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Asymptotics of Selberg-like integrals:
The unitary case and Newton’s interpolation formula

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We investigate the asymptotic behavior of the Selberg-like integral

\[ \frac{1}{N!} \int_{[0,1]^N} x^a \prod \Pi_{i<j} (x_i - x_j)^2 \prod_i x_i^{a-1}(1 - x_i)^{b-1} dx_i, \]

as \( N \to \infty \) for different scalings of the parameters \( a \) and \( b \) with \( N \). Integrals of this type arise in the random matrix theory of electronic scattering in chaotic cavities supporting \( N \) channels in the two attached leads. Making use of Newton’s interpolation formula, we show that an asymptotic limit exists and we compute it explicitly.
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I. INTRODUCTION

In his famous 1944 paper, Atle Selberg introduced and computed the integral

\[ S_N(a, b, \beta) := \int_{[0,1]^N} \prod_{1 \leq i < j \leq N} |x_i - x_j|^{2\beta} \prod_{i=1}^{N} x_i^{a-1}(1 - x_i)^{b-1} dx_i \]

\[ = \prod_{j=0}^{N-1} \frac{\Gamma(a + j\beta)\Gamma(b + j\beta)\Gamma(1 + (j + 1)\beta)}{\Gamma(a + b + (N + j - 1)\beta)\Gamma(1 + \beta)} \]

in the aim to solve a problem of Gelfond. Since the sixties, many generalizations and applications have been developed (see for an interesting review about Selberg integral). The scope of these investigations involves many areas of mathematics: random matrices, calculations of constant terms (see e.g.), symmetric functions (in particular, Jack and Macdonald polynomials), multivariate orthogonal polynomials, the value distribution of the Riemann \( \zeta \) function on the critical line among other applications. Since the seventies, many applications in physics have been found, especially in the theory of quantum Hall effect. The number of variables involved in the integrand is then interpreted as the number of particles. In this context it is interesting to study what happens when this number becomes very large.

In the field of random matrices, the integrand of Selberg’s integral corresponds also (for quantized values of \( \beta \)) to the joint probability density of eigenvalues of one of the classical random matrix ensembles, the Jacobi ensemble. Matrices from this ensemble can be generated in three different ways:

- as truncations of Haar orthogonal, unitary or symplectic matrices. For the case of unitary matrices, an important application arises in the theory of electronic transport in mesoscopic systems at low temperatures, as detailed below.

- as composition of Wishart matrices, with applications to multivariate statistics.

- as composition of projection matrices.

In the theory of quantum transport through mesoscopic devices (Landauer-Büttiker scattering approach), the wave function coefficients of the incoming and outgoing electrons in a cavity are related through the unitary scattering matrix \( S \) (\( 2N \times 2N \), if \( N \) is the number...
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of electronic channels that each lead supports):

\[
S = \begin{pmatrix}
  r & t' \\
  t & r'
\end{pmatrix}
\]

(2)

where the transmission \((t, t')\) and reflection \((r, r')\) blocks are \((N \times N)\) matrices encoding the transmission and reflection coefficients among different channels. Many quantities of interest for the experiments are \textit{linear statistics} on the eigenvalues of the hermitian matrix \(tt^\dagger\), \textit{i.e.}, are quantities of the form \(A = \sum_{i=1}^{N} f(T_i)\) where \(f(x)\) is a smooth function (not necessarily linear) and \(T_i\) are the eigenvalues of \(tt^\dagger\) (real numbers between 0 and 1), which have the intuitive interpretation as the probability that an electron gets transmitted through the \(i\)-th channel. For example, the dimensionless conductance and the shot noise are given respectively by \(G = \text{Tr}(tt^\dagger)^{20}\) and \(P = \text{Tr}[tt^\dagger(1 - tt^\dagger)]^{22,23}\). The random scattering theory models the scattering matrix \(S\) for the case of chaotic dynamics as a random unitary matrix uniformly distributed in the unitary group, \textit{i.e.}, it belongs to one of Dyson’s Circular Ensembles.

From this information, the joint probability density of the transmission eigenvalues \(\{T_i\}\) of the matrix \(tt^\dagger\), from which the statistics of interesting experimental quantities could be in principle derived, is readily recognized as the Selberg integrand (Jacobi measure) with \(b = 1\) and \(2\beta = 1, 2, 4^{24}\) depending on physical symmetries of the Hamiltonian (for recent results on the use of Selberg integral in the quantum transport problem, see\textsuperscript{25–28}). The importance of linear statistics, and their asymptotical properties when the number of channels grows to infinity, provides one of our main motivations for the present study.

This paper is the continuation of\textsuperscript{22} and we are interested in the following integrals

\[
\langle f(x_1, \ldots, x_N) \rangle_{a,b} = \frac{1}{N!} \int_{[0,1]^N} f(x_1, \ldots, x_N) \prod_{i<j} (x_i - x_j)^2 \prod_i x_i^{a-1}(1 - x_i)^{b-1} \, dx_i,
\]

denoting averages over the Jacobi probability density. Especially when \(f\) is a power sum (section \[II.B\])

\[
p_k := \sum_{i} x_i^k,
\]

or a Schur function (section \[II.A\]), we raise the question of its asymptotic behavior for \(N \rightarrow \infty\).

More precisely, using classical identities on symmetric functions we show that this problem reduces to the calculation of an inverse binomial transform.
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This paper is the first step towards a combinatorial interpretation of the asymptotic behavior of Selberg-like integrals. Note that J.-Y. Thibon with one of the authors have already investigated the links between combinatorics and Selberg integrals\(^{30–32}\).

The plan of the paper is as follows. In section \(\text{II}\) we give an expression of the integrals as a rational function in the numbers of variables \(N\). In section \(\text{III}\) we investigate some properties of the binomial transform which will be used to compute the limit values in section \(\text{IV}\). Section \(\text{V}\) deals with several special cases related to combinatorics. Finally, in section \(\text{VI}\) we provide concluding remarks, and in appendix \(\text{A}\) an alternative approach to corollary \(\text{II.3}\).

II. SOME SELBERG-LIKE INTEGRALS

A. Schur functions

Here we give an expression of \(\langle s_\lambda(x_1,\ldots,x_N)\rangle_{a,b}\), where

\[ s_\lambda(x_1,\ldots,x_N) = \det(x_i^{\lambda_i+N-j})_{1\leq i,j \leq N} \prod_{1 \leq i < j \leq N}(x_i - x_j) \]

denotes a Schur function (see e.g.\(^8,9\)). A rather classical formula which can be found in the book of Macdonald\(^9\) (this is a special case of the exercise 7 p385) gives

\[ \langle s_\lambda(x_1,\ldots,x_N)\rangle_{a,b} = \prod_{i<j}(\lambda_i - \lambda_j + j - i) \prod_{i=1}^N \frac{\Gamma(\lambda_i + a + N - i)\Gamma(b + N - i)}{\Gamma(\lambda_i + a + b + 2N - i - 1)}. \]

Hence,

\[ \frac{\langle s_\lambda(x_1,\ldots,x_N)\rangle_{a,b}}{\langle 1 \rangle_{a,b}} = \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i} \times \prod_{i=1}^N \frac{\Gamma(\lambda_i + a + N - i)\Gamma(a + b + 2N - i - 1)}{\Gamma(\lambda_i + a + b + 2N - i - 1)\Gamma(a + N - i)} \]

First, we remark that the number of factors of

\[ \prod_{i=1}^N \frac{\Gamma(\lambda_i + a + N - i)\Gamma(a + b + 2N - i - 1)}{\Gamma(\lambda_i + a + b + 2N - i - 1)\Gamma(a + N - i)} = \prod_{i=1}^{\ell(\lambda)} \prod_{j=0}^{\lambda_i - 1} \frac{a + N - i + j}{a + b + 2N - i + j - 1}, \]

where \(\ell(\lambda)\) is the length of the partition \(\lambda\).
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depends only on the partition \( \lambda \) and not on the number of variables \( N \). In the same way, one has

\[
\prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i} = \prod_{i=1}^{\ell(\lambda)} \left[ \prod_{j=i+1}^{\ell(\lambda)} \frac{\lambda_i - \lambda_j + j - i}{j - i} \right] \prod_{j=\ell(\lambda)+1}^{N} \frac{\lambda_i + j - i}{j - i}.
\]

One needs the following lemma which is obtained by reorganizing the factors and simplifying the resulting expression.

**Lemma II.1** For \( N \) large enough\(^1\), one has

\[
\prod_{i=a+1}^{N} \frac{b+i}{b+c+i} = \prod_{i=1}^{c} \frac{a+b+i}{b+N+i}
\]

where \( a, c \in \mathbb{N} \).

**Proof** First write

\[
\prod_{i=a+1}^{N} \frac{b+i}{b+c+i} = \prod_{i=0}^{N-a-1} \frac{b+i+a+1}{b+c+i+a+1} \prod_{i=0}^{c-1} \frac{(b+i+a+1)}{(b+c+i+a+1)} \prod_{i=N-c-a}^{N} \frac{(b+c+i+a+1)}{(b+N+i+1)}
\]

But

\[
\prod_{i=c}^{N} (b+i+a+1) = \prod_{i=0}^{N-c} (b+c+i+a+1),
\]

and

\[
\prod_{i=N-c-a}^{N} (b+c+i+a+1) = \prod_{i=0}^{c-1} (b+N+i+1).
\]

Hence, by substituting these two identities in (3), one recovers the result.\( \square \).

If one applies lemma II.1 to \( \prod_{j=\ell(\lambda)+1}^{N} \frac{\lambda_i + j - i}{j - i} \), one finds

\[
\prod_{j=\ell(\lambda)+1}^{N} \frac{\lambda_i + j - i}{j - i} = \prod_{j=1}^{\ell(\lambda)} \frac{j + N - i}{j - i}.
\]

Hence,

\( ^1 \) The condition “\( N \) large enough” can be omitted provided that we use the notation \( \prod_{i=a}^{b} f(i) = \prod_{i=a}^{b} f(i)^{-1} \).
Proposition II.2  One has

\[
\langle s_\lambda(x_1, \ldots, x_N) \rangle_{a,b} = \prod_{i=1}^{\ell(\lambda)} \prod_{j=i+1}^{\ell(\lambda)} \frac{\lambda_i - \lambda_j + j - i}{j - i} \prod_{j=0}^{\lambda_j-1} \frac{(j + N - i + 1)(a + N - i + j)}{(\ell(\lambda) + j - i + 1)(a + b + 2N - i + j - 1)}
\]

(4)

If \( a = a_1 N + a_0 \) and \( b = b_1 N + b_0 \) are two linear functions of \( N \), this implies that

\[
\frac{\langle s_\lambda(x_1, \ldots, x_N) \rangle_{a,b}}{N^{\lambda}} \bigg|_{N \to \infty} = \left(1 + \frac{a_1}{2 + a_1 + b_1}\right) \prod_{i=1}^{\ell(\lambda)} \prod_{j=i+1}^{\lambda_j-1} \frac{1}{j - i} \prod_{j=0}^{\lambda_j-1} \frac{1}{\ell(\lambda) + j - i + 1},
\]

where \(|\lambda| = \sum_i \lambda_i\).

B. Selberg-like integrals with a power sum in the integrand

In this section, we study the integral

\[
I_k := \frac{\langle p_k(x_1, \ldots, x_N) \rangle_{a,b}}{\langle 1 \rangle_{a,b}},
\]

where \( p_k = \sum_i x_i^k \).

One uses the formula (see e.g. \( 8,9 \)):

\[
p_k = \sum_{i=0}^{k-1} (-1)^i s_{[(k-i)1^i]},
\]

(5)

where \( [(k-i)1^i] \) denotes the partition \( [(k-i), 1, \ldots, 1] \). Hence,

\[
I_k = \sum_{i=0}^{k-1} (-1)^i \frac{\langle s_{[(k-i)1^i]} \rangle_{a,b}}{\langle 1 \rangle_{a,b}}.
\]

(6)

From proposition II.2, one has

Corollary II.3  For each \( k > 0 \), one has

\[
I_k = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \prod_{j=-i}^{k-i-1} \frac{(N + j)(a + N + j - 1)}{a + b + 2N + j - 2}.
\]

Proof  From Eq (1), one has

\[
\frac{\langle s_{[(k-i)1^i]} \rangle_{a,b}}{\langle 1 \rangle_{a,b}} = \prod_{p=2}^{i+1} \frac{k - i + p - 2}{p - 1} \prod_{p=0}^{k-i-1} \frac{(p + N)(a + N - 1 + p)}{(i + p + 1)(a + b + 2N + p - 2)} \times
\]

(first part of the partition)

\[
\times \prod_{j=2}^{i+1} \frac{(N - j + 1)(a + N - j)}{(i - j + 2)(a + b + 2N - j - 1)}
\]

(other parts of the partition)

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But,
\[ \prod_{p=2}^{i+1} \frac{k - i + p - 2}{p - 1} = \prod_{p=0}^{i-1} \frac{k - i + p}{p + 1} = \binom{k - 1}{i}, \]
\[ \prod_{p=0}^{k-i-1} \frac{1}{(i + p + 1)} \prod_{j=2}^{i+1} \frac{1}{(i - j + 2)} = \prod_{p=i+1}^{k} \frac{1}{p} \prod_{j=1}^{i} \frac{1}{j} = \frac{1}{k!}, \]
and
\[ \prod_{j=2}^{i+1} \frac{a + b + 2N - j - 1}{a + b + 2N - j + 1} = \prod_{j=-i}^{k-1} \frac{(N + j)(a + N + j - 1)}{a + b + 2N + j - 2}. \]

Using these equalities in (7), one finds
\[ \langle s \rangle^{(k-i)1} \rangle_{a,b}^{(1)} = \frac{1}{k!} \binom{k - 1}{i} \prod_{j=-i}^{k-1} \frac{(N + j)(a + N + j - 1)}{a + b + 2N + j - 2}. \]

The result is obtained by replacing \( \langle s \rangle^{(k-i)1} \rangle_{a,b}^{(1)} \) with its value in (6).

The expression in corollary II.3 is the starting point in computing the limit \( \lim_{N \to \infty} \frac{I_k}{N} \) (in section IV), using the tools we are going to introduce in the following section.

III. INVERSE BINOMIAL TRANSFORM

A. Inverse binomial transform and Newton’s interpolation formula

In this section, we shall use widely the inverse binomial transform (see e.g.\( ^{33,34} \)) that operates on a sequence of polynomials \( F = (f_i(x)) \) by
\[ \mathcal{B}_k^{-1}[F] = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f_i(x) \]

**Proposition III.1** Let \( F(y) = \sum_{i=0}^{p} \alpha_i(x) y^i \) be the unique polynomial in \( y \) with coefficients in \( \mathbb{C}[x] \) of degree \( p \) (in \( y \)) interpolating the points
\[ (0, f_0(x)) \ldots (p, f_p(x)). \]

We have:
\[ \mathcal{B}_k^{-1}[F] = k! \sum_{i=k}^{p} S_{i,k} \alpha_i(x), \]
where \( S_{i,k} \) is a Stirling number of the second kind.
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**Proof** By linearity, it suffices to show the result for $F(y) = y^p$. We remark that in this case

$$\mathfrak{B}_k^{-1}[F] = \sum_{i=0}^{k} (-1)^{k-i} \left( \begin{array}{l} k \\ i \end{array} \right) i^p = k! S_{p,k},$$

by means of the well known formula

$$S_{p,k} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^{k-i} \left( \begin{array}{l} k \\ i \end{array} \right) i^p.$$  

□

Consider the divided difference operator $\partial_{y_1y_2}$ acting on the right of any expression $f$ in $\{y_1, y_2\}$ by

$$f \partial_{y_1y_2} = \frac{f_{\sigma_{y_1y_2}} - f}{y_2 - y_1}$$

where $\sigma_{y_1y_2}$ permutes $y_1$ and $y_2$ in $f$.

**Remark III.2** Note that, assuming that $k \leq p$, the degree of $y_0^p \partial_{y_0y_1} \cdots \partial_{y_{k-1}y_k}$ is equal to $p-k$. Therefore, if $g(x, y)$ is a polynomial of degree $p$ in $x$ and $y$, the degree of $\mathfrak{B}_k^{-1}[(g(x, i))_i]$ equals $p-k$.

This operator is the main tool to describe the Newton interpolation. Indeed, consider a one variable function $f(y)$ and a set of interpolating variables $\{y_0, \ldots, y_k\}$. One has

$$f(y) = f(y_0) + f(y_0) \partial_{y_0y_1}(y - y_0) + \cdots + f(y_0) \partial_{y_0y_1} \cdots \partial_{y_{k-1}y_k} (y - y_0) \cdots (y - y_{k-1}) + R(y)$$

with $R(y_i) = 0$ for each $i = 0, \ldots, k$.

We will denote

$$f \partial_{m\ldots n} = f(y_m) \partial_{y_my_{m+1}} \cdots \partial_{y_{n}y_{n+1}} |_{y_i = i},$$

for each pair of integers $m \leq n$, with the special case $f \partial_{m\ldots m} = f(m)$. Set, also, $\partial_{i} := \partial_{i\ldots i+1}$.

With this notation, the polynomial $f$ of degree $n - m$ interpolating the points

$$(m, f(m)), \ldots, (n, f(n))$$

becomes

$$f(y) = \sum_{j=m}^{n} f \partial_{m\ldots j}(y - m) \cdots (y - (j - 1)).$$
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Remark that the Stirling numbers appear when we write $y^p$ in terms of falling factorials,
\[(y)_k := y(y - 1) \ldots (y - k + 1),\]
\[y^p = \sum_{k=0}^{p} S_{p,k}(y)_k.\]
This means that the Stirling numbers are the coefficients in the Newton interpolation of $y^p$
at $y = 0, \ldots, p$. With our notations this reads
\[S_{p,k} = y^p \partial_0 \ldots k+1.\] (9)
Hence, by linearity, one obtains immediately the following result from proposition III.1:

**Corollary III.3** Let $k < p$ be two integers. Let $F$ be the unique polynomial in $y$ with coefficients in $\mathbb{C}[x]$ of degree $p$ (in $y$) interpolating the points
\[(0, f_0(x)) \ldots (p, f_p(x)).\]
We have:
\[\mathcal{B}^{-1}_k [F] = k! F \partial_0 \ldots p. \square\]
When acting by $\partial_0 \ldots k+1$ on $y^p$, one observes the following (shifted) induction.

**Proposition III.4**
\[y^p \partial_0 \ldots k+1 = (y + 1)^{p-1} \partial_0 \ldots k\]

**Proof** Since
\[y^p \partial_{g_0 y_1} \ldots \partial_{g_k y_{k+1}} = \sum_{i=0}^{p-1} y_1^{p-i-1} \partial_{g_1 y_2} \ldots \partial_{g_k y_{k+1}} y_i^i,\]
the specialization gives
\[y^p \partial_0 \ldots k+1 = y^{p-1} \partial_1 \ldots k+1.\]
We conclude the proof by noting that the coefficients in the Newton interpolation of any $f(y)$
at $y = 1, \ldots, k + 1$ are, respectively, equal to the coefficients in the Newton interpolation of$f(y + 1)$ at $y = 0, \ldots, k. \square
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B. An example of generalized (inverse) binomial transform

As an application, consider the polynomials

\[ P_i^k(x; a, b) := \prod_{j=0}^{k-i-1} (x + j + a) \prod_{j=0}^{i-1} (x - j + b). \]

For simplicity, we will denote \( P_i^k := P_i^k(x; a, b) \) when there is no ambiguity.

**Proposition III.5** When \( p \leq k \), one has

\[ \mathfrak{B}^{-1}_{k-p} [P^k_p, \ldots, P^k_k] = \prod_{i=0}^{p-1} (x + b - i) \prod_{i=0}^{k-p-1} (b - a - p - i). \]

**Proof** First remark that \( y_0^i \partial_{y_0y_1} \cdots \partial_{y_jy_{j+1}} \) is a symmetric polynomial in \( \{y_0, \ldots, y_{j+1}\} \). Hence, we can permute the variables in the expression and obtain

\[ y_0^i \partial_{y_0y_1} \cdots \partial_{y_jy_{j+1}} = y_j^i \partial_{y_jy_{j-1}} \cdots \partial_{y_1y_0} \partial_{y_0y_{j+1}}. \]

Applying the same argument to \( y_j^i \partial_{y_jy_{j-1}} \cdots \partial_{y_1y_0} \) which is symmetric in the variables \( \{y_0, \ldots, y_j\} \) one gets

\[ y_j^i \partial_{y_jy_{j-1}} \cdots \partial_{y_1y_0} \partial_{y_0y_{j+1}} = y_0^j \partial_{y_0y_1} \cdots \partial_{y_{j-1}y_j} \partial_{y_0y_{j+1}}. \]

By definition of \( \partial_{y_0y_{j+1}} \) one obtains

\[ y_0^i \partial_{y_0y_1} \cdots \partial_{y_jy_{j+1}} = y_{j+1}^i \partial_{y_{j+1}y_j} \partial_{y_1y_2} \cdots \partial_{y_jy_{j-1}} y_j - y_0^i \partial_{y_0y_1} \cdots \partial_{y_jy_{j-1}y_j}. \]

(10)

Again,

\[ y_{j+1}^i \partial_{y_{j+1}y_j} \partial_{y_1y_2} \cdots \partial_{y_{j-1}y_j} = y_0^i \partial_{y_1y_2} \cdots \partial_{y_{j-1}y_j}, \]

and eq. (10) becomes

\[ y_0^i \partial_{y_0y_1} \cdots \partial_{y_jy_{j+1}} = \frac{y_{j+1}^i \partial_{y_1y_2} \cdots \partial_{y_jy_{j+1}} y_j - y_0^i \partial_{y_0y_1} \cdots \partial_{y_{j-1}y_j}}{y_{j+1} - y_0}. \]

(11)

Let us prove the result by induction on \( k \). Note that if \( k = p \), the result is straightforward.

Denote by \( P(y) \) the unique polynomial of degree \( k - p + 1 \) in \( y \) such that \( P(i) = P_{i-p}^k \) for each \( i = 0 \ldots k - p \). By linearity eq. (11) gives

\[ P(y_0) \partial_{y_0y_1} \cdots \partial_{y_{k-p-1}y_{k-p}} = \frac{P(y_j) \partial_{y_1y_2} \cdots \partial_{y_{k-p-1}y_{k-1}} - P(y_0) \partial_{y_0y_1} \cdots \partial_{y_{k-p-2}y_{k-p-1}}}{y_{k-p} - y_0}. \]
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Specializing at $y_i = i$, one obtains

$$ P \partial_{0...k-p} = \frac{P \partial_{1...k-p} - P \partial_{0...k-p-1}}{k - p}. \tag{12} $$

By definition of $P$ and using corollary III.3, one has

$$ \mathfrak{B}^{-1}_{k-p}[P_p^k, \ldots, P_k^k] = \mathfrak{B}^{-1}[P(0), \ldots, P(k - p)] = (k - p)! P \partial_{0...k-p}. $$

Hence, eq. (12) yields

$$ \mathfrak{B}^{-1}_{k-p}[P_p^k, \ldots, P_k^k] = (k - p - 1)! (P \partial_{1...k-p} - P \partial_{0...k-p-1}). $$

And then,

$$ \mathfrak{B}^{-1}_{k-p}[P_p^k, \ldots, P_k^k] = \mathfrak{B}^{-1}_{k-p-1}[P_{p+1}^k, \ldots, P_k^k] - \mathfrak{B}^{-1}_{k-p-1}[P_p^k, \ldots, P_{k-1}^k]. \tag{13} $$

Remark that if $i < k$ one has

$$ P_i^k(x; a, b) = (x + a) P_i^{k-1}(x; a + 1, b) $$

and if $i > 0$

$$ P_{i+1}^k(x; a, b) = (x + b) P_i^{k-1}(x; a, b - 1), $$

eq. (13) becomes

$$ \mathfrak{B}^{-1}_{k-p}[P_p^k, \ldots, P_k^k] = (x + b) \mathfrak{B}^{-1}_{k-p-1}[P_{p+1}^{k-1}(x; a, b - 1), \ldots, P_{p+1}^{k-1}(x; a, b - 1)] $$

$$ - (x + a) \mathfrak{B}^{-1}_{k-p-1}[P_p^{k-1}(x; a + 1, b), \ldots, P_p^{k-1}(x; a + 1, b)]. $$

One recovers the result by applying the induction hypothesis. □

Note that, setting $p = 0$ in the previous proposition, one obtains

$$ \mathfrak{B}_k^{-1}[P_0^k, \ldots, P_k^k] = \prod_{i=0}^{k-1} (b - a + i). \tag{14} $$

Remark III.6 Remark III.3 and proposition III.5 imply that the (Newton) polynomial interpolating the point $(0, P_0^k), \ldots, (k, P_k^k)$ (i.e. the polynomial $P$ of degree $k$ in $y$ such that $P(x, y) = P_i^k$) is a bi-variate polynomial (in $x$ and $y$) whose degree is $k$. 

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C. Leading coefficients

Let \( f(x, y) \) be a bivariate polynomial of degree \( m \). In this section, we investigate the coefficient of the leading term of the binomial transform \( \mathfrak{B}^{-1}_k[(f(x, i)i^p)_i] \), that is the scalar

\[
\mathfrak{L}_{k,p}(f) = \left[ x^{p+m-k}f(x, y_0)y_0^p\partial_0\ldots\partial_{k-1}\right]_{y_i=i}
\]

multiplied by \( k! \), where \( [x^p]P(x) = \alpha_i \) if \( P(x) = \sum_i \alpha_i x_i \). The following result explains how to manage the shift induced by the multiplication by \( i^p \):

**Proposition III.7** One has

\[
\mathfrak{L}_{k,p}(f) = \begin{cases} 
\mathfrak{L}_{k-p,0}(f_p) & \text{if } p \leq k \\
0 & \text{otherwise},
\end{cases}
\]

where \( f_p(x, y) = f(x, y + p) \).

**Proof** Proposition III.4 yields

\[
f(x, y)y^p\partial_0\ldots k + 1 = f(x, y + 1)(y + 1)^{p-1}\partial_0\ldots k.
\]

Note that, since \( f(x, y)(y + 1)^p \) is a bivariate polynomial of degree \( p + m - 1 \),

\[
[x^{p+m-k}]f(x, y + 1)(y + 1)^{p-1} = [x^{p+m-k}]f(x, y + 1)(y)^{p-1} + R(y),
\]

where \( R \) is a polynomial of degree at most \( k - 1 \). Hence, since the operator \( \partial_{y_0y_1}\ldots y_{k-1}y_k \) lowers the degree of \( k \) in the \( y_i \)'s, one observes

\[
[x^{p+m-k}]f(x, y + 1)(y + 1)^{p-1}\partial_0\ldots k = [x^{p+m-k}]f(x, y + 1)(y)^{p-1}\partial_0\ldots k,
\]

or, equivalently,

\[
\mathfrak{L}_{k,p}(f) = \begin{cases} 
\mathfrak{L}_{k-1,p-1}(f_1) & \text{if } 1 \leq k \\
0 & \text{otherwise},
\end{cases}
\]

Iterating the process, one proves the claim. \( \square \)

If \( (a_i(x))_i \) is a sequence of polynomials, one defines

\[
\mathfrak{X}^{a,b}_k[(a_i(x))_i] := (-1)^k \mathfrak{B}^{-1}_k\left[ (P^k_i a_i(x))_i \right].
\]
Selberg-like integrals

Remark III.8 From remark III.6 if $f(x, y)$ is a bivariate polynomial of degree $m$, its transform, $\Sigma^a_b[f(x, 0), \ldots, f(x, k)]$, is a polynomial in $x$ whose degree is also (at most) $m$. Indeed, the polynomial $F$ of degree $k$ in $y$ interpolating the points $(0, f(x, 0)P^0_k), \ldots, (k, f(x, k)P^k_k)$ is a bivariate polynomial whose degrees in $x$ and in $\{x, y\}$ are $m + k$. Acting by $\partial_0, \ldots, \partial_k$, one obtains a polynomial in $x$ which is a linear combination of coefficients $[y^i]F$ (whose degree in $x$ is $m + k - i$). We conclude by noting that, when $i < k$, $[y^i]F$ gives no contribution since $y^i\partial_{y^0}y_1 \ldots \partial_{y_{k-1}}y_k$ is a polynomial of degree $i - k$.

In particular, when $f(x, y) = y^p$, eq. (14) and proposition III.7 imply

$$[x^p]\Sigma^a_b[fp_i] = \begin{cases} (-1)^k \frac{k!}{(k-p)!} \prod_{i=0}^{k-p-1} (b - a - p - i) & \text{if } p \leq k \\ 0 & \text{otherwise.} \end{cases}$$

The following result allows us to compute the leading term in the action of $\Sigma^a_b$ on a product of linear factors of the form $cx - dy + e$.

Corollary III.9

$$[x^p]\Sigma^a_b \left[ \prod_{j=0}^{p-1} (c_jx - d_ji + e_j) \right] = \sum_{j=0}^{k} \frac{k!}{(k-j)!} \left( \sum_{(c_1, \ldots, c_{p-j})} d_{s_1} \ldots d_{s_j} c_{t_1} \ldots c_{t_{p-j}} \right) \prod_{i=0}^{k-j-1} (a - b + j + i).$$

Proof First note that the variables $e_j$ give no contribution:

$$[x^p]\Sigma^a_b \left[ \prod_{j=0}^{p-1} (c_jx - d_ji + e_j) \right] = \left[ x^p \right] \Sigma^a_b \left[ \prod_{j=0}^{p-1} (c_jx - d_ji) \right].$$

Indeed, the coefficient of $e_{j_1} \ldots e_{j_s}$ in $\prod_{j=0}^{p-1} (c_jx - d_jy + e_j)$ is a polynomial $f(x, y)$ whose degree is $p - s$. From remark III.2, the degree of $\Sigma_k [(f(x, i))]$, being at most $p - s$, it follows immediately that $[x^p]\Sigma^a_b [(f(x, i))] \neq 0$ only if $s = 0$. 
Selberg-like integrals

Now, to obtain our result, it suffices to expand $\prod_{j=0}^{p-1} (c_j x - d_j y)$ as a polynomial in $x$ and $y$:

$$[x^p] \mathcal{I}_k \left[ \left( \prod_{j=0}^{p-1} (c_j x - d_j i + e_j) \right) \right] =$$

$$[x^p] \sum_{j=0}^{p} (-1)^{p-j} \left( \sum_{\{s_1, \ldots, s_j\} \cup \{t_1, \ldots, t_{p-j}\} = \{0, \ldots, p-1\}} \mathcal{I}_k \left[ \left( x^{p-j} \right) \right] \right) [x^{p-j}] \mathcal{I}_k \left[ \left( i^{p-j} \right) \right].$$

Applying eq. (15), one recovers our result. \(\square\)

In the next section, we use the results of this section in the aim to investigate the asymptotic behavior of the integrals $I_k/N$. In particular, we show that its convergence (section IV) is a direct consequence of remark III.8 and we compute explicitly the limit (section IV B) by means of corollary III.9.

IV. ASYMPTOTIC BEHAVIOR OF $I_k/N$

In this section, we use the tools described in the previous section to prove the convergence of the integral $I_k/N$ and compute the limit.

A. Convergence

Suppose now that $a = a(N)$ and $b = b(N)$ are linear function of $N$. One has:

**Theorem IV.1**

$$\lim_{N \to \infty} \frac{I_k}{N} < +\infty$$

**Proof** We start from Corollary II.3 and write

$$\frac{I_k}{N} = \frac{1}{k! N \prod_{j=-k+1}^{k-1} (a(N) + b(N) + 2N + j - 2)} \mathfrak{N}_k(N)$$

where

$$\mathfrak{N}_k(N) := \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \prod_{j=-k+1}^{k-1} (2N + a(N) + b(N) + j - 2) \times$$

$$\times \prod_{j=-1}^{k-1} (N + j)(N + a(N) + j - 1) \prod_{j=k-i}^{k-1} (2N + a(N) + b(N) + j - 2).$$

(16)
Selberg-like integrals

We need the following lemma:

**Lemma IV.2** The degree in $N$ of the polynomial $\mathfrak{N}_k(N)$ is $2k$.

**Proof** For convenience, set $a = a_1N + a_0$ and $b = b_1N + b_0$, and write

$$
\prod_{j=-k+1}^{i-1} (2x + a(x) + b(x) + j - 2) \prod_{j=k-i}^{k-1} (2x + a(x) + b(x) + j - 2),
$$

$$
\prod_{j=0}^{k-i-2} ((2 + a_1 + b_1)x + j + a_0 + b_0 - 1 - k) \prod_{j=0}^{i-1} ((2 + a_1 + b_1)x + a_0 + b_0 - j + k - 3).
$$

With the notation of the previous section, one recognizes

$$
\prod_{j=0}^{k-i-2} ((2 + a_1 + b_1)x + j + a_0 + b_0 - 1 - k) \prod_{j=0}^{i-1} ((2 + a_1 + b_1)x + a_0 + b_0 - j + k - 3) =
$$

$$
P_i^{k-1}((2 + a_1 + b_1); a_0 + b_0 - k - 1, a_0 + b_0 + k - 3).
$$

If one sets

$$
Q_k(x, y) := \prod_{j=0}^{k-1} \left( \frac{x}{2 + a_1 + b_1} + j - y \right) \left( \frac{1 + a_1}{2 + a_1 + b_1} x + a_0 + j - 1 - y \right),
$$

the following holds

$$
\mathfrak{N}_k(N) = \sum_{k-1}^{a_0+b_0-1-k,a_0+b_0+k-3} [(Q_k(x, i))_{i \in \mathbb{N}}] |x=(a_1+b_1+2)N. \tag{17}
$$

Hence, from remark III.8, the degree of $\mathfrak{N}_k(N)$ equals the degree of $Q_k(x, y)$ that is $2k$. □.

The degree in $N$ of the denominator

$$
N \prod_{j=-k+1}^{k-1} (a(N) + b(N) + 2N + j - 2)
$$

of $\frac{L}{N}$ is $2k$. From lemma [IV.2] $\frac{L}{N}$ is a rational fraction in $N$ whose numerator and denominator have the same degree $2k$. Hence $\frac{L}{N}$ converges. □

**B. Computation of the limit**

Let $\mathbb{F}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(x, y) = \prod_{j=0}^{p-1} (\alpha_1x + j - y + \beta_1)(\alpha_2x + j - y + \beta_2)$. And set, for convenience,

$$
\mathbb{F}_{\alpha_1,\beta_1;\alpha_2,\beta_2}^{p}(x) = (\mathbb{F}_{\alpha_1,\beta_1;\alpha_2,\beta_2}(x, i))_{i \in \mathbb{N}}
$$

One has:
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**Proposition IV.3** The coefficient of $x^{2p}$ in $\Sigma_k^b \left[ \frac{p}{2} \alpha_p^b \right]_{\alpha_1, \alpha_2, \beta_2} (x)$ does not depend on $\beta_1$ and $\beta_2$. More precisely, one has:

$$[x^{2p}] \Sigma_k^b \left[ \frac{p}{2} \alpha_p^b \right]_{\alpha_1, \alpha_2, \beta_2} (x) = \sum_{j=0}^{k} \frac{k!}{(k-j)!} \sum_{i=0}^{p} \binom{p}{i} \left( 2p - j - i \right) \alpha_1^{i} \alpha_2^{2p-i-j} \prod_{i=0}^{k-j-1} (a - b + j + i).$$

**Proof** This equality is obtained from corollary [II.9] setting $c_j = \alpha_1$, $c_{j+p} = \alpha_2$, $d_j = d_{j+p} = 1$, $e_j = j + \beta_1$ and $e_{j+p} = \beta_2$ for each $j = 0, 1, \ldots, p - 1$. □

Using this result, one finds:

**Theorem IV.4** Setting $a = \alpha_1 N + \alpha_0$ and $b = \beta_1 N + \beta_0$, one has

$$\lim_{N \to \infty} \frac{I_k}{N} = \frac{1 + \alpha_1}{k(2 + \alpha_1 + \beta_1)^k} \sum_{j=0}^{k-1} (-1)^j \binom{1 + \alpha_1}{2 + \alpha_1 + \beta_1} j \left( j + k - 1 \right) \sum_{i=0}^{k-1-j} \frac{(1 + \alpha_1)^i}{i + j + 1} \binom{k}{i}.$$  \hspace{1em}(18)

**Proof** We have seen (see eq (17)) that

$$\lim_{N \to \infty} \frac{I_k}{N} = \frac{1}{k! (2 + \alpha_1 + \beta_1)^{2k-1}} \times \left[ N^{2k} \Sigma_{k-1}^{a_0 + b_0 - 1 - k, \alpha_0 + b_0 + k - 3} \left[ \frac{p}{2} \alpha_p^b \right]_{\alpha_1, \alpha_2, \beta_2} (x) \right]_{x = (2 + \alpha_1 + \beta_1)N}.$$  \hspace{1em}(19)

By proposition [IV.3] one obtains

$$\lim_{N \to \infty} \frac{I_k}{N} = \frac{2 + \alpha_1 + \beta_1}{k} \sum_{i=0}^{k-1} \left( 1 + \alpha_1 \right)^i \prod_{j=0}^{k-i-2} \left( 2(1 - k) + j + i \right) \frac{1}{(k - i - 1)!} \times \sum_{j=0}^{k} \binom{k}{j} \binom{k}{2k - i - j} \left( \frac{1 + \alpha_1}{2 + \alpha_1 + \beta_1} \right)^{2k-j}.$$  \hspace{1em}(18)

One recognizes

$$\prod_{j=0}^{k-i-2} \left( 2(1 - k) + j + i \right) \frac{1}{(k - i - 1)!} = (-1)^{i+k-1} \binom{2(k - 1) - i}{k - 1}.$$  \hspace{1em}(19)

And, slightly rearranging the terms of the sum, we show the theorem. □

Let us rewrite (18) as

$$\lim_{N \to \infty} \frac{I_k}{N} = \frac{1 + \alpha_1}{(2 + \alpha_1 + \beta_1)^k} \left( (1 + \alpha_1)^{k-1} + \sum_{i=0}^{k-2} \left( \frac{1 + \alpha_1}{2 + \alpha_1 + \beta_1} \right)^i \sum_{j=0}^{k-1-i} \left( \frac{1 + \alpha_1}{2 + \alpha_1 + \beta_1} \right)^j \left( \frac{1 + \alpha_1}{i + j + 1} \right) \binom{k}{i} \binom{k-1-i}{j} \right).$$  \hspace{1em}(19)

For convenience, we set $a_1 = \ell_1 - 1$ and $b_1 = \frac{1}{\ell_2} - \ell_1 - 1$. With this notation, equation (19) reads

$$\lim_{N \to \infty} \frac{I_k}{N} = \ell_1 \ell_2 \left( \ell_1^{k-1} + \sum_{i=0}^{k-2} \frac{\ell_1}{k - i - 1} \binom{k}{i} \sum_{j=0}^{k-1-i} \left( \ell_1 \ell_2 \right)^j \binom{k-1-i}{j} \right).$$
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Rearranging the sum, one obtains

\[
\lim_{N \to \infty} I_k^N = \ell_1 \ell_2^k \left( 1 + \sum_{j=1}^{k-1} \frac{\ell_j}{j} \binom{k}{j+1} \left( \sum_{i=1}^{j} (-1)^i \binom{j}{i} (i+k-1) \ell_i^2 \right) \right),
\]

(20)
or equivalently, in terms of binomial transform one has

\[
\lim_{N \to \infty} I_k^N = \ell_1 \ell_2^k \left( 1 + \sum_{j=1}^{k-1} \frac{\ell_j}{j} \binom{k}{j+1} B - 1_j \left[ \left( \sum_{i=1}^{j} (-1)^i \binom{j}{i} \ell_i \right)_{i+1} \right] \right).
\]

(21)

V. SOME SPECIAL CASES RELATED TO COMBINATORICS

A. Simplest cases

**Corollary V.1** 1. If \( a_1 = -1 \) and \( b_1 \neq -1 \) then \( \lim_{N \to \infty} \frac{I_k}{N} = 0 \).

2. If \( a_1 \neq 1 \) and \( b_1 = -1 \) then \( \lim_{N \to \infty} \frac{I_k}{N} = 1 \).

**Proof** The assertion (1) is straightforward from (18).

To prove the second assertion, we need the following lemma

**Lemma V.2** Let \( a, b \) and \( c \) be three integers and denote

\[
\{ \frac{c}{b} \}_{a} := \sum_{j=0}^{a} (-1)^j \binom{a}{j} \binom{c+j}{b+j}.
\]

With this notation, one has

\[
\{ \frac{c}{b} \}_{a} = (-1)^a \binom{c}{a+b}.
\]

**Proof** By induction, remarking that

\[
\{ \frac{c}{b} \}_{a} = \left\{ \frac{c-1}{b} \right\}_{a} + \left\{ \frac{c-1}{b-1} \right\}_{a}.
\]

\[\square\]

Under the specialization \( b_1 = -1 \) (or equivalently \( \ell_2 = \ell_1 \)) and using the notation of lemma [V.2], formula (20) reads

\[
\lim_{N \to \infty} \frac{I_k}{N} = \ell_1^{k-1} \left( \ell_1^{1-k} + \sum_{i=0}^{k-2} \frac{\ell_1^{-i}}{k-i-1} \binom{k}{i} \left\{ \frac{k-1}{i+1} \right\}_{k-i-1} \right).
\]

But lemma [V.2] yields \( \left\{ \frac{k-1}{i+1} \right\}_{k-i-1} = \binom{k-1}{k} = 0 \). This implies our result. \[\square\]
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B. Central binomial coefficients and the specialization $a_1 = b_1 = 0$

Under this specialization, (18) reads

$$\lim_{N \to \infty} \frac{I_k}{N} = \frac{1}{2^k k} \sum_{j=0}^{k-1} \left( \frac{-1}{2} \right)^i \binom{j+k-1}{j} \sum_{i=0}^{k-1-j} \binom{k}{i+j+1} \binom{k}{i}.$$ 

But, one has

$$\sum_{i=0}^{k-1-j} \binom{k}{i+j+1} \binom{k}{i} \left( \frac{-1}{2} \right)^i \binom{j+k-1}{j} \left( \frac{2k}{k+j+1} \right).$$ (22)

Indeed, formula (22) follows from a well known equality

$$\sum_{j=0}^{\infty} \left( \begin{array}{c} a \\ j \end{array} \right) \left( \begin{array}{c} b \\ c+j \end{array} \right) = \left( \begin{array}{c} a+b \\ c \end{array} \right),$$

which can be proved by a straightforward induction on $b$. Hence, from (22), one obtains

$$\lim_{N \to \infty} \frac{I_k}{N} = \frac{1}{2^k k} \sum_{j=0}^{k-1} \left( \frac{-1}{2} \right)^i \binom{j+k-1}{j} \left( \frac{2k}{k+j+1} \right).$$

We need the following lemma:

Lemma V.3 One has

$$\langle n \rangle := \sum_{j=0}^{n-m} (-2)^{-j} \binom{n-m+j}{j} \binom{2n}{n+m+j} = \frac{2^{m-n}}{\binom{2n}{m}}.$$ 

Proof Set $F_{n,j} := (-2)^{-j} \binom{n-m+j}{j} \binom{2n}{n+m+j}$. We compute the associated Gosper sequence:

$$G_{n,j} := -2 \frac{j(2n+1)}{m-n+j-1} F_{n,j},$$

and we check that the two sequences verify the equality

$$(2n+1)F_{n,j} + (m-n-1)F_{n+1,j} = G_{n,j+1} - G_{n,j}.$$ 

It follows that

$$\langle n \rangle = \frac{n+1}{m} \langle n \rangle.$$ 

Hence, the result is obtained by induction. □

From [V.2] one obtains

Corollary V.4

$$\lim_{N \to \infty} \frac{I_k}{N} = \frac{1}{2^k k} \binom{k}{1} = \frac{1}{2^{2k}} \binom{2k}{k}.$$ 

This result is well-known in the quantum scattering setting in symmetric cavities [35–37].
Selberg-like integrals

C. Catalan triangle and the specialization \( a_1 = 0 \)

Let us set for convenience \( b_1 = \ell - 1 \). Under this specialization eq (18) reads

\[
\lim_{N \to \infty} \frac{I_k}{N} = \frac{1}{k(1+\ell)^k} \sum_{j=0}^{k-1} (-1)^j \left( \frac{1}{1+\ell} \right)^j \binom{j+k-1}{j} \sum_{i=0}^{k-1-j} \binom{k}{i+j+1} \binom{k}{i}.
\]

Using formula (22), we obtain

\[
\lim_{N \to \infty} \frac{I_k}{N} = \frac{1}{k(1+\ell)^{2k-1}} \sum_{j=0}^{k-1} (-1)^j \left( \frac{1}{1+\ell} \right)^j \binom{j+k-1}{j} \left( \frac{2k}{k+j+1} \right),
\]

or equivalently

\[
\lim_{N \to \infty} \frac{I_k}{N} = \frac{1}{k(1+\ell)^{2k-1}} \sum_{j=0}^{k-1} \left( -1 \right)^j \left( 1+\ell \right)^{k-j-1} \binom{j+k-1}{j} \left( \frac{2k}{k+j+1} \right) \ell^i.
\]

By rearranging the factor of the products appearing in the coefficient of each \( \ell^i \), one restates this expression in terms of inverse binomial transform:

\[
\lim_{N \to \infty} \frac{I_k}{N} = \frac{(2k)!}{k!(1+\ell)^{2k-1}} \sum_{i=0}^{k-1} \left( -1 \right)^{k-1-i} \frac{1}{i!(k-1-i)} \mathfrak{B}^{-1}_{k-1} \left[ \left( \frac{1}{(j+k)(j+k+1)} \right) \right] \ell^i 
\]

We need the following lemma:

**Lemma V.5**

\[
\mathfrak{B}^{-1}_m \left[ \left( \frac{1}{(p+i)(p+i+1)} \right) \right] = \frac{(-1)^m(m+1)!}{\prod_{i=0}^{m+1}(p+i)}.
\]

**Proof** First remark that

\[
\frac{1}{(p+i)(p+i+1)} = \frac{P^m_{m-i}(p;0,m+1)}{\prod_{i=0}^{m+1}(p+i)}.
\]

Hence,

\[
\mathfrak{B}^{-1}_m \left[ \left( \frac{1}{(p+i)(p+i+1)} \right) \right] = \frac{\mathfrak{B}^{-1}_m \left[ P^m_0(p;0,m+1) \ldots P^m_m(p;0,m+1) \right]}{\prod_{i=0}^{m+1}(p+i)}.
\]

We conclude by using eq. (14). □

Now using lemma [V.5] in equality (23), one finds
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Proposition V.6

\[
\lim_{N \to \infty} \frac{I_k}{N} = \sum_{i=0}^{k-1} \frac{k-i}{k} \binom{2k}{i} \ell^i \quad \frac{1}{(1+\ell)^{2k-1}}
\]

The triangle \( \mathcal{D} := \left( \frac{k-i}{k} \binom{2k}{i} \right)_{k,i \in \mathbb{N}} \) is sometimes called Catalan triangle (see e.g. sequences A008315, A050166 and A039598 in \(^{38}\)).

Note that these numbers are related to many combinatorial objects. For example, they appear in the expansion of odd power of \( x \) in terms of orthogonal Chebyshev polynomials \( U_k(x) \) of the second kind (see e.g. \(^{39}\) p.796), since

\[
x^{2k-1} = \frac{1}{2^{2k-1}} \sum_{i=0}^{k-1} \mathcal{D}_{k,i} U_{2(k-i)-1}(x).
\]

Another example is given by R.K. Guy in \(^{40}\): he showed that the number of walks in a lattice with \( k \) steps (each in direction N, S, E or W) starting at \((0,0)\) and at a distance \( i \) from the \( x \)-axis equals \( \mathcal{D}_{k+1-i,k+1} \).

D. Symmetric Dyck paths counted by number of peaks and the specialization \( b_1 = 0 \)

For convenience, let us set \( a_1 = \ell - 1 \). This case has been already computed by M. Novaes in \(^{35,36}\). With our notation, he proved the following formula

\[
\lim_{N \to \infty} \frac{I_k}{N} = (\ell + 1) \sum_{i=1}^{k} \frac{(-1)^{i-1}}{i} \binom{k-1}{i-1} \binom{2(i-1)}{i-1} \left( \frac{\ell}{(1+\ell)^2} \right)^i. \tag{24}
\]

Our goal is to identify the coefficient \( \alpha_{i,k} \) such that

\[
\lim_{N \to \infty} \frac{I_k}{N} = \frac{\sum_{i} \alpha_{i,k} \ell^i}{(1+\ell)^{2k-1}}.
\]

Let us sketch the proof of the following result, announced by two of us in \(^{29}\).

Proposition V.7

\[
\lim_{N \to 0} \frac{I_k}{N} = \frac{\ell}{(1+\ell)^{2k-1}} \sum_{i=0}^{2(k-1)} \left( \frac{k-1}{\lfloor \frac{k}{2} \rfloor} \right) \left( \frac{k-1}{\lceil \frac{k}{2} \rceil} \right) \ell^i
\]
Selberg-like integrals

**Proof** First rewrite eq (24) as

\[
\lim_{N \to \infty} \frac{I_k}{N} = \frac{1}{2k-1} \sum_{i=1}^{k} \frac{(-1)^{i-1}}{i} \left( \begin{array}{c} k-1 \\ i-1 \end{array} \right) \left( \frac{2(i-1)}{i-1} \right) \ell^i (1 + \ell)^{2(k-i)}.
\]

Expanding \((1 + \ell)^{2(k-i)}\) and rearranging the sum one obtains

\[
\lim_{N \to \infty} \frac{I_k}{N} = \frac{\ell}{(1 + \ell)^{2k-1}} \sum_{i=0}^{2(k-1)} \left( \sum_{j=0}^{k-1} (-1)^j \left( \begin{array}{c} 2(k-j-1) \\ i-j \end{array} \right) \frac{2j}{j+1} \right) \ell^i
\]

Hence, we need to prove:

\[
\mathcal{J}_{k,i} := \sum_{j=0}^{k-1} (-1)^j \left( \begin{array}{c} 2(k-j) \\ i-j \end{array} \right) \frac{2j}{j+1} = \left( \begin{array}{c} k \\ \left\lfloor \frac{i}{2} \right\rfloor \end{array} \right) \left( \begin{array}{c} k \\ \left\lceil \frac{i}{2} \right\rceil \end{array} \right).
\]

Using Gosper algorithm and the Zeilberger method (see e.g.42), we find that the sequence \(\mathcal{J}_{k,i}\) is completely determined by the following three terms relation:

\[-(2k+1-i)(2k-i)\mathcal{J}_{k,i} + (2+2i-2k)\mathcal{J}_{k,i+1} + (i+3)(i+2)\mathcal{J}_{k,i+2} = 0, \quad (25)\]

and the initial conditions

\[
\mathcal{J}_{k,0} = 1, \quad \mathcal{J}_{k,1} = k.
\]

A straightforward calculation shows that the numbers \(\left( \begin{array}{c} k \\ \left\lfloor \frac{i}{2} \right\rfloor \end{array} \right) \left( \begin{array}{c} k \\ \left\lceil \frac{i}{2} \right\rceil \end{array} \right)\) verify also eq (25) and (26). This concludes the proof. \(\square\)

Remark that the numbers \(\mathcal{J}_{k,i}\) have an interesting combinatorial interpretation since it is the number of Dyck paths of odd semi-length \(2k-1\) with \(i\) peaks (see e.g.33 and sequence A088855 in48).

**VI. CONCLUSION**

This work is the first step towards a combinatorial interpretation for the asymptotic behavior of the integrals

\[
\langle \lambda \rangle^\sharp_N := \langle x_1^{\lambda_1} \ldots x_N^{\lambda_N} \rangle^\sharp := \frac{\langle x_1^{\lambda_1} \ldots x_N^{\lambda_N} \rangle^c_{a,b}}{\langle 1 \rangle^c_{a,b}}
\]

for any partition \(\lambda\) where

\[
\langle f(x_1, \ldots, x_N) \rangle^c_{a,b} := \int_{[0,1]^N} f(x_1, \ldots, x_N) \prod_{i<j} |x_i - x_j|^{2c} \prod_i x_i^{a_i-1}(1 - x_1)^{b-1} dx_i.
\]

21
Selberg-like integrals

In particular, our goal is to find an algebraic proof of the following conjecture suggested by numerical evidences.

**Conjecture VI.1** One has:

\[
\lim_{N \to \infty} \langle \frac{1}{N^{\ell(\lambda)}} p_{\lambda}(x_1, \ldots, x_N) \rangle^2 = \prod_{i=1}^{\ell(\lambda)} \lim_{N \to \infty} \langle \frac{1}{N} p_{\lambda_i}(x_1, \ldots, x_N) \rangle^2,
\]

(28)

where \( p_{\lambda} \) stands for the product \( p_{\lambda_1} \cdots p_{\lambda_{\ell(\lambda)}} \).

The limitations of the method described in this article is that we need an explicit expression for the integrals. This quickly becomes tedious in the general case. For testing equality (28), we first obtain an expression of a product of power sums in terms of Jack polynomials, we implement an identity similar to (4) for the integral associated to each of Jack polynomials and finally we take the limit after simplifying the sum.

The link between conjecture VI.1 and the integral (27) is the following. First, two of the authors remarked in that integral (27) can be restated in terms of monomial symmetric functions

\[
\langle \lambda \rangle_N^2 = \frac{\langle m_{\lambda} \rangle_N^2}{P_{\lambda}(N)},
\]

where \( P_{\lambda}(N) \) is an explicit polynomial whose degree is exactly \( \ell(\lambda) \). Remarking that

\[
m_{\lambda} = \alpha_{\lambda} p_{\lambda} + \sum_{\ell(\mu) < \ell(\lambda)} \alpha_{\mu} p_{\mu},
\]

and assuming conjecture VI.1, we see that \( \lim_{N \to \infty} \langle \lambda \rangle_N^2 \) equals (up to an explicit multiplicative coefficient) \( \lim_{N \to \infty} \langle \frac{1}{N^{\ell(\lambda)}} p_{\lambda}(x_1, \ldots, x_N) \rangle^2 \).

**ACKNOWLEDGMENTS**

The authors are grateful to M. Novaes for fruitful discussions on the factorization conjecture. This paper is partially supported by the ANR project PhysComb, ANR-08-BLAN-0243-04.

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2 At the time of writing, we are aware of a proof, based on other principles and deferred to a forthcoming paper, which contains eq. 18 as a special case and allows us to relate it to the combinatorics of symmetric Dyck paths.
Selberg-like integrals

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Appendix A: Calculation of $I_k$ (corollary II.3) via spectral density

We sketch here the derivation of an alternative formula for $I_k$ (Corollary II.3) via spectral density of the Jacobi ensemble.

The spectral density of the Jacobi ensemble is the marginal of the joint density function

$$
$$
Selberg-like integrals

(integrand of the Selberg integral), defined as:

$$\rho_N(x_1; A, B) = N \int_0^1 dx_2 \cdots dx_N \prod_{j<k} |x_j - x_k|^2 \prod_{i=1}^N x_i^A (1 - x_i)^B$$  \hspace{1cm} (A1)$$

and it is normalized to \(N\), \(\int_0^1 dx_1 \rho_N(x_1; a, b) = N\).

Such density is known analytically for all \(N\) in terms of Jacobi polynomials \(P_j^{(A,B)}(z)\) as:

$$\rho_N(x; A, B) = x^A (1 - x)^B \sum_{j=0}^{N-1} c_j(A, B)\left[P_j^{(A,B)}(1 - 2x)\right]^2$$  \hspace{1cm} (A2)$$

where:

$$c_j(A, B) = \frac{(2j + A + B + 1) \Gamma(j + 1) \Gamma(j + A + B + 1)}{\Gamma(j + A + 1) \Gamma(j + B + 1)}$$  \hspace{1cm} (A3)$$

The quantity \(I_k\) in Corollary II.3 is a linear statistics on the eigenvalues of the Jacobi ensemble. As such, it can be computed as a 1-fold integral over the density:

$$I_k = \int_0^1 dx \, x^k \, \rho_N(x; A, B)$$  \hspace{1cm} (A4)$$

where \(A = a - 1, B = b - 1\).

The integral (A4) can be evaluated in terms of nested finite sums as:

$$I_k = \sum_{j=0}^{N-1} c_j(A, B) \sum_{m,\ell=0}^j d_{mj}(A, B) f_{mj}^{(k)}(A, B) g_{mj\ell}^{(k)}(A, B)$$  \hspace{1cm} (A5)$$

where:

$$d_{mj}(A, B) = \frac{(-j)_m (A+B+j+1)_m (A+m+1)_{j-m}}{m!}$$

$$f_{mj}^{(k)}(A, B) = \frac{\Gamma(A+j+1) \Gamma(A+k+m+1) \Gamma(B+1)}{j! \Gamma(A+1) \Gamma(A+B+k+m+2)}$$

$$g_{mj\ell}^{(k)}(A, B) = \frac{(-j)_\ell (j+A+B+1)_\ell (A+k+m+1)_\ell}{\ell! \Gamma(A+1) \ell! (A+B+k+m+2)_{\ell}}$$

where \((x)_n = \Gamma(x + n)/\Gamma(x)\) is a Pochhammer symbol.

The identity (A5) is obtained straightforwardly by first expanding one of the two Jacobi polynomials using the definition in 41 and then computing the remaining integral using formula 7.392.1 in 43. The equivalence between formula (A5) and corollary II.3 can then be proved by means of elementary but lengthy algebraic steps.