INEQUIVALENT LEFSCHETZ FIBRATIONS ON RATIONAL AND RULED SURFACES

R. İNANÇ BAYKUR

Abstract. In this short note, we give an explicit construction of inequivalent Lefschetz pencils and fibrations of same genera on blow-ups of all rational and ruled surfaces. This complements our earlier results in [4], concluding that every symplectic 4–manifold, after sufficiently many blow-ups, admits inequivalent Lefschetz pencils and fibrations, which cannot be obtained from one another even via any sequence of fibered Luttinger surgeries.

1. Introduction

A pair of Lefschetz pencils or fibrations\(^1\) on a closed oriented smooth 4–manifold are said to be equivalent (or isomorphic) if there is an orientation-preserving self-diffeomorphism of the total space which commutes the two maps. Simon Donaldson’s ground-breaking work in [8] provides arbitrarily large genera pencils on any symplectic 4–manifold. A fundamental problem then is to determine if there are relatively-minimal inequivalent pencils or fibrations of the same genus and number of base points on a given symplectic 4–manifold. For brevity, we refer to them as inequivalent pencils and inequivalent fibrations, where it should be understood that their genera and number of base points are already the same.

This note is an amendment to our results in [4], where we proved that any symplectic 4-manifold, which is not a rational or a ruled surface, after a number of blow-ups, admits arbitrarily many inequivalent Lefschetz pencils and fibrations. Moreover, they are not related via fibered Luttinger surgeries [4]. Here we will give similar examples of pencils and fibrations on rational and ruled surfaces. Recall that rational and ruled surfaces are birationally equivalent to the complex projective plane and \(S^2\times\Sigma_h\), which are known to have unique symplectic structures up to symplectomorphisms and deformations.

Our examples in [4] were obtained from Donaldson pencils via partial doubling sequences, which involve blow-ups and degree doublings of pencils introduced in [19, 3]. These pencils were distinguished by looking at the collection of exceptional spheres, which arise as multisections [9, 5] of different degrees in respective pencils. Thus, blowing-up all the base points, we obtain inequivalent Lefschetz fibrations. However, the same strategy does not work for rational or ruled surfaces, since in this case, one is not guaranteed to have disjoint multisections representing a given collection of exceptional classes [4].

\(^1\)We assume that a pencil, unlike a Lefschetz fibration, always has base points. We also assume that a Lefschetz pencil or fibration always has critical points, and each singular fiber contains only one critical point and no exceptional spheres (i.e., it is relatively-minimal).
We will instead apply partial doublings to hand-picked pencils on rational and ruled surfaces in order to derive inequivalent pencils which have different numbers of reducible fibers versus irreducible ones. The point here is that “doublings are not created equal”; depending on which subcollections of base points we pick, we can arrive at pencils with topologically different singular fibers. Since the distinction comes from the number of reducible fibers, we get inequivalent fibrations after blowing-up all the base points. Our main result is:

**Theorem 1.** Any rational or ruled surface, after sufficiently many blow-ups, admits inequivalent relatively-minimal pencils and fibrations of the same genera and number of base points, which cannot be obtained from one another via any sequence of fibered Luttinger surgeries.

Last part of the theorem comes easy for our examples: performing a fibered Luttinger surgery is equivalent to conjugating a subword of the corresponding monodromy factorization of a pencil by some Dehn twist; see [1, 4]. This can be done whenever the twist curve is fixed, up to isotopy, by the mapping class given by the subword. Clearly, the number of reducible fibers is unchanged under this operation, so the examples with different numbers of reducible fibers we produce are also inequivalent up to fibered Luttinger surgeries.

There are many other examples of inequivalent Lefschetz pencils and fibrations. A pair of inequivalent fibrations on a blow-up of $T^2 \times \Sigma_2$, whose fibers have different divisibility in homology, was discussed by Ivan Smith in [18]. Several inequivalent fibrations on homotopy elliptic surfaces, distinguished by their monodromy groups, were discovered by Jongil Park and Ki-Heon Yun in [16, 17]. As for inequivalent pencils, first examples were constructed by Kenta Hayano and the author on 4–manifolds homeomorphic to blow-ups of the K3 surface [5]. The construction in [5] uses pairs of lantern substitutions which modify pencils differently, but land on the same 4–manifold. Notably, Noriyuki Hamada recently constructed pairs of inequivalent pencils on irrational ruled surfaces $S^2 \times \Sigma_h$, which only differ by the configuration of their base points on the reducible fibers, so they do not yield inequivalent fibrations after blow-ups.

With our aforementioned result from [4], we arrive at the following conclusion, which demonstrates how diverse the Donaldson-Gompf correspondence between symplectic 4–manifolds and Lefschetz pencils and fibrations is:

**Corollary 2.** Any closed symplectic 4–manifold, possibly after blow-ups, admits inequivalent relatively-minimal pencils and fibrations of the same genera and number of base points, which cannot be obtained from one another via any sequence of fibered Luttinger surgeries.

### 2. Preliminaries

Here we will quickly review the basic notions and background results on Lefschetz pencils / fibrations, positive Dehn twist factorizations, fibered Luttinger surgeries and partial conjugations, and doublings of Lefschetz pencils. For further details on these topics, the reader can turn to [11] and [2, 3, 4].

Let $X$ be a closed, oriented 4-manifold, and $B = \{b_j\}$, $C = \{p_i\}$ be finite, non-empty sets of points in $X$. A Lefschetz pencil $(X, f)$ is given by a surjective map
function $f: X \setminus B \to S^2$, which is a submersion on $X \setminus C$, such that around each base point $b_j$ and each critical point $p_i$, it conforms to local complex models $(z_1, z_2) \mapsto z_1/z_2$ and $(z_1, z_2) \mapsto z_1z_2$, respectively. A Lefschetz fibration is defined similarly for $B = \emptyset$. After blowing-up every base point $b_j$ of a pencil, we obtain a Lefschetz fibration with disjoint exceptional spheres $S_j$ as its sections. We say $(X, f)$ is a genus-$g$ pencil/fibration, for $g$ the genus of the regular fiber $F$. We will assume that every fiber contains at most one critical point, which can always be achieved after a small perturbation. A singular fiber has a nodal singularity at the critical point $p_i$, and is obtained by shrinking a simple loop $a_i$ on $F$, called the vanishing cycle. We have a reducible fiber (with two connected components) if $a_i$ is separating $F$, and irreducible if $a_i$ is non-separating.

Let $\Sigma^m_g$ denote a compact, oriented surface of genus $g$ with $m$ boundary components, where $\Sigma^0_g = \Sigma_g$. The mapping class group $\Gamma^m_g$ is the group of orientation-preserving self-diffeomorphisms of $\Sigma^m_g$ which restrict to identity along $\partial \Sigma^m_g$, modulo isotopies of the same type. Let $t_a \in \Gamma^m_g$ denote the positive (right-handed) Dehn twist along a simple loop $a$ on $\Sigma^m_g$. A genus-$g$ Lefschetz pencil $(X, f)$ with $m$ base points prescribes a positive factorization of the boundary multi-twist (or monodromy factorization), given by a relation

$$t_{\delta_1} \cdots t_{\delta_m} = t_{a_1} \cdots t_{a_k}$$

in $\Gamma^m_g$, where $\delta_1, \ldots, \delta_m$ are curves parallel to distinct boundary components of $\Sigma^m_g$ (which we will also denote the boundary component by), $a_i$ is a vanishing cycle corresponding to the critical point $p_i$ of $f$, and $\ell = |C|$. Conversely, given a positive factorization as above, one can construct such a Lefschetz pencil $(X, f)$. For $g \geq 2$, we indeed get a one-to-one correspondence between equivalence classes of Lefschetz pencils and positive factorizations up to Hurwitz moves (trading subwords $t_{a_i}t_{a_{i+1}}$ and $t_{a_i}(a_{i+1})t_{a_i}$) and global conjugations (replacing every $t_{a_i}$ in the factorization with $t_{\phi(a_i)}$, for the same $\phi \in \Gamma^m_g$). All goes the same for Lefschetz fibrations for $m = 0$.

By the pioneering works of Donaldson and Gompf [8, 10], every symplectic 4-manifold admits a Lefschetz pencil whose fibers are symplectic, and conversely, the total space of a Lefschetz pencil or a fibration (recall $C \neq \emptyset$) can always be equipped with a symplectic form for which all regular fibers are symplectic.

As observed in [2], Luttinger surgery (where one removes a neighborhood of a Lagrangian torus in a symplectic 4-manifold and glues it back in differently so as to produce a new symplectic 4-manifold) can be performed in a way compatible with the pencil structure, when the restriction of the pencil map to the torus is a circle bundle over a loop on the base, away from the critical values [1, 2, 4]. This surgery is equivalent to modifying the corresponding positive factorization by replacing a subword $t_{a_1} \cdots t_{a_k}$ with a conjugate $t_{\phi(a_1)} \cdots t_{\phi(a_k)}$ for some $\phi = t_b^\pm$, which can be done if and only if the mapping class $t_{a_1} \cdots t_{a_k}$ fixes $b$, up to isotopy [1].

Another way to produce a new pencil from a given one, and on the same 4-manifold, is the doubling procedure, discussed for holomorphic pencils and Donaldson pencils in [14], and for pencils obtained via branched coverings of $\mathbb{CP}^2$ in [3]. In general, by [4, Lemma 3.1], one can double any (topological) genus-$g$ pencil $(X, f)$ with $m$ base points, provided $g \geq 2$ and $m \leq 2g - 2$. The result is a genus-$g'$ pencil $(X, f')$ with $m'$ base points, where $g' = 2g + m - 1$ and $m' = 4m$. If the
original pencil is compatible with a given symplectic form, then the resulting one is also compatible with a deformation equivalent symplectic form.

Topologically, the doubling procedure can be interpreted roughly as follows: First break up the symplectic pencil \((X, f)\) into two pieces; the regular neighborhood of a smooth fiber (the “concave piece”), which is a symplectic disk bundle of degree \(m\) over a genus-\(g\) surface, and its complement (the “convex piece”). Describe a standard higher genus pencil on the former piece, based on \(g\) and \(m\) —the model for which comes from holomorphic pencils. Then extend it over the convex piece, which now becomes a subpiece of the new pencil \((X, f')\). By the enterprise of \([3]\), the monodromy of \(f'\) is explicitly determined by that of \(f\) \([3]\) [Theorem 4]. In particular, we observe the following: any separating Dehn twists in the positive factorization of \(f'\) comes, in a one-to-one fashion, from a separating Dehn twist in the positive factorization of \(f\), which bounds a subsurface on \(F\) that does not contain a base point of \(f\). (See Figure 1 for an example.)

3. Constructions

It is time to spell out our recipe in full detail. Take a positive factorization for a genus \(g \geq 2\) pencil \((X, f)\) with base locus \(B\) and regular fiber \(F\), whose separating vanishing cycles are \(a_1, \ldots, a_r\). Let \(B' = \{b'_j\}\) and \(B'' = \{b''_j\}\) be two subsets of the base locus \(B\) of \(f\) (where the base points correspond to boundary components in the positive factorization), so that \(|B'| = |B''| \leq 2g - 2\). Set \(K'_i, K''_i \in \{0, 1\}\) to be the number of components in \(F \setminus a_i\) which do not intersect \(B'\) (resp. \(B''\)). If \(K' = \sum_{i=1}^{r'} K'_i\) is unequal to \(K'' = \sum_{i=1}^{r''} K''_i\), we are game.

We blow-up all the base points of \(f\) in \(B \setminus B'\) (resp. \(B \setminus B''\)) and double the resulting pencil on \(\tilde{X} = X \# |B \setminus B'| \mathbb{CP}^2\) with \(m = |B'|\) base points, which naturally come from the inclusion of \(X\) minus the blown-up base points into \(\tilde{X}\). In this way, we arrive at a new pencil \((\tilde{X}, f')\) (resp. \((\tilde{X}, f'')\)). As we reviewed earlier, each one of the pencils \(f'\) and \(f''\) on \(\tilde{X}\) has genus \(2g + m - 1\) and \(4m\) base points, whereas the number of reducible fibers they have are \(K'\) and \(K''\), respectively. (Figure 1 illustrates 2 distinct partial doublings of sorts.) For each \(0 \leq m_0 < m\), we get a pair of inequivalent pencils by blowing-up \(m_0\) base points of both pencils, and for \(m_0 = m\), we get a pair of inequivalent fibrations.

\[\text{Figure 1. The lifts of a separating vanishing cycle (in blue) under doubling of a genus–2 pencil with 2 different pairs of base points, marked with \(\times\) in the figure.}\]
It remains to find the main ingredient for our recipe: a pencil \((X, f)\) as prescribed above, on (blow-ups of) rational and ruled surfaces.

3.1. Inequivalent pencils and fibrations on rational surfaces.

There is the following positive factorization of the boundary multi-twist in \(\Gamma^3_2\), which was found by Sinem Onaran in [15]:

\[
t_{\delta_1} t_{\delta_2} t_{\delta_3} = t_{a_3} t_{b_3} (t_{a_1} t_{a_2} t_{a_3} t_{b_1})^2 t_{a_1} t_{a_5} t_{b_2} t_{a_4} t_{a_3} t_{b_2} t_{a_1} t_{a_6}.
\]

Here the twist curves \(a_1, b_1\) and \(a_4\) are as in Figure 2 and \(\psi = t_{a_3} t_{a_1}^{-1}\). (See [15][Figure 6]. This is a lift of a well-known positive factorization of identity in \(\Gamma_2 \) [8], which corresponds to a hyperelliptic Lefschetz fibration on \(\mathbb{CP}^2 \# 13\mathbb{CP}^2\).)

![Figure 2. Twist curves \(a_i, b_j\) and \(a'_4\) on the surface \(\Sigma^3_3\). The boundary components are \(\delta_1, \delta_2, \delta_3\). The new curves \(x, y\) are given in the second copy of \(\Sigma^3_3\), where the lantern sphere sits in the middle. The separating curve \(x\) is in blue.](image)

We rewrite the above expression as:

\[
t_{\delta_1} t_{\delta_2} t_{\delta_3} = t_{a_3} t_{b_3} t_{a_1} t_{a_2} t_{a_3} t_{b_1} t_{a_1} t_{a_2} t_{a_3} t_{a_4} t_{a_6} \psi (t_{a_3} t_{b_1} t_{a_3} t_{b_2}) t_{a_3} t_{b_2} t_{a_1} t_{a_6}
\]

and then apply a lantern substitution along the underlined subword to obtain a new positive factorization:

\[
t_{\delta_1} t_{\delta_2} t_{\delta_3} = t_{a_3} t_{b_1} t_{a_3} t_{b_2} t_{a_3} t_{b_3} t_{a_3} t_{b_4} t_{a_3} t_{b_5} t_{a_3} t_{b_6} t_{a_3} t_{b_2} t_{a_1} t_{a_6},
\]

which involves new twist curves \(x, y\) given in Figure 2. Note that \(x\) is the only separating twist curve here. Importantly, the “lantern sphere” we utilized did not contain the base points. The final positive factorization we obtained above describes a genus–2 Lefschetz pencil \((X, f)\) with 3 base points.

The Seiberg-Witten adjunction inequality implies that in any closed symplectic 4–manifold, except for rational or ruled surfaces, we have \(2g(F) - 2 = -\varepsilon(F) \geq |F|^2\) for any connected symplectic surface \(F\) [20] [13]. However, \((X, f)\), equipped with the Gompf-Thurston symplectic form, has a symplectic fiber \(F\) of genus \(g(F) = 2\) and self-intersection \(|F|^2 = 3\) (the number of base points), violating the inequality. So we conclude that \(X\) is a rational or ruled surface. We can then determine its exact diffeomorphism type through a simple calculation of its algebraic topological invariants. The first integral homology group \(H_1(X)\), which is isomorphic to the quotient of \(H_1(F)\) by the subgroup normally generated by homology classes of vanishing cycles, is easily seen to be trivial: the vanishing cycles \(a_1, b_1, a_3\) and
$b_2$ already kill the whole group. Since the Euler characteristic of $X$ is given by $e(X) = 4 - 4g + |C| - |B|$, where $C$ and $B$ are the critical locus and the base locus, we calculate $e(X) = 12$. It follows that $X \cong \mathbb{CP}^2 \# 9 \mathbb{CP}^2$.

Now let $b_j$ be the base point of $(X, f)$ corresponding to the boundary components $\delta_j$ in the factorization, for $j = 1, 2, 3$. For $B' = \{b_2, b_3\}$ and $B'' = \{b_1, b_2\}$, our recipe generates a pair of inequivalent genus–5 pencils $f'$ and $f''$ on the rational ruled surface $\tilde{X} \cong \mathbb{CP}^2 \# 10 \mathbb{CP}^2 \cong \mathbb{CP}^2 \# 8 \mathbb{CP}^2$ with 8 base points, where $f'$ has 1 reducible fiber but $f''$ has none. (As illustrated in Figure 1.)

### 3.2. Inequivalent pencils and fibrations on irrational ruled surfaces.

We next consider the following positive factorization of the boundary multi-twist in $\Gamma_{3h}$, for each $h \geq 1$, due to Noriyuki Hamada [11]:

\[
t_{\delta_1} t_{\delta_2} t_{\delta_3} = t_{B_{0,1}} t_{B_{1,1}} \cdots t_{B_{h,1}} t_{C_1} t_{B_{0,2}} t_{B_{1,2}} \cdots t_{B_{h,2}} t_{C_2},
\]

where the curves $B_{i,k}, C_k$ for $i = 0, 1, \ldots, h$ and $k = 1, 2$ are as in Figure 3 (This is the positive factorization “$W_{IIA}$ for even genus” in [11] [Figure 23] with only 3 boundary components we picked and relabeled here. It is a lift of a well-known positive factorization of identity in $\Gamma_2$ [14, 12, 7], which corresponds to a hyperelliptic Lefschetz fibration on $S^2 \times \Sigma_h \# 4 \mathbb{CP}^2$.) Corresponding to this positive factorization is a genus–2 $h$ pencil $(X, f)$ with 3 base points, where $X \cong S^2 \times \Sigma_h \# 4 \mathbb{CP}^2$. Note that $f$ has 2 separating vanishing cycles.

Once again, let $b_j$ be the base point of $(X, f)$ corresponding to the boundary component $\delta_j$ in the factorization, for $j = 1, 2, 3$. We see that for $B' = \{b_2, b_3\}$ and $B'' = \{b_1, b_2\}$, our recipe generates a pair of inequivalent genus–$(4h + 1)$ pencils $f'$ and $f''$ on the irrational ruled surface $\tilde{X} \cong S^2 \times \Sigma_h \# 2 \mathbb{CP}^2 \cong S^2 \times \Sigma_h \# 2 \mathbb{CP}^2$ with 8 base points, where $f'$ has 2 reducible fibers but $f''$ has 1.

### 3.3. Generalizations.

We end with a few remarks on our recipe for constructing inequivalent pencils and fibrations in general:

1. From each pair of inequivalent pencils $f'$ and $f''$ we obtained, we can easily derive infinitely many more pairs of inequivalent pencils and fibrations of arbitrarily
high genera. This is achieved by taking further doubles of \( f' \) and \( f'' \); the base points of the new pencil will lie only on one component of each reducible fiber in \( f' \) or \( f'' \) (see the monodromy in \( [3] \) [Theorem 4]), so the number of reducible fibers will remain different.

2. One can produce more than just a pair of inequivalent pencils and fibrations, whenever there is an input pencil \((X, f)\) with several subcollections of base points \(B', B'', \ldots, B^{(n)}\) of equal rank, which have different numbers \(K', K'', \ldots, K^{(n)}\) of reducible fiber components they miss. For instance, a further lift of Hamada’s positive factorization to \( \Gamma^{4}_{2h} \) in \([11]\) can be used to generate yet another pencil \( f''' \) with no reducible fibers.

3. Our recipe can be improved in several ways. For example, we only compared the number of reducible fibers, yet one can as well compare the topological types of reducible fibers (determined by the genera of the subsurfaces they bound). It seems plausible that one can produce arbitrarily many inequivalent fibrations and pencils by employing doubling sequences as in \([4]\) for pencils on blow-ups of the same manifold, which contain different topological types of reducible fibers.

Acknowledgments. These examples of inequivalent pencils and fibrations came to life during my visit to Seoul, Korea in 2015, and were first announced during my lectures at the highly stimulating 2015 Seoul National University Topology Winter School. This work is supported by the NSF Grant DMS-1510395.

REFERENCES

[1] D. Auroux, *Mapping class group factorizations and symplectic 4-manifolds: some open problems*, Problems on mapping class groups and related topics, 123–132, Proc. Sympos. Pure Math. 74, Amer. Math. Soc., Providence, RI, 2006.
[2] D. Auroux, S. K. Donaldson, L. Katzarkov, *Luttinger surgery along Lagrangian tori and non-isotopy for singular symplectic plane curves*, Math. Ann. 326 (2003), 185–203.
[3] D. Auroux and L. Katzarkov, *A degree doubling formula for braid monodromies and Lefschetz pencils*, Pure Appl. Math. Q. 4 (2008), no. 2, part 1, 237–318.
[4] R. I. Baykur, *Inequivalent Lefschetz fibrations and surgery equivalence of symplectic 4-manifolds*, J. Symp. Geom. 14 (2016), no. 3, 671–686.
[5] R. I. Baykur and K. Hayano, *Multisections of Lefschetz fibrations and topology of symplectic 4-manifolds*, Geom. Topol. 20 (2016), No. 4, 2335–2395.
[6] J. Birman and H. Hilden, *On the mapping class group of closed surface as covering spaces*, in “Advances in the theory of Riemann surfaces”, Ann. of Math. Studies, no. 66 (1971), 81-115. Princeton Univ. Press, Princeton, N.J., 1971.
[7] C. Cadavid, *A remarkable set of words in the mapping class group*, Ph.D. dissertation, Univ. of Texas at Austin, 1998.
[8] S. K. Donaldson, *Lefschetz pencils on symplectic manifolds*, J. Differential Geom. 53 (1999), no.2, 205–236.
[9] S. K. Donaldson and I. Smith, *Lefschetz pencils and the canonical class for symplectic four-manifolds*, Topology 42 (2003), no. 4, 743–785.
[10] R. E. Gompf and A. I. Stipsicz, *4-Manifolds and Kirby Calculus*, Graduate Studies in Mathematics 20, American Mathematical Society, 1999.
[11] N. Hamada, *Sections of the Matsumoto-Cadavid-Korkmaz Lefschetz fibration*, preprint; https://arxiv.org/abs/1610.08458.
[12] M. Korkmaz, *Noncomplex smooth 4-manifolds with Lefschetz fibrations*, Internat. Math. Res. Notices 2001, no. 3, 115–128.
[13] T.-J. Li and A. Liu, *Symplectic structures on ruled surfaces and a generalized adjunction formula*, Math. Res. Lett. 2 (1995), 453–471.

[14] Y. Matsumoto, *Lefschetz fibrations of genus two: A topological approach*, in Topology and Teichmüller Spaces (Katinkulta, Finland, 1995), World Sci., River Edge, New Jersey, 1996, 123-148.

[15] S. C. Onaran, *On sections of genus two Lefschetz fibrations*, Pacific J. Math. 248 (2010), no. 1, 203-216.

[16] J. Park and K.-H. Yun, *Nonisomorphic Lefschetz fibrations on knot surgery 4–manifolds*, Math. Ann. 345 (2009), no. 3, 581–597.

[17] J. Park and K.-H. Yun, *Lefschetz fibrations on knot surgery 4–manifolds via Stallings twist*, preprint; [http://arxiv.org/abs/1503.06272](http://arxiv.org/abs/1503.06272).

[18] I. Smith, *Symplectic geometry of Lefschetz fibrations*, Ph.D. thesis, University of Oxford (1998).

[19] I. Smith, *Lefschetz pencils and divisors in moduli space*, Geom. Topol. 5 (2001), 579–608.

[20] C. H. Taubes, *The Seiberg–Witten and Gromov invariants*, Math. Res. Lett. 2 (1995), no. 2, 221–238.

Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003-9305, USA

E-mail address: baykur@math.umass.edu