NOVEL NON-INVOLUTIVE SOLUTIONS OF THE YANG-BAXTER EQUATION FROM (SKEW) BRACES
ANASTASIA DOIKOU AND BERNARD RYBOLOWICZ

Abstract. We produce novel non-involutive solutions of the Yang-Baxter equation coming from (skew) braces. These solutions are generalisations of the known ones coming from braces and skew braces, and surprisingly in the case of braces they are not necessarily involutive. In the case of two-sided (skew) braces one can assign such solutions to every element of the set. Novel bijective maps associated to the inverse solutions are also introduced. Moreover, we show that the recently derived Drinfeld twists of the involutive case are still admissible in the non-involutive frame and we identify the twisted $r$-matrices and twisted coproducts. We observe, as in the involutive case that the underlying quantum algebra is not a quasi-triangular bialgebra, as one would expect, but a quasi-triangular quasi-bialgebra. The same applies to the quantum algebra of the twisted $r$-matrices, contrary to the involutive case.

Introduction
The idea of set-theoretic solutions of the Yang-Baxter equation (YBE) [3, 39] was first introduced and studied by Drinfeld in early 90s [17] and ever since a significant progress has been made on this topic (see for instance [20, 21]). In 2007 Rump [37] identified certain algebraic structures called left braces and showed that with every left brace one can associate an involutive solution of the set-theoretic Yang-Baxter equation, and conversely from every involutive solution one can construct a left brace, such that the solution given by the brace, restricted to an appropriate subset, is a set-theoretic solution. This generated an increased research activity on left braces and set-theoretic of the YBE (see for instance [1, 2, 10, 12, 11]), and in 2017 Guarnieri and Vendramin [27] extended Rump’s construction to left skew braces in order to produce non-degenerate, non-involutive solutions. This generalization led to a trend of relaxing more conditions of braces to produce yet more general classes of solutions (see e.g. [8, 9, 29, 30, 34, 35, 38, 23, 25]).

In the first part of the present study (Sections 1 and 2) we introduce a new way of constructing solutions of the YBE from left skew braces. In contrast to already known methods, this one allows one to associate to left (skew) braces more than one solution, not necessarily involutive. To every two-sided (skew) brace we can associate as many solutions as there are elements. These solutions are not necessarily all different, but it is quite common for them to be distinct, (see Example 2.14.

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Due to the dependence of the solution on the choice of an element of the set, we obtain parameter dependent families of solutions. The striking observation here is that even in the case of braces one may obtain non-involutive solutions! In fact, we show that for generic values of the parameter there is no map to relate our parametric solutions with the known solutions appearing in [37, 27].

In the second part (Sections 3 and 4) we study the underlying quantum algebras associated to non-involutive set-theoretic solutions and discuss the notion of admissible Drinfeld twist. Admissible twists $F$, which link distinct (quasi-triangular) Hopf or quasi-Hopf algebras were introduced by Drinfeld in [19]. Whenever the notion of the twist is discussed in Drinfeld’s original work and in the literature in general a restrictive action of the co-unit on the twist is almost always assumed, i.e. $(\text{id} \otimes \epsilon) F = (\epsilon \otimes \text{id}) F = 1_A$ ($A$ is the associated quantum algebra). We should note however that Drinfeld in [19] uses certain simple twists without this restricted counit action to twist quasi-bialgebras with nontrivial unit constraints to quasi-bialgebras with trivial unit constraints.

Recently it was shown in [15] that all involutive, set-theoretic solutions of the YBE can be obtained from the permutation operator via a suitable twist that was explicitly derived and its admissibility was proven. It was noted in [15], using a special class of set-theoretic solutions, that the corresponding quantum algebra was not co-associative and the related associator was derived. This was indeed the first hint that set-theoretic solutions of the YBE give rise to quasi-bialgebras. Later the idea of non-co-associativity for involutive solutions was further explored in [16] and the generic action of the co-unit on the twist was considered, leading to the conclusion that the underlying quantum algebra for involutive set-theoretic solutions (see also [13, 14]) is a quasi-triangular quasi-bialgebra. We extend these findings here to the non-involutive case.

We describe below in more detail the outline and the main findings of this study.

1. In Section 1 we recall basic definitions and known results about (skew) braces and set-theoretic solutions of the Yang-Baxter equation.
2. In Section 2 we introduce novel set-theoretic solutions of the YBE associated to generalized left skew braces $B$, that depend on an extra parameter $z \in B$, (Theorem 2.6) i.e. we obtain a parameter dependent family of solutions. For $z = 1$, one recovers the solution given by Guarnieri and Vendramin in [27]. We also show that our $z$-deformed solution is involutive if and only if the skew brace is an abelian additive group and $z$ belongs to the socle of the brace, that is the solution reduces to Rump’s solution given in [37]. One of the most striking findings in this section is Corollary 3.5, which states that if $B$ is a skew brace with certain properties, then for any element $z \in B$, we can associate a $z$-deformed solution. Another key observation, as already mentioned, is that even in the case of braces one may obtain non-involutive solutions. We also introduce novel bijective maps associated to the inverse $r$-matrices (also solutions of the YBE). We conclude the section with several examples which show that not all deformed solutions are different, but it is not common for
two deformed solutions to be the same. We are able to present examples of non-involutive solutions associated to braces.

(3) In Section 3 we briefly review key definitions about the notions of the quasi-triangular quasi-bialgebras and Drinfeld twists as well as some of the main findings of [16], useful for the analysis of Section 4.

(4) In Section 4 we discuss the quantum algebras and the notion of Drinfeld twists for non-degenerate, non-involutive solutions of the YBE. Specifically, we first provide a brief review on the tensor notation for set-theoretic $r$-matrices (see also e.g. [13]) as well as the main results on admissible Drinfeld twists and the associated quasi-bialgebras for involutive set-theoretic solutions [15, 16]. We then move one to our aim, which is the generalization of the results of [15, 16] to the non-involutive scenario and the characterization of the associated quantum algebra as a quasi-bialgebra. This is achieved by introducing certain families of elements, which after suitable twisting become group-like elements. The form and the coproduct structure of these elements are inspired by tensor representations of the so called RTT algebra [22, 15]. The existence of these families of elements is fundamental in allowing us to conclude that for any set-theoretic solution of the YBE, the underlying quantum algebra is a quasi-bialgebra.

1. Preliminaries

We begin the section by recalling basic definitions and ideas about set-theoretic solutions of the Yang-Baxter equation and (skew) braces. For a set-theoretic solution of the braid equation, we will use the notation $(X, \tilde{r})$, instead of the usual notation $(X, r)$, to be consistent with the notation used in quantum integrable systems.

Let $X = \{x_1, \ldots, x_n\}$ be a set and $\tilde{r} : X \times X \rightarrow X \times X$. Denote

$$\tilde{r}(x, y) = (\sigma_x(y), \tau_y(x)).$$

(1.1)

We say that $\tilde{r}$ is non-degenerate if $\sigma_x$ and $\tau_y$ are bijective maps, and $(X, \tilde{r})$ is a set-theoretic solution of the braid equation if

$$(\tilde{r} \times \text{id})(\text{id} \times \tilde{r})(\tilde{r} \times \text{id}) = (\text{id} \times \tilde{r})(\tilde{r} \times \text{id})(\text{id} \times \tilde{r}).$$

(1.2)

The map $\tilde{r}$ is called involutive if $\tilde{r}^2 = \text{id}$.

The following notion of homomorphism and isomorphism between solutions will be useful.

**Definition 1.1.** [12, Section 3] Let $X, S$ be sets and $\tilde{r} : S \times S \rightarrow S \times S$ and $\tilde{r} : X \times X \rightarrow X \times X$ be two solutions of the braid equation. Then a function $f : S \rightarrow X$ is called a homomorphism of Yang-Baxter solutions if it satisfies the following equality

$$(f \times f)\tilde{r} = \tilde{r}(f \times f).$$

We say that $f$ is an isomorphism if $f$ is a bijection.
We also introduce the map \( r : X \times X \rightarrow X \times X \), such that \( r = r\pi \), where \( \pi : X \times X \rightarrow X \times X \) is the flip map: \( \pi(x, y) = (y, x) \). Hence, \( r(y, x) = (\sigma_x(y), \tau_y(x)) \), and it satisfies the YBE:

\[
\begin{align*}
    r_{12} \ r_{13} \ r_{23} &= r_{23} \ r_{13} \ r_{12},
\end{align*}
\]

where we denote \( r_{12}(y, x, z) = (\sigma_x(y), \tau_y(x), z) \), \( r_{23}(z, y, x) = (z, \sigma_x(y), \tau_y(x)) \) and \( r_{13}(y, z, x) = (\sigma_x(y), z, \tau_y(x)) \).

We note that a function satisfying (1.2), is usually called in literature a set-theoretic solution of the Yang-Baxter equation or the pair \( (X, \bar{r}) \) is called a braided set \( [20, 24, 25] \). Also, a function satisfying (1.3), can be found in literature under the name set-theoretic solution of the quantum Yang-Baxter equation (QYBE) \([20, 36, 37]\). In this paper a pair \( (X, \bar{r}) \), which satisfies (1.2) is called a set-theoretic solution of the braid equation, whereas a pair \( (X, r) \), which satisfies (1.3) is called a set-theoretic solution of the YBE.

We recall the definitions of the algebraic structures that provide set-theoretic solutions of the braid equation, such as left skew braces and braces. We also present some key properties associated to these structures that will be useful when formulating some of the main findings of the present study, summarized in Section 4.

**Definition 1.2** ([12, 27]). A **left skew brace** is a set \( B \) together with two group operations +, o : \( B \times B \rightarrow B \), the first is called addition and the second is called multiplication, such that for all \( a, b, c \in B \),

\[
    a \circ (b + c) = a \circ b - a + a \circ c.
\]

If + is an abelian group operation \( B \) is called a **left brace**. Moreover, if \( B \) is a left skew brace and for all \( a, b, c \in B \) \( (b + c) \circ a = b \circ a - a + c \circ a \), then \( B \) is called a **skew brace** or more commonly in literature **two-sided skew brace**. Analogously if + is abelian and \( B \) is a skew brace, then \( B \) is called a **brace**.

The additive identity of a left skew brace \( B \) will be denoted by 0 and the multiplicative identity by 1. In every left skew brace \( 0 = 1 \).

Rump showed the following powerful theorem for involutive set-theoretic solutions.

**Theorem 1.3.** (Rump’s theorem, [36, 37, 12]). Assume \((B, +, o)\) is a left brace. If the map \( \bar{r}_B : B \times B \rightarrow B \times B \) is defined as \( \bar{r}_B(x, y) = (\sigma_x(y), \tau_y(x)) \), where \( \sigma_x(y) = x \circ y - x \), \( \tau_y(x) = t \circ x - t \), and \( t \) is the inverse of \( \sigma_x(y) \) in the circle group \((B, o)\), then \((B, \bar{r}_B)\) is an involutive, non-degenerate solution of the braid equation.

Conversely, if \((X, \bar{r})\) is an involutive, non-degenerate solution of the braid equation, then there exists a left brace \((B, +, o)\) (called an underlying brace of the solution \((X, \bar{r})\)) such that \( B \) contains \( X, \bar{r}_B(X \times X) \subseteq X \times X \), and the map \( \bar{r} \) is isomorphic to the restriction of \( \bar{r}_B \) to \( X \times X \). Both the additive \((B, +)\) and multiplicative \((B, o)\) groups of the left brace \((B, +, o)\) are generated by \( X \).

The following fact was also noticed by Rump.
Remark 1.4. Let \((N, +, \cdot)\) be an associative ring. If for \(a, b \in N\) we define
\[ a \circ b = a \cdot b + a + b, \]
then \((N, +, \circ)\) is a brace if and only if \((N, +, \cdot)\) is a radical ring.

Guarnieri and Vendramin [27], extended Rump’s result to left skew braces and non-degenerate, non-involutive solutions.

**Theorem 1.5** ([27] Theorem 3.1). Let \(B\) be a left skew brace, then the map \(\tilde{r}_{GV}: B \times B \to B \times B\) given for all \(a, b \in B\) by
\[ \tilde{r}_{GV}(a, b) = (-a + a \circ b, (-a + a \circ b)^{-1} \circ a \circ b) \]
is a non-degenerate solution of the braid equation.

2. Extended non-involutive solutions of the YBE from (skew) braces

We are going to consider in this section some generalized version of the set-theoretic solution (1.1) by introducing some kind of “\(z\)-deformation”. Indeed, let \(z \in X\) be fixed, then we denote
\[ \tilde{r}_z(x, y) = (\sigma^z_x(y), \tau^z_y(x)). \] (2.1)
We say that \(\tilde{r}\) is non-degenerate if \(\sigma^z_x\) and \(\tau^z_y\) are bijective maps. We review below the constraints arising by requiring \((X, \tilde{r}_z)\) to be a solution of the braid equation ([17, 20, 36, 37]). Let,
\[ (\tilde{r} \times \text{id})(\text{id} \times \tilde{r})(\text{id} \times \tilde{r})(\eta, x, y) = (L_1, L_2, L_3), \]
\[ (\text{id} \times \tilde{r})(\tilde{r} \times \text{id})(\text{id} \times \tilde{r})(\eta, x, y) = (R_1, R_2, R_3), \]
where, after employing expression (2.1) we identify:
\[ L_1 = \sigma^z_y(\sigma^z_x(y)), \quad L_2 = \tau^z_{\sigma^z_y(y)}(\sigma^z_x(x)), \quad L_3 = \tau^z_y(\tau^z_{\sigma^z_x(x)}(\eta)), \]
\[ R_1 = \sigma^z_x(y), \quad R_1 = \sigma^z_{\tau^z_{\sigma^z_x(x)}(\eta)}(\tau^z_{\sigma^z_x(x)}(x)), \quad R_3 = \tau^z_{\tau^z_{\sigma^z_x(x)}(\eta)}(\tau^z_{\sigma^z_x(x)}(y)). \]
And by requiring \(L_i = R_i, i \in \{1, 2, 3\}\) we obtain the following fundamental constraints for the associated maps:
\[ \sigma^z_y(\sigma^z_x(y)) = \sigma^z_{\sigma^z_y(x)}(\sigma^z_{\tau^z_{\sigma^z_x(x)}(\eta)}(y)), \] (2.2)
\[ \tau^z_y(\tau^z_{\sigma^z_x(x)}(\eta)) = \tau^z_{\sigma^z_{\tau^z_{\sigma^z_x(x)}(\eta)}(\sigma^z_x(x))}(\tau^z_{\sigma^z_x(x)}(y)), \] (2.3)
\[ \tau^z_{\sigma^z_{\tau^z_{\sigma^z_x(x)}(\eta)}(\sigma^z_x(x))}(\tau^z_{\sigma^z_x(x)}(y)) = \sigma^z_{\tau^z_{\sigma^z_x(x)}(\eta)}(\tau^z_{\sigma^z_x(x)}(x)). \] (2.4)
Note that the constraints above are essentially the ones for the set-theoretic solution (1.1), given that \(z\) is a fixed element of the set, i.e. for different elements \(z\) we obtain in principle distinct solutions of the braid equation.

We will introduce in what follows suitable algebraic structures that satisfy the fundamental constraints above, i.e. provide solutions of the braid equation and generalize the findings of Rump.
and Guarnieri & Vendramin. Before we state and prove our main Theorem 2.6 we first show some useful properties of left skew braces.

Lemma 2.1. Let $B$ be a set with two group operations $+,$ $\circ$ with the same neutral element $1$. Then the condition (1.4) is equivalent to the following condition:

$$a \circ (b - c + d) = a \circ b - a \circ c + a \circ d,$$

for all $a, b, c, d \in B$

Proof. This follows from [4]. Indeed, let us assume that (1.4) holds, and recall that in any skew brace $0 = 1$, then we observe

$$a \circ 1 = a \Rightarrow a \circ (x - x) = a \Rightarrow a \circ (-x) = a - a \circ x + a,$$

and consequently $\forall a, b, c, d \in B$, we have

$$a \circ (b - c + d) = a \circ (b - c) - a + a \circ d = a \circ b - a + a \circ (-c) - a + a \circ d$$

$$= a \circ b - a - a \circ c + a - a + a \circ d = a \circ b - a \circ c + a \circ d.$$

Conversely, we observe that

$$a \circ (b + c) = a \circ (b - 1 + c) = a \circ b - a + a \circ c,$$

so (1.4) holds.

Remark 2.2. Similarly, one can rephrase the right distributivity in the brace, i.e. for all $a, b, c, d \in B$,

$$(b + c) \circ a = b \circ a - a + c \circ a \iff (b - c + d) \circ a = b \circ a - c \circ a + d \circ a.$$

We introduce below the notion of uniformly distributive of $u$-distributive elements.

Definition 2.3. Let $B$ be a left skew brace. We say that $z \in B$ is $u$-distributive, if for all $a, b, c \in B$,

$$(a - b + c) \circ z = a \circ z - b \circ z + c \circ z.$$  (2.5)

Remark 2.4. Notice that if $B$ is a skew brace, then every element of $B$ is $u$-distributive.

Proposition 2.5. Let $B$ be a left skew brace and $z \in B$ be $u$-distributive. For every $a \in B$, let $\sigma^z_a : B \to B$ and $\tau^z_a : B \to B$ be the maps defined as

$$\sigma^z_a (b) := a \circ b - a \circ z + z \quad \text{and} \quad \tau^z_a (a) := \sigma^z_a (b)^{-1} \circ a \circ b, \text{ for all } b \in B,$$  (2.6)

where $\sigma^z_a (b)^{-1}$ is the inverse of $\sigma^z_a (b)$ in $(B, \circ)$. Then the following conditions are satisfied for all $a, b, c, d \in B$:

1. The set $\{ z \in B \mid \forall a, b, c \in B \ (a - b + c) \circ z = a \circ z - b \circ z + c \circ z \}$ is a subgroup of $(B, \circ),$
2. $\sigma^z_a (b - c + d) = \sigma^z_a (b) - \sigma^z_a (c) + \sigma^z_a (d),$
3. $\sigma^z_a (\sigma^z_a (c)) = \sigma^z_a (c),$
4. $a \circ \sigma^z_a (c) = \sigma^z_a (c) - z + a \circ z,$
5. $\sigma^z_a (b) \circ \tau^z_a (a) = a \circ b$
(6) \( \sigma^z_a(b) \circ \sigma^z_{\tau^z_b(a)}(c) = \sigma^z_a(\sigma^z_{\tau^z_c(c)}(\tau^z_c(b))) \).

(7) The maps \( \sigma^z_a \) and \( \tau^z_a \) are bijective for all \( a \in B \).

Proof. Let \( a, b, c, d \in B \), and we recall Lemma 2.1:

\[
\sigma^z_a(b - c + d) = a \circ (b - c + d) - a \circ z + z = a \circ b - a \circ c + a \circ d - a \circ z + z
\]

\[
= a \circ b - a \circ z + a \circ z - a \circ c + a \circ d - a \circ z + z
\]

\[
= a \circ b - a \circ z + z - a \circ z - a \circ c + a \circ d - a \circ z + z
\]

\[
= a \circ b - a \circ z + z - (a \circ c - a \circ z + z) + a \circ d - a \circ z + z
\]

\[
= \sigma^z_a(b) - \sigma^z_a(c) + \sigma^z_a(d),
\]

\[
\sigma^z_a(\sigma^z_b(c)) = \sigma^z_a(b \circ c - b \circ z + z) = a \circ (b \circ c - b \circ z + z) - a \circ z + z
\]

\[
= a \circ b \circ c - a \circ b \circ z + a \circ z - a \circ z + z
\]

\[
= a \circ b \circ c - a \circ b \circ z + z - a \circ z + z = \sigma^z_{\alpha \circ b}(c),
\]

\[
a \circ \sigma^z_b(c) = a \circ (b \circ c - b \circ z + z) = a \circ b \circ c - a \circ b \circ z + a \circ z
\]

\[
= a \circ b \circ c - a \circ b \circ z + z - a \circ z + z = \sigma^z_{\alpha \circ b}(c) - z + a \circ z,
\]

\[
(a - b + c) \circ z^{-1} = (a \circ z^{-1} \circ z - b \circ z^{-1} \circ z + c \circ z^{-1} \circ z) \circ z^{-1}
\]

\[
= (a \circ z^{-1} - b \circ z^{-1} + c \circ z^{-1}) \circ z \circ z^{-1}
\]

\[
= (a \circ z^{-1} - b \circ z^{-1} + c \circ z^{-1}),
\]

\[
\sigma^z_a(b) \circ \tau^z_b(a) = \sigma^z_a(b) \circ \sigma^z_a(b)^{-1} \circ a \circ b = a \circ b,
\]

thus, properties (1)-(5) hold.

To show (6) we observe that using Lemma 2.1 and the fact that multiplying by \( z \) distributes from the right side, we get for all \( a, b, c \in B \)

\[
\sigma^z_a(b) \circ \sigma^z_{\tau^z_b(a)}(c) = \sigma^z_a(b) \circ (\tau^z_b(a) \circ c - \tau^z_b(a) \circ z + z)
\]

\[
= \sigma^z_a(b) \circ \tau^z_b(a) \circ c - \sigma^z_a(b) \circ \tau^z_b(a) \circ z + \sigma^z_a(b) \circ z
\]

\[
= a \circ b \circ c - a \circ b \circ z + \sigma^z_a(b) \circ z
\]

\[
= a \circ b \circ c - a \circ b \circ z + (a \circ b - a \circ z + z) \circ z
\]

\[
= a \circ b \circ c - a \circ z \circ z + z \circ z,
\]

By substituting \( b \) with \( \sigma^z_b(c) \) and \( c \) with \( \tau^z_c(b) \), by using Proposition 2, we immediately get

\[
\sigma^z_a(\sigma^z_b(c)) \circ \sigma^z_{\tau^z_{\tau^z_c(b)}}(\tau^z_{\tau^z_c(b)}) = a \circ \sigma^z_b(c) \circ \tau^z_c(b) - a \circ z \circ z + z \circ z = a \circ b \circ c - a \circ z \circ z + z \circ z.
\]
Thus, \( \sigma^z_B(b) \circ \sigma^z_{\sigma^z_B(a)}(c) = \sigma^z_B(\sigma^z_B(c)) \circ \sigma^z_{\sigma^z_B(c)(a)}(\tau^z_B(b)) \), and (6) holds.

For (7), observe that both maps are injective as

\[
\sigma^z_B(y_1) = \sigma^z_B(y_2) \iff x \circ y_1 - x \circ z + z = x \circ y_2 - x \circ z + z \iff y_1 = y_2,
\]

\[
\tau^z_B(x_1) = \tau^z_B(x_2) \iff t_1 \circ x_1 \circ z = t_2 \circ x_2 \circ z \iff t_1 \circ x_1 = t_2 \circ x_2,
\]

where recall \( t_i = \sigma^z_{\tau^z_i}(y) \), \( i \in \{1, 2\} \). Thus, \( t_1 \circ x_1 = t_2 \circ x_2 \iff x_1^{-1} \circ t_1^{-1} = x_2^{-1} \circ t_2^{-1} \), which leads to \( y - z + x_1^{-1} \circ z = y - z + x_2^{-1} \circ z \), and hence \( x_1 = x_2 \), so both maps are injective.

To show that the maps are surjective, we observe that for all \( c \in B \), fixed \( y, z \in B \), and \( h = (y^{-1} - z^{-1} + c^{-1} \circ z^{-1}) \),

\[
\sigma^z_B(x^{-1} \circ (c - z + x \circ z)) = c - z + x \circ z - x \circ z + z = c
\]

\[
\tau^z_B(h^{-1} \circ y^{-1}) = (h^{-1} \circ y^{-1} - y - h^{-1} \circ y^{-1} \circ z + z) \circ h^{-1} \circ y^{-1} \circ y
\]

\[
= (h^{-1} \circ (y^{-1} \circ z + h \circ z)) \circ h^{-1}
\]

\[
= (-y^{-1} \circ z + h \circ z) \circ h \circ h^{-1} = (-y^{-1} \circ z + h \circ z)^{-1}
\]

\[
= (-y^{-1} \circ z + (y^{-1} - z^{-1} + c^{-1} \circ z^{-1}) \circ z)^{-1}
\]

\[
= (-y^{-1} \circ z + y^{-1} \circ z - 1 + c^{-1})^{-1} = (c^{-1})^{-1} = c.
\]

Thus both \( \tau^z_B \) and \( \sigma^z_B \) are bijections. \( \square \)

We should note that even though the definition of \( \sigma^z_B \) might seem like a lucky guess, there is some intuition behind it. The map is closely related to trusses, see [4]. The map \( \sigma^z_B \) appears in [5], denoted by \( \lambda^z \), as a map that defines a paragon, that is a congruence class, see [6]. Thus, these solutions connect, in some particular way not yet clear to us, the structure of quotient of a brace with the Yang Baxter-equation.

We may now proceed in proving the following main theorem.

**Theorem 2.6.** Let \( B \) be a left skew brace, let \( z \in B \) be \( u \)-distributive, and let \( \sigma^z_B : B \to B \), and \( \tau^z_B : B \to B \) be the maps (2.6) defined in Proposition 2.5. Suppose the map \( \tau_z : B \times B \to B \times B \) is defined by

\[
\tau_z(a, b) = (\sigma^z_B(b), \tau^z_B(a)), \quad a, b \in B.
\]

Then \( \tau_z \) is a non-degenerate solution of the braid equation.

**Proof.** To prove this we need to show that the maps \( \sigma, \tau \) satisfy the constraints (2.2)–(2.4). To achieve this we use Lemma 2.1 and properties (2), (5) and (6) from Proposition 2.5.

Indeed, from Proposition 2.5 (2) and (5), it follows that (2.2) is satisfied, i.e.

\[
\sigma^z_B(\sigma^z_B(y)) = \sigma^z_{\sigma^z_B(x)}(\sigma^z_{\tau^z_B(\eta)}(y)).
\]

We observe that

\[
\tau^z_B(\tau^z_B(\eta)) = T \circ \tau^z_B(\eta) \circ b = T \circ t \circ \eta \circ a \circ b = T \circ t \circ \eta \circ \sigma^z_B(b) \circ \tau^z_B(a),
\]

where
where $T = \sigma^z_{\tau^y(x)(\eta)}(b)^{-1}$ and $t = \sigma^y_b(a)^{-1}$ (the inverse in the circle group). Due to (6), (2), (5) of Proposition 2.5, we then conclude that

$$\tau^z_b(\tau^z_a(\eta)) = \tau^z_{\tau^z_b(a)}(\tau^z_{\sigma^z_b(b)}(\eta)),$$

so (2.3) is also satisfied.

To prove (2.4), we first employ (6) of Proposition 2.5 and then use the definition of $\tau$,

$$\sigma^z_{\tau^z_x(y)}(\tau^z_y(x)) = \sigma^z_{\eta\circ x}(y)^{-1} \circ \sigma^z_y(x) \circ \sigma^z_{\tau^z_x(y)}(y) = \tau^z_{\sigma^z_{\tau^z_x(y)}(y)}(\sigma^z_y(x)).$$

Thus, (2.4) is satisfied, and $\tilde{\tau}_z(a, b) = (\sigma^z_a(b), \tau^z_b(a))$ is a solution of braid equation.

The non-degeneracy follows by the Proposition 2.5 (7). \hfill \Box

**Corollary 2.7.** Let $B$ be a skew brace. Then for all $z \in B$, $\tilde{\tau}_z$ defined as in Theorem 2.6 is a set-theoretic solution of the braid equation.

**Remark 2.8.** Notice that for $z = 1$, Guarnieri-Vedramin skew braces are recovered; if in addition $(B, +)$ is an abelian group Rump’s braces are recovered and $\tilde{\tau}_{z=1}$ becomes involutive, due to:

$$\sigma^z_{\sigma^z_x(y)}(\tau^z_y(x)) = x \text{ and } \tau^z_{\tau^z_y(x)}(\sigma^z_x(y)) = y.$$ In the general case $z \neq 1$ the solutions are not involutive anymore given that, although still $\sigma^z_x(y) \circ \tau^z_y(x) = x \circ y$ holds, $\sigma^z_{\sigma^z_x(y)}(\tau^z_y(x)) \neq x$, $\tau^z_{\tau^z_y(x)}(\sigma^z_x(y)) \neq y$.

The following Lemmata explain when our solutions are involutive.

**Lemma 2.9.** Let $B$ be a left skew brace, let $z \in B$ be $u$-distributive. Then

$$\sigma^z_a(b) = \sigma^1_a(b), \text{ for all } a, b \in B$$

if and only if $a \circ z = z + a$ for all $a \in B$. In this case the solution $(B, \tilde{\tau}_z)$ coincides with $(B, \tilde{\tau})$, the canonical solution on $B$.

**Proof.** Observe that for all $a, b \in B$ such that $a \circ z = z + a$,

$$\sigma^z_a(b) = a \circ b - a \circ z + z = a \circ b - (z + z) + z = a \circ b - a - z + z = a \circ b - a = \sigma^1_a(b).$$

Conversely, if $\sigma^z_a(b) = \sigma^1_a(b)$, then

$$a \circ z = z - \sigma^1_a(b) + a \circ b = z - (a \circ b - a) + a \circ b = z + a - a \circ b + a \circ b = z + a. \hfill \Box$$

**Lemma 2.10.** Let $B$ be a left skew brace. The solution $\tilde{\tau}_z$ is involutive if and only if $B$ is a left brace and for all $a, b \in B$, $\sigma^z_a(b) = \sigma^1_a(b)$.

**Proof.** Let $B$ be a left brace and $\sigma^z_a(b) = \sigma^1_a(b)$ for all $a, b \in B$. Then

$$\sigma^z_{\sigma^z_x(a)}(\tau^z_y(a)) = \sigma^1_{\sigma^1_x(a)}(\tau^z_y(a)) = a \circ b - \sigma^1_a(b) = a \circ b - (a \circ b - a) = a.$$ Thus, we get that $\tilde{\tau}^2_z = (a, a^{-1} \circ a \circ b) = (a, b)$, i.e. $\tilde{\tau}_z$ is involutive.

Conversely, let us assume that $\tilde{\tau}_z$ is involutive, that is, for all $a, b \in B$,

$$\sigma^z_{\sigma^z_a(b)}(\tau^z_b(a)) = a. \hfill \Box$$
Proof. Proposition 2.12. Let \( \hat{\tau}_x \) be the corresponding bijective maps.

Thus, \( \hat{\sigma}_a^1(\tau_a^1(a)) = a \circ b - (a \circ b - a) = a \),
and \( a \circ b + a = a + a \circ b \), for all \( a, b \in B \). Now, let \( b = a^{-1} \circ c \) for any \( c \in B \), then \( a + c = c + a \), \((B, +)\) is an abelian group, and thus \( B \) is a left brace.

Remark 2.11. In the case of a brace, Lemma 2.10 and Lemma 2.9 say that \( \hat{\tau}_x \) is involutive if and only if \( x \in \text{Soc}(B) := \{ b \in B \mid \forall a \in B \ a \circ b = a + b \} \). Moreover, for all \( z, z' \in \text{Soc}(B) \) \( \hat{\tau}_x = \hat{\tau}_x' \), that is there is only one involutive solution associated that way with a brace.

Note that although \( \hat{\tau}_x \) is not involutive, \( \hat{\tau}_x \hat{\tau}_x^T = \hat{\tau}_x^T \hat{\tau}_x = \text{id} \), where \( T \) denotes total transposition, i.e., \( \hat{\tau}_x^T : (\sigma_x^z(y), \tau_y^x(x)) \mapsto (x, y) \); similarly \( \hat{\tau}_x \hat{\tau}_x^T = \hat{\tau}_x^T \hat{\tau}_x = \text{id} \). Recall that \( \hat{\tau}_x \) satisfies the braid equation and \( \hat{\tau}_x \) satisfies the YBE.

In the following Proposition we provide the explicit expressions of the inverse \( \hat{\tau}_x \)-matrices as well as the corresponding bijective maps.

**Proposition 2.12.** Let \( \hat{\tau}_x \), \( \hat{\tau}_x^*: X \times X \) be solutions of the braid equations, such that \( \hat{\tau}_x : (x, y) \mapsto (\hat{\sigma}_x^z(y), \hat{\tau}_y^x(x)), \hat{\tau} : (x, y) \mapsto (\sigma_x^z(y), \tau_y^x(x)) \).

(1) \( \hat{\tau}_x^* = \hat{\tau}_x^{-1} \) if and only if
\[
\hat{\sigma}_x^z(y)(\hat{\tau}_y^x(x)) = x, \quad \hat{\tau}_x^z(\hat{\sigma}_x^z(y)(\hat{\tau}_y^x(x))) = x, \quad \tau_x^z(\hat{\sigma}_x^z(y)(\hat{\tau}_y^x(x))) = y. \tag{2.7}
\]

(2) If \( \sigma_x^z(y) = x \circ y - x \circ z + z \), \( \tau_x^z(x) = \sigma_x^z(y)^{-1} \circ x \circ y, \) then \( \hat{\sigma}_x^z(y) = -x \circ z^{-1} + x \circ y \circ z^{-1} \), \( \hat{\tau}_x^z(y) = \hat{\sigma}_x^z(y)^{-1} \circ x \circ y \).

**Proof.** The proof is straightforward. We prove the two parts of Proposition 2.12 below:

(1) If \( \hat{\tau}_x^* = \hat{\tau}_x^{-1} \), then \( \hat{\tau}_x \hat{\tau}_x^* = \hat{\tau}_x^* \hat{\tau}_x = \text{id} \) and \( \hat{\tau}_x \hat{\tau}_x^*(x, y) = (\hat{\sigma}_x^z(y)(\hat{\tau}_y^x(x)), \hat{\tau}_x^z(\hat{\sigma}_x^z(y)(\hat{\tau}_y^x(x)))) \). Thus \( \sigma_x^z(y)(\hat{\tau}_y^x(x)) = x, \quad \tau_x^z(\hat{\sigma}_x^z(y)(\hat{\tau}_y^x(x))) = y. \) And vice versa if \( \hat{\sigma}_x^z(y)(\hat{\tau}_y^x(x)) = x, \quad \tau_x^z(\hat{\sigma}_x^z(y)(\hat{\tau}_y^x(x))) = y, \) then it automatically follows that \( \hat{\tau}_x^* = \hat{\tau}_x^{-1} \). Similarly, \( \hat{\tau}_x^* \hat{x} = (x, y) \) leads to \( \hat{\sigma}_x^z(y)(\hat{\tau}_y^x(x)) = x, \quad \hat{\tau}_x^z(\hat{\sigma}_x^z(y)(\hat{\tau}_y^x(x))) = y, \) and vice versa.

(2) For the second part of the Proposition it suffices to show (2.7). Indeed,
\[
\hat{\sigma}_x^z(y)(\tau_y^x(x)) = -\sigma_x^z(y) \circ z^{-1} + \sigma_x^z(y) \circ \tau_y^x(x) \circ z^{-1} = -\sigma_x^z(y) \circ z^{-1} + x \circ y \circ z^{-1} = -(x \circ y - x \circ z + z) \circ z^{-1} + x \circ y \circ z^{-1} = -z \circ z^{-1} + x \circ z \circ z^{-1} - x \circ y \circ z^{-1} + x \circ y \circ z^{-1} = x.
\]
Also, \( \hat{\tau}^z_{\hat{r}_G(x)}(\sigma^z_{\hat{r}_G(x)}(y)) = x^{-1} \circ \sigma^z_{\hat{r}_G(x)}(y) \circ \hat{\tau}^z_{\hat{r}_G(x)}(x) = y \). Similarly, we show
\[
\sigma^z_{\hat{r}_G(x)}(\hat{\tau}^z_{\hat{r}_G(x)}(x)) = \hat{\sigma}^z_{\hat{r}_G(x)}(y) \circ \hat{\tau}^z_{\hat{r}_G(x)}(x) - \hat{\sigma}^z_{\hat{r}_G(x)}(y) \circ z + z
\]
\[
= x \circ y - (-x \circ z^{-1} + x \circ y \circ z^{-1}) \circ z + z
\]
\[
= x \circ y - x \circ y + x - z + z = x.
\]

And as above we immediately conclude that \( \tau^z_{\hat{r}_G(x)}(\sigma^z_{\hat{r}_G(x)}(y)) = y \). \( \square \)

**Remark 2.13.** Let \( \hat{r}(x, y) = (\sigma_x(y), \tau_y(x)) \) and \( \hat{r}_{GV}(x, y) = (\hat{\sigma}_x(y), \hat{\tau}_y(x)) \), such that:
\[
\sigma_x(y) = x \circ y - x, \quad \tau_y(x) = \sigma_x(y)^{-1} \circ x \circ y \quad \text{and} \quad \hat{\sigma}_x(y) = -x + x \circ y, \quad \hat{\tau}_y(x) = \hat{\sigma}_x(y)^{-1} \circ x \circ y
\]
i.e. this is the special case \( z = 1 \) (\( \hat{r}_{GV} \) is the solution of [27]). Then via Proposition 2.12 we immediately obtain that \( \hat{r}_{GV} = \hat{r}^{-1} \). Note that the solution \( \hat{r} \) is associated to opposite skew braces (see Theorem 4.1. in [33]).

**Example 2.14** (See [7] Example 5.6 or [6] Example 3.15). Let us consider a set \( \text{Odd} := \{\frac{2n+1}{2k+1} \mid n, k \in \mathbb{Z}\} \) together with two binary operations \( (a, b) \stackrel{+}{\mapsto} a - 1 + b \) and \( (a, b) \stackrel{\circ}{\mapsto} a \cdot b \), where \( +, \cdot \) are addition and multiplication of rational numbers, respectively. The triple \( (\text{Odd}, +, \circ) \) is a brace. By Lemma 2.10 the solution \( \hat{r}_z \) is involutive if and only if \( a \cdot b - a \cdot z + z = a \cdot b - a + 1 \) if and only if \( (z - 1) \cdot (1 - a) = 0 \), for all \( a, b \in B \). Therefore, for all \( z \neq 1 \), \( \hat{r}_z \) is non-involutive. Moreover, \( \hat{r}_z = \hat{r}_w \) if and only if \( -a \cdot z + z = -a \cdot w + w \), that is if \( z = w \).

**Example 2.15** (See [6] Corollary 3.14). Let \( U(\mathbb{Z}/2^n\mathbb{Z}) \) denote a set of invertible integers modulo \( 2^n \), for some \( n \in \mathbb{N} \). Then a triple \( (U(\mathbb{Z}/2^n\mathbb{Z}), +_1, \circ) \) is a brace, where \( a +_1 b = a - 1 + b \) for all \( a, b \in U(\mathbb{Z}/2^n\mathbb{Z}) \), \( + \) and \( \circ \) are addition and multiplication of integer numbers modulo \( 2^n \), respectively. Observe that \( \hat{r}_z = \hat{r}_w \) if and only if \( (a - 1) \circ (w - z) = 0 \) (mod \( 2^n \)), for all \( a \in U(\mathbb{Z}/2^n\mathbb{Z}) \).

**Example 2.16** (See [7] Example 5.7). Let us consider a ring \( \mathbb{Z}/8\mathbb{Z} \). A triple
\[
(\text{OM} := \left\{ \left\{ \begin{array}{cc} a & b \\ c & d \end{array} \right\} \mid a, d \in \{1, 3, 5, 7\}, \ b, c \in \{0, 2, 4, 6\} \right\}, +_0, \circ)
\]
is a brace, where \( (A, B) \stackrel{+_0}{\mapsto} A + 1 + B, \ (A, B) \stackrel{\circ}{\mapsto} A \cdot B \), and \( +, \cdot \) are addition and multiplication of two by two matrices over \( \mathbb{Z}/8\mathbb{Z} \), respectively. Moreover one can easily check that two solutions \( \hat{r}_A \) and \( \hat{r}_B \) are equal if and only if \( (D - 1) \cdot (B - A) = 0 \) (mod 8), for all \( D \in \text{OM} \).

**Example 2.17.** Let \( B \) be a two-sided brace and \( S \) be a left skew brace. Then the product \( B \times S \) is a left skew brace with operations given on coordinates. In this case, for all elements of the form \( (b, 1) \in B \times S \), we can associate a solution, as in Theorem 2.6 by defining \( \hat{r}_{(b, 1)}((a, c), (d, e)) = ((\sigma^b_{\hat{r}_A}(d), \sigma^c_{\hat{r}_B}(e)), (\tau^b_{\hat{r}_A}(a), \tau^c_{\hat{r}_B}(c))) \), for all \( (a, c), (d, e) \in B \times S \).

An interesting observation is that in contrast to the solutions given by skew braces and \( z = 1 \), in our parameter dependent solutions, \( \tau^z_b \) or \( \sigma^z_a \) are not necessarily right group actions, which follows from Lemma 2.18 below. Non-involutive parametric solutions are obtained even in the case that
the underlying algebraic structure is a brace. Furthermore, there does not necessarily have to exist a homomorphism of Yang-Baxter solutions (see Definition 1.1) between $\tilde{r}_z$ and $\tilde{r}_{GV}$, where $\tilde{r}_{GV}$ is the canonical solution given by a left skew brace, as shown in Proposition 2.19. All this make our solutions of particular interest, and they certainly merit further investigation.

**Lemma 2.18.** Let $B$ be a left skew brace and $\tilde{r}_z(x, y) = (\sigma_z^x(y), \tau_z^x(x))$ be a solution defined in the Theorem 2.6. Then $\tau^z : (B, \circ) \rightarrow \text{Aut}(B)$, $b \mapsto \tau^z_b$ is a group action if and only if $a \circ z = z + a$. Similarly, let $\tilde{r}^* = (\sigma^*_z(y), \tau^*_z(x))$ be the inverse solution of Proposition 2.12, then $\sigma^*_z : (B, \circ) \rightarrow \text{Aut}(B)$, $a \mapsto \sigma^*_a$ is a group action if and only if $a \circ z^{-1} = z^{-1} + a$.

**Proof.** If $x \circ z = z + x$ for all $x \in B$, then $\tau^1 = \tau^1$ and one can easily check that $\tau^1$ is an action. In the opposite direction, let $\tau^z_b \tau^z_c = \tau^z_{bc}$. We first compute

\[
\tau^z_c(\tau^z_b(a)) = \sigma^z_{\tau^z_b(a)}(c) - 1 \circ \tau^z_b(a) \circ c = \sigma^z_a(c) \circ \sigma^z_b(b^{-1}) \circ a \circ b \circ c,
\]

but $\sigma^z_a(b) \circ \sigma^z_{\tau^z_b(a)}(c) = \sigma^{zo}_a(b \circ c)$ and hence $\tau^z_c(\tau^z_b(a)) = \tau^{zo}_a(b)$. But due to our assumption that $\tau^z_b$ is a group action the following should hold for all $a, b \in B$

\[
\tau^{zo}_b(a) = \tau^z_b(a) \Rightarrow \sigma^{zo}_a(b) = \sigma^z_a(b)
\]

which leads to $-a \circ z + z = -a \circ z \circ z \circ z$. For $a = z^{-1}$ we get $z \circ z = z + z \Leftrightarrow z = -z^{-1}$. Then

\[
-a \circ z + z = -a \circ z \circ z \circ z \Rightarrow
\]

\[
-a \circ z + z = -a \circ (z + z) + z + z \Rightarrow
\]

\[
a - a \circ z + z = 0 \Rightarrow a \circ z = z + a.
\]

Similar proof holds for the second part of the Lemma. 


**Proposition 2.19.** Let $(B, \circ, +)$ and $(S, \bullet, +_s)$ be left skew braces, and let $z \in B$ be $u$-distributive. Let also $\tilde{r} : S \times S \rightarrow S \times S$ be the solution $r_{GV}(a, b) = (-a + s a \bullet b, (-a + s a \bullet b)^{-1} \bullet a \bullet b)$ ($-a$ is the inverse with respect to $+s$, $a^{-1}$ the inverse with respect to $\bullet$) and $\tilde{r}_z : B \times B \rightarrow B \times B$ be the solution $r_z(a, b) = (-a \circ z^{-1} + a \circ b \circ z^{-1}, (-a \circ z + a \circ b \circ z^{-1} \circ a \circ b)$. If there exists a map $f : S \rightarrow B$ such that

\[
(f \times f)\tilde{r}_{GV} = \tilde{r}_z(f \times f),
\]

\[
(2.8)
\]

\[
1, z \in \text{Im}(f)\) and $f(1) + z = z + f(1)$, then $z = -z^{-1}$.

**Proof.** From (2.8) it follows

\[
f(-a + s a \bullet b) = -f(a) \circ z^{-1} + f(a) \circ f(b) \circ z^{-1}
\]

(2.9)

By setting $(i) a = 1$, and $(ii) b = 1$ we obtain equalities:

\[(i) f(b) = -f(1) \circ z^{-1} + f(1) \circ f(b) \circ z^{-1}
\] and \[(ii) f(1) = -f(a) \circ z^{-1} + f(a) \circ f(1) \circ z^{-1}.
\]

(1) If $f(a) = z$ in equality $(i)$, then $f(1) \circ z = z \circ f(1)$. 


(2) If \( f(b) = z \) in equality (ii), then \( f(1) \circ z^{-1} = f(1) - z \).

(3) If \( f(a) = 1 \) in equality (i), then \( f(1) \circ z^{-1} = z^{-1} + f(1) \).

From the last two expressions above and the fact that \( f(1) + z = z + f(1) \), we conclude that \( z = -z^{-1} \). That is for generic values of \( z \) there exists no homorphism between the two distinct solutions.

\[ \blacksquare \]

**Proposition 2.20.** Let \((B, +, \circ), (S, +_S, \bullet)\) be left skew braces, and let \( z \in B \) be \( u \)-distributive. Then we can consider the following two solutions \( \tilde{r}_1 : S \times S \to S \times S \) \( r_z : B \times B \to B \times B \) from the Proposition 2.12. Let \( \eta \in B \) be such that \( z + \eta \neq \eta \circ z \) and \( f : S \to B \) be a function such that \( \eta, 1 \in \operatorname{Im}(f) \), and \( f(1) + \eta = \eta + f(1) \), then \( f \) is not a homomorphism of the Yang-Baxter solutions, i.e.

\[ (f \times f)\tilde{r}_1 \neq r_z(f \times f). \] (2.10)

**Proof.** Let us assume ad absurdum that \( f \) is a homomorphism of the Yang-Baxter solutions. Then

\[
(f \times f)\tilde{r}_1 = r_z(f \times f),
\]

\[
f(-s a + s a \bullet b) = f(a) \circ f(b) - f(a) \circ z + z,
\]

\[
f((\sigma_a(b))^{-1} \bullet a \bullet b) = \sigma_f^z(f(b))^{-1} \circ f(a) \circ f(b),
\]

for all \( a, b \in S \). Observe that \( f(1) \) commutes with all \( f(a) \), for \( a \in S \),

\[
f(a) = f(\tau_1(a)) = \tau_{f(1)}^z(f(a)) = \sigma_{f(a)}^z(f(1))^{-1} \circ f(a) \circ f(1) = f(\sigma_a(1))^{-1} \circ f(a) \circ f(1) = f(1)^{-1} \circ f(a) \circ f(1),
\]

and thus \( f(1) \circ f(a) = f(a) \circ f(1) \). Moreover, by taking \( b = 1 \), we get that

\[
f(1) = f(a) \circ f(1) - f(a) \circ z + z \& - f(a) \circ f(1) + f(1) = -f(a) \circ z + z.
\]

Thus, \( \sigma^z = \sigma^f(1) \) and \( \tau^z = \tau^f(1) \). Then for all \( a \in S \),

\[
f(a) = f(\sigma_1(a)) = \sigma_{f(1)}^f(f(a)) = f(1) \circ f(a) - f(1)^2 + f(1),
\]

and \(-f(1) \circ f(a) + f(a) = -f(1)^2 + f(1)\), which for \( f(a) = 1 \) gives \( f(1)^2 = f(1) + f(1) \). By simple substitution we get \(-f(1) \circ f(a) + f(a) = -f(1)\), and thus \( f(a) + f(1) = f(1) \circ f(a) = f(a) \circ f(1) \).

Thus, finally,

\[
f(1) = f(a) \circ f(1) - f(a) \circ z + z \implies f(1) = f(a) + f(1) - f(a) \circ z + z,
\]

but for \( f(a) = \eta \), we have that \( f(1) + \eta = \eta + f(1) \), and

\[
\eta \circ z = z + \eta,
\]

which contradicts with the assumption. Thus \( f \) is not a homomorphism of Yang-Baxter solutions.

\[ \blacksquare \]
Corollary 2.21. Observe that if $B$ is a left brace and $z + a \neq a \circ z$ for all $a \in B$, then there does not exist a skew brace $S$ such that $r_1 : S \times S \to S \times S$ is isomorphic to $r_2 : B \times B \to B \times B$, that is every surjection on the left brace satisfies all the assumptions of the map from Proposition 2.20.

We conclude the section by presenting the following example of a solution in which $a \mapsto \tau_i^7$ is not a right action of $(B, \circ)$, and the solution is not isomorphic to any other solution with parameter 1.

Example 2.22. Let us consider a two-sided brace $U(\mathbb{Z}/16\mathbb{Z})$ as in Example 2.15. Observe that in this case $\tilde{r}_7$ is not equivalent to $\tilde{r}_1$ as $5 - 1 + 7 = 11 \pmod{16}$ and $5 \circ 7 = 3 \pmod{16}$. One can easily compute that

$$
\tau_{15}^7(5) = 5 \pmod{16} \quad \& \quad \tau_{23}^7 \tau_{15}^7(5) = 13 \pmod{16}.
$$

3. Preliminaries on Quasi-bialgebras & Drinfeld twists

3.1. Quasi-bialgebras. In this subsection we recall fundamental definitions on quasi-bialgebras and the notion of quasi-triangularity as well as some recent relevant results presented in [16].

Definition 3.1. A quasi-bialgebra $(A, \Delta, \epsilon, \Phi, c_l, c_r)$ is a unital associative algebra $A$ over some field $k$ with the following algebra homomorphisms:

- the co-product $\Delta : A \to A \otimes A$
- the co-unit $\epsilon : A \to k$

together with the invertible element $\Phi \in A \otimes A \otimes A$ (the associator) and the invertible elements $c_l, c_r \in A$ (unit constraints), such that:

1. $(\text{id} \otimes \Delta)\Delta(a) = \Phi\left( (\Delta \otimes \text{id})\Delta(a) \right)\Phi^{-1}, \forall a \in A$.
2. $\left( (\text{id} \otimes \text{id} \otimes \Delta)\Phi \right)\left( (\Delta \otimes \text{id} \otimes \text{id})\Phi \right) = \left( 1 \otimes \Phi \right)\left( (\text{id} \otimes \Delta \otimes \text{id})\Phi \right)\left( \Phi \otimes 1 \right)$.
3. $(\epsilon \otimes \text{id})\Delta(a) = c_l^{-1}ac_l$ and $(\text{id} \otimes \epsilon)\Delta(a) = c_r^{-1}ac_r, \forall a \in A$.
4. $(\text{id} \otimes \epsilon \otimes \text{id})\Phi = c_r \otimes c_l^{-1}$.

In the special case where $\Phi = 1 \otimes 1 \otimes 1$ one recovers a bialgebra, i.e. co-associativity is restored.

Using the axioms of Definition 3.1 further counit formulas for the associator and unit constraints can be derived [16].

Lemma 3.2. (Lemma [16]) Let $(A, \Delta, \epsilon, \Phi, c_r, c_l)$ be a quasi-bialgebra, then:

$$(\epsilon \otimes \text{id} \otimes \text{id})\Phi = \Delta(c_l^{-1})(c_l \otimes 1), \quad (\text{id} \otimes \text{id} \otimes \epsilon)\Phi = (1 \otimes c_r^{-1})\Delta(c_r) \quad \& \quad \epsilon(c_l) = \epsilon(c_r).$$

We introduce here some useful notation. Let $\pi : A \otimes A \to A \otimes A$ be the “flip” map, such that $a \otimes b \mapsto b \otimes a \forall a, b \in A$, then we set $\Delta^{(op)} := \pi \circ \Delta$. A quasi-bialgebra is called cocommutative if $\Delta^{(op)} = \Delta$. We also consider the general element $A = \sum a_j \otimes b_j \in A \otimes A$, then in the “index” notation we denote: $A_{12} := \sum a_j \otimes b_j \otimes 1$, $A_{23} := \sum 1 \otimes a_j \otimes b_j$ and $A_{13} := \sum a_j \otimes 1 \otimes b_j$. 
The notion of quasi-triangularity for bialgebras extends to quasi-bialgebras [19, 32].

**Definition 3.3.** A quasi-bialgebra \((A, \Delta, \epsilon, \Phi)\) is called quasi-triangular (or braided) if an invertible element \(R \in A \otimes A\) (universal \(R\)-matrix) exists, such that

1. \(\Delta^{(op)}(a)R = R\Delta(a), \forall a \in A\).
2. \((\text{id} \otimes \Delta)R = \Phi_{231}^{-1}\Phi_{132}\Phi_{213}\Phi_{123}^{-1}\).
3. \((\Delta \otimes \text{id})R = \Phi_{312}\Phi_{132}^{-1}\Phi_{23}^{-1}\Phi_{123}\).

From the axioms (1)-(3) of Definition 3.3 and condition (3) of Definition 3.1 one deduces that \((\epsilon \otimes \text{id})R = c_l^{-1}c_r\) and \((\text{id} \otimes \epsilon)R = c_r^{-1}c_l\). Moreover, by means of conditions (1)-(3) of Definition 3.3 it follows that \(R\) satisfies a non-associative version of the quantum Yang Baxter equation (QYBE) \(R_{12}\Phi_{312}\Phi_{132}^{-1}\Phi_{23}\Phi_{123} = \Phi_{321}\Phi_{23}\Phi_{213}\Phi_{123}\Phi_{123}\).

In the case of \(\Phi = 1 \otimes 1 \otimes 1\) one recovers the familiar QYBE and a quasi-triangular bialgebra.

In the following proposition we consider two special cases of the general algebraic setting for quasi-triangular quasi-bialgebras as described above. This setup, introduced in [16], will be particularly relevant to the findings of the next section.

**Proposition 3.4.** *(Proposition [16])* Let \((A, \Delta, \epsilon, \Phi, R)\) be a quasi-triangular quasi-bialgebra, then the following two statements hold:

1. Suppose \(\Phi\) satisfies the condition (in the index notation) \(\Phi_{213}\Phi_{123} = \Phi_{123}\Phi_{213}\), then
   \[
   (\text{id} \otimes \Delta)R = \Phi_{231}^{-1}\Phi_{132}\Phi_{123}, \quad (\Delta \otimes \text{id})R = \Phi_{132}\Phi_{23}\Phi_{123},
   \]
   and the universal \(R\) matrix satisfies the usual YBE. Also, \((\epsilon \otimes \text{id})R = c_l\), \((\text{id} \otimes \epsilon)R = c_l^{-1}\) and \(c_r = 1_A\).
2. Suppose \(\Phi\) satisfies the condition \(\Phi_{132}\Phi_{23} = \Phi_{23}\Phi_{123}\), then
   \[
   (\text{id} \otimes \Delta)R = \Phi_{132}\Phi_{123}^{-1}, \quad (\Delta \otimes \text{id})R = \Phi_{312}\Phi_{132}\Phi_{23},
   \]
   and the universal \(R\) matrix satisfies the usual YBE. Also, \((\epsilon \otimes \text{id})R = c_r^{-1}\), \((\text{id} \otimes \epsilon)R = c_r\) and \(c_l = 1_A\).

The detailed proof of the proposition above is given in [16].

**3.2. Drinfeld twists.** We recall here basic facts about the twisting of quasi-bialgebras and the generalized results obtained in [16]. Drinfeld showed in [18, 19] that the property of being quasi-triangular (quasi-)bialgebra is preserved by using a suitable twist \(F \in A \otimes A\). Usually whenever the notion of Drinfeld twist is discussed in the literature a trivial action of the co-unit on the twist is almost always assumed, i.e. \((\epsilon \otimes \text{id})F = (\text{id} \otimes \epsilon)F = 1\). In [16] this condition is relaxed and the most general scenario is examined. We should note that in [19] certain types of simple twists without this
According to Proposition 3.4 we distinguish two cases: let the conditions of Proposition 3.4 hold. We recall also the useful notation: Lemma 3.6. Let \( \Phi \), \( \Delta \), \( \epsilon \), \( \Phi \), \( \mathcal{R} \) be a quasi-triangular quasi-bialgebra and let \( \mathcal{F} \in A \otimes A \) be an invertible element, such that
\[
\begin{align*}
\Delta_{\mathcal{F}}(a) &= \mathcal{F}\Delta(a)\mathcal{F}^{-1}, \quad \forall a \in A \\
\Phi_{\mathcal{F}}(\mathcal{F} \otimes 1)((\Delta \otimes \text{id})\mathcal{F}) &= (1 \otimes \mathcal{F})(\text{id} \otimes \Delta)\mathcal{F} \Phi \\
\mathcal{R}_{\mathcal{F}} &= \mathcal{F}^{(op)}\mathcal{R}^{-1}
\end{align*}
\]
where \( \mathcal{F}^{(op)} := \pi(\mathcal{F}) \), (recall \( \pi \) is the flip map). Then \( (A, \Delta_{\mathcal{F}}, \epsilon, \Phi_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}}) \) is also a quasi-triangular quasi-bialgebra.

For the proof of Proposition 3.5 the general twist \( \mathcal{F} = \sum_{j} f_j \otimes g_j \) is considered (details on the proof of the proposition can be found in [16]). In terms of the invertible elements \( v := \sum_{j} \epsilon(f_j)g_j \), \( w := \sum_{j} \epsilon(g_j)f_j \), we have \( (\epsilon \otimes \text{id})\mathcal{F} = v \), \( (\text{id} \otimes \epsilon)\mathcal{F} = w \), so the trivial constraint is relaxed and indeed the most general scenario is regarded.

We introduce now a general frame, which will be appropriate when examining quantum algebras emerging from non-degenerate, set-theoretic solutions of the YBE presented in the next section, compatible also with the analysis in [15, 16]. The following lemma is a generalization of Proposition 1.5 and Remark 1.9 of [16], suitable for the purposes of the next section.

Lemma 3.6. Let \( (A, \Delta, \epsilon, \Phi, \mathcal{R}) \) and \( (A, \Delta_{\mathcal{F}}, \epsilon, \Phi_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}}) \) be quasi-triangular quasi-bialgebras and let the conditions of Proposition 3.4 hold. We recall also the useful notation: \( \mathcal{F}_{12,3} := (\text{id} \otimes \Delta)\mathcal{F} \), \( \mathcal{F}_{12,3} := (\Delta \otimes \text{id})\mathcal{F} \), and by the quasi-bialgebra axioms \( \mathcal{F}_{21,3}\mathcal{R}_{12} = \mathcal{R}_{12}\mathcal{F}_{12,3}, \mathcal{F}_{13,2}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{F}_{1,23} \). According to Proposition 3.4 we distinguish two cases:

1. If the associators satisfy
\[
\Phi_{21,3}\mathcal{R}_{12} = \mathcal{R}_{12}\Phi_{12,3}, \quad \Phi_{21,3}\mathcal{R}_{12} = \mathcal{R}_{12}\Phi_{12,3},
\]
and \( \Phi_{\mathcal{F}} \) commutes with \( \mathcal{F}_{12} \), then (in the index notation) the condition (3.2) can be re-expressed as \( \mathcal{F}_{12,3} := \mathcal{F}_{23}\mathcal{F}_{1,23} = \mathcal{F}_{23}\mathcal{F}_{12,3} \). We also deduce that \( \mathcal{F}_{21,3}\mathcal{R}_{12} = \mathcal{R}_{12}\mathcal{F}_{12,3} \). This is compatible also with the first part of Proposition 3.4.

2. If the associator satisfies
\[
\Phi_{13,2}\mathcal{R}_{23} = \mathcal{R}_{23}\Phi_{12,3}, \quad \Phi_{13,2}\mathcal{R}_{23} = \mathcal{R}_{23}\Phi_{12,3},
\]
and \( \Phi_{\mathcal{F}} \) commutes with \( \mathcal{F}_{23} \), then then condition (3.2) is re-expressed as \( \mathcal{F}_{12,3} := \mathcal{F}_{23}\mathcal{F}_{1,23} = \mathcal{F}_{12,3}\mathcal{R}_{12,3} \). We also deduce that \( \mathcal{F}_{13,2}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{F}_{1,23} \). This is compatible with the second part of Proposition 3.4.
Proof. The proof is straightforward due to Proposition 3.5 and constraints (3.4), (3.5). □

4. Non-involutive solutions of the YBE & Drinfeld twists

We move on to the study of non-involutive set-theoretic solutions of the YBE, admissible Drinfeld twists and the associated quasi-bialgebras. From now on we work over the field \( k = \mathbb{C} \). We first review some fundamental results on the admissible Drinfeld twists for involutive set-theoretic solutions of the YBE derived in [15] and we use these twists to extend these results to the non-involutive case.

Let \( X \) be a set with \( n \) elements and \( \tilde{r}_z : X \times X \rightarrow X \times X \) be a solution of the set-theoretic braid equation, given in the previous section. It is convenient for the purposes of this section to consider a free vector space \( V = \mathbb{C}^X \) of dimension equal to the cardinality of \( X \). Let \( B = \{ e_x \}_{x \in X} \) be the basis of \( V \) and \( B^* = \{ e^*_x \}_{x \in X} \) be the dual basis: \( e^*_x e_y = \delta_{x,y} \). Let also \( f \in V \otimes V \) be expressed as \( f = \sum_{x,y \in X} f(x,y) e_x \otimes e_y \), then the set-theoretic solution is a map \( \tilde{r}_z : V \otimes V \rightarrow V \otimes V \), such that (\( \tilde{r}_z f \))(x, y) = \( f(\sigma^z_x(y), \tau^z_y(x)) \), \( \forall x, y \in X \); equivalently, \( \tilde{r}_z = \sum_{x,y \in X} e_{x,\sigma^z_x(y)} \otimes e_{y,\tau^z_y(x)} \),

where \( e_{x,y} \) are defined as \( e_{x,y} = e_x e^*_y \). In this construction \( e_x \) corresponds to a \( n \)-column vector with 1 at the \( x \) position and 0 elsewhere, \( e^*_x = e^T \) (\( T \) denotes transposition) and \( e_{x,y} \) is an \( n \times n \) matrix with one identity entry in \( x \)-row and \( y \)-column, and zeros elsewhere, i.e \( (e_{x,y})_{z,w} = \delta_{x,z} \delta_{y,w} \), and \( \tilde{r}_z \) is a \( n^2 \times n^2 \) matrix.

The matrix, \( \tilde{r}_z \) satisfies the braid equation:

\[
(id \otimes \tilde{r}_z)(\tilde{r}_z \otimes id)(id \otimes \tilde{r}_z) = (\tilde{r}_z \otimes id)(id \otimes \tilde{r}_z)(\tilde{r}_z \otimes id).
\]

We also recall \( r_z = \mathcal{P}\tilde{r}_z \), where \( \mathcal{P} = \sum_{x,y \in X} e_{x,y} \otimes e_{y,x} \) is the permutation operator, then \( r_z \) has the explicit form

\[
r_z = \sum_{x,y \in X} e_{y,\sigma^z_x(y)} \otimes e_{x,\tau^z_y(x)},
\]

i.e. \( (r_z f)(y,x) = f(\sigma^z_x(y), \tau^z_y(x)) \), and satisfies the YBE.

Henceforth, we will drop the \( z \)-index in \( \tilde{r}_z \), \( \sigma^z \), \( \tau^z \) for brevity, although it is always implied.

Before we start discussing the admissible twists we recall the definitions of two quadratic algebras \( \mathcal{A} \) and \( \mathcal{Q} \) associated to set-theoretic solutions, which arise from the FRT (Faddeev, Reshetikhin and Takhtajan) construction [22]. This will be useful for our considerations later in the text. Indeed, from the FRT construction we recall:

---

1This can be formally extended to the countably infinite case \( (n \rightarrow \infty) \), provided that finite norm elements of the space are considered. The infinite case and possible connections with orthogonal polynomials will be discussed in detail elsewhere.
Definition 4.1. Given a solution of the braid equation \( \hat{r} : V \to V \) \((V = \mathbb{C}X)\), for a finite set \( X \), the associated quantum algebra \( \mathcal{A} \) is a quotient of a free associative \( C \)-algebra, generated by \( \{L_{z,w} \mid x, w \in X\} \), and relations

\[
\hat{r}_{12} L_1 L_2 = L_1 L_2 \hat{r}_{12}, \quad (4.1)
\]

where \( L = \sum_{x,y \in X} e_{x,y} \otimes L_{x,y} \in \text{End}(V) \otimes \mathcal{A} \). Recall the index notation: \( \hat{r}_{12} = \hat{r} \otimes 1_{\mathcal{A}} \) and \( L_1 = \sum_{z,w \in X} e_{z,w} \otimes I \otimes L_{z,w}, \quad L_2 = \sum_{z,w \in X} I \otimes e_{z,w} \otimes L_{z,w} \).

From the fundamental relation (4.1) for the set-theoretic solution of the braid equation [20]:

\[
L_{x,z} L_{y,z} = L_{y,z} L_{x,z}, \quad (4.2)
\]

See also [13] for the algebra associated to the Baxterized \( r \)-matrix.

Definition 4.2. Given a solution of the YBE \( r : V \to V \), the quadratic algebra \( \mathcal{Q} \) is generated by \( \{q_x \mid x \in X\} \) and relations

\[
r_{12} q_1 q_2 = q_2 q_1, \quad (4.3)
\]

where \( q = e_x \otimes q_x \in V \otimes \mathcal{Q} \). Also, \( r_{12} = r \otimes 1_{\mathcal{A}} \), \( q_1 = \sum_{z,w \in X} e_{z,w} \otimes I \otimes q_x, \quad q_2 = \sum_{x \in X} I \otimes e_x \otimes q_x \).

The quadratic relation (4.3) for the set-theoretic solution of the YBE implies

\[
q_x q_y = q_{\sigma_x(y)} q_{\tau_y(x)}, \quad (4.4)
\]

also obtained in [20].

It is worth noting that if we conveniently re-express \( L = \sum_{x,y \in X} e_{y,\sigma_x(y)} \otimes L_{x,\tau_y(x)} \), imitating the form of the set-theoretic \( r \)-matrix, we conclude via (4.1) that the relations of the corresponding quantum algebra \( \mathcal{A} \) are,

\[
L_{\eta,\tau_y(x)} L_{x,\tau_y(x)} = L_{\eta,\tau_y(x)} L_{x,\tau_y(x)} \eta, \quad (4.5)
\]

subject to: \( \sigma_\eta(\sigma_x(y)) = \sigma_{\sigma_x(y)}(\sigma_y(x)) \) and \( \tau_\eta(\sigma_x(y)) = \sigma_{\tau_y(x)}(\tau_y(x)) \). The quantum algebras \( \mathcal{A} \) with relations (4.2), (4.5) are also closely related to the reflection algebra for set-theoretic solutions [14] and will be discussed in detail in future works.

We now briefly recall the notion of admissible Drinfeld twists for set-theoretic solutions. It was shown in [15] that all involutive set-theoretic solutions can be obtained from the permutation operator via suitable twists. Two distinct admissible twists \( F, \hat{F} \in \text{End}(V \otimes V) \) were identified in [15]. Indeed, consider the invertible elements \( \mathcal{V}_x = \sum_{y \in X} e_{\sigma_x(y),y} \) and \( \mathcal{W}_y = \sum_{x \in X} e_{\tau_y(x),x} \), then

\[
F = \sum_{x \in X} e_{x,x} \otimes \mathcal{V}_x, \quad \hat{F} = \sum_{y \in X} \mathcal{W}_y \otimes e_{y,y}. \quad (4.6)
\]

These are both admissible twists in the involutive case as was shown in [15].

We will show in what follows that both twists (4.6) are still admissible in the case of non-involutive solutions and we will identify the twisted \( r \)-matrices. Note that, similarly to the involutive case analysed in [16], the underlying algebra is not a quasi-triangular bialgebra, as one would expect, but a quasi-triangular quasi-bialgebra. We shall now prove the following useful lemma.
Lemma 4.3. Let \( \tilde{\tau} : V \otimes V \to V \otimes V \) be a non-degenerate, non-involutive, set-theoretic solution of the braid equation, \( \tilde{\tau} = \sum_{x,y \in X} e_{x,\sigma_x(y)} \otimes e_{y,\tau_y(x)} \). Suppose that for every \( x \in X \), the elements

\[
\mathbb{V}_x = \sum_{y \in X} e_{\sigma_x(y),y}, \quad \mathbb{W}_x = \sum_{\eta \in X} e_{\tau_x(\eta),\eta},
\]

satisfy,

\[
\Delta(\mathbb{V}_x) = \sum_{x,y \in X} e_{\sigma_y(x),x} \otimes e_{\sigma_{\tau_y}(\eta)(y),y}, \quad \Delta(\mathbb{W}_y) = \sum_{\eta,x \in X} e_{\tau_{\sigma_x}(y) (\eta),\eta} \otimes e_{\tau_y(x),x}.
\]

Then \( \Delta(\mathbb{V}_x) \tilde{\tau} = \tilde{\Delta}(\mathbb{V}_x) \), \( \mathbb{V}_x \in \{ \mathbb{V}_x, \mathbb{W}_x \} \).

Proof. It is computationally easier to show that \( [\Delta(\mathbb{V}_x^{-1}), \tilde{\tau}] = 0 \), where specifically \( \Delta(\mathbb{V}_x^{-1}) = \Delta(\mathbb{V}_x)^T \) (also \( \mathbb{V}_x^{-1} = \mathbb{V}_x^T \)) and \( T \) denotes transposition. We compute explicitly

\[
\tilde{\tau} \Delta(\mathbb{V}_x^{-1}) = \sum_{y,z \in X} e_{y,\sigma_z(y)} \otimes e_{y,\sigma_x(z) (\tau_x(y))},
\]

\[
\Delta(\mathbb{V}_x^{-1}) \tilde{\tau} = \sum_{y,z \in X} e_{y,\sigma_{\tau_y}(x) (z)} \otimes e_{\tau_y(x),z}.
\]

Due to conditions (2.2) and (2.4) for the set-theoretic solution \( \tilde{\tau} \) we conclude that for all \( x \in X \),

\[
[\tilde{\tau}, \Delta(\mathbb{V}_x^{-1})] = 0.
\]

Similarly, by conditions (2.3) and (2.4) we show that for all \( y \in X \),

\[
[\tilde{\tau}, \Delta(\mathbb{W}_y^{-1})] = 0,
\]

and this concludes our proof. \( \square \)

It is worth noting that the form and the coproduct structure of the elements \( \mathbb{V}_x, \mathbb{W}_x \) are inspired by tensor representations of the RTT algebra (4.1) (we refer the interested reader to [22] [15]). It is also worth recalling at this point the algebra \( Q \) generated by \( q_x, x \in X \) (4.1). It turns out that \( \mathbb{V}_x \) and \( \mathbb{W}_x^T \) (\( T \) denotes transposition), defined earlier, are \( n \)-dimensional representations of \( Q \), i.e. \( q_x \mapsto U_x \) , where \( U_x \in \{ \mathbb{V}_x, \mathbb{W}_x^T \} \), indeed:

Lemma 4.4. The \( n \times n \) matrices \( \mathbb{V}_x = \sum_{y \in X} e_{\sigma_x(y),y} \) and \( \mathbb{W}_x^T = \sum_{\eta \in X} e_{\eta,\tau_x(\eta)} \), for all \( x \in X \), satisfy the algebraic relations (4.4).

Proof. The proof is based on straightforward computation and use of the properties of skew braces:

\[
\mathbb{V}_x \mathbb{V}_y = \sum_{z \in X} e_{\sigma_x(y),z} \otimes e_{\sigma_{\tau_y}(z),x} = \mathbb{V}_x \mathbb{V}_y = \mathbb{V}_x \mathbb{V}_y = \mathbb{V}_x \mathbb{V}_y = \mathbb{V}_x \mathbb{V}_y = \mathbb{V}_x \mathbb{V}_y = \mathbb{V}_x \mathbb{V}_y.
\]

and similarly, \( \mathbb{W}_x^T \mathbb{W}_y^T = \mathbb{W}_x^T \mathbb{W}_y^T = \mathbb{W}_x^T \mathbb{W}_y^T = \mathbb{W}_x^T \mathbb{W}_y^T \). \( \square \)

Also, the coproducts \( \Delta(\mathbb{V}_x), \Delta(\mathbb{W}_x^T) \) defined in (4.8), satisfy the algebraic relations (4.4), i.e. \( \Delta \) is indeed an algebra homomorphism.

We may now extend the results of [15] [16] to the non-involutive case.
Proposition 4.5. (Proposition 15) We recall the following quantities introduced in 15:

\[ F_{1,23} = \sum_{x,y \in X} e_{\eta,\eta} \otimes e_{\sigma_\eta(x),x} \otimes e_{\sigma_\eta(y),y}, \quad F^*_{1,23} = \sum_{x,y \in X} e_{\eta,\eta} \otimes e_{x,x} \otimes e_{\eta,\eta(\sigma_\eta(y)),y} \]

\[ \hat{F}^*_{1,23} = \sum_{x,y \in X} e_{\eta,\eta} \otimes e_{x,x} \otimes e_{y,y}, \quad \hat{F}_{1,23} = \sum_{x,y \in X} e_{\eta,\eta} \otimes e_{\eta,\eta(\sigma_\eta(y)),x} \otimes e_{y,y}. \]

subject to \[(2.3)-(2.4).\] Let also \( \hat{r} : V \otimes V \rightarrow V \otimes V \) be the set-theoretic solution of the braid equation \( \hat{r} = \sum_{x,y \in X} e_{x,\sigma_\eta(y)} \otimes e_{y,\eta(x)}, \) then

\[ \hat{r}_{12} F^*_{123} = F^*_{12} \hat{r}_{12}, \quad \hat{r}_{23} F_{1,23} = F_{1,23} \hat{r}_{23} \quad \& \quad \hat{r}_{12} \hat{F}^*_{123} = \hat{F}^*_{123} \hat{r}_{12}, \quad \hat{r}_{23} \hat{F}_{1,23} = \hat{F}_{1,23} \hat{r}_{23}. \]

Proof. The proof goes along the same lines as in 15. Indeed, we first observe, recalling Lemma 4.3 and the definition of \( F, \) that \( F_{1,23} = (\text{id} \otimes \Delta) F, \) whereas \( F^*_{1,23} \neq (\Delta \otimes \text{id}) F.\) Hence, we immediately conclude that \( [F_{1,23}, \hat{r}_{23}] = 0; \) also by explicit computation and bearing in mind (2.2) we show that \( [F^*_{1,23}, \hat{r}_{12}] = 0. \)

Similarly, due to Lemma 4.3 and the definition of \( \hat{F}, \) \( \hat{F}_{1,23} = (\Delta \otimes \text{id}) \hat{F}, \) whereas \( \hat{F}^*_{1,23} \neq (\text{id} \otimes \Delta) \hat{F}.\) Then it immediately follows that \( [\hat{F}^*_{1,23}, \hat{r}_{12}] = 0; \) also by explicit computation and by recalling (2.2) we conclude that \( [\hat{F}^*_{1,23}, \hat{r}_{23}] = 0, \) (see also relevant Lemma 3.6).

The admissibility of the twist for involutive solutions was proven in Proposition 3.15 in 15. In the following proposition we generalize this result in the non-involutive scenario.

Proposition 4.6. (Proposition 15) Let \( T_{12} = T \otimes \text{id}, T_{23} = \text{id} \otimes T, \) where \( T \in \{ F, \hat{F} \} \) and \( F, \hat{F} \) are given in (4.6). Let also \( F^*_{1,23}, F_{1,23}, \hat{F}_{1,23}, \hat{F}^*_{1,23} \) defined in Proposition 4.5. Then

\[ F_{123} := F_{12} F^*_{12,3} = F_{23} F_{1,23}, \quad F^*_{123} := \hat{F}_{12} \hat{F}^*_{12,3} = \hat{F}_{23} \hat{F}^*_{1,23}. \]

Proof. The proof again goes along the same lines as in 15. By substituting the expressions for \( F_{12}, F_{23}, F^*_{12,3} \) and \( F_{1,23} \) and recalling that conditions (2.2)-(2.4) hold, we obtain,

\[ F_{123} := F_{12} F^*_{1,23} = F_{23} F_{1,23} = \sum_{\eta,x,y \in X} e_{\eta,\eta} \otimes e_{\sigma_\eta(x),x} \otimes e_{\sigma_\eta(y),y}. \]

Similarly, from expressions \( \hat{F}_{12}, \hat{F}_{23}, \hat{F}^*_1,23 \) and \( \hat{F}^*_1,23 \) and using conditions (2.2)-(2.4) we obtain,

\[ \hat{F}_{123} := \hat{F}_{12} \hat{F}_{1,23} = \hat{F}_{23} \hat{F}^*_1,23 = \sum_{\eta,x,y \in X} e_{\eta,\eta} \otimes e_{\eta,\eta(\sigma_\eta(y)),x} \otimes e_{y,y}. \]

Remark 4.7. We should note that no matter what the action of the counit on \( V_x, W_x \) is, we deduce that \( (\text{id} \otimes e) F = \sum_{\eta \in X} e(V_\eta) e_{\eta,\eta} \) (it becomes the \( n \times n \) identity matrix \( 1_n \) if \( e(V_\eta) = 1, \) for all \( \eta \in X \)) and \( (e \otimes \text{id}) F = \sum_{\eta \in X} e(\eta) V_\eta \neq 1_n. \) Similarly, \( (e \otimes \text{id}) \hat{F} = \sum_{x \in X} e(W_x) e_{x,x} \) (it becomes the \( n \times n \) identity matrix if \( e(W_x) = 1, \) for all \( x \in X \)) and \( (\text{id} \otimes e) \hat{F} = \sum_{x \in X} e(e_{x,x}) V_x \neq 1_n. \)
Remark 4.8. Propositions 4.5, 4.6 are essential in showing that if \( \hat{r} \) is a solution of the braid equation, then \( \hat{r}_F \) also is (see e.g. [17, 15] and Proposition 3.5). Indeed, if \( \hat{r} \) satisfies the braid equation, then by acting from the left of the braid equation with \( F_{123} \) and from the right with \( F_{123}^{-1} \) and recalling that \( F \hat{r} F^{-1} = \hat{r}_F \), as well as Propositions 4.5, 4.6 we conclude that \( \hat{r}_F \) also satisfies the braid equation.

We now identify the twisted \( \hat{r} \)-matrices, which are also solutions of the braid equation, as well as the associated twisted coproducts.

Lemma 4.9. Let \( \hat{r} : V \otimes V \to V \otimes V \) be the set-theoretic solution of the braid equation. Let also \( V_x = \sum_{y \in X} e_{\sigma_x(y),y} \) and \( W_x = \sum_{y \in X} e_{\tau_x(y),y} \), for all \( x \in X \), with coproducts defined in (4.8). Then \( \Delta_T(\hat{r}_x) = \hat{r}_T \Delta_T(\hat{r}_x) \), where \( \hat{r}_x \in \{ V_x, W_x \} \), \( T \in \{ F, \hat{F} \} \),

\[
\Delta_F(V_x) = V_x \otimes V_x, \quad \Delta_F(W_y) = \sum_{\eta, x \in X} e_{\tau_{\sigma_x(y),x}}(\eta) \otimes e_{\tau_{\sigma_x(y),x}}(\eta) \otimes e_{\delta_x(y),y}(x) \\
\Delta_{\hat{F}}(V_x) = \sum_{x, y \in X} e_{\sigma_{\tau_x(y),x}}(\eta) \otimes \tau_y(x) \otimes e_{\tau_x(y),y}(x) \quad \Delta_{\hat{F}}(W_y) = W_y \otimes W_y
\]

and the twisted matrices read as

\[
\hat{r}_F = \sum_{x, y \in X} e_{x, \sigma_x(y)} \otimes e_{x, \sigma_x(y)}(\tau_y(x)) \quad \& \quad \hat{r}_{\hat{F}} = \sum_{x, y \in X} e_{\tau_y(x), \tau_y(x)}(\tau_y(x)) \otimes e_{y, \tau_y(x)}.
\]

Proof. The proof immediately follows after recalling expressions \( V_x, W_x, F, \hat{F}, r, r, \Delta(V_x), \Delta(W_x) \), conditions (2.2)-(2.4) and by explicit computation.

Remark 4.10. Notice that in the involutive case both twisted \( \hat{r} \)-matrices reduce to the permutation operator as expected. Also, it follows from Lemma 4.9 that when we twist with \( F \) the invertible elements \( V_x \in \mathrm{End}(V) \) form a family of group like elements, i.e. \( \Delta_F(V_x) = V_x \otimes V_x \) and \( \epsilon(V_x) = 1 \), for all \( x \in X \). Similarly, when twisting with \( \hat{F} \) the invertible elements \( W_x \in \mathrm{End}(V) \) form a family of group like elements, i.e. \( \Delta_{\hat{F}}(W_x) = W_x \otimes W_x \) and \( \epsilon(W_x) = 1 \), for all \( x \in X \) (see Remark 4.7).

Remark 4.11. Recall that \( r = P \hat{r} \) is the set-theoretic solution of the QYBE. The established equalities \( \epsilon(V_x) = \epsilon(W_x) = 1 \), for all \( x \in X \), and the coproducts (4.8) given in Lemma 4.3 imply the following:

1. the coproducts (4.8) are not coassociative.
2. \( (\epsilon \otimes \epsilon) \Delta(V_y) = \sum_{x \in X} \epsilon(e_{\sigma_y(x),x}) y_V(x) \delta_y(\eta) \) and \( (\epsilon \otimes \epsilon) \Delta(W_y) = \sum_{x \in X} \epsilon(e_{\tau_y(x),x}) y_W(x) \delta_y(\eta) \).
3. \( (\Delta \otimes \epsilon) r \neq r_{13} r_{23} \) and \( (\epsilon \otimes \Delta) r \neq r_{13} r_{12} \).
4. The set-theoretic \( r \)-matrix satisfies the YBE.

All the above lead to the conclusion that, as in the involutive case 1.6, the underlying quantum algebra is a quasi-triangular quasi-bialgebra (see also Definitions 3.1, 3.3 and Proposition 3.4). Similar comments can be made for the twisted \( r \)-matrices.
Relevant results on the extension of the results of [15] about the admissibility of the twist $F$ to the non-involutive case are presented in [26]. And although an example of a twisted $r$-matrix is also presented neither the issue of the twisted coproducts of the families of operators $V_x, W_x$, which play a crucial role in characterising the associated quantum algebra as a quasi-bialgebra, nor the issue of the quantum algebra being a quasi-bialgebra are discussed. In fact, it is implicitly regarded that there is an underlying Hopf structure, but as we have shown this is not the case.

The next important step in this frame is to identify the associators $\Phi, \Phi_T$ and demonstrate the full structure of the underlying quasi-bialgebras for all set-theoretic solutions. This will indeed confirm the main conjecture of [16] that the underlying quantum algebra for involutive set-theoretic solutions is a quasi-triangular quasi-bialgebra. In the involutive case $\Phi_T = 1 \otimes 1 \otimes 1$, so the task in this situation is to identify $\Phi$, but this will be discussed in future works. We should also note that the notion of the antipode and the quasi-Hopf algebra is briefly discussed for the involutive case in [16] for some special set-theoretic solutions, however further study in this direction is required. The Baxterization of the non-involutive solutions and the derivation of a universal $R$-matrix are among the most challenging questions in this context and will be addressed in future investigations.

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**References**

[1] D. Bachiller, *Counterexample to a conjecture about braces*, J. Algebra, 453 (2016) 160–176.

[2] D. Bachiller, F. Cedó, E. Jespers and J. Okniński, *Iterated matched products of finite braces and simplicity; new solutions of the Yang-Baxter equation*, Trans. Amer. Math. Soc. 370 (2018) 4881–4907.

[3] R.J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press (1982).

[4] T. Brzeziński, *Trusses: Between braces and rings*, Trans. Amer. Math. Soc. 372 (2019), 4149–4176.

[5] T. Brzeziński, *Trusses: Paragons, ideals and modules*, J. Pure Appl. Algebra 224 (2020), 106258.

[6] T. Brzeziński & B. Rybołowicz, *Congruence classes and extensions of rings with an application to braces*, Comm. Contemp. Math. 23 (2021), 2050010.

[7] T. Brzeziński, S. Mereta and B. Rybołowicz, *From pre-trusses to skew braces*, Publicacions Matemàtiques (2022), In Press.

[8] F. Catino, I. Colazzo and P. Stefanelli, *Semi-braces and the Yang-Baxter equation*, J. Algebra, 483 (2017) 163–187.

[9] F. Catino, M. Mazzotta, P. Stefanelli, *Inverse semi-braces and the Yang-Baxter equation*, J. Algebra, 573 (2021) 576-619.

[10] F. Cedó, E. Jespers and C. Verwimp, *Structure monoids of set-theoretic solutions of the Yang-Baxter equation*. Publicacions Matemàtiques, (2021)

[11] F. Cedó, *Left braces: solutions of the Yang-Baxter equation*, Adv. Group Theory Appl., Vol. 5 (2018) 33–90.

[12] F. Cedó, E. Jespers, and J. Okniński, *Braces and the Yang-Baxter equation*. Comm. Math. Phys., 327(1) (2014) 101–116.

[13] A. Doikou and A. Smoktunowicz, *From Braces to Hecke algebras & Quantum Groups*, J. Algebra and Applications (2022) 2350179.

[14] A. Doikou and A. Smoktunowicz, *Set-theoretical Yang-Baxter and reflection equations & quantum group symmetries*, Lett. Math. Phys. 111 (2021) 105.
[15] A. Doikou, *Set-theoretic Yang–Baxter equation, braces and Drinfeld twists*, J. Phys A54 (2021), 415201.

[16] A. Doikou, A. Ghionis and B. Vlaar, *Quasi-bialgebras from set-theoretic type solutions of the Yang-Baxter equation*, Lett. Math. Phys. 112, 78 (2022).

[17] V.G. Drinfeld, *On some unsolved problems in quantum group theory*, Lecture Notes in Math., vol. 1510, Springer-Verlag, Berlin (1992) 1-8.

[18] V.G. Drinfel’d, *Hopf algebras and the quantum Yang–Baxter equation*, Soviet. Math. Dokl. 32 (1985) 254;
   *A new realization of Yangians and quantized affine algebras*, Soviet. Math. Dokl. 36 (1988) 212.

[19] V.G. Drinfeld, *Quasi-Hopf algebras*, Algebra i Analiz (1989) Volume 1, Issue 6, 114.
   *Quasi-Hopf algebras and Knizhnik-Zamolodchikov equations. In Problems of modern quantum field theory*, Springer, Berlin, Heidelberg (1989) 1-13.

[20] P. Etingof, T. Schedler and A. Soloviev, *Set-theoretical solutions to the quantum Yang–Baxter equation*, Duke Math. J. 100 (1999) 169–209.

[21] P. Etingof, *Geometric crystals and set-theoretical solutions to the quantum Yang–Baxter equation*, Comm. Algebra 31 (2003) 1961.

[22] L.D. Faddeev, N.Yu. Reshetikhin and L.A. Takhtajan, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. 1 (1990) 193.

[23] T. Gateva-Ivanova, *Binomial skew polynomial rings, Artin–Schelter regularity, and binomial solutions of the Yang-Baxter equation*, Serdica Mathematical Journal Volume: 30, Issue: 2-3 (2004) 431–470.

[24] T. Gateva-Ivanova, *A combinatorial approach to noninvolutive set-theoretical solutions of the Yang–Baxter equation*, preprint (2018). arXiv:1808.03938 [math.QA], (2018).

[25] T. Gateva-Ivanova, *Set-theoretical solutions of the Yang–Baxter equation, braces and symmetric groups*, Adv. Math., 388(7) (2018) 649–701.

[26] A. Ghibadi, *Drinfeld Twists on Skew Braces*, arXiv:2105.03286 [math.QA] (2021).

[27] L. Guarnieri and L. Vendramin, *Skew braces and the Yang–Baxter equation*, Math. Comp. 86(307) (2017) 2519–2534.

[28] J. Lu, M. Yan, Y. Zhu, *On the set-theoretical Yang-Baxter equation*, Duke Math. J. 104(1) (2000), 1-18.

[29] E. Jespers, E. Kubat and A. Van Antwerpen, *The structure monoid and algebra of a non-degenerate set-theoretic solution of the Yang-Baxter equation*, Trans. Amer. Math. Soc. 372 (2019) 7191–7223.

[30] E. Jespers, E. Kubat, A. Van Antwerpen and L. Vendramin, *Factorizations of skew braces*, Math. Ann. 375 (2019) no. 3-4, 1649–1663.

[31] E. Jespers and J. Okniński, *Monoids and groups of I-type*, Algebr. Represent. Theory 8 (2005) 709–729.

[32] C. Kassel, *Quantum Groups*, Graduate Texts in Mathematics Springer-Verlag (1995).

[33] A. Koch and P. J. Truman, *Opposite skew left braces and applications*, J. Algebra 546 (2020) 218–235.

[34] A. Konovalov, A. Smoktunowicz and L. Vendramin, *On skew braces and their ideals*, Experimental Mathematics, Volume 30 (2021) 95–104.

[35] V. Lebed and L. Vendramin, *On structure groups of set-theoretical solutions to the Yang-Baxter equation*, Proc. Edinburgh Math. Soc., Volume 62, Issue 3 (2019) 683 –717.

[36] W. Rump, *A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation*, Adv. Math., 193(1) (2005) 40–55.

[37] W. Rump, *Braces, radical rings, and the quantum Yang-Baxter equation*, J. Algebra 307(1) (2007) 153–170.

[38] A. Smoktunowicz and L. Vendramin, *On Skew Braces (with an appendix by N. Byott and L. Vendramin)*, Journal of Combinatorial Algebra Volume 2, Issue 1 (2018) 47-86.

[39] C.N. Yang, *Some exact results for the many-body problem in one dimension with repulsive delta-function interaction*, Rev. Lett. 19 (1967) 1312.