NOTE ON SOME MISINTERPRETATIONS OF GÖDEL’S INCOMPLETENESS THEOREMS

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Abstract. In this paper, I evaluate some formal and informal misinterpretations of Gödel’s incompleteness theorems from the literature and the folklore, as well as clarify some misunderstandings about Gödel’s incompleteness theorems based on the current research on Gödel’s incompleteness theorems in the literature.

1. Introduction

Gödel’s incompleteness theorems are one of the most remarkable and profound discoveries in the 20th century, an important milestone in the history of modern logic. Gödel’s incompleteness theorems have wide and profound influence on the development of logic, philosophy, mathematics, computer science and other fields, substantially shaping mathematical logic as well as foundations and philosophy of mathematics after 1931.

Gödel’s incompleteness theorems show certain weaknesses and limitations of one given formal system. For Gödel, his incompleteness theorems indicate the creative power of human reason. In Post’s celebrated words: mathematical proof is an essentially creative activity. The impact of Gödel’s incompleteness theorems is not confined to the community of mathematicians and logicians, and they have been very popular and widely used outside mathematics. For the impact of Gödel’s incompleteness theorems, Feferman said: “their relevance to mathematical logic (and its offspring in the theory of computation) is paramount; further, their philosophical relevance is significant, but in just what way is far from settled; and finally, their mathematical relevance outside of logic is very much unsubstantiated but is the object of ongoing, tantalizing efforts” (see [28], p.434).

Gödel’s incompleteness theorems raise a number of philosophical questions concerning the nature of logic and mathematics, as well as mind and machine. There are ample misinterpretations of Gödel’s incompleteness theorems from the literature and the folklore. Franzen’s [31] is a popular book about the use and misuse of Gödel’s incompleteness theorems in and outside mathematics and logic for a wider audience. In [31], Franzen comments on a fairly wide selection of the many invocations of the incompleteness theorems outside mathematics. In this paper, I will focus on some misinterpretations of Gödel’s incompleteness theorems in mathematics and logic.

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The structure of [31] is as follows: in Chapter 2 and 3, the author gave an introduction to the mathematics of incompleteness, along with a discussion of some basic philosophical issues; in Chapter 4-6 and 8, the author discussed some applications of the incompleteness
which are not covered in [31]. Both this paper and Franzen’s book have
discussed the mechanism thesis and Lucas-Penrose arguments. But the dis-
cussion in Franzen’s book is mainly from the philosophical perspective and
the discussion in this paper is mainly from the logical perspective based on
some recent work on Gödel’s disjunctive thesis.

The motivation of this paper is to review and evaluate some formal and
informal misinterpretations of Gödel’s incompleteness theorems and their
consequences from the literature and the folklore as well as to clarify some
confusions based on the current research on Gödel’s incompleteness theorems
in the literature. There are some good survey papers on Gödel’s incomple-
etness theorems in the literature, i.e. [96], [3], [59], [103]. This paper is not a
survey paper on the current status of research on incompleteness; instead,
in this paper I only focus on how recent research on incompleteness clarifies
some popular misinterpretations of Gödel’s incompleteness theorems.

This paper is structured as follows. In Section 2, I review some notions
and facts we will use in this paper. In Section 3, I discuss some misin-
terpretations of Gödel’s first incompleteness theorem (G1). In particular, I
will focus on the following interpretations or aspects of G1: the claim that
there is a truth that cannot be proved; the metaphorical application of G1
outside mathematics and logic; the claim that any consistent formal system
is incomplete; the claim that any consistent extension of PA is incomplete;
the dependence of incompleteness on the language of the theory; the dif-
ference between the theory of arithmetic and the theory of reals; the claim
that Gödel’s proof is paradoxical due to the use of the Liar Paradox; the
difference between the notion of provability in PA and the notion of truth
in the standard model; sentences of arithmetic independent of PA with real
mathematical content; and the theory with the minimal degree of interpret-
ation for which G1 holds. In Section 4, I discuss some misinterpretations
of Gödel’s second incompleteness theorem (G2). In particular, I will focus
on the following interpretations or aspects of G2: a delicate mistake for the
proof of G2; the vagueness of the consistency statement; the intensionality
of G2; the claim that the Drivability Conditions (see D1-D3 in Section 3)
are the necessary conditions to show that G2 holds for PA; the claim that
G2 holds for any consistent extension of PA; the claim that there are arith-
metic truths which cannot be proved in any formal theory in the language
of arithmetic; the claim that the consistency of PA can only be proved in
a stronger theory properly extending PA; and the claim that G2 refutes
Hilbert’s program. In Section 5, I discuss the popular interpretation that
Gödel’s incompleteness theorems show that the mechanism thesis fails.

2. Preliminaries

In this section, I review some basic notions and facts used in this paper.
Our notations are standard. For the theory of recursive function, we refer
to [22], [73], [88]. For details of Gödel’s incompleteness theorems, we refer
to [21], [73], [67], [95], [9]. For meta-mathematics of subsystems of PA, we
refer to [40]. For a standard textbook on reverse mathematics, we refer to

theorems outside mathematics and the philosophy of mathematics (e.g. in theology and
in the philosophy of mind as well as the philosophical claims of Gregory Chaitin).
A formal system has four components: a formal language, rules for the formation of formulas, a set of axioms and a set of reference rules. For a formal system $T$, let $L(T)$ denote the formal language of $T$. For formula $\phi$ in $L(T)$, $\phi$ is provable in $T$ (denoted by $T \vdash \phi$) iff there exists a finite sequence of formulas $\langle \phi_0, \cdots, \phi_n \rangle$ such that $\phi_n = \phi$, and for any $0 \leq i \leq n$, either $\phi_i$ is an axiom of $T$ or $\phi_i$ follows from some $\phi_j$ ($j < i$) by using one inference rule. $T$ is consistent if no contradiction is provable in $T$. $\phi$ is independent of $T$ if $T \nvdash \phi$ and $T \nvdash \neg \phi$. $T$ is complete if for any sentence $\phi$ in $L(T)$, either $T \vdash \phi$ or $T \vdash \neg \phi$; otherwise, $T$ is incomplete. An idea formal system should be both consistent (free of contradictions) and complete (representing all the truth). A theory is a set of sentences provable in a formal system. In this paper we focus on first order theory based on countable first order language, and always assume the arithmetization of the language of first order theory. Under arithmetization, any formula or finite sequence of formulas in first order theory can be coded by a natural number (this code is called Gödel’s number). In this paper, we use $^* \phi$ denote the numeral in $L(\text{PA})$ of the Gödel number of $\phi$. A theory $T$ is decidable if the set of Gödel numbers of sentences provable in $T$ is recursive; otherwise it is undecidable. In this paper, whenever we say a set of sentences has an arithmetic property we always mean that the set of Gödel numbers of sentences in this set has the corresponding arithmetic property. $T$ is recursively axiomatizable if it has a recursive set of axioms\footnote{I.e., there exists a sentence $\phi$ in $L(T)$ such that $\phi$ is independent of $T$.} $T$ is r.e. if it has a recursive enumerable set of axioms. $T$ is essentially undecidable if any recursively axiomatized consistent extension of $T$ is undecidable. $T$ is essentially incomplete if all recursively axiomatizable consistent extensions of $T$ are incomplete. It is well known that every consistent recursively axiomatizable complete theory is decidable; and every incomplete decidable theory has a consistent, decidable complete extension in the same language (see Corollary 3.1.8 and Theorem 3.1.9 in \cite{32}, p. 214-215). From these two facts, $T$ is essentially undecidable if and only if $T$ is essentially incomplete; and the theory of completeness/incompleteness is closely related to the theory of decidability/undecidability. $T$ is minimal essentially incomplete iff if deleting any axiom of $T$, the remaining theory is no longer essentially incomplete. For the definitions of $\Sigma_n$, $\Pi_n$ and $\Delta_n$ formulas in the arithmetic hierarchy, we refer to \cite{22} and \cite{33}.

A $n$-ary relation $R(x_1, \cdots, x_n)$ on $\mathbb{N}^n$ is representable in $T$ iff there is a formula $\phi(x_1, \cdots, x_n)$ such that $R(m_1, \cdots, m_n)$ holds, then $T \vdash \phi(m_1, \cdots, m_n)$ and if $R(m_1, \cdots, m_n)$ does not hold, then $T \vdash \neg \phi(m_1, \cdots, m_n)$. A theory $T$ is said to be $\omega$-consistent if there is no formula $\varphi(x)$ such that $T \vdash \exists x \varphi(x)$ and for any $n \in \mathbb{N}$, $T \vdash \neg \varphi(\bar{n})$; and $T$ is 1-consistent if there is no such a $\Delta_0^T$ formula $\varphi(x)$. We say that function $f(x_1, \cdots, x_n)$ on $\mathbb{N}^n$ is representable in $T$ if and only if there exists a formula $\varphi(x_1, \cdots, x_n, y)$ such that for any $a_1, \cdots, a_n \in \mathbb{N}$, $T \vdash \forall y(\varphi(\bar{a_1}, \cdots, \bar{a_n}, y) \leftrightarrow y = f(a_1, \cdots, a_n))$.

\footnote{I.e. the set of Gödel numbers of axioms of $T$ is recursive.}
\footnote{For $n \in \mathbb{N}$, $\bar{n}$ denotes the corresponding numeral in $L(\text{PA})$ for $n$.}
The notion of interpretation provides us a method to compare different theories in different languages. Generally, an interpretation of theory $T$ in theory $S$ is a mapping from formulas of $T$ to formulas of $S$ that maps all axioms of $T$ to sentences provable in $S$. For the premise definition of the notion of interpretation, we refer to [101], [102], [103]. If $T$ is interpretable in $S$, then all sentences provable (refutable) in $T$ are mapped, by the interpretation function, to sentences provable (refutable) in $S$. Let $T \triangleq S$ denote that $T$ is interpretable in $S$. $T \triangleq S$ denotes that $T \triangleq S$ but $S$ is not interpretable in $T$. We say that $T$ and $S$ are mutually interpretable if $T \triangleq S$ and $S \triangleq T$. Interpretability can be accepted as a measure of strength of first order theory. If $T \triangleq S$, then $T$ can be considered weaker than $S$; if $T$ and $S$ are mutually interpretable, then $T$ and $S$ are equally strong. Whenever we say that $S$ is weaker than $T$ w.r.t. interpretation, this means that $S \triangleq T$.

The following Theorem 2.1 provides us a method to prove the undecidability of theory via interpretation.

**Theorem 2.1.** ([97], Theorem 7, p.22) Let $T_1$ and $T_2$ be two theories such that $T_1$ is consistent and $T_2$ is interpretable in $T_1$. We then have:

1. if $T_2$ is essentially incomplete, then $T_1$ is also essentially incomplete;
2. if $T_2$ has a finitely axiomatizable sub-theory which is essentially incomplete, then so has $T_1$.

Peano Arithmetic (PA) is the first order theory of arithmetic with $L(PA) = \{0, S, +, \cdot, \}$. PA consists of axioms for first order logic and the following axioms for arithmetic:

1. $\forall x \forall y (Sx = Sy \rightarrow x = y)$;
2. $\forall x (Sx \neq 0)$;
3. $\forall x \forall y (x + 0 = x)$;
4. $\forall x \forall y (x + Sy = S(x + y))$;
5. $\forall x (x \cdot 0 = 0)$;
6. the scheme of induction: $(\phi(0) \land \forall x (\phi(x) \rightarrow \phi(Sx))) \rightarrow \forall x \phi(x)$, where $\phi$ is a formula in $L(PA)$ with at least one free variable $x$. In this paper, $\mathfrak{N} = (\mathbb{N}, 0, S, +, \cdot)$ denotes the standard model of PA. We say $\phi \in L(PA)$ is a true sentence if $\mathfrak{N} \models \phi$. Robinson’s arithmetic Q is a sub-theory of PA which consists of axioms (1)-(6) in the definition of PA plus the following axiom: $\forall x (x \neq 0 \rightarrow \exists y (x = Sy))$. $I\Sigma^0_n$ is a fragment of PA obtained by restricting the axiom scheme of induction to $\Sigma^0_n$ formulas. For the definition of the fragment $I\Sigma^0_n$, we refer to [10].

**Definition 2.2.** Let $R$ be the system consisting of schemes $R1-R5$ with $L(PA) = \{0, S, +, \cdot, \leq\}$ where $m, n \in \mathbb{N}$ and $\overline{m} = S^n(0)$.

- **R1:** $\overline{m + n} = \overline{m} + \overline{n}$
- **R2:** $\overline{m \cdot n} = \overline{m} \cdot \overline{n}$
- **R3:** $\overline{m} \neq \overline{n}$ if $m \neq n$
- **R4:** $\forall x (x \leq \overline{m} \rightarrow x = 0 \lor \cdots \lor x = \overline{m})$
- **R5:** $\forall x (x \leq \overline{m} \lor \overline{n} \leq x)$

$R$ is a sub-theory of $Q$. $Q$ is finitely axiomatizable but $R$ is not finitely axiomatizable. $Q$ is essentially incomplete and in fact minimal essentially incomplete (see [73], p.260).

In the following, we give a sketch of the main idea of Gödel’s proof of G1 and G2. Let $T$ be a recursively axiomatized consistent extension of PA in $L(PA)$. The three main ideas in Gödel’s proof of G1 and G2 are the arithmetization of the syntax of $T$, the representability of recursive functions in PA and the self-reference construction.
Firstly, Gödel gives a recursive arithmetization of axioms of PA. Then we could define some relations on natural numbers which express some metamathematical properties of T. For example, we could define a binary relation on \( \mathbb{N} \) as follows: \( \text{Prf}_T(m, n) \) if \( n \) is the Gödel’s number of a proof of the formula with Gödel number \( m \) in \( T \). Moreover, we can prove that the relation \( \text{Prf}_T(m, n) \) is recursive. Secondly, Gödel proves that every recursive relation is representable in \( \text{PA} \) and hence there is a formula \( \phi(x, y) \) which represents \( \text{Prf}_T(m, n) \) in \( \text{PA} \). From the representation formula \( \phi(x, y) \), we could naturally define the provability predicate \( \Pr_T(x) \) as follows: \( \Pr_T(x) = \exists y \phi(x, y) \). Thirdly, Gödel constructs a Gödel sentence \( G \) which asserts its own unprovability in \( T \), i.e. \( T \vdash G \leftrightarrow \neg \Pr_{\text{PA}}(\langle G \rangle) \). Finally, Gödel shows that if \( \text{PA} \) is consistent, then \( G \) is not provable in \( \text{PA} \); and if \( \text{PA} \) is \( \omega \)-consistent, then \( \neg G \) is not provable in \( \text{PA} \).

The provability predicate \( \Pr_T(x) \) satisfies the following conditions:

1. **D1**: If \( T \vdash \varphi \), then \( T \vdash \Pr_T(\langle \varphi \rangle) \);
2. **D2**: \( T \vdash \Pr_T(\langle \varphi \rangle) \rightarrow (\Pr_T(\langle \varphi \rightarrow \psi \rangle) \rightarrow \Pr_T(\langle \varphi \rangle)) \);
3. **D3**: \( T \vdash \Pr_T(\langle \varphi \rangle) \rightarrow \Pr_T(\langle \varphi \rightarrow \psi \rangle) \).

**D1-3** are called drivability conditions. For the proof of \( G_2 \), we first define the sentence in \( L(\text{PA}) \) which expresses the consistency of \( \text{PA} \) as follows:

\[ \text{Con}(\text{PA}) \equiv \neg \Pr_{\text{PA}}(\langle 0 = 1 \rangle). \]

From conditions **D1-3**, we can show that \( \text{PA} \vdash \text{Con}(\text{PA}) \leftrightarrow G \). So \( G_2 \) holds: if \( \text{PA} \) is consistent, then \( \text{PA} \nvdash \text{Con}(\text{PA}) \). For more details of Gödel’s proof of \( G_1 \) and \( G_2 \), see Chapter 2 in [73].

A first-order theory \( T \) containing \( \text{PA} \) is said to be reflexive iff for each finite sub-theory \( S \) of \( T \), \( T \vdash \text{Con}(S) \) where \( \text{Con}(S) \) is similarly defined as \( \text{Con}(\text{PA}) \). \( T \) is essentially reflective if any consistent extension of \( T \) in \( L(T) \) is reflective. Mostowski proved that \( \text{PA} \) is essentially reflective (see [73, Theorem 2.6.12]). In fact one can show that for every \( n \in \mathbb{N} \), \( \text{PA} \vdash \text{Con}(\text{PA}^0) \).

### 3. Some popular misinterpretations of Gödel’s first incompleteness theorem

The popular folklore version of \( G_1 \) says that any recursively axiomatized consistent theory containing a large enough fragment of arithmetic is incomplete. This statement is vague. In the following, I will reformulate and discuss some precise versions of \( G_1 \). Gödel proved his incompleteness theorems in [34] for a certain formal system \( P \) related to Russell-Whitehead’s Principia Mathematica and based on the simple theory of types over the natural number series and the Dedekind-Peano axioms (see [3], p.3). Gödel’s original first incompleteness theorem (34, Theorem VI) says that for formal theory \( T \) formulated in the language of \( P \) and obtained by adding a primitive recursive set of axioms to the system \( P \), if \( T \) is \( \omega \)-consistent, then

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5For the details of arithmetization, refer to Murawski [73]. Under Gödel’s arithmetization, the set of Gödel’s number of axioms of \( \text{PA} \) is recursive.

6Via arithmetization and representability, one can speak about the property of \( T \) in \( \text{PA} \) itself! This is the essence of Gödel’s idea of arithmetization.

7For a proof of this result, we refer to Hájek and Pudlák [40].
$T$ is incomplete. Rosser improved Gödel’s result by only assuming $T$ is consistent. The following theorem is a modern reformulation of Gödel-Rosser incompleteness theorem.

**Theorem 3.1. (Gödel-Rosser first incompleteness theorem (G1))** If $T$ is a recursively axiomatized consistent extension of $\text{PA}$, then $T$ is incomplete.\footnote{In fact, one can effectively find a true $\Pi^0_1$ sentence $G_T$ of arithmetic such that $G_T$ is independent of $T$. Gödel calls this the “incompleatability or inexhaustability of mathematics”.}

One motivation of Hilbert’s program is to formalize all mathematical statements and prove the completeness: all true mathematical statements can be proved in this formalism. Gödel’s work showed that it is not possible to formalize all of mathematics within a consistent formal system, as any attempt at such a formalism will omit some true mathematical statements: there is no complete recursively enumerable consistent extension of $\text{PA}$.

First of all, I discuss some misinterpretations of the applicability of Gödel’s incompleteness theorems outside mathematics. In ordinary language we also often use expressions such as “system”, “consistency”, “inconsistency”, “complete” and “incomplete”. It is not surprising that $\text{G1}$ has been thought to have a great many applications outside mathematics. $\text{G1}$ is about the consistency and completeness of formal theory containing enough arithmetic with precise mathematical meaning. So we should be careful when interpreting $\text{G1}$ outside logic. However, the implications of $\text{G1}$ are often overstated. Some argue that $\text{G1}$ also applies to the Bible, the U. S. Constitution, philosophical theory and human mind: since we can view the Bible, the U. S. Constitution, the theory of philosophy and the human mind as a formal system. However, $\text{G1}$ does not apply in contexts where there’s no formal system like the Bible, the U.S.Constitution, the informal theory of philosophy and the human mind, etc. Some argue that $\text{G1}$ applies to not only mathematics but also the whole world of science: since we can view scientific theory as a formal system, by $\text{G1}$, there is no such theory of everything in science. $\text{G1}$ only tells us the incompleteness of arithmetic, but whether or not the theory of science is complete as a description of the physical world, and what completeness might mean in this case, is not something that $\text{G1}$ tells us anything about.\footnote{See Chapter 4 in [31] for more discussions about misuses of the incompleteness theorems outside mathematics.} As Franzen argued in [31], these applications of the incompleteness theorems are at most analogies and metaphors at best. $\text{G1}$ only guarantees the existence of undecidable arithmetical statements in certain formal theory; but it says nothing about the existence of undecidable non-arithmetical statements. In a word, $\text{G1}$ has really no applications except metaphorical ones beyond the scope of formal theory in logic and mathematics.

It is often said that $\text{G1}$ shows that there is a truth that cannot be proved. This interpretation of $\text{G1}$ is not premise and correct: provability is always relative to a formal system; informally $\text{G1}$ only tells us that for a given formal system, if it is consistent and contains a large enough fragment of $\text{PA}$, then there is a true sentence which is independent of this system. $\text{G1}$ does not tell us that the independent sentence in one formal system is not provable in
any formal system. In fact, a problem unprovable in a given formal system can turn out to be provable in another stronger formal system.\(^{10}\)

A popular misinterpretation of G\(_1\) from folklore is that any consistent formal system is incomplete. G\(_1\) does not tell us this and informally G\(_1\) only tells us that any consistent formal system containing a large enough fragment of PA is incomplete. In fact there are many consistent formal theories which are complete. For example, the following formal theories are complete: first-order logic, the theory of dense linear orderings without endpoints (DLO), the theory of ordered divisible groups (ODG), the theory of algebraically closed fields of given characteristic (ACF\(_p\)), and the theory of real closed fields (RCF), etc.\(^{11}\)

It is a popular misinterpretation of G\(_1\) that any theory of arithmetic is incomplete. In fact, whether a theory of arithmetic is complete depends on the language of the theory. There are respectively recursively axiomatized complete arithmetic theories in the language of \(L(0, S)\), \(L(0, S, <)\) and \(L(0, S, <, +)\) (see Section 3.1-3.2 in [21]). But PA is not complete in the language of \(L(0, S, +, \cdot)\). Firstly, the condition in G\(_1\) that containing a large enough fragment of PA is essential. For example, the Euclidean geometry does not satisfy this condition: it is not about arithmetic but only about points, circles and lines in general; but the Euclidean geometry is complete as Tarski has proved. Secondly, containing the information about the arithmetic of multiplication is essential for the proof of G\(_1\). If the theory contains only the information about the arithmetic of addition without multiplication, then it could be complete. For example, Presburger arithmetic is the theory of arithmetic of addition, and its language only contains non-logical symbols \(0, S\) and \(+\); but Presburger arithmetic is complete (see [73] Theorem 3.2.2, p. 222). Finally, theory containing the arithmetic of multiplication is not sufficient for being incomplete. For example, there exists a complete recursively axiomatized theory in the language of \(L(0, S, \cdot)\) (see [73], p.230).

It is a popular misinterpretation of G\(_1\) that since PA is not complete and \(\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}\), theory about integers, rational numbers and real numbers are all incomplete. It is well known that \(Th(\mathbb{N}, +, \cdot)\) is interpretable in \(Th(\mathbb{Z}, +, \cdot)\) and \(Th(\mathbb{Q}, +, \cdot)\).\(^{12}\) Since \(Th(\mathbb{N}, +, \cdot)\) is undecidable and has a finitely axiomatizable incomplete sub-theory \(Q\), by Theorem 2.1 \(Th(\mathbb{Z}, +, \cdot)\) and \(Th(\mathbb{Q}, +, \cdot)\) are undecidable and hence not recursive axiomatizable but they respectively have a finitely axiomatizable incomplete sub-theory of integers and rational numbers. But \(Th(\mathbb{R}, +, \cdot)\) is decidable, recursively axiomatizable theory (even if not finitely axiomatizable) and \(Th(\mathbb{R}, +, \cdot) = RCF\) (the theory of real closed field) (see [23], p.320-321). Note that this fact does not contradict G\(_1\) since none of \(\mathbb{N}\), \(\mathbb{Z}\) and \(\mathbb{Q}\) is definable in the structure \((\mathbb{R}, +, \cdot)\).

\(^{10}\)Even if mathematicians can always use axiomatic-deductive method to organize and present their proofs, mathematicians do not work in one fixed formal system in their research practices; instead they could use various methods and means that would contribute to finding a solution to the problem they are working on.

\(^{11}\)We refer to [23] for details of these theories.

\(^{12}\)See chapter XVI in [23]. The key point is: \(\mathbb{N}\) is definable in \((\mathbb{Z}, +, \cdot)\) and \((\mathbb{Q}, +, \cdot)\).
A popular interpretation of $G_1$ is that any consistent extension of $\mathsf{PA}$ is incomplete. This interpretation is false based on the recent work in [50] on generalizing $G_1$ to arithmetically definable theory.

**Definition 3.2.** Let $T$ be a theory and $\Gamma$ be a class of formulas.

1. $T$ is $\Sigma_n^0$-definable iff there is a $\Sigma_n^0$ formula $\alpha(x)$ such that $n$ is the Gödel number of some sentence of $T$ if and only if $\mathcal{M} \models \alpha(n)$.
2. $T$ is $\Sigma_n^0$-sound if and only if for all $\Sigma_n^0$ sentences $\phi$, if $T \vdash \phi$, then $\mathcal{M} \models \phi$.
3. $T$ is $\Gamma$-decisive if and only if for all $\Gamma$ sentences $\phi$, either $T \vdash \phi$ or $T \vdash \neg \phi$ holds.

From $G_1$ and Craig’s trick, if theory $T$ is $\Sigma_1^0$-definable and consistent extension of $\mathsf{PA}$, then $T$ is not $\Pi^0_1$-decisive. In [50], Kikuchi and Kurahashi generalized $G_1$ to arithmetically definable theory.

**Theorem 3.3.** ([50, Theorem 4.8]) If $T$ is $\Sigma_{n+1}^0$-definable and $\Sigma_n^0$-sound extension of $\mathsf{PA}$, then $T$ is not $\Pi^0_{n+1}$-decisive.

The optimality of this generalization is shown by Salehi and Seraji in [89]; there exists a $\Sigma_{n+1}^0$-definable, $\Sigma_n^0$-sound ($n \geq 1$) and complete theory which contains $\mathsf{Q}$ (see Theorem 2.6 in [89]). So it is not true that any consistent extension of $\mathsf{PA}$ is incomplete.

Before Gödel’s work, it is often thought that for any arithmetic sentence $\phi$, $\phi$ is provable in $\mathsf{PA}$ iff $\phi$ is true in the standard model of arithmetic. $G_1$ reveals the difference between the notion of provability in $\mathsf{PA}$ and the notion of truth in the standard model. Define $\mathsf{Prov} = \{ \phi \in L(\mathsf{PA}) : \mathsf{PA} \vdash \phi \}$ and $\mathsf{Truth} = \{ \phi \in L(\mathsf{PA}) : \mathcal{M} \models \phi \}$. Tarski proved that $\mathsf{Truth}$ is not definable in $\mathcal{M}$. $G_1$ reveals that there are essential differences between the property of $\mathsf{Truth}$ and $\mathsf{Prov}$. $\mathsf{Prov} \subsetneq \mathsf{Truth}$ (i.e. there is true sentence of arithmetic which is independent of $\mathsf{PA}$); $\mathsf{Truth}$ is not definable in the standard model $\mathcal{M}$ but $\mathsf{Prov}$ is definable in $\mathcal{M}$; $\mathsf{Truth}$ is not arithmetic but $\mathsf{Prov}$ is recursive enumerable; $\mathsf{Truth}$ and $\mathsf{Prov}$ both are not recursive and not representable in $\mathsf{PA}$.13 Moreover, it is a surprising fact that we could use provability logic to characterize the difference between the notion of provability in $\mathsf{PA}$ and the notion of truth in $\mathcal{M}$ as the following theorem shows:

**Theorem 3.4 (Solovay).**

*Arithmetical completeness theorem for $\mathsf{GL}$:*

For any modal formula $\phi$ in $L(\mathsf{GL})$, $\mathsf{GL} \vdash \phi$ iff $\mathsf{PA} \vdash \phi'$ for every arithmetic interpretation $f$.

**Arithmetical completeness theorem for $\mathsf{GLS}$:** For any modal formula $\phi$, $\mathsf{GLS} \vdash \phi$ iff $\mathcal{M} \models \phi'$ for every arithmetic interpretation $f$.

Gödel’s incompleteness theorems are closely related to paradox. Gödel commented that “any epistemological antinomy could be used for a similar proof of the existence of undecidable propositions” (see [35, Note 14]).

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13For details of the properties of $\mathsf{Truth}$ and $\mathsf{Prov}$, we refer to [73] and [97].

14$\mathsf{GL}$ is a modal system consisting of the following schemes of axiom: (1) all tautologies; (2) $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$; (3) $\Box(\Box A \rightarrow A) \rightarrow \Box A$; as well as two inference rules: (1) if $\vdash A$ and $\vdash A \rightarrow B$, then $\vdash B$; (2) if $\vdash A$, then $\vdash \Box A$. An arithmetic interpretation is a function that maps each formula of $\mathsf{GL}$ to a sentence in $L(\mathsf{PA})$.

15$\mathsf{GLS}$ is a modal system consisting of all theorems of $\mathsf{GL}$ and instances of the following scheme of axiom: $\Box A \rightarrow A$. However, $\mathsf{GLS}$ has only one inference rule: the modus ponens.
In Gödel’s proof of \( \mathbf{G1} \), he imitated the Liar Paradox to construct a self-reference sentence called Gödel’s sentence \( \mathbf{G} \) which says that \( \mathbf{G} \) is not provable in \( \mathbf{PA} \). A popular misinterpretation of Gödel’s incompleteness theorems from the non-academic community says that since Gödel’s proof uses the Liar Paradox, his proof itself is paradoxical. However, the fact that we could imitate or formalize a logical paradox to prove incompleteness theorems does not imply that the incompleteness theorems themselves are paradoxical. Gödel’s sentence concerns the notion of provability but the liar sentence in the Liar Paradox concerns the notion of truth. But there is a big difference between the notion of provability and truth. Gödel’s sentence \( \mathbf{G} \) does not lead to a contradiction as the liar sentence. If \( \mathbf{G} \) is false, then \( \mathbf{G} \) is provable in \( \mathbf{PA} \) and hence \( \mathbf{G} \) is true. If \( \mathbf{G} \) is true, then \( \mathbf{G} \) is not provable in \( \mathbf{PA} \); if we can infer from this that \( \mathbf{\neg G} \) is provable in \( \mathbf{PA} \), then we will arrive at a contradiction since in this case \( \mathbf{G} \) is false. However, from \( \mathbf{G} \) is not provable in \( \mathbf{PA} \) we can not derive that \( \mathbf{\neg G} \) is provable.

Since \( \mathbf{PA} \vdash \mathbf{G} \iff \mathbf{Pr}_{\mathbf{PA}}(\mathbf{\langle G \rangle}) \), if \( \mathbf{PA} \vdash \mathbf{\neg G} \), then \( \mathbf{PA} \vdash \mathbf{Pr}_{\mathbf{PA}}(\mathbf{\langle G \rangle}) \). So \( \forall \mathbf{G} = \mathbf{Pr}_{\mathbf{PA}}(\mathbf{\langle G \rangle}) \) and hence \( \mathbf{PA} \vdash \mathbf{G} \) which contradicts the fact that \( \mathbf{PA} \) is consistent. Except for the Liar Paradox, many different paradoxes have been used to give a new proof of incompleteness theorems. For example, proofs of Gödel’s theorems from Berry’s Paradox ([8],[49]); from Unexpected Examination Paradox ([61]); from Yablo’s Paradox ([17],[62]) and from Grelling-Nelson’s Paradox ([16]), etc. All of these different proofs of incompleteness theorems via different paradoxes are not paradoxical even if the proofs use some logical paradoxes.

It is often claimed that the significance of \( \mathbf{G1} \) was diminished by the fact that the undecidable sentence Gödel constructed has no real mathematical content: Gödel’s sentence is of meta-mathematical nature and has no real mathematical content. However, after Gödel’s work people have found many examples of undecidable sentences with real mathematical contents. The first striking example of a mathematically natural independent statement of \( \mathbf{PA} \) with number-theoretical contents was Paris-Harrington principle proposed in [78] which generalizes the finite Ramsey theorem. Besides the Paris-Harrington principle, there are several mathematically natural independent statements of \( \mathbf{PA} \) with combinatorial contents in the literature: the Goodstein sequence ([52]), the Hercules-Hydra game ([52]), the Kanamori-McAlloon principle ([15]), the Worm principle ([4],[43]) and others. But does this phenomenon indicate that there is a gap between mathematics and meta-mathematics? However, it is interesting that all these naturally combinatorial independent principles with real mathematical contents are in fact provably equivalent in \( \mathbf{PA} \) to a certain metamathematical sentence. Consider the following reflection principle for \( \Sigma^0_1 \) sentences: for any \( \Sigma^0_1 \) sentence \( \phi \) in \( L(\mathbf{PA}) \), if \( \phi \) is provable in \( \mathbf{PA} \), then \( \phi \) is true. Using the arithmetization of syntax, one can write this principle as a sentence of \( L(\mathbf{PA}) \) and denote it by \( Rf_{n_{\Sigma^0_1}}(\mathbf{PA}) \). McAloon proved that \( \mathbf{PA} \vdash \varphi \iff Rf_{n_{\Sigma^0_1}}(\mathbf{PA}) \), where \( \varphi \) is the Paris-Harrington principle (similar equivalences can be shown for the above other independent principles).[16] These independent principles are

16Equivalently, all these principles are equivalent to the statement of 1-consistency of the arithmetic \( \mathbf{PA} \). See [3, p.36] and [73, p.301].
provable in some fragments of second order arithmetic but are more complex than Gödel’s sentence: Gödel’s sentence is equivalent to Con(PA) in PA; but all these principles are not only independent of PA but also independent of PA + Con(PA) (see [3] p.36 and [73] p.301). See [100]-[111] for more examples and discussions of mathematically independent statements. [10] is a good survey paper on unprovability theory as of autumn 2006. We refer to [32] for the new advances in Boolean Relation Theory for more examples of concrete mathematical incompleteness.

It is well known that for any recursively axiomatizable consistent theory S, if Q is interpretable in S, then S is incomplete. It is often thought that Q is the weakest base theory of arithmetic we need for the proof of G₁ w.r.t. interpretation. But this is wrong. Let us look at a general question about the limit of incompleteness: for the proof of G₁, exactly how much information of arithmetic is needed. To precisely reformulate this question, let us first introduce the notion “G₁ holds for T”.

Definition 3.5. G₁ holds for T iff for any recursively axiomatizable consistent theory S, if T is interpretable in S, then S is incomplete.

From G₁, PA is essentially incomplete and G₁ holds for PA. In fact, G₁ holds for many subsystems of PA. In the following, I give some such examples. We know that G₁ holds for Robinson’s Q (see [97]). Tarski, Mostowski and Robinson proved in [97] that if T is a consistent theory in which all recursive functions are representable, then T is essentially incomplete; and all recursive functions are representable in R. So R is essentially incomplete (see [97] theorem 9], p.60). Vaught essentially proved in [100] that G₁ holds for R. Let R₀ be the sub-theory of R which consists of axiom schemes R₁-R₄. R₀ is not essentially incomplete. Let R₁ be the system consisting of schemes R₁-R₃ and R₄’ where R₄’ is defined as follows:

\[ \forall x(x \leq \bar{n} \leftrightarrow x = \bar{0} \lor \cdots \lor x = \bar{n}) \]

R₁ is essentially incomplete since R is interpretable in R₁ (see [17], p. 62). However R₁ is not minimal essentially incomplete. Let R₂ be the system consisting of schemes R₂, R₃ and R₄’. From [17], R is interpretable in R₂ and hence R₂ is essentially incomplete. R₂ is minimal essentially incomplete in the sense that if we delete any axiom scheme of R₂, then the remaining system is not essentially incomplete. In a summary, R, R₀, R₁ and R₂ are mutually interpretable and essentially incomplete.

Visser proved in [102] Theorem 6] that R has the following maximality property: for any r.e. theory T, T is locally finite iff T is interpretable in R. (see [102] Theorem 6). As a corollary, Q is not interpretable in R and hence R ⪯ Q. It is often thought that R is the weakest theory for which

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17R₀ has a decidable complete extension given by the theory of reals with ≤ as the empty relation on reals.

18If we delete R₂, then the theory of natural numbers with x · y defined as x + y is a complete decidable extension; if we delete R₃, then the theory of models with only one element is a complete decidable extension; if we delete R₄’, then the theory of reals is a complete decidable extension.

19Theory T is locally finite iff any finite sub-theory of T has a finite model.

20If Q is interpretable in R, then by Visser’s theorem, Q is locally finite. But Q has no finite model.
G1 holds w.r.t. interpretation. A natural question is: could we find a theory $S$ such that G1 holds for $S$ and $S \prec R$? Recent progress on this problem shows that $R$ is not the weakest theory for which G1 holds with respect to interpretation. In fact, we could find many theory $S$ such that G1 holds for $S$ and $S \prec R$.

4. Some misinterpretations of Gödel's second incompleteness theorem

Gödel announced the second incompleteness theorem (G2) in an abstract published in October 1930: no consistency proof of systems such as Principia, Zermelo-Fraenkel set theory, or the systems investigated by Ackermann and von Neumann is possible by methods which can be formulated in these systems (see [113], p.431). For first order theory $T$, let $\text{Con}(T)$ denote the sentence in the language of arithmetic expressing the consistency of $T$ under Gödel's recursive arithmetization of $T$. We call $\text{Con}(T)$ the consistency statement of $T$. The following is a modern reformulation of G2:

**Theorem 4.1.** Let $T$ be a recursively axiomatized extension of PA. If $T$ is consistent, then $\text{Con}(T)$ is not provable in $T$.

I give two comments on G2. Firstly, note that for any extension $T$ of PA in $L(\text{PA})$, even if $\text{Con}(T)$ is not provable in $T$, since $T$ is reflective, $\text{Con}(S)$ is provable in $T$ for any finite sub-theory $S$ of $T$. Secondly, if $T$ is consistent, from G1, there is a sentence which is independent of $T$; but from G2, we cannot get that $\text{Con}(T)$ is independent of $T$. In fact, it is not provable in PA that if PA is consistent, then $\text{Con}(PA)$ is independent of PA. So it is not enough to show that $\neg\text{Con}(PA)$ is not provable in PA only assuming PA is consistent. But we could prove that $\text{Con}(PA)$ is independent of PA by assuming that PA is 1-consistent.

Now I examine a delicate mistake in the argument which claims that by an easy application of the compactness theorem we can show that for any axiomatization of a consistent theory $T$, $T$ can not prove its own consistency. Visser presented this argument in [104] as an interesting dialogue between Alcibiades and Socrates: “Suppose a consistent theory $T$ can prove its own consistency under some axiomatization. By compactness theorem, there must be a finitely axiomatized sub-theory $S$ of $T$ such that $S$ already proves the consistency of $T$. Since $S$ proves the consistency of $T$, it must also prove the consistency of $S$. So, we have a finitely axiomatized theory which proves its own consistency. But G2 applies to the finite axiomatization and we have a contradiction. It follows that $T$ can not prove its own consistency.”

The mistake in this argument is: from the fact that $S$ can prove the consistency of $T$ we cannot infer that $S$ can prove the consistency of $S$. Some may argue that since $S$ is a sub-theory of $T$ and $S$ can prove the

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21 [14] proves that for any Turing degree $0 < d < 0'$, there is a theory $U$ such that G1 holds for $U$, $U \prec R$ and $U$ has Turing degree $d$ where $0'$ is the Turing degree of the halting problem.

22 See [9, Theorem 4, p.97] for a modal proof in GL of this fact using the arithmetic completeness theorem for GL.

23 It is an easy fact that if PA is 1-consistent and $S$ is not a theorem of PA, then $\text{Pr}_{PA}(\ulcorner S \urcorner)$ is not a theorem of PA.
consistency of $T$, then of course $S$ can prove the consistency of $S$. As Visser pointed out in [104] that we should carefully distinguish three perspectives of theory $T$: our external perspective, the internal perspective of $S$ and the internal perspective. From each perspective, the consistency of the whole theory implies the consistency of its sub-theory. From $T$’s perspective, $S$ is a sub-theory of $T$; but from $S$’s perspective, $S$ may not be a sub-theory of $T$. From the fact that $T$ knows that $S$ is a sub-theory of $T$, we cannot infer that $S$ also knows that $S$ is a sub-theory of $T$ since $S$ is a finitely sub-theory of $T$ and may not know any information that $T$ knows. I.e., the sub-theory relation between theories is not absolute. Moreover, the notion of consistency is also not absolute. A theory may be consistent from the external perspective but inconsistent from the internal perspective. For example, let $T = \text{PA} + \neg \text{Con(PA)}$. From $G_2$, $T$ is consistent from the external perspective. But since $T \vdash \neg \text{Con(T)}$ $T$ is not consistent from the internal perspective of $T$.

A popular version of $G_2$ says that if a theory is sufficiently strong, then it does not prove its own consistency. The meaning of this statement is vague. What does “sufficiently strong” mean? The theory should contain Robinson’s Arithmetic $Q$, a very weak finitely axiomatized theory.\(^24\) Now we can reformulate a general version of $G_2$ in terms of the notion of interpretation:

**Theorem 4.2. (General version of G2, Visser [101])** There is no r.e. theory $T$ such that $Q + \text{Con}(T)$ is interpretable in $T$, i.e. $Q + \text{Con}(T) \not\equiv T$.\(^25\)

Bezboruah and Shepherdson proved in [7] that $G_2$ holds for $Q$: $Q \not\equiv \text{Con}(Q)$.\(^26\) A natural question is whether $G_2$ holds for other theories weaker than $Q$ w.r.t. interpretation (for example $R$). We do know that $R$ does interpret $R + \text{Con}(R)$ and Fedor Pakhomov recently produced a more natural example of the kind.\(^27\)

It is often thought that $\text{Con}(T)$ expresses that $T$ is consistent. However, the meaning of $\text{Con}(T)$ as well as Theorem 4.5 and Theorem 4.2 are vague. Now I examine the vagueness of the consistency statement. Detlefsen argued that it is possible that the unprovable sentence stating the consistency of the theory does not really express consistency of the theory (see [19], p.309). From the philosophical point of view, one could ask: what does it mean to say that a formal theory does not prove its own consistency? what a consistency statement of a theory is? when can we reasonably say that the arithmetic sentence $\text{Con}(T)$ does really express the consistency of $T$? how do we know that there are not entirely different statements, that are consistency statements and provable (see [101], p.545). These questions are difficult to answer and have been investigated by many logicians, among

\(^{24}\)Note that $\text{PA} \vdash \text{Pr}_{\text{PA}}(0 = 1) \rightarrow \text{Pr}_{\text{PA}}(\text{Pr}_{\text{PA}}(0 = 1) \rightarrow 0 = 1)$.

\(^{25}\)The notion “contain” is also vague: $\text{ZF}$ does not contain $Q$, but $Q$ is interpretable in $\text{ZF}$.

\(^{26}\)In fact, based on ideas of Solovay, Friedman, and Pudlák, Visser showed in [101] that for any consistent theory $U$, we have $S^1_2 + \text{Con}(U) \not\equiv U$ where $S^1_2$ is the weak arithmetic given in [11].

\(^{27}\)This result is a special case of Theorem 4.2 but the method used by Bezboruah and Shepherdson in [7] is different.

\(^{28}\)I would like to thank Prof. Albert Visser to point out this fact to me.
them Resnik [87], Detlefsen [20], Visser [103, 101], Feferman [24], Auerbach [2] and Franks [30].

Let $\text{CON}(T)$ denote the statement “$T$ is consistent”. Detlefsen asked in [20] when and how $\text{Con}(T)$ can plausibly be taken to “express” $\text{CON}(T)$. We may view $\text{Con}(T)$ as a sort of “replica” of $\text{CON}(T)$. Detlefsen argued that whether an arithmetic formula expresses the consistency of $T$ depends, in an essential way, upon what it is that one wants to show about the consistency of $T$ (see [20], p.133). If $\text{Con}(T)$ expresses $\text{CON}(T)$, how should we understand the “expression” relation between the arithmetic formula $\text{Con}(T)$ and the ordinary unarithmetized statement $\text{CON}(T)$. Detlefsen examined the general characterization of the “expression” relationship such that the replica of $\text{CON}(T)$ really expresses $\text{CON}(T)$. Detlefsen’s philosophical work on the characterization of the expression relation between the replica of $\text{CON}(T)$ and $\text{CON}(T)$ reveals the intensionability of the concept of consistency statement.[29]

Let $T$ be a recursively axiomatized consistent extension of $\text{PA}$. We say that $G_2$ holds for $T$ if the consistency statement of $T$ is not provable in $T$. From the mathematical point of view, the key question is how to eliminate the vagueness in the consistency statement in a precise way. Firstly, whether $G_2$ holds for $T$ depends on the definition of provability predicate. We could define provability predicate in the following general way. We say that a formula $\text{Prf}_T(x, y)$ is a proof predicate of $T$ iff it is $\Delta^0_1$ and satisfies the following conditions:[30]

1. For any formula $\phi, T \vdash \phi$ if and only if $T \vdash \text{Prf}_T(⌜\phi⌝, \pi)$ for some natural number $n$;
2. $T \vdash \forall x(\exists y \text{Prf}_T(x, y) \rightarrow \forall z \exists s > s \text{Prf}_T(x, s))$;
3. $T \vdash \forall y(\exists x \text{Prf}_T(x, y) \rightarrow \exists x \text{Prf}_T(x, y))$.

From the proof predicate $\text{Prf}_T(x, y)$, we could define the provability predicate $\text{Pr}T(x)$ as follows: $\text{Pr}T(x) \equiv \exists y \text{Prf}_T(x, y)$. From provability predicate $\text{Pr}T(x)$, we could define the consistency statement $\text{Con}(T)$ as $\neg \text{Pr}T(⌜0 = 1⌝)$. The derivability condition $D_1$ holds for any provability predicate. We say a provability predicate is standard if it satisfies condition $D_2-D_3$. Now we give a new formulation of $G_2$ via the standard provability predicate.

[29]Detlefsen pointed out the “replication” is supposed to work as follows (Detlefsen [20], p.133): “One is interested in determining whether $\text{CON}(T)$ has a certain property $F$ (e.g., the property of being finitistically provable, the property of being provable by means of a gainful proof, and so on). In order to make this determination, one inspects $\text{Con}(T)$ (the arithmetic “replica”) to see whether it has the property $G$ (e.g., the property of being provable within $T$, and so on). In each case, $\text{Con}(T)$ may be said to “express” $\text{CON}(T)$ when one has an appropriate means of transforming a determination of $\text{Con}(T)$’s $G$-ness (non-$G$-ness) into a determination of $\text{CON}(T)$’s $F$-ness (non-$F$-ness). This general characterization of the notion of “expression” is intentionally schematic, leaving open, as it does, the question of (i) what is to count as “an appropriate means of transforming”, (ii) what is to count as a “determination”, and (iii) what specific properties are to be substituted in the places of $F$ and $G$. Spelling out these variables in different ways will lead to different versions of the expression relation.”

[30]We can say that each proof predicate represents the relation “$y$ is a code of a proof of a formula with Gödel number $x$ in $T$.”
Theorem 4.3. Let $T$ be any recursively enumerable consistent extension of PA. If $Pr_T(x)$ is a standard provability predicate, then $T \nvdash \text{Con}(T)$.

As Visser argued in [103], being a consistency statement is not an absolute concept but a role w.r.t. a choice of the provability predicate (see Visser [103]). If $Pr_T(x)$ is not a standard provability predicate, then $\text{Con}(T)$ may be provable in $T$. For example, for the proof predicate $Pr_{PA}(x,y)$, we could define the Rosser provability predicate $Pr_{PA}^R(x)$ as the formula $\exists y (Pr_{PA}(x,y) \land \forall z \leq y \neg Pr_{PA}(\vec{\imath}(x),z))$, where $\vec{\imath}$ is a function symbol expressing a primitive recursive function calculating the code of $\neg \phi$ from the code of $\phi$. However, $\text{Con}^R(PA) \triangleq \neg Pr_{PA}^R(\vec{\imath}0 = 1)$ is provable in PA since for any sentence $\phi$, if $PA \vdash \neg \phi$, then $PA \vdash \neg Pr_{PA}^R(\vec{\imath}\phi)$ (see [51 Proposition 2.1]). So the Rosser provability predicate at least does not satisfy one of conditions D2 and D3. In the following, we give a more general definition of provability predicate for $T$ w.r.t. the numeration of $T$.

Definition 4.4. Let $T$ be any recursively axiomatized consistent extension of PA and $\alpha(x)$ be a formula in $L(T)$.

1. Define the formula $Prf_\alpha(x,y)$ saying “$y$ is the Gödel number of a proof of the formula with Gödel number $x$ from the set of all sentences satisfying $\alpha(x)$”.

2. Define the provability predicate $Pr_\alpha(x)$ of $\alpha(x)$ as $Pr_\alpha(x) \triangleq \exists y Prf_\alpha(x,y)$ and consistency statement $\text{Con}_\alpha(T)$ as $\triangleq \neg Pr_\alpha(\bot)$.

3. $\alpha(x)$ is a numeration of $T$ if for any $n$, $PA \vdash \alpha(nT)$ iff $n$ is the Gödel number of some $\phi \in T$.

Let $T$ be a recursively axiomatized consistent extension of PA. For each formula $\alpha(x)$, we have:

\[ D2' \quad PA \vdash Pr_\alpha(\vec{\imath}\phi) \rightarrow (Pr_\alpha(\vec{\imath}\phi \rightarrow \psi) \rightarrow Pr_\alpha(\vec{\imath}\psi)). \]

If $\alpha(x)$ is a numeration of $T$, then $Pr_\alpha(x)$ satisfies the following properties (see [63 Fact 2.2]):

1. D1': If $T \vdash \phi$, then $PA \vdash Pr_\alpha(\vec{\imath}\phi)$;
2. D3': If $\phi \in \Sigma_0^0$, then $PA \vdash \phi \rightarrow Pr_\alpha(\vec{\imath}\phi)$;

Now we give a new reformulation of G2 via numerations.

Theorem 4.5. Let $T$ be any recursively enumerable consistent extension of PA. If $\alpha(x)$ is any $\Sigma_1^0$ numeration of $T$, then $T \nvdash \text{Con}_\alpha(T)$.

It is often thought that G2 has the same status as G1. However, G2 is essentially different from G1 due to the intensionality of G2. The intensionality of G2 says that whether G2 holds for PA depends on the numeration of PA. G2 holds for $\Sigma_1^0$ numerations of PA, but fails for some $\Pi_1^0$ numerations of PA. For example, Feferman constructs in [24] a $\Pi_1^0$ numeration $\pi(x)$ of PA such that G2 fails: $\text{Con}_{\pi}(PA) \triangleq \neg Pr_{\pi}(\vec{\imath}0 = 1)$ is provable in PA.

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31 However, we could construct different Rosser provability predicates with different properties. For example, Guaspari and Solovay constructs in [39] a Rosser provability predicate which does not satisfy condition D2 and D3; Arai constructs in [1] a Rosser provability predicate which satisfies condition D2 and a Rosser provability predicate which satisfies condition D3.

32 Gödel’s consistency sentence $\text{Con}_{\pi}(PA)$ is not equivalent with $\text{Con}_{\tau}(PA)$. But PA does not know this fact, i.e. $PA \nvdash \neg (\text{Con}_{\pi}(PA) \leftrightarrow \text{Con}_{\tau}(PA))$ since $PA \nvdash \neg \text{Con}_{\tau}(PA)$.
Generally, Feferman showed in [24] that if $T$ is a $\Sigma^0_n$-definable extension of $\text{PA}$, then there is a $\Pi^1_1$ definition $\tau(u)$ of $T$ such that $T \vdash \text{Con}_{\tau}(T)$.

Now we examine the intensionality of $G_2$. The key point of the intensionality of $G_2$ lies in that the property of the provability predicate is intensional and depends on the numeration of the theory. Under different numerations of $\text{PA}$, the provability predicate may have different properties. It may happen that one theory $S$ has two numerations $\alpha(x)$ and $\beta(x)$ of $S$ such that $\text{Con}_{\alpha}(S)$ is not equivalent to $\text{Con}_{\beta}(S)$. For example, under Gödel’s recursive numeration and Feferman’s $\Pi^0_1$ numeration of $\text{PA}$, the corresponding consistency statements are not equivalent. Generally, Kikuchi and Kurahashi prove in [50, Corollary 5.11] that if $T$ is $\Sigma^0_{n+1}$-definable and not $\Sigma^0_n$-sound, then there are $\Sigma^0_{n+1}$ definitions $\sigma_1(x)$ and $\sigma_2(x)$ of $T$ such that $T \vdash \text{Con}_{\sigma_1}(T)$ and $T \vdash \neg \text{Con}_{\sigma_2}(T)$.

Due to the intensionability problem, $G_2$ is not coordinate-free (dependent on the numerations of $\text{PA}$). Visser addressed in [103] the important problem of formulating $G_2$ in a general way such that it is coordinate-free (independent of numerations of $\text{PA}$). One way to eliminate the intensionability of $G_2$ is to uniquely characterize the consistency statement in some sense. In [101], Visser proposed the interesting question of uniquely characterizing the consistency statement. Visser showed in [101] that the consistency statement can be pinned down as the unique solution of a certain equation modulo a suitable equivalence relation; especially, consistency for finitely axiomatized r.e. theories can be uniquely characterized modulo $\text{EA}$-proofable equivalence (see [101], p.543). But characterizing the consistency of infinitely axiomatized r.e. theories is more delicate and a big open problem in the current research on the intensionality of $G_2$.

Provability logic provides us a new way to examine the intensionality of the provability predicate. We could view the provability logic as the logic of the property of the provability predicate. Let $T$ be any recursively axiomatized consistent extension of $\text{PA}$ and $\alpha(x)$ be a numeration of $T$. Kurahashi defined in [63] that the provability logic $\text{PL}_\alpha(T)$ is the set of all modal principles which are verifiable in $T$ when the modal operator $\Box$ is interpreted as $\text{Pr}_\alpha(x)$.

**Theorem 4.6** (Solovay’s arithmetical completeness theorem). ([63, Theorem 2.5]) Let $T$ be any recursively axiomatized consistent extension of $\text{PA}$. If $T$ is $\Sigma^0_1$-sound, then for any $\Sigma^0_1$ numeration $\alpha(x)$ of $T$, the provability logic $\text{PL}_\alpha(T)$ is precisely $\text{GL}$.

However, under Feferman’s $\Pi^0_1$ numeration $\pi(x)$ of $\text{PA}$, since the consistency statement $\text{Con}_{\pi}(\text{PA})$ defined by using $\text{Pr}_\pi(x)$ is provable in $\text{PA}$, the provability logic $\text{PL}_\tau(\text{PA})$ of $\text{Pr}_\pi(x)$ is different from $\text{GL}$. An interesting research program is to classify the provability logic $\text{PL}_\alpha(T)$ according to the numeration $\alpha(x)$ of $T$. The provability logic $\text{PL}_\tau(T)$ of a $\Sigma^0_n$ numeration

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33Halbach and Visser examined the sources of intensionality in the construction of self referential sentences of arithmetic in [61, 62] and argued that corresponding to the three stages of the construction of self referential sentences of arithmetic there are at least three sources of intensionality: coding, expressing a property and self-reference. The three sources of intensionality are not independent of each other, and a choice made at an earlier stage will have influences on the availability of choices at a later stage.
\(\tau(x)\) of \(T\) is a normal modal logic. A natural question is: which normal modal logic is a provability logic \(\mathbf{PL}_\tau(T)\) of some \(\Sigma^0_n\) numeration \(\tau(x)\) of \(T\)? Kurahashi proved in [63] that for any recursively axiomatized consistent extension \(T\) of \(\mathbf{PA}\), there exists a \(\Sigma^0_2\) numeration \(\alpha(x)\) of \(T\) such that the provability logic \(\mathbf{PL}_{\alpha}(T)\) is \(K\): Kurahashi proved in [64] that for each \(n \geq 2\), there exists a \(\Sigma^0_2\) numeration \(\tau(x)\) of \(T\) such that the provability logic \(\mathbf{PL}_\tau(T)\) coincides with modal logic \(K + \Box(\Box^n p \rightarrow p) \rightarrow \Box p\); Kurahashi proved in [65] that there exists a Rosser provability predicate whose provability logic is exactly the normal modal logic \(\mathbf{KD}\). These provability logics based on different provability predicates reveal the intensionality of the provability predicate and \(G_2\).

It is often thought that \(D1-D3\) are the sufficient and necessary conditions to show that \(G2\) holds for \(\mathbf{PA}\). But this is not true. From Definition 4.34, conditions \(D1-D2\) hold for any numeration of \(\mathbf{PA}\). Whether the provability predicate satisfies the condition \(D3\) depends on the numeration of \(\mathbf{PA}\). For any \(\Sigma^0_1\)-numeration \(\alpha(x)\) of \(\mathbf{PA}\), the condition \(D3\) holds for \(\mathbf{Pr}_\alpha(x)\). In [62], Taishi Kurahashi constructed a \(\Sigma^0_2\)-numeration of axioms of \(\mathbf{PA}\) such that the provability logic for that numeration is precisely \(K\). Since \(K \not\vdash \neg \Box \bot\), as a corollary, under Taishi’s \(\Sigma^0_2\)-numeration \(\alpha(x)\) of \(\mathbf{PA}\), \(G2\) holds for \(\mathbf{PA}: \mathbf{Con}_\alpha(\mathbf{PA}) \triangleq \neg \mathbf{Pr}_\alpha(\forall 0 = 1)\) is not provable in \(\mathbf{PA}\). But the L"ob condition \(D3\) does not hold since \(K \not\vdash \Box \Box \bot\). This gives us an example of a \(\Sigma^0_2\) numeration \(\alpha(x)\) of \(\mathbf{PA}\) such that \(D3\) does not hold for \(\mathbf{Pr}_\alpha(x)\) but \(G2\) holds for \(\mathbf{PA}\). So \(D1-D3\) are the sufficient condition but not the necessary condition to show that \(G2\) holds for \(\mathbf{PA}\).

It is often said that \(G2\) holds for any consistent extension of \(\mathbf{PA}\). In fact, this is not true. From \(G2\), if \(T\) is a \(\Sigma^0_1\)-definable and consistent extension of \(\mathbf{PA}\), then \(T \not\vdash \mathbf{Con}_\sigma(T)\) for any \(\Sigma^0_1\) definition \(\sigma(u)\) of \(T\). \(G2\) is generalized in [18] by showing that any \(\Sigma^0_{n+1}\)-definable and \(\Sigma^0_n\)-sound extension of \(\mathbf{PA}\) cannot prove its own \(\Sigma^0_n\)-soundness (see [18, Theorem 2]). The optimality of the generalization is shown by presenting a \(\Sigma^0_{n+1}\)-definable and \(\Sigma^0_{n-1}\)-sound extension of \(\mathbf{PA}\) that proves its own \(\Sigma^0_{n-1}\)-soundness for any \(n > 0\) (see [18, Theorem 3]).

One folklore misinterpretation of \(G1\) and \(G2\) is that there are arithmetic truths which can not be proved in any formal theory in the language of arithmetic. However, this interpretation is false from Turing’s work in [99] and Feferman’s work in [25]. Turing’s work in [99] shows that any true \(\Pi_1^0\)-sentence of arithmetic can be provable in some transfinite iteration of \(\mathbf{PA}\); and Feferman’s work in [25] extends Turing’s work to any true sentence of arithmetic. Our following presentation of Turing and Feferman’s work follows closely [91] (see [91], p.285-290).

Now we define the transfinite iteration of \(\mathbf{PA}\). Given theory \(A\), let \(A^*\) consist of the axioms of \(A\) together with every instance of the following reflection principle

\[
\forall x \mathbf{Pr}_A(\forall \phi(x)^\gamma) \rightarrow \forall x \phi(x)
\]

where \(\mathbf{Pr}_A(\forall \phi(x)^\gamma)\) is a formula stating that the result of substituting the appropriate numeral for \(x\) in \(\phi(x)\) is provable in \(A\). Let \(A(0) = \mathbf{PA}\); for

\[34\mathbf{KD} = K + \neg \Box \bot.\]
each ordinal $\alpha$, if $A(\alpha)$ is defined, in Turing’s work let $A(\alpha + 1) = A(\alpha) + \text{Con}(A(\alpha))$ and in Feferman’s work let $A(\alpha + 1) = A(\alpha)^*$. If $\alpha$ is a limit ordinal, then let $A(\alpha) = \bigcup_{\beta < \alpha} A(\beta)$. Let us consider the transfinite sequence of theories $\langle A(\alpha) : \alpha \in \text{Ord} \rangle$. However, the theory $A(\alpha + 1)$ is not well-defined since the consistency statement and the above reflection principle is intensional: if $B$ is a theory, then the sentence $\text{Con}(B)$ and the theory $B^*$ depend not just on the theorems of $B$, but on how $B$ is given. At stage $\alpha + 1$, we need not just theorems of $A(\alpha)$ but also a description of those theorems, and this depends on a description of $\alpha$. If we have two different descriptions of $\alpha$, we can end up with two different theories $A(\alpha + 1)$. To overcome this problem, we use notations for recursive ordinals. Ordinals can be denoted by natural numbers. Let $O$ be the set of natural numbers that denote ordinals on this notation. If $m \in O$, let $|m|$ be the ordinal denoted by $m$. Let $R(1)$ be a standard enumeration of the theorems of $A(0)$. If $n \in O$, then let $R(2^n)$ be an enumeration of the result of applying the consistency statement to $R(n)$ (under that description). If $e$ is the Gödel number of a Turing machine that enumerates numbers denoting an increasing sequence $S$ of ordinals, then let $R(3 \cdot 5^e)$ be an uniform enumeration of the union of the sets $R(s)$ for $s \in S$. Note that for each $n \in O$, $R(n)$ is the theory $A(|n|)$ under that description. The notation makes the intentionality explicit, since the theorems of $R(n)$ depend not just on $|n|$ but also on $n$.

Turing [99] plans to overcome incompleteness by using theories like $R(n)$, with $n$ ranging over $O$. He showed that if $\phi$ is a true $\Pi^0_1$-sentence of arithmetic, then there is an $n \in O$ which can be found effectively from $\phi$ such that $|n| = \omega + 1$ and $\phi$ belongs to the theorems of $R(n)$. In [25], Feferman extended Turing’s work with his reflection principle $A^*$ and showed that for any true sentence of arithmetic, there is a number $n \in O$ such that $\phi$ belongs to the theorems of $R(n)$ and hence $\text{Truth} \subseteq \bigcup_{n \in O} R(n)$. So for any true sentence $\phi$ of arithmetic, there is a way to iterate the reflection principle and decide $\phi$.

One folklore interpretation of G2 is that since the consistency of $\text{PA}$ cannot be proved in $\text{PA}$, the consistency of $\text{PA}$ can only be proved in a stronger theory extending $\text{PA}$. This interpretation is wrong. Gentzen constructed a theory $T$ (primitive recursive arithmetic with the additional principle of quantifier-free transfinite induction up to the ordinal $\epsilon_0$) and proved that the consistency of $\text{PA}$ is provable over the theory $T$. Gentzen’s theory $T$ contains $Q$ but does not contain $\text{PA}$ since $T$ does not prove the ordinary mathematical induction for all formulas. Since Gentzen’s theory $T$ contains $Q$ and proves $\text{Con}(\text{PA})$, $T$ interprets $Q + \text{Con}(\text{PA})$. By the arithmetized completeness theorem, $Q + \text{Con}(\text{PA})$ interprets $\text{PA}$. Hence Gentzen’s theory $T$ interprets $\text{PA}$. But $\text{PA}$ does not interpret Gentzen’s theory $T$ since by Pudlák’s result, no consistent theory $T$ that contains

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35 The number 1 denotes the ordinal 0. If $n$ denotes an ordinal $\alpha$, then $2^n$ denotes its successor $\alpha + 1$. If $e$ is the Gödel number of a Turing machine that enumerates numbers denoting an increasing sequence of ordinals, then $3 \cdot 5^e$ denotes the limit of that sequence.

36 This astounding result shows that there is a way to iterate the Gödel construction on theories, beginning with $\text{PA}$, so that when we collect together the finite iterations and take one more Gödel sentence, $\phi$ is decided.

37 $\epsilon_0$ is the first ordinal $\alpha$ such that $\omega^\alpha = \alpha$. 
Robinson arithmetic $Q$ can interpret $Q + \text{Con}(T)$. So $\text{PA} \subset T$. I.e. Gentzen’s theory $T$ is stronger than $\text{PA}$ w.r.t. interpretation.

The driving goal in the original Hilbert’s program was to justify the implicit or explicit assumption of the “actual infinite” in mathematics by a reduction to purely finitary concepts and reasonings. A proof-theoretic reduction of a theory $T$ to a theory $S$ shows that, as far as a certain class of propositions is concerned, if $T$ proves a proposition, then $S$ proves it too and the proof of this fact is itself finitary (see [26], p.364). Hilbert’s program can then be seen to be a search for a proof-theoretic reduction of all of mathematics to finitary mathematics. G2 shows that it is impossible to reduce all of infinitistic mathematics to finitistic mathematics: to prove consistency of $\text{PA}$ one needs stronger methods than those available in $\text{PA}$. There are extensive literatures about the development of Hilbert’s program and its effect on mathematical logic (especially proof theory) and philosophy of mathematics (see Feferman [26], Franks [30], Murawski [73], Simpson [93], Zach [113]). Here, I will only give a brief summary of the status of the question from the literature: whether G2 refutes Hilbert’s program?

In the rest of this section, I will examine the following popular view of G2: G2 rejects Hilbert’s program since it demolished the goal of proving the consistency of mathematics by finitistic reasoning in Hilbert’s program. This failure of Hilbert’s program due to G2 is apparent if all finitary arguments can be formalized in $\text{PA}$.

In the literature, some argued that it is not the case that G2 showed the failure of Hilbert’s program. In [19], Detlefsen sketched an argument against the claim that G2 implies the failure of Hilbert’s Program for finding a finitistic consistency proof for the various theories of classic mathematics. The central claim of the argument is that $\text{Con}(T)$, the consistency formula shown to be unprovable in $T$ by G2, does not really “express” consistency in the sense of that term germane to an evaluation of Hilbert’s Program (see [19], p.309). Furthermore, Detlefsen convincingly pointed out in [19] (see [19], p.310): “G2, then, only seems to imply the failure of Hilbert’s Program so long as one ignores the fact that the logic of the finitistic proof theory of the classical $T$ and the logic of the classical $T$ itself are two quite different logics! Once this is recognized, the fact that $\text{Con}(T)$ is not provable in $T$ should come as no particular shock to those espousing Hilbert’s Program. If the logic of $T$ is expanded in a way that produces a scheme whose logic is in agreement with the logic of the finitistic proof theory of the classical $T$, then in at least some instances, $\text{Con}(T)$

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38 See Feferman [26] for a good survey of proof-theoretical results stemming from Hilbert’s program which are closely tied to various reductive foundational aims: reducing the countable infinite to the finitary; reducing the uncountable infinite to the countable infinite; reducing the impredicative to the predicative and reducing the nonconstructive to the constructive.

39 Note that the failure of Hilbert’s program for a certain formalized system of arithmetic need not be a failure of Hilbert’s program for elementary number theory if elementary number theory can be formalized in a system (much weaker than $\text{PA}$) which can be justified on finitist grounds. Research in reverse mathematics shows that elementary number theory in ordinary mathematics, as well as substantial tracts of algebra and analysis, can be developed in a theory proof-theoretically reducible to primitive recursive arithmetic ($\text{PRA}$) which is generally accepted as finitary reasoning.
becomes provable. The basic flaw of those using \(G_2\) to thwart Hilbert’s Program is that they fail to recognize that the logic of the arithmetized proof theory of \(T\) in \(G_2\) (since that arithmetized proof theory is itself embedded in \(T\)) is the logic of \(T\) itself, not the logic of the finitistic proof theory of \(T\) (which logic is not a subsystem of \(T\)’s logic)!

It is difficult to answer the question of whether there are finitary consistency proofs of \(\text{PA}\), mainly because there is no generally accepted formal characterization of the informal concept of finitary proof. If we regard finitary mathematics as being formalizable in \(\text{PA}\), then it is not possible to give finitary proofs of the consistency of \(\text{PA}\). However, Gödel’s work in [34] left open the possibility that there could be finitary methods which are not formalizable in these systems and which would yield the required consistency proofs. Gödel himself suggested the possibility of giving finitary consistency proofs by adopting the principle of transfinite induction on certain primitive recursive well-orderings, which cannot be formalized in \(\text{PA}\), but can be treated as finitistic (see [73], p.309-310). So Gödel adopts a more liberal view of what finitary methods might be allowed. It seems that Bernays was among the first to recognize the need for a generalization of Hilbert’s program by loosening the requirement of reduction to finitary methods, allowing reduction to constructive methods more generally. This is evidenced by his remark (see [26], p.365): “it thus became apparent that the finitary argument is not the only alternative to classical ways of reasoning and is not necessarily implied by the idea of proof theory. An enlarging of the methods of proof theory was therefore suggested: instead of a reduction to finitist methods of reasoning, it was required only that the arguments be of a constructive character, allowing us to deal with more general forms of inference.” The problem with such extended forms of Hilbert’s program is that there are many different styles of constructivity and the concept of constructivity in general is much less clear even than that of finitism (see [26], p.366). Gödel and Bernays’s ideas are realized by Gerhard Gentzen who proved the consistency of \(\text{PA}\) by introducing the principle of induction up to the ordinal \(\xi_0\). The only part of Gentzen’s proof that was not clearly finitary was the transfinite induction up to the ordinal \(\xi_0\). If this transfinite induction is accepted as a finitary method, then one can assert that there is a finitary proof of the consistency of \(\text{PA}\). The fact that induction up to \(\xi_0\) establishes the consistency of \(\text{PA}\), together with Gentzen’s result that for all \(\alpha < \xi_0\), \(\text{PA}\) proves the principle of induction up to the ordinal \(\alpha\) constitutes an ordinal analysis of \(\text{PA}\), and we say that \(\xi_0\) is the proof theoretic ordinal of \(\text{PA}\) (see [113], p.436). Gentzen’s work opened a new and productive direction to develop Hilbert’s program: finding the means necessary to prove the consistency of a given theory. More powerful subsystems of second order arithmetic have been given consistency proofs by Gaisi Takeuti and others, and the theories that have been proved consistent by these methods are quite strong and include most ordinary mathematics. Especially, Feferman

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has argued in [26] that most of mathematics needed for physics, for example, can be reduced to predicative systems which can be proof-theoretically characterized using ordinal notation systems albeit longer than \( \xi_0 \), but still of a small or manageable length.

Although the impact of Gödel’s incompleteness theorems for Hilbert’s program was recognized soon after its publication, Hilbert’s program was by no means abandoned. The program of so-called reverse mathematics developed by Friedman and Simpson contributes to providing us with a partial realization of Hilbert’s original program. Despite Gödel’s theorem, one can give a finitistic reduction for a substantial portion of infinitistic mathematics including many of the best-known nonconstructive theorems (see [93]). Reverse mathematics is a highly developed research program whose purpose is to investigate the role of strong set existence axioms in ordinary mathematics. In the face of Gödel’s results showing that not all of classical mathematics can be reduced to and justified by finitistic mathematics, one can ask: how much of classical mathematics can be so reduced? Reverse mathematics seeks to give a precise answer to this question by investigating which set existence axioms are needed in order to prove theorems of ordinary mathematics. Most of the work on reverse mathematics have been carried out in the context of subsystems of second order arithmetic (\( \mathbb{Z}_2 \)).

The usual pattern of mathematical reasoning is to deduce a theorem from some axioms. This might be called ”forward mathematics”. But in order to establish that the axioms are necessary for a proof of the theorem, reverse mathematics reverses the process and deduces the axioms from the theorem. Most of theorems of ordinary mathematics can be stated in the language of \( \mathbb{Z}_2 \) and proved in some subsystems of \( \mathbb{Z}_2 \). For many specific theorems \( T \), it turns out that there is a weakest natural subsystem of \( \mathbb{Z}_2 \) in which \( T \) is provable. Moreover, this weakest subsystem of \( \mathbb{Z}_2 \) is often one of five specific subsystems of \( \mathbb{Z}_2 \): \( \text{RCA}_0 \), \( \text{WKL}_0 \), \( \text{ACA}_0 \), \( \text{ATR}_0 \) and \( \Pi^1_1 \)-\( \text{CA}_0 \). In a word, even if \( G2 \) struck Hilbert’s program and revealed several difficulties in attempts to realize it, it is hard to say that \( G2 \) refutes Hilbert’s program or Hilbert’s program is killed by \( G2 \). Hilbert’s program has inspired various investigations in foundations and philosophy of mathematics, and it can be partially realized. For example, ordinal analysis, functional interpretations, proof theoretic reductions and reverse mathematics are the prominent areas most explicitly situated in the tradition of Hilbert’s program (see [113], p.440). As Zach concluded, although Gödel’s theorems show that Hilbert’s original expectations about what exactly can be analyzed in which way and with what restricted methods can not be fulfilled, Hilbert’s

\[ \text{RCA}_0 \] is the weakest of these specific systems and the others are listed in order of increasing strength. For the details of these five systems, we refer to the standard textbook [94].

\[ 41 \] In recent years, the program of reverse mathematics has been extended from analysis of ordinary mathematics in \( \mathbb{Z}_2 \) to higher order arithmetic. For works on higher order reverse mathematics, we refer to works by Kohlenbach [57, 58], Normann and Sanders [76, 77].

\[ 42 \] However, some important theorems about reals can be stated in \( \mathbb{Z}_2 \) but they are not provable in \( \mathbb{Z}_2 \) (see [13], [57], [76], [77]).

\[ 43 \] For the details of these five systems, we refer to the standard textbook [94].
NOTE ON SOME MISINTERPRETATIONS OF GÖDEL'S INCOMPLETENESS THEOREMS

Aims and proof theory more generally have been advanced tremendously over the last half-century (see [113], p.441).

5. Gödel’s theorem and the mechanism thesis

A popular interpretation of Gödel’s theorem is that it tells us that the mind cannot be mechanized in the sense that the mathematical outputs of the idealized human mind outstrip the mathematical outputs of any idealized finite machine. The mechanism thesis claims that the mind can be mechanized in our specific sense. It is well known that Turing proposed a convincing analysis of the vague and informal notion of “being computable by an idealized finite machine” in terms of the precise mathematical notion of “being computable by a Turing machine”. So we can replace the vague notion of “an idealized finite machine” with the mathematically precise notion of a Turing machine. In this paper, I will not examine the broad question of whether the mind can be mechanized, which has been extensively discussed in the literature; instead I will only examine the question of whether Gödel’s theorem implies that the mind cannot be mechanized in the sense that the mathematical outputs of the idealized human mind outstrip the mathematical outputs of any Turing machine. In the following, I give a concise overview of the current progress on this question based on Koellner’s work in [54], [55] and [56]. For more detailed discussion of the question of whether the mind can be mechanized, we refer to Koellner’s recent nice survey in [54] and [55].

Gödel did not argue that his incompleteness theorems imply that the mind cannot be mechanized understood in our specific sense; instead he argued that the incompleteness theorems imply a weaker conclusion: Gödel’s Disjunction (GD). The first disjunct says that the mind cannot be mechanized and the second disjunct says that there are absolutely undecidable statements in the sense that mathematical truth outstrips the idealized human mind (or there are mathematical truths that cannot be proved by the idealized human mind). GD says that either the first disjunct or the second disjunct holds. GD concerns the limit of mathematical knowledge and the possibility of the existence of mathematical truths that are inaccessible to the idealized human mind. The first disjunct expresses an aspect of the power of the idealized human mind, while the second disjunct expresses an aspect of its limitations.

44In this paper, we will not consider the performance of actual human minds, with their limitations and defects; but only consider the idealized human mind and look at what it can do in principle. See [55], p.2.

45See Penrose [80], Chalmers [13], Lucas [71], Lindström [69], Feferman [29], Shapiro [91], [92] and Koellner [54], [55], [56].

46The original version of GD was introduce by Gödel in [37], p. 310: “So the following disjunctive conclusion is inevitable: either mathematics is incompletable in this sense, that its evident axioms can never be comprised in a finite rule, that is to say, the human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine, or else there exist absolutely unsolvable diophantine problems of the type specified (where the case that both terms of the disjunction are true is not excluded, so that there are, strictly speaking, three alternatives)”.

47We refer to [45], a recent comprehensive research volume about GD, for more discussions of the status of GD.
What about Gödel’s view toward the first disjunct and the second disjunct? For Gödel, the first disjunct is true and the second disjunct is false; that is the mind cannot be mechanized and human mind is sufficiently powerful to capture all mathematical truths. Gödel believes that the distinctiveness of the human mind when compared to a Turing machine is evident in its ability to come up with new axioms and develop new mathematical theories. Gödel shared Hilbert’s belief expressed in 1926 in the words: “in mathematics there is no ignoramuses, we should know and we must know”. Based on his rationalistic optimism, Gödel believed that we are arithmetically omniscient and the second disjunct is false. However, Gödel admits that he cannot give a convincing argument for either the first disjunct or the second disjunct. Gödel thought that the most he could claim to have established was the disjunctive conclusion. For Gödel, GD is a “mathematically established fact” of great philosophical interest which follows from his incompleteness theorems, and it is “entirely independent from the standpoint taken toward the foundation of mathematics” (Gödel, [37]).

Let $K$ be the set of sentences in $L(\text{PA})$ that the idealized human mind can know. Gödel refers to Truth as objective mathematics and $K$ as subjective mathematics. We assume throughout this paper that $K \subseteq \text{Truth}$. However, from $G_1$, we have $\text{Prov} \subsetneq K$ since we can show that $\text{PA}$ is consistent and hence Gödel’s sentence is a true sentence of arithmetic but not provable in PA.

GD concerns the concepts of relative provability, absolute provability, and truth. Before our analysis of GD, let us first examine two key notions about provability: relative provability and absolute provability. The notion of relative provability is well understood and we have a precise definition of relative provability in a formal system. But the notion of absolute provability is much more ambiguous and we have no unambiguous formal definition of absolute provability as far as I know. The notion of absolute provability is intended to be intensionally different from the notion of relative provability in that absolute provability is not conceptually connected to a formal system. In contrast to the notion of relative provability, there is little agreement on what principles of the notion “absolute provability” should be adopted. In this paper, we equal the notion of “relatively provable with respect to a given formal system $F$” with the notion of “producible by a Turing machine $M$” (where $M$ is the Turing machine corresponding to $F$) and equal the notion of “absolute provability” with the notion of “what the idealized human mind

\[48\] For more discussions of the status of the second disjunct, we refer to Leon and Philip’s recent comprehensive research volume [45].

\[49\] In the literature there is a consensus that Gödel’s argument for GD is definitive, but until now we have no compelling evidence for or against any of the two disjuncts (Horsten and Welch, [45]).

\[50\] Let us take Fermat’s last theorem for another example. People have shown that Fermat’s last theorem is a true sentence of arithmetic but, as far as I know, it is still an open problem whether Fermat’s last theorem is provable in PA. So Fermat’s last theorem belongs to $K$ but it is open whether it belongs to $\text{Prov}$.

\[51\] Note that sentences relatively provable with respect to a given formal system $F$ can be enumerated by a Turing machine.
can know”²⁵. Under this assumption, \( K \) is just the set of sentences that are absolutely provable.

Let \( (M_e : e \in \mathbb{N}) \) be an enumeration of Turing machines and \( Th(M_e) \) be the theory enumerated by the Turing machine \( M_e \). In this paper, we assume without loss of generality that \( Q \subseteq Th(M_e) \) such that both \( G_1 \) and \( G_2 \) apply to \( Th(M_e) \). We say that a statement \( \phi \) is relatively undecidable w.r.t. theory \( Th(M_e) \) for some \( e \) if neither \( \phi \in Th(M_e) \) nor \( \neg \phi \in Th(M_e) \). We say that a statement \( \phi \) is absolutely undecidable if \( \phi \notin K \) nor \( \neg \phi \notin K \). We say that theory \( Th(M_e) \) is consistent if \( Th(M_e) \subseteq \text{Truth} \). Let us first examine what the incompleteness theorems tell us about the relationship between \( Th(M_e), K \) and \( \text{Truth} \).

\( G_1 \) tells us that for any sufficiently strong consistent theory \( F \) containing \( \text{PA} \), there are statements which are relatively undecidable with respect to \( F \). But as Gödel argued, these statements are not absolutely undecidable; instead one can always pass to higher systems in which the sentence in question is provable (see [38], p.35). For example, from \( G_2 \), \( \text{Con}(\text{PA}) \) is not provable in \( \text{PA} \); but \( \text{Con}(\text{PA}) \) is provable in second order arithmetic \( (\mathbb{Z}_2) \).²⁶ Since \( G_2 \) applies to \( \mathbb{Z}_2 \), the \( \Pi^0_1 \)-truth \( \text{Con}(\mathbb{Z}_2) \) is not provable in \( \mathbb{Z}_2 \). But \( \text{Con}(\mathbb{Z}_2) \) is provable in \( \mathbb{Z}_3 \) (third order arithmetic) which captures the \( \Pi^0_1 \)-truth that was missed by \( \mathbb{Z}_2 \). This pattern continues up through the orders of arithmetic and up through the hierarchy of set-theoretic systems; at each stage a missing \( \Pi^0_1 \)-truth is captured at the next stage (see [55], p. 13).

Now let us first examine the question whether the incompleteness theorems show that \( \text{GD} \) holds. From the literature, we have found a natural framework \( \text{EA}_T \) in which we can establish definitive results of the form: if the principles governing the fundamental concepts of relative provability, absolute provability and truth are such-and-such, then one can give a rigorous proof of \( \text{GD} \), vindicating Gödel’s claim that \( \text{GD} \) is a mathematically established fact.

Now I introduce two systems of epistemic arithmetic: \( \text{EA} \) and \( \text{EA}_T \). For the presentation of \( \text{EA} \) and \( \text{EA}_T \), I follow closely from [54] and [55]. The first is designed to deal with \( Th(M_e) \) and \( K \), and the second is designed to deal with \( Th(M_e) \), \( K \) and \( \text{Truth} \). For \( \text{EA}_T \), we only require a typed truth predicate.²⁷ The basic system \( \text{EA} \) of epistemic arithmetic has axioms of arithmetic and axioms of absolute provability, and the extended system \( \text{EA}_T \) has additional axioms of typed truth.²⁸ In \( \text{EA} \) and \( \text{EA}_T \), \( K \) is treated

\begin{footnotes}
²⁵Williamson [112] makes the similar definition that a mathematical hypothesis is absolutely decidable if and only if either it or its negation can in principle be known by a normal mathematical process; otherwise it is absolutely undecidable.

²⁶Recall that \( \text{Con}(\text{PA}) \) refers to the consistency statement of \( \text{PA} \) under a recursive enumeration of \( \text{PA} \).

²⁷A typed truth predicate is one that applies only to statements that do not themselves involve the truth predicate. In contrast, a type-free truth predicate is one which also applies to statements that themselves involve the truth predicate. The principles governing typed truth predicates are perfectly straightforward and uncontroversial, while the principles governing type-free truth predicates are much more delicate. See [55], p.18.

²⁸These systems were first introduced by Myhill [74], Reinhardt [85, 83, 80], and Shapiro [90], and then investigated by many others (e.g. Horsten [43], Leitgeb [60], Reinhardt [54, 85, 80], Carlson [12], Koellner [51, 55] and others).
\end{footnotes}
as an operator rather than a predicate. The basic axioms of absolute provability are:

- **K1**: Universal closures of formulas of the form $K\phi$ where $\phi$ is a first-order validity.
- **K2**: Universal closures of formulas of the form $(K(\phi \rightarrow \psi) \land K\phi) \rightarrow K\psi$.
- **K3**: Universal closures of formulas of the form $K\phi \rightarrow \phi$.
- **K4**: Universal closures of formulas of the form $K\phi \rightarrow KK\phi$.

The language $L(\text{EA})$ is $L(\text{PA})$ expanded to include an operator $K$ that takes formulas of $L(\text{EA})$ as arguments. The axioms of arithmetic are simply those of $\text{PA}$, only now the induction scheme is taken to cover all formulas in $L(\text{EA})$. For a collection $\Gamma$ of formulas in $L(\text{EA})$, let $K\Gamma$ denote the collection of formulas $K\phi$ where $\phi \in \Gamma$. The system $\text{EA}$ is the theory axiomatized by $\Sigma \cup K\Sigma$, where $\Sigma$ consists of the axioms of $\text{PA}$ in the language $L(\text{EA})$ and the basic axioms of absolute provability. The language $L(\text{EA}_T)$ of $\text{EA}_T$ is the language $L(\text{EA})$ augmented with a unary predicate $T$. The system $\text{EA}_T$ is the theory axiomatized by $\Sigma \cup K\Sigma$, where $\Sigma$ consists of the axioms of $\text{PA}$ in the language $L(\text{EA}_T)$, the basic axioms of absolute provability (in the language $L(\text{EA}_T)$), and the Tarskian axioms of truth for the language $L(\text{EA})$.

From the incompleteness theorems, Gödel made the following two claims about the relationship between $Th(M_e), K$ and $\text{Truth}$.

- **Claim One**: For any $e \in \mathbb{N}$, $K(Th(M_e) \subseteq \text{Truth}) \rightarrow Th(M_e) \subseteq K$.
- **Claim Two**: Either $\neg \exists e(Th(M_e) = K)$ or $\exists \phi \in \text{Truth} \land \phi \notin K$ and $\neg \phi \notin K$.

Gödel’s Claim One is formalizable and provable in $\text{EA}_T$. In fact, something stronger is provable in $\text{EA}$ as the following theorems show:

**Theorem 5.1.** (Reinhardt, [54]). Assume that $S$ includes $\text{EA}$. Suppose $F(x)$ is a formula with one free variable.

1. If for each sentence $\phi$, $S \vdash K(F(\Gamma \phi)) \rightarrow \phi$. Then there is a sentence $\phi$ such that $S \vdash K\phi \land K(\neg F(\Gamma \phi))$.
   
2. If for each sentence $\phi$, $S \vdash K(\phi \rightarrow F(\Gamma \phi))$. Then $S \vdash K \rightarrow K(\text{Con}(F))$.

56From results in Gödel [36], Myhill [74], Montague [72], Thomason [93], and others, if one formulates a theory of absolute provability with $K$ as a predicate then inconsistency may come. See [55], p.19.

57The basic conditions we will impose on knowability are: (1) if the idealized human mind knows $\phi$ and $\phi \rightarrow \psi$ then the idealized human mind knows $\psi$; (2) if the idealized human mind knows $\phi$ then $\phi$ is true; (3) if the idealized human mind knows $\phi$ then the idealized human mind knows that the idealized human mind knows $\phi$.

58K1-known as logical omniscience-says that $K$ holds of all first-order logical validities; $K2$ says that $K$ is closed under modus ponens, and so distributes across logical derivations; $K3$ says that $K$ is correct; and $K4$ says that $K$ is absolutely self-reflective.

59The informal proof of Claim One is as follows: Suppose $K(Th(M_e) \subseteq \text{Truth})$. Since it is knowable that $Th(M_e)$ is consistent, it is knowable that there is a true sentence of arithmetic which is not provable in $Th(M_e)$. So $Th(M_e) \not\subseteq K$.

60The informal proof of Claim Two is as follows: Suppose $Th(M_e) = K$ for some $e$. Since $Th(M_e)$ is R.E. but Truth is not arithmetic, $K \not\subseteq \text{Truth}$. So we can find some $\phi \in \text{Truth}$ but $\phi \notin K$ and $\neg \phi \notin K$. 

GD is also formalizable and provable in \( \text{EA}_T \) which confirms Gödel’s claim that GD is a mathematically established fact.\(^{61}\)

**Theorem 5.2.** (Reinhardt, [86]). Assume \( \text{EA}_T \). Then GD holds.

There has been a massive amount of literature on the arguments for the first disjunct due primarily to Lucas and Penrose (see Lucas [70], Penrose [80]) which claim that G1 shows that the human mind cannot be mechanized.\(^{62}\) Most philosophers and logicians believe that variants of the arguments of Lucas and Penrose are not fully convincing. In this paper, we focus on the following question: whether the first disjunct is also a mathematically established fact from Gödel’s incompleteness theorems.

The main point of Lucas’s argument is as follows: from G1, for any formal system \( T \) containing large enough fragment of \( \text{PA} \), we can construct Gödel’s sentence \( G \) such that \( G \) is not provable in \( T \); but it is knowable that \( G \) is true. So the human mind cannot be mechanized. This argument is invalid, in general we do not know whether \( G \) is true, we only know that \( G \) is true iff \( T \) is consistent, but \( \text{Con}(T) \rightarrow G \) is also provable in \( T \).\(^{63}\) Benacerraf correctly pointed out in [5] that Lucas’ argument does not exclude the possibility that the human mathematical mind is indeed a Turing machine, but it is not humanly knowable which one it is.

Following Reinhardt, we should distinguish three levels of the mechanistic thesis. (1) The weak mechanistic thesis (WMT): \( \exists e (K = Th(M_e)) \); (2) The strong mechanistic thesis (SMT): \( K \exists e (K = Th(M_e)) \); (3) The super strong mechanistic thesis (SSMT): \( \exists e K (K = Th(M_e)) \). WMT is just the first disjunct which says that there is a Turing machine which coincides with the idealized human mind in the sense that the two have the same outputs. SMT says that the idealized human mind knows that there is a Turing machine which coincides with the idealized human mind. SSMT says that there is a particular Turing machine such that the idealized human mind knows that that particular machine coincides with the idealized human mind.

Suppose WMT holds. Then \( K = Th(M_e) \) for some \( e \). But \( K \) can be enumerated by which machine? Let \( A = \{ e : K = Th(M_e) \} \). Then \( A \) is not recursive by Rice theorem.\(^{64}\) So there is no effective procedure such that given \( e \) we could decide whether \( K = Th(M_e) \) or not, even if we know that \( K = Th(M_e) \) for some \( e \).

The following theorem shows that we can prove in \( \text{EA}_T \) that there does not exist a particular Turing machine such that the idealized human mind

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\(^{61}\)It is a little delicate to formalize GD in \( \text{EA}_T \) since \( K \) is formalized as an operator in \( \text{EA}_T \) and so we are prohibited from quantifying into it. For the details, we refer to Reinhardt [86] and Koellner [55].

\(^{62}\)The arguments for the first disjunct began with Nagel and Newman in [75] and continued with Lucas’s publication in [70]. Nagel and Newman’s argument was criticized by Putnam in [83], while Lucas’s argument was much more widely criticized in the literature. See Feferman [29] for a historical account and Benacerraf [5] for an influential criticism of Lucas.

\(^{63}\)Putnam [83] correctly pointed out this and asked how can Lucas show that \( T \) is consistent.

\(^{64}\)Let \( C \) be any class of partial recursive functions. Rice theorem says that \( \{ n : \phi_n \in C \} \) is recursive if and only if \( C = \emptyset \) or \( C \) is the class of all partial recursive functions where \( \phi_n \) is the partial recursive function with index \( n \).
knows that that particular Turing machine coincides with the idealized human mind.

**Theorem 5.3.** (Reinhardt, [84]) $\text{EA}_T + \text{SSMT}$ is inconsistent.

The following theorem shows that, from the viewpoint of $\text{EA}_T$ it is possible that the idealized human mind is in fact a Turing machine. From Theorem 5.3 it just cannot know which one.

**Theorem 5.4.** (Reinhardt [85]) $\text{EA}_T + \text{WMT}$ is consistent.

Theorem 5.4 shows that the first disjunct is not provable in $\text{EA}_T$. Since $\text{EA}_T$ seems to embody all of the assumptions held by the proponents of the first disjunct, this shows that there is a fundamental obstacle to prove the first disjunct (see [55], p.28). But Gödel did think that one day we would be in a position to prove the first disjunct, and what was missing, as he saw it, was an adequate resolution of the paradoxes involving self-applicable concepts like the concept of truth. Gödel thought that “[i]f one could clear up the intensional paradoxes somehow, one would get a clear proof that mind is not machine”.

The following technical theorem from Carlson shows that, from the point of view of $\text{EA}_T$, it is possible that the idealized human mind knows that it is a Turing machine: it just cannot know which one.

**Theorem 5.5.** (Carlson, [12]) $\text{EA}_T + \text{SMT}$ is consistent.

Now I give a summary for the question whether Gödel’s incompleteness theorems imply the first disjunct. The incompleteness theorems imply that $\neg \exists e K(K = \text{Th}(M_e))$. But from Theorem 5.4 it does not follow that $\neg \exists e (K = \text{Th}(M_e))$; and from Theorem 5.5 it does not even follow that $\neg K \exists e (K = \text{Th}(M_e))$. The difference between $\exists e K$ and $K \exists e$ before $K = \text{Th}(M_e)$ is essential. Assuming the principles embodied in $\text{EA}_T$, it is possible to know that we are a Turing machine (i.e. $K \exists e (K = \text{Th}(M_e))$); it is just not possible for there to be a Turing machine such that we know that we are that Turing machine (i.e. $\exists e K(K = \text{Th}(M_e))$).

Penrose proposed a new argument for the first disjunct in [81] and [82]. Penrose’s new argument is the most sophisticated and promising argument for the first disjunct which has been extensively discussed and carefully analyzed in the literature (see Chalmers [13], Feferman [27], Lindström [68] [69], and Shapiro [91] [92], Gaifman [33] and Koellner [54] [56], etc). The question of whether Penrose’s new argument establishes the first disjunct is quite subtle. Penrose’s new argument involves treating truth as type-free, and so for the analysis and formalization of Penrose’s argument, we need to employ type-free notions of truth. However, we now have many type-free theories of truth and there is no consensus as to which option is best.

In the literature, we have found a framework $\text{DTK}$ which employs Feferman’s type-free theory of determinate truth $\text{DT}$ and some additional axioms governing $K$ to the axioms of $\text{DT}$ [14]. From [54] and [56], $\text{DTK}$ is consistent.
(see [54, Theorem 7.14.1]) and DTK proves GD (see [54, Theorem 7.15.3]). However, the particular argument Penrose gives for the first disjunct fails in the context of DTK (see [56, Theorem 4.1]). Moreover, even if we restrict the first and second disjunct to arithmetic statements, DTK can neither prove nor refute either the first disjunct or the second disjunct (see [54, Theorem 7.16.1-7.16.2]). From the point of view of DTK, it is in principle impossible to prove or refute either disjunct. Koellner argued that “these results may indicate that the concepts of absolute provability and knowability by the idealized human mind are not sharp enough for our questions whether the mind is not mechanized and whether there are absolutely undecidable problems to have definite sense and determinate truth values” (Koellner, p.31).

In our previous discussions about GD, the first disjunct and the second disjunct, we equate absolutely undecidability with knowability of the idealized human mind and define that $\phi$ is absolutely undecidable if $\phi \notin \text{K}$ and $\neg \phi \notin \text{K}$. Under this framework, the second disjunct is equivalent to “$\text{K}$ is not complete”. Under the assumption that $\text{K} \subseteq \text{Truth}$, the second disjunct is equivalent to “$\text{K} \nsubseteq \text{Truth}$”. However, $G1$ only tells us that $\text{Prov} \nsubseteq \text{Truth}$, and it does not tell us that $\text{K} \nsubseteq \text{Truth}$. Another natural informal definition of absolutely undecidability is: $\phi$ is absolutely undecidable if there is no consistent extension $T$ of $\text{ZFC}$ with well-justified axioms such that $\phi$ is provable in $T$. In this paper, we focus on whether the incompleteness theorems imply the second disjunct. In philosophy of set theory, there are extensive discussions about whether there exists an absolutely undecidable statement in set theory. For the detailed discussions about the question of absolutely undecidability in set theory and especially whether $\text{CH}$ is absolutely undecidable, we refer to Koellner [53].

We have shown that the popular view that $G1$ implies the first disjunct (mind cannot be mechanized) is problematic and not convincing. In the following, I introduce an effective version of $G1$ and examine whether this version of $G1$ refutes the first disjunct.

A general version of $G1$ says that for any recursive enumerable $A \subseteq \text{Truth}$, there is a sentence $\phi$ such that $\phi \in \text{Truth}$ but $\phi \notin A$. We say that a set of natural numbers $A$ is productive if there is a recursive function $f$ (a productive function for $A$) such that for any $e$, if $W_e \subseteq A$, then $f(e)$ is defined and $f(e) \in A \setminus W_e$ ($f(e)$ is the witness that $W_e \neq A$). Given $A,B \subseteq \mathbb{N}$, define $A \leq_m B$ if there exists a recursive function $f$ on $\mathbb{N}$ such that $n \in A$ iff $f(n) \in B$. We can prove that for $A,B \subseteq \mathbb{N}$, if $A \leq_m B$ and $A$ is productive, then $B$ is productive. Define $C = \{e : e \notin W_e\}$. Since $C \leq_m \text{Truth}$ and $C$ $\text{Prov} \nsubseteq \text{Truth}$.
is productive, we have Truth is productive. Let EG1 denote the following effective version of G1: there exists a recursive function $f$ such that for any $e$, if $W_e \subseteq \text{Truth}$, then $f(e)$ is defined and $f(e) \in \text{Truth} \setminus W_e$.

One misinterpretation of EG1 is that it refutes the first disjunct and supports the mechanism thesis. I would argue that the claim that EG1 refutes the first disjunct is problematic. Remember that $Th(M_n)$ is the set of sentences produced by the Turing machine $M_n$. From EG1, there exists a recursive function $f$ such that for every $n$, if $Th(M_n)$ is consistent then $f(n)$ is the Gödel number of a true sentence of arithmetic which is not in $Th(M_n)$. In the rest of this paper, let $f^*$ be the fixed recursive function as asserted in EG1.

From EG1, the mechanist may conclude that from the statement “I can find a limitation in any given machine”, it by no means follows that I am not a machine. The mechanist may use EG1 to claim that EG1 refutes the first disjunct. Firstly, the mechanist may claim that if the human mind is just the Turing machine which computes $f^*$ and simulates the procedure described above, then this machine is not really worse than we human being since for any Turing machine $M_n$, $f^*(n)$ picks up the true sentence of arithmetic not produced by $M_n$. However, this is a misinterpretation of EG1 which in fact says that, for any $n$, if $Th(M_n)$ is consistent then $f^*(n)$ is the Gödel number of a true sentence of arithmetic which is not in $Th(M_n)$. A natural question is: whether there exists an effective procedure such that we can decide whether $Th(M_n)$ is consistent. Define $B = \{ n : Th(M_n) \text{ is consistent} \}$. We can show that $B$ is not recursive. Since $B$ is undecidable, it is impossible to effectively distinguish the case that $Th(M_n)$ is consistent and the case that $Th(M_n)$ is not consistent. The claim that EG1 refutes the first disjunct is problematic: EG1 does not refute the first disjunct.

The mechanist may claim that $K$ can be enumerated by the Turing machine which computes $f^*$. Let $C = \{ f^*(n) : n \in \mathbb{N} \}$. Since $f^*$ is recursive, $C$ is recursive enumerable. A natural question is: could be that $K = C$? If $K = C$, then the first disjunct fails and the mechanism thesis holds. However, we can show that $C$ is inconsistent and hence $K \neq C$ since $K$ is consistent by our assumption.

In a summary, the claim that EG1 refutes the first disjunct and supports the mechanism thesis is problematic. We cannot consistently conclude from EG1 that the human mind is just the Turing machine which computes $f^*$.

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\[^{71}\] Any $e \in \mathbb{N}$. Let $h = f \circ g \circ t$. We show that $h$ is the productive function for $B$. Suppose $W_e \subseteq B$. Since $W_{t(e)} \subseteq A$, we have $g(t(e)) \in A \setminus W_{t(e)}$ and so $h(e) = f(g(t(e))) \in B \setminus W_e$.

\[^{72}\] $(W_e : e \in \mathbb{N})$ is the list of recursive enumerable subsets of $\mathbb{N}$ where $W_e = \{ n \in \mathbb{N} : \phi_e(n) \downarrow \}$.

\[^{73}\] Suppose $B$ is recursive. Let $C = \{ f^*(n) : n \in B \}$. Then $C$ is recursive enumerable. Suppose $C = Th(M_m)$ for some $m$. Note that $C \subseteq \text{Truth}$: for any $n \in B$, since $Th(M_m)$ is consistent we have $f^*(n) \in \text{Truth}$. So $C$ is consistent. By the definition of $B$, $m \in B$ and hence $f^*(m) \in C$. But since $Th(M_m)$ is consistent, by the definition of $f^*$, $f^*(m)$ is not in $Th(M_m)$ and hence $f^*(m) \notin C$ which leads to a contradiction.

\[^{74}\] Assume that $C$ is consistent. Suppose that $C = Th(M_k)$ for some $k \in \mathbb{N}$. Since $Th(M_k)$ is consistent, $f^*(k) \notin Th(M_k)$. But $f^*(k) \in C$ which leads to a contradiction. So under our assumption that $K$ is consistent, we have $K \neq C$. 
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NOTE ON SOME MISINTERPRETATIONS OF GÖDEL’S INCOMPLETENESS THEOREMS

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