Numerical Computation of Exponential Functions of Nabla Fractional Calculus

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Abstract: In this article, we illustrate the asymptotic behaviour of exponential functions of nabla fractional calculus. For this purpose, we propose a novel matrix technique to compute these functions numerically.

Key Words: Nabla fractional difference, exponential function, triangular strip matrix, general solution, asymptotic behaviour.

AMS Classification: 39A12.

1. Introduction & Preliminaries

Nabla fractional calculus is an integrated theory of arbitrary order sums and differences. The concept of nabla fractional difference traces back to the works of Miller & Ross [18], Gray & Zhang [13], Atici & Eloe [6], and Anastassiou [5]. During the past one decade, there has been an increasing interest in this field. For a detailed introduction on the evolution of nabla fractional calculus, we refer to [12] and the references therein.

We use the following notations, definitions and known results of nabla fractional calculus throughout the article. Denote by \( \mathbb{N}_a = \{a, a + 1, a + 2, \ldots\} \) and \( \mathbb{N}_a^b = \{a, a + 1, a + 2, \ldots, b\} \) for any \( a, b \in \mathbb{R} \) such that \( b - a \in \mathbb{N}_1 \). The backward jump operator \( \rho : \mathbb{N}_{a+1} \rightarrow \mathbb{N}_a \) is defined by

\[
\rho(t) = t - 1, \quad t \in \mathbb{N}_{a+1}.
\]

Define the \( \mu \)-th order nabla fractional Taylor monomial by

\[
H_\mu(t, a) = \frac{\Gamma(t - a + \mu)}{\Gamma(t - a)\Gamma(\mu + 1)}, \quad \mu \in \mathbb{R} \setminus \{\ldots, -2, -1\},
\]

provided the right-hand side of this equation is sensible. Here \( \Gamma(\cdot) \) denotes the Euler gamma function.

Lemma 1.1. [12] We observe the following properties of nabla fractional Taylor monomials.

1. \( H_\mu(t, a) = 0 \) for all \( \mu \in \{\ldots, -2, -1\} \) and \( t \in \mathbb{N}_a \).
2. \( H_\mu(t, \rho(t)) = 1 \) for all \( \mu \in \mathbb{R} \setminus \{\ldots, -2, -1\} \) and \( t \in \mathbb{N}_a \).
3. \( H_\mu(t, t) = 0 \) for all \( \mu \in \mathbb{R} \setminus \{\ldots, -2, -1\} \) and \( t \in \mathbb{N}_a \).

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Definition 1.1. Let \( u : \mathbb{N}_a \to \mathbb{R} \). The first order backward (nabla) difference of \( u \) is defined by
\[
(\nabla u)(t) = u(t) - u(t - 1), \quad t \in \mathbb{N}_{a+1}.
\]

Definition 1.2. Let \( u : \mathbb{N}_{a+1} \to \mathbb{R} \) and \( \nu > 0 \). The \( \nu \)-th order nabla sum of \( u \) based at \( a \) is given by
\[
(\nabla^{-\nu} u)(t) = \sum_{s=a+1}^{t} H_{\nu-1}(t, \rho(s))u(s), \quad t \in \mathbb{N}_a,
\]
where by convention \((\nabla^{-\nu} u)(a) = 0\).

Ahrendt et al. [4] showed that the definition of a fractional difference can be rewritten in a form similar to the definition of a fractional sum.

Theorem 1.2. Let \( u : \mathbb{N}_{a+1} \to \mathbb{R} \) and \( 0 < \nu < 1 \). Then,
\[
(\nabla_{a,\nu}^\nu u)(t) = \sum_{s=a+1}^{t} \frac{H_{\nu-1}(t, \rho(s))u(s)}{\Gamma(\nu)}, \quad t \in \mathbb{N}_{a+1}.
\]

Definition 1.3. Let \( u : \mathbb{N}_a \to \mathbb{R} \) and \( 0 < \nu \leq 1 \). The \( \nu \)-th order Riemann–Liouville nabla difference of \( u \) based at \( a \) is given by
\[
(\nabla^\nu_{a,\nu} u)(t) = \left(\nabla_{a,\nu}^{(1-\nu)} u\right)(t), \quad t \in \mathbb{N}_{a+1}.
\]

The following identity is useful in transforming the Caputo nabla fractional difference into the Riemann–Liouville nabla fractional difference.

Theorem 1.3. Let \( u : \mathbb{N}_a \to \mathbb{R} \) and \( 0 < \nu < 1 \). Then,
\[
(\nabla^\nu_{a,\nu} u)(t) = (\nabla^\nu_{a} u)(t) - H_{\nu}(t, a)u(a), \quad t \in \mathbb{N}_{a+1}.
\]

2. Exponential Functions of Nabla Fractional Calculus

Acar et al. [3] and Nagai [19] introduced the exponential functions of nabla fractional calculus as the unique solutions of the following initial value problems associated with the Riemann–Liouville and the Caputo nabla fractional differences:

\[
\begin{cases}
(\nabla^\nu_{\rho(0)} w)(t) = \lambda w(t), & t \in \mathbb{N}_1, \\
(\nabla^\nu_{\rho(0)} (1-\nu) w)(0) = w(0) = 1,
\end{cases}
\]

and

\[
\begin{cases}
(\nabla^\nu_{0} x)(t) = \lambda x(t), & t \in \mathbb{N}_1, \\
x(0) = 1,
\end{cases}
\]
where $0 < \nu < 1$ and $|\lambda| < 1$. The unique solutions of the initial value problems (2.1) and (2.2) are represented by $\hat{e}_{\nu,\nu}(\lambda, t_\nu)$ and $\hat{e}_\nu(\lambda, t_\nu)$, respectively, where

\[
\hat{e}_{\nu,\nu}(\lambda, t_\nu) = \sum_{k=0}^{\infty} \lambda^k H_{\nu k+\nu-1}(t, \rho(0)), \quad t \in \mathbb{N}_0,
\]

and

\[
\hat{e}_\nu(\lambda, t_\nu) = \sum_{k=0}^{\infty} \lambda^k H_{\nu k}(t, 0), \quad t \in \mathbb{N}_0.
\]

Atici et al. [7], Čermák et al. [9], Eloe et al. [11], Jia et al. [15] and Wu et al. [24] obtained the following asymptotic results of the discrete exponential functions.

\[
\lim_{t \to \infty} \hat{e}_{\nu,\nu}(\lambda, t_\nu) = 0, \quad \lambda \in (-1, 0],
\]

\[
\lim_{t \to \infty} \hat{e}_{\nu,\nu}(\lambda, t_\nu) = \infty, \quad \lambda \in (0, 1),
\]

\[
\lim_{t \to \infty} \hat{e}_\nu(\lambda, t_\nu) = 0, \quad \lambda \in (-1, 0),
\]

\[
\lim_{t \to \infty} \hat{e}_\nu(\lambda, t_\nu) = \infty, \quad \lambda \in (0, 1).
\]

Using triangular strip matrices, Podlubny [21] described a matrix approach to find numerical solutions of fractional differential equations. Motivated by this technique, we present a matrix method to compute the exponential functions (2.3) and (2.4) numerically.

2.1. Computation of (2.3): Let $m \in \mathbb{N}_1$ and consider the initial value problem associated with (2.1):

\[
\begin{cases}
(\nabla_{\rho(0)}^\nu w)(t) = \lambda w(t), & t \in \mathbb{N}_m^n, \\
(\nabla_{\rho(0)}^{(1-\nu)} w)(0) = w(0) = 1.
\end{cases}
\]

Rewriting the equation in (2.9) using Theorem 1.2 we have

\[
\sum_{s=0}^{t} H_{-\nu-1}(t, \rho(s)) w(s) = \lambda w(t), \quad t \in \mathbb{N}_1^n.
\]

Rearranging the terms in (2.10), we obtain

\[
(1 - \lambda) w(t) + \sum_{s=1}^{t-1} H_{-\nu-1}(t, \rho(s)) w(s) = -H_{-\nu-1}(t, \rho(0)) w(0), \quad t \in \mathbb{N}_1^n.
\]

Denote by $\hat{w} = [w(1), w(2), \cdots, w(m)]^T$. Then, the matrix form of (2.11) is given by

\[
\mathcal{L}\hat{w} = -\mathcal{B},
\]
where
\[ \mathcal{L} = \begin{pmatrix}
1 - \lambda & 0 & \cdots & \cdots & 0 & 0 \\
H_{\nu-1}(2, \rho(1)) & 1 - \lambda & \cdots & \cdots & 0 & 0 \\
H_{\nu-1}(3, \rho(1)) & H_{\nu-1}(3, \rho(2)) & \cdots & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
H_{\nu-1}(m, \rho(1)) & H_{\nu-1}(m, \rho(2)) & \cdots & \cdots & 1 - \lambda & 0 \\
H_{\nu-1}(m, \rho(1)) & H_{\nu-1}(m, \rho(2)) & \cdots & \cdots & H_{\nu-1}(m, \rho(m-1)) & 1 - \lambda
\end{pmatrix}_{m \times m} \]

is a lower triangular strip matrix and
\[ \mathcal{B} = \begin{pmatrix}
H_{\nu-1}(1, \rho(0)) \\
H_{\nu-1}(2, \rho(0)) \\
H_{\nu-1}(3, \rho(0)) \\
\vdots \\
H_{\nu-1}(m-1, \rho(0)) \\
H_{\nu-1}(m, \rho(0))
\end{pmatrix}_{m \times 1} \]

Since \( \mathcal{L} \) is non-singular, the exponential function (2.3) can be computed by the following numerical algorithm:
\[ \hat{e}_{\nu, \nu}(\lambda, \mathcal{L}^T) = -\mathcal{L}^{-1} \mathcal{B}, \quad t \in \mathbb{N}^m_1. \]

Here \( \mathcal{L} = [\mathcal{L}_{ij}]_{m \times m} \) and \( \mathcal{B} = [\mathcal{B}_i]_{m \times 1} \), where
\[ \mathcal{L}_{ij} = \begin{cases}
1 - \lambda, & i = j, \\
0, & i < j, \\
H_{\nu-1}(i, \rho(j)), & i > j,
\end{cases} \]
and
\[ \mathcal{B}_i = H_{\nu-1}(i, \rho(0)). \]

**Example 1.** Computation of \( \hat{e}_{0,5,0.5}(-0.5, \mathcal{L}^T) \) for \( t \in \mathbb{N}^m_1 \):

We have
\[ \mathcal{L} = \begin{pmatrix}
1.5000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.5000 & 1.5000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.1250 & -0.5000 & 1.5000 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.0625 & -0.1250 & -0.5000 & 1.5000 & 0 & 0 & 0 & 0 & 0 \\
-0.0391 & -0.0625 & -0.1250 & -0.5000 & 1.5000 & 0 & 0 & 0 & 0 \\
-0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 1.5000 & 0 & 0 & 0 \\
-0.0205 & -0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 1.5000 & 0 & 0 \\
-0.0161 & -0.0205 & -0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 1.5000 & 0 \\
-0.0131 & -0.0161 & -0.0205 & -0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 1.5000 \\
-0.0109 & -0.0131 & -0.0161 & -0.0205 & -0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 1.5000
\end{pmatrix}. \]
Example 2.

Computation of \( \hat{e}_{0.5,0.5}(t, 0.5) \) for \( t \in \mathbb{N}_1^{10} \).

We have

\[
L = \begin{pmatrix}
0.5000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.5000 & 0.5000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.1250 & -0.5000 & 0.5000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.0625 & -0.1250 & -0.5000 & 0.5000 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.0391 & -0.0625 & -0.1250 & -0.5000 & 0.5000 & 0 & 0 & 0 & 0 & 0 \\
-0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 0.5000 & 0 & 0 & 0 & 0 \\
-0.0205 & -0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 0.5000 & 0 & 0 & 0 \\
-0.0161 & -0.0205 & -0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 0.5000 & 0 & 0 \\
-0.0131 & -0.0161 & -0.0205 & -0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 0.5000 & 0 \\
-0.0109 & -0.0131 & -0.0161 & -0.0205 & -0.0273 & -0.0391 & -0.0625 & -0.1250 & -0.5000 & 0.5000
\end{pmatrix},
\]

\[
\hat{B} = \begin{pmatrix}
-0.5000 \\
-0.1250 \\
-0.0625 \\
-0.0391 \\
-0.0273 \\
-0.0205 \\
-0.0161 \\
-0.0131 \\
-0.0109 \\
-0.0093
\end{pmatrix}.
\]

Then, for \( t \in \mathbb{N}_1^{10} \),

\[
\hat{e}_{0.5,0.5}(t, 0.5) = -L^{-1} \hat{B} = \begin{pmatrix}
0.3333 \\
0.1944 \\
0.1343 \\
0.1009 \\
0.0798 \\
0.0654 \\
0.0550 \\
0.0472 \\
0.0411 \\
0.0362
\end{pmatrix}.
\]
Then, for \( t \in \mathbb{N}_{1}^{10} \),
\[
\hat{e}_{0.5,0.5}(0.5, t^{0.5}) = -\mathcal{L}^{-1}\mathcal{B} = \begin{pmatrix}
1 \\
1.2500 \\
1.6250 \\
2.1406 \\
2.8359 \\
3.7676 \\
5.0127 \\
6.6749 \\
8.8925 \\
11.8505
\end{pmatrix}.
\]

Example 3. The graphs of \( \hat{e}_{0.5,0.5}(-0.5, t^{0.5}) \) and \( \hat{e}_{0.5,0.5}(0.5, t^{0.5}) \) for \( t \in \mathbb{N}_{1}^{100} \) are shown in Figures 1 and 2, respectively.

![Figure 1](image)

2.2. Computation of (2.3): Let \( m \in \mathbb{N}_{1} \) and consider the initial value problem associated with (2.2):
\[
\begin{align*}
\left\{ \begin{array}{l}
(\nabla_{\nu} x)(t) = \lambda x(t), \quad t \in \mathbb{N}_{1}^{m}, \\
x(0) = 1.
\end{array} \right.
\end{align*}
\]
(2.12)

Rewriting the equation in (2.12) using Theorem 1.2 and Theorem 1.3 we have
\[
\sum_{s=1}^{t} H_{-\nu-1}(t, \rho(s))x(s) - H_{-\nu}(t, 0)x(0) = \lambda x(t), \quad t \in \mathbb{N}_{1}^{m}.
\]
(2.13)
Rearranging the terms in (2.13), we obtain
\[(1 - \lambda)x(t) + \sum_{s=1}^{t-1} H_{-\nu-1}(t, \rho(s))x(s) = H_{-\nu}(t, 0)x(0), \quad t \in \mathbb{N}_1^m. \tag{2.14}\]
Denote by \(\tilde{x} = [x(1), x(2), \cdots, x(m)]^T\). Then, the matrix form of (2.14) is given by
\[\mathcal{L}\tilde{x} = \mathcal{C},\]
where
\[
\mathcal{C} = \begin{pmatrix}
H_{-\nu}(1, 0) \\
H_{-\nu}(2, 0) \\
H_{-\nu}(3, 0) \\
\vdots \\
\vdots \\
H_{-\nu}(m - 1, 0) \\
H_{-\nu}(m, 0)
\end{pmatrix}_{m \times 1}.
\]
Since \(\mathcal{L}\) is non-singular, the exponential function (2.4) can be computed by the following numerical algorithm:
\[\hat{e}_\nu(\lambda, t^\nu) = \mathcal{L}^{-1}\mathcal{C}, \quad t \in \mathbb{N}_1^m.\]
Here \(\mathcal{L} = [\mathcal{L}_{ij}]_{m \times m}\) and \(\mathcal{C} = [C_i]_{m \times 1}\), where
\[
\mathcal{L}_{ij} = \begin{cases}
1 - \lambda, & i = j, \\
0, & i < j, \\
H_{-\nu-1}(i, \rho(j)), & i > j,
\end{cases}
\]
and
\[C_i = H_{-\nu}(i, 0).\]

**Example 4.** Computation of \(\hat{e}_{0.5}(-0.5, t^{0.5})\) for \(t \in \mathbb{N}_1^{10}\).
We have

\[
C = \begin{pmatrix}
1 \\
0.5000 \\
0.3750 \\
0.3125 \\
0.2734 \\
0.2461 \\
0.2256 \\
0.2095 \\
0.1964 \\
0.1855
\end{pmatrix}.
\]

Then, from Example 1, for \( t \in \mathbb{N}_1^{10} \),

\[
\hat{e}_{0.5}(-0.5, \hat{\nu}_{0.5}) = \mathcal{L}^{-1}C = \begin{pmatrix}
0.6667 \\
0.5556 \\
0.4907 \\
0.4460 \\
0.4124 \\
0.3857 \\
0.3639 \\
0.3456 \\
0.3299 \\
0.3162
\end{pmatrix}.
\]

**Example 5.** *Computation of \( \hat{e}_{0.5}(0.5, \hat{\nu}_{0.5}) \) for \( t \in \mathbb{N}_1^{10} \):*

We have

\[
C = \begin{pmatrix}
1 \\
0.5000 \\
0.3750 \\
0.3125 \\
0.2734 \\
0.2461 \\
0.2256 \\
0.2095 \\
0.1964 \\
0.1855
\end{pmatrix}.
\]
Then, from Example 2, for $t \in \mathbb{N}_{10}^1$,

$$\hat{e}_{0.5}(0.5, t^{0.5}) = \mathcal{L}^{-1} \mathcal{C} = \begin{pmatrix}
2 \\
3 \\
4.2500 \\
5.8750 \\
8.0156 \\
10.8516 \\
14.6191 \\
19.6318 \\
26.3067 \\
35.1992
\end{pmatrix}.$$ 

Example 6. The graphs of $\hat{e}_{0.5}(0.5, t^{0.5})$ and $\hat{e}_{0.5}(-0.5, t^{0.5})$ for $t \in \mathbb{N}_{100}^1$ are shown in Figures 3 and 4, respectively.

**Figure 3.**

### 3. Extensions

The method described in Section 2 can be extended to obtain numerical solutions of initial value problems involving linear non-homogeneous nabla fractional difference equations.

Let $0 < \nu < 1$ and $m \in \mathbb{N}_1$. Consider the initial value problem

$$\begin{cases}
(\nabla_{\rho(0)}^{\nu} u)(t) = a(t)u(t) + f(t), & t \in \mathbb{N}_1^m, \\
(\nabla_{\rho(0)}^{1-\nu} u)(0) = u(0) = c,
\end{cases} \tag{3.1}$$

where $a, f : \mathbb{N}_1^m \to \mathbb{R}$ such that

$a(t) \neq 1, \quad t \in \mathbb{N}_1^m.$
Denote by $\tilde{u} = [u(1), u(2), \ldots, u(m)]^T$ and $\mathcal{F} = [f(1), f(2), \ldots, f(m)]^T$. Then, the matrix form of (3.1) is given by

$$L_1 \tilde{u} = \mathcal{F} - c\mathcal{B},$$

where

$$L_1 = \begin{pmatrix}
1 - a(1) & 0 & \cdots & \cdots & 0 & 0 \\
H_{\nu-1}(2, \rho(1)) & 1 - a(2) & \cdots & \cdots & 0 & 0 \\
H_{\nu-1}(3, \rho(1)) & H_{\nu-1}(3, \rho(2)) & \cdots & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
H_{\nu-1}(m-1, \rho(1)) & H_{\nu-1}(m-1, \rho(2)) & \cdots & \cdots & 1 - a(m-1) & 0 \\
H_{\nu-1}(m, \rho(1)) & H_{\nu-1}(m, \rho(2)) & \cdots & H_{\nu-1}(m, \rho(m-1)) & 1 - a(m) & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}_{m \times m}$$

is a lower triangular strip matrix. Since $L_1$ is non-singular, the solution of (3.1) can be computed by the following numerical algorithm:

$$u(t) = L_1^{-1}[\mathcal{F} - c\mathcal{B}], \quad t \in \mathbb{N}_1^m.$$
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