Nonlinear relativistic electron Thomson Scattering for laser radiation with orbital angular momentum

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Abstract

The classical nonlinear incoherent Thomson Scattering (TS) power spectrum from free relativistic electrons moving in a laser beam with orbital angular momentum (OAM) is investigated. The main focus in this paper is on the TS process as a diagnostic technique for this type of beams. Linearly polarized incoming radiation and electrons of very low initial kinetic energy are considered. Averaged spectra from electrons randomly covering the transverse laser pattern have different shape in the case of a beam with OAM as compared with the TEM00 case (with vanishing net OAM). Hence, spectrally resolved measurements are needed to discriminate between both cases. If electrons are distributed over the laser spot as thin stripes at a given angle with respect to the polarization direction, computations show non-trivial angular dependencies of integrated power of a laser with OAM as compared with the TEM00 mode. An experimental test of the OAM state of a laser beam is proposed based on these results. The numerical code developed is general enough to deal with more complex polarization states of the laser beam and/or electrons having arbitrary initial kinetic energies.

1. Introduction

Since the invention of CPA technology [1] pulsed lasers have been developed dramatically. Now petawatt lasers are starting to be relatively common, and intensities of $10^{22}$ W cm$^{-2}$, or more, are possible in the near infrared domain. Such laser intensities ionize the outer electrons (and many of the inner shells ones also) and accelerate the released electrons to relativistic speeds. The onset threshold of relativistic motion for electrons in a near infrared intense laser pulse is about $10^{18}$ W cm$^{-2}$. This means that at $10^{22}$ W cm$^{-2}$ electron motion is highly relativistic. At those limits radiation reaction starts playing a role and the electron dynamics has to be calculated according to that [2].

Laser polarization is often associated to the spin angular momentum of the photon. Recently it has been clearly understood that laser beams may have, besides the intrinsic spin, orbital angular momentum (OAM): see for example the chapter by Allen and Padgett [3] and also [4–9] for some experiments with light in states having OAM. The generation of OAM beams is now clearly understood and the applications of those beams are remarkable because of their structure so different from the ordinary plane wave solutions. For example, it is now clear that photoelectric effect selection rules are valid for OAM photons provided that the photon orbital angular momentum is added in the selection rules [10], so the typical dipole transition rules are violated and an exchange of more that one unit angular momentum per photon is allowed.

Laser beams are very versatile, and allow very sophisticated shapes, including also polarization. The simplest type is the well known Gaussian mode (or TEM$_{00}$ mode) coming from a laser resonator. This has been the first approach for many descriptions of the laser-atom interaction. Any laser mode can be described as a superposition of plane waves, but this is not of great help in the present case because OAM beams are a combination of multiple plane waves. It is well known that the decomposition of any laser beam forward-
propagating and in the paraxial domain (Helmholtz equation) can be done in terms of an orthogonal set of modes, the Laguerre–Gauss modes (assuming circular boundary conditions). Laguerre–Gauss modes, having a well defined OAM provide, therefore, a very convenient realistic description of the propagating field.

With early petawatt laser sources, the only possibility was to consider the transverse mode structure given by the laser amplifier. However, it seems possible to include a high variety of structures that have a strong influence on the laser-atom or laser-electron interaction. As a result it seems feasible to study a general Laguerre–Gauss mode (with a step, or other, function to describe its time pulsed structure). Although it is now very tricky to get such OAM pulses at multi-joule level, techniques are going to advance very quickly so it seems very reasonable to consider various possible physical processes generated by the former: one of them is the motion of a fully relativistic electron in such OAM pulses.

One of the challenges of modern extreme pulses is the pulse diagnostic itself. At extreme powers the interaction with atoms is meaningless because atoms are going to be ionized during the turn on of the pulse. Therefore the only remaining particles are electrons and positive ions (protons or other ions). Ions move inside the laser field slower that electrons due to the mass/charge difference. Thus the acceleration of the electrons is the dominant process for light scattering.

Moreover, this is a self consistent process, particularly for high intensities. Laser driven electrons move relativistically and radiate due to Thomson scattering and in turn this scattering can be used to get information on the laser field itself. The information on the laser field is very valuable because it is impossible to place any detector on the focal region.

In this connection, incoherent Thomson Scattering (TS) appears as a natural diagnose technique for those laser pulses in interaction with essentially free electrons. As it is well known, incoherent Thomson Scattering with intense laser beams is one of the most powerful methods for diagnosis [11] in fusion plasmas (which are basically transparent at optical or near infrared frequencies). The basic understanding of this phenomenon is well documented, see for instance [12–18] and references therein. The usefulness of TS has remained as the temperatures of the plasmas have increased from the several keV range up to tens of keV and more, provided that the relative importance of relativistic effects (mainly, the overall blue shift of the spectrum, and the depolarization of scattered radiation) be taken into account accordingly. It is also clear that TS will continue to be a key diagnostic tool for the ITER Tokamak, where electron temperatures in the range of 40keV are expected when fully operational [19]. In experimental incoherent TS as commonly used in plasma fusion diagnosis the change in scattering frequency at the detector is mainly (or nearly completely) due to the initial distribution of velocities in the electron population (and hence, on the kinetic energies) and the scattering angle, and is not caused by the laser because, although intense, the lasers used in that connection are not powerful enough to substantially change the electron trajectories. This is in clear contradistinction with TS enacted by ultra-intense lasers, as they substantially and nonlinearly perturb electron motions.

The classical equations of motion for a relativistic electron interacting with an incoming (non-necessarily monochromatic) electromagnetic plane wave of arbitrary intensity have been solved in an analytical (although implicit) way [20], which provided the basis for subsequent studies of incoherent TS [21–25], by using the Liénard–Wiechert radiated fields [26]. Recently, part of the present authors extended those researches [27, 28].

Having motivated above the importance and the interest of both OAM laser beams and incoherent TS, an open issue arises naturally: to extend the previous studies on incoherent TS with an incoming plane wave by various authors and, very specifically, in [27, 28] to the new case of incoherent TS of relativistic electrons with an OAM beam. Such a generalization meets several genuine difficulties (since the classical equations of motion do not appear to be solvable in any analytical way) and, so, it is non-trivial and quite challenging.

The main purpose of this paper is to extend the computation of TS spectra to ultra-intense OAM laser beams in interaction with free electrons, and to spot the (possible) differences with the more standard plane-wave or TEM\textsubscript{00} mode approximations for the laser pulse, focusing in a dual aspect as mentioned above: either as a diagnostic technique for this type of beams, or as the realization of novel radiation sources based on OAM laser beams. The framework for this computation will be the numerical solution of the relativistic Lorentz force equation for the electron(s) in the OAM laser beam, and the subsequent calculation of the Liénard–Wiechert scattered fields at the position of a fiducial radiation detector. Fourier transform of radiated fields and (eventually) averaging over many electron trajectories would provide for the scattered spectra that are the focus of this paper. As it will become clear later on, the numerical study carried out by the authors imply laser powers which are of the order of \(2.0 \times 10^{19} \text{ W cm}^{-2}\), hence relativistic and nonlinear effects are essential ingredients of the computations, but we are still far from the radiation reaction effects power threshold, and so the latter effect has not been considered in the present work. It constitutes a very challenging scientific problem as the laser power increases.

The paper is organized as follows. Section II introduces the essentials of laser beams with orbital angular momentum to be considered, the classical relativistic equations of motion for the electron and the Liénard-Wiechert radiation fields. The issue of radiation reaction will be discussed shortly in subsection II.B and in
section IV. Section III presents the numerical techniques used in this work, and the main results regarding incoherent TS from lasers with OAM, and section IV summarizes the conclusions and prospects for future work.

2. Formulation

2.1. Incoming laser beam with OAM: vector potential and electromagnetic fields

The MKS system of units (see, for instance, [26]) will be used throughout the paper. We shall use the standard orthonormalized vectors \(i(= (1, 0, 0))\), \(j(= (0, 1, 0))\) and \(k(= (0, 0, 1))\) along the \(x\), \(y\), and \(z\) axis, respectively. The input beam field is chosen to be an electromagnetic wave in vacuum, which propagates along the \(z\) axis, from \(-\infty\) towards \(+\infty\) and corresponds to a Gauss-Laguerre mode in the \((x,y)\)-plane and, so, has some given orbital angular momentum (OAM). As it is well known, the fields of the OAM mode are not strictly transverse: they have small components in the direction of propagation (namely, the \(z\)-axis). See [10, 29–31]. The radiation (or Coulomb) gauge will be employed. Let \(A\) be the vector potential of the incoming electric \((E_i)\) and magnetic \((B_i)\) fields. \(E_i, B_i\) and \(A_i\) depend on a three-dimensional position \(y = (x, y, z)\) and on time \((t)\). One has in the radiation gauge:

\[
E_i = -\frac{\partial A_i}{\partial t}, \quad B_i = \nabla \times A_i.
\]  

By assumption, the (real) vector potential \(A_i\) lies in the \((x,y)\)-plane: \(A_i(y; t) = A_x i + A_y j\), with

\[
A_x = \zeta_0 A_0 LG_{lp}(\rho, z) \exp i\psi + c.c.
\]

\[
A_y = \zeta_i A_0 LG_{lp}(\rho, z) \exp i\psi + c.c.
\]

As usual, \(c.c.\) denotes the complex conjugate of the preceding term. \(A_0\) is a real amplitude, \(\rho = (x^2 + y^2)^{1/2}\), \(\zeta_0\) and \(\zeta_i\) are complex numbers, with \(|\zeta_0|^2 + |\zeta_i|^2 = 1\). We write:

\[
\zeta_0 = \pi_0 + i\eta_1
\]

\[
\zeta_i = \sigma_0 + i\sigma_1
\]

\(\pi_0, \pi_1, \sigma_0\) and \(\sigma_1\) are real constants fulfilling the condition \(\pi_0^2 + \pi_1^2 + \sigma_0^2 + \sigma_1^2 = 1\), and describing the polarization state of the field. Although we concentrate in the case of an incoming laser beam linearly polarized along the \(x\)-axis, other polarization states can be treated as well by a suitable selection of \(\pi_0, \pi_1, \sigma_0\) and \(\sigma_1\). The Laguerre–Gauss (LG) functions are defined as:

\[
LG_{lp}(\rho, z) = \left[\frac{2p!}{\pi(\rho + p)!}\right]^{1/2} \frac{w_0}{w(z)} \frac{\Gamma(p + 1)}{\Gamma(p + l)} \left(\frac{2\rho^2}{w(z)^2}\right)^{p/2} \exp\left[-\frac{\rho^2}{w(z)^2}\right]
\]

\[
\times \exp\left[-\frac{\rho^2}{w(z)^2}\right]^{l/2}
\]

(6)

describing some electromagnetic wave with OAM associated to the prescribed \(l, p\). \(w(z) = w_0(1 + (z/z_0)^2)^{1/2}\) and \(L_p^{(0)}(u)\) are the generalized Laguerre polynomials:

\[
L_p^{(0)}(u) = \sum_{m=0}^{p} (-1)^m \frac{(p + l)!}{(p - m)!(l + m)!m!} u^m
\]

(7)

On the other hand:

\[
\psi = k_0 z - \omega_0 t + \chi_0 + i\phi + \frac{k_0 \rho^2}{2R(z)} + \phi_G(z)
\]

(8)

\[
\frac{1}{R(z)} = r(z) = \frac{z}{z^2 + z_0^2}
\]

(9)

\[
\phi_G(z) = -\left(2p + |l| + 1\right)\arctan(z/z_0)
\]

(10)

with \(\phi = \arctan(y/x), \chi_0\) is a constant phase, \(w_0\) is the beam–waist parameter, and \(z_0\) is the Rayleigh range, defined by \(z_0 = k_0 w_0^2/2\) with \(k_0 = 2\pi/\lambda_0\), \(\omega_0\) (real and \(> 0\)) is the frequency, and \(\omega_0 = k_0 c\). \(c\) is the velocity of light in vacuum. Then:

\[
A_x = 2A_0 LG_{lp}(\rho, z) (\pi_0 \cos \psi - \eta_1 \sin \psi)
\]

(11)

\[
A_y = 2A_0 LG_{lp}(\rho, z) (\sigma_0 \cos \psi - \sigma_1 \sin \psi)
\]

(12)

We now turn to the corresponding electric and magnetic fields. We shall make use of equations (1), (11) and (12). We have:

\[
E_i(y; t) = E_x i + E_y j
\]

(13)
\[ \mathbf{B}(\mathbf{y}; t) = B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k} \]  
(14)

Then:
\[ E_x = -2\omega_0 A_0 L G_{i, p}(\rho, z)(\sigma_0 \sin \psi + \sigma_1 \cos \psi) \]  
(15)
\[ E_y = -2\omega_0 A_0 L G_{i, p}(\rho, z)(\sigma_0 \sin \psi + \sigma_1 \cos \psi) \]  
(16)

The analytical computation of the components of \( \mathbf{B}(\mathbf{y}; t) \) is far more cumbersome. One finds:
\[ B_x = -2k_0 A_0 \left[ \frac{1}{k_0} \frac{\partial}{\partial z} L G_{i, p}(\rho, z)(\sigma_0 \cos \psi - \sigma_1 \sin \psi) \right] \]
\[ - L G_{i, p}(\rho, z) \left[ 1 + \frac{\rho^2}{2(z_0/k_0)^2}(2p + |l| + 1) - \frac{\rho^2 \sigma^2}{(z^2 + z_0^2)^2} \right] \times (\sigma_0 \sin \psi + \sigma_1 \cos \psi) \]  
(17)
\[ B_y = 2k_0 A_0 \left[ \frac{1}{k_0} \frac{\partial}{\partial z} L G_{i, p}(\rho, z)(\sigma_0 \cos \psi - \sigma_1 \sin \psi) \right] \]
\[ - L G_{i, p}(\rho, z) \left[ 1 + \frac{\rho^2}{2(z_0/k_0)^2}(2p + |l| + 1) - \frac{\rho^2 \sigma^2}{(z^2 + z_0^2)^2} \right] \times (\sigma_0 \sin \psi + \sigma_1 \cos \psi) \]  
(18)
\[ B_z = 2A_0 \frac{\partial}{\partial \rho} L G_{i, p}(\rho, z) \left[ \cos \phi(\sigma_0 \cos \psi - \sigma_1 \sin \psi) - \sin \phi(\sigma_0 \cos \psi - \sigma_1 \sin \psi) \right] \]
\[ + 2k_0 A_0 L G_{i, p}(\rho, z) \frac{\rho z}{z^2 + z_0^2} \left[ -\cos \phi(\sigma_0 \sin \psi + \sigma_1 \cos \psi) + \sin \phi(\sigma_0 \sin \psi + \sigma_1 \cos \psi) \right] \]
\[ + 2k_0 A_0 L G_{i, p}(\rho, z) \frac{1}{\rho k_0} \left[ \cos \phi(\sigma_0 \sin \psi + \sigma_1 \cos \psi) + \sin \phi(\sigma_0 \sin \psi + \sigma_1 \cos \psi) \right] \]  
(19)

For an incoming laser beam linearly polarized along the x-axis, one has \( \sigma_1 = \sigma_0 = \sigma_1 = 0 \) and \( E_y = 0 \) for any \( t, (x, y, z) \). Then, \( A_x = 0, B_x = 0 \), while \( E_\infty, B_y \) and \( B_z \) are non-vanishing.

2.2. Classical equations of motion for the electron

We shall consider, in the infinite three-dimensional space, a classical relativistic electron with position vector \( \mathbf{x} = x(t) \), momentum \( \mathbf{p} = p(t) \), velocity \( \mathbf{v} = v(t) = dx/dt \) and normalized velocity \( \beta = \beta(t) = v/c \) at time \( t \), in vacuum. The electron interacts with the classical electromagnetic field in vacuum. The electromagnetic field is the sum of an incoming (subscript \( i \) ) field and of the dynamical field radiated by the electron itself (after its initial free motion is perturbed by the input beam field). The incoming electromagnetic field is described by \( \mathbf{E}_i \) and \( \mathbf{B}_i \), which, by assumption, correspond to the input radiation. Let \( \mathbf{E} \) and \( \mathbf{B} \) be the total electric and magnetic fields, respectively. All of them also depend on \( y \) and on \( t \).

In order to solve the dynamical problem, an approximation method is used, based on the assumption that \( \mathbf{E} \) and \( \mathbf{B} \) can be replaced, respectively, by \( \mathbf{E}_i \) and \( \mathbf{B}_i \). Then, the equations of motion of the relativistic electron, subject to the Lorentz force of the incoming electric and magnetic fields \( \mathbf{E}_i = \mathbf{E}_i(y = x(t), t) \) and \( \mathbf{B}_i = \mathbf{B}_i(y = x(t), t) \) read [20] \( (\gamma_0(t) = [1 - c^{-2}(dx/dt)^2]^{-1/2}) \):
\[ \mathbf{p} = m \gamma_0(t) \frac{d\mathbf{x}}{dt} \rightarrow \frac{d\mathbf{p}}{dt} = e \mathbf{E}_i + \frac{c d\mathbf{x}}{dt} \times \mathbf{B}_i \]  
(20)

or, equivalently\( (\beta = v/c) \):
\[ \frac{d\beta}{dt} = \frac{c}{mc} [1 - \beta^2]^{3/2} [\mathbf{E}_i + c \beta \times \mathbf{B}_i - (\beta \cdot \mathbf{E}_i) \beta] \]  
(21)

In the present case, in which \( \mathbf{E}_i \) and \( \mathbf{B}_i \) account for the incoming electromagnetic wave with OAM considered in subsection II.A, equation (21) will be more convenient.

Notice that \( dx/dt = c^2 \mathbf{p}/[m^2 c^4 + c^2 \mathbf{p}^2]^{1/2} \). The electron energy is:
\[ E = [m^2 c^4 + c^2 \mathbf{p}^2]^{1/2} = mc^2 + E_{\text{kin}} \]  
(22)

\( E_{\text{kin}} \) being the electron kinetic energy. With the above understanding for \( \mathbf{E}_i(y = x(t), t) \) and \( \mathbf{B}_i(y = x(t), t) \), the dynamical problem boils down to solve the non-linear equations (20) or (21) for the electron position \( x(t) \) and momentum \( \mathbf{p}(t) \) or velocity \( \mathbf{v}(t) \) at time \( t \). Let \( \beta_0 \) be some suitable normalized initial velocity of the electron (that is, \( c^{-1} \) times certain suitable initial velocity), before receiving and being affected by the incoming monochromatic plane wave created by the laser. We write \( \beta_0 = (\beta_{0x}, \beta_{0y}, \beta_{0z}) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \). We shall consider a general case in which \( \beta_0 \) is not orthogonal to \( \mathbf{A}_i ; \beta_0 \cdot \mathbf{A}_i \neq 0 \). Suitable initial position of the electron and \( \beta_0 \) will provide the initial conditions to solve equations (20) or (21).
In the approximation scheme used here, the radiated Liénard-Wiechert fields do not perturb the electric and magnetic fields responsible for the electron motion, which are always the laser fields only. In particular, this implies that the effect of radiation reaction and the possibility of run-away solutions (a controversial feature associated to the Abraham-Lorentz equation)\(^\text{[2, 20, 26, 32]}\) are excluded from the outset. It has been argued that radiation reaction effects could induce a significant alteration of the electron motion over a sufficiently long time interval; see\(^\text{[21]}\) and references therein. On the other hand, those effects are quantitatively small in a wide variety of situations\(^\text{[2, 20, 26, 32]}\), and one could also argue that the latter may include those having interest here, taking into account that laser peak powers dealt with (numerically) in this paper are well below the threshold for radiation reaction effects. To summarize, we shall be computing with relativistic electrons, including nonlinear effects in the electron-laser interaction (laser peak power of, say, \(10^{19}\) W cm\(^{-2}\)), and in this power range and for lasers in the near infrared spectral region, it is well known that radiation reaction effects play no significant role.

Run-away solutions have been shown to be absent in a consistent treatment: see\(^\text{[33]}\) (provided a suitable cut-off procedure be imposed) and references therein (for other treatments).

2.3. Liénard-Wiechert radiated fields

The Liénard-Wiechert radiation fields (subscript \(rr\)) \(E(y, t)_{rr}\) and \(B(y, t)_{rr}\) at position \(y\) and detection time \(t\) are given by\(^\text{[26, 27]}\):

\[
E(y, t)_{rr} = \frac{e}{4\pi\varepsilon_0 c^2} \left[ \frac{r}{c} \times \left( \frac{r - c \frac{d\mathbf{x}(t')}{dt}}{c} \times \frac{d^2\mathbf{x}(t')}{dt^2} \right) \right]
\]

\[
B(y, t)_{rr} = \frac{r}{cr} \times E(y, t)_{rr}
\]

with \(r = y - x(t')\) and \(t\) and \(t'\) fulfill the condition \(c(t - t') = \|y - x(t')\|\), where \(\|\|\) denotes the Euclidean norm of the corresponding vector \(\|\mathbf{C}\| = |\mathbf{C}|\|\mathbf{C}\|^{1/2}\), for any vector \(\mathbf{C}\). It is clear that \(E(y, t)_{rr}\) and \(B(y, t)_{rr}\) depend nonlinearly on \(x(t')\), \(v(t')\) (the solution of equations\(^\text{(20)}\) and\(^\text{(21)}\)) and, hence, nonlinearly on the input beam fields (the laser with OAM in our case).

In many radiating systems, it is a good approximation to consider that the observation point (the point where the detector lies) is located far away from the electron(s). Therefore, we assume that the electron is about or not far from the origin of coordinates initially, and does not separate too much from it in the course of its interaction with the laser. Then, equations\(^\text{(23)}\) and\(^\text{(24)}\) for \(E(y, t)_{rr} \simeq E(y, t)_{rr,\infty}\) and \(B(y, t)_{rr} \simeq B(y, t)_{rr,\infty}\) become for the asymptotic fields:

\[
E(y, t)_{rr,\infty} = \frac{e n}{4\pi\varepsilon_0 c^2 R} \times \left[ \mathbf{n} - \beta \right] \times \left( \frac{d^2\mathbf{x}(t')}{dt^2} \right)
\]

\[
B(y, t)_{rr,\infty} = \frac{e n}{c} \times E(y, t)_{rr,\infty}
\]

with \(R = \|y\|\), \(\mathbf{n} = R^{-1} y\) and \(\beta \equiv c^{-1} \left( d\mathbf{x}(t')/dt \right) = (\beta_1, \beta_2, \beta_3)\). \(\mathbf{n}\) (the scattering unit vector) indicates the direction at which a detector is located at \(y\). Both \(\beta\) and the acceleration of the relativistic electron \(d^2\mathbf{x}(t')/dt^2\) in equations\(^\text{(25)}\) and\(^\text{(26)}\) are taken at a time \(t'\) such that \(t' = c^{-1} \mathbf{n} \cdot \mathbf{x}(t') = t - c^{-1} R\) and \(R \gg \|\mathbf{x}(t')\|\). The asymptotic Poynting vector is \(\mu_0 E(y, t)_{rr,\infty} \times B(y, t)_{rr,\infty}\) \(\mu_0\) being the magnetic permeability of vacuum \((\mu_0, c^2 = 1)\). The flow of radiated energy per unit time and unit area, determined by the Poynting vector at \(y\) and along \(\mathbf{n}\), is\(^\text{[26]}\)[\(\text{see [27]}\) as mean power or scattered power spectrum of the radiated field in a, perhaps, loosely way]. It is understood that such an expression for the time average includes the sum over the two possible orthogonal polarizations of the radiated field, but they can be accounted for separately if needed.

3. Numerical techniques and results

3.1. Numerical techniques

In this subsection, we will describe the techniques used in the computation of the electron’s trajectories (subject to the electromagnetic field(s) with orbital angular momentum), the geometry of the interaction/observation point, the calculation of the Liénard-Wiechert radiation fields in the time domain, and their averaged spectra over many realizations of electron trajectories whose initial positions are chosen to randomly cover a substantial zone of the transverse laser pattern.

In all the results presented in this subsection, the initial kinetic energy of the electron is chosen to be zero, \(E_{\text{kin,0}} = 0\), and its initial position is chosen to lie near \(z_0 = 0\), and with initial \(x_0, y_0\) coordinates randomly and uniformly distributed in the square \((x_0, y_0) \in [-w_0, w_0] \times [-w_0, w_0]\), unless stated otherwise. The laser is
assumed to have the following parameters: $\lambda_0 = 800 \text{ nm}$, $w_0 = 100 \mu \text{m}$, $A_0 = 0.0050$ or $A_0 = 0.0032 \text{ V.s/m}$. The order of magnitude of the laser electric field is given by $A_0 \omega_0 \left( \omega_0 = 2\pi c / \lambda_0 \right)$, which turns out to be $1.2 \times 10^{13} \text{ V/m}$ for $A_0 = 0.0050 \text{ V.s/m}$; the corresponding peak power can be readily estimated to be of the order of $10^{19} \text{ W cm}^{-2}$.

The rationale behind the choice $E_{\text{kin0}} = 0$ is that the high-intensity laser first generates a free electron population (across its transverse pattern) of low (essentially zero) kinetic energy, and then the laser highly nonlinearly interacts with those free electrons in vacuum. Due to the high laser intensities dealt with, substantial perturbation of electron trajectories, acceleration to very high (actually relativistic) velocities and energy changes on them are to be expected, that ultimately lead to the Thomson Scattering spectra to be reported below.

As stated before, the potential vector $\mathbf{A}$ of the laser is chosen so that the corresponding electric field is linearly polarized along the $X$-axis, but it is important to note that the numerical implementation of the electron/laser interaction is general enough to deal with electrons of arbitrary initial kinetic energy and/or other states of polarization of the incoming radiation. The observation point (also named ‘detector’ for short) where the Liénard–Wiechert fields are computed in the time-domain is assumed to lay at a position given by $(x_{\text{det}}, y_{\text{det}}, z_{\text{det}}) = (0.0, 0.5, 0.0)$ in meters. State-of-the-art ultra-intense lasers are associated with spot size(s) of the order of $w_0 = 10 \mu \text{m}$. The choice of $w_0 = 100 \mu \text{m}$ for typical beam sizes in our paper seemed to be a natural starting point to begin analyzing the complexities implied by the simultaneous combination of ultra-intense fields and orbital angular momentum features. The reported investigations should be extended to beam sizes of $w_0 = 10 \mu \text{m}$ in order to cope with the most ultra-intense lasers available today, and that will certainly pose new and challenging problems for the numerical and/or analytical treatment of the problem due to large changes in the transverse fields sensed by relativistic electrons on small spot sizes.

The numerical integration of the relativistic Lorentz equation has been performed in MATHEMATICA, making explicit use of the directive MaxSteps → Infinity in the built-in NDSolve command of this programming language. This is a convenient way to keep control over the integration time step, and keeping the needed precision of the numerical integration, specially over long integration times. The integrator used in the numerical solution of the equations of motion is not a symplectic one. In fact the electromagnetic field considered by the authors is (in general) a rather complex one, inhomogeneous, anisotropic and involving the three spatial coordinates in a complex functional form for general values of the parameters $(l, p)$ describing the OAM state of the laser, and no obvious symmetries or conserved quantities are found in the dynamics of the electrons that would make the use of an specific symplectic integrator an asset versus a more standard technique like a Runge–Kutta integrator, for example. It is perhaps worth mentioning that any means taken to shorten the execution time of the code would help in the studies to be reported below, and in eventual future ones, but the bare integration of the equations of motion is not the most time-consuming task (a standard one takes about one second wall–clock time), but the subsequent computation of more or less long time histories of radiated EM fields, evenly sampled in terms of detector time (which is in general different from trajectory time due to the finite value of $c$) and the corresponding computation of power spectra. Those computations take typically five to ten times that of trajectory integration. Some comments on how to improve the performance of code, using the parallelization feature of new versions of MATHEMATICA and/or migrating the code to parallelized FORTRAN are made in section 4.

The Lorentz equation has been integrated for a total time equal to 32 optical cycles of the laser, and the interpolating object(s) obtained from NDSolve output have been subsequently used to obtain the electron’s position, velocity and acceleration on selected times, see figure 1 for a typical example (here and also in figure 3 initial conditions for the integration are given with nanometre resolution, to be able to exactly reproduce data in those figures, if needed). The velocity and acceleration shown in panels (b) and (c) of figure 1 can serve to partially and qualitatively explain some of the features in radiated spectra, namely the sharp spikes in acceleration, caused by the nonlinear effect of the EM laser field on electron(s), translate via the Liénard-Wiechert formulas to a radiated EM field at detector that shows also similar sharp spikes in time domain, hence a broad spectrum in frequency domain. Please, observe that figure 1(b) corresponds to the well known figure-of-eight described in [21]. The distinction between trajectory time (the time ‘label’ parametrizing the trajectory) and the time at detector corresponding to that precise trajectory time taking into account the finite value of $c$, has always been kept with the goal of computing Liénard-Wiechert fields at evenly spaced times at the detector. This is a critical step since the numerical computation of the power spectra (our final goal) needs an evenly sampled signal in the time domain. The radiation part of the Liénard-Wiechert fields at detector is found to be nearly perpendicular to the XZ plane. Hence, the components of the radiated electric field along the X- and Z-axis have been Fourier-transformed, squared and (eventually) averaged over realizations to compute a proxy to the spectral power density along two orthogonal quadratures, a quantity that has both theoretical and experimental relevance. Spectra are averaged over initial positions $(x_0, y_0, z_0)$ of the electron randomly chosen with $(x_0, y_0) \in [-w_0, w_0] \times [-w_0, w_0]$ and $z_0 \in [-\lambda_0/2, \lambda_0/2]$. 


3.2. Results
As stated in the Introduction, the main goal of this paper is to study the nonlinear Thomson Scattering features arising from beams with orbital angular momentum, and in this connection it is worth comparing the averaged spectra for cases where the beam has a net OAM ($l = 1, p = 0$ in our case) with the standard gaussian beam characterized by ($l = 0, p = 0$). The idea behind the average over realizations in twofold: on the one side, averaged spectra seem to be more robust to compare, and on the other hand, one can argue that the high-intensity laser beam can generate a random population of free electrons over its transverse pattern that will subsequently contribute to Thomson Scattering at detector. Figure 2 shows this comparison with spectra averaged over 16,384 realizations, and $A_0 = 0.0050 \text{ V.s/m}$ for ($l = 1, p = 0$), $A_0 = 0.0032 \text{ V.s/m}$ for ($l = 0, p = 0$). The $A_0$ values are chosen such that the maximum electric field in both cases is the same. The reason for

**Figure 1.** (a) Electron trajectory $k_0(x - x_0), k_0(y - y_0), k_0(z - z_0)$, (b) time evolution of normalized velocity components $v_x/c, v_y/c, v_z/c$, and (c) time evolution of normalized acceleration components $m a_x/q A_0 \omega_0, m a_y/q A_0 \omega_0, m a_z/q A_0 \omega_0$ ($a_x, a_y, a_z$ are the corresponding acceleration components), with $(x_0, y_0, z_0) = (62.75, 0.224, -0.40) [\mu\text{m}]$. Here and in subsequent figures, Red, Green and Blue colors will refer to the $x, y, z$ component of the corresponding magnitude.
choosing 16,384 realizations to cover the transverse pattern of the laser beam is just a compromise between obtaining averaged spectra that show no further major changes, i.e., are stable against adding more initial conditions, and reasonable execution times, which for that number of samples is of the order of 30 hours wall clock time. To make this statement quantitative, two different runs computing the averaged spectra in figure 2 have been computed, and the integrated power (the numerical integral under the curve(s) in figure 2(a)) is found to coincide to better than $2 \times 10^{-3}$ ($\int_{\omega_0}^{\omega_{\text{Max}}} S_2^\omega (\omega) \text{ d}\omega \approx 0.01905$, $\int_{\omega_0}^{\omega_{\text{Max}}} S_2^\omega (\omega) \text{ d}\omega = 0.01907$ (arb. units)).

In another run, 32,768 initial conditions have been used to compute the averaged spectra (under the same random sampling of the transverse pattern), the result being that in terms of shape and integrated power, the reproducibility is better than $4 \times 10^{-3}$.

It is perhaps interesting to note that averaged spectra, as reported in figure 2 are insensitive to the value of angle $\chi_0$ that determines the initial phase of the laser EM field. Also, in spite of the differences in shape between spectra in figures 2(a), (b), the integrated power (let us focus on the case of the X-component for definiteness) is the same in both cases to better than $3 \times 10^{-3}$. Hence a frequency (or equivalently, wavelength) resolved...
spectrum would be a better discriminator of the OAM state of the laser as compared to an integrated power measurement in this case.

In figure 3, some individual spectra are shown to give a flavor of their rich variety. Averaged spectra are perhaps more robust and amenable to experimental determination, but individual ones are useful to see the large variety of dynamical behaviors that can take place in the rather complex EM field of the laser.

The beam with \((l, p) = (1, 0)\) is anisotropic across the propagation direction (figure 4). This originally lead the authors to also consider the case of initial conditions that do not uniformly cover the transverse pattern of the laser, and the corresponding averaged spectra that result from them. In particular the following type of initial conditions have been studied: let us consider a line across the laser pattern, inclined to an angle \(\delta_0\) from the \(X\)-axis in the counterclockwise direction. Initial conditions are generated (typically 8192 samples have been used, in some cases 4096) as a thin stripe along this line, according to the equations

\[
\begin{align*}
x'_0 &= x_0 \cos(\delta_0) - y_0 \sin(\delta_0), \\
y'_0 &= x_0 \sin(\delta_0) + y_0 \cos(\delta_0), \\
x_0 \text{ randomly chosen } &\in [-w_0, w_0], \\
y_0 \text{ randomly } &\in [-\lambda_0/2, \lambda_0/2].
\end{align*}
\]

Averaged spectra have been computed for a range of \(\delta_0\) angles (from 0 to 180 degrees with increments of 15 degrees in the case of \((l, p) = (1, 0)\) and from 0 to 180 degrees in increments of 30 degrees for the case of \((l, p) = (0, 0)\)). Figure 5 illustrates the \((l, p) = (1, 0)\) case for \(\delta_0\) equal to 0 and 90 degrees. As apparent from this figure, the shape and also the total power (the numerical integral of the averaged spectra over frequencies) depend on \(\delta_0\), and this happens for either the case \((l, p) = (1, 0)\) or \((l, p) = (0, 0)\), but with different functional dependencies in \(\delta_0\); see figure 6 for a summary of numerical results. For the \((l, p) = (0, 0)\) mode, no dependence

![Figure 3. A selection of individual spectra (X-component of electric field at detector) from the 16,384 realizations included in the averaging of figure 2(a). (a) Spectrum from initial condition (in microns) \((x_0, y_0, z_0) = (37.835, -37.731, -0.374)\). (b), (c), and (d) the same for \((x_0, y_0, z_0) = (-16.682, 85.529, 0.335)\), \((x_0, y_0, z_0) = (-88.461, 74.560, -0.260)\), and \((x_0, y_0, z_0) = (-1.684, -20.226, 0.280)\) respectively.](image-url)
of power on $\delta_0$ angle should be expected on the basis of the purely transverse field structure. The observed dependence is fully accounted for by the non-zero longitudinal $B_z$ component (which has also in this case a non-trivial dependence on azimuthal $f$ coordinate). This has been checked numerically by setting $B_z$ explicitly equal to zero in the code, thereby finding that, indeed, no $\delta_0$ dependence appears in that case. Please note that due to the particular form of the initial conditions chosen, the integrated power must be a periodic function of $\delta_0$ with period equal to $\pi$. Hence, power ($s$) at 0 and 180 degrees should coincide within the statistical error of the underlying random sampling of initial conditions, and this is in agreement with numerical results. The conclusion also follows that the computed differences in power both as a function of angle $\delta_0$ and for the two states of OAM studied are significant, i.e., cannot in any way be attributed only to the sampling of initial conditions used in the computation.

As it occurred with the initial conditions uniformly covering the laser spot, the averaged spectra are also insensitive to initial phase angle $\chi_0$ for the type of non-uniform initial conditions reported here. A relevant conclusion from these calculations, to be further expanded in section 4, is that this particular type of initial conditions, and the power spectra associated with them, can serve as a quantitative measure of the OAM state of the laser beam.

4. Conclusions and final comments

The results presented in this paper contribute to a characterization of Thomson Scattering spectra emitted from electrons generated in the laser field of ultra-intense lasers with OAM.

The computation of averaged spectra from electrons uniformly covering the laser spot shows different spectral shapes (at essentially the same level of total radiated power) for the case $(l, p) = (1, 0)$ as compared with $(l, p) = (0, 0)$ provided the maximum electric field is the same in both cases. Hence, a spectrally resolved measurement of emitted radiation will be needed to ascertain differences in the OAM state of the beam in that case. The averaged spectra for $(l, p) = (1, 0)$, even at the maximum sampling of 32,768 initial conditions, still feature some discrete peaks over a more or less smooth continuum. This intriguing feature has not still been completely understood, although some investigations have been carried out regarding the relative influence of the different types of electron trajectories on the final spectrum. That shall be the subject of further inquiry by the authors.

In incoherent Thomson scattering, the scattering frequency at the detector can be shown to depend, in a quite general setting, on the laser frequency, the normalized initial electron velocity (and through it, on the initial kinetic energy), the normalized scattering vector (pointing towards the location of the detector) and also on the so-called (dimensionless) laser parameter $\alpha$; see reference [28] (equations (30), (31), (42) and (43)). If the laser intensity is sufficiently small, then the dependence on the laser parameter can be disregarded and the scattering

![Figure 4. Density plot and contour lines of $E_x$ normalized to $A_{0010}$ for an OAM beam with $l = 1, p = 0$ and $t = 0, z = 0, \chi_0 = 0$. Abcissas and ordinates in meters (w_0 = 100 $\mu$m). Red and green lines correspond to initial conditions in the form of thin stripes for $\delta_0$=0 and $\pi/2$ respectively; general stripe-like initial conditions are obtained by rotating the red line counterclockwise by an angle $\delta_0$.](image-url)
frequency is given by the standard Doppler formula. In order to illustrate that this general set-up or understanding is also valid in this work, see figure 7, in which the value of $A_0$ (the parameter setting the scale of laser electric field intensity, and directly proportional to $\alpha$) has been reduced from the standard value of $5.0 \times 10^{-3}$ (V.s/m) used in this paper to $5.0 \times 10^{-4}$ (V.s/m) (one order of magnitude reduction). The electric field associated with this lower value of $A_0$ is still very substantial, but the spectrum of scattered radiation is now concentrated close to $\omega/\omega_0 = 1$ (and a very small component about $\omega/\omega_0 = 2$ because some non-linearity still remains).

Initial conditions distributed over the laser spot in a non-uniform way (in the form of thin ‘stripes’ inclined at an angle $\delta_0$ with respect to the polarization direction of the electric field) have also been considered. They can have an experimental implementation as thin electron or gas beams crossing the laser spot at selected angles, and the results reported in the paper clearly show that the integrated power versus $\delta_0$ should unambiguously discriminate the OAM states $(l, p) = (1, 0)$ and $(l, p) = (0, 0)$. Hence the emitted spectra, adequately averaged over the laser spot through the motion of many electrons, are sensitive to the OAM state of the laser beam in a quantitative way that can be put to experimental test. Our results provides a rationale for a diagnose method to the OAM state of the beam.

Integrated power from initial conditions in the form of ‘stripes’ shows a non-trivial dependence of angle $\delta_0$ not only in the case $(l, p) = (1, 0)$, but also in the gaussian mode $(l, p) = (0, 0)$. The dynamical origin of this

Figure 5. Comparison of averaged TS spectra at detector (4,096 realizations) for (a) $l = 1, p = 0, \chi_0 = 0, A_0 = 0.0050$ [V.s/m], initial conditions a ‘stripe’ along X-axis (see main text for details) and (b) the same but initial conditions a ‘stripe’ along Y-axis.
effect, to be traced back to the differences on ponderomotive force on electrons depending on their relative location with respect to the polarization axis, the transverse and also the longitudinal ($B_z$ component of magnetic field) structure of the beam is perhaps a topic worth considering in further investigations. The different functional forms of integrated power versus $\delta_0$ found in this paper suggest an experimental test allowing to discriminate the OAM state of the laser field.

Averaged spectra are the result of many individual electron time histories, each of them (potentially) mapping a different region of the laser spot. A conclusion to be drawn from figures 2 and 3 is that individual spectra show a large variance with respect to averaged ones.

Figure 6. Integrated power as a function of $\delta_0$ for the cases $(l, p) = (1, 0)$ (red filled circles) and $(l, p) = (0, 0)$ (blue filled diamonds). Solid lines are spline fittings to the numerically obtained results.

Figure 7. Integrated power spectrum along the $X$-axis at detector for $A_0 = 5.0 \times 10^{-4}$ $(V.s/m)$, $(l, p) = (1, 0)$. This $A_0$ is one order or magnitude less than the standard value used in this paper. Spectrum is now essentially concentrated near to $\omega/\omega_0 = 1$ due to the associated reduction of non-linearity.
The complex functional form of the laser beams with OAM, and the, in general, highly nonlinear dynamics induced by them on electron motion make analytical studies rather difficult, and one has to heavily rely on numerical computations to understand the phenomenology of the Thomson Scattering emission in that case. This being true, it is perhaps valuable to spot some general features that contribute to a qualitative picture of the processes at hand. In our opinion, there are at least two main ingredients, namely, first the highly nonlinear influence of the laser beam on the electron, that makes acceleration to greatly deviate from a simple harmonic motion. Sharp spikes in the time domain are found, together with a red-shift of dynamical origin and a broadening of peaks caused by the averaging. Secondly, the inhomogeneous and anisotropic nature of the OAM laser beams considered in this paper makes that electrons that are ‘born’ (so to say) in different parts of the laser spot can experience substantially different peak values of the EM field, and so we must confront the presence of substantially different time histories for different electrons, giving rise, each of them, to a different radiated spectrum. These features partially explain some features of spectra at detector position, but authors acknowledge that his issue deserves a closer scrutiny to fully understand it.

It is perhaps interesting to note that in many cases, it is possible to find a close analogue of a given trajectory in a laser beam with \(l = 1, p = 0\) to another one with \(l = 0, p = 0\), provided the local EM fields of the laser in both cases be reasonably close to each other. It is also true that the dynamics of the electron in beams with strong transverse dependence of the laser intensity produces a large variety of qualitative behaviors whose elucidation would probably contribute to a better understanding of the averaged spectra that have been the main focus of this paper.

A valuable contribution to extend the computations in this paper would be to substantially shorten the execution time of the presently available MATHEMATICA code. If we keep the approximation of incoherent Thomson Scattering, the problem is a massively parallel computation in the sense that all the electrons contribute to the final averaged spectrum without having to consider any mutual influence among them. This opens the possibility of using the parallelization feature of new versions of MATHEMATICA and, or migrating the code to parallelized FORTRAN that can run on massively parallel machines.

The authors are aware of the added complexity (and also potentially interesting results) implied by the inclusion of an initial kinetic energy on the electron(s), and plan to pursue these investigation in the near future, with proposals for the study of the corresponding phenomena in electron populations created in laser pump-probe experiments, or in fusion plasmas where an electron population already exists with potentially substantial kinetic energies, for example. Also of potential interest would be the extension of the results presented in this paper to higher \((l, p)\) modes, for example \((l, p) = (2, 0)\) or \((l, p) = (1, 1)\), that can be tackled with the code developed here.

The transverse distribution of the radiation field is of considerable theoretical and practical interest to characterize nonlinear TS, but in order to treat this problem, extensions should be made in the code to allow for a more general location of the detector. Those extensions should include a more in-depth treatment of radiated Liénard-Wiechert fields to compute the two linearly independent components that would play the same role, for a general detector position, that X- and Z- components do in the particular detector location considered in the paper. Once this would be achieved, the code should be run a sufficient number of times for each of the potentially interesting spatial position(s) of detector to obtain the spatial pattern of scattered radiation.

The computations in this paper include relativistic electrons and nonlinear effects, but exclude so far radiation reaction corrections. That is, we are in a regime in which the laser pulse will be not so intense so as to reach, say, \(10^{22} \text{ W cm}^{-2}\). We consider the possibility of extending our computations so as to include such effects in the future.

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