A GEOMETRIC PROOF OF THE POINCARÉ-BIRKHOFF-WITT THEOREM

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Abstract. We use that the $n$-sphere for $n \geq 2$ is simply-connected to prove the Poincaré-Birkhoff-Witt Theorem.

There are several equivalent statements of the Poincaré-Birkhoff-Witt Theorem. The version we shall prove is as follows.

Theorem. Let $\mathfrak{g}$ be a Lie algebra. Define an equivalence relation on the tensor algebra $\bigotimes \mathfrak{g}$ by imposing the relations that

\[ a \otimes b - b \otimes a = [a, b] \]

as a two-sided ideal in $\bigotimes \mathfrak{g}$. Write the resulting associative algebra as $\mathcal{U}(\mathfrak{g})$ and write $ab \cdots d$ for the equivalence class of $a \otimes b \otimes \cdots \otimes d$. Pick a basis for $\mathfrak{g}$ and declare that an element $ab \cdots d \in \mathcal{U}(\mathfrak{g})$ is in ‘canonical form’ if and only if $a, b, \ldots, d$ are basis elements with $a \leq b \leq \cdots \leq d$ with respect to the ordering of the basis. Then elements in $\mathcal{U}(\mathfrak{g})$ may be consistently and uniquely written as linear combinations of elements in canonical form.

An algebraic proof may be found, for example, in [3]. The rest of this article is devoted to a geometric proof.

To understand what the Poincaré-Birkhoff-Witt Theorem says, let us consider the case of three elements $a, b, c \in \mathfrak{g}$, which we suppose are basis elements in this order $a \leq b \leq c$, and that we would like to rewrite the element $cba \in \mathcal{U}(\mathfrak{g})$ (given in the ‘wrong’ order) as a linear combination of canonically ordered elements. Certainly, we can use the equivalence relation $(\star \star \star)$ to try to reorder this element:

\[
\begin{align*}
    cba &= cab - c[a, b] \\
    &= acb - [a, c]b - c[a, b] \\
    &= abc - a[b, c] - [a, c]b - c[a, b],
\end{align*}
\]

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where we have firstly swopped $b$ and $a$ (and then followed our noses). The only problem is that one can firstly swop $c$ and $b$ instead:

\[
\begin{align*}
\text{c}{ba} &= \text{b}{ca} - [b, c]a \\
&= \text{b}{ac} - b[a, c] - [b, c]a \\
&= \text{a}{bc} - [a, b]c - b[a, c] - [b, c]a,
\end{align*}
\]

which is consistent if and only if the ‘second order’ remainder terms agree:

\[
a[b, c] + [a, c]b + c[a, b] = [a, b]c + b[a, c] + [b, c]a.
\]

Fortunately, this is exactly the Jacobi identity:

\[
[a, [b, c]] + [[a, c], b] + [c, [a, b]] = 0.
\]

We may arrange these calculations on a circle:

- Figure 1

where $\cdots$ denotes second order terms. Otherwise said, the Jacobi identity is exactly what is needed so that an excursion through the symmetric group $S_3$ on three letters

$abc \sim bac \sim bca \sim cba \sim cab \sim acb \sim abc$

is consistent in $\mathfrak{U}(g)$. One can think of this as saying that there is no ‘holonomy’ around the circle depicted in Figure 1.

If we attempt a similar proof for four basis element $a \leq b \leq c \leq d$, then we run into trouble because there is no ‘follow your nose’ method for reordering elements of the symmetric group $S_4$. Instead, we may
picture $\mathcal{S}_4$ as 24 countries in the plane arranged like this:

Also depicted is a typical excursion through $\mathcal{S}_4$ starting and finishing at $abcd$, namely

\[
abcd \leadsto abdc \leadsto adbc \leadsto acdb \leadsto cadb \leadsto cdab \leadsto dcab
\]

\[
\Downarrow
\]

\[
bcad \leadsto cbad \leadsto cbda \leadsto cdab \leadsto dcba \leadsto dbca \leadsto dcba
\]

\[
\Downarrow
\]

\[
bcad \leadsto dacb \leadsto dcab
\]

\[
\Downarrow
\]
We would like to see that this excursion is consistent. There are just 8 points in the plane where 6 countries come together. For example:

Figure 3

These are the eight points where countries of the form

\[a^*\] or \[b^*\] or \[c^*\] or \[d^*\] or \[a^*a\] or \[a^*b\] or \[a^*c\] or \[a^*d\]

meet at a vertex and these are marked by \(\bullet\) in Figure 2. The picture above is of the vertex \(\bullet\) and one recognises the circle from Figure 1 save that the elements \(a, b, c\) have been relabelled \(d, c, a\). We saw earlier that this circle corresponds to a consistent identity for three elements in \(\mathfrak{U}(g)\) and now we obtain a consistent identity for four elements in which \(b\) simply goes along for the ride. Geometrically, it means we may replace the path in Figure 3 by

Figure 3

...to obtain an alternative but simpler excursion through \(S_4\), which is consistent if and only if the original excursion is consistent. If we can similarly pull paths through the other 5 vertices where just four countries come together, then we can reduce any excursion through \(S_4\) to the trivial excursion (by a series of ‘simple jerks’ in the terminology of [H]) and our proof is complete. A typical example is

Figure 4

...
but vertices like this evidently have consistent holonomy

\[
\begin{align*}
   cbda - [b, c]ad - bc[a, d] + [b, c][a, d] \\
   bda - [b, c]da
\end{align*}
\]

without using the Jacobi identity. It is because we are transposing
the first two and the last two of four letters, and such transpositions
commute in \( \mathfrak{S}_4 \).

So now, we may consistently reorder any four elements in \( \mathfrak{U}(g) \) and
we ask about five elements and so on. We need a similar picture of the
symmetric groups \( \mathfrak{S}_N \) for all \( N \geq 4 \). To obtain such a picture, we now
admit that Figure 2 was obtained from a tessellation of the 2-sphere by
24 geodesic triangles with angles \((\pi/2, \pi/3, \pi/3)\). Specifically, it was
obtained by stereographic projection so that great circles on the sphere
are mapped to circles or straight lines on the plane whilst angles are
preserved.

Therefore, a better viewpoint on Figure 2 is as a triangulation of the
2-sphere. From this point of view there is one more ‘easy vertex,’ as in
Figure 4, out at infinity. The fact that one can contract any excursion
in \( \mathfrak{S}_4 \) to the trivial excursion is due to there being no obstructions
- at the 6 ‘easy vertices’ (commuting transpositions),
- at the 8 ‘tricky vertices’ (from the \( \mathfrak{S}_3 \) case),
and the fact that the 2-sphere is simply-connected. This triangulation
of the 2-sphere is well-known in a different guise. It is obtained by
letting the Weyl group of the \( A_3 \) root system act on \( \mathbb{R}^3 \), as described,
for example, in [2]. The triangulation is obtained by intersecting the 24 Weyl chambers with the unit sphere in $\mathbb{R}^3$. Since the Weyl group of $A_3$ may be identified with $\mathfrak{S}_4$, one can pick a triangle to be called the ‘fundamental triangle’ and use the Weyl group action to identify any element of $\mathfrak{S}_4$ with the triangle obtained as the corresponding image of the fundamental triangle. This is how Figure 2 was obtained.

It is evident how to extend this to $\mathfrak{S}_N$ for all $N \geq 4$ and, for the general pattern, it suffices to make sure that $\mathfrak{S}_5$ behaves as it should. The corresponding tessellation of the unit 3-sphere is by 120 tetrahedra with dihedral angles $(\pi/2, \pi/2, \pi/2, \pi/3, \pi/3, \pi/3)$ (each dihedral angle corresponds to a pair of vertices from the Dynkin diagram $\bullet \bullet \bullet \bullet \bullet$, which are either adjacent (angle $\pi/3$) or not (angle $\pi/2$)). To use the simple connectivity of the 3-sphere it now suffices to be able to move a path on the 3-sphere through any edge of this tessellation.

As on the 2-sphere, there are two cases. Firstly, there are the ‘easy edges,’ where just 4 tetrahedra meet at right angles. On the Dynkin diagram, edges of this type correspond to striking out all but two 2 non-adjacent nodes

$$\times \times \times \times \quad \bullet \times \times \times \quad \bullet \bullet \times \times,$$

in effect leaving the Weyl group of $A_1 \times A_1$ as in Figure 4. It is just the Abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The ‘tricky edges’ are when 6 tetrahedron meet at angle $\pi/3$. Tricky edges may be recorded on the Dynkin diagram by striking out all but two adjacent nodes

$$\times \times \bullet \bullet \leftrightarrow \text{permuting } ab** \text{ with } a, b \text{ held fixed},$$

$$\times \bullet \bullet \times \leftrightarrow \text{permuting } a**b \text{ with } a, b \text{ held fixed},$$

$$\bullet \bullet \times \times \leftrightarrow \text{permuting } **ab \text{ with } a, b \text{ held fixed}.$$

The tricky edges are not obstructed since the previous reasoning using the Jacobi identity applies (notice that we are left with $A_2 = \bullet \bullet \bullet \bullet \bullet$ and the Weyl group of $A_2$ is $\mathfrak{S}_3$). Looking back, we see that Figure 1 is the root diagram for $A_2$.

**References**

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