Statistically Efficient, Polynomial Time Algorithms for Combinatorial Semi Bandits

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Abstract

We consider combinatorial semi-bandits over a set of arms $\mathcal{X} \subset \{0,1\}^d$ where rewards are uncorrelated across items. For this problem, the algorithm ESCB yields the smallest known regret bound $R(T) = \mathcal{O}\left(\frac{d \ln m \ln(T)}{\Delta_{\min}}\right)$, but it has computational complexity $\mathcal{O}(|\mathcal{X}|)$ which is typically exponential in $d$, and cannot be used in large dimensions. We propose the first algorithm which is both computationally and statistically efficient for this problem with regret $R(T) = \mathcal{O}\left(\frac{d \ln m \ln(T)}{\Delta_{\min}}\right)$ and computational complexity $\mathcal{O}(T \text{poly}(d))$. Our approach involves carefully designing an approximate version of ESCB with the same regret guarantees, showing that this approximate algorithm can be implemented in time $\mathcal{O}(T \text{poly}(d))$ by repeatedly maximizing a linear function over $\mathcal{X}$ subject to a linear budget constraint, and showing how to solve this maximization problems efficiently.

1 Introduction

We consider the combinatorial bandit problem with semi-bandit feedback and independent rewards across items. Time is discrete, and at times $t = 1, \ldots, T$ a learner chooses a decision $x(t) \in \mathcal{X}$, where $\mathcal{X} \subset \{0,1\}^d$ is a combinatorial set which is known to the learner. The learner then receives a reward $Z^\top(t)x(t)$ and observes a feedback vector $Y(t) = (x_1(t)Z_1(t), \ldots, x_d(t)Z_d(t))$, where $Z(t) \in [0,1]^d$ is a random vector with mean $\theta \in \mathbb{R}^d$ and whose entries are independent.

The expected reward from decision $x \in \mathcal{X}$ is $\theta^\top x$, and the goal is to maximize the sum of expected rewards, or equivalently to minimize the regret:

$$R(T) = \sum_{t=1}^T \max_{x \in \mathcal{X}} \{\theta^\top x\} - E(\theta^\top x(t)).$$

The vector $\theta$ is unknown to the learner, and in order to minimize the regret one must discover the decision $x \in \mathcal{X}$ maximizing $\theta^\top x$, and in turn one must explore enough decisions to obtain enough information about $\theta$. This problem models a large amount of practically relevant online decision problems such as online shortest path routing, ad-display optimization and resource allocation.

In this paper we propose the first (to the best of our knowledge) algorithm with regret $R(T) = \mathcal{O}\left(\frac{d \ln m \ln(T)}{\Delta_{\min}}\right)$ and polynomial computational complexity in the problem dimension $d$ for a large family of combinatorial sets $\mathcal{X}$.

The rest of the paper is organised as follows. In section 2 we further highlight the model and describe combinatorial sets of interest. In section 3 we describe the related work on this problem, including state-of-the-art regret bounds and algorithms, and highlight our contribution. In section 4 we describe the proposed algorithm, and provide regret bounds. In section 5 we show that our algorithm may be implemented in polynomial time for a large class of combinatorial sets, and analyze its computational complexity in details. In section 6 we perform numerical experiments to complement. Section 7 concludes the paper.
2 Combinatorial Semi-Bandits

2.1 The Model

As said previously, we consider the following problem. Time is discrete, and at times $t = 1, ..., T$ a learner chooses a decision $x(t) \in \mathcal{X}$, where $\mathcal{X} \subset \{0, 1\}^d$ is a combinatorial set which is known to the learner. Set $\mathcal{X}$ may be any combinatorial set, including bases of a matroid, the set of paths in some graph, the set of matchings in some bipartite graph etc. The problem dimension is $d$, and we define $m = \max_{x \in \mathcal{X}} 1^\top x$ the size of the largest decision.

After selecting decision $x(t)$, the learner then receives a reward $Z(t)x(t)$ and observes a feedback vector $Y(t) = (x_1(t)Z_1(t), \ldots, x_d(t)Z_d(t))$, where $Z(t) \in [0, 1]^d$ is a random vector.

We assume that $(Z(t))_t$ are i.i.d. with mean $\theta \in [0, 1]^d$ and that the entries of $Z_i(t)$ are independent as well. Vector $\theta$ is initially unknown to the learner, and must be learnt by repetitively selecting decisions and observing subsequent feedback. For $i \in \{1, \ldots, d\}$ if $x_i(t) = 1$ then the learner obtains a noisy realization of $\theta_i$ and nothing otherwise, so that decisions must be carefully selected to obtain a good estimate of $\theta$. This is the so called "semi-bandit feedback" model.

Since $\theta$ is unknown to the learner, decision $x(t)$ must be selected solely as a function of the available feedback information available at time $t$, that is $Y(t-1), ..., Y(1)$.

The expected reward received by selecting decision $x \in \mathcal{X}$ is $\theta^\top x$, (i.e. rewards are linear in the decision), so that $\theta_i$ represents the amount of reward received by selecting $x_i = 1$. The optimal decision is $x^* = \arg \max x \in \mathcal{X} \{\theta^\top x\}$ (there may be several optimal decisions). We define the reward gap $\Delta_x = \theta^\top (x^* - x)$, the amount of regret incurred to the learner by selecting decision $x$ instead of $x^*$. We denote by $\Delta_{\min} = \min_{x: \Delta_x > 0} \Delta_x$ the smallest non-null gap.

The goal of the learner is to minimize the regret, which is simply the difference in terms of expected cumulative rewards between the learner and an oracle who knows the latent vector $\theta$ in advance and who always selects the optimal decision $x^*$, that is:

$$R(T) = \sum_{t=1}^T \mathbb{E}(\Delta_{x(t)}).$$

2.2 Combinatorial Sets of Interest

Of course, in order to devise algorithms with low regret and computational complexity, one must take into account the structure of the combinatorial set $\mathcal{X}$. We will consider several family of sets, which we present here. We denote by $e_i$ for $i = 1, \ldots, d$ the $i$-th canonical basis vector. When working with graphs, we identify sets of edges with binary vectors, namely given a graph $G = (V, E)$, we identify a subset of edges $E' \subset E$ with a binary vector $x = \{0, 1\}^{|E|}$ where $x_e = 1$ if $e \in E'$ and $x_e = 0$ otherwise.

$m$-sets The set of $m$ sets is the set of vectors $x \in \{0, 1\}^d$ which have at most $m$ non-null entries, namely:

$$\mathcal{X} = \{x \in \{0, 1\}^d : (1, ..., 1)^\top x \leq m\}.$$  

Spanning trees Consider $G = (V, E)$ a graph. A spanning tree $x \in \{0, 1\}^{|E|}$ is a subset of edges which covers each vertex $v \in V$ at least once and forms a tree. The dimension here is the number of edges: $d = |E|$.

$$\mathcal{X} = \{x \in \{0, 1\}^{|E|} : \sum_{e \in E} x_e \geq 1\forall v \text{ and } x \text{ is a tree} \}. $$

Matroids A matroid $\mathcal{I}$ over a set with $d$ elements is a set of vectors $x \in \{0, 1\}^d$ which verify two properties: (i) the inclusion property: if $x \leq x'$ and $x' \in \mathcal{I}$ then $x \in I$ and (ii) the exchange property: if $x \in \mathcal{I}$, $x' \in \mathcal{I}$ and $1^\top x \leq 1^\top x'$ then there exists $i$ such that $x_i = 0$, $x'_i = 1$ and $x + e_i \in \mathcal{I}$. Since $m$-sets and spanning trees of a graph form a matroid, any algorithm for matroids also applies to $m$-sets and spanning trees.

Source destination paths Consider $G = (V, E)$ a directed acyclic graph and $u$ and $v$ in $V$ two vertices. The set of paths between source $u$ and destination $v$ is defined as:

$$\mathcal{X} = \{x \in \{0, 1\}^{|E|} : \sum_{e \in \text{in}(u)} x_e - \sum_{e \in \text{out}(u)} x_e = 1\{w = u\} - 1\{w = v\}, w \in V\}. $$
where \( \text{in}(v) \) and \( \text{out}(v) \) are the set of ingoing and outgoing edges respectively of \( v \in V \). The dimension is \( d = |E| \) is the number of edges and \( m = \max_{x \in X} 1^\top x \) is the length of the longest path from \( u \) to \( v \).

Matchings Consider \( G = (V,E) \) a bipartite graph. A matching is a set of edges which cover each vertice \( v \in V \) at most once, and the set of matchings is:

\[
\mathcal{X} = \{ x \in \{0,1\}^{|E|} : \sum_{e \in E : v \in e} x_e \leq 1 \forall v \}.
\]

Intersection of two matroids Consider \( I \) and \( I' \) two matroids over a set with \( d \) elements, their intersection is

\[
\mathcal{X} = \{ x \in \{0,1\}^d : x \in I, x \in I' \}.
\]

Since the set of matchings of any bipartite graph is the intersection of two matroids, algorithms for intersection of two matroids also apply to matchings of a bipartite graph.

Knapsack-like sets Consider \( A \in \mathbb{N}^{d \times k} \) matrix and \( c \in \mathbb{N}^k \) a vector both with positive integer entries. The corresponding knapsack set is:

\[
\mathcal{X} = \{ x \in \{0,1\}^d : Ax \leq c \}.
\]

We call \( \mathcal{X} \) a knapsack-like set as its elements \( x \) verify \( k \) knapsack constraints \( \sum_{i=1}^{d} A_{\ell,i} x_i \leq c_i, \ell = 1,\ldots,k \). For \( k = 1 \), \( \mathcal{X} \) is the set of feasible solutions of a knapsack problem and the set of \( m \)-sets is a knapsack-like set.

2.3 Maximization Problems

In order to select a decision \( x(t) \) at time \( t \), most if not all known algorithms involve maximizing some function over the set of decisions \( \mathcal{X} \). Hence the computational complexity of these algorithms depends mostly on the complexity of these optimization problems. We consider \( a, b \) two vectors of \( \mathbb{R}^d \) with positive entries.

Linear Maximization Problem (\( P_1 \)) involves maximizing a linear function over \( \mathcal{X} \) and both CUCB and TS solve (\( P_1 \)) at each time step:

\[
\text{maximize } a^\top x \text{ subject to } x \in \mathcal{X} \quad (P_1)
\]

Index Maximization Problem (\( P_2 \)) involves maximizing the sum of a linear function and the square root of a linear function over \( \mathcal{X} \) and ESCB solves (\( P_2 \)) at each time step:

\[
\text{maximize } a^\top x + \sqrt{b^\top x} \text{ subject to } x \in \mathcal{X} \quad (P_2)
\]

Budgeted Linear Maximization Problem (\( P_3 \)) involves maximizing a linear function over \( \mathcal{X} \) subject to a linear budget constraint, and is solved several times by AESCB.

\[
\text{maximize } b^\top x \text{ subject to } x \in \mathcal{X} \text{ and } a^\top x \geq s \quad (P_3)
\]

3 Related Work and Contribution

The study of classical bandits dates back to [19] and [14]. The order optimal regret in this problem is \( R(T) = \mathcal{O}\left(\frac{d(\ln T)}{\Delta_{\min}}\right) \), which is attained by algorithms such as UCB1 [3] and KL-UCB [6]. Linear bandits extend classical bandits when the expected reward is a linear function of the decision [10]. When the set of decisions is a combinatorial set we have a combinatorial bandit which comes in two version: full bandit feedback and semi-bandit feedback [7].

We consider combinatorial semi-bandits with independent rewards across items. For such problems, the best known regret bounds are \( R(T) = \mathcal{O}\left(\frac{2m(\ln T)}{\Delta_{\min}}\right) \) for CUCB [13], and Thompson Sampling [21] \(^1\). The

\(^1\)The authors of [21] study a slightly more general problem, in appendix we show that their regret bound reduces to \( R(T) = \mathcal{O}\left(\frac{dm(\ln T)}{\Delta_{\min}}\right) \) in the problem we study.
| Algorithm | Best Regret Bound | Complexity |
|-----------|------------------|------------|
| CUCB      | $\mathcal{O}\left(\frac{d m (\ln T)}{\Delta_{\min}}\right)$ | $\mathcal{O}(\text{poly}(d))$ |
| TS        | $\mathcal{O}\left(\frac{d m (\ln T)}{\Delta_{\min}}\right)$ | $\mathcal{O}(\text{poly}(d))$ |
| ESCB      | $\mathcal{O}\left(\frac{d (\ln m)^2 (\ln T)}{\Delta_{\min}}\right)$ | $\mathcal{O}(|\mathcal{X}|)$ |
| AESCB     | $\mathcal{O}\left(\frac{d (\ln m)^2 (\ln T)}{\Delta_{\min}}\right)$ | $\mathcal{O}(T \text{poly}(d))$ |

Table 1: Regret and Complexity of Algorithms.

| Algorithm | CUCB & TS | AESCB |
|-----------|----------|-------|
| m-sets    | $\mathcal{O}(d \ln m)$ | $\mathcal{O}(\frac{m^2 t (\ln m)^3}{(\ln m)^2})$ |
| Spanning Trees | $\mathcal{O}(d \ln m)$ | $\mathcal{O}(m^2 t (1 + \frac{m (\ln d)^3}{(\ln m)^2}))$ |
| Matroids  | $\mathcal{O}(d \ln m)$ | $\mathcal{O}(m^2 t (1 + \frac{m (\ln d)^3}{(\ln m)^2}))$ |
| Paths     | $\mathcal{O}(d \ln m)$ | $\mathcal{O}(\frac{m t}{(\ln m)^2} + d \ln d)$ |
| Matchings | $\mathcal{O}(m^3)$ | $\mathcal{O}(\frac{m^5 d^3 t}{(\ln m)^2})$ |

Table 2: Complexity of Algorithms (at time step $t$).

ESCB algorithm was proposed in [9] where the authors prove a $R(T) = \mathcal{O}\left(\frac{d \sqrt{m (\ln T)}}{\Delta_{\min}}\right)$ regret bound. As a follow-up [11] propose OLS-USB which, in our problem, reduces to ESCB, and thereby prove that ESCB in fact achieves $R(T) = \mathcal{O}\left(\frac{d (\ln m)^2 (\ln T)}{\Delta_{\min}}\right)$. Therefore ESCB achieves the best known regret bound for the problem at hand, however, its computational complexity is not polynomial in the dimension. Namely ESCB must, at each step, solve optimization problem ($P_2$) which is in general NP-Hard [2]. There seems to be an interesting interplay between statistical efficiency (regret) and computational complexity, which is summarized in table [1].

We also highlight that there exists algorithms for particular combinatorial semi-bandits (for m-sets, spanning trees and more generally matroids) with both polynomial complexity and order optimal regret $R(T) = \mathcal{O}\left(\frac{d \ln m (\ln T)}{\Delta_{\min}}\right)$, see [1, 20]. However those algorithms do not extend to general combinatorial bandits.

Our Main Contribution We propose AESCB (Approximate-ESCB), the first algorithm which is both computationally and statistically efficient for this problem with regret $R(T) = \mathcal{O}\left(\frac{d (\ln m)^2 (\ln T)}{\Delta_{\min}}\right)$ and computational complexity $\mathcal{O}(T \text{poly}(d))$. We release the code (in Julia [5]) of AESCB for others to experiment with it. Code is available on GitHub [https://github.com/dourouc05/CombinatorialBandits.jl](https://github.com/dourouc05/CombinatorialBandits.jl).

4 Exact and Approximate ESCB Algorithms

We propose the AESCB algorithm for combinatorial semi-bandits, which is an approximate version of the ESCB algorithm. We prove that AESCB and ESCB both enjoy the same regret bound. In the subsequent sections we will also show that AESCB, unlike ESCB, can be implemented in polynomial time with respect to the dimension.
We define the following statistics, for $i = 1, \ldots, d$:

$$n_i(t) = \sum_{t' = 1}^{t-1} x_i(t')$$

$$\hat{\theta}(t) = \left\{ \frac{\sum_{t' = 1}^{t-1} x_i(t') Z_i(t')}{\max(1, \sum_{t' = 1}^{t-1} x_i(t'))} \right\}$$

$$\sigma_i^2(t) = \left\{ \begin{array}{ll}
   \frac{f(t)}{2n_i(t)} & \text{if } n_i(t) \geq 1 \\
   +\infty & \text{otherwise}
\end{array} \right.$$  

with $f(t) = \ln t + 4m \ln \ln t$, where, at time $t$, $n_i(t)$ is the number of samples obtained for $\theta_i$, $\hat{\theta}_i(t)$ is the estimate of $\theta_i$, and $\sigma_i^2(t)$ is proportional to the variance of estimate $\hat{\theta}_i(t)$. We denote by $n(t) = (n_i(t))_{i=1, \ldots, d}$, $\hat{\theta}(t) = (\hat{\theta}_i(t))_{i=1, \ldots, d}$ and $\sigma^2(t) = (\sigma_i^2(t))_{i=1, \ldots, d}$ the corresponding vectors.

### 4.1 The ESCB Algorithm

**Definition 4.1 (ESCB)** The ESCB algorithm is the policy which at any time $t \geq 1$ selects decision:

$$x(t) \in \arg \max_{x \in \mathcal{X}} \left\{ \hat{\theta}(t)^\top x + \sqrt{\sigma^2(t)^\top x} \right\}$$

where ties are broken arbitrarily.

The ESCB algorithm is an optimistic algorithm, where the index $\hat{\theta}(t)^\top x + \sqrt{\sigma^2(t)^\top x}$ serves as an upper confidence bound of the unknown reward of decision $x$ which is $\theta^\top x$. Also, ESCB is a natural extension of the UCB1 algorithm for classical bandits. In order to implement ESCB, one needs to solve the optimization problem ($P_2$) at each time step.

The regret of ESCB was analyzed by [9] and then improved by [11]. The regret bound is presented in Theorem 4.2 as the algorithm of [11] reduces to that of [9] when rewards are uncorrelated across items.

**Theorem 4.2 (Regret of ESCB)** The regret of ESCB admits the upper bound, for all $T \geq 1$:

$$R(T) \leq C_1 \frac{d(\ln m)^2(\ln T)}{\Delta_{\min}} + C_2(\theta, \mathcal{X})$$

with $C_1$ a universal constant and $C_2(\theta, \mathcal{X})$ is a positive number which does not depend on $T$.

The regret upper bound of Theorem 4.2 is the best known regret upper bound for combinatorial semi-bandits with independent rewards across items, so that ESCB is, to the best of our knowledge, the state-of-the-art algorithm for this problem in terms of regret. However, ESCB involves solving optimization problem ($P_2$) at each step, and this problem is NP-Hard [2], so one cannot implement it efficiently as is.

### 4.2 AESCB

We now propose AESCB (Approximate-ESCB), and algorithm which approximates ESCB and enjoys the same regret bound while being implementable with polynomial complexity. The AESCB algorithm requires two sequences $(\epsilon_t, \delta_t)$, which quantify the level of approximation at each time step.

**Definition 4.3 (AESCB)** The AESCB algorithm with approximation factors $(\epsilon_t, \delta_t)_{t \geq 1}$ is the policy which at any time $t \geq 1$ selects a decision verifying:

$$\arg \max_{x \in \mathcal{X}} \left\{ \hat{\theta}(t)^\top x + \sqrt{\sigma^2(t)^\top x} \right\} \leq \delta_t + \hat{\theta}(t)^\top x(t) + \frac{1}{\epsilon_t} \sqrt{\sigma^2(t)^\top x}$$

where ties are broken arbitrarily.

When $(\epsilon_t, \delta_t) = (1, 0)$ for all $t \geq 1$, AESCB reduces to ESCB. The rationale is that ESCB requires to solve optimization problem ($P_2$) at each time step, and while ($P_2$) is NP-Hard and cannot be solved exactly in polynomial time (unless $P=NP$), it can be approximated in polynomial time in many cases of interest, so that AESCB lends itself to polynomial time implementation. We show how to do this in section 5.
4.3 Regret Analysis of AESCB

Our first main result is Theorem 4.4 which provides a regret upper bound for AESCB. We show that if one chooses approximation parameters

\[(\epsilon_t, \delta_t) = \left(\epsilon, \frac{d(\ln m)^2}{t}\right)\]

with \(\epsilon\) some fixed number then AESCB verifies the same (state-of-the-art) regret bound as ESCB up to a multiplicative constant. For \(m\)-sets, knapsack sets and source destination paths we choose \(\epsilon = 1\). For spanning trees, matroids, matchings and matroid intersection we choose \(\epsilon = \frac{1}{2}\) (see Section 5). Note that this choice of parameters does not require any knowledge about the time horizon \(T\), or the unknown problem parameters \(\theta\) or the minimal gap \(\Delta_{\min}\). We will show that, for this choice of parameters, AESCB can be implemented in polynomial time (see Section 5). A sketch of proof for Theorem 4.4 is presented in the next subsection to further highlight the algorithm rationale, and the complete proof is presented in appendix.

**Theorem 4.4 (Regret of AESCB)** The regret of A-ESCB with parameters \((\epsilon_t, \delta_t)\) satisfies for all \(T \geq 1\),

\[R(T) \leq C_A(m) + \frac{2 d m^3}{\Delta_{\min}^2} + \frac{96 d f(T)}{(\min_{t \leq T} \epsilon_t)^3 \Delta_{\min}} \left[\ln m \frac{1.61}{T}\right]^2 + 4 \sum_{t=1}^{T} \delta_t,\]

with \(f(t) = \ln t + 4m \ln \ln t\) and \(C_A(m)\) is a positive number which solely depends on \(m\).

In particular, for \((\epsilon_t, \delta_t) = (\epsilon, \frac{d(\ln m)^2}{t})\) we have

\[R(T) = \mathcal{O}\left(d \left(\frac{\ln m}{\Delta_{\min}}\right)^2 \ln T\right) \quad \text{ when } \quad T \to \infty.\]

4.4 **Theorem 4.4** Sketch of Proof

The regret analysis of AESCB involves upper bounding the reward gap of the decision chosen at time \(t\), \(\Delta_{x(t)}\) by considering three cases. Define the following events

\[A_t = \left\{ \exists x \in X : |(\theta - \hat{\theta}(t))^\top x| \geq \sqrt{x(t)^\top \sigma^2(t)x} \right\}\]
\[B_t = \{\Delta_{x(t)} \leq 2\delta_t\}\]

If \(A_t\) occurs, since \(\theta \in [0,1]^d\) and \(1^\top x^* \leq m\):

\[\Delta_{x(t)} \leq \theta^\top x^* \leq m.\]

If \(B_t\) occurs, by definition:

\[\Delta_{x(t)} \leq 2\delta_t\]

If \(C_t = \overline{A_t} \cup \overline{B_t}\) occurs, the index of the optimal decision is greater than the optimal value so that:

\[\theta^\top x^* \leq \hat{\theta}(t)^\top x^* \leq \delta_t + \hat{\theta}(t)^\top x(t) + \frac{1}{\epsilon_t} \sqrt{x(t)^\top \sigma^2(t)} \leq \delta_t + \theta(t)^\top x(t) + \frac{2}{\epsilon_t} \sqrt{x(t)^\top \sigma^2(t)} \leq \frac{1}{2} \Delta_{x(t)} + \theta(t)^\top x(t) + \frac{2}{\epsilon_t} \sqrt{x(t)^\top \sigma^2(t)} \]

where we used the definition of AESCB, and the fact that \(A_t\) and \(B_t\) do not occur. So if \(C_t\) occurs:

\[\Delta_{x(t)} \leq \frac{4}{\epsilon_t} \sqrt{x(t)^\top \sigma^2(t)}\]

Putting it together, we get:

\[\Delta_{x(t)} \leq \Delta_{x(t)} \left(1(A_t) + 1(B_t) + 1(C_t)\right) \leq m1(A_t) + 2\delta_t + \Delta_{x(t)} 1 \left\{ \Delta_{x(t)} \leq \frac{4}{\epsilon_t} \sqrt{x(t)^\top \sigma^2(t)} \right\}.\]
Taking expectations and summing over $t$:

$$R(T) = \sum_{t=1}^{T} E(\Delta_{x(t)}) \leq \sum_{t=1}^{T} mP(A_t) + \sum_{t=1}^{T} 2\delta_t + \sum_{t=1}^{T} E\left(\Delta_{x(t)}1\left\{\Delta_{x(t)} \leq \frac{4}{\epsilon} \sqrt{x(t)^\top \sigma^2(t)}\right\}\right)$$

The first term is bounded by a constant, since, using a concentration inequality we may show that $A_t$ occurs with small probability. The last term can be bounded using similar counting arguments as in the analysis of ESCB. The complete proof is presented in appendix.

5 AESCB in Polynomial Time

5.1 General Technique

We now show a technique to implement AESCB which ensures polynomial complexity. While our technique is generic, the precise value of the computational complexity depends on the combinatorial set $\mathcal{X}$, and will be explained in details in sections 5.3 – 5.7. The technique involves three steps: rounding and scaling to ensure that the weights are integer, then solving the budgeted linear maximization ($P_3$) several times, and finally maximizing over the budget to obtain the result. Given time $t$, statistics $\hat{\theta}(t)$ and $\sigma^2(t)$ and approximation factors $(\epsilon_t, \delta_t)$, the method works as follows.

Step 1: Rounding and scaling Define $a(t)$ and $b(t)$:

$$a_i(t) = \lfloor \xi(t)\hat{\theta}_i(n) \rfloor, \ i \in \{1, \ldots, d\}$$

$$b_i(t) = \xi(t)^2\sigma_i^2(t), \ i \in \{1, \ldots, d\}$$

$$\xi(t) = \lceil m/\delta_t \rceil.$$

Step 2: Budgeted Linear Maximization For all $s \in \{0, \ldots, m\xi(t)\}$, compute $\bar{x}^s(t)$, an $\epsilon_t$-optimal solution to budgeted linear maximization problem ($P_3$):

$$\bar{x}^s(t) \geq \epsilon_t\left(\max_{x \in \mathcal{X}: a(t)^\top x \geq s} \{b(t)^\top x\}\right) \text{ and } a(t)^\top \bar{x}^s(t) \geq s.$$

Step 3: Optimizing over the Budget Return decision $x(t)$:

$$x(t) = \bar{x}^{s^*(t)}(t) \text{ with }$$

$$s^*(t) \in \arg \max_{s=0, \ldots, m\xi(t)} \left\{s + \frac{1}{\epsilon_t} \sqrt{b(t)^\top \bar{x}^s(t)}\right\}.$$

Theorem 5.1 (see proof in appendix) states that this technique returns the decision chosen by AESCB, in a time proportional to solving the optimization problem ($P_3$) at most $m\xi(t)$ times (note that $\xi(t)$ is chosen polynomial in $d$), and that the input parameters $a(t)$ and $b(t)$ of ($P_3$) are positive vectors and where the entries of $a(t)$ are in $\{1, \ldots, \xi(t)\}$. In the next subsections, we show that, for many combinatorial sets of interest, one can do this in time polynomial in the dimension, so AESCB is indeed implementable in polynomial time.

**Theorem 5.1** The above technique returns a decision $x(t) \in \mathcal{X}$ verifying the AESCB definition:

$$\arg\max_{x \in \mathcal{X}} \{\hat{\theta}(t)^\top x + \sqrt{\sigma^2(t)^\top x}\} \leq \delta_t + \hat{\theta}(t)^\top x(t) + \frac{1}{\epsilon_t} \sqrt{\sigma^2(t)^\top x}$$

And it does so by solving optimization problem ($P_3$) at most $m\xi(t)$ times with input parameters $a(t)$ and $b(t)$, and $a(t) \in \{1, \ldots, \xi(t)\}$.

5.2 Summary

Based on Theorem 5.1 we now highlight how to find $\epsilon$-optimal solutions to the optimization ($P_3$). The complexity at each time step of AESCB with recommended parameters $(\epsilon_t, \delta_t) = \left(\epsilon, \frac{d(\ln m)^2}{t}\right)$, so that
\[ \xi(t) = \left\lceil \frac{mt}{\ln(m)} \right\rceil \] is summarized in table 2. We now consider time \( t \) fixed and we drop the time index to simplify notation. We consider input parameters \( a \) and \( b \) for \((P_3)\) with \( a \in \{1,...,\xi\} \). We do so for each type of combinatorial set in sections 5.3 - 5.7. We provide a description of the algorithm, with pseudo code given in appendix.

### 5.3 M-sets

**Claim 5.2** Optimization problem \((P_3)\) with \( X \) the set of m-sets can be solved exactly (i.e. \( \epsilon = 0 \)) in time \( O(m^2d\xi) \) using algorithm below.

In fact, since m-sets are a particular case of a knapsack set with matrix \( A = (1,...,1)^T, k = 1 \) and \( c = (m) \), we can simply apply the algorithm for knapsack sets explained in section 5.4 below.

### 5.4 Knapsack sets

**Claim 5.3** Optimization problem \((P_3)\) with \( X \) a knapsack set can be solved exactly (i.e. \( \epsilon = 0 \)) in time \( O((\prod_{k=1}^c c\ell dm\xi) \) using algorithm below.

For \( i \in \{0,...,d\} \), define optimization problem

\[
\begin{align*}
\text{maximize} & \quad b^\top x \\
\text{subject to} & \quad Ax \leq c, \quad a^\top x \geq s, \\
& \quad \sum_{j=1}^i x_j = 0, \quad x \in \{0,1\}^d
\end{align*}
\]

and denote by \( V_4(s,c,i) \) its optimal value where we recall that \( c \in \mathbb{N}^k \) is a vector with integer entries. Since \((P_4(s,c,0))\) reduces to \((P_3)\), it is sufficient to solve \((P_4(s,c,i))\) for \( i \in \{0,...,d\} \). We do so using dynamic programming. Let \( x^* \) an optimal solution to \((P_4(s,c,i))\). If \( x^*_{i+1} = 1 \) then

\[
A(x^* - e_i) = Ax^* - Ae_i \leq c - Ae_i
\]

so \( x^* - e_i \) is an optimal solution to \((P_4(\max s - a_i, 0, c - Ae_i, i+1))\). If \( x^*_i = 0 \) then \( x^* \) is an optimal solution to \((P_4(s,c,i+1))\). Therefore

\[
V_4(s,c,i) = \max\{V_4(s,c,i+1), \\
\quad a_i + V_4(\max(s-a_i,0), c - Ae_i, i+1)\}
\]

By recursion over \( i, c \) and \( s \) we can to compute the value \( V_4(s,c',i) \) for \( s \in \{0,...,m\xi\} \), \( c' \in \mathbb{N}^k \) with \( c' \leq c \) and \( i \in \{0,...,d\} \) in time \( O((\prod_{k=1}^c c\ell dm\xi) \). The solution to \((P_3)\), denoted by \( x^* \), is then:

\[
x^*_i = \begin{cases} 
0 & \text{if } V_4(s,c,i) = V_4(s,c,i+1) \\
1 & \text{otherwise.}
\end{cases}
\]

So we can solve \((P_3)\) for all \( s \leq m\xi \) in time \( O((\prod_{k=1}^c c\ell dm\xi) \).

### 5.5 Source destination paths

**Claim 5.4** Optimization problem \((P_3)\) with \( X \) the set of paths between source \( u \) and destination \( v \) in \( G = (V,E) \) a directed acyclic graph can be solved exactly (i.e. \( \epsilon = 0 \)) in time \( O(m\xi|E| + |V| \ln |V|) \) using algorithm below.

Consider \( v \) fixed throughout, and denote by \((P_3(u,s))\) this optimization problem and \( V_3(u,s) \) its optimal value. If \( s \leq 0 \), \((P_3(u,s))\) is simply the problem of finding the path from \( u \) to \( v \) maximizing \( b^\top x \), since \( a \) has positive entries so that \( a^\top x \geq 0 \geq s \), for all \( x \in X \). Hence we can compute \((P_3(u,s))\) for all \( u \) by Dijkstra’s algorithm in time \( O(|E| + |V| \ln |V|) \), when Dijkstra’s algorithm is implemented with Fibonacci heaps [12]. If
s > 0, let \( x^* \) an optimal solution to \((P_3(u, s))\). Since \( x^* \) is a path from \( u \) to \( v \) there exists a unique \( w \in V \) such that \( x^*|_{(u,w)} = 1 \), and \( x^* - e(u,w) \) is a path from \( w \) to \( v \). In turn, we must have that \( x^* - e(u,w) \) is an optimal solution to \((P_3(w, \max(s-b(u,v),0)))\). Therefore, we have the dynamic programming equation:

\[
V_3(u, s) = \max_{w: (u,w) \in E} \{ b(u,w) + V_3(w, \max(s-a(u,w),0)) \}
\]

Recall that \( a \in \{1, ..., \xi \} \) so if \( V_3(u, s') \) is known for all \( u \in V \) and all \( s' \in \{0, ..., s-1\} \), then applying the above relationship enables us to compute \( V_3(u, s) \) for all \( u \in V \). By recursion we can compute \( V_3(u, s) \) for all \( s \in \{0, ..., m\xi \} \) in time \( O(m\xi |E| + |V| \ln |V|) \). The solution to \((P_3)\), denoted by \( x^* \), can also be computed by recursion. By the same dynamic programming principle, denote by \( x^*(u, s) \) the solution of \((P_3(u, s))\), we have:

\[
x^*(u, s) = e(u,v) + x^*(w(u,s), s - a(u,w,u,x)) \quad \text{with} \quad w(u,s) = \arg \max_{w: (u,w) \in E} \{ b(u,w) + V_3(w, \max(s-a(u,w),0)) \}
\]

By recursion, we can compute the solution to \((P_3)\) for all \( s \in \{0, ..., m\xi \} \) in time \( O(m\xi |E| + |V| \ln |V|) \).

### 5.6 Spanning Trees and Matroids

**Claim 5.5** Optimization problem \((P_3)\) with \( X \) the set of spanning trees of graph \( G = (V, E) \) can be solved with approximation ratio \( \epsilon = \frac{1}{2} \) in time \( O(|E|(|V|)^2 + |V|(|E|)^3) \) using the algorithm below. The same holds for any matroid.

The algorithm is made of four steps, and is similar to that of [18] (see this reference for further details).

**Step 1:** Lagrangian relaxation Define the Lagrangian relaxation of the problem:

\[
\max_x \{ b^T x + \lambda (a^T x - s) \} \quad \text{subject to} \quad x \in X
\]

Denote by \( M(z) \) its value and \( L(z) \) the set of optimal solutions. Define \( \lambda^* = \arg \min_{\lambda \geq 0} M(z) \). Computing \( M(z) \) can be done in time using the greedy algorithm, since it is maximizing a linear function over a matroid [17]. Furthermore, \( \lambda^* \) can be found using Meggido’s search technique [16] as it involves minimizing a piece-wise linear function.

**Step 2:** Candidate solutions For an arbitrarily small \( \epsilon > 0 \), if \( |z - z^*| < \epsilon \) we must have that \( L(z) \subset L(z^*) \). So, by solving the Lagrangian relaxation of the problem for \( z = z^* + \epsilon \) and \( z = z^* - \epsilon \), we obtain two solutions \( x^+ \) and \( x^- \) in \( L(z^*) \) with \( a^T x^+ \geq s \) and \( a^T x^- \leq s \).

**Step 3:** Refining the solutions We now use an iterative procedure in order to find a good solution using candidates \( x^+ \) and \( x^- \). Consider \( e, e' \in E \) such that \( x^+_e = x^-_{e'} = 1 \) and \( x^+_e = x^-_{e'} = 0 \) and define \( x = x^+ - e + e' \). If \( a^T x \geq s \) then replace \( x^+ \) by \( x \) and otherwise replace \( x^- \) by \( x \) and repeat this procedure until \( x^+ \) and \( x^- \) differ by exactly one element. Then return \( x^+ \). At each step of this procedure:

\[
(b + za)^T x = (b + za)^T x^+ = (b + za)^T x^-
\]

therefore \( x \in L(z^*) \). Denote by \( x^* \) the solution of \((P_3)\).

Since \( x^+ \) and \( x^- \) are in \( L(z^*) \):

\[
b^T x^- + z(a^T x^- - s) \geq b^T x^+ + z(a^T x^* - s)
\]

Since \( a^T x^- \leq s \leq a^T x^* \) we deduce that: \( b^T x^- \geq b^T x^* \). Since \( x^+ \) and \( x^- \) differ by at most one element:

\[
b^T x^+ \geq b^T x^- - \max_{e \in E} b_e \geq b^T x^* - \max_{e \in E} b_e
\]

So, after steps 1-3, we get \( x \) such that \( a^T x \geq s \) and \( b^T x \geq b^T x^* \).

**Step 4:** An \( \frac{1}{2} \) optimal solution Lastly, we search over the two edges with largest weight to obtain a constant multiplicative approximation factor. For all sets of two edges \( E'' \subset E \), \( |E''| = 2 \), define \( G' = (V, E') \) where

\[
E' = \{ e \in E \setminus E' \mid b_e \leq \min_{e' \in E''} b_{e'} \}
\]

and apply steps 1 to 3 to solve the problem where \( G \) and \( s \) are replaced by \( G' \) and \( s' = s - \sum_{e' \in E''} a_{e'v} \), where \( x'(E'') \) is the solution found by steps 1-3. Finally, return \( x(E'') = x'(E'') + \sum_{e' \in E''} e' \), for the value of \( E'' \) maximizing \( a^T x(E'') \). This yields a \( 1/2 \) optimal solution in time \( O(|E|^2(|V|)^2 + |V|(|E|)^3) \) by the same arguments as that used in [18] [4].
5.7 Matchings and Matroid Intersection

Claim 5.6 Optimization problem \((P_3)\) with \(X\) the set of matchings of a bipartite graph \(G = (V, E)\) can be solved with approximation ratio \(\epsilon = 1/2\) in time \(O(\left|V\right|^3 \left|E\right|^4)\) using the algorithm below. The same holds for any intersection of two matroids.

The algorithm is made of four steps and is very similar of that of [4], which itself is inspired by the algorithm for matroids of [18].

Step 1: Lagrangian relaxation Define the Lagrangian relaxation of the problem:

\[
\max_x \{b^\top x + \lambda(a^\top x - s)\} \text{ subject to } x \in X
\]

denote by \(M(\lambda)\) its value and \(L(\lambda)\) the set of optimal solutions. Define \(\lambda^* = \arg\min_{\lambda \geq 0} M(\lambda)\). Computing \(M(\lambda)\) can be done in time using the Hungarian algorithm, since it is maximizing a linear function over the set of matchings of a bipartite graph. Furthermore, \(\lambda^*\) can be found using Meggido’s parametric search technique [16] as it involves minimizing a piece-wise linear function.

Step 2: Candidate solutions For an arbitrarily small \(\epsilon > 0\), if \(|\lambda - \lambda^*| < \epsilon\) we must have that \(L(\lambda) \subseteq L(\lambda^*)\).

So, by solving the Lagrangian relaxation of the problem for \(\lambda = z^* + \epsilon\) and \(\lambda = \lambda^* - \epsilon\), we obtain two solutions \(x^+\) and \(x^-\) in \(L(\lambda^*)\) with \(a^\top x^+ \geq s\) and \(a^\top x^- \leq s\).

Step 3: Defining the solutions We now use an iterative procedure in order to find a good solution using candidates \(x^+\) and \(x^-\). Define their symmetric difference \(x' = x^+ \oplus x^-\). \(x'\) is made of a disjoint union of paths and cycles. Define \(x''\) one of such paths or cycles, and define \(x = x^- \oplus x''\). If \(a^\top x \geq s\) then replace \(x^+\) by \(x\) and otherwise replace \(x^+\) by \(x^-\). Repeat this procedure until \(x^+\) and \(x^-\) differ by at most two elements (note that the symmetric difference \(x^+ \oplus x^-\) decreases at each step). Then return \(x^+\). At each step of this procedure:

\[(b + \lambda^* a)^\top x = (b + \lambda^* a)^\top x^+ = (b + \lambda^* a)^\top x^-
\]

denotes \(x^+\) the solution of \((P_3)\).

Since \(x^+\) and \(x^-\) are in \(L(\lambda^*)\):

\[b^\top x^- + \lambda^*(a^\top x^- - s) \geq b^\top x^+ + \lambda^*(a^\top x^+ - s)
\]

Since \(a^\top x^- \leq s \leq a^\top x^+\) we deduce that: \(b^\top x^- \geq b^\top x^+\). Since \(x^+\) and \(x^-\) differ by at most two elements:

\[b^\top x^+ \geq b^\top x^- - 2 \max_{e \in E} b_e \geq b^\top x^+ - 2 \max_{e \in E} b_e
\]

So, after steps 1-3, we get \(x\) such that \(a^\top x \geq s\) and \(b^\top x \geq b^\top x^+ - 2 \max_{e \in E} b_e\).

Step 4: An \(1/2\) optimal solution Lastly, we search over the four edges with largest weight to obtain a constant multiplicative approximation factor. For all sets of four edges \(E'' \subset E\), \(|E''| = 4\), define \(G' = (V, E')\) where

\[E' = \{e \in E \setminus E'' : b_e \leq \min_{e' \in E''} \{b_{e'}\}\}
\]

and apply steps 1 to 3 to solve the problem where \(G\) and \(s\) are replaced by \(G'\) and \(s' = s - \sum_{e'' \in E''} a_{e''}\), where \(x'(E'')\) is the solution found by steps 1-3. Finally, return \(x(E'') = x'(E'') + \sum_{e'' \in E''} c_{e''}\), for the value of \(E''\) maximizing \(a^\top x(E'')\). This yields a 1/2 optimal solution in time \(O(\left|V\right|^3 \left|E\right|^4)\) by the same arguments as that used in [4].

6 Numerical Experiments

We evaluate the performance of TS, CUCB, ESCB and AESCB through numerical experiments. ESCB is implemented by casting optimization problem as a MISOC (Mixed Integer Second Order Cone Programming) and using a ISOC solver, see appendix for more details. Our code used to implement all algorithms is available along with this article. For \(m\)-sets, we choose \(m = \lfloor d/3 \rfloor\) and \(\theta_i = 0.8\) for \(i \leq d/2\) and \(\theta_i = 0.2\) for \(i > d/2\). For source destination paths we take \(G = (V, E)\) a complete directed acyclic graph, so that \((i, j) \in E\) if and only if \(i < j\). The source is 1 the destination is \(|V|\) and \(\theta_{(i, j)} = 1/(i + 1)\) for \((i, j) \neq (1, |V|)\) and
Figure 1: Expected regret of algorithms

\[ \theta_{(1,|V|)} = 1 - \frac{1}{4d+1} \]. We have \( d = |V|(|V| - 1)/2, m = |V| - 1 \) and the optimal path is \( 1 \rightarrow |V| \). For spanning trees we consider \( G = (V,E) \) a complete graph and \( \theta_{(i,j)} = \frac{1}{4d+1} \) for all \( (i,j) \) with \( i \neq 1 \) and \( \theta_{(1,j)} = 1 - \frac{1}{4d+1} \). We have \( d = |V|(|V| - 1)/2, m = |V| - 1 \) and that the optimal decision is a star network. For matchings we consider a complete bipartite graph \( G = (V,E) \) with \( V = V_1 \cup V_2 \) and \( |V_1| = |V_2| \), \( \theta_{(i,j)} = \frac{1}{4d+1} \) for all \( (i,j), i \neq j \) and \( \theta_{(i,i)} = 1 - \frac{1}{4d+1} \). The optimal decision is \( x^*_{(i,j)} = \mathbf{1}\{i = j\} \) and \( d = |V_1||V_2|, m = \min(|V_1|, |V_2|) \).

Regret: On figure 1, we present the expected regret of algorithms (with 95% confidence intervals) averaged over 10 sample paths. We observe that TS has the lowest regret, that AESEB and ESCB both perform equally as well (so that the approximation comes at virtually no cost in terms of regret) and that the performance of CUCB is comparable to that of ESCB.

Computation Time: On table 3, we present average the computation time required to select an arm at time \( t = 1000 \) for ESCB and AESCB (with 95% confidence intervals) averaged over 10 sample paths, as a function of the problem dimension \( d \). We observe that the computation time for AESCB seems indeed to grow ly in \( d \), and that the computation times for AESCB and ESCB seem of the same magnitude.
We propose AESCB, the first algorithm which enjoys both the state-of-the-art regret bound of ESCB and polynomial computational complexity. We believe our work opens two important research questions: (i) Since TS has generally polynomial complexity and seems to work better than AESCB numerically, can one prove that it also has $R(T) = \mathcal{O}\left(\frac{d\ln m}{\Delta_{\text{min}}}\ln T\right)$ regret in general? (ii) What is the optimal trade-off between regret and computational complexity in combinatorial bandits?

### 7 Conclusion

We propose AESCB, the first algorithm which enjoys both the state-of-the-art regret bound of ESCB and polynomial computational complexity. We believe our work opens two important research questions: (i) Since TS has generally polynomial complexity and seems to work better than AESCB numerically, can one prove that it also has $R(T) = \mathcal{O}\left(\frac{d\ln m}{\Delta_{\text{min}}}\ln T\right)$ regret in general? (ii) What is the optimal trade-off between regret and computational complexity in combinatorial bandits?

### 8 Regret of Thompson Sampling

The authors of [21] study a slightly more general problem than ours, and propose a regret bound for Thompson Sampling. Let us rephrase their bound with our notations. Their problem is a combinatorial semi bandit problem with a non linear reward function. Namely they consider decisions $x \in \mathcal{X} \subset \{0,1\}^d$ in a combinatorial set, and the expected reward of decision $x \in \mathcal{X}$ is given by a possibly non-linear function $r(\theta, x)$, where $\theta$ is a vector unknown to the learner. The expected reward function $r$ must satisfy the Lipschitz condition:

$$|r(\theta, x) - r(\theta', x)| \leq B|\theta^\top x - \theta'^\top x|, \forall \theta, \theta'$$

with $B$ the Lipschitz constant. In our setting the reward function is $r(\theta, x) = \theta^\top x$ so that $B = 1$.

Their main result is [21][Theorem 1], which is the regret upper bound for Thompson Sampling:

$$R(T) \leq C_5(\theta, \mathcal{X}, \epsilon) + 8B^2(\ln T) \sum_{i=1}^{d} \max_{x: x_i=1, \Delta_x > 0} \left( \frac{1^\top x}{\Delta_x - 2B(1^\top x + 2)\epsilon} \right) \forall \epsilon > 0$$

where $C_5(\theta, \mathcal{X}, \epsilon)$ is a positive number which does not depend on $T$. In our setting the reward function is $r(\theta, x) = \theta^\top x$ so that $B = 1$ and in the worse case, we will have for all $i$:

$$\max_{x: x_i=1, \Delta_x > 0} \frac{1^\top x}{\Delta_x} = \frac{m}{\Delta_{\text{min}}},$$

Hence the upper bound for the regret of Thompson Sampling provided by [21][Theorem 1] scales as:

$$R(T) = \mathcal{O}\left(\frac{d\ln m}{\Delta_{\text{min}}}\ln T\right), T \to \infty$$

and this bound does not match the (smaller) regret upper bound of algorithms such as ESCB and A-ESCB which is

$$R(T) = \mathcal{O}\left(\frac{d\ln m}{\Delta_{\text{min}}}2\ln T\right), T \to \infty$$
9 Casting Optimization problem (P2) as an MISOCP

As mentionned in the article, optimization problem (P2), whose definition is:

\[ \max_x \{ a^\top x + \sqrt{b^\top x} \} \text{ subject to } x \in X \]  

(P2)

can be cast as a Mixed Integer Second Order Cone Program (MISOCP), which enables one to solve it using an MISCOP solver. Indeed, the objective function feature a geometric mean which a special case of hyperbolic constraint \cite{15}. Problem (P2) can be rewritten as:

\[ \max_{(x,t)} \{ a^\top x + t \} \text{ subject to } x \in X \text{ and } t^2 \leq b^\top x \]

Applying the transformation proposed in \cite{15}[Section 2.3], the hyperbolic constraint can be written as a SOCP (Second Order Cone Program):

\[ \max_{(x,t)} \{ a^\top x + t \} \text{ subject to } x \in X \text{ and } \left\| \begin{bmatrix} 2t \\ b^\top x - 1 \end{bmatrix} \right\| \leq b^\top x + 1 \]

Even though the constraints defining \( X \) ensure that optimising a linear objective over \( X \) yields an integer solution, this is no more the case with the new formulation. Hence, integrality constraints must be added for the relevant variables.

\[ \max_{(x,t)} \{ a^\top x + t \} \text{ subject to } x \in X \text{ and } \left\| \begin{bmatrix} 2t \\ b^\top x - 1 \end{bmatrix} \right\| \leq b^\top x + 1 \quad \text{and} \quad x \in \{0, 1\}^d \]

This formulation is a MISOCP and can be readily solved to optimality by existing software such as CPLEX, Gurobi, Pajarito \cite{3}, Mosek, or Xpress.

10 Proof of Theorem 4.4

**Theorem 4.4.** The regret of A-ESCB with parameters \((\epsilon_t, \delta_t)\) satisfies for all \( T \geq 1, \)

\[ R(T) \leq C_4(m) + \frac{2d m^3}{\Delta_{\min}^2} + \frac{96d f(T)}{(\min_{t \leq T} \epsilon_t)^3 \Delta_{\min}} \left[ \frac{\ln m}{1.61} \right]^2 + 4 \sum_{t=1}^{T} \delta_t. \]

with \( f(t) = \ln t + 4 m \ln \ln t \) and \( C_4(m) \) is a positive number which solely depends on \( m. \)

In particular, for \((\epsilon_t, \delta_t) = (\epsilon, \frac{d (\ln m)^2}{t})\) we have

\[ R(T) = \mathcal{O}\left( \frac{d (\ln m)^2}{\Delta_{\min}} \frac{1}{\ln T} \right) \quad \text{when} \quad T \to \infty. \]

**Proof:** We decompose the regret based on two events:

- \( G_t \): the estimate \( \hat{\theta}(t) \) deviates abnormally from \( \theta \) so that:

\[ G_t = \left\{ \theta^\top x^* \geq \hat{\theta}(t)^\top x^* + \sqrt{\sigma^2(t)^\top x^*} \right\}. \]

- \( H_t \): the reward of the decision chosen at time \( t \) is poorly estimated, namely:

\[ H_{i,t} = \left\{ x_i(t) = 1, \quad |\hat{\theta}_i(t) - \theta_i| \geq \frac{\Delta_{\min}}{2 m} \right\} \text{ and } H_t = \bigcup_{i=1}^{d} H_{i,t}. \]

Of course, most of the time \( \overline{G_t} \) and \( \overline{H_t} \) occur, since both \( G_t \) and \( H_t \) have small probability. Therefore \( G_t \) and \( H_t \) cause only a constant regret as we shall see. For all \( x \in X \) and \( t \geq 1 \) we define the exploration bonus of decision \( x \) at time \( t. \)

\[ E_t(x) = \sqrt{\sigma^2(t)^\top x}. \]
**Generic regret bound.** Recall that the regret is by definition

\[
R(T) = \mathbb{E}\left\{\sum_{t=1}^{T} \Delta_{x(t)}\right\} = \mathbb{E}\left\{\sum_{t=1}^{T} \Delta_{x(t)}1\{x(t) \neq x^*\}\right\},
\]

Decomposing according to the occurrence of \(G_t\) and \(H_t\) we get:

\[
R(T) \leq \mathbb{E}\left\{\sum_{t=1}^{T} 1\{G_t\} \Delta_{x(t)}\right\} + \mathbb{E}\left\{\sum_{t=1}^{T} 1\{H_t\} \Delta_{x(t)}\right\} + \mathbb{E}\left\{\sum_{t=1}^{T} 1\{\overline{G_t}, \overline{H_t}, x(t) \neq x^*\} \Delta_{x(t)}\right\}.
\]

Define \(\tau_T = \min_{t \leq T} \epsilon_t\). The last term can be rewritten in terms of the following event:

\[
F_t = \left\{\Delta_{x(t)} \leq \frac{4}{\epsilon_T} E_t(x(t))\right\} \cup \left\{\Delta_{x(t)} \leq 4\delta_t\right\}.
\]

Let us prove that \((\overline{G_t} \cap \overline{H_t} \cap \{x(t) \neq x^*\}) \subset F_t\). Assume that \((\overline{G_t} \cap \overline{H_t} \cap \{x(t) \neq x^*\})\) occurs. Then

\[
\theta^\top x^* \leq \hat{\theta}(t)^\top x^* + E_t(x^*) \leq \max_{x \in \mathcal{X}} \{\hat{\theta}(t)^\top x + E_t(x)\} \leq \delta_t + \hat{\theta}(t)^\top x(t) + \frac{1}{\epsilon_t} E_t(x(t)) \leq \delta_t + \theta(t)^\top x(t) + \frac{\Delta_{x(t)}}{2} + \frac{1}{\epsilon_t} E_t(x(t))
\]

where we successively used the fact that \(\overline{G_t}\) occurs, the definition of AESCB, and the fact that \(\hat{\theta}(t)^\top x(t) \leq \theta^\top x(t) + \frac{\Delta_{x(t)}}{2}\) since \(\tau_T\) occurs. Therefore

\[
\frac{\Delta_{x(t)}}{2} \leq \delta_t + \frac{1}{\epsilon_t} E_t(x(t)) \leq \delta_t + \frac{1}{\epsilon_t} E_t(x(t)) \leq 2 \max\left\{\frac{1}{\epsilon_T} E_t(x(t)), \delta_t\right\}
\]

so that \(F_t\) indeed occurs.

This yields:

\[
\mathbb{E}\left\{\sum_{t=1}^{T} 1\{\overline{G_t}, \overline{H_t}, x(t) \neq x^*\} \Delta_{x(t)}\right\} \leq \mathbb{E}\left\{\sum_{t=1}^{T} 1\{\Delta_{x(t)} \leq \frac{4}{\epsilon_T} E_t(x(t))\} \Delta_{x(t)}\right\} + \mathbb{E}\left\{\sum_{t=1}^{T} 1\{\Delta_{x(t)} \leq 4\delta_t\} \Delta_{x(t)}\right\}
\]

\[
\leq \frac{4}{\epsilon_T} \mathbb{E}\left\{\sum_{t=1}^{T} 1\{\Delta_{x(t)} \leq \frac{4}{\epsilon_T} E_t(x(t))\} E_t(x(t))\right\} + 4 \sum_{t=1}^{T} \delta_t.
\]

So the regret is upper bounded by the sum of four terms:

\[
R(T) \leq \mathbb{E}\left\{\sum_{t=1}^{T} 1\{G_t\} \Delta_{x(t)}\right\} + \mathbb{E}\left\{\sum_{t=1}^{T} 1\{H_t\} \Delta_{x(t)}\right\} + \frac{4}{\epsilon_T} \mathbb{E}\left\{\sum_{t=1}^{T} 1\{\Delta_{x(t)} \leq \frac{4}{\epsilon_T} E_t(x(t))\} E_t(x(t))\right\} + 4 \sum_{t=1}^{T} \delta_t.
\]

**First term: poor reward estimation.** Recall that for any \(x\) we have \(\Delta_x \leq \theta^\top x^* \leq m\) since \(\theta \in [0,1]^d\) and \(\max_{x \in \mathcal{X}} 1^\top x = m\). Therefore, by applying [9][Theorem 3]:

\[
\mathbb{E}\left\{\sum_{t=1}^{T} 1\{G_t\} \Delta_{x(t)}\right\} \leq m \mathbb{E}\left\{\sum_{t=1}^{T} 1\{G_t\}\right\} = m \sum_{t=1}^{\infty} \mathbb{P}(G_t) \leq C_4(m).
\]

where \(C_4(m)\) is a positive number which only depends on \(m\), as stated by [9][Theorem 3].

**Second term: poor choice of item.** We turn to the second term, using a union bound

\[
\mathbb{P}(H_t) = \mathbb{P}\left(\bigcup_{i=1}^{d} H_{t,i}\right) \leq \sum_{i=1}^{d} \mathbb{P}(H_{t,i}).
\]
Using once again the fact that $\Delta x(t) \leq m$, the regret due to $H_t = \bigcup_{i=1}^{d} H_{t,i}$ is bounded by using a union bound

$$
\mathbb{E} \left\{ \sum_{t=1}^{T} \mathbb{I}\{H_t\} \Delta x(t) \right\} \leq m \mathbb{E} \left\{ \sum_{t=1}^{T} \mathbb{I}\{H_t\} \right\} = m \sum_{t=1}^{T} \mathbb{P}(H_t) \leq m \sum_{t=1}^{T} \sum_{i=1}^{d} \mathbb{P}(H_{t,i}).
$$

By definition of $H_{t,i}$ we get

$$
\mathbb{E} \left\{ \sum_{t=1}^{T} \mathbb{I}\{H_t\} \Delta x(t) \right\} \leq m \sum_{t=1}^{T} \sum_{i=1}^{d} \mathbb{P} \left( x_i(t) = 1, \ |\hat{\theta}_i(t) - \theta_i| \geq \frac{\Delta x(t)}{2m} \right)
$$

Using Hoeffding’s inequality this probability can be bounded by:

$$
\mathbb{E} \left\{ \sum_{t=1}^{T} \mathbb{I}\{H_t, x(t) \neq x^*\} \Delta x(t) \right\} \leq m \sum_{t=1}^{T} \sum_{i=1}^{d} \exp \left[ -t \left( \frac{\Delta_{\min}}{m} \right)^2 \right] \leq \frac{md}{1 - \exp \left( -\left( \frac{\Delta_{\min}}{m} \right)^2 \right)} \leq \frac{m^3 d (1 + \frac{\Delta_{\min}^2}{m^2})}{\Delta_{\min}^2} \leq \frac{2m^3 d}{\Delta_{\min}^2}
$$

where we recognize a geometric series and use the elementary inequality $e^z \geq 1 + z$ for all $z$ which gives $e^{-z} \leq (1 + z)^{-1}$ and $(1 - e^{-z})^{-1} \leq (1 + z)^{-1}$. We also used the fact that $\Delta_{\min} \leq m$.

**Third term: dominant term.** We now consider the event $\Delta x(t) \leq \frac{4}{\epsilon^2} E_t(x(t))$. Squaring we get:

$$
\Delta_{x(t)}^2 \leq \frac{16}{\epsilon^2} E_t(x(t))^2 = \frac{16}{\epsilon^2} \sigma^2(t)^T x(t) = \frac{8f(t)}{\epsilon^2} \sum_{i=1}^{d} \frac{x_i(t)}{n_i(t)}.
$$

where we used the definition of $E_t$ and that of $\sigma^2(t)$. If this event happens, it means that there exists a subset of indices $i = 1, ..., d$ such that the number of samples $n_i(t)$ is small. We further decompose this event as follows.

Consider $(\alpha_j)_{j \in \mathbb{N}}$ and $(\beta_j)_{j \in \mathbb{N}}$ two positive, non-increasing sequences verifying the following properties:

- $\lim_{j \to \infty} \alpha_j = \lim_{j \to \infty} \beta_j = 0$
- $\lim_{j \to +\infty} \frac{\beta_j}{\sqrt{\alpha_j}} = 0$
- $\beta_0 = 0$

We can define $j_0$ as the first integer $j$ such that

$$
\beta_j \leq \frac{1}{m},
$$

and define $l$ as the sum

$$
l = \frac{\beta_{j_0}}{\alpha_{j_0}} + \sum_{j=1}^{j_0} \frac{\beta_{j-1} - \beta_j}{\alpha_{j-1}}.
$$

Sequences $(\alpha_j)_{j \in \mathbb{N}}$ and $(\beta_j)_{j \in \mathbb{N}}$ are fixed, and their exact value will be specified later.

For all $j \in \mathbb{N}$, define the sets

$$
S_j^l = \left\{ i \in \{1, ..., d\} : x_i(t) = 1, n_i(t) \leq \alpha_j \frac{2f(t)}{\Delta_{x(t)}^2} g(m) \right\} \text{ if } j \geq 1 \text{ and } S_0^l = \{ i \in \{1, ..., d\} : x_i(t) = 1 \} \text{ otherwise}
$$

where $g(m)$ is defined as $g(m) = \frac{4md}{\epsilon^2}$ with $l$ a constant to be defined later.

Since $j \mapsto \alpha_j$ is decreasing and $\lim_{j \to \infty} \alpha_j = 0$ this implies that $S_j^l$ is a decreasing sequence for set inclusion and that there is an index $j_0$ such that $S_{j_0}^l = \emptyset$:

$$
\emptyset = S_{j_0}^l \subset S_{j_0}^{l-1} \subset \ldots \subset S_l^l \subset S_l^0
$$
Define the following event

$$A^j_t = \left\{ \left| S^j_t \right| \geq m \beta_j \quad \text{and} \quad \forall k < j, \; \left| S^j_k \right| < m \beta_k \right\}.$$ 

By assumption we have

$$\left| S^0_t \right| = m \beta_0 = m.$$ 

Finally, also define the events $A_t$ as the following unions:

$$A_t = \bigcup_{j=1}^{+\infty} A^j_t.$$ 

We have that $A_t$ is a finite union of events:

$$A_t = \bigcup_{j=1}^{j_0} A^j_t.$$ 

Indeed, for all $j > j_0$, due to $\beta_{j_0} \leq 1/m$ and the fact that $\beta_j$ is a decreasing sequence,

$$m \beta_j \leq \frac{1}{m}.$$ 

Thus, by definition of the event $A^j_t$, we have that

$$A^j_t = \left\{ \left| S^j_t \right| \geq 1 \quad \text{and} \quad \forall k < j_0, \; \left| S^j_k \right| < m \beta_j \quad \text{and} \quad \forall k \in [j_0, j-1], \; \left| S^j_k \right| = 0 \right\}.$$ 

However, the same set $S^j_t$ cannot be both empty and have at least one element. In other words, the event $A^j_t$ cannot happen for $j > j_0$.

Under event $\overline{A}_t$, the sum $\sum_{i=1}^{d} \frac{x_i(t)}{n_i(t)}$ can be bounded. The event $\overline{A}_t$ is, by De Morgan’s law, (recall that $j_0$ is finite):

$$\overline{A}_t = \bigcap_{j=1}^{j_0} \overline{A}^j_t$$

$$= \bigcap_{j=1}^{j_0} \left\{ \left| S^j_t \right| < m \beta_j \quad \text{or} \quad \exists k < j, \; \left| S^j_k \right| \geq m \beta_j \right\}, \quad \text{by definition of } A^j_t$$

$$= \bigcap_{j=1}^{j_0} \left[ \left\{ \left| S^j_t \right| < m \beta_j \right\} \cup \left\{ \bigcup_{k=1}^{j-1} \left| S^j_k \right| \geq m \beta_j \right\} \right]$$

$$= \bigcap_{j=1}^{j_0} \left\{ \left| S^j_t \right| < m \beta_j \right\}, \quad \text{as the latter events are not possible if the first holds}$$

$$= \bigcap_{j=1}^{j_0-1} \left\{ \left| S^j_t \right| < m \beta_j \right\} \cap \left\{ \left| S^{j_0}_t \right| = 0 \right\}.$$ 

Since $\beta_{j_0} \leq 1/m$, the last event can be written as $\left| S^{j_0}_t \right| < \frac{m}{m} = 1$. A set whose cardinality is strictly less than one must be empty, thus:

$$\overline{A}_t = \bigcap_{j=1}^{j_0-1} \left\{ \left| S^j_t \right| < m \beta_j \right\} \cap \left\{ \left| S^{j_0}_t \right| = 0 \right\}.$$
If event $\mathcal{A}_t$ happens, then

$$\mathcal{S}_t^i = \{i = 1, \ldots, d : x_i(t) = 1, i \not\in S_t^j\} = \{i = 1, \ldots, d : x_i(t) = 1, n_i(t) > \alpha_j \frac{2f(t)}{\Delta_{x(t)}} \},$$

$$\overline{\mathcal{S}}_t^j = \{i = 1, \ldots, d : x_i(t) = 1\}.$$

Indeed, due to the fact that $S_t^j$ is a decreasing sequence for set inclusion, the complement $\overline{\mathcal{S}}_t^j$ must be an increasing sequence for set inclusion. This implies that:

$$\{i = 1, \ldots, d : x_i(t) = 1\} = \bigcup_{j=1}^{j_0} \left(\overline{\mathcal{S}}_t^j \setminus \overline{\mathcal{S}}_t^{j-1}\right).$$

Thus,

$$\sum_{i=1}^{d} \frac{x_i(t)}{n_i(t)} = \sum_{j=1}^{j_0} \sum_{i \in S_t^j \setminus S_t^{j-1}} \frac{x_i(t)}{n_i(t)}.$$

Using the definition of $\overline{\mathcal{S}}_t^j$, one might write that, if $S_t^j$ holds, then

$$\sum_{i \in S_t^j \setminus S_t^{j-1}} \frac{x_i(t)}{n_i(t)} < \frac{\Delta_{x(t)}}{2f(t)} \frac{\alpha_j}{\Delta_{x(t)}} \sum_{i \in S_t^j \setminus S_t^{j-1}} x_i(t) = \frac{\Delta_{x(t)}}{2f(t)} \frac{\alpha_j}{\Delta_{x(t)}} |S_t^j \setminus S_t^{j-1}|.$$

This implies that the previous sum is bounded by:

$$\sum_{i=1}^{d} \frac{x_i(t)}{n_i(t)} = \sum_{j=1}^{j_0} \sum_{i \in S_t^j \setminus S_t^{j-1}} \frac{x_i(t)}{n_i(t)} \leq \frac{\Delta_{x(t)}}{2f(t)} \frac{\alpha_j}{\Delta_{x(t)}} \sum_{j=1}^{j_0} |S_t^j \setminus S_t^{j-1}|.$$

The inner sum can be decomposed as follows, by definition of $S_t^j$ and $\overline{\mathcal{S}}_t^j$:

$$\sum_{j=1}^{j_0} \left|S_t^j \setminus S_t^{j-1}\right| = \sum_{j=1}^{j_0} \frac{|S_t^j| - |S_t^{j-1}|}{\alpha_j},$$

dropping the complements

$$= \sum_{j=1}^{j_0} \frac{|S_t^j| - |S_t^{j-1}|}{\alpha_j}
= \frac{|S_t^{j_0}|}{\alpha_0} + \sum_{j=1}^{j_0} \left[|S_t^j| \left(\frac{1}{\alpha_j} - \frac{1}{\alpha_{j-1}}\right)\right],$$

by factoring the last term $j_0$

$$< \frac{m \beta_{j_0}}{\alpha_0} + \sum_{j=1}^{j_0} \left[m \beta_j \left(\frac{1}{\alpha_{j-1}} - \frac{1}{\alpha_j}\right)\right],$$

as $\mathcal{A}_t$ holds.

Finally, replacing $g$ and $l$ by their definition

$$\sum_{i=1}^{d} \frac{x_i(t)}{n_i(t)} < \frac{m \Delta_{x(t)}}{2f(t)} \frac{\beta_{j_0}}{\alpha_0} + \sum_{j=1}^{j_0} \beta_{j-1} - \beta_j = \frac{\Delta_{x(t)}^2 \sigma_T^2}{8f(t)}$$

We prove that event $\Delta_{x(t)} \leq \frac{4}{\sigma_T} E_t(x(t))$ implies $\mathcal{A}_t$. Indeed, if $\Delta_{x(t)} \leq \frac{4}{\sigma_T} E_t(x(t))$ and $\mathcal{A}_t$, then

$$\Delta_{x(t)}^2 \leq \frac{16}{\sigma_T^2} E_t(x(t))^2 = \frac{16}{\sigma_T^2} g^2(t) \sum_{i=1}^{d} \frac{x_i(t)}{n_i(t)} < \Delta_{x(t)}^2.$$
Thus, the previous bound can be written as:

\[ A_t^{j,i} = A_t^j \cap \left\{ x_i(t) = 1, n_i(t) \leq \frac{\alpha_j 2 f(T) g(m)}{\Delta_{x(t)}} \right\}. \]

Of course, the union over all \( i \) yields back \( A_t^j \):

\[ \bigcup_{i=1}^d A_t^{j,i} = A_t^j. \]

Since \( A_t^j \) implies that at least \( m \beta_j \) items have not yet been selected enough,

\[ \mathbb{I} \left\{ A_t^j \right\} \leq \frac{1}{m \beta_j} \sum_{i=1}^d \mathbb{I} \left\{ A_t^{j,i} \right\}. \]

The contribution to the regret is bounded by the items that are not selected frequently enough to ensure a good reward estimate:

\[ \sum_{t=1}^T \Delta_{x(t)} \mathbb{I} \left\{ \mathcal{H}_t \cap \mathcal{G}_t \right\} \leq \sum_{t=1}^T \Delta_{x(t)} \mathbb{I} \left\{ A_t \right\} \leq \sum_{t=1}^{T+\infty} \sum_{i=1}^d \Delta_{x(t)} \mathbb{I} \left\{ A_t^j \right\} \leq \sum_{t=1}^{T+\infty} \sum_{j=1}^d \sum_{i=1}^m \frac{\Delta_{x(t)}}{m \beta_j} \mathbb{I} \left\{ A_t^{j,i} \right\}. \]

For any \( i \), define the possible values of the gaps \( \Delta_x \) where \( x_i = 1 \), namely:

\[ \{ \Delta_x : x \in \mathcal{X}, x_i = 1 \} = \{ \Delta_{i,1}, ..., \Delta_{i,K_i} \} \]

where \( K_i \) is the number of possible values for the gap and we assume that the gaps are sorted in decreasing order

\[ \Delta_{i,1} > ... > \Delta_{i,K_i} \]

with the convention that \( \Delta_{i,0} = \infty \). We can then decompose the previous sum according to the value of the gap:

\[ \sum_{t=1}^T \Delta_{x(t)} \mathbb{I} \left\{ \mathcal{H}_t \cap \mathcal{G}_t \right\} \leq \sum_{t=1}^{T+\infty} \sum_{j=1}^d \sum_{i=1}^m \frac{\Delta_{x(t)}}{m \beta_j} \mathbb{I} \left\{ A_t^{j,i} \right\} \]

\[ \leq \sum_{t=1}^{T+\infty} \sum_{j=1}^d \sum_{k=1}^{K_i} \frac{\Delta_{i,k}}{m \beta_j} \mathbb{I} \left\{ A_t^{j,i}, \Delta_{x(t)} = \Delta_{i,k} \right\} \]

\[ \leq \sum_{t=1}^{T+\infty} \sum_{j=1}^d \sum_{i=1}^m \mathbb{H}_{i,k} \left\{ A_t^{j,i}, x_i(t) = 1, n_i(t) \leq \frac{\alpha_j f(T) g(m)}{2 \Delta_{i,k}^2}, \Delta_{x(t)} = \Delta_{i,k} \right\}, \]

by definition of \( A_t^{j,i} \). To simplify notation, let

\[ \tau_j = \frac{1}{2} (x_j f(T) g(m)). \]

Thus, the previous bound can be written as:

\[ \sum_{t=1}^T \Delta_{x(t)} \mathbb{I} \left\{ \mathcal{H}_t \cap \mathcal{G}_t \right\} \leq \sum_{t=1}^{T+\infty} \sum_{j=1}^d \sum_{i=1}^m \sum_{k=1}^{K_i} \frac{\Delta_{i,k}}{m \beta_j} \mathbb{I} \left\{ i : x_i(t) = 1, n_i(t) \leq \frac{\tau_j}{\Delta_{i,k}^2}, \Delta_{x(t)} = \Delta_{i,k} \right\}. \]

To simplify the developments, focus on the two sums, the one on the rounds \( t \) and the one on the gaps \( k \):

\[ \sum_{t=1}^T \sum_{k=1}^{K_i} \frac{\Delta_{i,k}}{m \beta_j} \mathbb{I} \left\{ x_i(t) = 1, n_i(t) \leq \frac{\tau_j}{\Delta_{i,k}^2}, \Delta_{x(t)} = \Delta_{i,k} \right\} \]
As the values of $\Delta_{i,k}$ are ordered, we can decompose as follows:

$$
\sum_{t=1}^{T} \sum_{k=1}^{K_i} \frac{\Delta_{i,k}}{m \beta_j} \mathbb{I} \left\{ x_i(t) = 1, \quad n_i(t) \leq \frac{\tau_j}{\Delta_{i,k}}, \quad \Delta_x(t) = \Delta_{i,k} \right\} \\
= \sum_{t=1}^{T} \sum_{k=1}^{K_i} \sum_{p=1}^{n} \frac{\Delta_{i,k}}{m \beta_j} \mathbb{I} \left\{ x_i(t) = 1, \quad n_i(t) \in \left( \frac{\tau_j}{\Delta_{i,p-1}}, \frac{\tau_j}{\Delta_{i,p}} \right], \quad \Delta_x(t) = \Delta_{i,k} \right\}
$$

The factor $\Delta_{i,k}$ can be moved within the summation over $j$ and be rewritten as $\Delta_{i,k}$, as it will be counted only once, when the step function is nonzero (when $j = k$):

$$
\sum_{t=1}^{T} \sum_{k=1}^{K_i} \frac{\Delta_{i,k}}{m \beta_j} \mathbb{I} \left\{ x_i(t) = 1, \quad n_i(t) \leq \frac{\tau_j}{\Delta_{i,k}}, \quad \Delta_x(t) = \Delta_{i,k} \right\} \\
\leq \sum_{t=1}^{T} \sum_{k=1}^{K_i} \sum_{p=1}^{n} \frac{\Delta_{i,p}}{m \beta_j} \mathbb{I} \left\{ x_i(t) = 1, \quad n_i(t) \in \left( \frac{\tau_j}{\Delta_{i,p-1}}, \frac{\tau_j}{\Delta_{i,p}} \right], \quad \Delta_x(t) = \Delta_{i,k} \right\}
$$

If the sum over $p$ goes to $K_i$, many new terms can be added, though:

$$
\sum_{t=1}^{T} \sum_{k=1}^{K_i} \frac{\Delta_{i,k}}{m \beta_j} \mathbb{I} \left\{ x_i(t) = 1, \quad n_i(t) \leq \frac{\tau_j}{\Delta_{i,k}}, \quad \Delta_x(t) = \Delta_{i,k} \right\} \\
\leq \sum_{t=1}^{T} \sum_{k=1}^{K_i} \sum_{p=1}^{K_i} \frac{\Delta_{i,p}}{m \beta_j} \mathbb{I} \left\{ x_i(t) = 1, \quad n_i(t) \in \left( \frac{\tau_j}{\Delta_{i,p-1}}, \frac{\tau_j}{\Delta_{i,p}} \right], \quad \Delta_x(t) = \Delta_{i,k} \right\}
$$

Again, if the solution $x(t)$ is not taken to be exactly $k$, but any suboptimal solution, many new terms now count in the summation. With this change, the sum over $k$ becomes irrelevant, as all gaps that may contribute to the regret are still counted in.

$$
\sum_{t=1}^{T} \sum_{k=1}^{K_i} \frac{\Delta_{i,k}}{m \beta_j} \mathbb{I} \left\{ x_i(t) = 1, \quad n_i(t) \leq \frac{\tau_j}{\Delta_{i,k}}, \quad \Delta_x(t) = \Delta_{i,k} \right\} \\
\leq \sum_{t=1}^{T} \sum_{k=1}^{K_i} \sum_{p=1}^{n} \frac{\Delta_{i,p}}{m \beta_j} \mathbb{I} \left\{ x_i(t) = 1, \quad n_i(t) \in \left( \frac{\tau_j}{\Delta_{i,p-1}}, \frac{\tau_j}{\Delta_{i,p}} \right], \quad \Delta_x(t) > 0 \right\} \\
\leq \frac{\tau_j}{m \beta_j} \left( \frac{1}{\Delta_{i,1}} + \sum_{p=2}^{K_i} \frac{1}{\Delta_{i,p}} \left( \frac{1}{\Delta_{i,p}} - \frac{1}{\Delta_{i,p-1}} \right) \right) \leq \frac{2 \tau_j}{m \beta_j \Delta_{\min}}
$$

where we used the following algebra, since $\Delta_{i,1} > ... > \Delta_{i,K_i}$:

$$
\frac{1}{\Delta_{i,1}} + \sum_{p=2}^{K_i} \Delta_{i,p} \left( \frac{1}{\Delta_{i,p}} - \frac{1}{\Delta_{i,p-1}} \right) = \frac{1}{\Delta_{i,K_i}} + \sum_{p=1}^{K_i-1} \frac{\Delta_{i,p} - \Delta_{i,p+1}}{\Delta_{i,p}^2} \leq \frac{1}{\Delta_{i,K_i}} + \sum_{p=1}^{K_i-1} \frac{\Delta_{i,p} - \Delta_{i,p+1}}{\Delta_{i,p} \Delta_{i,p+1}} \\
= \frac{1}{\Delta_{i,K_i}} + \sum_{p=1}^{K_i-1} \frac{1}{\Delta_{i,p+1}} - \frac{1}{\Delta_{i,p}} = \frac{2}{\Delta_{i,K_i}} - \frac{1}{\Delta_{i,1}} \leq \frac{2}{\Delta_{\min}}
$$
Injecting this result into the regret term bound,
\[ \sum_{t=1}^{T} \Delta x(t) \mathbb{1} \{ \Pi_t \cap \mathcal{G}_t \} \leq \sum_{t=1}^{T} \sum_{j=1}^{d} \sum_{i=1}^{K_j} \sum_{k=1}^{K_i} \frac{\Delta x(t)}{m_{\beta_j}} \mathbb{1} \left\{ x_i(t) = 1, \quad n_i(t) \leq \frac{\tau_j}{\Delta x(t, i, k)}, \quad \Delta x(t) = \Delta x_{i, k} \right\} \]
\[ \leq \sum_{j=1}^{\infty} \sum_{i=1}^{d} \frac{2 \tau_j}{m_{\beta_j} \Delta_{\min}} \]
\[ = \frac{f(T) dg(m)}{m_{\Delta_{\min}}} \left[ \sum_{j=1}^{\alpha_j} \frac{\beta_j}{\alpha_j} \right], \quad \text{by definition of } \tau_j \]
\[ = \frac{4ldf(T)}{\epsilon_{T} \Delta_{\min}} \left[ \sum_{j=1}^{\alpha_j} \frac{\beta_j}{\alpha_j} \right], \quad \text{by definition of } g. \]

Now set \( \alpha_i = \beta_i = \beta^i \), \( \beta \in (0, 1) \), which satisfies the previous assumptions. Since \( j_0 \) is the first integer \( j \) such that \( \beta_j \leq m^{-1} \) we have \( j_0 = \left\lfloor \frac{\ln m}{\ln \beta} \right\rfloor \). Also
\[ l \sum_{j=1}^{j_0} \frac{\alpha_j}{\beta_j} = l j_0, \quad \text{as } \alpha_i = \beta_i \]
\[ = j_0 \left( \frac{\beta_j}{\alpha_j} + \sum_{j=1}^{j_0} \frac{\beta_j - 1 - \beta_j}{\alpha_j - 1} \right), \quad \text{by definition of } l \]
\[ = j_0 \left( 1 + \sum_{j=1}^{j_0} \frac{\beta_j - 1 - \beta_j}{\beta} \right) \]
\[ = j_0 \left( 1 + \sum_{j=1}^{j_0} \frac{1 - \beta}{\beta} \right) \]
\[ = j_0 \left( 1 + \frac{j_0}{\beta} - j_0 \right) \]
\[ \leq j_0(1 + \frac{j_0}{\beta}). \]

Taking \( \beta = 1/5 \),
\[ j_0 = \left\lfloor \frac{\ln m}{\ln \beta} \right\rfloor \leq \left\lfloor \frac{\ln m}{1.61} \right\rfloor. \]

Injecting these into the regret term, we get:
\[ \sum_{t=1}^{T} \Delta x(t) \mathbb{1} \{ \Pi_t \cap \mathcal{G}_t \} \leq \frac{4ldf(T)}{\epsilon_{T} \Delta_{\min}} \left[ \sum_{j=1}^{\alpha_j} \frac{\beta_j}{\alpha_j} \right] \leq \frac{4df(T)}{\epsilon_{T} \Delta_{\min}} \left( \left\lfloor \frac{\ln m}{1.61} \right\rfloor + 5 \left( \left\lfloor \frac{\ln m}{1.61} \right\rfloor \right)^2 \right) \leq \frac{24df(T)}{\epsilon_{T} \Delta_{\min}} \left( \left\lfloor \frac{\ln m}{1.61} \right\rfloor \right)^2. \]

**Complete regret bound.** Gathering the results about the three terms of the regret decomposition, the regret can be bounded by:
\[ R(T) \leq C_4(m) + \frac{2d m^3}{\Delta_{\min}^2} + \frac{96df(T)}{\epsilon_{T}^3 \Delta_{\min}} \left( \left\lfloor \frac{\ln m}{1.61} \right\rfloor \right)^2 + 4 \sum_{t=1}^{T} \delta_t. \]

## 11 Proof of Theorem 5.1

The first step is to upper bound the value of problem \((P_2)\) with inputs \(a(t)\) and \(b(t)\) as a function of \(x(t)\).
Define the set:

\[ S(t) = \{0, ..., m\xi(t)\}. \]

By definition, \( a(t) \in \{0, ..., m\xi(t)\} \), and \( m = \max_{x \in \mathcal{X}} 1^\top x \), so:

\[ a(t)^\top x \in S(t), \forall x \in \mathcal{X}. \]

Therefore:

\[
\max_{x \in \mathcal{X}} \left\{ a(t)^\top x + \sqrt{b(t)^\top x} \right\} \\
= \max_{s \in S} \max_{x \in \mathcal{X}, a(t)^\top x = s} \left\{ s + \sqrt{b(t)^\top x} \right\} \\
\leq \max_{s \in S} \max_{x \in \mathcal{X}, a(t)^\top x \geq s} \left\{ s + \sqrt{b(t)^\top x} \right\} \\
\leq \max_{s \in S} \left\{ s + \frac{1}{\epsilon_t} \sqrt{b(t)^\top \pi^s(t)} \right\} \\
= s^*(t) + \frac{1}{\epsilon_t} \sqrt{b(t)^\top \pi^s(t)}(t) \\
\leq a(t)^\top \pi^s(t)(t) + \frac{1}{\epsilon_t} \sqrt{b(t)^\top \pi^s(t)}(t) \\
= a(t)^\top x(t) + \frac{1}{\epsilon_t} \sqrt{b(t)^\top x(t)}
\]

where we used the fact that by definition \( \pi^s(t) \) is an \( \epsilon_t^2 \) approximate solution to \( (P_3) \), the fact that by definition

\[ s^*(t) = \arg \max_{s \in S} \left\{ s + \frac{1}{\epsilon_t} \sqrt{b(t)^\top \pi^s(t)} \right\} \]

and the fact that \( a(t)^\top \pi^s(t) \geq s \) for all \( s \in S(t) \).

The second step is to relate the value of problem \( (P_2) \) with inputs \( a(t) \) and \( b(t) \) to the value of problem \( (P_2) \) with inputs \( \hat{\theta}(t) \) and \( \sigma^2(t) \). We recall that, by definition we have

\[ a_i(t) = [\xi(t)\hat{\theta}_i(n)], i = 1, ..., d \]

and

\[ b_i(t) = \xi^2(t)\sigma^2(t), i = 1, ..., d \]

where \( \xi(t) = [\frac{m}{\delta_t}] \). Therefore \( a(t) \in \{1, ..., \xi(t)\}^d \) and:

\[ \hat{\theta}(t) \leq \frac{1}{\xi(t)} a(t) \leq \frac{1}{\xi(t)} 1 + \hat{\theta}(t) \]

therefore for any \( x \in \mathcal{X} \):

\[ \hat{\theta}(t)^\top x \leq \frac{1}{\xi(t)} a(t)^\top x \leq \frac{1}{\xi(t)} 1^\top x + \hat{\theta}(t)^\top x \leq \delta_t + \hat{\theta}(t)^\top x \]

We have proven in the first step that:

\[
\max_{x \in \mathcal{X}} \left\{ a(t)^\top x + \sqrt{b(t)^\top x} \right\} \\
\leq a(t)^\top x(t) + \frac{1}{\epsilon_t} \sqrt{b(t)^\top x(t)}
\]
Then:

\[
\max_{x \in X} \{ \hat{\theta}(t)^\top x + \sqrt{\sigma^2(t)^\top x} \} \\
= \frac{1}{\xi(t)} \max_{x \in X} \{ \xi(t) \hat{\theta}(t)^\top x + \sqrt{\xi(t)^2 \sigma^2(t) x} \} \\
\leq \frac{1}{\xi(t)} \max_{x \in X} \{ a(t)^\top x + \sqrt{b(t)^\top x} \} \\
\leq \frac{1}{\xi(t)} a(t)^\top x(t) + \frac{1}{\xi(t) \epsilon t} \sqrt{b(t)^\top x(t)} \\
\leq \delta_t + \hat{\theta}(t)^\top x(t) + \frac{1}{\epsilon t} \sqrt{\sigma^2(t)^\top x(t)}.
\]

which is the announced result.

12 Pseudo-code to solve problem \((P_2)\)

In this section we provide the pseudo-code for the algorithms described in Section 5, for each family of combinatorial sets. We make use of several subroutines: Hungarian is the Hungarian algorithm which finds a maximum weighted matching in a bipartite graph, Greedy is the greedy algorithm which finds a maximum weighted spanning tree of a graph, Meggido is Meggido’s search algorithm which minimizes a piecewise linear function, Dijkstra is Dijkstra’s algorithm which finds the shortest path in a directed acyclic graph.
input: Number \( m \in \{1, \ldots, d \} \), rewards \( b \in \mathbb{R}^{|E|} \), weights \( a \in \mathbb{N}^{|E|} \)

Result: Solution to \( P_3(s) \) for all \( s \in \{0, 1, \ldots, d \max_i a_i \} \)

\( L \leftarrow \) empty array of size \((d \max_i a_i + 1, m, d)\)

\( S \leftarrow \) empty array of size \((d \max_i a_i + 1, m, d)\)

\( S^* \leftarrow \) empty array of size \((d \max_i a_i + 1)\) for \( s = 0, 1, \ldots, d \max_i a_i \) do

for \( \ell = 0, 1, \ldots, m \) do

for \( i = d, d - 1, \ldots, 0 \) do

if \( i = d \) then

if \( s = 0 \) then

\( L[s, \ell, i] \leftarrow 0 \)

\( S[s, \ell, i] \leftarrow \emptyset \)

else

\( L[s, \ell, i] \leftarrow -\infty \)

\( S[s, \ell, i] \leftarrow \# \)

end

else

if \( \ell = 0 \) then

\( L[s, \ell, i] \leftarrow L[s, \ell, i + 1] \)

\( S[s, \ell, i] \leftarrow S[s, \ell, i + 1] \)

else

\( L[s, \ell, i] \leftarrow \max\{b_i + L[\max(s - a_i, 0), \ell, i + 1], L[s, \ell, i + 1]\} \)

if \( L[s, \ell, i] = L[s, \ell, i + 1] \)

then

\( S[s, \ell, i] \leftarrow S[s, \ell, i + 1] \)

else

\( S[s, \ell, i] \leftarrow S[s, \ell, i + 1] \cup \{i\} \)

end

end

end

end

end

\( S^*[s] \leftarrow S[s, m, 0] \)

end

return \( S^* \)

Algorithm 1: Algorithm for the budgeted \( m \)-set problem.

input: A directed graph \( G = (V, E) \), source node \( u \in V \), destination node \( v \in V \), length of the longest path \( m \), rewards \( b \in \mathbb{R}^{|E|} \), weights \( a \in \mathbb{N}^{|E|} \)

Result: Solution to \( P_3(s) \) for all \( s \in \{0, 1, \ldots, m \max_i a_i \} \)

\( L \leftarrow \) empty array of size \((m \max_i a_i + 1, |V|)\)

\( S \leftarrow \) empty array of size \((m \max_i a_i + 1, |V|)\)

\( S^* \leftarrow \) empty array of size \((d \max_i a_i + 1)\) for \( s = 0, 1, \ldots, m \max_i a_i \) do

for \( w \in V \) do

if \( s = 0 \) then

\( L[s, w], S[s, w] \leftarrow \text{Dijkstra}(G, b, u, w) \)

else

\( x^* \leftarrow \arg \max_{x: (w, x) \in E} \{b_{(w, x)} + L[x, \max(s - a_{(w, x)}, 0)]\} \)

\( L[s, w] \leftarrow b_{(w, x^*)} + L[x^*, \max(s - a_{(w, x^*)}, 0)] \)

\( S[s, w] \leftarrow (w, x) \cup S[x^*, \max(s - a_{(w, x^*)}, 0)] \)

end

end

end

\( S^*[s] \leftarrow S[s, v] \)

end

return \( S^* \)

Algorithm 2: Algorithm for the budgeted source-destination path problem.
input: An undirected graph $G = (V, E)$, rewards $b \in \mathbb{R}^{|E|}$, weights $a \in \mathbb{N}^{|E|}$, a minimum budget $s \in \mathbb{N}$

Result: Solution to $P_3(s)$

$x \leftarrow \text{nothing}$

for all unordered pairs $(e_1, e_2)$ of distinct edges of $E$ do

$E' \leftarrow \{ e \in E \mid b_e \leq \min\{b_{e_1}, b_{e_2}\} \}$

$G' \leftarrow (V, E')$

$x^*, \lambda^* \leftarrow \text{Meggido(Greedy, } G', a + \lambda b)$

$\epsilon \leftarrow \text{arbitrary small value}$

$x^+ \leftarrow \text{Greedy}(G', a + (\lambda^* + \epsilon)b)$

$x^- \leftarrow \text{Greedy}(G', a + (\lambda^* - \epsilon)b)$

while $|x^+ \oplus x^-| > 1$ do

Find $e, e'$ such that $x^+_e = x^-_{e'} = 1$ and $x^+_e = x^-_{e} = 0$

$x^+ \leftarrow x^+ - e + e'$

if $a^T \hat{x} \geq s$ then

|x^+ \leftarrow \hat{x}$

else

|x^- \leftarrow \hat{x}$

end

end

if $x$ is nothing or $b^T \hat{x} > b^T x$ then

|x \leftarrow x^+$

end

end

return $x$

Algorithm 3: Approximation algorithm for the budgeted maximum spanning tree problem.

input: A bipartite graph $G = (V, E)$, rewards $b \in \mathbb{R}^{|E|}$, weights $a \in \mathbb{N}^{|E|}$, a minimum budget $s \in \mathbb{N}$

Result: Solution to $P_3(s)$

$x \leftarrow \text{nothing}$

for all unordered 4-tuples $(e_1, e_2, e_3, e_4)$ of distinct edges of $E$ do

$E' \leftarrow \{ e \in E \mid b_e \leq \min\{b_{e_1}, b_{e_2}, b_{e_3}, b_{e_4}\} \}$

$G' \leftarrow (V, E')$

$x^*, \lambda^* \leftarrow \text{Meggido(Hungarian, } G', a + \lambda b)$

$\epsilon \leftarrow \text{arbitrary small value}$

$x^+ \leftarrow \text{Hungarian}(G', a + (\lambda^* + \epsilon)b)$

$x^- \leftarrow \text{Hungarian}(G', a + (\lambda^* - \epsilon)b)$

while $|x^+ \oplus x^-| > 2$ do

$x' \leftarrow x^+ \oplus x^-$

$x'' \leftarrow \text{one path or one cycle from } x'$

$x^+ \leftarrow x^+ \oplus x''$

if $a^T \hat{x} \geq s$ then

|x^+ \leftarrow \hat{x}$

else

|x^- \leftarrow \hat{x}$

end

end

if $x$ is nothing or $b^T \hat{x} > b^T x$ then

|x \leftarrow x^+$

end

end

return $x$

Algorithm 4: Approximation algorithm for the budgeted maximum bipartite matching problem.
References

[1] Venkatachalam Anantharam, Pravin Varaiya, and Jean Walrand. Asymptotically efficient allocation rules for the multiarmed bandit problem with multiple plays-part i: iid rewards. Automatic Control, IEEE Transactions on, 32(11):968–976, 1987.

[2] Alper Atamtürk and Andrés Gómez. Maximizing a class of utility functions over the vertices of a polytope. Operations Research, 65(2):433–445, 2017.

[3] P. Auer, N. Cesa-Bianchi, Y. Freund, and R. E. Schapire. The nonstochastic multiarmed bandit problem. SIAM J. Comput., 32(1):48–77, January 2003.

[4] Andre Berger, Vincenzo Bonifaci, Fabrizio Gandoni, and Guido Schaefer. Budgeted matching and budgeted matroid intersection via the gasoline puzzle. Mathematical Programming, 2011.

[5] Jeff. Bezanson, Alan. Edelman, Stefan. Karpinski, and Viral B. Shah. Julia: A fresh approach to numerical computing. SIAM Review, 59(1):65–98, 2017.

[6] O. Cappé, A. Garivier, O. Maillard, R. Munos, and G. Stoltz. Kullback-leibler upper confidence bounds for optimal sequential allocation. Annals of Statistics, 41(3):516–541, June 2013.

[7] Nicolò Cesa-Bianchi and Gábor Lugosi. Combinatorial bandits. Journal of Computer and System Sciences, 78(5):1404–1422, 2012.

[8] Chris Coey, Miles Lubin, and Juan Pablo Vielma. Outer approximation with conic certificates for mixed-integer convex problems. arXiv preprint arXiv:1808.05290, 2018.

[9] Richard Combes, M. Sadegh Talebi, Alexandre Proutiere, and Marc Lelarge. Combinatorial bandits revisited. In Proc. of NIPS, 2015.

[10] V. Dani, T. P. Hayes, and S. M. Kakade. Stochastic linear optimization under bandit feedback. In Proc. of Conference On Learning Theory (COLT), pages 355–366, 2008.

[11] Remy Degenne and Vianney Perchet. Combinatorial semi-bandit with known covariance. In Proc. of NIPS, 2016.

[12] Robert E Fredman, Michael Lawrence; Tarjan. Fibonacci heaps and their uses in improved network optimization algorithms. In 25th Annual Symposium on Foundations of Computer Science, pages 338–346. IEEE, 1984.

[13] Branislav Kveton, Zheng Wen, Azin Ashkan, and Csaba Szepesvari. Tight regret bounds for stochastic combinatorial semi-bandits. In Proc. of AISTATS, 2015.

[14] T.L. Lai and H. Robbins. Asymptotically efficient adaptive allocation rules. Advances in Applied Mathematics, 6(1):4–2, 1985.

[15] Miguel Sousa Lobo, Lieven Vandenberghe, Stephen Boyd, and Hervé Lebret. Applications of second-order cone programming. Linear algebra and its applications, 284(1-3):193–228, 1998.

[16] N. Megiddo. Applying parallel computation algorithms in the design of serial algorithms. In 22nd Annual Symposium on Foundations of Computer Science (sfcs 1981), pages 399–408, Oct 1981.

[17] James G. Oxley. Matroid Theory (Oxford Graduate Texts in Mathematics). Oxford University Press, Inc., USA, 2006.

[18] R. Ravi and M. X. Goemans. The constrained minimum spanning tree problem. SWAT, 1996.

[19] H. Robbins. Some aspects of the sequential design of experiments. Bulletin of the American Mathematical Society, 58(5):527–535, 1952.

[20] Mohammad Sadegh Talebi and Alexandre Proutiere. An optimal algorithm for stochastic matroid bandit optimization. In Proc. of ICAAMS, 2016.

[21] Siwei Wang and Wei Chen. Thompson sampling for combinatorial semi-bandits. In Proc. of ICML, 2018.