Color-Kinematic Duality in ABJM Theory Without Amplitude Relations

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Abstract

We explicitly show that the Bern-Carrasco-Johansson color-kinematic duality holds at tree level through at least eight points in Aharony-Bergman-Jafferis-Maldacena theory with gauge group SU(N) x SU(N). At six points we give the explicit form of numerators in terms of amplitudes, displaying the generalized gauge freedom that leads to amplitude relations. However, at eight points no amplitude relations follow from the duality, so the diagram numerators are fixed unique functions of partial amplitudes. We provide the explicit amplitude-numerator decomposition and the numerator relations for eight-point amplitudes.
1 Introduction

Studies of scattering amplitudes have uncovered important insights into gauge and gravity theories. In particular, Bern, Carrasco, and Johansson (BCJ) found a surprising duality between the color factors and kinematic numerator factors that comprise diagrams in Yang-Mills theory [1]. The color-kinematic duality implies nontrivial amplitude relations. These amplitude relations have been studied in both field theory and string theory [2, 3, 4]. The color-kinematic duality appears to extend to loop level, as confirmed in a variety of examples with varying levels of supersymmetry [5, 6]. While there has been some progress in understanding the origin of the duality from a Lagrangian vantage point [7, 8, 9], further work is needed.

The color-kinematic duality reveals new structures in gravity through a surprisingly simple gauge-gravity correspondence. Kinematic numerators satisfying the color-kinematics duality provide the link: by replacing color factors with numerators that satisfy the duality, gauge-theory amplitudes are converted into gravity amplitudes [1, 5], revealing a double-copy structure of gravity. A connection between gravity and Yang-Mills theory has long been known at tree level from the Kawai-Lewellen-Tye relations [10], but the double-copy property of gravity reveals a more extensive correspondence, one that appears to extend to loop level. This property has advanced the study of supergravity’s properties, especially in uncovering unexpected ultraviolet cancellations at high loop orders [11]. More generally, the double-copy relation allows us to directly study the effects of any newly uncovered properties of gauge-theory amplitudes on corresponding gravity amplitudes.

In Yang-Mills theory, BCJ amplitude relations come from a residual generalized gauge freedom present in kinematic numerators even after imposing that the duality between color and kinematics is manifest [1, 4, 12, 13, 14]. There have also been string-theory studies to investigate the residual gauge invariance in the duality between color and kinematics in Yang-Mills theory [2, 3]. While there has been progress in understanding the underlying structure behind the duality and the residual gauge invariance [15, 8], further clarification is needed. To gain additional insight, it is important to study a wide variety of cases where
the duality holds.

In particular, the color-kinematic duality has been found in three-dimensional Chern-Simons-matter theories: the $\mathcal{N} = 8$ Bagger-Lambert-Gustavsson (BLG) theory and the $\mathcal{N} = 6$ Aharony-Bergman-Jafferis-Maldacena (ABJM) theory \cite{16, 17}. BLG theory turns out to be a special case of ABJM theory. These cases are quite interesting because a Lie three-algebra, not a Lie two-algebra, defines the gauge structure of these theories. These Chern-Simons-matter theories would appear to have rather different properties than gauge theory. The color-kinematic duality is governed by gauge-group relations, so the presence of the duality in ABJM-type theories shows that the duality is more general than previously appreciated.

We address the color-kinematic duality in ABJM theory. The first nontrivial example of the duality in this theory was first given in ref. \cite{16}: the six-point all-scalar amplitude. While this manuscript was in preparation, ref. \cite{18} noted that the duality holds up to ten points, but surprisingly found no BCJ amplitude relations at eight points or higher. Here we provide specifics about the eight-point amplitudes. The eight-point process is the simplest example in which the color-kinematic duality holds without the residual freedom that produces BCJ amplitude relations, warranting a detailed study.

Another curious feature that may be connected is that at lower points, the double-copy property holds \cite{16, 17}, but starting at eight points it does not \cite{18}: applying the double-copy procedure to the eight-point kinematic numerators does not produce the appropriate gravity numerators. The eight-point ABJM amplitude is the only known instance in which the color-kinematic duality holds but the double-copy property fails. Examining the eight-point amplitudes in some detail may therefore further illuminate the connection between the color-kinematic duality and the double-copy property.

In this note, we start by presenting relatively explicit forms of the six point numerators in ABJM theory that satisfy the duality between color and kinematics. This has not been given previously and is the first nontrivial case where the duality holds. (At four points the duality is trivial and for odd points the amplitudes vanish.) We then confirm that the
duality between color and kinematics holds at eight points for bosonic external states, but does not generate BCJ amplitude relations. In an ancillary online file [19], we present a set of independent eight-point amplitudes in terms of numerators, and the numerator relations implied by the color-kinematic duality.

This note is organized as follows. In Sec. 2, we review the necessary background. Sec. 2.1 describes the color-kinematic duality in Yang-Mills theory. Then, in Sec. 2.2 we explain the construction of partial amplitudes and how the color-kinematic duality leaves behind a residual freedom that gives rise to BCJ amplitude relations. Sec. 2.3 presents relevant properties of the three-algebra formulation of ABJM theory. Sec. 3 contains our results: demonstration of the color-kinematic duality through eight points. In Sec. 3.1, we show the trivial case: four points. We demonstrate the first non-trivial case – six points – and present numerator solutions in terms of amplitudes in Sec. 3.2. In Sec. 3.3 we detail the eight-point case. We explain how to construct the eight-point partial amplitudes from numerators, and confirm that the color-kinematic duality is satisfied but does not imply BCJ amplitude relations. In Sec. 4 we discuss the implications of the eight-point result and future directions.

2 Review

2.1 The color-kinematic duality in Yang-Mills theory

In Yang-Mills theory, tree amplitudes can in general be written as

$$A(1, 2, \ldots, n) = \sum_i c_i n_i \prod_{\alpha_i} s_{\alpha_i},$$

where the sum runs over diagrams with only cubic vertices. In general, any terms not of this form can be put into this form by multiplying and dividing by appropriate propagators. Here, $c_i$ are products of structure constants and we suppress the coupling constant and helicity labels. The $n_i$ are kinematic numerators: functions of momenta and polarization vectors. Each color factor $c_i$ is in one-to-one correspondence with a diagram with a specific
propagator structure. The $s_{\alpha_i}$ in the denominator are the Feynman propagators for the $i$-th diagram, where $s_{\alpha_i}$ are the kinematic invariants of the scattering process.

The $c_i$ are not an independent set and are related by Jacobi identities, as shown in fig. 1. BCJ proposed the color-kinematics duality, wherein the numerators $n_i$ obey the same relations and symmetries under relabelling $[1]$. For instance,

$$c_1 = c_2 + c_3 \Rightarrow n_1 = n_2 + n_3.$$  \hspace{1cm} (2)

$$c_1 \rightarrow -c_1 \Rightarrow n_1 \rightarrow -n_1.$$  \hspace{1cm} (3)

BCJ also noted that replacing $c_i$ with $n_i$ that satisfy the duality in (1) gives the scattering amplitude $M(1, 2, \ldots, n)$ in gravity. This connection between gravity and gauge theory is known as the double-copy property of gravity. At tree level, these properties have been proven in various ways $[7,4,20]$. The color-kinematic duality is a basic property of gauge and gravity theories that deserves further study, especially to understand the underlying symmetry.
2.2 Amplitude relations from the color-kinematic duality in Yang-Mills theory

The simplicity of the color-kinematic duality and double-copy property suggests a novel principle in gauge theories and gravity. Here, we review how amplitude relations follow from the color-kinematic duality in Yang-Mills theory. We also note that such amplitude relations arise from string theory [3, 15]. The color-kinematic duality has also been directly studied in string theory [2].

In Yang-Mills theory, we can write the color-decomposition in (1) in terms of color-ordered amplitudes and traces over gauge-group generators. At tree level, this trace decomposition is [21]

\[ A(1, 2, \ldots, n) = \sum_{\sigma \in S_n/Z_n} \text{Tr}(T^{\sigma_1}T^{\sigma_2} \ldots T^{\sigma_n})A(\sigma_1, \sigma_2, \ldots, \sigma_n), \]  

where \( S_n/Z_n \) is all leg orderings unrelated by cyclic permutation. The color-ordered amplitudes are in terms of numerators divided by propagators, as in (1). These amplitudes obey symmetry properties. They are invariant under cyclic permutations of indices. They have symmetry under reversal: \( A(1, 2, \ldots, n) = (-1)^n A(n, \ldots, 2, 1) \). These amplitudes obey the photon-decoupling identity:

\[ \sum_{\sigma \in \text{cyclic}} A(1, \sigma(2, 3, \ldots, n)) = 0, \]

and more generally the Kleiss-Kuijf (KK) relations [22].

\[ A(1, \{\alpha\}, n, \{\beta\}) = (-1)^{n\beta} \sum_{\{\sigma\} \in OP(\{\alpha\}, \{\beta^T\})} A(1, \{\sigma\}, n). \]

Here, \( OP(\{\alpha\}, \{\beta^T\}) \) is the set of permutations which preserve the order of each set. The set \( \{\beta^T\} \) is \( \{\beta\} \) but with reversed ordering. Using these identities, we can choose a set of \((n - 2)!\) color-ordered amplitudes as a KK-independent basis.

In eq. (1) the tree amplitudes are written in terms of numerators. The duality between
color and kinematics immediately implies that we can write the amplitude in terms of the same number of numerators as independent color factors. In principle, the \((n-2)!\) amplitudes and \((n-2)!\) numerators in this basis could uniquely specify each other. However, it turns out that the numerators are not unique – they can be shifted by kinematic functions in such a way that the basis amplitudes remain unchanged. See refs. [1, 7, 2] for details. This freedom is a residual generalized gauge freedom, or “residual freedom”. In other words, the \((n-2)!\) equations for the basis amplitudes in terms of basis numerators are non-invertible. Only \((n-3)!\) of the \((n-2)!\) numerators are independent, resulting in \((n-2)! - (n-3)!\) amplitude relations, known as BCJ amplitude relations.

2.3 The three-algebra formulation of ABJM theory

We will consider ABJM theory, a theory of M2 branes [23]. This theory admits a natural Lie three-algebra formulation analogous to the Lie two-algebra defining the gauge symmetry of Yang-Mills theory. ABJM theory is a Chern-Simons-matter gauge theory with gauge-group \(SU(N) \times SU(N)\). Matter fields are bi-fundamental, and map between the two gauge groups: they are written in terms of group elements \((T^a)^{bc} : V_1 \rightarrow V_2\) where \(V_1\) and \(V_2\) are the vector spaces of group elements from the first and second \(SU(N)\) groups. Indices are barred to distinguish between the two vector spaces. The index \(a\) is an adjoint index, while indices \(b, \bar{c}\) are fundamental and anti-fundamental. We do not use different notation for adjoint and (anti-)fundamental indices as we often show only the adjoint indices. The reader should keep in mind that multiplication of group elements involves contracting fundamental and

![Diagram](image-url)
anti-fundamental indices. The group elements \( T^a \) are related by a triple product:

\[
[T^a, T^b; \bar{T}^c] = f^{abc} \bar{T}^d,
\]

\[
[T^a, T^b; \bar{T}^c] = T^a \bar{T}^c T^b - T^b \bar{T}^c T^a.
\]  

(7)

Indices are raised and lowered with the metric \( \text{Tr}(T^a \bar{T}^b) = h^{ab} \). The unbarred and barred indices are antisymmetric amongst each other separately, but not together: they are adjoint indices in two different gauge groups. For example, \( f^{ab\bar{c}d} = -f^{ba\bar{c}d} = f^{ba\bar{d}c} \). One can write \( f^{ab\bar{d}} \) or \( f^{ab\bar{c}\bar{d}} \) as they are the same. The structure constants obey a generalized four-term Jacobi relation illustrated in fig. 2:

\[
f^{a\bar{b}\bar{c}e} f^{ef\bar{g}d} = f^{a\bar{f}\bar{g}e} f^{ef\bar{b}\bar{c}d} + f^{b\bar{f}\bar{g}e} f^{ef\bar{a}\bar{c}d} - f^{\bar{e}\bar{f}\bar{g}e} f^{a\bar{b}\bar{c}d}.
\]  

(8)

ABJM amplitudes are only nonzero for even numbers of external particles due to the theory’s bi-fundamental nature.

The color-ordering and numerator decomposition of ABJM amplitudes proceed along the same lines as in Yang-Mills theory. The ABJM numerator decomposition takes the same form \(^{11}\), but the color factors are now products of the four-index structure constants and the sum runs over diagrams with only quartic vertices. Unlike in Yang-Mills theory, generic numerators in the three-algebra formulation of ABJM theory are non-local. The structure constant has four indices, but the matter is coupled to the gauge field by a cubic interaction. The propagators in the color-kinematic decomposition of ABJM amplitudes \(^{11}\) specify a graph with four-point interactions, and so some of the propagators that specify three-point interactions must be absorbed into the numerator. The color factors in the numerator decomposition can be expanded into the trace over strings of generators, schematically \( \text{Tr}(T^a \bar{T}^b \ldots) \), by using the following identities:

\[
f^{ab\bar{c}\bar{d}} = \text{Tr}([T^a \bar{T}^\bar{c} T^b - T^b \bar{T}^\bar{c} T^a] \bar{T}^\bar{d}),
\]

(9)

\[
(T^a)^{i}_{\bar{j}} (\bar{T}^\bar{m})^{\bar{m}}_{\bar{n}} h_{ab} = \delta^\bar{i}_{\bar{j}} \delta^\bar{m}_{\bar{n}}.
\]  

(10)
The numerator decomposition can therefore be converted into a sum over color-ordered amplitudes, just as in Yang-Mills theory. Color-ordered 2m-point amplitudes are defined by

\[
A(\bar{1}, 2, \bar{3}, \ldots, \bar{2m} - 1, 2m) = \sum_{\sigma \in S'_{2m}/Z_{2m}} \text{Tr}(\bar{T}^{\sigma_1} T^{\sigma_2} \bar{T}^{\sigma_3} \ldots \bar{T}^{\sigma_{2m-1}} T^{\sigma_{2m}}) \\
\times A(\bar{\sigma}_1, \sigma_2, \bar{\sigma}_3, \ldots, \bar{\sigma}_{2m-1}, \sigma_{2m}).
\]  

(11)

The set \( S'_{2m}/Z_{2m} \) is all orderings that have alternating barred and unbarred legs and are unrelated by cyclic permutation. The color-ordered amplitudes can be written in terms of numerators. One color factor is the sum of different trace-strings with different signs, and these signs determine the relative signs of the numerators in the color-ordered amplitudes. For example, consider a color factor \( c_i \equiv f^{1321} = \text{Tr}([T^1 T^2 T^3 - T^3 T^2 T^1] T^4) \). The associated numerator \( n_i \) will enter \( A(1234) \) and \( A(3214) \), but with opposite signs. The color-ordered ABJM amplitudes have symmetries similar to those of Yang-Mills amplitudes upon inversion and cyclic permutation. For amplitudes with external bosonic states

\[
A_{2m}(\bar{1}, 2, \bar{3}, \ldots, \bar{2m} - 1, 2m) = A_{2m}(3, 4, \ldots, 2m - 1, 2m, \bar{1}, 2),
\]

(12)

\[
A_{2m}(\bar{1}, 2, \bar{3}, \ldots, \bar{2m} - 1, 2m) = (-1)^{2m-1} A(\bar{1}, 2m, 2m - 1, \ldots, \bar{3}, 2).
\]

(13)

The amplitudes obey KK-type identities, though these identities are not entirely understood beyond six points [18]. Such identities are linear relations between amplitudes with integer coefficients, just as in Yang-Mills theory. In the following, we sometimes denote barred and unbarred indices by even and odd particle labels. We do not need to keep track of ordering between the two types of indices. For a more detailed review of ABJM theory’s three-algebra formulation, see refs. [18, 24].
3 The color-kinematic duality in ABJM theory

Evidence for the color-kinematic duality in ABJM theory was recently found in tree-level scattering amplitudes [16]. Testing the duality in ABJM theory proceeds just as in Yang-Mills theory: requiring the numerators satisfy the duality between kinematics and color generates BCJ amplitude identities, which can be verified by using the explicit amplitudes. Details of how to calculate the explicit amplitudes, as well as some lower-point examples, are described in ref. [25].

We describe the color-kinematic duality at four, six, and eight points. The color-kinematic duality is trivially satisfied at four points. At six and eight points, we consider amplitudes with bosonic external states. The six-point case is the first non-trivial instance of the duality [18]. Next, we explicitly demonstrate the duality for six points as a warmup to our work at eight points. Duality-satisfying six-point numerators in terms of amplitudes are provided, as these do not appear elsewhere in the literature. At eight points, we show how to construct the amplitudes in terms of numerators. Numerical analysis shows that the numerators satisfy the color-kinematic duality but do not have any residual freedom – they are uniquely specified by amplitudes. We give explicit expressions for the generalized Jacobi identities the numerators satisfy and the eight-point amplitudes in terms of numerators in the attached files online [19]. All our analysis is for three dimensional on-shell momenta, consistent with the space-time dimension of the theory. As ref. [17] found, implications of the color-kinematic duality change when these conditions are relaxed: when momenta are off-shell or taken in more than three dimensions, the freedom that produces the BCJ amplitude relation at six points is no longer present.

3.1 Four points

In the three-algebra formulation of ABJM theory, the vertex associated with $f^{abcd}$ comes from a four-point diagram, illustrated in fig. 3 with a non-local numerator. At four points, there is only one independent color-ordered amplitude after accounting for amplitude symmetries. When assembling the indices on a structure constant, one must choose a convention: whether
to begin with an unbarred or barred index, and whether to move clockwise or counterclockwise. We chose to begin from an unbarred index and move clockwise in the diagram. We also use the convention that all momenta are incoming. With these conventions the four-point superamplitude is [25]

\[ A(1, \bar{2}, 3, \bar{4}) = \delta^{(3)}(P)\delta^{(6)}(Q) f_{a_1\bar{a}_2a_3\bar{a}_4}. \]

(14)

The delta functions conserve momentum \( P^{\alpha\beta} = \sum_i p_i^{\alpha\beta} \) and supermomentum \( Q^{\alpha I} = \sum_i q_i^{\alpha I} \).

As we have \( \mathcal{N} = 6 \) real supercharges, these can be grouped into 3 complex Grassmann-valued spinors. We have \( q_i^{\alpha I} = \lambda_i^{\alpha} \eta^I \) for the \( i \)-th particle. The label \( I \) is the index for the \( SO(6) \) R-symmetry, and \( \alpha \) labels the supercharge number, running from one to three [25] [24]. The amplitude is written using spinor-helicity formalism (see e.g. [21]).

### 3.2 Six points

At six points, the color-kinematic duality holds [16] [17]. We present explicit formulas showing how it holds. The six-point color factors are products of two structure constants, and obey the generalized Jacobi identity [8]. The full amplitude has nine independent channels:

\[ A(1, 2, 3, 4, 5, 6) = \frac{c_1n_1}{s_{123}} + \frac{c_2n_2}{s_{156}} + \frac{c_3n_3}{s_{126}} + \frac{c_4n_4}{s_{134}} + \frac{c_5n_5}{s_{125}} + \frac{c_6n_6}{s_{124}} + \frac{c_7n_7}{s_{136}} + \frac{c_8n_8}{s_{145}} + \frac{c_9n_9}{s_{146}}. \]

(15)
The kinematic invariants are $s_{ijk} = (p_i + p_j + p_k)^2$. The pole structure specifies the diagram, and so fixes the color factor up to an overall sign:

$$
\begin{align*}
  c_1 &= f_{123}^{a} f_{a}^{456}, & c_2 &= f_{156}^{a} f_{a}^{234}, & c_3 &= f_{345}^{a} f_{a}^{612}, \\
  c_4 &= f_{134}^{a} f_{a}^{562}, & c_5 &= f_{521}^{a} f_{a}^{436}, & c_6 &= f_{365}^{a} f_{a}^{124}, \\
  c_7 &= f_{163}^{a} f_{a}^{254}, & c_8 &= f_{541}^{a} f_{a}^{632}, & c_9 &= f_{325}^{a} f_{a}^{416}.
\end{align*}
$$

(16)

There are five amplitudes independent under the KK-type relations. We choose the following amplitudes as our basis amplitudes:

$$
\begin{align*}
  A(1, 2, 3, 4, 5, 6) &= \frac{n_1}{s_{123}} + \frac{n_2}{s_{156}} + \frac{n_3}{s_{126}}, & A(1, 4, 3, 6, 5, 2) &= \frac{n_4}{s_{134}} + \frac{n_5}{s_{125}} + \frac{n_6}{s_{124}}, \\
  A(1, 6, 3, 2, 5, 4) &= \frac{n_7}{s_{136}} + \frac{n_8}{s_{145}} + \frac{n_9}{s_{146}}, & A(1, 4, 3, 2, 5, 6) &= -\frac{n_4}{s_{134}} - \frac{n_2}{s_{156}} - \frac{n_9}{s_{146}}, \\
  A(1, 6, 3, 4, 5, 2) &= -\frac{n_7}{s_{136}} - \frac{n_5}{s_{125}} - \frac{n_3}{s_{126}}.
\end{align*}
$$

(17)

The relative sign of each numerator is conveniently determined by switching to the trace expansion of its color factor. Next, we require that the numerators satisfy the duality between color and kinematics. Each numerator must obey the same identities as its sibling.
color factor:

\[ n_5 = -n_2 + n_3 + n_4, \quad n_6 = n_1 - n_2 + n_4, \]

\[ n_8 = n_1 - n_3 + n_7, \quad n_9 = n_2 - n_3 + n_7. \]

We can choose \( \{n_1, n_2, n_3, n_4, n_7\} \) as a basis independent under the color-kinematic duality. There are now five independent numerators, the same number as the KK-independent amplitudes. Our set of KK-independent amplitudes are now

\[
\begin{align*}
A(1, 2, 3, 4, 5, 6) &\equiv A_1 = \frac{n_1}{s_{132}} + \frac{n_2}{s_{156}} + \frac{n_3}{s_{126}}, \\
A(1, 4, 3, 6, 5, 2) &\equiv A_2 = \frac{n_4}{s_{134}} + \frac{-n_2 + n_3 + n_4}{s_{125}} + \frac{n_1 - n_2 + n_4}{s_{124}}, \\
A(1, 6, 3, 2, 5, 4) &\equiv A_3 = \frac{n_7}{s_{136}} + \frac{n_1 - n_3 + n_7}{s_{145}} + \frac{n_2 - n_4 + n_7}{s_{146}}, \\
A(1, 4, 3, 2, 5, 6) &\equiv A_4 = -\frac{n_4}{s_{134}} - \frac{n_2}{s_{156}} - \frac{n_2 - n_3 + n_7}{s_{146}}, \\
A(1, 6, 3, 4, 5, 2) &\equiv A_5 = -\frac{n_7}{s_{136}} - \frac{-n_2 + n_3 + n_4}{s_{125}} - \frac{n_3}{s_{126}}. 
\end{align*}
\] (18)

We can solve for the numerators one by one and find that one numerator drops out of our equations. To be concrete, we begin by solving for \( n_1 \): choosing one of the amplitude-numerator equations above and solving for \( n_1 \) in terms of an amplitude and the remaining numerators, we substitute this expression for \( n_1 \) into the remaining amplitude equations. The equations now relate the five amplitudes to four numerators. We solve for \( n_2 \) and \( n_3 \) in the same way, leaving two equations that relate the five amplitudes to \( n_4 \) and \( n_7 \). When solving for \( n_4 \) in one equation and substituting the result into the remaining equation, we find that \( n_7 \) drops out. What remains is an equation relating the five amplitudes — this is the BCJ amplitude relation. One of the five numerators is arbitrary. In other words, the coefficient matrix of the numerators in the amplitude equations has rank four, while there are five numerators. This is similar to the situation in Yang-Mills theory [1].

We may therefore choose one numerator to have an arbitrary value. As the remaining numerators depend on this numerator, the remaining numerators depend on our choice.
A convenient choice is to simply set $n_7$ to zero and solve for the remaining numerators in terms of amplitudes. Imposing three-dimensional momentum conservation produces a lengthy expression, so we display the solution using the original kinematic invariants:

$$n_1 = s_{123} \left( A_1 + \frac{A}{s_{234}} + \frac{E}{D} \left( \frac{1}{s_{345}} - \frac{B}{C s_{234}} \right) \right), \quad n_2 = -A + \frac{BE}{CD},$$

$$n_4 = s_{134} \left( -A_4 + \left( A - \frac{BE}{CD} \right) \left( \frac{1}{s_{234}} + \frac{1}{s_{235}} - \frac{E}{D s_{235}} \right) \right), \quad n_3 = -\frac{E}{D}. \quad (19)$$

The quantities $A, B, C, D, E$ have been defined for convenience and are

$$A = -\frac{A_1}{s_{123}} + \frac{A_1}{s_{236}} - \frac{1}{s_{123} s_{235}} - \frac{1}{s_{234} s_{236}},$$

$$B = -\frac{1}{s_{235}} + \frac{1}{s_{236}} - \frac{1}{s_{236} s_{345}},$$

$$C = \frac{1}{s_{123} s_{235}} - \frac{1}{s_{234} s_{236}},$$

$$D = -B \left( -\frac{1}{s_{234}} + \frac{1}{s_{235}} \left( \frac{1}{s_{134}} - \frac{1}{s_{436}} - \frac{1}{s_{356}} \right) \right),$$

$$+ C \left( -\frac{1}{s_{134} s_{346}} + \frac{1}{s_{134} s_{235}} - \frac{1}{s_{134} s_{345} s_{356}} \right),$$

$$E = C \left( -\frac{A_4}{s_{134}} - A_4 \left( \frac{1}{s_{134}} - \frac{1}{s_{436}} - \frac{1}{s_{356}} \right) + \frac{A_1}{s_{134} s_{356}} \right),$$

$$+ A \left( -\frac{1}{s_{234}} + \frac{1}{s_{235}} \left( \frac{1}{s_{134}} - \frac{1}{s_{436}} - \frac{1}{s_{356}} \right) \right),$$

$$(20)$$

We now substitute the solutions for $n_1, n_2, n_3, n_4$ into the expression for $A_5$ in (18). Since $n_7$ drops out, it leaves behind a single nontrivial BCJ amplitude relation between the five partial amplitudes. We have confirmed that this relation holds numerically in the actual amplitudes by plugging in explicit values for the amplitudes, which were obtained in ref. [25]. The six-point case is discussed further in ref. [18], which gives the BCJ amplitude relation and checks it for the superamplitude. While in Yang-Mills theory the freedom to
adjust numerators may be used to keep the color-kinematic numerators local, in the three-

algebra formulation of ABJM theory numerators are inherently non-local. The freedom to
adjust $n_7$ cannot be used to make the remaining numerators local functions.

3.3 Eight points

![Diagrams](image-url)

Figure 5: The two diagrams that contribute numerators to the ABJM color-stripped amplitude. The left diagram has a color factor of the form $f^{abc}_m f^{m d n h} f^g ef$. The external indices in the middle are both barred or, for a different leg ordering than shown above, both unbarred. The right diagram has a color factor of the form $f^{abc}_m f^{m d n h} f^n def$. One middle index is barred while the other is not.

The eight-point case proceeds in the same fashion as the six-point case. The two distinct diagrams that contribute are shown in fig. [3]. For clarity, we provide one color-stripped amplitude from which all others are obtained by relabelling. In this amplitude, we label each numerator by the color factor it is associated with, suppressing the summed indices on the structure constants. For example, we denote the numerator associated with the color factor $f^{123}_a f^{a i s h}_b f^{576}_b$ as $n_{123,48,576}$. In assigning the color-factors, we use the convention that the summed indices of the color factors are as close together as possible, and the right-most structure constant has an index which is higher than the left-most structure constant’s indices. We suppress the bars and take odd and even labels to correspond to the barred and
unbarred indices separately. The color-stripped amplitude then is

\[
A(1, 2, 3, 4, 5, 6, 7, 8) = - \frac{n_{132,48,576}}{s_{123}s_{567}} - \frac{n_{354,62,718}}{s_{345}s_{781}} + \frac{n_{324,51,768}}{s_{234}s_{678}} + \frac{n_{546,73,182}}{s_{456}s_{812}} - \frac{n_{132,78,546}}{s_{123}s_{456}} - \frac{n_{354,12,768}}{s_{345}s_{567}} - \frac{n_{576,34,182}}{s_{567}s_{812}} - \frac{n_{718,56,324}}{s_{781}s_{234}} + \frac{n_{576,18,324}}{s_{567}s_{234}} + \frac{n_{718,32,546}}{s_{781}s_{456}} + \frac{n_{132,54,768}}{s_{123}s_{678}} + \frac{n_{354,76,182}}{s_{345}s_{812}}.
\]

(21)

All other color-stripped eight-point amplitudes can be obtained from this expression by relabelling. The signs of each numerator depends on the trace string’s contribution to each color factor, similar to the Yang-Mills case [1]. The eight-point amplitude has 216 different color factors, and therefore 216 corresponding numerators. We have checked that our numerator representation of the color-stripped amplitudes obey the correct cyclic permutation properties and KK-type identities as listed in ref. [18]. We repeat these symmetries below, using the bar-unbar notation for clarity. The symmetries are

\[
\text{Cyclic shift by } 2m: \quad A(12345678) = A(78123456), \quad (22)
\]

\[
\text{Reversal: } \quad A(12345678) = -A(18765432), \quad (23)
\]

where \( m \) is an arbitrary integer. An example of an eight-point KK-type identity satisfied by our amplitudes is

\[
- A(12345876) - A(14325876) + A(16385274) + A(16385472) + A(16783254) + A(12763854) + A(14763852) + A(16783452) + A(16723854) + A(16743852) + A(16327854) + A(16347852) + A(16387254) + A(16387452) + A(14367852) + A(12367854) = 0.
\]

(24)

There are 57 amplitudes independent under the KK-type identities. We call these the KK amplitudes. The basis was chosen by eliminating all linear dependence between amplitudes written in terms of numerators. As in Yang-Mills theory, we know the resulting linear ampli-
tude relations are not BCJ amplitude relations, because no generalized Jacobi relations have been used. Eliminating the linear dependence between amplitudes in this step corresponds to exhausting the KK-type amplitude relations. No closed form for the KK-type identities have been found, but the number of basis amplitudes agrees with the result in ref. [18].

We choose a set of KK amplitudes and present a sample below. The full set of equations is contained in an attached file online [19].

\[
A(12345876) = -\frac{n_1}{s_{123}^{876}} - \frac{n_2}{s_{123}^{857}} - \frac{n_7}{s_{123}^{845}} + \frac{n_{28}}{s_{587}^{612}} + \frac{n_{30}}{s_{458}^{612}} - \frac{n_{36}}{s_{345}^{8612}},
\]

\[
A(12365478) = -\frac{n_5}{s_{123}^{547}} - \frac{n_8}{s_{123}^{547}} + \frac{n_9}{s_{123}^{547}} + \frac{n_{48}}{s_{547}^{812}} - \frac{n_{49}}{s_{548}^{812}} - \frac{n_{51}}{s_{365}^{812}},
\]

\[
A(12365874) = -\frac{n_2}{s_{123}^{857}} - \frac{n_3}{s_{123}^{857}} - \frac{n_5}{s_{123}^{857}} + \frac{n_{11}}{s_{412}^{587}} + \frac{n_{12}}{s_{412}^{587}} - \frac{n_{18}}{s_{365}^{812}} - \frac{n_{100}}{s_{658}^{741}} - \frac{n_{103}}{s_{658}^{741}} + \frac{n_{106}}{s_{236}^{874}} + \frac{n_{170}}{s_{236}^{874}} + \frac{n_{171}}{s_{236}^{874}} + \frac{n_{212}}{s_{365}^{874}} - \frac{n_{212}}{s_{365}^{874}}.
\]

The four-term generalized Jacobi relations the numerators must satisfy according to the color-kinematic duality are also listed in an attached file online. These relations are solved, and all numerators are specified by the 57 basis numerators. After imposing the Jacobi relations, the 57 amplitude equations are expressed in terms of the 57 basis numerators. Here, the number of KK amplitudes and numerators independent under the generalized Jacobi relations are the same, just as in Yang-Mills theory. However, the surprise here is that these equations turn out to be invertible. Solving the system analytically is difficult, but it is straightforward to check invertibility numerically by using explicit values for the three-dimensional momentum that obey momentum conservation. In other words, the numerators have no residual freedom: each numerator is uniquely specified as a function of gauge-invariant amplitudes. Without residual freedom, the eight-point amplitudes have no BCJ amplitude relations. The lack of BCJ amplitude relations is a surprising result.

At eight points, the color-kinematic duality is automatically satisfied. Through the nu-
merator decomposition the amplitudes obey the KK-type identities manifestly, and unlike in Yang-Mills theory, there are no BCJ amplitudes relations to check.

4 Discussion

We have examined bosonic amplitudes through eight points in ABJM theory and found that the duality between color and kinematics holds, but at eight points attendant BCJ amplitude relations are not present. The color-kinematic duality specifies the eight-point numerators as unique functions of gauge-invariant partial amplitudes. We also have given explicit expressions necessary for analysis of ABJM amplitudes that do not appear in the literature. At six points, we have solved for a basis of color-kinematic numerators in terms of amplitudes. At eight points, we presented a set of amplitudes independent under the Kleiss-Kuijf relations in terms of numerators, as well as the relations the numerators satisfy according to the color-kinematic duality. The full eight-point expressions can be found in an ancillary online file [19].

The eight-point amplitude-numerator decomposition we provided can be used to study possible relations between ABJM amplitudes, or equivalently between ABJM numerators [18]. A recent twistor string construction for ABJM theory may also provide insight [26]. BLG theory is a special case of ABJM theory, and possesses the color-kinematic duality, residual freedom, and the double-copy property [18]. The mapping between BLG and ABJM amplitudes is straightforward at four and six points, but is not fully understood at eight points [18] and needs further investigation. While in ABJM theory the double-copy procedure leads to supergravity amplitudes at lower points, at eight points the double-copy property fails [18]. One might suspect that there is a connection between the simultaneous disappearances of the double-copy property and residual freedom. Further study of the eight point ABJM amplitudes may provide insight into the double-copy property through side-by-side comparison with BLG amplitudes.

Using the data we have provided, further analysis of the eight-point ABJM amplitudes may provide important clues about the color-kinematic duality, residual freedom, and the
double-copy property. The presence of the color-kinematic duality in ABJM theory without associated BCJ amplitude relations emphasizes the basic role numerators play in the duality compared to the amplitude relations. The eight-point ABJM amplitudes are the lowest-point amplitudes that allow the color-kinematic duality without possessing residual freedom to rearrange the numerators or yielding gravity amplitudes via the double-copy property. These amplitudes therefore provide an interesting avenue for further understanding the role of these properties in the duality between color and kinematics.

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