On the origin of divergences in massless $QED_2$

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Abstract

We show that ultraviolet divergences found in fermionic Green’s functions of massless $QED_2$ have an essentially non-perturbative nature. We investigate their origin both in gauge invariant formalism (the one where we introduce Wess-Zumino fields to restore quantum gauge invariance) and in gauge non-invariant formalism, mapping two different but equivalent mechanisms responsible for their appearance. We find the same results in both approaches, what contradicts a previous work of Jian-Ge, Qing-Hai and Yao-Yang, that found no divergences in the chiral Schwinger model considered in the gauge invariant formalism.

1 Introduction

Gauge theories are nowadays responsible for the description of elementary interactions [1]. One of the main requirements of the standard model is that of anomaly cancellation, without which it is not known how to perform perturbative calculations [2]. The phenomenologically achieved equilibrium between the number of families of quarks and leptons guarantees this cancellation. However, in practice, nothing prevents the discovery of new kinds of quarks or leptons, as higher energies are reached. This could threaten the theoretical consistency of the model and raise questions about the correctness of the gauge approach to elementary interactions.

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However, some features of gauge theories were discovered in the 80’s that could put the questions above under a more comfortable perspective. It was realized that, at least in 2 dimensional gauge theories, quantum consistency could be reached even for anomalous theories (the anomaly being understood as occurring in the gauge symmetry). A mechanism of symmetry restoration seemed to be operating in the background, becoming explicit through the natural introduction of a new set of degrees of freedom, available only at quantum level, the so-called Wess-Zumino fields. Since then, this mechanism has been intensively studied although the achievements have been little in four dimensions (see, however, [8]).

In two dimensions, two strategies have been mainly followed (taking advantage of the fact that, for this number of dimensions, exactly soluble models are well known). The first, already mentioned above, consists in studying the dynamics of the theory that emerges when one considers explicitly the Wess-Zumino fields, and is called gauge invariant formalism (GIF). The other, takes into account explicitly the dynamics of the longitudinal part of the gauge field, given by the anomaly. It is called gauge non-invariant formalism (GNIF). In both formalisms one ends with a gauge invariant theory, whether one integrates over the fermions and the Wess-Zumino fields (in GIF) or over the fermions and the longitudinal part of the gauge field (in GNIF). This is achieved regardless of the regularization method employed, which can preserve or not gauge invariance in intermediate computations.

This fact suggests that one should consider gauge theories in schemes wider than usual. As this mechanism of “restoration” of gauge symmetry is acting, there is a priori no reason to consider a fixed value for the Jackiw-Rajaraman parameter (that value which preserves gauge invariance in intermediate calculations) which appears precisely as a manifestation of regularization ambiguities. In fact, for theories involving chiral fermions, there is no value for this parameter that can preserve intermediate gauge invariance. However one ends up with an effective action which is explicitly gauge invariant.

Another well-known fact is that fermionic correlation functions are divergent if one considers a gauge non-invariant scheme for regularizing the theory. A fermion wave function renormalization is enough to render the theory finite, but several subtleties appear that make it very hard to be done exactly, making that the label “exactly soluble” be dependent on technical advances. The main problems are: 1) to identify precisely the origin of the divergences and 2) to learn how to deal with them. The second problem is treated in [13]. This paper addresses itself to the detailed examination of the origin of the divergences.

In particular, we found results that are in explicit contradiction to the ones found by Jian-Ge, Qing-Hai and Yao-Yang in [14]. In their paper, they found no
divergences in the chiral Schwinger model, when considered in the gauge invariant formalism. At the conclusions, we briefly comment their paper and the reasons that conducted them to this mistake.

This paper is organized as follows: in section 2 we review the GIF and the GNIF, applying the results to study the Schwinger model (in section 3) and the chiral Schwinger model (in section 4) perturbatively in both formalisms. In these sections we show that the divergences in fermionic Green’s functions have a non-perturbative nature, in both models and in both formalisms. We present our conclusions in section 5.

2 GIF and GNIF

The models to be studied are defined by the Lagrangian densities,

\[
\mathcal{L}[\psi, \bar{\psi}, A] = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\partial + eA^P)\psi,
\]

where

\[
P = \begin{cases} 
1, & \text{Schwinger model} \\
\pm, & \text{chiral Schwinger model}
\end{cases}
\]

The fermionic determinant is computed exactly in both cases,

\[
\det(i\partial + eA^P) = \exp(iW[A]) = \int d\psi d\bar{\psi} \exp \left(i \int dx \bar{\psi}(i\partial + eA^P)\psi \right).
\]

In general, the determinant has to be calculated through a regularization prescription. For the Schwinger model it is possible to do it in an gauge invariant way, but this is not the case for the chiral coupling. We will calculate the determinant using gauge non-invariant prescriptions, even though, for the Schwinger model, no authentic gauge anomalies are obtained in this way [15].

To define a free propagator for the field \( A_\mu \) from (1), it is usual to introduce a gauge fixing condition. However, the situation here is not the same as in usual gauge theories. As the theory after quantizing fermion fields is gauge non-invariant, the results of its quantization will depend, potentially, on the gauge fixing condition used. This prevents us from using a gauge fixing condition in (1). As we will see below, this problem can be easily bypassed, both in GIF and in GNIF.

Let us consider the functional integration over the fermion fields in (2), performing the following change in fermionic variables

\[
\frac{\psi}{\bar{\psi}} \rightarrow \frac{g\psi}{\bar{\psi}} = g\psi g^\dagger,
\]

(3)
where $g$ belongs to the gauge group under which the fermions transform. In general, we expect that the fermionic measure would not be invariant under this transformation, unless we use explicitly a gauge invariant prescription to define it (which is impossible, in the chiral Schwinger model). Then, we have in general,

$$
dψd\bar{ψ} = J[A_\mu, g]dψd\bar{ψ}g,
$$

where $J[g, A_\mu]$ is the Jacobian of the transformation. Having computed the fermion determinant, it is possible to obtain this Jacobian easily [6],

$$
J[A_\mu, g] = e^{i(W[A_\mu] − W[A_\mu^g])},
$$

where,

$$
A_\mu^g = g^{-1}A_\mu g + ig^{-1}∂_\mu g.
$$

We observe that, if we use a prescription which preserves the gauge symmetry of $W[A_\mu]$, we obtain $J[g, A_\mu] = 1$, according to our expectations.

These are the basic facts that lie below the formalisms that we are going to review in the next sections.

### 2.1 The gauge invariant formalism

The generating functional of the theory (1) is given by the following definition

$$
Z[\eta, \bar{\eta}, J] = N \int dA_\mu dψd\bar{ψ} \exp \left[ i \int dx \left( \mathcal{L}[ψ, \bar{ψ}, A] + \bar{ψ} \psi + \bar{ψ} η + J A \right) \right].
$$

We return to the problem of defining a free propagator for the field $A_\mu$. We notice that, if the theory were gauge invariant at quantum level, we should use Faddeev-Popov’s technique [15] to obtain a well defined functional integration. Harada and Tsutsui [12], and Babelon, Schaposnik e Viallet [11], observed that this is not necessary (in fact, it is redundant) when the theory is not gauge invariant at quantum level because, in this case, different gauge orbits of $A_\mu$ give different contributions to the effective action. However, Faddeev-Popov’s technique can still be applied, as it consists of multiplication by 1, expressed as

$$
1 = Δ_f[A_\mu] \int dg \ δ(f[A_\mu^g]).
$$

In the above formula, $dg$ represents the invariant measure over the gauge group $G$, $g ∈ G$ and $f[A_\mu]$ is the gauge fixing condition. Thus, as usual, we insert (3) in equation (7) and change integration variables in the bosonic sector $A_\mu → A_\mu^g$. 
(\(dA_\mu\) and \(\Delta_f[A_\mu]\) are gauge invariant by construction). The generating functional (7) becomes

\[
Z[\eta, \bar{\eta}, J] = N \int dA_\mu d\psi d\bar{\psi} d\psi dg \Delta_f[A_\mu] \delta(f[A_\mu]) \times \exp \left[ i \int dx \left( \mathcal{L}[\psi, \bar{\psi}, A^{g^{-1}}] + \bar{\eta} \psi + \bar{\psi} \eta + J\mu A_{\mu}^{g^{-1}} \right) \right].
\]

Now, we redefine the fermionic fields according to the rule

\[
\bar{\psi} \rightarrow \psi = g\psi^g, \quad \bar{\psi} \rightarrow \bar{\psi} = \bar{\psi} g^g, \quad (10)
\]

and we see that the Lagrangian returns to its original form. However, as the measure has not necessarily been defined in a gauge invariant way, it is not invariant under the transformation (10), but changes as we saw in (5),

\[
d\psi d\bar{\psi} = d\psi^g d\bar{\psi}^g e^{i\alpha[A_\mu, g^{-1}]},
\]

where \(\alpha[A_\mu, g^{-1}] = W[A^{g^{-1}}] - W[A_\mu]\) is the Wess-Zumino action. Thus, we obtain the following expression for the generating functional,

\[
Z[\eta, \bar{\eta}, J] = N \int dA_\mu d\psi d\bar{\psi} d\psi dg \Delta_f[A_\mu] \delta(f[A_\mu]) \exp \left( i \alpha[A_\mu, g^{-1}] \right) \exp \left[ i \int dx \left( \mathcal{L}[\psi, \bar{\psi}, A] + \bar{\eta} g \psi + \bar{\psi} g^{\dagger} \eta + J \cdot A^{g^{-1}} \right) \right].
\]

Now we can define a free propagator for the \(A_\mu\) field, by exponentiating the \(\delta\)–function as in the ordinary situation. The quantization of the theory is, thus, independent of the choice of the gauge fixing condition, as in the usual formulation of Faddeev-Popov. We will use the Lorentz gauge fixing condition, \(f[A_\mu] = \frac{1}{\sqrt{\xi}} \partial \cdot A\) and we will absorb \(\Delta_f[A]\) in the normalization constant (because the theories are Abelian and the Faddeev-Popov’s ghost fields decouple). Doing this, we arrive at

\[
Z[\eta, \bar{\eta}, J] = N' \int dA_\mu d\psi d\bar{\psi} dg \exp \left( i \alpha[A_\mu, g^{-1}] \right) \times \exp \left[ i \int dx \left( \mathcal{L}[\psi, \bar{\psi}, A] - \frac{1}{2\xi} (\partial \cdot A)^2 + \bar{\eta} g \psi + \bar{\psi} g^{\dagger} \eta + J \cdot A^{g^{-1}} \right) \right].
\]

This equation will be the starting point for the perturbative analysis of the theory defined by (1). As we are in a gauge non-invariant theory, the source \(J\mu\) will have its divergence \(\partial \mu J^\mu \neq 0\), in general. Another thing that must be indicated is that, in the process of defining the free propagator of \(A_\mu\), the theory acquires an additional degree of freedom, given by the Wess-Zumino field, which, in the end,
interacts with the fermion fields through the fermionic sources. This interaction is very complicated, and prevents the exact calculation of $Z[\eta, \overline{\eta}, J]$. However, the exact calculation of an arbitrary correlation function is possible, at least in principle (once one renormalizes the divergences to be found in the next sections). The possibility of defining correctly a free propagator for $A_\mu$ is essential to perform a perturbative analysis of the theory, and thus, to be able to see if the ultraviolet divergences that appear in the fermionic Green’s functions have a perturbative origin or not.

2.2 The gauge non-invariant formalism

Another approach to the perturbative problem is commonly called gauge non-invariant formalism [4, 10]. In this context, we use the fact that the classical decoupling of the longitudinal part of $A_\mu$ (that can be obtained with a gauge transformation of the fermion fields) does not keep the fermionic measure invariant, in general. This fact can be exploited to obtain a perturbative description of the theory, as we will see below.

Let us separate the field $A_\mu$ in its longitudinal and transverse parts, as usual,

$$eA_\mu = \partial_\mu \rho - \tilde{\partial}_\mu \phi,$$

and substitute the expression above into the generating functional [1]

$$Z[\eta, \overline{\eta}, J] = N \int d\rho d\phi d\bar{\psi} d\bar{\psi} \exp \left( i \int dx \left[ \mathcal{L}[\psi, \overline{\psi}, 1 \frac{e}{e} \partial_\mu \rho - \frac{1}{e} \tilde{\partial}_\mu \phi] \right) \right) \times \exp \left[ i \int dx \left( \overline{\eta} \psi + \overline{\psi} \eta + \frac{1}{e} J_\mu \partial_\mu \rho - \frac{1}{e} J_\mu \tilde{\partial}_\mu \phi \right) \right].$$

We see that, as the classical action is gauge invariant, the longitudinal part of the field $A_\mu$ (the field $\rho$) does not have a kinetical term, apparently appearing as an auxiliary field. Classically it is possible to remove completely this field from the sourceless part of the action through the following transformation

$$\psi \rightarrow g\psi', \quad \overline{\psi} \rightarrow \overline{\psi}'g^\dagger,$$

with $g = e^{i\rho P}$. If the measure were invariant under this transformation, we would have a linear dependence on $\rho$, that would render its integration undefined (in the non-anomalous case, that is why we have to use Faddeev-Popov’s method: to generate a kinetical term for $\rho$). However, the fermionic measure, as we saw, is not invariant under [10], but changes as

$$d\psi d\bar{\psi} = d\psi' d\bar{\psi}' e^{i\alpha[\rho, \phi]},$$

where $\alpha[\rho, \phi] = W[\frac{1}{e} \partial_\mu \rho - \frac{1}{e} \tilde{\partial}_\mu \phi] - W[-\frac{1}{e} \tilde{\partial}_\mu \phi]$.
The generating functional (15) then acquires the following form

\[
Z[\eta, \overline{\eta}, J] = N \int d\rho d\phi d\psi d\overline{\psi} \exp \left( i\alpha[\rho, \phi] + i \int dx \mathcal{L}[\psi, \overline{\psi}, -\frac{1}{e} \partial_\mu \phi] \right)
\]

(18)

\[
\times \exp \left[ i \int dx \left( \eta g \psi + \overline{\psi} g^\dagger \eta + \frac{1}{e} J_\mu \partial^\mu \rho - \frac{1}{e} J_\mu \partial^\mu \phi \right) \right].
\]

As we are going to see in the next two sections, the \( \alpha \) term contains a kinetical term for the \( \rho \) field which allows us to treat the theory perturbatively. We notice that the coupling of \( \rho \) to the fermion fields is done through the fermionic sources and is not minimal anymore. However, the theory in (18) now admits a perturbative analysis, as we will explicitly show for the two models mentioned in the beginning.

3 Schwinger model

3.1 Perturbative analysis in GIF

The Schwinger model is defined by setting \( P = 1 \) (see equation (3)). A typical gauge group element can be parameterized by a field \( \theta \) as \( g = e^{i\theta} \). The Wess-Zumino action is

\[
\alpha(A_\mu, \theta) = \left( a - 1 \right) \frac{1}{2\pi} \int dx \left( \frac{1}{2} \partial_\mu \theta \partial^\mu \theta - e \theta \partial_\mu A^\mu \right).
\]

(19)

Notice that, for \( a = 1 \), the Wess-Zumino action is zero and we have a Jacobian equal to one, which characterizes quantum gauge invariance. We are going to compute bosonic and fermionic Green’s functions perturbatively, looking for the appearance of divergences.

3.1.1 Photon Propagator

If we take two functional derivatives with respect to \( J_\mu(x) \) and \( J_\nu(y) \) of (13) we obtain,

\[
\langle 0|T A_\mu(x) A_\nu(y)|0 \rangle \equiv \delta \langle 0|T A_\mu(x) A_\nu(y)|0 \rangle_\theta + \frac{1}{e} \theta \langle 0|T A_\mu(x) \partial_\nu^\theta \theta(y)|0 \rangle_\theta + \cdots
\]

(20)

In the equation above we see that the photon propagator in the original theory is expressed as a sum of propagators referred to another theory given by an action \( S_\theta[\psi, \overline{\psi}, A_\mu, \theta] \) that includes the \( \theta \) field,
\[ S_\theta[\psi, \overline{\psi}, A_\mu, \theta] = \int dx \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2 \xi} (\partial \cdot A)^2 + \overline{\psi} (i\gamma^\mu + eA) \gamma^\mu \psi + \right. \\
\left. + \frac{(a - 1)}{4\pi} \partial_\mu \theta \partial^\mu \theta - \frac{(a - 1)}{2\pi} e\theta \partial_\mu A^\mu \right] \]

We will denote the expectation values in this theory by \( \langle \rangle_\theta \). From \( S_\theta[\psi, \overline{\psi}, A_\mu, \theta] \) we easily obtain Feynman rules and compute the necessary expectation values. In the case that we are considering here, it will be possible to add up the perturbative series and compare with the exact result. In the following we exhibit the relevant Feynman diagrams and their result after computation:

a) \( A_\mu \)-propagator, \( \theta\langle 0|T A_\mu(x) A_\nu(y)|0\rangle_\theta \):

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Computing these two contributions, we get

$$\theta\langle 0 | T \theta(p) \theta(-p) | 0 \rangle_\theta = i \left( \frac{2\pi}{a-1} \frac{1}{p^2} - \frac{\xi e^2}{p^4} \right).$$

(24)

c) Mixing terms, $\theta\langle 0 | T \theta(x) A_\mu(y) | 0 \rangle_\theta$ and $\theta\langle 0 | T A_\mu(x) \theta(y) | 0 \rangle_\theta$:
The two terms that contribute are:

$$\theta\langle 0 | T \theta(p) A_\mu(-p) | 0 \rangle_\theta = -\frac{\xi e p_\mu}{p^4}$$

(25)

$$\theta\langle 0 | T A_\mu(p) \theta(-p) | 0 \rangle_\theta = \frac{\xi e p_\mu}{p^4}$$

(26)

Adding all results as in (20), we obtain the full photon propagator for the theory,

$$i \langle 0 | T A_\mu(p) A_\nu(-p) | 0 \rangle = \frac{1}{p^2 - \frac{e^2}{2\pi} (a+1)} \left( g_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2} \right) - \frac{2\pi}{e^2(a-1)} \frac{p_{\mu}p_{\nu}}{p^2}.$$ 

(27)

This result agrees exactly with which is obtained by non-perturbative methods [18] (taking into account that the $\bar{a}$ parameter there is related to ours as $\bar{a} = (a-1)/2$).

3.1.2 Fermion propagator

From (13), we can take functional derivatives with respect to the fermionic sources $\bar{\eta}(x)$ e $\eta(y)$ and compute the fermion propagator as

$$\langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle = N' \int dA_\mu d\theta d\bar{\psi} d\psi \psi(x) \bar{\psi}(y)$$

$$\times \exp \left( i S_\theta[\psi, \bar{\psi}, A_\mu, \theta] + \int dz \, \theta(z) j(z, x, y) \right)$$

(28)

where $j(z, x, y) = \delta(z - x) - \delta(z - y)$. Integrating over the $\theta$ field, we obtain

$$\langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle = \exp \left\{ - \frac{2\pi i}{a-1} \int \frac{dk}{(2\pi)^2} \frac{1 - e^{-ik \cdot (x-y)}}{k^2} \right\} G_p(x-y),$$

(29)
where the exponential contains a divergence already found elsewhere [19, 18] which is generated by the integration of the field $\theta$. This divergence is not cancelled by the normalization factor $N'$, as it is induced by the presence of $j(z, x, y)$ (which, in turn, is generated by the functional derivations, absent in the normalization factor). $G_p(x - y)$ is defined from the remaining functional integration in terms of the fields $A_\mu$, $\psi$ and $\bar{\psi}$,

$$G_p(x - y) = N^\mu \int dA_\mu d\psi d\bar{\psi} \psi(x)\bar{\psi}(y)$$

$$\times \exp \left\{ i \int dz \left( \frac{1}{2} A_\mu H^\mu_\xi A_\nu + \bar{\psi}(i\partial + eA)\psi + eA_\mu \ell^\mu(z, x, y) \right) \right\}$$

where

$$H^\mu_\xi = g^\mu\nu \Box + \left( \frac{1}{\xi} - 1 \right) \partial^\mu \partial^\nu - \frac{e^2}{2\pi} (a - 1) \frac{\partial^\mu \partial^\nu}{\Box}.$$ (31)

From (31), we obtain an effective free propagator for $A_\mu$, that we call $h^\xi_{\mu\nu}$

$$h^\xi_{\mu\nu}(k) = -\frac{i}{k^2} \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) - \frac{i\xi k_\mu k_\nu}{k^2[k^2 + \xi^2 e^2 (a - 1)]}.$$ 

Its ultraviolet behavior is of the form $k^{-2}$. The Feynman rules to calculate the fermion self-energy are now given by the action appearing in the functional integration (30). It is now easy to calculate $G_p(x - y)$ to any desired loop order. We limit ourselves to the 1-loop contribution to the fermion self-energy,

$$-i\Sigma_p(p) = i \phi \left[ \frac{e^2}{4\pi p^2} - \frac{1}{2(a - 1)} \ln \left( 1 + \frac{\xi e^2}{2\pi p^2 (a - 1)} \right) \right],$$ (32)

that is finite, as well as all the other diagrams that enter in the computation of $G_p(x - y)$. So, the only source of divergences in (29) is the integration over the $\theta$ field, which is done exactly, outside the perturbative level.

A little further reflection shows quickly that the same is true for all fermionic Green’s functions: they all exhibit a divergence, originating in the integration over the Wess-Zumino field, being finite modulo this problem.

### 3.2 Perturbative analysis in GNIF

Now we start from (18), where

$$L[\psi, \bar{\psi}, -\frac{1}{e} \tilde{\partial}_\mu \phi] = \frac{1}{2e^2} \phi \Box^2 \phi + \bar{\psi}(i\partial - \tilde{\partial}\phi)\psi,$$ (33)
having \( g = e^{i\rho} \) and a Jacobian given by
\[
\alpha(\rho, \phi) = \exp \left( \frac{i(a - 1)}{4\pi} \int dx \partial_\mu \rho \partial^\mu \rho \right).
\]

### 3.2.1 Photon propagator

The propagator for the \( A_\mu \) field has to be expressed in terms of propagators for the \( \rho \) and \( \phi \) fields. In (18) we take two functional derivatives with respect to \( J_\mu \) and \( J_\nu \), put all the sources to zero and we are left with
\[
\langle 0 | T A_\mu(x) A_\nu(y) | 0 \rangle = \frac{1}{i^2} \left( i \langle 0 | T \partial_\mu \rho(x) \partial^\mu \rho(y) | 0 \rangle + i \langle 0 | T \tilde{\partial}_\mu \phi(x) \tilde{\partial}^\mu \phi(y) | 0 \rangle \right).
\]

\( i \langle | \rangle_t \) refers to expectation values calculated using the effective action
\[
S_t[\psi, \bar{\psi}, \rho, \phi] = \int dx \left( \frac{1}{2e^2} \phi \square \phi + \frac{(a - 1)}{4\pi} \partial_\mu \rho \partial^\mu \rho + \bar{\psi}(i\partial - \tilde{\partial})\phi \right).
\]

The photon propagator is split into a sum of two propagators of the fields \( \rho \) and \( \phi \), whose dynamics is described by the action above. We see in \( S_t[\psi, \bar{\psi}, \rho, \phi] \) that \( \rho \) is a free field, which implies that the mixed propagators \( i \langle 0 | T \partial_\mu \rho(x) \partial^\mu \phi(y) | 0 \rangle \) and \( i \langle 0 | T \tilde{\partial}_\mu \phi(x) \partial^\mu \rho(y) | 0 \rangle \) are null. With the Feynman rules generated from \( S_t[\psi, \bar{\psi}, \rho, \phi] \), we can calculate the photon propagator for the theory in a perturbative way. We show these results below:

- a) \( \phi \)-propagator, \( i \langle 0 | T \phi(x) \phi(y) | 0 \rangle_t \):

\[
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\]

The fermionic loop (37) is calculated with a Pauli-Villars regularization prescription, as before. Its contribution, in this case, is
\[
- \frac{i(a + 1)}{2\pi} p^2.
\]

Adding the series (37), we get the propagator of the \( \phi \) field
\[
i \langle 0 | T \phi(p) \phi(-p) | 0 \rangle_t = \frac{ie^2}{p^2(p^2 - e^2(a+1)/2\pi)}.
\]
b) \( \rho \)-propagator, \( i\langle 0| T \rho(x) \rho(y) |0 \rangle_i \): as the \( \rho \) field in (18) is a free field, its propagator is calculated directly, 

\[
i\langle 0| T \rho(p) \rho(-p) |0 \rangle_i = \frac{2\pi i}{(a-1)p^2}.
\]  

(39)

The photon propagator is then obtained from equation (35),

\[
i\langle 0| T A_\mu(p) A_\nu(-p) |0 \rangle = \frac{1}{k^2 - \frac{e^2(a+1)}{2\pi} \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right)} - \frac{2\pi}{e^2(a-1)} \frac{k_\mu k_\nu}{k^2}.
\]  

(40)

It coincides with the propagator computed by non perturbative calculation, and with the one calculated previously (27) in the gauge invariant formalism.

3.2.2 Fermion propagator

From (18), we take functional derivatives with respect to the fermion sources and we obtain,

\[
\langle 0| T \psi(x) \overline{\psi}(y) |0 \rangle = N \int d\rho d\phi d\psi d\overline{\psi} \, \psi(x) \overline{\psi}(y) \exp \left\{ iS_\mu[\psi, \overline{\psi}, \rho, \phi] + i \int dz \, \rho(z) j(z, x, y) \right\}.
\]  

(41)

The term involving \( j(z, x, y) = \delta(z - x) - \delta(z - y) \) is generated when we perform the above mentioned functional derivatives. The integration over the \( \rho \) field factorizes, but the presence of \( j(z, x, y) \) prevents the absorption of this integration in the normalization factor \( N \). Then (41) becomes, after \( \rho \) integration,

\[
\langle 0| T \psi(x) \overline{\psi}(y) |0 \rangle = \exp \left\{ -\frac{2\pi i}{a-1} \int \frac{dk}{(2\pi)^2} \frac{1 - e^{-i k \cdot (x-y)}}{k^2} \right\} G_\rho(x - y).
\]  

(42)

We observe the presence of an ultraviolet divergence in the exponential, that do not have perturbative origin (as it comes from the \( \rho \) integration) and coincides with the one calculated previously in the gauge invariant formalism (see equation (29)). The remaining functional integration in \( G_\rho(x - y) \), given by

\[
G_\rho(x - y) = N' \int d\phi d\psi d\overline{\psi} \, \psi(x) \overline{\psi}(y) \exp \left[ i \int dx \left( \frac{1}{2e^2} \phi \Box^2 \phi + \overline{\psi}(i\partial - \partial)\psi \right) \right],
\]  

(43)

is finite, that is, the Feynman diagrams generated from it do not show ultraviolet divergences. From (13), the 1-loop contribution to the fermion self-energy is given by

\[
- i \Sigma_\rho(p) = \frac{ie^2\phi}{4\pi p^2}.
\]  

(44)

Since it is finite, the contributions of higher loop order to the self-energy are also finite.
4 Chiral Schwinger model

4.1 Perturbative analysis in GIF

We will perform the same analysis for the chiral Schwinger model, defined by \( P = P_{\pm} \) (see, again, equation (2)). As before, \( g = e^{i\theta} \), but now the Wess-Zumino action is given by [17]

\[
\alpha(A, \theta) = \int dx \left\{ \frac{(a - 1)}{8\pi} \partial_\mu \theta \partial^\mu \theta - \frac{e\theta}{4\pi} \left[ (a - 1) \partial^\mu A_\mu - \tilde{\partial}^\mu A_\mu \right] \right\}
\] (45)

As opposed to the case of the Schwinger model, there is no value of \( a \) which can turn this action to zero, and this is the distinctive sign of the gauge anomaly. Again, we will compute Green’s functions perturbatively, observing similarities and differences in comparison to the vectorial coupling case.

4.1.1 Photon propagator

From (1), (13) and (45), we obtain the following expression for the photon propagator of the theory

\[
\langle 0 | T A_\mu(x) A_\nu(y) | 0 \rangle \equiv e^{\theta} \langle 0 | T A_\mu(x) \partial_\nu \theta(y) | 0 \rangle + \frac{1}{e^{2\theta}} \langle 0 | T \partial_\mu \theta(x) \partial_\nu \theta(y) | 0 \rangle \] (46)

The propagator of the original photon is again a sum of propagators which are referred to an effective theory \( S_{\theta}[\psi, \bar{\psi}, A_\mu, \theta] \) that includes the \( \theta \) field. We denote the expectation values in this theory by \( \theta \langle | \rangle \). This effective action \( S_{\theta}[\psi, \bar{\psi}, A_\mu, \theta] \) is given by

\[
S_{\theta}[\psi, \bar{\psi}, A_\mu, \theta] = \int dx \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial \cdot A)^2 + \bar{\psi}(i\gamma^\mu + e A_\mu P_+) \psi + \frac{(a - 1)}{8\pi} \partial_\mu \partial^\mu \theta - \frac{(a - 1)}{4\pi} e\theta \partial_\mu A^\mu + \frac{1}{4\pi} e\theta \tilde{\partial}^\mu A_\mu \right]
\] (47)

With Feynman rules obtained from \( S_{\theta}[\psi, \bar{\psi}, A_\mu, \theta] \), we will show, in what follows, the perturbative calculation of the propagators that appear in (46).

a) \( A_\mu \)-propagator

\[
\begin{align*}
p & \quad \text{propagator} \quad p & = & \quad \text{propagator} \quad p \\
\end{align*}
\] (48)
The fermionic loop in (48) is given by

\[-i \Pi_{\mu\nu}(p) = \frac{ie^2}{4\pi} \left[ (a+1) g_{\mu\nu} - \frac{2p_{\mu}p_{\nu}}{p^2} - \frac{p_{\mu}\tilde{p}_{\nu} + \tilde{p}_{\mu}p_{\nu}}{p^2} \right],\]

while the third graphic in (48) contributes as

\[ie^2 \left[ \frac{g_{\mu\nu}}{a-1} - \frac{1 + (a-1)^2 p_{\mu}p_{\nu}}{a-1} \frac{p_{\mu}\tilde{p}_{\nu} + \tilde{p}_{\mu}p_{\nu}}{p^2} \right].\]

Adding both contributions we obtain, to order \(e^2\)

\[\frac{ie^2 a^2}{4\pi (a-1)} \left[ g_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2} \right].\]  

(49)

It is easy to see that (48) is a geometrical series with an order \(n\) term \((n \geq 1)\)

\[\frac{-i}{p^2} \frac{e^2 a^2}{4\pi (a-1)p^2} \left( g_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2} \right)^n.\]

Adding this series (48), we get

\[\theta \langle 0 | TA_{\mu}(p) A_{\nu}(-p) | 0 \rangle_{\theta} = \frac{4\pi}{a-1} \frac{i}{p^2} - \frac{ie^2 a^2}{4\pi (a-1)} \left( g_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2} \right) - \frac{i\xi p_{\mu}p_{\nu}}{p^4}.\]  

(50)

b) \(\theta\)-propagator

\[\theta \langle 0 | T \theta(p) \theta(-p) | 0 \rangle_{\theta} = \frac{4\pi}{a-1} \frac{i}{p^2} - \frac{ie^2}{p^4} + \frac{ie^2 a^2}{p^2 (p^2 - \frac{e^2 a^2}{4\pi (a-1)})}.\]  

(51)

c) Mixed terms \(A_{\mu} - \theta\)

\[\theta \langle 0 | T \theta(p) A_{\nu}(-p) | 0 \rangle_{\theta} = \frac{-\xi e p_{\nu}}{p^4} + \frac{e}{(a-1)} \frac{\tilde{p}_{\nu}}{p^2 (p^2 - \frac{e^2 a^2}{4\pi (a-1)})},\]  

(52)
\[
\theta\langle0|TA_\mu(p)\theta(-p)|0\rangle_\theta = \frac{\xi e p_\mu}{p^4} - \frac{e}{(a-1)} \frac{\tilde{p}_\mu}{p^2(p^2 - \frac{e^2 a^2}{4\pi(a-1)})}
\]  

(53)

Adding all contributions in (46), we obtain the photon propagator of the theory

\[
i\langle0|TA_\mu(k)A_\nu(-k)|0\rangle = \frac{1}{k^2 - \frac{e^2 a^2}{4\pi(a-1)}} \left[ g_{\mu\nu} - \frac{k_\mu k_\nu}{a-1} \left( \frac{4\pi}{e^2} - \frac{2}{k^2} \right) + \frac{k_\mu \tilde{k}_\nu + \tilde{k}_\mu k_\nu}{(a-1)k^2} \right]
\]  

(54)

This is equal to the well known results in the literature [3]. Again we see that, apart from the regularization of the fermionic loop, there are no perturbatively induced divergences in this propagator. As in the vectorial case, only this regularization is enough to furnish a finite result for the photon propagator.

### 4.1.2 Fermion propagator

From (1), (13) and (45), we arrive at the following expression for the fermion propagator

\[
\langle0|T\psi(x)\bar{\psi}(y)|0\rangle = P_- G_F(x-y) + \langle0|T\psi_+(x)\bar{\psi}_+(y)|0\rangle,
\]  

(55)

with \(\psi_+ = P_+ \psi\). The left-handed fermion propagates freely, but the right-handed one interacts with the vector field \(A_\mu\) as

\[
\langle0|T\psi_+(x)\bar{\psi}_+(y)|0\rangle = N \int dA_\mu d\psi d\bar{\psi} d\theta \psi_+(x)\bar{\psi}_+(y)
\]

\[
\times \exp \left(i S_\theta[\psi,\bar{\psi}, A_\mu, \theta] + \int dz \theta(z) j(z,x,y) \right),
\]

with \(S_\theta[\psi, \bar{\psi}, A_\mu, \theta]\) given in the previous section and \(j(z,x,y) = \delta(z-x) - \delta(z-y)\). Integrating over the \(\theta\) field, we are left with

\[
\langle0|T\psi_+(x)\bar{\psi}_+(y)|0\rangle = \exp \left\{ -\frac{4\pi i}{a-1} \int \frac{dk}{(2\pi)^2} \frac{1 - e^{-ik\cdot(x-y)}}{k^2} \right\} G'_+(x-y).
\]  

(57)

We observe a logarithmic ultraviolet divergence in the exponential, as in the case of the anomalous Schwinger model. The remaining functional integration \(G'_+(x-y)\) is finite,

\[
G'_+(x-y) = N' \int dA_\mu d\psi d\bar{\psi} \psi_+(x)\bar{\psi}_+(y)
\]

\[
\times \exp \left[i \int dz \left( \frac{1}{2} A_\mu H^{\mu\nu} A_\nu + \bar{\psi}(i\partial + eA)\psi + eA_\mu l^\mu(z,x,y) \right) \right],
\]  

(58)
where

\[ H^{\mu\nu} = g^{\mu\nu} \left( \Box + \frac{e^2}{4\pi(a-1)} \right) + \left( \frac{1}{\xi} - 1 \right) \partial^\mu \partial^\nu + \frac{e^2}{4\pi(a-1)} \left[ (a-1)^2 + 1 \right] \frac{\partial^\mu \partial^\nu}{\Box} + \frac{e^2}{4\pi} \frac{\partial^\mu \tilde{\partial}^\nu + \tilde{\partial}^\mu \partial^\nu}{\Box}, \] (59)

and

\[ l^\mu(z, x, y) = \left( \partial^\mu z - \frac{\tilde{\partial}^\mu}{a-1} \right) \left[ D_F(z - x) - D_F(z - y) \right]. \] (60)

The \( A_\mu \)–propagator, which enters in (58), has an ultraviolet behavior as \( k^{-2} \). Then, the 1-loop contribution to the fermion self-energy is finite. This persists to all loop orders.

### 4.2 Perturbative analysis in GNIF

Here we start from (18), where

\[ L[\psi, \bar{\psi}, -\frac{1}{e} \tilde{\partial}_\mu \phi] = \frac{1}{2\epsilon^2} \phi \Box^2 \phi + \bar{\psi}(i\partial - \tilde{\partial} \phi P_+) \psi. \] (61)

Putting \( g = e^{i\rho P_+} \), we obtain the \( \alpha \) term from the Jacobian of this gauge transformation

\[ \alpha(\rho, \phi) = \exp \left[ i \int dx \left( \frac{a-1}{8\pi} \partial_\mu \rho \partial^\mu \rho - \frac{1}{4\pi} \partial_\mu \rho \partial^\mu \phi \right) \right]. \] (62)

#### 4.2.1 Photon Propagator

From equation (18), considering (61) and (62), we get the photon propagator

\[ \langle 0| T A_\mu(x) A_\nu(y) |0 \rangle = \frac{1}{\epsilon^2} \left( 1 \langle 0| T \partial_\mu^\rho \rho(x) \partial_\nu^\rho \rho(y) |0 \rangle_1 - i\langle 0| T \partial_\mu^\rho \rho(x) \tilde{\partial}_\nu^\phi \phi(y) |0 \rangle_1 + i\langle 0| T \tilde{\partial}_\mu^\rho \phi(x) \partial_\nu^\rho \rho(y) |0 \rangle_1 + i\langle 0| T \tilde{\partial}_\mu^\rho \phi(x) \tilde{\partial}_\nu^\phi \phi(y) |0 \rangle_1 \right). \] (63)

The dynamics is governed by the effective action \( S_l[\psi, \bar{\psi}, \rho, \phi] \), given by

\[ S_l[\psi, \bar{\psi}, \rho, \phi] = \int dx \left( \frac{1}{2\epsilon^2} \phi \Box^2 \phi + \bar{\psi}(i\partial - \tilde{\partial} \phi) \psi + \frac{(a-1)}{8\pi} \partial_\mu \rho \partial^\mu \rho - \frac{1}{4\pi} \partial_\mu \rho \partial^\mu \phi \right). \] (64)

and \( \langle \rangle_l \) refers to expectation values calculated in this dynamics. We proceed to the perturbative calculation of the relevant graphs.
a) $\phi$–propagator

\[
\phi - \text{propagator} = \frac{-i p^2}{4\pi} (a + 1),
\]

and the third graphic contribution is

\[
- \frac{i p^2}{4\pi (a - 1)}.
\]

The $\phi$–self-energy is

\[
-i \Sigma_{\phi}(p) = -\frac{ia^2 p^2}{4\pi (a - 1)}. \tag{66}
\]

Now, adding the series (65), we obtain

\[
i\langle 0 | T \phi(p) \phi(-p) | 0 \rangle_l = \frac{i e^2}{p^2 (p^2 + \frac{e^2 a^2}{4\pi (a - 1)})} \tag{67}
\]

b) $\rho$–propagator

\[
\rho - \text{propagator} = \frac{i 4\pi}{(a - 1)p^2} + \frac{1}{(a - 1)^2} \frac{i e^2}{p^2 (p^2 - \frac{e^2 a^2}{4\pi (a - 1)})} \tag{68}
\]

c) Mixed terms $\rho - \phi$

i) $\rho - \phi$

ii) $\rho - \phi$
\[
\begin{align*}
\langle 0 | T \rho(p) \phi(-p) | 0 \rangle & = \frac{1}{a - 1} \frac{ie^2}{p^2(p^2 - e^2a^2/4\pi(a-1))} \\
\langle 0 | T \phi(p) \rho(-p) | 0 \rangle & = \frac{1}{a - 1} \frac{ie^2}{p^2(p^2 - e^2a^2/4\pi(a-1))}
\end{align*}
\] (69)

Adding all the contributions we obtain

\[
\langle 0 | T A_{\mu}(k) A_{\nu}(-k) | 0 \rangle = \frac{-i}{k^2 - \frac{e^2a^2}{4\pi(a-1)}} \left[ g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{a - 1} \left( \frac{4\pi}{e^2} - \frac{2}{k^2} \right) + \frac{k_{\mu}\bar{k}_{\nu} + \bar{k}_{\mu}k_{\nu}}{(a - 1)k^2} \right]
\] (70)

### 4.2.2 Fermion propagator

From (18), we obtain the following expression for the fermionic propagator

\[
\langle 0 | T \psi(x) \overline{\psi}(y) | 0 \rangle = iP \overline{G}_F(x - y) + \langle 0 | T \psi_+(x) \overline{\psi}_+(y) | 0 \rangle
\] (71)

where

\[
\langle 0 | T \psi_+(x) \overline{\psi}_+(y) | 0 \rangle = N \int d\phi d\psi d\overline{\psi} \overline{\psi}_+(x) \overline{\psi}_+(y) \times \exp \left( iS_t[\psi, \overline{\psi}, \rho, \phi] + i \int dz \rho(z) j(z, x, y) \right).
\] (72)

After integration over the \(\rho\)-field, we find the same logarithmic ultraviolet divergence already found in (57),

\[
\langle 0 | T \psi_+(x) \overline{\psi}_+(y) | 0 \rangle = \exp \left\{ -\frac{4\pi i}{a - 1} \int \frac{dk}{(2\pi)^2} \frac{1 - e^{-ik \cdot (x - y)}}{k^2} \right\} G'_+(x - y).
\] (73)

The remaining functional integration in \(G'_+(x - y)\) involves the following terms

\[
G'_+(x - y) = N' \int d\phi d\psi d\overline{\psi} \overline{\psi}_+(x) \overline{\psi}_+(y) \exp \left[ i \int dz \overline{\psi}(i\partial - \overline{\partial}) P_+ \psi \right] \times \exp \left\{ i \int dz \left[ \frac{1}{2e^2} \phi(\Box + \frac{e^2}{4\pi(a-1)}) + \frac{1}{a - 1} \phi(z) j(z, x, y) \right] \right\}.
\] (74)

From (74) we easily see that the 1-loop contribution to the fermion self-energy is finite, as well as the contribution of the other loops. The ultraviolet divergence is entirely due to the longitudinal component of the photon, the \(\rho\) field.

### 5 Conclusions

We demonstrated the completely non-perturbative origin of divergences that occur in fermionic correlation functions in two dimensional massless Quantum Electrodynamics. This has been done explicitly, either by summing (wherever it was
possible to do it) the perturbative series or by giving arguments that showed the finiteness of individual terms. It resulted clear that the divergences are a consequence of the lack of gauge invariance (at least in intermediate steps) and that their structure is largely independent of the fact that the anomaly is a genuine one (as is the case for the chiral Schwinger model) or an artifact of regularization (as in the Schwinger model considered under a general regularization, not necessarily preserving gauge invariance). The only difference between the two cases is that, in the second case, the divergences could be circumvented by choosing a regularization that conducted to \( a = 1 \) (preservation of gauge invariance). Apart from this fact (which has its justification only on simplicity, not reflecting any fundamental principle of quantum field theory) there is no reason for choosing one or another value for \( a \) as, in the end, the effective action (the one obtained after integration over the fermions and the Wess-Zumino fields (GIF) or the longitudinal part of the gauge field (GNIF)) is gauge invariant [6].

Moreover, we performed this demonstration both in GIF and GNIF and found equal results. Although this may seem to be no surprise, as we were merely effecting the same integral by different means, it contradicts what is said by Jian-Ge, Qing-Hai and Yao-Yang in [14]. In their paper, the authors find that, when they add the Wess-Zumino term to the original action, there is no divergence in fermionic Green functions, as opposed to what they obtain in its absence, and they justify this exhibiting a different ultraviolet behaviour of the photon propagator in the two approaches. In fact, they missed a crucial point in their paper: they added the Wess-Zumino term before the introduction of the external sources, what is wrong. If one does this, one does not obtain the coupling between the fermions and the Wess-Zumino fields, and it is no surprise if no divergence appears. Also, they obtained two different photon propagators, one in the GIF and other in the GNIF, because they lost an additional coupling term between the photon and the Wess-Zumino field, that comes from the Faddeev-Popov procedure, and that is crucial for the final expressions for the photon propagators. The correct expression for the generating functional in GIF is given by our equation (13).

We would like to remember that the generating functional has to be a functional of something, so the starting expression has to include the external sources. This is just one instance where this kind of mistake can conduct one to completely wrong results, and it is not usually noticed (for an example of the crucial role played by external sources see [20]). If the external sources are carefully considered from the beginning, one finds exactly the same results in both formalisms.

The physical interpretation of this new type of divergence is still unknown for us. The origin of conventional ultraviolet divergences can be traced back to the requirement of relativistic covariance and non-triviality of the field theory under investigation [21]. This prevents a good definition for the field as an operator for all points of the support, introducing divergences when its powers appear. They manifest themselves in the diagonal of Green’s functions, thus allowing themselves to be renormalized through the well known ambiguity of these diagonals.
under local integrated polynomials in the fields (counterterms). In the new situation, although the cure may be similar (but significantly different [13]), the disease may not be the same. The new divergence that appeared multiplies an effective (and finite) two-point fermionic function. We called it *ultraviolet* just because of its form (look at the divergent integral appearing in (29), for example) but not because of its origin. After all, it has its origins in an integration over a quadratic portion of the action, which would contribute with linear terms in the equations of motion, not usually associated to divergences (they do not involve products of operators in the same point). There are evident connections between this divergence and the lack of gauge invariance, but they do not help to clarify the situation, as opposed to what was said above, about conventional UV divergences. Further investigation on this question is being done, and will be reported elsewhere.

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