TROPICAL EXPANSIONS AND TORIC VARIETY BUNDLES

FRANCESCA CAROCCI AND NAVID NABIJOU

ABSTRACT. A tropical expansion is a degeneration of a toroidal embedding induced by a polyhedral subdivision of its tropicalisation. Each irreducible component of a tropical expansion admits a collapsing map down to a stratum of the original variety. We study the relative geometry of this map. We give a necessary and sufficient polyhedral criterion for the map to have the structure of a toric variety bundle, and prove that this structure always exists over the interior of the codomain. We give examples demonstrating that this is the strongest statement one can hope for in general. In addition, we provide a combinatorial recipe for constructing the toric variety bundle as a fibrewise GIT quotient of an explicit split vector bundle. Our proofs make systematic use of Artin fans as a language for globalising local toric models.

CONTENTS

Introduction ................................................................. 1
1. Toroidal preliminaries ............................................. 3
2. Tropical expansions ............................................... 7
3. Toric variety bundles and fibrewise GIT ....................... 10
4. Expansion components ........................................... 14
References ................................................................. 25

INTRODUCTION

Given a toroidal embedding \((X|D)\), a tropical expansion is a multi-parameter degeneration of the pair induced by a polyhedral subdivision of the tropicalisation. In recent years these have come to occupy a central role in logarithmic enumerative geometry and moduli theory [NS06, MR20a, Ran22, MR20b].

0.1. Results. In this paper we establish structural results on the geometry of tropical expansions, and develop a combinatorial toolkit for working with them in practice.

The irreducible components of a tropical expansion are indexed by the vertices of the corresponding polyhedral subdivision. Given such a vertex \(v\), the component \(Y_v\) admits a natural collapsing map down to a stratum \(X_v\) of the original pair

\[ \rho_v: Y_v \to X_v. \]

Since \((X|D)\) is arbitrary the stratum \(X_v\) is equally arbitrary, and nothing particular can be said about it. Instead, our goal is to describe the relative geometry of \(\rho_v\).
The initial hope is that $\rho_v$ is a toric variety bundle, i.e. a fibrewise equivariant compactification of a principal torus bundle. Unfortunately, this is not always the case: $\rho_v$ can have reducible fibres, or can even fail to be flat.\footnote{These issues do not arise for expansions of smooth pairs. This accounts for much of the simplicity of relative enumerative geometry when compared with its logarithmic counterpart.} Examples are presented in Section 4.2.

Our first main result shows that $\rho_v$ is a toric variety bundle over the interior $X_v^\circ$ of the stratum $X_v$. We also provide a combinatorial recipe for constructing this bundle.

**Theorem A** (Theorem 4.11). Define $Y_v^\bullet := \rho_v^{-1}(X_v^\circ)$. Then the restricted collapsing morphism

$$\rho_v : Y_v^\bullet \to X_v^\circ$$

is a toric variety bundle. It can be constructed as the fibrewise GIT quotient of a direct sum of explicit line bundles on $X_v^\circ$.

This gives a complete description of the open subvariety

$$Y_v^\bullet \hookrightarrow Y_v.$$

If the stratum $X_v$ is minimal, then $X_v^\circ = X_v$ and so $Y_v^\bullet = Y_v$. This is often the case in applications, e.g. over torus-fixed loci in the moduli space of expanded stable maps $[\text{Ran22}]$. When $X_v$ is not minimal, Theorem A instead gives a cut-and-paste description of $Y_v \to X_v$ as a stratified union of toric variety bundles (Remark 4.13).

Our second main result is a criterion determining when the above description extends to the entire component. This generalises a well-known criterion in toric geometry $[\text{CLS11}, \text{Theorem 3.3.19}]$.

**Theorem B** (Theorem 4.14). The collapsing morphism $\rho_v : Y_v \to X_v$ is a toric variety bundle if and only if there is an isomorphism of cone complexes

$$\Upsilon/\omega_v \cong \Sigma/\sigma_v \times \Phi_v$$

commuting with the projections to $\Sigma/\sigma_v$. Here $\Upsilon/\omega_v$ and $\Sigma/\sigma_v$ are the tropicalisations of $Y_v$ and $X_v$ respectively, and $\Phi_v$ is the fibre fan of Definition 4.9. See Section 4 for details.

In this case, $Y_v$ is also a fibrewise GIT quotient of a direct sum of line bundles on $X_v$. When $X_v$ is smooth there is a combinatorial algorithm expressing these line bundles in terms of piecewise-linear functions on tropicalisations (Section 4.6.2).

These are the strongest statements one can hope for in general, as the morphism $Y_v \to X_v$ can be an arbitrary toroidal modification of a toric variety bundle. Given this, Theorem A allows us to completely describe its behaviour over the interior, and Theorem B provides a necessary and sufficient criterion for this description to persist over the boundary.

As necessary background, in Section 3 we study toric variety bundles. We establish several useful facts, including equivalences between three complementary approaches.

### 0.2. Context

The study of tropical expansions has a long history. They were introduced, for toric pairs $(X|D)$ and with one-dimensional base cone, in $[\text{NS06}]$. In this setting they form an important special case of the Mumford degeneration $[\text{Mum72}]$.

It has long been known that this construction extends beyond the toric case to the setting of toroidal embeddings. Despite this, there has been to our knowledge no systematic treatment of the geometry of such expansions, and in particular no general recipe for constructing $Y_v$ from $X_v$. Whereas in the toric setting all such components are again toric varieties with explicitly determined fans, this is not true in general, since the geometry of $(X|D)$ is unconstrained.
One of the fundamental insights of [Ran22, MR20b] is that it is sensible and profitable to consider tropical expansions in a universal fashion, i.e. over arbitrary base cones. This is a crucial step in the construction of logarithmic Gromov–Witten and Donaldson–Thomas invariants via expanded targets. Over arbitrary base cones, flatness of the degenerating family becomes much more delicate [AK00, Mol21].

The present paper is a companion to [CN21], which describes the higher-rank rubber torus acting on the tropical expansions which appear as targets in the moduli space of expanded stable maps. Taken together, these works provide an avenue for probing the recursive structure of these moduli spaces.

0.3. Acknowledgements. We thank Luca Battistella and Dhruv Ranganathan for numerous inspiring conversations. Heartfelt thanks are owed to the anonymous referee who, in addition to providing many expositional comments, supplied a key suggestion helping us to remove a superfluous smoothness hypothesis in Sections 4.5 and 4.6.

F.C. was supported by the EPFL Chair of Arithmetic Geometry (ARG), and N.N. was supported by the Herchel Smith Fund.

1. Toroidal preliminaries

We review the theory of toroidal embeddings, with a heavy bias towards tropicalisations and Artin fans. For further reading, see [KKM5D73] and [AK00, Section 1]. We assume familiarity with the basics of toric varieties [Ful93, CLS11].

1.1. Toroidal embeddings. Fix a normal variety $X$ and a reduced divisor $D \subseteq X$.

Definition 1.1. The pair $(X|D)$ is a (Zariski) toroidal embedding if, Zariski-locally, there exists a strictly convex rational polyhedral cone $\sigma$ and an isomorphism of pairs

\[(X|D)\big|_U \cong (U_\sigma|\partial U_\sigma)\]

where $U_\sigma = \text{Spec} \mathbb{k}[S_\sigma]$ is the associated affine toric variety and $\partial U_\sigma \subseteq U_\sigma$ is the toric boundary. The cone $\sigma$ is uniquely determined by the open set $U$ (however, there are choices for the isomorphism $U \cong U_\sigma$).

A toroidal morphism $f: (X|D) \to (Y|E)$ is a morphism $f: X \to Y$ such that $f^{-1}(D) \subseteq E$. Locally, a toroidal morphism restricts to a toric morphism on appropriate toric models.

Remark 1.2. An important class of toroidal embeddings are the simple normal crossings pairs. These consist of a smooth variety $X$ and a divisor $D = D_1 + \ldots + D_k \subseteq X$ with the $D_i$ smooth and intersecting transversely. The local toric model around the intersection of $l$ divisor components is

$$\mathbb{A}^l \times \mathbb{G}_m^{n-l}$$

where $n = \text{dim } X$.

Remark 1.3. The divisor $D$ defines a stratification of $X$. When $D = D_1 + \ldots + D_k$ is simple normal crossings, the locally-closed strata are precisely the connected components of the loci

$$D_I^\circ = \bigcap_{i \in I} D_i \setminus \bigcup_{i \notin I} D_i$$

for arbitrary $I \subseteq \{1, \ldots, k\}$.

Remark 1.4. Toroidal embeddings are usually defined with respect to the étale topology, which permits irreducible components of $D$ with self-intersections. We restrict to the Zariski topology to ensure that every closed stratum is again a toroidal embedding. This is necessary to describe the
relative geometry of tropical expansions. Any étale toroidal embedding can be transformed into a Zariski toroidal embedding via a toroidal modification (see Section 1.5).

1.2. Tropicalisations. A cone is a pair

\[ \sigma = (\sigma_\mathbb{R}, N_\sigma) \]

where \( N_\sigma \) is a lattice and \( \sigma_\mathbb{R} \subseteq N_\sigma \otimes \mathbb{R} \) is a strictly convex rational polyhedral cone. A morphism of cones \( \sigma_1 \to \sigma_2 \) is a lattice morphism \( N_{\sigma_1} \to N_{\sigma_2} \) sending \( \sigma_1\mathbb{R} \) into \( \sigma_2\mathbb{R} \).

An abstract cone complex is a collection of cones glued together along faces [ACP15, Section 2]. Formally, it as a diagram of cones connected by face maps. We prohibit non-identity isomorphisms, and require that there is at most one face map between a given pair of cones.

To every toroidal embedding we associate an abstract cone complex: the tropicalisation. This generalises the fan of a toric variety.

**Definition 1.5.** The tropicalisation \( \Sigma(X|D) \) is the following cone complex:

- Cones: Every locally-closed stratum \( S \subseteq X \) has an associated cone \( \sigma_S \) corresponding to the local toric model of \((X|D)\) in a Zariski neighbourhood of \( S \).\(^2\) We take the collection of all such cones \( \sigma_S \).
- Face maps: If \( S_2 \subseteq S_1 \) then there is an associated face inclusion \( \sigma_{S_1} \subseteq \sigma_{S_2} \). We take the collection of all such face inclusions.

Cones \( \sigma \in \Sigma(X|D) \) correspond bijectively to strata of \((X|D)\). This correspondence is inclusion-reversing, and the dimension of a cone is the codimension of the stratum.

If \((X|D)\) is a simple normal crossings pair with \( X \) connected, then \( \Sigma(X|D) \) coincides with the cone over the dual intersection complex of \( D \). In particular all the cones are smooth, i.e. unimodular. The apex corresponds to the open stratum \( X \setminus D \).

If \((X|D)\) is a toric pair then \( \Sigma(X|D) \) is isomorphic to the fan of \( X \). However, the tropicalisation does not come with a preferred embedding into a vector space; such an embedding depends on the choice of a torus action.

**Remark 1.6.** The cone \( \sigma_S \) associated to a locally-closed stratum \( S \) has the following alternative description. Consider the open neighbourhood \( U_S \subseteq X \) of \( S \) given as the union of locally-closed strata whose closures contain \( S \). Then \( \sigma_S \) is the cone dual to the monoid of effective divisors in \( U_S \) that are supported on \( U_S \cap D \). For example, the locally-closed stratum \( S = D_j^0 \) has

\[ U_S = X \setminus \cup_{j \neq i} D_j. \]

The monoid is \( \mathbb{N} \) with generator \( D_j^0 \), and so \( \sigma_S = \mathbb{R}_{\geq 0} \).

For further details see [AK00, Section 1.3].

1.3. Artin fans. Every cone complex has an algebro-geometric avatar: its Artin fan. This provides a bridge between the geometry of \((X|D)\) and the combinatorics of \( \Sigma(X|D) \). For detailed discussions, see [AW18, ACMW17, ACM+16].

**Definition 1.7.** An Artin cone is a quotient stack of an affine toric variety by its dense torus

\[ A_\sigma := [U_\sigma / T_\sigma]. \]

\(^2\)If the local toric model has torus factors, these are removed by quotienting by the subtorus of the big torus acting freely. This is equivalent to restricting to the lattice spanned by the cone \( \sigma_S \). See also Remark 1.6 for an alternative description.
The assignment \( \sigma \mapsto A_\sigma \) is functorial. When the morphism \( \sigma_1 \hookrightarrow \sigma_2 \) is a face inclusion, the morphism \( U_{\sigma_1} \to U_{\sigma_2} \) is an open embedding, and hence the same is true for the Artin cones
\[
A_{\sigma_1} \hookrightarrow A_{\sigma_2}.
\]
For example, the inclusion \( \mathbb{R}_{\geq 0} \hookrightarrow \mathbb{R}^2_{\geq 0} \) of the first coordinate ray gives the open embedding
\[
[A^1/\mathbb{G}_m] = ((A^1 \times \mathbb{G}_m)/\mathbb{G}_m^2) \hookrightarrow [A^2/\mathbb{G}_m^2].
\]

**Definition 1.8.** The Artin fan \( A_\Sigma \) of a cone complex \( \Sigma \) is the colimit, in the category of algebraic stacks, of the diagram consisting of

- Artin cones \( A_\sigma \) for \( \sigma \in \Sigma \),
- open embeddings \( A_{\sigma_1} \hookrightarrow A_{\sigma_2} \) for \( \sigma_1 \subseteq \sigma_2 \).

It is an irreducible stack of pure dimension zero. The Artin fan of a toroidal embedding \( (X|D) \) is the Artin fan of its tropicalisation
\[
A_{X|D} := A_{\Sigma(X|D)}.
\]
This can be thought of as a finite topological space, with points corresponding to the strata of \( (X|D) \). The topology is the order topology, induced by strata inclusions.

**Remark 1.9.** The assignment \( \Sigma \mapsto A_\Sigma \) gives an equivalence between the 2-categories of cone complexes (strictly speaking, cone stacks) and Artin fans [CCUW20, Theorem 3]. Happily, the distinction between a cone complex and its Artin fan has grown increasingly blurred.

The Artin fan \( A_{X|D} \) globalises the local toric models for \( (X|D) \). Given an open set \( U \) and a model \( (X|D)|_U \cong (U_\sigma|\partial U_\sigma) \), the composite morphism
\[
U \xrightarrow{\cong} U_\sigma \to A_\sigma \hookrightarrow A_{X|D}
\]
does not depend on the choice of isomorphism \( U \cong U_\sigma \). These local morphisms glue to produce a smooth structure morphism
\[
p: X \to A_{X|D}.
\]
This morphism encodes the toroidal structure on \( X \), since \( p^{-1}(\partial A_{X|D}) = D \). More generally, given a scheme \( X \) and a cone complex \( \Sigma \), an arbitrary morphism
\[
p: X \to A_\Sigma
\]
defines a **logarithmic structure** on \( X \) via pullback from \( A_\Sigma \). The resulting logarithmic scheme is logarithmically smooth if and only if \( p \) is smooth in the usual sense.

**Example 1.10.** Let \( (X|D) = (D_1 + \ldots + D_k) \) be a simple normal crossings pair and assume that all the strata of \( D \) are nonempty and connected. Then \( \Sigma(X|D) = \mathbb{R}^k_{\geq 0} \) and so
\[
A_{X|D} = [\mathbb{A}^k/\mathbb{G}_m^k].
\]
The structure morphism \( p \) encodes the pairs \( (O_X(D_1), s_{D_1}), \ldots, (O_X(D_k), s_{D_k}) \) corresponding to the irreducible components \( D_1, \ldots, D_k \).

For a general toroidal embedding, a choice of morphism
\[
A_{X|D} \to [\mathbb{A}^1/\mathbb{G}_m]
\]
determines a pair \( (L, s) \) on \( X \) via the composite
\[
X \to A_{X|D} \to [\mathbb{A}^1/\mathbb{G}_m].
\]
Morphisms \( A_{X|D} \to [\mathbb{A}^1/\mathbb{G}_m] \) correspond bijectively to morphisms \( \Sigma(X|D) \to \mathbb{R}_{\geq 0} \). Hence this construction generalises the correspondence between piecewise-linear functions on a fan and toric Cartier divisors on the associated toric variety.
1.4. **Strata in Artin fans.** In the toric context a cone \( \sigma \in \Sigma \) defines three loci in \( X \) : the affine open \( U_\sigma \), the locally-closed orbit \( O_\sigma \), and the closed stratum \( Z_\sigma \). We now discuss the analogues of these in the setting of Artin fans.

Fix a cone complex \( \Sigma \) and a cone \( \sigma \in \Sigma \). The associated affine open is the Artin cone

\[
A_\sigma \hookrightarrow A_\Sigma
\]

and the associated locally-closed stratum is the composite

\[
BT_\sigma \hookrightarrow A_\sigma \hookrightarrow A_\Sigma.
\]

For the associated closed stratum, first define the **isotropic star complex** as the set

\[
St(\sigma, \Sigma) := \{ \tau \in \Sigma \mid \sigma \subseteq \tau \}.
\]

This is not a cone complex: it is not closed under taking faces. Nevertheless, it is closed under intersections and has a unique minimal cone \( \sigma \) which functions as the apex. We refer to this structure as an **isotropic cone complex**.

Given \( \tau \in St(\sigma, \Sigma) \) let \( \tau/\sigma \) denote the image of \( \tau \) under the quotient morphism \( N_\tau \otimes \mathbb{R} \to (N_\tau/N_\sigma) \otimes \mathbb{R} \). This is strictly convex because \( \sigma \subseteq \tau \) is a face. There is a surjective homomorphism \( T_\tau \to T_{\tau/\sigma} \) inducing an action \( T_\tau \curvearrowright U_{\tau/\sigma} \). We define the **isotropic Artin fan** as the following colimit over cones \( \tau \in St(\sigma, \Sigma) \)

\[
B_{St(\sigma, \Sigma)} := \varprojlim \left[ U_{\tau/\sigma}/T_\tau \right].
\]

This has an open dense stratum \( BT_\sigma \) corresponding to the minimal cone \( \sigma \in St(\sigma, \Sigma) \).

For each \( \tau \in St(\sigma, \Sigma) \) there is an embedding \( U_{\tau/\sigma} \hookrightarrow U_\tau \) giving the closed stratum corresponding to the face \( \sigma \subseteq \tau \). This is \( T_\tau \)-equivariant and hence descends to a closed embedding \( [U_{\tau/\sigma}/T_\tau] \hookrightarrow A_\tau \). These glue to produce

\[
B_{St(\sigma, \Sigma)} \hookrightarrow A_\Sigma
\]

which gives the closed stratum corresponding to \( \sigma \in \Sigma \).

Define the **reduced star complex**

\[
\Sigma/\sigma := \{ \tau/\sigma \mid \tau \in St(\sigma, \Sigma) \}.
\]

For each \( \tau \in St(\sigma, \Sigma) \) there is a natural morphism \( [U_{\tau/\sigma}/T_\tau] \to [U_{\tau/\sigma}/T_{\tau/\sigma}] \). These glue to produce a global morphism

\[
B_{St(\sigma, \Sigma)} \to A_{\Sigma/\sigma}.
\]

This is a gerbe banded by \( T_\sigma \), as can be checked locally on the base (see also Lemma 4.15).

**Remark 1.11.** Suppose that all the cones of \( St(\sigma, \Sigma) \) are smooth. Then the above gerbe has a canonical trivialisation. Indeed, in this case every lattice \( N_\tau \) for \( \tau \in St(\sigma, \Sigma) \) has a natural basis given by the primitive ray generators. This defines a natural splitting \( N_\tau = N_\sigma \times N_{\tau/\sigma} \) which gives an isomorphism \( [U_{\tau/\sigma}/T_\tau] = A_{\tau/\sigma} \times BT_\sigma \). These glue to an isomorphism \( B_{St(\sigma, \Sigma)} = A_{\Sigma/\sigma} \times BT_\sigma \).

Now let \( (X|D) \) be a toroidal embedding with tropicalisation \( \Sigma = \Sigma(X|D) \). Let \( \sigma \in \Sigma \) be a cone and \( X_\sigma \hookrightarrow X \) the corresponding closed stratum. There is a cartesian square

\[
\begin{array}{ccc}
X_\sigma & \longrightarrow & X \\
\downarrow & \Box & \downarrow \\
B_{St(\sigma, \Sigma)} & \longrightarrow & A_\Sigma.
\end{array}
\]
The stratum $X_\sigma$ is itself a toroidal embedding, with boundary consisting of the intersection with the components of $D$ that do not contain $X_\sigma$. The map to its Artin fan is the composite
\[ X_\sigma \to B_{S^\ell(\sigma, \Sigma)} \to A_{\Sigma/\sigma}. \]

Now let $\rho: Y \to X$ be a morphism of toroidal embeddings. Consider a cone $\sigma_Y \in \Sigma_Y$ and let $\sigma_X \in \Sigma_X$ be the minimal cone containing the image of $\sigma_Y$. Then $\rho$ restricts to a morphism between the corresponding closed strata
\[ Y_{\sigma_Y} \to X_{\sigma_X}. \]
This is a toroidal morphism, i.e. there is a 2-commuting square
\[
\begin{array}{ccc}
Y_{\sigma_Y} & \longrightarrow & X_{\sigma_X} \\
\downarrow & & \downarrow \\
A_{\Sigma_Y/\sigma_Y} & \longrightarrow & A_{\Sigma_X/\sigma_X}.
\end{array}
\]

1.5. **Toroidal modifications.** A central theme in toroidal geometry is that toric constructions on the Artin fan pull back to toroidal constructions on $X$. We discuss an important instance of this phenomenon: toroidal modifications.

Following [MR20b, Definition 1.3.1] we permit subdivisions which are injective but not surjective on supports. This flexibility is useful for applications in enumerative geometry.

**Definition 1.12.** Let $\Sigma$ be a cone complex. An open subdivision of $\Sigma$ is a morphism of cone complexes
\[ \Sigma^\dagger \to \Sigma \]
such that the induced map on supports $|\Sigma^\dagger| \to |\Sigma|$ is injective, and such that the lattice of every cone in $\Sigma^\dagger$ is mapped onto a saturated sublattice of the lattice of a cone in $\Sigma$.

**Definition 1.13.** Consider a toroidal embedding $(X|D)$ and let $\Sigma = \Sigma(X|D)$. Given an open subdivision $\Sigma^\dagger \to \Sigma$, the associated open toroidal modification is defined as the fibre product
\[
\begin{array}{ccc}
X^\dagger & \longrightarrow & X \\
\downarrow & & \downarrow_{p^\dagger} \\
A_{\Sigma^\dagger} & \longrightarrow & A_\Sigma.
\end{array}
\]
The morphism $X^\dagger \to X$ is birational and an isomorphism over $X \setminus D$. It is proper and surjective if the open subdivision $\Sigma^\dagger \to \Sigma$ is surjective on supports, i.e. if it is a subdivision in the usual sense. We set $D^\dagger = (p^\dagger)^{-1}(\partial A_{\Sigma^\dagger})$ and thus obtain a toroidal morphism
\[ (X^\dagger|D^\dagger) \to (X|D). \]

Toroidal (and, more generally, logarithmic) modifications are central to the modern study of intersections on moduli spaces [RSPW19a, RSPW19b, BC20, Ran22, HPS19, Ran19, NR22, MPS23, MR21, BNR22, HMP+22].

2. **Tropical expansions**

Let $(X|D)$ be a toroidal embedding with tropicalisation $\Sigma = \Sigma(X|D)$. We consider tropical expansions of $\Sigma$ parametrised by an arbitrary base cone $\tau$. These have been studied by many authors, in various guises and levels of generality [Mum72, NS06, BGS11, FR16, MR20b].
Definition 2.1. A tropical expansion of $\Sigma$ over $\tau$ consists of an open subdivision of cone complexes
\[ \Upsilon = (\Sigma \times \tau)^\dagger \to \Sigma \times \tau \]
such that the projection $p : \Upsilon \to \tau$ satisfies the following conditions for every cone $\omega \in \Upsilon$

- $p$ maps $\omega$ surjectively onto a face of $\tau$ (p is combinatorially flat);
- $p$ maps $N_\omega$ surjectively onto the lattice of its image face (p is combinatorially reduced).

There is a diagram of cone complexes
\[ \begin{array}{ccc}
\Upsilon & \xrightarrow{p} & \Sigma \\
\downarrow & & \downarrow \\
\tau & & \\
\end{array} \]
with $p$ combinatorially flat and reduced. The open subdivision $\Upsilon \to \Sigma \times \tau$ gives rise to an open toroidal modification $\mathcal{X}_\Upsilon \to X \times U_\tau$ where $U_\tau = \text{Spec} \; k[S_\tau]$ is the corresponding affine toric variety. There is a diagram of schemes
\[ \begin{array}{ccc}
\mathcal{X}_\Upsilon & \xrightarrow{\rho} & X \\
\downarrow & & \downarrow \\
U_\tau. & & \\
\end{array} \]
The conditions on $p$ ensure that the morphism $\pi$ is flat with reduced fibres [AK00, Lemmas 4.1 and 5.2]. The morphism $\rho$ collapses every fibre of $\pi$ onto the unexpanded target $X$. We refer to this diagram as a tropical expansion of $(X|D)$ over $U_\tau$.

Tropical expansions encompass several familiar constructions, including the Mumford degeneration in toric geometry and the degeneration to the normal cone of a complete intersection. For further examples, see [CN21, Section 4] and Section 4.2 below.

2.1. Families of polyhedral subdivisions. Fix a tropical expansion $\Upsilon \to \Sigma \times \tau$. For every point $f \in |\tau|$ we intersect the fibre $p^{-1}(f) \subseteq |\Upsilon|$ with the cones of $\Upsilon$ to produce a polyhedral complex $\Upsilon_f$.

This polyhedral complex has support $|\Upsilon_f| = p^{-1}(f) \subseteq |\Sigma|$. We view $p$ as a family of open polyhedral subdivisions of $\Sigma$ parametrised by $f \in |\tau|$. The combinatorial type of $\Upsilon$ at $f$ is the data of the abstract polyhedral complex $\Upsilon_f$ together with:

- (1) For each polyhedron $P \in \Upsilon_f$ the minimal cone $\sigma_P \in \Sigma$ such that $|P| \subseteq |\sigma_P|$.
- (2) For each oriented edge $\vec{E} \in \Upsilon_f$ the integral slope $m_{\vec{E}} \in N_{\sigma_{\vec{E}}}$.

This data is constant on the relative interior of each face of $\tau$. The lengths of the boundary edges of the polyhedra in $\Upsilon_f$ depend on the precise choice of point $f$ in the base. Specialising $f$ to a face of $\tau$ collapses certain polyhedra in $\Upsilon_f$, giving rise to a simpler combinatorial type. In the limit $f = 0$ we obtain a cone complex
\[ \Sigma^\dagger = \Upsilon_0 \]
called the asymptotic cone complex. The restriction of $r$ to $\Upsilon_0$ exhibits $\Sigma^\dagger$ as an open (conical) subdivision of $\Sigma$. The associated open toroidal modification $X^\dagger \to X$ is the general fibre of $\pi$. 
2.2. **Polyhedral-conical dictionary.** The poset of cones \( \omega \in \Upsilon \) is equal to the poset of pairs \((\kappa, P)\) where \( \kappa \subseteq \tau \) is a face and \( P \) is a polyhedron in \( \Upsilon_f \) for \( f \in |\kappa| \) an arbitrary interior point. Under this correspondence we have \( p(\omega) = \kappa \). Moreover, \((\kappa_1, P_1) \leq (\kappa_2, P_2)\) if and only if \( \kappa_1 \subseteq \kappa_2 \) and \( P_1 \subseteq \bar{P}_2 \), where \( \bar{P}_2 \) is the limit of \( P_2 \) under the specialisation from the combinatorial type over \( \kappa_2 \) to the combinatorial type over \( \kappa_1 \). We have \( \dim \omega = \dim \kappa + \dim P \).

2.3. **Central fibre.** The central fibre of \( \pi \) over the torus-fixed point \( 0 \in U_\tau \) will be denoted \( Y_\Upsilon := \pi^{-1}(0) \).

Strata in \( Y_\Upsilon \) are indexed by cones \( \omega \in \Upsilon \) with \( p(\omega) = \tau \). By the polyhedral-conical dictionary, the poset of such cones is equal to the poset of polyhedra \( P \) in the polyhedral complex \( \Upsilon_f \) for \( f \in |\tau| \) an arbitrary interior point. For every such polyhedron \( P \) we let \( \omega_P \in \Upsilon \) denote the corresponding cone.

The irreducible components of \( Y_\Upsilon \) are the maximal strata, hence are indexed by the minimal polyhedra in \( \Upsilon_f \): the vertices. Let \( V \) denote the set of such vertices and for \( v \in V \) let \( Y_v \) denote the corresponding irreducible component, so that \( Y_\Upsilon = \bigcup_{v \in V} Y_v \).

For \( v \in V \) let \( \sigma_v \in \Sigma \) denote the unique minimal cone containing \( v \). By definition, this is the minimal cone through which the composition \( \omega_v \hookrightarrow \Upsilon \to \Sigma \) factors. Let \( X_v = X_{\sigma_v} \hookrightarrow X \) denote the corresponding closed stratum. The fibrewise collapsing morphism \( \rho : X_\Upsilon \to X \) restricts to a collapsing morphism \( \rho_v : Y_v \to X_v \).

In simple situations, \( \rho_v \) is a toric variety bundle. However, this is not the case in general. It can fail to be flat, or can be flat but with reducible fibres; see Section 4.2. The principal aim of this paper is to provide a combinatorial recipe to construct \( Y_v \) from \( X_v \).

2.4. **Tropical position maps.** For each \( v \in V \) there is a linear map \( \varphi_v : \tau \to \sigma_v \) recording the position of the vertex \( v \) in terms of the tropical parameters. This was introduced in [CN21, Definition 1.4] in order to describe rubber automorphisms of tropical expansions. Formally it is given by \( \varphi_v(f) = r(\omega_v \cap p^{-1}(f)) \) for \( f \in |\tau| \). The open subdivision \( \Upsilon \to \Sigma \times \tau \) restricts to an inclusion \( \omega_v \hookrightarrow \sigma_v \times \tau \) corresponding to an inclusion \( \tilde{N}_{\omega_v} \hookrightarrow \tilde{N}_{\sigma_v} \times N_\tau \). The following result originally appears in [CN21, proof of Theorem 1.5]. It is used heavily in Section 4.

**Proposition 2.2.** The tropical position map \( \varphi_v \) produces a natural isomorphism of lattices \( (N_{\sigma_v} \times N_\tau) / N_{\omega_v} = N_{\sigma_v} \).

**Proof.** Consider the map
\[
\text{id}_{\sigma_v} \times (-\varphi_v) : N_{\sigma_v} \times N_\tau \to N_{\sigma_v},
\]
This is surjective, with kernel the image of \( N_{\omega_v} \hookrightarrow N_{\sigma_v} \times N_\tau \). \( \square \)
We develop concrete techniques for constructing and manipulating toric variety bundles. These will be employed in Section 4 to describe the irreducible components of tropical expansions.

3.1. Toric variety bundles and the mixing construction. Let $X$ be a base scheme and $Z$ a toric variety. We consider the problem of constructing a toric variety bundle $Y \to X$ with fibre $Z$.

There are several inequivalent definitions of toric variety bundle in the literature, see e.g. [Hal03, HLY02, Hu08, Bro14, Oh21, KM19, HKM20]. These arise from different choices of subgroup of $\text{Aut}(Z)$ to contain the transition functions. The following definition is restrictive enough to retain desirable toric structures, but flexible enough that it still leads to an interesting theory.

**Definition 3.1.** A toric variety bundle over $X$ with fibre $Z$ consists of a diagram

$$
\begin{array}{ccc}
P & \to & Y \\
\downarrow && \downarrow \\
X & \to & X
\end{array}
$$

where $P \to X$ is a principal $T_Z$-bundle, and which Zariski-locally on $X$ restricts to the diagram

$$
\begin{array}{ccc}
X \times T_Z & \to & X \times Z \\
\downarrow && \downarrow \\
X & \to & X
\end{array}
$$

The following construction appears independently in [SU03] (and perhaps elsewhere).

**Construction 3.2** (Mixing construction). Consider the following input data:

1. $X$ an arbitrary base scheme.
2. $\Phi$ a fan in a lattice $N$ (the fibre fan).
3. $P$ a principal $T_N$-bundle over $X$ (the mixing collection).

Denote by $Z$ the toric variety corresponding to the fibre fan. Then the mixing bundle is the following scheme over $X$

$$
Y := (P \times Z)/T_Z
$$

where $T_Z = T_N$ is the dense torus of $Z$ acting diagonally on $P \times Z$.

**Remark 3.3.** A mixing collection is equivalently given by any of the following pieces of data:

1. A principal $T_N$-bundle over $X$.
2. A group homomorphism $M \to \text{Pic} \ X$ where $M = \text{Hom}(N, \mathbb{Z})$.
3. A morphism $X \to BT_N$.

Often $N$ will come with a preferred basis, in which case a mixing collection is a list of line bundles indexed by this basis.

**Lemma 3.4.** Definition 3.1 and Construction 3.2 are equivalent: the mixing construction produces a toric variety bundle, and every toric variety bundle arises from the mixing construction.
Proof. Start with a toric variety bundle $P \subseteq Y \to X$ and let $\{U_i\}_{i \in I}$ be a trivialising cover. On each double overlap $U_{ij} = U_i \cap U_j$ there is a commuting diagram of trivialisations

$$
\begin{array}{ccc}
U_{ij} \times T_Z & \longrightarrow & U_{ij} \times Z \\
\downarrow \varphi_j^{-1} & & \downarrow \psi_j^{-1} \\
P|_{U_{ij}} & \longrightarrow & Y'|_{U_{ij}} \\
\downarrow \varphi_i & & \downarrow \psi_i \\
U_{ij} \times T_Z & \longrightarrow & U_{ij} \times T_Z.
\end{array}
$$

(5)

The transition functions $\varphi_i \circ \varphi_j^{-1}$ and $\psi_i \circ \psi_j^{-1}$ correspond equivalently to morphisms $\varphi_{ij} : U_{ij} \to T_Z$, $\psi_{ij} : U_{ij} \to \text{Aut}(Z)$. To show that $Y \to X$ arises from the mixing construction, we will show that $\psi_{ij}$ factors through the subgroup $T_Z \leq \text{Aut}(Z)$. The action $T_Z \acts Z$ defines a unique extension of $\varphi_{ij}$ to an automorphism of $Z$. By (5), $\varphi_{ij}$ and $\psi_{ij}$ coincide on the dense open $T_Z$ and hence on all of $Z$. Therefore $\psi_{ij}$ factors through $T_Z$ as claimed.

Conversely, let $Y = (P \times Z)/T_Z \to X$ be obtained via the mixing construction. We have

$$P = (P \times T_Z)/T_Z$$

and so clearly there is an inclusion $P \hookrightarrow Y$ which locally restricts to $X \times T_Z \hookrightarrow X \times Z$. □

3.2. Identifying the mixing collection. Consider a toric variety bundle $P \subseteq Y \to X$, which by Lemma 3.4 we may express as a quotient

$$Y = (P \times Z)/T_Z$$

with $T_Z \subseteq Z$ the toric fibre. Since $T_Z \acts Z$ preserves the toric strata, there are well-defined fibrewise toric strata in $Y$ which map surjectively onto $X$. This is one of the major advantages of Definition 3.1, and will play a crucial role in the discussion of tropical expansions.

Let $\Phi$ be the fibre fan corresponding to $Z$ and let $\Phi(1)$ denote the set of rays. For $\rho \in \Phi(1)$ we let $D_\rho \subseteq Z$ denote the corresponding toric Weil divisor and

$$D_\rho = (P \times D_\rho)/T_Z \subseteq (P \times Z)/T_Z = Y$$

the horizontal divisor in the toric variety bundle. Let $M$ be the character lattice of $T_Z$. Recall that for each $m \in M$ there is a relation

$$\sum_{\rho \in \Phi(1)} \langle m, v_\rho \rangle D_\rho = 0$$

in $\text{Cl}(Z)$, where $v_\rho \in N$ is the primitive lattice generator of the ray. In the bundle $Y$, the analogous relations hold only up to line bundles pulled back from the base. The following lemma shows that these line bundles are equivalent to the data of the mixing collection.

Lemma 3.5. Let $\pi : Y \to X$ be a toric variety bundle obtained via the mixing construction (Construction 3.2), with fibre fan $\Phi$ and mixing collection encoded by a homomorphism

$$L : M \to \text{Pic} X.$$

Then for each $m \in M$ we have the following relation in $\text{Cl} Y$

$$\sum_{\rho \in \Phi(1)} \langle m, v_\rho \rangle D_\rho = \pi^* L(m).$$
Proof. Note that $M = \text{Cl}_{T_Z}(pt) = \text{Cl}(BT_Z)$. Consider the commuting (in fact cartesian) diagram

$$
\begin{array}{ccc}
\text{[Z/T_Z]} & \rightarrow & \text{BT}_Z \\
\downarrow & & \downarrow \\
\text{[(P × Z)/T_Z]} & \rightarrow & \text{[P/T_Z]}
\end{array}
$$

Fix $m \in M = \text{Cl}(BT_Z)$. Pulling back along the top route gives the left-hand side of (6). To pull back along the bottom route, we note that $[P/T_Z] = X$ and that the pullback of $m$ to $X$ is $L(m)$. Hence pulling back along the bottom route gives the right-hand side of (6). □

3.3. Fibrewise GIT. For this section only, we assume that the fibre fan $\Phi$ is simplicial and not contained in any proper linear subspace; equivalently, the associated toric variety $Z$ is $\mathbb{Q}$-factorial and contains no torus factors. In this context, it is well-known that $Z$ arises as a GIT quotient

$$Z = A_{\Phi(1)} \sslash G$$

where $G = \text{Hom}(\text{Cl} Z, \mathbb{G}_m)$ is a finite extension of an algebraic torus [Cox95b] (see also [CLS11, Chapters 5 and 14]) and $\Phi(1)$ is the set of rays. The exact sequence

$$0 \rightarrow M \rightarrow Z_{\Phi(1)} \rightarrow \text{Cl} Z \rightarrow 0$$

dualises, since $\mathbb{G}_m$ is divisible, to an exact sequence

$$(7) \quad 0 \rightarrow G \rightarrow \mathbb{G}_{\Phi(1)} \rightarrow T_Z \rightarrow 0.$$  

The inclusion $G \hookrightarrow \mathbb{G}_{\Phi(1)}$ defines an action $G \curvearrowright A_{\Phi(1)}$ of which $Z$ is the GIT quotient. The unstable locus

$$B(\Phi) \subseteq A_{\Phi(1)}$$

is a union of coordinate subspaces, corresponding to collections of toric divisors in $Z$ with empty intersection. It is induced by a character $\theta$ of $G$ which we fix once and for all (see e.g. [CI18, Section 4.2] for more details).

The GIT construction extends to the relative setting mutatis mutandis, as we now explain. This has already appeared at various points in the literature, see e.g. [Hal03, Hu08, Bro14, Oh21].

Construction 3.6 (Fibrewise GIT). Fix a fan $\Phi$ whose associated toric variety $Z$ has a GIT quotient presentation, as above. Let $X$ be an arbitrary base scheme and choose a homomorphism $K : Z_{\Phi(1)} \rightarrow \text{Pic} X$.

The inclusion $G \hookrightarrow \mathbb{G}_{\Phi(1)}$ produces an action of $G$ on the total space of the vector bundle

$$E = \bigoplus_{\rho \in \Phi(1)} K(e_\rho).$$

The character $\theta$ of $G$ produces a lifted action on the total space of the trivial line bundle over $E$

$$G \curvearrowright E \times \mathbb{A}^1$$

$$g(y, z) = (g \cdot y, \theta(g) \cdot z).$$

We let $O_E(\theta)$ denote this equivariant line bundle. The fibrewise GIT quotient is then defined as

$$E \sslash G := \text{Proj}_X \bigoplus_{k \geq 0} (\pi_* O(E(k\theta)))^G$$

where $\pi : E \rightarrow X$. Since the character $\theta$ is fixed throughout, we suppress it from the notation.
The following Theorems 3.7 and 3.8 together establish an equivalence between the mixing construction (Construction 3.2) and the fibrewise GIT construction (Construction 3.6).

**Theorem 3.7.** Every fibrewise GIT quotient is a toric variety bundle in the sense of Definition 3.1.

**Proof.** Fix a fibrewise GIT quotient $Y = E \sslash G$ as in Construction 3.6. Let $E^0 \subseteq E$ be the complement of all the coordinate subbundles. We claim that

$$P := E^0 \sslash G \hookrightarrow E \sslash G = Y$$

is the inclusion of a principal $T_Z$-bundle which locally restricts to $X \times T_Z \hookrightarrow X \times Z$. Over a trivialising open set $U \subseteq X$ for $E$ we have

$$E^0|_U \cong U \times \mathbb{G}^\Phi(1)_{m} \subseteq U \times \mathbb{A}^\Phi(1) \cong E|_U.$$

Note that $E^0$ is disjoint from the unstable locus in $E$, and moreover the action $G \rtimes E$ restricts to a free action $G \rtimes E^0$. Hence we obtain

$$(E^0 \sslash G)|_U = U \times (\mathbb{G}^\Phi(1)_{m}/G) = U \times T_Z$$

where the last equality holds by (7). This produces the desired inclusion $P := E^0 \sslash G \hookrightarrow Y$ which clearly restricts to $U \times T_Z \hookrightarrow U \times Z$ locally, as required. We note that $P$ is the principal $T_Z$-bundle corresponding to the composite

$$M \hookrightarrow Z^\Phi(1) \xrightarrow{K} \text{Pic} X.$$ 

□

**Theorem 3.8.** Consider a toric variety bundle $Y \to X$ with mixing collection encoded in a homomorphism $L: M \to \text{Pic} X$. Choose a homomorphism

$$K: Z^\Phi(1) \to \text{Pic} X$$

which restricts to $L$ under the natural inclusion $M \hookrightarrow Z^\Phi(1)$. Then $Y$ is equal to the fibrewise GIT quotient associated to $K$.

In particular, if $K_1: Z^\Phi(1) \to \text{Pic} X$ and $K_2: Z^\Phi(1) \to \text{Pic} X$ restrict to the same homomorphism $L: M \to \text{Pic} X$ then the fibrewise GIT quotients associated to $K_1$ and $K_2$ are isomorphic.

**Proof.** Let $E = \bigoplus_{\rho \in \Phi(1)} K(e_{\rho})$. On a trivialising open set we have $E|_U = U \times \mathbb{A}^\Phi(1)$ and the fibrewise GIT construction gives

$$(E \sslash G)|_U = U \times Z.$$

The transition functions for $E$ take values in $\mathbb{G}^\Phi(1)_{m}$ and hence the transition functions for $E \sslash G$ take values in

$$\mathbb{G}^\Phi(1)_{m}/G = T_Z.$$

Restricting to the principal bundle $P \subseteq Y$, the fact that $K$ restricts to $L$ implies that the transition functions for $P$ and for $E \sslash G$ coincide. Hence $Y = E \sslash G$ as claimed. □

**Remark 3.9.** The fact that the fibrewise GIT quotient only depends on the restriction $L$ generalises the well-known fact that

$$\mathbb{P}_X(E \otimes L) \cong \mathbb{P}_X(E)$$

for $E$ a vector bundle and $L$ a line bundle.

When the fibre fan is smooth the fibrewise GIT quotient admits homogeneous coordinates over the base, as in [Cox95a].
Lemma 3.10. Fix a base scheme $X$, a fibre fan $\Phi$, and a homomorphism \\
$K : \mathbb{Z}^{\Phi(1)} \to \text{Pic } X$ \\
with restriction $L : M \to \text{Pic } X$. Let $Y = E \sslash G$ denote the fibrewise GIT quotient of Construction 3.6. Suppose that $\Phi$ is smooth and let $S$ be an arbitrary test scheme. Then a morphism $S \to Y$ is equivalent to the following data:

1. a morphism $f : S \to X$;
2. for each $\rho \in \Phi(1)$ a line bundle and section $(A_\rho, u_\rho)$ on $S$;
3. an isomorphism \\
$$
\bigotimes_{\rho \in \Phi(1)} A_\rho \otimes ^{(m, v_\rho)} \rho \cong f^* L(m)
$$
for each $m \in M$, compatible with the group structure on $M$.

The tuple of sections $(u_\rho)_{\rho \in \Phi(1)}$ must avoid the unstable locus $B(\Phi) \subseteq \bigoplus_{\rho \in \Phi(1)} A_\rho$.

Proof. As in the proof of Lemma 3.5 there is a cartesian square

$$
\begin{array}{ccc}
Y & \longrightarrow & [Z/T] \\
\downarrow & & \downarrow \\
X & \longrightarrow & B T Z.
\end{array}
$$

Since $\Phi$ is smooth, the GIT quotient coincides with the stack quotient:

$$
Z = \left[ (\mathbb{A}^{\Phi(1)} \backslash B(\Phi))/G \right].
$$

Together with (7), this gives \\
$$
[Z/T] = \left[ (\mathbb{A}^{\Phi(1)} \backslash B(\Phi))/\mathbb{G}_m^{\Phi(1)} \right].
$$

The result now follows directly from (8). Note that the morphism $X \to B T Z$ precisely encodes the mixing collection $L$. \hfill \Box

4. Expansion Components

4.1. Setup. Fix a toroidal embedding $(X|D)$ with tropicalisation $\Sigma = \Sigma(X|D)$. Let $\tau$ be a cone and consider a combinatorial tropical expansion \\
$$
\Upsilon \xrightarrow{T} \Sigma \\
\downarrow \rho \downarrow \tau
$$
with associated geometric tropical expansion \\
$$
\mathcal{X}_\Upsilon \xrightarrow{\rho} X \\
\downarrow \pi \downarrow U_\tau
$$
where $U_\tau = \text{Spec } k[S_\tau]$ is the corresponding affine toric variety. Let $0 \in U_\tau$ be the torus-fixed point. As discussed in Section 2.3, the irreducible components of the central fibre $Y_\Upsilon = \pi^{-1}(0)$ are indexed by vertices $v$ in the corresponding polyhedral subdivision of $\Sigma$, \\
$$
Y_\Upsilon = \bigcup_{v \in V} Y_v.
$$
By the polyhedral-conical dictionary, a vertex $v$ corresponds to a cone $\omega_v \in \Upsilon$ mapped isomorphically onto $\tau$ by $p$, and $Y_v \hookrightarrow X\Upsilon$ is the closed stratum corresponding to $\omega_v$.

Let $\sigma_v \in \Sigma$ be the minimal cone containing $r(\omega_v)$ and let $X_v \hookrightarrow X$ be the closed stratum corresponding to $\sigma_v$. The morphism $\rho$ restricts to a collapsing morphism

$$\rho_v: Y_v \rightarrow X_v.$$ 

Since the pair $(X|D)$ is arbitrary the stratum $X_v$ is equally arbitrary, and nothing interesting can be said about it. Instead, our goal is to describe the relative geometry of the morphism $\rho_v$. As we will see, this is controlled entirely by the polyhedral combinatorics.

### 4.2. Cautionary examples.

The naive expectation is that $\rho_v$ should be a toric variety bundle. However, this is false: it may have reducible fibres (Example 4.1) or even fail to be flat (Example 4.2). This precludes a simple description of $\rho_v$. In the following sections we study a sequence of open subschemes

$$Y_v^\circ \hookrightarrow Y_v^\bullet \hookrightarrow Y_v$$

and proceed to describe the relative geometry of each over the base. For $Y_v^\circ$ and $Y_v^\bullet$ we obtain complete descriptions (Sections 4.4 and 4.5). For $Y_v$ we obtain a combinatorial criterion for $\rho_v$ to be a toric variety bundle; when this criterion is satisfied, we give a combinatorial recipe to construct $Y_v$ from $X_v$ (Section 4.6). If the stratum $X_v \hookrightarrow X$ is minimal then $Y_v^\bullet = Y_v$ and so the results of Section 4.5 suffice.

**Example 4.1.** Take $X$ a smooth variety and $D = D_1 + D_2$ the union of two smooth divisors with nonempty connected intersection. We have $\Sigma = \Sigma(X|D) = \mathbb{R}^2_{\geq 0}$. Consider the following tropical expansion over the base cone $\tau = \mathbb{R}_{\geq 0}$ with coordinate $e$.

On the left is the height-1 slice of the conical subdivision of $\Sigma \times \tau = \mathbb{R}^3_{\geq 0}$; on the right is the polyhedral subdivision of $\Sigma$ with parameter $e$. These subdivisions are bijective on supports.

The collapsing morphism $Y_{v_1} \rightarrow X_{v_1} = D_2$ is a flat family of rational curves, but is not a toric variety bundle: the general fibre is smooth but the fibre over $D_1 \cap D_2$ is nodal with two smooth components. Explicitly, $Y_{v_1}$ is the blowup of the $\mathbb{P}^1$-bundle $\mathbb{P}_{D_2}(\mathcal{O}(D_2) \oplus \mathcal{O})$, at the intersection of the infinity section with the fibre over $D_1 \cap D_2$.

**Example 4.2.** Take $X = \mathbb{A}^3$ with $D = D_1 + D_2 + D_3$ the three coordinate planes. Then $\Sigma = \mathbb{R}^3_{\geq 0}$ with coordinates $\ell_1, \ell_2, \ell_3$. Consider the following tropical expansion over the base cone $\tau = \mathbb{R}_{\geq 0}$ with coordinate $e$. 

On the left is the height-1 slice of the conical subdivision of $\Sigma \times \tau = \mathbb{R}^3_{\geq 0}$; on the right is the polyhedral subdivision of $\Sigma$ with parameter $e$. These subdivisions are bijective on supports.
This is an open subdivision: only the original cones of $\Sigma \times \tau$ and the new cones indicated in blue are included. We have $X_{v_1} = D_1$ and the morphism $Y_{v_1} \to D_1$ is toric, with fan map

$$\Sigma_{Y_{v_1}} \to \Sigma_{D_1}$$

given by the lattice morphism $N_T/N_{\omega_{v_1}} \to N_{\Sigma}/N_{\sigma_{v_1}}$. The ray

$$(\mathbb{R}_{\geq 0} v_1 + \mathbb{R}_{\geq 0} w)/(N_{\omega_{v_1}} \otimes \mathbb{R})$$

in $\Sigma_{Y_{v_1}}$ corresponds to the ray in the polyhedral subdivision given by $\{\ell_1 = e, \ell_2 = \ell_3\}$. Under (10) it is mapped onto the diagonal of $\Sigma_{D_1} = \mathbb{R}^2_{\geq 0}$. By [AK00, Lemma 4.1] we conclude that $Y_{v_1} \to D_1$ is not flat. Geometrically: the fibre over a general point of $D_1$ is 1-dimensional, but the fibre over the point $D_1 \cap D_2 \cap D_3$ is 2-dimensional. Combinatorial semistable reduction [AK00, Mol21, ALT20] can be used to flatten the morphism, by subdividing $\Sigma_{D_1}$ along the diagonal. This corresponds to blowing up $D_1$ at the point $D_1 \cap D_2 \cap D_3$.

4.3. Standard mixing collection. The locally-closed stratum $X^0_\circ \hookrightarrow X$ is obtained from the closed stratum $X_{v_1}$ by removing its intersection with all boundary components which do not contain $X_{v_1}$. It sits in a fibre diagram

$$
\begin{array}{ccc}
X^0_\circ & \longrightarrow & X \\
\downarrow & & \downarrow \\
BT_{\sigma_{v_1}} & \longrightarrow & A_{\Sigma}.
\end{array}
$$

The morphism $X^0_\circ \to BT_{\sigma_{v_1}}$ gives a mixing collection on $X^0_\circ$ with lattice $N_{\sigma_{v_1}}$. We refer to this as the **standard mixing collection**.

When $\sigma_{v_1}$ is smooth, the lattice $N_{\sigma_{v_1}}$ has a natural basis given by the primitive ray generators, and the mixing collection is equivalent to the list $O_X(D_\rho)|_{X^0_\circ}$ for $\rho \in \sigma_{v_1}(1)$. There is a natural splitting of the normal bundle

$$N_{X^0_\circ}|_X = \bigoplus_{\rho \in \sigma_{v_1}(1)} O_X(D_\rho)|_{X^0_\circ}. $$
4.4. Torus bundles \( Y_v^o \). We first consider the locally-closed stratum

\[ Y_v^o \hookrightarrow Y_v. \]

This is the open subvariety of \( Y_v \) obtained by removing its intersection with all boundary components of \( X_T \) which do not contain \( Y_v \).

**Proposition 4.3.** The restricted morphism

\[ \rho_v: Y_v^o \to X_v^o \]

is a principal torus bundle, with structure group \( T_{\sigma_v} \). It is the principal bundle corresponding to the standard mixing collection of Section 4.3.

**Remark 4.4.** The identification of the structure group with \( T_{\sigma_v} \) is part of [CN21, Theorem 1.8].

**Proof.** The cone \( \sigma_v \times \tau \in \Sigma \times \tau \) defines (Section 1.4) an open substack \( A_{\sigma_v \times \tau} \hookrightarrow A_{\Sigma \times \tau} \). The composition

\[ BT_{\sigma_v \times \tau} \hookrightarrow A_{\sigma_v \times \tau} \hookrightarrow A_{\Sigma \times \tau} \]

defines a locally-closed substack whose pullback is the corresponding locally-closed stratum

\[ X_v^o \hookrightarrow X \times U_\tau \]

Similarly the cone \( \omega_v \in \Upsilon \) defines a locally-closed substack \( BT_{\omega_v} \hookrightarrow A_T \) whose pullback is the corresponding locally-closed stratum \( Y_v^o \hookrightarrow X_T \). We obtain a diagram

\[
\begin{array}{ccc}
Y_v^o & \longrightarrow & X_T \\
\downarrow & & \downarrow \\
X_v^o & \longrightarrow & X \times U_\tau \\
\downarrow & & \downarrow \\
BT_{\omega_v} & \longrightarrow & A_T \\
\downarrow & & \downarrow \\
BT_{\sigma_v \times \tau} & \longrightarrow & A_{\Sigma \times \tau}.
\end{array}
\]

The back and right faces are cartesian, hence so is their composition. This coincides with the composition of the left and front faces. Since the front face is also cartesian, it follows that the left face is cartesian

\[
\begin{array}{ccc}
Y_v^o & \longrightarrow & BT_{\omega_v} \\
\downarrow & & \downarrow \\
X_v^o & \longrightarrow & BT_{\sigma_v \times \tau}.
\end{array}
\]

By Proposition 2.2 we have a natural isomorphism \( T_{\sigma_v \times \tau}/T_{\omega_v} = T_{\sigma_v} \). Applying Lemma 4.5 below, we obtain a cartesian square
Lemma 4.5. Consider a short exact sequence of algebraic tori
\[ 0 \to H \to G \to G/H \to 0. \]
Then the following square is cartesian
\[ \begin{array}{ccc}
BH & \to & B0 \\
\downarrow & & \downarrow \\
BG & \to & B(G/H)
\end{array} \]
where \( B0 = \text{Spec } k \to BT_{\sigma_v} \) is the universal principal bundle and \( X_v^0 \to BT_{\sigma_v} \) is the standard mixing collection of Section 4.3.

Proof. Fix an arbitrary test scheme \( S \) with a morphism \( S \to BG \) corresponding to a principal \( G \)-bundle \( P_G \). The short exact sequence
\[ 0 \to H \to G \to G/H \to 0 \]
duces a long exact cohomology sequence
\[ \cdots \to H^0(S,G/H) \to H^1(S,H) \to H^1(S,G) \to H^1(S,G/H) \to \cdots \]
Exactness at \( H^1(S,G) \) shows that the principal \( G/H \)-bundle \( PQ/H \) is trivial if and only if \( PQ \cong (PH \times G)/H \) for some principal \( H \)-bundle \( PH \). This proves that the square is cartesian. Note that exactness at \( H^1(S,H) \) shows that the set of lifts \( PH \) forms a torsor under \( G/H \). □

Remark 4.6. The above result has the following amusing formulation
\[ \frac{1/H}{1/G} = G/H. \]

4.5. Toric variety bundles \( Y_v^\bullet \). Consider the open subset of \( Y_v \) given by
\[ Y_v^\bullet := \rho_v^{-1}(X_v^0). \]
Note that if \( \sigma_v \in \Sigma \) is maximal then \( X_v^0 = X_v \) and so \( Y_v^\bullet = Y_v \). In general there is a sequence of open inclusions
\[ Y_v^\circ \hookrightarrow Y_v^\bullet \hookrightarrow Y_v. \]

Remark 4.7. The inclusion \( Y_v^\circ \hookrightarrow Y_v^\bullet \) is a fibrewise toric compactification of the principal torus bundle \( Y_v^\circ \to X_v^\circ \). If the open subdivision \( \Upsilon \to \Sigma \times \tau \) is bijective on supports then the morphism \( Y_v^\bullet \to X_v^\circ \) is proper. In contrast, \( Y_v^\circ \to X_v^\circ \) is only proper when it is an isomorphism.

In this section we prove that \( Y_v^\bullet \) is obtained from \( X_v^0 \) by applying the mixing construction (Construction 3.2). We first define the appropriate fibre fan. The subscheme \( Y_v^\bullet \hookrightarrow X_\Upsilon \) is the union of the locally-closed strata indexed by cones in the following set
\[ \Psi_v := \{ \omega \in \Upsilon \mid \omega_v \subseteq \omega \text{ and } r(\omega) \subseteq \sigma_v \}. \]
For \( \omega \in \Psi_v \) we have \( \tau = p(\omega_v) \subseteq p(\omega) \) and so \( p(\omega) = \tau \) which guarantees that \( \omega \) indexes a stratum of the central fibre. Similarly, the conditions \( \omega_v \subseteq \omega \) and \( r(\omega) \subseteq \sigma_v \) together imply that \( \sigma_v \in \Sigma \) is
the minimal cone containing $r(\omega)$. This guarantees that the locally-closed stratum corresponding to $\omega$ maps to $X_v^o$ under $\rho$.

**Remark 4.8.** The set $\Psi_v$ is not a subcomplex of $\Upsilon$ since it is not closed under taking faces. However, it is closed under intersections and contains a unique minimal cone $\omega_v$ which functions as the apex. In the language of Section 1.4, it is an isotropic cone complex.

For every $\omega \in \Psi_v$ the restriction of the open subdivision $r \times p : \Upsilon \to \Sigma \times \tau$ produces an inclusion

$$\omega \subseteq (N_{\sigma_v} \times N_{\tau}) \otimes \mathbb{R}.$$  

By Proposition 2.2 we have $(N_{\sigma_v} \times N_{\tau})/N_{\omega_v} = N_{\sigma_v}$. Let

$$\omega/\omega_v \subseteq N_{\sigma_v} \otimes \mathbb{R}$$

denote the image of $\omega$ in the quotient. This is strictly convex because $\omega_v \subseteq \omega$ is a face.

**Definition 4.9.** The fibre fan associated to $Y_v^\bullet \to X_v^o$ is the fan

$$\Phi_v := \{\omega/\omega_v \mid \omega \in \Psi_v\}$$

in the lattice $N_{\sigma_v}$.

**Remark 4.10.** The fibre fan is visible in the polyhedral complex $\Upsilon_f$. Restrict this complex to a small neighbourhood around the vertex $v$. Declare $v$ to be the origin, and extend all local polyhedra radially out from this point. This gives a fan in the lattice $N$. Intersecting this fan with the sublattice $N_{\sigma_v}$ gives $\Phi_v$.

We now establish the main result constructing $Y_v^\bullet$ from $X_v^o$.

**Theorem 4.11.** The morphism

$$\rho_v : Y_v^\bullet \to X_v^o$$

coincides with the output of the mixing construction (Construction 3.2), with fibre fan $\Phi_v$ (Definition 4.9) and the standard mixing collection (Section 4.3).

**Proof.** Recall the discussion in Section 1.4. The sequence of embeddings $Y_v^\bullet \hookrightarrow Y_v \hookrightarrow X^\tau$ corresponds to the sequence of isotropic Artin fans $B_{\Psi_v} \hookrightarrow B_{\St(\omega_v, \Upsilon)} \hookrightarrow A^\Upsilon$. By the definition of $Y_v^\bullet$ there is a cartesian square

$$
\begin{array}{ccc}
Y_v^\bullet & \hookrightarrow & X^\tau \\
\downarrow & & \downarrow \\
B_{\Psi_v} & \hookrightarrow & A^\Upsilon.
\end{array}
$$

For $\omega \in \Psi_v$ we have $\omega \subseteq \sigma_v \times \tau$ and so we obtain a homomorphism $T_\omega \to T_{\sigma_v \times \tau}$. The compositions $[U_{\omega/\omega_v}/T_\omega] \to BT_\omega \to BT_{\sigma_v \times \tau}$ glue along face inclusions to produce a global morphism

$$B_{\Psi_v} \to BT_{\sigma_v \times \tau}.$$
We obtain a cube

\[
\begin{array}{ccc}
Y_v \rightarrow & \mathcal{X}_\tau & \rightarrow X \times U_\tau \\
(12) & \downarrow & \downarrow \\
B_{\Psi_v} \rightarrow & A_\Phi_v & \rightarrow A_{\Sigma \times \tau} \\
& BT_{\sigma_v \times \tau} \rightarrow & \\
\end{array}
\]

as in the proof of Proposition 4.3, and a diagram chase again shows that the left face is cartesian.

We now show that there is a fibre square

\[
\begin{array}{ccc}
B_{\Psi_v} \rightarrow & A_\Phi_v & \\
\downarrow & \downarrow & \\
BT_{\sigma_v \times \tau} \rightarrow & BT_{\sigma_v} & \\
\end{array}
\]

(13)

where the map $BT_{\sigma_v \times \tau} \rightarrow BT_{\sigma_v}$ is the gerbe banded by $T_{\omega_v}$ induced by the isomorphism $(N_{\sigma_v} \times N_\tau)/N_{\omega_v} = N_{\sigma_v}$ (see Proposition 2.2 and Lemma 4.5).

For every $\omega \in \Psi_v$, there is a homomorphism of tori $T_\omega \rightarrow T_{\omega/\omega_v}$ through which the action $T_\omega \curvearrowright U_{\omega/\omega_v}$ is defined. This gives a morphism

\[
[U_{\omega/\omega_v}/T_\omega] \rightarrow [U_{\omega/\omega_v}/T_{\omega/\omega_v}]
\]

compatible along face inclusions. Globalising, we obtain $B_{\Psi_v} \rightarrow A_{\Phi_v}$ which we combine with the map $B_{\Psi_v} \rightarrow BT_{\sigma_v \times \tau}$ constructed above to obtain a morphism

\[
B_{\Psi_v} \rightarrow BT_{\sigma_v \times \tau} \times BT_{\sigma_v} A_{\Phi_v}.
\]

(14)

It can be checked locally on the target that the morphism $B_{\Psi_v} \rightarrow A_{\Phi_v}$ is a gerbe banded by $T_{\omega_v}$. On the other hand the morphism $BT_{\sigma_v \times \tau} \times BT_{\sigma_v} A_{\Phi_v} \rightarrow A_{\Phi_v}$ is a gerbe banded by $T_{\omega_v}$ by Lemma 4.5. A morphism between two gerbes banded by the same group is automatically an isomorphism (see e.g. [Bro21, proof of Proposition 5.11]). We conclude that (13) is cartesian (we thank the anonymous referee for suggesting this argument). Composing with the left face of (12) we obtain

\[
\begin{array}{ccc}
Y_v \rightarrow & A_\Phi_v & \\
\downarrow & \downarrow & \\
X_v \rightarrow & BT_{\sigma_v}. & \\
\end{array}
\]

Let $F_v$ be the toric variety corresponding to the fan $\Phi_v$. This carries the action of the dense torus $T_{\sigma_v} \curvearrowright F_v$. There is a natural isomorphism

\[
A_{\Phi_v} = [F_v/T_{\sigma_v}]
\]

since both arise as the colimit of quotients $A_{\omega/\omega_v} = [U_{\omega/\omega_v}/T_{\omega/\omega_v}]$ for $\omega \in \Psi_v$. For the right-hand side this relies on the following fact: if $\sigma \subseteq N_1 \otimes \mathbb{R}$ is a cone and $N_1 \subseteq N_2$ a lattice inclusion, the closed embedding $U_{\sigma,N_1} \hookrightarrow U_{\sigma,N_2}$ becomes an isomorphism after quotienting by dense tori

\[
[U_{\sigma,N_1}/T_{N_1}] = [U_{\sigma,N_2}/T_{N_2}]
\]
because on the right-hand side the larger torus cancels out the additional torus factors (in short: Artin cones do not countenance enlargements of the lattice). We obtain

\[
\begin{align*}
Y_v^* & \longrightarrow [F_v/T_{\sigma_v}] \\
\downarrow & \downarrow \\
X_v^0 & \longrightarrow BT_{\sigma_v}.
\end{align*}
\]

(15)

It follows that a morphism \( S \rightarrow Y_v^* \) consists of: a morphism \( g: S \rightarrow X_v^0 \), a principal \( T_{\sigma_v} \)-bundle \( Q \rightarrow S \) with an equivariant morphism \( Q \rightarrow F_v \), and an isomorphism \( Q \cong g^* P \) where \( P \) is the principal bundle on \( X_v^0 \) induced by the morphism \( X_v^0 \rightarrow BT_{\sigma_v} \). This data is equivalent to: a morphism \( g: S \rightarrow X_v^0 \), a principal \( T_{\sigma_v} \)-bundle \( Q \rightarrow S \) and an equivariant morphism

\[
Q \rightarrow g^* P \times F_v
\]

where \( T_{\sigma_v} \) acts diagonally on the target. This shows \( Y_v^* = (P \times F_v)/T_{\sigma_v} \) as claimed. \( \square \)

**Remark 4.12.** The morphism \( [F_v/T_{\sigma_v}] \rightarrow BT_{\sigma_v} \) appearing in the above proof encodes the equivariant divisor classes

\[
\sum_{\varphi \in \Phi_v(1)} \langle m, v_\varphi \rangle D_\varphi
\]

on \( F_v \), for \( m \in M_{\sigma_v} \). Commutativity of (15) then recovers the following identity in \( \text{Cl} Y_v^* \)

\[
\sum_{\varphi \in \Phi_v(1)} \langle m, v_\varphi \rangle D_\varphi = \rho_\varphi^* L(m)
\]

where \( L \) is the standard mixing collection; see Lemma 3.5 and Section 4.3.

**Remark 4.13.** Theorem 4.11 generalises: for every locally-closed stratum \( S \hookrightarrow X_v \) the restriction

\[
\rho_S: \rho_S^{-1}(S) \rightarrow S
\]

is a finite union of toric variety bundles, glued along toric strata. This provides a cut-and-paste description of the morphism \( Y_v \rightarrow X_v \), as a stratified union of toric variety bundles. See [HLY02, Proposition 2.1.4] for a parallel in the toric setting.

### 4.6. Components \( Y_v \).

We now consider the entire component \( Y_v \). We emphasise that if \( \sigma_v \) is maximal, then \( X_v = X_v^0 \) and \( Y_v = Y_v^* \). In this case Theorem 4.11 gives a complete description of \( Y_v \). The following discussion is significantly more delicate, and only required when \( \sigma_v \) is not maximal.

#### 4.6.1. Bundle criterion.

As discussed in Section 4.2, the morphism \( Y_v \rightarrow X_v \) is not always a toric variety bundle. This can be detected on the level of cone complexes.

**Theorem 4.14.** The morphism \( Y_v \rightarrow X_v \) is a toric variety bundle if and only if there is an isomorphism of cone complexes

\[
\Upsilon/\omega_v \cong \Sigma/\sigma_v \times \Phi_v
\]

commuting with the projections to \( \Sigma/\sigma_v \), where \( \Phi_v \) is the fibre fan of Definition 4.9.

**Proof.** If the morphism \( Y_v \rightarrow X_v \) is a toric variety bundle, every fibre must be isomorphic to the fibre over a point in \( X_v^0 \). By Theorem 4.11 it follows that the fibre fan is \( \Phi_v \). Examining the fibrewise toric strata, we conclude that there is an order-preserving bijection between strata of \( Y_v \) and strata of \( X_v \times F_v \). The corresponding cones are isomorphic, so we obtain an isomorphism \( \Upsilon/\omega_v \cong \Sigma/\sigma_v \times \Phi_v \).

Conversely, suppose we are given the isomorphism \( \Upsilon/\omega_v \cong \Sigma/\sigma_v \times \Phi_v \). For each \( \omega \in \text{St}(\omega_v, \Upsilon) \) this furnishes cones \( \sigma \in \text{St}(\sigma_v, \Sigma), \varphi \in \Phi_v \) and a fixed isomorphism

\[
\omega/\omega_v = \sigma/\sigma_v \times \varphi.
\]
From this we obtain the following diagram of lattices, with exact rows and columns:

\[
\begin{array}{ccccccccc}
0 & & & & & & & & 0 \\
\downarrow & & & & & & & & \downarrow \\
0 & \rightarrow & N_{\omega_v} & \rightarrow & N_{\omega} & \rightarrow & N_{\omega/\omega_v} & \rightarrow & 0 \\
\downarrow & & & & & & & & \downarrow \\
0 & \rightarrow & N_{\sigma_v \times \tau} & \rightarrow & N_{\sigma \times \tau} & \rightarrow & N_{\sigma/\sigma_v} & \rightarrow & 0 \\
\downarrow & & & & & & & & \downarrow \\
N_{\sigma_v} & \rightarrow & N_{(\sigma \times \tau)/\omega} & \rightarrow & 0 & & & & \\
\downarrow & & & & & & & & \downarrow \\
0 & & & & & & & & 0
\end{array}
\]

The Snake Lemma produces a short exact sequence

\[(16) \quad 0 \rightarrow N_{\sigma} \rightarrow N_{\sigma_v} \rightarrow N_{(\sigma \times \tau)/\omega} \rightarrow 0\]

while the isomorphism \(\omega/\omega_v = \sigma/\sigma_v \times \varphi\) gives a splitting of the right column

\[N_{\omega/\omega_v} = N_{\sigma/\sigma_v} \times N_{\varphi}.\]

Returning to the morphism \(Y_v \rightarrow X_v\), there is a diagram

\[
\begin{array}{ccc}
Y_v & \rightarrow & X_T \\
\downarrow & & \downarrow & & \downarrow \\
X_v & \rightarrow & X \times U_\tau \\
\downarrow & & \downarrow & & \downarrow \\
B_{\text{St}(\omega_v, \tau)} & \rightarrow & A_\tau \\
\downarrow & & \downarrow & & \downarrow \\
B_{\text{St}(\sigma_v \times \tau, \Sigma \times \tau)} & \rightarrow & A_{\Sigma \times \tau}
\end{array}
\]

and as in the proof of Proposition 4.3 we see that the left face is cartesian. It is therefore sufficient to prove that the morphism

\[(17) \quad \rho_v : B_{\text{St}(\omega_v, \tau)} \rightarrow B_{\text{St}(\sigma_v \times \tau, \Sigma \times \tau)}\]

is a toric variety bundle. We work locally on the target. This is covered by open substacks

\[\left[ U_{(\sigma \times \tau)/(\sigma_v \times \tau)}/T_{\sigma \times \tau} \right]\]

for \(\sigma \in \text{St}(\sigma_v, \Sigma)\). The preimage under \(\rho_v\) is covered by open substacks indexed by cones \(\omega \in \text{St}(\omega_v, Y)\) with \(r(\omega) = \sigma\). These are precisely those cones with \(\omega/\omega_v = \sigma/\sigma_v \times \varphi\) for some \(\varphi \in \Phi_v\). The corresponding open substack of the source is

\[\left[ U_{\omega/\omega_v}/T_\omega \right] = \left[ (U_{\sigma/\sigma_v} \times U_\varphi)/T_\omega \right].\]
The action of $T_\omega$ on the product is induced by the projection $T_\omega \to T_{\omega/\omega_v}$ and the splitting $T_{\omega/\omega_v} = \sigma_v \times T_\varphi$. We reduce to proving that the morphism

$$\lim_{\varphi \in \Phi_v} [(U_{\sigma_v} \times U_\varphi) / T_\omega] \to [U_{(\sigma \times \tau)/(\sigma_v \times \tau)} / T_{\sigma \times \tau}]$$

is a toric variety bundle. On the source, Lemma 4.15 below shows that the morphism

$$[(U_{\sigma_v} \times U_\varphi) / T_\omega] \to [(U_{\sigma_v} \times U_\varphi) / T_{\omega/\omega_v}] = [U_{\sigma_v} / T_{\sigma_v}] \times [U_\varphi / T_\varphi]$$

is a gerbe banded by $T_{\omega_v}$. On the target, we first prove that there is a natural splitting

$$N_{(\sigma \times \tau)/\omega_v} = N_{\sigma} \times N_{\sigma_v}.$$  

Indeed, multiplying the first two terms of the short exact sequence (16) by $N_{\sigma_v}$ gives

$$0 \to N_{\omega/\omega_v} \to N_{\sigma} \times N_{\sigma_v} \to N_{(\sigma \times \tau)/\omega} \to 0$$

while the second isomorphism theorem for groups gives

$$0 \to N_{\omega/\omega_v} \to N_{(\sigma \times \tau)/\omega_v} \to N_{(\sigma \times \tau)/\omega} \to 0$$

from which we obtain (18). The action $T_{\sigma \times \tau} \cap U_{(\sigma \times \tau)/(\sigma_v \times \tau)} = U_{\sigma_v}$ is induced by the composition $T_{\sigma \times \tau} \to T_{(\sigma \times \tau)/\omega_v} \to T_{\sigma_v}$. We conclude from Lemma 4.15 that the induced morphism

$$[U_{(\sigma \times \tau)/(\sigma_v \times \tau)} / T_{\sigma \times \tau}] \to [U_{(\sigma \times \tau)/(\sigma_v \times \tau)} / T_{(\sigma \times \tau)/\omega_v}] = [U_{\sigma_v} / T_{\sigma_v}] \times BT_{\sigma_v}$$

is a gerbe banded by $T_{\omega_v}$. We obtain a commuting square

$$[(U_{\sigma_v} \times U_\varphi) / T_\omega] \to [U_{\sigma_v} / T_{\sigma_v}] \times [U_\varphi / T_\varphi]$$

where both horizontal arrows are gerbes banded by $T_{\omega_v}$. As in the proof of Theorem 4.11 we conclude that this square is in fact cartesian, since a morphism between two gerbes banded by the same group is automatically an isomorphism. We obtain

$$[U_{\sigma_v} \times U_\varphi) / T_\omega] \to [U_\varphi / T_\varphi]$$

and passing to the limit over $\varphi \in \Phi_v$ gives

$$\lim_{\varphi \in \Phi_v} [(U_{\sigma_v} \times U_\varphi) / T_\omega] \to [F_\nu / T_{\sigma_v}]$$

The right vertical arrow is a toric variety bundle, as shown in the proof of Theorem 4.11. It follows that the left vertical arrow is a toric variety bundle. This shows that, locally on the target, (17) is a toric variety bundle.

These local bundles are compatible along overlaps: globally, the principal $T_{\sigma_v}$-bundle is the open substack of $B_{St(\omega, \nu)}$ corresponding to the set of cones $\omega \in St(\omega, \nu)$ with $\omega / \omega_v \cong \sigma / \sigma_v \times 0$. □
Lemma 4.15. Consider a short exact sequence of algebraic tori

\[ 0 \to H \to G \to G/H \to 0. \]

Given an algebraic stack \( X \) with an action \( G/H \rhd X \), let \( G \rhd X \) be the action induced by the homomorphism \( G \to G/H \). Then the following square is cartesian

\[
\begin{array}{ccc}
[X/G] & \to & [X/(G/H)] \\
\downarrow & & \downarrow \\
BG & \to & B(G/H)
\end{array}
\]

and the horizontal morphisms are gerbes banded by \( H \).

Proof. Fix an arbitrary test scheme \( S \). A morphism from \( S \) to the fibre product is given by the following data: a principal \( G \)-bundle \( P_G \to S \), a principal \((G/H)\)-bundle \( P_{G/H} \to S \), a \((G/H)\)-equivariant morphism \( P_{G/H} \to X \) and an isomorphism \( P_{G/H} \cong P_{G/H} \). The composition \( P_G \to P_{G/H} \to X \) is \( G \)-equivariant, producing a morphism from \( S \) to \([X/G]\). This shows that the square is cartesian. It follows from Lemma 4.5 that the horizontal morphisms are gerbes banded by \( H \). \( \square \)

Remark 4.16. If \( Y \to X \) is not a toric variety bundle, it can be transformed into one by toroidal modifications of the source and target. This is the strongest statement one can hope for in general: for instance if \( X = X \) then the morphism \( Y \to X \) can be an arbitrary toroidal modification.

4.6.2. Determining the mixing collection. Assume that the criterion of Theorem 4.14 is satisfied. The morphism \( Y \to X \) is a toric variety bundle and is therefore obtained by applying the mixing construction (Construction 3.2). The fibre fan is simply \( \Phi \), as given in Definition 4.9. The mixing collection is less explicit. It is constructed in the proof of Theorem 4.14: the morphisms

\[ [U_{(\sigma \times \tau)/(\sigma \times \tau)}/T_{\sigma \times \tau}] \to BT_{\sigma \times \tau} \]

glue to give a global morphism \( B_{\St(\sigma \times \tau, \Sigma \times \tau)} \to BT_{\sigma \times \tau} \) which pulls back to the mixing collection

\[ X \to BT_{\sigma \times \tau}. \]

When \( \Sigma \) is a smooth cone complex this mixing collection can be calculated explicitly. In this case the lattice \( N_{\sigma \times \tau} \) has a natural basis given by the primitive generators of the rays \( \rho \in \sigma \times \tau(1) \). The list of line bundles \( O_{X_{\tau}}(D_{\rho}) \) for \( \rho \in \sigma \times \tau(1) \) defines a mixing collection on \( X \). This is precisely the mixing collection corresponding to the composite (see Remark 1.11):

\[ X \to B_{\St(\sigma \times \tau, \Sigma \times \tau)} = \mathbb{A}_{\Sigma/\sigma \times \tau} \times BT_{\sigma \times \tau} \to BT_{\sigma \times \tau}. \]

We refer to this as the standard mixing collection. The mixing collection inducing the toric variety bundle \( Y \to X \) differs from the standard mixing collection in general. The difference is a sum of divisors supported on the boundary \( X \setminus X_0 \).

This difference can be calculated explicitly using Lemma 3.5. For \( \rho \in \sigma \times \tau(1) \) we must calculate the pullback of the line bundle \( O_{X_{\tau}}(D_{\rho}) \) along the projection \( Y \to X \). This can be expressed in terms of piecewise-linear functions on \( \Sigma \) and \( \tau \), as we now explain.

The divisor \( D_{\rho} \subseteq X \) corresponds to a piecewise-linear function on \( \Sigma \) which pulls back to a piecewise-linear function on \( \tau \). Translating by the pullback of a linear function on \( \tau \) we obtain a piecewise-linear function on \( \tau \) which is identically zero on \( \omega_{\nu} \); this is always possible because \( \omega_{\nu} \to \tau \) is an isomorphism.

The corresponding toroidal divisor in \( X_{\tau} \) intersects the stratum \( Y_{\nu} \) transversely, so we can directly describe its restriction to \( Y_{\nu} \). The result is a sum of horizontal toric divisors on \( Y_{\nu} \), together
with divisors pulled back from $X_v$. The former give the left-hand side of (6), the latter the right-hand side. Thus, the mixing collection is determined.

**Example 4.17.** Consider the following tropical expansion

$$
\ell_1 \ell_2 e
$$

By Theorem 4.14, the component $Y_v$ is a $\mathbb{P}^1$-bundle over $X_v = D_2$. To calculate the mixing collection, first name the toroidal divisors on $Y_v$ as follows

$$
E_{\infty} E_0 F
$$

where $E_0$ and $E_{\infty}$ are the horizontal toric divisors of the bundle, and $F$ is the fibre over $D_1 \cap D_2$. Note that $\mathcal{O}_{Y_v}(F) = \rho^*_v \mathcal{O}_{X_v}(D_1)$. Consider on $\Upsilon$ the following piecewise-linear functions, represented visually by their slopes along rays

$$
\ell_1 \ell_2 e
\begin{array}{c}
E_{\infty} \\
\downarrow \\
E_0 \\
\uparrow \\
F \\
\ell_1 = \ell_2 \ell_2 = \ell_1 + e
\end{array}
$$

Restricting $\rho^* D_2 - \pi^* 0$ from the total space $X_\Upsilon$ to the central fibre component $Y_v$ gives

$$
\rho^* \mathcal{O}_{X_v}(D_2) = \mathcal{O}_{Y_v}(E_{\infty} - E_0 + F)
$$

which we rearrange to obtain

$$
\mathcal{O}_{Y_v}(E_{\infty} - E_0) = \rho^*_v \mathcal{O}_{X_v}(D_2 - D_1).
$$

Combining Lemma 3.5 and Theorem 3.8 we conclude

$$
Y_v = \mathbb{P}_{X_v}(\mathcal{O}_{X_v}(D_2 - D_1) \oplus \mathcal{O}_{X_v}).
$$

**REFERENCES**

[ACM+16] D. Abramovich, Q. Chen, S. Marcus, M. Ulirsch, and J. Wise. Skeletons and fans of logarithmic structures. In Nonarchimedean and tropical geometry, Simons Symp., pages 287–336. Springer, 2016.

[ACMW17] D. Abramovich, Q. Chen, S. Marcus, and J. Wise. Boundedness of the space of stable logarithmic maps. J. Eur. Math. Soc. (JEMS), 19(9):2783–2809, 2017.

[ACP15] D. Abramovich, L. Caporaso, and S. Payne. The tropicalization of the moduli space of curves. Ann. Sci. Éc. Norm. Supér. (4), 48(4):765–809, 2015.
[NS06] T. Nishinou and B. Siebert. Toric degenerations of toric varieties and tropical curves. *Duke Math. J.*, 135(1):1–51, 2006.

[Oh21] J. Oh. Quasimaps to GIT fibre bundles and applications. *Forum Math. Sigma*, 9:Paper No. e56, 39, 2021.

[Ran19] D. Ranganathan. A note on cycles of curves in a product of pairs. *arXiv e-prints*, page arXiv:1910.00239, October 2019.

[Ran22] D. Ranganathan. Logarithmic Gromov-Witten theory with expansions. *Algebr. Geom.*, 9(6):714–761, 2022.

[RSPW19a] D. Ranganathan, K. Santos-Parker, and J. Wise. Moduli of stable maps in genus one and logarithmic geometry, I. *Geom. Topol.*, 23(7):3315–3366, 2019.

[RSPW19b] D. Ranganathan, K. Santos-Parker, and J. Wise. Moduli of stable maps in genus one and logarithmic geometry, II. *Algebra Number Theory*, 13(8):1765–1805, 2019.

[SU03] P. Sankaran and V. Uma. Cohomology of toric bundles. *Comment. Math. Helv.*, 78(3):540–554, 2003.

Francesca Carocci. EPFL, Switzerland. francesca.carocci@epfl.ch

Navid Nabijou. Queen Mary University of London, UK. n.nabijou@qmul.ac.uk