MINIMAL MASS BLOW-UP SOLUTIONS TO ROUGH NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We study the focusing mass-critical rough nonlinear Schrödinger equations, where the stochastic integration is taken in the sense of controlled rough path. We obtain the global well-posedness if the mass of initial data is below that of the ground state. Moreover, the existence of minimal mass blow-up solutions is also obtained in both dimensions one and two. In particular, these yield that the mass of ground state is exactly the threshold of global well-posedness and blow-up of solutions in the stochastic focusing mass-critical case. Similar results are also obtained for a class of nonlinear Schrödinger equations with lower order perturbations.

1. Introduction

We are concerned with the focusing mass-critical rough nonlinear Schrödinger equations with linear multiplicative conservative noise and the blow-up dynamics of their solutions. Precisely, consider the equation

\[ dX = i\Delta X dt + i|X|^4 X dt - \mu X dt + X dW(t), \]
\[ X(0) = X_0 \in H^1(\mathbb{R}^d). \]

Here

\[ W(t, x) = \sum_{k=1}^N i\phi_k(x)B_k(t), \quad x \in \mathbb{R}^d, \quad t \geq 0, \]

where \( \phi_k \in C^\infty_b(\mathbb{R}^d, \mathbb{R}), B_k \) are the standard \( N \)-dimensional real valued Brownian motions on a stochastic basis \( \{\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}\} \), \( 1 \leq k \leq N < \infty \), and \( \mu = \frac{1}{2} \sum_{k=1}^N \phi_k^2 \). The last term \( X dW(t) \) in (1.1) is taken in the sense of controlled rough path, see Definition 2.1 below. In particular, the rough integration coincides with the usual Itô integration if the corresponding processes are \( \{\mathcal{F}_t\} \)-adapted (see [24, Chapter 5]).

The physical significance of (1.1) is well known. In the conservative case considered here (i.e., \( \text{Re}W = 0 \)), \( ||X(t)||_{L^2}^2 \) is pathwisely conserved. Hence, with the normalization \( \|X_0\|_{L^2} = 1 \), the quantum system evolves on the unit ball of \( L^2 \) and so verifies the conservation of probability. Moreover, the noise in (1.1) can be viewed as a random potential acting in the system. In crystals it corresponds to scattering of excitons by phonons, because of thermal vibrations of molecules, see [1, 2] in the one dimensional case, and also [38] for the two dimensional case. Another important application can be found in the quantum measurement in open quantum systems ([10]). We also refer to [42] for more physical applications in the deterministic case, such as nonlinear optics, Bose-Einstein condensation and the Gross-Pitaevskii equation.

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The blow-up phenomena are extensively studied in the literature for the focusing mass-critical nonlinear Schrödinger equation (NLS):

\[(1.2) \quad i\partial_t u + \Delta u + |u|^{\frac{4}{d}} u = 0, \quad u(0) = u_0 \in H^1(\mathbb{R}^d).\]

As a matter of fact, equation (1.2) admits a number of symmetries and conservation laws. Precisely, it is invariant under the translation, scaling, phase rotation and Galilean transform, i.e., if \(u\) solves (1.2), then so does

\[(1.3) \quad v(t, x) = \lambda_0^{-\frac{d}{2}} u \left( \frac{t-t_0}{\lambda_0^2}, \frac{x-x_0}{\lambda_0} - \frac{\beta_0(t-t_0)}{\lambda_0} \right) e^{i \frac{\beta_0}{\lambda_0} (x-x_0) - i \frac{\beta_0^2}{2} (t-t_0) + i \theta_0},\]

with \(v(t_0, x) = \lambda_0^{-\frac{d}{2}} u_0 \left( \frac{x-x_0}{\lambda_0} \right) e^{i \frac{\beta_0}{\lambda_0} (x-x_0) + i \theta_0}\), where \((\lambda_0, \beta_0, \theta_0) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}\), \(x_0 \in \mathbb{R}^d\), \(t_0 \in \mathbb{R}\). In particular, the \(L^2\)-norm of solutions is preserved under the symmetries above, and thus (1.2) is called the mass-critical equation. Another important symmetry is related to the pseudo-conformal transformation in the pseudo-conformal space \(\Sigma := \{ u \in H^1(\mathbb{R}^d), \| x \|_{L^2(\mathbb{R}^d)} < \infty \}\),

\[(1.4) \quad \frac{1}{(-t)^{\frac{d}{2}}} u \left( 1 - \frac{x}{-t^{-\frac{1}{2}}} \right) e^{-i \frac{|x|^2}{4t}}, \quad t \neq 0.\]

The conservation laws related to (1.2) contain

\[(1.5) \quad \text{Mass : } M(u)(t) := \int_{\mathbb{R}^d} |u(t)|^2 dx = M(u_0),\]

\[(1.6) \quad \text{Energy : } E(u)(t) := \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(t)|^2 dx - \frac{d}{2d+4} \int_{\mathbb{R}^d} |u(t)|^{2+\frac{4}{d}} dx = E(u_0),\]

\[(1.7) \quad \text{Momentum : } \text{Mom}(u) := \text{Im} \int_{\mathbb{R}^d} \nabla u \bar{u} dx = \text{Im} \int_{\mathbb{R}^d} \nabla u_0 \bar{u}_0 dx.\]

It is well known that equation (1.2) is locally well posed and the solutions exist globally if the mass of initial data is below that of the ground state \(Q\), which is the unique positive spherically symmetric solution to the soliton equation

\[(1.8) \quad \Delta Q - Q + Q^{1+\frac{4}{d}} = 0.\]

However, the situation becomes much more delicate if \(\| u_0 \|_{L^2} = \| Q \|_{L^2}\). In this case, equation (1.2) has two special solutions: the solitary wave solution \(u(t, x) = Q(x)e^{it}\) that exists globally, and the so-called pseudo-conformal blow-up solution \(S_T(t, x), T \in \mathbb{R}\), obtained from the pseudo-conformal transformation (1.4) and the solitary wave solution:

\[(1.9) \quad S_T(t, x) := \frac{1}{(T-t)^{\frac{d}{2}}} Q \left( \frac{x}{T-t} \right) e^{i \left( t - \frac{d}{4} \right)|x|^2}, \quad x \in \mathbb{R}^d, \quad t < T.\]

Note that, \(\| S_T \|_{L^2} = \| Q \|_{L^2}\), and \(S_T\) blows up at time \(T\) with the blow-up speed \((T-t)^{-1}\).

In particular, the mass of ground state \(\| Q \|_{L^2}\) characterizes the threshold for the global well-posedness and blow-up of solutions to NLS, and \(S_T\) is exactly the minimal mass blow-up solution.

It should be mentioned that, minimal mass blow-up solutions are of significant importance in the study of blow-up phenomena. A remarkable result proved by Merle (33) is that, up to the symmetries (1.3), the pseudo-conformal blow-up solution is the unique minimal mass blow-up solution to NLS. Moreover, for the inhomogeneous NLS, a sufficient condition for the nonexistence of minimal mass blow-up solutions was proved in (32).
Under suitable degenerate conditions, Banica, Carles and Duyckaerts [34] constructed the minimal mass blow-up solutions. The sharp non-degenerate conditions for the existence of minimal mass blow-up solutions, as well as the strong rigidity theorem of uniqueness, have been proved by Raphaël and Szeftel [37].

For minimal mass blow-up solutions of other dispersive equations, see, e.g., [28] for the mass-critical fractional NLS, [13] for the inhomogeneous Hartree equation, [40, 41] for the nonlinear Schrödinger system, and [30] for the mass-critical gKdV equation. We also refer to [12, 21, 34, 35] for the case of supercritical mass (i.e., \( \|u_0\|_{L^2} > \|Q\|_{L^2} \)), where a new type of blow-up solutions with log-log blow-up rate has been investigated.

For stochastic Schrödinger equations (with Itô’s integration), there are many works devoted to the well-posedness results. See, e.g., [6, 16, 27, 11] for the subcritical case, and also the recent works [22, 23, 46] for the defocusing mass- and energy-critical cases. As regards the blow-up phenomena, de Bouard and Debussche [15, 17] first studied the noise effect on blow-up. Quite surprisingly, the conservative noise has the effect to accelerate blow-up immediately with positive probability in the focusing mass-supercritical case, namely, the nonlinearity in (1.1) has the exponent \( \alpha \in (1 + \frac{4}{d}, 1 + \frac{4}{d} - \frac{2}{d}) \) ([15, 17]).

The proof is based on the fact that, the non-degenerate noise is able to push the solution to a blow-up regime, in which the energy is sufficiently negative to formalize singularity. Furthermore, in the focusing mass-critical case, the numerical results in [18, 19, 20] suggest that the colored conservative noise has a tendency to delay blow-up, and the white noise may even prevent blow-up. We also refer to [8] for the damped effect on blow-up of the non-conservative noise (i.e., Re\(W\) ≠ 0) in the mass-(super)critical case.

However, to the best of our knowledge, quite few results are known for the quantitative description of blow-up dynamics in the stochastic setting. It is unclear whether \( \|Q\|_{L^2} \) still serves as the threshold for the global well-posedness and blow-up of solutions in the stochastic situation. One main difficulty here lies in the loss of the symmetries (1.3) and the conservation law of energy (1.6), which are, actually, completely destroyed because of the presence of noise.

Here we obtain the global well-posedness of equation (1.1) if the mass of initial data is below that of the ground state. Moreover, in both dimensions one and two, we construct the minimal mass blow-up solutions to equation (1.1), which indeed evolve asymptotically like the pseudo-conformal blow-up solutions near the blow-up time. In particular, these results yield that the mass of the ground state is exactly the threshold for the global well-posedness and blow-up of solutions to (1.1).

Furthermore, similar results are also obtained for a class of nonlinear Schrödinger equations with lower order perturbations, which has not been well studied because of the loss of symmetries (1.3) and the conservation law of energy.

The idea of proof is mainly based on the rescaling approach as in [5, 6, 9] and on the modulation method developed in [37], which includes a bootstrap device and the backward propagation from the singularity. The latter enables to reduce the analysis of blow-up solutions of (1.1) to that of the dynamics of finite dimensional geometrical parameters and a small remainder corresponding to a random nonlinear Schrödinger equation with lower order perturbations (see (2.7) below). In particular, it provides quantitative descriptions of blow-up dynamics in the absence of symmetries and conservation of energy.

It should be mentioned that, because of the backward propagation procedure, the blow-up solution constructed is no longer adapted, and thus the last term in equation (1.1) should not be taken in the sense of Itô. It turns out that it can be appropriately interpreted in the sense of controlled rough path. Similar situation arises also in the random three dimensional vorticity equation in [4, 39].
A key role is then played by the equivalence between solutions to the rough equation \((2.1)\) and the random equation \((2.7)\). Quite differently from \([5, 6]\), the proof here requires finer descriptions of time regularities of solutions and, actually, the local smoothing space is also applied to measure the spatial regularity of solutions in the two dimensional case.

**Notations.** For any \(x = (x_1, \ldots, x_d) \in \mathbb{R}^d\) and any multi-index \(\nu = (\nu_1, \cdots, \nu_d)\), let 
\[
|\nu| = \sum_{j=1}^d \nu_j, \quad (x) = (1 + |x|^2)^{1/2}, \quad \partial_x^n = \partial_{x_1}^{n_1} \cdots \partial_{x_d}^{n_d}, \quad \text{and } \nabla = (I - \Delta)^{1/2}.
\]

For \(1 \leq p \leq \infty\), \(L^p = L^p(\mathbb{R}^d)\) is the space of \(p\)-integrable (complex-valued) functions endowed with the norm \(|| \cdot ||_{L^p}\), and \(W^{s,p}\) denotes the standard Sobolev space, \(s \in \mathbb{R}\). In particular, \(L^2(\mathbb{R}^d)\) is the Hilbert space endowed with the scalar product 
\[
\left< v, w \right> = \int_{\mathbb{R}^d} v(x)\bar{w}(x)dx, \quad \text{and } H^s := W^{s,2}.
\]

As usual, \(L^2(0, T; L^p)\) means the space of all integrable \(L^p\)-valued functions \(f : (0, T) \to L^p\) with the norm \(|| \cdot ||_{L^2(0, T; L^p)}\), and \(C([0, T]; L^p)\) denotes the space of all \(L^p\)-valued continuous functions on \([0, T]\) with the sup norm over \(t\). We also use the local smoothing spaces defined by \(L^2(I; H^s_0) = \{ u \in \mathcal{S}' : \int \int (x)^{2\beta} |\nabla|^\alpha u(t,x)^2dxdt < \infty \}\), where \(\alpha, \beta \in \mathbb{R}\).

For any Hölder continuous function \(f \in C^\alpha(I), I \subseteq \mathbb{R}^+\), we write \(\delta f_{st} := f(t) - f(s), s, t \in I\), and \(||f||_{\alpha, I} := \sup_{s, t \in I, s \neq t} |\frac{\delta f_{st}}{|s - t|^{\alpha}}|\). Let \(C^\infty_c\) be the space of all compactly supported smooth functions on \(\mathbb{R}^d\). We also set \(g_t := \frac{d}{dt} g\) for any \(C^1\) functions.

The symbol \(u = O(v)\) means that \(|u/v|\) stays bounded, and \(v_n = o(1)\) means that \(|v_n|\) tends to zero as \(n \to \infty\). Throughout this paper, we use \(C\) for various constants that may change from line to line.

2. Formulation of main results

To begin with, we first present the precise definition of solutions to equation \((1.1)\).

**Definition 2.1.** We say that \(X\) is a solution to \((1.1)\) on \([0, \tau^*]\), where \(\tau^* \in (0, \infty)\) is a random variable, if \(\mathbb{P}\)-a.s. for any \(\varphi \in C^\infty_c\), \(t \mapsto (X(t), \varphi)\) is continuous on \([0, \tau^*]\) and for any \(0 < s < t < \tau^*\),

\[
\langle X(t) - X(s), \varphi \rangle - \int_s^t \langle iX, \Delta \varphi \rangle + \langle i|X|^2X, \varphi \rangle - \langle \mu X, \varphi \rangle dr = \sum_{k=1}^N \int_s^t \langle i\phi_k X, \varphi \rangle dB_k(r).
\]

Here the integral \(\int_s^t \langle i\phi_k X, \varphi \rangle dB_k(r)\) is taken in the sense of controlled rough path with respect to the rough paths \((B, \mathbb{B})\), where \(\mathbb{B} = (\mathbb{B}_{jk})\), \(\mathbb{B}_{jk, st} := \int_s^t \delta B_{j, sr} dB_k(r)\) with the integration taken in the sense of \(\text{Itô}\). That is, \(\langle i\phi_k X, \varphi \rangle \in C^\alpha([s, t])\),

\[
(2.1) \quad \delta(\langle i\phi_k X, \varphi \rangle)_{st} = -\sum_{j=1}^N (\phi_j \phi_k X(s), \varphi) \delta B_{j, st} + \delta R_{k, st},
\]

and \(||\langle \phi_j \phi_k X, \varphi \rangle||_{\alpha, [s, t]} < \infty, \quad ||R_k||_{2\alpha, [s, t]} < \infty||

We assume that

(A0) \textit{(Asymptotical flatness)} For \(1 \leq k \leq N\), \(\phi_k\) satisfies that for any multi-index \(\nu \neq 0\),

\[
\lim_{|x| \to \infty} (x)^2 |\partial_x^n \phi_k(x)| = 0.
\]

(A1) \textit{(Flatness at the origin)} For \(1 \leq k \leq N\) and for any multi-index \(0 \leq |
u| \leq 5\),

\[
\partial_x^n \phi_k(0) = 0.
\]
Remark 2.2. Assumption (A0) is related to the asymptotical behavior at infinity of the spatial functions of noise and guarantees the well-posedness of equation (1.1). While, Assumption (A1) characterizes the local behavior near the origin and is mainly imposed for the construction of minimal mass blow-up solutions.

Theorem 2.3 summarizes the local well-posedness and blow-up alternative results.

Theorem 2.3. Let \( d \geq 1 \). Assume (A0). Then, for each \( X_0 \in H^1 \), \( \mathbb{P} \)-a.s. there exists a unique solution \( X \) to (1.1) on \([0, \tau^*)\) in the sense of Definition 2.1 where \( \tau^* \in (0, \infty) \) is a random variable, such that \( \mathbb{P}\)-a.s. for any \( T < \tau^* \),

\[
X |_{[0,T]} \in C([0, T]; H^1) \cap L^r(0, T; W^{1, \rho}) \cap L^2(0, T; H^{\frac{3}{2}}),
\]

where \((\rho, \gamma)\) is any Strichartz pair, i.e., \( \frac{2}{\gamma} = d \left( \frac{1}{2} - \frac{1}{\rho} \right) \), \( (\rho, \gamma, d) \neq (\infty, 2, 2) \).

Moreover, we have that for \( \mathbb{P}\)-a.e. \( \omega \), either \( \tau^*(\omega) = \infty \), or
\[
\lim_{t \to \tau^*(\omega)} \| X(\omega, t) \|_{H^1} = \infty.
\]

Remark 2.4. The proof of Theorem 2.3 is similar to that in [6], based on the Strichartz estimates for Schrödinger equations with lower order perturbations and the equivalence between solutions to equations (1.1) and (2.1) in Theorem 2.10 below.

The next result is concerned with the global well-posedness of (1.1) if the mass of initial data is below that of the ground state.

Theorem 2.5. Let \( d \geq 1 \). Assume (A0). If \( \| X_0 \|_{L^2} < \| Q \|_{L^2} \), then the corresponding solution \( X \) to (1.1) exists globally almost surely.

The main result concerning the existence of minimal mass blow-up solutions is formulated below.

Theorem 2.6. Let \( d = 1, 2 \). Assume Assumptions (A0) and (A1) to hold. Then, for \( \mathbb{P}\)-a.e. \( \omega \) there exists \( \tau^*(\omega) \in (0, \infty) \), such that for any \( T \in (0, \tau^*(\omega)) \) there exists \( X_0(\omega) \in H^1 \) satisfying that \( \| X_0(\omega) \|_{L^2} = \| Q \|_{L^2} \) and the corresponding solution \( X(\omega) \) to (1.1) blows up at time \( T \). Moreover, there exist \( \delta, C(\omega, T) > 0 \) such that for \( t \) close to \( T \),
\[
\| X(\omega, t) - e^{W(\omega, t)} S_T(t) \|_{H^1} \leq C(\omega, T)(T - t)^{\delta}.
\]

Remark 2.7. (i) Theorems 2.3 and 2.6 show that, in dimensions one and two, the mass of the ground state \( \| Q \|_{L^2} \) is indeed the threshold for the global well-posedness and blow-up of solutions to the rough nonlinear Schrödinger equation (1.1).

(ii) The asymptotic behavior (2.6) yields that, up to a phase shift related to the noise, the pseudo-conformal blow-up solutions are exactly the main blow-up profile near the blow-up time. We also would like to mention that, it is possible to construct solutions that blow up at finitely many points, which will be done in the forthcoming work.

(iii) It is interesting to see whether uniqueness holds for the minimal mass blow-up solutions to (1.1), which, however, is still unclear. One difficulty arises from the loss of the conservation law of energy due to the presence of noise.

The first step of the proof is based on the rescaling approach. More precisely, equation (1.1) can be formally transformed, via \( u = e^{-W} X \), to the random equation
\[
i\partial_t u + e^{-W} \Delta (e^W u) + |u|^{\frac{4}{d}} u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d,
\]
\[
u(0, x) = u_0(x)
\]
with \( u_0 = X_0 \) and \( e^{-W} \Delta(e^W u) = \Delta u + b \cdot \nabla u + cu \), where

\[
(2.8) \quad b(t, x) = 2\nabla W(t, x) = 2\sum_{k=1}^{N} \nabla \phi_k(x) B_k(t),
\]

\[
(2.9) \quad c(t, x) = \sum_{j=1}^{d} (\partial_j W(t, x))^2 + \Delta W(t, x)
\]

\[
= -\sum_{j=1}^{d} \left( \sum_{k=1}^{N} \partial_j \phi_k(x) B_k(t) \right)^2 + i \sum_{k=1}^{N} \Delta \phi_k(x) B_k(t).
\]

The transformation \( u = e^{-W} X \) is known as the Doss-Sussman transformation in finite dimensional case. It turns out to be quite robust also in the study of stochastic equations in infinite dimensional spaces. One advantage of the rescaling approach is, that it enables one to reduce the problem of stochastic equations to that of equations with random coefficients, to which one is able to obtain sharp estimates in a pathwise manner. See, e.g., [4] for stochastic logarithmic Schrödinger equations, [26] for the scattering behavior and [9, 47] for optimal control problems. See also [11, 39] for random three dimensional vorticity equations.

The solutions to the random equation (2.7) can be defined similarly as in [6].

**Definition 2.8.** We say that \( u \) is a solution to (2.7) on \([0, \tau^*)\), where \( \tau^* \in (0, \infty) \) is a random variable, if \( \mathbb{P} - \text{a.s.} \, u \in C([0, \tau^*]; H^1), \, |u|^2 u \in L^1(0, \tau^*; H^{-1}) \), and \( u \) satisfies

\[
(2.10) \quad u(t) = u(0) + \int_{0}^{t} i e^{-W(s)} \Delta(e^W(u(s)) + i|u(s)|^2 u(s))ds, \quad t \in [0, \tau^*),
\]

as an equation in \( H^{-1} \).

**Remark 2.9.** Under Assumption (A0), the solution to (2.7) in the sense of Definition 2.8 is equivalent to that taken in the mild sense below

\[
u(t) = e^{it\Delta} u(0) + \int_{0}^{t} e^{i(t-s)\Delta} (i|u(s)|^2 u(s) + i(b(s) \cdot \nabla + c(s))u(s))ds, \quad \forall t \in [0, \tau^*),
\]

as an equation in \( H^{-1} \). Moreover, similarly to Theorem 2.3 by [6] one also has the local well-posedness, blow-up alternative results and that for any \( 0 < T < \tau^* \) and for any Strichartz pair \( (\rho, \gamma) \),

\[
(2.11) \quad \|u\|_{C([0,T]; H^1)} + \|u\|_{\mathbb{L}^7(0,T;W^{1,\rho})} + \|u\|_{\mathbb{L}^2(0,T;H^\frac{1}{2})} < \infty, \quad \mathbb{P} - \text{a.s.}
\]

An important role here is played by the equivalence between solutions to the rough equation (1.1) and the random equation (2.7).

**Theorem 2.10.** (i). Let \( u \) be the solution to (2.7) on \([0, \tau^*)\) with \( u(0) = u_0 \in H^1 \) in the sense of Definition 2.8, where \( \tau^* \in (0, \infty) \) is a random variable. Then, \( \mathbb{P} - \text{a.s.} \), \( X := e^W u \) is the solution to equation (1.1) on \([0, \tau^*)\) with \( X(0) = u_0 \) in the sense of Definition 2.7.

(ii). Let \( X \) be the solution to equation (1.1) on \([0, \tau^*)\) with \( X(0) = X_0 \in H^1 \) in the sense of Definition 2.7 satisfying that \( \mathbb{P} - \text{a.s.} \), \( \|X\|_{C([0,T]; H^1)} + \|X\|_{\mathbb{L}^2(0,T;H^\frac{1}{2})} < \infty, \quad T \in (0, \tau^*), \) and

\[
\|e^{-it\Delta} e^{-W(t)} X(t) - e^{-is\Delta} e^{-W(s)} X(s)\|_{L^2} \leq C(t)(t-s), \quad \forall 0 \leq s < t < \tau^*.
\]

Then, \( u := e^{-W} X \) solves equation (2.7) on \([0, \tau^*)\) with \( u(0) = X_0 \) in the sense of Definition 2.8.
Similar results were proved in \cite{33} where Itô’s integration was considered. Quite differently, the proof of Theorem 2.10 requires finer descriptions of the time regularity of solutions, and the local smoothing space will be also applied to measure the spatial regularity of solutions in the two dimensional case. The proof is contained in Section 6.

By virtue of Theorem 2.11, we can now reduce the proof of Theorems 2.5 and 2.6 to that of the following two results corresponding to the random equation (2.7).

**Theorem 2.11.** Let $d \geq 1$. Assume (A0). If $\|u_0\|_{L^2} < \|Q\|_{L^2}$, then the corresponding solution $u$ to (2.7) exists globally almost surely.

**Theorem 2.12.** Let $d = 1, 2$. Assume Assumptions (A0) and (A1) to hold. Then, for $\mathbb{P}$-a.e. $\omega$ there exists $\tau^*(\omega) \in (0, \infty)$ such that for any $T \in (0, \tau^*(\omega)]$, there exists $u_0(\omega) \in H^1$ satisfying $\|u_0(\omega)\|_{L^2} = \|Q\|_{L^2}$ and the corresponding solution $u(\omega)$ to (2.7) blows up at time $T$. Moreover, there exist $\delta, C(\omega, T) > 0$ such that for $t$ close to $T$,

$$
\|u(\omega, t) - S_T(t)\|_{H^1} \leq C(\omega, T)(T - t)^{\delta}.
$$

**Theorem 2.11** is proved by the analysis of the evolution of energy and the sharp Gagliardo-Nirenberg inequality obtained in \cite{44}.

Moreover, the strategy of the proof of Theorem 2.12 is mainly inspired by the modulation method in the recent work \cite{37} and the compactness arguments in \cite{33}. It should be mentioned that, the main challenge here lies in the absence of the symmetries and the conservation of energy.

As a matter of fact, the proof in the original work \cite{33} relies crucially on the pseudo-conformal symmetry, which, however, is lost in equation (2.7) because of the lower order perturbations. Moreover, quite different from the exponential decay setting in \cite{33}, a polynomial decay problem arises from the error term if one linearizes equation (2.7) around the pseudo-conformal blow-up solution.

Here, we apply the framework recently developed by Raphaël and Szeftel \cite{37} in the setting of inhomogeneous nonlinear Schrödinger equations. This framework is, actually, quite robust and applies to more general situations, in which the symmetries may be lost. However, unlike in \cite{37}, the conservation law of energy is also destroyed in equation (2.7). This leads us to a finer control of the energy and to the spatial flatness condition (2.3). It turns out that, the flatness at the origin of the spatial functions of noise is sufficient to decouple the interactions between the lower order perturbations and the main blow-up profile. This also reflects the nature of spatial localization in the construction of minimal mass blow-up solutions.

Below let us give a brief outline of the main steps in the proof of Theorem 2.12.

We first construct a sequence of approximating solutions $u_n$ with $u(t_n) = S_T(t_n)$, where $t_n$ is any sequence converging to $T(> 0)$. The key point here is that each $u_n$ admits a geometrical decomposition

$$
u_n(t, x) = \lambda_n^{-\frac{d}{2}}(Q_{\mathcal{P}_n} + \varepsilon_n)(t, x - \frac{\alpha_n}{\lambda_n})e^{\gamma_n}(=: w_n(t, x) + R_n(t, x)),$$

where $Q_{\mathcal{P}_n}(t, y) := Q(y)e^{i\beta_n(t)\cdot y - i\frac{2\alpha_0(t)}{\lambda_n}|y|^2}$ is a two parameters deformation of the ground state $Q$, and the geometrical parameters $\mathcal{P}_n := (\lambda_n(t), \alpha_n(t), \beta_n(t), \gamma_n(t), \theta_n(t))$ and the remainder $R_n$ satisfy the appropriate orthogonality conditions in (4.1) below, which are closely related to the generalized kernels of the linearized operators $L_+ = -\Delta + I - (1 + \frac{1}{d})Q^\frac{2}{d}$ and $L_- = -\Delta + I - Q^\frac{2}{d}$.

Hence, the analysis of blow-up dynamics is now reduced to that of the finite dimensional geometrical parameters and the remainder term. Plugging (2.13) into (2.7) we then obtain
the equation for the remainder

\[ i\partial_t R_n + \Delta R_n + (|w_n + R_n|^2(w_n + R_n) - |w_n|^2 w_n) + b \cdot \nabla R_n + c R_n = -\eta_n, \]

where

\[ \eta_n = i\partial_t w_n + \Delta w_n + |w_n|^2 w_n + b \cdot \nabla w_n + c w_n. \]

(See also the equation of \( \varepsilon_n \) in (4.14) below.) Then, taking the inner product of equation (2.14) with the directions spanning the generalized null space of \( L_{\pm} \), one is able to obtain the crucial estimates of modulation equations driving the geometrical parameters. See Section 4 below.

The next step in Section 5 is devoted to the key uniform estimates of approximating solutions, to which a bootstrap device and the backward propagation from the singularity have been applied. It should be mentioned that, the crucial quadratic terms of \( R_n \) are controlled by the energy (1.6) and a Lyapounov functional (5.34) involving a local Morawetz type term. Moreover, the orthogonality conditions enable us to obtain the negligible errors of the linear terms of \( R_n \), while the terms of order higher than two can be easily controlled by using Gagliardo-Nirenberg’s inequality.

Finally, in Section 6 using compactness arguments we are able to extract a limit \( u_0 \) at the initial time 0, which then yields a solution by the well-posedness result. In view of the fact that the pseudo-conformal blow-up solutions are the main profile in the construction of approximating solutions, we thus obtain that the resulting solution is exactly a minimal mass blow-up solution.

It should be mentioned that, because of the backward propagation procedure above, the existence of the limit \( u_0 \) requires the information of the whole sample paths of Brownian motions on \( [0, T] \), and thus the resulting blow-up solution is no longer \( \{ \mathcal{F}_t \} \)-adapted. Instead of Itô’s integration, the last term in equation (1.1) can be reinterpreted in the sense of controlled rough path. Actually, these two notions of stochastic integrations coincide with each other if the adaptedness of processes is fulfilled. The proof of the key equivalence between solutions to equations (1.1) and (2.7) is also contained in Section 6.

The remainder of this paper is structured as follows. Section 3 contains some preliminaries, the coercivity of linearized operators, the expansion formulas and basic notions of controlled rough path. Section 4 and 5 are devoted to the approximating solutions and constitute the most technical part of this paper. In Section 4, we obtain the geometrical decomposition of approximating solutions and also the estimates of modulation equations. Then, in Section 5, we derive the key uniform estimates of approximating solutions, including the analysis of the energy and the Lyapounov functional involving a local Morawetz type term. The proof of main results are contained in Section 6. Finally, some technical proofs are postponed to the Appendix, i.e., Section 7 for simplicity.

3. Preliminaries

This section contains the preliminaries needed in the proof, including the coercivity of linearized operators, the expansion formulas and basic notions of controlled rough path.

3.1. Coercivity of linearized operators. We denote \( Q \) the ground state that solves the soliton equation (1.8). It follows from [14, Theorem 8.1.1] that \( Q \) is smooth and decays at infinity exponentially fast, i.e., there exist \( C, \delta > 0 \) such that for any multi-index \( |\nu| \leq 2 \),

\[ |\partial^\nu_x Q(x)| \leq Ce^{-\delta|x|}, \quad x \in \mathbb{R}^d. \]
Let \( L = (L_+, L_-) \) be the linearized operator around the ground state, defined by
\[
L_+ := -\Delta + I - (1 + \frac{4}{d})Q^2, \quad L_- := -\Delta + I - Q^2. \tag{3.2}
\]
The generalized null space of operator \( L \) is spanned by \( \{Q, xQ, |x|^2Q, \nabla Q, \Lambda Q, \rho\} \), where \( \Lambda := \frac{d}{2}I + x \cdot \nabla \), and \( \rho \) is the unique \( H^1 \) spherically symmetric solution to the equation
\[
L_+ \rho = -|x|^2Q, \tag{3.3}
\]
which satisfies the exponential decay property (see, e.g., [29, (B.1), (B.10), (B.15)]), i.e., for some \( C, \delta > 0, \)
\[
|\rho(x)| + |\nabla \rho(x)| \leq Ce^{-\delta|x|}. \]
Moreover, we have (see, e.g., [15, (B.1), (B.10), (B.15)])
\[
\begin{align*}
L_+ \nabla Q &= 0, & L_+ \Lambda Q &= -2Q, & L_+ \rho &= -|x|^2Q, \\
L_- Q &= 0, & L_- xQ &= -2\nabla Q, & L_- |x|^2Q &= -4\Lambda Q. \tag{3.4}
\end{align*}
\]
For any complex valued \( H^1 \) function \( f = f_1 + if_2 \) in terms of the real and imaginary parts, we set
\[
(Lf, f) := \int f_1 L_+ f_1 dx + \int f_2 L_- f_2 dx. \tag{3.5}
\]
Let \( K \) denote the set of all complex valued \( H^1 \) functions \( f = f_1 + if_2 \) satisfying the orthogonality conditions below
\[
\begin{align*}
\int Qf_1 dx &= 0, \quad \int xQf_1 dx = 0, \quad \int |x|^2Qf_1 dx = 0, \\
\int \nabla Qf_2 dx &= 0, \quad \int \Lambda Qf_2 dx = 0, \quad \int \rho f_2 dx = 0. \tag{3.6}
\end{align*}
\]
The coercivity property below is crucial in the proof of Theorem 2.12

**Lemma 3.1.** ([15, Theorem 2.5]) We have that for some \( \nu > 0, \)
\[
(Lf, f) \geq \nu \|f\|^2_{H^1}, \quad \forall f \in K. \tag{3.7}
\]
As a consequence we have

**Corollary 3.2.** There exist positive constants \( \nu_1, \nu_2 > 0, \) such that
\[
(Lf, f) \geq \nu_1 \|f\|^2_{H^1} - \nu_2 \left( \langle f_1, Q \rangle^2 + \langle f_1, xQ \rangle^2 + \langle f_1, |x|^2Q \rangle^2 + \langle f_2, \nabla Q \rangle^2 + \langle f_2, \Lambda Q \rangle^2 + \langle f_2, \rho \rangle^2 \right), \quad \forall f \in H^1, \tag{3.8}
\]
where \( f_1 \) and \( f_2 \) are the real and imaginary parts of \( f \), respectively.

The following localized version of the coercivity property will be also useful.

**Corollary 3.3.** (Localized coercivity) Let \( \Phi \) be a positive smooth radial function on \( \mathbb{R}^d \), such that \( \Phi(x) = 1 \) for \( |x| \leq 1 \), \( \Phi(x) = e^{-|x|} \) for \( |x| \geq 2 \), \( 0 < \Phi \leq 1 \), and \( \frac{\nabla \Phi}{\Phi} \leq C \) for some \( C > 0 \). Set \( \Phi_A(x) := \Phi \left( \frac{x}{A} \right), \) \( A > 0 \). Then, for \( A \) large enough we have
\[
\int |\nabla f|^2\Phi_A + |f|^2 - (1 + \frac{4}{d})Q^2 f_1^2 - Q^2 f_2^2 dx \geq \nu \int (|\nabla f|^2 + |f|^2)\Phi_A dx, \quad \forall f \in K, \tag{3.9}
\]
where \( \nu > 0, \) and \( f_1, f_2 \) are the real and imaginary parts of \( f \), respectively.

The proofs of Corollaries 3.2 and 3.3 are postponed to the Appendix for simplicity.
3.2. Expansion of the nonlinearity. Let $d = 1, 2$. Let $f(z) := |z|^\frac{1}{d}z$, $z \in \mathbb{C}$, and for $v, R \in \mathbb{C}$ set

$$f'(v) \cdot R := \partial_z f(v) R + \partial_v f(v) R = (1 + \frac{2}{d})|v|^\frac{1}{d} R + \frac{2}{d} |v|^\frac{2}{d} v^2 R,$$

$$N_{f, 2}(v, R) := \frac{1}{2} \partial_z f(v) R^2 + \partial_v f(v) |R|^2 + \frac{1}{2} \partial_v f(v) R^2$$

$$= \frac{1}{d} (1 + \frac{2}{d}) |v|^\frac{2}{d} R^2 + \frac{2}{d} (1 + \frac{2}{d}) |v|^\frac{2}{d} v^2 R^2 + \frac{1}{d} (\frac{2}{d} - 1) |v|^\frac{4}{d} v^2 R^2.$$ 

Then, expanding $f(v + R)$ around $v$ we have

$$f(v + R) = f(v) + f'(v) \cdot R + N_{f, 2}(v, R) + O(\sum_{k=3}^{1+\frac{4}{d}} |v|^{1+\frac{4}{d} - k} |R|^k).$$

Similarly, for $F(z) := \frac{d}{2d} |z|^{\frac{1}{d} + \frac{4}{d}}$, $z \in \mathbb{C}$, we have the expansion

$$F(u) = F(v) + \frac{1}{2} |v|^{\frac{1}{d} v^2} R + \frac{1}{2} |v|^{\frac{4}{d} v R}$$

$$+ \frac{1}{2d} (1 + \frac{2}{d}) |v|^\frac{2}{d} R^2 + \frac{2}{d} (1 + \frac{2}{d}) |v|^\frac{2}{d} v^2 R^2 + \frac{1}{2d} |v|^\frac{4}{d} v^2 R^2 + O(\sum_{k=3}^{2+\frac{4}{d}} |v|^{2+\frac{4}{d} - k} |R|^k).$$

The following lemma contains the well-known Gagliardo-Nirenberg inequality, which is useful to control the remainder terms in the expansion formulas.

**Lemma 3.4.** ([14] Theorem 1.3.7]) Let $d \geq 1$ and $2 \leq p < \infty$. Then, we have

$$\|f\|_{L^p} \leq C \|f\|_{L^2}^{1 - d} \|\nabla f\|_{L^2}^{d}, \quad \forall f \in H^1,$$

where $C > 0$. Moreover, in the case where $d = 1$,

$$\|f\|_{L^p} \leq C \|f\|_{H^1}, \quad \forall f \in H^1.$$

3.3. Controlled rough path. Below we briefly recall the basic notions of controlled rough path that are needed in the proof. For more details of the theory of (controlled) rough path we refer to [25, 31] and the references therein.

Given a path $X \in \mathcal{C}^{\alpha}([0, T]; \mathbb{R}^N), 0 < T < \infty$, we say that $Y \in \mathcal{C}^{\alpha}([0, T]; \mathbb{R}^N)$ is controlled by $X$ if there exists $Y' \in \mathcal{C}^{\alpha}([0, T]; \mathbb{R}^{N \times N})$ such that the remainder term $R^Y$ implicitly given by

$$\delta Y_{j, st} = \sum_{k=1}^{N} Y'_{jk}(s) \delta X_{k, st} + \delta R^Y_{j, st}$$

satisfies $\|R^Y\|_{2 \alpha, [s, t]} < \infty, 1 \leq j \leq N$. This defines the controlled rough path $(Y, Y')$ in $\mathcal{D}^{2\alpha}([0, T]; \mathbb{R}^N)$, and $Y'$ is the so-called Gubinelli's derivative.

Note that, the $N$-dimensional Brownian motions $B = (B_j)_{j=1}^N$ can be enhanced to a rough path $\mathbb{B} = (B, \mathbb{B})$, where $\mathbb{B}_{j, k, st, r} := \int_s^t \delta B_{jr, s} dB_k(r)$ with the integration taken in the sense of Itô. It is known ([24, Section 3.2]) that $\|B\|_{\alpha, [0, T]} < \infty, \|B\|_{2 \alpha, [s, t]} < \infty, \mathbb{P}$-a.s., where $\frac{1}{2} < \alpha < \frac{3}{2}$.
Given a path $Y$ controlled by the $N$-dimensional Brownian motion, i.e., $Y \in \mathcal{D}_{\mathbb{R}}^{2\alpha}(\mathbb{R}^N)$, $0 < S < T < \infty$, we can define the rough integration of $Y$ against $B = (B, \mathbb{R})$ as follows (see [24] Theorem 4.10), for each $1 \leq k \leq N$,

$$
\int_{S}^{T} Y_k(r)dB_k(r) := \lim_{|\mathcal{P}| \to 0} \sum_{i=0}^{n-1} (Y_k(t_i) - Y_k(t_{i+1})) \delta B_k(t_i, t_{i+1}),
$$

where $\mathcal{P} = \{ t_0, t_1, \ldots, t_n \}$ is a partition of $[S, T]$ so that $t_0 = S$, $t_n = T$, and $|\mathcal{P}| := \max_{0 \leq i \leq n-1} |t_{i+1} - t_i|$.

4. Approximating solutions and modulation parameters

4.1. Approximating solutions. Below we fix $T \in (0, \infty)$, which may be taken sufficiently small and will be determined later. Let $t_n$, $n \geq 1$, be an increasing sequence such that $\lim_{n \to \infty} t_n = T$ and $u_n$ be the corresponding solution to the equation

$$
i\partial_t u_n + \Delta u_n + |u_n|^4 u_n + b \cdot \nabla u_n + cu_n = 0,
\quad u_n(t_n) = S_T(t_n),
$$

where $b, c$ are given by (2.8) and (2.9) respectively.

Proposition [4.1] below contains the key geometrical decomposition of $u_n$, which enables us to reduce the analysis of blow-up dynamics to that of finite dimensional modulation parameters and a small remainder term.

**Proposition 4.1** (Geometrical decomposition). For any $t_n$ sufficiently close to $T$, there exist $t_n^* < T$ and unique modulation parameters $\mathcal{P}_n := (\lambda_n, \alpha_n, \beta_n, \gamma_n, \theta_n) \in C^1((t_n^*, t_n); \mathbb{R}^{2d+3})$, such that $u_n \in C([t_n^*, t_n^*], H^1)$, $u_n$ has the geometrical decomposition

$$
u_n(t, x) = \lambda_n^{-\frac{d}{2}} (Q_{\mathcal{P}_n} + \epsilon_n)(t, \frac{x - \alpha_n}{\lambda_n}) e^{i\theta_n}(=: w_n(t, x) + R_n(t, x)),
$$

with

$$
Q_{\mathcal{P}_n}(t, y) := Q(y) e^{i\beta_n(t)y - i\frac{\lambda_n(t)}{4}|y|^2}, \quad t \in [t_n^*, t_n], \; y \in \mathbb{R}^d,
$$

and the following orthogonality conditions hold on $[t_n^*, t_n]$:

$$
\begin{align*}
\operatorname{Re} \int (x - \alpha) w_n(t) \overline{\rho_n(t)}dx &= 0, \quad \operatorname{Re} \int |x - \alpha|^2 w_n(t) \overline{\rho_n(t)}dx = 0, \\
\operatorname{Im} \int \nabla w_n(t) \overline{\rho_n(t)}dx &= 0, \quad \operatorname{Im} \int \nabla w_n(t) \overline{\rho_n(t)}dx = 0, \\
\operatorname{Re} \int \lambda w_n(t) \overline{\rho_n(t)}dx &= 0, \quad \operatorname{Im} \int \lambda w_n(t) \overline{\rho_n(t)}dx = 0,
\end{align*}
$$

where

$$
\rho_n(t, x) = \lambda_n^{-\frac{d}{2}} (t) \rho_{\mathcal{P}_n} (t, \frac{x - \alpha_n(t)}{\lambda_n(t)}) e^{i\theta_n(t)}, \quad \text{with } \rho_{\mathcal{P}_n}(t, y) = \rho(y) e^{i\beta_n(t)y - i\frac{\lambda_n(t)}{4}|y|^2},
$$

and $\rho$ is given by (3.3).

The proof of Proposition 4.1 is based on Lemma 4.2 below.

**Lemma 4.2.** Let $\mathcal{U}_r(0) := \{ u \in H^1; \| u - u_0 \|_{H^1} \leq r \}$, $\mathcal{U}_r(P_0) := \{ P \in \mathbb{R}^{2d+3}; |P - P_0| \leq r \}$, where $P_0 := (\lambda_0, \alpha_0, \beta_0, \gamma_0, \theta_0)$ and $r > 0$. Assume that $u_0 \in H^1$ has the decomposition

$$
u_0(x) = \lambda_0^{-\frac{d}{2}} Q \left( \frac{x - \alpha_0}{\lambda_0} \right) e^{i(\beta_0 \cdot \frac{x - \alpha_0}{\lambda_0} - \frac{\gamma_0}{4}|\frac{x - \alpha_0}{\lambda_0}|^2 + \theta_0)} + R_0 (=: w_0 + R_0), \; x \in \mathbb{R}^d,
\[ \|R_0\|_{L^2} \leq C\lambda_0^2 T \] and the orthogonality conditions in (1.1) hold with \( P_0, w_0 \) and \( R_0 \) replacing \( \mathcal{P}_n, w_n \) and \( R_n \), respectively. Then, there exist \( T_0, r_0 > 0 \) and a unique \( C^1 \) map \( \Psi : \mathcal{U}_n(u_0) \mapsto \mathcal{U}_n(P_0) \) such that for any \( 0 < T \leq T_0 \) and for any \( u \in \mathcal{U}_n(P_0) \),

\[
\frac{d}{dt}u = w + R,
\]

where \( w \) and \( R \) are as in Proposition 4.1 with \( \Psi(u) \) replacing \( \mathcal{P}_n \), and the following orthogonality conditions hold:

\[
\begin{align*}
\Re \int (x - \alpha)w dx &= 0, \quad \Re \int |x - \alpha|^2 w dx = 0, \\
\Im \int \frac{d}{2} w + (x - \alpha) \cdot \nabla w dx &= 0, \quad \Im \int \nabla w dx = 0, \quad \Im \int \phi \overline{w} dx = 0.
\end{align*}
\]

The proof of Lemma 4.2 is postponed to the Appendix for simplicity.

**Proof of Proposition 4.1.** Let \( u_n \) be the solution to (1.1), \( P_0 := (\lambda_0, \alpha_0, \beta_0, \gamma_0, \theta_0) = (T - t_n, 0, 0, T - t_n, \frac{1}{12}) \) and \( w_0 := S_T(t_n) \). Let \( \mathcal{U}_n(P_0) \) be as in Lemma 4.2.

Then, \( R_0 = 0 \), and thus Lemma 4.2 yields that there exists \( r > 0 \), depending on \( S_T(t_n) \), and a unique \( C^1 \) map \( \Psi \) from \( \mathcal{U}_n(u_0) \) to \( \mathcal{U}_n(P_0) \) such that (4.7) and (4.8) hold. Moreover, the \( H^1 \)-continuity of \( u_n \) implies that for \( t_n^* \) close to \( t_n \), \( u_n(t) \in \mathcal{U}_n(u_0) \) for any \( t \in [t_n^*, t_n] \). Thus, letting \( w_n(t) \) be defined as in (1.7), \( \Psi(u_n(t)) \) replacing \( \mathcal{P}_n \) and \( R_n(t) := u_n(t) - w_n(t) \), we obtain that for any \( t \in [t_n^*, t_n] \), \( \mathcal{P}_n(t) := \Psi(u_n(t)) \in \mathcal{U}_n(P_0) \) and (4.7) and (4.8) hold with \( u_0(t), w_n(t), R_n(t) \) and \( \mathcal{P}_n(t) \) replacing \( u, w, R \) and \( \Psi(u) \), respectively, which immediately yields (4.2) and (4.4).

As regards the \( C^1 \)-regularity of modulation parameters, in view of the \( C^1 \)-regularity of \( \Psi \) and equation (1.1), it suffices to show that \( u_n \in C([t_n^*, t_n]; \mathcal{H}^3) \).

For this purpose, we set \( \mathcal{X}(I) := C(I; L^2) \cap L^{2+\frac{4}{5}}(I \times \mathbb{R}^d) \cap L^2(I; H^\frac{1}{2-1}) \) and let \( \mathcal{X}'(I) \) be its dual space. Then, similarly to (2.11), we have \( \|u\|_{\mathcal{X}(t_n^*, t_n)} + \|\nabla u\|_{\mathcal{X}(t_n^*, t_n)} < \infty \). In particular, we take a finite partition \( \{I_j\}_{j=1}^{L} \) of \([t_n^*, t_n]\) such that \( \|u\|_{L^{2+\frac{4}{5}}(I_j \times \mathbb{R}^d)} \leq \varepsilon \), where \( I_j := [t_n - j\Delta t, t_n - (j - 1)\Delta t] \), \( 1 \leq j \leq L - 1 \), \( I_L := [t_n^*, t_n - (L - 1)\Delta t] \) and \( L < \infty \).

For any multi-index \( |\nu| = 2 \), we obtain from (1.1) that

\[
i \partial_x \partial_x \nu u + \Delta \partial_x \nu u + (b \cdot \nabla + c) \partial_x \nu u + \left( \frac{2}{d} + 1 \right) |u|^\frac{2}{d} \partial_x ^\nu u + \frac{2}{d} |u|^\frac{2}{d} u^2 \partial_x ^\nu \overline{u} + G(u) = 0,
\]

where

\[
G(u) := \sum_{|\mu_1 + \mu_2| = 2, |\mu_2| = 1} (\partial_x ^{\mu_1} \cdot \nabla + \partial_x ^{\mu_2} u) \partial_x ^{\mu_1} u + \sum_{|\mu_1 + \ldots + \mu_j + \frac{3}{4}| = 2, |\mu_j| \leq 1} \prod_{j=1}^{1+\frac{3}{4}} \partial_x ^{\mu_j + \frac{3}{4} + \frac{i}{4}},
\]

Then, using the Strichartz and local smoothing estimates (see, e.g., [46, Theorem 3.3]) we obtain that for each \( 0 \leq j \leq L - 1 \),

\[
\|\partial_x ^{\nu} u\|_{\mathcal{X}(I_{j+1})} \leq C(\|\partial_x ^{\nu} u(t_n - j\Delta t)\|_2 + \|u\|^{\frac{4}{d}} \|\partial_x ^{\nu} u\|_{L^{2+\frac{4}{5}}(I_{j+1} \times \mathbb{R}^d)} + \|G(u)\|_{\mathcal{X}'(I_{j+1})}),
\]

which along with the asymptotically flatness of \( \partial_x ^{\mu_1} \cdot \nabla, \partial_x ^{\mu_1} u \) and the Hölder inequality yields

\[
\|\partial_x ^{\nu} u\|_{\mathcal{X}(I_{j+1})} \leq C \left( \|\partial_x ^{\nu} u(t_n - j\Delta t)\|_2 + \|u\|^{\frac{4}{d}} \|\partial_x ^{\nu} u\|_{L^{2+\frac{4}{5}}(I_{j+1} \times \mathbb{R}^d)} \right) + C \left( \|\partial_x ^{\nu} u(t_n - j\Delta t)\|_2 + \varepsilon \|\partial_x ^{\nu} u\|_{\mathcal{X}'(I_{j+1})} \right)
\]

\[
\leq C \left( \|\partial_x ^{\nu} u(t_n - j\Delta t)\|_2 + \varepsilon \|\partial_x ^{\nu} u\|_{\mathcal{X}'(I_{j+1})} + 1 \right).
\]
Hence, for \( \varepsilon \) small enough we obtain that for any \( |\nu| = 2 \),
\[
\|\partial^j_x u\|_{X(t_{j+1})} \leq C(\|\partial^j_x u(t_n - j\Delta t)\|_2 + 1), \quad 0 \leq j \leq L - 1.
\]

Thus, taking into account \( u(t_n) = S_T(t_n) \in H^m \) for any \( m \geq 0 \), we can use (4.9) and inductive arguments to obtain that \( \|\partial^j_x u\|_{X(t_n)} < \infty \) for any \( |\nu| = 2 \), which, in particular, yields that \( u \in C([t^*_n, t_n]; H^2) \). One can also use similar arguments as above to show that \( u \in C([t^*_n, t_n]; H^3) \) and so obtain the \( C^1 \)-regularity of modulation parameters.

Therefore, the proof is complete. \( \square \)

4.2. Modulation parameters. This subsection contains the analysis of the dynamics of modulation parameters and the remainder. Below we set \( g_t := \frac{d}{dt}g \) for any \( C^1 \) function \( g \) for simplicity. Set
\[
P_n(t) := |\lambda_n(t)| + |\alpha_n(t)| + |\beta_n(t)| + |\gamma_n(t)|,
\]
and denote the vector of modulation equations by
\[
\text{Mod}_n(t) := |\lambda_n| + |\alpha_n| + |\beta_n| + |\gamma_n|,
\]
\[
\text{Mod}_n(s) := |\lambda_{n,s}| + |\beta_n(s) + \gamma_n| + |\alpha_{n,s}| - 2|\beta_n| + |\beta_n + \gamma_n| + |\alpha_{n,s} - 1 - |\beta_n|^2|.
\]

Note that, since \( S_T \) is the main profile for the approximating solutions near the blow-up time, one has \( \text{Mod}_n(t) \approx 0 \) for \( t \) close to \( T \), i.e., the leading order ODE system driving modulation parameters takes the form
\[
\lambda \lambda_t + \gamma = 0, \quad \lambda^2 \gamma_t + \gamma^2 = 0, \quad \lambda \alpha_t - 2\beta = 0, \quad \lambda^2 \beta_t + \beta \gamma = 0, \quad \lambda^2 \theta_t - 1 - |\beta|^2 = 0.
\]

As a matter of fact, the modulation equations arise from the equation of remainder \( \varepsilon_n \) formulated in (4.10) below, which is obtained by linearizing the equation of (2.7) around the main profile \( w_n \) that is close to \( S_T \) near the blow-up time. Precisely, plugging the decomposition (4.2) into (4.1) and using the identities
\[
\Delta Q_{\varepsilon_n} - Q_{\varepsilon_n} + f(Q_{\varepsilon_n}) = |\beta_n - \gamma_n| \quad Q_{\varepsilon_n} - i\gamma_n \Lambda Q_{\varepsilon_n} + 2i \beta_n \cdot \nabla Q_{\varepsilon_n},
\]
where \( f(z) = |z|^2z \) for any \( z \in \mathbb{C} \), and
\[
\partial_t Q_{\varepsilon_n} = i(\beta_{n,s} \cdot y - \frac{\gamma_{n,s}}{4}|y|^2)Q_{\varepsilon_n},
\]
we obtain the equation of \( \varepsilon_n \) below
\[
i \partial_t \varepsilon_n - M(\varepsilon_n) - \frac{\lambda_n}{\lambda_{n,s}} \Lambda \varepsilon_n - i \frac{\alpha_n}{\lambda_{n,s}} \cdot \nabla \varepsilon_n - (\theta_{n,s} - 1) \varepsilon_n + N_f(Q_{\varepsilon_n}, \varepsilon_n)
\]
\[
= - (\lambda \beta_n \cdot \nabla + \lambda^2 \varepsilon_n)(Q_{\varepsilon_n} + \varepsilon_n)
\]
\[
+ (\beta_{n,s} + \beta_n \gamma_n) \cdot y Q_{\varepsilon_n} + i \left( \frac{\lambda_n}{\lambda_{n,s}} + \gamma_n \right) \Lambda Q_{\varepsilon_n} + i \left( \frac{\alpha_n}{\lambda_{n,s}} - 2\beta_n \right) \cdot \nabla Q_{\varepsilon_n}
\]
\[
+ (\theta_{n,s} - 1 - |\beta_n|^2) Q_{\varepsilon_n} - \frac{\gamma_n^2 + \gamma_{n,s}^2}{4}|y|^2 Q_{\varepsilon_n}.
\]
Here, \( M(\varepsilon_n) \) denotes the linearized operator around \( Q_{\varepsilon_n} \) with respect to \( \varepsilon_n \)
\[
M(\varepsilon_n) := -\Delta \varepsilon_n + \varepsilon_n - \left( 1 + \frac{2}{d} \right)|Q_{\varepsilon_n}|^2 \varepsilon_n - \frac{2}{d}|Q_{\varepsilon_n}|^2 \varepsilon_n, \quad \varepsilon_n = \|\varepsilon_n\|_{L^2}.
\]
the term \( N_f(Q_{\nu_n}, \varepsilon_n) \) contains the nonlinear terms of \( \varepsilon_n \), i.e.,

\[
N_f(Q_{\nu_n}, \varepsilon_n) := f(Q_{\nu_n} + \varepsilon_n) - f(Q_{\nu_n}) - f'(Q_{\nu_n}) \cdot \varepsilon_n
\]

\[
(4.22)
\]

(4.22)

\[
= |Q_{\nu_n} + \varepsilon_n|^\frac{4}{d}(Q_{\nu_n} + \varepsilon_n) - |Q_{\nu_n}|^\frac{4}{d}Q_{\nu_n} - (1 + \frac{2}{d})|Q_{\nu_n}|^\frac{4}{d} \varepsilon_n - \frac{2}{d}|Q_{\nu_n}|^\frac{4}{d-2}Q_{\nu_n}^2 \varepsilon_n,
\]

and the coefficients of lower order perturbations \( N \) below, as the other four modulation equations can be estimated similarly. For simplicity, Proposition 4.3 below contains the key estimate for the modulation equations.

\[
\text{Let } \text{Proposition 4.3.}
\]

\[
\text{and the real and imaginary parts. We have that for some }\lambda_n \text{, independent of } n \text{, such that for any } t \in [t_n^*, t_n],
\]

\[
\text{We also note that, by the orthogonality conditions (4.4),}
\]

\[
\begin{align*}
\text{Re } &\int y Q_{\nu_n}(t) \overline{\sigma_n(t)} dy = 0, \quad \text{Re } \int |y|^2 Q_{\nu_n}(t) \overline{\sigma_n(t)} dy = 0, \\
\text{Im } &\int \Lambda Q_{\nu_n}(t) \overline{\sigma_n(t)} dy = 0, \quad \text{Im } \int \nabla Q_{\nu_n}(t) \overline{\sigma_n(t)} dy = 0, \quad \text{Im } \int \rho_{\nu_n}(t) \overline{\sigma_n(t)} dy = 0.
\end{align*}
\]

\[
(4.23)
\]

Proposition 4.3 below contains the key estimate for the modulation equations.

**Proposition 4.3.** Let \( t_n^* \) be close to \( t_n \) such that \( \mathcal{P} + \| \varepsilon \|_{L^2} \) is sufficiently small on \([t_n^*, t_n]\). Then, there exists \( C > 0 \), independent of \( n \), such that for any \( t \in [t_n^*, t_n] \),

\[
\begin{align*}
\text{Mod}_n(t) &\leq C(P_n^2(t)\|\varepsilon_n(t)\|_{L^2} + \|\varepsilon_n(t)\|_{H^1}^2 + \sum_{k=3}^{1+\frac{4}{d}} \|\varepsilon_n(t)\|_{H^k}^{2} + P_n^0(t)).
\end{align*}
\]

**Proof of Proposition 4.3.** We mainly prove in detail the estimate for \( \frac{\lambda_n}{\gamma_n} + \gamma_n \) below, as the other four modulation equations can be estimated similarly. For simplicity, the dependence on \( n \) is omitted below.

In the estimates below we shall use the estimates \((4.17)\) and \((4.18)\) and the orthogonality conditions \((4.21)\) above. We will also frequently use the fact that

\[
(1 + |y|^n)(Q + \nabla Q) \in L^\infty \cap L^2, \quad n \geq 1.
\]

\[
(4.23)
\]
Now, taking the inner product of (4.14) with $|y|^2 Q_P$ and then taking the imaginary part we obtain that, if $\varepsilon := \varepsilon_1 + i \varepsilon_2$ in terms of real and imaginary parts,
\[
\partial_s \{\langle \varepsilon_1, |y|^2 \Sigma \rangle + \langle \varepsilon_2, |y|^2 \Theta \rangle \} - \langle \varepsilon_1, |y|^2 \Sigma_s \rangle - \langle \varepsilon_2, |y|^2 \Theta_s \rangle = 0
\]
\[
= \langle M_1(\varepsilon), |y|^2 \Theta \rangle - \langle M_2(\varepsilon), |y|^2 \Sigma \rangle - 2\beta \{\langle \nabla \varepsilon_1, |y|^2 \Sigma \rangle + \langle \nabla \varepsilon_2, |y|^2 \Theta \rangle \}
+ \gamma \{\langle \Lambda \varepsilon_1, |y|^2 \Sigma \rangle + \langle \Lambda \varepsilon_2, |y|^2 \Theta \rangle \} + |\beta|^2 \{\langle \varepsilon_1, |y|^2 \Theta \rangle - \langle \varepsilon_2, |y|^2 \Sigma \rangle \}
+ \text{Im} \langle N_f(Q_P, \varepsilon), |y|^2 Q_P \rangle
= - \text{Im} \langle \tilde{\alpha} \cdot \nabla + \Lambda^2 \varphi \rangle (\varepsilon + Q_P), |y|^2 Q_P \rangle
\]
\[
+ \left( \frac{\lambda_s}{\lambda} + \gamma \right) \{\langle \Lambda \Sigma, |y|^2 \Sigma \rangle + \langle \Lambda \Theta, |y|^2 \Theta \rangle + \langle \Lambda \varepsilon_1, |y|^2 \Sigma \rangle + \langle \Lambda \varepsilon_2, |y|^2 \Theta \rangle \}
+ \left( \frac{\alpha_s}{\lambda} - 2\beta \right) \{\langle \nabla \Theta, |y|^2 \Theta \rangle + \langle \nabla \Sigma, |y|^2 \Sigma \rangle + \langle \nabla \varepsilon_1, |y|^2 \Sigma \rangle + \langle \nabla \varepsilon_2, |y|^2 \Theta \rangle \}
\]
\[
\langle \varepsilon_1, |y|^2 \Theta \rangle - \langle \varepsilon_2, |y|^2 \Sigma \rangle \}
\]
Here, the left-hand side above is arranged according to the orders of $\varepsilon_1$ and $\varepsilon_2$, while the right-hand side contains the lower order perturbations and the modulation equations.

The key point is, that the orthogonality condition (4.21) enables us to gain the negligible smallness $\mathcal{O}(P^2 ||\varepsilon||_{L^2})$ for the linear terms of $\varepsilon_1$ and $\varepsilon_2$, while the nonlinear terms can be estimated easily by using (4.17) and (4.18). The flatness condition (2.3) will be also used to control the lower order perturbations. Moreover, the coefficients of modulation equations on the right-hand side of (4.24) are all negligible, except the one $\langle \Lambda \Sigma, |y|^2 \Sigma \rangle$ which gives the non-small coefficient $-||yQ||_{L^2}^2$ and thus yields the bound of $|\frac{\alpha_s}{\lambda} + \gamma|$.

To be precise, using the orthogonality condition (4.21) we have
\[
\partial_s \{\langle \varepsilon_1, |y|^2 \Sigma \rangle + \langle \varepsilon_2, |y|^2 \Theta \rangle \} = \partial_s \text{Re} \int |y|^2 Q_P \varphi dy = 0,
\]
and by (4.18),
\[
\langle \varepsilon_1, |y|^2 \Sigma_s \rangle + \langle \varepsilon_2, |y|^2 \Theta_s \rangle \leq C (\text{Mod} + P^2) ||\varepsilon||_{L^2},
\]
Moreover, by (4.17), (4.19) and (4.20),
\[
\langle M_1(\varepsilon), |y|^2 \Theta \rangle - \langle M_2(\varepsilon), |y|^2 \Sigma \rangle
= \langle L_+ \varepsilon_1 + \frac{1}{4} Q_4 \varepsilon_1, |y|^2 \Theta \rangle - \langle L_- \varepsilon_2, |y|^2 \Sigma \rangle + \mathcal{O}(P^2 ||\varepsilon||_{L^2})
= \langle L_+ \varepsilon_1, (\beta \cdot y - \frac{\gamma}{4} |y|^2) |y|^2 Q \rangle - \langle L_- \varepsilon_2, |y|^2 Q \rangle + \mathcal{O}(P^2 ||\varepsilon||_{L^2}).
\]
This yields that the second and third lines on the left-hand side of (4.24) equal to
\[
|\langle L_+ \varepsilon_1, (\beta \cdot y - \frac{\gamma}{4} |y|^2) |y|^2 Q \rangle - \langle L_- \varepsilon_2, |y|^2 Q \rangle + \gamma (\Lambda \varepsilon_1, |y|^2 Q) - \langle L_- \varepsilon_2, |y|^2 Q \rangle|
+ \mathcal{O}(P^2 ||\varepsilon||_{L^2})
= |\langle \varepsilon_1, (\beta \cdot y - \frac{\gamma}{4} |y|^2) L_+ |y|^2 Q \rangle - \langle \varepsilon_2, L_- |y|^2 Q \rangle| + \mathcal{O}(P^2 ||\varepsilon||_{L^2}),
\]
\[
= |4 \langle \varepsilon_1, (\beta \cdot y - \frac{\gamma}{4} |y|^2) \Lambda Q \rangle - 4 \langle \varepsilon_2, \Lambda Q \rangle| + \mathcal{O}(P^2 ||\varepsilon||_{L^2}),
\]
where the last step is due to the identity $L_- |y|^2 Q = -4 \Lambda Q$ in (3.4). We claim that
\[
4 \langle \varepsilon_1, (\beta \cdot y - \frac{\gamma}{4} |y|^2) \Lambda Q \rangle - 4 \langle \varepsilon_2, \Lambda Q \rangle = \mathcal{O}(P^2 ||\varepsilon||_{L^2}).
\]
To this end, since \( Q_{\mathcal{P}} = \Sigma + i\Theta \), using (4.17) we have
\[
\Re \Lambda Q_{\mathcal{P}} = \Lambda Q + \mathcal{O}(P^2 e^{-\delta|\nu|}),
\]
\[
\Im \Lambda Q_{\mathcal{P}} = (\beta \cdot y - \frac{\gamma}{4}|y|^2)\Lambda Q + (\beta \cdot y - \frac{\gamma}{2}|y|^2)Q + \mathcal{O}(P^2 e^{-\delta|\nu|}).
\]

Taking into account (4.17) again we get
\[
4\langle \varepsilon_1, (\beta \cdot y - \frac{\gamma}{4}|y|^2)\Lambda Q \rangle - 4\langle \varepsilon_2, \Lambda Q \rangle
\]
\[
= 4[\langle \varepsilon_1, \Im \Lambda Q_{\mathcal{P}} \rangle - \langle \varepsilon_2, \Re \Lambda Q_{\mathcal{P}} \rangle] - 4\beta \langle \varepsilon_1, \Re y Q_{\mathcal{P}} \rangle + 2\gamma \langle \varepsilon_1, \Re |y|^2 Q_{\mathcal{P}} \rangle + \mathcal{O}(P^2\|\varepsilon\|_2)
\]
\[
= 4\Im \int \Lambda Q_{\mathcal{P}} \bar{\varepsilon} dy - 4\beta \Re \int y Q_{\mathcal{P}} \bar{\varepsilon} dy + 2\gamma \Re \int |y|^2 Q_{\mathcal{P}} \bar{\varepsilon} dy + \mathcal{O}(P^2\|\varepsilon\|_2).
\]

Then, applying the orthogonality condition (4.22) we obtain (4.28), as claimed.

Hence, putting together (4.25), (4.26), (4.27) and (4.28) we obtain that the linear terms on the left-hand side of (4.24) is bounded by
\[
C(\text{Mod}(t) + P^2)\|\varepsilon\|_{L^2}.
\]

The nonlinear term involving \( N_f(Q_{\mathcal{P}}, \varepsilon) \) can be bounded directly by using Lemma 3.14 and (4.23). Actually, using (3.14), (4.23) and Gagliardo-Nirenberg’s inequality we get
\[
|\Im \langle N_f(Q_{\mathcal{P}}, \varepsilon), |y|^2 Q_{\mathcal{P}} \rangle| \leq C \sum_{k=2}^{1+\frac{\delta}{4}} \int |\varepsilon|^k dy \leq C(\|\varepsilon\|_{L^2}^2 + \sum_{k=3}^{1+\frac{\delta}{4}} \|\varepsilon\|_{H^k}^k).
\]

Next we treat the right-hand side of (4.24). Set \( \partial_y^\nu \tilde{\phi}_k(y) := (\partial_y^\nu \phi_k)(\lambda y + \alpha) \). We have
\[
\Im (\lambda \tilde{b} \cdot \nabla + \lambda^2 \tilde{c})(\varepsilon + Q_{\mathcal{P}}), |y|^2 Q_{\mathcal{P}}
\]
\[
= 2\lambda \sum_{k=1}^N B_k \{ \langle \nabla \tilde{\phi}_k \cdot (\nabla \Sigma + \nabla \varepsilon_1), |y|^2 \Sigma \rangle + \langle \nabla \tilde{\phi}_k \cdot (\nabla \Theta + \nabla \varepsilon_2), |y|^2 \Theta \rangle \}
\]
\[
+ \lambda^2 \sum_{k=1}^N B_k \{ \langle \Delta \tilde{\phi}_k (\Sigma + \varepsilon_1), |y|^2 \Sigma \rangle + \langle \Delta \tilde{\phi}_k (\Theta + \varepsilon_2), |y|^2 \Theta \rangle \}
\]
\[
- \lambda^2 \{ \sum_{j=1}^d \sum_{k=1}^N \partial_j \tilde{\phi}_k B_k (\theta + \varepsilon_2), |y|^2 \Sigma \} - \lambda^2 \{ \sum_{j=1}^d \sum_{k=1}^N \partial_j \tilde{\phi}_k B_k (\Sigma + \varepsilon_1), |y|^2 \Theta \}\}
\]
\[
(4.31)
\]
Using Taylor’s expansion, the flatness condition (2.3) and that \( \partial_y^\nu \tilde{\phi}_k \in L^\infty \) for any multi-index \( \nu \), we have
\[
|\langle \partial_y^\nu \tilde{\phi}_k(y) \rangle| \leq C(\lambda y + \alpha)^6-|\nu| \leq C P^6-|\nu| (1 + |y|^6), \quad 0 \leq |\nu| \leq 5.
\]

Then, using the integration by parts formula and (4.23) again we have
\[
|2\lambda B_k \langle \nabla \tilde{\phi}_k \cdot (\nabla \Sigma + \nabla \varepsilon_1), |y|^2 \Sigma \rangle| \leq C P^6(1 + \|\varepsilon\|_{L^2})
\]
\[
(4.33)
\]
We can estimate the other terms on the right-hand side of (4.31) similarly and so obtain
\[
|\Im (\lambda \tilde{b} \cdot \nabla + \lambda^2 \tilde{c})(\varepsilon + Q_{\mathcal{P}}), |y|^2 Q_{\mathcal{P}}| \leq C P^6(1 + \|\varepsilon\|_{L^2})
\]
\[
(4.34)
\]
It remains to treat the coefficients of modulation equations on the right-hand side of (4.24). Straightforward computations yield that
\[
\langle \Lambda \Sigma, |y|^2 \Sigma \rangle = -\int |y|^2 |\Sigma|^2 dy = -\|y Q\|^2_{L^2} + \mathcal{O}(P^2),
\]
(4.35)
where in the last step we used (4.17). Moreover, since $\nabla Q$ is odd while $Q$ is even, we have $\langle \nabla Q, |y|^{2}Q \rangle = 0$, which along with (4.17) implies that
\[
\langle \nabla \Sigma, |y|^{2}\Sigma \rangle = \langle \nabla Q, |y|^{2}Q \rangle + \mathcal{O}(P^{2}) = \mathcal{O}(P^{2}).
\]
The remaining coefficients of $(\lambda + \gamma)$ can be easily bounded by $C(\|\epsilon\|_{L^{2}} + P^{2})$, by using (4.17) and (4.23). Thus, the right-hand side of (4.24) is bounded by
\[
-\|yQ\|_{L^{2}}^{2}(\frac{\lambda}{\chi} + \gamma) + CMod(\|\epsilon\|_{L^{2}} + P^{2}) + CP^{6}(1 + \|\epsilon\|_{L^{2}}).
\]
Therefore, plugging (4.29), (4.30) and (4.36) into equation (4.24) we obtain
\[
\|yQ\|_{L^{2}}^{2}(\frac{\lambda}{\chi} + \gamma) \leq C[(P + \|\epsilon\|_{L^{2}})Mod + P^{2}\|\epsilon\|_{L^{2}} + \|\epsilon\|_{L^{2}}^{2} + \sum_{k=3}^{1+\delta}\|\epsilon\|_{H^{k}}^{k} + P^{6}(1 + \|\epsilon\|_{L^{2}})).
\]
Similar arguments apply also to the other four modulation equations. Actually, taking the inner products of equation (4.14) with $yQ_{P}$, $i\Lambda Q_{P}$, $i\nabla Q_{P}$, $i\rho_{P}$, respectively, then taking the imaginary parts and using analogous arguments as above, we can obtain the same bounds for $\frac{1}{2}\|Q\|_{L^{2}}^{2}|\frac{\lambda}{\chi} - 2\beta|$, $\frac{1}{2}\|yQ\|_{L^{2}}^{2}|\gamma_{n} + \gamma^{2}|$, $\frac{1}{4}\|Q\|_{L^{2}}^{2}|\beta + \beta\gamma|$ and $\frac{1}{2}\|yQ\|_{L^{2}}^{2}|\theta_{n} - 1 - |\beta|^{2}|$, respectively.
Therefore, taking $P + \|\epsilon\|_{L^{2}}$ small enough such that $C(P + \|\epsilon\|_{L^{2}}) \leq \frac{1}{2}$ we obtain (4.22) and finish the proof. □

5. Uniform estimates of approximating solutions

5.1. Main results. This section is devoted to the key uniform estimates of approximating solutions. The main result is formulated in Theorem 5.1 below.

Theorem 5.1 (Uniform estimates). Let $0 < \delta < \frac{1}{12}$. Then, there exists small $\tau^{*} > 0$, such that for any $T \in (0, \tau^{*}]$ and for $n$ large enough, $u_{n}$ admits the unique geometrical decomposition $u_{n} = \omega_{n} + R_{n}$ on $[0, t_{n}]$ as in (1.2), and the following estimates hold:

(i) For the reminder term,
\[
\|\nabla R_{n}(t)\|_{2} \leq (T - t)^{2}, \quad \|R_{n}(t)\|_{2} \leq (T - t)^{3}.
\]

(ii) For the modulation parameters,
\[
|\lambda_{n}(t) - (T - t)| + |\gamma_{n}(t) - (T - t)| \leq (T - t)^{3+\delta},
\]
\[
|\alpha_{n}(t)| + |\beta_{n}(t)| \leq (T - t)^{2+\delta},
\]
\[
|\theta_{n}(t) - \frac{1}{T - t}| \leq (T - t)^{1+\delta}.
\]

The proof of Theorem 5.1 is based on continuity arguments and Proposition 5.2 below.

Proposition 5.2 (Bootstrap). Fix $0 < \delta < \frac{1}{12}$ and $n$ large enough. Suppose that there exists $t^{*} \in (0, t_{n})$ such that $u_{n}$ admits the unique geometrical decomposition (1.2) on $[t^{*}, t_{n}]$ and the estimates (5.1)–(5.4) hold.
Lemma 5.5. Assume estimates \( C > 0 \) exist a constant \( \eta \) forwardly from the explicit expression of (5.18). Mod\(n\)

Lemma 5.3. Assume estimates (5.6)-(5.8) have that for some

Lemma 5.4. Assume estimates (5.12) and (5.16) using (5.1).

By virtue of estimates (5.9)-(5.13), we have the refined estimates below.

In order to prove Proposition 5.2, we apply Lemma 4.2 with \( u_0 \) and \( P_0 \) to obtain that there exists \( t_0 < 0 \) such that \( u_0 \) has the geometrical decomposition \( u_0 = w_0 + R_0 \) on \( [t_0, t_0] \) as in (4.2).

Moreover, using the local well-posedness theory and the \( C^1 \)-regularity of modulation parameters, we can take \( t_0 \) sufficiently close to \( t^* \) such that for any \( t \in [t_0, t_0] \),

In particular, estimate (5.10) yields that if \( T^{2+\delta} \leq \frac{1}{4} \),

By virtue of estimates (5.9)-(5.13), we have the refined estimates below.

**Lemma 5.3.** Assume estimates (5.9)-(5.12) to hold. Then, there exists \( C > 0 \) independent of \( n \) such that for all \( t \in [t_0, t_0] \),

We also note that the estimate of modulation equations in (5.12) still holds on \( [t_0, t_0] \). Then, using Lemma 5.3 we have the refined estimate of modulation equations below.

**Lemma 5.4.** Assume estimates (5.9)-(5.12) to hold with \( T \) sufficiently small. Then, we have that for some \( C > 0 \) independent of \( n \),

Lemma 5.5 below contains the estimates of \( \eta \) given by (2.15), which follow straightforwardly from the explicit expression of \( \eta \) and the estimate (4.32).

**Lemma 5.5.** Assume estimates (5.9)-(5.12) to hold with \( T \) sufficiently small. There exists a constant \( C > 0 \) independent of \( n \), such that for all \( t \in [t_0, t_0] \),

\[
\| \partial^\nu \eta_n(t) \|_{L^2} \leq \frac{C}{\lambda_n^{2+|\nu|}} (\text{Mod}_n(t) + (\lambda_n(t))^6) \leq C \lambda_n^{3-|\nu|} \quad \forall 0 \leq |\nu| \leq 2.
\]
Below Subsections 5.2 and 5.3 contain the key analysis of the energy (1.6) and the monotonicity formula of the Lyapounov functional (5.34), respectively. We shall assume that estimates (5.9)-(5.12) hold on \([t_\ast, t_n]\) with \(T\) sufficiently small throughout Subsections 5.2 and 5.3.

### 5.2. Energy estimate

Because of the presence of lower order perturbations, the energy (1.6) of solutions to equation (2.7) is no longer conserved. Here, under Assumption (A1), we are able to control the variation of energy up to a suitable order, which is a key ingredient to obtain the refined estimate (5.7) for the modulation parameters \(\alpha_n\) and \(\beta_n\).

**Theorem 5.6.** There exists \(C > 0\), independent of \(n\), such that for any \(t \in [t_\ast, t_n]\),

\[
|E(u_n)(t) - E(u_n)(t_\ast)| \leq C(T - t)^3.
\]

**Proof.** Using (4.11) and the integration by parts formula we have

\[
\begin{align*}
\frac{d}{dt}E(u_n) &= - \text{Im} \int (\overline{b} \cdot \nabla u_n + \overline{u_n})(\nabla u_n + |u_n|^4u_n)\,dx \\
&= - 2 \sum_{k=1}^{N} B_k \text{Re} \int \nabla^2 \phi_k(\nabla u_n, \nabla u_n)\,dx + \frac{1}{2} \sum_{k=1}^{N} B_k \int \Delta^2 \phi_k |u_n|^2\,dx \\
&+ \frac{2}{d + 2} \sum_{k=1}^{N} B_k \int \Delta \phi_k |u_n|^{2+\frac{4}{d}}\,dx - \sum_{j=1}^{d} \text{Im} \int \nabla \left( \sum_{k=1}^{N} \partial_j \phi_k B_k \right)^2 \cdot \nabla u_n u_n\,dx.
\end{align*}
\]

Then, by (4.2),

\[
\left| \frac{d}{dt}E(u_n) \right| \leq \frac{C}{\lambda_n^2} \sum_{k=1}^{N} \int (|\nabla^2 \phi_k| + |\Delta \phi_k| + |\partial_j \phi_k \nabla \partial_j \phi_k|)(\lambda_n y + \alpha_n)(|\nabla Q|^2 + Q^{2+\frac{4}{d}})(y)\,dy
\]

\[
+ \sum_{k=1}^{N} \sum_{j=1}^{d} \int (|\Delta \phi_k| + |\partial_j \phi_k \nabla \partial_j \phi_k|)(\lambda_n y + \alpha_n)Q^2(y)\,dy
\]

\[
+ C(\|R\|_{H^1}^2 + \|R\|_{L^{2+\frac{4}{d}}}^{2+\frac{4}{d}}) =: K_1 + K_2 + K_3.
\]

Using (1.23) and (1.32) we have

\[
K_1 + K_2 \leq C\lambda_n^2.
\]

Moreover, applying the Gagliardo-Nirenberg inequality (3.16) and using (5.15) we get

\[
K_3 \leq C(\|R\|_{H^1}^2 + \|R\|_{L^{2+\frac{4}{d}}}^{2+\frac{4}{d}}) \leq C\lambda_n^4.
\]

Thus, plugging (5.22) and (5.23) into (5.21) and using (5.13) we obtain

\[
\left| \frac{d}{dt}E(u_n) \right| \leq C(T - t)^2,
\]

which yields (5.19) and finishes the proof. □

As a consequence of Theorem 5.6 and the coercivity of operator \(L\), we have

**Lemma 5.7.** There exists \(C > 0\) independent of \(n\) such that for all \(t \in [t_\ast, t_n]\),

\[
|\beta_n(t)|^2 \leq C|\lambda_n^2(t) - \gamma_n^2(t)| + C(T - t)^5.
\]
Proof. Let \( F(z) := \frac{d}{dz} |z|^{\frac{d+2}{2}}, \) \( z \in \mathbb{C} \). We suppress the \( n \) dependence below. First note that, since \( M(u)(t) = M(u)(t_n) = \|Q\|_{L^2}^2 \),

\[(5.26) \quad 2 \text{Re} \int Q_P \bar{\varepsilon} dy + \|\varepsilon(t)\|_{L^2}^2 = 0.\]

The identity \((5.26)\) allows to gain an extra factor \( \|\varepsilon\|_{L^2} \) when estimating \( \text{Re} \int Q_P \bar{\varepsilon} dy \).

Now, rewriting \( E(u) \) in the renormalized variables and using \((5.26)\) we have

\[(5.27) \quad E(u) = \frac{1}{2\lambda^2} \int |\nabla Q_P + \nabla \varepsilon|^2 dy - \frac{1}{\lambda^2} \int F(Q_P + \varepsilon) dy + \frac{1}{\lambda^2} \text{Re} \int Q_P \bar{\varepsilon} dy + \frac{1}{2\lambda^2} \int |\varepsilon|^2 dy.\]

Then, using the identities \( E(Q) = 0 \), \( |\nabla Q_P|^2 = |\nabla Q|^2 + |\beta - \frac{\gamma}{2} y|^2 |Q|^2 \) and the expansion \((3.15)\) and then rearranging each term according to the order of \( \varepsilon \) we come to

\[(5.28) \quad E(u) = \frac{1}{2\lambda^2} \int |\beta - \frac{\gamma}{2} y|^2 |Q|^2 dy - \frac{1}{\lambda^2} \text{Re} \int (\Delta Q_P - Q_P + |Q_P|^2 Q_P) \bar{\varepsilon} dy + \frac{1}{2\lambda^2} \int |\nabla \varepsilon|^2 + |\varepsilon|^2 - (1 + \frac{2}{d}) |Q_P|^2 |\varepsilon|^2 - \frac{2}{d} |Q_P|^4 Q_P^2 \bar{\varepsilon}^2 dy + O\left(\frac{1}{\lambda^2} \sum_{k=3}^{2+\frac{d}{2}} \|\varepsilon\|_{L^1}^k\right).\]

In order to estimate the linear term of \( \varepsilon \) in \((5.28)\) above, using \((4.12)\) and \((4.21)\) we get

\[(5.29) \quad \text{Re} \int (\Delta Q_P - Q_P + |Q_P|^2 Q_P) \bar{\varepsilon} dy = \text{Im} \int (\gamma \Delta Q_P - 2\beta \cdot \nabla Q_P) \bar{\varepsilon} dy + \text{Re} \int |\beta - \frac{\gamma}{2} y|^2 |Q_P| \bar{\varepsilon} dy = \mathcal{O}(P^2 \|\varepsilon\|_{L^2}).\]

Moreover, the quadratic interactions of \( \varepsilon \) can be bounded from below by using the coercivity of operator \( L \). Precisely, using \((4.17)\) we get that if \( \varepsilon := \varepsilon_1 + i\varepsilon_2 \),

\[(5.30) \quad \text{Re} \int |\nabla \varepsilon|^2 + |\varepsilon|^2 - (1 + \frac{2}{d}) |Q_P|^2 |\varepsilon|^2 - \frac{2}{d} |Q_P|^4 |Q_P^2 \bar{\varepsilon}^2 dy = \langle L+\varepsilon_1, \varepsilon_1 \rangle + \langle L-\varepsilon_2, \varepsilon_2 \rangle + \mathcal{O}(P \|\varepsilon\|_{L^2}^2) \geq \nu \|\varepsilon\|_{H^1}^2 + \mathcal{O}(P \|\varepsilon\|_{L^2}^2),\]

where the last step is due to Corollary \((3.2)\) and the estimates

\[(5.31) \quad \langle Q, \varepsilon_1 \rangle = \frac{1}{2} \|\varepsilon\|_{L^2}^2 + \mathcal{O}(P \|\varepsilon\|_{L^2}), \quad \langle yQ, \varepsilon_1 \rangle = \mathcal{O}(P \|\varepsilon\|_{L^2}), \quad \langle y^2 Q, \varepsilon_1 \rangle = \mathcal{O}(P \|\varepsilon\|_{L^2}), \quad \langle y^4 \varepsilon_1 \rangle = \mathcal{O}(P \|\varepsilon\|_{L^2}), \quad \langle \rho, \varepsilon_2 \rangle = \mathcal{O}(P \|\varepsilon\|_{L^2}),
\]

which follow from \((3.17), (4.21)\) and \((5.26)\).

Hence, plugging \((5.29)\) and \((5.30)\) into \((5.27)\) we obtain that, for \( \|\varepsilon\|_{L^2} \) small enough (or equivalently, \( t_n \) close to \( t_n \)),

\[
\lambda^2 E(u) \geq \frac{1}{2} |Q|^2 \beta^2 + \frac{1}{8} \|yQ\|_{L^2}^2 \gamma^2 + \frac{\nu}{2} \|\varepsilon\|_{H^1}^2 - C(P^2 \|\varepsilon\|_{L^2} + P \|\varepsilon\|_{L^2}^2 + \sum_{k=3}^{2+\frac{d}{2}} \|\varepsilon\|_{H^1}^k).\]
Lemma 5.10. For every $E(u(t_n)) = E(S_T(t_n)) = \frac{1}{8}\|yQ\|_L^2$ we arrive at
\[ \frac{1}{2}\|Q\|_L^2|\beta(t)|^2 + \frac{\nu}{2}\|\varepsilon(t)\|_H^2 \leq \frac{1}{8}\|yQ\|_L^2|\lambda^2(t) - \gamma^2(t)| + \lambda^2(t)|E(u)(t) - E(u)(t_n)| \]
\[ + C(P^2\|\varepsilon\|_L^2 + P\|\varepsilon\|_L^2 + \sum_{k=3}^{2+4}\|\varepsilon\|_H^1), \]
which yields (5.28), due to (5.13), (5.15) and (5.19). The proof is complete. \qed

5.3. Monotonicity of generalized energy. Let $\chi(x) = \psi(|x|)$ be a smooth radial function on $\mathbb{R}^d$, where $\psi$ satisfies $\psi'(r) = r$ if $r \leq 1$, $\psi'(r) = 2 - e^{-r}$ if $r \geq 2$, and
\[ \frac{|\psi''(r)|}{|\psi'(r)|} \leq C, \quad \psi'(r) - \psi''(r) \geq 0. \]
Let $\chi_A(y) := A^2\chi(\frac{y}{A})$, $A > 0$, $y \in \mathbb{R}^d$, and $F(u)$, $f(u)$, $f'(v) \cdot R$ and $N_f(v, R)$ be as in Subsection 3.2. As in (37), we define the generalized energy by
\[ I_n(t) := \frac{1}{2}\int |\nabla R_n|^2 + \frac{|R_n|^2}{\lambda^2_n}dx - \text{Re} \int F(u_n) - F(w_n) - f(w_n)\bar{R}_ndx \]
\[ + \frac{\gamma n}{2\lambda_n} \text{Im} \int (\nabla \chi_A)(\frac{x - \alpha}{\lambda_n}) \cdot \nabla R_n\bar{R}_ndx. \]
The main result of this subsection is formulated below.

Theorem 5.8. For $n$ large enough and for any $t \in [t_*, t_n]$, we have
\[ \frac{dI_n}{dt} \geq C_1 \lambda_n \int (|\nabla R_n|^2 + \frac{|R_n|^2}{\lambda^2_n})e^{-\frac{|x-a|}{\lambda_n}} dx - C_2(A)\lambda^4_n, \]
where $C_1, C_2(A) > 0$ are independent of $n$.

Remark 5.9. Theorem 5.8 enables us to obtain the order $O(\lambda^5_n)$ of the generalized energy, which in turn gives us the important refined estimate (5.5) for the remainder $R_n$. See Subsection 3.4 below.

In order to prove Theorem 5.8, we write $I_n$ into two parts $I_n = I_n^{(1)} + I_n^{(2)}$, where
\[ I_n^{(1)} := \frac{1}{2}\int |\nabla R_n|^2 + \frac{|R_n|^2}{\lambda^2_n}dx - \text{Re} \int F(u_n) - F(w_n) - f(w_n)\bar{R}_ndx, \]
\[ I_n^{(2)} := \frac{\gamma n}{2\lambda_n} \text{Im} \int \nabla \chi_A(\frac{x - \alpha}{\lambda_n}) \cdot \nabla R_n\bar{R}_ndx. \]
The evolution formulas of $I_n^{(i)} := \frac{d}{dt}I_n^{(i)}$, $i = 1, 2$, are contained in Lemmas 5.10 and 5.11 below, respectively. See also (37) for similar computations for the inhomogeneous nonlinear Schrödinger equations in dimension two.

In the sequel, we omit the dependence on $n$ for simplicity.

Lemma 5.10. For every $t \in [t_*, t_n]$, we have
\[ I_t^{(1)} = -\frac{\gamma}{\lambda^2} \text{Re} \int ((1 + \frac{2}{d})|w|^\frac{2}{d} |R|^2 + \frac{1}{2}(1 + \frac{2}{d})|w|^\frac{4}{d} w^2 R^2 + \frac{1}{2}(1 + \frac{2}{d})|w|^\frac{2}{d} w^2 R^2)dx \]
\[ - \frac{\gamma}{\lambda} \text{Re} \int (\frac{x - \alpha}{\lambda}) \cdot \nabla w \left\{ \frac{2}{d}(1 + \frac{2}{d})|w|^\frac{4}{d} w|R|^2 + \frac{1}{d}(1 + \frac{2}{d})|w|^\frac{2}{d} w^2 R^2 \right\} dx + \frac{\gamma n}{\lambda^4} |R|^2_{L^2} + O(\lambda^4). \]
Thus, taking into account (5.15) and (5.18) we obtain
\[ (5.42) \]
\[ (5.43) \]
we have
\[ (5.44) \]
\[ (5.45) \]
\[ (3.11) \]
with
\[ w \]
\[ (5.45) \]
\[ (5.46) \]
\[ (5.47) \]
\[ (5.48) \]
\[ (5.49) \]
\[ (5.50) \]
\[ (5.51) \]
\[ (5.52) \]
\[ (5.53) \]
\[ (5.54) \]

Proof. Since \( \partial_t F(u) = \text{Re} (f(u) \partial_t \vec{w}) \), \( \partial_t f(u) = (1 + \frac{3}{2}) |u|^2 \partial_t u + \frac{3}{2} |u|^3 - u^2 \partial_t \vec{w} \), we have
\[ (5.39) \]
\[ I_t^{(1)} = \text{Re} (-\Delta R + \frac{1}{\lambda^2} R - f(u) + f(w), \partial_t R) - \text{Re} (N_f(w, R), \partial_t w) - \frac{\lambda_t}{\lambda^2} \| R \|_{L^2}^2. \]

Then, using equation (2.14) we obtain
\[ I_t^{(1)} = - \text{Im} \int (\Delta R - \frac{1}{\lambda^2} R + f'(w) \cdot \bar{R}) dx - \text{Re} \int N_f(w, R) \partial_t \bar{w} dx \]
\[ - \frac{1}{\lambda^2} \text{Im} \int \frac{2}{d} |w|^2 |R|^2 dx - \text{Im} \int N_f(w, R) \bar{w} dx - \frac{1}{\lambda^2} \text{Im} \int N_f(w, R) \bar{R} dx \]
\[ - \lambda_t \| R \|_{L^2}^2 - \text{Im} \int (\Delta R - \frac{1}{\lambda^2} R + f(u) - f(w)) (\bar{b} \cdot \nabla \bar{R} + \bar{c} \bar{R}) dx \]
\[ (5.40) \]
\[ = \sum_{j=1}^{7} I_{t,j}^{(1)}. \]

Below we estimate each term on the right-hand side above separately.

(i) Estimate of the first order term \( I_{t,1}^{(1)} \). Using Hölder’s inequality we have
\[ |I_{t,1}^{(1)}| \leq \| \nabla \eta \|_{L^2} \| \nabla R \|_{L^2} + \frac{1}{\lambda^2} \| \eta \|_{L^2} \| R \|_{L^2} + (1 + \frac{4}{d}) \int |\eta| |\bar{w}|^2 |R| dx. \]

Note that, by the pointwise bound \( |w| \leq C \lambda^{-\frac{d}{2}} \) and Hölder’s inequality,
\[ (5.41) \]
\[ \int |\eta| |\bar{w}|^2 |R| dx \leq \frac{1}{\lambda^2} C \| \eta \|_{L^2} \| R \|_{L^2}. \]

Thus, taking into account (5.15) and (5.18) we obtain
\[ (5.42) \]
\[ |I_{t,1}^{(1)}| \leq C (\lambda^2 \| \nabla R \|_{L^2} + \lambda \| R \|_{L^2}) \leq C \lambda^4. \]

(ii) Estimates of the nonlinear terms \( I_{t,j}^{(1)} \), \( 2 \leq j \leq 6 \). Let \( N_{f, 2}(w, R) \) be defined as in (3.11) with \( w \) replacing \( v \) and write
\[ I_{t,2}^{(1)} = - \text{Re} \int N_{f, 2}(w, R) \partial_t \bar{w} dx - \text{Re} \int (N_f(w, R) - N_{f, 2}(w, R)) \partial_t \bar{w} dx \]
\[ (5.43) \]
\[ = I_{t,21}^{(1)} + I_{t,22}^{(1)}. \]

Letting \( y = \frac{\bar{v} - u}{\lambda} \) and using the identities
\[ \partial_t Q_P(t, y) = (i \beta_t \cdot y - i \frac{\gamma_t}{4} |y|^2) Q_P(t, y), \quad \nabla_x w(t, x) = (\lambda(t))^{-\frac{d}{2} - 1} (\nabla_y Q_P)(t, y) e^{i\theta(t)}, \]

we have
\[ (5.44) \]
\[ \partial_t w = - \frac{d \lambda_t}{2 \lambda} w + i \beta_t \cdot y w - i \frac{\gamma_t}{4} |y|^2 w - \lambda_t y \cdot \nabla_x w - \alpha_t \cdot \nabla_x w + i \theta_t w, \]

which along with (5.11)-(5.12) and Lemma 5.4 implies that
\[ (5.45) \]
\[ \partial_t w = \frac{i}{\lambda^2} w + \frac{d \gamma}{2 \lambda^2} w + \frac{\gamma}{\lambda} y \cdot \nabla w - 2 \beta \frac{\lambda}{\lambda} \nabla_x w + O((1 + |y|^2)(|w| + |\nabla w|)). \]
This yields that for the quadratic terms of $R$,

\[
I_{t,21}^{(1)} + I_{t,3}^{(1)} = -\frac{\gamma}{\lambda^2} \text{Re} \int (1 + \frac{2}{d})|w|^{\frac{1}{d}}|R|^2 \cdot \frac{2}{d - 1} |w|^{\frac{1}{d} - 2} w^2 R^2 \, dx \\
- \frac{\gamma}{\lambda} \text{Re} \int \left( \frac{x - \alpha}{\lambda} \right) \cdot \nabla \left\{ \frac{2}{d} (1 + \frac{2}{d}) |w|^{\frac{1}{d} - 2} w |R|^2 + \frac{1}{d} (1 + \frac{2}{d}) |w|^{\frac{1}{d} - 2} \text{Re} \left\{ w^2 R^2 \right\} \right\} \, dx + O\left( \frac{1}{\lambda^2} \|R\|^2_{L^2} \right).
\]

As regards the remaining higher order terms contained in $I_{t,22}^{(1)}$, using the pointwise bounds $|w| \leq C\lambda^{-\frac{d}{2}}$, $|\nabla w| \leq C\lambda^{-\frac{d}{2} - 1}$, the expansion (3.13) and equation (5.45) we have

\[
|I_{t,22}^{(1)}| \leq \frac{C}{\lambda^2} \sum_{k=3}^{1 + \frac{4}{d}} \int |w|^{2 + \frac{d}{2} - k} |R|^k \, dx + \frac{C}{\lambda^2} \sum_{k=3}^{1 + \frac{4}{d}} \int \left( 1 + \left| \frac{x - \alpha}{\lambda} \right| \right) |\nabla w| |w|^{1 + \frac{d}{2} - k} |R|^k \, dx \\
+ C \lambda^{\frac{4}{d} - 4 - d} \left\| R \right\|_{L^k}^2.
\]

Hence, we obtain

\[
|I_{t,22}^{(1)}| \leq \frac{1 + \frac{4}{d}}{\lambda^2} \sum_{k=3} \lambda^{\frac{4}{d}k - 4 - d} \lambda^{d + 3k - \frac{4}{d}k} \leq \frac{1 + \frac{4}{d}}{\lambda^2} \sum_{k=3} \lambda^{3k - 4} \leq C\lambda^4.
\]

Note that, Gagliardo-Nirenberg’s inequality and (5.15) yield that

\[
\|R\|_{L^k}^k \leq C \|R\|_{L^2}^{k + d - \frac{4}{d}k} \|\nabla R\|_{L^2}^{\frac{4}{d}k - d} \leq C\lambda^{d + 3k - \frac{4}{d}k}.
\]

Similarly, using again (5.15), (5.18) and Gagliardo-Nirenberg’s inequality we get

\[
|I_{t,4}^{(1)}| \leq C \sum_{k=2}^{1 + \frac{4}{d}} \lambda^{\frac{4}{d}k - 4} \|\nabla \|_{L^2}^2 \|R\|_{L^2}^6 \leq C \sum_{k=2}^{1 + \frac{4}{d}} \lambda^{\frac{4}{d}k - 4} \lambda^{3k} \lambda^{3k - \frac{4}{d}k} \leq C\lambda^4,
\]

and

\[
|I_{t,5}^{(1)}| \leq C \sum_{k=2}^{1 + \frac{4}{d}} \lambda^{-2} \int \left| w \right|^{1 + \frac{4}{d} - k} |R|^{1 + k} \, dx \leq C \sum_{k=2}^{1 + \frac{4}{d}} \lambda^{3k - 4} \leq C\lambda^4.
\]

Moreover, by (5.15) and (5.17),

\[
\left| \frac{\lambda \lambda_t + \gamma}{\lambda^4} \|R\|_{L^2}^2 - \frac{\lambda \lambda_t + \gamma}{\lambda^4} \|R\|_{L^2}^2 \right| \leq \frac{M \|t \|}{\lambda^4} \|R\|_{L^2}^2 \leq C\lambda^4,
\]

which implies that

\[
I_{t,6}^{(1)} = \frac{\gamma}{\lambda^4} \|R\|_{L^2}^2 - \frac{\lambda \lambda_t + \gamma}{\lambda^4} \|R\|_{L^2}^2 = \frac{\gamma}{\lambda^4} \|R\|_{L^2}^2 + O(\lambda^4).
\]
Thus, we conclude from the estimates above that

\[
\sum_{j=2}^{6} I_{t,j}^{(1)} = -\frac{\gamma}{\lambda^2} \text{Re} \int \left( (1 + \frac{2}{d}) |w|^\frac{4}{3} |R|^2 + \frac{1}{2} \left( 1 + \frac{2}{d} \right) |w|^\frac{4-2}{3} \omega^2 R^2 + \frac{1}{2} \left( \frac{2}{d} - 1 \right) |w|^\frac{4-2}{3} \omega^2 R^2 \right) dx
- \frac{\gamma}{\lambda} \text{Re} \int \left( \frac{x - \alpha}{\lambda} \cdot \nabla w \left\{ \frac{2}{d} \left( 1 + \frac{2}{d} \right) |w|^\frac{4-2}{3} w |R|^2 + \frac{1}{d} \left( 1 + \frac{2}{d} \right) |w|^\frac{4-2}{3} \omega^2 R^2 \right\} dx + \frac{\gamma}{\lambda^4} \| R \|^2_{L^2} + O(\lambda^4).
\]

(5.54)

(iii) Estimates of the last term \( I_{t,b}^{(1)} \). By the integration by parts formula and \( \text{(4.3)} \),

\[
|\text{Im} \langle \Delta R - \lambda^{-2} R + f(u) - f(w), b \cdot \nabla R \rangle| \leq C \sum_{k=1}^{N} \left( |\text{Re} \int B_k \Delta R \nabla \phi_k \cdot \nabla R dx| + \frac{1}{\lambda^2} |\text{Re} \int B_k R \nabla \phi_k \cdot \nabla R dx| \right.
+ \int |B_k| R |^\frac{4}{3} R \nabla \phi_k \cdot \nabla R | dx + | \int B_k (f(u) - f(w) - |R|^\frac{4}{3} R) \nabla \phi_k \cdot \nabla R dx| \right)
\]

(5.55) \(:= \sum_{j=1}^{4} I_{t,bj}^{(1)}.

We estimate each term \( I_{t,bj}^{(1)}, 1 \leq j \leq 4 \), separately below. First, using the integration by parts formula to shift the derivatives of \( R \) onto \( \phi_k \) we get

\[
\text{Re} \int \Delta R \nabla \phi_k \cdot \nabla R dx = -\text{Re} \int \nabla^2 \phi_k (\nabla R, \nabla R) dx + \frac{1}{2} \int \Delta \phi_k |\nabla R|^2 dx,
\]

\[
\text{Re} \int R \nabla \phi_k \cdot \nabla R dx = -\frac{1}{2} \int \Delta \phi_k |R|^2 dx.
\]

Then, taking into account the boundedness of \( \| \partial_x^\nu \phi_k \|_{L^\infty} \) and \( \sup_{0 \leq t \leq T} \| B_k(t) \|, 1 \leq |\nu| \leq 2 \), and using the Gagliardo-Nirenberg inequality we obtain that, if \( p := 1 + \frac{4}{d} \),

\[
I_{t,b1}^{(1)} + I_{t,b2}^{(1)} + I_{t,b3}^{(1)} \leq C \left( \| \nabla R \|^2_{L^2} + \frac{1}{\lambda^2} \| R \|^2_{L^2} + \| \nabla R \|^2_{L^2} \| R \|^p_{L^p} \right)
\]

(5.56) \(:= C \left( \| \nabla R \|^2_{L^2} + \frac{1}{\lambda^2} \| R \|^2_{L^2} + \| R \|^p_{L^p} \right) \leq C \lambda^4,
\]

where we also used \( \text{(4.3)} \) and \( \text{(5.4)} \). Moreover, we infer from \( \text{(4.3)} \) and \( \text{(5.32)} \) that

\[
I_{t,b4}^{(1)} \leq C \sum_{k=1}^{\frac{d}{4}} \lambda^d \int \left| \nabla \phi_k (\lambda y + \alpha) (\nabla R R^k)(\lambda y + \alpha) w^{1 + \frac{4}{d} - k} (\lambda y + \alpha) \right| dy
\]

\[
\leq C \sum_{k=1}^{\frac{d}{4}} \lambda^{5+d} \int \left( 1 + |y|^6 \right) \left| \nabla R R^k(\lambda y + \alpha) w^{1 + \frac{4}{d} - k} (\lambda y + \alpha) \right| dy
\]

\[
\leq C \sum_{k=1}^{\frac{d}{4}} \lambda^5 \int \left( 1 + \left| \frac{x - \alpha}{\lambda} \right|^6 \right) \left| w^{1 + \frac{4}{d} - k} (x) \right| \nabla R R^k(x) dx.
\]

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Since $\text{sup}_x (1 + |z|^6)|w|^{d+4-k} \leq C\lambda^{\frac{d}{2}+\frac{k}{2}-2}$, by (5.48).

\begin{equation}
I_{t,\beta}^{(1)} \leq C \sum_{k=1}^{\frac{4}{d}} \lambda^{\frac{\alpha}{\lambda} + \frac{k}{2}} \|\nabla R\|_{L^2} \|R\|_{L^{2k}} \leq \sum_{k=1}^{\frac{4}{d}} \lambda^{5+3k} \leq C\lambda^4.
\end{equation}

The estimate of the term involving the coefficient $c$ is easier. Actually, since $|f(u) - f(w)| \leq C \sum_{k=1}^{\frac{4}{d}} |w|^{1+4/d-k} |R|^{k}$, taking into account (5.48) and the boundedness of $\|\partial_x^\nu |x|^\alpha R\|_{L^\infty}$ and $\text{sup}_{0 \leq t \leq T} |B_k(t)|$, $0 \leq |\nu| \leq 2$, we get

\[ |\text{Im}(\Delta R - \lambda^{-2} R + f(u) - f(w), cR)| \leq C(\|\nabla R\|_{L^2}^2 + |R|_{L^2}^2 + \sum_{k=1}^{1+\frac{4}{d}} \int |w|^{1+4/k} |R|^{k+1} dx) \leq C(\lambda^4 + \sum_{k=1}^{1+\frac{4}{d}} \lambda^{3k+1}) \leq C\lambda^4. \]

Thus, we conclude from the estimates above that

\begin{equation}
|I_{t,\gamma}^{(1)}| \leq C\lambda^4.
\end{equation}

Therefore, plugging (5.42), (5.54) and (5.58) into (5.40) we obtain (5.38). \hfill \Box

**Lemma 5.11.** For all $t \in [t_*, t_n]$, we have

\begin{equation}
I_{t}^{(2)} \geq -\frac{\gamma}{4\lambda^4} \int \Delta^2 \chi_A \left( \frac{x - \alpha}{\lambda} \right) |R|^2 dx + \frac{\gamma}{\lambda^2} \text{Re} \int \nabla^2 \chi_A \left( \frac{x - \alpha}{\lambda} \right) (\nabla R, \nabla R') dx + \frac{\gamma}{\lambda^2} \int \text{Re} \left\{ \frac{2}{d}(1 + \frac{2}{d}) |w|^\frac{d}{2} - 2 |R|^2 + \frac{1}{d}(1 + \frac{2}{d}) |w|^\frac{d}{2} - 2 |R| dx \right\} dx - C(A)\lambda^4.
\end{equation}

**Proof.** Using (2.14), (2.15), (5.37) and integrating by parts formula we first see that

\begin{equation}
I_{t}^{(2)} = -\frac{\lambda\gamma}{2\lambda^2} \text{Im} \int \nabla \chi_A \left( \frac{x - \alpha}{\lambda} \right) \cdot \nabla R dx + \frac{\gamma}{\lambda} \text{Re} \int \partial_t (\nabla \chi_A \left( \frac{x - \alpha}{\lambda} \right)) \cdot \nabla R dx + \frac{\gamma}{2\lambda} \text{Im} \int \partial_t (\nabla \chi_A \left( \frac{x - \alpha}{\lambda} \right)) \cdot \nabla R dx =: \sum_{j=1}^{3} I_{t,j}^{(2)}.
\end{equation}

Since $\text{sup}_y |\nabla^2 \chi_A (y) (1 + |y|)| \leq C(A)$, by Lemmas 5.3 and 5.4

\begin{equation}
|\partial_t \nabla \chi_A \left( \frac{x - \alpha}{\lambda} \right)| = |\nabla^2 \chi_A \left( \frac{x - \alpha}{\lambda} \right) \cdot (\frac{x - \alpha}{\lambda} \cdot \frac{\lambda t - \gamma}{\lambda^2} - (\frac{x - \alpha}{\lambda} \cdot \frac{\gamma}{\lambda^2} + \frac{\lambda t - 2\beta}{\lambda^2} + \frac{\gamma}{\lambda^2})| \leq \frac{1}{\lambda^2} (\text{Mod} + P) C(A) \leq \frac{1}{\lambda} C(A),
\end{equation}

which along with $|\frac{\lambda\gamma - \lambda\nu}{4\lambda^2}| \leq C\frac{\text{Mod}(a)}{\lambda^2} \leq C\lambda^2$ and (5.15) yields the bound

\begin{equation}
|I_{t,1}^{(2)} + I_{t,2}^{(2)}| \leq C(A)\lambda^4.
\end{equation}
Moreover, using equation (5.64), the expansion (3.12) and integrating by parts we get

\[ I_{t,3}^{(2)} = -\frac{\gamma}{4\lambda^2} \Re \int \Delta^2 \chi_A \left(\frac{x-\alpha}{\lambda}\right)|R|^2 \, dx + \frac{\gamma}{\lambda^2} \Re \int \nabla^2 \chi_A \left(\frac{x-\alpha}{\lambda}\right)(\nabla R, \nabla \overline{R}) \, dx \\
- \Re \left(\frac{\gamma}{2\lambda^2} \Delta \chi_A \left(\frac{x-\alpha}{\lambda}\right) R + \frac{\gamma}{\lambda^2} \nabla \chi_A \left(\frac{x-\alpha}{\lambda}\right) \cdot \nabla R, f'(w) \cdot R \right) \\
- \Re \left(\frac{\gamma}{2\lambda^2} \Delta \chi_A \left(\frac{x-\alpha}{\lambda}\right) R + \frac{\gamma}{\lambda^2} \nabla \chi_A \left(\frac{x-\alpha}{\lambda}\right) \cdot \nabla R, N_f(w, R) \right) \\
- \Re \left(\frac{\gamma}{2\lambda^2} \Delta \chi_A \left(\frac{x-\alpha}{\lambda}\right) R + \frac{\gamma}{\lambda^2} \nabla \chi_A \left(\frac{x-\alpha}{\lambda}\right) \cdot \nabla R, (b \cdot \nabla + c) R \right) \\
- \Re \left(\frac{\gamma}{2\lambda^2} \Delta \chi_A \left(\frac{x-\alpha}{\lambda}\right) R + \frac{\gamma}{\lambda^2} \nabla \chi_A \left(\frac{x-\alpha}{\lambda}\right) \cdot \nabla R, \eta \right) \\
=: -\frac{\gamma}{4\lambda^2} \Re \int \Delta^2 \chi_A \left(\frac{x-\alpha}{\lambda}\right)|R|^2 \, dx + \frac{\gamma}{\lambda^2} \Re \int \nabla^2 \chi_A \left(\frac{x-\alpha}{\lambda}\right)(\nabla R, \nabla \overline{R}) \, dx \\
(5.63) + \sum_{j=1}^{4} I_{t,3j}^{(2)}. \]

Using the integration by parts formula we have

\[
\Re \langle \nabla \chi_A \left(\frac{x-\alpha}{\lambda}\right) \cdot \nabla R, f'(w) \cdot R \rangle = - \frac{1}{2\lambda} \Re \int \Delta \chi_A \left(\frac{x-\alpha}{\lambda}\right) \Re f'(w) \cdot R \, dx \\
- \frac{1}{2} \Re \int \nabla \chi_A \left(\frac{x-\alpha}{\lambda}\right) \cdot \nabla w((\partial_{ww} f(w) + \partial_{w} f(w))|R|^2 + \partial_{ww} f(w) \Re^2 + \partial_{w} f(w) R^2). \]

Then, using the explicit expression (3.11) we obtain

\[
I_{t,31}^{(2)} = \frac{\gamma}{\lambda} \Re \int \nabla \chi_A \left(\frac{x-\alpha}{\lambda}\right) \cdot \nabla w \left\{ \frac{2}{d}(1 + \frac{2}{d})|w|^{\frac{d}{2}-2} w |R|^2 \\
+ \frac{1}{d}(1 + \frac{2}{d})|w|^{\frac{d}{2}-2} w R^2 + \frac{1}{d}(\frac{2}{d} - 1)|w|^{\frac{d}{2}-4} w^3 R^2 \right\} \, dx. \]

For the nonlinear term \( I_{t,32}^{(2)} \), similar arguments as in the estimate (5.49) lead to

\[
|I_{t,32}^{(2)}| \leq C(A) \sum_{k=2}^{1+\frac{d}{2}} \int |w|^{1+\frac{d}{2}-k} |R|^k (\lambda^{-1} |R| + |\nabla R|) \, dx \leq C(A) \lambda^4. \]

Similarly, we have

\[
|I_{t,33}^{(2)}| \leq \frac{1}{\lambda} C(A)(\|R\|_{L^2} \|\nabla R\|_{L^2} + \|R\|_{L^2}^2) + C(A)(\|\nabla R\|_{L^2}^2 + \|R\|_{L^2} \|\nabla R\|_{L^2}) \leq C(A)(\|\nabla R\|_{L^2}^2 + \frac{1}{\lambda^2} \|R\|_{L^2}^2) \leq C(A) \lambda^4. \]

Moreover, by Lemmas 5.4 and 5.5

\[
|I_{t,34}^{(2)}| \leq \frac{1}{\lambda} C(A) \|\eta\|_{L^2} \|R\|_{L^2} + C(A) \|\nabla R\|_{L^2} \|\eta\|_{L^2} \leq C(A) \lambda^4. \]

Therefore, plugging (5.64) - (5.67) into (5.63) and using (5.62) we obtain (5.59). \( \square \)
Proof of Theorem 5.5. We infer from (5.38) and (5.59) that for all \( t \in [t_s, t_n] \),
\[
\frac{dI}{dt} \geq \frac{\gamma}{\lambda^2} \text{Re} \int \nabla^2 \chi \left( \frac{x - \alpha}{\lambda} \right) (\nabla R, \nabla R) dx + \frac{\gamma}{\lambda^4} \int |R|^2 dx
\]
\[
- \frac{\gamma}{\lambda^2} \text{Re} \int \left( (1 + \frac{2}{d}) |w|^{\frac{2}{d}} |R|^2 + \frac{1}{2} (1 + \frac{2}{d}) |w|^{\frac{2}{d} - 2} w^2 R^2 + \frac{1}{2} (\frac{2}{d} - 1) |w|^{\frac{2}{d} - 2} w^2 R^2 \right) dx
\]
\[
+ \frac{\gamma}{\lambda^4} \int (\nabla \chi \left( \frac{x - \alpha}{\lambda} \right) - (\frac{x - \alpha}{\lambda})) \cdot \nabla w |w|^{\frac{2}{d} - 1} w^2 R^2 dx
\]
\[
- \frac{\gamma}{4 \lambda^4} \int \nabla^2 \chi \left( \frac{x - \alpha}{\lambda} \right) |R|^2 dx - C(A) \lambda^4
\]
with \( C(A) > 0 \) independent of \( n \), which, via (4.3), can be reformulated as follows
\[
\frac{dI}{dt} \geq \frac{\gamma}{\lambda^2} \int \nabla^2 \chi \left( \frac{y}{A} \right) (\nabla \epsilon, \nabla \bar{\epsilon}) dy + \int |\epsilon|^2 dy - \int ((1 + \frac{4}{d}) Q^2 \epsilon_1^2 + Q^2 \epsilon_2^2) dy
\]
\[
- \frac{1}{4A^2} \int \nabla^2 \chi \left( \frac{y}{A} \right) |\epsilon|^2 dy + \frac{2}{d \lambda^2} \int (A \nabla \chi \left( \frac{y}{A} \right) - y) \cdot \nabla Q^2 \chi \left( \frac{y}{A} \right) dy - C(A) \lambda^4.
\]

We claim that
\[
\int \nabla^2 \chi \left( \frac{y}{A} \right) (\nabla \epsilon, \nabla \bar{\epsilon}) dy \geq \int \psi''(|\frac{y}{A}|) |\nabla \epsilon|^2 dy.
\]

To this end, since the case where \( d = 1 \) is obvious, we only need to treat the case where \( d = 2 \) below. Since \( \chi(y) = \psi(|y|) \), we have
\[
\partial_{y_i y_j} \chi(y) = \psi''(|y|) \frac{y_i y_j}{|y|^2} + \psi'(|y|) \delta_{ij} - \psi'(|y|) \frac{y_i y_j}{|y|^3}, \quad i, j = 1, 2.
\]

Then, by the condition (5.33), we have
\[
\int \nabla^2 \chi \left( \frac{y}{A} \right) (\nabla \epsilon, \nabla \bar{\epsilon}) dy = \int \psi''(|\frac{y}{A}|) |\nabla \epsilon|^2 dy
\]
\[
+ \int \left( \frac{A}{|y|} \psi'\left( \frac{|y|}{A} \right) - \psi'\left( \frac{|y|}{A} \right) \right) \frac{y_i^2}{|y|^2} |\partial_2 \epsilon|^2 + \frac{y_i^2}{|y|^2} |\partial_1 \epsilon|^2 - \frac{2 y_i y_j}{|y|^2} \text{Re}(\partial_1 \epsilon \partial_2 \bar{\epsilon}) dy
\]
\[
\geq \int \psi''\left( \frac{|y|}{A} \right) |\nabla \epsilon|^2 dy,
\]
which implies (5.69), as claimed.

Thus, applying Corollary 5.3 with \( \Phi(x) := \psi''(|x|) \) we obtain that for some \( \nu > 0 \)
\[
\frac{dI}{dt} \geq \nu \frac{\gamma}{\lambda^2} \int \psi''\left( \frac{|y|}{A} \right) (|\epsilon|^2 + |\nabla \epsilon|^2) dy - \frac{1}{4A^2} \frac{\gamma}{\lambda^4} \int \nabla^2 \chi \left( \frac{y}{A} \right) |\epsilon|^2 dy
\]
\[
+ \frac{2}{d \lambda^2} \int (A \nabla \chi \left( \frac{y}{A} \right) - y) \cdot \nabla Q^2 \chi \left( \frac{y}{A} \right) dy - C(A) \lambda^4.
\]

Note that, since \( \psi''\left( \frac{|y|}{A} \right) \to 1 \) and \( \frac{1}{A^2} \nabla^2 \chi \left( \frac{y}{A} \right) \to 0 \) as \( A \to \infty \), we infer that for \( A \) large enough,
\[
\frac{1}{4A^2} |\nabla^2 \chi \left( \frac{y}{A} \right)| \leq \frac{\nu}{4} \psi''\left( \frac{|y|}{A} \right).
\]
Similarly, since \( A \nabla \chi(\frac{\psi}{A}) = y \) for \(|y| \leq A\) and \(Q\) is exponentially decay at infinity, we have that \(|A \nabla \chi(\frac{\psi}{A}) - y| \leq A(1 + |y|) e^{-\delta A}\), which yields that for \(A\) large enough
\[
\frac{2}{d} (2 + \frac{4}{d}) |A \nabla \chi(\frac{\psi}{A}) - y||\nabla QQ^{\frac{1}{2}}| \leq \frac{\nu}{4} \psi''(|\frac{\psi}{A}|).
\]

Therefore, in view of \(\psi''(r) \geq \delta e^{-r}\) for some \(\delta > 0\), we obtain that for \(A\) large enough
\[
\frac{dI_2}{dt} \geq \frac{\nu}{2} \frac{\gamma^2}{\lambda^4} \int \psi''(|\frac{\psi}{A}|)(|\varepsilon|^2 + |\nabla \varepsilon|^2)dy - C(A) \lambda^4
\]

(5.72)
\[
\geq \frac{\nu \delta}{2} \frac{\gamma^2}{\lambda^4} \int e^{-\frac{|\psi|^2}{A}}(|\varepsilon|^2 + |\nabla \varepsilon|^2)dy - C(A) \lambda^4,
\]

which yields (5.35). The proof of Theorem 5.8 is complete. \(\square\)

5.4. Proof of Proposition 5.2. (i) Estimate of \(R_n\). By (4.12) and (5.34), for \(t \in [t_*, t_n]\),
\[
I_n(t) = \frac{1}{2 \lambda^2} \int |\nabla \varepsilon|^2 + |\varepsilon|^2 dy - \frac{1}{\lambda^2} \text{Re} \int F(QP_n + \varepsilon_n) - F(QP_n) - f(QP_n)\varepsilon_n dy
\]

(5.73) 
\[
+ \mathcal{O}(\|\nabla R_n(t)\|_{L^2}\|R_n(t)\|_{L^2}).
\]

This along with (5.15) yields that
\[
\frac{1}{\lambda^2} \text{Re} \int F(QP_n + \varepsilon_n) - F(QP_n) - f(QP_n)\varepsilon_n dy
\]

\[
= \frac{1}{2 \lambda^2} \text{Re} \int Q^{\frac{1}{2}} (2 \varepsilon_n^2 + (1 + \frac{2}{d})|\varepsilon_n|^2) dy + o\left(\frac{1}{\lambda^2} \|\varepsilon_n\|_{H^1}^2\right),
\]

which yields immediately that
\[
I_n(t) = \frac{1}{2 \lambda^2} L(\varepsilon_n, \varepsilon_n) + \frac{1}{2 \lambda^2} o(\|\varepsilon_n\|_{H^1}^2) + \mathcal{O}(\|\nabla R_n(t)\|_{L^2}\|R_n(t)\|_{L^2}).
\]

(5.74)

Then, using Corollary 3.2 and similar arguments as in the proof of (5.30) we obtain that for some \(\nu > 0\) and for \(T\) small enough,
\[
I_n(t) \geq \frac{\nu}{2 \lambda^2} \|\nabla \varepsilon_n\|_{H^1}^2 - C(A)(\|\nabla R_n(t)\|_{L^2}\|R_n(t)\|_{L^2}) \geq \frac{\nu}{4} (\|\nabla R_n(t)\|_{L^2}^2 + \frac{1}{\lambda^2} \|R_n(t)\|_{L^2}^2).
\]

Moreover, Theorem 5.8 yields that for any \(t \in [t_*, t_n]\),
\[
\frac{dI_n}{dt} \geq -C(A) \lambda_n^4.
\]

(5.75)

Thus, we infer from (5.74) and (5.75) that for any \(t \in [t_*, t_n]\),
\[
\frac{\nu}{4} (\|\nabla R_n(t)\|_{L^2}^2 + \frac{1}{\lambda^2} \|R_n(t)\|_{L^2}^2) \leq I_n(t) = I_n(t_n) - \int_{t_n}^{t} \frac{dI_n}{dr}(r) dr
\]

\[
\leq I_n(t_n) + C(A) \int_{t_n}^{t} \lambda_n^4(r) dr.
\]

Taking into account \(I_n(t_n) = 0\) and using (5.15) we obtain that for \(T\) small enough,
\[
\|\nabla R_n(t)\|_{L^2}^2 + \frac{1}{\lambda^2} \|R_n(t)\|_{L^2}^2 \leq \frac{4C(A)}{\nu} \int_{t}^{T} (T - r)^4 dr \leq \frac{1}{2}(T - t)^4,
\]

which yields (5.5).
(ii) Estimates of $\lambda_n$ and $\gamma_n$. First note that, by Lemma 5.4,
\begin{equation}
|\gamma_n| = \left| \lambda_n^2 \gamma - \lambda_n \alpha_n \gamma_n \right| \leq 2 \frac{|\text{Mod}_n(t)|}{\lambda^3_n} \leq C \lambda^2_n,
\end{equation}
which along with (5.13) and (5.78) yields that for $T$ small enough,
\begin{equation}
|\gamma_n(t) - 1| \leq \int_t^{t+1} |\gamma_n| \, dr \leq C(T - t)^3 \leq \frac{1}{2} (T - t)^{2+\delta}.
\end{equation}
Then, taking into account (5.13) and (5.17) we obtain
\begin{equation}
|\lambda_n - (T - t))| = |\alpha_n + \gamma_n + 1 - \gamma_n| \leq \frac{\text{Mod}_n}{\lambda} + \frac{1}{2} (T - t)^{2+\delta} \leq C(T - t)^{2+\delta},
\end{equation}
which implies that for $T$ possibly even smaller, such that $CT^4 \leq \frac{1}{2}$,
\begin{equation}
|\gamma_n(t) - 1| \leq \int_t^{t+1} |\gamma_n - (T - r)| \, dr \leq \frac{1}{2} (T - t)^{3+5\delta}.
\end{equation}
Similarly, by (5.13), (5.17) and (5.78),
\begin{equation}
|\lambda_n - (T - t)| \leq \int_t^{t+1} |\lambda_n - (T - r)| \, dr \leq \frac{1}{2} (T - t)^{3+5\delta}.
\end{equation}
Thus, we obtain (5.7).

(iii) Estimates of $\beta_n$ and $\alpha_n$. By (5.13) and (5.25),
\begin{equation}
|\beta_n|^2 \leq C|\lambda_n^2 - \gamma_n^2| + C(T - t)^5 \leq C \lambda_n^2 (1 - \gamma_n) + C \lambda_n^5,
\end{equation}
which along with (5.13) and (5.78) yields that for $T$ small enough,
\begin{equation}
|\beta_n(t)| \leq C \lambda_n |1 - \frac{\gamma_n}{\lambda_n}|^2 + C \lambda_n^5 \leq C \lambda_n^{2+5\delta} \leq \frac{1}{2} (T - t)^{2+\delta}.
\end{equation}
Moreover, using again (5.17) and (5.82) we have
\begin{equation}
|\alpha_n| = \left| \lambda_n \alpha_n + 2 \beta_n \right| \leq \frac{\text{Mod}_n}{\lambda} + \frac{2 \beta_n}{\lambda} \leq C \lambda_n^{1+\delta},
\end{equation}
which yields that for sufficiently small $T$,
\begin{equation}
\alpha_n(t) \leq \int_t^{t+1} \alpha_n \, dr \leq \frac{1}{2} (T - t)^{2+\delta},
\end{equation}
thereby yielding (5.7).

(iv) Estimate of $\theta_n$. Using (5.13), (5.17), (5.79) and (5.82) we get
\begin{equation}
\left( \theta_n - \frac{1}{T - t} \right) = \theta_n - \frac{1}{T - t} = \frac{\lambda_n^2 \theta_n}{\lambda_n^2} - \frac{1 - |\beta_n|^2}{\lambda_n^2} + \frac{|\beta_n|^2}{\lambda_n^2} + \frac{1}{2} \frac{1}{(T - t)^2}.
\end{equation}
Note that, by (5.13) and (5.79),
\begin{equation}
\frac{1}{\lambda_n^2} - \frac{1}{(T - t)^2} \leq \frac{|\lambda_n - (T - t)| \lambda_n + (T - t)|}{\lambda_n^2 (T - t)^2} \leq C \lambda_n^{5\delta},
\end{equation}
which along with (5.13), (5.17), (5.82) and (5.85) yields that
\begin{equation}
|\theta_n - \frac{1}{T - t}| \leq \frac{\text{Mod}_n}{\lambda_n^2} + \frac{|\beta_n|^2}{\lambda_n^2} + C \lambda_n^{5\delta} \leq C(T - t)^{5\delta}.
\end{equation}
Thus, integrating both sides above and taking $T$ very small we obtain
\begin{equation}
|\theta_n - \frac{1}{T - t}| \leq \int_t^{t+1} |\theta_n - \frac{1}{T - r}| \, dr \leq \frac{1}{2} (T - t)^{1+4\delta}.
\end{equation}
which implies (5.8). Therefore, the proof of Proposition 5.2 is complete.

5.5. Proof of Theorem 5.1. Since the bounds in the proof of Proposition 5.2 are uniform of \( n \), we may take a universal sufficiently small \( \tau^* \) such that the estimates in Proposition 5.2 hold. Below we take \( T \in (0, \tau^*] \) fixed.

Let \( H(t) \) denote the statement that the geometrical decomposition (4.2) and estimates (5.1)-(5.4) hold on \([t, t_n]\), and let \( C(t) \) denote the statement that the decomposition (4.2) and estimates (5.5)-(5.8) hold on \([t, t_n]\), \( 0 \leq t \leq t_n \).

It is clear that \( H(t_n) \) is true. By Lemma 4.2 and the continuity of \( u_n \) and \( P_n \), we also have that if \( C(t) \) is true for some \( t \in [0, t_n] \), then \( H(t') \) holds for all \( t' \) in a neighborhood of \( t \). Moreover, Proposition 5.2 yields that if \( H(t) \) is true for some \( t \in [0, t_n] \), then \( C(t) \) is also true.

Furthermore, we claim that if \( \tilde{t}_m, m \geq 1 \), is a sequence in \([0, t_n]\) which converges to another \( \tilde{t}_s \in [0, t_n] \) and \( C(\tilde{t}_m) \) is true for all \( \tilde{t}_m, m \geq 1 \), then \( C(\tilde{t}_s) \) is also true.

To this end, by virtue of estimates (5.6)-(5.8), Lemma 5.4 and (5.15), we infer that the derivatives of the modulation parameters \( \dot{P}_n(t_m) \) are uniformly bounded on \([\tilde{t}_s, \tilde{t}_m]\), \( m \geq 1 \), which yields that there exists a unique \( \dot{P}_n^* \) such that \( \lim_{m \to \infty} \dot{P}_n(t_m) = \dot{P}_n^* \). Hence, letting \( P_n(\tilde{t}_s) := \dot{P}_n^* \) we have that \( P_n \) is continuous on \([\tilde{t}_s, t_n]\). Taking into account the decomposition (4.2) and estimates (5.1)-(5.4) we infer that \( w_n(\tilde{t}_m) \to w_n(\tilde{t}_s) \) in \( H^1 \) and \( |x|^2w_n(\tilde{t}_m) \), \( \dot{\rho}_n(\tilde{t}_m) \) and \( R_n(\tilde{t}_m) \) converge to \( |x|^2w_n(\tilde{t}_s) \), \( \dot{\rho}_n(\tilde{t}_s) \) and \( R_n(\tilde{t}_s) := u_n(\tilde{t}_s) - w_n(\tilde{t}_s) \), respectively, in the space \( L^2 \) (see similar arguments as in the proof of (5.30) below). So, the decomposition (4.2) and the orthogonality condition (4.3) also hold at time \( \tilde{t}_s \). Moreover, in view of the continuity of \( u_n \) and \( P_n \), we also infer that the estimates (5.5)-(5.8) hold on \([\tilde{t}_s, t_n]\). Thus, we conclude that the statement \( C(\tilde{t}_s) \) is also true, as claimed.

Therefore, by virtue of the abstract bootstrap principle (see [43], Proposition 1.21), we prove Theorem 5.1.

6. PROOF OF MAIN RESULTS

Let us start with the global well-posedness result in Theorem 2.11.

Proof of Theorem 2.11. In view of Remark 2.9 we have that \( \mathbb{P} \)-a.s. there exists a unique solution \( u \) to equation (2.7) on \([0, \tau^*) \) with \( u(0) = u_0 \in H^1 \), where \( \tau^* \in (0, \infty) \) is some positive random variable. Hence, we only need to prove that \( \tau^* = \infty, \mathbb{P} \)-a.s.

For this purpose, similarly to (5.20), we have

\[
\frac{d}{dt} E(u) = -2 \sum_{k=1}^{N} B_k \operatorname{Re} \int \nabla^2 \phi_k(\nabla u, \nabla u) dx + \frac{1}{2} \sum_{k=1}^{N} B_k \int \Delta^2 \phi_k |u|^2 dx \\
+ \frac{2}{d+2} \sum_{k=1}^{N} B_k \int \nabla \phi_k |u|^{2+\frac{4}{d}} dx - \sum_{j=1}^{d} \int \sum_{k=1}^{N} \nabla \left( \partial_j \phi_k B_k \right)^2 \cdot \nabla u \bar{u} dx.
\]

(6.1)

Then, for any \( T \in (0, \tau^*) \), since \( \phi_k \in C^\infty_b \) and \( B_k \in C([0, T]) \), \( \mathbb{P} \)-a.s., \( 1 \leq k \leq N \), using Hölder’s inequality we obtain that \( \mathbb{P} \)-a.s. for any \( t \in [0, T] \),

\[
E(u(t)) \leq E(u_0) + C(\tau^*) \int_0^t \left( ||u(s)||^2_{L^2} + ||\nabla u(s)||^2_{L^2} + ||u(s)||^{2+\frac{4}{d}}_{L^{2+\frac{4}{d}}} \right) ds.
\]

(6.2)

Note that, \( ||u(s)||^2_{L^2} = ||u_0||^2_{L^2} \), and by the Gagliardo-Nirenberg inequality (3.16),

\[
||\nabla u(s)||^{2+\frac{4}{d}}_{L^{2+\frac{4}{d}}} \leq C ||u(s)||^{\frac{4}{d}}_{L^2} ||\nabla u(s)||^2_{L^2} = C ||u_0||^{\frac{4}{d}}_{L^2} ||\nabla u(s)||^2_{L^2}.
\]

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Thus, we obtain

$$E(u(t)) \leq C(\tau^*) + C(\tau^*) \int_0^t \|\nabla u(s)\|_{L^2}^2 ds, \quad 0 < t \leq T.$$  

Moreover, applying the sharp Gagliardo-Nirenberg inequality (cf. [31, (III.5)], [36, (4)]) we get

$$\left(1 - \left(\frac{\|u_0\|_{L^2}^2}{\|Q\|_{L^2}^2}\right)^\frac{4}{d}\right)\|\nabla u(t)\|_{L^2}^2 = \left(1 - \left(\frac{\|u(t)\|_{L^2}^2}{\|Q\|_{L^2}^2}\right)^\frac{4}{d}\right)\|\nabla u(t)\|_{L^2}^2 \leq 2E(u(t)).$$  

Thus, in view of $\|u_0\|_{L^2} < \|Q\|_{L^2}$, putting together (6.3) and (6.4) and using Gronwall’s inequality we obtain $\mathbb{P}$-a.s.

$$\sup_{0 \leq t \leq T} \|\nabla u(t)\|_{L^2}^2 \leq C(\tau^*) < \infty,$$

which yields the boundedness of $\sup_{0 \leq t < \tau^*} \|\nabla u(t)\|_{L^2}^2 < \infty$ by letting $T \to \tau^*$. Therefore, using the blow-up alternative result we obtain $\tau^* = \infty$, $\mathbb{P}$-a.s., and finish the proof. \hfill \Box

**Proof of Theorem 2.12.** Let $\tau^*$ be as in Theorem 5.1 and $T \in (0, \tau^*]$ fixed below. By virtue of Theorem 5.1 we have

$$\nabla u_n(t, x) = \nabla w_n(t, x) + R_n(t, x) = \lambda_n^{\frac{d-2}{2}}(t)Q_n(t, \frac{x - \alpha_n(t)}{\lambda_n(t)})e^{i\theta_n(t)} + R_n(t, x), \quad \forall t \in [0, t_n],$$

with the modulation parameters

$$\mathcal{P}_n(t_n) := (\lambda_n(t_n), \alpha_n(t_n), \beta_n(t_n), \gamma_n(t_n), \theta_n(t_n)) = (T - t_n, 0, 0, T - t_n, \frac{1}{T - t_n}),$$

and so

$$u_n(t_n) = S_T(t_n), \quad R_n(t_n) = 0,$$

where $S_T$ is the pseudo-conformal blow-up solution given by (1.9).

Moreover, the estimates (5.1)-(5.4) hold for all $t \in [0, t_n]$. In particular,

$$\frac{1}{2}(T - t) \leq |\lambda_n(t)| \leq 3(T - t), \quad \forall t \in [0, t_n].$$

Since

$$\|w_n(0)\|_{L^2} = \|Q\|_{L^2}, \quad \|\nabla w_n(0)\|_{L^2} \leq \frac{C}{T}(\|Q\|_{H^1} + \|Q\|_{L^2}),$$

where $C$ is independent of $n$, taking into account the uniform $H^1$-boundedness of $R_n$ in (5.1) we infer that $\{u_n(0)\}$ is uniformly bounded in $H^1$. This yields that up to a subsequence (still denoted by $\{n\}$), for some $u_0 \in H^1$,

$$u_n(0) \rightharpoonup u_0, \text{ weakly in } H^1, \text{ as } n \to \infty.$$  

We claim that

$$u_n(0) \to u_0, \text{ in } L^2, \text{ as } n \to \infty.$$  

In particular, since $\|u_n(0)\|_{L^2} = \|u_n(t_n)\|_{L^2} = \|Q\|_{L^2}$, we have that $\|u_0\|_{L^2} = \|Q\|_{L^2}$.

For this purpose, since $u_n(0) \to u_0$ in $L^2_{loc}(\mathbb{R}^d)$ by the compactness imbedding, it suffices to prove the uniform integrability of $\{u_n(0)\}$, i.e.,

$$\lim_{A \to \infty} \sup_{n \geq 1} \|u_n(0)\|_{L^2(|x| > 2A)} = 0.$$  

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In order to prove \((6.13)\), we take a nonnegative function \(\varphi \in C^\infty\) such that \(\varphi(x) = 0\) if \(|x| \leq 1\), \(\varphi(x) = 1\) if \(|x| \geq 2\), \(|\nabla \varphi(x)| \leq 2\) for \(x \in \mathbb{R}^d\). Set \(\varphi_A(\cdot) := \varphi(\frac{\cdot}{A})\), \(A > 0\). Then,
\[
(6.14) \quad |\int \varphi_A |u_n(0)|^2 dx| \leq \int \varphi_A |u_n(t)|^2 dx + \int_0^t \frac{d}{dt} \int \varphi_A |u_n(t)|^2 dx dt.
\]
Since \(u_n(t_n) = S_T(t_n)\), we have that as \(A \to \infty\),
\[
(6.15) \quad \sup_{n \geq 1} |\int \varphi_A |u_n(t)|^2 dx| \leq C \sup_{n \geq 1} \int_{|y| \geq \frac{1}{A}} |Q(y)|^2 dy \leq C \int_{|y| \geq \frac{1}{A}} |Q(y)|^2 dy \to 0.
\]
Moreover, by \((2.8)\), \((2.9)\) and \((4.1)\),
\[
\text{Moreover, since } |\nabla |u_n(t)|^2 dx| = 2 \text{Im} \int \nabla \varphi_A \cdot \nabla u_n(t) \bar{w}_n(t) dx + \sum_{k=1}^N \int B_k \nabla \varphi_A \cdot \nabla \phi_k |u_n|^2(t) dx \leq \frac{C}{A} (\|u_n(t)\|_{L^2(\{x|>A\})}^2 + \|\nabla u_n(t)\|_{L^2(\{x|>A\})}^2).
\]
Note that, by \((5.1)\), \((6.6)\) and \((6.10)\),
\[
(6.16) \quad \|u_n(t)\|_{L^2(\{x|>A\})} \leq \|u_n\|_{L^2} + \|R_n\|_{L^2} \leq C(\|Q\|_{L^2} + T^3) < \infty.
\]
Moreover, using \((5.1)\) and \((6.6)\) again we have
\[
\|\nabla u_n(t)\|_{L^2(\{x|>A\})} \leq C(\|\nabla w_n(t)\|_{L^2(\{x|>A\})}^2 + \|\nabla R_n(t)\|_{L^2(\{x|>A\})}^2) \leq \frac{C}{\lambda_n^2(t)} \int_{|y| \geq \frac{1}{A-\lambda_n(t)}} |\nabla Q(y)|^2 + (1 + |y|^2)Q^2 dy + C.
\]
Then, taking into account the exponential decay of \(Q\) we obtain
\[
(6.18) \quad \|\nabla u_n(t)\|_{L^2(\{x|>A\})} \leq \frac{C}{\lambda_n^2(t)} \int_{|y| \geq \frac{1}{A-\lambda_n(t)}} e^{-\delta |y|} dy + C \leq C \left( \frac{1}{\lambda_n^2(t)} e^{-\frac{\delta(A-1)}{2\lambda_n(t)}} + 1 \right) \leq C,
\]
where \(C, \delta > 0\) are independent of \(n\).

Thus, plugging \((6.17)\) and \((6.18)\) into \((6.16)\) we obtain
\[
(6.19) \quad \sup_{n \geq 1} \left| \frac{d}{dt} \int \varphi_A |u_n(t)|^2 dx \right| \leq \frac{C}{A} \to 0, \quad \text{as } A \to \infty,
\]
which along with \((6.14)\) and \((6.15)\) implies \((6.13)\), thereby yielding \((6.12)\), as claimed.

Now, since \(u_n\) solves the equation \((2.7)\) on \([0, t_n]\) with \(\lim_{n \to \infty} t_n = T\), and by \((6.12)\), \(u_n(0)\) converge strongly to \(u_0\) in the space \(L^2\) as \(n \to \infty\), the \(L^2\) local well-posedness theory yields that there exists a unique \(L^2\)-solution \(u\) to \((2.7)\) on \([0, T]\) satisfying
\[
(6.20) \quad \lim_{n \to \infty} \|u_n(t) - u(t)\|_{L^2} = 0, \quad t \in [0, T).
\]
Moreover, since \(u_0 \in H^1\), the preservation of \(H^1\)-regularity implies that \(u(t) \in H^1\) for any \(t \in [0, T]\).

Below we prove that for some \(\delta > 0\),
\[
(6.21) \quad \|u(t) - S_T(t)\|_{H^1} \leq C(T - t)^\delta,
\]
which yields that \(u\) behaves asymptotically like the pseudo-conformal blow-up solution \(S_T\) as \(t \to T\). In particular, \(u\) blows up at time \(T\).

In order to prove \((6.21)\), using \((6.6)\) we have
\[
(6.22) \quad \|u_n(t) - S_T(t)\|_{H^1} \leq \|R_n(t)\|_{H^1} + \|w_n(t) - S_T(t)\|_{H^1}, \quad \forall t \in [0, T).
\]
The uniform estimate (5.1) yields immediately that
\[ 6.23 \quad \| R_n(t) \|_{H^1} \leq C(T-t)^2, \quad \forall t \in [0,T). \]

Moreover, since
\[ 6.24 \quad S_T(t) = \lambda_0^{-\frac{d}{2}}(t)Q_{p_0}(t, \frac{x - \alpha_0(t)}{\lambda_0(t)})e^{i\theta_0(t)}, \]
where \( Q_{p_0}(t,y) := Q(y)e^{i(\beta_0 - \frac{d}{2}y^2)} \) and \( P_0 := (\lambda_0(t), \alpha_0(t), \beta_0(t), \gamma_0(t), \theta_0(t)) = (T - t, 0, 0, T - t, \frac{1}{d}) \), taking into account (6.6) we get
\[ 6.25 \quad \| w_n(t) - S_T(t) \|_{L^2} = \| \lambda_0^{-\frac{d}{2}}(t)Q_{p_0}(t, \frac{x - \alpha_0(t)}{\lambda_0(t)}) - \lambda_0^{-\frac{d}{2}}(t)Q_{p_0}(t, \frac{x - \alpha_0(t)}{\lambda_0(t)}) \|_{L^2} \]
\[ \leq \lambda_0^{-\frac{d}{2}}(t) \| Q_{p_0}(t, \frac{x - \alpha_0(t)}{\lambda_0(t)}) - Q_{p_0}(t, \frac{x - \alpha_0(t)}{\lambda_0(t)}) \|_{L^2} \]
\[ + \lambda_0^{-\frac{d}{2}}(t) \| Q_{p_0}(t, \frac{x - \alpha_0(t)}{\lambda_0(t)}) - Q_{p_0}(t, \frac{x - \alpha_0(t)}{\lambda_0(t)}) \|_{L^2} \]
\[ + (\lambda_0^{-\frac{d}{2}}(t)\| \lambda_0^{-\frac{d}{2}}(t) - \lambda_0^{-\frac{d}{2}}(t) \| + |\theta_n(t) - \theta_0(t)|) \| Q \|_{L^2}. \]

Note that, by the change of variables,
\[ 6.26 \quad \lambda_0^{-\frac{d}{2}}(t) \| Q_{p_0}(t, \frac{x - \alpha_0(t)}{\lambda_0(t)}) - Q_{p_0}(t, \frac{x - \alpha_0(t)}{\lambda_0(t)}) \|_{L^2} \]
\[ \leq \lambda_0^{-\frac{d}{2}}(t) \| Q_{p_0}(t, \frac{x - \alpha_0(t)}{\lambda_0(t)}) - Q_{p_0}(t, \frac{x - \alpha_0(t)}{\lambda_0(t)}) \|_{L^2} \]
\[ = \frac{\lambda_0(t)}{\lambda_0(t)} \| Q_{p_0}(t, \frac{x - \alpha_0(t)}{\lambda_0(t)}) - Q_{p_0}(t, \frac{x - \alpha_0(t)}{\lambda_0(t)}) \|_{L^2} \leq C \left( \frac{\lambda_0(t)}{\lambda_0(t)} - 1 \right) \| \beta_n(t) - \beta(t) \| + |\gamma_n(t) - \gamma_0(t)| \right). \]

Moreover, using the change of variables again and the mean value theorem we get
\[ 6.27 \quad \lambda_0^{-\frac{d}{2}}(t) \| Q_{p_0}(t, \frac{x - \alpha_0(t)}{\lambda_0(t)}) - Q_{p_0}(t, \frac{x - \alpha_0(t)}{\lambda_0(t)}) \|_{L^2} \]
\[ \leq \| Q(\lambda_0(t))y + \frac{\alpha_0(t) - \alpha_0(t)}{\lambda_0(t)} - Q(y) \|_{L^2} \leq C \left( \frac{\lambda_0(t)}{\lambda_0(t)} - 1 \right) \| \frac{\alpha_0(t) - \alpha_0(t)}{\lambda_0(t)} \| \right). \]

Hence, plugging (6.26) and (6.27) into (6.25) we obtain
\[ 6.28 \quad \| w_n(t) - S_T(t) \|_{L^2} \leq C \left( \frac{\lambda_0(t)}{\lambda_0(t)} - \frac{\lambda_0(t)}{\lambda_0(t)} \right) + \frac{\alpha_0(t) - \alpha_n(t)}{\lambda_0(t)} + \left( \frac{\lambda_0(t)}{\lambda_0(t)} \right) \beta_n(t) - \beta_0(t) \]
\[ + \left( \frac{\lambda_0(t)}{\lambda_0(t)} \right) \| \gamma_n(t) - \gamma_0(t) \| + |\theta_n(t) - \theta_0(t)| + \left( \frac{\lambda_0(t)}{\lambda_0(t)} - 1 \right), \]
which, via (5.2)-(5.4), yields immediately that
\[ 6.29 \quad \| w_n(t) - S_T(t) \|_{L^2} \leq C(T-t)^{1+\frac{d}{2}}, \]
where \( C \) is independent of \( n \).
Moreover, in view of the boundedness of $\sup_{0 \leq s \leq t} |B_k(s)|$, $1 \leq k \leq N$, we also have

$$\int_s^t e^{-ir\Delta}(b(r) \cdot \nabla + c(r))u(r)dr \lesssim \|b(r) \cdot \nabla + c(r)\|_{L^2} \leq C(t)\|u\|_{C([0,t];H^1)}(t-s).$$

Thus, combining the estimates above we obtain (6.33).

The remaining part of this section is devoted to the proof of Theorem 2.12. Let us first state the time regularity of $u$ below.

**Lemma 6.1.** Let $u$ be the solution to (2.7) on $[0, \tau^*)$ with $u(0) = u_0 \in H^1$ in the sense of Definition 2.8. Then, $P$-a.e. for any $0 \leq s < t < \tau^*$, we have that

$$\|e^{-it\Delta}u(t) - e^{-is\Delta}u(s)\|_{L^2} \leq C(t)|t-s|,$$

where $C(t)$ depends on $\|u\|_{C([0,t];H^1)}$ and $\sup_{0 \leq s \leq t} |B_k(s)|$, $1 \leq k \leq N$.

**Proof.** We reformulate equation (2.7) in the mild form

$$u(t) = e^{it\Delta}u_0 + \int_0^t e^{i(t-s)\Delta}(i|u(s)|^{\frac{4}{n}}u(s) + i(b(s) \cdot \nabla + c(s))u(s))ds,$$

where $b, c$ are given by (2.8) and (2.9), respectively. This yields that

$$\|e^{-it\Delta}u(t) - e^{-is\Delta}u(s)\|_{L^2} = \| \int_s^t e^{-ir\Delta}(|u(r)|^{\frac{4}{n}}u(r) + (b(r) \cdot \nabla + c(r))u(r))dr\|_{L^2}.$$ 

Note that, by the Sobolev embedding $H^1 \hookrightarrow L^{2+\frac{2}{n}}$, $\int_s^t \|u(r)\|_{L^{2+\frac{2}{n}}}dr \leq C\|u\|_{C([0,t];H^1)}(t-s)$. Moreover, in view of the boundedness of $\sup_{0 \leq s \leq t} |B_k(s)|$, $1 \leq k \leq N$, we also have

$$\int_s^t \|e^{-ir\Delta}(b(r) \cdot \nabla + c(r))u(r)dr\|_{L^2} \leq \int_s^t \|(b(r) \cdot \nabla + c(r))u(r)\|_{L^2}dr \leq C(t)\|u\|_{C([0,t];H^1)}(t-s).$$

Thus, combining the estimates above we obtain (6.33).
Proof of Theorem 2.10. We mainly prove the first assertion (i), as the second assertion (ii) can be proved similarly. Below, we fix any \( T \in (0, \tau^*) \) and recall that \( B_k \in C^\nu([0, T]) \) for any \( \nu \in (\frac{1}{2}, 1) \), \( 1 \leq k \leq N, \mathbb{P}\text{-a.s.}, \) and by Remark 2.9
\[
\|u\|_{C^\nu([0, T]; H^1)} + \|u\|_{L^2(0, T; H^\frac{3}{2})} < \infty.
\]

Note that for any \( \varphi \in C_c^\infty \) and any \( 0 \leq s < t \leq T, \)
\[
\langle \delta X_{st}, \varphi \rangle = \langle (\delta e^W)_{st}u(s), \varphi \rangle + \langle e^W(s)\delta u_{st}, \varphi \rangle + \langle (\delta e^W)_{st}\delta u_{st}, \varphi \rangle.
\]
We shall estimate each term on the right-hand side above separately below.

(i) Estimate of \( \langle (\delta e^W)_{st}u(s), \varphi \rangle \). Using Taylor’s expansion we have
\[
(\delta e^W)_{st} = e^W(s) - \mu(t - s) + \sum_{j=1}^N i\phi_k \delta B_{k, st} - \frac{1}{2} \sum_{j,k=1}^N \phi_j \phi_k \delta B_{j, st} \delta B_{k, st} + o(t - s).
\]
Then, taking into account (see [24, Section 3.3], [39, p.9])
\[
\delta B_{j, st} \delta B_{k, st} = \mathbb{B}_{jk, st} + \mathbb{B}_{kj, st} + \delta_{jk}(t - s),
\]
we obtain
\[
(\delta e^W)_{st} = e^W(s)(-\mu(t - s) + \sum_{k=1}^N i\phi_k \delta B_{k, st} - \sum_{j,k=1}^N \phi_j \phi_k \mathbb{B}_{jk, st}) + o(t - s),
\]
which yields that
\[
\langle (\delta e^W)_{st}u(s), \varphi \rangle = \langle -\mu(e^W(s)u(s)), \varphi \rangle(t - s) + \sum_{k=1}^N \langle i\phi_k (e^W(s)u(s)), \varphi \rangle \delta B_{k, st}
\]
\[
- \sum_{j,k=1}^N \langle \phi_j \phi_k (e^W(s)u(s)), \varphi \rangle \mathbb{B}_{jk, st} + o(t - s).
\]

(ii) Estimate of \( \langle e^W(s)\delta u_{st}, \varphi \rangle \). Let \( f(u) := |u|^\frac{4}{3} u \). We claim that
\[
\langle e^W(s)\delta u_{st}, \varphi \rangle = \langle i\Delta(e^W(s)u(s)), \varphi \rangle(t - s) + \langle i f(e^W(s)u(s)), \varphi \rangle(t - s) + O((t - s)^{1+\nu}).
\]
In order to prove (6.39), by (2.10), we have
\[
\langle e^W(s)\delta u_{st}, \varphi \rangle = \langle e^W(s) \int_s^t i e^{-W(r)} \Delta(e^W(r)u(r))dr, \varphi \rangle + \langle e^W(s) \int_s^t i f(u(r))dr, \varphi \rangle.
\]
\[
=: K_1 + K_2.
\]
We shall treat \( K_1 \) and \( K_2 \) separately below. For simplicity, we set \( L(r)u := (b(r) \cdot \nabla + c(r))u, \ v \in H^1, \ r \in [s, t] \). Then, since \( e^{-W(r)} \Delta(e^W(r)u(r)) - e^{-W(s)} \Delta(e^W(s)u(s)) = \Delta(u(r) - u(s)) + (L(r)u(r) - L(s)u(s)) \) and \( \overline{W} = e^{-W} \), we have
\[
K_1 = \langle i\Delta(e^W(s)u(s)), \varphi \rangle(t - s) + \int_s^t \langle u(r) - u(s), (i)\Delta(e^{-W(s)}\varphi) \rangle dr
\]
\[
+ \int_s^t \langle L(r)u(r) - L(s)u(s), (i)e^{-W(s)}\varphi \rangle dr
\]
\[
=: (i\Delta(e^W(s)u(s)), \varphi)(t - s) + K_{11} + K_{12}.
\]
Note that, integration by parts formula yields that
\[
K_{11} = \int_s^t \langle e^{-ir\Delta} u(r) - e^{-is\Delta} u(s), (-i)e^{-is\Delta}(e^{-W(s)}\varphi) \rangle dr
+ \int_s^t \langle u(r), (-i)(1 - e^{i(r-s)\Delta})\Delta(e^{-W(s)}\varphi) \rangle dr
\]
\[
\leq \int_s^t \|e^{-ir\Delta} u(r) - e^{-is\Delta} u(s)\|_{L^2} \|\Delta(e^{-W(s)}\varphi)\|_{L^2} dr
+ \int_s^t \|u(r)\|_{L^2} \|(1 - e^{i(r-s)\Delta})\Delta(e^{-W(s)}\varphi)\|_{L^2} dr.
\]
Since \(\sup_{s \leq T} \|\partial_x^\nu (e^{-W(s)}\varphi)\|_{L^2} \leq C(T), \forall 0 \leq |\nu| \leq 4\), and \(\|(1 - e^{i(r-s)\Delta}\varphi)\|_{L^2} \leq C(r - s)\|\Delta\varphi\|_{L^2}\) for any \(\varphi \in C_c^\infty\), taking into account (6.33) we obtain
\[
(6.42) \quad K_{11} \leq C(T)(t - s)^2.
\]
Moreover, using the integration by part formula again we have
\[
K_{12} = \int_s^t \langle e^{-ir\Delta} u(r) - e^{-is\Delta} u(s), (-i)e^{-is\Delta}L(s)^*(e^{-W(s)}\varphi) \rangle dr
+ \int_s^t \langle u(r), (-i)(1 - e^{i(r-s)\Delta})L(s)^*(e^{-W(s)}\varphi) \rangle dr
+ \int_s^t \langle (L(r) - L(s))u(r), (-i)e^{-W(s)}\varphi \rangle dr.
\]
Since \(\|(L(r) - L(s))u(r)\|_{L^2} \leq C(T)\max_{1 \leq k \leq N} |B_k(r) - B_k(s)| \leq C(T)(r - s)^\nu\), similarly to (6.42), we have
\[
(6.43) \quad K_{12} \leq C(T) \int_s^t (r - s) + (r - s)^\nu dr \leq C(T)(t - s)^{1+\nu}.
\]
Thus, plugging (6.42) and (6.43) into (6.41) we obtain
\[
(6.44) \quad K_1 = \langle i\Delta(e^{W(s)}u(s)), \varphi \rangle (t - s) + O((t - s)^{1+\nu}).
\]
Next we treat the delicate term \(K_2\) in (6.40). We see that
\[
(6.45) \quad K_2 = \langle if(e^{W(s)}u(s)), \varphi \rangle (t - s) + \int_s^t \langle f(u(r)) - f(u(s)), (-i)e^{-W(s)}\varphi \rangle dr.
\]
Setting \(u_\tau(r) := \tau u(r) + (1 - \tau)u(s), \tau \in [0, 1]\), we have
\[
f(u(r)) - f(u(s)) = \int_0^1 \partial_z f(u_\tau(r)) d\tau \delta u_{sr} + \int_0^1 \partial_z f(u_\tau(r)) d\tau \delta u_{sr}
= h_1(u_\tau(r)) \delta u_{sr} + h_2(u_\tau(r)) \delta u_{sr},
\]
where \(\partial_z f(z) = (1 + \frac{2}{\delta})|z|^{\frac{3}{2}}, \partial_z f(z) = \frac{2}{\delta^2}|z|^{\frac{3}{2}} - 2z, z \in \mathbb{C}\). Then, by (6.45),
\[
K_2 = \langle if(e^{W(s)}u(s)), \varphi \rangle (t - s) + \int_s^t \langle u(r) - u(s), (-i)h_1(u_\tau(r)) e^{-W(s)}\varphi \rangle dr
+ \int_s^t \langle u(r) - u(s), (-i)h_2(u_\tau(r)) e^{-W(s)}\varphi \rangle dr.
\]
In order to estimate the right-hand side above, we take the third term involving \( h_2 \) for example, the second term containing \( h_1 \) can be estimated similarly. Note that

\[
\int_s^t \langle u(r) - u(s), (-i)h_2(u_r(r))e^{-W(s)\varphi}\rangle dr
\]

\[
= \int_s^t \langle e^{ir\Delta}u(r) - e^{is\Delta}u(s), (-i)e^{is\Delta}h_2(u_r(r))e^{-W(s)\varphi}\rangle dr
\]

\[
+ \int_s^t \langle (1 - e^{i(r-s)\Delta})u(r), (-i)h_2(u_r(r))e^{-W(s)\varphi}\rangle dr
\]

(6.47)

\[
=: K_{21} + K_{22}.
\]

Since by the Hölder inequality and the Sobolev embedding \( H^1 \hookrightarrow L^{\frac{5}{2}} \),

\[
\|h_2(u_r(r))e^{-W(s)\varphi}\|_{L^2} \leq C\|\varphi\|_{L^\infty} \int_s^1 \|u_r(r)\|_{L^{\frac{5}{2}}}^\frac{3}{2} dr \leq C\|u\|_{H^1(0, T ; H^1)}
\]

which along with (6.33) yields that

(6.48)  \[
K_{21} \leq \int_s^t \|e^{-ir\Delta}u(r) - e^{-is\Delta}u(s)\|_{L^2} \|h_2(u_r(r))e^{-W(s)\varphi}\|_{L^2} dr \leq C(T)(t - s)^2.
\]

As regards \( K_{22} \), we consider the cases where \( d = 1 \) and \( d = 2 \) separately below. In the case where \( d = 1 \), we have

\[
K_{22} \leq \int_s^t \|1 - e^{-i(r-s)\Delta}u(r)\|_{H^{-1}} \|h_2(u_r(r))\|_{H^1} \|e^{-W(s)\varphi}\|_{W^{1, \infty}} dr
\]

\[
\leq C(T) \int_s^t (r - s) \|h_2(u_r(r))\|_{H^1} dr.
\]

Note that, by Hölder’s inequality and Sobolev’s embedding \( H^1 \hookrightarrow L^{\frac{5}{2}} \cap L^\infty \),

\[
\|h_2(u_r)\|_{H^1} \leq C \int_s^1 \|u_r^4\|_{L^2} + \|\nabla(u_r^3 u_r)\|_{L^2} dr
\]

\[
\leq C \int_s^1 \|u_r^4\|_{L^8} + \|u_r\|_{L^\infty}^3 \|\nabla u_r\|_{L^2} dr \leq C\|u\|_{L^2(0, T ; H^1)}
\]

This yields that, in the case where \( d = 1 \),

(6.49)  \[
K_{22} \leq C(T) \int_s^t (r - s) dr \leq C(T)(t - s)^2.
\]

Moreover, in the case where \( d = 2 \) we shall measure the spatial regularity of \( u \) in the local smoothing space. Precisely, since \( \varphi \in C_c^\infty \), setting \( D := \text{supp}(\varphi) \) we have

\[
\overline{K}_{22} = \int_s^t \langle (1 - e^{-i(r-s)\Delta})u(r), i\nabla(\frac{3}{2}h_2(u_r(r))e^{W(s)\varphi})\rangle dr
\]

\[
\leq \int_s^t \|\chi_D(\nabla)^{-\frac{1}{2}}((1 - e^{-i(r-s)\Delta})u(r))\|_{L^2} \|h_2(u_r(r))\|_{H^\frac{1}{2}} \|e^{W(s)\varphi}\|_{W^{1, \infty}} dr
\]

\[
\leq C(T) \int_s^t \|u(r)\|_{H^{-\frac{1}{2}}} \|h_2(u_r(r))\|_{H^\frac{1}{2}} dr,
\]

where \( \chi_D \) is the characteristic function of \( D \), and we also used the inequalities \( \|\chi_D(\nabla)^{\frac{1}{2}}\psi\|_{L^2} \leq \|\psi\|_{L^2} \) and \( \|\psi_1 \psi_2\|_{H^\frac{1}{2}} \leq C\|\psi_1\|_{W^{1, \infty}} \|\psi_2\|_{H^\frac{1}{2}} \). Note that, by the Leibniz law for fractional
derivatives and the Sobolev embedding $H^1 \hookrightarrow W^{1,4}$,
\[
\|h_2(u_r(r))\|_{H^\frac{1}{2}} \leq \int_0^1 \|\langle \nabla \rangle^{\frac{1}{2}}(u_r(r))^2\|_{L^2} d\tau \leq \int_0^1 \|\langle \nabla \rangle^{\frac{1}{2}}u_r(r)\|_{L^4}^2 d\tau \leq C\|u\|_{C([0,T];H^1)}^2.
\]
This yields that, in the case where $d = 2$,
\[
|K_{22}| \leq C(T)\|u\|_{C([0,T];H^1)}^2 \int_s^t (r - s)\|u(r)\|_{H^\frac{3}{2}}^2 dr \\
\leq C(T)\|u\|_{C([0,T];H^1)}^2\|u\|_{L^2(0,T;H^\frac{3}{2})}^2 (t - s)^\frac{3}{2}.
\]
Thus, plugging (6.48), (6.49) and (6.50) into (6.47) we obtain
\[
\int_s^t \langle u(r) - u(s), (-i)h_2(u_r(r))e^{-W(s)}\varphi \rangle dr \leq C(T)(t - s)^\frac{3}{2}.
\]
The term involving $h_1(u_r(r))$ on the right-hand side of (6.46) can be estimated similarly. Thus, we conclude that
\[
K_2 = \langle if(e^{W(s)}u(s)), \varphi \rangle(t - s) + O((t - s)^\frac{3}{2}).
\]
Therefore, plugging (6.44) and (6.52) into (6.40) we obtain (6.39), as claimed.

(iii) Estimate of $\langle (\delta e^W)_{st}\delta u_{st}, \varphi \rangle$. Using the integration by parts formula and Hölder’s inequality we have
\[
\langle (\delta e^W)_{st}\delta u_{st}, \varphi \rangle = \int_s^t \langle u(r), (-i)e^{-W(r)}\Delta(e^W(r)(\delta e^W)_{st}\varphi) \rangle + \langle f(u(r)), (-i)(\delta e^W)_{st}\varphi \rangle dr \\
\leq \int_s^t \|u(r)\|_{L^2}\|e^W(r)(\delta e^W)_{st}\|_{H^2}\|\varphi\|_{W^{2,\infty}} + \|u(r)\|_{L^2}^{1+\frac{3}{2}}\|\delta e^W_{st}\|_{L^2}\|\varphi\|_{L^\infty} dr.
\]
Since for any multi-index $|\nu| \geq 0$,
\[
\|\partial_{x}^\nu(\delta e^W)_{st}\|_{L^\infty} \leq C(T)(t - s)^{\nu},
\]
taking into account Sobolev’s embedding $H^1 \hookrightarrow L^{2+\frac{3}{2}}$, we obtain
\[
\langle (\delta e^W)_{st}\delta u_{st}, \varphi \rangle \leq C(T)(t - s)^{1+\nu}.
\]
Now, plugging (6.38), (6.39) and (6.54) into (6.34) and using $X = e^W u$ we obtain
\[
\langle \delta X_{st}, \varphi \rangle = \langle i\Delta X(s) + if(X(s)) - \mu X(s), \varphi \rangle(t - s) \\
+ \sum_{k=1}^N \langle i\phi_k X(s), \varphi \rangle \delta B_{k, st} - \sum_{j,k=1}^N \langle \phi_j \phi_k X(s), \varphi \rangle \mathbb{B}_{jk, st} + o(t - s).
\]
This yields that for any $\varphi \in C_c^{\infty}$, $X, \varphi \in D_B^{2\alpha}([0,T];\mathbb{R})$ with the Gubinelli derivative $Y^\varphi = \langle i\phi_k X, \varphi \rangle$, $1 \leq k \leq N$. In particular, we infer that $X := e^W u$ satisfies equation (1.1) in the sense of Definition 2.1 and (2.1) holds. Therefore, the proof is complete. □

Now, by virtue of Theorem 2.10 we obtain Theorems 2.5 and 2.7 from Theorems 2.11 and 2.12 respectively.
7. Appendix

Proof of Corollary 3.3.2. For any \( f_1, f_2 \in H^1 \), let

\[
\hat{f}_1 = f_1 + a_1 \cdot \nabla Q + b_1 \Lambda Q + c_1 \rho, \quad \hat{f}_2 = f_2 + a_2 Q + b_2 \cdot xQ + c_2 |x|^2 Q,
\]

where \( a_1 = (a_{1,j}) \) and \( b_2 = (b_{2,j}) \) are vectors.

Let \( \tilde{f} := \hat{f}_1 + i \hat{f}_2 \) be such that \( \tilde{f} \in \mathcal{K} \). This yields that, for \( 1 \leq j \leq d \),

\[
a_{1,j} = \frac{2}{\|Q\|_{L^2}^2} \langle f_1, xQ \rangle, \quad b_1 = \frac{1}{\|xQ\|_{L^2}^2} \langle f_1, |x|^2 Q \rangle + 2 \frac{\rho_1 |x|^2 Q}{\|xQ\|_{L^2}^4} \langle f_1, Q \rangle, \quad c_1 = \frac{2}{\|xQ\|_{L^2}^2} \langle f_1, Q \rangle,
\]

\[
a_{2,j} = \frac{2}{\|xQ\|_{L^2}^2} \langle f_2, \partial_j Q \rangle + 2 \frac{\rho_1 |x|^2 Q}{\|xQ\|_{L^2}^4} \langle f_2, Q \rangle, \quad b_2 = \frac{2}{\|xQ\|_{L^2}^2} \langle f_2, \partial_j Q \rangle, \quad c_2 = \frac{1}{\|xQ\|_{L^2}^2} \langle f_2, Q \rangle.
\]

By direct calculation we have

\[
\langle L\tilde{f}, \tilde{f} \rangle = (Lf, f) - 4b_1 \langle f_1, Q \rangle - 2c_1 \langle f_1, |x|^2 Q \rangle + 2b_1c_1 \|xQ\|_{L^2}^2 - c_1^2 \langle \rho, |x|^2 Q \rangle - 4 \langle f_2, b_2 \cdot \nabla Q \rangle - 8c_2 \langle f_2, \Lambda Q \rangle + |b_2|^2 \|Q\|_{L^2}^2 + 4c_2^2 \|xQ\|_{L^2}^2,
\]

which along with the inequality \( ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2 \) and Lemma 3.1 yields

\[
\langle Lf, f \rangle \geq \nu \|\tilde{f}\|_{H^1}^2 - \varepsilon \|f\|_{H^1}^2 - C(\|a_1\|_2^2 + b_1^2 + c_1^2 + a_2^2 + |b_2|^2 + c_2^2).
\]

Moreover, expanding \( \|\tilde{f}\|_{H^1}^2 \) by (7.1) and then using the inequality \( \langle f_j, g \rangle_{H^1} \leq \varepsilon \|f_j\|_{H^1}^2 + \varepsilon^{-1} \|g\|_{H^1}^2 \) for any \( g \in H^1 \), \( j = 1, 2 \), we obtain that for some \( C_1, C_2 > 0 \),

\[
\|\tilde{f}\|_{H^1}^2 \geq C_1 \|f\|_{H^1}^2 - C_2 (|a_1|^2 + b_1^2 + c_1^2 + a_2^2 + |b_2|^2 + c_2^2).
\]

Therefore, combining (7.2) and (7.3) and taking \( \varepsilon \) small enough we obtain (3.8). \( \Box \)

Proof of Corollary 3.3.3. Let \( \tilde{f} := f \Phi_A^{\frac{1}{2}} \). Since \( \nabla f \Phi_A^{\frac{1}{2}} = \nabla \tilde{f} - \frac{\nabla \Phi_A^{\frac{1}{2}}}{2 \Phi_A} \tilde{f} \), we have

\[
\int |\nabla f|^2 \Phi_A + |f|^2 - (1 + \frac{4}{d}) Q^2 \tilde{f}_1^2 - Q^2 \tilde{f}_2^2 dx
\]

\[
= \int |\nabla \tilde{f}|^2 + |\tilde{f}|^2 - (1 + \frac{4}{d}) Q^2 \tilde{f}_1^2 - Q^2 \tilde{f}_2^2 dx
\]

\[
+ \int (1 - \Phi_A) (|f|^2 - (1 + \frac{4}{d}) Q^2 \tilde{f}_1^2 - Q^2 \tilde{f}_2^2) dx
\]

\[
+ \frac{1}{4} \int \frac{\nabla \Phi_A}{\Phi_A} |\tilde{f}|^2 dx - \operatorname{Re} \int \frac{\nabla \Phi_A}{\Phi_A} \cdot \nabla \tilde{f} dx =: \sum_{i=1}^d K_i.
\]

Since \( \Phi_A(x) = 1 \) for \( |x| \leq A \) and by (3.6), \( \int Q f_1 dx = 0 \), we infer that

\[
\int Q \tilde{f}_1 dx = \int_{|x| > A} Q (\tilde{f}_1 - f_1) dx = \int_{|x| > A} \tilde{f}_1 (Q - Q \Phi_A^{\frac{1}{2}}) dx.
\]

Note that, for \( A \leq |x| \leq 2A, \Phi_A^{\frac{1}{2}}(x) \leq C \) and so, by (3.1), \( |Q(x) - Q \Phi_A^{\frac{1}{2}}(x)| \leq C e^{-|x|} \) for some \( \delta > 0 \). Moreover, for \( |x| \geq 2|A| \), since \( \Phi_A^{\frac{1}{2}}(x) = e^{\frac{|x|}{A}} \), taking \( A \) sufficiently large such that \( \delta - \frac{1}{2A} \geq \frac{\delta}{2} > 0 \), we obtain that \( Q \Phi_A^{\frac{1}{2}}(x) \leq C e^{-(\delta - \frac{1}{2A})|x|} \leq C e^{-\frac{\delta}{2}|x|} \). Thus, we
conclude that $|Q(x) - Q\Phi_A^{-\frac{1}{2}}(x)| \leq Ce^{-\frac{4}{d}|x|}$ for $|x| \geq A$. This yields that

$$|\int Q\tilde{f}_1 dx| \leq \|\tilde{f}_1\|_{L^2}(\int_{|x|>A}|Q - Q\Phi_A^{-\frac{1}{2}}|^2 dx)^{\frac{1}{2}}$$

(7.6)

$$\leq C\|\tilde{f}_1\|_{L^2}(\int_{|x|>A}e^{-\delta|x|} dx)^{\frac{1}{2}} =: \delta_A\|\tilde{f}_1\|_{L^2},$$

where $\delta_A \to 0$ as $A \to \infty$. In view of the orthogonal conditions and the exponential decay of $Q$ and $\rho$, similar arguments as above also yield that the remaining five inner products in the second part on the right-hand side of (3.8) can be also bounded by $\delta_A\|\tilde{f}\|_{L^2}$ with $\delta_A \to 0$ as $A \to \infty$. Thus, Corollary 3.2 yields that

(7.7)

$$K_1 \geq \nu_1\|\tilde{f}\|^2_{H^1} - \nu_2\delta_A\|\tilde{f}\|^2_{L^2}.$$

Moreover, by (3.1), we have that for $A$ large enough,

(7.8)

$$K_2 = \int_{|x|>A} (1 - \Phi_A)(|f|^2 - (1 + \frac{4}{d})Q^2f_1^2 - Q^2f_2^2) dx > 0.$$

We also note that, since $|\nabla\Phi_A| \leq CA^{-1}$, by Hölder’s inequality,

(7.9)

$$K_3 + K_3 \leq CA\|\tilde{f}\|^2_{H^1}.$$

Thus, plugging (7.7), (7.8) and (7.9) into (7.4), we obtain that for $A$ large enough

(7.10)

$$\int |\nabla f|^2\Phi_A + |f|^2 - (1 + \frac{4}{d})Q^2f_1^2 - Q^2f_2^2 dx \geq \frac{\nu_1}{2}\|\tilde{f}\|^2_{H^1}.$$

Using again $\nabla \tilde{f} = \nabla f\Phi_A^{-\frac{1}{2}} + \frac{\nabla \Phi_A}{2\Phi_A}\tilde{f}$ we see that

(7.11)

$$\frac{\nu_1}{2}\|\tilde{f}\|^2_{H^1} = \frac{\nu_1}{2}\int |f|^2\Phi_A dx + \frac{\nu_1}{2}\int |\nabla f\Phi_A^{-\frac{1}{2}} + \frac{\nabla \Phi_A}{2\Phi_A}\tilde{f}|^2 dx$$

$$= \frac{\nu_1}{2}\int (|f|^2 + |\nabla f|^2)\Phi_A dx + \frac{\nu_1}{2}\int \text{Re}(\nabla f\Phi_A^{-\frac{1}{2}}\frac{\nabla \Phi_A}{\Phi_A}\tilde{f}) dx + \frac{\nu_1}{8}\int |\frac{\nabla \Phi_A}{\Phi_A}\tilde{f}|^2 dx.$$

By Hölder’s inequality and $|\nabla\Phi_A| \leq CA^{-1},$

(7.12)

$$|\frac{\nu_1}{2}\int \text{Re}(\nabla f\Phi_A^{-\frac{1}{2}}\frac{\nabla \Phi_A}{\Phi_A}\tilde{f}) dx| \leq \frac{\nu_1C}{2A} \left( \int |\nabla f|^2\Phi_A dx \right)^{\frac{1}{2}} \left( \int |f|^2\Phi_A dx \right)^{\frac{1}{2}}$$

$$\leq \frac{\nu_1}{4}\int |\nabla f|^2\Phi_A dx + \frac{4\nu_1C^2}{A^2}\int |f|^2\Phi_A dx.$$

Moreover, we have

(7.13)

$$\frac{\nu_1}{8}\int \left| \frac{\nabla \Phi_A}{\Phi_A}\tilde{f} \right|^2 dx \leq \frac{\nu_1C^2}{2A^2}\int |f|^2\Phi_A dx.$$

Therefore, putting together (7.10)-(7.13) and taking $A$ large enough we obtain (3.3).
Proof of Lemma 4.2. The proof is based on the implicit function theorem. Let \( w \) and \( R \) be as in (1.7). Set \( \mathcal{P} := (\lambda, \alpha, \beta, \gamma, \theta) \) and define the functionals
\[
\begin{align*}
  f_{1,j}(u, \mathcal{P}) &= \text{Re} \int (x_j - \alpha_j)wRdx, \quad f_2(u, \mathcal{P}) = \text{Re} \int |x - \alpha|^2wRdx, \\
  f_3(u, \mathcal{P}) &= \text{Im} \int (\frac{d}{2}w + (x - \alpha) \cdot \nabla w)Rdx, \quad f_{4,j}(u, \mathcal{P}) = \text{Im} \int \partial_{x_j} wRdx,
\end{align*}
\]
\[
f_5(u, \mathcal{P}) = \text{Im} \int \partial_\rho Rdx, \quad 1 \leq j \leq d.
\]

Note that, by the definition of \( u_0 \), \( f_j(u_0, \mathcal{P}_0) = 0 \), \( 1 \leq j \leq 5 \). Moreover, the orthogonality conditions and straightforward computations show that for \( 1 \leq j, k \leq d \),
\[
\begin{align*}
  \partial_{\alpha_k} f_{1,j}(u_0, \mathcal{P}_0) &= -\frac{\delta_{jk}}{2} \|Q\|^2_{L^2} + \mathcal{O}(\|R_0\|_{L^2}), \quad \partial_{\lambda} f_2(u_0, \mathcal{P}_0) = -\lambda_0 \|xQ\|^2_{L^2} + \mathcal{O}(\|R_0\|_{L^2}), \\
  \partial_{\alpha_k} f_3(u_0, \mathcal{P}_0) &= -\frac{\beta_{0,k}}{2\lambda_0^2} \|Q\|^2_{L^2} + \mathcal{O}(\|R_0\|_{L^2}), \quad \partial_{\lambda} f_4(u_0, \mathcal{P}_0) = \frac{1}{4} \|xQ\|^2_{L^2} + \mathcal{O}(\|R_0\|_{L^2}), \\
  \partial_{\alpha_k} f_{4,j}(u_0, \mathcal{P}_0) &= \frac{\beta_{0,k}}{2\lambda_0^2} \|Q\|^2_{L^2} + \mathcal{O}(\|R_0\|_{L^2}), \quad \partial_{\lambda} f_4(u_0, \mathcal{P}_0) = -\frac{\delta_{jk}}{2\lambda_0^2} \|Q\|^2_{L^2} + \mathcal{O}(\|R_0\|_{L^2}), \\
  \partial_{\alpha_k} f_5(u_0, \mathcal{P}_0) &= \frac{\gamma_0}{2\lambda_0} \langle \rho, |x|^2Q \rangle + \mathcal{O}(\|R_0\|_{L^2}), \quad \partial_{\lambda} f_5 = \frac{\beta_{0,k}}{2\lambda_0} \|xQ\|^2_{L^2} + \mathcal{O}(\|R_0\|_{L^2}),
\end{align*}
\]

and the remaining partial differentials are bounded at most by \( C\lambda_0^{-2} \|R_0\|_{L^2} = \mathcal{O}(T) \) for some \( C > 0 \). This yields that the Jacob determinant
\[
(7.14) \quad \left. \frac{\partial(f_1, f_2, f_3, f_4, f_5)}{\partial \mathcal{P}} \right|_{(u_0, \mathcal{P}_0)} = 2^{-2d-3} \lambda_0^{1-d} \|Q\|_{L^2}^4 \|xQ\|_{L^2}^6 + \mathcal{O}(T),
\]
which is positive for \( T \) small enough.

Thus, the implicit function theorem yields the existence of \( r_0 > 0 \) and a \( C^1 \) map \( \Psi \) from \( \mathcal{U}_{r_0}(u_0) \) to \( \mathcal{U}_{r_0}(\mathcal{P}_0) \) and the orthogonality conditions in (4.8) hold.

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