Bi-gyrogroup: The Group-like Structure Induced by Bi-decomposition of Groups

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Abstract

The decomposition $\Gamma = BH$ of a group $\Gamma$ into a subset $B$ and a subgroup $H$ of $\Gamma$ induces, under general conditions, a group-like structure for $B$, known as a gyrogroup. The famous concrete realization of a gyrogroup, which motivated the emergence of gyrogroups into the mainstream, is the space of all relativistically admissible velocities along with a binary operation given by the Einstein velocity addition law of special relativity theory. The latter leads to the Lorentz transformation group $SO(1, n)$, $n \in \mathbb{N}$, in pseudo-Euclidean spaces of signature $(1, n)$. The study in this article is motivated by generalized Lorentz groups $SO(m, n)$, $m, n \in \mathbb{N}$, in pseudo-Euclidean spaces of signature $(m, n)$. Accordingly, this article explores the bi-decomposition $\Gamma = H_L BH_R$ of a group $\Gamma$ into a subset $B$ and subgroups $H_L$ and $H_R$ of $\Gamma$, along with the novel bi-gyrogroup structure of $B$ induced by the bi-decomposition of $\Gamma$. As an example, we show by methods of Clifford algebras that the quotient group of the spin group $Spin(m, n)$ possesses the bi-decomposition structure.

Keywords: bi-decomposition of group; bi-gyrogroup; gyrogroup; spin group; pseudo-orthogonal group.

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1 Introduction

Lorentz transformation groups $\Gamma = \text{SO}(1, n)$, $n \in \mathbb{N}$, possess the decomposition structure $\Gamma = BH$, where $B$ is a subset of $\Gamma$ and $H$ is a subgroup of $\Gamma$ [26]. The decomposition structure of $\Gamma$ induces a group-like structure for $B$. This group-like structure was discovered in 1988 [26] and became known as a gyrogroup [27, 28]. Subsequently, gyrogroups turned out to play a universal computational role that extends far beyond the domain of Lorentz groups $\text{SO}(1, n)$ [32, 33], as noted by Chatelin in [4, p. 523] and in references therein. In fact, gyrogroups are special loops that, according to [17], are placed centrally in loop theory.

The use of Clifford algebras to employ gyrogroups as a computational tool in harmonic analysis is presented by Ferreira in the seminal papers [9, 10]. The use of Clifford algebras to obtain a better understanding of gyrogroups is found, for instance, in [7, 8, 11, 20, 24].

Generalized Lorentz transformation groups $\Gamma = \text{SO}(m, n)$, $m, n \in \mathbb{N}$, possess the so-called bi-decomposition structure $\Gamma = H_LBH_R$, where $B$ is a subset of $\Gamma$ and $H_L$ and $H_R$ are subgroups of $\Gamma$. The bi-decomposition structure of $\Gamma$ induces a group-like structure for $B$, called a bi-gyrogroup [34]. The use of Clifford algebras that may improve our understanding of bi-gyrogroups is found in [12]. Clearly, the notion of bi-gyrogroups extends the notion of gyrogroups. Accordingly, “gyro-language”, the algebraic language crafted for gyrogroup theory is extended to “bi-gyro-language” for bi-gyrogroup theory.

As a first step towards demonstrating that bi-gyrogroups play a universal computational role that extends far beyond the domain of generalized Lorentz groups $\text{SO}(m, n)$, the aim of the present article is to approach the study of bi-gyrogroups from the abstract viewpoint.

The article is organized as follows. In Section 2 we give the definition of a bi-gyrogroupoid. In Section 3 we show that the bi-transversal decomposition of a group with additional properties yields a highly structured type of bi-gyrogroupoids. In Section 4 we introduce the notion of bi-gyrodecomposition of groups and prove that any bi-gyrodecomposition of a group gives rise to a bi-gyrogroup. Finally, in Sections 5 and 6 we demonstrate that the pseudo-orthogonal group $\text{SO}(m, n)$ and the quotient group of the spin group Spin $(m, n)$ possess the bi-gyrodecomposition structure.

2 Bi-gyrogroupoids

We begin with the abstract definition of a bi-gyrogroupoid, which is modeled on the groupoid $\mathbb{R}^{n \times m}$ of all $n \times m$ real matrices with bi-gyroaddition studied in detail in [34]. We recall that a groupoid $(B, \oplus_b)$ is a non-empty set $B$ with a binary operation $\oplus_b$. An automorphism of a groupoid $(B, \oplus_b)$ is a bijection from $B$ to itself that preserves the groupoid operation. The group of all automorphisms of $(B, \oplus_b)$ is denoted by Aut $(B, \oplus_b)$ or simply Aut $(B)$.
Definition 2.1 (Bi-gyrogroupoid). A groupoid \((B, \oplus)\) is a bi-gyrogroupoid if its binary operation satisfies the following axioms.

(BG1) There is an element \(0 \in B\) such that \(0 \oplus a = a \oplus 0 = a\) for all \(a \in B\).

(BG2) For each \(a \in B\), there is an element \(b \in B\) such that \(b \oplus a = 0\).

(BG3) Each pair of \(a\) and \(b\) in \(B\) corresponds to a left automorphism \(\text{lg}r[a, b]\) and a right automorphism \(\text{rg}r[a, b]\) in \(\text{Aut} (B, \oplus)\) such that for all \(c \in B\),

\[
(a \oplus b) \oplus_b \text{lg}r[a, b]c = \text{rg}r[b, c]a \oplus_b (b \oplus_b c).
\] (2.1)

(BG4) For all \(a, b \in B\),

(a) \(\text{rg}r[a, b] = \text{rg}r[\text{lg}r[a, b], a \oplus b]\), and

(b) \(\text{lg}r[a, b] = \text{lg}r[\text{lg}r[a, b], a \oplus b]\).

(BG5) For all \(a \in B\), \(\text{lg}r[a, 0]\) and \(\text{rg}r[a, 0]\) are the identity map of \(B\).

A concrete realization of Axioms (BG1) through (BG5) will be presented in Section 5.

Roughly speaking, any bi-gyrogroupoid is a groupoid that comes with two families of automorphisms, called left and right automorphisms or, collectively, bi-automorphisms. Note that if bi-automorphisms of a bi-gyrogroupoid \((B, \oplus)\) reduce to the identity automorphism of \(B\), then \((B, \oplus)\) forms a group.

Let \(\text{lg}r^{-1}[a, b]\) and \(\text{rg}r^{-1}[a, b]\) be the inverse map of \(\text{lg}r[a, b]\) and \(\text{rg}r[a, b]\), respectively. Let \(\circ\) denote function composition and let \(\text{id}_X\) denote the identity map on a non-empty set \(X\). The following theorem asserts that bi-gyrogroupoids satisfy a generalized associative law.

Theorem 2.2. Any bi-gyrogroupoid \(B\) satisfies the left bi-gyroassociative law

\[
a \oplus_b (b \oplus_b c) = (\text{rg}r^{-1}[b, c]a \oplus_b b) \oplus_b \text{lg}r[\text{rg}r^{-1}[b, c]a, b]c
\] (2.2)

and the right bi-gyroassociative law

\[
(a \oplus_b b) \oplus_b c = \text{rg}r[b, \text{lg}r^{-1}[a, b]c]a \oplus_b b \oplus_b \text{lg}r^{-1}[a, b]c
\] (2.3)

for all \(a, b, c \in B\).

Proof. Let \(a, b, c \in B\) be arbitrary. Since \(\text{rg}r[b, c]\) is surjective, there is an element \(d \in B\) for which \(\text{rg}r[b, c]d = a\). By (BG3),

\[
a \oplus_b (b \oplus_b c) = \text{rg}r[b, c]d \oplus_b (b \oplus_b c) = (d \oplus_b b) \oplus_b \text{lg}r[d, b]c.
\]

Since \(d = \text{rg}r^{-1}[b, c]a\), (2.2) is obtained. One obtains (2.3) in a similar way. \(\square\)

Lemma 2.3. Any bi-gyrogroupoid \(B\) has a unique two-sided identity element.

Proof. By Definition 2.1, \(B\) has a two-sided identity element. Suppose that \(e\) and \(f\) are two-sided identity elements of \(B\). As \(e\) is a left identity, \(e \oplus_b f = f\). As \(f\) is a right identity, \(e \oplus_b f = e\). Hence, \(e = e \oplus_b f = f\). \(\square\)
Following Lemma 2.3, the unique two-sided identity of a bi-gyrogroupoid will be denoted by 0. Let $B$ be a bi-gyrogroupoid and let $a \in B$. We say that $b \in B$ is a left inverse of $a$ if $b \oplus_b a = 0$ and that $c \in B$ is a right inverse of $a$ if $a \oplus_b c = 0$. To see that each element of a bi-gyrogroupoid has a unique two-sided inverse, we investigate some basic properties of a bi-gyrogroupoid.

**Theorem 2.4.** Let $B$ be a bi-gyrogroupoid. The following properties are true.

1. For all $a, b \in B$, $lgyr[a, b]0 = 0$ and $rgyr[a, b]0 = 0$.

2. For all $a \in B$, $lgyr[a, a] = id_B$ and $rgyr[a, a] = id_B$.

3. If $a$ is a left inverse of $b$, then $lgyr[a, b] = id_B$ and $rgyr[a, b] = id_B$.

4. For all $b, c \in B$, if $a$ is a left inverse of $b$, then $rgyr[b, c]a \oplus_b (b \oplus_b c) = c$.

5. For all $a \in B$, if $b$ is a left inverse of $a$, then $b$ is a right inverse of $a$.

**Proof.** (1) Let $a, b \in B$. Let $c \in B$ be arbitrary. Since $lgyr[a, b]$ is surjective, $c = lgyr[a, b]d$ for some $d \in B$. Then

$$c \oplus_b lgyr[a, b]0 = lgyr[a, b]d \oplus_b lgyr[a, b]0 = lgyr[a, b](d \oplus_b 0) = lgyr[a, b]d = c.$$ 

Similarly, $(lgyr[a, b]0) \oplus_b c = c$. Hence, $lgyr[a, b]0$ is a two-sided identity of $B$. By Lemma 2.3, $lgyr[a, b]0 = 0$. Similarly, one can prove that $rgyr[a, b]0 = 0$.

(2) Setting $b = 0$ in (BG4a) gives $rgyr[a, a] = rgyr[a, 0] = id_B$ by (BG5). Similarly, setting $b = 0$ in (BG4b) gives $lgyr[a, a] = id_B$.

(3) Let $b \in B$ and let $a$ be a left inverse of $b$. By (BG4a) and (BG5),

$$rgyr[a, b] = rgyr[lgyr[a, b]a, a \oplus_b b] = rgyr[lgyr[a, b]a, 0] = id_B.$$ 

Similarly, $lgyr[a, b] = id_B$ by (BG4b) and (BG5).

(4) Let $b, c \in B$ and let $a$ be a left inverse of $b$. From Identity (2.1) and Item (3), we have $rgyr[b, c]a \oplus_b (b \oplus_b c) = (a \oplus_b b) \oplus_b lgyr[a, b]c = 0 \oplus_b c = c$.

(5) Let $a \in B$ and let $b$ be a left inverse of $a$. By (BG2), $b$ has a left inverse, say $b$. From Items (4) and (3), we have

$$a = rgyr[b, a]\tilde{b} \oplus_b (b \oplus_b a) = rgyr[b, a]\tilde{b} \oplus_b 0 = rgyr[b, a]b = b.$$ 

It follows that $a \oplus_b b = \tilde{b} \oplus_b b = 0$, which proves $b$ is a right inverse of $a$.

**Theorem 2.5.** Any element of a bi-gyrogroupoid $B$ has a unique two-sided inverse in $B$.

**Proof.** Let $a \in B$. By (BG2), $a$ has a left inverse $b$ in $B$. By Theorem 2.4 (5), $b$ is also a right inverse of $a$. Hence, $b$ is a two-sided inverse of $a$. Suppose that $c$ is a two-sided inverse of $a$. Then $a$ is a left inverse of $c$. By Theorem 2.4 (3)-(4), $c = rgyr[a, c]b \oplus_b (a \oplus_b c) = rgyr[a, c]b \oplus_b 0 = rgyr[a, c]b = b$, which proves the uniqueness of $b$. 

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Following Theorem 2.5, if $a$ is an element of a bi-gyrogroupoid, then the unique two-sided inverse of $a$ will be denoted by $\ominus_b a$. We also write $a \ominus_b b$ instead of $a \ominus (\ominus_b b)$. As a consequence of Theorems 2.4 and 2.5, we derive the following theorem.

**Theorem 2.6.** Let $B$ be a bi-gyrogroupoid. The following properties are true for all $a, b, c \in B$:

1. $\ominus_b (\ominus_b a) = a$;
2. $\lgyr[a, b](\ominus_b c) = \ominus_b \lgyr[a, b]c$ and $\rgyr[a, b](\ominus_b c) = \ominus_b \rgyr[a, b]c$;
3. $\lgyr[a, \ominus_b a] = \lgyr[\ominus_b a, a] = \rgyr[a, \ominus_b a] = \rgyr[\ominus_b a, a] = \text{id}_B$.

Any bi-gyrogroupoid satisfies a generalized cancellation law, as shown in the following theorem.

**Theorem 2.7.** Any bi-gyrogroupoid $B$ satisfies the left cancellation law

\[
\ominus_b \rgyr[a, b]a \oplus_b (a \oplus_b b) = b \tag{2.4}
\]

and the right cancellation law

\[
(a \oplus_b b) \ominus_b \lgyr[a, b]b = a \tag{2.5}
\]

for all $a, b \in B$.

**Proof.** Identity (2.4) follows from Theorem 2.4 (4) and Theorem 2.6 (2). Identity (2.5) follows from (BG3) with $c = \ominus_b b$. \(\square\)

**Definition 2.8 (Bi-gyrocommutative bi-gyrogroupoid).** A bi-gyrogroupoid $B$ is **bi-gyrocommutative** if it satisfies the bi-gyrocommutative law

\[
a \oplus_b b = (\lgyr[a, b] \circ \rgyr[a, b])(b \ominus_a a) \tag{2.6}
\]

for all $a, b \in B$.

**Definition 2.9 (Automorphic inverse property).** A bi-gyrogroupoid $B$ has the **automorphic inverse property** if

\[
\ominus_b (a \ominus_b b) = (\ominus_b a) \ominus_b (\ominus_b b)
\]

for all $a, b \in B$.

**Definition 2.10 (Bi-gyration inversion law).** A bi-gyrogroupoid $B$ satisfies the **bi-gyration inversion law** if

\[
\lgyr^{-1}[a, b] = \lgyr[b, a] \quad \text{and} \quad \rgyr^{-1}[a, b] = \rgyr[b, a]
\]

for all $a, b \in B$. 
Under certain conditions, the bi-gyrocommutative property and the automorphic inverse property are equivalent, as shown in the following theorem.

**Theorem 2.11.** Let $B$ be a bi-gyrogroupoid such that

1. $\lg yr[a, b] \circ \rg yr[a, b] = \rg yr[a, b] \circ \lg yr[a, b]$;
2. $\lg yr^{-1}[a, b] = \lg yr[⊖b, ⊖a]$ and $\rg yr^{-1}[a, b] = \rg yr[⊖b, ⊖a]$;
3. $⊖b(a ⊕ b) = (\lg yr[a, b] \circ \rg yr[a, b])(⊖b ⊕ b)$

for all $a, b \in B$. If $B$ is bi-gyrocommutative, then $B$ has the automorphic inverse property. The converse is true if $B$ satisfies the bi-gyration inversion law.

**Proof.** Suppose that $B$ is bi-gyrocommutative and let $a, b \in B$. Then $b ⊕ a = \lg yr[a, b] \circ \rg yr[a, b]$ (2.7)

Hence, $B$ is bi-gyrocommutative.

### 3 Bi-transversal decomposition

In this section we study the bi-decomposition $\Gamma = BH_L BH_R$ of a group $\Gamma$ into a subset $B$ and subgroups $H_L$ and $H_R$ of $\Gamma$. The bi-decomposition $\Gamma = BH_L BH_R$ leads to a bi-gyrogroupoid $B$, and under certain conditions, a group-like structure for $B$, called a bi-gyrogroup. Further, in the special case when $H_L$ is the trivial subgroup of $\Gamma$, the bi-decomposition $\Gamma = BH_L BH_R$ descends to the decomposition studied in [14]. It turns out that the bi-gyrogroup $B$ induced by the bi-decomposition of $\Gamma$ forms a gyrogroup, a rich algebraic structure extensively studied, for instance, in [7, 9–11, 18, 22–25, 28–31].

**Definition 3.1 (Bi-transversal).** A subset $B$ of a group $\Gamma$ is said to be a bi-transversal of subgroups $H_L$ and $H_R$ of $\Gamma$ if every element $g$ of $\Gamma$ can be written uniquely as $g = h_L b h_R$, where $h_L \in H_L$, $b \in B$, and $h_R \in H_R$. 

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Let $B$ be a bi-transversal of subgroups $H_L$ and $H_R$ in a group $\Gamma$. For each pair of elements $b_1$ and $b_2$ in $B$, the product $b_1b_2$ gives unique elements $h_\ell(b_1, b_2) \in H_L$, $b_1 \odot b_2 \in B$, and $h_r(b_1, b_2) \in H_R$ such that
\[ b_1b_2 = h_\ell(b_1, b_2)(b_1 \odot b_2)h_r(b_1, b_2). \tag{3.1} \]

Hence, any bi-transversal $B$ of $H_L$ and $H_R$ gives rise to

1. a binary operation $\odot$ in $B$, called the bi-transversal operation;
2. a map $h_\ell: B \times B \to H_L$, called the left transversal map;
3. a map $h_r: B \times B \to H_R$, called the right transversal map.

The pair $(B, \odot)$ is called the bi-transversal groupoid of $H_L$ and $H_R$.

We will see shortly that the left and right transversal maps of the bi-transversal groupoid $(B, \odot)$ generate automorphisms of $(B, \odot)$, called left and right gyrations or, collectively, bi-gyrations. Accordingly, left and right gyrations are also called left and right gyroautomorphisms.

**Definition 3.2 (Bi-gyration).** Let $B$ be a bi-transversal of subgroups $H_L$ and $H_R$ in a group $\Gamma$. Let $h_\ell$ and $h_r$ be the left and right transversal maps, respectively. The left gyration $lgyr[b_1, b_2]$ of $B$ generated by $b_1, b_2 \in B$ is defined by
\[ lgyr[b_1, b_2]b = h_r(b_1, b_2)b h_r(b_1, b_2)^{-1}, \quad b \in B. \tag{3.2} \]

The right gyration $rgyr[b_1, b_2]$ of $B$ generated by $b_1, b_2 \in B$ is defined by
\[ rgyr[b_1, b_2]b = h_\ell(b_1, b_2)^{-1}b h_\ell(b_1, b_2), \quad b \in B. \tag{3.3} \]

**Remark.** In Definition 3.2, left gyrations are associated with the right transversal map $h_r$, and right gyrations are associated with the left transversal map $h_\ell$.

We use the convenient notation $x^h = h x h^{-1}$ and denote conjugation by $h$ by $\alpha_h$. That is, $\alpha_h(x) = x^h = h x h^{-1}$. With this notation, the left and right gyrations in Definition 3.2 read
\[ lgyr[a, b] = \alpha_{h_r(a, b)} \quad \text{and} \quad rgyr[a, b] = \alpha_{h_\ell(a, b)^{-1}}, \tag{3.4} \]
for all $a, b \in B$. Let $B$ be a non-empty subset of a group $\Gamma$. We say that a subgroup $H$ of $\Gamma$ normalizes $B$ if $h Bh^{-1} \subseteq B$ for all $h \in H$.

**Definition 3.3 (Bi-gyrotransversal).** A bi-transversal $B$ of subgroups $H_L$ and $H_R$ in a group $\Gamma$ is a bi-gyrotransversal if

1. $H_L$ and $H_R$ normalize $B$, and
2. $h_\ell h_r = h_r h_\ell$ for all $h_\ell \in H_L, h_r \in H_R$. 

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**Proposition 3.4.** If $B$ is a bi-gyrotransversal of subgroups $H_L$ and $H_R$ in a group $\Gamma$, then $H_LH_R$ is a subgroup of $\Gamma$ with normal subgroups $H_L$ and $H_R$. If $B$ contains the identity $1$ of $\Gamma$, then $H_L \cap H_R = \{1\}$. In this case, $H_LH_R$ is isomorphic to the direct product $H_L \times H_R$ as groups.

*Proof.* Since $H_LH_R = H_RH_L$, $H_LH_R$ forms a subgroup of $\Gamma$ by Proposition 14 of [5, Chapter 3]. If $g \in H_LH_R$, then $g = h_\ell h_r$ for some $h_\ell \in H_L$ and $h_r \in H_R$. For any $h \in H_L$, $h_\ell h = hh_\ell^{-1} \in H_L$. Hence, $gH_Lg^{-1} \subseteq H_L$. This proves $H_L \leq H_LH_R$. Similarly, $H_R \leq H_LH_R$.

Suppose that $1 \in B$ and that $h \in H_L \cap H_R$. The unique decomposition of $1$, $1 = hh^{-1} = h_1h^{-1}$, implies $h = 1$. Hence, $H_L \cap H_R = \{1\}$. It follows from Theorem 9 of [5, Chapter 5] that $H_LH_R \cong H_L \times H_R$ as groups. $\square$

**Theorem 3.5.** Let $B$ be a bi-gyrotransversal of subgroups $H_L$ and $H_R$ in a group $\Gamma$. If $h \in H_LH_R$, then conjugation by $h$ is an automorphism of $(B, \circ)$.

*Proof.* Note first that $H_LH_R$ normalizes $B$. In fact, if $h = h_\ell h_r$ with $h_\ell \in H_L$ and $h_r \in H_R$, then $hBh^{-1} = h_\ell(h_rBh^{-1}_r)h^{-1}_r \subseteq B$ for $H_R$ and $H_L$ normalize $B$.

Let $h \in H_LH_R$. Since $H_LH_R$ normalizes $B$, $\alpha_h$ is a bijection from $B$ to itself. Next, we will show that $(x \circ y)^h = x^h \circ y^h$ for all $x, y \in B$. Employing (3.1), we have

$$(xy)^h = (h_\ell(x, y)(x \circ y)h_r(x, y))^h = h_\ell(x, y)^h(x \circ y)^h h_r(x, y)^h.$$

Since $x^h, y^h \in B$, we also have

$$x^h y^h = h_\ell(x^h, y^h)(x^h \circ y^h) h_r(x^h, y^h).$$

Note that $h_\ell(x, y)^h \in H_L$ and $h_r(x, y)^h \in H_R$ because $H_L$ and $H_R$ are normal in $H_LH_R$. Thus, $(xy)^h = x^h y^h$ implies

$$h_\ell(x, y)^h = h_\ell(x^h, y^h), \quad (x \circ y)^h = x^h \circ y^h, \quad \text{and} \quad h_r(x, y)^h = h_r(x^h, y^h),$$

which completes the proof. $\square$

**Corollary 3.6.** Let $B$ be a bi-gyrotransversal of subgroups $H_L$ and $H_R$ in a group $\Gamma$. Then $lgyr[a, b]$ and $rgyr[a, b]$ are automorphisms of $(B, \circ)$ for all $a, b \in B$.

*Proof.* This is because $lgyr[a, b] = \alpha_{h_e(a, b)}$ and $rgyr[a, b] = \alpha_{h_e(a, b)^{-1}}$. $\square$

The next theorem provides us with commuting relations between conjugation automorphisms of the bi-transversal groupoid $(B, \circ)$ and its bi-gyrations.

**Theorem 3.7.** Let $B$ be a bi-gyrotransversal of subgroups $H_L$ and $H_R$ in a group $\Gamma$. The following commuting relations hold.
(1) \( \operatorname{lgyr}[a, b] \circ \operatorname{rgyr}[c, d] = \operatorname{rgyr}[c, d] \circ \operatorname{lgyr}[a, b] \) for all \( a, b, c, d \in B \).

(2) \( \alpha_h \circ \operatorname{lgyr}[a, b] = \operatorname{lgyr}[\alpha_h(a), \alpha_h(b)] \circ \alpha_h \) for all \( h \in H_L H_R \) and \( a, b \in B \).

(3) \( \alpha_h \circ \operatorname{rgyr}[a, b] = \operatorname{rgyr}[\alpha_h(a), \alpha_h(b)] \circ \alpha_h \) for all \( h \in H_L H_R \) and \( a, b \in B \).

**Proof.** Item (1) follows from the fact that \( h_{\ell} h_r = h_r h_{\ell} \) for all \( h_{\ell} \in H_L \) and \( h_r \in H_R \) and that \( \alpha_{gh} = \alpha_g \circ \alpha_h \) for all \( g, h \in \Gamma \).

Let \( h \in H_L H_R \) and let \( a, b \in B \). As in the proof of Theorem 3.5, \( h_r(a, b)^h = h_r(a^h, b^h) \). Hence, \( \alpha_h \circ \operatorname{lgyr}[a, b] \circ \alpha_h^{-1} = \operatorname{lgyr}[a^h, b^h] \) and Item (2) follows. Similarly, \( h_{\ell}(a, b)^h = h_{\ell}(a^h, b^h) \) implies Item (3). \( \square \)

As a consequence of Theorem 3.7, left gyrations are invariant under right gyrations, and vice versa. In fact, we have the following two theorems.

**Theorem 3.8.** Let \( B \) be a bi-gyrotransversal of subgroups \( H_L \) and \( H_R \) in a group \( \Gamma \). If \( \rho \) is a finite composition of right gyrations of \( B \), then

\[
\operatorname{lgyr}[a, b] = \operatorname{lgyr}[\rho(a), \rho(b)]
\]

for all \( a, b \in B \). If \( \lambda \) is a finite composition of left gyrations of \( B \), then

\[
\operatorname{rgyr}[a, b] = \operatorname{rgyr}[\lambda(a), \lambda(b)]
\]

for all \( a, b \in B \).

**Proof.** By assumption, \( \rho = \operatorname{rgyr}[a_1, b_1] \circ \operatorname{rgyr}[a_2, b_2] \circ \cdots \circ \operatorname{rgyr}[a_n, b_n] \) for some \( a_i, b_i \in B \). Since \( \operatorname{rgyr}[a_i, b_i] = \alpha_{h_i(a_i, b_i)}^{-1} \) for all \( i \), it follows that \( \rho = \alpha_h \), where \( h = h_{\ell}(a_1, b_1)^{-1} h_{\ell}(a_2, b_2)^{-1} \cdots h_{\ell}(a_n, b_n)^{-1} \). As \( \rho = \alpha_h \) and \( h \in H_L \), Theorem 3.7 (2) implies \( \rho \circ \operatorname{lgyr}[a, b] = \operatorname{lgyr}[\rho(a), \rho(b)] \circ \rho \). Since \( \rho \) and \( \operatorname{lgyr}[a, b] \) commute, we have (3.5). One obtains similarly that \( \lambda = \alpha_h \) for some \( h \in H_R \), which implies (3.6) by Theorem 3.7 (3). \( \square \)

**Theorem 3.9.** Let \( B \) be a bi-gyrotransversal of subgroups \( H_L \) and \( H_R \) in a group \( \Gamma \). If \( \rho \) is a finite composition of right gyrations of \( B \), then

\[
\rho \circ \operatorname{rgyr}[a, b] = \operatorname{rgyr}[\rho(a), \rho(b)] \circ \rho
\]

for all \( a, b \in B \). If \( \lambda \) is a finite composition of left gyrations of \( B \), then

\[
\lambda \circ \operatorname{lgyr}[a, b] = \operatorname{lgyr}[\lambda(a), \lambda(b)] \circ \lambda
\]

for all \( a, b \in B \).

**Proof.** As in the proof of Theorem 3.8, \( \rho = \alpha_h \) for some \( h \in H_L \). Hence, (3.7) is an application of Theorem 3.7 (3). Similarly, (3.8) is an application of Theorem 3.7 (2). \( \square \)
The associativity of $\Gamma$ is reflected in its bi-gyrotransversal decomposition $\Gamma = H_LBH_R$, as shown in the following theorem.

**Theorem 3.10.** Let $B$ be a bi-gyrotransversal of subgroups $H_L$ and $H_R$ in a group $\Gamma$. For all $a, b, c \in B$,

$$(a \circ b) \circ \lg yr[a, b|c] = \rg yr[b, c]|a \circ (b \circ c).$$

**Proof.** Let $a, b, c \in B$. Set $a_r = \rg yr[b, c]|a$ and $c_l = \lg yr[a, b]|c$. Then $a_r \in B$ and $c_l \in B$. By employing (3.1),

$$a(bc) = a(h_l(b, c)(b \circ c)h_r(b, c))$$

$$= h_l(b, c)(h_l(b, c)^{-1}ah_l(b, c))(b \circ c)h_r(b, c)$$

$$= h_l(b, c)a_r(b \circ c)h_r(b, c)$$

$$= [h_l(b, c)h_r(a_r, b \circ c)][a_r \circ (b \circ c)][h_r(a_r, b \circ c)h_r(b, c)]$$

and, similarly, $(ab)c = [h_l(a, b)h_l(a \circ b, c_l)][(a \circ b) \circ c_l][h_r(a \circ b, c_l)h_r(a, b)]$. Since $a(bc) = (ab)c$, it follows that $(a \circ b) \circ c_l = a_r \circ (b \circ c)$, which was to be proved. □

**Proposition 3.11.** Let $B$ be a bi-gyrotransversal of subgroups $H_L$ and $H_R$ in a group $\Gamma$. For all $a, b, c \in B$,

1. $\rg yr[\rg yr[b, c]|a, b \circ c] \circ \rg yr[b, c] = \rg yr[a \circ b, \lg yr[a, b]|c] \circ \rg yr[a, b]$, and
2. $\lg yr[a \circ b, \lg yr[a, b]|c] \circ \lg yr[a, b] = \lg yr[\rg yr[b, c]|a, b \circ c] \circ \lg yr[b, c]$.

**Proof.** As we have computed in the proof of Theorem 3.10,

$$h_l(b, c)h_l(a_r, b \circ c) = h_l(a, b)h_l(a \circ b, c_l),$$

where $a_r = \rg yr[b, c]|a$ and $c_l = \lg yr[a, b]|c$. Thus, Item (1) is obtained. Similarly, $h_r(a_r, b \circ c)h_r(b, c) = h_r(a \circ b, c_l)h_r(a, b)$ gives Item (2). □

**Twisted subgroups**

Twisted subgroups abound in group theory, gyrogroup theory, and loop theory, as evidenced, for instance, from [1–3, 6, 13, 14, 18]. Here, we demonstrate that a bi-gyrotransversal decomposition $\Gamma = H_LBH_R$ in which $B$ is a twisted subgroup gives rise to a highly structured type of bi-gyrogroupoids and, eventually, a bi-gyrogroup. We follow Aschbacher for the definition of a twisted subgroup.

**Definition 3.12 (Twisted subgroup).** A subset $B$ of a group $\Gamma$ is a **twisted subgroup** of $\Gamma$ if the following conditions hold:

1. $1 \in B$, $1$ being the identity of $\Gamma$;
2. if $b \in B$, then $b^{-1} \in B$;
3. if $a, b \in B$, then $aba \in B$. 

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Theorem 3.13. Let $B$ be a $bi$-gyrotransversal of subgroups $H_L$ and $H_R$ in a group $\Gamma$. If $B$ is a twisted subgroup of $\Gamma$, then the following properties are true for all $a, b \in B$.

(1) $1 \circ b = b \circ 1 = b$.

(2) $b^{-1} \in B$ and $b^{-1} \circ b = b \circ b^{-1} = 1$.

(3) $\text{lgyr}[1, b] = \text{lgyr}[b, 1] = \text{rgyr}[1, b] = \text{rgyr}[b, 1] = \text{id}_B$.

(4) $\text{lgyr}[b^{-1}, b] = \text{lgyr}[b, b^{-1}] = \text{rgyr}[b^{-1}, b] = \text{rgyr}[b, b^{-1}] = \text{id}_B$.

(5) $\text{lgyr}^{-1}[a, b] = \text{lgyr}[b^{-1}, a^{-1}]$ and $\text{rgyr}^{-1}[a, b] = \text{rgyr}[b^{-1}, a^{-1}]$.

(6) $(a \circ b)^{-1} = (\text{lgyr}[a, b] \circ \text{rgyr}[a, b])(b^{-1} \circ a^{-1})$.

Proof. (1) As $b = b(h_\ell(1, b) (1 \circ b) h_r(1, b))$, we have $h_\ell(1, b) = 1$, $1 \circ b = b$, and $h_r(1, b) = 1$. Similarly, $b = b1$ implies $b \circ 1 = b$. 

(2) Let $b \in B$. Since $B$ is a twisted subgroup, $b^{-1} \in B$. Further,

$$1 = b^{-1}b = h_\ell(b^{-1}, b)(b^{-1} \circ b)h_r(b^{-1}, b)$$

implies $h_\ell(b^{-1}, b) = 1$, $b^{-1} \circ b = 1$, and $h_r(b^{-1}, b) = 1$. Similarly, $bb^{-1} = 1$ implies $b \circ b^{-1} = 1$.

(3) We have $h_\ell(1, b) = h_\ell(b, 1) = h_r(1, b) = h_r(b, 1) = 1$, as computed in Item (1). Hence, Item (3) follows.

(4) We have $h_\ell(b^{-1}, b) = h_\ell(b, b^{-1}) = h_r(b^{-1}, b) = h_r(b, b^{-1}) = 1$, as computed in Item (2). Hence, Item (4) follows.

(5) Let $a, b \in B$. Then $a^{-1}, b^{-1} \in B$. On the one hand, we have

$$(ab)^{-1} = (h_\ell(a, b)(a \circ b)h_r(a, b))^{-1} = h_r(a, b)^{-1}(a \circ b)^{-1}h_\ell(a, b)^{-1},$$

and on the other hand we have $b^{-1}a^{-1} = h_\ell(b^{-1}, a^{-1})(b^{-1} \circ a^{-1})h_r(b^{-1}, a^{-1})$. Since $(ab)^{-1} = b^{-1}a^{-1}$, it follows that

$$(a \circ b)^{-1} = h_\ell(a, b)h_\ell(b^{-1}, a^{-1})(b^{-1} \circ a^{-1})h_r(b^{-1}, a^{-1})h_\ell(a, b)$$

$$= h_\ell(b^{-1}, a^{-1})h_r(a, b)(b^{-1} \circ a^{-1})h_\ell(a, b)h_r(b^{-1}, a^{-1}) \quad (3.9)$$

$$= h_\ell(b^{-1}, a^{-1})h_\ell(h_r(a, b)h_r(a, b)h_r(b^{-1}, a^{-1}),$$

where $\hat{b} = \text{lgyr}[a, b](\text{rgyr}[a, b](b^{-1} \circ a^{-1}))$. Because $(a \circ b)^{-1}$ and $\hat{b}$ belong to $B$, we have from the extreme sides of (3.9) that

$$h_r(a, b)h_r(b^{-1}, a^{-1}) = 1 \quad \text{and} \quad h_\ell(b^{-1}, a^{-1})h_\ell(a, b) = 1.$$ 

Hence, $h_r(a, b)^{-1} = h_r(b^{-1}, a^{-1})$, which implies $\text{lgyr}^{-1}[a, b] = \text{lgyr}[b^{-1}, a^{-1}]$.

Similarly, $h_\ell(a, b) = h_\ell(b^{-1}, a^{-1})^{-1}$ implies $\text{rgyr}^{-1}[a, b] = \text{rgyr}[b^{-1}, a^{-1}]$.

(6) As in Item (5), $(a \circ b)^{-1} = \hat{b} = \text{lgyr}[a, b](\text{rgyr}[a, b](b^{-1} \circ a^{-1}))$. \hfill $\square$
Remark. Note that we do not invoke the third defining property of a twisted subgroup in proving Theorem 3.13.

At this point, we have shown that any bi-gyrotransversal decomposition $\Gamma = H_LBH_R$ in which $B$ is a twisted subgroup of $\Gamma$ gives the bi-transversal groupoid $B$ that satisfies all the axioms of a bi-gyrogroupoid except for (BG4). In order to complete this, we have to impose additional conditions on the left and right transversal maps, as the following lemma indicates.

**Lemma 3.14.** If $B$ is a bi-transversal of subgroups $H_L$ and $H_R$ in a group $\Gamma$ such that $h_l(a, b)^{-1} = h_l(b, a)$ and $h_r(a, b)^{-1} = h_r(b, a)$ for all $a, b \in B$, then

\[
\text{lgyr}^{-1}\left[a, b\right] = \text{lgyr}\left[b, a\right] \quad \text{and} \quad \text{rgyr}^{-1}\left[a, b\right] = \text{rgyr}\left[b, a\right]
\]

for all $a, b \in B$.

**Proof.** Note first that $\alpha_{h_l^{-1}} = \alpha_{h_l}$ for all $h \in \Gamma$. From this we have $\text{lgyr}\left[b, a\right] = \alpha_{h_r(a, b)} = \alpha_{h_r(a, b)^{-1}} = \alpha_{h_r(a, b)} = \text{lgyr}^{-1}\left[a, b\right]$. One can prove in a similar way that $\text{rgyr}^{-1}\left[a, b\right] = \text{rgyr}\left[b, a\right]$. ∎

**Theorem 3.15.** Let $B$ be a bi-gyrotransversal of subgroups $H_L$ and $H_R$ in a group $\Gamma$. If $B$ is a twisted subgroup of $\Gamma$ such that $h_l(a, b)^{-1} = h_l(b, a)$ and $h_r(a, b)^{-1} = h_r(b, a)$ for all $a, b \in B$, then the following relations hold for all $a, b \in B$:

1. $\text{rgyr}\left[a, b\right] = \text{rgyr}\left[\text{lgyr}\left[a, b\right]a, a \circ b\right]$;
2. $\text{lgyr}\left[a, b\right] = \text{lgyr}\left[\text{lgyr}\left[a, b\right]a, a \circ b\right]$;
3. $\text{rgyr}\left[a, b\right] = \text{rgyr}\left[\text{rgyr}\left[a, b\right]a, b \circ a\right]$;
4. $\text{lgyr}\left[a, b\right] = \text{lgyr}\left[\text{rgyr}\left[a, b\right]a, b \circ a\right]$.

**Proof.** Let $a, b \in B$. Set $a_l = \text{lgyr}\left[a, b\right]a$. Employing (3.1), we obtain

\[
(ab)a = (h_l(a, b)(a \circ b)h_r(a, b))a
= h_l(a, b)(a \circ b)a(h_r(a, b)
= [h_l(a, b)h_r(a, b)]((a \circ b) \circ a_l)[h_r(a, b)h_r(a, b)]h_r(a, b)
\]

Since $(ab)a \in B$, the extreme sides of (3.10) imply

\[
h_l(a, b)h_l(a, b) = 1 \quad \text{and} \quad h_r(a, b)h_r(a, b) = 1.
\]

The first equation of (3.11) implies $h_l(a \circ b, \text{lgyr}\left[a, b\right]a) = h_l(a, b)^{-1}$. Hence, $\text{rgyr}^{-1}\left[a \circ b, \text{lgyr}\left[a, b\right]a\right] = \text{rgyr}\left[a, b\right]$. From Lemma 3.14, we have

\[
\text{rgyr}\left[a, b\right] = \text{rgyr}\left[\text{lgyr}\left[a, b\right]a, a \circ b\right].
\]

The second equation of (3.11) implies $h_r(a, b) = h_r(a \circ b, \text{lgyr}\left[a, b\right]a)^{-1}$. Hence, $\text{lgyr}\left[a, b\right] = \text{lgyr}\left[\text{lgyr}\left[a, b\right]a, a \circ b\right]$. This proves Items (1) and (2). Items (3) and (4) can be proved in a similar way by computing the product $a(ba)$. ∎
Theorem 3.16. Let $B$ be a bi-gyrotransversal of subgroups $H_L$ and $H_R$ in a group $\Gamma$. If $B$ is a twisted subgroup of $\Gamma$ such that $h_\ell(a, b)^{-1} = h_\ell(b, a)$ and $h_r(a, b)^{-1} = h_r(b, a)$ for all $a, b \in B$, then left and right gyrations of $B$ are even in the sense that

\[ \text{lgyr}[a^{-1}, b^{-1}] = \text{lgyr}[a, b] \quad \text{and} \quad \text{rgyr}[a^{-1}, b^{-1}] = \text{rgyr}[a, b] \]

for all $a, b \in B$.

Proof. This theorem follows directly from Theorem 3.13 (5) and Lemma 3.14. \qed

4 Bi-gyrodecomposition and bi-gyrogroups

Taking the key features of bi-gyrotransversal decomposition of a group given in Section 3, we formulate the definition of bi-gyrodecomposition and show that any bi-gyrodecomposition leads to a bi-gyrogroup, which in turn is a gyrogroup. Most of the results in Section 3 are directly translated into results in this section with appropriate modifications.

Definition 4.1 (Bi-gyrodecomposition). Let $\Gamma$ be a group, let $B$ be a subset of $\Gamma$, and let $H_L$ and $H_R$ be subgroups of $\Gamma$. A decomposition $\Gamma = H_LBH_R$ is a bi-gyrodecomposition if

1. $B$ is a bi-gyrotransversal of $H_L$ and $H_R$ in $\Gamma$;
2. $B$ is a twisted subgroup of $\Gamma$; and
3. $h_\ell(a, b)^{-1} = h_\ell(b, a)$ and $h_r(a, b)^{-1} = h_r(b, a)$ for all $a, b \in B$,

where $h_\ell$ and $h_r$ are the bi-transversal maps given below Definition 3.1.

Theorem 4.2. If $\Gamma = H_LBH_R$ is a bi-gyrodecomposition, then $B$ equipped with the bi-transversal operation forms a bi-gyrogroupoid.

Proof. Axiom (BG1) holds by Theorem 3.13 (1), where the identity 1 of $\Gamma$ acts as the identity of $B$. Axiom (BG2) holds by Theorem 3.13 (2), where $b^{-1}$ acts as a left inverse of $b \in B$ with respect to the bi-transversal operation. Axiom (BG3) holds by Corollary 3.6 and Theorem 3.10. Axiom (BG4) holds by Theorem 3.15. Axiom (BG5) holds by Theorem 3.13 (3). \qed

It is shown in Section 3 that any bi-transversal decomposition $\Gamma = H_LBH_R$ gives rise to a bi-transversal groupoid $(B, \circ)$. Theorem 4.2 asserts that in the special case when the decomposition is a bi-gyrodecomposition, the bi-transversal groupoid $(B, \circ)$ becomes the bi-gyrogroupoid $(B, \oplus_b)$ described in Definition 2.1. Hence, in particular, the binary operations $\oplus_b$ and $\circ$ share the same algebraic properties. Further, the identity of the bi-gyrogroupoid $B$ coincides with the group identity of $\Gamma$ and $\oplus_b b = b^{-1}$ for all $b \in B$. 

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Theorem 4.3 (Bi-gyration invariant relation). Let $\Gamma = H_L BH_R$ be a bi-gyrodecomposition. If $\rho$ is a finite composition of right gyrations of $B$, then

$$\text{lgyr}[a, b] = \text{lgyr}[\rho(a), \rho(b)]$$

for all $a, b \in B$. If $\lambda$ is a finite composition of left gyrations of $B$, then

$$\text{rgyr}[a, b] = \text{rgyr}[\lambda(a), \lambda(b)]$$

for all $a, b \in B$.

Proof. The theorem follows immediately from Theorem 3.8. \qed

Theorem 4.4 (Bi-gyration commuting relation). Let $\Gamma = H_L BH_R$ be a bi-gyrodecomposition. If $\rho$ is a finite composition of right gyrations of $B$, then

$$\rho \circ \text{rgyr}[a, b] = \text{rgyr}[\rho(a), \rho(b)] \circ \rho$$

for all $a, b \in B$. If $\lambda$ is a finite composition of left gyrations of $B$, then

$$\lambda \circ \text{lgyr}[a, b] = \text{lgyr}[\lambda(a), \lambda(b)] \circ \lambda$$

for all $a, b \in B$.

Proof. The theorem follows immediately from Theorem 3.9. \qed

Theorem 4.5 (Trivial bi-gyration). If $\Gamma = H_L BH_R$ is a bi-gyrodecomposition, then for all $a \in B$,

$$\begin{align*}
\text{lgyr}[0, a] &= \text{lgyr}[a, 0] = \text{id}_B \\
\text{lgyr}[a, \ominus a] &= \text{lgyr}[\ominus a, a] = \text{id}_B \\
\text{rgyr}[0, a] &= \text{rgyr}[a, 0] = \text{id}_B \\
\text{rgyr}[a, \ominus a] &= \text{rgyr}[\ominus a, a] = \text{id}_B \\
\text{lgyr}[a, a] &= \text{rgyr}[a, a] = \text{id}_B.
\end{align*}$$

(4.5)

Proof. The theorem follows from Theorem 2.4 (2) and Theorem 3.13 (3)–(4). \qed

Theorem 4.6 (Bi-gyration inversion law). If $\Gamma = H_L BH_R$ is a bi-gyrodecomposition, then

$$\begin{align*}
\text{lgyr}^{-1}[a, b] &= \text{lgyr}[b, a] \\
\text{rgyr}^{-1}[a, b] &= \text{rgyr}[b, a]
\end{align*}$$

for all $a, b \in B$.

Proof. The theorem follows immediately from Lemma 3.14. \qed
Theorem 4.7 (Even bi-gyration). If $\Gamma = H_LBH_R$ is a bi-gyrodecomposition, then left and right gyrations of $B$ are even:

$$\text{lgyr}[\otimes_b a, \otimes_b b] = \text{lgyr}[a, b] \quad \text{and} \quad \text{rgyr}[\otimes_b a, \otimes_b b] = \text{rgyr}[a, b]$$

for all $a, b \in B$.

Proof. The theorem follows immediately from Theorem 3.16.

Theorem 4.8 (Left and right cancellation laws). If $\Gamma = H_LBH_R$ is a bi-gyrodecomposition, then $B$ satisfies the left cancellation law

$$\otimes_b \text{rgyr}[a, b] a \oplus_b (a \oplus_b b) = b \quad (4.6)$$

and the right cancellation law

$$(a \oplus_b b) \otimes_b \text{lgyr}[a, b] b = a \quad (4.7)$$

for all $a, b \in B$.

Proof. The theorem follows from Theorem 2.7.

Theorem 4.9 (Left and right bi-gyroassociative laws). If $\Gamma = H_LBH_R$ is a bi-gyrodecomposition, then $B$ satisfies the left bi-gyroassociative law

$$a \oplus_b (b \oplus_b c) = (\text{rgyr}[c, b] a \oplus_b a) \oplus_b \text{lgyr}[\text{rgyr}[c, b] a, b] c \quad (4.8)$$

and the right bi-gyroassociative law

$$(a \oplus_b b) \oplus_b c = \text{rgyr}[b, \text{lgyr}[b, a] c] a \oplus_b (b \oplus_b \text{lgyr}[b, a] c) \quad (4.9)$$

for all $a, b, c \in B$.

Proof. The theorem follows from Theorems 2.2 and 4.6.

Theorem 4.10 (Left gyration reduction property). If $\Gamma = H_LBH_R$ is a bi-gyrodecomposition, then

$$\text{lgyr}[a, b] = \text{lgyr}[\text{rgyr}[b, a] a, b \oplus_b a] \quad (4.10)$$

and

$$\text{lgyr}[a, b] = \text{lgyr}[a \oplus_b b, \text{rgyr}[a, b] b] \quad (4.11)$$

for all $a, b \in B$.

Proof. Identity (4.10) follows from Theorem 3.15 (4). Identity (4.11) is obtained from (4.10) by applying the bi-gyration inversion law (Theorem 4.6) followed by interchanging $a$ and $b$. 

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Theorem 4.11 (Right gyration reduction property). If $\Gamma = H_LBH_R$ is a bi-gyrodecomposition, then
\[
\text{rgyr}[a, b] = \text{rgyr}[\text{rgyr}[a, b]a, a \oplus b] \tag{4.12}
\]
and
\[
\text{rgyr}[a, b] = \text{rgyr}[b \oplus a, \text{rgyr}[b, a]b] \tag{4.13}
\]
for all $a, b \in B$.

Proof. Identity (4.12) follows from Theorem 3.15 (1). Identity (4.13) is obtained from (4.12) by applying the bi-gyration inversion law followed by interchanging $a$ and $b$. \qed

Theorem 4.12 (Bi-gyration reduction property). If $\Gamma = H_LBH_R$ is a bi-gyrodecomposition, then
\[
\text{lgyr}[a, b] = \text{rgyr}[\text{lgyr}[a, b]a, a \oplus b] \tag{4.14}
\]
and
\[
\text{rgyr}[a, b] = \text{rgyr}[a \oplus b, \text{lgyr}[a, b]b] \tag{4.15}
\]
for all $a, b \in B$.

Proof. Identity (4.14) follows from Theorem 3.15 (2). Identity (4.15) is obtained from Theorem 3.15 (3) by applying the bi-gyration inversion law followed by interchanging $a$ and $b$. \qed

Theorem 4.13 (Left and right gyration reduction properties). If $\Gamma = H_LBH_R$ is a bi-gyrodecomposition, then
\[
\begin{align*}
\text{rgyr}[a, b] &= \text{rgyr}[\ominus b\text{rgyr}[a, b]b, a \oplus b] \tag{4.16} \\
\text{lgyr}[a, b] &= \text{lgyr}[\ominus b\text{lgyr}[a, b]b, a \oplus b]
\end{align*}
\]
and
\[
\begin{align*}
\text{rgyr}[a, b] &= \text{rgyr}[a \oplus b, \ominus b\text{rgyr}[a, b]a] \\
\text{lgyr}[a, b] &= \text{lgyr}[a \oplus b, \ominus b\text{rgyr}[a, b]a] \tag{4.17}
\end{align*}
\]
for all $a, b \in B$.

Proof. Setting $c = \ominus b$ in Proposition 3.11 (1)–(2) followed by using the bi-gyration inversion law gives (4.16). Setting $a = \ominus b$ in the same proposition followed by using the bi-gyration inversion law gives
\[
\begin{align*}
\text{rgyr}[b, c] &= \text{rgyr}[b \oplus c, \ominus b\text{rgyr}[b, c]b] \\
\text{lgyr}[b, c] &= \text{lgyr}[b \oplus c, \ominus b\text{rgyr}[b, c]b].
\end{align*}
\]
Replacing $b$ by $a$ and $c$ by $b$, we obtain (4.17). \qed
Theorem 4.14 (Left and right gyration reduction properties). If $\Gamma = H_LBH_R$ is a bi-gyrodecomposition, then
\[
\begin{align*}
\text{lgyr}[a, b] &= \text{lgyr}[	ext{rgyr}[b, a](a \oplus_b b), \ominus_b a] \\
\text{rgyr}[a, b] &= \text{rgyr}[	ext{rgyr}[b, a](a \oplus_b b), \ominus_b a]
\end{align*}
\] (4.18)
for all $a, b \in B$.

Proof. From the second equation of (4.17), we have
\[
\text{lgyr}[a, b] = \text{lgyr}[a \oplus_b b, \ominus_b \text{rgyr}[a, b]a].
\]
Applying Theorem 4.3 to the previous equation with $\rho = \text{rgyr}[b, a]$ gives
\[
\text{lgyr}[a, b] = \text{lgyr}[a \oplus_b b, \ominus_b \text{rgyr}[a, b]a] = \text{lgyr}[\text{rgyr}[b, a](a \oplus_b b), \ominus_b a].
\]
We obtain the last equation since $\text{rgyr}[b, a] = \text{rgyr}^{-1}[a, b]$.

Similarly, the first equation of (4.17) and Identity (4.3) together imply
\[
\begin{align*}
\text{id}_B &= \text{rgyr}^{-1}[a, b] \circ \text{rgyr}[a \oplus_b b, \ominus_b \text{rgyr}[a, b]a] \\
&= \text{rgyr}[a, b] \circ \text{rgyr}[a \oplus_b b, \ominus_b \text{rgyr}[a, b]a] \\
&= \text{rgyr}[\text{rgyr}[b, a](a \oplus_b b), \ominus_b a][\text{rgyr}[b, a](a \oplus_b b), \ominus_b a] \circ \text{rgyr}[b, a] \\
&= \text{rgyr}[\text{rgyr}[b, a](a \oplus_b b), \ominus_b a] \circ \text{rgyr}[b, a].
\end{align*}
\] (4.19)
The extreme sides of (4.19) imply $\text{rgyr}[a, b] = \text{rgyr}^{-1}[a, b]$.  

Bi-gyrogroups

We are now in a position to present the formal definition of a bi-gyrogroup.

Definition 4.15 (Bi-gyrogroup). Let $\Gamma = H_LBH_R$ be a bi-gyrodecomposition. The bi-gyrogroup operation $\oplus$ in $B$ is defined by
\[
a \oplus b = \text{rgyr}[b, a](a \oplus_b b), \quad a, b \in B.
\] (4.20)
Here, $\oplus_b$ is the bi-transversal operation induced by the decomposition $\Gamma = H_LBH_R$.
The groupoid $(B, \oplus)$ consisting of the set $B$ with the bi-gyrogroup operation $\oplus$ is called a bi-gyrogroup.

Throughout the remainder of this section, we assume that $\Gamma = H_LBH_R$ is a bi-gyrodecomposition and let $(B, \oplus)$ be the corresponding bi-gyrogroup.

Proposition 4.16. The unique two-sided identity element of $(B, \oplus)$ is 0. For each $a \in B$, $\ominus_b a$ is the unique two-sided inverse of $a$ in $(B, \oplus)$. 

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Proof. Let \( a \in B \). Since \( \text{rgyr}[a, 0] = \text{rgyr}[0, a] = \text{id}_B \), we have

\[
a \oplus 0 = \text{rgyr}[0, a](a \ominus_b 0) = a = \text{rgyr}[a, 0](0 \ominus_b a) = (0 \oplus a).
\]

Hence, 0 is a two-sided identity of \((B, \oplus)\). The uniqueness of 0 follows, as in the proof of Lemma 2.3.

Since \( \text{rgyr}[a, \ominus_b a] = \text{rgyr}[\ominus_b a, a] = \text{id}_B \), we have

\[
a \ominus (\ominus_b a) = \text{rgyr}[\ominus_b a, a](a \ominus_b a) = 0 = \text{rgyr}[a, \ominus_b a](\ominus_b a \ominus_b a) = (\ominus_b a) \oplus a.
\]

Hence, \( \ominus_b a \) acts as a two-sided inverse of \( a \) with respect to \( \oplus \). Suppose that \( b \) is a two-sided inverse of \( a \) with respect to \( \oplus \). Then \( 0 = a \ominus b = \text{rgyr}[b, a](a \ominus_b b) \), which implies \( a \ominus_b b = 0 \). Similarly, \( b \ominus a = 0 \) implies \( b \ominus_b a = 0 \). This proves that \( b \) is a two-sided inverse of \( a \) with respect to \( \ominus_b \). Hence, \( b = \ominus_b a \) by Theorem 2.5.

Following Proposition 4.16, if \( a \) is an element of \( B \), then the unique two-sided inverse of \( a \) with respect to \( \ominus \) will be denoted by \( \ominus a \). Further,

\[
\ominus a = \ominus_b a
\]

for all \( a \in B \). We also write \( a \ominus b \) instead of \( a \ominus (\ominus b) \). The following theorem asserts that left and right gyrations of the bi-transversal groupoid \((B, \ominus_b)\) ascend to automorphisms of the bi-gyrogroup \((B, \oplus)\).

**Theorem 4.17.** If \( \lambda \) is a finite composition of left gyrations of \((B, \ominus_b)\), then

\[
\lambda(a \ominus b) = \lambda(a) \ominus \lambda(b)
\]

for all \( a, b \in B \). If \( \rho \) is a finite composition of right gyrations of \((B, \ominus_b)\), then

\[
\rho(a \ominus b) = \rho(a) \ominus \rho(b)
\]

for all \( a, b \in B \).

**Proof.** Let \( a, b \in B \). By Theorem 3.7 (1), \( \lambda \) and \( \text{rgyr}[b, a] \) commute. Hence,

\[
\lambda(a \ominus b) = (\lambda \circ \text{rgyr}[b, a])(a \ominus_b b) = (\text{rgyr}[b, a] \circ \lambda)(a \ominus_b b) = \text{rgyr}[\lambda(b), \lambda(a)](\lambda(a) \ominus_b \lambda(b)) = \lambda(a) \ominus \lambda(b).
\]

We have the third equation since \( \lambda \) is a finite composition of left gyrations; the forth equation from (4.2); and the last equation from Definition 4.15. Similarly, (4.22) is obtained from (4.3). \( \square \)
Lemma 4.18. In the bi-gyrogroup $B$,
\[ \text{rgyr}[c, a \oplus b] \circ \text{rgyr}[b, a] = \text{rgyr}[b \oplus c, a] \circ \text{rgyr}[c, b] \]
for all $a, b, c \in B$.

Proof. By Theorem 4.6 and Proposition 3.11 (1),
\[
\text{rgyr}[b, a] \circ \text{rgyr} [\text{lgyr}[a, b]c, a \oplus b] = (\text{rgyr}[a \oplus b, \text{lgyr}[a, b]c] \circ \text{rgyr}[a, b])^{-1} = (\text{rgyr}[\text{rgyr}[b, c]a, b \oplus b] \circ \text{rgyr}[b, c])^{-1} = \text{rgyr}[c, b] \circ \text{rgyr}[b \oplus b c, \text{rgyr}[b, c]a].
\]
By Identity (4.3) and Theorem 4.6, the extreme sides of (4.23) imply
\[ \text{rgyr}[c, \text{rgyr}[b, a](a \oplus b)] \circ \text{rgyr}[b, a] = \text{rgyr} [\text{rgyr}[c, b](b \oplus b c), a] \circ \text{rgyr}[c, b]. \]
By Definition 4.15, the previous equation reads
\[ \text{rgyr}[c, a \oplus b] \circ \text{rgyr}[b, a] = \text{rgyr}[b \oplus c, a] \circ \text{rgyr}[c, b], \]
which completes the proof.

Theorem 4.19 (Bi-gyroassociative law in bi-gyrogroups). The bi-gyrogroup $B$ satisfies the left bi-gyroassociative law
\[ a \oplus (b \oplus c) = (a \oplus b) \oplus (\text{lgyr}[a, b] \circ \text{rgyr}[b, a])(c) \]
and the right bi-gyroassociative law
\[ (a \oplus b) \oplus c = a \oplus (b \oplus (\text{lgyr}[b, a] \circ \text{rgyr}[a, b])(c)) \]
for all $a, b, c \in B$.

Proof. From Theorem 3.10, we have
\[ (a \oplus b) \oplus b \text{lgyr}[a, b]c = \text{rgyr}[b, c]a \oplus b (b \oplus b c). \]
Applying $\text{rgyr}[c, b]$ followed by applying $\text{rgyr}[b + c, a]$ to the previous equation gives
\[ (\text{rgyr}[b \oplus c, a] \circ \text{rgyr}[c, b])(a \oplus b) \oplus b \text{lgyr}[a, b]c = a \oplus (b \oplus c). \]
On the other hand, we compute
\[
(a \oplus b) \oplus (\text{lgyr}[a, b] \circ \text{rgyr}[b, a])(c)
= (a \oplus b) \oplus (\text{rgyr}[b, a] \circ \text{lgyr}[a, b])(c)
= [\text{rgyr}[b, a](a \oplus b) \oplus [\text{rgyr}[b, a] \circ \text{lgyr}[a, b]c]
= \text{rgyr}[b, a](a \oplus b) \oplus \text{lgyr}[a, b]c
= (\text{rgyr}[b, a] \circ \text{lgyr}[a, b]c, a \oplus b) \oplus (a \oplus b) \oplus b \text{lgyr}[a, b]c
= (\text{rgyr}[c, \text{rgyr}[b, a](a \oplus b)] \circ \text{rgyr}[b, a])(a \oplus b) \oplus b \text{lgyr}[a, b]c
= (\text{rgyr}[c, a \oplus b] \circ \text{rgyr}[b, a])(a \oplus b) \oplus b \text{lgyr}[a, b]c.
\]
We obtain the first equation from Theorem 3.7 (1); the third equation from (4.22); the fifth equation from Identity (4.3) and Theorem 4.6.

By the lemma, \( \text{rgyr}[b \oplus c, a] \circ \text{rgyr}[c, b] = \text{rgyr}[c, a \oplus b] \circ \text{rgyr}[b, a] \). Hence, (4.26) and (4.27) together imply \( a \oplus (b \oplus c) = (a \oplus b) \oplus (\text{rgyr}[a, b] \circ \text{rgyr}[b, a])(c) \).

Replacing \( c \) by \( (\text{rgyr}[b, a] \circ \text{rgyr}[a, b])(c) \) in (4.24) followed by commuting \( \text{rgyr}[b, a] \) and \( \text{rgyr}[a, b] \) gives (4.25).

**Theorem 4.20 (Left gyration reduction property of bi-gyrogroups).** The bi-gyrogroup \( B \) has the left gyration left reduction property

\[ \text{lgyr}[a, b] = \text{lgyr}[a \oplus b, b] \] (4.28)

and the left gyration right reduction property

\[ \text{lgyr}[a, b] = \text{lgyr}[a, b \oplus a] \] (4.29)

for all \( a, b \in B \).

**Proof.** From (4.11), (4.1) with \( \rho = \text{rgyr}[b, a] \), and Theorem 4.6, we have the following series of equations

\[
\text{lgyr}[a, b] = \text{lgyr}[a \oplus b, \text{rgyr}[a, b]b] \\
= \text{lgyr}[\text{rgyr}[a, b],(a \oplus b), \text{rgyr}[b, a](\text{rgyr}[a, b]b)] \\
= \text{lgyr}[a \oplus b, b],
\]

thus proving (4.28). One obtains similarly that

\[
\text{lgyr}[a, b] = \text{lgyr}[\text{rgyr}[b, a], b \oplus a] \\
= \text{lgyr}[\text{rgyr}[a, b](\text{rgyr}[b, a]a), \text{rgyr}[a, b]b, \text{rgyr}[b, a]a] \\
= \text{lgyr}[a, b \oplus a].
\]

**Theorem 4.21 (Right gyration reduction property of bi-gyrogroups).** The bi-gyrogroup \( B \) satisfies the right gyration left reduction property

\[ \text{rgyr}[a, b] = \text{rgyr}[a \oplus b, b] \] (4.30)

and the right gyration right reduction property

\[ \text{rgyr}[a, b] = \text{rgyr}[a, b \oplus a] \] (4.31)

for all \( a, b \in B \).
Proof. From (4.15), (4.3) with \( \rho = \rgyr[b,a] \), and Theorem 4.6, we have the following series of equations

\[
id_B = \rgyr[b,a] \circ \rgyr[\oplus b, \rgyr[a,b]] = \rgyr[\rgyr[b,a](\oplus b), \rgyr[\rgyr[a,b]]] \circ \rgyr[b,a] \quad (4.32)
\]

Hence, the extreme sides of (4.32) imply \( \rgyr[a,b] = \rgyr[\oplus b, b] \). Applying the bi-gyration inversion law to (4.30) followed by interchanging \( a \) and \( b \) gives (4.31).

Let \((B, \oplus)\) be the corresponding bi-gyrogroup of a bi-gyrodecomposition \( \Gamma = H_LBH_R \). By Theorem 4.17, left and right gyrations of \((B, \oplus)\) preserve the bi-gyrogroup operation. This result and Theorem 4.19 motivate the following definition.

**Definition 4.22 (Gyration of bi-gyrogroups).** Let \( \Gamma = H_LBH_R \) be a bi-gyrodecomposition and let \((B, \oplus)\) be the corresponding bi-gyrogroup. The gyrorator is the map

\[\text{gyr}: B \times B \rightarrow \text{Aut}(B, \oplus)\]

defined by

\[\text{gyr}[a,b] = \text{rgyr}[a,b] \circ \text{rgyr}[b,a] \quad (4.33)\]

for all \( a, b \in B \).

**Theorem 4.23.** For all \( a, b \in B \), \( \text{gyr}[a,b] \) is an automorphism of the bi-gyrogroup \( B \).

**Proof.** The theorem follows from Theorem 4.17.

**Theorem 4.24 (Gyroassociative law in bi-gyrogroups).** The bi-gyrogroup \( B \) satisfies the left gyroassociative law

\[a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a,b]c \quad (4.34)\]

and the right gyroassociative law

\[(a \oplus b) \oplus c = a \oplus (b \oplus \text{gyr}[b,a]c) \quad (4.35)\]

for all \( a, b, c \in B \).

**Proof.** The theorem follows directly from Theorem 4.19 and Definition 4.22.

**Theorem 4.25 (Gyration reduction property in bi-gyrogroups).** The bi-gyrogroup \( B \) has the left reduction property

\[\text{gyr}[a,b] = \text{gyr}[a \oplus b, b] \quad (4.36)\]
and the right reduction property

\[ \text{gyr}[a, b] = \text{gyr}[a, b \oplus a] \] (4.37)

for all \( a, b \in B \).

Proof. From (4.28) and (4.31), we have the following series of equations

\[
\begin{align*}
\text{gyr}[a \oplus b, b] &= \text{lgyr}[a \oplus b, b] \circ \text{rgyr}[b, a \oplus b] \\
&= \text{lgyr}[a, b] \circ \text{rgyr}[b, a] \\
&= \text{gyr}[a, b],
\end{align*}
\]

thus proving (4.36). Similarly, (4.29) and (4.30) together imply (4.37). \( \square \)

Theorems 4.24 and 4.25 indicate that any bi-gyrogroup is indeed a gyrogroup. Therefore, we recall the following definition of a gyrogroup.

Definition 4.26 (Gyrogroup, [29]). A groupoid \((G, \oplus)\) is a gyrogroup if its binary operation satisfies the following axioms.

(G1) There is an element \(0 \in G\) such that \(0 \oplus a = a\) for all \(a \in G\).

(G2) For each \(a \in G\), there is an element \(b \in G\) such that \(b \oplus a = 0\).

(G3) For all \(a, b \in G\), there is an automorphism \(\text{gyr}[a, b] \in \text{Aut}(G, \oplus)\) such that

\[ a \oplus (b \oplus c) = (a \oplus b) \oplus \text{gyr}[a, b]c \]

for all \(c \in G\).

(G4) For all \(a, b \in G\), \(\text{gyr}[a, b] = \text{gyr}[a \oplus b, b]\).

Definition 4.27 (Gyrocommutative gyrogroup, [29]). A gyrogroup \((G, \oplus)\) is gyrocommutative if it satisfies the gyrocommutative law

\[ a \oplus b = \text{gyr}[a, b](b \oplus a) \]

for all \(a, b \in G\).

Theorem 4.28. Let \(\Gamma = H_LBH_R\) be a bi-gyrodecomposition and let \((B, \oplus)\) be the corresponding bi-gyrogroup. Then \(B\) equipped with the bi-gyrogroup operation forms a gyrogroup.

Proof. Axioms (G1) and (G2) are validated in Proposition 4.16. Axiom (G3) is validated in Theorems 4.23 and 4.24. Axiom (G4) is validated in Theorem 4.25. \( \square \)

Definition 4.29. A bi-gyrodecomposition \(\Gamma = H_LBH_R\) is bi-gyrocommutative if its bi-transversal groupoid is bi-gyrocommutative in the sense of Definition 2.8.
Theorem 4.30. If $\Gamma = H_LBH_R$ is a bi-gyrocommutative bi-gyrodecomposition, then $B$ equipped with the bi-gyrogroup operation is a gyrocommutative gyrogroup.

Proof. Let $a, b \in B$. We compute

$$a \circ b = \text{rgyr}[b, a](a \circ b)$$
$$= \text{rgyr}[b, a](\text{lgyr}[a, b] \circ \text{rgyr}[a, b](b \circ a))$$
$$= (\text{lgyr}[a, b] \circ \text{rgyr}[b, a])(\text{rgyr}[a, b](b \circ a))$$
$$= \text{gyr}[a, b](b \circ a),$$

thus proving $B$ satisfies the gyrocommutative law.

We close this section by proving that having a bi-gyrodecomposition is an invariant property of groups.

Theorem 4.31. Let $\Gamma_1$ and $\Gamma_2$ be isomorphic groups via an isomorphism $\phi$. If $\Gamma_1 = H_LBH_R$ is a bi-gyrodecomposition, then so is $\Gamma_2 = \phi(H_L)\phi(B)\phi(H_R)$.

Proof. The proof of this theorem is straightforward, using the fact that $\phi$ is a group isomorphism from $\Gamma_1$ to $\Gamma_2$.

Theorem 4.32. Let $\Gamma_1$ and $\Gamma_2$ be isomorphic groups via an isomorphism $\phi$. If $\Gamma_1 = H_LBH_R$ is a bi-gyrocommutative bi-gyrodecomposition, then so is $\Gamma_2 = \phi(H_L)\phi(B)\phi(H_R)$.

Proof. This theorem follows from the fact that

$$\text{rgyr}[\phi(b_1), \phi(b_2)]\phi(b) = \phi(\text{rgyr}[b_1, b_2]b)$$
$$\text{lgyr}[\phi(b_1), \phi(b_2)]\phi(b) = \phi(\text{lgyr}[b_1, b_2]b)$$

for all $b_1, b_2 \in B$.

Theorem 4.33. Let $\Gamma_1$ and $\Gamma_2$ be isomorphic groups via an isomorphism $\phi$ and let $\Gamma_1 = H_LBH_R$ be a bi-gyrodecomposition. Then the bi-gyrogroups $B$ and $\phi(B)$ are isomorphic as gyrogroups via $\phi$.

Proof. By Theorem 4.28, $B$ forms a gyrogroup whose gyrogroup operation is given by $a \oplus b = \text{rgyr}[b, a](a \odot_1 b)$ for all $a, b \in B$, and $\phi(B)$ forms a gyrogroup whose gyrogroup operation is given by $c \oplus d = \text{rgyr}[d, c](c \odot_2 d)$ for all $c, d \in \phi(B)$. Let $a, b \in B$. We compute

$$\phi(a \oplus b) = \phi(\text{rgyr}[b, a](a \odot_1 b))$$
$$= \text{rgyr}[\phi(b), \phi(a)]\phi(a \odot_1 b)$$
$$= \text{rgyr}[\phi(b), \phi(a)](\phi(a) \odot_2 \phi(b))$$
$$= \phi(a) \oplus \phi(b).$$

Hence, the restriction of $\phi$ to $B$ acts as a gyrogroup isomorphism from $B$ to $\phi(B)$.
5 Special pseudo-orthogonal groups

In this section, we provide a concrete realization of a bi-gyrocommutative bi-gyrodecomposition.

A pseudo-Euclidean space \( \mathbb{R}^{m,n} \) of signature \((m, n)\), \( m, n \in \mathbb{N} \), is an \((m + n)\)-dimensional linear space with the pseudo-Euclidean inner product of signature \((m, n)\). The special pseudo-orthogonal group, denoted by \( \text{SO}(m, n) \), consists of all the Lorentz transformations of order \((m, n)\) that leave the pseudo-Euclidean inner product invariant and that can be reached continuously from the identity transformation in \( \mathbb{R}^{m,n} \). Denote by \( \text{SO}(m) \) the group of \( m \times m \) special orthogonal matrices and by \( \text{SO}(n) \) the group of \( n \times n \) special orthogonal matrices.

Following [34], \( \text{SO}(m) \) and \( \text{SO}(n) \) can be embedded into \( \text{SO}(m, n) \) as subgroups by defining

\[
\rho: O_m \mapsto \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix}, \quad O_m \in \text{SO}(m),
\]

\[
\lambda: O_n \mapsto \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix}, \quad O_n \in \text{SO}(n).
\]

Let \( \beta \) be the map defined on the space \( \mathbb{R}^{n\times m} \) of all \( n \times m \) real matrices by

\[
\beta: P \mapsto \begin{pmatrix} \sqrt{I_m + P^tP} \\ P \end{pmatrix}, \quad P \in \mathbb{R}^{n\times m}.
\]

It is easy to see that \( \beta \) is a bijection from \( \mathbb{R}^{n\times m} \) to \( \beta(\mathbb{R}^{n\times m}) \).

Note that

\[
\rho(\text{SO}(m)) = \left\{ \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} : O_m \in \text{SO}(m) \right\}
\]

\[
\lambda(\text{SO}(n)) = \left\{ \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} : O_n \in \text{SO}(n) \right\}
\]

\[
\beta(\mathbb{R}^{n\times m}) = \left\{ \begin{pmatrix} \sqrt{I_m + P^tP} \\ P \end{pmatrix} : P \in \mathbb{R}^{n\times m} \right\}.
\]

It follows from Examples 22 and 23 of [34] that \( \lambda(\text{SO}(n)) \) and \( \rho(\text{SO}(m)) \) are subgroups of \( \text{SO}(m, n) \). Further, \( \text{SO}(m) \) and \( \rho(\text{SO}(m)) \) are isomorphic as groups via \( \rho \), and \( \text{SO}(n) \) and \( \lambda(\text{SO}(n)) \) are isomorphic as groups via \( \lambda \).

We will see shortly that

\[
\text{SO}(m, n) = \rho(\text{SO}(m))\beta(\mathbb{R}^{n\times m})\lambda(\text{SO}(n))
\]

is a bi-gyrocommutative bi-gyrodecomposition.

By Theorem 8 of [34], \( \beta(\mathbb{R}^{n\times m}) \) is a bi-transversal of subgroups \( \rho(\text{SO}(m)) \) and \( \lambda(\text{SO}(n)) \) in the pseudo-orthogonal group \( \text{SO}(m, n) \). From Lemma 6 of [34], we
have

\[ \rho(O_m) \beta(P) \rho(O_m)^{-1} = \beta(PO_m^{-1}) \]
\[ \lambda(O_n) \beta(P) \lambda(O_n)^{-1} = \beta(O_n P) \]

for all \( O_m \in \text{SO}(m) \), \( O_n \in \text{SO}(n) \), and \( P \in \mathbb{R}^{n \times m} \). Hence, \( \rho(\text{SO}(m)) \) and \( \lambda(\text{SO}(n)) \) normalize \( \beta(\mathbb{R}^{n \times m}) \). Setting \( P = 0_{n,m} \) in the third identity of (77) of [34], we have

\[ \lambda(O_n) \rho(O_m) = \rho(O_m) \lambda(O_n) \]

for all \( O_m \in \text{SO}(m), O_n \in \text{SO}(n) \) because \( \beta(P) = \beta(0_{n,m}) = I_{m+n} \). Thus, \( \beta(\mathbb{R}^{n \times m}) \) is a bi-gyrotransversal of \( \rho(\text{SO}(m)) \) and \( \lambda(\text{SO}(n)) \) in \( \text{SO}(m,n) \).

In Theorem 13 of [34], the bi-gyroaddition, \( \oplus_U \), and bi-gyrations in the parameter bi-gyrogroupoid \( \mathbb{R}^{n \times m} \) are given by

\[ P_1 \oplus_U P_2 = P_1 \sqrt{I_m + P_2^2 P_2} + \sqrt{I_n + P_1 P_1^T} \]
\[ \text{lgyr}[P_1, P_2] = \sqrt{I_n + P_1 P_1^T} \{ P_1 P_2^T + \sqrt{I_n + P_1 P_1^T} \sqrt{I_m + P_2^T P_2} \}^{-1} \]
\[ \text{rgyr}[P_1, P_2] = \{ P_1 P_2 + \sqrt{I_m + P_1^T P_1 \sqrt{I_n + P_2 P_2}} \} \sqrt{I_m + P_1 P_2}^{-1} \]

for all \( P_1, P_2 \in \mathbb{R}^{n \times m} \) and \( P_{1,2} = P_1 \oplus_U P_2 \).

From (74) of [34], we have \( I_{m+n} = B(0_{n,m}) \in \beta(\mathbb{R}^{n \times m}) \). From Theorem 10 of [34], we have \( \beta(P)^{-1} = \beta(-P) \) \( \in \beta(\mathbb{R}^{n \times m}) \) for all \( P \in \mathbb{R}^{n \times m} \). From Equations (179) and (184) of [34], we have

\[ \beta(P_1) \beta(P_2) \beta(P_1) = \beta((P_1 \oplus_U P_2) \oplus_U \text{lgyr}[P_1, P_2]) \]

Hence, \( \beta(P_1) \beta(P_2) \beta(P_1) \in \beta(\mathbb{R}^{n \times m}) \) for all \( P_1, P_2 \in \mathbb{R}^{n \times m} \). This proves that \( \beta(\mathbb{R}^{n \times m}) \) is a twisted subgroup of \( \text{SO}(m,n) \).

By (104) of [34],

\[ \beta(P_1) \beta(P_2) = \rho(\text{rgyr}[P_1, P_2]) \beta(P_1 \oplus_U P_2) \lambda(\text{lgyr}[P_1, P_2]) \]  \hspace{1cm} (5.4)

for all \( P_1, P_2 \in \mathbb{R}^{n \times m} \). Hence, the left and right transversal maps induced by the decomposition \( \text{SO}(m,n) = \rho(\text{SO}(m)) \beta(\mathbb{R}^{n \times m}) \lambda(\text{SO}(n)) \) are given by

\[ h_\ell(\beta(P_1), \beta(P_2)) = \rho(\text{rgyr}[P_1, P_2]) \]  \hspace{1cm} (5.5)

and

\[ h_r(\beta(P_1), \beta(P_2)) = \lambda(\text{lgyr}[P_1, P_2]) \]  \hspace{1cm} (5.6)

for all \( P_1, P_2 \in \mathbb{R}^{n \times m} \).

By (162b) of [34], \( \text{rgyr}^{-1}[P_1, P_2] = \text{rgyr}[P_2, P_1] \). Hence,

\[ h_\ell(\beta(P_1), \beta(P_2))^{-1} = \rho(\text{rgyr}^{-1}[P_1, P_2]) = \rho(\text{rgyr}[P_2, P_1]) = h_\ell(\beta(P_2), \beta(P_1)) \]

Similarly, (162a) of [34] implies \( h_r(\beta(P_1), \beta(P_2))^{-1} = h_r(\beta(P_2), \beta(P_1)) \). Combining these results gives
Theorem 5.1. The decomposition

\[ \text{SO}(m, n) = \rho(\text{SO}(m)) \beta(\mathbb{R}^{n \times m}) \lambda(\text{SO}(n)) \]  

is a bi-gyrodecomposition.

By (5.4), the bi-transversal operation induced by the decomposition (5.7) is given by

\[ \beta(P_1) \oplus_b \beta(P_2) = \beta(P_1 \oplus_U P_2) \]  

for all \( P_1, P_2 \in \mathbb{R}^{n \times m} \).

Note that \( \text{rgyr}[P_1, P_2] \) is an \( m \times m \) matrix and \( \text{lgyr}[P_1, P_2] \) is an \( n \times n \) matrix, while \( \text{rgyr}[\beta(P_1), \beta(P_2)] \) and \( \text{lgyr}[\beta(P_1), \beta(P_2)] \) are maps. By (3.4), the action of left and right gyrations on \( \beta(\mathbb{R}^{n \times m}) \) is given by

\[ \text{lgyr}[^{\beta(P_1)} \beta(P_2)] \beta(P) = \beta[^{\beta(P)} \beta(P_1) \beta(P_2)] \]  

and

\[ \text{rgyr}[\beta(P_1), \beta(P_2)] \beta(P) = \beta[^{\beta(P)} \beta(P_1) \beta(P_2)] \]  

for all \( P_1, P_2, P \in \mathbb{R}^{n \times m} \). Using (5.9) and (5.10), together with Theorem 25 of [34], we have

Theorem 5.2. The bi-gyrodecomposition

\[ \text{SO}(m, n) = \rho(\text{SO}(m)) \beta(\mathbb{R}^{n \times m}) \lambda(\text{SO}(n)) \]

is bi-gyrocommutative.

By Theorem 52 of [34], the space \( \mathbb{R}^{n \times m} \) of all \( n \times m \) real matrices forms a gyrocommutative gyrogroup under the operation \( \oplus_U \) given by

\[ P_1 \oplus_U P_2 = (P_1 \oplus_U P_2) \text{rgyr}[P_2, P_1], \quad P_1, P_2 \in \mathbb{R}^{n \times m}. \]  

(5.11)

Theorem 5.3. The set

\[ \beta(\mathbb{R}^{n \times m}) = \left\{ \left( \begin{array}{c} \sqrt{I_m + PP^T} \\ \sqrt{P^T \mathbb{R}^{n \times m}} \end{array} \right) : P \in \mathbb{R}^{n \times m} \right\} \]

together with the bi-gyrogroup operation \( \oplus \) given by

\[ \beta(P_1) \oplus \beta(P_2) = \beta((P_1 \oplus_U P_2) \text{rgyr}[P_2, P_1]) \]

is a gyrocommutative gyrogroup isomorphic to \( (\mathbb{R}^{n \times m}, \oplus_U) \).

Proof. The theorem follows from Theorems 5.1, 5.2, 4.28, and 4.30. Further, the bi-gyrogroup operation \( \oplus \) is given by

\[ \beta(P_1) \oplus \beta(P_2) = \text{rgyr}[\beta(P_2), \beta(P_1)] \beta(P_1 \oplus_U P_2) \]

\[ = \text{rgyr}[\beta(P_2), \beta(P_1)] \beta(P_1 \oplus_U P_2) \]

\[ = \beta((P_1 \oplus_U P_2) \text{rgyr}[P_2, P_1]). \]

From (5.11), we have \( \beta(P_1) \oplus \beta(P_2) = \beta(P_1 \oplus_U P_2) \). Hence, \( \beta \) acts as a gyrogroup isomorphism from \( \mathbb{R}^{n \times m} \) to \( \beta(\mathbb{R}^{n \times m}) \).
6 Spin groups

We establish that the spin group of the Clifford algebra of pseudo-Euclidean space \( \mathbb{R}^{m,n} \) of signature \((m, n)\) has a bi-gyrocommutative bi-gyrodecomposition. For the basic notion of Clifford algebras, the reader is referred to [15, 16, 19, 21].

Let \((V, B)\) be a real quadratic space. That is, \(V\) is a linear space over \(\mathbb{R}\) together with a non-degenerate symmetric bilinear form \(B\). Let \(Q\) be the associated quadratic form given by \(Q(v) = B(v, v)\) for \(v \in V\). Denote by \(\text{Cl}(V, Q)\) the Clifford algebra of \((V, B)\). Define

\[
\Gamma(V, Q) = \{ g \in \text{Cl}(V, Q) : \forall v \in V, \hat{g}vg^{-1} \in V \}. \tag{6.1}
\]

Here, \(\hat{\cdot}\) stands for the unique involutive automorphism of \(\text{Cl}(V, Q)\) such that \(\hat{v} = -v\) for all \(v \in V\), known as the grade involution. If \(V\) is finite dimensional, then \(\Gamma(V, Q)\) is indeed a subgroup of the group of units of \(\text{Cl}(V, Q)\), called the Clifford group of \(\text{Cl}(V, Q)\). In this case, any element \(g\) of \(\Gamma(V, Q)\) induces the linear automorphism \(T_g\) of \(V\) given by

\[
T_g(v) = \hat{g}vg^{-1}, \quad v \in V. \tag{6.2}
\]

Since \(T_g \circ T_h = T_{gh}\) for all \(g, h \in \Gamma(V, Q)\), the map \(\pi : g \mapsto T_g\) defines a group homomorphism from \(\Gamma(V, Q)\) to the general linear group GL\((V)\), known as the twisted adjoint representation of \(\Gamma(V, Q)\). The kernel of \(\pi\) equals \(\mathbb{R}^1 := \{ \lambda : \lambda \in \mathbb{R}, \lambda \neq 0 \}\). By the Cartan-Dieudonné theorem, \(\pi\) maps \(\Gamma(V, Q)\) onto the orthogonal group \(O(V, Q)\).

Recall that, in the Clifford algebra \(\text{Cl}(V, Q)\), we have \(v^2 = Q(v)1\) for all \(v \in V\). Hence, if \(v \in V\) and \(Q(v) \neq 0\), then \(v\) is invertible whose inverse is \(v/Q(v)\). Further, we have an important identity \(uv + vu = 2B(u, v)1\) for all \(u, v \in V\). Using this identity, we obtain

\[
-vuv^{-1} = u - (uv + vu)v^{-1} = u - (2B(u, v)1) \left( \frac{v}{Q(v)} \right) = u - \frac{2B(u, v)}{Q(v)} v, \]

which implies \(\hat{v}uv^{-1} = -vuv^{-1} \in V\) for all \(u \in V\). Hence, if \(v \in V\) and \(Q(v) \neq 0\), then \(v \in \Gamma(V, Q)\). In fact, \(T_v\) is the reflection about the hyperplane orthogonal to \(v\).

We also have the following important subgroup of the Clifford group of \(\text{Cl}(V, Q)\):

\[
\text{Spin}(V, Q) = \{ v_1v_2 \cdots v_r : r \text{ is even}, v_i \in V, \text{ and } Q(v_i) = \pm 1 \}, \tag{6.3}
\]

known as the spin group of \(\text{Cl}(V, Q)\).

The following theorem is well known in the literature. Its proof can be found, for instance, in Theorem 2.9 of [19].

**Theorem 6.1.** The restriction of the twisted adjoint representation to the spin group of \(\text{Cl}(V, Q)\) is a surjective group homomorphism from \(\text{Spin}(V, Q)\) to the special orthogonal group \(\text{SO}(V, Q)\) of \(V\). Its kernel is \(\{1, -1\}\).
**Corollary 6.2.** The quotient group \( \operatorname{Spin}(V, Q)/\{1, -1\} \) and the special orthogonal group \( \operatorname{SO}(V, Q) \) are isomorphic.

As \( V \) is a linear space over \( \mathbb{R} \), we can choose an ordered basis for \( V \) so that
\[
Q(v) = v_1^2 + v_2^2 + \cdots + v_{m+1}^2 - v_{m+2}^2 - \cdots - v_{m+n}^2
\]
for all \( v = (v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+n}) \in \mathbb{R}^{m+n} \), [16, Theorem 4.5]. Hence, \( \operatorname{SO}(V, Q) \equiv \operatorname{SO}(m, n) \) and \( \operatorname{Spin}(V, Q) \equiv \operatorname{Spin}(m, n) \). Corollary 6.2 implies that
\[
\operatorname{Spin}(m, n)/\{1, -1\} \cong \operatorname{SO}(m, n).
\] (6.4)

Hence, we have the following theorem.

**Theorem 6.3.** The quotient group
\[
\operatorname{Spin}(m, n)/\{1, -1\}
\]
has a bi-gyrocommutative bi-gyrodecomposition.

**Proof.** This theorem follows directly from (6.4) and Theorems 4.32 and 5.2. \( \square \)

## 7 Conclusion

A gyrogroup is a non-associative group-like structure in which the non-associativity is controlled by a special family of automorphisms called gyrations. Gyractions, in turn, result from the extension by abstraction of the relativistic effect known as Thomas precession. In this paper we generalize the notion of gyrogroups, which involves a single family of gyrations, to that of bi-gyrogroups, which involves two distinct families of gyrations, collectively called bi-gyrations.

The bi-transversal decomposition \( \Gamma = H_LBH_R \), studied in Section 3, naturally leads to a groupoid \((B, \odot)\) that comes with two families of automorphisms, left and right ones. This groupoid is related to the bi-gyrogroupoid \((B, \oplus_b)\), studied earlier in Section 2. Bi-gyrogroupoids \((B, \oplus_b)\) form an intermediate structure that suggestively leads to the desired bi-gyrogroup structure \((B, \oplus)\). The bi-transversal operation \( \odot \) arises naturally from the bi-transversal decomposition (3.1). Under the natural conditions of Definition 4.1, the bi-transversal operation \( \odot \) becomes the bi-gyrogroupoid operation \( \oplus_b \). The latter operation leads to the desired bi-gyrogroup operation \( \oplus \) by means of (4.20).

As we have shown in Section 4, any bi-gyrodecomposition \( \Gamma = H_LBH_R \) of a group \( \Gamma \) induces the bi-gyrogroup structure on \( B \), giving rise to a bi-gyrogroup \((B, \oplus)\) along with left gyrations \( \lgry[a, b] \) and right gyrations \( \rgyr[a, b], a, b \in B \). Further, in the case where \( H_L \) is the trivial subgroup of \( \Gamma \), the bi-gyrodecomposition reduces to the decomposition \( \Gamma = BH \) studied in [14]. The bi-gyrogroup \((B, \oplus)\) induced by a bi-gyrodecomposition of a group is indeed an abstract version of the bi-gyrogroup \( \mathbb{R}^{n \times m} \) of all \( n \times m \) real matrices studied in [34].
Bi-gyrogroups are group-like structures. For instance, they satisfy the bi-gyroassociative law (Theorem 4.19), which descends to the associative law if their left and right gyrations are the identity automorphism. A concrete realization of a bi-gyrogroup is found in the special pseudo-orthogonal group SO(m, n) of the pseudo-Euclidean space \( \mathbb{R}^{m,n} \) of signature \((m, n)\), as shown in [34] and in Section 5. Moreover, bi-gyrogroups arise in the group counterpart of Clifford algebras as we establish in Section 6 that the quotient group Spin \((m, n)\)/\{1, -1\} of the spin group possesses a bi-gyrodecomposition.

By Theorem 4.28, any bi-gyrogroup is a gyrogroup. Yet, in general, the bi-gyrostructure of a bi-gyrogroup is richer than the gyrostructure of a gyrogroup. To see this clearly, we note that gyrations \(\text{gyr}[a, b]\) of a gyrogroup \((B, \oplus)\), \(a, b \in B\), are completely determined by the gyrogroup operation according to the gyrator identity in Theorem 2.10 (10) of [29]:

\[
\text{gyr}[a, b]x = \ominus(a \oplus b) \oplus (a \oplus (b \oplus x)) \tag{7.1}
\]

for all \(a, b, x\) in the gyrogroup \((B, \oplus)\). In contrast, the bi-gyrator identity analogous to (7.1) is

\[
(\text{lgyr}[a, b] \circ \text{rgyr}[b, a])(x) = \ominus(a \oplus b) \oplus (a \oplus (b \oplus x)) \tag{7.2}
\]

for all \(a, b, x\) in a bi-gyrogroup \((B, \oplus)\). Here, the bi-gyrogroup operation completely determines the composite automorphism \(\text{lgyr}[a, b] \circ \text{rgyr}[b, a]\). However, it does not determine straightforwardly each of the two automorphisms \(\text{lgyr}[a, b]\) and \(\text{rgyr}[a, b]\). Thus, the presence of two families of gyrations in a bi-gyrogroup, as opposed to the presence of a single family of gyrations in a gyrogroup, significantly enriches the bi-gyrostructure of bi-gyrogroups.

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