TWO WEAK FORMS OF COUNTABILITY AXIOMS
IN FREE TOPOLOGICAL GROUPS

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Abstract. Given a Tychonoff space $X$, let $F(X)$ and $A(X)$ be respectively the free topological group and the free Abelian topological group over $X$ in the sense of Markov. For every $n \in \mathbb{N}$, let $F_n(X)$ (resp. $A_n(X)$) denote the subspace of $F(X)$ (resp. $A(X)$) that consists of words of reduced length at most $n$ with respect to the free basis $X$. In this paper, we discuss two weak forms of countability axioms in $F(X)$ or $A(X)$, namely the $csf$-countability and $snf$-countability. We provide some characterizations of the $csf$-countability and $snf$-countability of $F(X)$ and $A(X)$ for various classes of spaces $X$. In addition, we also study the $csf$-countability and $snf$-countability of $F_n(X)$ or $A_n(X)$, for $n = 2, 3, 4$. Some results of Arhangel’skiĭ in [4] and Yamada in [22] are generalized. An affirmative answer to an open question posed by Li et al. in [11] is provided.

1. Introduction

In 1941, Markov [16] introduced the concepts of the free topological group $F(X)$ and the free Abelian topological group $A(X)$ over a Tychonoff space $X$, respectively. Since then, free topological groups have been a source of various examples and also an interesting topic of study in the theory of topological groups, see [3]. From the algebraic point of view, the structure of $F(X)$ or $A(X)$ is very simple - it is the free algebraic group over the set $X$. However, the topological structure of $F(X)$ and $A(X)$ is rather complicated even for simple spaces $X$. For example, it is a well known fact that if $X$ is a non-discrete space, then neither $F(X)$ nor $A(X)$ is Fréchet-Urysohn, and hence first countable, see [1]. This fact motivates researchers to investigate free topological groups in two directions. The first direction of the research on free topological groups is to study some weak forms of countability axioms in $F(X)$ and $A(X)$ over certain classes of spaces $X$. In this line, Arhangel’skiĭ et al. [4] considered the following questions on $F(X)$ and $A(X)$ over a metrizable space $X$: For which spaces $X$, is $F(X)$ or $A(X)$ a k-space? When is the tightness of $F(X)$ or $A(X)$ countable? They proved that $F(X)$ is a $k$-space iff $X$ is locally compact separable or discrete; $A(X)$ is a $k$-space iff $X$ is locally compact and $X'$ is separable, where $X'$ is the derived set of $X$. Furthermore, the tightness of $F(X)$ is countable iff $X$ is separable or discrete, and the tightness of $A(X)$ is countable iff $X'$ is separable.

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The other direction of research on free topological groups is to study (weak) countability axioms of $F_n(X)$ or $A_n(X)$, where $F_n(X)$ (resp. $A_n(X)$) stands for the subset of $F(X)$ (resp. $A(X)$) formed by all words whose reduced length is at most $n$. Indeed, Yamada showed that for a metrizable space $X$, $F_3(X)$ or $A_3(X)$ is Fréchet-Uryshon iff $X'$ is compact, and $F_4(X)$ is Fréchet-Uryshon iff $X$ is compact or discrete. As applications, characterizations of a metrizable space $X$ are given such that $A_n(X)$ is Fréchet-Uryshon for each $n \geq 3$, and $F_n(X)$ is Fréchet-Uryshon for each $n \geq 3$ except for $n = 4$. The subspaces $F_3(X)$ and $A_4(X)$ are very special cases. In [22], Yamada proved that for a metrizable space $X$, the following are equivalent: (i) $F_n(X)$ is metrizable for each $n \in \mathbb{N}$; (ii) $F_n(X)$ is first-countable for each $n \in \mathbb{N}$; (iii) $F_4(X)$ is metrizable; (iv) $F_4(X)$ is first-countable; (v) $X$ is compact or discrete. In the same paper, Yamada also studied the first countability of $F_n(X)$ and $A_n(X)$ for $n = 2, 3$. It is proved that for a metrizable space $X$, the following are equivalent: (i) $F_3(X)$ is metrizable; (ii) $F_3(X)$ is first-countable; (iii) $F_2(X)$ is metrizable; (iv) $F_2(X)$ is first-countable; (v) $X'$ is compact. Furthermore, for a metrizable space $X$, the following are also equivalent: (i) $A_2(X)$ is first-countable; (ii) $A_2(X)$ is metrizable; (iii) $A_n(X)$ is first-countable for each $n \in \mathbb{N}$; (iv) $A_n(X)$ is metrizable for each $n \in \mathbb{N}$; (v) $X'$ is compact.

Recently, Li et al. [11] continued the study of $F(X)$ and $A(X)$ along the aforementioned first direction. They studied several weak forms of countability axioms of $F(X)$ and $A(X)$ defined by networks over some classes of generalized metric spaces $X$. More precisely, they studied the concepts of $sn$-networks, $cs$-networks, $cs^*$-networks in $F(X)$, $A(X)$, and their subspaces $F_n(X)$ and $A_n(X)$. Two types of countability axioms defined by these concepts, namely $snf$-countability and $csf$-countability, were considered. Among many other things, Li et al. established the following results: For a metrizable and crowded space $X$, $F(X)$ or $A(X)$ is $csf$-countable iff $X$ is separable; For a stratifiable $k$-space $X$, $F(X)$ is $snf$-countable iff $X$ is discrete. However, the authors of [11] did not consider the $snf$-countability and $csf$-countability of $F_n(X)$ and $A_n(X)$.

In the paper, we continue the study of free topological group $F(X)$ and the free Abelian topological group $A(X)$ in the afore-mentioned two directions. In particular, we investigate the $csf$-countability and the $snf$-countability of $F(X)$, $A(X)$, $F_n(X)$ and $A_n(X)$ over various classes of generalized metric spaces $X$. In Section 2, we introduce the necessary notation and terminologies which are used for the rest of the paper. In Section 3, we investigate the $snf$-countability of free (Abelian) topological groups. First, we provide some characterizations of the $snf$-countability of $F(X)$, $A(X)$, $F_n(X)$ and $A_n(X)$ over certain classes of topological spaces. The main theorem in this section generalizes a result of Yamada in [22]. Section 4 is devoted to the study of the $csf$-countability of $F(X)$, $A(X)$, $F_n(X)$ and $A_n(X)$. It is shown that for a non-discrete Ľasnev space $X$, $F_4(X)$ is $csf$-countable iff $F_4(X)$ is an $\aleph_0$-space. This result gives an affirmative answer to an open question in [11]. It is also shown that for a sequential $\mu$-space $X$, if $X'$ has a countable $cs^*$-network in $X$, then $F_3(X)$ and $A(X)$ are $csf$-countable. Finally, we pose several interesting open questions in the last section.

Throughout this paper, all topological spaces are assumed to be at least Tychonoff, unless explicitly stated otherwise.

2. Notation and Terminologies

In this section, we introduce the necessary notation and terminologies. First of all, let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote the sets of all positive integers, all integers and all real numbers, respectively. For undefined terminologies, the reader may refer to [3], [6] and [8].

Let $X$ be a topological space $X$ and $A \subseteq X$ be a subset of $X$. The closure of $A$ in $X$ is denoted by $\overline{A}$ and the diagonal of $X$ is denoted by $\Delta(X)$. The subset $A$ is called
**C*-embedded in X** if every bounded continuous real-valued function f defined on A has a bounded continuous extension over X. Moreover, A is called **bounded** if every continuous real-valued function f defined on A is bounded. If the closure of every bounded set in X is compact, then X is called a µ-space. The **derived set** of X is denoted by X'. We say that X is **crowded** if X = X'. Recall that X is said to have a **Gδ-diagonal** (resp. regular **Gδ-diagonal**) if ∆(X) is a Gδ-set (resp. regular Gδ-set) in X × X. A pseudometric d on X is said to be **continuous** if d is continuous as a mapping from the product space X × X to \( \mathbb{R} \). The space X is called a k-space provided that a subset C ⊆ X is closed in X if C ∩ K is closed in K for each compact subset K of X. If there exists a family of countably many compact subsets \{K_n : n ∈ \mathbb{N}\} of X such that each subset F of X is closed in X provided that F ∩ K_n is closed in K_n for each n ∈ \mathbb{N}, then X is called a \( k_\omega \)-space. Note that every \( k_\omega \)-space is a k-space. In addition, X is called a **cf-space** if every compact subset of X is finite. A subset P of X is called a **sequential neighborhood** \( [7] \) of x ∈ X, if each sequence converging to x is eventually in P. A subset U of X is called **sequentially open** if U is a sequential neighborhood of each of its points. The space X is called a **sequential space** if each sequentially open subset of X is open. Let \( k \) be an infinite cardinal. For each \( \alpha \in k \), let \( T_\alpha \) be a sequence converging to \( x_\alpha \notin T_\alpha \). Let \( T := \bigoplus_{\alpha \in k} (T_\alpha \cup \{x_\alpha\}) \) be the topological sum of \( \{T_\alpha \cup \{x_\alpha\} : \alpha \in k\} \). Then \( S_k := \{x\} \cup \bigcup_{\alpha \in k} T_\alpha \) is the quotient space obtained from T by identifying all the points \( x_\alpha \in T \) to the point x.

Let \( \mathcal{P} \) be a family of subsets of X. Then, \( \mathcal{P} \) is called a **csf-network** \([9]\) at a point \( x \in X \) if for every sequence \( \{x_n : n \in \mathbb{N}\} \) converging to x and an arbitrary open neighborhood \( U \) of x in X there exist an \( m \in \mathbb{N} \) and an element \( P \in \mathcal{P} \) such that
\[
\{x\} \cup \{x_n : n \geq m\} \subseteq P \subseteq U.
\]
The space X is called **csf-countable** if X has a countable csf-network at each point \( x \in X \). We call \( \mathcal{P} \) a **csf-network** \([14]\) of X if for every sequence \( \{x_n : n \in \mathbb{N}\} \) converging to a point \( x \) and an arbitrary open neighborhood \( U \) of x in X there is an element \( P \in \mathcal{P} \) and a subsequence \( \{x_{n_i} : i \in \mathbb{N}\} \) of \( \{x_n : n \in \mathbb{N}\} \) such that \( \{x\} \cup \{x_{n_i} : i \in \mathbb{N}\} \subseteq P \subseteq U \). Furthermore, \( \mathcal{P} \) is called a **k-network** \([17]\) if for every compact subset \( K \) of X and an arbitrary open set \( U \) containing \( K \) in X there is a finite subfamily \( \mathcal{P}' \subseteq \mathcal{P} \) such that \( K \subseteq \bigcup \mathcal{P}' \subseteq U \). Recall that a space X is an \( \aleph_0 \)-space (resp. \( R_0 \)-space) if X has a σ-locally finite (resp. countable) k-network. Let \( \mathcal{P} \) be a cover of X such that (i) \( \mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x \); (ii) for each point \( x \in X \), if \( U, V \in \mathcal{P}_x \), then \( W \subseteq U \cap V \) for some \( W \in \mathcal{P}_x \); and (iii) for each point \( x \in X \) and each open neighborhood \( U \) of x there is some \( P \in \mathcal{P}_x \) such that \( x \in P \subseteq U \). Then, \( \mathcal{P} \) is called an **sn-network** \([13]\) for X if for each point \( x \in X \), each element of \( \mathcal{P}_x \) is a sequential neighborhood of x in X, and X is called **snf-countable** \([13]\) if X has an sn-network \( \mathcal{P} \) and \( \mathcal{P}_x \) is countable for all \( x \in X \). The following implications follow directly from definitions:

\[
\text{first countable} \Rightarrow \text{snf-countable} \Rightarrow \text{csf-countable}.
\]

Note that none of the above implications can be reversed. It is well known that \( S_\omega \) is csf-countable but not snf-countable. Moreover, any space without non-trivial convergent sequences is snf-countable, see Example \([3,4]\).

Given a group G, let \( e_G \) denote the neutral element of G. If no confusion occurs, we simply use e instead of \( e_G \) to denote the neutral element of G. Let \( N : G \to \mathbb{R} \) be a function. We call \( N \) a **pre-norm** on G if the following conditions are satisfied for all \( x, y \in G \): (i) \( N(e) = 0 \); (ii) \( N(xy^{-1}) \leq N(x) + N(y) \). If \( G \) is a topological space and \( N \) is continuous, then we say that \( N \) is a **continuous pre-norm** on \( G \).

Let \( X \) be a non-empty Tychonoff space. Throughout this paper, \( X^{-1} := \{x^{-1} : x \in X\} \) and \( -X := \{-x : x \in X\} \), which are just two copies of \( X \). For every \( n \in \mathbb{N} \), \( F_n(X) \) denotes the subspace of \( F(X) \) that consists of all words of reduced length at most \( n \).
with respect to the free basis $X$. The subspace $A_n(X)$ is defined similarly. We always use $G(X)$ to denote topological groups $F(X)$ or $A(X)$, and $G_n(X)$ to $F_n(X)$ or $A_n(X)$ for each $n \in \mathbb{N}$. Therefore, any statement about $G(X)$ applies to $F(X)$ and $A(X)$, and $G_n(X)$ applies to $F_n(X)$ and $A_n(X)$. Let $e$ be the neutral element of $F(X)$ (i.e., the empty word) and $0$ be that of $A(X)$. For every $n \in \mathbb{N}$ and an element $(x_1, x_2, \cdots, x_n)$ of $(X \bigoplus X^{-1} \bigoplus \{e\})^n$ we call $g = x_1 x_2 \cdots x_n$ a form. In the Abelian case, $x_1 + x_2 + \cdots + x_n$ is also called a form for $(x_1, x_2, \cdots, x_n) \in (X \bigoplus X^{-1} \bigoplus \{0\})^n$. This word $g$ is called reduced if it does not contains $e$ or any pair of consecutive symbols of the form $x^{-1} x$. It follows that if the word $g$ is reduced and non-empty, then it is different from the neutral element $e$ of $F(X)$. In particular, each element $g \in F(X)$ distinct from the neutral element can be uniquely written in the form $g = x_1^n x_2^n \cdots x_n^n$, where $n \geq 1$, $r_i \in \mathbb{Z} \setminus \{0\}$, $x_i \in X$, and $x_i \neq x_{i+1}$ for each $i = 1, \cdots, n-1$, and the support of $g = x_1^n x_2^n \cdots x_n^n$ is defined as $\text{supp}(g) := \{x_1, \cdots, x_n\}$. Given a subset $K$ of $F(X)$, we put $\text{supp}(K) := \bigcup_{g \in K} \text{supp}(g)$. Similar assertions (with the obvious changes for commutativity) are valid for $A(X)$. For every $n \in \mathbb{N}$, let $i_n : (X \bigoplus X^{-1} \bigoplus \{e\})^n \to F_n(X)$ be the natural mapping defined by $i_n(x_1, x_2, \ldots, x_n) = x_1 x_2 \cdots x_n$ for each $(x_1, x_2, \ldots, x_n) \in (X \bigoplus X^{-1} \bigoplus \{e\})^n$. We also use the same symbol in the Abelian case, that is, it means the natural mapping from $(X \bigoplus X^{-1} \bigoplus \{0\})^n$ onto $A_n(X)$. Clearly, each $i_n$ is a continuous mapping.

3. The $snf$-Countability of Free Topological Groups

In this section, we discuss the $snf$-countability of $G(X)$ and $G_n(X)$ for a given topological space $X$. First, we provide some general characterizations for the $snf$-countability of $G(X)$. Then, we give some particular classes of spaces $X$ for which those characterizations hold. Finally, we characterize the $snf$-countability of $F_2(X)$ for a specific class of spaces $X$, namely $k$-spaces with a $G_3$-diagonal.

The following theorem generalizes Corollary 4.14 in [1].

**Theorem 3.1.** For a space $X$, the following statements are equivalent:

(i) $G(X)$ is $snf$-countable.

(ii) Each $G_n(X)$ contains no non-trivial convergent sequences.

(iii) $G(X)$ contains no non-trivial convergent sequences.

**Proof.** The implications $(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are obvious.

$(ii) \Rightarrow (iii)$. Assume that each $G_n(X)$ contains no non-trivial convergent sequences. If $G(X)$ contains a non-trivial convergent sequence $S$, there is an $n \in \mathbb{N}$ such that $S \subseteq G_n(X)$. This implies that $G_n(X)$ contains a non-trivial convergent sequence, which is a contradiction.

$(i) \Rightarrow (iii)$. Let $G(X)$ be $snf$-countable with a countable $sn$-network $\{U_n : n \in \mathbb{N}\}$ at $e$, where $U_{n+1} \subseteq U_n$ for each $n \in \mathbb{N}$. Assume that $G(X)$ contains a non-trivial convergent sequence $\{x_i : i \in \mathbb{N}\}$ converging to $e$. Without loss of generality, we assume that $x_i \neq e$ for each $i \in \mathbb{N}$. Then there exists $2 \leq n_0 \in \mathbb{N}$ such that $\{x_i : i \in \mathbb{N}\} \subseteq G_{n_0}(X)$. We may further assume that $\{x_i : i \in \mathbb{N}\} \subseteq G_{n_0}(X) \setminus G_{n_0-1}(X)$.

We first consider the case of $F(X)$. For each $i \in \mathbb{N}$, let $x_i(1) \in X \cup X^{-1}$ be the first letter of $x_i$. Put $A := \{i \in \mathbb{N} : x_i(1) \in X\}$ and $B := \{i \in \mathbb{N} : x_i(1) \in X^{-1}\}$. Obviously, we have $A \cup B = \mathbb{N}$. Without loss of generality, we may assume that $|A| = \omega$. We further assume that $A = \mathbb{N}$. Take an arbitrary point $x \in X$. For each $m \in \mathbb{N}$, let $L_m := \{2^{m-n}x : i \in \mathbb{N}\}$. Then $L_m \cap L_n = \emptyset$ for any $m \neq n$. Indeed, take arbitrary $m, n \in \mathbb{N}$ with $m < n$, and then pick any $g \in L_m$ and $h \in L_n$. Since each $x_i$ belongs to $G_{n_0}(X) \setminus G_{n_0-1}(X)$, we have

$$\ell(g) > 2^{m+1}n_0 - n_0 \geq 2^m n_0 + n_0 \geq 2^{n+1} n_0 + n_0 \geq \ell(h).$$
Hence \( g \neq h \), where \( \ell(g) \) and \( \ell(h) \) denote the lengths of \( g \) and \( h \), respectively. It is evident that each \( \{x^{2^n} \hat{x}x^{-2^n} : i \in \mathbb{N} \} \) converges to \( e \). For each \( n \in \mathbb{N} \), pick a point \( y_n \in U_n \cap L_n \). Then, the sequence \( \{y_n : n \in \mathbb{N} \} \) converges to \( e \). However, since \( |\{y_n : n \in \mathbb{N} \} \cap F_m(X)\} < \omega \) for each \( m \in \mathbb{N} \), by Corollary 7.4.3 in \[3\], the set \( \{y_n : n \in \mathbb{N} \} \) is closed and discrete in \( F(X) \). A contradiction occurs.

Now, we consider the case of \( A(X) \). Let \( L_m := \{mx_i : i \in \mathbb{N} \} \). It is easy to see that \( L_m \cap L_n = \emptyset \) for any \( m \neq n \). Obviously, each \( \{mx_i : i \in \mathbb{N} \} \) converges to \( e \). We can derive a contradiction by a proof similar to that for the case of \( F(X) \).

Next, we consider what conditions on a space \( X \) can guarantee \( G(X) \) to be \( \text{snf}-\text{countable} \). In the light of Theorem 3.2.1 one may conjecture that \( G(X) \) is \( \text{snf}-\text{countable} \) if \( X \) contains no non-trivial convergent sequences. Unfortunately, this is not true, since there is a countably compact and separable \( cf \)-space \( X \) such that \( A(X) \) and \( F(X) \) contain a non-trivial convergent sequence, refer to Theorem 3.5 in \[20\]. Nevertheless, by applying Theorem 3.3.1 to some special classes of topological spaces, we obtain the following result.

**Theorem 3.2.** For a space \( X \), \( G(X) \) is \( \text{snf}-\text{countable} \) if one of the following holds:

(i) \( X \) is a \( cf \)- and \( \mu \)-space;

(ii) Every countable discrete subset of \( X \) is \( C^* \)-embedded.

**Proof.** (i) It suffices to show that \( G(X) \) is a \( cf \)-space. If \( K \subseteq G(X) \) is compact, then \( K \subseteq G_n(X) \) for some \( n \in \mathbb{N} \). Let \( Z := \text{supp}(K) \). Then \( Z \) is a compact subset in \( X \), as \( X \) is a \( \mu \)-space. Hence \( Z \) is finite. Since \( K \subseteq i_n((Z \oplus Z^{-1} \oplus \{e\})^n) \) (in Abelian case, \( K \subseteq i_n((Z \oplus -Z \oplus \{0\})^n) \)), \( K \) must be finite. Therefore, \( G(X) \) is \( \text{snf}-\text{countable} \).

(ii) Since \( X \) is a space in which every countable discrete subset is \( C^* \)-embedded, it follows from Proposition 2.4 in \[20\] that \( G(X) \) contains no non-trivial convergent sequences. Therefore, by Theorem 3.3.1 \( G(X) \) is \( \text{snf}-\text{countable} \).

A topological space \( X \) is \textit{extremely disconnected} if the closure of every open subset is open. Since every countable discrete subset of an extremely disconnected space \( X \) is \( C^* \)-embedded, by Theorem 3.2.2 \( G(X) \) is \( \text{snf}-\text{countable} \). As a particular example, the Stone-\v{C}ech compactification \( \beta D \) of any discrete space \( D \) is extremely disconnected, and hence \( G(\beta D) \) is \( \text{snf}-\text{countable} \).

If \( X \) is a topological group, then we can characterize the \( \text{snf}-\text{countability} \) of \( G(X) \) in terms of the property that \( X \) contains no non-trivial convergent sequences, as it is shown in the next result.

**Theorem 3.3.** For a topological group \( X \), \( G(X) \) is \( \text{snf}-\text{countable} \) if and only if \( X \) contains no non-trivial convergent sequences.

**Proof.** The necessity is obvious by Theorem 3.3.1. To show the sufficiency, suppose that \( X \) contains no non-trivial convergent sequences. By Theorem 3.3.1 it suffices to show that \( G(X) \) contains no non-trivial convergent sequences. We only consider the case of \( F(X) \), since the proof for the case of \( A(X) \) is quite similar. Assume that \( F(X) \) contains a non-trivial convergent sequence. It follows from \[20\] Proposition 2.1 that there are two sequences \( \{x_n : n \in \mathbb{N} \} \) and \( \{y_n : n \in \mathbb{N} \} \) in \( X \) such that \( \{x_n : n \in \mathbb{N} \} \) is infinite, \( x_n \neq y_n \) for each \( n \in \mathbb{N} \) and for every continuous pseudometric \( d \) on \( X \), and \( d(x_n, y_n) \geq 1 \) for at most finitely many \( n \in \mathbb{N} \). Let \( S := \{x_n^{-1}y_n : n \in \mathbb{N} \} \). Then \( S \) is a non-trivial sequence.

We claim that \( S \) is convergent to \( e \) in \( X \). If not, there exists an open neighborhood \( U \) of \( e \) in \( X \) and an infinite set \( A \subseteq \mathbb{N} \) such that \( x_n^{-1}y_n \not\in U \) for each \( n \in A \). By Theorem 3.3.9 in \[20\], there exists a continuous pre-norm \( N \) on \( X \) such that \( \{g \in X : N(g) < 1\} \subseteq U \). Define a continuous pseudometric \( d \) on \( X \) by \( d(x, y) = N(x^{-1}y) \), for all \( x, y \in X \). It follows that \( d(x_n, y_n) \geq 1 \) for all \( n \in A \). However, this is impossible, since by the construction of \( \{x_n : n \in \mathbb{N} \} \) and \( \{y_n : n \in \mathbb{N} \} \), \( d(x_n, y_n) \geq 1 \) holds only for at most
To see how Theorem 3.3 can be applied, we need to identify some classes of topological groups that contain no non-trivial convergence sequences. By a result in [5], every hereditarily normal topological group without a $G_\delta$-diagonal contains no non-trivial convergence sequences, and thus $G(X)$ is snf-countable for such a topological group $X$. On the other hand, every infinite compact group $X$ contains a non-trivial convergent sequence, as it contains a copy of the Cantor cube $\{0, 1\}^{w(X)}$, where $w(X)$ is the weight of $X$. Hence, $G(X)$ is not snf-countable for any infinite compact group $X$. Note that there are infinite pseudocompact topological groups containing no non-trivial convergence sequences, refer to [19] for the existence of such a topological group.

Next, we discuss the snf-countability of subspaces $F_n(X)$ of $F(X)$ for a space $X$. Recall that a subspace $Y$ of a space $X$ is said to be $P$-embedded in $X$ if each continuous pseudometric on $Y$ admits a continuous extension over $X$.

**Theorem 3.4.** Let $X$ be a space with a regular $G_\delta$-diagonal. If $F_4(X)$ is snf-countable, then $X$ is either a cf-space or compact.

**Proof.** Suppose that $X$ contains an infinite compact subset $C$. We show that $X$ must be compact. Since $X$ has a regular $G_\delta$-diagonal, then $C$ must be a metrizable subspace. Without loss of generality, we may assume that $C$ is a non-trivial convergent sequence with its limit point $x$. Since by a result in [2] any pseudocompact space with a regular $G_\delta$-diagonal is metrizable and compact, we only need to show that $X$ is pseudocompact. Assume that $X$ is not pseudocompact. Then $X$ contains an infinite and discrete sequence of open subsets $\{U_n : n \in \mathbb{N}\}$. Note that $\{\overline{U}_n : n \in \mathbb{N}\}$ is also discrete in $X$. Therefore, $\bigcup_{n \in \mathbb{N}} \overline{U}_n$ is closed in $X$. Since $C$ is compact, we may assume that $C \cap (\bigcup_{n \in \mathbb{N}} \overline{U}_n) = \emptyset$. It follows that $C \cup \{U_n : n \in \mathbb{N}\}$ is discrete in $X$. For each $n \in \mathbb{N}$, take a point $x_n \in U_n$. Then, $Y := C \cup \{x_n : n \in \mathbb{N}\}$ is closed, $\sigma$-compact and $P$-embedded in $X$. By a result in [21], the subgroup $F(Y, X)$ of $F(X)$ generated by $Y$ is topologically isomorphic to $F(Y)$. Obviously, $F(Y)$ is a $k_\omega$-space. We claim that $F_4(Y)$ contains a closed copy of $S_\omega$. For each $n \in \mathbb{N}$, put $C_n := x_n C x^{-1} x_n^{-1}$. Let $Z := \bigcup_{n \in \mathbb{N}} C_n$. Then $Z \subseteq F(Y)$ is closed. Since $F(Y)$ is a $k_\omega$-space, $Z$ is also a $k_\omega$-subspace. Take an arbitrary infinite subset $P := \{x_{n_m} c_m x^{-1} x_{n_m}^{-1} : m \in \mathbb{N}, c_m \in C \setminus \{x\}\}$. Then $P$ is a discrete closed subset of $Z$, since $Z$ is a $k_\omega$-space. Therefore, the claim is verified. It follows that $S_\omega$ must be snf-countable, which is a contradiction. \[\Box\]

The following theorem generalizes Theorem 4.9 in [22].

**Theorem 3.5.** For a k-space $X$ with a regular $G_\delta$-diagonal, the following are equivalent:

(i) $F_4(X)$ is snf-countable.

(ii) Each $F_n(X)$ is snf-countable.

(iii) $F_4(X)$ is metrizable.

(iv) Each $F_n(X)$ is metrizable.

(v) $X$ is discrete or compact.

**Proof.** Obviously, we have (ii) $\Rightarrow$ (i), (iv) $\Rightarrow$ (iii), (iv) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i). It suffices to show that (i) $\Rightarrow$ (v) and (v) $\Rightarrow$ (iv).

(i) $\Rightarrow$ (v). Since $X$ has a regular $G_\delta$-diagonal, it follows from Theorem 3.4 that either each compact subset of $X$ is finite or $X$ is compact and metrizable. Moreover, it is easy to check that a k-space is discrete if each compact subset is finite. Therefore, $X$ is discrete or compact.
(v) \Rightarrow (iv). If \( X \) is discrete, then \( F(X) \) is discrete, hence each \( F_n(X) \) is metrizable. If \( X \) is compact, then \( X \) is a compact metrizable space since a compact space with a \( G_\delta \)-diagonal is metrizable [3]. Moreover, since each \( i_n \) is continuous, each \( F_n(X) \) is compact. Therefore, each \( F_n(X) \) is metrizable. □

The next example shows that the condition that \( X \) is “a \( k \)-space” in Theorem 3.5 cannot be dropped.

Example 3.6. There is an infinite, non-discrete, cf-space \( X \) with a regular \( G_\delta \)-diagonal such that \( F_4(X) \) is snf-countable. Let \( \beta \mathbb{N} \) be the Stone-Čech compactification of \( \mathbb{N} \) (equipped with the discrete topology). Take an arbitrary point \( p \in \beta \mathbb{N} \setminus \mathbb{N} \), and consider the subspace \( X := \mathbb{N} \cup \{ p \} \) of \( \beta \mathbb{N} \). It is well known that \( X \) is not compact. Indeed, \( X \) is a non-discrete cf-space with a regular \( G_\delta \)-diagonal. However, as shown in [11 Example 3.12], \( F(X) \) is snf-countable. Thus \( F_4(X) \) is also snf-countable.

By Theorem 4.12 in [22], it is easy to see that “\( F_4(X) \)” in Theorem 3.5 cannot be replaced by “\( F_3(X) \)” However, we have the following result.

Theorem 3.7. Let \( X \) be a stratifiable \( k \)-space. If \( G_2(X) \) is snf-countable, then \( X' \) is compact.

Proof. We only consider the case of \( A_2(X) \), as the proof of the \( F_2(X) \) case is quite similar. Suppose that \( X' \) is not compact. Then, \( X' \) is not countably compact since \( X \) is stratifiable. Therefore, there is a closed, countable, infinite and discrete subset \( \{ x_n : n \in \mathbb{N} \} \) in \( X' \). Since \( X \) is paracompact, we can choose a discrete family \( \{ U_n : n \in \mathbb{N} \} \) of open subsets in \( X \) such that \( x_n \in U_n \) for each \( n \in \mathbb{N} \). For each \( n \in \mathbb{N} \), since \( X \) is sequential and \( \{ x_n : n \in \mathbb{N} \} \subseteq X' \), we can choose a non-trivial sequence \( \{ x_n(m) : m \in \mathbb{N} \} \) such that \( \{ x_n(m) : m \in \mathbb{N} \} \) converges to \( x_n \) and \( \{ x_n(m) : m \in \mathbb{N} \} \subseteq U_n \). Let \( Y_n := \{ x_n(m) : m \in \mathbb{N} \} \) and \( C := \bigcup \{ C_n : n \in \mathbb{N} \} \cup \{ 0 \} \). Obviously, each sequence \( \{ x_n(m) - x_n : m \in \mathbb{N} \} \) converges to 0 in \( A_2(X) \) and \( \{ Y_n : n \in \mathbb{N} \} \) is a discrete family in \( X \).

We claim that the subspace \( C \) is a copy of \( S_{\omega} \). Indeed, the subspace \( S \), defined by

\[
S := \{ (x_n(m), -x_n) : m, n \in \mathbb{N} \} \cup \{ (x_n, -x_n) : n \in \mathbb{N} \},
\]

is closed in \((X \cup -X) \times (X \cup -X)\). Since \( X \) is paracompact, it follows from Proposition 4.8 in [22] that \( i_2 \) is a closed map. Then \( i_2 \mid S \) is a quotient mapping. Therefore, \( C \subseteq A_2(X) \) is homeomorphic to \( S_{\omega} \), and this verifies the claim. Since \( A_2(X) \) is snf-countable, then \( C \) is snf-countable. This contradicts with the fact that \( S_{\omega} \) is not snf-countable. □

Note that in general, the converse of Theorem 3.7 does not hold. To see this, consider the space \( S_{\omega} \). It is easy to check that \( S_{\omega} \) is a stratifiable \( k \)-space whose set of non-isolated points is compact. However, neither \( S_{\omega} \) nor \( G_2(S_{\omega}) \) is snf-countable.

4. The csf-Countability of Free Topological Groups

In this section, we discuss the csf-countability of \( G(X) \) and \( G_n(X) \) for a given space \( X \). First of all, we have the following simple observation.

Proposition 4.1. For a space \( X \), \( G(X) \) is csf-countable if and only if \( G_n(X) \) is csf-countable for all \( n \in \mathbb{N} \).

Proof. It is clear that if \( G(X) \) is csf-countable, then each \( G_n(X) \) is csf-countable.

Conversely, if each \( G_n(X) \) is csf-countable, then \( G(X) \) is csf-countable.
In the light of Proposition 4.1, one of our purposes in this section is to identify those classes of spaces \( X \) for which the csf-countability of \( G_n(X) \) at certain level \( n \in \mathbb{N} \) will be adequate to guarantee the csf-countability of \( G(X) \).

**Theorem 4.2.** For a paracompact, crowded, \( k \)- and \( \aleph_1 \)-space \( X \), the following are equivalent:

(i) \( G(X) \) is csf-countable.

(ii) \( G_2(X) \) is csf-countable.

(iii) \( X \) is separable.

**Proof.** Since (i) \( \Rightarrow \) (ii) is trivial, we only need to show (ii) \( \Rightarrow \) (iii) and (iii) \( \Rightarrow \) (i). Further, we only consider \( A(X) \), as the proof of the case of \( F(X) \) is quite similar.

(ii) \( \Rightarrow \) (iii). Suppose \( X \) is not separable. Since \( X \) is an \( \aleph_1 \)-space, there is a closed, uncountable and discrete subset \( \{x_\alpha : \alpha < \omega_1\} \) in \( X \). Since \( X \) is paracompact, we can choose a discrete family of open subsets \( \{U_\alpha : \alpha < \omega_1\} \) in \( X \) such that \( x_\alpha \in U_\alpha \) for each \( \alpha < \omega_1 \). For each \( \alpha < \omega_1 \), since \( X \) is sequential, we can choose a non-trivial sequence \( \{x_n(\alpha) : n \in \mathbb{N}\} \subseteq U_\alpha \), convergent to \( x_\alpha \). Put

\[
Y_\alpha := \{x_n(\alpha) : n \in \mathbb{N}\} \cup \{x_\alpha\}, \quad C_\alpha := \{x_n(\alpha) - x_\alpha : n \in \mathbb{N}\}
\]

and let \( C := \bigcup \{C_\alpha : \alpha < \omega_1\} \cup \{0\} \). Obviously, \( \{x_n(\alpha) - x_\alpha : n \in \mathbb{N}\} \) is convergent to 0 in \( A_2(X) \) and \( \{Y_\alpha : \alpha < \omega_1\} \) is a discrete family in \( X \). We claimed that \( C \) is a copy of \( S_{\omega_1} \). Indeed, let

\[
S := \{(x_n(\alpha) - x_\alpha) : n \in \mathbb{N}, \alpha < \omega_1\} \cup \{(x_\alpha, -x_n) : \alpha < \omega_1\}.
\]

Then \( S \) is closed in \( (X \cup -X) \times (X \cup -X) \). Since \( X \) is paracompact, by Proposition 4.8 in [22], \( i_2 \) is a closed map. It follows that \( i_2 \upharpoonright S \) is a quotient mapping, and thus \( C \subseteq A_2(X) \) is homeomorphic to \( S_{\omega_1} \). Since \( A_2(X) \) is csf-countable, then \( C \) is csf-countable. However, \( S_{\omega_1} \) is not csf-countable, which is a contradiction.

(iii) \( \Rightarrow \) (i). Let \( X \) be separable. Then \( X \) is an \( \aleph_0 \)-space. By Theorem 4.1 in [4], \( A(X) \) is an \( \aleph_0 \)-space. Thus, \( A(X) \) is csf-countable. \( \square \)

The following theorem provides an affirmative answer to Question 3.9 in [11]. Recall that a topological space \( X \) is said to be \( La\'snev \) if it is the closed image of some metric space.

**Theorem 4.3.** Let \( X \) be a non-discrete \( La\'snev \) space. Then \( F_4(X) \) is csf-countable if and only if \( F(X) \) is an \( \aleph_0 \)-space.

**Proof.** The sufficiency is obvious. To show the necessity, let \( F_4(X) \) be csf-countable. Then \( X \) contains no copy of \( S_{\omega_1} \), and hence \( X \) is an \( \aleph_1 \)-space. By Theorem 4.1 in [4], it suffices to show that \( X \) is an \( \aleph_0 \)-space. Since \( X \) is an \( \aleph_1 \)-space, the proof will be completed if we can show that each closed and discrete subset of \( X \) is at most countable. Assume that \( X \) contains an uncountable, closed and discrete subset \( D = \{d_\alpha : \alpha < \omega_1\} \). Since \( X \) is a non-discrete \( La\'snev \) space, there exists a non-trivial convergent sequence \( \{x_n : n \in \mathbb{N}\} \) with the limit point \( x \) in \( X \) such that \( D \bigoplus S \) is a closed copy of \( X \), where \( S := \{x_n : n \in \mathbb{N}\} \cup \{x\} \). Since \( F_4(D \bigoplus S) \) is a closed subspace of \( F_4(X) \), \( F_4(D \bigoplus S) \) is csf-countable.

Next, we shall show that \( F_4(D \bigoplus S) \) contains a copy of \( S_\omega \). To this end, for each \( \alpha < \omega_1 \), define \( C_\alpha := \{d_\alpha x_n x_n^{-1} d_\alpha^{-1} : n \in \mathbb{N}\} \). Then \( C_\alpha \) converges to \( e \). Let \( C := \{e\} \cup \bigcup_{\alpha < \omega_1} C_\alpha \). Obviously, \( C \subseteq F_4(D \bigoplus S) \), which implies that \( C \) is csf-countable. Then \( C \) has a countable cs-network \( \{P_n : n \in \mathbb{N}\} \) at \( e \). Put

\[
N_1 := \{n \in \mathbb{N} : |\{\alpha < \omega_1 : P_n \cap C_\alpha \neq \emptyset\}| \leq \aleph_0\}
\]
By Theorem 4.1 in [4], we have (i)

Proof. {\alpha < \omega_1: C_{\alpha} \cap P_n \neq \emptyset \text{ for some } n \in N_1}\{.

Clearly, $N \setminus N_1$ is a countable infinite set and $B$ is a countable set. It is easy to see that $\{P_n : n \in N \setminus N_1\}$ is a countable cs-network at $e$ in

$$D_1 := \{e\} \cup \{d_\alpha x_n x^{-1} d_\alpha^{-1} : \alpha \in \omega_1 \setminus B, n \in N\}.$$ 

Moreover, for each $n \in N \setminus N_1$, the set $P_n$ intersects uncountably many $C_\alpha$. Inductively, we can find a countable infinite subset $R := \{\alpha_n \in \omega_1 \setminus B : n \in N \setminus N_1\}$ of $\omega_1$ such that $\alpha_n \neq \alpha_m$ if $n \neq m$ and $P_n \cap C_{\alpha_n} \neq \emptyset$ for each $n \in N \setminus N_1$. Let $Y := \{e\} \cup \bigcup_{\alpha \in R} C_\alpha$. It is clear that $Y \subseteq F_3(D \bigoplus S)$. We claim that $Y$ is homeomorphic to $S_\omega$. Indeed, let $Z = \text{supp}(Y)$. Then, $Z$ is a countable, infinite, locally compact and closed subspace in $D \bigoplus S$. Hence, $F(Z)$ is a sequential space by Theorem 7.6.36 in [3]. Moreover, $F(Z)$ is a subspace of $F(D \bigoplus S)$, since $D \bigoplus S$ is metrizable and $Z$ is closed in $D \bigoplus S$. Assume that $Y$ is not a copy of $S_\omega$. Then there is a non-trivial convergent sequence $\{d_{\alpha_n} x_m x^{-1} d_{\alpha_n}^{-1} : k \in N\}$ in $Y$. Since $Z$ is paracompact, the closure of $\text{supp}\{d_{\alpha_n} x_m x^{-1} d_{\alpha_n}^{-1} : k \in N\}$ in $Z$ is compact. However, $\{d_{\alpha_n} : k \in N\}$ is an infinite, closed and discrete subspace in $Z$, which is a contradiction. Therefore, the claim is verified.

Finally, pick an arbitrary point $a_n \in P_n \cap C_\alpha$ for each $n \in N \setminus N_1$. Since the family $\{P_n : n \in N \setminus N_1\}$ is a countable cs-network at $e$ in $Y$, $e$ is a cluster point of the set $\{a_n : n \in N \setminus N_1\}$, which is a contradiction. Therefore, each closed and discrete subset of $X$ is at most countable. 

**Corollary 4.4.** For a non-discrete metrizable space $X$, the following are equivalent:

(i) $X$ is separable.
(ii) $F(X)$ is an $\aleph_0$-space.
(iii) $F(X)$ is csf-countable.
(iv) $F_3(X)$ is csf-countable.

**Proof.** By Theorem 4.1 in [4], we have (i) $\Leftrightarrow$ (ii). Moreover, (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are obvious. Finally, (iv) $\Rightarrow$ (ii) follows from Theorem 1.3. 

**Remark 4.5.** (i) Theorem 4.3 does not hold for the Abelian case, refer to Theorem 4.5 in [22].

(ii) In Corollary 4.4, $F_3(X)$ cannot be replaced by $F_3(X)$. Let $X := D \bigoplus S$, where $D$ is an uncountable discrete space and $S$ is a non-trivial convergent sequence with its limit. Then, $F_3(X)$ is metrizable by Theorem 4.12 in [22], hence it is csf-countable. However, $X$ is not separable.

(iii) The conclusion of Corollary 4.4 does not hold for the snf-countability. Consider $\mathbb{R}$ with the usual Euclidean topology. By Corollary 1.3, $F(\mathbb{R})$ is an $\alpha_0$-space. However, it follows from Theorem 3.3 that $F_3(\mathbb{R})$ is not snf-countable.

Next, we shall consider the question when $G_3(X)$ is csf-countable for a given space $X$. First, we recall an important lemma in [13].

**Lemma 4.6 (13).** If $\mathcal{P}$ is a countable cs*-network at $x \in X$, then the family

$$\left\{ \bigcup \mathcal{F} : \mathcal{F} \subseteq \mathcal{P}, \mathcal{F} \text{ is finite} \right\}$$

is a countable cs-network at $x$.

**Theorem 4.7.** Let $X$ be a sequential and $\mu$-space. If $X'$ has a countable cs*-network in $X$, then $F_3(X)$ is csf-countable.
Proof. Let $\mathcal{P}$ be a countable $cs^*$-network for $X'$ in $X$. By Lemma 1.6, $X$ is $csf$-countable. Hence, $(X \oplus X^{-1} \oplus \{e\})^n$ is $csf$-countable for each $n \in \mathbb{N}$. Moreover, since $F_3(X) \setminus F_1(X)$ is open in $F_3(X)$ and homeomorphic to a subspace of $(X \oplus X^{-1} \oplus \{e\})^3$, $F_3(X)$ is $csf$-countable at each point of $F_3(X) \setminus F_1(X)$. In what follows, we shall show that $F_3(X)$ is also $csf$-countable at each point of $F_1(X)$. To this end, take an arbitrary point $g \in F_1(X)$. By Lemma 1.6, we are done if we can prove that $F_3(X)$ has a countable $cs^*$-network at $g$. We divide the proof into three cases.

Case 1. The point $g$ is isolated in $X \cup X^{-1}$.

In this case, let

$$\mathcal{B}(g) := \{gP_1^iP_3^2, P_1^iP_2^2g : P_1, P_2 \in \mathcal{P} \cup \{(g)\}, \varepsilon_1, \varepsilon_2 \in \{1, -1\}, \varepsilon_1 \neq \varepsilon_2\}.$$  

Obviously, $|\mathcal{B}(g)| \leq \omega$. We verify that $\mathcal{B}(g)$ is a $cs^*$-network for $F_3(X)$ at $g$. Take an arbitrary sequence $\{x_n : n \in \mathbb{N}\}$ converging to $g$ in $F_3(X)$ and an open neighborhood $U$ of $g$ in $F_3(X)$. Without loss of generality, we may assume that $\{x_n : n \in \mathbb{N}\}$ is a non-trivial convergent sequence such that $x_n \neq x_m$ whenever $n \neq m$. By Theorem 1.5 in [1], $\text{supp}(\{x_n : n \in \mathbb{N}\} \cup \{g\})$ is bounded in $X$. Since $X$ is a $\mu$-space, $\text{supp}(\{x_n : n \in \mathbb{N}\} \cup \{g\})$ is compact in $X$. Let

$$Z := \left(\text{supp}\{x_n : n \in \mathbb{N}\} \cup \{g\}\right) \cup \left(\text{supp}\{x_n : n \in \mathbb{N}\} \cup \{g\}\right)^{-1}.$$  

Then $Z$ is sequentially compact. For each $n \in \mathbb{N}$, pick a point $y_n \in Z^3 \cap \mathfrak{i}^{-1}_3(x_n)$. Then it follows from the sequential compactness of $Z$ that $\{y_n : n \in \mathbb{N}\}$ has a subsequence $\{(z_1(i), z_2(i), z_3(i)) : i \in \mathbb{N}\}$ converging to $z = (z_1, z_2, z_3)$ for some $z \in Z^3$. Since $x_n \neq x_m$ for any $n \neq m$, the sequence $\{y_n : n \in \mathbb{N}\}$ is a non-trivial sequence. Hence, we have that $z \in \mathfrak{i}_3^{-1}(g)$, $z_i \in (X' \cup \{g\}) \cup (X' \cup \{g\})^{-1}$ for each $i \in \{1, 2, 3\}$, $\{i : i \leq 3, z_i \neq g\}$ is an even number, $z_1 = g$ or $z_3 = g$, and $z_2 \in X' \cup (X')^{-1}$. Then $z \in \{(g, z_2, z_3^{-1}), (z_2^{-1}, z_2, g)\}$. Without loss of generality, we assume that $z_1 = g, z_2 \in X'$ and $z = (g, z_2, z_2^{-1})$. Pick an open neighborhood $V$ of $z_2$ in $X$ such that $\{g\} \times V \times V^{-1} \subseteq \mathfrak{i}^{-1}_3(U)$. Since $g$ is isolated in $X \cup X^{-1}$ and the sequence $\{z_1(i) : i \in \mathbb{N}\}$ converges to $g$ in $X \cup X^{-1}$, $z_1(1) = g$ for all but finitely many $i$. Then, since $\mathcal{P}$ is a $cs^*$-network for $X'$, it is easy to see that there exist $P_1, P_2 \in \mathcal{P}$ such that $z_2 \in P_1 \subseteq V$, $z_2^{-1} \in P_2^{-1} \subseteq V^{-1}$ and $\{g\} \times P_1 \times P_2^{-1}$ contains a subsequence of $\{(z_1(i), z_2(i), z_3(i)) : i \in \mathbb{N}\}$. Hence, $g \in gP_1P_2^{-1} \subseteq gVV^{-1} \subseteq U$ and $gP_1P_2^{-1}$ contains a subsequence of $\{x_n : n \in \mathbb{N}\}$. 

Case 2. The point $g$ is non-isolated in $X \cup X^{-1}$.

Without loss of generality, we assume that $g \in X$. In this case, let

$$\mathcal{B}(g) := \{gP_1^iP_3^2, P_1^iP_2^2P_3^3 : P_1, P_2, P_3 \in \mathcal{P} \cup \{(e)\}, \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{1, -1\}, \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1\}.$$  

Obviously, $|\mathcal{B}(g)| \leq \omega$. We verify that $\mathcal{B}(g)$ is a $cs^*$-network for $F_3(X)$ at $g$. Take an arbitrary sequence $\{x_n : n \in \mathbb{N}\}$ converging to $g$ in $F_3(X)$ and an open neighborhood $U$ of $g$ in $F_3(X)$. Next we shall show that there exists a $B \in \mathcal{B}(g)$ such that $g \in P \subseteq U$ and $P$ contains a subsequence of $\{x_n : n \in \mathbb{N}\}$. Without loss of generality, we assume that $\{x_n : n \in \mathbb{N}\}$ is a non-trivial convergent sequence such that $x_n \neq x_m$ whenever $n \neq m$.

Subcase 2.1. The sequence $\{x_n : n \in \mathbb{N}\}$ contains a subsequence $\{x_{n_i} : i \in \mathbb{N}\}$ which is contained in $X$.

Since $\mathcal{P}$ is a $cs^*$-network for $X'$ in $X$, there exists a $P \in \mathcal{P}$ such that $P$ contains a subsequence of $\{x_{n_i} : i \in \mathbb{N}\}$ and $g \in P \subseteq U$. Hence, $P = P\{e\} \in \mathcal{B}(g)$ contains a subsequence of $\{x_{n_i} : i \in \mathbb{N}\}$ and $g \in P \subseteq U$.

Subcase 2.2. The sequence $\{x_n : n \in \mathbb{N}\}$ does not contain any subsequence which is contained in $X$.
Without loss of generality, we may assume that \( \{x_n : n \in \mathbb{N}\} \subseteq F_3(X) \setminus F_2(X) \), since \((F_2(X) \setminus F_1(X)) \cup \{e\}\) is clopen in \(F_3(X)\). Similar to the proof of Case 1, let
\[
Z := \left(\text{supp}\{x_n : n \in \mathbb{N}\} \cup \{g\}\right) \cup \left(\text{supp}\{x_n : n \in \mathbb{N}\} \cup \{g\}\right)^{-1}.
\]
Then \(Z^3\) is sequentially compact. For each \(n \in \mathbb{N}\), pick a point \(y_n \in Z^3 \cap i_3^{-1}(x_n)\). Then \(\{y_n : n \in \mathbb{N}\}\) has a subsequence \(\{(z_i(1), z_i(2), z_i(3))\}_{i \in \mathbb{N}}\) converging to \(z = (z_1, z_2, z_3)\) for some \(z \in Z^3\). Since \(g\) is assumed to be non-isolated in \(X \cup X^{-1}\), we have
\[
(X' \cup \{g\}) \cup (X' \cup \{g\})^{-1} = X' \cup (X')^{-1}.
\]
Moreover, since \(x_n \neq x_m\) for any \(n \neq m\), the sequence \(\{y_n : n \in \mathbb{N}\}\) is a non-trivial sequence. Hence, we have \(z \in i_3^{-1}(g)\), \(z_i \in X' \cup (X')^{-1}\) for each \(i \in \{1, 2, 3\}\), \(z_1 = g\) or \(z_3 = g\), and \(z_2 \in X' \cup (X')^{-1}\). Then \(z \in \{(g, z_2, z_2^{-1}), (z_2^{-1}, z_2, g)\}\). Without loss of generality, we assume that \(z_1 = g, z_2 \in X'\) and \(z = (g, z_2, z_2^{-1})\). Pick open neighborhoods \(V_1, V_2\) of \(g\) and \(z_2\) in \(X\), respectively, such that \(V_1 \times V_2 \times V_2^{-1} \subseteq i_3^{-1}(U)\). Since \(\mathcal{P}\) is a \(cs^*\)-network for \(X'\) in \(X\), it is easy to see that there exist \(P_1, P_2, P_3 \in \mathcal{P}\) such that \(g \in P_1 \subseteq V_1, z_2 \in P_2 \subseteq V_2, z_2^{-1} \in P_3^{-1} \subseteq V_2^{-1}\) and \(z \in P_1 \times P_2 \times P_3^{-1}\) contains a subsequence of \(\{(z_i(1), z_i(2), z_i(3)) : i \in \mathbb{N}\}\). Hence, \(g \in P_1 P_2 P_3^{-1} \subseteq V_1 V_2 V_2^{-1} \subseteq U\) and \(P_1 P_2 P_3^{-1}\) contains a subsequence of \(\{x_n : n \in \mathbb{N}\}\).

**Case 3:** \(g = e\).

In this case, let
\[
B(e) := \left\{P_1^{-1} P_2^2 : P_1, P_2 \in \mathcal{P} \cup \{\{e\}\}, \varepsilon_1, \varepsilon_2 \in \{1, -1\}, \varepsilon_1 + \varepsilon_2 = 0\right\}.
\]
Obviously, \(|B(g)| \leq \omega\). We verify that \(B(g)\) is a \(cs^*\)-network for \(F_3(X)\) at \(e\). Take an arbitrary sequence \(\{x_n : n \in \mathbb{N}\}\) converging to \(e\) and an open neighborhood \(U\) of \(e\) in \(F_3(X)\). Without loss of generality, we assume that \(\{x_n : n \in \mathbb{N}\}\) is a non-trivial convergent sequence such that \(x_n \neq x_m\) whenever \(n \neq m\). Since \((F_2(X) \setminus F_1(X)) \cup \{e\}\) is clopen in \(F_3(X)\), we assume that \(\{x_n : n \in \mathbb{N}\} \subseteq F_2(X) \setminus F_1(X)\).

By an argument similar to that in Case 2, we can show that there exist \(P_1, P_2 \in B(e)\) such that \(e \in P_1^{-1} P_2^2 \subseteq U\) and \(P_1^{-1} P_2^2\) contains a subsequence of \(\{x_n : n \in \mathbb{N}\}\). □

As shown by the following result, for the Abelian case, the conclusion of Theorem 4.7 can be strengthened significantly.

**Theorem 4.8.** Let \(X\) be a sequential and \(\mu\)-space. If \(X'\) has a countable \(cs^*\)-network in \(X\), then \(A(X)\) is \(cs^\text{countable}\).

**Proof.** Since \(A(X)\) is a topological group, we only need to prove that \(A(X)\) has a countable \(cs^\text{-network at 0}\). Let \(\mathcal{P}\) be a countable \(cs^*\)-network for \(X'\) and
\[
B(0) := \left\{\sum_{i=1}^{2k} \varepsilon_i P_i \in \mathcal{P} \cup \{0\}, \varepsilon_i \in \{1, -1\}, i \leq 2k, \sum_{j=1}^{2k} \varepsilon_j = 0\right\}.
\]
Obviously, \(|B(0)| \leq \omega\). Next, we verify that \(B(0)\) is a \(cs^*\)-network for \(A(X)\) at 0.

Take an arbitrary sequence \(\{x_n : n \in \mathbb{N}\}\) converging to 0 in \(A(X)\) and an arbitrary open neighborhood \(U\) of 0 in \(A(X)\). Without loss of generality, we may assume that \(\{x_n : n \in \mathbb{N}\}\) is a non-trivial convergent sequence. Further, we may assume that \(x_n \neq x_m\) for any \(n \neq m\). Obviously, we have \(\{x_n : n \in \mathbb{N}\} \cup \{0\} \subseteq A_l(X)\) for some \(l \in \mathbb{N}\). Then there exists some \(m \leq l\) such that \(A_m \setminus A_{m-1}\) contains a subsequence of \(\{x_n : n \in \mathbb{N}\}\). Therefore, we may assume that \(\{x_n : n \in \mathbb{N}\}\) is contained in \(A_m \setminus A_{m-1}\). Let
\[
Z := \left(\text{supp}\{x_n : n \in \mathbb{N}\} \cup \{0\}\right) \cup \left(-\text{supp}\{x_n : n \in \mathbb{N}\} \cup \{0\}\right).
\]
By the proof of Theorem 4.7, $Z^m$ is sequentially compact. For each $n \in \mathbb{N}$, pick a point $y_n \in Z^m \cap i_m^{-1}(x_n)$. Then $\{y_n : n \in \mathbb{N}\}$ has a subsequence $\{(z_i(1), z_i(2), ..., z_i(m)) : i \in \mathbb{N}\}$ converging to $z = (z_1, z_2, ..., z_m)$ for some $z \in Z^m$. Obviously, $z \in i_m^{-1}(0)$ and $z_j \in X' \cup (-X)'$ for each $j \leq m$. Pick an open subset $V_j$ in $X \oplus -X \oplus \{0\}$ for each $j \leq m$ such that $V_1 \times V_2 \times \cdots \times V_m \subseteq i_m^{-1}(U)$.

For each $z_j \in X' \cup (-X)'$, choose a $P_j \in \mathcal{P}$ inductively such that $z_j \in \varepsilon_j P_j \subseteq V_j$, $\varepsilon_j P_j$ contains a subsequence $\{z_{ik}(j) : k \in \mathbb{N}\}$ of $\{z_i(j) : i \in \mathbb{N}\}$ and $\{(z_{ik}(1), z_{ik}(2), ..., z_{ik}(m)) : k \in \mathbb{N}\}$ is a subsequence of $\{(z_i(1), z_i(2), ..., z_i(m)) : i \in \mathbb{N}\}$, where each $\varepsilon_j = 1$ or $-1$. Let $B := \varepsilon_1 P_1 + \varepsilon_2 P_2 + \cdots + \varepsilon_{2k} P_{2k}$.

Then $0 \in B \subseteq U$, which verifies that $\mathcal{B}(0)$ is a cs*-network at 0. By Lemma 4.6 $A(X)$ is csf-countable.

**Corollary 4.9.** For a Lašnev space $X$, if $G_2(X)$ is csf-countable, then so is $G_3(X)$.

**Proof.** Since $X$ is a Lašnev space, it is sequential. By a proof analogous to that of the implication (ii) $\Rightarrow$ (iii) in Theorem 4.2, one can show that $X'$ is a separable subspace of $X$. Hence it follows from [10] that $X$ is a paracompact $\aleph_0$-space, since $G_2(X)$ is csf-countable and $X$ is a Lašnev space. Since $X'$ is closed and separable in $X$, $X'$ is Lindelöf, which shows that $X'$ is an $\aleph_0$-space in $X$. Hence $X$ has a countable cs*-network in $X$. Therefore, $F_3(X)$ and $A_3(X)$ are csf-countable by Theorems 4.7 and 4.8 respectively.

**Remark 4.10.** (i) Let $X := D \oplus K$, where $D$ is an uncountable discrete space and $K$ is a compact metric space. Then $A(X)$ is csf-countable. However, $F_3(X)$ and each $A_n(X)$ are first-countable by Theorem 4.5 and Theorem 4.12 in [22], and $F_4(X)$ is not csf-countable by Corollary 4.4.

(ii) In general, the converse of Theorem 4.7 or Theorem 4.8 does not hold. Indeed, let $X$ be an uncountable pseudocompact topological group containing no nontrivial convergent sequences, as given in [19]. By Theorem 3.3, $G(X)$ is snf-countable, and hence $G(X)$ is csf-countable. Since $X$ is an uncountable pseudocompact topological group, $X'$ is uncountable. Then $X'$ must not have a countable cs*-network in $X$. Indeed, assume the contrary that $X'$ has a countable cs*-network in $X$. Then $X' = X$ is an $\aleph_0$-space, hence it is submetrizable. Since a pseudocompact submetrizable space is metrizable, it follows that $X$ is metrizable, which is a contradiction with the assumption.

### 5. Open Questions

We conclude this paper by posing some open questions. Our first open question concerns about the Abelian case of Theorem 4.5. Theorem 3.3 establishes relationships among the snf-countability and the metrizability of $F_4(X)$ and each $F_n(X)$, as well as some properties of $X$ for a fairly large class of topological spaces. It is nice to know whether a similar result on $A_4(X)$ and $A_n(X)$ holds for the same class of spaces. Thus, the following question is of interest.

**Question 5.1.** Let $X$ be a $k$-space with a regular $G_3$-diagonal. If $A_4(X)$ is snf-countable, must every $A_n(X)$ be snf-countable?

In the light of Theorem 4.5 and Theorem 4.12 in [22], Theorem 3.3 and Theorem 4.3 it is natural to pose the following two questions.

**Question 5.2.** Let $X$ be a $k$-space with a regular $G_3$-diagonal. If $G_2(X)$ is snf-countable, must $G_3(X)$ be snf-countable?
Question 5.3. Let $X$ be a non-discrete Lašnev space. If $A_4(X)$ is $csf$-countable, must each $A_n(X)$ be $csf$-countable?

Note that the answers to Question 5.2 and Question 5.3 are not known even when $X$ is a metrizable space. Our last open question concerns about how to characterize the $csf$-countability of $F(X)$ and $A(X)$ in term of properties of $X$.

Question 5.4. Let $X$ be a space. Is there a topological property $\mathcal{A}$ of $X$ which characterizes the $csf$-countability of $F(X)$ or $A(X)$?

References

[1] A.V. Arhangel’skiı, On relations between the invariants of topological groups and their subspaces, Russian Math. Surveys, 35 (1980), 1–23.
[2] A.V. Arhangel’skiı and D.K. Burke, Spaces with a regular $G_δ$-diagonal, Topology Appl., 153 (2006), 1917–1929.
[3] A. Arhangel’skiı and M. Tkachenko, Topological groups and related structures, Atlantis Press, Paris; World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2008.
[4] A.V. Arhangel’skiı, O. G. Okunev and V. G. Pestov, Free topological groups over metrizable spaces, Topology Appl., 33 (1989), 63–76.
[5] R.Z. Buzyakova, On hereditarily normal topological groups, Fund. Math., 219 (2012), 245–251.
[6] R. Engelking, General topology (revised and completed edition), Heldermann Verlag, Berlin, 1989.
[7] S.P. Franklin, Spaces in which sequences suffice, Fund. Math., 57 (1965), 107–115.
[8] G. Gruenhage, Generalized metric spaces, In: K. Kunen, J. E. Vaughan(Eds.), Handbook of set-theoretic topology, North-Holland, Amsterdam, 1984, 423–501.
[9] J.A. Guthrie, A characterization of $R_0$-spaces, General Topology Appl., 1 (1971), 105–110.
[10] H.J. Junnila, Z. Yun, $\aleph_0$-spaces and spaces with a $\sigma$-hereditarily closure preserving $k$-network, Topology Appl., 192 (1999), 209–215.
[11] Z Li, F. Lin and C. Liu, Networks on free topological groups, Topology Appl., 180 (2015), 186–198.
[12] F. Lin and C. Liu, $S_\omega$ and $S_\omega$ on free topological groups, Topology Appl., 176 (2014), 10–21.
[13] S. Lin, On sequence-covering $s$-maps, Adv. in Math. (China), 25 (1996), 548–551.
[14] S. Lin and Y. Tanaka, Point-countable $k$-networks, closed maps, and related results, Topology Appl., 59 (1994), 79–86.
[15] V.I. Malykhin and L.B. Shapiro, Pseudocompact groups without convergent sequences, Math. Notes, 37 (1985), 59–62.
[16] A.A. Markov, On free topological groups, Amer. Math. Soc. Translation, 8 (1962), 195–272.
[17] P. O’Meara, On paracompactness in function spaces with the compact-open topology, Proc. Amer. Math. Soc., 29 (1971), 183–189.
[18] M. Sakai, Function spaces with a countable cs*-network at a point, Topology Appl., 156 (2008), 117–123.
[19] S.M. Sirotá, A product of topological groups, and extremal disconnectedness, Math. USSR-Sb., 8 (1969), 169–180.
[20] M. Tkachenko, More on convergent sequences in free topological groups, Topology Appl., 160 (2013), 1206–1213.
[21] V. Uspenskii, Free topological groups of metrizable spaces, Math. USSR-Izv., 37 (1991), 657–680.
[22] K. Yamada, Metrizable subspaces of free topological groups on metrizable spaces, Topology Proc., 23 (1998), 379–409.
[23] K. Yamada, Fréchet-Urysohn spaces in free topological groups, Proc. Amer. Math. Soc., 130 (2002), 2461–2469.

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