An output-sensitive algorithm for all-pairs shortest paths in directed acyclic graphs

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Abstract. A straightforward dynamic programming method for the single-source shortest paths problem (SSSP) in an edge-weighted directed acyclic graph (DAG) processes the vertices in a topologically sorted order. Yen, by decomposing the input edge-weighted directed graph in two DAGs, could use this method iteratively to improve the time complexity of the SSSP Bellman-Ford algorithm for edge-weighted directed graphs by a constant factor.

First, we similarly iterate this method alternatively in a breadth-first search sorted order and the reverse order on an input directed graph with both positive and negative real edge weights, \( n \) vertices and \( m \) edges. For a positive integer \( t \), after \( O(t) \) iterations in \( O(tm) \) time, we obtain for each vertex \( v \) a path distance from the source to \( v \) not exceeding that yielded by the shortest path from the source to \( v \) among the so called \( t \)-light paths. A directed path between two vertices is \( t \)-light if it contains at most \( t \) more edges than the minimum edge-cardinality directed path between these vertices. After \( O(n) \) iterations, we obtain an \( O(nm) \)-time solution to SSSP in directed graphs with real edge weights matching that of Bellman and Ford.

Our main result is an output-sensitive algorithm for the all-pairs shortest paths problem (APSP) in DAGs with positive and negative real edge weights. It runs in time 

\[
O\left( \min\{n^\omega, nm n^{1/2} \log n \} + \sum_{v \in V} \text{indeg}(v) |\text{leaf}(T_v)| \right),
\]

where \( n \) is the number of vertices, \( m \) is the number of edges, \( \omega \) is the exponent of fast matrix multiplication, \( \text{indeg}(v) \) stands for the indegree of \( v \), \( T_v \) is a tree of lexicographically-first shortest directed paths from all ancestors of \( v \) to \( v \), and \( \text{leaf}(T_v) \) is the set of leaves in \( T_v \). Note that if \( T_v \) is a path the term \( O(\text{indeg}(v) |\text{leaf}(T_v)|) \) equals \( O(\text{indeg}(v)) \) while if \( T_v \) is a star with \( v \) as a sink the term becomes \( O(\text{indeg}(v) |T_v|) \). It also follows that if \( \max_{v \in V} |\text{leaf}(T_v)| = O(n^\alpha) \) then our APSP algorithm for DAGs runs in \( O(n^\omega + mn^\alpha) \) time. Similarly, if \( \sum_{v \in V} |\text{leaf}(T_v)| n = O(n^\beta) \) then the algorithm runs in \( O(n^\omega + n^{1/2} + n^\beta) \) time.

Next, we discuss an extension of hypothetical improved upper time-bounds for APSP in non-negatively edge-weighted DAGs to include directed graphs with a polynomial number of large directed cycles.

Finally, we present experimental comparisons of our SSSP algorithm with the Bellman-Ford one and our output-sensitive APSP algorithm for edge-weighted DAGs with the standard APSP algorithm for edge-weighted DAGs. In particular, they show that our SSSP algorithm converges to the true distances on dense edge-weighted pseudorandom graphs faster than the Bellman-Ford algorithm does.
1 Introduction

The length of a path in an edge-weighted graph is the sum of the weights of edges on the path. A shortest path between two vertices in a graph has minimal length among all paths between these vertices. The distance between vertices \( v \) and \( u \) is the length of a shortest path from \( v \) to \( u \). If the graph is directed, the paths are supposed to be also directed.

Shortest path problems, in particular the single-source shortest paths problem (SSSP) and the all-pairs shortest paths problem (APSP), belong to the most basic and important problems in graph algorithms [5,16]. There are several variants of SSSP and APSP depending among other things on the restrictions on edge weights and the input graphs. The input to these problems is a directed or an undirected edge-weighted graph. The output is a representation of shortest paths between the source and all other vertices or between all pairs of vertices in the graph, respectively.

In the general case of directed graphs (without negative cycles), when both positive and negative real edge weights are allowed, the difference between the best known asymptotic upper time-bounds for SSSP and APSP respectively is surprisingly small. Namely, if the input directed graph has \( n \) vertices and \( m \) edges with real weights, then the best known SSSP algorithm due to Bellman [3], Ford [7], and Moore [11] runs in \( O(nm) \) while the APSP can be solved already in \( O(nm + n^2 \log n) \) time [10,16]. The APSP solution uses Johnson’s \( O(nm) \)-time reduction of the general edge weight case to the non-negative edge case and then it runs Dijkstra’s algorithm [6] \( n \) times [10,16]. The latter upper time-bound for APSP with arbitrary real edge-weights has been more recently improved to \( O(nm + n^2 \log \log n) \) by Pettie in [13]. Note that the aforementioned best asymptotic upper time bounds for SSSP and APSP are different only for sparse graphs with \( o(n \log \log n) \) edges. Interestingly, when edge weights are integers, the best known upper time-bound for APSP just in terms of \( n \) is \( n^3 / 2^{O(\sqrt{\log n})} \) [4].

The situation alters dramatically when the input directed graph is acyclic, i.e., when it does not contain directed cycles. Then, a simple dynamic programming algorithm processing vertices in a topologically sorted order solves the SSSP problem in \( O(n + m) \) time [5], an \( O(n(n + m)) \)-time solution to the APSP problem in this case follows.

In fact, Yen could use the aforementioned method for SSSP in DAGs iteratively in order to improve the time complexity of Bellman-Ford algorithm for directed graphs by a constant factor [14]. Bellman-Ford algorithm runs in \( n - 1 \) iterations. In each iteration, for each edge \( e \), the current distance (from the source) at the head of \( e \) is compared to the sum of the current distance at the
tail of $e$ and the weight if $e$. If the sum is smaller the distance at the head of $e$ is updated. To achieve the improvement, Yen imposes a linear order on the vertices of the input directed graph which yields a decomposition of the graph into two DAGs. Next, the SSSP method for DAGs is run on each of the two DAGs instead of an iteration of Bellman-Ford algorithm [14]. Bannister and Eppstein obtained a further improvement of the time complexity of Bellman-Ford algorithm by a constant factor using a random linear order [2].

A pair of vertices in an edge weighted undirected or directed graph can be connected by several paths, in particular several shortest paths. Beside the length of a path, the number of edges forming it can be an important characteristic. For example, Zwick provided several exact and approximation algorithms for all pairs lightest (i.e., having minimal number of edges) shortest paths in directed graphs with restricted edge weights in [17].

In this paper, first we consider $t$-light paths, i.e., directed paths that have at most $t$ more edges than the paths with the same endpoints having the minimal number of edges. In part following [14], we iterate $O(t)$ times the SSSP method for DAGs on two implicit DAGs yielded by an extension of the BFS partial order to a linear order. The iterations alternatively process the vertices in a breadth-first sorted order and the reverse order. In result, we obtain path distances from the source to all other vertices that are not greater than the corresponding shortest-path distances for $t$-light paths. It takes $O(tm)$ time totally. For $t = n - 2$, our method matches that of Bellman-Ford for SSSP in directed graphs with real edge weights.

A vertex $v$ is an ancestor (direct ancestor, respectively) of a vertex $u$ in a DAG if there is a directed path (edge, respectively) from $v$ to $u$ in the DAG.

Our main result is an output-sensitive algorithm for the APSP problem in DAGs. It runs in time $O(n^\omega, nm + n^2 \log n) + \sum_{v \in V} \text{indeg}(v)|\text{leaf}(T_v)|)$, where $n$ is the number of vertices, $m$ is the number of edges, $\omega$ is the exponent of fast $n \times n$ matrix multiplication\(^4\) indeg($v$) stands for the indegree of $v$, $T_v$ is a tree of lexicographically-first shortest directed paths from all ancestors of $v$ to $v$, leaf($T_v$) is the set of leaves in $T_v$, and for a set $X$, $|X|$ stands for its size. Note that if $T_v$ is a path the term $O(\text{indeg}(v)|\text{leaf}(T_v)|)$ equals $O(\text{indeg}(v))$ while when $T_v$ is a star with $v$ as a sink the term becomes $O(\text{indeg}(v)|T_v|)$. Thus, the running time of the APSP algorithm can be so low as $O(n^\omega)$ and so high as $O(n^{\omega + nm})$. It follows also that if $\alpha$ is defined by $\max_{v \in V} |\text{leaf}(T_v)| = O(n^\alpha)$ then the algorithm runs in $O(n^{\omega + mn^\alpha})$ time. Similarly, if $\beta$ is defined by $\frac{\sum_{v \in V} |\text{leaf}(T_v)|}{n} = O(n^\beta)$ then the algorithm runs in $O(n^{\omega + n^2 + \beta})$ time.

\(^4\) $\omega$ is not greater than 2.3729 [1].
Next, we provide an extension of hypothetical, improved upper time-bounds for APSP in DAGs with non-negative edge weights to include directed graphs with a polynomial number of large directed cycles.

Finally, we present experimental comparisons of our SSSP algorithm with the Bellman-Ford one and our output-sensitive APSP algorithm for edge-weighted DAGs with the standard APSP algorithm for edge-weighted DAGS. In particular, they show that our SSSP algorithm converges to the true shortest-path distances on dense edge-weighted pseudorandom graphs faster than the Bellman-Ford algorithm does. On the other hand, they exhibit only a slight time-performance advantage of our APSP algorithm over the standard APSP algorithm on dense edge-weighted pseudorandom DAGs. Presumably, the shortest-path trees in the aforementioned DAGs have large number of leaves.

1.1 Paper organization

In the next section, we provide our solution to the SSSP problem in directed graphs with real edge weights based on the SSSP method for DAGs and the BFS partial order in terms of $t$-light paths. Section 3 is devoted to our output-sensitive algorithm for the APSP problem in DAGs with real edge weights and its analysis. In Section 4, we discuss the extension of hypothetical, improved bounds for APSP in DAGs with non-negatively weighted edges to directed graphs with a polynomial number of large directed cycles. Section 5 presents our experimental results. We conclude with final remarks.

2 An application of the SSSP method for DAGs

The SSSP problem for directed acyclic graphs can be solved by topologically sorting the DAG vertices and applying straightforward dynamic programming. For consecutive vertices $v$ in the sorted order, the distance $\text{dist}(v)$ of $v$ from the source is set to the minimum of $\text{dist}(u) + \text{weight}(u,v)$ over all direct ancestors $u$ of $v$, where $\text{weight}(u,v)$ stands for the weight of the edge $(u,v)$. It takes linear (in the size of the DAG) time. Yen used the dynamic programming method iteratively to improve the time complexity of Bellman-Ford algorithm for directed graphs by a constant factor in [14]. Interestingly, we can similarly apply this method iteratively to determine shortest-path distances among paths using almost the minimal number of edges. To formulate our algorithm (Algorithm 1), we need the following definition and two procedures.

**Definition 1.** A directed path from a vertex $u$ to a vertex $v$ in a directed graph is lightest if it consists of the smallest possible number of edges. A path from $u$ to $v$ is $t$-$t$-light if it includes at most $t$ more edges than a lightest path from $u$ to $v$. 
procedure SSSPDAG(G, D)
Input: A directed graph \((V, E)\) with real edge weights, linearly ordered vertices \(v_1, \ldots, v_n\), and a 1-dimensional table \(D\) of size \(n\) with upper bounds on the distances from \(v_1\) to all vertices in \(V\).
Output: Improved upper bounds on the shortest-path distances from \(v_1\) to all vertices in \(V\) in the table \(D\).

for \(j = 2, \ldots, n\) do
    For each edge \((v_i, v_j)\) where \(i < j\)
    \(D(v_j) \leftarrow \min\{D(v_j), D(v_i) + \text{weight}(v_i, v_j)\}\)

procedure reverseSSSPDAG(G, D)
Input and output: the same as in SSSPDAG(G, D)

for \(j = n - 1, \ldots, 1\) do
    For each edge \((v_i, v_j)\) where \(i > j\)
    \(D(v_j) \leftarrow \min\{D(v_j), D(v_i) + \text{weight}(v_i, v_j)\}\)

Algorithm 1
Input: A directed graph \((V, E)\) with \(n\) vertices, real edge weights and a distinguished source vertex \(s\), and a positive integer \(t\).
Output: Upper bounds on the shortest-path distances from \(s\) to all other vertices in \(V\) not exceeding the corresponding shortest-path distances constrained to \(t\) light paths.

1. Run BFS from the source \(s\).
2. Order the vertices of \(G\) extending the BFS partial order according to the levels of the tree, i.e., \(s\) comes first, then the vertices reachable by direct edges from \(s\), then the vertices reachable by paths composed of two edges and so on. We may assume w.l.o.g. that all vertices are reachable from \(s\) or alternatively extend the aforementioned order with the non-reachable vertices arbitrarily.
3. Initialize a 1-dimensional table \(D\) of size \(n\), setting \(D(v_1) \leftarrow 0\) and \(D(v_j) \leftarrow \infty\) for \(1 < j \leq n\)
4. SSSPDAG(G, D)
5. for \(k = 1, \ldots, t\) do
   (a) reverseSSSPDAG(G, D)
   (b) SSSPDAG(G, D)

Theorem 1. Let \(G\) be a directed graph with \(n\) vertices, \(m\) real-weighted edges, and a distinguished source vertex \(s\). For all vertices \(v\) of \(G\) different from \(s\), an upper bound on their distance from the source vertex \(s\), not exceeding the length of a shortest path among \(t\) light paths from \(s\) to \(v\), can be computed in \(O((t + 1)(m + n))\) total time.
Proof. Consider Algorithm 1 and in particular the ordering of the vertices specified in its second step. We shall refer to an edge \((v_i, v_j)\) as forward if \(i < j\) otherwise we shall call it backward. Note that the vertices at the same level of the BFS tree can be connected both by forward as well as backward edges. See also Fig. 1. Let \(\ell\) be the number of (forward) edges in a lightest path from \(s\) to a given vertex \(v\). It follows that any path from \(s\) to \(v\), in particular a shortest \(t+\)light one, has to have at least \(\ell\) forward edges.

Consider the BFS tree from the source \(s\). Define the level of a vertex in the tree as the number of edges on the path from \(s\) to the vertex in the tree. Thus, in particular, \(\text{level}(s) = 0\) while \(\text{level}(v) = \ell\). Recall that the linear order extending the partial BFS order used in Algorithm 1 is non-decreasing with respect of the levels of vertices. Also, if \((u, w)\) is a forward edge then \(\text{level}(u) \leq \text{level}(w) \leq \text{level}(u) + 1\) and if \((u, w)\) is a backward edge then \(\text{level}(u) \geq \text{level}(w)\). Hence, any path from \(s\) to \(v\) has to have at least \(\ell\) forward edges, each increasing the level by one.

Consequently, a shortest \(t+\)light path from \(s\) to \(v\) can have at most \(t\) backward edges. Thus, it can be decomposed into at most \(2t + 1\) maximal fragments of consecutive edges of the same type (i.e., forward or backward, respectively), where the even numbered fragments consist of backward edges. Thus, the at most \(2t + 1\) calls of the procedures \(\text{SSSPDAG}(G, s, D)\), \(\text{reverseSSSPDAG}(G, s, D)\) in the algorithm are sufficient to detect a distance from \(s\) to \(v\) not exceeding the length of a shortest path among \(t+\)light paths from \(s\) to \(v\). The asymptotic running time of the algorithm is dominated by the aforementioned procedure calls. Hence, it is \(O((t + 1)(m + n))\).

We can obtain a representation of directed paths achieving the upper bounds on the distances from the source provided in Theorem 1 in a form of a tree of paths emanating from the source by backtracking. By setting \(t = n - 2\) in this theorem, we can match the best known SSSP algorithm for directed graphs with positive and negative real edge weights, i.e., the Bellman-Ford algorithm and its constant factor improvements \([10][16]\), running in \(O(nm)\) time. Similarly as in the case of Bellman-Ford algorithm, by calling additionally \(\text{reverseSSSPDAG}(G, D)\) and \(\text{SSSPDAG}(G, D)\) after the last iteration in Algorithm 1, we can detect the existence of negative cycles.

Comparing our algorithm with the Bellman-Ford one, note that if the lightest path from the source to a vertex \(v\) has \(\ell\) edges then \(\ell + t\) iterations in the Bellman-Ford algorithm may be needed to obtain an upper bound on the distance of \(v\) from the source comparable to that obtained after \(O(t)\) iterations in Algorithm 1.
Fig. 1. An example of a graph with a BFS vertex numbering and the two DAGs implied by forward and backward edges, respectively.

3 An output-sensitive APSP algorithm for DAGs

The APSP problem in DAGs with both positive and negative real edge weights can be solved in $O(n(n + m))$ time by running $n$ times the SSSP algorithm for DAGs. It is an intriguing open problem if there exist substantially more efficient algorithms for APSP in edge-weighted DAGs. In this section, we make a progress on this question by providing an output-sensitive algorithm for this problem. Its running time depends on the structure of shortest path trees. Although in the worst-case it does not break the $O(nm)$ barrier it seems to be substantially more efficient in the majority of cases.

The standard algorithm for APSP for DAGs just runs the SSSP algorithm for DAGs for each vertex of the DAG as a source separately. Our APSP algorithm does everything in one sweep along the topologically sorted order. Its main idea is for each vertex $v$ to compute the tree of lexicographically-first shortest paths from the ancestors $u$ of the currently processed vertex $v$ to $v$, in the topologically sorted order. In case the tree of lexicographically-first shortest paths from the already considered ancestors of $v$ includes $u$ (as some intermediate vertex) then we are done as for $u$. Otherwise, we have to find the direct ancestor of $v$ on the lexicographically-first shortest path $P$ from $u$ to $v$ and add an initial fragment of $P$ to the tree. By the topologically sorted order in which the ancestors $u$ of
If $v$ are considered, this can happen only when $u$ is a leaf of the (final) tree of lexicographically-first shortest paths from the ancestors of $v$ to $v$. The direct ancestor of $v$ on $P$ can be found by comparing the lengths of shortest paths from $u$ to $v$ with different direct ancestors of $v$ as the next to the last vertex on the paths in time proportional to the indegree of $v$. In turn, the initial fragment of $P$ to add can be found by using the link to the lexicographically-first shortest path from $u$ to the direct ancestor of $v$ that is on $P$. The correctness of the algorithm is immediate. The issues are an implementation of these steps and an estimation of the running time.

To specify our output-sensitive algorithm (Algorithm 2) more exactly, we need the following definition.

**Definition 2.** Assume a numbering of vertices in an edge-weighted DAG extending the topological partial order. A shortest (directed) path $P$ from $v_k$ to $v_i$ in the DAG is first in a lexicographic order if the direct ancestor $v_j$ of $v_i$ on $P$ has the lowest number $j$ among all direct ancestors of $v_i$ on shortest paths from $v_k$ to $v_i$ and the subpath of $P$ from $v_k$ to $v_j$ is the lexicographically-first shortest path from $v_k$ to $v_j$. For a vertex $v_i$ in the DAG, the tree $T_{v_i}$ of (lexicographically-first) shortest paths is the union of lexicographically-first paths from all ancestors of $v_i$ to $v_i$. Note that the vertex $v_i$ is a sink of $T_{v_i}$. It is assumed to be the root of $T_{v_i}$ and leaf($T_{v_i}$) stands for the set of leaves of $T_{v_i}$.

**Algorithm 2**

*Input:* A DAG $(V, E)$ with real edge weights.

*Output:* For each vertex $v \in V$, the tree $T_v$ of lexicographically-first shortest paths from all ancestors of $v$ to $v$ given by the table $NEXT_v$, where for each ancestor $u$ of $v$, $NEXT_v(u)$ is the direct successor of $u$ in the tree $T_v$, (i.e., the head of the unique directed edge having $u$ as the tail in the tree).

1. Determine the source vertices, topologically sort the remaining vertices in $V$, and number the vertices in $V$ accordingly, assigning to the sources the lowest numbers.
2. Set $n$ to $|V|$ and $r$ to the number of sources in $G$.
3. Initialize an $n \times n$ table $dist$ by setting $dist(u, u) = 0$ and $dist(u, v) = \infty$ for $u, v \in V, u \neq v$.
4. **for** $i = r + 1, \ldots, n$ **do**
   (a) Compute the set $A(v_i)$ of ancestors of $v_i$.
   (b) Initialize a 1-dimensional table $NEXT_{v_i}$ of size $|A(v_i)|$, setting $NEXT_{v_i}(v_j)$ to 0 for $v_j \in A(v_i)$.
   (c) **for** $v_k \in A(v_i)$ in increasing order of the index $k$ **do**
      i. **if** $NEXT_{v_i}(v_k) \neq 0$ **then** proceed to the next iteration of the interior for block.
ii. Determine a direct ancestor \( v_j \) of \( v_i \) that minimizes the value of \( \text{dist}(v_k, v_j) + \text{weight}(v_j, v_i) \). In case of ties the vertex \( v_j \) with the smallest index \( j \) is chosen among those yielding the minimum.

iii. \( v_{\text{current}} \leftarrow v_k \)

iv. \textbf{while} \( v_{\text{current}} \neq v_j \) \& \( \text{NEXT}(v_{\text{current}}, v_i) = 0 \) \textbf{do}

\[
\begin{align*}
\text{dist}(v_{\text{current}}, v_i) &\leftarrow \text{dist}(v_{\text{current}}, v_j) + \text{weight}(v_j, v_i) \\
\text{NEXT}_{v_i}(v_{\text{current}}) &\leftarrow \text{NEXT}_{v_j}(v_{\text{current}}) \\
v_{\text{current}} &\leftarrow \text{NEXT}_{v_i}(v_{\text{current}})
\end{align*}
\]

v. \textbf{if} \( \text{NEXT}_{v_i}(v_j) = 0 \) \textbf{then} \( \text{dist}(v_j, v_i) \leftarrow \text{weight}(v_j, v_i) \) \& \( \text{NEXT}_{v_i}(v_j) \leftarrow v_i \)

Lemma 1. Steps 4.c.iii-v add the missing fragments of a lexicographically shortest path from \( v_k \) to \( v_i \) and set the distances from vertices in the fragments to \( v_i \) in time proportional to the number of vertices added to \( T_{v_i} \).

\textbf{Proof.} Follow the path from \( v_k \) to \( v_j \) in \( T_{v_j} \) extended by \((v_j, v_i)\) until a vertex \( v_q \in T_{v_i} \) is encountered. This is done in Steps 4.c.iii-v. The membership of \( v_{\text{current}} \) in \( T_{v_i} \) is verified by checking whether or not \( \text{NEXT}_{v_i}(v_{\text{current}}) = 0 \). Also, if \( v_{\text{current}} \) is not yet in \( T_{v_i} \) then its distance to \( v_i \) is set by \( \text{dist}(v_{\text{current}}, v_i) \leftarrow \text{dist}(v_{\text{current}}, v_j) + \text{weight}(v_j, v_i) \) and it is added to \( T_{v_i} \) by \( \text{NEXT}_{v_i}(v_{\text{current}}) \leftarrow \text{NEXT}_{v_j}(v_{\text{current}}) \) in Step 4.c.iv. By the inclusion of \( v_q \) in \( T_{v_i} \), a whole shortest path \( Q \) from \( v_q \) to \( v_i \) is already included in \( T_{v_i} \) by induction on the number of steps performed by the algorithm. We claim that \( Q \) exactly overlaps with the final fragment of the extended path starting from \( v_q \). To see this encode \( Q \) and the aforementioned fragment of the extended path by the indices of their vertices in the reverse order. By our rule of resolving ties in Step 4.c.ii both encodings should be first in the lexicographic order so we have an exact overlap. For this reason, it is sufficient to add the initial fragment of the extended path ending at \( v_q \) to \( T_{v_i} \) and if necessary also the edge \((v_j, v_i)\) to \( T_{v_i} \), and to update the distances from vertices in the added fragment to \( v_i \), i.e., to perform Steps 4.c.iii-v.

Theorem 2. The APSP algorithm for a DAG \((V, E)\) with \( n \) vertices, \( m \) edges and real edge weights (Algorithm 2) runs in time \( O(\min\{n^\omega, nm + n^2 \log n\} + \sum_{v \in V} \text{indeg}(v)|\text{leaf}(T_v)|) \).

\textbf{Proof.} The sets of ancestors can be determined in Step 4.a by computing the transitive closure of the input DAG in \( O(\min\{n^\omega, nm\}) \) time by using fast matrix multiplication \([12]\) or BFS \([5]\), first. In fact, to implement the loop in Step 4.c, we need the sets of ancestors to be ordered according to the numbering of vertices provided in Step 1. If the transitive closure matrix is computed such an ordered set of ancestors can be easily retrieved in \( O(n) \) time. Otherwise,
additional preprocessing sorting the unordered sets of ancestors is needed. The
total cost of the additional preprocessing is $O(n^2 \log n)$.

All the remaining steps, excluding Steps 4.c.ii-v for vertices $v_k$ not yet in
$T_{v_i}$, can be done in total (i.e., over all iterations) time $O(\sum_{v \in V} (1 + |A(v)|)) = O(n^2)$, where $A(v)$ stands for the set of ancestors of $v$ in the DAG. The time
taken by Step 4.c.ii, when $v_k$ is not yet in the current $T_{v_i}$, is $O(\text{indeg}(v_i))$.
Suppose that $v_k$ is not a leaf of the final tree $T_{v_i}$. Then, there must exist some
leaf $v_p$ of the final tree such that there is path from $v_p$ via $v_k$ to $v_i$ in this tree.
By the numbering of vertices extending the partial topological order, we have
$p < k$. We infer that the aforementioned path is already present in the current
$T_{v_i}$. Thus, in particular the vertex $v_k$ is in the current tree. Hence, the total time
taken by Step 4.c.ii is $O(\sum_{v \in V} \text{indeg}(v)|\text{leaf}(T_v)|)$. Finally, the total time taken
by Steps 4.c.iii-v is $O(\sum_{v \in V} (1 + |A(v)|))$ by Lemma[1]

Note that the following inequalities hold:

$$\sum_{v \in V} \text{indeg}(v)|\text{leaf}(T_v)| \leq m \max_{v \in V} |\text{leaf}(T_v)|,$$

$$\sum_{v \in V} \text{indeg}(v)|\text{leaf}(T_v)| \leq n^2 \sum_{v \in V} \frac{|\text{leaf}(T_v)|}{n}.$$

They immediately yield the following corollary from Theorem[2]

**Corollary 1.** Let $G = (V, E)$ be an $n$-vertex DAG with $n$ vertices and $m$
edges with real edge weights. Suppose $\max_{v \in V} |\text{leaf}(T_v)| = O(n^\alpha)$ and
$\sum_{v \in V} |\text{leaf}(T_v)| = O(n^\beta)$. The APSP problem for $G$ is solved by Algorithm 2 in
time $O(\min\{n^\omega, nm + n^2 \log n\} + \min\{mn^\alpha, n^{2+\beta}\})$.

Observe that $|\text{leaf}(T_v)|$ is equal to the minimum number of directed paths
covering the tree $T_v$. Hence, $\alpha < 1$ if the maximum of the minimum number
of paths covering $T_v$ over $v$ is substantially sublinear. Similarly, $\beta < 1$ if the
average of the minimum number of paths covering $T_v$ over $v$ is substantially
sublinear.

To illustrate the superiority of Algorithm 2 over the standard $O(n(n + m))$-
time method for APSP in DAGs, consider the following simple, extreme example.
Suppose $M$ is a positive integer. Let $D$ be a DAG with vertices $v_1, v_2, ..., v_n$, and
edges $(v_i, v_j)$, where $i < j$, such that the weight of $(v_i, v_j)$ is $-1$ if $j = i + 1$
and $M$ otherwise.
It is easy to see the tree $T_{v_i}$ is just the path $v_1, v_2, ..., v_i$ and hence $|\text{leaf}(T_{v_i})| = 1$. Consequently, Algorithm 2 on the DAG $D$ runs in $O(n^{\omega})$ time while the
standard method requires $O(n^3)$ time. If $M = 1$, one could also run Zwick’s
APSP algorithm for directed graphs with edge weights in \{-1, 0, 1\} on this example in \(O(n^{2.575})\) time \([15]\).

To refine Theorem 2, we need the following definition.

**Definition 3.** For an edge weighted DAG \(G\), let \(\tilde{G}\) be the edge weighted DAG resulting from reversing the direction of edges in \(G\). For a vertex \(v_i\) in the DAG \(G\), the tree \(U_{v_i}\) of (lexicographically-first in reversed order) shortest paths from \(v_i\) to all descendants of \(v_i\) in \(G\) is just the tree resulting from the tree \(T_{v_i}\) in \(\tilde{G}\) by reversing the edge directions. Note that the vertex \(v_i\) is a source of \(U_{v_i}\). It is assumed to be the root of \(U_{v_i}\) and leaf\((U_{v_i})\) stands for the set of leaves of \(U_{v_i}\).

The APSP for edge weighted DAGs can be solved by providing the trees \(U_v\) instead of the trees \(T_v\). Hence, we obtain immediately the following strengthening of Theorem 2 by symmetry.

**Theorem 3.** The APSP problem for a DAG \((V, E)\) with \(n\) vertices, \(m\) edges and real edge weights can be solved in time

\[
O(\min\{n^\omega, nm + n^2 \log n\} + \min \{\sum_{v \in V} \text{ indeg}(v) | \text{ leaf}(T_v), \sum_{v \in V} \text{ outdeg}(v) | \text{ leaf}(U_v)\}).
\]

**Proof.** Alternate the steps of Algorithm 2 run on the input DAG \(G\) with those of Algorithm 2 run on the DAG \(\tilde{G}\). When any of the two runs finishes we are basically done. In case the run of Algorithm 2 on the DAG \(\tilde{G}\) finishes first, we obtain the trees \(U_v\) in \(G\) from the trees \(T_v\) in \(\tilde{G}\) by reversing the direction of edges. The upper time bound follows from Theorem 2 and the fact that the indegree of a vertex in \(\tilde{G}\) is equal to its outdegree in \(G\).

### 4 A potential extension to digraphs with large cycles

As we have already noted the APSP problem in DAGs with both positive and negative real edge weights can be solved in \(O(n(n + m))\) time. It is also an interesting open problem if one can derive substantially more efficient algorithms for APSP in DAGs than the \(O(n(n + m))\)-time method in case of restricted edge weights, e.g., non-negative edge weights etc. In this section, under the assumption of the existence of such substantially more efficient algorithms for DAGs with non-negative edge weights, we show that they could be extended to include directed graphs having a polynomial number of large cycles.

The idea of the extension is fairly simple, see Fig. 2. We pick uniformly at random a sample of vertices of the input directed graph that hits all the directed cycles with high probability (cf. [15]). Here, we use the assumption on the minimum size of the cycles and on the polynomially bounded number of
the cycles. Next, we remove the vertices belonging to the sample and run the hypothetical fast algorithm for APSP in DAGs on the resulting subgraph of the input graph which is acyclic with high probability. In order to take into account shortest path connections using the removed vertices, we run the Dijkstra’s SSSP algorithm from each vertex in the sample on the original input graph two times. In the second run we reverse the directions of the edges in the input graph. Finally, we update the shortest path distances appropriately. See Algorithm 3 for a more detailed description.

**Algorithm 3**

**Input:** A directed graph $(V, E)$ with $n$ vertices, $m$ non-negatively weighted edges and a polynomial number of directed cycles, each with at least $d$ vertices.

**Output:** The shortest-path distances for all ordered pairs of vertices in $V$.

1. Initialize an $n \times n$ array $D$ by setting all its entries outside the main diagonal to $+\infty$ and those on the diagonal to zero.
2. Uniformly at random pick a sample $S$ of $O(n \ln n/d)$ vertices from $V$.
3. Run the hypothetical APSP algorithm for DAGs on the graph $(V \setminus S, E \cap \{(u, v) | u, v \in V \setminus S\})$ and for each pair $u, v \in V \setminus S$, set $D(u, v)$ to the distance determined by the algorithm.
4. For each $s \in S$, run the Dijkstra’s SSSP algorithm with $s$ as the source in $(V, E)$ and for all $v \in V \setminus \{s\}$ update the $D(s, v)$ entries respectively.
5. For each $s \in S$, run the Dijkstra’s SSSP algorithm with $s$ as the source on the directed graph resulting from reversing the directions of the edges in $(V, E)$, and for all $v \in V \setminus \{s\}$ update the $D(v, s)$ entries respectively.
6. For all pairs $u, v$ of distinct vertices in $V \setminus S$, and for all vertices $s \in S$, set $D(u, v) = \min\{D(u, v), D(u, s) + D(s, v)\}$.

**Fig. 2.** An example of a directed cycle that can be broken by removing the encircled vertex belonging to the sample. To find shortest-path connections passing through this vertex two SSSP from it are performed, in the original and the reversed edge directions, respectively.
Theorem 4. Let \( t(n, m) \) be the time required by APSP in DAGs with \( n \) vertices and \( m \) non-negatively weighted edges. Algorithm 3 solves the APSP problem for a directed graph with \( n \) vertices, \( m \) non-negatively weighted edges and a polynomial number of directed cycles, each with at least \( d \) vertices, in \( O(t(n, m) + n^3 \ln n/d) \) time with high probability.

Proof. Suppose that the number of directed cycles in the input graph \((V, E)\) is \( O(n^c) \). By picking enough large constant for the expression \( n \ln n/d \) specifying the size of the sample \( S \), the probability that a given directed cycle in \( G \) is not hit by \( S \) can be made smaller than \( n^{-c-1} \). Hence, the probability that the graph resulting from removing the vertices in \( S \) is not acyclic becomes smaller than \( n^{-1} \). It follows that Algorithm 3 is correct with high probability. It remains to estimate its running time. Steps 1, 2 can be easily implemented in \( O(n^2) \) time. Step 3 takes \( t(n, m) \) time. Steps 4, 5 can be implemented in \( O((n \ln n/d) \times m + n^2 \ln^2 n/d) \) time [5]. Finally, Step 6 takes \( O(n^3 \ln n/d) \) time.

Note that because of the term \( n^3 \ln n/d \) in the upper time-bound given by Theorem 4, the upper bound can be substantially subcubic only when \( d = \Omega(n^{\delta}) \) for some \( \delta > 0 \).

5 Experimental results

We have implemented Algorithm 1 and the Bellman-Ford algorithm in order to compare the quality of their estimation of the shortest-path distances after corresponding iterations. We have also implemented Algorithm 2 and the standard APSP algorithm for DAGs \((n - 1) \) runs of of the SSSP dynamic programming algorithm for DAGs) in order to compare their running times.

For the comparison sake, we used Erdős–Rényi \( G(n, p) \) random graph model, and generated 100 pseudorandom graphs for \( n \in \{10, 100, 1000\} \) and \( p \in \{0.2, 0.4, 0.6, 0.8\} \). We used \texttt{mt19937} implementation of Mersenne Twister pseudorandom number generator from GNU C++ Standard Library version 10.2. Pseudorandom integer weights from the interval \([-1000, 1000]\) were assigned to the edges. In case of the APSP algorithms for DAGs, the generated pseudorandom graphs were converted into DAGs simply by directing each edge \( \{v_i, v_j\} \), where \( i < j \), from \( v_i \) to \( v_j \).

All four algorithms were implemented in C++ and Google Benchmark library was used to measure the CPU time. High-resolution clock with nanosecond precision was used for time measurement. The code was compiled with \(-O2\) optimization flag using GNU C++ Compiler version 10.2, and was executed on a PC with Intel Core i5-2557M 2.7 GHz CPU and 4 GB RAM running Linux kernel version 5.11.15.
5.1 Algorithm 1

We have compared the quality of estimations of shortest-path distances in initial iterations of Algorithm 1 and the Bellman-Ford algorithm. We count Step 4 as the first iteration, and then each performance of Steps 5.a and 5.b as consecutive iterations of Algorithm 1. In an iteration of the Bellman-Ford algorithm, for each edge $e$, the current distance (from the source) at the head of $e$ is compared to the sum of the current distance at the tail of $e$ and the weight if $e$. If the sum is smaller the distance at the head of $e$ is updated.

Figures 4 and 5 (see Appendix) show the proportions between the numbers of vertices for which Algorithm 1 or the Bellman-Ford algorithm respectively provides a sharper estimation of the shortest-path distance from the source in corresponding iterations for pseudorandom graphs on 10 and 100 vertices. The figures support the claim that Algorithm 1 provides reasonable estimation substantially faster than the Bellman-Ford algorithm does.

5.2 Algorithm 2

In one of the initial steps of Algorithm 2, the transitive closure of the input DAG is computed. For dense DAGs, the computation of the transitive closure involves fast matrix multiplication algorithm known to have huge overhead. Since we run Algorithm 2 on relatively small DAGs where the aforementioned overhead could shadow the time performance of the core of the algorithm, we do not account the time taken by the transitive closure step in our results. See Figure 3.

| $p$   | Baseline $n = 10$ | Baseline $n = 100$ | Baseline $n = 1000$ | Algorithm 2 $n = 10$ | Algorithm 2 $n = 100$ | Algorithm 2 $n = 1000$ |
|-------|------------------|-------------------|-------------------|---------------------|---------------------|---------------------|
| 0.2   | 0.000371 ± 0.000035 | 0.285 ± 0.012     | 236 ± 2           | 0.000221 ± 0.000040 | 0.162 ± 0.009      | 94 ± 2              |
| 0.4   | 0.000504 ± 0.000047 | 0.433 ± 0.016     | 461 ± 4           | 0.000353 ± 0.000063 | 0.230 ± 0.015      | 150 ± 4             |
| 0.6   | 0.000646 ± 0.000053 | 0.557 ± 0.018     | 697 ± 7           | 0.000496 ± 0.000076 | 0.277 ± 0.0230     | 196 ± 7             |
| 0.8   | 0.000789 ± 0.000052 | 0.705 ± 0.018     | 945 ± 7           | 0.000607 ± 0.000089 | 0.309 ± 0.029      | 233 ± 7             |

Fig. 3. Mean and standard deviation for CPU time in milliseconds
6 Final remarks

In the absence of substantial asymptotic improvements to the time complexity of basic shortest-path algorithms, often formulated at the end of 50s, like the Bellman-Ford algorithm and Dijkstra’s algorithm, the results presented in this paper should be of interest. Our output-sensitive algorithm for the general APSP problem in DAGs possibly could lead to an improvement of the asymptotic time complexity of this problem in the average case. A probabilistic analysis of the number of leaves in the lexicographically-first shortest-path trees is an interesting open problem.

In the vast literature on shortest path problems, there are several examples of output-sensitive algorithms. For instance, Karger et al. [8] and McGeoch [9] could orchestrate the \(n\) runs of Dijkstra’s algorithm in order to solve the APSP problem for directed graphs with non-negative edge weights in \(O(m^*n + n \log n)\) time, where \(m^*\) is the number of (essential) edges that participate in shortest paths.

Finally, note that DAGs have several important scientific and computational applications in among other things scheduling, data processing networks, biology (phylogenetic networks, epidemiology), sociology (citation networks), and data compression. For these reasons, efficient algorithms for shortest paths in DAGs are of not only theoretical interest.

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Fig. 4. A comparison of Algorithm 1 with the Bellman-Ford algorithm on pseudorandom graphs with 10 vertices. The proportions between the numbers of vertices for which Algorithm 1 or the Bellman-Ford algorithm respectively gives better estimation of the shortest-path distance from the source in corresponding iterations are visualized with the colors.
Fig. 5. An analogous comparison of Algorithm 1 with the Bellman-Ford algorithm on pseudorandom graphs with 100 vertices.