SEQUENTIAL BETHE VECTORS AND THE QUANTUM ERNST SYSTEM

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We give a brief review on the use of Bethe ansatz techniques to construct solutions of recursive functional equations which emerged in a bootstrap approach to the quantum Ernst system. The construction involves two particular limits of a rational Bethe ansatz system with complex inhomogeneities. First, we pinch two insertions to the critical value. This links Bethe systems with different number of insertions and leads to the concept of sequential Bethe vectors. Second, we study the semiclassical limit of the system in which the scale parameter of the insertions tends to infinity.

1 Functional equations for matrix elements in the quantum Ernst system

In [1] we proposed a bootstrap approach to describe the quantum theory descending from the Ernst equation of general relativity [2]. In upshot, the quantum theory is described in terms of matrix elements e.g. of the metric operator between spectral-transformed multi-vielbein configurations. Functional equations for these matrix elements were derived from an underlying quadratic algebra similar to the way the form factor axioms [3] may be derived from an underlying algebra [4]. Eventually the mathematical problem consists in finding sequences of vector-valued functions

\[ f_A(\theta) := f_{a_N \ldots a_1}(\theta_N, \ldots, \theta_1), \quad N \geq N_0, \quad (1) \]

obeying the following system of functional equations:

\[ \mathcal{T}(\theta_0|\theta)f_B(\theta) = \tau(\theta_0|\theta)f_A(\theta), \quad (2) \]

\[ f_A(\theta) = L_k(\theta_{k+1},k)^A_B f_B(\sigma_k \theta), \quad (3) \]

\[ \text{Res}_{\theta_{k+1} = \theta_k + i\hbar} f_A(\theta) = \tau(\theta_k|p_k \theta) C_{a_{k+1}a_k} f_{p_k A}(p_k \theta), \quad (4) \]

\[ \text{Res}_{\theta_{k+1} = \theta_k - i\hbar} C^{a_k a_{k+1}} f_A(\theta) = \tau(\theta_k - i\hbar|p_k \theta) f_{p_k A}(p_k \theta). \quad (5) \]

Let us briefly describe these equations and the objects featuring. Throughout, we use the rational \( R \)-matrix

\[ R(\theta)_{ab}^{cd} := \frac{r(\theta)}{\theta - i\hbar} (\theta \delta_a^c \delta_b^d - i\hbar \delta_a^d \delta_b^c), \quad (6) \]

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where $r(\theta)$ satisfies $r(\theta) r(\theta - i\hbar) = 1 - i\hbar/\theta$. Equation (3) results in the diagonalization of the operator $\mathcal{T}$, for a fixed number of arguments $N$. It is basically the familiar transfer matrix

\[ \mathcal{T}(\theta_0|\theta) = \Gamma_a^b R_{c_Na_N}^{b_N} (\theta_{N,0}) R_{c_{N-1}a_{N-1}}^{b_{N-1}} (\theta_{N-1,0}) \ldots R_{a_1a_1}^{b_1} (\theta_{1,0}) . \]  

(7)

The matrix $\Gamma$ here denotes a traceless $SL(2,\mathbb{C})$ matrix which for simplicity we assume to be diagonalized: $\Gamma := \text{diag}(i, -i)$. The operator $\mathcal{T}$ descends from the central quantum current [5] in the Yangian double at the critical value of the central extension; the latter appears as part of the underlying quantized algebra of conserved charges [6].

Equation (3) describes the behavior of the eigenvectors $f_A$ under permutation of the arguments; $L_k$ acts as

\[ L_k (\theta_{k+1,k})^B = \delta_{a_N}^{b_N} \ldots R_{a_k+1a_k}^{b_k,b_{k+1}} (\theta_{k+1,k}) \ldots \delta_{a_1}^{b_1} , \]  

(8)

and $(\sigma_k \theta) = (\theta_N, \ldots, \theta_k, \theta_{k+1}, \ldots, \theta_1)$, where by $\theta_k l$ we denote the difference $\theta_k - \theta_l$. Solutions to (2) constructed by the Bethe ansatz can naturally be made compatible with (3). In this note we mainly focus on the recursive equations (4) and (5). There, $C_{a_b}$ denotes the $sl(2)$ invariant antisymmetric tensor and we have adopted the following notation for contraction: $p_k \theta = (\theta_N, \ldots, \theta_{k+2}, \theta_{k-1}, \ldots, \theta_1)$, $p_k A = (a_N, \ldots, a_{k+2}, a_{k-1}, \ldots, a_1)$. These two equations link solutions of different eigenvector problems (2), with $N$ and $N-2$ arguments, respectively, under the pinching $\theta_{k+1} \rightarrow \theta_k \pm i\hbar$. For the eigenvalues $\tau(\theta_0)$ they imply the compatibility condition

\[ \tau(\theta_0|\theta) \bigg|_{\theta_{k+1} = \theta_k \pm i\hbar} = \tau(\theta_0|p_k \theta) . \]  

(9)

In the following, we subsequently construct solutions to the equations (2)–(5). Equation (3) amounts to diagonalization of the transfer matrix, which is a well studied problem and can be solved by Bethe ansatz techniques [7, 8]. Joint solutions of equations (2), (3) can be obtained from them by a symmetrization procedure. Equations (4), (5) lead to the concept of sequential Bethe vectors, connecting Bethe roots with different number of arguments. From a technical viewpoint the latter might also offer a new recursive approach (cf. [9]) to issues like completeness of the Bethe vectors. Finally, we discuss the semi-classical limit $\hbar \rightarrow 0$ of the solutions. For details and further references we refer to [1].

2 Bethe ansatz

In this section we describe the solutions of equation (2). The spectrum of $\mathcal{T}$ is conveniently organized by the $SO(2)$ symmetry that leaves $\mathcal{T}$ invariant:

\[ \mathcal{T}(\theta_0|\theta)^B_A (\sum_k \Gamma_k)^C = (\sum_k \Gamma_k)^B_A \mathcal{T}(\theta_0|\theta)^C_B , \]  

(10)

with

\[ (\Gamma_k)^B_A = \delta_{a_N}^{b_N} \ldots \Gamma^b_{a_k} \ldots \delta_{a_1}^{b_1} . \]
The eigenvalue problem \( \mathcal{T}(\theta_0|\theta)^B_{\Lambda} f_{e:B}(\theta) = \tau_e(\theta_0|\theta) f_{e:A}(\theta) \), \( (\Gamma k)^B_{\Lambda} f_{e:B} = i e f_{e:A} \), \( k=1,2 \) hence decomposes into decoupled sectors

\[
\mathcal{T}(\theta_0|\theta)^B_{\Lambda} f_{e:B}(\theta) = \tau_e(\theta_0|\theta) f_{e:A}(\theta) , \quad (\Gamma k)^B_{\Lambda} f_{e:B} = i e f_{e:A} ,
\]

where \( e = N, N-2, \ldots, -N+2, -N \), denotes the \( SO(2) \) charge. We denote by \( \pm \) the \( sl(2) \) indices in the “charged” basis of eigenvectors. For small \( N \) the eigenvalue problem \( \mathcal{T}(\theta_0|\theta)^B_{\Lambda} \) can be solved by brute force but for generic \( N \) it is useful to employ the standard techniques of the Bethe ansatz (see e.g. [1],[8]) and to parameterize the solutions in terms of the roots of the Bethe equations.

Transferred to the present context this construction may be outlined as follows: Denote by \( \Omega_A := \delta^+ \cdots \delta^+ \) the lowest weight vector of the \( N \)-fold tensor product of the fundamental representation of \( sl(2) \). Following the Bethe Ansatz procedure, candidate eigenstates are generated from \( \Omega \) by the repeated action of

\[
B(t|\theta)^B_{\Lambda} := \Gamma^e R^+_N b_{N,0}(\theta_{N,0}) R^c_{N-1,a_{N-1}}(\theta_{N-1,0}) \cdots R^c_{a_1}(\theta_{1,0}) .
\]

The matrix operators \( B(t|\theta) \) are commuting for different values of \( t \) and each \( B(t_\alpha|\theta) \) lowers the \( SO(2) \) charge \( e \) of a candidate eigenstate of \( \mathcal{T} \) by two units. The candidate eigenstates can be made proper eigenstates by turning the parameters \( t_\alpha \) into judiciously chosen functions of the \( \theta_j \). In upshot one obtains eigenvectors

\[
w_e(\theta) = \prod_{\alpha=1}^{\Lambda} B(t_\alpha|\theta) \Omega , \quad \Lambda := \frac{1}{2}(N-e) ,
\]

with eigenvalues

\[
\tau_e(\theta_0|\theta) = i \prod_{\alpha} \frac{\theta_0-t_\alpha+i\hbar/2}{\theta_0-t_\alpha+3i\hbar/2} \prod_j r^{-1}(\theta_{0j}) - i \prod_{\alpha} \frac{\theta_0-t_\alpha+5i\hbar/2}{\theta_0-t_\alpha+3i\hbar/2} \prod_j r(\theta_{0j}+i\hbar) ,
\]

where the Bethe roots \( t_\alpha \) are solutions of the following Bethe Ansatz equations (BAE)

\[
\prod_{j=1}^{N} \frac{\theta_j-t_\alpha-i\hbar/2}{\theta_j-t_\alpha+i\hbar/2} = -\prod_{\beta \neq \alpha} \frac{t_\beta-t_\alpha-i\hbar}{t_\beta-t_\alpha+i\hbar} , \quad \alpha = 1, \ldots, \Lambda .
\]

The only modification of the BAE as compared to the standard case \( \Gamma = I \) is the sign on the r.h.s. which comes from the ratio of the eigenvalues of \( \Gamma \). This seemingly innocent modification turns out to have nontrivial consequences in the classical limit \( \hbar \to 0 \) to be described in section 3.

Let us now return to the eigenvectors (13). Clearly any eigenvector is only determined up to multiplication by an arbitrary scalar function. The Bethe eigenvectors as constructed by (13) will in general not obey the exchange relations (5). However, it is not difficult to modify them so that they do. Due to the symmetry \( \tau_e(\theta_0|\sigma \theta) = \tau_e(\theta_0|\theta) \), for all permutations \( \sigma \in \Sigma_N \), a joint solution of (2), (3) can be obtained simply by symmetrizing with the \( R \)-matrix. In brief, for any given Bethe eigenvector (13) the product

\[
f_{e:A}(\theta) \propto \prod_{k,l>j} \frac{i \psi(\theta_{kl})}{\theta_{kl}^2 - (i\hbar)^2} w_e(\theta) ,
\]

(16)
solves both (2) and (3). Here, the function $\psi$ satisfies $\psi(\theta) = r(\theta)\psi(-\theta)$, and $\psi(-\theta)\psi(\theta - i\hbar) = -1$ and is explicitly given by

$$\psi(\theta) = \tanh \frac{\pi \theta}{2\hbar} \exp \left\{ \int_0^\infty \frac{dt}{t} \frac{e^{-t/2} + e^{-t}}{1 + e^{-t}} \frac{\sin t}{\cosh \frac{\pi}{2}} \right\}.$$

The proportionality sign in (16) indicates that this eigenvector may still be multiplied with a scalar function $\phi(\theta)$ completely symmetric in $\theta_N, \ldots, \theta_1$. This freedom will mostly be fixed by further imposing the pinching equations (4), (5).

3 Sequential Bethe roots and vectors

Next let us examine the behavior of the Bethe ansatz equations and their solutions under pinching $\theta_{k+1} \to \theta_k \pm i\hbar$ of the arguments. The relations (11), (12) imply that the $SO(2)$ charge $e$ of the eigenvectors is conserved under $\theta_{k+1} \to \theta_k \pm i\hbar$, i.e.

$$N \to N - 2, \quad e \to e, \quad \Lambda \to \Lambda - 1.$$

This suggests that the Bethe roots describing these (special) sequences of eigenvectors should likewise be related. Indeed, the BAE (13) are consistent with the following $N \to N - 2$ reduction of their solutions

$$t_\Lambda(\theta) \bigg|_{\theta_{k+1} = \theta_k \pm i\hbar} = \theta_k \pm \frac{1}{2} i\hbar, \quad t_\alpha(\theta) \bigg|_{\theta_{k+1} = \theta_k \pm i\hbar} = t_\alpha(p_k \theta), \quad \text{for } \alpha < \Lambda.$$

Since the Bethe roots are symmetric in all $\theta_j$, it suffices to verify (18) for the $\theta_{k+1} = \theta_k + i\hbar$ case. It is easy to check that with (18) the BAE (13) for $\alpha < \Lambda$ reduce to the BAE with $N - 2$ insertions for the $t_\alpha(p_k \theta)$. The equation for $\alpha = \Lambda$ is slightly more subtle as it requires to specify the limit in which the pinched configuration is approached. Entering with the ansatz

$$t_\Lambda(\theta) = \theta_k + \frac{i\hbar}{2} + \delta/Z(\theta) + O(\delta^2), \quad \text{for } \theta_{k+1} = \theta_k + i\hbar + \delta,$$

into the $\alpha = \Lambda$ BAE one obtains at order $\delta^0$ a linear equation for $Z(\theta)$. This can be taken to define $Z(\theta)$ and shows that the reduction rule for $t_\Lambda(\theta)$ is consistent as $\delta \to 0$.

Of course, not every solution of the BAE will satisfy (18), in fact the vast majority won’t. The argument shows however that under the same genericity assumptions under which solutions exist at all, there also exists at each recursion step $N - 2 \to N$ at least one $\Lambda$-tuple of Bethe roots enjoying the property (18). We call a solution of the BAE a “sequential” tuple of Bethe roots, if all roots are distinct and satisfy (18). To justify the terminology one may easily verify that (18) with (14) implies the compatibility equation (9) for the eigenvalue $\tau_c(\theta_0|\theta)$. The corresponding eigenvectors will satisfy equations (4), (5) up to a scalar function. The construction is completed by determining the symmetric function $\phi_c(\theta)$ multiplying (16) such that (4), (5) are identically satisfied; c.f. (11).
Finally we study the limit $\hbar \to 0$ of the joint solutions of (3). In the context of the quantized Ernst system this corresponds to the semiclassical limit of the matrix elements. The limit of the transfer matrix $T$ is given by:

$$T(\theta_0|\theta) = i\hbar \sum_k \frac{\Gamma_k}{\theta_{ok}} + (i\hbar)^2 \left( - \sum_k \frac{\Gamma_k}{2\theta_{0k}} + \sum_k H_k \right) + \mathcal{O}(\hbar^3),$$

(20)

with $H_k = \sum_{l \neq k} \frac{\Omega_{kl} (\Gamma_k + \Gamma_l)}{\theta_{kl}}$, $(\Omega_{kl})^B_A = \delta_{a_N} \cdots \delta_{a_{1}} - \frac{1}{2} \delta_{a_k} \delta_{a_l} \cdots \delta_{a_{1}}$.

This expansion is valid either as a formal power series in $\hbar$ or, with a numerical $\hbar$, in the region $\text{Im} \theta_{0k} \gg \hbar, \text{Im} \theta_{lk} \gg \hbar, l \neq k$, in order to prevent a mixing of different powers of $\hbar$. The absence of a term of order $\hbar^0$ in (20) is due to the tracelessness of $\Gamma$ and distinguishes this case from the usual situation $\Gamma = I$. The matrices $\Gamma_k$ and Hamiltonians $H_k$ form a family of mutually commuting operators. Simultaneous diagonalization of the $\Gamma_k$ yields eigenvectors with only one nonvanishing component $(\epsilon_N, \ldots, \epsilon_1)$.

On these eigenvectors the $H_k$ act diagonally. Thus the first terms in the semiclassical expansion of the eigenvalues $\tau$ are

$$\tau(\theta_0|\theta) = -\hbar \sum_k \frac{\epsilon_k}{\theta_{0k}} + i\hbar^2 \left( \frac{1}{2} \sum_k \frac{\epsilon_k}{\theta_{0k}^2} - \sum_{k \neq l} \frac{\epsilon_k}{\theta_{0k} \theta_{kl}} \right) + \mathcal{O}(\hbar^3).$$

(22)

This phenomenon can also be understood in terms of the Bethe ansatz. In the limit $\hbar \to 0$, the symmetry of the solutions of (15) in $\theta_N, \ldots, \theta_1$ gets lost. Rather, the Bethe roots turn out to behave like

$$t_\alpha(\theta) = \theta_{j(\alpha)} + (i\hbar)^2 s_\alpha(\theta) + \mathcal{O}(\hbar^3),$$

(23)

for some $j(\alpha) \in \{1, \ldots, N\}$ with $j(\alpha) \neq j(\beta)$ for $\alpha \neq \beta$ and uniquely defined functions $s_\alpha(\theta)$. Generally one can show that the Bethe roots admit a power series expansion in $\hbar$ (in the region $\text{Im} \theta_{0k} \gg \hbar, k \neq l$) whose coefficients are uniquely determined by the assignment $\alpha \to j(\alpha)$ in (23). The limiting behavior (23) drastically differs from the standard case $\Gamma = I$, where no minus sign appears in the r.h.s. of the BAE (13) and the latter turn into an identity for $\hbar \to 0$. The eigenvector (21) corresponding to (23) is given by

$$w^{\text{cl}}_{e;A}(\epsilon) = 0 \quad \text{unless} \quad (a_N, \ldots, a_1) = (\epsilon_N, \ldots, \epsilon_1), \quad \sum_j \epsilon_j = e.$$

(24)
Summarizing, the semiclassical limit of the eigenvectors is given by
\[ f_{e;A}(\theta) = h^A f^{cl}_{e;A}(\theta) + \mathcal{O}(h^{A+1}) \, , \text{ with } \ f^{cl}_{e;A}(\theta) = \phi^{cl}_e(\theta) w^{cl}_{e;A}(\epsilon) \prod_{k>l} 1 \theta_{kl} \ . \tag{25} \]
As shown, this expansion refers to a fixed relative size of the variables \( \theta_N, \ldots, \theta_1 \), say \( \theta_N > \ldots > \theta_1 \). The results for other orderings then are compatible with the classical limit of the exchange relations in (3), i.e.
\[ f^{cl}_{e;A}(\theta) = f^{cl}_{e;\sigma_k A}(\sigma_k \theta) \ . \tag{26} \]
It remains the natural question for a classical counterpart of the recursive relations (4), (5), i.e. about commutativity of the two limits which we have described in this and the foregoing section. Indeed, it may be shown on the level of the Bethe roots that the classical limit of the \( t_\alpha \) (23) and the pinching operation (18) commute in the relevant situations. The final relation is
\[ \text{Res}_{\theta_{k+1}=\theta_k} f^{cl}_{e;A}(\theta) = c_0 C_{e_{k+1} e_k} f^{cl}_{e;\epsilon_k A}(p_k \epsilon_k \theta) \left( \sum_{j \neq k+1} c_j \theta_{kj} \right) \ , \tag{27} \]
with a constant \( c_0 \). The last factor on the right hand side, when restricted to \( \theta_{k+1} = \theta_k \) and \( \epsilon_{k+1} = -\epsilon_k \), equals the leading term in the \( h \) expansion (22) of the transfer matrix eigenvalues. This is the consistency condition on (27) analogous to (6).

In summary, the solutions of the functional equations (2)–(5) admit a consistent semi-classical expansion. The leading term (25) of this expansion has, for a given ordering of \( \theta_N, \ldots, \theta_1 \), only one non-vanishing component; different orderings being related by (26). Further these terms are themselves linked by the recurrence relation (27). It should be interesting to see whether these leading terms have a direct interpretation in the classical theory.

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