The \textit{J}-invariant, Tits algebras and triality

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Abstract

In the present paper we set up a connection between the indices of the Tits algebras of a semisimple linear algebraic group \( G \) and the degree one indices of its motivic \( J \)-invariant. Our main technical tools are the second Chern class map and Grothendieck’s \( \gamma \)-filtration.

As an application we provide lower and upper bounds for the degree one indices of the \( J \)-invariant of an algebra \( A \) with orthogonal involution \( \sigma \) and describe all possible values of the \( J \)-invariant in the trialitarian case, i.e., when degree of \( A \) equals 8. Moreover, we establish several relations between the \( J \)-invariant of \((A, \sigma)\) and the \( J \)-invariant of the corresponding quadratic form over the function field of the Severi-Brauer variety of \( A \).

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Introduction

The notion of a \textit{Tits algebra} was introduced by Jacques Tits in his celebrated paper on irreducible representations \cite{Tits71}. This invariant of a linear algebraic group \( G \) plays a crucial role in the computation of the \( K \)-theory of twisted flag varieties by Panin \cite{Panin94} and in the index reduction formulas by Merkurjev, Panin and Wadsworth \cite{MPW96}. It has important applications to the classification of linear algebraic groups, and to the study of the associated homogeneous varieties.

Another invariant of a linear algebraic group, the \textit{J-invariant}, has been recently defined in \cite{PSZ08}. It extends the \( J \)-invariant of a quadratic form which was studied during the last decade, notably by Karpenko, Merkurjev, Rost and Vishik. The \( J \)-invariant is a discrete invariant which describes the motivic behavior of the variety of Borel subgroups of \( G \). It plays an important role in the classification of generically split projective homogeneous varieties and in studying of cohomological invariants of \( G \) (see \cite{GPS10}, \cite{PS10}). Apart from this, it plays a crucial role in the solution of a problem posed by Serre about compact Lie groups of type \( E_8 \) (see \cite{Sem09}).

The main goal of the present paper is to set up a connection between the indices of the Tits algebras of a group \( G \) and the elements of its motivic \( J \)-invariant corresponding to degree 1 generators. The main results are Cor. 4.2.
and Thm. 4.7, which consist of inequalities relating those integers. As a crucial ingredient, we use Panin’s computation of $K_0(\mathcal{X})$, where $\mathcal{X}$ is the variety of Borel subgroups of $G$. The result is obtained using Grothendieck’s $\gamma$-filtration on $K_0(\mathcal{X})$, and relies on the properties of the Steinberg basis and on Lemma 4.10 which describes Chern classes of rational bundles of the first two layers of the $\gamma$-filtered group $K_0(\mathcal{X})$.

Let $(A, \sigma)$ be a central simple algebra endowed with an involution of orthogonal type and trivial discriminant. Its automorphism group is a group of type $D_n$, and the $J$-invariant in this setting provides a discrete motivic invariant of $(A, \sigma)$. The most interesting case is the degree 8 case, treated in Thm. 6.3, where the proof is based on triality, and precisely on its consequences on Clifford algebras (see [KMRT, §42.A]). In this case using results of Section 4, Dejaiffe’s direct sum of algebras with involutions, and Garibaldi’s “orthogonal sum lemma” we give a list of all possible values of the $J$-invariant.

The $J$-invariant of $(A, \sigma)$ is an $r$-tuple of integers $(j_1, j_2, \ldots, j_r)$ with $0 \leq j_i \leq k_i$ for some explicit upper bounds $k_i$ (see Section 3). Moreover, the Steenrod operations provide additional restrictions on values of the $J$-invariant (see [PSZ08, 4.12] and Appendix below for a precise table for algebras with orthogonal involutions). For quadratic forms, this was already noticed by Vishik [Vi05, §5], who also checked that these restrictions are the only ones for quadratic forms of small dimension (loc. cit. Question 5.13). As opposed to this, we prove in Cor. 6.1 that some values for the $J$-invariant, which are not excluded by the Steenrod operations, do not occur. This happens already in the trialitarian case, and follows from a classical result on algebras with involution [KMRT, (8.31)], due to Tits and Allen.

The paper is organized as follows. In Sections 1 and 3 we introduce notation and explain known results. Sections 2 and 4 are devoted to the inequalities relating Tits algebras and the $J$-invariant. In Sections 5 and 6 we give applications to algebras with orthogonal involutions. Finally, we study in Section 7 the properties of the quadratic form attached to an orthogonal involution $\sigma$ after generic splitting of the underlying algebra $A$. In particular, when this form belongs to the $s$-th power of the fundamental ideal of the Witt ring, we get interesting consequences on the $J$-invariant of $(A, \sigma)$ in Thm. 7.2.

Note that Junkins in [Ju11] has successfully applied and extended our results for prime $p = 3$ to characterize the behavior of Tits indices of exceptional groups of type $E_6$.

### 1 Preliminaries. Notation.

#### 1.1 Roots and weights.

We work over a base field $F$ of characteristic different from 2. Let $G_0$ be a split semisimple linear algebraic group of rank $n$ over $F$. We fix a split maximal torus $T_0 \subset G_0$, and a Borel subgroup $B_0 \supset T_0$, and we let $\hat{T}_0$ be the character group of $T_0$. We let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a set of simple roots with respect to $B_0$, and \{$\omega_1, \omega_2, \ldots, \omega_n$\} the respective set of fundamental weights, so that $\alpha_i^\vee(\omega_j) = \delta_{ij}$. The roots and weights are always
numbered as in Bourbaki [Bou].

Recall that \( \Lambda_r \subset T_0 \subset \Lambda_\omega \), where \( \Lambda_r \) and \( \Lambda_\omega \) are the root and weight lattices, respectively. The lattice \( T_0 \) coincides with \( \Lambda_r \) (respectively \( \Lambda_\omega \)) if and only if \( G_0 \) is adjoint (resp. simply connected).

1.2 The Picard group. We let \( \mathfrak{x}_0 \) be the variety of Borel subgroups of \( G_0 \), or equivalently of its simply connected cover \( G_0^{sc} \). The Picard group \( \text{Pic}(\mathfrak{x}_0) \) can be computed as follows. Since any character \( \lambda \in T_0 \) extends uniquely to \( B_0 \), it defines a line bundle \( \mathcal{L}(\lambda) \) over \( \mathfrak{x}_0 \). Hence, there is a natural map \( \hat{T}_0 \to \text{Pic}(\mathfrak{x}_0) \), which is an isomorphism if \( G_0 \) is simply connected. So, we can identify the Picard group \( \text{Pic}(\mathfrak{x}_0) \) with the weight lattice \( \Lambda_\omega \) (this fact goes back to Chevalley, see also [MT95, Prop. 2.2]).

1.3 Inner forms. Throughout the paper, \( G \) denotes a twisted form of \( G_0 \), and \( T \subset G \) is the corresponding maximal torus. We always assume that \( G \) is an inner twisted form of \( G_0 \), and even a little bit more, that is \( G = \xi G_0 \) for some cocycle \( \xi \in Z^1(F,G_0) \). Note that not every twisted form of \( G_0 \) can be obtained in this way. For instance, if \( G_0 \) is simply connected, then \( G \) is a strongly inner twisted form of \( G_0 \).

We denote by \( \mathfrak{x} = \xi \mathfrak{x}_0 \) the corresponding twisted variety. Observe that \( \mathfrak{x} \) is the variety of Borel subgroups of \( G \) and, hence, is a projective homogeneous \( G \)-variety (see e.g. [MPW96, §1]). The varieties \( \mathfrak{x} \) and \( \mathfrak{x}_0 \) are defined over \( F \), and they are isomorphic over a separable closure \( F_s \) of \( F \).

1.4 Tits algebras. Consider the simply connected cover \( G_0^{sc} \) of \( G_0 \) and the corresponding twisted group \( G^{sc} = \xi G_0^{sc} \), \( \xi \in Z^1(F,G_0) \). We denote by \( \Lambda_\omega^{sc} \subset \Lambda_\omega \) the cone of dominant weights. Since \( G \) is an inner twisted form of \( G_0 \), for any \( \omega \in \Lambda_\omega^{sc} \) the corresponding irreducible representation \( G_0^{sc} \to \text{GL}(V) \), viewed as a representation of \( G^{sc} \times F_s \), descends to a representation \( G^{sc} \to \text{GL}(A_\omega) \), where \( A_\omega \) is a central simple algebra over \( F \), called a Tits algebra of \( \xi \) (cf. [Tits71, §3.4] or [KMRT, §27]). In particular, to any fundamental weight \( \omega \) corresponds a Tits algebra \( A_\omega \).

Taking Brauer classes, the assignment \( \Lambda_\omega^{+} \ni \omega \mapsto A_\omega \) induces a homomorphism \( \beta: \Lambda_\omega^{+}/T_0 \to \text{Br}(F) \) known as the Tits map [Tits71].

For any \( \omega \in \Lambda_\omega \) we denote by \( \bar{\omega} \) its class in \( \Lambda_\omega/T_0 \), by \( i(\omega) \) the index of the Brauer class \( \beta(\bar{\omega}) \), that is the degree of the underlying division algebra. For fundamental weights, \( i(\omega) \) is the index of the Tits algebra \( A_\omega \).

1.5 Algebras with involution. We refer to [KMRT] for definitions and classical facts on algebras with involution. Throughout the paper, \( (A,\sigma) \) always stands for a central simple algebra of even degree \( 2n \), endowed with an involution of orthogonal type with trivial discriminant. In particular, this implies that the Brauer class \( [A] \) of the algebra \( A \) is an element of order 2 of the Brauer group \( \text{Br}(F) \). Because of the discriminant hypothesis, the Clifford algebra of \( (A,\sigma) \), endowed with its canonical involution, is a direct product \( (\text{Cl}(A,\sigma),\sigma) = (\text{Cl}_+,\sigma_+) \times (\text{Cl}_-,\sigma_-) \) of two central simple algebras. If moreover \( n \) is even, the involutions \( \sigma_+ \) and \( \sigma_- \) are also of orthogonal type.
1.6 Hyperbolic involutions. We refer to [KMRT, §6] for the definition of isotropic and hyperbolic involutions. In particular, recall that $A$ has a hyperbolic involution if and only if it decomposes as $A = M_2(A')$ for some central simple algebra $A'$ over $F$. When this occurs, $A$ has a unique hyperbolic involution $\sigma_0$ up to isomorphism. Moreover, $\sigma_0$ has trivial discriminant, and if additionally the degree of $A$ is divisible by 4, then its Clifford algebra has a split component by [KMRT, (8.31)].

1.7 The cocenter for $D_n$. The connected component of the automorphism group of $(A, \sigma)$ is denoted by $\text{PGO}^+(A, \sigma)$. Since the involution has trivial discriminant, it is an inner twisted form of $\text{PGO}^+_{2n}$ (see 1.3). Both groups are adjoint of type $D_n$. We recall from Bourbaki [Bou] the description of their cocenter $\Lambda_\omega / \Lambda_r$ in terms of the fundamental weights, for $n \geq 3$:

If $n = 2m$ is even, then $\Lambda_\omega / \Lambda_r \simeq \mathbb{Z} / 2\mathbb{Z} \times \mathbb{Z} / 2\mathbb{Z}$, and the three non-trivial elements are the classes of $\omega_1$, $\omega_{2m-1}$ and $\omega_{2m}$ if $m \geq 2$.

If $n = 2m + 1$ is odd, then $\Lambda_\omega / \Lambda_r \simeq \mathbb{Z} / 4\mathbb{Z}$, and the generators are the classes of $\omega_{2m}$ and $\omega_{2m+1}$. Moreover, the element of order 2 is the class of $\omega_1$.

1.8 Fundamental relations. The Tits algebras $A_{\omega_1}$, $A_{\omega_{2m-1}}$ and $A_{\omega_{2m}}$ of the group $G = \text{PGO}^+(A, \sigma)$ are respectively the algebra $A$ and the two components $C_+$ and $C_-$ of the Clifford algebra of $(A, \sigma)$ (see [KMRT, §27.B]). Applying the Tits map, and taking into account the description of $\Lambda_\omega / \hat{T}_0 = \Lambda_\omega / \Lambda_r$, we get the so-called fundamental relations [KMRT, (9.12)] relating their Brauer classes, namely:

- If $n = 2m$ is even, that is $\deg(A) \equiv 0 \mod 4$, then $[C_+]$ and $[C_-]$ are of order at most 2, and $[A] + [C_+] + [C_-] = 0 \in \text{Br}(F)$. In other words, any of those three algebras is Brauer equivalent to the tensor product of the other two.
- If $n = 2m + 1$ is odd, that is $\deg(A) \equiv 2 \mod 4$, then $[C_+]$ and $[C_-]$ are of order dividing 4, and $[A] = 2[C_+] = 2[C_-] \in \text{Br}(F)$.

2 Characteristic maps and restriction maps

2.1 Characteristic map for Chow groups. Let $\text{CH}^*(-)$ be the graded Chow ring of algebraic cycles modulo rational equivalence. Since $X_0$ is smooth projective, the first Chern class induces an isomorphism between the Picard group $\text{Pic}(X_0)$ and $\text{CH}^1(X_0)$ [Ha, Cor. II.6.16]. Combining with the isomorphism $\Lambda_\omega \simeq \text{Pic}(X_0)$ of [1.2], we get an isomorphism, which is the simply connected degree 1 characteristic map:

$$c^{(1)}_{\omega} : \Lambda_\omega \rightarrow \text{CH}^1(X_0).$$

Hence, the cycles $h_i := c_1(\mathcal{L}(\omega_i))$, $i = 1 \ldots n$, form a $\mathbb{Z}$-basis of the group $\text{CH}^1(X_0)$.

In general, the degree 1 characteristic map is the restriction of this isomorphism to the character group of $T_0$,

$$c^{(1)} : \hat{T}_0 \rightarrow \Lambda_\omega \rightarrow \text{CH}^1(X_0).$$
Hence, it maps $\lambda = \sum_{i=1}^{n} a_i \omega_i \in \tilde{T}_0$, where $a_i \in \mathbb{Z}$, to $c_1(\mathcal{L}(\lambda)) = \sum_{i=1}^{n} a_i h_i$. For instance, in the adjoint case, the image of $\mathcal{C}^{(1)}$ is generated by linear combinations $\sum_j c_{ij} h_j$, where $c_{ij} = \alpha_i^j(\alpha_j)$ are the coefficients of the Cartan matrix.

The degree 1 characteristic map extends to a characteristic map

$$c : S^*(\tilde{T}_0) \rightarrow CH^*(\mathcal{X}_0),$$

where $S^*(\tilde{T}_0)$ is the symmetric algebra of $\tilde{T}_0$ (see [Gr58, §4], [De74, §1.5]). Its image $\text{im}(c)$ is generated by the elements of codimension one, that is by the image of $\mathcal{C}^{(1)}$.

2.2 Example. We let $p = 2$ and consider the Chow group with coefficients in $\mathbb{F}_2 \text{Ch}^1(\mathcal{X}_0) = \text{CH}^1(\mathcal{X}_0) \otimes_\mathbb{Z} \mathbb{F}_2$. Assume $G_0$ is of type $D_4$. Using the simply connected characteristic map, we can identify the degree 1 Chow group modulo 2 with the $\mathbb{F}_2$-lattice $\text{Ch}^1(\mathcal{X}_0) = F_2 h_1 \oplus F_2 h_2 \oplus F_2 h_3 \oplus F_2 h_4$.

In the adjoint case the image of the characteristic map $\mathcal{C}^{(1)}_{\text{ad}}$ with $\mathbb{F}_2$-coefficients is the subgroup $\text{im}(\mathcal{C}^{(1)}_{\text{ad}}) = F_2 h_2 \oplus F_2 (h_1 + h_3 + h_4) \subset \text{Ch}^1(\mathcal{X}_0)$.

In the half-spin case, that is when one of the two weights $\omega_3, \omega_4$ is in $\tilde{T}_0$, say $\omega_3 \in \tilde{T}_0$, we get $\text{im}(\mathcal{C}^{(1)}_{\text{hs}}) = F_2 h_2 \oplus F_2 h_3 \oplus F_2 (h_1 + h_4) \subset \text{Ch}^1(\mathcal{X}_0)$.

2.3 Restriction map for Chow groups. Let $G$ and $\xi \in Z^1(F; G_0)$ be as in [1.3] so that $G = \xi G_0$. The cocycle $\xi$ induces an identification $\mathcal{X} \times_F F_s \simeq \mathcal{X}_0 \times_F F_s$. Moreover, since $\mathcal{X}_0$ is split, $\text{CH}^*(\mathcal{X}_0 \times_F F_s) = \text{CH}^*(\mathcal{X}_0)$. Hence the restriction map can be viewed as a map

$$\text{res}_{\text{CH}} : \text{CH}^*(\mathcal{X}) \rightarrow \text{CH}^*(\mathcal{X} \times_F F_s) \simeq \text{CH}^*(\mathcal{X}_0).$$

2.4 Definition. A cycle of $\text{CH}^*(\mathcal{X}_0)$ is called rational if it belongs to the image of the restriction $\text{res}_{\text{CH}}$.

In [KM06] Thm. 6.4(1)], it is proven that, under the hypothesis [1.3], any cycle in the image of the characteristic map $c$ is rational, i.e. $\text{im}(c) \subseteq \text{im}(\text{res}_{\text{CH}})$ (See [KM06] §7 to compare their $\tilde{\varphi}_c$ with our characteristic map.)

2.5 Remark. Note that the image of the restriction map does not depend on the choice of $G$ in its isogeny class, while the image of the characteristic map does.

For a split group $G_0$, the restriction map is an isomorphism, and this inclusion is strict, except if $H^1(F, G_0)$ is trivial. On the other hand, generic torsors are defined as the torsors for which it is an equality:

2.6 Definition. A cocycle $\xi \in Z^1(F, G_0)$ defining the twisted group $G = \xi G_0$ is said to be generic if any rational cycle is in $\text{im}(c)$, so that $\text{im}(c) = \text{im}(\text{res}_{\text{CH}})$.

Observe that a generic cocycle always exists over some field extension of $F$ by [KM06] Thm. 6.4(2)].

2.7 Characteristic map for $K_0$. Using the identification between $\Lambda_0$ and $\text{Pic}(\mathcal{X}_0)$ of [1.2] one also gets a characteristic map for $K_0$ (see [De74, §2.8]),

$$c_K : \mathbb{Z}[\tilde{T}_0] \rightarrow K_0(\mathcal{X}_0),$$
where \( \mathbb{Z}[\hat{T}_0] \subset \mathbb{Z}[\Lambda_\omega] \) denotes the integral group ring of the character group \( \hat{T}_0 \). Any generator \( e^\lambda, \lambda \in \hat{T}_0 \), maps to the class of the associated line bundle \([\mathcal{L}(\lambda)] \in K_0(\mathfrak{X}_0)\).

Combining a theorem of Pittie \cite{Pi72} (see also \cite{Pa94} §0), and Chevalley’s description of the representation rings of the simply connected cover \( G_0^{sc} \) of \( G_0 \) and its Borel subgroup \( B_0^{sc} \), one can check that \( K_0(\mathfrak{X}_0) \) is isomorphic to the tensor product \( \mathbb{Z}[\Lambda_\omega] \otimes \mathbb{Z}[\Lambda_\omega] \mathbb{Z} \). That is, the simply-connected characteristic map
\[
  c_{K,sc} : \mathbb{Z}[\Lambda_\omega] \to K_0(\mathfrak{X}_0)
\]
is surjective, and its kernel is generated by the elements of the augmentation ideal that are invariant under the action of the Weyl group \( W \).

2.8 The Steinberg basis. Consider the weights \( \rho_w \) defined for every \( w \) in the Weyl group \( W \) by
\[
  \rho_w = \sum_{\{ \alpha_k \in \Pi, \ w^{-1}(\alpha_k) \in \Phi^- \}} w^{-1}(\omega_k),
\]
where \( \Phi^- \) denotes the set of negative roots with respect to \( \Pi \). The elements
\[
  g_w := c_{K,sc}(e^{\rho_w}) = [\mathcal{L}(\rho_w)], \quad w \in W,
\]
form a \( \mathbb{Z} \)-basis of \( K_0(\mathfrak{X}_0) \), called the Steinberg basis (see \cite{St75} §2 and \cite{Pa94} §12.5). Note that if \( w \) is the reflection \( w = s_i, 1 \leq i \leq n \), associated to the root \( \alpha_i \), we get
\[
  \rho_{s_i} = \sum_{\{ \alpha_k \in \Pi, \ s_i(\alpha_k) \in \Phi^- \}} s_i(\omega_k) = s_i(\omega_i) = \omega_i - \alpha_i.
\]

2.9 Definition. The elements of the Steinberg basis \( g_i = [\mathcal{L}(\rho_{s_i})], i = 1 \ldots n \), will be called special elements.

2.10 Restriction map for \( K_0 \) and the Tits algebras. As we did for Chow groups, we use the identification \( \mathfrak{X} \times_F F_s \simeq \mathfrak{X}_0 \times_F F_s \) to view the restriction map for \( K_0 \) as a morphism
\[
  \text{res}_{K_0} : K_0(\mathfrak{X}) \to K_0(\mathfrak{X}_0) = \bigoplus_{w \in W} \mathbb{Z} \cdot g_w.
\]

By Panin’s theorem \cite{Pa94} Thm. 4.1, the image of the restriction map, whose elements are called rational bundles, is the sublattice with basis
\[
  \{ i(\rho_w) \cdot g_w, \ w \in W \},
\]
where \( i(\rho_w) \) is the index of the Brauer class \( \beta(\bar{\rho}_w) \), that is the index of any corresponding Tits algebra (see §1.3).

Note that since the Weyl group acts trivially on \( \Lambda_\omega / \hat{T}_0 \), we have
\[
  \bar{\rho}_w = \sum_{\{ \alpha_k \in \Pi, w^{-1}(\alpha_k) \in \Phi^- \}} \bar{\omega}_k.
\]

Therefore, the corresponding Brauer class is given by
\[
  \beta(\bar{\rho}_w) = \sum_{\{ \alpha_k \in \Pi, w^{-1}(\alpha_k) \in \Phi^- \}} \beta(\bar{\omega}_k).
\]
In particular, for special elements we get \( \beta(\bar{\rho}_{s_i}) = \beta(\bar{\omega}_i) \), so that \( i(\rho_{s_i}) \) is the index of the Tits algebra \( A_{\omega_i} \).
2.11 Rational cycles versus rational bundles. Since the total Chern class of a rational bundle is a rational cycle, the graded-subring $B_2$ of $\text{CH}^*(\mathfrak{X}_0)$ generated by Chern classes of rational bundles consists of rational cycles. We use Panin’s description of rational bundles to compute $B_2$. The total Chern class of $i(\rho_w) \cdot g_w$ is given by

$$c(i(\rho_w) \cdot g_w) = (1 + c_1(L(\rho_w)))^{i(\rho_w)} = \sum_{k=1}^{i(\rho_w)} \binom{i(\rho_w)}{k} c_1(L(\rho_w))^k$$

Therefore, $B^*$ is generated as a subring by the homogeneous elements

$$\binom{i(\rho_w)}{k} c_1(L(\rho_w))^k, \text{ for } w \in W, \ 1 \leq k \leq i(\rho_w).$$

Let $p$ be a prime number, and denote by $i_w$ the $p$-adic valuation of $i(\rho_w)$, so that $i(\rho_w) = p^iwq$ for some prime-to-$p$ integer $q$. By Luca’s theorem [Di, p. 271] the binomial coefficient $\binom{i(\rho_w)}{p^iw}$ is congruent to $q$ modulo $p$. Hence its image in $\mathbb{F}_p$ is invertible. Considering the image in the Chow group modulo $p$ of the rational cycle $c_1(L(\rho_w))^p$ we get:

2.12 Lemma. Let $p$ be a prime number. For any $w$ in the Weyl group, the cycle

$$c_1(L(\rho_w))^p \in \text{CH}(\mathfrak{X}_0) \otimes \mathbb{F}_p$$

is rational.

3 The $J$-invariant

In this section, we briefly recall the definition and key properties of the $J$-invariant following [PSZ08].

3.1 The Chow ring of $G_0$. Let us denote by $\pi: \text{CH}^*(\mathfrak{X}_0) \to \text{CH}^*(G_0)$ the pull-back induced by the natural projection $G_0 \to \mathfrak{X}_0$, where $\mathfrak{X}_0$ is the variety of Borel subgroups of $G_0$. By [Gr58, §4, Rem. 2], $\pi$ is surjective and its kernel is the ideal $I(\mathfrak{c}) \subset \text{CH}^*(\mathfrak{X}_0)$ generated by non-constant elements in the image of the characteristic map (see Section 2). Therefore, there is an isomorphism of graded rings

$$\text{CH}^*(\mathfrak{X}_0)/I(\mathfrak{c}) \simeq \text{CH}^*(G_0).$$

In particular, in degree 1, we get

$$\text{CH}^1(G_0) \simeq \text{CH}^1(\mathfrak{X}_0)/(\text{im } c^{(1)}) \simeq \Lambda_w/T_0. \quad (1)$$

By [Ke85, Thm. 3] the Chow ring of $G_0$ with $\mathbb{F}_p$-coefficient is isomorphic as an $\mathbb{F}_p$-algebra (and even as a Hopf algebra) to

$$\text{Ch}^*(G_0) \simeq \mathbb{F}_p[x_1, \ldots, x_r]/(x_1^{k_1}, \ldots, x_r^{k_r}) \quad (2)$$

for some integers $r$, $k_i$ and degrees $d_i$ of $x_i$ such that $d_1 \leq \ldots \leq d_r$. Below in 3.3 we explain, how we choose this isomorphism in $D_{2m}$ case. The number of generators of degree 1 of $\text{Ch}^*(G_0)$, denoted by $s$, is the dimension over $\mathbb{F}_p$ of the vector space $\Lambda_w/T_0 \otimes \mathbb{F}_p$. 

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3.2 Motivic decomposition. Let $G$ be an inner form of $G_0$ that is $G = \xi G_0$ for some $\xi \in Z^1(F, G_0)$. Consider the variety $X$ of Borel subgroups of $G$. Recall that $X \cong \xi X_0$. The main result (Thm. 5.13) in [PSZ08] asserts that the Chow-motive $M(X)$ splits as a direct sum of twisted copies of some indecomposable motive $R_p(G)$ and the Poincaré polynomial of $R_p(G)$ over a separable closure of $F$ (see [PSZ08, §1.3]) is given by
\[
P(R_p(G) \times_F F_s, t) = \prod_{i=1}^r \frac{1 - t^{d_i}x_{i_j}}{1 - t^d}, \quad \text{for some } 0 \leq j_i \leq k_i.
\] (3)

The parameters $r$, $d_i$ and $k_i$ for $i = 1, \ldots, r$ are the same as in the Chow ring of $G_0$, but the integers $j_i$ depend on $\xi$.

In this way we obtain a multiset of pairs
\[
\{(d_1, j_1), \ldots, (d_r, j_r)\}
\] (4)
with $d_1 \leq \ldots \leq d_r$ and $0 \leq j_i \leq k_i$ for each $i = 1 \ldots r$. Recall from [PSZ08, §4] that $j_i = k_i$ for each $i = 1 \ldots r$ if $\xi$ is a generic cocycle, and $j_1 = \ldots = j_r = 0$ if and only if $X$ has a point of degree prime to $p$, or equivalently, if $\xi$ splits over a $p$-primary closure of the base field $F$. Moreover, as explained in [PSZ08, 4.7] the integers $j_i$ can only decrease after extension of the base field.

Observe that the multiset $\{(d_1, j_1), \ldots, (d_r, j_r)\}$ depends only on the group $G$ and can be viewed as an invariant of $G$.

Let now $G_{an}$ denote the semisimple anisotropic kernel of $G$ and let
\[
\{(d'_1, j'_1), \ldots, (d'_m, j'_m)\}
\] be its (unordered) $J$-invariant. It follows from [PSZ08, Cor. 5.19] and formula (3) that the multisets $\{(d_i, j_i) \mid j_i \neq 0\}$ and $\{(d'_i, j'_i) \mid j'_i \neq 0\}$ are equal, i.e. the non-zero entries in the $J$-invariants of $G$ and $G_{an}$ are the same.

3.3 The $J$-invariant. Choosing a cocycle $\xi \in Z^1(F, G_0)$ and an isomorphism of the $J$-invariant allows us to compute the integers $j_i$ from (4) as follows: Let $R_\xi$ denote the subring
\[
\text{Im}(\text{Ch}^*(\xi X_0) \xrightarrow{\pi \circ \text{regCh}} \text{Ch}^*(G_0) \xrightarrow{[2]} \mathbb{F}_p[x_1, \ldots, x_r]/(x_1^{p^{j_1}}, \ldots, x_r^{p^{j_r}})),
\] (5)
where the restriction map is defined in [2,3]. Following [PSZ08, Definition 4.6] we introduce the deglex order on the set of generators $\{x_1, \ldots, x_r\}$ with $x_1 < \ldots < x_r$. Define $j_i$ to be the smallest non-negative integer $a$ such that $x_1^a \mod$ smaller terms lies in $R_\xi$. We get an ordered $r$-tuple of integers $(j_1, \ldots, j_r)$, whose elements precisely are the indices $j_1, \ldots, j_r$ of (4). We call this tuple the $J$-invariant of $\xi$ with respect to (4). In particular, the first element $j_1$ of the $J$-invariant is the smallest non-negative integer $a$ such that $x_1^a$ belongs to $R_\xi$. Hence, $j_1 = 0$ if and only if $x_1 \in R_\xi$.

3.4 Example. One can check from the values given in the table [Kc85, Table II] (see also [PSZ08, §4]), that for simple groups $G$, except if $p = 2$ and

8
determined by a choice of generators $x_j$ of the group $G$ which doesn’t depend on a choice of a cocycle with $G \simeq \xi G_0$ and an isomorphism \((\ref{eq:iso})\). We denote it by $J_p(G)$.

3.5 Example. Assume that $p = 2$ and $G_0$ is adjoint of type $D_{2m}$, with $m \geq 1$, that is $G_0 = \text{PGO}^+_1$. In this case $s = 2$ and an isomorphism \((\ref{eq:iso})\) is uniquely determined by a choice of generators $x_1$ and $x_2$ of degree one. In view of \((\ref{eq:iso})\), choosing two degree 1 generators for $\text{Ch}^*G_0$ amounts to the choice of two generators of the cocenter $\Lambda_\omega/\Lambda_r = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ of the group. We set

$$x_1 = \pi(h_1), \quad x_2 = \pi(h_{2m}) \text{ if } m \neq 1 \text{ and } x_1 = \pi(h_1 + h_2), \quad x_2 = \pi(h_2) \text{ if } m = 1,$$

where $h_i = c_1(L(\omega_i))$ as in Section \ref{sec:psz}. Observe that this is compatible with \cite{PSZ}, since the relations $x_1^{x_1} = x_2^{x_2} = 0$ are fulfilled. By definition $j_2$ is then the smallest integer $b$ such that

$$x_2^b + \sum_{0 < i \leq 2^b} a_i x_1^i x_2^{2^b-i} \in R_\xi \text{ for some } a_i \in F_2.$$

3.6 The $J$-invariant of an algebra with involution. Let $G_0 = \text{PGO}^+_1$ and consider $\xi \in Z^1(F, G_0)$. By \cite{KMRT} p. 409 the class of $\xi$ corresponds to a central simple algebra $A$ of degree $4m$ with orthogonal involution $\sigma$ and a designation of the two components of its Clifford algebra. Note that, if $m \geq 3$, $A$ is the Tits algebra $A_{\omega_1}$. Consider the two cocycles $\xi^+ = \xi$ and $\xi^-$ corresponding to the opposite designation of the components of the Clifford algebra.

We claim that the $J$-invariants of $\xi^+$ and $\xi^-$ with respect to \((\ref{eq:iso})\) are equal. Indeed, since the two cocycles lead to the same group $\text{PGO}^+(A, \sigma)$, the motivic point of view recalled in § \ref{sec:psz} shows that the two tuples can only differ by a permutation of $j_1$ and $j_2$, which both correspond to generators of degree 1. Hence, it suffices to compute the first entry $j_1$ of the $J$-invariants of $\xi^+$ and $\xi^-$. The images of the restriction maps defined by $\xi_+$ and $\xi_-$ differ by an automorphism of $\text{Ch}^*(X_0)$, which permutes $h_{2m-1}$ and $h_{2m}$ and, hence, leaves the ideal $I(\xi)$ invariant. This induces a ring automorphism of $\text{Ch}^*(G_0)$, which fixes $x_1$. Hence, $x_1^{x_1} \in R_{\xi_+}$ if and only if $x_1^{2^b} \in R_{\xi_-}$ and, thus, $j_1$ for $\xi^+$ and $\xi^-$ are equal.

So the tuple $(j_1, \ldots, j_r)$ is an invariant of the algebra with involution $(A, \sigma)$, denoted by $J(A, \sigma)$. If we are not in the trialitarian case, i.e., if $m \neq 2$, the algebra with involution $(A, \sigma)$ is uniquely determined by its automorphism group; therefore, $(j_1, \ldots, j_r)$ also is an invariant of the group $G$, which does not depend on the choice of a cocycle, and denoted by $J_2(G)$.

3.7 Remark. We could also take $x_1 = \pi(h_1)$ and $x_2 = \pi(h_{2m-1})$ in \((\ref{eq:iso})\), this would not affect the value of the $J$-invariant.

3.8 The trialitarian case. In the trialitarian case, i.e., if $m = 2$, the twisted group $G = \xi G_0$ can be described as the connected component of the automorphism group of three possibly non-isomorphic degree 8 algebras with orthogonal
involution, which are the Tits algebras of the weights $\omega_1$, $\omega_3$ and $\omega_4$ (see [KMR11 §42])

$$G \simeq \text{PGO}^+(A, \sigma_A) \simeq \text{PGO}^+(B, \sigma_B) \simeq \text{PGO}^+(C, \sigma_C).$$

Therefore, the $J$-invariant does depend in this case of the choice of a cocycle. Precisely, picking a cocycle determines an ordering of those three algebras with involution, and the $J$-invariant of the cocycle is the $J$-invariant of the first algebra with involution in the corresponding triple. Nevertheless, since the automorphism group is the same, the variety $X$ and its motive do not depend on this choice. Hence, the multiset $\{(1,j_1), (1,j_2), (2,j_3)\}$ is the same for all three algebras with involution, so that if $J(A, \sigma_A) = (j_1, j_2, j_3)$, then

$$J(B, \sigma_B), J(C, \sigma_C) \in \{(j_1, j_2, j_3), (j_2, j_1, j_3)\}.$$

In Theorem 6.3 and Example 6.8 below, we give a more precise statement, and provide explicit examples of algebras with involution having isomorphic automorphism groups and different $J$-invariants.

### 4 The $J$-invariant in degree one and indices of the Tits algebras

In this section, we prove the main results of the paper, which give connections between the indices of the $J$-invariant corresponding to generators of degree 1 and indices of Tits algebras of the group $G$ (cf. [PS10 §4]).

#### 4.1 Notation.

From now on, we let $s$ be the dimension over $\mathbb{F}_p$ of $\Lambda_{\omega}/\hat{T}_0 \otimes \mathbb{F}_p \simeq \text{Ch}^1(G_0)$, and we fix $G_0$ and a prime $p$ so that $s \geq 1$. We fix a cocycle $\xi$, and a presentation (2) of the Chow group of $G_0$. The $J$-invariant in this section refers to the $J$-invariant of $\xi$ with respect to (2). Moreover, we assume throughout this section that the degree 1 generators are given by $x_\ell = \pi(h_{i_\ell}) \in \text{Ch}^1(G_0)$ for some integers $i_1, \ldots, i_s$, chosen so that the classes $\omega_{i_\ell}$, $\ell = 1, \ldots, s$, generate $\Lambda_{\omega}/\hat{T}_0 \otimes \mathbb{F}_p$.

Note that one can always choose generators $x_\ell$ in such a way, though this conflicts the convention we made in (6) to define the $J$-invariant of a degree 4 algebra with involution.

Consider the special elements $g_i$, $i = 1 \ldots n$ of the Steinberg basis of $K_0(\mathcal{X}_0)$ (see Definition 2.9). Since $c_1(g_i) = h_i - c_1(\mathcal{L}(\alpha_i)) \in \text{Ch}^1(\mathcal{X}_0)$, we have

$$\pi(c_1(g_i)) = \pi(h_i) - \pi(c_1(\mathcal{L}(\alpha_i))) = \pi(h_i) \in \text{Ch}^1(G_0).$$

Hence $x_\ell = \pi(c_1(g_{i_\ell}))$. In view of the isomorphism (1), it follows that for any $g \in \text{Pic}(\mathcal{X}_0)$ its Chern class modulo $p$ can be written as

$$c_1(g) = \sum_{\ell=1}^s a_\ell c_1(g_{i_\ell}) \mod \text{im}(\mathcal{C}^{(1)}) \in \text{Ch}^1(\mathcal{X}_0) \quad (7)$$

As an immediate consequence of rationality of cycles introduced in Lemma 2.12, we obtain a different proof of the first part of [PS10 Prop. 4.2]:

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[1] Kottwitz, M., Rapoport, M.: Reductive groups and Shimura varieties. Cambridge University Press, Cambridge (2004).  
[2] Prasad, D., Rapoport, M.: On the Hasse principle for $G_0(A)^+$-torsors. J. Reine Angew. Math. 500, 65–89 (1998).
4.2 Corollary. For \( \ell = 1, \ldots, s \) the first entries \( j_\ell \) of the \( J \)-invariant are bounded

\[ j_\ell \leq i_{i_\ell}, \]

by the \( p \)-adic valuation \( i_{i_\ell} \) of the index of the Tits algebra \( A_{\omega_{i_\ell}} \) associated with \( \omega_{i_\ell} \).

Proof. We apply lemma 2.12 to the weight \( \rho_{s_{i_\ell}} = \omega_{i_\ell} - \alpha_{i_\ell} \). As noticed in 2.10 the index \( i(\rho_{s_{i_\ell}}) \) is equal to the index \( i(\omega_{i_\ell}) \) of the Tits algebra \( A_{\omega_{i_\ell}} \). Hence, the cycle \( c_1(g_{i_\ell})^{p^{i_{i_\ell}}} \) is rational, and its image \( x_\ell^{p^{i_{i_\ell}}} \in \text{Ch}^*(G_0) \) belongs to \( R := R_\xi \). The inequality then follows from the definition of \( j_\ell \) (see 3.3).

Assume now that \( p = 2 \) and \( G_0 \) is adjoint of type \( D_{2m} \) with \( m \geq 2 \). As explained in 3.5, we take \( i_1 = 1 \), \( i_2 = 2m \), and define \( x_1 \) and \( x_2 \) as in [9].

4.3 Corollary. If \( p = 2 \) and \( G \) is adjoint of type \( D_{2m} \) with \( m \geq 2 \), we have \( s = 2 \),

\[ j_1 \leq i_1 \text{ and } j_2 \leq \min\{i_{2m-1}, i_{2m}\}, \]

where \( i_\ell \) is the \( p \)-adic valuation of the index of the Tits algebra \( A_{\omega_\ell} \).

The next result, which gives an inequality in the other direction, uses the notion of common index, which we introduce now.

4.4 Definition. Consider the Tits algebras \( A_{\omega_{i_\ell}} \) associated to the fundamental weights \( \omega_{i_\ell} \), for \( 1 \leq \ell \leq s \), where \( i_\ell \) are as in 4.1. We define their common index \( i_{J} \) to be the \( p \)-adic valuation of the greatest common divisor of all the indices \( \text{ind}(A_{\omega_{i_1}} \otimes \ldots \otimes A_{\omega_{i_s}}) \), where at least one of the \( a_i \) is coprime to \( p \).

4.5 Example. If \( s = 1 \), then \( i_{J} \) is the \( p \)-adic valuation \( i_{i_1} \) of the index of the Tits algebra \( A_{\omega_{i_1}} \). Assume for instance that \( G \) is adjoint of type \( D_{2m+1} \). As recalled in [7] we may take \( i_1 = 2m \) or \( i_1 = 2m + 1 \), so that \( i_{J} \) is the \( 2 \)-adic valuation of any component \( C_+ \) or \( C_- \) of the Clifford algebra of \( (A, \sigma) \). From the fundamental relations [8] we know that the two components have the same index.

4.6 Example. If \( p = 2 \) and \( G_0 \) is adjoint of type \( D_{2m} \) with \( m \geq 2 \), then \( s = 2 \). Using [8] one can check that \( i_{J} \) is the \( p \)-adic valuation of the greatest common divisor of the indices of \( A_{\omega_{1}}, A_{\omega_{2m-1}} \) and \( A_{\omega_{2m}} \), that is

\[ i_{J} = \min\{i_1, i_{2m-1}, i_{2m}\}. \]

We will prove:

4.7 Theorem. Let \( i_{J} \) be the common index of the Tits algebras \( A_{\omega_{i_\ell}}, \ell = 1 \ldots s \).

If \( i_{J} > 0 \), then \( j_\ell > 0 \) for every \( \ell, 1 \leq \ell \leq s \).

If \( i_{J} > 1 \) and \( p = 2 \), then for every \( \ell \) such that \( k_\ell > 1 \) we have \( j_\ell > 1 \).
Proof. Consider the ideal $I(\text{res}_{\text{Ch}})$ of $\text{Ch}^i(\mathcal{X}_0)$ generated by non-constant rational elements. For any integer $i$, we let $I(\text{res}_{\text{Ch}})^{(1)} \subset \text{Ch}^i(\mathcal{X}_0)$ be the homogeneous part of degree $i$. Since the image of the characteristic map consists of rational elements, we have $I(c) \subset I(\text{res}_{\text{Ch}})$. The theorem follows immediately from the following lemma:

4.8 Lemma. If $i_j > 0$, then $I(\text{res}_{\text{Ch}})^{(1)} = I(c)^{(1)} \subset \text{Ch}^1(\mathcal{X}_0)$.
If $i_j > 1$ and $p = 2$, then $I(\text{res}_{\text{Ch}})^{(2)} = I(c)^{(2)} \subset \text{Ch}^2(\mathcal{X}_0)$.

Indeed, let us assume first that $i_j > 0$. By the lemma, any element in $\text{im}(\text{res}_{\text{Ch}})^{(1)}$ belongs to $I(c)^{(1)}$, which is in the kernel of $\pi$. Therefore, the image of the composition

$$R_{\xi}^{(1)} = \text{im}(\text{Ch}^i(\mathcal{X}) \xrightarrow{\text{res}_{\text{Ch}}} \text{Ch}^1(\mathcal{X}_0) \xrightarrow{\pi} \text{Ch}^1(G_0))$$

is trivial, $R_{\xi}^{(1)} = \{0\}$. From the definition 3.3 of $j_1$ and $j_2$, this implies that they are both strictly positive.

The proof of the second part follows the same lines. We write it in details for $s = 2$ and $k_1, k_2 > 1$. Assume that $i_j > 1$. Since the image $\text{im}(\text{res}_{\text{Ch}})^{(2)}$ is contained in $I(\text{res}_{\text{Ch}})^{(2)}$, the lemma again implies that $R_{\xi}^{(2)} = \{0\}$. On the other hand, the hypothesis on $k_1$ and $k_2$ guarantees that in the truncated polynomial algebra $\mathbb{F}_2[x_1, x_2]/(x_1^{2k_1}, x_2^{2k_2}) \subset \text{Ch}^i(G_0)$, the elements $x_1^2 + a_1 x_1 x_2 + a_2 x_2^2$ are all non-trivial. Hence they do not belong to $R_{\xi}$, and we get $j_1, j_2 > 1$.

The rest of the section is devoted to the proof of Lemma 4.8. The main tool is the Riemann-Roch theorem, which we now recall.

4.9 Filtrations of $K_0$ and the Riemann-Roch Theorem. Let $X$ be a smooth projective variety over $F$. Consider the topological filtration on $K_0(X)$ given by

$$\tau^i K_0(X) = \langle [\mathcal{O}_V], \text{codim } V \geq i \rangle,$$

where $\mathcal{O}_V$ is the structure sheaf of the closed subvariety $V$ in $X$. There is an obvious surjection

$$\psi: \text{CH}^i(X) \rightarrow \tau^{i/i+1} K_0(X) = \tau^i K_0(X)/\tau^{i+1} K_0(X),$$

given by $V \mapsto [\mathcal{O}_V]$. By the Riemann-Roch theorem without denominators [Ful §15], the $i$-th Chern class induces a map in the opposite direction

$$c_i: \tau^{i/i+1} K_0(X) \rightarrow \text{CH}^i(X)$$

and the composite $c_i \circ \psi$ is the multiplication by $(-1)^{i-1}(i-1)!$. In particular, it is an isomorphism for $i \leq 2$ (see [Ful Ex. 15.3.6]).

The topological filtration can be approximated by the so-called $\gamma$-filtration. Let $c_{\gamma}^{-K_0}$ be the $i$-th Chern class with values in $K_0$ (see [Ful Ex. 3.2.7(b)],
or [Ka98 §2]. We use the convention $c^1_1([L]) = 1 - [L^u]$ for any line bundle $L$, where $L^u$ is the dual of $L$. Similarly, one can compute the second Chern class

$$c_2(c^1_1([L_1])c^1_1([L_2])) = -c_1(L_1)c_1(L_2). \quad (8)$$

The $\gamma$-filtration on $K_0(X)$ is given by the subgroups (cf. GZ10 §1)

$$\gamma^i K_0(X) = \langle c^1_{n_1}(b_1) \cdots c^1_{n_m}(b_m) \mid n_1 + \ldots + n_m \geq i, \; b_i \in K_0(X) \rangle,$$

(see [Ex1 Ex.15.3.6], [FL Ch.3,5]). We let $\gamma^{i+1}(K_0(X)) = \gamma^i K_0(X)/\gamma^{i+1} K_0(X)$ be the respective quotients, and $\gamma^* (X) = \bigoplus_{i \geq 0} \gamma^{i+1}(K_0(X))$ the associated graded ring.

By [Ka98 Prop. 2.14], $\gamma^* (K_0(X))$ is contained in $\tau^i (K_0(X))$, and they coincide for $i \leq 2$. Hence, by the Riemann-Roch theorem, the Chern class $c_i$ with values in $CH^i(X)$ vanishes on $\gamma^{i+1}(K_0(X))$, and induces a map

$$c_i : \gamma^{i+1}(K_0(X)) \to CH^i(X).$$

In codimension 1 we get an isomorphism

$$c_1 : \gamma^{1/3}(K_0(X)) \xrightarrow{\cong} CH^1(X)$$

which sends for a line bundle $L$ the class $c^1 K_0(L)$ to $c_1(L)$. In codimension 2 the map

$$c_2 : \gamma^{2/3}(K_0(X)) \to CH^2(X),$$

is surjective and has torsion kernel [Ka98 Cor. 2.15].

Let us now apply this to the varieties $X_0$ and $X$ of Borel subgroups of $G_0$ and $G$ respectively. Since $K_0(X_0)$ is generated by the line bundles $g_w = [L(p_w)]$ for $w \in W$, one can check that $\gamma^{i+1}(X_0)$ is generated by the products

$$\{ c^1_{K_0}(g_{w_1}) \cdots c^1_{K_0}(g_{w_t}) \mid w_1, \ldots, w_t \in W \}.$$

Moreover, the restriction map commutes with Chern classes, so it induces

$$\text{res}_\gamma : \gamma^* (X) \to \gamma^* (X_0).$$

Using Panin’s description of the image of the restriction map $\text{res}_{K_0}$ we obtain that the image of $\text{res}_{K_0}^{1/3} : \gamma^{1/3}(X) \to \gamma^{1/3}(X_0)$ is generated by the elements $c^1_{K_0}(i(p_w) g_w) = i(p_w)c^1_{K_0}(g_w)$, for any $w \in W$, while the image of $\text{res}_{K_0}^{2/3}$ is generated by

$$i(p_{w_1})i(p_{w_2})c^1_{K_0}(g_{w_1})c^1_{K_0}(g_{w_2}) \text{ and } c^2_{K_0}(i(p_w) g_w) \text{ for } w_1, w_2, w \in W.$$

If the index $i(p_w)$ is 1, then $c^2_{K_0}(i(p_w) g_w) = 0$. Otherwise, the Whitney sum formula gives

$$c^2_{K_0}(i(p_w) g_w) = \binom{i(p_w)}{2} c^1_{K_0}(g_w)^2.$$
Lemma. The subgroup $c_1(\text{im}(\text{res}_{c_1}^{(1)})) \in \text{CH}^1(X_0)$ is generated by $i(\rho_w) c_1(g_w)$, for all $w \in W$. The subgroup $c_2(\text{im}(\text{res}_{c_2}^{(2)})) \in \text{CH}^2(X_0)$ is generated by the elements $i(\rho_{w_1}) i(\rho_{w_2}) c_1(g_{w_1}) c_1(g_{w_2})$ and $\left(\frac{i(\rho_w)}{2}\right) c_1(g_w)^2$ for all $w_1, w_2, w \in W$.

Proof of Lemma 4.8. Since the image of the characteristic map consists of rational elements (see 2.3), we already know that $I(\mathcal{E}) \subset I(\text{res}_{\text{Ch}})$. We now prove the reverse inclusions for the homogeneous parts of degree 1 and 2 under the relevant hypothesis on the common index $i_J$. Note that since $c_1$ and $c_2$ are both surjective, and commute with restriction maps, one has

$$\text{im}(\text{res}_{c_1}^{(1)}) = c_k(\text{im}(\text{res}_{c_k}^{(k)})), \text{ for } k = 1, 2.$$ 

In degree 1, we have $I(\text{res}_{\text{Ch}})^{(1)} = \text{im}(\text{res}_{\text{Ch}}^{(1)})$, so to prove the first part of the lemma, we have to prove that if $i_J > 0$, then for any $w \in W$, the element $i(\rho_w) c_1(g_w)$ belongs, after tensoring with $F_p$, to $I(\mathcal{E})^{(1)} = \text{im} \mathcal{E}^{(1)}$. Let us write

$$c_1(g_w) = \sum_{\ell=1}^{s} a_\ell c_1(g_\ell) \mod \mathcal{E}^{(1)},$$

as in (7). If all the $a_\ell \in F_p$ are trivial, we are done, so we may assume at least one of them is invertible in $F_p$. The weights $\rho_w$ and $\rho_\ell$ satisfy the same relation

$$\rho_w = \sum_{\ell=1}^{s} a_\ell \rho_\ell \mod \hat{T}_0 \otimes \mathbb{Z}/p.$$ 

Applying the morphism $\beta$, we get that the $p$-primary part of the Brauer class $\beta(\hat{T}_0)$ coincides with the $p$-primary part of the Brauer class of $\otimes_{\ell=1}^{s} A_{w_\ell}^{a_\ell}$ (see 2.10). The hypothesis on $i_J$ guarantees that this index of this algebra is divisible by $p$. Hence $i(\rho_w)$, which is the index of $\beta(\hat{T}_0)$, is also divisible by $p$, so that $i(\rho_w) c_1(g_w) = 0$ in the Chow group $\text{CH}^1(X_0)$ modulo $p$, and we are done.

Let us now assume that $p = 2$ and $i_J > 1$. The homogeneous part $I(\text{res}_{\text{Ch}})^{(2)}$ decomposes as

$$I(\text{res}_{\text{Ch}})^{(2)} = \text{im}(\text{res}_{\text{Ch}}^{(1)}) \text{CH}^1(X_0) + \text{im}(\text{res}_{\text{Ch}}^{(2)}).$$

By the first part of the Lemma, we already know that

$$\text{im}(\text{res}_{\text{Ch}}^{(1)}) \text{CH}^1(X_0) \subset I(\mathcal{E}).$$

Hence it remains to prove that $\text{im}(\text{res}_{\text{Ch}}^{(2)}) = c_2(\text{im}(\text{res}_{\text{Ch}}^{(2)})) \subset I(\mathcal{E})^{(2)}$. The proof for the degree 1 part already shows that $i(\rho_{w_1}) i(\rho_{w_2}) c_1(g_{w_1}) c_1(g_{w_2})$ belongs to $I(\mathcal{E})^{(2)}$. The same argument extends to $\left(\frac{i(\rho_w)}{2}\right) c_1(g_w)^2$. Indeed, if the coefficients $a_\ell$ are not all trivial modulo 2, the condition on the common index now implies that 4 divides $i(\rho_w)$, so that $\left(\frac{i(\rho_w)}{2}\right)$ is zero modulo 2. \qed
5 Applications to quadratic forms and algebras with orthogonal involutions

Let \( \varphi \) be a non-degenerate quadratic form of even dimension \( 2n \). We always assume that \( \varphi \) has trivial discriminant, so that its special orthogonal group \( O^+(\varphi) \) satisfies condition 1.3. We define the \( J \)-invariant of \( \varphi \) as

\[
J(\varphi) = J_2(O^+(\varphi)).
\]

Let \( \varphi_0 \) be any non-degenerate subform of \( \varphi \) of codimension 1. Since \( \varphi \) has trivial discriminant, \( \varphi \) and \( \varphi_0 \) have the same splitting fields. In particular, each of them splits over the function field of the maximal orthogonal Grassmannian of the other. Therefore by the comparison lemma [PSZ08, 5.18(iii)], the corresponding indecomposable motives \( R_2(O^+(\varphi_0)) \) and \( R_2(O^+(\varphi)) \) are isomorphic. Hence they have the same Poincaré polynomial, and by (3), it follows that \( O^+(\varphi_0) \) and \( O^+(\varphi) \) have the same \( J \)-invariant. Since any odd-dimensional form can be embedded in an even-dimensional form with trivial discriminant, we only consider the even-dimensional case in the sequel.

Theorem 4.7 immediately implies:

5.1 Corollary. Let \( \varphi \) be a \( 2n \)-dimensional quadratic form with trivial discriminant. The 2-adic valuation \( i_S \) of its Clifford algebra and the first index \( j_1 \) of its \( J \)-invariant are related as follows:

1. \( j_1 \leq i_S \);
2. If \( n \geq 2 \), and \( i_S > 0 \), then \( j_1 > 0 \);
3. If \( n \geq 3 \) and \( i_S > 1 \), then \( j_1 > 1 \).

Let now \( (A,\sigma) \) be a degree \( 2n \) central simple algebra over \( F \), endowed with an involution of orthogonal type and trivial discriminant. In particular, this implies that \( A \) has exponent 2, so that it has index \( 2^{i_A} \) for some integer \( i_A \). The connected component \( PGO^+(A,\sigma) \) of the automorphism group of \( (A,\sigma) \) is an adjoint group of type \( D_n \). Because of the discriminant hypothesis, it is an inner twisted form of \( PGO_{2n} \).

The \( J \)-invariant \( J(A,\sigma) \) was defined in section 3. From the table [PSZ08, 4.13] (see also the appendix below), \( J(A,\sigma) \) is an \( r \)-tuple \( (j_1, j_2, \ldots, j_r) \), with \( r = m + 1 \) if \( n = 2m \) and \( r = m \) if \( n = 2m + 1 \). Note that our notation slightly differs from the notation in the table, where in the \( n \)-odd case, there is an additional index, but which is bounded by \( k_1 = 0 \). So, for \( n \) odd, our \( (j_1, \ldots, j_r) \) coincides with \( (j_2, \ldots, j_{r+1}) \) in [PSZ08]. In particular, the indices corresponding to generators of degree 1 are \( j_1 \) if \( n \) is odd and \( j_1 \) and \( j_2 \) if \( n \) is even.

Since \( \sigma \) has trivial discriminant, its Clifford algebra splits as a direct product \( C(A,\sigma) = C_+ \times C_- \) of two central simple algebras over \( F \). We let \( i_A \) (respectively \( i_+, i_- \)) be the 2-adic valuation of the index of \( A \) (respectively \( C_+, C_- \)). From Examples 4.5 and 4.6, the common index \( i_J \) is

\[
i_J = \begin{cases} 
  i_+ = i_- & \text{if } n \text{ is odd}, \\
  \min\{i_A, i_+, i_-\} & \text{if } n \text{ is even}.
\end{cases}
\]
Hence, Corollaries 4.2 and 4.3 and Theorem 4.7 translate as follows:

5.2 Corollary. Depending on the parity of $n = \deg(A)/2$ we have

| $n$ is even, $n \neq 2$ and $i_j = \min\{i_A, i_+, i_-\}$ | $n$ is odd and $i_S = i_+ = i_-$ |
|---|---|
| (1) $j_1 \leq i_A$; | (1) $j_1 \leq i_S$; |
| (2) $j_2 \leq \min\{i_+, i_-\}$; | (2) If $i_S > 0$, then $j_1 > 0$; |
| (3) If $i_j > 0$, then $j_1 > 0$ and $j_2 > 0$. | (3) If $\deg(A) \geq 6$ and $i_S > 1$, then $j_1 > 1$. |
| (4) If $\deg(A) \equiv 0[8]$ and $i_j > 1$, then $j_1 > 1$. | |
| (5) If $\deg(A) \geq 8$ and $i_j > 1$, then $j_2 > 1$. | |

The additional conditions on the degrees are obtained from the table \textit{PSZ08} 4.13, and guarantee that $k_1 > 1$ or $k_2 > 1$.

We say that an algebra $(A, \sigma)$ with $\deg(A) \equiv 0[4]$ is half-spin, if one of the components of its Clifford algebra is split. As explained in \textit{Ga09} 4.1, this happens if and only if one of the cocycles $\xi \in Z^1(F, PGO_4^+)$ associated to $(A, \sigma)$ lifts to the half-spin group. Therefore, we can refine the inequalities in this case by applying Theorem 4.7 to the corresponding twisted group $\text{Spin}^+(A, \sigma)$. The common index is $i_j = i_A$, and we get:

5.3 Corollary. If $\deg(A) \equiv 0[4]$ and $(A, \sigma)$ is half-spin, then:

(1) If $i_A > 0$, then $j_1 > 0$. 

(2) If $i_A > 1$, then $j_1 > 1$.

5.4 Remark. Using \textit{PS10} Prop. 4.2 one can show that $(A, \sigma)$ is half-spin iff $j_2 = 0$, and $A$ is split iff $j_1 = 0$.

6 The trialitarian case

From now on, we assume that $(A, \sigma)$ has degree 8. The $J$-invariant of $(A, \sigma)$ is a triple $J(A, \sigma) = (j_1, j_2, j_3)$ with $0 \leq j_1, j_2 \leq 2$ and $0 \leq j_3 \leq 1$. In this section, we will explain how to compute $J(A, \sigma)$. As a consequence of our results, we will prove:

6.1 Corollary. (i) There is no algebra of degree 8 with orthogonal involution having $J$-invariant equal to $(1, 2, 0)$, $(2, 1, 0)$ or $(2, 2, 0)$.

(ii) All other possible values do occur.

In particular, this shows that the restrictions described in the table \textit{PSZ08} 4.13 (see also \textit{KMRT} 8.10), which were obtained by applying the Steenrod operations on $\text{Ch}^*(G_0)$ (loc. cit. 4.12), are not the only ones.

Recall that the group $\text{PGO}^+(A, \sigma)$ is of type $D_4$. To complete the classification in this case, we need to understand the action of the symmetric group $S_3$ on the $J$-invariant (see \textit{KMRT} 8.8). Let $(B, \tau)$ and $(C, \gamma)$ be the two components of the Clifford algebra $C(A, \sigma)$, each endowed with its canonical involution. It follows from the structure theorems \textit{KMRT} (8.10 and 8.12) that both are degree 8 algebras with orthogonal involutions. The triple $((A, \sigma), (B, \tau), (C, \gamma))$ is a trialitarian triple in the sense of loc. cit. \S42A, and in particular, the Clifford algebra of any of those three algebras with involution is the direct product of
the other 2. Hence, if one of them, say \((A, \sigma)\) is split, then the other two are half-spin.

6.2 Definition. The trialitarian triple \(\left((A, \sigma), (B, \tau), (C, \gamma)\right)\) is said to be ordered by indices if the indices of the algebras \(A, B\) and \(C\) satisfy

\[
\text{ind}(A) \leq \text{ind}(B) \leq \text{ind}(C).
\]

In the next theorem we compute the \(J\)-invariant of such a triple. We remark that we don’t know any elementary proof, which does not use both inequalities from Theorem 4.7.

6.3 Theorem. Let \(\left((A, \sigma), (B, \tau), (C, \gamma)\right)\) be a trialitarian triple ordered by indices, so that \(i_A \leq i_B \leq i_C\). The \(J\)-invariants are given by

\[
J(A, \sigma) = (j, j', j_3) \quad \text{and} \quad J(B, \tau) = J(C, \gamma) = (j', j, j_3),
\]

where \(j = \min\{i_A, 2\}\) and \(j' = \min\{i_B, i_C, 2\} = \min\{i_B, 2\}\).

Moreover, the third index \(j_3\) is 0 if the involution is isotropic and 1 otherwise.

6.4 Remark. (i) The first index of the \(J\)-invariant of a degree 8 algebra with involution \((D, \rho)\) is \(\min\{i_D, 2\}\) if \(D\) is not of maximal index in its triple. But if \(\text{ind}(D)\) is maximal, then \(j_1\) might be strictly smaller. In Example 6.9 below, we will give an explicit example where \(j_1 < i_D = 2\).

(ii) By 3.8, we already know that \(j_3\) does not depend on the choice of an element of the triple. On the other hand, as explained in [Ga99], the involutions \(\sigma, \tau\) and \(\gamma\) are either all isotropic or all anisotropic. The triple is said to be isotropic or anisotropic accordingly.

Proof. To start with, let us compute the first two indices \(j_1\) and \(j_2\) of the \(J\)-invariant of \((A, \sigma)\). Since we are in degree 8, they are both bounded by 2. Moreover, the triple being ordered by indices, the common index is given by \(i_J = i_A\). So the equality \(j_1 = j\) follows directly from the inequalities of Corollary 5.2. If additionally \(j' = j\), the very same argument gives \(j_2 = j'\). Assume now that \(j\) and \(j'\) are different, that is \(j < j'\). If so, \(j = 0\) or \(j = 1\).

In the first case, we have \(i_A = 0\) so that the algebra \(A\) is split, and the result follows from Corollary 5.1.

The only remaining case is \(j = i_A = 1\) and \(i_B \geq 2\), so that \(j' = 2\). Consider the function field \(F_A\) of the Severi-Brauer variety of \(A\), which is a generic splitting field of \(A\). By the fundamental relations [1.8], the algebra \(C\) is Brauer equivalent to \(A \otimes B\). Hence Merkurjev’s index reduction formula [Me91] says

\[
\text{ind}(B_{F_A}) = \min\{\text{ind}(B), \text{ind}(B \otimes A)\} = \text{ind}(B).
\]

So the values of \(i_B\) and \(j'\) are the same over \(F\) and \(F_A\). We know the result holds over \(F_A\) by reduction to the split case. Since the index \(j_2\) can only decrease under scalar extension, we get \(j_2 \geq j' = 2\), which concludes the proof in this case.

So the \(J\)-invariant of \((A, \sigma)\) is given by \(J(A, \sigma) = (j, j', j_3)\) for some integer \(j_3\). Let us now compute the \(J\)-invariant of \((B, \tau)\) and \((C, \gamma)\). Recall from 3.8
that \((j, j', j_3)\) and \((j', j, j_3)\) are the only possible values. So, if \(j = j'\), there is no choice and we are done. Again, there are two remaining cases. Assume first that \(j = i_A = 0\) and \(j' \geq 1\), so that \(J(A, \sigma) = (0, j', j_3)\). Since \(A\) is split, \((B, \tau)\) and \((C, \gamma)\) are half-spin, so they have trivial \(j_2\) and this gives the result. Assume now that \(j = 1\) and \(j' = 2\), so that \(J(A, \sigma) = (1, 2, j_3)\). By the previous case, over the field \(F_A\), both \((B, \tau)\) and \((C, \gamma)\) have \(J\)-invariant \((2, 0, j_3)\). So the value over \(F\) has to be \((2, 1, j_3)\).

To conclude the proof, it only remains to compute \(j_3\). If the involution is anisotropic, then by \([\text{Ka00}]\) in the division case, by \([\text{Si05}, \text{Prop. 3}]\) in index 4, and by \([\text{PPS01}, \text{Cor. 3.4}]\) in index 2 (see also \([\text{Ka09}]\)) the triple remains anisotropic after scalar extension to a generic splitting field \(F_A\) of the algebra \(A\), and the \(J\)-invariant over \(F_A\) is \((0, *, *)\). Now extending scalars to a generic splitting field \(F_C\) of the Clifford algebra, by \([\text{La96}, \text{Thm. 4}]\) the respective quadratic form still remains anisotropic, and the \(J\)-invariant equals \((0, 0, 1)\). Hence \(j_3\) is equal to 1 over the generic splitting fields, and this implies \(j_3 = 1\) over \(F\).

If \(\sigma\) is isotropic, then obviously \(J(A, \sigma)\) equals \((0, 2, 0)\), if the semisimple anisotropic kernel of the respective group is of type \(A_3\), equals \((1, 1, 0)\), if the anisotropic kernel is of type \(3A_1\), and equals \((0, 1, 0)\), if it is of type \(2A_1\). □

The first part of Corollary \([6.1]\) follows from Theorem \([6.3]\). Indeed, if one of \(j_1, j_2\) is 2 and the other one is \(\geq 1\), then the algebras \(A, B\) and \(C\) are all three non-split, and \(B\) and \(C\) have index \(\geq 4\). By \([1.6]\) since \(A\) and \(B\) are non-split, the involution \(\gamma\) on \(C\) is not hyperbolic, so it is anisotropic, and the theorem gives \(j_3 = 1\).

**Explicit examples**

We now prove the second part of Corollary \([6.1]\). Obviously, if \(A\) is split, and \(\sigma\) is adjoint to a quadratic form \(\varphi\), then \(J(A, \sigma) = (0, J(\varphi))\), and any triple with \(j_1 = 0\) is obtained for a suitable choice of \(\varphi\). Considering the components of the even Clifford algebra of those quadratic forms, we also obtain all triples with \(j_2 = 0\) by Theorem \([6.3]\). The maximal value \((2, 2, 1)\) is obtained from a generic cocycle; such a cocycle exists by \([\text{KM00}, \text{Thm. 6.4(iii)}]\). Hence, it only remains to prove that the values \((1, 1, 0)\), \((1, 1, 1)\), \((1, 2, 1)\) and \((2, 1, 1)\) occur. For each of those, we will produce an explicit example, inspired by the trialitarian triple constructed in \([\text{QT10}, \text{Lemma 6.2}]\).

Our construction uses the notion of direct sum for algebras with involution, which was introduced by Dejaiffe \([\text{Dej98}]\). Consider two algebras with involution \((E_1, \theta_1)\) and \((E_2, \theta_2)\) which are Morita-equivalent, that is \(E_1\) and \(E_2\) are Brauer equivalent and the involutions \(\theta_1\) and \(\theta_2\) are of the same type. Dejaiffe defined a notion of Morita equivalence data, and explains how to associate to any such data an algebra with involution \((A, \sigma)\), which is called a direct sum of \((E_1, \theta_1)\) and \((E_2, \theta_2)\). In the split orthogonal case, if \(\theta_1\) and \(\theta_2\) are respectively adjoint to the quadratic forms \(\varphi_1\) and \(\varphi_2\), any direct sum of \((E_1, \theta_1)\) and \((E_2, \theta_2)\) is adjoint to \(\varphi_1 \oplus (\lambda)\varphi_2\) for some \(\lambda \in F^*\), and the choice of a Morita-equivalence data precisely amounts to the choice of a scalar \(\lambda\). In general, as the split case shows,
there exist non-isomorphic direct sums of two given algebras with involution. We will use the following characterization of direct sums [QT10] Lemma 6.3:

6.5 Lemma. The algebra with involution \((A, \sigma)\) is a direct sum of \((E_1, \theta_1)\) and \((E_2, \theta_2)\) if and only if there is an embedding of the direct product \((E_1, \theta_1) \times (E_2, \theta_2)\) in \((A, \sigma)\) and \(\deg(A) = \deg(E_1) + \deg(E_2)\).

Slightly extending Garibaldi’s ‘orthogonal sum lemma’ [Ga01] Lemma 3.2], we get:

6.6 Proposition. Let \(Q_1, Q_2, Q_3\) and \(Q_4\) be quaternion algebras such that \(Q_1 \otimes Q_2\) and \(Q_3 \otimes Q_4\) are isomorphic. If \((A, \sigma)\) is a direct sum of \((Q_1, -) \otimes (Q_2, -)\) and \((Q_3, -) \otimes (Q_4, -)\), then one of the two components of the Clifford algebra of \((A, \sigma)\) is a direct sum of \((Q_1, -) \otimes (Q_3, -)\) and \((Q_2, -) \otimes (Q_4, -)\), while the other is a direct sum of \((Q_1, -) \otimes (Q_4, -)\) and \((Q_2, -) \otimes (Q_3, -)\).

6.7 Remark. If one of the four quaternion algebras is split, as we assumed in [QT10], then all three direct sums have a hyperbolic component. Hence they are uniquely defined. This is not the case anymore in the more general setting considered here. The algebra with involution \((A, \sigma)\) does depend on the choice of an equivalence data. Nevertheless, once such a choice is made, its Clifford algebra is well defined. So the equivalence data defining the other two direct sums are determined by the one we have chosen.

Proof. Denote \((E_1, \theta_1) = (Q_1, -) \otimes (Q_2, -)\) and \((E_2, \theta_2) = (Q_3, -) \otimes (Q_4, -)\). By [KMRT] (15.12)], their Clifford algebras with canonical involution are respectively \((Q_1, -) \times (Q_2, -)\), and \((Q_3, -) \times (Q_4, -)\). The embedding of the direct product \((E_1, \theta_1) \times (E_2, \theta_2)\) in \((A, \sigma)\) induces an embedding of the tensor product of their Clifford algebras in the Clifford algebra of \((A, \sigma)\):

\[
((Q_1, -) \times (Q_2, -)) \otimes ((Q_3, -) \times (Q_4, -)) \hookrightarrow (C(A, \sigma), \sigma).
\]

This tensor product splits as a direct product of four tensor products of quaternion algebras with canonical involution; for degree reasons, two of them embed in each component of \(C(A, \sigma)\). To identify them, it is enough to look at their Brauer classes. From the hypothesis, we have Brauer equivalences \(Q_1 \otimes Q_3 \approx Q_2 \otimes Q_4\) and \(Q_1 \otimes Q_4 \approx Q_2 \otimes Q_3\). If \(Q_1 \otimes Q_3\) and \(Q_1 \otimes Q_4\) are not Brauer equivalent, that is if \(A\) is non-split, this concludes the proof. Otherwise, all four tensor products are isomorphic, and the result is still valid. 

With this in hand, we now give explicit examples of algebras with involution having \(J\)-invariant \((1, 2, 1), (2, 1, 1), (1, 1, 1)\), and \((1, 1, 0)\).

6.8 Example. Let \(F = K(x, y, z, t)\) be a function field in 4 variables over a field \(K\), and consider the following quaternion algebras over \(F\):

\[
Q_1 = (x, zt), \ Q_2 = (y, zt), \ Q_3 = (xy, z) \text{ and } Q_4 = (xy, t).
\]

We let \((A, \sigma)\) be a direct sum of \((Q_1, -) \otimes (Q_2, -)\) and \((Q_3, -) \otimes (Q_4, -)\) as in Proposition 6.6 and denote by \((B, \tau)\), and respectively \((C, \gamma)\), the component of
\( C(A, \sigma) \) Brauer equivalent to \( Q_1 \otimes Q_3 \sim (x, t) \otimes (y, z) \) and \( Q_1 \otimes Q_4 \sim (x, z) \otimes (y, t) \). The algebras \( A, B \) and \( C \) have index 2, 4 and 4, so that \( ((A, \sigma), (B, \tau), (C, \gamma)) \) is a trialitarian triple ordered by indices. By Theorem 6.3 we get \( J(A, \sigma) = (1, 2, j_3) \) and \( J(B, \tau) = J(C, \gamma) = (2, 1, j_3) \) for some \( j_3 \). Finally, assertion (i) of Corollary 6.1 implies \( j_3 = 1 \); in other words, this triple is anisotropic.

6.9 Example. This example is obtained from the previous one by scalar extension. Consider the Albert form \( \varphi = \langle x, t, -xt, -y, -z, yz \rangle \) associated to the biquaternion algebra \( Q_1 \otimes Q_3 \). We let \( F' \) be its function field, \( F' = F(\varphi) \), and denote by \( (A', \sigma'), (B', \tau'), (C', \gamma') \) the extensions of \( (A, \sigma), (B, \tau), (C, \gamma) \) to \( F' \). Since \( B \) is Brauer equivalent to \( Q_1 \otimes Q_3 \), the algebra \( B' \) has index 2. On the other hand, it follows from Merkurjev’s index reduction formula [Me91, Thm. 3] that the indices of \( A \) and \( C \) are preserved by scalar extension to \( F' \), so that \( A' \) and \( C' \) have indices 2 and 4 respectively. Hence \( ((A', \sigma'), (B', \tau'), (C', \gamma')) \) is again a trialitarian triple ordered by indices and Theorem 6.3 now gives \( J(A', \sigma') = J(B', \tau') = J(C', \gamma') = (1, 1, j_3) \) for some \( j_3 \). The same argument as in the proof of the first assertion of Corollary 6.1 applies here: since \( A' \) and \( B' \) are non-split and \( C' \) has index 4, the involutions are anisotropic and Theorem 6.3 gives \( j_3 = 1 \). Note that, in particular, we have \( J(C', \gamma') = (1, 1, 1) \), even though \( C' \) has index 4 = 2^2.

6.10 Example. We now produce another example of an anisotropic trialitarian triple having \( J \)-invariant \( (1, 1, 1) \) in which all three algebras have index 2. Namely, consider the \( F \)-quaternion algebras

\[
Q_1 = (x, y), \quad Q_2 = (x, z), \quad Q_3 = (x, t) \quad \text{and} \quad Q_4 = (x, yzt).
\]

Pick an arbitrary orthogonal involution \( \rho \) on \( H = (x, yz) \) over \( F \). Since \( Q_1 \otimes Q_2 \) is isomorphic to 2 by 2 matrices over \( H \), the tensor product of the canonical involutions of \( Q_1 \) and \( Q_2 \) is adjoint to a 2-dimensional hermitian form \( h_{12} \) over \( (H, \rho) \). Similarly, \( (Q_3, -) \otimes (Q_4, -) \) is isomorphic to \( M_2(H) \) endowed with the adjoint involution with respect to some hermitian form \( h_{34} \). Since \( h_{12} \) and \( h_{34} \) are both anisotropic, the hermitian form \( h = h_{12} \oplus \langle u \rangle h_{34} \) over \( H'' = H \oplus F(u) \), for some indeterminate \( u \), is also anisotropic. We define

\[
(A, \sigma) = (M_4(H''), \text{ad}_h).
\]

It is clear from the definition that \( (A, \sigma) \) is a direct sum of \( (Q_1, -) \otimes (Q_2, -) \) and \( (Q_3, -) \otimes (Q_4, -) \). Hence, by Proposition 6.6 the two components \( (B, \tau) \) and \( (C, \gamma) \) of its Clifford algebra are Brauer equivalent to \( (x, yt) \) and \( (x, zt) \). This shows that all three algebras have index 2. Since the involutions are anisotropic, by Theorem 6.3 their \( J \)-invariant is \( (1, 1, 1) \).

6.11 Remark. Note that there are many other examples, and not all of them can be described as in Proposition 6.6. In particular, any triple which includes a division algebra cannot be obtained from this proposition. Consider for instance the algebra with involution \( (A, \sigma) \) described in [QT02, Example 3.6], and let \( (B, \tau) \) and \( (C, \gamma) \) be the two components of its Clifford algebra. As explained there, \( A \) is a indecomposable division algebra, and one component of
its Clifford algebra, say $B$, has index 2. Since $A$ is Brauer equivalent to $B \otimes C$, its indecomposability guarantees that $C$ is division, and we get $J(A, \sigma) = J(C, \gamma) = (2, 1, 1)$ and $J(B, \tau) = (1, 2, 1)$.

To produce examples of algebras with involution having $J$-invariant $(1, 1, 0)$, we now construct examples of isotropic non-split and non-half-spin triples. As opposed to the previous examples, they can always be described using Proposition 6.6. Indeed, we get the following explicit description for such triples (cf. Garibaldi’s [Ga98, Thm. 0.1])

6.12 Proposition. If $((A, \sigma), (B, \tau), (C, \gamma))$ is an isotropic trialitarian triple with $A, B$ and $C$ non-split, then there exists division quaternion algebras $Q_1, Q_2$ and $Q_3$ such that $Q_1 \otimes Q_2 \otimes Q_3$ is split and the triple is described as in Proposition 6.6 with $Q_4 = M_2(k)$.

Proof. Since $B$ and $C$ are non-split, the involution $\sigma$ is not hyperbolic by 1.6. Hence $A$ has index 2, $A = M_4(Q_1)$ for some quaternion algebra $Q_1$ over $F$. Fix an orthogonal involution $\rho_1$ on $Q_1$; the involution $\sigma$ is adjoint to a hermitian form $h = h_0 \oplus h_1$ over $(Q_1, \rho_1)$, with $h_0$ hyperbolic, $h_1$ anisotropic and both of dimension 2 and trivial discriminant. Therefore, $(A, \sigma)$ is a direct sum of $(M_2(Q_1), \mathrm{ad}_{h_0})$ and $(M_2(Q_1), \mathrm{ad}_{h_1})$. Since the first summand is hyperbolic, it is isomorphic to $(M_2(k), -) \otimes (Q_1, -)$. The second is $(Q_2, -) \otimes (Q_3, -)$, where $Q_2$ and $Q_3$ are the two components of the Clifford algebra $\mathrm{ad}_{h_1}$, and this concludes the proof.

We refer the reader to [QT10, §6] for a more precise description of those triples. They are the only ones for which the $J$-invariant is $(1, 1, 0)$.

7 Generic properties

In the present section we investigate the relationship between the values of the $J$-invariant of an algebra with involution $(A, \sigma)$ and the $J$-invariant of the respective adjoint quadratic form $\varphi_\sigma$ over the function field $F_A$ of the Severi-Brauer variety of $A$, which is a generic splitting field of $A$.

7.1 Definition. We say $(A, \sigma)$ is generically Pfister if $\varphi_\sigma$ is a Pfister form. Observe that in this case $\deg A$ is always a power of 2 and the $J$-invariant over $F_A$ has the form:

$$J((A, \sigma)_{F_A}) = (0, \ldots, 0, \ast)$$

(all zeros except possibly the last entry which is 0 or 1).

We say $(A, \sigma)$ is in $I^s$, $s > 2$, if $\varphi_\sigma$ belongs to the $s$-th power $I^s(F_A)$ of the fundamental ideal $I(F_A) \subset W(F_A)$ of the Witt ring of $F_A$.

7.2 Theorem. Let $(A, \sigma)$ be an algebra of degree $2n$ with orthogonal involution with trivial discriminant.

(a) If $(A, \sigma)$ is in $I^s$, $s > 2$, then $J(A, \sigma) = (j_1, 0, \ldots, 0, \ast, \ldots, \ast)$, $2^{s-2} - 1$ times
(b) In particular, if \((A, \sigma)\) is generically Pfister, then \(J(A, \sigma) = (\ast, 0, \ldots, 0, \ast)\).

**Proof.** (a) Let \(X = \text{D}_n/P_i\) be the variety of maximal parabolic subgroups of type \(i := 2 \cdot \frac{n+1}{2} - 2^{s-1} + 1\) (For parabolic subgroups we use notation from [PS10 2.1]). Since \(i\) is odd, \(A_F(X)\) splits, and therefore the quadratic form \(\varphi_\sigma\) is defined over \(F(X)\). By assumption \(\varphi_\sigma \in I^s(F(X))\). The Witt index of \(\varphi_\sigma\) is at least \(i\). Therefore the anisotropic part of \(\varphi_\sigma\) has dimension at most \(2(n - i) < 2^s\). Thus, by the Arason-Pfister theorem \(\varphi_\sigma\) is hyperbolic. In particular, the variety \(X\) is generically split. Therefore by [PS11, Theorem 2.3] we obtain the desired expression for the \(J\)-invariant.

(b) Finally, if \((A, \sigma)\) is generically Pfister, then \(\varphi_\sigma \in I^s(F(X))\), where \(2^s = 2n\) and (b) follows from (a).

7.3 **Remark.** Let \((j_2, \ldots, j_r)\) be the \(J\)-invariant of \(\varphi_\sigma\) over \(F_A\), \(r = \left[\frac{n+2}{2}\right]\). In view of the theorem one can conjecture that the \(J\)-invariant of \((A, \sigma)\) is obtained from \(J(\varphi_\sigma)\) just by adding an arbitrary left term, i.e.

\[
J(A, \sigma) = (\ast, j_2, \ldots, j_r).
\]

For example, if \(\varphi_\sigma\) is excellent, then the \(J\)-invariant should be equal to

\[
J(A, \sigma) = (\ast, 0, \ldots, 0, \ast, 0, \ldots, 0),
\]

where the second \(\ast\) has degree \(2^s - 1\) for some \(s\) and equals either 0 or 1.

By the results of Section 6, observe that this holds for algebras of degree 8.

8 **Appendix**

8.1 The following table provides the values of the parameters of the \(J\)-invariant for all orthogonal groups (here \(p = 2\)).

| \(G_0\)         | \(r\)   | \(d_i\)      | \(k_i\)       | restrictions on \(j_i\) |
|------------------|---------|--------------|---------------|-------------------------|
| \(O_n^+\)       | \(\frac{n+1}{4}\) | 2\(i - 1\)  | \(\log_2 \frac{n-1}{d_i}\) | if \(d_i + l = 2^sd_m\) and \(2 \not| (\frac{n}{d_i})\), then \(j_m \leq j_i + s\) |
| \(\text{Spin}_{2n}, 2 | n\) | \(\frac{n}{2}\) | \(1, i = 1\) | \(2^{k_i} \| n\) | the same restrictions |
| \(\text{Spin}_{n}\) | \(\frac{n-n}{4}\) | 2\(i + 1\)  | \(\log_2 \frac{n-1}{d_i}\) | the same restrictions |
| \(\text{PGO}_{2n}^+\) | \(\frac{n+2}{2}\) | 1, \(i = 1, 2\) | \(2^{k_i} \| n\) | the same restrictions assuming \(i, m \geq 2\) |

Note that this table coincides with [PSZ08 Table 4.13] except of the last column which in our case contains more restrictive conditions. For \(s = 0\) and \(1\) the restrictions in the last column are equivalent to those in [PSZ08 Table 4.13].

The conditions of the last column are simply translation of [Vi05 Prop. 5.12] from the language of Vishik’s \(J\)-invariant to ours.

All values of the \(J\)-invariant which satisfy the restrictions given in the table are called admissible.
8.2 In his paper [Ho98], Hoffmann classified quadratic forms of small dimension in terms of their splitting pattern. Using his classification, one can give a precise description of quadratic forms of dimension 8 with trivial discriminant, depending on the value of their $J$-invariant. The results are summarized in the table below.

The notation $J_v(\varphi)$ stands for Vishik’s $J$-invariant, as defined in [EKM, §88]. The index $i$ is the 2-adic valuation of the greatest common divisor of the degrees of the splitting fields of $\varphi$. In the explicit description, $Pf_k$ stands for a $k$-fold Pfister form, $s_{i/k}(Pf_2)$ for the Scharlau transfer of a 2-fold Pfister form with respect to a quadratic field extension, and $Al_6$ for an Albert form.

| $J(\varphi)$ | $J_v(\varphi)$ | $i_S$, $i$ | Splitting Pattern | Description |
|--------------|---------------|------------|------------------|-------------|
| (0)          | $\emptyset$  | $i_S = i = 0$ | (4)              | hyperbolic  |
| (1,0)        | {1}           | $i_S = i = 1$ | (2,4)            | $Pf_2 \perp 2\mathbb{H}$ |
| (2,0)        | {1, 2}        | $i_S = i = 2$ | (1,2,4)          | $Al_6 \perp \mathbb{H}$ |
| (0,1)        | {3}           | $i_S = 0$; $i = 1$ | (0,4)            | $Pf_3$ |
| (1,1)        | {1, 3}        | $i_S = i = 1$ | (0,2.4)          | $q = \langle 1, -a \rangle \otimes q'$ |
| (2,1)        | {1, 2, 3}     | $i_S = i = 2$ | (0,1,2,4)       | $Pf_2 \perp Pf_2$ or $s_{i/k}(Pf_2)$ |
|              |               | $i_S = i = 3$ |                  | "generic" |

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