RESTRICTING SCHUBERT CLASSES

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Introduction

The goal of the present note is twofold. Firstly, we correct some points in the paper (quoted in the following as [P]):

P. Pragacz: “A generalization of the Macdonald-You formula”, Journal of Algebra 204, 573–587 (1998).

Secondly, by comparing the formula appearing in the title of [P] with results of Stembridge [St] and some other combinatorial results, we deduce some new identities in Propositions 2–6. They concern: restrictions of Schubert classes to the cohomology of Lagrangian Grassmannians as well as relations between $Q$-functions, Stembridge’s coefficients, and various “hook numbers”. Also, we provide some examples illustrating [P] and the formulas given in the present note.

In this note, any unexplained notation or quotation stems from [P]. However, in order to make the notation maximally compatible with that used in [St] (which is our principal reference here), we label strict partitions by $\lambda$, and $\mu$ usually denotes an ordinary partition, contrariwise to [P].

1. Erratum to [P]

Due to some bugs in the computer system SCHUR [Sch], [P, Example 3(b)] was miscalculated: the quadratic expression in $Q$-functions displayed there, written as a $\mathbb{Z}$-linear combination of $Q$-functions, contains no negative summands. Consequently the sentence on p.585, lines 6–7 from the bottom, is to be withdrawn from [P]. (These corrections do not affect other results of [P], in particular the main formulas.)

2. Nonnegativity of the restriction coefficients

In fact, if

\begin{equation}
    i^*(\sigma_\mu) = \sum_\lambda c_{\lambda\mu}\sigma_\lambda',
\end{equation}

with $c_{\lambda\mu} \in \mathbb{Z}$, then all the coefficients $c_{\lambda\mu}$ are nonnegative. Perhaps the easiest way to see this, is the following. Let for $a \in H^*(G;\mathbb{Z})$, $\int_G a$ stand for the degree of the

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top codimensional component of $a$, and define similarly $\int_{G'} b$ for $b \in H^*(G'; \mathbb{Z})$. Given a strict partition $\lambda \subset (n, n-1, \ldots, 1)$, we denote by $\lambda^\vee$ the strict partition whose parts complement those of $\lambda$ in $\{1, \ldots, n\}$. We record the following property [P2]:

**Lemma 1 (Duality).** The basis $\{\sigma'_\lambda\}$ of the group $H^{2p}(G'; \mathbb{Z})$ and the basis $\{\sigma'_{\lambda^\vee}\}$ of the group $H^{n(n+1)-2p}(G'; \mathbb{Z})$ are dual under the pairing $(a, b) \mapsto \int_{G'} a \cdot b$ of Poincaré duality.

Now, if $i^*(\sigma_\mu) = \sum_\lambda c_{\lambda \mu} \sigma'_\lambda$, with $c_{\lambda \mu} \in \mathbb{Z}$, then it follows from the duality property that

$$c_{\lambda \mu} = \int_{G'} i^*(\sigma_\mu) \cdot \sigma'_{\lambda^\vee}.$$  

Using the projection formula for $i$, this is rewritten as

$$c_{\lambda \mu} = \int_G \sigma_\mu \cdot i_*(\sigma'_{\lambda^\vee}).$$

Regard $G$ as a homogeneous space $GL(V)/P$, where $P$ is a suitable parabolic subgroup of $GL(V)$. Let $\Omega \subset G$ be a Schubert variety representing $\sigma_\mu$ and let $\Omega' \subset G' \subset G$ be a Schubert variety representing $\sigma'_{\lambda^\vee}$. Using e.g. Kleiman’s theorem on a general translate [K], we can replace $\Omega$ by a translate by an element $g \in GL(V)$ such that $g \cdot \Omega$ and $\Omega'$ meet properly, and this intersection is represented as a nonnegative zero-cycle. This shows that $c_{\lambda \mu} \geq 0$.

A similar property holds in the following more general setting. Let now $G \supset P \supset B$ be a semisimple linear algebraic group, a parabolic subgroup, and a Borel subgroup. In a generalized flag variety $G/P$, one has Schubert varieties $BwP/P$ and their Schubert classes in $H^*(G/P; \mathbb{Z})$ indexed by a corresponding subset of the Weyl group. These Schubert classes enjoy a similar duality property. In an analogous way, using a general translate argument, one shows that the fundamental class of any subscheme of $G/P$ is a $\mathbb{Z}$-linear combination of the Schubert classes in $H^*(G/P; \mathbb{Z})$ with nonnegative coefficients. Combining this with a well-known fact about pulling back the class of a Cohen-Macaulay subscheme (see, e.g., Lemma on p.108 in [F-P]), we get the following result (also implying the nonnegativity of the above $c_{\lambda \mu}$):

**Proposition 1.** Let $f : G/P \to Y$ be morphism to a nonsingular variety $Y$. Let $Z$ be a pure-dimensional closed Cohen-Macaulay subscheme of $Y$. Then $f^*([Z])$ is a $\mathbb{Z}$-linear combination of the Schubert classes in $H^*(G/P; \mathbb{Z})$ with nonnegative coefficients.

3. Stembridge’s coefficients

We recall (see the discussion after [P, Proposition 8]) that the coefficients appearing in (1) and those appearing in:

$$\eta(s_\mu) = \sum g_{\lambda \mu} Q_\lambda$$

are related by:

$$g_{\lambda \mu} = \int_{G'} i^*(\sigma_\mu) \cdot \sigma'_{\lambda^\vee}.$$
satisfy \( c_{\lambda \mu} = g_{\lambda \mu} \). Here, we take sufficiently large Grassmannians \( i : G' \hookrightarrow G \).

To be more precise, this means that given \( \mu \), we take \( n \geq |\mu| \) so that any strict partition \( \lambda \) with \( |\lambda| = |\mu| \) is contained in \( (n, n-1, \ldots, 1) \). Consequently, all the coefficients \( g_{\lambda \mu} \) are nonnegative. But this result, together with a combinatorial interpretation of the \( g_{\lambda \mu} \)’s, was already established by Stembridge in [St].\(^1\) Indeed, the last displayed (unnumbered) equality before [St, Theorem 9.3]:

\[
S_\mu = \sum_{\lambda \in DP_n} g_{\lambda \mu} Q_\lambda
\]

is identical with (4) because \( S_\mu \) in the notation of [St] (and [M]) is equal to \( \eta(s_\mu) \) in our notation.\(^2\) In [St], (5) is a consequence of the equality

\[
P_\lambda = \sum_{|\mu| = |\lambda|} g_{\lambda \mu} s_\mu,
\]

where \( P_\lambda = 2^{-l(\lambda)} Q_\lambda \), and comparison of the canonical scalar products on the ring of all symmetric functions with that on the ring of \( Q \)-functions. To the nonnegativity of \( g_{\lambda \mu} \) is given, in loc. cit., several interpretations in representation theory, some of which go back to Morris and Stanley.

(Observe that (4) and (6) yield the following expression for \( \eta(P_\lambda) \):

\[
\eta(P_\lambda) = \sum_{|\mu| = |\lambda|} g_{\lambda \mu} \eta(s_\mu) = \sum_{|\mu| = |\lambda|} \sum_{|\nu| = |\lambda|} g_{\lambda \mu} g_{\nu \mu} Q_\nu.
\]

Stembridge [St] also established a combinatorial interpretation of the numbers \( f_{\mu \nu}^\lambda \) appearing as coefficients in the expansion:

\[
P_\mu P_\nu = \sum_\lambda f_{\mu \nu}^\lambda P_\lambda,
\]

where \( \mu, \nu, \) and \( \lambda \) denote now strict partitions. It will be convenient to set

\[
e_{\mu \nu}^\lambda := 2^{l(\mu)+l(\nu)-l(\lambda)} f_{\mu \nu}^\lambda.
\]

There exists a geometric analogue of (8): in the cohomology ring \( H^*(G'\Z) \) of a sufficiently large Lagrangian Grassmannian,

\[
\sigma'_\mu \cdot \sigma'_\nu = \sum_\lambda e_{\mu \nu}^\lambda \sigma'_\lambda.
\]

(See [P2, Sect.6].)

Stembridge’s combinatorial description of the above \( f_{\mu \nu}^\lambda \) and \( g_{\lambda \mu} \) can be summarized by the following:

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\(^1\)The fact that this result was already established by Stembridge, has been learned by the author only in June 1999.

\(^2\)Note that the map denoted in [P] and here by \( \eta \), is denoted by \( \phi \) in [M].
Theorem [St]. (i) The coefficient $f_{\lambda\mu}^\nu$ is equal to the number of marked shifted tableaux $T$ of shape $\lambda/\mu$ and weight (or content) $\nu$ such that:

(a) The word $w(T)$ associated with $T$ ([St, Sect.8] and [M, p.258]) has the lattice property in the sense of loc.cit.;
(b) for each $k \geq 1$, the rightmost occurrence of $k'$ in $w(T)$ precedes the last occurrence of $k$.

(ii) The coefficient $g_{\lambda\mu}$ is equal to the number of unshifted marked tableaux $T$ of shape $\mu$ and weight $\lambda$ satisfying (a) and (b) above.

For all unexplained here combinatorial notions, we refer the reader to [St, Sect.6 and 8], [P2, Sect.4], and to [M, III.8 pp.255–259]. We make no attempt to make a complete survey here. Some examples of the coefficients $g_{\lambda\mu}$ will be given below.

Summarizing the content of this section, we record:

Proposition 2. We have for a partition $\mu \subset (n^n)$

\[
i^*(\sigma_\mu) = \sum_\lambda g_{\lambda\mu} \sigma'_\lambda,\]

where $\lambda$ runs over strict partitions contained in $(n,n-1,\ldots,1)$, and $g_{\lambda\mu}$ is the Stembridge coefficient described in Theorem (ii).

4. Quadratic relations between $Q$-functions

We pass now to some applications of the generalized Macdonald-You formula ([L-L2], [P, Corollary 2]):

\[
2^n \eta(s_\mu) = \sum Q_{(a_1,\ldots,a_k)} \cdot Q_{A\#B \setminus (a_1,\ldots,a_k)}.\]

Recall that here, for $\mu = (\alpha_1,\ldots,\alpha_n|\beta_1,\ldots,\beta_n)$ in Frobenius notation,

\[
A = (a_1,\ldots,a_n) := (\alpha_1+1,\ldots,\alpha_n+1), \quad B := (\beta_1,\ldots,\beta_n),
\]

and the sum is over all sequences $1 \leq i_1 < \cdots < i_k \leq n$ and $k = 0,1,\ldots,n$.

Since $\eta(e_i) = \eta(h_i)$, where $h_i$ is the $i$th complete homogeneous symmetric function, we have for a partition $\mu$

\[
\eta(s_{\mu^\sim}) = \eta(s_\mu),
\]

where $\mu^\sim = (\beta_1,\ldots,\beta_n|\alpha_1,\ldots,\alpha_n)$ is the conjugate partition of $\mu$. We set in addition

\[
C = (c_1,\ldots,c_n) := (\beta_1+1,\ldots,\beta_n+1), \quad D := (\alpha_1,\ldots,\alpha_n).
\]

Then (12) and (14) imply the following:
Proposition 3. We have
\[ \sum Q(a_{i_1}, \ldots, a_{i_k}) \cdot Q_{A\#B \setminus (a_{i_1}, \ldots, a_{i_k})} = \sum Q(c_{i_1}, \ldots, c_{i_k}) \cdot Q_{C\#D \setminus (c_{i_1}, \ldots, c_{i_k})}, \]
where the sums are over all sequences \(1 \leq i_1 < \cdots < i_k \leq n\) and \(k = 0, 1, \ldots, n\).

The relations (16), regarded from the side of \(Q\)-functions, seem to be rather nontrivial. For instance, for \(\mu = (5^3 31^3) = (432|621)\), so \(A = (5, 4, 3)\), \(B = (6, 2, 1)\), \(C = (7, 3, 2)\), and \(D = (4, 3, 2)\), we get the equation
\[ Q_{654321} - Q_5 \cdot Q_{64321} + Q_4 \cdot Q_{65321} - Q_3 \cdot Q_{65421} - Q_54 \cdot Q_{6321} \]
\[ + Q_5 \cdot Q_{6421} - Q_3 \cdot Q_{5621} + Q_54 \cdot Q_{621} \]
\[ = Q_{53} \cdot Q_{7432} + Q_{732} \cdot Q_{432}. \]

Using a (hopefully) debugged version of SCHUR, (16) is expressed as the following \(\mathbb{Z}\)-linear combination of the \(Q_{\lambda}\)’s:
\[
\begin{align*}
8Q_{11 64} + 8Q_{11 631} + 8Q_{11 541} + 8Q_{11 532} + 8Q_{10 74} \\
+ 8Q_{10 731} + 8Q_{10 65} + 24Q_{10 641} + 24Q_{10 632} + 24Q_{10 542} \\
+ 8Q_{10 5321} + 8Q_{975} + 16Q_{9741} + 16Q_{9732} + 16Q_{9651} \\
+ 48Q_{9642} + 16Q_{96321} + 16Q_{9543} + 16Q_{95421} + 8Q_{8751} \\
+ 24Q_{8742} + 8Q_{87321} + 24Q_{8652} + 24Q_{8643} + 24Q_{86421} \\
+ 8Q_{85431} + 8Q_{7653} + 8Q_{76521} + 8Q_{76431}
\end{align*}
\]

Consequently, taking sufficiently large Grassmannians \(i : G' \hookrightarrow G\), we have
\[
i^*(\sigma_{5^3 31^3}) =
\begin{align*}
\sigma'_{11 64} + \sigma'_{11 631} + \sigma'_{11 541} + \sigma'_{11 532} + \sigma'_{10 74} \\
+ \sigma'_{10 731} + \sigma'_{10 65} + 3\sigma'_{10 641} + 3\sigma'_{10 632} + 3\sigma'_{10 542} \\
+ \sigma'_{10 5321} + \sigma'_{975} + 2\sigma'_{9741} + 2\sigma'_{9732} + 2\sigma'_{9651} \\
+ 6\sigma'_{9642} + 2\sigma'_{96321} + 2\sigma'_{9543} + 2\sigma'_{95421} + \sigma'_{8751} \\
+ 3\sigma'_{8742} + \sigma'_{87321} + 3\sigma'_{8652} + 3\sigma'_{8643} + 3\sigma'_{86421} \\
+ \sigma'_{85431} + \sigma'_{7653} + \sigma'_{76521} + \sigma'_{76431}
\end{align*}
\]

So e.g. we have: \(g_{(5^3 31^3)}(11 64) = 1\), \(g_{(5^3 31^3)}(10 641) = 3\), \(g_{(5^3 31^3)}(9741) = 2\), and \(g_{(5^3 31^3)}(9642) = 6\).

5. Linear relations between Stembridge’s coefficients

Combining (4), (12), and (16), we have in the above notation, associated with a fixed \(\mu\)
\[ \sum Q(a_{i_1}, \ldots, a_{i_k}) \cdot Q_{A\#B \setminus (a_{i_1}, \ldots, a_{i_k})} \]
\[ = \sum Q(c_{i_1}, \ldots, c_{i_k}) \cdot Q_{C\#D \setminus (c_{i_1}, \ldots, c_{i_k})} = 2^n \sum \lambda g_{\lambda \mu} Q_{\lambda}, \]
where the first two sums are over all sequences \(1 \leq i_1 < \cdots < i_k \leq n\) and \(k = 0, 1, \ldots, n\).

The equalities (18) imply linear relations between the \(e^\lambda_{\mu}\)’s and \(g_{\lambda \mu}\)’s. Given a sequence of different positive integers \(K = (k_1, \ldots, k_l)\), there is a permutation \(w = w_K \in S_l\) such that \(k_{w(1)} > \cdots > k_{w(l)} > 0\). Denote this last-mentioned strict partition by \(< K>\). Then given strict partitions \(\mu, \lambda\) and a sequence \(K\) as above, we set
\[ e^\lambda_{\mu} < K> := \text{sgn}(w_K) e^\lambda_{\mu < K>}. \]

From (18) and (16) we get the following result:
Proposition 4. For a fixed partition $\mu$ and strict partition $\lambda$ with $|\mu| = |\lambda|$, we have in the above notation associated with $\mu$

$$2^n g_{\lambda\mu} = \sum e_{(a_1, \ldots, a_{i_k})}^\lambda, A^\#B \setminus (a_1, \ldots, a_{i_k})$$

$$= \sum e_{(c_1, \ldots, c_{i_k})}^\lambda, C^\#D \setminus (c_1, \ldots, c_{i_k})$$

(20)

where the sums are over all sequences $1 \leq i_1 < \cdots < i_k \leq n$ for which $A^\#B \setminus (a_1, \ldots, a_{i_k})$ (resp. $C^\#D \setminus (c_1, \ldots, c_{i_k})$) is a sequence of different integers, and $k = 0, 1, \ldots, n$.

For instance, for any strict partition $\lambda$ with $|\lambda| = 21$, and for $\mu = (5^31^3) = (432|621)$, we get the equations:

$$2^3 g_{\lambda (5^31^3)} = e_{(654321)}^\lambda \cdot (\emptyset) - e_{(5)}^\lambda (64321) + e_{(4)}^\lambda (65321) - e_{(3)}^\lambda (65421) - e_{(54)}^\lambda (6321)$$

$$+ e_{(53)}^\lambda (6421) - e_{(43)}^\lambda (6521) + e_{(543)}^\lambda (621)$$

$$= e_{(32)}^\lambda (7432) + e_{(732)}^\lambda (432).$$

6. The class $i_\ast(\sigma'_\lambda)$ as a $\mathbb{Z}$-linear combination of the $\sigma_{\mu}$'s

By reasoning similarly as in §4, one shows that for any proper morphism $f : X \to G/P$ from a scheme $X$ to a generalized flag variety $G/P$, and for any irreducible subscheme $Y \subset X$, $f_\ast([Y])$ is a $\mathbb{Z}$-linear combination of Schubert classes in $H^\ast(G/P; \mathbb{Z})$ with nonnegative coefficients.

The next proposition will give $i_\ast(\sigma'_\lambda)$ as an explicit $\mathbb{Z}$-linear combination of the $\sigma_{\mu}$'s. Given a partition $\mu \subset (n^n)$, we set $\mu^\ast := (n-\mu_n, \ldots, n-\mu_1)$. The following duality property is a well-known result of Schubert calculus [F]:

Lemma 2. The basis $\{\sigma_{\mu}\}$ of the group $H^{2p}(G; \mathbb{Z})$ and the basis $\{\sigma_{\mu^\ast}\}$ of the group $H^{2(n^2-p)}(G; \mathbb{Z})$ are dual under the pairing $(a, b) \mapsto \int_G a \cdot b$ of Poincaré duality.

We now state:

Proposition 5. For a fixed strict partition $\lambda \subset (n, n-1, \ldots, 1)$, we have

$$i_\ast(\sigma'_\lambda) = \sum_{|\mu|=|\lambda|+n(n-1)/2} g_{\lambda^\ast, \mu^\ast} \sigma_{\mu},$$

(21)

where $\mu$ runs over partitions contained in $(n^n)$ and $g_{\lambda^\ast, \mu^\ast}$ is the Stembridge coefficient described in Theorem (ii).

Indeed, if $i_\ast(\sigma'_\lambda) = \sum \mu m_{\lambda\mu} \sigma_{\mu}$, with $m_{\lambda\mu} \in \mathbb{Z}$ (so that $|\mu| = |\lambda| + n(n-1)/2$ ), then it follows from Lemma 2 that

$$m_{\lambda\mu} = \int_G (i_\ast(\sigma'_\lambda)) \cdot \sigma_{\mu^\ast}.$$

(22)

Using the projection formula for $i$, this is rewritten as

$$m_{\lambda\mu} = \int \sigma'_\lambda \cdot i^\ast(\sigma_{\mu^\ast}).$$

(23)
In turn, using the description of $i^*(\sigma_{\mu^*})$ from Proposition 2, (23) is rewritten as

$$m_{\lambda\mu} = \int_{G'} \sigma'_\lambda \cdot \left( \sum_{\nu} g_{\nu\mu^*} \sigma'_\nu \right) = \int_{G'} \sum_{\tau} \sum_{\nu} e_{\lambda\nu}^\tau \ g_{\nu\mu^*} \sigma'_\tau = g_{\lambda^\nu,\mu^*}$$

because only $\tau = (n, n-1, \ldots, 1)$ and $\nu = \lambda^\nu$ give a nonzero contribution (note that for such $\tau$ and $\nu$, we have $e_{\lambda\nu}^\tau = 1$).

7. Relations between the degrees of the ordinary and projective representations of the symmetric groups

For a partition $\mu$, we set

$$f^\mu = \prod_{x \in \mu} \frac{1}{h(x)},$$

where $h(x)$ is the hook-length of $\mu$ at $x = (i, j)$ defined by $h(x) = h(i, j) = \mu_i + \mu_j - i - j + 1$. If $|\mu| = m$ then $f^\mu := m!$ is the degree of the irreducible representation of $S_m$ corresponding to $\mu$. Equivalently, $f^\mu$ is the number of standard tableaux of shape $\mu$, obtained by labeling the squares of the diagram of $\mu$ with the numbers 1, 2, $\ldots$, $m$. We refer to [F] for a detailed discussion of these facts.

For a strict partition $\lambda$, we set

$$g^\lambda = \prod_{x \in S(\lambda)} \frac{1}{h(x)},$$

where $S(\lambda)$ is the shifted diagram associated with $\lambda$ [M, p.255], and for each square $x \in S(\lambda)$ the hook-length $h(x)$ is defined to be the hook-length at $x$ in the “double diagram” $(\lambda_1, \lambda_2, \ldots | \lambda_1 - 1, \lambda_2 - 1, \ldots)$, containing $S(\lambda)$. If $|\lambda| = m$, $g^\lambda := m!$ is the number of shifted standard tableaux of shape $S(\lambda)$, obtained by labeling the squares of $S(\lambda)$ with the numbers 1, 2, $\ldots$, $m$ with strict increase along each row and down each column. The numbers $g^\lambda$ also admit an interpretation as the degree of suitable projective representations of $S_m$. We refer to [H-H] for a detailed discussion of these results.

One has the following formulas, in terms of parts, for $f^\mu$ [M, I.1 Example 1] and $g^\lambda$ [M, III.8 Example 12]:

$$f^\mu = \frac{\prod_{i<j}(\mu_i - \mu_j - i + j)}{\prod_{i \geq 1}(\mu_i + n - i)!},$$

$$g^\lambda = \frac{1}{\prod_{i \geq 1} \lambda_i} \ \prod_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$
Lemma 3. (i) Under the specialization $e_i := \frac{1}{i!}$, $s_\mu$ becomes $\mathcal{F}_\mu^\mu$.
(ii) Under the specialization $Q_i := \frac{1}{i!}$, $Q_\lambda$ becomes $\mathcal{F}_\lambda$.

(For assertion (i), see [M, I.3 Example 5]. Assertion (ii) stems from [DC-P, Proposition 6].)

Given a partition $\mu$, we want to apply formulas (12) and (16), so we adopt the notation of §5. Also, we follow the notation of §6 associated with a sequence $K$. For such a sequence, we set

$$g^K := \text{sgn}(w_K) g^{<K>}.$$  

From Lemma 3, (12), and (16), we get

Proposition 6. For a fixed partition $\mu$, we have

$$2^n g^\mu = \sum \mathcal{F}(a_{i_1}, \ldots, a_{i_k}) \mathcal{F}^{A\#B \setminus (a_{i_1}, \ldots, a_{i_k})}$$

$$= \sum \mathcal{F}(c_{i_1}, \ldots, c_{i_k}) \mathcal{F}^{C\#D \setminus (c_{i_1}, \ldots, c_{i_k})},$$

where the sums are over all sequences $1 \leq i_1 < \cdots < i_k \leq n$ for which $A\#B \setminus (a_{i_1}, \ldots, a_{i_k})$ (resp. $C\#D \setminus (c_{i_1}, \ldots, c_{i_k})$) is a sequence of different integers, and $k = 0, 1, \ldots, n$.

For instance, for $\mu = (5^3 \cdot 3^1) = (432|621)$, we get the equations:

$$2^3 g^{5^3 \cdot 3^1} = g^{(654321)} - g^{(5)} g^{(64321)} + g^{(4)} g^{(65321)} - g^{(3)} g^{(65421)} - g^{(2)} g^{(621)}$$

$$+ g^{(32)} g^{(7432)} - g^{(33)} g^{(6521)} + g^{(543)} g^{(621)}$$

$$= g^{(32)} g^{(7432)} + g^{(732)} g^{(432)}.$$  

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