SPACES OF POLYNOMIALS WITH CONSTRAINED REAL DIVISORS, II. (CO)HOMOLOGY & STABILIZATION

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Abstract. In the late 80s, V. Arnold and V. Vassiliev initiated the topological study of the space of real univariate polynomials of a given degree which have no real roots of multiplicity exceeding a given positive integer. Expanding their studies, we consider the spaces $\mathcal{P}_d^\Theta$ of real monic univariate polynomials of degree $d$ whose real divisors avoid given sequences of root multiplicities. These forbidden sequences are taken from an arbitrary poset $\Theta$ of compositions that are closed under certain natural combinatorial operations. We reduce the computation of the homology $H_*(\mathcal{P}_d^\Theta)$ to the computation of the homology of a differential complex, defined purely combinatorially in terms of the given closed poset $\Theta$. We also obtain the stabilization results about $H^*(\mathcal{P}_d^\Theta)$, as $d \to \infty$.

These results are deduced from our description of the homology of spaces $\mathcal{B}_d^\Theta$ whose points are binary real homogeneous forms, considered up to projective equivalence, with similarly $\Theta$-constrained real divisors. In particular, we exhibit differential complexes that calculate the homology of these spaces and obtain some stabilization results for $H^*(\mathcal{B}_d^\Theta)$, as $d \to \infty$. In particular, we compute the homology of the discriminants of projectivized binary real forms and of their complements in $\mathcal{B}_d \cong \mathbb{R}P^d$.

1. Introduction

In this paper, we conduct parallel investigations of the topology of spaces of real univariate polynomials and of projectivized spaces of real bilinear forms with some constraints...

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on their real divisors. The constraints we impose on the real divisors are of a quite general combinatorial nature. Denoting the collection of such constraints by $\Theta$, we are able to describe differential complexes that compute the (co)homology of the former spaces of polynomials/forms entirely in terms of the combinatorics $\Theta$. These differential complexes, which mimic possible bifurcations of real roots, seem to be of independent interest. The rich combinatorics, arising as a byproduct of our constructions, calls for computer experiments. The results of such experiments are assembled in several tables, spread through the text.

In [KSW1], part I of the present paper, we have obtained results about the fundamental groups of spaces of real polynomials with the $\Theta$-restricted root patterns. These results are in the spirit of [Theorem A] stated below.

In what follows, theorems, conjectures etc., labelled by letters, are borrowed from the existing literature, while those, labelled by numbers, are hopefully new.

1.1. Preliminary results. In [Ar], V. Arnold proved Theorems A and B below. Later, these results of Arnold were generalized by V. Vassiliev. In particular, he described the multiplication in the cohomology of the relevant spaces, see [Va, Va0]. All those papers study the topology of spaces of smooth functions/polynomials, which Arnold calls functions/polynomials with "moderate singularities." These works of Arnold and Vassiliev are the main sources of inspiration for our study of functions/polynomials with "less moderate singularities."

In the formulation of Arnold’s theorems, we keep the original notation of [Ar], which we will abandon later on. For $1 \leq k \leq d$, let $G^d_k$ be the space of real monic polynomials $x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{R}[x]$ with no real roots of multiplicity $\geq k$.

**Theorem A.** If $k \leq d < 2k-1$, then $G^d_k$ is diffeomorphic to the product of a sphere $S^{k-2}$ by an Euclidean space. In particular,

$$\pi_i(G^d_k) \cong \pi_i(S^{k-2}) \text{ for all } i.$$  

An analogous result holds for the space of polynomials whose sum of roots vanishes, i.e., the polynomials with the vanishing coefficient $a_{d-1}$.

**Theorem B.** The homology groups with integer coefficients of the space $G^d_k$ are nonzero only for dimensions which are the multiples of $k-2$ and less than or equal to $d$. For $(k-2)r \leq d$, we have

$$H_{r(k-2)}(G^d_k) \cong \mathbb{Z}.$$  

Besides the studies of V. Arnold [Ar] and V. Vassiliev [Va, Va0], the second major motivation for this paper comes from results of the first author, connecting the cohomology of spaces of real polynomials, that avoid the real root patterns from $\Theta$, with the theory of traversing flows on manifolds with boundary in [Ka1], [Ka2], and [Ka3]. For traversing flows (see [Ka1], [Ka2] for the definition) on compact manifolds $X$ that avoid a given collection of tangency patterns $\Theta$ of their trajectories to the boundary $\partial X$, the spaces of polynomials, avoiding the root patterns from $\Theta$, play the fundamental role of classifying spaces. This
role is quite similar to the one played by Graßmannians in the category of vector bundles [Ka4], [Ka5].

1.2. Our set-up. To make this paper independent of [KSW1], we repeat below some basic definitions, notations, and results from [KSW1].

Let $P_d$ denote the space of real monic univariate polynomials of degree $d$. Given a polynomial $P(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in P_d$, we define its real divisor $D_R(P)$ as the multiset

$$x_1 = \cdots = x_{i_1} < x_{i_1+1} = \cdots = x_{i_2+i_1} < \cdots < x_{i_{d-1}+1} = \cdots = x_{i_d}$$

of the real roots of $P(x)$. The ordered $\ell$-tuple $\omega = (\omega_1, \ldots, \omega_\ell)$ of natural numbers describes the real root multiplicity pattern of $P(x)$. Let $R^\omega_d$ be the set of all real polynomials with a real root multiplicity pattern $\omega$, and let $R^\omega_d$ be the closure of $R^\omega_d$ in $P_d$.

For a given collection $\Theta$ of root multiplicity patterns, we consider the union $P^\Theta_d \subset P_d$ of the subspaces $R^\omega_d$, taken over all $\omega \in \Theta$. We denote by $P^\Theta_{d} := P_d \setminus P^\Theta_d$ its complement. Our studies are restricted to the case when $P^\Theta_d$ is closed in $P_d$ and we call such collections $\Theta$ closed.

As has been observed in [KSW1] Lemma 1.2, for any collection $\Theta$ of root multiplicity patterns which contains the pattern $(d)$, the space $P^\Theta_d$ is contractible. Thus topologically interesting situation occurs if one considers the one-point compactification $\bar{P}^\Theta_d$ of $P^\Theta_d$. For a closed $\Theta$, the latter is the union of the one-point compactifications $R^\omega_d$ of the spaces $R^\omega_d$, taken over $\omega \in \Theta$ with points at infinity identified. By Alexander duality in $\bar{P}_d \cong S^d$, we get the relation

$$\tilde{H}^j(P^\Theta_d; \mathbb{Z}) \cong \tilde{H}_{d-j-1}(\bar{P}^\Theta_d; \mathbb{Z})$$

in reduced (co)homology, which implies that the spaces $P^\Theta_{d}$ and $\bar{P}^\Theta_{d}$ carry equivalent (co)homological information.

In particular, for $\Theta$ comprising all $\omega$s such that at least one entry of $\omega$ is greater than or equal to $k$, we have that $\Theta$ is closed and $P^\Theta_{d} \cong C^d_k$ (see [KSW1] Example 1.2).

1.3. Outline of the main results. In §2 we study the cellular structure of $P_d$, given by the cells $R^\omega_d$. In parallel, we investigate a cellular structure on the space $B_d$ of classes of binary homogeneous forms of degree $d$ with real coefficients up to projective equivalence. The cells of $B_d$ are indexed by pairs consisting of a root pattern $\omega$ and a non-negative integer $\kappa$. The integer $\kappa$ represents the root multiplicity at $\infty$. As in the polynomial case we consider for a set $\Theta$ of pairs $(\omega, \kappa)$ the space $B^\Theta_d$ of forms satisfying one of the root multiplicity pattern from $\Theta$ and its complement $B^\Theta_{d} = B_d \setminus B^{\Theta}_d$.

In §3 we turn to (co)homology of the compact spaces $\bar{P}^\Theta_d$, $B^\Theta_d$, and their complements $P^\Theta_{d}$, $B^\Theta_{d}$. In Theorem 3.3 we construct a combinatorial differential complex of free $\mathbb{Z}$-modules that, for a given closed $\Theta$, calculates the homology of $B^\Theta_d$.

Further, we can identify the univariate polynomial $a_0 + \cdots + a_{d-1}x^{d-1} + x^d$ of degree $d$ with the class of the homogeneous binary form $a_0 y^d + \cdots + a_{d-1} x^{d-1}y + x^d$ of degree $d$ with no (real) roots at infinity. Hence we can identify $\bar{P}_d$ with a quotient of $B_d$ by the closed
subspace $B_{d-1}$. Based on this identification, for a given closed $\Theta$, Corollary 3.5 introduces the combinatorial complex that calculates the homology of the space of $P_{d}^\Theta$. The latter complex is the restriction of the combinatorial complex from Theorem 3.3 (which computes the homology of $B_{d}^\Theta$) to the cells corresponding to univariate degree $d$ polynomials. Via Alexander duality, this complex also computes the cohomology groups of $P_{d}^\Theta$.

In § 4, we discuss what happens to the homology of $B_{d}^\Theta$ and $P_{d}^\Theta$ when $\Theta$ is fixed and $d \to \infty$. Our main stabilization results are Theorem 4.9, Theorem 4.11, Theorem 4.17 and Theorem 4.19. For a closed $\Theta$ and a $k$ bounded by a function of $d$, it claims the stabilization of the $k$th homology groups $H_{k}(B_{d}^\Theta; \mathbb{Z})$ and $H_{k}(P_{d}^\Theta; \mathbb{Z})$. For so called profinite $\Theta$ the result holds for any $k$ and $d$ large enough.

Finally, in § 5, we present the results of a variety of computer experiments calculating the homology of $\bar{P}_{\Theta}^d$ for various $\Theta$ and $d$.

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2. Combinatorics and geometry of the cell structure on spaces of real polynomials and real binary forms

Let $B_d$ denote the space of non-zero real bivariate homogeneous polynomials of the form

$$a_0 x^0 y^d + \cdots + a_d x^d y^0,$$

being considered up to a non-zero real scalar factor.

We have already associated to a polynomial $P(x) \in \mathbb{R}[x]$ its real root multiplicity pattern $(\omega_1, \ldots, \omega_\ell)$. Next, we define the combinatorial structure which will govern the cell decomposition of the space $B_d$. Topologically $P_d$ is a real $d$-dimensional affine space, while $B_d$ is the real $d$-dimensional projective space $\mathbb{R}P^d$. For $d' \leq d$, the map $\varphi$, given by

$$a_0 + \cdots + a_{d'-1} x^{d'-1} + x^{d'} \mapsto a_0 x^0 y^d + \cdots + a_{d'-1} x^{d'-1} y^{d-d'+1} + x^{d'} y^{d-d'} \in P_{d'} \subseteq B_d,$$

is a homeomorphism of $P_{d'}$ with a subspace of $B_d$. The images for different $d'$ are disjoint. We obtain the standard decomposition $B_d = P_d \sqcup P_{d-1} \sqcup P_{d-2} \sqcup \cdots \sqcup P_0$. Here $P_0$ can be identified with the one-point space of constant non-zero polynomials up to a non-zero factor. Let us define the combinatorial structures which capture and refine this natural stratification.
The space \( \mathcal{P}_d \) may be also identified with the space of positive degree \( d \) divisors on the complex plane \( \mathbb{C} \), invariant under the complex conjugation \( J : z \to \bar{z} \). At the same time, \( \mathcal{B}_d \) may be identified with the space of positive degree \( d \) divisors on the Riemann sphere \( \mathbb{CP}^1 \) invariant under the complex conjugation \( J : [z : w] \to [\bar{z} : \bar{w}] \) where \([z : w]\) are homogeneous coordinates on \( \mathbb{CP}^1 \).

**Definition 2.1.** An arbitrary sequence \( \omega = (\omega_1, \ldots, \omega_\ell) \) of positive integers is called a composition. We say that \( \omega \) is a composition of the number \( |\omega| := \omega_1 + \cdots + \omega_\ell \). We also allow the empty composition \( \omega = () \) of the number \(|()| = 0 \). We call the \( \omega_i \), \( i = 1, \ldots, \ell \), the parts of the composition \( \omega \).

Similarly, a pair \((\omega, \kappa)\) is called a marked composition if \( \omega \) is a composition and \( \kappa \) is a non-negative integer. For \( \omega = (\omega_1, \ldots, \omega_\ell) \), we say that \((\omega, \kappa)\) is a marked composition of the number \(|(\omega, \kappa)| := |\omega| + \kappa \).

For a composition \( \omega \), (resp., a marked composition \((\omega, \kappa)\)) we call \(|\omega| \) (resp., \(|(\omega, \kappa)|\)) its norm and \(|\omega'| := |\omega| - \ell \) (resp., \(|(\omega, \kappa)'| = |\omega'| + \kappa \)) its reduced norm.

If \( \kappa = 0 \), then we often identify the marked composition \((\omega, \kappa)\) with \( \omega \) and speak of the norm \(|\omega| = |(\omega, 0)|\) and the reduced norm \(|\omega'| = |(\omega, 0)'|\) of \( \omega \).

For \( f = a_0x^0y^d + \cdots + a_dx^dy^0 \in \mathcal{B}_d \), we define its real degree \( d' \) as \( \max\{i \mid a_i \neq 0\} \). As in (2.1), \( f \) can be identified with a polynomial \( f^1 \) in \( \mathcal{P}_{d'} \) such that \( \varphi(f^1) = f \).

Let \([x : y] \in \mathbb{RP}^1\) be a real root of \( f \). If \( y \neq 0 \), then we can choose \( y = 1 \) and \( x \) as a real root of \( f^1 \). If \( y = 0 \), then we can choose \( x = 1 \) and identify \((1 : 0) \) with the point at infinity in \( \mathbb{RP}^1 \). Thus we can describe the root multiplicities of \( f \) by a marked composition \((\omega, \kappa)\), where \( \omega \) is a composition with \(|\omega| \leq d' = d - \kappa \) and \( \kappa \) is the multiplicity of the root at infinity. Any polynomial in \( \mathcal{P}_0 \subseteq \mathcal{B}_d \) is associated with the marked composition \(((), d)\).

Let \((\omega, \kappa)\) be a marked composition for which \( d - |(\omega, \kappa)| \) is even and non-negative. Using \( \varphi \) from (2.1), we set \( \hat{\mathcal{R}}_{d'}^{(\omega, \kappa)} := \hat{\mathcal{R}}_{d'}^{\omega} \subseteq \mathcal{P}_{d'} \subseteq \mathcal{B}_d \) where \( d' = d - \kappa \). Analogously to the univariate case, we write \( \hat{\mathcal{R}}_d^{(\omega, \kappa)} \) for the closure of \( \hat{\mathcal{R}}_{d'}^{(\omega, \kappa)} \) in \( \mathcal{B}_d \). Evidently, for a given composition \( \omega \), the stratum \( \hat{\mathcal{R}}_d^{\omega} \) is empty if and only if either \(|\omega| > d \), or \(|\omega| \leq d \) and \( d - |\omega| \) is odd. Thus if \((\omega, \kappa)\) is a marked composition, then \( \hat{\mathcal{R}}_d^{(\omega, \kappa)} \) is empty if and only if either \(|(\omega, \kappa)| > d \) or \(|(\omega, \kappa)| \leq d \) and \( d - |(\omega, \kappa)| \) is odd. Note that for a composition \( \omega \), we have \( \hat{\mathcal{R}}_d^{\omega} = \hat{\mathcal{R}}_d^{(\omega, 0)} \). However, in general, \( \hat{\mathcal{R}}_d^{\omega} \neq \hat{\mathcal{R}}_d^{(\omega, 0)} \) since the closures are taken in different spaces.

For a given collection \( \Theta \) of marked compositions, consider the union \( \mathcal{B}_d^{\Theta} \) of the subspaces \( \hat{\mathcal{R}}_d^{(\omega, \kappa)} \) over all \( (\omega, \kappa) \in \Theta \). We denote its complement by \( \mathcal{B}_d^{\complement \Theta} := \mathcal{B}_d \setminus \mathcal{B}_d^{\Theta} \). Similarly, for a given collection of compositions \( \Theta \), we write \( \mathcal{P}_d^{\Theta} \) for the union of the subspaces \( \hat{\mathcal{R}}_d^{\omega} \) over all \( \omega \in \Theta \). Further, set \( \mathcal{P}_d^{\complement \Theta} := \mathcal{P}_d \setminus \mathcal{P}_d^{\Theta} \).

We restrict our studies to the case when either \( \mathcal{B}_d^{\Theta} \) is closed in \( \mathcal{B}_d \) or when \( \mathcal{P}_d^{\Theta} \) is closed in \( \mathcal{P}_d \). In such situations, we call the respective collection \( \Theta \) closed.

As observed in [KSW1, Lemma 1.2], for any collection \( \Theta \) of compositions containing \((d)\), the space \( \mathcal{P}_d^{\Theta} \) is contractible. Thus \( \mathcal{P}_d^{\Theta} \) is contractible for any closed \( \Theta \). As a consequence, we concentrate on the one-point compactification \( \mathcal{P}_d^{\Theta} = \mathcal{P}_d^\Theta \sqcup \infty \) of \( \mathcal{P}_d^{\Theta} \). It has a non-trivial
set of all compositions, \( \Omega \), are closed in \( B \) mod 2, and topology related to \( P \).

We denote by \( \Omega^\infty \) the set of all marked compositions \((\omega, \kappa)\). For a given positive integer \( d \), we denote by \( \Omega^\infty_{(d)} \) (resp., \( \Omega^\infty_d \)) the set of all marked compositions \((\omega, \kappa)\), such that \(|\omega, \kappa| \leq d \) and \(|\omega, \kappa| \equiv d \) mod 2 (resp., \(|\omega, \kappa| = d \)). Analogously we write \( \Omega \) for the set of all compositions, \( \Omega_{(d)} \) for the set of all compositions \( \omega \) for which \(|\omega| \leq d \) and \(|\omega| \equiv d \) mod 2, and \( \Omega_d \) for the set of all compositions \( \omega \) with \(|\omega| = d \).

Let us define two kinds operations on \( \Omega \) and on \( \Omega^\infty \), the merge and the insert operations, which will be instrumental in what follows.

For a marked composition \((\omega_1, \ldots, \omega_{\ell}) \in \Omega \) and an integer \( 1 \leq j \leq \ell - 1 \), we define \( M_j(\omega) \) as

\[
M_j(\omega) = (M_j(\omega)_1, \ldots, M_j(\omega)_{\ell-1}),
\]

with

\[
M_j(\omega)_i = \omega_i \quad \text{if} \quad i < j,
M_j(\omega)_j = \omega_j + \omega_{j+1},
M_j(\omega)_i = \omega_{i+1} \quad \text{if} \quad i+1 < j \leq \ell - 1.
\]

For a marked composition \((\omega, \kappa)\), we set

\[
M^\infty_j((\omega, \kappa)) = (M_j(\omega), \kappa) \text{ for } 1 \leq j \leq \ell - 1,
\]

\[
M^\infty_0((\omega, \kappa)) = ((\omega_2, \ldots, \omega_{\ell}), \kappa + \omega_1), \text{ and } M^\infty_\ell((\omega, \kappa)) = ((\omega_1, \ldots, \omega_{\ell-1}), \kappa + \omega_{\ell}).
\]

We call \( M_j, M^\infty_j \) the merge operations on compositions and marked compositions.

In parallel, we define the insert operations \( I_j \). For a composition \( \omega = (\omega_1, \ldots, \omega_{\ell}) \) and a number \( 1 \leq j \leq \ell + 1 \) we set \( I_j(\omega) = (I_j(\omega)_1, \ldots, I_j(\omega)_{\ell+1}) \), where

\[
I_j(\omega)_i = \omega_i \quad \text{if} \quad i < j,
I_j(\omega)_j = 2,
I_j(\omega)_i = \omega_{i-1} \quad \text{if} \quad j < i \leq \ell + 1.
\]

We extend \( I_j \) to marked compositions by defining \( I^\infty_j((\omega, \kappa)) = (I_j(\omega), \kappa) \).

Example 2.2. For \((\omega, \kappa) = ((122243), 2)\) we have

\[
M^\infty_0((\omega, \kappa)) = ((2243), 3), \quad M^\infty_1(\hat{\omega}) = ((3243), 2), \quad M^\infty_2(\hat{\omega}) = ((4343), 2),
M^\infty_3((\omega, \kappa)) = ((1263), 2), \quad M^\infty_4(\hat{\omega}) = ((2127), 2), \quad M^\infty_5(\hat{\omega}) = ((2224), 5),
\]

\[
I^\infty_0((\omega, \kappa)) = ((212242), 2), \quad I^\infty_1(\hat{\omega}) = ((122242), 2), \quad I^\infty_2(\hat{\omega}) = ((222242), 2),
I^\infty_3((\omega, \kappa)) = ((122242), 2), \quad I^\infty_4(\hat{\omega}) = ((122242), 2).
\]

Note that for example for \( d = 14 \) we have \(|I_j((\omega, \kappa))| = 16 \) and hence \( R_d^j((\omega, \kappa)) = \emptyset \) for all \( 1 \leq j \leq \ell + 1 \) while all for all \( 1 \leq j \leq \ell - 1 \) we have \( R_d^{M_j((\omega, \kappa))} \neq \emptyset \).
The next proposition collects some basic properties of the cells $\hat{R}_d^\omega$ and of their closures $\bar{R}_d^\omega$ in $\mathcal{P}_d$, see [Ka1] Theorem 4.1 for details. In Proposition C we will write $\mathbb{H}$ to denote the upper half-plane of the complex numbers with positive imaginary part, $\text{Sym}^m(X)$ for the $m$-fold symmetric product of a space $X$, and $\Omega_m$ for the open cone $\{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_1 < \cdots < x_m\}$. For later use, we also introduce $\text{Conf}^m(X)$ as a notation for the configuration space of $m$ distinct unordered points in a space $X$. Note that $\Omega_m \cong \text{Conf}^m(\mathbb{R})$.

**Proposition C.** Take $d \geq 1$ and $\omega = (\omega_1, \ldots, \omega_\ell) \in \Omega$ such that $|\omega| \leq d$ and $|\omega| \equiv d \mod 2$. Then $\bar{R}_d^\omega \subset \mathcal{P}_d$ is homeomorphic to $\Pi_\ell \times \text{Sym}^{d-|\omega|/2}\mathbb{H}$. In particular, it is an open cell of codimension $|\omega|'$. Moreover, $\bar{R}_d^\omega$ is the union of the cells $\hat{R}_d^\omega$, taken over all $\omega'$ that are obtained from $\omega$ by a sequence of merge and insert operations. In particular,

(a) The cell $\hat{R}_d^\omega$ has the maximal dimension $d$ if and only if, for $0 \leq \ell \leq d$ and $\ell \equiv d \mod 2$, $\omega = (1,1,\ldots,1)$. 
(b) The cell $\hat{R}_d^\omega$ has the minimal possible dimension 1 if and only if $\omega = (d)$. In this case, $\bar{R}^{(d)} = R^{(d)} = \{(x-a)^d \mid a \in \mathbb{R}\}$.

From a geometric perspective, if a point in $\bar{R}_d^\omega$ approaches the boundary $\bar{R}_d^\omega \setminus \hat{R}_d^\omega$, then either there exist at least one value of the index $j$ such that the distance between the $j$th and $(j+1)$th distinct real roots goes to 0, or there is a value of $j$ such that two complex-conjugate non-real roots converge to a real root (which is then the $j$th largest). The first situation corresponds to the application of the merge operation $M_j$ to $\omega$, and the second one to the application of the insertion operation $I_j$.

For $d' \leq d$, under the embedding $\varphi : \mathcal{P}_{d'} \hookrightarrow \mathcal{B}_d$, the cell $\bar{R}_d^{\omega'}$ is sent to $\hat{R}_d^{(\omega,d-d')}$. The image $\varphi(\mathcal{P}_{d'})$ lies in the closed subspace $\mathcal{B}_{d'} \cong \mathbb{R}^{d'}$ of $\mathcal{B}_d$ of codimension $d-d'$. As described in Proposition C, in addition to the cells $\hat{R}_d^{(\omega',d-d')}$, which are the $\varphi$-images of the cells $\bar{R}_d^{\omega'}$, the boundary of $\bar{R}_d^{\omega'}$ additionally contains only the cells which lie in the $\varphi$-image of $\mathcal{P}_{d'}$ for some $d'' < d'$. The latter arise when the smallest or the largest real root of some $f \in \mathcal{P}_{d'}$ approaches infinity. Since the latter situation is described by the merge operations $M_0^\infty$ and $M_\ell^\infty$ acting on marked compositions $((\omega_1, \ldots, \omega_\ell), \kappa)$, we obtain the following result.

**Corollary D.** Let $d \geq 1$, $\omega = (\omega_1, \ldots, \omega_\ell) \in \Omega$, and let $(\omega,\kappa) \in \Omega^\infty$ be such that $|\omega,\kappa| \leq d$ and $|\omega,\kappa| \equiv d \mod 2$. Then $\hat{R}_d^{(\omega,\kappa)} \subset \mathcal{B}_d$ is homeomorphic to $\Pi_\ell \times \text{Sym}^{d-|\omega,\kappa|/2}\mathbb{H}$. In particular, $\hat{R}_d^{(\omega,\kappa)}$ is an open cell of codimension $|\omega,\kappa|'$.

Moreover, $\hat{R}_d^{(\omega,\kappa)}$ is the union of cells $\hat{R}_d^{(\omega',\kappa')}$, taken over all $(\omega',\kappa')$ that are obtained from $(\omega,\kappa)$ by a sequence of merge and insert operations. In particular,

(a) the cell $\hat{R}_d^{(\omega,\kappa)}$ has the maximal dimension $d$ if and only if $\omega = (1,1,\ldots,1)$; for some $0 \leq \ell \leq d$, $\ell \equiv d \mod 2$ and $\kappa = 0$.
(b) the cell $\hat{R}_d^{(\omega,\kappa)}$ has the minimal possible dimension 0 and only if $\omega = ()$ and $\kappa = d$. In this case, $\bar{R}_d^{(0,d)} = \mathcal{P}_0 = \{a \mid a \in \mathbb{R} \setminus \{0\}\}/\sim$. 


Note that the norms $|\omega|$ and $|(\omega, \kappa)|$ are preserved under the merge operations for cells both in $\mathcal{P}_d$ and in $\mathcal{B}_d$. In both cases, the insert operations increase the norm by 2 and thus preserve its parity.

The merge and insert operations can be used to define a natural *partial order* “$\prec$” on the sets $\Omega$ and $\Omega^\infty$. It reflects the cellular structure, described in Proposition C and Corollary D.

**Definition 2.3.** For $\omega, \omega' \in \Omega$, we set $\omega' \prec \omega$, if $\omega'$ can be obtained from $\omega$ by a sequence of merge operations, $M_j$, $j \geq 1$, and insert operations $I_j$, $j \geq 0$.

Analogously, for $(\omega, \kappa), (\omega', \kappa') \in \Omega^\infty$, we set $(\omega', \kappa') \prec (\omega, \kappa)$, if $(\omega', \kappa')$ can be obtained from $(\omega, \kappa)$ by a sequence of merge operations, $M_j^\infty$, $j \geq 0$, and insert operations $I_j^\infty$, $j \geq 0$.

If $\omega' \prec \omega$ or $(\omega', \kappa') \prec (\omega, \kappa)$, then we say that $\omega'$ is smaller than $\omega$ or that $(\omega', \kappa')$ is smaller than $(\omega, \kappa)$.

One can easily check that “$\prec$” defines partial orders on $\Omega$ and $\Omega^\infty$. From now on, we will consider a subset $\Theta \subseteq \Omega$ or $\Theta \subseteq \Omega^\infty$ as a partially ordered set, *poset* for short, ordered by ”$\prec$.” Proposition C and Corollary D imply instantly the following two statements.

**Corollary E.** For a subset $\Theta \subseteq \Omega_{[d]}$,

(i) $\mathcal{P}_d^\Theta$ is closed in $\mathcal{P}_d$ if and only if, for any $\omega \in \Theta$ and $\omega' \prec \omega$, we have $\omega' \in \Theta$,

(ii) if $\mathcal{P}_d^\Theta$ is closed in $\mathcal{P}_d$, then $\mathcal{P}_d^\Theta$ carries the structure of a compact CW-complex with open cells $R_d^\omega$, labeled by $\omega \in \Theta$, and the unique 0-cell, represented by the point $\infty$.

**Corollary F.** For a subset $\Theta \subseteq \Omega_{d}$,

(i) $\mathcal{B}_d^\Theta$ is closed in $\mathcal{B}_d$ if and only if, for any $(\omega, \kappa) \in \Theta$ and $(\omega', \kappa') \prec (\omega, \kappa)$, we have $(\omega', \kappa') \in \Theta$,

(ii) if $\mathcal{B}_d^\Theta$ is closed in $\mathcal{B}_d$, then $\mathcal{B}_d^\Theta$ carries the structure of a compact CW-complex with open cells $R_d^{(\omega, \kappa)}$, labeled by $(\omega, \kappa) \in \Theta$.

Corollary E and Corollary F motivate the following definition.

**Definition 2.4.** A subposet $\Theta \subseteq \Omega$ is called *closed* if, for any $\omega' \prec \omega$ and $\omega \in \Theta$, we have $\omega' \in \Theta$.

Similarly, a subposet $\Theta \subseteq \Omega^\infty$ is called *closed* if, for any $(\omega', \kappa') \prec (\omega, \kappa)$ and $(\omega, \kappa) \in \Theta$, we have $(\omega', \kappa') \in \Theta$.

Revisiting the beginning of §1, the closed posets $\Theta$ yield exactly the spaces $\mathcal{P}_d^\Theta$, $\mathcal{P}_d^{\infty}$ and $\mathcal{B}_d^\Theta$, $\mathcal{B}_d^{\infty}$ we intend to study.

In §3 below, given any closed poset $\Theta \subseteq \Omega_{[d]}$ or $\Theta \subseteq \Omega^\infty_{[d]}$, we introduce a *combinatorial model* for the cellular chain complex that calculates the reduced homology of $\bar{\mathcal{P}}_d^\Theta$ or $\bar{\mathcal{B}}_d^\Theta$.

However, for a general closed poset $\Theta \subseteq \Omega_{(d)}$, it is impossible to obtain a *closed formula* description of the homology $H_*(\bar{\mathcal{P}}_d^\Theta; \mathbb{Z})$. To justify this “metaphysical” claim, consider the closed subposet $\Omega_d \subseteq \Omega_{[d]}$ of all $\omega \in \Omega_{[d]}$ with $|\omega| = d$. The space $\mathcal{P}_d^{\Omega_d}$ is the space of
all real monic polynomials of degree \(d\), having only real roots. We can identify \(\mathcal{P}_d^{\Omega_d}\) with 
\[
\{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1 < \cdots < x_d\},
\]
the closure of \(\Pi_d\).

As a result, the cellular structure of \(\mathcal{P}_d^{\Omega_d}\) is the facial structure of the product of a 
\((d - 1)\)-dimensional cone \(\mathcal{C}\Delta^{n-2}\) over a closed simplicial base \(\Delta^{n-2}\) with a copy of the 
real line \(\mathbb{R}\). The compositions in \(\Omega_d\) are in an order-preserving bijection with the open 
faces of the simplicial cone \(\mathcal{C}\Delta^{n-2}\); so that \(\hat{\mathcal{R}}^d\) is the face corresponding to \(\omega\). Since 
the face poset of \(\mathcal{C}\Delta^{n-2}\) is the Boolean lattice of subsets of \(\{1, \ldots, d - 1\}\), it follows that one 
can identify simplicial complexes \(K\) with vertex set contained in \(\{1, \ldots, d - 1\}\) and closed 
subposets of \(\Omega_d\). Then, for a closed subposet \(\Theta \subseteq \Omega_d\), the one-point compactification \(\mathcal{P}_d^{\Theta}\) is 
a double suspension the corresponding simplicial complex \(K\) whose vertexes are among 
\(\{1, \ldots, d - 1\}\). This situation is considered in details in earlier paper of the second and the 
third author [SW].

In particular, for arbitrary closed subposets \(\Theta \subseteq \Omega_d\), the spaces \(\mathcal{P}_d^{\Theta}\) can, up to a shift 
by 2 in the (co)homological dimension, carry any homology that a simplicial complex on 
\(d - 1\) vertices can carry. Since \(S^d \cong \mathcal{P}_d\), Alexander duality shows that the cohomology of 
\(\mathcal{P}_d^{\Theta}\) can have a similarly arbitrary complex structure! In particular, arbitrary torsion may 
occur.

However, let us stress that in this example, the dimension of the simplicial complex, cor-
responding to \(\Theta\), and the degree \(d\) of the polynomials under consideration are closely linked. 
When it is possible to loosen this link, then, as we shall see in §4, the (co)homological 
stabilization takes place, and quite “tame” answers for the limiting homology \(\hat{H}_*(\mathcal{P}_d^{\Theta}, \mathbb{Z})\) or 
\(\hat{H}_*(\mathcal{B}_d^{\Theta}, \mathbb{Z})\) emerge as \(d \to \infty\).

Before we further pursue our topological investigation, we would like to present a few 
combinatorial facts about the number of cells in \(\mathcal{P}_d\) and \(\mathcal{B}_d\). Let \(p^d_j\) be the number of cells in 
\(\mathcal{P}_d\) of dimension \(j\), and let \(b^d_j\) be the number of cells in \(\mathcal{B}_d\) of dimension \(j\). By Proposition C 
and Corollary D, we know that \(p^d_0 = 1\) and, for \(j \geq 1\), \(p^d_j\) counts the compositions \(\omega \in \Omega_{\langle d \rangle}\) 
with \(d - |\omega| = j\). Similarly, for \(j \geq 0\), \(b^d_j\) counts the marked compositions \((\omega, \kappa) \in \Omega^\infty_{\langle d \rangle}\) 
with \(d - |(\omega, \kappa)| = j\).

We introduce the generating \(t\)-polynomials

\[
G(\mathcal{P}_d; t) = \sum_{j=0}^{d} p^d_j t^j; \quad \text{and} \quad G(\mathcal{B}_d; t) = \sum_{j=0}^{d} b^d_j t^j.
\]

Interpreting the value of the generating functions at \(t = -1\) as the Euler characteristics of 
\(\mathcal{P}_d \cong S^d\) and \(\mathcal{B}_d \cong \mathbb{P}\mathbb{R}^d\), we get that \(G(\mathcal{P}_d, -1) = 1 + (-1)^d\) and \(G(\mathcal{B}_d, -1) = \frac{1}{2}(1 + (-1)^d)\). 

Proposition C (a) implies that \(p^d_0 = \left\lfloor \frac{d}{2} \right\rfloor + 1\).

Since \(\mathcal{B}_d = \mathcal{P}_d \sqcup \mathcal{B}_{d-1}\), we get for \(d \geq 1\) the recurrence relation

\[
G(\mathcal{B}_d; t) = G(\mathcal{B}_{d-1}; t) + (G(\mathcal{P}_d; t) - 1).
\]
This recurrence implies the following relation for the \((t, x)\)-generating functions
\[
\sum_{d \geq 0} G(B_d, t) x^d = 1 + x \sum_{d \geq 0} G(B_d, t) x^d + \sum_{d \geq 0} G(\mathcal{P}_d, t) x^d - 2 - \frac{x}{1 - x}.
\]

Note that the 2 in the recursion comes from \(\mathcal{P}_0\) consisting of 2 points. In particular,
\[
\sum_{d \geq 0} G(B_d, t) x^d = \frac{1}{1 - x} \left( -1 + \sum_{d \geq 0} G(\mathcal{P}_d, t) x^d - \frac{x}{1 - x} \right).
\]

Now, for \(j \geq 1\), \(p^d_j\) equals the number of compositions of a number \(d' \leq d\) with even \(d - d'\) and \(j - (d - d')\) parts. Thus for \(j \geq 1\),
\[
p^d_j = \sum_{\substack{d - j \leq d' \leq d \\ d - d' \text{ even}}} \left( \frac{d' - 1}{j - d + d' - 1} \right)^{\min\{\left\lfloor \frac{j - 1}{2} \right\rfloor, \left\lfloor \frac{d - 1}{2} \right\rfloor\}} \sum_{k=0}^\infty \left( \frac{d - 1 - 2k}{j - 1 - 2k} \right).
\]

The latter observations implies via some standard calculations of generating functions the following result.

**Lemma 2.5.** The number of cells in \(\mathcal{P}_d\) and \(B_d\) is given by the following generating functions:

(i) \[
\sum_{d=0}^\infty G(\mathcal{P}_d; t) x^d = \frac{t^3 x^3 + t^2 x^3 - x^2 t^2 - tx + x^2 - 3 x + 2}{(1 - t^2 x^2) (1 - x - tx) (1 - x)}
\]

(ii) \[
\sum_{d=0}^\infty G(B_d, t) x^d = \frac{1}{(1 - t^2 x^2) (1 - x - tx)}
\]

(iii) For any positive odd \(d\), \(G(\mathcal{P}_d; t) - 1 = t \cdot G(B_{d-1}; t)\), while for any positive even \(d\),
\(G(\mathcal{P}_d; t) - 1 = t \cdot G(B_{d-1}) + t^d\). \(\diamondsuit\)

**Example 2.6.** \(G(B_0; t) = 1\), \(G(B_1; t) = 1 + t\), \(G(B_2; t) = 1 + 2t + 2t^2\), \(G(B_3; t) = 1 + 3t + 4t^2 + 2t^3\), \(G(B_4; t) = 1 + 4t + 7t^2 + 6t^3 + 3t^4\),
\(G(B_5; t) = 1 + 5t + 11t^2 + 13t^3 + 9t^4 + 3t^5\).
\(G(\mathcal{P}_1; t) = 1 + t\), \(G(\mathcal{P}_2; t) = 1 + t + 2t^2\), \(G(\mathcal{P}_3; t) = 1 + t + 2t^2 + 2t^3\),
\(G(\mathcal{P}_4; t) = 1 + t + 3t^2 + 4t^3 + 3t^4\), \(G(\mathcal{P}_5; t) = 1 + t + 4t^2 + 7t^3 + 6t^4 + 3t^5\),
\(G(\mathcal{P}_6; t) = 1 + t + 5t^2 + 11t^3 + 13t^4 + 9t^5 + 4t^6\). \(\diamondsuit\)

The coefficient sequences of \(G(\mathcal{P}_d, t)\) and \(G(B_d, t)\), up to a shift in indexing and removals of trailing or leading 1s, also has appeared in other contexts (see A035317 in [SI]).
3. Combinatorial differential complexes for $\tilde{H}_*(B^\Theta_d;\mathbb{Z})$ and $\tilde{H}_*(\tilde{P}^\Theta_d;\mathbb{Z})$

In this section, we use the natural CW-structure on $B_d$ and $\tilde{P}_d$, described in [Corollary D] and [Proposition C], to construct combinational differential complexes that calculate the homology of $B^\Theta_d$ and $\tilde{P}^\Theta_d$ for any given closed subposet $\Theta \subset \Omega^\infty_{(d)}$ or $\Theta \subset \Omega_{(d)}$.

Recall that the cells $\tilde{R}_d^{(\omega,\kappa)}$ of the CW-complex $B_d$ are indexed by marked compositions $(\omega, \kappa) \in \Omega^\infty_{(d)}$ and that the dimension of the cell $\tilde{R}_d^{(\omega,\kappa)}$ equals $d - |(\omega, \kappa)|' = d - |\omega|' - \kappa$.

Next, we provide an explicit description of the cellular chain complex of the CW-complex $B^\Theta_d$. From now on, for $\Theta \subseteq \Omega_{(d)}$, or $\Theta \subseteq \Omega^\infty_{(d)}$, we write $\mathbb{Z}[\Theta]$ for the abelian group or equivalently $\mathbb{Z}$-module, freely generated by the elements of $\Theta$. For $\Theta \subseteq \Omega^\infty$ we define $\Theta|_{\sim|' \leq k}$ (resp., $\Theta|_{\sim|' = k}$) as the set of all (marked) compositions $(\omega, \kappa) \in \Theta$ with $|(\omega, \kappa)|' \leq k$ (resp., $|(\omega, \kappa)|' = k$). We use the analogous conventions for $\Theta \subseteq \Omega_d$.

For a composition $\omega = (\omega_1, \ldots, \omega_l)$, we denote by $s_\omega$ the number of its parts, i.e. $s_\omega = \ell$. Using the merge operators $M$ and the insert operators $I$, introduced in §2, we define two homomorphisms which act on $\mathbb{Z}[\Theta]$. They are given by

$$\partial^\infty_M((\omega, \kappa)) := -\sum_{k=0}^{s_\omega} (-1)^k M_k((\omega, \kappa))$$

and

$$\partial^\infty_I((\omega, \kappa)) := \sum_{k=0}^{s_\omega} (-1)^k I_k((\omega, \kappa)).$$

Next, we define a homomorphism

$$\partial^\infty = \partial^\infty_M + \partial^\infty_I : \mathbb{Z}[\Theta] \to \mathbb{Z}[\Theta]$$

by the formula

$$\partial^\infty((\omega, \kappa)) := \begin{cases} -\sum_{k=0}^{s_\omega} (-1)^k M_k((\omega, \kappa)) + \sum_{k=0}^{s_\omega} (-1)^k I_k((\omega, \kappa)), & \text{for } |\omega| < d, \\ -\sum_{k=0}^{s_\omega} (-1)^k M_k((\omega, \kappa)), & \text{for } |\omega| = d. \end{cases}$$

**Lemma 3.1.** For any closed $\Theta \subseteq \Omega_{(d)}$, the homomorphisms $\partial^\infty_M, \partial^\infty_I : \mathbb{Z}[\Theta] \to \mathbb{Z}[\Theta]$ are anticommuting differentials, i.e. $(\partial^\infty_I)^2 = (\partial^\infty_M)^2 = \partial^\infty_M \partial^\infty_I + \partial^\infty_I \partial^\infty_M = 0$.

Thus, $\partial^\infty = \partial^\infty_M + \partial^\infty_I$ is a differential as well, and $(\mathbb{Z}[\Theta], \partial)$ is a graded differential complex, whose $j$-th graded part is $\mathbb{Z}[\Theta|_{\sim|' = d-j}]$.

**Proof.** Let us first show that $(\partial^\infty_I)^2 = (\partial^\infty_M)^2 = \partial^\infty_M \partial^\infty_I + \partial^\infty_I \partial^\infty_M = 0$. Consider a marked composition $(\omega, \kappa)$ with $\omega = (\omega_1, \ldots, \omega_l)$. Then for $\ell \geq 3$, the expression for $\partial^\infty_M((\omega, \kappa))$ will, in particular, include the terms of the form

$$(-1)^k((\omega_1, \ldots, \omega_{k-1}, \omega_k + \omega_{k+1}, \omega_{k+2}, \ldots, \omega_l), \kappa) + (-1)^{k+1}(\omega_1, \ldots, \omega_k, \omega_{k+1} + \omega_{k+2}, \omega_{k+3}, \ldots, \omega_l), \kappa).$$

Thus $(\partial^\infty_M)^2((\omega, \kappa))$ will contain the vanishing sum

$$(-1)^{2k}((\omega_1, \ldots, \omega_{k-1}, \omega_k + \omega_{k+1} + \omega_{k+2}, \omega_{k+3}, \ldots, \omega_l), \kappa) + (-1)^{2k+1}((\omega_1, \ldots, \omega_{k-1}, \omega_k + \omega_{k+1} + \omega_{k+2}, \omega_{k+3}, \ldots, \omega_l), \kappa) = 0.$$
For $\ell \geq 2$, the homomorphism $\partial^\infty_M((\omega, \kappa))$ will also contain the terms

$$((\omega_2, \ldots, \omega_\ell), \omega_1 + \kappa) - ((\omega_1 + \omega_2, \omega_3, \ldots, \omega_\ell), \kappa),$$

which yields

$$((\omega_1, \ldots, \omega_\ell), \omega_1 + \omega_2 + \kappa) - ((\omega_3, \ldots, \omega_\ell), \omega_1 + \omega_2 + \kappa) = 0$$

in $(\partial^\infty_M)^2((\omega, \kappa))$. For $\ell = 1$, $\partial^\infty_M((\omega_1), \kappa) = 0$ by definition.

To show that $(\partial^\infty)^2 = 0$, write

$$\partial^\infty((\omega)) = \cdots + (-1)^k((\omega_1, \ldots, \omega_k, 2, \omega_{k+1}, \ldots, \omega_\ell), \kappa) + \ldots$$

Thus $(\partial^\infty)^2((\omega, \kappa))$ will only consist of the vanishing sums of the form

$$(-1)^{2k}(\omega_1, \ldots, \omega_k, 2, \omega_{k+1}, \ldots, \omega_\ell), \kappa) + (-1)^{k+1}(\omega_1, \ldots, \omega_k, 2, \omega_{k+1}, \ldots, \omega_\ell), \kappa) = 0.$$

Finally, let us compute $(\partial^\infty I^\infty_M + \partial^\infty_M I^\infty)((\omega, \kappa))$. Observe that $\partial^\infty((\omega, \kappa))$ consists of the sums of the form

$$(-1)^k((\omega_1, \ldots, \omega_k, 2, \omega_{k+1}, \ldots, \omega_\ell), \kappa) +$$

$$(-1)^{k+1}((\omega_1, \ldots, \omega_{k+1}, 2, \omega_{k+2}, \ldots, \omega_\ell), \kappa) + (-1)^{k+2}((\omega_1, \ldots, \omega_{k+2}, 2, \omega_{k+3}, \ldots, \omega_\ell), \kappa).$$

Cancelling terms in $\partial^\infty I^\infty_M((\omega, \kappa))$, we are left with expressions

$$(-1)^{2k+2}((\omega_1, \ldots, \omega_k, 2, \omega_{k+1} + \omega_{k+2}, \omega_{k+3}, \ldots, \omega_\ell), \kappa)) +$$

$$(-1)^{2k+3}((\omega_1, \ldots, \omega_k, \omega_{k+1} + \omega_{k+2}, 2, \omega_{k+3}, \ldots, \omega_\ell), \kappa).$$

In a similar computation of $\partial^\infty \partial^\infty_M((\omega, \kappa))$, we will obtain the contributions

$$(-1)^{2k+1}((\omega_1, \ldots, \omega_k, 2, \omega_{k+1} + \omega_{k+2}, \omega_{k+3}, \ldots, \omega_\ell), \kappa) +$$

$$(-1)^{2k+2}((\omega_1, \ldots, \omega_k, \omega_{k+1} + \omega_{k+2}, 2, \omega_{k+3}, \ldots, \omega_\ell), \kappa).$$

Adding them together, we get that $(\partial^\infty I^\infty_M + \partial^\infty_M \partial^\infty)((\omega, \kappa)) = 0$.

The claim that $\mathbb{Z}[\Theta]$ is a graded differential complex now follows from the fact that $\partial^\infty$ is a differential and that $\partial^\infty_M$ and $\partial^\infty$ respect the grading.

**Proposition 3.2.** For any closed subposet $\Theta \in \Omega^\infty_{(d)}$, the graded differential complex $(\mathbb{Z}[\Theta], \partial^\infty)$ coincides with the cellular chain complex of $B^\Theta_d$. In particular, $(\mathbb{Z}[\Theta], \partial^\infty)$ calculates the homology of $B^\Theta_d$.

**Proof.** By Corollary D, the topological boundary $\partial R_d^{(\omega, \kappa)}$ of $R_d^{(\omega, \kappa)}$ coincides with

$$\left( \bigcup_{k=0}^{\omega} R_d^{M_k^\infty((\omega, \kappa))} \right) \bigcup \left( \bigcup_{k=0}^{\omega} R_d^{L_k^\infty((\omega, \kappa))} \right).$$

Therefore, the algebraic boundary of $R_d^{(\omega, \kappa)}$ in the cellular chain complex of $B^\Theta_d$ is given by a sum of the form

$$\sum_{k=0}^{\omega} a_k R_d^{M_k^\infty((\omega, \kappa))} + \sum_{k=0}^{\omega} b_k R_d^{L_k^\infty((\omega, \kappa))},$$

where $a_k$ and $b_k$ are some integer coefficients.
In order to determine the coefficients \(a_k\) and \(b_k\), we proceed as follows. By (3.2) if a path \(\gamma = \{P_t\}_{t \in [0,1]}\) is such that \(\gamma \setminus \gamma(1) \subset \hat{R}_d^{(\omega,\kappa)}\) and \(\gamma(1)\) belongs to the boundary of \(R_d^{(\omega,\kappa)}\), then at least either two consecutive real roots of the polynomials \(P_t\) approach each other, or two complex-conjugate roots approach each other at a point of \(\mathbb{R}\), as \(t \to 1\). These situations are encoded in the merge and insert operations, respectively. Thus in order to calculate \(a_k\) and \(b_k\), we need to understand what happens along paths in \(\hat{R}_d^{(\omega,\kappa)}\) that approach the boundary cells \(R_d^{M_k^\infty((\omega,\kappa))}\) and \(R_d^{l_k^\infty((\omega,\kappa))}\), respectively. Note that we can face a situation when, for \(k \neq k'\), \(M_k^{\infty}((\omega,\kappa)) = M_{k'}^{\infty}((\omega,\kappa))\) or \(l_k^\infty((\omega,\kappa)) = l_{k'}^\infty((\omega,\kappa))\). Nevertheless, we will show that each operation \(M_k\) or \(l_k\) corresponds to a unique local homotopy type of paths \(\gamma\) in the vicinity of \(\gamma(1)\) and hence yields a path–independent contributions \(a_k\) and \(b_k\), respectively.

The preferred orientation of \(R_d^{(\omega,\kappa)}\) is induced by the canonical orientation of the open cell \(\hat{R}_d^{(\omega,\kappa)}\). Recall that we can identify a class of binary forms from \(\hat{R}_d^{(\omega,\kappa)}\) with a univariate polynomial of degree \(d = \kappa\). By Corollary D we know that \(\hat{R}_d^{(\omega,\kappa)} \cong \Pi_{\omega,\kappa} \times \text{Sym}^{m_\omega}(\mathbb{H})\) for \(\omega = (\omega_1, \ldots, \omega_{s_\omega})\) and \(m_\omega := \frac{d - (\omega,\kappa)}{2}\).

The orientation of the open cell \(\text{Sym}^{m_\omega}(\mathbb{H})\) is canonically induced by its complex structure, while the orientation of \(\Pi_{\omega,\kappa}\) is induced from \(\mathbb{R}^{s_\omega}\). In other words, the orientation of \(\hat{R}_d^{(\omega,\kappa)} \cong \hat{R}_d^{\omega,\kappa}\) is given by the volume form
\[
\rho_\omega := (dx_1 \wedge \cdots \wedge dx_{s_\omega}) \wedge \left(\frac{i}{2}\right)^{m_\omega} (dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{m_\omega} \wedge d\bar{z}_{m_\omega}),
\]
considered on the product \(\Pi_{\omega,\kappa} \times \text{Sym}^{m_\omega}(\mathbb{H})\).

Claim 1: In the formula (3.3) for the boundary operator we have \(a_k = (-1)^{k+1}\).

Note that, if for some \(k\) and \(l\), we have \(M_k^{\infty}((\omega,\kappa)) = M_l^{\infty}((\omega,\kappa))\), then either \(k = l\) or \(\{k, l\} = \{0, s_\omega\}\) and all the entries of \(\omega\) are identical. In the latter case, \(M_l^{\infty}((\omega,\kappa)) = M_0^{\infty}((\omega,\kappa))\) and the corresponding cell \(\hat{R}_d^{(\omega,\kappa)}\) is adjacent to the cell \(\hat{R}_d^{M_0^{\infty}((\omega,\kappa))}\) “on both sides.”

Case: \(k \neq s_\omega, 0\)

In this situation \(M_k^{\infty}((\omega,\kappa)) \neq M_l^{\infty}((\omega,\kappa))\) for \(l \neq k\). Let \(P \in \hat{R}_d^{M_k^{\infty}((\omega,\kappa))}\) be a polynomial of degree \(d - \kappa\). Consider a one-parameter family \(\{P_t \subset \hat{R}_d^{(\omega,\kappa)}\}_{t \geq 0}\) such that \(\lim_{t \to 0} P_t = P\). Then, for all sufficiently small \(t > 0\), the \(k\)th and \((k+1)\)st largest real roots \(x_k(P_t)\) and \(x_{k+1}(P_t)\) of \(P_t\) converge to the \(k\)th largest real root \(x_k(P)\) of \(P\). Moreover, for any two paths \(P_t, Q_t \subset \hat{R}_d^{(\omega,\kappa)}\) such that \(\lim_{t \to 0} P_t = \lim_{t \to 0} Q_t = P\), their germs at \(P\) can be deformed into one another by a small homotopy in \(\hat{R}_d^{(\omega,\kappa)}\). Let us explain this claim.

We say that \(\epsilon > 0\) is small enough for \(P\) if the \(\epsilon\)-disks around distinct roots of \(P\) are disjoint in \(\mathbb{C}\) and the \(\epsilon\)-disks around the non-real roots do not intersect the real line. Let \(U_\epsilon(P)\) denote the union of such \(\epsilon\)-disks.

For any \(\epsilon > 0\) which is small enough for \(P\), there exists \(t_\epsilon > 0\) such that for \(0 < t < t_\epsilon\), each root of \(P_t\) and of \(Q_t\) resides in \(U_\epsilon(P)\). This implies that, with the exception of the two merging roots \(x_k(P_t)\) and \(x_{k+1}(P_t)\), which share the same \(\epsilon\)-ball centered at the \(k\)th
root of $P$, any other real root of $P_t$ resides in a single $\epsilon$-ball contained in $U$. The same conclusion is valid for $Q_t$, $t < t_\epsilon$. The non-real roots of $P_t$ and $Q_t$ require more careful treatment, since more than two non-real roots of $P_t$ or $Q_t$ may belong the the same $\epsilon$-disk $D_\epsilon(z)$ centered on a non-real root $z$ of multiplicity $m(z) \geq 2$ of the limiting polynomial $P$. As a result, there is no natural correspondence between the roots of $P_t$ and $Q_t$ that reside in $D_\epsilon(z)$.

Now, we can deform $P_t$ into $Q_t$ in $\hat{R}_d^{(\omega,\kappa)}$ so that, in the process of deformation, the roots of the intermediate polynomials do not exit $U_\epsilon(P)$. This may be accomplished by the linear homotopy of the corresponding real roots of $P_t$ and $Q_t$, being combined with the following deformation of the non-real roots of $P_t$ and $Q_t$ that reside in each disk $D_\epsilon(z)$. We first contract radially all the roots of $P_t$ in $D_\epsilon(z)$ to $z$, counted with the multiplicity $m(z)$, and then expand $m(z)z$ radially into the divisor of $Q_t$ that is supported in $D_\epsilon(z)$. Thus the desired deformation of $P_t$ into $Q_t$ takes place not only in the open cell $\hat{R}_d^{(\omega,\kappa)}$, but also in the vicinity of $P$. Therefore, the germ of $\hat{R}_d^{(\omega,\kappa)}$ at $P \in \hat{R}_d^{M_\infty((\omega,\kappa))}$ has a single connected component contained in $\hat{R}_d^{(\omega,\kappa)}$. As a result, the incidence index $a_k$ of the cell $\hat{R}_d^{(\omega,\kappa)}$ with the cell $\hat{R}_d^{M_\infty((\omega,\kappa))}$ equals $\pm 1$.

In order to determine the sign, it suffices to consider a family of polynomials $P_t \subset \hat{R}_d^{\omega}$ such that:

1. $\lim_{t \to 0} P(t) = P \in \hat{R}_d^{M_\infty((\omega,\kappa))}$,
2. the polynomials $P(t)$ share the roots with $P$, except for the roots $x_k(P_t), x_{k+1}(P_t)$ and $x_k(P)$,
3. $x_k(P_t) = x_k(P) - t$, $x_{k+1}(P_t) = x_k(P) + t$.

Since the complex roots of $P_t$ and $P$ coincide, we can restrict our attention to $\Pi_\omega$. The tangent vector to the path $P(t)$ at $t = 0$ is given by $w = (0, \ldots, 0, -1, 1, 0, \ldots)$. This vector is the inward normal to the face $\{x_k = x_{k+1}\}$ of the polyhedron $\Pi_\omega \subset \mathbb{R}^{s_\omega}$.

The inner product “$\cdot$” of $w$ with the volume form in $\mathbb{R}^{s_\omega}$ is given by

$$w \cdot (dx_1 \land \cdots \land dx_k \land dx_{k+1} \land \cdots \land dx_{s_\omega}) = (-1)^{k+1}(dx_1 \land \cdots \land dx_k \land dx_{k+2} \land \cdots \land dx_{s_\omega}).$$

Therefore, the orientation of the boundary $\partial \Pi_\omega$, induced by the orientation of its interior $\Pi_\omega^\circ$, differs from the preferred orientation of $\Pi_\omega^{M_\infty((\omega,\kappa))}$ exactly by the factor $(-1)^{k+1}$.

Case: $k = s_\omega := |\omega| - |\omega'|$

If a real polynomial $P$ of degree $d - \kappa$ has its largest real root $x_k(P)$ of multiplicity $\omega_k$, then $P$ produces, with the help of the map $\phi$ from (2.1) a bilinear form $\phi(P)$ of degree $d$ in the homogeneous variables $[z : w]$. In fact, $\phi(P) = (z - x_k \cdot w)^{\omega_k}w^\kappa \times Q(z, w)$, where $Q(z, w)$ is a real binary form of degree $d - \omega_k - \kappa$ such that $Q(z, 1) = P(z) \cdot (z - x_k)^{-\omega_k}$.

Let us examine the result of the merge operation $M_\infty^{\omega}(\omega, \kappa)$ on the real zero divisors of the binary form $\phi(P)$.

First we choose an open interval $(\alpha, \beta) \subset \mathbb{R}$ that contains all the real roots $x_1(P), \ldots, x_k(P)$ of $P$. Then we choose an affine chart $U_\alpha := \mathbb{R}P^1 \setminus [\alpha : 1]$ and a projective transformation $A \in \text{PGL}(2, \mathbb{R})$ which maps the ordered sequence $[x_1 : 1], \ldots, [x_k : 1]$ in the
chart \( U_\infty := \{ [z : 1] \} \) to an ordered sequence \( \mathcal{A}( [x_1 : 1] , \ldots , \mathcal{A}( [x_k : 1] ) \) in the chart \( U_\alpha \). In particular, \( \mathcal{A} \) maps \( \infty = [1 : 0] \) to a point \([a : 1]\), where \( a > \mathcal{A}( [\beta , 1] ) \). In the original affine chart \( U_\infty \), the projective transformation \( \mathcal{A} \) is given by the rational function \( F(z) = a - \frac{b}{z^{2}a} \), where \( F(\beta) < a \). Now, as \( x_k \to \infty \), \( F(x_k) \to a \). The real zero locus \( \omega_1 \cdot x_1 + \cdots + \omega_k \cdot x_k + \kappa \cdot \infty \) of the binary form \( \wp(P) \) is mapped by \( \mathcal{A} \) to the real zero locus \( \omega_1 \cdot \mathcal{A}(x_1) + \cdots + \omega_k \cdot \mathcal{A}(x_k) + \kappa \cdot a \) of the bilinear form \( \mathcal{A}^* (\wp(P)) \). Under the transformation \( \mathcal{A} \), the combinatorial pattern \((\omega, \kappa)\) of \( \wp(P) \) is transformed into the combinatorial pattern \((\omega_1 , \ldots , \omega_k , \kappa) \) of \( \mathcal{A}^* (\wp(P)) \) in \( U_\alpha \). Now set \( \omega^\infty := (\omega_1 , \ldots , \omega_k , \kappa) \). Notice that under \( \mathcal{A}^* \), the merge operation \( M_k^\infty ((\omega, \kappa)) \) is transformed into the operation \( M_k(\omega^\infty) = M_k^\infty (\omega^\infty, 0) \). The latter case does not involve \( \infty \) and has been analyzed above.

**Case:** \( k = 0 \)

This case can be treated analogously to the case \( k = s_\omega \).

**Claim 2:** In the formula \( (3.3) \) for the boundary operator we have \( b_k = (-1)^k \).

This is a more delicate situation since multiple sheets of \( \tilde{R}_d^{(\omega, \kappa)} \) may appear in the neighborhood of \( P \in \tilde{R}_d^{(\omega)} \). The simplest example of such phenomenon occurs when \( P \) has two consecutive real roots, \( x_i \) and \( x_{i+1} \), each of multiplicity 2. Denote by \( P_1 \) and \( P_2 \) two polynomials obtained by resolving the first and, respectively, the second double root into a pair of simple complex-conjugate roots. Then the combinatorial patterns of the real root multiplicities of \( P_1 \) and \( P_2 \) are identical. Note that, although \( P_1 \) and \( P_2 \) are close to each other in \( \tilde{R}_d^{(\omega, \kappa)} \), there is no short path in \( \tilde{R}_d^{(\omega, \kappa)} \) connecting them in a neighborhood of \( P \).

Let \( x_1(P) < \cdots < x_{s_k+1}(P) \) be the distinct real roots of \( P \in \tilde{R}_d^{(\omega, \kappa)} \), and let \( \left\{ (z_1^t, z_2^t) \right\} \) be the unordered collection of non-real roots of \( P \). By our assumptions on \( P \), we have that \( x_{k+1}(P) \) is a root of multiplicity 2. Let \( P_t \subset \tilde{R}_d^{(\omega, \kappa)} \) be a path in \( \tilde{R}_d^{(\omega, \kappa)} \) such that \( \lim_{t \to 0} P_t = P \). We may assume that

- \( x_j (P_t) = x_j (P) \) for all \( j \leq k \) and \( x_j (P_t) = x_{j+1} (P) \) for all \( j > k + 1 \);
- \( (x_{k+1}(P) + it, x_{k+1}(P) - it) \) is a pair of simple complex-conjugate roots for \( P_t \) and all the other non-real roots of \( P_t \) and \( P \) coincide.

With this convention and choosing a fixed ordering of the complex roots with positive imaginary parts, we may consider \( P_t \) as a curve in the the ordered “root space” \( \mathbb{R}^{8_\omega} \times \mathbb{C}^{m_\omega} \). Here the vector \( w = (0, \ldots , 0, i, 0, \ldots , 0) \) is tangent to the curve \( P_t \) at \( P \).

The volume form \( \rho_\omega \) can be written as \( \rho_\omega^R \wedge \rho_\omega^C \), where

\[
\rho_\omega^R := dx_1 \wedge \cdots \wedge dx_{s_\omega}
\]

and

\[
\rho_\omega^C := \left( \frac{i}{2} \right)^{m_\omega} (dz_1 \wedge d\bar{z}_1) \wedge \cdots \wedge (dz_{m_\omega} \wedge d\bar{z}_{m_\omega}).
\]

Since \( i \left( \frac{1}{2} dz \wedge d\bar{z} \right) = (0, 1) \left( dx \wedge dy \right) = -dx \), we get

\[
w \left( \rho_\omega^R \wedge \rho_\omega^C \right) = (-1)^{s_\omega} \rho_\omega^R \wedge (w \left( \rho_\omega^C \right) = (-1)^{s_\omega} \rho_\omega^R \wedge dx_{k+1} \wedge \rho_{k}^C
\]

\[
= (-1)^{2s_\omega+k} \rho_{k}^R \wedge \rho_{k}^C = (-1)^k \rho_{k}^R.
\]
This calculation implies that the part \( R_d^{\infty} \rho_k(\omega) \) of the boundary \( \delta^{\infty}(R_d^\rho) \), being approached via the path \( P_t \times \omega := P_t \subset R_d^\omega \) as above, acquires an orientation that differs from its \( \rho_k(\omega) \)-induced orientation by the factor \( b_k = (-1)^k \).

Thus we have shown that, the \((E^2, d^2)\)-term of the \( H_* \)-homological spectral sequence, associated to the filtration of the space \( B^\Theta_d \) by its skeleta, coincides with the graded differential complex complex \( \partial^{\infty} : Z(\Theta) \to Z(\Theta) \), defined by the formula (3.1) \( \square \)

As an immediate consequence of Proposition 3.2 and of the previous discussions, we obtain one of our main results — the combinatorial differential complex that calculates the homology of \( B^\Theta_d \).

**Theorem 3.3.** Let \( \Theta \subset \Omega^\infty_{(d)} \) be a closed subposet. For all \( j \geq 0 \), the homology of \( B^\Theta_d \) is given by

\[
H_j(B^\Theta_d; \mathbb{Z}) \cong \frac{\ker\{ \partial^{\infty} : Z[\Theta|_{\sim'} = d-j] \to Z[\Theta|_{\sim'} = d-j+1] \}}{\text{im}\{ \partial^{\infty} : Z[\Theta|_{\sim'} = d-j-1] \to Z[\Theta|_{\sim'} = d-j] \}},
\]

where the differential \( \partial^{\infty} \) is given by formula (3.1) \( \diamondsuit \).

For a later use, let us also consider a relative version of this differential complex. Namely, given a closed subposet \( \Theta \subset \Omega^\infty_{(d)} \), consider the exact sequence of differential complexes

\[
0 \to (Z(\Theta), \partial^{\infty}) \to (Z[\Omega^\infty_{(d)}], \partial^{\infty}) \to (Z[\Omega_{(d)} \setminus \Theta], \partial^{\#}) \to 0,
\]

where the first homomorphism is the obvious inclusion and the last term in the exact sequence is, by definition, the differential quotient complex. Its differential \( \partial^{\#} \) is still given by formulas (3.1) in which all terms \( M_k^\infty((\omega, \kappa)) \) (resp. \( I_k^\infty((\omega, \kappa)) \)) with \( \Omega^\infty_{(d)}(\omega, \kappa) \in \Theta \) (resp. \( I_k^\infty((\omega, \kappa)) \in \Theta \)) are replaced by 0.

Since all \( \mathbb{Z} \)-modules in (3.5) are free, we get a short exact sequence of dual complexes:

\[
0 \to (Z[\Omega_{(d)} \setminus \Theta]^*, (\partial^{\#})^*) \to (Z[\Omega^\infty_{(d)}]^*, (\partial^{\infty})^*) \to (Z[\Theta]^*, (\partial^{\infty})^*) \to 0,
\]

where \( Z[\sim]^* \) denotes \( \text{Hom}_{\mathbb{Z}}(Z[\sim], \mathbb{Z}) \), and \( (\partial^{\#})^*, (\partial^{\infty})^* \) are the dual differentials.

For any closed subposet \( \Theta \subset \Omega^\infty_{(d)} \), the following claim is an immediate consequence of Theorem 3.3 and of the fact that \( B^\Theta_d \) is a subcomplex of the CW-complex \( B_d \).

**Corollary 3.4.** Let \( \Theta \subset \Omega^\infty_{(d)} \) be a closed subposet. Then for all \( j \geq 0 \), the reduced homology of the quotient \( B_d/B^\Theta_d \) is given by

\[
H_j(B_d/B^\Theta_d; \mathbb{Z}) \cong \frac{\ker\{ \partial^{\#} : Z([\Omega_{(d)} \setminus \Theta]|_{\sim'} = d-j] \to Z([\Omega_{(d)} \setminus \Theta]|_{\sim'} = d-j+1] \}}{\text{im}\{ \partial^{\#} : Z([\Omega_{(d)} \setminus \Theta]|_{\sim'} = d-j-1] \to Z([\Omega_{(d)} \setminus \Theta]|_{\sim'} = d-j] \}}.
\]

The reduced cohomology of the quotient space \( B_d/B^\Theta_d \) is isomorphic to the homology of the dual differential complex \( (Z[\Omega_{(d)} \setminus \Theta]^*, (\partial^{\#})^*) \) introduced in (3.6) \( \diamondsuit \).

Now let us consider of the one-point compactification \( \tilde{P}_d^\Theta \) for a closed subposet \( \Theta \subset \Omega^\infty_{(d)} \). Analogously to the case of a closed subposet \( \Theta \subset \Omega^\infty_{(d)} \), we introduce two homomorphisms on \( \mathbb{Z}[\Theta] \) by the formula
\[ \partial_M(\omega) := -\sum_{k=1}^{s_\omega-1} (-1)^k M_k(\omega) \quad \text{and} \quad \partial_l(\omega) := \sum_{k=0}^{s_\omega} (-1)^k l_k(\omega). \]

Next, we define a homomorphism
\[ \partial = \partial_M + \partial_l : \mathbb{Z}[\Theta] \rightarrow \mathbb{Z}[\Theta] \]
by
\[
(3.8) \quad \partial(\omega) := \begin{cases} 
- \sum_{k=1}^{s_\omega-1} (-1)^k M_k(\omega) + \sum_{k=0}^{s_\omega} (-1)^k l_k(\omega), & \text{for } \omega \neq \infty, |\omega| < d, \\
- \sum_{k=1}^{s_\omega-1} (-1)^k M_k(\omega), & \text{for } \omega \neq \infty, (d), |\omega| = d, \\
0, & \text{for } \omega = (d).
\end{cases}
\]

**Corollary 3.5.** Let \( \Theta \subset \Omega_{(d)} \) be a closed subposet. The for all \( j \geq 0 \), the reduced homology of the one-point compactification \( \mathcal{P}_d^\Theta \) of \( \mathcal{P}_d^\Theta \) is given by
\[
(3.9) \quad \tilde{H}_j(\mathcal{P}_d^\Theta; \mathbb{Z}) \cong \frac{\ker \{ \partial : \mathbb{Z}[\Theta_{\sim'} = d-j] \rightarrow \mathbb{Z}[\Theta_{\sim'} = d-j+1] \}}{\im \{ \partial : \mathbb{Z}[\Theta_{\sim'} = d-j-1] \rightarrow \mathbb{Z}[\Theta_{\sim'} = d-j] \}},
\]
where the differential \( \partial \) is given by formula (3.8).

**Proof.** Consider the decomposition \( \mathcal{B}_d = \mathcal{P}_d \sqcup \cdots \sqcup \mathcal{P}_0 = \mathcal{P}_d \sqcup \mathcal{B}_{d-1} \). For a closed \( \Theta \subset \Omega_{(d)} \), consider the smallest closed subposet \( \Theta^\infty \subseteq \Omega_{(d)} \) that contains \( \Theta \times \{0\} \). Let \( \Theta_\ell \) be the subset of all \( \omega \) such that \( (\omega, d-\ell) \in \Theta^\infty \). (In particular, \( \Theta = \Theta_d \).) Then the above decomposition induces decomposition \( \mathcal{B}_d^\Theta = \mathcal{P}_d^\Theta \sqcup \cdots \sqcup \mathcal{P}_0^\Theta \).

Since \( \Theta_{d-1} \times \{1\} \sqcup \cdots \sqcup \Theta_0 \times \{d\} \) is closed, we get that \( Y = \mathcal{P}_{d-1}^\Theta \sqcup \cdots \sqcup \mathcal{P}_0^\Theta \) is a closed CW-subcomplex of \( \mathcal{B}_d \). Thus \( \mathcal{P}_d^\Theta \) and \( \mathcal{B}_\infty^\Theta / \mathcal{Y} \) are isomorphic as CW-complexes. In particular, the cellular differentials of \( \mathcal{P}_d^\Theta \) and \( \mathcal{B}_\infty^\Theta / \mathcal{Y} \) coincide. Now the assertion follows from Corollary 3.4 and our definition of the differential \( \partial^\# \). \( \square \)

**Corollary 3.5** allows us to compute the (co)homology of \( \mathcal{P}_d^{c\Theta} := \mathcal{P}_d \setminus \mathcal{P}_d^\Theta = S^d \setminus \mathcal{P}_d^\Theta \) using the Alexander duality.

**Corollary 3.6.** Let \( \Theta \subset \Omega_{(d)} \) be a closed subposet. Then for all \( j \geq 0 \), the homology of \( \mathcal{P}_d^{c\Theta} \) is given by
\[
(3.10) \quad H_j(\mathcal{P}_d^{c\Theta}; \mathbb{Z}) \cong H_{d-j-1}(\partial : \mathbb{Z}[\Theta] \rightarrow \mathbb{Z}[\Theta]) := \frac{\ker \{ \partial : \mathbb{Z}[\Theta_{\sim'} = j+1] \rightarrow \mathbb{Z}[\Theta_{\sim'} = j+2] \}}{\im \{ \partial : \mathbb{Z}[\Theta_{\sim'} = j] \rightarrow \mathbb{Z}[\Theta_{\sim'} = j+1] \}},
\]
With the combinatorial complex that calculates the cohomology of \( \mathcal{P}_d^{c\Theta} \) in place, it is natural to look for a similar complex that would calculate the cohomology of the complement \( \mathcal{B}_d^{c\Theta} := \mathcal{B}_d \setminus \mathcal{B}_d^\Theta \) for any closed \( \Theta \subseteq \Omega_{(d)}^\infty \). Since \( \mathcal{B}_d \cong \mathbb{R}\mathbb{P}^d \), we need to consider the Alexander duality for the real projective spaces. With this goal in mind, let us remind the reader of a few standard constructions and notions regarding the Poincaré duality on non-simply-connected manifolds (see [W]).
Let $X$ be a $d$-dimensional compact connected manifold or, more generally, a Poincaré CW-complex. Let $\Lambda$ be the group ring $\mathbb{Z}[\pi_1(X)]$ of the fundamental group $\pi_1(X)$. An element of $\Lambda$ is a finite combination $\sum g n_g g$, where $g \in \pi_1(X)$ and $n_g \in \mathbb{Z}$.

We denote by $\tilde{X}$ the universal cover of $X$ and by $C_*(\tilde{X}, \mathbb{Z})$ the cellular chain complex of $\tilde{X}$, viewed as a right $\Lambda$-module under the free $\pi_1(X)$-action on $\tilde{X}$. We define the homology and cohomology of $X$ with coefficients in a right $\Lambda$-module $\Lambda$ (a local system of coefficients) by

$$H^*(X; \Lambda) := H(\text{Hom}_\Lambda(C_*\tilde{X}, \Lambda)), \quad H^*_\Lambda(X; \Lambda) := H(C_*\tilde{X}, \mathbb{Z}) \otimes \Lambda^\Lambda.\$$

The operation $\otimes_\Lambda$ relies on converting $\Lambda$ into a left $\Lambda$-module $\Lambda^\Lambda$ via the formula $\lambda a := a\tilde{\lambda}$, where $a \in \Lambda$ and $\lambda \in \Lambda$. Here, for any $\lambda = \sum n_g g$, we denote by $\tilde{\lambda} := \sum w(g)n_g g^{-1}$, where the homomorphism $w : \pi_1(X) \to \mathbb{Z}_2$ is defined by the first Stiefel-Whitney class, an element of $H^1(X; \mathbb{Z}_2)$. The special case $\Lambda = \mathbb{Z}$ is central for us.

Similar definitions of $H^*(X, \partial X; \Lambda)$ and $H^*_\Lambda(X, \partial X; \Lambda)$ are available for a manifold $X$ with boundary $\partial X$. They also make sense for any pair $X \supset Y$, where $Y$ is a compact subcomplex of a Poincaré complex $X$ (see [W]). In such a case, the cap product with the fundamental cycle $[X] \in H_d(X; \mathbb{Z}')$ delivers the Poincaré duality isomorphism

$$[X]\cap : H^j(X, K; \Lambda) \xrightarrow{\cong} H_{d-j}(X \setminus K; \Lambda'),$$

where $\Lambda'$ is a right $\mathbb{Z}[\pi_1(X \setminus K)]$-module. As before, in order to convert $\Lambda$ into a left $\mathbb{Z}[\pi]$-module $\Lambda'$, we use the Stiefel-Whitney map

$$w : \pi_1(X \setminus K) \to \pi_1(X) \to \mathbb{Z}_2,$$

that describes whether the local orientation of $X$ is preserved or reversed along a loop in $X \setminus K$.

Now consider the natural homomorphism $H^j(K; \Lambda) \xrightarrow{\delta^*} H^j+1(K, \partial X; \Lambda)$ from the long exact sequence of the pair $(X, K)$ and compose it with the Poincaré duality $D_{X \setminus K} = [X]\cap \sim$. This composition produces the Alexander homomorphism

$$\mathcal{A}^j : H^j(K; \Lambda) \xrightarrow{\delta^*} H^{j+1}(X, K; \Lambda) \xrightarrow{\cong D_{X \setminus K}} H_{d-j-1}(X \setminus K, K)^\Lambda.$$

Obviously, if $\delta^*$ is an isomorphism (monomorphism/epimorphism), then so is $\mathcal{A}^j$. Similarly, we get a homomorphism

$$\mathcal{A}_j : H^{d-j-1}(X \setminus K; \Lambda) \xrightarrow{\cong D_{X \setminus K}} H_{j+1}(X, K; \Lambda^\Lambda) \xrightarrow{\partial_\Lambda^\Lambda} H_j(K; \Lambda^\Lambda).$$

Again if $\partial^\Lambda$ is an isomorphism (monomorphism/epimorphism), then so is $\mathcal{A}_j$.

We will apply $\mathcal{A}$ in the case when $X = \mathbb{R}P^d$, $K = B^d_\Theta$, $\Lambda = \mathbb{Z}$, and $\Lambda' = \mathbb{Z}_2'$ is the local coefficient system, defined by the homomorphism $w : \pi_1(\mathbb{R}P^d \setminus B^d_\Theta) \to \pi_1(\mathbb{R}P^d) \cong \mathbb{Z}_2$.

**Theorem 3.7.** Given a closed subposet $\Theta \subset \Omega^\infty_{(d)}$, set $c\Theta := \Theta \setminus \Omega^\infty_{(d)}$. Then for any $j \geq 0$, we get isomorphisms:

$$H^j(B^\Theta_c; \mathbb{Z}) \cong \frac{\ker\{\partial^{\#}_{d-j} : \mathbb{Z}[c\Theta_{\sim^d_j}] \to \mathbb{Z}[c\Theta_{\sim^d_j+1}]\}}{\im\{\partial^{\#}_{d-j+1} : \mathbb{Z}[c\Theta_{\sim^d_{j+1}}] \to \mathbb{Z}[c\Theta_{\sim^d_{j+1}}]\}}.$$
with the Poincare dualities:

\[
\ker\{ (\partial_{d-j}^*)(c\Theta_{\sim|\sim|}\rightarrow Z[c\Theta_{\sim|\sim|}]=\downarrow Z[c\Theta_{\sim|\sim|}]
\right).
\]

Proof. The above isomorphisms follow from Theorem 3.3 and Corollary 3.4, being combined with the Poincare dualities:

\[
\mathcal{PD} : H^j(B_d^0; Z) \cong H_{d-j}(B_d, B_d^0; Z') \cong H_{d-j}(B_d/B_d^0; Z'),
\]

\[
\mathcal{PD}^{-1} : H_j(B_d^0; Z') \cong H^{d-j}(B_d, B_d^0; Z) \cong H^{d-j}(B_d/B_d^0; Z).
\]

Classical example: the discriminant of real binary forms To the best of our knowledge, only very few of the spaces $B_d^0$ or $B_d^0$ have been considered in the literature before. The most significant result, albeit in a different setting, is again due to Vassiliev. He considered Arnold’s situation from Theorem B for singularities $[Va1, Va2]$. Following Vassiliev’s notation, let $\mathcal{H}_d \cong \mathbb{R}^{d+1}$ denote the space of real binary homogeneous forms $Q(x, y)$ of degree $d$. Thus $B_d$ is the projective space of $\mathcal{H}_d$. For $k \geq 2$ let $\Sigma_k \subset \mathcal{H}_d$ be the subset consisting of all forms $Q$ vanishing with multiplicity at least $k$ on some line in $\mathbb{R}^2$. The main theorem of $[Va2]$, formulated below, describes the reduced homology $\tilde{H}^*(\mathcal{H}_d \setminus \Sigma_k; Z)$ for any $2 \leq k \leq d$.

**Theorem G.** Fix a number $k \in [2, d]$.

(i) For $k$ even, the reduced cohomology group $\tilde{H}^*(\mathcal{H}_d \setminus \Sigma_k; Z)$ is a free abelian group of rank $2[d/k] + 1$ with generators in degrees $k - 2, k - 1, 2(k - 2), 2(k - 2) + 1, \ldots, [d/k](k - 2), [d/k](k - 2) + 1,$ and $d - 2[d/k]$.

(ii) For $k$ odd and $d$ not divisible by $k$, $\tilde{H}^*(\mathcal{H}_d \setminus \Sigma_k; Z)$ is a direct product of the following groups:

(a) for any $p = 1, 2, \ldots, [d/k]$ such that $d - pk$ is odd, $Z$ in dimension $p(k - 2)$ and $\mathbb{Z}$ in dimension $p(k - 2) + 1$;

(b) for any $p = 1, 2, \ldots, [d/k]$ such that $d - pk$ is even, $\mathbb{Z}/2\mathbb{Z}$ in dimension $p(k - 2) + 1$;

(c) $\mathbb{Z}$ in dimension $d - 2[d/k]$.

(3) For $k$ odd and $d$ divisible by $k$, the answer is almost the same as in Case 2, but the summand $\mathbb{Z}/2\mathbb{Z}$ in dimension $d - 2(d/k) + 1$ disappears.

As an example of application of Theorem 3.7 and Theorem 3.3, let us now compute the reduced homology of the discriminant variety and the homotopy type of its complement in our projective setting. Proposition 3.10 should be compared with Theorem G for $k = 2$, in which case, the latter simplifies greatly.

Let $\mathcal{D}_d \subset B_d$ be the discriminant variety that consists of all classes of binary degree $d$ forms with at least one real root of multiplicity $\geq 2$. Note that $\mathcal{D}_d = B_d^{\text{disc}}$ for

\[
\Theta_{\text{disc}} = \left\{ ((\omega_1, \ldots, \omega_t), \kappa) \in \Omega_d^{\infty} \mid \exists 1 \leq i \leq \ell : \omega_i \geq 2 \text{ or } \kappa \geq 2 \right\}.
\]

The following result is contained in Example 1 and Lemma 1 of $[Va2]$.
Proposition 3.8. The connected components of $B_d^{\ell,\text{disc}} = B_d \setminus D_d$ are labeled by marked compositions $((1, \ldots, 1), 0)$, where $\ell \geq 0$ and $d - \ell \geq 0$ is even. If $d$ is odd, then each connected component of $B_d^{\ell,\text{disc}} = B_d \setminus D_d$ is homotopy equivalent to a circle; if $d$ is even, then each connected component of $B_d^{\ell,\text{disc}} = B_d \setminus D_d$ is homotopy equivalent to a circle, except for the component labeled by $((1, \ldots, 1), 0)$, which is contractible.

Proof. Let $\ell$ be such that $d - \ell$ is even. Then, for $\ell \neq 0$ and $\hat{\omega}_1 := ((1, \ldots, 1), 0)$, $\hat{\omega}_2 := (M_\ell^\infty(\hat{\omega}_1)) = (M_0^\infty(\hat{\omega}_1)) = ((1, \ldots, 1), 1)$, the union $C_\ell,d = \hat{R}_d^{\hat{\omega}_1} \cup \hat{R}_d^{\hat{\omega}_2}$ is a connected component of $B_d \setminus D_d$. If $d$ is even and $\ell = 0$, then $C_{1,d} = \hat{R}_d^{(1,0)}$ is a connected component. Since every cell of $B_d$ that is not contained in $D_d$ is contained in some $C_\ell,d$, these are all the connected components.

The space $C_{0,d} = \hat{R}_d^{(1,0)}$ is an open cell and hence contractible. Further, for $\ell \neq 0$ and even $d - \ell$, the polynomials in the corresponding connected components have $\ell$ simple real roots (including the root at infinity) and $\frac{d - \ell}{2}$ complex-conjugate pairs of roots in $\mathbb{C}P^1 \setminus \mathbb{R}P^1$, counted with their multiplicities. Then $C_{\ell,d}$ is the Cartesian product of the configuration space $\text{Conf}^\ell(\mathbb{R}P^1)$ and $\text{Sym}^{\frac{d - \ell}{2}}(\mathbb{H})$. Since the upper half-plane $\mathbb{H}$ is contractible, $\text{Sym}^{\frac{d - \ell}{2}}(\mathbb{H})$ is contractible as well. It is well-known that $\text{Conf}^\ell(\mathbb{R}P^1) = \text{Conf}^\ell(S^1)$ is homotopy equivalent to a circle $\text{Mor}$. As in $\text{Mor}$, there is a fibration $\text{Conf}^\ell(S^1) \to S^1$ whose fiber is the open simplicial cone $\Pi_{\ell-1}$. Since the latter is contractible, we conclude that $\text{Conf}^\ell(S^1)$ is homotopy equivalent to a circle, which completes the proof.

The following lemma is the crucial step for the determination of the homology of $D_d$.

Lemma 3.9.

(i) $H_i(B_d, D_d; \mathbb{Z}) = 0$ for $i \leq d - 2$.

(ii) If $d$ is odd, then $H_d(B_d, D_d; \mathbb{Z}) \cong \mathbb{Z}^{(d+1)/2}$ and $H_{d-1}(B_d, D_d; \mathbb{Z}) \cong \mathbb{Z}^{(d+1)/2}$.

(iii) If $d$ is even, then $H_d(B_d, D_d; \mathbb{Z}) \cong \mathbb{Z}$ and $H_{d-1}(B_d, D_d; \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{d/2}$.

We provide two proofs of the lemma in order to demonstrate the applicability of both Theorem 3.7 and of Theorem 3.3.

Proof of Lemma 3.9 based on Theorem 3.7. By the Poincaré duality and Theorem 3.7

$$H_j(B_d, D_d; \mathbb{Z}) \cong H^{d-j}(B_d \setminus D_d; \mathbb{Z})^\prime,$$

where, for $d$ odd, the local system of coefficients $\mathbb{Z}^\ell$ is constant, and, for $d$ even, the local coefficient system $\mathbb{Z}^\ell$ is twisted by the monodromy along the generator of $\pi_1(B_d) \cong \mathbb{Z}/2\mathbb{Z}$.

For an odd $d$, by Proposition 3.8, the space $B_d \setminus D_d$ is homotopy equivalent to a disjoint union of $(d + 1)/2$ circles, which implies that $H^0(B_d \setminus D_d; \mathbb{Z}) \cong \mathbb{Z}^{(d+1)/2}$, and $H^1(B_d \setminus D_d; \mathbb{Z}) \cong \mathbb{Z}^{(d+1)/2}$, and all other cohomology groups vanish. As a result, for $d$, odd we get $H_d(B_d, D_d; \mathbb{Z}) \cong \mathbb{Z}^{(d+1)/2}$, and $H_{d-1}(B_d, D_d; \mathbb{Z}) \cong \mathbb{Z}^{(d+1)/2}$, and all other homology groups vanish.
For an even \( d \), by Proposition 3.8 the space \( \mathcal{B}_d \setminus \mathcal{D}_d \) is homotopy equivalent to a disjoint union of a point and \( d/2 \) circles. By the twist in the local coefficient system \( \mathbb{Z}^t \), we get \( H^0(\mathcal{B}_d \setminus \mathcal{D}_d; \mathbb{Z}^t) \cong \mathbb{Z} \) and \( H^1(\mathcal{B}_d \setminus \mathcal{D}_d; \mathbb{Z}^t) \cong (\mathbb{Z}/2\mathbb{Z})^{d/2} \). All other cohomology groups \( H^i(\mathcal{B}_d \setminus \mathcal{D}_d; \mathbb{Z}^t) \) vanish. Thus, for \( d \) even, again by the Poincaré duality and Theorem 3.7, we get \( H_d(\mathcal{B}_d, \mathcal{D}_d; \mathbb{Z}) \cong \mathbb{Z} \), \( H_{d-1}(\mathcal{B}_d, \mathcal{D}_d; \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{d/2} \), and all other homology groups vanish.

**Proof of Lemma 3.9 based on Theorem 3.3.** Since in our cellulation the cells in \( \mathcal{B}_d \setminus \mathcal{D}_d \) are of dimension \( d \) and \( d - 1 \), the cellular chain complex of the pair \( (\mathcal{B}_d, \mathcal{D}_d) \), arising from Theorem 3.3, has trivial chain groups in dimensions \( \neq d, d - 1 \). This instantly implies (i).

From the preceding arguments it also follows that the group of cycles in dimension \( d - 1 \) of the cellular chain complex of \( (\mathcal{B}_d, \mathcal{D}_d) \) is freely generated by the cells of dimension \( d - 1 \) in \( \mathcal{B}_d \setminus \mathcal{D}_d \).

For \( d \) odd, the \( d \)-cells in \( \mathcal{B}_d \setminus \mathcal{D}_d \) are labeled by \((1, \ldots, 1, 0)\), where \( j \in [0, \frac{d-1}{2}] \), and the \((d - 1)\)-dimensional cells by \((1, \ldots, 1, 1)\), where \( j \in [1, \frac{d-1}{2}] \). By Theorem 3.3, the differential of the cells of dimension \( d \) is fully contained in \( \mathcal{D}_d \). Hence \( H_d(\mathcal{B}_d, \mathcal{D}_d; \mathbb{Z}) \) is freely generated by the cells of dimension \( d \) and \( H_{d-1}(\mathcal{B}_d, \mathcal{D}_d; \mathbb{Z}) \) is freely generated by the cells of dimension \( d - 1 \). A simple counting argument then yields (ii).

For \( d \) even, the \( d \)-cells in \( \mathcal{B}_d \setminus \mathcal{D}_d \) are labeled by \((1, \ldots, 1, 0)\), where \( j \in [0, \frac{d}{2}] \), and the \((d - 1)\)-dimensional cells by \((1, \ldots, 1, 1)\), where \( j \in [1, \frac{d}{2}] \). By Theorem 3.3, the differential of the cell, labeled by \((1, \ldots, 1, 0)\), consists only of cells from \( \mathcal{D}_d \) if \( j = 0 \), and twice the cell, labeled by \((1, \ldots, 1, 1)\), plus some terms which correspond to cells from \( \mathcal{D}_d \) when \( j \in [1, \frac{d}{2}] \). By simple counting and linear algebra, we then have \( H_d(\mathcal{B}_d, \mathcal{D}_d; \mathbb{Z}) = \mathbb{Z} \) and \( H_{d-1}(\mathcal{B}_d, \mathcal{D}_d; \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{d/2} \). This validates (iii).

Finally, we are in position to compute \( \tilde{H}_*(\mathcal{D}_d; \mathbb{Z}) \).

**Proposition 3.10.** The non-zero reduced integral homology groups \( \tilde{H}_i(\mathcal{D}_d; \mathbb{Z}) \) of the discriminant \( \mathcal{D}_d \subset \mathcal{B}_d \) have the following description:

(i) if \( d \) is odd, then

\[
\tilde{H}_{d-1}(\mathcal{D}_d; \mathbb{Z}) \cong \mathbb{Z}^{(d-1)/2}, \quad \tilde{H}_{d-2}(\mathcal{D}_d; \mathbb{Z}) \cong \mathbb{Z}^{(d+1)/2}, \quad \text{and}
\]

\[
\tilde{H}_{d-4}(\mathcal{D}_d; \mathbb{Z}) \cong \tilde{H}_{d-6}(\mathcal{D}_d; \mathbb{Z}) \cong \cdots \cong \tilde{H}_1(\mathcal{D}_d; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.
\]

The rest of the homology vanishes.

(ii) if \( d \) is even, then

\[
\tilde{H}_{d-1}(\mathcal{D}_d; \mathbb{Z}) \cong \mathbb{Z}, \quad \tilde{H}_{d-2}(\mathcal{D}_d; \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{(d/2)-1},
\]
$0 \cong H_d(D_d; \mathbb{Z}) \longrightarrow H_d(B_d; \mathbb{Z}) \longrightarrow H_d(B_d, D_d; \mathbb{Z})$

$H_{d-1}(D_d; \mathbb{Z}) \longrightarrow H_{d-1}(B_d; \mathbb{Z}) \longrightarrow H_{d-1}(B_d, D_d; \mathbb{Z})$

$\cdots \cdots \cdots \cdots$

$H_0(D_d; \mathbb{Z}) \longrightarrow H_0(B_d; \mathbb{Z}) \longrightarrow H_0(B_d, D_d; \mathbb{Z}) \cong 0.$

**Figure 1.** Long exact homology sequence of the pair $(B_d, D_d)$.

and $\bar{H}_{d-i}(D_d; \mathbb{Z}) \cong \bar{H}_{d-5}(D_d; \mathbb{Z}) \cong \cdots \cong \bar{H}_1(D_d; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. The rest of the homology vanishes.

**Proof.** Since $D_d$ has no cells in dimension $d$ by Theorem 3.3 we have $H_d(D_d; \mathbb{Z}) = 0$.

Consider the long exact homology sequence of the pair $D_d \subset B_d$ with coefficients in $\mathbb{Z}$:

Basic algebraic topology [Ha] tells us that the homology of $\mathbb{R}P^d$ is given by

$$H_p(\mathbb{R}P^d; \mathbb{Z}) = \begin{cases} 
\mathbb{Z} & \text{for } p = 0 \text{ and, when } d \equiv 1 \text{ mod } 2, \text{ for } p = d \\
\mathbb{Z}/2\mathbb{Z}, & p \text{ odd, } 0 < p < d \\
0, & \text{otherwise.}
\end{cases}$$

(3.15)

By [Lemma 3.9] (i) and the long exact sequence of the pair $(B_d, D_d)$ from Figure 1 we get that $H_i(D_d; \mathbb{Z}) \cong \bar{H}_i(\mathbb{R}P^2; \mathbb{Z})$ for $i < d - 2$.

It remains to determine $\bar{H}_i(D_d; \mathbb{Z})$ for $i = d - 1, d - 2$. For $d$ odd, by [Lemma 3.9] and the homology long exact sequence from Figure 1 we obtain two short exact sequences.

(3.16) $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}^{(d+1)/2} \rightarrow H_{d-1}(D_d; \mathbb{Z}) \rightarrow 0$

(3.17) $0 \rightarrow \mathbb{Z}^{(d+1)/2} \rightarrow H_{d-2}(D_d; \mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0.$

In (3.16) $H_{d-1}(D_d; \mathbb{Z})$ is the homology of $D_d$ in homological dimension $\dim(D_d)$ and hence torsion free. From the exactness of the sequence it then follows that $H_{d-1}(D_d; \mathbb{Z}) \cong \mathbb{Z}^{(d+1)/2 - 1} = \mathbb{Z}^{(d-1)/2}$.

For $0 \leq \ell \leq d - 1$ even, the boundary of the cell labeled $((1, \ldots, 1), 1)$ lies completely in $D_d$. In the expansion $\gamma_\ell$ of the boundary in the cellular chain complex the cell labeled $((1, \ldots, 1), 2)$ has coefficient $-2$. As a boundary each $\gamma_\ell$ is a cycle in the cellular chain complex of $D_d$. Since the cell labelled $((1, \ldots, 1), 2)$ does not appear in the boundary of any
cell of $D_d$ it follows that each $\gamma_\ell$ represents a non-zero homology class of $D_d$. Again by the fact that none of the cells labeled $((1,\ldots,1),2)$ appear in the boundary of any cell of $D_d$ it follows that a linear combination of the $\gamma_\ell$ is zero in homology if and only if all cells labeled $((1,\ldots,1),2)$ cancel. But the cell labeled $((1,\ldots,1),2)$ appears only in $\gamma_\ell$. It follows that over $\mathbb{Z}$ the collection of all $\gamma_\ell$ forms $\frac{d+1}{2}$ linearly independent non-trivial homology cycles. As a consequence they generate $\mathbb{Z}\frac{d+1}{2}$ inside $H_{d-2}(D_d;\mathbb{Z})$. A simple calculation shows that in the sum $\gamma_0 + \cdots + \gamma_{d-1}$ all non-zero coefficients are $\pm 2$. Hence $\gamma_0 + \cdots + \gamma_{d-1}$ is twice a cycle $\gamma'$ in the cellular chain complex of $D_d$. In $\gamma'$ each cell labelled $((1,\ldots,1),2)$ has coefficient $-1$. From the proof of Lemma 3.9 we know that the classes of the cells labelled $((1,\ldots,1),\ell)$ generate $H_{d-1}(\mathcal{B}_d,D_d;\mathbb{Z})$. Hence in (3.17) their boundaries $\gamma_\ell$ generate the image of $\mathbb{Z}\frac{d+1}{2}$ in $H_{d-2}(D_d;\mathbb{Z})$. The arguments above show that $\gamma'$ does not lie in the boundary and hence it follows from (3.17) that $H_{d-2}(D_d;\mathbb{Z}) \cong \mathbb{Z}\frac{d+2}{2}$.

For $d$ even, by Lemma 3.9 and the homology long exact sequence in Figure 1 we obtain the exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow H_{d-1}(D_d;\mathbb{Z}) \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow (\mathbb{Z}/2\mathbb{Z})^{d/2} \rightarrow H_{d-2}(D_d;\mathbb{Z}) \rightarrow 0.$$  

(3.18)

The copy of $\mathbb{Z}$ in (3.18) is the top homology of the pair $(\mathcal{B}_d,D_d)$. By the proof of Lemma 3.9 in our cellulation, the cell labeled $((),0)$ represents a class that generates this homology group. The image of this class, under the connecting homomorphism, is the class of its boundary, which consists only of the cell labeled by $((),0)$. This is a top-dimensional cell of $D_d$. Its boundary in the cellular chain complex of $D_d$ is 0. Since the the cell, labeled by $((),0)$, is a basis element of the top-dimensional cellular chain group of $D_d$, it follows that its class generates a $\mathbb{Z}$-summand of the homology group. Since this $\mathbb{Z}$-summand is the image $H_d(\mathcal{B}_d,D_d;\mathbb{Z})$ under the connecting homomorphism, it follows from the exactness of (3.18) that $H_{d-1}(D_d;\mathbb{Z}) \cong \mathbb{Z}$ and (3.18) reduces to

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow (\mathbb{Z}/2\mathbb{Z})^{d/2} \rightarrow H_{d-2}(D_d;\mathbb{Z}) \rightarrow 0.$$  

This exact sequence can be seen as an exact sequence of vector spaces. Hence the sequence splits and $H_{d-2}(D_d;\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^{(d/2)-1}$. \hfill \qed

At the moment, we are unable to extend Proposition 3.10 to the case of the discriminant of classes binary forms in $\mathcal{B}_d$ with at least one root of multiplicity $k \geq 3$. One should mention that the latter case does not immediately follow from Vassiliev’s Theorem G, where the case of actual (non-projectivized) binary forms has been settled; this is related to a non-trivial action of $\mathbb{Z}/2\mathbb{Z}$ on the homology, when taking the projectivization. To illustrate this phenomenon, consider the table in Figure 2. It shows the results of our computer calculations for the projectivized discriminant in case $k = 3$, using the cellular chain complex from Theorem 3.3. Note the appearance of the higher torsion group $\mathbb{Z}/4\mathbb{Z}$ in the table, while in Vassiliev’s case all the torsion groups are sums of copies of $\mathbb{Z}/2\mathbb{Z}$.
| $d \setminus i$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  |
|-----------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 3               | $\mathbb{Z}$ |     |     |     |     |     |     |     |     |     |     |     |
| 4               | $\mathbb{Z}^2$ | $\mathbb{Z}$ |     |     |     |     |     |     |     |     |     |     |
| 5               | $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | 0   |     |     |     |     |     |     |     |     |     |
| 6               | $\mathbb{Z}/2\mathbb{Z}$ | 0   | $\mathbb{Z}^2$ | $\mathbb{Z}$ |     |     |     |     |     |     |     |     |
| 7               | $\mathbb{Z}/2\mathbb{Z}$ | 0   | $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ | 0   |     |     |     |     |     |     |     |
| 8               | $\mathbb{Z}/2\mathbb{Z}$ | 0   | $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |     |     |     |     |     |     |
| 9               | $\mathbb{Z}/2\mathbb{Z}$ | 0   | $\mathbb{Z}/2\mathbb{Z}$ | 0   | $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ | 0   |     |     |     |     |     |
| 10              | $\mathbb{Z}/2\mathbb{Z}$ | 0   | $\mathbb{Z}/2\mathbb{Z}$ | 0   | $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |     |     |     |     |
| 11              | $\mathbb{Z}/2\mathbb{Z}$ | 0   | $\mathbb{Z}/2\mathbb{Z}$ | 0   | $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/4\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | 0   |     |     |
| 12              | $\mathbb{Z}/2\mathbb{Z}$ | 0   | $\mathbb{Z}/2\mathbb{Z}$ | 0   | $\mathbb{Z}/2\mathbb{Z}$ | 0   | $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}/2\math{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ |     |     |
| 13              | $\mathbb{Z}/2\mathbb{Z}$ | 0   | $\mathbb{Z}/2\mathbb{Z}$ | 0   | $\mathbb{Z}/2\mathbb{Z}$ | 0   | $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $\mathbb{Z}$ | 0   |

**Figure 2.** Reduced homology groups $\{H_i(\mathcal{B}_d^{\max \omega \geq 3}; \mathbb{Z})\}$ of the projectivized space of binary forms with at least one root (i.e., a line in the plane) of multiplicity $\geq 3$. The rows of the table correspond to the fixed values of $d$. 
4. (Co)homological stabilization in the univariate and bivariate situations

In what follows, we either consider closed subposets $\Theta \subseteq \Omega^\infty$ such that, for all $(\omega, \kappa) \in \Theta$, the norms $|\omega|^{\Theta}$ have the same parity, or we consider closed subposets $\Theta \subseteq \Omega$ such that, for all $\omega \in \Theta$, the norms $|\omega|$ have the same parity. We call these posets $\Theta$ closed equal parity subposets. We say that a closed equal parity poset $\Theta$ is of parity $d$ or equivalently $d$ is of parity $\Theta$ if the norm of all elements of $\Theta$ has the same parity as $d$. Note that posets of mixed parity do not have natural geometric interpretation.

We say that $\Theta$ is generated by a subset $\Theta' \subseteq \Theta$ if $\Theta$ is the smallest closed subposet of either $\Omega^\infty$ or $\Omega$ that contains $\Theta'$. If $\Theta'$ can be chosen to be finite, then we say that $\Theta$ is finitely generated. It is easily seen that, for any $\Theta' \subseteq \Omega^\infty$ or $\Theta' \subseteq \Omega$, there exists a unique smallest closed subposet $\Theta$ that contains $\Theta'$. Moreover, if $\Theta$ is generated by $\Theta'$, then one can always assume that $\Theta'$ consists of maximal elements of $\Theta$ with respect to "$\prec"$.

In particular, being finitely generated is equivalent to having only finitely many maximal elements.

If $\Theta$ is a closed equal parity poset, then we denote $\Theta_{(d)} := \Theta \cap \Omega^\infty_{(d)}$ (resp., $\Theta_{(d)} := \Theta \cap \Omega_{(d)}$). We also introduce $\Theta_{d} := \Theta \cap \Omega^\infty_{d}$ (resp., $\Theta_{d} := \Theta \cap \Omega_{d}$). (These constructions can only lead to non-empty posets if $\Theta$ is not of parity $d$.

**Definition 4.1.** We call a closed poset $\Theta \subseteq \Omega$ (resp., $\Omega^\infty$) profinite if, for all integers $q \geq 0$, there exist only finitely many elements $(\omega, \kappa) \in \Theta$ (resp. $\omega \in \Theta$)) such that $|\langle(\omega, \kappa)\rangle^{\prime} \leq q$ (resp., $|\omega|^{\prime} \leq q$).

Obviously, every finite $\Theta$ is profinite. In particular, for any closed $\Theta \subseteq \Omega$ (resp., $\Theta \subseteq \Omega^\infty$), we have that $\Theta_{(d)} = \Theta \cap \Omega^\infty_{(d)}$ (resp., $\Theta_{(d)} = \Theta \cap \Omega_{(d)}$) is finite and hence profinite.

**Lemma 4.2.** Let $\Theta$ be a closed finitely generated equal parity subposet of $\Omega^\infty$ (resp., of $\Omega$). Let $n$ be the largest norm of the maximal elements in $\Theta$.

Then, for any $q \geq 0$, we have inclusions

$$\{(\omega, \kappa) \in \Theta \mid |\langle(\omega, \kappa)\rangle^{\prime} \leq q\} \subseteq \Theta_{(n+2q)} \quad \text{and} \quad \{\omega \in \Theta \mid |\omega|^{\prime} \leq q\} \subseteq \Theta_{(n+2q)}.$$

In particular, every closed finitely generated equal parity subposet is profinite.

**Proof.** Consider the case $\Theta \subseteq \Omega^\infty$ and assume that $(\omega, \kappa)$ is one of its maximal elements. Let $(\omega, \kappa)$ be such that $|\langle(\omega, \kappa)\rangle^{\prime} \leq q$. By definition any $(\omega', \kappa') \leq (\omega, \kappa)$ can be obtained from $(\omega, \kappa)$ by a sequence of merge and insert operations. Both operations increase $|\sim \sim^\prime$ by one. In particular, at most $q - |\langle(\omega, \kappa)\rangle^{\prime} \geq 0$ insertion operations can be applied to $(\omega, \kappa)$ without $|\sim \sim^\prime$ exceeding $q$. Each merge operation preserves the norm and each insertion operation increases the norm by 2. In particular, if $(\omega', \kappa') \leq (\omega, \kappa)$ and $|\langle(\omega', \kappa')\rangle^{\prime} \leq q$ then

$$|\langle(\omega', \kappa')\rangle^{\prime} = |\langle(\omega, \kappa)\rangle^{\prime} + 2(q - |\langle(\omega, \kappa)\rangle^{\prime}) = |\langle(\omega, \kappa)\rangle^{\prime} + 2q \leq n + 2q.$$

The proof in case $\Theta \subseteq \Omega$ is analogous.

The last assertion now follows from the fact that $\Theta_{(n+2q)}$ is finite. \qed

Let $\Theta \subseteq \Omega^\infty$ (resp., $\Theta \subseteq \Omega$) be a closed equal parity subposet of parity $d$. Since each $\Theta_{(d)}$ is finite, it has only finitely many maximal elements. Let $n_1, \ldots, n_{d(d)}$ be the norms of
the maximal elements in $\Theta_{(d)}$ and $c_1, \ldots, c_{\ell(d)}$ be the corresponding reduced norms. Now we define for $d$ of parity $\Theta$

\begin{equation}
(4.1) \quad \eta_\Theta(d) := \max_{i=1, \ldots, \ell(d)} (n_i - 2c_i).
\end{equation}

Note that, if $\Theta$ is finitely generated, then $\eta_\Theta(d)$ is constant for all $d$ greater than or equal to the largest norm of a maximal element in $\Theta$. In general, if $(\omega, \kappa)$ is maximal in $\Theta_{(d)}$, then it is also maximal in $\Theta_{[\ell]}$ for any $\ell \geq d$ of parity $\Theta$. This implies that $\eta_\Theta$ is a weakly increasing function on arguments of parity $\Theta$.

Further, using (4.1), for any number $d$ of the parity $\Theta$, we define the function

\begin{equation}
(4.2) \quad \psi_\Theta(d) := \frac{1}{2}(d + \eta_\Theta(d)) = \frac{1}{2}(d + \max_{i=1, \ldots, \ell(d)} \{n_i - 2c_i\}).
\end{equation}

Since the function $\eta_\Theta(d)$ is weakly increasing as a function for arguments $d$ of parity $\Theta$, it follows that $\psi_\Theta$ is strictly increasing. In particular, we get that $\lim_{d \to -\infty} \psi_\Theta(d) = \infty$, the limit being taken over $d$ of parity $\Theta$. Nevertheless, the growth of $\psi_\Theta$ depends on whether $\Theta$ is a finite, a profinite, or a general subposet.

**Lemma 4.3.** Let $\Theta \subseteq \Omega^\infty$ (resp., $\Omega^\infty$) be a closed profinite equal parity poset. Then $\lim_{d \to +\infty}(d - \psi_\Theta(d)) = +\infty$.

**Proof.** If $\Theta$ is finitely generated, then $\eta_\Theta(d) = c' \in \mathbb{Z}$ is constant for $d \gg 0$. Elementary manipulations validate the claim.

Assume $\Theta$ is not finitely generated and let $\{n_i\}_{i \in \mathbb{N}}, \{c_i\}_{i \in \mathbb{N}}$ be the sequences of norms and the corresponding reduced norms of the maximal elements of $\Theta$ ordered such that the norms are weakly increasing. Since $\Theta$ is profinite, for any $q \geq 0$ there are only finitely many $c_i$ with $c_i \leq q$. It follows that $\lim_{i \to \infty} c_i = \infty$.

Thus, for $d \gg 0$, we have $\psi_\Theta(d) = \frac{1}{2}(d + n_i - 2c_i)$ for some $i$ such that $n_i \leq d$. Note that $i$ may depend on $d$.

It follows that $d - \psi_\Theta(d) = \frac{1}{2}d - \frac{1}{2}(n_i - 2c_i) \geq 2c_i$. The claim now follows from the property $\lim_{i \to \infty} c_i = \infty$. \qed

**Example 4.4.** Let $\ell \geq 1$ be a number and $\Theta(\ell) \subseteq \Omega^\infty$ be generated by the multiplicity patterns $\{\gamma_k = ((3, \ldots, 3, 1, \ldots, 1, 0)\}_{k \geq \ell}$. Then $|\gamma_k| = k(6 + 2\ell)$ and $|\gamma_k'| = 4k$. It follows that $\Theta(\ell)$ is profinite, but not finitely generated. If for some $k$ we have $d = k(6 + 2\ell)$, then $\eta_{\Theta(\ell)}(d) = \max_{k' \leq k} \{2k'(\ell - 2)\} \leq \frac{1}{3}d$ and $\psi_{\Theta(\ell)}(d) \leq \frac{5}{6}d$. It follows that $d - \psi_{\Theta(\ell)}(d) \geq \frac{1}{6}d \to +\infty$. \diamondsuit

We now show that the function $\psi_\Theta$ plays a fundamental role in the homological stabilization of spaces $B^\Theta_d$ and $P^\Theta_d$.

Let $\Theta$ be a closed equal parity poset in either $\Omega$ or $\Omega^\infty$. For a number $d$ of the parity $\Theta$, we introduce the homomorphism

$$
\text{trunc} : \mathbb{Z}[\Theta_{(d+2)}] \to \mathbb{Z}[\Theta_{(d)}]
$$
that is the identity on the elements of \( \Theta_{d+2} \) of norm \( \leq d \) and sends all elements of norm \( d + 2 \) to 0. A simple calculation shows that \( \text{trunc} \) commutes with the differentials \( \partial^\infty \) (resp., \( \partial \)) and thus defines a homomorphism of differential complexes \((\mathbb{Z}[\Theta_{d+2}], \partial^\infty)\) and \((\mathbb{Z}[\Theta_{d}], \partial)\) (resp., \((\mathbb{Z}[\Theta_{d+2}], \partial)\) and \((\mathbb{Z}[\Theta_{d}], \partial)\)). It also lowers the homological degree by 2. Note that in the univariate case (i.e. for \( \Theta \subset \Omega \)), the strata that correspond to compositions in \( \Omega_{d+2} \) with norm \( d+2 \) are exactly the cells that consists of real polynomials whose roots are all real.

**Proposition 4.5.** Let \( \Theta \) be a closed equal parity poset in \( \Omega \) or \( \Omega^\infty \) and \( d \geq 0 \) a number of parity \( \Theta \). Then

(i) the operation \( \text{trunc} \) generates the short exact sequences of differential complexes

\[
0 \to (\mathbb{Z}[\Theta_{d+2}], \partial_M) \to (\mathbb{Z}[\Theta_{d+2}], \partial) \xrightarrow{\text{trunc}} (\mathbb{Z}[\Theta_{d}], \partial) \to 0,
\]

or

\[
0 \to (\mathbb{Z}[\Theta_{d+2}], \partial^\infty_M) \to (\mathbb{Z}[\Theta_{d+2}], \partial^\infty) \xrightarrow{\text{trunc}} (\mathbb{Z}[\Theta_{d}], \partial^\infty) \to 0,
\]

(ii) if, for some \( k \), the complex \((\mathbb{Z}[\Theta_{d+2}], \partial_M)\) or the complex \((\mathbb{Z}[\Theta_{d+2}], \partial^\infty_M)\) is acyclic in dimensions \( \geq k \), then for all \( j > k \), the homomorphism \( \text{trunc} \) induces a homological isomorphisms

\[
H_{j+2}(\mathbb{Z}[\Theta_{d+2}], \partial) \xrightarrow{\text{trunc}} H_j(\mathbb{Z}[\Theta_{d}], \partial), \quad \text{or} \quad H_{j+2}(\mathbb{Z}[\Theta_{d+2}], \partial^\infty) \xrightarrow{\text{trunc}} H_j(\mathbb{Z}[\Theta_{d}], \partial^\infty).
\]

**Proof.** First note that on \( \mathbb{Z}[\Theta_{d+2}] \) the differential \( \partial \) (resp., \( \partial^\infty \)) coincides with \( \partial_M \) (resp., \( \partial^\infty_M \)). Now the fact that the two three-term sequences of differential complexes from (i) are exact follows instantly from definition of \( \text{trunc} \). The long exact sequences in their homology, induced by the short exact sequences of differential complexes, then imply (ii). \( \square \)

In **Theorem 4.9** we will find out which posets \( \Theta \) and integers \( k \) satisfy the hypotheses of **Proposition 4.5**(ii). Prior to that, we need to investigate the geometry that leads to the (co)homological stabilization for the spaces \( \mathcal{B}_d^{\Theta}, \mathcal{B}_d^{\infty} \) and \( \mathcal{P}_d^{\Theta}, \mathcal{P}_d^{\infty} \) when \( d \) increases.

For each \((\omega, \kappa) \in \Omega^\infty_{[d]}\), consider the set \( \mathcal{K}^{(\omega, \kappa)}_d \subseteq \mathcal{B}_d \) of classes of binary forms of degree \( d \) with all zeros being real with root multiplicity pattern \((\omega', \kappa') \preceq (\omega, \kappa)\). More precisely, we set

\[
\mathcal{K}^{(\omega, \kappa)}_d := \bigcup_{(\omega', \kappa') \preceq (\omega, \kappa) \leq (\omega, \kappa)} \mathcal{R}_d^{(\omega', \kappa')}.
\]

Analogously, for a closed equal parity poset \( \Theta \subseteq \Omega^\infty_{[d]} \), we define

\[
\mathcal{K}^{\Theta}_d := \bigcup_{(\omega, \kappa) \in \Theta} \mathcal{K}^{(\omega, \kappa)}_d.
\]

The main idea of our arguments in the next three lemmas is captured by **Figure 3**. Here is their sketch. Any cell \( \mathcal{R}_d^{(\omega, \kappa)} \) in \( \mathcal{B}_d^{\Theta} \) produces a unique cell \( \mathcal{R}_{d+2}^{(\omega, \kappa)} \) in \( \mathcal{B}_d^{\infty_{d+2}} \). The portion of the boundary \( \partial \mathcal{R}_{d+2}^{(\omega, \kappa)} \) that does not intersect \( \mathcal{B}_d \) is exactly the locus \( \mathcal{K}^{(\omega, \kappa)}_{d+2} \). In the case
Figure 3. A cell $R_{d+2}^{(\omega,\kappa)}$, $|(\omega,\kappa)| \leq d$, in $B_d \cong \mathbb{R}^d$ and the corresponding cell $R_{d+2}^{(\omega,\kappa)}$ in $B_{d+2} \cong \mathbb{R}^{d+2}$. The set $K_d^\Theta$ is shown as a graph in the plane $\mathbb{R}^d$. The normal 2-bundle $\nu_2(R_{d+2}^{(\omega,\kappa)})$ is denoted by $R_{d+2}^{(\omega,\kappa)}$. In (a) the portion $K_{d+2}^{(\omega,\kappa)}$ of the boundary $\partial R_{d+2}^{(\omega,\kappa)}$ has dimension $\dim(R_{d+2}^{(\omega,\kappa)}) - 1$, while (b) $K_{d+2}^{(\omega,\kappa)}$ has dimension smaller than $\dim(R_{d+2}^{(\omega,\kappa)}) - 1$.

of Figure 3(a), that portion has the dimension $\dim(R_{d+2}^{(\omega,\kappa)}) - 1$ and contributes a term to the boundary operator of $R_{d+2}^{(\omega,\kappa)}$ in the cellular chain complex. In case Figure 3 (b) the portion $K_{d+2}^{(\omega,\kappa)}$ has dimension strictly smaller than $\dim(R_{d+2}^{(\omega,\kappa)}) - 1$ and does not contribute a term. The function $\psi_\Theta(d+2)$ (see (4.2)) gives the dimension of $K_{d+2}^{(\omega,\kappa)}$ and thus helps to discriminate between the "bad case" (a) and the "good case" (b). If $\dim(R_{d+2}^{(\omega,\kappa)}) - 1 = \dim(R_{d+2}^{(\omega,\kappa)}) + 1 > \psi_\Theta(d + 2)$, we are dealing with the good case (b) in which the algebraic boundary of $R_{d+2}^{(\omega,\kappa)}$ is faithfully represented by the algebraic boundary of $R_{d}^{(\omega,\kappa)}$.

Lemma 4.6.

(i) For $(\omega,\kappa) \in \Omega_{[d]}^\infty$ or $\omega \in \Omega$, we have

$$\dim(K_d^{(\omega,\kappa)}) = \frac{1}{2} \left( d + (|(\omega,\kappa)| - 2(|(\omega,\kappa)|') \right)$$

or

$$\dim(K_d^{\omega}) = \frac{1}{2} \left( d + (|\omega| - 2|\omega'|) \right).$$

(ii) For $d \geq 0$ and a closed equal $d$-parity subposet $\Theta \subseteq \Omega_{[d]}^\infty$ or $\Theta \subseteq \Omega$, we have

$$\dim(K_d^{\Theta}) = \psi_\Theta(d).$$

Proof. It suffices to consider marked composition and $\Theta \subseteq \Omega_{[d]}^\infty$. To settle item (i), we need to find the marked composition $(\omega',\kappa') \preceq (\omega,\kappa)$ with the minimal reduced norm $|(\omega',\kappa')'|$ and the norm $|(\omega',\kappa')| = d$. Since $d - |(\omega,\kappa)|$ is even, any sequence of $\frac{1}{2}(d - |(\omega,\kappa)|)$ insert
operations, being applied to \((\omega, \kappa)\), does the job. The resulting \((\omega', \kappa')\) has the reduced norm \(|(\omega', \kappa')'| = |(\omega, \kappa)'| + \frac{1}{2}(d - |(\omega, \kappa)|)\). Hence the dimension of \(\mathcal{K}_{d(\omega', \kappa')}\) is given by

\[
d - \frac{1}{2}(d - |(\omega, \kappa)|) + 2|(\omega, \kappa)'| = \frac{1}{2}(d + |(\omega, \kappa)| - 2|(\omega, \kappa)'|).
\]

Item (ii) follows immediately from the definition of \(\psi_{\Theta}(d)\) and (i).

The next few lemmas prepare for [Theorem 4.9] below, which, in the spirit of [Lemma 3.1], represents a geometrization of the trunc operator. Recall that for \(\Theta \subseteq \Omega^\infty\) and a number \(\ell \geq 0\) we denote by \(\Theta|_{\sim'} = \ell\) all \((\omega, \kappa) \in \Theta\) such that \(|(\omega, \kappa)|' = \ell\).

**Lemma 4.7.** Let \(\Theta \subseteq \Omega^\infty\) (resp., \(\Theta \subseteq \Omega\)) be a closed equal parity poset, \(d \geq 0\) of parity \(\Theta\). For \(j \geq \psi_{\Theta}(d + 2)\), we have an inclusion \(\Theta|_{\sim'} = d - j \cap \Omega^\infty_{d+2} \subseteq \Theta(d)\). (resp., \(\Theta|_{\sim'} = d - j \cap \Omega_{d+2} \subseteq \Theta(d)\).

**Proof.** It again suffices to consider the case \(\Theta \subseteq \Omega^\infty\). Consider \((\omega, \kappa)\) such that \(|(\omega, \kappa)| \leq d + 2\) and \(|(\omega, \kappa)'| = d - j\), where \(j \geq \psi_{\Theta}(d + 2)\). Contrary to the lemma’s claim, assume that \(|(\omega, \kappa)| = d + 2\). Then

\[
\dim(R^{(\omega, \kappa)}_{d+2}) = d + 2 - (d - j) = j + 2^{j+1 \geq \psi_{\Theta}(d+2)} > \psi_{\Theta}(d + 2).
\]

By our assumptions, \(R^{(\omega, \kappa)}_{d+2} \subseteq \mathcal{K}^\Theta_{d+2}\). From [Lemma 4.6], (ii), it follows that

\[
\dim(R^{(\omega, \kappa)}_{d+2}) \leq \dim(\mathcal{K}^\Theta_{d+2}) = \psi_{\Theta}(d + 2).
\]

The latter yields a contradiction and thus we conclude that \((\omega, \kappa) \in \Theta(d)\).

The next lemma claims that, in dimensions greater than \(\dim(\mathcal{K}^\Theta_{d+2}) + 1\), the boundary operators \(\partial^\infty\) from (3.1) carry the same information in degrees \(d\) and \(d + 2\).

**Lemma 4.8.**

(i) Let \(\Theta \subseteq \Omega^\infty\) be a closed equal parity poset, \(d \geq 0\) of the parity \(\Theta\). Then the following diagram is a commutative. Its horizontal arrows are the boundary operators \(\partial^\infty\) and its vertical arrows are isomorphisms for \(j \geq \psi_{\Theta}(d + 2)\).

\[
\begin{array}{ccc}
Z[\Omega^\infty_{(d+2)} \cap \Theta|_{\sim'} = d + 2 - (j + 3)] & \rightarrow & Z[\Omega^\infty_{(d+2)} \cap \Theta|_{\sim'} = d + 2 - (j + 2)] \\
\downarrow \text{trunc.} & & \downarrow \text{trunc.} \\
Z[\Omega^\infty_{d} \cap \Theta|_{\sim'} = d - (j + 1)] & \rightarrow & Z[\Omega^\infty_{d} \cap \Theta|_{\sim'} = d - (j - 1)]
\end{array}
\]

(ii) Let \(\Theta \subseteq \Omega\) be a closed equal parity poset, \(d \geq 0\) of the parity \(\Theta\). Then the following diagram is a commutative. Its horizontal arrows are the boundary operators \(\partial\) and its vertical arrows are isomorphisms for \(j \geq \psi_{\Theta}(d + 2)\).

Proof. It suffices to prove (i). By Lemma 4.7 and the inequality \( j + 2 = (j + 1) + 1 > j \geq \psi_\Theta(d + 2) \), we have \( \Theta_{|\sim'|=d+2-(j+3)} = \Theta_{|\sim'|=d-(j+1)} \subseteq \Theta_{|d|} \).

The same lemma shows that the inequality \( j + 1 > j \geq \psi_\Theta(d + 2) \) implies that \( \Theta_{|\sim'|=d+2-(j+2)} = \Theta_{|\sim'|=d-j} \subseteq \Theta_{|d|} \).

Finally, this argument also shows that the inequality \( (j - 1) + 1 = j \geq \psi_\Theta(d + 2) \) implies that \( \Theta_{|\sim'|=d+2-(j+1)} = \Theta_{|\sim'|=d-(j-1)} \subseteq \Theta_{|d|} \).

Thus by definition, the vertical maps \( \text{trunc}_e \) in the diagram are isomorphisms of \( \mathbb{Z} \)-modules.

Again by Lemma 4.7 for any \( (\omega, \kappa) \) in \( \Theta_{|\sim'|=d-j-1} \), \( \Theta_{|\sim'|=d-j} \), or \( \Theta_{|\sim'|=d-j+1} \), we conclude that \( R^{(\omega', \kappa')}_{d+2} \) appears with coefficient \( a_{(\omega', \kappa')} \) in the expansion of the boundary \( \partial R^{(\omega, \kappa)}_{d+2} \) if and only if \( R^{(\omega', \kappa')}_{d} \) appears with coefficient \( a_{(\omega', \kappa')} \) in the expansion of the boundary \( \partial R^{(\omega, \kappa)}_{d} \). Therefore the diagram is commutative.

Now we are in position to establish our main stabilization results.

**Theorem 4.9 (Short stabilization \( d \to d + 2 \), the projective case).** Let \( \Theta \subseteq \Omega^\infty \) be a closed equal parity poset and \( d \) of parity \( \Theta \).

(i) For all \( j \geq \psi_\Theta(d + 2) - 1 \), there is an isomorphism

\[
T_j : H_j(B^\Theta_d; \mathbb{Z}) \cong H_{j+2}(B^\Theta_{d+2}; \mathbb{Z}).
\]

(ii) For all positive \( j > \psi_\Theta(d + 2) - 1 \), there is an isomorphism

\[
T^\#_j : H_j(B_d, B^\Theta_d; \mathbb{Z}) \cong H_{j+2}(B_{d+2}, B^\Theta_{d+2}; \mathbb{Z}).
\]

Both isomorphisms \( T_j \) and \( T^\#_j \) are delivered by the inverses \( (\text{trunc}_e)^{-1} \) of truncations \( \text{trunc}_e \).

Proof. Using that \( \mathbb{Z}[\Omega^\infty_{|d+2|} \cap \Theta_{|\sim'|=d-j+1}] \) is the chain group of cellular chain complex of \( B^\Theta_{d+2} \) in dimension \( j' \) and \( \mathbb{Z}[\Omega^\infty_{|d|} \cap \Theta_{|\sim'|=d-j+1}] \) is the chain group of cellular chain complex of \( B^\Theta_d \) in dimension \( j' \), assertion (i) follows directly from Lemma 4.8.

For (ii), consider the new poset \( \Theta' \), consisting of all the elements of \( \Omega^\infty \) with the same parity as \( d \). Of course, for that poset, we have \( B^\Theta_{d+2} = B_d \) and \( B^\Theta_{d+2} = B_{d+2} \). The maximal elements of \( \Theta'_{|d+2|} \) are \((\{1, \ldots, 1\}, 0)\) for \( 0 \leq \ell \leq d + 2 \) of the same parity as \( d \).

The reduced norm of each of those marked compositions is 0. Hence \( \psi_{\Theta'}(d + 2) \) is either 1 or 0, depending on the parity of \( d \). Thus by (i), for \( j > 0 \), \( \text{trunc}_e \) induces an isomorphism \( H_{j+2}(B_d; \mathbb{Z}) \cong H_j(B_{d+2}; \mathbb{Z}) \). Consider the long exact homology sequences of the two pairs \( (B_{d+2}, B^\Theta_{d+2}) \) and \( (B_d, B^\Theta_d) \) and the homomorphisms \( \text{trunc}_e \) connecting them.
By (i), the two leftmost homomorphisms \( \text{trunc}_* \) in each row are isomorphisms for all \( j + 1 > \psi_{\Theta}(d + 2) \). Thus, by the Five Lemma, the assertion (ii) follows.

If true, the conjecture below would deliver a better geometric understanding of the spaces that appear in the preceding proof.

**Conjecture 4.10.** For any closed \( \Theta \), the quotient \( \mathcal{B}_{d+2}^\Theta / \mathcal{K}_{d+2}(\Theta) \) is the Thom space of an orientable topological 2-dimensional microbundle over \( \mathcal{B}_d^\Theta \).

The same proof a the proof of Theorem 4.9 yields the following stabilization result for spaces of polynomials.

**Theorem 4.11 (Short stabilization \( d \Rightarrow d + 2 \), the polynomial case).** Let \( \Theta \subseteq \Omega \) be a closed equal parity poset and \( d \) of parity \( \Theta \).

(i) For all \( j \geq \psi_{\Theta}(d + 2) - 1 \), there is an isomorphism

\[
T_j : H_j(\overline{\mathcal{P}}_d^\Theta; \mathbb{Z}) \cong H_{j+2}(\overline{\mathcal{P}}_{d+2}^\Theta; \mathbb{Z}).
\]

(ii) For all positive \( j > \psi_{\Theta}(d + 2) - 1 \), there is an isomorphism

\[
T_j^\#: H_j(\overline{\mathcal{P}}_d, \overline{\mathcal{P}}_d^\Theta; \mathbb{Z}) \cong H_{j+2}(\overline{\mathcal{P}}_{d+2}, \overline{\mathcal{P}}_{d+2}^\Theta; \mathbb{Z}).
\]

Both isomorphisms \( T_j \) and \( T_j^\# \) are delivered by the inverses \( (\text{trunc}_*)^{-1} \) of truncations \( \text{trunc}_* \).

Next we give two examples where the stabilization bound \( \psi_{\Theta} \) from Theorem 4.9 and Theorem 4.11 can be explicitly determined.

**Example 4.12.** For Theorem 4.11 we give in \( \S \ 5 \) a long list of homology computations for \( \Theta \subseteq \Omega \) which are generated by a single composition. In these cases, for \( d \geq |\omega| \), we have \( \psi_{\Theta}(d) = d + |\omega| - 2|\omega|' \). The reader is welcome to verify the results from Theorem 4.11 in the tables.
Figure 5. Homology of the space \( B_d^{\Theta_{\ell,q}} \), the \((d-2)\)-skeleton of \( B_d \). The poset \( \Theta_{\ell,2} \) is defined in (4.3).

Interesting examples, where the stabilization bound from Theorem 4.9 is non-trivial can be derived from Example 4.4 or cases where \( \Theta \) is finitely generated. The next examples will not provide interesting applications of Theorem 4.9 but show that the stabilization bound in it is sharp.

Example 4.13. For \( \ell = 0 \) or 1, the two closed equal parity posets given by

\[
\Theta_{\ell,q} = \Omega_{|\ell| \geq q}^{\infty} := \{ (\omega, \kappa) \in \Omega^{\infty} \mid |(\omega, \kappa)| \geq q, |(\omega, \kappa)| \equiv \ell \mod 2 \};
\]

are easily seen to be non-profinite. The space \( B_d^{\Theta_{\ell,q}} \) is the \((d-q)\)-skeleton of \( B_d \).

By definition \( \eta_\Theta(d) \), we get \( \eta_\Theta(d) = d - 2q \). Thus \( \psi_\Theta(d) = d - q \) and \( \psi_\Theta(d+2) = d+2-q \). Now Theorem 4.9 shows that, if \( j + 1 \geq d + 2 - q \) or, equivalently, \( j \geq d + 1 - q \), then \( H_j(B_{d}^{\Theta_{\ell,q}}; \mathbb{Z}) \cong H_{j+2}(B_{d+2}^{\Theta_{\ell,q}}; \mathbb{Z}) \). For \( q = 2 \), the homology of \( B_d^{\Theta_{\ell,q}} \) is listed in Figure 5 for small \( d \) and \( \ell \equiv d \mod 2 \). The list is the result of our computer-assisted calculations.

Note that the stabilization from Theorem 4.9 appears for \( j \geq d - 1 \), which is just right above the top non-vanishing homology group. On the other hand, for \( j = d - 2 \), Figure 5 testifies that there is no stabilization. In fact, for \( j < d - 2 \), the homology is stable, since here we are considering the skeleta of the real projective space.

Example 4.14. Fix a parity and a number \( k \geq 2 \). Consider the closed equal parity poset \( \Theta = \Theta_{\max \geq k} \) containing all \((\omega, \kappa)\) of the chosen parity for which there is a part of \( \omega \) that is \( \geq k \) or \( \kappa \geq k \). Thus \( B_d^\Theta \) is the space of classes of binary forms of degree \( d \) with at least one \( k \)-fold root (see Theorem B and Theorem G for some results on analogously defined spaces).

The maximal elements of \( \Theta \) are the \((\omega, 0)\) where \( \omega \) has one part of size \( k \) and all other being 1. It follows that \( |(\omega, 0)|' = k - 1 \). Hence \( \eta_\Theta(d) = d - 2(k - 1) \). Therefore \( \psi_\Theta(d) = d - k + 1 \) and \( \psi_\Theta(d+2) = d - k + 3 \). Again, Theorem 4.9 shows that, if \( j + 1 \geq d - k + 3 \) or, equivalently, \( j \geq d - k + 2 \), then \( H_j(B_d^\Theta; \mathbb{Z}) \cong H_{j+2}(B_{d+2}^\Theta; \mathbb{Z}) \). Looking at Figure 2...
one sees that the stabilization bound is again right about the top non-vanishing homology group. However, it seems that here a stabilization below that bound is present.

Let us describe, in terms of divisors, the effect of increasing the degree $d \to d + 2$ on the spaces of projectivized binary forms.

Let $\epsilon : \mathcal{B}_d \to \mathcal{B}_{d+2}$ be the map which sends the class of a binary form $f(x, y)$ in $\mathcal{B}_d$ to the class of the binary form $(x^2 + y^2)f(x, y)$ in $\mathcal{B}_{d+2}$. Since $x^2 + y^2$ has no real roots in $\mathbb{C}\mathbb{P}^1$, for any $(\omega, \kappa) \in \Omega_{(d)}$, the image of $\mathcal{R}_d^{(\omega, \kappa)}$ under $\epsilon$ is a subset of $\mathcal{R}_{d+2}^{(\omega, \kappa)}$.

Consider the 2-sphere $\mathbb{C}\mathbb{P}^1$ and the involution $J : \mathbb{C}\mathbb{P}^1 \to \mathbb{C}\mathbb{P}^1$, given by the complex conjugation $J[z : w] = [\bar{z} : \bar{w}]$. The fixed point set of $J$ is the circle $\bar{\mathbb{R}} = \mathbb{R}\mathbb{P}^1$. We can identify $\mathcal{B}_d$ with the space positive $J$-symmetric divisors on $\mathbb{C}\mathbb{P}^1$ of degree $d$. Then $\epsilon$ translates into a map $\epsilon^* : [Sym^d(\mathbb{C}\mathbb{P}^1)]^J \to [Sym^{d+2}(\mathbb{C}\mathbb{P}^1)]^J$ that takes each divisor $D$ to the divisor $D + [i : 1] + [-i : 1]$. Thus the real divisors $\epsilon^*(D)_{\mathbb{R}}$ and $D_{\mathbb{R}}$ share the same combinatorial type $(\omega, \kappa)$.

Out next goal is to give alternative pure homological definitions of the maps $(\text{trunc})^*$, $(\text{trunc})_*$. The results will allow us the study stabilization in cohomology of the spaces $\mathcal{B}_d^{c\Theta}$. With this goal in mind, let us describe a list of hypotheses, needed for such a definition.

**Hypotheses A.** Let $M$ be a closed $n$-dimensional manifold $M$, equipped with a structure of a finite CW-complex, and $N \subseteq M$ a closed $(n - k)$-dimensional submanifold whose normal bundle is orientable. Assume that the intersections of $N$ with the cells of $M$ induce a structure of a cellular complex on $N$.

Let $K \subset M$ be a compact CW-subcomplex of $M$. Put $L := K \cap N$ and let $M \setminus K$ and $N \setminus L$ be connected.

Assume that, for each open cell $e$ of $M$, the intersection $e \cap N$ is either a single cell such that $\dim(e) = \dim(e \cap N) + k$, or $e \cap N = \emptyset$. We assume also that $M$ admits a triangulation $T$ that is consistent with its cellular structure: so that $K$, $L$, $N$ and their cellular skeletons are closed subcomplexes of this triangulation.

Put $\Lambda := \mathbb{Z}[\pi_1(M \setminus K)]$. Let $\Lambda$ be a right $\Lambda$-module. The embedding $\epsilon : N \setminus L \subset M \setminus K$, with the help of the homomorphism $\epsilon_* : \pi_1(N \setminus L) \to \pi_1(M \setminus K)$, converts the $\Lambda$-module $\Lambda$ into a $\mathbb{Z}[\pi_1(N \setminus L)]$-module $\epsilon^*(\Lambda)$.

Let $[M] \in H_n(M ; \mathbb{Z}_i)$ and $[N] \in H_{n-k}(M ; \mathbb{Z}_i)$ be the fundamental classes of $M$ and $N$, respectively.

We denote by $C_*(\sim)$ and $C^*(\sim)$ the $\mathbb{Z}$-modules of cellular chains of a cellular complex. \(\Diamond\)

**Lemma 4.15.** Under Hypotheses A, the following diagram is commutative:

$$
\begin{array}{ccc}
H^j(N, L ; \epsilon^*(\Lambda)) & \xrightarrow{(\text{trunc})^*} & H^{j+k}(M, K ; \Lambda) \\
\cong \downarrow [N] \cap & & \cong \downarrow [M] \cap \\
0 & & 0
\end{array}
$$

(4.4)

where the vertical isomorphisms are the Poincaré duality maps, $D_N = [N] \cap \sim$ and $D_M = [M] \cap \sim$. The horizontal homomorphisms $(\text{trunc})^*$ are induced by the duals of the chain homomorphisms $\text{trunc} : C_{*+k}(M, K ; \Lambda) \to C_*(N, L ; \epsilon^*(\Lambda))$, the latter ones are produced by
the intersection of cells \( \{e\} \) in \( M \) with \( N \). The proof below contains the exact definition of the cohomological transfer \((\text{trunc})^*\) from the diagram (4.4).

**Proof.** We use the second barycentric subdivision of the triangulation \( T \) to form regular neighborhoods of closed subcomplexes of \( T \).

Since, for each open cell \( e \) of \( M \), the intersection \( e \cap N \) is either a single cell such that \( \dim(e) = \dim(e \cap N) + k \), or \( e \cap N = \emptyset \), the restriction of the cellular structure on \( M \) to \( N \) gives rise to a cellular structure on \( N \) and produces the correspondence \( \text{trunc} : e \sim e \cap N \). Since \( \nu(N, M) \) is orientable, its orientation helps to use the orientation of \( e \) to orient \( e \cap N \). The correspondence \( \text{trunc} \) leads to a well-defined homomorphisms on the cellular chain level:

\[
\text{trunc} : C_{j+k}(K; \mathbb{A}) \rightarrow C_j(L; \epsilon^*(\mathbb{A})), \text{trunc} : C_{j+k}(M, K; \mathbb{A}) \rightarrow C_j(N, L; \epsilon^*(\mathbb{A}))
\]

We use the identity \([\text{trunc}^*(\alpha)](e) := \alpha(e \cap N)\) for any cochain \( \alpha \in C^j(L; \epsilon^*(\mathbb{A})) \) and a \((j + k)\)-dimensional cell \( e \subset K \) to introduce the dual operator \((\text{trunc})^*\).

Our next goal is to define the truncation \( H^j(N, L; \epsilon^*(\mathbb{A})) \xrightarrow{(\text{trunc})^*} H^{j+k}(M, K; \mathbb{A}) \) in pure homological terms, since the cap products in (4.4) are not directly amenable to the cellular structure \( \{e\} \) on \( M \).

Let \( M_p, K_p \) denote the \( p \)-skeletons of \( M, K \) in the cellular structure \( \{e\} \), and \( N_p, L_p \) denote the \( p \)-skeletons of \( N, L \) in the cellular structure \( \{N \cap e\} \). To simplify the notations, put

\[
K_p^\sharp := K_p \cup L, \quad M_p^\sharp := M_p \cup K, \quad N_p^\sharp := N_p \cup L.
\]

We omit the relevant coefficient systems \( \mathbb{A}, \mathbb{A}^t \) and \( \epsilon^*(\mathbb{A}), \epsilon^*(\mathbb{A}^t) \) in our notations.

Consider the following filtrations by closed subsets:

\[
N := N_n^\sharp \supset \ldots \supset N_p^\sharp \supset N_{p-1}^\sharp \supset \ldots \supset N_0^\sharp, \quad M := M_{n+k}^\sharp \supset \ldots \supset M_p^\sharp \supset M_{p-1}^\sharp \supset \ldots \supset M_0^\sharp.
\]

These filtrations give rise to two spectral sequences, whose \( E_1 \)-pages are:

\[
E_1^{p,q}(N, L) = H^{p+q}(N_p^\sharp, N_{p-1}^\sharp) \quad \text{converges to} \quad \Rightarrow H^*(N, L),
\]

\[
E_1^{p,q}(M, K) = H^{p+q}(M_p^\sharp, M_{p-1}^\sharp) \quad \text{converges to} \quad \Rightarrow H^*(M, K),
\]

Therefore, in terms of the given cellular structures \( \{e \cap N\} \) in \( N \) and \( \{e\} \) in \( M \), we get \( E_1^{p,0}(N, L) \cong C^p(N, L) \) and \( E_1^{p,0}(M, K) = 0 \) for all \( q > 0 \) since the cohomology with compact support of each open \( p \)-cell arise only in dimension \( p \). Similarly, \( E_1^{p,0}(M, K) \cong C^p(M, K) \), and \( E_1^{p,q}(M, K) = 0 \) for all \( q > 0 \).

Note that \( N_p^\sharp \setminus N_{p-1}^\sharp = N_p \setminus (N_{p-1} \cup L_p) \) and \( M_p^\sharp \setminus M_{p+k}^\sharp = M_{p+k} \setminus (M_{p+k-1} \cup K_{p+k}) \) are unions of open \( p \)-cells and \((p + k)\)-cells, respectively. So we have a homomorphism \((\text{trunc}^*)^{p,q} : E_1^{p,q}(N, L) \rightarrow E_1^{p+k,q}(M, K)\), defined by:

\[
(4.5) \quad H^p(N_p^\sharp, N_{p-1}^\sharp) \cong C^p(N, L) \xrightarrow{(\text{trunc})^*} C^{p+k}(M, K) \cong H^{p+k}(M_{p+k}^\sharp, M_{p+k-1}^\sharp).
\]
It relies on the coherent orientations of the pairs \((e, e \cap N)\) of cells and amounts to attaching the value \(\alpha(e \cap N)\) to each \((p + k)\)-cell \(e\) in \(M\) and each \(p\)-cochain \(\alpha\) that represents an element of \(H^p(N^2_p, N^2_{p-1})\).

It remains to verify that \(\text{trunc}^* : E^{p,q}_1(N, L) \to E^{p+k,q}_1(M, K)\) commutes with the differentials of the two spectral sequences. This verification is in line with the standard arguments that show the equivalence of singular and cellular (co)homology. We incorporate the differentials from both spectral sequences in the commutative diagrams as on page 227 of [Mu] (with homology being replaced with cohomology and the arrows being reversed). The top of the two diagrams is based on the relative cohomology of different pairs from the quadruple

\[
M^{q}_{p+k+1} \supset M^{q}_{p+k} \supset M^{q}_{p+k-1} \supset M^{q}_{p+k-2},
\]

and the bottom one on on the relative cohomology of different pairs from the quadruple

\[
N^{q}_{p+1} \supset N^{q}_{p} \supset N^{q}_{p-1} \supset N^{q}_{p-2}.
\]

The degree raising differentials \(\delta^{p-1}_N : H^p(N^2_{p-1}, N^2_{p-2}) \to H^p(N^2_p, N^2_{p-1})\) and \(\delta^{p}_N : H^p(N^2_p, N^2_{p-1}) \to H^p(N^2_{p+1}, N^2_p)\) are taken from the long exact cohomology sequence of the triples \(\{N^2_p \supset N^2_{p-1} \supset N^2_{p-2}\}\) and \(\{N^2_{p+1} \supset N^2_p \supset N^2_{p-1}\}\), respectively. Similarly, we have the differentials \(\delta^{p+k-1}_M : H^{p+k-1}(M^2_{p+k-1}, M^2_{p+k-2}) \to H^{p+k}(M^2_{p+k}, M^2_{p+k-1})\) and \(\delta^{p+k}_M : H^{p+k}(M^2_{p+k}, M^2_{p+k-1}) \to H^{p+k+1}(M^2_{p+k+1}, M^2_{p+k})\). By the lemma hypotheses “\(\partial(e) \cap N = \partial(e \cap N)\)” these differentials coincide with the coboundary operators on the cochain level. By the diagrams as on page 227 of [Mu], the differentials

\[
\delta^{p-1}_N = \partial^{p-1} : C^{p-1}(N, L) \to C^p(N, L), \quad \delta^p_N = \partial^p : C^p(N, L) \to C^{p+1}(N, L),
\]

\[
\delta^{p+k-1}_M = \partial^{p+k-1} : C^{p+k-1}(M, K) \to C^{p+k}(M, K), \quad \delta^{p+k}_M = \partial^{p+k} : C^{p+k}(M, K) \to C^{p+k+1}(M, K)
\]

commute with the \(\text{trunc}^*\) maps: that is, \(\delta^{p-1}_N \circ \text{trunc}^* = \text{trunc}^* \circ \delta^{p+k-1}_M\) and \(\delta^p_N \circ \text{trunc}^* = \text{trunc}^* \circ \delta^{p+k}_M\). Therefore (see [Mu], Theorems 39.4 and 39.5) the map from (4.5) gives rise to a homomorphism

\[
\text{trunc}^* : H^p(N, L) = \frac{\ker(\delta^p_N)}{\text{im}(\delta^{p-1}_N)} \to \frac{\ker(\delta^{p+k}_M)}{\text{im}(\delta^{p+k-1}_M)} = H^{p+k}(M, K),
\]

which is now well-defined on the level of cohomology (and not only on the cellular level). As a result, all the maps in diagram (4.4) are well-defined homologically.

Thus to validate the commutativity of (4.4), we seek to verify that, for a given \(\alpha \in H^p(N, L; e^*(\mathbb{A}))\), we get \([M] \cap \text{trunc}^*(\alpha) = \epsilon_*([N] \cap \alpha)\).

Let \(U(N) \subset M\) be a regular closed neighborhood of \(N\) in \(M\) (defined via the second barycentric subdivision of \(T\)) and \(\pi : U \to N\) its canonical retraction; thanks to the properties of the triangulation \(T\), \(\pi^{-1}(L) = K \cap U\). Let \(u \in H^k(U, \partial U)\) be a relative cocycle. Poincaré-dual in \(U\) to the fundamental cycle \(\epsilon_*([N]) \in H_n(U; \mathbb{Z}^t) \cong H_n(N; \mathbb{Z}^t)\). Then \([M] \cap u = \epsilon_*([N])\) and \(\pi^*(\alpha) \cup u\) is a relative cocycle of dimension \(p+k\) in \((U, \partial U)\). Since
\( \alpha \) vanishes on singular chains on \( L \), \( \pi^*(\alpha) \cup u \) vanishes on singular chains on \( \pi^{-1}(L) \cup \partial U = (K \cap U) \cup \partial U \). So \( \pi^*(\alpha) \cup u \) extends to a relative cocycle on \((M, K)\) that vanishes on \( K \cup (M \setminus \text{int}(U)) \). Abusing notations, we denote this extension by \( \pi^*(\alpha) \cup u \) as well.

For any \((p + k)\)-cell \( e \), we denote by \( e_U \) the intersection \( e \cap U \). Note that the interior of \( e_U \) is an open \((p + k)\)-cell, and \( e_U \) is a regular neighborhood of \( e \cap N \) in \( e \). Thanks to the constructions of \( T \) and \( U \), it is equipped with the canonical projection \( \pi : e_U \to e \cap N \) whose fiber is a \( k \)-dimensional \( \text{PL} \)-ball. Then, for any relative \( p \)-cocycle \( \alpha \) that vanishes on the boundary \( \partial(e \cap N) \), we get \((u \cup \pi^*(\alpha))(e_U) = \alpha(e \cap N) \). Thus we conclude that the restrictions \((u \cup \pi^*(\alpha))|_{e_U} = \text{trunc}^*(\alpha)|_{e_U} \).

Next, we represent the fundamental class \([M] \) as a \((n + k)\)-chain \( \mathcal{M} = \sum_{\{e: \dim(e) = n + k\}} e \), where each \((n + k)\)-cell \( e \) is viewed, with the help of \( T \), as a singular \((n + k)\)-chain, a sum of singular \((n + k)\)-simplicies.

Then, using the basic properties of cap and cup products (\( \cap \) and \( \cup \)) (see \([Sp]\)), we get

\[
\mathcal{M} \cap \text{trunc}^*(\alpha) = \sum_{\{e: \dim(e) = n + k\}} e \cap (\pi^*(\alpha) \cup u)
\]

\[
= \sum_{\{e: \dim(e) = n + k\}} (e \cap u) \cap \pi^*(\alpha) = \sum_{\{e: \dim(e) = n + k\}} e_u \cap \pi^*(\alpha),
\]

where the \( n \)-chain \( e_u := e \cap u \). Applying \( \pi_* \) to the last expression, we get

\[
\pi_* \left( \sum_{e} e_u \cap \pi^*(\alpha) \right) = \sum_{e} \pi_* (e_u \cap \pi^*(\alpha)) = \sum_{e} \left( \pi_* (e_u) \cap \alpha \right) = \left( \sum_{e} \pi_* (e_u) \right) \cap \alpha = [N] \cap \alpha.
\]

Since \( \pi : U \to N \) is a homotopy and thus homology equivalence, the cycles \([M] \cap \text{trunc}^*(\alpha)\) and \( e_*([N] \cap \alpha) \) are homologous in \( H_{n-j}(M \setminus K; \mathbb{A}^t) \). This completes the proof of the lemma. \( \square \)

The commutativity of diagram \((4.4)\) implies instantly the following fact.

**Corollary 4.16.** Under the hypotheses and notations of **Lemma 4.15**, the transfer \( H^j(N, L; \mathbb{A}) \xrightarrow{\text{trunc}^*} H^{j+k}(M, K; \mathbb{A}) \) is an isomorphism if and only if the natural homomorphism \( H_{n-j}(N \setminus L; \epsilon^*(\mathbb{A}^t)) \xrightarrow{\epsilon_*} H_{n-j}(M \setminus K; \mathbb{A}^t) \) is. \( \diamond \)

Now we are in position to consider stabilization in cohomology for the spaces \( \mathcal{B}^\Theta_d \).

**Theorem 4.17 (Dual short stabilization \( d \Rightarrow d + 2 \), projective case).** Let \( \Theta \subseteq \Omega^\infty \) be a closed closed equal parity poset.

Then for \( d \) of parity \( \Theta \) the embedding \( \epsilon : \mathcal{B}^\Theta_d \subseteq \mathcal{B}^\Theta_{d+2} \), induces an isomorphism

\[
\epsilon_* : H_k(\mathcal{B}^\Theta_d; \epsilon^*(\mathbb{Z}^t)) \cong H_k(\mathcal{B}^\Theta_{d+2}; \mathbb{Z}^t)
\]

for all \( k \leq d + 2 - \psi_\Theta(d + 2) \).

If \( d \equiv 1 \mod 2 \), then one can replace in \((4.6)\) the local coefficient system \( \mathbb{Z}^t \) with the trivial \( \mathbb{Z}[\pi_1(\mathcal{B}^\Theta_{d+2})] \)-module \( \mathbb{Z} \).
Proof. We apply Lemma 4.15 and Corollary 4.16 to the spaces of $\mathcal{B}_d$ and $\mathcal{B}^\Theta_d$, where $N = \mathcal{B}_d$, $M = \mathcal{B}_{d+2}$, $K = \mathcal{B}^\Theta_{d+2}$, and $L = \mathcal{B}^\Theta_d$. In order to apply Lemma 4.15 we use the fundamental fact that stratified algebraic sets admit triangulations [Jo], [Har], [Har1], consistent with their stratifications.

Let $A = \mathbb{Z}$ in Corollary 4.16 We then get that

$$\epsilon_* : H_{d-j}(\mathcal{B}^\Theta_d; \epsilon^*(\mathbb{Z}^l)) \to H_{d-j}(\mathcal{B}^\Theta_{d+2}; \mathbb{Z}^l)$$

is an isomorphism if and only if

$$\text{trunc}^*: H^j(\mathcal{B}_d, \mathcal{B}^\Theta_d; \mathbb{Z}) \to H^{j+2}(\mathcal{B}_d, \mathcal{B}^\Theta_d; \mathbb{Z})$$

is an isomorphism. By Theorem 4.9 $H_{j+2}(\mathcal{B}_{d+2}, \mathcal{B}^\Theta_{d+2}; \mathbb{Z}) \to H^j(\mathcal{B}_d, \mathcal{B}^\Theta_d; \mathbb{Z})$ is an isomorphism for all $j + 1 > \psi_\Theta(d + 2)$. By the universal coefficient theorem, $H^j(\mathcal{B}_d, \mathcal{B}^\Theta_d; \mathbb{Z}) \to H^j(\mathcal{B}_d, \mathcal{B}^\Theta_d; \mathbb{Z})$ is an isomorphism for all $j + 1 > \psi_\Theta(d + 2)$. Therefore, by Corollary 4.16 $\epsilon_* : H_k(\mathcal{B}^\Theta_d; \epsilon^*(\mathbb{Z}^l)) \cong H_k(\mathcal{B}^\Theta_{d+2}; \mathbb{Z}^l)$ is an isomorphism for all $k = d - 2 - j$ and $j + 1 > \psi_\Theta(d + 1)$, which yields $k \leq d - 2 - \psi_\Theta(d + 2)$.

For $d > 1$, $\epsilon_* : \pi_1(\mathbb{R}^d) \to \pi_1(\mathbb{R}^{d+2})$ is an isomorphism between two copies of $\mathbb{Z}_2$. Note that the Stiefel-Whitney homomorphism $w : \pi_1(\mathcal{B}^\Theta_d) \to \mathbb{Z}_2$ factors through the isomorphism $\tilde{w} : \pi_1(\mathbb{R}^d) \to \mathbb{Z}_2$, and the Stiefel-Whitney homomorphism $w^# : \pi_1(\mathcal{B}^\Theta_{d+2}) \to \mathbb{Z}_2$ factors through the isomorphism $\tilde{w}^# : \pi_1(\mathbb{R}^{d+2}) \to \mathbb{Z}_2$. Therefore, $\epsilon^*(\mathbb{Z}^l) \cong \mathbb{Z}^l$ as $\mathbb{Z}[\pi_1(\mathcal{B}^\Theta_d)]$-modules. Hence, if $d \equiv 1 \mod 2$, then one can replace in (4.6) the local coefficient system $\mathbb{Z}^l$ with the trivial $\mathbb{Z}[\pi_1(\mathcal{B}^\Theta_{d+2})]$-module $\mathbb{Z}$.

**Corollary 4.18 (Dual long stabilization $d \Rightarrow \infty$, the projective case).** Let $\Theta \subseteq \Omega^\infty$ be a profinite (see Definition 4.1) closed equal parity poset.

Then, for each $j > 0$, there is a sufficiently big number $d$ of parity $\Theta$ such that the homomorphism $((\epsilon_{d,d'})_* : H_j(\mathcal{B}^\Theta_d; \epsilon^*(\mathbb{Z}^l)) \cong H_{j'}(\mathcal{B}^\Theta_{d'}; \mathbb{Z}^l)$ is an isomorphism for all $d' \geq d$ of parity $\Theta$.

**Proof.** The assertion follows immediately from Lemma 4.3 and Theorem 4.17

Next, we derive the analogues of these stabilizations for the spaces of polynomials.

**Theorem 4.19 (Dual short stabilization, $d \Rightarrow d + 2$, the polynomial case).** Let $\Theta$ be a closed equal parity subposet of $\Omega$. Then for $d$ of parity $\Theta$ the embedding $\epsilon : \mathcal{P}^\Theta_d \subset \mathcal{P}^\Theta_{d+2}$ induces an isomorphism

$$\epsilon_* : H_k(\mathcal{P}^\Theta_d; \mathbb{Z}) \cong H_k(\mathcal{P}^\Theta_{d+2}; \mathbb{Z})$$

for all $k \leq d + 2 - \psi_\Theta(d + 2)$.

**Proof.** The long exact cohomology sequences of the pairs $(\mathcal{P}_{d+2}, \mathcal{P}^\Theta_{d+2})$ and $(\mathcal{P}_d, \mathcal{P}^\Theta_d)$, linked by the vertical (trunc)$^*$ homomorphisms lead to a commutative diagram analogous to the one in Figure 4. Then combining Corollary 3.6 with the Five Lemma validates the first claim. Applying the diagram (4.4) from Lemma 4.15 proves the second claim.
**Corollary 4.20** (Dual long stabilization \(d \Rightarrow \infty\), the polynomial case). For any profinite closed \(\Theta\) and for each \(j > 0\), there is a sufficiently big number \(d\) of parity \(\Theta\) such that the homomorphism \((\epsilon_{d,d'})_*: H_j(\bar{P}^c_\Theta; \mathbb{Z}) \to H_j(\bar{P}^c_{d'}; \mathbb{Z})\) is an isomorphism for all \(d' \geq d\) of parity \(\Theta\).

**Proof.** The assertion follows immediately from [Lemma 4.3 and Theorem 4.19](#).

We conclude by presenting a special case where we can describe the homotopy type of \(P^c_\Theta\).

**Proposition 4.21.** For \(k > 2\) and a fixed parity let \(\Theta\) be the closed poset of compositions from \(\Omega|_{|\omega'| \geq k}\) of that parity. Then \(P^c_{d\Theta}\) has the homotopy type of a bouquet of \((k - 1)\)-spheres. The number of spheres in the bouquet equals the absolute value of the reduced Euler characteristic of \(\bar{P}^c_\Theta\).

**Proof.** Since by its definition \(\bar{P}^c_\Theta\) is the \((d - k)\)-skeleton of \(\bar{P}_d \cong S^d\), the cohomology of \(\bar{P}^c_\Theta\) is torsion-free and is concentrated in a single dimension \(d - k\). By the Alexander duality, the homology of \(P^c_{d\Theta}\) is torsion-free and is concentrated in a single dimension \(k - 1\). The space \(P^c_{d\Theta}\) is simply-connected for \(k > 2\). By [Ha], Theorem 4C.1, this implies that \(P^c_{d\Theta}\) has the homotopy type of a bouquet of \((k - 1)\)-spheres. By Alexander duality the number of spheres in the bouquet equals the absolute value of the reduced Euler characteristic of \(\bar{P}^c_\Theta\). \(\square\)

5. **Computational results**

In conclusion, let us state somewhat surprising results of one computer-assisted computation.

For a given \(\omega \in \Omega\), we denote by \(\langle \omega \rangle\) the minimal closed poset that contains \(\omega\). Below, for \(d \leq 13\), we list all compositions \(\omega\) for which the space \(P^c_{d\omega}\) is homologically nontrivial. In fact, for all \(d \leq 13\), every homologically nontrivial \(P^c_{d\omega}\) is a homology sphere! Moreover, at least for all \(\omega\)’s with \(|\omega'| > 2\), all spaces \(P^c_{d\omega}\) are homotopy spheres. Unfortunately, the reason for such phenomena is unknown to us...
| $d$ | Codimension | $\omega$ | $i$ with $\tilde{H}_i = \mathbb{Z}$ | $\omega$ | $i$ with $\tilde{H}_i = \mathbb{Z}$ |
|-----|-------------|---------|-----------------------------------|---------|-----------------------------------|
| 4   | 0           | $(1^2)$ | 3                                 | $(1^2)$ | 5                                 |
|     | 1           | (2)     | 3                                 | (2)     | 5                                 |
|     | 3           | (4)     | 1                                 | (1,3)   | 3                                 |
| 5   | 0           | $(1^3)$ | 4                                 | $(1^2)$ | 5                                 |
|     | 4           | (5)     | 1                                 | (1,3)   | 3                                 |
| 6   | 0           | $(1^4)$ | 5                                 | (1,2,1) | 3                                 |
|     | 1           | (2)     | 5                                 | (3,1)   | 3                                 |
|     | 5           | (6)     | 1                                 | (1,2,1) | 3                                 |
| 7   | 0           | $(1^5)$ | 6                                 | (1^3)   | 6                                 |
|     | 6           | (7)     | 1                                 | (1,2,1) | 3                                 |
| 8   | 0           | $(1^6)$ | 7                                 | (1^4)   | 7                                 |
|     | 1           | (1,2,1) | 3                                 | (2)     | 7                                 |
|     | 2           | (1,3)   | 3                                 | (3,1)   | 3                                 |
|     | 3           | (4)     | 3                                 | (1,2,1) | 3                                 |
|     | 7           | (8)     | 1                                 | (1,2,1) | 3                                 |
| 9   | 0           | $(1^7)$ | 8                                 | $(1^5)$ | 8                                 |
|     | 1           | $(1^2)^3$ | 8                                | (1^5)   | 8                                 |
|     | 2           | $(1^2,2,1)$ | 4                             | (1^2,1^2) | 4                                |
|     | 8           | (9)     | 1                                 | (1,3,1) | 4                                 |
| 10  | 0           | $(1^8)$ | 9                                 | $(1^6)$ | 9                                 |
|     | 1           | $(1^2,2,1)$ | 5                             | $(1^2,2,1^2)$ | 5                             |
|     | 2           | $(1^3,3)$ | 5                             | $(1^2,3,1)$ | 5                             |
|     | 9           | (10)    | 1                                 | $(3,1^3)$ | 5                             |

**Figure 6.** The list of the $\omega$'s and of the corresponding unique homological degrees $i = i(\omega)$ for which $\tilde{H}_i(P_d^{(\omega)}; \mathbb{Z}) = \mathbb{Z}$ in case $d \leq 10$. For $\omega$'s and $i$'s not listed above the homology vanishes.
| $d$ | Codimension | $\omega$ | $i$ with $\widetilde{H}_i = \mathbb{Z}$ | $\omega$ | $i$ with $\widetilde{H}_i = \mathbb{Z}$ |
|---|---|---|---|---|---|
| 11 | 0 | $(1^9)$ | 10 | $(1^7)$ | 10 |
|   |   | $(1^{15})$ | 10 | $(1^3)$ | 10 |
|   | 1 | $(1^{14}, 2, 1)$ | 6 | $(1^3, 2, 1^2)$ | 6 |
|   |   | $(1^2, 2, 1^3)$ | 6 | $(1, 2, 1^4)$ | 6 |
|   | 2 | $(1^4, 3)$ | 6 | $(1^3, 3, 1)$ | 6 |
|   |   | $(1^2, 3, 1^2)$ | 6 | $(1, 3, 1^3)$ | 6 |
|   |   | $(3, 1^4)$ | 6 | $(1, 2, 1, 2, 1)$ | 4 |
|   | 3 | $(1, 2, 1, 3)$ | 4 | $(3, 1, 2, 1)$ | 4 |
|   | 4 | $(3, 1, 3)$ | 4 |   |   |
|   | 10 | $(11)$ | 1 |   |   |
| 12 | 0 | $(1^{10})$ | 11 | $(1^8)$ | 11 |
|   |   | $(1^6)$ | 11 | $(1^4)$ | 11 |
|   |   | $(1^2)$ | 11 |   |   |
|   | 1 | $(1^5, 2, 1)$ | 7 | $(1^4, 2, 1^2)$ | 7 |
|   |   | $(1^3, 2, 1^3)$ | 7 | $(1^2, 2, 1^4)$ | 7 |
|   |   | $(1^2, 1, 1^5)$ | 7 | $(1, 2, 1)$ | 5 |
|   |   | $(2)$ | 11 |   |   |
|   | 2 | $(1^5, 3)$ | 7 | $(1^3, 3, 1)$ | 7 |
|   |   | $(1^3, 3, 1^2)$ | 7 | $(1^2, 3, 1^3)$ | 7 |
|   |   | $(1, 3, 1^4)$ | 7 | $(3, 1^5)$ | 7 |
|   |   | $(1, 2^2, 2, 1)$ | 5 | $(1, 2, 1, 2, 1^2)$ | 5 |
|   |   | $(1, 2^2, 1)$ | 3 | $(1, 3)$ | 5 |
|   |   | $(3, 1)$ | 5 |   |   |
|   | 3 | $(1^2, 2, 1, 3)$ | 5 | $(1, 2, 1^2, 3)$ | 5 |
|   |   | $(1, 2, 1, 3, 1)$ | 5 | $(1, 3, 1, 2, 1)$ | 5 |
|   |   | $(3, 1^2, 2, 1)$ | 5 | $(3, 1, 2, 1^2)$ | 5 |
|   |   | $(1, 2, 3)$ | 3 | $(3, 2, 1)$ | 3 |
|   |   | $(1, 4, 1)$ | 3 | $(4)$ | 5 |
|   | 4 | $(1, 3, 1, 3)$ | 5 | $(3, 1, 3, 1)$ | 5 |
|   |   | $(1, 5)$ | 3 | $(5, 1)$ | 3 |
|   |   | $(3^2)$ | 3 |   |   |
|   | 5 | $(6)$ | 3 |   |   |
|   | 11 | $(12)$ | 1 |   |   |

**Figure 7.** The list of the $\omega$’s and of the corresponding unique homological degrees $i = i(\omega)$ for which $\widetilde{H}_i(\overline{P}_d^{(\omega)}; \mathbb{Z}) = \mathbb{Z}$ for $d = 11, 12$. For $\omega$’s and $i$’s not listed above, the homology vanishes.
| $d$ | Codimension | $\omega$ | $i$ with $\tilde{H}_i = \mathbb{Z}$ | $\omega$ | $i$ with $\tilde{H}_i = \mathbb{Z}$ |
|-----|-------------|---------|-----------------|---------|-----------------|
| 13  | 0           | $(11)$  | 12              | $(19)$  | 12              |
|     |             | $(17)$  | 12              | $(15)$  | 12              |
|     |             | $(13)$  | 12              |         |                 |
| 1   |             | $(16, 2, 1)$ | 8         | $(15, 2, 1^2)$ | 8       |
|     |             | $(11, 2, 1^3)$ | 8         | $(13, 2, 1^4)$ | 8       |
|     |             | $(12, 2, 1^5)$ | 8         | $(1, 2, 1^6)$  | 8       |
|     |             | $(12, 2, 1)$  | 6          | $(1, 2, 1^2)$  | 6       |
| 2   |             | $(16, 3)$  | 8          | $(15, 3, 1)$   | 8       |
|     |             | $(14, 3, 1^2)$ | 8         | $(13, 3, 1^3)$ | 8       |
|     |             | $(12, 3, 1^4)$ | 8         | $(1, 3, 1^5)$  | 8       |
|     |             | $(3, 1^6)$  | 8          | $(1^2, 3)$     | 6       |
|     |             | $(1, 3, 1)$  | 6          | $(3, 1^2)$     | 6       |
|     |             | $(13, 2, 1, 2, 1)$ | 6         | $(1^2, 2, 1^2, 2, 1)$ | 6 |
|     |             | $(12, 2, 1, 2, 1^2)$ | 6 | $(1^2, 2, 1^3, 2, 1)$ | 6 |
|     |             | $(1, 2, 1^2, 2, 1^2)$ | 6 | $(1, 2, 1, 2, 1^3)$ | 6 |
|     |             | $(1^2, 2^2, 1)$  | 4          | $(1, 2^2, 1^2)$ | 4       |
| 3   |             | $(1^3, 2, 1, 3)$ | 6         | $(1^2, 2, 1^2, 3)$ | 6       |
|     |             | $(12, 2, 1, 3, 1)$ | 6         | $(1^2, 3, 1, 2, 1)$ | 6       |
|     |             | $(1, 2, 1^2, 3, 1)$ | 6         | $(1, 2, 1, 3, 1^2)$ | 6       |
|     |             | $(1, 3, 1^2, 2, 1)$ | 6         | $(1, 3, 1, 2, 1^2)$ | 6       |
|     |             | $(3, 1^2, 2, 1^2)$  | 6          | $(3, 1, 2, 1^3)$ | 6       |
|     |             | $(1^2, 2, 3)$  | 4          | $(1, 2, 3, 1)$  | 4       |
|     |             | $(1, 3, 2, 1)$  | 4          | $(3, 2, 1^2)$  | 4       |
|     |             | $(1^2, 4, 1)$  | 4          | $(1, 4, 1^2)$  | 4       |
| 4   |             | $(1^2, 3, 1, 3)$ | 6         | $(1, 3, 1, 3, 1)$ | 6       |
|     |             | $(3, 1^3, 1^2)$  | 6          | $(1, 3^2)$     | 4       |
|     |             | $(3^2, 1)$  | 4          | $(1^2, 5)$     | 4       |
|     |             | $(1, 5, 1)$  | 4          | $(5, 1^2)$     | 4       |
| 12  |             | $(13)$  | 1          |         |                 |

Figure 8. The list of the $\omega$'s and of the corresponding unique homological degrees $i = i(\omega)$ for which $H_i(\mathcal{P}_d^{(\omega)}; \mathbb{Z}) = \mathbb{Z}$ for $d = 13$. For the rest of $\omega$'s and $i$'s, the homology vanishes.

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