Optimal Plastic Analysis of Structures under Uncertain Conditions

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Abstract. In this study optimum plastic shakedown analysis of framed steel structures with semi-rigid connections (SRC) between beam and columns subjected to multi-parameter static loading presented. Since shakedown analysis does not provide information concerned with the accumulated residual displacements, complementary strain energy of the residual force (CSERF) considered as constrain to evaluate the post yield behavior of framed steel structures. The constraints on the CSERF are modelled using probabilistic and deterministic methods. To evaluate the maximum shakedown load-multipliers a numerical example is introduced, shakedown load-multipliers are calculated and safe loading domains of the framed steel structure are illustrated. The numerical results show that the constraints on CSERF, SRC and probabilistic given constraints can influence on the value of the shakedown load-multipliers.

1. Introduction
Generally two methods have been used to calculate the uncertainties in structural engineering. The first approach is the deterministic method, where a safety factor is applied. The second method is probabilistic analysis and design, in which the design information are known within a certain bound and probability distributions are given. The deterministic method despite being favorably, the accurate safety is not clear for assumed safety factor. In the probabilistic method, the uncertainties are given randomly, where the most frequent value of the random variable are assumed to the highest number in the probability density function. In mechanical engineering the uncertainties have an extensive performance and requires precise computations. Reliability based design has a long history with important books and papers. Several international courses and meetings provide a proper tool about the practical and theoretical problems in the field of mechanical engineering like stochastic load modelling and stochastic computational mechanics, reliability-based optimization response surface methodology and reliability-based design (Melchers and Beck [1]; Wang and Katafygiotis [2]; Lógó [3]; Marti [4] and Marti and Stöckl [5]) proposed reliability programming for the solution method of different stochastic problems.

In structural optimization plastic analysis and design methods have a significant role since they provide information in respect to the post yield behaviour of structures which lead to have an economical design. However, as a consequence of this interest, large accumulated residual deformations might occur, which lead to the collapse of the structures. To avoid this undesirable situation in the literature several methods proposed which provide appropriate tools to control the residual deformations of the structures (see e.g. Atkociunas et al. [6]; Liepa et al. [7]; Simon and Weichert [8]; Levy et al. [9]; Weichert and Maier [10]; Tin-Loi [11]; König [12]; Polizzotto [13]; Kaneko and Maier [14]; Corradi [15]; Ponter [16] and Maier [17]). Along with others Kaliszyk and Lógó [18-19] proposed that the post yield behavior of
the structures can be limited by introducing a bound on CSERF. In this study, a parametric investigation is introduced to evaluate the influence of SRC and constraints on the shakedown load-multiplier. The SRC of beam-column are modelled using appropriate internal springs at the connections. Furthermore, the constraints on CSERF are modelled using probabilistic and deterministic methods.

2. Bounded plastic deformation

2.1. Deterministic problem
Consider an elastic-plastic solid body subjected to external force $P_0$. Consequently, plastic force $Q^p$ occurs inside the body. When the external force is reduced elastic deformations appear and consequently elastic force $-Q^e$ occurs inside the body. Accordingly, after full unloading the residual force $Q^r$ remains inside the body.

$$Q^r = Q^p - Q^e$$

where $Q^e$ is:

$$Q^e = T^{-1}GK^{-1}P_0$$

here $K$: stiffness matrix; $G$: geometrical matrix and $T$: flexibility matrix. Using positive definite function, CSERF can be calculated as follow:

$$C_r = \frac{1}{2} \sum_{i=1}^{n} \left[ \frac{1}{S_i} \int_0^{l_i} \left( Q_i^r(s) - Q_i^e(s) \right)^2 ds \right] \geq 0$$

applying equation (1):

$$C_r = \frac{1}{2} \sum_{i=1}^{n} \frac{1}{S_i} \int_0^{l_i} \left( Q_i^r(s) \right)^2 ds \geq 0$$

here $S_i$ defines flexural and tensile stiffnesses of beam and trusses elements. The residual deformations constrained by considering a limit value on the amount of CSERF:

$$\frac{1}{2} \sum_{i=1}^{n} \frac{1}{S_i} \int_0^{l_i} \left( Q_i^r(s) \right)^2 ds - C_{r0} \leq 0$$

where $C_{r0}$ is a proper bound for CSERF, $C_r$.

In the case of truss elements the CSERF can be introduced as follows:

$$C_r = \frac{1}{2E} \sum_{i=1}^{n} \frac{l_i}{A_i} \left( N_i^r \right)^2$$

where $N_i^r$: residual normal force of the truss elements; $A_i$: cross sectional area of the truss elements; $l_i$: the length of the elements; $E$: the elastic modulus. ($i = 1, 2, ..., n$): number of elements.

For beam members the CSERF can be defined as follows:

$$C_r = \frac{1}{2E} \sum_{i=1}^{n} \frac{l_i}{I_i} \int_0^{l_i} \left( M_i^r(s) \right)^2 ds$$

here $M_i^r$: residual moment of the beam elements; $I_i$: the moment of inertia for beam elements. Supposing residual moments $M_{1z}^r$ and $M_{2z}^r$ acting at the two ends of a beam element the integral part in equation (7) can be defined as:

$$\int_0^{l_i} \left( M_i^r(s) \right)^2 = \frac{1}{3} \left[ (M_{1z}^r)^2 + (M_{2z}^r)(M_{2z}^r) + (M_{2z}^r)^2 \right].$$

Then the plastic deformations are constrained by introducing an appropriate limit value for $C_{r0}$.

$$\frac{1}{6E} \sum_{i=1}^{n} \frac{l_i}{I_i} \left[ (M_{1z}^r)^2 + (M_{2z}^r)(M_{2z}^r) + (M_{2z}^r)^2 \right] - C_{r0} \leq 0$$

The limit functions for truss elements can be introduced by equation (10):
\[ C_r = \frac{1}{2E} \sum_{i=1}^{n} \frac{l_i}{(N_i^r)^2} - C_{r0} \leq 0 \]  

### 2.2. Probabilistic problem

Introducing the basic concepts of the static theorem, the structural failure can be expressed by \( S_R \leq S_s \). Here \( S_R \) expresses the bound for statically admissible forces \( S_s \) and \( f_R(S_R) \) and \( f_s(S_s) \) are corresponding distribution function.

The failure function can be defined by the following equation:

\[ P_f = P[S_R \leq S_s] = \int_{S_R \leq S_s} f_R(S_R) f_s(S_s) dS_R dS_s. \]  

The above problem alternatively can be expressed in term of the limit state function (LSF):

\[ G(S_R, S_s) = S_R - S_s \leq 0. \]  

Assuming that \( G \leq 0 \) shows the failure event, the probability that a failure occurs can be defined by

\[ P_f = F_G(0) \]  

here \( F_G(0) \) is the cumulative distribution function (CDF) of the LSF. Assuming both \( S_R \) and \( S_s \) have the normal distribution \( N(\mu_{S_R}, \sigma_{S_R}^2) \) and \( N(\mu_{S_s}, \sigma_{S_s}^2) \) respectively. In this case, \( G \) also has normal distribution with the mean value equal to \( \mu_G = \mu_{S_R} - \mu_{S_s} \) and standard deviation \( \sigma_G^2 = \sigma_{S_R}^2 + \sigma_{S_s}^2 \), hence,

\[ P_f = \Phi \left( -\frac{\mu_G}{\sigma_G} \right) = \Phi(-\beta) \]  

here \( \beta = \mu_G / \sigma_G \) is safety index and \( \Phi(.) \): the normal cumulative density function.

The failure probability of the structural system is computed by the following integral:

\[ P_f = P[X \in D_f] = P[G(x) \leq 0] = \int_{D_f} f_X(x) dx. \]  

\( D_f \): domain of failure. Assuming the bound on the CSERF is given randomly and \( C_{r0} \) considered as mean value and standard deviation: \( \sigma_w \). Then equation (15) can be expressed from the following equation:

\[ P_f = P[G(S) \leq 0] = \int_{D_f} f(C_{r0}, \sigma_w) dS. \]  

### 3. Determination of the loading domain

In most practical cases structures are subjected to general loading which consists of a set of proportional loads which can vary independently of each other. In this general case, the load carrying capacity of a structure cannot be characterized by a single collapse load multiplier but all the combinations of the independent loads under which the structure will collapse have to be defined.

To illustrate the problem of general loading, consider a structure which is subjected to forces \( P_0^{(j)} \) at points \( j = 1, 2, 3, \ldots, s \) with given locations and directions. Assume that the forces can be divide into groups which form independent proportional loads such that:

\[ P_j = m_1 P_0^{(1)} + m_2 P_0^{(2)} + \ldots + m_s P_0^{(s)} = \sum_{j=1}^{s} m_j P_0^{(j)} \]  

Here \( P_0^{(j)}, j = 1, 2, 3, \ldots, s \) denotes the independent group of forces and \( m_j \) are the associated load multipliers. These multipliers can vary independently but are limited by relationship

\[ L(m_j) \leq 0 \]  

Which defines all the possible combination of the external loads which can occur during the loading process. In an s-dimensional load space with the coordinate axes \( m_1, m_2, \ldots, m_s \) the function \( L(m_j) = \]
0 can be defined by a closed hyper-surface bounding, the domain of possible load, as shown for the case \(s=2\) in Figure 1 (by the conditions \(m_1 \geq 0\) and \(m_2 \geq 0\)).

![Figure 1](image)

**Figure 1.** Safe domain in the case of 2 load-multipliers

4. Shakedown analysis
During the shakedown analysis all the possible distributions of the residual forces should be considered. These residual internal forces form a self-equilibrating system. The solution mechanism can be expressed by introducing the static principal of shakedown theorem, hence equilibrium equation will be defined from equation (19):

\[
G Q^r = 0
\]

(19)

where \(G\) is the equilibrium matrix, \(Q^r\) denotes the residual bar forces or the residual bending moments.

The yields condition can be defined as follow:

\[
-Q^p \leq Q^r + \max Q^e \leq Q^p
\]

(20)

here \(Q^p\) is plastic limit force. Equation (21) defines elastic internal force:

\[
Q^e = T^{-1}GK^{-1}m_{sh}P_0
\]

(21)

where \(P_0\): constant load; \(m_{sh}\): shakedown multiplier; \(K\): stiffness matrix and \(T\): flexibility matrix.

In the case of deterministic problem shakedown load-multiplier will be maximized when accumulated residual displacements are controlled with applying \(C_{r0}\):

\[
\left\{ \begin{array}{l}
m_{sh} = \max \\
\quad G Q^r = 0 \\
\quad -Q^p \leq Q^r + \max Q^e \leq Q^p \\
\quad Q^e = T^{-1}GK^{-1}m_{sh}P_0 \\
\quad \frac{1}{2} \sum_{i=1}^{n} \int_0^1 (Q^e_i(s))^2 ds - C_{r0} \leq 0
\end{array} \right\}
\]

(22)

Assuming residual moments \(M_1^r\) and \(M_2^r\) acting at the ends of a beam element equation (22) can be written as follow:

\[
\left\{ \begin{array}{l}
m_{sh} = \max \\
\quad GM^r = 0 \\
\quad -M^p \leq M^r + \max M^e \leq M^p \\
\quad M^e = T^{-1}GK^{-1}m_{sh}P_0 \\
\quad \frac{1}{6E} \sum_{i=1}^{n} \int_0^{l_i} [(M_1^f)^2 + (M_1^r)(M_2^r) + (M_2^r)^2] \quad - C_{r0} \leq 0
\end{array} \right\}
\]

(23)
Supposing probabilistic problem the bound on the CSERF is given randomly as explained in section 2.2.

5. Numerical example
To evaluate the method proposed above a simple frame with SRC is considered (Figure 2). The constant load values are \( P_0^{(1)} = 10 \text{KN} \) and \( P_0^{(2)} = 15 \text{KN} \). At the beam to column connections joints the steel frame has semi-rigid connection. \( EI_{column} = 8172.4 \cdot 10^6 \text{KNcm}^2 \), \( EI_{beam} = 4080.3 \cdot 10^6 \text{KNcm}^2 \). First moment of area of columns and beam are \( S_{c0} = 215 \text{cm}^3 \) and \( S_{b0} = 130 \text{cm}^3 \), respectively. The influence of SRC are considered by using spring coefficient in the stiffness matrix for beam elements. The aim is to find maximum shakedown load multiplier for given SRC and bounds on the CSERF.

![Figure 2. Frame structure with SRC between beam and columns](image)

The numerical results using deterministic and probabilistic problems, are presented in Figures 3 and 4, respectively. In Figure 3 the variation of the shakedown load-multiplier is presented in function of the connection rigidities and limit values on the CSERF. One can see that the limit values on the CSERF and the stiffnesses of the SRC influences significantly the plastic behavior of the frame structure.

In Figure 4 safe domain of the structure for five different load combinations is illustrated. For each loading case \([m_1P_1, m_2P_2]\) a shakedown load-multiplier \( m_{sh} \) can be determined. Using these load-multipliers a safe domain, for five load combinations, can be generated in \( m_1 \) and \( m_2 \) plane. Mean value is \( \bar{C}_{r0} = 1.5 \), standard deviation \( \sigma_C = 0.1 \) and failure probability of the structure \( (P_f = 0.00069; 0.000013; 10 \times 10^{-7}) \). The numerical results are in agreement with the expectation that the increase of the probability of failure results in bigger load-multipliers.

![Figure 3. Variation of the shakedown load-multiplier in function of the connection rigidities and limit values on the CSERF.](image)
6. Conclusions

In this research optimal plastic shakedown analysis introduced to evaluate the maximum shakedown load-multipliers. Furthermore to control the post yield behaviour and residual displacements of the structures deterministic and probabilistic constraints are applied on the CSERF. The SRC between the beam and columns is described using spring coefficient in the stiffness matrix. Shakedown load-multipliers are calculated and safe domain of the structure for deterministic and probabilistic conditions under different load combinations is illustrated, reliable bound on the given probability of failures are presented. The numerical results show that the constraints on CSERF, stiffness of the SRC and probabilistic given conditions can influence on the magnitude of the shakedown load-multipliers.

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