What Data Augmentation Do We Need for Deep-Learning-Based Finance?

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Abstract

The main task we consider is portfolio construction in a speculative market, a fundamental problem in modern finance. While various empirical works now exist to explore deep learning in finance, the theory side is almost non-existent. In this work, we focus on developing a theoretical framework for understanding the use of data augmentation for deep-learning-based approaches to quantitative finance. The proposed theory clarifies the role and necessity of data augmentation for finance; moreover, our theory motivates a simple algorithm of injecting a random noise of strength $\sqrt{r_{t-1}}$ to the observed return $r_t$. This algorithm is shown to work well in practice.

1 Introduction

There is an increasing interest in applying machine learning methods to problems in the finance industry. This trend has been expected for almost forty years [Fama, 1970], when well-documented and fine-grained (minute-level) data of stock market prices became available. In fact, the essence of modern finance is fast and accurate large-scale data analysis [Goodhart and O’Hara, 1997], and it is hard to imagine that machine learning should not play an increasingly crucial role in this field. In contemporary research, the central theme in machine-learning based finance is to apply existing deep learning models to financial time-series prediction problems [Imajo et al., 2020, Ito et al., 2020, Buehler et al., 2019, Jay et al., 2020, Imaki et al., 2021, Jiang et al., 2017, Fons et al., 2020, Lim et al., 2019, Zhang et al., 2020], which have demonstrated the hypothesized usefulness of deep learning for the financial industry.

However, one major existing gap in this interdisciplinary field of deep-learning finance is the lack of a theory relevant to both finance and machine learning. The goal of this work is to propose such a framework, where machine learning practices are analyzed in a traditional financial-economic utility theory setting. We then demonstrate the success and the applicability of such a theoretical framework for designing a simple yet effective data augmentation technique for financial time series. To summarize, our main contributions are (1) to demonstrate how we can use utility theory to analyze practices of deep-learning-based finance, (2) to theoretically study the role of data augmentation in the deep-learning-based portfolio construction problem and (3) to propose a novel and theoretically-motivated machine learning technique for portfolio construction problems. Organization: the next section discusses the main related works; Section 3 provides the requisite finance background for understanding this work; Section 4 presents our theoretical contributions, which is a framework for understanding machine-learning practices in the portfolio construction problem; Section 5 proposes a theoretically motivated algorithm for the task; section 6 validates the proposed theory and methods with experiments. Also see the beginning of the Appendix for a table of contents.

2 Related Works

Existing deep learning finance methods. In recent years, various empirical approaches to apply state-of-the-art deep learning methods to finance have been proposed [Imajo et al., 2020, Ito et al., 2020, Buehler et al., 2019, Jay et al., 2020, Imaki et al., 2021, Jiang et al., 2017, Fons et al., 2020]. The interested readers are referred to [Ozbayoglu et al., 2020] for detailed descriptions of existing works. However, we notice that
one crucial gap is the complete lack of theoretical analysis or motivation in this interdisciplinary field of AI-finance. This work makes one initial step to bridge this gap. One theme of this work is that finance-oriented prior knowledge and inductive bias is required to design the relevant algorithms; Ziyin et al. [2020] shows that incorporating prior knowledge into architecture design is key to the success of neural networks and applied neural networks with periodic activation functions to the problem of financial index prediction. Imajo et al. [2020] shows how to incorporate no-transaction prior knowledge into network architecture design when transaction cost is incorporated.

In fact, most generic and popular machine learning techniques are proposed and have been tested for standard ML tasks such as image classification or language processing. Directly applying the ML methods that work for image tasks is unlikely to work well for financial tasks, where the nature of the data is different. See Figure 1, where we show the performance of a neural network directly trained to maximize wealth return on MSFT during 2019-2020. Using popular, generic deep learning techniques such as weight decay or dropout does not result in any improvement over the baseline. In contrast, the proposed method does. Combining the proposed method with weight decay has the potential to improve the performance a little further, but the improvement is much lesser than the improvement of using the proposed method over the baseline. This implies that a generic machine learning method is unlikely to capture well the inductive biases required to tackle a financial task. The present work proposes to fill this gap by showing how finance knowledge can be incorporated into algorithm design.

**Data augmentation.** Consider a training loss function of the additive form $L = \frac{1}{N} \sum_i \ell(x_i, y_i)$ for $N$ pairs of training data points $\{ (x_i, y_i) \}_{i=1}^N$. Data augmentation amounts to defining an underlying data-dependent distribution and generating new data points stochastically from this underlying distribution. A general way to define data augmentation is to start with a datum-level training loss and transform it to an expectation over an augmentation distribution $P(z|x_i, y_i)$ [Dao et al., 2019], $\ell(x_i, y_i) \rightarrow E_{(z_i, g_i)} \cdot P(z, g|(x_i, y_i)) \left[ \ell(z_i, g_i) \right]$, and the total training loss function becomes

$$L_{aug} = \frac{1}{N} \sum_{i=1}^N E_{(z_i, g_i)} \cdot P(z, g|(x_i, y_i)) \left[ \ell(z_i, g_i) \right].$$

One common example of data augmentation is injecting isotropic gaussian noise to the input [Shorten and Khoshgoftaar, 2019, Fons et al., 2020], which is equivalent to setting $P(z, g|(x_i, y_i)) \sim \delta(g - y_i) \exp \left[ -(z - x_i)^T (z - x_i) / (2\sigma^2) \right]$ for some specified strength $\sigma^2$. Despite the ubiquity of data augmentation in deep learning, existing works are often empirical in nature [Fons et al., 2020, Zhong et al., 2020, Shorten and Khoshgoftaar, 2019, Antoniou et al., 2017]. For a relevant example, Fons et al. [2020] empirically evaluates the effect of different types of data augmentation in a financial series prediction task. Dao et al. [2019] is one major recent theoretical work that tries to understand modern data augmentation theoretically; it shows that data augmentation is approximately learning in a special kernel; He et al. [2019] argues that data augmentation can be seen as an effective regularization. However, no theoretically motivated data augmentation method for finance exists yet. One major challenge and achievement of this work is to develop a theory that bridges the traditional finance theory and machine learning methods. In the next section, we introduce the portfolio theory.

### 3 Background: Markowitz Portfolio Theory

How to make optimal investment in a financial market is the central concern of portfolio theory. Consider a market with an equity (a stock) and a fixed-interest rate bond (a government bond). We denote the price of the equity at time step $t$ as $S_t$, and the price return is defined as $r_t = \frac{S_{t+1} - S_t}{S_t}$, which is a random variable with variance $C_t$, and the expected return $g_t := E[r_t]$. Our wealth at time step $t$ is $W_t = M_t + n_t S_t$, where $M_t$ is the amount of cash we hold, and $n_t$ the shares of stock we hold for the $i$-th stock. As in the standard

![Figure 1: Performance (measured by the Sharpe ratio) of various algorithms on MSFT (Microsoft) from 2018-2020. Directly applying generic machine learning methods, such as weight decay, fails to improve the vanilla model. The proposed method show significant improvement.](image-url)
finance literature, we assume that the shares are infinitely divisible; usually, a positive $n$ denotes holding (long) and a negative $n$ denotes borrowing (short). The wealth we hold initially is $W_0 > 0$, and we would like to invest our money on the equity; we denote the relative value of the stock we hold as $\pi_t = \frac{S_t}{W_t}$; $\pi$ is called a portfolio; the central challenge in portfolio theory is to find the best $\pi$. At time $t$, our wealth is $W_t$; after one time step, our wealth changes due to a change in the price of the stock (setting the interest rate to be 0): $\Delta W_t := W_{t+1} - W_t = W_t \pi_t r_t$. The goal is to maximize the wealth return $G_t := \pi_t \cdot r_t$ at every time step where minimizing risk\(^1\). The risk is defined as the variance of the wealth change:

$$R_t := R(\pi_t) := \text{Var}_{r_t}[G_t] = (\mathbb{E}[r_t^2] - g_t^2) \pi_t^2 = \pi_t^2 C_t.$$  

(2)

The standard way to control risk is to introduce a “risk regularizer” that punishes the portfolios with a large risk [Markowitz, 1959, Rubinstein, 2002].\(^2\) Introducing a parameter $\lambda$ for the strength of regularization (the factor of 1/2 appears for convention), we can now write down our objective:

$$\pi_t^* = \arg \max_{\pi_t} U(\pi) := \arg \max_{\pi_t} \left[ \pi_t^T G_t - \frac{\lambda}{2} R(\pi) \right].$$  

(3)

Here, $U$ stands for the utility function; $\lambda$ can be set to be the desired level of risk-aversion. When $g_t$ and $C_t$ is known, this problem can be explicitly solved. However, one main problem in finance is that its data is highly limited and we only observe one particular realized data trajectory, and $g_t$ and $C_t$ are hard to estimate. This fact motivates for the necessity of data augmentation and synthetic data generation in finance [Assefa, 2020]. In this paper, we treat the case where there is only one asset to trade in the market, and the task of utility maximization amounts to finding the best balance between cash-holding and investment. The equity we are treating is allowed to be a weighted combination of multiple stocks (a portfolio of some public fund manager, for example), and so our formalism is not limited to single-stock situations. In section E.1, we discuss portfolio theory with multiple stocks.

4 Portfolio Construction as a Training Objective

Recent advances have shown that the financial objectives can be interpreted as training losses for an appropriately inserted neural-network model [Ziyin et al., 2019, Buehler et al., 2019]. It should come as no surprise that the utility function (3) can be interpreted as a loss function. When the goal is portfolio construction, we parametrize the portfolio $\pi_t = \pi_w(x_t)$ by a neural network with weights $w$, and the utility maximization problem becomes a maximization problem over the weights of the neural network. The time-dependence is modeled through the input to the network $x_t$, which possibly consists of the available information at time $t$ for determining the future price\(^3\). The objective function (to be maximized) plus a pre-specified data augmentation transform $x_t \rightarrow z_t$ with underlying distribution $p(z|x_t)$ is then

$$\pi_t^* = \arg \max_w \left\{ \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{z_t} \left[ G_t(\pi_w(z_t)) \right] - \lambda \text{Var}_{z_t} \left[ G_t(\pi_w(z_t)) \right] \right\},$$  

(4)

where $\mathbb{E}_z := \mathbb{E}_{z_t \sim p(z|x_t)}$. In this work, we abstract away the details of the neural network to approximate $\pi$. We instead focus on studying the maximizers of this equation, which is a suitable choice when the underlying model is a neural network because one primary motivation for using neural networks in finance is that they are universal approximators and are often expected to find such maximizers [Buehler et al., 2019, Imaki et al., 2021].

The ultimate financial goal is to construct $\pi^*$ such that the utility function is maximized with respect to the true underlying distribution of $S_t$, which can be used as the generalization loss (to be maximized):

$$\pi_t^* = \arg \max_{\pi_t} \{ \mathbb{E}_{S_t} \left[ G_t(\pi) \right] - \lambda \text{Var}_{S_t} \left[ G_t(\pi) \right] \}.$$  

(5)

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\(^1\)It is important to not to confuse the price return $r_t$ with the wealth return $G_t$.

\(^2\)In principle, any concave function in $G_t$ can be a risk regularizer from classical economic theory [Von Neumann and Morgenstern, 1947], and our framework can be easily extended to such cases; one common alternative would be $R(G) = \log(G)$ [Kelly Jr, 2011]: our formalism can be extended to such cases.

\(^3\)It is helpful to imagine $x_t$ as, for example, the prices of the stocks in the past 10 days.
Figure 2: Effect of data augmentation by different noise injections. We see that the proposed data augmentation scheme preserves the structures in the original financial data, while the baseline methods erase such structures.

First Row: APPLE from 2017 to 2020. Left: Raw Return \( r_t \). Mid Left: Noise proportional to \( r_t S_t^2 \) (the proposed strength). Mid Right: Noise proportional to 1 (additive noise). Right: Noise proportional to \( S_t^2 \) (naive multiplicative noise).

Second Row: Effect on the autocorrelation of the price returns. Left: APPLE daily. Mid Left: BITCOIN daily. Mid Right: NASDAQ daily. Right: TESLA minutely.

Note the difference in taking the expectation between Eq (4) and (5) is that \( E_t \) is computed with respect to the training set we hold, while \( E_{S_t} := E_{S_t \sim p(S_t)} \) is computed with respect to the underlying distribution of \( S_t \) given its previous prices. We used the same short-hands for \( \text{Var}_t \) and \( \text{Var}_{S_t} \). Technically, the true utility we defined is an in-sample counterfactual objective, which roughly evaluates the expected utility to be obtained if we restart from yesterday, which is a relevant measure for financial decision making. In Section 4.5, we also analyze the out-of-sample performance when the portfolio is static.

### 4.1 Standard Models of Stock Prices

The expectations in the true objective Equation (5) need to be taken with respect to the true underlying distribution of the stock price generation process. In general, the price follows the following stochastic process

\[
\Delta S_t = f(\{S_{i}\}_{i=1}^{t}) + g(\{S_{i}\}_{i=1}^{t}) \eta_t
\]

for a zero-mean and unit variance random noise \( \eta_t \); the term \( f \) reflects the short-term predictability of the stock price based on past prices, and \( g \) reflects the extent of unpredictability in the price. A key observation in finance is that \( g \) is non-stationary (heteroskedastic) and price-dependent (multiplicative). One model is the geometric Brownian motion (GBM)

\[
S_{t+1} = (1 + r) S_t + \sigma_t S_t \eta_t,
\]

which is taken as the minimal standard model of the motion of stock prices [Mandelbrot, 1997, Black and Scholes, 1973]; this paper also assumes the GBM model as the underlying model. Here, we note that the theoretical problem we consider can be seen as a discrete-time version of the classical Merton's portfolio problem [Merton, 1969]. The more flexible Heston model [Heston, 1993] takes the form

\[
dS_t = r S_t dt + \sqrt{\nu_t} S_t dW_t,
\]

where \( \nu_t \) is the instantaneous volatility that follows its own random walk, and \( \eta_t \) is drawn from a Gaussian distribution. By definition, GBM is a special case of the Heston model with \( \kappa = \xi = 0 \). Despite the simplicity of these models, the statistical properties of these models agree well with the known statistical properties of the real financial markets [Drăgulescu and Yakovenko, 2002]. The readers are referred to [Karatzas et al., 1998] for a detailed discussion about the meaning and financial significance of these models.

### 4.2 No Data Augmentation

In practice, there is no way to observe more than one data point for a given stock at a given time \( t \). This means that it can be very risky to directly train on the raw observed data since nothing prevents the model from overfitting to the data. Without additional assumptions, the risk is zero because there is no randomness in the training set conditioning on the time \( t \). To control this risk, we thus need data augmentation. One can formalize this intuition through the following proposition, whose proof is given in Section E.3.
Proposition 1. (Utility of no-data-augmentation strategy.) Let the price trajectory be generated with GBM in Eq. (6) with initial price $S_0$, then the true utility for the no-data-augmentation strategy is

$$ U_{\text{no-aug}} = \left[ 1 - 2\Phi(-r/\sigma) \right] r - \frac{\lambda}{2} \sigma^2 $$

(7)

where $U(\pi)$ is the utility function defined in Eq. (3); $\Phi$ is the c.d.f. of a standard normal distribution.

This means that, the larger the volatility $\sigma$, the smaller is the utility of the no-data-augmentation strategy. This is because the model may easily overfit to the data when no data augmentation is used. In the next section, we discuss the case when a simple data augmentation is used.

4.3 Additive Gaussian Noise to the Training Set

While it is still far from clear how the stock price is correlated with the past prices, it is now well-recognized that $\text{Var}_{S_t}[S_t|S_{t-1}] \neq 0$ [Mandelbrot, 1997, Cont, 2001]. This motivates a simple data augmentation technique to add some randomness to the financial sequence we observe, $\{S_1, ..., S_{T+1}\}$. This section analyzes a vanilla version of data augmentation of injecting simple Gaussian noise, compared to a more sophisticated data augmentation method in the next section. Here, we inject random Gaussian noise $\xi_t \sim N(0, \rho^2)$ to $S_t$ during the training process such that $z_t = S_t + \xi$. Note that the noisified return needs to be carefully defined since noise might also appear in the denominator, which may cause divergence; to avoid this problem, we define the noisified return to be $\tilde{r}_t := \frac{z_t - S_{t-1}}{\sigma_t}$, i.e., we do not add noise to the denominator. Theoretically, we can find the optimal strength $\rho^*$ of the gaussian data augmentation to be such that the true utility function is maximized for a fixed training set. The result can be shown to be

$$ (\rho^*)^2 = \frac{\sigma^2}{2r} \frac{\sum_t (r_t S_{t-1})^2}{\sum_t \sigma_t^2}. $$

(8)

The fact the $\rho^*$ depends on the prices of the whole trajectory reflects the fact that time-independent data augmentation is not suitable for a stock price dynamics prescribed by Eq. (6), whose inherent noise $\sigma S_t \epsilon_t$ is time-dependent through the dependence on $S_t$. Finally, we can plug in the optimal $\rho^*$ to obtain the optimal achievable strategy for the additive Gaussian noise augmentation. As before, the above discussion can be formalized, with the true utility given in the next proposition (proof in Section E.4).

Proposition 2. (Utility of additive Gaussian noise strategy.) Under additive Gaussian noise strategy, and let other conditions the same as in Proposition 1, the true utility is

$$ U_{\text{Add}} = \frac{r^2}{2\lambda \sigma^2 T} \mathbb{E}_{S_t} \left[ \frac{(\sum_t r_t S_t)^2}{\sum_t (r_t S_t)^2} \Theta \left( \sum_t r_t S_t \right) \right], $$

(9)

where $\Theta$ is the Heaviside step function.

4.4 Multiplicative Gaussian Noise Data Augmentation

In this section, we derive a general kind of data augmentation for the price trajectories specified by the GBM and the Heston model. From the previous discussions, one might expect that a better kind of augmentation should have $\rho = \rho_0 S_t$, i.e., the injected noise should be multiplicative; however, we do not start from imposing $\rho \rightarrow \rho S_t$; instead, we consider $\rho \rightarrow \rho^*$, i.e., a general time-dependent noise. In the derivation, one can find an interesting relation for the optimal augmentation strength:

$$ (\rho^*_{t+1})^2 + (\rho^*_t)^2 = \frac{\sigma^2}{2r} \sigma_t^2. $$

(10)

The following proposition gives the true utility of using this data augmentation (derivations in Section E.5).

Proposition 3. (Utility of general multiplicative Gaussian noise strategy.) Under general multiplicative noise augmentation strategy, and let other conditions the same as in Proposition 1, then the true utility is

$$ U_{\text{mult}} = \frac{r^2}{2\lambda \sigma^2} \left[ 1 - \Phi(-r/\sigma) \right]. $$

(11)
Remark. Combining the above propositions, one can quickly obtain that, if $\sigma \neq 0$, then $U_{\text{mult}} > U_{\text{add}}$ and $U_{\text{mult}} > U_{\text{no-aug}}$ with probability 1 (Proof in Section E.6).

Heston Model and Real Price Augmentation. We also consider the more general Heston model. The derivation proceeds similarly by replacing $\sigma^2 \rightarrow \nu_i^2$; one arrives at the relation for optimal augmentation: $(\rho_{i+1})^2 + (\rho_i)^2 = \frac{1}{2} \nu_i^2 r_1 S_i^2$. One quantity we do not know is the volatility $\nu_i$, which has to be estimated by averaging over the neighboring price returns. One central message from the above results is that one should add noises with variance proportional to $r_1 S_i^2$ to the observed prices for augmenting the training set. Qualitatively, one can immediately check the effectiveness of this proposed technique. See Figure 2 and its related discussion in Section 6.1.

4.5 Stationary Portfolio

In the previous sections, we have discussed the case when the portfolio is dynamic (time-dependent). One slight limitation of the previous theory is that one can only compare the in-sample counterfactual performance of a dynamic portfolio. Here, we alternatively motivate the proposed data augmentation technique when the model is a stationary portfolio. One can show that, for a stationary portfolio, the proposed data augmentation technique gives the overall optimal performance.

Theorem 1. Under the multiplicative data augmentation strategy, the in-sample counterfactual utility and the out-of-sample utility is optimal among all stationary portfolios.

Remark. See Section E.8 for a detailed discussion and the proof. Stationary portfolios are important in financial theory and can be shown to be optimal even among all dynamic portfolios in some situations [Cover and Thomas, 2006, Merton, 1969]. While restricting to stationary portfolios allows us to also compare on out-of-sample performance, the limitation is that a stationary portfolio is less relevant for a deep learning model than the dynamical portfolios considered in the previous sections.

4.6 General Theory

So far, we have been analyzing the data augmentation for specific examples of the utility function and the data augmentation distribution to argue that certain types of data augmentation is preferable. Now we outline how this formulation can be generalized to deal with a wider range of problems, such as different utility functions and different data augmentations. For a general utility function $U = U(x, \pi)$ for some data point $x$ that describes the current state of the market, and $\pi$ that describes our strategy in this market state, we would like to ultimately maximize

$$\max_\pi V(\pi), \text{ for } V(\pi) = \mathbb{E}_x[U(x, \pi)] \quad (12)$$

However, only observing finitely many data points, we can only optimize the empirical loss with respect to some $\theta$-parametrized augmentation distribution $P_\theta$:

$$\hat{\pi}(\theta) = \arg \max_\pi \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}_{z_i \sim p_\theta(z|x_i)}[U(z_i, \pi_i)]. \quad (13)$$

The problem we would like to solve is to find the effect of using such data augmentation on the true utility $V$, and then, if possible, compare different data augmentations and identify the better one. Surprisingly, this is achievable since $V = V(\hat{\pi}(\theta))$ is now also dependent on the parameter $\theta$ of the data augmentation. Note that the true utility has to be found with respect to both the sampling over the test points and the sampling over the $N$-sized training set:

$$V(\hat{\pi}(\theta)) = \mathbb{E}_{x \sim p(x)} \mathbb{E}_{(x_i) \sim p_N(x)}[U(x, \hat{\pi}(\theta))] \quad (14)$$

In principle, this allows one to identify the best data augmentation for the problem at hand:

$$\theta^* = \arg \max_\theta V(\hat{\pi}(\theta)) = \arg \max_\theta \mathbb{E}_{x \sim p(x)} \mathbb{E}_{(x_i) \sim p_N(x)} \left[ U(x, \arg \max_\pi \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}_{z_i \sim p_\theta(z|x_i)}[U(z_i, \pi_i)]) \right], \quad (15)$$
and the analysis we performed in the previous sections is simply a special case of obtaining solutions to this maximization problem. Moreover, one can also compare two different parametric augmentation distributions; let their parameter be denoted as $\theta_\alpha$ and $\theta_\beta$ respectively, then we can say that data augmentation $\alpha$ is better than $\beta$ if and only if $\max_{\theta_\alpha} V(\hat{\pi}(\theta_\alpha)) > \max_{\theta_\beta} V(\hat{\pi}(\theta_\beta))$. This general formulation can also have applicability outside the field of finance because one can interpret the utility $U$ as a standard machine learning loss function and $\pi$ as the model output. This procedure also mimics the procedure of finding a Bayes estimator in the statistical decision theory [Wasserman, 2013], with $\theta$ being the estimator we want to find; we outline an alternative general formulation to find the “minimax” augmentation in Section E.2.

5 Algorithms

Our results strongly motivate for a specially designed data augmentation for financial data. For a data point consisting purely of past prices $(S_t, ..., S_{t+L})$ and the associated returns $(r_t, ..., r_{t+L-1}, r_{t+L})$, we use $x = (S_t, ..., S_{t+L})$ as the input for our model $f$, possibly a neural network, and use $S_{t+L+1}$ as the unseen future price for computing the training loss. Our results suggests that we should randomly noisify both the input $x$ and $S_{t+L+1}$ at every training step by

$$
\begin{align*}
S_i &\rightarrow S_i + \epsilon_i \\
S_i &\rightarrow S_i + \epsilon_i \\
S_{t+L+1} &\rightarrow S_{t+L+1} + \epsilon_i \\
S_{t+L+1} &\rightarrow S_{t+L+1} + \epsilon_i \\
\end{align*}
$$

where $\epsilon_i$ are i.i.d. samples from $\mathcal{N}(0, 1)$, and $c$ is a hyperparameter to be tuned. While the theory suggests that $c$ should be $1/2$, it is better to make it a tunable-parameter in algorithm design for better flexibility; $\hat{\sigma}_t$ is the instantaneous volatility, which can be estimated using standard methods in finance [Degiannakis and Floros, 2015]. One might also assume $\hat{\sigma}$ into $c$.

5.1 Using return as inputs

Practically and theoretically, it is better and standard to use the returns $x = (r_t, ..., r_{t+L-1}, r_{t+L})$ as the input, and the algorithm can be applied in a simpler form:

$$
\begin{align*}
S_i &\rightarrow S_i + c\sqrt{\hat{\sigma}^2_t |r_i|} \epsilon_i \\
S_i &\rightarrow S_i + c\sqrt{\hat{\sigma}^2_t |r_i|} \epsilon_i \\
S_{t+L+1} &\rightarrow S_{t+L+1} + c\sqrt{\hat{\sigma}^2_t |r_i|} \epsilon_i \\
S_{t+L+1} &\rightarrow S_{t+L+1} + c\sqrt{\hat{\sigma}^2_t |r_i|} \epsilon_i \\
\end{align*}
$$

5.2 Equivalent Regularization on the output

One additional simplification can be made by noticing the effect of injecting noise to $r_{t+L}$ on the training loss is equivalent to a regularization. We show in Section D that, under the GBM model, the training objective can be written as

$$
\arg\max_{b_t} \left\{ \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_x \left[ G_t(\pi) \right] - \lambda c^2 \hat{\sigma}^2_t |r_i| \right\},
$$

where the expectation over $x$ is now only taken with respect to the input. This means that the noise injection on the $r_{t+L}$ is equivalent to adding a $L_2$ regularization on the model output $\pi_t$. This completes the main proposed algorithm of this work. We discuss a few potential variants in Section D. Also, it is well known that the magnitude of $|r_t|$ has strong time-correlation (i.e., a large $|r_t|$ suggests a large $|r_{t+1}|$) [Lux and Marchesi, 2000, Cont and Bouchaud, 1997, Cont, 2007], and this suggests that one can also use the average of the neighboring returns to smooth the $|r_t|$ factor in the last term for some time-window of width $\tau$: $|r_t| \rightarrow \hat{|r_t|} = \frac{1}{\tau} \sum_{\tau} |r_{t-\tau}|$. In our S&P500 experiments, we use this smoothing technique with $\tau = 20$.

6 Experiments

We demonstrate the strength of the proposed method experimentally. We start with a qualitative comparison and then move on to benchmark out-of-sample comparisons both on synthetic and real data. We also perform a classical finance theory based analysis of the proposed method to demonstrate how the proposed method can be relevant in a finance-theoretic setting.
6.1 Qualitative Results

See Figure 2, where we compare the augmented data with the raw price return sequence of APPLE from 2017 to 2020. The plotted lines are normalized to have the same overall variance. We see that the theoretically motivated noise (Mid Left) injection strategy generates an augmented data that is visually closest to the original, where the well-known structures of volatility clustering [Cont, 2007, Lux and Marchesi, 2000, 1999] (high-density bumps in the return trajectory) are preserved. In comparison, additive Gaussian noise creates an imbalance in the data, creating larger randomness early in time (when the price is low). We also compare with a naive way of making multiplicative noise injection, which only makes the noise proportional to $S_t^2$ but not $r_t$ (related discussion and derivation in Section E.7); we see that the generated data shrinks the trajectory towards the center and blurs the original structures in the data significantly.

6.2 Benchmark Comparisons

We use the Sharpe ratio as the performance metric (the larger the better). Sharpe ratio is defined as

$$SR_t = \frac{E[\Delta W_t]}{\sqrt{\text{Var}[\Delta W_t]}}$$

which is a measure of the profitability per risk. We choose this metric because it is widely accepted as the best single metric for measuring the performance of a financial strategy in theory and practice [Sharpe, 1966]. Theoretically, it is a classical result in classical financial research that all optimal strategies must have the same Sharpe ratio [Sharpe, 1964] (also called the efficient capital frontier). Practically, the reason for comparing on Sharpe ratio is that it is also the most important indicator of success for a fund manager in the financial industry [Bailey and De Prado, 2014]. For the synthetic tasks, we can generate arbitrarily many test points to compare the Sharpe ratios unambiguously. We then move to experiments on real stock price series; the limitation is that the Sharpe ratio needs to be estimated and involves one additional source of uncertainty.

6.2.1 Geometric Brownian Motion

We first start from experimenting with stock prices generated with a GBM, as specified in Eq. (6), and we generate a fixed price trajectory with length $T = 400$ for training; each training point consists of a sequence of past prices $(S_t, ..., S_{t+9}, S_{t+10})$ where the first ten prices are used as the input to the model, and $S_{t+10}$ is used for computing the loss. We use a feedforward neural network with the number of neurons $10 \rightarrow 64 \rightarrow 64 \rightarrow 1$ with ReLU activations. Training proceeds with the Adam optimizer with a minibatch size of 64.

Results and discussion. See Figure 3. The proposed method is plotted in blue. The middle figure compares the proposed method with simple training with no data augmentation and training with weight decay. We see that the proposed method outperforms the baseline methods, suggesting that the proposed method has incorporated the correct prior knowledge of the problem. The right figure compares the proposed method with the other two baseline data augmentations we studied in this work. As the theory expects, the proposed method performs better. We also experiment with the Heston model in the B.5, and similar results are obtained.

6.2.2 S&P 500 Prices

This section demonstrates the applicability of the proposed algorithm to real market data. In particular, We use the data from S&P500 from 2016 to 2020, with 1000 days in total. We test on the 365 stocks that existed on S&P500 from 2000 to 2020. We use the first 800 days as the training set and the last 200 days...
is roughly 0 a better MCL than the original stocks. The slope of the SP500 MCL lack.
used to visualize and better understand the machine learning methods example also shows that how tools in classical finance theory can be yield from 2018 to 2020.
is to the upper left in the return-risk plane. See Figure 4. The risk-free return and lower risk and is considered to be better than an MCL that such as the government bond; an MCL with smaller slope means better pricing model [Sharpe, 1964], a foundational theory in classical finance. In this section, we link the result we obtained in the previous section does not perform well either. This is because the market during the time 2019 – 2020 is volatile and quite different from the previous years, and a stationary portfolio cannot capture the nuances in the change of the market condition. This shows that it is also important to leverage the flexibility and generalization property of the modern neural networks, along side the financial prior knowledge.

### 6.3 Market Capital Lines

In this section, we link the result we obtained in the previous section with the concept of market capital line (MCL) in the capital asset pricing model [Sharpe, 1964], a foundational theory in classical finance. The MCL of a set of portfolios denotes the line of the best return-risk combinations when these portfolios are combined with a risk-free asset such as the government bond; an MCL with smaller slope means better return and lower risk and is considered to be better than an MCL that is to the upper left in the return-risk plane. See Figure 4. The risk-free rate \( r_f \) is set to be 0.01, roughly equal to the average 1-year treasury yield from 2018 to 2020.\(^4\) We see that the learned portfolios achieves a better MCL than the original stocks. The slope of the SP500 MCL is roughly 0.53, while that of the proposed method is 0.35, i.e., much better return-risk combinations can be achieved using the proposed method. For example, if we specify the acceptable amount of risk to be 0.1, then the proposed method can result in roughly 10% more gain in annual return than investing in the best stock in the market. This example also shows that how tools in classical finance theory can be used to visualize and better understand the machine learning methods that are applied to finance, a crucial point that many previous works lack.

![Figure 4: Available portfolios and the market capital line (MCL). The black dots are the return-risk combinations of the original stocks; the orange dots are the learned portfolios. The MCL of the proposed method is lower than that of the original stocks, suggesting improved return and lower risk.](https://www.treasury.gov/resource-center/data-chart-center/interest-rates/pages/textview.aspx?data=yield.)

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\(^4\)https://www.treasury.gov/resource-center/data-chart-center/interest-rates/pages/textview.aspx?data=yield.

| Industry Sectors          | # Stock | Merton no aug. | weight decay | additive aug. | naive mult. | proposed |
|---------------------------|--------|----------------|--------------|---------------|-------------|-----------|
| Communication Services    | 39     | −0.09 ± 0.04   | −0.06 ± 0.04 | −0.09 ± 0.27  | 0.22 ± 0.18 | 0.29 ± 0.21 | 0.33 ± 0.16 |
| Consumer Discretionary    | 39     | −0.01 ± 0.03   | −0.07 ± 0.03 | −0.06 ± 0.10  | 0.18 ± 0.10 | 0.14 ± 0.09 | 0.64 ± 0.08  |
| Consumer Staples          | 37     | 0.05 ± 0.03    | 0.24 ± 0.03  | 0.23 ± 0.11   | 0.36 ± 0.08 | 0.34 ± 0.09 | 0.35 ± 0.07  |
| Energy                    | 17     | 0.07 ± 0.03    | 0.03 ± 0.03  | −0.02 ± 0.12  | 0.70 ± 0.09 | 0.52 ± 0.10 | 0.38 ± 0.09  |
| Financials                | 46     | −0.57 ± 0.04   | −0.61 ± 0.03 | −0.61 ± 0.09  | −0.06 ± 0.10| −0.13 ± 0.09| 0.18 ± 0.08  |
| Health Care               | 44     | 0.23 ± 0.04    | 0.60 ± 0.04  | 0.64 ± 0.11   | 0.86 ± 0.09 | 0.81 ± 0.09 | 0.83 ± 0.07  |
| Industrials               | 44     | −0.09 ± 0.03   | −0.11 ± 0.03 | −0.11 ± 0.08  | 0.36 ± 0.08 | 0.28 ± 0.08 | 0.48 ± 0.08  |
| Information Technology    | 41     | 0.41 ± 0.04    | 0.41 ± 0.04  | 0.41 ± 0.11   | 0.07 ± 0.10 | 0.74 ± 0.11 | 0.79 ± 0.09  |
| Materials                 | 19     | 0.07 ± 0.03    | 0.06 ± 0.03  | 0.03 ± 0.14   | 0.47 ± 0.13 | 0.43 ± 0.13 | 0.53 ± 0.10  |
| Real Estate               | 22     | −0.14 ± 0.04   | −0.39 ± 0.03 | −0.30 ± 0.12  | −0.01 ± 0.06| −0.00 ± 0.06| 0.19 ± 0.07  |
| Utilities                 | 24     | −0.29 ± 0.02   | −0.29 ± 0.02 | −0.28 ± 0.07  | −0.01 ± 0.06| −0.00 ± 0.06| 0.15 ± 0.04  |
| **S&P500 Avg.**           | 365    | −0.02 ± 0.04   | −0.00 ± 0.04 | −0.01 ± 0.04  | 0.39 ± 0.03 | 0.39 ± 0.03 | 0.51 ± 0.03  |
7 Outlook

In this work, we have presented a theoretical framework relevant to finance and machine learning to understand and analyze methods related to deep-learning-based finance. The result is a machine learning algorithm incorporating prior knowledge about the underlying financial processes. The good performance of the proposed method agrees with the standard expectation in machine learning that performance can be improved if the right inductive biases are incorporated. The limitation of the present work is obvious; we only considered the kinds of data augmentation that takes the form of noise injection. Other kinds of data augmentation may also be useful to the finance; for example, [Fons et al., 2020] empirically finds that magnify [Um et al., 2017], time warp [Kamycki et al., 2020], and SPAWNER [Le Guennec et al., 2016] are helpful for financial series prediction, and there is yet no theoretical understanding of why these methods suit the financial tasks; a correct theoretical analysis of these methods is likely to advance both the deep-learning based techniques for finance and our fundamental understanding of the underlying financial and economic mechanisms. Moreover, A lot is known about finance in the form of “stylized facts” [Cont, 2001], and these universal empirical facts should also serve as a basis for inspiring more finance-oriented algorithm design. Meanwhile, our understanding of the underlying financial dynamics is also rapidly advancing; we foresee better methods to be designed, and it is likely that the proposed method will be replaced by better algorithms soon. There is potentially positive social effects of this work because it is widely believed that designing better financial prediction methods can make the economy more efficient by eliminating arbitrage [Fama, 1970]; the cautionary note is that this work is only for the purpose of academic research, and should not be taken as an advice for monetary investment, and the readers should evaluate their own risk when applying the proposed method. Lastly, It is the sincere hope of the authors that this work can attract more attention to the rapidly growing field of AI-finance.

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A Experiments

This section describes the additional experiments and the experimental details in the main text. The experiments are all done on a single TITAN RTX GPU. The S&P500 data is obtained from Alphavantage. The code will be released on github.

B Why not Other Metrics?

This paper has used the Sharpe Ratio as the primary metric of objective comparison. A plethora of alternative metrics in previous works. We discuss a few of the mainly used alternatives and explain why we think they are unfit for our problem:

**Prediction Accuracy of Up-Down Motion of Price.** Portfolio construction is not the same as predicting future price, and it, therefore, does not apply. Even if one cannot forecast future price, portfolio construction remains a meaningful task for safe investment. Even for the works that do involve direct price forecasting, we do not recommend using accuracy as the main metric because most of the existing works, to our knowledge, cannot go beyond 52% in accuracy, and it is quite unconvincing when one tries to compare an accuracy of 51.5% to an accuracy of 51.7%, for example.

**Total Wealth Return:** Some existing works compare on total wealth return, but we do not think it is an appropriate metric for portfolio construction. It is a fundamental result in finance that high return cannot be achieved simultaneously with low risk, and therefore it is theoretically unjustified and meaningless to compare total wealth return.

B.1 Dataset Construction

For all the tasks, we observe a single trajectory of a single stock prices $S_1, ..., S_T$. For the toy tasks, $T = 400$; for the S&P500 task, $T = 800$. We then transform this into $T - L$ input-target pairs $\{(x_i, y_i)\}_{i=1}^{T-L}$, where

$$
\begin{align*}
x_i &= (S_i, ..., S_{L-1}); \\
y_i &= S_L.
\end{align*}
$$

(19)

$x_i$ is used as the input to the model for training; $y_i$ is used as the unseen future price for calculating the loss function. For the toy tasks, $L = 10$; for the S&P500 task, $L = 15$. In simple words, we use the most recent $L$ prices for constructing the next-step portfolio.

B.2 Sharpe Ratio for S&P500

The empirical Sharpe Ratios are calculated in the standard way (for example, it is the same as in [Ito et al., 2020, Imajo et al., 2020]). Given a trajectory of wealth $W_1, ..., W_T$ of a strategy $\pi$, the empirical Sharpe ratio is estimated as

$$
R_i = \frac{W_{i+1}}{W_i} - 1;
$$

(20)

$$
\hat{M} = \frac{1}{T} \sum_{i=1}^{T-1} R_i;
$$

(21)

$$
\hat{SR} = \frac{\hat{M}}{\sqrt{\frac{1}{T} \sum_{i=1}^{T} R_i^2 - \hat{M}^2}} \quad \text{average wealth return} \quad \text{std. of wealth return},
$$

(22)

and $\hat{SR}$ is the reported Sharpe Ratio for S&P500 experiments.

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5https://www.alphavantage.co/documentation/
Figure 5: Daily return $r_t$ for BITCOIN from 2018 to 2021.

Figure 6: Daily return $r_t$ for NASDAQ from 2018 to 2021.

Figure 7: Effect of data augmentation by different noise injections. **Left:** Raw minutely return $r_t$ for TESLA from 2021−4−9 to 2021−4−13. **Mid Left:** Noise proportional to $r_t S^2_t$ (the proposed strength). **Mid Right:** Noise proportional to 1. **Right:** Noise proportional to $S^2_t$.

### B.3 Variance of Sharpe Ratio

We do not report an uncertainty for the single stock Sharpe Ratios, but one can easily estimate the uncertainties. The Sharpe Ratio is estimated across a period of $T$ time steps. For the S&P500 stocks, $T = 200$, and by the law of large numbers, the estimated mean $\hat{M}$ has variance roughly $\sigma^2/T$, where $\sigma$ is the true volatility, and so is the estimated standard deviation. Therefore, the estimated Sharpe Ratio can be written as

$$\hat{SR} = \frac{\hat{M}}{\sqrt{\frac{1}{T} \sum_{i=1}^{T} R_i^2 - \hat{M}^2}}$$

$$= \frac{M + \frac{\sigma}{\sqrt{T}} \epsilon}{\sigma + \frac{\sigma}{\sqrt{T}} \eta}$$

$$\approx \frac{M + \frac{\sigma}{\sqrt{T}} \epsilon}{\sigma} = \frac{M}{\sigma} + \frac{1}{\sqrt{T}} \epsilon$$

where $\epsilon$ and $\eta$ are zero-mean random variables with unit variance. This shows that the uncertainty in the estimated $\hat{SR}$ is approximately $1/\sqrt{T} \approx 0.07$ for each of the single stocks, which is often much smaller than the difference between different methods.
Figure 8: Experiment on Heston model; $S_0 = 1$, $r = 0.005$, $\sigma = 0.01$, $\kappa = 0.25$, $\theta = 0.04$, $\rho = 0$. **Left:** Examples of prices trajectories in green; the black line shows the expected the price at the time. **Mid:** Comparison of the proposed data augmentation against other baselines. **Right:** Comparison against other alternative noise-based data augmentations. Result obtained after averaging over 500 independent price trajectories.

**B.4 More on Qualitative Comparison**

Here, we show more visual comparisons of the effects of data augmentation. See Figure 5 for an application to the daily data on BITCOIN from 2020 to 2021. Again, we see that additive Gaussian noise creates an imbalance on the trajectory; naive multiplicative noise blurs the structures in the original data, while the theoretically motivated augmentation is close to the original in structure.

The same conclusion can be reached for the daily return of the NASDAQ index from 2018 to 2021 (last 1000 days) and the minutely data for TESLA. See Figure 6 and Figure 7.

**B.5 Portfolio Construction Experiments with Heston Model**

Here we compare the case when the underlying price trajectory is generated by a Heston model. As in [Buehler et al., 2019], we generate the price trajectories with the discretized version of the Heston model. We generate a fixed price trajectory with length $T = 400$ for training; each training point consists of a sequence of past prices $(S_t, ..., S_{t+9}, S_{t+10})$ where the first ten prices are used as the input to the model, and $S_{t+10}$ is used for computing the loss. We use a feedforward neural network with the number of neurons $10 \rightarrow 64 \rightarrow 64$ to 1 with ReLU as the activation function. Training proceeds with the Adam optimizer with minibatch size 64. This setting is the same as in Section 6.2.1.

**Results and discussion.** As can be seen from Figure 8, the result is qualitatively the same as the GBM case. The proposed method outperforms the baseline methods steadily.

**B.6 S&P500 Experiment**

This section gives more results and discussion of the S&P500 experiment.

**B.6.1 Underperformance of Weight Decay**

This section gives the detail of the comparison made in Figure 1. The experimental setting is the same as the S&P500 experiments. For illustration and motivation, we only show the result on MSFT (Microsoft). Choosing most of the other stocks would give a qualitatively similar plot.

See Figure 1, where we show the performance of directly training a neural network to maximize wealth return on MSFT during 2018-2020. Using popular, generic deep learning techniques such as weight decay or dropout does not improve the baseline. In contrast, the proposed method does. Combining the proposed method with weight decay has the potential to improve the performance a little further, but the improvement is much lesser than the improvement of using the proposed method over the baseline. This implies that generic machine learning is unlikely to capture the inductive bias required to process a financial task.

In the plot, we did not interpolate the dropout method between a large $p$ and a small $p$. The result is similar to the case of weight decay in our experiments.
Table 2: Sharpe ratio on S&P 500 when short sell is forbidden. We show the beginning 30 stocks on our list due to space constraints.

| SYMBOL | Merton’s Portfolio | no augmentation | weight decay | additive aug. | naive multiplicative | proposed |
|--------|--------------------|------------------|--------------|---------------|----------------------|-----------|
| A      | 0.57               | 0.57             | 0.57         | 0.82          | 0.79                 | 0.80      |
| APL    | 1.55               | 1.55             | 1.55         | 2.56          | 2.14                 | 1.97      |
| ABC    | 0.54               | 0.54             | 0.54         | 0.72          | 0.73                 | 0.65      |
| ABT    | 0.50               | 0.50             | 0.50         | 0.73          | 0.56                 | 0.66      |
| ADBE   | 1.35               | 1.35             | 1.35         | 0.30          | 1.41                 | 1.47      |
| ADI    | 0.16               | 0.16             | 0.16         | 0.57          | 0.48                 | 0.63      |
| ADM    | -0.41              | -0.41            | -0.41        | 0.10          | -0.01                | 0.21      |
| ADP    | -0.45              | -0.45            | -0.45        | -0.10         | -0.18                | 0.12      |
| ADSK   | 1.11               | 1.11             | 1.11         | 1.12          | 1.47                 | 1.53      |
| AEE    | -0.09              | -0.09            | -0.09        | -0.02         | -0.02                | 0.07      |
| AEP    | -0.33              | -0.33            | -0.33        | -0.14         | -0.21                | -0.02     |
| AES    | -0.17              | -0.17            | -0.17        | 0.43          | 0.40                 | 0.32      |
| AFL    | -0.80              | -0.80            | -0.80        | -0.14         | -0.34                | 0.14      |
| AIG    | -1.27              | -1.30            | -1.30        | -0.15         | 0.14                 | 0.31      |
| AIV    | -0.70              | -0.70            | -0.70        | -0.18         | -0.12                | 0.17      |
| AJG    | 0.31               | 0.31             | 0.31         | 0.53          | 0.51                 | 0.79      |
| AKAM   | 0.48               | 0.48             | 0.48         | 0.62          | 0.62                 | 0.45      |
| ALB    | -0.23              | 0.59             | 0.55         | 0.87          | 0.77                 | 0.92      |
| ALL    | -0.02              | -0.02            | -0.02        | 0.17          | 0.28                 | 0.61      |
| ALXN   | 0.01               | 0.16             | 0.05         | 0.62          | 0.48                 | 0.53      |
| AMAT   | 0.57               | 0.57             | 0.57         | 1.01          | 0.86                 | 1.07      |
| AMD    | 1.84               | 1.84             | 1.84         | 0.25          | 2.54                 | 0.51      |
| AME    | 0.06               | 0.06             | 0.06         | 0.30          | 0.27                 | 0.65      |
| AMG    | 0.36               | 0.37             | 0.24         | 0.33          | 0.41                 | 0.43      |
| AMGN   | 1.00               | 1.00             | 1.00         | 1.34          | 1.30                 | 1.19      |
| AMT    | 0.41               | 0.41             | 0.41         | 0.43          | 0.59                 | 0.45      |
| AMZN   | 1.40               | 1.40             | 1.40         | 0.39          | 1.39                 | 0.97      |
| ANSS   | 0.97               | 0.97             | 0.97         | 0.55          | 1.17                 | 1.20      |
| AON    | 0.30               | 0.30             | 0.30         | 0.75          | 0.62                 | 0.73      |
| AOS    | -0.32              | -0.32            | -0.32        | -0.02         | -0.13                | 0.10      |
| APA    | 0.09               | 1.32             | 1.03         | 1.79          | 1.60                 | 1.70      |
| APD    | 0.57               | 0.57             | 0.57         | 0.83          | 0.87                 | 0.66      |
| APH    | 0.04               | 0.04             | 0.04         | 0.55          | 0.39                 | 0.77      |
| ARE    | 0.05               | 0.05             | 0.05         | 0.28          | 0.36                 | 0.31      |
| ATV1   | 1.29               | 1.29             | 1.29         | 0.94          | 1.20                 | 1.08      |

S&P500  | -0.02 ± 0.04        | -0.00 ± 0.04     | -0.01 ± 0.04 | 0.39 ± 0.03   | 0.35 ± 0.03          | 0.51 ± 0.03|

B.6.2 Additional Results

In this section, we show more results for the S&P500 experiment. See Figure 2. Due to space constraints, we show the thirty beginning stocks on our list (see the next section). The augmentation strength hyperparameters for the three kinds of data augmentation is set to be the theoretical optimal values found in the theory section, given by Eq (8), (97) and (10) respectively. The weight decay hyperparameter is searched for in the range \(10^{-1}, 10^{-2}, ..., 10^{-6}\). For Merton’s portfolio, we do not forbid short selling since it is a closed-form solution. As shown, using weight decay does not seem to improve the no-data-augmentation baseline beyond a chance level. We also show the result when short selling is allowed in Figure 3. We see that the proposed method still performs the best, but the gap and the performance drop. This is possibly because the market is on average in a growing trend, so forbidding short selling might be a good inductive bias. In Warren Buffett’s words, “you can’t make big money shorting because the risk of big losses means you can’t make big bets.”
Table 3: Sharpe ratio on S&P 500 when short sell is allowed. We show the beginning 30 stocks on our list due to space constraints.

| SYMBOL | no augmentation | weight decay | additive aug. | naive multiplicative | proposed |
|--------|-----------------|--------------|----------------|-----------------------|-----------|
| A      | 0.57            | 0.57         | 0.19           | -0.10                 | 0.39      |
| AAPL   | 1.55            | 1.55         | 2.53           | 2.18                  | 1.65      |
| ABC    | 0.54            | 0.54         | 0.17           | -0.03                 | -0.36     |
| ABT    | 0.50            | 0.50         | 0.30           | 0.48                  | 0.37      |
| ADBE   | 1.35            | 1.35         | 1.32           | 1.17                  | 0.30      |
| ADI    | 0.16            | 0.16         | 0.44           | 0.37                  | 0.81      |
| ADM    | -0.41           | -0.41        | 0.42           | 0.10                  | 0.88      |
| ADP    | -0.45           | -0.45        | -0.56          | -0.49                 | -0.19     |
| ADSK   | 1.11            | 1.11         | 0.90           | 0.59                  | -0.26     |
| AEE    | -0.09           | -0.09        | -0.21          | -0.24                 | 0.22      |
| AEP    | -0.33           | -0.33        | -0.35          | -0.37                 | 0.64      |
| AES    | -0.17           | -0.17        | -0.25          | -0.00                 | -0.24     |
| AFL    | -0.80           | -0.80        | -0.34          | -0.68                 | -0.36     |
| AIG    | -1.29           | -1.29        | 2.15           | -0.72                 | 0.03      |
| AIV    | -0.70           | -0.70        | -0.16          | -0.29                 | 0.32      |
| AJG    | 0.31            | 0.31         | 0.26           | 0.20                  | 0.75      |
| AKAM   | 0.48            | 0.48         | 0.44           | 0.58                  | 0.41      |
| ALB    | -0.02           | -0.07        | 0.26           | 0.31                  | 0.45      |
| ALL    | -0.02           | -0.02        | -0.73          | -0.54                 | -0.38     |
| ALXN   | 0.01            | 0.15         | 0.73           | 0.28                  | 0.53      |
| AMAT   | 0.57            | 0.57         | 0.37           | 0.43                  | 0.73      |
| AMD    | 1.84            | 1.84         | -0.49          | -0.86                 | 0.22      |
| AME    | 0.06            | 0.06         | -0.01          | -0.02                 | 0.15      |
| AMG    | 0.44            | 0.46         | 1.84           | 0.85                  | 1.23      |
| AMGN   | 1.00            | 1.00         | 0.04           | -0.12                 | -0.68     |
| AMT    | 0.41            | 0.41         | 0.39           | 0.40                  | 0.49      |
| AMZN   | 1.40            | 1.40         | 1.48           | 1.18                  | 1.14      |
| ANSS   | 0.97            | 0.97         | 0.69           | 0.97                  | 0.58      |
| AON    | 0.30            | 0.30         | 0.32           | 0.16                  | 1.32      |
| AOS    | -0.32           | -0.32        | 0.66           | 0.09                  | 1.58      |
| APA    | 2.46            | 1.71         | 2.36           | 2.80                  | 2.27      |
| APD    | 0.57            | 0.57         | 0.23           | 0.30                  | 0.16      |
| APH    | 0.04            | 0.04         | 0.01           | -0.13                 | 0.62      |
| ARE    | 0.05            | 0.05         | 0.40           | 0.08                  | -0.26     |
| ATVI   | 1.29            | 1.29         | -1.17          | 1.02                  | 1.24      |
| S&P500 | 0.05 ± 0.04     | 0.08 ± 0.04  | 0.32 ± 0.04    | 0.16 ± 0.04           | 0.38 ± 0.04 |
Figure 9: Case study of the performance of the model on MSFT from 2019 August to 2020 May. We see that the model learns to invest less and less as the price of the stock rises to an unreasonable level, thus avoiding the high risk of the market crash in February 2020.

B.6.3 Case Study

In this section, we qualitatively study the behavior of the learned portfolio of the proposed method. The model is trained as in the other S&P500 experiments. See Figure 9. We see that the model learns to invest less and less as the stock price rises to an excessive level, thus avoiding the high risk of the market crash in February 2020. This avoidance demonstrates the effectiveness of the proposed method qualitatively.

B.6.4 List of Symbols for S&P500

The following are the symbols we used for the S&P500 experiments, separated by quotation marks.

[A 'AAPL' 'ABC' 'ABT' 'ADBE' 'ADI' 'ADM' 'ADP' 'ADSK' 'AEE' 'AEP' 'AFL' 'AIG' 'AIV' 'AJG' 'AKAM' 'ALB' 'ALL' 'ALXN' 'AMAT' 'AMD' 'AME' 'AMG' 'AMGN' 'AMT' 'AMZN' 'ANSS' 'AON' 'AOS' 'APA' 'APD' 'APH' 'ARE' 'AVTN' 'AVB' 'AVY' 'AXP' 'AZO' 'BA' 'BAC' 'BAX' 'BBT' 'BBY' 'BDX' 'BEN' 'BII'B 'BK' 'BKG' 'BLK' 'BLM' 'BRK.B' 'BSX' 'BWA' 'BXP' 'C' 'CAG' 'CAH' 'CAT' 'CB' 'CCI' 'CCL' 'CDNS' 'CERN' 'CHD' 'CHRW' 'CI' 'CINF' 'CL' 'CLX' 'CMA' 'CMCSA' 'CMI' 'CMS' 'CNP' 'COF' 'COG' 'COO' 'COP' 'COST' 'CPB' 'CSCO' 'CSX' 'CTAS' 'CTL' 'CTSH' 'CTXS' 'CVS' 'CVX' 'D' 'DE' 'DGX' 'DHI' 'DM' 'DIS' 'DLR' 'DOV' 'DRE' 'DRI' 'DT' 'DVN' 'EA' 'EBAY' 'ECL' 'ED' 'EFX' 'EIX' 'EL' 'EMN' 'EMR' 'EOG' 'EQR' 'EQT' 'ES' 'ESS' 'ETFC' 'ETN' 'ETR' 'EW' 'EXC' 'EXPD' 'F' 'FAST' 'FCX' 'FDX' 'FE' 'FFIV' 'FISV' 'FITB' 'FL' 'FLIR' 'FLS' 'FMC' 'FRT' 'GD' 'GE' 'GILD' 'GIS' 'GLW' 'GPC' 'GPS' 'GS' 'GT' 'GWW' 'HAL' 'HAS' 'HBAN' 'HCP' 'HD' 'HES' 'HIG' 'HOG' 'HOLX' 'HON' 'HP' 'HPQ' 'HRB' 'HRL' 'HRS' 'HSIC' 'HST' 'HSY' 'HUM' 'IBM' 'IDXX' 'IFF' 'INCY' 'INTC' 'INTU' 'IP' 'IPG' 'IRM' 'IT' 'ITW' 'IVZ' 'JBHT' 'JCI' 'JEC' 'JNJ' 'JNPR' 'JPM' 'JWN' 'K' 'KEY' 'KLAC' 'KMB' 'MKM' 'KO' 'KR' 'KSS' 'KU' 'L' 'LB' 'LEG' 'LEN' 'LF' 'LMT' 'LNC' 'LNT' 'LOW' 'LRCX' 'LUV' 'M' 'MAA' 'MAC' 'MAR' 'MAS' 'MAT' 'MCD' 'MCHP' 'MCK' 'MCO' 'MDT' 'MET' 'MGM' 'MHK' 'MKC' 'MLM' 'MMC' 'MMT' 'MO' 'MOS' 'MRK' 'MRO' 'MS' 'MSFT' 'MSI' 'MTB' 'MTD' 'MU' 'MYL' 'NB' 'NEE' 'NEM' 'NI' 'NKE' 'NKTR' 'NOC' 'NOV' 'NSC' 'NTAP' 'NTRS' 'NUE' 'NVDA' 'NWL' 'O' 'OKE' 'OMC' 'ORCL' 'ORLY' 'OXY' 'PAYX' 'PBCT' 'PCAR' 'PCG' 'PEG' 'PEP' 'PE' 'PG' 'PGR' 'PH' 'PHM' 'PKG' 'PKI' 'PLD' 'PNG' 'PNR' 'PNW' 'PPG' 'PPL' 'PRGO' 'PSA' 'PVH' 'PWR' 'PXD' 'QCOM' 'RCL' 'RE' 'REG' 'REGN' 'RF' 'RFJ' 'RL' 'RMD' 'ROL' 'ROP' 'ROST' 'RRC' 'RSN' 'SBAC' 'SBUX' 'SCHW' 'SEE' 'SHW' 'SIVB' 'SLM' 'SLG' 'SNA' 'SNPS' 'SO' 'SPG' 'SRCL' 'SRE' 'STT' 'STZ' 'SWK' 'SWKS' 'SYK' 'SYMC' 'SYY' 'T' 'TAP' 'TGT' 'TIF' 'TXJ' 'TMK' 'TMO' 'TROW' 'TRV' 'TSCK' 'TSN' 'TTWO' 'TXN' 'TXT' 'UDR' 'UHS' 'UNH' 'UNP' 'UPS' 'URI' 'USB' 'UTX' 'VAR' 'VFC' 'VLO' 'YM' 'YMC' 'YNO' 'YRNS' 'VRTX' 'VTR' 'VZ' 'WAT' 'WBA' 'WDC' 'WEC' 'WFC' 'WHR' 'WM' 'WMB' 'WMT' 'WY' 'XEL' 'XLN' 'XOM' 'XRAY' 'XR' 'YUM' 'ZION']
C Related works

Classical finance and portfolio theory. The optimal strategy of investing is the central concern of modern portfolio theory. Modern finance theory originates from the 1952 Markowitz paper that formalizes the problem of portfolio construction as a constrained utility maximization problem under risk constraint [Markowitz, 1959]. Sharpe then developed the capital asset pricing model (CAPM), which shows that all optimal portfolios should have the same Sharpe ratio [Sharpe, 1964]. Moreover, it is then empirically shown that the Sharpe ratio has a strong correlation with the performance of the financial companies [Sharpe, 1966]. Later developments and more detailed discussions can be found in any standard textbook [Karatzas et al., 1998].

D Additional Discussion of the Proposed Algorithm

D.1 Derivation

We first derive Equation 18. The original training loss is

$$\frac{1}{T} \sum_{t=1}^{T} E_t [G_t(\pi)] - \lambda \text{Var}_t[G_t(\pi)].$$

(26)

The last term can be written as

$$\lambda \text{Var}_t[G_t(\pi)] = E_{z_1, \ldots, z_t}[z_t^2 \pi_t^2] - E_{z_1, \ldots, z_t}[z_t \pi_t]^2$$

(27)

$$= E_{z_t}[z_t^2]E_{z_1, \ldots, z_{t-1}}[\pi_t^2] - E_{z_t}[z_t]^2E_{z_1, \ldots, z_{t-1}}[\pi_t]^2$$

(28)

$$= \lambda \sigma_t^2 E_{z_1, \ldots, z_{t-1}}[\pi_t^2] + \lambda c^2 \sigma_t^2 |r_t| E_{z_1, \ldots, z_{t-1}}[\pi_t] - \lambda \sigma_t^2 E_{z_1, \ldots, z_{t-1}}[\pi_t]^2$$

(29)

$$= \lambda \sigma_t^2 \text{Var}_{z_1, \ldots, z_{t-1}}[\pi_t] + \lambda c^2 \sigma_t^2 |r_t| E_{z_1, \ldots, z_{t-1}}[\pi_t]^2$$

(30)

Plug in, this leads to the following maximization problem, which is the desired equation.

$$\arg \max_{b_t} \left( \frac{1}{T} \sum_{t=1}^{T} E_t [G_t(\pi)] - \lambda \sigma_t^2 \text{Var}_{z_1, \ldots, z_{t-1}}[\pi_t] - \lambda c^2 \sigma_t^2 |r_t| \pi_t^2 \right),$$

(31)

where we have given each term a name for reference; the expectation is taken with respect to the augmented data points $z_t$:

$$r_i = z_i = r_i + c \sqrt{\sigma_t^2 |r_t|} \epsilon_i \quad \text{for} \quad r_i \in x.$$

(32)

Under the GBM model (or when the optimal portfolio only weakly depends on $z_1, \ldots, z_{t-1}$), the optimal $\pi_t$ does not depend on $z_1, \ldots, z_{t-1}$, and so the objective can be further simplified to be

$$\arg \max_{b_t} \left( \frac{1}{T} \sum_{t=1}^{T} E_t [G_t(\pi)] - \lambda c^2 \sigma_t^2 |r_t| \pi_t^2 \right);$$

(33)

the first term is the data augmentation for wealth gain, and the second term is a regularization for risk control. Most of the experiments in this paper use this equation for training. When it does not work well, the readers are encouraged to try the full training objective in Equation 31.

D.2 Extension to Multi-Asset Setting

It is possible and interesting to derive the data augmentation for a multi-asset setting. However, this is hindered by the lack of a standard model to describe the co-motion of multiple stocks. For example, it is unsure what the geometric Brownian motion should be for a multi-stock setting. In this case, we tentatively suggest the following form of the formula for the injected noise, whose effectiveness and theoretical
justification are left for future work. Let $S_t = (S_{1,t}, ..., S_{N,t})$ be the prices of the stocks viewed as an $N$-dimensional vector. The return is assumed to have covariance $\Sigma$, then, by analogy with the discovery of this work:

$$S_{i,t} \rightarrow S_{i,t} + c \sqrt{\sum_j \Sigma_{ij}r_j S_j^2 \epsilon_t}$$

(34)

for some white gaussian noise $\epsilon$ and some tunable parameter $c$. The matrix $\Sigma$ has to be estimated by the data using standard methods of estimating multi-stock volatility.

D.3 Non-Gaussian Noise

While the proposed algorithm proposes to use Gaussian noise for injection; the theory developed in this work only requires the noise to have a finite second moment and, therefore, any other distribution works as well for the particular kind of utility function we specified in Eq. (5). This is because this utility function only contains the second moments of the wealth return and is indifferent to higher moments. Therefore, there is one caveat to the present theory: when the utility function involves higher moments of the wealth return, the utility of a certain type of noise injection is not indifferent to the choice of the injection distribution. The practitioners are recommended to analyze the specific utility function they use in our framework and decide on the strength and distribution of the injected noise.
E Additional Theory and Proofs

This section contains all the additional theoretical discussions and the proofs.

E.1 Background: Classical Solution to the Portfolio Construction Problem

Consider a market with $N$ stocks, we denote the price of all these stocks as $S_t \in \mathbb{R}^N$, and one can likewise define the stock price return as

$$ r_t = \frac{S_{t+1} - S_t}{S_t}. $$

(35)

which is a random variable with the covariance matrix $C_t := \text{Cov}[r_t]$ and the expected return $g_t := \mathbb{E}[r_t]$. Our wealth at time step $t$ is defined as $W_t = M_t + \sum_{i=1}^N n_{t,i}S_t$ where $M_t$ is the amount of cash we hold, and $n_{t,i}$ the shares of stock we hold for the $i$-th stock; we also defined vectors $n_t := (n_{t,1}, ..., n_{t,N}, 1)$ and $S_t := (S_{t,1}, ..., S_{t,N}, M_t)$ where the cash is included in the definition of $n_t$. As in the standard finance literature, we assume that the shares are infinitely divisible; usually, a positive $n_t$ denotes holding (long) and a negative $n$ denotes borrowing (short). The wealth we hold initially is $W_0 > 0$, and we would like to invest our money on $N$ stocks; we denote the relative value of each stock we hold also as a vector $\pi_{t,i} = \frac{n_{t,i}S_t}{W_t} \in \mathbb{R}^{N+1}$; $\pi$ is called a portfolio; the central challenge in portfolio theory is to find the best $\pi$.

At time $t$, our relative wealth is $W_t$; after one time step, our wealth changes due to a change in the price of the stocks: $\Delta W_t := W_{t+1} - W_t = W_t \pi_t \cdot r_t$. The standard goal is to maximize the wealth return $G_t := \pi_t \cdot r_t$ at every time step while minimizing risk.$^6$. The risk is defined as the variance of the wealth change$^7$:

$$ R_t := R(\pi_t) := \text{Var}_{r_t}[G_t] = \pi_t^T \left( \mathbb{E}[r_t r_t^T] - g_t g_t^T \right) \pi_t = \pi_t^T C_t \pi_t. $$

(36)

The standard way to control risk is to introduce a “risk regularizer” that punishes the portfolios with a large risk [Markowitz, 1959, Rubinstein, 2002]. Introducing a parameter $\lambda$ for the strength of regularization (the factor of 1/2 appears for convention), we can now write down our objective:

$$ \pi_t^* = \arg \max_{\pi} U(\pi) := \arg \max_{\pi} \left[ \pi_t^T G_t - \frac{\lambda}{2} R(\pi) \right]. $$

(37)

Here, $U$ stands for the utility function; $\lambda$ can be set to be the desired level of risk-aversion. When $g_t$ and $C_t$ is known, this problem can be explicitly solved (see Section E.1). However, one main problem in finance is that its data is very limited and we only observe one particular realized data trajectory, and, therefore, $g_t$ and $C_t$ cannot be accurately estimated.

Eq. (37) can be solved directly by taking derivative and set to 0; we can obtain

$$ \begin{cases} \pi_t^* = \frac{1}{\lambda} C_t^{-1} g_t; \\ R_t(\pi_t^*) = \frac{1}{2\lambda} g_t^T C_t^{-1} g_t. \end{cases} $$

(38)

This formula is the standard formula to use when both $g_t$ and $C_t$ are known or can be accurately estimated [Bouchaud and Potters, 2009]. Meanwhile, when one finds difficulty estimating $g_t$ or, more importantly, $C_t$, then the above formula can go arbitrarily wrong. Let $\hat{C}$ denote our estimated covariance and $\hat{g}$ the estimated mean$^8$, then the in-sample risk and the true is respectively given by

$$ \begin{cases} \tilde{R}_t(\pi_t^*) = \frac{1}{\lambda} \hat{g}_t^T \hat{C}_t^{-1} \hat{g}_t; \\ \hat{R}_t(\pi_t^*) = \frac{1}{2\lambda} \hat{g}_t^T \hat{C}_t^{-1} \hat{C}_t \hat{g}_t. \end{cases} $$

(39)

The readers are encouraged to examine the differences between the two equations carefully.

---

$^6$It is important to not to confuse the price return $r_t$ with the wealth return $G_t$.

$^7$In principle, any concave function in $G_t$ can be a risk regularizer from classical economic theory [Von Neumann and Morgenstern, 1947], and our framework can be easily extended to such cases; one common alternative would be $R(G) = \log(G)$.

$^8$For example, using some Bayesian machine learning model.


| Financial Terms | Statistical Terms |
|-----------------|-------------------|
| Utility $U$     | Loss function $L$ |
| Expected Utility $V$ | Risk $R$ |
| Data Augmentation Parameter $\theta$ | Estimator $\hat{\theta}$ |
| True Parameter $\Omega$ | True Parameter $\theta$ |
| Prior $p(\Omega)$ | Prior $p(\theta)$ |

Table 4: Correspondence table between the concepts in our general theory and the classical statistical decision theory.

### E.2 Analogy to Statistical Decision Theory and the Minimax Formulation

In Section 4.6, we mentioned that the procedure we used is analogous to the process of finding a Bayesian estimator in a statistical decision theory. Here, we explain this analogy a little more (but keep in mind that this is only an analogy, not a rigorous equivalence relation). Equation 14 of empirical utility can be seen as an equivalent of the statistical risk function $R$; finding the optimal data augmentation strength is similar to finding the best Bayesian estimator. To make an exact agreement with the statistical decision theory, we also need to define a prior over the risk in Equation 14:

$$r := E_{p(\Omega)}[V(\hat{\pi}(\theta))] = E_{p(\Omega)}E_{x \sim p(x;\Omega)}E_{\{x_i\} \sim p^n(x)}[U(x, \hat{\pi}(\theta))]$$

(40)

where we have written the distributions $p(x; \Omega)$ as a function of the true parameters in our underlying model. In the main text, we have effectively assumed that $p(\Omega)$ is a Dirac delta distribution, but, in the more general case, it is possible that the true parameter is not known or cannot be accurately estimated, and it makes sense to assign a prior distribution to them. One can then find the optimal data augmentation with respect to $r$: $\theta^* = \arg \max_{\theta} r$.

See table 4 for the list of correspondences. This analogy breaks down at the following point: the Bayesian estimator tries to find $\hat{\theta}$ that is as close to $\theta$ as possible, while, in our formulation, the goal is not to make data augmentation $\theta$ as close as possible to $\Omega$.

One might also hope to establish an analogous “minimax” theory for the portfolio construction problem. This can be done simply by replacing the expectation over $p(\Omega)$ with a minimization over $\Omega$:

$$r_{\text{minimax}} := \min_{\Omega} V(\hat{\pi}(\theta))$$

(41)

and the best augmentation parameter can be found as the maximizer of this risk: $\theta^* = \max_\theta \min_\Omega V$.

### E.3 Proof for no data augmentation

When there is no data augmentation, $E_t[G_t(\pi)] = b_t r_t$ and $\text{Var}_t[G_t(\pi)] = 0$. One immediately see that the utility is then maximized at

$$\pi^*_t = \begin{cases} 1, & \text{if } r_t \geq 0 \\ -1, & \text{if } r_t < 0. \end{cases}$$

(42)

We restate the theorem here.

**Proposition 4.** (Utility of no-data augmentation strategy.) Let the strategy be as specified in Eq. (42), and let the price trajectory be generated with GBM in Eq. (6) with initial price $S_0$, then the true utility is

$$U = [1 - 2 \Phi(-r/\sigma)]r - \frac{\lambda}{2} \sigma^2$$

(43)

where $U(\pi)$ is the utility function defined in Eq. (3).

---

9For example, in the GBM model, the true parameters are the growth rate $r$ and the volatility $\sigma$, and so $\Omega = (r, \sigma)$, and $p(\Omega) = p(r, \sigma)$.
Proof. For a time-dependent strategy $\pi_t^*$, the true utility is defined as\textsuperscript{10}

$$U(\pi^*) = \mathbb{E}_{S_0', S_1', \ldots, S_T'} \left[ \frac{1}{T} \sum_{t=1}^{T} \pi_t^* r_t^* - \left( \frac{\lambda}{2T} \sum_{t=1}^{T} (\pi_t^* r_t^*)^2 - \mathbb{E}_{S_0', S_1', \ldots, S_T'} (r_t^*)^2 \right) \right]$$  \hspace{1cm} (44)

where $S_0', S_1', \ldots, S_T'$ is an independently sampled distribution for testing, and $r_t^* := \frac{S_{t+1}' - S_t'}{S_t'}$ are their respective returns. Now, we note that we can write the price update equation (the GBM model) in terms of the returns:

$$S_{t+1} = (1 + r_t) S_t + \sigma_t \eta_t \rightarrow r_t = r + \sigma \eta_t$$  \hspace{1cm} (45)

which means that $r_t \sim \mathcal{N}(r, \sigma^2)$ obeys a Gaussian distribution. Therefore,

$$U(\pi^*) = \frac{r}{T} \sum_{t=1}^{T} \pi_t^* - \frac{\lambda \sigma^2}{2T} \sum_{t=1}^{T} (\pi_t^*)^2.$$  \hspace{1cm} (46)

Now we would like to average over $\pi_t^*$, because we also want to average over the realizations of the training set to make the true utility independent of the sampling of the training set (see Section 4.6 for an explanation).

Recall that the strategy is defined as

$$\pi_t^* = \begin{cases} 1, & \text{if } r_t \geq 0 \\ -1, & \text{if } r_t < 0. \end{cases} = \Theta(r_t \geq 1) - \Theta(r_t < 1)$$  \hspace{1cm} (47)

for a training set $\{S_0, \ldots, S_T\}$, and $\Theta$ is the Heaviside step function. We thus have that

$$\mathbb{E}_{S_1, \ldots, S_T} [\pi_t^*]^2 = \mathbb{E}_{S_1, \ldots, S_T} [\Theta(r_t \geq 0) - \Theta(r_t < 0)] = 1 - 2\Phi(-r/\sigma)$$  \hspace{1cm} (48)

where $\Phi$ is the Gauss c.d.f. We can use this to average the utility over the training set; noticing that the training set and the test set are independent, we can obtain

$$U = \mathbb{E}_{S_1, \ldots, S_T} [U(\pi^*)]$$  \hspace{1cm} (49)

$$= \frac{1}{T} \sum_{t=1}^{T}[1 - 2\Phi(-r/\sigma)]r - \frac{\lambda}{2} \frac{1}{T} \sum_{t=1}^{T} \sigma^2$$  \hspace{1cm} (50)

$$= [1 - 2\Phi(-r/\sigma)]r - \frac{\lambda}{2} \sigma^2.$$  \hspace{1cm} (51)

This finishes the proof. \qed

\textbf{E.4 Proof for Additive Gaussian noise}

Before we prove the proposition, we first prove that the strategy is indeed the one given in Eq. (52):

\textbf{Lemma 1.} \textit{The maximizer of the utility function in Eq. 4 with additive gaussian noise is}

$$\pi_t^*(\rho) = \begin{cases} \frac{r_t S_t^2}{2\lambda \rho^2}, & \text{if } -1 < \frac{r_t S_t^2}{2\lambda \rho^2} < 1; \\ \text{sgn}(r_t), & \text{otherwise.} \end{cases}$$  \hspace{1cm} (52)

\textit{Proof.} With additive Gaussian noise, we have

$$\mathbb{E}_t [G_t(\pi)] = \pi_t \mathbb{E}_t [\tilde{r}_t] = \pi_t \mathbb{E}_t \left[ \frac{S_{t+1} + \rho r_t + \rho \epsilon_t - S_t - \rho \epsilon_t}{S_t} \right] = \pi_t \frac{S_{t+1} - S_t}{S_t} = \pi_t r_t;$$

$$\text{Var}_t [G_t(\pi)] = \pi_t^2 \text{Var}_t [\tilde{r}_t] = \pi_t^2 \text{Var}_t \left[ \frac{\tilde{r}_t + \epsilon_t - \rho \epsilon_t}{S_t} \right] = \frac{2\rho^2 \pi_t^2}{S_t^2},$$

\textsuperscript{10}While we mainly use $\Theta(x)$ as the Heaviside step function, we overload this notation a little. When we write $\Theta(x > 0)$, $\Theta$ is defined as the indicator function. We think that this is harmless because the difference is clearly shown by the argument to the function.
where the last line follows from the definition for additive Gaussian noise that $\rho_t = ... \rho_T = \rho$. The training objective becomes

$$\pi^*_t = \arg \max_{\pi_t} \left\{ \frac{1}{T} \sum_{t=1}^{T} E_t \left[ G_t(\pi) \right] - \frac{\lambda}{2} \text{Var}_t[\pi_t(\pi)] \right\}$$

$$= \arg \max_{\pi_t} \left\{ \frac{1}{T} \sum_{t=1}^{T} \pi_t r_t - \frac{\lambda}{2} \rho^2 \frac{\pi^2_t}{S_t^2} \right\}.$$  \hspace{1cm} (54)

This maximization problem can be maximized for every $t$ respectively. Taking derivative and set to 0, we find the condition that $\pi^*_t$ satisfies

$$\frac{\partial}{\partial \pi_t} \left( \pi_t r_t - \lambda \frac{\rho^2 \pi^2_t}{S_t^2} \right) = 0$$

$$\rightarrow \pi^*_t(\rho) = \frac{r_t S_t^2}{2 \lambda \rho^2}.$$  \hspace{1cm} (56)

By definition, we also have $|\pi_t| \leq 1$, and so

$$\pi^*_t(\rho) = \begin{cases} \frac{r_t S_t^2}{2 \lambda \rho^2}, & \text{if } -1 < \frac{r_t S_t^2}{2 \lambda \rho^2} < 1; \\ \text{sgn}(r_t), & \text{otherwise}, \end{cases}$$  \hspace{1cm} (57)

which is the desired result. □

We would like to comment that, although we paid special attention to enforcing the constraint $|\pi_t| \leq 1$ it is often not needed in practice because investors tend to be quite risk-averse, and it is hard to imagine that any investor would invest all his or her money in the financial market such that $\pi_t = 1$. Mathematically, this means that it is often the case that $\lambda \geq \frac{|r_t| S_t^2}{2 \lambda \rho^2}$. Therefore, for what comes, we assume that $\lambda \geq \frac{|r_t| S_t^2}{2 \lambda \rho^2}$ for all $t$ for notational simplicity; note that, even without assumption, the conclusion that a multiplicative noise is the better kind of data augmentation will not change. Now we are ready to prove the proposition.

**Proposition 5.** (Utility of additive Gaussian noise strategy.) Let the strategy be as specified in Eq. (52), and other conditions the same as in Proposition 1, then the true utility is

$$U_{\text{Add}} = \frac{r^2}{2 \sigma^2 T} \mathbb{E}_t \left[ \frac{(\sum_{t=1}^{T} r_t S_t)^2}{\sum_{t=1}^{T} (r_t S_t^2)^2} \Theta(\sum_{t=1}^{T} r_t S_t^2) \right].$$  \hspace{1cm} (59)

**Proof.** The beginning of the proof is similar to the case with no data augmentation. Following the same procedure, we obtain an equation that is the same as Eq. (46):

$$U(\pi^*) = \frac{r}{T} \sum_{t=1}^{T} \pi^*_t - \frac{\lambda \sigma^2}{2T} \sum_{t=1}^{T} \left( \pi^*_t \right)^2.$$  \hspace{1cm} (60)

Plug in the preceding lemma, we have

$$U(\pi^*) = \frac{r}{T} \sum_{t=1}^{T} \frac{r_t S_t^2}{2 \lambda \rho^2} - \frac{\lambda \sigma^2}{2T} \sum_{t=1}^{T} \left( \frac{r_t S_t^2}{2 \lambda \rho^2} \right)^2.$$  \hspace{1cm} (61)

This utility is a function of the data augmentation strength $\rho$. For a fixed training set, we would like to find the best $\rho$ that maximizes the above utility. Note that the maximizer of the utility is different depending on the sign of $\sum_{t=1}^{T} r_t S_t^2$. Taking derivative and set to 0, we obtain that

$$(\rho^*)^2 = \begin{cases} \frac{\lambda \sigma^2}{2r} \sum_{t=1}^{T} (r_t S_t^2)^2, & \text{if } \sum_{t=1}^{T} r_t S_t^2 > 0 \\ \infty, & \text{otherwise}. \end{cases}$$  \hspace{1cm} (62)

Plug in to the previous lemma, we have

$$\pi^*_t(\rho^*) = \frac{r_t S_t^2}{2 \lambda (\rho^*)^2} = \frac{r_t S_t^2}{\sigma^2} \sum_{t=1}^{T} \frac{r_t S_t^2}{\sum_{t=1}^{T} (r_t S_t^2)^2} \Theta(\sum_{t=1}^{T} r_t S_t^2).$$  \hspace{1cm} (63)
One thing to notice is that the optimal strength is independent of $\lambda$, which is an arbitrary value and dependent only on the investor’s psychology. Plug into the utility function and take expectation with respect to the training set, we obtain

$$U_{\text{add}} = E_{S_1, \ldots, S_{T+1}} \left[ U(\pi^*(\rho^*)) \right]$$

$$= U(\pi^*) = \frac{r}{T} \sum_{t=1}^{T} \pi^*_t(\rho^*) - \frac{\lambda \sigma^2}{2T} \sum_{t=1}^{T} \left( \pi^*_t(\rho^2) \right)^2$$

$$= \frac{r^2}{2\lambda \sigma^2 T} E_{S_1, \ldots, S_{T+1}} \left[ \left( \sum_t r_t S_t \right)^2 \right]$$

This finishes the proof. $\square$

**Remark.** Notice that the term in the expectation generally depends on $T$ in a non-trivial way and cannot be obtained explicitly. However, it does not cause a problem since the final goal is to compare it with the result in the next section.

### E.5 Proof for General Multiplicative Gaussian noise

Before we prove the proposition, we first find the strategy for this case. Note that the term $\rho^2 + \rho_{t+1}^2$ appears repetitively in this section, and so we define a shorthand notation for it: $\frac{1}{2} (\rho^2 + \rho_{t+1}^2) = \gamma^2_t$.

**Lemma 2.** The maximizer of the utility function in Eq. 4 with multiplicative gaussian noise is

$$\pi^*_t(\rho) = \begin{cases} \frac{r_t S^2_t}{2\lambda \rho^2 + \rho_{t+1}^2} & \text{if } -1 < \frac{r_t S^2_t}{2\lambda \rho^2 + \rho_{t+1}^2} < 1; \\ \text{sgn}(r_t) & \text{otherwise.} \end{cases}$$

**Proof.** With additive Gaussian noise, we have

$$\begin{align*}
\mathbb{E}_i [G_i(\pi)] &= \pi_i \mathbb{E}_i \left[ \hat{r}_t \right] = \pi_i \mathbb{E}_i \left[ \frac{S_t + \rho_t + \rho_{t+1} - S_t - \rho_t}{S_t} \right] = \pi_i \frac{S_{t+1} - S_t}{S_t} = \pi_i r_t; \\
\mathbb{Var}_i [G_i(\pi)] &= \pi_i^2 \mathbb{Var}_i \left[ \hat{r}_t \right] = \pi_i^2 \mathbb{Var}_i \left[ \frac{\rho_t + \rho_{t+1} - \rho_t}{S_t} \right] = \frac{(\rho_t^2 + \rho_{t+1}^2)}{S_t^2} \pi_t^2.
\end{align*}$$

We see that $\frac{1}{2} (\rho^2 + \rho_{t+1}^2) = \gamma^2_t$ replaces the role of $2\rho^2$ for additive Gaussian noise. The training objective becomes

$$\pi^*_t = \arg \max_{\pi_t} \left\{ \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_i [G_i(\pi)] - \frac{\lambda}{2} \mathbb{Var}_i [G_i(\pi)] \right\}$$

$$= \arg \max_{\pi_t} \left\{ \frac{1}{T} \sum_{t=1}^{T} \pi_t r_t - \frac{\lambda}{2} \frac{\pi_t^2}{\gamma^2_t} \right\}.$$ 

This maximization problem can be maximized for every $t$ respectively. Taking derivative and set to 0, we find the condition that $\pi^*_t$ satisfies

$$\frac{\partial}{\partial \pi_t} \left( \pi_t r_t - \frac{\lambda}{2} \frac{\pi_t^2}{\gamma^2_t} \right) = 0$$

$$\rightarrow \pi^*_t(\gamma_t) = \frac{r_t S^2_t}{2\lambda \gamma^2_t}.$$

By definition, we also have $|\pi_t| \leq 1$, and so

$$\pi^*_t(\rho) = \begin{cases} \frac{r_t S^2_t}{2\lambda \rho^2 + \rho_{t+1}^2} & \text{if } -1 < \frac{r_t S^2_t}{2\lambda \rho^2 + \rho_{t+1}^2} < 1; \\ \text{sgn}(r_t) & \text{otherwise}, \end{cases}$$

which is the desired result. $\square$
Remark. For fair comparison with the previous section, we also assume that \( \lambda \geq \frac{|r|\sigma^2}{2\gamma_t^2} \). Again, this is the same as assume that the investors are reasonably risk-averse and is the correct assumption for all practical circumstances.

Proposition 6. (Utility of general multiplicative Gaussian noise strategy.) Let the strategy be as specified in Eq. (52), and other conditions the same as in Proposition 1, then the true utility is

\[
U_{\text{mult}} = \frac{\lambda^2}{2\gamma_t^2} \left[ 1 - \Phi(-r/\sigma) \right].
\]  

Proof. Most of the proof is similar to the Gaussian case by replacing \( \rho^2 \) with \( \gamma_t^2 \). Following the same procedure, We have:

\[
U(\pi^*) = \frac{r}{T} \sum_{t=1}^T \pi_t^* - \frac{\lambda^2}{2T} \sum_{t=1}^T (\pi_t^*)^2.
\]  

Plug in the preceding lemma, we have

\[
U(\pi^*) = \frac{r}{T} \sum_{t=1}^T \pi_t^* - \frac{\lambda^2}{2T} \sum_{t=1}^T \left( \frac{r_t S_t^2}{2\gamma_t^2} \right)^2.
\]  

This utility is a function of the data augmentation strength \( \gamma_t \), and, unlike the additive Gaussian case, can be maximized term by term for different \( t \). For a fixed training set, we would like to find the best \( \gamma_t \) that maximizes the above utility. Note that the maximizer of the utility is different depending on the sign of \( r_t S_t^2 \). Taking derivative and set to 0, we obtain that

\[
(\gamma_t^*)^2 = \begin{cases} 
\frac{\pi^2}{2\lambda \gamma_t^2}, & \text{if } r_t S_t^2 > 0 \\
\infty, & \text{otherwise.}
\end{cases}
\]  

Plug in to the previous lemma, we have

\[
\pi_t^*(\gamma_t^*) = \frac{r_t S_t^2}{2\lambda (\gamma_t^*)^2} = \frac{r}{\lambda \sigma^2} \Theta(r_t).
\]  

One thing to notice that the optimal strength is independent of \( \lambda \), which is an arbitrary value and dependent only on the psychology of the investor. Plug into the utility function and take expectation with respect to the training set, we obtain

\[
U_{\text{add}} = \mathbb{E}_{S_1, \ldots, S_T} [U(\pi^*(\rho^*))]
\]

\[
= U(\pi^*) = \frac{r}{T} \sum_{t=1}^T \pi_t^* \gamma_t^* - \frac{\lambda^2}{2T} \sum_{t=1}^T (\pi_t^*)^2
\]

\[
= \frac{r^2}{2\lambda \sigma^2} [\Theta(r_t) - \Phi(-r/\sigma)]
\]

\[
= \frac{r^2}{2\lambda \sigma^2} \Phi(r/\sigma)
\]  

This finishes the proof. \( \square \)

This result can be directly compared to the results in the previous section, and the following remark shows that the multiplicative noise injection is the best kind of noise.

Remark. (Infinite augmentation strength.) For all of the theoretical results, there is a corner case when the optimal injection strength is equal to infinity, which leads to a non-investing portfolio \( \pi = 0 \). This case requires special interpretation. This corner case is due to the fact the underlying model we use has a constant, positive expected price return equal to \( r \), and so it leads to the bizarre data augmentation which effectively amounts to throwing away all the training points with \( < 0 \) return. This is unnatural for a real market. It is possible for the real market to have short-term negative return when conditioned on the previous prices, and so one should not simply discard the negative points. Therefore, in our algorithm section, we recommend treating the training points with positive and negative return equally by taking the absolute value of the data augmentation strength and ignoring \( \infty \) case.

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E.6 Proof of Remark

Remark. Combining the above propositions, one can quickly obtain that, if $\sigma \neq 0$, then $U_{\text{mult}} > U_{\text{add}}$ and $U_{\text{mult}} > U_{\text{no-aug}}$ with probability 1 (Proof in Appendix).

Proof. We first show that $U_{\text{mult}} > U_{\text{add}}$. Recall that

$$U_{\text{Add}} = \frac{r^2}{2\lambda\sigma^2 T} E_S \left[ \frac{(\sum_t r_t S_t)^2}{\sum_t (r_t S_t)^2} \right] \Theta \left( \sum_t r_t S_t^2 \right)$$

(84)

$$\leq \frac{r^2}{2\lambda\sigma^2 T} E_S \left[ \frac{(\sum_t r_t S_t \Theta(r_t > 0))^2}{\sum_t (r_t S_t)^2} \right] \Theta \left( \sum_t r_t S_t^2 \right)$$

(85)

$$\leq \frac{r^2}{2\lambda\sigma^2 T} E_S \left[ \frac{(\sum_t r_t S_t \Theta(r_t > 0))^2}{\sum_t (r_t S_t)^2} \right]$$

(86)

$$\leq \text{(Cauchy Inequality)} \frac{r^2}{2\lambda\sigma^2 T} E_S \left[ \sum_t (r_t S_t \Theta(r_t > 0))^2 \right]$$

(87)

$$= \frac{r^2}{2\lambda\sigma^2 T} \sum_t \Theta(r_t > 0)$$

(88)

$$= \frac{r^2}{2\lambda\sigma^2 T} \sum_t \mathbb{P}(r_t > 0)$$

(89)

$$= \frac{r^2}{2\lambda\sigma^2} \Phi(r/t) = U_{\text{mult}}.$$  

(90)

The Cauchy equality holds if and only if $S_1 = \ldots = S_{T+1}$; this event has probability measure 0, and so, with probability 1, $U_{\text{add}} < U_{\text{mult}}$.

Now we prove the second inequality. Recall that

$$U_{\text{no-aug}} = [1 - 2\Phi(-r/\sigma)]r - \frac{1}{2} \frac{\lambda}{\sigma^2}.$$  

(91)

We divide into 2 subcases. Case 1: $\lambda > \frac{r}{\sigma}$. We have

$$U_{\text{no-aug}} < -2r\Phi(-r/\sigma) < 0 < U_{\text{mult}}.$$  

(92)

Case 2: $0 < \lambda \leq \frac{r}{\sigma}$. We have

$$U_{\text{no-aug}} < [1 - 2\Phi(-r/\sigma)]r < \Phi(r/\sigma)r \leq \frac{r^2}{2\lambda\sigma^2} \Phi(r/t) = U_{\text{mult}}.$$  

(93)

(94)

(95)

This finishes the proof. □

E.7 Augmentation for a naive multiplicative noise

In the discussion and experiment sections in the main text, we also mentioned a “naive” version of the multiplicative noise. The motivation for this kind of noise is simple, since the underlying noise in the theoretical models are all of the form $\sigma^2 S^2_t$, and so it is tempting to also inject noise that mimicks the underlying noise. It turns out that this is not a good idea.

In this section, we let $\rho_t = \rho_0 S_t$ for some positive, time-independent $\rho_0$. Our goal is to find the optimal $\rho_0$. With the same calculation, one finds that the learned portfolio is given by the same formula in Lemma 2:

$$\pi^*_t(\rho) = \frac{\rho_t S^2_t}{2\lambda\gamma^2_t}.$$  

(96)

With this strategy, one can find the optimal noise injection strength to be given by the following proposition.
Proposition 7. Let the portfolio be given by Eq. (96) and let the price be generated by the GBM, then the optimal noise strength is

\[(\rho_0^*)^2 = \begin{cases} \frac{\sum_t r_t}{\Sigma_t^{-1} r_t}, & \text{if } \sum_t r_t > 0 \\ \infty, & \text{otherwise.} \end{cases} \]  

(97)

Proof. As before, We have:

\[U(\pi^*) = r T_T - \frac{\lambda}{2T} \sum_t (\pi_t^*)^2. \]  

(98)

Plug in the portfolio, we have

\[U(\pi^*) = \frac{r}{T} \sum_t S_t - \frac{\lambda}{2T} \sum_t \left( \frac{r_t S_t^2}{2\lambda \gamma_t^2} \right)^2. \]  

(99)

Plug in \(\gamma_t^2 = \rho_0^2 S_t^2\) and take derivative, we obtain that

\[(\rho_0^*)^2 = \begin{cases} \frac{\sum_t r_t}{\Sigma_t^{-1} r_t}, & \text{if } \sum_t r_t > 0 \\ \infty, & \text{otherwise.} \end{cases} \]  

(100)

This finishes the proof. □

E.8 Data Augmentation for a Stationary Portfolio

While the main theory focused on the case with a dynamic portfolio that is updated through time, we also present a study of the stationary portfolio in this section. While this kind of portfolio is less relevant for deep-learning-based finance, we study this case to show that, even in this setting, there is still strong motivation to inject noise of strength \(r_t S_t^2\). Now we state the formal definition of a stationary portfolio.

Definition 1. A portfolio \(\{\pi_t\}_{t=1}^T\) is said to be stationary if \(\pi_t = \pi\) for some constant \(\pi\) for all \(t\).

In the language of machine learning, this corresponds to choosing our model as having only a single parameter, whose output is input independent:

\[f(x) = \pi. \]  

(101)

In traditional finance theory, stationary portfolios have been very important. In practice, most portfolios are “approximately” stationary, since most portfolio managers tend to not to change their portfolio weight at a very short time-scale unless the market is very unstable due to market failure or external information injection. For a stationary portfolio, one still would like to maximize the utility function given in Eq. 4.

For conciseness, we only examine the case when we inject a general time-dependent noise. The curious readers are encouraged to examine the cases with no data augmentation and with constant data augmentation. As before, the following lemma gives the portfolio of the empirical utility. Again, we use the shorthand: \(\frac{1}{2}(\rho_t^2 + \rho_{t+1}^2) = \gamma_t^2\). For illustrative purpose, we have ignore the corner cases of \(\pi\) being greater than 1 or smaller than -1.

Lemma 3. The stationary portfolio that maximizes the utility function in Eq. 4 with multiplicative gaussian noise is

\[\pi_t^*(\rho) = \frac{\sum_t r_t}{2\lambda \sum_t (\gamma_t^2 / S_t^2)} \]  

(102)

Proof sketch. The proof follows almost the same as the previous sections. With a slight difference that \(\pi\) is no more time-dependent and can be taken out of the sum.

Proposition 8. Let the portfolio be given in Eq. (102), then an augmentation strength satisfying the relation

\[(\gamma_t^*)^2 = c r_t S_t^2 \]  

(103)

is an optimal data augmentation for constant \(c = \frac{2\sigma^2}{r}\).
Proof Sketch. The proof is also simple and very similar to the proofs for the dynamic portfolio case. One first solve for the optimal augmentation strength $\gamma^*_t$ and find that it satisfies the following relation

$$
\sum_t \left( \frac{\gamma^*_t}{S^2_t} \right)^2 = c \sum_t r_t
$$

(104)

with $c = \frac{2\sigma^2}{r}$, and then it suffices to check that the following is one solution

$$
(\gamma^*_t)^2 = cr_t S^2_t.
$$

(105)

This finishes the proof sketch. □

The curious readers are encouraged to check the intermediate steps. We see that, even for a stationary portfolio case, there is still strong motivation for using a augmentation with strength proportional to $r_t S^2_t$.

We would like to compare this with the best achievable stationary portfolio, which is solved by the following proposition.

**Proposition 9. (Optimal Stationary Portfolio).** The optimal stationary portfolio for GBM is $\pi^*_{stat} = \frac{r}{\lambda \sigma^2}$, i.e., for any other portfolio $\pi$, $U(\pi^*_{stat}) \geq U(\pi)$.

**Proof.** It suffices to find the maximizer portfolio of the true utility:

$$
\pi^*_{stat} = \arg \max_{\pi} \left\{ \pi r - \frac{\lambda}{2} \pi^2 \sigma^2 \right\}.
$$

(106)

The solution is simple and given by

$$
\pi^*_{stat} = \frac{r}{\lambda \sigma^2}.
$$

(107)

This completes the proof. □

Note that the above optimality result holds for both in-sample counterfactual utility and out-of-sample utility. This proposition can be seen as the discrete-time version of the famous Merton’s portfolio solution [Merton, 1969], where the optimal stationary portfolio is also found to be $\frac{r}{\lambda \sigma^2}$. In fact, it is well-known that, for a static market, the stationary portfolios are optimal, but this is beyond the scope of this work [Cover and Thomas, 2006].

Combining the above two propositions, one obtain the following theorem.

**Theorem 2.** The stationary portfolio obtained by training with data augmentation strength in given in Proposition 8 is optimal, i.e., it is no worse than any other stationary portfolio.

**Proof.** Plug in $\gamma = cr_t S^2_t$, we have that the trained portfolio is $\pi^* = \frac{r}{\lambda \sigma^2}$, which is equivalent to the optimal stationary portfolio, and we are done. □

This shows that, even for a stationary portfolio, it is useful to use the proposed data augmentation technique.