ABSTRACT

In settings varying from transport networks to communication networks, a fundamental goal of network design has been to achieve high throughput. While myriad network designs have been proposed, particularly in the data center and HPC networking communities, a consistent and accurate method of predicting a design’s throughput performance and comparing it to past proposals is conspicuously absent.

In this work, we develop a framework to benchmark the throughput of network topologies and apply this methodology to reveal insights about network structure. First, we show that despite being commonly used, cut-based metrics such as bisection bandwidth are the wrong metrics: they yield incorrect conclusions about the throughput performance of networks. We therefore measure flow-based throughput directly and show how to evaluate topologies with nearly-worst-case traffic matrices. Next, we use these metrics to analyze the throughput performance of a variety of networks. We evaluate networks used in data centers and high performance computing, and find that random graphs outperform the carefully-structured networks. We also evaluate several real-world complex networks and observe that throughput varies widely, and is not captured by other graph metrics such as betweenness or clustering coefficient. Finally, we study whether proposed models of complex networks are able to capture the throughput of real networks.

We have made our evaluation framework freely available to facilitate future work on rigorously designing and evaluating networks, and reproducing results.

1. INTRODUCTION

Networks move materials and information across physical and digital landscapes – transport networks carry people and goods, neurological networks carry electrochemical signals, social networks carry interactions between participants, computer networks carry data, etc. A fundamental property of a network is its carrying capacity: how much of the payload can be transported between the desired end-points per unit time? Across networks ranging from transport to communication, high capacity has been a core goal of network design. In the communication networks domain, particularly in the application areas of data centers and high performance computing (HPC), recent developments have reinvigorated research on the subject. So much so, that large number of network topologies have been proposed in the past few years [4, 23, 25, 42, 26, 59, 16, 53, 51, 12, 11] to achieve high capacity at low cost.

However, there is little order to this large and ever-growing set of network topologies. There is little clarity on which topology a network operator should pick, because we lack a broad comparison of topologies, and there is no open, public framework available for testing and comparing topology designs. The absence of a well specified benchmark also complicates research on network design, making it difficult to evaluate a new design against the numerous past proposals.

In fact, not only are we lacking throughput comparisons across a spectrum of topologies, we argue that the situation is worse: the community has been using the wrong metrics for measuring throughput. Cut-based metrics such as bisection bandwidth and sparsest cut are commonly used to estimate throughput, because minimum cuts are assumed to measure worst-case throughput [44]. However, while this is true for the case where the network carries only one flow, it does not hold for general traffic matrices. In the latter case, the cut is only an upper bound on throughput. The resulting gap between the cut-metric and the throughput leaves open the possibility that in a comparison of two topologies, the cut-metric could be larger in one topology, while the throughput could be greater in the other. Later in this paper, we shall demonstrate that this discrepancy indeed occurs in practice.

Further, given that the community’s approach to measuring throughput is flawed, it should not be surprising that our understanding of what gives a topology high throughput is also lacking. How do we identify topologies that can be expected to achieve high throughput? The idea of ‘nature-inspired design’ is becoming popular in several domains [24, 29, 21]: does nature offer any inspiration for network design? How do natural graphs compare to engineered designs and to each other? All of these questions remain largely unexplored so far.

The goals of this work are precisely to overcome the above described problems: (a) to allow a fair, consistent, and accurate evaluation of network throughput to facilitate future
network design efforts; (b) to benchmark existing network
design proposals to inform the choice of topologies for ap-
lications; (c) to examine the relationship of throughput with
other graph metrics across several natural topologies to un-
derstand what gives a topology high throughput. We make
the following contributions towards meeting these goals:

(1) Establishing a throughput metric. We show that the
commonly used cut-based metrics (bisection bandwidth and
sparsest cut) are flawed: for instance, topology $A$ can have
a higher cut-metric than topology $B$, but yield lower through-
put. Consequently, we make the case that there is no reason
to use these unreliable and inefficient proxies; we propose
to measure flow-based throughput directly, which is both
more accurate and more efficiently computable. Through-
put measurements depend on the traffic workload presented,
so we test multiple workloads, including a heuristic hard-to-
route case, and traffic matrices with more uniform spread.
To make topology comparisons fair with regards to cost,
we compare each topology tested against a random graph
constructed with precisely the same equipment as the given
topology, and use the throughput relative to the random graph
as our measure for the topology’s capacity.

(2) Analysis of computer network topologies. We evaluate
networks used in data centers and high performance com-
puting, and find that for the hard-to-route traffic, the 2008
fat-tree design \cite{2} significantly outperforms even the more
recent proposals. For random matching, all the proposals we
test perform quite similarly. The exception in both cases is
the random graph topology which outperforms all networks
significantly.

(3) Understanding throughput. With the objective of un-
derstanding what gives a topology high throughput, we eval-
uate throughput in a variety of natural graphs including bio-
logical networks, social networks, food-webs, and ISP net-
works. We find that while most of the natural networks we
test have very low throughput compared to random graphs
with the same degree distribution, a few have higher through-
put. We study how throughput correlates with other graph
metrics such as betweenness, clustering coefficient, aver-
age shortest path distance, and assortativity. We find that
throughput varies widely across topologies, and is not cap-
tured by the other graph metrics, underscoring its position
as a fundamental property of topologies. Further, we evalu-
ate several models of natural graphs and find that only one
model reproduces the throughput property accurately.

To our knowledge, this work is the most expansive com-
parison of network topologies on throughput, and the only
one to use an accurate and consistent method of compari-
on. Our evaluation framework and the set of topologies we
tested are freely available \cite{52}. We hope these tools will fa-
cilitate future work on rigorously designing and evaluating
networks, and allow quick reproduction of research results
in this space.

2. METRICS FOR THROUGHPUT

In this section, we investigate the metrics currently used
for estimating throughput, identify their shortcomings and
propose a new metric which is more efficient and computa-
tionally tractable.

2.1 Throughput defined

Our focus in this paper is on network throughput. Fur-
thermore, our focus is on abstract network topology, not on
systems-level design like routing and congestion control\footnote{There are many performance measures for networks that are not considered here, and higher-layer systems obviously affect throughput experienced by applications. But our goal is a compre-
hensive understanding of the fundamental goal of high-throughput
network topology, which as we will see is sufficient for a single
paper. At this stage, systems-level issues would be a distraction.}

Therefore, the metric of interest is end-to-end throughput
supported by a network in a fluid-flow model with optimal
routing. We next define this more precisely.

A **network** is a graph $G = (V, E)$ with capacities $c(u, v)$
for every edge $(u, v) \in E_G$. Among the nodes $V$ are servers,
which send and receive traffic flows, connected through non-
terminal nodes called switches. Each server is connected to
one switch, and each switch is connected to zero or more
servers, and other switches. Unless otherwise specified, for
switch-to-switch edges $(u, v)$, we set $c(u, v) = 1$, while
server-to-switch links have infinite capacity. This allows
us to stress-test the network topology itself, rather than the
servers.

A **traffic matrix** (TM) $T$ defines the traffic demand: for
any two servers $v$ and $w$, $T(v, w)$ is an amount of requested
flow from $v$ to $w$. We assume without loss of generality that
the traffic matrix is normalized so that it conforms to the
“hose model”: each server sends and receives at most 1 unit
of traffic ($\forall v, \sum_w T(v, w) \leq 1$ and $\sum_v T(w, v) \leq 1$).

The **throughput** of a network $G$ with TM $T$ is defined as
the maximum value $t$ for which $T \cdot t$ is feasible in $G$. That
is, we seek the maximum $t$ for which there exists a feasible
multicommodity flow that routes flow $T(v, w) \cdot t$ through the
network from each $v$ to each $w$, subject to the link capacity
and the usual flow conservation constraints. This can be for-
mulated in a standard way as a linear program (which we
omit for brevity) and is thus computable in polynomial time.
If the nonzero traffic demands $T(v, w)$ have equal weight,
as they will throughout this paper, this is equivalent to max-
imizing the minimum throughput of any of the requested
end-to-end flows. This maximization is a standard problem
called **maximum concurrent flow** \cite{48}.

Note that we could alternately maximize the **total** through-
put of all flows. We avoid this because it would allow the
network to pick and choose among the TM’s flows, giving
high bandwidth only to the “easy” ones (e.g., short flows
that do not cross a bottleneck). The formulation above is
equivalent to maximizing total throughput subject to a fair-
ness constraint across flows.
We now have a precise definition of throughput, but it depends on the choice of traffic matrix. How can we evaluate a topology independent of assumptions about the traffic?

2.2 Bisection bandwidth defined

Bisection bandwidth is by far the most commonly-used attempt to provide an evaluation of a topology’s performance independent of a specific TM. Since any cut in the graph upper-bounds the flow across the cut, if we find the minimum cut then we can bound the worst-case performance of the network. The intuition is that this corresponds to the unfortunate case that all communicating pairs are located on opposite sides of this cut. Now, of course, the smallest cuts will usually just slice off a single node, while we are interested in larger-scale bottlenecks; so the bisection bandwidth requires cuts which split the nodes into two equal-sized groups. Specifically, bisection bandwidth of a graph \( G \) is typically (see [11], p. 974) defined as

\[
BB(G) = \min_{S \subseteq V, |S| = \frac{n}{2}} c(S, \bar{S}),
\]

where \( \bar{S} \) is the complement of \( S \) and \( c(S, \bar{S}) \) is the total capacity of edges crossing the bisection \( (S, \bar{S}) \).

2.3 Bisection bandwidth is the wrong metric

Bisection bandwidth does give some insight into the capacity of a network. It provides an upper-bound on worst-case network performance, is simple to state, and can sometimes be “eyeballed” for simple networks. However, it has several limitations. First, it is NP-complete to compute [19]; the best approximation algorithm has a polylogarithmic approximation factor [17]. Even if one could use specialized algorithms to calculate it for structured networks, no such methods are known for unstructured networks (such as ones recently proposed for computer networks [11][51]). This makes automated comparisons across networks difficult.

Second, the insistence on splitting the network in half means that bisection bandwidth may not uncover the true bottleneck. In the graph \( G \) of Figure 1 any bisection must split off at least \( \frac{n}{2} \) of the nodes in the large clique, and each of these have \( \Theta(n) \) neighbors, meaning that \( BB(G) = \Theta(n^2) \) while the graph actually has a bottleneck consisting of just a single link!

This is, of course, a rather trivial problem. It may not arise in practice for highly symmetric networks typically encountered in the HPC community where bisection bandwidth has historically been used. However, it is worth mentioning here since (1) the above definition of bisection bandwidth has been used extensively, (2) the above issue means bisection bandwidth is not a sufficiently robust metric to handle general irregular and heterogeneous networks including those we consider in this paper. We also note that recently-proposed data center networks are both heterogeneous [23] and unstructured [51][49], so bisection bandwidth may be unsuitable for them.

Next, we will see how this particular problem with bisection bandwidth can be fixed, but how cut-based metrics (including bisection bandwidth) are subject to a more fundamental problem.

2.4 Sparsest cut defined

There are several ways to fix bisection bandwidth’s overly rigid requirement of exact bisection, such as balanced partitioning [5] and sparsest cut. Our overall conclusions are not sensitive to the distinction; we use sparsest cut here.

The uniform sparsest cut weights the cut by the number of separated vertex pairs. That is, the uniform sparsity of a cut \( S \subseteq V \) is

\[
\phi(S) = \frac{c(S, \bar{S})}{|S|(|S| - |\bar{S}|)},
\]

where \( c(S, \bar{S}) \) is the total capacity of edges crossing the cut. The uniform sparsest cut is the cut of sparsity \( \min_{S \subseteq V} \phi(S) \).

To motivate this definition, it helps to generalize it. The (nonuniform) sparsity of a cut weights the cut by the amount of traffic demand across the cut. That is, for a given TM \( T \), the sparsity of a cut is

\[
\phi(S, T) = \frac{c(S, \bar{S})}{T(S, \bar{S})},
\]

where \( T(S, \bar{S}) \) is the traffic demand crossing the cut, i.e., \( T(S, \bar{S}) = \sum_{v \in S, w \in \bar{S}} T(v, w) \). The nonuniform sparsest cut for TM \( T \) is then the cut of sparsity \( \min_{S \subseteq V} \phi(S, T) \).

Observe that the uniform sparsest cut is a special case of the nonuniform sparsest cut when we take \( T \) to be the complete traffic matrix \( (T(v, w) = 1 \forall v, w) \). Also observe that bisection bandwidth is equivalent to the uniform sparsest cut if it happens that the uniform sparsest cut has \( \frac{n}{2} \) nodes on either side. The reader can assume sparsest cut refers to the uniform case unless otherwise specified.

2.5 Sparsest cut is the wrong metric

The improved cut metric will now succeed in finding the true bottleneck in Figure 1 unlike bisection bandwidth. But have we found the right metric for worst-case throughput? We argue that the answer is no, for three reasons.

(1) Sparsest cut and bisection bandwidth are not actually TM-independent, contrary to our original goal of evaluating a topology independent of traffic. As discussed above, bisection bandwidth and the uniform sparsest cut correspond to the worst cuts for the complete (all-to-all) traffic matrix,
so they have a hidden implicit assumption of this particular TM.

**2. Even for a specific traffic matrix, computing sparsest cut is NP-hard** for most TMs, and it is believed that there is no efficient constant factor approximation algorithm [8]. In contrast, throughput is computable in polynomial time. Sparsest cut’s difficulty is both practically inconvenient and strong evidence that cut-based metrics are not computing the same physical quantity as throughput.

**3. While cuts upper-bound throughput, it is only a loose upper bound.** This may be counterintuitive, if our intuition is guided by the well-known max-flow min-cut theorem. In a network with a single \( v \rightarrow w \) traffic flow, the theorem states that the maximum value of the \( v \rightarrow w \) flow is equal to the minimum capacity over all cuts separating \( v \) and \( w \) [18, 15]. Hu [31] extended these results and showed that maximum concurrent flow is equal to the minimum cut for two-commodity flows. But this result is not valid for multi-commodity flows when the number of flows is greater than two, which is of course the case of primary interest in computer networks. Specifically, in a multi-commodity flow with uniform demands, the maximum flow throughput can be an \( O(\log n) \) factor lower than the sparsest cut [36]. Hence, the maximum flow in the network is not directly captured by minimum cuts in the network.

Figure 2 depicts the relationship between sparsest cut and throughput. The flow (throughput) in the network cannot exceed the upper bound imposed by the worst-case cut. On the other hand, the cut cannot be more than a factor \( O(\log n) \) greater than the flow [36]. Thus, any graph and an associated traffic matrix can be represented by a unique point in the region bounded by these limits.

While this distinction is of course well-established in the theory literature [36], we further argue here that it can lead to incorrect decisions when evaluating networks. Specifically, we will exhibit a pair of graphs \( A \) and \( B \) such that, as shown in Figure 2, \( A \) has higher throughput but \( B \) has higher sparsest cut. If sparsest cut is the metric used to choose a network, graph \( B \) will be wrongly assessed to have better performance than graph \( A \), while in fact it has a factor \( \Omega(\sqrt{\log n}) \) worse performance!

**Graph A** is a clustered random graph adapted from previous work [50] with the same \( n \) nodes and degree \( 2d \). A is composed of two equal-sized clusters with \( n/2 \) nodes each. Each node in a cluster is connected by degree \( \alpha \) to nodes inside its cluster, and degree \( \beta \) to nodes in the other cluster, such that \( \alpha + \beta = 2d \). \( A \) is sampled uniformly at random from the space of all graphs satisfying these constraints. We can pick \( \alpha \) and \( \beta \) such that \( \beta = \Theta\left(\frac{\alpha}{\log n}\right) \). Then, as per [50] (Lemma 3), the throughput of \( A \), \( T_A \) and its sparsest cut, \( \Phi_A \) are both \( \Theta\left(\frac{1}{n \log n}\right) \).

Let graph \( G \) be any \( 2d \)-regular expander on \( N = \frac{n}{p} \) nodes, where \( d \) is a constant and \( p \) is a parameter we shall adjust later. **Graph B** is constructed by replacing each edge of \( G \) with a path of length \( p \). It is easy to see that \( B \) has \( n \) nodes. We prove in Appendix A the following theorem.

**Theorem 1.** \( T_B = O\left(\frac{1}{n \log n}\right) \) and \( \Phi_B = \Omega\left(\frac{1}{np^2}\right) \).

In the above, setting \( p = 1 \) corresponds to the ‘expanders’ point in Figure 2, both \( A \) and \( B \) have the same throughput (within constant factors), but the \( B \)’s cut is larger by \( O(\log n) \). Increasing \( p \) creates an asymptotic separation in both the cut and the throughput such that \( \Phi_A < \Phi_B \), while \( T_A > T_B \). For instance, if \( p = \sqrt{\log n} \), \( T_B = O\left(\frac{1}{n(\log n)^{3/2}}\right) \) and \( \Phi_B = \Omega\left(\frac{1}{n \sqrt{\log n}}\right) \). Further, if \( p = \Theta(\log n) \), we can tune the constants such that \( T_A > T_B \Theta(\log n) \) even though \( \Phi_A < \Phi_B \).

**Intuition.** The reason that throughput may be smaller than sparsest cut is that in addition to being limited by bottlenecks, the network is limited by the total volume of “work” it has to accomplish within its total link capacity. That is, if the TM has equal-weight flows,

\[
\text{Throughput per flow} \leq \frac{\text{Total link capacity}}{\text{# of flows} \cdot \text{Average path length}}
\]

where the total capacity is \( \sum_{(i,j) \in E} c(i,j) \) and the average path length is computed over the flows in the TM. This “volumetric” upper bound may be tighter than a cut-based bound. For example, [50] observed that the cut between two halves of a random graph could be reduced significantly without affecting throughput, indicating that the throughput of a random graph is limited by the volumetric upper bound, rather than by the sparsest cut. Our Graph \( A \) corresponds to the extreme point of that process, where the cut is as small as possible without impeding throughput.

**2.6 Aside: Are cut-based metrics useful at all?**

While throughput is the focus of this paper, it is not the only graph property of interest, and cut-based metrics do capture important graph properties. One obvious example is reliability: cut-based metrics directly capture the difficulty of partitioning a network.

A less obvious example is physical network layout. Consider two networks \( A, B \) with the same throughput, but with \( B \) having greater bisection bandwidth. Suppose we desire a physical realization of these networks grouping them into two equal-sized clusters, such that shorter, cheaper cables.
will be used to create links within clusters and longer, more expensive cables must be used across clusters. Since $A$ has lower bisection bandwidth, there is a way to partition it which uses fewer long cables. In other words, in this case, $B$’s higher bisection bandwidth is actually bad — it yields no performance benefit and corresponds to greater difficulty in physical cabling! — so we actually wish to minimize bisection bandwidth.

2.7 Towards a throughput metric

Having exhausted cut-based metrics, we return to our original metric of throughput defined in \cite{cite} and suggest a simple solution: network topologies should simply be evaluated directly in terms of throughput (via LP optimization software), for specific traffic matrices.

The key, of course, is how to choose the TM. If we can find a worst-case TM, this would fulfill the goal of evaluating topologies independent of assumptions about traffic. To the best of our knowledge, this is an unsolved, non-trivial problem. Here, we offer practical, efficient heuristics to find a bad-case TM which can be used to benchmark topologies.

We begin with the complete or all-to-all TM $T_{A2A}$ which for all $v, w$ has $T_{A2A}(v, w) = \frac{1}{n}$. We observe that $T_{A2A}$ is in fact within $2 \times$ of the worst case TM. While this fact is simple, we have not seen it previously noted. (Please refer to Appendix \ref{app} for the proof.)

**Theorem 2.** Let $G$ be any graph. If $T_{A2A}$ is feasible in $G$ with throughput $t$, then any hose model traffic matrix is feasible in $G$ with throughput $\geq t/2$.

Can we get closer to the worst case TM? In our experience, TMs with a smaller number of “elephant” flows are more difficult to route than TMs with a large number of small flows, like $T_{A2A}$. This suggests a random matching TM in which we have only one outgoing flow and one incoming flow per server, chosen uniform-randomly among servers. We can further stress the network topology by decreasing the number of servers attached to each switch, so there are even fewer flows originating from each switch. These TMs actually lack the factor of 2 near-worst-case guarantee of $T_{A2A}$, but in practice we have found they consistently result in lower throughput than $T_{A2A}$.

Can we get even closer to the worst-case TM? Intuitively, the all-to-all and random matching TMs will tend to find cuts, but only have average-length paths. Drawing on the intuition that throughput decreases with average flow path length, we seek to produce traffic matrices that force the optimal-flow-router to use as long paths as possible. To do this, given a network $G$ we compute all-pairs shortest paths and create a complete bipartite graph $H$, whose nodes represent all sources and destinations in $G$, and for which the weight of edge $v \rightarrow w$ is the length of the shortest $v \rightarrow w$ path in $G$. We then find (via another LP optimization) the maximum weight matching in $H$. The resulting matching corresponds to the pairing of servers which maximizes average flow path length, resulting in a longest matching TM.

Figure \ref{fig} shows the resulting throughput of these TMs in three topologies: hypercubes, random regular graphs, and fat trees \cite{cite}. Note that a comparison of absolute throughput across topologies is meaningless since these are not built with the same “equipment” (node degree, number of servers, etc.); the meaningful comparison here is of the various TMs within each topology. In all cases, $A2A$ traffic has the highest throughput; throughput decreases or does not change as we move to a random matching TM with $d$ servers per switch (where $d$ is the switch-to-switch node degree), and then to a random matching TM with 1 server per switch, and finally to the longest matching TM. We also plot a lower bound: $T_{A2A}/2$.

These three topologies were chosen to illustrate cases when our approach is most helpful, somewhat helpful, and least helpful at finding near-worst-case TMs. In the hypercube, the longest matching TM is extremely close to the worst-case performance. To see why, note that the longest paths have length $d$ in a $d$-dimensional hypercube, twice as long as the mean path length; and the hypercube has $n \cdot d$ unidirectional links. The total flow in the network will thus be $(\# \text{ flows} \cdot \text{average flow path length}) = n \cdot d$ is $n \cdot d$. Thus, all links will be perfectly utilized if the flow can be routed, which empirically it is. In the random graph, there is less variation in end-to-end path length, but across our experiments the longest matching is always within $1.5 \times$ of the provable lower bound (which itself may not be tight). In the fat tree, which is here built as a three-level nonblocking topology, there is essentially no variation in path length since asymptotically nearly all paths go three hops up to the core switches and three hops down to the destination. Here, our TMs are no worse than all-to-all, and the simple $T_{A2A}/2$ lower bound is off by a factor of 2 from the true worst case (which is throughput of 1 as this is a nonblocking topology).

2.8 Summary and implications

Cut-based metrics can be safely abandoned for the purposes of topology performance evaluation. They are not in fact independent of assumptions on the traffic matrix; they are NP-complete to compute; and they are not always an accurate measure of worst-case throughput, which as we showed can lead to incorrect performance evaluations. These facts call into question work which has optimized networks based on bisection bandwidth \cite{cite} and work which compared networks on “equal footing” by attempting to approximately equalize their bisection bandwidth \cite{cite}.

Having surmounted that moment of catharsis, we actually obtain some relief: directly evaluating throughput with particular TMs using LP optimization is both more accurate and more tractable than cut-based metrics. In choosing a TM to evaluate, both “average case” and near-worst-case TMs are reasonable choices. We suggest that a random matching TM with several servers per switch provides a reasonable “average case”, because it is similar to uniform all-to-all traffic, yet is faster to compute as it reduces the number of flows in
the network from $\Theta(n^2)$ to $\Theta(n)$ when there are $n$ servers. For near-worst-case traffic, we developed a practical heuristic that often succeeds in substantially worsening the TM compared with all-to-all.

We note that measuring throughput in terms of flow is not in itself a novel idea – however, prior work [51, 45] has used it in an ad-hoc fashion, lacking the rigorous analysis we use to support this choice, with only uniform traffic, and to compare a small number of networks. In what follows, we compare a large number of networks using both the average case and bad-case traffic.

3. EVALUATION OF TOPOLOGIES

Having established our metric for throughput, we use it to evaluate the performance of 5 types of computer networks, 66 natural networks, 4 models of natural networks, and a family of expander graphs\footnote{We realize that there are always more topologies available to test, so we have made our benchmarking tools available [52] for other researchers to easily add to these comparisons any topologies we might have missed.}. The networks we test are as diverse in their characteristics as they are in their origins; the size and degree distribution varies widely. Our evaluation addresses the following questions:

- How do we fairly compare networks with different sizes and degree distributions?
- Which communication networks achieve highest throughput?
- Does throughput vary significantly across naturally occurring networks? This could potentially indicate environmental factors that govern their structure.
- How well do other common graph metrics correlate with throughput? High correlations could suggest improvements in network design, while small correlations would position throughput as an independent property of interest.
- Do models of natural networks faithfully reproduce throughput?

3.1 Methodology

The networks we evaluate fit into three broad categories – structured computer networks, naturally occurring networks and models of natural networks. Our high-level approach to evaluating a network is to: (i) compute the network’s throughput; (ii) build a random graph with precisely the same degree distribution and compute its throughput under the same traffic model; (iii) normalize the network’s throughput with the random graph’s throughput for comparison against other networks.

**Traffic workload:** In structured computer networks with restrictions on server-locations, servers are added only at the locations prescribed by the models. For example, in fat-trees, servers are attached only to the lowest layer. For all other networks, we add $k$ servers to each switch with degree $k$. The traffic matrices are either random matching or the more difficult longest matching.

**Throughput computation:** Throughput is computed as a solution to a linear program whose objective is to maximize the minimum flow across all flow demands, as explained in $\mathbb{E}$ This objective function captures the notion of fairness among flows. We use the Gurobi [27] solver to compute throughput. We normalize the obtained throughput value to a comparable random graph’s throughput to make the comparison of networks with diverse degree distributions and sizes meaningful. This normalized value is referred to as relative throughput. Unless otherwise stated, each data-point is an average across 10 iterations, and all error bars are 95% two-sided confidence intervals.

3.2 Computer networks

High throughput is a critical requirement for computer networks. Here we compare 5 types of computer networks on their throughput performance under two traffic matrices – random matching and longest matching – across a variety of sizes and configurations. Figure 4 shows the results of this comparison. In the random matching case, performance is fairly similar, but our longest matching TM reveals more differences among networks, with some (such as BCube) taking a large performance hit. Among structured networks, DCell and flattened butterflies achieve highest throughput for random matching, but for longest matching, the fat-tree
achieves highest throughput at the largest scale. In comparison with random graphs, performance degrades for all networks with increasing size. We discuss each network’s performance in more detail below.

**Fat Tree** [57] based designs are used in modern data centers and high performance computing. A three-level fat-tree constructed from degree-$k$ switches consists of $\frac{3}{4}k^2$ switches and supports $k^3$ hosts. In comparison, at the largest scale in Figure 4, a fat tree achieves only 70% of a random graph’s throughput under random matching, and 85% with longest matching. While the fat-tree’s non-blocking design is an overkill for the random matching, compared to networks other than the random graph, it is advantageous for the longest matching. In fact, the fat-tree’s *relative* throughput is higher in the longest matching TM, because moving from random to longest matching degrades performance more in the random graph than in the fat-tree.

**Hypercubes** [1] are another family of structured graphs used in computer architecture to interconnect processors. A $d$-dimensional hypercube consists of $2^d$ nodes and has a bisection bandwidth of $2^{d-1}$. The hypercube achieves 70% of the random graph’s throughput under random matching, and 51% under longest matching.

**Flattened butterfly** [33] is another topology used in on-chip networks. A $k$-ary $n$-fly topology consists of $kn^2$ switches. At the largest scale, the 4-ary flattened butterfly achieves 86% of the random graphs throughput under random matching, and only 58% under longest matching. Beyond a large enough scale, the larger the $k$ for a $k$-ary butterfly, the better its performance.

**BCube** [25], designed for modular data center networks, differs from other structured computer networks in its server-centric design. $BCube_k$ consists of $(k+1)$ levels with each level containing $n^k$ $n$-port switches. Each server in $BCube_k$ has degree $(k+1)$ and is connected to one switch in each of the $(k+1)$ levels. $(k+1)\times n^k$ switches in BCube can support only $n^{k+1}$ hosts. The 3-ary BCube achieves 88% of the random graph’s throughput under random matching, but only 29% under longest matching. Across topologies, this is the largest degradation we see in performance.

**DCell** [26] is also designed for data center networks, and like BCube is server-centric. It is constructed as a recursive design starting with a building block where a switch is connected to a small number of servers. 6-ary DCell achieves 97% of the random graph’s throughput under random matching, and 79% under longest matching.

### 3.3 Expander Graphs

Expanders [22] are sparse graphs with the seemingly contradictory property of high connectivity. By definition, every vertex subset of an expander graph has a significant number of edges leaving it. Thus, expander graphs are very robust and are generally used in the design of communication networks. Random regular graphs are known to be good expanders [51]. In addition, here we evaluate another class of expanders – Ramanujan graphs. We constructed instances of these graphs using the method introduced by Lubotzky [38]. The throughput of Ramanujan graphs is within 2% of that of the corresponding random graphs. (Figure omitted for space.) However, the construction of Ramanujan graph is restrictive in what sizes are permitted.

### 3.4 Natural networks

Despite following a diverse set of constraints through their evolution, natural networks have been observed to share some structural properties. Biological networks such as protein interaction networks and food-webs, social networks such as patterns of friendship, and technological networks such as the Internet, have all been found to have heavy-tailed degree distributions and high clustering coefficients. The question we address here is whether such networks also share throughput behavior – for instance, do natural networks with similar degree distributions have similar throughput? Do any characteristics of natural networks predict high throughput? For instance, do natural networks with high clustering coefficients have high or low throughput? Answering these questions could aid network design – if a certain property correlates with high throughput, we could attempt to design for that property. If we do not find such correlations, it implies that throughput is an independent property of graphs that reveals structure not captured by other metrics.

The 66 natural networks we collected from various datasets...
Discussion ties among 31 physicians participating in Columbia University Drug Study [13],

AS 3257 core topology with fringe - radius 1 around backbone and gateways [54],

AS 1755 core topology - backbone and gateways [54],

Food web - Chesapeake Bay Mesohaline Net [13],

Relationship between university freshmen [57],

Dining table partners in a dormitory [41] [13],

Food web - Chesapeake Bay in Summer [13],

Food web - Lower Chesapeake Bay in Summer [13],

Food web - Upper Chesapeake Bay in Summer [13],

Food web - Middle Chesapeake Bay in Summer [13],

Drug Study1: Discussion ties among 31 physicians participating in Columbia University Drug Study [13],

Drug Study2: Discussion ties among 31 physicians participating in Columbia University Drug Study [13],

Drug Study3: Discussion ties among 31 physicians participating in Columbia University Drug Study [13],

Drug Study: Discussion ties between 17 physicians who adopted the new drug in Galesburg [31],

Dutch Literature: Critical attention among a set of Dutch literary authors and critics in 1976 [13],

Hi-Tech Firm: Friendship ties among the employees in a hi-tech firm [12],

Mexican Politics: Core network of Mexican political elite [13],

Math Diffusion: The diffusion network of a new mathematics method in the 1950s [13],

Saw Mill Strike: Communication structure among the employees in a saw-mill during a strike [13],

High-Tech Managers: Krackhardt’s high-tech managers [60],

Food web-Chesapeake: Food web - Chesapeake Bay Mesohaline Net [13],

Food web-Ches.lower: Food web - Lower Chesapeake Bay in Summer [13],

Food web-ChesMiddle: Food web - Middle Chesapeake Bay in Summer [13],

Food web-ChesUpper: Food web - Upper Chesapeake Bay in Summer [13],

Food web-CypressDry: Food web - Cypress, Dry Season [13],

Food web-CypressWet: Food web - Cypress, Wet Season [13],

Food web-Mapaolomas: Food web - Charca de Maquilpalomas [13],

Food web-StMarkRiver: Food web - St. Marks River (Florida Flow network) [13],

Food web-MangroveDry: Food web - Mangrove Estuary, Dry Season [13],

Food web-MangroveWet: Food web - Mangrove Estuary, Wet Season [13],

Food web-FllBayDry: Food web - Florida Bay, Dry Season [13],

Food web-FllBayWet: Food web - Florida Bay, Wet Season [13],

Food web-FloridaBay: Food web - Florida Bay Trophic Exchange Matrix [13],

Food web-YiYuan: Food web - YiYuan Estuary [41],

1755-edges: AS 1755 topology [54],

1755-r0-edges: AS 1755 core topology - backbone and gateways [54],

1755-r1-edges: AS 1755 core topology with fringe - radius 1 around backbone and gateways [54],

3257-edges: AS 3257 topology [54],

3257-r0-edges: AS 3257 core topology with fringe - radius 1 around backbone and gateways [54],

4755-r0-edges: AS 4755 core topology - backbone and gateways [54],

4755-r1-edges: AS 4755 core topology with fringe - radius 1 around backbone and gateways [54],

6461-edges: AS 6461 topology [54],

1239-r0-edges: AS 1239 core topology - backbone and gateways [54],

7018-r0-edges: AS 7018 core topology - backbone and gateways [54],

Table 1: Throughput of natural networks with random matching traffic matrix
are shown in Table 1. In our analysis, we categorize topologies into biological networks, food-webs, social networks, ISP networks, and other networks. Figure 6 shows for each topology, the correlation of throughput with other graph metrics. Table 2 lists the Pearson correlation coefficient (PCC) values for the same data. Results for each metric are discussed below.

Table 2: Pearson correlation coefficient for natural networks

| X           | Y             | Correlation |
|-------------|---------------|-------------|
| Abs. Thru.  | Assortativity | -0.057      |
| Abs. Thru.  | Clustering    | 0.439       |
| Abs. Thru.  | Avg. Distance | -0.792      |
| Abs. Thru.  | Max. Centrality | -0.401    |
| Abs. Thru.  | Avg. Degree   | 0.692       |
| Rel. Thru.  | Assortativity | -0.176      |
| Rel. Thru.  | Rel. Clustering | -0.434    |
| Rel. Thru.  | Rel. Distance | -0.584      |
| Rel. Thru.  | Max. Centrality | -0.188    |
| Rel. Thru.  | Avg. Degree   | 0.307       |

Average degree is the average number of neighbors of a node in the graph. It represents the density of the graph. The correlation of absolute throughput with high density (Figure 6(c)) should not be surprising — graphs with more connectivity per node have higher throughput. Relative throughput, by comparing relative to a random graph built with the same “equipment”, largely factors this out, and shows a significantly weaker correlation (not plotted, but PCC= 0.307, compared to 0.692 for absolute throughput.)

Clustering coefficient is the probability of two neighboring nodes having a common neighbor. In other words, it measures the probability of occurrence of triangles in a graph. A low clustering coefficient implies that new links from a node enable its neighbors to reach nodes not directly connected to them. Hence, one might expect throughput to increase as clustering coefficient decreases. However, we see a wide range of clustering coefficients ranging from 0 to 0.66 (Figure 6(c)) and throughput does not seem to show a high correlation with clustering coefficient. When the clustering coefficients are also normalized to the clustering in the random graph, we find that indeed, most natural networks have higher clustering than random networks (x-axis > 1 in Figure 6(d)). However, we do not observe a strong correlation between relative clustering and throughput either.

Assortativity is a measure of how often nodes in a network connect to other nodes which are similar to themselves. Generally, node degrees are used as a similarity measure [43]. Assortativity is positive if similar vertices connect to each other in a graph, i.e., high-degree nodes tend to connect with other high-degree nodes and low-degree nodes have more connections with other low-degree nodes. The networks used in this analysis exhibit a wide range of assortativity values (Figure 6(e), but throughput was found to be independent of assortativity in the network (PCC nearly 0).

Edge betweenness centrality of an edge is the number of paths traversing it when all pairs of shortest paths in the network are considered. It is normalized with respect to the total number of shortest paths. Maximum centrality is the largest edge betweenness centrality value across edges in the network. Maximum betweenness centrality might be expected to be closely related to throughput: it captures effects of both path length and bottlenecks in the network, and in fact is similar to (an inverse of) throughput if flows were constrained to take only exact shortest paths. Somewhat surprisingly, it does not correlate highly with either relative or absolute throughput (Figures 6(h) and 6(i)). In fact, the PCC with relative throughput (Table 2) is statistically insignificant (at p-value = 5%).

Average distance is computed as the average of shortest path lengths between all pairs of nodes in the network. Intuitively, low average distance (particularly in the absence of small cuts) implies each flow consumes resources over fewer hops in the network, thus leading to higher throughput. This intuition is borne out by the visible correlation between average distance and absolute throughput (Figure 6(e). The relationship is less clear with relative throughput (Figure 6(f)). The PCC values for absolute and relative throughput with average distance are −0.792 and −0.584 respectively, indicating the inverse relationship between average distance and throughput. Even though throughput is somewhat correlated with average distance, it is not entirely explained by it.
**Summary:** None of the metrics we investigated explain the throughput behavior of topologies very well. Average distance does correlate with throughput, and this fact has been exploited recently for the high throughput network design – the Jellyfish proposal [51] explored the use of degree-diameter graphs [10] in this context. These graphs are optimized for low diameter rather than low average distance, but also have low average distance. One of the findings of the Jellyfish work was that degree-diameter graphs achieve even higher throughput than random graphs.

We also note that the category of food-webs stands out for the similarity of the metrics across instances of graphs from this category. Most of these graphs have higher than average degree and throughput, and lower than average edge-betweenness centrality; almost all have relative clustering close to 1, and average distances close to 2. It could be an interesting subject of investigation as to what factors drive these similarities within the category and set them apart from the other networks we tested.

Given that none of the graph metrics completely explain throughput, it is interesting to examine whether models of natural networks faithfully reproduce the throughput behavior. This is the subject we tackle next.

### 3.5 Models of natural networks

Models of natural networks are categorized as either causality-aware or causality-oblivious. Causality-aware models take into account the factors that influence the nature of the graph. Causality-oblivious models blindly emulate the properties of the network without considering the factors driving them. We evaluate representative models from both categories.

#### 3.5.1 Causality-aware models

Causality-aware models of natural networks reproduce power-law distributions by expanding graphs according to well-defined rules. We use NetworkX [28] to generate graphs from three commonly used causality-aware models: (a) the Barabasi-Albert scale-free model [7] recreates power-law degree distributions; (b) The Watts-Strogatz small-world model [61] produces networks with high clustering; and (c) The Holme-Kim model [30] reproduces both power-law degree distributions and high clustering.

**Barabasi-Albert:** In this *preferential attachment* model, graph generation involves attaching new nodes preferentially to ex-
isting high degree nodes. We find that the throughput of this model replicates the throughput of equivalent random graphs to within the variance in throughput. (Result plot omitted for space.) Given that natural networks can differ greatly in throughput from random graphs with identical degree distributions (Figure 6), we conclude that this model does not capture throughput characteristics of natural networks.

Watts-Strogatz: This model starts with a regular network and rewires any link with probability \( p \) choosing its target at random. \( p = 0 \) leads to a completely regular structure, while \( p = 1 \) leads to a random graph. Even for small \( p \), these networks exhibit small-world phenomena. NetworkX \([23]\) uses a ring structure with each node connected to a fixed number of neighbors as the underlying regular structure. To test whether this model fits the throughput characteristics of natural networks, we use the 29 smallest of our 66 natural networks (in the interest of time). We attempted to fit each natural network under consideration to the model as best as possible: we used the same number of nodes and average degree; then we tried 100 \( p \)-values and picked the one with the closest clustering coefficient. For each of the 29 natural networks, we plot in Figure 8(a) the model graph’s throughput versus the natural network’s throughput. Figure 8(a) repeats this exercise by by calibrating the model to match the natural network’s average distance instead of its clustering. As the results show, the model does not faithfully replicate throughput in either case – the model graph’s throughput is more than 40% off in both cases for at least 10 natural networks. Note that attempts to match both clustering and average distance fail for almost all of the natural graphs.

Holme-Kim model: This model allows us to vary the clustering in a graph where nodes obey the power-law distribution. The models’ parameter specifies the probability of adding a triangle while adding a random edge, although this does not entirely determine the resulting graph’s clustering coefficient – in our experiments, the clustering coefficient varied over a smaller range (approximately 0.2 to 0.7) as we varied the parameter from 0 to 1. From Figure 7 we observe that the throughput of the Holme-Kim graph is very close to that of the random graph at low parameter values, and throughput drops at very high values of the parameter. This drop-point depends on network size. This behavior does not fit our observations with natural networks, where a wide range of throughput is seen across all values of clustering.

In summary, while a calibrated Watts-Strogatz has some correlation with throughput, none of these causality-aware models faithfully reproduce the throughput of the natural networks.

3.5.2 Causality-oblivious models

Causality-oblivious models attempt to replicate the features of a given graph at varying levels of detail, without regard to the process which might have led to that structure.

dK-Series \([40]\) model of a graph captures the degree correlations in \( d \)-sized subgraphs of the given graph. A model with higher value of \( d \) reproduces more features of the graph. More specifically, \( d = 0 \) implies that the model has the same average degree as the original graph, while \( d = 1 \) has the same degree distribution, \( d = 2 \) considers joint degree distribution of nodes, etc. We evaluate throughput for \( d = 1, 2, 3 \) for 37 natural networks. (In the interest of time, the smallest 37 of the 66 networks were used.) As Figure 9 shows, at \( d = 3 \), the throughput of the natural network is faithfully replicated. Thus, while degree distribution and joint degree distribution fail to capture the throughput characteristics, considering all 3-node subgraphs is enough to replicate throughput.

Across all network models, only the dK-series graphs faithfully replicate throughput.

4. RELATED WORK

The literature on network topology design is large and growing quickly, with a number of designs having been proposed in the past few years \([4, 23, 25, 42, 26, 59, 16, 53, 51, 12, 11]\). However, each of these research proposals only makes a comparison with one or two other past proposals, with no standard benchmarks for the comparison. There has been no independent, rigorous evaluation of a large number of topologies.

The only significant work in the space is from Popa et al. \([45]\). They assess 4 topologies to determine the one that incurs least expense while achieving a target level of performance under a specific workload (all-to-all traffic). Their attempts to equalize performance required careful calibration, and approximations still had to be made. Accounting for the different costs of building different topologies is also an inexact process. We sidestep that issue by using the random graph as a normalizer: instead of attempting to match performance, for each topology, we build a random graph with identical equipment, and then compare throughput performance of the topology with that of the random graph. Each topology is compared to the random graph and thus the problem of massaging structured designs into roughly equivalent configurations is alleviated. This also makes it easy for others to use our tools, and to test arbitrary workloads. Apart
from comparing topologies, our work also argues the superiority of flow-metrics to cuts. Our analysis of 66 natural networks also furthers the community’s understanding of the relationship of throughput with other graph metrics.

Other work on comparing topologies is more focused on reliability and cuts in the topology [35]. Several researchers have used bisection bandwidth and sparsest cut as proxies for throughput performance. Further, the usage of these two terms is not consistent across the literature. For instance, REWIRE [11] explicitly optimizes its topology designs for high sparsest cut, although it refers to the standard sparsest cut metric as bisection bandwidth. Tomic [56] builds topologies with the objective of maximizing bisection bandwidth (in a particular class of graphs). Webb et. al [62] use bisection bandwidth to pick virtual topologies over the physical topology. An interesting point of note is that they consider all-to-all traffic ‘a worst-case communication scenario’, while our results show that other traffic patterns can be significantly worse. PAST [55] tests 3 data center network proposals with the same bisection bandwidth. The hardness of computing bisection bandwidth [19] causes those authors to use a lower bound on Jellyfish’s bisection bandwidth in the comparison, making it unfair to Jellyfish, thus highlighting one of the difficulties with using bisection bandwidth as a metric. Even more importantly, PAST’s own results show that the throughput performance of topologies with the same bisection bandwidth is different, raising questions about the usefulness of such a comparison; one must either build topologies of the same cost and compare them on throughput (as we do), or build topologies with the same performance and compare cost (as Popa et. al [45] do). These observations underscore the community’s lack of clarity on the relationship between bisection bandwidth, sparsest cut, and throughput. A significant component of our work in this paper tackles this subject.

5. CONCLUSION

Network capacity is a core goal of network design. Despite this, as we have shown here, the community’s approach to even its measurement is flawed. In this work, we have attempted to argue rigorously for an improved benchmark for network capacity – a direct, flow-based computation of throughput. Further, we have used this metric to compare a large number of networks with the objective of helping network operators choose topologies, and network design researchers to be able to improve their designs. We have also evaluated natural networks, and the relationship of throughput with other graph metrics in these networks. Our work also raises a number of interesting directions for future work:

- A small number of natural networks have throughput greater than the random graph (Figure 6). What gives these networks high throughput? What can network designers learn from such graphs?
- The food-web category of graphs shows greater similarity of metrics across its members than other categories (Figure 6). Is there a specific process of development which causes this?
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APPENDIX

A. PROOF OF THEOREM 1

We revisit the maximum concurrent flow problem, based on which we defined throughput in [2]. Given a network \( G = (V, E_H) \) with capacities \( c(u, v) \) for every edge \((u, v) \in E_G\), and a collection (not necessarily disjoint) of pairs \((s_i, t_i)\), \(i = 1, \ldots, k\) each having a unit flow demand, we are interested in maximizing the minimum flow. Instead of the traffic matrix (TM) formulation of [2] for the following discussion, it will be convenient to think of the pairs of vertices that require flow between them as defining a demand graph, \( H = (V, E_H) \). Thus, given \( G \) and \( H \), we want the maximum throughput. As we noted in [2] this problem can be formulated as a standard linear program, and is thus computable in polynomial time.

We are interested in comparing our suggested throughput metric with sparsest cut. We first prove the following theorem.

THEOREM 3. The dual of the linear program for computing throughput is a linear programming relaxation for sparsest cut.

PROOF. We shall use a formulation of the throughput linear program that involves an exponential number of variables but for which is easier to derive the dual. We denote by \( P_{s,t} \) the set of all paths from \( s \) to \( t \) in \( G \) and we introduce a variable \( x_p \) for each path \( p \in P_{s,t} \), for each \((s, t) \in E_H\), corresponding to how many units of flow from \( s \) to \( t \) are routed through path \( p \).

\[
\begin{align*}
\text{max} & \quad y \\
\text{subject to} & \quad \sum_{p \in P_{s,t}} x_p \geq y \quad \forall (s, t) \in E_H, \\
& \quad \sum_{p: (u, v) \in p} x_p \leq c(u, v) \quad \forall (u, v) \in E_G, \\
& \quad x_p \geq 0 \quad \forall p, \\
& \quad y \geq 0.
\end{align*}
\]

The dual of the above linear program will have one variable \( w(s, t) \) for each \((s, t) \in E_H\) and one variable \( z(u, v) \) for each \((u, v) \in E_G\).

\[
\begin{align*}
\text{min} & \quad \sum_{u, v} z(u, v) c(u, v) \\
\text{subject to} & \quad \sum_{(s, t) \in E_H} w(s, t) \geq 1 \\
& \quad \sum_{(u, v) \in p} z(u, v) \geq w(s, t) \quad \forall (s, t) \in E_H, p \in P_{s,t} \\
& \quad w(s, t) \geq 0 \quad \forall (s, t) \in E_H \\
& \quad z(u, v) \geq 0 \quad \forall (u, v) \in E_G.
\end{align*}
\]

It is not hard to realize that in an optimal solution, without loss of generality, \( w(s, t) \) is the length of the shortest path from \( s \) to \( t \) in the graph weighted by the \( z(u, v) \). We can also observe that in an optimal solution we have \( \sum w(s, t) = 1 \). These remarks imply that the above dual is equivalent to the following program, where we introduce a variable \( l(x, y) \) for every pair or vertices in \( E_G \cup E_H \).

\[
\begin{align*}
\text{min} & \quad \sum_{u, v} l(u, v) c(u, v) \\
\text{subject to} & \quad \sum_{(s, t) \in E_H} l(s, t) = 1 \\
& \quad \sum_{(u, v) \in p} l(u, v) \geq l(s, t) \quad \forall (s, t) \in E_H, p \in P_{s,t} \\
& \quad l(u, v) \geq 0 \quad \forall (u, v) \in E_G \cup E_H.
\end{align*}
\]

The constraints \( \sum_{(u, v) \in p} l(u, v) \geq l(s, t) \) can be equivalently restated as triangle inequalities. This means that we require \( l(u, v) \) to be a metric over \( V \). These observations give us one more alternative formulation:

\[
\min_{(\cdot, \cdot) \text{metric}} \frac{\sum_{(u, v) \in E_G} c(u, v) \cdot l(u, v)}{\sum_{(s, t) \in E_H} l(s, t)} \tag{1}
\]

We can finally see that the above formulation is a linear programming relaxation for a cut problem. More specifically, the sparsest cut problem is asking to find a cut \( S \) that minimizes the ratio

\[
\sum_{(u, v) \in E_G \text{ cut by } S} c(u, v) \left\| \right\| \sum_{\text{edges } \in E_H \text{ cut by } S} l(s, t) \tag{2}
\]

This is equivalent to minimizing ratio \([1]\) over \( \ell_1 \) metrics only.

If we take \( E_H \) to be the complete graph (corresponding to all-to-all demands), we get the standard sparsest cut definition:

\[
\sum_{(u, v) \in E_G \text{ cut by } S} c(u, v) \left\| \right\| S \| S \| \tag{3}
\]

Before we prove Theorem [1] from [2] we shall demonstrate the following claim.
Claim 4. If $G$ is a $d$-regular expander graph on $N$ nodes and $H$ is the complete graph, the value of the linear program for throughput is $O\left(d \log d \cdot \frac{N}{N \log N}\right)$. The value of the sparsest cut is $\Omega\left(\frac{d}{\sqrt{N}}\right)$.

Proof. Let us denote by $T$ the optimal value of expression $[1]$. Note that this is the optimal value of the dual for the linear program for throughput, therefore equal to the optimal throughput. By taking $l(\cdot, cdot)$ to be the shortest path metric on $G$, we calculate:

$$T \leq \frac{\sum_{i,j \in E_G} l(i,j)}{\sum_{i,j \in V} l(i,j)} \leq \frac{d/2 |V|}{\Theta(N^2 \log N)} \leq O(d \log d)$$

Here, the first inequality follows from the fact that for $d$-regular graphs, each node can reach no more than $d^i$ nodes in $i$ hops. This means that given a vertex $v$, there exist $\Theta(N)$ nodes with distance at least $\log N$ from it, which means that the total distance between all pairs of nodes is $\Theta(N^2 \log N)$.

Let $\Phi$ denote the minimum value of ratio $[3]$ for $G$. Since $G$ is an expander, this ratio is

$$\Phi \geq \Omega\left(\frac{d \cdot |S|}{|S||V-S|}\right) = \Omega\left(\frac{d}{N}\right)$$

Theorem 1. Let graph $G$ be any $2d$-regular expander on $N = \frac{n}{p}$ nodes, where $d$ is a constant and $p$ is a free parameter. Let graph $B$ be constructed by replacing each edge of $G$ with a path of length $p$. Then, $B$ has throughput $T_B = O\left(\frac{1}{np \log n}\right)$ and sparsest cut $\Phi_B = \Omega\left(\frac{1}{np}\right)$.

Proof. Let $(S_1, S_2)$ be the sparsest cut in $B$. Let $(S_1', S_2')$ be the corresponding cut in $G$. Namely, if an edge was cut in $B$ by $(S_1, S_2)$ that belonged to a path $p_e$, then $(S_1', S_2')$ cuts $e$. Let $\Phi_B$ be the value of the cut $(S_1, S_2)$ in $A$ and $\Phi_G$ the value of $(S_1', S_2')$ in $G$. Then

$$\Phi_B = \frac{E(S_1, S_2)}{|S_1||S_2|} \geq \frac{E(S_1', S_2')}{|S_1'||S_2'|} \geq \frac{\Phi_G}{p^2}$$

by equation $[5]$ we have $\Phi_G \geq \Omega\left(\frac{1}{N}\right) = \Omega\left(\frac{1}{n^2}\right)$ which gives us

$$\Phi_B \geq \Omega\left(\frac{1}{np}\right)$$

On the other hand, let $T_B$ be the value of the throughput of $B$. We follow a similar reasoning as we did in equation $[4]$.

$$T_B \leq \frac{\sum_{i,j \in E_G} l(i,j)}{\sum_{i,j \in V} l(i,j)} \leq O\left(\frac{N dp}{(Np)^2 \log N}\right) \leq O\left(\frac{1}{Np^2 \log n}\right) = O\left(\frac{1}{np \log n}\right)$$