Algebraic symplectic reduction and quantization of singular spaces

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Abstract. The algebraic method of singular reduction is applied for non regular group action on manifolds which provides singular Poisson spaces. For some examples of singular Poisson spaces the deformation quantization is explicitly constructed. It is shown that for the flat phase space with the classical moment map and the orthogonal group action the deformation quantization converges for the class of entire functions.

Non free
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1 Introduction

The problem of systems with constraints in the quantum field theory comes to Dirac [3]. The general method of Meyer-Marsden-Weinstein provides the reduction of a symplectic manifold with constraint and a free group action. If the group action is not free and the constraint locus is singular. The singular points are often the most interesting because they have smaller orbits and larger symmetry. Sniatycki and Weinstein [4] applied a pure algebraic method for symplectic reduction in a modelling case. The problem of singular symplectic reduction of the angular momentum was studied in [5], [6] by geometric methods. Batalin-Vilkovisky-Fradkin’s method [11], [12] was proposed for gauge systems. In [8] the BRST method was developed based on the rather complicated homological construction including ghosts fields.

The method of algebraic singular reduction can be applied to any algebraic Poisson manifold \((X, q)\) with an algebraic momentum map and action of an algebraic group \(G\). It ends up to an affine Poisson algebraic variety \((X_{\text{red}}, q_{\text{red}})\) with the algebra sheaf \(O_{\text{red}}\) of \(G\) invariant functions restricted to the constraint locus. This variety is singular if the group action is not free. This is the case of the Yang-Mills theory and general relativity where the constraint locus has quadratic singularities and the reduced space \(X_{\text{red}}\) is singular [14]. We give here construction of deformation quantization of some singular spaces \(X_{\text{red}}\). Other examples are some singular K3 surfaces. Our method is based on the Grönewold-Moyal formula. In the simplest cases the flat phase space associative product locally converges for entire holomorphic arguments.

The problem of quantization of spaces with singularities was rased by Kontse-
vich [7]. To my best knowledge there is no examples of deformation quantization of singular spaces so far. See [13] for a survey on quantization deformation and [9], [10] for basics of the theory of associative deformations of singular spaces.

2 Singular reduction

We use the construction of singular reduction which is close to that of [4]. Let \( X \) be a real (or complex) algebraic variety endowed with a Poisson bracket \( q \) defined on the algebra of rational real or complex functions on \( X \). In a more general setting let \( (X, \mathcal{O}_X) \) be a real algebraic scheme with a Poisson biderivation \( q : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X \). An algebraic group \( G \) is defined on \( X \) such that the bracket \( q \) is \( G \) covariant. Let \( \mathcal{O}_{X/G} \) be the subsheaf of \( \mathcal{O}_X \) of \( G \) invariant germs. It is a sheaf of algebras defined on the space of orbits \( X/G \). The invariant Poisson bracket \( q \) can be lifted to a Poisson bracket \( q_{G} \) on \( X/G \).

Let \( J : X \to \mathfrak{g}^* \) be an algebraic momentum map, where \( \mathfrak{g}^* \) is the dual space to the Lie algebra \( \mathfrak{g} \) of \( G \). The set \( Y = J^{-1}(0) \) is a subscheme of \( \mathcal{O}_X \) (called constraint locus) with the structure sheaf \( \mathcal{O}_Y = \mathcal{O}_X/(J) \), where \( (J) \) denotes the ideal in \( \mathcal{O}_X \) generated by the coordinates of \( J \). We suppose that \( J \) is equivariant that is \( J(gx) = \text{Ad}_gJ(g^{-1}x) \) for \( g \in G \). It follows that \( (J) \) is \( G \) invariant and \( J \) can be lifted to a mapping \( J_G \) defined on \( Y/G \) making the diagram commutative:

\[
\begin{array}{ccc}
Y & \to & X \\
\downarrow & & \downarrow \\
X_{\text{red}} = Y/G & \to & X/G \\
\end{array}
\]

We assume further that the action is hamiltonian that is for any \( v \in \mathfrak{g} \) and any \( a \in \mathcal{O}_X \), we have

\[
q((v, J), a) = d_{G}A(v)(a)
\]

where \( A : X \times G \to X \) denotes the group action and \( d_{G}A : \mathfrak{g} \to \mathfrak{b}(T(X)) \) is the tangent map.

**Proposition 1** The bracket \( q \) can be lifted to a biderivation \( q_{\text{red}} \) on \( X_{\text{red}} \cong Y/G \). This is a Poisson bracket.

**Proof.** Check that inclusion \( q(j, b) \in (J) \) holds for any \( j \in (J) \) and arbitrary \( b \in \mathcal{O}_{X/G} \). Let \( j = \langle v, J \rangle a \) for some \( a \in \mathcal{O}_X \) and \( v \in \mathfrak{g} \). We have

\[
q(j, b) = \langle v, J \rangle q(a, b) + aq((v, J), b)
\]

because \( q \) is biderivation. The first term belongs to \( (J) \) and by (1)

\[
q((v, J), b) = d_{G}A(v)(b) = 0
\]

since \( b \) is constant on any orbit and the field \( d_{G}A(v) \) is tangent to orbits of \( G \). Finally \( q(j, b) \in (J) \).

The Poisson variety \( (X_{\text{red}}, \mathcal{O}_{X/G}, q_{\text{red}}) \) will be called singular symplectic reduction of \( (X, q, G, J) \). This construction is translated to the category of sheaves of smooth functions on \( X \) with obvious modifications.
3 Poisson bracket from hamiltonian fields

Let \( \mathcal{A} \) be a unitary commutative algebra over a field \( K \) of zero characteristic.

**Proposition 2** Let \( q \) be a Poisson bracket on \( \mathcal{A} \). If \( q(a, b, \cdot) = 0 \) for some \( a, b \in \mathcal{A} \), then the hamiltonian fields \( A(\cdot) = q(\cdot, a) \) and \( B(\cdot) = q(\cdot, b) \) commute.

This follows from the Jacobi identity. \( \blacktriangleright \)

For derivations \( A, B \) on \( \mathcal{A} \), we define the biderivation \( (A \wedge B)(a, b) = A(a)B(b) - B(a)A(b), \quad a, b \in \mathcal{A} \). For a biderivation \( q \), we denote

\[
\text{Jac} [q](a, b, c) = q(q(a, b), c) + q(q(b, c), a) + q(q(c, a), b)
\]

and have \( \text{Jac} [q] = 0 \) if \( q \) is the Poisson bracket.

**Proposition 3** If \( A_i, B_j, \ i, j = 1, ..., n \) are commuting fields on \( \mathcal{A} \) then the bracket \( q = \sum A_i \wedge B_i \) satisfies the Jacobi identity.

*Proof.* For \( n = 1 \) this identity can be checked by a direct computation. In the general case, we set \( U = \sum t^i A_i, V = \sum t^n B_i \) where \( t \) is a real parameter. The field \( U \) and \( V \) commute, hence \( \text{Jac} [U \wedge V] = 0 \). The left hand side is a polynomial in \( t \) which vanishes identically. In particular the term with \( t^n \) vanishes which implies the statement. \( \blacktriangleright \)

We say that a subalgebra \( \mathcal{B} \) of \( \mathcal{A} \) is dense, if any derivation \( \delta : \mathcal{A} \rightarrow \mathcal{A} \) such that \( \delta \mid \mathcal{B} = 0 \) vanishes on \( \mathcal{A} \).

**Proposition 4** Let \( q \) be a Poisson bracket defined on \( \mathcal{A} \). If there exist elements \( a_i, b_i \in \mathcal{A}, \ i = 1, ..., n \) such that

\[
q(a_i, a_j) = q(b_i, b_j) = 0 = q(a_i, b_j) = 0, \quad i \neq j, \quad q(a_i, a_i) = 1, \ i = 1, ..., n \quad (2)
\]

and \( a_i, b_i \) generate the dense subalgebra \( \mathcal{B} \) of \( \mathcal{A} \) then

\[
q(\cdot, \cdot) = \sum_{k=1}^{n} q(\cdot, b_k) \wedge q(a_k, \cdot). \quad (3)
\]

*Proof.* Proposition 4 implies commutativity of any pair of the fields \( q(\cdot, b_k), \ q(a_k, \cdot) \), \( i, j = 1, 2, ..., n \). By (2) the biderivation

\[
[\cdot, \cdot] = \sum_{k=1}^{n} q(\cdot, b_k) \wedge q(a_k, \cdot) = \sum_{k=1}^{n} q(\cdot, b_k) q(a_k, \cdot) - \sum_{k=1}^{n} q(\cdot, a_k) q(b_k, \cdot)
\]

fulfills

\[
[a_i, b_j] = \sum_{k=1}^{n} q(a_i, b_k) q(a_k, b_j) - \sum_{k=1}^{n} q(b_j, b_k) q(a_k, a_i) = \delta_{ij} q(a_i, b_j)
\]

that is \( [a_i, b_j] = q(a_i, b_j) \). Therefore \( [P, Q] = q(P, Q) \) for any polynomials \( P, Q \in \mathcal{B} \). The subalgebra \( \mathcal{B} \) is dense in \( \mathcal{A} \) by the assumption. This implies that the brackets coincide on \( \mathcal{A} \). \( \blacktriangleright \)
4 The Grönwold-Moyal star product

Theorem 5 For any Poisson bracket \( q \) on \( A \) and any elements \( a_i, b_j \) as in Proposition 4, the Grönwold-Moyal (GM) product

\[
(f * g)(t) = fg + \sum_{k=1}^{\infty} \frac{t^k}{k!} Q_k(f, g)
\]

defined on \( A \) is a deformation quantization of this bracket where \( Q_1 = q \) and for any \( k = 2, 3, \ldots \)

\[
Q_k(f, g) = \sum_{j=0}^{k} (-1)^j \frac{k!}{j!(k-j)!} \sum_{i_1=1}^{n} A_{i_1} \ldots A_{i_j} B_{i_{j+1}} \ldots B_{i_k} (f) B_{i_1} \ldots B_{i_j} A_{i_{j+1}} \ldots A_{i_k} (g),
\]

\( A_k = q(\cdot, b_k), B_k = q(a_k, \cdot), k = 1, \ldots, n. \)

Proof. The fields \( A_i, B_j \) commute for \( i, j = 1, \ldots, n \) since of the Jacobi identity and (3) coincides with (5) for \( a_i = x^i, b_i = \xi_i \). Therefore (4) is the associative product which has the same form as the classical Grönwold-Moyal series. ▶

5 Invariant quantization of a flat phase space

The phase space \( T^* (\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{R}^n \) is supplied with the Poisson bracket

\[
q (a, b) = \sum \frac{\partial a}{\partial x^i} \frac{\partial b}{\partial \xi_i} - \frac{\partial a}{\partial \xi_i} \frac{\partial b}{\partial x^i},
\]

and the classical momentum map

\( J : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \wedge^2 \mathbb{R}^n, J(x, \xi) = x \wedge \xi. \)

The action of the orthogonal group \( O(n) \) on \( \mathbb{R}^n \times \mathbb{R}^n : (x, \xi) \mapsto (Ux, U\xi) \) preserves the Poisson bracket and \( J \) is equivariant. The constraint locus \( Y = J^{-1}(0) \) consists of pairs \( (x, \xi) \) of proportional vectors \( x \) and \( \xi \). For elements \( e_{jk} = y^j \partial / \partial y^k - y^k \partial / \partial y^j, j \neq k = 1, \ldots, n \) of the Lie algebra of the group \( O(n) \), we have \( \langle e_{jk}, J \rangle = x^j \xi_k - x^k \xi_j \) and equation

\[
q (\langle e_{jk}, J \rangle, a) = \xi_k \frac{\partial a}{\partial \xi_j} - \xi_j \frac{\partial a}{\partial \xi_k} + x^j \frac{\partial a}{\partial x^k} - x^k \frac{\partial a}{\partial x^j} = d_G A (e_{jk})(a)
\]

implies (1). By Proposition 4 the bracket \( q \) is lifted to the Poisson bracket \( q_{\text{red}} \) in \( Y/G \).

Let \( \mathcal{A} \) be the algebra of real polynomials on \( \mathbb{R}^n \). The algebra \( \mathcal{A}_{X/G} \) of polynomials on \( X = \mathbb{R}^n \times \mathbb{R}^n \) invariant with respect to the action of is generated by

\[
s_1 = |x|^2, s_2 = |\xi|^2, s_3 = \langle x, \xi \rangle.
\]
The restrictions of the generators on $Y$ fulfil equation $f(s) = s_3^2 - s_1 s_2 = 0$, which implies $A_{Y/G} \cong \mathbb{R}[s_1, s_2, s_3]/(f)$ and we have
$$q(s_1, s_2) = 4s_3, \quad q(s_1, s_3) = 2s_1, \quad q(s_2, s_3) = -2s_2$$
or equivalently
$$q_{\text{red}} = 4s_3 \partial_1 \wedge \partial_2 - 2s_2 \partial_2 \wedge \partial_3 - 2s_1 \partial_3 \wedge \partial_1. \quad (6)$$

The elements $a_1 = \sqrt{s_1}, \ b_1 = \sqrt{s_2}$ belong to the quadratic extension $A^*$ of the polynomial algebra $A_{Y/G}$. We have
$$q(a_1, b_1) = 4s_3 \sqrt{s_1^2} \sqrt{s_2^2} = 1.$$

Therefore the elements fulfil conditions of Proposition 4 for $n = 1$. It follows that the bracket $q_{\text{red}}$ admits the quantization of GM type on the algebra $A^*$. The algebra $B$ of polynomials of $a_1$ and $b_1$ is dense in $A^*$.

6 Convergence of the Grönewold-Moyal series

**Theorem 6** The terms $Q_m$ of the GM quantization of (6) are bidifferential operators with polynomial coefficients of degree $\leq m$ in each argument.

**Proof.** Denote $A = q(\cdot, b_1), \ B = q(a_1, 0)$. The bracket (6) has linear coefficients. For an arbitrary even $k$, we can write
$$Q_k(a, b) = \sum_{i+j=k/2} \frac{k!}{2^i! 2^j!} A^{2i} B^{2j}(a) B^{2i} A^{2j}(b)$$
$$- \sum_{i+j+1=k/2} \frac{k!}{(2i+1)! (2j+1)!} A B A^{2i} B^{2j}(a) B A^{2j} B^{2i}(b) \quad (7)$$
since the fields
$$A = q(\cdot, \sqrt{s_2}) = 2 \sqrt{s_1} \partial_1 + \sqrt{s_2} \partial_3,$$  
$$B = q(\cdot, \sqrt{s_1}) = 2 \sqrt{s_2} \partial_2 + \sqrt{s_1} \partial_3$$
vanish on $f$, commute and $A \wedge B = q$. The bidifferential operators are composed from the operators
$$A^2 = 4s_1 \partial_1^2 + 2 \partial_1 + s_2 \partial_3^2, \quad B^2 = 4s_2 \partial_2^2 + 2 \partial_2 + s_1 \partial_3,$$  
$$BA = AB = 4s_3 \partial_1 \partial_2 + 2s_1 \partial_1 \partial_3 + 2s_2 \partial_2 \partial_3 + s_3 \partial_3^2$$
which are second order differential operators with linear coefficients. For odd $k$,
$$Q_k(a, b) = \sum_{i+j=k-1} (-1)^j \frac{(k-1)!}{i!j!} q \left( (A^i B^j(a), A^j B^i(b)) \right).$$
Theorem 7 For arbitrary holomorphic functions \(a, b\) on \(\mathbb{C}^\sigma\) of exponential type \(\sigma\), the GM series for the Poisson bracket (6) converges for \(s\) in the ball \(\{|s| < 1/4\sigma\}\) for \(t\) satisfying \(|t| < 1/9\sigma^{1/2}\).

Proof. Denote
\[
\|a_m\| = \max_{|s|=1} |a_m(s)|
\]
for a polynomial \(a_m\) of degree \(m\). Note that \(\|\partial_i a_m\| \leq m \|a_m\|, \|s_i a_m\| = \|a_m\|, i = 1, \ldots, n\). If
\[
p(s, D) = \sum p_{ijk} s_i \partial_j \partial_k,
\]
then degree of the polynomial \(p(s, D) a_m\) is equal to \(m - 1\) and
\[
\|p(s, D) a_m\| \leq |s|^m \frac{m!}{(m-2)!} \|p\| \|a_m\|, \|p\| = \sum |p_{ijk}|.
\]
For any \(i, j, A^{2i}B^{2j}(a_m)\) is a polynomial of degree \(m - i - j\) and
\[
\|AB(a_m)\| \leq 9^2 \frac{m! (m-1)!}{(m-2)! (m-3)!} \|a_m\| \leq 9^2 \left( \frac{m!}{(m-2)!} \right)^2 \|a_m\|
\]
\[
\|A^i B^j(a_m)\| \leq \frac{m! (m-1)!}{(m-i-j)! (m-i-j-1)!} \|a_m\|
\]
\[
\leq 9^{i+j} \left( \frac{m!}{(m-i-j)!} \right)^2 \|a_m\|, \ i + j \geq 2
\]
since
\[
\max(\|AB\|, \|A^2\|, \|B^2\|, \|q\|) \leq 9.
\]
It follows that for an arbitrary homogeneous polynomial \(b_n\) of degree \(n\), and any even \(k\),
\[
|Q_k(a_m, b_n)| \leq 9^k \sum_{i+j=k} \frac{k!}{i!j!} \left( \frac{m!}{(m-k/2)! (n-k/2)!} \right)^2 \|a_m\| \|b_n\| |s|^{m+n-k/2}
\]
\[
\leq (18)^k |s|^{-k/2} (k/2)!^4 4^{n+n} |s|^{m+n} \|a_m\| \|b_n\|
\]
\[
\leq C9^k |s|^{-k/2} k! (k/2)!^2 (4 |s|)^{m+n} \|a_m\| \|b_n\|
\]
since \((k/2)!^2 \leq 2\pi^{1/2}2^{-k}k!\). Note that \(Q_k(a, b) = 0\) if \(k/2 > \min\{m, n\}\). The similar estimate holds for any odd \(k\). Let
\[
a = \sum a_m, \ b = \sum b_n
\]
for some series of homogeneous polynomials \(a_m, b_n\). By the condition both series fulfill
\[
\|a_m\| \leq C \varepsilon^m m!, \ |b_n| \leq C \varepsilon^n n! \quad (8)
\]
for arbitrary \( \varepsilon > \sigma \) and some constant \( C_\varepsilon \) that does not depend on \( m \) and \( n \). For arbitrary polynomials \( a_m \), and \( b_n \) satisfying \([S]\), we finally obtain the inequality for \( |s| < 1/4\varepsilon \) and \( |t| < \varepsilon^{-1/2}/9 \):

\[
\sum_k \frac{t^k}{k!} |Q_k (a, b) (s)| \leq C'_\varepsilon \sum_k (9 |t| |s|^{-1/2})^k \sum_{\min\{m,n\} > k/2} (4\varepsilon |s|)^{m+n} \frac{(k/2)!}{m!n!} \]

\[
\leq \frac{C'_\varepsilon}{1-4\varepsilon |s|} \sum_{k \geq 0} (9 |t| |s|^{-1/2})^k \sum (4\varepsilon |s|)^{k/2} \]

\[
= \frac{C'_\varepsilon}{(1-4\varepsilon |s|) (1-9\varepsilon^{-1/2} |t|)}.
\]

It follows that the series converges for any \( s \) and \( t \) such that \( |s| < 1/4\sigma \) and \( |t| < 1/9\sigma^{1/2} \).

### 7 Commuting matrices

Let \( M_2 \) be the space of \( 2 \times 2 \)-matrices with complex entries. The manifold \( X = M_2 \times M_2 \) is endowed with the Poisson bracket

\[
q = \sum_{k=1}^4 \frac{\partial}{\partial a_k} \wedge \frac{\partial}{\partial b_k}. \tag{9}
\]

where

\[
A = \begin{pmatrix} a_1 & a_3 \\ a_4 & a_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & b_3 \\ b_4 & b_2 \end{pmatrix}
\]

are coordinates in \( X \). The group \( SL(2, \mathbb{C}) \) acts diagonally by

\[
g : (A, B) \mapsto (gAg^{-1}, gBg^{-1}).
\]

Let \( J : (A, B) \mapsto [A, B] \) be the momentum map on \( X \); the constraint locus is the cone

\[
Y = \{(A, B) : b_3(a_1-a_2) - a_3(b_1-b_2) = 0, \ b_4(a_1-a_2) - a_4(b_1-b_2) = 0\}. \tag{10}
\]

Condition \((1)\) is easy to check. The polynomials

\[
\alpha_1 = \text{tr} A, \quad \alpha_2 = \det A, \quad \beta_1 = \text{tr} B, \quad \beta_2 = \det B, \quad \gamma = \text{tr} AB
\]

generate the algebra \( A_{X/G} \) of invariant polynomials on \( X \). The reduced Poisson bracket equals

\[
q_{\text{red}} = 2 \frac{\partial}{\partial \alpha_1} \wedge \frac{\partial}{\partial \beta_1} + \beta_1 \frac{\partial}{\partial \alpha_1} \wedge \frac{\partial}{\partial \beta_2} + \alpha_1 \frac{\partial}{\partial \alpha_2} \wedge \frac{\partial}{\partial \beta_1} + \gamma \frac{\partial}{\partial \alpha_2} \wedge \frac{\partial}{\partial \beta_2} + \frac{\partial}{\partial \gamma} \tag{11}
\]

\[
+ \left( \alpha_1 \frac{\partial}{\partial \alpha_1} - \beta_1 \frac{\partial}{\partial \beta_1} + 2\alpha_2 \frac{\partial}{\partial \alpha_2} - 2\beta_2 \frac{\partial}{\partial \beta_2} \right) \wedge \frac{\partial}{\partial \gamma}.
\]
**Proposition 8** The algebra $A_{Y/G}$ of invariant polynomials of algebra restricted to $Y$ is isomorphic to $B/\langle \rho \rangle$, where $B = \mathbb{R}[\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma]$ and

$$
\rho = \gamma^2 - \alpha_1 \beta_1 \gamma + \alpha_2 \left( \beta_1^2 - 2 \beta_2 \right) + \beta_2 \left( \alpha_1^2 - 2 \alpha_2 \right) = \left( \gamma - \frac{1}{2} \alpha_1 \beta_1 \right)^2 - 4\pi, \quad (12)
$$

$$
\pi = \tilde{\alpha} \tilde{\beta}, \quad \tilde{\alpha} \equiv \alpha - \frac{1}{2} \alpha_1^2, \quad \tilde{\beta} \equiv \beta - \frac{1}{2} \beta_1^2.
$$

**Proof.** Check that $\rho = 0$ on $Y$. For any pair $(A, B) \in Y$, there exists $g \in \text{Sl}(2, \mathbb{C})$ such that both matrices $gAg^{-1}$ and $gBg^{-1}$ have Jordan form. This is easy to prove by means of (10). Let $(a_1, a_2)$ and $(b_1, b_2)$ be its diagonal elements, respectively. Then

$$
\alpha_1 = a_1 + a_2, \quad \beta_1 = b_1 b_2, \quad \gamma = a_1 b_1 + a_2 b_2
$$

and (12) can be checked directly. It is easy to show that this equation generates all algebraic relations. ▷

It follows that the spectrum of the algebra $A_{Y/G}$ is a two-fold covering of $K^4$ ramified over the discriminant set $\{ \pi = 0 \}$.

**Conclusion 9** The singular symplectic reduction of the variety $(X, O(2), q)$ is the singular hypersurface $X_{\text{red}} = \{ \rho = 0 \}$ with coordinate functions $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma$ defined by (12) with the Poisson bracket $q_{\text{red}}$ as in (11).

Let $A^*$ be the extension of the algebra $A_{X/G}$ by means of the element $\pi^{-1/4}$.

**Proposition 10** Elements

$$
a_1 = \frac{1}{\sqrt{2}} \alpha_1, \quad b_1 = \frac{1}{\sqrt{2}} \beta_1,
$$

$$
a_2 = \frac{\tilde{\alpha}}{\pi^{1/4}}, \quad b_2 = \frac{\tilde{\beta}}{\pi^{1/4}}
$$

of algebra $A^*$ fulfil (3) with $n = 2$.

**Proof.** Obviously $\{a_1, b_1\} = 1$. We have

$$
q\left( \tilde{\alpha}, \tilde{\beta} \right) = \gamma - \frac{1}{2} \{ \alpha_2, \beta_1 \} \beta_1 = \gamma - a_1 b_1. \quad (13)
$$

By (12)

$$
\gamma - a_1 b_1 = 2\pi^{1/2} \quad (14)
$$

on $X_{\text{red}}$ hence

$$
q\left( a_2, b_2 \right) = \left\{ \frac{\tilde{\alpha}}{\pi^{1/4}}, \frac{\tilde{\beta}}{\pi^{1/4}} \right\} = \frac{1}{2\pi^{1/2}} q\left( \tilde{\alpha}, \tilde{\beta} \right) = 1. \quad ▷
$$
This implies that the elements \( a_k, b_k \in \mathcal{A}^* \) fulfill conditions (2). By Proposition 4 bracket \( q_{\text{red}} \) admits a quantization by means of the Hamiltonian fields \( A_k = q (\cdot, b_k), B_k = q (a_k, \cdot) \). These fields are well defined on since they vanish on the polynomial \( \rho \). The explicit forms are

\[
\begin{align*}
A_1 &= \sqrt{2} \frac{\partial}{\partial \alpha_1} + \frac{1}{\sqrt{2}} \alpha_1 \frac{\partial}{\partial \alpha_2} + \frac{1}{\sqrt{2}} \beta_1 \frac{\partial}{\partial \gamma}, \\
B_1 &= \sqrt{2} \frac{\partial}{\partial \beta_1} + \frac{1}{\sqrt{2}} \beta_1 \frac{\partial}{\partial \beta_2} + \frac{1}{\sqrt{2}} \alpha_1 \frac{\partial}{\partial \gamma}, \\
A_2 &= \frac{3}{2} \pi^{-1/4} \frac{\partial}{\partial \alpha_2} + \pi^{-1/4} \beta \frac{\partial}{\partial \gamma} + \frac{1}{2} \pi^{-3/4} \alpha \frac{\partial}{\partial \beta_2}, \\
B_2 &= \frac{3}{2} \pi^{-1/4} \frac{\partial}{\partial \beta_2} + \pi^{-1/4} \beta \frac{\partial}{\partial \gamma} + \frac{1}{2} \pi^{-3/4} \alpha \frac{\partial}{\partial \alpha_2}.
\end{align*}
\]

*Conjecture 11* Any term \( Q_k, k = 1, 2, \ldots \) of the GM series is a bidifferential operator of degree \( \leq (k, k) \) with rational coefficients and the denominator \( \pi^{k-1} \).

This is obvious for \( k = 1 \) since \( Q_1 = q \). The direct calculation of \( Q_2 \) supports the conjecture.

**Other groups.** The above method works for the conjugacy action of the orthogonal group \( \text{O}(2) \) on the space of pairs of real symmetric \( 2 \times 2 \) matrices as well for action of the unitary group \( \text{SU}(2) \) on the space of pairs of Hermitian \( 2 \times 2 \) matrices. The algebra of invariants is generated by the same five symmetric polynomials. This bracket can be quantized in a similar way.

### 8 Quantization of K3 surfaces

K3 surfaces are topologically trivial Calabi-Yau 2-manifolds. A smooth variety \( X_f \) given in \( \mathbb{C}P^3 \) by a polynomial equation \( f = 0 \) of degree 4 is a K3 surface. The Poisson bracket on \( \mathcal{O}(\mathbb{C}P^3) / (f) \) is equal (up to a constant factor) to \( x_0^{-1} q_0 \) on the chart \( X_0 = \{ x_0 \neq 0 \} \) where

\[
g_0 (a, b) = \det \begin{pmatrix}
\partial_1 a & \partial_2 a & \partial_3 a \\
\partial_1 b & \partial_2 b & \partial_3 b \\
\partial_1 f & \partial_2 f & \partial_3 f
\end{pmatrix}, \tag{15}
\]

\( \partial_i = \partial / \partial x_i, \ i = 1, 2, 3 \) and \( x_0, x_1, x_2, x_3 \) are arbitrary homogeneous coordinates on \( \mathbb{C}P^3 \) in \( CP \). Below we consider two examples where Theorem 5 can be applied.

1. The variety \( X_f \) is a nonsingular K3 surface for \( f = 1/4 \left( x_0^4 + x_1^4 + x_2^4 + x_3^4 \right) \). The canonical Poisson bracket defined on \( X_f \) is given by \( q_f = x_0^2 \partial_1 \wedge \partial_2 + x_0^2 \partial_2 \wedge \partial_3 + x_0^2 \partial_1 \wedge \partial_1 \) on the chart \( X_0 = \{ x_0 = 1 \} \). We set \( a = \varphi (x_0, x_3) x_1, b = \varphi (x_0, x_3) x_2 \) for an unknown function \( \varphi \) and solve the equation

\[
q_f (a, b) = q_f (\varphi (x_3) x_1, \varphi (x_3) x_2) = 1.
\]
It is to check that $\varphi$ can be found in the form
$$
\varphi^2 (x_3) = \frac{2}{\sqrt{1 + x_3^4}} \int_{x_3}^{x_3} \frac{dy}{\sqrt{1 + y^4}}.
$$

Singular surfaces in $\mathbb{C}P^3$ of degree 4.

II. Hypersurface $f = x_0 x_3^3 - x_1^2 x_2^2$ has singularities at four points where both terms $x_0 x_3^3$, $x_1^2 x_2^2$ vanish. The bracket
$$
q_f = 3x_0 x_3^2 \partial_1 \wedge \partial_2 - 2x_1 x_2^2 \partial_2 \wedge \partial_3 - 2x_1^2 x_2 \partial_3 \wedge \partial_1
$$
is quantized on $X_0$ by the functions
$$
a = \frac{x_2}{x_3 \sqrt{x_0}}, \quad b = \frac{x_1}{x_3 \sqrt{x_0}}.
$$

It is easy to check that $q_f (a, b) = 1$ which implies that the hamiltonian fields
$$
B = q_f (a, r) = -\frac{1}{\sqrt{x_0}} \left( x_0 x_3 \partial_1 + 2x_1 x_2^2 x_3^{-2} \partial_2 + 2x_1^2 x_2^{-1} \partial_3 \right),
$$
$$
A = q_f (r, b) = -\frac{1}{\sqrt{x_0}} \left( 2x_1^3 x_2 x_3^{-2} \partial_1 + x_0 x_3 \partial_2 + 2x_1^2 x_2 x_3^{-1} \partial_3 \right)
$$
generate the quantization of the GM type.

III. If $f = x_0^2 x_3^2 - x_1^2 x_2^2$ we have
$$
q_f = 2x_0^2 x_3 \partial_1 \wedge \partial_2 + 2x_1 x_2^2 \partial_2 \wedge \partial_3 + 2x_1^2 x_2 \partial_3 \wedge \partial_1
$$
and have $q_f (a_1, b_1) = 1$ if we take
$$
a = \frac{x_1}{2x_0 \sqrt{x_3}}, \quad b = \frac{x_2}{2x_0 \sqrt{x_3}}.
$$

IV. For $f = x_3^4 - x_1^2 x_2^2$ we have $q_f (a, b) = 1$ for the elements
$$
a = \frac{x_1}{x_3 \sqrt{x_3}}, \quad b = -\frac{x_2}{x_3 \sqrt{x_3}}.
$$

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