Inexact reduced gradient methods in smooth nonconvex optimization

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Consider the optimization problem of the form

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad x \in \mathbb{R}^n$$

(1)

with a continuously differentiable ($C^1$-smooth) objective function $f : \mathbb{R}^n \to \mathbb{R}$. 

Gradient descent finds stationary points

A necessary condition for $\bar{x} \in \mathbb{R}^n$ to be a minimizer of $f$ is

$$\nabla f(\bar{x}) = 0.$$  \hfill (2)

The point $\bar{x}$ satisfies (2) is called a stationary point. To find such a point the gradient descent methods construct the iterative procedure

$$x^{k+1} := x^k - t_k \nabla f(x^k) \quad \text{for all} \quad k \in \mathbb{N},$$  \hfill (3)

where $t_k \geq 0$ is a stepsize at the $k^{th}$ iteration, and where $\nabla f(x^k)$ is the gradient of $f$ at $x^k$. 
The stepsize sequence \( \{ t_k \} \) satisfies the **Armijo rule** if there exist a scalar \( \beta \in (0, 1) \) and a reduction factor \( \gamma \in (0, 1) \) such that for all \( k \in \mathbb{N} \) we have the representation

\[
t_k = \max_{t \in \{1, \gamma, \gamma^2, \ldots \}} \left\{ t \mid f(x^k - t \nabla f(x^k)) - f(x^k) \leq -\beta t \| \nabla f(x^k) \|^2 \right\}. \tag{4}
\]
Stepsize selections - Constant and Diminishing

$L-$Lipschitz continuity

\[ \| \nabla f(x) - \nabla f(y) \| \leq L \| x - y \| \text{ for all } x, y \in \mathbb{R}^n. \]

**Constant stepsize:** \( t_k = \tau \in \left( 0, \frac{2}{L} \right) \) for all \( k \in \mathbb{N} \). A smaller modulus \( L \) gives a broader range of selections for \( \tau \).

**Diminishing stepsize:** \( t_k \downarrow 0 \) and \( \sum_{k=1}^{\infty} t_k = \infty \), e.g., \( t_k = \frac{1}{k} \).

For more discussions on gradient descent methods and its variants, see [Ber16, IS14, Nes18, NW16, Pol87, Rus06].
Advantages and disadvantages of stepsize selections

| Method          | Class of functions      | Compute $L$ | Speed  | fval |
|-----------------|-------------------------|-------------|--------|------|
| Backtracking    | $C^1$                   | No          | Moderate | Yes  |
| Constant        | $C^1 + \nabla f \leq L$  | Yes         | Fast   | No   |
| Diminishing     | $C^1 + \nabla f \leq L$  | No          | Slow   | No   |

**Table:** Comparison between stepsize selections
A relaxation of \( L \)-Lipschitz continuity of \( \nabla f \)

The \( L \)-Lipschitz continuity of \( \nabla f \) yields the \( L \)-descent condition

\[
f(y) \leq f(x) + \langle \nabla f(x), y - x \rangle + \frac{L}{2} \|y - x\|^2 \quad \text{for all } x, y \in \mathbb{R}^n. \tag{5}
\]

Quadratic function \( f_{x,L}(y) \) with amplitude \( \frac{L}{2} \)

**Figure:** An \( L \)-descent function that does not have Lipschitz gradient
A relaxation of $L$–Lipschitz continuity of $\nabla f$

Consider the univariate function $f$, where

$$f(x) = \begin{cases} 
\frac{3}{4}x^2 & \text{if } |x| < \frac{2}{3}, \\
-\frac{3}{2}x^2 + 3x - 1 & \text{if } \frac{2}{3} \leq x \leq 1, \\
-\frac{3}{2}x^2 - 3x - 1 & \text{if } -1 \leq x \leq -\frac{2}{3}, \\
|x| - \frac{x^2}{2} & \text{if } |x| > 1.
\end{cases}$$

Then $\nabla f$ is $3$–Lipschitz while $f$ satisfies the $3/2$-descent proper.
Gradient descent methods with errors

- When the function $f$ is a black-box function, i.e., does not have an analytic form [ACL11].
- When the function $f$ is noisy, [GK95].
- When the function $f$ is a smoothing version (Moreau envelope/Forward-backward envelope) of another nonsmooth function $g$ [RW98,STP17,THP20].
Main concerns

**Question 1**

*Does the convergence results hold when errors appear in the calculation $\nabla f(x^k)$?*

*Answer 1 (Ber16, Section 1.2)*

When the $C^1$-smooth function $f: \mathbb{R}^n \to \mathbb{R}$ has a Lipschitz gradient, the gradient descent method with diminishing step-size has stationary accumulation points.

**Question 2**

How about the other types of step-size and the general class of $C^1$-smooth functions?
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Question 2

How about the other types of step-size and the general class of $C^1$-smooth functions?
Main contributions

Design the novel **Inexact reduced gradient** (IRG) methods with 3 stepsize selections:

1. $C^1$—smooth functions: backtracking stepsizes.
2. $C^1$-smooth functions that satisfy $L$—descent condition: constant and diminishing step stepsizes.

What we have achieved?

- Stationary accumulation points for all methods.
- *Global convergence* for all methods under the *Kurdyka-Łojasiewicz (KL) property* of the objective functions.
- *Linear convergence rates* for methods using backtracking stepsizes and constant stepsize.
Our ideas

1. Calculate an arbitrary inexact gradient $g^k$ satisfying
   \[ \left| \nabla f(x^k) - g^k \right| \leq \delta_k. \]

2. Choose $\tilde{g}^k$ near $g^k$ that have a better property than $g^k$.

3. Choose $d^k = -\tilde{g}^k$ as a descent direction.
\[ \langle \nabla f(x^1), d^1 \rangle \leq -\|d^1\|^2 \]

Note: \( \nabla f(x^{2,3}) := \nabla f(x^2) = \nabla f(x^3) \), \( \nabla f(x^1), \nabla f(x^{2,3}) \) are unknown.
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General framework

Step 0. (initialization) Select an initial point $x^1 \in \mathbb{R}^n$, initial radii $\varepsilon_1, r_1 > 0$, radius reduction factors $\mu, \theta \in (0, 1)$, and a sequence of gradient errors $\{\rho_k\} \downarrow 0$.

Step 1. (inexact gradient and stopping criterion) Choose $g^k$ such that

$$\left\| g^k - \nabla f(x^k) \right\| \leq \min \{\rho_k, \varepsilon_k\}. \quad (6)$$

If $\|g^k\| = \rho_k = 0$, then stop.

Step 2. (radius update) If $\|g^k\| \leq r_k + \varepsilon_k$, then set

$$r_{k+1} := \mu r_k, \quad \varepsilon_{k+1} := \theta \varepsilon_k, \quad d^k := 0,$$

and go to Step 3. Otherwise, set

$$r_{k+1} := r_k, \quad \varepsilon_{k+1} := \varepsilon_k,$$

and

$$d^k := -\text{Proj}(0, B(g^k, \varepsilon_k)) = -\frac{\|g^k\| - \varepsilon_k}{\|g^k\|} g^k. \quad (7)$$

Step 3. (stepsize) Choose $t_k > 0$ by a specific rule.

Step 4. (iteration update) Set $x^{k+1} := x^k + t_k d^k$. Increase $k$ by 1 and go back to Step 1.
Deriving the following assertions:

1. Every accumulation point of \( \{x^k\} \) is a stationary point of \( f \).
2. The sequence \( \{x^k\} \) is convergent.
Backtracking step-size for inexact gradient

Choose a line search scalar $\beta \in (0, 1)$, a reduction factor $\gamma \in (0, 1)$, and an artificial stepsize at stagnant iterations $\tau \in (0, 1)$.

**Backtracking stepsize**

If $d^k = 0$, then put $t_k := \tau$. Otherwise, we set

$$t_k := \max \left\{ t \mid f(x^k + td^k) \leq f(x^k) - \beta t \|d^k\|^2, \ t = 1, \gamma, \gamma^2, \ldots \right\}.$$  \hspace{1cm} (8)
 Lemma 1 (KMT22)

Let \( \{x^k\} \) and \( \{d^k\} \) be sequences satisfying

\[
\sum_{k=1}^{\infty} \|x^{k+1} - x^k\| \cdot \|d^k\| < \infty. \tag{9}
\]

If \( \bar{x} \) is an accumulation point of \( \{x^k\} \) and if 0 is an accumulation point of \( \{d^k\} \), then there exists an infinite set \( J \subset \mathbb{N} \) such that

\[
x^k \xrightarrow{J} \bar{x} \text{ and } d^k \xrightarrow{J} 0. \tag{10}
\]
Stationary accumulation points

When \( f \) is \( C^1 \)—smooth and step-size is backtracking, or when \( f \) satisfies \( L \)—descent condition and step-size is constant or diminishing.

**Theorem 2 (KMT22)**

(i) Every accumulation point of \( \{x^k\} \) is a stationary point of \( f \).

(ii) If the sequence \( \{x^k\} \) is bounded, then the set of accumulation points of \( \{x^k\} \) is nonempty, compact, and connected.

(iii) If \( \{x^k\} \) has an isolated accumulation point, then the entire sequence \( \{x^k\} \) converges to this point.
Definition 3 (AMA6)

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable function. We say that $f$ satisfies the KL property at $\bar{x} \in \mathbb{R}^n$ if there exist a number $\eta > 0$, a neighborhood $U$ of $\bar{x}$, and a nondecreasing function $\psi : (0, \eta) \rightarrow (0, \infty)$ such that the function $1/\psi$ is integrable over $(0, \eta)$ and we have

$$\|\nabla f(x)\| \geq \psi(f(x) - f(\bar{x}))$$

for all $x \in U$ with $f(\bar{x}) < f(x) < f(\bar{x}) + \eta$.

Remark 1

The KL property is satisfied at every point $x \in \mathbb{R}^n$ when $f$ is either

- analytic
- semi-algebraic (graph is built up by polynomial inequalities)
- definable in o-minimal structures
From now, assume that the radius reduction factors satisfy $\theta < \mu$. When $f$ is $C^1$—smooth and step-size is backtracking, or when $f$ satisfies $L$—descent condition and step-size is constant or diminishing.

**Theorem 4 (KMT22)**

Assume that $f$ satisfies the **KL property at some accumulation point** $\bar{x}$ of $\{x^k\}$. Then $x^k \to \bar{x}$. 
Definition 5

The $k^{th}$ iteration of Algorithm 1 is called a null iteration if $x^{k+1} = x^k$. The set of all null iterations is denoted by

$$\mathcal{N} := \left\{ k \in \mathbb{N} \mid x^{k+1} = x^k \right\}.$$  

Remark 2

If the set of non-null iterations is finite, $\{x^k\}$ converges finitely to some stationary point $\bar{x}$. So we consider the case this set is infinite.

Assumption 1

The non-null iterations sequence $\{z^k\}$ has an accumulation point $\bar{z}$, that $f$ satisfies the KL property at $\bar{z}$ with $\psi(t) = Mt^q$ for some $M > 0$ and $q \in [0, 1)$. 

Theorem 6 (KMT22)

Suppose that the step-size considered is backtracking. Assume further that $f$ is bounded from below, and $\nabla f$ is locally Lipschitzian around $\bar{z}$. Then

$$z^k \rightarrow \bar{x} \text{ R-linearly or Q-linearly.}$$
Theorem 6 (KMT22)

Suppose that the step-size considered is backtracking. Assume further that \( f \) is bounded from below, and \( \nabla f \) is \textit{locally Lipschitzian} around \( \bar{z} \). Then

\[ z^k \rightarrow \bar{x} \quad \text{R-linearly or Q-linearly}. \]

Theorem 7 (KMT22)

The same rate of convergence holds when \( f \) satisfies the \( L \)–\textit{descent condition} and step-size is constant.
Numerical experiments

Compare the efficiency of our new IRG methods with the reduced gradient (RG) methods and the gradient descent (GD) method, and then check the sensitivity of the IRG methods with respect to error accumulations in the following settings:

1. The accuracy of inexact gradient $g^k$ is low, i.e., $||g^k - \nabla f(x^k)|| \leq \delta_k$, where $\delta_k$ is not too small relative to $||\nabla f(x^k)||$.

2. The accuracy $||\nabla f(x^{last})|| \leq \nu$ required for the solution is increasing.

3. The dimension of the objective function is increasing.
## Testing functions

| Test number | Problem name | Dimension | Accuracy |
|-------------|--------------|-----------|----------|
| 1           | Beale        | 2         | 0.01     |
| 2           | Branin       | 2         | 0.01     |
| 3           | Camel        | 2         | 0.01     |
| 4           | Gol          | 2         | 0.01     |
| 5           | Himmel       | 2         | 0.01     |
| 6           | Beale        | 2         | 0.001    |
| 7           | Branin       | 2         | 0.001    |
| 8           | Camel        | 2         | 0.001    |
| 9           | Gol          | 2         | 0.001    |
| 10          | Himmel       | 2         | 0.001    |
| Test number | Problem name   | Dimension | $\nu$  |
|-------------|----------------|-----------|--------|
| 11          | Dixon 20       | 20        | 0.01   |
| 12          | Dixon 500      | 500       | 0.01   |
| 13          | Dixon 2000     | 2000      | 0.01   |
| 14          | Rosen 20       | 20        | 0.01   |
| 15          | Rosen 500      | 500       | 0.01   |
| 16          | Rosen 2000     | 2000      | 0.01   |
| 17          | Dixon 20       | 20        | 0.001  |
| 18          | Dixon 500      | 500       | 0.001  |
| 19          | Dixon 2000     | 2000      | 0.001  |
| 20          | Rosen 20       | 20        | 0.001  |
| 21          | Rosen 500      | 500       | 0.001  |
| 22          | Rosen 2000     | 2000      | 0.001  |
Figure: The ratio between number of iterations of IRGB and GD by tests
**Figure:** The ratio between number of iterations of IRGB when $\nu = 0.001$ and when $\nu = 0.01$ by functions
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