A Generalization of Tepper’s Identity

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Abstract

In this paper, we first give a simple combinatorial proof of Tepper’s identity. Then, as a by product of this interesting identity we present another proof of the well-known Wilson’s identity in number theory. Finally, we obtain a generalization of Tepper’s identity for any polynomial with real coefficients.
1 Introduction

There is no doubt that Pascal’s triangle is one of the most beautiful triangular numerical array in mathematics. One can explore many algebraic, geometric and number-theoretic patterns inside this well-known numeric array.

The Newton’s binomial identity is one of the famous algebraic identity that one may encounter. That is, for example, for the third row of Pascal’s triangle, we have

\[ 1x^0 + 3x^1 + 3x^2 + 1x^3 = (1 + x)^3. \]

By a more careful inspection, we may notice that, for example, for the third row, we can see

\[ 1(x - 0)^3 - 3(x - 1)^3 + 3(x - 2)^3 - 1(x - 3)^3 = 3!. \]

The above observation may lead us to the important question whether the above identity is a coincidence or it is just a special case of a general identity. Indeed, one can conjecture that the inner product of the \( n \)th row of Pascal triangle with signs alternating between + and − and the row-vector

\[ ((x - 0)^n, (x - 1)^n, \ldots, (x - n)^n), \]

is equal to \( n! \) for any non-negative integer number \( n \). In other words, we get the following identity

\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x - k)^n = n!. \quad (1) \]
This identity is known as *Tepper’s identity*, as M. Tepper conjectured this result from a consideration of the numerical data which he gave in [1]. In the same year, C. T. Long gave a proof of formula (1) in [2]. Later, F. J. Papp gave another proof of it by mathematical induction in [3]. This result also *implicitly* derived in problem 20 by Feller [4].

The paper organization is as follows. We first give a simple proof of Tepper’s identity by a *combinatorial argument*. Next, by using this beautiful identity, we present another proof of *Wilson’s theorem*. Finally, we generalize Tepper’s identity for any *polynomial* with real coefficients.

## 2 A Combinatorial Method

The following interesting combinatorial problem has an important role for the investigation of the *Stirling* numbers of the *first kind*. As the proof of Tepper’s identity and it’s generalization is closely related to this problem, we review a *variant* of the original problem and it’s solution here [5].

**Problem (a).** Let a train have $n$ wagons. Now, if we *randomly* choose a wagon, compute the number of ways in which *exactly* $r$ wagons will be occupied.

**Problem (b).** Using the solution of the above problem compute the following *summation*:

$$
\binom{n}{1}1^p - \binom{n}{2}2^p + \binom{n}{3}3^p \cdots + (-1)^{n-1}\binom{n}{n}n^p, \quad (1 \leq p \leq n). \quad (2)
$$

**Remark 2.1** *The above statements are equivalent to the following interesting problem in physics. A sensor contains $n$ receivers and receives a flux of $p$
particles. If the probability of receiving particles for any receiver is the same, then compute the probability that these particles will hit exactly \( r \) receivers.

**Solution (a).** Suppose \( A_i \) is the number of ways in which the \( i \)th wagon can be empty (\( 1 \leq i \leq n \)). Then the number undesirable ways are \( |A_1 \cup A_2 \cup \cdots \cup A_r| \). Hence, by the inclusion - exclusion principle, we obtain

\[
|A_1 \cup A_2 \cup \cdots \cup A_r| = \sum_i |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^{n-1} |A_1 \cap A_2 \cdots \cap A_r|
\]

\[
= \sum_i (r-1)^p - \sum_{i<j} (r-2)^p + \cdots + (-1)^{n-1} (r-r)^p
\]

\[
= \binom{r}{1} (r-1)^p - \binom{r}{2} (r-2)^p + \cdots + (-1)^{r-2} \binom{r}{r-1} 1^p.
\]

Since the number of all possible cases is \( r^p \), then the number of desirable ways is equal to

\[
r^p - \binom{r}{1} (r-1)^p + \binom{r}{2} (r-2)^p - \cdots + (-1)^{r-1} \binom{r}{r-1} 1^p.
\]

Finally, since the number of ways of choosing \( r \) wagons is \( \binom{n}{r} \), then the total number of the desirable ways is

\[
\binom{n}{r} \left( r^p - \binom{r}{1} (r-1)^p + \binom{r}{2} (r-2)^p - \cdots + (-1)^{r-1} \binom{r}{r-1} 1^p \right).
\]  

(3)

**Solution (b).** Let \( r = n \) in (3). If \( p < n \), the number of desirable ways computed in part (a) is equal to zero and considering the well-known identity \( \binom{n}{r} = \binom{n}{n-r} \), we obtain

\[
\binom{n}{1} (1)^p - \binom{n}{2} (2)^p + \cdots + (-1)^{n-1} \binom{n}{n} n^p = 0.
\]  

(4)
Now if we choose \( n = p \) in (3), then for every passenger we have exactly one wagon and consequently the number of desirable ways is \( n! \). Finally, considering \( n = p \) and \( r = p \) in (3) and after simplifications, we have

\[
\binom{p}{1}p^p - \binom{p}{2}2^p + \cdots + (-1)^{p-1}\binom{p}{p-1}p^{p-1} = (-1)^{p-1}p!.
\] (5)

3 A Simple Proof of Tepper’s Identity

In this section, we give a simple proof of Tepper’s identity, using the results of the previous section.

By substituting the Newton binomial expansion for \((x - k)^n\) in the left-hand side of (1), we have

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x - k)^n = \sum_{j=0}^{n} x^j \left( \sum_{k=0}^{n} \binom{n}{k} (-k)^{n-j} \right). \tag{6}
\]

Now, relations (4) and (5) lead us to

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} (-k)^{n-j} = 0, \quad (1 \leq j \leq n). \tag{7}
\]

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} (-k)^n = n!. \tag{8}
\]

Thus, by substituting the above relations in (6), we obtain Tepper’s identity. As an immediate consequence of Tepper’s identity, we obtain another proof of the Wilson’s theorem.

**Theorem 3.1 (Wilson’s theorem)** Let \( p \) be an odd prime number. Then, we have

\[(p - 1)! \equiv -1 \pmod{p}.\]
Proof:

Put $x = 0$ and $n = p - 1$ in Tepper’s identity (1). Then, we get

$$(p - 1)! = \sum_{k=0}^{p-1} (-1)^k \binom{p-1}{k} k^{p-1}. \tag{9}$$

Now, using Fermat’s little theorem, we obtain

$$(p - 1)! \equiv \sum_{k=1}^{p-1} (-1)^k \binom{p-1}{k} \pmod{p}, \tag{10}$$

or equivalently,

$$ (p - 1)! \equiv -\binom{p-1}{0} + \sum_{k=0}^{p-1} (-1)^k \binom{p-1}{k} \pmod{p}, \tag{11}$$

which leads us to

$$(p - 1)! \equiv -1 \pmod{p},$$

as required.

4 A Generalization of Tepper’s Identity

As we already saw in the previous section, the Newton’s binomial expansion of $(x - k)^n$ had a key role in proving Tepper’s identity. Now, we show that the identity is also valid for any polynomial of degree $n$. To do this, we first prove the following lemma by the same argument stated in [6].

**Lemma 4.1** For any real number $x$, and natural numbers $m$ and $n$ such that, $0 \leq m < n$, we have

$$ \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x - k)^m = 0. \tag{12}$$
Proof: By considering the coefficient $x^j$, in the binomial expansion of $(x - k)^m$ for $0 \leq j \leq m$, on the left-hand side of (12) and relation (4) we get the desired result.

Remark 4.2 The above lemma is also true for any polynomial $P(x)$ of degree $m$, provided that $m < n$; that is,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} P(x - k) = 0. \quad (13)$$

Theorem 4.3 (The Generalized Tepper’s Identity) Suppose $n$ is a natural number and $P(x)$ is any polynomial of degree $n$, as follows

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

in which $a_i \in \mathbb{R}$ for $(0 \leq i \leq n)$. Then, we have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} P(x - k) = a_n n!.. \quad (14)$$

Proof: By considering Tepper’s identity and lemma (12), we have

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} P(x - k) = \sum_{i=0}^{n} \left( \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x - k)^i \right)$$

$$= a_n \left( \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x - k)^i \right)$$

$$= a_n n!.$$

Remark 4.4 Another interesting proof of Tepper’s identity and lemma (12) is given in [7], using Pascal’s functional matrix. Also, several interesting
combinatorial identities are given in [8]. We also come up with the following conjecture, as a generalization of Tepper’s identity

**Conjecture 4.5** For any positive integer \( l \) and any real polynomial \( P(x) \) of degree \( n \) with leading coefficient \( a_n \), we have

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} P(x - lk) = a_n l^n n!.
\]  

\( (15) \)

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