Polynomial Relations in the Centre of $\mathcal{U}_q(sl(N))$

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When the parameter of deformation $q$ is a root of unity, the centre of $\mathcal{U}_q(sl(N))$ contains, besides the usual $q$-deformed Casimirs, a set of new generators, which are basically the $m$-th powers of all the Cartan generators of $\mathcal{U}_q(sl(N))$. All these central elements are however not independent. In this letter, generalising the well-known case of $\mathcal{U}_q(sl(2))$, we explicitly write polynomial relations satisfied by the generators of the centre. Application to the parametrization of irreducible representations and to fusion rules are sketched.
1. Symmetric polynomials

In the course of our study of Casimirs for $\mathcal{U}_q(sl(N))$ we shall repeatedly encounter special symmetric polynomials of $N$ variables denoted by $x_1, \cdots, x_N$. In the classical theory of Casimirs of $sl(N)$, symmetric polynomials in $N$ variables play a special role since the Weyl group of $sl(N)$ is the symmetric group on $N$ objects. The Weyl group acts on the Cartan torus and on its Lie algebra $\mathfrak{g}$ and a well-known theorem of Harish–Chandra says that there is natural isomorphism between the centre of $U(sl(N))$ and the Weyl-invariant elements of $U(\mathfrak{g})$. The more precise corresponding statements in the case of $U_q(sl(N))$ when $q$ is a root of unity will be given below.

The elementary symmetric polynomials $c_1, \cdots, c_N$ are defined by the identity

$$\prod_{i=1}^{N}(1 - tx_i) = 1 - c_1 t + c_2 t^2 - \cdots + (-1)^N c_N t^N \equiv G(t). \quad (1.1)$$

Hence for $i = 1, \cdots, N$

$$c_i = \sum_{1 \leq j_1 < \cdots < j_i \leq N} x_{j_1} \cdots x_{j_i} \quad (1.2)$$

and it is an old theorem attributed to Newton that any symmetric polynomial in $x_1, \cdots, x_N$ (with coefficients in a ring) is a polynomial in $c_1, \cdots, c_N$ with coefficients in the same ring.

The polynomials of interest to us in the sequel are generalisations of the elementary ones obtained by replacing the variables $x_i$ by their $m$th power. Hence we define $P_{i,m}^{(N)}(c_1, \cdots, c_N)$ for $i = 1, \cdots, N$ and $m = 1, 2, \cdots$ by the identity

$$\prod_{i=1}^{N}(1 - tx_i^m) \equiv 1 - P_{1,m}^{(N)} t + P_{2,m}^{(N)} t^2 - \cdots + (-1)^N P_{N,m}^{(N)} t^N. \quad (1.3)$$

It is useful to have expressions displaying these polynomials directly in terms of the elementary symmetric polynomials $c_i$ (and not in terms of the variables $x_1, \cdots, x_N$). A method that works nicely for fixed $m$ is to remark that for any primitive $m$th root of unity $q$

$$1 - t^m x_i^m = \prod_{l=1}^{m}(1 - q^l t x_i) , \quad (1.4)$$

from which we deduce that

$$\prod_{i=1}^{N}(1 - t^m x_i^m) = \prod_{l=1}^{m} G(q^l t) . \quad (1.5)$$

Finally we obtain the desired result

$$1 - P_{1,m}^{(N)} t^m + P_{2,m}^{(N)} t^{2m} - \cdots + (-1)^N P_{N,m}^{(N)} t^{Nm} = \prod_{l=1}^{m} G(q^l t). \quad (1.6)$$
This formula makes the computation of the $P_{i,m}$s for reasonable values of $N$ and $m$ tractable, at least with the help of a computer.

The polynomials $P_{i,m}^{(N)}$ will play a distinguished role in what follows. The generating function

$$\sum_{m=1}^{\infty} P_{i,m}^{(N)} \frac{t^m}{m}$$

is easy to express in terms of $c_1, \ldots, c_N$, because

$$-\log(1 - tx_i) = \sum_{m=1}^{\infty} x_i^m \frac{t^m}{m},$$

leading to

$$\sum_{m=1}^{\infty} P_{i,m}^{(N)} \frac{t^m}{m} = -\log G(t).$$

Let us end this section with some examples of these polynomials. First note that for our purpose we will have to consider only the particular case $c_N = 1$.

In the case of $U_q(sl(2))$, we will need $P_{i,m}^{(2)}$, which is closely related to the $m$th Chebichev polynomial of the first kind.

In the case of $U_q(sl(3))$ and $m = 5$, the polynomials of interest are

$$P_{1,5}^{(3)}(c_1, c_2) = c_1^5 - 5c_1^3c_2 + 5c_1c_2^2 + 5c_1^2 - 5c_2$$
$$P_{2,5}^{(3)}(c_1, c_2) = c_2^5 - 5c_1^3c_2 + 5c_1c_2^2 + 5c_2^2 - 5c_1.$$

In the case of $U_q(sl(4))$ and $m = 5$, we will need

$$P_{1,5}^{(4)}(c_1, c_2, c_3) = c_1^5 - 5c_1^3c_2 + 5c_1c_2^2 + 5c_1^2c_3 - 5c_2c_3 - 5c_1$$
$$P_{2,5}^{(4)}(c_1, c_2, c_3) = c_2^5 - 5c_1^3c_2 + 5c_1c_2^2 + 5c_2^2c_3 - 5c_1c_3^2 + 5c_2c_3 + 5c_1^2c_2$$
$$- 5c_2 - 5c_1^3c_3 - 5c_1c_2c_3 + 5c_3^2 + 5c_1^2 + 5c_2$$
$$P_{3,5}^{(4)}(c_1, c_2, c_3) = c_3^5 - 5c_2c_3^3 + 5c_2^2c_3 + 5c_1c_3^2 - 5c_1c_2 - 5c_3.$$

2. $U_q(sl(N))$ at roots of unity

Let $\{\alpha_1, \ldots, \alpha_{N-1}\}$ be the set of simple roots of $sl(N)$. We define vectors $\epsilon_1, \ldots, \epsilon_N$ by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ and $\sum_{i=1}^{N} \epsilon_i = 0$. 
The “simply connected” quantum group $\mathcal{U}_q(sl(N))$ is defined by the generators $e_i$, and $f_i$, for $i = 1, \ldots, N - 1$, and $k_{\pm e_i}$ for $i = 1, \ldots, N$, and the relations

$$\begin{align*}
\begin{cases}
  k_{\beta_1}k_{\beta_2} = k_{\beta_1 + \beta_2}, \\
  k_{\epsilon_i}e_jk_{\epsilon_i}^{-1} = q^{\delta_{ij} - \delta_{i-1,j}}e_j, \\
  k_{\epsilon_i}f_jk_{\epsilon_i}^{-1} = q^{-\delta_{ij} + \delta_{i-1,j}}f_j, \\
  [e_i, f_j] = \delta_{ij}k_{\alpha_i - \alpha_i}^{-1}, \\
  [e_i, e_j] = [f_i, f_j] = 0 \quad \text{for} \quad |i - j| \geq 2, \\
  e_i^2e_i\pm1 - (q + q^{-1})e_i e_i \pm 1 e_i + e_i \pm 1 e_i^2 = 0, \\
  f_i^2f_i\pm1 - (q + q^{-1})f_i f_i \pm 1 f_i + f_i \pm 1 f_i^2 = 0.
\end{cases}
\end{align*}$$

(2.1)

Let $\mathcal{U}^0$ be the subalgebra generated by the $k_{\epsilon_i}$’s, and $\mathcal{U}^+, \mathcal{U}^-$ the subalgebras generated by the $e_i$’s, $f_i$’s, respectively.

Two sets of quantum analogues of the roots vectors are inductively defined as

$$\begin{align*}
\begin{cases}
  e_{i,i+1} = \tilde{e}_{i,i+1} \equiv e_i \quad \text{for} \quad i = 1, \ldots, N - 1 \\
  e_{i,j+1} = e_{ij}e_j - q^{-1}e_je_{ij} \quad \text{for} \quad i < j \\
  \tilde{e}_{i,j+1} = \tilde{e}_{ij}e_j - qe_je_{ij} \quad \text{for} \quad i < j.
\end{cases}
\end{align*}$$

(2.2)

as are the $f_{ij}$ and $\tilde{f}_{ij}$.

Quantum analogues of Poincaré–Birkhoff–Witt bases can be built with ordered monomials in these generators [4].

When $q$ is not a root of unity, there exists a quantum analogue of Harish–Chandra theorem [23]: there exists an algebra isomorphism $h$ from $Z$, the centre of $\mathcal{U}_q(sl(N))$, to the algebra of symmetric polynomials in the $k_{2\epsilon_i}$. This isomorphism $h$ can be written as $h = \gamma^{-1} \circ h'$, with the following notations: $h'$ is the projection on $\mathcal{U}^0$, within the direct sum $\mathcal{U} = \mathcal{U}^0 \oplus (\mathcal{U}^- \mathcal{U}^+ \mathcal{U}^+)$, with $\mathcal{U} \equiv \mathcal{U}_q(sl(N))$; $\gamma$ is the automorphism of $\mathcal{U}^0$ given by $\gamma(k_{2\epsilon_i}) = q^{N+1-2i}k_{2\epsilon_i}$.

A set of generators of $Z$ is given by

$$\{C_i = h^{-1}(c_i(k_{2\epsilon_1}, \ldots, k_{2\epsilon_N}))\}_{i=1,\ldots,N-1}. \quad (2.3)$$

An expanded expression for these generators (denoted there by $\tilde{c}_k$) appears in [4] in the form (up to slight changes of convention and normalization):

$$\begin{align*}
C_i = q^{i(N-i)}N_i(q^{-2})^{-1}N_{N-i}(q^{-2})^{-1} \sum_{\sigma, \sigma' \in \mathcal{S}(N)} (-q^{-1})^{l(\sigma) + l(\sigma')} \prod_{\sigma_i, \sigma_i'} f_{\sigma_i, \sigma_i'} f_{\sigma_i + 1, \sigma_i + 1} \quad \text{for} \quad i = 1, \ldots, N - 1.
\end{align*}$$

(2.4)
where \( \mathcal{N}_i(x) = \prod_{n=1}^{i} (1 + \cdots + x^{n-1}) \), where \( l(\sigma) \) is the length of the shortest expression of the permutation \( \sigma \) in terms of simple transpositions, and where

\[
\begin{align*}
    l_{ii}^{(+)} &= \left( l_{ii}^{(-)} \right)^{-1} = k_{\epsilon_i} \\
    l_{ij}^{(+)} &= l_{ji}^{(-)} = 0 \quad \text{for} \quad i > j \\
    l_{ij}^{(+)} &= (q - q^{-1})^{-1} \tilde{f}_{ij} k_{\epsilon_i} \quad \text{for} \quad i < j \\
    l_{ij}^{(-)} &= (q - q^{-1})^{-1} k_{-\epsilon_i} \tilde{e}_{ij} \quad \text{for} \quad i > j.
\end{align*}
\]

The first and last of these Casimirs are explicitly given by

\[
\mathcal{C}_1 = \sum_{i=1}^{N} q^{N+1-2i} k_{2\epsilon_i} + (q - q^{-1})^2 \sum_{1 \leq i < j \leq N} (-1)^{j-i-1} q^{N-1-i-j} \tilde{f}_{ij} \tilde{e}_{ij} k_{\epsilon_i+\epsilon_j} \quad (2.5)
\]

and

\[
\mathcal{C}_{N-1} = \sum_{i=1}^{N} q^{-N+1-2i} k_{-2\epsilon_i} + (q - q^{-1})^2 \sum_{1 \leq i < j \leq N} (-1)^{j-i-1} q^{-N+1+i-j} \tilde{f}_{ij} \tilde{e}_{ij} k_{-\epsilon_i-\epsilon_j}. \quad (2.6)
\]

When \( q \) is a root of unity, the image \( Z_1 \) of \( h \) is still a well-defined central subalgebra of \( \mathcal{U}_q(\text{sl}(N)) \) \([3]\), but it does not generate the whole centre. Let \( Z_0 \) be the subalgebra of \( \mathcal{U}_q(\text{sl}(N)) \) generated by the elements \( f_{ij}, e_{ij} \) and \( k_{m\epsilon_i} \). (We could also replace \( f_{ij} \) by \( \tilde{f}_{ij} \), or \( e_{ij} \) by \( \tilde{e}_{ij} \), this would lead to the same \( Z_0 \).) When \( m' \) is odd, these elements are central, and the centre \( Z \) of \( \mathcal{U}_q(\text{sl}(N)) \) is actually generated by \( Z_0 \) and \( Z_1 \) \([3]\).

### 3. Relations in the centre of \( \mathcal{U}_q(\text{sl}(N)) \)

**Theorem:** If \( m' \) is odd, the following relations are satisfied in the centre of \( \mathcal{U}_q(\text{sl}(N)) \),

\[
\begin{align*}
P^{(N)}_{1,m} (\mathcal{C}_1, \ldots, \mathcal{C}_{N-1}) &= \sum_{i=1}^{N} q^{m(N+1)} k_{2m\epsilon_i} \\
&+ (q - q^{-1})^{2m} \sum_{1 \leq i < j \leq N} (-1)^{m(j-i-1)} q^{m(N+1-i-j)} \tilde{f}_{ij} \tilde{e}_{ij} k_{m\epsilon_i+m\epsilon_j} \quad (3.1)\end{align*}
\]

and

\[
\begin{align*}
P^{(N)}_{N-1,m} (\mathcal{C}_1, \ldots, \mathcal{C}_{N-1}) &= \sum_{i=1}^{N} q^{-m(N+1)} k_{-2m\epsilon_i} \\
&+ (q - q^{-1})^{2m} \sum_{1 \leq i < j \leq N} (-1)^{m(j-i-1)} q^{-m(N-1+i+j)} \tilde{f}_{ij} \tilde{e}_{ij} k_{-m\epsilon_i-m\epsilon_j} \quad (3.2)\end{align*}
\]
Remark 1: Actually, all the powers of $q$ are equal to 1 since $m'$ is odd, but we conjecture that these formulae remain true for even $m'$. [In this case, the terms $\tilde{f}_ij^m$, $e_i^m$ and $k_{m\epsilon_i+m\epsilon_j}$ are not individually central, but their products are.]

Remark 2: To get the right-hand sides of these relations, one simply replaces each term (including numerical factors) in the expression of $C_1$ (resp. $C_{N-1}$) by its $m$th power. This remarkable relationship seems to hold between $P_{1,m}^{(N)}(C_1, ..., C_{N-1})$ and $C_i$ for the other values of $i$ as well, if $C_i$ is written in a suitable Poincaré–Birkhoff–Witt basis.

Proof of the theorem:

a. We first apply the relations (2.1) and (2.2) in order to write (3.1) and (3.2) and the $C_i$’s in the Poincaré–Birkhoff–Witt basis. Then

$$h\left(P_{1,m}^{(N)}(C_1, ..., C_{N-1})\right) = P_{1,m}^{(N)}(h(C_1), ..., h(C_{N-1})) = \sum_{i=1}^{N} q^{m(N+1)}k_{2m\epsilon_i}$$

(and the corresponding formula with $P_{N-1,m}^{(N)}$). This follows from the definitions of the first section. It then appears that this projection belongs to $Z_0$, and hence so does the whole result ([3] Prop. 6.3.c). This part of the proof also applies to $P_{i,m}^{(N)}(C_1, ..., C_{N-1})$ for $1 < i < N-1$, whereas the second part is limited to the cases $i = 1$ or $i = N-1$.

b. We can then use considerations on the degrees of the monomials appearing in $P_{1,m}^{(N)}$ (and $P_{N-1,m}^{(N)}$) to complete the proof. The term of highest degree of $P_{1,m}^{(N)}$ (resp. $P_{N-1,m}^{(N)}$) is indeed $C_1^m$ (resp. $C_{N-1}^m$), and it is also the only term of degree $m$. According to the form of the $C_i$ (2.4), only monomials of degree at least equal to $m$ can contribute to non-trivial terms belonging to $Z_0$: a necessary condition is indeed that the products of root vectors they contain correspond to an element of the root lattice $R$ belonging to $mR$. For the same reason, the contribution of the monomial of degree $m$ is precisely the second part of the right-hand side of (3.1) (resp. (3.2)).

Relations (3.1), (3.2) differ, for $N > 2$, from the equation in the last remark of [3]. In particular, the degree of the polynomial is different. In the case of $U_q(sl(2))$, the relation (3.1) was already given in [3].

4. Applications

a. Parametrization of generic irreducible representations:

We know from [3] that generic irreducible representations of $U_q(sl(N))$ are characterized by the values of the central elements on them. Once the values of the elements of $Z_0$ are determined, a choice between $m^{N-1}$ values for $C_1, ..., C_N$ remain. A nice way to parametrize them is to write, for a representation $\rho$,

$$\rho(C_1) = c_1(\zeta_1, ..., \zeta_N)$$

(4.1)
with $c_i$ defined in (4.2) and $\prod_{i=1}^{N} \zeta_i = 1$. (Note the absence of $h^{-1}$, by comparison with (2.3).) The $m^{N-1}$ irreducible representations on which the elements of $Z_0$ take the same value simply correspond to the parameters

$$q^{p_1} \zeta_1, \ldots, q^{p_N} \zeta_N,$$

with $p_1, \ldots, p_N \in \mathbb{Z}$ and $\sum_{1}^{N} p_i = 0 \mod m$. Since

$$\rho \left( P_{i,m}^{(N)}(C_1, \ldots, C_{N-1}) \right) = c_i(\zeta_1^m, \ldots, \zeta_N^m) \quad (4.3)$$

for $1 \leq i \leq N - 1$, these sets of parameters indeed correspond to the sets of solutions for the $C_i$'s, to the system of $N - 1$ equations including (3.1) and (3.2).

With this parametrization, the $\zeta_i$ become powers of $q$ when the central elements $e_{ij}^m$, $f_{ij}^m$ and $k_{2m\epsilon_i}$ take the values 0, 0 and 1 respectively. In this highly non-generic case, a finite number of irreducible representations is related to the same parametrization. These representations are $q$-deformations of classical representations.

b. Application to fusion rules:

We suggest that these relations and the above parametrization could help in the study of fusion of unrestricted (generic) irreducible representations of $U_q(sl(N))$, as in [4] in the case of $U(sl(2))$. The strategy would be the following: to evaluate the values of the elements of $Z_0$ in the tensor product of two irreducible representations (they are scalar); find then a solution for the parameters $\zeta_i$ compatible with these values. Then all the irreducible representations characterized by the parameters (4.2) should appear in the fusion rule, with multiplicity 1 in the case generic $\otimes$ minimal–periodic, and with multiplicity $m^{(N-1)(N-2)/2}$ in the case generic $\otimes$ generic.

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