THE BLOB ALGEBRA AND THE PERIODIC TEMPERLEY-LIEB ALGEBRA
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Abstract
We determine the structure of two variations on the Temperley-Lieb algebra, both used for
dealing with special kinds of boundary conditions in statistical mechanics models. The first
is a new algebra, the ‘blob’ algebra (the reason for the name will become obvious shortly!).
We determine both the generic and all the exceptional structures for this two parameter
algebra. The second is the periodic Temperley-Lieb algebra. The generic structure and part
of the exceptional structure of this algebra have already been studied. Here we complete the
analysis, using results from the study of the blob algebra.

1 Introduction
There has been much recent interest in two dimensional Potts models and related models in the case
of toroidal boundary conditions [1, 2]. In this paper we determine the structure of two variations
on the ordinary Temperley-Lieb algebras, which appear in the transfer matrices of these models.

In the next section we introduce a new two parameter algebra, the ‘blob’ algebra $b_n(q, q')$ (it
turns out to be a particular quotient of the affine Hecke algebra). Using diagrams, some ideas from
category theory, and experience gained from analysing the ordinary T-L algebras, we determine
both the generic and all the exceptional structures (depending on $q, q'$ and also on their relationship)
for this algebra. In the subsequent section we analyse the periodic Temperley-Lieb algebra. The
generic structure and part of the exceptional structure of this algebra has already been studied
[3, 4, 5, 6]. Here we complete the analysis, relying heavily on results from the study of the blob
algebra.

A striking result already emphasized in [3] is the analogy of certain special representations
of the periodic T-L algebra with the sum of representations of left and right Virasoro algebra
$\sum Vir_{ra} Vir_{rb}$ (where the labels refer to the highest weights $h_{ra}, h_{rb}$ of the Virasoro represen-
tations) [7]. We will develop the technology to examine how this analogy goes further, that is,
representations that are not in the special part of the Bratteli diagram already studied can also be
put in correspondence with representations of the Virasoro algebra, but now “outside” the minimal
Kac table.

This work has some overlap with the papers of Levy [4, 5]. However we believe that the
natural graphical representations we use provide a clearer and more complete point of view on
the representation theory. In conclusion we also mention some interesting possibilities for further
generalizations.

2 The blob algebra
2.1 Definition and general results

Recall the well known diagrammatic realisation of the Temperley-Lieb algebra $T_n(q) [8, 9]$ in
which the generators are drawn as $n$ non-overlapping strings on a rectangular frame

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and composition is by identification of the bottom of one diagram with the top of the other (the exterior rectangles are construction lines only, and can be ignored in composition). The relations are

\[ 1 = \quad \text{Diagram 1} \]

\[ U_i = \quad \text{Diagram 2} \]

(where \( x = q + q^{-1} \) and \( q \) refers to the usual \( U_qsl(2) \) quantum algebra) and equivalence under end point preserving isotopy, e.g.

\[ \quad = \quad \text{Diagram 3} \]

It is convenient to refer to lines which travel from top to bottom as propagating lines, and those that double back to the same edge as loop lines (so, for example, the pictures above each have ten propagating lines).

The following generalisation has several applications, which we will discuss later.
For $q'$ an invertible complex parameter we define the BLOB algebra $b_n = b_n(q, q')$ as the generalisation obtained by including an additional idempotent 'blob' generator $e = \bullet$ and additional relations given by $\bullet = \bullet\bullet$ (idempotency) and $\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bubble
Corollary 1.1. As an algebra bimodule the algebra is filtered through a sequence of invariant subspaces

\[ \emptyset \subset b_1^0 \subset b_2^0 \subset \ldots \subset b_n^0 = b_n \quad (n \text{ odd}) \]

or

\[ \emptyset \subset b_0^1 \subset b_1^1 \subset b_2^1 \subset \ldots \subset b_n^1 = b_n \quad (n \text{ even}). \]

The subquotients \( b_n^h / b_n^{h-2} \) in turn are each a direct sum of a part with a propagating \( e \) and a part with a propagating \( f \) (apart from \( b_0^0 \) which has no invariant subspaces). A basis for \( b_n^h \), the propagating \( e \) part, is \( \{ D \in B_n^* : |D| = h \text{ and } |D'| > 0 \} \).

2.2 The structure of \( b_n(q, q') \)

As in the TL case there is a diadic notation for diagrams. Cutting each propagating line at its midpoint (for definiteness \( \bullet \) is written \( \bullet\bullet \) and cut between the blobs) the diagrams may be separated uniquely into a top part and a bottom part, for example:

\[
\begin{array}{c}
\text{Diagram 1} = \[
\begin{array}{c}
\text{Top Part} \\
\text{Bottom Part}
\end{array}
\]
\end{array}
\]

Since no braiding is allowed, the recombination of such diagrams is unique within each subquotient \( b_n^h / b_n^{h-2} \). We denote the (mutually top-bottom inverted but otherwise isomorphic) sets of upper and lower half diagrams \( R_n^h \), and note that the set \( R_n^h \) \((h > 0)\) may be split into a propagating \( e \) part (called \( e \)-type diagrams) \( R_e^h \) and a propagating \( f \) part (\( f \)-type) \( R_f^h \). We have

\[ B_n^h \rightarrow (R_e^h \times R_e^h) \oplus (R_f^h \times R_f^h) \rightarrow B_n^h. \]

We denote the upper and lower half diagrams extracted from a diagram \( D \) by \( |D| > \) and \( < D| \):

\[ D \mapsto |D| > < D| \mapsto [d_1 > < d_2] = D \]  

It is useful to make a distinction between notations \(|D| > \) and \(|d_1 >| \). The latter, as indicated in our picture above, denotes a specific half diagram \( d_1 \in R_n^h \) realized as an upper half diagram. We will continue to use the notion of propagating lines for cut lines, and hence of index \( h \) for half diagrams with \( h \) cut lines.

For a given half diagram \( d \in R_n^h \) we define \( R_d \) as the set of all diagrams of lower index (actually index \( \leq h - 2 \)) which can be obtained from \( d \) by connecting some or all of the \( h \) propagating lines in pairs to form loop lines. In other words

\[ R_d = \{ < D| : |D| < h \text{ and } D \in b_n[d > < d] \}. \]

Similarly define the subset

\[ R'_d = \{ < D| : |D| = h - 2 \text{ and } D \in b_n[d > < d] \}. \]

Consider the algebra product

\[ D_1 D_2 = X(D_1, D_2) D_3, \]
where $X$ is a scalar function of $y_e, y_f$. Since the loop lines at the top of $D_1$ and bottom of $D_2$ are not affected by the product, and $X(D_1, D_2)$ just depends on the number and nature of closed loops produced, we have the diadic version

$$[a >< b] [c >< d] = < b | c > [a' >< d']$$  \hspace{1cm} (4)

where $a', d'$ are either $a, d$ or are in $R_a, R_d$ respectively. An (proper) invariant subspace of the left $b_n$ module $b_n[d >< d]$ (say) is thus

$$M_d = \bigoplus_{d' \in R_d} b_n[d' >< d'] \in b_{n,h}^{h-2}$$

(we could make the sum over all $R_d$, but then it is not a direct sum - the summands as written here may overlap at index $< (h - 2)$).

The bimodule subquotient $b_n/h_{n,h}^{h-2}$ of $b_n$ decomposes via the diadic structure as a direct sum of left modules, denoted

$$b_n(b_n/h_{n}^{h-2}) = \bigoplus_{t \in \{e,f\}} \bigoplus_{d \in R_t^{n,h}} ((b_n[d >< d])/M_d)$$  \hspace{1cm} (5)

where all the summands inside the second sum are isomorphic.

The half diagrams of given index $h$ and type $t$ ($t = e, f$) thus provide a basis for representations of $b_n$, with action and inner product both defined by the equation

$$[a >< b] [c >< d] = < b | c > [a >< d] \quad \text{(mod. $b_{n,h}^{h-2}$)}.$$  \hspace{1cm} (6)

In other words the ‘ket’ (‘bra’) vectors give a basis for left (right) modules (which in our pictures means that the diagrams act from the top (bottom)). Abusing symmetric group notation \cite{11} we call the corresponding left modules Specht modules - i.e. for $a \in R_t^{n,h}$

$$S_t^{n,h} \sim b_n[a >< a] / M_a$$

(independently of which $a \in R_t^{n,h}$ is chosen).

**Definition 3** For index $h$ and type $t$ we define a Gram matrix $g_t^{n,h}$ for the inner product by $(g_t^{n,h})_{ab} = < a | b >$.

For example, with $n = 3, h = 1$ and a suitable order of the bases (see table 3 below)

$$g_f^{3,1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & y_f \\ 0 & y_e & 0 \end{pmatrix} \quad g_e^{3,1} = \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & y_e \\ 0 & 0 & x \end{pmatrix}.$$

Noting that $g_t^{n,h}$ is symmetric, and non-singular for $q, q'$ indeterminate, and that $[a >< a]$ is (up to normalisation) a primitive idempotent in the appropriate subquotient, it follows that these representations are inequivalent irreducible for $q, q'$ indeterminate.

There is a natural inclusion of $b_n \subset b_{n+1}$ (add a propagating line on the right). Hence

**Proposition 2 (Induction/Restriction Diagram)** The dimensions, induction and restriction rules and bases for these Specht modules are given by the following generalised Pascal triangle for
the iterative construction of ‘ket’ (or ‘bra’) basis states. Starting with $n = 1$ on the top layer:

\[
\begin{array}{c}
\uparrow \\
\uparrow \quad \downarrow \\
\downarrow \\
\uparrow \\
\end{array}
\]

and so on.

Note that each new basis (with index $h$, say) is obtained by taking the elements of the immediately above left and right bases (which, unless $h = 0$, are of indices $h \pm 1$), adding a new line on the right of each element (giving diagrams with indices $h$ and $h + 2$), and then in the index $h + 2$ cases connecting the new line to a previous line to form a loop (and so reduce the number of propagating lines from $h + 2$ to $h$). Note that such a connection is unique - as no propagating line may be trapped by a cup, the rightmost propagating line must be used.

Proof: From the diagrams, or as follows:

Let $[d_n ] < d_n ]$ denote the inclusion of $[d ] < d ]$ in $b_{n+1}$ (i.e. $d_n$ has one extra propagating line on the right, making $k + 1$ altogether). Then each of the summands in equation (7) induces a left $b_{n+1}$-module

\[
b_{n+1}[d_n ] < d_n ] \bigoplus_{d' \in R_{d_n}} b_{n+1}[d'_n ] < d'_n ]
\]

In $b_{n+1}$ the quotient subspace sum is not quite over all of $R_{d_n}$, since this includes the case in which the rightmost line becomes looped back into the next such line (let’s call it $d''$, with $((h + 1) - 2)$ propagating lines). Consequently, as a vector space the induced module from the index $k$ Specht module is a direct sum of an index $k + 1$ and an index $k - 1$ Specht module -

\[
\left((b_{n+1}[d_n ] < d_n ] + (b_{n+1}[d'' ] < d'' ]) \bigoplus_{d' \in R_{d_n}} b_{n+1}[d'_n ] < d'_n ]\right)
\]
This we now discuss in more details. Irreducibles may be determined by a sequence of subtractions of dimensions of invariant subspaces.

But this kernel is just simple two dimensional module based on such that $F G_{n-1}$ is an isomorphism of unital algebras.

**Proof:** Compare $U_{n-1} B_n U_{n-1}$ and $U_{n-1} B_n^{-2}$. This has a standard corollary \[12, 13, 14\]

**Corollary 3.1** There exist functors on the categories of left modules

$$ (b_{n-2} - \text{mod}) \xrightarrow{G} (b_n - \text{mod}) \xrightarrow{F} (b_{n-2} - \text{mod}) $$

such that $FG$ is the identity map, and $GF(b_n) = b_n U_{n-1} b_n$.

The kernel of $GF$ determines the extent to which it fails to be an isomorphism of categories. But this kernel is just $b_n/b_n U_{n-1} b_n \sim e + f$ by proposition \[6\] (i.e. $b_n U_{n-1} b_n = b_{n-2}$), so exactly two simple modules are missed in treating $b_{n-2} - \text{mod}$ as $b_n - \text{mod}$. Now $b_1 = C e + C f$ and $b_2$ has three simple modules by explicit computation. It follows that the Pascal diagram above gives bases for a complete list of generic irreducible representations, since there two new nodes appear in going from level $(n - 2)$ to level $n$.

It also follows that at each level $n$ the only morphisms between modules with a trivial image at level $(n - 2)$ are those involving $b_n/b_{n-2} \sim e \oplus f$. Therefore we can build up the details of the exceptional structure by looking at these morphisms at each level, and adding them to the (known) morphisms from level $(n - 2)$. The new morphisms can be determined from the zeros of the Gram matrices $q_{n,k}$, together with Frobenius reciprocity.

In what follows we use the symmetry between the roles of types $e$ and $f$. For generic type $t$ we will then use $t'$ for the other type. It follows from the construction of bases in table \[7\] that (where $||$ stands for determinant)

$$ |g_t^{2,0}| = y f y_e, $$

$$ |g_t^{3,1}| = y f (y_e x - 1), \text{ and for } n \geq 4 $$

$$ |g_t^{n,n-2}| = x |g^{n-1,n-3}_e| - |g^{n-2,n-4}_e| $$

and similarly for the type $f$ cases. This is a recursion familiar from the Temperley-Lieb case \[13\].

If we regard $|g_t^{n,n-2}|$ as a polynomial in $y_e$, then there are various points at which it has (typically order 1) zeros. These correspond to the occurrence of a non-trivial algebra homomorphism at level $n$

$$ 0 \to S_e^{n,n} \to S_e^{n,n-2}. $$

For example equation \[8\] has a zero at $y_e = 0$ corresponding to

$$ \uparrow \quad \Rightarrow \quad \bullet $$

(and similarly with the roles of $e$ and $f$ interchanged). Frobenius reciprocity then determines a cascade of morphisms at higher level, $m (> n)$ say, which exhaust the morphisms involving $S_e^{m,m}$, the one dimensional module based on $e$ (similarly $f$). All other morphisms follow by corollary \[3.1\].

These morphisms determine the structure of the algebra. In particular the dimensions of irreducibles may be determined by a sequence of subtractions of dimensions of invariant subspaces. This we now discuss in more details.
\[ 2.3 \text{ The exceptional cases} \]

Introducing the standard sequence of polynomials

\[ P_t^1 = 1, \quad P_t^2 = \frac{y_t}{x}, \quad P_t^n = P_t^{n-1} - x^{-2} P_t^{n-2} \]  

one finds therefore

\[ |g_t^{n-2} | = \frac{y_t}{x} x^n P_t^n \]  

It is useful to parametrize

\[ y_c = \frac{q - q^{-1} e^{2i\eta}}{1 - e^{2i\eta}}, \quad y_f = \frac{q^{-1} - q e^{2i\eta}}{1 - e^{2i\eta}} \]  

in which case the recursion relation is solved by

\[ P_{n+1}^c = x^{-n} \frac{q^{-n} e^{2i\eta} - q^n}{e^{2i\eta} - 1} \]  

\[ P_{n+1}^f = x^{-n} \frac{q^n e^{2i\eta} - q^{-n}}{e^{2i\eta} - 1} \]  

The zeroes of these polynomials occur at values

\[ \eta = \pm n \gamma + m \pi \quad m \text{ integer} \]  

respectively, where we set \( q = e^{i\gamma} \).

For given \( \eta, \gamma \) there are various possibilities to consider:

Firstly if there are no solutions to equation \( 4 \) then the algebra is semi-simple (generic) for all \( n \). This is because by corollary 3.1 the first occurrence (as we increase \( n \)) of an invariant subspace in a generic indecomposable module \( S_t^{n,h} \) (for some \( h \)) must give a homomorphism from the \( t \)-trivial module (that is, if \( t = e \) say, the rightmost module in level \( n \) of the Bratteli diagram)

\[ 0 \to S_t^{n,n} \to S_t^{n,h}. \]  

But since \( S_t^{n,h} \) restricts to \( S_t^{n,h+1} + S_t^{n,h+1} \) Frobenius reciprocity \( 10 \) implies a morphism from one of these to \( S_t^{n-1,n-1} \) - a contradiction unless \( h = n - 2 \).

Secondly if equation \( 4 \) is satisfied for some sign and a pair of integers \( m, n = m_c, n_c \), but there are no other solutions (i.e. \( q \) not a root of unity), then by the same argument as above this signals the first occurrence of an invariant subspace in a generically irreducible module. All subsequent homomorphisms are determined by Frobenius reciprocity and corollary 3.1. In particular all modules \( S_t^{n,h} \) with \( h > n_c \) remain irreducible (suppose there is a first one which does not, then it must have an invariant subspace isomorphic to \( S_t^{n,n} \) by co.3.1, but then by Frobenius reciprocity there must either be an earlier one with an invariant subspace or else \( h = n - 2 \) - either way we have a contradiction). Further, for each positive \( l \) such that \( n_c + l \leq n \) we have

\[ 0 \to S_t^{n,n_c+l} \to S_t^{n,n_c-l} \]  

where the cases \( n_c + l < n \) are given by corollary 3.1, and for \( l = n - n_c \) the morphism follows by Frobenius reciprocity, and there can be no other morphisms. Note that when \( n_c - l < 0 \) we have \( S_t^{n,n_c-l} = S_t^{n_c-l,n_c} \) (a notational convenience from the generic Bratteli diagram).

In this case the dimensions of the new irreducibles may be computed by subtracting \( \dim(S_t^{n,n_c+l}) \) from the generic dimension, or noting that for these irreducibles, call them \( I_t^{n,h} \), the induction and restriction rules are the same as before, except that \( I_t^{n,n_c-1} \) restricts (where defined) to \( I_t^{n-1,n_c-2} \) (and not \( I_t^{n-1,n_c-2} + I_t^{n-1,n_c} \) as usual); and \( I_t^{n,n_c} \) restricts (where defined) to \( I_t^{n-1,n_c-1} + I_t^{n-1,n_c+1} + I_t^{n-1,n_c+1} \) of the Bratteli diagram).
Finally, if there is a solution to equation 14 and \( q \) is a root of unity then \( \gamma/\pi = m_1/m_2 \) for some coprime integers \( m_1, m_2 \) and there is another solution for each integer \( m \) such that \( (m-m_c)\pi/\gamma \in \mathbb{Z} \) (i.e. \( m = m_c \mod m_1 \)), at \( n = |n_c + km_2| \) (integer \( l \)). Without loss of generality let us assume that the lowest \( n \) solution, at \( n = n_c \), occurs when the positive sign occurs in equation 14, i.e. a solution to \( P_{c}^{n+1} = 0 \). The next (or equal) lowest \( n \) solution will be to \( P_{f}^{n+1} = 0 \), at \( n = m_2 - n_c \), and so on.

As before, for \( n \leq n_c \) the algebra is generic. At \( n = n_c + 1 \) the only morphism is as in equation 16 (with \( l = 1 \)). For \( n \leq n_c + m_2 \) the structure is as in the single solution case above, except that at \( n = m_2 + 1 \) there are two invariant subspace (one coming from the \( e \) type, and one from the \( f \) type) maps from right to left, and all other \( e \) type morphisms map from left to right (corollary 3.1). Thus by Frobenius reciprocity then forces a new series of morphisms, together with the new solution there is also one from \( S_{l}^{n, m_2, n_c + m_2} \) to \( S_{l}^{n, m_2, 0} \) and \( S_{l}^{n, m_2, n_c + m_2 + 2} \). Frobenius reciprocity then forces a new series of morphisms, together with corollary 3.1 altogether giving

\[
0 \rightarrow S_{l}^{n, m_2 - n_c + l} \rightarrow S_{l}^{n, m_2 - n_c - l}
\]

where \( S_{l}^{n, m_2 - n_c + l} \) is simple, i.e.

\[
I_{l}^{n, m_2 - n_c + l} = S_{l}^{n, m_2 - n_c + l}/I_{\ast}
\]

\( I_{\ast} \) the maximal proper invariant subspace of \( S_{l}^{n, m_2 - n_c + l} \). Indeed, since \( I = S \) far enough out in the Bratteli diagram we might as well write all our morphisms as in equation 20.

In general there are many morphisms in and out of each generically irreducible module, but all \( e \) type morphisms map from right to left, and all \( f \) type from left to right (corollary 3.1). Thus by working in from the edges in a suitable order all the dimensions of simple modules can be computed by subtractions of known dimensions of invariant subspaces for the generic case. The induction and restriction rules for the new simple modules may be worked out directly from this.

Alternatively the dimensions of simple modules may be thought of in terms of subsets of walks from the top of the diagram to the position of the module in question. The simple dimension for a module between two critical lines on the same side is the number of walks which do not touch the innermost line on the other side, and which do not touch the outer line of the two after the last
time they they touch the inner one. For a module on a critical line all walks which never touch the first critical line on the other side are allowed. For a module between the two innermost lines (one on each side) only walks which never touch either line are allowed. Thus for this innermost sector in particular we may summarize:

If there is an integer $n$ (the smallest) for which $P_{e}^{n+1}$ vanishes we may truncate the Bratteli diagram on the right (e part) to keep only connectivities $h \leq n - 1$. Similarly if $P_{f}^{n'+1}$ vanishes we truncate it on the left to keep only connectivities $h \leq |1 - n'|$. If there is a pair of integers $n, n'$ such that $P_{e}^{n}$ and $P_{f}^{n'}$ vanish, this implies that $q$ is a $(n + n')^{th}$ root of unity. In this case the Bratteli diagram may be truncated on both sides.

Our results in this sector are in correspondence with the ones of Levy [3] who studied an algebra $Y(\tau, a, b, c, N)$ generated by $1, e_{1}, \ldots, e_{N-1}; x_{1}$ with usual TL relations $e_{i}^{2} = e_{i}, e_{i}e_{i\pm 1}e_{i} = \tau e_{i}, [e_{i}, e_{j}] = 0, |i - j| \geq 2$ and $x_{i}^{2} = bx_{i} + c, e_{1}x_{1}e_{1} = ac_{1}$. Besides a simple rescaling and renaming of the various generators, the correspondence with us is $\tau = x^{2}$ and

$$
\begin{align*}
\varepsilon &= \frac{\mu x_{1} + 1}{\mu b + 2}, \\
y &= x, \\
\mu &= \frac{b \pm \sqrt{b^{2} + 4c}}{2c}
\end{align*}
$$

Our results agree with those of [4] in this sector, but we believe the representation theory is much more transparent our way, this belief being reinforced by the fact that we obtain the whole structure, not just the innermost part.

### 2.4 Discussion

Note that $\varepsilon$ can be braid translated (conjugated by the usual braid generator $g_{1} = 1 - qU_{1}$ of the Temperley-Lieb algebra) so that the blob appears in other places besides the first strand. Leaving it on the first strand is just a prescription to ensure linear independence of diagrams. The blob can be thought of as a trick for introducing a cohomological ‘seam’ into the system - the first step in generalizing to periodic boundary conditions. As such only one blob is required, but the seam can occur anywhere in the chain.

### 3 Application to the periodic Temperley Lieb algebra

We now wish to apply the above results to the study of the periodic Temperley Lieb algebra $T_{A_{n-1}}$. Rename first the generators of $T_{2n-1}(q)$ as $U_{1}, U_{12}, \ldots, U_{n}$. Then $T_{A_{n-1}}$ (denoted simply by $T$ in the following) is the unital algebra over the complex numbers generated by these generators and an additional one - $U_{n1}$ - that satisfies the relations

$$
U_{n1}^{2} = xU_{n1}
$$

$$
U_{n}U_{n1}U_{n} = U_{n}, \\
U_{1}U_{n1}U_{1} = U_{1}, \\
U_{n1}U_{n}U_{n1} = U_{n1}, \\
U_{n1}U_{1}U_{n1} = U_{n1}
$$

$$
[U_{n1}, U_{i}] = 0 \text{ if } i \neq 1 \text{ or } n, [U_{n1}, U_{i}, i+1] = 0
$$

(21)

It is an infinite dimensional algebra.

Setting $I_{0} = \prod_{i=1}^{n}(U_{i}/\sqrt{q})$ consider first the left ideal $TI_{0}$. Recall from [5] that all irreducible representations may be found by considering the quotients

$$
(U_{1}U_{2} \ldots U_{n})U_{12}U_{23} \ldots U_{n1}U_{1}U_{2} \ldots U_{n} = \alpha (U_{1}U_{2} \ldots U_{n})
$$

(22)

*Recall that the $A_{n}$ diagram has $n + 1$ vertices.*
for some parameter $\alpha$. \(T_0\) modulo (22) is indecomposable. It has a natural basis of words in the generators, and we call the representation induced from this basis \(T_0(z)\) where we have set

\[
\alpha = (z^{1/2} + z^{-1/2})^2
\]

There is an algebra homomorphism from the periodic Temperley-Lieb algebra into \(T_{2n-1}(q)\) given by

\[
P : T_{\hat{A}_{n-1}}(q) \to T_{2n-1}(q)
\]

\[
P : U_i \mapsto U_i, \quad (i = 1, 2, ..., n - 1)
\]

(similarly \(U_{i+1}\)) and, recalling the "braid translator" introduced in [6], by

\[
P : U_{n1} \mapsto \left( \prod_{i=1}^{n-1} g_i g_{ii+1} \right)^{-1} U_1 \left( \prod_{i=1}^{n-1} g_i g_{ii+1} \right)
\]

(24)

where \(g^{\pm 1} = 1 - q^{\pm 1}U\). It is easy to check that this realization of \(U_{n1}\) satisfies the relations (21). Moreover (22) holds with \(\alpha = x^2\). Hence we were able in [8] to induce representations of \(T\) with that particular value of \(\alpha\) from representations of \(T_{2n-1}\).

We can now generalize the braid translator by considering blob and squares decorations. This should allow an algebra homomorphism from the periodic algebra into the blob algebra. For given parameters \(x\) and \(y_f\) consider now trying to build a homomorphism of the form

\[
U_{n1} \mapsto (af + 1) \left( \prod_{i=1}^{n-1} g_i g_{ii+1} \right)^{-1} U_1 \left( \prod_{i=1}^{n-1} g_i g_{ii+1} \right) (bf + 1)
\]

(25)

It is easy to check that for any \(y_f\) this generator satisfies the relations (22) provided the following conditions hold

\[
a + b + ab = 0
\]

\[
aq^{-1} + bq - y_f(a + b) = 0
\]

(26)

where the first equation follows from idempotency of \(U_{n1}/x\) and the second from the relations involving three \(U\)'s. Beside the trivial solution \(a = b = 0\) used in [8] another possibility is

\[
a = \frac{q - q^{-1}}{q^{-1} - y_f}
\]

\[
b = \frac{q^{-1} - q}{q - y_f}
\]

(27)

In that case (22) holds with

\[
\alpha = x^2 + \frac{x^2 - 4}{y_f^2 - xy_f + 1} yf(x - yf)
\]

(28)

Setting

\[
y_f = \frac{q - q^{-1} e^{2i\eta}}{1 - e^{2i\eta}}
\]

(29)

one finds

\[
z = \exp(2i\eta)
\]

(30)

(all choices of phases for \(z, q\) give isomorphic results up to \(e, f\) interchange). Therefore we can establish an isomorphism between \(T_n(z)\) as introduced in [8] (sec. 4.3) and representations of
the blob algebra $S_{2n,0}^f(y_f)$. From the results of the previous paragraph we deduce immediately that $\mathcal{T}_n(\rho)$ is irreducible for $z$ generic, with dimension the number of paths of $2n$ steps from the origin to the point of horizontal coordinate zero on the Pascal triangle (here for convenience we use horizontal coordinates that are equal to the number of connectivities on the $e$ side, and minus it on the $f$ side), i.e $C_{2n}^n$. It is reducible when $\exp i\eta = q^{n_c}$ for $n_c$ an integer, that is $z = q^{2k}$. By symmetry we can restrict to the case $n_c$ positive. The representation contains then an irreducible component with dimension the number of paths with same characteristics but on a Pascal triangle that is truncated on the left to include only points with horizontal coordinates greater or equal to $1 - n_c$, i.e $C_{2n}^n - C_{2n}^{n-n_c}$. In [11] since we restricted to the case without blob, we could use the braid translator only in the case $n_c = 1$. This recovers the results of prop.16 in [11].

The process generalizes to the case $TI_h/IT_{h-1}T$ where $I_h = \prod_{i=1}^{n-h} (U_i/\sqrt{Q})$ (that is the sector with $2h$ propagating lines. Notice that connectivities as they have been defined so far are half of the connectivities defined in [6] where they referred to “clusters” rather than “boundaries”). The relevant quotient relations are obtained by taking the word $I_h$, rotating the top once around the cylinder clockwise holding the bottom fixed, and equating this new word with $\alpha_h I_h$. The same result with $\alpha_h^{-1}$ holds then for counterclockwise rotation. In the case $h=1$ an additional quotient has to be taken

$$ (U_1, U_2, \ldots U_{(n-1)_1}) U_{12} U_{23} \ldots U_{n1} (U_1, U_2, \ldots U_{(n-1)_1}) = 0 \quad (31) $$

Quotienting $TI_h/IT_{h-1}T$ by these relations one obtains the representation $\mathcal{T}_n(\alpha_h)$. On the other hand from braid translating the blob algebra one gets the same relations with the parameter

$$ \alpha_h = q^{2h} \exp(2i\eta) \quad (32) $$

so we have isomorphism with $S_{2n,2h}^f(y_f)$. The results of [11] immediately follow. For $\alpha_h$ generic, $\mathcal{T}_n(\alpha_h)$ is irreducible with dimension the number of paths of $2n$ steps going from the origin to the point of horizontal coordinate $2h$ on the Pascal triangle, i.e $C_{2n}^{2h}$. The representation is reducible for $\alpha_h = q^{2k}$ where $k = h + n_c$, $n_c = 1, \ldots, n$. In that case it contains an irreducible component $\rho_n(h+n_h)$ of dimension the number of paths with same characteristics but on a diagram truncated on the left to contain only points with horizontal coordinates greater or equal to $1 - n_c$, i.e $C_{2n}^{n-h} - C_{2n}^{n-h-n_c}$. Notice this coincides with the number of paths with same characteristics but on a diagram truncated on the right to contain only points of horizontal coordinate lower or equal to $2h + n_c - 1$, as well as the number of paths of $2n$ steps on a half Pascal triangle (with only positive coordinates) that go from a point of horizontal coordinate $n_c-1$ to a point of horizontal coordinate $2h + n_c - 1$ see figure 1). This recovers the results of prop.18 in [11].

The connection with the blob algebra allows us as well to study the representation theory of $T$ when $q$ is a root of unity. This was not straightforward using the Gram determinants results of [11] due to the existence of multiple zeroes. Suppose $m_2$ is the smallest integer such that $q^{m_2} = \pm 1$. Then $\rho_n(h+n_h)$ is further reducible. It contains an irreducible component $\rho_{ab}(n)$ where $a = n_c - 1, b = 2h + n_c - 1 (0 \leq a, b \leq m_2 - 2)$ of dimension the number of paths of $2n$ steps that go from the origin to the point of horizontal coordinate $2h$ on a Pascal triangle that is truncated on the left and on the right so as to include only points of horizontal coordinate greater or equal to $1 - n_c$ and smaller or equal to $p-2+1-n_c$ (or horizontal coordinate greater or equal to $2-p+2h+n_c-1$ smaller or equal to $2h + n_c - 1$). This is as well the number of paths of $2n$ steps on a half Pascal triangle truncated on the right to contain points of coordinate smaller or equal to $p-2$, that go from a point of horizontal coordinate $n_c-1$ to a point of horizontal coordinate $2h + n_c - 1$ (see figure 2). The explicit expression of this dimension is, for $(a,b) \neq (m_2/2-1, m_2/2-1)$

$$ \dim \rho_{ab} = \sum_{i \in \mathbb{Z}} \left( n - \frac{a+b}{2} + im_2 \right) - \left( n - \frac{a+b}{2} - 1 + im_2 \right) \quad (33) $$

n where the sum truncates for negative arguments in the binomial coefficients. When $(a,b) = (m_2/2-1, m_2/2-1)$ the dimension is half of the above expression. The representations $\rho_{ab}$ and $\rho_{m_2-a-2,m_2-b-2}$ are isomorphic by $e,f$ interchange symmetry.
These results were conjectured first in [3]. In this latter reference, the decomposition of the reducible representations \( R \) of \( T \) provided by solid on solid models on Dynkin diagrams \( D \) [17] were also given. Recall for instance in the simplest case of the Ising model (\( A_3 \)) and the 3 state Potts model (\( D_4 \))

\[
R^{A_3} = \rho_{00} + \rho_{11}, \quad R^{D_4} = \rho_{00} + 2\rho_{22} + \rho_{04}
\]  

(34)

### 4 Conclusion

As well as giving a complete analysis of the blob algebra this paper completes the study of the representation theory of the periodic Temperley Lieb algebra [3, 5, 6]. The analogy between representations of this algebra and those of left-right Virasoro algebra goes actually further than the “minimal set” that was mainly discussed so far [3]: representations that are not in the innermost part of the Bratteli diagram can as well be put in correspondence with representations of the Virasoro algebra that are “outside” the minimal Kac table. This is easily checked as in [3] for instance by calculating traces of the physical Hamiltonian in the continuum limit and comparing them with the known Virasoro characters.

An interesting physical question is whether there are lattice models of restricted solid on solid type that use only the outside representations, the way the Andrews Baxter Forrester [18] models use only the innermost part of the Bratteli diagram?. By conformal invariance analogy one expects there are no such models that lead to modular invariant partition functions. This is because, for a conformal field theory, if one representation outside the minimal Kac table appears, then all the ones inside must appear as well by the effect of modular transformations [19]. Indeed, it is easy to build SOS models that use only outside representations by a limiting process on the interacting round a face form of the Temperley Lieb generators. But the constraint that the paths must not touch the outer line after the last time they touched the inner one makes such model very different in space and time direction, and likely enough cannot lead to a modular invariant partition function.

An interesting question concerns the nature and properties of “physical” representations of the blob algebra. For instance the vertex model representation of the Temperley Lieb algebra is well known [2], with basis provided by \( n \) tensored copies of \( \mathbb{C}^2 \) and \( U \) matrices acting between two neighboring copies (\( U_i \) acts on the \( i^{th} \) and \( (i + 1)^{th} \) copies) as the matrix

\[
U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]  

(35)

It is immediate then to find a representation of \( e \) acting on the left most copy by the matrix

\[
e = \frac{1}{a + a^{-1}} \begin{pmatrix} a^{-1} & -1 \\ -1 & a \end{pmatrix}
\]  

(36)

with the value

\[
y_e = \frac{aq + a^{-1}q^{-1}}{a + a^{-1}}
\]  

(37)

Note that this matrix breaks the ‘charge conservation’ property which allows the vertex model representation to be immediately broken up into blocks (q-analogues of permutation blocks in the symmetric group). The use of this representation is not completely clear to us.

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Figure 1: Three ways of counting the number of states in $\rho_{n=3}(k=4)$. This number coincides with the number of paths of 6 steps on any of the diagrams that go form the top (with coordinate 0) to the bottom cross (with coordinate 2).

Figure 2: Same as the first diagram above, but for $q$ a root of unity with $m_2 - 2 = 6$
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