Research Article

Spectral Collocation Methods for Fractional Integro-Differential Equations with Weakly Singular Kernels

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In this paper, we propose and analyze a spectral approximation for the numerical solutions of fractional integro-differential equations with weakly kernels. First, the original equations are transformed into an equivalent weakly singular Volterra integral equation, which possesses nonsmooth solutions. To eliminate the singularity of the solution, we introduce some suitable smoothing transformations, and then use Jacobi spectral collocation method to approximate the resulting equation. Later, the spectral accuracy of the proposed method is investigated in the infinity norm and weighted $L^2$ norm. Finally, some numerical examples are considered to verify the obtained theoretical results.

1. Introduction

Fractional integro-differential equations (FIDEs) have been frequently utilized in modeling real phenomena in earthquake engineering, statistical mechanics, thermal systems, control theory, astronomy, turbulence, and other application fields and have attracted more and more attention among researchers. Note that, obtaining an analytical solution of FIDEs is very difficult and sometimes even impossible. Therefore, effective numerical methods have been widely used for solving this kind of equations in recent years, such as fractional differential transform methods [1], Taylor expansion [2], operational Tau methods [3], Adomian decomposition methods [4], spline collocation methods [5], wavelets [6–8], piecewise polynomial collocation methods [9], and Laplace decomposition methods [10]. Recently, the kernels methods have also received much attention, for the detail, see [11–13]. It is well known, that fractional differential operators are nonlocal and have weakly singular kernels, and so global methods; for example, spectral methods, could be better suited for solving numerically FDEs. In the past decades, some works are devoted to the spectral approximation of FDEs with smooth kernels. [14] introduced a Chebyshev spectral collocation method for solving a general form of nonlinear FDEs with linear functional arguments. In [15], a Chebyshev pseudospectral method was developed for FDEs. Yang et al. proposed and analyzed a Jacobi spectral collocation approximation for FDEs of Volterra type or Fredholm-Volterra type in [16, 17], and constructed a general spectral and pseudospectral Jacobi-Galerkin method for FDEs of Volterra type in [18]. All of these works carried out the analyses under the assumption that the underlying solutions are smooth. However, even if the input functions are sufficiently smooth, the solutions of FDEs are usually not smooth and will exhibit some weak singularity. And, Yang considered the case of nonsmooth solutions of FDEs with smooth kernels in [19], where some smoothing transformations were introduced to eliminate the singularity of the solutions, then the Jacobi spectral collocation method was used to solve the transformed equation. Inspired by the work [19], in the present one, we intend to apply spectral methods to solve FDEs with weakly kernels and with nonsmooth solutions. We will justify that the proposed numerical methods can achieve spectral accuracy in the infinity norm and weighted $L^2$
norm. Here, we consider the following initial value problems for FIDEs in the form:

\[ D^{\mu_1} y(t) = f(t) + a(t) y(t) + \int_0^t \frac{1}{\Gamma(n - \mu_1)} (t - s)^{n - \mu_1 - 1} y^{(n)}(s) ds, \quad \mu_1 > 0, \quad 0 < \mu_2 < 1, \quad t \in [0, T], \]  

(1)

\[ y^{(i)}(0) = y_0^{(i)}, \quad i = 0, 1, \ldots, n - 1, \]  

(2)

where \( y(t) \) is the unknown function to be determined, \( f(t), a(t) \) and \( K(t, s) \) are known smooth functions on their respective domains, \( y_0^{(i)} (i = 0, 1, \ldots, n - 1) \) are given real numbers, \( n = [\mu_1] \) is the smallest integer which is bigger than the real number \( \mu_1 \). In this paper, \( D^{\mu_1} y(t) \) denotes the Caputo fractional derivative of order \( \mu_1 \) defined as follows:

\[ D^{\mu_1} y(t) = J^{\mu_1} y(t) - \sum_{k=0}^{\lfloor \mu_1 \rfloor - 1} \frac{t^k}{k!} y^{(k)}(0), \]  

(3)

where \( \Gamma(\cdot) \) denotes the Gamma function, and

\[ J^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t - s)^{\mu - 1} f(s) ds, \]  

(4)

is called the Riemann–Liouville fractional integral of order \( \mu \).

Using the same methods as in the proof of Lemma 2 in [9], we can transform the original Equations (1) and (2) into an equivalent weakly singular Volterra integral equation

\[ y(t) = \mathcal{J}(t) + \int_0^t (t - \sigma)^{-\mu} \mathcal{K}(t, \sigma) y(\sigma) d\sigma, \]  

(5)

where \( \mu = [\mu_1] - \mu_1 \) and

\[ \mathcal{J}(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\mu_1)} \int_0^t (t - \sigma)^{\mu_1 - 1} f(\sigma) d\sigma, \]  

(6)

From [20], we known that the \( m \)-th derivative of the solution \( y(t) \) of (5) behaves like \( y^{(m)}(t) \sim t^{1 - m - \mu} \) as \( t \to 0 \), which indicates that \( y \notin C^m[0, T] \). To eliminate this singularity of the solution, we introduce the smoothing transformations (see [21–23])

\[ t = \gamma(z) = z^q, \quad \sigma = \gamma(w) = w^\rho, \]  

(7)

where \( q \) is a positive integer number, then equation (5) becomes

\[ Y(z) = \mathcal{J}(z) + \int_0^1 (z - w)^{-\mu} \mathcal{K}(z, w) Y(w) dw, \]  

(8)

where \( z \in (0, \sqrt{T}) \), \( Y(z) = y(\gamma(z)) \), \( \mathcal{J}(z) = \mathcal{J}(\gamma(z)) \) and

\[ \mathcal{K}(z, w) = \begin{cases} \left( \frac{\gamma(z) - \gamma(w)}{z - w} \right)^{-\mu} \gamma'(w) \mathcal{K}(\gamma(z), \gamma(w)), & z \neq w, \\ \left( \frac{\gamma'(z)}{\gamma(z) - \gamma(w)} \right)^{-\mu} \gamma'(w) \mathcal{K}(\gamma(z), \gamma(w)), & z = w. \end{cases} \]  

(9)

It is easy to see that the solution of Equation (8) satisfies \( Y^{(m)}(z) \sim z^{(1-m-\mu)} \) as \( z \to 0 \). Then, by choosing a suitable \( q \), we can obtain the regularity of \( Y(z) \) as we like. Therefore, spectral methods can be applied for solving the resulting equation.

The structure of this paper is as follows: In Section 2, we construct a Jacobi spectral collocation approximation for Equation (8). Some elementary definitions and lemmas will be presented in Section 3, and convergence analysis of the proposed approximation will be carried out in Section 4. The numerical experiments are carried out in Section 5, which will be used to verify the theoretical results obtained in Section 4. Finally, Section 6 outlines the conclusions.

2. Jacobi Spectral Collocation Method

In this section, we derive the Jacobi spectral collocation method with the numerical implementation. Set \( a^\alpha\beta(x) = (1-x)^\alpha (1+x)^\beta \) be a weight function in the usual sense, for \( \alpha, \beta > -1 \). As described in [24], the set of Jacobi polynomials \( \{ P_n^{\alpha\beta}(x) \}_{n=0}^{\infty} \) forms a complete \( L^2_{a^\alpha\beta}((-1, 1)) \)-orthogonal system, where \( L^2_{a^\alpha\beta}((-1, 1)) \) is a weighted space defined by the following equation:
\[ L^2_{\omega, \beta} (-1, 1) = \{ v: v \text{ is measurable and } \| v \|_{\omega, \beta} < \infty \}, \] (10)

which is equipped with the following inner product and norm:

\[ (u, v)_{\omega, \beta} = \int_{-1}^{1} u(x)v(x)\omega(x)\beta(x)dx, \]
\[ \| u \|_{\omega, \beta} = (u, u)_{\omega, \beta}^{1/2}. \] (11)

For the sake of implementing the spectral methods naturally, we take the linear transformations:

\[ z = \frac{\sqrt{T}}{2} (1 + x), \quad -1 \leq x \leq 1, \]
\[ w = \frac{\sqrt{T}}{2} (1 + \xi), \quad -1 \leq \xi \leq x, \] (12)

and so (8) read as follows:

\[ u(x) = f(x) + \int_{-1}^{1} (x - \xi)^{-\mu} K(x, \xi) u(\xi)d\xi. \] (13)

Here,

\[ u(x) = Y \left( \frac{\sqrt{T}}{2} (1 + x) \right), \]
\[ f(x) = \tilde{f} \left( \frac{\sqrt{T}}{2} (1 + x) \right), \] (14)
\[ K(x, \xi) = \left( \frac{\sqrt{T}}{2} \right)^{-\mu} K \left( \frac{\sqrt{T}}{2} (1 + x), \frac{\sqrt{T}}{2} (1 + \xi) \right). \]

For a given positive integer \( N \), we denote the collocation points by \( \{ x_i \}_{i=0}^{N} \) which is the set of \( (N + 1) \) Jacobi–Gauss points corresponding to the weight functions \( \omega(x) \). In this paper, we take the special collocation points \( \{ x_i \}_{i=0}^{N} \) which are respect to the case that \( \alpha = \beta = -\mu \). Obviously, (13) holds at \( x_i, i = 0, \ldots, N \), i.e.,

\[ u(x_i) = f(x_i) + \int_{-1}^{1} (x_i - \xi)^{-\mu} K(x_i, \xi) u(\xi)d\xi. \] (15)

Using the linear transformation

\[ \xi = \xi_i(\theta) \]
\[ = \frac{1 + x_i}{2} \theta + \frac{x_i - 1}{2}, \quad -1 \leq \theta \leq 1. \] (16)

yields

\[ u(x_i) = f(x_i) + \left( \frac{1 + x_i}{2} \right)^{-\mu} \int_{-1}^{1} (1 - \theta)^{-\mu} K(x_i, \xi_i(\theta)) u(\xi_i(\theta))d\theta. \] (17)

Applying the Jacobi-Gauss integration formula, we have the following equation:

\[ u(x_i) = \tilde{f}(x_i) + \left( \frac{1 + x_i}{2} \right)^{-\mu} \sum_{k=0}^{N} \tilde{K}(x_i, \xi_i(\theta_k)) u(\xi_i(\theta_k))\omega_k. \] (18)

Where \( \{ \theta_k \}_{k=0}^{N} \) and \( \{ \omega_k \}_{k=0}^{N} \) denote the Jacobi–Gauss points and the weights with respect to the weight function \( \omega^{-\mu} \), respectively.

Let \( P_N \) denote the space of all polynomials of degree not exceeding \( N \). We use \( u_i \) to indicate the approximate values for \( u(x_i), 0 \leq i \leq N \). Then, the Jacobi collocation method is to seek an approximate solution \( u_N(x) \in P_N \) of the following form:

\[ u(x) \approx u_N(x) = \sum_{j=0}^{N} u_i F_j(x), \] (19)

where \( \{ F_j(x) \}_{j=0}^{N} \) are the Lagrange interpolation basis functions associated with \( \{ x_i \}_{i=0}^{N} \), and \( \{ u_i \}_{i=0}^{N} \) are determined by the following discrete collocation equations:

\[ u_i = \tilde{f}(x_i) + \left( \frac{1 + x_i}{2} \right)^{-\mu} \sum_{j=0}^{N} u_j \left[ \tilde{K}(x_i, \xi_i(\theta_k)) F_j(\xi_i(\theta_k)) \omega_k \right]. \] (20)

### 3. Some Useful Lemmas

In this section, we introduce some important definitions and lemmas, which will be used to study the properties of the proposed numerical method later.

For an integer \( m \geq 1 \), we introduce a Sobolev space

\[ H^m_{\omega, \beta} (-1, 1): = \{ v(x): v^{(p)}(x) \in L^2_{\omega, \beta} (-1, 1), 0 \leq p \leq m \}, \] (21)

equipped with the norm

\[ \| v \|_{m, \omega, \beta} = \left( \sum_{p=0}^{m} \| v^{(p)}(x) \|_{\omega, \beta}^2 \right)^{1/2}, \] (22)

and seminorm

\[ \| v \|_{m; N, \omega, \beta} = \left( \sum_{p=\min(m,N+1)}^{m} \| v^{(p)}(x) \|_{\omega, \beta}^2 \right)^{1/2}. \] (23)

Hereafter, we use \( C \) to denotes a positive constant which is independent of \( N \) and may have different values in different occurrences.

**Lemma 1** (see [24]). Assume that Gauss quadrature formula relative to the Jacobi weight is used to integrate the product \( \psi \phi \), where \( v \in H^m_{\omega, \beta} (-1, 1) \), for some \( m \geq 1 \) and \( \psi \in P_N \), then

\[ \| (v, \psi)_{m; \omega, \beta} - (v, \psi)_{N} \| \leq CN^{-m} \| v \|_{m; N, \omega, \beta} \| \psi \|_{m, \omega, \beta}, \] (24)

where \( (\cdot, \cdot)_{N} \) represents the discrete inner product in \( L^2_{\omega, \beta} (-1, 1) \) space and
\[(v, \phi)_N = \sum_{k=0}^{N} v(x_k) \psi(x_k) \omega_k. \tag{25}\]

Lemma 2 (see [25]). Assume that \( \{F_j(x)\}_{j=0}^N \) is the set of \( N \)-th Lagrange interpolation polynomials associated with the Jacobi-Gauss points \( \{x_i\}_{i=0}^N \), then

\[
\left\| f^{a,b}_N \right\|_{\infty} := \max_{x \in [-1,1]} \sum_{j=0}^{N} |F_j(x)| = \begin{cases} \Theta(\log N), & -1 < \alpha, \beta \leq - \frac{1}{2} \\ \Theta(N^{1+\alpha/2}), & \lambda = \max\{\alpha, \beta\}, \text{ otherwise.} \end{cases} \tag{26}\]

Lemma 3 (see [26]). Assume that \( v \in H^{m}\omega^{\phi} (-1,1) \) and denote by \( F^{a,b}_N \omega^{\phi} (x) \) its interpolation polynomial associated with the Jacobi-Gauss points \( \{x_i\}_{i=0}^N \), namely,

\[
f^{a,b}_N v(x) = \sum_{i=0}^{N} v(x_i) F_i(x). \tag{27}\]

Then, the following estimates hold:

\[
\left\| v - f^{a,b}_N \omega^{\phi} \right\|_{\omega^{\phi}} \leq CN^{-m}\log N|v|^{m,N}_{\omega^{\phi}}, \quad \left\| v - f^{a,b}_N \omega^{\phi} \right\|_{\infty} \leq \begin{cases} CN^{1/2-m}\log N|v|^{m,N}_{\omega^{\phi}}, & -1 < \alpha, \beta \leq - \frac{1}{2} \\ CN^{1+\lambda-m}\log N|v|^{m,N}_{\omega^{\phi}}, & \lambda = \max\{\alpha, \beta\}, \text{ otherwise,}\end{cases} \tag{28}\]

where \( \omega^{\phi} = \omega^{-1/2, -1/2} \).

Lemma 4 (see [27, 28]). Suppose \( L \geq 0, 0 < \mu < 1 \), and let \( v(t) \) be a non-negative and locally integrable function defined on \([0, T]\) satisfying

\[
v(t) \leq w(t) + L \int_0^t (t-s)^{-\mu} v(s) ds. \tag{29}\]

Then, we have the following equation:

\[
v(t) \leq w(t) + C \int_0^t (t-s)^{-\mu} \omega(s) ds. \tag{30}\]

Lemma 5 (see [29]). For any measurable function \( f \geq 0 \), the following generalized Hardy’s inequality

\[
\left( \int_a^b |(\mathcal{F} f)(x)|^q \omega(x) dx \right)^{1/q} \leq C \left( \int_a^b |f(x)|^q \omega(x) dx \right)^{1/p}, \tag{31}\]

holds if and only if

\[
\sup_{a \leq x \leq b} \left( \int_a^b \omega(t) dt \right)^{1/(\mu(q-1)')} < \infty, \quad p' = \frac{p}{p-1} \tag{32}\]

for the case \( 1 < p \leq q < \infty \). Here, \( \mathcal{F} \) is an operator of the form

\[
(\mathcal{F} f)(x) = \int_a^x R(x,t) f(t) dt, \tag{33}\]

with a given kernel \( R(x,t) \), weight functions \( \omega, \bar{\omega} \), and \( -\infty \leq a < b \leq \infty \).

Lemma 6 (see [30]). For every bounded function \( v(x) \), we have the following equation:

\[
\sup_{N} \left\| f^{a,b}_N \omega^{\phi} \right\|_{\omega^{\phi}} \leq C\|v(x)\|_{\infty}. \tag{34}\]

4. Convergence Analysis

In this section, we devote to analyzing the convergence of the approximation method (20). The goal is to show that the proposed method possesses spectral accuracy in the infinity norm and weighted \( L^2 \) norms.

Theorem 1. Let \( u(x) \) be the exact solution of equation (13). Assume that \( u_N(x) \) is the numerical solution obtained by the proposed spectral method. If \( u \in H^m_{\omega^{\phi}, \alpha, \lambda} (-1,1) \cap H^m_{\omega^{\phi}} (-1,1) \), then for sufficiently large \( N \),

\[
\left\| u - u_N \right\|_{\infty} \leq \begin{cases} CN^{1/2-m}K_0 \|u\|_{\infty} + N^{1/2} |u|^{m,N}_{\omega^{\phi}}, & 0 < \mu < \frac{1}{2} \\ CN^{-m} \log N(K_0 \|u\|_{\infty} + N^{1/2} |u|^{m,N}_{\omega^{\phi}}), & \frac{1}{2} \leq \mu < 1, \end{cases} \tag{36}\]

where
where $\mathcal{M}e = \int_{-1}^{x} (x - \xi)^{-\mu} \tilde{K}(x, \xi) e(\xi) d\xi$.

Consequently,

$$e(x) = \int_{-1}^{x} (x - \xi)^{-\mu} \tilde{K}(x, \xi) e(\xi) d\xi + \sum_{i=1}^{3} J_i(x),$$

where

$$J_1(x) = \sum_{i=0}^{N} J_i^1 F_i(x),$$

$$J_2(x) = u(x) - I_N^{1-\mu} u(x),$$

$$J_3(x) = I_N^{1-\mu} \tilde{K}(x_i, \xi_j) - \mathcal{M}e - \mathcal{M}e.$$

It follows from (44) and Lemma 4 that

$$\|e\|_{\infty} \leq C \sum_{i=1}^{3} \|J_i\|_{\infty}. \quad (46)$$

By applying Lemma 2 and the estimate (40), we obtain that

$$\|J_i^1\|_{\infty} = \sum_{i=0}^{N} |J_i^1| F_i(x) \leq C \max_{1 \leq i \leq 3N} \sum_{j=0}^{N} |F_j(x)| \leq \begin{cases} C N^{1/2 - \mu} \max_{1 \leq i \leq 3N} \|\tilde{K}(x_i, \xi_j)\|_{\infty}^{m_N} (\|e\|_{\infty} + \|u\|_{\infty}), & 0 < \mu < \frac{1}{2}, \\ C N^{-\mu} \max_{1 \leq i \leq 3N} \|\tilde{K}(x_i, \xi_j)\|_{\infty}^{m_N} (\|e\|_{\infty} + \|u\|_{\infty}), & \frac{1}{2} \leq \mu < 1. \end{cases} \quad (47)$$

Using Lemma 3 gives the following equation:

$$\|J_2\|_{\infty} = \|u(x) - I_N^{1-\mu} u(x)\|_{\infty} \leq \begin{cases} C N^{1-\mu} |u|_{\infty}^{m_N}, & 0 < \mu < \frac{1}{2}, \\ C N^{(1/2) - \mu} \log N |u|_{\infty}^{m_N}, & \frac{1}{2} \leq \mu < 1. \end{cases} \quad (48)$$

From [33], we known that $\mathcal{M}$ are linear and compact operators from $C[-1,1]$ into $C^0[-1,1]$. This implies that for any function $v \in C[-1,1]$, there exists a positive constant $C$ such that
\[
\|v_{0,x}\|_N = C\|v\|_\infty, \quad 0 < \kappa < 1 - \mu. \tag{49}
\]

Hence, it follows from (35) and Lemma 2 that
\[
\|J_3\|_\infty = \|T_N^{\mu-\mu} \mathcal{M} e - \mathcal{M} e\|_\infty
\leq \|T_N^{\mu-\mu} - 1\|_\infty \|\mathcal{M} e - \mathcal{F}_N \mathcal{M} e\|_\infty
\leq (1 + \|T_N^{\mu-\mu}\|_\infty) \|\mathcal{M} e - \mathcal{F}_N \mathcal{M} e\|_\infty
\leq (1 + \|T_N^{\mu-\mu}\|_\infty) CN^{-\kappa} \|\mathcal{M} e\|_{0,x} \tag{50}
\]

\[
\|u - u_N\|_{w^{\kappa,\nu}} \leq \left\{
\begin{aligned}
CN^{-m} \left(K^* \|u\|_{\infty} + |u|_{w^{\kappa,\nu}}^{m,N} + N^{1-\kappa-\mu} |u|_{w^{\kappa,\nu}}^{m,N}\right), & \quad 0 < \mu < \frac{1}{2}, \\
CN^{-m} \left(K^* \|u\|_{\infty} + |u|_{w^{\kappa,\nu}}^{m,N} + \frac{1}{2} - \kappa \log N |u|_{w^{\kappa,\nu}}^{m,N}\right), & \quad \frac{1}{2} \leq \mu < 1, 0 < \kappa < 1 - \mu.
\end{aligned}\right.
\tag{51}
\]

**Proof.** By Lemma 4 and Lemma 5, it follows from (44) that
\[
\|e\|_{w^{\kappa,\nu}} \leq C \sum_{i=1}^N \|f_i(x)\|_{w^{\kappa,\nu}}. \tag{52}
\]

Now, using (40) and Lemma 6 gives the following equation:
\[
\|J_1\|_{w^{\kappa,\nu}} = \|\sum_{i=0}^N f_i(x)\|_{w^{\kappa,\nu}} \leq C \max_{1 \leq i \leq N} |f_i'| \leq CN^{-m} \max_{1 \leq i \leq N} |K(x, \xi, \cdot)|_{w^{\kappa,\nu}}^{m,N} \left(\|e\|_{\infty} + \|u\|_{\infty}\right).
\tag{53}
\]

Using Lemma 3, we obtain that
\[
\|J_2\|_{w^{\kappa,\nu}} = \|u(x) - T_N^{\mu-\mu} u\|_{w^{\kappa,\nu}} \leq CN^{-m} |u(x)|_{w^{\kappa,\nu}}^{m,N}. \tag{54}
\]

Finally, it follows from (35), (49) and Lemma 6 that
\[
\begin{align*}
\|J_3\|_{w^{\kappa,\nu}} &= \|T_N^{\mu-\mu} \mathcal{M} e - \mathcal{M} e\|_{w^{\kappa,\nu}} \leq \|T_N^{\mu-\mu} - 1\|_{w^{\kappa,\nu}} \|\mathcal{M} e - \mathcal{F}_N \mathcal{M} e\|_{w^{\kappa,\nu}} \leq \|T_N^{\mu-\mu}\|_{w^{\kappa,\nu}} \|\mathcal{M} e - \mathcal{F}_N \mathcal{M} e\|_{w^{\kappa,\nu}} \leq C \|\mathcal{M} e - \mathcal{F}_N \mathcal{M} e\|_{\infty} \leq CN^{-\kappa} \|\mathcal{M} e\|_{0,x} \leq CN^{-\kappa} \|\mathcal{M} e\|_{0,x} \tag{55}
\end{align*}
\]

Therefore, the estimate (51) is obtained by combining (36), (52)–(55), provided that \(N\) is sufficiently large.

### 5. Numerical Experiments

In this section, we present some numerical experiments to confirm the efficiency and accuracy of the suggested spectral method.

**Example 1.** Consider the following fractional integro-differential equation with weakly kernels:

\[
D^{\alpha/2} y(t) = \Gamma\left(\frac{\alpha}{2}\right) + t^{1/3} - \frac{\Gamma(5/3)\Gamma(199/200)}{\Gamma(1597/600)} t^{997/600} - t^3 y(t) + \int_0^t (t-s)^{-1/200} y(s) ds, t \in [0, 10],
\tag{56}
\]

Equation (56) has nonsmooth solution \(y(t) = t^{2/3}\). We introduce the smoothing transformations \(t = z^2\), \(\sigma = w^3\) to the equivalent Volterra integral equation and implement the spectral collocation method to solve the transformed
equation. The obtained $L^\infty$ and $L_{\omega-\mu}$ errors are presented in Table 1. We also plotted the numerical errors in Figure 1. As we can see from Table 1 and Figure 1, the proposed spectral method converges rapidly, which is confirmed by spectral accuracy. This is in accordance with our theoretical results.

**Example 2.** Consider the following fractional integro-differential equation with weakly kernels:

$$D^{11/6} y(t) = \Gamma\left(\frac{17}{6}\right) t^{3/6} - \frac{\Gamma(29/30) \Gamma(17/6)}{\Gamma(19/5)} t^{14/5} - t^2 y(t) + \int_0^t (t-s)^{-1/30} y(s) ds, \quad t \in [0,1],$$

$$y(0) = 0, y'(0) = 0.$$  \hfill (57)

The exact solution of above equation is given by $y(t) = t^{11/6}$. It is very clear, $y(t)$ is not smooth at $t = 0$. Similar to the previous example, we introduce the smoothing transformations $t = z^6, \sigma = w^6$ to the

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**Table 1:** The $L^\infty$ and $L_{\omega-\mu}$ errors for Example 1.

| $N$ | $\|u - u_N\|_\infty$ | $\|u - u_N\|_{\omega-\mu}$ |
|-----|----------------------|---------------------------|
| 4   | 1.3305e-03           | 7.0244e-06                |
| 8   | 1.1124e-03           | 4.623e-06                 |
| 16  | 2.1050e-06           | 1.2920e-06                |
| 32  | 8.8371e-07           | 5.2854e-07                |
| 64  | 5.9416e-07           | 3.2                        |

**Figure 1:** The $L^\infty$ and $L_{\omega-\mu}$ errors for Example 1.

**Table 2:** The $L^\infty$ and $L_{\omega-\mu}$ errors for Example 2.

| $N$ | $\|u - u_N\|_\infty$ | $\|u - u_N\|_{\omega-\mu}$ |
|-----|----------------------|---------------------------|
| 4   | 1.2413e-03           | 7.0244e-06                |
| 8   | 3.8887e-03           | 4.623e-06                 |
| 16  | 3.1362e-04           | 1.0465e-05                |
| 32  | 2.9163e-06           | 4.6884e-07                |
| 64  | 5.9416e-07           | 1.4909e-07                |
| 128 | 5.9416e-07           | 3.2                        |

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corresponding Volterra integral equation, and apply the spectral collocation method to approximate the transformed equation. The numerical errors are demonstrated in Table 2 and Figure 2 for different values of \( N \). Again we can see the spectral approximation gives spectral accuracy.

6. Conclusion

In this work, we have elaborated a Jacobi spectral collocation approximation for fractional integro-differential equation with weakly kernels and with nonsmooth solutions. The converge analysis in \( L^\infty \) norm and \( L^2_{\omega^{-\mu}} \) norm was established for the approximation method. Two numerical test examples with nonsmooth solutions was presented to illustrate the spectral accuracy of the proposed method.

Data Availability

The author declares that the data supporting the findings of this study are available within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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References

[1] A. Atikoglu and I. Ozkol, “Solution of fractional integro-differential equations by using fractional differential transform method,” Chaos, Solitons & Fractals, vol. 40, no. 2, pp. 521–529, 2009.

[2] L. Huang, X. F. Li, Y. Zhao, and X. Y. Duan, “Approximate solution of fractional integro-differential equations by Taylor expansion method,” Computers & Mathematics with Applications, vol. 62, no. 3, pp. 1127–1134, 2011.

[3] S. K. Vanani and A. Aminataei, “Operational Tau approximation for a general class of fractional integro-differential equations,” Computational and Applied Mathematics, vol. 30, no. 3, pp. 655–674, 2011.

[4] S. Momani and R. Qaralleh, “An efficient method for solving systems of fractional integro-differential equations,” Computers & Mathematics with Applications, vol. 52, no. 3-4, pp. 459–470, 2006.

[5] A. Pedas, E. Tamme, and M. Vikerpuur, “Numerical solution of linear fractional weakly singular integro-differential equations with integral boundary conditions,” Applied Numerical Mathematics, vol. 149, 2019.

[6] Z. Meng, L. Wang, H. Li, and W. Zhang, “Legendre wavelets method for solving fractional integro-differential equations,” Advances in Difference Equations, vol. 2017, pp. 27–16, 2017.

[7] M. Yi, L. Wang, and J. Huang, “Legendre wavelets method for the numerical solution of fractional integro-differential equations with weakly singular kernel,” Applied Mathematical Modelling, vol. 40, no. 4, pp. 3422–3437, 2016.

[8] J. Zhao, J. Xiao, and N. J. Ford, “Collocation methods for fractional integro-differential equations with weakly singular kernels,” Numerical Algorithms, vol. 65, no. 4, pp. 723–743, 2014.

[9] C. Yang and J. Hou, “Numerical solution of Volterra integro-differential equations of fractional order by Laplace decomposition method,” International Journal of Mathematical, Computer, Natural and Physical Engineering, vol. 7, pp. 549–553, 2013.

[10] J. C. Guella, “Operator valued positive definite kernels and differentiable universality,” Analysis and Applications, vol. 20, no. 04, pp. 681–735, 2020.
[12] R. Wang, S. An, W. Liu, and L. Li, “Fixed-point Algorithms for inverse of residual rectifier neural networks,” *Mathematical Foundations of Computing*, vol. 4, no. 1, pp. 31–44, 2021.

[13] D. X. Zhou, “Deep distributed convolutional neural networks: Universality,” *Analysis and Applications*, vol. 16, no. 06, pp. 895–919, 2018.

[14] K. K. Ali, M. A. Abd El Salam, E. M. H. Mohamed, B. Samet, S. Kumar, and M. S. Osman, “Numerical solution for generalized nonlinear fractional integro-differential equations with linear functional arguments using Chebyshev series,” *Advances in Difference Equations*, vol. 2020, pp. 494–523, 2020.

[15] N. H. Sweilam and M. M. Khader, “A Chebyshev pseudospectral method for solving fractional-order integro-differential equations,” *ANZIAM Journal*, vol. 51, pp. 464–475, 2011.

[16] Y. Yang, Y. Chen, and Y. Huang, “Spectral-collocation method for fractional Fredholm integro-differential equations,” *Journal of the Korean Mathematical Society*, vol. 51, no. 1, pp. 203–224, 2014.

[17] Y. Yang, Y. Chen, and Y. Huang, “Convergence analysis of the Jacobi spectral-collocation method for fractional integro-differential equations,” *Acta Mathematica Scientia*, vol. 34, no. 3, pp. 673–690, 2014.

[18] Y. Yang, “Jacobi spectral Galerkin methods for fractional integro-differential equations,” *Calcolo*, vol. 52, no. 4, pp. 519–542, 2015.

[19] Y. Yang, S. Kang, and V. I. Vasil’ev, “The Jacobi spectral collocation method for fractional integro-differential equations with non-smooth solutions,” *Electronic Research Archive*, vol. 28, no. 3, pp. 1161–1189, 2020.

[20] H. Brunner, *Collocation Methods for Volterra Integral and Related Functional Equations*, Cambridge University Press, New York, NY, USA, 2004.

[29] A. Kufner and L. Persson, *Weighted Inequalities of Hardy Type*, World Scientific, New York, NY, USA, 2003.

[30] P. Neval, “Mean convergence of Lagrange interpolation, III. Trans,” *Journal of the American Mathematical Society*, vol. 282, pp. 669–698, 1984.

[31] D. Ragozin, “Polynomial approximation on compact manifolds and homogeneous space,” *Transactions of the American Mathematical Society*, vol. 150, 1970.

[32] D. L. Ragozin, “Constructive polynomial approximation on spheres and projective spaces,” *Transactions of the American Mathematical Society*, vol. 162, pp. 157–170, 1971.

[33] D. Colton and R. Kress, *Inverse Coasitic and Electromagnetic Scattering Theory*, Springer, Berlin, Germany, 2nd edition, 1998.