Seiberg Duality for Quiver Gauge Theories

David Berenstein† and Michael R. Douglas‡

† School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540, USA
‡ Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855-0849, USA
§ Isaac Newton Institute for Mathematical Sciences, Cambridge, CB3 0EH, U.K.
¶ I.H.E.S., Le Bois-Marie, Bures-sur-Yvette, 91440 France

ABSTRACT: A popular way to study $\mathcal{N} = 1$ supersymmetric gauge theories is to realize them geometrically in string theory, as suspended brane constructions, D-branes wrapping cycles in Calabi-Yau manifolds, orbifolds, and otherwise. Among the applications of this idea are simple derivations and generalizations of Seiberg duality for the theories which can be so realized.

We abstract from these arguments the idea that Seiberg duality arises because a configuration of gauge theory can be realized as a bound state of a collection of branes in more than one way, and we show that different brane world-volume theories obtained this way have matching moduli spaces, the primary test of Seiberg duality.

Furthermore, we do this by defining “brane” and all the other ingredients of such arguments purely algebraically, for a very large class of $\mathcal{N} = 1$ quiver supersymmetric gauge theories, making physical intuitions about brane-antibrane systems and tachyon condensation precise using the tools of homological algebra.

These techniques allow us to compute the spectrum and superpotential of the dual theory from first principles, and to make contact with geometry and topological string theory when this is appropriate, but in general provide a more abstract notion of “noncommutative geometry” which is better suited to these problems. This makes contact with mathematical results in the representation theory of algebras; in this language, Seiberg duality is a tilting equivalence between the derived categories of the quiver algebras of the dual theories.
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1. Introduction

One of the more striking results in the modern study of $\mathcal{N} = 1$ supersymmetric gauge theory was Seiberg’s discovery \[53\] of an IR duality between two QCD-like theories, both with $N_f$ flavors of quarks (fundamental chiral superfields), but with different gauge groups $SU(N_c)$ and $SU(N_f - N_c)$. The simplest prediction of this duality is that the moduli spaces of supersymmetric vacua of the two theories are the same. Other nontrivial agreements between the theories are the ’t Hooft anomaly matching conditions, and the behavior under adding relevant operators. This duality was extended to many $\mathcal{N} = 1$ theories, and various physical derivations of it were proposed, as reviewed in \[40, 11, 35\].

Perhaps the simplest and most suggestive derivation was in terms of the Hanany-Witten-Diaconescu suspended brane construction \[37, 16\]. As in all such constructions, one obtains $D$-dimensional gauge theory as the world-volume theory of a set of Dirichlet branes extending in $D$ dimensions; the embedding in the additional or “internal” $10 - D$ dimensions determines the spectrum and other structure. Here, the D-branes are strings in the internal dimensions, with one or both ends attached to NS5-branes. Taking $N_c$ finite length strings between 5-branes produces pure $SU(N_c)$ gauge theory, and adding $N_f$ semi-infinite strings produces $N_f$ quarks. The duality is then obtained by moving one 5-brane around the other to exchange their positions, which reverses the orientation of the finite strings. If $N_f \geq N_c$, by moving and reconnecting the strings, one obtains a theory with $N_f$ semi-infinite and $N_f - N_c$ finite strings, the dual theory. \[26\]

This construction provides a very intuitive physical picture for the duality and shows clearly where it is relevant in compactification of string theory: since the inverse coupling constant is identified with the length of the string, it will arise if one varies the gauge coupling (by varying a moduli field) through infinity. The result of this is to turn branes into their antibranes, leading to a very simple derivation of the flip $N_c \rightarrow -N_c$.

The main disadvantage of this construction is that certain steps are somewhat ad hoc. In particular, reproducing the moduli spaces of vacua requires postulating rules such as the “s-rule” of \[37\] on allowed brane configurations. Although this particular rule can be justified independently \[35\], one is left with a qualitative approach which works only in simple examples. One requires more precision to treat more complicated examples, and ultimately only a procedure which can produce explicit superpotentials can be considered completely satisfactory.

Of course gauge theories can be embedded in string and M theory in many ways. A large class of $\mathcal{N} = 1$ examples is provided by the quiver gauge theories of D-branes at $\mathbb{C}^3/\Gamma$ orbifold singularities \[21\]. This approach is both highly computable, and has a direct geometric interpretation in terms of branes wrapping cycles in the internal space obtained by resolving the singularity.

Recently, several groups \[6, 13, 29\] have shown that Seiberg dual pairs or even sets of theories can be obtained in this framework, using the procedure of “partial resolution” \[50\]. Many interesting singularities are not orbifold singularities, but can be obtained by partial resolution of an orbifold singularity. Physically, the gauge theory associated to the singularity is obtained by starting with the orbifold quiver theory, turning on certain
Fayet-Iliopoulos terms, and integrating out any matter which becomes massive. One can make various choices of FI terms and their signs, and these choices lead to Seiberg dual theories.

The intuitive picture is very similar to the work on branes in flat space. As before, one can take a gauge coupling through infinity through a continuous deformation which changes the sign of FI terms. This leads to a topologically distinct but birationally equivalent resolution. The identification of “fractional brane” to brane wrapping cycle depends on this choice, so one gets different geometric identifications related by “brane-antibrane” transitions of the sort described above. This work is thus one part of the motivation for the present work.

In fact, one does not need to bring in the geometry of orbifold resolution in this discussion, and one can discuss Seiberg duality for general quiver theories, not just those obtained from orbifolds. A more direct motivation for the present work is an observation of B. Fiol, that Seiberg duality is related to the “reflection functor” of the theory of quiver representations. We will show that this is indeed what underlies classical duality in Seiberg’s original example, and provides a one-to-one map between configurations.

In simple cases, this map can be understood quite simply as translating a bound state formed from a particular basis $B_i$ of branes, to a new but simply related basis $B'_i$, copying over all the structure of the bound state. For example, the suspended brane argument is the change of basis from $(B_1, B_2)$ the semi-infinite and finite strings, to $(B_1 + B_2, -B_2)$ a new basis of a semi-infinite string obtained by combining $B_1$ and $B_2$, with the orientation reversed finite strings $B_2$.

However, our map is defined in a completely algebraic way, explicitly computing the superpotential of the dual theory, and mapping specific field configurations to field configurations, without requiring other ad hoc input. For purely bifundamental matter, the relation between superpotentials is indeed the one postulated by [6, 13, 29]. However, our argument generalizes to a much larger class of quiver theories, and we will demonstrate this by treating the adjoint theories of [17]. We feel our arguments both explain the duality concretely, and reduces the general problem of finding dualities to mathematics, in a sense we will describe.

As in the suspended brane arguments, our basic results will be for $U(N)$ gauge theories with large Fayet-Iliopoulos terms. This justifies treating the theories classically, and our basic results will be to match the moduli space of pairs of classical theories. We will also compute the superpotential for the dual theory in terms of the original theory. Again this will be for large FI terms, but if we make the natural assumption (which can be proven in some cases) that the superpotential is independent of the FI terms, then this is the general result. In this sense we can say we have proven Seiberg duality for this class of theories.

Of course one cannot avoid discussing quantum corrections in general. We will not much discuss the quantum aspects, not because we think this is unimportant or because they cannot be addressed, but more because at the moment we only see how to do this on a case by case basis, as is already well discussed in the literature.

To summarize what follows, in section 2 we give a very explicit argument for duality in the original case of supersymmetric QCD (with $U(N)$ gauge group), to make the ideas
clear. We will be able to make almost all of the argument in purely field theoretic terms, without explicitly bringing in string theory. In section 3 we generalize this to arbitrary $U(N)$ quiver theories with a superpotential.

In section 4 we explain how many of these arguments can be based on the theory of Dirichlet branes in Calabi-Yau manifolds. This will allow us to clean up some points in the previous argument and serve as a simple introduction to the derived category, which underlies the discussion. We then reconsider our examples and move on to treat the case with an adjoint field.

In section 5 we discuss related mathematics. In particular, there is a theorem \[ \text{[52]} \] which, in a sense we will explain, shows that all Seiberg-like dualities are examples of “tilting equivalences,” a generalization of the reflection functor. This will allow us to give a simple general argument that toric dualities are indeed Seiberg dualities.

We also go somewhat more deeply into the underlying formalism, to make the point that the homological algebra we are using does not depend \textit{a priori} on realizing the gauge theories using branes in a Calabi-Yau, but can be defined directly, just from the quiver gauge theory.

Finally we give conclusions in section 6.

2. Seiberg duality as change of basis

In this section we consider the original example of $SU(N_c)$ gauge theory with $N_f$ flavors of massless quarks, i.e. $N_f$ chiral multiplets in the fundamental $N_c$ and $\bar{N}_c$ representations. Classically, this theory has global symmetry $U(N_f) \times U(N_f)$ acting on the $N_c$ and $\bar{N}_c$ quarks respectively.

Our arguments will essentially be classical, so we ignore anomalies for the time being. While we are at it, we can even take the numbers of $N_c$ and $\bar{N}_c$ quarks to differ. Furthermore, we ignore for now the difference between $SU(N)$ and $U(N)$, and take $U(N)$ gauge group.

We write the resulting continuous symmetry group as

$$U(N_1) \times U(N_2) \times U(N_3),$$

and the matter is chiral multiplets $\tilde{Q}$ and $Q$ transforming as

$$\tilde{Q} \in (\bar{N}_1, N_2, 1); \quad Q \in (1, \bar{N}_2, N_3).$$

This theory can be summarized in the diagram in figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{quiver.png}
\caption{Quiver diagram corresponding to the $(N_1, N_2, N_3)$ theory.}
\end{figure}
Clearly these theories can be obtained by embedding or wrapping Dirichlet branes in a variety of ways. The standard (non-chiral) theory can be obtained from any configuration containing two distinct types of brane, $B_c$ and $B_f$, such that the massless fermionic open strings between $B_c$ and $B_f$ are precisely one left and one right chirality Weyl fermion. The generalized theories can be obtained by postulating branes $B_1$, $B_2$ and $B_3$, with a single Weyl fermion between the pair $(B_1, B_2)$ and between the pair $(B_2, B_3)$. The spectrum, general properties of D-branes and supersymmetry lead directly to the classical quiver theory (diagram). We will sometimes denote the theory with brane content $N_1B_1 + N_2B_2 + N_3B_3$ as the theory “$(N_1 \ N_2 \ N_3)$”, and refer to this vector as the “charge.”

We will only consider quiver theories, with bifundamental and adjoint matter, for which the notation (2.2) is somewhat cumbersome. We now switch to denote the vector space of chiral multiplets charged

$$(\ldots, \bar{N}_E, \ldots, N_{E'}, \ldots)$$

for two branes $E$ and $E'$ as

$$\text{Ext}(E, E').$$

In other words, a theory with $k$ such chiral multiplets would have $\text{Ext}(E, E') = \mathbb{C}^k$, and the consequent $U(k)$ global symmetry (which could of course be broken by the superpotential) would act linearly on this vector space. A similar notation to denote the complex gauge invariances or gauginos is

$$\text{Hom}(E, E').$$

Normally one has $\dim \text{Hom}(E, E') = \delta_{E,E'}$ for a collection of branes each carrying a $U(1)$ world-volume gauge field.

In this notation, $\hat{Q} \in \text{Ext}(B_1, B_2)$ and $Q \in \text{Ext}(B_2, B_3)$. At this point, this is just an alternate way to describe the field content of the world-volume theory. Once we define bound states of branes, we will use the same notations to denote the linear spaces of chiral multiplets and gauge symmetries between any pair of branes.

2.1 BPS bound states and duality as change of basis

We now ask, what are the simple BPS bound states of branes in this theory, meaning combinations of branes, possibly with vevs of chiral fields, which preserve supersymmetry and break the gauge symmetry to $U(1)$.

It is fairly obvious for the theory (2.1), and can even be proven rigorously, that there are three such bound states, with charges

$$[B_4] = (1 \ 1 \ 0); [B_5] = (0 \ 1 \ 1); [B_6] = (1 \ 1 \ 1).$$

(2.3)

These supersymmetric gauge theories have one (for $B_4$ and $B_5$) or two (for $B_6$) chiral multiplets, which must be set non-zero to break the gauge symmetry to $U(1)$. Their precise values, up to gauge equivalence, are completely determined by the FI terms, which must be set non-zero (in an obvious way) for any of these to be supersymmetric vacua. Thus, all three bound states are “rigid,” meaning that they themselves have no moduli, and are described at low energy by pure supersymmetric $U(1)$ gauge theory.
We can ignore the dependence on FI terms for now by defining a "holomorphic configuration" of the gauge theory as a configuration of the chiral multiplets up to complexified gauge equivalence, here with complex gauge group $\prod_i GL(N_i, \mathbb{C})$. Thus we define (for example) the brane $B_4$ as the unique holomorphic configuration of $(1 1 0)$ quiver gauge theory with unbroken $GL(1)$ gauge symmetry. This is the precise sense in which we define a brane $B_i$ as a bound state of branes (but not antibranes as yet). We will then use the notation $[B_1]$ to denote the charge of the brane $B_i$.

More generally, a theory with given charge might have more than one gauge equivalence class of supersymmetric configuration. In general we will use the notation $B_i$ for a brane, to refer to a particular holomorphic configuration of the theory with particular charges.

The main idea is now that, since the bound states (2.3) share all of the properties of the original branes $B_1$, $B_2$ and $B_3$, it should be possible to describe any configuration made out of the original branes equally well as a bound state of the new branes, or as a bound state of some combination of the old and new branes. Each such choice of a generating set of branes will lead to an a priori different supersymmetric gauge theory. Since we are merely reexpressing the same brane configuration as a bound state in a different way, if in each theory there is a single way to get each configuration as a bound state (this is what we will mean by a set of branes forming a "basis"), this must give a one-to-one map between gauge theory configurations. This idea can be made quite precise and will lead to our map.

Suppose we try to go from the basis $(B_1, B_2, B_3)$ to the basis $(B_1, B_2, B_5)$. Following the intuition that the ranks of the gauge groups are charges, and inverting to get $[B_3] = [B_5] - [B_2]$, we can identify the new charges $N'_i$ in terms of the old $N_i$ as

$$N'_1 = N_1; N'_5 = N_3; N'_2 = N_2 - N_3.$$  \hspace{1cm} (2.4)

Evidently there are two cases, $N_2 \geq N_3$ and $N_2 \leq N_3$. We can try to proceed either way, but in the second case $N_2 \leq N_3$, we should instead of $B_2$ use its antibrane $\bar{B}_2$, with multiplicity

$$N'_2 = N_3 - N_2.$$ 

2.2 Dual world-volume theory

The next step is to derive the world-volume theory of a collection of branes from the new basis. Since these are bound states of the original branes, everything should be determined by straightforward computation.

The idea behind this is that the massless fields associated to a pair of branes $E$ and $E'$ with charges $N_i$ and $N'_i$, each a specific bound state, can be found by computation in the joint $\prod_i U(N_i + N'_i)$ theory describing the pair. For the case with no superpotential, this is described in some detail in the appendix to [20], and we give the basic idea here. A more general discussion covering some superpotentials is in [24].

Consider the combinations of $B_5$ with $B_1$ and $B_2$ relevant for the problem at hand. We first combine $B_1$ and $B_5$. Together, these sit in the gauge theory $(1 1 1)$. Upon combining this pair of branes, there is an extra massless field, inherited from that between $B_1$ and $B_2$, and transforming under the unbroken gauge groups of $B_1$ and $B_5$. This can be summarized
in the result

\[ \dim \text{Ext}(B_1, B_5) = 1. \]  \hspace{1cm} (2.5)

For \( B_2 \) and \( B_5 \), one must consider the theory \((0 \ 2 \ 1)\). Upon bringing this pair together, we also gain an extra matter field, now from \( \text{Ext}(B_2, B_3) \). However, unlike the first case, the vev of this field can be gauged away by \( GL(2) \), so it does not contribute a physical degree of freedom. In fact there are no extra massless chiral multiplets:

\[ \dim \text{Ext}(B_2, B_5) = \dim \text{Ext}(B_5, B_2) = 0. \]  \hspace{1cm} (2.6)

An equivalent way to say this is to note that the combined theory has two extra vector multiplets, off-diagonal in \( GL(2) \). If the vev of the \( \text{Ext}(B_2, B_3) \) chiral multiplet had been zero, i.e. \( q = 0 \), the \( GL(2) \) would be unbroken, and these would transform as a \( \text{Hom}(B_2, B_5) \) and a \( \text{Hom}(B_5, B_2) \). On the other hand, in the bound state \( B_5 \), \( q \neq 0 \). This vev Higgses the \( \text{Hom}(B_2, B_5) \) gauge boson, and both chiral and vector multiplet are lifted. It is Higgsed by the matter, and both become massive. However, the \( \text{Hom}(B_5, B_2) \) vector remains massless.

The resulting massless spectrum is

\[ \dim \text{Hom}(B_5, B_2) = 1 \]  \hspace{1cm} (2.7)

with all other dimensions zero.

This is the basic data we now need to incorporate in a new, presumably dual, quiver gauge theory. We have one node for each of \( B_1, B_2 \) and \( B_5 \). The result (2.5) is easy to incorporate as this tells us that we have a single bifundamental chiral multiplet in the \((\tilde{N}_1', N_5')\). Encouragingly, this has the right quantum numbers to be the “meson” field \( M \) of [53].

On the other hand, the result (2.6) is confusing, for various reasons. It tells us that when we put together two different branes, \( B_2 \) and \( B_5 \), we get an enhanced gauge symmetry. On the other hand, one knows one does not get enhanced gauge symmetry when one puts two different D-branes together.

Furthermore, the asymmetry between \( B_2 \) and \( B_5 \) leads to a non-unitary gauge group, which might be disturbing. There are various reasons why this does not contradict general theorems, but the most pertinent is that this configuration cannot solve the D-flatness conditions (for any choice of FI terms) and thus cannot be realized as a vacuum. \(^1\) In the brane language, \( B_2 \) and \( B_5 \) cannot form a bound state.

We will come back to this point later, but indeed it is true that this makes it difficult to describe this combination of three branes directly as a supersymmetric field theory. Of course, we know that not all combinations of branes can be so described, for example combinations of a brane with its own antibrane are problematic.

\(^1\)A related observation is that, in the physical theory, a vev for the field \( Q \) implies a vev for its hermitian conjugate \( Q^\dagger \). This will give a mass to the other off-diagonal vector particle and lead to a unitary gauge group. In our holomorphic definitions, we are only giving a vev to \( Q \).
However, one can describe the combination \((B_1, \bar{B}_2, B_5)\) with supersymmetric field theory. To see this, one needs a definition of the antibrane \(\bar{B}_2\). Usually this is explained in terms of the string world-sheet, or in terms of embeddings of branes. For example, in the D-brane context, an antibrane is defined by reversing the GSO projection. This exchanges vector and chiral multiplets, and will relate the \(\text{Hom}(B_5, B_2)\) of \((2.7)\) to a chiral multiplet \(\text{Ext}(B_5, \bar{B}_2)\). This has the right quantum numbers to be one of the "dual quarks" of \([53]\), so clearly this is a step in the right direction.

In fact, one can define antibranes without any reference to string theory or a higher dimensional embedding, by using the language of homological algebra, as we will now explain.

### 2.3 The reflection functor

We now explain how bound states of \(B_2\) and \(B_3\) map in a one-to-one way to bound states of \(\bar{B}_2\) and \(B_5\). This construction, known as a "reflection functor" in the theory of quiver representations \([10]\), will underlie the map between configurations in the general case.

The change of basis \((B_2, B_3) \rightarrow (\bar{B}_2, B_5)\) does not directly involve \(B_1\), so for now let us just consider \(U(N_2) \times U(N_3)\) gauge theory with a single chiral field \(B\) in the \((\bar{N}_2, N_3)\). We assume \(N_3 \geq N_2\) and let \(N = N_3\).

The basic problem we face is to give a definition to the antibrane \(\bar{B}_2\), using only the most general features of string theory. Now the defining property of our new basis is that \(B_3\) can be obtained as a bound state of \(\bar{B}_2\) and \(B_5\). Given the result \((2.7)\), there is a well-motivated mathematical way to do this: we realize \(B_3\) as a complex. This is a construction of homological algebra in which we take a finite length sequence of "objects" (branes for us) \(E_i\), \(i\) integer, and postulate a series of linear maps \(d_i : E_i \rightarrow E_{i+1}\) between successive terms in the complex, satisfying \(d_i \cdot d_{i+1} = 0\). The standard notation for such a complex is

\[
E_{-m} \rightarrow E_{-m+1} \rightarrow \cdots \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_n.
\]

(2.8)

The underline is used to indicate the zero position in the complex.

At least in a first approximation, we will interpret the cohomology of \(d\) as the physical brane represented by the complex. The most elementary example of this is a complex of the form

\[
B \rightarrow B
\]

(2.9)

where the two terms are the same object \(B\), and \(d = 1\). We could also consider the direct sum of this with another complex. In either case, this combination completely cancels out of the cohomology. We will use this to represent the physical fact that such a \(BB\) pair can annihilate to the vacuum via tachyon condensation \([54]\), and that the vacuum is the unique supersymmetric configuration of this type.

To define this cohomology in more complicated problems, one of course needs some definition of the underlying objects. Here our definition is holomorphic configurations of a quiver gauge theory, as in \([20]\). For D-branes on Calabi-Yau, one might use coherent sheaves on the Calabi-Yau as the underlying objects instead. We will discuss this point
further in sections 4 and 5. Actually, much of the discussion can be made without knowing what the objects are, only that they satisfy the axioms of an abelian category. If this is too abstract for the reader’s taste, he is encouraged to ignore it.

To represent $B_3$ as a brane-antibrane bound state, we will use a two term complex with $E_0 = B_5$ and $E_1 = B_2$, and $d = \alpha$ a non-zero element of Hom($B_5, B_2$). We denote this as $$B_3 = [B_5 \xrightarrow{\alpha} B_2].$$

The cohomology does not depend on the overall magnitude of $d$ and since in our example $\dim \text{Hom}(B_5, B_2) = 1$ there is a unique such bound state. More generally one gets a bound state with $\dim \text{Hom}(B_5, B_2) - 1$ parameters or world-volume chiral multiplets.

There are various ways to interpret this construction physically. One is that $d$ represents a brane-antibrane tachyon, and we are constructing the bound state $B_3$ by tachyon condensation. Another interpretation is that $d$ forms part of the BRST operator in topological open string theory, as in [19]. The two interpretations are closely related in the case of a two-term complex, but the BRST interpretation also makes sense for the case of longer complexes.

For now, we postpone these subtleties and will not rely on the detailed physics of the construction, but use only its most basic property: namely, that the complex (2.9) is equivalent to the vacuum configuration.

Having this definition of the change of basis, we now seek a map which takes a configuration of this theory, in other words a vacuum expectation value for $Q$ up to complex gauge equivalence, and produces a unique corresponding configuration of $U(N_3 - N_2) \times U(N_3)$ gauge theory with a single chiral field in the $(N_3 - N_2, \bar{N}_3)$, call this $\tilde{q}$.

The starting configuration can now be realized as the complex

$$\begin{align*}
\mathbb{C}^{N_2} \otimes B_2 & \quad \downarrow Q \\
\mathbb{C}^N \otimes B_5 & \xrightarrow{1 \otimes \alpha} \mathbb{C}^N \otimes B_2
\end{align*}$$

Here $1_N$ is the $N \times N$ identity matrix, and the vertical axis is just direct sum (e.g., the map $d$ is the direct sum of the maps indicated by both arrows).

This complex is obviously not the minimal complex we could use to describe this state; a complex containing fewer branes would be obtained by annihilating $B_2 \bar{B}_2$ pairs. If we assume that $Q$ has maximal rank (we discuss this below), we can apply a $GL(N)$ transformation $g$ on the left,

$$g(Q \ 1_N) = (1_{N_2} \ g) = \begin{pmatrix} 1_{N_2} & v \\ 0 & \tilde{q} \end{pmatrix}$$

(2.11)

to obtain a complex with $N_2 \ B_2 \bar{B}_2$ pairs of the form (2.9), an $N \times (N - N_2)$ matrix $\tilde{q}$, and an an $N \times N_2$ matrix $v$. We then annihilate the $B_2 \bar{B}_2$ pairs,\(^2\) to obtain an equivalent complex

$$\begin{align*}
\mathbb{C}^N \otimes B_5 & \xrightarrow{\tilde{q}} \mathbb{C}^{N - N_2} \otimes B_2
\end{align*}$$

\(^2\)Among the subtleties in defining this, one is particularly noteworthy. Part of $d$ maps from the $B_5$’s
This is the representation of the same configuration in the “dual” theory.

One should ask whether the step of making a \( GL(N) \) transformation leads to some ambiguity in this procedure. To see that this procedure takes a gauge equivalence class of configurations of \( Q \) to a single gauge equivalence class of configurations \( \tilde{q} \), we can rephrase the map in the following equivalent way. We add another arrow to (2.10),

\[
\begin{array}{ccc}
\mathbb{C}^{N_2} \otimes B_2 & \xrightarrow{Q} & \mathbb{C}^{N_5} \otimes B_2 \\
\downarrow & & \downarrow \\
\mathbb{C}^N \otimes B_5 & \xrightarrow{1 \otimes \alpha} & \mathbb{C}^N \otimes B_2,
\end{array}
\]

to get a commutative diagram (notice that we are allowed to do this because the map \( 1 \otimes \alpha \) is onto). This property will of course be preserved by the transformation (2.11). Annihilating the \( B_2 \bar{B}_2 \) pairs then restricts the product \( \tilde{q} \cdot Q \) to an \( N-N_2 \)-dimensional subspace, the quotient space \( \mathbb{C}^N/\text{im}Q \). By definition however this is the cokernel of \( Q \), so one has the gauge invariant statement that

\[
\tilde{q} \cdot Q = 0.
\]  

(2.12)

This can be regarded as a set of linear equations which given \( Q \) determines \( \tilde{q} \) up to gauge equivalence. One can say the same thing by writing the exact sequence

\[
0 \longrightarrow \mathbb{C}^{N_2} \xrightarrow{Q} \mathbb{C}^{N_3} \xrightarrow{\tilde{q}} \mathbb{C}^{N_3-N_2} \longrightarrow 0.
\]  

(2.13)

All this assumed that \( Q \) has maximal rank. On the other hand, if \( Q \) did not have maximal rank, we would not be able to find a \( GL(N) \) transformation \( g \) of the form (2.11). This does not prevent us from making the change of basis, but doing it results in a complex containing terms \( B_2 \bar{B}_2 \) which are “non-annihilated brane-antibrane pairs.”

Brane-antibrane annihilation can be defined more precisely by using the formalism of the derived category. In the derived category it will be true that the map between configurations is one-to-one in all cases, but will not always take configurations of supersymmetric gauge theory to supersymmetric gauge theory. The configurations with less than maximal rank typically (not always) leave unbroken nonabelian gauge symmetry, and thus these cases obtain non-trivial quantum corrections. We will return to this later.

We can go on to derive the value of \( M \) in the dual theory for a given starting configuration, by putting in \( B_1 \) and its map to the other branes. Adding this to the complex (2.10) produces the commutative diagram

\[
\begin{array}{ccc}
B_1 \otimes \mathbb{C}^{N_1} & \xrightarrow{\tilde{Q}} & B_2 \otimes \mathbb{C}^{N_2} \\
\downarrow M & & \downarrow Q \\
B_5 \otimes \mathbb{C}^{N_3} & \xrightarrow{1} & B_2 \otimes \mathbb{C}^{N_3},
\end{array}
\]

to the \( B_2 \)'s which will be annihilated (this is \( v \neq 0 \)), and in doing the annihilation, we are losing this information. Physically, this makes the annihilation process like an RG flow (as indeed it is in boundary CFT). Mathematically, it is formalized in the statement that the equivalence between the two configurations is a \textit{quasi-isomorphism}, a chain map which preserves homology.
so clearly

\[ M = Q \cdot \tilde{Q}. \]  

(2.14)

This is gauge equivalent using \((2.11)\) to

\[
\begin{array}{ccc}
B_1 \otimes \mathbb{C}^{N_1} & \xrightarrow{\tilde{Q}} & B_2 \otimes \mathbb{C}^{N_2} \\
M & \downarrow & 1 \\
B_5 \otimes \mathbb{C}^{N_3} & \xrightarrow{g} & B_2 \otimes \mathbb{C}^{N_3}.
\end{array}
\]

The same argument which led to \((2.12)\) then leads to

\[ \tilde{q} \cdot M = 0. \]  

(2.15)

Given the constraints that \(Q\) and \(\tilde{q}\) are maximal rank, the map from \((\tilde{Q}, Q)\) to \((M, \tilde{q})\) is one to one. In the forward direction this is clear. In the reverse direction, this follows by reinterpreting \((2.12)\) as a set of linear equations determining \(Q\) from \(\tilde{q}\).

There is a very similar transformation which maps configurations of \(B_1\) and \(B_2\) to configurations of \(B_4\) and \(\bar{B}_2\). The analysis can be reduced to the previous case by the device of reversing all the arrows, and leads to a map from \(\tilde{Q}\) to \(q\) defined by

\[
0 \rightarrow \mathbb{C}^{N_1-N_2} \xrightarrow{q} \mathbb{C}^{N_1} \xrightarrow{\tilde{Q}} \mathbb{C}^{N_2} \rightarrow 0.
\]  

(2.16)

### 2.4 The role of the FI terms

Continuing the discussion of the change of basis to \((B_1, \bar{B}_2, B_3)\), we next need to derive the massless spectrum between \(B_1\) and \(\bar{B}_2\), i.e. the other dual quark. Again there are string theory arguments for this, which we will return to, but again we would like to see if this can be done purely algebraically.

Naively, one might think we could just repeat what we did to get the first dual quark, as the final dual theory is symmetric under exchanging their roles. In fact this is completely wrong. Our procedure treats the two very asymmetrically, because we replaced \(B_3\) with \(B_5\), keeping \(B_1\) in the new basis. We could have made the alternate choice of keeping \(B_3\) and replacing \(B_1\) with \(B_4\), but this is a different change of basis.

In fact this asymmetry is inherent in the problem, in the classical limit in which we are working. This limit is best justified by the device of turning on large Fayet-Iliopoulos terms, which completely break the gauge symmetry at high energy. This makes quantum effects arbitrarily small, and in fact the exact moduli space agrees with the classical result. However, in turning on FI terms, one is making a choice which leads to the asymmetry. Indeed, varying FI terms will lead to naturally to variations of the basis.

The D-flatness conditions for SQCD are

\[ \tilde{Q}^\dagger \tilde{Q} - QQ^\dagger = \zeta \cdot 1 \]
where \( \zeta \) is the FI parameter. If we take this large and positive, the vacuum expectation value of \( \tilde{Q} \) is forced to take its maximal rank \( N_c \). There is a vacuum with \( Q = 0 \); using \( U(N_f) \) symmetry it can be brought to the form

\[
\tilde{Q} = \left( \sqrt{\zeta} \cdot 1_{N_2} \ 0_{N-N_2} \right).
\]

Giving a vev to \( Q \) increases the magnitude of \( \tilde{Q} \), which always has maximal rank.

Conversely, if \( \zeta \) had been large and negative, we would find supersymmetric vacua in which \( Q \) always had maximal rank.

The reflection functor is a one-to-one map between gauge theory configurations precisely when the morphisms involved have maximal rank, so we now see that we can obtain a duality in either of these limits, but by using two different changes of basis.

The change of basis to \( (B_1, \bar{B}_2, B_3) \) is appropriate when \( \zeta \gg 0 \). The relations (2.13) and (2.14) give us a map from \( (\tilde{Q}, Q) \) to \( (M, \tilde{q}) \), satisfying (2.15),

\[
\tilde{q} \cdot M = 0. 
\] (2.17)

The existence of these vacua tells us that the dual theory also has large FI terms. These nonzero FI parameters are also implicit in brane treatments, as we discuss in the next section. In this context one can independently vary the FI parameters for each node. The original theory then corresponds to the regime

\[
\zeta_1 < \zeta_2 < \zeta_3
\] (2.18)

while the dual we discussed corresponds to

\[
\zeta_1 < \zeta_5 < \zeta_2.
\] (2.19)

2.5 Superpotential and completing the argument

To interpret these as fields in a dual gauge theory, the relation (2.15) must follow from the classical equations of motion or other constraints in this theory. There is only one way to accomplish this, which is as an F-flatness condition.\(^3\) Thus we postulate a superpotential \( W \) and an additional chiral field \( q \) in \( \text{Ext}(\bar{B}_2, B_1) \) such that

\[
\frac{\partial W}{\partial q} = 0 
\] (2.20)

implies (2.15). The simplest candidate is

\[
W = \text{tr} \ Mq\tilde{q}.
\] (2.21)

This step may seem less well motivated than the previous ones, but in fact making it requires more input than just algebra. We need to assume that the F-flatness conditions (quiver relations) in the dual theory follow from a superpotential, or else a statement

\(^3\)The relation lifts fermions, so it cannot be a D-flatness condition.
from which this follows. In section 4, we will derive the existence of the field \( q \) and the superpotential (2.21) in the weakly coupled D-brane context.

One can write more complicated superpotentials which lead to (2.15). In fact, we suspect that general arguments should not produce a unique answer unless we specify additional conditions or give a particular UV definition of the theory. The simplest additional condition which determines (2.21) is to insist that the spaces of supersymmetric vacua be the same on both sides. If we maintain the field content we discussed, the general gauge-invariant superpotential is \( W = \text{tr} f(M\bar{q}q) \) for a holomorphic function \( f \). Adding higher order terms to this always leads to additional vacua. The overall coefficient of the linear term can of course be absorbed into the normalization of the dual quarks.

Finally, given (2.21), we have the additional F-flatness conditions

\[
Mq = q\bar{q} = 0. \tag{2.22}
\]

We can solve these and still have a one-to-one map by postulating

\[
q = 0. \tag{2.23}
\]

This is also natural given the FI terms (2.19).

Thus the matter content of the dual theory is reproduced exactly.

To summarize the results: the change of basis to \((B_1, B_2, B_3)\) leads to a new gauge theory with quiver as in 2. Moreover, when the dual ranks (2.4) are non-negative, there is a one-to-one map between supersymmetric configurations (with large FI term) given by (2.13), (2.14) and (2.23).

![Figure 2: Dual quiver diagram for theory \((N_1, N_2, N_3)\).](image)

### 2.6 Other aspects of the duality

The argument above may have struck the reader as long-winded, especially as it only led to a subset of the original results of [53]. However, the concepts we just defined allow generalizing the same argument to a large class of quiver theories.

We will do this in the next section, but let us discuss some other aspects of Seiberg duality in this example, which one might also try to generalize. Our point is not to make a strong claim that our classical discussion can “prove” quantum Seiberg duality, but to try to see how much can be captured at this level, and by general arguments.

The main point of [53] was of course that in a pair of dual theories, one theory would be preferred because it gave a weakly coupled description. The direct physical implications
of this for $\mathcal{N} = 1$ superconformal field theories are not so obvious. On the other hand, one can add small mass terms to get more conventional theories with a particle spectrum, and then the two different gauge groups and sets of chiral fields would correspond to two candidate particle spectrums. Deciding which is preferred in general requires knowing the low energy gauge couplings. There are general results on this for quiver theories [13, 31], but this is somewhat tangential to our concerns here.

Other tests of the duality include matching of anomalies and chiral rings. For $U(N)$ quiver theories, anomaly matching for the explicit $U(N_i)$ symmetry groups is fairly simple, as almost all of it follows from the simple condition that each node have the same number of fundamentals minus antifundamentals before and after the duality. On the other hand, anomaly matching for other global symmetries, involving the $U(1)_R$ symmetry, is not manifest. There are general arguments which prove anomaly matching for certain pairs of theories with the same IR moduli space [13]. On the other hand, in general “naive” anomaly matching can fail, because a symmetry visible on one side can be an accidental symmetry of the IR limit not manifest in the classical Lagrangian, or because of strong coupling corrections to anomalous dimensions (for the $U(1)_R$ anomaly matching). These subtleties indeed appeared for the toric dual theories considered by [6]. Given that anomaly matching is not manifest, it is not clear at present in what sense one should try to “prove” it.

The chiral ring by definition is the algebra defined by multiplying operators modulo the F-flatness relations $W' = 0$. One can of course make field redefinitions and so the equivalence of chiral rings should be formulated in a more geometric way which allows for this. We believe that such an equivalence should generally follow from the equivalence of classical moduli spaces (with source terms added for the fields). We will not try to study this here but only remark that the simplest picture of the situation is the standard mathematical philosophy which considers a space to be equivalent information to the algebra of functions on that space. Here, the space is the space of supersymmetric configurations, and the algebra is the chiral ring.

Our general philosophy, that Seiberg duality is the description of the same configuration using different bases of branes, suggests that one should have duality in some sense for any ranks of the gauge and symmetry groups, not just those which lead to superconformal theories. Let us consider this point.

Our arguments did not assume $N_f \leq 3N_c$, and work for any $N_f > N_c$. However we need to discuss the other cases. For $0 < N_f < N_c$ there are no supersymmetric configurations. At zero FI term this is because of quantum corrections to the superpotential, while at nonzero FI term it is because one cannot solve the D-flatness conditions. We would say that there is still a dual theory in this case, but it is a theory with $N'_c = N_f - N_c < 0$ which has no supersymmetric vacua.

The case of $N_f = N_c$ is the trickiest as there are supersymmetric configurations, whose quantum corrected moduli space depends explicitly on the quantum scale $\Lambda$. It would seem rather artificial to try to reproduce this in classical terms, but let us make one comment about this.

According to the definitions we are giving the dual is a $U(N_c)$ theory with $N_c = 0$. 
It is not clear what an FI term would mean for such a theory, to say the least. But it is here that one needs to look, for the following reason. The usual discussion is for $SU(N)$ theory. We neglected this distinction (as is usually done in brane arguments) because the $U(1)$ sector only has trivial dynamics. One still has extra baryonic operators in the $SU(N)$ theories. However, very generally, one can trade these invariants for the $U(N)$ FI terms, as discussed in [50, 6].

In some more general phrasing of the problem, this relation between baryonic operators and FI terms might make sense for $N_c = 0$ as well, and describe the quantum corrections. Such a phrasing might be more natural in string theory, where the FI terms are themselves controlled by dynamical fields.

3. General case – a physical approach

The arguments we just gave generalize very straightforwardly to arbitrary theories with bifundamental matter. We start with the case of no initial superpotential, and then incorporate a superpotential.

The ideas generalize directly to any theory of “finite representation type” [3, 33, 39], i.e. a theory with finitely many simple brane bound states. In particular this includes theories with adjoint matter, as long as there is a superpotential which lifts the moduli space of simple branes. We begin the discussion of SQCD with one adjoint along these lines, but eventually explicit computations along the previous lines become tedious, at which point we break off to develop more formalism.

3.1 The general reflection functor

The reflection functor can be defined more generally, on a quiver with a node $E$ (playing the role of $B_2$ above) with $k$ arrows leaving $E$; let their targets be $F_i$ for $1 \leq i \leq n \leq k$, with $n_i$ arrows from $E$ terminating on $F_i$. It can also be applied to a subquiver of this form in a larger quiver. Let the numbers of branes involved be $N_E E + \sum_i N_i F_i$.

The change of basis is now

$$E \to \bar{E}; \quad F_i \to \hat{F}_i \equiv F_i + n_i E.$$ 

A more precise definition of the second line is given by the exact sequence

$$0 \to F_i \to \hat{F}_i \xrightarrow{f_i} \text{Ext}(E,F_i) \otimes E \to 0 \quad (3.1)$$

where $f_i$ is the “tautological Ext,” essentially the direct sum over a basis for $\text{Ext}(E,F_i)$. This leads to the change of rank

$$N_E \to \sum_i N_i n_i - N_E.$$ 

Considerations similar to those above (and which will be made precise in section 4) then lead to

$$\dim \text{Ext}(\hat{F}_i,E) = n_i$$
and a quiver with the same form as before but with arrows reversed.

Let the chiral multiplets be $\phi_\alpha$; the generalization of (2.13) is then
\[
0 \rightarrow \mathbb{C}^N_{\phi} \phi^N \rightarrow \mathbb{C} \sum N_i n_i \rightarrow \mathbb{C} \sum N_i n_i - N_E \rightarrow 0.
\] (3.2)

This multibrane generalization can also be applied to a quiver containing a node $E$ and a set of nodes $G_i$ with arrows from $G_i$ to $E$, by reversing arrows in the previous argument.

This reflection functor allows us to analyze Seiberg duality in a general theory with nodes $B_a$ and no superpotential $W$. We pick one node, call it $E$, with $J$ incoming and $I$ outgoing arrows, and call the corresponding fields $\tilde{Q}^j$ and $Q^i$, where the indices run $1 \leq j \leq J$ and $1 \leq i \leq I$. We then just repeat the same arguments ignoring the fact that different arrows may terminate on different nodes $B_a$, to get a dual theory with $Q$ and $\tilde{Q}$ replaced by $q$, $\tilde{q}$ and $M$ and a cubic superpotential.

This leads to a quiver with nodes $\tilde{F}_i$, $\tilde{E}$, and dual quarks obtained by “reversing all the arrows” from $E$ to $F_i$. The arguments which led to (2.14) also go through straightforwardly to give an $I \times J$ matrix of mesons
\[
M^{ij} = Q^i \cdot \tilde{Q}^j
\]
satisfying
\[
\sum_j \tilde{q}_j M^{ij} = 0.
\] (3.3)

Continuing the line of reasoning above, we require dual quarks $q^i$ to enforce these relations, and we postulate the natural cubic superpotential.

### 3.2 An initial superpotential

We next generalize this argument to deal with a superpotential, say $W(\phi, Q, \tilde{Q})$ where $\phi$ are all the fields which are singlet under the $U(N_E)$ gauge group and not mentioned in the previous discussion.

The shortest way to do this is to note that the original F-flatness conditions pick out a subset of the original configurations, defined by equations
\[
0 = \frac{\partial W}{\partial \phi_\alpha} = F_\alpha(\phi, M) \tag{3.4}
\]
\[
0 = \frac{\partial W}{\partial Q^i} = \tilde{Q}^j G_{ij}(\phi, M) \tag{3.5}
\]
\[
0 = \frac{\partial W}{\partial \tilde{Q}^j} = G_{ij}(\phi, M) Q^i, \tag{3.6}
\]
where
\[
G_{ij} = \frac{\partial}{\partial M_{ij}} W|_{M^{\tilde{j}j} = Q^{\tilde{i}} i}.
\]

This would be the same as $\frac{\partial^2 W}{\partial Q^i \partial \tilde{Q}^j}$ except that we insist that the two matrices $Q$ and $\tilde{Q}$ appear in succession.

Let us first assume that $G_{ij}$ does not depend on $M_{ij}$, i.e. if $W$ is linear in $Q$ and $\tilde{Q}$, and return later to the general case. Given this assumption, we can argue separately for
each configuration of the fields $\phi$. In any particular configuration, $G_{ij}$ will take a definite value, which we now treat as fixed.

There are two general cases. If the matrix $G$ has rank greater than $N_f - N_c$, no choice of $Q$ and $\tilde{Q}$ can lead to a supersymmetric vacuum, because the number of remaining massless fermions is $N_f < N_c$. Without FI terms, this is a consequence of quantum effects. It also follows purely classically if we turn on a large FI term for all of the $Q$ or all of the $\tilde{Q}$.

This leaves us with rank $G \leq N_f - N_c$, which combined with a large FI term force either $Q_i$ or $\tilde{Q}_j$ to take maximal rank $N_c$. The relations $GQ = \tilde{Q}G = 0$ generally do not force $G = 0$.

We need to postulate a new superpotential which leads to the combination of the relations $F = 0$ and (3.3). This can be done by postulating the sum superpotential

$$W_{\text{new}} = W|_{Q,\tilde{Q}_j=\delta_{ij}} + \text{tr } Mq\tilde{q}$$

as in [8, 13, 29], which leads to $F = \frac{\partial W}{\partial \phi^\alpha} = 0$ and

$$\frac{\partial W}{\partial M_{ij}} = G_{ij} = q_j\tilde{q}_i; \quad (3.7)$$

$$Mq = 0; \quad (3.8)$$

$$\tilde{q}M = 0. \quad (3.9)$$

We now need to postulate a general map. Suppose the FI terms are positive and $Q$ is maximal rank (otherwise we reverse all the arrows). We again use (2.13) to determine $\tilde{q}$ from $Q$, which will have maximal rank, and (2.14) to determine $M$ from $\tilde{Q}$. As before, this clearly satisfies (3.9).

We then use (3.7) to determine $q$ from $\tilde{q}$ and $G$. This is an overdetermined system of equations, but given that $G$ has rank at most $N_f - N_c$ it will have a unique solution up to gauge invariance. The relation (3.8) then follows from $\tilde{Q} \cdot G = 0$.

Conversely, given $(G,q,\tilde{q},M)$ we determine $Q$ from $\tilde{q}$ and $\tilde{Q}$ from $M$ as before. $G \cdot Q = 0$ then follows from (3.7), while $\tilde{Q} \cdot G = 0$ follows from (3.8) and (3.7). We have not used $q$, but because of (3.7) this is redundant information if we know $G$.

Finally, since the map was one-to-one before putting the additional constraints, then given that the images in both directions satisfy the constraints, it must be one-to-one between the spaces satisfying the constraints.

This shows that the new superpotential reproduces the original moduli space. In some cases one can then integrate out fields to get a simpler theory with the same moduli space.

A special case of what we just discussed is the verification that the duality is an involution in the usual way, by taking $G_{ij}$ constant. This gives the electric theory with a mass term, and we conclude that its magnetic dual is

$$W_{\text{new}} = \text{tr } G_{ij}M_{ij} + Mq\tilde{q}$$
We can then dualize again, and eliminate the additional field, to arrive back at the original electric theory. The additional ingredient added by the above is the check that the map on configurations is the identity map.

We finally turn to the case in which the superpotential is not linear in $M$, and thus the matrix $G^{ij}$ depends on $M$. Actually, for the problem of finding classical supersymmetric vacua, this does not make any difference. What we have already proven is that we have a one-to-one equivalence between configurations $(G, Q, \tilde{Q})$ and configurations $(G, M, q, \tilde{q})$ for any $G$, in which $M = Q\tilde{Q}$. The effect of letting $G$ depend on $M$ is that supersymmetric vacua will be a subset of pairs $(G, M)$ defined by this relationship. This subset is the same on both sides in the obvious way, so the same map will be one-to-one in this case as well.

### 3.3 Adjoint matter

The previous arguments assumed that the nodes $F_i$ were all distinct from $E$, in other words that there was no adjoint matter associated to the node $E$, or $\text{Ext}(E, E) = 0$.

Suppose we have adjoint matter $\text{Ext}(E, E)$ with no superpotential, call it $\chi$. A physical way to think about this is that the brane $E$ now comes in a family with continuous parameters. Call these $E_\chi$ where $\chi$ can now be thought of as an eigenvalue of the field $\chi$.

If there is no superpotential, there will be bound states between the branes $F_i$ and an arbitrary number of $E_\chi$ with differing $\chi$. For example, we have the holomorphic configurations

$$Q = (1 \ 0 \ldots); \quad \chi \sim \begin{pmatrix} 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

for arbitrary rank matrices. These can also satisfy the D term constraints,

$$D_2 = Q^\dagger Q + [\chi^\dagger, \chi] > 0$$

So, there is no maximal bound state with which we can apply our arguments. At best, the style of argument we gave will lead to a theory with infinite gauge groups. Mathematically, this is not a quiver of finite representation type, meaning (by definition) that there are an infinite number of simple bound states.

This problem can be avoided by starting with a superpotential which only allows a finite number of $E_\chi$, or a finite number of bound states. A case discussed in the literature is to add the superpotential

$$W = \text{tr} \; P(\chi)$$

for some degree $k+1$ polynomial $P$ [15, 17]. Now there are $k$ supersymmetric configurations of the brane $E$, labelled by the roots $\chi_i$ of $P'(\chi) = 0$.

Let us now restrict attention to SQCD with this additional field; we again dualize the $B_2$ node (so $E = B_2$). Let $B_2^i$ with $1 \leq i \leq k$ be the $k$ different configurations of $B_2$. If
the roots $\chi_i$ are all distinct, we can handle this case by successively dualizing each of these branes along the same lines as before.

The new possibility is that some of the roots $\chi_i$ coincide. Let us take $P = \chi^{k+1}$ for definiteness. Now there are a series of bound states $E_t$ with charges $(0 \ t \ 1)$, with field configuration (3.10). Among these, one expects $E_k$ to be a maximal bound state, which can play the role of $B_5$ above.

Changing basis from $(B_1, B_2, B_3)$ to $(B_1, \bar{B}_2, B_5)$ now leads to $N'_2 = kN_3 - N_2$, the same as the rank of the dual theory found in [45, 46, 47].

We now need to derive the morphisms in the new basis. We should also verify that $B_5 = E_k$ is maximal. One could do this by hand as we did before, but at this point some mathematical technology will be welcome, so we postpone this to the next section.

4. Dirichlet brane realization

The previous arguments can be embedded into the discussion of D-branes in the weak string coupling limit, which provides a complete definition including branes and antibranes. Indeed, this is how they were found. We summarize the most relevant aspects of this theory from [15] (other discussions of D-brane categories can be found in [4, 17, 48]).

We consider type IIb D-branes embedded in a Calabi-Yau threefold $\mathcal{M}$ and extending in the $3 + 1$ Minkowski dimensions, such that their world-sheet theory has at most $\mathcal{N} = 1$ supersymmetry. We say at most because a generic collection of branes will break all supersymmetry. This should be thought of as a spontaneous breaking, and the (low energy) world-volume Lagrangian satisfies the constraints of $\mathcal{N} = 1$ supersymmetry.

In translating from geometry to the $\mathcal{N} = 1$ language, the holomorphic structure of the CY and brane embedding becomes holomorphic data (spectrum and superpotential), while Kähler data is related to gauge couplings, D-flatness conditions and FI terms. The discussion is best phrased in this two-step way, as we did in the previous section. In particular, most of the subtleties involving branes and antibranes only appear in this second step.

In world-sheet terms, the holomorphic structure of the theory is all visible in the topologically twisted open string theory. D-branes are boundary conditions in this theory. The open string states are precisely the massless Ramond states of the physical theory, while the physical superpotential is the generating function for their correlation functions. The explicit definitions are simplest in the B model in the large volume limit, in which boundary conditions are holomorphic bundles, massless Ramond states are Dolbeault cohomology, and the superpotential is the holomorphic Chern-Simons action. Other starting points such as $(2, 2)$ CFT or orbifold CFT can be used as well.

Independent of the starting point, the structure of open string theory allows a direct generalization to use complexes as defined in (2.8) as boundary conditions, by including the information of the $d$ operator in the BRST charge. One can then argue that all configurations related by brane-antibrane annihilation as in (2.9) are quasi-isomorphic, and that passing from the original configurations to their derived category incorporates all possible brane-antibrane bound states in a systematic way.
The choice of which of these objects are branes and which are antibranes then depends on Kähler moduli. The basic results relevant for us are the following. First, the holomorphic structure of an object is independent of Kähler moduli, and this will guarantee that we can take a specific configuration from one basis to another, which was the underlying basis of our argument.

Next, let us compare a brane $B$ extending in the $3+1$ Minkowski dimensions, with a brane in IIa theory wrapping the same cycle (and entirely the same in the CY), but which is a BPS particle in $3+1$ dimensions. This particle will have a central charge $Z(B)$ depending on Kähler moduli, and computable using mirror symmetry. A brane and its antibrane have $Z(B) = -Z(\bar{B})$. More generally problems in which all $Z(B_i)$ are roughly aligned in the complex plane can be treated using supersymmetric gauge theory of “branes,” while more general combinations involve brane-antibrane subtleties.

The same central charge enters into many physical quantities in the IIb world-volume theory. Let us first consider a collection of branes whose central charges are all close to real numbers (possibly after some overall phase rotation), we have

$$Z(B) = \frac{1}{g_B^2} + i\zeta_B$$

where $g_B$ is the Yang-Mills coupling for its $U(N)$ gauge group, and $\zeta_B$ is its FI term (and we work with $\alpha' = 1$). This result has numerous consequences. For example, the question of whether the boson partner to a massless fermion is tachyonic or massive, and thus whether the two branes form a bound state, is determined by the relative phase of the central charges.

The condition that central charges are almost real is not necessary; the more general result is

$$\frac{1}{g_B^2} = |Z(B)|; \quad \zeta_B = \varphi(B) = \frac{1}{\pi} \arg Z(B).$$  \hspace{1cm} (4.1)

We will use the notation $\varphi(B)$ or “grade” below as it is more precise (the relation to $\zeta_B$ is only precise in field theory), and explain how this enters our considerations.

Now, it is clear from the brane discussions \cite{26,25} that Seiberg duality provides a field theory description of the result of going through “infinite gauge coupling.” This corresponds to going through zero volume or $Z(B) = 0$ in the present discussion, which is singular in CFT and of course takes us out of weak string coupling.

Of course, because the central charge is complex, we can avoid this region, going around a point $Z(B) = 0$ on a path with $|Z(B)| >> 1$. In field theory, this amounts to turning on large FI terms, as in our previous discussion. The result takes (say) positive real $Z(B)$ to negative real $Z(B)$. To turn this into a problem involving only branes and thus describable by supersymmetric gauge theory, we instead use $\bar{B}$, which by the discussion we just made now counts as a brane: $Z(\bar{B}) > 0$. We then need to do one of the changes of basis discussed above to get a new basis whose objects have minimal $Z(B_i)$, and in terms of which the configuration is again a bound state of positively many constituents.

Note that in this discussion, there are two paths by which one can continue around $Z(B) = 0$, taking $\Im Z(B) > 0$ or taking $\Im Z(B) < 0$. These are distinct physical operations.
and correspond to the two changes of basis \((2,13)\) and \((2,10)\). Of course they lead to the same point in Kähler moduli space and thus the same dual theory. However they differ by a full loop around \(Z(B) = 0\), in other words by a monodromy in the Kähler moduli space. Such a monodromy relates different presentations of the same physical theory in terms of different bases of branes, and thus the set of physical configurations of the two theories must be precisely the same.

This should be contrasted with the Seiberg duality itself which does vary the Kähler moduli and thus need not preserve the entire spectrum of classical solutions of F-flatness and D-flatness. We concentrated on the subset of configurations which are preserved, but in general there are others which only exist on one side of the duality. The situation becomes more symmetric after considering quantum effects, as we discuss below.

In the language above, the Seiberg dualities arise on performing “partial” monodromies. As we discuss below (and as pointed out in \([13]\)), in the CY examples, these can be identified with mutations on the exceptional collection of bundles corresponding to the fractional branes.

### 4.1 D-brane categories

Let us consider two D-branes \(B_1\) and \(B_2\) in a smooth Calabi-Yau manifold \(X\) which are BPS in the large volume limit. The simplest case to start with is branes \(B_1\) and \(B_2\) wrapping \(X\), with a holomorphic vector bundle on its worldvolume. The spectrum of massless fermionic strings stretching between branes \(B_1\) and \(B_2\) is then given by elements of Dolbeault cohomology, \(H^{0,q}(X, B_1^* \otimes B_2)\). These are just holomorphic \(q\)-forms, which can be multiplied (by wedge product).

We can next add to this branes wrapping lower dimensional cycles carrying bundles. However the set of these does not form an abelian category, because kernels and cokernels of maps between bundles are not necessarily bundles. Rather, a more general and algebraic description is that the branes \(B_i\) correspond to coherent sheaves on \(X\). This is a standard mathematical construction (e.g. see \([36, 34]\)) which includes the original bundles (as locally free sheaves of sections of bundles), bundles with singularities, and objects supported on lower dimensional holomorphic cycles. The spectrum of massless fermionic strings stretching between branes \(B_1\) and \(B_2\) is then given by elements of \(\text{Ext}^q(B_1, B_2)\).

The coherent sheaves form an abelian category and in this sense are a “complete” set of objects which one expects to appear as branes at large volume. Superficially, this looks very different from the abelian categories of quiver representations discussed earlier. On a deeper level, they are very similar. Indeed, one can find quiver categories which “represent” them in the sense that large subcategories of the two are literally the same, the original example being \([7]\). We will come back to this point later as it is quite relevant.

The further discussion relies on properties of the world-sheet \(N = 2\) algebra. Each open string has a \(U(1)\) charge \(Q\) (in the topologically twisted string, this is the ghost charge). For branes which are bundles on \(X\), this is just \(q\) in \(H^{0,q}\). the physical string, this sector is obtained by acting with \(q\) fermions \(\psi^0_i\) on the ground state. On its NS partner state, these are \(\psi^i_{-1/2}\), so the mass of the NS state is \(m^2_{NS} = (q - 1)/2\).
We also have to remember that in the physical theory we need to impose the GSO projection, and that there are also fermionic oscillators transverse to the Calabi-Yau manifold. At this point we can distinguish branes and antibranes. For a pair of branes, elements of $\text{Ext}^1(B,B)$ give rise to massless matter. Geometrically, these deform the connection of the associated holomorphic bundle, and explore the moduli space of coherent sheaves. Elements of $\text{Ext}^0 = \Gamma(\text{Hom}(B,B))$ are global holomorphic sections of the sheaf. To get a physical vertex operator, the GSO projection forces us to act with an additional oscillator transverse to the Calabi-Yau, so these give rise to vector particles in $\mathbb{R}^{3,1}$, with $m^2 = q/2$.

For a brane and antibrane, the GSO projection is reversed. Thus an $\text{Ext}^0(B_1,B_2)$ becomes the tachyon between a brane and anti-branes (this state is projected out by the GSO projection for a brane), now with $m^2 = (q - 1)/2 = -1/2$ for $q = 0$.

The category of sheaves also has the Serre duality functor, which on a Calabi-Yau $d$-fold takes the form

$$\text{Ext}^q(A,B) = \text{Ext}^{d-q}(B,A)^*. \quad (4.2)$$

In each open string sector, the GSO projection will eliminate half of the states, either the even or odd $q$. Since for us $d = 3$ is odd, we can implement this by considering both $\text{Ext}(A,B)$ and $\text{Ext}(B,A)$, but restricting attention to the states with $q < d/2$.

We represent the resulting chiral matter spectrum in a quiver diagram as follows. Each brane is represented by a node. Between two branes $A$ and $B$, we have an arrow $A \rightarrow B$ for each basis vector of $\text{Ext}^\text{odd}(A,B)$, and an arrow from $B$ to $A$ for each basis vector of $\text{Ext}^\text{even}(A,B)$.

All this assumed that $U(1)$ charges are integral. However, a general combination of branes, breaking supersymmetry, will have open strings with non-integral $q$. This is true even in the large volume limit, as can be seen by considering the system of a $Dp$ and $Dp-2$-brane. This is because $q$ and $m^2$ receive a correction related to the NS ground state energy (for example, two branes with real relative dimension $n$, i.e. $n$ “DN” boundary conditions, have a vacuum with $m^2 = n/4 - 1/2$). This effect can be summarized in terms of the gradings of the branes, $\varphi_i = \frac{1}{\pi} \arg(Z(D_i))$: one then has $q = q_0 + \varphi_2 - \varphi_1$, where $q_0$ is the grading of the $\text{Ext}^q$ in the standard definitions.

We will mostly be concerned with field theory limits, in which $q$ is always near an integer, but let us discuss the general case. The formula $m_{NS}^2 = (Q - 1)/2$ still applies; for example, the $Dp-Dp-2$ tachyon has $m^2 = -1/4$, and $q = 1/2$. In general, the $U(1)$ charges at a given point in Kähler moduli space can be obtained by “flow of gradings.” Between a point $x$ with gradings $\varphi_x$ and another point $y$ with gradings $\varphi_y$, writing $\Delta \varphi(B) = \varphi_y(B) - \varphi_x(B)$, the charges are related as

$$\text{Ext}^q(B_1,B_2) \rightarrow \text{Ext}^{q + \Delta \varphi(B_2) - \Delta \varphi(B_1)}(B_1,B_2).$$

The GSO projection does not vary along such a flow. A more convenient notation for the same thing is

$$\text{Hom}(B_1,B_2[q]) \rightarrow \text{Hom}(B_1[\Delta \varphi(B_1)],B_2[\Delta \varphi(B_2)]).$$
This leads to the corresponding variation
\[ m^2 \rightarrow m^2 + \Delta \varphi(B_2) - \Delta \varphi(B_1) \]
(just as for varying FI terms) and this can in general lead to \( m^2 < 0 \) if the phases \( \varphi(B_i) \) vary appropriately. Therefore, via motion in moduli space, a massive field can become massless or tachyonic, which can force one collection of D-branes to condense into a bound state.

We will use this to infer the result of a flow in moduli space along which a brane “turns into an antibrane,” more precisely along which \( Z(E) \) goes from positive real to negative real. As we discussed, this can happen in two ways, which take \( \varphi(E) \rightarrow \varphi(E) + 1 \) or \( \varphi(E) \rightarrow \varphi(E) - 1 \). Let us consider the first; then matter \( \text{Ext}^1(B, E) \) turns into “partial gauge invariances” \( \text{Hom}(E, B_i) \), which are somewhat problematic from the point of view of gauge theory. The antibrane \( \bar{E} \) has spectrum obtained by reversing the GSO projection; the same maps are interpreted as tachyonic matter \( \text{Hom}(\bar{E}, B_i) \), which in some sense lead to the formation of bound states which make up the new basis.

In the new basis, the shift takes \( \text{Hom}(B_i', E) \) to \( \text{Ext}^1(B_i', \bar{E}) \), and shows that this dual quark is a massless chiral multiplet. Similarly the original \( \text{Ext}^1(B, E) \) for a brane \( B \) not transformed under the duality becomes \( \text{Ext}^2(B, \bar{E}) \), which is massive; its Serre dual \( \text{Ext}^1(E, B) \) would become a massive vector, but its antibrane version \( \text{Ext}^1(\bar{E}, B) \) is a chiral multiplet, the other dual quark.

### 4.2 D-branes and quiver categories

Given two branes \( B_1 \) and \( B_2 \) defined as sheaves, quiver representations, or whatever, the \( \text{Ext}^1(B_2, B_1) \) group classifies all of the branes which can be defined via short exact sequences
\[
0 \rightarrow B_2 \xrightarrow{f} B \xrightarrow{g} B_1 \rightarrow 0 \tag{4.3}
\]
We think of \( B \) as a deformation of \( B_2 \oplus B_1 \), given by condensing a chosen field in \( \text{Ext}^1(B_1, B_2) \). The arrows here are \( f \in \text{Hom}(B_2, B) \) and \( g \in \text{Hom}(B, B_1) \). They reflect the fact that \( B_1 \) and \( B_2 \) are constituents of \( B \), and that this can be seen in the appearance of “partial gauge invariances.”

In section 2 we implicitly used exact sequences as above for our computations. Indeed, the results presented there can be rewritten in this language, where the procedure becomes more natural. The main computational tool in this description is the long exact sequence in cohomology
\[
\ldots 0 \rightarrow \text{Ext}^0(M, B_2) \rightarrow \text{Ext}^0(M, B) \rightarrow \text{Ext}^0(M, B_1) \rightarrow \text{Ext}^1(M, B_2) \rightarrow \text{Ext}^1(M, B) \rightarrow \text{Ext}^1(M, B_1) \ldots . \tag{4.4}
\]
This may be familiar to the reader from algebraic topology, and as an abstract statement it works the same way here [51]. There is a similar sequence for \( \text{Ext}(B, M) \) obtained by reversing arrows.
\[
\ldots 0 \rightarrow \text{Ext}^0(B_1, M) \rightarrow \text{Ext}^0(B, M) \rightarrow \text{Ext}^0(B_2, M) \rightarrow \text{Ext}^1(B_1, M) \rightarrow \text{Ext}^1(B, M) \rightarrow \text{Ext}^1(B_2, M) \ldots . \tag{4.5}
\]
We are going to use this shortly, but first let us remark on the underlying definitions. The overall context of this section is B-type brane realizations in CY, for which the underlying definition of the abelian category is sheaf cohomology. The category of coherent sheaves is an abelian category with Serre duality, and the “decoupling statement” of [12] (or general considerations of topological open string theory, if one likes) tells us that this is the abelian category containing the bundles we started with.

One can make the following arguments precise by choosing a specific geometry and wrapping D-branes to realize the following quiver theories on their world-volumes; the Ext groups we discuss are then those of sheaf cohomology. To be a bit more precise, suppose we are interested in a quiver theory with superpotential, call it $T$. Let Coh$M$ be the category of coherent sheaves on a Calabi-Yau $M$. We choose some $M$ such that there is some full subcategory $C(T) \subset$ Coh$M$ in which the branes of our discussion are contained, meaning that the relevant physical computations (e.g. spectrum and superpotential) reproduce $T$. Then, we are working in $C(T)$. Of course there might be many $C(T)$’s which could play this role.

We are not going to make this step explicit, because our philosophy is somewhat different. Rather, a quiver theory $T$ by itself gives rise to an abelian category, call this Cat$T$. For the theories that arise from geometry, this will be the same as the geometric category $C(T)$, no matter what geometry we derive it from. A particular quiver theory might $a$ $p$riori not even have a geometric realization. It is this category Cat$T$ which we have in mind in the following discussion. If we believe this claim, then the dualities we are discussing again do not follow from string theory or even geometry, but are simply properties of supersymmetric field theory.

We will explain this point in section 5. If one is familiar with the geometric approach, one can keep a specific $C(T)$ in mind in reading the following arguments. If one is familiar with quivers, one may be bothered by the fact that these categories do not in general have Serre duality. However, we will show in section 5 that if the relations follow from a superpotential, then under certain assumptions the category will have Serre duality.

### 4.3 Seiberg duality via long exact sequences

We now return to the theory $(N_1, N_2, N_3)$ of section 2. We recall that $\dim(\text{Ext}^1(B_2, B_3)) = \dim(\text{Ext}^1(B_1, B_2)) = 1$, and $\dim(\text{Hom}(B_i, B_j)) = \dim(\text{Ext}^0(B_i, B_j)) = \delta_{ij}$.

The bound state $B_5$ between branes $B_2, B_3$ is defined by a short exact sequence $0 \to B_3 \to B_5 \to B_2 \to 0$. Since there is only one matter field, $B_5$ has no moduli; in other words there is a unique non-trivial extension.

Now, let us calculate the spectrum between brane $B_1$ and brane $B_5$. Chasing the long
exact sequence in homology we find

\[ 0 \rightarrow 0 \rightarrow \text{Hom}(B_1, B_5) \rightarrow 0 \rightarrow \]

\[ 0 \rightarrow \text{Ext}^1(B_1, B_5) \rightarrow \text{Ext}^1(B_1, B_2) \rightarrow \]

\[ 0 \rightarrow \text{Ext}^2(B_1, B_5) \rightarrow 0 \rightarrow \]

\[ 0 \rightarrow \text{Ext}^3(B_1, B_5) \rightarrow 0 \rightarrow \]

So we find that \( \dim(\text{Ext}^1(B_1, B_5)) = 1 \), and that it is derived from \( \tilde{Q} \in \text{Ext}(B_1, B_2) \). This is of course the meson \( M \) in the dual theory.

To get \( \text{Ext}^p(B_5, B_2) \), we can use the long exact sequence

\[ 0 \rightarrow \text{Hom}(B_2, B_2) \rightarrow \text{Hom}(B_5, B_2) \rightarrow 0 \rightarrow \]

\[ 0 \rightarrow \text{Ext}^1(B_5, B_2) \rightarrow 0 \rightarrow \]

\[ 0 \rightarrow \text{Ext}^2(B_5, B_2) \rightarrow \text{Ext}^2(B_3, B_2) \rightarrow \]

\[ \text{Ext}^3(B_2, B_2) \rightarrow \text{Ext}^3(B_5, B_2) \rightarrow 0 \rightarrow \]

We want to see that the nature of the extension is such that there is a connecting homomorphism between lines 3 and 4, so we find that in the end the only non-trivial ext group is \( \text{Hom}(B_5, B_2) \approx \text{Hom}(B_2, B_2) \).

This is most easily seen with the dual exact sequence

\[ 0 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \text{Hom}(B_2, B_2) \rightarrow . \]

\[ \text{Ext}^1(B_2, B_3) \rightarrow \text{Ext}^1(B_2, B_5) \rightarrow 0 \rightarrow \]

\[ 0 \rightarrow \text{Ext}^2(B_2, B_5) \rightarrow 0 \rightarrow \]

\[ 0 \rightarrow \text{Ext}^3(B_2, B_5) \rightarrow \text{Ext}^3(B_2, B_2) \]

Actually both of these sequences contain the same information, all the groups are related by Serre duality. However, in this second exact sequence we know that we are using a
non-trivial (canonical) extension which identifies $\text{Hom}(B_2, B_2) \sim \text{Ext}^1(B_2, B_3)$. This is in accordance with the field theory: giving a vev to a field reduces the number of massless degrees of freedom.

In the bound state $B_5$, the connecting map pairs $\text{Ext}^1(B_2, B_3)$ with $\text{Hom}(B_2, B_2)$, leaving $\text{Ext}^1(B_2, B_5) = 0$. This is really the same argument we gave in section 2, that the extra matter is Higgsed in this case. If one had considered the trivial extension $B_2 \oplus B_3$, the connecting map would have been zero, and both gauge and matter fields would remain massless.

Dualizing everything, one finds that $\dim \text{Hom}(B_5, B_2) = 1$, as before. In fact $\text{Hom}(B_5, B_2) \cong \text{Hom}(B_2, B_2)$.

Now we use this to obtain the morphisms to the antibrane of $B_2$, defined by shifting the ghost charge by $\pm 1$. By the general principle that gradings are non-negative, we must take $\bar{B}_2 = B_2[-1]$, i.e. shift its ghost charge by $-1$. Thus this morphism is also an $\text{Ext}^1(B_5, \bar{B}_2)$. This should be drawn as an arrow beginning at $B_5$ and ending at $B_2$, so we see that we reverse the arrows in the quiver. This shift also relates $\text{Ext}^1(B_1, B_2)$ to $\text{Ext}^2(B_1, \bar{B}_2)$ which is dual to $\text{Ext}^1(\bar{B}_2, B_1)$.

In terms of variation of Kähler moduli, this shift is induced by varying the central charge $Z(B_2) \sim e^{i\pi \varphi}$ from $\varphi = 1$ to $\varphi = 0$ above $Z = 0$. The path below $Z = 0$ would produce $\bar{B}_2 = B_2[1]$ and we would need to use $B_1$ instead of $B_5$ to get a sensible basis.

For completeness, we should check that $B_5$ is simple, i.e. $\text{Ext}^1(B_5, B_5) = 0$. This can be done by a succession of exact sequences, but follows more simply from the original gauge theory definitions.

Let us now consider a slightly more general case, with three nodes $B_1, B_2, B_3$, but now with $n_{12}$ arrows from node $B_1$ to $B_2$ and $n_{23}$ arrows departing from node $B_2$ to node $B_3$, as shown in the figure 3.

![Figure 3: Quiver with more arrows](image)

Again, the basic duality will be obtained by taking $Z(B_2) \rightarrow -Z(B_2)$. We then want to go to a new basis including $\bar{B}_2$ and a supersymmetric bound state $B_5$ with $(0, m, 1)$ with $m$ as large as possible. This could be justified by asking for branes with minimal central charge, or equivalently asking that all central charges can be realized as positive combinations of those from the basis.

Clearly $B_5 = (0, n_{23}, 1)$ as for larger $m$ there are not enough matter fields to break $U(m)$ gauge symmetry. The same constraint arises by asking to solve the D-flatness conditions with large FI terms.

There is a unique bound state of this form, given by the tautological exact sequence

$$0 \rightarrow \text{Ext}^1(B_3, B_2)^* \otimes B_2 \rightarrow B_5 \rightarrow B_3 \rightarrow 0.$$  \hfill (4.11)
In components, we set the fields \( Q^i_\alpha = \delta^i_\alpha \), where \( 1 \leq i \leq n_{23} \) is the flavor index and \( \alpha \) is the gauge index.

Chasing diagrams, we find that \( \dim(\text{Ext}^1(B_5, B_1)) = n_{12}n_{23} \), and that all other arrows are reversed with respect to the original diagram.

The same ideas can be pushed to handle the general theory with bifundamentals and no superpotential, reproducing the results for dualizing a node. The main point where input about the specific branes enters is in the connecting homomorphisms. As we saw, the maps \( \text{Hom} \rightarrow \text{Ext}^1 \) and the dual \( \text{Ext}^2 \rightarrow \text{Ext}^3 \) correspond to the Higgs mechanism as giving mass to the ‘extra’ quark field.

### 4.4 Connecting maps from the superpotential

The connecting maps \( \text{Ext}^1 \rightarrow \text{Ext}^2 \) correspond to lifting matter by the superpotential. As a simple illustration, we check that applying the duality twice returns us to the same theory.

Actually, we had two derivations of the duality, distinguished by whether \( B_2 \) bound with \( B_3 \) or \( B_1 \). As we explained, these were associated to the shifts \( B_2 \rightarrow B_2[-1] \) and \( B_2 \rightarrow B_2[1] \), respectively. The more interesting case is to shift again in the same direction, say to \( B_2[-2] \), which turns the half monodromy into a full monodromy. This is done by changing to a basis with a brane \( B_7 \) defined by the exact sequence \( B_1 \rightarrow B_7 \rightarrow B_2[-1] \), while we keep brane \( B_5 \). At the level of brane charge we have that \( [B_2] \rightarrow [B_2] \), \( [B_3] \rightarrow [B_3] + [B_2] \), and \( [B_1] \rightarrow [B_1] - [B_2] \).

The nontrivial part of this test is that the dual mesons must be lifted in the double dual theory. Also, the dual theory has a superpotential which affects the massless spectrum of the theory when we give vevs. This can be seen by considering the arrows between brane \( B_5 \) and brane \( B_7 \). This gives us

\[
\begin{align*}
0 &\rightarrow 0 \rightarrow \text{Hom}(B_5, B_7) \rightarrow 0 \rightarrow \\
0 &\rightarrow \text{Ext}^1(B_5, B_7) \rightarrow \text{Ext}^1(B_5, B_2[-1]) \rightarrow \\
\text{Ext}^2(B_5, B_1) &\rightarrow \text{Ext}^2(B_5, B_7) \rightarrow 0 \rightarrow \\
0 &\rightarrow \text{Ext}^3(B_5, B_7) \rightarrow 0
\end{align*}
\]

If things worked in parallel to the first duality, the dual quarks \( \text{Ext}^1(B_5, B_2[-1]) \) would give rise to doubly dual mesons \( \text{Ext}^1(B_5, B_7) \). However, the dual theory has nonzero \( \text{Ext}^2(B_5, B_1) \), the Serre dual to its mesons, and \( \text{Ext}^1(B_5, B_2[-1]) \) instead pairs through the connecting map to these. Thus there are no dual dual mesons.

In fact, we recover the original quiver diagram, but with basis a different set of branes. In mathematical terms, dualizing twice is an autoequivalence of the derived category.
4.5 Generating the superpotential

The previous argument was nice, but it would be better to just derive the dual superpotential. It turns out that this can be done directly if we have a realization of our quiver category which has Serre duality.

If we have realized our branes as B-type D-branes in a CY, we can base this on the standard result that the superpotential is the holomorphic Chern-Simons action \[57\],

\[
W = \int \Omega \wedge (\bar{A} \bar{\partial} \bar{A} + \frac{2}{3} \bar{A}^3). \tag{4.13}
\]

For holomorphic forms \(\bar{A}\), the derivative term can be dropped.

In the notations we are using now, we regard \(A\) as a sum of elements in \(\text{Ext}^1(A, B)\) for all \(A\) and \(B\). Since these are morphisms in a category, we know how to multiply them. We then write

\[
W = \text{tr} \ A^3
\]

where \(\text{tr} \ \text{Ext}^3(A, A) = 1\) for all \(A\) and is zero on anything else. This formula should be interpreted in the following sense: \(\text{Ext}^3(A, A) = \text{Hom}(A, A)^*\). Now, in \(\text{Hom}(A, A)\) we have the identity homomorphism \(1_A\), so we should interpret \(\text{tr} (a)\) for \(a \in \text{Ext}^3(A, A)\) as given by the canonical pairing \(\text{Hom}(A, A) \times \text{Hom}(A, A)^* \to \mathbb{C}, \ <1_A, a>\), which is a complex number.

We then need to adapt this to a system of branes and antibranes. Actually, if we identify an antibrane as a brane with shifted grading, such that the morphisms are the same with shifted grading, then the same formula applies, since the superpotential is independent of these shifts (they only affect Kähler moduli). The only generalization we need is to allow \(\bar{A}\) to be an arbitrary sum of \(p\)-forms. (See \([56, 17]\) for (4.13) written out explicitly this way, but it is just standard mathematical formalism.) So, if we had a term involving three \(\text{Ext}^1\)'s like

\[
\int \bar{A}_{1,2} \wedge A_{2,3} \wedge A_{3,1},
\]

shifting the grade of \(B_2\) produces a term \(\text{Ext}^2 \times \text{Hom} \times \text{Ext}^1\), etc.\(^4\)

The most direct application of this is to derive cubic terms involving a brane-antibrane tachyon. We work with our SQCD quiver but now include the antibrane to \(B_2\), defined \(\bar{B}_2 = B_2[-1]\), as in figure [3].

The quiver contains two tachyons, which are shifts of \(\text{Ext}^{0,3}(B_2, B_2)\). One is \(T \in \text{Ext}^1(B_2, B_2[-1])\), and the other is \(\bar{T} \in \text{Ext}^{-1}(B_2[-1], B_2)\).

We also include the fields \(Q_0 \in \text{Ext}^3(B_3, B_2[-1])\) (the shift of the Serre dual of \(Q\), \(Q^* \in \text{Ext}^2(B_3, B_2)\). It’s dual is \(Q_3\) in the figure, keeping with the convention that we only have \(\text{Ext}^{odd}\), and \(\bar{Q}_2 \in \text{Ext}^1(B_2[-1], B_1)\), the shift of the Serre dual \(\bar{Q}^* \in \text{Ext}^2(B_2, B_1)\).

\(^4\)There is a slightly confusing point here which is that the coefficients stay the same, which means that we can violate the naive sign conventions. For example, if this product started out antisymmetric (as one would expect for one-forms), it stays antisymmetric when shifted to even forms. This is not contradicting anything as the algebra is not commutative or supercommutative in general, but should be kept in mind.
Figure 4: Quiver with antibranes. The ghost charges of the arrows are explicitly written as a subindex.

By Serre duality, the original superpotential included the terms \( \text{tr } Q \cdot 1 \cdot Q^* \) and \( \text{tr } \tilde{Q}^* \cdot 1 \cdot \tilde{Q} \). The shift turns these into

\[
W = \text{tr } Q \tilde{T} Q_3 + \tilde{Q}_1^* T \tilde{Q}_1.
\]

with the notation as in figure 4.

Since the “tachyon” is just obtained by shifting the unit operator, it enters in a canonical way.

Looking back at our derivation of the dual spectrum, we can identify \( \tilde{q} \) as a tachyon between \( \bar{B}_2 \) and the \( B_2 \) constituent of \( B_5 \), and \( M \) as a string from \( B_1 \) to the \( B_2 \) constituent of \( B_5 \), leaving \( q \) as its “Serre dual” and justifying the identification of one of these cubic terms with our dual superpotential.

This argument can be made a little more directly by just computing

\[
\text{tr } \text{Ext}^1(B_1, B_5) \cdot \text{Hom}(B_5, B_2) \cdot \text{Ext}^2(B_2, B_1)
\]

as a product of linear transformations between the original quiver representations. Tracing through the definitions, one sees that

\[
\text{Ext}^1(B_1, B_5) \cdot \text{Hom}(B_5, B_2) \in \text{Ext}^1(B_1, B_2),
\]

is nonzero (it is the original defining Ext). By definition this has a nonzero trace with its Serre dual. Shifting leads directly to the superpotential in the dual theory.

To repeat, understanding Serre duality is the key to getting the superpotential in these arguments. We will argue in section 5 that even this property of branes on CY is not necessarily geometric, but is a more general property of quiver gauge theories.

4.6 Dualizing theories with adjoints

We are now prepared to consider the situation where we take the original example we started with and we add an adjoint superfield for node \( B_2 \). Here we expect new phenomena, based on the results obtained by [45, 47], who found a nontrivial relation between the original and dual superpotentials.

As discussed earlier, we need a superpotential, and we use \( W = \frac{1}{k+1} \text{tr } (\chi^{k+1}) \), which forces \( \chi^k = 0 \). We then described a candidate maximal bound state \( B_5 \), with \( |B_5| = \)
\[ k[B_2] + [B_3], \] as an explicit field configuration. This configuration will be the bound state of \( B_3 \) with the maximum possible number of branes of type \( B_2 \).

In the language of exact sequences, \( B_5 \) can be built by a sequence of extensions, successively producing the bound states \( E_t \) with charges \((0 \ t \ 1)\), starting with

\[ 0 \to B_2 \to E_1 \to B_3 \to 0. \quad (4.14) \]

The exact sequences we used earlier now show that \( \dim \text{Hom}(B_2, E_1) = 1 \) as before, but also \( \dim(\text{Ext}^{1,2,3}(E_1, B_2)) = 1 \). This is just the statement that the brane \( B_2 \) inside \( E_1 \) has an adjoint which does not get lifted when we give a vev to \( Q \). In the exact sequence this is read from \( \text{Ext}^{1,2}(E_1, B_2) \simeq \text{Ext}^{1,2}(B_2, B_2) \).

The best way to think about this calculation is by considering the system as describing the arrows between branes \((0, 1, 1)\) and \((0, 1, 0)\); so it arises from the total brane system \((0, 2, 1)\). When we turn on the vev for the quark field, it breaks the \( U(2) \times U(1) \) symmetry to a \( U(1) \times U(1) \) symmetry. We have an action of the \( U(2) \) on a vector space of dimension 2, and let us label it’s basis as \( e_1, e_2 \). We choose \( e_1 \) so that it spans the image of the quark field when considered as a map between the vector space associated to brane \( B_3 \), and the one associated to \( B_2 \). A basis of \( \text{Ext}^{1,2}(B_5, B_2) \) counts massless fields transforming as bifundamentals between the gauge groups of the bound state brane \( B_5 \) and \( B_2 \). These are the off-diagonal matrix elements of the field \( \chi \). One can not turn on the element \( \chi_{11} \) because the superpotential forbids it. However this field is not massive. The obstruction is at a higher order. We will choose now to set \( \chi_{ii} = 0 \) for all diagonal elements (eventually all of these fields will become massive, but for the time being this follows from enforcing the equations of motion \( \chi^k = 0 \)).

We can now continue, with either \( \text{Ext}^1(E_1, B_2) \) or \( \text{Ext}^1(B_2, E_1) \). This amounts to the choice of which matrix element of \( \chi \) we turn on, \( \chi_{1,2} \) or \( \chi_{2,1} \). The choice we take is determined by the structure of the D-terms, as these two will contribute with opposite signs. In our conventions we will turn on \( \chi_{2,1} \). The way extensions are made successively is determined by the first step \( 4.14 \), we need to keep extending with brane \( B_2 \) always on the same side of the exact sequence. In any case, we will have an extension (either to the left or right) as follows

\[ 0 \to B_2 \to E_t \to E_{t-1} \to 0. \quad (4.15) \]

Since \( \text{Ext}^1(E_1, B_2) \neq 0 \), we can find a non-trivial canonical extension \( 0E_2 \to E_1 \to B_2 \to 0 \), which has charge \( B_3 + 2B_2 \). In general we can build successively \( E_t \) via a sequence when \( \dim(\text{Ext}(E_t, B_2)) \neq 0 \). As we have argued already, chasing the exact sequences will take care of both the Higgs mechanism, and of terms that appear in the superpotential. It can be proven that for general \( E_t \) with \( t < k \), that \( \dim(\text{Ext}^{1,2,3}(E_t, B_2)) = 1 \). The exact
sequence will look like

\[ 0 \longrightarrow \text{Hom}(E_t, B_2) \longrightarrow \text{Hom}(B_2, B_2) \rightarrow \]

\[ \begin{array}{c}
\text{Ext}^1(E_{t-1}, B_2) \\
\text{Ext}^2(E_{t-1}, B_2) \\
\text{Ext}^3(E_{t-1}, B_2)
\end{array} \longrightarrow \begin{array}{c}
\text{Ext}^1(E_t, B_2) \\
\text{Ext}^2(E_t, B_2) \\
\text{Ext}^3(E_t, B_2)
\end{array} \longrightarrow \begin{array}{c}
\text{Ext}^1(B_2, B_2) \\
\text{Ext}^2(B_2, B_2) \\
\text{Ext}^3(B_2, B_2)
\end{array} \rightarrow \]

If we write just the dimension of the vector spaces involved we get

\[ 0 \longrightarrow \text{Hom}(E_t, B_2) \longrightarrow 1 \rightarrow \]

\[ 1 \longrightarrow \text{Ext}^1(E_t, B_2) \longrightarrow 1 \rightarrow \]

\[ 1 \longrightarrow \text{Ext}^2(E_t, B_2) \longrightarrow 1 \rightarrow \]

\[ 1 \longrightarrow \text{Ext}^3(E_t, B_2) \longrightarrow 1 \]

The canonical extension provides us with a connecting homomorphism between lines one and two, which can be traced to the higgs mechanism. This should also apply between lines three and four, because \( \text{Ext}^3(E_t, B) \) comes from the vector fields associated to the \( B_2 \) branes inside \( E_t \). However, we can only get a connecting homomorphism between lines two and three if they arise from the superpotential of the theory.

In terms of a basis, we need to consider for \( E_t \) the theory \((0, t - 1, 1) + (0, 1, 0)\). The extra field takes the basis \( e_1, e_2, \ldots, e_{t-1} \) and adds one extra vector. Now we need to turn on an extension, which corresponds to the \( \chi_{t,0} \) components, as these are the fields that transform as bifundamentals of the unbroken gauge group for the \( E_t + B_2 \) brane configuration. However, for \( \alpha < t - 1 \) these can be gauged away, so the only extension up to gauge invariance is given by the field \( \chi_{t,t-1} \) which we can set equal to one. Notice how the extension is providing the field configurations written in equation (3.10).

The sequence above tells us that for \( E_t \) there is one degree of freedom coming from \( \chi \) which does not participate in the higgs mechanism. We need to know when this particular degree of freedom needs to be integrated out. Each time we perform a step, we add a column and a row to the matrix \( \chi \). The quadratic spectrum of fluctuations receives a mass term \( \text{tr}(\chi^{k+1}) \sim \text{tr}(<\chi>^{k-1} \delta \chi \delta \chi) \) exactly when we reach the \( k \)-th step in the extension. One can see that this happens because a further extension will give us a matrix \( \chi \) such that \( \chi^k \neq 0 \), thus one would not be able to satisfy the F-term constraints and then the
associated field is massive. At the $k$-th step we find that $\text{Ext}^{1,2}(E_k, B_2) = 0$, and it is not possible to do any further extensions; we have arrived at a situation where the only non-trivial Ext group is $\text{Ext}^{3}(E_k, B_2)$.

When we use this particular D-brane and shift to obtain $\bar{B}_2$, we see that in the dual quiver the arrows between branes $B_5$ and $B_2$ are reversed with respect to those of the original quiver, between branes $B_2$ and $B_3$. The same is true for the arrows between $B_2$ and $B_1$. Again, $B_1$ can not be extended with the choice of D-terms we made, because the quarks contribute with opposite sign to the D-term constraint, and it is not possible to get a positive definite matrix. This statement is reflected in the exact sequences because we can not extend $B_1$ to the left with copies of $B_2$, however we can extend it to the right.

Now, let us analyze the mesons. We want to use the exact sequence for $E_t$ as follows

$$
\begin{align*}
0 & \longrightarrow \text{Hom}(E_t, B_1) \longrightarrow 0 \\
\text{Ext}^1(E_{t-1}, B_1) & \longrightarrow \text{Ext}^1(E_t, B_1) \longrightarrow \text{Ext}^1(B_2, B_1) \\
\text{Ext}^2(E_{t-1}, B_2) & \longrightarrow \text{Ext}^2(E_t, B_2) \longrightarrow 0 \\
\text{Ext}^3(E_{t-1}, B_2) & \longrightarrow \text{Ext}^3(E_t, B_2) \longrightarrow 0
\end{align*}
$$

From here it is easy to show by induction that $\text{Ext}^{0,2,3}(E_t, B_1) = 0$, and that $\text{dim}(\text{Ext}^1(E_t, B_1)) = t$. We find that the dual quiver has $k$ meson arrows corresponding to $\text{Ext}^1(B_5, B_1) = \text{Ext}^1(E_k, B_1)$. This is exactly the matter content predicted by Kutasov [13]. It is very clear that we can tie the mesons to a particular step in the extension, so this determines their index structure with respect to the basis $e_1, \ldots, e_t$.

For completeness we also need to verify that the brane $B_5$ is rigid, namely , that there are no massless adjoints available for brane $B_5$. This is most easily done in the field theory and will be left as an exercise for the reader.

Now let us analyze the superpotential of the dual theory. As we have seen in subsection 4.3, this is best done by examining the original theory in a quiver with branes and antibranes included simultaneously. For this, we will consider a theory of branes $B_1 + B_2 + B_5$, where $B_5$ is written in terms of the original branes, $kB_2 + B_3$.

The new ingredient in the construction is that we have extra arrows between the branes and the antibranes, which arise from shifting $\text{Ext}^{1,2}(B_2, B_2)$. These are depicted in the figure 5.

The new fields to consider are the field $Y_{1,3}$ and $\tilde{\chi}$. One can argue via the graded Chern Simons theory that the term $\text{tr} \left( \chi^{k+1} \right)$ needs to be completed to a graded version of this term, which includes a term of the form $\text{tr} \left( \tilde{\chi}^{k+1} \right)$. This is a term expected in the superpotential of the dual theory.
Similarly we find that there are terms in the potential required by tachyon condensation which are of the form
\[ \text{tr} \left( \chi_1 Y_1 T_1 + T_1 Y_1 \tilde{\chi}_1 + \tilde{Q}^*_1 T_1 \tilde{Q}_1 \right) \] (4.19)

In the dual theory we know that most of these fields are not part of the spectrum. Some are eaten by the Higgs mechanism, and some others are eaten by masses generated in the superpotential. From this point of view, it is important to notice that when we give a vev to \( \chi_1 \), we generate mass terms for \( T_1 \) and \( Y_1 \) jointly. Not all of the terms of \( T_1 \) get a mass however, because \( \chi_1 \) does not have maximal rank. Now we need to eliminate the fields that acquire a mass and integrate them out.

In particular here we will find that we need to satisfy the equations of motion for \( Y_1, T_1 < \chi_1 > = \tilde{\chi}_1 T_1 \). Once we choose the vacuum solution \( < \chi_1 > \sim \delta_{i,j+1} \), we find that \( (T_1)_i = \tilde{\chi}_1 (T_1)_{i-1} \), so we can solve for all of the components of the tachyon \( (T_1)_i = (T_1)_i (\tilde{\chi}_1)^{i-1} \), where we are only making explicit the gauge indices corresponding to the brane \( B_2 \), but not it’s antibrane. Once we integrate out \( Y_1, T_1 \), we are left with the effective superpotential
\[ \text{tr} \left( \sum_i \tilde{Q}^i \tilde{Q}^*_1 \tilde{\chi}^{i-1} (T_1)_1 \right) \] (4.20)

In the above, one can follow the exact sequences to realize that in the dual theory we need to identify \( (Q_1)_i \sim M_i, \tilde{Q}^*_1 \sim \tilde{q}_i, (T_1)_1 \sim q \). Thus the above equation reproduces exactly the superpotential written in [45], namely
\[ W = \text{tr} \left( \sum_i q M_i \tilde{q}^{-i} \right) + \frac{1}{k+1} \text{tr} \left( \tilde{\chi}^{k+1} \right) \] (4.21)

This formula is consistent with our previous results when \( k = 1 \): here, the adjoint field has a mass term and is integrated out, this can be done before or after performing the duality. One might worry that one needs to integrate out other fields, so that this procedure might

\[ \text{Figure 5:} \quad \text{Quiver with antibranes in the presence of adjoints. The ghost charges are explicitly labeled.} \]
spoil the form of the superpotential. The only dubious point of our derivation is that the equation of motion of $Y_1$ that we used can contain additional terms. These are indeed proportional to $\chi^{s_1} Y_3 \chi^{s_2-s_1-1}$ and are obtained from the completion of the tr $(\chi^{k+1})$ term so that it is covariant with respect to the graded gauge group for the $B_2 \bar{B}_2$ brane anti-brane system. Notice that these extra terms involve fields with ghost charge different than one. We expect these to have a mass of the order of the string scale, and to be completely irrelevant in the infrared, where we are taking the decoupling limit $\alpha' \to 0$.

The configuration that we studied can be generalized easily to a situation where we have more than one arrow going between branes $B_1, B_2$ (let’s say $n_{12}$ of them), and between brane $B_2$ and $B_3$, ($n_{23}$ of them). The maximal extension of brane $B_3$ by branes of type $B_1$ will have charge $(0, n_{23} k, 1)$. We will find that there are $n_{12} n_{23} k$ meson fields. The arrows that leave brane $\bar{B}_2$ in the dual quiver are reversed, and the superpotential is as above, summing over different types of quark fields.

4.7 The general case

Now we can consider a general case with adjoints and extra superpotential terms, and dualize one node $E$. For each node $B_i$ we need to consider in the dual theory a change of basis where $\hat{B}_i$ is obtained by the maximal canonical extension of node $B_i$ by nodes of type $E$. This is, we take

$$\text{Ext}^1(B_i, E)^\ast \otimes E \to (B_i)^{(1)} \to B_i$$

and repeat extending $(B_i)^{(k)}$ until the process stops. If the process does not stop, then there is no dual theory. Thus the original theory must be such that the extension process is finite for the node we are dualizing.

The brane charge of the bound state $\hat{B}_i$ is $[\hat{B}_i] = [B_i] + m_i[E]$. If the original theory is given by $\sum N_i B_i + N_E E$, corresponding to a gauge group $\prod U(N_i) \times U(N_E)$, then the dual theory will have the gauge group

$$\prod U(N_i) \times U(\sum N_i m_i - N_E)$$

in the new basis. This is just the brane charge vector written in the new basis.

The matter fields between the nodes in the dual theory are obtained by following the change of basis by chasing exact sequences in the quiver. This is equivalent to finding the massless spectrum of fermions between the bound states in the field theory between branes $(\hat{B}_i)$ and $(\hat{B}_j)$. For the spectrum between brane $\hat{B}_i$ and the node $\hat{E}$ the arrows are reversed with respect to the original theory. This follows from chasing exact sequences. In particular the matter content between these two usually involves tachyons between the branes of type $E$ that form the bound state $\hat{B}_i$ and $\hat{E}$. Also the node $\hat{E}$ has the same number of adjoints and superpotential for adjoints than the original brane $E$.

To construct the full superpotential of the dual theory one needs to consider adding the terms required by tachyon condensation and integrating out all massive fields. Also one might need to covariantize the superpotential with respect to the graded group structure between branes and anti-branes. We have found no need to do that so far in the examples.
we have studied, so it is not clear that this is required beyond the introduction of the
tachyon (super)potential.

To summarize, we have described a systematic derivation of Seiberg dual theories for
any quiver theory of finite representation type, which produces an explicit superpotential
and proves equivalences of moduli spaces of supersymmetric vacua between the two theories.
Interesting examples of other dualities that can be analyzed with these methods can be
found in [1, 35, 40].

5. A more mathematical approach

One can go on to ask to what extent all Seiberg-like dualities can be discussed in this
language. In fact the dualities we are discussing are known in mathematics, as “tilting
equivalences.” They relate algebras which are not Morita equivalent (do not have identical
representation theory) but for which large subcategories of representations are equivalent.
We will quote Rickard’s theorem, which gives a sense in which all such dualities are tilting
equivalences.

We also return to the point we mentioned in section 4, that the supersymmetric field
theory does not need an explicit CY embedding to use these ideas. Rather, all of the
information for our discussion of duality is already contained in a given quiver diagram
and superpotential. The concepts we are explaining then allow direct comparison to branes
on CY, and systematic study of the question of which quiver theories can be realized by
embedding branes on CY.

5.1 Quiver categories

Notice that in section 4 all of our manipulations depended on formal properties of the
derived category of coherent sheaves, but made very little use of specific configurations
of branes in a given CY geometry, although we were implicitly assuming that this was
the situation we were in. This was also true of the arguments in section 2 in which we
abstracted the dynamics of brane-anti-brane systems as far as possible in a less formal
setting, but we made no explicit mention of a particular Calabi-Yau geometry.

We can ask the question in a different way: where do derived categories come from?
Axiomatically, one can produce a derived category starting from an abelian category. The
simplest examples of abelian categories are given by (subcategories of) Mod – A for some
ring A. Also, coherent sheaves are locally of the form Mod – A where A is the local ring
of holomorphic functions on some Calabi-Yau space. At least formally, we can build a
D-brane category if we are given a ring R, thus one can ask if there is a canonical ring A
associated to a given supersymmetric quiver field theory.

In fact, the standard construction of an associative algebra (over C) associated to a
quiver with relations [33] provides the ring A for the corresponding quiver gauge theory.
In the physics literature, this has appeared in the appendix of [20], and in the work [8] which
considers the problem of reconstructing singularities from the quiver data of a quantum
field theory. We now review this construction.
5.2 Path algebras

We assume the usual translation, already reviewed in section 2, from quiver diagrams to the field content and gauge symmetry of a supersymmetric field theory.

The path algebra \( \mathcal{P}(Q) \) associated to a quiver \( Q \) has the following generators:

- A projector \( P_i \) for each node \( i \), satisfying \( P_i^2 = P_i \). We also have \( \sum_i P_i = 1 \).
- A generator \( \phi^\alpha_{ij} \) for each arrow of the quiver from the node \( j \) to the node \( i \). (We index the arrows by \( \alpha \) in this general discussion; of course one could give names or other labels to these generators). This satisfies

\[
\phi^\alpha_{ij} P_k = \delta_{jk} \delta^\alpha_{ij}; \tag{5.1}
\]
\[
P_k \phi^\alpha_{ij} = \delta_{ik} \phi^\alpha_{ij}. \tag{5.2}
\]

As usual, products of these generators not constrained by the relations we just gave are considered to be new, linearly independent elements of the algebra. If we have no superpotential, we are done; this defines the free path algebra \( \mathcal{P}_0(Q) \).

If we have a superpotential \( W \) which is a single trace of products of the fields, we can regard it as a cyclically symmetric function of the noncommuting variables \( \phi^\alpha \) in an obvious way. The relevant algebra \( \mathcal{P}_W(Q) \) is then defined in terms of the free path algebra by imposing all of the F-flatness conditions \( \partial W / \partial \phi^i = 0 \) as additional relations. Since \( W \) is a single trace, these will be linear relations, and one can phrase the result as the quotient

\[
\mathcal{P}_W(Q) = \frac{\mathcal{P}_0(Q)}{\mathcal{I}(W' = 0)}
\]

where \( \mathcal{I}(W' = 0) \) is the two-sided ideal given as sums of terms from \( W' \) multiplied by elements of \( \mathcal{P}_0(Q) \) on both sides.

A field theory configuration which solves the F-flatness conditions now provides a finite dimensional representation of this path algebra, i.e. a map \( R \) from elements of \( \mathcal{P}_W(Q) \) to linear transformations acting on a vector space \( V(R) \cong \mathbb{C}^N \), respecting the relations. The operator \( P_i \) defines a projection onto a subspace \( V_i(R) \cong \mathbb{C}^{N_i} \in V(R) \) which is acted on by the \( U(N_i) \) gauge group, while the representation matrices of the \( \phi^\alpha_{ij} \) are the explicit matter configuration.

Normally we are only interested in complex gauge equivalence classes of configurations, and this translates into the statement that we are interested in (a certain class of) modules \( M \) over \( A \), i.e. those which can be obtained by quotienting finite dimensional free modules. Modules are defined more abstractly than representations, by prescribing a set of generators and a product \( A \times M \to M \), and formulating the theory in these terms makes it far more general. For the specific application to quiver gauge theories, however, one can think in terms of gauge equivalence classes of representations.

5.3 Homological algebra

The physics reader might consider the foregoing to be obvious restatement of what he has been doing for a long time in finding solutions of supersymmetric gauge theory (which
may bring to mind Monsieur Jourdain’s remark). However there are some crucial constructions one can make for the particular case of quiver theories, which are well-known mathematically, but to learn about them there one needs to know the translation we just gave.

The main point is that one can easily show that representations of a path algebra $P_W(Q)$ as defined above form an abelian category. The basic definition is the following: a homomorphism between quiver representations, $\rho \in \text{Hom}(R, S)$, is a linear map $\rho$ from $V_R$ to $V_S$, satisfying

$$S(a)\rho = \rho R(a) \quad \forall a \in P_W(Q).$$

(5.3)

Expanding out the definitions, one finds that this is a collection of $N_i(S) \times N_i(R)$ matrices, which live in the upper diagonal block of a collection of $GL(N_i(S) + N_i(R))$ gauge transformations. These should be thought of as “partial gauge transformations” which preserve the configuration of the joint $U(N_i(S) + N_i(R))$ gauge theory obtained by direct sum.

There are several axioms to check in saying that the representations $R$ and the morphisms $\text{Hom}(R, S)$ make up an abelian category. All are obvious, if we think of them as finite dimensional linear maps: the morphisms have an associative multiplication, and they have kernels and cokernels. They are less obvious if one regards $\text{Mod} - A$ as the definition and one must put appropriate conditions; for the concrete application at hand we assume the appropriate conditions.

This is the general definition of the $\text{Hom}(R, S)$ which we used in section 2, for any quiver gauge theory. What we will not explain here (it is standard mathematics; some of this is in [24]) is that this definition leads to definitions of $\text{Ext}^p(R, S)$ and many of the other concepts in our arguments, all the way up to the derived category of finite dimensional representations of the quiver algebra.

Let us denote this category of representations arising from a theory $T$ with quiver $Q_T$ and superpotential $W_T$, $\text{Mod} - P_W(Q_T)$, simply as $\text{Quiv}T$.

In examples, these definitions of $\text{Ext}^p(R, S)$ satisfy the simple physical interpretations known for $p = 1$ and $p = 2$ from geometry, branes and other frameworks. In particular, $\text{Ext}^1(R, S)$ corresponds to variations of bifundamental matter from $R$ to $S$ which is not gauge and not lifted by the superpotential, and $\text{Ext}^2(R, S)$ corresponds to inactive superpotential constraints. The higher $\text{Ext}^p(R, S)$ encode information about redundancies of superpotential constraints (e.g. see [24]).

This is one underlying mathematical context for quiver gauge theory. As we have seen, it is the natural context in which to discuss Seiberg duality. Let us expand a bit on this point.

In our previous discussion, we described classical dualities as the matching of moduli spaces of supersymmetric vacua of the two theories. Let us compare to this, the statement that this categorical structure matches between two quiver theories. In other words, the set of F-flat configurations match, and the partial gauge equivalences and higher Ext’s match. This type of relation is known as an equivalence between categories.

One obvious difference is that we need to talk about the D-flatness conditions or FI terms to make the first statement, while we do not in the second. In fact one can find
necessary and sufficient conditions to solve the D-flatness conditions, which only depend on the categorical structure \[12\]. Thus the second statement already contains the information needed to check the first statement, for any choice of FI terms. In this sense, the second statement is stronger.

One can go on to ask whether equivalence of categories is stronger than matching solutions of F-flatness in the sense of matching off-shell information about the superpotential. This is an interesting question which we will not address here.

5.4 Tilting equivalences

In section 2, we saw that the basic mathematical concept underlying Seiberg’s original duality (for \(U(N)\) and with all the caveats we mentioned) was the reflection functor of \[10\]. This was defined in 1973 and the mathematicians were not idle in the meantime.

By the previous discussion, we want to rephrase the problem of classical duality between quiver gauge theories \(T\) and \(T'\), as the problem of finding an equivalence between the categories of modules of their respective path algebras. This brings to mind Morita equivalence: two algebras are Morita equivalent if they have equivalent categories of modules.

However, we did not find dualities for all ranks of the gauge groups, just a large subset. For example, if the dual theory would have had \(N_c < 0\), obviously we do not have a dual theory. We excused this fault in the previous discussion by saying that the original theory would not have solved the D-flatness conditions for large FI terms, but now we are not imposing D-flatness and we do not have this out.

In fact, the two path algebras are not Morita equivalent. Rather, the duality corresponds to an equivalence between two large subcategories of \(\text{Quiv}T\) and \(\text{Quiv}T'\), not the full categories, which is weaker.

A nice introduction to the general theory of such equivalences can be found in the textbook \[43\] (which applies it to rather different problems coming from modular representation theory of groups). In fact, the first example they discuss, that of equivalences between algebras (in their notation) \(A_1\) and \(A_3\), is just the original \(U(N)\) Seiberg duality we discussed.

We will not go much further into the details of this, but just cite the general results of the theory. First of all, one wants to see that these equivalences of large subcategories of modules, imply that the derived categories of the two module categories are equivalent. This was found to be true for the explicit equivalences discovered by mathematicians and one might take it as axiomatic. In discussing theories which arise from D-branes, we can appeal to the topological open string construction of \[19\] to justify this. In any case, the intuition is that the derived category is a universal construction in which the formal manipulations we were doing before, always lead to specific configurations of a dual theory, no matter what we started with (and thus lead to a complete equivalence), while retaining enough information to force the specific equivalences of representations in the subcategories.

The main result in this theory is Rickard’s theorem \[52\]: any equivalence between two derived categories \(D(\text{Mod} - A)\) and \(D(\text{Mod} - B)\), can be realized as a tilting equivalence.
There are various definitions of tilting equivalence; let us start with the simplest, and later give a more sophisticated one (the theorem as stated only holds with the second definition).

A simple definition is the following: we need a tilting module $T$, an $A$-module such that

- The projective dimension of $T$ is zero or one;
- $T$ does not have self-extensions, i.e. $\text{Ext}^i(T, T) = 0$ for $i > 0$; and
- $A$ (as a module) has a finite resolution whose terms are direct summands in some number of copies of $T$.

Clearly explaining even what this simple definition means is going to get us into a long discussion, but physically the idea is more or less that $T$ is a set of branes from which no bound states can form, but if we add to them their antibranes we can generate the whole category as bound states.

One then has a tilting equivalence between $A$ and $B = \text{Hom}(T, T)$, the endomorphism ring of $T$. This is also hard to visualize physically, but let us see how the original example works. For this, we take $T$ to be the direct sum $T = B_1 \oplus B_3 \oplus B_6$ in the notations of (2.3). The no bound state condition is clear. The endomorphism algebra is now generated by $\alpha \in \text{Hom}(B_3, B_6)$ and $\beta \in \text{Hom}(B_6, B_1)$. These turn out to satisfy the single relation $\beta \alpha = 0$.

This is precisely the path algebra of the Seiberg dual theory, with the relation (2.14) (but leaving out the other dual quark which we put in to have a superpotential). Classifying representations of this algebra will then lead us to the brane content of this theory.

This argument strikes us as both very short and to the point, and very mysterious. Anyways, the tilting equivalence for this simplest example of duality is relatively simple, so this might be a viable framework in which to look for new dualities.

The axioms for a tilting module are rather restrictive, and to get the general result we cited, one needs a more general definition, involving a tilting complex. We will not give the axioms this must satisfy, but quote the actual transformation one gets: if $T$ is a tilting complex, then the functor $F_T$ from $D(\text{Mod} - A)$ to $D(\text{Mod} - B)$ defined by

$$X \xrightarrow{F} X \otimes_A^L T$$

is an equivalence (and all equivalences can be written this way).

This is rather formidable in general but does reduce to something more concrete (at least, comparable to the first definition) in cases in which the modules exist on both sides. Granting the earlier point that classical Seiberg dualities must give equivalences of derived categories, this provides a complete answer to the problem of finding such dualities, in principle. However, we suspect the physics reader who tries to follow this up will soon find himself seeking professional help.
The main reason we cited this was to make the point that this transformation is formally very similar to the Fourier-Mukai transforms which describe autoequivalences of $D(\text{Coh}M)$ for Calabi-Yaus, again reinforcing the theme that the structures identified in this context are more generally present in supersymmetric quiver gauge theory.

5.5 The Calabi-Yau condition

It is interesting to ask what characterizes the quiver theories which actually come out of string compactification on a Calabi-Yau. The most basic property of these categories of coherent sheaves (and the derived categories) is a particularly simple form for Serre duality (since the canonical sheaf is trivial), namely

$$\text{Ext}^{d-i}(S_1, S_2) \cong \text{Ext}^i(S_2, S_1)^*$$
on a Calabi-Yau $d$-fold. The corresponding statement in the derived category has been suggested as the definition of a “Calabi-Yau category” [44].

This is not a general property of quiver categories. If there are no relations, one can show that $\text{Ext}^p(A, B) = 0$ for all $p > 1$, so it could only be true for $d = 1$. In fact only the theory with a single adjoint superfield would pass this test. This is in fact the theory describing branes on $T^2$, so imposing the Calabi-Yau condition does lead to a sensible constraint in this case.

One might ask if it is a general property of quivers with relations which follow from a superpotential. One can even be more specific and ask for $d = 3$, on the grounds that “the superpotential is a three-form.” This somewhat cryptic comment is justified by the idea that relations $\frac{\partial W}{\partial \phi} = 0$ are naturally dual to fields (elements of $\text{Ext}^1$), but also naturally live in $\text{Ext}^2$.

Let us see more specifically how one might arrive at this conclusion from a purely algebraic point of view. To do this more precisely, we will need a working definition of $\text{Ext}$ for modules over algebras.

The most economical definition is in terms of projective resolutions. A projective resolution of a module $M$ is an exact sequence of modules

$$\mathcal{P}_i \xrightarrow{\delta_i} \mathcal{P}_{i-1} \to \cdots \to \mathcal{P}_0 \to M \tag{5.4}$$

where each $\mathcal{P}_i$ is a projective module for the algebra $\mathcal{A}$. These resolutions are also important to give a noncommutative definition of smoothness for singularities [4]. These modules are essentially determined by a projector $p_i \in M_n(\mathcal{A})$ for some $n$.

The definition of $\text{Ext}$ is then the homology of the sequence

$$\text{Ext}^p(M, N) = H^p(\text{Hom}(\mathcal{P}, N)) \tag{5.5}$$

which can be shown to be independent of the choice of projective resolution.

The advantage of writing the $\text{Ext}$ in this form is that we have a large class of choices for projectors in our algebra, namely the $\mathcal{P}_i$, so that the modules $\mathcal{A}\mathcal{P}_i$ are projective.
Now, we also want to associate to each node in the quiver diagram a particular representation of the group. Under the assumption that the quiver diagram has superpotentials of quadratic order or higher, this is explicitly given by the following

\[ B_k : (P_i = \delta_{ik}, \phi^\alpha = 0) \]  \hspace{1cm} (5.6)

We will choose this representation to be giving us an explicit module with action by \( A \) on the right. We choose this, because we want to be able to make maps between modules by multiplication on the left by elements of \( A \).

Now, we want to build a projective resolution for \( B_i \). Let us consider all the fields \( \phi^a_{ji} \) that begin at node \( i \) with \( j \) variable. Then it is easy to see that we can begin to write a projective resolution in the following form

\[ \oplus \phi^a_{ik} (P_k A) \xrightarrow{\delta_1} P_i A \xrightarrow{\delta_2} B_i \]  \hspace{1cm} (5.7)

where \( \delta_1 (a_k) = \sum (\phi^a_{ik} a_k) \). This map produces all the possible polynomials in the generators with at least one \( \phi \), so the cokernel of this map is exactly the only term in \( P_i A \) that is not of this form, \( P_i \). This gives us the first two terms of the resolution, and one has exactness in the first term.

Now, we can try to see what relations will appear when we consider the superpotential. To do this we consider the following composite fields

\[ M_{jk}^{ba} = \phi^b_{ji} \phi^a_{ik} \]  \hspace{1cm} (5.8)

and the quantities

\[ W_{ab}^{kij} = \frac{\partial W}{\partial M_{jk}^{ba}} = \frac{DL}{D\phi^a_{ik}} \frac{\partial W}{\partial \phi^b_{ji}} \]  \hspace{1cm} (5.9)

where the term \( \frac{DL}{D\phi^b_{ji}} \) indicates that we only take a derivative of the equation if the field \( \phi^b \) appears in the leftmost (rightmost) term of the polynomial \( \frac{\partial W}{\partial \phi^a_{ik}} \), which encodes the superpotential relations. The idea is that

\[ W_b' = \sum a \phi^a W_{ab} \]  \hspace{1cm} (5.10)

gives us the polynomial relations associated to the derivative of the superpotential with respect to the field \( \phi^b \). Thus we can try as a second term in the resolution, a sum over all fields that go into that node,

\[ \oplus \phi^b_{ik} (P_k A) \xrightarrow{\delta_2} \oplus \phi^a_{ij} (P_j A) \xrightarrow{\delta_1} P_i A \xrightarrow{\delta_2} B_i \]  \hspace{1cm} (5.11)

The map \( \delta_2 \) acts by \( \delta_2 (a_k) = (\sum W_{ab}^{kij} a_k) \). One can easily show that \( \delta_1 \circ \delta_2 = 0 \), because \( \sum W_{ab}^{kij} a_k \) are exactly the relations derived from the superpotential. Similarly we can consider the next term in the exact sequence

\[ P_i A \xrightarrow{\delta_1} \oplus \phi^b_{ik} (P_k A) \xrightarrow{\delta_2} \oplus \phi^a_{ij} (P_j A) \xrightarrow{\delta_1} P_i A \xrightarrow{\delta_2} B_i \]  \hspace{1cm} (5.12)
where $\delta_3 a = (\phi^h_{ki} a)$. Again, one can show that $\sum_b W_{ab} \phi^b$ are exactly the superpotential relations for $\phi^a$, so we obtain $\delta_2 \circ \delta_3 = 0$.

We thus have a complex (5.12), whose terms are projective. For this to be a projective resolution, we need it to be exact. This condition depends on the choice of superpotential. For example, it will fail for $W = 0$. It is also clear that every field must appear in the superpotential.

On the other hand, there are many superpotentials for which this is a resolution. The case which is best understood mathematically is if the path algebra is Koszul (e.g. see [32]), essentially meaning that there is a projective resolution in which the maps are linear in the generators. This includes the $\mathbb{C}^3/\Gamma$ McKay quiver path algebras [41], so in this case the CY condition is satisfied.

A Koszul algebra must have purely quadratic relations and a purely cubic superpotential. On the other hand, as is well known, branes on CY need not have cubic superpotentials, despite naive appearances from (4.13). Thus we do not assume this in formulating the CY condition. This point is explained in [13, 55, 22] and will be explained further elsewhere.

Given that (5.12) is a projective resolution, we believe the quiver category has Serre duality, and that this is a CY category (we are not at this point claiming that this is necessary). It is easy to check that the morphisms between fractional branes (simple representations) correspond to those we used in section 4, and satisfy Serre duality.

A bit more explicitly, one first checks that $\dim(\text{Hom}(P_i A, B_k)) = \delta_{ik}$. If the superpotential is cubic or higher order then each term in the $\delta_\alpha$ will be multiplying by polynomials in the $\phi$ of degree equal to one or higher, so in the homology sequence of of $\text{Hom}([P \to B_k], B_j)$ all the chain maps are zero. With these conditions one finds that $\dim(\text{Ext}^0(B_i, B_k)) = \delta_{ik}, \dim(\text{Ext}^1(B_i, B_k)) = n_{ki}, \dim(\text{Ext}^2(B_i, B_k)) = n_{ik}, \dim(\text{Ext}^3(B_i, B_k)) = \delta_{ik}$; where $n_{ki}$ are the number of arrows in the quiver begining at node $i$ and ending at node $k$.

One can now ask when do we expect the sequence above to have homology, so that it is not a resolution. The simplest example is $W = 0$, for which $\delta_2 = 0$ and there is homology in the third term of the resolution.

A simple argument also shows that if the complex written above is a resolution for every $B_k$ then the algebra needs to be infinite dimensional over the complex numbers. To show this we need to consider the brane given by $B = \oplus B_k$ over all possible elementary fractional branes. If the algebra is finite dimensional, then all of the terms of the resolutions have finite dimension. This permits us to show that the (virtual) dimension of the resolution of $B$, and hence the dimension of $B$ is zero. This follows because $P_i A$ appears repeated as each term is counted for $\text{Ext}^{1,2}$ either when arrows depart from a node, or when they arrive at the node. For $\text{Ext}^{0,3}$ this pairing is obvious.

For quotient singularities the algebras involved are always infinite dimensional, as their center is the ring of the quotient variety [1], and it is indeed the case that in this situation the algebra satisfies the CY condition.

### 5.6 Non CY examples

Not all of the theories we considered in section 4 satisfy the CY condition. For example, the
original SQCD had no superpotential. On the other hand, it can be obtained by wrapping branes on CY.

It is also true that “tilting equivalence” and many of the other structures make sense for general quivers, not just CY quivers. On the other hand one does not expect that in general the dual algebra can be described using a superpotential; the relations might be more complicated.

One might conjecture that when this can be done, it is because the original quiver is a subquiver of a larger quiver which does satisfy the CY property, meaning that by eliminating some of the nodes and all arrows to these nodes one reduces the larger quiver to the smaller quiver. One is not allowed to eliminate arrows between nodes which remain in the subquiver, so this relation is a priori nontrivial.

\[ \begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\uparrow & & \uparrow \\
N_1 & & N_2 \\
\end{array} \]

\[ \begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
N_2 & & N_3 \\
\end{array} \]

Figure 6: \( \mathbb{Z}_n \times \mathbb{Z}_m \) quiver diagram: the pattern repeats itself to get an \( n \times m \) periodic lattice

For example, one can embed the SQCD quiver in the \( \mathbb{C}^3/\mathbb{Z}_m \times \mathbb{Z}_n \) McKay quiver, as in figure 6. One sets all \( N_i = 0 \) except for \( N_1, N_2, N_3 \) as marked on the figure.

We must now distinguish the Ext groups for the small algebra and the larger CY algebra; these are in general different because extra nodes appear as data in the projective resolution of the fractional branes in the big quiver. Our arguments strictly speaking (and whenever we discuss superpotentials) apply to the larger quiver.

This raises another general question: can all quivers can be embedded in CY quivers; if not, what is the constraint under which this is possible.

5.7 Partial resolutions

Many singularities can be built from partial resolutions of an orbifold singularity, giving us access to a large class of field theories other than orbifolds. These techniques have been used to geometrically engineer other singularities and make them tractable from a D-brane analysis; and have led to the notion of toric duality \[27, 28, 29, 30, 6\] by arriving to different field theories associated to the same singularity by taking different paths to get there. Here we will explain how this procedure of partial resolutions can be used in a more abstract setting.

The procedure for generating these partial resolutions is as follows (see for example \[27, 1\])

- Give large FI terms to some nodes in a known quiver diagram.
• One decides that some fields (arrows) in the quiver get large vevs.

• One reduces to the unbroken gauge group. This fuses the nodes that are connected by large vevs into one.

• Integrate out massive fields.

We have seen very similar procedures in our discussion of duality. We have turned on large FI terms to be able to go around singularities, turning on large vevs forces branes to form bound states, and the procedure of integrating out massive fields (which can include tachyons) gives rise to the superpotentials of the dual theory. Thus all of these operations fit naturally into the discussion of this paper.

We want to argue that the procedure above is like localization and that it can be described in more algebraic terms. The most important point in the above is the second one. We choose some fields to get large vevs and we force certain branes to form bound states, so we should throw away certain brane configurations where this does not happen.

In a geometrically engineered theory, the branes we will throw away have wrapping number around the cycle that is resolved, so they are not local branes at the leftover singularity. At an abstract level however, we need to be able to make these statements without geometry.

Giving large vevs to some fields forces the arrows we condense $\phi_{ij}^a$ to give isomorphisms between certain gauge groups and they break the gauge group to a common diagonal.

At the level of representation theory of the original quiver this means we need to focus on brane configurations where $N_i = N_j$. The vev for $\phi_{ij}^a$ being large means that it is essentially invertible. This is, one can find in each of these representations a matrix $\beta_{ji}^a$ such that

$$\beta^a \phi^a = P_j$$
$$\phi^a \beta^a = P_i$$

and we can always choose $\beta$ such that $\beta^a P_k = \delta_{ik} \beta^a$ and $P_k \beta^a = \delta_{jk} \beta^a$. If the original $\beta$ does not satisfy this condition then $\beta' = \beta \phi \beta$ does.

Let $A$ be the original quiver algebra. The representations above are representations of the algebra where we adjoin $\beta$ as a generator of the algebra with the relations imposed above, let us call this bigger algebra $A[\beta]$.

Since we have an embedding of algebras $A \subset A[\beta]$, every representation of the latter algebra is automatically a representation of $A$. Given $R_1, R_2$ representations of $A[\beta]$ one can prove by a matrix argument that $\text{Hom}_A(R_1, R_2) = \text{Hom}_{A[\beta]}(R_1, R_2)$. In particular kernels and cokernels of maps between representations of $A[\beta]$ will be also such type of representations. Thus, if we consider the category of the representations of $A[\beta]$ as a subcategory of the representations of $A$, then this category is closed under taking kernels and cokernels.

Also one can prove that given any short exact sequence $0 \to R_1 \to R \to R_2 \to 0$ of $A$ modules, where $R_1$ and $R_2$ are $A[\beta]$ modules, then $R$ is a representation of $A[\beta]$ as well.
The category $\text{Mod}(A[\beta])$ is therefore a full (abelian) subcategory of $\text{Mod}(A)$: it is closed under the operation of taking kernels, cokernels. The $\text{Hom}$ coincide, and the category is closed under extensions. The above arguments have shown that $\text{Ext}^{0,1}_A(R_1, R_2) = \text{Ext}^{0,1}_{A[\beta]}(R_1, R_2)$, so that the matter content between bound states of branes in the partial resolution is completely equivalent to the matter content between the same bound states as seen from the original quiver diagram. The process of integrating out massive fields is exactly what is captured by the homological algebra of the $\text{Ext}$ functors.

In the end we should have $\text{Ext}^i_A(R_1, R_2) = \text{Ext}^i_{A[\beta]}(R_1, R_2)$. This is exactly the behavior that one expects for the $\text{Ext}$ functors in an algebraic geometry for coherent sheaves (branes) whose support is entirely contained in an open set (localization) of some known geometry.

In particular, if some quiver algebra satisfies the CY condition, then any partial resolution of the algebra as given above should also satisfy the CY condition.

This also poses the question of how the duality transformations between a quiver and its partial resolutions are related.

Geometrically, the partial resolution produces a space $X$, which can be further resolved to produce $\tilde{X}$ which embeds into $Y$, where $Y$ is a resolution of $\mathbb{C}^3/\Gamma$. The process of going from $\mathbb{C}^3/\Gamma$ to $X$ blows up some cycles to infinite size and thus reduces the local K-theory group (the coherent sheaves of compact support). The quiver category $D(\text{Mod} - Q_\Gamma) \cong D(\text{Mod}Y)$ via the McKay correspondence. The classes (of compact support) that disappear geometrically are the classes that are wrapped on the blown up cycles. These are identified in the field theory as the classes for which $N_i \neq N_j$.

We have the inclusion $D(\text{Mod}Q_T) \subset D(\text{Mod} - Q_\Gamma)$, as we described above, for the quiver category of the partial resolution field theory, and $D(\text{Mod}\tilde{X}) \subset D(\text{Mod}Y)$. These are both subcategories of the same category, and they span the same K-theory lattice, so these two coincide. We can then identify $D(\text{Mod}Q_T) \cong D(\text{Coh}\tilde{X})$. This will hold whenever one can resolve completely the singularity. This is the case for toric singularities: turning on generic D-terms resolves all the singularities, and $X$ is the moduli space of the field theory with $N_i = 1$ for all the nodes of $Q_T$. It should be said that there are situations in which this condition will not hold, e.g. on orbifolds with discrete torsion.

5.8 Toric duality is generalized Seiberg duality

It is not true that all partial resolutions of a quiver theory are Seiberg dual. Rather, the claim of [1, 13, 23] is that two partial resolutions of the same orbifold theory which lead to theories $T_1$ and $T_2$ which describe the same space $X$, i.e. the moduli space of supersymmetric configurations with all $N_i = 1$ is $X$ in both cases, are Seiberg dual.

The previous considerations give us a very short argument to this effect. As we discussed, partial resolution of the orbifold theory $\Gamma$ leads to a gauge theory $T$ such that $D(\text{Mod} - Q_\Gamma) \cong D(\text{Coh}X)$. The assumption that two such theories produce the same $X$ then implies that $D(\text{Mod} - Q_{T_1}) \cong D(\text{Mod} - Q_{T_2})$.

By Rickard’s theorem, any Seiberg duality would have to induce such an equivalence. In this sense, toric duality must be a generalized Seiberg duality – it will be a tilting equivalence, but it is not obvious whether the equivalence can be realized by some sequence
of Seiberg dualities, each of which acts on a single node. In the examples of \([6, 13, 29]\), this tended to be true, but there might be more general dualities which could be obtained this way.

6. Conclusions and further directions

Seiberg duality is an important aspect of \(\mathcal{N} = 1\) supersymmetric gauge theory. After its original discovery in supersymmetric QCD, it has been generalized to a very large class of theories with multiple gauge groups and varied matter content.

There have been various derivations, each of which works in some class of theory and suggests some underlying origin of the phenomenon. The arguments of \([2]\) apply to deformed \(\mathcal{N} = 2\) theories. Suspended brane arguments \([37, 35, 26]\) work in theories that can be realized as brane webs, a certain class of quiver theories and orientifolded quiver theories. These seem to be a special case of the larger class of quiver theories obtained by placing branes near a partially resolved orbifold singularity. This allows more contact with geometry but is still local on the CY. Not all combinations of cycles in a CY can be so realized, and one can further extend the class of geometrically engineered theories by allowing more general combinations of cycles. In the stringy regime, the very definition of cycle has to be generalized \([19, 3]\). Eventually this line of development merges with the general discussion of \(\mathcal{N} = 1\) string/M theory compactification. On the other hand, it seems unlikely that all \(\mathcal{N} = 1\) theories arise from string and M theory, making one wonder whether such explanations are somehow missing some simpler point.

All of these constructions work by embedding into quantum theories with some independent definition and thus provide some explanation of the duality at the quantum level. This does not necessarily mean that one can get the explicit quantum effective theory, however. The best understood cases remain those in which quantum corrections are either absent or operate by destabilizing certain supersymmetric vacua, while other components of the moduli space are the same as in the classical limit.

In this work, we gave a rather different type of argument: namely, we abstracted the essential features of the brane arguments at the classical level, and discussed how they could be applied to any quiver theory. The essential idea is that in a given “geometry,” one can consider the set of all possible supersymmetric configurations of branes and antibranes. This set can be described as the moduli space of vacua of the combined world-volume theory of the branes, but making such an explicit description requires a choice of basis, a finite set of elementary branes in terms of which each configuration can be constructed in one way. There are many possible choices of basis, and each leads to a different gauge theory with the same set of configurations. The relation between two such descriptions is Seiberg duality.

Making this precise requires being able to work with branes and antibranes at the same time, which is not possible in conventional supersymmetric gauge theory. This is why previous arguments relied on particular realizations of the branes in string theory. However, in the context of quiver theories, one can make a more abstract definition of antibrane in terms of its defining property, that tachyon condensation between a brane...
and its antibrane leads to the vacuum configuration, using homological algebra techniques. This agrees with the string theory definition coming from D-branes when this makes sense, but does not presuppose that the theory can actually be realized in terms of D-branes. Indeed, these arguments do not make use of any conventional geometric definition of a space with branes in it, suggesting that quantum Seiberg duality does not require such a picture either.\footnote{One can regard the quiver point of view as a working definition of “noncommutative geometry.”}

It is a very interesting question, which quiver theories have a geometric interpretation, meaning that they can be realized by D-branes on a Calabi-Yau threefold, and which do not. It is hard to believe that all of them do, and if this were true, it would probably have dire consequences for the predictive power of string theory. Assuming that it is not true, one would like to know the necessary and sufficient criteria for a geometric realization.

One test which might be applied is that the center of the algebra should be a Calabi-Yau threefold (in some sense). Another test is that the category of quiver representations must have homological dimension 3 and admit Serre duality. Both conditions come directly out of the theory of branes on CY$_3$ but are not at this point obvious for general quiver theories. In fact, we believe that the second condition is rather weak and will return to this point in subsequent work.

Of course we know of phenomena such as discrete torsion that require generalizing our idea of “geometry” in any case, and there might be non-geometric string theory compactifications as well. However, it is hard to formulate an accessible question about which theories can come from branes in string theory at this point.

Comparing to the work of \cite{27, 28, 29, 30, 6} on toric duality, our arguments seem to us conceptually simpler than basing the discussion on an underlying geometry. In particular, they completely bypass the complicated question of deriving the quiver from the geometry. On the other hand, we believe one could obtain a fairly complete explanation of toric duality from this point of view, based on the framework of the generalized McKay correspondence, which provides a direct (though somewhat abstract) relation between the geometry of orbifold resolution and quivers. The outline of this is already present in existing mathematical work – the various partial resolutions of the orbifold lead to birationally equivalent spaces, which are connected by performing flops; each such flop can be realized as a known transformation on the derived category which would be a tilting equivalence between the categories of quiver representations on both sides. It will be interesting to develop this understanding.

Perhaps the most interesting direct application of the relation to tilting equivalences is that it provides a way to look for new Seiberg dualities, not realized as a succession of the known dualities which act on a single node. Much is known about the case of “tame” algebras (i.e. finitely many indecomposable representations), which might be relevant.

It is interesting to note that the key step in the derivation, namely (2.11), is formally a gauge transformation on the larger system of branes and antibranes we are using to represent a configuration. In this sense, we find that Seiberg duality is itself a gauge symmetry. This observation can be pushed further; the key step leading to the derived
category framework in which these considerations naturally fit, is to allow equivalences involving brane-antibrane annihilation (quasisomorphisms) to play the same role as the original gauge transformations (homomorphisms). In this sense, brane-antibrane annihilation becomes a sort of generalized gauge equivalence. Although we are seeing it here in the specific context of quiver theories, it seems to us that this idea should be valid more generally, and could potentially be very fruitful.

We only derived equivalences between classical moduli spaces, but we believe this is the most nontrivial test of the duality and that this captures the heart of the phenomenon. We made some comments in section 2.6 about issues related to quantum effects in four dimensions. Another consequence/test of the idea that the origin of the duality is effectively classical, is that similar dualities should exist for quantum theories in dimensions 1, 2 and 3 (and even 0 if one considers matrix integrals). This is known in 3 dimensions and indeed many brane constructions do extend to this case [35]. From the point of view here, it would be particularly interesting to study dimension 1 (quantum mechanics) as in this case one is not comparing moduli spaces. It seems very plausible that these dualities would act as symmetries on the BPS spectrum; an example (with eight supercharges) in which this is known to be true is the action of the Weyl group on the affine Lie algebras of [35].

It is important to try to push this understanding beyond quiver theories. A next step in complexity is to incorporate the other classical groups. As is well known, these can be expressed in quiver language by starting with a $U(N)$ theory and restricting to fixed points of a $Z_2$ action on the fields (the usual orientifold construction). One can in principle represent exceptional groups and more general matter by imposing analogous non-linear conditions; the value of this is not yet clear. We would not claim at this point that all phenomena in string/M theory can be understood in terms of branes, but we do believe that branes have much more to teach us.

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