A New Method of Constructing Black Hole Solutions in Einstein and 5D Gravity

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Abstract

It is formulated a new 'anholonomic frame' method of constructing exact solutions of Einstein equations with off–diagonal metrics in 4D and 5D gravity. The previous approaches and results [1–4] are summarized and generalized as three theorems which state the conditions when two types of ansatz result in integrable gravitational field equations. There are constructed and analyzed different classes of anisotropic and/or warped vacuum 5D and 4D metrics describing ellipsoidal black holes with static anisotropic horizons and possible anisotropic gravitational polarizations and/or running constants. We conclude that warped metrics can be defined in 5D vacuum gravity without postulating any brane configurations with specific energy momentum tensors. Finally, the 5D and 4D anisotropic Einstein spaces with cosmological constant are investigated.

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I. INTRODUCTION

During the last three years large extra dimensions and brane worlds attract a lot of attention as possible new paradigms for gravity, particle physics and string/M–theory. As basic references there are considered Refs. [5], for string gravity papers, the Refs. [6], for extra dimension particle fields, and gravity phenomenology with effective Plank scale and [7], for the simplest and comprehenive models proposed by Randall and Sundrum (in brief, RS; one could also find in the same line some early works [8] as well to cite, for instance, [9] for further developments with supersymmetry, black hole solutions and cosmological scenario).

The new ideas are based on the assumption that our Universe is realized as a three dimensional (in brief, 3D) brane, modeling a 4D pseudo–Riemannian spacetime, embedded in the 5D anti–de Sitter ($AdS_5$) bulk spacetime. It was proved in the RS papers [7] that

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in such models the extra dimensions could be not compactified (being even infinite) if a nontrivial warped geometric configuration is defined. Some warped factors are essential for solving the mass hierarchy propoblem and localization of gravity which at low energies can ”bound” the matter fields on a 3D subspace. In general, the gravity may propagate in extra dimensions.

In connection to modern string and brane gravity it is very important to develop new methods of constructing exact solutions of gravitational field equations in the bulk of extra dimension spacetime and to develop new applications in particle physics, astrophysics and cosmology. This paper is devoted to elaboration of a such method and investigation of new classes of anisotropic black hole solutions.

In higher dimensional gravity much attention has been paid to the off–diagonal metrics beginning the Salam, Strathee and Peracci works [10] which showed that including off–diagonal components in higher dimensional metrics is equivalent to including $U(1), SU(2)$ and $SU(3)$ gauge fields. They considered a parametrization of metrics of type

$$g_{\alpha\beta} = \begin{pmatrix} g_{ij} + N_i^a N_j^b h_{ab} & N_j^v h_{ae} \\ N_i^e h_{be} & h_{ab} \end{pmatrix}$$

(1)

where the Greek indices run values 1, 2, ..., $n + m$, the Latin indices $i, j, k, ...$ from the middle of the alphabet run values 1, 2, ..., $n$ (usually, in Kaluza–Klein theories one put $n = 4$) and the Latin indices from the beginning of the alphabet, $a, b, c, ...$, run values $n + 1, n + 2, ..., n + m$ taken for extra dimensions. The local coordinates on higher dimensional spacetime are denoted $u^\alpha = (x^i, y^a)$ which defines respectively the local coordinate frame (basis), co–frame (co–basis, or dual basis)

$$\partial_\alpha = \frac{\partial}{\partial u^\alpha} = \left( \partial_i = \frac{\partial}{\partial x^i}, \partial_a = \frac{\partial}{\partial y^a} \right),$$

(2)

$$d^\alpha = du^\alpha = \left( d^i = dx^i, d^a = dy^a \right).$$

(3)

The coefficients $g_{ij} = g_{ij}(u^\alpha), h_{ab} = h_{ab}(u^\alpha)$ and $N_i^a = N_i^a(u^\alpha)$ should be defined by a solution of the Einstein equations (in some models of Kaluza–Klein gravity [11] one considers the Einstein–Yang–Mills fields) for extra dimension gravity.

The metric (1) can be rewritten in a block $(n \times n) \oplus (m \times m)$ form

$$g_{\alpha\beta} = \begin{pmatrix} g_{ij} & 0 \\ 0 & h_{ab} \end{pmatrix}$$

(4)

with respect to some anholonomic frames (N–elongated basis), co–frame (N–elongated co–basis),

$$\delta_\alpha = \frac{\delta}{\delta u^\alpha} = \left( \delta_i = \partial_i - N_i^b \partial_b, \delta_a = \partial_a \right),$$

(5)

$$\delta^\alpha = \delta u^\alpha = \left( \delta^i = dx^i, \delta^a = dy^a + N_i^a dx^i \right),$$

(6)

which satisfy the anholonomy relations

$$\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = w_{\alpha\beta}^\gamma \delta_\gamma.$$
with the anholonomy coefficients computed as
\[ w_{ij}^k = 0, w_{aj}^b = 0, w_{ab}^k = 0, w_{ij}^a = \delta_i^a N_j^a - \delta_j^a N_i^a, w_{ja}^b = -w_{aj}^b = \partial_a N_j^b. \] (7)

In Refs. [10] the coefficients \( N_i^a \) (hereafter, N–coefficients) were treated as some \( U(1) \), \( SU(2) \) or \( SU(3) \) gauge fields (depending on the extra dimension \( m \)). There are another classes of gravity models which are constructed on vector (or tangent) bundles generalizing the Finsler geometry [12]. In such approaches the set of functions \( N_i^a \) were stated to define a structure of nonlinear connection and the variables \( y^a \) were taken to parametrize fibers in some bundles. In the theory of locally anisotropic (super) strings and supergravity, and gauge generalizations of the so–called Finsler–Kaluza–Klein gravity the coefficients \( N_i^a \) were suggested to be found from some alternative string models in low energy limits or from gauge and spinor variants of gravitational field equations with anholonomic frames and generic local anisotropy [3].

The Salam, Strathee and Peracci [10] idea on a gauge field like status of the coefficients of off–diagonal metrics in extra dimension gravity was developed in a new fashion by applying the method of anholonomic frames with associated nonlinear connections just on the (pseudo) Riemannian spaces [1,2]. The approach allowed to construct new classes of solutions of Einstein’s equations in three (3D), four (4D) and five (5D) dimensions with generic local anisotropy (e.g. static black hole and cosmological solutions with ellipsoidal or toroidal symmetry, various soliton–dilaton 2D and 3D configurations in 4D gravity, and wormhole and flux tubes with anisotropic polarizations and/or running on the 5th coordinate constants with different extensions to backgrounds of rotation ellipsoids, elliptic cylinders, bipolar and toroidal symmetry and anisotropy).

Recently, it was shown in Refs. [4] that if we consider off–diagonal metrics which can be equivalently diagonalized with respect to corresponding anholonomic frames, the RS theories become substantially locally anisotropic with variations of constants on extra dimension coordinate or with anisotropic angular polarizations of effective 4D constants, induced by higher dimension and/or anholonomic gravitational interactions.

The basic idea on the application of the anholonomic frame method for constructing exact solutions of the Einstein equations is to define such N–coefficients when a given type of off–diagonal metric is diagonalized with respect to some anholonomic frames (5) and the Einstein equations, re–written in mixed holonomic and anholonomic variables, trasform into a system of partial differential equations with separation of variables which admit exact solutions. This approach differs from the usual tetradic method where the differential forms and frame bases are all 'pure' holonomic or 'pure" anholonomic. In our case the N–coefficients and associated N–elongated partial derivatives (5) are chosen as to be some undefined values which at the final step are fixed as to separate variables and satisfy the Einstein equations.

The first aim of this paper is to formulate three theorems (and to suggest the way of their proof) for two off–diagonal metric ansatz which admit anholonomic transforms resulting in a substantial simplification of the system of Einstein equations in 5D and 4D gravity. The second aim is to consider four applications of the anholonomic frame method in order to construct new classes of exact solutions describing ellipsoidal black holes with anisotropies and running of constants. We emphasize that is possible to define classes of warped on the extra dimension coordinate metrics which are exact solutions of 5D vacuum gravity. We
analyze basic physical properties of such solutions. We also investigate 5D spacetimes with anisotropy and cosmological constants.

We use the term 'locally anisotropic' spacetime (or 'anisotropic' spacetime) for a 5D (4D) pseudo-Riemannian spacetime provided with an anholonomic frame structure with mixed holonomic and anholonomic variables. The anisotropy of gravitational interactions is modeled by off–diagonal metrics, or, equivalently, by theirs diagonalized analogs given with respect to anholonomic frames.

The paper is organized as follow: In Sec. II we formulate three theorems for two types of off–diagonal metric ansatz, construct the corresponding exact solutions of 5D vacuum Einstein equations and illustrate the possibility of extension by introducing matter fields (the necessary geometric background and some proofs are presented in the Appendix). We also consider the conditions when the method generates 4D metrics. In Sec. III we construct two classes of 5D anisotropic black hole solutions with rotation ellipsoid horizon and consider subclasses and reparametization of such solutions in order to generate new ones. Sec. IV is devoted to 4D ellipsoidal black hole solutions. In Sec. V we extend the method for anisotropic 5D and 4D spacetimes with cosmological constant, formulate two theorems on basic properties of the system of field equations and theirs solutions, and give an example of 5D anisotropic black solution with cosmological constant. Finally, in Sec. VI, we conclude and discuss the obtained results.

II. OFF–DIAGONAL METRICS IN EXTRA DIMENSION GRAVITY

The bulk of solutions of 5D Einstein equations and their reductions to 4D (like the Schwarzschild solution and brane generalizations [13], metrics with cylindrical and toroidal symmetry [14], the Friedman–Robertson–Worlker metric and brane generalizations [15]) were constructed by using diagonal metrics and extensions to solutions with rotation, all given with respect to holonomic coordinate frames of references. This Section is devoted to a geometrical and nonlinear partial derivation equations formalism which deals with more general, generic off–diagonal metrics with respect to coordinate frames, and anholonomic frames. It summarizes and generalizes various particular cases and ansatz used for construction of exact solutions of the Einstein gravitational field equations in 3D, 4D and 5D gravity [1–4].

A. The first ansatz for vacuum Einstein equations

Let us consider a 5D pseudo–Riemannian spacetime provided with local coordinates \( u^\alpha = (x^i, y^4 = v, y^5) \), for \( i = 1, 2, 3 \). Our aim is to prove that a metric ansatz of type (1) can be diagonalized by some anholonomic transforms with the N–coefficients \( N^i_a = N^i_a (x^i, v) \) depending on variables \( (x^i, v) \) and to define the corresponding system of vacuum Einstein equations in the bulk. The exact solutions of the Einstein equations to be constructed will depend on the so–called holonomic variables \( x^i \) and on one anholonomic (equivalently, anisotropic) variable \( y^4 = v \). In our further considerations every coordinate from a set \( u^\alpha \) can be stated to be time like, 3D space like or extra dimensional.
For simplicity, the partial derivatives will be denoted like \( a^x = \partial a/\partial x^1 \), \( a^* = \partial a/\partial x^2 \), \( a' = \partial a/\partial x^3 \), \( a^* = \partial a/\partial v \).

We begin our approach by considering a 5D quadratic line element

\[
ds^2 = g_{\alpha\beta}(x^i, v) \, du^\alpha du^\beta
\]

with the metric coefficients \( g_{\alpha\beta} \) parametrized (with respect to the coordinate frame (3)) by an off–diagonal matrix (ansatz)

\[
\begin{bmatrix}
g_1 + w_1^2 h_4 + n_1^2 h_5 & w_1 w_2 h_4 + n_1 n_2 h_5 & w_1 w_3 h_4 + n_1 n_3 h_5 & w_1 h_4 & n_1 h_5 \\
w_1 w_2 h_4 + n_1 n_2 h_5 & g_2 + w_2^2 h_4 + n_2^2 h_5 & w_2 w_3 h_4 + n_2 n_3 h_5 & w_2 h_4 & n_2 h_5 \\
w_1 w_3 h_4 + n_1 n_3 h_5 & w_2 w_3 h_4 + n_2 n_3 h_5 & g_3 + w_3^2 h_4 + n_3^2 h_5 & w_3 h_4 & n_3 h_5 \\
w_1 h_4 & w_2 h_4 & w_3 h_4 & h_4 & 0 \\
n_1 h_5 & n_2 h_5 & n_3 h_5 & 0 & h_5
\end{bmatrix}
\]

where the coefficients are some necessary smoothly class functions of type:

\[
g_1 = \pm 1, \quad g_{2,3} = g_{2,3}(x^2, x^3), \quad h_{4,5} = h_{4,5}(x^i, v),
\]

\[
w_i = w_i(x^i, v), \quad n_i = n_i(x^i, v).
\]

**Lemma 1** The quadratic line element (8) with metric coefficients (9) can be diagonalized,

\[
\delta s^2 = [g_1(dx^1)^2 + g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\delta v)^2 + h_5(\delta y^5)^2],
\]

with respect to the anholonomic co–frame \((dx^i, \delta v, \delta y^5)\), where

\[
\delta v = dv + w_i dx^i \quad \text{and} \quad \delta y^5 = dy^5 + n_i dx^i
\]

which is dual to the frame \((\delta_i, \partial_i, \partial_5)\), where

\[
\delta_i = \partial_i + w_i \partial_4 + n_i \partial_5.
\]

In the Lemma 1 the \(N\)–coefficients from (3) and (4) are parametrized like \( N_i^4 = w_i \) and \( N_i^5 = n_i \).

The proof of the Lemma 1 is a trivial computation if we substitute the values of (11) into the quadratic line element (10). Re-writing the metric coefficients with respect to the coordinate basis (3) we obtain just the quadratic line element (8) with the ansatz (9).

In the Appendix A we outline the basic formulas from the geometry of anholonomic frames with mixed holonomic and anholonomic variables and associated nonlinear connections on (pseudo) Riemannian spaces.

Now we can formulate the

**Theorem 1** The nontrivial components of the 5D vacuum Einstein equations, \( R^\beta_\alpha = 0 \), (see (A4) in the Appendix) for the metric (11) given with respect to anholonomic frames (12) and (13) are written in a form with separation of variables:
where

\[ \alpha_i = \partial_i h_5^* - h_5^* \partial_i \ln \sqrt{|h_4 h_5|}, \beta = h_5^{**} - h_5^{*} \left[ \ln \sqrt{|h_4 h_5|} \right]^*, \gamma = 3h_5^{**}/2h_5 - h_4^*/h_4. \] (17)

Here the separation of variables means: 1) we can define a function \( g_2(x^2, x^3) \) for a given \( g_3(x^2, x^3) \), or inversely, to define a function \( g_2(x^2, x^3) \) for a given \( g_3(x^2, x^3) \), from equation (13); 2) we can define a function \( h_4(x^1, x^2, x^3, v) \) for a given \( h_5(x^1, x^2, x^3, v) \), or inversely, to define a function \( h_5(x^1, x^2, x^3, v) \) for a given \( h_4(x^1, x^2, x^3, v) \), from equation (14); 3-4) having the values of \( h_4 \) and \( h_5 \), we can compute the coefficients (17) which allow to solve the algebraic equations (13) and to integrate two times on \( v \) the equations (16) which allow to find respectively the coefficients \( w_i(x^k, v) \) and \( n_i(x^k, v) \).

The proof of Theorem 1 is a straightforward tensorial and differential calculus for the components of Ricci tensor (A8) as it is outlined in the Appendix A. We omit such cumbersome calculations in this paper.

B. The second ansatz for vacuum Einstein equations

We can consider a generalization of the constructions from the previous subsection by introducing a conformal factor \( \Omega(x^i, v) \) and additional deformations of the metric via coefficients \( \zeta_i(x^i, v) \) (indices with ‘hat’ take values like \( i = 1, 2, 3, 5 \)). The new metric is written like

\[ ds^2 = \Omega^2(x^i, v) \hat{g}_{\alpha\beta} \left( x^i, v \right) du^\alpha du^\beta, \] (18)

were the coefficients \( \hat{g}_{\alpha\beta} \) are parametrized by the ansatz

\[
\begin{bmatrix}
  g_1 + (w_1^2 + \zeta_1^2)h_4 + n_1^2 h_5 & (w_1 w_2 + \zeta_1 \zeta_2)h_4 + n_1 n_2 h_5 & (w_1 w_3 + \zeta_1 \zeta_3)h_4 + n_1 n_3 h_5 & (w_1 w_4 + \zeta_1 \zeta_4)h_4 + n_1 n_4 h_5 & (w_1 + \zeta_1)h_4 & n_1 h_5 \\
(w_1 w_2 + \zeta_1 \zeta_2)h_4 + n_1 n_2 h_5 & g_2 + (w_2^2 + \zeta_2^2)h_4 + n_2^2 h_5 & (w_2 w_3 + \zeta_2 \zeta_3)h_4 + n_2 n_3 h_5 & (w_2 w_4 + \zeta_2 \zeta_4)h_4 + n_2 n_4 h_5 & (w_2 + \zeta_2)h_4 & n_2 h_5 \\
(w_1 w_3 + \zeta_1 \zeta_3)h_4 + n_1 n_3 h_5 & (w_2 w_3 + \zeta_2 \zeta_3)h_4 + n_2 n_3 h_5 & g_3 + (w_3^2 + \zeta_3^2)h_4 + n_3^2 h_5 & (w_3 w_4 + \zeta_3 \zeta_4)h_4 + n_3 n_4 h_5 & (w_3 + \zeta_3)h_4 & n_3 h_5 \\
(w_1 + \zeta_1)h_4 & (w_2 + \zeta_2)h_4 & (w_3 + \zeta_3)h_4 & (w_4 + \zeta_4)h_4 & h_4 & 0 \\
(n_1 + \zeta_1)h_5 & (n_2 + \zeta_2)h_5 & (n_3 + \zeta_3)h_5 & (n_4 + \zeta_4)h_5 & 0 & h_5 + \zeta_5 h_4
\end{bmatrix}. \] (19)

Such 5D pseudo–Riemannian metrics are considered to have second order anisotropy (312). For trivial values \( \Omega = 1 \) and \( \zeta_i = 0 \), the squared line interval (18) transforms into (8).

Lemma 2 The quadratic line element (18) with metric coefficients (19) can be diagonalized,

\[ \delta s^2 = \Omega^2(x^i, v) [g_1(dx^1)^2 + g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\delta v)^2 + h_5(\delta y^5)^2], \] (20)
with respect to the anholonomic co-frame \((dx^i, \hat{\delta}v, \delta y^5)\), where

\[
\delta v = dv + (w_i + \zeta_i)dx^i + \zeta_5 dy^5 \quad \text{and} \quad \delta y^5 = dy^5 + n_idx^i
\]  

which is dual to the frame \((\hat{\delta}_i, \partial_4, \hat{\partial}_5)\), where

\[
\hat{\delta}_i = \partial_i - (w_i + \zeta_i)\partial_4 + n_i\partial_5, \quad \hat{\partial}_5 = \partial_5 - \zeta_5\partial_4.
\]  

In the Lemma 2 the \(N\)-coefficients from (2) and (5) are parametrized in the first order anisotropy (with three anholonomic, \(x^i\), and two anholonomic, \(y^4\) and \(y^5\), coordinates) like \(N_i^4 = w_i\) and \(N_i^5 = n_i\) and in the second order anisotropy (on the second 'shell', with four anholonomic, \((x^i, y^5)\), and one anholonomic, \(y^4\), coordinates) with \(N_i^5 = \zeta_i\), in this work we state, for simplicity, \(\zeta_i = 0\).

The Theorem 1 can be extended as to include the generalization to the second ansatz:

**Theorem 2** The nontrivial components of the 5D vacuum Einstein equations, \(R^\beta_\alpha = 0\), (see (A10) in the Appendix) for the metric (20) given with respect to anholonomic frames (21) and (22) are written in the same form as in the system (13)–(16) with the additional conditions that

\[
\hat{\delta}_i h_4 = 0 \quad \text{and} \quad \hat{\delta}_i \Omega = 0
\]  

and the values \(\zeta_i = (\zeta_i, \zeta_5 = 0)\) are found as to be a unique solution of (23); for instance, if \(\Omega^{q_1/q_2} = h_4\) \((q_1 \text{ and } q_2 \text{ are integers})\),

\[
\zeta_i \text{ satisfy the equations}
\]

\[
\partial_i \Omega - (w_i + \zeta_i)\Omega^* = 0.
\]  

The proof of Theorem 2 consists from a straightforward calculation of the components of the Ricci tensor (A11) as it is outlined in the Appendix. The simplest way is to use the calculus for Theorem 1 and then to compute deformations of the canonical d–connection (A1). Such deformations induce corresponding deformations of the Ricci tensor (A11). The condition that we have the same values of the Ricci tensor for the (9) and (19) results in equations (A3) and (A5) which are compatible, for instance, if \(\Omega^{q_1/q_2} = h_4\). There are also another possibilities to satisfy the condition (A3), for instance, if \(\Omega = \Omega_1 \Omega_2\), we can consider that \(h_4 = \Omega_1^{q_1/q_2} \Omega_2^{q_2/q_4}\) for some integers \(q_1, q_2, q_3\) and \(q_4\).

**C. General solutions**

The surprising result is that we are able to construct exact solutions of the 5D vacuum Einstein equations for both types of the ansatz (9) and (19):
Theorem 3 The system of second order nonlinear partial differential equations (13), (16) and (23) can be solved in general form if there are given some values of functions \( g_3(x^2, x^3), h_4(x^i, v) \) (or \( h_5(x^i, v) \)) and \( \Omega (x^i, v) \):

- The general solution of equation (13) can be written in the form
  \[
  \varpi = g_{[0]} \exp[a_2 x^2 (x^2, x^3) + a_3 x^3 (x^2, x^3)],
  \]
  were \( g_{[0]}, a_2 \) and \( a_3 \) are some constants and the functions \( x^{2,3} (x^2, x^3) \) define coordinate transforms \( x^{2,3} \rightarrow \bar{x}^{2,3} \) for which the 2D line element becomes conformally flat, i.e.
  \[
  g_2(x^2, x^3)(dx^2)^2 + g_3(x^2, x^3)(dx^3)^2 \rightarrow \varpi \left[ (d\bar{x}^2)^2 + \varepsilon (d\bar{x}^3)^2 \right].
  \]

- The equation (14) relates two functions \( h_4(x^i, v) \) and \( h_5(x^i, v) \). There are two possibilities:
  a) to compute
  \[
  \sqrt{|h_5|} = h_{5[1]}(x^i) + h_{5[2]}(x^i) \int \sqrt{|h_4(x^i, v)|} dv, \quad h_4^*(x^i, v) \neq 0;
  \]
  \[
  = h_{5[1]}(x^i) + h_{5[2]}(x^i) v, \quad h_4^*(x^i, v) = 0,
  \]
  for some functions \( h_{5[1,2]}(x^i) \) stated by boundary conditions;
  b) or, inversely, to compute \( h_4 \) for a given \( h_5(x^i, v) \), \( h_5^* \neq 0,
  \]
  \[
  \sqrt{|h_4|} = h_{[0]}(x^i) \left( \sqrt{|h_5(x^i, v)|} \right)^*,
  \]
  with \( h_{[0]}(x^i) \) given by boundary conditions.

- The exact solutions of (13) for \( \beta \neq 0 \) is
  \[
  w_k = \partial_k \ln[\sqrt{|h_4h_5|}/|h_5^*|]/\partial_v \ln[\sqrt{|h_4h_5|}/|h_5^*|],
  \]
  with \( \partial_v = \partial/\partial v \) and \( h_5^* \neq 0 \). If \( h_5^* = 0 \), or even \( h_5^* \neq 0 \) but \( \beta = 0 \), the coefficients \( w_k \) could be arbitrary functions on \( (x^i, v) \). For vacuum Einstein equations this is a degenerated case which imposes the the compatibility conditions \( \beta = \alpha_i = 0 \), which are satisfied, for instance, if the \( h_4 \) and \( h_5 \) are related as in the formula (22) but with \( h_{[0]}(x^i) = \text{const.} \).

- The exact solution of (16) is
  \[
  n_k = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int [h_4/(\sqrt{|h_5|})^3] dv, \quad h_5^* \neq 0;
  \]
  \[
  = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int h_4 dv, \quad h_5^* = 0;
  \]
  \[
  = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int [1/(\sqrt{|h_5|})^3] dv, \quad h_4^* = 0,
  \]
  for some functions \( n_{k[1,2]}(x^i) \) stated by boundary conditions.
The exact solution of (25) is given by some arbitrary functions \( \zeta_i = \zeta_i(x^i, v) \) if both \( \partial_i \Omega = 0 \) and \( \Omega^* = 0 \), we chose \( \zeta_i = 0 \) for \( \Omega = \text{const} \), and

\[
\zeta_i = -w_i + (\Omega^*)^{-1} \partial_i \Omega, \quad \Omega^* \neq 0,
\]

\[
= (\Omega^*)^{-1} \partial_i \Omega, \quad \Omega^* \neq 0, \text{ for vacuum solutions.}
\]

We note that a transform (27) is always possible for 2D metrics and the explicit form of solutions depends on chosen system of 2D coordinates and on the signature \( \epsilon = \pm 1 \). In the simplest case the equation (13) is solved by arbitrary two functions \( g_2(x^3) \) and \( g_3(x^2) \). The equation (14) is satisfied by arbitrary pairs of coefficients \( h_4(x^i, v) \) and \( h_{5|0}(x^i) \).

The proof of Theorem 3 is given in the Appendix B.

D. Consequences of Theorems 1–3

We consider three important consequences of the Lemmas and Theorems formulated in this Section:

Corollary 1 The non–trivial diagonal components of the Einstein tensor, \( G^\alpha_{\beta} = R^\alpha_{\beta} - \frac{1}{2} R \delta^\alpha_{\beta} \), for the metric (10), given with respect to anholonomic N–bases, are

\[
G^1_1 = - \left( R^2_2 + S^4_4 \right), G^2_2 = G^3_3 = - S^1_1, G^4_4 = G^5_5 = - R^2_2.
\]

(33)

So, the dynamics of the system is defined by two values \( R^2_2 \) and \( S^1_1 \). The rest of non–diagonal components of the Ricci (Einstein tensor) are compensated by fixing corresponding values of N–coefficients.

The formulas (33) are obtained following the relations for the Ricci tensor (13)–(16).

Corollary 2 We can extend the system of 5D vacuum Einstein equations (13) –(16) by introducing matter fields for which the energy–momentum tensor \( \Upsilon^\alpha_{\beta} \) given with respect to anholonomic frames satisfy the conditions

\[
\Upsilon^1_1 = \Upsilon^2_2 + \Upsilon^4_4, \Upsilon^2_2 = \Upsilon^3_3, \Upsilon^4_4 = \Upsilon^5_5.
\]

(34)

We note that, in general, the tensor \( \Upsilon^\alpha_{\beta} \) for the non–vacuum Einstein equations,

\[
R^\alpha_{\beta} - \frac{1}{2} g^\alpha_{\beta} R = \kappa \Upsilon^\alpha_{\beta},
\]

is not symmetric because with respect to anholonomic frames there are imposed constraints which makes non symmetric the Ricci and Einstein tensors (the symmetry conditions hold only with respect to holonomic, coordinate frames; for details see the Appendix and the formulas (A9)).

For simplicity, in our further investigations we shall consider only diagonal matter sources, given with respect to anholonomic frames, satisfying the conditions.
\[ \kappa \Upsilon_2^2 = \kappa \Upsilon_3^3 = \Upsilon_2, \kappa \Upsilon_4^4 = \kappa \Upsilon_5^5 = \Upsilon_4, \text{ and } \Upsilon_1 = \Upsilon_2 + \Upsilon_4, \] (35)

where \( \kappa \) is the gravitational coupling constant. In this case the equations (13) and (14) are respectively generalized to

\[ R_2^2 = R_3^3 = -\frac{1}{2g_2g_3} [g^{\bullet \bullet} - \frac{g^{\bullet \bullet}_2 g^{\bullet \bullet}_3}{2g_2} - \frac{(g_{2}^\prime)^2}{2g_3} + g_{2}^\prime - \frac{g_{2}^\prime g_{3}^\prime}{2g_2} - \frac{(g_{3}^\prime)^2}{2g_2}] = -\Upsilon_4 \] (36)

and

\[ S_4^4 = S_5^5 = -\frac{\beta}{2h_4 h_5} = -\Upsilon_2. \] (37)

**Corollary 3** The class of metrics (13) satisfying vacuum Einstein equations (13)–(16) and (25) contains as particular cases some solutions when the Schwarzschild potential \( \Phi = -M/(M_p^2 r) \), where \( M_p \) is the effective Planck mass on the brane, is modified to

\[ \Phi = -\frac{M \sigma_m}{M_p^2 r} + \frac{Q \sigma_q}{2r^2}, \]

where the ‘tidal charge’ parameter \( Q \) may be positive or negative.

As proofs of this corollary we can consider the Refs [4] where the possibility to modify anisotropically the Newton law via effective anisotropic masses \( M \sigma_m \), or by anisotropic effective 4D Plank constants, renormalized like \( \sigma_m/M_p^2 \), and with “effective” electric charge, \( Q \sigma_q \) was recently emphasized (see also the end of Section III in this paper). For diagonal metrics, in the locally isotropic limit, we put the effective polarizations \( \sigma_m = \sigma_q = 1 \).

**E. Reduction from 5D to 4D gravity**

The above presented results are for generic off–diagonal metrics of gravitational fields, anholonomic transforms and nonlinear field equations. Reductions to a lower dimensional theory are not trivial in such cases. We give a detailed analysis of this procedure.

The simplest way to construct a \( 5D \rightarrow 4D \) reduction for the ansatz (9) and (19) is to eliminate from formulas the variable \( x^1 \) and to consider a 4D space (parametrized by local coordinates \((x^2, x^3, v, y^5)\)) being trivially embedded into 5D space (parametrized by local coordinates \((x^1, x^2, x^3, v, y^5)\)) with \( g_{11} = \pm 1, g_{1\alpha} = 0, \alpha = 2, 3, 4, 5 \) with further possible conformal and anholonomic transforms depending only on variables \((x^2, x^3, v)\). We admit that the 4D metric \( g_{\alpha \beta} \) could be of arbitrary signature. In order to emphasize that some coordinates are stated just for a such 4D space we underline the Greek indices, \( \alpha, \beta, \ldots \) and the Latin indices from the meadle of alphabet, \( i, j, \ldots = 2, 3 \), where \( u^\alpha = (x^2, y^3, y^4, y^5) \).

In result, the analogs of Lemmas 1 and 2, Theorems 1-3 and Corollaries 1-3 can be reformulated for 4D gravity with mixed holonomic–anholonomic variables. We outline here the most important properties of a such reduction.
• The line element (8) with ansatz (9) and the line element (8) with (19) are respectively transformed on 4D space to the values:

The first type 4D quadratic line element is taken

\[ ds^2 = g_{\alpha\beta} (x^\alpha, v) \, du^\alpha du^\beta \]  

with the metric coefficients \( g_{\alpha\beta} \) parametrized (with respect to the coordinate frame (3) in 4D) by an off–diagonal matrix (ansatz)

\[
\begin{bmatrix}
  g_2 + w_2^2 h_4 + n_2^2 h_5 & w_2 w_3 h_4 + n_2 n_3 h_5 & w_2 h_4 & n_2 h_5 \\
  w_2 w_3 h_4 + n_2 n_3 h_5 & g_3 + w_3^2 h_4 + n_3^2 h_5 & w_3 h_4 & n_3 h_5 \\
  w_2 h_4 & w_3 h_4 & h_4 & 0 \\
  n_2 h_5 & n_3 h_5 & 0 & h_5 \\
\end{bmatrix},
\]  

(39)

where the coefficients are some necessary smoothly class functions of type:

\[
\begin{align*}
g_{2,3} &= g_{2,3} (x^2, x^3), \quad h_{4,5} = h_{4,5} (x^4, v), \\
w_2 &= w_2 (x^2, v), \quad n_2 = n_2 (x^4, v); \quad i, k = 2, 3.
\end{align*}
\]

The anholonomically and conformally transformed 4D line element is

\[ ds^2 = \Omega^2 (x^\xi, v) \hat{g}_{\alpha\beta} (x^\xi, v) \, du^\alpha du^\beta, \]  

(40)

were the coefficients \( \hat{g}_{\alpha\beta} \) are parametrized by the ansatz

\[
\begin{bmatrix}
  g_2 + (w_2^2 + \zeta_2^2) h_4 + n_2^2 h_5 & (w_2 w_3 + \zeta_2 \zeta_3) h_4 + n_2 n_3 h_5 & (w_2 + \zeta_2) h_4 & n_2 h_5 \\
  (w_2 w_3 + \zeta_2 \zeta_3) h_4 + n_2 n_3 h_5 & g_3 + (w_3^2 + \zeta_3^2) h_4 + n_3^2 h_5 & (w_3 + \zeta_3) h_4 & n_3 h_5 \\
  (w_2 + \zeta_2) h_4 & (w_3 + \zeta_3) h_4 & h_4 & 0 \\
  n_2 h_5 & n_3 h_5 & 0 & h_5 + \zeta_5 h_4 \\
\end{bmatrix},
\]  

(41)

where \( \zeta_i = \zeta_i (x^\xi, v) \) and we shall restrict our considerations for \( \zeta_5 = 0 \).

• In the 4D analog of Lemma 1 we have

\[ \delta s^2 = [g_2 (dx^2)^2 + g_3 (dx^3)^2 + h_4 (\delta v)^2 + h_5 (\delta y^5)^2], \]  

(42)

with respect to the anholonomic co–frame \((dx^\xi, \delta v, \delta y^5)\), where

\[ \delta v = dv + w_2 dx^\xi \quad \text{and} \quad \delta y^5 = dy^5 + n_2 dx^\xi \]  

(43)

which is dual to the frame \((\delta^\xi, \partial_4, \partial_5)\), where

\[ \delta^\xi = \partial^\xi + w_2 \partial_4 + n_2 \partial_5. \]  

(44)
In the conditions of the 4D variant of Theorem 1 we have the same equations (13)–(16) were we must put $h_4 = h_4(x^k, v)$ and $h_5 = h_5(x^k, v)$. As a consequence we have that $\alpha_i(x^k, v) \rightarrow \alpha_i(x^k, v), \beta = \beta(x^k, v)$ and $\gamma = \gamma(x^k, v)$ which result that $w_i = w_i(x^k, v)$ and $n_i = n_i(x^k, v)$.

The respective formulas from Lemma 2, for $\zeta_5 = 0$, transform into

$$\delta s^2 = \Omega^2(x^i, v)[g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\delta v)^2 + h_5(\delta y^5)^2],$$

with respect to the anholonomic co-frame $(dx^i, \delta v, \delta y^5)$, where

$$\delta v = dv + (w_i + \zeta_i)dx^i$$

and

$$\delta y^5 = dy^5 + n_idx^i$$

which is dual to the frame $(\delta_i, \partial_4, \partial_5)$, where

$$\delta_i = \partial_i - (w_i + \zeta_i)\partial_4 + n_i\partial_5, \delta_5 = \partial_5.$$  

The formulas (23) and (25) from Theorem 2 must be modified into a 4D form

$$\hat{\delta}_i h_4 = 0$$

and

$$\hat{\delta}_i \Omega = 0$$

and the values $\zeta_i = (\zeta_i, \zeta_5 = 0)$ are found as to be a unique solution of (23); for instance, if

$$\Omega^{q_1/q_2} = h_4 (q_1 \text{ and } q_2 \text{ are integers}),$$

$\zeta_i$ satisfy the equations

$$\partial_i \Omega - (w_i + \zeta_i)\Omega^* = 0.$$  

One holds the same formulas (23)–(31) from the Theorem 3 on the general form of exact solutions with that difference that their 4D analogs are to be obtained by reductions of holonomic indices, $\mathbb{I} \rightarrow i$, and holonomic coordinates, $x^i \rightarrow x^\mathbb{I}$, i. e. in the 4D solutions there is not contained the variable $x^1$.

The formulae (33) for the nontrivial coefficients of the Einstein tensor in 4D stated by the Corollary 1 are written

$$G_2^2 = G_3^3 = -S_4^4, G_4^4 = G_5^5 = -R_2^2.$$  

For symmetries of the Einstein tensor (50) we can introduce a matter field source with a diagonal energy momentum tensor, like it is stated in the Corollary 2 by the conditions (34), which in 4D are transformed into

$$\Upsilon_2^2 = \Upsilon_3^3, \Upsilon_4^4 = \Upsilon_5^5.$$  

• In the conditions of the 4D variant of Theorem 1 we have the same equations (13)–(16) were we must put $h_4 = h_4(x^k, v)$ and $h_5 = h_5(x^k, v)$. As a consequence we have that $\alpha_i(x^k, v) \rightarrow \alpha_i(x^k, v), \beta = \beta(x^k, v)$ and $\gamma = \gamma(x^k, v)$ which result that $w_i = w_i(x^k, v)$ and $n_i = n_i(x^k, v)$.

• The respective formulas from Lemma 2, for $\zeta_5 = 0$, transform into

$$\delta s^2 = \Omega^2(x^i, v)[g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\delta v)^2 + h_5(\delta y^5)^2],$$

with respect to the anholonomic co-frame $(dx^i, \delta v, \delta y^5)$, where

$$\delta v = dv + (w_i + \zeta_i)dx^i$$

and

$$\delta y^5 = dy^5 + n_idx^i$$

which is dual to the frame $(\delta_i, \partial_4, \partial_5)$, where

$$\delta_i = \partial_i - (w_i + \zeta_i)\partial_4 + n_i\partial_5, \delta_5 = \partial_5.$$  

• The formulas (23) and (25) from Theorem 2 must be modified into a 4D form

$$\hat{\delta}_i h_4 = 0 \text{ and } \hat{\delta}_i \Omega = 0$$

and the values $\zeta_i = (\zeta_i, \zeta_5 = 0)$ are found as to be a unique solution of (23); for instance, if

$$\Omega^{q_1/q_2} = h_4 (q_1 \text{ and } q_2 \text{ are integers}),$$

$\zeta_i$ satisfy the equations

$$\partial_i \Omega - (w_i + \zeta_i)\Omega^* = 0.$$  

• One holds the same formulas (23)–(31) from the Theorem 3 on the general form of exact solutions with that difference that their 4D analogs are to be obtained by reductions of holonomic indices, $\mathbb{I} \rightarrow i$, and holonomic coordinates, $x^i \rightarrow x^\mathbb{I}$, i. e. in the 4D solutions there is not contained the variable $x^1$.

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• For symmetries of the Einstein tensor (50) we can introduce a matter field source with a diagonal energy momentum tensor, like it is stated in the Corollary 2 by the conditions (34), which in 4D are transformed into

$$\Upsilon_2^2 = \Upsilon_3^3, \Upsilon_4^4 = \Upsilon_5^5.$$  

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In 4D Einstein gravity we are not having violations of the Newton law as it was state in Corollary 3 for 5D. Nevertheless, off–diagonal and anholonomic frames can induce an anholonomic particle and field dynamics, for instance, with deformations of horizons of black holes, which can be modeled by an effective anisotropic renormalization of constants if some conditions are satisfied [1,2].

There were constructed and analyzed various classes of exact solutions of the Einstein equations (both in the vacuum, reducing to the system (13), (14), (15) and (16) and non–vacuum, reducing to (36), (37), (15) and (16), cases) in 3D, 4D and 5D gravity [1,4]. The aim of the next Sections III – V is to prove that such solutions contain warped factors which in the vacuum case are induced by a second order anisotropy. We shall analyze some classes of such exact solutions with running constants and/or their anisotropic polarizations induced from extra dimension gravitational interactions.

### III. 5D ELLIPSOIDAL BLACK HOLES

Our goal is to apply the anholonomic frame method as to construct such exact solutions of vacuum 5D Einstein equations as they will be static ones but, for instance, with ellipsoidal horizon for a diagonal metric given with respect to some well defined anholonomic frames. If such metrics are redefined with respect to usual coordinate frames, they are described by some particular cases of off–diagonal ansatz of type (9), or (19) which results in a very sophisticate form of the Einstein equations. That why it was not possible to construct such solutions in the past, before elaboration of the anholonomic frame method with associated nonlinear connection structure which allows to find exact solutions of the Einstein equations for very general off–diagonal metric ansatz.

By using anholonomic transforms the Schwarzschild and Reissner-Nördstrom solutions were generalized in anisotropic forms with deformed horizons, anisoropic polarizations and running constants both in the Einstein and extra dimension gravity (see Refs. [1,4]). It was shown that there are possible anisotropic solutions which preserve the local Lorentz symmetry. and that at large radial distances from the horizon the anisotropic configurations transform into the usual one with spherical symmetry. So, the solutions with anisotropic rotation ellipsoidal horizons do not contradict the well known Israel and Carter theorems [19] which were proved in the assumption of spherical symmetry at asymptotics. The vacuum metrics presented here differ from anisotropic black hole solutions investigated in Refs. [1,4].

#### A. The Schwarzschild solution in ellipsoidal coordinates

Let us consider the system of isotropic spherical coordinates \((\rho, \theta, \varphi)\), where the isotropic radial coordinate \(\rho\) is related with the usual radial coordinate \(r\) via the relation
\[
r = \rho \left(1 + r_g/4\rho\right)^2
\]
for \(r_g = 2G_{[4]}m_0/c^2\) being the 4D gravitational radius of a point particle of mass \(m_0\), \(G_{[4]} = 1/M_{P[4]}^2\) is the 4D Newton constant expressed via Plank mass \(M_{P[4]}\) (following modern string/brane theories, \(M_{P[4]}\) can be considered as a value induced from extra dimensions). We put the light speed constant \(c = 1\). This system of coordinates is considered for the so-called isotropic representation of the Schwarzschild solution [17].
\[ dS^2 = \left( \frac{\tilde{\rho} - 1}{\tilde{\rho} + 1} \right)^2 dt^2 - \rho_g^2 \left( \frac{\tilde{\rho} + 1}{\rho} \right)^4 \left( d\tilde{\rho}^2 + \tilde{\rho}^2 d\theta^2 + \tilde{\rho}^2 \sin^2 \theta d\phi^2 \right), \] (52)

where, for our further considerations, we re-scaled the isotropic radial coordinate as \( \tilde{\rho} = \rho / \rho_g \), with \( \rho_g = r_g / 4 \). The metric (52) is a vacuum static solution of 4D Einstein equations with spherical symmetry describing the gravitational field of a point particle of mass \( m_0 \).

We also introduce the rotation ellipsoid coordinates (in our case considered as alternatives to the isotropic radial coordinates) \( (u, \lambda, \varphi) \) with \( 0 \leq u < \infty, 0 \leq \lambda \leq \pi, 0 \leq \varphi \leq 2\pi \), where \( \sigma = \cosh u \geq 1 \) are related with the isotropic 3D Cartesian coordinates

\[ (\tilde{x} = \tilde{\rho} \sinh u \sin \lambda \cos \varphi, \tilde{y} = \tilde{\rho} \sinh u \sin \lambda \sin \varphi, \tilde{z} = \tilde{\rho} \cosh u \cos \lambda) \] (53)

and define an elongated rotation ellipsoid hypersurface

\[ \left( \tilde{x}^2 + \tilde{y}^2 \right) / (\sigma^2 - 1) + \tilde{z}^2 / \sigma^2 = \tilde{\rho}^2. \] (54)

with \( \sigma = \cosh u \). The 3D metric on a such hypersurface is

\[ dS^2_{(3D)} = g_{uu} du^2 + g_{\lambda\lambda} d\lambda^2 + g_{\varphi\varphi} d\varphi^2, \]

where

\[ g_{uu} = g_{\lambda\lambda} = \tilde{\rho}^2 \left( \sinh^2 u + \sin^2 \lambda \right), g_{\varphi\varphi} = \tilde{\rho}^2 \sinh^2 u \sin^2 \lambda. \]

We can relate the rotation ellipsoid coordinates \( (u, \lambda, \varphi) \) from (53) with the isotropic radial coordinates \( (\tilde{\rho}, \theta, \varphi) \), scaled by the constant \( \rho_g \), from (52) as

\[ \tilde{\rho} = 1, \sigma = \cosh u = \tilde{\rho} \]

and deform the Schwarzschild metric by introducing ellipsoidal coordinates and a new horizon defined by the condition that vanishing of the metric coefficient before \( dt^2 \) describe an elongated rotation ellipsoid hypersurface (54).

\[ dS^2_{(S)} = \left( \frac{\cosh u - 1}{\cosh u + 1} \right)^2 dt^2 - \rho_g^2 \left( \frac{\cosh u + 1}{\cosh u} \right)^4 (\sinh^2 u + \sin^2 \lambda) \]

\[ \times [du^2 + d\lambda^2 + \frac{\sinh^2 u}{\sinh^2 u + \sin^2 \lambda} \sin^2 \lambda d\varphi^2]. \] (55)

The ellipsoidally deformed metric (55) does not satisfy the vacuum Einstein equations, but at long distances from the horizon it transforms into the usual Schwarzschild solution (52).

For our further considerations we introduce two Classes (A and B) of 4D auxiliary pseudo–Riemannian metrics, also given in ellipsoid coordinates, being some conformal transforms of (53), like

\[ dS^2_{(S)} = \Omega_{A,B} (u, \lambda) dS^2_{(A,B)} \]

but which are not supposed to be solutions of the Einstein equations:
• Metric of Class A:

\[ ds^2_{(A)} = -du^2 - d\lambda^2 + a(u, \lambda)d\varphi^2 + b(u, \lambda)dt^2, \]  
\( (56) \)

where

\[ a(u, \lambda) = -\frac{\sinh^2 u \sin^2 \lambda}{\sinh^2 u + \sin^2 \lambda} \text{ and } b(u, \lambda) = -\frac{(\cosh u - 1)^2 \cosh^4 u}{\rho_g^2 (\cosh u + 1)^6 (\sinh^2 u + \sin^2 \lambda)^4}, \]

which results in the metric \((55)\) by multiplication on the conformal factor

\[ \Omega_A (u, \lambda) = \rho_g^2 (\cosh u + 1)^4 (\sinh^2 u + \sin^2 \lambda). \]  
\( (57) \)

• Metric of Class B:

\[ ds^2 = g(u, \lambda) (du^2 + d\lambda^2) - d\varphi^2 + f(u, \lambda)dt^2, \]  
\( (58) \)

where

\[ g(u, \lambda) = -\frac{\sinh^2 u + \sin^2 \lambda}{\sinh^2 u \sin^2 \lambda} \text{ and } f(u, \lambda) = \frac{(\cosh u - 1)^2 \cosh^4 u}{\rho_g^2 (\cosh u + 1)^6 \sinh^2 u \sin^2 \lambda}, \]

which results in the metric \((55)\) by multiplication on the conformal factor

\[ \Omega_B (u, \lambda) = \rho_g^2 (\cosh u + 1)^4 \sinh^2 u \sin^2 \lambda. \]

Now it is possible to generate exact solutions of the Einstein equations with rotation ellipsoid horizons and anisotropic polarizations and running of constants by performing corresponding anholonomic transforms as the solutions will have an horizon parametrized by a hypersurface like rotation ellipsoid and gravitational (extra dimensional or nonlinear 4D) renormalization of the constant \(\rho_g\) of the Schwarzschild solution, \(\rho_g \to \bar{\rho}_g = \omega \rho_g\), where the dependence of the function \(\omega\) on some holonomic or anholonomic coordinates depend on the type of anisotropy. For some solutions we can treat \(\omega\) as a factor modeling running of the gravitational constant, induced, induced from extra dimension, in another cases we may consider \(\omega\) as a nonlinear gravitational polarization which model some anisotropic distributions of masses and matter fields and/or anholonomic vacuum gravitational interactions.

B. Ellipsoidal 5D metrics of Class A

In this subsection we consider four classes of 5D vacuum solutions which are related to the metric of Class A \((56)\) and to the Schwarzschild metric in ellipsoidal coordinates \((55)\).

Let us parametrize the 5D coordinates as \((x^1 = \chi, x^2 = u, x^3 = \lambda, y^4 = v, y^5 = p)\), where the solutions with the so-called \(\varphi\)-anisotropy will be constructed for \((v = \varphi, p = t)\) and the solutions with \(t\)-anisotropy will be stated for \((v = t, p = \varphi)\) (in brief, we shall write respective \(\varphi\)-solutions and \(t\)-solutions).
1. Class A solutions with ansatz (9):

We take an off–diagonal metric ansatz of type (9) (equivalently, (8)) by representing

\[ g_1 = \pm 1, g_2 = -1, g_3 = -1, h_4 = \eta_4(x^i, v) h_{4(0)}(x^i) \text{ and } h_5 = \eta_5(x^i, v) h_{5(0)}(x^i), \]

where \( \eta_4,5(x^i, v) \) are corresponding ”gravitational renormalizations” of the metric coefficients \( h_{4,5(0)}(x^i) \). For \( \varphi \)-solutions we state \( h_{4(0)} = a(u, \lambda) \) and \( h_{5(0)} = b(u, \lambda) \) (inversely, for \( t \)-solutions, \( h_{4(0)} = b(u, \lambda) \) and \( h_{5(0)} = a(u, \lambda) \)).

Next we consider a renormalized gravitational 'constant' \( \overline{\rho}_g = \omega \rho_g \), were for \( \varphi \)-solutions the receptivity \( \omega = \omega(x^i, v) \) is included in the gravitational polarization \( \eta_5 \) as \( \eta_5 = [\omega(x^i, \varphi)]^{-2} \), or for \( t \)-solutions is included in \( \eta_4 \), when \( \eta_4 = [\omega(x^i, t)]^{-2} \). We can construct an exact solution of the 5D vacuum Einstein equations if, for explicit dependencies on anisotropic coordinate, the metric coefficients \( h_4 \) and \( h_5 \) are related by formula (29) with \( h_{4(0)}(x^i) = h_{5(0)} = const \) (see the Theorem 3, with statements on formulas (29) and (30)), which in its turn imposes a corresponding relation between \( \eta_4 \) and \( \eta_5 \),

\[ \eta_4 h_{4(0)}(x^i) = h_{5(0)}^2 h_{5(0)}(x^i) \left( \left( \sqrt{|\eta_5|} \right)^* \right)^2. \]

In result, we express the polarizations \( \eta_4 \) and \( \eta_5 \) via the value of receptivity \( \omega \),

\[ \eta_4(\chi, u, \lambda, \varphi) = h_{4(0)}^2 b(u, \lambda) \left\{ \left[ \omega^{-1}(\chi, u, \lambda, \varphi) \right]^* \right\}^2, \eta_5(\chi, u, \lambda, \varphi) = \omega^{-2}(\chi, u, \lambda, \varphi), \] (59)

for \( \varphi \)-solutions , and

\[ \eta_4(\chi, u, \lambda, t) = \omega^{-2}(\chi, u, \lambda, t), \eta_5(\chi, u, \lambda, t) = h_{5(0)}^2 b(u, \lambda) \left[ \int dt \omega^{-1}(\chi, u, \lambda, t) \right]^2, \] (60)

for \( t \)-solutions, where \( a(u, \lambda) \) and \( b(u, \lambda) \) are those from (53).

For vacuum configurations, following the discussions of formula (30) in Theorem 3, we put \( u_i = 0 \). The next step is to find the values of \( n_i \) by introducing \( h_4 = \eta_4 h_{4(0)} \) and \( h_5 = \eta_5 h_{5(0)} \) into the formula (31), which, for convenience, is expressed via general coefficients \( \eta_4 \) and \( \eta_5 \), with the functions \( n_{k[2]}(x^i) \) redefined as to contain the values \( h_{4(0)}^2, a(u, \lambda) \) and \( b(u, \lambda) \)

\[ n_k = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int [\eta_4/|\eta_5|^3] dv, \eta_5^* \neq 0; \] (61)
\[ = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int \eta_4 dv, \eta_5^* = 0; \]
\[ = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int [1/|\eta_5|^3] dv, \eta_4^* = 0. \]

By introducing the formulas (59) for \( \varphi \)-solutions (or (60) for \( t \)-solutions) and fixing some boundary condition, in order to state the values of coefficients \( n_{k[1,2]}(x^i) \) we can express the ansatz components \( n_k(x^i, \varphi) \) as integrals of some functions of \( \omega(x^i, \varphi) \) and \( \partial_\varphi \omega(x^i, \varphi) \) (or, we can express the ansatz components \( n_k(x^i, t) \) as integrals of some functions of \( \omega(x^i, t) \) and \( \partial_t \omega(x^i, t) \)). We do not present an explicit form of such formulas because they depend on
the type of receptivity $\omega = \omega (x^i, v)$, which must be defined experimentally, or from some quantum models of gravity in the quasi classical limit. We preserved a general dependence on coordinates $x^i$ which reflect the fact that there is a freedom in fixing holonomic coordinates (for instance, on ellipsoidal hypersurface and its extensions to 4D and 5D spacetimes). For simplicity, we write that $n_i$ are some functionals of $\{x^i, \omega (x^i, v), \omega^* (x^i, v)\}$

$$n_i \{x, \omega, \omega^* \} = n_i \{x^i, \omega (x^i, v), \omega^* (x^i, v)\}.$$  

In conclusion, we constructed two exact solutions of the 5D vacuum Einstein equations, defined by the ansatz (9) with coordinates and coefficients stated by the data:

$$\varphi\text{-solutions} : (x^1 = \chi, x^2 = u, x^3 = \lambda, y^4 = v = \varphi, y^5 = p = t), g_1 = \pm 1,$$
$$g_2 = -1, g_3 = -1, h_{4(0)} = a(u, \lambda), h_{5(0)} = b(u, \lambda), \text{see } (56);$$
$$h_4 = \eta_4 (x^i, \varphi) h_{4(0)} (x^i), h_5 = \eta_5 (x^i, \varphi) h_{5(0)} (x^i),$$
$$\eta_4 = h_{5(0)}^{-1} (u, \lambda) \left\{ \left[ \omega^{-1} (x, u, \lambda, \varphi) \right]^* \right\}^2, \eta_5 = \omega^{-2} (x, u, \lambda, \varphi),$$
$$w_i = 0, n_i \{x, \omega, \omega^* \} = n_i \{x^i, \omega (x^i, \varphi), \omega^* (x^i, \varphi)\}. \quad (62)$$

and

$$t\text{-solutions} : (x^1 = \chi, x^2 = u, x^3 = \lambda, y^4 = v = t, y^5 = p = \varphi), g_1 = \pm 1,$$
$$g_2 = -1, g_3 = -1, h_{4(0)} = b(u, \lambda), h_{5(0)} = a(u, \lambda), \text{see } (56);$$
$$h_4 = \eta_4 (x^i, t) h_{4(0)} (x^i), h_5 = \eta_5 (x^i, t) h_{5(0)} (x^i),$$
$$\eta_4 = \omega^{-2} (x, u, \lambda, t), \eta_5 = h_{5(0)}^{-2} (u, \lambda) \left\{ \int dt \omega^{-1} (x, u, \lambda, t) \right\}^2,$$
$$w_i = 0, n_i \{x, \omega, \omega^* \} = n_i \{x^i, \omega (x^i, t), \omega^* (x^i, t)\}. \quad (63)$$

Both types of solutions have a horizon parametrized by a rotation ellipsoid hypersurface (as the condition of vanishing of the "time" metric coefficient states, i. e. when the function $b(u, \lambda) = 0$). These solutions are generically anholonomic (anisotropic) because in the locally isotropic limit, when $\eta_4, \eta_5, \omega \rightarrow 1$ and $n_i \rightarrow 0$, they reduce to the coefficients of the metric (50). The last one is not an exact solution of 4D vacuum Einstein equations, but it is a conformal transform of the 4D Schwarzschild solution with a further trivial extension to 5D. With respect to the anholonomic frames adapted to the coefficients $n_i$ (see (11)), the obtained solutions have diagonal metric coefficients being very similar to the Schwarzschild metric (53) written in ellipsoidal coordinates. We can treat such solutions as black hole ones with a point particle mass put in one of the focuses of rotation ellipsoid hypersurface (for flattened ellipsoids the mass should be placed on the circle described by ellipse’s focuses under rotation; we omit such details in this work which were presented for 4D gravity in Ref. [4]).

The initial data for anholonomic frames and the chosen configuration of gravitational interactions in the bulk lead to deformed "ellipsoidal" horizons even for static configurations. The solutions admit anisotropic polarizations on ellipsoidal and angular coordinates $(u, \lambda)$ and running of constants on time $t$ and/or on extra dimension coordinate $\chi$. Such renormalizations of constants are defined by the nonlinear configuration of the 5D vacuum
gravitational field and depend on introduced receptivity function $\omega(x^i, v)$ which is to be considered an intrinsic characteristics of the 5D vacuum gravitational ‘ether’, emphasizing the possibility of nonlinear self-polarization of gravitational fields.

Finally, we note that the data (22) and (33) parametrize two very different classes of solutions. The first one is for static 5D vacuum black hole configurations with explicit dependence on anholonomic coordinate $\varphi$ and possible renormalizations on the rest of 3D space coordinates $u$ and $\lambda$ and on the 5th coordinate $\chi$. The second class of solutions are similar to the static solutions but with an emphasized anholonomic time running of constants and with possible anisotropic dependencies on coordinates $(u, \lambda, \chi)$.

2. Class A solutions with ansatz (13):

We construct here 5D vacuum $\varphi$- and $t$-solutions parametrized by an ansatz with conformal factor $\Omega(x^i, v)$ (see (19) and (20)). Let us consider conformal factors parametrized as $\Omega = \Omega_{[0]}(x^i)\Omega_{[1]}(x^i, v)$. We can generate from the data (22) (or (33)) an exact solution of vacuum Einstein equations if we are satisfied the conditions (24) and (22), i.e.

$$\Omega_{[0]}^{q_1/q_2}\Omega_{[1]}^{q_1/q_2} = \eta_4 h_{4(0)},$$

for some integers $q_1$ and $q_2$, and there are defined the second anisotropy coefficients

$$\zeta_i = \left( \partial_i \ln |\Omega_{[0]}| \right) \left( \ln |\Omega_{[1]}| \right)^* + \left( \Omega_{[1]}^* \right)^{-1} \partial_i \Omega_{[1]}.$$

So, taking a $\varphi$- or $t$-solution with corresponding values of $h_4 = \eta_4 h_{4(0)}$, for some $q_1$ and $q_2$, we obtain new exact solutions, called in brief, $\varphi_c$- or $t_c$-solutions (with the index “c” pointing to an ansatz with conformal factor), of the vacuum 5D Einstein equations given in explicit form by the data:

$\varphi_c$-solutions : $(x^1 = \chi, x^2 = u, x^3 = \lambda, x^4 = v = \varphi, x^5 = p = t), g_1 = \pm 1,$

$$g_2 = -1, g_3 = -1, h_{4(0)} = a(u, \lambda), h_{5(0)} = b(u, \lambda), \text{see (56)};$$

$$h_4 = \eta_4(x^i, \varphi)h_{4(0)}(x^i), h_5 = \eta_5(x^i, \varphi)h_{5(0)}(x^i),$$

$$\eta_4 = h_2^2 \frac{b(u, \lambda)}{a(u, \lambda)} \left[ \omega^{-1}(\chi, u, \lambda, \varphi) \right]^2, \eta_5 = \omega^{-2}(\chi, u, \lambda, \varphi),$$

$$w_i = 0, n_i \{x, \omega, \omega^*\} = n_i \{x^i, \omega(x^i, \varphi), \omega^*(x^i, \varphi)\}, \Omega = \Omega_{[0]}(x^i)\Omega_{[1]}(x^i, \varphi)$$

$$\zeta_i = \left( \partial_i \ln |\Omega_{[0]}| \right) \left( \ln |\Omega_{[1]}| \right)^* + \left( \Omega_{[1]}^* \right)^{-1} \partial_i \Omega_{[1]}, \eta_4 a = \Omega_{[0]}^{q_1/q_2}(x^i)\Omega_{[1]}^{q_1/q_2}(x^i, \varphi).$$

and

$t_c$-solutions : $(x^1 = \chi, x^2 = u, x^3 = \lambda, y^4 = v = t, y^5 = p = \varphi), g_1 = \pm 1,$

$$g_2 = -1, g_3 = -1, h_{4(0)} = b(u, \lambda), h_{5(0)} = a(u, \lambda), \text{see (56)};$$

$$h_4 = \eta_4(x^i, t)h_{4(0)}(x^i), h_5 = \eta_5(x^i, t)h_{5(0)}(x^i),$$

$$\eta_4 = \omega^{-2}(\chi, u, \lambda, t), \eta_5 = h_2^{-2} \frac{b(u, \lambda)}{a(u, \lambda)} \left[ \int dt \omega^{-1}(\chi, u, \lambda, t) \right]^2,$$

$$w_i = 0, n_i \{x, \omega, \omega^*\} = n_i \{x^i, \omega(x^i, t), \omega^*(x^i, t)\}, \Omega = \Omega_{[0]}(x^i)\Omega_{[1]}(x^i, t)$$

$$\zeta_i = \left( \partial_i \ln |\Omega_{[0]}| \right) \left( \ln |\Omega_{[1]}| \right)^* + \left( \Omega_{[1]}^* \right)^{-1} \partial_i \Omega_{[1]}, \eta_4 a = \Omega_{[0]}^{q_1/q_2}(x^i)\Omega_{[1]}^{q_1/q_2}(x^i, t).$$

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These solutions have two very interesting properties: 1) they admit a warped factor on the 5th coordinate, like $\Omega_{[0]}^{q_1/q_2} \sim \exp[-k|\chi|]$, which in our case is constructed for an anisotropic 5D vacuum gravitational configuration and not following a brane configuration like in Refs. [7, 2] we can impose such conditions on the receptivity $\omega(x^i, v)$ as to obtain in the locally isotropic limit just the Schwarzschild metric (55) trivially embedded into the 5D spacetime.

Let us analyze the second property in details. We have to chose the conformal factor as to be satisfied three conditions:

$$\Omega_{[0]}^{q_1/q_2} = \Omega_A, \Omega_{[1]}^{q_1/q_2} \eta_4 = 1, \Omega_{[1]}^{q_1/q_2} \eta_5 = 1,$$

were $\Omega_A$ is that from (24). The last two conditions are possible if

$$\eta_4^{-q_1/q_2} \eta_5 = 1,$$

which selects a specific form of receptivity $\omega(x^i, v)$. Putting into (67) the values $\eta_4$ and $\eta_5$ respectively from (64), or (65), we obtain some differential, or integral, relations of the unknown $\omega(x^i, v)$, which results that

$$\omega(x^i, \varphi) = (1 - q_1/q_2)^{-1-q_1/q_2} \left[h_{(0)}^{-1} \sqrt{|a/b|} \varphi + \omega_{[0]}(x^i)\right], \text{ for } \varphi_c\text{-solutions;}$$

$$\omega(x^i, t) = \left[(q_1/q_2 - 1) h_{(0)} \sqrt{|a/b|} t + \omega_{[1]}(x^i)\right]^{1-q_1/q_2}, \text{ for } t_c\text{-solutions},$$

for some arbitrary functions $\omega_{[0]}(x^i)$ and $\omega_{[1]}(x^i)$. So, recepтивities of particular form like (68) allow us to obtain in the locally isotropic limit just the Schwarzschild metric.

We conclude this subsection by the remark: the vacuum 5D metrics solving the Einstein equations describe a nonlinear gravitational dynamics which under some particular boundary conditions and parametrizations of metric’s coefficients can model anisotropic solutions transforming, in a corresponding locally isotropic limit, in some well known exact solutions like Schwarzschild, Reissner-Nördstrom, Taub NUT, various type of wormhole, solitonic and disk solutions (see details in Refs. [1, 2, 4]). Here we emphasize that, in general, an anisotropic solution (parametrized by an off–diagonal ansatz) could not have a locally isotropic limit to a diagonal metric with respect to some holonomic coordinate frames. By some boundary conditions and suggested type of horizons, singularities, symmetries and topological configuration such solutions model new classes of black hole/tori, wormholes and another type of solutions which defines a generic anholonomic gravitational field dynamics and has not locally isotropic limits.

C. Ellipsoidal 5D metrics of Class B

In this subsection we construct and analyze another two classes of 5D vacuum solutions which are related to the metric of Class B (58) and which can be reduced to the Schwarzschild metric in ellipsoidal coordinates (55) by corresponding parametrizations of receptivity $\omega(x^i, v)$. We emphasize that because the function $g(u, \lambda)$ from (58) is not a solution of equation (13) we introduce an auxiliary factor $\varpi (u, \lambda)$ for which $\varpi g$ becomes a
such solution, then we consider conformal factors parametrized as \( \Omega = \varpi^{-1} \Omega_{[2]}(x^i, v) \) and find solutions parametrized by the ansatz (13) and anholonomic metric interval (20).

Because the method of definition of such solutions is similar to that from previous subsection, in our further considerations we shall omit intermediate computations and present directly the data which select the respective configurations for \( \varphi_c \)-solutions and \( t_c \)-solutions.

The Class B of 5D solutions with conformal factor are parametrized by the data:

\[
\varphi_c \text{-solutions} : \begin{align*}
(x^1 = \chi, x^2 = u, x^3 = \lambda, y^4 = v = \varphi, y^5 = p = t), \quad g_1 = \pm 1,
g_2 = g_3 = \varpi(u, \lambda)\varpi(u, \lambda), h_{4(0)} = -\varpi(u, \lambda), h_{5(0)} = \varpi(u, \lambda)f(u, \lambda), \text{ see (58)};
\varpi = g^{-1}\varpi_0 \exp[a_2u + a_3\lambda], \quad \varpi_0, a_2, a_3 = \text{const}; \quad \text{see (26)}
h_4 = \eta_4(x^i, \varphi)h_{4(0)}(x^i), \quad h_5 = \eta_5(x^i, \varphi)h_{5(0)}(x^i),
\eta_4 = -h_{2(0)}^2 \exp(f(u, \lambda)\left[\left(\omega^{-1}(\chi, u, \lambda, \varphi)\right)^{\frac{1}{2}}\right]^2), \quad \eta_5 = \omega^{-1}(\chi, u, \lambda, \varphi), \quad (69)
\end{align*}
\]

\[
\lambda_i = \partial_i \ln|\varpi| \left(\ln|\Omega_{[2]}|\right)^* + \left(\Omega_{[2]}^*\right)^{-1} \partial_i \Omega_{[2]}, \quad \eta_4 = -\varpi^{-\frac{q_1+q_2}{q_2}}(x^i)\Omega_{[2]}^* \Omega_{[2]}^{q_1/q_2}(x^i, \varphi),
\]

and

\[
t_c \text{-solutions} : \begin{align*}
(x^1 = \chi, x^2 = u, x^3 = \lambda, y^4 = v = t, y^5 = p = \varphi), \quad g_1 = \pm 1,
g_2 = g_3 = \varpi(u, \lambda)\varpi(u, \lambda), h_{4(0)} = \varpi(u, \lambda)f(u, \lambda), h_{5(0)} = -\varpi(u, \lambda), \text{ see (58)};
\varpi = g^{-1}\varpi_0 \exp[a_2u + a_3\lambda], \quad \varpi_0, a_2, a_3 = \text{const}; \quad \text{see (26)}
h_4 = \eta_4(x^i, t)h_{4(0)}(x^i), \quad h_5 = \eta_5(x^i, t)h_{5(0)}(x^i),
\eta_4 = \omega^{-2}(\chi, u, \lambda, t), \quad \eta_5 = -h_{2(0)}^2 \exp(f(u, \lambda)\int dt \omega^{-1}(\chi, u, \lambda, t)^2), \quad (70)
\lambda_i = \partial_i \ln|\varpi| \left(\ln|\Omega_{[2]}|\right)^* + \left(\Omega_{[2]}^*\right)^{-1} \partial_i \Omega_{[2]}, \quad \eta_4 = -\varpi^{-\frac{q_1+q_2}{q_2}}(x^i)\Omega_{[2]}^* \Omega_{[2]}^{q_1/q_2}(x^i, t),
\end{align*}
\]

where the coefficients \( n_i \) can be found explicitly by introducing the corresponding values \( \eta_4 \) and \( \eta_5 \) in formula (63).

By a procedure similar to the solutions of Class A (see previous subsection) we can find the conditions when the solutions (63) and (70) will have in the locally anisotropic limit the Schwarzschild solutions, which impose corresponding parametrizations and dependencies on \( \Omega_{[2]}(x^i, v) \) and \( \omega(x^i, v) \) like (56) and (58). We omit these formulas because, in general, for aholonomic configurations and nonlinear solutions there are not hard arguments to prefer any holonomic limits of such off–diagonal metrics.

Finally, in this Section, we remark that for the considered classes of ellipsoidal black hole solutions the so–called \( tt \)-components of metric contain modifications of the Schwarzschild potential

\[
\Phi = -\frac{M}{M_{P[4]}^2 r} \quad \text{into} \quad \Phi = -\frac{M \omega(x^i, v)}{M_{P[4]}^2 r^2},
\]

where \( M_{P[4]} \) is the usual 4D Plank constant, and this is given with respect to the corresponding aholonomic frame of reference. The receptivity \( \omega(x^i, v) \) could model corrections
warped on extra dimension coordinate, $\chi$, which for our solutions are induced by anholonomic vacuum gravitational interactions in the bulk and not from a brane configuration in $AdS_5$ spacetime. In the vacuum case $k$ is a constant which characterizes the receptivity for bulk vacuum gravitational polarizations.

**IV. 4D ELLIPSOIDAL BLACK HOLES**

For the ansatz (39), without conformal factor, some classes of ellipsoidal solutions of 4D Einstein equations were constructed in Ref. [1] with further generalizations and applications to brane physics [4]. The goal of this Section is to consider some alternative variants, both with and without conformal factors and for different coordinate parametrizations and types of anisotropies. The bulk of 5D solutions from the previous Section are reduced into corresponding 4D ones if one eliminates the 5th coordinate $\chi$ from the formulas and the off–diagonal ansatz (39) and (41) are considered.

### A. Ellipsiodal 5D metrics of Class A

Let us parametrize the 4D coordinates as $(x^2, y^a) = (x^2 = u, x^3 = \lambda, y^4 = v, y^5 = p)$; for the $\varphi$–solutions we shall take $(v = \varphi, p = t)$ and for the solutions $t$–solutions we shall consider $(v = t, p = \varphi)$. Following the prescription from subsection IIE we can write down the data for solutions without proofs and computations.

1. **Class A solutions with ansat (39):**

   The off–diagonal metric ansatz of type (39) (equivalently, (8)) with the data

   \[
   g_2 = -1, g_3 = -1, h_{4(0)} = a(u, \lambda), h_{5(0)} = b(u, \lambda), \quad \text{see (56)};
   \]

   \[
   h_4 = \eta_4(u, \lambda, \varphi) h_{4(0)}(u, \lambda), h_5 = \eta_5(u, \lambda, \varphi) h_{5(0)}(u, \lambda),
   \]

   \[
   \eta_4 = \frac{b(u, \lambda)}{a(u, \lambda)} \left\{ \left[ \omega^{-1}(u, \lambda, \varphi) \right]^* \right\}^2, \quad \eta_5 = \omega^{-2}(u, \lambda, \varphi),
   \]

   \[
   w_i = 0, n_i \{x, \omega, \omega^*\} = n_i \{u, \lambda, \omega(u, \lambda, \varphi), \omega^*(u, \lambda, \varphi)\}. \quad (71)
   \]

   and

   \[
   t$–solutions : \quad (x^2 = u, x^3 = \lambda, y^4 = v = t, y^5 = p = \varphi)\]

   \[
   g_2 = -1, g_3 = -1, h_{4(0)} = b(u, \lambda), h_{5(0)} = a(u, \lambda), \quad \text{see (56)};
   \]

   \[
   h_4 = \eta_4(u, \lambda, t) h_{4(0)}(u, \lambda), h_5 = \eta_5(u, \lambda, t) h_{5(0)}(u, \lambda),
   \]

   \[
   \eta_4 = \omega^{-2}(u, \lambda, t), \quad \eta_5 = \frac{b(u, \lambda)}{a(u, \lambda)} \left[ \int dt \omega^{-1}(u, \lambda, t) \right]^2,
   \]

   \[
   w_i = 0, n_i \{x, \omega, \omega^*\} = n_i \{u, \lambda, \omega(u, \lambda, t), \omega^*(u, \lambda, t)\}. \quad (72)
   \]

where the $n_i$ are computed
These solutions have the same ellipsoidal symmetries and properties stated for their 5D analogs (52) and (53) with that difference that there are not any warped factors and extra dimension dependencies. We emphasize that the solutions defined by the formulas (71) and (72) do not result in a locally isotropic limit into an exact solution having diagonal coefficients with respect to some holonomic coordinate frames. The data introduced in this subsection are for generic 4D vacuum solutions of the Einstein equations parametrized by off–diagonal metrics. The renormalization of constants and metric coefficients have a 4D nonlinear vacuum gravitational origin and reflects a corresponding anholonomic dynamics.

2. Class A solutions with ansatz (44):

The 4D vacuum \( \varphi \)– and \( t \)–solutions parametrized by an ansatz with conformal factor \( \Omega(u, \lambda, v) \) (see (41) and (43)). Let us consider conformal factors parametrized as \( \Omega = \Omega_{[0]}(u, \lambda)\Omega_{[1]}(u, \lambda, v) \). The data are

\[
\begin{align*}
\varphi_c &- \text{solutions: } (x^2 = u, x^3 = \lambda, y^4 = v = \varphi, y^5 = p = t) \\
&= g_2 = -1, g_3 = -1, h_{4(0)} = a(u, \lambda), h_{5(0)} = b(u, \lambda), \text{see (56)}; \\
h_4 &= \eta_4(u, \lambda, \varphi)h_{4(0)}(u, \lambda), h_5 = \eta_5(u, \lambda, \varphi)h_{5(0)}(u, \lambda), \\
\eta_4 &= h_{2(0)}^2 \frac{b(u, \lambda)}{a(u, \lambda)} \left[ \omega^{-1}(u, \lambda, \varphi) \right]^* \omega^* \eta_5 = \omega^{-2}(u, \lambda, \varphi), \\
w_i &= 0, n_i = \{x, \omega, \omega^*\} = n_i \{u, \lambda, \omega(u, \lambda, \varphi), \omega^*(u, \lambda, \varphi)\}, \Omega = \Omega_{[0]}(u, \lambda)\Omega_{[1]}(u, \lambda, \varphi), \\
\zeta_i &= (\partial_i \ln |\Omega_{[0]}|) \left( \ln |\Omega_{[1]}| \right)^* + (\Omega_{[1]}^*)^{-1} \partial_i \Omega_{[1]}, \eta_4 = \Omega_{[0]}^{q_1/q_2}(u, \lambda)\Omega_{[1]}^{q_1/q_2}(u, \lambda, \varphi).
\end{align*}
\]

and

\[
\begin{align*}
t_c &- \text{solutions: } (x^2 = u, x^3 = \lambda, y^4 = v = t, y^5 = p = \varphi) \\
&= g_2 = -1, g_3 = -1, h_{4(0)} = b(u, \lambda), h_{5(0)} = a(u, \lambda), \text{see (56)}; \\
h_4 &= \eta_4(u, \lambda, t)h_{4(0)}(u, \lambda), h_5 = \eta_5(u, \lambda, t)h_{5(0)}(u, \lambda), \\
\eta_4 &= \omega^{-2}(u, \lambda, t), \eta_5 = h_{2(0)}^2 \frac{b(u, \lambda)}{a(u, \lambda)} \left[ \int dt \omega^{-1}(u, \lambda, t) \right]^2, \\
w_i &= 0, n_i = \{x, \omega, \omega^*\} = n_i \{u, \lambda, \omega(u, \lambda, t), \omega^*(u, \lambda, t)\}, \Omega = \Omega_{[0]}(u, \lambda)\Omega_{[1]}(u, \lambda, t) \\
\zeta_i &= (\partial_i \ln |\Omega_{[0]}|) \left( \ln |\Omega_{[1]}| \right)^* + (\Omega_{[1]}^*)^{-1} \partial_i \Omega_{[1]}, \eta_4 = \Omega_{[0]}^{q_1/q_2}(u, \lambda)\Omega_{[1]}^{q_1/q_2}(u, \lambda, t),
\end{align*}
\]

where the coefficients the \( n_i \) are given by the same formulas (43).

Contrary to the solutions (71) and (72) theirs conformal anholonomic transforms, respectively, (44) and (43), can be subjected to such parametrizations of the conformal factor and conditions on the receptivity \( \omega(u, \lambda, v) \) as to obtain in the locally isotropic limit.
just the Schwarzschild metric (53). These conditions are stated for $\Omega_{[0]}^{q_1/q_2} = \Omega_A$, $\Omega_{[1]}^{q_1/q_2} \eta_4 = 1$, $\Omega_{[1]}^{q_1/q_2} \eta_5 = 1$, where $\Omega_A$ is that from (24), which is possible if $\eta_4^{-q_1/q_2} \eta_5 = 1$, which selects a specific form of the receptivity $\omega$. Putting the values $\eta_4$ and $\eta_5$, respectively, from (74), or (75), we obtain some differential, or integral, relations of the unknown $\omega(x^i, v)$, which results in

$$
\omega(u, \lambda, \varphi) = (1 - q_1/q_2)^{-1-q_1/q_2} \left[ h_{(0)}^{-1} \sqrt{[a/b] [\varphi + \omega_{[0]}(u, \lambda)]} \right], \text{ for } \varphi_c\text{-solutions;}
$$

$$
\omega(u, \lambda, t) = \left[ (q_1/q_2 - 1) h_{(0)} \sqrt{[a/b] t + \omega_{[1]}(u, \lambda)} \right]^{1-q_1/q_2}, \text{ for } t_c\text{-solutions,}
$$

for some arbitrary functions $\omega_{[0]}(u, \lambda)$ and $\omega_{[1]}(u, \lambda)$. The obtained formulas for $\omega(u, \lambda, \varphi)$ and $\omega(u, \lambda, t)$ are 4D reductions of the formulas (56) and (58).

### B. Ellipsoidal 4D metrics of Class B

We construct another two classes of 4D vacuum solutions which are related to the metric of Class B (58) and which can be reduced to the Schwarzschild metric in ellipsoidal coordinates (53) by corresponding parametrizations of receptivity $\omega(u, \lambda, v)$. The solutions contain a 2D conformal factor $\varpi(u, \lambda)$ for which $\varpi g$ becomes a solution of (13) and a 4D conformal factor parametrized as $\Omega = \varpi^{-1} \Omega_{[2]}(u, \lambda, v)$ in order to set the constructions into the ansatz (11) and anholonomic metric interval (15).

The data selecting the 4D configurations for $\varphi_c$-solutions and $t_c$-solutions:

**$\varphi_c$-solutions**

$(x^2 = u, x^3 = \lambda, y^4 = v = \varphi, y^5 = p = t)$

$$
g_2 = g_3 = \varpi(u, \lambda) g(u, \lambda), h_{4(0)} = -\varpi(u, \lambda), h_{5(0)} = \varpi(u, \lambda) f(u, \lambda), \text{ see (58)};$$

$$
\varpi = g^{-1} \varpi_0 \exp[a_2 u + a_3 \lambda], \varpi_0, a_2, a_3 = \text{const}; \text{ see (26)}
$$

$$
h_4 = \eta_4(u, \lambda, \varphi) h_{4(0)}(u, \lambda), h_5 = \eta_5(u, \lambda, \varphi) h_{5(0)}(u, \lambda),$$

$$
\eta_4 = -h_{(0)}^{-2} f(u, \lambda) \left[ \left[ \omega^{-1}(u, \lambda, \varphi) \right]^{-1} \right], \eta_5 = \omega^{-2}(u, \lambda, \varphi),
$$

$$
w_i = 0, n_i \{ x, \omega, \omega^* \} = n_i \{ u, \lambda, \omega(u, \lambda, \varphi), \omega^*(u, \lambda, \varphi) \}, \Omega = \varpi^{-1} (u, \lambda) \Omega_{[2]}(u, \lambda, \varphi)$$

$$
\zeta_i = \partial_i \left( \ln |\varpi| \right) \left( \ln |\Omega_{[2]}| \right)^* + \left( \Omega_{[2]}^* \right)^{-1} \partial_i \Omega_{[2]}, \eta_4 = -\varpi^{-q_1/q_2}(u, \lambda) \Omega_{[2]}^{q_1/q_2}(u, \lambda, \varphi).
$$

and

**$t_c$-solutions**

$(x^2 = u, x^3 = \lambda, y^4 = v = t, y^5 = p = \varphi)$

$$
g_2 = g_3 = \varpi(u, \lambda) g(u, \lambda), h_{4(0)} = -\varpi(u, \lambda) f(u, \lambda), h_{5(0)} = -\varpi(u, \lambda), \text{ see (58)};$$

$$
\varpi = g^{-1} \varpi_0 \exp[a_2 u + a_3 \lambda], \varpi_0, a_2, a_3 = \text{const}, \text{ see (26)}
$$

$$
h_4 = \eta_4(u, \lambda, t) h_{4(0)}(x^i), h_5 = \eta_5(u, \lambda, t) h_{5(0)}(x^i),$$

$$
\eta_4 = \omega^{-2}(u, \lambda, t), \eta_5 = -h_{(0)}^{-2} f(u, \lambda) \left[ \int dt \omega^{-1}(u, \lambda, t) \right]^2,
$$

$$
w_i = 0, n_i \{ x, \omega, \omega^* \} = n_i \{ u, \lambda, \omega(u, \lambda, t), \omega^*(u, \lambda, t) \}, \Omega = \varpi^{-1} (u, \lambda) \Omega_{[2]}(u, \lambda, t)$$

$$
\zeta_i = \partial_i \left( \ln |\varpi| \right) \left( \ln |\Omega_{[2]}| \right)^* + \left( \Omega_{[2]}^* \right)^{-1} \partial_i \Omega_{[2]}, \eta_4 = -\varpi^{-q_1/q_2}(u, \lambda) \Omega_{[2]}^{q_1/q_2}(u, \lambda, t).
$$
where the coefficients \( n_i \) can be found explicitly by introducing the corresponding values \( \eta_4 \) and \( \eta_5 \) in formula (61).

For the 4D Class B solutions one can be imposed some conditions (see previous subsection) when the solutions (70) and (71) have in the locally anisotropic limit the Schwarzschild solution, which imposes some specific parametrizations and dependencies on \( \Omega_{[2]}(u, \lambda, v) \) and \( \omega(u, \lambda, v) \) like (66) and (68). We omit these considerations because for anholonomic configurations and nonlinear solutions there are not arguments to prefer any holonomic limits of such off–diagonal metrics.

We conclude this Section by noting that for the considered classes of ellipsoidal black hole 4D solutions the so–called \( t \)–component of metric contains modifications of the Schwarzschild potential

\[
\Phi = -\frac{M}{M_{P[4]}^2} \quad \text{into} \quad \Phi = -\frac{M\omega(u, \lambda, v)}{M_{P[4]}^2},
\]

where \( M_{P[4]} \) is the usual 4D Plank constant; the metric coefficients are given with respect to the corresponding anholonomic frame of reference. In 4D anholonomic gravity the receptivity \( \omega(u, \lambda, v) \) is considered to renormalize the mass constant. Such gravitational self–polarizations are induced by anholonomic vacuum gravitational interactions. They should be defined experimentally or computed following a model of quantum gravity.

V. THE COSMOLOGICAL CONSTANT AND ANISOTROPY

In this Section we analyze the general properties of anholonomic Einstein equations in 5D and 4D gravity with cosmological constant and construct a 5D exact solution with cosmological constant.

A. 4D and 5D Anholnomic Einstein spaces

There is a difference between locally anisotropic 4D and 5D gravity. The first theory admits an ”isotropic” 4D cosmological constant \( \Lambda_{[4]} = \Lambda \) even for anisotropic gravitational configurations. The second, 5D, theory admits extensions of vacuum anistoropic solutions to those with a cosmological constant only for anisotropic 5D sources parametrized like \( \Lambda_{[5]\alpha\beta} = (2\Lambda g_{11}, \Lambda g_{\alpha\beta}) \) (see the Corollary 4 below). We emphasize that the conclusions from this subsection refer to the two classes of ansatz (9) and (19).

The simplest way to consider a source into the 4D Einstein equations, both with or not anistoropy, is to consider a gravitational constant \( \Lambda \) and to write the field equations

\[
G_{\alpha\beta} = \Lambda_{[4]} \delta_{\alpha\beta} \tag{78}
\]

which means that we introduced a ”vacuum” energy–momentum tensor \( \kappa \gamma_{\alpha\beta} = \Lambda_{[4]} \delta_{\alpha\beta} \) which is diagonal with respect to anholonomic frames and the conditions (61) transforms into \( \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \kappa^{-1}\Lambda \). According to A. Z. Petrov [20] the spaces described by solutions of the Einstein equations

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are called the Einstein spaces. With respect to anisotropic frames we shall use the term anholonomic (equivalently, anisotropic) Einstein spaces.

In order to extend the equations (78) to 5D gravity we have to take into consideration the compatibility conditions for the energy–momentum tensors (34).

Corollary 4 We are able to satisfy the conditions of the Corollary 2 if we consider a 5D diagonal source $\Upsilon^\alpha_\beta = \{2\Lambda, \Upsilon^\alpha_2 = \Lambda \delta^\alpha_2\}$, for an anisotropic 5D cosmological constant source $(2\Lambda g_{11}, \Lambda g_{\alpha\beta})$. The 5D Einstein equations with anisotropic cosmological “constants”, for ansatz (2) are written in the form

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta}, \Lambda = \text{const}$$

These equations without coordinate $x^1$ and $g_{11}$ hold for the (39). We can extend the constructions for the ansatz with conformal factors, (19) and (41) by considering additional coefficients $\zeta_i$ satisfying the equations (25) and (49) for non vanishing values of $w_i$.

The proof follows from Corollaries 1 and 2 formulated respectively to 4D and 5D gravity (see formulas (50) and (51) and, correspondingly, (33) and (34)).

Theorem 4 The nontrivial components of the 5D Einstein equations with anisotropic cosmological constant, $R_{11} = 2\Lambda g_{11}$ and $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$, for the ansatz (19) and anholonomic metric (20), given with respect to anholonomic frames (21) and (22) are written in a form with separation of variables:

$$R_{33} = S_4^2 = -\Lambda.$$ (79)

The proof follows from Corollaries 1 and 2 formulated respectively to 4D and 5D gravity (see formulas (50) and (51) and, correspondingly, (33) and (34)).

Theorem 4 The nontrivial components of the 5D Einstein equations with anisotropic cosmological constant, $R_{11} = 2\Lambda g_{11}$ and $R_{\alpha\beta} = \Lambda g_{\alpha\beta}$, for the ansatz (19) and anholonomic metric (20), given with respect to anholonomic frames (21) and (22) are written in a form with separation of variables:

$$g_{33}^{**} - \frac{g_{22}g_{33}}{2g_2} - \frac{(g_3')^2}{2g_3} + g_2 - \frac{g_2g_3}{2g_3} - \frac{(g_2')^2}{2g_2} = 2\Lambda g_2g_3,$$ (80)

$$h_5^{**} - h_5^3[\ln |h_4h_5|^* = 2\Lambda h_4h_5,$$ (81)

$$\beta + \alpha_i = 0,$$ (82)

$$n_i^{**} + \gamma n_i^* = 0,$$ (83)

$$\partial_i \Omega - (\omega_i + \zeta_i)\Omega^* = 0.$$ (84)

where

$$\alpha_i = \partial_i h_5^3 - h_5^3 \partial_i \ln |h_4h_5|, \beta = 2\Lambda h_4h_5, \gamma = 3h_5^3/2h_5 - h_4^3/h_4.$$ (85)

The Theorem 4 is a generalization of the Theorem 2 for energy–momentum tensors induced by the an anisotropic 5D constant. The proof follows from (13)–(16) and (25), revised as to satisfy the formulas (36) and (37) with that substantial difference that $\beta \neq 0$ and in this case, in general, $w_i \neq 0$. We conclude that in the presence of a nonvanishing cosmological constant the equations (13) and (14) transform respectively into (80) and (81) which have a more general nonlinearity because of the $2\Lambda g_2g_3$ and $2\Lambda h_4h_5$ terms. For instance, the solutions with $g_2 = \text{const}$ and $g_3 = \text{const}$ (and $h_4 = \text{const}$ and $h_5 = \text{const}$) are not admitted. This makes more sophisticate the procedure of definition of $g_2$ for a given $g_3$ (or inversely, of definition of $g_3$ for a given $g_2$) from (80) [similarly of construction $h_4$ for a
given \( h_5 \) from (81) and inversely], nevertheless, the separation of variables is not affected by introduction of cosmological constant and there is a number of possibilities to generate new exact solutions.

The general properties of solutions of the system (80)-(84) are stated by the

**Theorem 5** The system of second order nonlinear partial differential equations (80)-(83) and (84) can be solved in general form if there are given some values of functions \( g_2(x^2,x^3) \) (or \( g_3(x^2,x^3) \)), \( h_4(x^i,v) \) (or \( h_5(x^i,v) \)) and \( \Omega(x^i,v) \):

- The general solution of equation (80) is to be found from the equation

  \[
  \varpi \varpi^{**} - (\varpi^*)^2 + \varpi \varpi'' - (\varpi')^2 = 2\Lambda \varpi^3. \tag{86}
  \]

  for a coordinate transform coordinate transforms \( x^{2,3} \to \tilde{x}^{2,3} \) for which

  \[
  g_2(u,\lambda)(du)^2 + g_3(u,\lambda)(d\lambda)^2 \to \varpi \left[(dx^2)^2 + \epsilon(dx^3)^2 \right], \epsilon = \pm 1
  \]

  and \( \varpi^* = \partial\varpi/\partial\tilde{x}^2 \) and \( \varpi' = \partial\varpi/\partial\tilde{x}^3 \).

- The equation (81) relates two functions \( h_4(x^i,v) \) and \( h_5(x^i,v) \) with \( h_5^* \neq 0 \). If the function \( h_5 \) is given we can find \( h_4 \) as a solution of

  \[
  h_4^* + 2\Lambda \tau (h_4)^2 + 2 \left( \tau^* - \tau \right) h_4 = 0, \tag{87}
  \]

  where \( \tau = h_5^*/2h_5 \).

- The exact solutions of (82) for \( \beta \neq 0 \) is

  \[
  w_k = -\alpha_k/\beta, \tag{88}
  \]

  \[
  = \partial v \ln[\sqrt{|h_4h_5|}/|h_5^*|] / \partial v \ln[\sqrt{|h_4h_5|}/|h_5^*|],
  \]

  for \( \partial v = \partial/\partial v \) and \( h_5^* \neq 0 \).

- The exact solution of (83) is

  \[
  n_k = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int |h_4/|h_5|^{3}|dv, \tag{89}
  \]

  \[
  = n_{k[1]}(x^i) + n_{k[2]}(x^i) \int [1/(|h_5|^{3}]dv, \ h_4^* = 0,
  \]

  for some functions \( n_{k[1,2]}(x^i) \) stated by boundary conditions.

- The exact solution of (25) is given by

  \[
  \zeta_i = -w_i + (\Omega^*)^{-1}\partial_i\Omega, \quad \Omega^* \neq 0, \tag{90}
  \]
We note that by a corresponding re-parametrizations of the conformal factor $\Omega(x^i, v)$ we can reduce (86) to

$$\varpi \varpi^{*} - (\varpi^*)^2 = 2\Lambda \varpi^3$$

(91)

which has an exact solution $\varpi = \varpi(\bar{x}^2)$ to be found from

$$(\varpi^*)^2 = \varpi^3 \left( C \varpi^{-1} + 4\Lambda \right), C = \text{const},$$

(or, inversely, to reduce to

$$\varpi \varpi'' - (\varpi')^2 = 2\Lambda \varpi^3$$

with exact solution $\varpi = \varpi(\bar{x}^3)$ found from

$$(\varpi')^2 = \varpi^3 \left( C \varpi^{-1} + 4\Lambda \right), C = \text{const}).$$

The inverse problem of definition of $h_5$ for a given $h_4$ can be solved in explicit form when $h_4^* = 0$, $h_4 = h_{4(0)}(x^i)$. In this case we have to solve

$$h_5^{**} + \frac{(h_5^*)^2}{2h_5^*} - 2\Lambda h_{4(0)}h_5 = 0,$$

(92)

which admits exact solutions by reduction to a Bernulli equation.

The proof of Theorem 5 is outlined in Appendix C.

The conditions of the Theorem 4 and 5 can be reduced to 4D anholonomic spacetimes with ”isotropic” cosmological constant $\Lambda$. To do this we have to eliminate dependencies on the coordinate $x^1$ and to consider the 4D ansatz without $g_{11}$ term as it was stated in the subsection II E.

B. 5D anisotropic black holes with cosmological constant

We give an example of generalization of anisotropic black hole solutions of Class A, constructed in the Section III, as they will contain the cosmological constant $\Lambda$; we extend the solutions given by the data (64).

Our new 5D $\varphi$– solution is parametrized by an ansatz with conformal factor $\Omega(x^i, v)$ (see (19) and (20)) as $\Omega = \varpi^{-1}(u)\Omega_{0}[0](x^i)\varpi^{-1}(u)\Omega_{1}[1](x^i, v)$. The factor $\varpi(u)$ is chosen to be a solution of (91). This conformal data must satisfy the conditions (24) and (32), i. e.

$$\varpi^{-q_1/q_2}\Omega_{0}^{q_1/q_2}\Omega_{1}^{q_1/q_2} = \eta_4 \varpi h_{4(0)}$$

for some integers $q_1$ and $q_2$, where $\eta_4$ is found as $h_4 = \eta_4 \varpi h_{4(0)}$ is a solution of equation (87). The factor $\Omega_{0}[0](x^i)$ could be chosen as to obtain in the locally isotropic limit and $\Lambda \rightarrow 0$ the Schwarzchild metric in ellipsoidal coordinates (55). Putting $h_5 = \eta_5 \varpi h_{5(0)}, \eta_5 h_{5(0)}$ in the ansatz for (64), for which we compute the value $\tau = h_{5}^*/2h_5$, we obtain from (87) an equation for $\eta_4$. 


$$\eta_4^* + \frac{2\Lambda}{\tau} \omega h_{4(0)}(\eta_4)^2 + 2 \left( \frac{\tau^*}{\tau} - \tau \right) \eta_4 = 0$$

which is a Bernulli equation \cite{18} and admit an exact solution, in general, in non explicit form, \( \eta_4 = \eta_4^{[\text{bern}]}(x^i, v, \Lambda, \omega, a, b) \), were we emphasize the functional dependencies on functions \( \omega, \omega, a, b \) and cosmological constant \( \Lambda \). Having defined \( \eta_4^{[\text{bern}]} \), \( \eta_5 \) and \( \omega \), we can compute the \( \alpha_i \), \( \beta_i \), and \( \gamma_i \)-coefficients, expressed as \( \alpha_i = \alpha_i^{[\text{bern}]}(x^i, v, \Lambda, \omega, a, b) \), \( \beta_i = \beta_i^{[\text{bern}]}(x^i, v, \Lambda, \omega, a, b) \) and \( \gamma_i = \gamma_i^{[\text{bern}]}(x^i, v, \Lambda, \omega, a, b) \), following the formulas \cite{55}.

The next step is to find

$$w_i = w_i^{[\text{bern}]}(x^i, v, \Lambda, \omega, a, b)$$

and

$$n_i = n_i^{[\text{bern}]}(x^i, v, \Lambda, \omega, a, b)$$

for the general solutions \cite{55} and \cite{56}.

At the final step we are able to compute the the second anisotropy coefficients

$$\zeta_i = -w_i^{[\text{bern}]} + (\partial_i \ln |\omega^{-1}\Omega_0|) \left( \ln |\Omega_1| \right)^* + (\Omega_1^*)^{-1} \partial_i \Omega_1,$$

which depends on an arbitrary function \( \Omega_0(u, \lambda) \). If we state \( \Omega_0(u, \lambda) = \Omega_A \), as for \( \Omega_A \) from \cite{55}, see similar details with respect to formulas \cite{56}, \cite{7} and \cite{55}.

The data for the exact solutions with cosmological constant for \( v = \varphi \) can be stated in the form

\begin{equation}
\varphi_c \text{–solutions : } (x^1 = \chi, x^2 = u, x^3 = \lambda, y^4 = v = \varphi, y^5 = p = t), g_1 = \pm 1,
\end{equation}

\begin{equation}
g_2 = \omega(u), g_3 = \omega(u), h_{4(0)} = a(u, \lambda), h_{5(0)} = b(u, \lambda), \text{see } (56) \text{ and } (91); \nonumber
\end{equation}

\begin{equation}
h_4 = \eta_4(x^i, \varphi) \omega(u) h_{4(0)}(x^i), h_5 = \eta_5(x^i, \varphi) \omega(u) h_{5(0)}(x^i), \nonumber
\end{equation}

\begin{equation}
\eta_4 = \eta_4^{[\text{bern}]}(x^i, v, \Lambda, \omega, a, b), \eta_5 = \omega^{-2}(\chi, u, \lambda, \varphi), \nonumber
\end{equation}

\begin{equation}
w_i = w_i^{[\text{bern}]}(x^i, v, \Lambda, \omega, a, b), n_i(x^i, \omega, \omega^*) = n_i^{[\text{bern}]}(x^i, v, \Lambda, \omega, a, b), \nonumber
\end{equation}

\begin{equation}
\Omega = \omega^{-1}(u) \Omega_0(x^i) \Omega_1(x^i, \varphi), \eta_4 a = \Omega_0^{n_4/q_4} \Omega_1^{n_4/q_4} = \Omega_1^{n_4/q_4} \nonumber
\end{equation}

\begin{equation}
\zeta_i = -w_i^{[\text{bern}]} + (\partial_i \ln |\omega^{-1}\Omega_0|) \left( \ln |\Omega_1| \right)^* + (\Omega_1^*)^{-1} \partial_i \Omega_1. \nonumber
\end{equation}

We note that a solution with \( v = t \) can be constructed as to generalize \cite{55} in order to contain \( \Lambda \). We can not present such data in explicit form because in this case we have to define \( \eta_5 \) by integrating an equation like \cite{51} for \( h_5 \), for a given \( h_4 \), with \( h_4^* \neq 0 \) which can not be integrated in explicit form.

The solution \cite{53} has has the same the two very interesting properties as the solution \cite{51}: 1) it admits a warped factor on the 5th coordinate, like \( \Omega_1^{n_4/q_4} \sim \exp[-k|\chi|] \), which in this case is constructed for an anisotropic 5D vacuum gravitational configuration with anisotropic cosmological constant and does not follow from a brane configuration like in Refs. \cite{7}; 2) we can impose such conditions on the receptivity \( \omega(x^i, \varphi) \) as to obtain in the locally isotropic limit just the Schwarzschild metric \cite{55} trivially embedded into the 5D spacetime (the procedure is the same as in the subsection IIIB).

Finally, we note that in a similar manner like in the Sections III and IV we can construct another classes of anisotropic black holes solutions in 5D and 4D spacetimes with cosmological constants, being of Class A or Class B, with anisotropic \( \varphi \)-coordinate, or anisotropic \( t \)-coordinate. We omit the explicit data which are some nonlinear anholonomic generalizations of those solutions.
VI. CONCLUSIONS

We formulated a new method of constructing exact solutions of Einstein equations with off–diagonal metrics in 4D and 5D gravity. We introduced anholonomic transforms which diagonalize metrics and simplify the system of gravitational field equations. The method works also for gravitational configurations with cosmological constants and for non–trivial matter sources. We constructed different classes of new exact solutions of the Einstein equations is 5D and 4D gravity which describe a generic anholonomic (anisotropic) dynamics modeled by off–diagonal metrics and anholonomic frames with mixed holonomic and anholonomic variables. They extend the class of exact solutions with linear extensions to the bulk 5D gravity \[21\].

We emphasized such exact solutions which can be associated to some black hole like configurations in 5D and 4D gravity. We consider that the constructed off–diagonal metrics define anisotropic black holes because they have a static horizon parametrized by a rotation ellipsoid hypersurface, they are singular in focuses of ellipsoid (or on the circle of focuses, for flattened ellipsoids) and they reduce in the locally anisotropic limit, with holonomic coordinates, to the Schwarzshild solution in ellipsoidal coordinates, or to some conformal transforms of the Schwarzshild metric.

The new classes of solutions admit variations of constants (in time and extra dimension coordinate) and anholonomic gravitational polarizations of masses which are induced by nonlinear gravitational interactions in the bulk of 5D gravity and by a constrained (anholonomic) dynamics of the fields in the 4D gravity. There are possible solutions with warped factors which are defined by some vacuum 5D gravitational interactions in the bulk and not by a specific brane configuration with energy–momentum tensor source. We emphasized anisotropies which in the effective 4D spacetime preserve the local Lorentz invariance but the method allows constructions with violation of local Lorentz symmetry like in Refs. \[22\]. In order to generate such solutions we should admit that the metric coefficients depends, for instance, anisotropically on extra dimension coordinate.

It should be noted that the anholonomic frame method deals with generic off–diagonal metrics and nonlinear systems of equations and allows to construct substantially nonlinear solutions. In general, such solutions could not have a locally isotropic limit with a holonomic analog. We can understand the physical properties of such solutions by analyzing both the metric coefficients stated with respect to an adapted anholonomic frame of reference and by a study of the coefficients defining such frames.

There is a subclass of static anisotropic black holes solutions, with static ellipsoidal horizons, which do not violate the well known Israel and Carter theorems \[19\] on spherical symmetry of solutions in asymptotically flat spacetimes. Those theorems were proved in the radial symmetry asymptotic limit and for holonomic coordinates. There is not a much difference between 3D static spherical and ellipsoidal horizons at long distances. In other turn, the statements of the mentioned theorems do not refers to generic off–diagonal gravitational metrics, anholonomic frames and anholonomic deformations of symmetries.

Finally, we note that the anholonomic frame method may have a number of applications in modern brane and string/M–theory gravity because it defines a general formalism of constructing exact solutions with off–diagonal metrics. It results in such prescriptions on anholonomic ”mappings” of some known locally isotropic solutions from a gravity/string...
theory that new types of anisotropic solutions are generated:

A vacuum, or non-vacuum, solution, and metrics conformally equivalent to a such solution, parametrized by a diagonal matrix given with respect to a holonomic (coordinate) base, contained in a trivial form of ansatz (9), or (12), can be generalized to an anisotropic solution with similar but anisotropically renormalized physical constants and diagonal metric coefficients given with respect to adapted anholonomic frames; the new anholonomic metric defines an exact solution of a simplified form of the Einstein equations (13)–(16) and (25); such solutions are parametrized by off–diagonal metrics if they are re–defined with respect to coordinate frames.

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APPENDIX A: ANHOLONOMIC FRAMES AND NONLINEAR CONNECTIONS

For convenience, we outline here the basic formulas for connections, curvatures and, induced by anholonomic frames, torsions on (pseudo) Riemannian spacetimes provided with N–coefficient bases (3) and (4). The N–coefficients define an associated nonlinear connection (in brief, N–connection) structure. On (pseudo)–Riemannian spacetimes the N–connection structure can be treated as a ”pure” anholonomic frame effect which is induced if we are dealing with mixed sets of holonomic–anholonomic basis vectors. When we are transferring our considerations only to coordinate frames (2) and (3) the N–connection coefficients are removed into both off–diagonal and diagonal components of the metric like in (9). In some cases the N–connection (anholonomic) structure is to be stated in a non–dynamical form by definition of some initial (boundary) conditions for the frame structure, following some prescribed symmetries of the gravitational–matter field interactions, or , in another cases, a subset of N–coefficients have to be treated as some dynamical variables defined as to satisfy the Einstein equations.

1. D–connections, d–torsions and d–curvatures

If a pseudo–Riemannian spacetime is enabled with a N–connection strucutre, the components of geometrical objects (for instance, linear connections and tensors) are distinguished into horizontal components (in brief h–components, labeled by indices like $i, j, k, ...$) and vertical components (in brief v–components, labeled by indices like $a, b, c, ..$). One call such objects, distinguished (d) by the N–connection structure, as d–tensors, d–connections, d–spinors and so on.
a. D–metrics and d–connections:

A metric of type (10), in general, with arbitrary coefficients $g_{ij}(x^k, y^a)$ and $h_{ab}(x^k, y^a)$ defined with respect to a N–elongated basis (6) is called a d–metric.

A linear connection $D_\delta \alpha \delta = \Gamma_\alpha ^{\beta \gamma}(x, y) \delta_\alpha$, associated to an operator of covariant derivation $D$, is compatible with a metric $g_{\alpha \beta}$ and N–connection structure on a 5D pseudo–Riemannian spacetime if $D_\alpha g_{\beta \gamma} = 0$. The linear d–connection is parametrized by irreducible h–v–components, $\Gamma_\alpha ^{\beta \gamma} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc})$, where

$$L^i_{jk} = \frac{1}{2} g^{in} (\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}),$$

$$L^a_{bk} = \partial_b N^a_k + \frac{1}{2} h^{ac} \left( \delta_k h_{bc} - h_{dc} \partial_b N^d_k - h_{db} \partial_c N^d_k \right),$$

$$C^i_{jc} = \frac{1}{2} g^{ik} \partial_c g_{jk}, \quad C^a_{bc} = \frac{1}{2} h^{ad} \left( \partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc} \right).$$

This defines a canonical linear connection (distinguished by a N–connection, in brief, the canonical d–connection) which is similar to the metric connection introduced by Christoffel symbols in the case of holonomic bases.

b. D–torsions and d–curvatures:

The anholonomic coefficients $W_\alpha \beta ^{\gamma}$ and N–elongated derivatives give nontrivial coefficients for the torsion tensor, $T(\delta_\gamma, \delta_\beta) = T^\alpha _{\beta \gamma} \delta_\alpha$, where

$$T^\alpha _{\beta \gamma} = \Gamma^\alpha _{\beta \gamma} - \Gamma^\alpha _{\gamma \beta} + w^\alpha _{\beta \gamma},$$

and for the curvature tensor, $R(\delta_\tau, \delta_\gamma) \delta_\beta = R^\alpha _{\beta \gamma \tau} \delta_\alpha$, where

$$R^\alpha _{\beta \gamma \tau} = \delta_\tau \Gamma^\alpha _{\beta \gamma} - \delta_\gamma \Gamma^\alpha _{\beta \tau} + \Gamma^\phi _{\beta \gamma} \Gamma^\alpha _{\phi \tau} - \Gamma^\phi _{\beta \tau} \Gamma^\alpha _{\phi \gamma} + \Gamma^\alpha _{\beta \phi} w^\phi _{\gamma \tau}.$$ (A3)

We emphasize that the torsion tensor on (pseudo) Riemannian spacetimes is induced by anholonomic frames, whereas its components vanish with respect to holonomic frames. All tensors are distinguished (d) by the N–connection structure into irreducible h–v–components, and are called d–tensors. For instance, the torsion, d–tensor has the following irreducible, nonvanishing, h–v–components, $T^\alpha _{\beta \gamma} = \{ T^i_{jk}, C^i_{ja}, S^a_{bc}, T^a_{ji}, T^a_{bi} \}$, where

$$T^i_{jk} = T^i_{jk} = L^i_{jk} - L^i_{kj}, \quad T^j_{ia} = C^i_{ja}, \quad T^i_{aj} = -C^i_{ja},$$

$$T^i_{ja} = 0, \quad T^a_{bc} = S^a_{bc} = C^a_{bc} - C^a_{cb},$$

$$T^a_{ij} = -\Omega^a _{ij}, \quad T^a_{bi} = \partial_b N^a_i - L^a_{bi}, \quad T^a_{ib} = -T^a_{bi}.$$ (A4)

(the d–torsion is computed by substituting the h–v–components of the canonical d–connection (A1) and anholonomic coefficients (7) into the formula for the torsion coefficients (A2)), where

$$\Omega^a _{ij} = \delta_j N^a_i - \delta_i N^a_j.$$
is called the N–connection curvature (N–curvature).

The curvature d-tensor has the following irreducible, non-vanishing, h–v–components
\[ R^\alpha_{\beta \gamma \tau} = \{ R^i_{h,jk}, R^i_{k,jh}, P^i_{j,k\alpha}, P^i_{\beta,k} a, S^i_{j,bc}, S^i_{b,cd} \}, \]
where
\[ R^i_{h,jk} = \delta_k L^i_{hj} - \delta_j L^i_{hk} + L^m_{hj} L^i_{mk} - L^m_{hk} L^i_{mj} - C^i_{hka} \Omega^a_{jk}, \]  
\[ R^a_{b,jk} = \delta_k L^a_{bj} - \delta_j L^a_{bk} + L^c_{bj} L^a_{ck} - L^c_{bk} L^a_{cj} - C^a_{b,c} \Omega^c_{jk}, \]  
\[ P^i_{j,k\alpha} = \delta_{\alpha} L^i_{jk} + C^i_{jk} T^\delta_{k\alpha} - (\delta_k C^i_{ja} + L^i_{jk} C^c_{j \alpha} - L^i_{ja} C^c_{j \alpha} - L^i_{jk} C^c_{j \alpha}), \]  
\[ P^i_{\beta,k} a = \delta_{a} L^i_{\beta k} + C^i_{\beta d} T^\delta_{d k} - (\delta_k C^i_{ba} + L^e_{d k} C^c_{d} - L^e_{d k} C^c_{d} - L^e_{d k} C^c_{d}), \]  
\[ S^i_{j,bc} = \delta_{c} C^i_{j b} - \delta_{b} C^i_{j c} + C^h_{jbc} C^i_{c} - C^h_{jbc} C^i_{c}, \]  
\[ S^i_{b,cd} = \delta_{d} C^i_{b,c} - \delta_{c} C^i_{b,d} + C^a_{b,cd} C^i_{a} - C^a_{b,cd} C^i_{a}. \]

(the d–curvature components are computed in a similar fashion by using the formula for curvature coefficients (A3)).

2. Einstein equations with holonomic–anholonomic variables

In this subsection we write and analyze the Einstein equations on 5D (pseudo) Riemannian spacetimes provided with anholonomic frame structures and associated N–connections.

a. Einstein equations with matter sources

The Ricci tensor \( R^{i}_{\beta \gamma} = R^{a}_{\beta \gamma \alpha} \) has the d–components
\[ R_{ij} = R_{i,j}, \quad R_{ia} = -2 P_{ia} = -P_{i,ka}, \]  
\[ R_{ai} = P_{a,i} = P_{a,ib}, \quad R_{ab} = S_{a,cd}. \]

In general, since \( P_{ai} \neq P_{ia} \), the Ricci d-tensor is non-symmetric (this could be with respect to anholonomic frames of reference). The scalar curvature of the metric d–connection, \( \hat{R} = g^{\beta \gamma} R_{\beta \gamma} \), is computed
\[ \hat{R} = G^{\alpha \beta} R_{\alpha \beta} = \hat{R} + S, \]
where \( \hat{R} = g^{ij} R_{ij} \) and \( S = h^{ab} S_{ab} \).

By substituting (A6) and (A7) into the 5D Einstein equations
\[ R_{\alpha \beta} = \frac{1}{2} g_{\alpha \beta} R = \kappa \Upsilon_{\alpha \beta}, \]
where \( \kappa \) and \( \Upsilon_{\alpha \beta} \) are respectively the coupling constant and the energy–momentum tensor we obtain the h–v–decomposition by N–connection of the Einstein equations
\[ R_{ij} = \frac{1}{2} \left( \hat{R} + S \right) g_{ij} = \kappa \Upsilon_{ij}, \]  
\[ S_{ab} = \frac{1}{2} \left( \hat{R} + S \right) h_{ab} = \kappa \Upsilon_{ab}, \]  
\[ 1 P_{ai} = \kappa \Upsilon_{ai}, \quad 2 P_{ia} = \kappa \Upsilon_{ia}. \]

The definition of matter sources with respect to anholonomic frames is considered in Refs. 34.
b. 5D vacuum Einstein equations

The vacuum 5D, locally anisotropic gravitational field equations, in invariant h– v– components, are written

$$R_{ij} = 0, S_{ab} = 0,$$

$$^1P_{ai} = 0, ^2P_{ia} = 0.$$  \hspace{1cm} (A10)

The main ‘trick’ of the anholonomic frames method for integrating the Einstein equations in general relativity and various (super) string and higher / lower dimension gravitational theories is to find the coefficients $N^a_j$ such that the block matrices $g_{ij}$ and $h_{ab}$ are diagonalized \[3,1,4\]. This greatly simplifies computations. With respect to such anholonomic frames the partial derivatives are N–elongated (locally anisotropic).

APPENDIX B: PROOF OF THE THEOREM 3

We prove step by step the items of the Theorem 3.

The first statement with respect to the solution of (13) is a connected with the well known result from 2D (pseudo) Riemannian gravity that every 2D metric can be redefined by using coordinate transforms into a conformally flat one.

The equation (14) can be treated as a second order differential equation on variable $v$, with parameters $x^i$, for the unknown function $h_5(x^i, v)$ if the value of $h_4(x^i, v)$ is given (or inversely as a first order differential equation on variable $v$, with parameters $x^i$, for the unknown function $h_4(x^i, v)$ if the value of $h_5(x^i, v)$ is given). The formulas (29) and (28) are consequences of integration on $v$ of the equation (14) being considered also the degenerated cases when $h^*_5 = 0$ or $h^*_4 = 0$.

Having defined the values $h_4$ and $h_5$, we can compute the values the coefficients $\alpha_i, \beta$ and $\gamma$ (17) and find the coefficients $w_i$ and $n_i$. The first set (30) for $w_i$ is a solution of three independent first order algebraic equations (13) with known coefficients $\alpha_i$ and $\beta$. The second set of solutions (31) for $n_i$ is found after two integrations on the anisotropic variable $v$ of the independent equations (13) with known $\gamma$ (the variables $x^i$ being considered as parameters). In the formulas (31) we distinguish also the degenerated cases when $h^*_5 = 0$ or $h^*_4 = 0$.

Finally, we note that the formula (32) is a simple algebraic consequence from (25). The Theorem 3 has been proven.

APPENDIX C: PROOF OF THEOREM 5

We emphasize the first two items:

- The equation (80) imposes a constraint on coefficients of a diagonal 2D metric parametrized by coordinates $x^2 = u$ and $x^3 = \lambda$. By coordinate transforms $x^{2,3} \rightarrow \tilde{x}^{2,3} (u, \lambda)$, see for instance, \[20\] we can reduce 2D every metric

$$ds^2_{[2]} = g_2(u, \lambda)du^2 + g_3(u, \lambda)d\lambda^2$$
to a conformally flat one

\[ ds_{[2]}^2 = \varpi(\bar{x}^2, \bar{x}^3) \left[ (d\bar{x}^2)^2 + \epsilon d(\bar{x}^3)^2 \right], \epsilon = \pm 1. \]

with conformal factor \( \varpi(\bar{x}^2, \bar{x}^3) \), for which (80) transforms into (86) with new 'dot' and 'prime' derivatives \( \varpi^{\bullet} = \partial \varpi / \partial \bar{x}^2 \) and \( \varpi^{'} = \partial \varpi / \partial \bar{x}^3 \). It is not possible to find an explicit form of the general solution of (86). If we approximate, for instance, that \( \varpi = \varpi(\bar{x}^2) \), the equation

\[ \varpi \varpi^{\bullet\bullet} - (\varpi^{'})^2 \epsilon = 2\Lambda \varpi^3 \]

has an exact solution (see 6.127 in [18]) which can be found from a Bernulli equation

\[ (\varpi^{'})^2 = \varpi^3 \left( C\varpi^{-1} + 4\Lambda \right), C = \text{const}, \]

which allow us to find \( \bar{x}^2(\varpi) \), or, in non explicit form \( \varpi = \varpi(\bar{x}^2) \). We can chose a such solution as a background one and by using conformal factors \( \Omega(\bar{x}^2, \bar{x}^3) \), transforming \( \varpi(\bar{x}^2, \bar{x}^3) \) into \( \varpi(\bar{x}^2) \) we can generate solutions of the 5D Einstein equations with anisotropic cosmological constant by inducing second order anisotropy \( \zeta_i \). The case when \( \varpi = \varpi(\bar{x}^3) \) is to be obtained in a similar manner by changing the 'dot' derivative into 'prime' derivative.

- The equation (81) das not admit \( h_5^* = 0 \) because in this case we must have \( h_5 = 0 \). For a given value of \( h_5 \), introducing a new variable \( \tau = h_5^*/2h_5 \) we can transform (81) into a first order nonlinear equation for \( h_4 \) (C1), which can be transformed [18] to a Ricatti, then to a Bernulli equation which admits exact solutions. We note that the holonomic coordinates are considered as parameters. The inverse problem, to find \( h_5 \) for a given \( h_4 \) is more complex because is connected with solution of a second order nonlinear differential equation

\[ h_5^{\bullet\bullet} + \frac{(h_5^*)^2}{2h_5} - \frac{h_4^*}{2h_4} h_5^* - 2\Lambda h_4 h_5 = 0, \]  

which can not integrated in general form. Nevertheless, a very general class of solutions can be found explicitly if \( h_4^* = 0 \), i.e. if \( h_4 \) depend only on holonomic coordinates. In this case the equation (C1) can be reduced to a Bernulli equation [18] which admits exact solutions.

- The formulas (88), (89) and (90) solving respectively (82), (83) and (84) are proven similarly as for the Theorem 3 with that difference that in the presence of the cosmological term \( h_5^* \neq 0, \beta \neq 0 \) and, in general, \( w_i \neq 0 \).

The Theorem 5 has been proven.
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