PARTIALLY MULTIPLICATIVE QUANDLES AND SIMPLICIAL HURWITZ SPACES

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ABSTRACT. We introduce the notion of partially multiplicative quandle (PMQ), generalising both classical notions of partial monoid and of quandle. We set up the basic theory of PMQs, in analogy with classical group theory. For a complete PMQ \(\mathcal{Q}\), and for a braided closed monoidal category \(\mathcal{A}\), we introduce a braided monoidal category of \(\mathcal{Q}\)-crossed objects in \(\mathcal{A}\), in analogy with the classical notion of \(G\)-crossed objects, depending on a discrete group \(G\). We use the previous to give a definition of Hurwitz spaces \(\text{Hur}^\mathcal{Q}(\Delta)\) of configurations of points in the plane with monodromies in a PMQ; the definition has a simplicial flavour, and gives rise to topological spaces with a natural cell stratification; it recovers the classical definition of Hurwitz spaces in the case of a discrete group \(G\) considered as a PMQ with trivial multiplication. We give a criterion to detect whether a PMQ \(\mathcal{Q}\) is Poincare, i.e. whether all connected components of \(\text{Hur}^\mathcal{Q}(\Delta)\) are topological manifolds. Finally, we analyse thoroughly the PMQ \(S^{d\alpha}_{\mathcal{Q}}\), for \(d \geq 2\), and we compute its enveloping group and its completion as PMQ.

1. INTRODUCTION

The main goal of the article is to introduce the algebraic notion of partially multiplicative quandle (PMQ), which generalises both notions of quandle and of partial monoid:

- a quandle is a set with a binary operation behaving like the conjugation in a group; quandles were introduced by Joyce [Joy82], and are classically used to define knot invariants;
- a partial monoid is a set with a partially defined binary operation behaving like the product in an associative monoid;
- a PMQ is a set \(\mathcal{Q}\) with both structures simultaneously, satisfying some compatibility conditions (see Definition 2.4).

We use the notion of PMQ to develop a theory of generalised Hurwitz spaces. Classically, Hurwitz spaces are defined by fixing an integer \(k \geq 0\) and a group \(G\): a configuration in \(\text{hur}_k(G)\) is the datum \((P, \varphi)\) of a set \(P = \{z_1, \ldots, z_k\}\) of \(k\) points in the unit square \((0, 1)^2\), together with a monodromy \(\varphi\): one considers
(0, 1)² as a subspace of the complex plane C, and the monodromy is defined to be a group homomorphism \( \varphi : \pi_1(\mathbb{C} \setminus P) \to G \). The topology on Hurwitz spaces is usually defined leveraging on the topology of configuration spaces, in such a way that the forgetful map \((P, \varphi) \mapsto P\) induces a covering map \(\text{hur}_k(G) \to C_k((0, 1)^2)\), with target the unordered configuration space of \(k\) points in \((0, 1)^2\). The classical Fox-Neuwirth-Fuchs cell stratification on \(C_k((0, 1)^2)\) can then be lifted to a cell stratification on \(\text{hur}_k(G)\).

Our strategy to define Hurwitz spaces \(\text{Hur}^\Delta(Q)\), with monodromies in a PMQ \(Q\), consists in first mimicking the cell stratification. One novelty of the construction is the absence of the parameter \(k\) in the notation “\(\text{Hur}^\Delta(Q)\)”: the partial product on the PMQ \(Q\) is used to allow collisions between points of a configuration in a controlled way, and in particular the cardinality of \(P\) is no longer a continuous invariant of configurations, i.e., an invariant of connected components, but rather gives a stratification of configuration spaces.

Our motivating examples of PMQs are the family of PMQs \(S^\text{geo}_d\), for \(d \geq 1\): the PMQ \(S^\text{geo}_d\) is obtained from the symmetric group \(S_d\) by keeping the binary operation of conjugation, and by restricting the binary operation of product to certain pairs of permutations, satisfying a “geodesic” requirement. In related work [Bia21c], we will establish a connection between these PMQs, with their associated Hurwitz spaces, and moduli spaces of Riemann surfaces with boundary.

1.1. Statement of results. We begin this paper by discussing several constructions and results about PMQs and their relation to groups, developing a basic theory of PMQs which is in many respects parallel to classical quandle theory and group theory. We prove in particular the following results. The following is Proposition 2.22; just like a partial monoid \(M\) admits a completion \(\hat{M}\), which is the initial monoid receiving a map of partial monoids from \(M\), also a PMQ \(Q\) admits a completion \(\hat{Q}\), which is the initial PMQ receiving a map from \(Q\) such that the product is defined for all pairs of elements in \(\hat{Q}\). Among other things, the completion \(\hat{Q}\) of \(Q\) will play a crucial role in classifying connected components of generalised Hurwitz spaces with monodromies in \(Q\).

**Proposition 1.1.** The canonical map from a PMQ \(Q\) to its completion \(\hat{Q}\) is injective.

This is similar to the classical result stating that a partial monoid \(M\) injects into its completion \(\hat{M}\). Note that, instead, the canonical map from \(Q\) to its enveloping group \(G(Q)\) need not be injective: the enveloping group of \(Q\) is the initial group receiving a map of PMQs from \(Q\).

An important notion, analogous to the one given in [Joy82, Definition 9.1] in the context of quandles, is the one of PMQ-group pair: roughly speaking, it is a pair \((Q, G)\) of a PMQ \(Q\) and a group \(G\) that are interrelated with each other (see Definition 2.15). Our main results regarding PMQ-group pairs are concerned with the example of \((F^kQ^l, F^k)\), for \(0 \leq l \leq k\): here \(F^k\) is the free group on \(k\) generators, and \(F^kQ^l \subset F^k\) is the sub-PMQ containing the neutral element and the first \(l\) conjugacy classes of generators of \(F^k\). The following is an informal rephrasing of Theorem 3.3.

**Theorem 1.2.** The PMQ-group pair \((F^kQ^l, F^k)\) is a free object in the category of PMQ-group pairs, on \(l\) generators of PMQ-type and \(k - l\) generators of group-type.
To state the next result, let $0 \leq r \leq l \leq k$ and consider the Artin action of the braid group $\mathcal{B}_r$ on the subgroup $\mathbb{F}^r \subset \mathbb{F}^k$ spanned by the first $r$ generators of $\mathbb{F}^k$; note that the product $f_1 \ldots f_r \in \mathbb{F}^k$ is fixed by this action. Let $\mathcal{B}_r$ act on the entire $\mathbb{F}^k$ by acting trivially on the last $k - r$ generators. The following is a rephrasing of Proposition 3.7.

**Proposition 1.3.** The Artin action of $\mathcal{B}_r$ is transitive on the set of decompositions of the element $f_1 \ldots f_r \in \mathbb{F}^k$ as a product of $r$ elements in $\mathbb{F}^k$.

These last two results are a bit technical in nature, but will play an important role in future work [Bia21a]. We include them in this article because they are results in the algebraic theory of PMQs, independently of their application to the study of Hurwitz spaces.

In analogy with the construction of the group ring $R[G]$ of a group $G$, we introduce the PMQ-ring $R[Q]$ of a PMQ $Q$ with coefficients in a commutative ring $R$ (see Definition 4.26). This ring will play an important role in the computation of the stable homology of Hurwitz spaces in future work [Bia21b]. The following is Theorem 4.28, translating properties of a PMQ into properties of the associated ring.

**Theorem 1.4.** Let $Q$ be a PMQ with the following three properties: it is maximally decomposable (Definition 4.17), coconnected (Definition 4.20), and pairwise determined (Definition 4.24). Then $R[Q]$ is a quadratic $R$-algebra.

The following is Lemma 1.5; its proof is quite elementary, but the statement is a little surprising, considering that $R[Q]$ is in general not a commutative ring.

**Lemma 1.5.** The subring $A(Q) \subseteq R[Q]$ of all elements which are invariant under conjugation is a commutative ring.

In order to make the definition of simplicial Hurwitz spaces more conceptual, we introduce some categorical constructions. For a complete PMQ $\hat{Q}$ and a category $A$ we introduce the category $X_A(\hat{Q})$ of $\hat{Q}$-crossed objects in $A$; this notion parallels and generalises the more classical one of $G$-crossed objects, for a group $G$. The following proposition is a consequence of the discussion of Subsection 6.1, leading in particular to Definition 6.5.

**Proposition 1.6.** Let $A$ be a braided closed monoidal category, and let $\hat{Q}$ be a complete PMQ. Then the category $X_A(\hat{Q})$ of $\hat{Q}$-crossed objects in $A$ is endowed with a braided monoidal structure.

The machinery of $\hat{Q}$-crossed objects is used to define simplicial Hurwitz spaces with monodromies in a PMQ $Q$: leveraging on the fact that each PMQ $Q$ embeds into its completion $\hat{Q}$, we will first define a Hurwitz space $\text{Hur}^A(\hat{Q})$ and then identify $\text{Hur}(Q)$ as a subspace of the latter. The decoration “$\Delta$” reminds that the definition relies on a double bar construction, and is simplicial in flavour.

As already mentioned, our main motivation to introduce PMQs and to generalise Hurwitz spaces comes from the family of PMQs denoted $\mathcal{S}_d^{\text{geo}}$, for $d \geq 1$. Our last, main result of the article is the following theorem, which combines Lemma 7.2 and Proposition 7.13.

**Theorem 1.7.** Let $d \geq 2$. The enveloping group of $\mathcal{S}_d^{\text{geo}}$ coincides with the index 2 subgroup of $\mathbb{Z} \times \mathcal{S}_d$ of pairs $(r, \sigma)$ such that $r$ and $\sigma$ have the same parity.
The completion $\hat{S}_d^{\text{geo}}$ of $S_d^{\text{geo}}$ as a PMQ contains all sequences

$$(\sigma; \mathcal{P}_1, \ldots, \mathcal{P}_\ell; r_1, \ldots, r_\ell),$$

consisting of a permutation $\sigma \in S_d$, an unordered partition $\mathcal{P}_1, \ldots, \mathcal{P}_\ell$ of the set $\{1, \ldots, d\}$, and a system of weights $r_1, \ldots, r_\ell \geq 0$ on the pieces of the partition, satisfying the following properties:

1. $\sigma$ preserves each piece $\mathcal{P}_j$ of the partition;
2. $r_j + 1 - |\mathcal{P}_j|$ is greater or equal to the number of cycles of the restricted permutation $\sigma|_{\mathcal{P}_j} \in S_{\mathcal{P}_j}$, for all $1 \leq j \leq \ell$;
3. $r_j$ has the same parity as the restricted permutation $\sigma|_{\mathcal{P}_j} \in S_{\mathcal{P}_j}$, for all $1 \leq j \leq \ell$.

1.2. Outline of the article. In Section 2 we introduce partially multiplicative quandles (PMQs) and the related notions of PMQ-group pair, completion of a PMQ, and enveloping group of a PMQ. We prove the aforementioned Proposition 2.22.

In Section 3 we study the PMQs $FQ_k^l$ and their relation to free groups $F_k^l$; we prove the aforementioned Theorem 3.3 and Proposition 3.7.

In Section 4 we introduce the notion of norm on a PMQ, and study several combinatorial properties that a PMQ can enjoy. We also introduce the PMQ-ring $R[Q]$ associated with a PMQ $Q$. We study how properties of the PMQ $Q$ are reflected into properties of the ring $R[Q]$, and in particular we prove the aforementioned Theorem 4.28 and Lemma 4.31.

In Section 5 we describe, in the generality of a braided monoidal category $A$, a procedure taking as input a morphism $f : A \to B$ of commutative algebras in $A$, and giving as output a bisimplicial object in $A$, denoted $B_{\bullet,\bullet}(A, B, f)$ and called the double bar construction. The material of this section is largely standard and is given in detail only to make the paper self-contained.

In Section 6 we associate with an augmented PMQ $Q$ a bisimplicial set $\text{Arr}(Q)$ endowed with a sub-bisimplicial set $\text{NAdm}(Q)$; the simplicial Hurwitz spaces $\text{Hur}^A(Q)$ is then defined as the difference of the geometric realisations of the two bisimplicial sets. We describe the construction using the notion of $Q$-crossed topological spaces, where $Q$ is the completion of $Q$. Similarly, the relative chain complex of the pair of bisimplicial sets $(\text{Arr}(Q), \text{NAdm}(Q))$ is described as a $Q$-crossed chain complex. In this section we also introduce the notion of Poincare PMQ, which will play a key role in related work on the homology of Hurwitz spaces.

In Section 7 we give a detailed analysis of the family of PMQs $S_d^{\text{geo}}$. Together with the aforementioned Lemma 7.2 and Proposition 7.13, we mention Proposition 7.14 in which we show that $S_d^{\text{geo}}$ is a Koszul PMQ, i.e. $R[S_d^{\text{geo}}]$ is a Koszul quadratic $R$-algebra. The results of this section are partially contained in [Bia20, Subsection 8.1.3].

Finally, in the Appendix A we discuss how the theory of PMQs changes if we relax the notion of PMQ to the notion of partially multiplicative rack: in particular we explain the failure of Theorem 3.3 in the context of partially multiplicative racks, thus motivating our focus on the more restrictive notion of PMQ.

1.3. Motivation. This is the first article in a series about Hurwitz spaces. The scope we have in mind, for this and the following articles, is to define and study
generalised Hurwitz spaces $\text{Hur}(\mathcal{X}; \mathcal{Q})$ with monodromies in a PMQ and with a suitable subspace $\mathcal{X} \subset \mathbb{C}$ as background.

In this article we will achieve a first, simplicial definition of a generalised Hurwitz space with monodromies in $\mathcal{Q}$, which we denote $\text{Hur}^\Delta(\mathcal{Q})$; this space comes with a stratification by open cells which reminds of the Fox-Neuwirth-Fuchs stratification of the classical configuration spaces $\mathcal{C}_k(\mathbb{C})$ of the plane; we remark that recently [ETW17] a similar cell stratification has been used in the study of classical Hurwitz spaces.

The simplicial definition of Hurwitz spaces is, in a certain sense, not coordinate-free, in the sense that it uses dramatically the two standard, Euclidean coordinates of the plane $\mathbb{C}$. One of the main achievements of the second article of the series [Bin21a] will be a coordinate-free definition of the generalised Hurwitz spaces, allowing for more flexible manipulations. The results of this article will give the algebraic input for the second article.

The third article in the series [Bin21b] will use the algebraic input of the first article, together with the topological input of the second, to study Hurwitz spaces as topological monoids, and compute their deloopings.

Finally, the fourth article in the series [Bin21c] will apply the entire machinery of generalised Hurwitz spaces to the study of moduli spaces $\mathcal{M}_{g,n}$ of Riemann surfaces of genus $g$ with $n \geq 1$ ordered and parametrised boundary curves: the PMQs $\mathcal{S}_{g,\infty}$ will play a prominent role and motivate the very definition of PMQs, but we prefer to set up the theory more generally to allow other applications and to make the exposition more conceptual.

The attempt to generalise Hurwitz spaces to PMQs should also be seen as an attempt to unify two classical notions in topology: classical Hurwitz spaces, and configuration spaces with summable labels. Both notions deal with decorated configurations of points in a background space $\mathcal{X}$. The decoration of a configuration $P \subset \mathcal{X}$ is

- a monodromy with values in a group $G$ (or more generally a quandle $\mathcal{Q}$) and defined on certain loops in $\mathbb{C} \setminus P$, for classical Hurwitz spaces (here $\mathcal{X} = \mathbb{C}$ is the complex plane, but often one requires $P$ to be confined in a region, such as $(0,1)^2$);
- a labeling with values in an abelian group $G$ (or more generally in a partial abelian monoid $M$) and defined on the points of $P$, for configuration spaces with summable labels (here $\mathcal{X}$ can be any topological space).

Neither of these two classical notions is more general than the other; in particular there are two aspects under which the two notions can be compared:

- a commutativity property is required for the labels of a configuration space with summable labels (i.e. the group or partial monoid must be abelian); differently, classical Hurwitz spaces can take monodromies in a non-abelian group (or, more generally, a non-abelian quandle or even in a rack);
- collisions between points of a configuration are not allowed in the classical setting of Hurwitz spaces; differently, collisions between points of a configuration with summable labels is allowed in certain circumstances: if the labels are summable, we replace the points with a single point, whose label is the sum of the old labels.

The “intersection” of these two classical notions only contains configuration spaces of points in $\mathbb{C}$ with labels in a set $S$: these spaces are the topic of the classical theory of (coloured) braids, and they are well understood.
1.4. A brief history of Hurwitz spaces. The notion of Hurwitz spaces goes back to Clebsch [Cle72] and Hurwitz [Hur91]. For a fixed $k \geq 0$ and a discrete group $G$, the classical Hurwitz space $\text{hur}_k(G)$ contains configurations of the form $(P, \varphi)$, where

- $P = \{z_1, \ldots, z_k\}$ is a collection of $k$ distinct points in $\mathbb{C}$;
- $\varphi : \pi_1(\mathbb{C} \setminus P) \to G$ is a group homomorphism.

A homotopy theoretic characterisation, which can be found in [EVW16 Subsection 1.3] and [RW19 Section 4], is the following. There is a natural action of the braid group $\mathcal{B}r_k$ on the set $G^k$: the standard generator $b_i$, for $1 \leq i \leq k - 1$, sends the $k$-tuple $(g_1, \ldots, g_k)$ to the $k$-tuple

$$(g_1, \ldots, g_{i-1}, g_{i+1}, g_i^{-1} g_{i+1} g_i + 1, g_{i+2}, \ldots, g_k);$$

the homotopy type of the classical Hurwitz space $\text{hur}_k(G)$ is then that of the homotopy quotient $G^k//\mathcal{B}r_k$. This second definition has the advantage of admitting a straightforward extension to the case in which the group $G$ is replaced by a rack. Recall that a rack is a set $\mathcal{R}$ with a binary operation $\mathcal{R} \times \mathcal{R} \to \mathcal{R}$, $(a, b) \mapsto a^b$, satisfying the relation $(a^b)^c = (a^c)^b$ for all $a, b, c \in \mathcal{R}$; the action of $b_i \in \mathcal{B}r_k$ on $\mathcal{R}^k$ sends the $k$-tuple $(g_1, \ldots, g_k)$ to $(g_1, \ldots, g_{i-1}, g_{i+1}, g_i^{-1} g_{i+1} g_i + 1, g_{i+2}, \ldots, g_k)$. The most familiar example of rack is a group $G$, and the reader will note that, according to this second description, the homotopy type of $\text{hur}_k(G)$ only depends on the underlying structure of rack that every group $G$ has; in particular, $\text{hur}_k(G)$ remembers very little about the multiplication of $G$.

The notion of quandle is slightly more restrictive than the notion of rack: a rack $\mathcal{R}$ is a quandle if $a^a = a$ for all $a \in \mathcal{R}$. See Definition 2.1 for more details, [FR92] for a classical account on the history of these notions, and the very recent preprints [DRS21] and [LS21] for an updated account.

Classical Hurwitz spaces have been shown to admit a structure of algebraic variety, and have been employed to study the geometry of the moduli space of curves $\mathcal{M}_g$ in different characteristics (see [RW06] for an account on the history of applications of Hurwitz spaces in algebraic geometry). More recently, Hurwitz spaces have been employed as a topological tool to obtain results about the Cohen-Lestrade heuristics and Malle’s conjecture over function fields [EVW16, ETW17, RW19].

Finally, a version of Hurwitz spaces with coefficients in a space has been introduced in [EVW12], and further investigated in [PT21] and [PT20].

1.5. A brief history of configuration spaces with summable labels. Another classical notion is that of configuration space with summable labels $C(\mathcal{X}; M)$, depending on a topological space $\mathcal{X}$ and on a partial abelian monoid $M$. This notion was originally considered by Dold and Thom [DT56] in the special case in which $M$ is an abelian group, under the familiar name of symmetric product; later McCord [McC69] considered the case of an abelian monoid, Kallel [Kal01] the case of a partial abelian monoid, and Salvatore [Sal01] the case of a partial (framed) $E_n$-algebra, in the assumption that $\mathcal{X}$ is a framed manifold (a manifold). Similar, classical constructions occur in [Seg73] and [McD73].

The peculiarity of the notion of configuration space with summable labels is that collisions between points in a configuration are allowed under controlled circumstances: the simplified rule is that whenever two or more points collide, they produce a single new point whose label is the sum (in $M$) of the old labels. The Dold-Thom theorem, expressing the reduced homology groups of a based, connected
space $\mathcal{X}$ as the homotopy groups of the relative version $\text{SP}(\mathcal{X},*;\mathbb{Z})$ of the infinite symmetric product, is perhaps the most famous application of configuration spaces with summable labels.

The case of configuration spaces with labels in a space (without a partial abelian structure) is also classical [May72, Sna74, Böd87].

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2. Algebraic properties of partially multiplicative quandles

We usually denote by $G$ a discrete group, with neutral element $1 = 1_G$.

2.1. Basic definitions.

**Definition 2.1.** A quandle (with unit) is a set $Q$ with a marked element $1 \in Q$, called unit, and a binary operation $Q \times Q \rightarrow Q$, denoted $(a, b) \rightarrow a^b$ and called conjugation, such that:

1. for all $a \in Q$ the map $(-)^a : Q \rightarrow Q$ is bijective;
2. for all $a \in Q$ we have $1^a = 1$ and $a^a = a$;
3. for all $a \in Q$ we have $a^a = a$;
4. for all $a, b, c \in Q$ we have $(a^b)^c = (a^c)^{(b^c)}$.

We denote by $(-)^{-1} : Q \rightarrow Q$ the inverse map of $(-)^a : Q \rightarrow Q$.

The conjugacy class of an element $a \in Q$, denoted by $\text{conj}(a)$, is the smallest subset $S \subset Q$ which contains $a$ and is closed under the operations $(-)^b$ and $(-)^{-1}$ for all $b \in Q$. We denote by $\text{conj}(Q)$ the set of conjugacy classes of $Q$. A quandle $Q$ is abelian, if $(-)^c$ is the identity of $Q$ for all $c \in Q$.

A morphism of quandles is a morphism of the underlying sets that preserves unit and conjugation. Quandles form a category $\text{Qnd}$.

Note that for all $c$ in a quandle $Q$, the map $(-)^c : Q \rightarrow Q$ is automatically an automorphism of $Q$ as a quandle.

The usual definition of the word “quandle” in the literature differs from Definition 2.1 in that no unit $1 \in Q$ is required, and condition (2) is dropped. Note however that if $Q$ is a quandle without unit, then the set $Q \cup \{1\}$ can be given a unique structure of quandle, such that $1$ is the unit and the inclusion $Q \subset Q \cup \{1\}$ preserves conjugation. Throughout the article we will use the word “quandle” in the sense to Definition 2.1.

If instead we drop condition (3) from Definition 2.1 we obtain the classical definition of rack (with unit). In the Appendix A we will briefly discuss the possibility to extend the results of this article to the generality of racks, and we will describe what difficulties arise. Since condition (3) holds in all applications we have in mind, in this article and the following ones we will focus on quandles.

**Example 2.2.** Let $G$ be a group with unit $1$, and let $1 \in Q \subset G$ be a conjugation invariant subset. Then $Q$ is a quandle with the conjugation given by $a^b := b^{-1}ab$, for all $a, b \in Q$. 
Note that an abelian quandle only contains the information of its underlying pointed set: more precisely, there is a fully faithful functor $\text{Set}_* \to \text{Qnd}$ with essential image given by abelian quandles.

**Definition 2.3.** A partial monoid $M$ is a set $M$ with a marked element $\mathbb{1} \in M$, called unit, a subset $D \subseteq M \times M$ and a map $D \to M$, denoted $(a, b) \mapsto ab$ and called partial product. We say that the product $ab$ is defined if $(a, b) \in D$. The following properties must hold:

1. for all $a \in M$ both $\mathbb{1}a$ and $a\mathbb{1}$ are defined and equal to $a$;
2. for all $a, b, c \in M$, the following two conditions are equivalent:
   - $ab$ is defined and $(ab)c$ is defined;
   - $bc$ is defined and $a(bc)$ is defined;
   moreover if both $(ab)c$ and $a(bc)$ are defined, then they are equal.

A partial monoid $M$ is abelian if for all $a, b \in M$ either of the following holds:

- both products $ab$ and $ba$ are not defined;
- both products $ab$ and $ba$ are defined, and $ab = ba$.

A partial monoid $M$ has a trivial product if for all $a, b \in M$ the product $ab$ is defined if and only if either $a$ or $b$ is equal to $\mathbb{1}$.

A morphism of partial monoids $M \to M'$ is a map of the underlying sets such that $\mathbb{1}_M \mapsto \mathbb{1}_{M'}$, and the following holds: whenever $a \mapsto a'$, $b \mapsto b'$ and $ab$ is defined in $M$, then $a'b'$ is defined in $M'$ and $(ab) \mapsto (a'b')$. Partial monoids form a category $\text{PMon}$.

We can amalgamate Definitions 2.1 and 2.3 into the following one.

**Definition 2.4.** A partially multiplicative quandle (PMQ) is a set $Q$ with a marked element $\mathbb{1} \in Q$, called unit, such that $Q$ is both a quandle and a partial monoid, the unit is $\mathbb{1}$ in both cases and for all $a, b, c \in Q$ the following equalities hold:

- $ab$ is defined if and only if $b(ab)$ is defined, and whenever both $ab$ and $b(ab)$ are defined we have $ab = b(ab)$; we usually write $ba^b$ for $b(ab)$;
- $a^{(bc)} = (a^b)^c$, whenever the product $bc$ is defined;
- $ab$ is defined if and only if $(a^c)(b^c)$ is defined, and whenever both $ab$ and $(a^c)(b^c)$ are defined we have $(ab)^c = (a^c)(b^c)$.

A PMQ $Q$ is abelian if the underlying quandle is abelian: the first equality implies immediately that also the underlying partial monoid of $Q$ is abelian. A PMQ has a trivial product if the underlying partial monoid has a trivial product.

A morphism of PMQs is a map of sets that is both a morphism of quandles and of partial monoids; the category of PMQs is denoted $\text{PMQ}$.

The following are simple examples of PMQs.

**Example 2.5.** Let $G$ be a group; then $G$ is a PMQ by setting $a^b = b^{-1}ab$ for all $a, b \in G$ and by using the usual product, defined on the entire $G \times G$; the unit is $\mathbb{1} \in G$. This construction defines a forgetful functor $\text{Grp} \to \text{PMQ}$ from the category of groups to the category of PMQs.

**Example 2.6.** Let $Q$ be a quandle; then $Q$ is a PMQ with trivial product. This construction defines a functor $\text{Qnd} \to \text{PMQ}$, which is the left adjoint to the forgetful functor $\text{PMQ} \to \text{Qnd}$ forgetting the partial product; in particular this construction makes every pointed set $S$ into an abelian PMQ with trivial product.
Example 2.7. Let $M$ be a partial abelian monoid; then $M$ is an abelian PMQ by setting $a^b = a$ for all $a, b \in M$: this construction gives an equivalence between the category of abelian PMQs and the category of partial abelian monoids.

The following definition gives a method to obtain PMQs as subsets of groups.

**Definition 2.8.** Let $G$ be a group and let $1 \in S \subseteq G$ be a conjugation invariant subset of $G$ satisfying the following property: for all $r \geq 3$, if $a_1, \ldots, a_r$ are elements in $S$ and the product $a_1 \ldots a_r$ lies in $S$, then also the products $a_i a_{i+1}$ lie in $S$, for all $1 \leq i \leq r - 1$. Then $S$ inherits from $G$ a structure of PMQ as follows:

- the unit is $1$;
- the conjugation is defined as in $G$;
- given two elements $a, b \in S$, if their product $ab \in G$ lies in $S$, then we declare $ab$ to be also their product in $S$ as PMQ; otherwise the product $ab$ is not defined.

Note that if $S$ satisfies the hypotheses of Definition 2.8 and $a_1 \ldots a_r$ is a product of elements of $S$ lying in $S \subseteq G$, then for all $1 \leq i < j \leq r$ the product $a_i \ldots a_j$ also lies in $S$. Note also that if $S \subseteq G$ inherits from $G$ the structure of PMQ as in the previous definition, then the inclusion $S \hookrightarrow G$ is a map of PMQs.

2.2. Enveloping group of a PMQ. Conversely as in Example 2.5 we can construct a group from a PMQ as follows.

**Definition 2.9.** Let $Q$ be a PMQ. We define its enveloping group $G(Q)$ as the group with the following presentation:

- **Generators** For all $a \in Q$ there is a generator $[a]$.
- **Relations**
  - $[b]^{-1}[a][b] = [a^b]$ for all $a, b \in Q$.
  - $[a][b] = [ab]$ for all $a, b \in Q$ such that $ab$ is defined in $Q$.

The assignment $Q \mapsto G(Q)$ gives the left adjoint of the forgetful functor $\text{Grp} \to \text{PMQ}$ from Example 2.5. We denote by $\eta = \eta_Q : Q \rightarrow G(Q)$ the unit of the adjunction: it is the map of PMQs defined by $a \mapsto [a]$ for all $a \in Q$.

In general $\eta$ is not injective, as we see in the following (compare also with [Joy82 Section 6]).

**Definition 2.10.** Let $G$ be a group acting on right on a set $S$. We define a PMQ $G \ltimes S$. The underlying set is $G \sqcup S$; the neutral element is $1 \in G$; the quandle structure is given as follows:

- $a^s = a$ for all $a \in G \ltimes S$ and $s \in S$;
- $h^g = g^{-1}hg$ for all $g, h \in G$;
- $s^g = s \cdot g$ for all $g \in G$ and $s \in S$.

The partial product is defined only on couples $(a, b)$ of elements in $G$, and it coincides with the group product.

Note that for $g \in G$ and $s \in S$ the following equalities hold in $G(G \ltimes S)$:

- $[g]^{-1}[s][g] = [s \cdot g]$
- $[s]^{-1}[g][s] = [g]$.

Putting them together one obtains the equality $[s] = [s \cdot g] \in G(G \ltimes S)$. Hence the map $\eta : G \ltimes S \rightarrow G(G \ltimes S)$ identifies the elements $s$ and $s \cdot g$ of $S$, and is not injective unless $G$ acts trivially on $S$. 
2.3. Adjoint action and PMQ-group pairs.

Notation 2.11. For a PMQ \( Q \) we denote by \( \text{Aut}_{\text{PMQ}}(Q) \) the group of automorphisms of \( Q \) as a PMQ; we use the classical convention that automorphisms, as functions in general, act on left. We denote by \( \text{Aut}_{\text{PMQ}}(Q)^{\text{op}} \) the opposite group, whose elements are the same functions \( Q \to Q \) (and can thus be evaluated on elements of \( Q \)), but whose composition is reversed.

By definition of PMQ there is a map of PMQs \( Q \to \text{Aut}_{\text{PMQ}}(Q)^{\text{op}} \) given by \( a \mapsto (-)^a \). This map gives rise to a homomorphism of groups \( \rho: \mathcal{G}(Q) \to \text{Aut}_{\text{PMQ}}(Q)^{\text{op}} \), i.e., to a right action of \( \mathcal{G}(Q) \) on \( Q \): we call this the adjoint action.

Note that the map \( \eta: Q \to \mathcal{G}(Q) \) (see Definition 2.11) is \( \mathcal{G}(Q) \)-equivariant if we consider the right action of \( \mathcal{G}(Q) \) on itself by conjugation.

Notation 2.12. We denote by \( \mathcal{K}(Q) \subseteq \mathcal{G}(Q) \) the kernel \( \ker(\rho) \).

Lemma 2.13. The subgroup \( \mathcal{K}(Q) \) is contained in the centre of \( \mathcal{G}(Q) \).

Proof. Let \( g \in \mathcal{G}(Q) \) be an element with \( \rho(g) = \text{Id}_Q \). Then conjugation by \( g \) fixes the image of \( \eta: Q \to \mathcal{G}(Q) \), which contains all generators of \( G \); hence conjugation by \( g \) fixes \( \mathcal{G}(Q) \), i.e., \( g \) is central in \( \mathcal{G}(Q) \).

In general equality does not hold, as we see in the following example. Let \( G \) be a group acting on a set \( S \). Using Definitions 2.9 and 2.10 it is immediate to see that \( \mathcal{G}(G \ltimes S) \) is isomorphic to \( G \times \mathbb{Z} \langle S/G \rangle \), i.e., the direct product of \( G \) and the free abelian group on the orbits of the action of \( G \) on \( S \). If we assume that \( G \) has non-trivial centre \( Z(G) \) and acts faithfully on a set \( S \), then \( \mathcal{K}(G \ltimes S) \) is isomorphic to \( \bigoplus_{S/G} \mathbb{Z} \), the free abelian group on the set \( S/G \), whereas the centre \( Z(\mathcal{G}(G \ltimes S)) \) is the strictly larger group \( Z(G) \oplus \bigoplus_{S/G} \mathbb{Z} \).

Consider now a finite PMQ \( Q \); then the group \( \text{Aut}_{\text{PMQ}}(Q) \) is also finite, and therefore we have a central extension of groups with finite cokernel

\[
1 \longrightarrow \mathcal{K}(Q) \longrightarrow \mathcal{G}(Q) \longrightarrow \text{Im}(\rho) \longrightarrow 1.
\]

In particular the quotient \( \mathcal{G}(Q)/\mathcal{K}(Q) \) is a finite group in this case.

Since \( \mathcal{G} \) is a functor, it transforms automorphisms of PMQs into automorphisms of groups, hence \( \mathcal{G} \) gives a map of groups \( \text{Aut}_{\text{PMQ}}(Q) \to \text{Aut}_{\text{Grp}}(\mathcal{G}(Q)) \). We obtain the following lemma.

Lemma 2.14. There is a natural sequence of maps of PMQs:

\[
Q \xrightarrow{\eta} \mathcal{G}(Q) \xrightarrow{\rho} \text{Aut}_{\text{PMQ}}(Q)^{\text{op}} \xrightarrow{\mathcal{G}} \text{Aut}_{\text{Grp}}(\mathcal{G}(Q))^{\text{op}}.
\]

The following definition generalises the situation of Lemma 2.14. Compare also with [Joy82, Definition 9.1].

Definition 2.15. A PMQ-group pair consists of a PMQ \( Q \), a group \( G \), a map of PMQs \( \epsilon: Q \to G \) and a right action \( \tau: G \to \text{Aut}_{\text{PMQ}}(Q)^{\text{op}} \) of \( G \) on \( Q \), such that the composition \( \tau \circ \epsilon: Q \to \text{Aut}_{\text{PMQ}}(Q)^{\text{op}} \) is equal to the map \( \rho \circ \eta \), and such that the map \( \epsilon \) is \( G \)-equivariant if \( G \) acts on \( Q \) by \( \tau \) and on itself by right group conjugation.

We usually denote by \((Q, G, \epsilon, \tau)\) a PMQ-group pair, or just by \((Q, G)\), leaving the maps \( \epsilon \) and \( \tau \) implicit. A map of PMQ-group pairs \((Q, G, \epsilon, \tau) \to (Q', G', \epsilon', \tau')\)
is given by a couple \((\Psi, \Phi)\), where \(\Psi: Q \to Q'\) is a map of PMQs and \(\Phi: G \to G'\) is a map of groups, such that the following diagram of PMQs commutes:

\[
\begin{array}{ccc}
Q & \xrightarrow{\epsilon} & G \\
\downarrow \Psi & & \downarrow \Phi \\
Q' & \xrightarrow{\epsilon'} & G',
\end{array}
\]

and such that for all \(g \in G\) the following diagram of PMQs commutes:

\[
\begin{array}{ccc}
Q & \xrightarrow{r(g)} & Q \\
\downarrow \Psi & & \downarrow \Psi \\
Q' & \xrightarrow{r'(\Phi(g))} & Q'.
\end{array}
\]

We obtain a category \(\text{PMQGrp}\) of PMQ-group pairs.

For instance Lemma 2.14 implies that \((Q, G(Q), \eta, \rho)\) is a PMQ-group pair for all PMQ \(Q\); since \(\rho\) factors through the quotient by \(K(Q)\) (see Notation 2.12), we also have that \((Q, G(Q)/K(Q))\) is naturally a PMQ-group pair. Moreover, if \(S\) and \(G\) are as in Definition 2.8, then \((S, G)\) is a PMQ-group pair in a natural way.

**Definition 2.16.** Let \((Q, G, \epsilon, r)\) and \((Q', G', \epsilon', r')\) be PMQ-group pairs. We define the product \((Q, G) \times (Q', G')\) as the PMQ-group pair \((Q \times Q', G \times G', \epsilon \times \epsilon', r \times r')\), where:

- the conjugation on \(Q \times Q'\) is defined component-wise, and the product \((a, a')(b, b')\) is defined if and only if the products \(ab \in Q\) and \(a'b' \in Q'\) are defined: in this case \((a, a')(b, b') = (ab, a'b')\);
- \(G \times G'\) is given the product group structure, and \(\epsilon \times \epsilon': (a, a') \mapsto (\epsilon(a), \epsilon'(a'))\);
- for \((g, g') \in G \times G'\) we set \(r \times r': (g, g') \mapsto (\epsilon(g)(a), \epsilon'(g')(a'))\).

The product \((Q, G) \times (Q', G')\) is the categorical product of \((Q, G)\) and \((Q', G')\) in \(\text{PMQGrp}\), and its projections are denoted \(p_{(Q, G)}: (Q, G) \times (Q', G') \to (Q, G)\) and \(p_{(Q', G')}: (Q, G) \times (Q', G') \to (Q', G')\).

**Notation 2.17.** We will denote by \((1, 1)\) the unique PMQ-group pair \((Q, G)\) for which both \(Q\) and \(G\) consist of the only element \(1\).

2.4. Complete PMQs.

**Definition 2.18.** A PMQ \(Q\) is **complete** if the product is defined for all couples of elements. A PMQ-group pair \((Q, G)\) is complete if \(Q\) is complete.

Complete PMQs are also called **multiplicative quandles** and form a full subcategory \(\text{MQ} \subset \text{PMQ}\).

**Definition 2.19.** The inclusion functor \(\text{MQ} \to \text{PMQ}\) admits a left adjoint: given a PMQ \(Q\), we construct its **completion** \(\hat{Q}\) as follows:

- as a monoid, \(\hat{Q}\) is freely generated by elements \(\hat{a}\) for \(a \in Q\), with generating relations given by \(\hat{1} = 1\), \(\hat{a}\hat{b} = \hat{ab}\) whenever the product \(ab\) is defined in \(Q\), and \(\hat{a}\hat{b} = \hat{b}\hat{a}\) for all \(a, b \in Q\);
- there is a natural map of partial monoids \(Q \to \hat{Q}\) given by \(a \mapsto \hat{a}\); the map of partial monoids \(Q \to \text{Aut}_{\text{PMQ}}(Q)^{op}\) given by conjugation extends to a map of monoids \(\hat{Q} \to \text{Aut}_{\text{PMQ}}(Q)^{op}\); we can compose the latter with the natural maps \(\text{Aut}_{\text{PMQ}}(Q)^{op} \hookrightarrow \text{Aut}_{\text{PMon}}(Q)^{op} \to \text{Aut}_{\text{Mon}}(\hat{Q})^{op} \hookrightarrow \text{Aut}_{\text{Set}}(\hat{Q})^{op}\).
• the adjoint $\hat{\mathcal{Q}} \times \hat{\mathcal{Q}} \to \hat{\mathcal{Q}}$ of the map $\hat{\mathcal{Q}} \to \text{Aut}_{\text{Set}}(\hat{\mathcal{Q}})^{\text{op}}$ makes $\hat{\mathcal{Q}}$ into a quandle; all axioms of PMQ are satisfied.

Our next aim is to prove that the map of PMQs $\mathcal{Q} \to \hat{\mathcal{Q}}$ is injective, and the subset $\mathcal{Q} \setminus \mathcal{Q} \subset \mathcal{Q}$ is an ideal, in the following sense.

**Definition 2.20.** Let $\mathcal{Q}$ be a PMQ and let $I \subset \mathcal{Q}$ be a subset. We say that $I$ is an ideal if the following hold:

- $I$ is conjugation invariant, i.e. for all $a \in I$ and $b \in \mathcal{Q}$ we have $a^b, a^{b^{-1}} \in I$;
- $I$ absorbs products when they are defined, i.e. for all $a \in I$ and $b \in \mathcal{Q}$, if $ab$ is defined then it lies in $I$, and if $ba$ is defined then it lies in $I$.

Note that if $\Psi: \mathcal{Q} \to \mathcal{Q'}$ is a map of PMQs and $I' \subset \mathcal{Q'}$ is an ideal, then also $I := \Psi^{-1}(I') \subset \mathcal{Q}$ is an ideal.

**Definition 2.21.** Let $(Q, G, \epsilon, \tau)$ be a PMQ-group pair: we define a new PMQ-group pair $(Q \ltimes G, G, \epsilon, \tau)$. As a set, we put $Q \ltimes G = G \sqcup G$; for $a \in Q$ and $g \in G$ we let $a^g$ be the corresponding elements in $Q \ltimes G$.

To define the conjugation of the PMQ $Q \ltimes G$, for $a, b \in Q$ and $g, h \in G$ we set $a^g b = a^g b^g$, $a^g = \tau(g)(a)$, $g^a = \tau(a^{-1}) g \epsilon(a)$ and $g^h = h^{-1} g h$.

The product of the PMQ $Q \ltimes G$ is defined for all couples of elements: for $a, b \in Q$ and $g, h \in G$ we set $\bar{g} h = gh$, $a \bar{g} = \bar{\epsilon}(a) g$, $\bar{g} a = g \epsilon(a)$; moreover, if $ab$ is defined in $Q$ we set $\bar{a} \bar{b} = \bar{a} \bar{b}$, otherwise we set $\bar{a} \bar{b} = \bar{\epsilon}(a) \bar{\epsilon}(b)$. The unit of $Q \ltimes G$ is $1 \in Q$.

For $a \in Q$ and $g, h \in G$, the map $\bar{\epsilon}: Q \ltimes G \to G$ is given by $\bar{\epsilon}(a) = \epsilon(a)$ and $\bar{\epsilon}(g) = g$; the automorphism $\bar{\epsilon}(g) \in \text{Aut}_{\text{PMQ}}(Q \ltimes G)^{\text{op}}$ sends $\bar{a} \mapsto \bar{\epsilon}(g)(a)$ and $\bar{h} \mapsto g^{-1} h g$.

Let now $\mathcal{Q}$ be any PMQ and fix a PMQ-group pair of the form $(Q, G)$, e.g. $(Q, \mathcal{G}(Q))$. Note that the natural inclusion $Q \subset Q \ltimes G$ is a map of PMQs, extending with the identity of $G$ to a map of PMQ-group pairs. Note also that $Q \ltimes G$ is a complete PMQ; therefore the inclusion $Q \subset Q \ltimes G$ induces a map of complete PMQs $\Psi: \hat{\mathcal{Q}} \to \hat{\mathcal{Q}} \ltimes G$. Since the composition $\mathcal{Q} \to \hat{\mathcal{Q}} \xrightarrow{\Psi} \mathcal{Q} \ltimes G$ is injective, we deduce that $\mathcal{Q} \to \hat{\mathcal{Q}}$ is also injective, so we can regard $\mathcal{Q}$ as a subset of $\hat{\mathcal{Q}}$.

Moreover $G \subset Q \ltimes G$ is an ideal, hence $\Psi^{-1}(G) \subset \hat{\mathcal{Q}}$ is also an ideal. We claim that $\Psi^{-1}(G) = Q \setminus \mathcal{Q}$. Since $\Psi(Q) = \mathcal{Q}$, we have the inclusion $\Psi^{-1}(G) \subset \hat{\mathcal{Q}} \setminus \mathcal{Q}$. On the other hand an element $w \in \Psi^{-1}(G) = \hat{\mathcal{Q}} \setminus \mathcal{Q}$ can be represented by a word $\hat{a}_1 \ldots \hat{a}_r$ such that the product $a_1 \ldots a_r$ is not defined in $\mathcal{Q}$ (otherwise $w$ would lie in $Q \subset \hat{\mathcal{Q}}$). By definition of the product on $Q \ltimes G$ we have $\Psi(w) \in G$. We state the previous result as a proposition.

**Proposition 2.22.** Let $\mathcal{Q}$ be a PMQ. Then the natural map of PMQs $\mathcal{Q} \to \hat{\mathcal{Q}}$ is injective, and $\mathcal{Q} \setminus \mathcal{Q}$ is an ideal of $\hat{\mathcal{Q}}$.

To conclude the subsection, note that if $\mathcal{Q}$ is already a complete PMQ, then the natural map of PMQs $\mathcal{Q} \to \hat{\mathcal{Q}}$ is an isomorphism (both in PMQ and in MQ). In particular every complete PMQ is in the essential image of the completion functor $\text{PMQ} \to \text{MQ}$, and thus, when we want to consider a complete PMQ, we can assume that it takes the form $\mathcal{Q}$ for some PMQ $\mathcal{Q}$. For this reason we shall often abuse notation and denote by $\hat{\mathcal{Q}}$ a generic multiplicative quandle, even if no “underlying” PMQ $\mathcal{Q}$ is specified, whose completion is $\hat{\mathcal{Q}}$. In fancier language, one can
say that \( \text{MQ} \) is a full subcategory of \( \text{PMQ} \) and the completion functor is a (left) localisation functor.

**Notation 2.23.** For a PMQ \( \mathcal{Q} \) we denote by \( \mathcal{J}(\mathcal{Q}) \) the ideal \( \hat{\mathcal{Q}} \setminus \mathcal{Q} \) of \( \hat{\mathcal{Q}} \). Since the natural map \( \mathcal{Q} \to \hat{\mathcal{Q}} \) is injective, we will abuse notation and denote by \( \hat{a} \) corresponding to \( a \in \mathcal{Q} \) the element \( \hat{a} \) in \( \hat{\mathcal{Q}} \).

3. **Free groups and associated PMQs**

In this section we study certain PMQs arising as subsets of free groups.

3.1. **Free sub-PMQs.** We fix natural numbers \( 0 \leq l \leq k \) throughout the section.

**Notation 3.1.** We denote by \( \mathbb{F}^k \) the free group on \( k \) generators \( f_1, \ldots, f_k \). The abelianisation of \( \mathbb{F}^k \) is identified with \( \mathbb{Z}^k \), generated by the classes of the generators \( f_1, \ldots, f_k \). We denote by \( \text{ab}: \mathbb{F}^k \to \mathbb{Z}^k \) the abelianisation map.

**Definition 3.2.** Let \( 0 \leq l \leq k \). We denote by \( \mathbb{F}^k_l \subset \mathbb{F}^k \) the union of \( \{1\} \) and the conjugacy classes of the generators \( f_1, \ldots, f_l \). The set \( \mathbb{F}^k_l \) inherits from \( \mathbb{F}^k \) the structure of PMQ in the sense of Definition 2.8, and we call it the free sub-PMQ of \( \mathbb{F}^k \) on the first \( l \) generators.

To check that the conditions from Definition 2.8 are satisfied, note that each non-trivial element of the set \( \mathbb{F}^k_l \) is mapped under the map \( \text{ab} \) to a vector in \( \mathbb{Z}^k \) with one entry (among the first \( l \)) equal to 1, and all other entries equal to 0: hence the product in \( \mathbb{F}^k \) of two or more non-trivial elements in \( \mathbb{F}^k_l \) does not lie in \( \mathbb{F}^k_l \), and thus \( \mathbb{F}^k_l \) inherits from \( \mathbb{F}^k \) a structure of trivial PMQ (see Definition 2.4). It follows that \( (\mathbb{F}^k, \mathbb{F}^k_l) \) is a PMQ-group pair (see Definition 2.15).

The following Theorem generalises [Joy82, Theorem 4.1].

**Theorem 3.3.** Let \( (\mathcal{Q}, G, e, \tau) \) be a PMQ-group pair, let \( a_1, \ldots, a_l \in \mathcal{Q} \) and let \( g_1, \ldots, g_k \in G \). Then there are unique maps \( \varphi: \mathbb{F}^k \to G(\mathcal{Q}) \) of groups and \( \psi: \mathbb{F}^k_l \to \mathcal{Q} \) of PMQs such that \( \psi: f_i \mapsto a_i \) for all \( 1 \leq i \leq l \), \( \varphi: f_i \mapsto g_i \) for all \( l + 1 \leq i \leq k \) and \( (\psi, \varphi): (\mathbb{F}^k_l, \mathbb{F}^k) \to (\mathcal{Q}, G) \) is a map of PMQ-group pairs.

Before proving Theorem 3.3 we introduce some notation.

**Notation 3.4.** Let \( w \in \mathbb{F}^k \). The reduced expression of \( w \) as a word in the letters \( f_1^{\pm 1}, \ldots, f_k^{\pm 1} \) takes the form \( w = f_j^{\varepsilon_1} \cdots f_j^{\varepsilon_m} \), where \( \varepsilon_i = \pm 1 \) for all \( 1 \leq i \leq m \), and no two consecutive letters cancel out in \( \mathbb{F}^k \). The number \( m \) is also called word-length norm of \( w \) and denoted by \( \|w\| \).

**Proof of Theorem 3.3** The existence and uniqueness of \( \varphi \) is granted by the fact that \( \mathbb{F}^k \) is a free group on \( f_1, \ldots, f_k \): we set \( \varphi(f_i) = e(a_i) \) for all \( 1 \leq i \leq l \) and \( \varphi(f_i) = g_i \) for \( l + 1 \leq i \leq k \). To show existence of \( \psi \), start by setting \( \psi(1) = 1 \). Let \( g \neq 1 \) be an element in \( \mathbb{F}^k_l \); then there are unique \( w \in \mathbb{F}^k \) and \( 1 \leq \nu \leq l \) such that the following hold:

1. \( g = w^{-1}f_\nu w \) in \( \mathbb{F}^k \);
2. if \( w = f_j^{\varepsilon_1} \cdots f_j^{\varepsilon_m} \) is the reduced expression of \( w \) (see Notation 3.4), then either \( m = 0 \) or \( f_j \neq f_\nu \).

For \( a \in \mathcal{Q} \) and \( g \in G \) denote by \( a^g \in \mathcal{Q} \) the image of \( a \) under the map \( \tau(g): \mathcal{Q} \to \mathcal{Q} \). Using the notation above, we set

\[
\psi(g) = a_\nu^{\varphi(w)}.
\]
This defines a map of sets \( \psi : FQ^k_l \to Q \), with \( \psi(f_i) = a_i \) for all \( 1 \leq i \leq l \).

If we drop condition (2), the choice of \( w \) and \( \nu \) fails to be unique only because of the following ambiguity: one can replace \( w \) by \( f_i^e w \), for some \( e \in \mathbb{Z} \). Note however that \( a_{\nu} = a_{\nu}^e(a_{\nu})^{-1} \) : this follows from the assumption that \( Q \) is a quandle, and not only a rack, hence \( a_{\nu} = a_{\nu}^{e\nu} = a_{\nu}^{e\nu^{-1}} \) (see Definition 2.1). Therefore \( \psi \) is well-defined by the formula given above even if we drop condition (2) in the choice of \( w \) and \( \nu \). By construction \( \psi \) is a map of PMQs and \((\psi, \varphi)\) is a map of PMQ-group pairs.

Conversely, let \( \psi' : FQ^k_l \to Q \) be a map of PMQs such that \((\psi', \varphi)\) is a map of PMQ-group pairs, and such that \( \psi' : f_i \mapsto q_i \) for all \( 1 \leq i \leq l \). Then \( \psi' \) must satisfy the formula above for any given \( g \in FQ^k_l \), and hence \( \psi' = \psi \) : this shows uniqueness of \( \psi \).  

In the proof of Theorem 3.3 we see for the first time why it is convenient to work with quandles instead of racks, see the discussion after Definition 2.1 Theorem 3.3 motivates the use of the word “free” in Definition 3.2.

3.2. Decompositions of elements in free groups. In the rest of the section we study the problem of decomposing elements \( g \in F^k \) as products of elements in \( FQ^k_l \) in different ways. The rather technical Proposition 3.7 ensures that if \( g \) has a particularly nice form, then there is essentially only one such decomposition.

**Definition 3.5.** Let \( g \in F^k \); a decomposition of \( g \) with respect to \( FQ^k_l \) is a sequence \( g = (g_1, \ldots, g_r) \) of elements in \( FQ^k_l \) such that the product \( g_1 \cdots g_r \) is equal to \( g \).

In general, if an element \( g \in F^k \) admits a decomposition with respect to \( FQ^k_l \), this decomposition is not unique: for example, if \( g \) can be decomposed as \( g_1 \cdot g_2 \), then it can also be decomposed as \( g_2 \cdot g_1^e \) or \( g_2^e \cdot g_1 \).

However we note that the number \( r \) of factors appearing in any decomposition \( g \) of \( g \) with respect to \( FQ^k_l \) is the same for any decomposition. To see this, consider again the map \( ab \) from Notation 2.1 then \( ab(g) = ab(g_1) + \cdots + ab(g_r) \), and each summand \( ab(g_i) \) is a vector with one entry equal to 1 and all other entries equal to 0; hence \( r \) depends only on \( g \) and is equal to the sum of the entries in \( ab(g) \).

We introduce the notion of standard moves, which allows us to modify a sequence of elements in a group, or more generally in a quandle.

**Definition 3.6.** Let \( Q \) be a quandle; a standard move on a sequence of elements \( (a_1, \ldots, a_r) \) replaces, for some \( 1 \leq i \leq r - 1 \), the couple of consecutive elements \((a_i, a_{i+1})\) with either \((a_{i+1}, a_i^{a_{i+1}})\) or \((a_{i+1}, a_i^{a_{i+1}^{-1}})\).

If one applies, after one other, two standard moves on the same couple of indices \((i, i + 1)\), using once each of the two different rules \((a_i, a_{i+1}) \mapsto (a_{i+1}, a_i^{a_{i+1}})\) and \((a_i, a_{i+1}) \mapsto (a_{i+1}^{-1}, a_i)\), one recovers the original decomposition: in this sense the two rules are inverse of each other.

In the case \( Q = F^k \), the reader will notice the connection between standard moves and Artin’s action of the braid group \( B_t_n \) on the free group \( F^n \): for \( 1 \leq i \leq n - 1 \), the standard generator \( b_i \in B_t_n \) acts on \( F^n \) by mapping the list of generators \((f_1, \ldots, f_k)\) to the list of generators \((f_1, \ldots, f_{i-1}, f_{i+1}, f_i^{f_{i+1}}, f_{i+2}, \ldots, f_k)\), i.e., by applying a standard move.
Proposition 3.7. Let $g = f_1 \ldots f_r$ for some $1 \leq r \leq l$, and let $(g_1, \ldots, g_r)$ be a decomposition of $g$ with respect to $FQ_k$. Then it is possible to pass from the decomposition $(g_1, \ldots, g_r)$ to the decomposition $(f_1, \ldots, f_r)$ by applying a suitable sequence of standard moves.

We will prove Proposition 3.7 in the rest of the section.

3.3. Generalised decompositions and straightforward computations.

Definition 3.8. A generalised decomposition (gd) in $F^k$ is a formal, structured iteration of the operations of conjugation by an element $f_i^{\pm 1}$ and of product, using only the letters $f_1, \ldots, f_k$ as elementary inputs and taking the associativity of the product into account. More precisely, the set of all gds is constructed recursively by the following:

- for all $1 \leq i \leq k$ we have a gd $f_i$;
- if $x$ and $y$ are gds, then also $x \cdot y$ is a gd;
- if $x$ is a gd, then for all $1 \leq i \leq k$ both $(x)^{f_i}$ and $(x)^{f_i^{-1}}$ are gds.

Associativity of the product is formally taken into account, i.e. for any three gds $x_1, x_2, x_3$ the two gds $x_1 \cdot (x_2 \cdot x_3)$ and $(x_1 \cdot x_2) \cdot x_3$ are equivalent, and we write them without parentheses as $x_1 \cdot x_2 \cdot x_3$.

The weight of a gd $x$, denoted by $|x|$, is defined recursively by

- $|f_i| = 1$ for all $1 \leq i \leq k$;
- if $x$ and $y$ are gds, then $|x \cdot y| = |x| + |y|$;
- if $x$ is a gd and $1 \leq i \leq k$, then $|(x)^{f_i}| = |(x)^{f_i^{-1}}| = |x| + 2$.

Definition 3.9. Each gd $x$ gives rise to an element $\varpi \in F^k$ by straightforward computation, i.e. by interpreting product and conjugation inside $F^k$. We first define recursively a formal straightforward computation associating with every gd $x$ a word in the letters $f_1^{\pm 1}, \ldots, f_k^{\pm 1}$:

- the formal straightforward computation of the gd $x$ is one-letter word ($f_i$);
- let $x$ be a gd and suppose that the formal straightforward computation of $x$ is $(f_{\nu_1}^{\pm 1}, \ldots, f_{\nu_\lambda}^{\pm 1})$; then the formal straightforward computation of $(x)^{f_i}$ is
  
  $$(f_i^{-1}, f_{\nu_1}^{\pm 1}, \ldots, f_{\nu_\lambda}^{\pm 1}, f_i)$$

and the formal straightforward computation of $(x)^{f_i^{-1}}$ is

$$(f_i, f_{\nu_1}^{\pm 1}, \ldots, f_{\nu_\lambda}^{\pm 1}, f_i^{-1});$$

- let $x$ and $y$ be gds, then the formal straightforward computation of $x \cdot y$ is the concatenation of the formal straightforward computations of $x$ and of $y$.

If the formal straightforward computation of a gd $x$ is $(f_{\nu_1}^{\pm 1}, \ldots, f_{\nu_\lambda}^{\pm 1})$, we set

$$\varpi = f_{\nu_1}^{\pm 1} \cdots f_{\nu_\lambda}^{\pm 1} \in F^k.$$  

We say that the straightforward computation of the gd $x$ involves no cancellation if, using the notation above, no cancellation between two consecutive occurrences of $f_i^{\pm 1}$ occurs in the product $f_{\nu_1}^{\pm 1} \cdots f_{\nu_\lambda}^{\pm 1}$. For an element $g \in F^k$ we say that $x$ is a gd of $g$ if $g = \varpi$.

Example 3.10. For $g = f_1 f_2 f_3 \in F^4$ the following are gds of $g$:

- $f_1 \cdot f_2 \cdot f_3$, having weight 3;
• $f_2 \cdot (f_1)^{f_2} \cdot f_3$, having weight 5;
• $f_3 \cdot (f_2 \cdot (f_1)^{f_2})^{f_3}$, having weight 7;
• $f_3 \cdot (f_2)^{f_3} \cdot ((f_1)^{f_2})^{f_3}$, having weight 9;
• $\left( f_3 \cdot (f_2 \cdot (f_1)^{f_2})^{f_3} \right)^{f_3^{-1}}$, having weight 11.

Note that the weight of a gd is the length of its formal straightforward computation. Clearly for any gd $x$ of $\tilde{g} \in F^k$ we have $|x| \geq |\tilde{g}|$, where $|x|$ is the weight of $x$ and $|\tilde{g}|$ is the word-length norm of $\tilde{g} \in F^k$ (see Notation 3.4). The equality occurs exactly when the straightforward computation of $\tilde{g}$ from $x$ involves no cancellation.

**Lemma 3.11.** Let $x$ be a gd of $\tilde{g} \in F^k$ and suppose that computing $\tilde{g}$ from $x$ by straightforward computation involves some cancellation. Then $x$ contains a sub-gd that has one of the following forms, where $y_1$ and $y_2$ are gds and $1 \leq i \leq k$:

1. $(y_1)^{f_i} \cdot (y_2)^{f_i}$;
2. $(y_1)^{f_i^{-1}} \cdot (y_2)^{f_i^{-1}}$;
3. $f_i \cdot (y_1)^{f_i}$ or $(f_i \cdot y_1)^{f_i}$;
4. $(y_1)^{f_i^{-1}} \cdot f_i$ or $(y_1 \cdot f_i)^{f_i^{-1}}$;
5. $\left((y_1)^{f_i}\right)^{f_i^{-1}}$ or $\left(((y_1)^{f_i^{-1}})^{f_i}\right)$;
6. $(y_1 \cdot (y_2)^{f_i})^{f_i^{-1}}$;
7. $(y_1)^{f_i} \cdot (y_2)^{f_i}$;
8. $(y_1 \cdot (y_2)^{f_i})^{f_i^{-1}}$;
9. $(y_1)^{f_i^{-1}} \cdot (y_2)^{f_i}$;
10. $(y_1 \cdot (y_2)^{f_i})^{f_i^{-1}}$.

**Proof.** We start the straightforward computation of $x$ from the innermost operations, and we continue until the first cancellation occurs.

• Suppose that the first cancellation occurs after a conjugation, taking the form $(-)^{f_i}$ or $(-)^{f_i^{-1}}$; i.e. there is a sub-gd $y$ in $x$ such that the straightforward computation of $y$ gives no cancellation, but the straightforward computation of $(y)^{f_i}$ or $(y)^{f_i^{-1}}$ gives some cancellation. Then we are cancelling one instance of $f_i$ with one instance of $f_i^{-1}$, and one of these two letters is the first or the last letter of the straightforward computation of $y$. Either the gd $y$ is obtained by conjugating once a smaller gd (then we are in case (5)), or $y$ is obtained by multiplying two smaller gds (then we are in one of cases (3),(4),(6),(7),(8) or (9)), or $y$ is $f_i$ (then we are in case (10)).

• Suppose that the first cancellation occurs after a product, i.e. there are sub-gds $y_1$ and $y_2$ in $x$ such that the straightforward computations of $y_1$ and $y_2$ give no cancellation, but the straightforward computation of $y_1 \cdot y_2$ gives some cancellation. Then we are cancelling the last letter of the straightforward computation $y_1$ with the first letter of the straightforward computation of $y_2$; note that, up to reducing the size of $y_1$ and $y_2$ and using the associativity of the product built in Definition 3.8, we can assume that neither $y_1$ nor $y_2$ is itself obtained as a product of two smaller gds. If either $y_1$ or $y_2$ is equal to $f_i$, we are in one of cases (3) or (4); if both $y_1$ and $y_2$ are obtained by conjugating a smaller gd, we are in one of cases (1) or (2).
Notation 3.12. Let \( \tilde{\mathbf{g}} = (\tilde{g}_1, \ldots, \tilde{g}_\lambda) \) be a decomposition of an element \( \tilde{g} \in \mathbb{F}^k \) with respect to \( \mathbb{F}Q^k \) (see Definition 3.1). For each \( 1 \leq i \leq \lambda \) there is an element \( w_i \in \mathbb{F}^k \) and a generator \( f_{v_i} \), such that \( \tilde{g}_i = f_{v_i}^{w_i} \) and either \( w_i = 1 \) or the first letter appearing in the reduced expression of \( w_i \) is not \( f_{v_i}^{\pm 1} \) (see Notation 3.4) and compare with \( w \) and \( f_{v} \) in the proof of Theorem 3.3).

We associate with \( (\tilde{g}_1, \ldots, \tilde{g}_\lambda) \) the following gd of \( \tilde{g} \):

\[
(f_{v_1})^{w_1} \cdot (f_{v_2})^{w_2} \cdots (f_{v_\lambda})^{w_\lambda},
\]

where by \((-)^{w_i}\) we abbreviate the iteration of \( |w_i| \) conjugations, one for every letter in \( w_i \). We thus consider each decomposition \( \tilde{g} \) of \( \tilde{g} \) with respect to \( \mathbb{F}Q^k \) also as a gd of \( \tilde{g} \).

Using Notation 3.12, we have \( |\tilde{\mathbf{g}}| = \lambda + 2|w_1| + \cdots + 2|w_\lambda| \), where \( \tilde{\mathbf{g}} \) is regarded as a gd as in Notation 3.12.

Definition 3.13. Given a gd \( x \) of some element \( \tilde{g} \in \mathbb{F}^k \), we can find a decomposition \( \tilde{\mathbf{g}} = (\tilde{g}_1, \ldots, \tilde{g}_\lambda) \) of \( \tilde{g} \) with respect to \( \mathbb{F}Q^k \) as follows:

- we first make a list \( (f_{v_1}, \ldots, f_{v_\lambda}) \) of all sub-gds of \( x \) of the elementary form \( f_v \), reading \( x \) from left to right;
- we change the previous list as follows: for all \( 1 \leq i \leq \lambda \) we apply to the \( i \)th element \( f_{v_i} \), in the natural order, all conjugations \( (-)^{\pm 1} \) which in \( x \) conjugate a sub-gd containing \( f_{v_i} \).

We say that \( \tilde{\mathbf{g}} \) is the decomposition of \( \tilde{g} \) with respect to \( \mathbb{F}Q^k \) associated with \( x \).

Consider again the element \( \tilde{\mathbf{g}} \) from Example 3.10 and the given list of gds: the corresponding decomposition of \( \tilde{g} \) with respect to \( \mathbb{F}Q^4 \) are, respectively:

- \((f_1, f_2, f_3)\);
- \( (f_2, f_1^2, f_3) \);
- \( (f_3, f_2^3, f_1 f_2 f_3) \) for the last three dgs.

Note that if we start from a decomposition \( \tilde{\mathbf{g}} \) of an element \( \tilde{g} \in \mathbb{F}^k \) with respect to \( \mathbb{F}Q^k \), consider \( \tilde{\mathbf{g}} \) as a gd of \( \tilde{g} \) according to Notation 3.12 and then take again the associated decomposition with respect to \( \mathbb{F}Q^k \) in the sense of Definition 3.13 we recover precisely \( \tilde{\mathbf{g}} \); here we use also the inclusion \( \mathbb{F}Q^k \subset \mathbb{F}Q^k \).

3.4. Proof of Proposition 3.7. The decomposition \((g_1, \ldots, g_r)\) of \( g \in \mathbb{F}^k \) can be seen as a gd \( x_0 \) of \( g \) as in Notation 3.12.

Suppose that \( x_0 \) contains a sub-gd that has one of the forms (1)-(10) listed in Lemma 3.11. Then we can obtain a new gd \( x_1 \) of \( g \) by replacing the given sub-gd respectively by:

\[
\begin{align*}
(1) & \quad (y_1 \cdot y_2)^{f_1}; & (6) & \quad (y_1)^{f_{i-1}} \cdot y_2; \\
(2) & \quad (y_1 \cdot y_2)^{f_i^{-1}}; & (7) & \quad y_1 \cdot (y_2)^{f_i^{-1}}; \\
(3) & \quad y_1 \cdot f_i; & (8) & \quad (y_1)^{f_i} \cdot y_2; \\
(4) & \quad f_i \cdot y_1; & (9) & \quad y_1 \cdot (y_2)^{f_i}; \\
(5) & \quad y_1; & (10) & \quad f_i.
\end{align*}
\]
Note that $|x_1| < |x_0|$ in all cases. We iterate such replacements until it is possible, obtaining a sequence of gds $x_0, x_1, x_2, \ldots$; since the weight drops at each replacement, we will reach a generalised decomposition $x_n$ of $g$ which contains no sub-decomposition of forms (1)-(10) listed in Lemma 3.11.

By Lemma 3.11 the straightforward computation of $x_n$ yields $g = f_1 \ldots f_r$ without cancellations. Since the reduced expression of $g = f_1, \ldots, f_r \in F^k$ does not contain letters of the form $f_i^{-1}$, no conjugation can occur in $x_n$, and we conclude that $x_n$ is just $f_1 \cdot f_2 \cdots \cdot f_r$.

Let now, for $1 \leq \nu \leq n$, $\mathfrak{g}_\nu = (g_{\nu,1}, \ldots, g_{\nu,r})$ the decomposition of $g$ with respect to $FQ^k$ associated with the gd $x_\nu$ of $g$ (see Definition 3.13); then for all $0 \leq \nu \leq n-1$ the following holds:

- if passing from $x_\nu$ to $x_{\nu+1}$ we have used a replacement of type (1)-(10) which is not of type (3) or (4), then $\mathfrak{g}_\nu = \mathfrak{g}_{\nu,1}$;
- if passing from $x_\nu$ to $x_{\nu+1}$ we have used a replacement of type (3) or (4), then $\mathfrak{g}_{\nu,1}$ is obtained from $\mathfrak{g}_\nu$ by a standard move.

It now suffices to note that $\mathfrak{g}_0 = (g_1, \ldots, g_r)$ and $\mathfrak{g}_n = (f_1, \ldots, f_r)$; a posteriori we also note that all decompositions $\mathfrak{g}_\nu$ of $g$ are actually with respect to $FQ^k \subset FQ^k$.

4. Tameness properties for PMQs and the PMQ-ring

In this section we introduce the notion of norm for a PMQ, and discuss several properties that a PMQ may enjoy, such as being augmented and locally finite, and, in the normed case, being maximally decomposable, cocompact, pairwise determined, and Koszul. We also define the PMQ-ring $R[Q]$ of a PMQ $Q$ with coefficients in a commutative ring $R$, and study its basic properties.

4.1. Normed PMQs and normed groups.

Definition 4.1. A norm on a PMQ $Q$ is a map of PMQs $N: Q \to \mathbb{N}$ with the property that $N^{-1}(0) = \{1\} \subset Q$. Here we treat the abelian monoid $\mathbb{N}$ as an abelian PMQ (see Definition 2.4).

A PMQ $Q$ is normed if it is endowed with a norm $N: Q \to \mathbb{N}$.

In most cases the precise norm $N$ that we consider on a PMQ $Q$ will be evident from the context and will be omitted from the notation; we will then only say that $Q$ is normed. Note however that being normed is not a property that a PMQ may or may not satisfy, but rather it is an additional piece of structure. The following are simple examples of PMQs with or without norms.

Example 4.2. The natural numbers $\mathbb{N}$ form a normed PMQ, with unit $1_\mathbb{N} = 0$ and norm the identity.

Example 4.3. If $Q$ is a PMQ with norm $N: Q \to \mathbb{N}$, we can extend $N$ to a map of PMQs $N: Q \to \mathbb{N}$, using that $\mathbb{N}$ is a complete PMQ and the universal property of the complete PMQ $\bar{Q}$. The map $N: \bar{Q} \to \mathbb{N}$ turns out to be a norm on $\bar{Q}$: if $w \in \bar{Q}$ we can represent $w$ as a product $a_1 \ldots a_r$ with $a_i \in Q$; then $N(w) = N(a_1) + \cdots + N(a_r)$, so if $N(w) = 0$ we must have $N(a_i) = 0$ for all $1 \leq i \leq r$, i.e. $a_i = 1$ because $N: Q \to \mathbb{N}$ was assumed to be a norm, and hence $w = 1 \in \bar{Q}$.

Similarly as in the previous example, if $Q$ is a PMQ with norm $N$, there is an induced map between the enveloping groups $\mathcal{G}(N): \mathcal{G}(Q) \to \mathcal{G}(\mathbb{N}) = \mathbb{Z}$. 
Example 4.4. The same PMQ $Q$ may have different norms. For instance, if $S$ is a pointed set, with $1 \in S$, then $S$ can be considered as a trivial abelian PMQ as in Example 2.6. Any map of sets $N : S \to \mathbb{N}$ with $N^{-1}(0) = \{1\}$ will automatically be a norm.

Example 4.5. A PMQ $Q$ may not admit any norm. For instance, if $G$ is a non-trivial group, then $G$ can be considered as a PMQ as in Example 2.3. Then there exists no norm $N$ on $G$, as for all $g \neq 1 \in G$ we would have $N(g) + N(g^{-1}) = N(1) = 0$, but both $N(g)$ and $N(g^{-1})$ should be strictly positive integers.

We use the word “norm” in Definition 4.1 because of the relation with the following, standard definition.

Definition 4.6. A norm on a group $G$ is a function of sets $N : G \to \mathbb{N}$ satisfying the following properties:

• $N(g) + N(h) \geq N(gh)$ for all $g, h \in G$;
• $N(g) = 0$ if and only if $g = 1$.

A norm is conjugation invariant if moreover $N(g) = N(h^{-1}gh)$ for all $g, h \in G$.

The link between normed groups and normed PMQs is given by the following definition.

Definition 4.7. Let $G$ be a group with a conjugation invariant norm $N$. We define a PMQ $G^{geo} = G_N^{geo}$, called the geodesic PMQ associated with $G$. The underlying quandle of $G^{geo}$ is the underlying quandle of $G$; the partial product of $G^{geo}$ is only defined on pairs $(a, b)$ of elements of $G$ such that $N(ab) = N(a) + N(b)$, and coincides with the product $ab$ in $G$. The norm $N : G^{geo} \to \mathbb{N}$ is defined, as a map of sets, by the norm $N : G \to \mathbb{N}$.

The triangular inequality and conjugation invariance for $N$ ensure that all conditions in Definitions 2.3 and 2.4 are indeed satisfied. The fact that $N : G^{geo} \to \mathbb{N}$ is a map of PMQs, and in particular of partial monoids, follows from the fact that the products $(a, b) \mapsto ab$ allowed in $G^{geo}$ are precisely those for which $N$ is additive.

One can consider Definition 4.7 as a particular instance of Definition 2.8: we can indeed consider the group $G \times \mathbb{Z}$, and let $S \subset G \times \mathbb{Z}$ be the subset containing all couples of the form $(g, N(g))$. Then $S$ contains the unit $(1, 0)$ of $G \times \mathbb{Z}$ and inherits from $G \times \mathbb{Z}$ a structure of PMQ, which is isomorphic to $G^{geo}$.

Note also that $(G^{geo}, G)$ is naturally a PMQ-group pair (see Definition 2.13), by considering $\text{Id}_G$ as a map (of PMQs) $G^{geo} \to G$, and by letting $G$ act on $G^{geo}$ by right conjugation.

Definition 4.8. The map of PMQs $\text{Id}_G : G^{geo} \to G$ gives rise to a map of groups $\varepsilon^{geo} = \mathcal{G}(\text{Id}_G) : \mathcal{G}(G^{geo}) \to G$.

In Section 7 we will study in detail the PMQ $S^{geo}_d$ arising from the symmetric group $S_d$, endowed with a suitable norm.

4.2. Augmented and locally finite PMQs. A necessary condition, for a PMQ to admit norms, is that it is augmented.

Definition 4.9. Recall Definition 2.20: A PMQ $Q$ is augmented if $Q \setminus \{1\}$ is an ideal of $Q$. For an augmented PMQ $Q$ we denote by $Q_+ = Q \setminus \{1\}$. If $Q$ and $Q'$ are augmented PMQs, a map of PMQs $\Psi : Q \to Q'$ is augmented if $\Psi(Q_+) \subset Q'_+$. 


Example 4.10. Let \( \{1, 0\} \) be the abelian monoid with unit \( 1 \), such that \( 0 \cdot 0 = 0 \), and regard \( \{1, 0\} \) as an abelian (complete) PMQ. Then \( \{1, 0\} \) is augmented, as \( \{0\} \) is an ideal.

In fact, for a generic PMQ \( Q \), the following are equivalent:

- \( Q \) is augmented;
- the map of sets \( Q \to \{1, 0\} \), given by \( a \mapsto 0 \) for \( a \in Q \setminus \{1\} \) and \( 1 \mapsto 1 \), is a map of PMQs.

This explains the use of the word augmented in Definition 4.9: we think of the map \( Q \to \{1, 0\} \) as being an augmentation.

Example 4.11. Every PMQ with trivial product is augmented. More generally, a normed PMQ \( Q \) is augmented: the set \( Q \setminus \{1\} \) contains all elements of strictly positive norm and is thus an ideal.

Definition 4.12. A PMQ \( Q \) is locally finite if for every \( a \in Q \) there are finitely many sequences \( (a_1, \ldots, a_r) \) of elements of \( Q \setminus \{1\} \) with \( a = a_1 \ldots a_r \).

Example 4.13. We give an example of a complete and normed PMQ which is not locally finite. Let \( Q = \mathbb{F}^2 \) be the free group on two elements, and consider it just as a quandle, i.e. as a PMQ with trivial multiplication. Then we can define \( N: \mathbb{F}^2 \to Q \) by declaring \( N(a) = 1 \) for all \( a \neq 1 \in Q \). Since every element \( a \in Q \) can be factored only as \( a 1 \) or \( 1 a \), we have that \( Q \) is locally finite.

The completion \( \hat{Q} \) is however not locally finite: for instance, if \( f_1, f_2 \) denote the generators of \( \mathbb{F}^2 \), then the element \( w = \hat{f}_1 \hat{f}_2 \in \hat{Q} \) can be rewritten in infinitely many ways as a product of elements of norm 1:

\[
\hat{f}_1 \hat{f}_2 = \hat{f}_2 \hat{f}_1 = \hat{f}_1 \hat{f}_2 \hat{f}_1 \hat{f}_2 = \hat{f}_2 \hat{f}_1 \hat{f}_2 \left( \hat{f}_1 \hat{f}_2 \right) \hat{f}_2 \hat{f}_1 \hat{f}_2 = \ldots
\]

Example 4.14. Let \( G \) be a finite, non-trivial group and consider \( G \) as a PMQ with full product. Then for all \( a \in G \) there are finitely many couples \( (b, c) \in G \) such that \( a = bc \); nevertheless \( G \) is not locally finite, since every element \( a \) can be written in infinitely many ways as a product \( a_1 \ldots a_r \), for \( r \) arbitrarily large.

A similar same argument shows in fact that a locally finite PMQ must be augmented: if a PMQ \( Q \) is not augmented, there exist \( a, b \in Q \setminus 1 \) with \( ab = 1 \); then \( 1 \) can be decomposed as \( ab = abab = ababab = \ldots \), in infinitely many ways, showing that \( Q \) is not locally finite.

Example 4.15. Let \( Q \) be a finite, normed PMQ with norm \( N \): then \( Q \) is locally finite. Indeed for all \( n \geq 0 \) there are finitely many sequences \( (a_1, \ldots, a_r) \) of elements of \( Q_+ \) with \( N(a_1) + \cdots + N(a_r) \leq n \); a fortiori each element \( a \in Q \) admits infinitely many decompositions as product of elements \( \neq 1 \).

4.3. Coconnected PMQs. If \( Q \) is a PMQ with norm \( N \) and \( a \in Q \), we may attempt to factor \( a \) as a product \( a_1 \ldots a_r \) in \( Q_+ \), where \( r = N(a) \) and each \( a_i \) has norm 1.

Notation 4.16. For a normed PMQ \( Q \) and \( r \geq 0 \) we denote by \( Q_r \subset Q \) the subset of elements of \( Q \) of norm \( r \).

Definition 4.17. Recall Definition 4.9 and let \( a \) be an element of a normed PMQ \( Q \). A decomposition of \( a \) with respect to \( Q_1 \) is a (possibly empty) sequence
(a_1, \ldots, a_r) of elements of \mathcal{Q}_1, where r = N(a), such that the product a_1 \ldots a_r is defined in \mathcal{Q} and is equal to a. We say that \mathcal{Q} is maximally decomposable if every element a \in \mathcal{Q} admits a decomposition with respect to \mathcal{Q}_1.

Not every normed PMQ is maximally decomposable: for instance the normed PMQ \mathcal{S} from Example 4.14 is maximally decomposable if and only if all elements in \mathcal{S} \setminus \{1\} have norm 1.

**Definition 4.18.** Let \mathcal{Q} be an augmented PMQ. An element a \in \mathcal{Q}_+ is irreducible if it cannot be written as a product bc in \mathcal{Q} with b, c \in \mathcal{Q}_+. For a generic element a \in \mathcal{Q} we set h(a) \in \mathbb{N} \cup \{\infty\} to be the supremum of r \geq 0 such that a admits a decomposition a = a_1 \ldots a_r with a_1, \ldots, a_r \in \mathcal{Q}_+: by convention h(1) = 0 and h(a) = 1 if a is irreducible. If h(a) is finite for all a \in \mathcal{Q}, we say that h : \mathcal{Q} \rightarrow \mathbb{N} is the intrinsic pseudonorm of \mathcal{Q}.

A condition ensuring that h(a) is finite for all a \in \mathcal{Q} is that \mathcal{Q} is locally finite. Note that for all a, b \in \mathcal{Q} we have h(ab) \geq h(a) + h(b), and h(a) = 0 if and only if a = 1. Note also that if h(a) is finite and a = a_1 \ldots a_{h(a)} is a decomposition witnessing the value h(a), then the elements a_1, \ldots, a_{h(a)} must be irreducible.

We can now characterise those augmented PMQs which admit a norm N for which they are maximally decomposable. These are precisely those augmented PMQs satisfying all of the following properties:

- for all a \in \mathcal{Q}, h(a) is finite;
- for all a \in \mathcal{Q} and all decompositions a = a_1 \ldots a_r of a into irreducibles, we have r = h(a).

**Definition 4.19.** If \mathcal{Q} satisfies the above properties, then the intrinsic pseudonorm h from Definition 4.18 is in fact norm making \mathcal{Q} into a maximally decomposable PMQ, with \mathcal{Q}_1 being the set of irreducible elements; moreover h is the unique such norm. We will say that h is the intrinsic norm on the augmented PMQ \mathcal{Q}.

If \mathcal{Q} does not satisfy the above properties, then there is no norm N : \mathcal{Q} \rightarrow \mathbb{N} making \mathcal{Q} into a maximally decomposable PMQ.

We have discussed existence of maximal decompositions, let us now turn to uniqueness. Recall Definition 4.6 and note that if (a_1, \ldots, a_r) is a decomposition of a \in \mathcal{Q} with respect to \mathcal{Q}_1 and if r \geq 2, then we can apply a standard move to it and obtain a possibly different decomposition, e.g. (a_2, a_1^2, a_3, \ldots, a_r).

**Definition 4.20.** A normed PMQ \mathcal{Q} is coconnected if it is maximally decomposable and for all a \in \mathcal{Q} and for every pair of decompositions (a_1, \ldots, a_r) and (a'_1, \ldots, a'_r) of a with respect to \mathcal{Q}_1, there is a sequence of standard moves connecting the first decomposition to the second.

**Example 4.21.** The abelian monoid \mathbb{N}, regarded as a normed, abelian PMQ, is coconnected. More generally, for n \geq 0 the subset \mathcal{Q} = \{0, \ldots, n\} \subset \mathbb{N} can be regarded as an abelian PMQ by virtue of Definition 2.8 with norm given by the inclusion in \mathbb{N}: then \mathcal{Q} is coconnected. In fact, all abelian, coconnected PMQs have one of these forms.

**Example 4.22.** An example of a PMQ which is maximally decomposable but not coconnected is \mathcal{Q} = \mathbb{N} \cup \{1'\} = \{0, 1, 1', 2, 3, \ldots\}, regarded as an abelian, complete PMQ as follows: switching to additive notation, we use the usual sum for couples of
elements of $\mathbb{N} \subset Q$ and we set $1'+0 = 0+1' = 1'$, $1'+1' = 2$ and $n+1' = 1'+n = n+1$ for all $n \in \mathbb{N} \setminus \{0\}$. The norm $N: Q \to \mathbb{N}$ is the identity on $\mathbb{N} \subset Q$, and $N(1') = 1$.

Note that $2 \in Q$ has four decompositions with respect to $Q_1 = \{1,1'\}$, namely $(1,1)$, $(1',1')$, $(1,1')$ and $(1',1)$; the last two are connected by a standard move, whose effect is just swapping the two entries; however there is, for instance, no sequence of standard moves connecting any two of the first three decompositions.

We conclude the subsection by giving a convenient description of the completion of a coconnected PMQ $Q$. Let $Q_{\leq 1} = \{1\} \cup Q_1 \subset Q$ be the subset containing all elements of norm $\leq 1$. Then $Q_{\leq 1}$ can be considered as a PMQ with trivial multiplication. The inclusion $Q_{\leq 1} \subset Q$ is a map of PMQs, inducing a map between the completions $\hat{Q}_{\leq 1} \to \hat{Q}$.

**Lemma 4.23.** In the hypotheses above, the natural map $\hat{Q}_{\leq 1} \to \hat{Q}$ is an isomorphism.

**Proof.** The monoid $\hat{Q}$ is generated by the elements of $Q$, which in turn can be obtained as products of elements in $Q_{\leq 1}$, since $Q$ is maximally decomposable. This implies that the map $\hat{Q}_{\leq 1} \to \hat{Q}$ is surjective.

To show injectivity, let $(a_1, \ldots, a_r)$ and $(a'_1, \ldots, a'_r)$ be sequences of elements of $Q_1$ such that the products $a_1 \ldots a_r \in \hat{Q}_{\leq 1}$ and $a'_1 \ldots a'_r \in \hat{Q}_{\leq 1}$ are sent to the same element of $\hat{Q}$. The equality

$$a_1 \ldots a_r = a'_1 \ldots a'_r \in \hat{Q}$$

implies the following: there exist $n \geq 1$ and, for all $1 \leq i \leq n$ a sequence $(b_{i,1}, \ldots, b_{i,r})$ of elements of $Q$, such that the following hold:

1. $(b_{1,1}, \ldots, b_{1,r_1}) = (a_1, \ldots, a_r)$ and $(b_{n,1}, \ldots, b_{n,r_n}) = (a'_1, \ldots, a'_r)$;
2. for all $1 \leq i \leq n-1$, the sequence $(b_{i+1,1}, \ldots, b_{i+1,r_{i+1}})$ can be obtained from $(b_{i,1}, \ldots, b_{i,r_i})$ by one of the following moves (or their inverses):
   (1) replace two consecutive entries $(b_{i,j}, b_{i,j+1})$ by their product $b_{i,j}b_{i,j+1}$, provided that this product exists in $Q$;
   (2) replace two consecutive entries $(b_{i,j}, b_{i,j+1})$ by the two consecutive entries $(b_{i,j+1}, b_{i,j}^{-1})$;
   (3) replace two consecutive entries $(b_{i,j}, b_{i,j+1})$ by the two consecutive entries $(b_{i,j+1}, b_{i,j})$.

The hypothesis that $Q$ is coconnected implies that, without loss of generality, we can assume that no move of type (1) takes place, at the cost of increasing $n$ and inserting more standard moves, i.e. those of type (2)-(3).

If no move of type (1) takes place, all elements $b_{i,j}$ belong to $Q_1$, so the entire sequence of replacements of sequences witnesses that also the products $a_1 \ldots a_r \in \hat{Q}_{\leq 1}$ and $a'_1 \ldots a'_r \in \hat{Q}_{\leq 1}$ are equal.

\[\square\]

Lemma 4.23 can be interpreted as follows: all coconnected PMQs can be obtained as sub-PMQs of completions of PMQs with trivial product.

### 4.4. Pairwise determined PMQs.

Classically, a partial abelian monoid $M$ is **pairwise determined** if for every $r \geq 3$ and every sequence $(a_1, \ldots, a_r)$ of elements of $M$, either the sum $a_1 + \cdots + a_r$ is defined in $M$, or there are indices $1 \leq i < j \leq r$
such that the sum $a_i + a_j$ is not defined. Note that if $M$, considered as an abelian PMQ, is normed and maximally decomposable, then it is equivalent to require the previous dichotomy for all $r \geq 3$ and all sequences $(a_1, \ldots, a_r)$ of elements of $M$ of norm 1. We generalise this notion to PMQs.

**Definition 4.24.** Let $Q$ be a normed and maximally decomposable PMQ. We say that $Q$ is **pairwise determined** if the following holds: for every $r \geq 3$ and every sequence $(a_1, \ldots, a_r)$ of elements of $Q$, either the product $a_1 \cdots a_r$ is defined in $Q$, or there is a sequence of standard moves connecting $(a_1, \ldots, a_r)$ to a new sequence $(a'_1, \ldots, a'_r)$ such that the product $a'_1 a'_2$ is not defined in $Q$.

An example of a PMQ which is pairwise determined but not coconnected is any abelian monoid which is not coconnected, for instance the abelian monoid $\mathbb{N} \cup \{1'\}$ from Example 4.22.

**Example 4.25.** Let $n \geq 1$ and let $Q = \{0, \ldots, n\} \subseteq \mathbb{N}$, regarded as an abelian PMQ by virtue of Definition 4.23 with norm given by the inclusion into $\mathbb{N}$. Note that every element admits a unique maximal decomposition as sum of 1's, hence $Q$ is coconnected; on the other hand, if $n \geq 2$, the sequence $(1,1,\ldots,1)$ of length $n+1$ is fixed by any standard move, is not summable, yet its first two elements are summable: hence for $n \geq 2$ the PMQ $Q$ is not pairwise determined.

4.5. **PMQ-ring of a PMQ.** For a group $G$ and a commutative ring $R$, there is a classical notion of group ring $R[G]$. We generalise this notion to PMQs.

**Definition 4.26.** For a PMQ $Q$ and a commutative ring $R$ we denote by $R[Q]$ the **PMQ-ring** of $Q$ with coefficients in $R$. It is an associative $R$-algebra; as an $R$-module it is free with standard basis given by elements $[a]$ for $a \in Q$; for $a,b \in Q$ the product $[a][b] \in R[Q]$ is equal to $[ab]$ if $ab$ is defined in $Q$, and we have $[a][b] = 0$ otherwise.

Note that the definition only depends on the structure of partial monoid that a PMQ has. If $Q$ is augmented (see Definition 4.19), then $R[Q]$ is an augmented algebra: the augmentation $\varepsilon: R[Q] \to R$ sends $[1] \mapsto 1$ and $[a] \mapsto 0$ for all $a \in Q \setminus \{1\}$.

If $Q$ is endowed with a norm $N$, then $R[Q]$ is a graded algebra: for $\nu \geq 0$ the degree-$\nu$ part of $R[Q]$, denoted $R[Q]_\nu$, is spanned by the elements $[a]$ for $a \in Q_\nu$.

Note that if $Q$ is maximally decomposable, then $R[Q]$ is generated in degree 1: every basis element $[a]$ can be written as a product of elements of degree 1 $[a_1] \ldots [a_r]$, where $(a_1, \ldots, a_r)$ is a decomposition of $a$ with respect to $Q_1$.

**Definition 4.27.** Let $R$ be a commutative ring; a graded $R$-algebra $A$ is **quadratic** if it is generated in degree 1 and related in degree 2. More precisely, $A$ is the quotient of the free tensor algebra $T_\bullet A_1$ by a (bilateral) ideal $I \subset T_\bullet A_1$ generated by elements in $(A_1)^{\otimes 2}$. Here $A_1 \subset A$ denotes the degree 1 part of $A$, and $T_\bullet A_1 = \bigoplus_{i \geq 0} (A_1)^{\otimes i}$, where tensor products are taken over $R$.

**Theorem 4.28.** Let $Q$ be a maximally decomposable, coconnected and pairwise determined PMQ; then $R[Q]$ is a quadratic $R$-algebra.

**Proof.** Let $R \langle Q_1 \rangle$ denote the free associative $R$-algebra on the set $Q_1$, which is a graded $R$-algebra by putting the generators in degree 1. There is a natural map of graded $R$-algebras $u: R \langle Q_1 \rangle \to R[Q]$ given by $a \mapsto [a]$ for $a \in Q_1$. Since $Q$ is
maximally decomposable, the map $u$ is surjective; note also that $u$ is bijective in degrees 0 and 1.

We want to show that the kernel of $u$ is generated, as a bilateral ideal of $R \langle Q_1 \rangle$, by elements of degree 2. Let $\ker(u)_\nu$ denote the degree-$\nu$ part of $\ker(u)$. Then $\ker(u)_\nu$ is generated as $R$-module by the following elements:

1. monomials $\langle a_1, \ldots, a_h \rangle$, such that the product $a_1 \ldots a_h$ is not defined in $Q$;
2. differences of monomials $\langle a_1, \ldots, a_h \rangle - \langle a'_1, \ldots, a'_h \rangle$, such that the products $a_1 \ldots a_h$ and $a'_1 \ldots a'_h$ are defined and equal in $Q$.

Let $I \subset R \langle Q_1 \rangle$ be the bilateral ideal generated by $\ker(u)_2$: we want to prove that the inclusion $I \subseteq \ker(u)$ is an equality.

First, let $(a_1, \ldots, a_r)$ and $(a'_1, \ldots, a'_r)$ be two sequences of elements of $Q_1$ that differ by a standard move, swapping the elements in positions $j$ and $j + 1$ (in particular $a_i = a'_i$ for all $i \neq j, j + 1$). Then $\langle a_j, a_{j+1} \rangle - \langle a'_j, a'_{j+1} \rangle \in \ker(u)_2$, in both of the following cases:

- if $a_j a_{j+1}$ is defined, then $\langle a_j, a_{j+1} \rangle - \langle a'_j, a'_{j+1} \rangle$ is an element of type (2));
- if $a_j a_{j+1}$ is not defined, then $\langle a_j, a_{j+1} \rangle - \langle a'_j, a'_{j+1} \rangle$ is a difference of elements of type (1).

It follows that the difference $\langle a_1, \ldots, a_r \rangle - \langle a'_1, \ldots, a'_r \rangle$ is equal to

$$\langle a_1, \ldots, a_{j-1} \rangle \left( \langle a_j, a_{j+1} \rangle - \langle a'_j, a'_{j+1} \rangle \right) \langle a_{j+2}, \ldots, a_r \rangle \in I.$$

Using that $Q$ is coconnected, we can express every element of $\ker(u)_2$ of type (2) as a linear combination (in fact a sum) of elements of the form $\langle a_1, \ldots, a_r \rangle - \langle a'_1, \ldots, a'_r \rangle$ with $(a_1, \ldots, a_r)$ and $(a'_1, \ldots, a'_r)$ differing for a single standard move; this shows that all elements of type (2) lie in $I$.

By the previous argument and the hypothesis that $Q$ is pairwise determined, every element $\langle a_1, \ldots, a_r \rangle$ of type (1) can be written as a sum of an element in $I$ and another element of the form $\langle a'_1, \ldots, a'_r \rangle$, where we can assume that the product $a'_1 a'_2$ is not defined in $Q$. Then $\langle a'_1, a'_2 \rangle \in \ker(u)_2$, and thus

$$\langle a'_1, \ldots, a'_h \rangle = \langle a'_1, a'_2 \rangle \langle a'_3, \ldots, a'_r \rangle \in I.$$

\square

4.6. Invariants of the adjoint action. Recall from Subsection 2.3 the adjoint action of $G(Q)$ on $Q$: it induces an action of $G(Q)$ on $R[Q]$ by ring automorphisms.

**Definition 4.29.** We define $A(Q)$ as the sub-$R$-algebra $R[Q]^G(Q) \subseteq R[Q]$ of invariants under the adjoint action.

Note that at least the copy of $R \subset R[Q]$ spanned by the element $[1]$ is contained in $A(Q)$; therefore $A(Q)$ is a unital $R$-algebra.

**Definition 4.30.** Let $Q$ be a PMQ, and recall from Definition 2.1 the notion of conjugacy class. If $S \in \text{conj}(Q)$ is a finite conjugacy class, we denote by $[S]$ the element

$$[S] := \sum_{a \in S} [a] \in R[Q].$$

Note that the elements $[S]$, for $S$ ranging among finite conjugacy classes of $Q$, exhibit $A(Q)$ as a free $R$-module. This follows from the observation that a generic element $x \in R[Q]$ takes the form $x = \sum_{a \in Q} \lambda_a [a]$, where all but finitely many of the coefficients $\lambda_a \in R$ are zero. The element $x$ belongs to $A(Q)$ if and only if
\( \lambda_a = \lambda_b \) whenever \( \text{conj}(a) = \text{conj}(b) \); hence for every conjugacy class \( S \) there is a (unique) \( \lambda_S \in R \) such that \( x = \sum_{S \in \text{conj}(Q)} \sum_{a \in S} \lambda_S [a] \). Since \( x \) is a finite linear combination of element \([a]\), we have that \( \lambda_S \) must vanish whenever \( S \) is infinite, and moreover all but finitely many coefficients \( \lambda_S \) vanish.

**Lemma 4.31.** The ring \( A(Q) \) is contained in the centre of \( R[Q] \); in particular \( A(Q) \) is a commutative ring.

**Proof.** It suffices to prove that for all finite conjugacy class \( S \in \text{conj}(Q) \) and all \( b \in Q \) the equality \([S][b] = [b][S]\) holds in \( R[Q] \). The first product is equal to \( \sum_{a \in S}[a][b] \); by Definition 4.26 and the axioms of PMQ, we have \([a][b] = [b][a]^{ab}\) for all \( a, b \in Q \), so we may rewrite the first product as \( \sum_{a \in S}[b][a]^{ab} \). It suffices now to remember that \((-)^b : Q \to Q \) is a bijection and restricts to a bijection \((-)^b : S \to S \), hence the latter formula equals the second product. \( \square \)

### 4.7. Koszul PMQs

Let \( A \) be a graded, associative \( R \)-algebra, and assume that \( A \) is a finitely generated, free \( R \)-module for all \( i \geq 0 \). The algebra \( A \) is connected if \( A_0 = R \); in this case \( A \) admits a canonical augmentation \( A \to R \), making in particular \( R \) into a left and right \( A \)-module concentrated in degree 0. The cohomology groups \( \text{Ext}^*_{\mathcal{A}}(R, R) \) inherit a grading from \( A \), so that for all \( j \geq 0 \) there is a decomposition \( \text{Ext}^j_{\mathcal{A}}(R, R) = \oplus_{a \geq 0} \text{Ext}^{j+a}_{\mathcal{A}}(R, R) \). Classically, a connected \( R \)-algebra \( A \) is Koszul if \( \text{Ext}^j_{\mathcal{A}}(R, R) \) is a finitely generated, free \( R \)-module concentrated in degree \( j \) for all \( j \geq 0 \), i.e. \( \text{Ext}^j_{\mathcal{A}}(R, R) = \text{Ext}^{j+1}_{\mathcal{A}}(R, R) \). Recall that if \( A \) is Koszul, then it is also quadratic.

**Definition 4.32.** A normed, locally finite PMQ \( Q \) is Koszul (over \( R \)) if \( R[Q] \), as a graded, connected \( R \)-algebra, is Koszul.

By Theorem 4.28 if \( Q \) is a normed, locally finite, coconnected, pairwise determined and Koszul PMQ, then \( R[Q] \) is a (quadratic) Koszul \( R \)-algebra and therefore \( \text{Ext}^*_{R[Q]}(R, R) \) is isomorphic, as a graded \( R \)-algebra, to the dual quadratic algebra of \( R[Q] \). More precisely, on the one hand \( R[Q] \) is isomorphic to the free associative \( R \)-algebra with following generators and relations:

- **Generators** For all \( a \in Q_1 \) there is a generator \([a]\) in degree 1.
- **Relations** For all \((a, b) \in Q_1 \times Q_1 \) there is a relation \([a][b] = [b][a]^{ab} \):
  - if \( ab \) is not defined in \( Q \), there is also a relation \([a][b] = 0 \).

On the other hand \( \text{Ext}^*_{R[Q]}(R, R) \) is isomorphic to the free associative \( R \)-algebra with the following generators and relations:

- **Generators** For all \( a \in Q_1 \) there is a generator \([a]'\) in degree 1.
- **Relations** For all \( c \in Q_2 \) there is a relation \( \sum [a]''[b]' = 0 \): here the sum is extended over all pairs \((a, b) \in Q_1 \times Q_1 \) satisfying \( ab = c \).

Note that the sum \( \sum [a]''[b]' = 0 \) is finite because we assume \( Q \) locally finite.

**Example 4.33.** Let \( Q = \mathbb{N} = \{0, 1, 2, \ldots \} \) with the identity norm; then \( Q \) is a Koszul PMQ, as \( R[\mathbb{N}] \cong R[x] \) is a Koszul algebra.

**Example 4.34.** Let \( n \geq 1 \) and let \( Q = \{0, 1, \ldots, n\} \subset \mathbb{N} \), regarded as a PMQ in light of Definition 2.3, let the norm on \( Q \) be given by the inclusion in \( \mathbb{N} \). Then \( R[Q] \cong R[x]/x^{n+1} \) is not a quadratic algebra unless \( n = 1 \); for \( n = 1 \) we have that \( R[x]/x^2 \) is a Koszul algebra. Therefore \( Q \) is Koszul if and only if \( n = 1 \).
Example 4.35. Let $Q = \{1, a, a', b, b', c\}$ be the abelian PMQ in which the only non-trivial, defined multiplications are $ab = a'b' = c$. Define a norm $N$ on $Q$ by setting $N(a) = N(a') = N(b) = N(b') = 1$ and $N(c) = 2$. Then $R(Q) \cong R[x, y, x', y']/(xy - x'y')$ is isomorphic to the Segre subalgebra $R[s_1t_1, s_1t_2, s_2t_1, s_2t_2]$ of the polynomial ring $R[s_1, s_2, t_1, t_2]$ in 4 variables, which is known to be Koszul. Hence $Q$ is Koszul.

5. Double bar constructions in braided monoidal categories

In this section we collect some general facts about algebra objects in braided monoidal categories. The main goal is to define, for a pair of commutative algebras $(A, B, f)$ in a braided monoidal category $A$, the double bar construction $B_{\bullet, \bullet}(A, B, f)$, which is a bisimplicial object in $A$. The material of this section is likely to be standard, and is included here for the sake of completeness.

**Notation 5.1.** For a category $A$ we denote by $sA$ the category of simplicial objects in $A$ (i.e. functors $\Delta^{op} \to A$); similarly $ssA$ denotes the category of bisimplicial objects in $A$.

### 5.1. Algebras in monoidal categories

In this subsection we denote by $A$ a monoidal category. We will denote by $- \otimes -$ the monoidal product, and by $1$ the unit object. We shall neglect all issues related to associators and unitors.

**Definition 5.2.** We denote by $\text{Alg}(A)$ the category of algebras (or unital monoid objects) in $A$. An algebra $A$ is endowed with a multiplication $\mu_A : A \otimes A \to A$ and a unit $\eta_A : 1 \to A$, and the following identities are required:

- $\mu_A \circ (\mu_A \otimes \text{Id}_A) = \mu_A \circ (\text{Id}_A \otimes \mu_A) : A \otimes A \otimes A \to A$;
- $\mu_A \circ (\eta_A \otimes \text{Id}_A) = \mu_A \circ (\text{Id}_A \otimes \eta_A) = \text{Id}_A : A \to A$.

A morphism of algebras $f : A \to B$ satisfies the following identities:

- $\mu_B \circ (f \otimes f) = f \circ \mu_B : A \otimes A \to B$;
- $f \circ \eta_A = \eta_B : 1 \to B$.

An algebra pair in $A$ is the datum $(A, B, f)$ of two algebras $A, B \in \text{Alg}(A)$ and a morphism of algebras $f : A \to B$. Algebra pairs form a category $\text{Alg}(A)^{[0,1]}$, which is the arrow category of $\text{Alg}(A)$.

Let $A \in \text{Alg}(A)$. A left $A$-module $B$ is endowed with a multiplication $\mu_{A,B} : A \otimes B \to B$, and the following identities are required:

- $\mu_{A,B} \circ (\mu_A \otimes \text{Id}_B) = \mu_{A,B} \circ (\text{Id}_A \otimes \mu_{A,B}) : A \otimes A \otimes B \to B$;
- $\mu_{A,B} \circ (\eta_A \otimes \text{Id}_B) = \text{Id}_B : B \to B$.

The notion of right $A$-module is defined in an analogous way.

Finally, an $A$-bimodule $B$ is endowed with both left and right $A$-module structures, and the following identity is required:

- $\mu_{B,A} \circ (\mu_{A,B} \otimes \text{Id}_A) = \mu_{A,B} \circ (\text{Id}_A \otimes \mu_{B,A}) : A \otimes B \otimes A \to B$.

**Notation 5.3.** If $(A, B, f) \in \text{Alg}(A)^{[0,1]}$ is an algebra pair, then we also consider $B$ as an $A$-bimodule with structure maps $\mu_{A,B} := \mu_B \circ (f \otimes \text{Id}_B)$ and $\mu_{B,A} := \mu_B \circ (\text{Id}_B \otimes f)$.

### 5.2. Algebras in braided monoidal categories

Assume now that $A$ is braided monoidal; for two objects $A, B \in A$ we denote by $\text{br}_{A,B} : A \otimes B \to B \otimes A$ the braiding. The following construction makes $\text{Alg}(A)$ into a monoidal category.
Definition 5.4. Let $\mathbf{A}$ be a braided monoidal category with tensor product $\otimes$, unit object $1_\mathbf{A}$ and braiding $\text{br}_{-, -}$, and let $A, B \in \text{Alg}(\mathbf{A})$. The tensor product $A \otimes B$ in $\mathbf{A}$ can be endowed with a natural structure of algebra in $\mathbf{A}$, where the product

$$\mu_{A \otimes B} : (A \otimes B) \otimes (A \otimes B) \to A \otimes B$$

is the composition

$$A \otimes B \otimes A \otimes B \xrightarrow{\text{Id}_A \otimes \text{br}_{B,A} \otimes \text{Id}_B} A \otimes A \otimes B \otimes B \xrightarrow{\mu_A \otimes \mu_B} A \otimes B.$$

Here $\mu_A : A \otimes A \to A$ and $\mu_B : B \otimes B \to B$ denote the products of $A$ and $B$ respectively. Similarly, the unit

$$\eta_{A \otimes B} : 1_\mathbf{A} \to A \otimes B$$

is defined as $1_\mathbf{A} \cong 1_\mathbf{A} \otimes 1_\mathbf{A} \xrightarrow{\eta_A \otimes \eta_B} A \otimes B$, where $\eta_A : 1_\mathbf{A} \to A$ and $\eta_B : 1_\mathbf{A} \to B$ are the units of $A$ and $B$.

If $f : A \to A'$ and $g : B \to B'$ are morphisms in $\text{Alg}(\mathbf{A})$, then the morphism $f \otimes g : A \otimes B \to A' \otimes B'$, which is defined as a morphism in $\mathbf{A}$, is automatically a morphism in $\text{Alg}(\mathbf{A})$.

The previous definition makes $\text{Alg}(\mathbf{A})$ into a monoidal category. As a consequence, if $(A, B, f)$ and $(A', B', f')$ are algebra pairs in $\mathbf{A}$, then $f \otimes f' : A \otimes A' \to B \otimes B'$ is a morphism of algebras, hence $(A \otimes A', B \otimes B', f \otimes f')$ is also an algebra pair. In fact we obtain a monoidal structure on the category $\text{Alg}(\mathbf{A})^{[0,1]}$: it is actually a general fact that the arrow category of a monoidal category (in this case $\text{Alg}(\mathbf{A})$) is also a monoidal category.

Assuming that $\mathbf{A}$ is braided monoidal allows us also to define the notion of commutative algebra (and similarly of commutative algebra pair).

Definition 5.5. Let $\mathbf{A}$ be a braided monoidal category and $A \in \text{Alg}(\mathbf{A})$. We say that $A$ is *commutative* if the following diagram commutes, where $\mu_A$ denotes the multiplication of $A$:

$$\begin{array}{ccc}
A \otimes A & \xrightarrow{\text{br}_{A,A}} & A \otimes A \\
& \mu_A \downarrow & \mu_A \\
& A & \\
\end{array}$$

Similarly, let $(A, B, f) \in \text{Alg}(\mathbf{A})^{[0,1]}$. We say that $(A, B, f)$ is commutative if both $A$ and $B$ are commutative.

5.3. Single bar construction. We will define the bar construction in the setting of a monoidal category $\mathbf{A}$ and an algebra pair $(A, B, f)$ in $\mathbf{A}$: the most general construction would use a left $A$-module and a right $A$-module which are possibly distinct, but we will not need this level of generality in this article.

Definition 5.6. Let $\mathbf{A}$ be a monoidal category and let $(A, B, f)$ be an algebra pair in $\mathbf{A}$. Recall Notation $\square$. We define a simplicial object $B_\bullet(A, B, f) \in s\mathbf{A}$ as follows:

- for $n \geq 0$, the $n$th object $B_n(A, B)$ is given by $B \otimes A^\otimes n \otimes B$;
- for $n \geq 1$ and $1 \leq i \leq n − 1$, the $i$th face map $d_i : B_n(A, B) \to B_{n-1}(A, B)$ is given by $\text{Id}_B \otimes \text{Id}_{A}^{\otimes i-1} \otimes \mu_A \otimes \text{Id}_{A}^{\otimes n-i-1} \otimes \text{Id}_B$; we also set $d_0 = \mu_{B,A} \otimes \text{Id}_{A}^{\otimes n-1} \otimes \text{Id}_B$ and $d_n = \text{Id}_B \otimes \text{Id}_{A}^{\otimes n-1} \otimes \mu_{A,B}$. 

for \( n \geq 0 \) and \( 0 \leq i \leq n \), the \( i \text{th} \) degeneracy map \( s_i : B_n(A, B) \to B_{n+1}(A, B) \) is given by \( \text{Id}_B \otimes \text{Id}_A^{n+1} \otimes \eta_A \otimes \text{Id}_A^{n-i} \otimes \text{Id}_B \).

Note that by Definition 5.4 the objects \( B_n(A, B, f) = B \otimes A^\otimes_n \otimes B \) can be naturally regarded as algebras in \( B \). However, in order to enhance \( B_n(A, B, f) \) to a simplicial object in \( \text{Alg}(A) \), we need some additional hypothesis, e.g.\( \text{that} \ (A, B, f) \text{ is commutative.} \)

**Lemma 5.7.** Let \( A \) be a braided monoidal category and \( (A, B, f) \) be a commutative algebra pair; then \( B_n(A, B, f) \) can be naturally regarded as an object in \( \text{sAlg}(A) \).

Moreover the assignment
\[
B_n(f, \text{Id}_B) := \text{Id}_B \otimes f^\otimes_n \otimes \text{Id}_B : B_n(A, B, f) \to B_n(B, B, \text{Id}_B)
\]
gives a morphism in \( \text{sAlg}(A) \), so that we obtain a simplicial algebra pair
\[
(B_n(A, B, f), B_n(B, B, \text{Id}_B), B_n(f, \text{Id}_B)).
\]

**Proof.** We first have to check that all face maps \( d_i : B_n(A, B) \to B_{n-1}(A, B) \) and degeneracy maps \( s_i : B_n(A, B) \to B_{n+1}(A, B) \), which a priori are only morphisms in \( A \), are in fact morphisms in \( \text{Alg}(A) \). First, note that since \( \mu_A \) is associative and is invariant under precomposition with \( \text{br}_{A,A} \), the following diagram commutes:
\[
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{\mu_A \otimes A} & A \otimes A \\
\downarrow{\mu_A \otimes \mu_A} & & \downarrow{\mu_A} \\
A \otimes A & \xrightarrow{\mu_A} & A.
\end{array}
\]

This, together with the observation \( \mu_A \circ \eta_A \otimes A = \eta_A \), implies that \( \mu_A : A \otimes A \to A \) is a map of algebras in \( A \). Similarly one can check that \( \eta_A : 1 \to A \), \( \mu_{A,B} : A \otimes B \to B \), \( \mu_{B,A} : B \otimes A \to B \) are morphisms in \( \text{Alg}(A) \), using that \( B \) is commutative and that, by the naturality of the braiding, also the following diagram commutes:
\[
\begin{array}{ccc}
A \otimes A & \xrightarrow{\text{br}_{A,A}} & A \otimes A \\
\downarrow{f \otimes f} & & \downarrow{f \otimes f} \\
B \otimes B & \xrightarrow{\text{br}_{B,B}} & B \otimes B.
\end{array}
\]

It follows now directly from definition 5.4 that both the face map \( d_i : B_n A \to B_{n-1} A \) and the degeneracy map \( s_i : B_n A \to B_{n+1} A \) are monoidal products (in the monoidal category \( \text{Alg}(A) \)) of morphisms in \( \text{Alg}(A) \), and hence they are morphisms in \( \text{Alg}(A) \) as well, i.e. maps of algebras. This makes \( B_n(A, B) \) into a simplicial algebra in \( A \), or equivalently into an algebra in \( \text{sAlg} A \); the category \( \text{sAlg} A \) is braided monoidal, with levelwise monoidal product and braiding.

The same argument shows that \( B_n(B, B, \text{Id}_B) \) is a simplicial algebra in \( A \). To check that \( B_n(f, \text{Id}_B) \) is a morphism of simplicial algebras, it suffices to check that for all \( n \geq 0 \) the morphism \( B_n(f, \text{Id}_B) \) is a morphism of algebras: this is evident, as \( B_n(f, \text{Id}_B) \) is a tensor product of morphisms of algebras. \( \square \)

### 5.4. Double bar construction

Having a simplicial algebra pair in \( A \), we can apply again, levelwise, the bar construction from Definition 5.5 and obtain the following.

**Definition 5.8.** Let \( (A, B, f) \in \text{Alg}(A)^{[0,1]} \) be a commutative algebra pair in a braided monoidal category. Regard \( (B_n(A, B, f), B_n(B, B, \text{Id}_B), B_n(f, \text{Id}_B)) \) as an
object in $s\text{Alg}(A)^{[0,1]}$. We denote by $B_{\bullet,\bullet}(A,B,f)$ the bisimplicial object in $A$ obtained by applying levelwise the bar construction: explicitly, we have

$$B_{p,q}(A,B,f) = B_{p}(B_{q}(A,B,f), B_{q}(B,B,\text{Id}_{B}), B_{q}(f,\text{Id}_{B})), $$

where we use that $(B_{q}(A,B,f), B_{q}(B,B,\text{Id}_{B}), B_{q}(f,\text{Id}_{B})) \in \text{Alg}(A)^{[0,1]}$.

For $0 \leq i \leq p$ and $q \geq 0$ with $p \neq 0$, we denote by $d_{i}^{\text{hor}} : B_{p,q}A \to B_{p-1,q}A$ the $i$th horizontal face map; for $p \geq 0$ and $0 \leq j \leq q$ with $q \neq 0$, we denote by $d_{j}^{\text{ver}} : B_{p,q}A \to B_{p,q-1}A$ the $j$th vertical face map. We use a similar notation $s_{i}^{\text{hor}}$ and $s_{j}^{\text{ver}}$ for the horizontal and vertical degeneracy maps.

Note that, even if $(A,B,f)$ is a commutative algebra pair, in general $B \otimes A^{\otimes n} \otimes B$ and $B^{\otimes n+2}$ are not commutative algebras: thus, in general, $B_{\bullet}(A,B,f)$ is not a simplicial object in commutative algebra pairs, hence $B_{\bullet,\bullet}$ is in general not a bisimplicial object in $\text{Alg}(A)$, hence in general one cannot further iterate the bar construction.

For completeness, we remark that if $(A,B,f) \in \text{Alg}(A)$ is a commutative algebra pair and the following equalities hold

- $\text{br}_{A,A} \circ \text{br}_{A,A} = \text{Id}_{A \otimes A}$;
- $\text{br}_{A,B} \circ \text{br}_{B,A} = \text{Id}_{B \otimes B}$;
- $\text{br}_{B,A} \circ \text{br}_{A,B} = \text{Id}_{A \otimes B}$,

then $(B_{\bullet}(A,B,f), B_{\bullet}(B,B,\text{Id}_{B}), B_{\bullet}(f,\text{Id}_{B}))$ is a commutative algebra pair in $sA$ (which is endowed with the levelwise braided monoidal structure) satisfying the analogous of three equalities above, and in fact one can iterate arbitrarily many times the bar construction. For instance, if $A$ is symmetric monoidal, then the three equalities above are satisfied. However we will be most interested in examples in which the three equalities above do not hold (in fact, the first does not hold), so we will only focus on the double bar construction.

6. Simplicial Hurwitz spaces

We fix a PMQ $Q$ throughout the section, and denote by $\hat{Q}$ its completion. In this section we will construct a CW-complex $\lvert \text{Arr}(Q) \rvert$, obtained as geometric realisation of a bisimplicial set $\text{Arr}(Q)$. The definition of $\text{Arr}(Q)$ is based on arrays of elements of $Q$ and is an instance of a double bar construction.

We then assume that $Q$ is augmented, and define a sub-bisimplicial set $\text{NAdm}(Q)$ of $\text{Arr}(Q)$, containing the non-admissible part of $\text{Arr}(Q)$. The relative cellular chain complex $\text{Ch}_{\bullet}(\lvert \text{Arr}(Q) \rvert, \lvert \text{NAdm}(Q) \rvert)$ has a rather simple, combinatorial description: in fact, it can also be regarded as the (reduced) total chain complex associated with a certain bisimplicial abelian group: again, this bisimplicial abelian group can be obtained through a double bar construction.

For an augmented PMQ $Q$, the simplicial Hurwitz space $\text{Hur}^\Delta(Q)$ is then defined as the difference of spaces $\lvert \text{Arr}(Q) \rvert \setminus \lvert \text{NAdm}(Q) \rvert$. This is the first definition of generalised Hurwitz space with coefficients in a PMQ that we give in this series of articles. The space $\text{Hur}^\Delta(Q)$ has the advantage of having a purely combinatorial definition in terms of the PMQ $Q$: its main disadvantages are the following.

- The space $\text{Hur}^\Delta(Q)$ arises as a difference of CW-complexes, and as such it is not immediately endowed with a CW structure; as a consequence, we do not have a simple model for the singular chain complex of $\text{Hur}^\Delta(Q)$, which can be used, for instance, for homology computations. In the special case in which all
connected components of $\text{Hur}^\Delta(Q)$ happen to be manifolds, possibly of different dimensions, then we can compute the homology of $\text{Hur}^\Delta(Q)$ using the compactly supported cellular cochain complex $\text{Ch}_{\text{cpt}}(|\text{Arr}(Q)|, |\text{NAdm}(Q)|)$, by means of Poincare-Lefschetz duality. We will introduce in this article the notion of Poincare PMQ and give some examples, but we will postpone to the next articles the thorough study of this property.

- Points in the space $\text{Hur}^\Delta(Q)$ can be interpreted as configurations $P$ of points in the open, unit square $(0,1)^2$ with the additional information of a "monodromy" with values in $Q$. The construction has an evident functoriality in the PMQ $Q$, but there is no functoriality in the ambient space (which is fixed to be $(0,1)^2$, with its product structure $(0,1) \times (0,1)$). This prevents the possibility to define operations on Hurwitz spaces that combine two or more configurations, or deform conveniently one configuration into another. This issue will be addressed in the next articles.

6.1. $\hat{Q}$-crossed objects. The following definition extends the classical notion of $G$-crossed objects in a category (see for instance [FY89] Section 4.2) from the specific case of a group $G$ to the more general case of a complete PMQ $\hat{Q}$.

**Definition 6.1.** We define a (small) category $\hat{Q}\mathbb{//}\hat{Q}$. Its set of objects is $\hat{Q}$; for $a, b \in \hat{Q}$, a morphism $a \to b$ is given by an element $c \in \hat{Q}$ such that $b = a^c$. Composition of morphisms is given by multiplication in $\hat{Q}$.

If $A$ is any category, we denote by $X\mathbb{A}(\hat{Q})$ the category of functors $\hat{Q}\mathbb{//}\hat{Q} \to A$, with natural transformations as morphisms. An object in $X\mathbb{A}(\hat{Q})$ is also called a $\hat{Q}$-crossed object in $A$.

Concretely, an object $X \in X\mathbb{A}(\hat{Q})$ consists of objects $X(a)$ for all $a \in \hat{Q}$, together with maps $(-)^b : X(a) \to X(a^b)$ for all $a, b \in \hat{Q}$.

In all examples we will consider, $A$ will be a category with coproducts; in this case we have a forgetful functor

$$U : X\mathbb{A}(\hat{Q}) \to A, \quad X \mapsto \coprod_{a \in \hat{Q}} X(a);$$

we will abuse notation and regard $X \in X\mathbb{A}(\hat{Q})$ as the object $\coprod_{a \in \hat{Q}} X(a) \in A$, which is endowed with a decomposition into subobjects $X(a)$ as well as with a right action of $\hat{Q}$, i.e. a map of PMQs $\rho : \hat{Q} \to \text{Aut}_A(X)^{op}$ with the property that $\rho_b$ restricts to a map $X(a) \to X(a^b)$ for all $a, b \in \hat{Q}$. The object $X(a)$ will called the part of $X$ of $\hat{Q}$-grading equal to $a$. The map $\rho_b$ is also denoted $(-)^b$.

Note also that, since $\text{Aut}_A(X)^{op}$ is a group, the map $\rho : \hat{Q} \to \text{Aut}_A(X)^{op}$ extends to a map of groups $\mathcal{G}(\rho) : \mathcal{G}(\hat{Q}) \to \text{Aut}_A(X)^{op}$, so a $\hat{Q}$-crossed object in $A$ is naturally endowed with an action of the group $\mathcal{G}(\hat{Q})$.

The examples of $\hat{Q}$-crossed categories that we will consider are the following:

- the category $X\text{Set}(\hat{Q})$ of $\hat{Q}$-crossed sets;
- the category $X\text{Top}(\hat{Q})$ of $\hat{Q}$-crossed spaces;
- for a commutative ring $R$, the category $X\text{Mod}_R(\hat{Q})$ of $\hat{Q}$-crossed $R$-modules.

In the special case $\hat{Q} = G$, for a group $G$, the category $X\text{Mod}_R(\hat{Q})$ was already considered in [ETW17] under the name of $G$-Yetter-Drinfeld modules.
Example 6.2. The set $\hat{Q}$ is a $\hat{Q}$-crossed set, by declaring $a \in \hat{Q}$ to lie in $\hat{Q}$-grading $a$, and by using the action of $\hat{Q}$ on itself by conjugation. In fact $\hat{Q}$ is final in the category $\textbf{XSet}(\hat{Q})$: every $\hat{Q}$-crossed set admits precisely one map of $\hat{Q}$-crossed sets towards $\hat{Q}$.

If $Q$ is an augmented PMQ and $\hat{Q}$ still denotes the completion of $Q$, we can regard $\hat{Q}$ as the disjoint union of three $\hat{Q}$-crossed sets, namely $\mathcal{J}(Q) = \hat{Q} \setminus Q$ (see Notation 2.23), $Q_+ = Q \setminus \{1\}$ (see Definition 4.9) and $\{1\}$.

Example 6.3. The PMQ-rings $R[Q]$ and $R[\hat{Q}]$ (see Definition 4.26) are $\hat{Q}$-crossed $R$-modules, by declaring the generator $[a]$ to lie in $\hat{Q}$-grading $a \in Q$, and by defining $[a]^b = [a^b]$ for all $b \in \hat{Q}$. Similarly $R[Q]$ is a $\hat{Q}$-crossed $R$-module.

Definition 6.4. Let $A$ be a monoidal category which admits coproducts and for which the monoidal product $- \otimes -$ is distributive with respect to coproducts (e.g., $A$ is closed monoidal). Denote by $1_A$ the monoidal unit of $A$.

We define a monoidal product on $\textbf{XA}(\hat{Q})$, denoted also $- \otimes -$, such that the forgetful functor $U: \textbf{XA}(\hat{Q}) \to A$ is strong monoidal. Given $X,Y \in \textbf{XA}$, we set

$$(X \otimes Y)(a) = \prod_{b,c \in \hat{Q}: bc=a} X(b) \otimes Y(c).$$

The action of $\hat{Q}$ is diagonal: for $d \in \hat{Q}$ the map $(-)^d: (X \otimes Y)(a) \to (X \otimes Y)(ad)$ restricts to the map $(-)^d \otimes (-)^d: X(b) \otimes Y(c) \to X(bd) \otimes Y(cd)$ whenever $bc = a$. Note that $X(bd) \otimes Y(cd)$ is indeed one of the objects occurring in the coproduct defining $(X \otimes Y)((bc)d)$, because of the equality $b^d c^d = (bc)^d$.

We obtain a monoidal structure on $\textbf{XA}(\hat{Q})$: associativity of the tensor product follows from the associativity of the product of $\hat{Q}$. The unit object of the monoidal category $\textbf{XA}(\hat{Q})$ is $1_{\textbf{XA}(\hat{Q})}$, defined by setting:

- $1_{\textbf{XA}(\hat{Q})}(1) = 1_A$, which is endowed with the trivial $\hat{Q}$-action;
- $1_A(a) = \emptyset_A$, for all $a \in \hat{Q} \setminus \{1\}$, where $\emptyset_A$ denotes the initial object (or empty coproduct) in $A$.

Note that by definition we have a chain of isomorphisms of objects in $A$

$$U(X) \otimes U(Y) = \left( \prod_{b \in \hat{Q}} X(b) \right) \otimes \left( \prod_{c \in \hat{Q}} Y(c) \right) \cong \prod_{b,c \in \hat{Q}} X(b) \otimes Y(c) \cong \prod_{a \in \hat{Q}} (X \otimes Y)(a) = U(X \otimes Y),$$

which is natural in $X$ and $Y$, so that the forgetful functor $U: \textbf{XA}(\hat{Q}) \to A$ is strong monoidal.

Definition 6.5. Let $A$ be a category as in Definition 6.4 and assume further that $A$ is braided monoidal, with braiding denoted $	ext{br}_{-,-}$.

We enhance the monoidal structure on $\textbf{XA}(\hat{Q})$ to a braided monoidal structure, by defining a braiding on $\textbf{XA}(\hat{Q})$, also denoted $	ext{br}_{-,-}$. For $X,Y \in \textbf{XA}(\hat{Q})$ and for $a \in \hat{Q}$, the braiding $\text{br}_{X,Y}(a): (X \otimes Y)(a) \to (Y \otimes X)(a)$ restricts, for all decompositions $a = bc$ in $\hat{Q}$, to the isomorphism $X(b) \otimes Y(c) \to Y(c) \otimes X(b^c)$ given
by the composition

\[ X(b) \otimes Y(c) \xrightarrow{br_{X(b),Y(c)}} Y(c) \otimes X(b) \xrightarrow{Id_{Y(c)} \otimes (-)} Y(c) \otimes Y(b'). \]

We check explicitly that the braidingings \( br_{-, -} \) on \( XA(\hat{Q}) \) satisfies the braid relation. Let \( X, Y, Z \in XA(\hat{Q}) \); we have to check for all \( a \in \hat{Q} \) the equality of the two following compositions of morphisms \( (X \otimes Y \otimes Z)(a) \to (Z \otimes Y \otimes X)(a) \):

\[
(b_{Y,Z} \otimes \text{Id}_X)(a) \circ (\text{Id}_Y \otimes b_{X,Z})(a) \circ (b_{X,Y} \otimes \text{Id}_Z)(a);
(\text{Id}_Z \otimes b_{X,Y})(a) \circ (b_{X,Z} \otimes \text{Id}_Y)(a) \circ (\text{Id}_X \otimes b_{Y,Z})(a).
\]

For all \( b, c, d \in \hat{Q} \) satisfying \( bcd = a \), both compositions restrict to the same morphism \( X(b) \otimes Y(c) \otimes Z(d) \to Z(d) \otimes Y(c'^d) \otimes X(b'^cd) \), namely

\[
(\text{Id}_{Z(d)} \otimes (-d) \otimes (-cd)) \circ b_{X(b),Y(c),Z(d)} : X(b) \otimes Y(c) \otimes Z(d) \to Z(d) \otimes Y(c'^d) \otimes X(b'^cd),
\]

where \( b_{X(b),Y(c),Z(d)} \) denotes either of the following compositions in \( A \):

\[
(b_{Y,c,Z}(d) \otimes \text{Id}_b)(a) \circ (b_{X,Y}(c) \otimes \text{Id}_{Z(d)})(a) \circ (b_{X,Z}(c) \otimes \text{Id}_{Y(d)})(a),
(\text{Id}_{Z(d)} \otimes b_{X,b,Y}(c))(a) \circ (\text{Id}_{Z(d)} \otimes b_{Y,c}(d))(a) \circ (\text{Id}_{X(b)} \otimes b_{Y,Z}(d))(a).
\]

This uses that the braiding of \( A \) satisfies the braid relation, together with the naturality of the braiding of \( A \) and the equality \( cd = dc \).

We leave to the reader to check that the other axioms of braided monoidal category are satisfied.

### 6.2. The bisimplicial set of arrays

Recall Example 6.2 and Definition 5.5. The monoid structure of the complete PMQ \( \hat{Q} \) makes \( \hat{Q} \) into a commutative algebra in \( XSet(\hat{Q}) \).

**Definition 6.6.** Let \( Q \) be a PMQ and let \( \hat{Q} \) denote its completion. We denote by \( \text{Arr}(Q) = B_{*, *}(\hat{Q}, \hat{Q}, \text{Id}_\hat{Q}) \). It is a bisimplicial \( \hat{Q} \)-crossed set, and by the levelwise forgetful functor

\[ \text{ssU} : \text{ssXSet}(\hat{Q}) \to \text{ssSet} \]

it can be regarded as a bisimplicial set.

Note that \( \text{Arr}(Q) \) only depends on the completion \( \hat{Q} \) of \( Q \). In the next subsection we will define a bisimplicial sub-\( \hat{Q} \)-crossed set \( \text{NAdm}(Q) \subset \text{Arr}(Q) \); the latter will depend indeed on \( Q \), and we will be mainly interested in the couple of bisimplicial \( \hat{Q} \)-crossed sets \( (\text{Arr}(Q), \text{NAdm}(Q)) \).

Note that the \( \hat{Q} \)-crossed set \( \text{Arr}_{p,q}(\hat{Q}) = B_{p,q}(\hat{Q}, \hat{Q}, \text{Id}_\hat{Q}) \) is isomorphic to the \( \hat{Q} \)-crossed set \( \hat{Q}^{\otimes (p+2)(q+2)} \), whose underlying set is \( \hat{Q}^{(p+2)(q+2)} \): we thus regard an element of \( \text{Arr}_{p,q}(\hat{Q}) \) as an array of size \( (p + 2) \times (q + 2) \) with entries in \( \hat{Q} \). More precisely, a generic array \( \underline{a} = (a_{i,j})_{0 \leq i \leq p+1, 0 \leq j \leq q+1} \) of size \( (p + 2) \times (q + 2) \) with entries in \( \hat{Q} \) consists of \( p + 2 \) columns \( \underline{a}_0, \ldots, \underline{a}_{p+1} \), containing each the entries \( a_{i,j} \) for a fixed value of \( i \); similarly, the \( \hat{Q} \)-crossed set \( B_{p,q}(\hat{Q}, \hat{Q}, \text{Id}_\hat{Q}) = (\hat{Q}^{\otimes p+2}) \otimes (q+2) \) can be regarded as a set under the forgetful functor \( U \): the set \( U(B_{p,q}(\hat{Q}, \hat{Q}, \text{Id}_\hat{Q})) \) is in canonical bijection with the set \( (\hat{Q}^{\otimes p+2})^{p+2} \), containing \( (p + 2) \)-tuples of \( (q + 2) \)-tuples of elements of \( \hat{Q} \).

The horizontal and vertical face and degeneracy maps of \( \text{Arr}(Q) \) can be described explicitly.
Notation 6.7. For an array $\underline{a} \in \text{Arr}_{p,q}(\mathcal{Q})$ and for $0 \leq i \leq p + 1$ we denote by $\underline{a}_i = (a_{i,0}, \ldots, a_{i,q+1})$ the $i$th column, which is a sequence of $q + 2$ elements in $\mathcal{Q}$.

For $0 \leq j \leq q + 2$ and for any sequence $\underline{b} = (b_0, \ldots, b_{q+1})$ of $q + 2$ elements in $\mathcal{Q}$ we denote by $c(\underline{b})_j$ the product $b_0 \ldots b_{j-1} \in \mathcal{Q}$; by convention we set $c(\underline{b})_0 = \mathbb{1}$.

The following lemma is a direct consequence of the definitions.

Lemma 6.8. Let $\underline{a} \in \text{Arr}_{p,q}(\mathcal{Q})$ for some $p, q \geq 0$.

- for $0 \leq i \leq p$, the degeneracy map $s^\text{hor}_i: \text{Arr}_{p,q}(\mathcal{Q}) \to \text{Arr}_{p+1,q}(\mathcal{Q})$ acts on $\underline{a}$ by adding an additional column made of $\mathbb{1}$’s, between the $i$th and $(i + 1)^{st}$ columns of $\underline{a}$;

- similarly, for $0 \leq j \leq q$, the degeneracy map $s^\text{ver}_j: \text{Arr}_{p,q}(\mathcal{Q}) \to \text{Arr}_{p,q+1}(\mathcal{Q})$ acts on $\underline{a}$ by adding an additional row made of $\mathbb{1}$’s, between the $j$th and $(j + 1)^{st}$ rows of $\underline{a}$;

- for $p \geq 1$ and $0 \leq i \leq p$, the face map $d^\text{hor}_i: \text{Arr}_{p,q}(\mathcal{Q}) \to \text{Arr}_{p-1,q}(\mathcal{Q})$ acts on $\underline{a}$ as follows: the columns of $d^\text{hor}_i(\underline{a})$ are obtained from those of $\underline{a}$ by replacing the $i$th and $(i + 1)^{st}$ columns of $\underline{a}$ by the following sequence of $q + 2$ elements in $\mathcal{Q}$:

$$
\begin{pmatrix}
a_{i,0}a_{i+1,0}, a_{i,1}^{a_{i+1,1}}, a_{i+1,1}, \ldots, a_{i+1,j}^{c(\underline{a}_{i+1,j})}, a_{i+1,j}, \ldots, a_{i+1,q+1}^{c(\underline{a}_{i+1,q+1})}, a_{i+1,q+1}+1
\end{pmatrix}.
$$

- for $q \geq 1$ and $0 \leq j \leq q$, the face map $d^\text{ver}_j: \text{Arr}_{p,q}(\mathcal{Q}) \to \text{Arr}_{p,q-1}(\mathcal{Q})$ acts on $\underline{a}$ as follows: the rows of $d^\text{ver}_j(\underline{a})$ are obtained from those of $\underline{a}$ by replacing the $j$th and $(j + 1)^{st}$ rows by the following sequence of $p + 2$ elements in $\mathcal{Q}$:

$$
\begin{pmatrix}
a_{0,j}a_{0,j+1}, a_{1,j}a_{1,j+1}, \ldots, a_{i,j}a_{i,j+1}, \ldots, a_{p+1,j}a_{p+1,j+1}
\end{pmatrix}.
$$

6.3. Non-degenerate arrays. In this subsection we assume for the first time that $\mathcal{Q}$ is augmented: under this hypothesis we will show that non-degenerate arrays in $\text{Arr}(\mathcal{Q})$ form a semi-bisimplicial $\mathcal{Q}$-crossed set.

Notation 6.9. An array $\underline{a} \in \text{Arr}_{p,q}(\mathcal{Q})$ is degenerate if it lies in the image of a degeneracy map $s^\text{hor}_i$ or $s^\text{ver}_j$ of the bisimplicial $\mathcal{Q}$-crossed set $\text{Arr}(\mathcal{Q})$. For all $p, q \geq 0$ we denote by $\text{Arr}^\text{ndeg}_{p,q}(\mathcal{Q}) \subset \text{Arr}_{p,q}(\mathcal{Q})$ the subset of non-degenerate arrays.

Visualising elements of $\text{Arr}(\mathcal{Q}) = B_{\bullet \mathcal{Q}}$ as arrays of elements in $\mathcal{Q}$ is very helpful: for instance, an array $\underline{a}$ is degenerate if and only if it has an inner row or an inner column made of $\mathbb{1}$’s, i.e. if either of the following holds:

- there is $1 \leq i \leq p$ such that for all $0 \leq j \leq q + 1$ we have $a_{i,j} = 1$;

- there is $1 \leq j \leq q$ such that for all $0 \leq i \leq p + 1$ we have $a_{i,j} = 1$.

Lemma 6.10. The $\mathcal{Q}$-crossed sets $\text{Arr}^\text{ndeg}_{p,q}(\mathcal{Q})$ assemble into a semi-bisimplicial $\mathcal{Q}$-crossed set $\text{Arr}^\text{ndeg}(\mathcal{Q})$, whose horizontal and vertical face maps are the restrictions of those of the bisimplicial $\mathcal{Q}$-crossed set $\text{Arr}(\mathcal{Q})$.

Proof. Let $\underline{a} \in \text{Arr}_{p,q}(\mathcal{Q})$ be an array, and let $0 \leq i \leq p + 1$ and $0 \leq j \leq q + 1$. We need to check that $d^\text{hor}_i(\underline{a})$ and $d^\text{ver}_j(\underline{a})$ are non-degenerate arrays, as soon as $\underline{a}$ is non-degenerate. Applying Lemma 6.8 we obtain the following:

- Suppose that $\underline{a}' := d^\text{hor}_i\underline{a} \in \text{Arr}_{p-1,q}(\mathcal{Q})$ is degenerate; then there are two possibilities.
The $(i')$th column of $a'$ is made of 1's, for some $1 \leq i' \leq p - 1$: if $i' \neq i$, we have that the $(i')$th column of $a'$ is also an inner column in $\tilde{a}$, witnessing that $\tilde{a}$ is degenerate; if $i' = i$, by Lemma 6.8 we have that the $i$th column of $a'$ contains the elements $a_{i,j+1,j'} a_{i+1,j'}$, for varying $0 \leq j' \leq q + 1$, and if all these elements are equal to 1, using that $\tilde{Q}$ is augmented we obtain that both the $i$th and the $(i + 1)$st columns of $\tilde{a}$ are made of 1's, witnessing even twice that $\tilde{a}$ is degenerate.

The $(j')$th row of $a'$ is made of 1's, for some $1 \leq j' \leq q$: by Lemma 6.8 the $(j')$th row of $a'$ contains the elements $a_{i,j',j}$, for varying $0 \leq i' \leq p + 1$ with $i' \neq i, i + 1$, together with the element $a_{i,j+1,j'} a_{i+1,j'}$: again, if all elements in the $(j')$th row of $a'$ are 1's, then also all elements in the $(j')$th row of $\tilde{a}$ are 1's, witnessing that $\tilde{a}$ is degenerate.

Suppose that $a' := d^\text{cr}_p a \in \text{Arr}_{p,q-1}(\tilde{Q})$ is degenerate; then there are two possibilities.

The $(j')$th row of $a'$ is made of 1's, for some $1 \leq j' \leq q - 1$: if $j' \neq j$, we have that the $(j')$th row of $a'$ is also an inner row in $\tilde{a}$, witnessing that $\tilde{a}$ is degenerate; if $j' = j, i + 1$, together with the element $a_{i,j+1,j+1}$; again, if all elements in the $(j')$th row of $a'$ are 1's, then also all elements in the $(j')$th row of $\tilde{a}$ are 1's, witnessing that $\tilde{a}$ is degenerate.

This shows that a horizontal or vertical face of a non-degenerate array is again non-degenerate. \hfill \Box

In particular, the geometric realisation $|\text{Arr}(\tilde{Q})|$ of $\text{Arr}(\tilde{Q})$ as a bisimplicial set is homeomorphic to the thick geometric realisation $\|\text{Arr}^{\text{ndeg}}(\tilde{Q})\|$ of $\text{Arr}^{\text{ndeg}}(\tilde{Q})$ as a semi-bisimplicial set.

This has a mild advantage when constructing $\text{Ch}_n(|\text{Arr}(\tilde{Q})|)$, the cellular chain complex of $|\text{Arr}(\tilde{Q})|$: its generators are in bijection with cells of $|\text{Arr}(\tilde{Q})|$, which are in bijection with non-degenerate arrays (this would be true also in the setting of a non-augmented PMQ $\tilde{Q}$); moreover the differential in $\text{Ch}_n(|\text{Arr}(\tilde{Q})|)$ is given by the usual alternating sum of vertical and horizontal face maps, where no term has to be skipped (usually one skips terms corresponding to degenerate faces of a bisimplex).

The fact that $\text{Arr}(\tilde{Q})$ is a bisimplicial $\tilde{Q}$-crossed set (and similarly $\text{Arr}^{\text{ndeg}}(\tilde{Q})$ is a semi-bisimplicial $\tilde{Q}$-crossed set) enhances the topological space $|\text{Arr}(\tilde{Q})| \cong |\text{Arr}^{\text{ndeg}}(\tilde{Q})|$ to a $\tilde{Q}$-crossed space, i.e. an object in $\text{XTop}(\tilde{Q})$.

### 6.4. Non-admissible arrays.

Also in this subsection we assume that $\tilde{Q}$ and hence $\tilde{Q}$ are augmented.

**Definition 6.11.** Let $p,q \geq 0$; an array $\tilde{a} \in \text{Arr}_{p,q}(\tilde{Q})$ is admissible if both the following conditions hold:
satisfies condition (2) as must lie in \( \in \). An array is non-admissible if it is not admissible. We denote by \( \text{NAdm}_{p,q}(Q) \subset \text{Arr}_{p,q}(Q) \) the subset of non-admissible arrays. By definition \( \text{NAdm}_{p,q}(Q) \) arises as the (non-disjoint) union of two sets \( \text{NAdm}_{p,q}(Q) \) and \( \text{NAdm}_{p,q}(Q) \), containing arrays for which condition (1), respectively condition (2), fails.

Note that \( \text{NAdm}_{p,q}(Q) \) is closed under the action of \( \hat{Q} \) by conjugation, hence \( \text{NAdm}_{p,q}(Q) \) can be regarded as a \( \hat{Q} \)-crossed set. The same remark holds, for \( \star = 1, 2 \), for the subset \( \text{NAdm}_{p,q}(Q) \).

**Lemma 6.12.** Let \( \star = 1, 2 \); then the \( \hat{Q} \)-crossed sets \( \text{NAdm}_{p,q}(Q) \), for varying \( p, q \geq 0 \), assemble into a sub-\( \hat{Q} \)-crossed bisimplicial set \( \text{NAdm}(Q) \subset \text{Arr}(Q) \). As a consequence also the sets \( \text{NAdm}_{p,q}(Q) \) assemble into a sub-\( \hat{Q} \)-crossed bisimplicial set \( \text{NAdm}(Q) \subset \text{Arr}(Q) \).

**Proof.** We will prove that all face and degeneracy maps of \( \text{Arr}(Q) \) reflect each of condition (1) and (2) in Definition 6.11: this means, for example, that if \( \vec{a} \in \text{Arr}_{p,q}(Q) \), and if \( \hat{d}_{ij}^{\text{ver}}(\vec{a}) \) satisfies condition (2), then \( \vec{a} \) satisfies condition (2) as well.

We start from condition (1). Here the argument is analogous as the one in the proof of Lemma 6.10 so we omit it.

We consider now condition (2). Let \( \vec{a} \in \text{Arr}_{p,q}(Q) \) for some \( p, q \geq 0 \) and let \( 0 \leq i \leq p \). The degeneracy map \( s_i^{\text{hor}} \) acts on \( \vec{a} \) by adding an additional inner column made of 1’s; in particular the entries of \( \vec{a} \) are all contained in the set of entries of \( s_i^{\text{hor}}(\vec{a}) \), and thus if \( s_i^{\text{hor}}(\vec{a}) \) satisfies condition (2), then so does also \( \vec{a} \).

A similar argument shows that the vertical degeneracy maps \( s_j^{\text{ver}} \) reflect condition (2).

Let now \( 0 \leq i \leq p \) and suppose that \( \hat{d}_{i}^{\text{hor}}(\vec{a}) \) satisfies condition (2). The columns of \( \hat{d}_{i}^{\text{hor}}(\vec{a}) \) are obtained from those of \( \vec{a} \) by replacing the \( i \)-th and \( (i + 1) \)-st column by the sequence of elements \( a_{i,j}a_{i,j+1} \), for \( 0 \leq j \leq q + 1 \); in particular all entries of \( \vec{a} \) occur (up to conjugation) as factors of entries of \( \hat{d}_{i}^{\text{hor}}(\vec{a}) \). Since \( \hat{J}(Q) = Q \setminus Q \) is an ideal, if all entries of \( \hat{d}_{i}^{\text{hor}}(\vec{a}) \) lie in \( Q \), then also all entries of \( \vec{a} \) must lie in \( Q \). Hence condition (2) is reflected by horizontal face maps.

Let finally \( 0 \leq j \leq p \) and suppose that \( \hat{d}_{j}^{\text{ver}}(\vec{a}) \) satisfies condition (2). The rows of \( \hat{d}_{j}^{\text{ver}}(\vec{a}) \) are obtained from those of \( \vec{a} \) by replacing the \( j \)-th and \( (j + 1) \)-st row by the sequence of elements \( a_{i,j}a_{i,j+1} \), for \( 0 \leq i \leq p + 1 \); again we notice that all entries of \( \vec{a} \) occur as factors of entries of \( \hat{d}_{j}^{\text{ver}}(\vec{a}) \), and the same argument used before shows that condition (2) is reflected by vertical face maps. \( \square \)

### 6.5. Configurations with monodromy

Combining Lemmas 6.10 and 6.12 we obtain that also the intersection \( \text{NAdm}^\text{adeg}(Q) = \text{NAdm}(Q) \cap \text{Arr}^\text{adeg}(Q) \) is a semi-bisimplicial \( \hat{Q} \)-crossed set, and we obtain a homeomorphism of \( \hat{Q} \)-crossed space \( |\text{NAdm}(Q)| \cong |\text{NAdm}^\text{adeg}(Q)| \). We also have an inclusion of \( \hat{Q} \)-crossed spaces \( |\text{NAdm}(Q)| \subset |\text{Arr}(Q)| \).

**Definition 6.13.** The simplicial Hurwitz space with coefficients in \( Q \), denoted \( \text{Hur}^\Delta(Q) \), is the \( \hat{Q} \)-crossed space \( |\text{Arr}(Q)| \setminus |\text{NAdm}(Q)| \).
Recall Example 6.2, and note that \( \hat{Q} \), considered as a discrete space, is a final object also in \( \mathbf{XTop}(\hat{Q}) \). We denote by \( \omega: \text{Hur}^\Delta(\hat{Q}) \to \hat{Q} \) the unique map of \( \hat{Q} \)-crossed spaces, and call it the \textit{total monodromy}.

The right action of \( \hat{Q} \) (and hence of \( \mathcal{G}(\hat{Q}) = \mathcal{G}(Q) \)) on \( \text{Hur}^\Delta(Q) \) is called the \textit{action by global conjugation}.

A point in \( \text{Hur}^\Delta(Q) \) can be interpreted as a configuration of points in the unit square \((0,1)^2\) with the additional information of a monodromy with values in \( Q \). We briefly describe this idea in the following, and refer to [Bia21a] for the precise construction; see also Figure 1.

![Figure 1](image-url)

**Figure 1.** A configuration in \( \text{Hur}^\Delta(Q) \) of the form \((a; s, t)\), where \( a \in \text{Arr}_{p,q}(\hat{Q}) \) is an admissible array of size \( 4 \times 5 \) whose only entries in \( Q_+ \) are \( a_{1,1}, a_{1,2}, a_{2,2} \), and \( a_{2,3} \). We have \( I(a) = \{(1,1),(1,2),(2,2),(2,3)\} \), and \( P \) consists of four points \( z_{1,1}, z_{1,2}, z_{2,2}, z_{2,3} \). The monodromy \( \psi \) associates with the element \( [\gamma_{i,j}] \in \pi_1(C \setminus P; *) \) the element \( a_{i,j} \in Q_+ \). Since \( [\gamma] = [\gamma_{2,2}]^{[\gamma_{1,1}]^{-1}} \) in \( \pi_1(C \setminus P; *) \), the monodromy \( \psi \) associates with \( [\gamma] \) the element \( a_{2,2}^{(a_{1,1}[a_{1,2}])^{-1}} \in Q \), where we use the adjoint action of \( \mathcal{G}(Q) \) on \( Q \).

Let \( a \) be a non-degenerate, admissible array in \( \text{Arr}_{p,q}(\hat{Q}) \), for some \( p, q \geq 0 \), and let \( s = (0 = s_0 < s_1 < \cdots < s_{p+1} = 1) \) and \( t = (0 = t_0 < t_1 < \cdots < t_{q+1} = 1) \) be the coordinates of a point in the interior of \( \Delta^p \times \Delta^q \). The datum of \( a, s, t \) identifies a point in \( \text{Hur}^\Delta(Q) \), which we shall denote by \((a; s, t)\).

Let \( I(a) \subset \{0, \ldots, p+1\} \times \{0, \ldots, q+1\} \) denote the set of pairs \((i, j)\) with \( a_{i,j} \in Q_+ = Q \setminus \{1\} \), and note that in fact \( I(a) \subset \{1, \ldots, p\} \times \{1, \ldots, q\} \). Let
subsection we describe the cellular chain complex $Ch^\ast (\pi_1(\mathbb{C}\setminus P, \ast))$ represented by a simple loop the algebra pair $(\mathbb{Z}_{\text{MOD}})$ unit of function $A_{\gamma}$ a commutative algebra in these spaces are one of the central objects of study in [EVW16, ETW17, RW19].

extend this norm to the completion $\hat{G}$ subset, and consider is the union of the components of $\hat{G}$ crossed space $\hat{Q}$ of $\hat{Q}$-grading (or total monodromy) equal to some $a \in \hat{Q}$ of norm $n$.

The space $\text{Hur}_{\hat{Q}}(\hat{Q})_n$ is homeomorphic to the classical Hurwitz space $\text{Hur}_{G,n}$ of branched $G$-coverings of $(0,1)^2$ with $n$ branch points and local monodromies in $c$; these spaces are one of the central objects of study in [EVW16, ETW17, RW19].

Example 6.15. Let $A$ be a partial abelian monoid (with unit $\mathbb{1} \in A$), and consider $A$ as an abelian PMQ. Then $\text{Hur}^A$ is weakly equivalent to the space $C((0,1)^2; A)$ of configurations of points in $(0,1)^2$ with labels in $A$; these spaces are a special case of the main construction in [Sa01].

6.6. The relative cellular chain complex. Let $R$ be a commutative ring. In this subsection we describe the cellular chain complex $Ch_*([\text{Arr}(\hat{Q})], [\text{NAdm}(\hat{Q})]; R)$. Note that this chain complex is obtained from the $\hat{Q}$-crossed pair of CW-complex $([\text{Arr}(\hat{Q})], [\text{NAdm}(\hat{Q})])$, and as such is a $\hat{Q}$-crossed chain complex in $R$-modules.

We will compare this chain complex with the double bar construction applied to the algebra pair $(R[\hat{Q}], R, \varepsilon_{\hat{Q}}) \in \text{Alg}(\text{XMod}_R(\hat{Q}))^{[0,1]}$, where $R[\hat{Q}]$ is regarded as a commutative algebra in $\text{XMod}_R(\hat{Q})$ as in Example 6.3. $R$ denotes the monoidal unit of $\text{XMod}_R(\hat{Q})$, and is thus an algebra in this category; and $\varepsilon_{\hat{Q}}: R[\hat{Q}] \to R$ is the augmentation given by $[a] \mapsto 0$ for all $a \in \hat{Q}$ and $[\mathbb{1}, \hat{Q}] \mapsto 1$. 
Theorem 6.16. The cellular chain complex $\text{Ch}_*([\text{Arr}(\hat{Q})], |\text{NAdm}(\hat{Q})|; R)$ is isomorphic, as $\hat{Q}$-crossed chain complex in $R$-modules, to the reduced total chain complex associated with the bisimplicial $\hat{Q}$-crossed $R$-module $B_{*,*}(R[\hat{Q}], R, \varepsilon_{\hat{Q}})$.

The proof of Theorem 6.16 is the content of the rest of the subsection. First we note that $\text{Ch}_*([\text{Arr}(\hat{Q})]; R)$ can be obtained from the bisimplicial $\hat{Q}$-crossed sets $\text{Arr}(\hat{Q})$ in a two-step process.

The first step of the process is to replace each $\text{Arr}_{p,q}(\hat{Q}) \in \text{XSet}(\hat{Q})$ with the corresponding $R$-linearisation $R[\text{Arr}_{p,q}(\hat{Q})] \in \text{XMod}_R$: more precisely, we consider the free $R$-module functor $R[-]: \text{Set} \to \text{Mod}_R$, which induces first a levelwise free $R$-module functor $R[-]: \text{XSet} \to \text{XMod}_R(\hat{Q})$, and then a free $R$-module functor $R[-]: \text{ssXSet}(\hat{Q}) \to \text{ssXMod}_R(\hat{Q})$. We apply the latter functor to $\text{Arr}(\hat{Q})$.

In a similar fashion we can apply the functor $R[-]$ to $\text{NAdm}(\hat{Q}) \in \text{ssXMod}_R(\hat{Q})$. The inclusion of bisimplicial $\hat{Q}$-crossed sets $\text{NAdm}(\hat{Q}) \to \text{Arr}(\hat{Q})$ induces an inclusion of bisimplicial $\hat{Q}$-crossed $R$-modules $R[\text{NAdm}(\hat{Q})] \to R[\text{Arr}(\hat{Q})]$; note now that $\text{ssXMod}_R(\hat{Q})$ is an abelian category, because it is a category of functors into $\text{Mod}_R$; hence we can consider the quotient

$$R[\text{Arr}(\hat{Q})]/R[\text{NAdm}(\hat{Q})] \in \text{ssXMod}_R(\hat{Q}),$$

given levelwise, for $p, q \geq 0$, by the $\hat{Q}$-crossed $R$-module

$$(R[\text{Arr}(\hat{Q})]/R[\text{NAdm}(\hat{Q})])_{p,q} = R[\text{Arr}(\hat{Q})]_{p,q}/R[\text{NAdm}(\hat{Q})]_{p,q}.$$ We have an isomorphism of free $R$-modules $R[\text{Arr}(\hat{Q})]_{p,q} \cong (R[\hat{Q}])^{\otimes (p+2) \times (q+2)}$; in other words, $R[\text{Arr}(\hat{Q})]_{p,q}$ can be regarded as the free $R$-module on the set of all arrays of size $(p+2) \times (q+2)$ with entries in $\hat{Q}$; the submodule $R[\text{NAdm}(\hat{Q})]_{p,q}$ is then spanned by the non-admissible arrays of size $(p+2) \times (q+2)$.

Keeping the previous analysis in mind, recall Definition 5.2 and note that there is a map $(R[\hat{Q}], R[\hat{Q}], \text{Id}_{R[\hat{Q}]}) \to (R[\hat{Q}], R, \varepsilon_{\hat{Q}})$ of algebra pairs in $\text{XMod}_R(\hat{Q})$, given by the horizontal arrows in the commutative square

$$\begin{array}{ccc}
R[\hat{Q}] & \overset{-/\mathcal{J}(\hat{Q})}{\longrightarrow} & R[\hat{Q}] \\
\downarrow \text{Id}_{R[\hat{Q}]} & & \downarrow \varepsilon_{\hat{Q}} \\
R[\hat{Q}] & \overset{\varepsilon_{\hat{Q}}}{\longrightarrow} & R.
\end{array}$$

Here $\varepsilon_{\hat{Q}}: R[\hat{Q}] \to R$ denotes the augmentation sending $a \mapsto 0$ for $a \in \hat{Q}_+$ and $1_{\hat{Q}} \mapsto 1$; note that $\varepsilon$ is a map of algebras in $\text{XMod}_R(\hat{Q})$, since $\hat{Q}$ is augmented. Similarly we define the map of algebras $\varepsilon_{\hat{Q}}: R[\hat{Q}] \to R$. The map $-/[\mathcal{J}(\hat{Q})]$ is the quotient by the bilateral ideal of $R[\hat{Q}]$ generated by the elements $[a]$ for $a \in \mathcal{J}(\hat{Q}) = \hat{Q} \setminus \hat{Q}$.

Applying the double bar construction we obtain a morphism of bisimplicial $\hat{Q}$-crossed $R$-modules

$$(-/\mathcal{J}(\hat{Q}), \varepsilon_{\hat{Q}}): R[\text{Arr}(\hat{Q})] = B_{*,*}(R[\hat{Q}], R[\hat{Q}], \text{Id}_{R[\hat{Q}]}) \to B_{*,*}(R[\hat{Q}], R, \varepsilon_{\hat{Q}}),$$
which for fixed \( p, q \geq 0 \) restricts to a morphism of \( \hat{Q} \)-crossed \( R \)-modules

\[
\left( -/\lfloor \mathcal{J}(Q) \rfloor, \varepsilon_{\hat{Q}} \right)_{p,q} : R[\text{Arr}(Q)]_{p,q} \xrightarrow{\cong} B_{p,q}(R[Q], R, \varepsilon_Q)
\]

\[
\cong \quad \cong \quad R[\hat{Q}]^{(p+2) \times (q+2)} \quad R[Q]^{\otimes p \times q}.
\]

The rightmost isomorphism comes from interpreting \( B_{p,q}(R[Q], R, \varepsilon_Q) \) as the free \( R \)-module on the set of admissible arrays of size \((p + 2) \times (q + 2)\). The map \((-/\lfloor \mathcal{J}(Q) \rfloor, \varepsilon_{\hat{Q}})_{p,q}\) is then precisely the quotient by the submodule spanned by non-admissible arrays, and therefore we have a short exact sequence in \( ssX\text{Mod}_R(\hat{Q}) \)

\[
0 \to R[N\text{Adm}(Q)] \to R[\text{Arr}(Q)] \to B_{\bullet, \bullet}(R[Q], R, \varepsilon_Q) \to 0.
\]

The second step replaces a bisimplicial \( R \)-module \( M = M_{\bullet, \bullet} \) with its reduced, total chain complex \( \hat{\text{Ch}}(M) \); we set

\[
\hat{\text{Ch}}_n(M) = \oplus_{i+j=n} \hat{M}_{i,j},
\]

where \( \hat{M}_{i,j} \) is the quotient of \( M_{i,j} \) by the sum of all images of all degeneracy maps in \( M \) with target \( M_{i,j} \), both horizontal and vertical. The differential \( \partial_n : \hat{\text{Ch}}_n(M) \to \hat{\text{Ch}}_{n-1}(M) \) is induced on the summand \( \hat{M}_{i,j} \) by the formula \( \sum_{i' = 0}^i (-1)^i d^0_{i', j} + (-1)^{i+1} \sum_{j' = 0}^j (-1)^j d^1_{i, j'} \).

We call \( \hat{\text{Ch}}_n(M) \) the reduced, total chain complex associated with the bisimplicial \( \hat{Q} \)-crossed \( R \)-module \( M \); it is a \( \hat{Q} \)-crossed chain complex in \( R \)-modules.

In particular we have an isomorphism of chain complexes

\[
\hat{\text{Ch}}_n(R[\text{Arr}(Q)]/R[N\text{Adm}(Q)]) \cong \hat{\text{Ch}}_n(B_{\bullet, \bullet}(R[Q], R, \varepsilon_Q)),
\]

and the left hand side is naturally identified with \( \hat{\text{Ch}}_n([\text{Arr}(Q)], [N\text{Adm}(Q)]; R) \).

### 6.7. Functoriality of Hurwitz spaces in the PMQ.

Let \( Q \) and \( Q' \) be augmented PMQs, and let \( \psi : Q \to Q' \) be an augmented map of PMQs, i.e. \( \psi(Q_+) \subset Q'_+ \), and let \( \hat{\psi} : \hat{Q} \to \hat{Q}' \) denote the corresponding map between completions.

We can consider \( \text{Arr}(Q) \) and \( \text{Arr}(Q') \) as plain bisimplicial sets, under the forgetful functors \( U : ssX\text{Set}(\hat{Q}) \to ss\text{Set} \) and \( U : ssX\text{Set}(\hat{Q}') \to ss\text{Set} \). Then the assignment

\[
\bar{a} = (a_{i,j})_{i,j} \mapsto \hat{\psi}(\bar{a}) = (\psi(a_{i,j}))_{i,j}
\]

defines a map of bisimplicial sets \( \psi_* : U\text{Arr}(Q) \to U\text{Arr}(Q') \). This map sends admissible bisimplices to admissible bisimplices and non-degenerate simplices to non-degenerate simplices; in particular it induces a map of spaces \( \psi_* : [U\text{Arr}(Q)] \to [U\text{Arr}(Q')] \) which restricts to a map between Hurwitz spaces

\[
\psi_* : \text{Hur}^\Delta(Q) \to \text{Hur}^\Delta(Q').
\]

This map behaves well with respect to the total monodromy, i.e. \( \psi_* \) sends the subspace \( \text{Hur}^\Delta(Q)(a) \) inside \( \text{Hur}^\Delta(Q')(\hat{\psi}(a)) \) for all \( a \in \hat{Q} \).

Note that in general it is not true that \( \psi_* : U\text{Arr}(Q) \to U\text{Arr}(Q') \) restricts to a map \( U\text{NAdm}(Q) \to U\text{NAdm}(Q') \); this is for instance almost never the case when \( Q' = \hat{Q} \).
6.8. Poincare PMQs. Let $Q$ be an augmented PMQ. As a space, $\text{Hur}^\Delta(Q)$ is the difference of two CW-complexes $|\text{Arr}(Q)| \setminus |\text{Adm}(Q)|$, so in particular it has no evident CW-structure; this prevents us to find, for instance, a simple, combinatorial chain complex to study the homology of $\text{Hur}^\Delta(Q)$, as for example a cellular chain complex would be. In a particular situation, which is described in the following definition, we can circumvent this problem.

**Definition 6.17.** Let $Q$ be a locally finite PMQ with completion $\hat{Q}$. We say that $Q$ is *Poincare* if, for all $a \in \hat{Q}$, the space $\text{Hur}^\Delta(Q)(a)$ is a topological manifold of some dimension.

The hypothesis that $Q$ is locally finite in Definition 6.17 is to ensure that the space $\text{Hur}^\Delta(Q)$ is locally compact: if $Q$ is locally finite, then every *admissible, non-degenerate* array $a \in \text{Arr}_{p,q}(Q)$ is the image of finitely many *admissible, non-degenerate* arrays in $\text{Arr}(Q)$ under iterated horizontal and vertical face maps: as a consequence of this observation, $\text{Hur}^\Delta(Q)$ is locally compact.

**Example 6.18.** Let $Q$ be a PMQ with trivial multiplication, define the norm $N: Q \to \mathbb{N}$ by setting $N(a) = 1$ for all $a \in \mathbb{Q}_+$, let $Q$ be the completion of $Q$ and let $N: \hat{Q} \to \mathbb{N}$ be the extension of the norm.

Let $a \in \hat{Q}$ and set $n = N(a)$: then $\text{Hur}^\Delta(Q)(a)$ is a covering space of the $n$th unordered configuration space of $n$ points in $(0,1)^2$, which we denote by $\text{Conf}_n((0,1)^2)$. Since $\text{Conf}_n((0,1)^2)$ is a manifold of (real) dimension $2n$, the same holds for the space $\text{Hur}^\Delta(Q)(a)$. It follows that $Q$ is Poincare. Note also that the intrinsic pseudonorm $h: Q \to \mathbb{N}$ is in fact a norm and coincides with $N$.

**Example 6.19.** Let $Q = \mathbb{N}$ or $Q = \{0,1,\ldots,n\} \subset \mathbb{N}$ as in Example 4.34. Then the completion $\hat{Q}$ of $Q$ is canonically identified with $\mathbb{N}$; for all $n \in \mathbb{N}$ the space $\text{Hur}^\Delta(Q)(n)$ is homeomorphic to the $n$-fold symmetric product $\mathbb{S}^n((0,1)^2)$, which is homeomorphic to $\mathbb{R}^{2n}$. It follows that $Q$ is Poincare. Note also that the intrinsic pseudonorm $h: Q \to \mathbb{N}$ is a norm and coincides with the natural inclusion of $Q$ into $\mathbb{N}$. In the case $Q = \{0,1,\ldots,n\}$ with $n \geq 2$ we obtain an example of a PMQ that is Poincare but not Koszul.

**Example 6.20.** Let $Q$ be as in Example 4.35. Then $\text{Hur}^\Delta(Q)(c)$ is homeomorphic to the union of two copies of $(0,1)^2 \times (0,1,1)^2$ along their diagonal subspace $(0,1)^2 \subset (0,1)^2 \times (0,1,1)^2$. In fact, $\text{Hur}^\Delta(Q)(c)$ can be regarded as the union of $\text{Hur}^\Delta(\{1, a, b, c\})(c)$ and $\text{Hur}^\Delta(\{1, a', b', c\})(c)$ along $\text{Hur}^\Delta(\{1, c\})(c)$.

If $c \in \text{Hur}^\Delta(Q)(c)$ lies in the subspace $\text{Hur}^\Delta(\{1, c\})(c)$, then the local homology group

$$H_i(\text{Hur}^\Delta(Q)(c), \text{Hur}^\Delta(Q)(c) \setminus \{c\}; R)$$

is isomorphic to $R$ for $i = 3$ and to $R^2$ for $i = 4$. It follows that $Q$ is not Poincare, though it is Koszul.

**Proposition 6.21.** Let $Q$ be a Poincare PMQ. Then $Q$ is maximally decomposable and admits an intrinsic norm $h: Q \to \mathbb{N}$ in the sense of Definitions 4.18 and 4.19.

Moreover for all $b \in \hat{Q}$ the space $\text{Hur}^\Delta(Q)(b)$ is an oriented topological manifold of dimension $2h(b)$, where we extend the intrinsic norm to $h: \hat{Q} \to \mathbb{N}$.

**Proof.** Let $b \in \hat{Q}$. Since $Q$ is locally finite, $b$ can be decomposed in finitely many ways as a product $b_1\ldots b_r$ of elements of $\mathbb{Q}_+$. Fix a maximal decomposition $b =
\(\bar{b}_1 \ldots \bar{b}_r\), with all \(\bar{b}_i \in \mathbb{Q}_+\) irreducible, for some \(r \geq 0\); then we can define an admissible, non-degenerate array \(a \in \text{Arr}_{r,r}(\mathbb{Q})(b)\) by setting \(a_{i,j} = \bar{b}_i\) for all \(1 \leq i \leq r\), and all other entries \(a_{i,j} = 1\).

The array \(a\) is of maximal dimension among the admissible, non-degenerate arrays in \(\text{Arr}(\mathbb{Q})(b)\), so it corresponds to a maximal open cell of the cell stratification of \(\text{Hur}^\Delta(\mathbb{Q})(b)\). It follows that \(\text{Hur}^\Delta(\mathbb{Q})(b)\) is a manifold of dimension \(2r\). The same equality holds for every other maximal decomposition of \(b\) in elements of \(\mathbb{Q}_+:\) this implies both that \(\text{Hur}^\Delta(\mathbb{Q})(b)\) is a manifold of dimension \(2h(b)\) (in the sense of Definition 4.13), and that \(h: \mathbb{Q} \to \mathbb{N}\) satisfies the conditions of Definition 4.19 and is thus a norm on \(\mathbb{Q}\).

Let \(h: \mathbb{Q} \to \mathbb{N}\) be the extension of the intrinsic norm, and let \(b \in \mathbb{Q}\); we can again fix a maximal decomposition \(b = \bar{b}_1 \ldots \bar{b}_{h(b)}\), yielding by the same argument a maximal cell in the cell stratification of \(\text{Hur}^\Delta(\mathbb{Q})(b)\). This implies that \(\text{Hur}^\Delta(\mathbb{Q})(b)\), which is a manifold, must be a manifold of dimension \(2h(b)\).

We are left to check orientability of \(\text{Hur}^\Delta(\mathbb{Q})(b)\) for \(b \in \mathbb{Q}\). Let \(\mathcal{Q}_{\leq 1} \subset \mathbb{Q}\) be the subset of elements of norm \(\leq 1\), and note that \(\mathcal{Q}_{\leq 1}\) has a natural structure of PMQ with trivial multiplication. The inclusion \(\mathcal{Q}_{\leq 1} \subset \mathbb{Q}\) is a map of augmented PMQs, so it induces a map of Hurwitz spaces \(\text{Hur}^\Delta(\mathcal{Q}_{\leq 1}) \to \text{Hur}^\Delta(\mathbb{Q})\).

This map is injective, since the map of bisimplicial sets \(\text{Arr}(\mathcal{Q}_{\leq 1}) \to \text{Arr}(\mathbb{Q})\) restricts to an injective map between admissible arrays; note however that the completion \(\mathcal{Q}_{\leq 1}\) of \(\mathcal{Q}_{\leq 1}\) only surjects and need not be in bijection with \(\mathcal{Q}\), so in general \(\text{Arr}(\mathcal{Q}_{\leq 1}) \to \text{Arr}(\mathcal{Q})\) is only surjective. The complement of the image of \(\text{Hur}^\Delta(\mathcal{Q}_{\leq 1}) \to \text{Arr}(\mathcal{Q})\) is the geometric realisation of the sub-bisimplicial complex of \(\text{Arr}(\mathbb{Q})\) spanned by all arrays with at least one entry of norm \(\geq 2\). It follows that we can regard \(\text{Hur}^\Delta(\mathcal{Q}_{\leq 1})\) as an open subspace of \(|\text{Arr}(\mathbb{Q})|\), and hence as an open subspace of \(\text{Hur}^\Delta(\mathbb{Q})\).

Let \(\text{Hur}^\Delta(\mathcal{Q}_{\leq 1})(b)\) denote the intersection of \(\text{Hur}^\Delta(\mathcal{Q}_{\leq 1})\) with \(\text{Hur}^\Delta(\mathbb{Q})(b)\). Note that this is an abuse of notation, since \(b\) is an element of \(\mathcal{Q}\) and not of \(\mathcal{Q}_{\leq 1}\); in fact there are possibly several \(b' \in \mathcal{Q}_{\leq 1}\) mapping to \(b\) under the natural map \(\mathcal{Q}_{\leq 1} \to \mathcal{Q}\).

Note now that all maximal cells of the cell stratification of \(\text{Hur}^\Delta(\mathbb{Q})(b)\) correspond to arrays \(a \in \text{Arr}_{h(b),h(b)}(\mathbb{Q})(b)\) of the following special form: there exists a maximal decomposition \(b = \bar{b}_1 \ldots \bar{b}_{h(b)}\) of \(b\) into irreducible elements of \(\mathcal{Q}_{\leq 1}\), and there exists a permutation \(\sigma \in \mathfrak{S}_{h(b)}\), such that \(a_{i,h(i)} = \bar{b}_i\) for all \(1 \leq i \leq h(b)\), and all other entries \(a_{i,j}\) are equal to \(1\). All such arrays are in the image of \(\text{Arr}(\mathcal{Q}_{\leq 1}) \to \text{Arr}(\mathbb{Q})\), because all \(\bar{b}_i\) belong to \(\mathcal{Q}_{\leq 1}\); hence \(\text{Hur}^\Delta(\mathcal{Q}_{\leq 1})(b) \subset \text{Hur}^\Delta(\mathbb{Q})\) is a dense open subset.

Finally, note that in fact all cells of \(\text{Hur}^\Delta(\mathbb{Q})(b)\) of dimension \(\geq 2h(b) - 1\) correspond to arrays \(a \in \text{Arr}_{h(b),h(b)-1}(\mathbb{Q})\) or \(a \in \text{Arr}_{h(b)-1,h(b)}(\mathbb{Q})\) with a similar description as above, but allowing precisely one inner row or one inner column with two entries different from \(1\). In particular the complement of \(\text{Hur}^\Delta(\mathcal{Q}_{\leq 1})(b)\) in \(\text{Hur}^\Delta(\mathbb{Q})(b)\) has codimension \(\geq 2\) in the manifold \(\text{Hur}^\Delta(\mathbb{Q})(b)\).

Thus it suffices to check that \(\text{Hur}^\Delta(\mathcal{Q}_{\leq 1})(b)\) is orientable, and this follows from Example 6.18.

If \(\mathbb{Q}\) is a Poincare PMQ and \(R\) is a commutative ring, then for all \(a \in \mathbb{Q}\) we can apply Poincare-Lefschetz duality and obtain an isomorphism

\[
H_*(\text{Hur}^\Delta(\mathbb{Q})(a); R) \cong H_{\text{cpt}}^{2h(a)-*}(\text{Hur}^\Delta(\mathbb{Q})(a); R);
\]
Here $H_c^{p,q}(b)\circledast$ denotes cohomology with compact support. This can be computed, in principle, using the compactly supported cellular cochain complex of the couple $\langle |\text{Arr}(Q)|, |\text{NAdm}(Q)| \rangle$: the cochain complex

$$\text{Ch}_{\text{cpt}}^c (\langle |\text{Arr}(Q)|, |\text{NAdm}(Q)|; R)$$

consists of free $R$-modules, and has a generator $\varphi^*$ in degree $p + q - b(b)$ for every admissible, non-degenerate array $\varphi \in \text{Arr}_{p,q}(Q)(b)$.

We conclude with two other applications of the strategy of the proof of Proposition 6.21.

**Proposition 6.22.** Let $Q$ be a Poincare PMQ. Then $Q$ is coconnected.

**Proof.** Let $Q_{\leq 1}$ be as in the proof of Proposition 6.21, let $b \in Q$ and consider the open, dense subspace $\text{Hur}^\Delta(Q_{\leq 1})(b) \subset \text{Hur}^\Delta(Q)(b)$. We observe the following:

- The connected components of $\text{Hur}^\Delta(Q_{\leq 1})(b)$ are in bijection with the equivalence classes of decompositions $(b_1, \ldots, b_h(b))$ of $b$ with respect to $Q_1$, where two decompositions are considered equivalent if they are connected by a sequence of standard moves (see Definition 3.6 and compare with Definition 4.20).

- The space $\text{Hur}^\Delta(Q)(b)$ is connected: this follows from the fact that there is a unique admissible array of minimal dimension in $\text{Arr}(Q)(b)$, namely the array $\varphi \in \text{Arr}_{1,1}(Q)$ with $a_{1,1} = b$ and all other entries equal to $0$; this array is the image of every other admissible array in $\text{Arr}(Q)(b)$ under iterated horizontal and vertical face maps.

As argued in the proof of Proposition 6.21, the open inclusion $\text{Hur}^\Delta(Q_{\leq 1})(b) \subset \text{Hur}^\Delta(Q)(b)$ has complement of codimension $\geq 2$, hence it is a bijection on $\pi_0$. □

The second application is the definition of a canonical fundamental homology class for the couple of spaces $\langle |\text{Arr}(Q)(b)|, |\text{NAdm}(Q)(b)| \rangle$, for all $b \in Q$, assuming that $Q$ is a locally finite and coconnected PMQ. In the following we understand integral coefficients for homology.

Let $Q_{\leq 1}$ be as in the proof of Proposition 6.21, let $b \in Q$, and consider again the open subspace $\text{Hur}^\Delta(Q_{\leq 1})(b) \subset \text{Hur}^\Delta(Q)(b)$.

By Example 6.18, the space $\text{Hur}^\Delta(Q_{\leq 1})(b)$ is a finite covering of the unordered configuration space $\text{Conf}_{b(b)}((0,1)^2)$, in particular it admits a natural structure of complex manifold and thus a canonical orientation. Moreover, since $Q$ is coconnected, $\text{Hur}^\Delta(Q_{\leq 1})(b)$ is connected, i.e. there is a canonical ground class in $H_0(\text{Hur}^\Delta(Q_{\leq 1})(b))$. By Poincare-Lefschetz duality we get a canonical fundamental class in the homology group $H_{2h(b)}(\langle |\text{Arr}(Q)(b)|, |\text{Arr}(Q)(b)| \setminus \text{Hur}^\Delta(Q_{\leq 1})(b) \rangle)$.

Note now that there is a canonical isomorphism of homology groups

$$H_{2h(b)}(\langle |\text{Arr}(Q)(b)|, |\text{NAdm}(Q)(b)| \rangle)$$

$$\xrightarrow{\cong}$$

$$H_{2h(b)}(\langle |\text{Arr}(Q)(b)|, |\text{Arr}(Q)(b)| \setminus \text{Hur}^\Delta(Q)(b) \rangle)$$

due to the fact that all bisimplices of $|\text{Arr}(Q)(b)|$ of dimension $\geq 2h(b) - 1$ are contained in $\text{Hur}^\Delta(Q')(b) \subset \text{Hur}^\Delta(Q)(b)$. 

Definition 6.23. Let \( \mathcal{Q} \) be a cocompact PMQ and \( b \in \mathcal{Q} \). We denote by 
\[ \text{Arr}(\mathcal{Q})(b), \text{NAdm}(\mathcal{Q})(b) \in H_{2h(b)}(\text{Arr}(\mathcal{Q})(b), |\text{NAdm}(\mathcal{Q})(b)|; \mathbb{Z}) \]
the fundamental class constructed above. It is a generator of the homology group 
\[ H_{2h(b)}(\text{Arr}(\mathcal{Q})(b), |\text{NAdm}(\mathcal{Q})(b)|; \mathbb{Z}) \]; more generally, for any commutative ring \( R \), 
changing the coefficients from \( \mathbb{Z} \) to \( R \) yields a canonical fundamental class 
\[ \text{Arr}(\mathcal{Q})(b), \text{NAdm}(\mathcal{Q})(b) \in H_{2h(b)}(\text{Arr}(\mathcal{Q})(b), |\text{NAdm}(\mathcal{Q})(b)|; R) \cong R, \]
generating the top homology group as a free \( R \)-module of rank 1.

7. Geodesic PMQs associated with symmetric groups

The most interesting examples of PMQs that we study in this series of articles 
come from symmetric groups, considered as normed groups. In this section, for 
\( d \geq 2 \), we consider the PMQ \( \mathcal{S}^\text{geo}_d \), coming from the symmetric group on \( d \) letters 
endowed with the word length norm with respect to all transpositions. We will 
show that \( \mathcal{S}^\text{geo}_d \) is cocompact, pairwise determined and Koszul. In fact \( \mathcal{S}^\text{geo}_d \) is 
also Poincaré, but the proof of this fact will be given in the following articles.

The importance of the PMQs \( \mathcal{S}^\text{geo}_d \) relies on the connection between the Hurwitz 
spaces \( \text{Hur}^d(\mathcal{S}^\text{geo}_d) \), for varying \( d \geq 2 \), and the moduli spaces \( \mathcal{M}_{g,n} \) of Riemann 
surfaces of genus \( g \) with \( n \) ordered and parametrised boundary curves.

7.1. The PMQ \( \mathcal{S}^\text{geo}_d \) and its enveloping group.

Definition 7.1. For all \( d \geq 2 \) we will denote by \( \mathfrak{S}_d \) the group of permutations of the set \( \{1, \ldots, d\} \). For \( i \neq j \in \{d\} \) we denote \((i, j) \in \mathfrak{S}_d \) the transposition that exchanges \( i \) and \( j \). We consider on \( \mathfrak{S}_d \) the word length norm \( N \) with respect to the 
generating set of all transpositions: for \( \sigma \in \mathfrak{S}_d \), \( N(\sigma) \) is the smallest \( m \geq 0 \) 
such that there exist transpositions \( t_1, \ldots, t_m \in \mathfrak{S}_d \) with \( \sigma = t_1 \cdots t_m \).

Note that the sign of a permutation \( \sigma \) (i.e. the image of \( \sigma \) under the unique 
surjective map of groups \( \mathfrak{S}_d \to \{\pm 1\} \)) can be written as \((-1)^{N(\sigma)} \). By Definition 
4.7 we obtain a PMQ \( \mathcal{S}^\text{geo}_d \). Our first aim is to compute the group \( \mathcal{G}(\mathcal{S}^\text{geo}_d) \).

Lemma 7.2. Let \( d \geq 2 \) and let \( \mathfrak{S}_d \subseteq \mathbb{Z} \times \mathfrak{S}_d \) be the index 2 subgroup containing 
couples \((r, \sigma) \in \mathbb{Z} \times \mathfrak{S}_d \) such that \( r \) has the same parity as \( \sigma \).

Then the norm and the map \( \varepsilon^\text{geo} \) from Definition 4.8 give an injection of groups 
\( (\mathcal{G}(N), \varepsilon^\text{geo}) : \mathcal{G}(\mathcal{S}^\text{geo}_d) \to \mathbb{Z} \times \mathfrak{S}_d \)
with image the subgroup \( \mathfrak{S}_d \).

Proof. The group \( \mathcal{G}(\mathcal{S}^\text{geo}_d) \) is generated by all elements of the form \([\sigma] \), where \( \sigma \in \mathcal{S}^\text{geo}_d \). We have \( (\mathcal{G}(N), \varepsilon^\text{geo})([\sigma]) = (N(\sigma), \sigma) \in \mathfrak{S}_d \), hence the image of \( (N, \varepsilon) \) is 
contain in \( \mathfrak{S}_d \). Moreover, if \( t \in \mathfrak{S}_d \) is any transposition, then \( (\mathcal{G}(N), \varepsilon^\text{geo})([t]^2) = (2, 1) \), and the family of elements of the forms \((N(\sigma), \sigma) \), together with \((2, 1) \), 
generate \( \mathfrak{S}_d \).

Next we show that \((\mathcal{G}(N), \varepsilon^\text{geo}) \) is injective. First, note that the equality \( 1 \cdot 1 = 1 \) in \( \mathcal{S}^\text{geo}_d \) gives an equality \([1] = [1]^2 \) in \( \mathcal{G}(\mathcal{S}^\text{geo}_d) \), so that \([1] = 1 \) in \( \mathcal{G}(\mathcal{S}^\text{geo}_d) \). Note 
also that every \( \sigma \in \mathcal{S}^\text{geo}_d \) with \( \sigma \neq 1 \) can be written as a product \( t_1 \cdots t_{N(\sigma)} \) in 
\( \mathcal{S}^\text{geo}_d \), hence the generator \([\sigma] \in \mathcal{G}(\mathcal{S}^\text{geo}_d) \) is also equal to \([t_1] \cdots [t_{N(\sigma)}] \).

This shows that the elements \([t_i] \), for \( t \) varying in the transpositions of \( \mathfrak{S}_d \), 
generate \( \mathcal{G}(\mathcal{S}^\text{geo}_d) \).
Second, we show that for all transpositions \(t, t' \in \mathfrak{S}_d\), the equality \([t]^2 = [t']^2\) holds in \(\mathcal{G}(\mathfrak{S}_d^{geo})\). Let \(\sigma \in \mathfrak{S}_d\) with \(t' = \sigma t^{-1}\). Then \([t']^2 = [\sigma][t]^2[\sigma]^{-1}\), using twice the relation \([t'] = [\sigma][t][\sigma]^{-1}\). On the other hand we have \([t][\sigma][t]^{-1} = [\sigma^t]\) and \([t][\sigma^t][t]^{-1} = [(\sigma^t)^t] = [\sigma]\), hence \([t]^2\) and \([\sigma]\) commute. This shows that \([t]^2 = [t']^2\).

We denote by \(\mathfrak{S}_d\) the element \([t]^2 \in \mathcal{G}(\mathfrak{S}_d^{geo})\), which is the same element for any transposition \(t \in \mathfrak{S}_d\) and is central in \(\mathcal{G}(\mathfrak{S}_d^{geo})\). Note also that \(\mathfrak{S}_d\) has infinite order, since \((\mathcal{G}(N), e^{geo})/(\mathfrak{S}_d) = (2, 1)\). We have a commutative diagram of central extensions:

\[
\begin{array}{ccc}
1 & \longrightarrow & \langle \mathfrak{S}_d^{geo} \rangle \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \langle \langle 2, 1 \rangle \rangle \\
\end{array}
\]

The left vertical map is an isomorphism of infinite cyclic groups. The right vertical map is also an isomorphism: first note that there is an isomorphism \(\mathfrak{S}_d/\langle \langle 2, 1 \rangle \rangle \cong \mathfrak{S}_d\) induced by projection on the second coordinate; second, recall that \(\mathfrak{S}_d\) can be given a presentation with the following generators and relations:

**Generators**  
For all distinct \(i, j \in [d]\) there is a generator \((i, j) = (j, i)\).

**Relations**  
\(\bullet (i, j)^2 = 1\) for all distinct \(i, j\).
\(\bullet (k, l)(i, j)(k, l)^{-1} = (i, j)\) for all distinct \(i, j, k, l\).
\(\bullet (i, j)(j, k)(i, j)^{-1} = (j, k)(i, j)(j, k)^{-1}\) for all distinct \(i, j, k\).

The relations of second and third type also hold between the elements \([[i, j]]\), that generate \(\mathcal{G}(\mathfrak{S}_d^{geo})\); by quotienting \(\mathcal{G}(\mathfrak{S}_d^{geo})\) by the subgroup \(\langle \mathfrak{S}_d \rangle\), we impose also the relations of first type: hence also \(\mathcal{G}(\mathfrak{S}_d^{geo})/\langle \mathfrak{S}_d \rangle\) is isomorphic to \(\mathfrak{S}_d\), and the right vertical map is an isomorphism. By the five lemma, the middle vertical map is an isomorphism as well. \(\square\)

In particular the group \(\mathcal{K}(\mathfrak{S}_d^{geo})\) (see Notation \([2.12\]) can be identified with the subgroup \(2\mathbb{Z} \subset \mathfrak{S}_d\).

### 7.2. Some classical results about symmetric groups.

In this subsection we recall some classical facts, most of which go back to Clebsch [Cle 72] and Hurwitz [Hur 91], about sequences of transpositions and standard moves. We do this for the sake of completeness, and some details in the proofs are omitted and left to the reader.

**Lemma 7.3.** Let \(\sigma \in \mathfrak{S}_d\) be a permutation having a cycle with \(k\) cycles \(c_1, \ldots, c_k\), where fixpoints of \(\sigma\) count as cycles of length 1; then \(N(\sigma) = d - k\).

**Proof.** Any collection of transpositions \(\{t_1, \ldots, t_r\} \subset \mathfrak{S}_d\) generates a subgroup of \(\mathfrak{S}_d\) of the form \(\mathfrak{S}_{\mathfrak{P}_1} \times \ldots \times \mathfrak{S}_{\mathfrak{P}_c} \subset \mathfrak{S}_d\), for some partition \(\mathfrak{P}_1, \ldots, \mathfrak{P}_c\) of \([d]\). The pieces of the partition \(\mathfrak{P}\) are the connected components of the graph having as vertices the elements of \([d]\), and having one edge between \(j\) and \(j'\) for any transposition \(t_i = (j, j')\). The number \(\ell\) of connected components of this graph is at least \(d - r\), so we get the inequality \(r \geq d - \ell\).

Let now \(\sigma = t_1 \ldots t_r\) be a factorisation of \(\sigma\) exhibiting \(r = N(\sigma)\). Then each cycle \(c_i\), considered as a subset of \([d]\), is contained in some piece of the partition \(\mathfrak{P}\) of \([d]\) associated with the set of transpositions \(\{t_1, \ldots, t_r\}\), since the action of \(\sigma\) is transitive on the set \(c_i\) and since \(\sigma \in \mathfrak{S}_{\mathfrak{P}_1} \times \ldots \mathfrak{S}_{\mathfrak{P}_c}\). This implies
that $k \geq \ell$, hence $d - \ell \geq d - k$. Putting together the two inequalities, we obtain $N(\sigma) \geq d - k$.

On the other hand, one can factor each cycle $c_i$ into $|c_i| - 1$ transpositions, and get in this way a factorisation of $\sigma$ into precisely $\sum_{i=1}^k (|c_i| - 1) = d - k$ transpositions. See for instance the monotone decomposition described below. □

**Lemma 7.4.** Let $\sigma \in S_d$ be a permutation and $t = (i, i') \in S_d$ be a transposition. Then $N(t\sigma) = N(\sigma) + 1$ if $i$ and $i'$ belong to different cycles of the cycle decomposition of $\sigma$, and $N(t\sigma) = N(\sigma) - 1$ otherwise.

**Proof.** Let $c$ and $c'$ be the cycles containing $i$ and $i'$ respectively. If $c$ and $c'$ are distinct cycles, then the cycle decomposition of $t\sigma$ consists of all cycles of $\sigma$ different from $c$ and $c'$, plus a cycle $\hat{c}$ which is a concatenation of $c$ and $c'$:

$$\hat{c} = (i, \sigma(i), \sigma^2(i), \ldots, \sigma^{-1}(i), j, \sigma(j), \ldots, \sigma^{-1}(j)).$$

The statement follows in this case from the formula for $N$ in terms of the number of cycles, given in Lemma 7.3.

If instead $c = c'$, then the cycle decomposition of $t\sigma$ consists of all cycles of $\sigma$ different from $c$, plus two cycles $\check{c}$ and $\check{c'}$ giving a *splitting* of $c$, and containing $i$ and $i'$ respectively:

$$\check{c} = (i, \sigma(i), \ldots, \sigma^{-1}(j)); \quad \check{c'} = (j, \sigma(j), \ldots, \sigma^{-1}(i)).$$

Again the statement follows from the formula for $N$ from Lemma 7.3. □

The same result holds in Lemma 7.4 if we consider the product $t\sigma$ instead of $t\sigma$. The following corollary is a direct generalisation of Lemma 7.4.

**Corollary 7.5.** Let $\sigma, \tau \in S_d$ and suppose that $N(\sigma\tau) = N(\sigma) + N(\tau)$. Let $c$ be a cycle in the cycle decomposition of $\sigma$, and consider $c$ as a subset of $[d]$. Then $c$ is contained in some cycle $c'$ of the cycle decomposition of $\sigma\tau$.

**Proof.** Let $\tau = t_1 \ldots t_r$ be a minimal decomposition of $\tau$ in transpositions, with $r = N(\tau)$. Then the hypothesis $N(\sigma\tau) = N(\sigma) + N(\tau)$ and the triangular inequality imply that, for all $0 \leq j \leq r$ we also have $N(\sigma t_1 \ldots t_j) = N(\sigma) + j$. In particular for $1 \leq j \leq r$ we have the equality $N(\sigma t_1 \ldots t_{j-1}) + 1 = N(\sigma t_1 \ldots t_j)$, which by Lemma 7.3 implies that each cycle of $\sigma t_1 \ldots t_{j-1}$ is contained in some cycle of $\sigma t_1 \ldots t_j$. In particular each cycle of $\sigma$ is contained in some cycle of $\sigma\tau$. □

**Definition 7.6.** The *height* of a permutation $\sigma \in S_d$, denoted by $\text{ht}(\sigma)$, is the greatest index $i \in [d]$ such that $\sigma(i) \neq i$.

Each permutation $\sigma \in S_d$ admits a unique decomposition $\sigma = t_1 \ldots t_{N(\sigma)}$ into transpositions with $\text{ht}(t_1) < \cdots < \text{ht}(t_{N(\sigma)})$. We call this the monotone decomposition of $\sigma$ into transpositions.

The monotone decomposition can be computed recursively by setting $t_{N(\sigma)} := (\text{ht}(\sigma), \sigma^{-1}(\text{ht}(\sigma)))$, and noting that $\sigma' := \sigma t_{N(\sigma)}$ is a permutation of norm $N(\sigma') = N(\sigma) - 1$ and height $\text{ht}(\sigma') < \text{ht}(\sigma)$. In fact the transposition $(\text{ht}(\sigma), \sigma^{-1}(\text{ht}(\sigma)))$ is the unique possible choice to ensure that $\text{ht}(\sigma') < \text{ht}(\sigma)$.

**Notation 7.7.** We denote by $(t_1^\sigma, \ldots, t_{N(\sigma)}^\sigma)$ the monotone decomposition of a permutation $\sigma \in S_d$. It is the empty sequence for $\sigma = 1$. 
Lemma 7.8. Let \((t_1, \ldots, t_r)\) be a decomposition of \(\sigma \in \mathfrak{S}_d\) into transpositions, witnessing \(r = N(\sigma)\). Then there is a sequence of standard moves transforming \((t_1, \ldots, t_r)\) into \((t'_1, \ldots, t'_r)\).

Proof. It suffices to find a sequence of standard moves transforming \((t_1, \ldots, t_r)\) into a decomposition \((t'_1, \ldots, t'_r)\) of \(\sigma\) with \(ht(t'_i) = ht(\sigma)\) and \(ht(t'_i) < ht(\sigma)\) for all \(1 \leq i \leq r - 1\); then the permutation \(\sigma' = \sigma t'_i = t'_1 \cdots t'_{i-1}\) satisfies \(ht(\sigma') < ht(\sigma)\), whence \(t'_i = t'_r\); we can then proceed by induction on \(\sigma'\), which has both smaller height and smaller norm than \(\sigma\).

First, note that by Lemma 7.4 each transposition \(t_i\) satisfies \(ht(t_i) \leq ht(\sigma)\). Suppose that there is an index \(i\) with \(ht(t_i) > ht(t_{i+1})\); then we can replace \((t_i, t_{i+1})\) by \((t_{i+1}, t^{i+1}_i)\), thus moving the transposition with maximal height to right.

By repeating this procedure, we can assume that there is an index \(1 \leq i \leq r\) such that \(t_1, \ldots, t_{i-1}\) have height strictly less than \(ht(\sigma)\), whereas \(t_i, \ldots, t_r\) have height equal to \(ht(\sigma)\). Suppose \(i < r\) and note that \(t_i \neq t_{i+1}\), otherwise their product would be \(1 \in \mathfrak{S}_d\) and thus the norm of \(\sigma\) would be at most \(r - 2\). By a standard move we can replace \((t_i, t_{i+1})\) by \((t_i^{t_{i+1}}, t_i)\) (here we rewrite as \(t_i\) the exponent, which should be \(t_i^{-1}\)). The crucial observation is that \(t^{t_{i+1}_i}\) has height strictly less than \(ht(t_i) = ht(t_{i+1}) = ht(\sigma)\).

By repeating the last procedure, we can assume that the only transposition \(t_i\) satisfying \(ht(t_i) = ht(\sigma)\) is \(t_r\), as desired.

Lemma 7.9. Let \((t_1, \ldots, t_r)\) be a sequence of transpositions in \(\mathfrak{S}_d\) that generate \(\mathfrak{S}_d\) as a group. Let \((j, j')\) be any transposition in \(\mathfrak{S}_d\). Then there is a sequence of standard moves transforming \((t_1, \ldots, t_r)\) into a sequence \((t'_1, \ldots, t'_r)\) with \(t'_r = (j, j')\).

Proof. The statement is obvious for \(d = 2\), so we assume \(d \geq 3\). Note that it suffices to prove the statement in the case \(j' = d\), as the statement is invariant under conjugation of permutations in \(\mathfrak{S}_d\). So we wish to achieve a sequence of transpositions ending with \((d, j)\), for a fixed \(1 \leq j \leq d - 1\).

By using the procedures described in the proof of Lemma 7.8 we can operate a sequence of standard moves and reach a sequence \((t_1, \ldots, t_r)\) for which there exists an index \(1 \leq i \leq r - 1\) satisfying the following:
- the transpositions \(t_1, \ldots, t_i\) have height < \(d\);
- \(t_{i+1} = \cdots = t_r = (d, J)\) for some \(1 \leq J \leq d - 1\).

If \(J = j\) we are done; otherwise we use the inductive hypothesis on the sequence \(t_1, \ldots, t_i\), which generates \(\mathfrak{S}_{d-1}\): possibly after a suitable sequence of standard moves on these \(i\) transpositions, we can assume that \(t_i = (j, J)\). We can now replace \((t_i, t_{i+1}) = ((J, J), (J, d))\) with \(((J, d), (d, j))\); we can then move \((d, j)\) to the end of the sequence, replacing thus each further occurrence of \((J, d)\) with \((j, J)\).

Lemma 7.10. Let \(\sigma \in \mathfrak{S}_d\) be a permutation of height \(d\), let \((t_1, \ldots, t_r)\) be a sequence of transpositions generating \(\mathfrak{S}_d\), such that \(\sigma = t_1 \cdots t_r\). Then there is a sequence of standard moves transforming \((t_1, \ldots, t_r)\) into a sequence \((t'_1, \ldots, t'_r)\) with \(t_1, \ldots, t'_{r-1}\) of height < \(d\), and \(t'_r = (d, \sigma^{-1}(d))\).

Proof. The statement is obvious for \(d = 2\), so assume \(d \geq 3\). Use the procedure of Lemmas 7.8 and 7.9 and note that the new sequence \((t_1, \ldots, t_r)\) satisfies \(t_{i+1} = \cdots = t_r = (d, J)\) for some \(1 \leq J \leq d - 1\). Then there is a sequence of standard moves transforming \((t_1, \ldots, t_r)\) into a sequence \((t'_1, \ldots, t'_r)\) with \(t_1, \ldots, t'_{r-1}\) of height < \(d\), and \(t'_r = (d, \sigma^{-1}(d))\).
\[\cdots = t_i = (d, \sigma^{-1}(d))\] for some \(i < r\) with \(r - i\) odd. If \(r - i = 1\) we are done, so assume \(r - i \geq 3\).

Fix any \(j < d\) with \(j \neq \sigma^{-1}(d)\). By Lemma 7.10 we can assume \(t_i = (j, \sigma^{-1}(d))\), possibly after applying a sequence of standard moves on \((t_1, \ldots, t_i)\): in fact the transpositions \(t_1, \ldots, t_i\) generate \(S_{d-1}\). By a suitable sequence of 4 standard moves we can then replace
\[(t_i, t_{i+1}, t_{i+2}) = ((j, \sigma^{-1}(d)), (d, \sigma^{-1}(d)), (d, \sigma^{-1}(d)))\]
with \((j, \sigma^{-1}(d)), (d, j), (d, j))\). We obtain a sequence \((t_1, \ldots, t_r)\) such that \(t_1, \ldots, t_i\) have height \(< d\), whereas \(t_{i+1}, \ldots, t_r\) have height \(d\) and are not all equal.

We can now repeat the second part of the procedure of Lemma 7.8 to the sequence \((t_1, \ldots, t_r)\), obtaining a sequence \((t'_1, \ldots, t'_r)\) such that \(t'_1, \ldots, t'_d\) have height \(< d\), and \(t'_{d+1}, \ldots, t'_r\) are equal and have height \(d\), for some \(d' \geq i + 2\). Possibly after iterating the entire procedure, we can assume \(d' = r - 1\).

Proposition 7.11. Let \((t_1, \ldots, t_r)\) and \((t'_1, \ldots, t'_r)\) be two sequences of transpositions of the same length \(r\), suppose \((t_1, \ldots, t_r) = \langle t'_1, \ldots, t'_r \rangle = S_d\), and also assume that there is \(\sigma \in S_d\) with \(\sigma = t_1 \cdots t_r = t'_1 \cdots t'_r\). Then there is a sequence of standard moves transforming \((t_1, \ldots, t_r)\) into \((t'_1, \ldots, t'_r)\).

Proof. The statement is obvious for \(d = 2\), so we assume \(d \geq 3\). If \(ht(\sigma) = d\), by Lemma 7.10 we can assume, up to changing both sequences of transpositions by suitable sequences of standard moves, that \(t_r = t'_r = (d, \sigma^{-1}(d))\) and all other transpositions \(t_i\) and \(t'_i\) have height \(< d\). We then have \(t_1 \cdots t_{r-1} = t'_1 \cdots t'_{r-1}\) and \(\langle t_1, \ldots, t_{r-1} \rangle = \langle t'_1, \ldots, t'_{r-1} \rangle = S_{d-1}\), so we can conclude by inductive hypothesis on \(r\) and \(d\).

If \(ht(\sigma) < d\), by Lemma 7.9 we may assume \(t_r = t'_r = (d, d-1)\). Let \(\sigma' := \sigma \cdot (d, d-1) = t_1 \cdots t_{r-1} = t'_1 \cdots t'_{r-1}\) and note that \(d - 1\) and \(d\) belong to the same cycle of the cycle decomposition of \(\sigma'\). We claim that \(\langle t_1, \ldots, t_{r-1} \rangle = S_d\).

Suppose instead that \(\langle t_1, \ldots, t_{r-1} \rangle = S_{\Psi_1} \times \cdots \times S_{\Psi_r}\), for a proper partition \(\langle \Psi_1, \ldots, \Psi_r \rangle\) of \([d]\). Then the elements \(d - 1\) and \(d\) belong to the same cycle of \(\sigma'\), which is entirely contained in one piece of the partition: it follows that \(t_1, \ldots, t_r\) also generate \(S_{\Psi_1} \times \cdots \times S_{\Psi_r}\), contradicting the assumption that they generate the entire \(S_d\).

In a similar way \(\langle t'_1, \ldots, t'_{r-1} \rangle = S_d\). We conclude by induction on \(r\).

7.3. Consequences for \(S_d^{geo}\). Using the results of the previous subsection we can prove the following properties of \(S_d^{geo}\).

Lemma 7.12. The PMQ \(S_d^{geo}\) is coconnected and pairwise determined.

Proof. Being coconnected is the statement of Lemma 7.8. To prove that \(S_d\) is pairwise determined, we note that a sequence of elements \(t_1, \ldots, t_r \in (S_d^{geo})_1\) does not admit a product in \(S_d^{geo}\) precisely when the product permutation \(\sigma := t_1 \cdots t_r \in S_d\) satisfies \(N(\sigma) < r\); on the other hand a product of two transpositions \(t, t'\) is not defined in \(S_d^{geo}\) if and only if \(t = t'\). The statement of the lemma is obvious for \(d = 2\), so we assume \(d \geq 3\) henceforth.

If no transposition \(t_i\) has height \(d\), the inductive hypothesis on \(d - 1\) ensures that there is a sequence of standard moves transforming the sequence \((t_1, \ldots, t_r)\) into a sequence ending with two equal transpositions. Otherwise assume that some \(t_i\) has height \(d\). We can use on \((t_1, \ldots, t_r)\) the procedure of Lemma 7.8 thus
transforming the sequence into a sequence \((t'_1, \ldots, t'_r)\) with \(t'_1, \ldots, t'_r\) of height \(< d\) and \(t'_{i+1} = \cdots = t'_r\) having height \(d\), for some \(1 \leq i \leq r - 1\). If \(i \leq r - 2\) we have reached a situation with two equal transpositions, so we can assume \(i = r - 1\).

In particular, if \(i = r - 1\), we have that \(ht(\sigma) = d\) and \(t'_r = (d, \sigma^{-1}(d)) = t^*_N(\sigma)\). It follows that \(\sigma' := \sigma \cdot t'_r = t'_1 \cdots t'_r\) has norm \(N(\sigma') = N(\sigma) - 1\), and \(\sigma'\), as well as all transpositions \(t'_1, \ldots, t'_{r-1}\) can be regarded as permutations in \(S_{d-1}\). We conclude by inductive hypothesis on the sequence \(t_1, \ldots, t_{r-1}\) of elements of norm \(1\) in \(S^geo_{d-1}\).

\[\square\]

We can in fact give a complete description of the completion \(\hat{S}^geo_d\) of \(S^geo_d\).

**Proposition 7.13.** The complete PMQ \(\hat{S}^geo_d\) is the set of all sequences

\[(\sigma; \Psi_1, \ldots, \Psi_\ell; r_1, \ldots, r_\ell),\]

where \(\sigma \in S_\ell\), \((\Psi_1, \ldots, \Psi_\ell)\) is a partition of \([d]\) and \(r_1, \ldots, r_\ell \geq 0\), satisfying the following properties:

1. \(\sigma \in S_\Psi_1 \times \cdots \times S_\Psi_\ell\);
2. \(r_j \geq 2|\Psi_j| - N(\sigma|_{\Psi_j}) - 2\) for all \(1 \leq j \leq \ell\);
3. \(r_j\) has the same parity as \(N(\sigma|_{\Psi_j}) \in S_\Psi_j\), for all \(1 \leq j \leq \ell\).

Conjugation by \((\sigma; \Psi_1, \ldots, \Psi_\ell; r_1, \ldots, r_\ell)\) sends

\[(\sigma'; \Psi'_1, \ldots, \Psi'_\ell; r'_1, \ldots, r'_\ell) \mapsto ((\sigma')^\sigma; \Psi'_1, \ldots, \Psi'_\ell; r_1, \ldots, r_\ell).

The product of \((\sigma; \Psi_1, \ldots, \Psi_\ell; r_1, \ldots, r_\ell)\) and \((\sigma'; \Psi'_1, \ldots, \Psi'_\ell; r'_1, \ldots, r'_\ell)\) is the sequence \((\sigma'; \Psi''_1, \ldots, \Psi''_\ell; r''_1, \ldots, r''_\ell)\), where \((\Psi''_1, \ldots, \Psi''_\ell)\) is the finest partition which is coarser than both \((\Psi_1, \ldots, \Psi_\ell)\) and \((\Psi'_1, \ldots, \Psi'_\ell)\), and where, for all \(1 \leq j \leq \ell\),

\[r''_j = \sum_{j : \Psi_j \subset \Psi'_j} r_j + \sum_{j' : \Psi_j \subset \Psi'_j} r'_j.

In the statement of Proposition 7.13 the partition \((\Psi_1, \ldots, \Psi_\ell)\) is unordered, as well as the sequence of numbers \(r_1, \ldots, r_\ell\); still each piece \(\Psi_j\) of the partition is associated with its corresponding number \(r_j\).

**Proof of Proposition 7.13** By Lemma 7.12 the PMQ \(\hat{S}^geo_d\) is coconnected, hence its completion \(\hat{S}^geo_d\) can be computed as the completion of \((\hat{S}^geo_d)_1\); thus an element of \(\hat{S}^geo_d\) is an equivalence class of sequences \((t_1, \ldots, t_r)\) of elements in \((\hat{S}^geo_d)_1\), where two sequences are equivalent if they can be transformed into another by standard moves. With a sequence \((t_1, \ldots, t_r)\) we can associate the following, which are invariants under standard moves:

- the product \(\sigma := t_1 \cdots t_r \in S_d\);
- the unordered partition \((\Psi_1, \ldots, \Psi_\ell)\) of \([d]\) such that \((t_1, \ldots, t_r) = \Psi_1 \times \cdots \times \Psi_\ell\);
- for each partition piece \(\Psi_j\), the number \(r_j\) of transpositions \(t_i\) belonging to the symmetric group \(S_\Psi_j\).

Properties (1)-(3) in the statement of the Proposition ensure that, viceversa, a sequence \((\sigma; \Psi_1, \ldots, \Psi_\ell; r_1, \ldots, r_\ell)\) can be achieved from a suitable sequence of transpositions \((t_1, \ldots, t_r)\). Indeed, if (1)-(3) are satisfied, one can let \(r := r_1 + \cdots + r_\ell\) and define \((t_1, \ldots, t_r)\) as the concatenation of \((t'_1, \ldots, t'_{N(\sigma)})\) with a choice of \((r - N(\sigma))/2\) couples of equal transpositions \((t, t)\), chosen in such a way that, for all \(1 \leq j \leq \ell\), there are precisely \(r_j - N(\sigma|_{\Psi_j})/2\) couples with \(t \in S_\Psi_j\), since
\[ r_j - N(\sigma|_{\mathfrak{P}_j})/2 \geq |\mathfrak{P}_j| - N(\sigma|_{\mathfrak{P}_j}) - 1, \]
and since the last expression is 1 less than the number of cycles of \( \sigma|_{\mathfrak{P}_j} \), we can also ensure that \( t_1, \ldots, t_r \) generate precisely the subgroup \( \mathcal{S}_{\mathfrak{P}_j} \times \cdots \times \mathcal{S}_{\mathfrak{P}_j} \).

Vice versa, if two sequences of transpositions \( (t_1, \ldots, t_r) \) and \( (t'_1, \ldots, t'_{r'}) \) give rise to the same sequence \( (\sigma; \mathfrak{P}_1, \ldots, \mathfrak{P}_l; r_1, \ldots, r_l) \), then \( r = \sum_{j=1}^{l} r_j = r' \); since transpositions lying in two different factors \( \mathcal{S}_{\mathfrak{P}_j} \) and \( \mathcal{S}_{\mathfrak{P}_j'} \) commute, up to operating suitable sequences of standard moves we can assume that both \( (t_1, \ldots, t_r) \) and \( (t'_1, \ldots, t'_{r'}) \) are concatenations of smaller sequences in the following way:

- for all \( 1 \leq j \leq \ell \) there is a sequence of transpositions \( (t_j, 1, \ldots, t_{j,r_j}) \) in \( \mathcal{S}_{\mathfrak{P}_j} \), and
- \( (t_1, \ldots, t_r) \) is the concatenation of \( (t_1, 1, \ldots, t_{1,r_1}) \), \( (t_2, 1, \ldots, t_{2,r_2}) \), \ldots, \( (t_\ell, 1, \ldots, t_{\ell,r_\ell}) \);

- for all \( 1 \leq j \leq \ell \) there is a sequence of transpositions \( (t'_j, 1, \ldots, t'_{j,r_j}) \) in \( \mathcal{S}_{\mathfrak{P}_j} \), and
- \( (t'_1, \ldots, t'_r) \) is the concatenation of \( (t'_1, 1, \ldots, t'_{1,r_1}) \), \( (t'_2, 1, \ldots, t'_{2,r_2}) \), \ldots, \( (t'_\ell, 1, \ldots, t'_{\ell,r_\ell}) \).

We can then apply Proposition 7.14 to each of the \( \ell \) pairs of corresponding sequences \( (t_j, 1, \ldots, t_{j,r_j}) \) and \( (t'_j, 1, \ldots, t'_{j,r_j}) \), showing that these sequences of transpositions are connected by a sequence of standard moves. Concatenating, we obtain that also \( (t_1, \ldots, t_r) \) and \( (t'_1, \ldots, t'_r) \) are connected by a sequence of standard moves.

In this way we have shown that \( \mathcal{S}_d^{geo} \) is in bijection with the set of sequences \( (\sigma; \mathfrak{P}_1, \ldots, \mathfrak{P}_l; r_1, \ldots, r_l) \) satisfying properties (1)-(3). The description of conjugation and product in light of this bijection is straightforward.

\[ \square \]

Let \( R \) be a commutative ring. It follows from Theorem 7.12 that the PMQ-ring \( R(\mathcal{S}_d^{geo}) \) is isomorphic to the free associative \( R \)-algebra with the following generators and relations:

**Generators** \[ \text{For all } x < y \in [d] \text{ there is a generator } [xy] = [yx]. \]

**Relations** \[ [xy]^2 = 0 \text{ for all distinct } x, y \in [d], \]
[\[ xy \] [yz] = [yz] [zx] = [zx] [xy] \text{ for all distinct } x, y, z \in [d]. \]
[\[ xy \] [zw] = [zw] [xy] \text{ for all distinct } x, y, z, w \in [d]. \]

The following Proposition is a reformulation of a Theorem of Visy [Vis11]. The proof is taken from [Bia20, page 108].

**Proposition 7.14.** Let \( d \geq 2 \); then the normed PMQ \( \mathcal{S}_d^{geo} \) is Koszul over any commutative ring \( R \).

**Proof.** We will prove that \( R(\mathcal{S}_d^{geo}) \) admits a Poincaré-Birkhoff-Witt (PBW) basis: a result by Priddy [Pri70] then ensures that \( R(\mathcal{S}_d^{geo}) \) is Koszul; see also Theorem 3.1 in [PP05, Chapter 4].

We give a total order \( \prec \) on the generators of \( R(\mathcal{S}_d^{geo}) \): let \( [xy] \) and \( [x'y'] \) be two different generators, with \( x < y \) and \( x' < y' \); then \( [xy] \prec [x'y'] \) if and only if \( y < y' \), or \( y = y' \) and \( x < x' \). We give the the lexicographic order, also denoted by \( \prec \), to the set \( \{([xy], [x'y'])\} \) of couples of generators.

For a transposition \( t = (x, y) \in \mathcal{S}_d \) we denote by \( [xy] \) the corresponding generator of \( R(\mathcal{S}_d^{geo}) \). More generally, for \( \sigma \in \mathcal{S}_d \) we denote by \( \{\sigma\} \in R(\mathcal{S}_d^{geo}) \) the generator corresponding to \( \sigma \); recall that the elements \( \{\sigma\} \), for varying \( \sigma \in \mathcal{S}_d \), form a basis of \( R(\mathcal{S}_d^{geo}) \) as a free \( R \)-module.

Define \( \mathcal{S} \subset \{ ([xy], [x'y']) \} \) as the subset containing couples \( ([t], [t']) \) such that the product \( [t][t'] \in R(\mathcal{S}_d^{geo}) \) cannot be expressed as a linear combination \( \sum_{i=1}^{p} \lambda_i [t_i][t'_i] \), with \( ([t], [t']) \prec ([t_i], [t'_i]) \) for all \( i \). A PBW-monomial in the generators \( [xy] \) is then a monomial \( [t_1] \cdots [t_p] \) with \( ([t], [t_{i+1}]) \in \mathcal{S} \) for all \( 1 \leq i \leq p - 1 \), and our aim is to prove that PBW-monomials form a basis of \( R(\mathcal{S}_d^{geo}) \) as a free \( R \)-module, called PBW-basis.
By the relations in the presentation of \( R[\mathcal{S}_d^{\text{geo}}] \) it is straightforward to see that \( ([t],[t']) \in \mathcal{S} \) if and only if \( \text{ht}(t) < \text{ht}(t') \).

Recall that every permutation \( \sigma \in \mathfrak{S}_d \) has a unique monotone decomposition \( t_1^\sigma \cdots t_n^\sigma \) with \( \text{ht}(t_1) < \cdots < \text{ht}(t_n) \); viceversa every product \( t_1 \cdots t_p \), with \( \text{ht}(t_1) < \cdots < \text{ht}(t_p) \) gives a permutation \( \sigma \in \mathfrak{S}_d \) of norm \( p \).

This shows that PBW-monomials form precisely the standard basis of elements \([\sigma]\) of \( R[\mathcal{S}_d^{\text{geo}}] \).

\[ \Box \]

\section*{Appendix A. A brief discussion of partially multiplicative racks}

A rack with unit is as a set \( \mathfrak{R} \) with a marked element \( 1 \in \mathfrak{R} \), called unit, and a binary operation \( \mathfrak{R} \times \mathfrak{R} \rightarrow \mathfrak{R} \), called conjugation and denoted \( (a,b) \mapsto a^b \), such that all properties of Definition 2.4 are satisfied, except possibly property (3). A morphism of racks is required to preserve the unit and the conjugation.

A partially multiplicative rack (PMR) is then a set \( \mathfrak{R} \) with compatible structures of partial monoid and rack, i.e. satisfying all properties of Definition 2.1 with the word “quandle” replaced by the word “rack”. PMRs form a category \( \text{PMR} \), with morphisms those maps of sets that preserve unit, conjugation and partial multiplication.

All definitions from Subsection 2.1 extend naturally to the setting of racks. We have a fully faithful inclusion of categories \( \text{PMQ} \subset \text{PMR} \); the next example shows that it is not an equivalence.

\begin{example}
Let \( \mathfrak{R} = \{1, a, b\} \) be a rack with conjugation given by \( a^b = a^a = b \) and \( b^b = b^a = a \). Then \( \mathfrak{R} \) is not a quandle. If we consider \( \mathfrak{R} \) as a PMR with trivial multiplication, we obtain an example of a PMR which is not a PMQ.
\end{example}

In fact, all definitions and results in Sections 2, 4, 5 and 6 can be generalised to the context of PMRs. In particular:

- every PMR \( \mathfrak{R} \) can be completed to a complete PMR (or multiplicative rack) \( \hat{\mathfrak{R}} \);
- for every complete PMR \( \hat{\mathfrak{R}} \) and every category \( A \) we have a category \( XA(\hat{\mathfrak{R}}) \) of \( \hat{\mathfrak{R}} \)-crossed object in \( A \), which is a functor \( \hat{\mathfrak{R}}//\hat{\mathfrak{R}} \rightarrow A \), where \( \hat{\mathfrak{R}}//\hat{\mathfrak{R}} \) is defined precisely as in Definition 6.1, replacing \( \mathcal{Q} \) by \( \mathfrak{R} \);
- if \( A \) is (braided) monoidal, then \( XA(\mathfrak{R}) \) is also (braided) monoidal;
- for an augmented PMR \( \mathfrak{R} \) we can define a simplicial Hurwitz space \( \text{Hur}^A(\mathfrak{R}) \), arising as the difference of the geometric realisations of the bisimplicial sets \( \text{Arr}(\mathfrak{R}) \) and \( N\text{Adm}(\mathfrak{R}) \), which are defined precisely as in Definitions 6.6 and 6.11;
- for an augmented PMR \( \mathfrak{R} \) the double bar construction \( B_{**}(R[\mathfrak{R}], R, \varepsilon) \) gives rise, after passing to the associated total chain complex, to the cellular chain complex of the pair \((\text{Arr}(\mathfrak{R})), |N\text{Adm}(\mathfrak{R})|)\).

The only results which do not admit a direct generalisation to PMRs are those in Section 3 in particular Theorem 3.3. In the following we discuss what complications arise. First, we note that the fully faithful inclusion of categories \( \text{PMQ} \subset \text{PMR} \) has a right adjoint.

\begin{definition}
Let \( \mathfrak{R} \) be a PMR. An element \( a \in \mathfrak{R} \) is quandle-like if \( a^a = a \). We denote by \( \mathfrak{R}^{\text{maxPMQ}} \subset \mathfrak{R} \) the subset of quandle-like elements.
\end{definition}

\begin{lemma}
Let \( \mathfrak{R} \) be a PMR. Then \( \mathfrak{R}^{\text{maxPMQ}} \) inherits from \( \mathfrak{R} \) a structure of PMQ, such that the inclusion \( \mathfrak{R}^{\text{PMQ}} \subset \mathfrak{R} \) is a map of PMRs. Moreover for any PMQ \( Q \), any map of PMRs \( \psi : Q \rightarrow \mathfrak{R} \) has image contained in \( \mathfrak{R}^{\text{PMQ}} \).
\end{lemma}
If $Q$ of $C$ the definition is quite general, and it includes, for each PMQ $Q$ finite subset and $\psi$ the most straightforward way, as the set of couples $(a, b)$: the space $\text{Hur}(\{0, 1\}^2, \mathbb{R})$ with the set $\text{Hur}(\{0, 1\}^2, \mathbb{R})$ is involved in the definition of the set $\text{Hur}(\{0, 1\}^2, \mathbb{R})$. This means that only the quandle-like part of $\mathcal{R}$ is involved in the definition of the set $\mathcal{R}(\{0, 1\}^2, \mathbb{R})$, or, in other words, the definition of the set $\mathcal{R}(\{0, 1\}^2, \mathbb{R})$ is only interesting when $\mathcal{R}$ is a PMQ.

The previous lemma shows that the assignment $\mathcal{R} \rightarrow \mathcal{R}(\{0, 1\}^2, \mathbb{R})$ gives the right adjoint $(-)^{\mathcal{PMQ}}: \text{PMR} \rightarrow \text{PMQ}$ to the inclusion $\text{PMQ} \subseteq \text{PMR}$.

In [Bia21a] we will introduce a “coordinate-free” definition of Hurwitz spaces: the definition is quite general, and it includes, for each PMQ $\mathcal{Q}$, the definition of a space $\text{Hur}(\{0, 1\}^2, \mathcal{Q})$, containing configurations $P \subset \{0, 1\}$ with the additional datum of a monodromy $\psi$, defined on certain loops of $\mathbb{C} \setminus P$ and with values in $\mathcal{Q}$.

More precisely, a configuration in $\text{Hur}(\{0, 1\}^2, \mathbb{R}_+)$ takes the form of a couple $(P, \psi)$, where $P \subset \{0, 1\}$ is a finite subset, and $\psi: \Omega(P) \rightarrow \mathcal{Q}$ is a map of PMRs. Here we define $\Omega(P) \subset \pi_1(\mathbb{C} \setminus P, \ast)$ to be the union of $\{1\}$ and all conjugacy classes corresponding to simple closed curves in $\mathbb{C} \setminus P$ spinning clockwise around precisely one point of $P$. The set $\Omega(P)$ is a PMQ with trivial multiplication. As in Subsection 6.5, we use the basepoint $\ast = -\sqrt{-1} \in \mathbb{C} \setminus P$ when defining the fundamental group of $\mathbb{C} \setminus P$.

Suppose now that $\mathcal{R}$ is any PMR, and let us define the set $\mathcal{H}(\{0, 1\}^2, \mathcal{R})$, in the most straightforward way, as the set of couples $(P, \psi)$, where $P \subset \{0, 1\}$ is a finite subset and $\psi: \Omega(P) \rightarrow \mathcal{R}$ is a morphism of PMRs; then by Lemma A.3 the image of $\psi$ is contained in $\mathcal{R}(\{0, 1\}^2, \mathcal{R})$, and in fact the set $\mathcal{H}(\{0, 1\}^2, \mathcal{R})$ is in bijection with the set $\mathcal{R}(\{0, 1\}^2, \mathcal{R})$. This means that only the quandle-like part of $\mathcal{R}$ is involved in the definition of the set $\mathcal{H}(\{0, 1\}^2, \mathcal{R})$, or, in other words, the definition of the set $\mathcal{H}(\{0, 1\}^2, \mathcal{R})$ is only interesting when $\mathcal{R}$ is a PMQ.

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