Existence of Incompressible and Immiscible Flows in Critical Function Spaces on Bounded Domains

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Communicated by Y. Giga

Abstract. We study global existence and uniqueness of solutions to inhomogeneous incompressible Navier–Stokes equations on bounded domains of $\mathbb{R}^n$, $n \geq 2$, with initial velocity in the Besov space $B^0_{q,\infty}(\Omega)$, $q \geq n$, and piecewise constant initial density. Existence of solutions is proved when $B^0_{q,\infty}$-norm of initial velocity and initial density difference are small, and for uniqueness we require that $q > n$. The proof of existence of solutions is done via an iterative scheme based on maximal $L^\infty_\gamma$-regularity of the Stokes operator in little Nikolskii spaces and on solvability for transport equations in the spaces of pointwise multipliers for little Nikolskii spaces, while the proof of uniqueness is done via a Lagrangian approach using the result of an time-evolutionary Stokes system with nonzero divergence obtained in this paper.

Mathematics Subject Classification. 35Q30, 35B35, 76D03, 76D07, 76E99.

Keywords. Existence, Uniqueness, Inhomogeneous Navier–Stokes equations, Immiscible flow, Divergence problem.

1. Introduction and Main Result

In this paper, we consider the initial boundary value problem for inhomogeneous incompressible Navier–Stokes equations

$$
\rho_t + (u \cdot \nabla)\rho = 0 \quad \text{in} \quad (0, T) \times \Omega,
$$

$$
\rho u_t - \mu \Delta u + \rho (u \cdot \nabla)u + \nabla P = 0 \quad \text{in} \quad (0, T) \times \Omega,
$$

$$
\text{div} u = 0 \quad \text{in} \quad (0, T) \times \Omega,
$$

$$
u = 0 \quad \text{on} \quad (0, T) \times \partial \Omega,
$$

$$\rho(0, x) = \rho_0, \quad u(0, x) = u_0 \quad \text{in} \quad \Omega,$$

where $0 < T \leq \infty$ and $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a bounded domain of $C^2$-class, $\rho$ is density of the fluid, $\mu$ is the dynamic viscosity, and $u$, $P$ are, respectively, the velocity and pressure.

The system (1.1) describes the motion of viscous incompressible flows with variable density. In particular, the motion of a mixture of several immiscible and incompressible fluids can be modeled by (1.1).

Kazhikhov [23] considered (1.1) in whole $\mathbb{R}^n$, $n = 2, 3$, and proved the global existence of weak solutions in energy space and, in addition, strong solution for small initial velocity.

Ladyzhenskaya and Solonnikov considered unique solvability for (1.1) in bounded domain of $\mathbb{R}^n$, $n = 2, 3$, in [25], where existence and uniqueness of solutions are proved when $\rho_0 \in C^4(\Omega)$ is positive away from 0 and the norm of $u_0$ in $W^{2-2/q,q}(\Omega)$, $q > n$, is small enough. Similar result has been obtained by Danchin [12], assuming less regularity on initial data.

If one considers the case where variable density is close to a constant (say $\bar{\rho}$) and writes $\bar{\rho} = a + 1$, then it is easy to check that $(a, u)$ solves the system:
\[ a_t + (u \cdot \nabla) a = 0 \quad \text{in} \quad (0, T) \times \Omega, \]
\[ u_t + (u \cdot \nabla) u + (1 + a)(-\nu \Delta u + \nabla p) = 0 \quad \text{in} \quad (0, T) \times \Omega, \]
\[ \text{div} \, u = 0 \quad \text{in} \quad (0, T) \times \Omega, \]
\[ u = 0 \quad \text{on} \quad (0, T) \times \partial \Omega, \]
\[ a(0, x) = a_0, \quad u(0, x) = u_0 \quad \text{in} \quad \Omega, \]

where here and in what follows
\[ \nu \equiv \frac{\mu}{\rho} \quad \text{and} \quad p \equiv \frac{P}{\rho}. \]

The equations of (1.2) are invariant under the scaling
\[
a_\lambda(t, x) = a(\lambda^2 t, \lambda x), \quad u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x), \quad p_\lambda(t, x) = \lambda^2 p(\lambda^2 t, \lambda x),
\]
\[ a_{0\lambda} = a_0(\lambda x), \quad u_{0\lambda} = \lambda u_0(\lambda x), \quad \lambda > 0, \]

and hence it is very important to show the existence of solutions to (1.2) in critical spaces, i.e., the spaces with norms invariant under the scaling (1.3).

In [11], Danchin investigated unique solvability for (1.2) in the whole space case with \( n \geq 2 \) in some scaling invariant homogeneous Besov spaces; more precisely, he showed that if \((a_0, u_0) \in (\dot{B}^{n/2}_2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)) \times \dot{B}^{n/2-1}_{2,1}(\mathbb{R}^n) \) \((r = \infty \text{ for } n = 3 \text{ and } r = 1 \text{ for } n = 2)\) and \( \|a_0\|_{\dot{B}^{n/2}_2} + \|u_0\|_{\dot{B}^{n/2-1}_{2,1}} \) is small enough, then (1.2) has a unique solution \((\rho, u)\) such that
\[
a \in BC([0, T), \dot{B}^{n/2}_2(\mathbb{R}^n)) \cap L^\infty(0, T; L^\infty(\mathbb{R}^n)),
\]
\[ u \in BC([0, T), \dot{B}^{n/2-1}_{2,1}(\mathbb{R}^n)) \cap L^1(0, T; \dot{B}^{n/2+1}_{2,1}(\mathbb{R}^n)). \]

This result was generalized by Abidi [1] to the case where the space \((\dot{B}^{n/2}_2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)) \times \dot{B}^{n/2-1}_{2,1}(\mathbb{R}^n)\) for \((a, u)\) in [11] is replaced by \(\dot{B}^{n/q}_q(\mathbb{R}^n) \times \dot{B}^{-1+n/q}_q(\mathbb{R}^n)\), \(1 < q < 2n\), and showed existence for \(1 < q < 2n\) and uniqueness for \(n < q < 2n\); the gap in the uniqueness for \(n < q < 2n\) was filled by Danchin and Mucha [14] via Lagrangian approach. Furthermore, the smallness assumption for initial density variation \(a_0\) was relaxed by Abidi, Gui and Zhang in [2,3]. In [13], Danchin and Mucha proved global well-posedness of (1.1) in half space \(\mathbb{R}^n_+\) under the assumption that \(\rho_0\) is close enough to a constant in \(L^\infty(\mathbb{R}^n_+) \cap \dot{W}^1_1(\mathbb{R}^n_+)\) and the norm of \(u_0 \in \dot{B}^{0}_{q,1}(\mathbb{R}^n_+)\) is small enough.

On the other hand, very recently, an attempt to study (1.2) (equivalently (1.1)) in critical function spaces, just assuming initial density merely bounded positively away from 0, has been made, cf. [15,16,22]. Huang et al. [22] proved existence of a global solution to (1.2) in whole \(\mathbb{R}^n\) under a smallness condition of \(a_0 \in L^\infty(\mathbb{R}^n)\) and \(u_0 \in \dot{B}^{-1+n/q}_{q,1}(\mathbb{R}^n)\), \(q \in (1, n), r \in (1, \infty)\), and uniqueness of such solution under a slightly higher regularity assumption on initial velocity \(u_0\); this result is extended to the half-space setting by Danchin and Zhang [16]. The main ideas of [16] and [22] are to employ, for existence, maximal \(L^p\)-regularity for Stokes operator in Lebesgue spaces and, for uniqueness, a Lagrangian approach which was exploited in [14]. In [15], the case of bounded domains with \(C^2\)-boundary was considered under the same assumptions on the initial density and \(u_0 \in \dot{B}^{2-2/r}_{q,r}(\Omega), \ n < q < \infty, \ 1 < r < \infty, \ 1/q \neq 2 - 2/r\).

Here, we recall the famous result by Kato [24] (see also [21] for the bounded domain case) for solvability of the classical homogeneous Navier–Stokes equations \((\rho \equiv \text{const})\) with small initial velocity
\[
u_0 \in L^n(\mathbb{R}^n), \quad \text{div} \, u_0 = 0. \]

Natural question arises whether an initial condition which guarantees well-posedness for homogeneous Navier–Stokes equations will still do for inhomogeneous Navier–Stokes equations (1.1). By above mentioned previous results for inhomogeneous Navier–Stokes equations (1.1) or (1.2), the initial value \(u_0\) is allowed to be taken in some Besov spaces, which, at least, do not include \(L^n\)-space. On the other hand, in the theory of Navier–Stokes equations, it is always a question whether a statement for the whole space case can be extended to the cases of domains with boundaries, and vice versa.
Therefore, in this paper, we show the existence of a solution for (1.1) on bounded domains when
the initial velocity is in inhomogeneous Besov spaces $B_{q,\infty}^0(\Omega)$, $q \geq n$, and initial density is a piecewise
constant function. Here we recall that $L^n(\Omega) \subset B_{n,\infty}^0(\Omega)$.

Before presenting the main result of the paper we need to introduce some notations. For a linear
normed space $X$ the notation $X'$ stands for the dual space of $X$. We always denote the conjugate number
of $q \in (1,\infty)$ by $q'$, i.e. $q' = q/(q-1)$. Let $[\cdot,\cdot]_r$, $(\cdot,\cdot)_{q,r}$ and $(\cdot,\cdot)_{q,\infty}$ for $\theta \in (0,1)$, $1 \leq r \leq \infty$ be complex,
real and continuous interpolation functors, respectively, see [6,28] for real and complex interpolation
functors, and see e.g. [5], §§2.4.4, §2.5, [28] §§1.11.2, page 69 for continuous interpolation functors. We use
standard notation $L^q, H^s_q, B_{q,r}$ for Lebesgue spaces, Bessel potential spaces and Besov spaces, respectively,
without distinguishing whether or not it is the space of scalar-valued functions or vector-valued functions.
For $1 < q < \infty$ let $L^q_0(\Omega) := \{g \in L^q(\Omega) : \int_{\Omega} g \, dx = 0\}$ and let $b^q_0(\Omega)$ for $s \in \mathbb{R}$ be the little Nikolskii space
defined by the completion of $H^s_q(\Omega)$ in $B^q_{\infty}(\Omega)$. For $s > 0$, $1 < q < \infty$, let $H^s_{q,0}(\Omega)$, $B^s_{q,r,0}(\Omega)$,
$1 \leq r \leq \infty$, and $b^q_{0,0,0}(\Omega)$ denote the closures of $C_0^\infty(\Omega)$ in $H^s_q(\Omega)$, $B^s_{q,r}(\Omega)$ and $b^q_{q,\infty}(\Omega)$, respectively.
Let $L^q_{q}(\Omega)$, $H^1_{q,0,0}(\Omega)$, $1 < q < \infty$, be the closure of $C_0^{\infty}(\Omega) := \{u \in (C^{\infty}_0(\Omega))^n : \text{div } u = 0\}$ in $L^q$-norm,$H^1_q$-norm, respectively, and

$$B^0_{q,0,0,0}(\Omega) := ((H^1_{q',0,0}(\Omega)', H^1_{q,0,0}(\Omega)))_{1/2,\infty}.$$  

It is known in [26] that $B^0_{q,0,0,0}(\Omega)$ for $1 < q < \infty$ is a closed subspace of solenoidal functions of $B^0_{q,\infty}(\Omega)$
and $L^q_q(\Omega) \subset B^0_{q,0,0,0}(\Omega)$. Note that

$$B^0_{q,1} \subset L^q \subset b^0_{q,\infty} \subset B^0_{q,\infty}, \quad 1 < q < \infty.$$  

Given $\gamma \in (0,1]$, $0 < T \leq \infty$ and Banach space $X$, we introduce $L^\infty(0,T; X) := \{f : t^{1-\gamma} f \in L^\infty(0,T; X)\}$ with norm $\|f\|_{L^\infty(0,T; X)} := \|t^{1-\gamma} f(t)\|_{L^\infty(0,T; X)}$.
For a set $G$ of $\mathbb{R}^n$, $C_{\text{Lip}}(G)$ denotes the set of all Lipschitz continuous functions on $G$ and $\chi_G$ the characteristic function for $G$. We denote the
tensor product of two tensors $a, b$ by $a \otimes b$ and by $A : B = \sum_{i,j} a_{ij} b_{ij}$ for two matrices $A = (a_{ij})_{1 \leq i,j \leq n}$
and $B = (b_{ij})_{1 \leq i,j \leq n}$.

**Definition 1.1.** Let $2 \leq n \leq q < \infty$ and $0 < T \leq \infty$. We say that a pair of functions $(\rho, u)$ is a solution
to (1.1) if it satisfies the followings:

(i) $\rho \in L^\infty(0,T; L^\infty(\Omega)), \ u \in L^\infty(0,T; B^0_{q,\infty}(\Omega)) \cap L^{\infty}_{s/2}(0,T; B^{2-s}_{q,1}(\Omega))$ \hfill (1.5)

for some $s \in (0,1)$, $u|_{\partial \Omega} = 0$ and $\text{div } u = 0$.

(ii) Two identities

$$\int_0^T \int_{\Omega} (\rho \psi_t + \rho u \cdot \nabla \psi) \, dx \, dt + \int_{\Omega} \rho_0 \psi(0,\cdot) \, dx = 0, \quad \forall \psi \in C^1_0(\{0,T\} \times \Omega),$$ \hfill (1.6)

and

$$\int_0^T \int_{\Omega} [\rho u \cdot \varphi_t + \mu u \cdot \Delta \varphi + \rho u \otimes u : \nabla \varphi] \, dx \, dt + \int_{\Omega} \rho_0 u_0 \cdot \varphi(0,\cdot) \, dx = 0,$$

$$\forall \varphi \in C^\infty_0(\{0,T\} \times \Omega)^n \ (\text{div } \varphi = 0),$$ \hfill (1.7)

hold true.

If $(\rho, u)$ is a solution to (1.1) in the sense of Definition 1.1, then, by standard argument using De-
Rham’s lemma, it follows that there is a distribution $P$, associated pressure, in $\Omega$ such that $\rho, u$ and $P$
satisfy the momentum equations of (1.1) in the sense of distribution and the initial condition in (1.1)
is satisfied in a weak sense. Hence, if necessary in the below, the triple $(\rho, u, \nabla P)$ will also be called a
solution to (1.1).

The main result of the paper is stated as follows:
Theorem 1.1. Let $\Omega \subset \mathbb{R}^{n}$, $n \geq 2$, be a bounded domain of $C^{2}$-class and $0 < T \leq \infty$. Let
\[ \rho_{0}(x) = \rho_{01}\chi_{\Omega_{1}}(x) + \rho_{02}\chi_{\Omega_{2}}(x), \quad x \in \Omega, \quad \rho_{02} > \rho_{01} > 0, \]
with $\Omega_{1}$ a Lipschitz sub-domain of $\Omega$ and $\Omega_{2} = \Omega \setminus \bar{\Omega_{1}}$, and let $u_{0} \in B^{0}_{q,n,\infty}(\Omega)$ for some $q \geq n$. Then, for any $s \in \left(0, 1 - \frac{2}{q}\right)$ there are some constants $\delta_{i} = \delta_{i}(q, n, s, \Omega) > 0$, $i = 1, 2$, independent of $T$ and $\Omega_{i}$, such that if
\[ \frac{\rho_{02} - \rho_{01}}{\rho_{01}} < \delta_{1}, \quad \|u_{0}\|_{B^{0}_{q,n,\infty}(\Omega)} < \frac{\mu\delta_{2}}{\rho_{02}}, \]
then (1.1) has a solution $(\rho, u, \nabla P)$ satisfying (1.5)–(1.7) and, in addition,
\[ u_{t}, \nabla P \in L_{q/2}^{\infty}(0, T; B^{-s}_{q,1}(\Omega)). \]

The solution $(\rho, u, \nabla P)$ is unique in the class of functions satisfying (1.5) and (1.9) for $q > n$ and $s < \frac{q-n}{2q-n}$.

Remark 1.2. (i) If $(\rho, u)$ is a solution to (1.1) for $q > n$, then it follows that
\[ u \in L_{s/2}^{\infty}(0, T; B^{-s}_{q,1}(\Omega)) \subset L_{1}^{1}(0, T; W^{1,\infty}(\Omega)), \quad q > n, \]
and $\rho(t, x) = \rho_{01}\chi_{\Omega_{1}}(t) + \rho_{02}\chi_{\Omega_{2}}(t), \quad t \in (0, T), \quad x \in \Omega$, where $\Omega_{i}(t) := \{X(t, y) : y \in \Omega_{i}\}, \quad i = 1, 2$, and
\[ X(t, y) = y + \int_{0}^{t} u(\tau, X(\tau, y)) d\tau, \quad t \in (0, T), \quad y \in \Omega. \]
Note that $X(t, \cdot)$ is a $C^{1}$-diffeomorphism over $\Omega$ for each $t \in (0, T)$. Therefore, it follows by Theorem 1.1 that, if $q > n$, Lipschitz regularity ($C^{1}$-regularity, if assumed) of the initial interface of two different fluids persists for the whole time $(0, T)$. However, when $q = n$, it is not obvious for the solution whether the initial regularity of the interface will persist or not.

(ii) For the existence of a solution, the initial interface of two immiscible different fluids may be allowed to be even a fractal set, for example, a so-called $d$-set with $d \in (0, n)$ (cf. [9]), which can be verified by Remark 3.3 and by checking the proof arguments for the existence part of Theorem 1.1. In that case, a solution $(\rho, u)$ satisfying (1.5) for $s \in (0, 1 - \frac{n-d}{q})$ exists under the smallness condition (1.8).

In order to prove the main result, we shall first consider momentum equations with a prescribed variable density and transport equations in Sects. 2 and 3, respectively. In Sect. 2, existence for time-evolutionary Stokes system with nonzero divergence is proved (Theorem 2.5) relying on results of the divergence problem (Sect. 2.1) and of maximal $L^{\infty}_q$-regularity of the Stokes operator in $b^{0}_{q,\infty}(\Omega)$, $\alpha \in \mathbb{R}$, exploited in [26]. Then, unique solvability for the nonlinear momentum equations with prescribed density
\[ u_{t} - \nu\Delta u + \nabla p = a(\nu\Delta u - \nabla p) - (u \cdot \nabla)u \quad \text{in} \quad (0, T) \times \Omega, \]
\[ \text{div} u = 0 \quad \text{in} \quad (0, T) \times \Omega, \]
\[ u = 0 \quad \text{on} \quad (0, T) \times \partial\Omega, \]
\[ u(0, x) = u_{0} \quad \text{in} \quad \Omega, \]
is considered via a fixed point argument using linearization when the norms of $u_{0}$ in $B^{0}_{n,\infty}(\Omega)$ and $a \in L^{\infty}(0, T; \mathcal{M}(b^{-s}_{q,\infty}(\Omega))$ are small (Theorem 2.7). Here and in what follows, $\mathcal{M}(b^{-s}_{q,\infty}(\Omega))$ denotes the space of all pointwise multipliers for $b^{-s}_{q,\infty}(\Omega)$, more precisely,
$$\mathcal{M}(b^{-s}_{q,\infty}(\Omega)) := \{f : f\varphi \in b^{-s}_{q,\infty}(\Omega), \varphi \in b^{-s}_{q,\infty}(\Omega)\},$$
$$\|f\|_{\mathcal{M}(b^{-s}_{q,\infty}(\Omega))} := \sup_{\|\varphi\|_{b^{-s}_{q,\infty}(\Omega)} \leq 1} \|f\varphi\|_{b^{-s}_{q,\infty}(\Omega)}.$$
In Sect. 3, existence for a transport equation in $L^{\infty}(0, T; L^{\infty}(\Omega)) \cap L^{\infty}(0, T; \mathcal{M}(b^{-s}_{q,\infty}(\Omega)))$ is given (Proposition 3.2).

The proof of the main result is given in Sect. 4. An iterative scheme for (1.2) is constructed to prove existence part of Theorem 1.1 (Sect. 4.1), while uniqueness part of Theorem 1.1 is proved by Lagrangian
approach similarly as in [14, 16, 22], but we heavily use our result of Theorem 2.5 on the time-evolutionary Stokes system with nonzero divergence and the result of Lemma 4.6 on pointwise multiplication in little Nikolskii spaces \( b^{-s}_{q,\infty}(\Omega), s > 0 \) (Sect. 4.2).

Throughout the paper, we denote the estimate constants appearing inequalities by the same symbol \( c \) or \( C \) as long as no confusion arises.

2. Existence for Momentum Equations

In this section we consider existence for the momentum equations (2.24) with variable density fixed. Prior to this, we consider time-evolutionary Stokes problem with generally a nonzero divergence so that it can be used for the proof of uniqueness for (1.1) as well. To this end, we need to study the divergence problem, first.

2.1. Divergence problem

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n, n \geq 2 \), with \( C^2 \)-boundary \( \partial \Omega \). The divergence problem

\[
\text{div } u = g \quad \text{in } \Omega, \quad u|_{\partial \Omega} = 0,
\]

(2.1)

furnishes important basic tools for the theory of Navier–Stokes equations and is considerably studied in some references, see e.g. [7, 8, 18–20].

In [7], a solution operator for (2.1), so-called Bogovskii’s operator, in a star-shaped domain is constructed and, moreover, existence of a solution operator \( B \) for (2.1) in bounded Lipschitz domains satisfying

\[
\text{div } Bg = g \quad \text{and} \quad B \in \mathcal{L}(L^3_0(\Omega), H^1_{q,0}(\Omega)), 1 < q < \infty,
\]

is proved. Moreover, if \( \partial \Omega \) is smooth enough,

\[
B \in \mathcal{L}(H^{m}_{q,0}(\Omega) \cap L^3_0(\Omega), H^{m+1}_{q,0}(\Omega)), m \in \mathbb{N},
\]

(see e.g. [19], Chapter 3), and, furthermore, if \( 0 < s < 1 - 1/q \), then \( B \) has a unique extension in \( \mathcal{L}(H^{s+1}_{q,0}(\Omega)'), (H^s_{q,0}(\Omega)') \) (cf. [20]).

Note that solutions to (2.1) are not unique. In [18], Theorem 1.2 it was shown that, given \( f \in L^q(\Omega) \) and \( g \in H^1_q(\Omega) \cap L^3_0(\Omega) \), the problem

\[
\begin{align*}
-\Delta u + \nabla p &= f \quad \text{in } \Omega, \\
\text{div } u &= g \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(2.2)

has a unique solution \( u \in H^2_q(\Omega) \cap H^1_{q,0}(\Omega) \) satisfying

\[
\|u\|_{H^2_q(\Omega) \cap H^1_{q,0}(\Omega)} \leq C(\Omega)(\|f\|_{L^q(\Omega)} + \|g\|_{H^1_q(\Omega)})
\]

and, if \( f = 0 \),

\[
\|u\|_{L^q(\Omega)} \leq C(\Omega)\|g\|_{(H^1_{q,0}(\Omega) \cap L^3_0(\Omega)')}.
\]

Therefore, when \( f = 0 \) in (2.2), the operator \( \mathcal{R} : g \mapsto u \) defines another solution operator for (2.1). On the other hand, it is easily seen by standard argument that

\[
\|\mathcal{R}g\|_{H^1_{q,0}(\Omega)} \leq c\|g\|_{L^3_0(\Omega)}
\]
starting from the existence result for $L^q$-weak solution to (2.2) with $g = 0$, see [19], Theorem IV. 6.1 (b). Thus,
\[
\mathcal{R} \in \mathcal{L}(H^1_q(\Omega) \cap L^q_0(\Omega), H^2_q(\Omega) \cap H^1_{q,0}(\Omega)),
\mathcal{R} \in \mathcal{L}(L^q_0(\Omega), H^1_{q,0}(\Omega)),
\mathcal{R} \in \mathcal{L}((H^1_q(\Omega) \cap L^q_0(\Omega)), L^q_0(\Omega)).
\]

(2.3)

We shall show that $\mathcal{R}$ can be continuously extended as an operator from $(H^{s+1}_q(\Omega) \cap L^q_0(\Omega))'$ to $(H^q_s(\Omega))'$ and for any $s > 0$. More precisely, we have

**Proposition 2.1.** Let $\Omega \subset \mathbb{R}^n, n \geq 2$, be a bounded domain with $\partial \Omega \in C^2$ and $q \in (1, \infty)$. Let $F^q_s$ for $s > 0$ denote $H^s_q$ or $B^q_{s,r}(1 \leq r < \infty)$, or $b^q_{s,\infty}$ case by case. Then
\[
\mathcal{R} \in \mathcal{L}(F^q_s \cap L^q_0(\Omega), F^{1+\alpha}_q(\Omega) \cap H^1_{q,0}(\Omega)), \alpha \in (0, 1),
\]

(2.4)

If $\partial \Omega \in C^{2+s}, s > 0$, then the solution operator $\mathcal{R}$ for (2.1) can be uniquely extended as
\[
\mathcal{R} \in \mathcal{L}((F^{s+1}_q(\Omega) \cap L^q_0(\Omega))', (F^q_s(\Omega))').
\]

(2.5)

Moreover, if $\partial \Omega \in C^{3+s}$, then
\[
\mathcal{R} \in \text{Isom}((H^{s+1}_q(\Omega) \cap L^q_0(\Omega))', (H^q_s(\Omega))').
\]

(2.6)

**Proof.** We get from the first relation of (2.3), using interpolation, that
\[
\mathcal{R} \in \mathcal{L}((L^q_0(\Omega), H^1_q(\Omega) \cap L^q_0(\Omega))_\alpha, (H^2_q(\Omega), H^1_{q,0}(\Omega))_\alpha), \forall \alpha \in (0, 1),
\]

where $(\cdot, \cdot)_\alpha$ is any interpolation functor. Since $L^q_0(\Omega)$ is a subspace of $L^q(\Omega)$, which is the range of the projection operator $P^q_\alpha$ defined on $L^q(\Omega)$ by $P^q_\alpha := g - \int_{\Omega} g \, dx$, one gets by interpolation theory, see e.g. [28], Theorem 1.17.1/1, that
\[
(L^q_0(\Omega), H^1_q(\Omega) \cap L^q_0(\Omega))_\alpha = (L^q(\Omega), H^1_q(\Omega))_\alpha \cap L^q_0(\Omega), \alpha \in (0, 1).
\]

Then, by taking $(\cdot, \cdot)_\alpha$, $\alpha \in (0, 1)$, as complex, real and continuous interpolation functors, respectively, we get (2.4) in view of $[H^s_q, H^{s+1}_q]_{\alpha} = H^{1-\alpha s_1 + \alpha s_2}_q$, $(H^s_q, H^{s+1}_q)_{\alpha, r} = B^{(1-\alpha) s_1 + \alpha s_2}_q, 1 \leq r \leq \infty$, and
\[
(H^s_q, H^{s+1}_q)_{\alpha, \infty} = b^{(1-\alpha) s_1 + \alpha s_2}_q.
\]

Next, let us prove (2.5). The proof relies on a duality argument based on the regularity for the Stokes system (2.2). Consider the problem
\[
-\Delta z + \nabla \zeta = \varphi \quad \text{in} \quad \Omega,
\text{div} \, z = 0 \quad \text{in} \quad \Omega,
\]
\[
z = 0 \quad \text{on} \quad \partial \Omega.
\]

(2.7)

By the well-known regularity theory for stationary Stokes systems, see [19], for any $\varphi \in H^s_q(\Omega)$ (2.7) has a unique solution $\{z, \zeta\} \in (H^{2+s}_q(\Omega) \cap H^1_{q,0}(\Omega)) \times (H^{1+s}_q(\Omega) \cap L^q_0(\Omega))$ satisfying
\[
\|z\|_{H^{2+s}_q(\Omega)} + \|\zeta\|_{H^{1+s}_q(\Omega)} \leq c\|\varphi\|_{H^s_q(\Omega)}.
\]

Now, define the operator
\[
S \in \mathcal{L}(H^s_q(\Omega), H^{1+s}_q(\Omega) \cap L^q_0(\Omega)), \quad S \varphi := \zeta.
\]

(2.8)

If $u \in H^1_{q,0}(\Omega)$ is the solution to (2.2) with $f = 0$, $g \in L^q_0(\Omega)$, then
\[
\langle u, \varphi \rangle_{(H^s_q(\Omega))', H^s_q(\Omega)} = \langle u, \varphi \rangle_{L^q(\Omega), L^q(\Omega)} = \langle u, -\Delta z + \nabla \zeta \rangle_{L^q(\Omega), L^q(\Omega)} = \langle -\Delta u + \nabla \varphi, z \rangle_{H^{-1}_q(\Omega), H^1_{q,0}(\Omega)} - \langle \text{div} \, u, \zeta \rangle_{L^q(\Omega), L^q(\Omega)}
\]
\[
= \langle f, z \rangle_{H^{-1}_q(\Omega), H^1_{q,0}(\Omega)} - \langle g, S \varphi \rangle_{L^q(\Omega), L^q(\Omega)} = -\langle g, S \varphi \rangle_{(H^{s+1}_q(\Omega) \cap L^q_0(\Omega))', (H^{s+1}_q(\Omega) \cap L^q_0(\Omega))}, \forall \varphi \in H^s_q(\Omega);
\]

(2.9)
here we used that
\[
H^{s_1}_q(\Omega) \cap L^0_0(\Omega) \overset{d} \to L^0_0(\Omega) = (L^q_0(\Omega))' \overset{d} \to (H^{s_2}_q(\Omega) \cap L^0_0(\Omega))'
\]
(2.10)
for any \(s_1, s_2 > 0\), where \(\overset{d} \to\) means continuous and dense embedding. From (2.9) we have \(u = S'g\), where
\[
S' \in \mathcal{L}((H^{1+s}_q(\Omega) \cap L^q_0(\Omega))', (H^{s}_q(\Omega))')
\]
is the dual operator of \(S\). Thus we have \(\mathcal{R} = S'|_{L^0_0(\Omega)}\). It is clear from (2.10) that \(S'\) is the unique extension of \(\mathcal{R}\), in other words, \(\mathcal{R}\) can be uniquely extended as (2.5) with \(F^s_q \equiv H^{s}_q\). Then remaining cases of (2.5) with \(F^s_q \equiv B^s_q, r (1 \leq r < \infty)\) or \(F^s_q \equiv b^s_q, \infty\) then directly follows from the case \(F^s_q \equiv H^{s}_q\) by real and continuous interpolations, respectively, in view of the property of dual interpolation
\[
((X_1, X_2)_{\theta, r})' = (X_1', X_2')_{\theta, r}, 1 \leq r < \infty, (X_1, X_2)_{\theta, \infty}' = (X_1', X_2')_{\theta, 1}, \theta \in (0, 1),
\]
see [6], Theorem 3.7.1 or [28], Theorem 1.11.2, or [5], §2.6.

Finally, let us prove (2.6). Now, assume that \(\partial \Omega \in C^{3+s}\). By the well-known uniqueness of regular solution to (2.7) (cf. [19,27]), the operator \(S\) in (2.8) is injective. Therefore, if we prove that \(S\) is surjective, then (2.6) is proved. Given arbitrary \(y \in H^{s+1}_q(\Omega) \cap L^0_0(\Omega)\), let \(\eta_0 \in H^{3+s}_q(\Omega)\) be the (unique) solution to the problem
\[
-\Delta \eta_0 = \Delta y \text{ in } \Omega, \quad \eta_0|_{\partial \Omega} = 0,
\]
and let \(z_0 \in H^{2+s}_q(\Omega) \cap H^{1}_q,0(\Omega)\) be the (unique) solution to
\[
-\Delta z_0 = \nabla (\eta_0 - y) \text{ in } \Omega, \quad z_0|_{\partial \Omega} = 0.
\]
Note that \(\text{div } z_0 \in H^{1+s}_q(\Omega) \cap L^0_0(\Omega)\) and \(\text{div } z_0 \neq 0\), in general. Let \(z_1 = \nabla \eta_1\), where \(\eta_1 \in H^{3+s}_q(\Omega)\) is the solution to
\[
-\Delta \eta_1 = -\text{div } z_0 \text{ in } \Omega, \quad \eta_1|_{\partial \Omega} = 0,
\]
and \(z_2 \in H^{2+s}_q(\Omega)\) the solution to
\[
-\Delta z_2 + \nabla \pi = 0 \text{ in } \Omega, \quad \text{div } z_2 = 0, \quad z_2|_{\partial \Omega} = -(z_0 + z_1)|_{\partial \Omega}.
\]
Then, \(z := z_0 + z_1 + z_2\) and \(y\) solve the system (2.7) with \(\varphi = \nabla \eta_0 - \Delta z_1 - \Delta z_2 \in H^s_q(\Omega)\), that is, \(S\varphi = y\).

The proof of the lemma is complete. \(\square\)

Remark 2.2. By (2.6) of Proposition 2.1 and its proof, we can conclude that, if \(\partial \Omega\) is smooth enough, then \(\mathcal{R} = S' \in \text{Isom}(H^{s+1}_q(\Omega) \cap L^0_0(\Omega)'), (H^{s}_q(\Omega))')\), where \(S'\) is dual of the bounded linear operator \(S\) defined by (2.8). Moreover,
\[
\mathcal{R}^{-1} \in \text{Isom}((H^{s}_q(\Omega)'), (H^{s+1}_q(\Omega) \cap L^0_0(\Omega))').
\]
Note that if \(s > 1 - 1/q\), then \((H^{s}_q(\Omega))'\) is not included in the space of Schwartz distributions. Hence, \(\mathcal{R}^{-1}u\) for \(u \in (H^{s}_q(\Omega))'\), \(s > 1 - 1/q\), may not be regarded as \(\text{div } u\) in the sense of distribution.

Proposition 2.3. For \(q \in (1, \infty)\) and \(s \in (0, 1 - 1/q)\) one has
\[
\| \mathcal{R} \text{div } h \|_{b^{'s}_{q, \infty}(\Omega)} \leq c(\Omega, q)\| h \|_{b^s_{q, \infty}(\Omega)}, \forall h \in b^s_{q, \infty}(\Omega).
\]
(2.11)

Proof. For any \(h \in H^1_{q,0}(\Omega)\) and \(\psi \in H^{s+1}_q(\Omega) \cap L^0_q(\Omega)\) one has
\[
|\langle \text{div } h, \psi \rangle_{\Omega}| = |\langle h, \nabla \psi \rangle_{\Omega}| \leq \| h \|_{(H^s_q(\Omega))'} \| \psi \|_{H^{s+1}_q(\Omega) \cap L^0_q(\Omega)},
\]
which yields
\[
\| \text{div } h \|_{(H^{s+1}_q(\Omega) \cap L^0_q(\Omega))'} \leq \| h \|_{(H^s_q(\Omega))'}, \forall h \in (H^s_q(\Omega))',
\]
in view of denseness of $H^1_{q,0}(\Omega)$ in $(H^s_q(\Omega))'$. Here, $(H^s_q(\Omega))' = H^{-s}_q(\Omega)$ follows by denseness of $C^\infty_0(\Omega)$ in $H^s_q(\Omega)$ due to $s \in (0,1/q')$, cf. [28]. Therefore, by Proposition 2.1 we have
\[
\|\mathcal{R} \text{div } h\|_{H^{-s}_q(\Omega)} \leq c(\Omega, q)\|\text{div } h\|_{(H^{-s}_{q,1}(\Omega) \cap L^q_q(\Omega))'} \\
\leq c(\Omega, q)\|h\|_{H^{-s}_q(\Omega)}, \forall h \in H^{-s}_q(\Omega),
\]
(2.12)
with $c = c(\Omega, q)$. Then, (2.11) follows from (2.12) by continuous interpolation in view of
\[(H^{-s}_q(\Omega), H^{-s'_q}(\Omega))^{0,\infty} = b^{-s}_{q,\infty}(\Omega), \ s = (1-\theta)s_1 + \theta s_2, s_1, s_2 \in (0,1/q'), \theta \in (0,1).
\]
The proof of the proposition is complete.

\[
\square
\]

2.2. Time-Evolutionary Stokes Problem

Consider the initial boundary value problem for time-evolutionary Stokes system
\[
\begin{align*}
\partial_t u - \nu \Delta u + \nabla p &= f & \text{in } (0,T) \times \Omega, \\
\partial_t u &= g & \text{in } (0,T) \times \Omega, \\
\partial_n u &= 0 & \text{on } (0,T) \times \partial\Omega, \\
u(0,x) &= u_0 & \text{in } \Omega,
\end{align*}
\]
(2.13)
where $\Omega$ is a bounded domain of $\mathbb{R}^n$, $n \geq 2$, and $0 < T \leq \infty$.

**Lemma 2.4.** Let $\Omega$ be a bounded domain of $\mathbb{R}^n$, $n \geq 2$, with $C^1$-boundary, $1 < q < \infty$ and $s \in (0,1)$. Then,
\[
\|g\|_{b^{-s}_{q,\infty}(\Omega)} \leq c(\Omega, q)\|\nabla g\|_{b^{-s}_{q,\infty}(\Omega)}, \forall g \in b^{-s}_{q,\infty}(\Omega) \cap L^q_q(\Omega).
\]
(2.14)
**Proof.** By Poincaré’s inequality, one gets
\[
\|g\|_{H^s_q(\Omega)} \leq c(\Omega, q)\|\nabla g\|_{L^q_q(\Omega)}, \forall g \in H^s_q(\Omega) \cap L^q_q(\Omega).
\]
(2.15)
On the other hand, one gets
\[
\|g\|_{L^q_q(\Omega)} \leq c(\Omega, q)\|\nabla g\|_{H^{-s}_q(\Omega)}.
\]
(2.16)
In fact, for any $\psi \in L^q_q(\Omega)$ there is some $h \in H^1_{q,0}(\Omega)$ such that
\[
\text{div } h = \psi, \quad \|\nabla h\|_{q'} \leq c(q, \Omega)\|\psi\|_{q'}
\]
(see e.g. [19], Theorem III.3.1). Therefore,
\[
|\langle g, \psi \rangle| = |\langle \nabla g, h \rangle| \leq \|\nabla g\|_{H^{-1}_q(\Omega)}\|h\|_{H^{-1}_{q,0}(\Omega)} \leq c(q, \Omega)\|\nabla g\|_{H^{-1}_q(\Omega)}\|\psi\|_{q'},
\]
which yields (2.16).

Thus, using continuous interpolation
\[(L^q_q(\Omega), H^1_q(\Omega))_{1-s,\infty}^{0} = b^{-s}_{q,\infty}(\Omega), \quad (H^{-1}_q(\Omega), L^q_q(\Omega))_{1-s,\infty}^{0} = b^{-s}_{q,\infty}(\Omega),
\]
the assertion of the lemma follows from (2.15) and (2.16).

**Theorem 2.5.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain of $C^2$-class, $0 < T \leq \infty$ and $u_0 \in H^0_q(\Omega)$. Let $f \in L^q_{s/2}(0,T; b^{-s}_{q,\infty}(\Omega))$, $g \in L^q_{s/2}(0,T; b^{1-s}_{q,\infty}(\Omega) \cap L^q_q(\Omega))$, $q \in (1, \infty)$, $s \in (0,1-1/q')$, and $g = \text{div } R$ with some distribution $R = R(t, \cdot)$ such that $R_1 \in L^q_{s/2}(0,T; b^{s}_{q,\infty}(\Omega))$, $R(0) = 0$. Then the problem (2.13) has a unique solution $(u, \nabla p)$ such that
\[
\begin{align*}
u \|
abla^2 u, \nabla p\|_{L^q_{s/2}(0,T;b^{s}_{q,\infty}(\Omega))} + \nu^{s/2}\|u\|_{L^q(0,T;B^0_{q,\infty}(\Omega))} \\
\leq c(\|f, \nu \nabla g, R_1\|_{L^q_{s/2}(0,T;b^{s}_{q,\infty}(\Omega))} + \nu^{s/2}\|u_0\|_{B^0_{q,\infty}(\Omega)})
\]
(2.17)
with constant $c > 0$ depending only on $q, n, s, \Omega$ and independent of $T$.

Moreover, the solution $u$ to (2.13) satisfies

$$\|u\|_{L^\infty_{t,0,x}}(0, T; B^q_{q,1}(\Omega)) \leq c \|f\|_{L^\infty_{t,0,x}}(0, T; B^q_{q,1}(\Omega)) + \|u_0\|_{L^2_{t,0,x}}(\Omega),$$

$$\forall \theta \in (s/2, 1),$$

(2.18)

with constant $c > 0$ depending only on $q, n, s, \Omega$ and independent of $T$ and $\nu$.

**Proof.** First let us prove the theorem for $\nu = 1$.

The assertion for the case $g = 0$ follows by [26], Corollary 4.14 (ii) (with $\gamma = s/2, \alpha = -s/2$), where $n \geq 3$ is assumed, but the result of course holds true for $n = 2$ as well.

Let $g \in L^\infty_{t,0,x}(0, T; B^{1-s}_q(\Omega))$ be identically not 0 and $w(t) = Rg(t), t \in (0, T)$, where $R$ is the solution operator for (2.1) constructed through the solution to (2.2) with $f = 0$. Then, $\text{div} w(t) = g(t)$ for all $t \in (0, T)$ and it is easily checked by Proposition 2.1, (2.4) with $F^\alpha_q \equiv b^\alpha_q(\Omega), \alpha = 1 - s$, that

$$w \in L^\infty_{t,0,x}(0, T; B^{2-s}_q(\Omega) \cap H^1_{q,0}(\Omega)),$$

$$\|w\|_{L^\infty_{t,0,x}(0, T; b^{2-s}_q(\Omega))} \leq c \|g\|_{L^\infty_{t,0,x}(0, T; b^{1-s}_q(\Omega))}.$$

(2.19)

Moreover, since $w_t = Rg_t = R\text{div} R_t$ and $R_t \in L^\infty_{t,0,x}(0, T; b^{1-s}_q(\Omega))$, we get by Proposition 2.3 that

$$w_t \in L^\infty_{t,0,x}(0, T; b^{1-s}_q(\Omega)),$$

$$\|w_t\|_{L^\infty_{t,0,x}(0, T; b^{1-s}_q(\Omega))} \leq c \|R_t\|_{L^\infty_{t,0,x}(0, T; b^{1-s}_q(\Omega))}.$$

(2.20)

Furthermore, since $R(0) = 0$ and $R_t \in L^\infty_{t,0,x}(0, T; b^{1-s}_q(\Omega))$, we have $R(t) = \int_0^t \text{R}_s d\tau = \int_0^t \tau^{1-s/2}(\tau^{1-s/2}R_s) d\tau$, and

$$\|R(t)\|_{b^{1-s}_q(\Omega)} \leq \frac{2t^{s/2}}{s} \|R_t\|_{L^\infty_{t,0,x}(0, T; b^{1-s}_q(\Omega))}, \forall t \in (0, T).$$

Hence,

$$t^{-s/2}R \in L^\infty_{t,0,x}(0, T; b^{1-s}_q(\Omega)),$$

$$\|t^{-s/2}R\|_{L^\infty(0, T; b^{1-s}_q(\Omega))} \leq \frac{2}{s} \|R_t\|_{L^\infty_{t,0,x}(0, T; b^{1-s}_q(\Omega))}.$$

which yields by Proposition 2.3, (2.11) that

$$t^{-s/2}w = t^{-s/2}R \text{div} R \in L^\infty_{t,0,x}(0, T; b^{1-s}_q(\Omega)),$$

$$\|t^{-s/2}w\|_{L^\infty(0, T; b^{1-s}_q(\Omega))} \leq \frac{2}{s} \|R_t\|_{L^\infty_{t,0,x}(0, T; b^{1-s}_q(\Omega))}.$$

(2.21)

Therefore, by using complex interpolation, it follows from (2.19) and (2.21) that

$$w \in L^{1-s}_t(0, T; B^{2-s}_q(\Omega), B^{s}_q(\Omega))(2-s/2) \subset L^{1-s}_t(0, T; B^0_{q,\infty}(\Omega)),$$

$$\|w(t)\|_{L^\infty_{t,0,x}(\Omega)} \leq c(s) \|w(t)\|_{L^{2-s}_q(\Omega)} \|w(t)\|_{L^{1-s}_q(\Omega)} \leq c(s) \|g\|_{L^\infty_{t,0,x}(0, T; b^{1-s}_q(\Omega))} \|R_t\|_{L^{1-s}_t(0, T; b^{1-s}_q(\Omega))} \|R_t\|_{L^{1-s}_t(0, T; b^{1-s}_q(\Omega))},$$

$$= c(s) \|g\|_{L^\infty_{t,0,x}(0, T; b^{1-s}_q(\Omega))} \|R_t\|_{L^{1-s}_t(0, T; b^{1-s}_q(\Omega))} \text{ for a.a. } t \in (0, T),$$

where we used the interpolation relation $[B^{2-s}_q(\Omega), B^{s}_q(\Omega)](2-s/2) \subset B^0_{q,\infty}(\Omega)$ in view of $(2-s)(1-\frac{2-s}{2}) - s \cdot \frac{2-s}{2} = 0$. Thus,

$$w \in L^\infty_{t,0,x}(0, T; B^0_{q,\infty}(\Omega)),$$

$$\|w\|_{L^\infty_{t,0,x}(0, T; b^{0}_q(\Omega))} \leq c(s) \|g\|_{L^\infty_{t,0,x}(0, T; b^{1-s}_q(\Omega))} \|R_t\|_{L^{1-s}_t(0, T; b^{1-s}_q(\Omega))},$$

(2.22)
Now, introducing the new unknown $U = u - w$, the problem (2.13) is reduced to a divergence-free problem, that is,

$$
U_t - \Delta U + \nabla p = F \quad \text{in} \quad (0, T) \times \Omega,
$$

$$
div U = 0 \quad \text{in} \quad (0, T) \times \Omega,
$$

$$
U = 0 \quad \text{on} \quad (0, T) \times \partial \Omega,
$$

$$
U(0, x) = U_0 \quad \text{in} \quad \Omega,
$$

where $F := f - w_t - \Delta w \in L^{2}_{s/2}(0, T; b_{q, \infty}^{-s}(\Omega))$, $U_0 := u_0 - w(0) = u_0 \in B^{0}_{q, \infty, 0, \sigma}(\Omega)$. Thus, by the assertion of the theorem for the divergence-free case, we get that

$$
U_t, \nabla^2 U, \nabla p \in L^{2}_{s/2}(0, T; b_{q, \infty}^{-s}(\Omega)), \quad U|_{\partial \Omega} = 0, \quad U \in L^\infty(0, T; B^{0}_{q, \infty}(\Omega)),
$$

$$
||U_t, \nabla^2 U, \nabla p||_{L^{2}_{s/2}(0, T; b_{q, \infty}^{-s}(\Omega))} + ||U||_{L^\infty(0, T; B^{0}_{q, \infty}(\Omega))} \leq c(\|F\|_{L^{2}_{s/2}(0, T; b_{q, \infty}^{-s}(\Omega))} + \|U_0\|_{B^{0}_{q, \infty}(\Omega)}).
$$

Thus, in view of (2.19), (2.20), (2.22) and Lemma 2.4, we get (2.17) for $u = U + w$.

The uniqueness of solution to (2.13) is clear from (2.17).

The proof of the theorem is complete.

The following lemma for the proof of Theorem 2.7, which is the main result of this section.

**Lemma 2.6.** Let $\Omega$ be a bounded domain of $\mathbb{R}^n$, $n \in \mathbb{N}$, of $C^2$-class, then for $q \in \{n, \infty\}$ and $s \in (0, 1)$

$$
H^{2(1-a)}_q(\Omega) \cdot H^{2n-s-1}_n(\Omega) \hookrightarrow H^{-s}_q(\Omega), \quad \forall \alpha \in \left(1 - \frac{n}{2q}, 1\right),
$$

$$
H^{2(1-\beta)}_n(\Omega) \cdot H^{2\beta-s-1}_q(\Omega) \hookrightarrow H^{-s}_q(\Omega), \quad \forall \beta \in \left(1 - \frac{1}{2q}, \frac{1}{2} \frac{1 + s + n/q}{2}\right).
$$

(2.25)
Proof. By Sobolev embedding theorem, for \( \alpha \in (1 - \frac{n}{2q}, 1) \) we have

\[
H^{2(1-\alpha)}_q(\Omega) \hookrightarrow L^{p_1}(\Omega), \quad H^{2\alpha-s-1}_n(\Omega) \hookrightarrow L^{p_2}(\Omega), \quad H^{s}_{q',0}(\Omega) \hookrightarrow L^{r}(\Omega),
\]

where \( 2(1 - \alpha) - \frac{n}{q} = -\frac{n}{p_1}, 2\alpha - s - 1 - \frac{n}{n} = -\frac{n}{p_2}, s - \frac{n}{q} = -\frac{n}{r} \). Note that \( \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{r} = 1 \). Hence, by Hölder’s inequality we get the first relation of (2.25).

Similarly, the second relation of (2.25) follows. \( \square \)

**Theorem 2.7.** Let \( \Omega \subset \mathbb{R}^n, n \geq 2 \), be a bounded domain of \( C^2 \)-class and let \( 0 < T \leq \infty \). Let \( q \geq n \) and \( u_0 \in B^0_{q,0,0,0,0}(\Omega) \).

(i) If \( a \in L^\infty(0,T; \mathcal{M}(b^{-s}_{q,\infty}(\Omega)) \cap \mathcal{M}(b^{-s}_{n,\infty}(\Omega))) \) for some \( s \in (0,1 - \frac{1}{q}) \), then there are some constants \( \delta = \delta(q,n,\Omega,s) > 0 \) and \( M = M(q,n,\Omega,s) > 0 \) independent of \( T \) such that if

\[
\|a\|_{L^\infty(0,T; \mathcal{M}(b^{-s}_{q,\infty}(\Omega)) \cap \mathcal{M}(b^{-s}_{n,\infty}(\Omega)))} + \nu^{-1}\|u_0\|_{B^0_{q,0,0,0}} < \delta,
\]

then (2.24) has a solution \((u, \nabla p)\) satisfying

\[
u^{s/2}\|u\|_{L^\infty(0,T; B^0_{q,\infty}(\Omega))} + \nu^q\|u\|_{L^\infty(0,T; B^{2s-\alpha}_{q,1}(\Omega))} + \|u_t \cdot \nabla u\|_{L^\infty(0,T; B^{s}_{q,1}(\Omega))} \leq M
\]

for all \( \theta \in (s/2,1) \). The solution \((u, \nabla p)\) is unique in the class of functions satisfying (2.26) with sufficiently small \( M = M(q,n,\Omega,s) > 0 \) on the right-hand side.

(ii) If, in addition, \( a \in L^\infty(0,T; \mathcal{M}(b^{-\tilde{s}}_{q,\infty}(\Omega)) \cap \mathcal{M}(b^{-\tilde{s}}_{n,\infty}(\Omega))) \) for all \( \tilde{s} \in (0,1 - \frac{1}{q}) \), then there are constants \( \eta = \eta(q,n,\Omega,s) > 0 \) and \( N = N(q,n,\Omega,s) > 0 \) independent of \( T \) such that if

\[
\|a\|_{L^\infty(0,T; \mathcal{M}(b^{-\tilde{s}}_{q,\infty}(\Omega)) \cap \mathcal{M}(b^{-\tilde{s}}_{n,\infty}(\Omega)))} + \nu^{-1}\|u_0\|_{B^0_{q,0,0,0}} < \eta
\]

for all \( \tilde{s} \in (s - \alpha, s + \alpha) \) with sufficiently small \( \alpha > 0 \), then (2.24) has a solution \((u, \nabla p)\) satisfying

\[
\nu^{s/2}\|u\|_{L^\infty(0,T; B^0_{q,\infty}(\Omega))} + \nu^q\|u\|_{L^\infty(0,T; B^{2s-\alpha}_{q,1}(\Omega))} + \|u_t \cdot \nabla u\|_{L^\infty(0,T; B^{s}_{q,1}(\Omega))} \leq N
\]

for all \( \theta \in (s/2,1) \), with sufficiently small \( \alpha > 0 \), then (2.24) has a solution \((u, \nabla p)\) satisfying

\[
\nu^{s/2}\|u\|_{L^\infty(0,T; B^0_{q,\infty}(\Omega))} + \nu^q\|u\|_{L^\infty(0,T; B^{2s-\alpha}_{q,1}(\Omega))} + \|u_t \cdot \nabla u\|_{L^\infty(0,T; B^{s}_{q,1}(\Omega))} \leq N
\]

Proof. The proof is based on linearization and a fixed point argument. For convenience, we will omit \( \Omega \) in writing function spaces in norms.

First, let us prove (i). Consider the linear system (2.13) with arbitrarily fixed \( f \in L^\infty_s(0,T; b^{-s}_{q,\infty}(\Omega)) \)

and \( u_0 \in B^0_{q,0,0,0}(\Omega) \). Then, by Theorem 2.5 the system (2.13) has a unique solution \((u, \nabla p)\) such that

\[
\|u\|_{L^\infty(0,T; B^0_{q,\infty}(\Omega))} + \|u_t \cdot \nabla u\|_{L^\infty(0,T; B^{s}_{q,1}(\Omega))} \leq \hat{C}
\]

(2.28)
with constant $\tilde{C} > 0$ depending on $q, n, \Omega$ and $s$. Now, given $u_0 \in B^0_{q,\infty,0,\sigma}(\Omega)$, define the mapping $\Phi$ from $Y := L^\infty_{s/2}(0,T;b^{-q}_{n,\infty}(\Omega))$ to itself by

$$
\Phi f := -(u_f \cdot \nabla) u_f + a(\nu \Delta u_f - \nabla p_f),
$$

where $(u_f, \nabla p_f)$ is the unique solution to $(2.13)$ corresponding to $u_0$ and $f$.

For $f_1, f_2 \in Y$ and almost all $t \in (0,T)$ we get, using (2.28) and the first relation of (2.25) of Lemma 2.6, that

$$
\| (u_{f_1} \cdot \nabla) u_{f_2}(t) \|_{b^{-q}_{n,\infty}} \leq c \| u_{f_1}(t) \|_{H^q_{s/2}} \| \nabla u_{f_2}(t) \|_{H^{2n-q-1}_{s/2}},
$$

where

$$
\| (u_{f_2} \cdot \nabla) u_{f_2}(t) \|_{b^{-q}_{n,\infty}} \leq c \nu^{-1-s/2} t^{-1+s/2} \| f_1 \|_{L^\infty_{s/2}(0,T;b^{-q}_{n,\infty})} + \nu^{s/2} \| u_0 \|_{B^0_{q,\infty}}.
$$

Therefore,

$$
\| (u_{f_1} \cdot \nabla) u_{f_2}(t) \|_{b^{-q}_{n,\infty}} \leq c \nu^{-1-s/2} t^{-1+s/2} \| f_1 \|_{L^\infty_{s/2}(0,T;b^{-q}_{n,\infty})} + \nu^{s/2} \| u_0 \|_{B^0_{q,\infty}}.
$$

(2.29)

with $c = c(q,n,s,\Omega)$. Moreover, by (2.28), we have

$$
\| a(\nu \Delta u_f - \nabla p_f) \|_Y \leq \| a \|_{L^\infty(0,T;M(b^{-q}_{n,\infty}))} \| \nu \Delta u_f - \nabla p_f \|_{L^\infty_{s/2}(0,T;b^{-q}_{n,\infty})} \leq c \nu^{-1-s/2} K + \nu^{-1} \| u_0 \|_{B^0_{q,\infty}}.
$$

(2.30)

If we show that $\Phi$ has a fixed point $\tilde{f} \in Y$, then $(u_f, \nabla p_f)$ is obviously a solution to the system (2.24). Let

$$
G_K := \{ f \in Y : \| f \|_{L^\infty_{s/2}(0,T;b^{-q}_{n,\infty})} \leq K \}, \quad K > 0.
$$

Obviously, $G_K$ is a closed subset of $Y$ since $b^{-q}_{n,\infty}(\Omega) \hookrightarrow b^{-q}_{n,\infty}(\Omega)$. If $f \in G_K$, then, by (2.28), (2.29) and (2.30) with $q = n$,

$$
\| \Phi(f) \|_{L^\infty_{s/2}(0,T;b^{-q}_{n,\infty})} \leq C_1(K + \nu^{s/2} \| u_0 \|_{B^0_{q,\infty}}) \| a \|_{L^\infty(0,T;M(b^{-q}_{n,\infty}))} + \nu^{-1-s/2} K + \nu^{-1} \| u_0 \|_{B^0_{q,\infty}},
$$

(2.31)

where $C_1 = C_1(n,s,\Omega)$.

On the other hand, if $f_1, f_2 \in G_K$, then

$$
\Phi(f_1) - \Phi(f_2) = \left[ -((u_{f_1} - u_{f_2}) \cdot \nabla) u_{f_1} - (u_{f_2} \cdot \nabla)(u_{f_1} - u_{f_2}) \right] + [a(\nu \Delta (u_{f_1} - u_{f_2}) - \nabla (p_{f_1} - p_{f_2})] =: (I) + (II).
$$

(2.32)

Note that $u_{f_1} - u_{f_2}$ is a solution to (2.13) with zero initial value and right-hand side $f_1 - f_2$. Hence we get by (2.29) that

$$
\| (I) \|_Y \leq 2C_2 \nu^{-1-s/2} \| f_1 - f_2 \|_Y (K + \nu^{s/2} \| u_0 \|_{B^0_{q,\infty}})
$$

and, by (2.28),

$$
\| (II) \|_Y \leq C_2 \| a \|_{L^\infty(0,T;M(b^{-q}_{n,\infty}))} \| f_1 - f_2 \|_Y,
$$

where $C_2 = C_2(q,n,s,\Omega)$. Finally, we have

$$
\| \Phi(f_1) - \Phi(f_2) \|_Y \leq C_2(2\nu^{-1-s/2} K + \| a \|_{L^\infty(0,T;M(b^{-q}_{n,\infty}))} + 2\nu^{-1} \| u_0 \|_{B^0_{q,\infty}}) \| f_1 - f_2 \|_Y.
$$

(2.33)
Now, in view of (2.31) and (2.33), consider the inequality
\[
\begin{cases}
C_0 (K + \nu^{s/2} \| u_0 \|_{B^0_{\infty,n}}) (\nu^{-1-s/2}K + \|a\|_{L^\infty(0,T;\mathcal{M}(b^{-\infty,0}_{n,\infty}))} + \nu^{-1} \| u_0 \|_{B^0_{\infty,n}}) < K, \\
C_0 (2\nu^{-1-s/2}K + \|a\|_{L^\infty(0,T;\mathcal{M}(b^{-\infty,0}_{q,\infty}))} + 2\nu^{-1} \| u_0 \|_{B^0_{\infty,n}}) < 1,
\end{cases}
\]
(2.34)
where \( C_0 := \max\{C_1, C_2\} \). By elementary calculations, it follows that, if
\[
\epsilon := \|a\|_{L^\infty(0,T;\mathcal{M}(b^{-\infty,0}_{q,\infty}))} + \nu^{-1} \| u_0 \|_{B^0_{\infty,n}} < \frac{1}{4C_0},
\]
(2.35)
then for any
\[
K \in (k_1, k_2)
\]
(2.36)
with
\[
k_1 = \frac{1 - 2\epsilon C_0 - \sqrt{1 - 4\epsilon C_0}}{2C_0\nu^{-1-s/2}}, \quad k_2 = \frac{1 - 2\epsilon C_0}{2C_0\nu^{-1-s/2}}
\]
(2.37)
the inequality (2.34) holds true. In other words, if (2.35) and (2.36) are satisfied, then \( \Phi(G_K) \subset G_K \) and \( \Phi : G_K \mapsto G_K \) is a contraction mapping. Thus, by the Banach fixed point theorem \( \Phi \) has a fixed point \( \tilde{f} \) in \( G_K \), which is unique in \( G_K \), and \( u = u_f \) is a solution to (2.24).

Note that \( K < \frac{\nu^{1+s/2}}{2C_0} \). It follows from (2.28), (2.35) and (2.36) that the solution \( u \) satisfies (2.26) with \( M \equiv \frac{\tilde{C}}{8C_0} \).

Let \( K' = \frac{K}{3} \) and the norm of solution \( (u, \nabla p) \) in (2.26) is bounded by \( K' \). Then \( \|u_t - \nu\Delta u + \nabla p\|_{L^\infty(0,T;b^{-\infty,0}_{q,\infty})} \leq K \). Hence, in view of the uniqueness of the linear Stokes problem, \( (u, \nabla p) \) must be the only solution satisfying the inequality (2.26) with \( K' \) on the right-hand side.

Thus, the proof of (i) is complete.

Let us prove (ii). Tracking the above proofs, we can infer that the constant \( C_0 = C_0(q,n,\Omega,s) \) in (2.34) and hence the constants \( k_{1,2} \) in (2.37) are continuously dependent on \( s \). Denote \( k_{1,2} \) in (2.37) by \( k_{1,2}(s) \). Then, one can choose sufficiently small \( \alpha = \alpha(q,n,\Omega,s) > 0 \) so that \( \max\{k_1(s - \alpha), k_1(s + \alpha)\} < \min\{k_2(s - \alpha), k_2(s + \alpha)\} \). Let
\[
X := \left. L^\infty(s_{-\alpha}/2)(0,T;b^{-\infty,0}_{q,\infty}(\Omega)) \cap L^\infty(s_{+\alpha}/2)(0,T;b^{-\infty,0}_{q,\infty}(\Omega)) \right.
\]
where the norm is given as maximum of the two norms. We construct the mapping \( \Psi : X \mapsto X \) by
\[
\Psi f := -(u_f \cdot \nabla)u_f + a(\nu\Delta u_f - \nabla p_f),
\]
where \( (u_f, \nabla p_f) \) is the unique solution to (2.13) corresponding to \( u_0 \) and \( f \in X \). Then, it is obvious from the argument of the proof of (i) (see (2.33)–(2.36)) that \( \Psi \) maps
\[
B_K := \{ g \in X : \|g\|_{L^\infty(s_{-\alpha}/2)(0,T;b^{-\infty,0}_{q,\infty}(\Omega))} \leq K, \|g\|_{L^\infty(s_{+\alpha}/2)(0,T;b^{-\infty,0}_{q,\infty}(\Omega))} \leq K \}
\]
into \( B_K \) (note that \( B_K \) is closed in \( X \)) and becomes a contraction mapping on \( B_K \) provided that
\[
\|a\|_{L^\infty(0,T;\mathcal{M}(b^{-\infty,0}_{q,\infty}) \cap \mathcal{M}(b^{-\infty,0}_{q,\infty}))} + \nu^{-1} \| u_0 \|_{B^0_{\infty,n}} < \eta^\pm,
\]
where \( \eta^\pm = \eta^\pm(q,n,\Omega,s) > 0 \), see (2.35), and that
\[
K \in \left( \max\{k_1(s - \alpha), k_1(s + \alpha)\}, \min\{k_2(s - \alpha), k_2(s + \alpha)\} \right),
\]
see (2.36). Hence, \( \Psi \) has a unique fixed point \( \tilde{f} \) and \( (u_f, \nabla p_f) \) becomes a solution to (2.24) satisfying (2.26) with \( s = s \pm \alpha \) (and possibly with different \( M \)).

Thus, the assertion (ii) follows from (2.26) with \( s = s \pm \alpha \) by the real interpolation relation \((\cdot, \cdot)_{1/2,1}\). Here, we recall the fact that
\[
(L^\infty_\gamma(0,T;X_1), L^\infty_\gamma(0,T;X_2))_{\theta,1} \hookrightarrow L^\infty_\gamma(0,T;(X_1, X_2)_\theta,1), \gamma = (1 - \theta)\gamma_1 + \theta\gamma_2, 0 < \theta < 1,
\]
for Banach interpolation couple \((X_1, X_2)\) and that
\[
(b^{\kappa-\varepsilon}_{q,\infty}(\Omega), b^{\kappa+\varepsilon}_{q,\infty}(\Omega))_{1/2,1} = (b^{\kappa-\varepsilon}_{q,\infty}(\Omega), b^{\kappa+\varepsilon}_{q,\infty}(\Omega))_{1/2,1} = B^\kappa_{q,1}(\Omega), \kappa \in \mathbb{R},
\]
for bounded Lipschitz domain Ω which follows by [6], Theorem 3.4.2 (d), Theorem 6.4.5 (1) and Sobolev extension theorem (cf. [28], Theorem 4.3.1/1).

Thus, the assertion (ii) is proved. □

3. Transport Equation

In this section, we shall prove an existence result for the transport equation with piecewise constant initial values.

We know that the characteristic function χ(Ω') of any Lipschitz subset Ω' of Ω can be a pointwise multiplier of Besov spaces $B^s_{q,r}(Ω)$ and $b^s_{q,∞}(Ω)$, respectively, with $q ∈ (1,∞), r ∈ [1,∞), s ∈ (-1 + 1/q, 1/q)$, cf. [29], Proposition 5.1, 5.3; cf. also [30], Theorem 1.41. Here, the point is that for $q,r,s$ satisfying the above conditions the set $C^0_0(Ω')$ is dense in $B^s_{q,r}(Ω')$ and $b^s_{q,∞}(Ω')$, thus extension of $f ∈ B^s_{q,r}(Ω')$ or $f ∈ b^s_{q,∞}(Ω')$ by 0 in $Ω \\setminus Ω'$ defines a linear continuous extension with norm 1 from $B^s_{q,r}(Ω')$ to $B^s_{q,r}(Ω)$ and $b^s_{q,∞}(Ω')$ to $b^s_{q,∞}(Ω)$, respectively.

**Lemma 3.1.** Let $Ω$ be a domain of $\mathbb{R}^n$, $n ∈ \mathbb{N}$, and let $1 < q < ∞$ and $s ∈ (-1 + 1/q, 1/q)$. Then, for any Lipschitz subdomain $Ω'$ of $Ω$

$$∥χ_{Ω'} u∥_Y ≤ c∥u∥_Y, \quad ∀u ∈ Y,$$

with $c = c(q,s,Ω) > 0$ independent of $Ω'$, where $Y = B^s_{q,r}(Ω)$, $1 ≤ r < ∞$, or $Y = b^s_{q,∞}(Ω)$.

**Proof.** First, suppose that $s ∈ (0,1/q)$. Define the operator $E_0 : L^q(Ω') \to \tilde{L}^q(Ω)$ by $E_0 f := \tilde{f}$, where $\tilde{f}$ is the extension of $f$ by zero on $Ω \\setminus Ω'$. Then, obviously,

$$∥E_0 f∥_{L^q(Ω')} ≤ ∥f∥_{L^q(Ω')}, \quad ∀f ∈ L^q(Ω'),$$

$$∥E_0 f∥_{H^1_{q,0}(Ω')} ≤ ∥f∥_{H^1_{q,0}(Ω')}, \quad ∀f ∈ H^1_{q,0}(Ω').$$

On the other hand, for $r_{Ω'}$ being the restriction operator onto $Ω'$, we have

$$∥r_{Ω'} f∥_{L^q(Ω')} ≤ ∥f∥_{L^q(Ω)}, \quad ∀f ∈ L^q(Ω'),$$

$$∥r_{Ω'} f∥_{H^1_{q,0}(Ω')} ≤ ∥f∥_{H^1_{q,0}(Ω')}, \quad ∀f ∈ H^1_{q,0}(Ω').$$

Therefore, in view of $χ_{Ω'} f = E_0 r_{Ω'} f$, we have

$$∥χ_{Ω'} f∥_{L^q(Ω')} ≤ ∥f∥_{L^q(Ω)}, \quad ∀f ∈ L^q(Ω'),$$

$$∥χ_{Ω'} f∥_{H^1_{q,0}(Ω')} ≤ ∥f∥_{H^1_{q,0}(Ω')}, \quad ∀f ∈ H^1_{q,0}(Ω').$$

Note that, due to $s ∈ (0,1/q)$, we have

$$(L^q(Ω), H^1_{q,0}(Ω))_{s,r} ≅ B^s_{q,r}(Ω), \quad (L^q(Ω), H^1_{q,0}(Ω))_{s,∞} ≅ b^s_{q,∞}(Ω),$$

where “” means the equality with norm equivalence, see [4], Theorem 2.2. Thus, we get the assertion of the lemma from (3.1) using real interpolation $(\cdot,\cdot)_{s,r}, 1 ≤ r < ∞, and (\cdot,\cdot)_{s,∞}$.

The assertion of the lemma for the case $s ∈ (-1 + 1/q,0)$ follows by duality argument using the assertion for $s ∈ (0,1/q)$.

Finally, the assertion for the case $s = 0$ directly follows by interpolation. □

Based on Lemma 3.1, we can prove the following statement, where we treat not only bounded domains but also whole and half space since it may be of independent significance for the theory of transport equations.

**Proposition 3.2.** Let $Ω ⊂ \mathbb{R}^n, n ≥ 2$, be whole or half space, or a bounded domain with boundary of $C^2$-class, and let $u ∈ L^1(0,T;W^{1,∞}(Ω)), 0 < T ≤ ∞, div u = 0$ and $u|_{Ω_2} = 0$. Let $Ω_1$ be a Lipschitz subdomain of $Ω$ and $Ω_2 = Ω \\setminus Ω_1$ and $ρ_0(x) = ρ_{01}χ_{Ω_1}(x) + ρ_{02}χ_{Ω_2}(x), x ∈ Ω, 0 < ρ_{01} < ρ_{02}$. Then, the transport equation

$$ρ_t + u \cdot \nabla ρ = 0 \quad in \ (0,T) \times Ω, \quad ρ(0) = ρ_0 \quad in \ Ω,$$  \quad (3.2)
has a unique solution $\rho$ such that for all $q \in (1, \infty)$, $s \in (-1 + 1/q, 1/q)$

$$\rho \in L^\infty(0, T; L^\infty(\Omega)) \cap L^\infty(0, T; \mathcal{M}(Y))$$

and

$$\|\rho(t)\|_{L^\infty(\Omega)} = \|\rho_0\|_{L^\infty(\Omega)}, \quad \forall t \in (0, T),$$

$$\|\rho\|_{L^\infty(0, T; \mathcal{M}(Y))} \leq c \rho_{02}, \quad \|a\|_{L^\infty(0, T; \mathcal{M}(Y))} \leq c \left(\frac{\bar{\rho} - \rho_{01}}{\rho_{01}} + \frac{\rho_{02} - \bar{\rho}}{\rho_{02}}\right),$$

(3.3)

where $Y = B^{q, r}_{q, \infty}(\Omega)$, $1 \leq r < \infty$, or $Y = b^{q, \infty}_{q, \infty}(\Omega)$, $a = \frac{\bar{\rho}}{\rho} - 1$ and $c = c(q, s, \Omega)$ is independent of $\Omega_i$, $i = 1, 2$.

**Proof.** Unique existence of solution in $L^\infty(0, T; L^\infty(\Omega))$ is already proved in [10], Theorem II.3 under the assumption of the proposition; more precisely, the unique solution $\rho$ to (3.2) is expressed by $\rho(t, x) = \rho_0(X(t, \cdot)^{-1}(x))$, $\forall t \in (0, T)$, where

$$X(t, y) = y + \int_0^t u(\tau, X(\tau, y)) d\tau, \quad t \in (0, T), y \in \Omega.$$  

(3.4)

Note that $X(t, \cdot)$ for each $t > 0$ is $C^1$-diffeomorphism and measure preserving over $\Omega$ due to $u \in L^1(0, T; C_{Lip}(\Omega))$ and $\text{div } u = 0$.

Since $\rho_0(x) = \rho_01 \chi_{\Omega_1}(x) + \rho_{02} \chi_{\Omega_2}(x)$, we have

$$\rho(t, x) = \rho_{01} \chi_{\Omega_1(t)}(x) + \rho_{02} \chi_{\Omega_2(t)}(x),$$

$$a(t, x) = \frac{\bar{\rho} - \rho_{01}}{\rho_{01}} \chi_{\Omega_1(t)}(x) + \frac{\rho_{02} - \bar{\rho}}{\rho_{02}} \chi_{\Omega_2(t)}(x), \quad t \in (0, T), x \in \Omega,$$

(3.5)

where

$$\Omega_i(t) = \{X(t, y) : y \in \Omega_i\}, i = 1, 2.$$  

(3.6)

Here we used that $\Omega_1(t) \cap \Omega_2(t) = \emptyset$ for all $t \in (0, T)$ due to Lipschitz conditions on the vector field $u$. Therefore we get (3.3) by Lemma 3.1. \hfill \Box

**Remark 3.3.** In Lemma 3.1 the boundary of $\Omega'$ is allowed to be so-called a d-set with some $d \in (0, n)$ and

$$s \in \left(- (n - d)/q', (n - d)/q\right).$$

(3.7)

We recall the definition of a d-set for $0 < d \leq n$; a closed non-empty set $\Gamma$ of $\mathbb{R}^n$ is called a d-set if

$$\exists c_1, c_2 > 0 : \forall x \in \Gamma, \forall r \in (0, 1], c_1 r^d \leq \mathcal{H}^d(B(x, r) \cap \Gamma) \leq c_2 r^d,$$

where $\mathcal{H}^d$ denotes the d-dimensional Hausdorff measure on $\mathbb{R}^n$ and $B(x, r)$ stands for the open ball centered at $x$ with radius $r$ (cf. e.g. [9]). In fact, if $\Omega' \subset \mathbb{R}^n$ is a d-set with some $d \in (0, n)$, then, by [9], Corollary 2.7 it follows that $C_0^\infty(\Omega')$ is dense in $B^{q, r}_{q, \infty}(\Omega')$, $1 < q, r < \infty$ for all $s \in (0, (n - d)/q)$ and hence, by duality, for all $s$ satisfying (3.7). Consequently, by denseness of $B^{q, r}_{q, \infty}(\Omega')$ in $b^{q, \infty}_{q, \infty}(\Omega')$, we get that $C_0^\infty(\Omega')$ is dense in $b^{q, \infty}_{q, \infty}(\Omega')$.

Therefore, Proposition 3.2 will still hold true while assuming (3.7) instead of $s \in (-1 + 1/q, 1/q)$ and allowing the initial interface $\partial \Omega_1 \cap \partial \Omega_2$ to be a d-set. Then, the interface evolves with time remaining as d-sets due to the fact that $X(t, \cdot), t \geq 0$, in (3.4) is a diffeomorphism over $\Omega$ and that

$$\mathcal{H}^d(f(A)) \leq (\text{Lip}(f))^d \mathcal{H}^d(A), \quad A \subset \mathbb{R}^n,$$

for Lipschitz function $f$ holds true, where Lip($f$) is the Lipschitz constant of $f$, see e.g. [17], Section 2.4, Theorem 1.

4. **Proof of Theorem 1.1**

The procedure to prove Theorem 1.1 is twofold, i.e., existence part and uniqueness part.
4.1. Proof of Existence

Let $\Omega$ be a bounded domain with $C^2$-boundary of $\mathbb{R}^n$, $n \geq 2$, $0 < T \leq \infty$, and let $\Omega_1$ be a subdomain of $\Omega$ with Lipschitz boundary and $\Omega_2 = \Omega \setminus \Omega_1$. Suppose that

$$
\rho_0(x) = \rho_{01} \chi_{\Omega_1}(x) + \rho_{02} \chi_{\Omega_2}(x), \quad x \in \Omega, \quad 0 < \rho_{01} < \rho_{02}, \quad u_0 \in B^0_{q,\infty,0,0}(\Omega), \quad q \geq n.
$$

(4.1)

Let $\eta_m \in C^\infty(\mathbb{R}^n), m \in \mathbb{N}$, be mollifiers such that

$$
\eta_m(x) \geq 0, \quad \eta(x) = \eta(-x), \quad \text{supp} \eta_m \subset \left\{ x : |x| < \frac{1}{m} \right\}, \quad \int_{\mathbb{R}^n} \eta_m(x) \, dx = 1.
$$

For $m = 1, 2, \ldots$, let us construct an iterative scheme for (1.1) as

$$
\begin{align*}
\rho_m + (u^{(m-1)} \cdot \nabla) \rho_m &= 0, \quad \rho_m(0,x) = \rho_0(x), \quad \text{in } (0,T) \times \Omega, \\
u m - \nu \Delta u_m + (u_m \cdot \nabla) u_m + \nabla p_m &= a_m(v \Delta u_m - \nabla p_m), \quad \text{in } (0,T) \times \Omega, \\
\text{div } u_m &= 0, \quad \text{in } (0,T) \times \Omega, \\
\frac{u_m}{\rho_m} &= 0, \quad \text{on } (0,T) \times \partial \Omega, \\
u m(x) &= u_0, \quad \text{in } \Omega,
\end{align*}
$$

(4.2)

where $\nu : \frac{u}{\rho}$ with $\tilde{\rho} \in (\rho_{01}, \rho_{02})$ fixed, $u^{(0)} \equiv 0$ and $a_m(t,x) := \frac{\tilde{\rho}}{\rho_m(t,x)} - 1$, $u^{(m)} = \eta_m \ast \bar{u}_m|_{\Omega}$, where “$\ast$” means the convolution,

$$
\bar{u}_m = \begin{cases} 
  u_m & \text{for } x \in \Omega \text{ with dist}(x, \partial \Omega) > \frac{1}{m} \\
  0 & \text{for else } x \in \mathbb{R}^n
\end{cases}
$$

for $m \in \mathbb{N}$. Obviously, $u^{(m)}|_{\partial \Omega} = 0, m \in \mathbb{N}$.

Remark 4.1. Let $\Omega$ be a bounded domain of $\mathbb{R}^n, n \in \mathbb{N}$, with Lipschitz boundary. Denoting by $\tilde{w}$ the extension of $w$ by zero in $\mathbb{R}^n \setminus \Omega$, for $w \in b^s_{q,\infty}(\Omega), q \in (1, \infty), s \in (-1+1/q, 1/q)$ one gets $\tilde{w} \in b^s_{q,\infty}(\mathbb{R}^n)$ and $\|\tilde{w}\|_{b^s_{q,\infty}(\mathbb{R}^n)} \leq c(q,s,\Omega)\|w\|_{b^s_{q,\infty}(\Omega)}$.

On the other hand, for $f \in H^1_q(\mathbb{R}^n)$ (or $H^{-1}_q(\mathbb{R}^n)$) it holds $\|\eta_m \ast f - f\|_{H^1_q(\mathbb{R}^n)} \to 0$ (or $\|\eta_m \ast f - f\|_{H^{-1}_q(\mathbb{R}^n)} \to 0$) as $m \to \infty$. Hence, by the Banach-Steinhaus theorem one gets uniform boundedness with respect to $m \in \mathbb{N}$ of the operator norms of convolution operators $\eta_m \ast$ in $L(H^1_q(\mathbb{R}^n)) \cap L(H^{-1}_q(\mathbb{R}^n))$ and, consequently, in $L(b^1_{q,\infty}(\mathbb{R}^n)) \cap L(b^{1-2\theta}_{q,\infty}(\mathbb{R}^n))$, $\theta \in (0,1)$, due to $(H^1_q(\mathbb{R}^n), H^{-1}_q(\mathbb{R}^n))_{\theta,\infty} = b^{1-2\theta}_{q,\infty}(\mathbb{R}^n)$ and $(H^1_q(\mathbb{R}^n), H^{-1}_q(\mathbb{R}^n))_0 = b^{1}_{q,\infty}(\mathbb{R}^n)$.

Therefore it follows that

$$
(w \mapsto (\eta_m \ast \tilde{w})|_{\Omega}) \in L(b^s_{q,\infty}(\Omega), b^s_{q,\infty}(\mathbb{R}^n)), \forall s \in (-1+1/q, 1/q),
$$

$$
(\eta_m \ast \tilde{w})|_{\Omega} \in C\|w\|_{b^s_{q,\infty}(\Omega)}, \forall w \in b^s_{q,\infty}(\Omega),
$$

(4.3)

with $C = C(q,s,\Omega) > 0$ independent of $m \in \mathbb{N}$.

Moreover, it follows that

$$
(\eta_m \ast \tilde{w})|_{\Omega} \to w \text{ in } b^s_{q,\infty}(\Omega), \forall w \in b^s_{q,\infty}(\Omega), -1 + 1/q < s < 1/q, \text{ (as } m \to \infty). \quad (4.4)
$$

In fact, if $w \in H^1_{q,0}(\Omega)$, then $(\eta_m \ast \tilde{w})|_{\Omega}$ tends to $w$ in $H^1_{q,0}(\Omega)$ and $H^{-1}_q(\Omega)$, respectively, hence in $b^s_{q,\infty}(\Omega), -1 < s < 1$, by continuous interpolation. Thus we get (4.4), in view of (4.3) and denseness of $H^1_{q,0}(\Omega)$ in $b^s_{q,\infty}(\Omega), -1 < s < 1/q$.

Furthermore, it follows that if $u_m \in L^\infty_{s/2}(0,T; b^{2-s}_{q,\infty}(\Omega)), q \geq n$, for some $s \in (0,2)$, then $u^{(m)}(t) \in C^\infty(\mathbb{R}^n)$, supp $u^{(m)}(t) \subset \bar{\Omega}$ for almost all $t \in (0,T)$, and, in particular,

$$
u^{(m)} \in L^1_{loc}(0,T, W^{1,\infty}(\Omega)).
$$

(4.5)

We have the following lemma.
Lemma 4.2. Let \( \Omega \) be a bounded domain with \( C^2\) boundary of \( \mathbb{R}^n \), \( n \geq 2 \), and \( 0 < T \leq \infty \). Suppose that (4.1) holds. Then for any \( s \in (0, 1 - \frac{1}{q}) \) there are some constants \( \delta_i = \delta_i(q, n, s, \Omega) > 0 \), \( i = 1, 2 \), and \( M = M(q, n, s, \Omega) > 0 \) independent of \( m \in \mathbb{N} \) and \( \Omega_i, i = 1, 2 \), with the following property: If

\[
\frac{\rho_{02} - \rho_{01}}{\rho_{01}} < \delta_1, \quad \|u_0\|_{B^0_{n, \infty}(\Omega)} < \delta_2 \nu,
\]

the iterative system (4.2) has a solution \( \{(\rho_m, u_m, \nabla p_m) : m \in \mathbb{N}\} \) satisfying

\[
\rho_m \in L^\infty(0, T; L^\infty(\Omega)) \cap L^\infty(0, T; \mathcal{M}(Y)),
\]

\[
\|\rho_m\|_{L^\infty(0, T; L^\infty(\Omega))} = \|\rho_0\|_{L^\infty(\Omega)},
\]

\[
\|\rho_m\|_{L^\infty(0, T; \mathcal{M}(Y))} \leq c(\bar{q}, \bar{s}, \Omega)\rho_{02},
\]

\[
\|a_m\|_{L^\infty(0, T; \mathcal{M}(Y))} \leq c(\bar{q}, \bar{s}, \Omega) \left( \frac{\hat{\rho} - \rho_{01}}{\rho_{01}} + \frac{\rho_{02} - \hat{\rho}}{\rho_{02}} \right),
\]

(4.6)

where \( Y = B_{\bar{q}_i, \bar{s}}(\Omega) \), \( 1 \leq r < \infty \), or \( Y = B_{\bar{q}_i, \bar{s}}(\Omega) \) for \( \bar{q} \in (1, \infty) \), \( \bar{s} \in (-1 + 1/\bar{q}, 1/\bar{q}) \), and

\[
u^{s/2} = \|u_m\|_{L^\infty(0, T; B^0_{q, \infty}(\Omega))} + \nu^{q} \|u_m\|_{L^\infty(0, T; B^2_{q, 1-s}(\Omega))} + \|u_m\|_{L^\infty(0, T; B^2_{q, 1-s}(\Omega))} \]

(4.7)

with the estimate

\[
\nu^{s/2} \leq M \nu^{1+s/2}, \quad \forall \theta \in (s/2, 1].
\]

(4.8)

Proof. This lemma follows directly by Theorem 2.7 and Proposition 3.2. \( \square \)

Remark 4.3. Let \( \Omega \) be a bounded domain with \( C^2\) boundary of \( \mathbb{R}^n \), \( n \geq 2 \), \( 0 < T \leq \infty \) and let \( s \in (0, 1 - \frac{1}{q}) \). Let \( v \in L^\infty_{1-\theta+s/2}(0, T; B^2_{q, 1-s}(\Omega)) \) for all \( \theta \in (s/2, 1) \). Then, for any \( q_1 \in (1, \frac{q(n+2)}{n}) \) and

\[
\theta_1 \in \left( \frac{s}{2} + \frac{1}{2} \left( \frac{n}{q} - \frac{n}{q_1} \right), \frac{s}{2} + \frac{n}{q(n+2)} \right),
\]

(4.9)

it follows that

\[
v \in L^\infty_{1-\theta_1+s/2}(0, T; B^2_{q_1, 1-s}(\Omega)) \hookrightarrow L^{q(n+2)/n}_{1-\theta_1+s/2}(0, T; B^2_{q_1, 1-s}(\Omega)) \hookrightarrow L^2(Q_{T'}),
\]

(4.10)

for any \( T' < T \), where \( Q_{T'} = (0, T') \times \Omega \), and

\[
\|v\|_{L^{q(n+2)/n}_{1-\theta_1+s/2}(0, T; B^2_{q_1, 1-s}(\Omega))} \leq c(T')\|v\|_{L^2_{1-\theta_1+s/2}(0, T; B^2_{q_1, 1-s}(\Omega))},
\]

(4.11)

in view of \( \frac{q(n+2)}{n} \left( \theta_1 - \frac{s}{2} \right) < \theta_1 \) and \( B^2_{q_1, 1-s}(\Omega) \hookrightarrow L^2(\Omega) \).

Similarly, for any \( q_2 \in (1, \frac{q(n+2)}{n+q}) \) and \( \theta_2 \in \left( \frac{s+1}{2} + \frac{1}{2} \left( \frac{n}{q} - \frac{n}{q_2} \right), \frac{s+1}{2} + \frac{n+q}{q(n+2)} \right) \), it follows that

\[
\nabla v \in L^{2^2}(Q_{T'}), \quad \|\nabla v\|_{L^{2^2}(Q_{T'})} \leq c(T')\|v\|_{L^{2^2}_{1-\theta_2+s/2}(0, T; B^{2^2}_{q_2, 1-s}(\Omega))},
\]

(4.12)

for any \( T' < T \). Since \( n \geq 2 \), one has \( \left( \frac{q(n+2)}{n} \right)^2 = \frac{q(n+2)}{q(n+2)-n} < \frac{q(n+2)}{n+q} \). Hence, it follows by (4.12) that

\[
\nabla v \in (L^{q(n+2)/n-\delta}(Q_{T'}))^\prime, \quad \|\nabla v\|_{(L^{q(n+2)/n-\delta}(Q_{T'}))^\prime} \leq c(T')\|v\|_{L^2_{1-\theta_2+s/2}(0, T; B^{2^2}_{q_2, 1-s}(\Omega))},
\]

(4.13)

for sufficiently small \( \delta > 0 \) and for any \( T' < T \).

Remark 4.4. Let \( q \geq n \geq 2 \), \( s \in (0, 1 - \frac{1}{q}) \). Let \( \{(\rho_m, u_m, \nabla p_m) : m \in \mathbb{N}\} \) be solutions to the iterative system (4.2) whose existence is guaranteed by Lemma 4.2. Then, since \( u_m \in L^\infty_{1-\theta+s/2}(0, T; B^2_{q_1, 1-s}(\Omega)) \), it follows by (4.3) of Remark 4.1 that

\[
u^{(m)} \in L^\infty_{1-\theta+s/2}(0, T; B^2_{q_1, 1-s}(\Omega)), \quad \forall \theta \in (s/2, 1], m \in \mathbb{N}.
\]
Therefore, using $\text{div } u^{(m-1)} = 0$, $u^{(m-1)}|_{\partial \Omega} = 0$, $m \in \mathbb{N}$, it follows that if $\theta \in (s/2, s/2 + 1/(2q))$, then for $\varphi \in C^\infty(\Omega)$ and almost all $t \in (0, T)$

$$\left| \int_{\Omega} (u^{(m-1)}(t) \cdot \nabla) \rho_m(t) \varphi \, dx \right| = \left| \int_{\Omega} \rho_m(t) u^{(m-1)}(t) \cdot \nabla \varphi \, dx \right|$$

$$\leq \|u^{(m-1)}(t)\|_{B_{q,1}^{2q-\theta}(\Omega)} \|\rho_m(t)\|_{B_{q,1}^{2q-\theta}(\Omega)} \|\nabla \varphi\|_{B_{q,1}^{2q-\theta}(\Omega)}$$

$$\leq \|u^{(m-1)}(t)\|_{B_{q,1}^{2q-\theta}(\Omega)} \|\rho_m(t)\|_{M(b_{q,1}^{2q-\theta}(\Omega))} \|\varphi\|_{b_{q,1}^{2q-\theta}(\Omega)};$$

here note that $(B_{q,1}^{2q-\theta}(\Omega))' = B_{q,1}^{2q-\theta}(\Omega)$ thanks to $0 < 2\theta - s < 1/q$ (cf. [28], Theorem 4.3.2/1 and (2.6.2) of [5], §2.6). Thus, in view of $\rho_{mt} = -(u^{(m-1)} \cdot \nabla) \rho_m$, $u^{(m-1)} \in L_{1-\theta+s/2}^{\infty}(0, T; B_{q,1}^{2q-\theta}(\Omega))$ and $\rho_m \in L^{\infty}(0, T; M(b_{q,1}^{2q-\theta}(\Omega)))$, it follows that $t^{\theta-s/2} \rho_{mt} \in L^{\infty}(0, T; H_{q,1}^{1,2q-\theta}(\Omega))$, $m \in \mathbb{N}$, and, in particular,

$$\rho_{mt} \in L^{p}_{\text{loc}}([0, T), H_{q,1}^{1,2q-\theta}(\Omega)), \forall \theta \in \left( \frac{s}{2}, \frac{s}{2} + \frac{1}{2q} \right), \forall p \in \left( 1, \frac{2q}{2\theta - s} \right), m \in \mathbb{N}. \quad (4.14)$$

**Proof of Theorem 1.1: Existence Part.** Let $s \in (0, 1 - 1/q)$ and let $\{\rho_m, u_m, \nabla p_m : m \in \mathbb{N}\}$ be the solutions to the iterative system (4.2), the existence of which is given by Lemma 4.2. Then, by Lemma 4.2, $\{u_m\}$ is bounded in

$$L^{\infty}(0, T; B_{q,1}^{0}(\Omega)) \cap L_{1-\theta+s/2}^{\infty}(0, T; B_{q,1}^{2q-\theta}(\Omega)), \forall \theta \in (s/2, 1],$$

$\{u_{mt}, \nabla p_m\}$ is bounded in $L_{s/2}^{\infty}(0, T; b_{q,1}^{s}(\Omega))$ and $\{\rho_m\}$ is bounded in $L^{\infty}(Q_T)$. Hence, it follows by standard arguments that $\{u_m\}$ and $\{\rho_m\}$ have some subsequences $\{u_{mk}\}$ and $\{\rho_{mk}\}$, respectively, such that

$$u_{mk} \rightharpoonup u \quad \text{in} \quad L^{\infty}(0, T; B_{q,1}^{0}(\Omega))^{n} \quad (\text{*-weakly as } k \to \infty),$$

$$u_{mk} \to u \quad \text{in} \quad L_{s/2}^{\infty}(0, T; B_{q,1}^{s}(\Omega))^{n} \quad (\text{-weakly as } k \to \infty),$$

$$\nabla^2 u_{mk} \to \nabla^2 u \quad \text{in} \quad L_{s/2}^{\infty}(0, T; B_{q,1}^{1}(\Omega))^{n} \quad (\text{-weakly as } k \to \infty),$$

$$u_{mt,k} \to u_t \quad \text{in} \quad L_{s/2}^{\infty}(0, T; B_{q,1}^{s}(\Omega))^{n} \quad (\text{-weakly as } k \to \infty),$$

$$\rho_{mk} \to \rho \quad \text{in} \quad L^{\infty}(0, T; L^{\infty}(\Omega)) \quad (\text{-weakly as } k \to \infty)$$

for some $u$, $\rho$ and distribution $P$, where note that $B_{q,1}^{0}(\Omega) = (B_{q,1}^{0}(\Omega))'$ and

$$B_{q,1}^{1}(\Omega) \cap B_{q,1}^{s}(\Omega) = (b_{q,1}^{s}(\Omega) + B_{q,1}^{s-1}(\Omega))',$$

in view of $B_{q,1}^{s}(\Omega) = (b_{q,1}^{s}(\Omega))'$ thanks to $s \in (0, 1/q)$ and $B_{q,1}^{s}(\Omega) = (b_{q,1}^{s}(\Omega))'$, see e.g. page 40–41 of [4]. Moreover, it follows from (4.10), (4.11) that

$$u_{mk} \rightharpoonup u \quad \text{in} \quad L^{q(n+2)/n}(0, T'; H_{q,1}^{2(q-\theta)}(\Omega))^{n} \quad (\text{weakly as } k \to \infty) \quad (4.16)$$

for all $\theta_1 \in (s/2, 1)$ satisfying (4.9) and for any $T' < T$.

We shall show that $\rho, u, \nabla P$ is a solution to (1.1) satisfying the assertion of Theorem 1.1.

By (4.15), obviously, $(\rho, u, \nabla P)$ satisfies (1.5) and (1.9).

Next, in order to prove (1.7), note that by (4.7), (4.8) the sequence $\{u_{mt,k}\}$ weakly converges in $L^{\alpha}(0, T'; H_{q,1}^{s}(\Omega))$ for some $\alpha > 1$ and for any $T' < T$. Therefore, in view of (4.16) and compact embedding $H_{q,1}^{s}(\Omega) \hookrightarrow L^{q(n+2)/n-\delta}(\Omega)$ for sufficiently small $\delta > 0$, we get by a compactness theorem ([27], Ch.3, Theorem 2.1) that

$$u_{mk} \to u \quad \text{in} \quad L^{q(n+2)/n-\delta}(Q_{T'}) \quad (4.17)$$

as $k \to \infty$. Moreover, we get from (4.13) that $\{\nabla u_{mk}\}$ is bounded in $(L^{q(n+2)/n-\delta}(Q_{T'}))'$.

Rewriting the second equation of the system (4.2), we have

$$u_{mt,k} - \frac{\mu}{\rho_{mk}} \Delta u_{mk} + (u_{mk} \cdot \nabla) u_{mk} + \frac{\rho}{\rho_{mk}} \nabla p_{mk} = 0,$$
which is equivalent to
\[ \rho_m u_{mk,t} - \mu \Delta u_{mk} + \rho_m (u_{mk} \cdot \nabla)u_{mk} + \bar{\rho} \nabla p_{mk} = 0 \] (4.18)
in view of
\[ \rho_m \in L^\infty(0,T; \mathcal{M}(b^{-s}_{q,\infty}(\Omega))), \ u_{mk,t}, (u_{mk} \cdot \nabla)u_{mk}, \nabla p_{mk} \in L^\infty_{s/2}(0,T; b^{-s}_{q,\infty}(\Omega)) \]
by Lemma 4.2 and \( B^{-s}_{q,1} \hookrightarrow b^{-s}_{q,\infty} \). In view of the fact that each term of (4.18) belongs to \( L^1_{loc}([0,T), b^{-s}_{q,\infty}(\Omega)) \) by (4.7) and \( (b^{-s}_{q,\infty}(\Omega))' = B^{s'}_{q',1}(\Omega) \), we get by testing (4.18) with arbitrary \( \varphi \in C^\infty_0([0,T) \times \Omega)^n \), \( \div \varphi = 0 \), that
\[ \int_0^T \langle \rho_m u_{mk,t} - \mu \Delta u_{mk} + \rho_m (u_{mk} \cdot \nabla)u_{mk}, \varphi \rangle_{b^{-s}_{q,\infty}(\Omega), B^{s'}_{q',1}(\Omega)} \, dt = 0. \]
Let \( \tilde{T} < T \) be such that \( \text{supp } \varphi \subset Q_{\tilde{T}} \). By (4.7) of Lemma 4.2 we have
\[ u_m \in L^\infty_{1-\tau+s/2}(0,T; B^{2\tau-s}_{q,1}(\Omega)), \ \forall m \in \mathbb{N}, \]
with \( \tau = s - \theta + 1/2 \) provided \( s - 1/2 < \theta < (s + 1)/2 \). Hence, if \( (\tau - s/2)p' < \frac{1}{2} \), then, if \( p > 2/(2\theta - s + 1) \),
\[ u_{mk} \in L^p(0,\tilde{T}; H^{1-2\theta-s}_{0,0}(\Omega)) \subset L^{p'}(0,\tilde{T}; H^{1-2\theta+s}_{q',0}(\Omega)). \] (4.19)
On the other hand, by (4.14), we have \( \rho_{mk} \varphi \in L^p(0,\tilde{T}; H^{1+2\theta-s}_{q,0}(\Omega)) \) for all \( p < 2/(2\theta - s) \), \( \theta \in (s/2, s/2 + 1/2q) \). Therefore, if
\[ p \in \left( \frac{2}{2\theta - s + 1}, \frac{2}{2\theta - s} \right), \ \theta \in \left( \frac{s}{2}, \frac{s + 1}{2} \right), \]
then we have
\[ \int_0^T \langle \rho_{mk} u_{mk,t}, \varphi \rangle_{b^{-s}_{q,\infty}(\Omega), B^{s'}_{q',1}(\Omega)} \, dt = -\langle u_{mk}, \rho_{mk} \varphi \rangle_{L^{p'}(0,\tilde{T}; H^{1-2\theta+s}_{q',0}(\Omega)), L^p(0,\tilde{T}; H^{-1+2\theta-s}_{q,0}(\Omega))}
\[ - \int_0^T \int_\Omega \rho_{mk} u_{mk} \cdot \varphi_t \, dx \, dt - \int_\Omega \rho_0 u_0 \cdot \varphi(0,\cdot) \, dx. \]
Moreover, for \( \rho \) and \( \theta \) satisfying (4.20) we have
\[ \int_0^T \langle (u_{mk} \cdot \nabla)u_{mk}, \varphi \rangle_{b^{-s}_{q,\infty}(\Omega), B^{s'}_{q',1}(\Omega)} \, dt = -\int_0^T \int_\Omega \rho_{mk} u_{mk} \otimes u_{mk} \cdot \nabla \varphi \, dx \, dt
\[ -\langle u_{mk}, (u_{mk} \cdot \nabla) \rho_{mk} \varphi \rangle_{L^{p'}(0,\tilde{T}; H^{1-2\theta+s}_{q',0}(\Omega)), L^p(0,\tilde{T}; H^{-1+2\theta-s}_{q,0}(\Omega))}, \]
in view of (4.19) and the fact that \( (u_m \cdot \nabla) \rho_m \in L^p(0,\tilde{T}; H^{1+2\theta-s}_{q,0}(\Omega)) \) holds true since \( (u_m \cdot \nabla) \rho_m = \div (\rho_m u_m) \) and \( \rho_m u_m \in L^p(0,\tilde{T}; H^{1+2\theta-s}_{q,0}(\Omega)) \) due to \( \rho_m \in L^\infty(0,T; \mathcal{M}(B^{2\theta-s}_{q,1}(\Omega))), u_m \in L^p(0,\tilde{T}; B^{2\theta-s}_{q,1}(\Omega)) \) by Lemma 4.2.

Therefore, we have
\[ 0 = \int_0^T \int_\Omega \left[ \rho_{mk} u_{mk} \cdot \varphi_t + \mu u_{mk} \cdot \Delta \varphi + \rho_{mk} u_{mk} \otimes u_{mk} \cdot \nabla \varphi \right] \, dx \, dt
\[ + \langle u_{mk}, (u_{mk} \cdot \nabla) \rho_{mk} \varphi \rangle_{L^{p'}(0,\tilde{T}; H^{1-2\theta+s}_{q',0}(\Omega)), L^p(0,\tilde{T}; H^{-1+2\theta-s}_{q,0}(\Omega))}
\[ = \int_0^T \int_\Omega \left[ \rho_{mk} u_{mk} \cdot \varphi_t + \mu u_{mk} \cdot \Delta \varphi + \rho_{mk} u_{mk} \otimes u_{mk} \cdot \nabla \varphi \right] \, dx \, dt
\[ + \langle u_{mk}, ((u_{mk} - u^{(m_1-1)} \cdot \nabla) \rho_{mk} \varphi \rangle_{L^{p'}(0,\tilde{T}; H^{1-2\theta+s}_{q',0}(\Omega)), L^p(0,\tilde{T}; H^{-1+2\theta-s}_{q,0}(\Omega))}. \] (4.21)
Lemma 4.5. Let $p$ and $\theta$ satisfy (4.20) and put

$$R_k := (u_{m_k} - u^{(m_k-1)}) \cdot \nabla \rho_{m_k} \varphi \big|_{L^p(0, T; H^{-1+2s}_\psi, _Q)}$$

where $\varphi \in C_0^\infty((0, T) \times \Omega)$, $\text{div} \varphi = 0$, supp $\varphi \subset Q_T$. Then $R_k$ tends to 0 as $k \to \infty$.

Proof. By Remark 4.3 and Lemma 4.2 we get that for any sufficiently small $\delta > 0$

$$|R_k| = \|\rho_{m_k} (u_{m_k} - u^{(m_k-1)}) \cdot \nabla (u_{m_k} \cdot \varphi)\|_{L^\infty(Q_T)}$$

$$\leq \|\rho_0\|_{L^\infty(\Omega)} \|u_{m_k} - u^{(m_k-1)}\|_{L^{q(n+2)/n-\delta}(Q_T)} \|\nabla (u_{m_k} \cdot \varphi)\|_{L^{q(n+2)/n-\delta}(Q_T)}$$

Then, since $\|\nabla (u_{m_k} \cdot \varphi)\|_{L^{q(n+2)/n-\delta}(Q_T)}$ is bounded with respect to $k \in \mathbb{N}$, see (4.13), the proof of the lemma is complete if we show that

$$\|u_{m_k} - u^{(m_k-1)}\|_{L^{q(n+2)/n-\delta}(Q_T)} \to 0 \quad (k \to \infty). \quad (4.22)$$

Note that

$$\|u_{m_k} - u^{(m_k-1)}\|_{L^{q(n+2)/n-\delta}(Q_T)} \leq \|u_{m_k} - u\|_{L^{q(n+2)/n-\delta}(Q_T)} + \|u^{(m_k-1)} - u\|_{L^{q(n+2)/n-\delta}(Q_T)},$$

where the first term on the right-hand side tends to 0 as $k \to \infty$ due to (4.17).

In order to show

$$\|u^{(m_k-1)} - u\|_{L^{q(n+2)/n-\delta}(Q_T)} \to 0 \quad (k \to \infty), \quad (4.23)$$

we write

$$u^{(m_k-1)} - u = \eta_{m_k-1} \ast (\tilde{u}_{m_k-1} - \tilde{u})|_{\Omega} + \eta_{m_k-1} \ast (\tilde{u} - \tilde{u})|_{\Omega}.$$ 

Here, $L^{q(n+2)/n-\delta}(Q_T)$-norm of $(\eta_{m_k-1} \ast \tilde{u} - \tilde{u})|_{\Omega}$ obviously tends to 0 as $k \to \infty$. The $L^{q(n+2)/n-\delta}(Q_T)$-norm of $\eta_{m_k-1} \ast (\tilde{u}_{m_k-1} - \tilde{u})|_{\Omega}$ goes to zero as $k \to \infty$ since $\|\eta_{m_k} \ast \|_{L^q(L^{q(n+2)/n-\delta}(Q_T))}$ is uniformly bounded with respect to $k \in \mathbb{N}$ and

$$\|\tilde{u}_{m_k} - \tilde{u}\|_{L^{q(n+2)/n-\delta}(Q_T)} \leq \|u_{m_k} - u\|_{L^{q(n+2)/n-\delta}(0, T) \times \Omega_{m_k}'} + \|u\|_{L^{q(n+2)/n-\delta}(0, T) \times (\Omega \setminus \Omega_{m_k}')} \to 0$$

as $k \to \infty$, where $\Omega_{m_k}' = \{x \in \Omega: \text{dist}(x, \partial \Omega) \geq \frac{1}{m_k}\}$. Thus, (4.23) and, consequently, (4.22) are proved. The proof of the lemma comes to end. \qed

Let us continue the proof of existence part of Theorem 1.1. In (4.21), we get easily that

$$\int_0^T \int_{\Omega} (\rho_{m_k} u_{m_k} \cdot \varphi_t + m u_{m_k} \cdot \Delta \varphi) \, dx \, dt \to \int_0^T \int_{\Omega} (\rho u \cdot \varphi_t + m u \cdot \Delta \varphi) \, dx \, dt$$

as $k \to \infty$ due to $*$-weak convergence $\rho_{m_k} \to \rho$ in $L^\infty(Q_T)$ and strong convergence $u_{m_k} \to u$ in $L^{q(n+2)/n-\delta}(Q_T)$ for any sufficiently small $\delta > 0$, see (4.15) and (4.17). Note that $q(n+2)/n - \delta \geq 2$ and hence $u_{m_k} \otimes u_{m_k} \to u \otimes u$ in $L^1(Q_T)$ as $k \to \infty$. Therefore,

$$\int_0^T \int_{\Omega} \rho_{m_k} (u_{m_k} \otimes u_{m_k}) \cdot \nabla \varphi \, dx \, dt \to \int_0^T \int_{\Omega} \rho (u \otimes u) \cdot \nabla \varphi \, dx \, dt \quad \text{as} \quad k \to \infty.$$ 

Thus, letting $k \to \infty$ in (4.21), it follows that $(\rho, u)$ satisfies (1.7).

Finally, in order to show (1.6), test the first equation of (4.2) with $\psi \in C_0^1((0, T) \times \Omega)$ to get

$$\int_0^T \int_{\Omega} (\rho_{m_k} \psi_t + \rho_{m_k} u^{(m_k-1)} \cdot \nabla \psi) \, dx \, dt + \int_\Omega \rho_0 \psi(0, \cdot) \, dx = 0, \quad \forall \psi \in C_0^1((0, T) \times \Omega). \quad (4.24)$$

Obviously,

$$\int_{Q_T} \rho_{m_k} \psi_t \, dx \, dt \to \int_{Q_T} \rho \psi_t \, dx \, dt \quad (k \to \infty).$$
Moreover, for all $\psi \in C^0_0([0,T) \times \Omega)$ we get in view of (4.23) and (4.15) that
\[
\left| \int_{Q_T} (\rho_{mk} u^{(m_k-1)} - \rho u) \cdot \nabla \psi \, dx \, dt \right| \\
\leq \left| \int_{Q_T} (u^{(m_k-1)} - u) \cdot (\rho_{mk} \nabla \psi) \, dx \, dt \right| + \left| \int_{Q_T} (\rho_{mk} - \rho) u \cdot \nabla \psi \, dx \, dt \right| \rightarrow 0
\]
as $k \to \infty$. Therefore, $(\rho, u)$ satisfies (1.6) in the limiting case $k \to \infty$ in (4.24).

Thus, the proof of existence part of Theorem 1.1 is completed. $\square$

### 4.2. Proof of Uniqueness

For the proof of uniqueness part of Theorem 1.1 we need the following statement.

**Lemma 4.6.** Let $\Omega$ be a bounded Lipschitz domain of $\mathbb{R}^n$, $n \in \mathbb{N}$, and let $q > n$ and $s \in (0, \frac{q-n}{2q-n})$. Then there holds the following statements:

(i) \[
\|f\varphi\|_{B_q^{1-s}((\Omega))} \leq c(q, s, \Omega) \|f\|_{B_q^{1-s}((\Omega))} \|\varphi\|_{B_q^{1-s}((\Omega))}, \quad \forall f \in B_q^{1-s}((\Omega)), \varphi \in B_q^{s,1}((\Omega)).
\]

(ii) \[
\|fg\|_{B_q^{1-s}((\Omega))} \leq k_0(q, s, \Omega) \|f\|_{B_q^{1,s}((\Omega))} \|g\|_{b_q^{1-s}((\Omega))}, \quad \forall f \in B_q^{1,s}((\Omega)), g \in b_q^{1-s}((\Omega)).
\]

(iii) \[
\|fg\|_{b_q^{1-s}((\Omega))} \leq k_1(q, s, \Omega) \|f\|_{b_q^{1,s}((\Omega))} \|g\|_{b_q^{1-s}((\Omega))}, \quad \forall f, g \in b_q^{1-s}((\Omega)).
\]

(iv) \[
\forall \tau \in (\max\{s, n/q\}, s + n/q), \|fg\|_{B_q^{1-s}((\Omega))} \leq c(q, s, \Omega) \|f\|_{B_q^{1,s}((\Omega))} \|g\|_{B_q^{0}((\Omega))}, \quad \forall f \in B_q^{1,s}((\Omega)), g \in B_q^{0}((\Omega)).
\]

**Proof.** - Proof of (i): Thanks to $q > n$, it is clear that
\[
H_q^1((\Omega)) \cdot H_q^1((\Omega)) \hookrightarrow H_q^1((\Omega)),
\]
and
\[
H_q^1((\Omega)) \cdot L^q((\Omega)) \hookrightarrow L^q((\Omega)).
\]

From (4.25) and (4.26) we get by bilinear interpolation that
\[
(H_q^1((\Omega)), H_q^1((\Omega)))_{s,1} \cdot (L^q((\Omega)), H_q^1((\Omega)))_{s,1} \hookrightarrow (L^q((\Omega)), H_q^1((\Omega)))_{s,1},
\]
i.e.,
\[
B_{q,1}^{1-s,1}(\Omega) \cdot B_{q,1}^{s,1}(\Omega) \hookrightarrow B_{q,1}^{s,1}(\Omega).
\]

By the way, we have $B_q^{1-s}((\Omega)) \hookrightarrow B_q^{1-s,1}(\Omega)$ since $\Omega$ is bounded and $1 - s > (1 - s)\alpha + s$ due to $\alpha < \frac{1}{2}$. Hence we get the conclusion.

- Proof of (ii): Let $f \in B_q^{1-s}((\Omega))$, $g \in b_q^{1-s}((\Omega))$. Since $0 < s < (q-n)/(2q-n) < 1/q'$, one has $(b_q^{1-s}((\Omega)))' = B_q^{s,1}((\Omega))$ and $(B_q^{1,s}((\Omega)))' = B_q^{s}((\Omega))$. Then, by the first assertion (i) of the lemma proved above, we have
\[
\langle \langle f g, \varphi \rangle \rangle = |\langle g, f \varphi \rangle_{b_q^{1-s}((\Omega)), b_q^{1-s}((\Omega))'}|
\leq \|g\|_{b_q^{1-s}((\Omega))} \|f \varphi\|_{B_q^{s,1}((\Omega))}
\leq c(q, s, \Omega) \|g\|_{b_q^{1-s}((\Omega))} \|f\|_{B_q^{1,s}((\Omega))} \|\varphi\|_{B_q^{s,1}((\Omega))}
\]
for any $\varphi \in B^{s,1}_{q',1}(\Omega)$. Hence we have $fg \in B^{s,-s}_{q,\infty}(\Omega)$ and
\[ \|fg\|_{B^{s,-s}_{q,\infty}(\Omega)} \leq k_0 \|f\|_{B^{1,-s}_{q,\infty}(\Omega)} \|g\|_{b^{s}_{q,\infty}(\Omega)} \] (4.27)
with some $k_0 = k_0(q,s,\Omega) > 0$.

Thus it remained to prove $fg \in b^{s}_{q,\infty}(\Omega)$ provided $f \in b^{1,-s}_{q,\infty}(\Omega)$. Let $\{f_m\}, \{g_m\} \subset C^\infty(\Omega)$ be sequences converging to $f$ and $g$ in $b^{1,-s}_{q,\infty}(\Omega)$ and $b^{s}_{q,\infty}(\Omega)$, respectively. Then, by (4.27) we have
\[ \|f_m g_m - f g\|_{b^{s}_{q,\infty}(\Omega)} \leq \epsilon \|f_m - f\|_{B^{1,-s}_{q,\infty}(\Omega)} \|g_m\|_{b^{s}_{q,\infty}(\Omega)} + \|f\|_{B^{1,-s}_{q,\infty}(\Omega)} \|g_m - g\|_{b^{s}_{q,\infty}(\Omega)} \to 0 \]
as $m \to \infty$. Hence, $fg \in b^{s}_{q,\infty}(\Omega)$ and the assertion follows.

**Proof of (iii):** By a similar argument to prove (ii) one can show that
\[ \|fg\|_{H_q^{1-s}(\mathbb{R}^n)} \leq c \|f\|_{H_q^{1-s}(\mathbb{R}^n)} \|g\|_{H_q^{1-s}(\mathbb{R}^n)}, \quad \forall f \in H_q^{1-s}(\mathbb{R}^n), \quad g \in f \in H_q^{1-s}(\mathbb{R}^n). \]
Therefore, in view of the fact that
\[ H_q^{1-s}(\mathbb{R}^n) = \{ u \in H_q^{1-s}(\mathbb{R}^n) : \nabla u \in H_q^{-s}(\mathbb{R}^n) \}, \]
we have
\[ \|fg\|_{H_q^{1-s}(\mathbb{R}^n)} \leq \|f\|_{H_q^{1-s}(\mathbb{R}^n)} \|g\|_{H_q^{1-s}(\mathbb{R}^n)}, \quad \forall f, g \in H_q^{1-s}(\mathbb{R}^n). \] (4.28)
Then, (4.28) for functions on general Lipschitz domains can be proved easily using the Sobolev extension theorem. Finally, the assertion follows by bilinear interpolation.

**Proof of (iv):** Let $\tau \in (\max\{s,n/q\}, s + n/q)$. By Sobolev embedding theorem and Hölder inequality, it is easily seen that
\[ B^\tau_{q,1}(\Omega) \cdot H_q^{s-\varepsilon_1}(\Omega) \hookrightarrow H_q^{s+\varepsilon_2}(\Omega) \]
for sufficiently small $\varepsilon_1 \in (0,s)$. Hence, if we show
\[ B^\tau_{q,1}(\Omega) \cdot H_q^{s+\varepsilon_2}(\Omega) \hookrightarrow H_q^{s+\varepsilon_2}(\Omega). \] (4.29)
for some $\varepsilon_2 > 0$, then by a suitable real interpolation of the type $(\cdot,\cdot)_{\theta,1}$ the assertion (iv) is proved. In fact, one can choose $\zeta$ and $\mu$ such that
\[ 1 < \mu < q < \zeta, \quad 0 < n/\zeta - n/q + \tau - s < n/\zeta, \quad 0 < n/\mu - n/q + \tau - s - 1 < n/\mu \]
thanks to the assumption on $\tau$. Moreover, $1 + s < n/q'$ holds due to $s \in (0, \frac{n}{2q-n})$ and $n \geq 2$. Hence, by Sobolev embedding theorem,
\[ B^{n/\zeta - n/q + \tau - s}_{\zeta,1}(\Omega) \cdot H_q^s(\Omega) \hookrightarrow L^{q'}(\Omega), \quad B^{n/\mu - n/q + \tau - s}_{\mu,1}(\Omega) \cdot H_q^{1+s}(\Omega) \hookrightarrow H_q^1(\Omega), \]
which yields (4.29) by complex interpolation $[\cdot,\cdot]_{\varepsilon_2,1}$ with $\frac{1-\varepsilon_2}{\zeta} + \frac{\varepsilon_2}{\mu} = \frac{1}{q'}$.

The proof of the lemma comes to end. \[ \square \]

**Proof of Theorem 1.1: uniqueness part.** The uniqueness proof relies on a Lagrangian coordinates approach, using, in principle, the same idea as that on pages 29–31 of [16] but being based on pointwise multipliers on little Nikol'ski spaces.

First let us recall some facts concerning Lagrangian coordinates. Let $u$ be a vector field such that
\[ u \in L^1_{\text{loc}}([0,\infty), C_{L^p}(\mathbb{R}^n)) \] (4.30)
and let $X(t,y)$ be the (unique) solution to the ordinary differential system:
\[ \frac{dX}{dt} = u(t,X) \quad t \in (0,\infty), \quad X(0) = y \in \mathbb{R}^n. \] (4.31)
The unique solution to (4.31)
\[ X(t,y) = y + \int_0^t u(\tau, X(\tau,y)) \, d\tau, \quad t \in (0,\infty), \] (4.32)
determines a unique continuous semiflow, i.e., \( t \rightarrow X(t,y) \) for each \( y \) is continuous and \( X(0, \cdot) = \operatorname{Id} \), \( X(t+s, y) = X(t, X(s,y)) \) for \( t, s > 0 \). Then, Eulerian coordinates \( x \) and Lagrangian coordinates \( y \) are related by

\[
x = X(t,y). \]

If

\[
u \in L^1_{\text{loc}}([0,T), C_{\text{Lip}}(\Omega)), \quad u|_{\partial \Omega} = 0 \tag{4.33}
\]

for \( 0 < T \leq \infty \) and \( \Omega \subset \mathbb{R}^n \) with \( \partial \Omega \in C^1 \) is assumed instead of (4.30), then \( X \) given by (4.32) satisfies \( X(t,y) \in \Omega \) for all \( y \in \Omega \) and \( t \in (0,T) \) and unique semiflow \( X \) over \( \Omega \) is generated. Note that \( W^{1,\infty}(\Omega) \subset C_{\text{Lip}}(\Omega) \). Moreover, if \( \text{div} \ u = 0 \), then \( X(t, \cdot) \) for each \( t > 0 \) is a \( C^1 \)-diffeomorphism and measure preserving due to the Jacobian \( |D_y X(t,y)| = 1 \). Let \( Y(t, \cdot) \) be the inverse mapping of \( X(t, \cdot) \), then

\[
D_x Y(t,x) = (D_y X(t,y))^{-1} =: A(t,y),
\]

where and in what follows we use the notation \((\nabla u)_{i,j} = (\partial_i u^j)_{1 \leq i,j \leq n}, Du = (\nabla u)^T \). Let \( v(t,y) := u(t, X(t,y)) \). Then,

\[
\nabla_x u(t,x) = A^T \nabla_y v(t,y), \quad \text{div}_x u(x,t) = \text{div}_y(Av(t,y)), \tag{4.34}
\]

see [14], Appendix. In view of (4.34), we use the notation

\[
\nabla_u w := A^T \nabla_y w, \quad \text{div}_u w := \text{div}_y(Aw), \quad \Delta_u w := \text{div}_u(\nabla_u w).
\]

Let \((\rho, u, \nabla P)\) be a solution to (1.1) satisfying (1.5) and (1.9) with \( q > n \) and \( s \in (0, \frac{q-n}{2q-n}) \). Then \( u \) satisfies (4.33) since \( s < \frac{q-n}{2q-n} < 1 - \frac{n}{q} \) and hence \( u \in L^\infty_s(T; B^{2-\frac{q}{2}}_{q,1}(\Omega)) \hookrightarrow L^1(0,T; W^{1,\infty}(\Omega)) \).

Let \( a(t,x) := \frac{\rho}{\rho_t(x)} - 1, \quad (t,x) \in (0,T) \times \Omega \), and

\[
(b, v, Q)(t,y) := (a, u, P)(t, X(t,y)), \quad (t,y) \in (0,T) \times \Omega,
\]

where \( X(t,y) \) is given by (4.32). Then we get from (1.6) that \( b_1 = 0 \) in the sense of distribution. In fact, given any \( \tilde{\psi} \in C^0_0(\Omega \times (0,T)) \), for \( \psi(t,x) := \tilde{\psi}(t, Y(t,x)) \) we have \( \tilde{\psi}(t,y) = \psi(t, X(t,y)) \) and \( \psi \in C^0_0(\Omega \times (0,T)) \). Hence, we have

\[
\int_{\Omega \times (0,T)} b \tilde{\psi}_t \, dy dt = \int_{\Omega \times (0,T)} a(t, X(t,y)) \partial_t \psi(t, X(t,y)) \, dy dt
\]

\[
= \int_{\Omega \times (0,T)} a(t, X(t,y)) \psi_t + u \cdot \nabla \psi(t, X(t,y)) \, dy dt
\]

\[
= \int_{\Omega \times (0,T)} a(t, x) \psi_t + u \cdot \nabla \psi(t, x) |D_y X(t,y)| \, dx dt
\]

\[
= \int_{\Omega \times (0,T)} a(t, x) \psi_t + u \cdot \nabla \psi(t, x) \, dx dt = 0.
\]

Therefore, \( b(t,y) \equiv b(0,y) \equiv a_0(y) \) for each \( y \in \Omega \). Assuming \( \mu = 1 \) without loss of generality, \( \{v, Q\} \) solves the system:

\[
v_t - (1+b)(\Delta_u v - \nabla_u Q) = 0 \quad \text{in} \quad (0,T) \times \Omega,
\]

\[
\text{div}_u v = 0 \quad \text{in} \quad (0,T) \times \Omega,
\]

\[
v = 0 \quad \text{on} \quad (0,T) \times \partial \Omega,
\]

\[
v(0,y) = u_0 \quad \text{in} \quad \Omega. \tag{4.35}
\]

Now, in order to prove uniqueness of solutions, let \((\rho_i, u_i, \nabla P_i), i = 1, 2, \) be two solutions to (1.1) which satisfy (1.5) and (1.9) with \( q > n \) and \( s \in (0, \frac{q-n}{2q-n}) \). For \( i = 1, 2, \) let \( X_i \) be the semiflow corresponding
to $u_i$ (see (4.32)) and let $(b_i, v_i, Q_i)$ be the corresponding density perturbation, velocity and pressure in the Lagrangian coordinates. Then, for $\delta v = v_1 - v_2$, $\delta Q = Q_1 - Q_2$ we get from (4.35) that

$$
(\delta v)_t - \Delta \delta v + \nabla \delta Q = a_0(\Delta \delta v - \nabla \delta Q) + \delta F \quad \text{in} \quad (0, T) \times \Omega,
$$

$$
div \delta v = \delta g \equiv div \delta R \quad \text{in} \quad (0, T) \times \Omega,
$$

$$
\delta v(0, y) = 0 \quad \text{on} \quad (0, T) \times \partial \Omega,
$$

where

$$
\delta F = \delta f_1 + \delta f_2,
$$

$$
\delta f_1 := (1 + a_0)\left([Id - A^T_{2j}]\nabla \delta Q - \delta A \nabla Q_1\right) \quad \text{with} \quad \delta A := A_1 - A_2,
$$

$$
\delta f_2 := (1 + a_0)\text{div} [A^T_{2j} A_2 - Id] \nabla \delta v - (A^T_{2j} A_2 - A^T_{2j} A_1) \nabla v_1, \quad \delta g := (Id - A^T_2) : \nabla \delta v - (\delta A)^T : \nabla v_1,
$$

$$
\delta R := (Id - A_2) \delta v - \delta A v_1. \quad (4.37)
$$

Note that $\delta R(0) = 0$. Therefore, by Theorem 2.5 we have

$$
\|\delta v_t, \nabla^2 \delta v, \nabla \delta Q\|_{L^{\infty/2}(0, t; b_{q, \infty}^{1-s}(\Omega))} + \|\delta v\|_{L^{\infty/2}(0, t; b_{q, \infty}^{1-s}(\Omega))} \leq K\|a_0(\Delta \delta v - \nabla \delta Q), \delta F, \nabla \delta g, (\delta R)_t\|_{L^{\infty/2}(0, t; b_{q, \infty}^{1-s}(\Omega))}, \forall t \in (0, T), \quad (4.38)
$$

with constant $K > 0$ independent of $t$ and $\theta \in (s/2, 1)$.

From now on, let us get estimate of the right-hand side of (4.38).

Thanks to $\nabla v_1 \in L^{\infty/2}(0, T; b_{q, \infty}^{1-s}(\Omega))$, $i = 1, 2$, and $b_{q, \infty}^{1-s}(\Omega) \hookrightarrow L^{\infty}(\Omega)$ there is some $m_0 > 0$ such that, if $|t_2 - t_1| < m_0$,

$$
\left\| \int_{t_1}^{t_2} \nabla v_i(t, y) \, dt \right\|_{L^{\infty}(\Omega)} < 1, \quad \int_{t_1}^{t_2} \|\nabla v_i(t, y)\|_{b_{q, \infty}^{1-s}((\Omega))} \, dt < \frac{1}{2k_2}, \quad i = 1, 2, \quad (4.39)
$$

where $k_2 := \max\{k_0(q, s, \Omega), k_1(q, s, \Omega)\}$ with constants $k_0$, $k_1$ appearing in Lemma 4.6. Throughout the proof, we assume that $0 < t < m_0$. Then we have

$$
A_i(t, y) = (Id + C_i(t))^{-1} = \sum_{k \geq 0} (-1)^k C_i(t)^k, \quad i = 1, 2, \quad (4.40)
$$

where $C_i(t) := \int_t^t Dv_i(\tau) \, d\tau$, $i = 1, 2$. Hence one has

$$
\delta A(t) = h_1(t) \int_0^t D\delta v(\tau) \, d\tau, \quad t \in (0, m_0), \quad (4.41)
$$

with

$$
h_1(t) := \sum_{k \geq 1} (-1)^k \sum_{j=0}^{k-1} C_1(t)^j C_2(t)^{k-1-j}, \quad t \in (0, m_0). \quad (4.42)
$$

Since

$$
\tilde{C}_i(t) := \|C_i\|_{L^{\infty}(0, t; b_{q, \infty}^{1-s}(\Omega))} < \frac{1}{2k_2(q, s, \Omega)}, \quad (4.43)
$$

due to (4.39), we get by Lemma 4.6 (ii) that

$$
\|h_1\|_{L^{\infty}(0, t; M(b_{q, \infty}^{1-s}(\Omega)))} \leq \sum_{k \geq 1} \sum_{j=0}^{k-1} \tilde{C}_1(t)^j \tilde{C}_2(t)^{k-1-j} < C \quad (4.44)
$$

with $C = C(q, s, \Omega)$. Hence, we get

$$
\|\delta A\|_{L^{\infty}(0, t; b_{q, \infty}^{1-s}(\Omega))} \leq ct^{s/2}\|\delta v\|_{L^{\infty/2}(0, t; b_{q, \infty}^{1-s}(\Omega))} \quad (4.45)
$$
with \( c = c(q, s, \Omega) > 0 \). In the same way, using Lemma 4.6 (iii) we can obtain
\[
\| \delta A \|_{L^\infty(0, t; b_{q_s}^{\infty}(\Omega))} \leq c t^{s/2} \| \delta v \|_{L_{t\rightarrow 2}^\infty(0, t; b_{q_s}^{\infty}(\Omega))}
\]  
(4.46)  
with \( c = c(q, s, \Omega) > 0 \).

Since \( a_0(y) = \frac{\rho_1 - \rho_0(y)}{\rho_0(y)} \), there is some \( \delta_1 = \delta_1(\Omega) > 0 \) such that if
\[
\frac{\rho_0 - \rho_1}{\rho_1} < \delta_1,
\]  
(4.47)  
then
\[
\| a_0(\Delta \delta v - \nabla \delta Q) \|_{L_{t\rightarrow 2}^\infty(0, t; b_{q_s}^{\infty}(\Omega))} \leq \frac{1}{2K} \| \nabla^2 \delta v, \nabla \delta Q \|_{L_{t\rightarrow 2}^\infty(0, t; b_{q_s}^{\infty}(\Omega))}.  
\]  
(4.48)  
Next, let us get estimate of \( \| \nabla \delta g \|_{L_{t\rightarrow 2}^\infty(0, t; b_{q_s}^{\infty}(\Omega))} \). From (4.37) one has
\[
\| \nabla \delta g \|_{L_{t\rightarrow 2}^\infty(0, t; b_{q_s}^{\infty}(\Omega))} \leq \| \nabla A_2^T \otimes \nabla \delta v, (\text{Id} - A_2^T) \otimes \nabla^2 \delta v, \nabla \delta A^T \otimes \nabla v_1, \delta A^T \otimes \nabla^2 v_1 \|_{L_{t\rightarrow 2}^\infty(0, t; b_{q_s}^{\infty}(\Omega))},
\]  
(4.49)  
where the right-hand side can be estimated as below.  
Since \( DA_i = \sum_{k \geq 1} (-1)^k C_i^{k-1} DC_i \) due to (4.40), we get by Lemma 4.6 (ii) and (4.43) that
\[
\| \nabla A_2^T \|_{L^\infty(0, t; b_{q_s}^{\infty}(\Omega))} \leq c \| \nabla^2 v_2 \|_{L^1(0, t; b_{q_s}^{\infty}(\Omega))}
\leq c t^{s/2} \| \nabla^2 v_2 \|_{L_{t\rightarrow 2}^\infty(0, t; b_{q_s}^{\infty}(\Omega))}
\]  
(4.50)  
and
\[
\| \nabla A_2^T \otimes \nabla \delta v \|_{L_{t\rightarrow 2}^\infty(0, t; b_{q_s}^{\infty}(\Omega))} \leq c \| \nabla A_2^T \|_{L^\infty(0, t; b_{q_s}^{\infty}(\Omega))} \| \nabla \delta v \|_{L_{t\rightarrow 2}^\infty(0, t; b_{q_s}^{\infty}(\Omega))}
\leq c t^{s/2} \| \nabla v_2 \|_{L_{t\rightarrow 2}^\infty(0, t; b_{q_s}^{\infty}(\Omega))} \| \nabla \delta v \|_{L_{t\rightarrow 2}^\infty(0, t; b_{q_s}^{\infty}(\Omega))}
\]  
(4.51)  
with \( c = c(q, s, \Omega) \). Furthermore, by Lemma 4.6 (ii) and (4.43) we have
\[
\| \text{Id} - A_i \|_{L^\infty(0, t; b_{q_s}^{\infty}(\Omega))} = \| \sum_{k \geq 1} (-1)^k C_i(t) \|_{L^\infty(0, t; b_{q_s}^{\infty}(\Omega))}
\leq c \| \nabla v_i \|_{L^1(0, t; b_{q_s}^{\infty}(\Omega))}, \ i = 1, 2,
\]  
(4.52)  
and, consequently,
\[
\| (\text{Id} - A_2^T) \otimes \nabla^2 \delta v \|_{L_{t\rightarrow 2}^\infty(0, t; b_{q_s}^{\infty}(\Omega))}
\leq c \| \text{Id} - A_2^T \|_{L^\infty(0, t; b_{q_s}^{\infty}(\Omega))} \| \nabla^2 \delta v \|_{L_{t\rightarrow 2}^\infty(0, t; b_{q_s}^{\infty}(\Omega))}
\leq c \| \nabla v_2 \|_{L^1(0, t; b_{q_s}^{\infty}(\Omega))} \| \nabla^2 \delta v \|_{L_{t\rightarrow 2}^\infty(0, t; b_{q_s}^{\infty}(\Omega))}
\leq c t^{s/2} \| \nabla v_2 \|_{L_{t\rightarrow 2}^\infty(0, t; b_{q_s}^{\infty}(\Omega))} \| \nabla^2 \delta v \|_{L_{t\rightarrow 2}^\infty(0, t; b_{q_s}^{\infty}(\Omega))}
\]  
(4.53)  
with \( c = c(q, s, \Omega) \).

Note that \( D\delta A(t) = h_1(t) \int_0^t D^2 \delta v(\tau) \, d\tau + h_2(t) \int_0^t D\delta v(\tau) \, d\tau \), where
\[
h_2 := Dh_1 = \sum_{k \geq 1} (-1)^k \sum_{j=0}^{k-1} (jC_1^{j-1}C_2^{k-1-j} DC_1 + (k - 1 - j)C_1^{j}C_2^{k-2-j} DC_2).
\]  
We get, in view of (4.39), that
\[
\| h_2 \|_{L^\infty(0, t; b_{q_s}^{\infty}(\Omega))} < c \| \nabla^2 v_1, \nabla^2 v_2 \|_{L^1(0, t; b_{q_s}^{\infty}(\Omega))} < C_1
\]
with \( C_1 = C_1(q, s, \Omega) \), which together with (4.44) yields
\[
\| D\delta A \|_{L^\infty(0, t; b_{q, \infty}^2(\Omega))} \leq \| h_1 \|_{L^\infty(0, t; M(b_{q, \infty}^2(\Omega)))} \int_0^t \| D^2 \delta v(\tau) \|_{L^\infty(0, t; b_{q, \infty}^2(\Omega))} d\tau \\
+ c \| h_2 \|_{L^\infty(0, t; b_{q, \infty}^2(\Omega))} \int_0^t \| D \delta v(\tau) \|_{L^\infty(0, t; b_{q, \infty}^2(\Omega))} d\tau \\
\leq c \| \delta v \|_{L^1(0, t; b_{q, \infty}^2(\Omega))} \leq c t^{s/2} \| \delta v \|_{L^1_{s/2}(0, t; b_{q, \infty}^2(\Omega))}
\]
with \( c = c(q, s, \Omega) \). Therefore, by Lemma 4.6 (ii) we have
\[
\| \nabla \delta A^T \otimes \nabla v_1 \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))} \leq c \| \nabla \delta A \|_{L^\infty(0, t; b_{q, \infty}^2(\Omega))} \| \nabla v_1 \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))} \\
\leq c t^{s/2} \| v_1 \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))} \| \delta v \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))}.
\]
(4.54)

By (4.45) we have
\[
\| \delta A^T \otimes \nabla^2 v_1 \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))} \leq c t^{s/2} \| v_1 \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))} \| \delta v \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))}.
\]
(4.55)

Thus, from (4.49), (4.51), (4.53), (4.54) and (4.55) we get
\[
\| \nabla \delta g \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))} \leq \eta_1(t) \| \delta v \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))},
\]
(4.56)

with \( \eta_1(t) := c t^{s/2} \| v_1, v_2 \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))} \to 0 \) as \( t \to +0 \).

Following the same procedure as the derivation of (4.56), we can get estimate
\[
\| \delta f_1 \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))} \leq c t^{s/2} \| \nabla^2 v_2, \nabla Q_1 \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))} \| \nabla^2 \delta v, \nabla \delta Q \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))}
\]
(4.57)

under the smallness assumption (4.47); we omit the details here.

On the other hand, we have
\[
\| \delta f_2 \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))} \leq \| \text{div} \left( (A^T_2 A_2 - \text{Id}) \nabla \delta v - (A^T_2 A_2 - A^T_1 A_1) \nabla v_1 \right) \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))}.
\]
(4.58)

Here,
\[
\| \text{div} \left( (A^T_2 A_2 - \text{Id}) \nabla \delta v \right) \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))} \\
\leq (\| \nabla (A^T_2 A_2) \|_{L^\infty(0, t; b_{q, \infty}^2(\Omega))} + \| A^T_2 A_2 - \text{Id} \|_{L^\infty(0, t; b_{q, \infty}^2(\Omega))}) \| \delta v \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))}.
\]

Note that, by Lemma 4.6 (ii), (iii), (4.43) and (4.50),
\[
\| \nabla (A^T_2 A_1) \|_{L^\infty(0, t; b_{q, \infty}^2(\Omega))} \leq 2 \| \nabla A_i \|_{L^\infty(0, t; b_{q, \infty}^2(\Omega))} \| A_i \|_{L^\infty(0, t; b_{q, \infty}^2(\Omega))} \\
\leq ct^{s/2} \| v_i \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))}^2, \quad i = 1, 2,
\]
and, by Lemma 4.6 (iii),
\[
\| A^T_2 A_2 - \text{Id} \|_{L^\infty(0, t; b_{q, \infty}^2(\Omega))} \leq ct^{s/2} \| v_2 \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))} (1 + \| v_2 \|_{L^\infty(0, t; b_{q, \infty}^2(\Omega))})
\]
in view of \( A^T_2(t) A_2(t) - \text{Id} = \int_0^t (Dv_2 + Dv_2^2) \ d\tau + \int_0^t Dv_2^2 \ d\tau \cdot \int_0^t Dv_2 \ d\tau \).

Therefore,
\[
\| \text{div} \left( (A^T_2 A_2 - \text{Id}) \nabla \delta v \right) \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))} \\
\leq ct^{s/2} \| v_1, v_2 \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))} (\| v_1, v_2 \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))} + 1) \| \delta v \|_{L^\infty_{s/2}(0, t; b_{q, \infty}^2(\Omega))}.
\]
(4.59)
Furthermore, using $A_T^2 A_2 - A_T^1 A_1 = A_T^2 \delta A + \delta A^T A_1$, (4.41), (4.44) and Lemma 4.6 (ii), we have

$$\| \text{div} [(A_T^2 A_2 - A_T^1 A_1) \nabla v_1] \|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))}$$

$$\leq \| \nabla (A_T^2 \delta A + \delta A^T A_1) \otimes \nabla v_1, (A_T^2 \delta A + \delta A^T A_1) \otimes \nabla^2 v_1 \|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))}$$

$$\leq c(\|\delta A\|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))} \| \nabla A_1 \otimes \nabla v_1, \nabla A_2 \otimes \nabla v_1 \|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))}$$

$$+ \|A_1, A_2\|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))} \| \nabla^2 v_1 \|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))} \| \nabla A \otimes \nabla v_1, \delta A \otimes \nabla^2 v_1 \|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))}$$

$$\leq c(\|\delta A\|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))} \| \nabla A_1 \otimes \nabla v_1, \nabla A_2 \otimes \nabla v_1, \nabla^2 v_1 \|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))}$$

$$+ \|A_1, A_2\|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))} \| \nabla^2 v_1 \|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))}$$

$$\leq c(\|\delta A\|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))} \| \nabla A_1, A_2 \|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))} \| \nabla^2 v_1 \|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))}$$

$$\leq c(\|\delta A\|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))} \| \nabla A_1, A_2 \|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))} \| \nabla^2 v_1 \|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))}$$

(4.60)

Thus, from (4.58)–(4.60) we have

$$\|\delta f_2\|_{L^{\gamma/2}_T(0,T;b_{q,\infty}^s(\Omega))}$$

$$\leq c_{s/2} \|v_1, v_2\|_{L^{\gamma/2}_T(0,T;b_{q,\infty}^s(\Omega))} (\|v_1, v_2\|_{L^{\gamma/2}_T(0,T;b_{q,\infty}^s(\Omega))) + 1) \|\delta v\|_{L^{\gamma/2}_T(0,T;b_{q,\infty}^s(\Omega))},$$

which together with (4.57) yields

$$\|\delta F\|_{L^\infty_T(0,T;b_{q,\infty}^s(\Omega))} \leq \eta_2(t) \|\nabla^2 \delta v, \nabla \delta Q\|_{L^\infty_T(0,T;b_{q,\infty}^s(\Omega))}$$

(4.61)

with some $\eta_2(t)$ such that $\eta_2(t) \to 0$ as $t \to +0$.

Finally, let us get estimate of $|\delta R_i|$. Recall that $\delta R = (\text{Id} - A_2) \delta v - \delta A v_1$ and hence $(\delta R)_i = -A_2 \delta v + (\text{Id} - A_2) \delta v_i - \delta A v_1$. Since we have

$$A_i(t, y) = -\text{Id} + \sum_{k \geq 2} (-1)^k k C^{k-1}_i(t), i = 1, 2,$$

due to (4.40), it follows that

$$\|A_2 \delta v\|_{L^{\gamma/2}_T(0,T;b_{q,\infty}^s(\Omega))} \leq \left(1 + \sum_{k \geq 2} k k^{-1} \|C_i(t)\|_{L^\infty(0,T)} \right) \|\delta v\|_{L^{\gamma/2}_T(0,T;b_{q,\infty}^s(\Omega))}$$

$$\leq c_{1-s/2} \|\delta v\|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))} \leq c t \|\delta v\|_{L^\infty_T(0,T;b_{q,\infty}^s(\Omega))}$$

(4.62)

by repeatedly applying Lemma 4.6 (ii) in view of (4.43). On the other hand, using (4.52), we have

$$\|\text{Id} - A_2\|_{L^{\gamma/2}_T(0,T;b_{q,\infty}^s(\Omega))} \leq c \|\text{Id} - A_2\|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))} \|\delta v_i\|_{L^{\gamma/2}_T(0,T;b_{q,\infty}^s(\Omega))}$$

$$\leq c \|\nabla v_i\|_{L^1(0,T;b_{q,\infty}^s(\Omega))} \|\delta v_i\|_{L^{\gamma/2}_T(0,T;b_{q,\infty}^s(\Omega))}$$

$$\leq c_{s/2} \|v_i\|_{L^\infty(0,T;b_{q,\infty}^s(\Omega))} \|\delta v_i\|_{L^{\gamma/2}_T(0,T;b_{q,\infty}^s(\Omega))}.$$
Therefore, by Lemma 4.6 (ii) and (4.44) we have
\[
\|\delta A_t v_1\|_{L^{\infty}_{T/2}(0,t;b_\gamma^s,*(\Omega))} \leq c\|h_1\|_{L^1(0,T;A(\delta b_\gamma^s,*(\Omega)))} \|D\delta v \otimes v_1\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} \\
+ c\|h_3\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} \|D\delta v \otimes v_1\|_{L^1(0,t;b_\gamma^s,*(\Omega))} \\
\leq c(1 + \|v_1, v_2\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))}) \|D\delta v \otimes v_1\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} \cap L^1(0,t;b_\gamma^s,*(\Omega)) \\
\leq c(1 + \|v_1, v_2\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))})(1 + t^{s/2}) \|D\delta v \otimes v_1\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega)).} \tag{4.64}
\]

On the right hand side of (4.64), we have
\[
\|D\delta v \otimes v_1\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} \leq \|D\delta v \otimes (v_1 - u_0)\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega)),}
\]
where
\[
\|D\delta v \otimes (v_1 - u_0)\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} \leq c\|D\delta v\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} \|v_1 - u_0\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} \\
\leq ct^{s/2}\|\delta v\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} \|v_1\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))}
\]
and, by Lemma 4.6 (iv) with \(\tau = 2\theta - 1\),
\[
\|D\delta v \otimes u_0\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} \leq c\|D\delta v\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} \|u_0\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} \\
\leq ct^{1-\theta}\|\delta v\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} \|u_0\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))}
\]
for \(\theta \in \left(\frac{1+s}{2}, \frac{1+s}{2} + \frac{n}{2q}\right)\). Therefore,
\[
\|\delta A_t v_1\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} \leq c(v_1, u_0)t^\zeta \left(\|\delta v\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} + \|\delta v\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))}\right), \tag{4.65}
\]
where \(\zeta := \min\{1 - \theta, s/2\}\).

Finally, in view of the expression \(\delta A(t) = h_1(t) \int_0^t D\delta v dt\) and (4.46), we get by Lemma 4.6 (ii) that
\[
\|\delta A v_1 t\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} \leq c\|\delta A\|_{L^\infty(0,T;B^s_{1,2}(\Omega))} \|v_1\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} \\
\leq ct^{s/2}\|\delta v\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} \|v_1\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))}, \tag{4.66}
\]
Thus, from (4.62)–(4.66) it follows that
\[
\|\delta R_t\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} \leq \eta_3(t) \left(\|\delta v\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} + \|\delta v\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))}\right) \tag{4.67}
\]
for \(\theta \in \left(\frac{1+s}{2}, \frac{1+s}{2} + \frac{n}{2q}\right)\) and some \(\eta_3(t)\) converging to 0 as \(t \to +0\).

Summarizing, we can conclude from (4.38), (4.48), (4.56), (4.61) and (4.67) that
\[
\|\delta v_t, \nabla^2 \delta v, \nabla \delta Q\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} + \|\delta v\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} \\
\leq \eta(t)(\|\delta v_t, \nabla \delta Q\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} + \|\delta v\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))}), \forall t \in (0,m_0),
\]
for some \(\eta(t)\) with \(\eta(t) \to 0\) as \(t \to 0\), which together with\(\frac{n}{2}\|\delta v\|_{L^\infty_{T/2}(0,t;b_\gamma^s,*(\Omega))} \leq \|\delta v_t, \nabla^2 v\|_{L^\infty_{T/2}(0,0,b_\gamma^s,*(\Omega))}\)
for \(0 \leq t < 1\), yields that \(\delta v(t) = 0, \delta Q(t) = 0\) for all \(t \in (0, T_1)\) with some \(T_1 > 0\). Then, by standard continuation argument, it can be shown that \(\delta v(t) = 0, \delta Q(t) = 0\) for all \(t \in (0, T)\).

Now, the proof of the uniqueness part of Theorem 1.1 comes to end. \(\square\)
Acknowledgements. P. Zhang is partially supported by NSF of China under Grants 11371347 and 11688101, Morningside Center of Mathematics of The Chinese Academy of Sciences and innovation grant from National Center for Mathematics and Interdisciplinary Sciences.

Compliance with ethical standards
Conflict of interest
The authors declare that they have no conflict of interest.

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(accepted: September 13, 2019; published online: September 26, 2019)