Outlier-robust sparse/low-rank least-squares regression and robust matrix completion

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Abstract

We consider high-dimensional least-squares regression when a fraction $\epsilon$ of the labels are contaminated by an arbitrary adversary. We analyze such a problem in the statistical learning framework with a subgaussian distribution and linear hypothesis class on the space of $d_1 \times d_2$ matrices. As such, we allow the noise to be heterogeneous. This framework includes sparse linear regression and low-rank trace-regression. For a $p$-dimensional $s$-sparse parameter, we show that a convex regularized $M$-estimator using a sorted Huber-type loss achieves the near-optimal subgaussian rate $\sqrt{s \log(ep/s)} + \sqrt{\log(1/\delta)/n} + \epsilon \log(1/\epsilon)$, with probability at least $1 - \delta$. For a $(d_1 \times d_2)$-dimensional parameter with rank $r$, a nuclear-norm regularized $M$-estimator using the same sorted Huber-type loss achieves the subgaussian rate $\sqrt{r d_1/n} + \sqrt{r d_2/n} + \sqrt{\log(1/\delta)/n} + \epsilon \log(1/\epsilon)$, again optimal up to a log factor. In a second part, we study the trace-regression problem when the parameter is the sum of a matrix with rank $r$ plus a $s$-sparse matrix assuming the "low-spikeness" condition. Unlike multivariate regression studied in previous work, the design in trace-regression lacks positive-definiteness in high-dimensions. Still, we show that a regularized least-squares estimator achieves the subgaussian rate $\sqrt{r d_1/n} + \sqrt{r d_2/n} + s \log(d_1 d_2)/n + \sqrt{\log(1/\delta)/n}$. Lastly, we consider noisy matrix completion with non-uniform sampling when a fraction $\epsilon$ of the sampled low-rank matrix is corrupted by outliers. If only the low-rank matrix is of interest, we show that a nuclear-norm regularized Huber-type estimator achieves, up to log factors, the optimal rate adaptively to the corruption level. The above mentioned rates require no information on $(s, r, \epsilon)$.

1 Introduction

Outlier-robust estimation has been a topic studied for many decades since the seminal work by Huber [44]. One of the objectives of the field is to devise estimators which are less sensitive to outlier sample contamination. The formalization of outlyingness and the construction of robust estimators matured in several directions. One common assumption is that the adversary can only change a fraction $\epsilon$ of the original sample. For an extensive overview we refer, e.g., to Hampel et al. [42], Maronna et al. [61], Huber and Ronchetti [45] and references therein.

Within a very general framework, the minimax optimality of several robust estimation problems have been recently obtained in a series of elegant works by Chen et al. [16, 17], Gao [40]. The construction, however, is based on Tukey’s depth, a hard computational problem in higher dimensions. A recent trend of research, initiated by Diakonikolas et al. [32], Lai et al. [52], has focused in obtaining optimality of robust estimators within the class of computationally tractable algorithms. The oblivious model assumes the contamination is independent of the original sample. The above mentioned works also establish
optimality within the adversary model where the outliers may depend arbitrarily on the sample. As an example, near optimal mean estimators within the adversarial model can now be computed in nearly-linear time [20, 34, 27, 23]. We refer to the recent survey [29] for an extensive survey.

In the realm of robust linear regression, two broad lines of investigations exist: (1) one in which only the response (label) is contaminated and (2) the more general model in which the covariate (feature) is also corrupted [30, 31]. Both have been considered within the adversarial or oblivious model. For instance, an interesting property of model (1) with oblivious contamination is the existence of consistent estimators, a property not shared by the adversary model. See for instance the recent papers [80, 6, 77, 41, 72]. Model (1), albeit being more restrictive, has been the object of numerous past and recent works. It has also some connection with the problems of robust matrix completion [14, 18, 87, 56, 50] and matrix decomposition [82, 14, 86, 43, 2], at least when only the low-rank parameter is of interest. We refer to Section 3 for further references and discussion.

In this work, we revisit adversarial outlier-robust least-squares regression. We are particularly interested in the high-dimensional scaling, where not only the sample is corrupted by $o$ outliers but its sample size $n$ is prohibitively smaller than the extrinsic dimension $p$. More precisely, we focus on robust least-squares regression with adversarial response contamination and subgaussian data, paying attention to the following points:

(a) **High dimensions.** We consider the general framework of trace-regression with matrix parameters in $\mathbb{R}^p := \mathbb{R}^{d_1 \times d_2}$ when $n \ll p := d_1 d_2$. It includes in particular $s$-sparse linear regression [79], $r$-low-rank noisy compressed sensing [67] and noisy robust matrix completion [50]. Of practical relevance in the established theory of high-dimensional estimation is adaptation to $(s, r)$. We wish to avoid knowledge of $(s, r, o)$, at least in the particular setting of label contamination.

(b) **Noise heterogeneity.** A large portion of the literature in outlier-robust sparse linear regression assumes the covariates and noise are independent. This model is relevant in its own right. However, we wish to assume no particular assumptions between noise and covariates and consider the statistical learning framework with the linear hypothesis class.

(c) **Subgaussian rates.** Significant effort has been paid recently in obtaining minimax rates also with respect to the failure probability $\delta$. Just to illustrate what this means, consider the challenging problem of estimating the mean of a heavy-tailed $p$-dimensional random vector with identity covariance. Minsker’s original bound [64] in this case is $\sqrt{p \log(1/\delta)/n}$. The “subgaussian” rate $\sqrt{p/n + \log(1/\delta)/n}$ was obtained recently after a series of works and include efficient estimators. For lack of space, we refer to the survey [60]. The relevant point of the subgaussian rate is that $\log(1/\delta)$ does not multiply the “effective dimension” $p$. In our specific setting, that is, linear regression with adversarial response contamination, subgaussian data and validity of (a)-(b), we wonder about similar goals.

(d) **Matrix decomposition.** With identity designs, this problem was first considered in the papers [82, 14, 86, 43] under the “incoherence” condition. Agarwal et al. [2] considered identity or random designs with the “low-spikeness” condition. For instance, their general framework includes multivariate regression with random designs and low-rank plus sparse components. In this setting, one fact used is that the design operator is positive definite (one has $n \geq d_1$ albeit $n \ll d_1 d_2$). Unfortunately, the same property does not hold in trace-regression. Motivated by the problem of noisy matrix compressed sensing [67], we wonder if a correspondent theory exists for the problem of trace-regression with low-rank plus sparse components.

The rest of the paper is organized as follows. We state our set-up in Section 2. Contributions and related work are discussed in Section 3. The main results are stated formally in Section 4. Proofs are given in Sections 5, 7 and 6. Useful technical results are stated in the Appendix.
2 Framework

2.1 Sparse and trace regression

Let \((\mathbf{X}, y) \in \mathbb{R}^p \times \mathbb{R}\) be a zero mean covariate-response pair. Within a statistical learning framework, we wish to explain \(y\) through \(\mathbf{X}\) via the linear class \(\mathcal{F} = \{ \langle \cdot, \mathbf{B} \rangle : \mathbf{B} \in \mathbb{R}^p \}\), where \(\langle \mathbf{A}, \mathbf{B} \rangle := \text{tr}(\mathbf{A}^\top \mathbf{B})\) denotes standard inner product. Precisely, giving a sample of \((\mathbf{X}, y)\), we wish to estimate

\[
\mathbf{B}^* \in \arg\min_{\mathbf{B} \in \mathbb{R}^p} \mathbb{E} \| y - \langle \mathbf{X}, \mathbf{B} \rangle \|^2.
\]

In particular, one has \(y = \langle \mathbf{X}, \mathbf{B}^* \rangle + \xi\) with a random noise \(\xi \in \mathbb{R}\) satisfying \(\mathbb{E}[\xi \mathbf{X}] = 0\). We will assume a contaminated sample according to the following model.

**Assumption 1** (Adversarial response contamination). Let \(\{(y^i, \mathbf{X}^i)\}_{i \in [n]}\) be an iid sample from the distribution of \((\mathbf{X}, y)\). We assume available a sample \(\{(y_i, \mathbf{X}_i)\}_{i \in [n]}\) such that \(\mathbf{X}_i = \mathbf{X}^i\) for all \(i \in [n]\) and \(o\) arbitrary outliers replace the response sample \(\{y^i\}_{i \in [n]}\). We denote the fraction of contamination by \(\epsilon := \frac{o}{n}\).

Let us define the design operator with components \(\mathcal{X}_i(\mathbf{B}) := \langle \mathbf{X}_i, \mathbf{B} \rangle\). Under Assumption 1, one may write

\[
y = \mathcal{X}(\mathbf{B}^*) + \sqrt{n}\mathbf{\theta}^* + \xi,
\]

where \(y = (y_i)_{i \in [n]}, \xi = (\xi_i)_{i \in [n]}\) is an iid copy of \(\xi\) and \(\mathbf{\theta}^* \in \mathbb{R}^n\) is an arbitrary and unknown corruption vector having at most \(o\) nonzero components.

**Example 1** (Sparse linear regression). In sparse linear regression [81, 8], we have \(\mathbf{x} := \mathbf{X} \in \mathbb{R}^p\) and \(\mathbf{b}^* := \mathbf{B}^* \in \mathbb{R}^p\) with at most \(s \ll p\) nonzero coordinates \((d_1 = p, d_2 = 1)\). One may write the design as \(\mathcal{X}(\mathbf{b}) := \mathcal{X}\mathbf{b}\) where \(\mathcal{X}\) is the design matrix with \(i\)th row \(\mathbf{x}_i^\top\).

**Example 2** (Trace-regression). In trace-regression, the parameter \(\mathbf{B}^* \in \mathbb{R}^p\) is assumed to have rank \(r \ll d_1 \wedge d_2\).

Given nonincreasing positive sequence \(\{\omega_i\}_{i \in [n]}\), the Slope norm [9] at a point \(\mathbf{u} \in \mathbb{R}^n\) is defined by

\[
\|\mathbf{u}\|_\omega := \sum_{i \in [n]} \omega_i |u_i|,
\]

where \(u_i^\downarrow \leq \ldots \leq u_n^\downarrow\) denotes the nonincreasing rearrangement of the absolute coordinates of \(\mathbf{u}\). For the rest of the paper, we will fix the sequence \(\omega := \{\omega_i\}_{i \in [n]}\) to be \(\omega_i := \log(2/i)\).

**Definition 1** (Sorted Huber regression). Given \(\tau > 0\), define the loss \(\rho_{\tau, \omega} : \mathbb{R}^n \rightarrow \mathbb{R}_+\) to be the optimal value of the proximal map associated to the Slope norm \(\tau\|\cdot\|_\omega\). Precisely,

\[
\rho_{\tau, \omega}(\mathbf{u}) := \min_{\mathbf{z} \in \mathbb{R}^n} \frac{1}{2}\|\mathbf{z} - \mathbf{u}\|_\omega^2 + \tau\|\mathbf{z}\|_\omega.
\]

For tuning parameters \(\tau, \lambda > 0\) and convex norm \(\mathcal{R}\) over \(\mathbb{R}^p\), let

\[
\hat{\mathbf{B}} \in \arg\min_{\mathbf{B} \in \mathbb{R}^p} \rho_{\tau, \omega} \left( \frac{y - \mathcal{X}(\mathbf{B})}{\sqrt{n}} \right) + \lambda\mathcal{R}(\mathbf{B}).
\]

Note that when equal weights are used \(\omega_1 = \ldots = \omega_n\), the estimator (2) corresponds to regression with the Huber loss. A well known fact in the literature is that linear regression with Huber’s loss
corresponds to a Lasso-type estimator in the augmented variable \([B; \theta] \in \mathbb{R}^p \times \mathbb{R}^n\) [75, 35]. Similarly, estimator (2) is equivalent to the following augmented least-squares estimator:

\[
(\hat{B}, \hat{\theta}) \in \arg\min_{[B, \theta] \in \mathbb{R}^p \times \mathbb{R}^n} \frac{1}{2n} \| y - X(B) - \sqrt{n} \theta \|^2_2 + \lambda \mathcal{R}(B) + \tau \| \theta \|_2.
\]

If \(\mathcal{R}\) is either the \(\ell_1\)-norm or the nuclear norm, one practical appeal of problem (3) is that it may be computed by alternated convex optimization using jointly Lasso or matrix-Lasso and Slope solvers [9].

### 2.2 Trace-regression with matrix decomposition

Agarwal et al. [2] considered a general framework for the matrix decomposition problem: to estimate a pair \([B^*; \Gamma^*] \in (\mathbb{R}^p)^2\) given a noisy linear observation of its sum. In the case of random design and noise, their model is

\[
Y = X(B^* + \Gamma^*) + W,
\]

where the design \(X : \mathbb{R}^p \to \mathbb{R}^{n_1 \times n_2}\) takes values on matrices with \(n_1\) iid rows and \(W \in \mathbb{R}^{n_1 \times n_2}\) is noise independent of \(X\) with \(n_1\) centered iid rows. One application with random design considered in [2] is multi-task learning with \(d_1\) “features” and \(d_2\) “tasks” and normal covariate \(x \in \mathbb{R}^{d_1}\). In this case, \(X(B) := B^\top x\).

Motivated by the correspondent problem in matrix “compressed sensing” [67], we investigate matrix decomposition in the trace regression problem within a statistical learning framework: to estimate the pair

\[
[B^*; \Gamma^*] \in \arg\min_{[B, \Gamma] \in (\mathbb{R}^p)^2} \mathbb{E}[\|y - \langle X, B + \Gamma \rangle\|^2],
\]

given sample of \((X, y) \in \mathbb{R}^p \times \mathbb{R}\). In particular, one has \(y = \langle X, B^* + \Gamma^* \rangle + \xi\) with \(\xi \in \mathbb{R}\) satisfying \(\mathbb{E}[\xi X] = 0\). In high-dimensions, one assumes that \(B^*\) has low-rank and \(\Gamma^*\) is sparse. If \(\{(y_i, X_i)\}_{i \in [n]}\) is an iid sample of \((y, X)\) and \(\{(\xi_i)_{i \in [n]}\) an iid copy of \(\xi\), one may write

\[
y = X(B^* + \Gamma^*) + \xi,
\]

where \(X : \mathbb{R}^p \to \mathbb{R}^n\) is as in Section 2.1, \(y = (y_i)_{i \in [n]}\) and \(\xi = (\xi_i)_{i \in [n]}\).

Following [2], we consider the assumption:

**Assumption 2 (Low-spikeness).** Assume \(X\) is isotropic, there is, \(\mathbb{E}[\|XV\|^2] = \|V\|_F^2\) for all \(V \in \mathbb{R}^p\). Moreover, assume there exists \(a^* > 0\) such that

\[
\|B^*\|_\infty \leq \frac{a^*}{\sqrt{n}}.
\]

**Remark 1.** The low-spikeness condition in [2] is \(\|B^*\|_\infty \leq \frac{a^*}{\sqrt{d_1 d_2}}\) for some \(a^* > 0\). In high-dimensions \((n \leq d_1 d_2)\) and assuming isotropy, it implies Assumption 2.

If \(\|\cdot\|_N\) and \(\|\cdot\|_1\) denote respectively the nuclear and \(\ell_1\)-norms in \(\mathbb{R}^p\), we consider the estimator

\[
(\hat{B}, \hat{\Gamma}) \in \arg\min_{[B, \Gamma] \in (\mathbb{R}^p)^2} \frac{1}{2n} \|y - X(B + \Gamma)\|^2_2 + \lambda \|B\|_N + \tau \|\Gamma\|_1
\]

s.t. \(\|B\|_\infty \leq \frac{a^*}{\sqrt{n}}\)
2.3 Robust matrix completion

The matrix completion problem consists in estimating the low-rank matrix $\mathbf{B}^* \in \mathbb{R}^p$ having sampled an incomplete subset of its entries. Several works have obtained statistical bounds for this problem using nuclear norm relaxation \cite{37, 76}, either with the “incoherence” condition \cite{13}, the “low-spikeness” condition \cite{68} or assuming an upper bound on the sup-norm of $\mathbf{B}^* \in \mathbb{R}^p$ \cite{51, 49}. Note that matrix completion may be equivalently seen as the trace-regression problem with $\mathbf{X}$ having a discrete distribution $\Pi$ supported on the canonical basis

$$\mathcal{X} := \{e_j \bar{e}_k^\top : j \in [d_1], k \in [d_2]\},$$

where $e_j$ is the $j$th canonical vector in $\mathbb{R}^{d_1}$ and $\bar{e}_k$ is the $k$th canonical vector in $\mathbb{R}^{d_2}$. The $i$th sample is thus $\mathbf{X}_i = e_j(i) \bar{e}_k(i)$.

In robust matrix completion, a fraction of the sampled entries are corrupted by outliers. Klopp et al. \cite{50} consider this problem in the noisy setting through the lens of the matrix decomposition model (4) where the corruption matrix $\mathbf{\Gamma}^* \in \mathbb{R}^p$ have at most $o$ nonzero entries.\footnote{They also consider the columnwise corruption model.} Optimal minimax bounds are derived on the class of parameters $[\mathbf{B}^*, \mathbf{\Gamma}^*]$ with bounded sup norm, $\mathbf{B}^*$ with low-rank and sparse $\mathbf{\Gamma}^*$.

If only the sampled low-rank matrix $\mathbf{B}^*$ is of interest, one may equivalently see the same problem as a response adversarial trace regression problem (1) under Assumption 1. In this work we use this point of view on the class of parameters $\mathbf{B}^*$ with low-rank and bounded sup norm. We consider the constrained estimator:

$$\hat{\mathbf{B}}, \hat{\theta} \in \arg\min_{\mathbf{B} \in \mathbb{R}^{d_1 \times d_2}, \theta \in \mathbb{R}^n} \frac{1}{2} \| \mathbf{y} - \mathcal{X}(\mathbf{B}) - \sqrt{n} \theta \|_2^2 + \lambda \| \mathbf{B} \|_\infty + \tau \| \theta \|_\Delta \quad \text{s.t.} \quad \| \mathbf{B} \|_\infty \leq a.$$ \hspace{1cm} (7)

3 Contributions and related work

3.1 Robust sparse least-squares regression

For sparse least-squares regression with adversarial response contamination and subgaussian $(\mathbf{x}, \xi)$, we show that estimator (2) achieves the subgaussian rate

$$r_n + \sqrt{\log(1/\delta)/n + \epsilon \log(1/\epsilon)}$$

with breakdown point $\epsilon \leq O(1)$. Here, $r_n = \sqrt{s \log p/n}$ taking $\mathcal{R}$ to be the $\ell_1$-norm and $r_n = \sqrt{s \log(ep/s)/n}$ taking $\mathcal{R}$ to be the Slope norm on $\mathbb{R}^p$ (see Theorem 1). This rate is optimal up to a $\sqrt{\log(1/\epsilon)}$ factor \cite{16, 17, 40}. These bounds are attained with no information on $(s, o)$ and within the statistical learning framework with the linear class. To the best of our knowledge, previous works on sparse linear regression with response contamination have assumed the noise independent of the features.

Sparse linear regression with response contamination has been the subject of numerous works. From a methodological point of view, the $\ell_1$-penalized Huber’s estimator has been considered in \cite{74, 75, 55}. Empirical evaluation for the choice of tuning parameters is comprehensively studied in these papers. As already observed in past work \cite{75, 35}, Huber’s estimator with $\ell_1$-penalization is equivalent to the
augmented estimator

$$\begin{align*}
(b, \theta) \in \arg\min_{(b, \theta) \in \mathbb{R}^p \times \mathbb{R}^n} \frac{1}{2n} \|y - Xb - \sqrt{p}\theta\|_2^2 + \lambda \|b\|_1 + \tau \|\theta\|_1.
\end{align*}$$

In the response adversarial model with Gaussian data, fast rates for such estimator have been obtained in [12, 53, 25, 26, 69]. The minimax optimality of estimator (8), up to log factors, was achieved only recently in [24], showing it satisfies the rate $\sqrt{s \log(p/\delta)}/n + \epsilon \log(n/\delta)$ and breakdown point $\epsilon \leq O(1/\log n)$. This was later refined and generalized in [22], where the subgaussian minimax rate was shown to hold for the $l_1$-penalized Huber’s estimator in the subgaussian set-up, albeit assuming knowledge of $(s, \alpha)$ and noise independent of features. We also remark that this estimator has been considered in different contamination models with dense bounded noise in [85, 56, 70, 38, 1]. This setting is also studied in [46] with the LAD-estimator [84]. Alternatively, a refined analysis of iterative thresholding methods were considered in [7], [6], [77], [65]. They obtain sharp breakdown points and consistency bounds for the oblivious model. Works on sparse linear regression with covariate contamination were considered early on by Chen et al. [19] and, more recently, in [3], albeit still attaining suboptimal rates. Works by Loh and Wainwright [57, 58] have also studied the optimality of sparse linear regression in models with error-in-variables and missing-data covariates.

Although out of scope, we mention for completeness that tractable algorithms for linear regression with covariate contamination have been intensively investigated in the low-dimensional scaling $(n \geq p)$, with initial works by Diakonikolas et al. [30, 31], Prasad et al. [73] and more recent ones in [28, 21, 71].

### 3.2 Robust low-rank trace-regression

**Trace-regression with subgaussian designs.** Negahban and Wainwright [67], Negahban et al. [66] proposed a general framework of $M$-estimators with decomposable regularizers. Among several different results, they obtain optimal rates for trace-regression with Gaussian designs for the first time. Precisely, they attain the minimax rate $\sqrt{rd_1/n} + \sqrt{rd_2/n}$ with failure probability $e^{-c(d_1 + d_2)}$ and noise independent of $X$. Inspired by their results and with the objectives (a)-(c) of Section 1 in mind, we consider trace-regression with response adversarial corruptions and subgaussian data. Within the statistical learning framework, we show that estimator (2) with nuclear norm regularization achieves the subgaussian rate

$$\sqrt{rd_1/n} + \sqrt{rd_2/n} + \sqrt{\log(1/\delta)/n} + \epsilon \log(1/\epsilon)$$

with failure probability $\delta$ and breakdown point $\epsilon \leq O(1)$ (see Theorem 2). This rate is optimal up to a $\sqrt{\log(1/\epsilon)}$ factor [16, 17, 40]. We are not aware of similar bounds with efficient estimators.

**Trace-regression with matrix decomposition.** Several works studied the problem of matrix decomposition with identity designs and the “incoherence” condition [14, 82, 43, 87]. Alternatively, Agarwal et al. [2] viewed such problem in a general framework assuming “low-spikeness”. Among several applications, multi-task learning is one of them. We consider the different problem of matrix compressed sensing [67] with $r$-low-rank plus $s$-sparse components satisfying the “low-spikeness” condition and an isotropic design (see Section 2.2). Unlike multi-task learning, the design is not positive definite in high-dimensions, so an alternative argument is required. Under Assumption 2 and subgaussian design, we show that estimator (5) attains the subgaussian rate

$$\sqrt{rd_1/n} + \sqrt{rd_2/n} + \sqrt{s \log p/n} + a^* \sqrt{s/n} + \sqrt{\log(1/\delta)/n}$$

with failure probability $\delta$ (see Theorem 3). This is valid within the statistical learning framework and with no information on $(r, s)$. We are not aware of previous work establishing such rates for this problem.

**Robust matrix completion.** The literature on matrix completion using nuclear norm relaxation [37, 76] is extensive. For instance, bounds for exact recovery were first obtained by Candès and Recht [13] where the notion of “incoherence” was introduced. Negahban and Wainwright [68] considers noisy matrix
completion with the different notion of “low-spikeness”. Several works on exact and noisy set-up exist. As we are mainly interested in the corrupted model, a complete overview is out of scope. We refer to further references in [49, 50] and the recent work [15] for a comprehensive review on matrix completion under the “incoherence” assumption.

More related to our work are [59, 33, 47, 48, 51, 49, 50, 11]. In these papers the main assumption is that an upper bound on the parameter sup-norm $\| \cdot \|_\infty$ is known. Assuming known $a > 0$ such that $a \geq \| B^* \|_\infty \vee \| \Gamma^* \|_\infty$, Klopp et al. [50] obtained optimal minimax bounds for noisy robust matrix completion within the model (4) over the class $A(r, o, a) = \{ B + \Gamma \in \mathbb{R}^p : \text{rank}(B) \leq r, \| \Gamma \|_0 \leq o, \| B \|_\infty \vee \| \Gamma \|_\infty \leq a \}$.

As usual, $\| A \|_0$ denotes the number of nonzero entries of matrix $A$. Assuming known $a > 0$ such that $a \geq \| B \|_\infty$, we consider noisy robust matrix completion within model (1) on the different class $\tilde{A}(r, o, a) = \{ [B; \theta] \in \mathbb{R}^p \times \mathbb{R}^n : \text{rank}(B) \leq r, \| \theta \|_0 \leq o, \| B \|_\infty \leq a \}$.

For simplicity let us assume $d_1 = d_2 = d$. Under similar distributional assumptions of [50], we show that estimator (7) attains, up to logs, the rate

$$r_n + a \sqrt{\log(1/\delta)/n} + a \sqrt{\epsilon \log(1/\epsilon)},$$

with $r_n = \sqrt{dr/n \log(1/\delta)}$ (see Theorem 4). This rate is near-optimal on the class $\tilde{A}(r, o, a)$. Finally, we note that our bounds are not fully comparable to Klopp et al. [50] whose aim is to estimate the pair $[B^*; \Gamma^*]$. If only the low-rank $B^*$ is of interest, however, our bounds show some improvement as they depend only on the sup norm of $B^*$ and no prior information on the corruption level is relevant.

We finish with some technical considerations. Related to our work is [24], who considered sparse linear regression in the Gaussian setting with independent noise. In establishing rates within the statistical learning framework, we rely on concentration inequalities for the multiplier process (MP) and product process (PP). Impressive general bounds for these processes were obtained by Mendelson [62] valid for classes having only a few bounded moments. In establishing subgaussian rates (see Section 1 item (c)), our arguments need concentration inequalities for the MP and PP with some improvement concerning the confidence level. Tailored specifically to subgaussian classes, these are proven in Theorems 9 and 10 in the Appendix. Their proof is based on the methods by Dirksen [33], Bednorz [4] for the quadratic process. The proof of Theorem 4 for robust matrix completion is inspired by [68, 50]. One difference is that we use a one-sided tail inequality by Bousquet in order to derive a sufficient lower bound. This lower bound readily suggests a cone in $\mathbb{R}^p \times \mathbb{R}^n$ at which a restricted eigenvalue condition is satisfied for the augmented design $\mathcal{M}(B; \theta) := X(B) + \sqrt{n} \theta$. Finally, our analysis is also inspired by the findings in Bellec et al. [5] regarding regularization with the Slope norm [9].

## 4 Main results

We first present some notation. We say that $a \overset{\leq}{\sim} b$ if $a \leq Cb$ for some absolute constant $C > 0$ and $a \overset{\sim}{\sim} b$ if $a \lesssim b$ and $b \lesssim a$. We denote the $\psi_2$-Orlicz norm by $\| \cdot \|_\psi$. Finally, $\| V \|_1 := \sqrt{\mathbb{E}[\langle X, V \rangle^2]}$ for any $V \in \mathbb{R}^p$.

The next three theorems assume the subgaussian framework.
Theorem 1. (Response adversarial sparse linear regression) In the framework of Section 2.1, suppose \( b^* \) in Example 1 is a \( s \)-sparse vector. Denote by \( \Sigma \) the covariance matrix of \( x \) and let \( \rho_1(\Sigma) := \max_{j \in [p]} \sqrt{\Sigma_{jj}} \).

Grant Assumptions 1 and 3 such that \( \sigma^2 L^2 c \log(1/\epsilon) \leq c \) for some universal constant \( c \in (0, 1) \). Define \( r_n := \sqrt{s \log p/n} \) and \( S = s \log p \).

In estimator (2), take \( \mathcal{R} \) to be the \( \ell_1 \)-norm and tuning parameters

\[
\tau \asymp \frac{\sigma}{\sqrt{n}}, \quad \gamma \asymp L \rho_1(\Sigma) \sqrt{\log p}.
\]

For absolute positive constants \( C, c_1 \) and \( c_2 \), constant \( \mu(b^*) \geq 0 \) and \( \delta \in (0, 1) \), assume that

\[
n \geq C \sigma^2 L^2 \rho_1^2(\Sigma) \mu^2(b^*) S, \quad \delta \geq \exp\left(-c_1 \frac{n}{L^2}\right) \sqrt{\exp\left(-c_2 \frac{n}{\sigma^2 L^2}\right)}.
\]

Then, with probability at least \( 1 - \delta \),

\[
\|\hat{b} - b^*\|_1 \leq L^3 \sigma^2 \rho_1^2(\Sigma) \mu^2(b^*) \cdot r_n^2 + L \rho_1(\Sigma) \mu(b^*) r_n
+ (L^3 \sigma^2 + L^2 \sigma^2) \frac{1 + \log(1/\delta)}{n} + L \sigma \frac{1 + \sqrt{\log(1/\delta)}}{\sqrt{n}}
+ \sigma \gamma \log(1/\epsilon).
\]

If one takes \( \mathcal{R} \) to be the Slope norm in \( \mathbb{R}^p \) with the sequence \( \hat{w}_j = 2 \log(p/j) \) and tuning \( \gamma \asymp L \rho_1(\Sigma) \), a similar bound holds but with \( S = s \log(ep/s) \) and \( r_n = \sqrt{s \log(ep/s)/n} \).

Theorem 2. (Response adversarial low-rank trace regression) In Example 2 in Section 2.1, suppose \( B^* \) has rank \( r \) and denote by \( \Sigma \) the covariance matrix of \( \text{vec}(X) \). Define the quantity \( \rho_N(\Sigma) := \sup_{\|x\|_2 = 1} \sqrt{\mathbb{E}[\|x^\top X v\|^2]} \).

Grant Assumptions 1 and 3 such that \( \sigma^2 L^2 c \log(1/\epsilon) \leq c \) for some universal constant \( c \in (0, 1) \). In estimator (2), take \( \mathcal{R} \) to be the nuclear norm and tuning parameters \( \tau \asymp \frac{\sigma}{\sqrt{n}} \) and

\[
\gamma := C L \rho_N(\Sigma)/(\sqrt{d_1} + \sqrt{d_2}).
\]

For absolute positive constants \( C, c_1 \) and \( c_2 \), constant \( \mu(B^*) \geq 0 \) and \( \delta \in (0, 1) \), assume that

\[
n \geq C \sigma^2 L^2 \rho_N^2(\Sigma) \mu^2(B^*) \cdot r(d_1 + d_2), \quad \delta \geq \exp\left(-c_1 \frac{n}{L^2}\right) \sqrt{\exp\left(-c_2 \frac{n}{\sigma^2 L^2}\right)}.
\]
Then, with probability at least $1 - \delta$,}
\[
\|\hat{B} - B^*\|_1 \lesssim L^3 \sigma^2 \rho_N^2(\Sigma) \mu^2(B^*) \left( \frac{r(d_1 + d_2)}{n} \right) + L \sigma \rho_N(\Sigma) \mu(B^*) \left( \sqrt{\frac{rd_1}{n}} + \sqrt{\frac{rd_2}{n}} \right)
\]
\[
+ (L^3 \sigma^2 + L^2 \sigma) \frac{1 + \log(1/\delta)}{n} + L \sigma \frac{1 + \sqrt{\log(1/\delta)}}{\sqrt{n}}
\]
\[+ L \sigma^2 \cdot \epsilon \log(\epsilon).\]

**Theorem 3** (Trace regression with matrix decomposition). In the problem of trace-regression with matrix decomposition of Section 2.2, suppose $B^*$ has rank $r$ and $\Gamma^*$ has at most $s$ nonzero entries. Grant Assumptions 2 and 3. In estimator (5), take tuning parameters
\[
\lambda \asymp \sigma L \left( \sqrt{\frac{d_1}{n}} + \sqrt{\frac{d_2}{n}} \right), \quad \tau \asymp \sigma L \frac{\log p}{n} + \frac{a^*}{\sqrt{n}}.
\]

For absolute positive constants $C, c_1$ and $c_2$, positive constants $\mu(B^*)$ and $\mu(\Gamma^*)$ and $\delta \in (0, 1)$, assume that
\[
n \geq C \left[ \mu^2(B^*) \sigma^2 L^2 \cdot r(d_1 + d_2) \right] \sqrt{\mu^2(\Gamma^*) \left( \sigma^2 L^2 \log p + (a^*)^2 \right) \cdot s},
\]
\[
\delta \geq \exp \left( -c_1 \frac{n}{L^2} \right) \exp \left( -c_2 \frac{n}{\sigma^2 L^2} \right).
\]

Then, with probability at least $1 - \delta$,
\[
\sqrt{\|\hat{B} - B^*\|_1^2 + \|\hat{\Gamma} - \Gamma^*\|_1^2} \lesssim \sigma L \mu(B^*) \left( \sqrt{\frac{rd_1}{n}} + \sqrt{\frac{rd_2}{n}} \right) + \sigma L \mu(\Gamma^*) \left( \sqrt{\frac{s \log p}{n}} + \frac{a^* \mu(\Gamma^*)}{\sqrt{n}} \right)
\]
\[+ \sigma L \frac{1 + \sqrt{\log(1/\delta)}}{\sqrt{n}} + \sigma^2 L^2 \frac{1 + \log(1/\delta)}{n}.
\]

We now state our results for the robust matrix completion problem under the following standard condition already assumed in [49, 50].

**Assumption 4.** Assume the random pair $(X, \xi) \in \mathbb{R}^p \times \mathbb{R}$ is such that

(i) $X$ has a discrete distribution $\Pi = \{(\sigma_{k, \ell})_{(k, \ell) \in [d_1] \times [d_2]}\}$ with support on $X$ defined in (6).
We define $d := d_1 + d_2$, $m := d_1 \wedge d_2$, $R_k := \sum_\ell \sigma_{k, \ell}$, $C_k := \sum_\ell \sigma_{k, \ell}$ and $L := \max_{k, \ell} \{R_k, C_k\}$.

(ii) $\sigma := \xi |_{\{\xi \in (1, \infty) \text{ and } \sigma_2^2 := E[\xi^2] \geq 1\}}$.

**Theorem 4.** (Robust matrix completion) In the robust matrix completion problem of Section 2.3, suppose $B^*$ has rank $r$. Grant Assumptions 1 and 4 and such that $\epsilon < 0.5$. Let $\delta \in (0, 1)$. In estimator (7), take $\mathcal{R}$ to be the nuclear norm, $a > 0$ such that $\|B^*\|_\infty \leq a$ and tuning parameters $\tau \asymp \frac{a}{\sqrt{n}}$ and
\[
\lambda \asymp \left( (\sigma \vee a) \sqrt{\frac{Lp}{mn}} \left( \log(4/\delta) + \log d \right) \right) \left( \frac{\sigma m}{\sigma_x^2} \sqrt{\frac{p}{n}} \cdot (\log(4/\delta) + \log d) \right).
\]
Assume that for absolute constants $C$, $c > 0$,

\[
    n \geq C \left( \frac{\log m}{m} \right) \sqrt{\frac{\log \log m}{\log d}},
\]

\[\delta \geq \exp \left(-cn\right).
\]

Then, for some constant $\mu(B^*) > 0$, with probability at least $1 - \delta$,

\[
    \sqrt{||\hat{B} - B^*||_2^2 + ||\theta - \theta^*||_2^2} \leq (a \vee \sigma) \mu(B^*) \sqrt{\frac{Lp}{mn}} \sqrt{\log(4/\delta) + \log d} + \frac{(a \vee \sigma) \log^{1/2} \left( \frac{\sigma m}{\sigma \varepsilon} \right) \mu(B^*)}{n} \cdot (\log(4/\delta) + \log d) + (a \vee \sigma) \frac{1 + \log(2/\delta)}{\sqrt{n}} + (a \vee \sigma) \varepsilon \log 1/\delta.
\]

Let us make a few comments about Theorem 4. A similar argument in [50] shows that the rate of Theorem 4 is optimal up to log factors over the class $\mathcal{A}(\sigma, \theta, a)$ in (9). If $\Pi$ is the uniform distribution then $L = 1$ and $\mu(B^*) = 1$. This rate is also meaningful over a large class of non-uniform sampling distributions $\Pi$ having $L$ and $\mu(B^*)$ of reasonable magnitudes (see Remark 2 in the following). Finally, the estimator and corresponding rate in [50] depend on an upper bound on the corruption sup-norm. In many applications, it is reasonable to expect that only the low-rank matrix $B^*$ is of interest. In that specific setting, Theorem 4 reveals that only an upper bound on $||B^*||_\infty$ is of relevance.

**Remark 2.** (Restricted eigenvalue constants) In Theorem 1, $\mu(b^*) = \sup_{\|V\|_F} \frac{\|V\|_F}{\|V\|_H}$ is the usual restricted eigenvalue constant, where $\mathcal{E}$ is a dimension reduction cone associated to the $\ell_1$-norm and the "sparsity support" of $b^*$ (see Section 5.1). Similarly, in Theorems 2 and 3 $\mu(B^*) = \sup_{\|V\|_F} \frac{\|V\|_F}{\|V\|_H}$ with $\mathcal{E}$ a dimension reduction cone associated to the nuclear norm and the "low-rank support" of $B^*$. Analogous comments apply to $\mu(\Gamma^*)$. In Theorem 4, $\mu(B^*) = \sup_{\|V\|_F} \frac{\|V\|_F}{\|V\|_H}$. For the uniform distribution, $\mu(B^*) = 1$ so $\mu(B^*)$ is the condition number measuring how far $\Pi$ is from the uniform distribution.

### 4.1 Basic notation

We will use the standard norm notations in $\mathbb{R}^p$: $\| \cdot \|_p$ for $\ell_p$-norm, $\| \cdot \|_\infty$ for the nuclear norm and $\| \cdot \|_{op}$ for the operator norm. The inner product in $\mathbb{R}^f$ is denoted by $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^p$. Throughout the paper, $[\ell] = \{1, \ldots, n\}$ and $A^{(\ell)} := \frac{1}{\sqrt{\ell}} A$ whenever $A$ is a number, vector or function and $\ell \in \mathbb{N}$. With respect to the Slope norm in $\mathbb{R}^n$ with sequence $\omega = (\omega_i)_{i \in [n]}$, we set $\Omega := \sqrt{\sum_{i=1}^n \omega_i^2}$. Recall that by Stirling formula, $\Omega \asymp o \log(n/o) \asymp n$ [5].

We use the notations $\Delta^B := B - B^*$, $\Delta^F := F - F^*$, $\Delta^\theta := \theta - \theta^*$. It will be useful to define the augmented design $\Xi(B; \theta) := \mathcal{X}(B) + \sqrt{\Omega} \theta$.

Letting $\Pi$ be the distribution of $X$, we define the bilinear form $\langle \langle V, W \rangle \rangle_{\Pi} := \mathbb{E} \langle X, V \rangle \langle X, W \rangle$ and $L^2(\Pi)$ pseudo-distance $||V||_\Pi := \mathbb{E} \langle V, V \rangle_{\Pi}^{1/2}$. We denote by $\mathcal{E}$ the covariance operator of $X$, that is, the self-adjoint linear operator on $\mathbb{R}^p$ satisfying $\mathbb{E} \langle \mathcal{E}(V), W \rangle = \langle \langle V, W \rangle \rangle_{\Pi}$ for all $V, W$. It will be useful to define the augmented pseudo-norms $||V, u||_\Pi := \sqrt{||V||_\Pi^2 + ||u||_F^2}$ and $||V, W||_\Pi := \sqrt{||V||_\Pi^2 + ||W||_\Pi^2}$.

We define the unit balls $B_{\Pi} := \{ V \in \mathbb{R}^p : \| V \|_\Pi \leq 1 \}, B_F := \{ V \in \mathbb{R}^p : \| V \|_F \leq 1 \}$, $B_\mathcal{E} := \{ u \in \mathbb{R}^k : \| u \|_\mathcal{E} \leq 1 \}$. With a slight abuse of notation, we denote by $B_\mathcal{R} := \{ V \in \mathbb{R}^p : \mathcal{R}(V) \leq 1 \}$ and $B_\mathcal{Q} := \{ u \in \mathbb{R}^n : \mathcal{Q}(u) \leq 1 \}$ the correspondent unit balls for norms $\mathcal{R}$ on $\mathbb{R}^p$ and $\mathcal{Q}$ on $\mathbb{R}^n$. All the corresponding unit spheres will take the symbol $S$.

Finally, the Gaussian width of a compact set $B \subset \mathbb{R}^{k \times f}$ is the quantity $\mathcal{W}(B) := \mathbb{E} \sup_{V \in B} \langle \langle V, E \rangle \rangle$, where $E \in \mathbb{R}^{k \times f}$ is random matrix with iid $N(0, 1)$ entries.
5 Proofs of Theorems 1 and 2

Recall estimator (3).

Throughout Sections 5.1 and 5.2, \( \| \cdot \|_1 \) can be regarded as a generic pseudo-norm. Moreover, \( \{(X_i, \xi_i)\}_{i\in[n]} \) can be regarded as deterministic satisfying (1) with \( \xi = (\xi_i)_{i\in[n]} \) and \( X \) and the design and augmented design operators associated to the sequence \( \{X_i\}_{i\in[n]} \). Probabilistic assumptions are used only in Sections 5.3, 5.4 and 5.5.

5.1 Design properties, cones and restricted eigenvalues

Next we present some structural properties for the design operator \( X : \mathbb{R}^p \to \mathbb{R}^n \).

**Definition 2** (Transfer principles). Let \( R \) be a norm over \( \mathbb{R}^p \), \( Q \) be a norm over \( \mathbb{R}^n \) and subsets \( C \subset \mathbb{R}^p \) and \( C' \subset \mathbb{R}^p \times \mathbb{R}^n \).

(i) Given positive numbers \( a_1 \) and \( a_2 \), we say that \( X \) satisfies TP\(_R\) \((a_1; a_2)\) on \( C \) if

\[
\forall V \in C, \quad \| X^{(n)}(V) \|_2 \geq a_1 \| V \|_1 - a_2 R(V).
\]

(ii) Given positive numbers \( b_1, b_2 \) and \( b_3 \), we say that \( X \) satisfies IP\(_{R, Q}\)((b_1; b_2; b_3)) on if

\[
\forall [V; u] \in \mathbb{R}^p, \quad |(u, X^{(n)}(V))| \leq b_1 \| V \|_1 \| u \|_2 + b_2 R(V) \| u \|_2 + b_3 \| V \|_1 Q(u).
\]

(iii) Given positive numbers \( c_1, c_2 \) and \( c_3 \), we say that \( X \) satisfies ATP\(_{R, Q}\)((c_1; c_2; c_3)) if

\[
\forall [V; u] \in C', \quad \| X^{(n)}(V) + u \|_2 \geq c_1 \| V \|_1 \| u \|_1 - c_2 R(V) - c_3 Q(u).
\]

(iv) Given positive numbers \( f_1, f_2 \) and \( f_3 \), we will say that \( (X, \xi) \) satisfies MP\(_{R, Q}\)((f_1; f_2; f_3)) if

\[
\forall [V; u] \in \mathbb{R}^p, \quad |\langle \xi^{(n)}, X^{(n)}(V) + u \rangle| \leq f_1 \| V \|_1 \| u \|_1 + f_2 R(V) + f_3 Q(u).
\]

If either \( C = \mathbb{R}^p \) or \( C' = \mathbb{R}^p \times \mathbb{R}^n \), we omit the reference to the set where the above properties hold.

TP is essentially “restricted strong convexity” [58], a well known fundamental property in high-dimensional estimation. Indeed, TP\(_R\)(\(a_1; 0\)) on \( C \) is strong convexity of \( X^{(n)} \) on \( C \) with respect to the pseudo-norm \( \| \cdot \|_1 \).\(^2\)MP\(_{f_1; f_2; 0}\) implies a bound on the “multiplier process” \( V \mapsto \frac{1}{n} \sum_{i\in[n]} \xi_i \langle X_i, V \rangle \), also an essential property used in high-dimensional estimation.

The “augmented” notions of ATP, MP and IP will be useful for robust estimation with subgaussian designs. We will show in Section 5.3 that these properties are satisfied with high probability with universal constants \( a_1, c_1 \in (0, 1) \). The IP is a consequence of Chevet’s inequality [83]. From Lemma 1 in the following, ATP follows from TP and IP.

**Lemma 1** (TP + IP \(\Rightarrow\) ATP, Lemma 7 in [24]). Let \( R \) be a norm over \( \mathbb{R}^p \) and \( Q \) a norm over \( \mathbb{R}^n \) with unit ball \( B_\mathbb{R}^p \). Suppose \( X : \mathbb{R}^p \to \mathbb{R}^n \) satisfies TP\(_R\)(\(a_1; a_2\) and IP\(_{R, Q}\)((b_1; b_2; b_3)) for some positive numbers \( a_1, a_2, b_1, b_2 \) and \( b_3 \). Then, for any \( \alpha > 0 \), \( X \) satisfies the ATP\(_{R, Q}\)((c_1; c_2; c_3)) with constants \( c_1 = \sqrt{a_1^2 + b_1 - \alpha^2}, c_2 = a_2 + b_2/\alpha \) and \( c_3 = b_3/\alpha \). Taking \( \alpha = a_1/2 \), we obtain that ATP\(_{R, Q}\)((c_1; c_2; c_3)) holds with constants \( c_1 = \sqrt{(3/4)a_1^2 + b_1}, c_2 = a_2 + 2b_2/a_1 \) and \( c_3 = 2b_3/a_1 \).

\(^2\)To be precise, \( a_1 \) is an absolute constant for general classes of designs. The usual notion of restricted eigenvalue, as e.g. in [8, 5], is to respect the Frobenius norm with \( a_1 \) representing a “condition number”.

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We now recall the definition of decomposable norms [66, 51].

**Definition 3** (Decomposable norm). A norm $\mathcal{R}$ over $\mathbb{R}^p$ is said to be decomposable if for all $\mathbf{B} \in \mathbb{R}^p$, there exist linear map $\mathbf{V} \mapsto \mathbf{P}_\mathbf{B}(\mathbf{V})$ such that, for all $\mathbf{V} \in \mathbb{R}^p$, defining $\mathbf{P}_\mathbf{B}(\mathbf{V}) := \mathbf{V} - \mathbf{P}_\mathbf{B}^\perp(\mathbf{V})$,

- $\mathbf{P}_\mathbf{B}(\mathbf{0}) = 0$,
- $\langle \mathbf{P}_\mathbf{B}(\mathbf{V}), \mathbf{P}_\mathbf{B}(\mathbf{V}) \rangle = 0$,
- $\mathcal{R}(\mathbf{V}) = \mathcal{R}(\mathbf{P}_\mathbf{B}(\mathbf{V})) + \mathcal{R}(\mathbf{P}_\mathbf{B}^\perp(\mathbf{V}))$.

In particular, $\|\mathbf{V}\|_p^2 = \|\mathbf{P}_\mathbf{B}(\mathbf{V})\|_p^2 + \|\mathbf{P}_\mathbf{B}^\perp(\mathbf{V})\|_p^2$. For $\mathbf{V} \in \mathbb{R}^p \setminus \{0\}$, we define

$$\Psi_\mathcal{R}(\mathbf{V}) := \frac{\mathcal{R}(\mathbf{V})}{\|\mathbf{V}\|_p}.$$

We omit the subscript $\mathcal{R}$ when it is clear in the context.

Two well known examples are the $\ell_1$ and nuclear norms.

**Example 3** ($\ell_1$-norm). Given $\mathbf{B} \in \mathbb{R}^p$ with sparsity support $\mathcal{S}(\mathbf{B}) := \{[j,k] : \mathbf{B}_{j,k} \neq 0\}$, the $\ell_1$-norm in $\mathbb{R}^p$ satisfies the above decomposability condition with the map $\mathbf{V} \mapsto \mathbf{P}_{\mathbf{S}(\mathbf{B})}(\mathbf{V}) := \mathbf{V}_{\mathbf{S}(\mathbf{B})}^\perp$ where $\mathbf{V}_{\mathbf{S}(\mathbf{B})}^\perp$ denotes the $d_1 \times d_2$ matrix whose entries are zero at indexes in $\mathcal{S}(\mathbf{B})$.

**Example 4** (Nuclear norm). Let $\mathbf{B} \in \mathbb{R}^p$ with rank $r := \text{rank}(\mathbf{B})$, singular values $\{\sigma_j\}_{j \in [r]}$ and singular vector decomposition $\mathbf{B} = \sum_{j \in [r]} \sigma_j \mathbf{u}_j \mathbf{v}_j^\top$. Here $\{\mathbf{u}_j\}_{j \in [r]}$ are the left singular vectors spanning the subspace $\mathcal{U}$ and $\{\mathbf{v}_j\}_{j \in [r]}$ are the right singular vectors spanning the subspace $\mathcal{V}$. The pair $(\mathcal{U}, \mathcal{V})$ is sometimes referred as the low-rank support of $\mathbf{B}$. Given subspace $\mathcal{S} \subset \mathbb{R}^r$ let $\mathbf{P}_{\mathcal{S}^\perp}$ denote the matrix defining the orthogonal projection onto $\mathcal{S}^\perp$. Then, the map $\mathbf{V} \mapsto \mathbf{P}_\mathbf{B}(\mathbf{V}) := \mathbf{P}_{\mathcal{U}^\perp} \mathbf{V} \mathbf{P}_{\mathcal{V}^\perp}$ satisfy the decomposability condition for the nuclear norm $\|\cdot\|_N$.

Decomposability is mainly useful because of the following well-known lemmas [66, 5].

**Lemma 2.** Let $\mathcal{R}$ be a decomposable norm over $\mathbb{R}^p$. Let $\mathbf{B}, \tilde{\mathbf{B}} \in \mathbb{R}^p$ and $\mathbf{V} := \tilde{\mathbf{B}} - \mathbf{B}$. Then, for any $\nu \in [0, 1]$,

$$\nu \mathcal{R}(\mathbf{V}) + \mathcal{R}(\mathbf{B}) - \mathcal{R}(\tilde{\mathbf{B}}) \leq \mathcal{R}(\mathbf{P}_\mathbf{B}(\mathbf{V})) - (1 - \nu) \mathcal{R}(\mathbf{P}_\mathbf{B}(\mathbf{V})).$$

**Lemma 3.** Let $\omega \in [n]$, $\mathbf{U}, \tilde{\mathbf{U}} \in \mathbb{R}^n$ such that $\|\mathbf{U}\|_0 \leq \omega$. Let $\mathbf{u} := \tilde{\mathbf{U}} - \mathbf{U}$. Then $\|\mathbf{u}\|_2^2 = \sum_{i=1}^\omega \omega_i \mathbf{u}_i^2 - \sum_{i=\omega+1}^n \omega_i \mathbf{u}_i^2$. In particular, for any $\nu \in [0, 1]$,

$$\nu \|\mathbf{u}\|_2^2 + \|\tilde{\mathbf{U}}\|_2^2 - \|\mathbf{U}\|_2^2 \leq (1 + \nu) \Omega \|\mathbf{u}\|_2 - (1 - \nu) \sum_{i=\omega+1}^n \omega_i \mathbf{u}_i^2.$$

**Definition 4.** Let $\mathcal{R}$ be a decomposable norm over $\mathbb{R}^p$. Given $\mathbf{B} \in \mathbb{R}^p$ and $c_0, \gamma, \eta > 0$, we define the following cones

$$\mathcal{C}_{\mathbf{B}, \mathcal{R}}(c_0) := \{ \mathbf{V} : \mathcal{R}(\mathbf{P}_{\mathbf{B}}(\mathbf{V})) \leq c_0 \mathcal{R}(\mathbf{P}_{\mathbf{B}}(\mathbf{V})) \},$$

$$\mathcal{C}_{\mathbf{B}, \mathcal{R}}(c_0, \gamma, \eta) := \{ \mathbf{V}, \mathbf{u} : \gamma \mathcal{R}(\mathbf{P}_{\mathbf{B}}(\mathbf{V})) + \sum_{i=\omega+1}^n \omega_i \mathbf{u}_i^2 \leq c_0 \mathcal{R}(\mathbf{P}_{\mathbf{B}}(\mathbf{V})) + \eta \|\mathbf{u}\|_2 \}.$$
Recall the subdifferential of a norm. Throughout this section, we introduce some notation and facts.

Lemma 4 (Dimension reduction). Suppose

(i) \( (X, \xi) \) satisfies the MP, \( \| \|_1 \) for some positive numbers \( f_1, f_2 \) and \( f_3 \).

(ii) \( X \) satisfies the ATP, \( \| \|_1 \) for some positive numbers \( c_1, c_2, c_3 \).

(iii) \( \lambda \gamma \geq 2[f_2 + (f_1/c_1)] \), \( \tau \geq 2[f_3 + (f_1/c_1)] \).

Then either \( [\Delta B, \hat{\theta}] \in C_B \cdot (6, \gamma, \Omega) \) or

\[
\| \Delta B : \Delta \|_2 \leq \frac{f_1}{c_1^2} + \left( \frac{c_2}{\lambda} \right) \frac{28f_2^2}{3c_1^2}.
\]

\[
\lambda R(\Delta B) + \tau \| \Delta \|_2 \leq \frac{28f_2^2}{3c_1^2}.
\]

Proof. Of course, for any \( i \in [n] \), \( (k, \ell) \in [d_1] \times [d_2] \), \( \frac{\partial R}{\partial x_{k \ell}} (B) = (X_{i \ell})_{k \ell} \). The first order condition of (3) at \( [\hat{B}, \hat{\theta}] \) is equivalent to the statement: there exist \( V \in \partial R(\hat{B}) \) and \( u \in \partial \| \hat{\theta} \|_1 \) such that for all \( [B, \theta] \),

\[
\sum_{i \in [n]} \left[ y_i^{(n)} - \hat{x}_i^{(n)}(\hat{B}) - \hat{\theta} \right] \langle X_i^{(n)}, B - B \rangle \geq \lambda \| V, B - B \|,
\]

\[
\left( y_i^{(n)} - \hat{x}_i^{(n)}(\hat{B}) - \hat{\theta} - \theta \right) \geq \tau \langle u, \hat{\theta} - \theta \rangle.
\]

Evaluating at \( [B^*, \theta^*] \) and using that \( y_i^{(n)} = X_i^{(n)}(B^*) + \theta^* + \xi_i^{(n)} \) we obtain

\[
\sum_{i \in [n]} \left[ x_i^{(n)}(\Delta B) + \Delta \right] \langle X_i^{(n)}, \Delta B \rangle \leq \sum_{i \in [n]} \xi_i^{(n)} \langle X_i^{(n)}, \Delta B \rangle - \lambda \| V, \Delta B \|,
\]

\[
\left( x_i^{(n)}(\Delta \theta) + \Delta \right) \langle \xi_i^{(n)}, \xi_i^{(n)} \rangle \leq \left( \xi_i^{(n)} - \tau u, \Delta \theta \right),
\]

so summing both equations we get

\[
\| \partial R^{(n)}(\Delta B, \Delta \theta) \|_2 \leq \left( \xi_i^{(n)} - \tau u, \Delta \theta \right) - \lambda \| V, \Delta B \| - \tau \langle u, \Delta \theta \rangle.
\]

There is \( U \) such that \( R^*(U) \leq 1 \) and \( \langle U, \hat{\theta} \rangle = R(\hat{B}) \). Hence, we get

\[
-\langle \Delta B, V \rangle = \langle B^* - B, V \rangle = \langle B^*, V \rangle - \langle B, V \rangle - R(\hat{B}) \leq R(B^*) - R(\hat{B}).
\]

\[\text{\footnotesize Recall the subdifferential of a norm } R \text{ at a point } W \text{ is } \partial R(W) = \{ U : R^*(U) \leq 1, \langle U, W \rangle = R(W) \}.\]
Similarly, \(-\langle \Delta^B, u \rangle \leq \|\theta^*\|_2 - \|\hat{\theta}\|_2\). From these bounds we obtain

\[
\begin{align*}
&\|\Psi_1(n) (\Delta^B, \Delta^\theta)\|_2 \leq f_1 \|\Delta^B; \Delta^\theta\|_2 + f_2 \mathcal{R}(\Delta^B) + f_3 \|\Delta^B\|_2 \\
&\quad + \lambda (\mathcal{R}(\mathcal{B}^*) - \mathcal{R}(\hat{\mathcal{B}})) + \tau (\|\theta^*\|_2 - \|\hat{\theta}\|_2) \\
&\leq f_1 \|\Psi_1(n) (\Delta^B, \Delta^\theta)\|_2 + \left( f_2 + \frac{f_3 c_3}{c_1} \right) \mathcal{R}(\Delta^B) + \left( f_3 + \frac{f_1 c_3}{c_1} \right) \|\Delta^B\|_2 \\
&\quad + \lambda (\mathcal{R}(\mathcal{B}^*) - \mathcal{R}(\hat{\mathcal{B}})) + \tau (\|\theta^*\|_2 - \|\hat{\theta}\|_2) \\
&\leq \frac{f_1}{c_1} \|\Psi_1(n) (\Delta^B, \Delta^\theta)\|_2 + \Delta,
\end{align*}
\]

where in last inequality we used the decomposability of \(\mathcal{R}\) and Lemmas 2-3 with \(\nu := 1/2\) and have defined

\[
\Delta := (\lambda/2) (\mathcal{R} \circ \mathcal{P}_{\mathcal{B}^*}) (\Delta^B) - (\lambda/2) (\mathcal{R} \circ \mathcal{P}_{\hat{\mathcal{B}}}) (\Delta^B) + (3\sigma/2) \|\Delta^B\|_2 - (\gamma/2) \sum_{i=0}^n \omega_i (\Delta^B)_i^2.
\]

Define also \(G := \|\Psi_1(n) (\Delta^B, \Delta^\theta)\|_2\) and \(H := (\lambda/2) (\mathcal{R} \circ \mathcal{P}_{\mathcal{B}^*}) (\Delta^B) + (3\sigma/2) \|\Delta^B\|_2\).

**Case 1:** \(\frac{f_1}{c_1} G \leq H\). In that case, from (13) we obtain \(c_{\mathcal{B}^*} (6, \gamma, \Omega)\).

**Case 2:** \(\frac{f_1}{c_1} G \geq H\). In that case we obtain \(G^2 \leq \frac{2f_1}{c_1} G \Rightarrow G \leq \frac{2f_1}{c_1}\). Therefore \(H \leq \frac{2f_1^2}{c_1^2}\). We first obtain the error bound on \(\gamma \mathcal{R}(\Delta^B) + \|\Delta^\theta\|_2\). Again from (13), we obtain that

\[
\lambda (\mathcal{R} \circ \mathcal{P}_{\hat{\mathcal{B}}}) (\Delta^B) + \tau \sum_{i=0}^n \omega_i (\Delta^\theta)_i^2 \leq 4\frac{f_1}{c_1} G \leq 8\frac{f_1^2}{c_1^2}.
\]

This fact and decomposability imply

\[
\lambda \mathcal{R}(\Delta^B) + \tau \|\Delta^\theta\|_2 \leq \frac{2}{3} H + \frac{c_2^2}{\lambda} \leq \frac{28f_1^2}{3c_1^2}.
\]

From the ATP in (iii) and the above bound, we obtain

\[
c_1 \|\Delta^B; \Delta^\theta\|_2 \leq \|\Psi_1(n) (\Delta^B, \Delta^\theta)\|_2 + c_2 \mathcal{R}(\Delta^B) + c_3 \|\Delta^\theta\|_2 \\
\leq \frac{2f_1}{c_1} + \left( \frac{c_2}{\lambda} \sqrt{\frac{c_3}{\tau}} \right) \frac{28f_1^2}{3c_1^2}.
\]

**Proposition 1.** Suppose that, in addition to (i)-(iii) in Lemma 4, the following condition holds:

**(iv)** For \(R := \Psi(\mathcal{P}_{\mathcal{B}^*} (\Delta^B)) \mu(C_{\mathcal{B}^*} (12))\), assume

\[
14 (c_2 \vee c_3) \left( R^2 + \frac{4\Omega^2}{\gamma^2} \right)^{1/2} \leq c_1.
\]
Then either (10)–(11) hold or

\[
\| [\Delta^B ; \Delta^\hat{\theta}] \|_\Pi \leq \frac{4}{c_1^2} f_1 + \frac{6}{c_1^2} \sqrt{\lambda^2 R^2 + 4 \tau^2 \Omega^2},
\]

\[
\lambda \mathcal{R}(\Delta^B) + \tau \| \Delta^\hat{\theta} \|_2 \leq \frac{4}{c_1^2} f_1 + \frac{6}{c_1^2} \lambda^2 R^2 + \frac{196}{c_1^2} \tau^2 \Omega^2.
\]

**Proof.** From Lemma 4, we only need to consider the case when \([\Delta^\hat{B} ; \Delta^\hat{\theta}] \in \mathcal{C}_B^*(6, \gamma, \Omega)\). A slight variation of the argument to establish (13) leads to

\[
\| \mathcal{R}^{(n)}(\Delta^B, \Delta^\hat{\theta}) \|_2^2 \leq f_1 \| [\Delta^B ; \Delta^\hat{\theta}] \|_\Pi + \Delta,
\]

where \(\Delta\) was already defined in the proof of Lemma 4. ATP as stated in item (iii) of Lemma 4 further leads to

\[
(16) \quad \left\{ c_1 \| [\Delta^\hat{B} ; \Delta^\hat{\theta}] \|_\Pi - c_2 \mathcal{R}(\Delta^B) - c_3 \| \Delta^\hat{\theta} \|_2 \right\} \leq \sqrt{f_1 \| [\Delta^B ; \Delta^\hat{\theta}] \|_\Pi + \Delta}.
\]

**Case 1:** \(12 \mathcal{R}(\mathcal{P}_B^* (\Delta^B)) \geq \mathcal{R}(\mathcal{P}_B^* (\Delta^\hat{B}))\). Hence \(\Delta^B \in \mathcal{C}_B^*(12)\). Decomposability of \(\mathcal{R}\) and \([\Delta^\hat{B} ; \Delta^\hat{\theta}] \in \mathcal{C}_B^*(6, \gamma, \Omega)\) further imply

\[
c_2 \mathcal{R}(\Delta^B) + c_3 \| \Delta^\hat{\theta} \|_2 \leq \left( \frac{c_2}{\lambda} \sqrt{\frac{c_3}{\tau}} \right) (\lambda \mathcal{R}(\Delta^B) + \tau \| \Delta^\hat{\theta} \|_2)
\]

\[
\leq \frac{7}{2} \left( \frac{c_2}{\lambda} \sqrt{\frac{c_3}{\tau}} \right) \left( \lambda^2 R^2 + 4 \tau^2 \Omega^2 \right)^{1/2} || [\Delta^B ; \Delta^\hat{\theta}] ||_\Pi.
\]

Similarly,

\[
\Delta \leq \left( \frac{3}{\lambda} / \mathcal{R}(\mathcal{P}_B^* (\Delta^\hat{B})) + (3 \tau^2 \Omega^2) \| \Delta^\hat{\theta} \|_2 \right)^{1/2}
\]

\[
\leq \left( \frac{3}{\lambda} \left( \lambda^2 R^2 + \tau^2 \Omega^2 \right)^{1/2} \right)^{1/2} \| [\Delta^B ; \Delta^\hat{\theta}] \|_\Pi.
\]

To ease notation, define \(x = \| [\Delta^B ; \Delta^\hat{\theta}] \|_\Pi\).

\(A = \frac{7}{2} \left( \frac{c_2}{\lambda} \sqrt{\frac{c_3}{\tau}} \right) \left( \lambda^2 R^2 + 4 \tau^2 \Omega^2 \right)^{1/2},\)

\(B = f_1 + \frac{7}{2} \left( \frac{c_2}{\lambda} \sqrt{\frac{c_3}{\tau}} \right) \left( \lambda^2 R^2 + \tau^2 \Omega^2 \right)^{1/2}\).

From (16), (17) and (18) we get

\[
c_1 x \leq Ax + \sqrt{Bx} \quad \iff \quad x \leq \frac{B}{(c_1 - A)^2}
\]

provided that \(A \leq c_1\). Assuming \(2A \leq c_1\), we get

\[
\| [\Delta^B ; \Delta^\hat{\theta}] \|_\Pi \leq \frac{4B}{c_1^2} = \frac{4}{c_1^2} f_1 + \frac{6}{c_1^2} (\lambda^2 R^2 + \tau^2 \Omega^2)^{1/2}.
\]
Case 2:

For deriving the bound on $\gamma \mathcal{R}(\Delta^B) + \|\Delta^\theta\|_1$, we again use the decomposability of $\mathcal{R}$ and $[\Delta^B; \Delta^\theta] \in C_{B^*}(6, \gamma, \Omega)$, obtaining

$$
\lambda \mathcal{R}(\Delta^B) + \tau \|\Delta^\theta\|_1 \leq (7 \lambda \mathcal{R}(P_{B^*}(\Delta^B)) + 6 \tau \Omega \|\Delta^\theta\|_2)
\leq 7 \left(\lambda^2 R^2 + \tau^2 \Omega^2\right)^{1/2} \|\Delta^B; \Delta^\theta\|_1
\leq \frac{28}{c_1^2} \left(\lambda^2 R^2 + \tau^2 \Omega^2\right)^{1/2} + \frac{42}{c_1} \left(\lambda^2 R^2 + \tau^2 \Omega^2\right)
\leq \frac{56}{c_1^2} \left(\lambda^2 R^2 + \tau^2 \Omega^2\right).
$$

This and decomposability of $\mathcal{R}$ imply

$$
c_2 \lambda \mathcal{R}(\Delta^B) + c_3 \|\Delta^\theta\|_1 \leq \left(\frac{c_2}{\lambda} \sqrt{\frac{c_3}{\tau}}\right) (\lambda \mathcal{R}(\Delta^B) + \tau \|\Delta^\theta\|_2)
\leq 7 \left(\frac{c_2}{\lambda} \sqrt{\frac{c_3}{\tau}}\right) (\lambda \mathcal{R}(P_{B^*}(\Delta^B)) + \tau \Omega \|\Delta^\theta\|_2)
\leq 14 \left(\frac{c_2}{\lambda} \sqrt{\frac{c_3}{\tau}}\right) \tau \Omega \|\Delta^\theta\|_2.
$$

Similarly,

$$
\Delta \leq (\frac{\lambda}{\tau}) \lambda \mathcal{R}(P_{B^*}(\Delta^B)) + (\frac{\lambda}{\tau}) \tau \Omega \|\Delta^\theta\|_2 \leq 3 \tau \Omega \|\Delta^\theta\|_2.
$$

Again it is convenient to define $x = \|\Delta^B; \Delta^\theta\|_1, A' = 14 \left(\frac{c_2}{\lambda} \sqrt{\frac{c_3}{\tau}}\right) \tau \Omega$ and $B' = f_1 + 3 \tau \Omega$.

From (16), (19) and (20),

$$
c_1 x \leq A' x + \sqrt{B' x} \implies x \leq \frac{B'}{(c_1 - A')^2} \leq \frac{4B'}{c_1^2}
$$

provided that $2A' \leq c_1$. In conclusion,

$$
\|\Delta^B; \Delta^\theta\|_1 \leq \frac{4f_1}{c_1^2} + \frac{12 \tau \Omega}{c_1^2},
$$

implying

$$
\lambda \mathcal{R}(\Delta^B) + \tau \|\Delta^\theta\|_1 \leq 7 \lambda \mathcal{R}(P_{B^*}(\Delta^B)) + 7 \tau \Omega \|\Delta^\theta\|_2
\leq 14 \tau \Omega \|\Delta^\theta\|_2
\leq \frac{56 f_1}{c_1^2} \tau \Omega + \frac{168 \tau^2 \Omega^2}{c_1^2}
\leq \frac{28}{c_1^2} \tau \Omega + \frac{196 \tau^2 \Omega^2}{c_1^2}.
$$
The proof is complete by noting that the bounds in the statement of the proposition are larger than the bounds we have just established in the above cases.

Lemma 5.

\[
\left\langle \mathcal{M}(n)(\Delta^B, \Delta^B), \bar{\chi}^{(n)}(\Delta^B) \right\rangle \leq \langle \xi^{(n)}, \bar{\chi}^{(n)}(\Delta^B) \rangle + \lambda \left( 2\mathcal{R}(\mathcal{P}_B^\top(\Delta^B)) - \mathcal{R}(\Delta^B) \right).
\]

Proof. As \( \hat{B}_n; \hat{\theta}_n \) is the minimizer of \( (3) \), in particular

\[
\hat{B}_n \in \arg \min_{B \in \mathbb{R}^{m \times m^2}} \left\{ \frac{1}{2} \| y^{(n)} - \chi^{(n)}(B) - \hat{\theta}_n \|_2^2 + \lambda \mathcal{R}(B) \right\}.
\]

The KKT conditions of the above minimization problem and the expression of the subdifferential of any norm imply that there exists \( V \in \mathbb{R}^{m \times m^2} \) with \( \mathcal{R}^*(V) \leq 1 \) and \( \langle V, \hat{B}_n \rangle = \mathcal{R}(\hat{B}_n) \) such that, for all \( B \in \mathbb{R}^{m \times m^2} \),

\[
0 \leq \sum_{i \in [n]} \left[ \hat{\chi}_i^{(n)}(\hat{B}_n) + \hat{\theta}_i - y_i^{(n)} \right] \langle \chi^{(n)}_i, B - \hat{B}_n \rangle + \lambda \| V, B - \hat{B}_n \|_2^2
\]

\[
= \sum_{i \in [n]} \left[ \hat{\chi}_i^{(n)}(\Delta^B) + \hat{\theta}_i - \hat{\chi}_i^{(n)} \right] \langle \chi^{(n)}_i, B - \hat{B}_n \rangle + \lambda \| V, B - \hat{B}_n \|_2^2.
\]

We can take \( B := B^* \) above and obtain

\[
0 \leq - \left( \chi^{(n)}(\Delta^B) + \hat{\theta}_n, \chi^{(n)}(\Delta^B) \right) + \langle \chi^{(n)}(\Delta^B), \hat{\chi}^{(n)} \rangle - \lambda \mathcal{R}(\Delta^B, V).
\]

Using that \( \langle V, \hat{B}_n \rangle = \mathcal{R}(\hat{B}_n) \) and \( \langle V, B^* \rangle \leq \mathcal{R}(B^*) \) (since \( \mathcal{R}^*(V) \leq 1 \)), we obtain that \( -\langle \Delta^B, V \rangle \leq \mathcal{R}(B^*) - \mathcal{R}(\hat{B}_n) \). Moreover, from the triangle inequality and the decomposability property for the norm \( \mathcal{R} \), one checks that

\[
\mathcal{R}(B^*) - \mathcal{R}(\hat{B}_n) \leq \mathcal{R}(\mathcal{P}_B^\top(\Delta^B)) - \mathcal{R}(\mathcal{P}_B^\top(\Delta^B)) = 2\mathcal{R}(\mathcal{P}_B^\top(\Delta^B)) - \mathcal{R}(\Delta^B).
\]

Combining the two previous displays finishes the proof.

Theorem 5 (Trace regression). Suppose the following condition holds: for some positive \( \gamma > 0 \) and positive real numbers \( \{a_i\}_{i=1}^2, \{b_i\}_{i=1}^2, \{c_i\}_{i=1}^3, \{d_i\}_{i=1}^3 \) and \( \{f_i\}_{i=1}^3 \),

(i) \( \langle X, \xi \rangle \) satisfies the MP \( \mathcal{M}_p \| \| (f_1, f_2, f_3) \) for some positive numbers \( f_1, f_2, f_3 \).

(ii) \( X \) satisfies the ATP \( \mathcal{A}_p \| \| (c_1, c_2, c_3) \) for some positive numbers \( c_1, c_2, c_3 \).

(iii) \( \lambda = \gamma \tau \geq 2[f_2 + (f_1/c_1)], \) and \( \tau \geq 2[f_3 + (f_1/c_3)] \).

(iv) \( X \) satisfies the TP \( \mathcal{P}_p \) \( \{a_1, a_2\} \).

(v) \( X \) satisfies the IP \( \mathcal{I}_p \| \| (b_1, b_2, b_3) \).

Suppose that \( \mu(C_B \cdot (12)) < \infty \). Let \( R := \Psi(\mathcal{P}_B^\top(\Delta^B)) \mu(C_B \cdot (12)) \) and suppose further that

\[
14 \left( c_2 \vee c_3 \right) \left( R^2 + \frac{4\Omega^2}{\gamma^2} \right)^{1/2} \leq c_1,
\]

\[
\frac{4}{c_1} f_1 + \frac{6}{c_1} \sqrt{\lambda^2 R^2 + 4\tau^2 \Omega^2} \leq \frac{\lambda}{6d_2}.
\]
Define the quantities □ := \( \frac{6}{3a_1^2} \), \( \sqrt{\frac{6}{3a_1}} \), \( \Delta := \frac{\omega}{\hat{\lambda}} \) and
\[
\Phi_{\square, \lambda} := \left( \frac{56}{c_{1}^2} + \frac{4.5}{c_{1}^2} \right) (\lambda R)^2 + \frac{1.5}{a_1^2} (\lambda R),
\]
\[
\Phi_{\square, b_1, f_1} := \left( \frac{56}{c_{1}^2} \right) (\lambda R) + \frac{6}{c_{1}^2} b_1 f_1 + \frac{4.5 b_1^2}{c_{1}^2} + \frac{1.5 f_1}{a_1^2}
\]
\[
\omega_{\square, \tau} := \left( \frac{196}{c_{1}^2} + 18 \right) (\tau \Omega)^2.
\]

Then
\[
\|\Delta^B\|_\Pi \leq \{ \Phi_{\square, \lambda}, \Phi_{\square, b_1, f_1} + \omega_{\square, \tau} \} \sqrt{\left\{ \frac{2 f_1}{c_{1}^2} + \Delta \frac{28R^2}{3c_{1}^2} \right\}}.
\]

**Proof.** Condition (21) and items (i)-(iii) imply that the claims of Proposition 1 hold. If (10)-(11) hold we have nothing to prove. Otherwise, (14)-(15) hold. In particular,
\[
(23) \quad \|\Delta^\theta\|_2 \leq \frac{4}{c_{1}^2} f_1 + \frac{6}{c_{1}^2} \sqrt{\lambda^2 R^2 + 4 \tau^2 \Omega^2} \leq \frac{\lambda}{6 b_2}.
\]

By Lemma 5 and MP as stated in item (i),
\[
\|X^{(n)}(\Delta^B)\|_2^2 \leq \|\Delta^\theta\|_2 + f_1 \|\Delta^B\|_\Pi + f_2 \|\Delta^B\|_\Pi + \frac{\tau}{2} \lambda R(\Delta^B) + \lambda \left( 2 R(P_B \cdot (\Delta^B)) - R(\Delta^B) \right)
\]
\[
\leq b_1 \|\Delta^B\|_\Pi \|\Delta^\theta\|_2 + b_3 \|\Delta^B\|_\Pi \|\Delta^\theta\|_2 + f_1 \|\Delta^B\|_\Pi + 2 \lambda R(P_B \cdot (\Delta^B)) - \frac{\lambda}{3} R(\Delta^B)
\]
\[
\leq b_1 \|\Delta^B\|_\Pi \|\Delta^\theta\|_2 + b_3 \|\Delta^B\|_\Pi \|\Delta^\theta\|_2 + f_1 \|\Delta^B\|_\Pi + 2 \lambda R(P_B \cdot (\Delta^B)) - \frac{\lambda}{3} R(\Delta^B).
\]

We now define the local variables \( x := \|\Delta^B\|_\Pi \) and
\[
A := b_1 \|\Delta^\theta\|_2 + b_3 \|\Delta^\theta\|_2 + f_1,
\]
\[
B := \left\{ 2 \lambda R(P_B \cdot (\Delta^B)) - \frac{\lambda}{3} R(\Delta^B) \right\}^+.
\]

On the one hand, combining the last inequality and the TP, as stated in item (i), we arrive at
\[
(a_1 x - a_2 R(\Delta^B))^2 \leq Ax + B.
\]

This implies that either \( x \leq \left( \frac{a_2}{a_1} \right) R(\Delta^B) \) or
\[
\left( a_1 x - a_2 R(\Delta^B) - \frac{A}{2a_1} \right)^2 \leq B + \frac{A^2}{4a_1^2} + \frac{A^2}{2a_1^2} R(\Delta^B).
\]

In both cases,
\[
x \leq \frac{a_2}{a_1} R(\Delta^B) + \frac{A}{2a_1^2} + \frac{1}{a_1} \left( B + \frac{A^2}{4a_1^2} + \frac{A^2}{2a_1^2} R(\Delta^B) \right)^{1/2}
\]
\[
\leq 1.5 \frac{a_2}{a_1} R(\Delta^B) + 1.5 \frac{A}{a_1^2} + \frac{B^{1/2}}{a_1}.
\]
On the other hand,

\[(25) \quad B \leq 2\lambda R (P_{B^\perp} (\Delta^B)) \leq 2\lambda Rx \leq \left( \frac{a_1 x}{2} + \frac{2\lambda R}{a_1^2} \right)^2.
\]

Combining (24) and (25), we get

\[
\frac{x}{2} \leq \frac{1.5a_2}{a_1^2} R (\Delta^B) + \frac{1.5}{a_1^2} + \frac{2\lambda R}{a_1^2}.
\]

Replacing \( A \) and \( x \) by their expressions, we arrive at

\[
\frac{1}{2} \| \Delta^\theta \|_2 \leq \frac{1.5a_2}{a_1} R (\Delta^B) + \frac{1.5b_1}{a_1} \| \Delta^\theta \|_2 + 1.5b_2 \| \Delta^\theta \|_2 + 1.5f_1 + 1.5\lambda R + \frac{1.5b_3}{a_1^2} + \frac{1.5\lambda R}{a_1^2} + \frac{1.5f_1}{a_1^2}.
\]

By (23),

\[
1.5b_1 \| \Delta^\theta \|_2 \leq \frac{6b_1 f_1}{c_1^2} + 9 \frac{b_1}{c_1} \sqrt{\lambda^2 R^2 + 4\tau^2} \leq \frac{6b_1 f_1}{c_1^2} + 4.5 \frac{b_1^2}{c_1^2} + 4.5 \frac{\lambda^2 R^2 + 4\tau^2}{c_1^2}.
\]

The two previous inequalities and (15) lead to the claimed rate on \( \| \Delta^B \|_2 \).

\[\square\]

### 5.3 Properties for subgaussian \((X, \xi)\)

In this section we prove that all properties of Definition 2 are satisfied with high-probability.

Throughout this section, we additionally assume \((X, \xi) \in \mathbb{R}^p \times \mathbb{R}\) is a centered (not necessarily independent) random pair satisfying Assumption 3 and \(\{(X_i, \xi_i)\}_{i \in [n]}\) is an iid copy of \((X, \xi)\). Moreover, \(\mathcal{R}\) is any norm on \(\mathbb{R}^p\) and \(Q\) is any norm on \(\mathbb{R}^n\).

The proof that \(L\)-subgaussian designs satisfy TP in Definition 2 will follow from a concentration result for the quadratic process due to Dirksen [33] and Bednorz [4] and a peeling argument in one parameter.

**Theorem 6** (Dirksen [33] Theorem 5.5, Bednorz [4] Theorem 1). *Let \(V\) be a compact subset of \(B_1\). Then, for universal constant \(C > 0\), for any \(n \geq 1\) and \(t > 0\), with probability at least \(1 - 2e^{-t^2}\),

\[
\sup_{\mathcal{V} \in V} \|X(\mathcal{V})\|^2 - n\|\mathcal{V}\|_{\mathcal{R}}^2 \leq C^2 \left[ \sigma^2 \left( \mathcal{G}^{1/2}(\mathcal{V}) \right) + L \sqrt{n} (\sigma \left( \mathcal{G}^{1/2}(\mathcal{V}) \right) ) + L^2 \max(t, \sqrt{nt}) \right].
\]

**Proposition 2** (TP). *Grant the assumptions of Theorem 6. Then, for the universal constant \(C > 0\) stated in Theorem 6, for all \(\epsilon \in (0, 1)\), \(\delta \in (0, 1]\) and \(n \in \mathbb{N}\), with probability at least \(1 - \delta\), the following property holds: for all \(\mathcal{V} \in \mathbb{R}^p\),

\[
\|X^{(n)}(\mathcal{V})\|_2 \geq \left( 1 + \frac{C^2 L^2 \epsilon}{4} \right)^{1/2} - CL \sqrt{\epsilon} - \frac{CL}{\sqrt{n}} \left[ 3 + \sqrt{\log(18/\delta)} \right] \|\mathcal{V}\|_{\mathcal{R}} - 1.2C \sqrt{\sigma \left( \mathcal{R}(\mathcal{V}) \mathcal{G}^{1/2}(\mathbb{R}) \cap \|\mathcal{V}\|_{\mathcal{R}} \right)}.
\]

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In addition, for all \( \epsilon \in (0, 1) \), \( \delta \in (0, 1) \) and \( n \in \mathbb{N} \), with probability at least \( 1 - \delta \), the following property holds: for all \( V \in \mathbb{R}^{m_1 \times m_2} \),

\[
\|X^{(n)}(V)\|_2 \leq \left( 1 - \frac{C^2 L^2 \epsilon}{4} \right)^{1/2} + C L \sqrt{\epsilon} + \frac{C L}{\sqrt{cm}} \left[ 3 + \sqrt{\log(18/\delta)} \right] \|V\|_1
\]

\[+ 1.2C \frac{\mathcal{G}(R(V) \Theta^{1/2}(B_R) \cap \|V\|_n B) \sqrt{cn}}{\sqrt{cn}}. \]

Proof. The proof is based on a peeling argument in one parameter. Let \( R_1 > 0 \) and define the set

\[ V_1 := \{ V \in \mathbb{R}^p : \|V\|_1 = 1, R(V) \leq R_1 \}. \]

Note that, \( \mathcal{G}(\Theta^{1/2}(V_1)) \leq R_1 \mathcal{G}(\Theta^{1/2}(B_R) \cap R^{-1}B) \). Define for convenience the function \( f(r) := r \mathcal{G}(\Theta^{1/2}(B_R) \cap r^{-1}B) \). By Theorem 6, there is universal \( C > 0 \) such that, for any \( R_1 > 0 \) and \( t \geq 0 \), with probability at least \( 1 - 2e^{-t} \),

\[ 1 - \inf_{V \in \mathcal{B}} \|X^{(n)}(V)\|_2^2 \leq C^2 \left[ \frac{f^2(R_1)}{n} + L \frac{f(R_1)}{\sqrt{n}} + L^2 \max \left( \frac{t}{n}, \sqrt{\frac{t}{n}} \right) \right]. \]

By dividing in the cases \( t \geq n \) and \( t \leq n \) and completing the squares, the above relation implies in particular that, for any \( \epsilon \in (0, 1) \) and \( \delta \in (0, 1/2) \), with probability at least \( 1 - 2\delta \),

\[ \inf_{V \in \mathcal{B}} \left[ \|X^{(n)}(V)\|_2^2 - \left( 1 + \frac{C^2 L^2 \epsilon}{4} \right)^{1/2} - \sqrt{\epsilon CL} \right] \geq -C \frac{f(R_1)}{\sqrt{cn}} - CL \sqrt{\frac{\log(1/\delta)}{cn}}. \]

We use the above property and the single-parameter peeling Lemma 15 with constraint set \( V := S_{11} \), functions \( M(V) := \|X^{(n)}(V)\|_2^2 - \left( 1 + \frac{C^2 L^2 \epsilon}{4} \right)^{1/2} - \sqrt{\epsilon CL} \), \( h(V) := R(V) \), \( g(r) := \mathcal{G}(\Sigma^{1/2}(B)) \) and constants \( c := 2 \) and \( b := \frac{CL}{\sqrt{cn}} \). Note that the claimed inequality trivially holds if \( \|V\|_1 = 0 \) by L-sub-Gaussianity. The desired inequality follows from Lemma 15 combined with the fact that \( V/\|V\|_n \in V \), for all \( V \in \mathbb{R}^{m_1 \times m_2} \) such that \( \|V\|_1 \neq 0 \) and the homogeneity of norms. The proof for the upper bound is similar. \( \square \)

We now show IP in Definition 2 for L-subgaussian designs. The proof follows from Chevet’s inequality [] followed by a peeling argument in two parameters. A high-probability version of Chevet’s inequality is suggested as an exercise in Vershynin [83]. We next give a proof for completeness.

Lemma 6. Let \( V \) be any bounded subset of \( \mathbb{S}_{11} \times \mathbb{S}_2^d \). Define \( V_1 := \{ V : \exists u \text{ s.t. } (V, u) \in V \} \) and \( V_2 := \{ u : \exists V \text{ s.t. } (V, u) \in V \} \).

Then, there exists universal numerical constant \( C > 0 \), such that, for any \( n \geq 1 \) and \( t > 0 \), with probability at least \( 1 - 2 \exp(-t^2) \),

\[ \sup_{[V, u] \in V} \langle u, X(V) \rangle \leq CL \mathcal{G}(\Sigma^{1/2}(V_1)) + \mathcal{G}(V_2) + t. \]

Proof. In the following, the numerical constant \( C > 0 \) may change from line to line. For each \( (V, u) \in V \), we define

\[ Z_{V,u} := \langle u, X(V) \rangle = \sum_{i \in [n]} u_i \langle X_i, V \rangle, \quad W_{V,u} := L(\langle V, \Theta^{1/2}(Z) \rangle + \langle u, \xi \rangle), \]

where \( Z \in \mathbb{R}^p \) and \( \xi \in \mathbb{R}^n \) are independent each one having iid \( \mathcal{N}(0, 1) \) entries. Therefore, \( (V, u) \mapsto W_{V,u} \) defines a centered Gaussian process indexed by \( V \).
We may easily bound the $\psi_2$-norm of the increments using rotation invariance of sub-Gaussian random variables. Indeed, using that $\{X_i\}$ is an iid sequence and Proposition 2.6.1 in [83], there is an universal numerical constant $C > 0$ such that, given $[V; u]$ and $[V'; u']$ in $V$,

$$|Z_{V,u} - Z_{V',u'}| \leq C \sum_{i \in [n]} \|X_i - u'_i V'\|^2 \psi_2$$

$$\leq 2C \sum_{i \in [n]} \|X_i - u'_i V'\|^2 \psi_2$$

$$\leq 2C \sum_{i \in [n]} \|X_i - u_i V\|^2 \psi_2 + 2C \sum_{i \in [n]} \|X_i - u'_i (V - V')\|^2 \psi_2$$

$$\leq 2C \|V - V'\|^2 \|V\| + 2C \|V - V'\|^2 \|V'\|^2 \leq 2C \|V - V'\|^2$$

with the pseudo-metric $d([V; u], [V'; u']) := \sqrt{\|u - u'\|^2 + \|V - V'\|^2}$, using that $\|V\| \leq 1$ and $\|u'\| \leq 1$. On the other hand, by definition of the process $W$ it is easy to check that

$$\mathbb{E}[(W_{V,u} - W_{V',u'})^2] = L^2(\|V - V'\|^2 + \|u - u'\|^2).$$

From (26),(27), we conclude that the processes $W$ and $Z$ satisfy the conditions of Talagrand’s majorization and minorization generic chaining bounds for sub-Gaussian processes (e.g. Theorems 8.5.5 and 8.6.1 in [83]). Hence, there is a universal numerical constant $C > 0$ such that, for any $t \geq 0$, with probability at least $1 - 2e^{-t^2}$,

$$\sup_{[V,u] \in V} |Z_{V,u}| \leq CL \left\{ \mathbb{E} \left[ \sup_{[V,u] \in V} W_{V,u} \right] + t \right\}.$$

In above we used that $Z_{V_0, u_0} = 0$ at $[V_0, u_0] = 0$ and that the diameter of $V \subset \mathbb{B}^{n_1 \times m_2} \times \mathbb{B}^{2n}$ under the metric $d$ is less than $2\sqrt{2}$. We also have

$$\mathbb{E} \left[ \sup_{[V,u] \in V} W_{V,u} \right] \leq \mathbb{E} \left[ \sup_{V \in V_1} \mathbb{E} \left[ \mathcal{G}(V) \right] \right] + \mathbb{E} \left[ \sup_{u \in V_2} \langle u, \xi \rangle \right] = \mathcal{G}(\mathcal{G}^{1/2}(V_1)) + \mathcal{G}(V_2).$$

Joining the two previous inequalities complete the proof of the claimed inequality. \qed

**Proposition 3 (IP).** There exists universal constant $C > 0$, such that for all $\delta \in (0, 1]$ and $n \in \mathbb{N}$, with probability at least $1 - \delta$, the following property holds: for all $[V; u] \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\mathbb{E}|\langle u, X^{(n)}(V) \rangle| \leq CL \left\{ \frac{1}{\sqrt{n}} \right\} \left\| V \right\|_1 \left\| u \right\|_2$$

$$+ \frac{CL \mathcal{G}(\mathcal{R}(V)) \mathcal{G}(B_{\mathbb{R}}) \cap \{ V \} \|B\|_F}{\sqrt{n}} \left\| u \right\|_2 + CL \left\{ \mathcal{G}(\mathcal{Q}(u) B_{\mathbb{R}} \cap \{ V \} \|B\|_F) \right\} \left\| V \right\|_1,$$

Proof. The proof is based on a peeling argument in two parameters. Let $R_1, R_2 > 0$ and define the sets

$$V_1 := \{ V \in \mathbb{R}^n : \|V\|_1 \leq 1, \mathcal{R}(V) \leq R_1 \},$$

$$V_2 := \{ u \in \mathbb{R}^n : \|u\|_2 = 1, \mathcal{Q}(u) \leq R_2 \}.$$

Note that,

$$\mathcal{G}(\mathcal{G}^{1/2}(V_1)) \leq R_1 \mathcal{G}(\mathcal{G}^{1/2}(B_{\mathbb{R}}) \cap R_1^{-1}B_{\mathbb{F}}), \quad \mathcal{G}(V_2) \leq R_2 \mathcal{G}(B_{\mathbb{F}} \cap R_2^{-1}B_{\mathbb{F}}).$$

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Define for convenience the functions
\[ g(r) := CL \frac{\mathcal{G}\left( \mathcal{E}^{1/2}(B_R) \cap r^{-1}B_F \right)}{\sqrt{n}} r, \quad \tilde{g}(\tilde{r}) := CL \frac{\mathcal{G}\left( \mathcal{B}_R^b \cap \tilde{r}^{-1}B_F^a \right)}{\sqrt{n}} \tilde{r}. \]

By Lemma 6, there is universal constant \( C > 0 \) such that, for any \( R_1, R_2 > 0 \) and \( \delta \in (0, 1] \), with probability at least \( 1 - \delta \), the following inequality holds:
\[
\sup_{[V;u] \in V_1 \times V_2} \langle u, X^{(n)}(V) \rangle \leq g(R_1) + \tilde{g}(R_2) + CL \sqrt{\frac{\log(1/\delta)}{n}}.
\]

We use the above property and the bi-parameter peeling Lemma 16 with constraint set \( V := B_{\Pi} \times B_{\Pi}^2 \), functions \( M(V, u) := -\langle u, X^{(n)}(V) \rangle \), \( h(V, u) := \mathcal{R}(V) \), \( h(V, u) := \mathcal{Q}(u) \), \( g \) and \( \tilde{g} \) and constants \( c := 1 \) and \( b := CL \sqrt{1/n} \). Note that the claimed inequality trivially holds if \( \|V\|_{\Pi} = 0 \) or \( u = 0 \). Indeed, since \( X_i \) is \( L \)-sub-Gaussian, \( \|V\|_{\Pi} = 0 \) implies that \( \langle X_i, V \rangle = 0 \) with probability 1. The desired inequality follows from Lemma 16 combined with the fact that \( [V;u] \in \mathbb{R}^{m_1 \times m_2} \times \mathbb{R}^n \) such that \( \|V\|_{\Pi} \neq 0 \) and \( u \neq 0 \) and the homogeneity of norms.

We now turn our attention to property MP in Definition 2. Of course,
\[
\langle \xi^{(n)}, M^{(n)}(V, u) \rangle = \frac{1}{n} \sum_{i \in [n]} \xi_i \langle V, X_i \rangle + \frac{1}{\sqrt{n}} \sum_{i \in [n]} \xi_i u_i.
\]

The control of the first term will follow from a bound for the multiplier process (Theorem 9 in the Appendix). As for the second, one may avoid chaining by using a standard symmetrization-contraction argument which we present for completeness.

**Lemma 7.** Let \( U \) be any bounded subset of \( B_{\Pi}^2 \).

Then, for any \( n \geq 1 \) and \( t > 0 \), with probability at least \( 1 - \exp(-t^2/2) \), we have
\[
\sup_{u \in U} \langle \xi, u \rangle \leq 8\sigma \left[ \mathcal{G}(V_2) + t \right].
\]

**Proof.** Let \( \epsilon \in \mathbb{R}^n \) be a vector whose components are iid Rademacher random variables. Let \( t \geq 0 \). The symmetrization inequality (e.g., Exercise 11.5 in [10]) yields
\[
\mathbb{E}\left[ \exp\left( t \sup_{u \in U} \langle \epsilon, z \rangle \right) \right] \leq \mathbb{E}\left[ \exp\left( t \sup_{u \in U} \langle \epsilon \circ \xi, z \rangle \right) \right],
\]

where \( \epsilon \circ \xi \) is the vector with components \( \{\epsilon_i \xi_i\}_{i \in [n]} \). For each \( i \in [n] \), \( \epsilon_i \xi_i \) is a symmetric sub-Gaussian random variable with \( \ell_2 \)-norm not greater than \( \sigma \). Let \( g \sim N(0, I_n) \) a standard normal vector in \( \mathbb{R}^n \). Hence, the following tail dominance holds:
\[
P(\langle \epsilon_i \xi_i, z \rangle > \tau) \leq 4P(\sigma \langle g, z \rangle > \tau) \text{ for all } i \in [n]
\]

and for all \( \tau > 0 \). From the contraction principle as stated in Lemma 4.6 in [54],
\[
\mathbb{E}\left[ \exp\left( t \sup_{u \in U} 2 \langle \epsilon \circ \xi, z \rangle \right) \right] \leq \mathbb{E}\left[ \exp\left( 8\sigma t \langle g, z \rangle \right) \right].
\]

Since \( U \subseteq B_{\Pi}^2 \), the function \( g \mapsto \sup_{u \in U} \langle g, z \rangle \) is \( 1 \)-Lipschitz under the \( \ell_2 \)-norm. By Theorem 5.5 in [10], the RHS of the previous inequality is upper bounded by
\[
\exp\left( 8\sigma t \mathcal{G}(U) + 32t^2 \sigma^2 \right).
\]

A standard Chernoff bound concludes the proof. \( \square \)
Proposition 4 (MP). For $t, s > 0$ define
\[
\Delta(t, s) := \frac{1}{\sqrt{n}} \sqrt{\log t} + \frac{1}{n} \log t + \frac{1}{n} \sqrt{\log t} \log s + \frac{1}{n} \sqrt{\log s} + \frac{1}{\sqrt{n}} [1 + \sqrt{\log s}].
\]

Then there exists universal constant $C > 0, c_0, c \geq 2$ such that for all $n \in \mathbb{N}$ and all $\delta \in (0, 1/c)$ and $\rho \in (0, 1/c_0)$, with probability at least $1 - \delta - \rho$, the following property holds: for all $[V; u] \in \mathbb{R}^p \times \mathbb{R}^n$,
\[
\langle \xi^{(n)}, \mathfrak{M}^{(n)}(V; u) \rangle \leq C \sigma L \cdot \Delta(1/s, 1/\rho) \cdot \|\|V; u\|\|_1
+ C \sigma L \left[ 1 + \frac{\log(1/\rho)}{\sqrt{n}} \right] \frac{\mathcal{G}(\mathbb{S}^{1/2}(B_\mathbb{R})) R_1}{\sqrt{n}} + C \sigma \frac{\mathcal{G}(B_\mathbb{R})}{\sqrt{n}} Q(u).
\]

Proof. Define the set
\[
V_0 := \{[V; u] : \|\|V; u\|\|_1 \leq 1, \mathcal{R}(V) \leq R_1, Q(u) \leq R_2\}.
\]

Theorem 9 in the Appendix (together with Talagrand’s majorization theorem), Lemma 7 and an union bound imply that, for universal constants $C > 0$ and $c, c_0 \geq 2$, for all $\delta \in (0, 1/c)$ and $\rho \in (0, 1/c_0)$, with probability at least $1 - \delta - \rho$,
\[
\sup_{[V; u] \in V_0} \langle \xi^{(n)}, \mathfrak{M}^{(n)}(V; u) \rangle \leq C \sigma L \left( \frac{\log(1/\rho)}{n} + 1 \right) \frac{\mathcal{G}(\mathbb{S}^{1/2}(B_\mathbb{R})) R_1}{\sqrt{n}} + C \sigma \left( \frac{\log(1/\delta)}{n} + \sqrt{\frac{\log(1/\delta) \log(1/\rho)}{n^2}} \right) + C \sigma \frac{\mathcal{G}(B_\mathbb{R})}{\sqrt{n}} R_2 + C \sigma \sqrt{\frac{\log(1/\delta)}{n}}.
\]

We will now apply Lemma 17 with set $V := \{[V; u] : \|\|V; u\|\|_1 \leq 1\}$, functions $M(V, u) := -\langle \xi^{(n)}, \mathfrak{M}^{(n)}(V; u) \rangle, h(V) := \mathcal{R}(V), h(u) := Q(u), g(R_1) := \mathcal{G}(\mathbb{S}^{1/2}(B_\mathbb{R})) R_1, g(R_2) := \mathcal{G}(B_\mathbb{R}) R_2$, and constant $b := C \sigma L$, where we recall $L \geq 1$. The result then follow from such lemma and homogeneity noting that $\frac{\langle V; u \rangle}{\|\|V; u\|\|_1} \in V_0$ for $[V; u]$ such that $\|\|V; u\|\|_1 \neq 0$.

\[\square\]

5.4 Proof of Theorem 1

We now set $\mathcal{R} := \|\|., \mathcal{Q} := \|\|., A$ a standard Gaussian maximal inequality implies that $\mathcal{G}(\mathbb{S}^{1/2}(B_n^p)) \leq \mathcal{B}(\mathbb{S}_n^p) \sqrt{\log n}$. If $B_n^p$ denotes the unit ball with respect to the Slope norm $\|\|_p$, Proposition E.2 in Beller, Lecuè and Tsybakov [5] implies that $\mathcal{G}(B_n^p) \leq 1$.

We next use Proposition 2 with $\epsilon = c_2 L \frac{n}{c_2 T^2}$ for sufficiently small $c \in (0, 1)$ and assuming
\[
(28) \quad \delta \geq \exp \left(-c_2 \frac{n}{L^2}\right)
\]
for large enough universal constant $c_1 > 0$. It follows that for $a_1 \in (0, 1)$ an universal constant and
\[
a_2 \asymp L \rho_1(\Sigma) \sqrt{\frac{\log p}{n}},
\]
on an event $\Omega_1$ of probability at least $1 - \delta/3$, property $\text{TP}_\|\|_1(a_1; a_2)$ is satisfied.

From Proposition 3, for every $\delta \in (0, 1)$ and
\[
b_1 \asymp L \frac{1 + \sqrt{\log(1/\delta)}}{\sqrt{n}}, \quad b_2 \asymp L \rho_1(\Sigma) \sqrt{\frac{\log p}{n}}, \quad b_3 \asymp \frac{L}{\sqrt{n}}
\]

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on an event $\Omega_2$ of probability at least $1 - \delta/3$, property $\text{IP}_{\|\cdot\|_1,\|\cdot\|_2}(b_1; b_2; b_3)$ is satisfied. From Lemma 1, by enlarging $c_1$ if necessary, for $c_1 \in (0, 1)$ an universal constant and

$$c_2 = L\rho_1(\Sigma) \sqrt{\frac{\log p}{n}}, \quad c_3 = \frac{L}{\sqrt{n}},$$

$\text{ATP}_{\|\cdot\|_1,\|\cdot\|_2}(c_1; c_2; c_3)$ is satisfied on $\Omega_1 \cap \Omega_2$.

We now use Proposition 4 (with $\delta = \rho$). By enlarging $c_1$ in (28) if necessary, if we take

$$f_1 = \sigma L \frac{1 + \sqrt{\log(1/\delta)}}{\sqrt{n}}, \quad f_2 = \sigma L\rho_1(\Sigma) \sqrt{\frac{\log p}{n}}, \quad f_3 = \frac{\sigma}{\sqrt{n}},$$

we have by Proposition 4 that on an event $\Omega_2$ of probability at least $1 - \delta/3$, $\text{MP}_{\|\cdot\|_1,\|\cdot\|_2}(f_1; f_2; f_3)$ is satisfied.

By an union bound and enlarging constants, for $\delta$ satisfying (28), on the event $\Omega_1 \cap \Omega_2 \cap \Omega_3$ of probability at least $1 - \delta$, all properties TP, IP, ATP and MP hold with constants as specified above. We now assume such event is realized. We now invoke Theorem 5. It is straightforward to check (iii) by the definitions of $\tau$ and $\lambda = \gamma \tau$ in Theorem 1. Note that that $R \leq \sqrt{\sigma}\mu(b^*)$ with $\mu(b^*) := \mu(C_{\sigma,\|\cdot\|_1}(12))$. In item (iv), conditions (21) and (22) require checking

$$C\sigma^2 L^2 \rho^2(\Sigma)\mu^2(b^*) \frac{8\log p}{n} + C\sigma^2 L^2 \epsilon \log(1/\epsilon) < 1/2,$$

and that $\frac{C}{c_2} < 1/2$. Assuming further that $\delta \geq \exp\left(-c_2 \frac{n}{\sigma^2L^2}\right)$ for universal constant $c_2 > 0$ and enlarging $C$ above, item (iv) is satisfied. With all conditions of Theorem 5 taking place, the rate in Theorem 1 follows.\footnote{Note that $\Box$ and $\Delta$ are bounded by a numerical constant of $O(L)$.}

**Remark 3.** The proof taking $\mathcal{R}$ to be the Slope norm in $\mathbb{R}^p$ with sequence $\tilde{w}_i = 2 \log(p/j)$ follows a similar path. One difference is that we need to consider the cones defined, for $c_0, \gamma > 0$, by

$$\mathcal{C}(c_0) := \left\{ v \in \mathbb{R}^p : \sum_{i = s + 1}^n \tilde{w}_j v^2_j \leq c_0 \|v\|_2 \right\},$$

$$\mathcal{C}(c_0, \gamma) := \left\{ [v; u] \in \mathbb{R}^p \times \mathbb{R}^n : \gamma \sum_{i = s + 1}^n \tilde{w}_j v^2_j + \sum_{i = s + 1}^n \omega_i u^2_i \leq c_0 [\gamma \|v\|_2 + \|u\|_2] \right\}.$$

In this setting, $\mu(b^*) = \mu(\mathcal{C}(12))$. We omit the details.

### 5.5 Proof of Theorem 2

The proof follows exact same guidelines as the Proof of Theorem 1. The changes are that for the nuclear norm $\mathcal{R} := \|\cdot\|_N$, we have $\mathcal{G}(\mathcal{S}^{1/2}(B_{1,\|\cdot\|_N})) \lesssim \rho_N(\Sigma)(\sqrt{d_1} + \sqrt{d_2})$ by Lemma H.1 in [67] and

$$R \leq \sqrt{\tau}\mu(B^*)$$

with $\mu(B^*) := \mu(C_{\beta,\|\cdot\|_1}(12))$.\footnote{Note that $\Box$ and $\Delta$ are bounded by a numerical constant of $O(L)$.}
6 Proof of Theorem 3

Recall estimator (5).

Throughout Sections 6.1 and 6.2, we see $\| \cdot \|_\Pi$ as a generic pseudo-norm and $\{(X_i, \xi_i)\}_{i \in [n]}$ as a deterministic sequence satisfying (4) where $\xi = (\xi_i)_{i \in [n]}$ and $X$ is the design operator associated to the sequence $\{X_i\}_{i \in [n]}$. Probabilistic assumptions are used only in Sections 6.3 and 6.4.

6.1 Additional design properties and cones

In the following, let $\mathcal{R}$ and $\mathcal{Q}$ be norms on $\mathbb{R}^p$. We first present a definition bounding the product process:

$$(29) \quad [V; W] \mapsto \frac{1}{n} \sum_{i \in [n]} \left( \langle X_i, V \rangle \langle X_i, W \rangle - \mathbb{E}[\langle X_i, V \rangle \langle X_i, W \rangle] \right).$$

For convenience, we use the notation $\langle V, W \rangle_n := \frac{1}{n} \sum_{i \in [n]} \langle X_i, V \rangle \langle X_i, W \rangle = \langle V, (X^{\ast} \circ X/n)(W) \rangle$.

**Definition 6** (PP). Given positive numbers $(b_1, b_2, b_3, b_4)$, we say that $X$ satisfies $PP_{\mathcal{R}, \mathcal{Q}}(b_1; b_2; b_3; b_4)$ if $\forall [V; W] \in \mathbb{R}^p,$

$$|\langle V, W \rangle_n - \langle V, W \rangle| \leq b_1 \|V\|_\Pi \|W\|_\Pi + b_2 \mathcal{R}(V) \|W\|_\Pi + b_3 \|V\|_\Pi \mathcal{Q}(W) + b_4 \mathcal{R}(V) \mathcal{Q}(W).$$

We will also need variations of ATP and MP in Definition 2 which will be used throughout Section 6.

**Definition 7** (ATP and MP). Given positive numbers $c_1$, $c_2$ and $c_3$, we say that $X$ satisfies $ATP_{\mathcal{R}, \mathcal{Q}}(c_1; c_2; c_3)$ if $\forall [V; W] \in \mathbb{R}^p,$

$$\|X^{(n)}(V + W)\| \geq \left[ c_1 \|V; W\|_\Pi - c_2 \mathcal{R}(V) - c_3 \mathcal{Q}(W) \right]^2 + 2|\langle V, W \rangle|_\Pi. $$

In addition, given positive numbers $f_1, f_2$ and $f_3$, we will say that $(X, \xi)$ satisfies $MP_{\mathcal{R}, \mathcal{Q}}(f_1; f_2; f_3)$ if $\forall [V; W] \in \mathbb{R}^p,$

$$|\langle \xi, X^{(n)}(V + W) \rangle| \leq f_1 \|V; W\|_\Pi + f_2 \mathcal{R}(V) + f_3 \mathcal{Q}(W).$$

The next lemma states that ATP is a consequence of TP and PP. We omit the proof as it follows similar reasoning of Lemma 1.

**Lemma 8** (TP + PP $\Rightarrow$ ATP). Let positive numbers $a_1$, $\bar{a}_1$, $a_2$, $\bar{a}_2$, $b_1$, $b_2$ and $b_3$ with $b_1 < a_1 \land \bar{a}_1$. Suppose that $X$ satisfies:

(i) TP$||\mathcal{R}(a_1; a_2).$
(ii) TP$||\mathcal{Q}(a_1; a_2).$
(iii) PP$||\mathcal{R}, \mathcal{Q}(b_1; b_2; b_3; \beta b_2 b_3)$ for some $\beta \in (0, 1].$
Then, for any \( \alpha > 0 \) such that \( \alpha^2 + b_1 \leq (a_1 \land \hat{a}_1)^2 \), \( \hat{X} \) satisfies the ATP with constants \( c_1 = \sqrt{(a_1 \land \hat{a}_1)^2 - b_1 - \alpha^2} \), \( c_2 = a_2 + b_2/\alpha \), \( c_3 = \hat{a}_2 + b_3/\alpha \). Taking \( \alpha = (a_1 \land \hat{a}_1)/2 \), we obtain that ATP\((c_1; c_2; c_3)\) holds with constants \( c_1 = \sqrt{(3/4)(a_1 \land \hat{a}_1)^2 - b_1} \), \( c_2 = a_2 + 2b_2/(a_1 \land \hat{a}_1) \) and \( c_3 = \hat{a}_2 + 2b_3/(a_1 \land \hat{a}_1) \).

We will need an additional cone definition.

**Definition 8.** Let \( R \) and \( Q \) be decomposable norms on \( \mathbb{R}^p \) (see Definition 3 in Section 5.1). Given \( [B, \Gamma] \in (\mathbb{R}^p)^2 \), let \( P_B \) and \( P_T \) denote the projection maps associated to \( (R, B) \) and \( (Q, \Gamma) \) respectively. Given \( c_0, \gamma > 0 \), we define

\[
c_B,\Gamma,R,Q(c_0, \gamma) := \{ [V; W] : \gamma R(P_B^+(V)) + Q(P_T^+(W)) \leq c_0 [\gamma R(P_B(V)) + Q(P_T(W))] \}.
\]

We will omit the subscripts \( R \) and \( Q \) when the norms are clear in the context.

### 6.2 Deterministic bounds

Throughout this section, we work with the nuclear norm \( \| \cdot \|_N \) and the \( \ell_1 \)-norm \( \| \cdot \|_1 \) on \( \mathbb{R}^p \). With some abuse of notation, we will denote by \( P_B^+ \) the projection associated to \((\| \cdot \|_N, B^+\)) and by \( P_T^+ \) the projection associated to \((\| \cdot \|_1, \Gamma^+)\) (see Definition 3 in Section 5.1).

**Lemma 9** (Dimension reduction). Grant Assumption 2 and suppose that:

(i) \( (X, \xi) \) satisfies the MP \((\| \cdot \|_N, \| \cdot \|_1) \) for some positive numbers \( f_1, f_2, f_3 \).

(ii) \( \hat{X} \) satisfies the ATP \((\| \cdot \|_N, \| \cdot \|_1) \) for some positive numbers \( c_1, c_2, c_3 \).

(iii) \( \lambda = \gamma \tau \geq 4(f_2 \lor c_2) \) and \( \tau \geq 4[c_3 \lor (f_3 + 2\gamma\sqrt{\pi})] \).

(iv) \( 2f_1 \leq c_1 \).

Define

\[
\Delta := (3\lambda/2)\|P_{B^+}(\Delta_B^+)\|_N - (\gamma/2)\|P_{\Gamma^+}(\Delta^+)\|_N + (3\tau/2)\|P_{\Gamma^+}(\Delta^+)\|_1 - (\gamma/2)\|P_{\Gamma^+}(\Delta^+)\|_1.
\]

Then either \([\Delta_B^+; \Delta^+] \in C_{B^+; \Gamma^+}(3, \gamma)\) and

\[
(30) \quad (c_1\|\Delta_B^+; \Delta^+\|_N - (\gamma/2)\|\Delta_B^+\|_N - (\gamma/2)\|\Delta^+\|_1)^2 \leq f_1\|\Delta_B^+; \Delta^+\|_N + \Delta.
\]

or \([\Delta_B^+; \Delta^+] \in C_{B^+; \Gamma^+}(6, \gamma)\) and

\[
(31) \quad \frac{c_1^2}{4}\|\Delta_B^+; \Delta^+\|_N^2 \leq f_1\|\Delta_B^+; \Delta^+\|_N + \Delta.
\]

or

\[
(32) \quad \|\Delta_B^+; \Delta^+\|_N \leq 80\frac{f_2^2}{c_1^2}.
\]

or

\[
(33) \quad \lambda\|\Delta_B^+\|_N + \tau\|\Delta^+\|_1 \leq 37.4\frac{f_2^2}{c_1^2}.
\]

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Proof. By the first order condition of (5) at \([\hat{B}, \hat{\Gamma}]\), there exist \(V \in \partial \hat{B} \|N\) and \(W \in \partial \hat{\Gamma} \|1\) such that for all \([B; \Gamma]\) satisfying \(\|B\|_\infty \leq \frac{\nu}{\sqrt{\nu}}\),

\[
\sum_{i \in [n]} \left[ y_i^{(n)} - X_i^{(n)}(\hat{B} + \hat{\Gamma}) \right] \langle X_i^{(n)}, \hat{B} - B \rangle \geq \lambda \ell \langle V, \hat{B} - B \rangle,
\]

\[
\sum_{i \in [n]} \left[ y_i^{(n)} - X_i^{(n)}(\hat{\Gamma} + \hat{\Gamma}) \right] \langle X_i^{(n)}, \hat{\Gamma} - \Gamma \rangle \geq \lambda \ell \langle W, \hat{\Gamma} - \Gamma \rangle.
\]

Evaluating at \([B^*; \Gamma^*]\), which satisfies \(\|B^*\|_\infty \leq \frac{\nu}{\sqrt{\nu}}\) by assumption, and using (4) we get

\[
\|X^{(n)}(\Delta^B + \Delta^F)\|_2^2 \leq \langle \xi^{(n)}, X^{(n)}(\Delta^B + \Delta^F) \rangle - \lambda \ell \langle V, \Delta^B \rangle - \tau \ell \langle W, \Delta^F \rangle.
\]

Using that \(-\langle \Delta^B, V \rangle \leq \|B\|_N - \|\hat{B}\|_N, -\langle \Delta^F, W \rangle \leq \|\Gamma\|_1 - \|\hat{\Gamma}\|_1\), and \(\text{MP}^\ast\) in item (i), we get

\[
\|X^{(n)}(\Delta^B + \Delta^F)\|_2^2 \leq f_1 \|\Delta^B; \Delta^F\|_1 + f_2 \|\Delta^B\|_N + f_3 \|\Delta^F\|_N + \lambda(\|B^*\|_N - \|\hat{B}\|_N) + \tau(\|\Gamma^*\|_1 - \|\hat{\Gamma}\|_1).
\]

For convenience, we define the quantity

\[
\tilde{H} := (c_2 \vee f_2) \|\Delta^B\|_N + \left[ c_3 \vee \left( f_3 + \frac{2a^*}{\sqrt{\nu}} \right) \right] \|\Delta^F\|_1.
\]

By norm duality and isotropy, \(|\langle \Delta^B, \Delta^F \rangle| = |\langle \Delta^B, \Delta^F \rangle| \|\Delta^F\|_1 \leq \frac{2a^*}{\sqrt{\nu}} \|\Delta^F\|_1\). This fact, the previous display and ATP in item (ii) imply

(34)

\[
(c_1 \|\Delta^B; \Delta^F\|_1 - \tilde{H})^2 \leq f_1 \|\Delta^B; \Delta^F\|_1 + \tilde{H} + \lambda(\|B^*\|_N - \|\hat{B}\|_N) + \tau(\|\Gamma^*\|_1 - \|\hat{\Gamma}\|_1).
\]

We now divide in two cases.

**Case 1:** \(\tilde{H} \geq f_1 \|\Delta^B; \Delta^F\|_1\).

We obtain that the RHS of (34) is upper bounded by

\[
2(c_2 \vee f_2) \|\Delta^B\|_N + 2 \left[ c_3 \vee \left( f_3 + \frac{2a^*}{\sqrt{\nu}} \right) \right] \|\Delta^F\|_1 + \lambda(\|B^*\|_N - \|\hat{B}\|_N) + \tau(\|\Gamma^*\|_1 - \|\hat{\Gamma}\|_1)
\]

\[
\leq (\gamma/2) \|\Delta^B\|_N + (\gamma/2) \|\Delta^F\|_1 + \lambda(\|B^*\|_N - \|\hat{B}\|_N) + \tau(\|\Gamma^*\|_1 - \|\hat{\Gamma}\|_1)
\]

\[
\leq (\lambda/2) \|P_{B^*}(\Delta^B)\|_N - (\lambda/2) \|P_{B^*}(\Delta^B)\|_N + (\alpha/2) \|P_{\Gamma^*}(\Delta^F)\|_1 - (\gamma/2) \|P_{\Gamma^*}(\Delta^F)\|_1,
\]

implying \([\Delta^B; \Delta^F] \in C_{B^*; F^*}(3, \gamma)\). The above bound also implies, again by (iii) and (34), inequality (30).

**Case 2:** \(\tilde{H} \leq f_1 \|\Delta^B; \Delta^F\|_1\).

A similar bound of Case 1, using (iii) and Lemma 2 with \(\nu = 1/4\), implies that \(\tilde{H} + \lambda(\|B^*\|_N - \|\hat{B}\|_N) + \tau(\|\Gamma^*\|_1 - \|\hat{\Gamma}\|_1)\) is upper bounded by

\[
(\lambda/4) \|P_{B^*}(\Delta^B)\|_N - (\lambda/4) \|P_{B^*}(\Delta^B)\|_N + (\alpha/4) \|P_{\Gamma^*}(\Delta^F)\|_1 - (\gamma/4) \|P_{\Gamma^*}(\Delta^F)\|_1 \leq \Delta.
\]

This fact, \(2f_1 \leq c_1\) in item (iv) and (34) imply

(35)

\[
\frac{c^2}{4} \|\Delta^B; \Delta^F\|_1^2 \leq f_1 \|\Delta^B; \Delta^F\|_1 + \Delta.
\]

Define \(H := (\lambda/2) \|P_{B^*}(\Delta^B)\|_N + (\alpha/2) \|P_{\Gamma^*}(\Delta^F)\|_1\).

We further consider two subcases.
Case 2.1: \( f_1 \| [\Delta^B; \Delta^f] \|_1 \leq H \).
In that case, we conclude from (35) that \([\Delta^B; \Delta^f] \in \mathcal{C}_{B^*} (6, \gamma)\).

Case 2.2: \( f_1 \| [\Delta^B; \Delta^f] \|_1 \geq H \).

Let \( G := \| [\Delta^B; \Delta^f] \|_1 \). In that case we obtain \( \frac{c^2}{4} G^2 \leq 2 f_1 G \Rightarrow G \leq \frac{8 f_1}{c^2} \). Therefore \( H \leq \frac{8 f_1^2}{c^2} \). From (35), we obtain that

\[
\| P_B^* (\Delta^B) \|_N + \| P_B^* (\Delta^f) \|_1 \leq 4 f_1 G \leq 32 \frac{f_1^2}{c^2},
\]

which further implies

\[
\lambda \| \Delta^B \|_N + \tau \| \Delta^f \|_1 \leq \frac{2}{3} H + 32 \frac{f_1^2}{c^2} \leq 37.4 \frac{f_1^2}{c^2}.
\]

Finally, from (35) and \( H \leq \frac{8 f_1^2}{c^2} \),

\[
\frac{c^2}{4} \| [\Delta^B; \Delta^f] \|_1 \leq f_1 \| [\Delta^B; \Delta^f] \|_1 + 8 \frac{f_1^2}{c^2},
\]

which implies (32) by Young’s inequality.

\( \square \)

**Theorem 7** (Trace-regression with matrix decomposition). Suppose that, in addition to (i)-(iv) in Lemma 9, the following condition holds:

(v) If we define

\[
R := \Psi_{\| \cdot \|_N} (P_B^* (\Delta^B)) : \mu (C_{B^*} \| \cdot \|_N (12)),
\]

\[
Q := \Psi_{\| \cdot \|_1} (P_{\Gamma^*} (\Delta^f)) : \mu (C_{\Gamma^*} \| \cdot \|_1 (12)),
\]

assume that

\[
\frac{20}{3} \sqrt{\lambda^2 R^2 + \tau^2 Q^2} \leq c_1.
\]

Then

\[
\| [\Delta^B; \Delta^f] \|_1 \leq 80 \frac{f_1^2}{c^2} \sqrt{\left[ \frac{4 f_1^2}{c^2} + \frac{28}{c^2} \sqrt{\lambda^2 R^2 + \tau^2 Q^2} \right] \sqrt{\left[ \frac{4 f_1^2}{c^2} + \frac{40}{c^2} (\lambda R) \vee (\tau Q) \right]}},
\]

\[
\lambda \| \Delta^B \|_N + \tau \| \Delta^f \|_1 \leq 37.4 \frac{f_1^2}{c^2} \sqrt{\left[ \frac{14 f_1^2}{c^2} + \frac{32}{c^2} \lambda^2 R^2 + \tau^2 Q^2 \right] \sqrt{\left[ \frac{10 f_1^2}{c^2} + \frac{20}{c^2} (\lambda^2 R^2) \vee (\tau^2 Q^2) \right]}}.
\]

**Proof.** In Lemma 9, if (32)-(33) hold there is nothing to prove. We thus need to consider the case (A) for which \([\Delta^B; \Delta^f] \in \mathcal{C}_{B^*} (3, \gamma)\) and (30) or case (B) for which \([\Delta^B; \Delta^f] \in \mathcal{C}_{B^*} (6, \gamma)\) and (31) hold.

Case (A). We first claim that, as \([\Delta^B; \Delta^f] \in \mathcal{C}_{B^*} (3, \gamma)\) we may assume that either \( \Delta^B \in \mathcal{C}_{B^*} \|_N (12) \) or \( \Delta^f \in \mathcal{C}_{\Gamma^*} \|_1 (12) \). Indeed, otherwise, \( 9 \gamma \| P_B^* (\Delta^B) \|_N + 9 \| P_{\Gamma^*} (\Delta^f) \|_1 \leq 0 \) implying \( \Delta^B = \Delta^f = 0 \).
Case 1: $\Delta^B \in C_{B^*} \parallel N_1 (12)$ and $\Delta^F \in C_{\Gamma^*} \parallel f_1 (12)$. Decomposability, $[\Delta^B; \Delta^F] \in C_{B^*} \Gamma^* (3, \gamma)$ and Cauchy-Schwarz imply

$$\frac{\Delta}{3} \leq (\gamma/2)\|\Delta^B\|_N + (\tau/2)\|\Delta^F\|_1 \leq \frac{5}{2} \sqrt{\lambda^2 R^2 + \tau^2 Q^2}\|[\Delta^B; \Delta^F]\|_n.$$ 

Assuming $5\sqrt{\lambda^2 R^2 + \tau^2 Q^2} \leq c_1$, a similar argument in Proposition 1 implies that

$$\frac{\lambda}{\Delta} \|\Delta^B\|_N + \frac{\tau}{\Delta} \|\Delta^F\|_1 \leq \frac{5}{2} \sqrt{\lambda^2 R^2 + \tau^2 Q^2}\|[\Delta^B; \Delta^F]\|_n.$$ 

Case 2: $\Delta^B \notin C_{B^*} \parallel N_1 (12)$ and $\Delta^F \in C_{\Gamma^*} \parallel f_1 (12)$. As $[\Delta^B; \Delta^F] \in C_{B^*} \Gamma^* (3, \gamma)$,

$$9\gamma \|P_{B^*} (\Delta^B)\|_N + \|P_{\Gamma^*} (\Delta^F)\|_1 \leq 3\|P_{\Gamma^*} (\Delta^F)\|_1,$$

implying

$$\frac{\Delta}{3} \leq (\gamma/2)\|\Delta^B\|_N + (\tau/2)\|\Delta^F\|_1 \leq 20\tau\|\Delta^F\|_1 \leq \frac{10}{3} \tau Q\|[\Delta^B; \Delta^F]\|_n.$$ 

Assuming $\frac{20}{3} \tau Q \leq c_1$, we obtain

$$\|[\Delta^B; \Delta^F]\|_n \leq \frac{4}{c_1} f_1 + \frac{40}{c_1^2} \tau Q,$$

$$\lambda\|\Delta^B\|_N + \tau\|\Delta^F\|_1 \leq \frac{10f_1^2}{3c_1} + \frac{270}{c_1} \tau^2 Q^2.$$ 

Case 3: $\Delta^B \in C_{B^*} \parallel N_1 (12)$ and $\Delta^F \notin C_{\Gamma^*} \parallel f_1 (12)$. Similarly to Case 2, assuming $\frac{20}{3} \lambda R \leq c_1$, we obtain

$$\|[\Delta^B; \Delta^F]\|_n \leq \frac{4}{c_1} f_1 + \frac{40}{c_1^2} \lambda R,$$

$$\lambda\|\Delta^B\|_N + \tau\|\Delta^F\|_1 \leq \frac{10f_1^2}{3c_1} + \frac{270}{c_1} \lambda^2 R^2.$$ 

Case (B). Without further conditions, $\|[\Delta^B; \Delta^F]\|_n \leq \frac{1}{c_1} (f_1 + \Delta)$. By dividing in subcases as in Case (A), one obtains the bounds

$$\|[\Delta^B; \Delta^F]\|_n \leq \left[\frac{4}{c_1} f_1 + \frac{28}{c_1^2} \sqrt{\lambda^2 R^2 + \tau^2 Q^2}\right] \vee \left[\frac{4}{c_1} f_1 + \frac{12}{c_1^2} \tau Q\right] \vee \left[\frac{4}{c_1} f_1 + \frac{12}{c_1^2} \lambda R\right],$$

$$\lambda\|\Delta^B\|_N + \tau\|\Delta^F\|_1 \leq \left[\frac{14}{c_1} f_1 + \frac{32}{c_1^2} (\lambda^2 R^2 + \tau^2 Q^2)\right] \vee \left[\frac{4}{c_1} f_1 + \frac{16}{c_1^2} \tau^2 Q^2\right] \vee \left[\frac{4}{c_1} f_1 + \frac{16}{c_1^2} \lambda^2 R^2\right].$$

The proof is finished. \(\square\)

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6.3 Properties for subgaussian \((X, \xi)\)

Throughout this section, we additionally assume \((X, \xi) \in \mathbb{R}^p \times \mathbb{R}\) is a centered (not necessarily independent) random pair satisfying Assumption 3 and \(\{(X_i, \xi_i)\}_{i \in [n]}\) is an iid copy of \((X, \xi)\). Moreover, \(\mathcal{R}\) is any norm on \(\mathbb{R}^p\) and \(Q\) is any norm on \(\mathbb{R}^n\).

In this section we prove that all properties of Definitions 6 and 7 are satisfied with high-probability. Recall that TP in Definition 2 was already proved in Proposition 2 in Section 5.3.

We first prove PP. We will use the next lemma which is immediate from Theorem 10 in the Appendix for the product process (29) over the linear class \(\mathcal{F} = \{\langle \cdot, V \rangle : V \in \mathbb{R}^p\}\).

**Lemma 10.** Let \(B_1\) and \(B_2\) be bounded subsets of \(\mathbb{B}_1\). There exists universal numerical constant \(C > 0\), such that, for any \(n \geq 1\) and \(t > 0\), with probability at least \(1 - e^{-t}\),

\[
\sup_{[V, W] \in B_1 \times B_2} \|\langle V, W \rangle_n - \langle V, W \rangle\| \leq C \frac{\|g(\mathcal{G}^{1/2}(B_1))\|\mathcal{G}(\mathcal{G}^{1/2}(B_2))}{\sqrt{n}} + C \frac{\mathcal{G}(\mathcal{G}(\mathcal{G}^{1/2}(B_1) + \mathcal{G}(\mathcal{G}^{1/2}(B_2)))}{\sqrt{n}} + C L^2 \left(\frac{t}{n} + \frac{\sqrt{t}}{n}\right).
\]

**Proposition 5 (PP).** There exists universal numerical constant \(C > 0\), such that for all \(\delta \in (0, 1]\) and \(n \in \mathbb{N}\), with probability at least \(1 - \delta\), the following property holds: for all \([V, W] \in (\mathbb{R}^p)^2\),

\[
|\langle V, W \rangle_n - \langle V, W \rangle\| \leq CL^2 \left(\frac{\log(1/\delta)}{n} + \frac{\log(1/\delta)}{n}\right) \|\mathcal{V}\| \|\mathcal{W}\| + C \frac{\mathcal{G}(\mathcal{G}(\mathcal{G}^{1/2}(B_1) \cap \|\mathcal{W}\| B_F))}{\sqrt{n}} \|\mathcal{V}\| + C \frac{\mathcal{G}(\mathcal{G}(\mathcal{G}^{1/2}(B_2) \cap \|\mathcal{W}\| B_F))}{\sqrt{n}} \|\mathcal{V}\| + C \frac{\mathcal{G}(\mathcal{G}(\mathcal{G}^{1/2}(B_1) \cap \|\mathcal{W}\| B_F))}{\sqrt{n}} \|\mathcal{V}\| + C \frac{\mathcal{G}(\mathcal{G}(\mathcal{G}^{1/2}(B_2) \cap \|\mathcal{W}\| B_F))}{\sqrt{n}} \|\mathcal{V}\| + C \frac{\mathcal{G}(\mathcal{G}(\mathcal{G}^{1/2}(B_1) \cap \|\mathcal{W}\| B_F))}{\sqrt{n}} \|\mathcal{V}\| + C \frac{\mathcal{G}(\mathcal{G}(\mathcal{G}^{1/2}(B_2) \cap \|\mathcal{W}\| B_F))}{\sqrt{n}} \|\mathcal{V}\|
\]

**Proof.** Let \(R_1, R_2 > 0\) and define the sets

\[V_1 := \{V : \|V\| \leq 1, R(V) \leq R_1\}, \quad V_2 := \{V : \|V\| \leq 1, Q(V) \leq R_2\}.\]

Note that,

\[\mathcal{G}(\mathcal{G}^{1/2}(V_1)) \leq R_1 \mathcal{G}(\mathcal{G}^{1/2}(\mathbb{B}_R) \cap R_1^{-1} \mathbb{B}_F), \quad \mathcal{G}(\mathcal{G}^{1/2}(V_2)) \leq R_2 \mathcal{G}(\mathcal{G}^{1/2}(\mathbb{B}_Q) \cap R_2^{-1} \mathbb{B}_F).\]

Define for convenience the functions

\[g(r) := \frac{\mathcal{G}(\mathcal{G}^{1/2}(\mathbb{B}_R) \cap R^{-1} \mathbb{B}_F)}{\sqrt{n}} r, \quad \tilde{g}(\tilde{r}) := \frac{\mathcal{G}(\mathcal{G}^{1/2}(\mathbb{B}_Q) \cap \tilde{R}^{-1} \mathbb{B}_F)}{\sqrt{n}} \tilde{r}.\]

By Lemma 10, there is universal constant \(C > 0\) such that, for any \(R_1, R_2 > 0\) and \(\delta \in (0, 1]\), with probability at least \(1 - \delta\),

\[
\sup_{[V, W] \in V_1 \times V_2} |\langle V, W \rangle_n - \langle V, W \rangle| \leq C g(R_1) \tilde{g}(R_2) + C L g(R_1) + \tilde{g}(R_2)] + C L^2 \left(\frac{\log(1/\delta)}{n} + \frac{\log(1/\delta)}{n}\right).
\]

We can now use a bi-parameter peeling lemma for subexponential tails (analogously to Lemma 16 for subgaussian tails) with set \(V := \mathbb{B}_1 \times \mathbb{B}_1\), functions \(M(V, u) := -\langle V, W \rangle, h_1(V, W) := R(V), h_2(V, W) := Q(W), g\ and \ \tilde{g}\ and constants \(c := 1\ and \ b_1 := CL^2(\frac{1}{\sqrt{n}} + \frac{1}{n}).\)

The claim follows from such lemma and the homogeneity of norms.
The next proposition follows from the concentration bound for the multiplier process (Theorem 9 in the Appendix) and a peeling lemma. The proof is similar to Proposition 4 so we omit the details.

**Proposition 6 (MP).** For \( t, s > 0 \), let \( \triangle(t, s) \) as defined in Proposition 4. There exists universal constant \( C > 0, c_0, c \geq 2 \) such that for all \( n \in \mathbb{N} \) and all \( \delta \in (0, 1/c] \) and \( \rho \in (0, 1/c_0] \), with probability at least \( 1 - \delta - \rho \), the following property holds: for all \( [V, W] \in (\mathbb{R}^p)^2 \),

\[
\langle \xi^{(n)}, x^{(n)}(V + W) \rangle \leq C \sigma L \cdot \triangle(t, s, 1/\rho) \cdot \|||V; W|||1
+ C\sigma L \left[ 1 + \frac{\sqrt{\log(1/\rho)}}{\sqrt{n}} \right] \| (\mathbb{S}^{1/2}(B_{\bar{Q}})) \|_{\mathcal{R}}(V)
+ C\sigma L \left[ 1 + \frac{\sqrt{\log(1/\rho)}}{\sqrt{n}} \right] \| (\mathbb{S}^{1/2}(B_{\bar{Q}})) \|_{Q}(W).
\]

### 6.4 Proof of Theorem 3

We will apply Sections 5.3 and 6.3 to the nuclear and \( \ell_1 \) norms in \( \mathbb{R}^p \). As before, \( \| \Sigma_{1/2}B_{\|\|_{\|\|_N}} \| \lesssim \rho N (\sqrt{d_1} + \sqrt{d_2}) \) and \( \| \Sigma_{1/2}B_{\|\|_{\|\|_1}} \| \lesssim \rho \sqrt{\log \rho} \), noting that \( \mathbb{S} \) is the identity operator by isotropy of \( \mathbf{X} \).

We first use Proposition 2 with \( \mathcal{R} = \| \cdot \|_{\|\|_N}, \epsilon = \frac{\sigma}{\sqrt{d_1} + \sqrt{d_2}} \) for sufficiently small \( c \in (0, 1) \) and assuming

\[
(36) \quad \delta \geq \exp \left( -c_1 \frac{n}{L^2} \right)
\]

for large enough constant \( c_1 > 0 \). It follows that for \( a_2 \in (0, 1) \) an universal constant and

\[
a_2 \asymp L \rho N \left( \sqrt{\frac{d_1}{n}} + \sqrt{\frac{d_2}{n}} \right),
\]

on an event \( \Omega_1 \) of probability at least \( 1 - \delta/4 \), property \( \text{TP}_{\|\|_N}(a_1; a_2) \) is satisfied. Similarly, assuming (36), we have that for universal constant \( \bar{a}_2 \in (0, 1) \) and

\[
\bar{a}_2 \asymp L \rho_1 \sqrt{\frac{\log \rho}{n}},
\]

on an event \( \Omega'_1 \) of probability at least \( 1 - \delta/4 \), property \( \text{TP}_{\|\|_1}(\bar{a}_1; \bar{a}_2) \) is satisfied.

From Proposition 5, for every \( \delta \in (0, 1) \) and taking

\[
b_1 \asymp L^2 \left( \sqrt{\frac{\log(1/\delta)}{n}} + \sqrt{\frac{\log(1/\delta)}{n}} \right), \quad b_2 \asymp L \rho N \left( \sqrt{\frac{d_1}{n}} + \sqrt{\frac{d_2}{n}} \right), \quad b_3 \asymp L \rho_1 \sqrt{\frac{\log \rho}{n}}
\]

and \( b_4 = \frac{b_3 b_4}{b_4} \) (recall \( L \geq 1 \)), we have that on an event \( \Omega_2 \) of probability at least \( 1 - \delta/4 \), property \( \text{PP}_{\|\|_N,\|\|_1}(b_1; b_2; b_3; b_4) \) is satisfied.

From Lemma 1, by enlarging \( c_1 \) in (36) if necessary, for \( c_1 \in (0, 1) \) an universal constant and

\[
c_2 \asymp L \rho N \left( \sqrt{\frac{d_1}{n}} + \sqrt{\frac{d_2}{n}} \right), \quad c_3 \asymp L \rho_1 \sqrt{\frac{\log \rho}{n}},
\]

\( \text{ATP}_{\|\|_N,\|\|_1}(c_1; c_2; c_3) \) is satisfied on \( \Omega_1 \cap \Omega'_1 \cap \Omega_2 \).
We now use Proposition 6 (with \( \delta = \rho \)). By enlarging \( c_1 \) in (36) if necessary, if we take

\[
f_1 \leq \sigma L \frac{1 + \sqrt{\log(1/\delta)}}{\sqrt{n}}, \quad f_2 \leq \sigma L \rho_n \left( \sqrt{\frac{d_1}{n}} + \sqrt{\frac{d_2}{n}} \right), \quad f_3 \leq \sigma L \rho_1 \sqrt{\frac{\log p}{n}},
\]

we have by Proposition 6 that on an event \( \Omega_3 \) of probability at least \( 1 - \delta/4 \), \( \text{MP} \| X \|_{\| \|_1} (f_1; f_2; f_3) \) is satisfied.

By an union bound and enlarging constants, for \( \delta \) satisfying (36), on the event \( \Omega_1 \cap \Omega_2 \cap \Omega_3 \) of probability at least \( 1 - \delta \), all properties TP, PP, ATP and MP hold with constants as specified above. We assume such event is realized and invoke Theorem 7. It is straightforward to check (iii) by the definitions of \( \tau \) and \( \lambda \) in Theorem 3. Item (iv) is tantamount requiring \( \delta \geq \exp(-c_2n/\sigma^2 L^2) \) for universal constant \( c_2 > 0 \). We now check item (v). In our setting, \( R \leq \sqrt{n} \mu(B^*) \) with \( \mu(B^*) := \mu(C_{B^*}, \| \|_{\| \|_N}(12)) \) and \( Q \leq \sqrt{n} \mu(\Gamma^* \| \|_\infty (12)) \). Condition in item (iv) requires

\[
C \sigma^2 L^2 \rho_n^2 \left( \frac{d_1}{n} + \frac{d_2}{n} \right) \mu^2(B^*) + C \left( \sigma^2 L^2 \rho_1^2 \frac{\log p}{n} + \frac{(a^*)^2}{n} \right) \mu^2(\Gamma^*) < 1,
\]

which holds by assumption. Similarly, condition (22) may be checked by enlarging \( C \) above and \( c_1 \) if necessary (so that \( \frac{d_1}{c_1} \) is small than an universal constant). With all conditions of Theorem 5 taking place, the rate in Theorem 1 follows.  

\section{Proof of Theorem 4}

Recall the estimator (7).

Throughout Sections 7.1 and 7.2, \( \| \cdot \|_1 \) can be regarded as a generic pseudo-norm. Moreover, \( \{ (X_i, \xi_i) \}_{i \in [n]} \) can be regarded deterministic satisfying (1) with \( \xi = \{ \xi_i \}_{i \in [n]} \) and \( X \) and \( \Omega \) the design and augmented design operators associated to the sequence \( \{ X_i \}_{i \in [n]} \). Probabilistic arguments are used only in Sections 7.3 and 7.4.

\subsection{Additional design properties and cones}

We recall definitions of Section 5.1. We will show in Section 7.3 that ATP(\( c_1; 0; 0 \)) is satisfied over a specific cone \( C \) for an universal constant \( c_1 \in (0, 1] \) and when \( Q \) is the Slope norm.

For the robust matrix completion problem, we will use a variation of MP of Definition 2. This will be used throughout Section 7. In the next definitions, \( R \) is a norm on \( \mathbb{R}^p \) and \( Q \) is a norm on \( \mathbb{R}^n \).

\begin{definition}[MP]
Given positive numbers \( f_1, f_2 \text{ and } f_3 \), we will say that \( (X, \xi) \) satisfies \( \text{MP}_{\| \|_1, Q}(f_1; f_2; f_3) \) if

\[
\forall \| V \| \in \mathbb{R}^p \quad |\langle \xi^{(n)}, X^{(n)}(V) + u \rangle| \leq f_1 \| u \|_2 + f_2 R(V) + f_3 Q(u).
\]

We will also need additional cone definitions.

\footnote{Note that \( \Box \text{ and } \triangle \) are bounded by a numerical constant of \( O(L) \).}

\end{definition}
Definition 10. Given $c_0, h > 0$, we define
\[
\mathcal{C}(c_0, h) := \{ V : h\| V \|_\infty \leq c_0\| V \|_\Omega \},
\]
\[
\mathcal{C}_R(c_0, h) := \{ V : h\| V \|_\infty R(V) \leq c_0\| V \|_\Omega \},
\]
\[
\mathcal{C}_\infty(c_0, h) := \{ [V; u] : h\| V \|_\infty \| R(V) \| \leq c_0\| [V; u] \|_\Omega \},
\]
\[
\mathcal{C}_Q(c_0, h) := \{ [V; u] : h\| V \|_\infty Q(u) \leq c_0\| [V; u] \|_\Omega \}.
\]

7.2 Deterministic bounds

Throughout this section, $R$ is a any decomposable norm on $\mathbb{R}^p$ (see Definition 3).

We first restate Lemma 4 to the setting suitable for the matrix completion problem.

Lemma 11 (Dimension reduction). Suppose that $\| B^* \|_\infty \leq a$ and

(i) $(X, \xi)$ satisfies the MP$_{R^{(p)}, \| \cdot \|_4}(f_1; f_2; f_3)$ for some positive numbers $f_1, f_2$ and $f_3$.

(ii) $\lambda = \gamma \tau \geq 2f_2$, and $\tau \geq 2f_3$.

Then
\[
\| W^{(\mathcal{X})}(\Delta^B, \Delta^\theta) \|_2^2 \leq f_2\| \Delta^\theta \|_2 + \Delta,
\]
where
\[
\Delta := \lambda/2)(R^{(p)} \circ \mathcal{P}_{B^*})(\Delta^\hat{B}) - (\lambda/2)(R^{(p)} \circ \mathcal{P}_{B^*})(\Delta^\hat{B}) + (\lambda \tau/2)\| \Delta^\theta \|_2 - (\tau/2)\| \sum_{i=1}^n \omega_i(\Delta^\theta) \|_2.
\]

In particular, $[\Delta^B; \Delta^\theta] \in \mathcal{C}_{B^*, R^{(p)}}(3\gamma, 2^{1/2} + \Omega)$.

Proof. There are only two minor changes in the proof of Lemma 4. First, the first order condition (12) of problem (7) has the additional constraint $\| B \|_\infty \leq a$. Since $\| B^* \|_\infty \leq a$ by assumption the argument still applies. Second, replacing MP in Definition 2 by Definition 9, a slight change in the argument in (13) yield the statement of the lemma.

Theorem 8 (Robust matrix completion). Define the cone
\[
\mathcal{C} := \mathcal{C}(\tilde{c}_1, h_1) \times \mathbb{R}^n \bigcap \mathcal{C}_{R^{(p)}}(\tilde{c}_3, h_3) \bigcap \mathcal{C}_{\| \cdot \|_2}(\tilde{c}_4, h_4),
\]
for universal constants $\tilde{c}_1, \tilde{c}_3, \tilde{c}_4 \in (0, 1)$ and positive constants $h_1, h_3$ and $h_4$.

(i) $(X, \xi)$ satisfies the MP$_{R^{(p)}, \| \cdot \|_4}(f_1; f_2; f_3)$ for some positive numbers $f_1, f_2$ and $f_3$.

(ii) $X$ satisfies the ATP$_{R^{(p)}, \| \cdot \|_4}(c_1; 0; 0)$ on $\mathcal{C}$ for some positive number $c_1$.

(iii) $\lambda = \gamma \tau \geq 2f_2$, and $\tau \geq 2f_3$.

Moreover, let
\[
R := \Psi \left( \mathcal{P}_{B^*}^{(p)}(\Delta^B) \right) \mu^{(p)}(\mathcal{C}_{B^*, R^{(p)}}(4)).
\]

Then either $\| \Delta^B \|_\Omega \leq \frac{c_0}{c_1} h_1$ or
\[
\| \Delta^B; \Delta^\theta \|_\Omega \leq \left[ \left( \frac{ah_1}{c_3 \lambda} + \frac{ah_1}{c_4 \tau} \right) \sqrt{64\lambda^2 R^2 + (32\tau \Omega + 20f_1)^2} \right] \sqrt{\left[ \frac{1}{c_1^2} \sqrt{2.25\lambda^2 R^2 + (6\tau \Omega + 4f_1)^2} \right]},
\]
\[
\lambda R^{(p)}(\Delta^B) + \tau \| \Delta^\theta \|_2 \leq \left[ \left( \frac{ah_1}{c_3 \lambda} + \frac{ah_1}{c_4 \tau} \right) (4\lambda^2 R^2 + (8\tau \Omega + 5f_1)^2) \right] \sqrt{\left[ \frac{1}{c_1^2} (7.5\lambda^2 R^2 + (16\tau \Omega + 10f_1)^2) \right]}.
\]
Proof. We will divide the proof in different cases. We recall that by Lemma 11, $[\Delta^B; \Delta^\theta] \in \mathcal{C}$ with $\mathcal{C} := C_{B^*, R^{(p)}}(3, \gamma, \Omega + 2\tan)$. 

**Case 1:** $\Delta^B \notin \mathcal{C}(\bar{c}_1, h_1)$.

By definition, $\bar{c}_1 \Vert \Delta^B \Vert _\Pi \leq h_1 \Vert \Delta^B \Vert _\infty \leq 2ah_1$ and we are done.

**Case 2:** $[\Delta^B; \Delta^\theta] \notin \mathcal{C}_{R^{(p)}}(\bar{c}_3, h_3)$.

In that case

$$\bar{c}_3 \Vert [\Delta^B; \Delta^\theta] \Vert _\Pi^2 \leq h_3 \Vert \Delta^B \Vert _\infty \mathcal{R}^{(p)}(\Delta^B) \leq 2a(h_3)\lambda\mathcal{R}^{(p)}(\Delta^B).$$

We now consider two cases.

**Case 2.1:** $4\mathcal{R}^{(p)}(P_{B^*}(\Delta^B)) \geq \mathcal{R}^{(p)}(P_{B^*}(\Delta^B))$. Hence $\Delta^B \in C_{B^*, R^{(p)}}(4)$. Decomposability of $\mathcal{R}$, $[\Delta^B; \Delta^\theta] \in \mathcal{C}$ and Cauchy-Schwarz further imply

$$\lambda\mathcal{R}^{(p)}(\Delta^B) + \tau \Vert \Delta^\theta \Vert _2 \leq 4\lambda\mathcal{R}^{(p)}(P_{B^*}(\Delta^B)) + (4\tau\Omega + 2f_1)\Vert \Delta^\theta \Vert _2$$

$$\leq \sqrt{16\lambda^2R^2 + (4\tau\Omega + 2f_1)^2} \Vert [\Delta^B; \Delta^\theta] \Vert _\Pi.$$ 

The above display and (38) imply

$$\Vert [\Delta^B; \Delta^\theta] \Vert _\Pi \leq 4\bar{c}_3(h_3)\sqrt{4\lambda^2R^2 + (2\tau\Omega + f_1)^2},$$

and

$$\lambda\mathcal{R}^{(p)}(\Delta^B) + \tau \Vert \Delta^\theta \Vert _2 \leq 8\bar{c}_3(h_3)[4\lambda^2R^2 + (2\tau\Omega + f_1)^2].$$

**Case 2.2:** $4\mathcal{R}^{(p)}(P_{B^*}(\Delta^B)) < \mathcal{R}^{(p)}(P_{B^*}(\Delta^B))$. As $[\Delta^B; \Delta^\theta] \in \mathcal{C}$,

$$\lambda\mathcal{R}^{(p)}(P_{B^*}(\Delta^B)) + \tau \sum_{i=0+1}^n \omega_i(\Delta^\theta)^2_i \leq (3\tau\Omega + 2f_1)\Vert \Delta^\theta \Vert _2.$$ 

This fact and decomposability of $\mathcal{R}$ imply

$$\lambda\mathcal{R}^{(p)}(\Delta^B) + \tau \Vert \Delta^\theta \Vert _2 \leq 4\lambda\mathcal{R}^{(p)}(P_{B^*}(\Delta^B)) + (4\tau\Omega + 2f_1)\Vert \Delta^\theta \Vert _2$$

$$\leq (16\tau\Omega + 10f_1) \Vert \Delta^\theta \Vert _2.$$ 

The above display and (38) imply

$$\Vert [\Delta^B; \Delta^\theta] \Vert _\Pi \leq 4\bar{c}_3(h_3)(8\tau\Omega + 5f_1),$$

and

$$\lambda\mathcal{R}^{(p)}(\Delta^B) + \tau \Vert \Delta^\theta \Vert _2 \leq 8\bar{c}_3(h_3)(8\tau\Omega + 5f_1)^2.$$

**Case 3:** $[\Delta^B; \Delta^\theta] \notin \mathcal{C}_{\Vert \Pi \Vert _2}(\bar{c}_4, h_4)$.

In that case

$$\bar{c}_4 \Vert [\Delta^B; \Delta^\theta] \Vert _\Pi \leq h_4 \Vert \Delta^B \Vert _\infty \Vert \Delta^\theta \Vert _2 \leq 2a(h_4/\tau)\tau \Vert \Delta^\theta \Vert _2.$$ 

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As in Case 2, by dividing in the two subcases and using the bounds (39) and (41) we get
\[
\left\| [\Delta^B; \Delta^\theta] \right\|_\Pi \leq \frac{4}{c_4} (h_{1/2}) \sqrt{4 \lambda^2 R^2 + (2 \tau \Omega + f_1)^2},
\]
\[
\lambda \mathcal{R}^{(p)}(\Delta^B) + \tau \| \Delta^\theta \|_2 \leq \frac{8}{c_4} (h_{1/2}) [4 \lambda^2 R^2 + (2 \tau \Omega + f_1)^2].
\]

or
\[
\left\| [\Delta^B; \Delta^\theta] \right\|_\Pi \leq \frac{4}{c_4} (h_{1/2})(8 \tau \Omega + 5f_1),
\]
\[
\lambda \mathcal{R}^{(p)}(\Delta^B) + \tau \| \Delta^\theta \|_2 \leq \frac{8}{c_4} (h_{1/2})(8 \tau \Omega + 5f_1)^2.
\]

Case 4: \([\Delta^B; \Delta^\theta] \in \mathbb{C}\).

As above, we further split in two cases.

Case 4.1: \(4 \mathcal{R}^{(p)}(\mathcal{P}_{\mathcal{B}^*}(\Delta^B)) \geq \mathcal{R}^{(p)}(\mathcal{P}_{\mathcal{B}^*}(\Delta^B))\). Hence \(\Delta^B \in \mathbb{C}^{\mathcal{B}^*, \mathcal{R}^{(p)}}(4)\). Decomposability of \(\mathcal{R}\), \([\Delta^B; \Delta^\theta] \in \mathcal{C}\) and Cauchy-Schwarz give
\[
f_1 \| \Delta^\theta \|_2 + \Delta \leq (3/2) \mathcal{R}^{(p)}(\mathcal{P}_{\mathcal{B}^*}(\Delta^B)) + (3 \tau \Omega/2 + f_1) \| \Delta^\theta \|_2
\]
\[
\leq (3/2) \left\{ \lambda^2 \theta^2 \left( \mathcal{P}_{\mathcal{B}^*}(\Delta^B) \right) \left[ \mu^{(p)}(\mathbb{C}^{\mathcal{B}^*, \mathcal{R}^{(p)}}(4)) \right] \right\}^{1/2} \left\| [\Delta^B; \Delta^\theta] \right\|_\Pi.
\]

From the above display, (37) and ATP\((c_1; 0; 0)\) as stated in Lemma 11(ii), we obtain
\[
\left\| [\Delta^B; \Delta^\theta] \right\|_\Pi \leq \frac{(3/2) \sqrt{\lambda^2 R^2 + (\tau \Omega + (2 f_1/3))^2}}{c_4^2}.
\]

Finally, (39) and the previous display imply
\[
\lambda \mathcal{R}^{(p)}(\Delta^B) + \tau \| \Delta^\theta \|_2 \leq \frac{3}{c_4^2} \sqrt{\lambda^2 R^2 + (\tau \Omega + (2 f_1/3))^2} \sqrt{4 \lambda^2 R^2 + (2 \tau \Omega + f_1)^2}
\]
\[
\leq \frac{1}{c_4^2} \left[ 7.5 \lambda^2 R^2 + 3(2 \tau \Omega + f_1)^2 \right].
\]

Case 4.2: \(4 \mathcal{R}^{(p)}(\mathcal{P}_{\mathcal{B}^*}(\Delta^B)) < \mathcal{R}^{(p)}(\mathcal{P}_{\mathcal{B}^*}(\Delta^B))\). As \([\Delta^B; \Delta^\theta] \in \mathcal{C}\), relation (40) holds. This fact and decomposability of \(\mathcal{R}\) imply
\[
f_1 \| \Delta^\theta \|_2 + \Delta \leq (3/2) \mathcal{R}^{(p)}(\mathcal{P}_{\mathcal{B}^*}(\Delta^B)) + (3 \tau \Omega/2 + f_1) \| \Delta^\theta \|_2 \leq (6 \tau \Omega + 4f_1) \| \Delta^\theta \|_2.
\]
The above display, (37) and ATP\((c_1; 0; 0)\) yield
\[
\left\| [\Delta^B; \Delta^\theta] \right\|_\Pi \leq \frac{6 \tau \Omega + 4f_1}{c_4^2}.
\]

Finally, (41) and the previous display imply
\[
\lambda \mathcal{R}^{(p)}(\Delta^B) + \tau \| \Delta^\theta \|_2 \leq \frac{(16 \tau \Omega + 10f_1)^2}{c_4^2}.
\]

To finish, we note that the bounds in the statement of the theorem is the maximum of all the bounds established in the above cases. \(\square\)
7.3 Properties for “heavy-tailed” discrete designs

Recall Definition 2. The aim of this section is to prove that, with high probability, MP holds everywhere and ATP(f_1; 0; 0) holds on the cone C defined in Theorem 8 for specific values of positive parameters \( h_1, h_3, h_4 \) and universal constants \( \tilde{c}_1, \tilde{c}_3, \tilde{c}_4 \).

Throughout this section, we additionally assume \( (X, \xi) \in \mathbb{R}^p \times \mathbb{R} \) satisfies Assumption 4 and \( \{(X_i, \xi_i)\}_{i \in [n]} \) is an iid copy of \( (X, \xi) \). Unless specified, \( R \) is any norm on \( \mathbb{R}^p \). We also define the following Gaussian complexity

\[
\tilde{G} := \mathbb{E} [\mathcal{G} (\sqrt{p} \cdot X (B_R))] .
\]

Our arguments will use that \( \forall V \in \mathbb{R}^p, \quad \|XV\| \leq \|X\|_\infty \), which follows from norm duality.

Next, we obtain a lower bound using a one-sided version of a Bernstein’s inequality for bounded processes due to Bousquet [10].

**Lemma 12.** Let \( B_{R, \infty} \) denote the sup-norm unit ball in \( \mathbb{R}^p \) and \( V \) be any bounded subset of \( S_{\mathbb{E}} \cap R B_{R, \infty}^p \) for some \( R > 0 \). Define

\[
\mathcal{G} := \mathbb{E} [\mathcal{G} (X(V))].
\]

Then, for any \( n \geq 1 \) and \( t \geq 0 \), with probability at least \( 1 - \exp(-t) \),

\[
\sup_{V \in \mathcal{V}} \frac{1}{n} \sum_{i \in [n]} \left[ 1 - \langle X_i, V \rangle \right]^2 \leq \frac{\sqrt{2\pi} \mathcal{G}}{n} R + 2 \sqrt{\frac{t}{n}} \left( \max \left\{ \frac{1}{n} R^2, 1 \right\} + \frac{\sqrt{2\pi} R \omega}{n^2} \right) + \frac{2t}{3n}.
\]

**Proof.** Define \( X_i, V := 1 - \langle X_i, V \rangle \). By the symmetrization inequality (e.g. Exercise 11.5 in [10]), we have

\[
\mathbb{E} \left[ \sup_{V \in \mathcal{V}} \sum_{i \in [n]} X_i, V \right] \leq 2 \mathbb{E} \left[ \sup_{V \in \mathcal{V}} \sum_{i \in [n]} \epsilon_i X_i, V \right],
\]

where \( \{\epsilon_i\}_{i \in [n]} \) are iid Rademacher variables independent of \( X := \{X_i\}_{i \in [n]} \). One may bound the Rademacher complexity by the Gaussian complexity as

\[
\mathbb{E} \left[ \sup_{V \in \mathcal{V}} \sum_{i \in [n]} \epsilon_i X_i, V \right] \leq \sqrt{\frac{\mathcal{G}}{2}} \mathbb{E} \left[ \sup_{V \in \mathcal{V}} \sum_{i \in [n]} g_i X_i, V \right],
\]

where \( \{g_i\}_{i \in [n]} \) is an iid \( \mathcal{N}(0, 1) \) sequence independent of \( \{X_i\}_{i \in [n]} \).

We will now use an standard argument via Slepian’s inequality over the randomness of \( \{g_i\}_{i \in [n]} \) for the process \( V \mapsto Z_V := \sum_{i \in [n]} g_i X_i, V \). One has

\[
\mathbb{E}[(Z_V - Z_{V'})^2 | X] = \sum_{i \in [n]} \langle X_i, V - V' \rangle^2 \langle X_i, V + V' \rangle^2 \leq 4R^2 \|X(V - V')\|_2^2.
\]

Define the Gaussian process \( W_V := 2R(X(V), \xi) \) with \( \xi \sim \mathcal{N}_n(0, I_n) \) independent of \( \{X_i\}_{i \in [n]} \). Under the above conditions, Slepian’s inequality implies

\[
\mathbb{E} \left[ \sup_{V \in \mathcal{V}} Z_V | X \right] \leq \mathbb{E} \left[ \sup_{V \in \mathcal{V}} W_V | X \right] = 2R \mathbb{E} \mathcal{G}(X(V)).
\]

Define \( Z := \sup_{V \in \mathcal{V}} \sum_{i \in [n]} X_i, V \). The previous displays imply \( \mathbb{E} Z \leq \sqrt{2\pi} R \mathbb{E} \mathcal{G}(X(V)) \).
We now establish concentration. For all $i \in [n]$ and $V \in V$, $1 - R_i^2 \leq X_i, V \leq 1$. Since, for each $V \in V$, $\{X_i, V\}_{i \in [n]}$ are independent and identically distributed, we may apply Bousquet’s inequality (e.g. Theorem 12.5 in []). We thus get that, for all $t \geq 0$, with probability at least $1 - \exp(-t)$,

$$Z \leq \mathbb{E}Z + \sqrt{2t[\sigma^2 + 2\mathbb{E}Z]} + \frac{2t}{3}$$

In above,

$$\sigma^2 := \sup_{V \in V} \sum_{i \in [n]} \mathbb{E}[X_i^2, V] \leq \max\{1, R_i^2 - 1\} \sup_{V \in V} \sum_{i \in [n]} \mathbb{E}[X_i, V]^2 - 1 \leq \max\{1, R_i^2 - 1\} 2n.$$

This finishes the proof. \qed

**Proposition 7.** There is universal constant $C > 0$ such that, for all $n \in \mathbb{N}$ and $\delta \in (0, 1)$, with probability at least $1 - \delta$, for all $V \in \mathbb{R}^p$,

$$\|X(V)\|_2^2 \geq \left[ 1 - \frac{\hat{C}_V}{n} - \frac{C_1}{\sqrt{n}} \left( 1 + \sqrt{\log(1/\delta)} \right) - \frac{C_2}{n} \log(1/\delta) \right] \|V\|_1^2$$

$$- \frac{C_1}{n} \mathcal{R}(V(\mathbb{R})) \|V\|_\infty - C_2 \sqrt{\log(1/\delta)} \|V\|_\infty \|V\|_1.$$

**Proof.** Given positive numbers $r_1, r_2 > 0$, define

$$V_1 := \{ V \in S_{11} : \mathcal{R}(V) \leq r_1, \|V\|_\infty \leq r_2 \}.$$

By definition, $\mathbb{E}[\mathcal{G}(X(V_1))] \leq r_1 \mathbb{E}[\mathcal{G}(\mathcal{G}(V(\mathbb{R}_{\mathbb{R}}))) = r_1 \hat{G}$. Hence, by Lemma 12, we have that, for any $n \in \mathbb{N}$ and $\delta \in (0, 1)$, with probability at least $1 - \delta$,

$$\sup_{V \in V_1} \left[ 1 - \|X(V)\|_2^2 \right] \leq 2\sqrt{2\pi} \hat{C}_V r_1 r_2 + 2 \sqrt{\log(1/\delta)} \left( \max\{1, r_2^2 - 1\} \frac{1}{n} + \frac{\sqrt{2\pi} \hat{C}_V r_1 r_2}{n^2} \right) + \frac{2\log(1/\delta)}{3n}$$

$$\leq 2\sqrt{2\pi} \hat{C}_V r_1 r_2 + 2 \sqrt{\log(1/\delta)} \max\{1, r_2^2 - 1\} + \frac{5\log(1/\delta)}{3n}$$

$$\leq 2\sqrt{2\pi} \hat{C}_V r_1 r_2 + 2r_2 \sqrt{\log(1/\delta)} + \frac{5\log(1/\delta)}{3n}.$$

Without loss on generality we may assume $\hat{G} \geq 1$.

We will now use the previous display with the peeling Lemma 18 for subexponential tails with $V := S_{11}$ and functions $M(V, u) := \|X(V)\|_2^2 - 1$, $h_1(V, u) := \mathcal{R}(V)$ and $h_2(V, u) := \|V\|_\infty$. Note that the claim is trivial for $V$ with $\|V\|_1 = 0$. From such lemma and the homogeneity of norms we finish the proof. \qed

Proposition 7 immediately implies the following restricted strong convexity condition with respect to $\| \cdot \|_1$.

**Proposition 8 (TP).** There are universal constants $c_1, \in (0, 1)$ and $C, c, \bar{c}_1, \bar{c}_2 > 0$ for which the following holds. Define the cone

$$C := c \left( \bar{c}_1, \frac{C}{\sqrt{n}} \right) \cap C_{\mathcal{R}(\mathbb{R})} \left( \bar{c}_2, \frac{C}{n} \right).$$
Given $\delta \in (0, 1)$, suppose that
\[
\begin{cases}
n \geq C \tilde{G}, \\
\delta \geq e^{-cn}.
\end{cases}
\]

With probability at least $1 - \delta$ the design operator $\hat{X}$ satisfies the restricted eigenvalue condition
\[
\inf_{V \in C} \frac{\|X^{(n)}(V)\|_2}{\|\hat{V}\|_\Pi} \geq c_1.
\]

The following result is an immediate consequence of the dual norm inequality.

**Lemma 13.** For all $[V; u] \in \mathbb{R}^p \times \mathbb{R}^n$,
\[
\langle X^{(n)}(V), u \rangle \leq \frac{\|V\|_{\infty}}{\sqrt{n}} \|u\|_2.
\]

**Proof.** Clearly
\[
\sum_{i \in [n]} u_i \langle X^{(n)}(V), V \rangle \leq \frac{1}{\sqrt{n}} \sum_{i \in [n]} u_i^2 \|X_i, V\|_2 \leq \frac{\|V\|_{\infty}}{\sqrt{n}} \|u\|_2,
\]

since $\min_{i \in [n]} \omega_i = \omega_n \geq 1$ as the sequence $\{\omega_i\}$ is non-increasing. \(
\)

The previous lemma and Proposition 7 imply the following augmented restricted strong convexity condition with respect to $\| \cdot \|_\Pi$.

**Proposition 9 (ATP).** There are universal constants $c_1 \in (0, 1)$ and $C, c, \bar{c}_1, \bar{c}_3, \bar{c}_4 > 0$ for which the following holds. Define the cone
\[
C := \left( \mathcal{C} \left( \bar{c}_1, \frac{C}{\sqrt{n}} \right) \times \mathbb{R}^n \right) \cap \mathcal{C}_R^{(p)} \left( \bar{c}_3, \frac{\tilde{G}}{n} \right) \cap \mathcal{C}_{\|\cdot\|_2} \left( \bar{c}_4, \frac{C}{\sqrt{n}} \right).
\]

Given $\delta \in (0, 1)$, suppose that
\[
\begin{cases}
n \geq C \tilde{G}, \\
\delta \geq e^{-cn}.
\end{cases}
\]

With probability at least $1 - \delta$ the augmented design operator $\mathcal{M}$ satisfies the restricted eigenvalue condition
\[
\inf_{V \in C} \frac{\|\mathcal{M}^{(n)}(V; u)\|_2}{\|\hat{V}, u\|_\Pi} \geq c_1.
\]

**Proof.** In the following suppose the event for which the statement of Proposition 7 is satisfied. Let $V \in C$. First, by Proposition 7, definitions of $\mathcal{C}(\bar{c}_1, \frac{C}{\sqrt{n}})$ and $\mathcal{C}_R^{(p)}(\bar{c}_3, \frac{\tilde{G}}{n})$ and conditions on sample size,
\[
\|X^{(n)}(V)\|_2^2 \geq \left( c' - \bar{c}_1 \right) \|V\|_\Pi^2 - \bar{c}_3 \|\mathcal{M}(V; u)\|_\Pi^2,
\]

for universal constant $c' \in (0, 1)$. By Lemma 13 and definition of $\mathcal{C}_{\|\cdot\|_2}(\bar{c}_4, \frac{C}{\sqrt{n}})$,
\[
2\|X^{(n)}(V), u\| \leq \bar{c}_4 \|\hat{V}; u\|_\Pi^2.
\]

By enlarging $C$ and $c$ if necessary, we thus conclude that
\[
\|X^{(n)}(V) + u\|_2^2 = \|X^{(n)}(V)\|_2^2 + \|u\|_2^2 + 2\langle X^{(n)}(V), u \rangle \geq c_1 \|\hat{V}; u\|_\Pi^2,
\]

for universal constant $c_1 \in (0, 1)$. \(\square\)
We now aim in proving MP with \( \mathcal{R} = \| \cdot \|_N \) the nuclear norm. We will use the following well-known result.\(^4\)

**Lemma 14** (Lemma 2 in [51], Lemma 5 in [49]). Suppose \( \{(\eta_i, \mathbf{X}_i)\}_{i \in [n]} \subset \mathbb{R} \times \mathcal{X} \) is a centered iid sequence where \( \eta \) has variance \( \sigma_\eta^2 > 0 \) and \( 0 < |\eta|_{\psi_2} < \infty \). For \( t > 0 \), define

\[
\Delta_\eta(t) := \max \left\{ \sigma_\eta \sqrt{p \max_{k,\ell} \{R_k, C_\ell\}} \frac{\log t + \log d}{n} \cdot \sqrt{p} |\eta|_{\psi_2} \log^{1/2} \left( \frac{|\eta|_{\psi_2} m}{\sigma_\eta^2} \right) \log t + \log d \right\}.
\]

Then for some absolute constant \( C > 0 \), for all \( \delta \in (0, 1) \), with probability at least \( 1 - \delta \),

\[
\left\| \frac{1}{n} \sum_{i \in [n]} \eta_i (\sqrt{p} \cdot \mathbf{X}_i) \right\|_{op} \leq C \Delta_\eta(2/\delta).
\]

Using the previous lemma with \( \eta = \xi \) (and norm duality), Lemma 7 (together with a peeling argument) and a union bound yield the following result. We omit the details.

**Proposition 10** (MP). Let \( C > 0 \) be the universal constant of Lemma 14. Then, for all \( \delta \in (0, 1] \) and \( n \in \mathbb{N} \), with probability of at least \( 1 - \delta \), for all \( \mathbf{V}, \mathbf{u} \in \mathbb{R}^p \times \mathbb{R}^n \),

\[
\langle \xi^{(n)}, \mathcal{G}^{(n)}(\mathbf{V}, \mathbf{u}) \rangle \leq C \Delta_\xi(4/\delta) \cdot \mathcal{R}^{(p)}(\mathbf{V}) + \sigma \frac{24 + 8 \sqrt{\log(18/\delta)}}{\sqrt{n}} \| \mathbf{u} \|_2 + 9.6 \sigma \frac{\mathcal{G}(\mathcal{B}^n_\| \cdot \|_N)}{\sqrt{n}} \| \mathbf{u} \|_2.
\]

### 7.4 Proof of Theorem 4

We now set \( \mathcal{R} := \| \cdot \|_N \). We have

\[
\hat{\mathcal{G}} = \mathbb{E} \left[ \mathcal{G} \left( \sqrt{p} \cdot \mathbf{X} \left( \mathbb{E} \| \cdot \|_N \right) \right) \right] \leq \mathbb{E} \left[ \left\| \sum_{i \in [n]} g_i (\sqrt{p} \cdot \mathbf{X}_i) \right\|_{op} \right],
\]

with \( \{g_i\}_{i \in [n]} \) iid \( \mathcal{N}(0, 1) \) independent of \( \{\mathbf{X}_i\}_{i \in [n]} \). The assumptions of Lemma 14 apply with \( \eta = \mathcal{N}(0, 1) \) so integration yields

\[
\hat{\mathcal{G}} \leq \max \left\{ \sqrt{p \max_{k,\ell} \{R_k, C_\ell\}} \frac{(\log d)}{n}, \sqrt{p} \left( \log m \right)^{1/2} \frac{\log d}{n} \right\}.
\]

As before, Proposition E.2 in Bellec, Lecué and Tsybakov [5] implies that \( \mathcal{G}(\mathcal{B}^n_\| \cdot \|_N) \leq 1 \).

By the conditions on the sample size (with sufficient large universal constants \( C, c > 0 \) and Proposition 9, we have that for universal constant \( c_1 \in (0, 1) \) and \( c_1, c_2, c_4 > 0 \) and constants \( h_1 \asymp \frac{1}{\sqrt{n}} \), \( h_2 \asymp \frac{\sqrt{n}}{\sqrt{m}} \), and

\[
h_3 \asymp \max \left\{ \sqrt{p \max_{k,\ell} \{R_k, C_\ell\}} \frac{\log d}{n}, \sqrt{p} \log^{1/2} \frac{m}{n} \log d \frac{d}{n} \right\},
\]

for all \( \delta \in (0, 1) \), with probability at least \( 1 - \delta/2 \), ATP \( \| \cdot \|_N \| \cdot \|_2 \) \((c_1; 0, 0)\) is satisfied on the cone

\[
C := (\mathcal{C} (\bar{c}_1, h_1) \times \mathbb{R}^n) \bigcap \mathcal{C}^{(p)}(\bar{c}_3, h_3) \bigcap \mathcal{C}_{\| \cdot \|_2} (\bar{c}_4, h_4).
\]

\(^4\)Note that \( \max_{k,\ell} \{R_k, C_\ell\} \geq \frac{1}{a_1 a_2} \).
By Proposition 10, for all $\delta \in (0, 1)$, if we take

$$f_1 \asymp \frac{1 + \sqrt{\log(2/\delta)}}{\sqrt{n}},$$

$$f_2 \asymp \max \left\{ \frac{\sigma_\xi}{\nu_\ell, d} \log \left( \frac{4/\delta + \log d}{\theta \log \left( \frac{4/\delta + \log d}{\theta} \right) \log(1/\delta)} \right), \right. \quad \text{with probability at least } 1 - \delta/2,$$

$$f_3 \asymp \frac{\sigma}{\sqrt{n}},$$

we have that with probability at least $1 - \delta/2$, $\text{MP}(\|\|\|_{N=1} + \|\|_{N=1})(f_1; f_2; f_3)$ is satisfied.

By an union bound, for any $\delta \in (0, 1)$, $\text{ATP}(c_1; 0, 0)$ and $\text{MP}(f_1; f_2; f_3)$ hold on such event implying the claimed rates, noting that $R \leq \sqrt{\nu} \mu(B^*)$ with

$$\mu(B^*) := \mu((C_{B^1,B^2}^{+} N=1))(4) = \mu((C_{B^1,B^2}^{+} N=1)(4)).$$

8 Appendix

8.1 Peeling lemmas

Lemma 15 ([24]). Let $g$ be right-continuous, non-decreasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ and $h$ be function from $V$ to $\mathbb{R}_+$. Assume that for some constants $b \in \mathbb{R}_+$ and $c \geq 1$, for every $r > 0$ and for any $\delta \in (0, 1/(c \vee 7))$, we have

$$A(r, \delta) = \left\{ v \in V : h(v) \leq r, \quad M(v) \geq -g(r) - b\sqrt{\log(1/\delta)} \right\},$$

with probability at least $1 - c\delta$.

Then, for any $\delta \in (0, 1/(c \vee 7))$, with probability at least $1 - c\delta$, we have for all $v \in V$,

$$M(v) \geq -1.2(g \circ h)(v) - b\left( 3 + \sqrt{\log(9/\delta)} \right).$$

Lemma 16 ([24]). Let $g, \hat{g}$ be right-continuous, non-decreasing functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ and $h, \hat{h}$ be functions from $V$ to $\mathbb{R}_+$. Assume that for some constants $b \in \mathbb{R}_+$ and $c \geq 1$, for every $r, \bar{r} > 0$ and for any $\delta \in (0, 1/(c \vee 7))$, we have

$$A(r, \bar{r}, \delta) = \left\{ v \in V : h(v) \leq r, \quad \hat{h}(v) \leq \bar{r}, \quad M(v) \geq -g(r) - \hat{g}(\bar{r}) - b\sqrt{\log(1/\delta)} \right\},$$

with probability at least $1 - c\delta$.

Then, for any $\delta \in (0, 1/(c \vee 7))$, with probability at least $1 - c\delta$, we have for all $v \in V$,

$$M(v) \geq -1.2(g \circ h)(v) - 1.2(\hat{g} \circ \hat{h})(v) - b\left( 4.8 + \sqrt{\log(81/\delta)} \right).$$

Lemma 17. Let $g, \hat{g}$ be right-continuous, non-decreasing functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ and $h, \hat{h}$ be functions from $V$ to $\mathbb{R}_+$. Let $b > 0$ be a constant and $c_0 \geq 2$ be universal constants. Assume that for every $r, \bar{r} > 0$ and every $\delta \in (0, 1/c)$ and every $\rho \in (0, 1/c_0)$, the event $A(r, \bar{r}, \delta, \rho)$ defined by the inequality

$$\inf_{v \in V : h(v) \leq r, \hat{h}(v) \leq \bar{r}} M(v) \geq - \left[ 1 + \frac{1}{\sqrt{n}} \sqrt{\log(1/\rho)} \right] \left\{ b \frac{g(r)}{\sqrt{n}} - b \frac{\hat{g}(\bar{r})}{\sqrt{n}} \right\}$$

$$- \frac{b}{\sqrt{n}} \sqrt{\log(1/\delta)} - \frac{b}{n} \log(1/\delta) - \frac{b}{n} \sqrt{\log(1/\delta) \log(1/\rho)},$$

hold on such event implying the claimed rates, noting that $R \leq \sqrt{\nu} \mu(B^*)$ with

$$\mu(B^*) := \mu((C_{B^1,B^2}^{+} N=1))(4) = \mu((C_{B^1,B^2}^{+} N=1)(4)).$$
has probability at least \( 1 - c\delta - c_0\rho \).

Then, for universal constant \( C > 0 \), with probability at least \( 1 - \delta - \rho \), we have that for all \( v \in V \),

\[
M(v) \geq -\frac{b}{\sqrt{n}} \left( \frac{\log(1/\delta)}{\sqrt{n}} - C_0 \frac{b^2}{n} \log(1/\delta) \log(1/\rho) \right) - \frac{c}{n} \sqrt{\log(1/\rho)}.
\]

**Proof.** The proof is an adaptation of the proof of Lemma 16 valid for subgaussian tails in [24]. The computations are slightly more involved in our case as \( \sqrt{\log(1/\rho)} \) multiples the peeled function \( g \). We only sketch the proof for the one-dimensional case \((g \equiv 0, \bar{h} \equiv 0, \bar{r} \equiv 0)\). During the proof, \( C \) is an universal constant that may change.

Let \( \eta, \epsilon > 0 \) be two parameters to be chosen later on. We set \( \mu_0 = 0 \) and, for \( k \geq 1 \), \( \mu_k := \mu \eta^{k-1} \).

For \( k \in \mathbb{N}^* \), we define\(^7\) \( \nu_k := g^{-1}(\mu_k) \) and the set

\[
V_k := \{ v \in V : \mu_k \leq (g \circ h)(v) < \mu_{k+1} \}.
\]

The union bound and the fact that \( \sum_{k \geq 1} k^{-1-\epsilon} \leq 1 + \epsilon^{-1} \) imply that the event

\[
A := \bigcap_{k=1}^{\infty} A \left( \nu_k, 0, \frac{\epsilon \delta}{(1+\epsilon)k^{1+\epsilon}}, \frac{\epsilon \rho}{(1+\epsilon)k^{1+\epsilon}} \right),
\]

has a probability at least \( 1 - c\delta - c_0\rho \). For convenience, we define \( \Delta(t) := \log\{1+(1+t)/(\epsilon t)\} \) and \( \Delta_k := (1+\epsilon)k \log k \). Throughout the proof, assume that this event is realized:

\[
(42) \quad \forall k \in \mathbb{N}^* \quad \begin{cases} 
\forall v \in V \text{ such that } h(v) \leq \nu_k, \text{ we have} \\
M(v) \geq -[1 + (\ell/\sqrt{n}) \sqrt{\Delta(\rho) + \Delta_k} b h(v) - (\ell/\sqrt{n}) \sqrt{\Delta(\delta) + \Delta_k} - (\ell/\nu \Delta(\delta) + \Delta_k)] - (\ell/\nu) \sqrt{\Delta(\delta) + \Delta_k} \Delta(k + \Delta_k).
\end{cases}
\]

For every \( v \in V \), there is \( \ell \in \mathbb{N}^* \) such that \( v \in V_{\ell} \). If \( \ell \geq 1 \), then \( h(v) \leq \nu_{\ell+1} \). This fact and (42) implies

\[
M(v) \geq -[1 + (\ell/\sqrt{n}) \sqrt{\Delta(\rho) + \Delta_k} b h(v) - (\ell/\sqrt{n}) \sqrt{\Delta(\delta) + \Delta_k} - (\ell/\nu \Delta(\delta) + \Delta_k)] - (\ell/\nu) \sqrt{\Delta(\delta) + \Delta_k} \Delta(k + \Delta_k),
\]

where

\[
\hat{\ell} := [1 + (\ell/\sqrt{n}) \sqrt{\Delta(\rho) + \Delta_k} b h(v) - (\ell/\sqrt{n}) \sqrt{\Delta(\delta) + \Delta_k} - (\ell/\nu \Delta(\delta) + \Delta_k)] + (\ell/\nu) \sqrt{\Delta(\delta) + \Delta_k} \Delta(k + \Delta_k).
\]

By appropriately choosing \( \mu > 0 \) and \( \eta > 1 \), a standard calculation shows that \( \sup_{\ell \geq 1} \hat{\ell} \leq C \sqrt{n} \frac{1}{1 + \sqrt{\Delta(\rho)}} \).

---

\(^7\)Here \( g^{-1} \) is the generalized inverse defined by \( g^{-1}(x) := \inf\{ a \in \mathbb{R}_+ : g(a) \geq x \} \).
If $\ell = 0$, then (42) with $k = 1$ and using $g(\nu) = \mu$ lead to

$$M(v) \geq -\left[1 + (\sqrt{\frac{\mu}{\nu}})\sqrt{\Delta(r)}(b\sqrt{\frac{\mu}{\nu}} - (\sqrt{\frac{\mu}{\nu}})\sqrt{\Delta(r)} - (\sqrt{\frac{\mu}{\nu}})\sqrt{\Delta(r)} - (\sqrt{\frac{\mu}{\nu}})\sqrt{\Delta(r)}\right].$$

Joining the two lower bounds establish the claim.

Lemma 18. Let $h, \bar{h}$ be functions from $V$ to $\mathbb{R}^+$. Let $b > 0$ and $\omega \geq 1$ be constants and $c \geq 1$ be a universal constant. Assume that for every $r, \bar{r} > 0$ and every $\delta \in (0, 1/c)$, the event $A(r, \bar{r}, \delta)$ defined by the inequality

$$\inf_{w \in V: (h, \bar{h})(v) \leq (r, \bar{r})} M(v) \geq -\frac{b\omega}{n}r\bar{r} - b\sqrt{\frac{\log(1/\delta)}{n}},$$

has probability at least $1 - c\delta$.

Then, for universal constant $C > 0$, with probability at least $1 - c\delta$, we have that for all $v \in V$,

$$M(v) \geq -Cb\omega\left[1 + h(v)\bar{h}(v)\right] - C\frac{b}{\sqrt{n}}\sqrt{\log(1/\delta)} \cdot \bar{h}(v) - C\frac{b}{\sqrt{n}}\left[1 + \sqrt{\log(1/\delta)}\right] - C\frac{b}{n}\log(1/\delta).$$

Proof. The proof is an adaptation of proof of Lemmas 16 and 17. One notable change is that the peeled functions $g(r) = r$ and $\bar{g}(\bar{r}) = \bar{r}$ are multiplied. We only give a sketch of the proof. In the following, $C$ is an universal constant that may change.

Let $k, \ell \in \mathbb{N}^*$, we define the set

$$V_{k, \ell} := \{v \in V: k \leq \mu(v) < k + 1, \quad k \leq \bar{h}(v) < k + 1\}.$$ 

The union bound and the fact that $\sum_{k, \ell \geq 1} (k\ell)^{1-\epsilon} \leq (1 + \epsilon^{-1})^2$ imply that the event

$$A = \bigcap_{k, \ell \geq 1} \left\{A\left(\sum_{k, \ell \geq 1} k\ell^2 \mu(\mu, \bar{\mu}) \leq \frac{e^2c\bar{\mu}^{1+\epsilon}}{(1 + \epsilon)(kk)^{1+\epsilon}}\right)\right\}$$

has probability at least $1 - c\delta$. To ease notation, set $\Delta(\delta) := \log\left\{(1 + \epsilon)^2/(e^2\delta)\right\}$ and $\Delta_{k, \ell} := (1 + \epsilon)\log(k\ell)$. We assume in the sequel that the event $A$ is realized, that is

$$\forall k, \ell \in \mathbb{N}^* \quad \forall v \in V \text{ such that } h, \bar{h} \leq (\mu, \bar{\mu}) \text{ we have } M(v) \geq -b\omega\mu k \leq (\sqrt{\frac{\mu}{\nu}})\sqrt{\Delta(\delta) + \Delta_{k, \ell}} \cdot \bar{\mu} k - (\sqrt{\frac{\bar{\mu}}{\nu}})\sqrt{\Delta(\delta) + \Delta_{k, \ell}}.$$

For every $v \in V$, there are $k, \ell \in \mathbb{N}^*$ such that $v \in V_{k, \ell}$. If $\ell \geq 1$ or $\ell \geq 1$, then $\bar{h}(v) \leq \mu_{\ell + 1}$ and $\bar{h}(v) \leq \mu_{\ell + 1}$. This fact and (43) imply

$$M(v) \geq -\frac{b\omega}{n}\mu^2\eta^2\eta^f - b\sqrt{\frac{\Delta(\delta) + \Delta_{\ell + 1, \ell + 1}}{\eta^f}} \cdot \mu\eta^f - b\frac{\eta}{n} \left[\Delta(\delta) + \Delta_{\ell + 1, \ell + 1}\right]$$

where

$$\quad \quad \Delta_{\ell + 1, \ell + 1} := \frac{b\omega}{n}\mu^2\eta^f + b\sqrt{\frac{\Delta(\delta) + \Delta_{\ell + 1, \ell + 1}}{\eta^f}} \cdot \mu\eta^f + b\frac{\eta}{n} \left[\Delta(\delta) + \Delta_{\ell + 1, \ell + 1}\right]$$

and

$$\quad \quad \Delta_{\ell + 1, \ell + 1} := \frac{b\omega}{n}\mu^2\eta^f + b\sqrt{\frac{\Delta(\delta) + \Delta_{\ell + 1, \ell + 1}}{\eta^f}} \cdot \mu\eta^f + b\frac{\eta}{n} \left[\Delta(\delta) + \Delta_{\ell + 1, \ell + 1}\right].$$

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By appropriately choosing $\mu, \eta > 1$, one shows that $\sup_{t, \ell \in \mathbb{N}^*} \hat{Q}_{t, \ell} \leq C \frac{1}{\sqrt{n}} [1 + \sqrt{\Delta(\delta)}]$ for a universal constant $C > 0$.

If $\ell = \hat{\ell} = 0$, then (43) with $k = \hat{k} = 1$ leads to

$$M(n) \geq -\frac{b^2}{n} \mu^2 - \left(\frac{b\sqrt{n}}{\sqrt{n}}\right) \sqrt{\Delta(\delta)} - \langle \nu/\mu \rangle \Delta(\delta).$$

Joining the two lower bounds establish the claim.

8.2 Multiplier process

Throughout this section, $(B, B, P)$ is a probability space, $(\xi, X)$ is a random (possibly not independent) pair taking values over $\mathbb{R} \times B$ and $X$ has marginal distribution $P$. $\{(\xi_i, X_i)\}_{i \in [n]}$ will denote an iid copy of $(\xi, X)$ and $\hat{P}$ be denotes the empirical measure associated to $\{X_i\}_{i \in [n]}$.

We are interested in concentration properties for the multiplier process [62]

$$M(f) := \frac{1}{n} \sum_{i \in [n]} (\xi_i f(X_i) - \mathbb{E}[f(X)])$$

defined over a subgaussian class of measurable functions $f : B \rightarrow \mathbb{R}$.

Let $L_{\psi_2} = L_{\psi_2}(P)$ be the family of measurable functions $f : B \rightarrow \mathbb{R}$ having finite $\psi_2$-norm

$$\|f\|_{\psi_2} := \|f(X)\|_{\psi_2} := \inf\{c > 0 : \mathbb{E}[\psi_2(f(X)/c)] \leq 1\}$$

where $\psi_2(t) := e^{t^2} - 1$. We will also assume that the $\psi_2$-norm of $\xi$, denoted also by $\|\xi\|_{\psi_1}$, is finite.

Given $f, g \in L_{\psi_2}$, we set $d_{\psi_2}(f, g) := \|f - g\|_{\psi_2}$, $\langle f, g \rangle_n := \hat{P} f g$ and $\|f\|_n := \sqrt{\langle f, f \rangle_n}$. We recall the Hölder-type inequality $\|fg\|_{\psi_1} \leq \|f\|_{\psi_2} \|g\|_{\psi_2}$.

One pioneering idea is of “generic chaining” developed by Talagrand [78] and recently refined by Dirksen, Bednorz, Mendelson and collaborators [63, 33, 4, 62]. The following notion of complexity is used in generic chaining bounds.

**Definition 11 ($\gamma_{2,p}$-functional).** Let $(T, d)$ be a pseudo-metric space. We say a sequence $(T_k)$ of subsets of $T$ is admissible if $|T_0| = 1$ and $|T_k| \leq 2^k$ for $k \in \mathbb{N}$ and $\bigcup_{k \geq 0} T_k$ is dense in $T$. Let $A$ denote the class of all such admissible subset sequences. Given $p \geq 1$, the $\gamma_{2,p}$-functional with respect to $(T, d)$ is the quantity

$$\gamma_{2,p}(T, d) := \inf_{(T_k) \in A} \sup_{T_k \in T} \sum_{k \geq \lfloor \log_2 p \rfloor} 2^{k/2} d(t, T_k).$$

We will say that $(T_k) \in A$ is optimal if it achieves the infimum above. Set $\gamma_{2}(T, d) := \gamma_{2,1}(T, d)$.

For the rest of this section, $d := d_{\psi_2}$. Given a subclass $F \subset L_{\psi_2}$, we let $\Delta(F) := \sup_{f, f' \in F} d(f, f')$, $\Delta\Delta(F) := \sup_{f \in F} \Delta(f, 0)$, $\gamma_{2,p}(F) := \gamma_{2,p}(F, d)$ and $\gamma_2(F) := \gamma_2(F, d)$.

We will prove the following theorem.

**Theorem 9 (Multiplier process).** There exist universal constants $c > 0$, such that for all $f_0 \in F$, $n \geq 1$, $u \geq 1$ and $v \geq 1$, with probability at least $1 - ce^{-u/4} - ce^{-u^2/2}$,

$$\sup_{f \in F} |M(f) - M(f_0)| \lesssim \left(\sqrt{u} + 1\right) \|\xi\|_{\psi_2} \gamma_2(F) / \sqrt{n} + \left(\frac{2u}{n} + \frac{u}{n} + \sqrt{\frac{nu}{n}}\right) \|\xi\|_{\psi_2} \Delta(F)$$
where we have used that $2k^2$ be defined by $F$. For simplicity, define the vector $\Omega$ already present in Talagrand’s original bound for the empirical process “subgaussian path”. In bounding the multiplier process, we additionally use a “lazy walked” chain, a technique already present in Talagrand’s original bound for the empirical process. The following lemma is proved similarly to Lemma 5.4 in [33] so we omit the proof.

Lemma 19. Let $f, f' \in L_{\psi^2}$. If for $k \in \mathbb{N}, 2k^2 \leq \sqrt{n}$, then for any $u > 0$, with probability at least $1 - 2\exp(-2k^2 + u))$,

$$|M(f) - M(f')| \leq \left[1 + \sqrt{2}\frac{2k^2}{\sqrt{n}} + \frac{2u}{n} + \frac{u}{n}\right] \|\xi\|_{\psi^2} \|f - f'\|_{\psi^2}.$$

If for $k \in \mathbb{N}, 2k^2 \geq \sqrt{n}$, then for any $u \geq 1$, with probability at least $1 - 2\exp(-2k^2 + u))$,

$$\|f - f'\|_n \leq (\sqrt{n} + 2k^2)\frac{2(1 + \sqrt{2}) + 1}{\sqrt{n}}d(f, f').$$

Proof of Theorem 9. Let $(F_k)$ be an optimal admissible sequences for $\gamma_{2,1}(F, d) = \gamma_2(F, d)$. Let $(F_k)$ be defined by $F_0 := F$ and $F_k := \cup_{j \leq k} F_j$ so that $|F_k| \leq 2|F_k| = 2k^2 + 1$. Set $k_0 := \min\{k \geq 1 : 2k^2 > \sqrt{n}\}$ and let us define $I := \{k \in \mathbb{N} : \ell < k < k_0\}$ and $J := \{k \in \mathbb{N} : k \geq k_0\}$. Given $k \in \mathbb{N}$ and $f \in F$, let $\Pi_k(f) \in \arg\min_{f' \in F_k} d(f, f')$. Given $f \in F$, we take some $\Pi_k(f) \in F$ and for any $j \in \mathbb{N}$, we define the “lazy walk” chain selection by:

$$k_j(f) := \inf\{j \geq k_{j-1}(f) : d(\Pi_j(f), f) \leq \frac{1}{2}d(\Pi_{k_{j-1}(f)}(f))\}.$$

For simplicity of notation, we will set $\pi_j(f) := \Pi_{k_j}(f)$. For $f \in F$, our proof will rely on the chain:

$$M(f) - M(\pi_0(f)) = \sum_{j : k_j(f) \in J} [M(\pi_{j+1}(f)) - M(\pi_j(f))] + \sum_{j : k_j(f) \in I} [M(\pi_j(f)) - M(\pi_{j-1}(f))],$$

where we have used that $\cup_{k \geq 0} F_k$ is dense on $F$.

Fix $u \geq 1$. Given any $k \in \mathbb{N}$, define the event $\Omega_{k, I, u}$ for which, for all $f, f' \in F_k$, we have

$$|M(f) - M(f')| \leq \left[1 + \sqrt{2}\frac{2k^2}{\sqrt{n}} + \frac{2u}{n} + \frac{u}{n}\right] \|\xi\|_{\psi^2} \|f - f'\|_{\psi^2}.$$

Define also the event $\Omega_{k, J, u}$ for which, for all $f, f' \in F_k$, we have

$$\|f - f'\|_n \leq (\sqrt{n} + 2k^2)\frac{2(1 + \sqrt{2}) + 1}{\sqrt{n}}\|f - f'\|_{\psi^2}.$$

For simplicity, define the vector $\xi := (\xi_i)_{i \in [n]}$ and $\|\xi\|_n := \frac{1}{\sqrt{n}}\|\xi\|_2$. Given $u \geq 1$, we define the event $\Omega_{\xi, u}$, for which

$$\|\xi\|_n \leq [2(1 + \sqrt{2}) + 1]^{1/2}\|\xi\|_{\psi^2} \sqrt{\gamma}.$$
By an union bound over all possible pairs \((\pi_{k-1}(f), \pi_k(f))\) we have \(|\Omega_{k,I,u}| \leq |F_{k-1}| |F_k| \leq 2^{k+1}
.
If \(\Omega_{I,u} := \cap_{k \in \mathbb{Z}} \Omega_{k,I,u}\), the first bound on Lemma 19 for \(k \in I\) and a standard union bound using the geometric series\(^8\) imply that there is universal constant \(c > 0\)

\[
\mathbb{P}(\Omega_{I,u}^c) \leq ce^{-u/4}.
\]

Similarly, the second bound in Lemma 19 for \(k \in \mathcal{J}\) imply that for the event \(\Omega_{\mathcal{J},u} := \cap_{k \in \mathcal{J}} \Omega_{k,\mathcal{J},u}\), we have

\[
\mathbb{P}(\Omega_{\mathcal{J},u}^c) \leq ce^{-u/4}.
\]

Using Bernstein’s inequality for \(\{\xi_i\}_{i \in [n]}\) we get \(\mathbb{P}(\Omega_{\mathcal{J},v}^c) \leq ce^{-vn}\). Hence, the event \(\Omega_{u,v} := \Omega_{I,u} \cap \Omega_{\mathcal{J},u} \cap \Omega_{\mathcal{J},v}\) has \(\mathbb{P}(\Omega_{u,v}^c) \leq ce^{-u/4} + ce^{-vn}\).

We next fix \(u \geq 2\) and \(v \geq 1\) and assume that \(\Omega_{u,v}\) always holds. We now bound the chain over \(I\) and \(\mathcal{J}\) separately.

The subgaussian path \(I\). Given \(j\) such that \(k_j(f) \in I\), since \(\pi_j(f), \pi_{j-1}(f) \in F_{k_j(f)}\), we may apply (45) to \(k := k_j(f)\) so that

\[
|M(\pi_j(f)) - M(\pi_{j-1}(f))| \leq \left(1 + \sqrt{2}\right) \sqrt{\frac{2}{n}} + \sqrt{\frac{2u}{n}} + \frac{u}{n} ||\xi||_{\psi_2} ||\pi_j(f) - \pi_{j-1}(f)||_{\psi_2}.
\]

We note that, by triangle inequality and minimality of \(k_{j-1}(f)\),

\[
||\pi_j(f) - \pi_{j-1}(f)||_{\psi_2} \leq d(f, F_{k_j(f)}) + d(f, F_{k_{j-1}(f)}) \leq d(f, F_{k_j(f)}) + 2d(f, F_{k_{j}(f)-1}),
\]

so that

\[
\sum_{j:k_j(f) \in I} 2^{k_j(f)/2} ||\pi_j(f) - \pi_{j-1}(f)||_{\psi_2} \leq (1 + \sqrt{2})\gamma_2(F).
\]

Moreover, by the definition of the lazy walked chain and a geometric series bound,

\[
\sum_{j:k_j(f) \in I} ||\pi_j(f) - \pi_{j-1}(f)||_{\psi_2} \leq 4d(f, F_u) \leq 4\Delta(F).
\]

We thus conclude that

\[
\left| \sum_{j:k_j(f) \in I} [M(\pi_j(f)) - M(\pi_{j-1}(f))] \right| \leq 4 \left(\sqrt{\frac{2u}{n}} + \frac{u}{n} \right) ||\xi||_{\psi_2} \Delta(F) + (1 + \sqrt{2})(1 + 2\sqrt{2}) ||\xi||_{\psi_2} \frac{\gamma_2(F)}{\sqrt{n}}.
\]

The subexponential path \(\mathcal{J}\). Let us denote by \(Q\) the joint distribution of \((\xi, X)\) and \(\hat{Q}\) the empirical distribution associated to \(\{(\xi_i, X_i)\}_{i \in [n]}\), so that \(M(f) = Q(\cdot)f - QQ(\cdot)f\). By Jensen’s and triangle inequalities,

\[
\left| \sum_{j:k_j(f) \in \mathcal{J}} [M(\pi_{j+1}(f)) - M(\pi_j(f))] \right| \leq \sum_{j:k_j(f) \in \mathcal{J}} Q(\cdot) ||\pi_{j+1}(f) - \pi_j(f)|| + \sum_{j:k_j(f) \in \mathcal{J}} Q\hat{Q}(\cdot) ||\pi_{j+1}(f) - \pi_j(f)||.
\]

For convenience, next we set \(\bar{T}_j := Q(\cdot) ||\pi_{j+1}(f) - \pi_j(f)||\).

\(^8\)See e.g. a modification of Lemma A.4 in [33].
Given \( j \) such that \( k_j(f) \in \mathcal{F} \), since \( \pi_{j+1}(f), \pi_j(f) \in \mathcal{F}_{k_{j+1}(f)} \), we may apply (46) to \( k := k_{j+1}(f) \). This fact, (47) and Cauchy-Schwarz yield

\[
\tilde{T}_j \leq \|\xi\|_n \|\pi_{j+1}(f) - \pi_j(f)\|_n \leq \sqrt{\alpha(2 + \sqrt{2} + 1)} \|\xi\|_2 (\sqrt{u + 2k_{j+1}(f)/n}) \frac{1}{\sqrt{n}} \|\pi_{j+1}(f) - \pi_j(f)\|_2.
\]

In a similar fashion, we can also state that with probability at least \( 1 - ce^{-u/4} \),

\[
\frac{\tilde{T}_j}{\|\xi\|_n} \leq (\sqrt{u} + 2^{k_{j+1}(f)/2}) \frac{2(1 + \sqrt{2}) + 1}{\sqrt{n}} \|\pi_{j+1}(f) - \pi_j(f)\|_2,
\]

so integrating the tail leads to

\[
\left\{ \mathbb{E}\left( \tilde{T}_j \|\xi\|_n \right)^2 \right\}^{1/2} \leq c2^{k_{j+1}(f)/2} \frac{2(1 + \sqrt{2} + 1)^{1/2}}{\sqrt{n}} \|\pi_{j+1}(f) - \pi_j(f)\|_2,
\]

and by Hölder’s inequality,

\[
\mathbb{P}(\tilde{T}_j) \leq \left\{ \mathbb{E}\|\xi\|_n^2 \right\}^{1/2} \left\{ \mathbb{E}\left( \tilde{T}_j \|\xi\|_n \right)^2 \right\}^{1/2} \leq c[2(1 + \sqrt{2} + 1)^{1/2}] \|\pi_{j+1}(f) - \pi_j(f)\|_2.
\]

We thus conclude from (51) and a analogous reasoning to (48) and (49) that

\[
\left| \sum_{j : k_j(f) \in \mathcal{F}} M(\pi_{j+1}(f)) - M(\pi_j(f)) \right| \leq c_1 \sqrt{\alpha} \|\xi\|_2 \sqrt{u} \frac{4\tilde{A}(F)}{\sqrt{n}} \tilde{A}(G) \tilde{A}(G) \sqrt{n} + (c_1^2 + c_1 \tilde{C}) \|\xi\|_2 \sqrt{1 + 2\sqrt{2}} \frac{\tilde{C}(F)}{\sqrt{n}},
\]

with \( c_1 := [2(1 + \sqrt{2}) + 1]^{1/2} \).

From the above bound, (50) and (44) we conclude that, for any \( u \geq 2 \) and \( v \geq 1 \), on the event \( \Omega_{u,v} \) of probability at least \( 1 - ce^{-u/4} - ce^{-uv} \), we have the bound stated in the theorem.

\[\square\]

### 8.3 Product process

With the same setting and definitions of Section 8.2, we are now interested in concentration bounds for the product process [62]

\[
F \times G \ni (f, g) \mapsto A(f, g) := \hat{P}(fg - Pf) = \frac{1}{n} \sum_{i \in [n]} \left\{ f(X_i)g(X_i) - \mathbb{E}f(X_i)g(X_i) \right\},
\]

defined over two distinct subgaussian classes \( F \) and \( G \) of measurable functions.

We will prove the following theorem.

**Theorem 10 (Product process).** Let \( F, G \) be subclasses of \( L_{\psi_2} \). For any \( 1 \leq p < \infty \),

\[
\left| \sup_{(f,g) \in F \times G} |A(f,g)| \right|_{p} \leq \frac{\gamma_2(F)\gamma_2(G)}{n} + \tilde{A}(F)\gamma_2(F)\frac{1}{\sqrt{n}} + \tilde{A}(G)\gamma_2(G)\frac{1}{\sqrt{n}} + \tilde{A}(F)\tilde{A}(G)\left( \sqrt{n} + \frac{p}{n} \right).
\]

In particular, there exist universal constants \( c, C > 0 \), such that for all \( n \geq 1 \) and \( u \geq 1 \), with probability at least \( 1 - e^{-u} \),

\[
\sup_{(f,g) \in F \times G} |A(f,g)| \leq C \left[ \frac{\gamma_2(F)\gamma_2(G)}{n} + \tilde{A}(F)\gamma_2(F)\frac{1}{\sqrt{n}} + \tilde{A}(G)\gamma_2(G)\frac{1}{\sqrt{n}} \right] + c \sup_{(f,g) \in F \times G} \|fg - Pf\|_{\psi_1} \left( \sqrt{n} + \frac{u}{n} \right).
\]
Remark 5. Again, Mendelson [62] proved general concentration inequalities for the product process for much more general classes having heavier tails (see Theorem 1.13 in [62]). Analogous observations to Remark 4 apply. The fact that in Theorem 10 the confidence parameter \( u > 0 \) does not multiply the complexity functionals \( \gamma_2(F) \) and \( \gamma_2(G) \) will be useful for our purposes.

The following lemma is proved similarly to Lemma 5.4 in [33] so we omit the proof.

**Lemma 20.** Let \( f, f' \) and \( g, g' \) in \( L_{\psi_2} \).

If for \( k \in \mathbb{N}, 2^{k/2} \leq \sqrt{n} \), then for any \( u \geq 1 \), with probability at least \( 1 - 2 \exp(-2^k u) \),

\[
|A(f, g) - A(f', g')| \leq 2(1 + \sqrt{2})\sqrt{\frac{2^{k/2}}{\sqrt{n}}} \| f - f' \|_{\psi_1}.
\]

If for \( k \in \mathbb{N}, 2^{k/2} \geq \sqrt{n} \), then for any \( u \geq 1 \), with probability at least \( 1 - 2 \exp(-2^k u) \),

\[
\| f - f' \|_n \leq \sqrt{n}2^{k/2} \left[ 2(1 + \sqrt{2}) + 1 \right]^{1/2} \text{d}(f, f').
\]

As in the proof of Theorem 9, we combine Dirksen’s method [33] with Talagrand’s [78] "lazy-walked" chain. One difference now is that we will explicitly need Dirksen’s bound for the quadratic process

\[
A(f) := \hat{P}(f^2 - Pf^2).
\]

The following proposition is a corollary of the proof of Theorem 5.5 in [33].

**Proposition 11 (Dirksen [33], Theorem 5.5).** Let \( F \subset L_{\psi_2} \). Given \( 1 \leq p < \infty \), set \( \ell := \lfloor \log_2 p \rfloor \) and

\[
k_0 := \min \{ k > \ell : 2^{k/2} > \sqrt{n} \}.
\]

Let \( (F_k) \) be an optimal admissible sequence for \( \gamma_{2,p}(F, d) \) and, for any \( f \in F \) and \( k \in \mathbb{N} \), let \( \pi_k(f) \in \arg\min_{f \in F_k} \text{d}(f, f') \). Then there exists universal constant \( c > 0 \) such that for all \( n \in \mathbb{N} \) and \( u \geq 2 \), with probability at least \( 1 - ce^{-pu/4} \),

\[
\sup_{f \in F} \sup_{k \geq k_0} |A(\pi_k(f))|^{1/2} - \sup_{f \in F} |A(\pi_{k_0}(f))|^{1/2} \leq \sqrt{\frac{25}{\sqrt{n}} \gamma_{2,p}(F, d) + \left( \frac{88}{\sqrt{n}} \Delta(F) \gamma_{2,p}(F, d) \right)^{1/2}}.
\]

Moreover, for all \( n \in \mathbb{N} \) and \( u \geq 1 \), with probability at least \( 1 - ce^{-pu/4} \),

\[
\sup_{f \in F} |A(\pi_{k_0}(f))|^{1/2} \leq \sqrt{\frac{4}{\sqrt{n}}(1 + \sqrt{2}) + 2\Delta(F)}.
\]

\( \Omega_{k_0, u} \) for which, for all \( f \in F \),

**Lemma 21** (Lemma A.3 in [33]). Fix \( 1 \leq p < \infty \), set \( \ell := \lfloor \log_2 p \rfloor \) and let \( (X_t)_{t \in T} \) be a finite collection of real-valued random variables with \( |T| \leq 2^{2\ell} \).

Then

\[
\left( \mathbb{E} \sup_{t \in T} |X_t|^p \right)^{1/p} \leq 2 \sup_{t \in T} (\mathbb{E}|X_t|^p)^{1/p}.
\]

**Lemma 22** (Lemma A.5 in [33]). Fix \( 1 \leq p < \infty \) and \( 0 < \alpha < \infty \). Let \( \gamma \geq 0 \) and suppose that \( \xi \) is a positive random variable such that for some \( c \geq 1 \) and \( u_* > 0 \), for all \( u \geq u_* \),

\[
\mathbb{P}(\xi > \gamma u) \leq c \exp(-pu_*/4).
\]

Then, for a constant \( \tilde{c}_\alpha > 0 \), depending only on \( \alpha \),

\[
(\mathbb{E}|\xi|^p)^{1/p} \leq \tilde{c}_\alpha c + u_*. 
\]
We now also define the event $\Omega$ where we have used that $P$ separately.

Similarily, the second bound in Lemma respectively. Set $\|F\|_2 = \sum_{k \in J} |\Pi_k(f) - \Pi_{k-1}(f)|$.

Define also the event $\Omega_\pi := A(\pi_k(f), \Pi_k(g))$ and $\Omega_{\pi} := \pi_k(f)\Pi_k(g)$. Our proof will rely on the chain:

$$A(f, g) = A(\pi_k(f), \Pi_k(g)) = \sum_{k \in I} |\Pi_{k+1}(f) - \Pi_k(f)| + \sum_{k \in I} |\Pi_k(f) - \Pi_{k-1}(f)|,$$

where we have used that $\cup_{k \geq 0} \mathcal{F}_k \times G_k$ is dense on $F \times G$.

Fix $u \geq 2$. Given any $k \in \mathbb{N}$, define the event $\Omega_{k, x, u, p}$ for which, for all $f \in F$ and $g \in G$, we have

$$|\Pi_k(f, g) - \Pi_{k-1}(f, g)| \leq 2(1 + \sqrt{2})u^{2k/2} \|\mathcal{P}_k(f, g) - \mathcal{P}_{k-1}(f, g)\|_\psi.$$

Define also the event $\Omega_{k, J, u, p}$ for which, for all $f \in F$ and $g \in G$, we have both inequalities:

$$\|\pi_{k+1}(f) - \pi_k(f)\|_\psi \leq \sqrt{2}u^{2k/2} \|\pi_{k+1}(f) - \pi_k(f)\|_\psi,$$

$$\|\Pi_{k+1}(g) - \Pi_k(g)\|_\psi \leq \sqrt{2}u^{2k/2} \|\Pi_{k+1}(g) - \Pi_k(g)\|_\psi.$$

By an union bound over all possible 4-tuples ($\pi_{k-1}(f), \pi_k(f), \Pi_{k-1}(g), \Pi_k(g)$) we have $|\Omega_{k, x, u, p}| \leq |\mathcal{F}_k - \mathcal{F}_k||G_k|G_k| \leq 2^{k/2}$. If $\Omega_{k, u, p} := \cap_{k \in I} \Omega_{k, x, u, p}$, the first bound on Lemma 20 for $k \in I$ and Lemma A.4 in [33] (using that $k > \ell$ over $I$) imply that there is universal constant $c > 0$

$$\mathbb{P}(\Omega_{x, u, p}) \leq c^{-pu^{2k}/4}.$$  

Similarly, the second bound in Lemma 20 for $k \in J$ and Lemma A.4 in [33] (using that $k > \ell$ over $J$) imply that for the event $\Omega_{J, u, p} := \cap_{k \in J} \Omega_{k, J, u, p}$, we have

$$\mathbb{P}(\Omega_{J, u, p}) \leq c^{-pu^{2k}/4}.$$  

We now also define the event $\Omega_{u, p}$ as the intersection of $\Omega_{I, u, p} \cap \Omega_{J, u, p}$ and the events for which both inequalities of Proposition 11 hold for both classes $F$ and $G$. Clearly, by such proposition and the two previous displays we have $\mathbb{P}(\Omega_{u, p}) \leq c^{-pu^{2k}/4}$ from an union bound.

We next fix $u \geq 2$ and assume that $\Omega_{u, p}$ always holds. We now bound the chain over $I$ and $J$ separately.

**Subgaussian path $I$.** From (53) and the inequality

$$\|\mathcal{P}_k(f, g) - \mathcal{P}_{k-1}(f, g)\|_\psi \leq \|\pi_k(f) - \pi_{k-1}(f)\|_\psi \|\Pi_k(g)\|_\psi + \|\pi_{k-1}(f)\|_\psi \|\Pi_k(g) - \Pi_{k-1}(g)\|_\psi + \Delta(G)(d(f, \pi_k(f)) + d(f, \pi_{k-1}(f))) + \Delta(F)(d(g, \Pi_k(g)) + d(g, \Pi_{k-1}(g)))$$

implying

$$\left|\sum_{k \in I} |\mathcal{P}_k(f, g) - \mathcal{P}_{k-1}(f, g)|\right| \leq 2(1 + \sqrt{2})u^{2k} \|\Delta(G)\|_\psi (\|G\|_\psi + \|G_2\|_\psi) .$$

**Subexponential path $J$.** Note that $A(f, g) = \hat{P}(f, g) - P(f, g)$ and thus, by Jensen’s and triangle inequalities,

$$\left|\sum_{k \in J} |\mathcal{P}_k(f, g) - \mathcal{P}_{k-1}(f, g)|\right| \leq \left|\sum_{k \in J} \hat{P} |\mathcal{P}_k(f, g) - \mathcal{P}_{k-1}(f, g)|\right| + \left|\sum_{k \in J} P |\mathcal{P}_k(f, g) - \mathcal{P}_{k-1}(f, g)|\right| \leq \left|\sum_{k \in J} \hat{P} |\mathcal{P}_k(f, g) - \mathcal{P}_{k-1}(f, g)|\right| + \left|\sum_{k \in J} P |\mathcal{P}_k(f, g) - \mathcal{P}_{k-1}(f, g)|\right| .$$

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Let us denote \( \hat{T}_k := \hat{P} \{ \mathcal{P}_{k+1}(f, g) - \mathcal{P}_k(f, g) \} \). We have the split
\[
\hat{T}_k \leq |\hat{P}\pi_k(f)\Pi_{k+1}(g) - \Pi_k(g)| + |\hat{P}\Pi_{k+1}(g)[\pi_{k+1}(f) - \pi_k(f)]|.
\]
By Cauchy-Schwarz,
\[
|\hat{P}\pi_k(f)\Pi_{k+1}(g) - \Pi_k(g)| \leq \|\pi_k(f)\|_n\|\Pi_{k+1}(g) - \Pi_k(g)\|_n
\]
which together with (54), bounds in Proposition 11 and \( \sqrt{u} \geq 1 \) give
\[
|\hat{P}\pi_k(f)\Pi_{k+1}(g) - \Pi_k(g)| \leq c_2u^{2k/2}\sqrt{n} \left[ c_1\hat{\Delta}(F) + 25\frac{\gamma_{2,p}(F)}{\sqrt{n}} + \left( \frac{85\hat{\Delta}(F)\gamma_{2,p}(F)}{\sqrt{n}} \right)^{1/2} \right] d(\Pi_{k+1}(g), \Pi_k(g))
\]
\[
\leq c_2u^{2k/2}\sqrt{n} \left[ c_1\hat{\Delta}(F) + c_4\frac{\gamma_{2,p}(F)}{\sqrt{n}} \right] d(\Pi_{k+1}(g), \Pi_k(g)),
\]
by Young’s inequality and constants \( c_1 := \{4(1 + \sqrt{2} + 2)\}^{1/2} + 1, c_2 := \{2(1 + \sqrt{2} + 1)\}^{1/2}, c_3 := c_1 + \frac{\sqrt{n}}{2} \) and \( c_4 := 25 + \frac{\sqrt{n}}{2} \). An identical bound gives
\[
|\hat{P}\Pi_{k+1}(g)[\pi_{k+1}(f) - \pi_k(f)]| \leq c_2u^{2k/2}\sqrt{n} \left[ c_3\hat{\Delta}(G) + c_4\frac{\gamma_{2,p}(G)}{\sqrt{n}} \right].
\]
We thus conclude that
\[
\hat{T}_k \leq c_2u^{2k/2}\sqrt{n} \left[ c_1\hat{\Delta}(F) + c_4\frac{\gamma_{2,p}(F)}{\sqrt{n}} \right] + c_2u^{2k/2}\sqrt{n} \left[ c_3\hat{\Delta}(G) + c_4\frac{\gamma_{2,p}(G)}{\sqrt{n}} \right].
\]

Note that, in fact, we have proved that the above bound on \( \hat{T}_k \) holds with probability at least \( 1 - c\exp(-pu/4) \) for any \( u \geq 2 \). Thus, from Lemma 22 we have, for some universal constant \( c_0 > 0 \),
\[
P\hat{T}_k \leq c_0c_2u^{2k/2}\sqrt{n} \left[ c_1\hat{\Delta}(F) + c_4\frac{\gamma_{2,p}(F)}{\sqrt{n}} \right] + c_0c_2u^{2k/2}\sqrt{n} \left[ c_3\hat{\Delta}(G) + c_4\frac{\gamma_{2,p}(G)}{\sqrt{n}} \right].
\]

Using the previous two bounds in (56) gives, after using the triangle inequality for \( d \), summing over \( k \in J \) and using the definition of \( \gamma_{2,p}(F) \) and \( \gamma_{2,p}(G) \) (recalling that \( k > \ell \)),
\[
\sum_{k \in J} |\mathcal{P}_{k+1}(f, g) - \mathcal{P}_k(f, g)| \leq (1 + c_0)(1 + 2^{-1/2})c_2u^{2k/2}\sqrt{n} \left[ c_0\hat{\Delta}(F) + c_4\frac{\gamma_{2,p}(F)}{\sqrt{n}} \right] + (1 + c_0)(1 + 2^{-1/2})c_2u^{2k/2}\sqrt{n} \left[ c_3\hat{\Delta}(G) + c_4\frac{\gamma_{2,p}(G)}{\sqrt{n}} \right].
\]

From the above bound, (55) and (32) we conclude that, for any \( u \geq 2 \), on the event \( \Omega_{u,p} \) of probability at least \( 1 - e^{-pu/4} \), we have
\[
\sup_{(f, g) \in F \times G} |A(f, g)|^{1/2} \leq \sup_{(f, g) \in F \times G} |A(\pi_\ell(f), \Pi_\ell(g))|^{1/2}
\]
\[
\leq \sqrt{u} \left[ \frac{c_1}{\sqrt{n}} \left( \hat{\Delta}(F)\gamma_{2,p}(G) + \hat{\Delta}(G)\gamma_{2,p}(F) \right) + c_0 \frac{\gamma_{2,p}(F)\gamma_{2,p}(G)}{\sqrt{n}} \right]^{1/2},
\]
with \( c_5 := 2(1 + \sqrt{2^2 + (1 + c_0)(1 + 2^{-1/2})c_2c_3} \) and \( c_6 := 2(1 + c_0)c_2c_4(1 + 2^{-1/2}) \). This and Lemma 22 (with \( \alpha = 2 \)) imply that

\[
\left| \sup_{(f,g) \in F \times G} |A(f, g)|^{1/2} - \sup_{(f,g) \in F \times G} |A(\pi_\ell(f), \Pi_\ell(g))|^{1/2} \right|_p \\
\leq c \left[ \frac{1}{\sqrt{n}} \left( \Delta(F)\gamma_{2,p}(G) + \Delta(G)\gamma_{2,p}(F) \right) + \frac{\gamma_{2,p}(F)\gamma_{2,p}(G)}{n} \right]^{1/2}.
\]

We also have from Lemma 21,

\[
\left( \frac{\mathbb{E}}{\sup_{(f,g) \in F \times G} |A(\pi_\ell(f), \Pi_\ell(g))|^{p/2}} \right)^{2/p} \leq 4 \sup_{(f,g) \in F \times G} \left( \mathbb{E}|A(\pi_\ell(f), \Pi_\ell(g))|^{p/2} \right)^{2/p} \\
\leq c \sup_{(f,g) \in F \times G} \left[ \| \mathcal{P}_\ell(f, g) - \mathbf{P} \mathcal{P}_\ell(f, g) \| \psi_1 \left( \sqrt{\frac{\pi n}{n} + \frac{p}{n}} \right) \right],
\]

where the second inequality follows from Bernstein's inequality for \( A(\pi_\ell(f), \Pi_\ell(g)) \) and Lemma A.2 in [33]. The two previous displays finish the proof.

References

[1] B. Adcock, A. Bao, J. Jakeman, and A. Narayan. Compressed sensing with sparse corruptions: Fault-tolerant sparse collocation approximations. *SIAM/ASA Journal on Uncertainty Quantification*, 6(4):1424–1453, 2018.

[2] A. Agarwal, S. Negahban, and M. Wainwright. Noisy matrix decomposition via convex relaxation: optimal rates in high dimensions. 2012.

[3] Sivaraman Balakrishnan, Simon S. Du, Jerry Li, and Aarti Singh. Computationally efficient robust sparse estimation in high dimensions. *COLT*, 2017.

[4] W. Bednorz. Concentration via chaining method and its applications. 2014.

[5] P. Bellec, G. Lecué, and A.B. Tsybakov. Slope meets lasso: improved oracle bounds and optimality. 2018.

[6] K. Bhatia, P. Jain, P. Kamalaruban, and P. Kar. Consistent robust regression. 2017.

[7] Kush Bhatia, Prateek Jain, and Purushottam Kar. Robust regression via hard thresholding. pages 721–729, 2015.

[8] P. J. Bickel, Y. Ritov, and A. B. Tsybakov. Simultaneous analysis of Lasso and Dantzig selector. *Ann. Statist.*, 37(4):1705–1732, 2009.

[9] M. Bogdan, E. van den Berg, C. Sabatti, W. Su, and E.J. Candès. Slope - adaptive variable selection via convex optimization. 2015.

[10] S. Boucheron, G. Lugosi, and P. Massart. *Concentration inequalities: a nonasymptotic theory of independence*. Oxford University Press, 2013.

[11] T. T. Cai and W.-X. Zhou. Matrix completion via max-norm constrained optimization. 2016.

[12] E. Candès and P. A. Randall. Highly robust error correction by convex programming. *IEEE Trans. Inform. Theory*, 54(7):2829–2840, 2008.

50
[13] Emmanuel Candès and Benjamin Recht. Exact matrix completion via convex optimization. 2009.

[14] Emmanuel Candès, Xiaodong Li, Yi Ma, and John Wright. Robust principal component analysis? 2011.

[15] Fan J. Ma C. Chen, Y. and Y. Yan. Bridging convex and nonconvex optimization in robust pca: Noise, outliers, and missing data. 2020.

[16] M. Chen, C. Gao, and Z. Ren. A general decision theory for huber’s $\epsilon$-contamination model. *Electronic Journal of Statistics*, 2016.

[17] M. Chen, C. Gao, and Z. Ren. Robust covariance and scatter matrix estimation under huber’s contamination model. *Annals of Statistics*, 2018.

[18] Xu H. Caramanis C. Chen, Y. and S. Sanghavi. Robust matrix completion and corrupted columns. 2011.

[19] Yudong Chen, Constantine Caramanis, and Shie Mannor. Robust sparse regression under adversarial corruption. 28(3):774–782, 2013.

[20] Y. Cheng, I. Diakonikolas, and R. Ge. High-dimensional robust mean estimation in nearly-linear time. *SODA*, 2019.

[21] Y. Cherapanamjeri, E. Aras, N. Tripuraneni, M.I. Jordan, N. Flammarion, and P.L. Bartlett. Optimal robust linear regression in nearly linear time. 2020.

[22] G. Chintot. Erm and rerm are optimal estimators for regression problems when malicious outliers corrupt the labels. 2019.

[23] A. Dalalyan and A. Minasyan. All-in-one robust estimator of the gaussian mean. 2020.

[24] A. Dalalyan and P. Thompson. Outlier-robust estimation of a sparse linear model using $\ell_1$-penalized huber’s $m$-estimator. *NIPS*, 2019.

[25] Arnak Dalalyan and Renaud Keriven. L1-penalized robust estimation for a class of inverse problems arising in multiview geometry. In *Advances in Neural Information Processing Systems 22*. http://nips.cc/, 2009.

[26] Arnak S. Dalalyan and Yin Chen. Fused sparsity and robust estimation for linear models with unknown variance. In *Advances in Neural Information Processing Systems 25: NIPS*, pages 1268–1276, 2012.

[27] J. Depersin and G. Lecué. Robust subgaussian estimation of a mean vector in nearly linear time. 2019.

[28] Jules Depersin. A spectral algorithm for robust regression with subgaussian rates. 2020.

[29] I. Diakonikolas and D. Kane. Recent advances in algorithmic high-dimensional robust statistics. 2019.

[30] I. Diakonikolas, W. Kong, and A. Stewart. Sever: A robust meta-algorithm for stochastic optimization. *ICML*, 2019.

[31] I. Diakonikolas, W. Kong, and A. Stewart. Efficient algorithms and lower bounds for robust linear regression. *SODA*, 2019.
[32] Ilias Diakonikolas, Daniel Kane, Sushrut Karmalkar, Eric Price, and Alistair Stewart. Robust estimators in high dimensions without the computational intractability. FOCS, 2016.

[33] S. Dirksen. Tail bounds via generic chaining. 2015.

[34] Y. Dong, S. B. Hopkins, and J. Li. Quantum entropy scoring for fast robust mean estimation and improved outlier detection. 2019.

[35] D. Donoho and A. Montanari. High dimensional robust m-estimation: asymptotic variance via approximate message passing. 2016.

[36] J. Romberg E. Candes and T. Tao. Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information. 2006.

[37] M. Fazel. Matrix rank minimization with applications. PhD thesis, Stanford University, 2002.

[38] Rina Foygel and Lester Mackey. Corrupted sensing: novel guarantees for separating structured signals. IEEE Trans. Inform. Theory, 60(2):1223–1247, 2014.

[39] S. Gaïffas and G. Lecué. Sharp oracle inequalities for high-dimensional matrix prediction. 2011.

[40] C. Gao. Robust regression via multivariate regression depth. 2020.

[41] C. Gao and J. Lafferty. Model repair: Robust recovery of over-parameterized statistical models. 2020.

[42] F. Hampel, E. Ronchetti, P. Rousseeuw, and W. Stahel. Robust statistics: the approach based on influence functions. 2005.

[43] D. Hsu, S. M. Kakade, and T. Zhang. Robust matrix decomposition with sparse corruptions. 2011.

[44] P. J. Huber. Robust estimation of a location parameter. 1964.

[45] Peter J. Huber and Elvezio M. Ronchetti. Robust statistics. 2011.

[46] S. Karmalkar and E. Price. Compressed sensing with adversarial sparse noise via l1 regression. 2018.

[47] O. Klopp. Rank penalized estimators for high-dimensional matrices. 2011.

[48] O. Klopp. Matrix completion with unknown variance of the noise. 2011.

[49] O. Klopp. Noisy low-rank matrix completion with general sampling distribution. 2014.

[50] O. Klopp, K. Lounici, and A.B. Tsybakov. Robust matrix completion. 2017.

[51] V. Koltchinskii, K. Lounici, and A.B. Tsybakov. Nuclear-norm penalization and optimal rates for noisy low-rank matrix completion. 2011.

[52] K. A. Lai, A. B. Rao, and S. Vempala. Agnostic estimation of mean and covariance. FOCS, 2016.

[53] J. N. Laska, M. A. Davenport, and R. G. Baraniuk. Exact signal recovery from sparsely corrupted measurements through the pursuit of justice. In Asilomar Conference on Signals, Systems and Computers, pages 1556–1560, 2009.

[54] M. Ledoux and M. Talagrand. Probability in Banach spaces. 1991.
[55] Yoonkyung Lee, Steven N. MacEachern, and Yoonsuh Jung. Regularization of case-specific parameters for robustness and efficiency. *Statist. Sci.*, 27(3):350–372, 08 2012.

[56] Xiaodong Li. Compressed sensing and matrix completion with constant proportion of corruptions. 2013.

[57] P.-L. Loh and M. J. Wainwright. Corrupted and missing predictors: Minimax bounds for high-dimensional linear regression. 2012.

[58] P.-L. Loh and M. J. Wainwright. High-dimensional regression with noisy and missing data: provable guarantees with nonconvexity. *Annals of Statistics*, 2012.

[59] K. Lounici. Optimal spectral norm rates for noisy low-rank matrix completion. 2011.

[60] G. Lugosi and S. Mendelson. Mean estimation and regression under heavy-tailed distributions - a survey. 2019.

[61] R. A. Maronna, D. R. Martin, and V. J. Yohai. *Robust Statistics: Theory and Methods*. 2006.

[62] S. Mendelson. Upper bounds on product and multiplier empirical processes. 2016.

[63] S. Mendelson, A. Pajor, and N. Tomczak-Jaegermann. Reconstruction and subgaussian operators in asymptotic geometric analysis. 2007.

[64] Stanislav Minsker. Geometric median and robust estimation in banach spaces. 2015.

[65] Bhaskar Mukhoty, Govind Gopakumar, Prateek Jain, and Purushottam Kar. Globally-convergent iteratively reweighted least squares for robust regression problems. 2019.

[66] S. Negahban, P. Ravikumar, M. Wainwright, and B. Yu. A unified framework for high-dimensional analysis of m-estimators with decomposable regularizers. 2012.

[67] S. Negahnan and M. Wainwright. Estimation of (near) low-rank matrices with noise and high-dimensional scaling. 2011.

[68] S. Negahnan and M. Wainwright. Restricted strong convexity and weighted matrix completion: optimal bounds with noise. 2012.

[69] N. H. Nguyen and T. D. Tran. Robust lasso with missing and grossly corrupted observations. *IEEE Trans. Inform. Theory*, 59(4):2036–2058, 2013.

[70] N. H. Nguyen and T. D. Tran. Exact recoverability from dense corrupted observations via ℓ_1-minimization. 2013.

[71] Ankit Pensia, Varun Jog, and Po-Ling Loh. Robust regression with covariate filtering: Heavy tails and adversarial contamination. 2020.

[72] S. Pesme and N. Flammarion. Online robust regression via sgd on the ℓ_1 loss. 2020.

[73] Adarsh Prasad, Arun Sai Suggala, Sivaraman Balakrishnan, and Pradeep Ravikumar. Robust estimation via robust gradient estimation. 2018.

[74] S. Sardy, P. Tseng, and A. Bruce. Robust wavelet denoising. *IEEE Transactions on Signal Processing*, 49(6):1146–1152, Jun 2001.

[75] Yiyuan She and Art B. Owen. Outlier detection using nonconvex penalized regression. *Journal of the American Statistical Association*, 106(494):626–639, 2011.
[76] N. Srebro. Learning with matrix factorizations. 2004.

[77] Arun Sai Suggala, Kush Bhatia, Pradeep Ravikumar, and Prateek Jain. Adaptive hard thresholding for near-optimal consistent robust regression. 2019.

[78] M. Talagrand. Upper and lower bounds for stochastic processes. 2014.

[79] R. Tibshirani. Lasso. 1998.

[80] E. Tsakonas, J. Jaldén, N. D. Sidiropoulos, and B. Ottersten. Convergence of the huber regression m-estimate in the presence of dense outliers. 2014.

[81] Sara A. van de Geer and Peter Bühlmann. On the conditions used to prove oracle results for the lasso. pages 1436–1462, 2009.

[82] S. Pablo A Parrilo Venkat Chandrasekaran, Sanghavi and Alan S Willsky. Rank-sparsity incoherence for matrix decomposition. 2011.

[83] R. Vershynin. High-Dimensional Probability, An Introduction with applications in Data Science. 2018.

[84] H. Wang, G. Li, and G. Jiang. Robust regression shrinkage and consistent variable selection through the lad-lasso. 2007.

[85] J. Wright and Y. Ma. Dense error correction via $\ell_1$-minimization. 2010.

[86] H. Xu, C. Caramanis, and S. Sanghavi. Robust pca via outlier pursuit. 2011.

[87] Sujay Sanghavi Yudong Chen, Ali Jalali and Constantine Caramanis. Low-rank matrix recovery from errors and erasures. 2013.