Direct-homotopy analysis for solving fredholm integro-differential equations

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Abstract. This research work will consider the problem of integro-differential equation with separable kernel. We propose the hybrid method of direct-homotopy analysis (DHAM) for solving Fredholm integro-differential equation. Convergence analysis to the exact solution of the proposed method will be established. Examples will be solved and comparison will be made with other existing methods to test the efficiency of the proposed method.

1. Introduction

Many engineering as well as some physical problems are formulated by functional equations in the form of integro-differential equations (IDEs), differential equations (DEs) and integral equations (IEs) [1, 2]. Finding solutions of some particular linear and many nonlinear form of these equations are difficult, especially the analytic ones [3]. Most mathematical formulation in applied sciences and engineering are govern by integro-differential equations. Various authors used different approach both numerical and analytical in order to solve integro-differential equations. Wazwaz [4] combined Laplace transform method with Adomian decomposition method (ADM) to handle nonlinear Voltera IDE. Variational iteration method (VIT) was compared with adomian decomposition in [5] and it was shown that VIT is more effective than ADM. Several methods were used to solve IDEs (see. [6–10]).

The homotopy analysis method (HAM) was introduced by Liao to handle nonlinear and linear equations. HAM method provides solution in series form. If the series solution obtained from HAM has a close form, the method provides exact solution. Otherwise, the solution is approximated to some degree of accuracy [11]. Since the introduction of homotopy analysis several authors applied the method to solve nonlinear and linear integral equation and integro-differential equation. In [13] HAM was applied to solve IDE. Comparison was further made between HAM with two other known methods, namely; sine-cosine wavelet (SCW) and homotopy perturbation method (HPM). It was found that HAM has advantage of fast convergence over SCW and HPM. Sadigh [14] solved weakly singular nonlinear IDE using HPM and HAM and have shown that the HAM converges more rapidly than HPM. In [15] HAM was combined with Laplace transform method to find the analytic solution of IDE. Other literatures which used homotopy analysis method to solve IDE can be found in [12, 16–19].
In this research, we proposed a new method which combined direct computational method with homotopy analysis method together to solve integro-differential equation of the form:

\[ y^{(n)}(x) = f(x) + \lambda \int_a^b k(x, t)y^{(l)}(t)dt, \quad y^{(r)}(0) = b_r, \quad 0 \leq l \leq r \leq n - 1, \quad (1) \]

where \( a, b \) and \( \lambda \neq 0 \) are constants, \( f(x) \) is a known function, \( k(x, t) \) is a known function of two variables called the kernel, \( u(x) \) is the unknown function to be determined and \( y^{(n)}(x) = \frac{d^n y(x)}{dx^n} \).

The kernel \( k(x, t) \) is separable i.e \( k(x, t) = g(x)h(t) \). Convergence analysis to the exact solution of the proposed method (DHAM) was established. The DHAM was tested with some examples. It was found that the method was efficient.

Section 2 of this paper provides the idea of HAM and Direct computational method (DCM). Section 3 gives the description the proposed method of DHAM. The convergence analysis to the exact solution is discussed in section 4. While section 5 presents the numerical results.

2. Concept of homotopy analysis and direct computational methods

Here we present the main idea behind homotopy analysis and direct computational methods as follows

2.1. Description of Homotopy Analysis method

Consider

\[ N[y(x)] = 0, \quad (2) \]

where \( N \) is a nonlinear operator, \( y(x) \) is the unknown function of independent variable \( x \). For simplicity we ignore all initial or boundary conditions. Liao [11] constructs the so called zero-order deformation equation as follows:

\[ (1 - q)L[\phi(x; q) - y_0(x)] = qhH(x)N[\phi(x; q)], \quad (3) \]

where \( q \in [0, 1] \) is the embedding parameter, \( h \neq 0 \) an auxiliary parameter, \( H(x) \neq 0 \) is an auxiliary function, \( L \) is a linear operator and \( y_0(x) \) is the initial guess of \( y(x) \). When \( q = 0 \) and \( q = 1 \) it holds that

\[ \phi(x; 0) = y_0(x) \text{ and } \phi(x; 1) = y(x), \quad (4) \]

respectively. Thus, from equation (4) as \( q \) increases from 0 to 1 \( \phi(x; q) \) varies from initial guess \( y_0(x) \) to solution \( y(x) \). According taylor’s theorem \( \phi(x; q) \) can be expanded in power of \( q \) as follows:

\[ \phi(x; q) = y_0(x) + \sum_{m=1}^{\infty} y_m(x)q^m, \quad (5) \]

where

\[ y_m = \frac{1}{m!} \frac{\partial^m \phi(x; q)}{\partial q^m} |_{q=0}. \quad (6) \]

If the auxiliary linear operator, the initial guess, the auxiliary parameter and the auxiliary function are properly chosen, the series equation (5) converges at \( q = 1 \). So, we obtain

\[ y(x) = y_0(x) + \sum_{m=1}^{\infty} y_m(x). \quad (7) \]

Define a vector

\[ \overline{y}_n = \{y_0(x), y_1(x), \cdots, y_n(x)\}. \quad (8) \]
Differentiating equation (3) \( m \) times with respect to the embedding parameter \( q \) and setting \( q = 0 \) then divide through by \( m! \) we obtain the \( m \)th order deformation equation as:

\[
L[y_m(x) - \chi_m y_{m-1}(x)] = hH(x)R_m(y_{m-1}),
\]

(9)

where

\[
R_m(y_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1}N[\phi(x; q)]}{\partial \phi^{m-1}} \bigg|_{q=0}
\]

(10)

and

\[
\chi_m = \begin{cases} 
0, & \text{for } m \leq 1 \\
1, & \text{for } m > 1
\end{cases}
\]

(11)

2.2. Direct computational method

If we consider Fredhom integro-differential equation below and substitute \( k(x, t) = g(x)h(t) \) we obtain

\[
y^{(n)}(x) = f(x) + \lambda g(x) \int_{a(x)}^{b(x)} h(t)y(t)dt
\]

(12)

as \( a(x) \) and \( b(x) \) are constant, the definite integral in equation (12) is constant. Setting

\[
\alpha = \int_{a(x)}^{b(x)} h(t)y(t)dt,
\]

(13)

therefore, equation (12) can be written as

\[
y^{(n)}(x) = f(x) + \lambda \alpha g(x).
\]

(14)

We shall now determine the constant \( \alpha \) by using equation (13) and equation (14). Integrating both side of equation (14) \( n \) times from 0 to \( x \) using the initial condition \( y^k = b_k, \ 0 \leq k \leq n-1 \), we obtain expression for \( y(x) \) in the form

\[
y(x) = p(x; \alpha),
\]

(15)

where \( p(x; \alpha) \) is the result obtained from integrating equation (14) using the given initial conditions. Substituting equation (15) into the right-hand side of equation (13), integrating and solving the resulting algebraic equation to determine \( \alpha \). The exact solution of equation (12) follows immediately upon substituting the value of \( \alpha \) into equation (15).

3. Direct-homotopy analysis method

The construction of the new method of DHAM is as follows:

By using equation (15), Integrating equation (1) \( n \) times from 0 to \( x \) we obtain

\[
y(x) = F(x) + \gamma + \lambda L^{-1}\left(g(x) \int_{a}^{b} h(t)g^{(l)}(t)dt\right),
\]

(16)

where

\[
L^{-1} = \int_{0}^{x} \int_{0}^{x} \cdots \int_{0}^{x} \text{\( n \) times} \ dt dx dx \cdots dx,
\]

(17)

\[
F(x) = L^{-1} f(x) \text{ and } \gamma = \sum_{r=1}^{n} b_{n-r} \frac{x^{n-r}}{(n-r)!} + b_0.
\]

(18)
From equation (2), we define the nonlinear operator as:

\[ N[y(x)] = y(x) - F(x) + \gamma - \lambda L^{-1}\left(g(x) \int_a^b h(t)y^{(l)}(t)dt\right). \tag{19} \]

The corresponding mth-order deformation equation is as follows

\[ L[y_m(x) - \chi_m y_{m-1}(x)] = hH(x)R_m(y_{m-1}(x)) \tag{20} \]

where

\[ R_m(y_{m-1}(x)) = y_{m-1}(x) - (F(x) + \gamma)(1 - \chi_m) - \lambda L^{-1}\left(g(x) \int_a^b h(t)y^{(l)}_{m-1}(t)dt\right). \tag{21} \]

choosing the auxiliary linear operator \( L[y] = y \), we obtain

\[ y_1(x) = hH(x)[y_0(x) - F(x) - \gamma - \lambda L^{-1}\left(g(x) \int_a^b h(t)y^{(l)}_0(t)dt\right)], \tag{22} \]

and

\[ y_m(x) = \chi_m y_{m-1}(x) + hH(x)\left[y_{m-1}(x) - \lambda L^{-1}\left(g(x) \int_a^b h(t)y^{(l)}_{m-1}(t)dt\right)\right], \tag{23} \]

\[ m = 2, 3, 4, \ldots. \]

With proper choice of initial guess \( y_0(x) \), auxiliary function \( H(x) \) and auxiliary parameter \( h \) the series equation (7) whose terms are the initial guess \( y_0(x) \) together with \( y_1(x) \) and \( y_m(x) \) obtained respectively in equation (22) and equation (23) converges to the solution. Therefore the solution is:

\[ y(x) = y_0(x) + \sum_{m=1}^{\infty} y_m(x). \tag{24} \]

4. Convergence analysis

**Theorem 4.1** Suppose that the series in equation (24) is convergent where \( y_0(x) \) is the initial guess, \( y_1(x) \) and \( y_m \) are obtained from equation (20) and equation (21) then it is the exact solution of equation (1)

**Proof of Theorem 4.1** Let

\[ y(x) = \sum_{m=0}^{\infty} y_m(x), \tag{25} \]

then it holds that

\[ \lim_{m \to \infty} y_m(x) = 0, \tag{26} \]

we can see that

\[ \sum_{m=1}^{n} [y_m(x) - \chi_m y_{m-1}(x)] = y_1 + (y_2 - y_1) + \ldots + (y_n - y_{n-1}) = y_n(x). \tag{27} \]
As \( n \to \infty \), we have

\[
\sum_{m=1}^{\infty} [y_m(x) - \chi_m y_{m-1}(x)] = \lim_{n \to \infty} y_n(x) = 0.
\] (28)

By definition of linear operator \( L \)

\[
\sum_{m=1}^{\infty} L[y_m(x) - \chi_m y_{m-1}(x)] = L \sum_{m=1}^{\infty} [y_m(x) - \chi_m y_{m-1}(x)] = 0.
\] (29)

But

\[
L[y_m(x) - \chi_m y_{m-1}(x)] = hH(x)R_{m-1}(y_{m-1}(x)).
\] (30)

Thus, we have from equation (29) and equation (30)

\[
\sum_{m=1}^{\infty} L[y_m(x) - \chi_m y_{m-1}(x)] = \sum_{m=1}^{\infty} hH(x)R_{m-1}(y_{m-1}(x)).
\] (31)

\[
= hH(x) \sum_{m=1}^{\infty} R_{m-1}(y_{m-1}(x)) = 0.
\] (32)

Since \( h \neq 0 \) and \( H(x) \neq 0 \) we have

\[
\sum_{m=1}^{\infty} R_{m-1}(y_{m-1}(x)) = 0,
\] (34)

from equation (21) and equation (34)

\[
0 = \sum_{m=1}^{\infty} R_{m-1}(y_{m-1}(x))
= \sum_{m=1}^{\infty} \left[ y_{m-1}(x) - (F(x) + \alpha)(1 - \chi_m) - \lambda L^{-1} \left( g(x) \int_a^b h(t)y_{m-1}(t)dt \right) \right]
= \sum_{m=0}^{\infty} y_{m-1}(x) - (F(x) + \alpha) - \lambda L^{-1} \left( g(x) \int_a^b h(t) \sum_{m=0}^{\infty} y_{m-1}(t)dt \right)
= \sum_{m=0}^{\infty} y_m(x) - (F(x) + \alpha) - \lambda L^{-1} \left( g(x) \int_a^b h(t) \sum_{m=0}^{\infty} y_{m}(t)dt \right).
\] (35)

Thus

\[
y(x) = F(x) + \alpha + \lambda L^{-1} \left( \int_a^b k(x,t)y^{(l)}(t)dt \right).
\] (36)

Differentiating equation (36) \( n \) times, \( y(x) \) must be the exact solution of equation (1) and

\[
y^n(x) = f(x) + \lambda \int_a^b k(x,t)y^{(l)}(t)dt.
\] (37)
5. Numerical results
Here we present some examples that uses the method of DHAM to solve integro differential equations with separable kernels. We choose the auxiliary parameter \( h = -1 \), the auxiliary function \( H(x) = 1 \) and linear operator \( L[y(x)] = y(x) \). The results of DHAM obtained are compared with the exact solutions and the absolute errors are given in the following examples.

Example 5.1 Consider the integro-differential equation of second kind:

\[
y'(x) = 1 - \frac{1}{3}x + \int_0^1 xty(t)dt, \quad y(0) = 0, \tag{38}
\]

the exact solution of the above equation is \( y_{\text{exact}}(x) = x \). From equation (22) and equation (23)

\[
F(x) = L^{-1}[f(x)] = \int_0^1 1 - \frac{\tau}{3}d\tau = x - \frac{x^2}{6}, \quad y(0) = b_0 = 0 \rightarrow \alpha = 0, \tag{39}
\]

we obtain

\[
y_1(x) = y_0(x) - x + \frac{x^2}{6} + \int_0^x \left( \tau \int_0^1 y_0(t)dt \right)d\tau, \tag{40}
\]

\[
y_m(x) = \int_0^x \left( \tau \int_0^1 y_{m-1}(t)dt \right)d\tau, \quad m \geq 2. \tag{41}
\]

Choosing \( y_0(x) = x - \frac{1}{6}x^2 \) as initial guess, we used maple to compute the table below:

| iterations values in \([0, 1]\) | Exact solutions \(y_{\text{exact}}(x)\) | Aprox. solutions of DHAM (22) and (23) | Absolute error \(|y_{\text{exact}} - y_m(x)|\) |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0.2 | 0.2 | 0.2000000000 | 5.3852E − 30 |
| 0.4 | 0.4 | 0.4000000000 | 2.1541E − 29 |
| 30 | 0.7 | 0.7000000000 | 6.5969E − 29 |
| 0.9 | 0.9 | 0.9000000000 | 1.0905E − 28 |
| 1 | 1 | 1.0000000000 | 1.3463E − 28 |

Table 1 provides the exact solution of example 5.1 and it is approximate solution obtained by DHAM equation (22) and equation (23). The error obtained for \( m = 30 \) indicated that the solution obtained by DHAM is accurate.

Figure 1: Exact and DHAM solutions for Example 5.1

Figure 2: Errors of Example 5.1
Figure 1 shows that the numerical results obtained with DHAM equation (22) and equation (23) coincide with the exact solution. While Figure 2 indicated the numerical convergence of the proposed method of DHAM since the error is on the $x-$axis.

**Example 5.2** Consider another integro-differential equation:

$$y'''(x) = \sin(x) - x - \int_0^x xty'(t)dt,$$

with initial conditions $y''(0) = -1$, $y'(0) = 0$ and $y(0) = 1$. The exact solution of this problem is $y_{exact}(x) = \cos(x)$. From equation (22) and equation (23)

$$F(x) = L^{-1}[f(x)] = \int_0^x \int_0^x \int_0^x \sin \tau - \tau d\tau d\tau d\tau = -1 + \frac{x^2}{2} + \cos x - \frac{x^4}{24},$$

$$b_2 = -1, b_1 = 0, b_0 = 1 \rightarrow \alpha = 1 - \frac{x^2}{2}$$

and

$$y_1(x) = -y_0(x) + \cos(x) - \frac{x^4}{24} - \int_0^x \int_0^x \int_0^x \left( \int_0^x y_o(t)dt \right) d\tau d\tau d\tau,$$

$$y_m(x) = -\int_0^x \int_0^x \int_0^x \left( \int_0^x y_{m-1}(t)dt \right) d\tau d\tau d\tau, \quad m \geq 2.$$  

Choosing $y_0(x) = \cos x - \frac{x^4}{24!}$ as initial guess, we used maple to compute the table below:

**Table 2:** The exact and approximate solution of example 5.2.

| Iterations $m$ | Values in $[0, \frac{\pi}{2}]$ | Exact solutions $y_{exact}(x)$ | Aprox. solutions of DHAM (22) and (23) $|y_{exact} - y_m(x)|$ |
|----------------|-------------------------------|-------------------------------|------------------------------------------|
| 0              | 1                             | 1                             | 0                                        |
| 0.1            | 0.9950041653                  | 0.9950041653                  | 3.7126$E-20$                            |
| 0.3            | 0.9553364891                  | 0.9553364891                  | 3.0072$E-18$                            |
| 30             | 0.5                           | 0.8775825619                  | 0.8775825619                             |
| 0.7            | 0.7568421873                  | 0.7648421873                  | 8.9140$E-17$                            |
| 0.9            | 0.6216099683                  | 0.6216099683                  | 2.4358$E-16$                            |
| 1              | 0.5403023059                  | 0.5403023059                  | 3.7126$E-16$                            |

Table 2 provides the exact solution of example 5.2 and it is approximate solution obtained by DHAM equation. (22) and equation (23). The error obtained for $m = 30$ indicate the accuracy in the solution obtained by DHAM.

**Figure 3:** Exact and DHAM solutions for Example 5.2

**Figure 4:** Error graph for Example 5.2
Figure 3 shows that the numerical results obtained with DHAM equation (22) and equation (23) coincide with the exact solution. While Figure 4 indicated that error of the proposed method of DHAM is small since the error is close to the $x-$axis.

**Example 5.3** Consider

$$y'(x) = (x + 1)e^x - x + \int_0^1 xy(t)dt, \quad y(0) = 0,$$  \hspace{1cm} (46)

whose exact solution is $y_{\text{exact}}(x) = xe^x$. From equation (22) and equation (23)

$$F(x) = L^{-1}[f(x)] = \int_0^x (\tau + 1)e^\tau - \tau d\tau = xe^x - \frac{x^2}{2}, \quad y(0) = b_0 = 0 \rightarrow \alpha = 0 \hspace{1cm} (47)$$

and

$$y_1(x) = y_0(x) + xe^x - \frac{x^2}{2} + \int_0^x \tau \left( \int_0^1 y_0(t)dt \right) d\tau, \hspace{1cm} (48)$$

$$y_m(x) = \int_0^x \tau \left( \int_0^1 y_{m-1}(t)dt \right) d\tau \quad m \geq 2. \hspace{1cm} (49)$$

Choosing guess to $y_0(x) = 1$ as initial guess, we used maple to compute the tables below:

**Table 3:** Comparison between $ADM,VIM,HPM$ and $DHAM$ for example 5.3.

| $x$  | $y_{\text{exact}}$ | $y_{ADM}$ | $y_{VIM}$ | $y_{HPM}$ | $y_{DHAM}$ |
|-----|---------------------|------------|-----------|-----------|------------|
| 0.1 | 0.1105170           | 0.1103782  | 0.1096837 | 0.1103782 | 0.1105170  |
| 0.2 | 0.2442805           | 0.2437249  | 0.2409472 | 0.2437249 | 0.2442805  |
| 0.3 | 0.4049576           | 0.4037076  | 0.3974576 | 0.4037076 | 0.4049576  |
| 0.4 | 0.5967298           | 0.5945076  | 0.5833965 | 0.5945076 | 0.5967298  |
| 0.5 | 0.8243606           | 0.8208884  | 0.8035273 | 0.8208884 | 0.8233606  |
| 0.6 | 1.0932712           | 1.0882712  | 1.0632712 | 1.0882712 | 1.0932712  |
| 0.7 | 1.4096268           | 1.4028213  | 1.3687935 | 1.4028213 | 1.4096268  |
| 0.8 | 1.7804327           | 1.7715438  | 1.7270994 | 1.7715438 | 1.7804327  |
| 0.9 | 2.2136428           | 2.2023928  | 2.1461428 | 2.2023928 | 2.2136428  |

In Table 3, the solution obtained with ADM, VIM, and HPM are given together with solution obtained with the proposed method of DHAM (each by taking $m = 4$) in equation (46).

**Figure 5:** Exact, ADM, VIM, HPM and DHAM solution from Table 3

**Figure 6:** Comparison of the absolute error obtained by ADM, VIM, HPM and DHAM in Table 3
Figure 5 shows that the numerical results obtained with DHAM equation (22) and equation (23) coincide with the exact, ADM, VIM and HPM solutions. While Figure 6 indicated that the proposed method of DHAM converges to the exact solution since it’s error graph is on the $x-$axis.

6. Conclusion
In this paper, new method of DHAM which is a powerful tool for solving problems of IDEs was developed. The method was tested with some examples and it was found that DHAM is effective and efficient in solving IDEs. The results obtained by DHAM are in line with the theoretical finding.

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