On the variety of Euclidean point sets

Gerald Kuba

Abstract. We construct a continuum of non-homeomorphic compact subspaces of \( \mathbb{R} \) without singleton components. Thus from the purely topological point of view the real line \( \mathbb{R} \) contains not only more closed sets than open sets but also more closures of open sets than open sets. On the other hand, we show that this discrepancy vanishes either if the topological point of view is sharpened in the metrical or in the order-theoretical direction, or if \( \mathbb{R} \) is replaced with \( \mathbb{R}^n \) for \( n \geq 2 \). Furthermore, we track down a continuum of topological types of closed and totally disconnected subsets of \( \mathbb{R} \). In doing so we also track down a continuum of metrical types of infinite, discrete subsets of \([0,1]\). (As a consequence, any countably infinite discrete space has a continuum of non-homeomorphic metrizable compactifications.)

1. Unions of intervals

Let \( U \) be the family of all open subsets of the Euclidean real line \( \mathbb{R} \). Motivated by the fact that each open subset of \( \mathbb{R} \) is a union of mutually disjoint open intervals, we define a family \( A \) of point sets so that \( X \in A \) if and only if \( X \) is a closed subset of \( \mathbb{R} \) which is a union of mutually disjoint nondegenerate closed intervals. In other words, the members of \( A \) are precisely the closed subspaces of \( \mathbb{R} \) where no component of any space is a singleton.

It is common opinion that the topological structure of an arbitrary closed subset \( \mathbb{R} \) may be more complicated than of any open subset of \( \mathbb{R} \), although grounds for this opinion are rather informal. Our first goal is to support this view by pointing out that the structural discrepancy in question is revealed by a clear cardinal discrepancy already when only point sets in the family \( U \cup A \) are considered. In fact, there are more topological types of members of \( A \) than of members of \( U \! \! \! \!. \)

Naturally, both families \( U \) and \( A \) have the cardinality \( c \) of the continuum \( \mathbb{R} \). But the family \( U \) contains only countably many topologically distinct members. Indeed, since each \( U \in U \) can be written as a union of countably many mutually disjoint open intervals, if \( \emptyset \neq U \in U \) then there is precisely one \( n \in \mathbb{N} \cup \{\infty\} \) so that \( U \) and \( \bigcup_{k=1}^{n} [k, k+1[ \) are homeomorphic subspaces of \( \mathbb{R} \). (In order to avoid potential misinterpretations, \( 0 \notin \mathbb{N} \), i.e. \( \mathbb{N} = \{1,2,3,...\} \).) In particular, each open subspace of \( \mathbb{R} \) with infinitely many components is homeomorphic to \( \mathbb{R} \setminus \mathbb{Z} \). On the other hand, the following theorem shows that there are \( c \) topologically distinct point sets in the family \( A \).

**Theorem 1.** There are \( c \) mutually non-homeomorphic compact subspaces of \( \mathbb{R} \) without singleton components.

The situation is different when, instead of topological types of point sets in \( \mathbb{R} \), metrical types are considered, which means that continuity is sharpened to uniform continuity. (The metric is the inherited Euclidean metric of \( \mathbb{R} \).) A fortiori, topologically distinct point sets are always metrically distinct. Thus the interiors of the \( c \) topologically distinct compact point sets given by Theorem 1 must be metrically distinct because every \( A \in A \) equipped with the Euclidean metric is a completion of the interior of \( A \) equipped with the Euclidean metric. Therefore, the total number of metrical types of the open point sets in \( \mathbb{R} \) is \( c \) and hence greater than the total number of their topological types. (As a consequence, there exists a collection of \( c \) metrically distinct and topologically similar open subsets of \( \mathbb{R} \).) Certainly, metrically distinct compact subspaces of \( \mathbb{R} \) cannot be homeomorphic. But, as the following proposition shows in an illustrative way, it is possible to track down \( c \) members of \( A \) which are metrically distinct and topologically similar. And there is also an illustrative stack of \( c \) metrically distinct and topologically similar open subsets of \( \mathbb{R} \).
Proposition 1. For each real number \( u \geq 2 \) define \( X_u \in A \) and \( X_u^c \in U \) via
\[
X_u := \bigcup_{n=1}^{\infty} [2^u n, 2^u n + u^n] \quad \text{and} \quad X_u^c := \bigcup_{n=1}^{\infty} [2^u n, 2^u n + u^n].
\]
If \( 2 \leq v < w \) then there is no uniformly continuous bijection from \( X_v \) onto \( X_w \) or from \( X_v^c \) onto \( X_w^c \).

Besides the topological and the metrical view there is a third natural way to look at the point sets in the families \( U \) and \( A \). Two sets \( X, Y \subset \mathbb{R} \) are order-isomorphic if and only if there exists a strictly increasing function from \( X \) onto \( Y \). Of course, order-isomorphic sets \( X, Y \subset \mathbb{R} \) need not be homeomorphic subspaces of \( \mathbb{R} \). (Consider for example \( X = [2, 3] \) and \( Y = \{1\} \cup [2, 3] \).) However, if \( X, Y \subset \mathbb{R} \) are open or closed then the spaces \( X, Y \) must be homeomorphic if the sets \( X, Y \) are order-isomorphic. Because it is plain that the Euclidean topology restricted to a closed or open set \( S \subset \mathbb{R} \) coincides with the order topology on \( S \) induced by the natural ordering of the real numbers in \( S \). And, naturally, any order isomorphism between two linearly ordered spaces is a homeomorphism with respect to their order topologies. In particular, topologically distinct sets in the family \( U \cup A \) are never order-isomorphic. On the other hand, the \( c \) metrically distinct open resp. closed sets in Proposition 1 are obviously order-isomorphic. It is also possible to establish a completely converse situation.

Proposition 2. There are \( c \) metrically (and hence topologically) similar sets in the family \( U \) and in the family \( A \), respectively, which are mutually not order-isomorphic.

Thus, other than concerning topological types and similarly as concerning metrical types, there is no discrepancy between the total numbers of the order types of open sets and of the order types of closed sets in the real line.

2. Unions of cubes

The cardinal discrepancy between all topological types of open and all topological types of closed sets in the realm of linear point sets already vanishes in the realm of planar point sets. Indeed, the following theorem shows that for arbitrary dimensions \( n \geq 2 \) the Euclidean space \( \mathbb{R}^n \) contains \( c \) topologically distinct open point sets whose closures have no singleton components and are topologically distinct as well. (As usual, \( \overline{X} \) denotes the closure of \( X \).)

Theorem 2. For each \( n \geq 2 \) there is a family \( \mathcal{F}_n \) of open subsets of the Euclidean space \( \mathbb{R}^n \) such that \( \mathcal{F}_n \) has cardinality \( c \) and neither \( U, V \) nor \( \overline{U}, \overline{V} \) are homeomorphic subspaces of \( \mathbb{R}^n \) whenever \( U, V \in \mathcal{F}_n \) and \( U \neq V \). Moreover, the family \( \mathcal{F}_n \) can be chosen so that for every \( X \in \mathcal{F}_n \) the set \( \overline{X} \) is a compact union of closed cubes of the form \([a_1, a_1 + h] \times \cdots \times [a_n, a_n + h] \) with \( h > 0 \) where the interiors of distinct cubes are always disjoint. Alternatively, the family \( \mathcal{F}_n \) can be chosen so that \( \overline{X} \) is a union of unit cubes \([k_1, k_1 + 1] \times \cdots \times [k_n, k_n + 1] \) with \( k_1, \ldots, k_n \in \mathbb{Z} \) for every \( X \in \mathcal{F}_n \).

It is impossible that every set \( U \) in the uncountable family \( \mathcal{F}_n \) is a union of mutually disjoint open cubes (or that \( \overline{U} \) is a union of mutually disjoint compact cubes for every \( U \in \mathcal{F}_n \).) Because two open subspaces of \( \mathbb{R}^n \) where each component is an open cube are homeomorphic if and only if the total numbers of components coincide. But if one sets a value on disjoint cubes, it is possible to achieve the following results.
Proposition 3. For each of the $c$ sets $S \subset \mathbb{N}$ define an open set $Y_S \subset [-1,1]^n$ via

$$Y_S := [-1,0[^n \cup \bigcup_{m=1}^{\infty} \left[ 2^{-2m}, 2^{-2m+1} \right] \cup \left( \bigcup_{s \in S} \left\{ 2^{-2s} + \frac{k}{s+1}2^{-2s} \mid k = 1, 2, \ldots, s \right\} \right) \big]$$

Obviously, all $Y_S$ are unions of infinitely many mutually disjoint open cubes and hence homeomorphic spaces, and $\overline{Y_S} = [-1,0[^n \cup \bigcup_{m=1}^{\infty} \left[ 2^{-2m}, 2^{-2m+1} \right] \big]^n$ for every $S \subset \mathbb{N}$.

But whenever $S \not= S'$, there is no bijection $f$ from $Y_S$ onto $Y_{S'}$ such that both $f$ and $f^{-1}$ are uniformly continuous.

Theorem 3. For each dimension $n$ there exists a family $\mathcal{V}_n$ of open subsets of $\mathbb{R}^n$ such that (i) $\mathcal{V}_n$ has the cardinality $c$; (ii) each $V \in \mathcal{V}_n$ is a union of mutually disjoint open cubes; (iii) $\overline{V}$ is a union of mutually disjoint nondegenerate compact cubes if $V \in \mathcal{V}_n$; (iv) all $V \in \mathcal{V}_n$ are metrically distinct but topologically similar; (v) all $\mathcal{V}(V \in \mathcal{V}_n)$ are mutually non-homeomorphic compact subspaces of $\mathbb{R}^n$.

3. Proof of Theorem 1

In the following we need Cantor derivatives but in order to keep the story simple we use only finite derivatives. If $P$ is a point set in a Hausdorff space then the first derivative $P'$ of $P$ is the set of all limit points of $P$. The first derivative of any set is closed. And $P$ is closed if and only if $P' \subset P$. Further, with $P^{(0)} = P$, for every $k = 1, 2, 3, \ldots$ the $k$-th derivative $P^{(k)}$ of $P$ is given by $P^{(k)} = (P^{(k-1)})'$. Consequently, all derivatives of a closed set $A$ are closed and $A = A^{(0)} \supset A^{(1)} \supset A^{(2)} \supset A^{(3)} \supset \cdots$. (And possibly but not necessarily, $A^{(k)} = A^{(m)}$ whenever $k \geq m$ for some $m \in \mathbb{N}$.) For abbreviation let $h(x) := \frac{2}{\pi} \arctan x$. (Then $h$ is a strictly increasing function from $[0, \infty[$ onto $[0,1[\,].$

In order to prove Theorem 1 we construct a compact subspace $X_S$ of $\mathbb{R}$ without point components for each infinite $S \subset \mathbb{N}$ so that $X_S, X_{S'}$ are never homeomorphic for distinct sets $S,S'$. Define for each $n \in \mathbb{N}$ a countable subset $K_n$ of the interval $[5n, 5n+1]$ in the following way. For arbitrary $X \subset [0,1]$ define $F(X) \subset [0,1]$ by

$$F(X) := h(\{ n - 1 + x \mid n \in \mathbb{N} \wedge x \in X \} \cup \{1\}$$

and starting with $A_1 = \{1 - \frac{1}{m} \mid m \in \mathbb{N}\} \cup \{1\}$ put $A_{n+1} = F(A_n)$ for $n = 1, 2, 3, \ldots$ and define $K_n := \{x + 5n \mid x \in A_n\}$ for each $n \in \mathbb{N}$. Obviously, $K_n$ is a closed subset of $[5n, 5n+1]$ with $\max K_n = 5n + 1$ for each $n \in \mathbb{N}$. Hence $K_n$ is always compact. Furthermore it is evident that $K_n$ is well-ordered by the natural ordering $\leq$. (Besides, one may realize that the order type of $(K_n, \leq)$ is $\omega^n + 1$.)

By construction, for each $n \in \mathbb{N}$ the $k$-th Cantor derivative $K_n^{(k)}$ is infinite whenever $k < n$ and empty whenever $k > n$ and $K_n^{(n)} = \{5n + 1\}$. For each $n \in \mathbb{N}$ let $g_n$ be the reflection in the point $5n + 2$, whence $g_n(x) = 10n + 4 - x$ for $x \in \mathbb{R}$ and $g_n([5n, 5n+1]) = [5n+3, 5n+4]$. For every $a \in K_n$ choose $0 < \epsilon(a) \leq 1$ such that $[a, a + \epsilon(a)] \cap K_n = \{a\}$ whenever $a \in K_n$ and put $\epsilon(5n+1) = 1$. (For example put $\epsilon(a) = (a'-a)/2$ where $a' = \min \{x \in K_n \mid x > a\}$ whenever $a \in K_n \setminus \{5n + 1\}$.) Finally, for each infinite set $S \subset \mathbb{N}$ define

$$X_S := h(\bigcup \left\{ \bigcup \{ [a, a + \epsilon(a)] \cup g_n([a, a + \epsilon(a)]) \mid a \in K_n \} \mid n \in S \right\} \cup [1,2].$$

It is plain that $X_S$ is always a closed and hence compact subset of $[0,2]$. Obviously, all components of the space $X_S$ are compact (and nondegenerate) intervals and $C_n := h([5n + 1, 5n+3])$ is a component of $X_S$ for every $n \in S$. Hence we can write $X_S = \bigcup \{[a_j, b_j] \mid j \in \mathbb{N}\}$ where always $a_j < b_j$ and the intervals $[a_j, b_j]$ are mutually disjoint.
Consider the point set $B_S := \{ a_j \mid j \in \mathbb{N} \} \cup \{ b_j \mid j \in \mathbb{N} \}$, which clearly is the boundary of the point set $X_S$ in the Euclidean space $\mathbb{R}$. The point set $B_S$ is also topologically determined within the subspace $X_S$ of $\mathbb{R}$ because $B_S$ equals the set of all points $x \in X_S$ so that for every component $C$ of the space $X_S$ the point set $C \setminus \{x\}$ remains connected in the space $X_S$. Let $L_S$ be the family of all components $C$ of $X_S$ such that $C$ contains precisely two limit points of $B_S$. (Thus the family $L_S$ is also topologically determined with respect to the space $X_S$.) By construction we have $L_S = \{ C_n \mid n \in \mathbb{N} \}$ for each infinite $S \subset \mathbb{N}$. (Any component $C$ of $X_S$ with $C \neq C_n$ for every $n \in S$ contains at most one limit point of $B_S$.) Moreover, if $S$ is any infinite subset of $\mathbb{N}$ and if $k \in \mathbb{N}$ and $n \in S$ then $B_S^{(k)} \cap C_n = \emptyset$ when $k > n$ and $B_S^{(k)} \cap C_n \neq \emptyset$ when $k \leq n$. Consequently,

$$S = \left\{ \min \{ m \in \mathbb{N} \mid B_S^{(m+1)} \cap C = \emptyset \} \mid C \in L_S \right\}$$

and hence the set $S$ is completely determined by the topology of the space $X_S$. Thus for distinct infinite sets $S, S' \subset \mathbb{N}$ the spaces $X_S, X_{S'}$ cannot be homeomorphic.

**Remark.** The clue in the previous proof is to approximate certain intervals from both the left and the right. The proof would not work with approximations, say, from the left. Because if we consider the compact spaces

$$\hat{X}_S := h\left( \bigcup \left\{ \bigcup \left\{ [a, a + \varepsilon(a)] \mid a \in K_n \right\} \mid n \in S \right\} \right) \cup [1, 2]$$

for arbitrary infinite $S \subset \mathbb{N}$ then all spaces are order-isomorphic and hence homeomorphic! (In fact, with the notation as in the proof of Proposition 2 below, for any infinite $S \subset \mathbb{N}$ the linearly ordered set $(C(\hat{X}_S), \prec)$ is well-ordered and order-isomorphic to the set of all ordinal numbers $\alpha \leq \omega^\omega$.)

### 4. Proof of Theorem 3

The proof of Theorem 1 can be adapted in order to verify Theorem 3. We replace each point set $X_S = \bigcup \{ [a_j, b_j] \mid j \in \mathbb{N} \}$ with $\hat{X}_S = \bigcup \{ [a_j, b_j]^n \mid j \in \mathbb{N} \}$ and claim that these compact subspaces of $\mathbb{R}^n$ are mutually non-homeomorphic also for arbitrary dimensions $n$. As a consequence, Theorem 3 is settled by defining $\mathcal{V}_n$ as the family of all open sets $\bigcup_{j=1}^\infty [a_j, b_j]^n$ corresponding to $X_S$ represented as above with $S$ running through the infinite subsets of $\mathbb{N}$. Indeed, (i), (ii), (iii) are obviously satisfied and (iv) follows from (v) since $\overline{V}$ is the completion of each metric space $V \in \mathcal{V}_n$.

Since an elimination of one point of a cube never destroys its connectedness, we cannot adopt the argumentation using the set $B_S$ in higher dimensions. But fortunately we can stay very close to the proof of dimension 1 by transforming the concept of Cantor derivatives from point sets of a topological space to families of components of the space in the following way.

Let $G$ be the family of all components of a Hausdorff space. For every $F \subset G$ define $F' = F^{(1)} := \{ G \in G \mid G \cap \bigcup(F \setminus \{G\}) \neq \emptyset \}$ and $F^{(k+1)} := (F^{(k)})'$ for every $k \in \mathbb{N}$. Now referring to $\hat{X}_S$ let $\hat{L}_S$ be the family of all components $C \in G$ such that $C \cap \hat{X}_S \setminus C$ contains precisely two points. Then, similarly as in the proof of Theorem 1, the set $S$ is topologically characterized via

$$S = \left\{ \min \{ m \in \mathbb{N} \mid C \notin G^{(m+1)} \} \mid C \in \hat{L}_S \right\}.$$
5. Proof of Theorem 2

In the following, if \( X \subset \mathbb{R}^n \) then \( \overline{X} \) is the closure of \( X \) in the space \( \mathbb{R}^n \) and if \( n = 2 \) then \( X^\circ \) is the interior of \( X \) in the plane \( \mathbb{R}^2 \). For abbreviation let \( I = [0, 1] \) and \( J = ]0, 1[ \) and let \( 2\mathbb{N} := \{2k \mid k \in \mathbb{N}\} \) be the set of all positive even numbers. Furthermore, if \( X \subset \mathbb{R}^2 \) and \( L \subset \mathbb{R} \) we regard \( X \times L^k \) as a subset of \( \mathbb{R}^{k+2} \) for every \( k \geq 0 \) where \( X \times L^k \) is identified with \( X \) if \( k = 0 \).

For each \( m \in \mathbb{N} \) let \( D_m := [2^{-m}, 2^{-m+1}]^2 \) and
\[
W_m := \bigcup_{k=1}^{m} \left[ 2^{-m} + \frac{2k-1}{2^m+1} \cdot 2^{-m}, 2^{-m} + \frac{2k}{2^m+1} \cdot 2^{-m} \right]^2.
\]
So \( D_m \) is a compact square area and \( W_m \) is a union of \( m \) disjoint compact square areas which all lie in the interior of \( D_m \). For \( S \subset 2\mathbb{N} \) put
\[
Z_S := [-1, 0]^2 \cup \bigcup_{m \in S} (D_m \setminus W_m)
\]
and define \( F_n := \{Z_S \times I^{n-2} \mid S \subset 2\mathbb{N}\} \) for each dimension \( n \geq 2 \). Clearly, we always have \( \overline{Z_S} = Z_S \) and hence \( \overline{Z_S \times I^{n-2}} = Z_S \times I^{n-2} \). Obviously, for every \( U \in F_n \) the closure \( \overline{U} \) is compact and a union of cubes \([a_1, a_1+h] \times \cdots \times [a_n, a_n+h]\) with \( h > 0 \) so that the interiors of distinct cubes are always disjoint. (Notice that \( (D_m \setminus W_m) \times I^{n-2} \) is a union of precisely \( ((2m+1)^2 - m)\langle\frac{1}{4}\rangle^{n-2} \) such cubes with edge length \( l = \frac{2^{-m}}{2m+1} \).)

In order to verify that the family \( F_n \) has the desired homeomorphism properties it is enough to investigate the components of the space \( Z_S \times I^{n-2} \) and \( Z_S \times I^{n-2} \) respectively. Clearly the components are always path connected spaces and so it is natural to determine their fundamental groups. (Two spaces \( X, Y \) cannot be homeomorphic if the fundamental group of some path component of \( X \) is not isomorphic to the fundamental group of any path component of \( Y \).)

For each \( S \subset 2\mathbb{N} \) the components of the space \( Z_S \times I^{n-2} \) resp. \( Z_S \times I^{n-2} \) are precisely \( (D_m \setminus W_m) \times I^{n-2} \) resp. \( (D_m \setminus W_m) \times I^{n-2} \) with \( m \in S \) and the one simply connected component \([-1, 0]^2 \times I^{n-2} \) resp. \([-1, 0]^2 \times I^{n-2} \).

For each \( m \in \mathbb{N} \) the fundamental group both of \( D_m \setminus W_m \) and of \( D_m \setminus W_m \) is free on \( m \) generators. This is enough since for \( n \geq 3 \) both \( I^{n-2} \) and \( J^{n-2} \) have trivial fundamental groups. (If \( X, Y \) are path connected spaces then the fundamental group of the product space \( X \times Y \) is isomorphic to the direct product of the fundamental groups of \( X \) and \( Y \).)

Finally, dispensing with compactness, it is plain to modify the definition of \( F_n \) so that each member of \( F_n \) is the interior of a union of cubes of the form \([k_1, k_1+1] \times \cdots \times [k_n, k_n+1] \) with \( k_1, \ldots, k_n \in \mathbb{Z} \). For example, for \( \emptyset \neq S \subset 2\mathbb{N} \) replace \( Z_S \times I^{n-2} \) with \( Y_S \times J^{n-2} \) where \( Y_S := \bigcup_{m \in S} \{(t_mx, t_my) \mid (x, y) \in D_m \setminus W_m\} \) with \( t_m := 4m(2m+1) \).

6. Proofs of Propositions 1 and 3

First we need two basic lemmas. A proof of Lemma 1 is an easy exercise and Lemma 2 is a consequence of a well-known theorem due to Sierpinski (cf. [1] 6.1.27).

**Lemma 1.** If \( g \) is an arbitrary injection from \( \mathbb{N} \) into \( \mathbb{N} \) then \( \{n \in \mathbb{N} \mid n \leq g(n)\} \) must be an infinite set.

**Lemma 2.** If \( a, b \in \mathbb{R} \) and \( a < b \) then for any family \( \mathcal{F} \) of mutually disjoint intervals \([x, y]\) with \( x < y \) the equality \( [a, b] = \bigcup \mathcal{F} \) is only possible in the trivial case \( \mathcal{F} = \{[a, b]\} \).
In order to prove Proposition 1 it is enough to settle the statement on the closed point sets \( X_u \) because each \( X_u \) is a completion of the metric space \( X_0^u \). Fix \( 2 \leq v < w \) and let \( I_n := [2^v, 2^w + v^n] \) and \( J_n := [2^w, 2^w + w^n] \) for every \( n \in \mathbb{N} \). So we have \( b_n - a_n = v^n \) and \( d_n - c_n = w^n \) for every \( n \in \mathbb{N} \) and \( 1 + b_n \leq a_{n+1} \) and \( 1 + d_n \leq c_{n+1} \) for every \( n \in \mathbb{N} \). Assume indirectly that \( f \) is a uniformly continuous bijection from \( X_v \) onto \( X_w \). We claim that for each \( n \in \mathbb{N} \) we must have \( f(I_n) = J_m \) for some \( m \in \mathbb{N} \). Indeed, choose \( m \) so that \( f(I_n) \cap J_m \neq \emptyset \) and define an equivalence relation on \( J_m \) via \( x \sim y \) if and only if \( f^{-1}(x), f^{-1}(y) \in I_k \) for some \( k \). Then, since \( f(I_k) \) is always compact and connected and since all point sets \( J_k \) are open and closed in the space \( X_w \), the family \( F \) of all equivalence classes must equal \( \{ f(I_k) \mid k \in K \} \) for some \( K \subset \mathbb{N} \) with \( n \in K \). Hence, in view of Lemma 2 we must have \( K = \{ n \} \) or, equivalently, \( f(I_n) = J_m \).

Consequently, there is a bijection \( g : \mathbb{N} \to \mathbb{N} \) such that \( f(I_n) = J_{g(n)} \) for every \( n \in \mathbb{N} \).

By Lemma 1, \( G := \{ n \in \mathbb{N} \mid n \leq g(n) \} \) is an infinite set. Let \( Y_n := \bigcup I_n \) and \( Y_w := \bigcup_{n \in G} J_{g(n)} \). Then \( f \) is a uniformly continuous function from the unbounded set \( Y_v \) onto the unbounded set \( Y_w \). Thus we may fix \( 0 < \delta < 1 \) so that \( |f(x) - f(y)| \leq 1 \) whenever \( x, y \in Y_v \) and \( |x - y| \leq \delta \). Naturally, \( f \) is strictly monotonic on each interval \( I_n (n \in G) \) and \( \{ f(a_n), f(b_n) \} = \{ c_{g(n)}, d_{g(n)} \} \) for every \( n \in G \). Now, for every \( n \in G \) we have

\[
|f(a_n) - f(a_n + \delta)| + |f(a_n + \delta) - f(a_n + 2\delta)| + \cdots + |f(a_n + k\delta) - f(b_n)| \leq k + 1
\]

where \( k \in \mathbb{N} \) is chosen so that \( a_n + k\delta < b_n \leq a_n + (k + 1)\delta \) or, equivalently, \( k\delta < v^n \leq (k + 1)\delta \). But then \( w^n - 1 \leq v^n/\delta \) for every \( n \) in the infinite set \( G \). This is impossible since \( \lim_{n \to \infty} w^n/v^n = \infty \) and so the proof of Proposition 1 is finished.

Remark. Concerning higher dimensions, in view of the preceding proof it is plain that the \( c \) closed resp. open point sets \( \bigcup_{m=1}^{\infty} [2^u, 2^u + u^m] \) resp. \( \bigcup_{m=1}^{\infty} [2^u, 2^u + u^m]^n \) (\( u \geq 2 \)) in \( \mathbb{R}^n \) are metrically distinct and topologically similar for arbitrary \( n \).

Now we are going to prove Proposition 3. As usual, the distance \( d(A, B) \) between two nonempty subsets \( A, B \) of \( \mathbb{R}^n \) is the infimum of all numbers \( d(a, b) \) with arbitrary \( a \in A \) and \( b \in B \) where \( d(x, y) \) denotes the Euclidean distance between \( x, y \in \mathbb{R}^n \). If \( U \) is an open subspace of \( \mathbb{R}^n \) and if \( G \) is the (countable) family of all components of \( U \), then let us call a finite subset \( F \) of \( G \) a chain if and only if there is an ordering \( F = \{ U_1, \ldots, U_m \} \) with \( U_i \neq U_j (i \neq j) \) and \( d(U_k, U_{k+1}) = 0 \) for every \( k < m \). The length of \( F \) is \( m \). A chain is maximal if it is not contained in a chain of greater length. For every \( S \subset \mathbb{N} \) all the components of \( Y_S \) are open cubes of the form \( [a_1, a_1 + h[x \times \cdots \times]a_n, a_n + h] \) and, evidently (by induction on the dimension \( n \)),

\[
Y_S = \bigcup_{s \in S} (\bigcup F_s) \cup \bigcup_{k=1}^{\alpha} C_k \quad (\alpha \in \mathbb{N} \cup \{ \infty \})
\]

where \( \{ C_k \} \) is always a maximal chain of length \( 1 \) and \( F_s \) is a maximal chain of length \( (s + 1)^n \) for every \( s \in S \) and \( d(\bigcup F_i, \bigcup F_j) > 0 \) whenever \( i, j \in S \) and \( i \neq j \).

Suppose that \( S, S' \subset \mathbb{N} \) and that \( f \) is a uniform homeomorphism from \( Y_S \) onto \( Y_{S'} \). Of course, \( f(C) \) is a component of \( Y_{S'} \) if and only if \( C \) is a component of \( Y_S \). Furthermore, for any \( \emptyset \neq A, B \subset Y_S \) we certainly have \( d(A, B) = 0 \) if and only if \( d(f(A), f(B)) = 0 \). Therefore, for every \( s \in S \) the set \( \{ f(C) \mid C \in F_s \} \) must be a maximal chain of length \( (s + 1)^n \) in the space \( Y_{S'} \), whence \( s \in S' \). Thus \( S \subset S' \). Similarly, \( S' \subset S \).
7. Proof of Proposition 2

For a nonempty set $X$ in the family $\mathcal{U} \cup \mathcal{A}$ let $\mathcal{C}(X)$ be the family of all components of the Euclidean subspace $X$ of $\mathbb{R}$. Since each member of $\mathcal{C}(X)$ is an open or closed interval, we may define a natural strict linear ordering $\prec$ of $\mathcal{C}(X)$ via $A \prec B$ for distinct (and hence disjoint) $A, B \in \mathcal{C}(Z)$ if and only if $a < b$ for some $(a, b) \in A \times B$ or, equivalently, if $a < b$ for every $(a, b) \in A \times B$.

Let $\emptyset \neq X,Y \subset \mathbb{R}$ and let $\varphi : X \to Y$ be an order isomorphism. Then $\varphi$ is a homeomorphism with respect to the order topologies of $(X, \prec)$ and $(Y, \prec)$. Moreover, if the sets $X,Y$ lie in the family $\mathcal{U} \cup \mathcal{A}$ then $\varphi$ is a homeomorphism between the Euclidean spaces $X$ and $Y$ and hence $A \mapsto \varphi(A)$ defines a bijection from $\mathcal{C}(X)$ onto $\mathcal{C}(Y)$ and it is evident that this bijection is an order isomorphism between $(\mathcal{C}(X), \prec)$ and $(\mathcal{C}(Y), \prec)$. Thus, $X,Y \in \mathcal{U} \cup \mathcal{A}$ are not order-isomorphic if the two families $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ are not order-isomorphic. (Conversely, if either $X,Y \in \mathcal{U}$ or $X,Y \in \mathcal{A}$ are compact then from any order isomorphism between $(\mathcal{C}(X), \prec)$ and $(\mathcal{C}(Y), \prec)$ we may easily construct an order isomorphism between $(X, \prec)$ and $(Y, \prec)$. This is not true for arbitrary sets $X,Y \in \mathcal{A}$ or for compact sets $X,Y \subset \mathbb{R}$. Consider, for example, $X = [0,1] \cup [2,3]$ and firstly $Y = [0,1] \cup [2,\infty[$ and secondly $Y = [0,1] \cup \{2\}$.)

So in order to settle Proposition 2 it is enough to find $c$ metrically similar sets $X$ in the family $\mathcal{U}$ resp. $\mathcal{A}$, such that the corresponding sets $\mathcal{C}(X)$ are mutually not order-isomorphic. Let $\mathcal{G}$ be the family of all functions $g$ from $\mathbb{N}$ to $\{0,1\}$ such that the set $g^{-1}(\{1\})$ is infinite. Clearly, the cardinal number of the family $\mathcal{G}$ is $c$. For every $n \in \mathbb{N}$ define

$$Z_n := \bigcup_{k=1}^{\infty} (6n + 1 + 3^{-2k}, 6n + 1 + 3^{-2k+1}] \cup ]6n + 2 - 3^{-2k+1}, 6n + 2 - 3^{-2k}[)$$

and for every $g \in \mathcal{G}$ define

$$U_g := \bigcup_{n=1}^{\infty} (Z_n \cup ]6n, 6n + 1[ \cup ]6n + 2, 6n + 3[) \cup \bigcup_{n \in g^{-1}(\{1\})} ]6n + 4, 6n + 5[.$$

It is plain that the $c$ open sets $U_g (g \in \mathcal{G})$ are metrically similar and that the $c$ closed sets $\overline{U_g} (g \in \mathcal{G})$ lie in the family $\mathcal{A}$ and are metrically similar too. Let $\zeta$ denote the order type of $Z_n$. Then $\zeta$ is also the order type of $\mathcal{C}(Z_n)$ for every $n \in \mathbb{N}$ and it is evident that for each $g \in \mathcal{G}$ both the order type of $\mathcal{C}(U_g)$ and the order type of $\mathcal{C}(\overline{U_g})$ equals

$$(1 + \zeta + 1) + g(1) + (1 + \zeta + 1) + g(2) + (1 + \zeta + 1) + g(3) + \cdots$$

$$= 1 + \zeta + (1 + g(1) + 1) + \zeta + (1 + g(2) + 1) + \zeta + (1 + g(3) + 1) + \cdots$$

where a nonnegative integer $k$ is always the order type of any linearly ordered set of precisely $k$ elements. (If $\alpha, \beta$ are order types of nonempty sets then $\alpha + 0 + \beta$ is just $\alpha + \beta$.) Naturally, for distinct $f, g \in \mathcal{G}$ the order types of $\mathcal{C}(U_f)$ and $\mathcal{C}(U_g)$ are distinct and this concludes the proof.

Remark. In view of the previous considerations it is easy to track down $c$ metrically similar compact subsets of $\mathbb{R}$ without singleton components which are mutually not order-isomorphic. (The existence of such sets follows from \[5\] Main Theorem 2.) Take for example (with $h$ as in the proof of Theorem 1) the $c$ point sets $h(\overline{U_g}) \cup [1,2]$ $(g \in \mathcal{G})$.
8. Totally disconnected point sets

So far we considered only point sets where no component is a singleton. Now we consider point sets where every component is a singleton, i.e. totally disconnected point sets. Since for every totally disconnected set \( T \subset \mathbb{R} \) the set \( T \times \{0\}^{n-1} \) is a totally disconnected subset of the Euclidean space \( \mathbb{R}^n \), for our purpose there will be no benefit of considering arbitrary dimensions and so we restrict to dimension \( n = 1 \) in the following. (In view of Theorem 4 below, notice also that every compact and totally disconnected subspace of \( \mathbb{R}^n \) is homeomorphic to some subspace of \( \mathbb{R} \).) In the real line \( \mathbb{R} \) a point set is totally disconnected if and only if it does not contain a nondegenerate interval. (In particular, no nonempty open set is totally disconnected.) The real line \( \mathbb{R} \) contains \( 2^\omega \) totally disconnected subsets, for example all subsets of \( \mathbb{R} \setminus \mathbb{Q} \). Among these \( 2^\omega \) totally disconnected spaces there must also be \( 2^\omega \) non-homeomorphic spaces because one cannot track down more than \( \omega \) homeomorphic subspaces of \( \mathbb{R} \). (For if \( X \subset \mathbb{R} \) then there are at most \( \omega \) continuous functions from \( X \) into \( \mathbb{R} \).)

What is the number of all topological types of totally disconnected closed point sets in \( \mathbb{R} \)?

The following theorem, which is a noteworthy counterpart to Theorem 1, gives the answer.

**Theorem 4.** There are \( c \) mutually non-homeomorphic compact and totally disconnected subspaces of the Euclidean unit interval \([0,1] \).

**Proof.** In any Hausdorff space \( X \) a point set \( A \) is dense in itself if and only if every point in \( A \) is a limit point of \( A \), i.e. \( A \subset A' \). Let \( \Delta(X) := \bigcup \{ A \subset X \mid A \subset A' \} \) denote the maximal dense-in-itself point set in the space \( X \). Define a sort of signature set of integers by

\[
\Sigma(X) := \{ k \in \mathbb{N} \mid (X \setminus \Delta(X))^{(k)} \setminus (X \setminus \Delta(X))^{(k+1)} \cap \Delta(X) \neq \emptyset \}.
\]

Let \( h \) be a strictly increasing function from \([0, \infty[ \) onto \([0,1] \), for example \( h(x) = \frac{\pi}{2} \arctan x \) as in the proof of Theorem 1. Let \( \mathbb{D} \subset [0,1] \) be the Cantor ternary set. (Notice that \( \mathbb{D}' = \mathbb{D} \).) As in the proof of Theorem 1, for every \( n \in \mathbb{N} \) let \( K_n \subset [5n,5n+1) \) be compact with \( \max K_n = 5n+1 \) such that the \( k \)-th derivative \( K_n^{(k)} \) is infinite whenever \( k < n \) and empty whenever \( k > n \) and a singleton when \( k = n \), namely \( K_n^{(n)} = \{5n+1\} \).

Now for each infinite set \( S \subset \mathbb{N} \) define

\[
D_S := \bigcup_{n \in S} \{ 5n+1+x \mid x \in \mathbb{D} \} \quad \text{and} \quad Y_S := h(D_S \cup \bigcup_{n \in S} K_n) \cup \{1\}.
\]

Of course, all point sets \( Y_S \subset [0,1] \) are compact and totally disconnected. Moreover, \( \Delta(Y_S) = h(D_S) \cup \{1\} \) and \( (K_n \setminus D_S)^{(k)} = K_n^{(k)} \) for every \( k \in \mathbb{N} \) and \( n \in S \). Therefore, similarly as in the proof of [3] Theorem 7.1 we always have \( \Sigma(Y_S) = S \) and hence \( Y_S \) and \( Y_T \) are never homeomorphic for distinct infinite sets \( S, T \subset \mathbb{N} \), q.e.d.

**Remark.** If in Theorem 4 the property perfect (dense-in-itself) is added then the variety of the spaces collapses. Indeed, it is well-known that any perfect, compact, zero-dimensional, second countable Hausdorff space is homeomorphic to \( \mathbb{D} \) (cf. [2]). (Note that a compact Hausdorff space is zero-dimensional if and only if it is totally disconnected, cf. [1] 6.2.9).

**Remark.** By a classic theorem due to Mazurkiewicz und Sierpinski [4] there are precisely \( \aleph_1 \) compact and countable Hausdorff spaces up to homeomorphism. (\( \aleph_1 \) is the least cardinal number greater than the cardinality \( \aleph_0 \) of a countably infinite set, whence \( \aleph_1 \leq c \).) As a consequence, since each countable metric space can be embedded in \( \mathbb{R} \) (see [1] 4.3.H.b), the space \( \mathbb{R} \) has uncountably many non-homeomorphic compact and totally disconnected subspaces. Theorem 4 is an improvement of this consequence because one cannot rule out...
the existence of an uncountable set whose cardinality is smaller than \( c \). (Actually, it is consistent with ZFC set theory that there exist \( c \) uncountable cardinal numbers smaller than \( c \).)

9. Discrete linear point sets

There is an interesting consequence of Theorem 4 concerning discrete point sets. A point set \( X \) in the real line is discrete if and only if for every \( x \in X \) the singleton \( \{x\} \) is open in the subspace \( X \) of \( \mathbb{R} \). Equivalently, for every \( x \in X \) there is \( \delta_x > 0 \) such that \([y - \delta_y, y + \delta_y] \cap [z - \delta_z, z + \delta_z] = \emptyset\) whenever \( y, z \in X \) and \( x \neq y \). Consequently, any discrete subset of \( \mathbb{R} \) is countable. Since \( \mathbb{R} \) has precisely \( c \) countable subsets and since \( x + \mathbb{Z} \) is discrete for \( 0 < x < 1 \), the real line contains precisely \( c \) discrete point sets. From the topological point of view, essentially there is precisely one infinite discrete point set in \( \mathbb{R} \). Indeed, if \( X \) is any infinite discrete point set in \( \mathbb{R} \) then \( X \) is obviously homeomorphic to the discrete space \( \mathbb{Z} \). On the other hand, from the metrical point of view there are very many discrete point sets.

**Theorem 5.** There are \( c \) metrically distinct infinite and discrete point sets in the unit interval \([0, 1]\).

**Proof.** Choose for every set \( Y_S \) in the proof of Theorem 4 a discrete set \( Z_S \subset [0, 1] \) such that \( Z_S \cap Y_S = \emptyset \) and \( Z'_S = Y_S \), whence \( Z_S \) is infinite and \( \overline{Z_S} \setminus Z_S = Y_S \). Suppose that for infinite sets \( S, T \subset \mathbb{N} \) there is a bijection \( f \) from \( Z_S \) onto \( Z_T \) such that \( f, f^{-1} \) are uniformly continuous. Then there is an expansion of \( f \) to a homeomorphism \( g \) from \( \overline{Z_S} \) onto \( \overline{Z_T} \) since the Euclidean metric space \( \overline{Z_S} \) resp. \( \overline{Z_T} \) is a completion of the Euclidean metric space \( Z_S \) resp. \( Z_T \). Then we must have \( S = T \) since \( g(Y_S) = g(\overline{Z_S} \setminus Z_S) = g(\overline{Z_S}) \setminus f(Z_S) = \overline{Z_T} \setminus Z_T = Y_T \) and \( Y_S, Y_T \) cannot be homeomorphic if \( S \neq T \). It is always possible to choose such sets \( Z_S \) and the proof of Theorem 5 is finished in view of the following lemma.

**Lemma 3.** Any compact and totally disconnected and nonempty set \( A \subset \mathbb{R} \) equals \( Z' \) for some discrete set \( Z \subset \left[\min A, \max A\right] \) with \( Z \cap A = \emptyset \).

**Proof.** Let \( a = \min A \) and \( b = \max A \). Clearly \( A \) must be nowhere dense. Hence \( A \) is the boundary of the open set \([a, b] \setminus A\). Write \([a, b] \setminus A = \bigcup_{j \in J} I_j\) with a countable index set \( J \) and where the sets \( I_j \) are mutually disjoint open intervals, \( I_j = [x_j, y_j] \) with \( x_j < y_j \). Define a countable set \( C_j \subset I_j \) by

\[
C_j := \{x_j + (y_j - x_j)2^{-k} \mid k \in \mathbb{N}\} \cup \{y_j - (y_j - x_j)2^{-k} \mid k \in \mathbb{N}\}
\]

for every \( j \in J \) and put \( Z = \bigcup_{j \in J} C_j \). Clearly, \( Z \) is discrete and \( Z \subset [a, b] \) and \( Z \cap A = \emptyset \). Since \( C'_j = \{x_j, y_j\} \) for every \( j \in J \), we have \( Z' = A \), q.e.d.

Since the discrete sets in Theorem 5 are all absolutely bounded, none of them is closed. Naturally, a closed and discrete subset of \( \mathbb{R} \) cannot be bounded and infinite, while a bounded and discrete subset of \( \mathbb{R} \) cannot be infinite and closed. Obviously, an infinite closed set \( A \subset \mathbb{R} \) is discrete if and only if \( A \cap [-k, k] \) is finite for every \( k \in \mathbb{N} \). How many infinite, closed, discrete point sets do exist from the metrical point of view? The following theorem gives the answer.

**Theorem 6.** There are \( c \) metrically distinct infinite, closed, discrete point sets in the real line \( \mathbb{R} \).

**Proof.** Let \( \mathbb{P} \) be the set of all primes. For \( p \in \mathbb{P} \) and \( n \in \mathbb{N} \) define a set of precisely \( p \) elements which is contained in an interval of length \( p^{-n} \) by

\[
G[p; n] := \{p^n + k^{-1}p^{-n} \mid k = 1, \ldots, p\}.
\]
For \( \emptyset \neq S \subset \mathbb{P} \) define \( A_S \subset \mathbb{R} \) by \( A_S := \bigcup_{p \in S} \bigcup_{n=1}^{\infty} G[p; n] \).

We always have \( G[p; n] \subset [p^n, p^n+1[ \) and, clearly, \( [p^n, p^n+1[ \cap ]q^m, q^m+1[ = \emptyset \) whenever \( p, q \in \mathbb{P} \) and \( n, m \in \mathbb{N} \) and \( (p, n) \neq (q, m) \). Thus \( A_S \) is always a closed and discrete point set. We claim that there is no uniform homeomorphism between any two of the \( c \) Euclidean metric spaces \( A_S (\emptyset \neq S \subset \mathbb{N}) \). Let \( S, T \subset \mathbb{P} \) and \( S \not\subset T \) and suppose indirectly that \( f \) is a bijection from \( A_S \) onto \( A_T \) such that \( f \) and \( f^{-1} \) are uniformly continuous. Then we can choose \( \delta > 0 \) so that

(i) \( |f(x) - f(y)| < \frac{1}{2} \) whenever \( x, y \in A_S \) and \( |x - y| < \delta \),

(ii) \( |f^{-1}(x) - f^{-1}(y)| < \frac{1}{2} \) whenever \( x, y \in A_T \) and \( |x - y| < \delta \).

Now choose \( p \) in \( S \setminus T \) and fix \( N \in \mathbb{N} \) so that \( p^{-N} < \delta \). In view of (i), for every \( n \geq N \) we can define a prime \( q_n \in T \) and a number \( m_n \in \mathbb{N} \) such that \( f(G[p; n]) \subset G[q_n; m_n] \). Since \( G[q_n; m_n] \) is finite, not both \( \{ q_n \mid n \geq N \} \) and \( \{ m_n \mid n \geq N \} \) can be finite sets. Hence we may choose \( n \geq N \) so that \( q_n^{-n} < \delta \). But then, in view of (ii) and \( f(G[p; n]) \subset G[q_n; m_n] \) we have \( f^{-1}(G[q_n; m_n]) = G[p; n] \) and this is impossible since \( G[q_n; m_n] \) has precisely \( q_n \) elements and \( G[p; n] \) has precisely \( p \) elements and \( q_n \neq p \), \( q.e.d. \).

Remark. From the order-theoretical point of view there are only countably many closed and discrete subsets of \( \mathbb{R} \). Firstly, two finite sets are order-isomorphic if and only if they are equipollent. Secondly, a moment’s reflection is sufficient to see that any infinite, closed, discrete subset of \( \mathbb{R} \) is order-isomorphic to \( \mathbb{Z} \) or to \( \mathbb{N} \) or to \( \mathbb{Z} \setminus \mathbb{N} \). On the other hand, there are \( c \) mutually not order-isomorphic discrete subsets of \( \mathbb{R} \). For example, let \( U_g, \mathcal{C}(U_g) (g \in \mathcal{G}) \) be as in the proof of Proposition 2 and let \( \varphi \) denote any choice function on the family \( \mathcal{U} \setminus \{ \emptyset \} \), i.e. \( \varphi(U) \in U \) whenever \( \emptyset \neq U \in \mathcal{U} \). Then for each \( g \in \mathcal{G} \) the set \( D_g := \{ \varphi(C) \mid C \in \mathcal{C}(U_g) \} \) is discrete and the order type of \( D_g \) equals the order type of \( \mathcal{C}(U_g) \), whence \( D_f \) and \( D_g \) are never order-isomorphic for distinct \( f, g \in \mathcal{G} \).

In a natural way the proof of Theorem 5 leads to the following noteworthy theorem.

**Theorem 7.** A countably infinite discrete space \( X \) has \( c \) mutually non-homeomorphic metrizable compactifications of size \( c \). There exist \( c \) incomplete metrics \( d \) on a countably infinite set \( X \) such that \( (X, d) \) is always a discrete topological space and the completions of the metric spaces \( (X, d) \) are compact of size \( c \) and topologically distinct.

**Proof.** For each infinite set \( S \subset \mathbb{N} \) let \( Y_S, Z_S \) be as in the proof of Theorem 5 and define a metric \( d_S \) on \( X \) such that the metric space \( (X, d_S) \) is an isometric copy of \( Z_S \) equipped with the Euclidean metric. Then the topology of the discrete space \( X \) is induced by the metric \( d_S \) which of course is not complete. The Euclidean metric space \( \overline{Z_S} = Y_S \cup Z_S \) is both a completion of the metric space \( (X, d_S) \) and a compactification of the discrete space \( X \). Two spaces \( \overline{Z_S}, \overline{Z_T} \) are never homeomorphic for distinct infinite sets \( S, T \subset \mathbb{N} \) since in view of \( Z'_S = Y_S \) we have \( \Sigma(\overline{Z_S}) = \Sigma(Y_S) = S \) for each \( S \), \( q.e.d. \).

Remark. The cardinality \( c \) in Theorem 7 is the largest possible in all cases. Indeed, there exist precisely \( c \) compact metrizable spaces up to homeomorphism and any compact metric space is either countable or of size \( c \) (cf.[3]). And in view of [4] a countably infinite discrete space has only \( \aleph_1 \) topologically distinct countable (and hence metrizable) compactifications. (Note that if \( \alpha \) is any countable ordinal number then in the compact space \( \omega^\alpha + 1 \) the successor ordinals form a discrete and dense subspace.)
References

[1] Engelking, R.: *General Topology*, revised and completed edition. Heldermann 1989.
[2] Kechris, A.: *Classical Descriptive Set Theory*. Springer 1995.
[3] Kuba, G.: *Counting metric spaces*. Arch.d.Math. 97, 569-578 (2011).
[4] Mazurkiewicz, S.; Sierpinski, W.: Contribution la topologie des ensembles dnombrables. *Fund. Math.* 1 (1920), 17-27.
[5] Winkler, R.: *How much must an order theorist forget to become a topologist?* Contributions to General Algebra 12, Proc. Vienna Conference (June 1999).

Author’s address: Institute of Mathematics,
University of Natural Resources and Life Sciences, Vienna, Austria.
E-mail: gerald.kuba@boku.ac.at

Additional References

[A1] Kuba, G.: *On the variety of Euclidean point sets*. Int. Math. News 228, 23-32 (2015).
[A2] Kuba, G.: *Counting ultrametric spaces*. Colloq. Math. 152, 217-234 (2018).

Up to Section 7 the present paper is essentially identical with [A1].

The proof of Theorem 5 is very similar to the proof of Proposition 2 in [A2]. Theorem 7 is a consequence of [A2] Corollary 2. In connection with Theorem 7 the following consequence of [A2] Corollary 3 is worth mentioning.

**Theorem 8.** The topology of an infinite discrete space $S$ can be generated by $2^{|S|}$ metrics $d$ such that the completions of the metric spaces $(S, d)$ are mutually non-homeomorphic metric spaces of size $|S|$.

The cardinality $2^{|S|}$ in Theorem 8 is the largest possible since, trivially, an infinite set $S$ cannot carry more metrics than the total amount of mappings from $S \times S$ to $\mathbb{R}$ which equals $|\mathbb{R}|^{2^{|S|}} = 2^{|S|}$. 