A remark on variational inequalities in small balls

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To Professor Franco Giannessi on his 85th birthday, with esteem and friendship

Abstract. In this paper, we prove the following result: Let \((H, \langle \cdot, \cdot \rangle)\) be a real Hilbert space, \(B\) a ball in \(H\) centered at 0 and \(\Phi : B \to H\) a \(C^{1,1}\) function, with \(\Phi(0) \neq 0\), such that the function \(x \to \langle \Phi(x), x - y \rangle\) is weakly lower semicontinuous in \(B\) for all \(y \in B\). Then, for each \(r > 0\) small enough, there exists a point \(x^* \in H\), with \(\|x^*\| = r\), such that

\[
\max\{\langle \Phi(x^*), x^* - y \rangle, \langle \Phi(y), x^* - y \rangle\} < 0
\]

for all \(y \in H \setminus \{x^*\}\), with \(\|y\| \leq r\).

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1. INTRODUCTION

In the sequel, \((H, \langle \cdot, \cdot \rangle)\) is a real Hilbert space. For each \(r > 0\), set

\[
B_r = \{x \in H : \|x\| \leq r\}
\]

and

\[
S_r = \{x \in H : \|x\| = r\}.
\]

Let \(\Phi : B_r \to H\) be a given function.

We are interested in the classical variational inequality associated to \(\Phi\): to find \(x_0 \in B_r\) such that

\[
\sup_{y \in B_r} \langle \Phi(x_0), x_0 - y \rangle \leq 0. \tag{1}
\]

If \(H\) is finite-dimensional, the mere continuity of \(\Phi\) is enough to guarantee the existence of solutions, in view of the classical result of Hartman and Stampacchia ([3]). This is no longer true when \(H\) is infinite-dimensional. Actually, in that case, Frasca and Villani ([2]) constructed a continuous affine operator \(\Phi : H \to H\) such that, for each \(r > 0\) and \(x \in B_r\), one has

\[
\sup_{y \in B_r} \langle \Phi(x), x - y \rangle > 0.
\]

We also mention the related wonderful paper [7].

Another existence result is obtained assuming the following condition:

(a) for each \(y \in B_r\), the function \(x \to \langle \Phi(x), x - y \rangle\) is weakly lower semicontinuous in \(B_r\).

Such a result is a direct consequence of the famous Ky Fan minimax inequality ([1]).
In particular, condition (a) is satisfied when $\Phi$ is continuous and monotone (i.e. $\langle \Phi(x) - \Phi(y), x - y \rangle \geq 0$ for all $x, y \in B_r$). Moreover, when $\Phi$ is so, (1) is equivalent to the inequality

\[
\sup_{y \in B_r} \langle \Phi(y), x_0 - y \rangle \leq 0 \tag{2}
\]

(see [6]).

On the basis of the above remarks, a quite natural question is to find non-monotone functions $\Phi$ such that there is a solution of (1) which also satisfies (2).

The aim of the present very short note is just to give a first contribution along this direction, assuming, besides condition (a), that $\Phi$ is of class $C^{1,1}$, with $\Phi(0) \neq 0$ (Theorem 2.3).

2. RESULTS

We first establish the following saddle-point result:

THEOREM 2.1 - Let $Y$ be a non-empty closed convex set in a Hausdorff real topological vector space, let $\rho > 0$ and let $J : B_\rho \times Y \to \mathbb{R}$ be a function satisfying the following conditions:

(a1) for each $y \in Y$, the function $J(\cdot, y)$ is $C^1$, weakly lower semicontinuous and $J^*_x(\cdot, y)$ is Lipschitzian with constant $L$ (independent of $y$);

(a2) $J(x, \cdot)$ is continuous and quasi-concave for all $x \in B_\rho$ and $J(x_0, \cdot)$ is sup-compact for some $x_0 \in B_\rho$;

(a3) $\delta := \inf_{y \in Y} \| J^*_x(0, y) \| > 0$.

Then, for each $r \in [0, \min \{ \rho, \frac{\delta}{2L} \}]$ and for each non-empty closed convex $T \subseteq Y$, there exist $x^* \in S_r$ and $y^* \in T$ such that

\[
J(x^*, y) \leq J(x^*, y^*) \leq J(x, y^*)
\]

for all $x \in B_r$, $y \in T$.

PROOF. Fix an increasing sequence $\{ r_n \}$ of positive numbers converging to $r$. Since $\inf_{y \in T} \| J^*_x(0, y) \| \geq \delta$, for each $n \in \mathbb{N}$, Corollary 2.4 of [5] ensures that

\[
\sup_{T} \inf_{B_{r_n}} J = \inf_{B_{r_n}} \sup_{T} J.
\]

By Proposition 2.1 of [4], this implies that

\[
\sup_{T} \inf_{\text{int}(B_r)} J = \inf_{\text{int}(B_r)} \sup_{T} J.
\]

Then, since $J(\cdot, y)$ is continuous, we have

\[
\inf_{B_r} \sup_{T} J \leq \inf_{T} \sup_{\text{int}(B_r)} J = \sup_{T} \inf_{\text{int}(B_r)} J = \sup_{B_r} \inf_{T} J
\]

and so

\[
\inf_{B_r} \sup_{T} J \leq \inf_{T} \sup_{B_r} J = \sup_{B_r} \inf_{T} J.
\]

Now, due the semicontinuity and compactness assumptions, there exist $x^* \in B_r$ and $y^* \in T$ such that

\[
J(x^*, y) \leq J(x^*, y^*) \leq J(x, y^*)
\]

for all $x \in B_r$, $y \in T$. Finally, observe that $x^* \in S_r$. Indeed, if $x^* \in \text{int}(B_r)$ we would have

\[
J^*_x(x^*, y^*) = 0
\]

and so

\[
\delta \leq \| J^*_x(0, y^*) \| \leq L \| x^* \| \leq \frac{\delta}{2},
\]

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an absurd. The proof is complete.

Here is our main theorem:

THEOREM 2.2. - Let \( \rho > 0 \) and let \( \Phi : B_\rho \to H \) be a \( C^1 \) function whose derivative is Lipschitzian with constant \( \gamma \). Moreover, assume that, for each \( y \in B_\rho \), the function \( x \to \langle \Phi(x), x - y \rangle \) is weakly lower semicontinuous. Set

\[
\theta := \sup_{x \in B_\rho} \| \Phi'(x) \|_{\mathcal{L}(H)} ,
\]

\[
M := 2(\theta + \rho \gamma)
\]

and assume also that

\[
\sigma := \inf_{y \in B_\rho} \sup_{\| u \| = 1} |\langle \Phi(0), u \rangle - \langle \Phi'(0)(0), y \rangle| > 0 .
\]

Then, for each \( r \in [0, \min \{ \rho, \frac{\sigma}{2\gamma} \}] \), there exists \( x^* \in S_r \) such that

\[
\max\{ \langle \Phi(x^*), x^* - y \rangle, \langle \Phi(y), x^* - y \rangle \} < 0
\]

for all \( y \in B_r \setminus \{ x^* \} \).

PROOF. Consider the function \( J : B_\rho \times B_\rho \to \mathbb{R} \) defined by

\[
J(x, y) = \langle \Phi(x), x - y \rangle
\]

for all \( x, y \in B_\rho \). Of course, for each \( y \in B_\rho \), the function \( J(\cdot, y) \) is \( C^1 \) and one has

\[
(J'_x(x, y), u) = \langle \Phi'(x)(u), x - y \rangle + \langle \Phi(x), u \rangle
\]

for all \( x \in B_\rho, u \in H \). Fix \( x, v \in B_\rho \) and \( u \in S_1 \). We then have

\[
|\langle J'_x(x, y), u \rangle - \langle J'_v(x, y), u \rangle| = |\langle \Phi(x) - \Phi(v), u \rangle + \langle \Phi'(x)(u), x - y \rangle - \langle \Phi'(v)(u), v - y \rangle|
\]

\[
\leq \| \Phi(x) - \Phi(v) \| + |\Phi'(x)(u) - \Phi'(v)(u)| + \| \Phi'(x)(u), x - y \|
\]

\[
\leq \theta \| x - v \| + 2\rho \| \Phi'(x) - \Phi'(v) \|_{\mathcal{L}(H)} + \theta \| x - y \|
\]

\[
\leq 2(\theta + \rho \gamma) \| x - v \| .
\]

Hence, the function \( J(\cdot, y) \) is Lipschitzian with constant \( M \). At this point, we can apply Theorem 2.1 taking \( Y = B_\rho \) with the weak topology. Therefore, for each \( r \in [0, \min \{ \rho, \frac{\sigma}{2\gamma} \}] \), there exist \( x^* \in S_r \) and \( y^* \in B_r \) such that

\[
\langle \Phi(x^*), x^* - y \rangle \leq \langle \Phi(x^*), x^* - y^* \rangle \leq \langle \Phi(x), x - y^* \rangle
\]

(3)

for all \( x, y \in B_r \). Notice that \( \Phi(x^*) \neq 0 \). Indeed, if \( \Phi(x^*) = 0 \), we would have

\[
\| \Phi(0) \| = \| \Phi(0) - \Phi(x^*) \| \leq \theta r
\]

and hence, since \( \sigma \leq \| \Phi(0) \| \), it would follow that

\[
r \leq \frac{\| \Phi(0) \|}{2M} < \frac{\| \Phi(0) \|}{\theta} \leq r .
\]

Consequently, the infimum in \( B_r \) of the linear functional \( y \to \langle \Phi(x^*), y \rangle \) is equal to \( -\| \Phi(x^*) \| r \) and attained only at the point \( -r \frac{\Phi(x^*)}{\| \Phi(x^*) \|} \). But, from the first inequality in (3), it just follows that \( y^* \) is the global minimum in \( B_r \) of the functional \( y \to \langle \Phi(x^*), y \rangle \), and hence

\[
y^* = -r \frac{\Phi(x^*)}{\| \Phi(x^*) \|} .
\]
Moreover, from (3) again (taking \( y = x^* \) and \( x = y^* \)), it follows that
\[
\langle \Phi(x^*), x^* - y^* \rangle = 0 .
\]
Consequently, we have
\[
\langle \Phi(x^*), x^* \rangle = \langle \Phi(x^*), y^* \rangle = \left\langle \Phi(x^*), -r \frac{\Phi(x^*)}{\|\Phi(x^*)\|} \right\rangle = -\|\Phi(x^*)\| r .
\]
Therefore, \( x^* \) is the global minimum in \( B_r \) of the functional \( y \to \langle \Phi(x^*), y \rangle \) and hence \( x^* = y^* \). Thus, (3) actually reads
\[
\langle \Phi(x^*), x^* - y \rangle \leq 0 \leq \langle \Phi(x), x - x^* \rangle
\]
for all \( x, y \in B_r \). Finally, fix \( u \in B_r \setminus \{x^*\} \). By what seen above, the inequality
\[
\langle \Phi(u), x^* - u \rangle < 0
\]
is clear. Moreover, from the proofs of Corollaries 2.1, 2.3, 2.4 and Theorem 2.1 of [5], it follows that, for each \( y \in B_r \), the function \( J(\cdot, y) \) has a unique global minimum in \( B_r \). But, the second inequality in (4) says that \( x^* \) is a global minimum in \( B_r \) of the function \( J(\cdot, x^*) \) and hence the inequality
\[
\langle \Phi(u), x^* - u \rangle < 0
\]
follows, and the proof is complete. \( \triangle \)

From Theorem 2.2 we obtain the following characterization:

**THEOREM 2.3.** - Let \( \rho > 0 \) and let \( \Phi : B_\rho \to H \) be a \( C^1 \) function, with Lipschitzian derivative, such that, for each \( y \in B_\rho \), the function \( x \to \langle \Phi(x), x - y \rangle \) is weakly lower semicontinuous.

Then, the following assertions are equivalent:

(i) for each \( r > 0 \) small enough, there exists \( x^* \in S_r \) such that
\[
\max\{\langle \Phi(x^*), x^* - y \rangle, \langle \Phi(y), x^* - y \rangle\} < 0
\]
for all \( y \in B_r \setminus \{x^*\} \); \( \Phi(0) \neq 0 \).

(ii) \( \Phi(0) \neq 0 \).

**PROOF.** The implication (i) \( \to (ii) \) is clear. So, assume that (ii) holds. Observe that the function \( y \to \sup_{\|u\|=1} |\langle \Phi(0), u \rangle - \langle \Phi'(0)(u), y \rangle| \) is continuous in \( H \) and takes the value \( \|\Phi(0)\| > 0 \) at 0. Consequently, for a suitable \( r^* \in [0, \rho] \), we have
\[
\inf_{y \in B_r} \sup_{\|u\|=1} |\langle \Phi(0), u \rangle - \langle \Phi'(0)(u), y \rangle| > 0 .
\]
At this point, we can apply Theorem 2.1 to the restriction of \( \Phi \) to \( B_{r^*} \), and (i) follows. \( \triangle \)

Finally, it is also worth noticing the following further corollary of Theorem 2.2:

**THEOREM 2.4.** - Let \( \rho > 0 \) and let \( \Psi : B_\rho \to H \) be a \( C^1 \) function whose derivative vanishes at 0 and is Lipschitzian with constant \( \gamma_1 \). Moreover, assume that, for each \( y \in B_\rho \), the function \( x \to \langle \Psi(x), x - y \rangle \) is weakly lower semicontinuous. Set
\[
\theta_1 := \sup_{x \in B_\rho} \|\Psi'(x)\|_{\mathcal{L}(H)} ,
\]
\[
M_1 := 2(\theta_1 + \rho\gamma_1)
\]
and let \( w \in H \) satisfy
\[
\|w - \Psi(0)\| \geq 2M_1 \rho .
\]
Then, for each \( r \in [0, \rho] \), there exists \( x^* \in S_r \) such that

\[
\max\{\langle \Psi(x^*) - w, x^* - y \rangle, \langle \Psi(y) - w, x^* - y \rangle\} < 0
\]

for all \( y \in B_r \setminus \{x^*\} \).

**Proof.** Set \( \Phi := \Psi - w \). Apply Theorem 2.1 to \( \Phi \). Since \( \Phi' = \Psi' \), we have \( M = M_1 \). Since \( \Phi'(0) = 0 \), we have \( \sigma = \|\Phi(0)\| \) and hence, by (5),

\[
\rho \leq \frac{\sigma}{2M}
\]

and the conclusion follows. \( \triangle \)

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