INVARIANT MEASURES FOR SOME PIECEWISE CONTINUOUS MAPS

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Abstract. We study some special classes of piecewise continuous maps on a finite smooth partition of the compact phase space and look for invariant measures for such maps. We show that in the simplest one-dimensional case (so-called interval translation maps), this measure exists for any map. Moreover, they can always be selected non-atomic. We use this result to demonstrate that any piecewise translation map is metrically equivalent to an interval exchange map. In higher dimensions, we demonstrate that the invariant measure exists if the number of domains in the partition does not exceed the dimension of the space plus one. Finally, we provide a general conditions on approximation of invariant measures of a piecewise continuous map by invariant measures of piecewise translation maps.

1. Introduction

The main objective of this research is to understand how the numerical methods could be used to study piecewise continuous dynamics. Recently, many papers appear where the authors make computer simulation of a discontinuous dynamics without any theoretical verification of numerical method they use. Modelling the discontinuous dynamics manifests much more difficulties that could be easily illustrated by the following example [11, Exercise 4.1.1].

The famous Krylov-Bogolyubov Theorem claims that any continuous transformation of a compact metric space admits a Borel probability invariant measure. A similar statement for discontinuous maps is wrong.

Example 1.1. Consider the map $T : [0, 1] \to [0, 1]$ given by the formula: $T(x) = x/2$ if $x > 0$; $T(0) = 1$. This map does not admit any Borel probability invariant measure.

So, the numerical methods may be inappropriate even in their weakest form – modelling invariant measures. Observe that for continuous maps of compact sets we have lower semicontinuity of the set of invariant measures that fails for discontinuous maps.

Applying numerical methods, usually one cuts the phase space into small indivisible pieces (pixels, finite elements etc) that do not change their shapes. Here we are face at least two questions:

Q1: do such piecewise isometries admit any invariant measures?
Q2: do these measures approximate any invariant measure of the initial system?

The first problem is well-known in the one-dimensional case.

Example 1.2. Consider the circle $T^1 := \mathbb{R}/\mathbb{Z}$. We represent it as a union of disjoint subsegments $M_j = [t_j, t_{j+1})$, $j = 1, \ldots, n$ and define the map $S$ by the formula

$$S(t) = t + c_j \mod 1, \quad t \in M_j.$$ 

Here $c_j$ are real values. Such map is called interval translation (ITM) or, if it is one-to-one it is called interval exchange (IEM).

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Similarly one may consider interval translation maps on the segment \([0, 1]\). Notice that in this paper we consider the orientation-preserving maps only.

For interval exchange maps the question Q1 is trivial: the Lebesgue measure is always invariant. Moreover, the map \( S \) admits at most \( n \) Borel probability invariant non-atomic ergodic measures (see [11, §14.5, §14.6] for the basic theory of Interval Exchange maps and [17] and [20] as surveys for deeper results).

The case of non-invertible ITMs, first considered by M. Boshernitzan and I. Kornfeld [3] is much more sophisticated. One of the principal problems for ITMs is their classification.

**Definition 1.1.** We say that the ITM \( S \) is of finite type if there exists a number \( m \in \mathbb{N} \) such that

\[
S^m(T^1) = \bigcap_{k=1}^{\infty} S^k(T^1).
\]

Otherwise, the map \( S \) is of infinite type.

In [3], authors have demonstrated that many ITMs are finite and thus may be reduced to interval exchange maps. However, there are examples with ergodic measures supported on Cantor sets. J. Schmelling and S. Troubetzkoy [13] provided some estimates on the number of minimal subsets for ITMs.

H. Bruin and S. Troubetzkoy [6] were studying ITMs of a segment of 3 intervals \((n = 3)\). It was shown that in this case typical ITM is of finite type. In any case, results on Hausdorff dimension for attractors and unique ergodicity are given. These results are generalised in [11] for ITMs with arbitrary many pieces. There is an uncountable set of parameters leading to type \( \infty \) interval translation maps but the Lebesgue measure of these parameters is zero. Furthermore conditions are given that imply that the ITMs have multiple ergodic invariant measures. H. Bruin and G. Clark [5] were studying the so-called double rotations \((n = 2 \text{ for maps of the circle})\). Almost all double rotations are of finite type. The parameters that correspond to infinite type maps, form a set of Hausdorff measure strictly between 2 and 3.

J. Buzzi and P. Hubert [2] were studying piecewise monotonous maps of zero entropy and no periodic points. Particularly, they demonstrated that orientation-preserving ITMs without periodic points may have at most \( n \) ergodic probability invariant measures where \( n \) is the number of intervals.

D. Volk [18] was studying ITMs of the segment. He demonstrated that almost all (w.r.t. Lebesgue measure on the parameters set) ITMs of 3 intervals may be reduced to a rotation or to a double rotation and, hence, are of finite type.

We specially notice the result by B. Pires [12] who proved that almost all ITMs admit a non-atomic invariant measure (he assumed that the map does not have any connections or periodic points). In this paper, we generalise the referred result proving the same result for all ITMs. Moreover, our techniques allow us to claim that all reductions of ITMs to supports of their invariant measures are metrically equivalent to an interval exchange map. A very similar statement was proved in [12]: any injective piecewise continuous map \( S : [0, 1] \to [0, 1] \) is semi-conjugate to an interval exchange transformation, possibly with flips.

The more general class of maps is the so-called piecewise translation maps (PWT), see the precise definition below. This is the multi-dimensional generalisation of ITMs. Such maps are widely used in applications: herding dynamics in Markov networks, second order digital filters, sigma-delta modulators, buck converters, three-capacitance models, error diffusion algorithm in digital printing, machine learning etc, please see [13], [16] and [18] for review and references.

A good survey on the results on the theory of PWTs and some brilliant original results were given by D. Volk [18]. Particularly, such properties as ergodicity,
finite type of the map and the structure of the attractor were discussed. A very interesting result was obtained in [14]. If the dimension of the phase space \( d \) and the number \( n \) of the elements \( M_j \) of the partition are such that \( n = d + 1 \), it was proved that any PWT with rationally independent shift vectors is of finite type and, consequently, the Lebesgue measure of the attractor is non-zero and, hence there exist a continuous invariant measure. This result was proved by the powerful techniques of reduction of the discontinuous multivalued map to the map on the set of compact subsets. A similar approach was used in the preprint [19].

The most general class of the maps is the so-called piecewise isometries. We split the phase space into a finite number of parts and assume that all reductions of the considered maps are isometries (that may include rotations).

J. Buzzi [1] demonstrated that piecewise isometries defined on a finite union of polytopes have zero topological entropy in any dimension. Xin-chu Fu and Jinqiao Duan [5] were studying this problem in dimension 2 (the so-called planar isometries). They provided sufficient conditions for existence of Mihor-type attractors. Zhan-he Chen and Rong-zhong Yu [7] have also considered planar piecewise isometries. They have proved that Hausdorff measures restricted to an almost invariant set w.r.t. Hausdorff measure is invariant.

The partition of the phase space engenders a naturally defined symbolic dynamics. A. Goetz [9, 10] demonstrated that for so-called regular partition (that is true in assumptions of our paper) symbolic dynamics of the isometry cannot embed subshifts of finite type with positive entropy. The condition of polynomial growth of symbolic words is given.

In this paper we do the following. We formulate conditions on the piecewise continuous maps that are sufficient for existence of Borel probability invariant measures. Particularly, we demonstrate that any ITM admits a non-atomic Borel probability invariant measure and is metrically equivalent to an interval exchange map. Finally, we study the case when invariant measures of piecewise continuous maps may be approximated by ones of approximating PWTs.

2. PIECEWISE ISOMETRIES.

Consider an \( d \)–dimensional compact Riemannian manifold \((M, \text{dist})\) or a bounded domain \(M \subset \mathbb{R}^d\) that can be represented as a finite union of closures of disjoint domains \(M_j (j = 1, \ldots, n)\) such that \(\text{Int} M_j \cap \text{Int} M_k = \emptyset\) if \(j \neq k\). Later on, we call all sets \(M_j\) domains.

**Definition 2.1.** Let the map \(S : M \to M\) be such that \(\text{dist}(S(x), S(y)) = \text{dist}(x, y)\) for any \(j \in \{1, \ldots, n\}\) and any \(x, y \in M_j\).

Let \(H = \bigcup_{j=1}^n \partial M_j\) be the discontinuity set for the map \(S\), using terminology of Piecewise Continuous Dynamics, we call \(H\) discontinuity set, \(\Omega := \bigcup_{k \geq 0} S^{-k}(H)\) be the set of all eventually discontinuity points.

On \(M\), we consider the Lebesgue probability measure \(\text{Leb}\). We assume that \(\text{Leb}(H) = 0\) that implies \(\text{Leb}(\Omega) = 0\). We call all points of \(M \setminus \Omega\) typical.

Let us study invariant measures and invariant sets of the map \(S\). We use ideas of [11], §14.5. There, properties of the so-called interval exchange maps are discussed. In fact, "regular" interval exchange maps are particular case of maps, we are studying. However, our case is much more difficult and many techniques cannot be applied directly.

Evidently, \(\text{Leb}(S(A)) = \text{Leb}(A)\) for any measurable set \(A\). However, the measure \(\text{Leb}\) is not invariant for \(S\) unless this map is invertible almost everywhere.

Moreover, generally speaking, it is not evident if all interval translation maps admit Borel probability invariant measures. The positive answer to this question is one of the main results of the paper.
3. Periodic points and domains

We spread classical concepts of periodic and eventually periodic points to subsets of $M$.

**Definition 3.1.** We say that a domain $Q \subset M$ is periodic if there is $k \in \mathbb{N}$ such that $S(Q) = Q$ and all iterations $S^l|_Q$ are continuous for $l \leq k$.

For piecewise translation maps (but not for piecewise isometries) this implies that all points of $Q$ are periodic.

**Definition 3.2.** We say that a domain $Q \subset M$ is eventually periodic if there is $m \in \mathbb{N}$ such that $S^m(Q)$ is a periodic domain.

**Lemma 3.1.** If $x_0 \in M$ is a periodic point of $S$, there exists a periodic domain $Q$ that contains the point $x_0$.

**Proof.** Evidently, $x_0 \notin \Omega$. Observe that for any point $x \in M \setminus \Omega$ and any $k \in \mathbb{N}$ there exists a ball $Q \ni x$ such that the reduction $S^k|_Q$ is continuous. Then the desired property is evident since the map $S^k$ must preserve the ball $Q$. □

A similar statement is true for eventually periodic domains.

**Definition 3.3.** We call a domain $J \subset M$ tough if all maps $S^k$, $k \in \mathbb{N}$ are continuous on $J$.

Evidently, this means that discontinuity points do not belong to $S^k(J)$ for all $j = 1, \ldots, n - 1$.

We call a tough domain maximal if it is not a proper subset of another tough domain. Evidently if $J_0$ is a tough domain then all set $S^k(J_0)$ are tough domains.

**Lemma 3.2.** For interval translation maps of Example 1.2 if all values $c_j$ are rational, all points of $M \setminus \Omega$ are eventually periodic and the set $M \setminus \Omega$ is a finite union of tough domains.

Now we go back to the general case of piecewise isometries.

**Proposition 3.1.** The following statements hold.

1. Let $J_\alpha$, $\alpha \in A$ be a sequence of tough domains such that

$$\bigcap_{\alpha \in A} J_\alpha \neq \emptyset.$$

Then

$$J = \bigcup_{\alpha \in A} J_\alpha$$

is a tough domain.

2. If $J_1$ and $J_2$ are tough domains such that $J_1 \cap J_2 \neq \emptyset$. Then $J_1 \cap J_2$ is a tough domain.

**Lemma 3.3.** Any tough domain $J_0$ is a subset of the uniquely defined maximal tough domain.

**Proof.** This domain may be defined as the union of all tough domains $U \supset J_0$ □

**Lemma 3.4.** Any tough domain is eventually periodic.

**Proof.** Consider a tough domain $J_0$. Since the Lebesgue measure of the set $M$ is finite and $\text{Leb}(S^k(J_0)) = \text{Leb}(J_0) > 0$ for all positive $k$, there exist $k,l \in \mathbb{N}$, $k > l$ such that $S^l(J_0) \cap S^k(J_0) \neq \emptyset$. Take $D$ – the maximal tough domain that contains $S^l(J_0)$. Then $S^{k-l}(D) \cup D$ is also a tough domain which means that $S^{k-l}(D) = D$. Since $S^{k-l}|D$ is a shift, this is an identical map. This finishes the proof. □

Of course, the converse statement is wrong, the counterexample is as simple as interval exchange map of two segments: $[0,0.5)$ and $[0.5,1)$. 

Definition 3.4. We call a map \( S \) generic if it does not have any periodic (tough) domains.

Given the partition \( \{ M_j \} \), we consider the space of all piecewise translation maps, specifying values of parameters \( c_k \). Observe that for a given set \( M \) and a given partition all admissible translation maps form a compact subset of a Euclidean space. Later on, discussing topological properties of such maps we refer to the introduced set.

Corollary 3.1. For the interval translation map of Example 1.2 given a fixed number of domains, for a generic map \( S \), the set of all non-typical point is dense in \( M \).

Proof. If non-typical points are not dense, there exists a nontrivial tough domain. So, some of shift vectors must be rationally dependent. The set of all parameters \( t_j \) and \( c_j \) with rationally independent \( c_j \) is generic. \( \square \)

4. Invariant measures for interval translation maps.

In this section, we consider interval translation maps only (Example 1.2). Particularly, we set \( M = T^1 \) here. The general case of piecewise isometries will be considered later on.

Later on we consider Borel probability invariant measures only. As usually, we say that a measure \( \mu \) is invariant with respect to \( S \) if \( \mu(S^{-1}(A)) = \mu(A) \) for any measurable set \( A \). Since the map \( S \) is discontinuous, we cannot appeal to Krylov-Bogolyubov theorem for existence of invariant measures. And, unlike classical interval exchange maps, we cannot say that the Lebesgue measure is invariant. However, if we demonstrate below, invariant measures exist for any maps \( S \).

Definition 4.1. A measure is called non-atomic if the measure of any singleton is zero.

Observe that if a map does not have any periodic points, any invariant measure is non-atomic. For any non-atomic measure, its support must be uncountable.

Theorem 4.1. Any interval translation map admits a non-atomic invariant measure.

Proof. Let map \( S \) be non-generic i.e. it has periodic points. Then there is a tough periodic domain \( J_0 \). Take the minimal positive value \( m \) such that \( S^m(J_0) = J_0 \). Then the measure \( \mu^* \) can be constructed as a renormalised reduction of the Lebesgue measure to the set \( J_0 \cup S(J_0) \cup \ldots \cup S^{m-1}(J_0) \).

So, we may assume that the map \( S \) is typical. Consider sequences \( \{ t^m_k \} \) and \( \{ c^m_k \} \) of rational numbers that converge to \( t_k \) and \( c_k \) respectively. Let \( S_m \) be corresponding mappings. We take sequences \( \{ t^m_k \} \) and \( \{ c^m_k \} \) so that the following statement holds.

Condition A. For any \( r \in \mathbb{N}, i, j = 0, \ldots, n \) such that \( S^r(t_i) = t_j \), there is such \( m_0 \in \mathbb{N} \) such that \( S^r_m(t_i) = t_j \) for any \( m \geq m_0 \). If \( S^r(t_i) = t_j \), there exists such \( m_0 \in \mathbb{N} \) such that \( S^r_m(t_i - 0) = t_j \) for any \( m \geq m_0 \).

Let \( \delta_m = \max_k |t^m_k - t_k| \to 0 \).

Let

\[
\Xi_m = \bigcap_{k=1}^{\infty} S^k_m(M).
\]

These sets have positive Lebesgue measures, since each of these sets is a union of a finite number of arcs. We introduce measures \( \mu_m \) as renormalisations of the Lebesgue measure, reduced to \( \Xi_m \). These measures are invariant w.r.t. mappings \( S_m \).
Lemma 4.1. For any continuous function $\varphi : [0, 1] \to \mathbb{R}$, we have
\[
\int_M (\varphi(S_m(t)) - \varphi(S(t))) \, d\mu_m(t) \to 0
\]
as $\mu \to \infty$.

Proof. Let
\[
t^m_{k,-} = \min\{t^m_k, t_k\}, \quad t^m_{k,+} = \max\{t^m_k, t_k\}.
\]
For any $\varepsilon > 0$, we can find $m_0$ such that $|\varphi(S_m(t)) - \varphi(S(t))| < \varepsilon$ for any $m \geq m_0$ and
\[
t \notin \bigcup_{k=1}^m [t^m_{k,-}, t^m_{k,+}].
\]
So, integrals of $\varphi(S_m(t)) - \varphi(S(t))$ are small on completion of the union of $[t^m_{k,-}, t^m_{k,+}]$. It suffices to prove that
\[
\int_{t^m_{k,-}}^{t^m_{k,+}} (\varphi(S_m(t)) - \varphi(S(t))) \, d\mu_m(t) \to 0
\]
for any $k = 0, \ldots, n$. Instead of this, we can prove that
\[
(4.1) \quad \mu_m([t^m_{k,-}, t^m_{k,+}]) \to 0
\]
If Eq. (4.1) is wrong, there exists a value $\varepsilon_0 > 0$ and a sequence $m_l \to \infty$ such that
\[
\mu_m(\Theta_{k,l}) := \mu_m([t^m_{k,-}, t^m_{k,+}]) \bigcap \Xi_{m_l} \geq \varepsilon_0.
\]
Let $K \in \mathbb{N}$ be such that $K\varepsilon_0 > 1$. Then there exists a value $r_0 \in \{1, \ldots, K\}$ such that $S_{r_0}^{m_l}(\Theta_{k,l}) \bigcap \Theta_{k,l} \neq \emptyset$. Here we notice the fact that maps $S_{r_0}$ are invertible on sets $\Xi_{m_l}$.

So, there exists a number $k \in \{0, \ldots, n\}$ and a sequence $\tau_{k,l} \to t_k$ such that
\[
(4.2) \quad |S_{r_0}^{m_l}(\tau_{k,l}) - \tau_{k,l}| \to 0 \quad \text{as} \quad l \to \infty
\]
Then, by Condition A, Eq. (4.2) implies that either $S_{r_0}(t_k) = t_k$ or
\[
(4.3) \quad S_{r_0}(t_k - 0) = t_k
\]
If Eq. (4.3) takes place, then $S_{r_0}(\theta) = \theta$ where $\theta$ is the greatest of discontinuity point of the map $S_{r_0}$ less than $t_k$. Anyway, the map $S$ has a periodic point that contradicts to our assumptions that $S$ is generic. So, Eq. (4.1) is correct which finishes the proof of the lemma. □

Now, we continue the proof of the theorem.

The set of Borel probability measures in $M$ is compact in the $*$-weak topology. Without loss of generality, we may assume that the sequence of invariant measures $\mu_k$ $*$-weakly converges to a measure $\mu^*$.

Let us prove that $\mu^*$ is invariant w.r.t. $S$.

Fix a continuous function $\varphi : M \to \mathbb{R}$ and prove that
\[
\int_M \varphi \, d\mu^* = \int_M \varphi \circ S \, d\mu^*.
\]
We have
\[
(4.4) \quad \left| \int_M (\varphi - \varphi \circ S) \, d\mu^* \right| \leq \left| \int_M (\varphi - \varphi \circ S) \, d\mu_m \right| + \left| \int_M (\varphi \circ S_m - \varphi \circ S) \, d\mu_m \right| + \left| \int_M (\varphi \circ S_m - \varphi) \, d\mu_m \right|
\]
The first term in the right hand side of Eq. (4.4) tends to 0 since measures $\mu_k$ are weakly converging to $\mu^*$, the second one is zero since measures $\mu_k$ are $S_k$ – invariant and the third one tends to zero by Lemma 3.3. So, the right hand side of Eq. (4.4) is zero that finishes the proof of the theorem.
The obtained measure is non-atomic if the map does not have any periodic points. Existence of periodic points implies existence of a periodic segment and, hence, existence of a continuous invariant measure. □

Let $\Xi = \bigcap_{k=1}^{\infty} S^k(M)$.

Corollary 4.1. For any map $S$ we have the set $\Xi$ is uncountable.

Proof. The support of the non-atomic invariant measure that exists by Corollary 11 is a subset of $\Xi$. □

Lemma 4.2. Let $\mu$ be the non-atomic invariant measure for an interval translation map $S$ that exists by Theorem 4.1. Then
\[
\mu(S(A)) = \mu(A)
\]
for any measurable set $A$.

Proof. Similarly to the proof of Theorem 4.1 we approximate the map $S$ by periodic maps $S_m$ and $*$-weakly approximate the measure $\mu$ by periodic continuous measures $\mu_m$. All maps $S_m$ are periodic and, hence, invertible on supports of their invariant measures. So,
\[
\mu_m(S_m(A)) = \mu_m(A)
\]
for any $m \in \mathbb{N}$ and any measurable set $A$. Owing to the $*$-weak convergence of measures, and by Lemma 4.1 we could proceed to limit in (4.6) and get (4.5). □

The following statement could be formulated as follows: endowed with a non-atomic invariant measure, any interval translation map is metrically equivalent to an interval exchange map of the segment $[0, 1]$.

Theorem 4.2. Let $\mu$ be the non-atomic invariant measure for an interval translation map $S$ that exists by Theorem 4.1. Then the reduction $S_{|\text{supp } \mu}$ is metrically equivalent to an interval exchange map $T : [0, 1] \to [0, 1]$ with the Lebesgue measure. The semi-conjugacy map $h : \text{supp } \mu \to [0, 1]$ is continuous. It is one-to-one everywhere, except a countable set.

Proof. Consider the function $h(x) = \mu([0, x])$. This function, reduced to supp $\mu$ is monotonous. The measure $\mu$ is non-atomic, so $h$ is continuous. The equality $h(x) = h(y)$ implies $\mu([x, y]) = 0$ so there is a countable set $D \subset \text{supp } \mu$ such that $h|\text{supp } \mu \setminus D$ is injective and $\#h^{-1}(y) = 2$ for any $y \in h(D)$. Now we define the map $h(x) := \max h^{-1}(x) : [0, 1] \to \text{supp } \mu$ that is a right inverse to $h$ and set
\[
T(x) := h(S(h(x))).
\]
Then, by definition we have the semi-conjugacy: $h \circ T = S \circ h$.

Now let $0 = t_0 < \ldots < t_n = 1$ be the points such that the maps $S|_{[t_j, t_{j+1}]}$ are translations: $S(x) = x + c_j$ for all $j = 0, \ldots, n$. For any $j$ we denote $\tau_j = \mu([0, t_j])$, so $\tau_j = h(t_j)$ if we naturally spread the map $h$ to $[0, 1]$.

Take two points $x < y$ in a segment $I_j := [\tau_j, \tau_{j+1})$ if the segment is non-empty. By (4.7) we have
\[
T(y) - T(x) = h(S(h(y)) - h(S(h(x)))
\]
The map $h$ is monotonous and $h(x), h(y)$ belong to the same segment $[t_j, t_{j+1})$ where $S$ is the shift. So, using Eq. (4.3), we have
\[
T(y) - T(x) = \mu[S(h(x)), S(h(y))] = \mu[h(x), h(y)] = h(h(y)) - h(h(x)) = y - x.
\]

So, $T$ is an interval translation map. To finish the proof, it suffices to demonstrate that $T$ preserves the Lebesgue measure. Take a segment $[x, y]$. Then
\[
\text{Leb } T^{-1}(x, y) = \text{Leb } h^{-1} \circ S^{-1} \circ h^{-1}((x, y)) = \text{Leb } (h(S^{-1} \circ h^{-1}((x, y)))) = \mu(S^{-1} \circ h^{-1}((x, y))) = \mu(h^{-1}((x, y))) = \mu(h(x), h(y)) = y - x.
\]
Then $\text{Leb } T^{-1}(A) = \text{Leb } (A)$ for any measurable set $A$. □

Let us recall some standard definitions from Topological Dynamics.
Definition 4.2. We call a point $x$ Poisson stable with respect to the map $S$ if there exists an increasing sequence \( \{m_k \in \mathbb{N} \} \) such that $S^{m_k}(x) \to x$.

Definition 4.3. We call a point $x$ nonwandering with respect to the map $S$ if for any neighbourhood $U$ of the point $x$ there is a point $p \in U$ and a number $k \in \mathbb{N}$ such that $p, S^k(p) \in U$.

Definition 4.4. Let $x_1, x_2, \ldots, x_n \in M$. Given an $\varepsilon > 0$, we call the sequence $\{x_j \}$ an $\varepsilon$-chain with respect to the map if

$\text{dist} (x_{j+1}, S(x_j)) < \varepsilon \quad \forall j = 1, \ldots, n - 1.$

Here we use the word ”chain” for finite sequences, reserving the word ”pseudo-trajectory” for infinite sequences with similar properties.

Definition 4.5. We call a point $x$ chain recurrent if for any $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that $\{x, p_1, \ldots, p_k\}$

such that $\text{dist} (x, p_1) < \varepsilon$ and $\text{dist} (x, p_k) < \varepsilon$.

In this case, we may additionally assume that $p_1 = p_k$.

Observe that the last two definitions work even for points of the discontinuity set where the map is undefined.

In other words, the point $x$ is $\omega$- limit for itself. Recall that the closure of the set of all Poisson stable points is a subset of the set $\Omega(S)$ of all non-wandering points. The next statement is well-known for continuous maps.

Theorem 4.3. Let $\mu^*$ be the invariant measure for the mapping $S$. Then Poisson stable points are dense in $\text{supp } \mu^*$. Particularly, the set $\Omega(S)$ is uncountable.

Proof. Let $\{U_j\}_{j \in \mathbb{N}}$ be a countable base of the topology in $M$. Consider the set $R$ that for any $j \in \mathbb{N}$ either $x \not\in U_j$ or there exists a sequence $m_l \to \infty$ such that $S^{m_l}(x) \in V$.

For any $j \in \mathbb{N}$ we consider $N_j = \{x \in U_j : \exists m_0 \in \mathbb{N} : S^m(x) \notin U_j \forall m \geq m_0\}$. So $N_j$ is the set of all points of $U_j$ whose iterations leave this set forever. Sets $R$ and $N_j$ are measurable. By Poincaré Recurrence Theorem, $\mu(N_j) = 0$ for any $j$. On the other hand,

$$\bigcup_{j \in \mathbb{N}} U_j = \mathbb{R}^d,$$

so

$$\mathbb{R}^d \setminus R = \bigcup_{j \in \mathbb{N}} N_j.$$ 

Then the measure of the completion of the set $R$ equals to 0 and hence the set $R$ must be dense in $M$. On the other hand, if $x \in R$, for any neighborhood $U$ of the point $x$ there exists a sequence $m_l \to +\infty$ such that $S^{m_l}(x) \in U$. Then the $\omega$- limit set $\omega_x$ of the point $x$ intersects with $U$. Since the set $\omega_x$ is closed and the neighbourhood $U$ is arbitrary, we have $x \in \omega_x$ that finishes the proof. \( \square \)

So, the set of recurrent points must be non-empty for interval translation maps.

5. Invariant Measures for Piecewise Translation Maps, the Case $n \leq d + 1$.

Unfortunately, the direct generalisation of the proof of Theorem 4.3 fail in dimensions greater than one. Sometimes, the existence of invariant measures can be proved using tools of Topological Dynamics (we do it in the next sections). However, there is another approach to prove existence of invariant measures in some special cases. Consider a piecewise translation map $S : M \to M$ where $M$ is a subset of $\mathbb{R}^d$. Let the power $N$ of the partition $\{M_j\}$ be such that $n = d + 1$. Then the piecewise
translation map $S$ with shift vectors can be reduced to a rotation $x_{k+1} = x_k + c_1$
of the torus

$$T := \mathbb{R}^d / (c_2 - c_1) \mathbb{Z} \oplus (c_3 - c_1) \mathbb{Z} \oplus \ldots \oplus (c_{d+1} - c_1) \mathbb{Z},$$

see [10] and [19]. The latter map is ergodic (and, also, uniquely ergodic) if vectors $c_1, \ldots, c_{d+1}$ are rationally independent e.g., any nontrivial linear combination of these vectors with integer coefficients is non-zero. The following statement is the main result of [10].

**Theorem 5.1.** Let $n = d + 1$ and the vectors $c_1, \ldots, c_{d+1}$ be rationally independent. Then the map $S$ is finite that is there exists $m \in \mathbb{N}$ such that

$$A := \bigcap_{k=1}^{\infty} S^k(M) = \bigcap_{k=1}^{m} S^k(M).$$

We notice that in this case the set $A$ (let us call it attractor) is of the positive Lebesgue measure and the reduction of the Lebesgue measure to $A$ is invariant.

We claim the following simple corollary of the above result.

**Corollary 5.1.** Let

$$n \leq d + 1$$

and the vectors $c_1, \ldots, c_n$ be rationally independent and

$$\text{Leb } H = \text{Leb } \left( \bigcup_{j=1}^{n} \partial M_j \right) = 0.$$

Then the map $S$ admits a Borel probability invariant measure.

**Proof.** If $n = d + 1$, the desired statement follows immediately from Theorem 5.1. Otherwise consider the linear space $\mathcal{L} := \text{Lin } (c_1, \ldots, c_n)$. Let $\mathcal{L}^\perp$ be the subspace of $\mathbb{R}^d$, orthogonal to $\mathcal{L}$. We consider Lebesgue measures in both subspaces $\mathcal{L}$ and $\mathcal{L}^\perp$. The space $M$ is foliated by subsets $M_x := M \cap (\mathcal{L} + x)$, $x \in \mathcal{L}^\perp$. All these sets are invariant with respect of the map $S$. By Fubini Theorem, there exists a value $x_0 \in \mathcal{L}^\perp$ such that $\text{Leb } \mathcal{L}/x_0 M_{x_0} > 0$ and $\text{Leb } \mathcal{L}/x_0 H \cap (\mathcal{L} + x_0) = 0$. If all intersections $M_j \cap (\mathcal{L} + x_0)$ are non-empty, we apply the case $n = d + 1$ to the reduction $S|_{\mathcal{L}/x_0}$. We consider new "domains" (these sets may be not connected)

$$\tilde{M}_j := M_j \cap (\mathcal{L} + x_0)$$
or, better say, those of that sets that are non-empty. Otherwise, we apply the same procedure to that reduction. Observe that inequality [5] can never be violated after we proceed to the reduced map. In this case the value $n$ decreases by one and the value $d$ cannot increase. After at most $d - 1$ steps we get one of two options: either $n = d + 1$ or $n = d = 1$. In the latter case the final reduction must be the identity map and definitely admits an invariant measure. □

### 6. Invariant measures for Piecewise Continuous maps

Here we formulate and prove the result on existence of invariant measures for piecewise continuous maps. Consider a map $T : M \to M$ where $M$ satisfies all above conditions and $T|_{M_j}$ are just continuous maps, not isometries.

Let $H$ be the discontinuity set of $T$ and $H_\varepsilon = \{ x \in M : \text{dist } (x, H) < \varepsilon \}$. Let $\Omega := \bigcup_{j=0}^{\infty} T^{-j}(H)$.

**Theorem 6.1.** Let there exist a point $x_0 \notin \Omega$ such that

$$\lim_{\varepsilon \to 0} \lim_{m \to \infty} \frac{1}{m} \# \{ k \leq n : T^k(x) \in H_\varepsilon \} = 0.$$

Then the map $T$ admits a Borel probability invariant measure.
Proof. We use an idea of the proof of Krylov-Bogolyubov theorem. We fix the point \( x \in M \setminus \Omega \) and consider the sequence of measures

\[
\mu_m := \frac{1}{m} \sum_{l=0}^{m-1} \delta(T^l(x))
\]

where \( \delta(y) \) is the Dirac measure at the point \( y \). All these measures are Borel probabilities on \( M \). The space of such measures is compact in the \( * \)– weak topology hence we could take a subsequence \( \mu_{m_k} \) weakly converging to a measure \( \mu^* \). The assumption (6.1) guarantees that \( \mu(H_\epsilon) \to 0 \) and hence \( \mu(H) = \mu(\Omega) = 0 \). Let \( T^# \) be the pushforward operator on the space of Borel probability measures: \( T^#(\mu) = \nu \) if \( \nu(A) = \mu(T^{-1}(A)) \) for any measurable set \( A \). Observe that for any \( m \)

\[
T^# \mu_m - \mu_m = \frac{1}{m} \left( \delta(T^m(x)) - \delta(x) \right).
\]

Proceeding to limit along the subsequence \( \{m_k\} \) we see that \( T^#(\mu^*) = \mu^* \) and hence the measure \( \mu^* \) is invariant. \( \square \)

Corollary 6.1. If a discontinuity set \( H \) of a map \( T \) does not contain any nonwandering points, the map \( T \) admits a Borel probability invariant measure.

Definition 6.1. Let \( \mu \) be a probability invariant measure for the piecewise continuous map \( T \). This map is numerically stable if for any sequence \( S_k \) of piecewise translation maps, approximating the map \( T \) point-wise, and any sequence \( \mu_k \) of Borel probability invariant measures for mappings \( S_k \), weakly converging to a measure \( \mu \), the measure \( \mu \) is \( T \)– invariant.

Theorem 6.2. Let

\[
(6.2) \quad H \bigcap \text{CR}(T) = \emptyset.
\]

Then any Borel probability invariant measure of the map \( T \) is numerically stable.

Proof. Indeed, the statement of Theorem 13 claims, that supports of all measures \( \mu_k \) are subsets of the closure of all Poincaré stable points for the map \( S_k \) (call these sets \( R_k \)). If the statement we, are proving, is wrong, the lower limit of the Hausdorff distance between sets \( R_k \) and the discontinuity set \( H \) equals 0. Since the discontinuity set is compact this gives a contradiction with (6.2). \( \square \)

7. Conclusion

Here we list once again the principal results of the paper.

(1) Any piecewise translation map defined as piecewise shifts on a smooth partition of a domain, admits a Borel non-atomic probability invariant measure.

(2) Interval translation maps endowed with this measure are metrically equivalent to interval exchange maps with the Lebesgue measure.

(3) Any piecewise translation map of \( N \) domains in the space \( \mathbb{R}^d \) with \( d \geq N - 1 \) and rationally independent translation vectors \( c_k \) admits an invariant measure. If \( d = N - 1 \) this measure is absolutely continuous with respect to the Lebesgue measure.

(4) Any piecewise continuous map without nonwandering points on the discontinuity set admits a Borel probability invariant measure.

(5) If a discontinuity set does not contain any chain recurrent points, any weakly converging sequence of invariant probability measures for piecewise translation maps, approximating the given piecewise continuous map converges to an invariant measure of the given map.

We finish the paper with a conjecture that appeals to the main result of the paper [16]. Any piecewise isometry may be represented as \( Ax + c \) where \( A \) is an orthogonal matrix and \( c \) is a vector.
Open Problem 7.1. Let the piecewise isometry $S$ of a domain $M \subset \mathbb{R}^d$ be such that $S(x) = A_j x + c_j$ for all $x \in M_j$. As usually, we assume that

$$M \subset \bigcup_{j=1}^n \mathbb{R}^d_j,$$

$M_j \subset M$ are disjoint domains and $\text{Leb}(\partial M_j) = 0$ for all $j$. Suppose that vectors $c_j$ are rationally independent. Is it true that the map $S$ is of finite type if all vectors $c_j$ are rationally independent?

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