Exponent of a finite group of odd order with an involutory automorphism

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Abstract. Let $G$ be a finite group of odd order admitting an involutory automorphism $\phi$. We obtain two results bounding the exponent of $[G, \phi]$. Denote by $G_{-\phi}$ the set $\{g, \phi | g \in G\}$ and by $G_\phi$ the centralizer of $\phi$, that is, the subgroup of fixed points of $\phi$. The obtained results are as follows.

1. Assume that the subgroup $\langle x, y \rangle$ has derived length at most $d$ and $x^e = 1$ for every $x, y \in G_{-\phi}$. Suppose that $G_\phi$ is nilpotent of class $c$. Then the exponent of $[G, \phi]$ is $(c, d, e)$-bounded.

2. Assume that $G_\phi$ has rank $r$ and $x^e = 1$ for each $x \in G_{-\phi}$. Then the exponent of $[G, \phi]$ is $(e, r)$-bounded.

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1. Introduction

Let $G$ be a finite group of odd order admitting an involutory automorphism $\phi$. Here the term “involutory automorphism” means an automorphism $\phi$ such that $\phi^2 = 1$. We denote by $G_{-\phi}$ the set $\{g, \phi | g \in G\}$ and by $G_\phi$ the centralizer of $\phi$, that is, the subgroup of fixed points of $\phi$. The subgroup generated by $G_{-\phi}$ will be denoted by $[G, \phi]$. It is well-known that $[G, \phi]$ is normal in $G$ and $\phi$ induces the trivial automorphism of $G/[G, \phi]$. The following theorem was proved in [1]:

Let $p$ be an odd prime and $G$ a finite $p$-group admitting an involutory automorphism $\phi$ such that $G_\phi$ is nilpotent of class $c$ and $x^p = 1$ for each $x \in G_{-\phi}$. Suppose that the derived length of $G$ is at most $d$. Then the nilpotency class of $[G, \phi]$ is $(c, d, p)$-bounded.

Throughout this note we use the term “$(a, b, c \ldots)$-bounded” to mean “bounded from above by some function depending only on the parameters $a, b, c \ldots$”. An immediate corollary of the above theorem is that under its hypotheses the exponent of $[G, \phi]$ is $(c, d, p)$-bounded as well. Recall that the exponent of a group $K$ is the minimal number $e$ such that $x^e = 1$ for each $x \in K$. In this note we address the question whether one can relax the hypotheses of the theorem while still being able to conclude that the exponent of $[G, \phi]$ is bounded in terms of relevant parameters. Let $\langle X \rangle$ denote the subgroup generated by the set $X$. We obtain the following result.

**Theorem 1.1.** Let $c, d, e$ be nonnegative integers and $G$ a finite group of odd order admitting an involutory automorphism $\phi$ such that $G_\phi$ is nilpotent of class $c$ and $x^e = 1$ for each $x \in G_{-\phi}$. Suppose that the subgroup $\langle x, y \rangle$ has derived length at most $d$ for every $x, y \in G_{-\phi}$. Then the exponent of $[G, \phi]$ is $(c, d, e)$-bounded.

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We do not know whether or not all hypotheses in this theorem are necessary. It is conceivable that the exponent of \([G, \phi]\) in Theorem 1.1 can be bounded only in terms of \(c\) and \(e\), or only in terms of \(d\) and \(e\). This seems to be a difficult problem.

Recall that a finite group \(G\) is said to be of rank \(r\), if \(r\) is the least number such that any subgroup of \(G\) is \(r\)-generator. Our next theorem shows that the exponent of \([G, \phi]\) can be bounded in terms of \(e\) and the rank of \(G_\phi\). In this case we do not even need to assume that \(G_\phi\) is nilpotent.

**Theorem 1.2.** Let \(e\) and \(r\) be nonnegative integers and \(G\) a finite group of odd order admitting an involutory automorphism \(\phi\) such that \(G_\phi\) has rank \(r\) and \(x^e = 1\) for each \(x \in G_\phi\). Then the exponent of \([G, \phi]\) is \((e, r)\)-bounded.

We do not know if our results can somehow be generalized to the cases where \(\phi\) is not necessarily of order two, or the assumption that \((|G|, |\phi|) = 1\) is dropped from the hypotheses.

Throughout the paper the Feit–Thompson Theorem [2] is used without explicit references.

## 2. Proof of Theorem 1.1

The next lemma is a collection of well-known facts about involutory automorphisms. In the sequel we will frequently use it without any reference.

**Lemma 2.1.** If \(G\) is a finite group of odd order admitting an involutory automorphism \(\phi\), then

1. \(G = G_\phi G_{-\phi} = G_{-\phi} G_\phi\), and each element \(x \in G\) can be written uniquely in the form \(x = gh\), where \(g \in G_{-\phi}\) and \(h \in G_\phi\);
2. If \(N\) is any \(\phi\)-invariant normal subgroup of \(G\) we have \((G/N)_\phi = G_\phi N/N\), and \((G/N)_{-\phi} = \{gN; g \in G_{-\phi}\}\);
3. If \(N\) is a \(\phi\)-invariant normal subgroup of \(G\) such that either \(N = N_{-\phi}\) or \(N = N_\phi\), then \([G, \phi]\) centralizes \(N\);
4. The normal closure of \(G_\phi\) contains \(G'\);  
5. \(G_\phi\) normalizes the set \(G_{-\phi}\).

**Proposition 2.2.** Let \(G\) be a finite group of odd order admitting an involutory automorphism \(\phi\) and assume that \(G\) is soluble with derived length \(d\). Suppose that \(G = [G, \phi]\) and \(([G, \phi])^e = 1\). Then \(G\) has \((d, e)\)-bounded exponent.

**Proof.** We use induction on \(d\). If \(d = 1\), the result is obvious, so we assume that \(d \geq 2\). Let \(M = G^{(d-1)}\) be the last nontrivial term of the derived series of \(G\). By induction, the exponent of \(G/M\) is \((d, e)\)-bounded. Since \(M = M_{-\phi} M_\phi\) and \((M_{-\phi})^e = 1\), taking into account that \(M\) is abelian, we deduce that \(M^e \leq M_\phi\). In view of Lemma 2.13, we conclude that \(M^e \leq Z(G)\). Hence, the exponent of \(G/Z(G)\) is \((d, e)\)-bounded. A theorem of Mann [8] now guarantees that \(G'\) has \((d, e)\)-bounded exponent. Since \(G\) is generated by elements of order dividing \(e\), the result follows.

**Lemma 2.3.** Let \(G\) be a finite group of odd order with an involutory automorphism \(\phi\) such that \(G = [G, \phi]\). Let \(S\) be the set of those elements \(h \in G_\phi\) for which there exist \(x, y \in G_{-\phi}\) such that \(h \in \langle x, y \rangle\). Then \(G_\phi = \langle S \rangle\).

**Proof.** Set \(H = \langle S \rangle\). Obviously, \(H \leq G_\phi\). So we need to prove that \(G_\phi \leq H\). Choose \(h \in G_\phi\). Since \(G = [G, \phi]\), we can write \(h = g_1 \cdots g_n\) with \(g_i \in G_{-\phi}\). We wish to prove that \(h \in H\). This will be shown by induction on \(n\). If \(n \leq 2\), then obviously \(h \in H\), so assume that \(n \geq 3\). Let \(K = \langle g_{n-1}, g_n \rangle\) and note that \(K\) is \(\phi\)-invariant. Write \(g_{n-1}g_n = g_0 h_0\) where \(g_0 \in K_{-\phi}\) and \(h_0 \in K_\phi\). Observe that \(K_\phi \leq H\) and therefore \(h_0 \in H\). We have \(h = g_1 \cdots g_{n-2}g_0 h_0\) and \(hh_0^{-1} = g_1 \cdots g_{n-2}g_0\). By induction we get \(hh_0^{-1} \in H\), whence \(h \in H\). Therefore, we conclude that \(H = G_\phi\).

The following theorem was proved in [11] (see also [12]). Its proof is based on a Lie-theoretical result of Zelmanov [13].
Theorem 2.4. Let \( e \) be a positive integer and \( G \) a finite group of odd order admitting an involutory automorphism \( \phi \) such that all elements in \( G_{\phi} \cup G_{-\phi} \) have order dividing \( e \). Then the exponent of \( G \) is \( e \)-bounded.

Recall that if \( T \) is a nilpotent group of class \( c \) generated by elements of order dividing \( e \), then the exponent of \( T \) divides \( e^c \) (see for example [4, Corollary 2.5.4]). We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. In view of Theorem 2.4 it is sufficient to show that \( G_\phi \) has \((c,d,e)\)-bounded exponent. Without loss of generality we can assume that \( e = \) exponent of \( \phi \). Let \( S \) be the set of those elements \( h \in G_{\phi} \) for which there exist \( x,y \in G_{-\phi} \) such that \( h \in (x,y) \). Lemma 2.3 says that \( G_{\phi} = \langle S \rangle \). Combining Proposition 2.2 with our hypotheses we conclude that the elements in \( S \) have \((c,d,e)\)-bounded order. Thus, \( G_{\phi} \) is a nilpotent group of class \( c \) generated by elements of bounded order. Hence, the exponent of \( G_{\phi} \) is \((c,d,e)\)-bounded, as required.

3. Proof of Theorem 1.2

In what follows we write \( r(K) \) for the rank of a group \( K \). We start this section with a theorem established in [10]. Its proof uses the theory of powerful \( p \)-groups, due to Lubotzky and Mann [7].

Theorem 3.1. Let \( G \) be a finite group of odd order admitting an involutory automorphism \( \phi \) such that \( r(G_{\phi}) = r \). Then \( r([G, \phi]) \) is \((e,r)\)-bounded.

Let \( p \) be a prime and \( K \) a \( p \)-group. Suppose that a subgroup of \( K \) generated by a subset \( X \subseteq K \) can be generated by \( m \) elements. In the sequel we will use without explicit references the well-known observation that \( (X) \) can be generated by \( m \) elements from the set \( X \). This fact can be easily deduced from the Burnside Basis Theorem (see [9, Theorem 5.3.2]). Another well-known fact that we will require is the following lemma.

Lemma 3.2. The order of a finite group \( G \) of exponent \( e \) and rank \( r \) is \((e,r)\)-bounded.

Proof. Indeed, if \( G \) is a \( p \)-group the result is immediate from [3, Corollary 11.21]. Since the order of \( G \) is the product of the orders of its Sylow subgroups, the general case follows from the observation that the prime divisors of the order of \( G \) are divisors of \( e \).

Proof of Theorem 1.2. Without loss of generality we may assume that \( G = [G, \phi] \). Let \( r' = r(G') \). Theorem 3.1 tells us that \( r' \) is \((e,r)\)-bounded. It follows that the Fitting height of \( G \) is \((e,r)\)-bounded as well (see for example [6, Lemma 2.4]). Arguing by induction on the Fitting height of \( G' \) we can assume that the exponent of \( G/F(G') \) is \((e,r)\)-bounded. Thus, there is an \((e,r)\)-bounded positive integer \( f \) such that \( G^f \leq F(G') \). Set \( M = G^f \). Note that both the rank and the exponent of \( (G/M)_\phi \) are \((e,r)\)-bounded. We conclude that the order of \( (G/M)_\phi \) is \((e,r)\)-bounded, too (Lemma 3.2). Now a theorem of Hartley and Meixner [3] says that \( G/M \) has a subgroup of bounded index which is nilpotent of class at most two. We conclude that the derived length of \( G/M \) is \((e,r)\)-bounded.

Assume that \( M \) is a \( p \)-group for some prime \( p \). Let \( A \) be a maximal normal abelian subgroup of \( M \). Note that \( A = C_M(A) \) (see [9, Theorem 5.2.3]). Denote by \( B \) the minimal characteristic subgroup of \( M \) containing \( A \). Since \( B \) is a product of at most \( r' \) subgroups of the form \( A^\alpha \), where \( \alpha \in \text{Aut } G \), it follows that the nilpotency class of \( B \) is at most \( r' \). Of course, \( C_M(B) \leq B \). Let \( D \) be the minimal normal subgroup of \( G \) containing \( B_{-\phi} \). This is a subgroup of class at most \( r' \) generated by elements of order dividing \( e \). It follows that the exponent of \( D \) is \((e,r)\)-bounded. Taking into account that the rank of \( D \) is at most \( r' \) we conclude that the order of \( D \) is \((e,r)\)-bounded, too. It follows that there is an \((e,r)\)-bounded number \( j \) such that \( D \) is contained in \( Z_j(M) \), the \( j \)th term of the upper central series of \( M \). Observe that \( \phi \) acts trivially on \( B/D \). In view of Lemma 2.1.3 we conclude that \( B/D \leq Z(G/D) \). Therefore \( B \) is contained in \( Z_{j+1}(M) \). It follows that the \((j+1)\)th term of the lower central series of \( M \) centralizes \( B \). Recall that \( C_M(B) \leq B \). Hence the \((j+1)\)th term of the lower central series of \( M \) is contained in \( B \). Taking into account that the nilpotency class of \( B \) is at most...
we deduce that the derived length of $M$ is $(e, r)$-bounded. Since also the derived length of $G/M$ is $(e, r)$-bounded, it follows that the derived length of $G$ is $(e, r)$-bounded. By Proposition 2.2 the exponent of $G$ is $(e, r)$-bounded.

Thus, the theorem is proved in the particular case where $M$ is a $p$-group. Note that in that case the bound $e_0$ on the exponent of $G$ does not depend on the prime $p$. In general $M$ is a direct product of its Sylow $p$-subgroups. The above paragraph shows that the exponent of $G/O_p'(M)$ divides $e_0$ for each prime divisor $p$ of the order of $M$. Since $\bigcap_{p \in \pi(M)} O_p'(M) = 1$, we conclude that the exponent of $G$ divides $e_0$. The proof is now complete. 

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