Eigenvalue Dynamics of a $\mathcal{PT}$-symmetric Sturm-Liouville Operator. Criteria of the Similarity to a Self-adjoint or Normal Operator.

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**Introduction.** The goal of the paper is to investigate the dynamics of the eigenvalues of the Sturm-Liouville operator

$$T(\varepsilon)y = -\varepsilon^{-1}y'' + p(x)y$$

(1)

on the finite interval $[-1, 1]$ as the parameter $\varepsilon > 0$ changes. For simplicity we consider the Dirichlet boundary conditions

$$y(-1) = y(1) = 0.$$ 

(2)

It is assumed in the sequel that the potential $p$ is summable and $\mathcal{PT}$-symmetric, i.e. $p(x) = p(-x)$. It is easily seen that this condition guarantees the symmetry with respect to the real axis of the spectrum of the operator $T(\varepsilon)$ defined by differential expression (1) and boundary conditions (2).

Plenty of papers are dedicated to the study of $\mathcal{PT}$-symmetric operators especially in the physical literature. We point out the papers [1]-[10] which we acquainted with, and which have stimulated our investigation. More details and references can be found in the review of Dorey, Dunning and Tateo [5]. The papers which are close to the topic of our paper are dedicated, basically, to the proof of the reality of the spectrum of the 1-d Schrodinger operator on the whole line with some concrete potentials, in particular, with cubic or polynomial potentials for particular parameter values.

In this work we pose the question not only about the realness of the spectrum, but also on the similarity of the operator to a self-adjoint or normal one. Our main goal is to find or estimate the values of the parameter $\varepsilon$, under which the spectrum of the operator $T(\varepsilon)$ is real and the operator itself is similar to a self-adjoint one.

The main results of the paper are associated with the study of the operator $T(\varepsilon)$ with specific $\mathcal{PT}$-symmetric potential $p(x) = ix$. It turns out that this case presents an exactly solvable model which allows us to trace the dynamics of the movement of the eigenvalues in all details and to find explicitly the critical parameter values, in particular, to specify precisely the number $\varepsilon_1$ such that for $0 < \varepsilon < \varepsilon_1$ the operator $T(\varepsilon)$ has a real spectrum and is similar to a self-adjoint one, but for $\varepsilon \geq \varepsilon_1$ this property is broken down. In the general case, of course, to find explicitly critical values is not possible. However, it is possible to get some estimates for such values of the parameter.

2. **General results.** The following statement shows that for sufficiently small values of the parameter $\varepsilon$ the operator $T(\varepsilon)$ is similar to a self-adjoint one.
Theorem 1. The spectrum of the operator $T(\varepsilon)$ consists of simple real eigenvalues, and the operator itself is similar to a self-adjoint one provided that any of the following conditions holds:

$$\varepsilon < C_{\infty}, \quad \varepsilon < C_1, \quad \varepsilon < C_2.$$  

Here $\| \cdot \|_\infty, \| \cdot \|_1, \| \cdot \|_2$ are the norms in spaces $L_\infty((-1,1)), L_1((-1,1)), L_2((-1,1))$, respectively (assuming that the potential $p$ belongs to the corresponding space). The constants in these inequalities are defined as follows

$$C_{\infty} = \frac{3\pi^2}{8};$$

$$C_1 = \frac{\pi^2}{4} \left( \frac{2}{3} + \sum_{k=2}^{\infty} \left( k^2 - \frac{5}{2} \right)^{-1} \right)^{-1};$$

$$C_2 = \frac{\pi^2}{4} \left( \frac{4}{9} + \sum_{k=2}^{\infty} \left( k^2 - \frac{5}{2} \right)^{-2} \right)^{-1/2}.$$  

The proof of this theorem uses estimates for the resolvent of the operator family $L(\varepsilon) = \varepsilon T(\varepsilon)$, which can be considered as a perturbation of the operator $L(0) = -d^2/dx^2$ with boundary conditions (2) (see [12, §7, 9]). It is also used the Dunford theorem on the unconditional basis property of eigenfunctions of the operator $T(\varepsilon)$ and theorems of Bari and Boas about the properties of unconditional bases [12, §6].

In the case of non-real $\mathcal{PT}$-symmetric potential $p$ the property of the operator to be self-adjoint breaks down as the parameter $\varepsilon$ is growing up. For the case when the potential $p$ is a non-real polynomial the paper [11] contains the result, which approves the localization of the spectrum along certain curves in the complex plane, the structure of which is determined by the coefficients of the polynomial. It follows from this result that for large values of the parameter $\varepsilon$ the non-real eigenvalues do appear and their number increases proportionally to $\varepsilon^{1/2}$ as $\varepsilon \to \infty$.

3. The complex Airy operator, as an explicitly solvable model. Further we study the model operator

$$T(\varepsilon) y = -\varepsilon^{-1} y'' + i \varepsilon y, \quad y(-1) = y(1) = 0. \quad (3)$$

The behavior of the spectrum of this operator as $\varepsilon \to +\infty$ was investigated in details in the papers [13, 14]. For large values of the parameter $\varepsilon$ the eigenvalues are concentrated along segments $[i, 1/\sqrt{3}], [-i, 1/\sqrt{3}]$ and the ray $[1/\sqrt{3}, +\infty)$, and together they form the so-called limit spectral graph. The point of $1/\sqrt{3}$ is called the knot point of this limit spectral graph. From theorem 1 we obtain that for the values $\varepsilon < \frac{3\pi^2}{8} \approx 3.7011$ the spectrum of the operator (3) is real and consists of simple positive eigenvalues $\lambda_k = \lambda_k(\varepsilon)$, which we assume to be numerated in the increasing order. In this case $\lambda_k(\varepsilon) \sim \varepsilon^{-1}(\pi k/2)^2$ as $\varepsilon \to 0$ and $k$ is fixed, and the same is true as $k \to \infty$ and $\varepsilon$ is fixed. However, the estimate $\varepsilon < \varepsilon_0 = 3\pi^2/8$ is too coarse. This estimate could be close to be sharp if the first two eigenvalues $\mu_1(\varepsilon)$ and $\mu_2(\varepsilon)$ of the operator family $L(\varepsilon) = \varepsilon T(\varepsilon)$ would move towards each other with increasing $\varepsilon \in (0, \varepsilon_0)$. But it is not the case.

In reality, the portrait of the movements of the eigenvalues is the following. Numerical calculations show (in Theorem 4 they get analytical confirmation) that starting from
small values of the parameter $\varepsilon$, all the eigenvalues of the operator family $T(\varepsilon)$ move with increasing $\varepsilon > 0$ from infinity to the left. When $\varepsilon \approx 5.1$ the first eigenvalue crosses the knot point $1/\sqrt{3} \approx 0.58$ and continues to move to the left. When $\varepsilon = \varepsilon_{1,\text{turn}} \approx 9.3$ the first eigenvalue reaches some point $\lambda_{1,\text{turn}} \approx 0.45$, stops at this point, and upon further increasing $\varepsilon > \varepsilon_{1,\text{turn}}$ begins to move in the opposite direction, while all other eigenvalues continue to move to the left.

Further, the first and second eigenvalues move towards each other approaching the knot point $1/\sqrt{3}$ and collide at this point when $\varepsilon_1 \approx 12.3$.

After the collision in the knot point the first and second eigenvalues come off in the complex plane at the right angle to the real axis, and upon further increasing $\varepsilon$, approach rapidly the segments $[1/\sqrt{3}, \pm i]$ and continue the movement, clinging to these segments, in the direction of the points $\pm i$.

Further, with growth of $\varepsilon$ the third eigenvalue crosses the knot point moving to the left to the point $\lambda_{2,\text{turn}} < 1/\sqrt{3}$, stopped at this point, and then moves in the opposite direction towards the fourth eigenvalue until the collision again at the knot point $1/\sqrt{3}$, and subsequently jumping in the complex plane. The fifth and sixth eigenvalues (and subsequent $2k - 1$ and $2k$th ones) repeat the same dynamics. For large $\varepsilon$ the eigenvalues accumulate on the real ray $[1/\sqrt{3}, +\infty)$, while all the non-real eigenvalues nestle to the segments $[1/\sqrt{3}, \pm i]$, moving to the points $\pm i$. 
Explicit formulae for the distribution of the eigenvalues in the intervals $[1/\sqrt{3}, \pm i]$ and the ray $[1/\sqrt{3}, +\infty)$ for large parameter $\varepsilon$ can be found in [13].

Further, the set $E$ of pairs $(\varepsilon, \lambda) \in \mathbb{R}^+ \times \mathbb{C}$ for which for a given parameter $\varepsilon > 0$ the number $\lambda$ belongs to spectrum of the operator $T(\varepsilon)$ we call the spectral locus of the operator family $T(\varepsilon)$. The subset $E_\mathbb{R} \subset E$, which corresponds to the real eigenvalues $\lambda$, is called the real spectral locus of the family $T(\varepsilon)$.

The following figure shows the real spectral locus of the model operator (the figure reflects the real computer calculations).

Applying the Weierstrass preparation theorem to the characteristic determinant of the eigenvalue problem for the operator $T(\varepsilon)$, we can prove the following statement.

**Theorem 2.** The real spectral locus of the family $T(\varepsilon)$ consists of regular analytic pairwise non-intersecting Jordan curves in the extended complex plane with the ends in the infinity point.

A similar theorem was proved by Eremenko and Gabrielov [10] while investigating the real spectrum of the cubic anharmonic oscillator on the whole axis.

From Theorem 2 taking into account the smoothness of the curves comprising the real part of the locus, and the absence of pairwise intersections, we get the following statement.

**Theorem 3.** With increasing $\varepsilon$ the real eigenvalues of the operator family (3) can collide only in pairs. Just before the moment of the collision the eigenvalues of a corresponding pair move towards each other, and immediately after the collision diverge in the complex plane at the right angles to the real axis.
A point \((\varepsilon_0, \lambda_0) \in \mathcal{E}\) of the spectral locus we call \textit{critical} if the eigenvalue \(\lambda_0\) of the operator \(T(\varepsilon_0)\) is not simple (i.e. algebraic multiplicity of this eigenvalue > 1). Parameter values \(\{\varepsilon_k\}_{k=1}^\infty\), for which at least one eigenvalue of the operator \(T(\varepsilon_k)\) is multiple, we also call \textit{critical} (it is easy to see that the set of such values is not more than countable).

4. The main results for the model operator.

Consider the classic Airy equation
\[ y'' = \xi \cdot y, \quad y = y(\xi), \]
and its two standard solutions — the functions \(\text{Ai}\) and \(\text{Bi}\) (see [15], for example). A remarkable role in the sequel play the following special solutions of the Airy equation:
\[ U_-(\xi) = -\sqrt{3}\text{Ai}(\xi) + \text{Bi}(\xi), \]
\[ U_+(\xi) = \sqrt{3}\text{Ai}(\xi) + \text{Bi}(\xi). \]

The following theorem is the main result of the present work.

\textbf{Theorem 4.} The zeros of the functions \(U_-\) and \(U_+\) are located on the rays \(\arg z = \pi/3 + 2\pi k/3, k = -1, 0, 1\) symmetrically with respect to the origin. Let \(\{\alpha_k\}_{k=1}^\infty\) and \(\{\beta_k\}_{k=1}^\infty\) be the modules of the zeros of the functions \(U_-\) and \(U_+\), respectively, numerated in the increasing order. These zeros interlace:
\[ \alpha_0 < \beta_1 < \alpha_1 < \beta_2 < \alpha_2 < \beta_3 < \ldots. \]

Denote
\[ \delta_k = \left(\beta_k \frac{\sqrt{3}}{2}\right)^3, \quad \varepsilon_k = \left(\alpha_k \frac{\sqrt{3}}{2}\right)^3, \quad k \in \mathbb{N} \]
(obviously, \(0 < \delta_1 < \varepsilon_1 < \delta_2 < \varepsilon_2 < \ldots\)). Then

- All the eigenvalues of the operator \(T(\varepsilon)\) are simple and this operator is similar to a self-adjoint one, provided that \(\varepsilon \in (0, \varepsilon_1)\). For all other values of \(\varepsilon > 0\) this property is broken down.

- All the critical points of the spectral locus coincide with the set
\[ \mathcal{M} = \left\{ \left(\varepsilon_k, \frac{1}{\sqrt{3}}\right) \right\}. \]

For all critical values of the parameter the knot point \(1/\sqrt{3}\) is 2-multiple eigenvalue of the operator \(T(\varepsilon_k)\), which meets the Jordan cell. For all \(\varepsilon \neq \varepsilon_k\), the operator \(T(\varepsilon)\) is similar to a normal one (i.e. to an operator, commuting with its adjoint).

- All the odd eigenvalues \(\lambda_{2k-1}(\varepsilon)\) move to the left (decrease) with increasing parameter \(\varepsilon\), pass through the knot point \(1/\sqrt{3}\) as the parameter takes the values \(\varepsilon = \delta_k\), and continue to move to the left until they reach some points \(\lambda_{2k-1,\text{turn}} < 1/\sqrt{3}\). Then, starting from these points, they turn back and move to the right before colliding with the even eigenvalues \(\lambda_{2k}\) in the knot point \(1/\sqrt{3}\) as the parameter takes the critical values \(\varepsilon = \varepsilon_k\).
• After the collision the eigenvalues move in the opposite directions to the complex plane perpendicular to the real axis, and subsequently never come back to the real axis. Outside of the real axis the eigenvalues are unable to face.

• For the values \( \varepsilon = \delta_k \) and \( \lambda_{2k-1} = 1/\sqrt{3} \) the eigenfunctions can be written down explicitly:
  \[
y(z) = U_+ \left( \varepsilon_k^{1/3} \left( \frac{1}{\sqrt{3}} - iz \right) \right).
\]

• For the values \( \varepsilon = \varepsilon_k \) and \( \lambda_{2k} = 1/\sqrt{3} \) the eigenfunctions are also written down explicitly:
  \[
y(z) = U_- \left( \varepsilon_k^{1/3} \left( \frac{1}{\sqrt{3}} - iz \right) \right).
\]

• The following asymptotics are valid as \( k \to \infty \):
  \[
  \varepsilon_k = \frac{\sqrt{3}}{4} \left( \frac{3}{2} \right)^3 \left( \pi k - \frac{\pi}{12} + O \left( \frac{1}{k} \right) \right)^2,
  \delta_k = \frac{\sqrt{3}}{4} \left( \frac{3}{2} \right)^3 \left( \pi k - \frac{5\pi}{12} + O \left( \frac{1}{k} \right) \right)^2.
\]

The following theorem gives an upper bound for the turning points of the eigenvalues \( \lambda_{2k-1,\text{turn}} \), \( k \geq 1 \).

**Theorem 5.** Let the modules of the complex zeros \( \{z_k\}_{k=1}^{\infty} \) of the function \( \text{Bi} \) lying in the first quadrant of the complex plane be numerated in the increasing order of their modulas. Then the following estimate for the turning points of the odd eigenvalues is valid:

\[
\lambda_{2k-1,\text{turn}} < \cot \arg z_k < 1/\sqrt{3}.
\]

From this statement, in particular, we have \( \lambda_{1,\text{turn}} < 0.457 \).

Let us say some words about the idea of the proof of Theorems 4 and 5. It turns out that the substitution \( \xi = \varepsilon^{1/3}(\lambda - ix) \) reduces the eigenvalue problem for the operator \( T(\varepsilon) \) to the study of the zero distribution of solutions of the Airy equation

\[
y'' = \xi y.
\]

The function \( V_a(\xi) = a \text{Ai}(\xi) + \text{Bi}(\xi) \) of the argument \( \xi \in \mathbb{C} \), with \( a \in \mathbb{C} \), up to a constant multiplier, describes all the solutions to this equation with the exception of \( \text{Ai} \).

If \( a \) is real, then the complex (non-real) zeros of \( V_a(\xi) \) specify an alternative parametrization of the real spectral locus of the original problem. Each point \((\varepsilon, \lambda)\) of this locus corresponds in a unique way to the pair of complex conjugate zeros of a real solutions of the Airy equation. Vice versa, if \( \xi_0 \) is zero of \( V_a(\xi) \) for some real \( a \), then the real point of the spectral locus: \( \varepsilon = |\text{Im} \xi_0|^3 \), \( \lambda = \text{Re} \xi_0/|\text{Im} \xi_0| \) is uniquely determined. The equation \( V_a(\xi) = 0 \) defines a countable number of implicit functions \( \xi_k = \xi_k(a) \), \( k \in \mathbb{Z} \), each of which admits an analytic continuation in a neighborhood of the real axis and takes the values in the first or fourth quadrants of the complex plane. The images \( \Gamma_k \) of the real axis of each of these functions are pairwise disjoint analytic Jordan arcs, unlimited with both ends.
Each of the ends of $\Gamma_k$, tending to infinity, approaches the real positive half-line.

If $\xi_k(a)$, $a \in \mathbb{R}$, is the parametrization of a curve $\Gamma_k$, then the point $\xi_{k0} = \xi_k(a_0)$ of the curve $\Gamma_k$ corresponds to the critical point of the spectral locus of the operator family $T(\varepsilon)$ if and only if $\xi_k'(a_0) \in \mathbb{R}$. This is true if and only if the tangent to the curve $\Gamma_k$ at the point $\xi_{k0}$ is parallel to the real axis. This is true if and only if $a = -\sqrt{3}$ for all $k$.

For each $\Gamma_k$ lying in the first quadrant of the complex plane, there exists the only point $\xi_{k0} = \xi_k(-\sqrt{3})$ and it coincides with the zero of the functions $U_-$, lying on the ray $l = \{\arg z = \pi/3\}$. Among all the points of the curve $\Gamma_k$, it has the largest imaginary part.

The point $\xi_{k0}$ divides $\Gamma_k$ on two arcs: $\gamma_{2k-1}$ and $\gamma_{2k}$, which correspond to the dynamics of the eigenvalues $\lambda_{2k-1}$ and $\lambda_{2k}$ of the original problem. The point $\xi_{k1} = \xi_k(\sqrt{3})$ lying on the ray $\arg z = \pi/3$ corresponds to passing of the $\lambda_{2k-1}$ through the knot point, and the $\xi_{k0}$ corresponds to collision of the $\lambda_{2k-1}$ and $\lambda_{2k}$.

The statements of the theorem concerning the behavior of complex eigenvalues (the impossibility of coming back to the real axis and the impossibility of collisions outside the real axis) are proved while investigating the characteristic determinant of the original problem in terms of the new variable $\xi$ and the parameter $\varepsilon$.

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