MINIMIZING MEASURES ON CONDENSERS OF INFINITELY MANY PLATES

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Abstract

The study deals with the interior capacities of condensers in a locally compact space, a condenser being treated here as a countable, locally finite collection of closed sets $A_i, i \in I$, with the sign $+1$ or $−1$ prescribed such that the oppositely signed sets are mutually disjoint. We are concerned with the minimal energy problem over the class of linear combinations $\sum_{i \in I} (\text{sign } A_i) \mu_i$, where $\mu_i$ is a nonnegative Radon measure supported by $A_i$ and normalized by $\int g_i d\mu_i = a_i$, $g_i$ and $a_i$ being given. For all positive definite kernels satisfying Fuglede’s condition of consistency between the strong and vague (= weak∗) topologies, we establish sufficient conditions for the existence of minimizers and analyze properties of their uniqueness, compactness, and continuity.

1 Introduction

The present work is devoted to further development of the theory of interior capacities of condensers in a locally compact space. A condenser will be treated here as a countable, locally finite collection $A$ of closed sets $A_i, i \in I$, with the sign $+1$ or $−1$ prescribed such that the oppositely signed sets are mutually disjoint.

For a background of the theory for condensers of finitely many plates we refer the reader to [11, 12], [20, Chap. VIII], and [21]–[24]; see also [18, Chap. 5] and [19], where the condensers were additionally assumed to be compact.

The reader is expected to be familiar with the principal notions and results of the theory of measures and integration on a locally compact space; its exposition can be found in [2, 3, 8] (see also [9, 22] for a brief survey).

In all that follows, $X$ denotes a locally compact Hausdorff space and $M = M(X)$ the linear space of all real-valued Radon measures $\nu$ on $X$ equipped with the vague (= weak∗) topology, i.e., the topology of pointwise convergence on the class $C_0(X)$ of all real-valued continuous functions $\varphi$ on $X$ with compact support.

A kernel $\kappa$ on $X$ is meant to be an element from $\Phi(X \times X)$, where $\Phi(Y)$ consist of all lower semicontinuous functions $\psi : Y \rightarrow (-\infty, \infty]$ such that $\psi \geq 0$ unless $Y$ is compact. Given $\nu, \nu_1 \in M$, the mutual energy and the potential with respect to a kernel $\kappa$ are defined respectively by

$$\kappa(\nu, \nu_1) := \int \kappa(x, y) d(\nu \otimes \nu_1)(x, y)$$

and

$$\kappa(\cdot, \nu) := \int \kappa(\cdot, y) d\nu(y).$$

(Here and in the sequel, when introducing notation, we always tacitly assume the corresponding object on the right to be well defined.) For $\nu = \nu_1$ the mutual energy $\kappa(\nu, \nu_1)$ defines the energy of $\nu$. Let $E$ consist of all $\nu \in M$ with $-\infty < \kappa(\nu, \nu) < \infty$.

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We shall be mainly concerned with a positive definite kernel \( \kappa \), which means that it is symmetric (i.e., \( \kappa(x, y) = \kappa(y, x) \) for all \( x, y \in X \)) and the energy \( \kappa(\nu, \nu) \), \( \nu \in \mathcal{M} \), is nonnegative whenever defined. Then \( \mathcal{E} \) forms a pre-Hilbert space with the scalar product \( \kappa(\nu, \nu_1) \) and the seminorm \( \|\nu\|_\mathcal{E} := \sqrt{\kappa(\nu, \nu)} \) (see [2]). A positive definite kernel \( \kappa \) is called strictly positive definite if the seminorm \( \| \cdot \|_\mathcal{E} \) is a norm.

Let \( \mathcal{M}^+(E) \) consist of all nonnegative measures \( \nu \in \mathcal{M} \) supported by \( E \), where \( E \) is a given closed subset of \( X \), and let \( \mathcal{E}^+(E) := \mathcal{M}^+(E) \cap \mathcal{E} \). We also write \( \mathcal{M}^+ := \mathcal{M}^+(X) \) and \( \mathcal{E}^+ := \mathcal{E}^+(X) \).

Given a condenser \( A = (A_i)_{i \in I} \), we consider the class \( \mathcal{M}(A) \) of all linear combinations \( \mu = \sum_{i \in I} \alpha_i \mu^i \), where \( \alpha_i := \text{sign} A_i \) and \( \mu^i \in \mathcal{M}^+(A_i) \), equipped with a relation of identity and a topology so that it becomes homeomorphic to the product space \( \prod_{i \in I} \mathcal{M}^+(A_i) \) (where every \( \mathcal{M}^+(A_i) \) is endowed with the vague topology). We call the corresponding topology on \( \mathcal{M}(A) \) the \( A \)-vague topology.

To introduce a proper notion of energy \( \kappa(\mu, \mu) \) of \( \mu \in \mathcal{M}(A) \), we observe that, due to the local finiteness of a condenser, there is a unique Radon measure \( R\mu \in \mathcal{M} \) such that \( R\mu(\varphi) = \sum_{i \in I} \alpha_i \mu^i(\varphi) \) for all \( \varphi \in C_0(X) \). Therefore, it is reasonable to set \( \kappa(\mu, \mu) := \kappa(R\mu, R\mu) \). This notion can equivalently be defined also by

\[
\kappa(\mu, \mu) = \sum_{i,j \in I} \alpha_i \alpha_j \kappa(\mu^i, \mu^j)
\]

(see Sec. 3.3), which is in agreement with an electrostatic interpretation of a condenser. Let \( \mathcal{E}(A) \) consist of all \( \mu \in \mathcal{M}(A) \) whose energy is finite.

Having fixed a vector-valued function \( g = (g_i)_{i \in I} \), where all \( g_i : X \to (0, \infty) \) are continuous, and a numerical vector \( a = (a_i)_{i \in I} \) with \( a_i > 0 \), we next define the interior capacity of a condenser \( A \) (with respect to \( \kappa \), \( a \), and \( g \)) as \( 1/\inf \kappa(\mu, \mu) \), the infimum being taken over all \( \mu \in \mathcal{E}(A) \) normalized by \( \int g_i \, d\mu^i = a_i, \ i \in I \). Along with its electrostatic interpretation, such a notion has found various important applications to approximation theory, geometrical function theory, and potential theory itself (see the books [13, 18, 20] and the references cited therein).

The main question we shall be interested in is whether minimizers \( \lambda_A \) in the above minimal energy problem exist. If \( A \) is finite, all \( A_i \) are compact, while \( \kappa(x, y) \) is continuous on \( A_i \times A_j \) whenever \( \alpha_i \neq \alpha_j \), then the existence of those \( \lambda_A \) can be easily established by exploiting the \( A \)-vague topology only (see [19] Th. 2.30; cf. also [11, 12, 18, 20], related to the logarithmic kernel in the plane). However, the question becomes rather nontrivial if any of these three assumptions is dropped.

To solve the problem on the existence of minimizers \( \lambda_A \) in the general case where \( A \) is infinite and (or) noncompact, we restrict ourselves to positive definite kernels and work out an approach based on the following arguments.

The set \( \mathcal{E}(A) \) forms a semimetric space with the semimetric

\[
\|\mu_1 - \mu_2\|_{\mathcal{E}(A)} := \|R\mu_1 - R\mu_2\|_\mathcal{E} = \left[ \sum_{i,j \in I} \alpha_i \alpha_j \kappa(\mu^1_i, \mu^1_j) \right]^{1/2},
\]

and, for rather general \( \kappa \), \( g \), and \( a \), the topological subspace of \( \mathcal{E}(A) \) consisting of all \( \mu \) with \( \int g_i \, d\mu^i < a_i, \ i \in I \), is complete (see Sec. 3.2).
Using these arguments, we obtain sufficient conditions for the existence of minimizers $\lambda A$ and establish statements on their uniqueness, $A$-vague compactness, and continuity under exhaustion of $A$ by $K$, $i \in I$. See Sec. 5.

The results obtained and the approach applied develop and generalize the corresponding ones from the author’s articles [22, 23, 24], related to the condensers of finitely many plates.

2 Preliminaries: topologies, consistent and perfect kernels

From now on, the kernel under consideration is always assumed to be positive definite. In addition to the strong topology on $E$, determined by the seminorm $\|\cdot\| := \|\cdot\|_E$, it is often useful to consider the weak topology on $E$, defined by means of the seminorms $\nu \mapsto |\kappa(\nu, \mu)|$, $\mu \in E$ (see [9]). The Cauchy-Schwarz inequality

$$|\kappa(\nu, \mu)| \leq \|\nu\| \|\mu\|,$$

implies immediately that the strong topology on $E$ is finer than the weak one.

In [9, 10], B. Fuglede introduced the following two equivalent properties of consistency between the induced strong, weak, and vague topologies on $E^+$:

(C$_1$) Every strong Cauchy net in $E^+$ converges strongly to every its vague cluster point;

(C$_2$) Every strongly bounded and vaguely convergent net in $E^+$ converges weakly to the vague limit.

Definition 2.1. Following Fuglede [9], we call a kernel $\kappa$ consistent if it satisfies either of the properties (C$_1$) and (C$_2$), and perfect if, in addition, it is strictly positive definite.

Remark 2.1. One has to consider nets or filters in $M^+$ instead of sequences, since the vague topology in general does not satisfy the first axiom of countability. We follow Moore’s and Smith’s theory of convergence, based on the concept of nets (see [17; cf. also [8, Chap. 0] and [15, Chap. 2]). However, if $X$ is metrizable and countable at infinity, then $M^+$ satisfies the first axiom of countability (see [9, Lemma 1.2.1]) and the use of nets may be avoided.

Theorem 2.1 (Fuglede [9]). A kernel $\kappa$ is perfect if and only if $E^+$ is strongly complete and the strong topology on $E^+$ is finer than the vague one.

Example 2.1. In $\mathbb{R}^n$, $n \geq 3$, the Newtonian kernel $|x - y|^{2-n}$ is perfect [4]. So are the Riesz kernel $|x - y|^{\alpha-n}$, $0 < \alpha < n$, in $\mathbb{R}^n$, $n \geq 2$ [5, 6], and the restriction of the logarithmic kernel $-\log |x - y|$ in $\mathbb{R}^2$ to an open unit ball (see [10]). Furthermore, if $D$ is an open set in $\mathbb{R}^n$, $n \geq 2$, and its generalized Green function $g_D$ exists (see, e.g., [14, Th. 5.24]), then $g_D$ is perfect as well [7].

Remark 2.2. As is seen from the above definitions and Theorem 2.1, the concept of consistent or perfect kernels is an efficient tool in minimal energy problems over nonnegative Radon measures with finite energy. Indeed, the theory of capacities
of sets has been developed in [9] exactly for those kernels. We shall show that this concept is efficient, as well, in minimal energy problems over (finite or infinite) linear combinations \( \mu \in \mathcal{E}(A) \), A being a condenser. This is guaranteed by the completeness of proper topological subspaces of \( \mathcal{E}(A) \), established in Sec. 6.2 below.

3 Condensers, associated measures, energies and potentials

We start by introducing and discussing basic concepts of the theory of interior capacities of condensers, some of which have already been briefly mentioned in the Introduction (cf. also [24]).

3.1 Condensers

Let \( I^+ \) and \( I^- \) be countable (finite or infinite) disjoint sets of indices \( i \in \mathbb{N} \), where the latter is allowed to be empty, and let \( I \) denote their union. Assume that to every \( i \in I \) there corresponds a nonempty, closed set \( A_i \subset X \).

**Definition 3.1.** A collection \( A = (A_i)_{i \in I} \) is called an \((I^+, I^-)\)-condenser (or simply a condenser) in \( X \) if every compact subset of \( X \) intersects with at most finitely many \( A_i \) and

\[
A_i \cap A_j = \emptyset \quad \text{for all} \quad i \in I^+, \; j \in I^-.
\]

The sets \( A_i, \; i \in I^+ \), and \( A_j, \; j \in I^- \), are called the positive and, respectively, negative plates of the condenser \( A \). Note that any two equally signed plates can intersect each other. Given \( I^+ \) and \( I^- \), let \( \mathcal{C} = \mathcal{C}(I^+, I^-) \) be the class of all \((I^+, I^-)\)-condensers in \( X \). A condenser \( A \in \mathcal{C} \) is said to be compact if so are all \( A_i, \; i \in I \), and finite if \( I \) is finite. Also the following notation will be used:

\[
A^+ := \bigcup_{i \in I^+} A_i, \quad A^- := \bigcup_{i \in I^-} A_i.
\]

Observe that \( A^+ \) and \( A^- \) might both be noncompact even for a compact \( A \).

3.2 Measures associated with a condenser. \( \mathcal{A} \)-vague topology

With the preceding notation, write

\[
\alpha_i := \begin{cases} 
+1 & \text{if } i \in I^+, \\
-1 & \text{if } i \in I^-.
\end{cases}
\]

Given \( A \in \mathcal{C} \), let \( \mathcal{M}(A) \) consist of all (finite or infinite) linear combinations

\[
\mu := \sum_{i \in I} \alpha_i \mu^i, \quad \text{where} \quad \mu^i \in \mathcal{M}^+(A_i).
\]

Any two \( \mu_1 \) and \( \mu_2 \) in \( \mathcal{M}(A) \) are regarded to be identical \( (\mu_1 \equiv \mu_2) \) if and only if \( \mu^i_1 = \mu^i_2 \) for all \( i \in I \). Then, under the relation of identity thus defined, the following correspondence is one-to-one:

\[
\mathcal{M}(A) \ni \mu \mapsto (\mu^i)_{i \in I} \in \prod_{i \in I} \mathcal{M}^+(A_i).
\]
We call $\mu \in \mathcal{M}(A)$ a measure associated with $A$, and $\mu^i$ its $i$-coordinate. For measures associated with a condenser, it is therefore natural to introduce the following concept of convergence, actually corresponding to the vague convergence by coordinates. Let $S$ denote a directed set of indices, and let $\mu_s, s \in S$, and $\mu_0$ be given elements of the class $\mathcal{M}(A)$.

**Definition 3.2.** A net $(\mu_s)_{s \in S}$ is said to converge to $\mu_0$ $A$-vaguely if

$$\mu^i_s \to \mu^i_0 \text{ vaguely for all } i \in I.$$  

Then $\mathcal{M}(A)$, equipped with the topology of $A$-vague convergence, becomes homeomorphic to the product space $\prod_{i \in I} \mathcal{M}^+(A_i)$, where every $\mathcal{M}^+(A_i)$ is endowed with the vague topology. Since $\mathcal{M}(X)$ is Hausdorff, so are both the spaces $\mathcal{M}(A)$ and $\prod_{i \in I} \mathcal{M}^+(A_i)$ (see, e.g., [15, Chap. 3, Th. 5]).

Similarly, a set $F \subset \mathcal{M}(A)$ is called $A$-vaguely bounded if all its $i$-projections are vaguely bounded — that is, if for every $\varphi \in C_0(X)$ and every $i \in I$,

$$\sup_{\mu \in F} |\mu^i(\varphi)| < \infty.$$  

**Lemma 3.1.** Any $A$-vaguely bounded part of $\mathcal{M}(A)$ is $A$-vaguely relatively compact.

**Proof.** Since by [2, Chap. III, § 2, Prop. 9] any vaguely bounded and closed part of $\mathcal{M}$ is vaguely compact, the lemma follows from Tychonoff’s theorem on the product of compact spaces (see, e.g., [15, Chap. 5, Th. 13]).

### 3.3 Mapping $R : \mathcal{M}(A) \to \mathcal{M}$. Relation of equivalency on $\mathcal{M}(A)$

Since each compact subset of $X$ intersects with at most finitely many $A_i$, for every $\varphi \in C_0(X)$ only a finite number of $\mu^i(\varphi)$ (where $\mu \in \mathcal{M}(A)$ is given), are nonzero. This yields that to every $\mu \in \mathcal{M}(A)$ there corresponds a unique Radon measure $R\mu$ such that

$$R\mu(\varphi) = \sum_{i \in I} \alpha_i \mu^i(\varphi) \quad \text{for all } \varphi \in C_0(X);$$

due to (3.1), positive and negative parts in Jordan’s decomposition of $R\mu$ can respectively be written in the form

$$R\mu^+ = \sum_{i \in I^+} \mu^i \quad \text{and} \quad R\mu^- = \sum_{i \in I^-} \mu^i.$$  

Of course, the mapping $R : \mathcal{M}(A) \to \mathcal{M}$ thus defined is in general non-injective, i.e., one may choose $\mu_1, \mu_2 \in \mathcal{M}(A)$ so that $\mu_1 \neq \mu_2$, while $R\mu_1 = R\mu_2$. We call $\mu_1, \mu_2 \in \mathcal{M}(A)$ equivalent in $\mathcal{M}(A)$, and write $\mu_1 \equiv \mu_2$, if their $R$-images coincide — or, which is equivalent, whenever $\sum_{i \in I} \mu^i_1 = \sum_{i \in I} \mu^i_2$.

Observe that the relation of equivalency in $\mathcal{M}(A)$ implies that of identity (and, hence, these two relations on $\mathcal{M}(A)$ are actually equivalent) if and only if all $A_i$, $i \in I$, are mutually disjoint.
Lemma 3.2. The $A$-vague convergence of $(\mu_s)_{s \in S}$ to $\mu_0$ implies the vague convergence of $(R\mu_s)_{s \in S}$ to $R\mu_0$.

Proof. This is obvious since the support of any $\varphi \in C_0(X)$ might have points in common with only a finite number of $A_i$. \hfill \Box

Remark 3.1. Lemma 3.2 in general cannot be inverted. However, if all $A_i$, $i \in I$, are mutually disjoint, then the vague convergence of $(R\mu_s)_{s \in S}$ to $R\mu_0$ implies the $A$-vague convergence of $(\mu_s)_{s \in S}$ to $\mu_0$. This can be seen by using the Tietze-Urysohn extension theorem (see, e.g., [8, Th. 0.2.13]).

3.4 Energies and potentials of measures associated with a condenser

To introduce energies and potentials of linear combinations $\mu \in \mathcal{M}(A)$, we start with the following two lemmas, the former one being well known (see, e.g., [9]).

Lemma 3.3. Let $Y$ be a locally compact Hausdorff space. If $\psi \in \Phi(Y)$ is given, then the map $\nu \mapsto \int \psi d\nu$ is vaguely lower semicontinuous on $\mathcal{M}^+(Y)$.

Lemma 3.4. Fix $\mu \in \mathcal{M}(A)$ and $\psi \in \Phi(X)$. If $\int \psi dR\mu$ is well defined, then

$$\int \psi dR\mu = \sum_{i \in I} \alpha_i \int \psi d\mu^i,$$

and $\int \psi dR\mu$ is finite if and only if the series on the right converges absolutely.

Proof. We can certainly assume $\psi$ to be nonnegative, for if not, we replace $\psi$ by a function $\psi' \geq 0$ obtained by adding to $\psi$ a suitable constant $c > 0$, which is always possible since a lower semicontinuous function is bounded from below on a compact space. Hence,

$$\int \psi dR\mu^+ \geq \sum_{i \in I^+, \, i \leq N} \int \psi d\mu^i \quad \text{for all } N \in \mathbb{N}.$$

On the other hand, the sum of $\mu^i$ over all $i \in I^+$ that do not exceed $N$ approaches $R\mu^+$ vaguely as $N \to \infty$; consequently, by Lemma 3.3,

$$\int \psi dR\mu^+ \leq \lim_{N \to \infty} \sum_{i \in I^+, \, i \leq N} \int \psi d\mu^i.$$

Combining the last two inequalities and then letting $N \to \infty$ yields

$$\int \psi dR\mu^+ = \sum_{i \in I^+} \int \psi d\mu^i.$$

Since the same holds true for $R\mu^-$ and $I^-$ instead of $R\mu^+$ and $I^+$, respectively, the lemma follows. \hfill \Box

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Corollary 3.1. Given $\mu, \mu_1 \in \mathcal{M}(\mathcal{A})$ and $x \in X$, we have

$$\kappa(x, R\mu) = \sum_{i \in I} \alpha_i \kappa(x, \mu^i),$$

(3.3)

$$\kappa(R\mu, R\mu_1) = \sum_{i,j \in I} \alpha_i \alpha_j \kappa(\mu^i, \mu^j_1),$$

(3.4)

each of the identities being understood in the sense that its right-hand side is well defined whenever so is the left-hand one and then they coincide. Furthermore, the left-hand side in (3.3) or in (3.4) is finite if and only if the corresponding series on the right converges absolutely.

Proof. Relation (3.3) is a direct consequence of (3.2), while (3.4) follows from Fubini’s theorem (cf. [3, § 8, Th. 1]) and Lemma 3.4 on account of the fact that $\kappa(x, \nu)$, where $\nu \in \mathcal{M}^+$ is given, is lower semicontinuous on $X$ (see, e.g., [9]).

Definition 3.3. Given $\mu, \mu_1 \in \mathcal{M}(\mathcal{A})$, we call $\kappa(\cdot, \mu) := \kappa(\cdot, R\mu)$ the potential of $\mu$ and $\kappa(\mu, \mu_1) := \kappa(R\mu, R\mu_1)$ the mutual energy of $\mu$ and $\mu_1$. For $\mu \equiv \mu_1$ we get the energy $\kappa(\mu, \mu) := \kappa(R\mu, R\mu) = \sum_{i,j \in I} \alpha_i \alpha_j \kappa(\mu^i, \mu^j_1).$

(3.5)

Corollary 3.2. For $\mu \in \mathcal{M}(\mathcal{A})$ to be of finite energy, it is necessary and sufficient that $\mu^i \in \mathcal{E}$ for all $i \in I$ and

$$\sum_{i \in I} \|\mu^i\|^2 < \infty.$$  

Proof. This follows from (3.5) due to the inequality $2\kappa(\nu_1, \nu_2) \leq \|\nu_1\|^2 + \|\nu_2\|^2$, where $\nu_1, \nu_2 \in \mathcal{E}$.

Remark 3.2. Observe that, for every $\mu \in \mathcal{M}(\mathcal{A})$, the series in (3.5) actually defines the energy of the corresponding infinitely dimensional vector measure $(\mu^i)_{i \in I}$ relative to the interaction matrix $(\alpha_i \alpha_j)_{i,j \in I}$; compare with [11, 12] and [18, Chap. 5, § 4]. Our approach, however, is essentially based on the fact that, due to the specific form of interaction matrix, that value can also be obtained as the energy of the corresponding scalar Radon measure $R\mu$.

Remark 3.3. Since we make no difference between $\mu \in \mathcal{M}(\mathcal{A})$ and $R\mu$ when dealing with their energies or potentials, we shall sometimes call a measure associated with $\mathcal{A}$ simply a measure — certainly, if this causes no confusion.

3.5 Strong topology on $\mathcal{E}(\mathcal{A})$

Let $\mathcal{E}(\mathcal{A})$ consist of all $\mu \in \mathcal{M}(\mathcal{A})$ of finite energy $\kappa(\mu, \mu)$. Since $\mathcal{M}(\mathcal{A})$ forms a convex cone, it follows from Corollary 3.2 that so does $\mathcal{E}(\mathcal{A})$.

Let us treat $\mathcal{E}(\mathcal{A})$ as a semimetric space with the semimetric

$$\|\mu_1 - \mu_2\| := \|\mu_1 - \mu_2\|_{\mathcal{E}(\mathcal{A})} := \|R\mu_1 - R\mu_2\|_{\mathcal{E}}, \quad \mu_1, \mu_2 \in \mathcal{E}(\mathcal{A});$$

(3.6)
then $\mathcal{E}(A)$ and its $R$-image become isometric. Similarly with the terminology in $\mathcal{E}$, the topology on $\mathcal{E}(A)$ defined by means of the semimetric (3.6) is called strong.

Two elements of $\mathcal{E}(A)$, $\mu_1$ and $\mu_2$, are called equivalent in $\mathcal{E}(A)$ if $\|\mu_1 - \mu_2\| = 0$. If, in addition, the kernel $\kappa$ is assumed to be strictly positive definite, then the equivalence in $\mathcal{E}(A)$ implies that in $\mathcal{M}(A)$ (namely, then $\mu_1 \equiv \mu_2$), and it implies the identity (i.e., then $\mu_1 \equiv \mu_2$) if, moreover, all $A_i$, $i \in I$, are mutually disjoint.

4 Interior capacities of condensers. Elementary properties

Given a set $\mathcal{H}$ in the semimetric space $\mathcal{E}(A)$, let us introduce the quantity

$$\|\mathcal{H}\|^2 := \inf_{\nu \in \mathcal{H}} \|\nu\|^2,$$

interpreted as $+\infty$ if $\mathcal{H}$ is empty. If $\|\mathcal{H}\|^2 < \infty$, one can consider the variational problem on the existence of $\lambda_{\mathcal{H}} \in \mathcal{H}$ with minimal energy $\|\lambda_{\mathcal{H}}\|^2 = \|\mathcal{H}\|^2$; such a problem will be referred to as the $\mathcal{H}$-problem. The $\mathcal{H}$-problem is called solvable if a minimizer $\lambda_{\mathcal{H}}$ exists.

4.1 Capacity of a condenser

Fix a vector-valued function $g = (g_i)_{i \in I}$, where all $g_i : X \to (0, \infty)$ are continuous, and a numerical vector $a = (a_i)_{i \in I}$ with $a_i > 0$. Given a condenser $A$, write

$$\mathcal{M}^+(A, a_i, g_i) := \left\{ \nu \in \mathcal{M}^+(A_i) : \int g_i d\nu = a_i \right\}, \quad i \in I,$$

and let $\mathcal{M}(A, a, g)$ consist of all $\mu \in \mathcal{M}(A)$ with $\mu^i \in \mathcal{M}^+(A_i, a_i, g_i)$ for all $i \in I$.

Given a kernel $\kappa$, also write

$$\mathcal{E}^+(A, a_i, g_i) := \mathcal{M}^+(A_i, a_i, g_i) \cap \mathcal{E}, \quad \mathcal{E}(A, a, g) := \mathcal{M}(A, a, g) \cap \mathcal{E}(A).$$

Definition 4.1. We shall call the value

$$\text{cap } A := \text{cap } (A, a, g) := \frac{1}{\|\mathcal{E}(A, a, g)\|^2}$$

the (interior) capacity of an $(I^+, I^-)$-condenser $A$ (with respect to $\kappa$, $a$, and $g$).

Here and in the sequel, we adopt the convention that $1/0 = +\infty$. Then, by the positive definiteness of the kernel, $0 \leq \text{cap } A \leq \infty$; necessary and (or) sufficient conditions for $0 < \text{cap } A < \infty$ to hold will be provided in Sec. 4.3 below (see also Corollary 7.2).

4.2 Continuity property

On $\mathcal{E} = \mathcal{E}(I^+, I^-)$, it is natural to introduce an ordering relation $\prec$ by declaring $A' \prec A$ to mean that $A'_i \subset A_i$ for all $i \in I$. Here, $A' = (A'_i)_{i \in I}$. Then $\text{cap } (\cdot, a, g)$ is a nondecreasing function of a condenser, namely

$$\text{cap } (A', a, g) \leq \text{cap } (A, a, g) \quad \text{whenever } A' \prec A. \quad (4.1)$$
Given \( A \in \mathcal{C} \), let us consider the increasing family \( \{ \mathbf{K} \}_A \) of all compact condensers 
\( \mathbf{K} = (K_i)_{i \in I} \in \mathcal{C} \) such that \( \mathbf{K} \prec A \).

**Lemma 4.1.** If \( \mathbf{K} \) ranges over \( \{ \mathbf{K} \}_A \), then
\[
\operatorname{cap} (A, a, g) = \lim_{\mathbf{K} \uparrow A} \operatorname{cap} (\mathbf{K}, a, g).
\] (4.2)

**Proof.** We can certainly assume \( \operatorname{cap} (A, a, g) \) to be nonzero, since otherwise (4.2) follows at once from (4.1). Then the set \( \mathcal{E}(A, a, g) \) must be nonempty; fix \( \mu \), one of its elements. Given \( \mathbf{K} \in \{ \mathbf{K} \}_A \) and \( i \in I \), let \( \mu^i_K \) denote the trace of \( \mu^i \) upon \( K_i \), i.e., \( \mu^i_K := \mu^i_{K_i} \). Applying Lemma 1.2.2 from \cite{9}, we conclude that
\[
\int g_i \, d\mu^i = \lim_{\mathbf{K} \uparrow A} \int g_i \, d\mu^i_K, \quad i \in I, \quad (4.3)
\]
\[
\kappa(\mu^i, \mu^j) = \lim_{\mathbf{K} \uparrow A} \kappa(\mu^i_K, \mu^j_K), \quad i, j \in I. \quad (4.4)
\]

Fix \( \varepsilon > 0 \). It follows from (4.3) and (4.4) that for every \( i \in I \) one can choose a compact set \( K^0_i \subset A_i \) so that
\[
\int g_i \, d\mu^i < 1 + \varepsilon^{-2}, \quad (4.5)
\]
\[
\|\mu^i\|^2 - \|\mu^i_{K^0_i}\|^2 < \varepsilon^2 i^{-4}. \quad (4.6)
\]
Having denoted \( K^0 := (K^0_i)_{i \in I}, \) for every \( \mathbf{K} \in \{ \mathbf{K} \}_A \) that follows \( K^0 \) we therefore have \( \int g_i \, d\mu^i_K \neq 0 \) and
\[
\hat{\mu}_K := \sum_{i \in I} \frac{\alpha_i a_i}{\int g_i \, d\mu^i_K} \mu^i_K \in \mathcal{E}(\mathbf{K}, a, g), \quad (4.7)
\]
the finiteness of the energy being obtained from (4.6) and Corollary 3.2. Thus,
\[
\|\hat{\mu}_K\|^2 \geq \|\mathcal{E}(\mathbf{K}, a, g)\|^2. \quad (4.8)
\]

We next proceed by showing that
\[
\|\mu\|^2 = \lim_{\mathbf{K} \uparrow A} \|\hat{\mu}_K\|^2. \quad (4.9)
\]
To this end, it can be assumed that \( \kappa \geq 0 \); for if not, then \( A \) must be finite since \( X \) is compact, and (4.9) follows from (4.3) and (4.4) when substituted into (3.5). Therefore, for every \( \mathbf{K} \) that follows \( K^0 \) and every \( i \in I \) we get
\[
\|\mu^i_K\| \leq \|\mu^i\| \leq \|R\mu^+ + R\mu^-\|, \quad (4.10)
\]
\[
\|\mu^i - \mu^i_K\| < \varepsilon i^{-2}, \quad (4.11)
\]
the latter being clear from (4.6) because of \( \kappa(\mu^i_K, \mu^i - \mu^i_K) \geq 0 \). Furthermore, by (3.5),

\[
||\mu||^2 - ||\hat{\mu}_K||^2 \leq \sum_{i,j \in I} \left[ \kappa(\mu_i, \mu^j) - \frac{a_i}{g_i d\mu_K} \frac{a_j}{g_j d\mu_K} \kappa(\mu^i_K, \mu^j_K) \right]
\]

\[
\leq \sum_{i,j \in I} \left[ \kappa(\mu_i^j - \mu^i_K, \mu^j) + \kappa(\mu^i_K, \mu^j - \mu^j_K) + \left( \frac{a_i}{g_i d\mu_K} \frac{a_j}{g_j d\mu_K} - 1 \right) \kappa(\mu^i_K, \mu^j_K) \right].
\]

When combined with (4.8), (4.10), and (4.11), this yields

\[
||\mu||^2 - ||\hat{\mu}_K||^2 \leq M \varepsilon \quad \text{for all } K \succ K_0,
\]

where \( M \) is finite and independent of \( K \), and the required relation (4.9) follows.

Substituting (4.8) into (4.9), in view of the arbitrary choice of \( \mu \in \mathcal{E}(A, a, g) \) we get

\[
||\mathcal{E}(A, a, g)||^2 \geq \lim_{K \uparrow A} ||\mathcal{E}(K, a, g)||^2.
\]

Since the converse inequality is obvious from (4.1), the proof is complete. \( \square \)

Let \( \mathcal{E}_0(A, a, g) \) denote the class of all \( \mu \in \mathcal{E}(A, a, g) \) such that, for every \( i \in I \), the support \( S(\mu^i) \) of \( \mu^i \) is compact and contained in \( A_i \).

**Corollary 4.1.** The capacity \( \text{cap}(A, a, g) \) remains unchanged if the class \( \mathcal{E}(A, a, g) \) in its definition is replaced by \( \mathcal{E}_0(A, a, g) \). In other words,

\[
||\mathcal{E}(A, a, g)||^2 = ||\mathcal{E}_0(A, a, g)||^2.
\]

**Proof.** We can assume that \( ||\mathcal{E}(A, a, g)||^2 < \infty \), since otherwise the corollary follows from \( \mathcal{E}_0(A, a, g) \subset \mathcal{E}(A, a, g) \). Then, by (4.1) and (4.2), for every \( \varepsilon > 0 \) there exists a compact condenser \( K \preccurlyeq A \) such that \( ||\mathcal{E}(K, a, g)||^2 \leq ||\mathcal{E}(A, a, g)||^2 + \varepsilon \). This proves the corollary when combined with the inequalities

\[
||\mathcal{E}(K, a, g)||^2 \geq ||\mathcal{E}_0(A, a, g)||^2 \geq ||\mathcal{E}(A, a, g)||^2.
\]

\( \square \)

### 4.3 When does \( 0 < \text{cap} A < \infty \) hold?

In all that follows it is required that

\[
\text{cap}(A, a, g) > 0. \quad (4.12)
\]

**Lemma 4.2.** For (4.12) to hold, it is necessary and sufficient that any of the following three conditions be satisfied:

(i) \( \mathcal{E}(A, a, g) \) is nonempty;

(ii) there exist \( \nu_i \in \mathcal{E}^+(A_i, a_i, g_i) \) for all \( i \in I \) such that \( \sum_{i \in I} ||\nu_i||^2 < \infty \);

(iii) \( \sum_{i \in I} ||\mathcal{E}^+(A_i, a_i, g_i)||^2 < \infty \).
Proof. The equivalency of (4.12) and (i) is obvious, while that of (i) and (ii) can be obtained directly from Corollary 3.2. If (iii) is true, then for every \( i \in I \) one can choose \( \nu_i \in \mathcal{E}^+(A_i, a_i, g_i) \) so that \( \|\nu_i\|^2 < \|\mathcal{E}^+(A_i, a_i, g_i)\|^2 + i^{-2} \), and (ii) follows. Since (ii) obviously yields (iii), the proof is complete. □

Let \( C(\cdot) \) denote the interior capacity of a set with respect to the kernel \( \kappa \). [9]

**Corollary 4.2.** For (4.12) to be satisfied, it is necessary that

\[
C(A_i) > 0 \quad \text{for all} \quad i \in I.
\]

If \( A \) is finite, then (4.12) and (4.13) are actually equivalent. \(^1\)

**Proof.** For Lemma 4.2, (ii) to hold, it is necessary that, for every \( i \in I \), there exists a nonzero nonnegative measure of finite energy, compactly supported in \( A_i \), which in turn is equivalent to (4.13) according to [9, Lemma 2.3.1]. Since the former implication can obviously be inverted if \( A \) is finite, the proof is complete. □

Let \( g_i,\inf \) and \( g_i,\sup \) be the infimum and the supremum of \( g_i \) over \( A_i \). Also write

\[
g_{\inf} := \inf_{i \in I} g_i,\inf, \quad g_{\sup} := \sup_{i \in I} g_i,\sup.
\]

**Corollary 4.3.** Assume \( 0 < g_{\inf} \leq g_{\sup} < \infty \). Then (4.12) holds if and only if

\[
\sum_{i \in I} \frac{a_i^2}{C(A_i)} < \infty.
\]

**Proof.** Lemma 4.2, (iii) implies the corollary when combined with the inequalities

\[
\frac{a_i^2}{g_{i,\sup}^2 C(A_i)} \leq \|\mathcal{E}^+(A_i, a_i, g_i)\|^2 \leq \frac{a_i^2}{g_{i,\inf}^2 C(A_i)}, \quad i \in I,
\]

(4.14)

to be proved below by reasons of homogeneity. To establish (4.14), fix \( i \in I \). One can certainly assume \( C(A_i) \) to be nonzero, for otherwise Corollary 4.2 with \( I = \{i\} \) shows that each of the three parts in (4.14) equals \( +\infty \). Consequently, there exists \( \hat{\theta}_i \in \mathcal{E}^+(A_i, 1, 1) \). Since

\[
\hat{\theta}_i := \frac{a_i \theta_i}{\int g_i \, d\theta_i} \in \mathcal{E}^+(A_i, a_i, g_i),
\]

we get

\[
a_i^2 \|\theta_i\|^2 \geq g_{i,\inf}^2 \|\hat{\theta}_i\|^2 \geq g_{i,\inf}^2 \|\mathcal{E}^+(A_i, a_i, g_i)\|^2,
\]

and the right-hand side in (4.14) is obtained by letting \( \theta_i \) range over \( \mathcal{E}^+(A_i, 1, 1) \). To verify the left-hand side, fix \( \omega_i \in \mathcal{E}^+(A_i, 1, 1) \). Then

\[
0 < a_i g_{i,\inf}^{-1} \leq \omega_i(X) \leq a_i g_{i,\inf}^{-1} < \infty.
\]

Hence, \( \omega_i/\omega_i(X) \in \mathcal{E}^+(A_i, 1, 1) \) and

\[
\|\omega_i\|^2 \geq a_i^2 g_{i,\sup}^{-2} \|\mathcal{E}^+(A_i, 1, 1)\|^2.
\]

In view of the arbitrary choice of \( \omega_i \in \mathcal{E}^+(A_i, a_i, g_i) \), this completes the proof. □

\(^1\)However, (4.12) and (4.13) are no longer equivalent if \( A \) is infinite — cf. Corollary 4.3.
In the following assertion, providing necessary conditions for \( \text{cap} \ A \) to be finite, it is assumed that \( g_{i,\text{inf}} > 0 \) for all \( i \in I \).

**Lemma 4.3.** If \( \text{cap} (A, a, g) < \infty \), then there exists \( j \in I \) with \( C(A_j) < \infty \).

**Proof.** Under the assumptions of the lemma, suppose \( C(A_i) = \infty \) for all \( i \in I \). Given \( \varepsilon > 0 \), for every \( i \in I \) one can choose \( \mu_i \in E^+(A_i, 1, 1) \) with compact support so that \( \| \mu_i \| \leq \varepsilon a_i^{-1} i^{-2} g_{i,\text{inf}} \). Since then 

\[
\hat{\mu} := \sum_{i \in I} \frac{\alpha_i a_i \mu_i}{g_i} d\mu_i \in E(A, a, g)
\]

and \( \| \hat{\mu} \| \leq \varepsilon \sum_{i \in I} i^{-2} \), we arrive at a contradiction by letting \( \varepsilon \to 0 \). \( \square \)

**Remark 4.1.** It will be shown by Corollary 7.2 below that, under certain additional restrictions, Lemma 4.3 can be inverted.

5 Minimizing measures on condensers: existence, uniqueness, A-vague compactness, continuity

Because of (4.12), we are naturally led to the \( E(A, a, g) \)-problem (cf. Sec. 4), i.e., the problem on the existence of \( \lambda = \lambda_A \in E(A, a, g) \) with minimal energy

\[
\| \lambda_A \|^2 = \| E(A, a, g) \|^2.
\]

Let \( \mathcal{S}(A, a, g) \) denote the class of all minimizers \( \lambda_A \).

5.1 Uniqueness properties of \( \lambda_A \)

We start by observing that any two minimizers (if exist) are *equivalent in \( E(A) \)*, i.e.,

\[
\| \lambda_1 - \lambda_2 \| = 0 \quad \text{for all} \quad \lambda_1, \lambda_2 \in \mathcal{S}(A, a, g).
\]

Indeed, this follows from the convexity of \( E(A, a, g) \) and the parallelogram identity in \( E \), applied to \( R\lambda_1 \) and \( R\lambda_2 \). Thus, \( \lambda_1 \cong \lambda_2 \) provided the kernel \( \kappa \) is strictly positive definite, and \( \lambda_1 \equiv \lambda_2 \) if, moreover, all \( A_i, i \in I \), are mutually disjoint.

What about the existence of minimizers? Assume for a moment that \( A \) is *finite and compact* and that \( \kappa \) is *continuous on* \( A^+ \times A^- \). Then \( \mathcal{M}(A, a, g) \) is \( A \)-vaguely compact while \( \| \mu \|^2 \) is \( A \)-vaguely lower semicontinuous on \( E(A) \) and, therefore, the solvability of the \( E(A, a, g) \)-problem follows. See [19, Th. 2.30]; cf. also [11, 12, 18, 20], related to the logarithmic kernel in the plane.

However, these arguments break down if any of the above three assumptions is dropped. In particular, \( \mathcal{M}(A, a, g) \) is no longer \( A \)-vaguely compact if \( A \) is noncompact.

To solve the problem on the existence of minimizers \( \lambda_A \) in the general case where a condenser \( A \) is infinite and (or) noncompact, we develop an approach based on both the \( A \)-vague and strong topologies in the semimetric space \( E(A) \), introduced for finite condensers in [22, 23, 24].
5.2 Standing assumptions

Unless explicitly stated otherwise, in all that follows it is required that the kernel \( \kappa \) is consistent and either \( I^- = \emptyset \), or the following conditions are both satisfied:

\[
\sum_{i \in I} a_i g_i^{-1} < \infty, \quad (5.1)
\]

\[
\sup_{x \in A^+, y \in A^-} \kappa(x, y) < \infty. \quad (5.2)
\]

**Remark 5.1.** These assumptions on a kernel are not too restrictive. In particular, they all are satisfied by the Newtonian, Riesz, or Green kernels in \( \mathbb{R}^n \), \( n \geq 2 \), provided the Euclidean distance between \( A^+ \) and \( A^- \) is nonzero.

5.3 Existence of minimizers \( \lambda_A \). A-vague compactness

A proposition \( R(x) \) involving a variable point \( x \in X \) is said to subsist nearly everywhere (n. e.) in \( E \), where \( E \) is a given subset of \( X \), if the set of all \( x \in E \) for which \( R(x) \) fails to hold is of interior capacity zero [9].

**Theorem 5.1.** For every \( i \in I \), assume that either \( g_i, \sup < \infty \), or there exist \( r_i \in (1, \infty) \) and \( \omega_i \in \mathcal{E} \) such that

\[
g^{r_i}(x) \leq \kappa(x, \omega_i) \quad \text{n. e. in } A_i. \quad (5.3)
\]

If, moreover, \( A_i \) either is compact or has finite capacity\(^2\)

\[
C(A_i) < \infty,
\]

then for any vector \( a \) the class \( \mathcal{S}(A, a, g) \) is nonempty and \( A \)-vaguely compact.

**Corollary 5.1.** If \( A = K \) is compact, then for any \( a \) and \( g \) the class \( \mathcal{S}(A, a, g) \) is nonempty and \( A \)-vaguely compact.

**Proof.** This is an immediate consequence of Theorem 5.1 since \( g_i \) is bounded on \( K \).

5.4 On continuity of minimizers

When approaching \( A \) by the increasing family \( \{K\}_A \) of the compact condensers \( K \prec A \), we shall always suppose all those \( K \) to be of capacity nonzero. This involves no loss of generality, which is clear from (4.12) and Lemma 4.1. Choose an arbitrary \( \lambda_K \in \mathcal{S}(A, a, g) \) — its existence has been ensured by Corollary 5.1.

**Theorem 5.2.** Let all the conditions of Theorem 5.1 be satisfied. Then every \( A \)-vague cluster point of \( \{\lambda_K\}_{K \in \{K\}_A} \) (such a cluster point exists) belongs to \( \mathcal{S}(A, a, g) \). Furthermore, if \( \lambda_A \in \mathcal{S}(A, a, g) \) is arbitrarily given, then

\[
\lim_{K \uparrow A} \|\lambda_K - \lambda_A\|^2 = 0.
\]

\(^2\)Note that a compact set \( K \subset X \) might be of infinite capacity; \( C(K) \) is necessarily finite provided the kernel is strictly positive definite [9]. On the other hand, even for the Newtonian kernel, sets of finite capacity might be noncompact (see [19]).
Thus, under the assumptions of Theorem 5.2 if moreover \( \kappa \) is strictly positive definite and all \( A_i, i \in I \), are mutually disjoint, then the (unique) minimizer \( \lambda_K \) approaches the (unique) minimizer \( \lambda_A \) both \( A \)-vaguely and strongly as \( K \uparrow A \).

The proofs of Theorems 5.1 and 5.2, to be given in Sec. 8 (see also Sec. 7 for certain crucial auxiliary notions and results), are based on a theorem on the strong completeness of proper subspaces of \( \mathcal{E}(A) \), which is a subject of the next section.

6 Strong completeness of measures associated with condensers

As always, assume all the standing assumptions, stated in Sec. 5.2, to hold. Having denoted
\[
\mathcal{M}(A, \leq a, g) := \{ \mu \in \mathcal{M}(A) : \int g_i \, d\mu_i \leq a_i \text{ for all } i \in I \},
\]
we treat \( \mathcal{E}(A, \leq a, g) := \mathcal{M}(A, \leq a, g) \cap \mathcal{E}(A) \) as a topological subspace of the semimetric space \( \mathcal{E}(A) \); the induced topology is likewise called the strong topology.

Our purpose is to show that \( \mathcal{E}(A, \leq a, g) \) is strongly complete.

6.1 Auxiliary assertions

Lemma 6.1. \( \mathcal{M}(A, \leq a, g) \) is \( A \)-vaguely bounded and, hence, \( A \)-vaguely compact.

Proof. Fix \( i \in I \), and let a compact set \( K \subset A_i \) be given. Since \( g_i \) is positive and continuous, the relation
\[
a_i \geq \int g_i \, d\mu_i \geq \mu_i(K) \min_{x \in K} g_i(x), \quad \text{where } \mu \in \mathcal{M}(A, \leq a, g),
\]
yields
\[
\sup_{\mu \in \mathcal{M}(A, \leq a, g)} \mu_i(K) < \infty.
\]
This implies that \( \mathcal{M}(A, \leq a, g) \) is \( A \)-vaguely bounded, and hence it is \( A \)-vaguely relatively compact by Lemma 3.1. Since it is \( A \)-vaguely closed in consequence of Lemma 3.3 the desired assertion follows.

Lemma 6.2. If a net \( (\mu_s)_{s \in S} \subset \mathcal{E}(A, \leq a, g) \) is strongly bounded, then its \( A \)-vague adherence is nonempty and contained in \( \mathcal{E}(A, \leq a, g) \).

Proof. According to Lemma 6.1 the \( A \)-vague adherence of \( (\mu_s)_{s \in S} \) is nonempty and contained in \( \mathcal{M}(A, \leq a, g) \). To establish the lemma, it is enough to prove that every its element \( \mu \) is of finite energy.

To this end, observe that \( (R\mu_s)_{s \in S} \) is strongly bounded by (3.3). We proceed by showing that so are the nets \( (R\mu_s^+)_{s \in S} \) and \( (R\mu_s^-)_{s \in S} \), i.e.,
\[
\sup_{s \in S} \| R\mu_s^\pm \|^2 < \infty.
\]
Proof. Fix a strong Cauchy net \((s)\) where \((s)\) is mutually disjoint. If moreover \((s)\) is complete. In more detail, if \((s)\) is a strong Cauchy net in \(\mathcal{E}(A, \leq a, g)\) and \(\mu\) is one of its \(A\)-vague cluster point (such a \(\mu\) exists), then \(\mu \in \mathcal{E}(A, \leq a, g)\) and
\[
\lim_{s \in S} \|\mu_s - \mu\|^2 = 0.
\]

Assume, in addition, that the kernel is strictly positive definite and all \(A_i, i \in I\), are mutually disjoint. If moreover \((s)\) converges strongly to \(\mu_0 \in \mathcal{E}(A, \leq a, g)\), then actually \(\mu_0 \in \mathcal{E}(A, \leq a, g)\) and \(\mu_s \rightharpoonup \mu_0\) \(A\)-vaguely.

Proof. Fix a strong Cauchy net \((s)\) in \(\mathcal{E}(A, \leq a, g)\). Since such a net converges strongly to every its strong cluster point, \((s)\) can certainly be assumed to be

\[
\sup_{s \in S} \mu_s(X) \leq a_i g_{i,\text{inf}}^{-1} \quad \text{for all } i \in I.
\]

Consequently, by (6.6),
\[
\sup_{s \in S} R\mu_s^\pm(X) \leq \sum_{i \in I} a_i g_{i,\text{inf}}^{-1} < \infty.
\]

Because of (6.2), this implies that \(\kappa(R\mu^+, R\mu^-)\) remains bounded from above on \(S\); hence, so do \(\|R\mu^+\|^2\) and \(\|R\mu^-\|^2\).

Now, if \((\mu_s)_{s \in \mathbb{D}}\) is a subnet of \((\mu_s)_{s \in S}\) that converges \(A\)-vaguely to \(\mu\), then
\[
\kappa(R\mu^+, R\mu^-) \geq C > -\infty,
\]
where \(C\) is independent of \(s\). Since this is obvious when \(\kappa \geq 0\), one can assume \(X\) to be compact. Then \(\kappa\), being lower semicontinuous, is bounded from below on \(X\) (say by \(-c\), where \(c > 0\)), while \(A\) is finite. Furthermore, then \(g_{i,\text{inf}} > 0\) for all \(i \in I\) and, hence, (6.2) holds. This implies that \(\kappa(\mu_i, \mu_j) \geq -a_i a_j g_{i,\text{inf}}^{-1} g_{j,\text{inf}}^{-1} c\) for all \(i, j \in I\), and (6.4) follows.

**Corollary 6.1.** If \((\mu_s)_{s \in S} \subset \mathcal{E}(A, \leq a, g)\) is strongly bounded, then
\[
\sup_{s \in S} \|\mu_s\|^2 < \infty, \quad i \in I.
\]

**Proof.** It is seen from (6.1) that the claimed relation (6.3) will be proved once we show that
\[
\sum_{i,j \in I} \kappa(\mu_i, \mu_j) \geq C > -\infty,
\]
where \(C\) is independent of \(s\). Since this is obvious when \(\kappa \geq 0\), one can assume \(X\) to be compact. Then \(\kappa\), being lower semicontinuous, is bounded from below on \(X\) (say by \(-c\), where \(c > 0\)), while \(A\) is finite. Furthermore, then \(g_{i,\text{inf}} > 0\) for all \(i \in I\) and, hence, (6.2) holds. This implies that \(\kappa(\mu_i, \mu_j) \geq -a_i a_j g_{i,\text{inf}}^{-1} g_{j,\text{inf}}^{-1} c\) for all \(i, j \in I\), and (6.4) follows.

**6.2 Strong completeness of \(\mathcal{E}(A, \leq a, g)\)**

**Theorem 6.1.** The semimetric space \(\mathcal{E}(A, \leq a, g)\) is complete. In more detail, if \((\mu_s)_{s \in S}\) is a strong Cauchy net in \(\mathcal{E}(A, \leq a, g)\) and \(\mu\) is one of its \(A\)-vague cluster point (such a \(\mu\) exists), then \(\mu \in \mathcal{E}(A, \leq a, g)\) and
\[
\lim_{s \in S} \|\mu_s - \mu\|^2 = 0.
\]
strongly bounded. Then, by Lemma 6.2 there exists an $A$-vague cluster point $\mu$ of $(\mu_s)_{s \in S}$, and
\[ \mu \in \mathcal{E}(A, \leq a, g). \] (6.6)

We next proceed by verifying (6.5). Of course, there is no loss of generality in assuming $(\mu_s)_{s \in S}$ to converge $A$-vaguely to $\mu$. Then, by Lemma 3.2 $(R\mu^+_s)_{s \in S}$ and $(R\mu^-_s)_{s \in S}$ converge vaguely to $R\mu^+$ and $R\mu^-$, respectively. Since, by (6.1), these nets are strongly bounded in $\mathcal{E}$, the property (C2) (see Sec. 2) shows that they approach $R\mu^+$ and $R\mu^-$, respectively, in the weak topology as well, and so $R\mu_s \to R\mu$ weakly. This gives, by (3.6),
\[ \|\mu_s - \mu\|^2 = \|R\mu_s - R\mu\|^2 = \lim_{l \in S} \kappa(R\mu_s - R\mu, R\mu_s - R\mu_l), \]
and hence, by the Cauchy-Schwarz inequality,
\[ \|\mu_s - \mu\|^2 \leq \|\mu_s - \mu\| \liminf_{l \in S} \|\mu_s - \mu_l\|, \]
which proves (6.5) as required, because $\|\mu_s - \mu_l\|$ becomes arbitrarily small when $s, l \in S$ are both sufficiently large.

Suppose now that $\kappa$ is strictly positive definite, while all $A_i$, $i \in I$, are mutually disjoint, and let the net $(\mu_s)_{s \in S}$ converge strongly to some $\mu_0 \in \mathcal{E}(A)$. Given an $A$-vague limit point $\mu$ of $(\mu_s)_{s \in S}$, we conclude from (6.5) that $\|\mu_0 - \mu\| = 0$, hence $\mu_0 \equiv \mu$ since $\kappa$ is strictly positive definite, and finally $\mu_0 \equiv \mu$ because $A_i$, $i \in I$, are mutually disjoint. In view of (6.6), this means that $\mu_0 \in \mathcal{E}(A, \leq a, g)$, which is a part of the desired conclusion. Moreover, $\mu_0$ has thus been shown to be identical to any $A$-vague cluster point of $(\mu_s)_{s \in S}$. Since the $A$-vague topology is Hausdorff, this implies that $\mu_0$ is actually the $A$-vague limit of $(\mu_s)_{s \in S}$ (cf. [1, Chap. I, § 9, n° 1, cor.]), which completes the proof.

**Remark 6.1.** In view of the fact that the semimetric space $\mathcal{E}(A, \leq a, g)$ is isometric to its $R$-image, Theorem 6.1 has thus singled out a strongly complete topological subspace of the pre-Hilbert space $\mathcal{E}$, whose elements are signed measures. This is of independent interest since, according to a well-known counterexample by H. Cartan [4], all the space $\mathcal{E}$ is strongly incomplete even for the Newtonian kernel $|x - y|^{2-n}$ in $\mathbb{R}^n$, $n \geq 3$.

**Remark 6.2.** Assume $\kappa$ is strictly positive definite (hence, perfect). If moreover $I^- = \emptyset$, then Theorem 6.1 remains valid for $\mathcal{E}(A)$ in place of $\mathcal{E}(A, \leq a, g)$ (cf. Theorem 2.1). A question still unanswered is whether this is the case if $I^+$ and $I^-$ are both nonempty. We can however show that this is really so for the Riesz kernels $|x - y|^{\alpha-n}$, $0 < \alpha < n$, in $\mathbb{R}^n$, $n \geq 2$ (cf. [24] Th. 1). The proof utilizes Deny’s theorem [5] stating that, for the Riesz kernels, $\mathcal{E}$ can be completed by making use of distributions of finite energy.

7 Extremal measures

To apply Theorem 6.1 to the $\mathcal{E}(A, a, g)$-problem, we next proceed by introducing a concept of extremal measure defined as a strong and, simultaneously, the $A$-vague
limit of a minimizing net. Since the energy of such a measure equals the infimum value \( \|E(A, a, g)\|^2 \), it serves as a minimizer if and only if it belongs to the class \( E(A, a, g) \). See below for the strict definition and related auxiliary results.

7.1 Minimizing nets; their A-vague and strong cluster points

**Definition 7.1.** We call a net \((\mu_s)_{s \in S}\) minimizing if \((\mu_s)_{s \in S} \subset E_0(A, a, g)\) and

\[
\lim_{s \in S} \|\mu_s\|^2 = \|E(A, a, g)\|^2.
\]  

(7.1)

Let \( M(A, a, g) \) consist of all minimizing nets, and let \( M(A, a, g) \) stand for the union of all their A-vague cluster sets. Note that \( M(A, a, g) \) is nonempty, which is clear from (4.12) in view of Corollary 4.1. Hence, according to Lemma 6.2, \( M(A, a, g) \) is nonempty as well, and

\[
M(A, a, g) \subset E(A, \leq a, g).
\]  

(7.2)

**Definition 7.2.** A measure \( \gamma \in E(A) \) is called extremal in the \( E(A, a, g) \)-problem if there exists \((\mu_s)_{s \in S} \in M(A, a, g)\) converging to \( \gamma \) both strongly and A-vaguely; such a net \((\mu_s)_{s \in S}\) is said to generate \( \gamma \). The collection of all extremal measures will be denoted by \( E(A, a, g) \).

7.2 Extremal measures: existence, uniqueness, and compactness

It follows from Definition 7.2 and relations (7.1) and (7.2) that an extremal measure \( \gamma \) (if exists) belongs to the class \( E(A, \leq a, g) \) and satisfies the identity

\[
\|\gamma\|^2 = \|E(A, a, g)\|^2.
\]  

(7.3)

**Lemma 7.1.** The following assertions hold true:

(i) From every minimizing net one can select a subnet generating an extremal measure; hence, \( E(A, a, g) \) is nonempty. Furthermore,

\[
E(A, a, g) = M(A, a, g).
\]  

(7.4)

(ii) Every minimizing net converges strongly to every extremal measure; consequently, \( E(A, a, g) \) is contained in an equivalence class in \( E(A) \).

(iii) The class \( E(A, a, g) \) is A-vaguely compact.

**Proof.** Fix \((\mu_s)_{s \in S}\) and \((\nu_t)_{t \in T} \in M(A, a, g)\). It is seen with standard arguments that

\[
\lim_{(s, t) \in S \times T} \|\mu_s - \nu_t\|^2 = 0,
\]  

(7.5)

where \( S \times T \) denotes the directed product of the directed sets \( S \) and \( T \) (see, e.g., [15 Chap. 2, § 3]). Indeed, by the convexity of the class \( E(A, a, g) \),

\[
2 \|E(A, a, g)\| \leq \|\mu_s + \nu_t\| \leq \|\mu_s\| + \|\nu_t\|,
\]
and hence, by (7.1),
\[
\lim_{(s,t)\in S\times T} \|\mu_s + \nu_t\|^2 = 4 \|\mathcal{E}(A, a, g)\|^2.
\]

Then the parallelogram identity in \(E\), applied to \(R\mu_s\) and \(R\nu_t\), gives (7.5).

Relation (7.5) implies that the net \((\mu_s)_{s\in S}\) is strongly fundamental. Therefore, if \(\mu_0\) is one of its \(A\)-vague cluster points (such a \(\mu_0\) exists), then \(\mu_s \to \mu_0\) strongly by Theorem 6.1. This means that \(\mu_0\) is an extremal measure and, hence, \(\mathcal{M}(A, a, g) \subset \mathcal{E}(A, a, g)\). Since the inverse inclusion is obvious, (7.4) follows.

To prove (ii), fix \((\mu_s)_{s\in S} \in \mathcal{M}(A, a, g)\) and \(\gamma \in \mathcal{E}(A, a, g)\). By Definition 7.2 one can choose a net in \(\mathcal{M}(A, a, g)\), say \((\nu_t)_{t\in T}\), converging to \(\gamma\) strongly. Repeated application of (7.5) shows that also \((\mu_s)_{s\in S}\) converges to \(\gamma\) strongly, as claimed.

To verify (iii), it is enough to show that \(\mathcal{M}(A, a, g)\) is \(A\)-vaguely compact. Fix \((\gamma_s)_{s\in S} \subset \mathcal{M}(A, a, g)\). It follows from (7.2) and Lemma 6.1 that there exists an \(A\)-vague cluster point \(\gamma_0\) of \((\gamma_s)_{s\in S}\); let \((\gamma_t)_{t\in T}\) be a subnet of \((\gamma_s)_{s\in S}\) that converges \(A\)-vaguely to \(\gamma_0\). Then, for every \(t \in T\), there exists \((\mu_{s_i})_{s_i \in S_i} \in \mathcal{M}(A, a, g)\) converging \(A\)-vaguely to \(\gamma_t\). Consider the Cartesian product \(\prod \{S_i : t \in T\}\) — that is, the collection of all functions \(\beta\) on \(T\) with \(\beta(t) \in S_t\), and let \(D\) denote the directed product \(T \times \prod \{S_i : t \in T\}\) (see, e.g., [15, Chap. 2, § 3]). Given \((t, \beta) \in D\), write \(\mu_{(t,\beta)} := \mu_{\beta(t)}\). Then the theorem on iterated limits from [15, Chap. 2, § 4] yields that the net \((\mu_{(t,\beta)})_{(t,\beta) \in D}\) belongs to \(\mathcal{M}(A, a, g)\) and converges \(A\)-vaguely to \(\gamma_0\). Thus, \(\gamma_0 \in \mathcal{M}(A, a, g)\) as was to be proved.

**Corollary 7.1.** Every minimizing measure \(\lambda = \lambda_A\) (if exists) is extremal, i.e.,
\[
\mathcal{E}(A, a, g) \subset \mathcal{E}(A, a, g)\) \quad (7.6)
\]

**Proof.** For every sufficiently large \(K \in \{K\}_A\), we define \(\lambda_K\) by (4.7) with \(\lambda\) instead of \(\mu\). Then \((\lambda_K)_{K \in \{K\}_A}\) belongs to \(\mathcal{M}(A, a, g)\), which is clear from (4.9) with \(\mu\) replaced by \(\lambda\). On the other hand, this net converges \(A\)-vaguely to \(\lambda\); hence, \(\lambda \in \mathcal{M}(A, a, g)\). Combined with (7.4), this completes the proof.

### 7.3 Central lemma

The following assertion will be central in the proof of Theorem 5.1.

**Lemma 7.2.** Fix \(i \in I\) and assume that either \(g_{i,\sup} < \infty\), or (5.3) holds for some \(r_i \in (1, \infty)\) and \(\omega_i \in \mathcal{E}\). If, moreover, \(A_i\) either is compact or has finite interior capacity, then
\[
\int g_i \, d\gamma^i = a_i \quad \text{for all } \gamma \in \mathcal{E}(A, a, g)\) \quad (7.7)
\]

**Proof.** Given \(\gamma \in \mathcal{E}(A, a, g)\), choose a net \((\mu_s)_{s\in S} \in \mathcal{M}(A, a, g)\) that converges to \(\gamma\) both strongly and \(A\)-vaguely. Taking a subnet if necessary, one can certainly assume \((\mu_s)_{s\in S}\) to be strongly bounded.

Of course, (7.7) needs to be proved only if the set \(A_i\) is noncompact; then its capacity has to be finite. Hence, by [9, Th. 4.1], for every \(E \subset A_i\) there exists a
measure \( \theta_E \in \mathcal{E}^+(\overline{E}) \), called an interior equilibrium measure associated with \( E \), which admits the properties

\[
\theta_E(X) = \|\theta_E\|^2 = C(E), \tag{7.8}
\]

\[
\kappa(x, \theta_E) \geq 1 \quad \text{n.e. in } E. \tag{7.9}
\]

Also observe that there is no loss of generality in assuming \( g_i \) to satisfy (5.3) for some \( r_i \in (1, \infty) \) and \( \omega_i \in \mathcal{E} \). Indeed, otherwise \( g_i \) has to be bounded from above (say by \( M_1 \)), which combined with (7.9) again gives (5.3) for \( \omega_i := M_1 r_i^1 \theta_\lambda_i \), \( r_i \in (1, \infty) \) being arbitrary.

To establish (7.7), we shall treat \( A_i \) as a locally compact space with the topology induced from \( X \). Let \( \chi_E \) denote the characteristic function of a given set \( E \subset A_i \) and let \( E^c := A_i \setminus E \). Further, let \( \{K_i\} \) be the increasing family of all compact subsets \( K_i \) of \( A_i \). Since \( g_i \chi_{K_i} \) is upper semicontinuous on \( A_i \) while \( (\mu_i)_{s \in S} \) converges to \( \gamma_i \) vaguely, from Lemma 3.3 we get

\[
\int g_i \chi_{K_i} \, d\gamma_i \geq \limsup_{s \in S} \int g_i \chi_{K_i} \, d\mu_i^s \quad \text{for every } K_i \in \{K_i\}.
\]

On the other hand, application of Lemma 1.2.2 from [9] yields

\[
\int g_i \, d\gamma_i = \lim_{K_i \in \{K_i\}} \int g_i \chi_{K_i} \, d\gamma_i.
\]

Combining the last two relations, we obtain

\[
a_i \geq \int g_i \, d\gamma_i \geq \limsup_{(s, K_i) \in S \times \{K_i\}} \int g_i \chi_{K_i} \, d\mu_i^s = a_i - \liminf_{(s, K_i) \in S \times \{K_i\}} \int g_i \chi_{K_i} \, d\mu_i^s,
\]

\( S \times \{K_i\} \) being the directed product of the directed sets \( S \) and \( \{K_i\} \). Hence, if we prove that

\[
\liminf_{(s, K_i) \in S \times \{K_i\}} \int g_i \chi_{K_i} \, d\mu_i^s = 0, \tag{7.10}
\]

the desired relation (7.7) follows.

Consider an interior equilibrium measure \( \theta_{K_i^c} \), where \( K_i \in \{K_i\} \) is given. Then application of Lemma 4.1.1 and Theorem 4.1 from [9] shows that

\[
\|\theta_{K_i^c} - \theta_{\tilde{K}_i^c}\|^2 \leq \|\theta_{K_i^c}\|^2 - \|\theta_{\tilde{K}_i^c}\|^2 \quad \text{provided } K_i \subset \tilde{K}_i.
\]

Furthermore, it is clear from (7.8) that the net \( \|\theta_{K_i^c}\|, K_i \in \{K_i\} \), is bounded and nonincreasing, and hence fundamental in \( R \). The preceding inequality thus shows that the net \( (\theta_{K_i^c})_{K_i \in \{K_i\}} \) is strongly fundamental in \( \mathcal{E} \). Since, clearly, it converges vaguely to zero, the property (C1) (see, Sec. 2) implies immediately that zero is also one of its strong limits and, hence,

\[
\lim_{K_i \in \{K_i\}} \|\theta_{K_i^c}\| = 0. \tag{7.11}
\]
Write \( q_i := r_i(r_i - 1)^{-1} \), where \( r_i \in (1, \infty) \) is a number involved in condition \( 5.3 \). Combining \( 5.3 \) with \( 7.9 \) shows that the inequality

\[
g_i(x) \chi_{K_i^c}(x) \leq \kappa(x, \omega_i)^{1/r_i} \kappa(x, \theta_{K_i^c})^{1/q_i}
\]

subsists n. e. in \( A_i \), and hence \( \mu_i^s \)-almost everywhere in \( A_i \) by virtue of Lemma 2.3.1 from \([9]\) and the fact that \( \mu_i^s \) is a measure of finite energy, compactly supported in \( A_i \). Having integrated this relation with respect to \( \mu_i^s \), we then apply the Hölder and, subsequently, the Cauchy-Schwarz inequalities to the integrals on the right. This gives

\[
\int g_i \chi_{K_i^c} \, d\mu_i^s \leq \left( \int \kappa(x, \omega_i) \, d\mu_i^s(x) \right)^{1/r_i} \left( \int \kappa(x, \theta_{K_i^c}) \, d\mu_i^s(x) \right)^{1/q_i}
\]

Taking limits here along \( S \times \{ K \} \) and using \( 6.3 \) and \( 7.11 \), we obtain \( 7.10 \) as desired.

**Corollary 7.2.** Assume that for every \( i \in I \) either \( g_i, \sup < \infty \), or \( 5.3 \) holds for some \( r_i \in (1, \infty) \) and \( \omega_i \in \mathcal{E} \). If moreover \( \kappa \) is strictly positive (hence, perfect), while \( C(A_j) < \infty \) for some \( j \in I \), then \( \text{cap} (A, a, g) \) is finite as well.

**Proof.** In consequence of Lemma 7.2, every extremal measure \( \gamma \) (whose existence has been ensured by Lemma 7.1, (i)) is nonzero; hence, due to the strict positive definiteness of the kernel, \( \| \gamma \|^2 \neq 0 \). When combined with \( 7.3 \), this yields \( \| \mathcal{E}(A, a, g) \|^2 \neq 0 \), as was to be proved.

**8 Proof of Theorems 5.1 and 5.2**

Basing on the results of Sec. 7, we are now in a position to complete the proofs of Theorems 5.1 and 5.2.

Assume that \( \kappa, A, a, \) and \( g \) satisfy all the restrictions of Theorem 5.1. Fix an arbitrary \( \gamma \in \mathcal{E}(A, a, g) \) — it exists due to Lemma 7.1 (i). Then, by Lemma 7.2 \( \gamma \in \mathcal{E}(A, a, g) \) holds for every \( i \in I \) and, consequently, \( \gamma \in \mathcal{E}(A, a, g) \). In view of \( 7.3 \), this shows that \( \gamma \) is actually a minimizer in the \( \mathcal{E}(A, a, g) \)-problem. Hence, the \( \mathcal{E}(A, a, g) \)-problem is solvable and, moreover, \( \mathcal{E}(A, a, g) \subset \mathcal{G}(A, a, g) \). When combined with \( 7.6 \), this gives \( \mathcal{E}(A, a, g) = \mathcal{G}(A, a, g) \). Therefore Lemma 7.1 (iii) yields the \( A \)-vague compactness of \( \mathcal{G}(A, a, g) \), and the proof of Theorem 5.1 is complete. Because of \( 7.4 \), the last identity also implies

\[
\mathcal{G}(A, a, g) = \mathcal{E}(A, a, g) = M(A, a, g).
\]

To prove Theorem 5.2 for every \( K \in \{ K \}_A \) fix an arbitrary \( \lambda_K \in \mathcal{G}(K, a, g) \) — its existence has been ensured by Corollary 5.1. According to Lemma 4.1

\[
(\lambda_K)_{K \in \{ K \}_A} \in \mathcal{M}(A, a, g).
\]

\[\text{Cf. Lemma } 4.3.\]
Therefore, by (8.1), every $A$-vague cluster point of $(\lambda_{\mathcal{K}})_{\mathcal{K} \in \{K\}_A}$ is an element of $\mathcal{S}(A,a,g)$, which is a part of the desired conclusion.

What is left is to show that $\lambda_{\mathcal{K}} \to \lambda_A$ strongly, where $\lambda_A \in \mathcal{S}(A,a,g)$ is arbitrarily given, but this is obtained directly from (8.1), (8.2), and Lemma 7.1 (ii).

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