EXACT SOLUTIONS FOR CORRELATION FUNCTIONS IN SOME 1+1 D FIELD THEORIES WITH BOUNDARY*

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Abstract

We consider 1+1 D theories which are free everywhere except for cosine and magnetic interactions on the boundary. These theories arise in dissipative quantum systems, open string theory, and, in special cases, tunneling in quantum Hall systems. These boundary systems satisfy an approximate SL(2,Z) symmetry as a function of flux per unit cell and dissipation. At special multicritical points, they also satisfy a set of reparametrization Ward identities and have homogeneous, piecewise-linear correlation functions in momentum space. In this paper, we use these symmetries to find exact solutions for some of the correlation functions. We also comment on the form of the correlation functions in general, and verify that the SL(2,Z) duality transformation between different critical points is satisfied exactly in all cases where the full solution is known.

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1. Introduction

Recently, much attention has been paid to 1+1 dimensional field theories with boundary [1,2]. Systems where the fields are free everywhere except for an interaction at the boundary can be used to describe dissipative quantum mechanics [3,4], tunneling of edge states in quantum Hall systems [5,6], the Kondo problem [7], and open string theory with one boundary [8,9,10]. In this paper we will consider the particular case when there are two fields, and at the boundary each field experiences a cosine potential and also a “magnetic” potential that induces an interaction between the two fields. In the dissipative quantum system, this describes a particle confined to two dimensions that is subject to a doubly periodic potential and a transverse magnetic field. Restricting to only one field (and setting the magnetic interaction to zero) this model is of further interest because it describes the tunneling in quantum Hall systems when the edge states are composed of only a single branch. As indicated in ref. [11], we expect this system to have rich behavior. It has an approximate SL(2,Z) duality symmetry as a function of magnetic flux per unit cell, \( \beta \), and dissipation per unit cell, \( \alpha \). At the critical points, it should not only be scale invariant, but also satisfy a set of Ward identities reflecting the symmetry under reparametrization of the associated open string boundary state [12]. The simplest of these just requires the dissipative quantum system to be SL(2,R) invariant at its critical points. In references [13], [14], and [15] it is shown that at special values of flux and dissipation, these reparametrization Ward identities are satisfied, but at many other proposed critical points we do not yet know that this is the case. These “special” values of flux and friction all occur at multicritical points on the proposed phase diagram in reference [11], and they have the property that the magnetic interaction between exponentials of the two fields vanishes.

The only critical point occurring when the magnetic field is zero is described by a \( c = 1 \) conformal field theory [14,15]. It can be described by a system of free fermions [17,11,16], but the simplest such representations do not properly obey the duality symmetry combined with the SL(2,R) invariance. In ref. [18], a more involved calculation, based on the idea of the fermionization, shows that at all the special points, the correlation functions in Fourier space must be homogeneous, piecewise-linear functions of the momentum.
In this paper we show how these various symmetries can be used to find exact solutions for the correlation functions at the special points. Because the duality transformation relates these points to the $c = 1$ conformal field theory with zero magnetic field, we expect that even at non-zero magnetic field the system should have a simple conformal field theory interpretation, and it is hoped that the results of this paper will be a guide in finding such theories.

Another issue we address in this paper is that, when the magnetic field is equal to zero, all the connected correlation functions other than the two-point function consist only of contact terms. The duality transformation takes these correlation functions to ones with non-zero magnetic field that are not contact terms. This suggests that the contact terms are physical and that the symmetries of the system will suffice to fix them. We have found that this is the case for many of the correlation functions, and it seems likely it is true for all of them. The final issue we address is to what extent the duality symmetries found in ref. [11] are exact. We find that whenever we can solve exactly for the correlation functions, the symmetry corresponding to $z \rightarrow z/(1 + i n z)$ is exact, where $z = \alpha + i \beta$. In addition, we identify the value of the strength of the cosine potential at which the system satisfies the conditions for self-duality under the transformation $z \rightarrow 1/z$.

In Section 2, we describe the boundary systems studied in this paper, and in Section 3 we review all the symmetries and properties of the system that were found in references [11], [18], and [13]. Because we are solving for contact terms, we find it more convenient to work in Fourier space, where the contact terms are all well-defined functions. Thus, in the following two sections we first show how the SL(2,R) symmetries and the Ward identities act on the system in Fourier space, and then derive properties that correlation functions in momentum space must have if they are to respect these symmetries. Finally, in Section 6 we find the exact solutions for several of the correlation functions and discuss the form of the general solution. However, in the general case we still cannot prove that the symmetries are enough to determine the solutions.

This paper largely explains the results given in reference [19] and gives the details of their derivations. While completing this paper, the author became aware of reference [13], which was partly motivated by this work.
2. Background

The dissipative Hofstadter model describes a quantum particle confined to two dimensions subject to a doubly periodic potential, a perpendicular magnetic field, and dissipation. When the Caldeira-Leggett \[3\] model is used to model the dissipation, the Euclidean action for this system is given by

\[
S = S_q + S_\eta + S_V, \tag{2.1}
\]

where \(S_q\) is the usual action of a particle in a constant magnetic field, \(S_\eta\) is a nonlocal kinetic term that gives the effect of the friction, and \(S_V\) is due to the periodic potential. \(S_q\) is given by

\[
S_q = \int_{-T/2}^{T/2} dt \left[ \frac{M \dot{x}^2}{2} + \frac{i e B}{2c} (\dot{x} y - \dot{y} x) \right]. \tag{2.2}
\]

In this equation, \(x(t)\) and \(y(t)\) are the coordinates of the particle, \(B\) is the strength of the magnetic field, and \(M\) is the mass of the particle. In the presence of the dissipation, \(M\) just acts as a regulator, so in the calculations of references \[13\] and \[18\] we set \(M\) to zero and chose a more convenient regulator. The part of the action due to the periodic potential is chosen to be

\[
S_V = \int_{-T/2}^{T/2} \left[ V_0 \cos\left( \frac{2\pi x(t)}{a} \right) + V_0 \cos\left( \frac{2\pi y(t)}{a} \right) \right], \tag{2.3}
\]

where \(V_0\) is the strength of the potential and \(a\) is the size of the unit cell. The dissipative term in the action, due to the particle’s interaction with its environment, has the form

\[
S_\eta = \frac{n}{4\pi} \int_{-T/2}^{T/2} \int_{-\infty}^{\infty} dt \ dt' \left( \frac{\bar{x}(t) - \bar{x}(t')}{} \right)^2. \tag{2.4}
\]

This term comes about by modeling the particle’s environment with a bath of harmonic oscillators which interact linearly with the particle. The functional integral over the oscillators induces the nonlocal term \(2.4\) in the action for \(\bar{x}(t)\).

Because we obtained this action by integrating over modes of free oscillators, it is not surprising that this action can also be obtained from a 1+1 dimensional system with a
boundary, where the fields are free in the bulk and interact only at the boundary \[4\]. The action for this 1+1 dimensional boundary theory is given by

\[ S_B = S_b + S_q + S_V, \]  \hspace{1cm} (2.5)

where

\[ S_b = \frac{\alpha}{8\pi} \int_{-T/2}^{T/2} \int_{-\infty}^0 d\sigma dt \left( (\partial_{\mu} x)^2 + (\partial_{\mu} y)^2 \right) \]  \hspace{1cm} (2.6)

is the bulk action;

\[ S_q = i\pi\beta \int_{-T/2}^{T/2} dt (\dot{x} y - \dot{y} x) \]  \hspace{1cm} (2.7)

is the boundary magnetic field term; and

\[ S_V = V_0 \int_{-T/2}^{T/2} \left[ \cos (x(t)) + \cos (y(t)) \right] \]  \hspace{1cm} (2.8)

is the boundary action from the cosine potential. In terms of the parameters of the dissipative system, \( \alpha \) is the friction/unit cell, given by

\[ 2\pi\alpha = \frac{\eta a^2}{\hbar}, \]  \hspace{1cm} (2.9)

\( \beta \) is the magnetic flux per unit cell, given by

\[ 2\pi\beta = \frac{eB}{\hbar c}a^2, \]  \hspace{1cm} (2.10)

and \( x(t) \) and \( y(t) \) have been rescaled by \( a/(2\pi) \). When \( \beta = 0 \) and we let \( \vec{x}(t) = x(t) \), then the action in equation (2.5) also describes the tunneling of edge states in the quantum Hall effect, where \( \alpha \) is related to the filling fraction.

In this paper, we are interested in calculating the correlation functions of \( \dot{x}(t) \) and \( \dot{y}(t) \). We will concentrate on the correlation functions with \( \dot{x} \) and \( \dot{y} \) located only on the boundary, because these are the correlation functions of interest in the dissipative quantum system, and because it is fairly straightforward to obtain the bulk correlation functions once we can calculate the boundary ones. Thus, we will be calculating functions of the form

\[ C^{\mu_1 \ldots \mu_m}(t_1, \ldots, t_m) = \langle \dot{x}^{\mu_1}(t_1) \ldots \dot{x}^{\mu_m}(t_m) \rangle \]

\[ = \int \mathcal{D}x(t) \prod_{i=1}^m \dot{x}^{\mu_i}(t_i) e^{-\frac{1}{\hbar} S}. \]  \hspace{1cm} (2.11)

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Because we expect many of the correlation functions to be contact terms, we will find it more convenient to work in Fourier space, where the contact terms are well-defined functions. Thus we will solve for the correlation functions of the form

$$\tilde{C}_{\mu_1\ldots\mu_m}(k_1, \ldots, k_m) = \frac{1}{T^m} \int_{-T/2}^{T/2} \prod_{j=1}^{m} dt_j \langle \dot{x}^{\mu_1}(t_1) \ldots \dot{x}^{\mu_m}(t_m) \rangle e^{-\frac{2\pi i}{T} k_1 t_1} \ldots e^{-\frac{2\pi i}{T} k_m t_m}. \quad (2.12)$$

Most of the calculations in references [18] and [13] were done for finite values of $T$, so that $k_i$ take on only integer values. However, for most of our final answers, we will take the limit as $T \to \infty$. Also, for convenience, we will appropriately rescale the $\dot{x}$'s and $V_0$ by $T$ so that the explicit dependence on $T$ drops out of all the equations.

In references [20,11,18,13], the cosine potential in equation (2.8) was treated perturbatively, so that the correlation functions in (2.11) become integrals over correlation functions with arbitrary numbers of the insertions of $e^{\pm i x(t)}$ and $e^{\pm i y(t)}$. These insertions of $e^{i q_x x(t)}$ and $e^{i q_y y(t)}$ behave like a Coulomb gas, with charges $q_x = \pm 1$ and $q_y = \pm 1$. The $x$-charges interact logarithmically and the $y$-charges interact logarithmically. The only interaction between the $x$ and $y$ charges is a phase, so that when the locations of the $x$ and $y$ charges are interchanged, the correlation function picks up the phase $\exp(\pm i 2\pi \frac{\beta}{\alpha^2 + \beta^2} q_x q_y)$. Whenever $\alpha/(\alpha^2 + \beta^2) = 1$ and $\beta/\alpha$ is an integer, the dimension of the charge operators, $e^{i q_x x(t)}$ and $e^{i q_y y(t)}$, is one and the phase is also equal to one. These special points are the multicritical points in the phase diagram of reference [11] that lie on the intersections of the critical circle with $\alpha/(\alpha^2 + \beta^2) = 1$ and all circles tangent to it. In this paper, we will concentrate on these special points. For the special point with $\beta = 0$, the model (with $\vec{x}(t) = x(t)$) is a $c = 1$ conformal field theory [14,16]. In the picture of the tunneling of edge states, this theory just corresponds to tunneling of free fermions. Because the duality transformation relates the other special critical points to the one at $\beta = 0$, and because the theory at these special critical points satisfies a set of reparametrization invariance Ward identities, we expect that these theories should also be fairly simple conformal field theories. However, unlike in the $\beta = 0$ case, the connected correlation functions should not consist only of contact terms [20], so the conformal field theories should not be as trivial as when $\beta = 0$. 

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The correlation function should depend on both the potential strength, $V_0$, and the value of $\alpha$ and $\beta$. Because each of the special points has a different value of the flux $\beta$, we will label the correlation functions by the value of $\beta$ at the point, and call the correlation function $\tilde{C}^{\mu_1...\mu_m}(k_1,\ldots,k_m;\beta)$. However, for convenience, we will omit the $V$ dependence in this notation.

In the following sections we will review and explain the results of references [18], [11], and [13] about the properties of the correlation functions. Most of the properties were derived using the Coulomb gas picture, and we refer the reader to those references for details of their derivations.

3. Symmetries of the System

In this paper we will use the various symmetries and properties of this system to find exact solutions for the correlation functions. We will begin by reviewing these symmetries which were derived in references [11], [18], and [13]. In the latter two references, all the calculations were done perturbatively to all orders in $\alpha'$ (where $\alpha' = 1/\alpha$), and the results are valid at every order in $V$.

3.1. Duality Symmetry

In ref. [11], we show that this system has an approximate duality symmetry under $z \to z + i$ and $z \to 1/z$, where $z = \alpha + i\beta$. Under these transformations, the coordinate correlation functions transform in a simple way. In particular, up to questions about renormalization, we expect the symmetry under $z \to z/(1 + inz)$ to be exact. In this case, the two-point function at the special multicritical points with $z = 1/(1 + n^2) + in/(1 + n^2)$ and potential strength $V_0$ can be obtained from the two-point function at zero magnetic field with $z = 1$ and potential strength $V_0$ by the following transformation [11]:

$$
\langle \dot{x}^\mu(k)\dot{x}^\nu(-k) \rangle(z, V_0) = |k| \left[ \left( \frac{\beta}{\alpha} \right)^2 \delta^{\mu\nu} + \frac{\beta}{\alpha} \epsilon^{\mu\nu} \mathrm{sign}(k) \right] + r^{\mu\rho}(k)r^{\nu\sigma}(-k)\langle \dot{x}^\rho(k)\dot{x}^\sigma(-k) \rangle(1, V_0),
$$

(3.1)

where

$$
r^{\mu\nu}(k) = \delta^{\mu\nu} - \frac{\beta}{\alpha} \mathrm{sign}(k) \epsilon^{\mu\nu}.
$$

(3.2)
Because the value of the potential remains the same, and because, for the multicritical points we are considering, $z$ is entirely determined by its imaginary part $\beta$, we will drop the argument $V_0$ and replace the argument $z$ by $\beta$. Also we will use the notation described in the previous section, $\tilde{C}^{\mu\nu}(k_1, k_2; \beta)$, for the two-point function. Then, because the two-point function has no off-diagonal terms when $\beta = 0$, we can write equation (3.1) as

$$\tilde{C}^{\mu\nu}(k, -k; \beta) = \left[ 2 \prod_{j=1}^2 r^{\mu_j x}(k_j) + 2 \prod_{j=1}^2 r^{\mu_j y}(k_j) \right] \tilde{C}(k, -k; 0) + |k| \left[ \left( \frac{\beta}{\alpha} \right)^2 \delta^{\mu\nu} + \frac{\beta}{\alpha} \epsilon^{\mu\nu} \text{sign}(k) \right].$$  

(3.3)

Similarly, for higher $m$-point functions, we expect that

$$\tilde{C}^{\mu_1 \cdots \mu_m}(k_1, \ldots, k_m; \beta) = \left[ \prod_{j=1}^m r^{\mu_j x}(k_j) + \prod_{j=1}^m r^{\mu_j y}(k_j) \right] \tilde{C}(k_1, \ldots, k_m; 0).$$  

(3.4)

(A similar transformation should hold for correlation functions at any $z$ and $z'$ that are related by $z' = z/(1 + iz)$. Once we are careful about how the theory is regulated, the $\tilde{C}(k_1, \ldots, k_m; 0)$ in these expressions must be replaced with

$$\tilde{C}(k_1, \ldots, k_m; 0) \rightarrow F(\vec{k}; \beta).$$  

(3.5)

Note that $F$ depends on the size of the magnetic field, $\beta$, but it is independent of the indices $\mu_1, \ldots, \mu_m$. For the original duality symmetry to be exact, $F(\vec{k}; \beta)$ must be independent of $\beta$. We will see that when the symmetries are enough to determine the correlation functions, then the solution for $F(\vec{k}; \beta)$ is unique up to a constant, regardless of the value of $\beta$.

The duality transformation under $z \rightarrow 1/z$ for an arbitrary $m$-point function can be found using the methods of reference [11]. Unlike the transformation for $z \rightarrow 1/(1 + iz)$, when $z$ goes to $1/z$ the value of $V_0$ also changes. The self-dual point under this transformation occurs at $z = 1$ and some particular value of $V_0$ that will be determined in Section 6.4. At the self-dual point, the correlation functions satisfy the following relation:

$$\left( 1 - \prod_{i=1}^m \text{sign}(k_i) \right) \tilde{C}(k_1, \ldots, k_m; 0) = 0.$$  

(3.6)

This means that at the self-dual point we expect $\tilde{C}(\vec{k})$ to vanish if an odd number of the $k_i$ are equal to zero. Although the derivation for the transformation under $z \rightarrow 1/z$ involved many approximations, we will see that at the “self-dual” value of $V_0$, equation (3.6) is satisfied exactly.
3.2. Piecewise Linearity

At the special multicritical points, the unregulated partition function can be written as one for a fermion gas with quadratic interactions, which leads us to expect the theory is solvable. Furthermore, when \( \beta = 0 \), the expressions for the correlation functions can also be written in terms of bilinears of the fermions, and we can obtain exact solutions for all the correlation functions of the \( \dot{x} \)'s and \( e^{ix(t)} \)'s.* The problem with these solutions is that the fermionization is valid only for large distance behavior; it does not necessarily give the correct short-distance behavior. This becomes a problem once we are trying to solve for correlation functions that contain contact terms. Also, when \( \beta \neq 0 \), we run into difficulties when considering correlation functions such as \( \langle \dot{x}(t)y(0) \rangle \), because these are just delta-functions of \( t \), and therefore only have short-distance behavior. To overcome these difficulties and find correlation functions that satisfy both the duality symmetries and the reparametrization invariance Ward identities, in reference [18] we started with the regulated version of the theory. Making use of the fact that the unregulated theory can be fermionized, we found that the coordinate correlation functions in the regulated theory are piecewise-linear, homogeneous functions of the momenta. In particular,

\[
F(\vec{k}; \beta) = \vec{a}_R(\vec{k}) \cdot \vec{k},
\]

(3.7)

where \( F(\vec{k}; \beta) \) is defined in the previous section by equations (3.4) and (3.5), and \( \vec{a}_R(\vec{k}) \) depends only on the signs of the sums

\[
\sum_{i \in S} k_i \quad \text{with} \quad S \subset \{1, \ldots, m\}.
\]

(3.8)

Also, from the calculations done in reference [18], we can show that, as a function over the reals, \( \tilde{C}(\vec{k}) \), and therefore \( F(\vec{k}; \beta) \), is continuous. It follows from equation (3.8) that the slope, \( \vec{a}_R(\vec{k}) \), changes only when one of the hyperplanes, given by

\[
\sum_{i \in S} k_i = 0 \quad \text{for} \quad S \subset \{1, \ldots, m\},
\]

(3.9)

* The subtleties of the fermionization of this system have also been considered in reference [16].
is crossed.

Because the functions are piecewise linear in momentum space, in real space we expect them only to include \( \delta \)-functions and factors of \( 1/(t_i - t_j) \). If a function is homogeneous in momentum space, then it has an additional reparametrization symmetry in real space. In particular, it tells us what happens under the transformation \( z \rightarrow z^2 \), where \( z = e^{2\pi i t/T} \).

We can derive this symmetry as follows. The linearity and homogeneity of the correlation functions in momentum space means that they satisfy the following relation:

\[
n\tilde{C}(k_1, \ldots, k_n) = \tilde{C}(nk_1, \ldots nk_n) \quad \text{for} \quad n \in \mathbb{Z}^+.
\] (3.10)

(For simplicity, in this section we are dropping the indices on \( C \)) Next, we can consider the following Fourier series

\[
\sum_{0 \leq j < n} C(z_1 e^{2\pi ij_1/n}, \ldots, z_m e^{2\pi ij_m/n}) = \sum_{0 \leq j < n} \tilde{C}(k_1, \ldots, k_m) \prod_{l=1}^{m} (z_l e^{2\pi i j_l/n})^{k_l}. \tag{3.11}
\]

Interchanging the order of sums and products, and using the relation

\[
\sum_{0 \leq j < n} (e^{2\pi ik/n})^j = \begin{cases} n & \text{for } k \in n\mathbb{Z} \\ 0 & \text{for } k \notin n\mathbb{Z} \end{cases}, \tag{3.12}
\]

the expression on the right-hand side of (3.11) becomes

\[
n^m \sum_{k_i \in n\mathbb{Z}} \tilde{C}(k_1, \ldots, k_m) z_1^{k_1} \ldots z_m^{k_m}. \tag{3.13}
\]

In this expression, we can replace the sum over \( k_i \) by a sum over \( l_i = k_i/n \) and apply equation (3.10) to obtain

\[
n^{m+1} \sum_{l_i \in \mathbb{Z}} \tilde{C}(l_1, \ldots, l_m) (z_1^n)^{l_1} \ldots (z_m^n)^{l_m}. \tag{3.14}
\]

Performing the sum, and setting the result equal to the left-hand side of equation (3.11), we find

\[
C(z_1^n, z_2^n, \ldots, z_m^n) = \frac{1}{n^{m+1}} \sum_{0 \leq j < n} C(z_1 e^{2\pi ij_1/n}, \ldots, z_m e^{2\pi ij_m/n}). \tag{3.15}
\]

Because each \( z_l e^{2\pi ij_l/n} \) is an \( n \)th root of \( z_l^n \), equation (3.13) gives the transformation of the real-space correlation functions when \( z \rightarrow z^n \).
3.3. Permutation and Inversion Symmetries and the Boundary Conditions

In the derivation of equations (3.4) and (3.5) for $\tilde{C}$ in terms of $F(\vec{k}; \beta)$, several other symmetries for $F$ also followed. The first is that $F(\vec{k}; \beta)$ is symmetric under interchanges of $k_i$ and $k_j$. The second is that $F(\vec{k}; \beta) = F(-\vec{k}; \beta)$. Also, whenever $\vec{k}$ has an odd number of components, we have the condition that $F(\vec{k}; \beta) = 0$.

There is an additional property of $\tilde{C}$ and $F$ that was found in reference [18]. Both $\tilde{C}(\vec{k})$ and $F(\vec{k})$ must equal zero when any one of the $k_i = 0$. As we shall see in Section 5.4, the SL(2,R) invariance requires $\tilde{C}(\vec{k}; \beta)$ to be continuous, for any value of $\beta$. However, according to the duality transformation, $\tilde{C}(\vec{k}; \beta)$ is obtained from $\tilde{C}(\vec{k}; 0)$ by multiplying it by factors of $\text{sign}(k_i)$. Therefore the boundary conditions at $k_i = 0$ turn out to be a necessary condition for the duality transformation to be satisfied along with SL(2,R) invariance.

3.4. The Reparametrization Invariance Ward Identities

The 1-D field theory describing dissipative quantum mechanics also describes a 2-D statistical theory with boundary. At the critical points, we expect the theory to be conformally invariant, and also to describe the boundary state in open string theory [4]. To be a conformal field theory, the theory must satisfy a set of reparametrization invariance Ward identities [12]. These reflect the fact that the boundary state must be invariant under reparametrizations of the boundary. Thus, the boundary state $|B\rangle$ must satisfy

$$\left(L_n - \tilde{L}_{-n}\right) |B\rangle = 0 \quad \text{for} \quad -\infty \leq n \leq \infty,$$

(3.16)

where the $L_n$ and $\tilde{L}_{-n}$ are the closed string Virasoro generators, given by

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \bar{\alpha}_{n-m} \cdot \bar{\alpha}_m,$$

(3.17)

and similarly for $\tilde{L}_{-n}$. For $-m < 0$, $\bar{\alpha}_{-m}$ and $\tilde{\alpha}_{-m}$ are the closed string creation operators. For $m \geq 0$, $\bar{\alpha}_m$ can be expressed as a derivative with respect to $\bar{\alpha}_{-m}$ as follows:

$$\alpha^\mu_m = m \frac{\partial}{\partial \bar{\alpha}_{-m}^\mu},$$

(3.18)
and similarly for $\tilde{\alpha}_m$. The boundary state is given by

$$|B\rangle = \exp\left(\sum_{m=1}^{\infty} \frac{1}{m} \tilde{\alpha}_m \cdot \tilde{\alpha}_m - W[\tilde{\alpha}, \tilde{\alpha}, \tilde{x}_0]\right) |0\rangle,$$

where $|0\rangle$ is the closed string vacuum state, $\tilde{x}_0$ is the zero mode of $\tilde{x}(t)$, and $W[\tilde{\alpha}, \tilde{\alpha}, \tilde{x}_0]$ is the connected generating functional for the 1-D system, given by

$$\exp\left(-W[\tilde{\alpha}, \tilde{\alpha}, \tilde{x}_0]\right) = \int [D\tilde{x}(s)]' \exp(-S_\eta - S_q - S_V - S_{LS}).$$

In this equation, the prime denotes that the integration over the zero mode of $x(t)$ is omitted. $S_\eta$, $S_q$ and $S_V$ are defined in Section 2, and $S_{LS}$ is the linear source term given by

$$S_{LS} = \sqrt{\frac{2}{\alpha'}} \int ds \tilde{\alpha}(s) \cdot \tilde{x}(s),$$

with

$$\alpha^\mu(s) = \sum_{m=1}^{\infty} i \left( \tilde{\alpha}_m^\mu e^{-ims} + \alpha_{-m}^\mu e^{ims} \right).$$

The $\alpha'$ in equation (3.21) is the string tension. Once we rescale $x(t)$ as we did in equations (2.6)-(2.8), it is given by $\alpha' = 1/\alpha$. By applying equation (3.16) to equation (3.19), Callan and Thorlacius showed [12] that the condition for reparametrization invariance translates to a condition on only the 1-D theory on the boundary, $W[\tilde{\alpha}, \tilde{\alpha}, \tilde{x}_0]$. In reference [13], we show that this identity is satisfied to all orders in $V$ for the cosine potential, as long as $\alpha/2 + \beta^2 = 1$ and $\beta/\alpha \in \mathbb{Z}$.

In the dissipative quantum system, we instead integrate over the zero mode, so that $W$ depends only on $\alpha_m$ and $\tilde{\alpha}_m$, and we have $\partial W/\partial x_0^\mu = 0$. We will find it convenient to define a source $J(t)$ that is coupled to $\dot{x}(t)$ by

$$J_m^\mu = \sqrt{\frac{2}{\alpha'}} \frac{1}{m} \tilde{\alpha}_m^\mu$$

$$J_{-m}^\mu = -\sqrt{\frac{2}{\alpha'}} \tilde{\alpha}_{-m}^\mu,$$

for $m > 0$. Then the generating function becomes

$$e^{W[J]} = \int [D\tilde{x}(t)] e^{-(S_\eta + S_q + S_V)} e^{\int J(t) \dot{x}(t) dt},$$

for $m > 0$. Then the generating function becomes

$$e^{W[J]} = \int [D\tilde{x}(t)] e^{-(S_\eta + S_q + S_V)} e^{\int J(t) \dot{x}(t) dt},$$

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$$e^{W[J]} = \int [D\tilde{x}(t)] e^{-(S_\eta + S_q + S_V)} e^{\int J(t) \dot{x}(t) dt},$$

for $m > 0$. Then the generating function becomes

$$e^{W[J]} = \int [D\tilde{x}(t)] e^{-(S_\eta + S_q + S_V)} e^{\int J(t) \dot{x}(t) dt},$$
and the connected correlation functions are given by

\[ \tilde{\mathcal{C}}^{\mu_1, \ldots, \mu_l}(m_1, \ldots, m_l) = \prod_{i=1}^l \left. \frac{\partial}{\partial J_{-m_i}} e^{W[J]} \right|_{J=0}. \]  (3.25)

The Ward identity then reduces to

\[
\frac{2}{\alpha'} \sum_{m=1}^{n-1} \left[ \frac{1}{2} \frac{\partial W}{\partial \tilde{J}_m} \cdot \frac{\partial W}{\partial \tilde{J}_{m+n}} - \frac{\partial^2 W}{\partial \tilde{J}_m \cdot \partial \tilde{J}_{m+n}} \right] + \sum_{m=-\infty}^{\infty} m \tilde{J}_{-m} \cdot \frac{\partial W}{\partial \tilde{J}_{m+n}} = 0. \]  (3.26)

and the results from reference [13] showing that the multicritical points satisfy these Ward identities still hold. The last term in equation (3.26) is precisely the change in \( W \) under reparametrizations, and the first two terms come from the fact that the \( \eta \) term in the original 1-D action is not reparametrization invariant.

4. Difference Equations from the Ward Identities

In this section we will use the reparametrization invariance Ward identities (3.26) to derive difference equations for the correlation functions in momentum space.

4.1. \( \text{SL}(2,\mathbb{R}) \) invariance

When \( n = 0, \pm 1 \), the reparametrizations in equation (3.16) are simply the \( \text{SL}(2,\mathbb{R}) \) transformations. (Strictly speaking, when \( T \) is finite and the \( k_i \) take on values in the integers, these reparametrizations are the \( \text{SU}(1,1) \) transformations. However, in the limit as \( T \to \infty \) they become the \( \text{SL}(2,\mathbb{R}) \) transformations instead.) For these three values of \( n \), the terms in the Ward identity due to the breaking of reparametrization invariance are identically zero. Thus the theory should be \( \text{SL}(2,\mathbb{R}) \) invariant, and we expect that under these transformations the \( \dot{x}(t) \)'s within the correlation functions should transform as dimension one operators. In this section, we will see what conditions the \( \text{SL}(2,\mathbb{R}) \) invariance of the theory imposes on the correlation functions in momentum space.

The \( n = 0 \) case of the Ward identity is

\[
\sum_{m=0}^{\infty} \sum_{m \neq 0} m \tilde{J}_m \tilde{J}_{-m} \frac{\partial W}{\partial \tilde{J}_m} + \sum_{m=0}^{\infty} \sum_{m \neq 0} m \tilde{J}_m \tilde{J}_{-m} \frac{\partial W}{\partial \tilde{J}_{-m}} = 0. \]  (4.1)
To obtain the correlation functions, we take \( N \) derivatives and then set the \( J \)'s to zero.

The derivatives are

\[
\prod_{i=1}^{N} \frac{\partial}{\partial J_{-k_i}^{\mu}}.
\] (4.2)

Acting on the expression in equation (4.1), they give

\[
\sum_{i=1}^{N} k_i \frac{\partial^N W}{\prod_{j=1}^{N} \partial J_{-k_j}^{\mu_j}} = 0,
\] (4.3)

which can be written in terms of \( \tilde{C} \) as

\[
\sum_{i=1}^{N} k_i \tilde{C}^{\mu_1 \cdots \mu_N} (k_1, \ldots, k_N) = 0.
\] (4.4)

This implies that

\[
\tilde{C}^{\mu_1 \cdots \mu_N} (k_1, \ldots, k_N) = 0 \quad \text{unless} \quad \sum_{i=1}^{N} k_N = 0.
\] (4.5)

This condition is the statement of conservation of momentum. It is just what we expect from the \( n = 0 \) reparametrization identity, which tells us that the system is translationally invariant.

For the \( n = 1 \) case, the Ward identity becomes

\[
\sum_{m=-\infty}^{\infty} m \left[ J_{m}^{x} \frac{\partial W}{\partial J_{1-m}^{x}} + J_{m}^{y} \frac{\partial W}{\partial J_{1-m}^{y}} \right] = 0.
\] (4.6)

Again we take the derivatives in equation (4.2) to obtain

\[
\sum_{i=1}^{N} k_i \frac{\partial W}{\prod_{j \neq i}^{N} \partial J_{-k_j}^{\mu_j} \partial J_{1-k_i}^{\mu_i}} = 0.
\] (4.7)

Setting the \( J \)'s to zero, we find that

\[
\sum_{i=1}^{N} k_i \tilde{C}^{\mu_1 \cdots \mu_N} (\vec{k} - \hat{e}_i) = 0,
\] (4.8)

where \( \vec{k} = (k_1, \ldots, k_N) \) and \( \hat{e}_i \) is the unit vector with a 1 in the \( i \)th component and zeroes everywhere else. According to the \( n = 0 \) equation, in this equation the \( k_i \)'s must satisfy \( \sum k_i = 1 \).
The \( n = -1 \) Ward identity similarly gives

\[
\sum_{i=1}^{N} k_i \tilde{C}^\mu_1 \cdots \mu_N (\vec{k} + \hat{e}_i) = 0.
\]  
(4.9)

In solving for the correlation functions, it will be much simpler to write everything in terms of the function \( F(\vec{k}; \beta) \). For convenience, we will drop the \( \beta \) in the argument of \( F \). According to the definition of \( F(\vec{k}) \), we have

\[
\tilde{C}^\mu_1 \cdots \mu_N (\vec{k}) = R^\mu_1 \cdots \mu_N (\vec{k}) F(\vec{k})
\]  
(4.10)

for \( N > 2 \) and

\[
\tilde{C}^\mu_1 \mu_2 (\vec{k}) = R^\mu_1 \mu_2 (\vec{k}) F(\vec{k}) + |k_1| \left[ \frac{\beta}{\alpha} \delta^\mu_1 \mu_2 + \frac{\beta}{\alpha} \epsilon^{\mu_1 \mu_2} \text{sign}(k_1) \right] \delta_{k_1, k_2}
\]  
(4.11)

for \( N = 2 \). In these equations, \( R \) is defined as

\[
R^\mu_1 \cdots \mu_N = \prod_{i=1}^{N} r^{\mu_i, x} + \prod_{i=1}^{N} r^{\mu_i, y},
\]  
(4.12)

where \( r^{\mu \nu} \) is defined in equation (3.2). Also, it is important to note that \( R \) is never equal to zero and to recall that \( F(\vec{k}) = 0 \) if any of the \( k_i = 0 \).

The \( n = 0 \) Ward identity in equation (4.3) then implies that

\[
F(\vec{k}) = 0 \quad \text{unless} \quad \sum k_i = 0.
\]  
(4.13)

The \( n = 1 \) Ward identity in equation (4.8) says that

\[
\sum_{i=1}^{N} k_i R^\mu_1 \cdots \mu_N (\vec{k} - \hat{e}_i) F(\vec{k} - \hat{e}_i) = 0.
\]  
(4.14)

(This equation is satisfied even when \( N = 2 \), because in that case the second term of the correlation function (4.11) satisfies the \( n = 1 \) Ward identity by itself.) According to the definition for \( R \), we have

\[
R^\mu_1 \cdots \mu_N (\vec{k} - \hat{e}_i) = R^\mu_1 \cdots \mu_N (\vec{k}),
\]  
(4.15)
as long as \( k_i \neq 1 \) and \( k_i \neq 0 \). When \( k_i = 1 \) or \( 0 \), we have \( k_i F(\vec{k} - \hat{e}_i) = 0 \), so even in this case we can replace \( R^{\mu_1 \ldots \mu_N}(\vec{k} - \hat{e}_i) \) with \( R^{\mu_1 \ldots \mu_N}(\vec{k}) \) in equation (4.14). With this substitution, equation (4.14) becomes

\[
R^{\mu_1 \ldots \mu_N}(\vec{k}) \sum_{i=1}^{N} k_i F(\vec{k} - \hat{e}_i) = 0. \tag{4.16}
\]

Because \( R \) is non-zero, we find that, for all \( \vec{k} \) with \( \sum_{i=1}^{N} k_i = 1 \),

\[
\sum_{i=1}^{N} k_i F(\vec{k} - \hat{e}_i) = 0. \tag{4.17}
\]

To summarize, in Fourier space the SL(2,R) symmetry implies that

i) \( F(\vec{k}) = 0 \) unless \( \sum_{i=1}^{N} k_i = 0 \).

ii) \( \sum_{i=1}^{N} k_i F(\vec{k} - \hat{e}_i) = 0 \),

and, similarly,

iii) \( \sum_{i=1}^{N} k_i F(\vec{k} + \hat{e}_i) = 0 \).

4.2. The Remaining Ward Identities

Now we will turn our attention to the Ward identities for \( |n| > 1 \). In general, after taking \( N \) derivatives, the last term in the Ward identity is

\[
\sum_{i=1}^{N} k_i \partial W \prod_{j=1}^{N} \frac{\partial J_{-k_j}^{\mu_j} \partial J_n^{\mu_i} - k_i \partial J_{y}^{\mu_i} j_{m}^{n-m}}{\partial J_{-k_j}^{\mu_j} \partial J_n^{\mu_i} - k_i \partial J_{y}^{\mu_i} j_{m}^{n-m}} = 0. \tag{4.18}
\]

When we set \( J \) equal to zero, this becomes

\[
\sum_{i=1}^{N} k_i \tilde{C}^{\mu_1 \ldots \mu_N}(\vec{k} - n\hat{e}_i) = 0. \tag{4.19}
\]

If we take \( N \) derivatives of the second term, we obtain

\[
-\frac{2}{\alpha'} \sum_{m=1}^{n-1} \left[ \frac{\partial^{N+2} W}{\prod_{i=1}^{N} \partial J_{-k_i}^{\mu_i} \partial J_n^{\mu_i} j_{m}^{n-m} + \prod_{i=1}^{N} \partial J_{-k_i}^{\mu_i} \partial J_n^{\mu_i} j_{m}^{n-m}} \right]. \tag{4.20}
\]

When \( J \) goes to zero, this reduces to

\[
-\frac{2}{\alpha'} \sum_{m=1}^{n-1} \left[ \tilde{C}^{\mu_1 \ldots \mu_N xx}(k_1, \ldots, k_N, -m, m - n) + \tilde{C}^{\mu_1 \ldots \mu_N yy}(k_1, \ldots, k_N, -m, m - n) \right]. \tag{4.21}
\]
The first term of the Ward identity is
\[
\frac{1}{\alpha'} \sum_{m=1}^{n-1} \left[ \frac{\partial W}{\partial J_{m}^{x}} \frac{\partial W}{\partial J_{n-m}^{x}} + \frac{\partial W}{\partial J_{m}^{y}} \frac{\partial W}{\partial J_{n-m}^{y}} \right].
\]  
(4.22)

After taking \(N\) derivatives and setting \(J\) to zero, we will obtain a product of two correlation functions, each with \(N\) or fewer vertices. If we let \(S = \{1, 2, \ldots, N\}\) and let \(s\) run through all the subsets of \(S\), then, after taking \(N\) derivatives, we can write the first term as
\[
\frac{1}{\alpha'} \sum_{m=1}^{n-1} \sum_{s \subseteq S} \left[ \frac{\partial^{\vert s \vert +1} W}{\prod_{i \in s} \partial J_{\mu_i - m_i}^{x}} \frac{\partial^{N-\vert s \vert +1} W}{\prod_{i \notin s} \partial J_{\mu_i - m_i}^{x}} + \text{same for } y \right],
\]  
(4.23)

where \(\vert s \vert\) is the number of elements in \(s\). When we set the \(J\)'s to zero, we obtain
\[
\frac{1}{\alpha'} \sum_{m=1}^{n-1} \sum_{s \subseteq S} \left[ \tilde{C}^{x}_{s} (\vec{k}_{s}, -m) \tilde{C}^{x}_{\bar{s}} (\vec{k}_{\bar{s}}, m - n) + \tilde{C}^{y}_{s} (\vec{k}_{s}, -m) \tilde{C}^{y}_{\bar{s}} (\vec{k}_{\bar{s}}, m - n) \right],
\]  
(4.24)

where \(\vec{k}_{s}\) contains all the \(k_{i}\) with \(i \in S\), and \(\vec{k}_{\bar{s}}\) contains all the remaining \(k_{i}\). In equation (4.24), \(\tilde{C}^{x}_{s}\) stands for \(\tilde{C}^{\mu_{j_1} \cdots \mu_{j_{\vert s \vert}}}_{s}^{x}\) with \(j_i \in s\), and \(\tilde{C}^{x}_{\bar{s}}\) has all the remaining indices. \(\tilde{C}^{y}_{s}\) and \(\tilde{C}^{y}_{\bar{s}}\) are defined similarly.

The full Ward identity is then
\[
\sum_{m=1}^{n-1} \left[ \tilde{C}^{\mu_1 \cdots \mu_N}_{s}^{xx} (\vec{k}, -m, m - n) + \tilde{C}^{\mu_1 \cdots \mu_N}_{s}^{yy} (\vec{k}, -m, m - n) \right] = \frac{1}{2} \sum_{m=1}^{n-1} \sum_{s \subseteq S} \left[ \tilde{C}^{x}_{s} (\vec{k}_{s}, -m) \tilde{C}^{x}_{\bar{s}} (\vec{k}_{\bar{s}}, m - n) + \tilde{C}^{y}_{s} (\vec{k}_{s}, -m) \tilde{C}^{y}_{\bar{s}} (\vec{k}_{\bar{s}}, m - n) \right]
\]  
\[ - \frac{\alpha'}{2} \sum_{i=1}^{N} k_i \tilde{C}^{\mu_1 \cdots \mu_N} (\vec{k} - n \hat{e}_i).
\]  
(4.25)

This equation gives an expression for the \((N + 2)\)-point functions in terms of \(N\)-point functions and products of two correlation functions, each with less than \(N + 2\) arguments. However, the left-hand side contains a sum of \((N + 2)\)-point functions. Because the sums with different values of \(n\) are not all linearly independent, we cannot use this equation to directly calculate all of the correlation functions of \(N + 2\) variables in terms of correlation functions with fewer variables. However, at the very least, it will give us the proper normalization of the higher correlation functions.
Again, we would like to write these equations in terms of the simpler function $F(\vec{k}; \beta)$, which is independent of all the indices. The expression on the left-hand side of the Ward identity can be written in terms of $F$ as

$$\sum_{m=1}^{n-1} \left[ R^{{\mu_1}...{\mu_N}xx}(\vec{k}, -m, m - n) + R^{{\mu_1}...{\mu_N}yy}(\vec{k}, -m, m - n) \right] F(\vec{k}, -m, m - n; \beta).$$  \hspace{1cm} (4.26)

Using the definition of $R$ and $r^{\mu\nu}$ and the relations $-m < 0$ and $m - n < 0$, we can simplify this expression to

$$\left(1 + \left(\frac{\beta}{\alpha}\right)^2\right) R^{{\mu_1}...{\mu_N}}(\vec{k}) \sum_{m=1}^{n-1} F(\vec{k}, -m, m - n; \beta).$$  \hspace{1cm} (4.27)

The first term on the right-hand side of equation (4.25) can be written in terms of $F$ as follows:

$$\frac{1}{2} \sum_{m=1}^{n-1} \sum_{s \subset S} \left\{ \prod_{i \in s} r^{\mu_i x}(k_i)r^{xx}(-m) + \prod_{i \in s} r^{\mu_i y}(k_i)r^{xy}(-m) \right\} \times \left[ \prod_{i \notin s} r^{\mu_i x}(k_i)r^{xx}(m - n) + \prod_{i \notin s} r^{\mu_i y}(k_i)r^{xy}(m - n) \right] \times F(\vec{k}_s, -m; \beta)F(\vec{k}_{\bar{s}}, m - n; \beta)$$

$$+ \text{same for } y \}.$$

To obtain this expression, we assumed that neither $F(\vec{k}_s, -m; \beta)$ nor $F(\vec{k}_{\bar{s}}, m - n; \beta)$ are two-point functions. They can be two-point functions only when $0 < k_i < n$ for some $k_i$, and we will return to this special case shortly. Again we can use the relations $-m < 0$ and $n - m < 0$ and the definition for $r^{\mu\nu}$ to simplify this expression. We obtain

$$\left(1 + \left(\frac{\beta}{\alpha}\right)^2\right) \frac{1}{2} R^{{\mu_1}...{\mu_N}}(\vec{k}) \sum_{m=1}^{n-1} \sum_{s \subset S} F(\vec{k}_s, -m; \beta)F(\vec{k}_{\bar{s}}, m - n; \beta).$$  \hspace{1cm} (4.29)

The last term of the Ward identity can be written in terms of $F$ as follows:

$$-\frac{\alpha'}{2} \sum_{i=1}^{N} k_i R^{{\mu_1}...{\mu_N}}(\vec{k} - n\hat{e}_i)F(\vec{k} - n\hat{e}_i; \beta).$$  \hspace{1cm} (4.30)
As long as $k_i \leq 0$ or $k_i \geq n$, we can replace $R^{\mu_1 \ldots \mu_N}(\vec{k} - n \hat{e}_i)$ with $R^{\mu_1 \ldots \mu_N}(\vec{k})$ to obtain

$$-\frac{1}{2} \left( 1 + \left( \frac{\beta}{\alpha} \right)^2 \right) R^{\mu_1 \ldots \mu_N}(\vec{k}) \sum_{i=1}^{N} k_i F(\vec{k} - n \hat{e}_i; \beta). \quad (4.31)$$

In this equation, we have made use of the identities $\alpha' = 1/\alpha$ and, on the critical circle, $\alpha/(\alpha^2 + \beta^2) = 1$.

Therefore, we find that as long as none of the $k_i$ have values $0 < k_i < n$, the Ward identity is

$$\left( 1 + \left( \frac{\beta}{\alpha} \right)^2 \right) R^{\mu_1 \ldots \mu_N}(\vec{k}) \sum_{m=1}^{n-1} F(\vec{k},-m,m-n;\beta) \left( 1 + \left( \frac{\beta}{\alpha} \right)^2 \right) \frac{1}{2} R^{\mu_1 \ldots \mu_N}(\vec{k}) \sum_{m=1}^{n-1} \sum_{s \in S} F(\vec{k}_s,-m;\beta)F(\vec{k}_s,m-n;\beta) - \sum_{i=1}^{N} F(\vec{k} - n \hat{e}_i; \beta).$$

Because $R$ is non-zero, the Ward identity reduces to

$$\sum_{m=1}^{n-1} F(\vec{k},-m,m-n;\beta) = \frac{1}{2} \sum_{m=1}^{n-1} \sum_{s \in S} F(\vec{k}_s,-m;\beta)F(\vec{k}_s,m-n;\beta) - \frac{1}{2} \sum_{i=1}^{N} k_i F(\vec{k} - n \hat{e}_i; \beta). \quad (4.33)$$

To finish, we must show that this equation remains true even when $0 < k_l < n$ for any $k_l$. In that case, the two-point functions $C^{\mu_1 \nu}(k_l,-m)$ and $C^{\mu_1 \nu}(k_l,m-n)$ are non-zero and have the form given in equation (4.11). Accordingly, they have a piece of the form we assumed in deriving equation (4.33), but also an additional piece, of the form

$$\left( \left( \frac{\beta}{\alpha} \right)^2 \delta^{\mu_1 \nu} + \frac{\beta}{\alpha} \text{sign}(k_l) e^{\mu_1 \nu} \right) |k_l|. \quad (4.34)$$

This additional piece will give a correction to the second term in the Ward identity whenever $0 < k_l < n$, which has the form

$$|k_l| \left( \left( \frac{\beta}{\alpha} \right)^2 \delta^{\mu_1 \nu} + \frac{\beta}{\alpha} \text{sign}(k_l) e^{\mu_1 \nu} \right) C_s(\vec{k}_s, k_l - n), \quad (4.35)$$

where, in this case, $s = \{1, \ldots, l-1, l+1, \ldots, N\}$. We can substitute the expression

$$\tilde{C}_s(\vec{k}_s, k_l - n) = \left[ r^{\mu \nu}(k_l - n) \prod_{i=1}^{N} r^{\mu_i \nu}(k_i) + r^{\nu \mu}(k_l - n) \prod_{i=1}^{N} r^{\mu_i \nu}(k_i) \right] F(\vec{k} - n \hat{e}_l) \quad (4.36)$$
for $C$ in equation (4.33). Then, if we simplify, using the definition of $r^{\mu\nu}$ and the relations $k_l - n < 0$ and $k_l > 0$, we find the following additional term in the Ward identity:

$$\frac{\beta}{\alpha} \left(1 + \left(\frac{\beta}{\alpha}\right)^2\right) k_l F(\vec{k} - n\hat{e}_l) \left[ \epsilon^{\mu_1 x} \prod_{i=1}^{N} r^{\mu_1 x}(k_i) + \epsilon^{\mu_1 y} \prod_{i=1}^{N} r^{\mu_1 y}(k_i) \right]. \tag{4.37}$$

We have also neglected what happens to the last term of the Ward identity when $0 < k_l < n$, for the $l$th term in the sum. We can write the contribution to the Ward identity from this term as follows:

$$-\frac{\alpha'}{2} k_l C^{\mu_1 \cdots \mu_N}(\vec{k} - n\hat{e}_l) = \mathcal{F}_1 + \mathcal{F}_2, \tag{4.38}$$

where

$$\mathcal{F}_1 = -\frac{\alpha'}{2} k_l R^{\mu_1 \cdots \mu_N}(\vec{k}) F(\vec{k} - n\hat{e}_l), \tag{4.39}$$

and

$$\mathcal{F}_2 = -\frac{\alpha'}{2} k_l \left[ R^{\mu_1 \cdots \mu_N}(\vec{k} - n\hat{e}_l) - R^{\mu_1 \cdots \mu_N}(\vec{k}) \right] F(\vec{k} - n\hat{e}_l). \tag{4.40}$$

$\mathcal{F}_1$ has the same form as the contribution when $k_l \leq 0$ or $k_l \geq n$, so it will lead to the identity given in equation (4.33). The expression $\mathcal{F}_2$ is the correction to the Ward identity. Simplifying this expression, we obtain

$$-\frac{\beta}{\alpha} \left(1 + \left(\frac{\beta}{\alpha}\right)^2\right) k_l F(\vec{k} - n\hat{e}_l) \left[ \epsilon^{\mu_1 x} \prod_{i=1}^{N} r^{\mu_1 x}(k_i) + \epsilon^{\mu_1 y} \prod_{i=1}^{N} r^{\mu_1 y}(k_i) \right]. \tag{4.41}$$

This cancels with the correction from the second term in the Ward identity (4.37). Therefore, even when $0 < k_l < n$, the Ward identity for $F$, given by equation (4.33), still holds.

When $N = 2$, there are additional corrections to the Ward identity, because now the last term also contains two-point functions, and the second term contains a product of two two-point functions. Once again, it is straightforward to show that these new corrections still cancel each other. Therefore, for any $N$ and $\vec{k}$, the Ward identity is given by equation (4.33).
Equation (4.33) is important because it gives a relation between \((N+2)\)-point functions and functions with fewer variables. Unfortunately, as mentioned earlier, the identities for different values of \(n\) are not independent, so they do not give enough information to solve for the correlation functions. Instead, all the information is contained in the \(SL(2, R)\) equations and the Ward identity with \(n = 2\), which has the form

\[
F(\vec{k}, -1, -1; \beta) = \frac{1}{2} \sum_{s \subset S} F(\vec{k}_s, -1; \beta) F(\vec{k}_{\bar{s}}, -1; \beta) - \frac{1}{2} \sum_{i=1}^{N} k_i F(\vec{k} - 2\hat{e}_i; \beta).
\] (4.42)

Another important feature of equations (4.33) and (4.42) is that their form does not depend on \(\beta\). This property will insure that if the symmetries do determine the correlation functions, then the exact form of the duality transformation is satisfied, up to one overall renormalization constant.

5. Consequences of \(SL(2, R)\) Invariance

When correlation functions are \(SL(2, R)\) covariant, it is well known that we can use the symmetry to fix three of the coordinates. This uniquely determines the 2 and 3-point functions up to normalizations, and the 4-point function up to an arbitrary function of the cross-ratio of the coordinates.

Once we work in momentum space, it is no longer so clear what the \(SL(2, R)\) symmetry implies. In this section we will show how the \(SL(2, R)\) symmetry, given by

\[
\sum_{i=1}^{N} k_i F(\vec{k} - \hat{e}_i) = 0 \quad \text{with} \quad \sum_{i=1}^{N} k_i = 1,
\] (5.1)

and

\[
\sum_{i=1}^{N} k_i F(\vec{k} + \hat{e}_i) = 0 \quad \text{with} \quad \sum_{i=1}^{N} k_i = -1,
\] (5.2)

restricts the possible forms of the correlation functions. In the following calculations, we will also assume that the symmetries and boundary conditions of Section 3.3 hold. The results for the general case are only a slight modification of the ones given in this section.
5.1. Exactly One Negative $k_i$

First we will show that $F(\vec{q}) = 0$ if one component of $\vec{q}$ is negative and all others are positive. Let $q_1$ be the component that is less than zero. Then we have $q_i \geq 0$ for $i \neq 1$. We will induct on $P$, the absolute value of the negative component of $\vec{q}$.

When $P = |q_1| = 1$, by translational invariance the only possibility for $\vec{q}$ is $\vec{q} = (-1, 1, 0, \ldots, 0)$, up to permutations of the $q_i$. In this case, $F(\vec{q}) = 0$ as long as $N > 2$. When $N = 2$, instead $F(\vec{q}) \neq 0$, which is what we expect because it gives us the two-point function. When $q_1 = -2$, the two possibilities for $\vec{q}$ are $\vec{q} = (-2, 2, 0, \ldots, 0)$ and $(-2, 1, 1, 0, \ldots, 0)$. For $N \geq 4$, each of these have a component equal to zero, so again $F(\vec{q}) = 0$.

Now suppose $F(\vec{q}) = 0$ for any vector $\vec{q}$ satisfying $|q_1| < P$, for some $P > 1$. We will consider $F(\vec{p})$ where the first component of $\vec{p}$ is $p_1 = -P$, and we will define $\vec{k}$ by $\vec{k} = \vec{p} + \hat{e}_1$. Then we can write equation (5.1) as

$$F(\vec{p}) = -\frac{1}{k_1} \sum_{i=2}^{N} k_i F(\vec{p}(i)),$$

(5.3)

where $\vec{p}(j) = \vec{k} - \hat{e}_j$. The first component of each $\vec{p}(i)$ is given by $p_1(i) = k_1$. Using the definition of $\vec{k}$, we find $p_1(i) = p_1 + 1$. Because $p_1$ is negative, this implies that the absolute value of the negative component of each $\vec{p}(i)$ on the right-hand side of equation (5.3) is given by $P - 1$. It follows that, by the induction hypothesis, $F(\vec{p}(i)) = 0$. Therefore, the SL(2,R) invariance implies that if one of the $k_i < 0$ and all the others are greater than zero, or vice versa, then $F(\vec{k}) = 0$.

5.2. General $\vec{k}$; $F(1, k_2, \ldots, k_{N-1}, -1)$

Next we will consider the remaining case, when at least two $q_i$ are positive and at least two $q_i$ are negative. We will show that in that case $F(\vec{q})$ for $\sum q_i = 0$ can always be written as a sum of $F(\vec{k}(j))$'s, where each $\vec{k}(j)$ has one component equal to one and another component equal to minus one.

Suppose $q_1$ is the smallest positive $q_i$. We will first induct on this smallest component and show that $F(\vec{q})$ can be written as a sum of $F(\vec{q}(j))$'s where each $\vec{q}(j)$ has its first component equal to one. When $q_i = 1$, this is automatically satisfied.
Now suppose there is some $P > 1$ such that, for any $q_1 < P$, we can write $F(\vec{q})$ as a sum of $F(\vec{q}(i))$’s where the first component of each $\vec{q}(i)$ is equal to one. Consider $F(\vec{p})$, with $p_1$ equal to $P$. We will define $\vec{k}$ by

$$\vec{k} = \vec{p} - \hat{e}_1. \quad (5.4)$$

Then we can write equation (5.2) as

$$F(\vec{p}) = -\frac{1}{k_1} \sum_{i=2}^{N} k_i F(\vec{p}(i)), \quad (5.5)$$

where

$$\vec{p}(j) = \vec{k} + \hat{e}_j. \quad (5.6)$$

On the right-hand side of equation (5.5), the first component of each $\vec{p}(i)$ is given by $p_1(i) = k_1$. According to our definition of $k_1$, we find that $p_1(i) = p_1 - 1 = P - 1$. By our induction hypothesis, we can then write each $F(\vec{p}(i))$ as a sum of $F(\vec{q}(i))$’s where each $q_1(i)$ is equal to one. Therefore, the same is true for $F(\vec{p})$. Note that in equation (5.3), for each $\vec{p}(i)$ we have $p_1(i) > p_i$. This implies that we cannot reduce any other component of $\vec{p}$ to one while keeping $p_1$ equal to one.

Instead, we will next show that any $F(\vec{q})$ where $\vec{q} = (1, q_2, \ldots, q_N)$ can be written as a sum of $F(\vec{q}(i))$ where each $\vec{q}(i)$ has one component equal to one and the other equal to minus one. We will now suppose that $q_N$ is the negative component of $\vec{q}$ with the smallest absolute value. If $q_N = -1$, we are done. For the general case, we will suppose that our hypothesis is true for any $\vec{q}$ with $|q_N| < P$, for some $P > 1$. Then we will consider $F(\vec{q})$ with $|q_N| = P$. This time we will define $\vec{k}$ and $\vec{p}(i)$ by $\vec{k} = \vec{q} + \hat{e}_N$ and $\vec{p}(i) = \vec{k} - \hat{e}_i$. According to equation (5.1), $F(\vec{q})$ satisfies

$$F(\vec{q}) = -\frac{1}{k_N} \sum_{i=1}^{N-1} k_i F(\vec{p}(i)). \quad (5.7)$$

Because $q_1 = 1$, the first component of each $\vec{p}(i)$ is $p_1(i) = 1$ for $i \neq 1$, and $p_1(1) = 0$. Thus, in the first case, $p_1(i)$ remains equal to one, as desired, and in the second case $F(\vec{p}(1)) = 0$, so it drops out. The last component of each $\vec{p}(i)$ is given by $p_N(i) = q_N + 1$. This implies
that \( |p_N(i)| = P - 1 \), which in turn implies that, by our induction hypothesis, each \( F(\vec{p}(i)) \)

\( N = p_1 = 1 \) and \( q_N = -1 \).

Therefore, SL(2,R) invariance tells us that if at least two \( k_i \) are positive and at least two are negative, we can always get one \( k_i \) equal to one and another \( k_i \) equal to minus one. In other words, all non-zero \( F(\vec{k}) \) can be written in terms of \( F(1, k_2, \ldots, k_{N-1}, -1) \)

for arbitrary \( (k_2, \ldots, k_{N-1}) \).

5.3. General \( \vec{k} \); \( F(a, a, k_3, \ldots, k_{N-2}, b, b) \)

The form we just found for \( F \) is not the unique way to fix \( F \) in momentum space. Often it is more convenient to express \( F(\vec{k}) \) in terms of \( \vec{k} \) which have the form \( \vec{k} = (a, a, k_3, \ldots, k_{N-2}, b, b) \), where \( k_1 = k_2 = a \) are the two largest positive components of \( \vec{k} \), and \( k_{N-1} = k_N = b \) are the two negative components of \( \vec{k} \) with the largest absolute value. We will show this is true using a proof similar to the previous two.

We will first show that \( F(\vec{q}) \) can always be written as a sum of \( F(\vec{q}(i)) \) where the two largest positive components of \( \vec{q}(i) \) are equal. To show this, we will induct on the difference between the two largest positive components of \( \vec{q} \). We will assume that the largest component is \( q_1 \) and the second largest is \( q_2 \). This can always be arranged by the permutation symmetry of \( F(\vec{k}) \). If \( q_1 = q_2 \), we are done. Next, we consider \( q_1 = q_2 + 1 \). We will define \( \vec{k} \) as in equation (5.4). In this case, \( \vec{k} \) can be written as \( \vec{k} = (k_1, k_1, k_3, \ldots, k_N) \).

Then equation (5.2) says that

\[
 k_1 F(\vec{q}) = -k_1 F(k_1, k_1 + 1, k_3, \ldots, k_N) - \sum_{i=3}^{N} k_i F(\vec{k} + \hat{e}_i). \tag{5.8}
\]

Because \( F \) is symmetric under interchange of the \( k_i \), this implies

\[
 k_1 F(\vec{q}) = -\frac{1}{2} \sum_{i=3}^{N} k_i F(\vec{p}(i)), \tag{5.9}
\]

where \( \vec{p}(i) = \vec{k} + \hat{e}_i \) and \( p_1(i) = p_2(i) = k_1 \). Therefore, our claim is true in this case.

Now suppose that there is a \( P > 1 \) such that, for any vector \( \vec{q} \) with \( q_1 - q_2 < P \), we can write \( F(\vec{q}) \) as a sum of \( F(\vec{q}(i)) \)'s where the two largest components of each \( F(\vec{q}(i)) \) are equal. We can consider \( F(\vec{p}) \) with \( p_1 - p_2 = P \) and define \( \vec{k} \) and \( \vec{p}(i) \) as in equations (5.4) and (5.6).
When we apply equation (5.5) to $F(\vec{p})$, on the right-hand side the first component of each $\vec{p}(i)$ will equal $p_1 - 1$, and all the other components of $p_i$ will remain constant or increase by one. Therefore, for each $i$, we have $p_i(1) - p_i(2) < P$. By the induction hypothesis, $F(\vec{p}(i))$ can then be written as a sum of $F(\vec{q}(i))$’s with the two largest components of each $\vec{q}(i)$ equal. Thus, the same is true of $F(\vec{q})$. (Similar calculations show it is impossible to decrease $q_1$ and $q_2$ any further while keeping them equal to each other.)

Because $F(\vec{q}) = F(-\vec{q})$, we can also write $F(\vec{q})$ as a sum of $F(\vec{q}(i))$’s where the two negative components with largest absolute value are equal. Now all that remains is to show that we can simultaneously set the two largest positive $q_i$ equal and the two largest negative $q_i$ equal. To show this, we will induct on $S$, given by

$$ S = \sum_{i=1}^{N} |q_i|. \quad (5.10) $$

Suppose $\vec{q}$ has $L$ positive components and $M$ negative components with $L + M = N$ and $L \geq M$. Then the smallest possible value of $S$ that will give a non-zero $F(\vec{q})$ is $S = 2L$. For this value of $S$, the vector $\vec{q}$ has the form $\vec{q} = (1, \ldots, 1, q_{L+1}, \ldots, q_N)$. We can always write this $F(\vec{q})$ as as sum of $F(\vec{q}(i))$’s where the largest two negative components of $\vec{q}(i)$, $q_{N-1}(i)$, and $q_N(i)$ are equal, and $q_j(i) = 1$ for $1 \leq j \leq L$. This is because, every time we apply equation (5.7) to reduce $|q_N|$ by one, the components of $\vec{q}$ that are one either remain one or become zero.

Next, we will consider $F(\vec{q})$ where $\vec{q} = (q_1, q_2, \ldots, q_{N-2}, q_N, q_N)$ with $|q_N| > |q_i|$ for $1 \leq i \leq N - 2$. Suppose all such $F(\vec{q})$ with $\sum_{i=1}^{N} |q_i| < S$ for some $S$ can be written in the desired form. Then take $\vec{q} = (q_1, q_2, \ldots, q_N, q_N)$ with $\sum_{i=1}^{N} |q_i| = S$. We can keep applying equation (5.2) or (5.1) to $F(\vec{q})$ until it is written as a sum of $F(\vec{p}(i))$’s with $p_1(i) = p_2(i)$. At this point, some of the $\vec{p}(i)$’s will no longer have $p_{N-1}(i) = p_N(i)$. However, for each such $\vec{p}(i)$, the sum of the absolute value of its components is smaller than the original value of $S$ for $\vec{q}$. Therefore, by our induction hypothesis, we are done.

We conclude that SL(2,R) invariance lets us set the two largest positive $k_i$ equal and the two largest negative $k_i$ equal.
5.4. Continuity of $F(\vec{k})$

The SL(2,R) symmetry combined with the homogeneity and piecewise linearity (of the form in equations (3.7), (3.8), and (3.9)) of $F(\vec{k})$ also requires $F(\vec{k})$ to be a continuous function when the $k_i$ take on real values. In this section we will show how the SL(2,R) invariance implies that when $\vec{k}$ crosses a simple boundary between a region where $F$ has one slope and a region where $F$ has another slope, $F(\vec{k})$ is continuous. We will not consider what happens when $\vec{k}$ lies in such a boundary and crosses intersections of these boundaries, since, although the basic idea is the same, it is much more complicated, and, also, the calculations of $F(\vec{k})$ in reference [18] can be used to directly show that it is continuous.

According to equations (3.7), (3.8), and (3.9) for $F(\vec{k})$, the slope of $F(\vec{k})$ jumps whenever $\sum_{i \in S} k_i = 0$, for any $S \subset \{1, \ldots, N\}$. We will consider the particular boundary $B$, given by $\sum_{i \in S} k_i = 0$ for some particular $S$ that contains 1 and does not contain $N$. The two regions, $R_+$ and $R_-$, on either side of the boundary have

$$\sum_{i \in S} k_i > 0 \quad \text{for all } \vec{k} \in R_+, \quad (5.11)$$

and

$$\sum_{i \in S} k_i < 0 \quad \text{for all } \vec{k} \in R_. \quad (5.12)$$

Let $\vec{v}$ be a vector that lies in the boundary. We can assume that all the partial sums other than $\sum_{i \in S} k_i$ are either greater than one or less than one. This is because if $\vec{v}$ is not in the intersection of boundaries, then only the one partial sum (and its complement) can equal zero. If any of the other partial sums equal 1 (for $k_i \in Z$), we can just multiply $\vec{v}$ by a constant and use the resulting vector instead. (This condition is necessary to guarantee that all the $\vec{p}(i)$ defined below lie only in $B$ or $R_+.$) We will define $\vec{k}$ by $\vec{k} = \vec{v} + \hat{e}_1$, and $\vec{p}(i)$ by $\vec{p}(i) = \vec{k} - \hat{e}_i$. Then for $i \in S$, the vector $\vec{p}(i)$ lies in the boundary because $\sum_{j \in S} p_j(i) = 0$. For $i \notin S$, the vector $\vec{p}(i)$ has $\sum_{j \in S} p_j(i) = 1$, so it lies in the region $R_+$.

The SL(2,R) equation for $F(\vec{v})$ is

$$(v_1 + 1)F(\vec{v}) + \sum_{i \in S, i \neq 1} v_i F(\vec{k} - \hat{e}_i) + \sum_{i \notin S} v_i F(\vec{k} - \hat{e}_i) = 0. \quad (5.13)$$
Now we can use the piecewise linearity of $F(\tilde{k})$. We have $F(\tilde{k}) = \tilde{a}_R \cdot \tilde{k}$, where $\tilde{a}_R$ depends only on the sign of the partial sums. Therefore, we can take $F(\tilde{k}) = \tilde{a} \cdot \tilde{k}$ in the region $R_+$, and $F(\tilde{k}) = \tilde{b} \cdot \tilde{k}$ in the boundary. Keeping track of the different regions, we can write the SL(2,R) equation as

$$ (v_1 + 1)\tilde{b} \cdot \tilde{v} + \sum_{i \in S} v_i \tilde{b} \cdot (\tilde{v} + \hat{e}_1 - \hat{e}_i) + \sum_{i \notin S} v_i \tilde{a} \cdot (\tilde{v} + \hat{e}_1 - \hat{e}_i) = 0. \quad (5.14) $$

If we expand this equation out and use the relation $\sum_{i \in S} v_i = \sum_{i \notin S} v_i = 0$, we find that

$$ \tilde{b} \cdot \tilde{v} - \sum_{i \in S} v_i b_i - \sum_{i \notin S} v_i a_i = 0. \quad (5.15) $$

We can combine the sum and dot product to obtain

$$ \sum_{i \notin S} v_i b_i = \sum_{i \notin S} v_i a_i. \quad (5.16) $$

Now we can repeat the same calculation, but with $\tilde{k} = (v_1, \ldots, v_{N-1})$, where $N \notin S$. This time we will define $\tilde{p}(i)$ by $\tilde{p}(i) = \tilde{k} + \hat{e}_i$. The components of $\tilde{p}(i)$ now satisfy $\sum_{j \in S} p_j(i) = 1$ for $i \in S$ and $\sum_{j \in S} p_j(i) = 0$ for $i \notin S$. This means that $\tilde{p}(i)$ is in $R_+$ for $i \in S$, and $\tilde{p}(i)$ is in the boundary for $i \notin S$. If we replace $S$ by its complement $\bar{S}$, instead we have $\tilde{p}(i) \in B$ for $i \in \bar{S}$ and $\tilde{p}(i) \in R_+$ for $i \notin \bar{S}$. Thus the roles of $S$ and $\bar{S}$ are reversed from the first calculation. In this new calculation, we must use the other SL(2,R) equation for $F(\tilde{v})$. However, because $F(\tilde{v}) = F(-\tilde{v})$, equation (5.2) is equivalent to equation (5.1) if we just replace $\tilde{v}$ with minus $\tilde{v}$. It follows that if we repeat the calculation with this new $\tilde{k}$, we will obtain the same result as in equation (5.10), except that $S$ is replaced with its complement and $\tilde{v}$ is replaced with $-\tilde{v}$. Thus, we have

$$ \sum_{i \in S} v_i b_i = \sum_{i \notin S} a_i v_i. \quad (5.17) $$

Adding equation (5.16) and (5.17), we obtain

$$ \tilde{b} \cdot \tilde{v} = \tilde{a} \cdot \tilde{v}. \quad (5.18) $$

For every vector $\tilde{v}$ in the boundary, this equation is satisfied by either $\tilde{v}$ or some multiple of $\tilde{v}$. Because in each region $F$ is a linear function, equation (5.18) says that if we let the
$v_i$ take on values in the reals, as $\vec{v}$ goes from $R_+$ to $B$ the limit of $F(\vec{v})$ defined on $R_+$ equals $F(\vec{v})$ defined on $B$. A similar calculation shows the same is true for $R_-$ and $B$. Therefore, $F$ is continuous when crossing the hyperplane $B$.

The proof that $F$ is continuous when $\vec{v}$ crosses from a boundary region (or intersection of boundary regions) to another boundary region (or intersection of boundary regions) uses the SL(2,R) invariance and linearity in the same way, but it is much more complicated to keep track of the regions.

6. Exact Solutions for Correlation Functions

In this section, we will use all the properties of $F(\vec{k})$ derived in the previous sections to solve for the correlation functions. These properties will give us the solution for the two-point function up to normalization, $\mu$, and they determine the four and six-point functions in terms of $\mu$. We can also solve for all $F(\vec{k})$ where two $k_i$ are positive and all others are negative, and vice versa.

6.1. SL(2,R) Functions Satisfying the Boundary Conditions

In this section we will begin by showing that the simplest expression that is continuous, piecewise-linear, SL(2,R) covariant, and that satisfies the boundary conditions given in Section 3.3, has the form

$$P(\vec{k}) = \sum_{s \subseteq S} (-1)^{|s|+1} \left| \sum_{i \in s} k_i \right|,$$

where $S = \{1, 2, \ldots, N\}$ and $|s|$ is the number of elements in $s$. First, we note that $F(\vec{v}) = \vec{a} \cdot \vec{v}$ and $F(\vec{v}) = |\sum_{i \in s} v_i|$ both satisfy the SL(2,R) equations. This can be verified by substitution into equation (5.1) or (5.13). In addition, when we set one $k_i$ to zero, we find $P(\vec{k}) = 0$, so the boundary conditions are also satisfied. Therefore $P(\vec{k})$ satisfies all the requirements on the correlation functions that we derived in the previous sections, except possibly the $n = 2$ Ward identity. The correlation functions given in references [18], [14], and [13] all have this form. However, in the more general case, this is not the only continuous, piecewise-linear function with the appropriate boundary conditions that satisfy the SL(2,R) equations.
To get an idea of what the other solutions are like, we will analyze $P(\vec{k})$ further. It has the form of a “symmetrization” of the $SL(2,R)$ invariant two-point function, $P(k, -k) = 2|k|$, over the $N$ components of $\vec{k}$. Also, if we multiply $P(\vec{k})$ by products of sign($k_i$), it is still continuous and satisfies the boundary conditions. The only correlation functions obtained from $P(\vec{k})$ that are not just contact terms contain exactly two $x$’s or exactly two $y$’s. Such correlation functions go as

$$\langle y(t_1)y(t_2)x(t_3)\ldots x(t_N) \rangle = c\frac{1}{(t_1 - t_2)^2} \prod_{j=3}^{N} \left( \frac{1}{t_j - t_1} - \frac{1}{t_j - t_2} \right),$$

where $c$ is a constant, independent of the $t_j$. This has the same form as the free correlation function $\langle \cos(x(t_1)) \cos(x(t_2)) \dot{x}(t_3)\ldots \dot{x}(t_N) \rangle$.

These remarks suggest what happens in the general case. First, if we take any continuous, piecewise-linear, $SL(2,R)$ covariant function of $M$ variables that vanishes when any one of the variables equals zero, and then “symmetrize” this function over $N$ variables, the resulting function will still have all of the desired properties. Similarly, any free correlation function of the form $\langle \prod_{j=1}^{M} \cos(x(t_j)) \prod_{j=M+1}^{N} \dot{x}(t_j) \rangle$ is always $SL(2,R)$ invariant and has a piecewise-linear Fourier transform. It is possible to add in some contact terms so that this Fourier transform also satisfies the boundary conditions. The questions that remain are whether the correlation functions that satisfy all the Ward identities are always made up of functions of the form described above, and to what extent the correlation functions are determined exactly by all the symmetry conditions derived in this paper. In the remaining part of this paper, we will address these issues by using the symmetries to derive various correlation functions. Since the work in this paper was done, the authors of [15] have shown in that paper that the correlation functions of $\dot{x}(t)$ and $\dot{y}(t)$ must indeed be given by free correlation functions of $e^{\pm ix(t)}$ and $\dot{x}(t)$.

6.2. Two-point Function

For the two-point function, the only homogeneous, piecewise-linear solution for $F$ that is even in $\vec{k}$ and is translation invariant is

$$F(\vec{k}) = \mu|k|. \quad (6.3)$$
In this equation, \( k = k_1 = -k_2 \) and \( \mu \) is an arbitrary constant that depends on \( V_0 \) and possibly on how the theory is renormalized. In dissipative quantum mechanics, \( \mu \) plays the role of the mobility of the particle \[21\]. The methods of calculation in this paper do not give us the value of \( \mu \) in terms of \( V_0 \). However, because the two-point function has the form given in (6.3) (with non-zero \( \mu \)), the particle is delocalized at the special multicritical points.

This solution for \( F \) in equation (6.3) also follows directly from SL(2,R) invariance combined with the symmetry under \( \vec{k} \rightarrow -\vec{k} \). If we substitute \( F(\vec{k}) \) for \( \tilde{C}(k, -k; 0) \) in equation (3.3) and simplify the resulting expression, then we find that the two-point function is given by

\[
\tilde{C}^{\mu\nu}(k, -k; \beta) = \delta^{\mu\nu} \left[ \left( 1 - \left( \frac{\beta}{\alpha} \right)^2 \right) \mu + \left( \frac{\beta}{\alpha} \right)^2 \right] |k| + \epsilon^{\mu\nu} \frac{\beta}{\alpha} (2\mu - 1) k.
\]

When \( V_0 = 0 \), \( \tilde{C}^{\mu\nu} \) must reduce to its value for the free theory, \( \tilde{C}^{\mu\nu} = \delta^{\mu\nu} |k| + \epsilon^{\mu\nu} (\beta/\alpha) k \). This implies that when \( V_0 = 0 \) (and also to order \( V_0^0 \) in perturbation theory) we have \( \mu = 1 \).

When we take the Fourier transform of this function for \( \tilde{C} \), we find that in real space, as \( T \rightarrow \infty \), the correlation function is given by

\[
C^{\mu\nu}(t_1 - t_2) = 2\mu_\alpha \frac{1}{(t_1 - t_2)^2} \delta^{\mu\nu} - \epsilon^{\mu\nu} \frac{\beta}{\alpha} \mu_\beta \delta'(t_1 - t_2),
\]

where \( \mu_\alpha = (1 - (\beta/\alpha)^2) \mu + (\beta/\alpha)^2 \) and \( \mu_\beta = 2(\beta/\alpha)(2\mu - 1) \). The first term is what we expect from the SL(2,R) symmetry, but the second term clearly also transforms properly under SL(2,R) transformations. Lastly, because the two-point function is fixed by the SL(2,R) invariance, equation (3.3) guarantees that the duality transformation will also be satisfied.

### 6.3. Four-point Function

Next, we calculate the four-point function. There are two different types of regions for \( \vec{k} \) we must consider. In each of these regions, it is straightforward to solve for the continuous, piecewise-linear functions, up to normalizations, given the boundary conditions...
\( F(\vec{k}) = 0 \) when \( k_i = 0 \). Then the \( n = 2 \) Ward identity fixes the normalization. The solution for \( F \) is given by

\[
F(k_1, k_2, k_3, k_4) = \delta_{k_1+k_2+k_3+k_4} \frac{1}{4}(\mu^2 - \mu) \sum_{s \subset \{1, 2, 3, 4\}} (-1)^{|s|+1} \left| \sum_{i \in s} k_i \right|,
\]

where \(|s|\) is the number of elements in \( s \). Thus \( F \) is proportional to \( P(\vec{k}) \), given by equation (6.1). This expression for \( F \) is equivalent to

\[
F(k_1, k_2, k_3, k_4) = \delta_{k_1+k_2+k_3+k_4} (\mu^2 - \mu) \min(|k_1|, |k_2|, |k_3|, |k_4|) \times \frac{1}{2} (1 + \text{sign}(k_1)\text{sign}(k_2)\text{sign}(k_3)\text{sign}(k_4)).
\]

Alternatively, the SL(2,R) invariance, the homogeneity of \( F \), and the \( n = 2 \) Ward identity are enough to uniquely determine \( F \) for the four-point function. This can be shown as follows. The first of the two types of regions for \( \vec{k} \) has \( k_1 k_2 k_3 k_4 < 0 \), which can only happen if either exactly one \( k_i \) is positive or exactly one \( k_i \) is negative. In both cases, the SL(2,R) symmetry requires \( F(\vec{k}) = 0 \). The second type of region has \( k_1 k_2 k_3 k_4 > 0 \), which means two \( k_i \) are positive and two are negative. For concreteness, suppose \( k_1, k_2 > 0 \) and \( k_3, k_4 < 0 \). We can use the SL(2,R) symmetry to set \( k_1 = k_2 = a \) and \( k_3 = k_4 = b \) for some \( a > 0 \) and \( b < 0 \). Momentum conservation then implies that \( a = -b \). Thus all \( F(\vec{k}) \) are determined once \( F(a, a, -a, -a) \) for all positive integers \( a \) is known. This function satisfies

\[
F(a, a, -a, -a) = aF(1, 1, -1, -1)
\]

because \( F \) is homogeneous in \( \vec{k} \). Therefore, the only unknown is \( F(1, 1, -1, -1) \) and permutations. We can solve for \( F(1, 1, -1, -1) \) in terms of the two-point function by using the \( n = 2 \) Ward identity. If we set \( \vec{k} = (1, 1) \) and \( N = 2 \), then equation (4.42) becomes

\[
F(1, 1, -1, -1) = F^2(1, -1) - \frac{1}{2} (F(-1, 1) + F(1, -1)).
\]

Substituting in \( F(1, -1) = \mu \), we obtain

\[
F(1, 1, -1, -1) = \mu^2 - \mu.
\]
Therefore, the four-point function is uniquely determined given the two-point function. Once we have this solution for \( F(1,1,-1,-1) \), we can “integrate up” the difference equations to obtain the solution (6.6) for \( F(\vec{k}) \).

The solution for \( \tilde{C} \) can be obtained by substituting the expression for \( F \) into equation (4.10). We note that \( F \) and \( \tilde{C} \) are identically zero whenever \( \mu = 0 \) or \( \mu = 1 \). When \( \mu = 0 \), the two-point function is zero, so all correlation functions vanish. The somewhat more interesting case when \( \mu = 1 \) just corresponds to the free theory, where we expect the two-point function to be non-zero, but all other correlations to vanish.

To better understand these solutions, we will transform some special cases back to real space. When \( \beta = 0 \), the correlation function is

\[
\langle \dot{x}(t_1)\dot{x}(t_2)\dot{x}(t_3)\dot{x}(t_4) \rangle = \frac{1}{2}(\mu^2 - \mu) \left[ \frac{1}{t_1 - t_2} \frac{1}{t_2 - t_3} \frac{2\pi \delta(t_3 - t_4)}{2\pi \delta(t_4 - t_1)} - \frac{1}{t_1 - t_2} \frac{2\pi \delta(t_2 - t_3)}{2\pi \delta(t_4 - t_1)} \right] + \text{permutations},
\]

(6.11)

where the only permutations that are included are the ones that treat points joined by delta-functions as indistinguishable. From this equation, we see that when \( \beta = 0 \) the correlation function contains only contact terms. For non-zero \( \beta \) some of the correlation functions do not contain only contact terms. To have finite long-time behavior, \( \tilde{C} \) must depend on three independent variables. This occurs only when two of the fields are \( x \)'s and two of them are \( y \)'s. Then, in Fourier space we have

\[
\tilde{C}^{xxyy}(\vec{k}) = G(k_1, k_2, k_3, k_4)
\]

\[
= \left( \frac{\beta}{\alpha} \right)^2 [\text{sign}(k_1)\text{sign}(k_3) + \text{sign}(k_2)\text{sign}(k_4)] F(\vec{k}).
\]

(6.12)

In real space, this goes as \( 1/((t_1 - t_2)(t_2 - t_3)(t_3 - t_4)(t_4 - t_1)) \). This is equivalent to the function given in equation (6.2), so it has the same form as the free correlation function of \( \dot{x}(t_1) \) and \( \dot{x}(t_3) \) with \( \cos(x(t_2)) \) and \( \cos(x(t_4)) \).

6.4. Six-point Function and the Self-Dual Point

For the six-point function, one can again solve for the continuous, piecewise-linear, homogeneous functions that have the correct boundary conditions, but it is already quite
tedious. Also, the SL(2,R) invariance, homogeneity and $n = 2$ Ward identity no longer appear to uniquely fix the six-point function. Instead, in this section we will first use the SL(2,R) invariance to restrict $\vec{k}$ to only two types of regions. Then we will use the linearity and boundary condition to solve for the function in these regions, and finally we will use the $n = 2$ Ward identity to set the normalization in each of these regions.

Because of the SL(2,R) invariance, the only choices for the signs of the $k_i$ that give non-zero $F(\vec{k})$ are either three $k_i$’s positive and three negative, or exactly two $k_i$’s negative. (By inversion symmetry, this last case also takes into account the case when exactly two $k_i$’s are positive.) The SL(2,R) invariance further implies that the only unknowns in these regions can be written as

$$\vec{k}_3 = (a, a, b, -c, -d, -d) \quad \text{with} \quad a \geq b \quad d \geq c,$$  \hspace{1cm} (6.13)

and

$$\vec{k}_2 = (a, a, b, c, -d, -d) \quad \text{with} \quad a \geq b \geq c,$$  \hspace{1cm} (6.14)

where $a, b, c, d > 0$. Because $F(\vec{k})$ is invariant under inversion, we can assume $d \geq a$ in equation (6.13). These orderings of the variables $a, b, c, \text{and} d$, combined with momentum conservation $\sum_{i=1}^{6} k_i = 0$, uniquely determine the signs of the partial sums for any $\vec{k}_2$ or any $\vec{k}_3$. Thus the $\vec{k}_3$’s given in equation (6.13) all lie in only one region, and similarly for the $\vec{k}_2$’s. When any of the equalities are satisfied in equation (6.13) or (6.14), then $\vec{k}_3$ or $\vec{k}_2$, respectively, lies in the boundary of the region. Both the region for $\vec{k}_3$ and the region for $\vec{k}_2$ also have the boundary where $c = 0$ (and $a, b, \text{and} d \neq 0$). This boundary region contains, for example, the two linearly independent vectors $(3, 3, 2, 0, -4, -4)$ and $(4, 4, 2, 0, -5, -5)$.

Because of the piecewise linearity of $F$, we know that for $\vec{k}_3$ the function $F$ has the form

$$F(\vec{k}_3) = \vec{A} \cdot \vec{k}_3,$$  \hspace{1cm} (6.15)

for some constant $\vec{A}$. For $\vec{k}_3$ given by equation (6.13), the momentum conservation implies $2a + b - c - 2d = 0$. Using this condition to eliminate $d$ from equation (6.13), we find that $F(\vec{k}_3)$ must have the form

$$F(\vec{k}_3) = A_a a + A_b b + A_c c = 0,$$  \hspace{1cm} (6.16)
for some constants $A_a$, $A_b$, and $A_c$. The boundary conditions require that $F(\vec{k}_3) = 0$ whenever $c = 0$. This implies that

$$F(\vec{k}_b) = A_a a + A_b b = 0,$$  \hspace{1cm} (6.17)

for any $\vec{k}_b$ in the boundary. Since we have at least two linearly independent vectors in the boundary, equation (6.17) implies that $A_a = A_b = 0$. Therefore, $F(\vec{k}_3)$ has the form

$$F(\vec{k}_3) = A_c c.$$  \hspace{1cm} (6.18)

The $n = 2$ Ward identity can then be used to fix $A_c$, so that $F(\vec{k}_3)$ is uniquely determined. A similar calculation shows that $F(\vec{k}_2)$ is also uniquely determined.

It follows that in each of these regions, $F$ must be proportional to $P(\vec{k})$. Using the Ward identity to fix the constants of proportionality, we find

$$F(\vec{k}) = -\frac{1}{8} (\mu^2 - \mu) P(\vec{k}) \quad \text{for} \quad \prod_{i=1}^{6} k_i > 0,$$  \hspace{1cm} (6.19)

and

$$F(\vec{k}) = \frac{1}{8} \mu (2\mu - 1)(\mu - 1) P(\vec{k}) \quad \text{for} \quad \prod_{i=1}^{6} k_i < 0.$$  \hspace{1cm} (6.20)

We first note that $F(\vec{k})$ is no longer simply proportional to $P(\vec{k})$, because the coefficient now depends on which “quadrant” $\vec{k}$ is in. However, it does fall under the general form described in Section 6.1. When $\beta = 0$ (or if all the variables are only $x$’s or only $y$’s) the correlation function in Fourier space, $\tilde{C}$, depends only on three independent variables, which implies that in real space it consists of contact terms. The only solution without contact terms is $\langle \dot{x}(t_1) \dot{x}(t_2) \dot{y}(t_3) \ldots \dot{y}(t_6) \rangle$ (and the same with $x$ and $y$ interchanged), and it has the form given in equation (6.2). Thus all the symmetries derived in this paper uniquely determine the six-point function and require it to have the form described in Section 6.1.

Lastly, we note that when $\mu = 1/2$, $F(\vec{k}) = 0$ for $\prod k_i < 0$. Thus, according to equation (3.6) we expect that the self-dual point should occur at $\mu = 1/2$. In fact, using the Ward identity, it is straightforward to show that $\mu = 1/2$ does behave like the self-dual
point; at this value of $\mu$ all correlation functions with $\prod k_i < 0$ vanish, which means the $z \to 1/z$ duality transformation is satisfied. This is the “self-dual” value of the mobility found by Schmid in reference [21], and it is rather remarkable that these exact results agree with his calculations, which involved many approximations. One property of the self-dual point that was noted in ref. [11] is that at this point the off-diagonal part of the two-point function vanishes, as can be seen from equation (6.4). Because the linear response to a transverse electric field is related to this correlation function, these results imply that the “Hall” current should vanish at the self-dual points.

6.5. Exactly 2 $k_i$ are positive

This time we can restrict the $k_i$ to be $(1, k_2, \ldots, k_{n-1}, -1)$, with $k_2 > 0$ and all others negative. Again, these vectors lie in only one region and its boundaries. In particular, they all lie in the boundary of the region where $k_1$ is less than one and can be taken to zero without crossing any more boundaries. Therefore, as in the case for the six-point function, $F(\vec{k}) = ak_1$. Because $k_1 = 1$ in the region we are considering, $F(\vec{k})$ is a constant in this region, and the Ward identity once again will uniquely determine it. Thus, we find that

$$F(\vec{k}) = (-\frac{1}{2})^{N/2} (\mu^2 - \mu) P(\vec{k}),$$

whenever exactly two $k_i$ are positive. This again has the form given in Section 6.1.

6.6. Eight-point function and beyond

For the eight-point function, we find that when exactly four $P(\vec{k})$ are positive, the function $P(\vec{k})$ no longer satisfies the Ward identity. Instead, we must also consider the functions made from appropriately symmetrizing the four-point function given in equation (6.12). This “symmetrized” function has the form

$$\sum_{s_1, s_2, s_3, s_4} (-1)^{|s_1| + |s_3|} G(\sum_{i \in s_1} k_i, \sum_{i \in s_2} k_i, \sum_{i \in s_3} k_i, \sum_{i \in s_4} k_i),$$

where $s_1, s_2, s_3,$ and $s_4$ are summed over all disjoint subsets of $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ whose union equals $S$. Unfortunately, once we reach eight variables, the piecewise-linear functions become quite complicated, and we cannot easily show that this is the unique
solution. However, we still conjecture that all correlation functions must have the form described in Section 6.1, and that the properties and symmetries derived in this paper are enough to determine all the correlation functions. Since this work was originally done, the first of these conjectures has been shown to be true in reference \[15\], but the second still remains an open question.

7. Conclusions

In this paper, we have used the symmetries of the boundary system with cosine and magnetic interactions to find exact solutions for the correlation functions at the special multicritical points. We have shown that the piecewise-linearity, homogeneity, reparametrization invariance Ward identities, and SL(2,Z) duality symmetry uniquely determine the 2-point, 4-point, and 6-point functions up to normalization, and also all the correlation functions with exactly one or two positive momenta. For the eight-point functions, it again appears likely that these symmetries determine the correlation functions, and the solution for the eight-point function suggests the form for all the remaining correlation functions. Once we get beyond eight variables, the piecewise-linear functions become rather complicated, so it is difficult to determine a basis for them. If the piecewise-linear functions with the required boundary condition all have the form given in Section 6.1, it seems likely that the symmetries will continue to uniquely fix the correlation functions. In any case, the symmetries combined with the long-time behavior do appear to fix the contact terms.

The results in this paper also verify that the approximate duality transformation under \( z \rightarrow z/(1 + inz) \) is an exact transformation. The other interesting check on the SL(2,Z) duality symmetry found in reference \[11\] is what happens under the transformation \( z \rightarrow 1/z \) when \( \beta = 1 \). According to references \[21\] and \[11\], the value of \( V_0 \) changes under this transformation, and many approximations were made in deriving this symmetry. Remarkably, we found that the theory at \( \beta = 1 \) is self-dual precisely at the value of the mobility predicted in these two references. This value of the mobility occurs at a particular value of the potential strength \( V_0 \). We have also found that the correlation functions at all the other special critical points with this value of \( V_0 \) also exactly satisfy the self-dual
condition. Thus, we expect this value of $V_0$ to play an important role; one interesting property of the special critical points of the dissipative Hofstadter model at this potential strength is that the analogue of the Hall current vanishes.

The consequences of the $\text{SL}(2,\mathbb{R})$ invariance and the boundary reparametrization invariance Ward identities in Fourier space are quite general, and may be useful for other boundary theories. In particular, we note that the equations we used to solve for the correlation functions have solutions for all values of $\beta/\alpha$, and not just the values at the special multicritical points. According to reference [11], we expect the dissipative Hofstadter model to have many other critical and multicritical points at other values of magnetic flux and friction. It is interesting to speculate on whether these other critical theories are described by the additional conformal theories we have found, or whether these critical theories have a more complicated structure instead.

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