Abstract

Let \( X \subset \mathbb{P}(V) \) be a projective variety, which is not contained in a hyperplane. Then every vector \( v \) in \( V \) may be written as a sum of vectors from the affine cone over \( X \). The minimal number of summands in such a sum is called the rank of \( v \). In this paper, we classify all equivariantly embedded homogeneous projective varieties \( X \subset \mathbb{P}(V) \) whose rank function is lower semi-continuous. Classical examples are: the variety of rank one matrices (Segre variety with two factors) and the variety of rank one quadratic forms (quadratic Veronese variety). In the general setting, \( X \) is the orbit in \( \mathbb{P}(V) \) of a highest weight line in an irreducible representation \( V \) of a reductive algebraic group \( G \). Thus, our result is a list of all irreducible representations of reductive groups, for which the corresponding rank function is lower semi-continuous.

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1 Introduction

Let $V$ be a finite dimensional vector space over an algebraically closed field $\mathbb{F}$ of characteristic 0. Let $X \subset \mathbb{P}(V)$ be a projective variety and $X \subset V$ be the affine cone over $X$. Suppose that $X$ is nondegenerate, i.e. it is not contained in a hyperplane. Then every vector $v \in V$ can be written as a linear combination of points of $X$ and we have a well-defined function $\text{rk} : \mathbb{P}(V) \to \mathbb{N}$, called rank (or $X$-rank), given by

$$\text{rk}[v] = \text{rk}_X[v] = \min\{r \in \mathbb{N} : v = x_1 + \ldots + x_r, \text{ with } x_j \in X\},$$

where $[v] \in \mathbb{P}(V)$ denotes the projective point corresponding to a non-zero vector $v$. The rank sets in $\mathbb{P}(V)$ with respect to $X$ are defined as

$$X_r = \{[v] \in \mathbb{P}(V) : \text{rk}[v] = r\}.$$

The secant varieties of $X$ are defined as the Zariski closure

$$\sigma_r(X) = \bigcup_{s \leq r} X_s.$$

The border rank of $[v] \in \mathbb{P}(V)$ is defined as

$$\text{rk}_b[v] = \text{rk}_X[v] = \min\{r \in \mathbb{N} : [v] \in \sigma_r(X)\}.$$

Let Aut$(X) \subset SL(V)$ denote the group of linear automorphisms of $X$. Then rank and border rank are Aut$(X)$-invariant, and hence Aut$(X)$ acts on $X_r$ and $\sigma_r(X)$.

**Definition 1.1.** We call $X$ rs-continuous if $\text{rk}_X$ is a lower semi-continuous function on $\mathbb{P}(V)$, i.e. if rank and border rank defined by $X$ coincide. Otherwise, we say that $X$ is r-discontinuous. We call $[v] \in \mathbb{P}(V)$ exceptional if $\text{rk}_X[v] \neq \text{rk}_b[v]$.

Our goal is to classify all rs-continuous varieties belonging to a certain class, namely, the homogeneous rs-continuous varieties. By a homogeneous projective variety we mean an equivariantly embedded variety $X \subset \mathbb{P}(V)$ whose automorphism group is transitive. In such a case the (linear) automorphism group $G$ is always semisimple and $X \cong G/P$, where $P$ is a parabolic subgroup of $G$. In other words, $X$ is a flag variety of $G$. Furthermore, when the embedding is nondegenerate, $V$ is an irreducible $G$-module, and so it is determined up to isomorphism by its highest weight, say $\lambda$, so that $V \cong V(\lambda)$. Then $X$ can be viewed as the orbit of a highest weight line $X = G[\nu^\lambda] \subset \mathbb{P}(V)$. Conversely, if $G$ is a semisimple algebraic group and $V = V(\lambda)$ is an irreducible $G$-module, then $\mathbb{P}(V)$ contains a unique closed $G$-orbit — the orbit of a highest weight line; we denote this orbit by $X(G,V)$. It is not always true that $G$ is the full automorphism group of $X(G,V)$. For instance, the symplectic group $Sp_{2n}$ acts transitively on the projective space $\mathbb{P}(\mathbb{F}^{2n})$, but the full linear automorphism group is $SL_{2n}$. Our approach is to classify all irreducible representations $(G,V)$ such that $X(G,V)$ is rs-continuous. Then the list of rs-continuous homogeneous projective varieties is obtained by dropping the redundancies. For brevity of expression, we shall call a representation $(G,V)$ rs-continuous, if the corresponding variety $X(G,V)$ is rs-continuous.

Before stating our classification theorem, let us mention some classical examples, where the terminology stems from.

Let $V = \mathbb{F}^m \otimes \mathbb{F}^n$ be the space of $m \times n$-matrices. On $V$ we have the classical notion of rank of a matrix. Let $X \subset \mathbb{P}(V)$ denote the variety of matrices of rank 1; $X$ can be defined by the vanishing of all $2 \times 2$-minors and is also known as the Segre variety of simple tensors $X = \text{Segre}(\mathbb{P}^{m-1} \times \mathbb{P}^{n-1})$. It is well-known that every matrix of rank $r$ can be written as a sum of $r$ matrices of rank 1. The set of matrices of rank $r$ or less is the common zero-locus of all $(r+1) \times (r+1)$-minors and hence is a closed set, which equals the secant variety $\sigma_r(X)$. Hence $X$ is rs-continuous. The automorphism group of $X$ equals $PSL_m \times PSL_n$ in this case.
A similar situation occurs for symmetric and for skew symmetric matrices. The corresponding varieties are the quadratic Veronese embedding \( \text{Ver}_2(\mathbb{P}^{n-1}) \subset \mathbb{P}(S^2\mathbb{F}^n) \) and the Grassmannian of planes \( \text{Gr}_2(\mathbb{F}^n) \subset \mathbb{P}(\Lambda^2\mathbb{F}^n) \). In both cases we have a linear action of \( SL_n \) preserving \( \mathbb{X} \) and yielding the full automorphism group of \( \mathbb{X} \). Both varieties are rs-continuous.

Perhaps the simplest r-discontinuous homogeneous variety is the twisted cubic curve: \( \mathbb{X} = \text{Ver}_3(\mathbb{P}^1) \subset \mathbb{P}(S^3\mathbb{F}^2) \). The respective automorphism group \( G = PSL_2 \) has three orbits in \( \mathbb{P}(S^3\mathbb{F}^2) \), which can be written as \( G[x^3], G[x^2y] \) and \( G[x^3 + y^3] \), where \( x, y \) is an arbitrary basis of \( \mathbb{F}^2 \). Indeed, an element of \( \mathbb{P}(S^3\mathbb{F}^2) \) can be written as a product \( [l_1l_2l_3] \), with \( [l_i] \in \mathbb{P}^1 \), and there are three possibilities: either \( [l_1] = [l_2] = [l_3] \), or \( [l_1] = [l_2] \neq [l_3] \), or all three are distinct. Since \( PSL_2 \) acts transitively on triples of distinct points in \( \mathbb{P}^1 \), we have three orbits in \( \mathbb{P}(S^3\mathbb{F}^2) \). It is easy to see that the ranks are 1, 3 and 2, respectively. However, we have \( \sigma_2(\mathbb{X}) = G[x^3 + y^3] = \mathbb{P}(S^3\mathbb{F}^3) \) and hence \( [x^2y] \) is exceptional and \( \mathbb{X} \) is r-discontinuous.

Let us notice that, in the above example, \( \overline{G[x^2y]} \) is the tangential variety of \( \mathbb{X} \), which is clearly contained in the secant variety \( \sigma_2(\mathbb{X}) \). In fact it is a general phenomenon, that exceptional points do appear in the tangential variety, whenever they exist. This fact is in the basis of our methods.

Now, we formulate our main result.

**Theorem 1.1.** Let \( G \) be a semisimple algebraic group and \( V \) be a finite dimensional irreducible \( G \)-module, with \( \dim V \geq 2 \). Then the closed \( G \)-orbit \( \mathbb{X}(G, V) \subset \mathbb{P}(V) \) is rs-continuous if and only if the pair \( (G, V) \) appears in the following table.

| Group \( G \) | Representation \( V \) | Highest weight of \( V \) |
|--------------|-----------------|-----------------|
| Simple classical groups |
| \( SL_n \) | \( \mathbb{F}^n, (\mathbb{F}^n)^*, (\Lambda^2\mathbb{F}^n), (\Lambda^2\mathbb{F}^n)^* \), \( S^2\mathbb{F}^n, (S^2\mathbb{F}^n)^*, \text{sl}_n \) | \( \pi_1, \pi_{n-1}, \pi_2, \pi_{n-2}, \pi_{2n-1}, 2\pi_{n-1}, \pi_1 + \pi_{n-1} \) |
| \( SO_n \) | \( \mathbb{F}^n, R\text{Spin}_n(n \leq 10) \) | \( \pi_1, \frac{\pi_2}{2}(2 | n), \frac{\pi_2}{2} - 1(2 | n), \pi_{n-1}(2 | n) \) |
| \( Sp_{2n} \) | \( \mathbb{F}^{2n}, \Lambda^2\mathbb{F}^{2n}, S^2\mathbb{F}^{2n} \cong \mathfrak{sp}_{2n} \) | \( \pi_1, \pi_2, 2\pi_1 \) |

Simple exceptional groups

| Group \( G \) | Representation \( V \) | Highest weight of \( V \) |
|--------------|-----------------|-----------------|
| \( E_6 \) | \( \mathbb{F}^{27}, (\mathbb{F}^{27})^* \) | \( \pi_1, \pi_5 \) |
| \( F_4 \) | \( \mathbb{F}^{26} \) | \( \pi_1 \) |
| \( G_2 \) | \( \mathbb{F}^{16} \) | \( \pi_1 \) |

Non-simple groups

| Group \( G \) | Representation \( V \) | Highest weight of \( V \) |
|--------------|-----------------|-----------------|
| \( SL_m \times SL_n \) | \( \mathbb{F}^m \otimes \mathbb{F}^n \) | \( \pi_1 \oplus \pi_1 \) |
| \( SL_m \times Sp_{2n} \) | \( \mathbb{F}^m \otimes \mathfrak{sp}_{2n} \) | \( \pi_1 \oplus \pi_1 \) |
| \( Sp_{2n} \times Sp_{2n} \) | \( \mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2n} \) | \( \pi_1 \oplus \pi_1 \) |

where by \( R\text{Spin}_n \) we denote (any) spinor representation of the simply connected cover of \( SO_n \), by \( \Lambda^2\mathbb{F}^{2n} \) we denote the second fundamental representation of \( Sp_{2n} \) (which is identified with a hyperplane in \( \Lambda^2\mathbb{F}^{2n} \)), by \( \mathbb{F}^{27} \) we denote any one of the two smallest fundamental representations of \( E_6 \), by \( \mathbb{F}^{26} \) we denote the smallest fundamental representation of \( F_4 \), by \( \mathbb{F}^7 \) we denote the smallest fundamental representation of \( G_2 \). In the third column we list the highest weight of \( V \) with respect to \( G' \) (we use here and throughout the paper the numbering convention of \([\text{VO90}, \text{p. 294}]\)).

Moreover, if \( (G, V) \) is r-discontinuous, then it contains an exceptional vector of border rank 2.

We give a proof of Theorem 1.1 in Section 3. As a corollary, we obtain the list of homogeneous projective varieties given below. The list of varieties is shorter, because, as mentioned above, in certain cases there are subgroups of the automorphism group of the variety acting transitively.

**Corollary 1.2.** The rs-continuous projective varieties \( \mathbb{X} \subset \mathbb{P}(V) \) with transitive linear automorphism group are the following:

...
It is natural to ask whether rs-continuous varieties admit another general characterization. In fact, the starting point of our study was a result by Buczyński and Landsberg, [BL13], based on previous work by Landsberg and Manivel, [LM03]. This result exhibits a remarkable class of rs-continuous homogeneous varieties — the subcominuscule varieties. Recall that a variety $X \subset \mathbb{P}(V)$ is called subcominuscule if it is the variety associated to the isotropy representation of an irreducible Hermitian symmetric space $S$, i.e. $X = \mathcal{X}(G, V)$, with $G$ being the complexification of the semisimple part of isotropy subgroup the isometry group of $S$, and $V$ being the tangent space. Then [BL13, Prop. 4.1] states that $\text{rk}_X = \text{rk}_g$ and, furthermore, the $G$-orbits in $\mathbb{P}(V)$ are exactly the rank sets $X_r$. It is then natural to ask: are there other rs-continuous homogeneous varieties besides the subcominuscule ones? There are. It was shown by Kaji and Yasukura, [KY00], that the adjoint variety $\mathcal{X}(G, g)$ of a simple Lie algebra $g$ is rs-continuous if and only if $g$ is of type $A_n$ or $C_n$; see also [Kaji98, BD04]. The adjoint variety of type $C_n$ is just the quadratic Veronese variety, which is subcominuscule with respect to its automorphism group. However, the adjoint variety of type $A_n$ is not subcominuscule. The non-subcominuscule varieties in our list are $\text{Fl}(1, n-1; \mathbb{F}^n)$, $\text{Gr}_\omega(2, \mathbb{F}^{2n})$ and $\text{F}^{15}$. All these exceptions are, however, hyperplane sections in subcominuscule varieties. Conversely, the homogeneous hyperplane section of subminuscule varieties are the above three and the quadric $Q^{n-1}$ viewed as a hyperplane section in $\text{Ver}_2(\mathbb{P}^n)$. This follows from a classification of homogeneous hyperplane sections in homogeneous projective varieties given by Watanabe [WT1]. Thus our theorem implies that, if $X$ is a homogeneous hyperplane section in a subcominuscule variety $\tilde{X}$ in its minimal projective embedding, then $\tilde{X}$ is rs-continuous. So we can formulate the following:

**Corollary 1.3.** A homogeneous projective variety $X \subset \mathbb{P}(V)$ is rs-continuous if and only if it is either a subcominuscule variety, or a hyperplane section in a subcominuscule variety $\tilde{X} \subset \mathbb{P}(\tilde{V})$ in its minimal projective embedding. In the latter case, the rank function of $X$ is equal to the restriction of the rank function of $\tilde{X}$ and the secant varieties of $X$ are equal to the intersections of the secant varieties of $\tilde{X}$ with $\mathbb{P}(V)$.

Since our approach is to start with a representation $(G, V)$ rather than with a homogeneous variety $X$, we have found the following observations useful. If $(G, V)$ is a representation such that $G$ acts spherically on the projective space $\mathbb{P}(V)$, then the variety $\mathcal{X}(G, V)$ is subcominuscule, i.e. $(\text{Aut}(X), V)$ is a subminuscule representation. This follows directly from the classification of spherical representations given by Kac, [Kac80], see also [Knop98].

**Remark 1.1.** Since spherical representations have finitely many orbits and, by the above corollary, rs-continuous representations are not far away from spherical, it makes sense to ask whether this class of representations is related to the well-known class of projective representations with finitely many orbits. The fact that twisted cubic is not rs-continuous together with the fact that $PSL_n(n \geq 3)$ has infinitely many orbits on $\mathbb{P}(\mathfrak{sl}_n)$ show that the class of rs-continuous representations neither contains, nor is contained in, the class of projective representations with finitely many orbits. Let us notice, however, that the automorphism group of an rs-continuous homogeneous projective variety has finitely many orbits in the projective space if and only if the variety is subcominuscule.
One of the most important questions, which is asked in the studies of secant varieties and rank is: what are the ideals of secant varieties? It is known, by a result of Kostant, that the ideal of $X$ is generated in degree 2, by the appropriate generalization of the Plücker equations; cf. [L12 Th. 16.2.2.6]. It was shown by Landsberg and Manivel, [LM03], that, for a subminuscule variety $X \subset \mathbb{P}(V)$, the ideal of the $r$-th secant variety $\sigma_r(X)$ is generated in degree $r+1$ by the $(r-1)$-th prolongation of the generating set of the ideal of $X$, which is defined as $I_2(X)^{(r-1)} = (I_2(X) \otimes S^{r-1}V^*) \cap S^{r+1}V^*$. Our theorem has the following

**Corollary 1.4.** If $X \subset \mathbb{P}(V)$ is an rs-continuous homogeneous variety, then the ideal of $\sigma_r(X)$ is generated in degree $r+1$ by the prolongation $I_2(X)^{(r-1)}$.

Let us emphasize, that the second secant variety $\sigma_2(X)$ plays a prominent role both in this paper and in the literature. Sometimes the second secant variety is called just “the secant variety”. This variety is much more accessible than the higher secant varieties: for example for a simple $G$-module the second secant variety has an open $G$-orbit [Zak93 Ch. III, Thm 1.4]. Furthermore, we have $\sigma_2(X) = X_2 \cup TX$, where $TX$ is the tangential variety of $X$, cf. [L12 §8.1]. It turns out that, if exceptional points exist, they are always present in $TX$.

For a systematic treatment, as well as an extensive bibliography, on secant varieties and rank we refer the reader to the recent book of Landsberg, [L12]. The general theory of secant varieties allows one to deduce rs-continuity for varieties of small codimension, see Corollary 2.3 here. However, this criterion is applicable to relatively few homogeneous varieties, and to none of the more difficult cases.

The paper is organized as follows. In Subsection 2.1, we recall the notions of secant varieties, rank and border rank, with their basic properties. In Subsection 2.2, we recall some basic notions about algebraic groups: Borel subgroup, Cartan subgroup, weight lattice, root system, Weyl chamber. We also introduce the notion of chopping (this is a simple combinatorial procedure) and provide some facts on $X_2(G,V)$ and $\sigma_2(X(G,V))$ playing a crucial role in this paper.

In Section 3, we present a plan of our proof of Theorem 1.1, the main theorem of our article. Essentially, this proof is a compilation of Propositions 4.6, 4.8 and Theorems 6.1 and 7.1. We prove Propositions 4.6, 4.8 in Section 4. We prove Theorems 6.1, 7.1 in Section 6 and 7 respectively.

In Section 5, we prove a strong necessary condition for rs-continuity of a representation in terms of its choppings. This is formulated in Proposition 5.1.

The main statement of Section 6 is Theorem 6.1. In this theorem we find out which fundamental representations of classical groups are rs-continuous and which are r-discontinuous. This is done in the following way: for the fundamental modules $\mathbb{F}^n, \Lambda^2\mathbb{F}^n$ for $SL_n$, $\Lambda^3\mathbb{F}^6$ for $SL_6$, $\Lambda^2\mathbb{F}^{2n}$ and $\Lambda^3\mathbb{F}^{2n}$ for $Sp_{2n}$, $
$ $\Lambda^2\mathbb{F}^n$ for $SO_n$, $RSpin_n$ for $Spin_n(n \leq 12)$

we check rs-continuity/r-discontinuity in a straightforward way. From these data we deduce r-discontinuity of all other modules using the notion of chopping.

The main statement of Section 7 is Theorem 7.1. In this theorem we find out which fundamental representations of exceptional groups are rs-continuous and which are r-discontinuous. This is done via case by case checking of the 27 fundamental representations of the 5 exceptional Lie algebras. For any such representation we find some arguments by which it is r-discontinuous/rs-continuous. For most of the representations the arguments are quite short, using chopping or a reference, but for three representations: $V(\pi_1), V(\pi_2)$ for $F_4$ and $V(\pi_1)$ for $E_7$

we are able to find only relatively long arguments presented in the corresponding subsections.
2 Preliminaries

In this section, we recall some definitions and elementary facts about secant varieties and rank. The goal is to introduce notation and perhaps help the unexperienced reader to become more familiar with these notions. We also fix some standard notation for reductive algebraic groups, their Lie algebras and their representations.

Throughout the paper we use the following notation. The letter $X$ is always used for a projective variety and $X$ denotes the affine cone over it. For any subset $S \subset V$ we denote by $\langle S \rangle \subset V$ the span of $S$. For any subset $S \in \mathbb{P}(V)$ we denote by $\langle S \rangle \subset V$ the span of the cone $S$ of $S$ in $V$. For any non-zero vector $v \in V$ we denote by $[v]$ the class of it in $\mathbb{P}(V)$. If $v = 0$, we set $[v] := 0$.

2.1 Secant varieties and rank: general definitions

Let $X \subset \mathbb{P}$ be an algebraic variety and $X \subset V$ denote the affine cone over $X$. We denote by $\mathbb{P}(X) = \mathbb{P}(\langle X \rangle)$ the corresponding projective subspace of $\mathbb{P}$. We say that $X$ spans $\mathbb{P}$ if $\mathbb{P}(X) = \mathbb{P}$; this is equivalent to the requirement that $X$ contains a basis of $V$. Assume that this is the case. Then every point in $V$ can be written as a linear combination of points in $X$. This allows us to define the notion of rank already given in the introduction: the rank of $[\psi] \in \mathbb{P}$ with respect to $X$ is the minimal number of elements of $X$ necessary to express $\psi$ as a linear combination. Thus, the space $\mathbb{P}$ is partitioned into the rank subsets,

$$\mathbb{P} = X_1 \cup X_2 \cup \ldots$$

Since $X$ spans $\mathbb{P}$, we have $X_r = \emptyset$ for $r > \dim \mathbb{P}$.

The following properties of varieties $X_r$ hold:

(i) $X_1 = X$.

(ii) There exists a maximal $r_m \in \{1, \ldots, \dim V\}$, such that $X_{r_m} \neq \emptyset$ and $X_r = \emptyset$ for $r > r_m$.

(iii) If $r \in \{1, \ldots, r_m\}$, then $X_r \neq \emptyset$.

(iv) The projective space $\mathbb{P}$ can be written as a disjoint union $\mathbb{P} = X_1 \cup \cdots \cup X_{r_m}$.

Let $r \in \{2, \ldots, r_m\}$. The subset $X_r \subset \mathbb{P}$ is not closed, because we have $X \subset X_r$ and $X \not\subset X_r$. (Here and in what follows we use $\overline{X}$ to denote the Zariski closure of a subset $S$ of some algebraic variety.)

The $r$-th secant variety of $X$ is defined as

$$\sigma_r(X) = \bigcup_{s \leq r} X_s \subset \mathbb{P}.$$ 

It can also be written as

$$\sigma_r(X) = \bigcup_{x_1, \ldots, x_r \in X} \mathbb{P}_{x_1, \ldots, x_r},$$

where $\mathbb{P}_{x_1, \ldots, x_r}$ stands for the projective subspace of $\mathbb{P}$ spanned by the points $x_1, \ldots, x_r$.

The following properties of secant varieties $\sigma_r(X)$ hold:

(i) $\sigma_1(X) = X_1 = X$.

(ii) $\sigma_r(X) \subset \sigma_{r+1}(X)$.

(iii) If $X$ is irreducible, then $\sigma_r(X)$ is also irreducible.

(iv) There exists a minimal number $r_g \in \{1, \ldots, r_m\}$ such that $\sigma_{r_g}(X) = \mathbb{P}$ and $\sigma_{r_g-1}(X) \neq \mathbb{P}$.

(v) For $r \in \{1, \ldots, r_g\}$ the rank subset $X_r$ is dense in $\sigma_r(X)$, i.e. we have $\sigma_r(X) = \overline{X_r}$.

Definition 2.1. The number $r_g$ from part (iv) of the above proposition is called the typical rank of $\mathbb{P}$ with respect to $X$.

Let $[\psi] \in \mathbb{P}$. The border rank of $[\psi]$ with respect to $X$ is defined as

$$\text{rk}_X[\psi] := \text{rk}_X[\psi] := \min \{ r \in \mathbb{N} : [\psi] \in \overline{X_r} \}.$$
Definition 2.2. Points $[\psi] \in \mathbb{P}$, for which $\text{rk}[\psi] \neq \text{rk}[\psi]$, are called exceptional.

Clearly, $\text{rk}[\psi] \geq \text{rk}[\psi]$ and $[\psi]$ is exceptional exactly when $\text{rk}[\psi] < \text{rk}[\psi]$. So, exceptional points are points which can be approximated by points of lower rank. Also, we have

$$\text{rk}[\psi] = \min\{r \in \mathbb{N} : [\psi] \in \sigma_r(X)\}.$$

This leads us to the next definition.

Definition 2.3. (i) The secant variety $\sigma_r(X)$ is called $r$-discontinuous if it contains an exceptional vector and rs-continuous if it does not.

(ii) The embedding $X \subset \mathbb{P}$ is called rs-continuous, if all secant varieties $\sigma_r(X)$ are rs-continuous. Equivalently, $X \subset \mathbb{P}$ is rs-continuous if $r$ is a lower semi-continuous function on $\mathbb{P}$. We say that $X \subset \mathbb{P}$ is $r$-discontinuous if $X \subset \mathbb{P}$ is not rs-continuous.

(iii) The embedding $X \subset \mathbb{P}$ is called $i$-continuous, if $\sigma_i(X)$ is rs-continuous. The embedding $X \subset \mathbb{P}$ is called $i$-continuous, if $\sigma_i(X)$ is r-discontinuous and $\sigma_s(X)$ is rs-continuous for $s < i$.

We record another list of simple statements, which are derived immediately from the above definitions.

1) The secant variety $\sigma_r(X)$ is r-discontinuous if and only if $\sigma_r(X) \neq X_1 \cup X_2 \cup \ldots \cup X_r$.

2) The embedding $X \subset \mathbb{P}$ is r-discontinuous if and only if $\text{rk}_X \neq \text{rk}_X$.

We denote by $X_r \subset V$ the cone over $X_r$ without 0 and by $\sigma_r(X) \subset V$ the cone over $\sigma_r(X)$ with 0. We denote by $TX$ the union of over all points of $X$ of tangent spaces to $X$ in $\mathbb{P}$. We say that $\sigma_2(X)$ is nondegenerate if $\dim \sigma_2(X) = 2 \dim X + 1$. We say that $TX$ is nondegenerate if $\dim TX = 2 \dim X$. The following propositions provide to us abstract tools which we use to study rs-continuity/r-discontinuity of varieties $X \subset \mathbb{P}$.

Proposition 2.1 ( [FH79 Corollary 4]). If $X$ is a smooth projective variety, then precisely one of the following must hold:

(i) $\dim TX = 2 \dim X$ and $\dim \sigma_2(X) = 2 \dim X + 1$, or

(ii) $TX = \sigma_2(X)$.

Proposition 2.2 (Zak’s theorem on linear normality [FL81], [Zak93], see also [L96 Theorem 1.1]). Assume that $X$ is smooth, nondegenerate and $\mathbb{P} \neq \sigma_2(X)$. Then $\text{codim}_\mathbb{P} X \geq \frac{\dim X}{2} + 2$.

Corollary 2.3. Assume that $X$ is smooth, nondegenerate and $\text{codim}_\mathbb{P} X < \frac{\dim X}{2} + 2$. Then $X$ is rs-continuous.

Proposition 2.4 (Landsberg-Roberts [L96 Theorem 10.3], [R71 Introduction]). Assume that $X$ is smooth and $\text{codim}_\mathbb{P} X > \left(\frac{\dim X + 1}{2}\right)$. Then $\sigma_2(X)$ is nondegenerate.

Some representations satisfy the conditions of Corollary 2.3 and thus their rs-continuity is checked by the general machinery. Nevertheless, to study all possible representations we need more methods. The general methods we employ to decide rs-continuity for nonfundamental representations are independent of codimension of the variety or degeneracy of its secant variety.

### 2.2 Irreducible representations of reductive groups

Here we fix some notation concerning semisimple or reductive algebraic groups and their representations. All notions from this theory used by us can be found in [GW09] (the notation, however, may differ).

Let $G$ be a connected reductive algebraic group over $\mathbb{F}$ and $\mathfrak{g}$ be its Lie algebra. We assume that the semisimple part $G'$ of $G$ is simply connected. Let $B \subset G$ be a Borel subgroup and $T \subset B$ be a maximal torus. Let $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ denote the respective subalgebras of $\mathfrak{g}$. Let $\Lambda \subset \mathfrak{h}^*$ be the integral
defines a simple representation of the group \( \lambda \) respectively. Note that positive and negative roots corresponding to the Borel subgroup \( w_\lambda \) with respect to the Bruhat order. The weights of the form \( \lambda \in \Delta^+ \subset \mathbb{R}^n \) are one-dimensional. Note that a chopping of a fundamental representation is either fundamental, or trivial.

Remark 2.1. Note that a chopping of a fundamental representation is either fundamental, or trivial.

The Dynkin diagram determines the Weyl group \( W \). We denote by \( w_0 \) the longest element of \( W \) with respect to the Bruhat order. The weights of the form \( w_\lambda \), with \( w \in W \), are called extreme weights of the module \( V(\lambda) \) and the corresponding weight vectors are called extreme weight vectors. For any \( w \in W \) we denote by \( v^{w_\lambda} \) the unique up to scaling vector of weight \( w_\lambda \). The weight \( w_0\lambda \) is called the lowest weight of \( V(\lambda) \).

For \( \lambda_1, \lambda_2 \in \Delta^+ \), a \( G \)-module \( V(\lambda_1) \otimes V(\lambda_2) \) contains a unique up to scaling vector of weight \( \lambda_1 + \lambda_2 \). This vector is contained in a simple \( G \)-submodule, which is isomorphic to \( V(\lambda_1 + \lambda_2) \). We call this submodule the Cartan component of \( V(\lambda_1) \otimes V(\lambda_2) \). It is well known that the Cartan component does not depend on a choice of Borel subgroup \( B \subset G \).

Let us fix \( \lambda \in \Delta^+ \) and put \( V = V(\lambda) \) and \( \mathbb{P} = \mathbb{P}(\lambda) := \mathbb{P}(V(\lambda)) \). The group \( G \) acts on the projective space \( \mathbb{P} \) and has a unique closed orbit therein, namely, the orbit through the highest weight line, to be denoted by \( \mathbb{X} = \mathbb{X}(\lambda) := G[v^{\lambda}] \). We have \( \mathbb{X} = G/P \), where \( P \) denotes the stabilizer of \( v^{\lambda} \) in \( \mathbb{P} \) in \( G \). \( P \) is a standard parabolic subgroup, i.e. a closed subgroup of \( G \) containing the fixed Borel subgroup \( B \). The cosets of \( G \) by parabolic subgroups are called the flag varieties of \( G \). Thus we have an equivariantly embedded flag variety \( \mathbb{X} = G/P \subset \mathbb{P} \). In fact, all equivariantly embedded homogeneous projective varieties are obtained in this fashion. Note, that the variety \( \mathbb{X} \) is the set of highest weight vectors with respect to all possible choices of Borel subgroups \( B \subset G \).

The irreducibility of \( V \) implies that \( \mathbb{X} \) spans \( \mathbb{P} \). Hence, we have well defined rank and border rank functions on \( \mathbb{P} \) with respect to \( \mathbb{X} \), as well as secant varieties \( \sigma_r(\mathbb{X}) \subset \mathbb{P} \). Since the group \( G \) acts on \( \mathbb{P} \) by invertible linear transformations, it follows immediately that rank and border rank are \( G \)-invariant functions. Hence, the rank sets \( \mathbb{X} \) and the secant varieties \( \sigma_r(\mathbb{X}) \) are preserved by \( G \).

Definition 2.5. Let \( V = V(\lambda) \), with \( \lambda \in \Lambda^+ \), be an irreducible representation of a reductive linear algebraic group \( G \). Let \( \mathbb{X} = \mathbb{X}(\lambda) \) be the unique closed \( G \)-orbit in \( \mathbb{P} = \mathbb{P}(\lambda) \). The \( G \)-module \( V \) is called \( rs \)-continuous (resp. \( r \)-discontinuous, \( i \)-continuous, \( i \)-discontinuous), if the variety \( \mathbb{X} \subset \mathbb{P} \) is \( rs \)-continuous (resp. \( r \)-discontinuous, \( i \)-continuous, \( i \)-discontinuous).
Our goal is to classify all rs-continuous irreducible representations of semisimple algebraic groups.

**Remark 2.2.** In some of our constructions we consider reductive groups, rather than semisimple groups, just because this simplifies some steps. The actions of $G$ and $G'$ (the commutant of $G$) on $\mathbb{P}$ coincide and we are concerned with properties of the embedding $X \subset \mathbb{P}$. Thus the classification of rs-continuous representations of reductive groups can be easily obtained from the one for semisimple groups.

Below we assume that $\lambda \neq 0$ (this corresponds to the inequality $\dim V \geq 2$, $V = V(\lambda)$, assumed in Theorem 1.1).

We denote by $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$ the simple simply connected algebraic groups with the corresponding Dynkin diagrams. We denote by $a_n, b_n, c_n, d_n, e_6, e_7, e_8, f_4, g_2$ the corresponding Lie algebras.

### 3 Plan of proof of Theorem 1.1

Theorem 1.1 follows from a number of propositions and theorems proved throughout the article. Thus the plan explains the role of different parts of this text.

In Proposition 4.6, we prove that, if an irreducible representation $V(\lambda)$ with highest weight $\lambda$ is rs-continuous, then $h(\lambda) < 3$. In Proposition 4.8, we classify all irreducible rs-continuous modules $V(\lambda)$ with $h(\lambda) = 2$.

The rs-continuous fundamental representations (i.e. irreducible $G$-modules $V(\lambda)$ with $h(\lambda) = 1$) are classified in Theorems 6.1 and 7.1. This is done in a case by case study. We consider separately representations of classical groups (Theorem 6.1), and representations of exceptional groups (Theorem 7.1). This completes the proof of Theorem 1.1.

Let us say a few words about the proofs of Theorems 6.1 and 7.1. The rs-continuous representations are considered individually and for each case we provide specific arguments. Most representations are r-discontinuous and thus we have to check r-discontinuity of a huge amount of cases. The number of cases to be considered is greatly reduced by Proposition 5.1, where we show that, if a $G$-representation $V$ is a chopping of a $G$-representation $V$ and $V$ is r-discontinuous, then $V$ is also r-discontinuous. Thus, it suffices to check r-discontinuity directly for only few basic cases, then we are able to deduce r-discontinuity for most of the fundamental representations.

### 4 Non-fundamental rs-continuous modules

The goal of this section is to classify all non-fundamental rs-continuous modules. First of all we provide in Lemma 4.3 of Subsection 4.1 a way to construct exceptional vectors in $V(\lambda)$. Using this construction we show in Proposition 4.6 of Subsection 4.2 that, if a $G$-module $V(\lambda)$ is 2-continuous, then it is either a fundamental $G$-module (i.e. $h(\lambda) = 1$) or is a Cartan component in a tensor product of two fundamental $G$-modules (i.e. $h(\lambda) = 2$). Further on, in Proposition 4.7 we shall show that, in the latter situation, each of the fundamental modules satisfies a very strict condition, related to the notion of HW-density introduced in Definition 4.2. Using the explicit description of all HW-dense modules which we present in Corollary 4.5, we are able to complete in Proposition 4.8 a classification of non-fundamental rs-continuous $G$-modules $V(\lambda)$ (i.e. modules $V(\lambda)$ such that $h(\lambda) = 2$).

#### 4.1 The varieties $\sigma_2(\mathbb{X}(\lambda))$ and $\overline{X}_2(\lambda)$

In some sense, in this article we study the difference between the variety $\sigma_2(\mathbb{X}(\lambda))$ and its open subset $\overline{X}_2(\lambda)$. Here we collect some of their basic features. First we recall that, if the generic rank of $V(\lambda)$ is greater than 1 (i.e. if $\mathbb{P}(\lambda) \neq \mathbb{X}(\lambda)$), then

$$\sigma_2(\mathbb{X}(\lambda)) = \overline{X}_2(\lambda).$$
We provide some kind of explicit description of the elements of $\mathcal{X}_2(\lambda)$ (Lemma 4.1) and exhibit some set of elements of $\sigma_2(\mathcal{X}(\lambda))$, which tend to be exceptional (Lemma 4.3).

**Lemma 4.1.** Let $V(\lambda)$ be an irreducible representation of a reductive group $G$. Then
a) any pair $([v_1],[v_2]) \in \mathcal{X}(\lambda) \times \mathcal{X}(\lambda)$ is $G$-conjugate to a pair $([v^\lambda],[v^w\lambda])$ for some $w \in W$,
b) any element of $\mathcal{X}_2(\lambda)$ is conjugate to $[v^\lambda + v^w\lambda]$ for some $w \in W$.

**Proof.** Fix $v_1,v_2 \in X(\lambda)$ such that $[v_1] \neq [v_2]$. Let $B_1,B_2$ be Borel subgroups of $G$ such that $v_1,v_2$ are the corresponding to $B_1,B_2$ highest weight vectors. It is known that $B_1 \cap B_2$ contains a maximal torus $T_{12}$ of $G$ and there exists $w \in W_{12} := N_G(T_{12})/T_{12}$ such that $B_1 = wB_2$. Thus $([v_1],[v_2])$ is conjugate to $([v^\lambda],[v^w\lambda])$ for some $w \in W$. This completes the proof of part a).

To prove part b) we observe that any element of $\mathcal{X}_2(\lambda)$ is a sum $v_1 + v_2$ for some $v_1,v_2 \in X(\lambda)$ such that $v_1 \neq v_2$. According to part a) the pair $([v_1],[v_2])$ is conjugate to $([v^\lambda],[v^w\lambda])$ for some $w \in W$. Thus $[v_1 + v_2]$ is conjugate to $[av^\lambda + bv^w\lambda]$ for some non-zero $a,b \in \mathbb{F}$. As $[v_1] \neq [v_2]$, $\lambda \neq w\lambda$. Hence $[v^\lambda + v^w\lambda]$ is conjugate to $[av^\lambda + bv^w\lambda]$ for any non-zero $a,b \in \mathbb{F}$. This completes the proof of b). \( \square \)

**Corollary 4.2.** Let $w_0$ be the longest Weyl group element. The orbit $G[v^\lambda + v^{w_0}\lambda]$ is open in both varieties $\sigma_2(\mathcal{X}(\lambda))$ and $\mathcal{X}_2(\lambda)$. (see also [Zak93] Ch. III, Thm 1.4)

**Lemma 4.3.** Fix $x \in X(\lambda)$ and $t \in g$. Then $[x + tx] \in \sigma_2(\mathcal{X}(\lambda))$.

**Proof.** By definition $\mathcal{X}_2(\lambda) \cup \mathcal{X}(\lambda)$ is the union of lines going through pairs of points of $\mathcal{X}(\lambda)$. Thus the tangent space $T_x \mathcal{X}(\lambda) \subset V$ to $X(\lambda)$ in $x$ belongs to $\mathcal{X}_2(\lambda) = \sigma_2(X(\lambda))$. On the other hand, $x + tx$ belongs to $T_x X(\lambda)$ as $tx$ is tangent to $X(\lambda)$. This completes the proof. \( \square \)

### 4.2 HW-density and 2-continuity

In this subsection, we analyze the notion of 2-continuity via the notion of HW-density given in Definition 4.1. This analysis allows to find out all rs-continuous modules which are not fundamental. Notice that 2-continuity is, a priori, weaker than rs-continuity, but is much simpler to check. A posteriori, it turns out that rs-continuity and 2-continuity are equivalent for the class of homogeneous projective varieties considered in this paper.

We proceed in the following way. We prove that, if an irreducible $G$-module $V$ is 2-continuous, then it is either a fundamental module of $G$ (i.e. only one mark of the highest weight of $V$ is distinct from zero and this mark equals 1) or is a Cartan component of the tensor product of two fundamental modules of $G$, see Proposition 4.6. In the latter case we prove that both fundamental modules in the product have to be HW-dense, see Proposition 4.7. It turns out that HW-density is a very strict condition as we show in Corollary 4.5. This result leads to Proposition 4.8 which lists all 2-continuous $G$-modules, which are not fundamental.

We start with the definition of HW-density followed by the statements of the results. The proofs of these results are given below until the end of the section.

**Definition 4.1.** Let $\mathbb{X}$ be a smooth subvariety of a projective space $\mathbb{P}$. We say that $\mathbb{X}$ is HW-dense, if for any point $x_1 \in \mathbb{X}$ there exists an open subset $U$ of the tangent space to $\mathbb{X}$ at $x_1$ such that for all $v \in U$ there exists $x_2 \in \mathbb{X}$ such that $v \in \langle x_1,x_2 \rangle$.

**Definition 4.2.** We say that a simple $G$-module $V(\lambda)$ is HW-dense, if $\mathbb{X}(\lambda)$ is HW-dense in $\mathbb{P}(\lambda)$.

We shall prove below in this subsection the following criterion of HW-density.

**Lemma 4.4.** The $G$-module $V(\lambda)$ is HW-dense if and only if one of the following equivalent conditions holds:
(a) The set $X(\lambda)$ of highest weight vectors is dense in $V(\lambda)$.
(b) All non-zero vectors of $V$ are highest weight vectors with respect to some choice of a Borel subgroup of $G$.
(c) $G$ acts transitively on $\mathbb{P}(V)$.
Proof. Assume on the contrary that \( \langle V \rangle \) is 2-dimensional for some \( v \). On the other hand, if \( \text{Im}(\text{SL}(V)) = \text{Im}(\text{Sp}(V)) \) then \( \text{GL}(V) \) is essentially a particular case of Proposition 4.6. Proposition 4.8 immediately imply the following proposition.

**Proposition 4.6.** Assume that \( V(\lambda) \) is rs-continuous. Then \( h(\lambda) < 3 \).

**Proposition 4.7.** Let \( \lambda_1, \lambda_2 \in \Lambda \) be non-zero weights. Assume that \( V(\lambda_1 + \lambda_2) \) is 2-continuous. Then both \( V(\lambda_1), V(\lambda_2) \) are HW-dense.

Proofs of Propositions 4.6, 4.7 are presented below in this subsection. Corollary 4.5 and Proposition 4.8 immediately imply the following proposition.

**Proposition 4.8.** Assume that \( V(\lambda) \) is an effective 2-continuous module and \( h(\lambda) = 2 \). Then \( (G, V(\lambda)) \) appears in the following list:

1) \( (\text{SL}(V_1) \times \text{SL}(V_2), V_1 \otimes V_2) \); 2) \( (\text{SL}(V_1) \times \text{Sp}(V_2), V_1 \otimes V_2) \); 3) \( (\text{Sp}(V_1) \times \text{Sp}(V_2), V_1 \otimes V_2) \);
4) \( (\text{SL}(V), S^2V) \); 5) \( (\text{SL}(V), \text{sl}(V)) \); 6) \( (\text{Sp}(V), \text{sp}(V)) = (\text{Sp}(V), S^2V) \)

(here \( \text{sl}(V), \text{sp}(V) \) denote the adjoint modules of the corresponding groups).

All modules listed in Proposition 4.8 are rs-continuous. Cases 1) and 4) of Lemma 4.8 are known to be rs-continuous. Cases 2), 3) have the same secant varieties as case 1), and hence are also rs-continuous. Cases 5) and 6) are rs-continuous due to [Kaj98]. Therefore Proposition 4.8 explicitly lists all non-fundamental rs-continuous modules.

The rest of the current subsection is dedicated to the proofs of Propositions 4.6, 4.7 and Lemma 4.4. We need the following lemma for the \( \text{GL}(V_1) \times \text{GL}(V_2) \times \text{GL}(V_3) \)-module \( V_1 \otimes V_2 \otimes V_3 \), where \( V_1, V_2, V_3 \) are finite-dimensional vector spaces.

**Lemma 4.9.** Let \( x_i, y_i \) be linearly independent vectors in \( V_i \) for \( i = 1, 2, 3 \). Then

\[
T := x_1 \otimes x_2 \otimes x_3 + y_1 \otimes x_2 \otimes x_3 + x_1 \otimes y_2 \otimes x_3 + x_1 \otimes x_2 \otimes y_3 \neq v_1 \otimes v_2 \otimes v_3 + w_1 \otimes w_2 \otimes w_3
\]

for all \( v_i, w_i \in V_i \) \( (i = 1, 2, 3) \). In other words, \( \text{rk} T > 2 \).

**Proof.** Assume on the contrary that

\[
T = v_1 \otimes v_2 \otimes v_3 + w_1 \otimes w_2 \otimes w_3
\]

for some \( v_i, w_i \in V_i \), \( i = 1, 2, 3 \). We have \( V_1 \otimes V_2 \otimes V_3 \cong \text{Hom}(V_2^*, V_3^*, V_1) \). Therefore for any \( x \in V_1 \otimes V_2 \otimes V_3 \) we can define \( \text{Im}_1(x) \subset V_1 \) as the image of the corresponding homomorphism from \( \text{Hom}(V_2^*, V_3^*, V_1) \). Similarly we define \( \text{Im}_2(x) \) and \( \text{Im}_3(x) \). We have

\[
\text{Im}_i(T) = \langle x_i, y_i \rangle, i = 1, 2, 3.
\]

On the other hand, if \( v_1 \otimes v_2 \otimes v_3 + w_1 \otimes w_2 \otimes w_3 \neq 0 \),

\[
\text{Im}_i(v_1 \otimes v_2 \otimes v_3 + w_1 \otimes w_2 \otimes w_3) \subset \langle v_i, w_i \rangle (i = 1, 2, 3).
\]

Therefore, \( \langle x_i, y_i \rangle = \langle v_i, w_i \rangle, i = 1, 2, 3 \). Hence, without loss of generality, we may assume that

\[
V_i = \langle x_i, y_i \rangle = \langle v_i, w_i \rangle (i = 1, 2, 3),
\]

i.e. that \( \dim V_i = 2 \) \( (i = 1, 2, 3) \). For two-dimensional spaces \( V_1, V_2, V_3 \) this lemma is well known; see e.g. [L12].

Note that for \( (G, V(\lambda)) = (\text{SL}(V_1) \times \text{SL}(V_2) \times \text{SL}(V_3), V_1 \otimes V_2 \otimes V_3) \) we have \( h(\lambda) = 3 \). Moreover, Lemma 4.9 is essentially a particular case of Proposition 4.6.
Proof of Proposition 4.6. Assume on the contrary that \( b(\lambda) \geq 3 \). Then there exist non-zero weights \( \lambda_1, \lambda_2, \lambda_3 \in \Lambda^+ \) such that \( \lambda = \lambda_1 + \lambda_2 + \lambda_3 \). Then
\[
v^{\lambda_1} \otimes v^{\lambda_2} \otimes v^{\lambda_3} \in V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3)
\]
is a highest weight vector of weight \( \lambda = \lambda_1 + \lambda_2 + \lambda_3 \). Therefore the smallest \( G \)-submodule of \( V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3) \) containing \( v^{\lambda_1} \otimes v^{\lambda_2} \otimes v^{\lambda_3} \) is isomorphic to \( V(\lambda) \). We identify \( V(\lambda) \) with this submodule of \( V(\lambda_1) \otimes V(\lambda_2) \otimes V(\lambda_3) \) and set
\[
v^\lambda := v^{\lambda_1} \otimes v^{\lambda_2} \otimes v^{\lambda_3}.
\]
By Lemma 4.3 we have
\[
v^\lambda + tv^\lambda \in \overline{X_2(\lambda) / \sigma_2(X(\lambda))}
\]
for any \( t \in \mathfrak{g} \). Therefore
\[
\text{rk}_{X(\lambda)}(v^\lambda + tv^\lambda) \leq 2 \tag{2}
\]
for any \( t \in \mathfrak{g} \). By the Leibnitz rule we have
\[
T_\lambda := v^\lambda + tv^\lambda = v^{\lambda_1} \otimes v^{\lambda_2} \otimes v^{\lambda_3} + tv^{\lambda_2} \otimes v^{\lambda_3} + v^{\lambda_1} \otimes tv^{\lambda_2} \otimes v^{\lambda_3} + v^{\lambda_1} \otimes v^{\lambda_2} \otimes tv^{\lambda_3}
\]
(note that \( T_\lambda \) is of the form \( T \) of Lemma 4.9). We claim that
\[
v^\lambda + tv^\lambda \notin X_2(\lambda) \cup X(\lambda) \cup \{0\}
\]
for all \( t \in U \) from some open subset \( U \) of \( \mathfrak{g} \). As \( \lambda_i \neq 0 \), there exists some open subset \( U \subset \mathfrak{g} \) such that \([tv^{\lambda_i}] \neq [v^{\lambda_i}] \) for all \( t \in U \). We fix \( t \in U \). We claim that
\[
\text{rk}(v^\lambda + tv^\lambda) \geq 3 \tag{3}
\]
Assume on the contrary that \( v^\lambda + tv^\lambda \in X_2(\lambda) \cup X(\lambda) \cup \{0\} \), then
\[
v^\lambda + tv^\lambda = g_1(\lambda_1) \otimes g_1(\lambda_2) \otimes g_1(\lambda_3) + g_2(\lambda_1) \otimes g_2(\lambda_2) \otimes g_2(\lambda_3)
\]
for some \( g_1, g_2 \in G \) and thus
\[
T_\lambda = v^\lambda + tv^\lambda = v_1 \otimes v_2 \otimes v_3 + w_1 \otimes w_2 \otimes w_3
\]
for some \( v_1, v_2, v_3, w_1, w_2, w_3 \in V(\lambda) \). This contradicts the statement of Lemma 4.9. Comparing (2) and (3) we see that \( v^\lambda + tv^\lambda \) is an exceptional vector of \( V(\lambda) \) and thus \( V(\lambda) \) is 2-discontinuous. \( \Box \)

Proof of Proposition 4.7. We use notation analogous to the one in Proposition 4.6. Fix \( \lambda := \lambda_1 + \lambda_2 \). Set
\[
v^\lambda := v^{\lambda_1} \otimes v^{\lambda_2} \in V(\lambda_1) \otimes V(\lambda_2).
\]
This defines a canonical embedding \( V(\lambda) \to V(\lambda_1) \otimes V(\lambda_2) \). As \( \lambda_i \neq 0 \), there exists some open subset \( U \subset \mathfrak{g} \) such that \([tv^{\lambda_i}] \neq [v^{\lambda_i}] \) for all \( t \in U \). We fix \( t \in U \). Repeating the argument preceding (2) we deduce that (2) holds in the present notation. Hence, since \( V(\lambda) \) is 2-continuous we have
\[
v^\lambda + tv^\lambda = g_1v^\lambda + g_2v^\lambda \text{ or } v^\lambda + tv^\lambda = g_1v^\lambda
\]
for some \( g_1, g_2 \in G \). Then
\[
\text{Im}_i(v^\lambda + tv^\lambda) = \langle v^{\lambda_i}, tv^{\lambda_i} \rangle \text{ (} i = 1, 2 \text{)}.
\]
(for the definition of \( \text{Im}_i \) see proof of Lemma 4.9) and, if \( g_1v^\lambda + g_2v^\lambda \neq 0 \),
\[
\text{Im}_i(g_1v^\lambda + g_2v^\lambda) = \langle g_1v^{\lambda_i}, g_2v^{\lambda_i} \rangle \text{ (} i = 1, 2 \text{)} \text{ or } \text{Im}_i(g_1v^\lambda + g_2v^\lambda) = \langle g_1v^{\lambda_i} \rangle.
\]
Hence \( g_1v^{\lambda_i}, g_2v^{\lambda_i} \in \langle v^{\lambda_i}, tv^{\lambda_i} \rangle \) and either
\[
[g_1v^{\lambda_i}] \neq [v^{\lambda_i}] \text{ or } [g_2v^{\lambda_i}] \neq [v^{\lambda_i}] \text{ (} i = 1, 2 \text{).}
Therefore both $V(\lambda_1)$ and $V(\lambda_2)$ are HW-dense. This completes the proof.

To prove Lemma 4.4 we need two technical lemmas. The first gives a reformulation of Definition 4.2.

**Lemma 4.10.** A simple $G$-module $V(\lambda)$ is HW-dense if and only if there exists an open subset $U \subset g$ such that, for all $t \in U$, there exists an element

$$v \in X(\lambda) \cap \langle v^\lambda, tv^\lambda \rangle$$

such that $[v] \neq [v^\lambda]$ (note that if such an element $v$ exists, then $[tv^\lambda] \neq [v^\lambda]$).

**Proof.** This is a fairly easy exercise for Lie algebras-Lie groups formalism. We omit it.

**Lemma 4.11.** Fix $v_1 \in X(\lambda)$. Assume that there exists $v_2 \in X(\lambda)$ such that $v_2 \in \langle v_1, gv_1 \rangle$. Then all non-zero vectors of $\langle v_1, v_2 \rangle$ belong to $X(\lambda)$.

**Proof.** Without loss of generality we assume that $[v_1] \neq [v_2]$. Then, by Lemma 4.10, the pair $(v_1, v_2)$ is conjugate to the pair $(v^\lambda, v^w\lambda)$ for some $w \in W$ and thus we can assume that

$$v_1 = v^\lambda \text{ and } v_2 = v^w\lambda$$

for the fixed maximal torus $T \in G$. The space $gv_1$ is clearly $T$-invariant and the weights of this space form a subset of the set $\lambda + \Delta$ (this is a point-wise sum). As $v^w\lambda \in gv^\lambda$, we have

$$w\lambda = \lambda + \beta \text{ for some } \beta \in \Delta$$

(note that $w\lambda \neq \lambda$ as $[v_1] \neq [v_2]$). Let $SL_2(\beta)$ be the $T$-stable $SL_2$-subalgebra corresponding to the root $\beta \in \Delta$. Then the space $\langle v^\lambda, v^w\lambda \rangle$ is a two-dimensional simple $SL_2(\beta)$-module and thus any two non-zero elements of $\langle v_1, v_2 \rangle$ are $SL_2(\beta)$-conjugate, and hence $G$-conjugate. Therefore all non-zero elements of $\langle v_1, v_2 \rangle$ belong to $X(\lambda)$.

**Proof of Lemma 4.4.** The equivalence of conditions (a), (b), (c) is clear. It is also immediate to verify that each of these conditions implies HW-density. It remains to show that, if the module $V(\lambda)$ is HW-dense, then it satisfies condition (a).

Assume that $V(\lambda)$ is HW-dense. Then according to Lemma 4.10 and Lemma 4.11 there exists an open subset $U \subset g$ such that for any non-zero $a \in F$ and any $t \in U$ we have

$$v^\lambda + atm^\lambda \in X(\lambda),$$

i.e. we have that $X(\lambda) \cap \langle v^\lambda, gv^\lambda \rangle$ is a dense subset of $\langle v^\lambda, gv^\lambda \rangle$. Note that $v^\lambda \in gv^\lambda$, as $\lambda \neq 0$, and hence

$$\langle v^\lambda, gv^\lambda \rangle = \langle gv^\lambda \rangle.$$

We have

$$\dim X(\lambda) = \dim Gv^\lambda = \dim gv^\lambda$$

and therefore

$$\overline{X(\lambda)} = gv^\lambda.$$

On the other hand

$$V(\lambda) = \langle X(\lambda) \rangle$$

and hence

$$V(\lambda) = \overline{X(\lambda)} = gv^\lambda.$$
5 Restriction to a Levi subgroup

The main result of this section is Proposition 5.1. This proposition provides a strong sufficient condition for r-discontinuity of a representation. We will apply this proposition to study rs-continuity/r-discontinuity of fundamental representations in the subsequent sections of this article.

**Proposition 5.1.** Let \( V \) be an irreducible \( G \)-module and \( \underline{V} \) be a \( \underline{G} \)-module, which is a chopping of \( V \). If \( \underline{V} \) is r-discontinuous, then \( V \) is r-discontinuous.

This proposition is an immediate corollary of Proposition 5.2. To state Proposition 5.2 we need more notation.

Recall that we have fixed Cartan and Borel subgroups \( H \subset B \subset G \) and that \( \Pi \) denotes the corresponding set of simple roots. Let \( \Pi \subset \Pi \) be a subset. Then \( \Delta = \Delta \cap (\Pi) \) is a root system having \( \Pi \) as a set of simple roots and \( \Delta^\pm = \Delta \cap \Delta^\pm \) as sets of positive and negative roots. Further, let \( \mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta}) \mathfrak{g}^\alpha \). Then \( \mathfrak{g} \) is a reductive subalgebra of \( \mathfrak{g} \); we call subalgebras of this form (reductive) Levi subalgebras. Let \( G \subset G \) be the corresponding Levi subgroup. We shall add underline to denote the attributes of \( G \) with the notational conventions already introduced for \( G \).

Note that \( G \) and \( \underline{G} \) have a common Cartan subgroup \( H \) and hence have the same weight lattice \( \Lambda \). However, the dominant Weil chambers do not coincide, unless \( \Pi = \Pi \), a case which is of no use for us. We have an inclusion \( \Lambda^+[\underline{G}] \subset \Lambda^+[G] \), so a weight \( \lambda \in \Lambda^+ \) can be regarded as a dominant weight for both \( G \) and \( \underline{G} \). Furthermore, since \( B = B \cap G \), the \( B \)-highest weight vectors are also \( B \)-highest weight vectors.

Fix \( \lambda \in \Lambda^+ \). There is a \( \underline{G} \)-equivariant inclusion of the corresponding representations

\[
\underline{V} = V(\lambda) = \mathcal{U}(\mathfrak{g})v^\lambda \subset V(\lambda) = V,
\]

where \( v^\lambda \) denotes the \( B \)-highest weight vector in \( V(\lambda) \). Let \( \underline{X} \) denote the unique closed \( \underline{G} \)-orbit in \( \mathbb{P}(\underline{V}) \) and, as before, let \( X \) denote the unique closed \( G \)-orbit in \( \mathbb{P}(V) \). We have

\[
\underline{X} = G[v^\lambda] \subset \underline{G}[v^\lambda] = X.
\]

For points in \( \mathbb{P}(\underline{V}) \), we have two well defined rank functions \( \text{rk}_X \) and \( \text{rk}_\underline{X} \) (notice that \( \underline{V} \) is a chopping of \( V \) according to Definition 2.4). We would like to compare these functions and prove the following.

**Proposition 5.2.** Let \( G \subset \underline{G} \) and \( V \subset \underline{V} \) be as above. If \( [\psi] \in \mathbb{P}(\underline{V}) \), then \( \text{rk}_\underline{X}[\psi] = \text{rk}_X[\psi] \).

**Proof.** First, observe that the multiplicity of the \( \underline{G} \)-module \( \underline{V}(\lambda) \) in \( V(\lambda) \) is 1. This holds because \( \underline{V}(\lambda) \) has a weight vector with weight \( \lambda \) and this weight has multiplicity 1 in \( V(\lambda) \). Consequently, there is a well-defined \( \underline{G} \)-equivariant projection

\[
\pi : V \to \underline{V}.
\]

Let \( P \subset G \) be the parabolic subgroup containing \( B \) and having \( \underline{G} \) as a Levi component; the roots of \( P \) are \( \Delta^+ \cup \Delta^- \). Let \( N_P \) be the unipotent radical of \( P \); the roots of \( N_P \) are \( \Delta^+ \setminus \Delta^+ \). Then \( N_P \) acts trivially on \( \underline{V} \).

**Lemma 5.3.** We have \( \pi(X \cup 0) = \underline{X} \cup 0 \).

**Proof.** Let \( N_P^- \) be the nilradical of the parabolic \( P^- \) opposite to \( P \), with respect to the given Cartan subgroup \( H \). In other words, \( N_P^- \) is the regular unipotent subgroup of \( N^- \) with roots \( \Delta(N_P^-) = -\Delta(N_P) \). We have

\[
X = Gv^\lambda = P^-v^\lambda = N_P^-(Gv^\lambda) = \overline{N_P^-X}.
\]
Thus, to prove the lemma it is sufficient to show that for all \( g \in N_P^- \) and all \( v \in X \) we have \( \pi(gv) \in X \). Let \( g \in N_P^- \) and \( v \in X \). Since the exponential map \( \exp : \mathfrak{n}_P^- \to N_P^- \) is surjective, we can write \( g = \exp(\xi) \) with \( \xi \in \mathfrak{n}_P^- \). Viewing \( \xi \) as an element of \( \mathfrak{gl}(V) \) we can write

\[
gv = (1 + \xi + \frac{1}{2} \xi^2 + \cdots)v = v + \xi v + \frac{1}{2} \xi^2 v + \cdots.
\]

Let \( V' = \ker(\pi) \), so that \( V = V \oplus V' \) as \( G \)-modules. Then, for \( \xi \in \mathfrak{n}_P^- \), we have \( \xi(V) \subset V' \). Hence \( \pi(gv) = v \).

Now, let \([\psi] \in \mathcal{P}(V)\). The inequality \( \text{rk}_X[\psi] \geq \text{rk}_X[\psi] \) is immediate. Let \( r = \text{rk}[\psi] \) and

\[
\psi = v_1 + \ldots + v_r
\]

be a minimal expression, with \([v_j] \in X\). Then we have

\[
\psi = \pi(\psi) = \pi(v_1) + \ldots + \pi(v_r)
\]

and, according to the above lemma, \([\pi(v_j)] \in X \cup 0 \) (this is a set). Hence \( \text{rk}_X[\psi] \leq \text{rk}_X[\psi] \) and so

\[
\text{rk}_X[\psi] = \text{rk}_X[\psi].
\]

6 Fundamental representations (classical groups)

In this subsection we prove Theorem 1.1 for fundamental modules of classical groups, i.e. we prove Theorem 6.1. The result follows directly from Propositions 6.2, 6.4, 6.5 of Subsections 6.1, 6.2, 6.3, where we consider the cases of \( SL_n, SO_n, Sp_{2n} \), respectively.

**Theorem 6.1.** Let \( V(\lambda) \) be a fundamental module of a simple classical group \( G \). Then \( V(\lambda) \) is \( rs \)-continuous if and only if the pair \((G,V(\lambda))\) appears in the following table.

| Group \( G \) | Representation \( V \) | Highest weight of \( V \) |
|--------------|----------------------|-------------------------|
| Classical groups |
| \( SL_n \) | \( \mathbb{F}^n, (\mathbb{F}^n)^*, (\Lambda^2 \mathbb{F}^n), (\Lambda^2 \mathbb{F}^n)^* \) | \( \pi_1, \pi_{n-1}, \pi_2, \pi_{n-2} \) |
| \( SO_n \) | \( \mathbb{F}^n, RSpin_n(n \leq 10) \) | \( \pi_1, \pi_\frac{n}{2}(2 \mid n), \pi_{\frac{n}{2}-1}(2 \mid n), \pi_{\frac{n-1}{2}}(2 \nmid n) \) |
| \( Sp_{2n} \) | \( \mathbb{F}^{2n}, \Lambda^2 \mathbb{F}^{2n} \) | \( \pi_1, \pi_2 \) |

where the notation is the same as in Theorem 1.1.

Moreover, all \( r \)-discontinuous fundamental representations of classical groups are \( 2 \)-discontinuous.

Our approach for classical groups is tensor-based and we often use symmetric/antisymmetric bilinear forms. To prove Theorem 6.1 we also need some sufficient condition of \( r \)-discontinuity for representations. Such a condition is provided in Proposition 5.1 of Section 5. In a similar way Proposition 5.1 will be very useful in Section 7, where we consider the fundamental representations of the exceptional groups.
6.1 \( G = SL_n \)

Recall that the fundamental representations of \( G = SL_n \) are obtained as exterior tensor powers of the natural representation, i.e. \( V(\pi_k) = \Lambda^k \mathbb{F}^n \), \( k = 1, \ldots, n-1 \). Furthermore, we have

\[(\Lambda^k \mathbb{F}^n)^* = \Lambda^{n-k} \mathbb{F}^n\]

as \( SL_n \)-modules.

**Proposition 6.2.** The fundamental representations of \( SL_n \) which are rs-continuous are exactly

\[\mathbb{F}^n, (\mathbb{F}^n)^*, \Lambda^2 \mathbb{F}^n, (\Lambda^2 \mathbb{F}^n)^*.\]

Moreover, all r-discontinuous fundamental representations of \( SL_n \) are 2-discontinuous.

**Proof.** The closed \( G \)-orbit \( X \subset \mathbb{P}(\Lambda^k \mathbb{F}^n) \) is the Grassmann variety \( \text{Gr}_k(\mathbb{F}^n) \) under its Plücker embedding. It is well known that a suitable isomorphism between \( \Lambda^k \mathbb{F}^n \) and \( \Lambda^{n-k} \mathbb{F}^n \) induces an isomorphism between the respective projective spaces, which carries \( \text{Gr}_k(\mathbb{F}^n) \) to \( \text{Gr}_{n-k}(\mathbb{F}^n) \). Hence, for our purposes, it is sufficient to consider \( k \leq n/2 \).

The fact that \( V(\pi_1) = \mathbb{F}^n \) and \( V(\pi_2) = \Lambda^2 \mathbb{F}^n \) are rs-continuous is well known. In fact, in the first case we have \( X = \mathbb{P}(\mathbb{F}^n) \), so all vectors have rank 1. In the second case, \( \Lambda^2 \mathbb{F}^n \) can be identified with the space of skew-symmetric matrices. Such a matrix has even rank (in the usual sense) and the \( SL_n \)-orbit \( X \) through a highest weight vector in \( \Lambda^2 \mathbb{F}^n \) consists of all matrices of rank 2. A skew-symmetric matrix \( \psi \) of rank 2 can be written as a sum of \( r \) skew-symmetric matrices of rank 2, and so \( \text{rk}_X[\psi] = r \). We can now see that the set

\[\{[\psi] \in \mathbb{P}(\Lambda^2 \mathbb{F}^n) : \text{rk}_X \psi \leq r \}\]

is closed for every \( r \). This completes the argument in this case.

We now turn to the remaining cases. Due to the duality \( \Lambda^k \mathbb{F}^n \leftrightarrow (\Lambda^{n-k} \mathbb{F}^n)^* \), it suffices to consider \( n \geq 6 \). Proposition 5.2 implies that, to show that \( \Lambda^k \mathbb{F}^n \) (\( 3 \leq k \leq n/2 \)) is 2-discontinuous, it is sufficient to show that \( \Lambda^3 \mathbb{F}^6 \) is 2-discontinuous.

**Lemma 6.3.** The representation of \( SL_6 \) on \( \Lambda^3 \mathbb{F}^6 \) is 2-discontinuous.

**Proof.** It is shown in [Za93] Ch. III, Thm 1.4, that \( \sigma_2(X(\Lambda^3 \mathbb{F}^6)) = \Lambda^3 \mathbb{F}^6 \) and therefore it is enough to show that \( \Lambda^3 \mathbb{F}^6 \) contains a vector of rank 3 or more. To do this we count the number of orbits of vectors of rank 0, 1, 2 and compare this number with the known number of orbits for the action of \( SL_6 \) on \( \Lambda^3 \mathbb{F}^6 \), see [Gu48] or [R07].

By definition there is one orbit of vectors of rank 0 and one orbit of vectors of rank 1. Let us consider the vectors of rank 2 in \( V \). We shall show that there are two orbits of such vectors. Any vector of rank 2 can be written as

\[\psi = v_1 \wedge v_2 \wedge v_3 + v_4 \wedge v_5 \wedge v_6,\]

with some \( v_j \in V \). The first possibility is that \( v_1, \ldots, v_6 \) form a basis of \( \mathbb{F}^6 \). This is indeed the generic situation. If suitable Borel and Cartan subgroups of \( SL_6 \) are chosen, the two summands of \( \psi \) are, respectively, the highest and lowest weight vectors in \( V \). The group \( GL_6 \) acts transitively on the set of all bases of \( \mathbb{F}^6 \); the group \( SL_6 \) acts transitively on the set of their projective images. Thus the points of the first type form a single \( G \)-orbit \( \mathcal{X}'_2 \), which is open in \( \mathbb{P}(\Lambda^3 \mathbb{F}^6) \). We denote by \( \mathcal{Z} \) the complement to this orbit in \( \mathbb{P}(\Lambda^3 \mathbb{F}^6) \). The second possibility is to have

\[\dim(\langle v_1, v_2, v_3 \rangle \cap \langle v_4, v_5, v_6 \rangle) = 1.\]

If this is the case, by changing the vectors if necessary, we may reduce to the situation where \( v_1 = v_4 \) and

\[\psi = v_1 \wedge (v_2 \wedge v_3 + v_5 \wedge v_6), \quad \text{with} \quad \langle v_2, v_3 \rangle \cap \langle v_5, v_6 \rangle = 0.\]
Since \( v_2 \wedge v_3 + v_5 \wedge v_6 \) has rank 2 in \( \Lambda^2 \mathbb{P}^6 \) (with respect to \( \text{Gr}_2(\mathbb{P}^6) \)), we deduce that \( \psi \) has indeed rank 2 in \( V \). The point \( \phi \) does not belong to \( \mathcal{X}_2' \), because the action of \( GL_6 \) respects linear dependencies. On the other hand, it is also clear that \( GL_6 \) acts transitively on the set \( \mathcal{X}_2'' \) of points of this second type, and hence \( SL_6 \) acts transitively on the set of their images in \( \mathbb{P} \). Note that

\[
\text{if } \dim(\langle v_1, v_2, v_3 \rangle \cap \langle v_4, v_5, v_6 \rangle) > 1, \text{ then } \text{rk}[\psi] = 1.
\]

We can conclude that there are exactly two \( G \)-orbits consisting of points of rank 2, namely

\[
\mathcal{X}_2' = \mathbb{P} \setminus Z, \quad \mathcal{X}_2'' = Z \cap \mathcal{X}_2.
\]

Thus there are four \( SL_6 \)-orbits of vectors of rank 0, 1, 2. It is known that \( SL_6 \) has five orbits in \( \Lambda^3 \mathbb{P}^6 \). Therefore \( \Lambda^3 \mathbb{P}^6 \) has a unique \( SL_6 \)-orbit of vectors of rank 3 or more and \( \Lambda^3 \mathbb{P}^6 \) is 2-discontinuous. An example of a vector of rank 3 is (see [Gu48])

\[
\Lambda^3 = v_1 \wedge v_2 \wedge v_4 + v_1 \wedge v_5 \wedge v_3 + v_6 \wedge v_2 \wedge v_3.
\]

### 6.2 \( G = SO_n \)

Let \( \ell = \text{rank}(G) = \lfloor \frac{n}{2} \rfloor \). In this section we prove the following proposition.

**Proposition 6.4.** The natural representation \( V(\pi_1) \) is rs-continuous.

1. If \( n \) is even, then, for \( j = 2, \ldots, \ell - 2 \), the representation \( V(\pi_j) \) is r-discontinuous.
2. If \( n \) is odd, then, for \( j = 2, \ldots, \ell - 1 \), the representation \( V(\pi_j) \) is r-discontinuous.
3. The spin representations \( V(\pi_{\ell-1}) \) and \( V(\pi_{\ell}) \) for even \( n \) and \( V(\pi_{\ell}) \) for odd \( n \) are rs-continuous if and only if \( n \leq 10 \).

Moreover, all fundamental representations of \( SO_n \) which are r-discontinuous are 2-discontinuous.

**Proof.** The first statement is well known. Indeed, the group \( SO_n \) has exactly two orbits in \( \mathbb{P}(\pi_1) \), namely, the quadric and its complement. The first one consists, by definition, of points of rank 1. The second one consists necessarily of points of rank 2.

The second and third statement in the proposition are concerned with fundamental representations of \( SO_n \), which are not the natural nor the spin representation. We handle the two statements at once. In the case \( j = 2 \), the representation is actually the adjoint representation, i.e. \( V(\pi_2) = \mathfrak{so}_n \). Here results of [KY00] show that \( \sigma_2(\mathbb{X}) \neq \mathbb{X} \cup \mathcal{X}_2 \). Thus the representation is 2-discontinuous. The remaining cases, \( j \geq 3 \), are reduced to the case \( j = 2 \) via Proposition 5.2.

Now, we turn to the last statement of the proposition, concerning the spin representations. First, recall that, for even \( n \), the geometric properties we are concerned with are the same for the two spin representations \( V(\pi_{\ell-1}) \) and \( V(\pi_{\ell}) \). Also, either one of these representations remains irreducible when restricted to \( Spin_{n-1} \) and, furthermore, \( Spin_{n-1} \) acts transitively on the closed orbit of \( Spin_{n} \) in \( \mathbb{P}(\pi_1) \). Thus, it is enough to check statement 3) for the representations \( V(\pi_{\ell}) \) of \( Spin_{2\ell} \). Let \( \mathbb{X} \) denote the closed orbit of \( Spin_{2\ell} \) in \( \mathbb{P}(\pi_1) \).

It is shown, in [Car97], Section 3.5, that for \( 2\ell = 12 \) the secant variety \( \sigma_2(\mathbb{X}) \) contains elements of rank 3. Thus the representation \( V(\pi_6) \) of \( Spin_{12} \) is 2-discontinuous. Using Proposition 5.1 we deduce that the representation \( V(\pi_{\ell}) \) of \( Spin_{2\ell} \) is r-discontinuous for all \( \ell \geq 6 \). So, according to the remarks made earlier in this proof, the spin representations of \( Spin_n \) are 2-discontinuous for \( n \geq 11 \).

It remains to verify that the spin representations are rs-continuous for even \( n \leq 10 \). This statement easily follows from Corollary 2.3. This completes the proof of the proposition.

\[ \square \]
6.3 $G = Sp_{2n}$

**Proposition 6.5.** The fundamental representations of $Sp_{2n}$ which are rs-continuous are exactly $V(\pi_1)$ and $V(\pi_2)$. All other fundamental representations of $Sp_{2n}$ are 2-discontinuous.

*Proof.* The representation $V(\pi_1)$ is simply the natural representation of $Sp_{2n}$ on $\mathbb{F}^{2n}$. The action of $Sp_{2n}$ on $\mathbb{P}(\mathbb{F}^{2n})$ is transitive, i.e. $X = \mathbb{P}(\mathbb{F}^{2n})$ and there is nothing more to prove here. The representations $V(\pi_2)$ and $V(\pi_k), k \geq 3$ are considered in Lemmas 6.8 and 6.7 respectively. \hfill \Box

**Lemma 6.6.** The representation $V(\pi_3)$ of $Sp_{2n}$ is 2-discontinuous for $n \geq 3$.

*Proof.* Let $n \geq 3$ and consider $Sp_{2n}$ to be defined with respect to the skew-symmetric form

$$(z_1 \land z_6 + z_2 \land z_5 + z_3 \land z_4) + (z_7 \land z_8 + \ldots + z_{2n-1} \land z_{2n})$$

on $\mathbb{F}^{2n}$, where $z_1, \ldots, z_{2n}$ is the dual basis corresponding to the basis $v_1, \ldots, v_{2n}$ of $\mathbb{F}^{2n}$.

Let $X$ be the set of points $[w_1 \land w_2 \land w_3]$ such that $w_1, w_2, w_3 \in \mathbb{F}^{2n}$ span a 3-dimensional isotropic subspace $\mathbb{F}^{2n}$. Let $\mathcal{X}$ be the affine cone over $X$. We set

$$\Lambda^3_0 \mathbb{F}^{2n} := (X).$$

The set $X$ is a single $Sp_{2n}$-orbit and thus $\Lambda^3_0 \mathbb{F}^{2n}$ is an irreducible $Sp_{2n}$-module. There is a unique up to scaling $Sp_{2n}$-isomorphism between $V(\pi_3)$ of $Sp_{2n}$ and $\Lambda^3_0 \mathbb{F}^{2n}$. We identify $V(\pi_3)$ with $\Lambda^3_0 \mathbb{F}^{2n}$. The varieties $X$ and $\mathcal{X}(\pi_3)$ coincide under this identification.

Note that if $\psi \in V(\pi_3)$ has rank 3 as a vector of the $SL_2$-module $\Lambda^3 \mathbb{F}^{2n}$, then $\psi$ has rank 3 or more as a vector of the $Sp_{2n}$-module $V(\pi_3)$.

Consider the tensor $A3 \in \Lambda^3 \mathbb{F}^6$ given at the end of the proof of Lemma 6.3 One has

$$[v_1 \land v_2 \land v_4], [v_1 \land v_5 \land v_3], [v_6 \land v_2 \land v_3] \in \mathcal{X}(\pi_3)$$

and thus $A3 \in \Lambda^3_0 \mathbb{F}^6 \subset \Lambda^3_0 \mathbb{F}^{2n}$. Moreover the rank of $A3$ is 3 or less. As $A3$ has rank 3 as an element of $SL_2$-module $\Lambda^3 \mathbb{F}^6$, $A3$ has rank 3 as an element of $\Lambda^3 \mathbb{F}^{2n}$ (see Proposition 6.2). Hence

$$\text{rk}_X A3 = 3,$$

where $A3$ is considered as an element of $\Lambda^3_0 \mathbb{F}^{2n}$.

It is shown in [Zak93, Ch. III, Thm 1.4], that $\sigma_2(X) = \mathbb{P}(\Lambda^3_0 \mathbb{F}^6)$. Thus $A3$ has border rank 2 or less as an element of $\Lambda^3_0 \mathbb{F}^6$ and hence

$$\text{rk}_X A3 \leq 2,$$

here $A3$ is considered as an element of $\Lambda^3_0 \mathbb{F}^{2n}$. Therefore the $Sp_{2n}$-module $V(\pi_3) = \Lambda^3_0 \mathbb{F}^{2n}$ is 2-discontinuous. \hfill \Box

**Lemma 6.7.** Fix $n \geq k \geq 3$. The representation $V(\pi_k)$ of $Sp_{2n}$ is 2-discontinuous.

*Proof.* If $n \geq k \geq 3$, the Dynkin diagram $C_n$ of $Sp_{2n}$ has a unique subdiagram $\mathcal{C}_{n,k}$ of type $C_{n-k+3}$. The chopping of $\pi_k$ to this diagram equals $\pi_3$. By Lemma 6.6, $V(\pi_3)$ is not rs-continuous for $\mathcal{C} = Sp_{2(n-k+3)}$ (this group corresponds to the Dynkin diagram $\mathcal{C}_{n,k}$) and thus $V(\pi_k)$ is not an rs-continuous $Sp_{2n}$-module by Proposition 5.1. \hfill \Box

We are now going to prove that $V(\pi_2)$ is rs-continuous for $Sp_{2n}$ ($n \geq 2$). To do this we set $V := \mathbb{F}^{2n}$ and fix a nondegenerate antisymmetric bilinear form $\omega$ on $V$. Note that the second fundamental module of $\mathfrak{sp}(V)$ is isomorphic to the set of vectors in $\Lambda^2 V$, which are annihilated by $\omega$ (here we consider $\omega$ as an element of $(\Lambda^2 V)^\ast$). We denote this space by $\Lambda^2_0 V$. To complete the proof of Proposition 6.5 we prove the following lemma.
Lemma 6.8. a) For any $\omega \in \Lambda^2 V$ the rank of $\omega$ as a bilinear coform is twice the rank of $\omega$ as a vector in an $Sp(V)$-module.

b) The $Sp(V)$-module $\Lambda^2_0 V$ is rs-continuous.

To prove Lemma 6.8 we introduce a notion related to bilinear coforms $\omega \in \Lambda^2 V$. A bilinear coform $\omega$ defines a map $V^* \to V$ by $v \to \omega(v, \cdot)$. We denote the image of this map by Supp$\omega$. We have natural inclusions

$$\Lambda^2 \operatorname{Supp}\omega \to \Lambda^2 V, \quad \Lambda^2_0 \operatorname{Supp}\omega \to \Lambda^2_0 V,$$

and if $\omega \in \Lambda^2_0 V$, then $\omega \in \Lambda^2_0 \operatorname{Supp}\omega$. Note that $\omega$ is nondegenerate as an element of $\Lambda^2 \operatorname{Supp}\omega$ and, in particular, defines a bilinear form $\omega^*$ on $\operatorname{Supp}\omega$ (there is no canonical way to extend $\omega^*$ to the whole $V$).

Lemma 6.8 follows from Lemma 6.9 below; a proof of Lemma 6.8 is presented after the proof of Lemma 6.9

Lemma 6.9. Let $\omega \in \Lambda^2_0 V$ be a bilinear coform of rank $2r$. Then there exist a set of elements $x_1, ..., x_r, y_1, ..., y_r \in \operatorname{Supp}\omega$ such that

$$\omega = x_1 \wedge y_1 + \ldots + x_r \wedge y_r, \text{ and } \omega(x_i, y_i) = 0 \text{ for all } i.$$

In turn, Lemma 6.9 follows from Lemma 6.10 below; a proof of Lemma 6.9 is presented after the proof of Lemma 6.10

Lemma 6.10. Let $\omega \in \Lambda^2_0 V$ be a bilinear coform of rank $2r$. If $r > 0$, then there exist elements $x_1, y_1 \in \operatorname{Supp}\omega$ such that

$$\operatorname{rank}(\omega - x_1 \wedge y_1) = 2r - 2, \text{ and } \omega(x_1, y_1) = 0,$$

where $\operatorname{rank}(\eta)$ denotes the usual rank of a bilinear coform $\eta$.

In turn, Lemma 6.10 follows from Lemma 6.11 below; a proof of Lemma 6.10 is presented after the proof of Lemma 6.11

Lemma 6.11. Let $\omega \in \Lambda^2_0 V$ be a bilinear coform of rank $2r$. If $r > 0$, then there exists an open subset $U \subset \operatorname{Supp}\omega$ such that for any $x_1 \in U$ there exists $y_1 \in \operatorname{Supp}\omega$ such that $\omega^*(x_1, y_1) \neq 0$ and $\omega(x_1, y_1) = 0$.

Proof. If a form $\omega$ is zero on $\operatorname{Supp}\omega$, then for any non-zero $x_1 \in \operatorname{Supp}\omega$ there exists $y_1 \in \operatorname{Supp}\omega$ such that $\omega^*(x_1, y_1) \neq 0$, because the form $\omega^*$ is nondegenerate on $\operatorname{Supp}\omega$. In this case $\omega(x_1, y_1) = 0$, because $\omega = 0$.

We assume that $\omega$ is non-zero on $\operatorname{Supp}\omega$. Since $\omega \in \Lambda^2_0 V$, the pairing of $\omega$ with $\omega$ equals 0. Thus $[\omega^*] \neq [\omega|_{\Lambda^2 \operatorname{Supp}\omega}]$. Hence, for some open subset $U \subset \operatorname{Supp}\omega$ and any $x_1 \in U$, both $\omega(x_1, \cdot), \omega^*(x_1, \cdot)$ are non-zero and

$$[\omega(x_1, \cdot)] \neq [\omega^*(x_1, \cdot)].$$

Therefore for any $x_1 \in U$ there exists $y_1 \in \operatorname{Supp}\omega$ such that $\omega^*(x_1, y_1) \neq 0$ and $\omega(x_1, y_1) = 0$. $\square$

Proof of Lemma 6.11. Let $(x_1, y_1)$ be a pair as in Lemma 6.11. We denote by $W_2$ the space spanned by $x_1, y_1$ and by $W_{2r-2}$ the orthogonal complement to $W_2$ in $\operatorname{Supp}\omega$ with respect to $\omega$. Thanks to the choice of $x_1, y_1$, the form $\omega^*$ is nondegenerate on $W_2$ and therefore $\operatorname{Supp}\omega = W_2 \oplus W_{2r-2}$. Then $\omega = \omega^2 + \omega^{2r-2}$ for uniquely determined coforms $\omega^2 \in \Lambda^2 W_2$ and $\omega^{2r-2} \in \Lambda^2 W_{2r-2}$. We have $\omega^2 = \lambda(x_1 \wedge y_1) = x_1 \wedge (\lambda y_1)$ for some $\lambda \in \mathbb{F}$. Therefore

$$\operatorname{rk}(\omega - x_1 \wedge (\lambda y_1)) = 2r - 2,$$

and $\omega(x_1, (\lambda y_1)) = 0$. $\square$
Proof of Lemma 6.9. To prove Lemma 6.9 we use induction.

The r-th statement of the induction is: Let \( \omega \in \Lambda^0 V \) be a bilinear coform of rank 2r. Then there exist a set of elements \( x_1, \ldots, x_r, y_1, \ldots, y_r \in \text{Supp } \omega \) such that

\[
\omega = x_1 \wedge y_1 + \ldots + x_r \wedge y_r , \quad \text{and } \omega(x_i, y_i) = 0 \text{ for all } i.
\]

Basis of the induction, for \( r = 1 \): Let \( \omega \in \Lambda^0 V \) be a bilinear coform of rank 2. Then there exist elements \( x_1, y_1 \in \text{Supp } \omega \) such that \( \omega = x_1 \wedge y_1 \), and \( \omega(x_1, y_1) = 0 \).

First, we check the basis of the induction. Let \( x_1, y_1 \) be basis of \( \text{Supp } \omega \). Then \( \omega = \lambda x_1 \wedge y_1 \) for some \( \lambda \in \mathbb{F}^\times \). As \( \omega \in \Lambda^0 V \), we have

\[
\omega(\omega) = \omega(\lambda x_1 \wedge y_1) = \omega(x_1, y_1) = 0.
\]

Then \( \omega = x_1 \wedge (\lambda y_1) \) and \( \omega(x_1, y_1) = 0 \). Therefore we finish with the basis of induction.

Now we prove that the r-th statement of the induction follows from the \((r-1)\)-th statement. We assume that the \((r-1)\)-th statement holds. According to Lemma 6.10 there exists \( x_r, y_r \in \text{Supp } \omega \) such that

\[
\text{rk}(\omega - x_r \wedge y_r) = 2r - 2, \quad \text{and } \omega(x_r, y_r) = 0.
\]

Note that \( \omega(x_r \wedge y_r) = \omega(x_r, y_r) = 0 \) and therefore \( \omega - x_r \wedge y_r \in \Lambda^0 V \). By hypothesis, there exist \( x_1, \ldots, x_{r-1}, y_1, \ldots, y_{r-1} \in \text{Supp}(\omega - x_r \wedge y_r) \subset \text{Supp } \omega \) such that

\[
\omega - x_r \wedge y_r = x_1 \wedge y_1 + \ldots + x_{r-1} \wedge y_{r-1} , \quad \text{and } \omega(x_i, y_i) = 0 \text{ for all } i.
\]

This completes the proof of Lemma 6.9 \( \square \)

Proof of Lemma 6.8. First note that a highest weight vector of the \( Sp(V) \)-module \( \Lambda^2 V \) is a wedge product of two \( \omega \)-orthogonal vectors of \( V \). Fix a coform \( \omega \in \Lambda^2 V \). A sum of \( r \) vectors from the \( Sp(V) \)-orbit of a highest weight vector has rank at most 2r as a bilinear coform. Hence the rank of \( \omega \) as a vector of an \( Sp(V) \)-module is not less than half the rank of \( \omega \) as a bilinear coform. On the other hand, Lemma 6.9 implies that the rank of \( \omega \) as a vector of an \( Sp(V) \)-module is not larger than half the rank of \( \omega \) as a bilinear coform. Therefore the rank of \( \omega \) as a vector of an \( Sp(V) \)-module is equal to half the rank of \( \omega \) as a bilinear coform. This proves part a) of Proposition 6.8.

The set of coforms of rank \( r \) or less is closed for all \( r \) and this finishes part b). \( \square \)

7 Fundamental representations (exceptional groups)

In this section we prove Theorem 1.1 for fundamental modules of exceptional groups, i.e. we prove Theorem 7.1. Essentially, we consider case-by-case all 27 fundamental representations of the 5 exceptional groups and provide some arguments for each case, by which the corresponding fundamental module is \( r \)-discontinuous or \( rs \)-continuous. The result is presented below.

Theorem 7.1. Assume that \( V(\lambda) \) is a fundamental effective \( G \)-module. Then \( V(\lambda) \) is \( rs \)-continuous if and only if the pair \((G, V(\lambda))\) appears in the following table.

| \( G \) | Representation \( V \) | Highest weight of \( V \) |
|--------|----------------|------------------|
| \( E_6 \) | \( \mathbb{F}^{24}, (\mathbb{F}^{24})^* \) | \( \pi_1, \pi_5 \) |
| \( F_4 \) | \( \mathbb{F}^{26} \) | \( \pi_1 \) |
| \( G_2 \) | \( \mathbb{F}^{16} \) | \( \pi_1 \) |

where the notation is the same as in Theorem 1.1.

Moreover, all \( r \)-discontinuous fundamental representations of exceptional groups are 2-discontinuous.

The types of arguments are presented in the following tables.
### Arguments for being rs-continuous

| Symbol | Argument for being rs-continuous | References |
|--------|----------------------------------|------------|
| SM     | the representation is reduced to a subminuscule representation | Section [BL13]  |
| F4C    | the representation is equivalent to $V(\pi_1)$ of $F_4$ | Prop. [7.2] Subsect. [7.1] |

### Arguments for being r-discontinuous

| Symbol | Argument for being r-discontinuous | References |
|--------|----------------------------------|------------|
| CC     | the representation is chopable to an r-discontinuous representation of some classical group | — |
| Ad     | the representation is adjoint | Section [Kaji98] |
| AdC    | the representation is chopable to the adjoint representation of some exceptional group | — |
| F4D    | the representation is equivalent to $V(\pi_2)$ of $F_4$ | Prop. [7.9] Subsect. [7.2] |
| E7D    | the representation is equivalent to the $E_7$-representation $V(\pi_1)$ | Prop. [7.13] Subsect. [7.3] |

In the following tables, we provide, for each fundamental representation of each exceptional group, an argument by which it is rs-continuous or r-discontinuous.

| F. weights of $E_6$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ |
|---------------------|---------|---------|---------|---------|---------|---------|
| Arguments           | SM      | CC      | CC      | SM or Ad |

| F. weights of $E_7$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ | $\pi_7$ |
|---------------------|---------|---------|---------|---------|---------|---------|---------|
| Arguments           | E7D     | CC      | CC      | CC      | Ad      | CC      |

| F. weights of $E_8$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ | $\pi_5$ | $\pi_6$ | $\pi_7$ | $\pi_8$ |
|---------------------|---------|---------|---------|---------|---------|---------|---------|---------|
| Arguments           | Ad      | CC      | CC      | CC      | CC      | AdC     | CC      |

| F. weights of $F_4$ | $\pi_1$ | $\pi_2$ | $\pi_3$ | $\pi_4$ |
|---------------------|---------|---------|---------|---------|
| Arguments           | F4C     | F4D     | CC      | Ad      |

| F. weights of $G_2$ | $\pi_1$ | $\pi_2$ |
|---------------------|---------|---------|
| Arguments           | SM      | Ad      |

For the representations $V(\pi_k)$ of exceptional groups $E_n(n = 6, 7, 8)$, for which argument CC is applicable, chopping of $V$ in the vertex with number $n - 1$ is a 2-discontinuous representation of a classical group of type $D_{n-1}$. To apply argument AdC one should chop vertex with number 1. For all representations of the exceptional group $F_4$, to apply argument CC one can always chop the vertex with number 1.

Let us first justify arguments SM, CC, Ad, AdC.

SM) It was shown [BL13] that any subminuscule representation is rs-continuous, i.e. that rank and border rank coincide for such representations.

CC) According to Proposition 5.1 if some chopping $V$ of a representation $V$ is 2-discontinuous, then $V$ is 2-discontinuous.

Ad) According to [Kaji98], all adjoint representations of exceptional groups are 2-discontinuous.

AdC) According to Proposition 5.1 and Ad), if some chopping $V$ of a representation $V$ is an adjoint representation of an exceptional group, then $V$ is 2-discontinuous.

The rest of this section is devoted to the justification of arguments F4C, F4D and E7W.
7.1 Rs-continuity of $V(\pi_1)$ for $F_4$

In this subsection we prove the following.

**Proposition 7.2.** The fundamental representation $V(\pi_1)$ of $F_4$ is rs-continuous.

*Proof.* It is known, [Zak93, p. 59], that the generic rank of $V(\pi_1)$ is three, so that 
$$\sigma_3(X(\pi_1)) = \mathbb{P}(V(\pi_1)).$$

In Lemmas 7.3 and 7.4 below, we show that $V(\pi_1)$ is 2- and 3-continuous, respectively, which implies that this module is rs-continuous.

**Lemma 7.3.** The $F_4$-module $V(\pi_1)$ is 2-continuous.

**Lemma 7.4.** The $F_4$-module $V(\pi_1)$ is 3-continuous.

The first fundamental representation $V(\pi_1)$ of $F_4$ is 26-dimensional and is the (nontrivial) representation of the smallest possible dimension for this group. The discussion which follows involves several representations of various groups. This would make the notation $V(\lambda)$ ambiguous. We have chosen to denote the representations spaces by indices corresponding to their dimension. The set of highest weight vectors, previously denoted by $X(\lambda)$, will be denoted by $X(V)$. We let $V_{26}$ denote the representation space of $(F_4, V(\pi_1))$ and $X(V_{26})$ be the set of highest weight vectors. To study $V_{26}$ we use the fact that it can be obtained as a generic hyperplane in the smallest, 27-dimensional representation of $E_6$, which we denote by $V_{27} = (E_6, V(\pi_1))$. We summarize some known results in the following lemma.

**Lemma 7.5.** (i) The algebra of $E_6$-invariant polynomials on $V_{27}$ is polynomial in one generator of degree 3, i.e. $F[V_{27}]^{E_6} = F[DET]$, where $DET \in S^3(V_{27})$.

(ii) The orbits of $E_6$ in $V_{27}$ are the following:
$$0, X(V_{27}), \{DET = 0\} \setminus X(V_{27}), \{DET = a\}, a \in \mathbb{F}^\times;$$

their dimensions are, respectively, 0, 17, 26, 26. The orbits of $E_6 \times \mathbb{F}^\times$ in $V_{27}$ are the following (lower indices indicate dimension):
$$O_0 = \{0\}, \ O_{17} = X(V_{27}), \ O_{26} = \{DET = 0\} \setminus X(V_{27}), \ O_{27} = \{DET \neq 0\}.$$

(iii) There are exactly three $E_6$ orbits in the projective space $\mathbb{P}(V_{27})$ and they are exactly the rank subsets with respect to $X(V_{27})$, namely,
$$X(V_{27}), \ X_2(V_{27}) = \{DET = 0\} \setminus X(V_{27}), \ X_3(V_{27}) = \{DET \neq 0\};$$

their dimensions are, respectively, 16, 25, 26. The secant varieties of $X(V_{27})$ are exactly the closures of the $E_6$-orbits in $\mathbb{P}(V_{27})$.

(iv) The stabilizer of any vector $v \in \{DET \neq 0\}$ is isomorphic to $F_4$. The orth-complement $v^\perp \subset V_{27}$ is an irreducible $F_4$-module isomorphic to $V_{26}$, i.e. $V_{27} \cong \langle v \rangle \oplus V_{26}$ as $F_4$-modules.

(v) The secant varieties of $X(V_{26})$ are obtained as intersections of the secant varieties of $X(V_{27})$ with the hyperplane $\mathbb{P}(V_{26})$, i.e.
$$X(V_{26}) = \mathbb{P}(V_{26}) \cap X(V_{27}), \ \sigma_2(X(V_{26})) = \mathbb{P}(V_{26}) \cap \sigma_2(X(V_{27})), \ \sigma_3(X(V_{26})) = \mathbb{P}(V_{26}).$$

*Proof.* Since the results are known, but are a compilation of the work of many authors, we confine ourselves to giving references (not necessarily the original ones) for the various parts of the lemma. Part (i) can be found in Table II in [Kac80]. As for part (ii), the fact that
$$\{DET = a\}, a \neq 0$$
is a single $E_6$-orbit, is proven in [Kac80 Proposition 1.1], while the enumeration of the orbits in the nullcone \{$\text{DET} = 0$\} is given in [Zak93 p. 59]. Part (iii) can also be deduced from the discussion on p. 59 of [Zak93] or can be seen to follow directly from the fact that $V_{27}$ is a subcominuscule representation and for such representations the rank sets are exactly the group orbits in the projective space, cf. [BL13 §4]. Parts (iv) and (v) are also quoted from [Zak93 p. 59-60].

The above proposition and, specifically, parts (iv) and (v) allow us to practically forget about the group $F_4$ and use only properties of $V_{27}$ and a generic hyperplane inside it. We shall need to understand the structure of $V_{27}$ with respect to a subgroup of $E_6$ of type $D_5$. Let $H \subset E_6$ be the regular subgroup whose root system is generated by the set of simple roots $S := \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ (we use the numbering of simple roots as in [VO90 p. 292]). It turns our that $H \cong Spin_{10}$.

**Lemma 7.6.** The decomposition, as an $H$-module, of the simple $E_6$-module $V_{27}$ is

$$V_{27} \cong V_1 \oplus RSpin_{10} \oplus V_{10},$$

where $V_1$ is a one-dimensional trivial $H$-module, $RSpin_{10}$ is the spinor $H$-module, and $V_{10}$ is the natural representation of $SO_{10}$ (recall that $Spin_{10}$ is a cover of $SO_{10}$).

**Proof.** The result is obtained by a straightforward consideration of the weights of the modules involved, using the fact that $H$ is a regular subgroup of $E_6$.

**Lemma 7.7.** The nonzero isotropic vectors in $V_{10}$ belong to $X(V_{27})$ and have rank 1 as elements of the $E_6$-module $V_{27}$. The non-isotropic vectors in $V_{10}$ belong to $O_{26}$ and have rank 2 as elements of the $E_6$-module $V_{27}$.

**Proof.** Since $H$ is a regular subgroup of $E_6$, the weight spaces for $E_6$ in $V_{27}$ are also weight spaces for $H$. Thus $V_{10}$ is a span of some of these weight spaces. The $E_6$-weights of $V_{27}$ are

$$\varepsilon_i \pm \varepsilon_i, -\varepsilon_i - \varepsilon_j \ (i \neq j).$$

The weights appearing in $V_{10}$ are

$$\varepsilon_i - \varepsilon_i, -\varepsilon_i - \varepsilon_i \ (i \neq 1).$$

The weight $-\varepsilon_6 - \varepsilon$ is the lowest weight of $V_{27}$ and thus any element of the corresponding weight space belongs to $O_{17}$. On the other hand, any element of the weight space of weight $-\varepsilon_6 - \varepsilon$ is isotropic. Since all isotropic vectors of $V_{10}$ are conjugate by $SO_{10}$, all isotropic vectors of $V_{10}$ belong to $O_{17}$.

It remains to show that all non-isotropic vectors of $V_{10}$ (they are all $SO_{10} \times \mathbb{F}^x$-conjugate) belong to $O_{26}$. To this end, we note that the weights of $V_{10}$

$$\varepsilon_2 - \varepsilon, -\varepsilon_1 - \varepsilon_2$$

are, respectively, the highest and the lowest weight of $V_{27}$ with respect to the set of simple roots of $E_6$

$$\Pi' = \{\varepsilon_2 - \varepsilon_1, \varepsilon_1 + \varepsilon_4 + \varepsilon_5 - \varepsilon, \varepsilon_6 - \varepsilon_4, \varepsilon_4 - \varepsilon_5, -\varepsilon_4 - \varepsilon_3 - \varepsilon_6 + \varepsilon_3 \varepsilon_3 - \varepsilon_6\}.$$

Thus $v^{\varepsilon_2 - \varepsilon} + v^{-\varepsilon_1 - \varepsilon_2} \in O_{26}$ [Zak93 Ch. III, Thm 1.4]. Since all non-isotropic vectors of $V_{10}$ are $SO_{10} \times \mathbb{F}^x$-conjugate, all non-isotropic vectors of $V_{10}$ belong to $O_{26}$.

**Proof of Lemma 7.3.** According to Lemma [7.5] to prove that $V_{26}$ is 2-continuous it suffices to show that for any $x \in V_{26} \cap O_{26}$ there exist $x_+, x_- \in \mathbb{V}_{26} \cap O_{17}$ such that $x = x_+ + x_-$. We fix $x \in V_{26} \cap O_{26}$. First, we note that there exists a Borel subalgebra $\mathfrak{b} \subset E_6$ with a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$ such that $x \in V_{10}$ (we use the notation of Lemma [7.7]). Then $x \in V_{10} \cap O_{26}$ and thus $x$ is a non-isotropic vector of $V_{10}$ by Lemma [7.7]. Let $x^\perp$ be the orthogonal complement to $x$ in $V_{10}$. Note that a nondegenerate symmetric bilinear form $(\cdot, \cdot)$ of $V_{10}$ is still nondegenerate after restriction to $x^\perp$. 

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Since $V_{26}$ is a 26 dimensional subspace of the 27 dimensional space $V_{27}$, we have
\[
\dim(V_{26} \cap x^\perp) \geq \dim(x^\perp) - 1 = 9.
\]
As $9 > \frac{1}{2} \dim V_{10}$, the restriction of $(\cdot, \cdot)$ to $x^\perp \cap V_{26}$ is non-zero. Hence there exists $y \in x^\perp \cap V_{26}$ such that $(y, y) = \frac{-(x, x)}{4}$. Set
\[
x_+ = \frac{\sqrt{2}}{2} + y, \quad x_- = \frac{\sqrt{2}}{2} - y.
\]
We have
\[
(x_+, x_+) = \frac{1}{4}(x, x) + (y, y) = 0 = (x_-, x_-) \quad \text{and} \quad x_+ + x_- = x,
\]
i.e. the vectors $x_\pm$ are isotropic and their sum is equal to $x$. Thus $x_\pm \in O_{17}$. Since $y \in V_{26}$, we have $x_\pm \in V_{26}$. Hence $x_\pm \in X(V_{26})$. Therefore $V_{26}$ is 2-continuous.

Before proceeding with the proof of Lemma 7.4, we need the following auxiliary result.

**Lemma 7.8.** Let $V$ be a finite-dimensional vector space. Let $Z_f$ be a hypersurface determined as the zero-locus of a non-zero homogeneous polynomial $f \in F[V]$ and let $X \in V$ be a conical subset spanning $V$. Then, $V = Z_f + X$, i.e. for every $v \in V$ there exist $v_2 \in Z_f$ and $x \in X$ such that $v = v_2 + x$.

**Proof.** Assume on the contrary, that there exists $v \in V$ such that $v \neq v_2 + x$ for any $v_2 \in Z_f$ and $x \in X$. Then $f(v + tx) \neq 0$ for any $x \in X$ and any $t \in F$. The function $f(v + tx)$ is polynomial and thus $f(v + tx) \neq 0$ for any $t \in F$ if and only if $f(v + tx)$ is a non-zero constant as a polynomial of $t$. Hence the first derivative of $f(v + tx)$ with respect to $t$ is zero for $t = 0$, i.e. the value of $df$ in the direction $x$ at the point $v$ is zero. As $X$ spans $V$, $df = 0$ at $v$.

We claim that $f(w) = 0$ for all points $w \in V$ such that $df = 0$ at $w$ (equivalent to: all partial derivatives of $f$ vanish at $w$). Indeed, the set of equations $df = 0$ determines some subvariety $Z_{df}$ of $V$ and it suffices to show that $f = 0$ at any smooth point of any irreducible component of $Z_{df}$. Obviously $df = 0$ on the smooth locus of any irreducible component of $Z_{df}$. Thus $f$ is constant on any irreducible component of $Z_{df}$. Since $f$ is homogeneous, $f(0) = 0$, and any irreducible component of $Z_{df}$ contains 0. Thus $f|_{Z_{df}} = 0$.

Compiling the previous two paragraphs, we obtain $f(v) = 0$. Thus $v = v + 0$, where $v \in Z_f$ and $0 \in X$. This completes the proof. \[\square\]

**Proof of Lemma 7.4.** The secant variety $\sigma_2(X(V_{26}))$ is the zero-locus of some homogeneous function $\det$ of degree 3 and $X(\pi_1)$ spans $V_{26}$. Thus, according to Lemma 7.8 any vector $x \in V_{26}\backslash \sigma_2(X(V_{26}))$ may be represented as $x = x_1 + x_2$, for some $x_1 \in X(\pi_1)$ and $x_2 \in \sigma_2(X(V_{26}))$. Therefore, by Lemma 7.3 any vector in $V_{26}\backslash \sigma_2(X(V_{26}))$ has rank 3. \[\square\]

### 7.2 R-discontinuity of $V(\pi_2)$ for $F_4$

The goal of this section is to prove the following.

**Proposition 7.9.** The fundamental representation $V(\pi_2)$ of $F_4$ is 2-discontinuous.

We deduce this proposition from the following three lemmas (we use the description of the corresponding roots and weights given in [VO90, p. 294–295]). In particular, the highest root of $f_4$ coincides with the fundamental weight $\pi_4$. In the rest of this section we use our standard notation applied to the representation $(F_4, V(\pi_2))$. 

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Lemma 7.10. The $F_4$-orbit of $[g^{-\pi_4}v^{\pi_2}]$ is open in $TX$, where by $g^{-\pi_4}$ we denote a nonzero root vector of $f_4$ with root $-\pi_4$.

Lemma 7.11. If all elements of $TX$ have rank two or less, then the $F_4$-orbit of $[v^{\pi_2}+v^{-\pi_2+\alpha_2}]$ is open in $TX$.

Lemma 7.12. The vectors $g^{-\pi_4}v^{\pi_2}$ and $v^{\pi_2}+v^{-\pi_2+\alpha_2}$ belong to different $F_4$-orbits.

We present the proofs of Lemmas 7.10 and 7.11 consecutively.

Proof of Lemma 7.10. It suffices to show that the orbit $P_{\pi_2}g^{-\epsilon_1-\epsilon_2}$ is open in the quotient $f_4/p_{\pi_2}$, where $P_{\pi_2}$ denotes the stabilizer in $F_4$ of $v^{\pi_2}$ and $p_{\pi_2}$ is the Lie algebra of $P_{\pi_2}$. This statement follows from the fact that

$$[p_{\pi_2}, g^{-\epsilon_1-\epsilon_2}] + p_{\pi_2} = f_4.$$ 

Proof of Lemma 7.11. First note that, by Proposition 2.3, we have $\dim \sigma_2(X) = 2\dim X + 1$ and $\dim TX = 2\dim X$. Assume that all points in $TX$ have rank two or less. This means that $TX$ is contained in $X_2 \cup X$, which, by definition, is the image of $(X \times X) \times \mathbb{P}^1$, where $(X \times X) \times \mathbb{P}^1$ is the complement of the diagonal in $X \times X$. Then the preimage of $TX$ has to be an $F_4$-stable divisor $D'$ of $(X \times X) \times \mathbb{P}^1$. It follows from Lemma 4.1 that $D'$ has to be a product of a divisor $D \subset (X \times X) \times \mathbb{P}^1$ and $\mathbb{P}^1$. Using the fact that $\pi_2 = -w_0\pi_2$ we see that there is only one $F_4$-stable divisor on $X \times X = F_4/P_{\pi_2} \times F_4/P_{\pi_2}$. This divisor has an open $F_4$-orbit and the image of this $F_4$-orbit in $\mathbb{P}(\pi_2)$ equals $F_4[v^{\pi_2} + v^{w_0\alpha_2}(\pi_2)]$, where $s_{\alpha_2}$ denotes the reflection with respect to the root $\alpha_2$. It remains notice that $s_{\alpha_2}\pi_2 = \pi_2 - \alpha_2$ and that $w_0 = -1$ for $F_4$.

Proof of Lemma 7.12. The proof is based on the following two facts. First, the $F_4$-module $V(\pi_1)$ has an invariant non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ and thus $F_4 \subset SO(V(\pi_1))$. Second, the decomposition of $\Lambda^2 V(\pi_1)$ as an $F_4$-module is

$$\Lambda^2 V(\pi_1) \cong V(\pi_2) \oplus V(\pi_4),$$

see [VO90 Table 5 on p. 305]. This allows us to represent the elements of $V(\pi_2)$ as anti-symmetric tensors and perform calculations. Essentially, we will show that

$$v^{\pi_2} + v^{-\pi_2+\alpha_2} \notin SO(V(\pi_1)) (g^{-\pi_4}v^{\pi_2}).$$

From now on, we consider $V(\pi_2)$ as a subspace of $\Lambda^2 V(\pi_1)$. To any $v \in \Lambda^2 V(\pi_1)$ we assign Supp $v$ as in Subsection 6.3. It is clear that, if $v_1, v_2 \in V(\pi_2)$ are $SO(V(\pi_1))$-conjugate, then the spaces Supp $v_1$ and Supp $v_2$ must be $SO(V(\pi_1))$-conjugate and in particular $\langle \cdot, \cdot \rangle$ restricted to Supp $v_1$ and Supp $v_2$ must have the same rank.

We have the following $\wedge$-decompositions for the vectors of $V(\pi_2) \subset \Lambda^2 V(\pi_1)$

$$v^{\pi_2} = v^{\epsilon_1} \wedge v^{\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4}/2 + v^{-\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4}/2.$$ 

We calculate

$$v^{\pi_2} + v^{-\pi_2+\epsilon_2} = v^{\epsilon_1} \wedge v^{\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4}/2 + v^{-\epsilon_1} \wedge v^{-\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4}/2,$$

$$\text{Supp}(v^{\epsilon_1} \wedge v^{\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4}/2 + v^{-\epsilon_1} \wedge v^{-\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4}/2) = \langle v^{\epsilon_1}, v^{\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4}/2, v^{-\epsilon_1}, v^{-\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4}/2 \rangle$$

and

$$g^{-\pi_4}v^{\pi_2} = g^{-\epsilon_1-\epsilon_2}(v^{\epsilon_1} \wedge v^{\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4}/2) = v^{-\epsilon_2} \wedge v^{\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4}/2 + v^{\epsilon_1} \wedge v^{-\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4}/2,$$

$$\text{Supp}(v^{-\epsilon_2} \wedge v^{\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4}/2 + v^{\epsilon_1} \wedge v^{-\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4}/2) = \langle v^{-\epsilon_2}, v^{\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4}/2, v^{\epsilon_1}, v^{-\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4}/2 \rangle$$

This implies that the space Supp$(g^{-\pi_4}v^{\pi_2})$ is $\langle \cdot, \cdot \rangle$-isotropic, while Supp$(v^{\pi_2} + v^{-\pi_2+\epsilon_2})$ is not. From this we conclude that $g^{-\pi_4}v^{\pi_2}$ and $v^{\pi_2}+v^{-\pi_2+\alpha_2}$ belong to different $SO(V(\pi_1))$-orbits and thus to different $F_4$-orbits.
7.3 R-discontinuity of $V(\pi_1)$ for $E_7$

This subsection is devoted to the proof of the following proposition.

**Proposition 7.13.** The representation $V(\pi_1)$ of $E_7$ is 2-discontinuous.

The specific notation used in this section is motivated by the guest appearance of the adjoint module of $E_8$ in the proof. We set $G = E_8$ and $G = E_7$. By $g$, $h$, $\pi_1$, ... and so on we denote attributes of $E_8$ and by $\mathfrak{g}, \mathfrak{h}, \pi_1$, ... we denote the corresponding attributes of $G = E_7$. We refer to [VO90, Table 1 on p. 293-295] for the roots and fundamental weights of $E_8$. Note that the fundamental weight $\pi_1$ of $E_8$ coincides with the highest root, i.e. $V(\pi_1) \cong \mathfrak{e}_8$ is the adjoint representation of $E_8$.

The idea of the proof of Proposition 7.13 is to identify $V(\pi_1)$ of $E_7$ with some subspace of $\mathfrak{e}_8$ and then prove the following two lemmas.

**Lemma 7.14.** a) For any $x \in X(\pi_1)$ we have dim $E_8x = 58$.

b) For any $x \in X_2(\pi_1)$ we have dim $E_8x \in \{58, 92, 114\}$.

**Lemma 7.15.** There exists $x \in V(\pi_1)$ such that dim $E_8x = 112$.

It is known that $\sigma_2(X(\pi_1)) = V(\pi_1)$ [Zak93, Ch. III, Thm. 1.4]. Therefore $V(\pi_1)$ is r-discontinuous if and only if there exists $x \in V(\pi_1)$ such that

$$x \notin X_2(\pi_1) \cup X(\pi_1) \cup 0.$$ 

According to Lemma 7.14 and Lemma 7.15 such elements $x \in V(\pi_1)$ exist and hence Lemmas 7.14 and 7.15 imply Proposition 7.13. We now present some explanation of Lemma 7.14 and Lemma 7.15 and then proceed with their proofs.

We note the amazing fact that the $V(\pi_1)$ has only finitely many $G$-orbits and we wish to say some words about it (see e.g. [V76]). A description of the $E_7$-orbits on $F^{56}(\text{dim } V(\pi_1) = 56)$ appears in [HI71]. The idea of the description used here comes from [BC76] and is related to the description of $\mathfrak{sl}_2$-triples in exceptional groups due to [D52]. There is a recently developed software, which allows, in principle, to solve such problems [GVY12].

In our proof of 2-discontinuity of $V(\pi_1)$ we use the fact that $V(\pi_1)$ is the 1-component of some grading of $\mathfrak{e}_8$ (representations which arise in such a way are called $\theta$-representations, see [V76], [Kac80]). We need more notation related to $\theta$-representations.

For any $t \in \mathfrak{h}^*$ we denote by $\mathfrak{g}_t \subset \mathfrak{g}$ the corresponding weight space (we note that $\mathfrak{g}_t \neq 0$ if and only if $t \in \Delta \cup 0$). We identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via the Cartan-Killing form and thus consider fundamental weights $\pi_1$ as elements of $\mathfrak{h}^*$. We set

$$\Delta_1 := \{\alpha \in \Delta \cup 0 \mid (\alpha, \pi_1) = i\}, \quad \mathfrak{g}_i := \bigoplus_{t \in \Delta_1} \mathfrak{g}_t \quad (i \in \mathbb{F}).$$

The spaces $\{\mathfrak{g}_t\}_{t \in \mathbb{F}}$ form a grading of $\mathfrak{g}$. The space $\mathfrak{g}_0$ is a Lie algebra and it acts in a natural way on $\mathfrak{g}_i$ for any $i \in \mathbb{F}$. By definition, a $\theta$-representation is the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$.

We have

$$\mathfrak{g}_i = 0 \quad \text{if} \quad i \notin \{-2, -1, 0, 1, 2\}, \quad \mathfrak{g}_0 \cong \mathfrak{e}_7 \oplus \mathbb{F},$$

$$\dim \mathfrak{g}_2 = \dim \mathfrak{g}_{-2} = 1, \quad \dim \mathfrak{g}_1 = \mathfrak{g}_{-1} = 56, \quad \dim \mathfrak{g}_0 = 134.$$ 

We identify $\mathfrak{g} = \mathfrak{e}_7$ with $[\mathfrak{g}_0, \mathfrak{g}_0]$. As $\mathfrak{e}_7$-modules both $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are isomorphic to $V(\pi_1)$. Further, we identify $V(\pi_1)$ with $\mathfrak{g}_1$.

The following lemma plays a key role in the proof of Lemma 7.14.

**Lemma 7.16.** Let $\alpha_1, \alpha_2$ be roots of $E_8$ such that $\alpha_1 \neq -\alpha_2$. Then $v^{\alpha_1} + v^{\alpha_2}$ is a nilpotent element and dim $E_8(v^{\alpha_1} + v^{\alpha_2}) \in \{58, 92, 114\}$. 

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Proof. If \(\alpha_1 = \alpha_2\), then \(v^{\alpha_1} + v^{\alpha_2}\) is conjugate to \(v^{\alpha_1}\). The nilpotent element \(v^{\alpha_1}\) is a generic nilpotent element of the corresponding Levi subalgebra with semisimple part isomorphic to \(A_1\). Therefore \(\dim E_8(v^{\alpha_1} + v^{\alpha_2}) = 58\).

If \(\alpha_1 \neq \pm \alpha_2\), the vector \(v^{\alpha_1} + v^{\alpha_2}\) is a nilpotent element of the Lie algebra \(l_{\alpha_1,\alpha_2}\) corresponding to the root system generated by \(\alpha_1, \alpha_2\). We have three possibilities: \((\alpha_1, \alpha_2) = 1, (\alpha_1, \alpha_2) = 0, (\alpha_1, \alpha_2) = -1\). In the first and third cases, we have \(l_{\alpha_1,\alpha_2} \cong \mathfrak{sl}_3 = A_2\). In the second case, we have \(l_{\alpha_1,\alpha_2} \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 = 2A_1\). For any of these Lie algebras of rank 2 it is easy to check that:

1) if \((\alpha_1, \alpha_2) = 1\), then \(v^{\alpha_1} + v^{\alpha_2}\) is conjugate in \(l_{\alpha_1,\alpha_2}\) to \(v^{\alpha_1}\) (and therefore to \(v^{\alpha_2}\)), and thus is a distinguished nilpotent element for some root subalgebra \(A_1\).
2) if \((\alpha_1, \alpha_2) = 0\), then \(v^{\alpha_1} + v^{\alpha_2}\) is a distinguished nilpotent element of \(l_{\alpha_1,\alpha_2} \cong 2A_1\).
3) if \((\alpha_1, \alpha_2) = -1\), then \(v^{\alpha_1} + v^{\alpha_2}\) is a distinguished nilpotent element of \(l_{\alpha_1,\alpha_2} \cong A_2\).

Hence \(\dim E_8(v^{\alpha_1} + v^{\alpha_2}) = 58, 92, 114\), respectively, for cases 1, 2, 3, see [CM93] 8.4, Table: nilpotent elements for \(E_8\). \(\blacksquare\)

Proof of Lemma [7.14] For any weight \(\alpha \in \Delta_1\) and any \(v^\alpha \in \mathfrak{g}_{\alpha}\) we have

\[ v^\alpha \in V(\overline{\pi_1}) \]

and \(v^\alpha\) is a highest weight vector with respect to some choice of Borel subalgebra of \(\mathfrak{g}\), i.e.

\[ v^\alpha \in X(\overline{\pi_1}) \]

On the other hand \(v^\alpha \in X(\pi_1)\) and thus

\[ \dim E_8v^\alpha = \dim X(\pi_1) = 58. \]

This completes part a).

We proceed to part b). By Lemma [1.1] any element of \(X_2(\pi_1)\) is \(G\)-conjugate to the sum of two weight vectors. In our case this means that any \(x \in X_2(\pi_1)\) is \(G\)-conjugate to

\[ v^{\alpha_1} + v^{\alpha_2} \]

for some \(\alpha_1, \alpha_2 \in \Delta_1\). From this statement and Lemma [7.16] part b) of Lemma [7.14] follows immediately. \(\blacksquare\)

Proof of Lemma [7.15] We shall construct an element \(x\) with \(\dim E_8x = 112\). First note that all roots of \(E_8\) are conjugate and that the Dynkin diagram of \(E_8\) has a unique subdiagram of type \(D_4\). Hence there exists roots \(\alpha_1, \alpha_2, \alpha_3\) such that the quadruple

\[ (-\pi_1, \alpha_1, \alpha_2, \alpha_3) \]

is a system of simple roots of Dynkin type \(D_4\), i.e.

1) \((-\pi_1, \alpha_i) = -1\) for \(i = 1, 2, 3,\)
2) \((\alpha_i, \alpha_j) = 0\) for \(i, j \in \{1, 2, 3\}, i \neq j\).

Condition 1) means that \(\alpha_1, \alpha_2, \alpha_3 \in \Delta_1\). The element \(v^{\alpha_1} + v^{\alpha_2} + v^{\alpha_3}\) is a distinguished element of \(3A_1\), where by \(3A_1\) we denote the subgroup of \(G\) corresponding to the root subsystem

\[ \bigcup_i \{-\alpha_i, \alpha_i\} \subset \Delta. \]

Therefore \(\dim E_8(v^{\alpha_1} + v^{\alpha_2} + v^{\alpha_3}) = 112\), see [CM93] 8.4, Table: nilpotent elements for \(E_8\). Hence, for \(x = (v^{\alpha_1} + v^{\alpha_2} + v^{\alpha_3}) \in V(\overline{\pi_1})\), we have \(\dim E_8x = 112\). This completes the proof. \(\blacksquare\)

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