alpha$.

Proof: For any $\psi \in \mathcal{N}(\bar{\partial}^*_{b,t})$,
\[ \langle \alpha, \psi \rangle_t = \| \alpha \|_0. \]

Showing that $F_t \psi \in \mathcal{N}(\bar{\partial}^*_{b})$ would conclude the proof of the lemma. By Equation 4.7 of [N], $\bar{\partial}^*_{b,t}$ and $\bar{\partial}^*_{b}$ are related via the operators $F_t$ and $G_t$ from Lemma 2.11 as follows:
\[ \bar{\partial}^*_{b,t} = \bar{\partial}^*_{b} + G_t [\bar{\partial}^*_{b}, F_t]. \]

This means
\[ \bar{\partial}^*_{b,t} \psi = 0 = \bar{\partial}^*_{b} \psi + G_t \bar{\partial}^*_{b} F_t \psi - \bar{\partial}^*_{b} \psi, \]
i.e. $\bar{\partial}^*_{b} F_t \psi = 0$ a.e. because the operators $F_t$ and $G_t$ give the correspondence between two equivalent norms, $\| \cdot \|_0$ and $\langle \cdot \rangle_t$, so their kernels consist only of those functions and forms that are zero almost everywhere. \(\square\)

Definition 2.16. Let $\mathcal{S}(M) = \{ f \in \text{Dom}(\bar{\partial}_b) \mid \bar{\partial}_b f = 0 \}$. Then the weighted Szegő projection $S_{b,t}$ is the operator that projects $L^2(M)$ to $\mathcal{S}(M)$ in the $\langle \cdot \rangle_t$ norm.

Just as in [K4], the weighted Szegő projection can be expressed as follows:

Lemma 2.17. Let $M$ be a compact, orientable, weakly pseudoconvex CR manifold of dimension 3 embedded in a complex space $\mathbb{C}^N$ and endowed with the induced CR structure. If the range of $\bar{\partial}_b$ is closed in $L^2(M)$, then
\[ S_{b,t} = \text{Id} - \bar{\partial}^*_{b,t} N_t \bar{\partial}_b, \]
where $\text{Id}$ is the identity operator.

Proof: We consider the two complementary cases, $f \in \mathcal{S}(M)$ and $f \perp \mathcal{S}(M)$, and prove this expression for each. First, if $f \in \mathcal{S}(M)$, then $(\text{Id} - \bar{\partial}^*_{b,t} N_t \bar{\partial}_b) f = f$ as expected. If $f \perp \mathcal{S}(M)$, then $\bar{\partial}^*_{b,t} N_t \bar{\partial}_b f = f$ since the ranges of $\bar{\partial}_b$ and $\bar{\partial}^*_{b,t}$ are closed. Thus, $S_{b,t} f = 0$. \(\square\)

We want to investigate the regularity of the operator $S_{b,t}$. 
3. Regularity of the Weighted Szegö Projection

We start by recalling two results from [N], which we state in the notation appropriate for dimension three. These results are both local, namely we restrict the discussion to a small neighborhood $U$ in which all the necessary properties hold, namely part (iv) of Proposition 2.6.

Let $\zeta$ be a cutoff function with support in $U$ and $\tilde{\zeta}$ a cutoff function which dominates it, whose support is contained in a slightly larger open set $U'$. $U'$ is small enough to allow the existence of the three pseudodifferential operators of order zero $\Psi^+_t$, $\Psi^0_t$, and $\Psi^-_t$ supported in $C^+_t$, $C^0_t$, and $C^-_t$ respectively, as detailed in the previous section. Moreover, let $\tilde{\Psi}^+_t$, $\tilde{\Psi}^0_t$, and $\tilde{\Psi}^-_t$ be pseudodifferential operators of order zero whose symbols dominate those of $\Psi^+_t$, $\Psi^0_t$, and $\Psi^-_t$ and are supported in $\tilde{C}^+_t$, $\tilde{C}^0_t$, and $\tilde{C}^-_t$ respectively, which are slightly larger than $C^+_t$, $C^0_t$, and $C^-_t$. We shall work with the energy form

$$Q_{b,t}(\phi, \phi) = \langle |\bar{\partial}_b \phi|^2 \rangle_t + \langle |\bar{\partial}_{b,t} \phi|^2 \rangle_t$$

and its equivalents for the $t$ norm and the $-t$ norm,

$$Q^l_{b,+t}(\phi, \phi) = ||\bar{\partial}_b \phi||^2_t + ||\bar{\partial}^{+}_t \phi||^2_t,$$

and

$$Q^l_{b,-t}(\phi, \phi) = ||\bar{\partial}_b \phi||^2_{-t} + ||\bar{\partial}^{-}_t \phi||^2_{-t}.$$

**Lemma 3.1.** Let $\varphi \in \text{Dom}(\bar{\partial}_b) \cap \text{Dom}(\bar{\partial}^{+}_b)$ be a $(0,1)$ form supported in $U'$, a neighborhood of a compact, three-dimensional, weakly-pseudoconvex CR-manifold $M$. Let there exist a basis $L, \overline{L}, T$ of $T(U)$ such that $[L, \overline{L}] = cT + aL + bL^{*,t} + te$, where $a, b,$ and $e$ are $C^\infty$ functions independent of $t$ and $c$ is the coefficient of the Levi form. Let $M$ be also endowed with a strongly CR plurisubharmonic function $\lambda$. Then there exists a constant $C$ independent of $t$, a $t$-dependent constant $C_t$ and a positive number $T_0$ such that for any $t \geq T_0$

$$Q^l_{b,+t}(\tilde{\zeta} \Psi^+_t \varphi, \tilde{\zeta} \Psi^+_t \varphi) + C_t ||\tilde{\zeta} \Psi^0_t \varphi||^2_0 \geq C t ||\tilde{\zeta} \Psi^+_t \varphi||^2_t$$

**Lemma 3.2.** Let $u$ be a function supported in $U'$ such that up to a smooth term its Fourier transform $\hat{u}$ is supported in $\tilde{C}^-$, and let $g$ be a non-negative function, then the following holds:

$$\Re\{(g(-T)u, u)_{-t}\} \geq t A (g u, u)_{-t} + O(||u||^2_{-t}) + O_t(||\tilde{\zeta} \tilde{\Psi}^0_t u||^2_0)$$

The crucial difference between the three-dimensional CR manifold case and the case of manifolds of dimension five and above is in the handling of the microlocalization on $C^-$, so we shall now obtain an estimate for $Q^l_{b,-t}(\cdot, \cdot)$ for functions:

**Lemma 3.3.** Let $u \in \text{Dom}(\bar{\partial}_b) \cap \text{Dom}(\bar{\partial}^{+}_b)$ be a function supported in $U'$, a neighborhood of a compact, three-dimensional, weakly-pseudoconvex CR-manifold $M$. Let there exist a basis $L, \overline{L}, T$ of $T(U)$ such that $[L, \overline{L}] = cT + aL + bL^{*,t} - te$, where $a, b,$ and $e$ are $C^\infty$ functions independent of $t$ and $c$ is the coefficient of the Levi form. Let $M$ be also endowed with a
strongly CR plurisubharmonic function $\lambda$. Then there exists a constant $C'$ independent of $t$, a $t$-dependent constant $C'_t$ and a positive number $T'_0$ such that for any $t \geq T'_0$

$$Q^t_{b,-}(\tilde{\Psi}^- u, \tilde{\Psi}^- u) + C'_t |||\tilde{\Psi}_t^0 u||^2_0 \geq C' t ||\tilde{\Psi}_t^- u||^2_{-t}$$

**Proof:** Define the pseudodifferential operators $\Psi^+_t$, $\Psi^0_t$, and $\tilde{\Psi}^+_t$ using a constant $A \geq A_0$, where $A_0$ is the CR plurisubharmonicity constant of $\lambda$.

$$Q^t_{b,-}(u, u) = ||\tilde{\Psi}_t u||^2_{-t} = (\tilde{\Psi}_t^0 u, \tilde{\Psi}_t u)_{-t} = (|\tilde{\Psi}_t|^{s} - L_t |\tilde{\Psi}_t^0 u, \tilde{\Psi}_t u)_{-t} = (|\tilde{\Psi}_t|^{s} - L_t ||u||^2_{-t} + ||\tilde{\Psi}_t||^2_{-t} + ||L_t||^{s} - L_t ||u||^2_{-t} - \Re\{\langle cT u, u \rangle_{-t} \}

+ O(||u||^2_{-t}) + t \Re\{\langle L \tilde{\Psi}(\lambda) u, u \rangle_{-t} + t \Re\{\langle c u, u \rangle_{-t} \}

for some $1 \gg \epsilon > 0$. Now, replace $u$ by $\tilde{\Psi}_t^- u$ in the previous expression, which has the property that its Fourier transform is supported in $C^-$ up to a smooth error term. It is now possible to apply the previous lemma to $-\Re\{\langle cT u, u \rangle_{-t} \}$ to conclude that

$$Q^t_{b,-}(\tilde{\Psi}_t^- u, \tilde{\Psi}_t^- u) \geq (1 - \epsilon) |||\tilde{\Psi}_t||^2_{-t} + t A_0(c \tilde{\Psi}_t^- u, \tilde{\Psi}_t^- u)_{-t} + O(|||\tilde{\Psi}_t^- u||^2_{-t})$$

$$+ O_t(|||\tilde{\Psi}_t^- ||^2_{-t}) + t \Re\{\langle L \tilde{\Psi}(\lambda) \tilde{\Psi}_t^- u, \tilde{\Psi}_t^- u \rangle_{-t} \}

+ t \Re\{\langle c \tilde{\Psi}_t^- u, \tilde{\Psi}_t^- u \rangle_{-t} \}.$$ Since the inner product is Hermitian, it is easily seen that

$$\Re\{\langle L \tilde{\Psi}(\lambda) \tilde{\Psi}_t^- u, \tilde{\Psi}_t^- u \rangle_{-t} \} = \Re\{\frac{1}{2} (\tilde{\Psi}_t^{+} + LL(\lambda)) \tilde{\Psi}_t^- u, \tilde{\Psi}_t^- u \rangle_{-t} \}.$$ Using this, the fact that $\lambda$ is CR plurisubharmonic, and that $|c| < \epsilon_G \ll 1$, we obtain that

$$Q^t_{b,-}(\tilde{\Psi}_t^- u, \tilde{\Psi}_t^- u) + O(|||\tilde{\Psi}_t^- u||^2_{-t}) + O_t(|||\tilde{\Psi}_t^- ||^2_{-t}) \geq t (C - \epsilon_G) |||\tilde{\Psi}_t^- u||^2_{-t},$$ for some constant $C \geq 1$. Take $T'_0$ to be the smallest value of $t$ for which $O(|||\tilde{\Psi}_t^- u||^2_{-t})$ can be absorbed on the right-hand side. The conclusion of the lemma then follows.

We recall here one more result from [N], which will be instrumental for the estimates on the elliptic part $C^0$ in the microlocalization.

**Lemma 3.4.** Let $\varphi$ be a function or a $(0, 1)$ form supported in $U_\nu$ for some $\nu$ such that up to a smooth term, $\tilde{\varphi}$ is supported in $\tilde{C}^0_\nu$. There exist positive constants $C' > 1$ and $\nu'$ independent of $t$ for which

$$C' Q_{b,t}(\varphi, g_t \varphi) + \nu' |||\varphi||^2_{0} \geq ||\varphi||^2_{t}.$$ Now we are ready to tackle the global case, namely to prove Theorem 1.1.

**Proof of Theorem 1.1.** For any $f \in H^s$, let $\tilde{\partial}_b f = \alpha$. As we have shown in the previous section, $\exists u_t \in \text{Dom}(\partial_b) \cap \tilde{\mathcal{N}}(\tilde{\partial}_b)$ such that

$$\tilde{\partial}_b u_t = \alpha = \tilde{\partial}_b f.$$ It follows that $S_{b,t} f = f - u_t$. The proof of the theorem proceeds in two steps as follows:
(1) For each $s > 0$, we show that there exists some $t$-dependent constant $C_t$ such that

$$||u_t||_s \leq C_t ||f||_{s+1};$$

(2) Using the estimate in Step 1, we construct the smooth solution for $\bar{\partial}_b$.

Remark: We believe that $||u_t||_s \leq C_t ||f||_s$ should hold, in other words, that the weighted Szegö projection maps $H^s$ to $H^s$, but this exact regularity statement cannot be obtained from the proof given here.

Step 1: It is sufficient to prove that there exists a positive $C_t$ such that

$$\langle|\Lambda^s u_t|\rangle_t \leq C_t \langle|\Lambda^s \alpha|\rangle_t.$$

This estimate is trivially true for $s = 0$, and by induction we assume

$$\langle|\Lambda^{s-1} u_t|\rangle_t \leq C_t \langle|\Lambda^{s-1} \alpha|\rangle_t.$$

Since $\sum_\mu \zeta_\mu^2 = 1$ and $(\Psi^+_{t,\mu})^* \Psi^+_{t,\mu} + (\Psi^0_{t,\mu})^* \Psi^0_{t,\mu} + (\Psi^-_{t,\mu})^* \Psi^-_{t,\mu} = Id$,

$$\langle|\Lambda^s u_t|\rangle_t^2 = \langle|\Lambda^s u_t, \Lambda^s u_t|\rangle_t = \sum_\mu \langle|\Lambda^s \zeta_\mu (\Psi^+_{t,\mu})^* \Psi^+_{t,\mu} \zeta_\mu u_t, \Lambda^s u_t|\rangle_t + \sum_\mu \langle|\Lambda^s \zeta_\mu (\Psi^0_{t,\mu})^* \Psi^0_{t,\mu} \zeta_\mu u_t, \Lambda^s u_t|\rangle_t + \sum_\mu \langle|\Lambda^s \zeta_\mu (\Psi^-_{t,\mu})^* \Psi^-_{t,\mu} \zeta_\mu u_t, \Lambda^s u_t|\rangle_t$$

Set $\sum_\mu \langle|\Lambda^s \zeta_\mu (\Psi^+_{t,\mu})^* \Psi^+_{t,\mu} \zeta_\mu u_t, \Lambda^s u_t|\rangle_t = I$, $\sum_\mu \langle|\Lambda^s \zeta_\mu (\Psi^0_{t,\mu})^* \Psi^0_{t,\mu} \zeta_\mu u_t, \Lambda^s u_t|\rangle_t = II$, and $\sum_\mu \langle|\Lambda^s \zeta_\mu (\Psi^-_{t,\mu})^* \Psi^-_{t,\mu} \zeta_\mu u_t, \Lambda^s u_t|\rangle_t = III$. We will consider each of these terms separately, and then we will put together the results.

From the previous section we know that there exists a $(0,1)$ form $\varphi_t \perp \mathcal{N}(\bar{\partial}^*_{b,t})$ such that $\bar{\partial}^*_{b,t} \varphi_t = u_t$. We want to manipulate the term $I$ in such a way that we expose $\varphi_t$ and apply Lemma 3.1 to it.
for some $1 \gg \epsilon > 0$, a constant $C$ independent of $t$, and a $t$-dependent constant $C_t$. Set

$$\sum_\mu \langle | \Lambda^s \zeta_\mu \Psi^+_{t,\mu} | \rangle_t ^2 = IV.$$ 

By the definition of the norm,

$$IV = \sum_{\nu,\mu} \langle \tilde{\zeta}_\nu \Psi^+_{t,\nu} \zeta_\nu \Lambda^s \zeta_\mu \Psi^+_{t,\mu} \zeta_\mu \varphi^\mu_t \rangle_t + \sum_{\nu,\mu} \langle \tilde{\zeta}_\nu \Psi^0_{t,\nu} \zeta_\nu \Lambda^s \zeta_\mu \Psi^+_{t,\mu} \zeta_\mu \varphi^\mu_t \rangle_t$$

$$+ \sum_{\nu,\mu} \langle \tilde{\zeta}_\nu \Psi^0_{t,\nu} \zeta_\nu \Lambda^s \varphi^\nu_t \rangle_t + O(\langle \varphi^\nu_t \rangle_t^2).$$

By Lemma 3.1,

$$t \sum_{\nu,\mu} \langle \tilde{\zeta}_\nu \Psi^+_{t,\nu} \zeta_\nu \Lambda^s \zeta_\mu \Psi^+_{t,\mu} \zeta_\mu \varphi^\mu_t \rangle_t ^2 \leq C_t \sum_{\nu} \langle \tilde{\zeta}_\nu \Psi^0_{t,\nu} \zeta_\nu \Lambda^s \varphi^\nu_t \rangle_t^2$$

$$+ C \sum_{\nu,\mu} Q^l_{b,+t} \langle \tilde{\zeta}_\nu \Psi^+_{t,\nu} \zeta_\nu \Lambda^s \zeta_\mu \Psi^+_{t,\mu} \zeta_\mu \varphi^\mu_t \rangle_t + \tilde{\zeta}_\nu \Psi^+_{t,\nu} \zeta_\nu \Lambda^s \zeta_\mu \Psi^+_{t,\mu} \zeta_\mu \varphi^\mu_t,$$

for all $t \geq T_0$. $Q^l_{b,+t} \langle \cdot, \cdot \rangle$ is not particularly easy to handle because $\tilde{\partial}_b^+ \varphi$ and $\tilde{\partial}_b^+ \varphi$ do not compare well, so we convert $Q^l_{b,+t} \langle \cdot, \cdot \rangle$ to $Q_{b,t} \langle \cdot, \cdot \rangle$ using Equation 4.2 from [N] which says
that for each \((0,1)\) form \(\beta\), there exist constants \(C, C_t > 0\) such that

\[
\sum_{\nu} Q_{b, +t}^{\nu} (\tilde{\zeta}_{\nu} \Psi_{t, \nu}^+ \zeta_{\nu} \beta^\nu, \tilde{\zeta}_{\nu} \Psi_{t, \nu}^+ \zeta_{\nu} \beta^\nu) + \sum_{\nu} Q_{b, 0,t}^{\nu} (\tilde{\zeta}_{\nu} \Psi_{t, \nu}^0 \zeta_{\nu} \beta^\nu, \tilde{\zeta}_{\nu} \Psi_{t, \nu}^0 \zeta_{\nu} \beta^\nu)
\]

\[
+ \sum_{\nu} Q_{b, -t}^{\nu} (\tilde{\zeta}_{\nu} \Psi_{t, \nu}^- \zeta_{\nu} \beta^\nu, \tilde{\zeta}_{\nu} \Psi_{t, \nu}^- \zeta_{\nu} \beta^\nu)
\]

\[
\leq C Q_{b,t}(\beta, \beta) + C_t \sum_{\nu} ||\tilde{\zeta}_{\nu} \Psi_{t, \nu}^+ \zeta_{\nu} \beta^\nu||_t^2 + O(||\beta||^2) + O_t(||\beta||^2).
\]

This means

\[
t \sum_{\nu, \mu} ||\tilde{\zeta}_{\nu} \Psi_{t, \nu}^+ \zeta_{\nu} \Lambda^s \zeta_{\mu} (\Psi_{t, \mu}^+)^* \Psi_{t, \mu}^+ \zeta_{\mu} \varphi_t^\mu||_t^2 \leq C \sum_{\nu, \mu} ||\tilde{\zeta}_{\nu} \Psi_{t, \nu}^0 \zeta_{\nu} \Lambda^s \varphi_t^\nu||_0^2
\]

\[
+ C \sum_{\nu, \mu} ||\tilde{\zeta}_{\nu} \Psi_{t, \nu}^+ \zeta_{\nu} \Lambda^s \zeta_{\mu} (\Psi_{t, \mu}^+)^* \Psi_{t, \mu}^+ \zeta_{\mu} \varphi_t^\mu||_t^2 + C_t \sum_{\nu, \mu} ||\tilde{\zeta}_{\nu} \Psi_{t, \nu}^- \zeta_{\nu} \Lambda^s \zeta_{\mu} (\Psi_{t, \mu}^+)^* \Psi_{t, \mu}^+ \zeta_{\mu} \varphi_t^\mu||_t^2
\]

\[
+ C \sum_{\nu, \mu} Q_{b,t}(\tilde{\zeta}_{\nu} \Psi_{t, \nu}^+ \zeta_{\nu} \Lambda^s \zeta_{\mu} (\Psi_{t, \mu}^+)^* \Psi_{t, \mu}^+ \zeta_{\mu} \varphi_t^\mu).
\]

We increase \(T_0\) in order to absorb \(C \sum_{\nu, \mu} ||\tilde{\zeta}_{\nu} \Psi_{t, \nu}^+ \zeta_{\nu} \Lambda^s \zeta_{\mu} (\Psi_{t, \mu}^+)^* \Psi_{t, \mu}^+ \zeta_{\mu} \varphi_t^\mu||^2_t\) on the left-hand side and conclude that for all \(t\) larger than this new \(T_0\)

\[
t \sum_{\nu, \mu} ||\tilde{\zeta}_{\nu} \Psi_{t, \nu}^+ \zeta_{\nu} \Lambda^s \zeta_{\mu} (\Psi_{t, \mu}^+)^* \Psi_{t, \mu}^+ \zeta_{\mu} \varphi_t^\mu||^2_t \leq C \sum_{\nu, \mu} ||\tilde{\zeta}_{\nu} \Psi_{t, \nu}^0 \zeta_{\nu} \Lambda^s \varphi_t^\nu||_0^2
\]

\[
+ C_t \sum_{\nu, \mu} ||\tilde{\zeta}_{\nu} \Psi_{t, \nu}^+ \zeta_{\nu} \Lambda^s \zeta_{\mu} (\Psi_{t, \mu}^+)^* \Psi_{t, \mu}^+ \zeta_{\mu} \varphi_t^\mu||_t^2
\]

\[
+ C \sum_{\nu, \mu} Q_{b,t}(\tilde{\zeta}_{\nu} \Psi_{t, \nu}^+ \zeta_{\nu} \Lambda^s \zeta_{\mu} (\Psi_{t, \mu}^+)^* \Psi_{t, \mu}^+ \zeta_{\mu} \varphi_t^\mu).
\]

Next, we commute \(\tilde{\partial}_{b,t}^s\) inside so that it hits \(\varphi\) and gives \(u_t\):

\[
t \sum_{\nu, \mu} ||\tilde{\zeta}_{\nu} \Psi_{t, \nu}^+ \zeta_{\nu} \Lambda^s \zeta_{\mu} (\Psi_{t, \mu}^+)^* \Psi_{t, \mu}^+ \zeta_{\mu} \varphi_t^\mu||^2_t \leq C \sum_{\nu, \mu} \langle \tilde{\partial}_{b,t}^s \tilde{\zeta}_{\nu} \Psi_{t, \nu}^+ \zeta_{\nu} \Lambda^s \zeta_{\mu} (\Psi_{t, \mu}^+)^* \Psi_{t, \mu}^+ \zeta_{\mu} \varphi_t^\mu \rangle_t^2
\]

\[
+ C \sum_{\nu, \mu} \langle \tilde{\partial}_{b,t}^s \tilde{\zeta}_{\nu} \Psi_{t, \nu}^0 \zeta_{\nu} \Lambda^s \varphi_t^\nu \rangle_t^2
\]

\[
+ C \sum_{\nu, \mu} ||\tilde{\zeta}_{\nu} \Psi_{t, \nu}^+ \zeta_{\nu} \Lambda^s \zeta_{\mu} (\Psi_{t, \mu}^+)^* \Psi_{t, \mu}^+ \zeta_{\mu} \varphi_t^\mu||_t^2
\]

Since the first order terms of \(\tilde{\partial}_{b,t}^s\) are independent of \(t\), we unwind the bracket and absorb its \(C^+\) top order terms on the left-hand side by increasing the threshold value of \(t\) to some appropriate
while absorbing the rest of the errors in the error terms already present in the expression:

\[ t \sum_{\nu,\mu} ||\tilde{\psi}_{t,\nu} \Lambda^s \mu (\Psi_{t,\mu}^*) \Psi_{t,\mu}^* \Lambda^s \varphi_t^\mu ||_0^2 \leq C \langle |\Lambda^s u_t| \rangle_t^2 + C_t \sum_{\nu} ||\tilde{\psi}_{t,\nu} \Lambda^s \varphi_t^\mu ||_0^2 \]

\[ + C_t \sum_{\nu,\mu} ||\tilde{\psi}_{t,\nu} \Lambda^s \mu (\Psi_{t,\mu}^*) \Psi_{t,\mu}^* \Lambda^s \varphi_t^\mu ||_0^2 , \]

for all \( t \geq T'_0 \). For \( t \) large, \( \frac{\epsilon}{t} \) is very small. Altogether, this implies there exist some \( 1 \gg \epsilon' > 0 \) and some \( T''_0 \geq T'_0 \) such that for all \( t \geq T''_0 \),

\[ I \leq \epsilon' \langle |\Lambda^s u_t| \rangle_t^2 + \frac{1}{\epsilon' \langle |\Lambda^s \alpha| \rangle_t^2} + C_t \sum_{\mu} \langle |\Lambda^s \mu (\Psi_{t,\mu}^*) \Psi_{t,\mu}^* \Lambda^s \varphi_t^\mu \rangle_t^2 + O(||\varphi_t||^2_0) \]

\[ + C_t \sum_{\nu} ||\tilde{\psi}_{t,\nu} \Lambda^s \varphi_t^\mu ||_0^2. \]

We now look at II:

\[ II = \sum_{\mu} \langle |\Lambda^s \mu (\Psi_{t,\mu}^*) \Psi_{t,\mu}^* \Lambda^s \varphi_t^\mu u_t^\mu, \Lambda^s u_t| \rangle_t \]

\[ \leq \epsilon \langle |\Lambda^s u_t| \rangle_t^2 + \frac{1}{\epsilon} \sum_{\mu} \langle |\Lambda^s \mu (\Psi_{t,\mu}^*) \Psi_{t,\mu}^* \Lambda^s \varphi_t^\mu \rangle_t^2 \]

\[ \leq \epsilon \langle |\Lambda^s u_t| \rangle_t^2 + C_t \sum_{\mu} ||\tilde{\psi}_{t,\nu} \Lambda^s u_t^\mu ||_0^2 , \]

for some \( 0 < \epsilon \ll 1 \). By Lemma 3.3,

\[ \sum_{\mu} ||\tilde{\psi}_{t,\nu} \Lambda^s u_t^\mu ||_0^2 \leq C Q_{b,t}(\Lambda^s u_t, G_t \Lambda^s u_t) + C \sum_{\nu} ||\tilde{\psi}_{t,\nu} \Lambda^s u_t^\nu ||_0^2 . \]

Since \( \tilde{\psi}_b u_t = \alpha \), and \( G_t \) depends on \( t \), but it is of order zero,

\[ Q_{b,t}(\Lambda^s u_t, G_t \Lambda^s u_t) = \langle |\Lambda^s \alpha u_t, \tilde{\psi}_b G_t \Lambda^s u_t| \rangle_t \]

\[ = \langle |\Lambda^s \alpha, G_t \Lambda^s \alpha| \rangle_t + \langle |\Lambda^s \alpha, [\tilde{\psi}_b, G_t \Lambda^s] u_t| \rangle_t \]

\[ + \langle [|\tilde{\psi}_b, \Lambda^s] u_t, G_t \Lambda^s \alpha| \rangle_t + \langle [|\tilde{\psi}_b, \Lambda^s] u_t, [\tilde{\psi}_b, G_t \Lambda^s] u_t| \rangle_t \]

\[ \leq C_t \langle |\Lambda^s \alpha| \rangle_t^2 + C_t \langle |\Lambda^s \alpha| \rangle_t^2 \]

Altogether,

\[ II \leq \epsilon \langle |\Lambda^s u_t| \rangle_t^2 + C_t \langle |\Lambda^s \alpha| \rangle_t^2 + C_t \langle |\Lambda^s \alpha| \rangle_t^2 . \]

Finally, let us analyze III:

\[ III = \sum_{\mu} \langle |\Lambda^s \mu (\Psi_{t,\mu}^*) \Psi_{t,\mu}^* \Lambda^s u_t^\mu, \Lambda^s u_t| \rangle_t \leq \epsilon \langle |\Lambda^s u_t| \rangle_t^2 + \frac{1}{\epsilon} \sum_{\mu} \langle |\Lambda^s \mu (\Psi_{t,\mu}^*) \Psi_{t,\mu}^* \Lambda^s u_t^\mu | \rangle_t^2 , \]
for some \( 0 < \varepsilon \ll 1 \). Set \( \sum_\mu \langle |\Lambda^s \zeta_\mu (\Psi_{t,\mu}^-)^* \Psi_{t,\mu} \zeta_\mu u_t^\mu| |^2 \rangle = V \). By the definition of the norm,

\[
V = \sum_{\nu,\mu} \left| \tilde{\zeta}_\nu \Psi_{t,\nu} \zeta_\nu \Lambda^s \zeta_\mu (\Psi_{t,\mu}^-)^* \Psi_{t,\mu} \zeta_\mu u_t^\mu \right|^2 + \sum_{\nu,\mu} \left| \tilde{\zeta}_\nu \tilde{\Psi}_{t,\nu}^0 \zeta_\nu \Lambda^s \zeta_\mu (\Psi_{t,\mu}^-)^* \Psi_{t,\mu} \zeta_\mu u_t^\mu \right|^2
\]

+ smooth errors

\[
\leq \sum_{\nu,\mu} \left| \tilde{\zeta}_\nu \Psi_{t,\nu} \zeta_\nu \Lambda^s \zeta_\mu (\Psi_{t,\mu}^-)^* \Psi_{t,\mu} \zeta_\mu u_t^\mu \right|^2 + C \sum_{\nu} \left| \tilde{\zeta}_\nu \tilde{\Psi}_{t,\nu}^0 \zeta_\nu \Lambda^s u_t^\mu \right|^2 + O(\|u_t\|^2).
\]

By Lemma 3.3 for any \( t \geq T_0 \)

\[
t \sum_{\nu,\mu} \left| \tilde{\zeta}_\nu \Psi_{t,\nu} \zeta_\nu \Lambda^s \zeta_\mu (\Psi_{t,\mu}^-)^* \Psi_{t,\mu} \zeta_\mu u_t^\mu \right|^2 \leq C_t \sum_{\nu} \left| \tilde{\zeta}_\nu \tilde{\Psi}_{t,\nu}^0 \zeta_\nu \Lambda^s u_t^\mu \right|^2
\]

\[
+ C \sum_{\nu,\mu} Q_{b,t} \tilde{\zeta}_\nu \Psi_{t,\nu} \zeta_\nu \Lambda^s \zeta_\mu (\Psi_{t,\mu}^-)^* \Psi_{t,\mu} \zeta_\mu u_t^\mu
\]

\[
\leq C_t \sum_{\nu} \left| \tilde{\zeta}_\nu \tilde{\Psi}_{t,\nu}^0 \zeta_\nu \Lambda^s u_t^\mu \right|^2 + C \sum_{\nu,\mu} \left| \tilde{\zeta}_\nu \Psi_{t,\nu} \zeta_\nu \Lambda^s \zeta_\mu (\Psi_{t,\mu}^-)^* \Psi_{t,\mu} \zeta_\mu u_t^\mu \right|^2
\]

\[
+ C \sum_{\nu,\mu,\rho} \left| \zeta_\rho (\tilde{\partial}_b \tilde{\zeta}_\nu \Psi_{t,\nu} \zeta_\nu \Lambda^s \zeta_\mu (\Psi_{t,\mu}^-)^* \Psi_{t,\mu} \zeta_\mu \right| u_t^\mu \right|^2
\]

\[
\leq C_t \sum_{\nu} \left| \tilde{\zeta}_\nu \tilde{\Psi}_{t,\nu}^0 \zeta_\nu \Lambda^s u_t^\mu \right|^2 + C \|\Lambda^s \zeta_\mu \|^2 + C \langle |\Lambda^s u_t| \rangle^2
\]

because \( \tilde{\partial}_b \) is independent of \( t \). Thus, there exists \( 0 < \varepsilon' \ll 1 \) such that

\[
III \leq \varepsilon' \langle |\Lambda^s u_t| \rangle^2 + C \langle |\Lambda^s \zeta_\mu | \rangle^2 + C_t \sum_{\nu} \left| \tilde{\zeta}_\nu \tilde{\Psi}_{t,\nu}^0 \zeta_\nu \Lambda^s u_t^\mu \right|^2 + O(\|u_t\|^2)
\]

by the estimates for \( \sum_{\nu} \left| \tilde{\zeta}_\nu \tilde{\Psi}_{t,\nu}^0 \zeta_\nu \Lambda^s u_t^\mu \right|^2 \) from above. Putting together the estimates for \( I, II, \) and \( III \), we conclude that there exist \( 0 < \varepsilon'' \ll 1 \) and \( T''_0 \geq 1 \) such that for all \( t \geq T''_0 \)

\[
\langle |\Lambda^s u_t| \rangle^2 \leq \varepsilon'' \langle |\Lambda^s u_t| \rangle^2 + C_t \langle |\Lambda^s \zeta_\mu | \rangle^2 + C_t \langle |\Lambda^s \phi_t| \rangle^2 + C_t \langle |\Lambda^{s,-1} \phi_t| \rangle^2
\]

\[
+ C_t \langle |\Lambda^{s,-1} u_t| \rangle^2 + C_t \sum_{\nu} \left| \tilde{\zeta}_\nu \tilde{\Psi}_{t,\nu}^0 \zeta_\nu \Lambda^s \phi_t^\nu \right|^2.
\]

By Lemma 3.4

\[
\sum_{\nu} \left| \tilde{\zeta}_\nu \tilde{\Psi}_{t,\nu}^0 \zeta_\nu \Lambda^s \phi_t^\nu \right|^2 \leq C \langle |\Lambda^{s,-1} \phi_t| \rangle^2 + C_t \sum_{\nu} \left| \tilde{\zeta}_\nu \tilde{\Psi}_{t,\nu}^0 \zeta_\nu \Lambda^{s,-1} \phi_t^\nu \right|^2
\]

An argument similar to the one used above for \( Q_{b,t}(\Lambda^{s,-1} u_t, \Lambda^{s,-1} u_t) \) shows that

\[
Q_{b,t}(\Lambda^{s,-1} \phi_t, \Lambda^{s,-1} \phi_t) = \langle |\tilde{\partial}_b \Lambda^{s,-1} \phi_t, \tilde{\partial}_b \Lambda^{s,-1} \phi_t| \rangle \leq C_t \langle |\Lambda^{s,-1} \phi_t| \rangle^2 + C_t \langle |\Lambda^{s,-1} u_t| \rangle^2.
\]

Thus,

\[
\langle |\Lambda^s u_t| \rangle^2 \leq C_t \langle |\Lambda^s \zeta_\mu | \rangle^2 + C_t \langle |\Lambda^s \phi_t| \rangle^2 + C_t \langle |\Lambda^{s,-1} u_t| \rangle^2.
\]
By the induction hypothesis, \(|\langle \Lambda^{s-1} u_t \rangle_t \| \leq C_t \langle \langle \Lambda^{s-1} \varphi_t \rangle \|_t \), which is equivalent to \(|\langle \Lambda^{s-1} \varphi_t \rangle \|_t \leq C_t \langle \langle \Lambda^{s-1} u_t \rangle \|_t \) by Theorem 2.14. It follows that

\(|\langle \Lambda^s u_t \rangle_t \| \leq C_t \langle \langle \Lambda^s \alpha \rangle \|_t \).

This concludes the proof of Step 1.

**Step 2:** The estimate in Step 1 shows that having that the range of \(\bar{\partial}_b\) is closed in \(L^2\) implies that the range of \(\bar{\partial}_b\) is also closed in \(H^s\) for all \(s > 0\). The same estimate implies that the weighted Szegö projection maps \(H^s\) to \(H^{s-1}\) for each \(s \geq 0\). Using these two facts and the method in [K2], we construct a smooth solution to \(\bar{\partial}_b\). Let \(\alpha\) be a closed \((0,1)\) form such that \(\alpha \in C^\infty(M)\). We want to find some \(u \in C^\infty(M)\) such that \(\bar{\partial}_b u = \alpha\).

Since the range of \(\bar{\partial}_b\) is closed in each \(H^s\) space for \(s \geq 0\), for each \(k = 1, 2, \ldots\), there exists some \(u_k \in H^k\) such that \(\bar{\partial}_b u_k = \alpha\). We will modify each \(u_k\) by an element of \(\mathcal{N}(\bar{\partial}_b)\) in order to construct a telescoping series that is in \(H^k\) for each \(k \geq 1\). To do so, we need to show first that \(H^s \cap \mathcal{N}(\bar{\partial}_b)\) is dense in \(H^k \cap \mathcal{N}(\bar{\partial}_b)\) for each \(s > k + 1\). Let \(g\) be any element of \(H^s \cap \mathcal{N}(\bar{\partial}_b)\). Smooth functions are dense in all \(H^k(M)\), so there exists a sequence \(\{g_i\}_i\) such that \(g_i \in C^\infty(M)\) and \(g_i \to g\) in \(H^k\). \(\bar{\partial}_b g = 0\) implies that

\[g - S_{b,t} g = \bar{\partial}_{b,t} N_t \bar{\partial}_b g = 0,
\]

so \(g = S_{b,t} g\). Let \(g'_i = S_{b,t} g_i\). \(g'_i \in H^k \cap \mathcal{N}(\bar{\partial}_b)\) since the Szegö projection is bounded as a map from \(H^s\) to \(H^{s-1}\) and for the same reason, \(g'_i \to g\) in \(H^k\). Thus indeed, \(H^s \cap \mathcal{N}(\bar{\partial}_b)\) is dense in \(H^k \cap \mathcal{N}(\bar{\partial}_b)\). Using this fact, we inductively construct a sequence \(\{\tilde{u}_k\}_k\) as follows:

\[\tilde{u}_1 = u_1,\]
\[\tilde{u}_2 = u_2 + v_2,\]

where \(v_2 \in H^3 \cap \mathcal{N}(\bar{\partial}_b)\) is such that

\[||\tilde{u}_2 - \tilde{u}_1||_1 \leq 2^{-1},\]

and in general,

\[\tilde{u}_{k+1} = u_{k+1} + v_{k+1},\]

where \(v_{k+1} \in H^{k+2} \cap \mathcal{N}(\bar{\partial}_b)\) is such that

\[||\tilde{u}_{k+1} - \tilde{u}_k||_k \leq 2^{-k}.\]

Clearly, \(\bar{\partial}_b \tilde{u}_k = \alpha\), so we set

\[u = \tilde{u}_J + \sum_{k=J}^{\infty} (\tilde{u}_{k+1} - \tilde{u}_k), \quad J \in \mathbb{N}.
\]

It follows that \(u \in H^k\) for each \(k \in \mathbb{N}\), hence that \(u \in C^\infty(M)\) and \(\bar{\partial}_b u = \alpha\). \(\square\)
References

[B] Burns, Daniel M. “Global Behavior of Some Tangential Cauchy-Riemann Equations.” Partial Differential Equations and Geometry (Proc. Conf., Park City, Utah, 1977). Dekker, New York, 1979: 51-6.

[H] Hörmander, Lars. “$L^2$ Estimates and Existence Theorems for the $\bar{\partial}$ Operator.” Acta Math. 113 (1965): 89-152.

[K1] Kohn, J.J. “Global Regularity for $\bar{\partial}_b$ on Weakly Pseudo-convex Manifolds.” Trans. Amer. Math. Soc. 181 (1973): 273-92.

[K2] —. “Methods of Partial Differential Equations in Complex Analysis.” Proc. Symposia Pure Math. 30 pt.I (1977): 215-37.

[K3] —. “The Range of the Tangential Cauchy-Riemann Operator.” Duke Math. J. 53 (1986): 525-45.

[K4] —. “A Survey of the $\bar{\partial}$-Neumann Problem.” Proc. Symposia Pure Math. 41 (1984): 137-45.

[KN] Kohn, J.J., and Louis Nirenberg. “Noncoercive Boundary Value Problems.” Comm. Pure Appl. Math. 18 (1965): 443-92.

[KR] Kohn, J.J., and Hugo Rossi. “On the Extension of Holomorphic Functions from the Boundary of a Complex Manifold.” Ann. Math. 81 (1965): 451-72.

[N] Nicoara, Andreea “Global Regularity for $\bar{\partial}_b$ on Weakly Pseudoconvex CR Manifolds.” To appear in Advances in Mathematics.

[R] Rossi, Hugo. “Attaching Analytic Spaces to an Analytic Space along a Pseudoconcave Boundary.” Proc. Conf. on Complex Manifolds, Minneapolis, 1964. Springer-Verlag, New York, 1965: 242-56.