On the equation $-\Delta u + e^u - 1 = 0$ with measures as boundary data

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Abstract If $\Omega$ is a bounded domain in $\mathbb{R}^N$, we study conditions on a Radon measure $\mu$ on $\partial\Omega$ for solving the equation $-\Delta u + e^u - 1 = 0$ in $\Omega$ with $u = \mu$ on $\partial\Omega$. The conditions are expressed in terms of Orlicz capacities.

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1 Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary and $\mu$ a Radon measure on $\partial\Omega$. In this paper we consider first the problem of finding a function $u$ solution of

$$-\Delta u + e^u - 1 = 0 \quad (1.1)$$

in $\Omega$ satisfying $u = \mu$ on $\partial\Omega$. Let $\rho(x) = \text{dist}(x, \partial\Omega)$, then this problem admits a weak formulation: find a function $u \in L^1(\Omega)$ such that $e^u \in L^1(\Omega)$ satisfying

$$\int_{\Omega} (-u\Delta \zeta + (e^u - 1)\zeta) \, dx = -\int_{\partial\Omega} \frac{\partial \zeta}{\partial \nu} \, d\mu \quad \forall \zeta \in W^{1,\infty}_0(\Omega) \cap W^{2,\infty}(\Omega), \quad (1.2)$$

where $\nu$ is the unit normal outward vector. This type of problem has been initiated by Grillot and Véron [15] in 2-dim in the framework of the boundary trace theory. Much works on boundary trace problems for equation of the type

$$-\Delta u + u^q = 0 \quad (1.3)$$

with $q > 1$, have been developed by Le Gall [18], Marcus and Véron [19], [20], Dynkin and Kuznetsov [9], [10], respectively by purely probabilistic methods, by purely analytic methods or by a combination of the preceding aspects. One of the
main features of the problem with power nonlinearities is the existence of a critical exponent \( q_c = \frac{N+1}{N-1} \) which is linked to the existence of boundary removable sets. Existence of boundary removable points have been discovered by Gmira and Véron [14]. Let us recall briefly the main results for (1.3):

(i) If \( 1 < q < q_c \), then for any \( \mu \in \mathcal{M}_+(\partial \Omega) \) there exists a unique function \( u \in L^1_+ (\Omega) \cap L^q_0 (\Omega) \) which satisfies (1.3) in \( \Omega \) and takes the value \( \mu \) on \( \partial \Omega \) in the following weak sense

\[
\int_{\Omega} (-u \Delta \zeta + u^q \zeta) \, dx = -\int_{\partial \Omega} \frac{\partial \zeta}{\partial \nu} \, d\mu, \quad \forall \zeta \in W^{1,\infty}_0 (\Omega) \cap W^{2,\infty}_0 (\Omega).
\] (1.4)

(ii) If \( q \geq q_c \), the above problem can be solved if and only if \( \mu \) vanishes on boundary Borel subsets with zero \( C_{\frac{q}{2},q'} - \)Bessel capacity. Furthermore a boundary compact set is removable if and only if it has zero \( C_{\frac{q}{2},q'} - \)capacity.

In this article we adapt some of the ideas used for (1.3) to problem

\[-\Delta u + e^u - 1 = 0 \quad \text{in } \Omega \]
\[u = \mu \quad \text{on } \partial \Omega.\] (1.5)

Following the terminology of [5] we say that a measure \( \mu \in \mathcal{M}(\partial \Omega) \) is good if (1.5) admits a weak solution. Let \( P^\Omega(x,y) \) (resp. \( G^\Omega(x,y) \)) be the Poisson kernel (resp. the Green kernel) in \( \Omega \) and \( \mathbb{P}^\Omega[\mu] \) the Poisson potential of a boundary measure \( \mu \) (resp. \( G^\Omega[\phi] \) the Green potential of a bounded measure \( \phi \) defined in \( \Omega \)). A boundary measure \( \mu \) which satisfies

\[
\exp(\mathbb{P}^\Omega[\mu]) \in L^1(\Omega; \rho dx).
\] (1.6)

is called admissible. Since for \( \mu \geq 0 \), \( \mathbb{P}^\Omega[\mu] \) is a supersolution for (1.1), an admissible measure is good (see [23]). Our first result which extends a previous one obtained in [15] is the following.

**Theorem A.** Suppose \( \mu \in \mathcal{M}(\partial \Omega) \) admits Lebesgue decomposition \( \mu = \mu_S + \mu_R \) where \( \mu_S \) and \( \mu_R \) are mutually singular and \( \mu_R \) is absolutely continuous with respect to the \( (N-1) \)-dim Hausdorff measure \( dH^{N-1} \). If

\[
\exp(\mathbb{P}^\Omega[\mu_S]) \in L^1(\Omega; \rho dx),
\] (1.7)

then \( \mu \) is good.

In order to go further in the study of good measures, it is necessary to introduce an Orlicz capacity modelized on the Legendre transform of \( r \mapsto p(r) := e^r - 1. \)
These capacities have been studied by Aissaoui and Benkirane [2] and they inherit most of the properties of the Bessel capacities. The capacity $C_{N,L,ln}$ associated to the problem is constructed later and it has strong connexion with Hardy-Littlewood maximal function. In this framework we obtain the following types of results:

**Theorem B.** Let $\mu \in \mathcal{M}_+(\partial \Omega)$ be a good measure, then $\mu$ vanishes on boundary Borel subsets $E$ with zero $C_{N,L,ln}$-capacity.

We also give below a result of removability of boundary singularities.

**Theorem C.** Let $K \subset \partial \Omega$ be a compact subset with zero $C_{N,L,ln}$-capacity. Suppose $u \in C(\Omega \setminus K) \cap C^2(\Omega)$ is a positive solution of (1.1) in $\Omega$ which vanishes on $K$, then $u$ is identically zero.

In the last part of this paper we apply this approach to the problem

$$-\Delta u + e^u - 1 = \mu,$$

where $\mu$ is a bounded measure, as well as removability questions for internal singularities of solutions of (1.1). In that case the capacity associated to the problem is

$$C_{\Delta,L,ln}(K) = \inf\{\|M[\Delta \eta]\|_{L^1} : \eta \in C^2_0(\Omega) : 0 \leq \eta \leq 1, \eta = 1 \text{ in a neighborhood of } K\}$$

where $M[.]$ denotes Hardy-Littlewood’s maximal function.

**Theorem D.** Let $\mu \in \mathcal{M}_{b+}(\Omega)$ be a bounded good measure, then $\mu$ vanishes on boundary Borel subsets $E$ with zero $C_{\Delta,L,ln}$-capacity.

A characterization of positive measures which have the property of vanishing on Borel subsets $E$ with zero $C_{N,L,ln}$-capacity is also provided. We also give below a result of removability of boundary singularities for sigma moderate solutions (see Definition 4.4).

**Theorem E.** Let $K \subset \Omega$ be a compact subset with zero $C_{\Delta,L,ln}$-capacity. Suppose $u \in C(\Omega \setminus K) \cap C^2(\Omega)$ is a positive sigma moderate solution of (1.1) in $\Omega \setminus K$ which vanishes on $\partial \Omega$, then $u$ is identically zero.

This note is derived from the preliminary report [25], written in 2004 and left escheated since this period. The author is grateful to the referee for his careful verification of the manuscript which enabled several improvements.
2 Good measures

Proof of Theorem A. For simplicity, we shall denote by $\mu_R$ both the regular part of $\mu$ and its density with respect to the Hausdorff measure on $\partial \Omega$. Thus for $k > 0$, we denote by $\mu_{R,k}$ the measure on $\partial \Omega$ with density $\mu_{R,k} = \inf \{k, \mu_R\}$ and denote by $u_k$ the solution of

$$
\begin{aligned}
-\Delta u_k + e^{u_k} - 1 &= 0 & \text{in } \Omega \\
u_k &= \mu_S + \mu_{R,k} & \text{on } \partial \Omega.
\end{aligned}
$$

(2.1)

Such a solution exists because

$$\exp(\mathbb{P}_{\Omega} [\mu_S + \mu_{R,k}]) \leq e^k \exp(\mathbb{P}_{\Omega} [\mu_S])$$

by the maximum principle, and (1.7) implies that $\exp(\mathbb{P}_{\Omega} [\mu_S + \mu_{R,k}]) - 1 \in L^1(\Omega; \rho dx)$. The sequence $u_k$ is nondecreasing. Since, for any $\zeta \in C^1_c(\bar{\Omega})$,

$$\int_{\Omega} (-u_k \Delta \zeta + (e^{u_k} - 1)\zeta) dx = -\int_{\partial \Omega} \frac{\partial \zeta}{\partial \nu} d(\mu_S + \mu_{R,k}),$$

if we take in particular for test function $\zeta$ the solution $\zeta_0$ of

$$
\begin{aligned}
-\Delta \zeta_0 &= 1 & \text{in } \Omega \\
\zeta_0 &= 0 & \text{on } \partial \Omega,
\end{aligned}
$$

(2.2)

we get

$$\int_{\Omega} (u_k + (e^{u_k} - 1)\zeta_0) dx = -\int_{\partial \Omega} \frac{\partial \zeta_0}{\partial \nu} d(\mu_S + \mu_{R,k}) \leq c \|\mu\|_{\mathcal{M}}. \tag{2.3}
$$

Thus $u = \lim_{k \to \infty} u_k$ is integrable,

$$\int_{\Omega} (u + (e^u - 1)\zeta_0) dx \leq c \|\mu\|_{\mathcal{M}},$$

and the convergence of $u_k$ and $e^{u_k}$ to $u$ and $e^u$ hold respectively in $L^1(\Omega)$ and $L^1(\Omega; \rho dx)$ and $u$ satisfies (1.2). \hfill \square

The proof of the next result is directly inspired by [5] where nonlinear Poisson equations are treated.

**Proposition 2.1** The following properties hold:

(i) If $\mu \in \mathcal{M}_+(\partial \Omega)$ is a good measure, then any $\tilde{\mu} \in \mathcal{M}_+(\partial \Omega)$ smaller than $\mu$ is good.

(ii) Let $\{\mu_n\}$ be an increasing sequence of good measures which converges to $\mu$ in the weak sense of measures. Then $\mu$ is good.

(iii) If $\mu \in \mathcal{M}_+(\partial \Omega)$ is a good measure and $f \in L^1_+(\partial \Omega)$, then $f + \mu$ is a good measure.
Proof. We denote by $\partial \Omega_t$ the set of $x \in \Omega$ such that $\rho(x) = t > 0$. Since $\Omega$ is $C^2$ there exists $t_0 > 0$ such that for any $0 < t \leq t_0$, the set $\Omega \setminus \Omega_t$ is diffeomorphic to $(0, t_0) \times \partial \Omega$ by the mapping $x \mapsto (t, \sigma(x))$ where $t = \text{dist}(x, \partial \Omega)$ and $\sigma(x) = \text{proj}_{\partial \Omega}(x)$. Then $x = \sigma(x) - t\nu_{\sigma(x)}$ where $\nu_a$ is the outward normal unit vector to $\partial \Omega$ at $a$. If $\eta$ is defined on $\partial \Omega$ we define a normal extension of $\eta$ at $x \in \partial \Omega_t$ by assigning it the value of $\eta$ at $\sigma(x)$. When there is no ambiguity, we denote this extension by the same notation.

(i) Let $u = u_\mu$ be the solution of (1.5) and $w = \inf\{u, \mathbb{P}^\Omega[\tilde{\mu}]\}$. Since $\mathbb{P}^\Omega[\tilde{\mu}]$ is a supersolution for (1.1), $w$ is a supersolution too. Furthermore $w$ is nonnegative and $e^w - 1 \in L^1(\Omega; \rho dx)$. By Doob’s theorem $w$ admits a boundary trace $\mu^* \in \mathcal{M}_+(\partial \Omega)$ and $\mu^* \leq \tilde{\mu} \leq \mu$. Let $w^*$ be the solution of

$$-\Delta w^* + e^{w^*} - 1 = 0 \quad \text{in } \Omega$$

$$w^* = \tilde{\mu} \quad \text{on } \partial \Omega.$$

then $u \geq w \geq w^*$ and [21],

$$\lim_{t \to 0} \int_{\partial \Omega_t} w^*(t, \cdot) \eta dS_t = \int_{\partial \Omega} \eta d\tilde{\mu} \quad \forall \eta \in C(\partial \Omega).$$

This implies that the boundary trace of $w^*$ is $\tilde{\mu}$ and thus $\mu^* = \tilde{\mu}$. Set $\Omega_t = \{x \in \Omega : \rho(x) > t\}$ and let $v_t$ the solution of

$$-\Delta v_t + e^{v_t} - 1 = 0 \quad \text{in } \Omega_t$$

$$v_t = w \quad \text{on } \partial \Omega_t.$$

Then $v_t \leq w$ in $\Omega_t$. Furthermore $0 < t' < t \implies v_{t'} \leq v_t$ in $\Omega_t$. Then $\tilde{u} = \lim_{t \to 0} v_t$ exists, the convergence holds in $L^1(\Omega)$ and $e^{v_t} \to e^{\tilde{u}}$ in $L^1(\Omega; \rho dx)$ (here we use the fact that $e^w \in L^1(\Omega; \rho dx)$). Because

$$\lim_{t \to 0} \int_{\partial \Omega_t} w(t, \cdot) \eta dS_t = \int_{\partial \Omega} \eta d\tilde{\mu} \quad \forall \eta \in C(\partial \Omega),$$

and $v_t = w$ on $\partial \Omega_t$, is follows that $\tilde{u}$ admits $\tilde{\mu}$ for boundary trace and thus $\tilde{u} = u_{\tilde{\mu}}$.

(ii) Let $u_n = u_{\mu_n}$ be the solutions of (1.5) with boundary value $\mu_n$. The sequence $\{u_n\}$ is increasing. Since

$$\int_{\Omega} (-u_n \Delta \zeta_0 + (e^{u_n} - 1)\zeta_0)dx = -\int_{\partial \Omega} \frac{\partial \zeta_0}{\partial \nu} d\mu_n \leq -\int_{\partial \Omega} \frac{\partial \zeta_0}{\partial \nu} d\mu,$$

we conclude as in the proof of Theorem 1, that $u_n$ increases and converges to a solution $u = u_\mu$ of (1.5) with boundary value $\mu$.

(iii) In the proof of (i) we have actually used the following result: Let $w$ be a nonnegative supersolution of (1.7) such that $e^w \in L^1(\Omega; \rho dx)$ and let $\mu \in \mathcal{M}_+(\partial \Omega)$
be the boundary trace of $w$. Then $\mu$ is good. Let $f \in L^1_+(\partial \Omega)$ and $\mu$ be an good measure. We denote by $u = u_\mu$ the solution of (1.5). For $k > 0$, set $f_k = \min\{k, f\}$. The function $w_k = u_\mu + \mathbb{P}^\Omega[f_k]$ is a nonnegative supersolution, and, since $\mathbb{P}^\Omega[f_k] \leq k$, $e^{w_k} \in L^1(\Omega; \rho dx)$. Furthermore the boundary trace of $w_k$ is $\mu + f_k$. Therefore $\mu + f_k$ is good. We conclude by II that $\mu + f$ is good.

Remark. The assertions (i) and (ii) in Theorem 1 are still valid if we replace $r \mapsto e^r - 1$ by any continuous nondecreasing function $g$ vanishing at 0.

3 The Orlicz space framework

3.1 Orlicz capacities

The set $\mathcal{M}^{exp}(\partial \Omega)$ of nonnegative measures $\mu$ on $\partial \Omega$ such that

$$\exp(\mathbb{P}^\Omega[\mu]) \in L^1(\Omega; \rho dx) \quad (3.1)$$

is not a linear space, but it is a convex subset of $\mathcal{M}_+(\partial \Omega)$. The role of this set comes from the fact that any measure in $\mathcal{M}^{exp}(\partial \Omega)$ is good. Put

$$p(t) = \text{sgn}(s)(e^s - 1), \quad P(t) = e|t| - 1 - |t|,$$

and

$$\bar{p}(s) = \text{sgn}(s) \ln(|s| + 1), \quad P^*(t) = (|t| + 1) \ln(|t| + 1) - |t|.$$ 

Then $P$ and $P^*$ are complementary functions in the sense of Legendre. Furthermore Young inequality holds

$$xy \leq P(x) + P^*(y) \quad \forall (x, y) \in \mathbb{R} \times \mathbb{R},$$

with equality if and only if $x = \bar{p}(y)$ or $y = p(x)$. It is classical to define

$$M_P(\Omega; \rho dx) = \{ \phi \in L^1_{loc}(\Omega) : P(\phi) \in L^1(\Omega; \rho dx) \}, \quad (3.2)$$

and

$$M_{P^*}(\Omega; \rho dx) = \{ \phi \in L^1_{loc}(\Omega) : P^*(\phi) \in L^1(\Omega; \rho dx) \}. \quad (3.3)$$

The Orlicz spaces $L_P(\Omega; \rho dx)$ and $L_{P^*}(\Omega; \rho dx)$ are the vector spaces spanned respectively by $M_P(\Omega; \rho dx)$ and $M_{P^*}(\Omega; \rho dx)$. They are endowed with the Luxemburg norms

$$\|\phi\|_{L_P} = \inf \left\{ k > 0 : \int_{\Omega} P\left( \frac{\phi}{k} \right) \rho dx \leq 1 \right\}. \quad (3.4)$$

and

$$\|\phi\|_{L_{P^*}} = \inf \left\{ k > 0 : \int_{\Omega} P^*\left( \frac{\phi}{k} \right) \rho dx \leq 1 \right\}. \quad (3.5)$$
Furthermore the Hölder-Young inequality asserts [16]

$$\left| \int_\Omega \phi \psi \, \rho \, dx \right| \leq \|\phi\|_{L^p_\rho} \|\psi\|_{L^p_\rho^*} \quad \forall (\phi, \psi) \in L^p(\Omega; \rho \, dx) \times L^{p^*}(\Omega; \rho \, dx). \quad (3.6)$$

Since $P^*$ satisfies the $\Delta_2$-condition, $M_P^*(\Omega; \rho \, dx) = L^p(\Omega; \rho \, dx)$ and $L^p(\Omega; \rho \, dx)$ is the dual space of $L^{p^*}(\Omega; \rho \, dx)$, (see [12], [2]). Furthermore, since

$$\frac{|a| \ln(1 + |a|)}{2} \leq P^*(a) \leq |a| \ln(1 + |a|) \quad \forall a \in \mathbb{R},$$

the space $L^p(\Omega; \rho \, dx)$ is associated with the class $L \ln L(\Omega; \rho \, dx)$ and to the Hardy-Littlewood maximal function (see [12]). We recall its definition: we consider a cube $Q_0$ containing $\Omega$, with sides parallel to the axes. If $f \in L^1(\Omega)$ we denote by $\hat{f}$ its extension by 0 in $Q_0 \setminus \Omega$ and put

$$M_{Q_0}[f](x) = \sup \left\{ \frac{1}{|Q|} \int_Q |f(y)\, dy : Q \in Q_x \right\}$$

where $Q_x$ denotes the set of all cubes containing $x$ and contained in $Q_0$, with sides parallel to the axes. Thus

$$\|f\|_{L^1(\Omega)} := \int_{Q_0} M_{Q_0}[f](x) \, \rho \, dx \approx \|f\|_{L^{p^*}_\rho}. \quad (3.7)$$

**Definition 3.1** The space of all measures on $\partial \Omega$ such that $\mathbb{P}^\Omega[\mu] \in L^p(\Omega; \rho \, dx)$ is denoted by $B^{exp}(\partial \Omega)$ and endowed with the norm

$$\|\mu\|_{B^{exp}} = \|\mathbb{P}^\Omega[\mu]\|_{L^{p^*}_\rho}. \quad (3.8)$$

The set $\mathbb{M}^{exp}(\partial \Omega)$ is a subset of $B^{exp}(\partial \Omega)$.

The following result follows from the definition of the Luxemburg norm.

**Proposition 3.2** If $\mu \in B^{exp}_+(\partial \Omega)$ there exists $a_0 > 0$ such that $a\mu \in \mathbb{M}^{exp}_+(\partial \Omega)$ for all $0 \leq a < a_0$. Conversely, if $\mu \in \mathbb{M}^{exp}_+(\partial \Omega)$, then $a\mu \in B^{exp}(\partial \Omega)$ for all $a > 0$.

The analytic characterization of $B^{exp}(\partial \Omega)$ can be done by introducing the space of normal derivatives of $L \ln L$ functions:

$$N^{L \ln L}(\partial \Omega) = \{ \eta : \rho^{-1} \Delta (\rho^* \mathbb{P}^\Omega[\eta]) \in L \ln L(\Omega; \rho \, dx) \}. \quad (3.9)$$

where $\rho^*$ is a the first eigenfunction of $-\Delta$ in $H^{1,2}_0(\Omega)$ with maximum 1 (and $\lambda$ is the corresponding eigenvalue). Then $c^{-1} \rho \leq \rho^* \leq cp$ for some $c = c(\Omega) > 0$, by Hopf lemma, and

$$\left| \int_{\partial \Omega} \eta \, d\mu \right| = \left| \int_{\Omega} \mathbb{P}^\Omega[\mu] \Delta (\rho^* \mathbb{P}^\Omega[\eta]) \, dx \right| \leq \|\mathbb{P}^\Omega[\mu]\|_{L^{p^*}_\rho} \|\rho^{-1} \Delta (\rho^* \mathbb{P}^\Omega[\eta])\|_{L^{p^*_\rho}}. \quad (3.10)$$
We take for norm on $N^{L \ln L}(\partial \Omega)$

$$\|\eta\|_{N^{L \ln L}} = \|\rho^{-1} \Delta (\rho^* \mathbb{P}^\Omega[\eta])\|_{L^p_{\rho^*}},$$  \hfill (3.11)

and define the $C_{N^{L \ln L}}$-capacity of a compact subset $K$ of $\partial \Omega$ by

$$C_{N^{L \ln L}}(K) = \inf \{\|\eta\|_{N^{L \ln L}} : \eta \in C^2(\partial \Omega), 0 \leq \eta \leq 1, \eta \geq 1 \text{ in a neighborhood of } K\}. \hfill (3.12)$$

Considering the bilinear form $\mathcal{H}$ on $L^p_{\rho^*}(\partial \Omega) \times L^p_{\rho^*}(\partial \Omega)$

$$\mathcal{H}(\eta, \mu) := -\int_{\Omega} \mathbb{P}^\Omega[\mu] \Delta (\rho^* \mathbb{P}^\Omega[\eta]) \, dx$$  \hfill (3.13)

then

$$\mathcal{H}(\eta, \mu) = -\int_{\Omega} \int_{\partial \Omega} P^\Omega(x, y) d\mu(y) \Delta (\rho^* \mathbb{P}^\Omega[\eta])(x) \, dx$$  \hfill (3.14)

It is classical to define

$$C_{N^{L \ln L}}^*(K) = \sup \{\mu(K) : \mu \in \mathfrak{M}_+(\partial \Omega), \mu(K^c) = 0, \|\mathbb{P}^\Omega[\mu]\|_{L^p_{\rho^*}} \leq 1\}. \hfill (3.15)$$

The following result due to Fuglede [13] (and to Aïssaoui-Benkirane in the Orlicz space framework [2]) is a consequence of the Kneser-Fan min-max theorem.

**Proposition 3.3**  For any compact set $K \subset \partial \Omega$, there holds

$$C_{N^{L \ln L}}^*(K) = C_{N^{L \ln L}}(K). \hfill (3.16)$$

As a direct consequence of (3.10), we have the following

**Proposition 3.4**  If $\mu \in B_{\exp}^+(\partial \Omega)$, it does not charge Borel subsets with $C_{N^{L \ln L}}$-capacity zero.

### 3.2 Good measures and removable sets

**Proof of Theorem B.**  If $K$ is compact and $C_{N^{L \ln L}}(K) = 0$, there exist a sequence $\{\eta_n\} \subset C^2(\partial \Omega)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in a neighborhood of $K$ and

$$\lim_{n \to \infty} \|\eta_n\|_{N^{L \ln L}} = \|\rho^{-1} \Delta (\rho^* \mathbb{P}^\Omega[\eta_n])\|_{L^p_{\rho^*}} = 0. \hfill (3.17)$$

Take $\rho^* \mathbb{P}^\Omega[\eta_n]$ as a test function, then

$$\int_{\Omega} (\rho^* \mathbb{P}^\Omega[\eta_n] + (e^u - 1)\rho^* \mathbb{P}^\Omega[\eta_n])] \, dx = -\int_{\partial \Omega} \frac{\partial (\rho^* \mathbb{P}^\Omega[\eta_n])}{\partial \nu} \, d\mu$$
Since \( -\frac{\partial (\rho^* P^\Omega [\eta_n])}{\partial \nu} = \eta_n \) and \( \mu > 0 \), there holds \( \int_{\partial \Omega} \frac{\partial (\rho^* P^\Omega [\eta_n])}{\partial \nu} \, d\mu \geq \mu(K) \).

Furthermore
\[
\left| \int_{\Omega} u \Delta (\rho^* P^\Omega [\eta_n]) \, dx \right| \leq \|u\|_{L^p_\rho} \|\rho^{-1} \Delta (\rho^* P^\Omega [\eta_n])\|_{L^{p_2}_\rho}. \tag{3.18}
\]

Then
\[
\mu(E) \leq \int_{\Omega} (e^u - 1) \rho^* P^\Omega [\eta_n] \, dx + \|u\|_{L^p_\rho} \|\rho^{-1} \Delta (\rho^* P^\Omega [\eta_n])\|_{L^{p_2}_\rho}.
\]

By the same argument as in [5], \( \lim_{n \to \infty} \rho^* P^\Omega [\eta_n] = 0 \), a.e. in \( \Omega \), and there exists a nonnegative \( L^1_\rho \)-function \( \Phi \) such that \( 0 \leq \rho^* P^\Omega [\eta_n] \leq \Phi \). By (3.17), (3.18) and Lebesgue’s theorem, \( \mu(K) = 0 \). \( \square \)

**Definition 3.5** A subset \( E \subset \partial \Omega \) is said removable for equation (1.1), if any positive solution \( u \in C^2(\Omega) \) of (1.1) in \( \Omega \), which is continuous in \( \Omega \setminus \bar{E} \) and vanishes on \( \partial \Omega \setminus E \), is identically zero.

**Proof of Theorem C.** Let \( u \in C(\overline{\Omega \setminus K}) \) be a solution of (1.1) which is zero on \( \partial \Omega \setminus K \). As a consequence of Keller-Osserman estimate (see e.g. [23]), there holds
\[
u(x) \leq 2 \ln \left( \frac{1}{\rho(x)} \right) + D, \tag{3.19}
\]
but since \( u \) vanishes on \( \partial \Omega \setminus K \), we can extend it by 0 in \( \Omega^c \); in order it becomes a subsolution and obtain, always by Keller-Osserman method, that \( \rho(x) \) can be replaced by \( \rho_K(x) := \text{dist} (x, K) \) in (3.19). Furthermore, for any open subset containing \( K \), there exists a constant \( c_G \) such that \( u(x) \leq c_G \rho(x) \) for all \( x \in \overline{\Omega \setminus G} \).

Let \( \{ \eta_n \} \subset C^2(\partial \Omega) \) such that \( 0 \leq \eta_n \leq 1 \), \( \eta_n = 1 \) in a relative neighborhood \( V = G \cap \partial \Omega \) of \( K \), where \( G \) is open. Put \( \theta_n = 1 - \eta_n \). The function \( \zeta_n = \rho^* P^\Omega [\theta_n] \) satisfies \( \Delta \zeta_n = -\lambda \zeta_n + 2 \nabla \rho^* \cdot \nabla P^\Omega [\theta_n] \). Therefore \( |\Delta \zeta_n| \) remains bounded in \( G \cap \Omega \) where there also holds \( \zeta_n(x) \leq c \rho^2(x) \). Using (3.19) and an easy approximation argument, we can take \( \zeta_n \) as a test function and obtain
\[
\int_{\Omega} (-u \Delta \zeta_n + (e^u - 1) \zeta_n) \, dx = 0.
\]

We derive
\[
-\int_{\Omega} u \Delta \zeta_n \, dx = -\int_{\Omega} \zeta_n^{-1} \Delta \zeta_n u \zeta_n \, dx \\
\geq -2^{-1} \int_{\Omega} (e^u - 1 - u) \zeta_n \, dx - \int_{\Omega} Q(\zeta_n^{-1} \Delta (\rho^* P^\Omega [\eta_n])) \zeta_n \, dx,
\]

9
where
\[ Q(r) = (|r| + 2^{-1}) \ln(2 |r| + 1) - |r| \leq C |r| \ln(|r| + 1) \quad \forall r \in \mathbb{R}. \quad (3.20) \]

Therefore
\[
\int_{\Omega} (e^u - 1 - u) \zeta_n \, dx \leq 2C \int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) \, dx, \quad (3.21)
\]

since \( \zeta_n^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \leq \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \). Furthermore
\[
\ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) = - \ln \rho + \ln(\rho + \rho^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) \\
\leq - \ln \rho + \ln(1 + \rho^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|)
\]

But (we can assume \( \rho \leq 1 \))
\[
\int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) \, dx \\
\leq - \int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln \rho \, dx + \int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln(1 + \rho^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) \, dx,
\]

and
\[
\int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln \rho^{-1} \, dx \\
= \int_{\{ |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \leq 1 \}} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln \rho^{-1} \, dx + \int_{\{ |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| > 1 \}} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln \rho^{-1} \, dx \\
\leq \int_{\{ |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \leq 1 \}} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln \rho^{-1} \, dx + \int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln(1 + \rho^{-1} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) \, dx
\]

By assumption \( C_{N,L,L} \( K \) = 0 \), then we take \( \{ \eta_n \} \) such that \( \| \eta_n \|_{N,L,L} \to 0 \) and
\[
\lim_{n \to \infty} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| = 0 \quad \text{a. e. in } \Omega,
\]
at least up to some subsequence. Thus
\[
\lim_{n \to \infty} \int_{\Omega} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])| \ln(1 + \rho^{-2} |\Delta(\rho^* \mathbb{P}^\Omega[\eta_n])|) \, dx = 0. \quad (3.22)
\]

Using (3.21), we derive \( u = 0 \).

Conversely, assume that \( C_{N,L,L} \( K \) > 0 \). By Proposition 3.3 there exists a non-negative non-zero measure \( \mu \in \mathcal{M}_+(\partial \Omega) \) such that \( \mu(\partial \Omega^c) = 0 \) in the space \( B^e_{\mathbb{P}^\partial \Omega} \). This means that \( \theta \mu \in M^e_{\mathbb{P}^\partial \Omega} \) for some \( \theta > 0 \). Thus problem (1.5) admits a
Several open problems can be posed

1- If a measure $\mu$ is good, does there exist an increasing sequence of measures $\{\mu_n\}$ which converges to $\mu$ such that $\theta_n \mu_n$ is admissible for some $\theta_n > 0$ ?

2- If a measure $\mu$, singular with respect to $\mathcal{H}^{N-1}$ is good does, it exist an increasing sequence of admissible measures $\{\mu_n\}$ converging to $\mu$ ?

3- If a measure $\mu$ does not charge Borel sets with $C^L \ln L$-capacity zero, doest it exist $\theta > 0$ such that $\theta \mu$ is admissible ?

4- If a singular measure $\mu$ is good, is $(1 - \delta) \mu$ admissible for any $\delta \in (0, 1)$ ?

### 3.3 More general nonlinearities

In the section we consider the problem

$$
\begin{align*}
-\Delta u + P(u) &= 0 \quad \text{in } \Omega \\
u &= \mu \quad \text{on } \partial \Omega,
\end{align*}
$$

(3.23)

where $P$ is a convex increasing function vanishing at 0 and such that $\lim_{r \to \infty} P(r)/r = \infty$. In Theorem A-P, (1.7) should be replaced by

$$
P(\mathbb{P}^\Omega[\mu_S]) \in L^1(\Omega; \rho dx).
$$

(3.24)

In Proposition 2.1-P, (i), (ii) and (iii) still hold. For simplicity we assume that $P$ is a $N$-function in the sense of Orlicz spaces i.e.

$$
P(r) = \int_0^r p(s)ds
$$

where $p$ is increasing, vanishes at 0 and tends to infinity at infinity. Let $P^*$ be the conjugate $N$-function, $L_P(\Omega; \rho dx)$ and $L_{P^*}(\Omega; \rho dx)$ the corresponding Orlicz spaces endowed with the Luxemburg norms. Then Proposition 3.4-P is valid, provided the space

$$
B^P(\partial \Omega) := \{ \mu \in \mathcal{M}(\partial \Omega) : \mathbb{P}^\Omega[\mu] \in L_P(\Omega; \rho dx) \}
$$

endowed with its natural norm replaces $B^{exp}(\partial \Omega)$ with the norm (1.10). We set

$$
N^{P^*}(\partial \Omega) = \{ \eta : \rho^{-1} \Delta (\rho^* \mathbb{P}^\Omega[\eta]) \in L_{P^*}(\Omega; \rho dx) \}
$$

with corresponding norm

$$
\| \eta \|_{N^{P^*}} = \left\| \rho^{-1} \Delta (\rho^* \mathbb{P}^\Omega[\eta]) \right\|_{L_{P^*}}
$$
and the corresponding capacity $C_{N-P^*}$. The proof of Proposition 3.4-P, consequence of Young inequality between Orlicz space is valid without modification. However, it appears that the full characterization of removable sets cannot be adapted without further properties of the function $P^*$ like the $\Delta_2$-condition. Some results in this directions have been obtained in [17] where a necessary and sufficient condition for removability of boundary set is given, under a very restrictive growth condition on $P$ which reduces the nonlinearity to power-like with limited exponent.

4 Internal measures

Several above techniques can be extended to the following types of problem in which $\mu \in M^+_b(\Omega)$:

$$-\Delta u + e^u - 1 = \mu \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial \Omega.$$  \hspace{1cm} (4.1)

For this specific problem many interesting results can be found in [3] where the analysis of $\mu$ is made by comparison with the Hausdorff measure in dimension $N-2$, $\mathcal{H}^{N-2}$. It is proved in particular that if a measure $\mu$ satisfies $\mu \leq 4\pi \mathcal{H}^{N-2}$, then problem (4.1) admits a solution, while if $\mu$ charges some Borel set $A$ with Hausdorff dimension less than $N-2$, no solution exists. The results we provide are different and in the Orlicz capacities framework.

We define the classes $M_P(\Omega)$ and $M_{P^*}(\Omega)$ similarly to $M_P(\Omega; \rho dx)$ and $M_{P^*}(\Omega; \rho dx)$ except that the measure $\rho dx$ is replaced by the Lebesgue measure $dx$. The Orlicz spaces $L_P(\Omega)$ and $L_{P^*}(\Omega)$ are defined from $M_P(\Omega)$ and $M_{P^*}(\Omega)$ and endowed with the respective Luxemburg norms $\| \|$ and $\| \|^*$. We put

$$\Delta^{L\ln L}(\Omega) := \{ \eta \in W^{1,1}_0(\Omega) : \Delta \eta \in L_{P^*}(\Omega) \},$$

with natural norm

$$\| \eta \|_{\Delta^{L\ln L}} := \| \eta \|_{L^1} + \| \Delta \eta \|_{L_{P^*}}.$$  \hspace{1cm} (4.2)

The norm in $M_{P^*}(\Omega)$ can be characterized using the Hardy-Littlewood maximal function $f \mapsto M_{Q_0}[f]$ since

$$\| f \|_{L\ln L} := \int_{Q_0} M_{Q_0}[f](x) \, dx \approx \| f \|_{L_{P^*}}.$$  \hspace{1cm} (4.3)

Since $P^*$ satisfies the $\Delta_2$-condition, $C_0^\infty(\Omega)$ is dense in $\Delta^{L\ln L}(\Omega)$. Inequality (3.10) becomes

$$\left| \int_{\Omega} \eta d\mu \right| = \left| \int_{\Omega} \eta \Delta G_\Omega[\mu] \, dx \right| = \left| \int_{\Omega} \Delta G_\Omega[\mu] \Delta \eta \, dx \right| \leq \| \nabla \Omega[\mu] \|_{L_P} \| \Delta \eta \|_{L_{P^*}},$$

\hspace{1cm} (4.5)
for $\eta \in C^{1,1}_c(\Omega)$. We define the $C_{\Delta L, \ln L}$-capacity of a compact subset $K$ of $\partial \Omega$ by

$$C_{\Delta L, \ln L}(K) = \inf\{\|\Delta \eta\|_{L^p} : \eta \in C^2_c(\Omega), 0 \leq \eta \leq 1, \eta = 1 \text{ in a neighborhood of } K\},$$

(4.6)

By the min-max theorem there holds

$$C_{\Delta L, \ln L}(K) = \sup\{\mu(K) : \mu \in M^b_+ + (\Omega), \mu(K^c) = 0, \|G^{\Omega}[\mu]\|_{L^p} \leq 1\}.$$  (4.7)

**Remark.** The characterization of the $C_{\Delta L, \ln L}$-capacity is not simple, however, by a result of [7, Th1], there holds

$$\|D^2 \eta\|_{L^{1,\infty}} \leq C \|\Delta \eta\|_{L^{\ln L}} \quad \forall \eta \in C^{1,1}_c(\Omega)$$

(4.8)

where $L^{1,\infty}(\Omega)$ denotes the weak $L^1$-space, that is the space of all measurable functions $f$ defined in $\Omega$ satisfying

$$\text{meas}\left\{x \in \Omega : |f(x)| > t\right\} \leq \frac{C}{t}, \quad \forall t > 0$$

(4.9)

and $\|f\|_{L^{1,\infty}}$ is the smallest constant such that (4.9) holds.

**Definition 4.1** The space of all bounded measures in $\Omega$ such that $G^{\Omega}[\mu] \in L^p(\Omega)$ is denoted by $B^{\text{exp}}(\Omega)$, with norm

$$\|\mu\|_{B^{\text{exp}}} = \|G^{\Omega}[\mu]\|_{L^p}.$$  (4.10)

The subset of nonnegative measures $\mu$ in $\Omega$ such that $\exp(G^{\Omega}[\mu]) \in L^1(\Omega)$ is denoted by $M^{\text{exp}}_+(\Omega)$.

Proposition 3.4 and Theorem B admit the following counterparts

**Proposition 4.2** If $\mu \in B^{\text{exp}}_+(\Omega)$, it does not charge Borel subsets with $C_{\Delta L, \ln L}$-capacity zero.

**Theorem 4.3** Let $\mu \in M^b_+(\Omega)$ be a good measure, then $\mu$ vanishes on Borel subsets $E$ with zero $C_{\Delta L, \ln L}$-capacity.

**Proof.** The proof of Proposition 4.2 is straightforward from the definition. For Theorem 4.3 we consider a solution $u$ of (4.1) and $K \subset \Omega$ a compact set. Then there exists a sequence $\{\eta_n\} \subset C^2_0(\Omega)$ satisfying $0 \leq \eta_n \leq 1, \eta_n = 1$ in a neighborhood $V$ of $K$ such that $\lim_{n \to \infty} \|\Delta \eta_n\|_{L^{p^*}} = 0$. Then

$$\int_\Omega (-u \Delta \eta_n^3 + (e^u - 1)\eta_n)\,dx = \int_\Omega \eta_n^3\,d\mu \geq \mu(K).$$

13
Since $u$ is positive and $-u\Delta\eta_n^3 \leq -u\Delta\eta_n$ we derive by Hölder-Young inequality (3.6)

$$3 \|u\|_{L^p} \|\Delta\eta_n\|_{L^p} + \int_\Omega (e^u - 1) \eta_n \, dx \geq \mu(K). \quad (4.11)$$

Notice that $u \in L_p(\Omega; \, dx)$ since $e^u \in L^1(\Omega)$. If $C_{\Delta L_{\infty L}}(K) = 0$, the sequence $\{\eta_n\}$ can be taken such that $\|\Delta\eta_n\|_{L_p} + \|\eta_n\|_{L^1} \to 0$. Therefore $\mu(K) = 0$. □

Following Dynkin [10] (although in a slightly different context) it is natural to introduce the notions of moderate and sigma-moderate solutions.

**Definition 4.4** Let $K \subset \Omega$ be compact. A positive solution $u$ of (2.1) in $\Omega \setminus K$ is called moderate if $e^u \in L^1(\Omega \setminus K)$. It is sigma-moderate if there exists an increasing sequence $\{u_n\}$ of moderate solutions in $\Omega \setminus K$ which converges to $u$ in $\Omega \setminus K$.

**Theorem 4.5** Let $K \subset \Omega$ be compact. A sigma-moderate solution of (1.1) in $\Omega \setminus K$ is a solution in $\Omega$ if and only if $C_{\Delta L_{\infty L}}(K) = 0$.

**Proof.** We first assume that $u$ is a moderate solution. Let $\{\eta_n\} \subset C^2_0(\Omega)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in a neighborhood $U$ of $K$ and $\|\Delta\eta_n\|_{L^p} + \|\eta_n\|_{L^1} \to 0$ when $n \to \infty$. If $\zeta \in C^2_0(\Omega)$, we set $\zeta_n = (1 - \eta_n)\zeta$. Then

$$\int_\Omega (-u\Delta\zeta_n + (e^u - 1)\zeta_n) \, dx = 0.$$

Therefore

$$\int_\Omega (-u(1 - \eta_n)\Delta\zeta + (e^u - 1)\zeta_n) \, dx = -\int_\Omega (\zeta\Delta\eta_n + 2\nabla\zeta \cdot \nabla\eta_n) \, u \, dx. \quad (4.12)$$

Since $e^u - 1 \in L^1(\Omega \setminus K)$ and $|K| = 0$, $e^u - 1 \in L^1(\Omega)$. But $0 \leq u \leq e^u - 1$, therefore $u \in L^1(\Omega)$. By Lebesgue’s theorem

$$\lim_{n \to \infty} \int_\Omega (-u(1 - \eta_n)\Delta\zeta + (e^u - 1)\zeta_n) \, dx = \int_\Omega (-u\Delta\zeta + (e^u - 1)\zeta) \, dx.$$

Furthermore

$$\left| \int_\Omega (\zeta\Delta\eta_n + 2\nabla\zeta \cdot \nabla\eta_n) \, u \, dx \right| \leq \left( \|\zeta\|_{L^\infty} \|\Delta\eta_n\|_{L^p} + 2 \|\nabla\zeta\|_{L^\infty} \|\nabla\eta_n\|_{L^p} \right) \|u\|_{L^p}.$$

By standard regularity $\|\nabla\eta_n\|_{L^r} \leq \|\Delta\eta_n\|_{L^1}$ for any $r \in (1, \frac{N}{N-1})$. Since

$$\int_\Omega |\nabla\eta_n| \ln(1 + |\nabla\eta_n|) \, dx \leq C \int_\Omega (|\nabla\eta_n|^r + |\nabla\eta_n|) \, dx,$$
the right-hand side of (4.12) tends to zero as $n \to \infty$ which implies that $u$ is a solution in whole $\Omega$. If $u$ is a sigma-moderate solution in $\Omega \setminus K$, it is the limit of an increasing sequence $\{u_n\}$ of positive moderate solutions in $\Omega \setminus K$. These solutions are solutions in whole $\Omega$, so is $u$. Finally, if $C_{\Delta \ln L}(K) > 0$, by the dual definition (4.7) there exists a positive bounded measure $\mu$ with support in $K$ such that $\mu(K) > 0$ and $\|G^\Omega[\mu]\|_{L_P} \leq 1$. For this measure problem (4.1) admits a solution and this solution is not a solution of (1.1) in whole $\Omega$. □

When the solution is not sigma-moderate we have a weaker result.

**Theorem 4.6** Let $K \subset \Omega$ be compact such that

$$\inf \left\{ \int_\Omega |\Delta \eta| + |\nabla \eta|^2dx : \eta \in C^2_c(\Omega), 0 \leq \eta \leq 1, \eta = 1 \text{ in a neighborhood of } K \right\} = 0.$$  

(4.13)

If $u$ is a positive solution of (1.1) in $\Omega \setminus K$, it can be extended as a solution in $\Omega$.

**Proof.** If $\psi \in C^\infty_c(\Omega)$ is nonnegative, there holds

$$\int_\Omega (e^u - 1)\psi dx = \int_\Omega u\Delta \psi dx = \int_\Omega u(\psi^{-1}\Delta \psi)\psi dx$$

$$\leq \frac{1}{2} \int_\Omega (e^u - 1 - u)\psi dx + c \int_\Omega Q(\psi^{-1}\Delta \psi)|\psi dx,$$

where $Q$ is defined in (3.20). Consider $\phi \in C^\infty_c(\Omega)$, $0 \leq \phi \leq 1$, $\phi = 1$ in a neighborhood $G$ of $K$ and a sequence of functions $\{\eta_n\} \subset C^\infty_c(\Omega)$ such that $0 \leq \eta_n \leq 1$, $\eta_n = 1$ in some neighborhood of $K$. We set $\psi = \psi_n^3 = \phi^3(1 - \eta_n)^3$ and derive

$$Q(\psi^{-1}|\Delta \psi|) \leq (3\psi_n^{-1}|\Delta \psi_n| + 6\psi_n^{-2}|\nabla \psi_n|^2) \ln(1 + 3\psi_n^{-1}|\Delta \psi_n| + 6\psi_n^{-2}|\nabla \psi_n|^2)$$

$$\leq 6\psi_n^{-1}|\Delta \psi_n| \ln(1 + 3\psi_n^{-1}|\Delta \psi_n|) + 12\psi_n^{-2}|\nabla \psi_n|^2 \ln(1 + 6\psi_n^{-2}|\nabla \psi_n|^2)$$

It follows from the Keller-Osserman estimate for this type of nonlinearity (see e.g. [23]) that $u$ is bounded on each compact subset of $\Omega \setminus K$; it is in particular the case of on $H := \text{supp}(\phi) \setminus G$. Using the fact that $\phi$ is constant on $G$, which implies $|\Delta \psi_n| \leq |\Delta \eta_n| + c_1$, we derive

$$6\psi_n^2|\Delta \psi_n| \ln(1 + 3\psi_n^{-1}|\Delta \psi_n|) \leq 6\psi_n^2|\Delta \psi_n| (\ln(\psi_n + 3|\Delta \psi_n|) - \ln \psi_n)$$

$$\leq 6|\Delta \psi_n| \ln(1 + |\Delta \psi_n|) + c_2|\Delta \psi_n| + c_3.$$ 

Similarly

$$12\psi_n|\nabla \psi_n|^2 \ln(1 + 6\psi_n^{-2}|\nabla \psi_n|^2) \leq 12\psi_n|\nabla \psi_n|^2 (\ln(\psi_n^2 + 6|\nabla \psi_n|^2) - 2 \ln \psi_n)$$

$$\leq 12|\nabla \psi_n|^2 (\ln(1 + |\nabla \psi_n|) + c_4|\nabla \psi_n|^2 + c_5),$$

15
where the \( c_j \) do not depend on \( n \). Since there always hold (as \( 0 \leq \eta_n \leq 1 \) and \( \Omega \) is bounded)
\[
e \int_{\Omega} \eta_n^2 dx \leq \int_{\Omega} |\nabla \eta_n|^2 dx \leq \int_{\Omega} |\Delta \eta_n| dx,
\]
we derive
\[
\int_G (e^u - 1 - u) dx \leq \limsup_{n \to \infty} \int_{\Omega} (e^u - 1 - u) \psi_n^3 dx \\
\leq 2 \limsup \int_{\Omega} Q(\psi_n^{-3} |\Delta \psi_n^3|) \psi_n^3 dx \leq |H|(c_3 + c_5).
\]
Therefore \( u \) is moderate and the conclusion follows from Theorem 4.5.

Remark. It is an open question whether all positive solutions of (1.1) in \( \Omega \setminus K \) are sigma-moderate.

4.1 More on good measures

The main characterization of good measures is the following

**Theorem 4.7** Assume \( \mu \) is a positive good measure, then there exists an increasing sequence \( \{ \mu_n \} \subset B_{+}^{\exp}(\Omega) \) which converges weakly to \( \mu \).

The proof will necessitate several intermediate results which are classical in the framework of Lebesgue measure or Bessel capacities, but appear to be new for Orlicz capacities.

**Lemma 4.8** Let \( K \subset \Omega \), then \( C_{\Delta_{L,\ln L}}(K) = 0 \) if and only if there exists \( \eta \in \Delta_{L,\ln L}(\Omega) \) such that \( \eta \geq 0 \) and \( K \subset \{ y \in \Omega : \eta(y) = \infty \} \).

**Proof.** By the definition of the capacity, for any \( \lambda > 0 \) and \( \eta \in \Delta_{L,\ln L}(\Omega) \), \( \eta \geq 0 \),
\[
C_{\Delta_{L,\ln L}} (\{ y \in \Omega : \eta(y) \geq \lambda \}) \leq \frac{1}{\lambda} \| \eta \|_{\Delta_{L,\ln L}}.
\]
This implies
\[
C_{\Delta_{L,\ln L}} (\{ y \in \Omega : \eta(y) = \infty \}) = 0.
\]

**Lemma 4.9** Suppose \( \{ \eta_j \} \) is a Cauchy sequence in \( \Delta_{L,\ln L}(\Omega) \). Then there exist a subsequence \( \{ \eta_{j_i} \} \) and \( \eta \in \Delta_{L,\ln L}(\Omega) \) such that
\[
\lim_{i_i \to \infty} \eta_{j_i} = \eta,
\]
uniformly outside an open subset of arbitrary small \( C_{\Delta_{L,\ln L}} \)-capacity.

16
Proof. By Lemma 4.8, $\eta_j$ and $\eta$ are finite outside a set $F$ with zero $C_{\Delta L^{\ln L}}$-capacity. There exists a subsequence $\{\eta_{j\ell}\}$ such that

$$\|\eta_{j\ell} - \eta\|_{\Delta L^{\ln L}} \leq 2^{-2\ell}.$$  

Put $E_\ell = \{y \in \Omega : \eta_{j\ell}(y) - \eta(y) \geq 2^{-\ell}\}$. By (4.14) $C_{\Delta L^{\ln L}}(E_\ell) \leq 2^{-\ell}$, and if $G_m = \bigcup_{\ell \geq m} E_\ell$, there holds $C_{\Delta L^{\ln L}}(G_m) \leq 2^{1-m}$. Therefore

$$C_{\Delta L^{\ln L}}(\cap_{m \geq 1} G_m) = 0.$$  

Since for any $y \notin G_m \cup F$, there holds

$$|(\eta_{j\ell} - \eta)(y)| \leq 2^{-\ell},$$

the claim follows. \hfill \square

Lemma 4.10 If $\eta \in \Delta L^{\ln L}(\partial\Omega)$ it has a unique quasi-continuous representative with respect to the capacity $C_{\Delta L^{\ln L}}$.

Proof. Uniqueness is clear as in the Bessel capacity case [1, Chap 6]. Let $\{\eta_j\} \subset C^2_0(\Omega)$ be a sequence which converges to $\eta$ in $\Delta L^{\ln L}(\Omega)$. Then there exists a subsequence $\{\eta_{j\ell}\}$ such that $\eta_{j\ell}$ converges to $\eta$ uniformly on the complement of an open set of arbitrarily small $C_{\Delta L^{\ln L}}$-capacity. This is the claim. \hfill \square

Proof of Theorem 4.7. The method is adapted from [11, Th 8], [4, Lemma 4.2]. By Lemma 4.10 we can define the functional $h$ on $\Delta L^{\ln L}(\Omega)$ by

$$h(\eta) = \int_\Omega \eta d\mu \quad \forall \eta \in \Delta L^{\ln L}(\Omega),$$

where $\eta$ stands for the $C_{\Delta L^{\ln L}}$-quasi-continuous representative of $\eta$. Notice that we can write

$$h(\eta) = -\int_\Omega G_\Omega[\mu]\eta dx = -\int_\Omega G_\Omega[\mu]\Delta\eta dx$$

The following steps are similar to the previous proofs:

Step 1- The functional $h$ is convex, positively homogeneous and l.s.c. The convexity and the homogeneity are clear. If $\eta_n \to \eta$ in $\Delta L^{\ln L}(\partial\Omega)$, then by Lemma 4.10 we can extract a subsequence which is converging everywhere except for a set with zero capacity. The conclusion follows from Fatou’s lemma.

Step 2- Since $L_P(\Omega)$ is the dual space of $L_{P^*}(\Omega)$, for any continuous linear form $\alpha$ on $\Delta L^{\ln L}(\Omega)$ there exists $\beta \in L_P(\Omega)$ such that

$$\alpha(\eta) = -\int_\Omega \beta\Delta\eta dx \quad \forall \eta \in \Delta L^{\ln L}(\Omega).$$
Therefore, in the sense of distributions there holds
\[ \alpha(\eta) = -\langle \Delta \beta, \eta \rangle \quad \forall \eta \in C_0^\infty(\Omega). \]

**Step 3** By the geometric Hahn-Banach theorem, \( h \) is the upper convex hull of the continuous linear functionals on \( \Delta L^\text{ln}(\partial \Omega) \) it dominates. Fix a function \( \eta_0 \in C_0^\infty(\Omega) \) and \( \epsilon > 0 \), there exists a continuous linear form \( \alpha \) on \( \Delta L^\text{ln}(\Omega) \) and constants \( a, b \) such that
\[
a + bt + \alpha(\eta) \leq 0 \quad \forall (\eta, t) \in \mathcal{E} := \{(\eta, t) \in \Delta L^\text{ln}(\Omega) \times \mathbb{R} : h(\eta) \leq t\},
\]
and
\[
a + b(h(\eta_0) - \epsilon) + \alpha(\eta_0) > 0.
\]
The same ideas as in [4, Lemma 4.2] yields successively to \( a = 0 \) and \( b < 0 \). If we put \( \sigma(\eta) = -b^{-1}\alpha(\eta) \) we derive \( \sigma(\eta) \leq h(\eta) \) for all \( \eta \in \Delta L^\text{ln}(\Omega) \). This implies in particular that \( \sigma(\eta) \leq 0 \) if \( \eta \leq 0 \), thus \( \sigma \) is a positive linear form on \( \Delta L^\text{ln}(\Omega) \).

Therefore there exist a Radon measure \( \nu \) on \( \Omega \) and \( \beta \in L^p(\Omega) \) such that
\[ -\Delta \beta = \nu, \quad 0 \leq \nu \leq \mu \text{ and } \int_\Omega \eta_0 d\mu \leq \epsilon + \int_\Omega \eta_0 d\nu. \]

**Step 4** Considering an increasing sequence of compact sets \( K_j \) such that \( K_j \subset K_{j+1} \) and \( \bigcup_j K_j = \Omega \), we construct for each \( j \in \mathbb{N}^* \) a Radon measure \( \nu_j \) and \( \beta_j \in L_P(\Omega) \) such that
\[ -\Delta \beta_j = \nu_j, \quad 0 \leq \nu_j \leq \mu \text{ and } \int_{K_j} d\mu \leq j^{-1} + \int_{K_j} d\nu_j. \]

At last we can assume that the sequence \( \{\nu_j\} \) is increasing since if \( -\Delta \beta_j = \nu_j \) for \( j = 1, 2 \), then
\[ -\Delta \beta_{1,2} = \sup\{\nu_1, \nu_2\} \leq \nu_1 + \nu_2 = -\Delta \beta_1 - \Delta \beta_2 \]
thus \( \beta_{1,2} \in L_P(\Omega) \). Iterating this process, we can replace the sequence \( \{\nu_j\} \) by \( \{\nu'_j\} := \{\nu_1, \sup\{\nu_1, \nu_2\}, \sup\{\nu_3, \sup\{\nu_1, \nu_2\}\}, \ldots\} \). The sequence \( \{\nu'_j\} \) is increasing, converges to \( \mu \) and since \( \beta_j = \mathbb{G}^\Omega[\nu'_j] \) with \( \beta_j \in L_P(\Omega) \), \( \nu'_j \) belongs to \( B^\exp(\Omega) \).

As a consequence of this result and the characterization of linear functionals over \( L \ln L(\Omega) \), the following result holds.

**Corollary 4.11** Assume \( \mu \) is a bounded positive good measure in \( \Omega \), then there exist an increasing sequence of positive measures \( \nu_j \) in \( \Omega \) and positive real numbers \( \theta_j \) such that \( \nu_j \to \mu \) in the weak sense of measures and \( \exp(\theta_j \mathbb{G}^\Omega[\nu_j]) \in L^1(\Omega) \).

18
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