Fréchet Derivative for Light-Like Wilson Loops

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Abstract

We address the equations of motion for the light-like QCD Wilson exponentials defined in the generalized loop space. We attribute an important class of the infinitesimal shape variations of the rectangular light-like Wilson loops to the Fréchet derivative associated to a diffeomorphism in loop space what enables the derivation of the law of the classically conformal-invariant shape variations. We show explicitly that the Fréchet derivative coincides (at least in the leading perturbative order) with the area differential operator introduced in the previous works to relate the rapidity evolution and geometrical properties of the light-like Wilson polygons.

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1. INTRODUCTION

Quadrilateral planar Wilson loop with light-like sides [1–3] can be considered as a “hydrogen atom” of the Wilson loop theory in generalized loop space. Wilson loops having cusps and light-like segments show more complex renormalization and conformal properties than smooth and/or fully off-light-cone functionals. Analysis of the geometrical and dynamical properties of the generalized loop space, which can include cusped light-like Wilson exponentials, will deliver important information on the renormalization properties and evolution of various gauge-invariant quantum correlation functions, such as transverse-momentum dependent quark and gluon densities, multi-gluon scattering amplitudes, jet quenching parameter, etc. (see, e.g., Refs. [4–23] and Refs. therein).

In the generalized loop space, the laws of “motion” are naturally formulated in terms of integro-differential equations for the Wilson loop which undergo certain variations of the underlying contours on which these path-ordered exponentials of the gauge fields are defined. The infinitesimal local variations of the contours give rise to the variations of the exponentials themselves, the latter being described by the infinite set of the Makeenko-Migdal loop equations [24–30]. On the other hand, physically meaningful transformations of the cusped light-like paths constitute a special class of motions in the generalized loop space which is not grasped straightforwardly by the Makeenko-Migdal approach. In this paper we show that the nonlocal area derivative of a Wilson loop which has been proposed in [31–35] can be (at least in the lowest order of perturbative expansion) mathematically correctly introduced as a Fréchet derivative associated to a diffeomorphism with specific choice of the generating variational vector field in a generalized loop space setting (for details see Ref. [36] and Refs. therein).

The paper is organized as follows. In Section 2 we formally introduce the Fréchet derivative and recapitulate some of the results from Ref. [36] to show how it links to diffeomorphisms with associated variational vector field. In Section 3 we apply this derivative to generic parallel transporters and Wilson loops. In Section 4 we address the derivative on a specific Wilson loop, the light-like quadrilateral, and show that the leading-order contribution, when taking vacuum expectation values, is consistent with our derivative from Ref. [31].

2. FRÉCHET DERIVATIVE AND DIFFEOMORPHISMS: MATHEMATICAL PRELIMINARIES

Given that the generalized loop space is Banach [36], one can define the Fréchet derivative acting in this space [37]. Let \( X, Y \) be Banach spaces and \( U \subset X \) be an open subset. Then we define:

**Definition 2.1 (Fréchet Derivative)**

A function \( f : U \to Y \) is called Fréchet differentiable at \( x \in U \) if there exists a bounded linear operator \( A_x : X \to Y \) such that

\[
\lim_{h \to 0} \frac{\| f(x + h) - f(x) - A_x(h) \|_Y}{\| h \|_X} = 0 ,
\]

where the limit is understood in the usual sense. If this limit exist, one defines \( Df(x) = A_x \) to be the Fréchet derivative. We call the function \( f \), \( C^1 \) if:

\[
Df : U \to B(X,Y) , \ x \mapsto Df(x) = A_x
\]
is continuous, where the $B$ highlights the fact that this a map between the Banach spaces.

Now we introduce the Chen iterated integrals $^{39,42}$ which are defined as an iterative extension of the usual line integrals

$$X(\gamma) = I_{i_1 \cdots i_p}(\gamma) = \int_a^b I_{i_1 \cdots i_{p-1}}(\gamma^t) \, dx_{i_p}(t) ,$$

where $\gamma$ denotes a path (integration contour) in the generalized path/loop space. After parametrization of the path $\gamma$ this becomes\footnote{In generalized loop space we assume reparametrization invariance, see also $^{43}$ for a detailed discussion.}:

$$X^{\omega_1 \cdots \omega_r}(\gamma) = \int_{\gamma} \omega_1 \cdots \omega_r = \int_0^1 \left( \int_{\gamma} \omega_1 \cdots \omega_{r-1} \right) \omega_r(t) \, dt ,$$

where $\omega_k(t) \equiv \omega_k(\gamma(t)) \cdot \dot{\gamma}(t)$ and $\gamma^t$ represents the part of the path for $t \in [0,1]$. Note that the operators $\omega_i$ are path-ordered under the integration, which will absorb the path-ordering operator $P$ when considering Wilson loops in what follows.

Considering now the generalized loop $\gamma \in \tilde{LM}_p$ at the point $p$, with tangent space $T_p LM_p$ to $\tilde{LM}_p$ at $\gamma$ which consists of sections of the pull-back bundle $\gamma^*TM$. Put otherwise, it consists of the vector fields along $\gamma$, that vanish on $p$. Now choose a tangent vector

$$V \in T_p \mathcal{P}M_p ,$$

and let $s \mapsto \gamma_s$ be a curve of paths in $\mathcal{P}M_p$, starting at $\gamma$. We have then

$$V(t) = \left. \frac{\partial}{\partial s} \right|_{s=0} \gamma_s(t) ,$$

from now on referred to as the variational vector field.

In Ref. $^{36}$, Tavares shows that the Fréchet derivative of $X^{\omega_1 \cdots \omega_r}(\gamma)$ at $\gamma$ can be written as follows

$$A_\gamma = D_V X^{\omega_1 \cdots \omega_r}(\gamma) = \sum_{i=1}^r \int_{\gamma} \omega_1 \cdots \omega_{i-1} \cdot \mathcal{J}_V(d\omega_i) \cdot \omega_{i+1} \cdots \omega_r + \sum_{i=2}^r \int_{\gamma} \omega_1 \cdots \omega_{i-2} \cdot \mathcal{J}_V(\omega_{i-1} \wedge \omega_i) \cdot \omega_{i+1} \cdots \omega_r + \left( \int_{\gamma} \omega_1 \cdots \omega_{r-1} \right) \cdot \omega_r(V(1)) ,$$

where for a closed path $V(0) = V(1) = 0$ and $\mathcal{J}_V$ is defined as the interior product $^{38}$

$$\mathcal{J}_V : \bigwedge^p(M) \to \bigwedge^{p-1}(M) ,$$

with $M$ a differentiable manifold and defined by:

$$\mathcal{J}_V \alpha^0 = 0, \quad \text{if } \alpha^0 \text{ is a 0-form}, \quad (9a)$$

$$\mathcal{J}_V \alpha^1 = \alpha(V), \quad \text{if } \alpha^1 \text{ is a 1-form}, \quad (9b)$$

$$\mathcal{J}_V \alpha^p(w_2, \cdots, w_p) = \alpha(V, w_2, \cdots, w_p), \quad \text{if } \alpha^p \text{ is a p-form}. \quad (9c)$$
Therefore one obtains
\[ D_V X^{\omega_1 ... \omega_r}(\gamma) = \sum_{i=1}^r \int_\gamma \omega_1 ... \omega_{i-1} \cdot J_V(d\omega_i) \cdot \omega_{i+1} ... \omega_r \]
\[ + \sum_{i=2}^r \int_\gamma \omega_1 ... \omega_{i-2} \cdot J_V(\omega_{i-1} \wedge \omega_i) \cdot \omega_{i+1} ... \omega_r . \]  

(10)

If one restricts the variational vector field \( V \) to be induced by a vector field \( Y \in X_p M \), i.e., \( V = Y \circ \gamma \) (for example, if \( \gamma \) is embedded), then we observe that the Fréchet derivative coincides with the derivative associated with a diffeomorphism of the manifold \( M \) that is infinitesimally generated by the vector field \( Y \), see Ref. [36].

3. FRÉCHET DERIVATIVE OF A WILSON LOOP

We define a Wilson loop \( \mathcal{W}_\gamma \) as a vacuum average of the traced operator-valued exponential
\[ U_{\gamma,t} = \exp \left( \int_\gamma A_\mu(x) \, dx^\mu \right), \]
where \( A \) belongs to the Lie algebra of the gauge group \( SU(N_c) \), that is
\[ \mathcal{W}_\gamma = \left\langle 0 \left\vert \frac{1}{N_c} \text{Tr} \ U_{\gamma,t} \right\vert 0 \right\rangle . \]  

(12)

Applying the operation (10) to the parallel transporter (11), one obtains for the logarithmic Fréchet derivative [36]
\[ D_V[U_{\gamma,t}] = U_{\gamma,t} \cdot \int_0^1 \frac{d \tau}{1} U_{\gamma,t} \cdot F_{\mu\nu}(t) \cdot [V^\mu(t) \wedge \dot{\gamma}^\nu(t)] \cdot U_{\gamma,t}^{-1} . \]  

(13)

where \( U_{\gamma,t} \) is interpreted now as the operator-valued parallel transporter (see also Eq. (11)) along the part of the path \( \gamma \) from the point 0 to \( t \), and the vector field \( V \), associated with the diffeomorphism flow, determines the direction of the variation of the loop.

From Eq. (13) it is now clear that this derivative is closely related to the area derivative of the parallel transporter around a loop \( \gamma \):
\[ \triangle^E_{(u,v \wedge v)}(p) \ U_{\gamma} = U_{\gamma} \cdot F_{\mu\nu}(u^\mu \wedge v^\nu) , \]  

(14)

which depends on the two independent vector fields \( \{u, v\} \) and where \( F_{\mu\nu} \) is the usual field strength tensor (or curvature tensor), by taking one of the vector fields to be the tangent to the loop and integrating over it along the loop.

Figures 1 and 2 visualise the relation between the two derivatives, where the arrows represent the vector fields, where in Fig. 2 one of the fields is tangent to the curve. Notice that in Fig. 2 the small “square” formed between the original, the deformed curve and the “normal” vector field arrows are actually pointed area derivatives (i.e. the area derivatives operating on specific points). Integration over these area derivatives then results in the deformed curve (the thick curve in Fig 2). In the next Section we show that the derivatives
\[
S_{12} \frac{\delta}{\delta S_{12}} = (2\ell_1 \cdot \ell_2) \frac{\delta}{\delta (2\ell_1 \cdot \ell_2)} = \ell_1^+ \frac{\delta}{\delta \ell_1^+} \\
S_{23} \frac{\delta}{\delta S_{23}} = (2\ell_2 \cdot \ell_3) \frac{\delta}{\delta (2\ell_2 \cdot \ell_3)} = \ell_2^- \frac{\delta}{\delta \ell_2^-}
\]

with \( S_{ij} \) being the adapted Mandelstam-like variables associated with the Wilson loop (with the parametrization shown in Fig. 3) defined in [31–34] and used in [35] are the lowest order contributions of the logarithmic Fréchet derivatives with the appropriate vector field \( V^\mu \) as generator for diffeomorphism transformation associated to the Fréchet derivative, as stated before.
4. CALCULATION OF THE LEADING-ORDER CONTRIBUTIONS

The perturbative expansion of the parallel transporter (11) written in terms of Chen iterated integrals [36, 39–42] is given by

\[ U^\gamma := 1 + \int A^\mu(x) \, dx^\mu + \int A^\mu(x) A^\nu(y) \, dx^\mu dy^\nu + \cdots , \]

(17)

where the operators \( A^\mu \cdots A^\nu \) are ordered as defined by the Chen integrals. For the inverse path (with reverse ordering and correct sign) one has

\[ U^{-1} := 1 - \int A^\mu(x) \, dx^\mu + \int A^\nu(y) A^\mu(x) \, dx^\mu dy^\nu - \cdots . \]

(18)

Given that the non-Abelian field strength tensor reads

\[ F^{\mu\nu} = \left( \frac{d}{dx^\mu} A^{\nu} \right) + g A^{\mu} \wedge A^{\nu} , \]

(19)

we expand Eq. (13) to lowest non-trivial order:

\[ D_V [W^\gamma]_{LO} = \]

\[ 1 \cdot \oint_0^1 dt \left[ \left( \oint_0^t A^\sigma(x(s)) \, \frac{dx^\sigma}{ds} \cdot \left( \partial_\mu A^\nu(y(t)) - \partial_\nu A^\mu(y(t)) \right) (V^\mu(y(t)) \wedge \partial^\nu y(t)) \right) \cdot 1 \right] \]

\[ - \left( 1 \cdot \left( \partial_\mu A^\nu(y(t)) - \partial_\nu A^\mu(y(t)) \right) (V^\mu(y(t)) \wedge \partial^\nu y(t)) \right) \cdot \oint_0^t A^\lambda(x(u)) \, \frac{dx^\lambda}{du} \, du \right] \]

\[ + \oint_0^1 A^\sigma(x) \, \frac{dx^\sigma}{ds} \cdot \oint_0^1 dt \left( \partial_\mu A^\nu(y(t)) - \partial_\nu A^\mu(y(t)) \right) (V^\mu(y(t)) \wedge \partial^\nu y(t)) \right) \cdot 1 , \]

(20)

where the term with the minus in the first contribution originates from the inverse path. Calculating the vacuum expectation value of the r.h.s. of Eq. (20), we have to Wick contract the different fields in the factors and terms to acquire the propagators. It is worth remarking that the partial derivatives \( \partial_\mu, \partial_\nu \) are defined with respect to the coordinate \( y \), i.e.

\[ \partial_\mu = \frac{\partial}{\partial y^\mu} , \quad \partial_\nu = \frac{\partial}{\partial y^\nu} . \]

Due to the path reduction property the lowest order contribution in the first term cancels, what was also checked by explicit calculations using the coordinate expression for the gluon propagator in the Feynman gauge:

\[ \langle 0|T[A^\mu(x) A^\nu(y)]|0 \rangle = D^{\mu\nu}_A(x - y) = \frac{(\mu^2 \pi)^\epsilon}{4 \pi^2} \Gamma(1 - \epsilon) \frac{g^\nu_\mu \delta^{ab}}{[-(x - y)^2]^{1-\epsilon}} . \]

(21)

The cancelation of these terms is graphically represented in Figs. 4 and 5.

\[ ^2 \text{Since the contributions have an opposite sign due to the inverse ordering on the inverse path.} \]
Before starting the explicit calculation of the remaining contribution we have to choose an appropriate vector field $V^\mu$ that will generate the same deformation as the $S_{12}\frac{\delta}{S_{12}}$ from [31]. Choosing $V^\mu := (\ell_1^+ \sigma, 0^-, 0_\perp)$, $\sigma \in [0, 1]$ we see immediately that this will restrict the possible contributions from the wedge product $V^\mu(y(\sigma)) \wedge \dot{\gamma}^\nu(y(\sigma))$ due to its anti-symmetric nature:

- Along $\ell_1$: $V^\mu \wedge \dot{\gamma}^\nu = 0$, what follows from the asymmetry of the wedge product and the fact that both vectors are parallel
- Along $\ell_2$: $V^\mu \wedge \dot{\gamma}^\nu = -\ell_1^+ \ell_2^- (\partial_+ \wedge \partial_-)$, due to (anti-)linearity of the wedge product
- Along $\ell_3$: $V^\mu \wedge \dot{\gamma}^\nu = 0$, what follows from the asymmetry of the wedge product and the fact that both vectors are parallel
- Along $\ell_4$: $V^\mu \wedge \dot{\gamma}^\nu = 0$, because we assume the vector field to be zero along the part of the path.

As a result we only need to consider the following Wick contractions:

- $A_\sigma^a(x(\sigma))\partial_\mu A_\nu^b(y(\sigma')) = \partial_\mu D_{\sigma\nu}^{ab}(x - y) = \delta^{ab}\partial_\mu D_{\sigma\nu}(x - y)$
- $A_\sigma^a(x(\sigma))\partial_\nu A_\mu^b(y(\sigma')) = \partial_\nu D_{\sigma\mu}^{ab}(x - y) = \delta^{ab}\partial_\nu D_{\sigma\mu}(x - y)$,

with the side note that $y$ is restricted to the top line in the diagram shown in Fig. 5. Each of these Wick contractions gives rise to four terms, one for each side of the quadrilateral in Fig. 3 so that we end up with a total of eight terms which we calculate below.
1. $\partial_\mu D_{\sigma\nu}(x-y) - \partial_\nu D_{\sigma\mu}(x-y)$ term with $x \in \ell_1$

Parametrizing the paths for $x$ and $y$ as (assuming that $x_1 = 0$):

\begin{align*}
  x &= \sigma \ell_1, \quad \sigma \in [0,1] \quad (22) \\
  y &= \ell_1 + \sigma' \ell_2, \quad \sigma' \in [0,1] ,
\end{align*}

we have:

\begin{align*}
  dx' &= \left( \frac{dx'}{d\sigma} \right) d\sigma = (\ell_1^+, 0^- , 0_\perp) d\sigma \\
  dy' &= \left( \frac{dy'}{d\sigma'} \right) d\sigma' = (0^+, \ell_2^- , 0_\perp) d\sigma' = \dot{\gamma}(\sigma') d\sigma' \\
  x - y &= (\sigma - 1)\ell_1 - \sigma' \ell_2 \\
  (x - y)^2 &= -2(\sigma - 1)\sigma' (\ell_1^+ \ell_2^-).
\end{align*}

For notational simplicity let us define:

\[ K_\epsilon := \left( \frac{\mu^2 \pi}{4\pi^2} \right)^\epsilon \Gamma(1 - \epsilon) . \quad (24) \]

Calculating this contribution:

\begin{align*}
  \int_0^1 d\sigma' d\sigma \frac{dx'}{d\sigma} \left( \frac{\partial}{\partial y_\mu} D_{\rho\nu}(x-y) - \frac{\partial}{\partial y_\nu} D_{\rho\mu}(x-y) \right) [V_\mu(y) \wedge \dot{\gamma}(y)] \\
  &= K_\epsilon \int_0^1 d\sigma' d\sigma \left( \frac{dy'}{d\sigma'} \left[ \frac{2(\epsilon - 1)g_{\rho\nu}(x-y)_\mu V_\mu(\sigma')}{{(x-y)^2}^{2-\epsilon}} \right] - \frac{dy'}{d\sigma'} \left[ \frac{2(\epsilon - 1)g_{\rho\mu}(x-y)_\nu V_\mu(\sigma')}{{(x-y)^2}^{2-\epsilon}} \right] \right) \\
  &= K_\epsilon \left[ \left( \frac{1 - \epsilon}{2} (-S_{12})^\epsilon \int_0^1 \frac{d\sigma d\sigma'}{{(\sigma - 1)^2}^{2-\epsilon}} \right) - \left( \frac{1 - \epsilon}{2} (-S_{12})^{\epsilon - 1}(\ell_1)^2 \int_0^1 \frac{d\sigma d\sigma'}{{(\sigma - 1)^2}^{2-\epsilon}} \right) \right] \\
  &= \frac{1}{2} K_\epsilon \frac{S_{12}^\epsilon}{\epsilon} ,
\end{align*}

where $S_{ij}$ represents the Mandelstam-like variable for the pair of vectors $\ell_{i,j}$. Which is exactly the same result as taking the derivative $\ell_1 \frac{\delta}{\delta \ell_1}$ of the original integral:

\[ \ell_1 \frac{\delta}{\delta \ell_1} K_\epsilon \int \frac{g_{\rho\nu} \frac{dx'}{d\sigma} \frac{dy'}{d\sigma'}}{{(x-y)^2}^{1-\epsilon}} = \ell_1 \frac{\delta}{\delta \ell_1} K_\epsilon \int \frac{(\ell_1 \ell_2) \frac{d\sigma d\sigma'}{{(\ell_1 \ell_2 (\sigma - 1)\sigma')^2}^{1-\epsilon}}}{{(2\ell_1 \ell_2 (\sigma - 1)\sigma')^2}^{1-\epsilon}} = \frac{1}{2} K_\epsilon \frac{S_{12}^\epsilon}{\epsilon} . \quad (26) \]

2. $\partial_\mu D_{\rho\nu}(x-y) - \partial_\nu D_{\rho\mu}(x-y)$ term with $x \in \ell_2$

This term is trivial since it reduces to a self-energy on the light-cone which in dimension regularization is formally zero.
3. \( \partial_\mu D_{\rho\nu}(x - y) - \partial_\nu D_{\rho\mu}(x - y) \) term with \( x \in \ell_3 \)

Making use of the symmetry \( 2\ell_1\ell_2 = -2\ell_2\ell_3 = S_{23} \), where now \( S_{23} \) is the second Mandelstam variable we can write down this contribution immediately:

\[
\int_0^1 d\sigma' \, d\sigma \frac{dx^\rho}{d\sigma} \left( \frac{\partial}{\partial y^\mu} D_{\rho\nu}(x - y) - \frac{\partial}{\partial y^\nu} D_{\rho\mu}(x - y) \right) \left[ V^\mu(y) \land \dot{\gamma}^\nu(y) \right] = 0 .
\]

which is again the same as taking the derivative \( \ell_1 \frac{d}{d\ell_1} \) since the original integral is formally independent of \( \ell_1 \) thus resulting in zero.

4. \( \partial_\mu D_{\rho\nu}(x - y) - \partial_\nu D_{\rho\mu}(x - y) \) term with \( x \in \ell_4 \)

This contribution is actually the most tricky to calculate, where the intricacies of the calculation are hidden in the combination of the integration and derivatives with respect to \( y \). So here we will apply a slightly different approach then in the derivations above. Instead of evaluating the integrals we will keep the integrals and show that the taking the derivative \( \ell_1 \frac{d}{d\ell_1} \) results in the same integrals as when we take the Fréchet derivative. Using the parametrization:

\[
\begin{align*}
x &= -(1 - \sigma)\ell_4, \quad \sigma \in [0, 1] , \\
y &= \ell_1 + \sigma'\ell_2 , \quad \sigma' \in [0, 1] ,
\end{align*}
\]

we start by splitting up the calculations in the contributions \( \partial_\mu D_{\rho\nu}(x - y) \) and \( -\partial_\nu D_{\rho\mu}(x - y) \). For the first term \( \partial_\mu D_{\rho\nu}(x - y) \) we proceed as before resulting in:

\[
-2(\epsilon - 1) \int_0^1 d\sigma' \, d\sigma \, [\ell_1 \cdot (\ell_1 + \sigma'\ell_2 + (1 - \sigma)\ell_4)] \frac{\ell_2 \cdot \ell_4}{(-(-1 - \sigma)\ell_4 - \ell_1)^{2-\epsilon}} ,
\]

the second term is the tricky one. If we look at the index of the derivative with respect to \( y \) (i.e. \( \nu \) one can see that then afterwards we integrate again over \( dy^\nu \), so that we might as well evaluate the original kernel \( \frac{1}{(-x - y)^{1-\epsilon}} \) between its boundary values as one would do by a normal integration. This results in:

\[
-\int_0^1 d\sigma' \, d\sigma \frac{dx^\rho}{d\sigma} \left( \frac{\partial}{\partial y^\rho} D_{\rho\mu}(x - y) \right) \left[ V^\mu(y) \land \dot{\gamma}^\nu(y) \right] =
\]

\[
- \int_0^1 d\sigma(\ell_1 \cdot \ell_4)\sigma' \left[ \frac{1}{(\ell_1 + \ell_2 + (1 - \sigma')\ell_4)^{2(1-\epsilon)}} - \frac{1}{(\ell_1 + \sigma'\ell_4)^{2(1-\epsilon)}} \right] = 0 ,
\]

\( (31) \)
where we used \((\ell_2 \cdot \ell_4) = 0\) and \(\ell_1 \ell_2 = -\ell_1 \ell_4\) making the two integrals equal which of course after subtraction results in the zero. Taking the \(\ell_1 \frac{\delta}{\delta \ell_1}\) of the original integral results in:

\[
\ell_1 \frac{\delta}{\delta \ell_1} \int_0^1 d\sigma' \, d\sigma \frac{dx^\sigma}{d\rho} \frac{dy^\mu}{d\sigma} (D_{\rho\mu}(x - y)) = \\
-2(\epsilon - 1) \int_0^1 d\sigma' \, d\sigma \, [\ell_1 \cdot (\ell_1 + \sigma' \ell_2 + (1 - \sigma)\ell_4)] \frac{(\ell_2 \cdot \ell_4)}{(-(\ell_1 + \sigma' \ell_2 + (1 - \sigma)\ell_4)^2)^{2-\epsilon}}, \tag{32}
\]

which is the same as Eq. (30) as desired.

Similar calculations with the variational vector field now chosen \((0^+, \ell_2^- , 0_\perp)\) and the point \(y\) restricted to the side \(\ell_3\) of the quadrilateral (due to the anti-symmetry of the wedge product) result in the contribution:

\[
\frac{1}{2} K, \frac{S_{23}^\epsilon}{\epsilon} - 2(\epsilon - 1) \int_0^1 d\sigma' \, d\sigma \, [\ell_4 \cdot (\ell_4 + \sigma' \ell_1 + (1 - \sigma)\ell_3)] \frac{(\ell_1 \cdot \ell_3)}{(-(\ell_4 + \sigma' \ell_1 + (1 - \sigma)\ell_3)^2)^{2-\epsilon}}, \tag{33}
\]

with \(S_{23} = 2(\ell_2 \cdot \ell_3)\).

Taking the trace over the color matrices then adds the color factor \(C_N\) an using the linearity of the wedge product in the vector field \(V^\mu\) we have the final result:

\[
\left(\ell_1 \frac{\delta}{\delta \ell_1} + \ell_2 \frac{\delta}{\delta \ell_2}\right) W_\gamma = D_V \, W_\gamma, \tag{34}
\]

with \(V^\mu = V_1^\mu + V_2^\mu = (\ell_1^+, \ell_2^- , 0_\perp)\) (see also figure 6). Taking into account the renormalization properties of the light-like Wilson quadrilateral loop [1–3, 31], we come to our final result:

\[
\mu \frac{d}{d\mu} \left[D_V \, W_\gamma\right] = - \sum \Gamma_{\text{cusp}}, \tag{35}
\]

where \(\Gamma_{\text{cusp}}\) is the light-cone cusp anomalous dimension [1, 4] and the summation runs over the number of cusps.

5. SUMMARY

After introducing classically the logarithmic Fréchet derivative as a diffeomorphism induced derivative with associated variational vector field \(V^\mu\) we have shown that its lowest order quantum field-theoretic contribution is equivalent to the derivative \(\ell_1 \frac{\delta}{\delta \ell_1} + \ell_2 \frac{\delta}{\delta \ell_2}\) we introduced in Ref. [31]. Therefore, we demonstrated explicitly that an important class of the “motions” (which apparently is not taken into account straightforwardly within the Makeenko-Migdal approach) in the generalized loop space can be described by using the mathematically consistently defined Fréchet derivative. Since diffeomorphisms cannot bring about new cusps, the number of cusps is diffeomorphism-invariant. We would expect then that the light-like Wilson polygonal loops having different number of cusps relate to different physical objects.

On the other hand, diffeomorphism-invariant transformations of the light-like loops find straightforward applications in the analysis of UV and rapidity evolution of gauge-invariant
correlation functions. In particular, a useful duality relation exists between this class of the
paths transformations in the generalised loop space and rapidity evolution of certain matrix
elements. Namely, rapidities attributed to the light-like vectors $\ell_{1,2}$ are formally infinite:

\[ y_{1,2} = \frac{1}{2} \ln \frac{\ell_{1,2}^+}{\ell_{1,2}^-} \sim \pm \frac{1}{2} \lim_{\eta^\pm \to 0} \frac{\ln ( \ell_1 \cdot \ell_2 )}{\eta^\pm} , \]  

where $\eta^\pm$ is a regulator and plus- and minus- components of a vector $a_\mu$ are given by the
scalar products $a^\pm = (a \cdot n^\mp)$ with $n^\mp \sim \ell_{2,1}$. Eq. (36) suggests, clearly, that

\[ \frac{d}{d \ln S_{ij}} \sim \pm \frac{d}{dy_{i,j}} . \]  

This suggests that the rapidity evolution of a given correlation function is dual to the area
transformations of a properly defined class of elements of the generalized loop space. Validity
of Eq. (35) in the higher orders of the perturbative expansion will be considered elsewhere
[44].

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