Normalized vacuum states in $\mathcal{N} = 4$ supersymmetric Yang–Mills quantum mechanics with any gauge group.

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Abstract

We study the question of existence and the number of normalized vacuum states in $\mathcal{N} = 4$ super–Yang–Mills quantum mechanics for any gauge group. The mass deformation method is the simplest and clearest one. It allowed us to calculate the number of normalized vacuum states for all gauge groups. For all unitary groups, $\#_{\text{vac}} = 1$, but for the symplectic groups [starting from $Sp(6)$], for the orthogonal groups [starting from $SO(8)$] and for all the exceptional groups, it is greater than one. We also discuss at length the functional integral method. We calculate the “deficit term” for some non–unitary groups and predict the value of the integral giving the “principal contribution”. The issues like the Born–Oppenheimer procedure to derive the effective theory and the manifestation of the localized vacua in the asymptotic effective wave functions are also discussed.

1 Introduction

Consider the theory obtained by the dimensional reduction of $\mathcal{N} = 4$, $D = 4$ super–Yang–Mills theory (which is obtained in turn by the dimensional reduction of $\mathcal{N} = 1$, $D = 10$ SYM theory to four dimensions) when the whole space is shrinked to a point and we are dealing with a supersymmetric quantum mechanical (SQM) system involving 16 real supercharges. In $(9+1)$–dimensional notations,

$$Q_{\alpha} = \frac{1}{\sqrt{2}} \left[ (\Gamma_I)_{\alpha\gamma} E^A_I + \frac{g}{2} (\Gamma_I \Gamma_J)_{\alpha\gamma} f^{ABC} A^B_I A^C_J \right] \lambda^A_\gamma$$

$$H = \frac{1}{2} E^A_I E^A_I + \frac{g^2}{4} f^{ABE} f^{CDE} A^A_I A^B_J A^C_K + \frac{ig}{2} f^{ABC} \lambda^A_\alpha (\Gamma_I)_{\alpha\beta} \lambda^B_\beta A^C_I ,$$

where $E^A_I = -i \partial / \partial A^A_I$, $A = 1, \ldots, \dim(G)$, $I, J = 1, \ldots, 9$; $f^{ABC}$ are structure constants; $\lambda^A_\alpha$ are Majorana spinors forming the $\mathbf{16}$–plet of $SO(9)$, they should be understood as
quantum fermion operators satisfying \( \{ \lambda^A_\alpha, \lambda^B_\beta \} = \delta^{AB} \delta_{\alpha\beta} \); \( \{ \Gamma_I, \Gamma_J \} = 2 \delta_{IJ} \).

The operators (1.1) and (1.2) act on the wave functions depending on \( A^A_I \) and holomorphic fermion variables \( \mu^\alpha_\tilde{\alpha} \), \( \tilde{\alpha} = 1, \ldots, 8 \). We may choose

\[
\begin{align*}
\mu^1_A &= (\lambda^A_1 + i\lambda^A_9)/\sqrt{2} \\
\ldots \\
\mu^8_A &= (\lambda^A_8 + i\lambda^A_{16})/\sqrt{2}
\end{align*}
\]

and

\[
\begin{align*}
\lambda^A_1 &= (\mu^A_1 + \bar{\mu}^A_1)/\sqrt{2} = \left( \mu^A_1 + \frac{\partial}{\partial \mu^A_1} \right)/\sqrt{2} \\
\ldots \\
\lambda^A_{16} &= -i(\mu^A_8 - \bar{\mu}^A_8)/\sqrt{2} = -i \left( \mu^A_8 - \frac{\partial}{\partial \mu^A_8} \right)/\sqrt{2}
\end{align*}
\]

The specifics of the \( \mathcal{N} = 4 \) theory (compared with \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) theories) is that the hamiltonian (1.2) does not conserve the fermion charge which is related to the fact that our holomorphic variables \( \mu^\alpha_\tilde{\alpha} \) do not form a representation of \( SO(9) \).

The dynamic variables \( A^A_I, E^A_I, \lambda^A_\alpha \) are dimensionless (if relating as we will do later the SQM model (1.1, 1.2) to a field theory placed in a small box, \( A^A_I \) are measured in the units of its inverse size \( L^{-1} \) and \( \lambda^A_\alpha \) are measured in the units of \( L^{-3/2} \)), but we have chosen not to rescale away the coupling constant \( g \). It sets up a characteristic energy scale \( E_{\text{char}} \approx g^{2/3} \).

The relation

\[
\{ Q_\alpha, Q_\beta \} = \delta_{\alpha\beta} H + \frac{g}{2} (\Gamma_I)_{\alpha\beta} A^A_I G^A
\]

holds where

\[
G^A = f^{ABC} \left( A^B_I E^C_I - \frac{i}{2} \lambda^B_\alpha \lambda^C_\alpha \right)
\]

is the Gauss law constraint. We are interested only in the gauge invariant states \( G^A|\Psi\rangle = 0 \) for which the second term in (1.4) vanishes, and we have the standard algebra of extended SQM. The dynamics of this theory (and more simple SQM theories with 8 and 4 real supercharges obtained by dimensional reduction of \( \mathcal{N} = 2 \) and \( \mathcal{N} = 1 \) SYM theories) was a subject of intense interest since the middle of eighties \([1, 2]\). As was first noted in \([2]\), the spectrum of the hamiltonian (1.2) is continuous and the band of delocalized states starts right from zero. The reason for that is very simple. The classical potential energy in the hamiltonian (1.2) goes to zero if

\[
f^{ABC} A^A_I A^B_J = 0
\]

for all \( I, J \). The condition (1.6) means that \( A^A_I t^A \) belong to the Cartan subalgebra. Up to a global gauge transformation,

\[
(A^A_I)_{\text{class. vac.}} = A^A_s,
\]

\( s = 1, \ldots, r \), where \( r \) is the rank of the gauge group. Back in 1982 Witten noticed that, in supersymmetric case, this valley is not lifted by quantum corrections \([3]\). As a result, the low energy wave functions tend to smear out along the valley. As the valley (or alias, the vacuum moduli space) (1.7) is not compact, the motion is infinite, the wave function is delocalized, and the spectrum is continuous.
One can make this statement more accurate, writing down the supercharges and hamiltonian describing the motion along the valley of slow variables in Born–Oppenheimer spirit. In the lowest order in the Born–Oppenheimer expansion parameter $1/(g|A|^3)$, the result is very simple \[3, 4\]

$$Q_{\alpha}^{\text{eff}} = \sum_{s=1}^{r} \frac{1}{\sqrt{2}} (\Gamma_I)_{\alpha\beta} \lambda_{\beta}^{s} E_{I}^{s},$$

$$H_{\text{eff}}^{\text{eff}} = \sum_{s=1}^{r} \frac{1}{2} E_{I}^{s} E_{I}^{s},$$ \hspace{1cm} (1.8)

if the orthonormal basis in the Cartan subalgebra is chosen (for clarity, the sum over $s$ is written explicitly). Thereby, the problem is reduced to the problem of free motion in the $(D - 1)r = (2N + 1)r$–dimensional flat space [with a certain discrete symmetry imposed; this symmetry will be discussed in details in the Appendix, and a detailed derivation of Eq.(1.8) will be given in Sect. 4]. The spectrum is obviously continuous.

The theory \[1.2\] is interesting by itself, but also because of its relations to brane dynamics. The hamiltonian \[1.2\] for the gauge group $SU(n)$ in the large $n$ limit just coincides with the mass operator of 2+1 supermembranes embedded in 9+1 - dimensional space \[3\]. The fact that the spectrum of \[1.2\] is continuous means that the supermembrane mass spectrum is continuous \[4, 6\]. The realization of this fact has quenched early attempts to build up a supermembrane theory (where supermembranes were treated as fundamental objects).

The revival of interest to the hamiltonian \[1.2\] was due to a recent discovery that on top of delocalized continuum spectrum states, the hamiltonian \[1.2\] enjoys also a normalized vacuum state. \[1\] The existence of such a state is very important for $D$–brane theory (in the modern approach where $D$–branes are not believed to be fundamental ingredients of the theory, but kind of solitons in the holy grail M–theory). In the following, we will not use the brane terminology, however, and will concentrate on studying the dynamics of the hamiltonian \[1.2\] as it is.

Originally, the existence of the normalized supersymmetric vacuum state was demonstrated when calculating carefully the Witten index

$$I_{W} = \lim_{\beta \rightarrow \infty} \text{Tr} \left\{ (-1)^F e^{-\beta H} \right\} = n_{B}^{0} - n_{F}^{0},$$ \hspace{1cm} (1.9)

for the hamiltonian $H$. For the systems where the spectrum is discrete and where all the states are localized, this is a rather straightforward method. As all the bosonic states with non-zero energy have their fermionic counterparts, the expression $\text{Tr} \left\{ (-1)^F e^{-\beta H} \right\}$ does not depend on $\beta$ in this case. One can present $\text{Tr} \left\{ (-1)^F e^{-\beta H} \right\}$ in the functional integral form and calculate it in the limit $\beta \rightarrow 0$ where the functional integral is reduced

\[1\] The effective hamiltonian in Eq.(1.8) does not enjoy such a normalized vacuum. There is no contradiction here because the supersymmetric vacuum state of the full hamiltonian \[1.2\] is localized in the region $g|A|^3 \sim 1$ which is just the region where the effective theory \[1.8\] makes no sense. An important remark, however, is that the existence of the normalized vacuum in the full theory \[1.2\] can be conjectured by analyzing the dynamics of the effective free theory \[1.8\]. We will discuss it in details in Sect. 5 of the paper.
to a finite–dimensional integral of \( \exp\{-\beta H_{cl}\} \) over the classical phase space \( \mathcal{I} \):

\[
I_W = \int \prod_n \frac{dx_n dp_n}{2\pi} \prod_a d\bar{\psi}_a \psi_a e^{-\beta H_{cl}(x_n,p_n; \bar{\psi}_a, \psi_a)}
\]  

(1.10)

For example, for the supersymmetric oscillator

\[
H = \frac{p^2}{2} + \omega^2 x^2 + \omega \bar{\psi} \psi,
\]

(1.11)

and

\[
I_W = \int \frac{dxdp}{2\pi} d\bar{\psi} \psi e^{-\beta H_{cl}(x,p; \bar{\psi}, \psi)} = 1
\]  

(1.12)

signalizing the presence of one bosonic vacuum state annihilated by the action of the supercharges and hamiltonian.

For systems with continuous spectrum, the situation is more intricate. Instead of the discrete sum

\[
\text{Tr} \left\{ (-1)^F e^{-\beta H} \right\} = \sum_n (-1)^F_n e^{-\beta E_n}
\]

(1.13)

we have continuous integrals involving boson and fermion spectral densities which, generally speaking, are not equal even though the hamiltonian is supersymmetric, and we cannot argue anymore that the supertrace \( \text{Tr}\{(-1)^F e^{-\beta H}\} \) is \( \beta \)-independent. We can write

\[
I_W = \text{Tr}_{\beta=\infty} = \text{Tr}_{\beta=0} + \int_0^\infty d\beta \frac{\partial}{\partial \beta} \text{Tr}_{\beta} \equiv I^p_W - I^d_W
\]

(1.14)

The first term here is called the “principal contribution” and the second term is known as the “deficit term”. In many cases, the second term is just zero even though the spectrum is continuous. The example of such “benign” system is the spectrum of massless Dirac operator on \( R^4 \) in the instanton background (as it is well known, the chiral symmetry of this problem can be presented as supersymmetry \( \text{[10]} \)). As the instanton field falls away rapidly at large distances, we have the continuum spectrum states with asymptotics of plane waves. But on top of that, we also have the normalized fermion zero modes. Their number is given by the Atiyah–Singer theorem, and the Atiyah–Singer index is nothing else as the Witten index in this particular problem. It can be presented in the form (1.14) where the deficit term is absent. The benign nature of the Dirac system is related to the fact that we can compactify \( R^4 \) on \( S^4 \) (so that the spectrum becomes discrete) while preserving the supersymmetry.

But it is not so for the problem under consideration. Both principal and deficit term contribute on equal footing here. Let us briefly comment first on the calculation of the principal term. To begin with, assume that the gauge group is \( SU(2) \) (the most simple case). One can show that the corresponding finite dimensional integrals (for \( \mathcal{N} = 1, \mathcal{N} = 2, \) and \( \mathcal{N} = 4 \) theories) have the form

\[
I_W = \frac{1}{8\pi^2} \left( \frac{\beta g^2}{2\pi} \right)^{3(2\mathcal{N}+1)/2} \int A_\mu^A d A_\mu \det \| i A_\mu^A \Gamma_{\mu} e^{ABC} \|
\]

\[
\exp \left\{ -\frac{\beta g^2}{4} e^{ABC} e^{CDE} A_\mu^A A_\nu^B A_\mu^C A_\rho^D \right\} = \frac{2^{2(\mathcal{N}-2)} \Gamma(\mathcal{N} - 1/2)}{\sqrt{\pi} \Gamma(\mathcal{N} + 1)}
\]

(1.15)
with \( \mu = 0, 1, 2, 3, \mu = 0, 1, \ldots, 5, \) and \( \mu = 0, 1, \ldots, 9 \) for \( \mathcal{N} = 1, \mathcal{N} = 2 \) and \( \mathcal{N} = 4, \) respectively. The integrals in the R.H.S. of Eq.(1.13) were first correctly calculated in Ref.[2]. However, the correct expressions (1.13) differ from the expressions quoted in Ref.[2] by the overall factor \( 1/4 \) in the case of \( \mathcal{N} = 1 \) theory, and by \( 1/8 \) in the case of \( \mathcal{N} = 2 \) and \( \mathcal{N} = 4 \) theories. Correspondingly, the correct results (found in [11]):

\[
(I_W^p)^{\mathcal{N}=1} = \frac{1}{4}, \quad (I_W^p)^{\mathcal{N}=2} = \frac{1}{4}, \quad (I_W^p)^{\mathcal{N}=4} = \frac{5}{4} \tag{1.16}
\]

differ from the results quoted in Refs.[2] by these factors.

The calculation of the principal contribution for more complicated groups is a rather intricate business. For higher unitary groups \( SU(n) \), it was done in recent [12]. The result is

\[
(I_W^p)^{\mathcal{N}=1} = (I_W^p)^{\mathcal{N}=2} = \frac{1}{n^2}, \quad (I_W^p)^{\mathcal{N}=4} = 1 + \sum_{m|n} \frac{1}{m^2} \tag{1.17}
\]

where the sum in the last formula runs over all integer divisors \( m \) of \( n \) including \( n \), but not including 1.

The results (1.16, 1.17) are fractional, but the number of supersymmetric normalizable vacuum states is, of course, integer. That means that in our case the “deficit term” cannot be zero. And it is not. A not so difficult calculation (which we will dwell upon in details later) displays

\[
(I_W^d)^{\mathcal{N}=1} = (I_W^d)^{\mathcal{N}=2} = \frac{1}{n^2}, \quad (I_W^d)^{\mathcal{N}=4} = \sum_{m|n} \frac{1}{m^2} \tag{1.18}
\]

for the \( SU(n) \) gauge group. Subtracting (1.18) from (1.17), we finally obtain

\[
(I_W)^{\mathcal{N}=1} = (I_W)^{\mathcal{N}=2} = 0, \quad (I_W)^{\mathcal{N}=4} = 1, \tag{1.19}
\]

i.e. the quantum mechanical systems obtained by the dimensional reduction of \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) supersymmetric Yang–Mills theories with \( SU(n) \) gauge group does not have a normalizable supersymmetric vacuum state at all, while the theory (1.2) has exactly one such state.

This is fine, but the method just outlined has two disadvantages. First, it is indirect and does not give a clue how the wave function of the normalized vacuum state looks like. Second, it is far from being obvious how to generalize this method to non-unitary groups. The calculation of the primary contribution is especially tricky. Already the paper [12] where the principal contribution was calculated for unitary groups was technically very

\[2\]This is not just an arithmetic error. The difference in normalization factor stems from different methods used when deriving (1.13). The authors of Ref.[1] implemented gauge invariance by imposing Gauss law constraint on the wave functions while the starting point in Ref.[2] was the hamiltonian with the constraints explicitly resolved on the classical level. The missing factors can be restored in this approach if taking into account the condition of discrete Weyl invariance for the wave functions. See Appendix for more details.
difficult. We have no idea how to generalize it to symplectic, orthogonal, or exceptional groups. What we are able to do is to calculate the deficit term for other groups. We will present these calculation in Sect. 4.

Our main message, however, is different. As has been observed in [13], the presence of the normalized vacuum state in $\mathcal{N} = 4$ theory can be established without coming to grips with difficult calculations of the integrals for Witten index. It suffices to deform the theory adding the mass term to the matter fields (we are thinking now of our theory in $\mathcal{N} = 1$ 4-dimensional terms where it involves the gauge multiplet and 3 chiral matter multiplets in the adjoint representation of the group). Establishing the supersymmetric vacua becomes now an almost trivial business of solving some simple algebraic equations. The number of quantum vacua just coincides with the number of (gauge–inequivalent) solutions of these classical equations. It turns out that for unitary groups, there is only one such solution. If the mass is large, the Born–Oppenheimer approximation works, and one can just write down the vacuum wave function explicitly. We are interested in the theory in the opposite limit $M \to 0$ where this cannot be done. But, by continuity, the normalizable vacuum state exists for any mass, however small it is. It is a natural hypothesis that the state remains to be normalizable also at the point $M = 0$. This hypothesis is confirmed by the indirect calculations of $I_W$.

The great advantage of this mass deformation method is that its generalization to higher groups is not difficult. It turns out that the problem is reduced to the problem of classification of the so called “distinguished” nilpotent elements of a complex simple Lie algebra the solution of which has been known to mathematicians for a long time. For higher symplectic [starting from $Sp(6)$], higher orthogonal [starting from $SO(8)$] and for all exceptional groups, the solution is not unique, and there are several supersymmetric normalized vacuum states.

In the next section, we will write down the equations determining the positions of the classical vacua in the large $M$ limit, solve them for the unitary groups, and discuss at some length the philosophy of this method. Sect. 3 is devoted to solving these equations for other groups. The final result for the counting of supersymmetric vacuum states presents the content of Theorem 6. In Sect. 4, we calculate the deficit term in Eq. (1.14) for some groups. In Sect. 5, we discuss a third way to deduce the existence of the normalized vacuum state(s) in $\mathcal{N} = 4$ SYM quantum mechanics: via studying the asymptotic solutions for the effective theory. We present a simple derivation for the asymptotic supersymmetric wave function in the $SU(2)$ case. Calculation of the principal contribution to the Witten index in the $SU(2)$ theory is the subject of the Appendix.

2 Mass deformation of the $\mathcal{N} = 4$ theory.

As we have mentioned before, the problem (1.1, 1.2) has exciting reverberations for D–branes and M–theory. However, the dynamics of this quantum mechanical model can be understood better if exploiting the other relation: the relation of (1.1, 1.2) and of the conventional 4–dimensional supersymmetric field theory. In 4–dimensional language, the variables $A_{1,2,3}^4$ present the zero Fourier harmonics of the gauge fields, while the variables
$A^4_{1,...,9}$ are associated with matter fields. Let us define

$$
\phi^A_1 = A^4_4 + i A^4_5, \quad \phi^A_2 = A^6_6 + i A^7_7, \quad \phi^A_3 = A^8_8 + i A^9_9
$$

(2.1)

These fields together with 12 (out of 16) components of of $\lambda^A_{\alpha}$ form 3 chiral matter multiplets $\Phi^A_f$ ($f = 1, 2, 3$ is the "flavor" index). In the matter sector, the lagrangian of the $\mathcal{N} = 4$ SYM theory presents the Wess–Zumino model with superpotential $\sim \epsilon_{fgh} f^{ABC} \Phi^A_f \Phi^B_g \Phi^C_h$. One can modify the superpotential adding the quadratic mass term:

$$
W^M = \frac{g}{6\sqrt{2}} \epsilon_{fgh} f^{ABC} \Phi^A_f \Phi^B_g \Phi^C_h - \frac{M}{2} \Phi^A_f \Phi^A_f
$$

(2.2)

The modified scalar field potential $U = |\partial W^M / \partial \phi^A_f|^2$ turns to zero when the F–terms vanish

$$
\epsilon_{fgh} f^{ABC} \phi^A_f \phi^B_g = \frac{2\sqrt{2}M}{g} \phi^C_h
$$

(2.3)

Our matter fields interact also with the gauge fields. Correspondingly, the $D$–term $\sim f^{ABC} \phi^A_f \phi^B_f$ is generated. In the vacuum, it also has to vanish. We have the equation system

$$
\epsilon_{fgh} f^{ABC} \phi^A_f \phi^B_g = C \phi^C_h, \\
f^{ABC} \phi^A_f \phi^B_f = 0
$$

(2.4)

$C = 2\sqrt{2}M/g$. Consider first the $SU(2)$ case. Besides the obvious solution $\phi = 0$, the system (2.4) enjoys a unique up to an overall gauge rotation solution [14]

$$
\phi^A_f = \frac{1}{2} C \delta^A_f
$$

(2.5)

The appearance of the Higgs average (2.3) breaks down the gauge invariance completely; all gauge fields and their superpartners acquire mass of order $M$. As the solution (2.3) is unique, the same applies to the matter fields irrespectively of whether we are at the vicinity of classical vacua with $\langle \phi \rangle_{\text{vac}} \sim C$ or $\langle \phi \rangle_{\text{vac}} = 0$. When mass is large $M \gg E_{\text{char}} \sim g^{2/3}$, the state (2.3) is separated from the sector $\langle \phi \rangle_{\text{vac}} = 0$ by a high barrier. In the limit $M \to \infty$, this barrier becomes unpenetrable, and if in the morning we wake up in the sector with $\langle \phi \rangle_{\text{vac}} \sim 0$, we are going to stay there also by the end of the day. The presence of heavy matter fields would not be felt and the dynamics would be the same as in $\mathcal{N} = 1$ 4–dimensional SYM theory.

On the other hand, when mass is small, the barrier disappears and the new vacuum state overlaps essentially with the conventional vacuum sector. In the limit $M \to 0$, the state (2.3) goes over into the celebrated localized supersymmetric vacuum state of the hamiltonian (1.2).

Let us emphasize that the final conclusion that yes, there is such a localized supersymmetric state is valid irrespectively of whether we are thinking in the language of the SQM system (1.1, 1.2) or in the language of the associated 4–dimensional field theory. In
the former case, one should speak about a deformation of the Hamiltonian (1.2), leaving only 4 of 16 real supercharges $Q_\alpha$ conserved. Even for non-zero $M$, the system involves the continuum spectrum associated with the (4-dimensional) gauge potentials $A_i^a$. For large $M$, the localized state is well separated from the continuum spectrum states but, in the limit $M \to 0$, it is kind of mixed up with them making the analysis difficult. Still, the true index, the number of the normalized vacuum states does not depend on $M$ and for the $SU(2)$ gauge group, is equal to 1.

If we are thinking in terms of 4-dimensional field theory, the most convenient way to treat it is to put it in a finite spatial volume. That makes the spectrum discrete and just removes all uncertainties connected with nonzero “deficit contribution” in Eq.(1.14). If you like, going from quantum mechanics to field theory defined in the box presents a convenient infrared regularization making the motion finite and preserving supersymmetry [15]. It plays the same role as the compactification $R^4 \to S^4$ for the Dirac operator in gauge field background. The only difference is that, for the problem (1.2), such a regularization brings about a lot of (infinitely many) new degrees of freedom, but as we know since [3] how to handle them in case when the spatial box is small, it is not a real problem.

In the field theory with large mass, we have one extra state at large values of Higgs average, and also two conventional vacuum states of $\mathcal{N} = 1$ SYM theory coming from the region $\phi \sim 0$ where the heavy matter fields decouple. In this approach, the answer $I_W = 1$ is not obtained as a difference $I_W = I_W^0 - I_W^1 = 5/4 - 1/4$, but rather as the difference $I_W = 3 - 2$, with 3 being the Witten index of the $\mathcal{N} = 4$ SYM field theory while 2 is the Witten index of $\mathcal{N} = 1$ SYM field theory. This reasoning emphasizes again that the separate terms like $5/4$, $1/4$ or $3,2$ have no particular physical meaning. Only the total answer $I_W = 1$ is meaningful.

One more remark is in order. As we see, when going from $M = \infty$ to $M = 0$, the new vacuum state appears in the physical spectrum, and it comes from infinity of configuration space. This phenomenon is well known. This happens e.g. in the models of Wess–Zumino type when the asymptotics of superpotential is changed. Various gauge SUSY models involving this phenomenon have been recently constructed. One example is the $\mathcal{N} = 1$ theory with the $G_2$ gauge group where superpotential for matter fields is modified by adding a cubic term [16]. Another classic example is $\mathcal{N} = 2$ supersymmetric QCD [17]. Some pecularity of $\mathcal{N} = 4$ theory is that we do not add here any unusually high power in superpotential, but just a quadratic mass term (while the cubic term was already there). It was conjectured in [3] that the Witten index in supersymmetric gauge theories with non-chiral matter content (so that the mass terms can be added) is the same as in the pure SYM theory. We see that it is not true in this case. But it is true e.g. in the $\mathcal{N} = 2$ theory where we have only one scalar field and no cubic superpotential.

Let us discuss now higher unitary groups. We have to solve again the equation system (2.4), but with an additional requirement: the Higgs average obtained should break the gauge invariance completely and give mass to all gauge fields (otherwise, the wave functions would smear out along the flat directions corresponding to the remaining massless fields, and the state would not be localized.

In mathematical language, that means that we are looking for the triples $\phi_f = \phi^A t^A$ belonging to our Lie algebra $\mathfrak{g}$ which satisfy the relations (2.4) and whose centralizer
is trivial (i.e. there is no such $g \in \mathfrak{g}$ that $[g, \phi_f] = 0$ for all $f = 1, 2, 3$). As was noted in Ref. [14], this problem is reduced to the mathematical problem of classifying the embeddings $su(2) \subset \mathfrak{g}$ with trivial centralizer factorized over the action of the complexified group $G$.

Let us prove it. Let $G$ be the complex connected simple Lie group with trivial center such that $\mathfrak{g}$ is its Lie algebra. The group $G$ acts on $\mathfrak{g}$ faithfully via the adjoint representation. Let $K$ be a maximal compact subgroup of $G$. Then there exists on $\mathfrak{g}$ a unique (up to a positive factor) positive definite Hermitian form such that

$$K = \{ g \in G | g^\dagger = g^{-1} \} .$$

Let

$$P = \{ g \in G | g^\dagger = g \text{ and } g \text{ is positive definite} \} .$$

Then $K \cap P = \{ 1 \}$ and one has the following well-known polar decomposition

$$G = KP .$$

In other words, in any representation, the matrix representing the element of the complex group $G$ can be written as a product of a unitary and a positively defined Hermitian matrix.

**Lemma 1.** If $k \in K$, $p \in P$, and $kp^{-1} \in K$, then $pk = kp$.

**Proof.** We have:

$$pk = k_1 p$$

for some $k_1 \in K$. Applying $^\dagger$ to both sides of (2.7), we get

$$pk_1 p^{-1} = k$$

Comparing (2.7) and (2.8), we get

$$p^2 k p^{-2} = k$$

Thus, $p^2$ commutes with $k$, but since $p$ is positive definite Hermitian, we deduce that $p$ commutes with $k$ as well.

**Corollary 1.** If $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are two real subalgebras of $Lie K \subset \mathfrak{g}$ isomorphic to $su(2)$ such that they are conjugate by an element $g \in G$, then they are conjugate by an element of $K$.

**Proof.** We write $g = kp$, where $k \in K$, $p \in P$. Then

$$(Ad p)\mathfrak{a}_1 = (Ad k^{-1})\mathfrak{a}_2 \subset Lie K$$

Denoting by $A_i$ the subgroup of $K$ whose Lie algebras are $\mathfrak{a}_i$, we deduce:

$$(Ad p)A_1 = (Ad k^{-1})A_2 \subset K .$$

It follows from Lemma 1 that $(Ad p)A_1 = A_1$, hence $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are conjugate by $k$.

**Theorem 1.** Triples of elements $(T_1, T_2, T_3)$ of $\mathfrak{g}$ satisfying the equations
\((i) [T_j, T_k] = i\epsilon_{jkl}T_l\)
\((ii) [T_j, T_j^\dagger] = 0\)
are conjugate by \(G\) if and only if they are conjugate by \(K\).

**Proof.** Condition \((i)\) means that \(T_j\) form a basis of \(su(2) \subset \mathfrak{g}\). In view of Corollary 1, it suffices to show that conditions \((i)\) and \((ii)\) imply that \(T_j = T_j^\dagger\).

We may assume that \(T_j \subset \text{Lie } K\) (since the maximal compact subgroups of \(G\) are conjugate). Thus, we have a homomorphism \(\phi : \mathfrak{sl}(2) \to \mathfrak{g}\) such that \(\phi[su(2)] \subset K\). Hence it suffices to show that equations \((i)\) and \((ii)\) on three elements \(T_j\) of \(\mathfrak{sl}(2)\) imply that \(T_j \in su(2)\).

Let \(\sigma_1, \sigma_2,\) and \(\sigma_3\) be an orthonormal basis of \(su(2)\). We have for some \(g \in SL(2) :\)

\[ gT_jg^{-1} = \sigma_j, \quad j = 1, 2, 3. \]

Writing \(g = kp\) (polar decomposition) and replacing \(\sigma_j\) by \(k^{-1}\sigma_jk\), equation \((ii)\) gives:

\[ \sum_j p^2(\sigma_jp^{-2}\sigma_j)p^2 = \sum_j \sigma_jp^2\sigma_j, \quad (2.10) \]

where \(p^2\) is a positive definite Hermitean matrix. Choosing a suitable basis, we may assume that \(\sigma_j\) are the Pauli matrices. A direct calculation shows that \((2.10)\) implies that \(p = 1\).

When \(G = SU(n)\), there is only one such embedding (we will prove it rigourously in the next section). It is sufficient to write down the generators of \(SU(2)\) in the representation with the spin \(j = (n - 1)/2\) and treat them as the elements of the \(su(n)\) algebra in the fundamental representation. For example, for \(su(3)\), the non–trivial triple is

\[
\phi_1 + i\phi_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \phi_1 - i\phi_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \phi_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (2.11)
\]

The existence and uniqueness of the solution means that the \(\mathcal{N} = 4\) theory with \(SU(n)\) gauge group has one and only one normalized vacuum state in agreement with \((1.19)\).

### 3 Distinguished \(sl(2)\) subalgebras in simple Lie algebras.

The problem is reduced to finding all the solutions of the equation system \((2.4)\) for an arbitrary gauge group. It is not a trivial problem but, fortunately, its solution can be easily derived from related problems that have actually already been solved by mathematicians.

Let us first introduce some basic notations and definitions. Let \(\mathfrak{g}\) be a complex simple Lie algebra and \(G\) be the corresponding complexified group. Choose a Cartan subalgebra \(\mathfrak{h}\) in \(\mathfrak{g}\). A convenient choice of basis in \(\mathfrak{g}\) is a union of the basis of \(\mathfrak{h}\) and the **root vectors** \(e_\alpha : [h, e_\alpha] = \alpha(h)e_\alpha\) for any \(h \in \mathfrak{h}\), \(\alpha\) (the linear forms on the Cartan subalgebra) being the **roots**. \(\Box\) For any root \(\alpha, -\alpha\) is also a root, and the whole set of roots \(\Delta\) can be

\(^3\)In other notation, \([h_i, e_\alpha] = \alpha_i e_\alpha\) for a particular basis \(h_i\) in \(\mathfrak{h}\).
decomposed into a set of positive roots \( \Delta_+ \) and a set of negative roots \( \Delta_- \). If \(-\alpha \in \Delta_-\), we will use the notation \( f_\alpha \) for \( e_{-\alpha} \). The commutator \([e_\alpha, f_\alpha] \equiv \alpha^\vee \) lies in the Cartan subalgebra. With the standard choice of normalization for the root vectors, \([\alpha^\vee, e_\alpha] = 2e_\alpha \) and \([\alpha^\vee, f_\alpha] = -2f_\alpha \), \( \alpha^\vee \) is called the coroot. For any coroot \( \alpha^\vee \), the identity

\[
\exp\{2\pi i\alpha^\vee\} = 1 \in G
\]

holds. For any \( \alpha, \beta \in \Delta \) with \( \alpha + \beta \neq 0 \), \([e_\alpha, e_\beta] \) is proportional to \( e_{\alpha+\beta} \) with non-zero coefficient if \( \alpha + \beta \in \Delta \), and \([e_\alpha, e_\beta] = 0 \) otherwise. A set of \( r \) simple roots \( \alpha^{(i)} \) and the corresponding simple root vectors \( e_i, f_i \) can be chosen so that all other root vectors are obtained from \( e_i, f_i \) by a number of subsequent commutations. The corresponding coroots \( \alpha^\vee_{(i)} \equiv h_i \) present a convenient basis in the Cartan subalgebra. The set \( \{e_i, f_i, h_i\} \) is called the Chevalley generators. An element \( \omega_i \in \mathfrak{h} \) commuting with all but one pair of simple root vectors, so that

\[
[\omega_i, e_j] = \delta_{ij} e_j, \quad [\omega_i, f_j] = -\delta_{ij} f_j,
\]

is called fundamental coweight.

An element \( x \) of a Lie algebra \( \mathfrak{g} \) is called nilpotent (resp. semisimple) if \( ad \ x \equiv [x, \cdot] \) is a nilpotent (resp. diagonalizable) operator on \( \mathfrak{g} \). A subalgebra of \( \mathfrak{g} \) is called reductive if it is a direct sum of simple subalgebras and a torus (i.e. commutative subalgebra consisting of semisimple elements).

The following result is due to Morozov, Jacobson, Dynkin, and Kostant. Its proof can be found in [18, 19, 20].

**Theorem 2.** Let \( e \) be a non-zero nilpotent element of a simple complex Lie algebra \( \mathfrak{g} \). Then

(a) There exist elements \( h, f \in \mathfrak{g} \) such that

\[
[h, e] = e, \quad [h, f] = -f, \quad [e, f] = h,
\]

i.e. \( C e + Ch + Cf \) is isomorphic to the 3-dimensional simple algebra \( sl(2, \mathbb{C}) \). In this case, the element \( f \) is nilpotent and element \( h \) is semisimple with integer or half-integer eigenvalues.

(b) If \( C e + Ch' + Cf' \) is another 3-dimensional simple algebra containing \( e \), then there exists \( g \in G \) such that \( g(e) = e, \ g(h) = h', \ g(f) = f' \).

(c) There is a bijective correspondence between conjugacy classes of non-zero nilpotent elements of \( \mathfrak{g} \) and conjugacy classes of 3-dimensional simple subalgebras of \( \mathfrak{g} \).

It follows from Theorem 2a that one has the eigenspace decomposition with respect to \( ad \ h \):

\[
\mathfrak{g} = \oplus_{j \in \mathbb{Z}/2} \mathfrak{g}_j, \quad [\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \quad e \in \mathfrak{g}_1, \ f \in \mathfrak{g}_{-1}
\]

In other words, \([h, x] = jx \) if \( x \in \mathfrak{g}_j \). Let \( \mathfrak{g}_+ = \oplus_{j > 0} \mathfrak{g}_j, \ \mathfrak{g}_- = \oplus_{j < 0} \mathfrak{g}_j \). The proof of the following result may be found in [21].

**Theorem 3.**

(a) The centralizer of \( e \) (resp. \( f \)) in \( \mathfrak{g} \) is a sum of a reductive subalgebra \( m^+ \) of \( \mathfrak{g}_0 \) and a subalgebra of \( \mathfrak{g}_+ \) (resp. \( \mathfrak{g}_- \)) consisting of nilpotent elements (of \( \mathfrak{g} \)).
(b) \([\mathfrak{m}^+, e] = \mathfrak{g}_1, [\mathfrak{m}^-, f] = \mathfrak{g}_{-1}\). In other words, the \(M^+\)-orbit of \(e\) (resp. \(M^-\)-orbit of \(f\)) in \(\mathfrak{g}_1\) (resp. \(\mathfrak{g}_{-1}\)) is open and dense.

(c) \(ad\ e: \mathfrak{g}_0 \to \mathfrak{g}_1\) and \(ad\ f: \mathfrak{g}_0 \to \mathfrak{g}_{-1}\) are surjective linear maps. In particular, \(\dim \mathfrak{g}_0 \geq \dim \mathfrak{g}_1\).

A nilpotent element \(e\) is called distinguished if \(\mathfrak{m}^+ = 0\). The following result is straightforward. Its proof may be found in [18, 21].

**Theorem 4.** A nilpotent element \(e\) is distinguished if one of the following equivalent properties holds:

(i) The centralizer of \(e\) in \(\mathfrak{g}\) lies in \(\mathfrak{g}_+\), i.e. it consists of nilpotent elements.

(ii) \(\dim \mathfrak{g}_0 = \dim \mathfrak{g}_1\).

(iii) \(\dim \mathfrak{g}_0 = \dim \mathfrak{g}_{-1}\).

A 3-dimensional simple subalgebra of \(\mathfrak{g}\) is called distinguished if its (unique up to conjugacy) non-zero nilpotent element is distinguished.

**Theorem 5.** A 3-dimensional simple subalgebra \(\mathfrak{a}\) of \(\mathfrak{g}\) is distinguished if and only if its centralizer in \(\mathfrak{g}\) is zero.

**Proof.** Let \(\mathfrak{a}\) be a distinguished subalgebra of \(\mathfrak{g}\) and let \(e\) be the corresponding (distinguished) nilpotent element. But the centralizer \(C(\mathfrak{a})\) of \(\mathfrak{a}\) in \(\mathfrak{g}\) is a reductive subalgebra which lies in the centralizer of \(e\), which consists of nilpotent elements due to Theorem 3a, hence \(C(\mathfrak{a}) = 0\).

Conversely, let \(\mathfrak{a} = C e + C h + C f\) be a 3-dimensional simple subalgebra of \(\mathfrak{g}\) with zero centralizer. Then with respect to the adjoint representation of \(\mathfrak{a}\) in \(\mathfrak{g}\), \(\mathfrak{g}\) decomposes into a direct sum of non-trivial irreducible submodules \(V_i\) such that \(\dim(V_i \cap \mathfrak{g}_j) = 1\) if \(|j| \leq s\) and \(s - j \in \mathbb{Z}\), and \(\dim(V_i \cap \mathfrak{g}_j) = 0\) otherwise [see (3.3)]. It follows that \(\dim \mathfrak{g}_0 = \dim \mathfrak{g}_1 = \dim \mathfrak{g}_{-1}\), hence \(e\) is distinguished by Theorem 4.

Choose Chevalley generators \(e_i, h_i, f_i\) \((i = 1, \ldots, r)\) of \(\mathfrak{g}\). The Dynkin diagram with dots on some of its nodes is called marked. Such a marking defines a \(\mathbb{Z}\)-gradation \(\mathfrak{g} = \bigoplus_j \mathfrak{g}_j\) if we let \(\deg e_i = -\deg f_i = 1\) if the \(i\)-th node is marked, and \(\deg e_i = \deg f_i = 0\) otherwise.

A marking is called distinguished if

\[\dim \mathfrak{g}_0 = \dim \mathfrak{g}_1\]  \hspace{1cm} (3.4)

According to Dynkin [18] (see also [21]), one has a bijective correspondence between conjugacy classes of distinguished nilpotent elements of \(\mathfrak{g}\) and distinguished markings of the Dynkin diagram of \(\mathfrak{g}\). Namely, given a distinguished nilpotent element \(e\) of \(\mathfrak{g}\), we construct a \(\mathbb{Z}\)-gradation (3.3) of \(\mathfrak{g}\) by \(ad\ h\) and choose a set of positive roots \(\Delta_+\) such that \([h, e_\alpha] \equiv \frac{1}{2} \alpha(h) e_\alpha\) if \(\alpha \in \Delta_+\). It turns out that \(\alpha(h) = 0\) or 2 if \(\alpha\) is a simple root, hence we get a \(\mathbb{Z}\)-gradation of \(\mathfrak{g}\) corresponding to a marked Dynkin diagram. Conversely, given such a gradation, we pick \(e \in \mathfrak{g}_1\) such that \([\mathfrak{g}_0, e] = \mathfrak{g}_1\) and take \(h \in \mathfrak{g}_0\) such that \(ad\ h\) defines this gradation. Due to (3.4), there exists a unique \(f \in \mathfrak{g}_{-1}\) such that \([e, f] = h\) giving a 3-dimensional simple subalgebra with zero centralizer.

**Example 1.** Look again at Eq. (2.11) defining the distinguished \(sl(2)\) subalgebra in the \(sl(3)\) algebra. We have

\[\phi_1 + i \phi_2 = e_\alpha + e_\beta \equiv e, \quad \phi_1 - i \phi_2 = f_\alpha + f_\beta \equiv f, \quad \phi_3 = \alpha'' + \beta'' \equiv h,\]  \hspace{1cm} (3.5)
where $\alpha$ and $\beta$ are two simple roots. The gradation defined by $h$ involves: (i) $g_0$ which coincides in this case with the Cartan subalgebra; (ii) $g_1$ (resp. $g_{-1}$) with the basis $e_\alpha$, $e_\beta$ (resp. $f_\alpha$, $f_\beta$); and (iii) $g_2$ (resp. $g_{-2}$) with the basis $e_{\alpha+\beta}$ (resp. $f_{\alpha+\beta}$). Obviously, the condition (3.4) is satisfied.

**Example 2.** Consider an arbitrary simple Lie algebra $g$. Choose an element $\rho \in h$ such that $[\rho, e_i] = e_i$ for all positive simple roots $e_i$. The element $\rho$ defines the canonical gradation of the Lie algebra (corresponding to the Dynkin diagram with all nodes marked) such that $g_0$ is the Cartan subalgebra; the basis of $g_1$ (of $g_{-1}$) is the system of positive (negative) simple root vectors; $g_2$ (resp. $g_{-2}$) is spanned by the root vectors of level 2 (resp. $-2$), etc. Obviously, $\dim g_0 = \dim g_1 = \dim g_{-1} = r$.

Let $\rho = \sum_{i=1}^{r} b_i h_i$ where $h_i$ are the simple coroots; then $b_i$ are positive numbers. The triple

$$e = \sum_{i=1}^{r} e_i \sqrt{b_i}, \quad f = \sum_{i=1}^{r} f_i \sqrt{b_i}, \quad h = \rho \quad (3.6)$$

form the distinguished $sl(2)$ subalgebra. In the $sl(n)$ case, Eq. (3.6) is reduced to the known distinguished embedding described at the end of the previous section.

We see thereby that, for any $g$, a solution of the equations (2.4) exists which provides us with at least one normalized vacuum state for any gauge group $G$. For most of simple groups, however, the solution is not unique. The simplest group involving more than one supersymmetric vacuum is the group $G_2$.

**Example 3.** The system of roots of $G_2$ is depicted in Fig.1a, the system of corresponding coroots in Fig.1b, and the Dynkin diagram is drawn in Fig. 1c. We have put the dot on the long simple root $\alpha$, but not on the short root $\beta$ (Here and in the following short roots will be denoted by smaller circles). Thereby, a certain non–trivial marking of the Dynkin diagram is defined. The corresponding $\mathbb{Z}$-gradation involves:

- $g_0$ with the basis $\alpha^\vee, \beta^\vee, e_\beta$, and $f_\beta$
- $g_1$ with the basis $e_\alpha, e_{\alpha+\beta}, e_{\alpha+2\beta}$, and $e_{\alpha+3\beta}$
- $g_{-1}$ with the basis $f_\alpha, f_{\alpha+\beta}, f_{\alpha+2\beta}$, and $f_{\alpha+3\beta}$
- $g_2$ with the basis $e_{2\alpha+3\beta}$
- $g_{-2}$ with the basis $f_{2\alpha+3\beta}$

The condition (3.4) is satisfied, and hence the marking in Fig. 1b is distinguished.

Let us construct now the corresponding distinguished $sl(2)$ subalgebra, the distinguished triple $(e, f, h)$. First, let us find the required element of the Cartan subalgebra

$$h : \quad [h, e_\alpha] = e_\alpha, \quad [h, e_\beta] = 0 \quad (3.8)$$

so that the gradation (3.7) is realized by the action of $ad h$. This is just the fundamental coweight corresponding to the node $\alpha$ of the Dynkin diagram. In our case, $h = (2\alpha + 4)$. The coroot corresponding to a long coroot is short and the coroot corresponding to a short coroot is long.
Figure 1:  

a) root system,  
b) coroot system, and  
c) non–trivial distinguished marked Dynkin diagram for the group $G_2$. 

3β\vee = 2α\vee + β\vee. To find explicitly the elements e, f of the triple, write e as a generic element of \(g_1\)

\[ e = a_1 e_\alpha + a_2 e_{\alpha+\beta} + a_3 e_{\alpha+2\beta} + a_4 e_{\alpha+3\beta}, \]

choose

\[ f \in g_{-1} = \bar{a}_1 f_\alpha + \bar{a}_2 f_{\alpha+\beta} + \bar{a}_3 f_{\alpha+2\beta} + \bar{a}_4 f_{\alpha+3\beta}, \]

and impose the requirement \([e, f] = h\). Substituting here the standard commutators

\[
\begin{align*}
[e_{\alpha+\beta}, f_{\alpha+\beta}] &= (\alpha + \beta)\vee = 3\alpha\vee + \beta\vee \\
[e_{\alpha+2\beta}, f_{\alpha+2\beta}] &= (\alpha + 2\beta)\vee = 3\alpha\vee + 2\beta\vee \\
[e_{\alpha+3\beta}, f_{\alpha+3\beta}] &= (\alpha + 3\beta)\vee = \alpha\vee + \beta\vee \\
[e_{\alpha+\beta}, f_\alpha] &= \frac{1}{2}[e_{\alpha+2\beta}, f_{\alpha+\beta}] = [e_{\alpha+3\beta}, f_{\alpha+2\beta}] = -e_\beta,
\end{align*}
\]

we obtain 3 equations for 4 complex parameters \(a_i\):

\[
\begin{align*}
|a_1|^2 + 3|a_2|^2 + 3|a_3|^2 + |a_4|^2 &= 2 \\
|a_2|^2 + 2|a_3|^2 + |a_4|^2 &= 1 \\
a_2\bar{a}_1 + 2a_3\bar{a}_2 + a_4\bar{a}_3 &= 0
\end{align*}
\]

(3.10)

Different solutions to this equation system are related to each other by conjugation. The convenient choice is \(a_1 = a_4 = 1, a_2 = a_3 = 0\) which gives the triple

\[ e = e_\alpha + e_{\alpha+3\beta}, \quad f = f_\alpha + f_{\alpha+3\beta}, \quad h = 2\alpha\vee + \beta\vee \]

(3.11)

The distinguished \(sl(2)\) subalgebra of the Lie algebra of \(G_2\) with the basis (3.11) is not equivalent by conjugation to the universal subalgebra (3.6). Therefore, the \(N = 4\) supersymmetric Yang–Mills quantum mechanics with the \(G_2\) gauge group has two different normalized supersymmetric vacuum states.

**Example 4.** Consider the marked Dynkin diagram for the \(Sp(6)\) group drawn in Fig. 2. The corresponding \(Z\)-gradation involves:

\[
\begin{align*}
g_0 & \text{ with the basis } \alpha\vee, \beta\vee, \gamma\vee, e_\beta, f_\beta, \\
g_1 & \text{ with the basis } e_\alpha, e_{\alpha+\beta}, e_\gamma, e_{\beta+\gamma}, e_{2\beta+\gamma}, \\
g_2 & \text{ with the basis } e_{\alpha+\beta+\gamma}, e_{\alpha+2\beta+\gamma}, \\
g_3 & \text{ with the basis } e_{2\alpha+2\beta+\gamma},
\end{align*}
\]

(3.12)
and the subalgebras $g_{-1}$, $g_{-2}$, $g_{-3}$ spanned by the corresponding negative root vectors. The condition (3.4) is satisfied, and hence the marking in Fig. 2 is distinguished.

The element $h$ of our distinguished triple realizing the gradation (3.12) is the sum of the fundamental coweights corresponding to the marked nodes:

$$h = \omega_\alpha + \omega_\gamma = \frac{3}{2} \alpha^\vee + 2\beta^\vee + \frac{5}{2} \gamma^\vee$$  (3.13)

The elements $e \in g_1$ and $f \in g_{-1}$ can be found in the same way as in the previous example. We have a system of 4 equations for 5 complex coefficients. One of the solutions has the form

$$e = \sqrt{\frac{3}{2}} e_\alpha + \frac{1}{\sqrt{2}} e_\gamma + \sqrt{2} e_{2\beta+\gamma}$$
$$f = \sqrt{\frac{3}{2}} f_\alpha + \frac{1}{\sqrt{2}} f_\gamma + \sqrt{2} f_{2\beta+\gamma}.$$  (3.14)

All other solutions of this equation system are equivalent to Eq. (3.14) by conjugation. The triples (3.14) and (3.6) present two inequivalent by conjugation distinguished sl(2) subalgebras of the Lie algebra of $Sp(6)$. This gives two different normalized supersymmetric vacuum states.

**Example 5.** Consider the marked Dynkin diagram for the $SO(8)$ group drawn in Fig. 3. The corresponding $\mathbb{Z}$–gradation involves:

- $g_0$ with the basis $\alpha^\vee$, $\beta^\vee$, $\gamma^\vee$, $\delta^\vee$, $e_\delta$, $f_\delta$
- $g_1$ with the basis $e_\alpha$, $e_\beta$, $e_\gamma$, $e_{\alpha+\delta}$, $e_{\beta+\delta}$, $e_{\gamma+\delta}$
- $g_2$ with the basis $e_{\alpha+\beta+\delta}$, $e_{\gamma+\beta+\delta}$, $e_{\alpha+\gamma+\delta}$
- $g_3$ with the basis $e_{\alpha+\beta+\gamma+\delta}$, $e_{\alpha+\beta+\gamma+2\delta}$

(3.15)

and the corresponding subalgebras $g_{-1}$, $g_{-2}$, $g_{-3}$. We have $\dim(g_0) = \dim(g_{\pm 1}) = 5$, and hence the marking in Fig. 3 is distinguished.

The distinguished triple can be constructed along the same lines as in the previous example. It is (up to a conjugation):

$$h = 2(\alpha^\vee + \beta^\vee + \gamma^\vee) + 3\delta^\vee,$$
$$e = e_\alpha + e^{\pi/3} e_\beta + e^{2\pi/3} e_\gamma + e_{\alpha+\delta} + e^{-\pi/3} e_{\beta+\delta} + e^{-2\pi/3} e_{\gamma+\delta}$$
$$f = f_\alpha + e^{-\pi/3} f_\beta + e^{-2\pi/3} f_\gamma + f_{\alpha+\delta} + e^{\pi/3} f_{\beta+\delta} + e^{2\pi/3} f_{\gamma+\delta}.$$  (3.16)

This triple together with the universal triple (3.6) gives us two different normalized supersymmetric vacuum states.

The full classification of distinguished markings for all algebras was done in Refs. [18, 21]. Translating it into our physical language gives immediately the following result:

**Theorem 6.** The number of inequivalent by conjugation solutions of Eq. (2.4) and hence the number $\#_{\text{vac}}[G]$ of different normalized supersymmetric vacua in the theory (1.2) with gauge group $G$ is

(a) $\#_{\text{vac}}[SU(n)] = 1$. 

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Figure 3: Non–trivial distinguished marked Dynkin diagram for the group $SO(8)$.

(b) $\#_{\mathrm{vac}}[Sp(2r)]$ coincides with the number of partitions of $r$ into distinct parts. 
(c) $\#_{\mathrm{vac}}[SO(n)]$ coincides with the number of partitions of $n$ into distinct odd parts.
(d) $\#_{\mathrm{vac}}[G_2] = 2$, $\#_{\mathrm{vac}}[F_4] = 4$, $\#_{\mathrm{vac}}[E_6] = 3$, $\#_{\mathrm{vac}}[E_7] = 6$, and $\#_{\mathrm{vac}}[E_8] = 11$.

A nilpotent element of $sp(n)$ [resp. $so(n)$] is distinguished iff, viewed as an element of $gl(n)$ it can be conjugated to a Jordan form with distinct even (resp. odd) sizes of Jordan blocks, and this Jordan form completely determines the conjugacy class. Denote by $s$ the number of these blocks; $s$ just coincides with the number of nonzero distinct parts in the partition of $r = n/2$ [resp. $n$].

Let $\bar{G} = G/Z$ ($Z$ — center of $G$) be the adjoint group. Let $sl(2) = Ce + Ch + Cf \subset g$ and let $\bar{G}_{e,h,f}$ be the centralizer of this $sl(2)$ in $\bar{G}$. By a theorem of Kostant [20], $\bar{G}_{e,h,f}$ is the maximal reductive subgroup of $\bar{G}_e$ (the centralizer of $e$ in $\bar{G}$). The $sl(2)$ is distinguished if and only if the group $\bar{G}_{e,h,f}$ is a finite group isomorphic to the group of components of $\bar{G}_e$. The groups $\bar{G}_{e,h,f}$ are always trivial in the $sl(n)$ case and are isomorphic to $\mathbb{Z}_2^{s-1}$ in the $sp(2r)$ and $so(2r + 1)$ cases and to $\mathbb{Z}_2^{s-2}$ in the $so(2r)$ case, where $s$ is the number of Jordan blocks of the nilpotent elements (see above). The nilpotent element $e$ in the universal distinguished triple (3.6) consists of just one block for $sp(2r)$ and $so(2r + 1)$ and of two blocks for $so(2r)$, and $\bar{G}_{e,h,f}$ is always trivial.

Let us illustrate it in the $Sp(6)$ example. $Sp(6)$ is a subgroup of $SU(6)$ leaving invariant the form $\psi_\alpha C^{\alpha\beta} \chi_\beta$ where $\psi_\alpha$ and $\chi_\alpha$ are some 6–plets of $SU(6)$ and the antisymmetric

---

5 For $r = 1$ or $r = 2$, we have only one universal solution (3.6). For $r = 3 = 3 = 2 + 1$, we have two inequivalent solutions, for $r = 6 = 6 = 5 + 1 = 4 + 2 = 1 + 2 + 3$, we have four solutions, etc.

6 The second solution appears starting from $n = 8 = 7 + 1 = 5 + 3$ and $n = 9 = 9 = 1 + 3 + 5$. 

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symplectic matrix $C$ can be chosen in the form

$$C = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{3.17}$$

Then the coroots of $Sp(6)$ are represented by the diagonal matrices $6 \times 6$:

$$\alpha^\vee = \text{diag}(1, -1, 0, 0, 1, -1),$$
$$\beta^\vee = \text{diag}(0, 1, -1, -1, 1, 0),$$
$$\gamma^\vee = \text{diag}(0, 0, 1, -1, 0, 0). \tag{3.18}$$

The triple (3.13), (3.14) acquires the form

$$h = \frac{1}{2}\text{diag}(3, 1, 1, -1, -1, -3), \quad e = \frac{1}{\sqrt{2}}\begin{pmatrix}
0 & \sqrt{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sqrt{3} \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad f = e^T \tag{3.19}$$

Indeed, we see that the nilpotent element $e$ viewed as a $6 \times 6$ matrix involves two Jordan blocks:

$$J_1 = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}, \quad J_2 = \begin{pmatrix}
0 & \sqrt{3} & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & -\sqrt{3} \\
0 & 0 & 0 & 0
\end{pmatrix}. \tag{3.20}$$

The block $J_1$ is formed by the “center” of the matrix (the columns and rows 3,4) and the block $J_2$ — by its “periphery”. The triple (3.13) corresponds to the partition $r = 3 = 2 + 1$. $2 \times 2 = 4$ and $2 \times 1 = 2$ are the dimensions of the Jordan blocks in (3.20). The centralizer of the triple (3.19) in $Sp(6)/\mathbb{Z}_2$ is $\mathbb{Z}_2$ whose nontrivial element is $\text{diag}(1, 1, -1, -1, 1, 1)$.

In the $SO(8)$ example discussed above, we have two Jordan blocks corresponding to the partition $8 = 5 + 3$, $s = 2$ and $\tilde{G}_{e,h,f}$ is trivial.

A. Alexeevski [22] showed that for exceptional Lie algebras, the groups $\tilde{G}_{e,h,f}$ are always isomorphic to one of the symmetric groups $S_m$ where $m = 1, 2, 3, 4$ or 5. We list all the distinguished marked Dynkin diagrams for exceptional Lie algebras and the corresponding values of $m$ in the table in Fig. [4]. Note that in all cases $m = 1$ for the universal distinguished triples.
Figure 4: Unbroken discrete subgroups $S_m$ of the exceptional groups for a supersymmetric Higgs vacuum state corresponding to a given distinguished marked Dynkin diagram.
Let us give the explicit construction of $\bar{G}_{e,h,f}$ for $G_2$. Consider the following elements of the group $G_2$:

$$a = \exp\{2\pi i \beta^\vee /3\}$$

$$b = \exp\left\{\frac{\pi i}{2}(e_\beta + f_\beta)\right\}\exp\{\pi i h/2\}$$

(3.21)

with $h = 2\alpha^\vee + \beta^\vee$ as in Eq.(3.11). The triple (3.11) is invariant under the action of $a$ and $b$. For the element $a$, it follows directly from the standard commutators $[\beta^\vee, e_\alpha] = -3e_\alpha$, $[\beta^\vee, e_{\alpha+3\beta}] = 3e_{\alpha+3\beta}$. The element $b$ realizes the automorphism

$$\beta^\vee \rightarrow -\beta^\vee, \quad \alpha^\vee \leftrightarrow (\alpha + 3\beta)^\vee, \quad (\alpha + \beta)^\vee \leftrightarrow (\alpha + 2\beta)^\vee,$$

$$e_\alpha \leftrightarrow e_{\alpha+3\beta}, \quad e_{\alpha+\beta} \leftrightarrow e_{\alpha+2\beta}, \quad f_\alpha \leftrightarrow f_{\alpha+3\beta}, \quad f_{\alpha+\beta} \leftrightarrow f_{\alpha+2\beta}$$

(3.22)

(the first line in Eq.(3.22) describes the action of $b$ on the Cartan subalgebra of $g_2$; it is just the Weyl reflection of the system of coroots in Fig. 1b with respect to the line orthogonal to $\beta^\vee$). Again, the triple (3.11) is invariant.

$a$ and $b$ satisfy the relations

$$a^3 = 1, \quad b^2 = 1, \quad ab = ba^2$$

(3.23)

The first two of them follow immediately from the definition $h = 2\alpha^\vee + \beta^\vee$ and from the property (3.1), and the third one is easily verified if leaving aside the common $U(1)$ factor $\exp\{\pi i h/2\}$ and writing the $SU(2)$ part in the matrix form:

$$a \sim \begin{pmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{-2\pi i/3} \end{pmatrix}, \quad b \sim \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

(3.24)

The elements $a, b$ satisfying the defining relations (3.23) generate the group $S_3$, the centralizer of the distinguished exceptional triple (3.11) in $G_2$.

In the physical language, the presence of non-trivial finite centralizers $\bar{G}_{e,h,f}$ means that our Higgs averages, the solutions to Eq.(2.4), in many case break down the gauge symmetry not completely, but a discrete subgroup of the original gauge group remains unbroken. Some isolated examples of the systems where this phenomenon takes place have been discovered before (see e.g. Ref.[24]), but our construction with 3 adjoint scalar fields is much more natural and gives a rich family of such examples.

4 Deficit term.

4.1 Generalities. $SU(2)$ case.

As we have seen, solving the equation system (2.4) is the most direct and the most simple way to obtain the answer. But the other ways to calculate the index, in particular the

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7A side remark is that $b$ is an “exceptional” element in the sense of Ref.[23]: the fundamental group of the centralizer $G_b = [SU(2)]^2 / \mathbb{Z}_2$ is non-trivial (more exactly, the relevant fact is that the fundamental group involves a non-trivial finite factor).
traditional way based on the formula (1.14) present also a considerable methodic interest.

The principal contribution has been calculated only for the unitary groups, and we do not
know now how to do it in other cases. Speaking of the deficit term, its calculation is much
more simple. In this section, we first describe following Refs.[11, 25] how the deficit term
is calculated for the unitary groups and generalize this calculation to some orthogonal,
symplectic, and exceptional groups.

The main idea is the following. Let us regularize the theory in infrared putting it in
the large but finite ball with rigid walls \((A_I^a)^2 \leq R^2\). Such a regularization breaks down
supersymmetry, and the supertrace \(\text{Tr}\{(-1)^F e^{-\beta H}\}\) acquires a certain
\(\beta\)–dependence so that the integral in the second term in Eq.(1.14) does not vanish. It turns out that this
gives a non–vanishing contribution also in the limit \(R \to \infty\) we are interested in!
It is more or less clear (we address the reader to Ref.[11] for details) that this is associated
with the behavior of the theory at large \(|A|\). However, as we have already noted, at
large \(|A| \gg g^{-1/3}\) the theory is greatly simplified. Basically, it is given by the effective
hamiltonian in Eq. (1.8).

Strictly speaking, the statement expressed in the last sentence is wrong and we will
see it soon. It is true, however, for, say, \(SU(2)\) or \(SU(3)\) gauge groups. It makes sense to
understand first the spirit of the argumentation in the simplest \(SU(2)\) example, and then
we will easily understand how to correct it in more complicated cases.

We will briefly describe now the Born–Oppenheimer approach suggested in [3], de-
veloped in details in [26, 27, 4] and rediscovered in recent [7, 8, 28]. If the gauge group
is \(SU(2)\) and \((A_I^a)^2\) is large, we can subdivide the physical bosonic variables (there are
altogether 24 such variables: \(3 \times 9\) modulo 3 gauge degrees of freedom) into two groups:
9 slow variables \(A^a_{\text{slow}} = c_I\) which describe the motion along the vacuum valley and 15
fast variables \(A^a_{\text{fast}}\) with \(A^a_{\text{fast}} c_I = 0, \ a = 1, 2,\).

A meticulous reader might have been perplexed by the fact that \(2 \times 9 - 2 = 16 > 15,\)
but it is just because not all the variables \(A^a_{\text{fast}}\) are physical. In \(SU(2)\) case, any solution
of the valley equation has the form \(A^a_I = \eta^a c_I.\) We have used just 2 gauge parameters
to bring it to the form \(\eta^a = \delta^a_3.\) One gauge parameter is left, and it corresponds to the
\(U(1)\) rotation of the fast variables \(A^a_{\text{fast}}\) around the third isotopic axis.

Typically, \(A_{\text{fast}} \ll A_{\text{slow}}.\) Using this, we can classify the terms in the full hamiltonian
by the powers of the formal parameter \(|A_{\text{fast}}|/|A_{\text{slow}}|\). The leading term \(H_0\) has the form

\[
H_0 = \left(\delta_{IJ} - \frac{c_I c_J}{c^2}\right) \left(-\frac{1}{2} \frac{\partial^2}{\partial A^a_I \partial A^a_J} + \frac{g^2 c^2}{2} A^a_I A^a_J\right) + \frac{ig}{2} c_I c^a b \lambda^a \Gamma_I \lambda^b
\]

(4.1)

It is just a supersymmetric oscillator (one should understand that the hamiltonian (1.1)
acts on fast variables and their superpartners, and the slow variables \(c_I\) play the role of parameters)
The ground supersymmetric state of \(H_0\) has zero energy and the gap

\footnote{Another way of reasoning not requiring a rigid ball regularization and not using Eq.(1.14) is the following: The functional integral for the index in the limit \(\beta \to 0\) includes not only the contribution of possible normalizable supersymmetric vacuum state, but is also “contaminated” by the low–lying states of the continuum spectrum. The deficit term is exactly this continuum–driven contamination whose dynamics depends on the large \(|A|\) region.}
separating it from excited states is $\sim g|c|$. The total wave function may be written as

$$\Psi(x_{\text{fast}}, x_{\text{slow}}) = \sum_n \chi_n(x_{\text{slow}})\psi_n(x_{\text{fast}}) \approx \chi_0(x_{\text{slow}})\psi_0(x_{\text{fast}}) + \text{contribution of excited states},$$

(4.2)

where $x_{\text{fast}}$ stand for $A_{\text{fast}}$ and their superpartners, $x_{\text{slow}}$ stand for $A_{\text{slow}}$ and their superpartners and $\psi_n(x_{\text{fast}}) = |n\rangle$ is the spectrum of $H_0$.

Let us find the explicit expression for $\psi_0(x_{\text{fast}})$. To this end, it is convenient to choose $\Gamma_I$ so that $\Gamma_I$ is diagonal with $\gamma_3$:

$$\Gamma_I = \text{diag}(\gamma_3, \gamma_3),$$

(4.3)

Then the Hamiltonian (4.1) conserves the “fast fermion charge”

$$F_{\text{fast}} = \mu_\alpha^a \tilde{\mu}_\alpha^a$$

(4.4)

with $\mu_\alpha^a$ defined as in Eq. (1.3). With this choice, $\mu_\alpha^a$ are naturally decomposed into 4 sets of variables $\mu_{1,2}^a, \mu_{3,4}^a, \mu_{5,6}^a,$ and $\mu_{7,8}^a$, each such set corresponding to a couple of 4-dimensional Weyl fermions. Let us introduce the antisymmetric $8 \times 8$ matrix of charge conjugation $C = \text{diag}(i\sigma_2, i\sigma_2, i\sigma_2, i\sigma_2)$ (it rises and lowers the indices for each 4D Weyl fermion). The vacuum wave function reads

$$|0\rangle \propto |c|^4 \exp\left\{ -\frac{g|c|}{2} A_I^a A_J^b \left( \delta_{IJ} - \frac{c_I c_J}{|c|^2} \right) \right\} |\mu^a C \mu^a + i\epsilon^{ab} \mu^a C \gamma_3 \mu^b|^4,$$

(4.5)

where the prefactor $|c|^4$ makes the normalization integral $\langle 0|0 \rangle$ $c$-independent. The effective supercharges are defined as

$$Q_\alpha^\text{eff} = \langle 0| Q_\alpha |0 \rangle,$$

(4.6)

where $Q_\alpha$ are the full supercharges (1.1). To find (4.6), note first that, as the wave function (4.5) has a definite fast fermion charge $F_{\text{fast}} = 8$, only the part of (1.1) with $F_{\text{fast}} = 0$ contributes. It has the form

$$Q_\alpha = \frac{1}{\sqrt{2}} ((\Gamma_I)_{\alpha\beta} E_I^3 + \frac{g}{2} (\Gamma_I \Gamma_J)_{\alpha\beta} \epsilon^{ab} A_I^a A_J^b) \lambda_\beta^3$$

(4.7)

Let us observe now that the second term in Eq. (4.7) has zero average over the vacuum state (4.5) of $H_0$ due to

$$\int \left( \prod_{aI} dA_I^a \right) [\epsilon^{ab} A_M^a A_N^b] \exp\left\{ -g|c| A_I^a A_J^b \left( \delta_{IJ} - \frac{c_I c_J}{|c|^2} \right) \right\} = 0,$$

where the prime in $\prod'$ means that the integral is done only over the 16 fast variables with $(A_I^a)_{\text{fast}} c_I = 0$.

The remaining first term just coincides with the supercharge in Eq. (1.8) with only one term in the sum. To show that $Q_\alpha^\text{eff}$ is, indeed, given by this expression, it is necessary,
however, to be convinced that the contribution of the term when the derivative $\partial/\partial c_I$ acts on the wave function (4.5) gives zero. As far as the bosonic part of Eq. (4.5) is concerned, we have

$$\int \left( \prod_{aI} dA^a_I \right) |c|^4 \exp \left\{ -g |c|/2 \ A^a_i A^a_j \left( \delta_{IJ} - \frac{c_I c_J}{|c|^2} \right) \right\} \frac{\partial}{\partial c_I} |c|^4 \exp \left\{ -g |c|/2 \ A^a_i A^a_j \left( \delta_{IJ} - \frac{c_I c_J}{|c|^2} \right) \right\} = 0. \quad (4.8)$$

The only thing which is left to understand is what happens when the derivative $\partial/\partial c_I$ acts on the fermion part of the wave function (4.5). The latter depends on $c_I$ because the choice (4.3) is possible only for some particular value of $c_I$. When $c_I$ is shifted, the fermion part of the hamiltonian (4.1) is modified and so is the fermion structure of its vacuum wave function. Let us first consider the $\mathcal{N} = 1$ theory with the hamiltonian written in Eq. (A.1). Slow bosonic variables present a 3–vector $c_i$ and we have just a couple of fast Weyl fermions $\lambda^\alpha_a$, $\alpha = 1, 2$. The fermion structure of the analog of Eq. (4.5) is

$$\psi_0(x^{\text{fast}}) \propto \lambda^{\alpha a} \lambda^a_\alpha + i \epsilon^{ab} \lambda^{\alpha a} (\sigma_k)_{\beta}^\gamma \lambda^b_\beta \frac{c_k}{|c|}, \quad (4.9)$$

where $\lambda^{\alpha a} = \epsilon^{\alpha \beta} \lambda^a_\beta = (i \sigma_2)^{\beta} \lambda^a_\beta$. Differentiating this over $c_k$ gives the structure

$$\propto \epsilon^{ab} \lambda^{\alpha a} (\sigma_j)_{\beta}^\gamma \lambda^b_\beta \left[ \delta_{kj} - \frac{c_k c_j}{|c|^2} \right]. \quad (4.10)$$

This structure is orthogonal to (4.9) and gives zero after averaging.

A certain complication of the $\mathcal{N} = 4$ case is due to the fact that $\mu^a_4$ do not form a representation of $SO(9)$. For a generic $c_I$, the hamiltonian $H_0$ does not conserve the fast fermion charge (4.4), and the form of the wave function is more complicated than that in Eq. (4.3). We can make use of the $SO(9)$ invariance, however, and assume a special form of the shift $\delta c_I$: $\delta c_4 = \ldots = \delta c_9 = 0$. In 4–dimensional language, that means that only the gauge field is shifted and scalar fields are not. In that case, the vacuum wave function involves the product of 4 factors like in Eq. (4.9) and its differentiating over $c_i$ produces the structure (4.10) with zero projection on the vacuum state.

Thus, we have shown that

$$\langle 0 | \partial/\partial c_I | 0 \rangle = \partial/\partial c_I$$

and the effective supercharge (4.4) is given by Eq. (1.8). We have derived it for the $SU(2)$ theory, but the derivation can be easily generalized for other groups. We will comment on that a bit later.

The effective hamiltonian (in the $SU(2)$ case, it is just the 9–dimensional laplacian) can be obtained as $(1/16) \{ Q_\alpha, Q_\alpha \}^\alpha_+ \text{ or also determined with the Born–Oppenheimer procedure (which is a little bit trickier than for supercharges because one should take into}
account also the contribution of excited states in Eq. (4.2) \[26, 27\]. \(Q^\alpha\) and \(H^\text{eff}\) act on (properly normalized) \(\chi_0(x^{\text{slow}})\).

The simple result (1.8) is specific for pure supersymmetric Yang–Mills theories and theories with non–chiral matter content. For chiral theories (like QED with one left chiral superfield of charge 2 and 8 right chiral superfields of charge 1, or the \(SU(5)\) theory with a left quintet and a right decuplet), the second term in the analog of (4.7) and also the term due to the action of the derivative \(\partial / \partial c_i\) onto \(\psi_0(x^{\text{fast}})\) do not vanish after averaging. A Berry phase (with singularity at \(|A| = 0\)) appears \[26\]. Also, for non–chiral theories, but in the next–to–leading Born–Oppenheimer order, the calculations are more involved and the expressions are less trivial, the vacuum moduli space acquires a (conformally flat, but not just flat) metric, etc \[27\]. All this is irrelevant in our case, however.

As the deficit term is related to the large values of \(|A|\) and as, for such large values, our original problem is equivalent to (1.8), the deficit term of the original hamiltonian should be equal to the deficit term of \(H^\text{eff}\). \(H^\text{eff}\) describes free motion and its spectrum consists of delocalized plane waves. Obviously, the total index (1.14) of \(H^\text{eff}\) is zero. And that means that the deficit term for \(H^\text{eff}\) (and hence the deficit term for original hamiltonian (1.2) should coincide with the principal contribution in \(H^\text{eff}\).

Naively, the latter seems to be zero. Indeed, the classical hamiltonian \(H^\text{eff} = P_j P_j / 2\) does not depend on fermion variables, and the fermion integrals in the analog of Eq. (1.12) should give zero. This is not true, however, and the reason is that not all eigenstates of \(H^\text{eff}\) are physical and contribute in the supertrace \(\text{Tr} \{ (-1)^{-1} e^{-\beta H} \}\). The matter is that the requirement of the gauge symmetry of the wave functions imposes the requirement of Weyl invariance on the eigenstates of the effective hamiltonian \[3\]. In the \(SU(2)\) case, this Weyl invariance corresponds just to the simultaneous sign reversal for \(c_I\) and \(\lambda_\alpha\). The wave functions should not change under such a transformation.

To be more precise, wave functions depend on \(c_I\) and holomorphic variables \(\mu_\tilde{\alpha}\) defined like in Eq. (1.3). In 4–dimensional language, 8 complex variables can be interpreted as 4 Weyl 4D spinors \(\psi_\alpha f\) with \(\alpha = 1, 2\) and \(f = 0, 1, 2, 3\) (\(\psi_\alpha 0\) is the abelianized version of gluino and \(\psi_\alpha(1,2,3)\) are related to the fermion components of chiral matter multiplets). Thus, we impose the requirement

\[
\Psi(-c_I, -\mu_\tilde{\alpha}) = \Psi(c_I, \mu_\tilde{\alpha}) \quad (4.11)
\]

It is not difficult to implement the discrete symmetry (4.11) in the path integral language. The supertrace projected on the states invariant under the action of the Weyl symmetry (4.11) can be presented as the integral

\[
I_W = \frac{1}{2} \int \prod_I dc_I \prod_\tilde{\alpha} d\mu_\tilde{\alpha} \exp \left\{ -\mu_\tilde{\alpha} \mu_\tilde{\alpha} \right\} \left[ \mathcal{K}(c_I, \mu_\tilde{\alpha}; c_I, \mu_\tilde{\alpha}; \beta) + \mathcal{K}(-c_I, -\mu_\tilde{\alpha}; c_I, \mu_\tilde{\alpha}; \beta) \right], \quad (4.12)
\]

where \(\mathcal{K}(\cdots)\) is the kernel of the evolution operator. In our case (free motion !), this kernel has the simple form

\[
\mathcal{K}(c'_I, \mu'_\tilde{\alpha}; c_I, \mu_\tilde{\alpha}; \beta) = \frac{1}{\sqrt{2\pi\beta}} \exp \left\{ \frac{(c'_I - c_I)^2}{2\beta} \right\} \quad (4.13)
\]
Substituting it in Eq.(4.12), we find that: \(i\) the first term \(\equiv\) supertrace for the system with no constraints imposed = 0; \(ii\) The only contribution comes from the second term, and it is non-zero:
\[
I^d_W = \frac{1}{2} \frac{2^8}{2^9} = \frac{1}{4}
\]
(4.14)
The factor \(2^8\) in the numerator comes from the fermion integrals and the factor \(2^9\) in the denominator — from the bosonic integrals.

Note that the result (4.14) holds universally for \(\mathcal{N} = 4\) theory and also for \(\mathcal{N} = 2\) and \(\mathcal{N} = 1\) theories. Indeed, for \(\mathcal{N} = 1\), we have just two fermionic variables and 3 bosonic variables, for \(\mathcal{N} = 2\), we have 4 fermionic variables and 5 bosonic variables. We obtain
\[
I^d_W = \frac{1}{2} \frac{2^4}{2^5} = \frac{1}{2} \frac{2^2}{2^3} = \frac{1}{4}
\]
for any \(\mathcal{N}\).

4.2 Proper deficit term.

What happens for other groups? Let us start with applying the same logics and calculate the deficit term in the original nonabelian theory as the principal contribution in the abelian effective theory \((1.8)\) where only the Weyl invariant states
\[
\Psi(wc^s_I, w\mu^s_\alpha) = \Psi(c^s_I, \mu^s_\alpha)
\]
are taken into account. Let us call this contribution the “proper deficit term” (we will see by the end of this section that it is not the whole story and, generically, some other contributions appear.)

The generalization of Eq.(4.12) can be easily written:
\[
I_W = \frac{1}{\# W} \int \prod_{ls} dc^s_I \prod_{\tilde{\alpha}s} d\bar{\mu}^s_{\tilde{\alpha}} d\mu^s_\alpha \exp \{-\bar{\mu}^s_{\tilde{\alpha}} \mu^s_\alpha\} \sum_{w \in W} K(wc^s_I, w\bar{\mu}^s_{\tilde{\alpha}}; c^s_I, \mu^s_\alpha; \beta).
\]
(4.16)
Substituting here the corresponding generalization of (4.13) and doing the integrals, we obtain
\[
I_W = \frac{1}{\# W} \sum_{\wtilde{w} \in W} \frac{1}{\det(1 - w)},
\]
(4.17)
where \(\Sigma'\) means that the sum is done over all elements of \(W\) with \(\det(1 - w) \neq 0\). Again, this is true for all \(\mathcal{N}\).

It is instructive to compare the result (4.17) with the formula
\[
I_W = \frac{1}{\# W} \sum_{w \in W} [\det(1 - w)]^2
\]
(4.18)
which counts the number of vacuum states within the sector of constant gauge potentials for the \(\mathcal{N} = 1\) SYM theory defined on small 3–torus. \(\Box\) We have here the same set of

\(\Box\) Note in passing that for higher orthogonal and for exceptional groups, there are also other vacuum states associated with certain non-trivial triples of commuting group elements \([29, 23]\).
variables $c_i^s$, $\psi_i^a$ ($i = 1, 2, 3$), but the motion in the space of $c_i^s$ is finite, and the spectrum is discrete. For vacuum states, the wave functions do not depend at all on $c_i^s$, but only on $\psi_i^a$. The index can be presented in the integral form (4.14) where the evolution operator depends now only on fermion variables (and there is no integral over $\prod dc_i^s$). Thus, instead of $\det^2(1 - w)/\det^3(1 - w)$ we have just $\det^5(1 - w)$ and Eq.(4.18). 

Similar formulae counting the number of vacuum states in 6D and 10D SYM theories defined on 5–dimensional and 9–dimensional spatial tori, respectively, can be written:

$$D = 6: \quad I_W = \frac{1}{\#W} \sum_{w \in W} [\det(1 - w)]^4$$

$$D = 10: \quad I_W = \frac{1}{\#W} \sum_{w \in W} [\det(1 - w)]^8 \quad (4.19)$$

The problem of counting the states on 3–torus is especially simple. One can show that the sum in the R.H.S. of Eq.(4.18) is equal to $r + 1$ for any group [23] which justifies the original Witten’s conjecture [3]. There is no such simple universal formula for the sum (4.17) however. One should calculate it case by case.

Consider first the higher unitary groups [25]. The Cartan subalgebra of $su(n)$ is realized by the matrices $\text{diag}(a_1, \ldots, a_n)$, $a_1 + \ldots + a_n = 0$. The Weyl group is the group $\mathcal{S}_n$ of permutations of $a_n$. Only the elements $w \in W$ with $\det(1 - w) \neq 0$ contribute in the sum (4.17). These are the so called Coxeter elements corresponding to the cyclic permutations of $\{a_n\}$. There are $(n - 1)!$ such elements (while $\#W = n!$). All they are equivalent by conjugation and have one and the same $\det(1 - w)$. Take the element

$$w_\pi = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \ldots & 1 & 0 \\
-1 & -1 & \ldots & \ldots & -1
\end{pmatrix} \quad (4.20)$$

(this is just the permutation $a_1 \rightarrow a_2 \rightarrow \ldots \rightarrow -a_1 - \ldots - a_{n-1} \rightarrow a_1$ written as a matrix in the basis $\{a_1, \ldots, a_{n-1}\}$). One can show that $\det(1 - w_\pi) = n$. Therefore, the proper deficit term for the $SU(n)$ gauge group is

$$I_W^{\text{prop. def.}}[SU(n)] = \frac{1}{n^2} \quad (4.21)$$

---

There are at least two other ways to derive (4.18). First, one can duly take into account the dependence of $\mathcal{K}$ on bosonic dynamical variables, but implementing besides (4.17) also the periodicity conditions

$$\Psi(c_i^s + L_i^s) = \Psi(c_i^s)$$

where $L_i^s$ is a set of 3 $s$–dimensional vectors leaving invariant the Weyl lattice. In other words, for the theory defined on the torus, global structure of our gauge group becomes important, and the Weyl symmetry involves not only rotations, but also translations [for $SU(2)$ we have, besides the symmetry $c_i, \lambda_s \rightarrow -c_i, -\lambda_s$, also the symmetry $c_i \rightarrow c_i + 2\pi n_i/(gL)$ with integer $n_i$]. One can be convinced that the effects brought about by the Weyl rotations $c_i^s \rightarrow \omega c_i^s$ and the Weyl translations $c_i^s \rightarrow c_i^s + L_i^s$ exactly cancel each other, and the answer (4.18) is reproduced.

Formula (4.18) is also well known to mathematicians. It gives the number of invariants of $W$ in the Grassman algebra over $\mathfrak{h} \times \mathfrak{h}$ which is equal to $r + 1$ due to [3] (see [23]). Incidentally, Eq.(4.18) with 2 replaced by 1 is equal to 1 since $W$ has no non–trivial invariants in the Grassman algebra over $\mathfrak{h}$ [3].

26
As a simple exercise, one can calculate the sums in Eqs. (4.18, 4.19) and derive

\begin{align*}
D = 4 &: I_W = n \\
D = 6 &: I_W = n^3 \\
D = 10 &: I_W = n^7
\end{align*}

(4.22)

Let us go over now to non-unitary groups. Consider first the groups $SO(2r + 1)$ and $Sp(2r)$ [the root systems for these groups are dual to each other, their Weyl groups coincide, and the proper deficit term (4.17) is the same]. It is easier to think in symplectic language. The elements of Cartan subalgebra of $sp(2r)$ can be represented as the diagonal matrices $\text{diag}(a_1, \ldots, a_r, -a_1, \ldots, -a_r)$. The Weyl group is the product of the group $S_r$ of permutations of $a_i$ and $r \ Z_2$ factors corresponding to the reflections $a_i \to -a_i$. The essential complication compared to the unitary case is that many different conjugacy classes contribute in the sum (4.17). In the simplest case of $Sp(4) = SO(5)$, $\# W = 8$, and 3 elements of two different conjugacy classes contribute:

\[ w = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]  

(4.23)

That gives

\[ I_W^{\text{prop. def.}}[SO(5)] = \frac{1}{8} \left( \frac{1}{4} + \frac{1}{2} + \frac{1}{2} \right) = \frac{5}{32} \]  

(4.24)

We have also made explicit calculations for $Sp(6)$, $SO(7)$, and $G_2$. The results are given in Table 2 below.

**4.3 Total deficit vs. proper deficit.**

Eq. (4.21) is the correct result for the deficit term for the unitary groups when $n$ is prime. But if $n$ is not prime, Eq. (4.21) is only one of the contributions. The total deficit term is given by the sum (4.18) over the divisors of $n$. In Ref. [25], the appearance of these extra terms was explained in terms of D–particles and D–instantons. We present here a conventional (or, better to say, a conservative) explanation.

Some problem appears already on the level of $SU(3)$. The vacuum valley is labelled by two 9–vectors $a_I$ and $b_I$ so that

\[(A^3)^{\text{slow}} t^A = \text{diag}(a_I, b_I - a_I, -b_I) \]  

(4.25)

Let us find now the $SU(3)$ version of (4.1). To this end, we substitute $A = A^{\text{slow}} + A^{\text{fast}}$ in the hamiltonian (4.2) and pick up the leading terms in $|A^{\text{fast}}|/|A^{\text{slow}}|$. The potential

\[^{11}\text{It is remarkable that the same result for the D=10 theory is obtained with the t’Hooft twisted boundary conditions.} ]
part of $H_0$ reads [3]

$$ V_0 = \frac{g^2(2a - b)^2}{2} A_I^{a=1,2} A_J^{a=1,2} \left( \delta_{IJ} - \frac{(2a - b)_I(2a - b)_J}{(2a - b)^2} \right) + \frac{g^2(a + b)^2}{2} A_I^{a=4,5} A_J^{a=4,5} \left( \delta_{IJ} - \frac{(a + b)_I(a + b)_J}{(a + b)^2} \right) + \frac{g^2(2b - a)^2}{2} A_I^{a=6,7} A_J^{a=6,7} \left( \delta_{IJ} - \frac{(2b - a)_I(2b - a)_J}{(2b - a)^2} \right) \quad (4.26) $$

Thus, there are 48 fast variables divided naturally into three groups: $A_I^{1,2}$ satisfying the condition $(2a - b)A_I^{1,2} = 0$, $A_I^{4,5}$ satisfying the condition $(a + b)A_I^{1,2} = 0$, and $A_I^{6,7}$ satisfying the condition $(2b - a)A_I^{1,2} = 0$. (six variables with nonzero projection $(2a - b)A_I^{1,2}$, etc. are the gauge degrees of freedom; two other gauge degrees of freedom are “hidden” in 48 variables: we should require that eigenstates of $H_0$ be annihilated by the Gauss constraints $G^3$ and $G^8$).

Again we have a supersymmetric oscillator. Or rather combination of several oscillators with frequencies $g|2a - b|$, $g|a + b|$ and $g|2b - a|$. The standard Born–Oppenheimer philosophy of Refs.[3, 4] works when these frequencies are much larger that the characteristic energy scale $g^{2/3}$, i.e. when

$$ g|2a - b|^3 \gg 1, \quad g|a + b|^3 \gg 1, \quad g|2b - a|^3 \gg 1 $$

In other words, the eigenvalues $a$, $b$, $-a$, and $-b$ should not be very small by absolute value, and also they should not be too close to each other. From mathematical viewpoint, the condition for the eigenvalues in Eq.(4.25) are different means that the centralizer of the generic element (4.25) in (9 copies of) $su(3)$ coincides with the (9 copies of) the Cartan subalgebra of $su(3)$.

When, say, $2a - b \sim 0$, the fields $A_I^{1,2}$ become massless. At the point $2a = b$, they form together with the fields $A_I^3$ the $SU(2)$ gauge multiplet. A very important point is that though the standard Born–Oppenheimer approach breaks down here, we still can treat the system in Born–Oppenheimer spirit, only the classification of the dynamic variables into fast and slow categories is modified. We have now 9 slow variables

$$(A_I^A)_{\text{slow}} = \text{diag}(a_I, a_I, -2a_I) \quad (4.27)$$

We still have “abelian” fast variables $A_I^{4,5}$ and $A_I^{6,7}$. There are 32 such variables ($A_I^{4,5}a = A_I^{6,7}a = 0$). They involve 31 physical variables and a gauge degree of freedom associated with the rotation around the color axis 8. Besides, we have 27 variables $A_I^{1,2,3}$ which can be called “semi-fast” (we will see very soon why). These 27 variables involve 24 physical semi-fast variables and 3 gauge degrees of freedom. [3] The total wave function can be written as [cf. Eq.(4.2)]

$$ \Psi(a, A^{1,2,3}, A^{4,7}) = \chi_0(a)\psi_0^{\text{non-ab}}(A^{1,2,3})\psi_0^{\text{osc}}(A^{4,5})\psi_0^{\text{osc}}(A^{6,7}) + \text{contribution of excited states} \quad (4.28) $$

All together: $g^{\text{slow}} + (31)^{\text{physical fast}} + (24)^{\text{physical semi-fast}} + 8^{\text{gauge}} = 72$ as it should be.
(we did not display here explicitly the dependence on the fermion superpartners, but one should remember that they are also present). $\psi_0^{\text{osc}}(A^{4,5})$ and $\psi_0^{\text{osc}}(A^{6,7})$ are the familiar oscillator wave functions \cite{15} with $c = 3a$. The characteristic scale of $|A^{4,5}|$, $|A^{6,7}|$ is $\sim 1/\sqrt{|g|a}$. And $\psi_0^{\text{non-ab}}(A^{1,2,3})$ is the wave function of the normalized vacuum state of the $SU(2)$ theory the existence of which we have established before! The characteristic scale of $|A^{1,2,3}|$ is $\sim g^{-1/3}$ which is still much smaller than $|a|$ if $g|a|^3 \gg 1$. The hierarchy

$$\frac{1}{\sqrt{|g|a}} \ll \frac{1}{g^{1/3}} \ll |a|$$

explains why we called the variables $A^{1,2,3}$ "semi-fast". But as far as the Born–Oppenheimer method is concerned, there is no distinction between "fast" and "semi-fast" variables. Once the condition $g|a|^3 \gg 1$ is satisfied, we are allowed to integrate out both $A^{4−7}$ and $A^{1,2,3}$ and write down the effective theory for $A^8 \propto a$. Again, this theory is just \cite{18} with only one term in the sum and describes free motion in 9-dimensional space.

A very important distinction compared with the $SU(2)$ case is, however, that we should not impose now the invariance requirement with respect to any kind of discrete symmetry for the effective wave functions $\chi_0(a_I, \lambda_\alpha)$. Indeed, non-trivial elements of the Weyl group do not leave the subspace \cite{27} invariant. Once we have fixed the gauge as in Eq.\cite{27}, there is no more freedom. As a result, the valley \cite{27} gives zero contribution to the index, and the only non-zero one comes from the generic valley \cite{25} with oscillator wave functions in the fast sector.

The first example where this effect of extra subvalleys with non-abelian fast sector provides a contribution in the index is the $SU(4)$ theory. Consider the subspace

$$(A_I^A)^{\text{slow}} = \text{diag}(a, a, -a, -a)$$

It presents 9 copies of a certain subalgebra $\mathfrak{h}_a = \text{diag}(a, a, -a, -a)$ of the Cartan subalgebra of $su(4)$. The centralizer of $\mathfrak{h}_a$ in $su(4)$ is $su(2) \times su(2) \times u(1)$ and involves non-abelian factors which support localized vacuum states. On the other hand, there exist now a non-trivial subgroup $W_a$ of the original Weyl group $W = S_4$ leaving the subalgebra $\mathfrak{h}_a$ invariant and acting on its elements faithfully. This is just $Z_2$ a non-trivial element $w_*$ of which corresponds to the sign reflection $a \rightarrow -a$. Thereby, we have to impose now the symmetry requirement like in Eq.\cite{13}. In the linear basis in $\mathfrak{h}_a$, $w_* = -1$ and hence $\det(1 - w_*) = 2 \neq 0$. As a result, the effective theory on the subvalley \cite{30} has a non-zero index ($=1/4$), and the total deficit term is

$$I_W^{\text{tot \ def}}[SU(4)] = \left(\frac{1}{16}\right)_{\text{generic valley}} + \left(\frac{1}{4}\right)_{\text{subvalley}} \quad \text{(4.31)}$$

For $SU(4)$, there is no other subvalley giving a non-trivial contribution in the index.

Let us have an arbitrary group $G$, its Lie algebra $\mathfrak{g}$ and the Cartan subalgebra $\mathfrak{h}$. Let us formulate the conditions on the subalgebra $\mathfrak{h}_a \subset \mathfrak{h}$ for the subvalley associated with $\mathfrak{h}_a$ provide a non-zero contribution in the index:

\begin{enumerate}
  \item The centralizer of $\mathfrak{h}_a$ in $\mathfrak{g}$ should involve a semi-simple factor.
\end{enumerate}
(ii). Consider a subgroup $W_a$ of the Weyl group $W$ leaving invariant $\mathfrak{h}_a$. There should be at least one element $w \in W_a$ such that $\det(1-w) \neq 0$ ($w$ is understood as a matrix in the basis on $\mathfrak{h}_a$, not $\mathfrak{h}$).

(iii) The center of the centralizer of $\mathfrak{h}_a$ in $\mathfrak{g}$ should coincide with $\mathfrak{h}_a$.

(i) is necessary for the presence of a localized vacuum state in the fast sector. \(^\text{(ii)}\) is required for the effective theory to have a non-zero index. The condition (iii) guarantees that our Born–Oppenheimer separation of the variables is justified and that all non-valley field variables are indeed fast. \(^\text{[3]}\)

**Theorem 7.** Let $\mathfrak{g} = su(n)$. The only subalgebras $\mathfrak{h}_a$ of its Cartan subalgebra satisfying the conditions (i) – (iii) are the Cartan subalgebras of $su(m)$, $m|n$. The corresponding subvalley gives the contribution $1/m^2$ to the deficit term in the index.

**Proof.** The conditions (i) and (iii) imply that $\mathfrak{h}_a$ presents a subalgebra of $\mathfrak{h}$ commuting with a certain non-trivial set of root vectors. For $su(n)$ an element of $\mathfrak{h}$ is represented by a traceless diagonal matrix. A subalgebra $\mathfrak{h}_a$ consists of such matrices for which some of the elements are equal, i.e. matrices of the form $(a, \ldots, a, b, \ldots, b, c, \ldots, c, \ldots)$ where $a$ is repeated $k_1$ times, $b$ repeated $k_2$ times, etc. To fulfill the condition (ii), a permutation which maps this $\mathfrak{h}_a$ in itself and has no fixed vectors should exist. That implies $k_1 = k_2 = \ldots = k$. The relevant subgroup $W_a$ is $S_m$ with $k = n/m$: it involves permutations of the set $(a, b, c, \ldots)$. The corresponding contribution to the index is $1/m^2$. Thereby, the result \(^\text{[1.13]}\) for the total deficit term for the unitary groups is proven.

The subalgebras $\mathfrak{h}_a$ can be found and their contribution to the deficit term can be calculated also for non-unitary groups. As was just noted, the property equivalent to (i), (iii) is that there exists a subset of roots $\underline{a} \subset \Delta$ such that $\mathfrak{h}_a = \{ h \in \mathfrak{h} | \alpha(h) = 0 \text{ for all } \alpha \in \underline{a} \}$. It is easy to see that, up to $W$–conjugacy (conjugacy by an element of the Weyl group), we may choose $\underline{a}$ to be a subset of simple roots. Given a subset $\underline{a}$ of the set of simple roots, let $W_a = \{ w \in W | w(\mathfrak{h}_a) \subset \mathfrak{h}_a \}$ and let

$$\left[ I_W^{\text{def}} \right]_a = \frac{1}{\# W_a} \sum_{w \in W_a} \frac{1}{\det(1-w)}.$$ \hspace{1cm} (4.32)

Then the total deficit term is given by the sum

$$I_W^{\text{tot def}} = \sum_{a \text{ mod } W} \left[ I_W^{\text{def}} \right]_a$$ \hspace{1cm} (4.33)

Here the summation is taken over all subsets $\underline{a}$ of the set of simple roots modulo $W$–equivalence. We will calculate the sum \(^\text{[4.33]}\) for the (non–unitary) groups of the second and of the third rank.

**sp(4).** A generic element of $\mathfrak{h}$ can be presented as a diagonal $su(4)$ matrix $\text{diag}(a, b, -b, -a)$. The Dynkin diagram is depicted in Fig.6a. The corresponding coroots 1) We want to emphasize here that nonzero contributions from subvalleys like in Eq.\(^\text{[4.33]}\) to the deficit term are specific for the $\mathcal{N} = 4$ theory. In the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases, such localized vacuum states do not appear, and the total deficit term coincides with the proper one. 14 An example of $\mathfrak{h}_a$ which fits the conditions (i), (ii), but does not fit the condition (iii) is $\mathfrak{h}_a = \text{diag}(0, 0, a, -a) \in \mathfrak{h}[su(4)]$. Here the variables $\text{diag}(b, b, -b, -b)$ are exactly as slow as $\text{diag}(0, 0, a, -a)$. That means that the system does not want to stay on the valley $\text{diag}(0, 0, a, -a)$, but smears out along a larger valley $\text{diag}(b, b, a-b, -a-b)$. This latter valley does not fit, however, the condition (ii).
Figure 5: Dynkin diagrams for some groups.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
set \(a\) & \(h_a\) & \(W_a\) & \(G_a\) & \(I_{W_a}^{\text{def}}\) \\
\hline
\(\alpha\) & \(\text{diag}(a,a,b,-b,-a,-a)\) & \(Z_2 \times Z_2\) & \(SU(2)\) & 1/16 \\
\(\gamma\) & \(\text{diag}(a,a,0,-b,a)\) & \(W_{sp(4)}\) & \(SU(2)\) & 5/32 \\
\(\{\alpha,\beta\}\) & \(\text{diag}(a,a,a,-a,-a,a)\) & \(Z_2\) & \(SU(3)\) & 1/4 \\
\(\{\alpha,\gamma\}\) & \(\text{diag}(a,a,0,0,-a,a)\) & \(Z_2\) & \([SU(2)]^2\) & 1/4 \\
\(\{\beta,\gamma\}\) & \(\text{diag}(a,a,0,0,-a)\) & \(Z_2\) & \(Sp(4)\) & 1/4 \\
\hline
\end{tabular}
\caption{Subalgebras \(h_a\) and their contributions in the deficit term for the \(sp(6)\) algebra.}
\end{table}

are \(\alpha^\vee = \text{diag}(1,-1,1,-1)\) and \(\beta^\vee = \text{diag}(0,1,-1,0)\) There are two different nontrivial subalgebras \(h_a = \text{diag}(a,a,-a,a)\) and \(h_a = \text{diag}(a,0,0,-a)\) corresponding to the choice \(a = \alpha\) and \(a = \beta\), respectively. In both cases, \(W_a = Z_2\) (the non–trivial element of \(W_a\) being \(w : \{a \rightarrow -a\}\) ) and the contribution (4.32) to the index is equal to 1/4. Adding it with the proper deficit term from the second line of Table 2, we obtain the result 21/32 quoted in the third line.

\(G_2\). Again, we have a long and a short simple root, and two different nontrivial subalgebras \(h_a\). Again, in both cases, \(W_a = Z_2\) giving the contribution 1/4. The total deficit term is 1/4 + 1/4 + 35/144 = 107/144.

\(sp(6)\). A generic element of \(h\) can be presented as a diagonal \(su(6)\) matrix \(\text{diag}(a,b,c,-c,-b,-a)\). The Dynkin diagram is depicted in Fig.\(b\). The coroots are listed in Eq.\((3.18)\). There are five different nontrivial subalgebras \(h_a\) listed in Table 1 together with the semi–simple parts \(G_a\) of the centralizers of \(h_a\) in \(G\). The total deficit term is 139/128.

\(so(7)\). A generic element of \(h\) has the form \((a,b,c) = aT_{12} + bT_{34} + cT_{56}\) where \(T_{ij}\) is the generator of rotation in the \(ij\) plane. The Dynkin diagram is depicted in Fig.\(c\). The corresponding coroots are \(\alpha^\vee = (1,-1,0), \beta^\vee = (0,1,-1),\) and \(\gamma^\vee = (0,0,2)\). Again, there are five different nontrivial subalgebras \(h_a\) associated with the sets \(\{\alpha\}\), \(\{\gamma\}\), \(\{\alpha, \beta\}\), \(\{\alpha, \gamma\}\), and \(\{\beta, \gamma\}\). Their contributions to the deficit term are exactly the same as in the \(sp(6)\) case (which is not surprising as the sets of roots of the algebras \(sp(6)\) and \(so(7)\) are dual to each other) and the total deficit term is 139/128.

Adding the total deficit term and the number of normalized vacua determined earlier with the mass deformation method, we obtain the predictions for the principal contribution to the index. It would be interesting to confirm them calculating directly the

\footnote{Note that, in the considered cases, the unbroken gauge group \(G_a\) supports only one localized vacuum state. When \(G\) is larger, the number \(#\text{vac}[G_a]\) can turn out to be be greater than 1 in which case the contribution of the corresponding subvalley to the deficit term should be multiplied by \(#\text{vac}[G_a]\).}
corresponding integrals. If not analytically (that must be a difficult task), then first numerically as it was earlier done for the unitary groups \[32\].

5 Asymptotic wave function.

Besides two methods discussed above, the mass deformation method and the functional integral method, there is also a third way to detect the presence of the localized supersymmetric vacuum state in the hamiltonian (1.2). One can study the solutions of the equation \( Q_\alpha \ket{\text{vac}} = 0 \) in the asymptotic region \( g \ket{A} \gg 1 \) where the dynamics is described by the effective theory (1.8).

To understand better the philosophy of this method, consider at first a toy model. Suppose we want to find the localized zero-energy \( s \)–wave solution of the Schrödinger equation

\[
\left[ -\frac{1}{2} \Delta + V(r) \right] \psi(r) = 0
\]

in \( d \)-dimensional space. We will assume that the spherically symmetric potential \( V(r) \) dies away at infinity as a power faster than \( 1/r^2 \). Then, at large \( r \), our equation \( \Delta \psi = 0 \) has formally two solutions: (i) \( \psi(r) = \text{const} \) and (ii) \( \psi(r) \propto r^{2-d} \). The first solution is not normalizable at infinity whereas the second one (the Green’s function of the laplacian in \( d \) dimensions) is if \( d \geq 5 \) (the measure is \( \int |\psi(r)|^2 r^{d-1} dr \)).

Let us choose \( d = 5 \). The solution of the free laplacian equation \( \psi(r) \propto r^{-3} \) is normalizable at infinity, but not at zero. Intuitively, it is rather clear that, if the potential \( V(r) \) is attractive and the well is deep enough, a zero energy solution with the required asymptotics may be found. Indeed, one can be easily convinced that the equation (5.1) with the potential

\[
V(r) = -\frac{15a^2}{2(r^2 + a^2)^2}
\]

Table 2: Deficit term for some groups.
has a nice normalized solution

\[ \psi(r) = \sqrt{\frac{2a}{\pi^3}} \frac{1}{(r^2 + a^2)^{3/2}} \]  (5.3)

Of course, the presence of the normalized at infinity zero energy solution of the free Schrödinger equation is just a necessary condition for the existence of a zero energy solution of the full Schrödinger equation normalized in the whole domain and does not guarantee it yet. If the potential has other form than that in Eq.(5.2), if it is e.g. repulsive, Eq.(5.1) has no solutions. If, however, this necessary condition is not satisfied, we can be sure that solutions are absent. This would be the case, for example, for the conventional 3-dimensional Schrödinger equation in the s-wave (The asymptotic normalizability condition for the function \( r^{-l-1} \) is satisfied for \( l \geq 1 \), and one can invent a 3-dim problem with the normalized zero energy solution in the p-wave, but it would not be a ground state: the s-wave states with negative energy would be present in the spectrum.)

Let us return now to our SYM quantum mechanics. Only the theory with the \( SU(2) \) gauge group has been analyzed with this method so far, and we will restrict ourselves to that case. The problem is supersymmetric and the vacuum wave function satisfies not only the Schrödinger equation \( H|\text{vac}\rangle = 0 \), but also the equation \( Q_\alpha|\text{vac}\rangle = 0 \). The necessary condition for a normalized solution to this equation to exist is that the equation

\[ Q^{\mu_\beta}_{\alpha_\delta} \chi^\text{slow}_0(c_I, \mu_\bar{\alpha}) = 0 \]  (5.4)

have a solution normalized at large \( |c| \).

Following Ref. [8] but using the explicit form of the effective supercharges in Eq.(1.8) which simplifies the reasoning and the derivation a lot, let us describe how such a solution for the \( \mathcal{N} = 4 \) theory can be constructed. The wave function

\[ \chi^\text{slow}_0(c_I, \mu_\bar{\alpha}) = a(c_I) + b_\alpha(c_I)\mu_\bar{\alpha} + c_{\bar{\alpha}\beta}(c_I)\mu_\bar{\alpha}\mu_\beta + \ldots \]  (5.5)

has altogether \( 2^8 = 256 \) components. As was mentioned above, \( \mu_\bar{\alpha} \) do not provide a representation of \( SO(9) \), but the set of components \( \{a(c_I), b_\alpha(c_I), \ldots\} \) does. Indeed, if acting on the wave function (5.5) by the operator of spin

\[ S_{IJ} = \frac{1}{4} \lambda_\alpha(\Gamma_I \Gamma_J)_{\alpha\beta}\lambda_\beta \]  (5.6)

[with \( \lambda_\alpha \) being expressed via \( \mu_\alpha \) and \( \bar{\mu}_\bar{\alpha} \) according to Eq.(1.3)], we will obtain again a function of the form (5.3).

This representation is reducible. To understand it, note first that, when substituting (1.3) in (5.6), we obtain generically the terms of three types: \( \propto \bar{\mu}_\bar{\alpha}\mu_\beta \), \( \propto \mu_\alpha\bar{\mu}_\beta \), and \( \propto \mu_\alpha\bar{\mu}_\beta \). That means that, though the fermion charge is not a good conserved quantity \footnote{The precise meaning of this statement is the following. \( i \) The full hamiltonian (1.2) does not commute with the full fermion charge. \( ii \) The effective hamiltonian in (1.3) commutes with the “slow fermion charge” \( F^\text{slow} = \mu_\bar{\alpha}\bar{\mu}_\alpha \), but it does not help much because we are in a position to solve Eq.(5.4) involving the supercharge rather than hamiltonian, and the commutator \( [Q_\alpha, F^\text{slow}] \) is a mess. The solution to Eq.(5.4) is not going to have a definite slow fermion charge.} the fermion parity operator \( (-1)^F \) is: it commutes with \( S_{IJ} \) and anticommutes with
the supercharge. Thus, a wave function involving only even powers of \( \mu \) preserves its form under the action of \( S_{IJ} \). And so does the wave function involving only odd powers of \( \mu \).

As it turns out, the latter presents an irreducible \( 128 \)–plet of \( SO(9) \). This is a kind of Rarita–Schwinger spin–vector \((128)_{I\alpha} \), \( I = 1, \ldots , 9; \alpha = 1, \ldots , 16 \) satisfying the constraints \((\Gamma_I)_{\alpha\beta}(128)_{I\beta} = 0 \). The remaining 128 components of the wave function with \((-1)^F = 1 \) split in two irreducible representations \( 44 + 84 \). The first one is the traceless symmetric tensor \((44)_{IJ} \) and the second one is the antisymmetric tensor \((84)_{IJK} \).

Let us pick up the symmetric \( 44 \)–plet \(^{17} \) and construct our asymptotic wave function as

\[
\chi_{slow}^{0}(c_I, \mu_\alpha) \propto (44)_{IJ} \partial_I \partial_J \frac{1}{|c|^7} \tag{5.7}
\]

It is an \( SO(9) \) singlet. Obviously, it is normalizable at infinity. Acting on it with the supercharge \( Q^\text{eff}_{\alpha} \), we obtain

\[
Q^\text{eff}_{\alpha} \chi_{slow}^{0}(c_I, \mu_\alpha) \propto (\Gamma_K)_{\alpha\beta\lambda}(44)_{IJ} \partial_I \partial_J \partial_K \frac{1}{|c|^7} \tag{5.8}
\]

The fermion structure in Eq.(5.8) is odd in \( \mu_\alpha \) and presents our Rarita–Schwinger \( 128 \)–plet. We may write

\[
(\Gamma_K)_{\alpha\beta\lambda}(44)_{IJ} = \delta_{IK}(128)_{J\alpha} + \delta_{JK}(128)_{I\alpha} + \delta_{IJ}(128)_{K\alpha} \tag{5.9}
\]

Substituting it in Eq.(5.8), we obtain zero due to the property \( \Delta(1/|c|^7) = 0 \).

The existence of the normalized at infinity solution to the equation \((5.4)\) is specific for the \( N = 4 \) theory. Let us show that no such solution exists in the \( N = 1 \) and \( N = 2 \) cases. Let first \( N = 1 \). The situation is much simpler than for \( N = 4 \) because the fermion variables \( \lambda_\alpha \) are complex spinors in the representation \( 2 \) of the \( SO(3) \) group. They are the holomorphic variables on which the wave function depends, and one need not bother to construct some other variables like \( \mu_\alpha \). The operator of the fermion charge \( F = \lambda_\alpha \lambda^\alpha \) commutes with the hamiltonian here and the properties

\[
[Q^\text{eff}_{\alpha}, F] = -Q^\text{eff}_{\alpha}, \quad [Q^\alpha \text{ eff}, F] = Q^\alpha \text{ eff} \tag{5.10}
\]

hold. \( 2^2 = 4 \) components of the wave function \( \chi_{slow}^{0}(c_i, \lambda_\alpha) \) are decomposed into two singlets with \( F = 0, 2 \) and a doublet with \( F = 1 \). We see that a construction like in Eq.(5.8) is impossible and all the solutions to the equation \((5.4)\) have the form \( \chi_{slow}^{0} = \text{const}(c_i) f(\lambda_\alpha) \) and are not normalizable.

In the \( N = 2 \) case, the fermion variables \( \lambda_\alpha, \alpha = 1, \ldots , 4 \) belong to the representation \( 4 \) of \( SO(5) \). Again, they are complex and can be chosen as holomorphic variables on which wave functions depend. Again, \( [H^\text{eff}, F] = 0, \) the properties \((5.10)\) are fulfilled, and one can look for the solutions of Eq.(5.4) in a sector with a particular \( F \). A 16–component wave function is decomposed into 2 singlets with \( F = 0 \) and \( F = 4 \), 2 quartets with \( F = 1, 3 \), and \( 5 \)–plet and a singlet with \( F = 2 \). We can in principle construct the function

\[
\mathcal{N} = 2 : \chi_{slow}^{0}(c_i, \lambda_\alpha) \propto \partial_j \frac{1}{|c|^3} \lambda_\alpha (C_{\gamma j})^{\alpha\beta} \lambda_\beta \tag{5.11}
\]

\(^{17}\)Its explicit form can be found in [8].
where $\gamma_j$ are 5-dimensional gamma-matrices and $C$ is the charge conjugation matrix which lowers and rises spinor indices [when recalling that $SO(5) \equiv Sp(4)$, $C$ is the antisymmetric skew-diagonal matrix $(3.17)$ defining the group $Sp(4)$]. When acting on it with the effective supercharge $\bar{Q}_\alpha^{\text{eff}} = \bar{\lambda}^\beta (\gamma_k)_{\beta}^\alpha \partial / \partial c_k$, we obtain

$$\partial_j \partial_k \frac{1}{|c|^3} \lambda_\gamma (\gamma_k)^\alpha_c \lambda_\delta (C \gamma_j)^\gamma \partial \gamma^c \lambda_\delta = -2 \partial_j \partial_k \frac{1}{|c|^3} \lambda_\gamma (C \gamma_j \gamma_k)^\alpha \propto \Delta \frac{1}{|c|^3} = 0.$$  

The result of the action of the supercharge $Q^{\text{eff}}_\alpha$ is also zero.

However, our best try $(5.11)$ is not an admissible solution because it does not satisfy the requirement of Weyl invariance $(4.15)$. Throwing it away, we are left with nothing. Speaking of the $N = 4$ effective wave function $(5.7)$, it is Weyl invariant, is annihilated by effective supercharges and normalizable at infinity. It is the asymptotic solution we were looking for.

Obviously, the function $(5.7)$ satisfies also the equation

$$H^{\text{eff}} \chi^{\text{slow}}_0 (c_I, \mu_\tilde{\alpha}) = 0 \quad (5.12)$$

In the $N = 4$ theory, there is a unique Lorentz–invariant function satisfying the equation $(5.12)$: this is how the solution $(5.7)$ was originally found $(7)$. Note, however, that for $N = 2$ the equation $(5.12)$ involves three extra Weyl– and Lorentz–invariant solutions: $\chi^{\text{slow}}_0 = 1 / |c|^3$, $\chi^{\text{slow}}_0 = \lambda_\alpha C^{\alpha \beta} \lambda_\beta / |c|^3$, and $\chi^{\text{slow}}_0 = (\lambda_\alpha C^{\alpha \beta} \lambda_\beta)^2 / |c|^3$. These effective wave functions are not annihilated by the effective supercharges and do not correspond to any normalized supersymmetric vacuum state in the full theory.

It would be interesting to generalize this analysis for other groups. In particular, for the groups $Sp(2n \geq 6)$, $SO(n \geq 8)$ and for the exceptional groups, the full theory has several normalized solutions and hence the asymptotic equations $(5.4)$, $(6.12)$ should have several normalized Weyl–invariant solutions. A good educated guess is that, for the supercharge equation, the inverse is also true and any Lorentz– and Weyl–invariant solution to the equation $(5.4)$ can be promoted up to a normalized supersymmetric vacuum in the full theory.

6 Conclusions.

We have discussed three different techniques which allow one to deduce the existence of the normalizable supersymmetric vacuum states in the $N = 4$ SYM quantum mechanics: (i) the method of mass deformation, (ii) the functional integral method, and (iii) the asymptotic wave function method. The mass deformation method is, of course, the most straightforward and the simplest one. It allowed us to obtain the result and determine the number of normalized states for all gauge groups.

But two other methods are also interesting and valuable. First, it is really thrilling to see how the completely different ways of reasoning give the identical results for the physical Witten index whenever the comparison is possible (at the moment, the existence of a supersymmetric vacuum state for the $SU(2)$ group is observed with all three methods and with the methods $(i)$ and $(ii)$ — for higher unitary groups.) Second, the method $(i)$
involves a possible weak point: we are sure of existence of the quantum supersymmetric vacuum (vacua) for large mass, but cannot prove that all the states we have found stay normalizable in the limit $M \to 0$. It might happen in principle that one or several or all such states become delocalized at this point. We do not find it probable, but independent confirmation of our result by other methods, especially with the functional integral method which is the most bullet-proof is highly desirable.

We are indebted to A. Elashvili, N. Nekrasov, and A. Vainshtein for illuminating discussions.

Note added. It appears that there is also the fourth method to calculate $\#_{\text{vac}}$ which uses the D–brane language and ideology. In the very recent [33], our results for $\#_{\text{vac}}$ in the case of symplectic and orthogonal gauge groups were reproduced in this way.

Appendix. Principal contribution for $SU(2)$.

We will describe here how the results (1.16) are obtained and explain the reasons of disagreement between the results of the old [2] and new [11] calculations.

The starting point of Ref.[2] was the Cecotti–Girardello formula (1.10) for the index. However, this formula cannot be directly applied to the gauge theories where the degrees of freedom forming the physical phase space are not explicitly singled out. To use Eq.(1.10), we have first to resolve the Gauss’ law constraints. Consider the simplest $\mathcal{N} = 1$ theory

$$H = \frac{1}{2} E_j^A E_j^A + \frac{g^2}{4} \left( \epsilon^{ABC} A_j^B A_k^C \right)^2 + ig \epsilon^{ABC} \lambda_\alpha^A (\sigma_j)^\alpha_\beta \bar{\lambda}_B^B A_j^C ,$$

where $j = 1, 2, 3$ and $\lambda_\alpha^A$, $\alpha = 1, 2$, are holomorphic fermion variables. The constraint is

$$G^A = \epsilon^{ABC} \left( A_j^B E_j^C - i \lambda_\alpha^B \bar{\lambda}_A^C \right) = 0 .$$

To separate the physical degrees of freedom, it is convenient to use the polar representation

$$A_j^A = U_{jk} \Lambda_k^{BA} ,$$

where $U$ and $V$ are orthogonal matrices and $\Lambda_k^B = \text{diag}(a, b, c)$. We have 6 physical variables: $a, b, c$, and 3 Euler angles $\theta, \phi, \psi$ of the matrix $U$, and 3 gauge degrees of freedom — the Euler angles of the matrix $V$. The hamiltonian (A.1) can be rewritten via the new variables as follows:

$$H = \frac{1}{2} \left\{ p_a^2 + p_b^2 + p_c^2 + \left( \frac{b^2 + c^2}{(b^2 - a^2)^2} (I_1^2 + J_1^2) + 4bcI_1J_1 \right) + \left( \frac{a^2 + c^2}{(a^2 - b^2)^2} (I_2^2 + J_2^2) + 4acI_2J_2 \right) + \left( \frac{a^2 + b^2}{(b^2 - a^2)^2} (I_3^2 + J_3^2) + 4abI_3J_3 \right) \right\}$$

$$+ \frac{1}{2} \left( c^2 b^2 + a^2 c^2 + b^2 c^2 \right) + ig \epsilon^{ABC} \lambda_\alpha^A (\sigma_j)^\alpha_\beta \bar{\lambda}_B^B A_j^C ,$$

where $p_a, p_b, p_c$ are the canonical momenta of the variables $a, b, c$; $I_i$ are the standard combinations representing the generators of $SO(3)$ which depend on the Euler angles.
\[ \theta, \phi, \psi \text{ and their canonical momenta:} \]
\[ I_1 = \sin \psi p_\theta + \cos \psi \left( \cot \theta p_\phi - \frac{p_\phi}{\sin \theta} \right), \]
\[ I_2 = \cos \psi p_\theta - \sin \psi \left( \cot \theta p_\phi - \frac{p_\phi}{\sin \theta} \right), \]
\[ I_3 = p_\psi, \]  
(A.5)

and \( J^A \) are the analogous combinations for the matrix \( V \).

The major advantage of the decomposition (A.3) is a great simplification of the constraints. When expressed via new variables, they acquire the following nice form
\[ J^A = i\epsilon^{ABC} \lambda^B \bar{\lambda}^C. \]  
(A.6)

Substituting it in Eq.(A.4), we obtain the gauge–fixed classical hamiltonian depending only on physical variables. We can go now with this hamiltonian to Eq.(1.10) where the integration is performed over the physical phase space
\[ \prod_n \frac{dx_n dp_n}{2\pi} \prod_a d\bar{\psi}_a d\psi_a = \frac{1}{(2\pi)^6} \int dU dA_0 \prod_{Aa} d\lambda^A_a d\lambda_\alpha^A, \]  
(A.7)

where \( dU = \sin \theta d\theta d\phi d\psi \). Defining carefully the proper range of integration over the variables \( a, b, c, \theta, \phi, \psi \), inserting into the integral the unity
\[ 1 = \int \prod_A dJ^A \delta(J^A - i\epsilon^{ABC} \lambda^B \bar{\lambda}^C) \]
\[ = \left( \frac{2\pi}{\beta g} \right)^3 \prod_A \int dJ^A dA_0^A \exp \left\{ i\beta g A_0^A (J^A - i\epsilon^{ABC} \lambda^B_\alpha \bar{\lambda}^C) \right\} \]  
(A.8)

(so that the non–dynamic variables \( A_0^A \) entering the original Yang–Mills lagrangian are restored), and calculating the fermion determinant, the integral can be expressed eventually in the symmetric form (1.15) (with \( \mathcal{N} = 1 \), but with the extra overall factor 4. Calculating the integral gives \( I_W = 1 \) which is 4 times larger than the correct value \( I_W = 1/4 \).

The resolution of this paradox is the following. \( I_W = 1 \) is the correct value for the Cecotti–Girardello integral (1.10) for the index of the gauge–fixed hamiltonian \( H^{g.f.} \) obtained when substituting Eq.(A.4) into Eq.(A.4). However, \( H^{g.f.} \) is not completely equivalent to the original hamiltonian (A.1) with the constraints (A.2). Note that \( H^{g.f.} \) enjoys the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) discrete symmetry \footnote{Using the low case letters for the indices of \( \lambda^a_{\alpha,b,c} \) reflects the fact that they are treated now as fermion superpartners to the bosonic variables \( a, b, c \) rather than as components of a colored vector. But, of course, \( \lambda^a_\alpha \equiv \lambda^4_\alpha \).}

\[
\begin{align*}
(a, b, c, \lambda^a_\alpha, \lambda^b_\alpha, \lambda^c_\alpha) & \rightarrow (-a, -b, c, -\lambda^a_\alpha, -\lambda^b_\alpha, \lambda^c_\alpha) \\
(a, b, c, \lambda^a_\alpha, \lambda^b_\alpha, \lambda^c_\alpha) & \rightarrow (-a, b, -c, -\lambda^a_\alpha, \lambda^b_\alpha, -\lambda^c_\alpha) \\
(a, b, c, \lambda^a_\alpha, \lambda^b_\alpha, \lambda^c_\alpha) & \rightarrow (a, -b, -c, \lambda^a_\alpha, -\lambda^b_\alpha, -\lambda^c_\alpha) .
\end{align*}
\]  
(A.9)
The symmetry (A.9) presents a remnant of the original gauge symmetry [it corresponds to multiplying $A_3^i$, $\lambda_0^i$ by the orthogonal matrices $V^{AB} = \text{diag}(-1, -1, 1)$, $\text{diag}(-1, 1, -1)$ and $\text{diag}(1, -1, -1)$] and plays exactly the same role as the Weyl symmetry of the effective hamiltonian in Eq. (18) discussed at length in Sect. 4\textsuperscript{19}. The states non-invariant under the symmetry (A.9) are not physical ones and should be discarded.

We know already from Sect. 4 how to implement such a discrete symmetry for the path integral for the index. In the full analogy with Eqs. (4.12, 4.16), we may write

$$
I_W = \frac{1}{4} \int \text{dadbcdU} \prod_\alpha d\bar{\lambda}^{a\alpha} d\lambda_0^a d\lambda^{b\alpha} d\lambda_0^b d\lambda^{c\alpha} d\lambda_0^c \exp \left\{ -\bar{\lambda}^{a\alpha} \lambda_0^a - \bar{\lambda}^{b\alpha} \lambda_0^b - \bar{\lambda}^{c\alpha} \lambda_0^c \right\} \exp \left[ K(a, b, c, \bar{\lambda}^{a\alpha}, \lambda_0^a, \bar{\lambda}^{b\alpha}, \lambda_0^b, \bar{\lambda}^{c\alpha}, \lambda_0^c; \beta) + \mathcal{K}(-a, -b, c, -\bar{\lambda}^{a\alpha}, -\lambda_0^a, \bar{\lambda}^{b\alpha}, -\lambda_0^b, -\bar{\lambda}^{c\alpha}, \lambda_0^c; \beta) + \text{two more terms} \right] \quad (A.10)
$$

Consider the second integral. The fermion part of the evolution operator $\mathcal{K}(..., -\bar{\lambda}^{a\alpha}, -\lambda_0^a, \bar{\lambda}^{b\alpha}, \lambda_0^b, -\bar{\lambda}^{c\alpha}, \lambda_0^c; \beta)$ involves the factor $\exp\{-\bar{\lambda}^{a\alpha} \lambda_0^a - \bar{\lambda}^{b\alpha} \lambda_0^b - \bar{\lambda}^{c\alpha} \lambda_0^c\}$ [cf. Eq. (1.13)] so that we obtain the overall factor $\propto \exp\{-2\bar{\lambda}^{a\alpha} \lambda_0^a - 2\bar{\lambda}^{b\alpha} \lambda_0^b\}$ in the measure. As it was the case for the calculation of the first term in Eq. (A.10), it is convenient to insert the unity as in Eq. (A.6) so that the Lorentz invariance is restored.

The fermion determinant is now

$$
\det \| \beta g(\sigma_\mu)^2 \epsilon^{ABC} A^C_\mu - 2\delta_\beta \text{diag}(1, 1, 0) \| \sim (\beta g)^4[(A^1_\mu)^2 + (A^2_\mu)^2]^2 \quad (A.11)
$$

The bosonic part of the evolution operator involves now the extra factor $\propto \exp\{-2(a^2 + b^2)/\beta\}$. Performing the same transformations as in [4], we arrive at the integral

$$
\sim \frac{(\beta g)^3(\beta g)^4}{\beta^{9/2}} \int \prod_\mu dA^1_\mu dA^2_\mu dA^3_\mu \left[(A^1_\mu)^2 + (A^2_\mu)^2\right]^2 \exp \left\{ -\frac{2}{\beta}[(A^1_\mu)^2 + (A^2_\mu)^2] \exp \left\{ -\frac{\beta^2}{2}[(A^3_\mu)^2](A^1_\mu)^2 + (A^2_\mu)^2\right\} \right\} \quad (A.12)
$$

The last factor comes from the potential where we have neglected the small terms $\sim (A^1_\mu)^4$.

This integral is estimated as $\propto \beta^{9/2}g^3$ which vanishes together with two other terms in Eq. (A.10) in the limit $\beta \to 0$. Thus, the final result for the index in the $\mathcal{N} = 1$ theory is

$$
I_W = 1 \quad \text{for } I_W = \frac{1}{4}[1 + 0 + 0 + 0] = \frac{1}{4} \quad (A.13)
$$

We see that the requirement of invariance of the wave functions with respect to the symmetry (A.9) has reduced the value for the index fourfold. This is related to the fact that the principal contribution alone does not count the number of normalized vacuum states, but is contaminated by the contribution of the continuum spectrum states.

In is instructive to consider a simple example of the system with discrete spectrum where this phenomenon does not happen. Take the oscillator with the hamiltonian

\textsuperscript{19}It that case the symmetry was not $\mathbb{Z}_2 \times \mathbb{Z}_2$, but just $\mathbb{Z}_2$ because we were interested with abelian classical vacuum configurations for which two of three elements of the diagonal matrix $A^B_k$ vanish.
The hamiltonian and the supercharges are invariant under the transformation
\((x \to -x, \psi \to -\psi)\). The index of the system involving only the states even under this
symmetry is given by the integral
\[
I_W = \frac{1}{2} \int dxd\bar{\psi}d\psi e^{-\bar{\psi}\psi} \left[ K(x, \bar{\psi}; x, \psi; \beta) + K(-x, -\bar{\psi}; x, \psi; \beta) \right]
\]
\[
= \frac{1}{2} + \frac{1}{2} \int \frac{dp}{2\pi} d\bar{\psi}d\psi e^{-2\bar{\psi}\psi} \exp \left\{ -\beta p^2 + 2ipx \right\} = \frac{1}{2} + \frac{1}{2} = 1,
\]
the same as with Eq.(1.12). That could not be otherwise, of course, because the value
\(I_W = 1\) corresponds to the presence of a supersymmetric bosonic vacuum state invariant
under the transformation \(x, \psi \to -x, -\psi\).

But for the system in interest, the terms coming from the integration of non–diagonal
\(K(\cdots)\) vanish and the result is given by Eq.(A.13).

This result was obtained in [11] by another method. Instead of resolving explicitly the
Gauss law constraints and then coming to grips with implementing carefully the residual
discrete gauge symmetry, we could write
\[
I_W = \frac{1}{8\pi^2} \int dV \int \prod_{A_j} dA_j^A \prod_{\lambda_\alpha} d\lambda_\alpha^A \int \mathcal{K}(A_j^B V^{BA}, \bar{\lambda}_\alpha^B V^{BA}; A_j^A, \lambda_\alpha^A; \beta),
\]
where \(\mathcal{K}(\cdots)\) is now the evolution operator of the unconstrained hamiltonian (A.1) (and
\(8\pi^2\) is the volume of the \(SO(3)\) gauge group). The integral (A.13) automatically takes
into account only the gauge–invariant states in the spectral decomposition of \(\mathcal{K}(\cdots)\). It
is reduced to the integral (A.15) with a correct prefactor.

For the \(N = 2\) and \(N = 4\) theories, the analogs of Eq.(A.13) lead to the result (1.15)
with the correct coefficient. And the same result can be obtained by resolving explicitly
the Gauss law constraints and implementing the discrete symmetry \(Z_2 \times Z_2\). We have
repeated the calculations of Ref.[2b] and obtained the results \(I_W[H^{g.f.}] = 1\) for the \(N = 2\)
theory and \(I_W[H^{g.f.}] = 5\) for the \(N = 4\) theory which differ from the original results of
Ref. [2b] by a factor of 2. Dividing it further by \(#(Z_2 \times Z_2) = 4\), we reproduce the
result (1.13).

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