DESTRUCTION OF INVARIANT CIRCLES FOR GEVREY AREA-PRESERVING TWIST MAPS

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Abstract. In this paper, we show that for exact area-preserving twist maps on annulus, the invariant circles with a given rotation number can be destroyed by arbitrarily small Gevrey-α perturbations of the integrable generating function in the $C^r$ topology with $r < 4 - \frac{2}{\alpha}$, where $\alpha > 1$.

Key words. invariant circle, minimal configuration, Peierls's barrier

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1. Introduction and main result

For exact area-preserving twist maps on annulus, it was proved by Herman in [H1] that invariant circles with given rotation numbers can be destroyed by $C^{3-\epsilon}$ arbitrarily small $C^\infty$ perturbations. Following the ideas and techniques developed by J.N.Mather in the series of papers [M1, M2, M3, M4], a variational proof of Herman’s result was provided in [W1]. In contrast with it, it has been shown that KAM invariant circles with certain rotation number persist under arbitrarily small perturbations in the $C^3$ topology ([H2]). For Hamiltonian systems with multi-degrees of freedom, the corresponding results were obtained by [Ch, CW] and [Pö]. A partial result on destruction of all invariant tori can be found in [W2].

On the other hand, for certain rotation numbers, it was obtained by Mather (resp. Forni) in [M4] (resp. [Fo]) that the invariant circles with that rotation numbers can be destroyed by small perturbations in finer topology respectively. More precisely, Mather considered Liouville rotation numbers and the topology of the perturbation induced by $C^\infty$ metric. Forni was concerned about more special rotation numbers which can be approximated by rational ones exponentially fast and the topology of the perturbation induced by the supremum norm of real-analytic function. Roughly speaking, there is a balance among the arithmetic property of the rotation number, the regularity of the perturbation and its topology.

Comparing the results on both sides, it is natural to ask what happens for perturbations of regularity between $C^\infty$ and $C^\omega$ (real-analytic). Gevrey-α ($\alpha \geq 1$) functions (see Definition 2.1) characterize that kind of regularity quantitatively. Gevrey Hamiltonians were considered in lots of works (see [MS1, MS2] and [Pö] for instance). Gevrey-1 functions correspond to “the best” $C^\infty$ functions, i.e. $C^\omega$ functions. Gevrey-∞ functions are equivalent to “the worst” $C^\infty$ functions. For $\alpha > 1$, there are compactly supported functions in the class that are not identically zero. This gives much more flexibility to construct examples and show non-existence of invariant circles. In this paper, we consider the following problem:
for every given rotation number \( \omega \) and \( \alpha \) (\( \alpha > 1 \)), what is the maximum value of \( r \) such that the invariant circle with \( \omega \) can be destroyed by an arbitrarily small Gevrey-\( \alpha \) perturbations of the integrable generating function in the \( C^r \) topology?

To state our result, we first introduce some terminology. An irrational number \( \omega \in \mathbb{R} \) is called \( \mu \)-approximated if there exists a positive number \( C > 0 \) as well as infinitely many integers \( p_n \in \mathbb{Z} \) and \( q_n \in \mathbb{N} \) such that

\[
|q_n \omega - p_n| < C q_n^{-1-\mu}.
\]

It follows from Dirichlet approximation that any irrational number is 0-approximated. In particular, \( \omega \) is called Liouville if it is \( \mu \)-approximated for all \( \mu > 0 \). Given a completely integrable system with the generating function

\[
h_0(x, x') = \frac{1}{2}(x - x')^2, \quad x, x' \in \mathbb{R},
\]

we solve the problem above partially. More precisely, we have the following theorem.

**Theorem 1.1** For exact area-preserving twist maps on annulus, the invariant circles with a given \( \mu \)-approximated rotation number can be destroyed by arbitrarily small Gevrey-\( \alpha \) (\( \alpha > 1 \)) perturbations of \( h_0 \) in the \( C^r \) topology with \( r < 2 + \left( \frac{2}{2} - \frac{2}{\alpha} \right)(1 + \mu) \). In particular, the invariant circles with a given Liouville rotation number can be destroyed by arbitrarily small Gevrey-\( \alpha \) (\( \alpha > 1 \)) perturbations of \( h_0 \) in the \( C^\infty \) topology.

Obviously, Theorem 1.1 implies Herman’s result (\[H1\]) and Mather’s result (\[M4\]). Unfortunately, we still don’t know whether our result is optimal in the class of Gevrey-\( \alpha \) (\( \alpha \geq 1 \)) perturbations. Some further developments of KAM theory are needed to verify the optimality.

For the proof of Theorem 1.1 our approach is parallel to an investigation of variational destruction of invariant circles under \( C^{4-\delta} \) arbitrarily small \( C^\infty \) perturbations of generating functions in \[W1\] (see also \[Fo, M4\]). Hence, some parts of the respective exposition are quite similar. But we decided to repeat them anyway such that the reader needs not refer to \[Fo, M4, W1\] for the essentials.

2. Preliminaries

2.1. Minimal configuration

Let \( F \) be a diffeomorphism of \( \mathbb{R}^2 \) denoted by \( F(x, y) = (X(x, y), Y(x, y)) \). Let \( F \) satisfy:

- **Periodicity:** \( F \circ T = T \circ F \) for the translation \( T(x, y) = (x + 1, y) \);
- **Twist condition:** the map \( \psi : (x, y) \mapsto (x, X(x, y)) \) is a diffeomorphism of \( \mathbb{R}^2 \);
- **Exact symplectic:** there exists a real valued function \( h \) on \( \mathbb{R}^2 \) with \( h(x + 1, y) = h(x, y) \) such that

\[
YdX - ydx = dh.
\]
Then $F$ induces a map on the cylinder denoted by $f: \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{T} \times \mathbb{R}$ ($\mathbb{T} = \mathbb{R}/\mathbb{Z}$). $f$ is called an exact area-preserving monotone twist map. The function $h: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is called a generating function of $F$, namely $F$ is generated by the following equations

$$\begin{align*}
y &= -\partial_1 h(x, x'), \\
y' &= \partial_2 h(x, x'),
\end{align*}$$

where $F(x, y) = (x', y')$.

The function $F$ gives rise to a dynamical system whose orbits are given by the images of points of $\mathbb{R}^2$ under the successive iterates of $F$. The orbit of the point $(x_0, y_0)$ is the bi-infinite sequence

$$\{..., (x_{-k}, y_{-k}), ..., (x_{-1}, y_{-1}), (x_0, y_0), (x_1, y_1), ..., (x_k, y_k), ...\},$$

where $(x_k, y_k) = F(x_{k-1}, y_{k-1})$. The sequence

$$(..., x_{-k}, ..., x_{-1}, x_0, x_1, ..., x_k, ...)$$

denoted by $(x_i)_{i \in \mathbb{Z}}$ is called a stationary configuration if it stratifies the identity

$$\partial_1 h(x_i, x_{i+1}) + \partial_2 h(x_{i-1}, x_i) = 0, \text{ for every } i \in \mathbb{Z}.$$

Given a sequence of points $(z_i, ..., z_j)$, we can associate its action

$$h(z_i, ..., z_j) = \sum_{i \leq s < j} h(z_s, z_{s+1}).$$

A configuration $(x_i)_{i \in \mathbb{Z}}$ is called minimal if for any $i < j \in \mathbb{Z}$, the segment $(x_i, ..., x_j)$ minimizes $h(z_i, ..., z_j)$ among all segments $(z_i, ..., z_j)$ of the configuration satisfying $z_i = x_i$ and $z_j = x_j$. It is easy to see that every minimal configuration is a stationary configuration. There is a visual way to describe configurations. A configuration $(x_i)_{i \in \mathbb{Z}}$ is a function from $\mathbb{Z}$ to $\mathbb{R}$. One can interpolate this function linearly and obtain a piecewise affine function $\mathbb{R} \rightarrow \mathbb{R}$ denoted by $t \rightarrow x_t$. The graph of this function is sometimes called the Aubry diagram of the configuration. By [Ba] (see also [Go]), minimal configurations satisfy a group of remarkable properties as follows:

- Two distinct minimal configurations seen as the Aubry diagrams cross at most once, which is so called Aubry’s crossing lemma.

- For every minimal configuration $x = (x_i)_{i \in \mathbb{Z}}$, the limit

$$\rho(x) = \lim_{n \rightarrow \infty} \frac{x_{i+n} - x_i}{n}$$

exists and doesn’t depend on $i \in \mathbb{Z}$. $\rho(x)$ is called the rotation number of $x$.

- For every $\omega \in \mathbb{R}$, there exists a minimal configuration with rotation number $\omega$. Following the notations of [Ba], the set of all minimal configurations with rotation number $\omega$ is denoted by $M^h_\omega$, which can be endowed with the topology induced from the product topology on $\mathbb{R}^\mathbb{Z}$. If $x = (x_i)_{i \in \mathbb{Z}}$ is a minimal configuration, considering the projection $pr: M^h_\omega \rightarrow \mathbb{R}$ defined by $pr(x) = x_0$, we set $A^h_\omega = pr(M^h_\omega)$. 

If \( \omega \in \mathbb{Q} \), say \( \omega = p/q \) (in lowest terms), then it is convenient to define the rotation symbol to detect the structure of \( M^h_{p/q} \). If \( x \) is a minimal configuration with rotation number \( p/q \), then the rotation symbol \( \sigma(x) \) of \( x \) is defined as follows:

\[
\sigma(x) = \begin{cases} 
p/q+, & \text{if } x_{i+q} > x_i + p \text{ for all } i, 
p/q, & \text{if } x_{i+q} = x_i + p \text{ for all } i, 
p/q-, & \text{if } x_{i+q} < x_i + p \text{ for all } i.
\end{cases}
\]

Moreover, we set

\[
M^h_{p/q+} = \{ x \text{ is a minimal configuration with rotation symbol } p/q \text{ or } p/q+ \},
\]

\[
M^h_{p/q-} = \{ x \text{ is a minimal configuration with rotation symbol } p/q \text{ or } p/q- \},
\]

then both \( M^h_{p/q+} \) and \( M^h_{p/q-} \) are totally ordered. Namely, every two configurations in each of them (seen as Aubry diagrams) do not cross. We denote \( pr(M^h_{p/q+}) \) and \( pr(M^h_{p/q-}) \) by \( A^h_{p/q+} \) and \( A^h_{p/q-} \) respectively.

- If \( \omega \in \mathbb{R} \setminus \mathbb{Q} \) and \( x \) is a minimal configuration with rotation number \( \omega \), then \( \sigma(x) = \omega \) and \( M^h_{\omega} \) is totally ordered.
- \( A^h_{\omega} \) is a closed subset of \( \mathbb{R} \) for every rotation symbol \( \omega \).

### 2.2. Peierls’s barrier

In [M3], Mather introduced the notion of Peierls’s barrier and gave a criterion of existence of invariant circle. Namely, the exact area-preserving monotone twist map generated by \( h \) admits an invariant circle with rotation number \( \omega \) if and only if the Peierls’s barrier \( P^h_{\omega}(\xi) \) vanishes identically for all \( \xi \in \mathbb{R} \). The Peierls’s barrier is defined as follows:

- If \( \xi \in A^h_{\omega} \), we set \( P^h_{\omega}(\xi) = 0 \).
- If \( \xi \not\in A^h_{\omega} \), since \( A^h_{\omega} \) is a closed set in \( \mathbb{R} \), then \( \xi \) is contained in some complementary interval \( (\xi^-, \xi^+) \) of \( A^h_{\omega} \) in \( \mathbb{R} \). By the definition of \( A^h_{\omega} \), there exist minimal configurations with rotation symbol \( \omega \), \( x^- = (x_i^-)_{i \in \mathbb{Z}} \) and \( x^+ = (x_i^+)_{i \in \mathbb{Z}} \) satisfying \( x_0^- = \xi^- \) and \( x_0^+ = \xi^+ \). For every configuration \( x = (x_i)_{i \in \mathbb{Z}} \) satisfying \( x_i^- \leq x_i \leq x_i^+ \), we set

\[
G_{\omega}(x) = \sum_I (h(x_i, x_{i+1}) - h(x_i^-, x_{i+1}^-)),
\]

where \( I = \mathbb{Z} \), if \( \omega \) is not a rational number, and \( I = \{0, ..., q-1\} \), if \( \omega = p/q \).

\( P^h_{\omega}(\xi) \) is defined as the minimum of \( G_{\omega}(x) \) over the configurations \( x \in \Pi = \prod_{i \in I} [x_i^-, x_i^+] \) satisfying \( x_0 = \xi \). Namely

\[
P^h_{\omega}(\xi) = \min_{x \in \Pi} \{ G_{\omega}(x) | x_0 = \xi \}.
\]

By [M3], \( P^h_{\omega}(\xi) \) is a non-negative periodic function of the variable \( \xi \in \mathbb{R} \) with the modulus of continuity with respect to \( \omega \) and its modulus of continuity with respect to \( \omega \) can be bounded from above. Due to the periodicity of \( P^h_{\omega}(\xi) \) with respect to \( \xi \), we only need to consider it in the interval \([0, 1]\).
2.3. Gevrey function

We fix $\alpha > 1$ and let $K$ be a closed interval in $\mathbb{R}$. We recall the definitions

$$
\|\phi\|_{\alpha,L} := \sum_{k \in \mathbb{N}} \frac{L!}{k!} \|\partial^k \phi\|_{C^0(K)}^{\alpha}
$$

$$
G^{\alpha,L}(K) = \{ \phi \in C^\infty(K) \|\phi\|_{\alpha,L} < \infty \}, \quad G^\alpha(K) = \bigcup_{L > 0} G^{\alpha,L}(K).
$$

Following [MS1], Gevrey-$\alpha$ function is defined as follow.

**Definition 2.1** A function $\phi$ is called Gevrey-$\alpha$ function on $K$ if $\phi \in G^\alpha(K)$.

From Leibniz rule, it follows that for $L > 0$, $\phi, \psi \in G^{\alpha,L}(K)$,

$$
\|\phi \psi\|_{\alpha,L} \leq \|\phi\|_{\alpha,L} \|\psi\|_{\alpha,L}.
$$

For the simplicity of notations, we don’t distinguish the constant $C$ in following different estimate formulas.

3. Construction of the generating functions

In order to destroy the invariant circle with a given rotation number of the completely integrable system

$$
h_0(x, x') = \frac{1}{2} (x - x')^2, \quad x, x' \in \mathbb{R},
$$

we construct the perturbation consisting of two parts. The first one is

$$
u_n(x) = \frac{1}{n^a} (1 - \cos(2\pi x)), \quad x \in \mathbb{R},
$$

where $n \in \mathbb{N}$ and $a$ is a positive constant independent of $n$.

We construct the second part of the perturbation in the following. First of all, for each $\lambda > 0$, we construct a function $f_\lambda \in C^\infty(\mathbb{R})$ as follow:

$$
f_\lambda(x) = \begin{cases} 0, & x \leq 0, \\ \exp \left( -\lambda \sqrt{2x - \pi} \right), & \text{otherwise}. \end{cases}
$$

**Lemma 3.1** There exists $\lambda > 0$ such that $f_\lambda(x)$ is a Gevrey-$\alpha$ function on $\mathbb{R}$.

**Proof** For the simplicity of notations, let $p = \frac{1}{\alpha - 1}$. Since $\alpha \in (1, \infty)$, $p \in (0, \infty)$ and

$$
f_\lambda(x) = \begin{cases} 0, & x \leq 0, \\ \exp \left( -\lambda \sqrt{2x - \pi} \right), & \text{otherwise}. \end{cases}
$$

Let $k \in \mathbb{N}$ and $x > 0$. We observe that $f_\lambda|\mathbb{R}^+$ can be extended to a holomorphic function on $\mathbb{C}\setminus(-\infty, 0]$. Let $\sigma := \frac{\pi}{4} \min\{1, \frac{\pi}{2}\}$ and $\Sigma_\sigma = \{ z \in \mathbb{C} | \arg z \leq \sigma \}$. The closed disk $D_x$ of center $x$ and radius $(x \sin \sigma)$ is the largest disk centered at $x$ contained in $\Sigma_\sigma$, and the Cauchy inequalities yield

$$
|f_\lambda^{(k)}(x)| \leq \frac{k!}{(x \sin \sigma)^k} \max_{D_x} |f_\lambda|.
$$
Let $z = re^{i\theta} \in D_z$. Since $|\theta| \leq \sigma = \frac{\pi}{4} \min\{1, \frac{1}{p}\}$, then $\Re e(z^{-p}) = r^{-p} \cos(p\theta) \geq \frac{1}{\sqrt{2}r^p}$ and $|z| \leq 2r$. Hence, we have

$$\max_{D_z} |f_\lambda| \leq \exp\left(\frac{-\lambda}{(2x)^p}\right).$$

It is easy to see that the maximum of $y \mapsto y^k e^{-\lambda y^p}$ is $\left(\frac{k}{\lambda p}\right)^{k/p}$, therefore

$$\left|f^{(k)}_\lambda(x)\right| \leq \left(\frac{2}{\sin \sigma}\right)^k \left(\frac{k}{\lambda p}\right)^{k/p} k!.$$  

By Stirling formula, we have that for any given $L > 0$, if $\lambda > (2L^\alpha / \sin \sigma)^p / p$, then

$$\sum_{k \in \mathbb{N}} \frac{L|\alpha|}{k!} \left|f^{(k)}_\lambda(x)\right|_{C^0(\mathbb{R})} < \infty.$$  

Therefore, there exists $\lambda > 0$ such that $f_\lambda(x)$ is a Gevrey-$\alpha$ function on $\mathbb{R}$.  

$v_n(x)$ is constructed as follows. For $x \in [0, 1]$, we let

$$v_n(x) = \begin{cases} \frac{1}{n^\alpha} f_\lambda \left(\frac{1}{8n^{\alpha/2}} - \frac{1}{2} + x\right) f_\lambda \left(\frac{1}{8n^{\alpha/2}} + \frac{1}{2} - x\right), & x \in \left[\frac{1}{2} - \frac{1}{8n^{\alpha/2}}, \frac{1}{2} + \frac{1}{8n^{\alpha/2}}\right], \\ 0, & \text{otherwise}. \end{cases}$$

More precisely, for $x \in \left[\frac{1}{2} - \frac{1}{8n^{\alpha/2}}, \frac{1}{2} + \frac{1}{8n^{\alpha/2}}\right]$,

$$v_n(x) = \frac{1}{n^\alpha} \exp\left(-\lambda \sqrt{2} \left(\left(\frac{1}{8n^{\alpha/2}} - \frac{1}{2} + x\right)^{-\frac{1}{\alpha - 1}} + \left(\frac{1}{8n^{\alpha/2}} + \frac{1}{2} - x\right)^{-\frac{1}{\alpha - 1}}\right)\right),$$

where $\lambda$ is a positive constant independent of $n$. Moreover, we extend $v_n(x)$ on $[0, 1]$ to be a periodic function on $\mathbb{R}$ by $v_n(x + 1) = v_n(x)$. By (2.2), $v_n(x)$ is a Gevrey-$\alpha$ function on $\mathbb{R}$. Based on the definition of $v_n$, it follows from a simple calculation that for $\alpha \in (1, \infty)$

$$\max_{[0, 1]} v_n(x) = v_n \left(\frac{1}{2}\right) \sim \frac{1}{n^\alpha} \exp \left(-C_n \frac{\alpha}{2^{\alpha - 1}}\right),$$

where $f \sim g$ means that $\frac{1}{C} g < f < C g$ holds for some constant $C > 0$. From (3.4), it follows that for $r > 0$, we have

$$\left\|f^{(r)}_{\lambda} \left(\frac{1}{8n^{\alpha/2}} - \frac{1}{2} + x\right)\right\|_{C^0} \leq C_1 \text{ and } \left\|f^{(r)}_{\lambda} \left(\frac{1}{8n^{\alpha/2}} + \frac{1}{2} - x\right)\right\|_{C^0} \leq C_1,$$

where $C_1$ is a positive constant independent of $n$. By Leibniz formula, we have

$$\|v^{(r)}_n(x)\|_{C^0} \leq \frac{1}{n^\alpha} \sum_{i=0}^r C_r^i \left\|f^{(i)}_{\lambda} \left(\frac{1}{8n^{\alpha/2}} - \frac{1}{2} + x\right)\right\|_{C^0} \cdot \left\|f^{(r-i)}_{\lambda} \left(\frac{1}{8n^{\alpha/2}} + \frac{1}{2} - x\right)\right\|_{C^0},$$

which together with (3.6) implies that for any fixed $r > 0$ and $n$ large enough, we have

$$\|v_n(x)\|_{C^r([0, 1])} \leq C_2 \frac{1}{n^\alpha}.$$
where $C_2$ is a positive constant independent of $n$.

So far, we complete the construction of the generating function of the nearly integrable system,

$$h_n(x, x') = h_0(x, x') + u_n(x') + v_n(x'),$$

where $n \in \mathbb{N}$.

4. Proof of Theorem 1.1

If $\omega \in \mathbb{Q}$, then the invariant circles with rotation number $\omega$ could be easily destroyed by an analytic perturbation arbitrarily close to zero. Therefore it suffices to consider the irrational $\omega$. Firstly, we prove the non-existence of invariant circles with a small enough rotation number. More precisely, we have the following Lemma:

**Lemma 4.1** For $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and $n$ large enough, the exact area-preserving monotone twist map generated by $h_n$ admits no invariant circle with the rotation number satisfying

$$|\omega| < n^{-\frac{n \alpha}{2(n-1)} - \delta},$$

where $\delta$ is a small positive constant independent of $n$.

First of all, we will estimate the lower bound of $P_{\bar{h}_n}$ at a given point. To achieve that, we need to estimate the distances of pairwise adjacent elements of the minimal configuration. More precisely, we have

**Lemma 4.2** Let $(x_i)_{i \in \mathbb{Z}}$ be a minimal configuration of $\bar{h}_n$ with rotation symbol $\omega > 0$, then

$$x_{i+1} - x_i \geq \frac{1}{2} n^{-\frac{\alpha}{2}}, \quad \text{for} \quad x_i \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

**Proof** Without loss of generality, we assume $x_i \in [0, 1]$ for all $i \in \mathbb{Z}$. By Aubry’s crossing lemma, we have

$$... < x_{i-1} < x_i < x_{i+1} < ....$$

We consider the configuration $(\xi_i)_{i \in \mathbb{Z}}$ defined by

$$\xi_j = \begin{cases} x_j, & j < i, \\ x_{j+1}, & j \geq i. \end{cases}$$

Since $(x_i)_{i \in \mathbb{Z}}$ is minimal, we have

$$\sum_{i \in \mathbb{Z}} \bar{h}_n(\xi_i, \xi_{i+1}) - \sum_{i \in \mathbb{Z}} \bar{h}_n(x_i, x_{i+1}) \geq 0.$$
Moreover,
\[ u_n(x_i) \leq (x_{i+1} - x_i)(x_i - x_{i-1}) \leq \frac{1}{4}(x_{i+1} - x_{i-1})^2. \]
Therefore,
\[
(4.1) \quad x_{i+1} - x_{i-1} \geq 2\sqrt{u_n(x_i)}.
\]
For \( x_i \in \left[\frac{1}{4}, \frac{3}{4}\right], \) \( u_n(x_i) \geq n^{-a}, \) hence,
\[
(4.2) \quad x_{i+1} - x_{i-1} \geq 2n^{-\frac{a}{2}}.
\]
Since \((x_i)_{i \in \mathbb{Z}}\) is a stationary configuration, we have
\[
x_{i+1} - x_i = -\partial h_n(x_i, x_{i+1}),
\]
\[
= \partial h_n(x_{i-1}, x_i),
\]
\[
= x_i - x_{i-1} + u'_n(x_i).
\]
Since \( u'_n(x) = \frac{2\pi}{n^a} \sin(2\pi x), \) it follows from (4.2) that
\[
x_{i+1} - x_i \geq \frac{1}{2}n^{-\frac{a}{2}}, \quad x_i \in \left[\frac{1}{4}, \frac{3}{4}\right].
\]
The proof of Lemma 4.2 is completed. \( \square \)

In order to estimate the lower bound of \( P_{h_n}^{0+} \) at a given point, we make a modification of \( v_n \) by changing the axis of symmetry of its support into \( \eta \). We denote \( v_{n,\eta}(x) := v_n(x - (\eta - 1/2)) \). Then we have
\[
\text{supp} \ v_{n,\eta} = \left[ \eta - \frac{1}{8n^{a/2}}, \eta + \frac{1}{8n^{a/2}} \right]
\]
By a similar calculation as (3.5), we have
\[
(4.3) \quad v_{n,\eta}(\eta) = \max v_{n,\eta}(x) \sim \frac{1}{n^a} \exp \left( -Cn^{\frac{a}{2(a-1)}} \right).
\]
Let \((x_i)_{i \in \mathbb{Z}}\) be the minimal configuration of \( \bar{h}_n(x_i, x_{i+1}) = h_0(x_i, x_{i+1}) + u_n(x_{i+1}) \) with rotation symbol \( 0^+ \), then from Lemma 4.2, we have
\[
x_{i+1} - x_i \geq \frac{1}{2}n^{-\frac{a}{2}}, \quad x_i \in \left[\frac{1}{4}, \frac{3}{4}\right].
\]
Hence, there exists \( \eta \in \left[\frac{3}{8}, \frac{5}{8}\right] \) such that
\[
(x_i)_{i \in \mathbb{Z}} \cap \text{supp} \ v_{n,\eta} = \emptyset.
\]
Moreover, for all \( i \in \mathbb{Z}, \)
\[
v_{n,\eta}(x_i) = 0.
\]
Based on [M4] (p.207-208), the Peierls’s barrier \( P_{0^+}^{h_n}(\eta) \) could be defined as follows
\[
P_{0^+}^{h_n}(\eta) = \min_{\xi_0 = \eta} \sum_{i \in \mathbb{Z}} h_n(\xi_i, \xi_{i+1}) - \min_{i \in \mathbb{Z}} h_n(z_i, z_{i+1}),
\]
where \((\xi_i)_{i \in \mathbb{Z}}\) and \((z_i)_{i \in \mathbb{Z}}\) are monotone increasing configurations limiting on 0, 1. Let \((\xi_i)_{i \in \mathbb{Z}}\) and \((z_i)_{i \in \mathbb{Z}}\) be minimal configurations of \(h_n\) defined by (3.3) with rotation symbol \(0^+\) satisfying \(\xi_0 = \eta\) and \(\). Then we have

\[
\sum_{i \in \mathbb{Z}} (h_n(\xi_i, \xi_{i+1}) - h_n(z_i, z_{i+1})) \\
\geq v_{n, \eta}(\eta) + \sum_{i \in \mathbb{Z}} h_n(\xi_i, \xi_{i+1}) - \sum_{i \in \mathbb{Z}} h_n(z_i, z_{i+1}), \\
\geq v_{n, \eta}(\eta) + \sum_{i \in \mathbb{Z}} h_n(x_i, x_{i+1}) - \sum_{i \in \mathbb{Z}} h_n(z_i, z_{i+1}), \\
\geq v_{n, \eta}(\eta) + \sum_{i \in \mathbb{Z}} h_n(x_i, x_{i+1}) - \sum_{i \in \mathbb{Z}} h_n(x_i, x_{i+1}), \\
= v_{n, \eta}(\eta) - \sum_{i \in \mathbb{Z}} v_{n, \eta}(x_{i+1}),
\]

where the first inequality holds since \(v_{n, \eta} \geq 0\), the second one since \((x_i)_{i \in \mathbb{Z}}\) is a minimal configuration of \(\tilde{h}_n\), the third one since \((z_i)_{i \in \mathbb{Z}}\) is a minimal configuration of \(h_n\) and the last one since \(v_{n, \eta}(x_i) = 0\) for all \(i \in \mathbb{Z}\). Moreover, we have

\[
P^h_{0^+}(\eta) \geq v_{n, \eta}(\eta).
\]

It follows that

\[
P^h_{0^+}(\eta) \geq \frac{C_1}{n^a} \exp \left( -C_2 n^{\frac{a}{\alpha + 1}} \right).
\]

Second, following a similar argument as [W1], one can obtain the improvement of modulus of continuity of Peierls’s barrier based on the hyperbolicity of \(h_n\). More precisely, we have the following lemma.

**Lemma 4.3** For every irrational rotation symbol \(\omega\) satisfying \(0 < \omega < n^{-\frac{a}{\alpha + 1} - \delta}\), we have

\[
\left| P^h_{0^+}(\eta) - P^h_{\omega}(\eta) \right| \leq C_1 \exp \left( -C_2 n^{\frac{a}{\alpha + 1} + \frac{\delta}{2}} \right),
\]

where \(\eta \in [3/8, 5/8]\) and \(\delta\) is a small positive constant independent of \(n\).

**Proof** If \(\eta \in \mathcal{A}^h_{\omega}\), then \(P^h_{0^+}(\eta) = 0\). Hence, it suffices to consider the case with \(\eta \not\in \mathcal{A}^h_{\omega}\) to destroy invariant circles. Since the proof of Lemma 4.3 is similar to Lemma 5.1 in [W1], we only give a sketch of the proof to show some main differences between them. For the simplicity of notations, we denote \(\kappa := \frac{a}{2(\alpha + 1)}\) and \(\epsilon_n := \exp(-n^{\kappa + \frac{\delta}{2}})\). The proof is proceeded by three steps as follows.

In the first step, we will show that each of the intervals \([0, \epsilon_n]\) and \([1 - \epsilon_n, 1]\) contains a large number of elements of the minimal configuration \((x_i)_{i \in \mathbb{Z}}\) of \(h_n\) with irrational rotation symbol \(0 < \omega < n^{-\kappa - \frac{\delta}{2}}\) for \(n\) large enough. Let

\[
\Sigma_n = \{i \in \mathbb{Z} \mid x_i \in [\epsilon_n, 1 - \epsilon_n]\},
\]

then it follows from a similar argument as Lemma 5.2 in [W1] that

\[
\|\Sigma_n \leq C n^{\kappa + \frac{\delta}{2} + \frac{\delta}{2}},
\]

\[
\|\Sigma_n \leq C n^{\kappa + \frac{\delta}{2} + \frac{\delta}{2}},
\]
where $\| \Sigma_n \|$ denotes the number of elements in $\Sigma_n$. Let $I$ be an interval of length 1. We denote $\Delta_\omega := \{ i \in \mathbb{Z} \mid x_i \in I \}$. Since $(x_i)_{i \in \mathbb{Z}}$ is a minimal configuration with rotation number $\omega \in \mathbb{R} \setminus \mathbb{Q}$, then it follows from Lemma 5.3 in [W1] that

\[
\frac{1}{\omega} - 1 < \frac{x_\omega}{\omega} \leq \frac{1}{\omega} + 1.
\]

which together with (4.6) implies

\[
\frac{x_\omega}{\omega} \geq Cn^{\frac{\omega}{2} + \delta} \gg Cn^{\frac{\omega}{2} + \delta} \geq \| \Sigma_n \|.
\]

Hence, one can obtain that each of the intervals $[0, \epsilon_n]$ and $[1 - \epsilon_n, 1]$ contains a large number of elements of the minimal configuration $(x_i)_{i \in \mathbb{Z}}$ of $h_n$ with irrational rotation symbol $0 < \omega < n^{-\kappa - \frac{1}{2} - \delta}$ for $n$ large enough (see Lemma 5.4 in [W1]).

In the second step, we approximate $P_{h_n}^\omega(\eta)$ for $\eta \in [3/8, 5/8]$ by the difference of the actions of the segments with a given length, where we consider the number of the elements in a segment of the configuration as the length of the segment. Let $(\xi^-, \xi^+)$ be the complementary interval of $A_{\omega}^{h_n}$ in $\mathbb{R}$ and contains $\eta$. Let $(\xi_i^\pm)_{i \in \mathbb{Z}}$ be the minimal configurations with rotation symbol $\omega$ satisfying $\xi_0^\pm = \xi^\pm$ and let $(\xi_i)_{i \in \mathbb{Z}}$ be a minimal configuration of $h_n$ with rotation symbol $\omega$ satisfying $\xi_0 = \eta$ and $\xi_i^- \leq \xi_i \leq \xi_i^+$. We denote $d(x) := \min\{|x|, |x - 1|\}$. By Step 1, there exist $i^-, i^+$ such that

\[
d(\xi_i^-) < \epsilon_n \quad \text{and} \quad \xi_{i+1}^- - \xi_i^- \leq \epsilon_n \quad \text{for} \quad i = i^-, i^+.
\]

Thanks to Aubry’s crossing lemma, we have $\xi_i^- \leq \xi_i \leq \xi_i^+ \leq \xi_i+1$. Hence,

\[
\xi_i - \xi_i^- \leq \epsilon_n \quad \text{for} \quad i = i^-, i^+.
\]

We define the following configuration:

\[
y_i = \begin{cases} 
\xi_i, & i^- < i < i^+; \\
\xi_i^-, & i \leq i^-, i \geq i^+.
\end{cases}
\]

Since $\eta \in [3/8, 5/8] \subset [\epsilon_n, 1 - \epsilon_n]$ for $n$ large enough, then $\xi_0 = \eta$ is contained in $(y_i)_{i \in \mathbb{Z}}$ up to the rearrangement of the index $i$. By a direct calculation (see (11)-(15) in [W1]), we have

\[
P_{\omega}^{h_n}(\eta) \leq \sum_{i \in \mathbb{Z}} (h_n(y_i, y_{i+1}) - h_n(\xi_i^-, \xi_{i+1}^-)) \leq P_{\omega}^{h_n}(\eta) + C\epsilon_n^2.
\]

In the third step, we will compare $P_{0^+}^{h_n}(\eta)$ with $\sum_{i \in \mathbb{Z}} (h_n(y_i, y_{i+1}) - h_n(\xi_i^-, \xi_{i+1}^-))$. By [M1], we have

\[
P_{0^+}^{h_n}(\eta) = \min_{\xi_0 = \eta} \sum_{i \in \mathbb{Z}} h_n(\xi_i, \xi_{i+1}) - \min_{i \in \mathbb{Z}} \sum_{i \in \mathbb{Z}} h_n(z_i, z_{i+1}),
\]

where $(\xi_i)_{i \in \mathbb{Z}}$ and $(z_i)_{i \in \mathbb{Z}}$ are monotone increasing configurations limiting on 0, 1. We denote

\[
\begin{cases}
K(\eta) = \min_{\xi_0 = \eta} \sum_{i \in \mathbb{Z}} h_n(\xi_i, \xi_{i+1}), \\
K = \min \sum_{i \in \mathbb{Z}} h_n(z_i, z_{i+1}).
\end{cases}
\]
By a direct calculation (see (17)-(28) in [W1]), we have

$$\sum_{i \in \mathbb{Z}} h_n(\xi_i^-, \xi_{i+1}^-) - K \leq C_1 n^2$$ and $$\sum_{i \in \mathbb{Z}} h_n(y_i, y_{i+1}) - K(\eta) \leq C_1 n^2.$$  

Finally, from (4.9) and (4.10), we obtain

$$\left| P_{\omega}^{h_n}(\eta) - P_{0+}^{h_n}(\eta) \right| \leq \left| \sum_{i \in \mathbb{Z}} h_n(y_i, y_{i+1}) - \sum_{i \in \mathbb{Z}} h_n(\xi_i^-, \xi_{i+1}^-) + K - K(\xi) \right| + C_1 \epsilon_n^2,$$

$$\leq \left| \sum_{i \in \mathbb{Z}} h_n(y_i, y_{i+1}) - K(\xi) \right| + \left| \sum_{i \in \mathbb{Z}} h_n(\xi_i^-, \xi_{i+1}^-) - K \right| + C_1 \epsilon_n^2,$$

$$\leq C_1 n^2.$$  

Recalling $$\kappa := \frac{a}{2(\alpha - 1)}$$ and $$\epsilon_n := \exp(-n^{\frac{a}{2(\alpha - 1)}}),$$ we have

$$\left| P_{\omega}^{h_n}(\eta) - P_{0+}^{h_n}(\eta) \right| \leq C_1 \exp \left( -C_2 n^{\frac{a}{2(\alpha - 1)}} + \frac{\delta}{2} \right).$$  

which completes the proof of Lemma 4.3. \( \square \)

Based on the preparations above, it is easy to prove Lemma 4.4. We assume that there exists an invariant circle with rotation number $$0 < \omega < n^{-\frac{1}{2(\alpha - 1)}}$$ for $$h_n,$$ then $$P_{\omega}^{h_n}(\xi) \equiv 0$$ for every $$\xi \in \mathbb{R}.$$ By Lemma 4.3, we have

$$\left| P_{0+}^{h_n}(\eta) \right| \leq C_1 \exp \left( -C_2 n^{\frac{a}{2(\alpha - 1)}} + \frac{\delta}{2} \right).$$  

On the other hand, (4.11) implies that

$$P_{0+}^{h_n}(\eta) \geq \frac{C_1}{n^a} \exp \left( -C_2 n^{\frac{a}{2(\alpha - 1)}} \right).$$

Hence, we have

$$\frac{C_1}{n^a} \exp \left( -C_1 n^{\frac{a}{2(\alpha - 1)}} \right) \leq C_2 \exp \left( -C_2 n^{\frac{a}{2(\alpha - 1)}} \right).$$

It is an obvious contradiction for $$n$$ large enough. Therefore, there exists no invariant circle with rotation number $$0 < \omega < n^{-\frac{1}{2(\alpha - 1)}}.$$ For $$-n^{-\frac{1}{2(\alpha - 1)}} < \omega < 0,$$ by comparing $$P_{\omega}^{h_n}(\xi)$$ with $$P_{0+}^{h_n}(\xi),$$ the proof is similar. We omit the details. Therefore, the proof of Theorem 4.1 is completed. \( \square \)

The case with a given irrational rotation number can be easily reduced to the one with a small enough rotation number. More precisely,

**Lemma 4.4** Let $$h_P$$ be a generating function as follow

$$h_P(x, x') = h_q(x, x') + P(x'),$$

where $$P$$ is a periodic function of periodic 1. Let $$Q(x) = q^{-2} P(qx),$$ then the exact area-preserving monotone twist map generated by $$h_Q(x, x') = h_q(x, x') + Q(x')$$ admits an invariant circle with rotation number $$\omega \in \mathbb{R} \setminus \mathbb{Q}$$ if and only if the exact area-preserving monotone twist map generated by $$h_P$$ admits an invariant circle with rotation number $$q \omega - p, p \in \mathbb{Z}.$$
We omit the proof and for more details, see [11]. For the sake of simplicity of notations, we denote $Q_n$ by $Q_n$ and the same to $u_n, v_n$ and $h_n$. Let

$$Q_n(x) = q_n^{-2}(u_n(q_n x) + v_n(q_n x)),$$

where $(q_n)_{n \in \mathbb{N}}$ is a sequence satisfying (1.1) (4.13)

$$|q_n \omega - p_n| < \frac{C}{q_n^{1+\mu}},$$

where $p_n \in \mathbb{Z}$ and $q_n \in \mathbb{N}$. Since $\omega \in \mathbb{R} \setminus \mathbb{Q}$, we say $q_n \to \infty$ as $n \to \infty$. Let $\tilde{h}_n(x, x') = h_0(x, x') + Q_n(x')$, we prove Theorem 1.1 for $(\tilde{h}_n)_{n \in \mathbb{N}}$ as follow:

Proof Based on Lemma 4.1 and (4.13), it suffices to take

$$\frac{C}{q_n^{1+\mu}} \leq \frac{1}{q_n^{2\alpha - a - 1 + \delta}},$$

which implies

$$a \leq \left( 2 - \frac{2}{\alpha} \right) (1 + \mu) - \epsilon,$$

where $\epsilon = 2\delta(\frac{2}{\alpha})$ and $\delta$ is a small positive constant independent of $n$. From the constructions of $u_n$ and $v_n$, it follows from (3.1) and (3.7) that

$$\|\tilde{h}_n(x, x') - h_0(x, x')\|_{C^r}$$
$$= \|Q_n(x')\|_{C^r},$$
$$\leq q_n^{-2}(\|u_n(q_n x')\|_{C^r} + \|v_n(q_n x')\|_{C^r}),$$
$$\leq q_n^{-2}(q_n^{-a}(2\pi)^r q_n^r + C_1 q_n^{-a} q_n^r),$$
$$\leq C_2 q_n^{-a-2},$$

where $C_1, C_2$ are positive constants only depending on $r$.

To complete the proof, it is enough to make $r - a - 2 < 0$, which together with (4.14) implies

$$r < a + 2 \leq 2 + \left( 2 - \frac{2}{\alpha} \right) (1 + \mu) - \epsilon.$$

This completes the proof of Theorem 1.1 if we take $a = \left( 2 - \frac{2}{\alpha} \right) (1 + \mu) - \epsilon$ and $r = 2 + \left( 2 - \frac{2}{\alpha} \right) (1 + \mu) - 2\epsilon$. \hfill \Box

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