STRING TOPOLOGY WITH GRAVITATIONAL DESCENDANTS, AND PERIODS OF LANDAU-GINZBURG POTENTIALS

DMITRY TONKONOG

Abstract. This paper introduces new operations on the string topology of a smooth manifold: gravitational descendants of its cotangent bundle, which are augmentations of the Chas-Sullivan $L_\infty$ algebra structure of the loop space. The definition extends to Liouville domains. Descendants of the $n$-torus are computed.

To a monotone Lagrangian torus in a symplectic manifold, one associates a Laurent polynomial called the Landau-Ginzburg potential, by counting holomorphic disks. The following quantum periods theorem is proved: the constant terms of the powers of an LG potential are equal to descendant Gromov-Witten invariants of the ambient manifold. As an application, this allows to classify the LG potentials of monotone Lagrangian tori in $\mathbb{C}P^2$, and unlocks a symplectic approach towards the quantum Lefschetz hyperplane theorem.

1. Overview

1.1. Landau-Ginzburg potential. Given a smooth Fano variety considered as a symplectic manifold, one question that can be asked about it is what monotone Lagrangian tori does it contain. This paper explains how Lagrangian tori shed light on the enumerative geometry of the Fano variety, and vice versa.

The interest in monotone Lagrangian tori generally stems from mirror symmetry; concretely, recent constructions of such reveal exciting algebraic and combinatorial patterns which seem to correctly capture the structure of cluster charts of the mirror variety. The constructions include those of Lagrangian tori in $\mathbb{C}P^2$ (indexed by Markov triples, i.e. solutions of the Markov equation $a^2 + b^2 + c^2 = 3abc$) and del Pezzo surfaces by Vianna [102, 103, 104]; tori in $\mathbb{R}^6$ by Auroux [7]; and higher-dimensional mutations of Lagrangian tori in toric Fano varieties by Pascaleff and the author [85].

One is naturally interested in monotone Lagrangian tori up to Hamiltonian isotopy. Given such a torus $L \subset X$ with a fixed basis of $H_1(L; \mathbb{Z}) \cong \mathbb{Z}^n$, one defines the Landau-Ginzburg potential of $L$ to be the Laurent polynomial

$$W_L \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}],$$

where $x_1, \ldots, x_n$ are formal variables associated with the basis of $H_1(L; \mathbb{Z})$, in the following way. The LG potential counts $J$-holomorphic Maslov index 2 disks $(D, \partial D) \subset (X, L)$ whose boundary passes through a specified point on $L$, for some fixed compatible almost complex structure $J$. There is a finite number of such disks; the potential also retains the information about their boundary homology classes. Namely, to a disk $D$ with $[\partial D] = (v^1, \ldots, v^n) \in \mathbb{Z}^n = H_1(L; \mathbb{Z})$ one associates the monomial $\pm x_1^{v^1} \ldots x_n^{v^n}$, and $W_L$ is obtained as the sum of these monomials over all

This work was partially supported by the Simons Foundation grant #385573, Simons Collaboration on Homological Mirror Symmetry.
holomorphic disks as above; see e.g. [24, 6, 50]. The signs arise from the orientation on the moduli space of holomorphic disks.

This definition is not specific to tori but the requirement that \( L \subset X \) be monotone is essential: it guarantees that \( W_L \) is invariant of the choice of \( J \) and Hamiltonian isotopies of \( L \). Recall that a Lagrangian submanifold \( L \subset X \) is called monotone if the following two maps:

\[
\omega: H_2(X, L; \mathbb{Z}) \to \mathbb{R}, \quad \mu: H_2(X, L; \mathbb{Z}) \to \mathbb{Z},
\]

the symplectic area and the Maslov index, are positively proportional to each other. This in particular implies that \( \omega \in H^2(X; \mathbb{Z}) \) and \( c_1(X) \in H^2(X; \mathbb{Z}) \) are positively proportional, that is, \( X \) is a monotone symplectic manifold. Fano varieties constitute the main class of monotone symplectic manifolds.

**Example 1.1.** The product torus in \( \mathbb{R}^4 = \mathbb{C}^2 \) given by \( \{|z_1| = r_1, \ |z_2| = r_2\} \) is monotone if and only if \( r_1 = r_2 \). Its potential equals \( W = x_1 + x_2 \).

**Example 1.2.** Let \( X \) be a (smooth) compact toric Fano variety, \( \Delta \) its moment polytope and \( \pi: X \to \Delta \) the moment map. The preimages of interior points of \( \Delta \) are Lagrangian tori in \( X \), and there is a unique point in \( \Delta \) (the origin using the common normalisation of \( \pi \)) whose preimage is monotone. The potential of this monotone torus has been computed by Cho and Oh [24]: it is equal to the standard Hori-Vafa potential [69], cf. [10]. For example, for \( \mathbb{C}P^2 \) this is \( x_1 + x_2 + x_1^{-1}x_2^{-1} \).

1.2. **Quantum periods theorem.** Let \( W \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) be a Laurent polynomial. Its \( d \)th period \( \phi_d(W) \in \mathbb{C} \) is the constant term of the \( d \)th power \( W^d \).

**Example 1.3.** One has \( \phi_3(x_1 + x_2 + x_1^{-1}x_2^{-1}) = 6 \).

Let \( X \) be a compact monotone symplectic manifold. The \( d \)th quantum period of \( X \) is the number

\[
\langle \tau_{d-2 \text{ pt}} \rangle_{X,d} \in \mathbb{Q},
\]

the gravitational descendant one-pointed Gromov-Witten invariant of the point class \( \text{pt} \in H^{2n}(X) \), see Section 2 or [34, 57, 58] for a reminder.

**Theorem 1.1** (Quantum periods theorem). Let \( X \) be a closed monotone symplectic \( 2n \)-manifold and \( L \subset X \) a monotone Lagrangian torus with LG potential \( W_L \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \). Then for all \( d \geq 2 \),

\[
\phi_d(W_L) = d! \langle \tau_{d-2 \text{ pt}} \rangle_{X,d}.
\]

**Corollary 1.2.** Let \( X \) be a monotone symplectic manifold. For each \( d \), the \( d \)th periods of the LG potentials of all monotone Lagrangian tori in \( X \) are equal to each other.

Instead of working with descendant Gromov-Witten invariants directly, this paper mainly deals with the following enumerative version of them. Fix a point \( y \in X \), a compatible almost complex structure on \( X \) which is integrable in a neighbourhood \( U \) of \( y \), and a germ \( Y \subset U \) of a \( J \)-complex hypersurface passing through \( y \). Following Cieliebak and Mohnke [30], one defines the gravitational Gromov-Witten invariant

\[
\langle \tau_{d-2 \text{ pt}} \rangle^{\bullet}_{X,d} \in \mathbb{Z}
\]
to be the count of $J$-holomorphic Chern number $d$ spheres in $X$ which pass through $y \in X$ and have intersection multiplicity $d - 1$ with $Y$ at that point. See Section 2 for more details. The proof of Theorem 1.1 will actually show that

$$\phi_d(W_L) = d(d - 1) \cdot \langle \tau_{d-2} \text{pt} \rangle_{X,d},$$

whereas the comparison between the two versions of descendant GW invariants is a separate lemma proved in Section 2.

**Lemma 1.3.** For any Fano variety $X$ and any $d \geq 2$ it holds that

$$\langle \tau_{d-2} \text{pt} \rangle_{X,d} = (d - 2)! \langle \tau_{d-2} \text{pt} \rangle_{X,d}.$$

The background placing Theorem 1.1 and Corollary 1.2 into context will be surveyed soon, along with their applications. The following generalisation of Theorem 1.1 is proved as well.

**Theorem 1.4** (=Theorem 5.1). Let $X$ be a closed monotone symplectic manifold, and $M \subset X$ a monotone Liouville subdomain admitting a non-negatively graded Floer complex. Then

$$\langle BS|\ldots|BS \rangle_M = d! \langle \tau_{d-2} \text{pt} \rangle_{X,d}.$$

Here is a quick explanation of the statement, without going into the forthcoming details. A monotone Liouville domain $M \subset X$ is a notion generalising the Weinstein neighbourhood of a monotone Lagrangian submanifold; the definition is simple but does not seem to have appeared in the literature before. The Borman-Sheridan class $BS \in SH^0(M)$ accordingly generalises the Landau-Ginzburg potential. Here $SH^0(M)$ is the symplectic cohomology; for example, it holds that $SH^0(T^n) = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$

One says that $M$ has non-negatively graded Floer complex if $\partial M$ admits a contact structure such that the Floer complex $CF^*(M)$ computing the symplectic cohomology is concentrated in non-negative degrees up to an arbitrarily high action truncation: for example, cotangent bundles of manifolds admitting a metric with non-positive sectional curvature (like the $n$-torus) have this property.

Finally, the brackets from the left hand side of the identity from Theorem 1.4 are gravitational descendant operations

$$\langle \cdot | \ldots | \cdot \rangle_M : SH^0(M)^{\otimes d} \to \mathbb{C}$$

introduced in this paper. These operations are only defined at the cohomology level when $M$ admits non-negatively graded Floer complex. The main step towards the proof of Theorem 1.1 is to compute descendants of $M = T^n$; the statement is found in Theorem 4.5 and the proof occupies Section 6.

For a general Liouville domain, gravitational descendants are chain-level operations; they are quickly overviewed next.

**1.3. Gravitational string topology.** Let $M$ be a Liouville domain (see the references in Section 3); the cotangent bundle $M = T^*L$ of a smooth manifold $L$ is already a very interesting case for this discussion. The Floer chain complex $CF^*(M)$ computing symplectic cohomology has a natural structure of an $L_\infty$ algebra. It means that there is a sequence of operations

$$l^k : CF^*(M)^{\otimes k} \to CF^{*+3-2k}(M)$$
satisfying the $L_\infty$ relations appearing in Section 4. For each $m, k \geq 1$, this paper introduces a gravitational descendant operation which is a degree $2 - 2k$ map

$$\tau_{m-1}^k: CF^*(M)^{\otimes k} \to \mathbb{C}[-2m]$$

where $\mathbb{C}[-2m]$ is a copy of $\mathbb{C}$ of grading $-2m$. In other words the operation $\tau_{m-1}^k$ vanishes unless the degrees of the inputs sum to $2k - 2 - 2m$, and otherwise returns a number.

![Figure 1. Left: a gravitational descendant operation. Right: the bubbling showing part of the $L_\infty$ augmentation relation ($k = 4$, $r = 2$).](image)

Here is a rough definition of $\tau_{m-1}^k$. These operations count holomorphic maps $\mathbb{C}P^1 \setminus \{z_1, \ldots, z_k\} \to M$ with an additional marked point $z_0 \in \mathbb{C}P^1$, asymptotic to the given input orbits $x_i \in CF^*(M)$ at the punctures $z_i$, and passing through a specified point $y \in M$ at $z_0$ with intersection multiplicity $m$ with a fixed germ $Y$ of a complex hypersurface. The last condition is analogous to the one from the definition of $\langle \tau_{d-2\text{ pt}} \rangle$; see Figure 1, left. The precise definition also uses asymptotic markers explained in Section 4.

The stated degree of $\tau_{m-1}^k$ means that one is counting the moduli spaces which are zero-dimensional. Taking the boundary of 1-dimensional moduli spaces defined in the same way, see Figure 1, right, one obtains the following identities between gravitational descendants and the $L_\infty$ structure:

$$\sum_{1 \leq i \leq \ell, \sigma \in S_k} (-1)^\ell \frac{1}{\ell!(\ell-r)!} \tau_{m-1}^{k+1-r}(l^r(x_{\sigma_1}, \ldots, x_{\sigma_r}), x_{\sigma_{r+1}}, \ldots, x_{\sigma_k}) = 0.$$ 

These identities may be rephrased to say that for each $m \geq 1$, the collection of maps $\tau_{m-1} = \{\tau_{m-1}^k\}_{k \geq 1}$ is an $L_\infty$ morphism from the $L_\infty$ algebra $CF^*(M)$ to the shifted one-dimensional vector space $\mathbb{C}[-2m]$ considered as the trivial $L_\infty$ algebra. Yet another equivalent formulation is that $\tau_{m-1} = \{\tau_{m-1}^k\}_{k \geq 1}$ is a shifted augmentation of the $L_\infty$ algebra $CF^*(M)$.

Let $L$ be a smooth orientable spin manifold, and $\mathcal{L}L$ its free loop space. In view of the Viterbo isomorphism $SH^*(T^*L) \cong H_{n-*}(\mathcal{L}L)$ and the results extending it, the Floer complex $CF^*(T^*L)$ is understood as a model for the string topology of $L$, that is, for the space of chains $C_*(\mathcal{L}L)$ on the loop space together with the wealth of operations it carries (see the references in Section 4). For example, the symplectic $L_\infty$ algebra is conjecturally quasi-isomorphic to the Chas-Sullivan $L_\infty$ algebra structure on $C_*(\mathcal{L}L)$. 

Gravitational descendants \( \tau^{k}_{m-1} \) of \( T^*L \) are therefore new string topology operations. Unlike the previously known ones, descendants have no obvious explicit interpretation as operations defined geometrically on the chains on the loop space; indeed, it is not clear how to reinterpret the tangency condition for the holomorphic curves in terms of chains on \( L \). It would be interesting to obtain such a description, which might involve looking at the strata of intersections of loop cycles where the intersections happen less generically than transversally. It might also be possible to formulate these operations within the open conformal field theory framework for string topology [97, 59].

1.4. Periods and wall-crossing. For a Laurent polynomial \( W \) in \( n \) variables, a version of the Cauchy formula reads

\[
\phi_d(W) = \frac{1}{(2\pi i)^n} \cdot \int_{|x_1|=\ldots=|x_n|=1} W^d(x_1, \ldots, x_n) \frac{dx_1}{x_1} \wedge \ldots \wedge \frac{dx_n}{x_n}
\]

justifying the name period for \( \phi_d(W) \). Here \( x_i \) are considered as complex variables. This formula makes it obvious that the periods of \( W \) do not change if \( W \) is modified by a birational substitution of variables

\[
\begin{align*}
x_1 & \mapsto x_1, \\
\vdots & \\
x_{n-1} & \mapsto x_{n-1}, \\
x_n & \mapsto x_n \cdot f(x_1, \ldots, x_{n-1}),
\end{align*}
\]

where \( f \) is a holomorphic function. Provided such a substitution transforms \( W \) into another Laurent polynomial, this is called a mutation of \( W \), or a wall-crossing of \( W \), see e.g. [87, 73, 53, 4, 35, 85]. In particular the lemma below is easily seen from the Cauchy formula.

Lemma 1.5. The periods of a Laurent polynomial remain unchanged under mutation.

There exist geometric modifications of monotone Lagrangian tori in monotone symplectic manifolds which are also called mutations; they have been studied by Vianna [102] and Shende, Treumann and Williams [96] in dimension four, and by Pascaleff and the author [85] for higher-dimensional toric Fanos. The wall-crossing formula for such mutations was proved in [85]; it states that the Landau-Ginzburg potential of a monotone torus changes by a specific wall-crossing (1.1) when a Lagrangian torus is mutated geometrically. (In these known wall-crossing formulas, the functions \( f \) from (1.1) are of the form \( 1 + x_1 + \ldots + x_k \) for appropriately chosen \( H_1 \)-bases for the tori.) Therefore the known geometric mutations of Lagrangian tori do not change the periods of their Landau-Ginzburg potentials, in agreement with Theorem 1.1.

1.5. Mirror symmetry. Given a Laurent polynomial \( W \), one introduces the classical period of \( W \), or its non-regularised constant term series, by

\[
\pi_W = \sum_{d \geq 0} \frac{1}{d!} \phi_d(W) \cdot t^d.
\]

Let \( X \) be a monotone symplectic manifold. Givental’s \( J \)-series of \( X \) is a generating function for its Gromov-Witten invariants with gravitational descendants. Restricting to descendant Gromov-Witten invariants of the point class \( pt \in H_0(X) \),
one defines the quantum period of $X$, or the fundamental term of Givental’s $J$-series to be the following power series in the formal variable $t$:

$$G_X = 1 + \sum_{d \geq 2} (\tau_{d-2}\text{pt})_{X,d} \cdot t^d.$$ 

One says that $W$ is a mirror dual to $X$, see e.g. [31, Definition 4.9] if

$$\pi_W = G_X.$$ 

Surveys of this topic are found in [31, 32]; it has also been studied in e.g. [87, 73, 88, 89, 52] where a mirror dual potential is called a very weak Landau-Ginzburg model. In these terms, Theorem 1.1 can be reformulated as follows.

**Corollary 1.6.** Let $L \subset X$ be a monotone Lagrangian torus. Its LG potential is mirror dual to $X$, that is, $\pi_{W_L} = G_X$.

This corollary can be seen as a vast generalisation of the following theorem of Givental. It follows from his proof of mirror symmetry for toric varieties, specifically from his computation of their $J$-function [58]; see [32, Corollary C.2], [31] for details, and also recall Example 1.2. (Sample computations extracting quantum periods from the $J$-function are also found in [34, Example 10.1.3.1] and [65, Example 5.13].)

**Theorem 1.7.** Let $X$ be a toric Fano variety and $W$ its standard toric potential. Then $W$ is mirror dual to $X$. \hfill \Box

**Example 1.4.** The Laurent polynomial

$$W = x_1 + \ldots + x_{n-1} + x_1^{-1} \ldots x_{n-1}^{-1}$$

is mirror dual to $\mathbb{C}P^{n-1}$. The periods of $W$ are easily computed: for each $r \geq 1$

$$\phi_{nr}(W) = \binom{nr}{n,n,\ldots,n} = \frac{(nr)!}{(n!)^r}$$

where the middle term is a multinomial coefficient. The quantum periods of $\mathbb{C}P^{n-1}$ are

$$\langle \tau_{nr-2}\text{pt} \rangle_{\mathbb{C}P^n, nr} = \frac{1}{(n!)^r}.$$ 

The definition of a Laurent polynomial dual to a Fano variety has the following motivation. Under suitable assumptions, the series $G_X$ and $\pi_W$ are solutions to the following differential equations, respectively: the flatness equation with respect to the Dubrovin connection and the Picard-Fuchs equation. The classical mirror symmetry conjecture for variations of Hodge structures predicts that the two differential equations are equivalent; in particular for a true mirror potential $W$ it should hold that $G_X = \pi_W$.

The existence of mirror dual potentials has been established through explicit computation for important classes of Fanos like toric varieties and their complete intersections, and for certain other examples like Fano 3-folds where the proofs in a sense rely on a case-by-case analysis, see e.g. [33, 88, 70, 32]. Given a general Fano variety, little is known about the existence of a mirror dual potential, let alone mirror symmetry. The following is a conjecture.

**Conjecture 1.8.** For every smooth compact Fano variety, there exists a Laurent polynomial which is its mirror dual.
Theorem 1.1 reduces this to the problem of finding a monotone Lagrangian torus in $X$, without ever having to compute the periods explicitly. One generally expects that any Fano variety admitting a toric degeneration contains such a torus.

**Corollary 1.9.** Conjecture 1.8 holds for any Fano variety that contains a monotone Lagrangian torus. $\square$

1.6. **Quantum Lefschetz.** One envisions another application of the periods theorem to enumerative geometry. Given a smooth divisor $Y \subset X$ where both $X$ and $Y$ are Fano, the quantum Lefschetz theorem of Coates and Givental [33] relates their quantum periods to each other; in [32] one finds a more explicit version of this formula when $Y$ is toric. Theorem 1.1 should lead to a symplectic-geometric understanding of this result.

Consider a Fano hypersurface $Y \subset X$ whose homology class is proportional to the anticanonical class. Assuming $Y$ contains a monotone Lagrangian torus $L \subset Y$, one can write down an explicit formula relating the potential of $L \subset Y$ with the potential of its Biran lift [12, 11] to a monotone torus in $X$, cf. [13] and the ongoing project [39]. Comparing the periods of the two potentials yields a version of the quantum Lefschetz theorem: the result has been checked to agree with [32] in several illuminating examples. One hopes that the strong requirement on $Y$ above may be removed; the details will appear in a forthcoming work.

1.7. **Lagrangian tori and mirror cluster charts.** Mirror symmetry predicts that any compact Fano variety $X$ has a mirror Landau-Ginzburg model $(\tilde{X}, W)$ where $\tilde{X}$ is a complex variety, and $W: \tilde{X} \to \mathbb{C}$ is a holomorphic function whose fibres are compact Calabi-Yau varieties. Here $\tilde{X}$ is a complex variety which, without the potential $W$, is expected to be mirror to $X \setminus \Sigma$ where $\Sigma \subset X$ is a smooth anticanonical divisor. Mirror symmetry is a package of several interrelated conjectures, including mirror symmetry for variations of Hodge structures and homological mirror symmetry; they have been established in a significant number of cases.

The periods theorem and the results about mutations suggest the following very geometric conjecture within the framework of mirror symmetry; cf. [100, Section 1.2].

**Conjecture 1.10.** Let $X$ be a compact Fano variety. The set of embeddings $(\mathbb{C}^*)^n \subset \tilde{X}$ (‘cluster charts’) is in bijection with the set of monotone Lagrangian tori in $X$ up to Hamiltonian isotopy. For a chart $(\mathbb{C}^*)^n_L \subset \tilde{X}$ corresponding via this bijection to a monotone Lagrangian torus $L \subset X$, it holds that

$$W|_{(\mathbb{C}^*)^n_L} = W_L \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Above, the left hand side is the restriction of $W$ from $\tilde{X}$ onto $(\mathbb{C}^*)^n_L$; since this is a regular function on $(\mathbb{C}^*)^n_L$, it can be written as a Laurent polynomial. The right hand side is the LG potential of $L$.

Rather than considering cluster charts in the complex variety $\tilde{X}$, one can formulate an analogous conjecture by looking at the chambers in the mirror constructed using the Gross-Siebert algorithm [63, 64]; the papers of Carl, Pumperla and Siebert [20] and Prince [86] explain this point of view in detail.

This conjecture seems to be currently out of reach even for $X = \mathbb{CP}^2$. This is because it is extremely hard to classify Lagrangian tori up to Hamiltonian isotopy even in the simplest manifolds. Basically, the only known non-trivial classifications
are for $T^*T^2$ \[38\] and the neighbourhood of the Whitney immersion \[36\]. It is an open problem whether every monotone Lagrangian torus in $\mathbb{C}P^2$ is Hamiltonian isotopic to a Vianna torus; and whether a monotone Lagrangian torus in $\mathbb{R}^4$ is Hamiltonian isotopic to either the Clifford or the Chekanov one.

What Theorem 1.1 makes more tractable is an algebraic version of Conjecture 1.10: that the set of LG potentials of all possible monotone Lagrangian tori in $X$ is in bijection with the restrictions of the mirror LG model to the charts,

$$\{W_L : L \subset X\} / GL(n, \mathbb{Z}) = \{W_\mathcal{C}^n : (\mathcal{C}^*)^n \subset \mathcal{X}\} /GL(n, \mathbb{Z}).$$

Given a sufficiently complete construction of Lagrangian tori in $X$, Theorem 1.1 can reduce this question to a combinatorial one. For example, the answer to the relevant combinatorial question about $\mathbb{C}P^2$ has been given by Akhtar and Kasprczyk \[5\].

**Theorem 1.11.** Let $W(x, y)$ be a Laurent polynomial in two variables. If it has the same periods as $x + y + 1/xy$, that is $\pi W = \pi_{x+y+1/xy}$, then $W$ is equal to the potential of a Vianna torus. \[□\]

Now Theorem 1.1 implies the following.

**Corollary 1.12.** Every monotone Lagrangian torus in $\mathbb{C}P^2$ has the LG potential of a Vianna torus. \[□\]

### 1.8. Structure and proof ideas

The proof of Theorem 1.1 begins with a beautiful idea of Cieliebak and Mohnke which applies SFT stretching around $L \subset X$ to the holomorphic spheres computing $\langle \tau_{d-2 \text{pt}} \rangle_{X,d}$ choosing the point constraint to lie on $L$. The argument, sketched in Figure 2, is recalled in Section 2 together with some extra details about the invariant $\langle \tau_{d-2 \text{pt}} \rangle_{X,d}$. Sections 3, 4 analyse the two broken moduli spaces arising from the stretching; they compute the concurrently introduced Borman-Sheridan class and gravitational descendants, respectively. The short Section 5 revisits the stretching argument from Section 2 to show that it essentially amounts to Theorem 1.4. Section 6 contains the central computation of the paper: it determines descendant invariants for the cotangent bundle of the $n$-torus. The computation of descendants and Theorem 1.4 imply Theorem 1.1.

The computation in Section 6 starts with an identity for descendants of $T^*T^n$ implied by the $L_\infty$ relations and the computed Chas-Sullivan bracket on the loop space of the torus. The same identity is also shown by a completely alternative wall-crossing argument. The section finishes with an induction carefully devised to reduce all descendants to the standard one corresponding to the first period for $\mathbb{C}P^N$.

**Figure 2.** Stretching argument of Cieliebak and Mohnke.
Acknowledgements. I am deeply grateful to Denis Auroux, Tobias Ekholm and Ivan Smith for creating environments where doing mathematics turns into a truly engaging process. I learned a lot from them, and benefited many times from their support.

This work was partially supported by the Simons Foundation grant #385573, Simons Collaboration on Homological Mirror Symmetry.

2. Stretching closed-string descendants

2.1. Closed-string descendants. Let $X$ be a compact monotone symplectic manifold, e.g. a smooth Fano variety with a Kähler symplectic form. Fix a point $y \in X$, a (compatible) almost complex structure $J$ on $X$ which is integrable in a neighbourhood of $y$, and a germ $Y$ of a $J$-complex hypersurface defined in a neighbourhood of $y$ so that $y \in Y$. Following [30] consider the moduli space

$$M_1(\tau_{m-1} \text{pt})_d = \left\{ \begin{array}{l} u: \mathbb{C}P^1 \to X, \bar{\partial}u = 0, \ c_1(u) = d, \\ u(0) = y, \ u \text{ has local intersection} \\ \text{multiplicity } m \text{ with } Y \text{ at } u(0) = y \end{array} \right\} / \text{Aut}(\mathbb{C}P^1, 0).$$

The intersection multiplicity condition appearing above is sometimes called a \textit{tangency condition} of order $m-1$. The dimension of this moduli space is

$$2d - 2(m + 1).$$

Indeed, it can be computed as

$$2n + 2d - 4 - 2n - 2(m - 1).$$

Here $2n + 2d - 4$ is the dimension of the space of Chern number $d$ holomorphic spheres with one marked points; the $-4$ summand is due to the quotient by $\text{Aut}(\mathbb{C}P^1, 0)$. The $(-2n)$ and the $-(m-1)$ summands account for the incidence condition with $y$ and the intersection multiplicity, respectively.

The moduli space is 0-dimensional whenever $m = d - 1$, and for each $d \geq 2$ the \textit{gravitational Gromov-Witten invariant} is defined by:

$$\langle \tau_{d-2} \text{pt} \rangle_{X,d}^* := \# M_1(\tau_{d-2} \text{pt})_d \in \mathbb{Z}.$$  

(The appearance of $X$ in the subscript will be frequently omitted.) To make sure that this invariant is well-defined, one has to argue that the moduli space (2.2) is regular for generic $J$. As usual in Gromov-Witten theory, the regularity for generic $J$ is easily ensured for \textit{simple} curves; and it turns out that all curves in this moduli space are in fact simple.

\textbf{Lemma 2.1.} For any $d \geq 2$, every curve in $M_1(\tau_{d-2} \text{pt})_d$ is simple for a generic $J$.

\textbf{Proof.} Suppose a curve $u \in M_1(\tau_{d-2} \text{pt})_d$ is a degree $p$ branched cover over a simple curve $\tilde{u}$, and the cover $u \to \tilde{u}$ has ramification order $q \leq p$ at the origin (the tangency point). Then $c_1(\tilde{u}) = d/p$ and $\tilde{u}$ intersects $Y$ with multiplicity $(d - 1)/q$ at the origin. Hence $\tilde{u}$ belongs to the space

$$M_1(\tau_{(d-1)/q-1} \text{pt})_{d/p}.$$
Since simple curves are regular for generic $J$, the dimension of this moduli space has to be $\geq 0$. In view of (2.2) this means

$$\frac{d}{p} - \frac{(d - 1)}{q} - 1 \geq 0$$

or equivalently

$$\frac{(d - p)}{q} \geq \frac{(d - 1)}{p}.$$ 

But $q \leq p$ so this implies $d - p \geq d - 1$, or equivalently $p = 1$. Hence $u$ is simple. $\square$

In what follows, the stable map compactification $\overline{M}_1(\tau_{m-1} \pt)_d$ shall mean the following. Consider the moduli space $M_1(pt)_d$ of all Chern number $d$ spheres with one marked point passing through $y \in X$, without any tangency condition. This space has the usual Kontsevich stable map compactification $\overline{M}_1(pt)_d$. There is an inclusion

$$\mathcal{M}_1(\tau_{m-1} \pt)_d \subset M_1(pt)_d,$$

and by $\overline{\mathcal{M}}_1(\tau_{m-1} \pt)_d$ one denotes its closure in $\overline{M}_1(pt)_d$.

**Lemma 2.2.** All compactification strata of $\overline{M}_1(\tau_{m-1} \pt)_d$ are of complex codimension 1 and higher.

**Proof.** The curves in $\mathcal{M}_1(\tau_{m-1} \pt)_d$ can undergo bubbling which may or may not involve the marked point of tangency. In the first case, it follows from monotonicity in a standard way that the bubbled configurations are of codimension 1 and higher.

The second type of bubbling has one special case when the stable curve inheriting the tangency condition becomes contained in $Y$ near the tangency point $y$. If $Y$ were a globally defined divisor, this would have been an issue, but for a generic germ $Y$ of a hypersurface and generic $J$, there are no non-constant rational curves in $X$ that pass through $y$ and are contained in $Y$ in the neighbourhood of $y$. (This is ensured by perturbing $J$ in a non-integrable way in a bigger neighbourhood of $y$, keeping it integrable in a smaller one.)

So it remains to consider the case when the component of the stable curve inheriting the tangency condition is a constant curve at $y$. Consider such a stable map $u: C \to X$ modelled on a tree $T$ and denote by:

- $e \in T$ the edge corresponding to the component containing $z_0$,
- $T' \subset T$ the maximal subtree containing $e$ and consisting of components that are all contracted to the point $y$, and $C_0$ the corresponding sub-curve,
- $C_1, \ldots, C_l$ the sub-curves of the domain curve corresponding to the connected components of $T \setminus T'$,
- and $z_i \in C_i$ the marked points attached to the curves in $T'$ so that $u(z_i) = y$.

By [29, Lemma 7.2], the intersection multiplicity $m$ with $Y$ gets distributed between the curves $C_i$ namely:

$$\sum_{i=1}^l u_{z_i} Y \geq m$$

where $u_{z_i} Y$ is the intersection multiplicity at $z_i$. Suppose that $u_\epsilon \in \mathcal{M}_1(\tau_{m-1} \pt)_d$ is a sequence of elements Gromov converging to a stable map $u \in \overline{M}_1(pt)_d$ as $\epsilon \to 0$. Using the notation above, let $m_i = u_{z_i} Y$ and $d_i = c_1(C_i)$. One has $\sum d_i = d$. The dimension of the moduli space containing $C_i$ is at most

$$2d_i - 2m_i - 2.$$
where the equality is achieved when $C_i$ has a single component (no extra bubbles).

The sum of these dimensions is

$$2d - 2m - 2l.$$  

The dimension of the moduli space of the ghost component $C$ mapping to the point $y$ is

$$2(l + 1) - 6 = 2l - 4;$$

this is the dimension of the moduli space $\mathcal{M}_{l+1}$ of $l + 1$ marked points on $\mathbb{C}P^1$.

The resulting total sum of dimensions is at most

$$2d - 2m - 4 = 2d - 2(m + 1) - 2,$$

as claimed. □

**Lemma 2.3.** The number $\langle \tau_{d-2} \text{pt} \rangle^*_d$ is invariant of the choice of $y$, $Y$ and $J$ as above.

**Proof.** Consider a 1-dimensional moduli space $\mathcal{M}_1(\tau_{d-2} \text{pt})_d$ corresponding to a generic homotopy of the data $y, Y, J$ in a 1-parametric family. By Lemma 2.2, it undergoes no bubbling. □

### 2.2. Comparison with the psi-class definition.

As before, let $X$ be a compact monotone symplectic manifold; fix a compatible almost complex structure $J$. Let $\mathcal{M}_1(X, d)$ be the Kontsevich stable map compactification of the space of rational $J$-holomorphic curves $(u: \mathbb{C}P^1 \to X, z \in \mathbb{C}P^1)$ of Chern number $d$ and with one marked point. Let $L \to \mathcal{M}_1(X, d)$ be the line bundle whose geometric fibre at a stable map $(u: C \to X, z)$ is $T^*_z C$. The standard definition of the gravitational descendant Gromov-Witten invariant of the point class is

$$\langle \tau_d \text{pt} \rangle_d := \int_{[\mathcal{M}_1(X, d)\setminus]} c_1(L)^{d-2} \cup \text{ev}^\ast([\text{pt}]),$$

see [34, Definition 10.11], [74, Section 4]. Recall that since $X$ is monotone, $\mathcal{M}_1(X, d)$ compactifies $\mathcal{M}_1(X, d)$ by strata of positive complex codimension, and the fundamental class $[\mathcal{M}_1(X, d)]$ is defined in a straightforward way [74]. Above, the point class $[\text{pt}] \in H^{2n}(X)$ corresponds to the point constraint $y$ as in (2.1). The class $c_1(L)$ is frequently denoted by $\psi$ in the literature.

Returning to the moduli space (2.1) with a tangency condition at $y$, in the algebro-geometric world one would typically let $Y \subset X$ be a globally defined complex divisor. In this case it is generally hard to relate Gromov-Witten invariants with tangency conditions at $Y$ (also known as relative Gromov-Witten invariants) to gravitational descendant Gromov-Witten invariants due to sphere bubbles that fall into $Y$, cf. [62]. But in the actual setup of the moduli problem (2.1), where $Y$ is taken to be a germ of a hypersurface defined in a neighbourhood of a point $y \in X$ and the almost complex structure is otherwise generic, only constant bubbles in $Y$ are possible, which makes the comparison from Lemma 1.3 possible. The same statement holds true if $Y$ a globally defined complex hypersurface which is not uniruled so that one can pick a point $y \in Y$ not incident to any holomorphic sphere in $Y$.

**Proof.** The proof elaborates on the discussion found in the paper of Gathmann [56], cf. Vakil [101]. Pick $y, Y, J$ as in the setting of the tangency moduli problem (2.1). The strata of the stable map compactification $\overline{\mathcal{M}}_1(\tau_{m-1} \text{pt})_d$ are all of complex
codimension 1 and higher, and come in two types: those strata where the component of the stable curve inheriting the tangency marked point is not a constant map to the point \( y \), and those where it is. (The stable maps belonging to the strata of the second type were described in the proof of Lemma 2.2.) Denote by

\[
\mathcal{M}(m) \supset \mathcal{M}_1(\tau_{m-1} \text{ pt})_d,
\]

the union of \( \mathcal{M}_1(\tau_{m-1} \text{ pt})_d \) with the strata of the first type, and by

\[
D(m) \subset \mathcal{M}_1(\tau_{m-1} \text{ pt})_d
\]

the union of strata of the second type so that

\[
\mathcal{M}(m) \sqcup D(m) = \mathcal{M}_1(\tau_{m-1} \text{ pt})_d.
\]

For \( \mathcal{M}(m) \), the number \( m \) means the intersection multiplicity with \( Y \) at the marked point; for \( D(m) \), this number means the sum of intersection multiplicities at \( y \) for all non-constant curves attached to the constant tangency component.

Note that \( D(1) = D(2) \) because every curve in \( D(1) \) has at least two components, each having intersection multiplicity at least 1 with \( Y \). Whenever \( m \geq 2 \), one has \( \dim D(m) = \dim \mathcal{M}(m) - 1 \).

It is shown in [56, Section 2] that there is a section of \( \sigma_m \) of \( L \otimes m \) defined on \( \mathcal{M}_1(\tau_{m-1} \text{ pt})_d \) whose zero-locus inside \( \mathcal{M}(m) \) equals \( \mathcal{M}(m+1) \). However \( \sigma_m \) also vanishes identically on \( D(m) \). This section computes the \( m \)th jet normal to \( Y \) at the marked point, and an important step in [56] is to compute the vanishing multiplicities of \( \sigma_m \) on various irreducible components of \( D(m) \). For the present proof these multiplicities are inessential; without determining them one can write

\[
mc_1(L) \cdot [\mathcal{M}_1(\tau_{m-1} \text{ pt})] = [\mathcal{M}(m+1)] + [\hat{D}(m)]
\]

where \( [\hat{D}(m)] \) is a cycle supported on \( D(m) \), cf. [56, Remark 3.2].

From this one sees that the zero loci of the sections \( \sigma_1, \sigma_2, \ldots \) are not generic enough for computing (2.4) using their intersections. The non-genericity is due to the first two sections, \( \sigma_1 \) and \( \sigma_2 \), which both vanish on \( D(1) = D(2) \). Hence there is an obstruction class \( \sigma \in H^2(D(1)) \) such that

\[
c_1(L) \cdot c_1(L) : [\mathcal{M}_1(\text{pt})] = [\mathcal{M}(3)] + \sigma \cdot [\hat{D}(2)]
\]

Intersecting with the zero loci of \( \sigma_3, \ldots, \sigma_{m-2} \) one arrives at

\[
(m - 2)! \cdot c_1(L)^{m-2} : [\mathcal{M}_1(\text{pt})] = [\mathcal{M}(m-1)] + \sigma \cdot [\hat{D}(m-2)]
\]

where \( [\hat{D}(m-2)] \) is a cycle supported on \( D(m-2) \). Now take \( m = d \) so that \( \dim \mathcal{M}(d-1) = \dim D(d-2) = 0 \); then the last summand from the previous display formula vanishes because the application of \( \sigma \) drops the dimension of the cycle to \(-2\). The factor \( (m - 2)! \) above is due to the fact that \( \sigma_i \) is a section of \( L \otimes i \) rather than \( L \). Summarising, it has been shown that

\[
(d - 2)! \cdot c_1(L)^{d-2} : [\mathcal{M}_1(\text{pt})] = [\mathcal{M}(d-1)].
\]

This amounts to Lemma 1.3. \( \square \)
2.3. Two stabilisations. The domains of the curves from the moduli space (2.1) are unstable: they only have one marked point. Although it was shown above that this is not a problem for the regularity of the moduli space, for an argument in Section 5 it will be convenient to work with curves over stable domains, i.e. domains having at least 3 marked points. Two natural ways of stabilising the moduli problem (2.1) will be discussed, with a spelt out numerical difference. Either way is suitable for the purposes of Section 5.

Fix an integer $p \geq 2$. Let $\Sigma_i \subset X, i = 1, \ldots, p,$ be auxiliary oriented smooth codimension 2 submanifolds Poincaré dual to $N_{c_1}(X)$. One does not require the $\Sigma_i$ to be complex or even symplectic. Consider the moduli space

$$M_{p+1}(\tau_{m-1} \text{pt}, \Sigma_1, \ldots, \Sigma_p)_d = \left\{ \begin{array}{l} u: \mathbb{C}P^1 \to X, \quad \partial u = 0, \quad c_1(u) = d, \\ u(0) = \text{pt}, \\ u(z_i) \in \Sigma_i, \quad i = 1, \ldots, p, \\ u \text{ has local intersection multiplicity } m \text{ with } Y \text{ at } u(0) = y \end{array} \right\}. $$

Here $z_i \in \mathbb{C}P^1$ are distinct marked points which are free to vary in the domain. Recall that $Y$ is a germ of a hypersurface as earlier. For any $p$ this moduli space has the same dimension as $M_{p+1}(\tau_{d-2} \text{pt})_d$ given by (2.2). For any $d \geq 2, p \geq 2$ and $N \geq 1$ one has:

$$\langle \tau_{d-2} \text{pt} \rangle_d^\bullet = \frac{1}{(Nd)^p} \# M_{p+1}(\tau_{d-2} \text{pt}, \Sigma_1, \ldots, \Sigma_p)_d. $$

Indeed, consider the forgetful map

$$M_{p+1}(\tau_{d-2} \text{pt}, \Sigma_1, \ldots, \Sigma_p)_d \to M_1(\tau_{d-2} \text{pt})_d.$$

It has degree $(Nd)^p$ because the intersection number between every curve $u \in M_1(\tau_{d-2} \text{pt})_d$ and $\Sigma_i$ equals $Nd$. Moreover the preimages of those intersection points in $\mathbb{C}P^1$ are different for all $i$ for generic $J$, since $\Sigma_i \cap \Sigma_j \subset X$ is a codimension 4 submanifold. Hence marking the preimage of any such intersection by $z_i$ recovers the full preimage of the forgetful map, and its degree is evidently $(Nd)^p$.

A similar argument is used to prove that the right hand side of (2.6), i.e. the count of the moduli space (2.5), is invariant under generic choices of $y, Y, J$ and $\Sigma_i$. For this one needs to rule out bubbling happening to (2.5) when these data are changed in a 1-parametric family. Using monotonicity one easily rules out any non-constant sphere bubbles, and the remaining case to consider is when two different marked points $z_i, z_j$ collide producing a constant bubble. Forgetting the constant bubble, the main part of the curve (the one inheriting the tangency condition) acquires an incidence condition to $\Sigma_i \cap \Sigma_j$ which is a codimension 4 submanifold; this makes the Fredholm index of the moduli problem drop by 2, so this does not generically happen.

For the second stabilisation scheme, suppose $X$ is a Fano manifold with a Kähler symplectic form. Fix an integer $p \geq 2$ and a single oriented smooth complex codimension 1 divisor $\Sigma \subset X$ Poincaré dual to $N_{c_1}(X)$. Equip $X$ with an almost complex structure $J$ which preserves $\Sigma$ and is integrable in a neighbourhood of $\Sigma$. 
Consider the moduli space

\begin{equation}
\mathcal{M}_{p+1}(\tau_{m-1} \text{ pt}, \Sigma, \ldots, \Sigma)_d = \left\{ u : \mathbb{CP}^1 \to X, \partial u = 0, ~ c_1(u) = d, \right. \\
\left. u(0) = \text{pt}, \quad u(z_i) \in \Sigma, \quad i = 1, \ldots, p, \right. \\
\left. u \text{ has local intersection multiplicity } m \text{ with } Y \text{ at } u(0) = y \right\}.
\end{equation}

As earlier, \( z_i \in \mathbb{CP}^1 \) are distinct marked points which are free to vary in the domain. For any \( p \) this moduli space again has the same dimension as \( \mathcal{M}_{p+1}(\tau_{m-1} \text{ pt})_d \) given by (2.2). For any \( d \geq 2, N \geq 1 \) and \( Nd \geq p \geq 2 \) one has:

\begin{equation}
\langle \tau_{d-2} \text{ pt} \rangle_d^* = \frac{1}{(Nd)(Nd-1)\ldots(Nd-p+1)} \cdot \# \mathcal{M}_{p+1}(\tau_{d-2} \text{ pt}, \Sigma, \ldots, \Sigma)_d.
\end{equation}

This is to be compared with (2.6) which has a different numerical factor. Indeed, consider the forgetful map as earlier

\[
\mathcal{M}_{p+1}(\tau_{d-2} \text{ pt}, \Sigma, \ldots, \Sigma)_d \to \mathcal{M}_1(\tau_{d-2} \text{ pt})_d.
\]

This time, it has degree \( (Nd)(Nd-1)\ldots(Nd-p+1) \). This is true because any curve \( u \in \mathcal{M}_1(\tau_{d-2} \text{ pt})_d \) intersects \( \Sigma \) at precisely \( Nd \) points by positivity of intersections, and any ordered subcollection of \( p \) points among them (without repetitions) can be labelled by the \( z_i \).

As earlier, it is to be explained why the collision of marked points \( z_i = z_j \) does not generically occur. This is again a codimension 2 phenomenon but for a reason slightly different than in the previous setting. This time, assuming \( z_i \) and \( z_j \) collide creating a constant bubble, the main part of the curve acquires an incidence condition with \( \Sigma \) with a tangency, i.e. local intersection multiplicity with \( \Sigma \) at least two \( \text{[29 Lemma 7.2]} \). The tangency condition drops the Fredholm index of the corresponding moduli problem by 2, much like in the previous case.

2.4. Stretching after Cieliebak and Mohrke. A beautiful stretching argument due to Cieliebak and Mohrke appearing in [30] is now recalled. Let \( X \) be a closed monotone symplectic \( 2n \)-manifold and \( L \subset X \) a monotone Lagrangian torus or, more generally, a monotone Lagrangian submanifold admitting a metric of non-positive sectional curvature.

Consider the moduli space (2.2) choosing the fixed point \( y \) to lie in \( L: \ y \in L \). As above, choose a \( J \) which is integrable in a neighbourhood of \( y \), and a germ \( Y \) of a hypersurface near \( y \). For each \( r > 0 \), consider the almost complex structure \( J_r \) which is obtained by stretching \( J \) along the boundary of a fixed Weinstein neighbourhood of \( L \), using \( r \) as the stretching parameter. See e.g. [45] for the general stretching procedure, [30] for the specific argument being explained, and e.g. [66, 67, 37, 38] for neck-stretching applied in related problems. Since the \( J_r \) are unmodified near \( y \), one may consider the rigid moduli spaces \( \mathcal{M}_1(\tau_{d-2} \text{ pt}; J_r)_d \) as in (2.3) for each \( J_r \).

The SFT compactness theorem [45] states that as \( r \to +\infty \), the curves in the moduli space \( \mathcal{M}_1(\tau_{d-2} \text{ pt}; J_r)_d \) converge to broken holomorphic buildings composed of punctured holomorphic curves in \( T^*L \), \( S^*L \times \mathbb{R} \), and \( X \setminus L \). Here \( S^*L \) is the unit cotangent bundle of \( L \).

There is a component \( u \) of the limiting holomorphic building which lies in \( T^*L \) and inherits from (2.2) the incidence condition at the fixed point \( y \in L \) having intersection multiplicity \( d - 1 \) with \( Y \). Let \( p \) be the number of punctures of \( u \) and
\[ \gamma_1, \ldots, \gamma_p \text{ its (unparametrised) asymptotic Reeb orbits which correspond to closed geodesics on } L \text{ with respect to a metric chosen in advance.} \]

Following [30], a simple action argument shows that the lengths of the geodesics that can potentially arise as asymptotic Reeb orbits from the above stretching are bounded by an apriori constant which depends on the size of the Weinstein neighbourhood of \( L \) embeddable into \( X \). By the non-negative sectional curvature assumption, there exists a metric on \( L \) all of whose closed geodesics up to any given length satisfy [30]

\[ \mu(\gamma_i) \leq n - 1. \]

Here \( \mu \) stands for the Conley-Zehnder index, compare with Section 3.2 below. It also holds that \( 0 \leq \mu(\gamma_i) \), but this will not be used.

Accounting for the tangency condition, the Fredholm index of the moduli problem satisfied by \( u \) equals [30]

\[ (n - 3)(2 - p) + \sum_i \mu(\gamma_i) - (2n - 2) - 2(d - 2) = p(3 - n) + \sum_i \mu(\gamma_i) \geq 0; \]

it must be positive for regular curves. The regularity will be treated later in Section 5; now it is taken for granted. In the above formula, \(-2(n - 2)\) accounts for the condition of passing through the fixed point \( y \in T^*L \), and \(-2(d - 2)\) accounts for the tangency condition.

Next, observe that \( p \leq d \).

Indeed, \( T^*L \) has no contractible Reeb orbits so every puncture of \( u \) gives rise to at least one holomorphic cap in \( X \setminus L \) which topologically corresponds to a disk in \((X, L)\) of Maslov index at least 2, by the monotonicity of \( L \). Since the original curve before the breaking was a Chern number \( d \) sphere, the limiting building has no more than \( d \) caps.

Combining the inequalities in the three display formulas above, the only possible solution is found to be:

\[ p = d, \quad \mu(\gamma_i) = n - 1. \]

Furthermore it follows that the whole broken building looks as shown in Figure 2 right. It consists of precisely \( d + 1 \) components where one component is the curve \( u \subset T^*L \) with the tangency condition appearing above, and the remaining \( d \) curves \( w_1, \ldots, w_d \subset X \setminus L \) are holomorphic planes in \( X \setminus D \) topologically corresponding to Maslov index 2 disks in \( X \setminus D \); note that the curves \( w_i \) are automatically simple. The \( \gamma_i \) is the asymptotic orbit of \( w_i \).

This is where the relevant part of the argument of Cieliebak and Mohnke stops: it proves at this point that \( L \) bounds a Maslov index 2 disk \( w_1 \), establishing the Audin conjecture which was one of the main goals of [30]. The next two sections give a second and more careful look at the corresponding moduli spaces; this leads to the definition of the Borman-Sheridan class and gravitational descendants of Liouville domains.
3. Borman-Sheridan class

3.1. Introduction. Liouville domains are a certain type of symplectic manifolds whose boundary \( \partial M \) is convex, hence is a contact manifold \([15][15]\). Each Liouville domain has an associated completion: the result of attaching an infinite collar to \( \partial M \). For brevity, this paper usually does not distinguish between a Liouville domain and its completion; hopefully this does not create any confusion.

Example 3.1. Two important examples of Liouville domains are the cotangent bundle \( T^*L \) of a smooth manifold \( L \), and the complement \( X \setminus \Sigma \) to an ample (not necessarily smooth) divisor in a compact Kaehler manifold \( X \).

Let \( X \) be a closed monotone symplectic manifold, for example a compact Fano variety. Let \( M \subset X \) be a symplectically embedded Liouville domain. For example, one may take \( M = X \setminus \Sigma \) as in the previous example, or \( M \) could be a Weinstein neighbourhood of a Lagrangian submanifold \( L \subset X \). Suppose \( M \) satisfies the condition called monotonicity introduced below which generalises the notion of a monotone Lagrangian submanifold. Then one can define the Borman-Sheridan class

\[ BS \in SH^0(M). \]

Roughly speaking, the Borman-Sheridan class controls the deformation of holomorphic curve theory as one passes from \( M \) to \( X \). Its name originates from the ongoing work \([14]\), and related ideas can be traced back to the earlier works of Fukaya \([49]\) and Cieliebak and Latschev \([27]\). The first definition of the Borman-Sheridan class that appeared in the literature is due to Seidel \([95]\), and was given in the Calabi-Yau context.

In the present setup, the Borman-Sheridan class has been defined by the author \([100]\) using the Hamiltonian framework of symplectic cohomology (under a condition slightly stronger than monotonicity, although very similar). This section gives a brief definition of the Borman-Sheridan class in the SFT framework. The SFT version is slightly easier to define, but assumes that a suitable virtual perturbation scheme for holomorphic curves has been fixed.

In the special case when \( \partial M \) has no contractible Reeb orbits, the transversality difficulties of the SFT framework disappear and virtual perturbations are not required. This is completely enough for the aims of this paper: the proof of Theorem 1.1 only uses the Borman-Sheridan class for \( M = T^*T^n \) which has no contractible Reeb orbits.

3.2. Symplectic cohomology. Let \( M \) be a Liouville domain with \( c_1(M) = 0 \). An important symplectic invariant of \( M \) is its symplectic cohomology \( SH^*(M) \), which is a \( \mathbb{Z} \)-graded algebra over \( \mathbb{C} \). There are two definitions of symplectic cohomology, using the Hamiltonian and the Symplectic Field Theory frameworks; both are quickly reminded.

Hamiltonian definition. The classical definition of \( SH^*(M) \) takes the direct limit of the Floer cohomologies of a cofinal family of time-perturbed Hamiltonians with fast linear growth near \( \partial M \). The references include \([47][25][78][92][26][18][90]\). This paper adopts the cohomological grading convention of \([90][92]\): the Floer differential has degree +1, there is a natural map \( H^*(M) \to SH^*(M) \), and the Viterbo isomorphism \([105][91][1][2][3]\) reads

\[ SH^*(T^*L) \cong H_{n-*}(\mathcal{L}L) \]
where \( \mathcal{L}L \) is the free loop space of \( L \). For any Hamiltonian in the cofinal family of sufficiently high slope, its 1-periodic orbit are called the generators of the Floer complex \( CF^*(M) \).

The generators of \( CF^*(M) \) have the following geometric description: every unparametrised periodic Reeb orbit \( \gamma \subset \partial M \) gives rise to two Hamiltonian orbits \( \hat{\gamma}, \check{\gamma} \). Additionally, there are constant orbits corresponding to the critical points of the Hamiltonian on \( M \). The degree in \( CF^*(M) \) is denoted by \( |\cdot| \), and one has

\[
|\gamma| = |\check{\gamma}| + 1.
\]

The symbols \( x_i \) will be commonly used to denote generators of the Floer complex:

\[
x_i \in CF^*(M).
\]

This way, \( x_i \) can be a \( \hat{\gamma}, \check{\gamma} \) or a constant orbit.

**SFT definition.** The second definition of \( SH^*(M) \), using the SFT framework, is given in [19, 15, 44], see also [40, 43, 42]. In [19] the symplectic cohomology is denoted by \( \text{NCH}^{\text{lin}}_*(M) \) and called filled non-equivariant linearised contact homology; while [15, 44] keep the notation \( SH^*(M) \) which is adopted here. In this version the complex \( CF^*(M) \) is generated by the critical points of a Morse function on \( M \), and two formal generators \( \hat{\gamma}, \check{\gamma} \) associated to each periodic Reeb orbit \( \gamma \subset \partial M \) (no longer thought of as being 1-periodic orbits of any specific Hamiltonian). The differential counts purely holomorphic cylinders between the Reeb orbits augmented by holomorphic planes, with respect to a \( J \) which is cylindrical at infinity of a domain. This is the version of the definition being used in the present paper, although the gradings conventions are adopted from above.

For a general Liouville domain \( M \), the SFT definition of \( SH^*(M) \) requires a choice of a virtual perturbation scheme making the moduli spaces regular; see [51, 68, 9, 81]. An example of a Liouville domain whose SFT version of symplectic cohomology can be defined using elementary transversality arguments is \( \mathcal{T}^*L \) where \( L \) is a smooth manifold admitting a metric of non-positive curvature, for instance, the torus [17]. This is enough for the purpose of proving Theorem 1.1.

The domains of curves whose counts define various operations on the symplectic cohomology, like the Floer differential or the product, by definition carry asymptotic markers at each puncture. This is true both in the Hamiltonian and the SFT setups. In the Hamiltonian setup the markers are required to define the Floer equation with a time-dependent Hamiltonian. In the SFT setup, asymptotic markers are not needed to define the holomorphic equation \textit{per se}; the way they appear in moduli problems is as follows. Whenever \( \check{\gamma} \) is the input asymptotic of a holomorphic curve, or \( \hat{\gamma} \) is an output, the asymptotic marker on the curve at that cylindrical end must go to a given point on the geometric Reeb orbit \( \gamma \), fixed in advance. In the other two scenarios, when \( \hat{\gamma} \) is an input or \( \check{\gamma} \) is the output, the asymptotic marker is unconstrained.

The grading conventions for symplectic cohomology in the SFT setup [19] are different from those used in the Hamiltonian one. It is worth spelling out the difference. One denotes the Conley-Zehnder index of an unparametrised Reeb orbit \( \gamma \subset \partial M \) by

\[
\text{CZ}(\gamma) = \mu(\gamma).
\]

When \( M = \mathcal{T}^*L \) and \( \gamma \) corresponds to a geodesic in \( L \), \( \mu(\gamma) \) is the Morse index of that geodesic. When \( L \) has non-negative curvature, the Morse indices lie within the range \([0, n-1]\), \( n = \dim L \). The SFT grading of \( \gamma \) is given by \( \bar{\mu}(\gamma) \) where

\[
\bar{\mu}(\gamma) = \mu(\gamma) + n - 3,
\]
if \( \dim M = 2n \). For example, \( \bar{\mu}(\gamma) \) is the dimension of the moduli space of unparametrised holomorphic planes in \( M \) asymptotic to \( \gamma \). Finally, one has

\[
\begin{align*}
|\gamma| &= n - \mu(\gamma) = 2n - 3 - \bar{\mu}(\gamma), \\
|\hat{\gamma}| &= n - 1 - \mu(\gamma) = 2n - 4 - \bar{\mu}(\gamma).
\end{align*}
\]

where \( |\cdot| \) is the cohomological degree in the Hamiltonian setup, used in this paper.

3.3. **Monotone Liouville subdomains.** Let \((X, \omega)\) be a monotone symplectic manifold, and \((M, \theta) \subset X\) a (non-completed) Liouville domain symplectically embedded into \( X \), where \( \theta \) is a Liouville form on \( M \). Assume that \( c_1(M) = 0 \) and choose a trivialisation of the canonical bundle \( K_M \). This trivialisation gives rise to the relative first Chern class

\[ c_1^{rel} \in H^2(X, M) \]

analogous to (twice) the Maslov class of a Lagrangian submanifold. Define

\[ \omega^{rel} \in H^2(X, M) \]

by its value on any relative homology class \( A \in H_2(X, M) \) as follows:

\[
(3.2) \quad \omega^{rel} = \int_A \omega - \int_{\partial A} \theta.
\]

**Definition 3.1.** A symplectic embedding \((M, \theta) \subset X\) of a Liouville domain with boundary into a closed symplectic manifold is called **monotone** if \( c_1^{rel} \) and \( \omega^{rel} \) are positively proportional.

**Remark 3.2.** If \((M, \theta) \subset X\) is monotone and \( \theta' \) is a different Liouville form on \( M \) such that \( \theta - \theta' \) is an exact 1-form, then \((M, \theta') \subset X\) is also monotone.

**Example 3.3.** Suppose \( L \subset X \) is a monotone Lagrangian submanifold, and let \( M = T^*L \) be a neighbourhood of the zero-section of its cotangent bundle embedded into \( X \) as a Weinstein neighbourhood of \( L \), equipped with the Liouville form \( \theta \) making the zero section exact. Recall the agreement to blur the distinction between a Liouville domain and its completion. Then \( M \subset X \) is a monotone embedding. Indeed, a class \( A \in H_2(X, M) \) can be uniquely extended to a class \( A' \in H_2(X, L) \) so that \( \omega^{rel}(A) = \omega(A') \) and \( c_1^{rel}(A) = 2\mu(A') \).

**Example 3.4.** Suppose \( \Sigma \subset X \) is a (not necessarily smooth) anticanonical divisor, then \( M = X \setminus \Sigma \) is monotone in \( X \) when equipped with the standard trivialisation of \( K_M \) and the standard Liouville form \( \theta \) having a simple pole along \( \Sigma \). Indeed, a version of the Stokes formula gives

\[ \int_A \omega = \int_{\partial A} \theta + A \cdot \Sigma \]

and \( A \cdot \Sigma = c_1^{rel}(A) \).

3.4. **Borman-Sheridan class.** Consider the complex plane \( \mathbb{C} \), viewed as a \( \mathbb{C}P^1 \) with one puncture. Equip \( \mathbb{C} \) with a fixed asymptotic marker at the puncture, which one can simply take to be the real positive direction \( \mathbb{R}_+ \subset \mathbb{C} \). The domain \((\mathbb{C}, \mathbb{R}_+)\) with this marker has real 3-dimensional space of automorphisms

\[
(3.3) \quad \text{Aff}_+(\mathbb{C}) = \{ z \mapsto az + b \mid a \in \mathbb{R}_+, b \in \mathbb{C} \}.
\]
Let \((M, \theta)\) be a monotone Liouville subdomain in a closed symplectic manifold \(X\). For a Reeb orbit \(\gamma \subset \partial M\), consider the corresponding generator \(\hat{\gamma} \in CF^*(M)\) of degree zero: \(|\hat{\gamma}| = 0\). Set
\[
W = X \setminus M,
\]
with \(\partial M\) being its negative contact boundary. In this subsection is better to distinguish the notation between \(W\) and its completion
\[
\hat{W} = ((-\infty, 0] \times \partial M) \sqcup (X \setminus M)
\]
with the symplectic form
\[
\tilde{\omega} = \begin{cases} 
d(e^r\lambda) & \text{on } (-\infty, 0] \times \partial M \\
\omega & \text{on } X \setminus M,
\end{cases}
\]
where \(\lambda = \theta|_{\partial M}\) is the contact form on \(\partial M\), \(r \in (-\infty, 0]\) is the standard collar co-ordinate and \(\omega\) is the initial symplectic form on \(X\).

Assuming that \(\partial M\) has no contractible Reeb orbits, define
\[
M_{1|0}(\hat{\gamma})_W = \left\{ u: C \to \hat{W}, \ c_1^{rel}(u) = 1, \ u \text{ is asymptotic to } \hat{\gamma} \text{ at } \infty \text{ as output} \right\} / \text{Aff}_+ (\mathbb{C}).
\]
According to Section 3.2 the asymptotic condition includes the requirement that \(u|_{\mathbb{R}_+}\) converges to a specified initial point of \(\gamma\).

When \(\partial M\) has contractible Reeb orbits, the moduli space \(M_{1|0}(\hat{\gamma})_W\) is defined to consist of curves as above which are additionally augmented, i.e. they may have arbitrarily many additional punctures augmented by holomorphic planes in \(M\). The details follow the usual framework of augmented curves [19, 15] and are omitted.

The moduli space \(M_{1|0}(\hat{\gamma})_W\) is zero-dimensional. In general, the dimension formula for \(M_{1|0}(\hat{\gamma})_W\) letting \(|\hat{\gamma}|\) and \(c_1^{rel}(u)\) be arbitrary, would be
\[2c_1^{rel}(u) + |\hat{\gamma}| - 2.\]

The Borman-Sheridan class is defined as follows:
\[
BS = \sum_{\hat{\gamma} \in CF^0(M)} (\# M_{1|0}(\hat{\gamma})_W) \cdot \hat{\gamma} \in SH^0(M).
\]
It is closed because its Floer differential counts the boundary points of a one-dimensional moduli space of the same type, which causes the differential to vanish. This class depends on \(M\) and \(W\), i.e. on \(M\) and its embedding into \(X\). The monotonicity condition implies the invariance of the Borman-Sheridan class; this is an analogue of the statement that the count of Maslov index 2 disks with boundary on a monotone Lagrangian submanifold is an invariant. A preliminary lemma is necessary first.

**Lemma 3.2.** Consider a holomorphic curve \(u: C \to \hat{W}\) asymptotic to a Reeb orbit and its relative homology class
\[
[u] \in H^2_{BM}(\hat{W}) \cong H_2(X, M)
\]
where BM stands for Borel-Moore homology, i.e. the homology of locally compact chains. Under this identification, it holds that
\[
\tilde{\omega}(u) = \omega^{rel}(u)
\]
where the two symplectic forms are from (3.4) and (3.2).
Proof. Consider a collar $C \subset X \setminus M$ of $\partial M$ and identify it with

$$C = (0, \epsilon) \times \partial M, \quad \omega|_C = d(e^r\lambda)$$

where $r \in (0, \epsilon)$ and $\lambda = \theta|_{\partial M}$. The same collar embeds into $\hat{W}$ extending the infinite end of $\hat{W}$ to an embedding

$$\hat{C} = (-\infty, \epsilon) \times \partial M \subset \hat{W}, \quad \hat{\omega}|_{\hat{C}} = d(e^r\lambda).$$

Fix a monotone bijective function $(-\infty, \epsilon) \to (0, \epsilon)$ which is the identity in a neighbourhood of $\epsilon$. It defines a diffeomorphism $\hat{C} \to C$ which extends by the identity to the diffeomorphism $\phi: \hat{W} \to X \setminus M$. The claim is that for any relative homology class $A \in H_2(X, M) = H_2(X \setminus M, \partial M)$, it holds that

$$\phi^*\hat{\omega}(A) = \omega^\text{rel}(A).$$

To see this, realise $A$ as a chain and break it into a union $A' \sqcup A''$ of chains in $C$ and $(X \setminus M) \setminus C$. One has $\int_{A''} \omega = \int_{A''} \phi_*\hat{\omega}$ because the two forms coincide on $(X \setminus M) \setminus C$. Next one has $\int_{A'} \omega = \int_{A'} \phi_*\hat{\omega} + \int_{\partial A} \lambda$ by the Stokes formula. But $\int_{\partial A} \lambda = \int_{\partial A} \theta$, which justifies (3.5). This equality immediately implies the statement of the lemma. □

Lemma 3.3. If $M \subset X$ is monotone, the Borman-Sheridan class $BS \in SH^0(M)$ does not depend on the choice of $J$ on $\hat{W}$ which is compatible with $\hat{\omega}$ and cylindrical at infinity.

Proof. Any holomorphic curve $u: C \to \hat{W}$ satisfies $\hat{\omega}(u) > 0$; by Lemma 3.2, $\omega^\text{rel}(u) > 0$. So the monotonicity condition guarantees that $c_1^\text{rel}(u) \geq 1$. This implies that 1-dimensional moduli spaces (3.4) with varying $J$ do not undergo any SFT breaking except for the breaking of augmented Floer cylinders. Indeed, any other broken building would have at least two components mapping into $\hat{W}$, and this would contradict the additivity of $c_1^\text{rel}$ and the fact that the Borman-Sheridan class is defined using curves with $c_1^\text{rel} = 1$. □

Figure 3. Bubbling responsible for the failure of the invariance of the Borman-Sheridan class in the non-monotone case. The numbers indicate $c_1^\text{rel}$.

Remark 3.5. For a non-monotone Lagrangian submanifold $L \subset M$, a Maslov index 2 disk can break into a Maslov index 2 and a Maslov index 0 disk inside a 1-parametric family. A similar phenomenon for the Borman-Sheridan class is the breaking of a (relative) Chern number 1 plane into a Chern number 1 plane and a Chern number 0 plane. It may be useful to take a look at a model for this breaking, shown in Figure 3.
Remark 3.6. The definition of the Borman-Sheridan class using Hamiltonians appearing in [100] uses a slightly stronger assumption on the embedding $M \subset X$ than monotonicity. The condition is that there is a smooth Donaldson divisor $\Sigma \subset X$ away from $M$, $M \subset X \setminus \Sigma$ is a Liouville embedding, and $c_1^{rel} \in H^2(X, M)$ is positively proportional to the intersection number with $\Sigma$.

3.5. The torus case. Consider a monotone Lagrangian torus $L \subset X$, and let $M = T^*L$ be embedded into $X$ as a Weinstein neighbourhood of $L$. In this case the Borman-Sheridan class is nothing but a reformulation of the LG potential. One has

$$SH^0(T^*T^n) \cong H_n(\mathcal{L}T^n) \cong \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

The first isomorphism is the Viterbo isomorphism, and the second one is understood by looking at the Serre fibration

$$\Omega T^n \to \mathcal{L}T^n \to T^n$$

where $\Omega T^n$ is the based loop space of $T^n$. Consider the map induced by intersection with the fibre of this fibration:

$$i_t: H_n(\mathcal{L}T^n) \to H_0(\Omega T^n).$$

This is easily seen to be an isomorphism, and

$$H_0(\Omega T^n) = \mathbb{Z}[\pi_1 T^n] \cong \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Lemma 3.4. Let $L \subset X$ be a monotone Lagrangian torus, $W_L$ its LG potential, and $BS \in SH^0(T^*L)$ the Borman-Sheridan class for the Weinstein neighbourhood of $L \subset X$. Under the identification (3.6), it holds that:

$$BS = W_L \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Proof. The Viterbo isomorphism is proven by considering moduli spaces of semi-infinite cylinders $[0, +\infty) \times S^1 \to T^*L$ asymptotic to generators $x \in CF^*(T^*L)$ and having free Lagrangian boundary condition on the 0-section $L$, giving rise to a quasi-isomorphism $CF^*(L) \to C_{n-\ast}(L)$. This holds for a general $L$, and now recall that $L$ is a torus. Fix a point $p \in L$ and a generator $\hat{\gamma} \in SH^0(T^*L)$ corresponding to a homology class $[\gamma] \in H_1(L; \mathbb{Z})$. It follows from the Viterbo isomorphism that the count of the cylinders with input $\hat{\gamma}$ and boundary passing through $p$, equals 1, moreover the boundary homology class of those cylinders is $[\gamma]$. Gluing these cylinders (see e.g. [80] for the gluing in the regular setting) to the moduli space $M_{1,0}(\hat{\gamma})_W$, one gets a sign-preserving bijection onto the moduli space of Maslov index 2 holomorphic disks in $(X, L)$ passing through $p \in L$ and having boundary homology class $[\gamma]$. The latter spaces count $W_L$ by definition.

4. Gravitational descendants of Liouville domains

This section defines gravitational descendants of a Liouville domain which are chain-level operations on its Floer complex; recalls the $L_\infty$ structure on the same complex; explains its relation with gravitational descendants; states the theorem computing gravitational descendants for the cotangent bundle of the $n$-torus; and mentions the string topology perspective of the story.
4.1. **Compactifying spaces of marked curves.** The first step is to introduce
the moduli spaces of domains used in the definitions of the $L_\infty$ structure and
gravitational descendants. Recall the notation

\[
\text{Aff}(\mathbb{C}) = \{ z \mapsto az + b \mid a \in \mathbb{C}, b \in \mathbb{C} \},
\]
\[
\text{Aff}_+(\mathbb{C}) = \{ z \mapsto az + b \mid a \in \mathbb{R}^+, b \in \mathbb{C} \}.
\]

For $k \geq 2$ consider the space $\mathcal{M}_{k+1}$ of $k+1$ distinct numbered marked points

\[ z_0, z_1, \ldots, z_k \in \mathbb{C} \mathbb{P}^1 \]

up to $\text{Aut}(\mathbb{C} \mathbb{P}^1)$. An *marker* at a point $z_i$ is the choice of a real ray in $T_{z_i} \mathbb{C} \mathbb{P}^1$; automorphisms of $\mathbb{C} \mathbb{P}^1$ naturally act on markers.

By finding an automorphism of $\mathbb{C} \mathbb{P}^1$ sending $z_0$ to the fixed point $\infty \in \mathbb{C} \mathbb{P}^1$ one identifies

\[ \mathcal{M}_{k+1} = \text{Conf}(\mathbb{C}, k) / \text{Aff}(\mathbb{C}), \]

the configuration space of $k$ distinct points in $\mathbb{C}$ up to automorphisms of $\mathbb{C}$. The following is an observation from [44].

**Lemma 4.1.** The choice of a marker at $z_0$ across the space $\mathcal{M}_{k+1}$ canonically induces a marker at each other point $z_i$.

**Proof.** For each curve $u \in \mathcal{M}_{k+1}$ with a given marker at $z_0$, consider any automorphism of $\mathbb{C} \mathbb{P}^1$ taking $z_0$ to $\infty \in \mathbb{C} \mathbb{P}^1$ and the marker to $\mathbb{R}^+$. Once such an automorphism has been applied, assign the marker $\mathbb{R}^+$ to each other marked point $z_i$. Automorphisms of $\mathbb{C} \mathbb{P}^1$ preserving $\infty$ with the marker $\mathbb{R}^+$ form the group $\text{Aff}_+(\mathbb{C})$. The action of this group preserves the horizontal direction, which ensures that the above assignment of markers is well-defined. $\square$

The space $\mathcal{M}_{k+1}$ has the classical Deligne-Mumford compactification $\overline{\mathcal{M}}_{k+1}$ by stable curves. The boundary strata of this compactification are the unions of $\mathcal{M}_j$s for $j \leq k$; they are of real codimension 2 and higher.

Fix the markers at all points $z_i$ across $\mathcal{M}_{k+1}$ as in Lemma 4.1. The Deligne-Mumford compactification is not compatible with these markers in the sense that there is no natural way of extending the choice of markers to the extra marked points of the stable curves. This is made possible by a different compactification of $\mathcal{M}_{k+1}$. There seems to be no widely used notation for it although the compactification itself is rather well known, see [16, Section 8], [74], [44]; here it will be denoted by $\overline{\mathcal{M}}_{k+1}$ and a definition appears shortly.

Denote

\[ \mathcal{R}_{k+1} = \text{Conf}(\mathbb{C}, k) / \text{Aff}_+(\mathbb{C}) \cong S^1 \times \mathcal{M}_{k+1}. \]

**Remark 4.1.** It is useful to think of this space as the moduli space of $k+1$ distinct points in $\mathbb{C} \mathbb{P}^1$ equipped with a marker at the first point $z_0$. This way, a section of the $S^1$-bundle $\mathcal{R}_{k+1} \to \mathcal{M}_{k+1}$ means a choice of marker at $z_0$ across $\mathcal{M}_{k+1}$ as in Lemma 4.1. The markers induced by Lemma 4.1 at all other points are simply $\mathbb{R}^+$ when $\mathcal{R}_{k+1}$ is viewed as in (4.2).

**Remark 4.2.** For the $L_\infty$ operations that will be recalled soon, one considers the $k$ points in $\mathbb{C}$ from (4.2) as input punctures equipped with the asymptotic marker $\mathbb{R}^+$, and the infinity as the output puncture with an asymptotic marker varying freely in the $S^1$-family.
It is convenient to begin with the compactification of $\mathcal{R}_{k+1}$ denoted by $\overline{\mathcal{R}}_{k+1}$. The geometric picture of the compactification is very natural: as two or more marked points in $\mathcal{R}_{k+1}$ approach each other, they create a bubble with a canonically induced marker at the attaching point which points in the direction $\mathbb{R}_+$. For example, the collision of two points together is a codimension 1 phenomenon because this collision can happen in an $S^1$-worth of ways with respect to the marker at infinity, see Figure 4. More than two points may also collide in codimension 1.

![Figure 4. An $S^1$ worth of ways for two marked points to collide with respect to the asymptotic marker at infinity.](image)

Formally, the compactification strata of $\overline{\mathcal{R}}_{k+1}$ are indexed by rooted trees $T$ with $k$ labelled leaves (not counting the root). A stratum corresponding to such a tree is the product of the spaces $\mathcal{R}_{k_i+1}$ across the vertices $v_i \in T$, where $k_i + 1$ is the valency of $v_i$:

$$\overline{\mathcal{R}}^T_{k+1} = \prod_{v_i \in T} \mathcal{R}_{k_i+1}.$$  

Very similarly, $\overline{\mathcal{M}}_{k+1}$ is a compactification of $\mathcal{M}_{k+1}$ whose strata are indexed by rooted trees with $k$ leaves exactly as above. If $T$ is a tree and $v_0 \in T$ its root vertex, the corresponding stratum is

$$\overline{\mathcal{M}}^T_{k+1} = \mathcal{M}_{k_0+1} \times \prod_{v_i \in T, i \neq 0} \mathcal{R}_{k_i+1}.$$  

Remark 4.3. There is a natural identification $\overline{\mathcal{R}}_{k+1} \cong S^1 \times \overline{\mathcal{M}}_{k+1}$ extending (4.2), since for any tree $T$

$$\overline{\mathcal{R}}^T_{k+1} = (S^1 \times \mathcal{M}_{k_0+1}) \times \prod_{v_i \in T, i \neq 0} \mathcal{R}_{k_i+1} = S^1 \times \overline{\mathcal{M}}^T_{k+1}.$$  

4.2. $L_\infty$ structure on the Floer complex. Let $M$ be a Liouville domain with $c_1(M) = 0$, and $CF^*(M)$ be the Floer complex computing $SH^*(M)$ graded cohomologically as in Section 3.2. For each $k \geq 1$ there is an operation

$$l^k : CF^*(M)^{\otimes k} \to CF^*(M), \quad \deg l^k = 3 - 2k,$$

where $l^1$ is the Floer differential. These operations turn $CF^*(M)$ into an $L_\infty$ algebra, which means that they satisfy the following two properties. The first one is graded-commutativity: for a permutation $\sigma \in S_k$, one has

$$l^k(x_1, \ldots, x_k) = (-1)^\dagger l^k(x_{\sigma_1}, \ldots, x_{\sigma_k})$$

where

$$\dagger = \sum_{i<j, \sigma(i) > \sigma(j)} |x_i| \cdot |x_j|.$$
The second property is the $L_\infty$ relations:

\begin{equation}
\sum_{1 \leq r \leq k, \sigma \in S_k} (-1)^{\varpi} \frac{1}{r!(k-r)!} l^{k+1-r}(I^r(x_{\sigma_1}, \ldots x_{\sigma_r}), x_{\sigma_{r+1}}, \ldots, x_{\sigma_k}) = 0
\end{equation}

where an explicit formula for the sign number $\varpi$ is rather lengthy; the easiest way to specify the sign is to reformulate the $L_\infty$ relations in terms of a single equation on the bar complex in the way it is done in e.g. \cite{75, 51}, cf. \cite{48}. The references for this $L_\infty$ structure are \cite{49, 75, 46, 44}. The latter reference defines the dual $L_\infty$ coalgebra, but the $L_\infty$ algebra definitions are analogous. See also \cite{94, 84} for $\ell^2$, the Lie bracket.

Consider the moduli space

$$M_{1|k}(x_0; x_1, \ldots, x_k)$$

of holomorphic curves in $M$ with domain an element of $R_{k+1}$, equipped with punctures and asymptotic markers as in Remark 4.2 which are asymptotic to $x_i \in CF^*(M)$ at the input punctures and to $x_0$ at the output. The dimension formula reads

$$\dim M_{1|k}(x_0; x_1, \ldots, x_k) = \dim R_{k+1} + |x_0| - \sum_{i=1}^k |x_i| = 2k - 3 + |x_0| - \sum_{i=1}^k |x_i|.$$ 

The operations $l^k$ are the counts of these moduli spaces when they are 0-dimensional. The graded-commutativity is immediate and the $L_\infty$ relations are derived by looking at the boundaries of the 1-dimensional moduli spaces above, compactified over $\overline{R}_{k+1}$. The orientation analysis responsible for the signs has been explained in \cite{44}, in the dual coalgebra setting.

4.3. Gravitational descendants. As above, let $M$ be a Liouville domain with $c_1(M) = 0$; fix the data defining the symplectic cohomology complex $CF^*(M)$. As in Section 2.1 fix a point $y \in M$ and assume that $J$ is integrable in its neighbourhood; fix a germ $Y$ of a hypersurface defined in the neighbourhood. As in Section 4.1 choose a marker at $z_0$ across the moduli space $M_{k+1}$ and use it to canonically induce markers at the other marked points $z_i$ as in Lemma 4.1.

For a domain $(\mathbb{CP}^1, z_0, \ldots, z_k) \in M_{k+1}$ equipped with markers in the way just specified, consider the corresponding curve with punctures

$$D = \mathbb{CP}^1 \setminus \cup_{i=1}^k \{z_i\}$$

equipped with cylindrical ends with asymptotic markers. Note that the point $z_0$ is not being punctured. For any $k \geq 1$, $m \geq 1$, and $x_i \in CF^*(M)$, consider the gravitational descendant moduli spaces

\begin{equation}
M_{1|k}(\tau_{m-1} \text{pt}, x_1, \ldots, x_k) = \{(D, u) : D \in M_{k+1}, u : D \to M, \\
\hat{\partial}u = 0, u(z_0) = y, \\
u \text{ has local intersection multiplicity } m \text{ with } Y \text{ at } u(z_0) = y, \\
u \text{ is asymptotic to } x_i, i = 1, \ldots, k, \\
at z_i \text{ considered as an input puncture.}
\end{equation}

One exception is the case $k = 1$ where one takes quotient by the remaining automorphism group $R_+$. 
The regularity of the moduli space (4.4), before any compactification is addressed, is easily achieved by using a domain-dependent \( J \), localised in a neighbourhood of \( z_0 \) in the domain and in a neighbourhood of \( y \) in the target; it has to be done invariantly under the permutations of inputs. This forces the curves in (4.4) to be simple, and hence generically regular.

The dimension of (4.4) equals

\[
2n + 2(k + 1) - 6 - 2m - 2(m - 1) - \sum_i |x_i|
\]

or equivalently

\[
2(k - 1) - 2m - \sum_i |x_i|.
\]

Let \( \mathbb{C}[-2m] \) be the 1-dimensional graded vector space concentrated in degree \(-2m\). In view of the above, for each \( m, k \geq 1 \) one has the gravitational descendant operations

\[
\tau_{m-1}^k : (CF^*(M))^\otimes k \to \mathbb{C}[-2m], \quad \deg \tau_{m-1}^k = 2 - 2k.
\]

By definition of this being a graded map, it vanishes unless (4.5) holds, in which case it is set to count

\[
\tau_{m-1}^k(x_1, \ldots, x_k) := \# M_{1|[x]}(\tau_{m-1} pt, x_1, \ldots, x_k).
\]

(The right hand side has been defined on the generators of \( CF^*(M) \), and it is extended by linearity to their combinations.) It is clear from the definition that these operations are invariant under permutations of the inputs up to sign, and orientation analysis similar to [44] reveals that

\[
\tau_{m-1}^k(x_1, \ldots, x_k) = (-1)^\dagger \tau_{m-1}^k(x_{\sigma_1}, \ldots, x_{\sigma_k})
\]

with \( \dagger \) is as above.

**Example 4.4.** Let \( \int : CF^*(M) \to \mathbb{C} \) be the projection to the constant orbit generator corresponding to the minimum of the Morse function on \( M \). This generator has zero degree and is a chain-level representative of the unit \( 1 \in SH^0(M) \). Then

\[
\tau_0^0(x, y) = \int (x \cdot y) \in \mathbb{Z},
\]

where \( \cdot \) is the standard product on symplectic cohomology. Indeed, the simple incidence condition \( (m = 1) \) at the fixed point \( y \) corresponds to applying \( \int \). The other descendant operations do not reduce to previously known algebraic structures on symplectic cohomology.

The curves in higher-dimensional moduli spaces (4.5) can undergo the following types of bubbling:

- bubbling resulting from domain degenerations, when several marked points \( z_i \) come together. The combinatorics of the limiting holomorphic buildings is governed by the underlying compactification \( \mathcal{M}_{k+1} \);
- bubbling of (augmented) Floer cylinders attached to inputs which happen independently of domain degenerations;
- the bubbling of a stable constant sphere at the tangency point \( y \) which results in the splitting of a curve at \( y \) similarly to the way described in the proof of Lemma 2.2. Analogously to that lemma, this is a codimension 2 phenomenon.

For generic 1-dimensional descendant moduli spaces, only the first two types of bubbling happens, which leads to the next theorem.
Theorem 4.2. For each $m \geq 1$, the collection of maps
\[ \tau_{m-1} = \{ \tau_{m-1}^k \}_{k \geq 1} \]
defines an $L_\infty$ morphism from the $L_\infty$ algebra $CF^*(M)$ to the 1-dimensional vector space $\mathbb{C}[-2m]$ considered as the trivial $L_\infty$ algebra. In other words, the equations below hold.
\[ \sum_{1 \leq r \leq k, \sigma \in S_k} (-1)^r \frac{1}{r!(k-r)!} \tau_{m-1}^{k+1-r}(l'_{r}(x_{\sigma_1}, \ldots, x_{\sigma_r}), x_{\sigma_{r+1}}, \ldots, x_{\sigma_k}) = 0. \]

Remark 4.5. One can equivalently say that $\tau_{m-1}$ is a shifted $L_\infty$ augmentation of $CF^*(M)$.

The second type of bubbling is responsible for the appearance of the $l_1$-terms (the Floer differential) in Theorem 4.2, and the first type of bubbling is responsible for the $l_r$, $r \geq 2$. The proof of Theorem 4.2 is entirely analogous to the proof of the $L_\infty$ relations stated in Section 4.2, and can be e.g. be obtained by a modification of [44].

4.4. Non-negatively graded domains. In view of Theorem 4.2, the descendant operations $\tau_{m-1}^k$ from (4.6) do not generally give rise to operations on symplectic cohomology. On the other hand, recall that the domain needed for proving Theorem 1.1 is $M = T^*T^n$. In this case one can arrange that $CF^*(M) = SH^*(M)$ up to an arbitrarily high action truncation, so the operations $\tau_{m-1}^k$ do automatically become cohomological operations. The definition below is a natural generalisation of this observation.

Definition 4.3. A Liouville domain $(M, \theta)$ is said to have non-negatively graded Floer complex if for any $A > 0$, there is a Liouville 1-form $\theta'$ on $M$ such that $\theta' - \theta = df$ and such that in the Floer complex $CF^*(M)$ defined using $\theta'$, every generator of action smaller than $A$ has non-negative degree. Equivalently, $\theta'|_{\partial M}$ is a contact form all of whose Reeb orbits $\gamma$ of action smaller than $A$ have Conley-Zehnder index satisfying $\mu(\gamma) \leq n - 1$.

Example 4.6. The cotangent bundle of a manifold with a non-positive sectional curvature has non-negatively graded Floer complex [30, Lemma 2.2]. If the manifold has negative sectional curvature, one can find a $\theta$ making all orbits have non-negative symplectic cohomology degree, otherwise this is only achieved up to an arbitrarily high action truncation.

Example 4.7. Let $X$ be a compact Fano variety and $\Sigma$ a smooth divisor in the class $N_{\mathcal{C}_1}(X)$ for $N \geq 1$. Then $X \setminus \Sigma$ has non-negatively graded Floer complex [55], in this case there is Liouville form $\theta$ all of whose Reeb orbits have non-negative symplectic cohomology degree.

Consider a Liouville manifold $M$ with a non-negatively graded Floer complex. For any $k \geq 2$ and generators $x_i \in CF^0(M)$, consider the counts
\[ \langle x_1 | \ldots | x_k \rangle_M = \tau_{k-2}^{k}(x_1, \ldots, x_k) \in \mathbb{Z}. \]

Remark 4.8. For any Liouville domain with non-negatively graded Floer complex, the degree 0 orbits $x_i$ are all of type $\hat{\gamma}$, not $\check{\gamma}$. 


In the rest of the paper, it will be assumed without further notice that all generators appearing in (4.7) have action $< A$ for a sufficiently large $A$, and that a corresponding Liouville form $\theta'$ from Definition 4.3 has been fixed.

**Proposition 4.4.** Consider a Liouville domain $M$ with non-negatively graded Floer complex. Then (4.7) descend to cohomological operations $SH^0(M)^\otimes k \to \mathbb{Z}$ invariant of the choices of $y$ and $Y$, and of compactly-supported homotopies of $J$.

**Proof.** Consider a moduli space (4.4) computing (4.7) where the data $y$, $Y$ or $J$ vary in a 1-parametric family. The moduli space becomes 1-dimensional as well, and curves in this space can undergo the bubbling outlined above.

The claim is that bubbling arising from domain degenerations does not happen. Suppose it happens; consider the part of the broken building which inherits the tangency condition. It belongs to a moduli space

$M_{1|k_0}(\tau_{m-1} \text{ pt}, x'_1, \ldots, x'_{k_0})$

for some $k_0 < k$ and $x'_i \in CF^*(M)$; according to Definition 4.3 holds that $|x'_i| \geq 0$. By (4.5) the dimension of this moduli space is $\leq -2$, hence such bubbling cannot happen.

The only bubbling that can happen in codimension 1 is, therefore, the bubbling of Floer cylinders from the input asymptotics, but they cancel when the inputs $x_i$ are Floer cocycles. \hfill $\Box$

**Remark 4.9.** If $M$ has a positively graded Floer complex without contractible Reeb orbits (e.g. the cotangent bundle of a torus), the definition and the well-definedness of the invariants (4.7) only use classical transversality techniques because one can use a domain-dependent $J$ as described above. Indeed, the proof of Proposition 4.4 only used the regularity for the main component of any bubbled curve; that component inherits a domain-dependent $J$ after any bubbling.

**4.5. Descendants of the torus.** Recall that

$$SH^0(T^*T^n) = H_n(LT^n) \cong \mathbb{Z}[H_1(T^n; \mathbb{Z})] \cong \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

The following notation for the elements of $SH^0(T^*T^n)$ shall be adopted: for $v_i \in H_1(T^n; \mathbb{Z}) \cong \mathbb{Z}^n$, the generator corresponding to it is denoted by

$$x^{v_i} = x_1^{v_{i1}} \cdots x_n^{v_{in}} \in SH^0(T^*T^n).$$

Here $v_{ij}$ are the co-ordinates of $v_i \in \mathbb{Z}^n$. The theorem below is the core computation of this paper; Section 6 is devoted to its proof.

**Theorem 4.5.** Suppose $v_1, \ldots, v_k \in H_1(T^n; \mathbb{Z})$. Then

$$\langle x^{v_1} | \cdots | x^{v_k} \rangle_{T^*T^n} = \begin{cases} (k-2)! & \text{if } v_1 + \ldots + v_k = 0, \\ 0 & \text{otherwise}. \end{cases}$$

The left hand side is the descendant invariant (4.7). It is clear that the descendant is zero unless $v_1 + \ldots + v_k = 0$ because the curves computing it can only connect orbits whose sum is null-homologous. A quick reflection persuades that the rest of the statement is not obvious. Clearly the descendants from the theorem are invariant under the diagonal action of $SL(n, \mathbb{Z})$ on $(\mathbb{Z}^n)^k$, but this action has infinitely many orbits on $k$-tuples of vectors summing to zero.
4.6. A reminder on symplectic cohomology operations. This is a good occasion to recall some basic operations on the symplectic cohomology \( SH^* (M) \); they will be used in Section 6. They are the product

\[ - \cdot - : SH^* (M) \otimes SH^* (M) \to SH^* (M), \]

the bracket

\[ l^2 : SH^* (M) \otimes SH^* (M) \to SH^{*-1} (M), \]

and the BV operator

\[ \Delta : SH^* (M) \to SH^{*-1} (M). \]

The bracket \( l^2 \) is part of the \( L_\infty \) structure defined above, and descends to the cohomology by (4.3). In the orientation framework being used for symplectic cohomology, both the product and the bracket are graded commutative:

\[ x \cdot y = (-1)^{|x||y|} y \cdot x, \quad l^2 (x, y) = (-1)^{|x||y|} l^2 (y, x). \]

The three operations are related by the identity analogous to (4.8).

\[ l^2 (x, y) = \Delta (x \cdot y) - \Delta (x) \cdot y - (-1)^{|x|} x \cdot \Delta (y), \]

see e.g. [94, 95]. It is reminded that the BV operator acts at chain level by \( \Delta (\tilde{\gamma}) = \hat{\gamma} \) and \( \Delta (\hat{\gamma}) = 0 \); \( \Delta \) also vanishes on the constant orbits.

4.7. String topology. Let \( L \) be a smooth orientable spin \( n \)-manifold, and \( M = T^* L \). The homology \( H_* (\mathcal{L} L) \) of its free loop space, and more generally the space of chains \( C_* (\mathcal{L} L) \) on the free loop space, carry an abundance of algebraic structures. Their study goes under the general name of string topology and was pioneered by Chas and Sullivan [23]. Examples of cohomological operations include the Chas-Sullivan product

\[ - \cdot - : H_* (\mathcal{L} L) \otimes H_* (\mathcal{L} L) \to H_{*-n} (\mathcal{L} L), \]

the Chas-Sullivan bracket

\[ [-, -] : H_* (\mathcal{L} L) \otimes H_* (\mathcal{L} L) \to H_{*-n+1} (\mathcal{L} L), \]

and the BV operator

\[ \Delta : H_* (\mathcal{L} L) \to H_{*+1} (\mathcal{L} L). \]

These three are related by the identity analogous to (4.8).

Remark 4.10. This paper uses the convention where the sign behaviour of the Chas-Sullivan bracket matches the symplectic cohomology one. In [23], the graded-commutativity property of the bracket, and the signs in a version of (4.8), are different. One brings them to match the symplectic cohomology signs by redefining \([x, y] \mapsto (-1)^{n+|x|} [x, y]\).

There exist other cohomological operations, see e.g. [59, 61]. And importantly, one expects there to be vastly more operations defined at the chain level, i.e. operations with inputs and outputs in \( C_* (\mathcal{L} L) \). For example, the differential and the chain-level Chas-Sullivan bracket are expected to extend to a sequence of operations which turn \( C_* (\mathcal{L} L) \) into an \( L_\infty \) algebra, called the Chas-Sullivan algebra. This is sketched in [23, 97], but in general the definition of chain-level string topology operation meets a technical obstacle: the natural geometric definitions usually work only when the inputs are sufficiently transverse chains of loops, and the issue lies in extending them to all chains. There are recently proposed solutions to this issue [71, 72].
Now, it is a fundamental fact that the Floer complex $CF^*(M)$ is a model for the chains on the free loop space of $L$. Indeed, the Viterbo theorem says that $SH^*(M) \cong H_{n-*}(\mathcal{LL})$, and one expects there to be a complete correspondence between the algebraic structures on $CF^*(M)$ defined using Floer theory, and the string topology operations on $C_{n-*}^{}(\mathcal{LL})$.

For example, it is known that the Viterbo isomorphism intertwines the symplectic cohomology product and the BV operator on symplectic cohomology with the corresponding string topology appearing above. The statement about the product is due to Abbondandolo and Schwarz [2], and the monograph of Abouzaid [3] also includes the BV structure.

Theorem 4.6. *The Viterbo isomorphism $SH^*(T^*L) \cong H_{n-*}(\mathcal{LL})$ is an isomorphism of BV algebras.*

The references given for the above theorem use the Hamiltonian framework for symplectic cohomology but the continuation isomorphism [17] between the SFT and Hamiltonian versions of symplectic cohomology is easily shown to intertwine the bracket, cf. [44], so Corollary 4.7 holds in the SFT framework as well.

Given a rigorous definition of the Chas-Sullivan $L_\infty$ algebra, one expects that it is $L_\infty$ quasi-isomorphic to the symplectic $L_\infty$ algebra of $T^*L$ explained above. As discussed in the introduction, this point of view implies that gravitational descendents $\tau_{m-1}$ of $T^*L$ are new string topology operations.

5. FROM GROMOV-WITTEN TO LANDAU-GINZBURG

This section gathers the results from Sections 2, 3 and 4 together to prove Theorem 1.1. It is reminded that the discussion in Sections 3 and 4 is more general than actually needed: it concerns general Liouville domains while the proof of Theorem 1.1 only requires working with the domain $T^*T^n$.

5.1. Proof of Theorem 1.1. One picks up from the outcome of Section 2, where the curves computing the closed-string Gromov-Witten invariant $\langle \tau_{d-2} pt_{d}^{*} \rangle$ were stretched around a monotone Lagrangian torus $L \subset X$ to holomorphic buildings $(u, w_1, \ldots, w_d)$. Recall that they consist of $u \subset T^*L$ and holomorphic planes $w_1, \ldots, w_d \subset X \setminus L$. Denote $M = T^*L$, embedded into $X$ as the Weinstein neighbourhood of $L$ which was used for stretching. The compition of $X \setminus M$ is naturally isomorphic to the completion of $X \setminus L$.

Choose an initial point for each Reeb orbit $\gamma \subset T^*L$; equip the domain $C$ of each curve $w_i$ with a marker at the puncture at infinity in such a way that the marker is asymptotic to that initial point. Then each $w_i$ becomes a curve computing the Borman-Sheridan class, see Section 3.

Next consider the curve $u$. Its domain has $d+1$ marked points: the point $z_0$ which carries the tangency condition, and $d$ points $z_1, \ldots, z_d$ which are the punctures. As in Lemma 4.1 choose a marker at $z_0$ over the space $\mathcal{M}_{d+1}$ in an arbitrary way and induce markers at all other points $z_i$. Equip each curve $u$ with markers coming from the corresponding element of $\mathcal{M}_{d+1}$. The markers at its inputs do not necessarily
match the initial points of the asymptotic orbits \( \gamma_i \), so one sets the input to be \( \hat{\gamma}_i \), see Section 3.2. Note that
\[
|\hat{\gamma}_i| = 0,
\]
by rephrasing (2.9) and (3.1). This way each \( u \) becomes an element of the moduli space computing gravitational descendants (4.7).

The curves \( w_i \) are simple; assuming all \( u \)-curves are also simple, the SFT compactness [16] and gluing [80] theorems imply that up to a reordering of the inputs, there is a sign-preserving bijection between the closed curves computing \( \langle \tau_{d-2 \text{ pt}} \rangle_d \), and the buildings \((u, w_1, \ldots, w_d)\):

\[
\langle \tau_{d-2 \text{ pt}} \rangle_d^\bullet = \frac{1}{d!} \cdot \langle BS \rangle \cdots |BS\rangle_{T^*L}
\]

where the right hand side brackets are from (4.7). The \( 1/d! \) factor is due to the fact that the punctures of the \( u \)-curve arising from stretching are not numbered while the punctures of the curves computing the descendants (4.4) are.

Recalling that \( BS = W_L \) by Lemma 3.4, write it as an abstract Laurent polynomial in \( n \) variables with monomials indexed by a finite set \( I \):

\[
BS = W_L = \sum_{i \in I} x^i.
\]

By the computation of gravitational descendants of the torus in Theorem 4.5,

\[
\langle BS \rangle \cdots |BS\rangle_{T^*L} = \sum_{(v_1, \ldots, v_d) \in \mathbb{N}^d} (d-2)! = (d-2)! \cdot \phi_d(W_L).
\]

Together with Lemma 1.3, this proves Theorem 1.1 under the assumption that the curves \( u \) are simple.

It remains to be explained how to achieve this last property. Recall that the curves computing descendants of Liouville domains (4.5) were made simple and regular by a domain-dependent \( J \) near the tangency point; this arrangement was possible because these curves have stable domain. On the other hand the domains of curves defining the closed descendant (2.1) are not stable so cannot be equipped with a domain-dependent \( J \); hence the curves \( u \) arising from the stretching argument in Section 2 inherit a domain-independent \( J \).

The issue is resolved by stabilising the closed descendant curves from Section 2.3 before the stretching. One can use either stabilisation framework; for concreteness consider the first one with two stabilising divisors \( \Sigma_1, \Sigma_2 \). Recall that for every monotone Lagrangian submanifold \( L \), there exists a smooth manifold \( \Sigma \subset X \) which is disjoint from \( L \) and dual to \( NC_1(X) \in H^2(X) \). Strictly than that, one can ensure that \( \Sigma \) is dual to \( 2N\mu_L \in H^2(X,L) \) where \( \mu_L \) is the Maslov class; see e.g. [21]. One chooses two such divisors \( \Sigma_1, \Sigma_2 \); one can be a smooth perturbation of the other one. Recall that

\[
\langle \tau_{d-2 \text{ pt}} \rangle_d^\bullet = \frac{1}{(Nd)^2} \cdot \# \mathcal{M}_3(\tau_{d-2 \text{ pt}}, \Sigma_1, \Sigma_2)_d.
\]

The curves from the right-hand side moduli space now have stable domain, and can be equipped with a domain-dependent \( J \) localised at the tangency point. One applies the Cieliebak-Mohnke stretching argument to them; they break into buildings \((u, w_1, \ldots, w_d)\) as previously with the following difference. Now there are two extra marked points distributed between the \( w_1, \ldots, w_d \) which map to \( \Sigma_1, \Sigma_2 \) respectively. There are \( d^2 \) ways of distributing the marked points between the \( w_i \) curve.
and each \( w_i \)-curve has intersection number \( N \) with \( \Sigma_i \). The stretching argument now shows that
\[
\#M_3(\tau_{d-2} pt, \Sigma_1, \Sigma_2)_d = \frac{1}{d! N^2 d^2} \cdot \langle BS | \ldots | BS \rangle_{T^* L}.
\]
Now the curve \( u \) inherits a domain-dependent \( J \) near the tangency point so all such curves are simple and regular. The proof is concluded as previously. \( \square \)

5.2. Generalisation to Liouville subdomains. The theorem below generalises Theorem 1.1 which is its special case when \( M \) is taken to be the Weinstein neighbourhood of a monotone Lagrangian torus, see Example 3.3 and Example 4.6.

**Theorem 5.1.** Let \( X \) be a closed monotone symplectic manifold, and \( M \subset X \) a monotone Liouville subdomain (Definition 3.1) admitting a non-negatively graded Floer complex (Definition 4.3). Then
\[
\frac{1}{d!} \langle BS | \ldots | BS \rangle_d = \langle \tau_{d-2} pt \rangle_{X, d}^\bullet,
\]
where \( BS \) is the Borman-Sheridan class from Section 3, the left-hand side descendants are from (4.7), and the right-hand side Gromov-Witten invariant is from (2.3).

**Proof.** The stretching argument from Section 2 may be performed around \( M \) rather than around a neighbourhood of \( L \). Once again the actions of the Reeb orbits \( \gamma_i \) arising from the stretching are bounded by an a priori constant determined by the size of a Liouville collar of \( M \) embeddable into \( X \), and one arranges using Definition 4.3 that the Conley-Zehnder indices of all such orbits satisfy \( \mu(\gamma_i) \leq n - 1 \). The argument in Section 2 only needs this assumption, and the assumption of monotonicity, to yield exactly the same structure of broken buildings. (In the case when \( \partial M \) has contractible Reeb orbits, the curves \( w_i \) are additionally augmented by holomorphic planes in \( M \) in accordance with the definition of the Borman-Sheridan class.) The rest of the proof is analogous to the proof of Theorem 1.1 given in this section. \( \square \)

5.3. Remarks. It is possible to give an alternative proof of Theorem 5.1 (and of Theorem 1.1) working with the Hamiltonian framework for symplectic cohomology. This is done by adopting the Hamiltonian stretching technique from [100]. In this case, a slightly stronger condition on the stabilising divisor is required: one assumes that \( \Sigma \) is a smooth symplectic divisor as in Section 2.3 \( M \subset X \setminus \Sigma \) is exact and \( c_1^{rel} \in H^2(X, M) \) is proportional to the intersection number with \( \Sigma \). This ensures the positivity of intersections with \( \Sigma \) which is important for the argument. When \( M \) is a neighbourhood of a monotone Lagrangian \( L \), such a divisor can always be found [8, 21].

Assume for simplicity that \( \Sigma \) is anticanonical, so that \( c_1^{rel} \) is equal to the intersection number with \( \Sigma \). One stabilises the closed curves computing \( \langle \tau_{d-2} pt \rangle^\bullet_d \) using the second stabilisation scheme from Section 2.3 by adding the maximal number of \( d \) extra marked points. Then one performs Hamiltonian stretching around each of those marked points as in [100]; this results in precisely the same structure of broken curves as encountered in Section 2.

As a final remark notice that the non-negatively graded assumption on \( M \), or the non-positive sectional curvature assumption on \( L \), was used crucially in the stretching argument to guarantee that Figure 2 is combinatorially the only possible
configuration of broken curves. If $\text{CF}^*(M)$ has negative degree generators, different combinatorial types of broken curves become possible. For example, the stretching of $c_1 = 3$ spheres around $M \subset X$ may result in either of the broken curves shown in Figure 5 (these two possibilities may not be exhaustive). This tallies with the fact that gravitational descendants of $M$ are no longer cohomological operations since they can undergo domain degenerations. It would be interesting to obtain a version of Theorem 5.1 in this general case, involving chain-level descendants.

6. Gravitational descendants of the torus

This section presents a proof of Theorem 4.5, the core computation of the paper.

6.1. Basic relations. Consider the free loop space $\mathcal{L}T^n$ of the $n$-torus, a vector $v \in \mathbb{Z}^n = H_1(T^n; \mathbb{Z})$ and recall the notation for the corresponding symplectic cohomology class $x^v = x_1^v \ldots x_n^v \in SH^0(T^*T^n) \cong H_n(\mathcal{L}T^n) \cong \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Here $v^i$ are the co-ordinates of $v$. Now let $v_1, \ldots, v_k \in \mathbb{Z}^n$ be $k$ vectors; the aim is to compute descendant invariants

$$\langle x^{v_1} | \ldots | x^{v_k} \rangle_{T^*T^n} \in \mathbb{Z}$$

from (4.7). The subscript $T^*T^n$ will be dropped when not crucial.

The computation will rely on several basic properties for these invariants collected below, among which Proposition 6.6 is the most important one.

Definition 6.1. A collection of vectors $v_1, \ldots, v_k$ is said to be balanced if $v_1 + \ldots + v_k = 0$.

Lemma 6.2. Descendants of the torus vanish on non-balanced inputs:

$$(6.1) \quad \langle x^{v_1} | \ldots | x^{v_k} \rangle = 0 \quad \text{unless} \quad \sum_{i=1}^k v_i = 0.$$

Proof. Any holomorphic curve contributing to a descendant invariant provides a nullhomology for the sum of its input Reeb orbits. □

Lemma 6.3. Descendants of the torus are invariant under the action of $GL(n, \mathbb{Z})$, that is, for any $g \in GL(n, \mathbb{Z})$ one has

$$\langle x^{v_1} | \ldots | x^{v_k} \rangle = \langle x^{g(v_1)} | \ldots | x^{g(v_k)} \rangle.$$

Proof. This is true because the $GL(n, \mathbb{Z})$-action on the torus lifts to symplectomorphisms of its cotangent bundle. □
Lemma 6.4. Descendants of the torus are invariant under stabilisation, that is, if \( v_1, \ldots, v_k \in \mathbb{Z}^n \subset \mathbb{Z}^{n+1} \), then
\[
\langle x^{v_1} | \ldots | x^{v_k} \rangle_{T^*T^n} = \langle x^{v_1} | \ldots | x^{v_k} \rangle_{T^*T^{n+1}}.
\]

Proof. The usual technical issue with this type of statements is that a product almost complex structure on \( T^*T^n \times T^*S^1 \) is not cylindrical, calling for a workaround. The one used here is inspired by [38, Section 4].

Let the tangency condition point \( y \in T^*T^{n+1} \) defining gravitational descendants belong to the zero-section \( T^{n+1} \). Consider the splitting \( T^{n+1} = T^n \times S^1 \) and a perturbation of the flat metric on it with two totally geodesic tori of the form \( T^n \times \{ p \}, T^n \times \{ P \} \), the first one containing geodesics of symplectic cohomology degree 0. All inputs \( x^{v_1} \) correspond to closed geodesics within \( T^n \times \{ p \} \). Assume that \( y \in T^n \times \{ p \} \). There is a compatible cylindrical almost complex structure \( J \) on \( T^*T^{n+1} \) such that \( \Sigma = T^*(T^n \times \{ p \}) \) is a complex hypersurface.

Let \( D \subset T^*T^{n+1} \) be the image of a curve \( u \) contributing to the count of the descendant from the statement; it is enough to show that \( D \subset \Sigma \) so assume that this is not the case. Then for a generic such \( J \), \( D \cap \Sigma \) is discrete and non-empty because \( y \in D \cap \Sigma \). So the intersection number \( D \cap \Sigma \) is positive, since both \( D \) and \( \Sigma \) are \( J \)-holomorphic. This intersection number, by definition, only counts interior intersections and ignores the fact that \( D \) is asymptotic to \( \Sigma \) at infinity.

The projection \( \pi: T^*T^{n+1} \to T^*S^1 \) to the last factor contracts the asymptotic orbits of \( D \) to the point \( p \subset T^*S^1 \), and with this contraction the map \( \pi|_D \) is null-homotopic because \( T^*S^1 \) is aspherical. So \( \pi|_D \) has zero degree. The claim is that, on the other hand, \( D \cap \Sigma \) equals that degree, hence the contradiction. The claim holds because, since \( \pi \) contracts the asymptotics of \( D \) to \( p \), the restriction of \( \pi \) to a neighbourhood of each asymptotic of \( D \) has a well-defined and vanishing degree, and the fact that the asymptotics of \( D \) are contained in \( \Sigma \) may be ignored for the purpose of computing the total degree as the intersection number \( D \cap \Sigma \).

\[\square\]

Lemma 6.5. Let \( x_1, \ldots, x_n \) be the variables corresponding to a basis of \( H_1(T^n) \). Then
\[
\langle x_1 | \ldots | x_n | x_1^{-1} \ldots x_n^{-1} \rangle_{T^*T^n} = (n + 1)!.
\]

Proof. Consider the LG potential of the Clifford torus in \( CP^n \),
\[
W = x_1 + \ldots + x_n + \frac{1}{x_1 \ldots x_n}.
\]

It holds that
\[
(n - 1)! = \langle \tau_{n-1} \text{pt} \rangle_{CP^n, n+1} = \frac{1}{(n + 1)!} \langle W | \ldots | W \rangle_{T^*T^n};
\]
the first equality is the computation of Cieliebak and Mohrke [30, Proposition 3.4] and the second one follows from Theorem [1.1] and Lemma [3.4]. If one expands the right hand side linearly, the only balanced summands in the sum turn out to be \( \langle x_1 | \ldots | x_n | x_1^{-1} \ldots x_n^{-1} \rangle \) and all its permutations, therefore
\[
\frac{1}{(n + 1)!} \langle W | \ldots | W \rangle_{T^*T^n} = \langle x_1 | \ldots | x_n | x_1^{-1} \ldots x_n^{-1} \rangle_{T^*T^n}
\]
which proves the lemma.

\[\square\]
Denote 
\[ \Lambda^2(Z^n) = \Lambda^2(H^1(T^n; \mathbb{Z})) = H^2(T^n; \mathbb{Z}) \cong H_{2n-2}(T^n; \mathbb{Z}); \]
this space can be seen as the space of skew-symmetric bilinear maps \( Z^n \otimes Z^n \to Z \). Below is the last and most important property which will allow to compute descendants of the torus.

**Proposition 6.6.** Consider a collection of vectors \( v_1, \ldots, v_k \in Z^n \) and any \( \Omega \in \Lambda^2(Z^n) \), \( u \in Z^n \).
The following relation holds, where \( \Omega(u, v_i) \in Z \) are the pairings.

\[
\sum_{i=1}^{k} \Omega(u, v_i) \cdot \langle x^{v_i} | \ldots | x^{v_{i-1}} x^{v_{i+u}} | x^{v_{i+1}} | \ldots | x^{v_k} \rangle = 0.
\]

**Remark 6.1.** The inputs in (6.2) are simultaneously either all balanced or not. In the latter case the proposition is obvious. The balanced case in Proposition 6.6 implies Proposition 6.6 indeed, assuming Theorem 4.5 (6.2) rewrites as

\[
\sum_{i=1}^{k} \Omega(u, v_i) \cdot (k-2)! = (k-2)! \cdot \Omega(u, \sum_{i=1}^{k} v_i) = (k-2)! \cdot \Omega(u, -u) = 0.
\]

The goal will be to prove the converse, also using the previous properties.

**Example 6.2.** Let
\[ v_1 = (-2, -1), \quad v_2 = (1, 0), \quad v_3 = (1, 0). \]
Denoting the variables by \( x, y \) instead of \( x_1, x_2 \) for convenience, the corresponding monomials \( x^{v_i} \) are
\[ x^{-2} y^{-1}, \quad x, \quad x. \]
Take \( u = (0, 1) \) so that \( x^u = y \), and take \( \Omega = dy \wedge dx \). The relation from Proposition 6.6 gives
\[
-2 \langle x^{-2} | x | x \rangle + \langle x^{-2} y^{-1} | xy | x \rangle + \langle x^{-2} y^{-1} | x | xy \rangle = 0
\]
The last two terms are equal to 1 by Lemmas 6.5 and 6.3 because \( x, \ xy \) are monomials corresponding to a basis in \( Z^2 \). Therefore Proposition 6.6 proves that
\[ \langle x^{-2} | x | x \rangle = 1. \]
Observe that no two inputs here form a basis in \( Z^2 \) so this is a new identity justifying Theorem 4.5 which does not follow from Lemma 6.5 directly.

The way Theorem 4.5 will be proved is by applying Proposition 6.6 in a systematic way to compute the descendants for all balanced inputs, reducing all computations to the standard case of Lemma 6.5. The implementation of this idea will also use dimension stabilising and a carefully set up induction to systematise the relations.

Two proofs of Proposition 6.6 will be given. The first one shows (6.2) as a consequence of the \( L_\infty \) relations from Theorem 4.2 together with a computation of the string Lie bracket on the torus. The second proof does not rely on the \( L_\infty \) relations or string topology, but uses the fact that descendants are well-behaved under Viterbo restriction maps and uses a version of the wall-crossing formula, cf. [82, 83, 85, 100]. (The second proof assumes that \( u \) is a primitive vector, which
is enough for the proof of Theorem 4.5. Given that the two proofs are notably different, it seems reasonable to include both of them for completeness.

6.2. Lie bracket and a string topology approach. Recall that

\[ H_s(\mathcal{L}T^n) \cong H_s(T^n) \otimes \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \]

this isomorphism is realised as follows. The part \( \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \) keeps track of the homology class of the loops in a given cycle, and the part \( H_s(T^n) \) is obtained by evaluating at the origin of each loop. It is convenient to identify \( H_s(T^n) \cong \Lambda^s(H_1(T^n)) \cong \Lambda^{n-s}(H^1(T^n)) \) where \( \Lambda^s \) is the exterior algebra. For \( a \in \Lambda^{n-s}(H^1(T^n)) \) and \( u \in H_1(T^n) \), denote the corresponding element of \( H_s(\mathcal{L}T^n) \) by

\[ a \otimes x^u \in H_s(\mathcal{L}T^n). \]

Recall that the Chas-Sullivan product

\[ - \cdot - : H_s(\mathcal{L}T^n) \otimes H_s(\mathcal{L}T^n) \to H_{s-n}(\mathcal{L}T^n) \]

is defined as follows. Consider two cycles \( x, y \) of parametrised loops on \( L \). One evaluates the loops in the \( x \)-cycle and in the \( y \)-cycle at \( 0 \in S^1 \), and concatenates them pairwise over the fibre product of these evaluations. For the torus, the product is easily computed:

\[ (a \otimes x^u) \cdot (b \otimes x^v) = (a \wedge b) \otimes x^{u+v}. \]  

Next recall the BV operator

\[ \Delta : H_s(\mathcal{L}T^n) \to H_{s+1}(\mathcal{L}T^n). \]

It modifies cycles of loops by letting the parametrisations of loops sweep a once-around turn. One computes in the adopted notation

\[ \Delta(a \otimes x^u) = \iota_u a \otimes x^u \]

where \( \iota_u a \in \Lambda^{[u]-1}(H^1(T^n)) \) is the interior product, or the result of insertion.

**Lemma 6.7.** Consider a 2-form \( \Omega \in \Lambda^2(H^1(T^n)) \) and vectors \( u, v \in H_1(T^n) \). It holds that

\[ [\Delta(\Omega \otimes x^u), x^v] = \Omega(u, v) \cdot x^{u+v}. \]

**Proof.** Recall that the Chas-Sullivan bracket is expressed in terms of the product and the BV operator via (4.8). Specifically,

\[ [\Delta(\Omega \otimes x^u), x^v] = \Delta(\Delta(\Omega \otimes x^u) \cdot x^v) - \Delta(\Delta(\Omega \otimes x^v) \cdot x^u - \Delta(\Omega \otimes x^u) \cdot \Delta(x^v)). \]

The second term vanishes because \( \Delta^2 = 0 \), and the third term vanishes because according to the notation being used, \( x^v \in H_n(\mathcal{L}L) \) so the image of \( \Delta \) lands in \( H_{n+1}(\mathcal{L}L) = 0 \). Therefore

\[ [\Delta(\Omega \otimes x^u), x^v] = \Delta(\Delta(\Omega \otimes x^u) \cdot x^v). \]

The computation is continued using (6.3) and (6.4).

\[ \Delta(\Delta(\Omega \otimes x^u) \cdot x^v) = \Delta(\Delta(u \Omega \otimes x^{u+v}) \cdot x^v) = \Delta(\Delta(u \Omega \otimes x^{u+v}) \cdot x^v) = \Omega(u, u + v) \cdot x^{u+v} = \Omega(u, v) \cdot x^{u+v}. \]

The result is as claimed. \( \Box \)
Remark 6.3. Before the emergence of string topology, Goldman [60] discovered a Lie bracket on the space of free loops on a surface. Let \( \hat{\pi} \) be the set of free homotopy classes of oriented unparametrised loops on the genus \( g \) surface \( L = \Sigma_g \) (i.e. the set of conjugacy classes of its fundamental group). The Goldman bracket is an operation

\[
[-, -]_G: \mathbb{Z}[\hat{\pi}] \otimes \mathbb{Z}[\hat{\pi}] \to \mathbb{Z}[\hat{\pi}].
\]

For two transverse unparametrised oriented loops \( \alpha, \beta \), one puts

\[
[\alpha, \beta]_G = \sum_{p \in \alpha \cap \beta} \epsilon(p) \alpha \ast_p \beta
\]

where \( \epsilon(p) \) is the intersection sign and \( \alpha \ast_p \beta \) is the concatenation of the two loops at \( p \). From the point of view of string topology, the Goldman bracket is equivalent to the Chas-Sullivan bracket on the \( S^1 \)-equivariant loop space homology,

\[
[-, -]: H^0_S(\mathcal{L}L) \otimes H^0_S(\mathcal{L}L) \to H^0_S(\mathcal{L}L),
\]

but it can be rephrased in terms of non-equivariant string topology using the BV operator. The Goldman bracket on the torus admits the following straightforward computation, see [98, 99, 22]. Identify \( \mathbb{Z}^2 = H_1(T^2) \) with the set of free homotopy classes of loops on \( T^2 \); then for any \( u, v \in \mathbb{Z}^2 \), the Goldman bracket is given by

\[
[u, v]_G = (u^1v^1 - u^2v^2)(u + v).
\]

Here \( u^i, v^i \) are the co-ordinates of \( u \) resp. \( v \). Lemma 6.7 is a close relative of that.

Returning to the symplectic cohomology bracket on the \( n \)-torus, its part

\[
(\mathcal{L}^2 : SH^1(T^*T^n) \otimes SH^0(T^*T^n) \to SH^0(T^*T^n)).
\]

will be of particular interest. Using the Viterbo isomorphism and Corollary 4.7, Lemma 6.7 translates into

\[
(\mathcal{L}^2(\Delta(\Omega \otimes x^u), x^v)) = \Omega(u, v) \cdot x^{u+v}
\]

where

\[
\Omega \otimes x^u \in SH^2(T^*T^n), \quad \Delta(\Omega \otimes x^u) \in SH^1(T^*T^n), \quad x^v \in SH^0(T^*T^n).
\]

Proof of Proposition 6.6. This follows by combining the \( L_\infty \) relations from Theorem 4.2. Recall that the Floer differential on \( CF^*(T^*T^n) \) vanishes so there is a natural identification \( CF^*(T^*T^n) = SH^*(T^*T^n) \). Consider the sequence of elements in \( CF^*(T^*T^n) \) below:

\[
\Delta(\Omega \otimes x^u), x^{v_1}, \ldots, x^{v_k}.
\]

They have degrees 1, 0, \ldots, 0 respectively. Apply the \( L_\infty \) morphism equation from Theorem 4.2. The only non-trivial \( \mathcal{L}^2 \) operations that can be applied to a subcollection of the above inputs are \( (\mathcal{L}^2(\Delta(\Omega \otimes x^u), x^{v_i}), x^{v_i}) \), since all other possible applications of an \( \mathcal{L}^2 \) land in negative degree and vanish.

So by Theorem 4.2,

\[
\sum_{i=1}^k \frac{1}{2^{(k-i)!}} \langle (\mathcal{L}^2(\Delta(\Omega \otimes x^u), x^{v_i}), x^{v_i}) \rangle \cdots |x^{v_{i+1}}| x^{v_{i+1}} \cdots |x^{v_k}| = 0.
\]
The common factor may be dropped, and the first term may be moved to the $i$th position. Finally (6.6) gives
\[
\sum_{i=1}^{k} (x^{v_1}| \ldots |x^{v_{i-1}}| l^2 (\Delta(\Omega \otimes x^u), x^{v_i}) |x^{v_{i+1}}| \ldots |x^{v_k}) \\
= \sum_{i=1}^{k} \Omega(u, v_i) \cdot (x^{v_1}| \ldots |x^{v_{i-1}}| x^{v_i+u}|x^{v_{i+1}}| \ldots |x^{v_k})
\]
which proves Proposition 6.6.

6.3. Wall-crossing approach. This subsection gives an alternative proof of Proposition 6.6. The proof of the lemma below follows the standard arguments, using the fact that the non-negatively graded condition prevents the domains of the curves computing descendants from degenerating.

Lemma 6.8. Suppose $M$ is a Liouville subdomain of $N$, and both admit non-negatively graded Floer complex in the sense of Definition 4.3. Consider the Viterbo map
\[ \text{Vit}: SH^0(N) \rightarrow SH^0(M). \]
It holds that
\[ \langle x_1| \ldots |x_k \rangle_N = \langle \text{Vit}(x_1)| \ldots |\text{Vit}(x_k) \rangle_M \]
for all $x_1 \ldots, x_k \in SH^0(N)$. □

Consider the Weinstein manifold $N$ whose Lagragian skeleton is the union of the 2-torus $T^2$ and a 2-disk attached cleanly to a simple closed curve in $T^2$ in the homology class of $(1, 0) \in \mathbb{Z}^2$. This space is Liouville deformation equivalent to the complement of a conic in $\mathbb{C}^2$:
\[ N \cong \mathbb{C}^2 \setminus \{xy = 1\}. \]
By [82], $N$ has non-negatively graded Floer complex, in particular there are well-defined gravitational descendant operations on $SH^0(N)$.

Theorem 6.9. The ring isomorphism below holds.
\[ SH^0(N) \cong \mathbb{C}[p, q, t^{\pm 1}]/(pq = 1 - t). \]
The manifold $N$ contains two exact Lagrangian tori, the original torus $T = T^2$ and a torus $T' \subset N$ obtained by mutation of $T$ along the disk [6, 83]. (The tori $T$ and $T'$ are called the Clifford and the Chekanov torus.) Equip $T, T'$ with the standard spin structures and orientations. For some choices of bases of $H_1(T; \mathbb{Z})$ and $H_1(T'; \mathbb{Z})$ and the corresponding identifications
\[ SH^0(T^*T) = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}], \quad SH^0(T^*T') = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}], \]
the Viterbo maps from $SH^0(N)$ are given by the formulas below. □

\[
\text{Vit}: SH^0(N) \rightarrow SH^0(T^*T) : \begin{cases} 
p \mapsto x_1, \\
q \mapsto x_1^{-1}(1 + x_2), \\
t \mapsto -x_2. 
\end{cases}
\]
\[
\text{Vit}: SH^0(N) \rightarrow SH^0(T^*T') : \begin{cases} 
p \mapsto x_1(1 + x_2), \\
q \mapsto x_1^{-1}, \\
t \mapsto -x_2. 
\end{cases}
\]
In the above theorem, the computation of the ring structure on $SH^0(N)$ is due to Pascaleff [82], and the computation of the Viterbo map will appear in [41]. The composition of the birational inverse of the first Viterbo map with the second Viterbo map,

$$SH^0(T^*T) \rightarrow SH^0(N) \rightarrow SH^*(T^*T'),$$

is

$$x_1 \mapsto x_1(1 + x_2),$$

$$x_2 \mapsto x_2,$$

This transformation is called the wall-crossing map; in this context it has been discussed in [93, Section 11] and [85]. The next lemma states that descendants of the torus are preserved by wall-crossing maps.

**Lemma 6.10.** Consider the wall-crossing map

$$x_1 \mapsto x_1(1 + x_2),$$

$$x_2 \mapsto x_2,$$

...$$x_n \mapsto x_n.$$

Descendants of the torus are preserved if this map is applied to all inputs simultaneously. It means that

$$\langle x^{v_1} | \ldots | x^{v_k} \rangle = \langle x^{v_1}(1 + x_2)^{v_1} | \ldots | x^{v_k}(1 + x_2)^{v_k} \rangle.$$

In the case when some $v_i < 0$, the expression $(1 + x_2)^{v_i}$ is understood as the infinite Taylor expansion $(1 + x_2)^{v_i} = 1 + v_i x_2 + \ldots$; the equality from the display formula is meaningful because its right hand side involves a finite number of balanced terms.

**Remark 6.4.** Writing all monomials in full form, the identity from Lemma 6.10 becomes

$$\langle x^{v_1} | \ldots | x^{v_k} \rangle = \langle x_1^{v_1} x_2^{v_2} \ldots x_n^{v_n} | \ldots | x_1^{v_1} x_2^{v_2} \ldots x_n^{v_n} \rangle.$$

**Proof.** Consider the product splitting $T^*T^n = T^*T^2 \times T^*T^{n-2}$, the Liouville inclusion $T^*T^2 \subset N$ from Theorem 6.9 and the resulting inclusion

$$T^*T^n \subset M := N \times T^*T^{n-2}.$$

By the Künneth formula for symplectic cohomology [79, 84] and Theorem 6.9

$$SH^0(M) \cong \mathbb{C}[p,q,t^{\pm 1}] \otimes \mathbb{C}[x_3^{\pm 1}, \ldots , x_n^{\pm 1}].$$

Viterbo maps respect Künneth isomorphisms, so in particular the Viterbo map $SH^0(M) \rightarrow SH^0(T^*T^n)$ acts on the generators $p,q,t$ as described in Theorem 6.9 and acts by the identity on the generators $x_2, \ldots , x_n$. Pick a monomial $x^{v_i}$ seen as an element of $SH^0(T^*T^n)$, and consider the two cases below.

If the first co-ordinate $v_1$ of the vector $v_i$ is positive, then $x^{v_i}$ belongs to the image of the Viterbo map:

$$\text{Vit}(r_i) = x^{v_i} \quad \text{for} \quad r_i = p^{v_1} t^{v_2} x_3^{v_3} \ldots x_n^{v_n} \in SH^0(M),$$

as follows from Theorem 6.9.
If the first co-ordinate $v_i^1$ is negative, then $x^{v_i}$ does not belong to the image of the Viterbo map. It falls into the image after being multiplied by the binomial $(1 + x_2)^{-v_i^1}$ namely:

$$Vit(r_i) = x^{v_i}(1 + x_2)^{-v_i^1} \text{ for } r_i = q^{-v_i^1}t^q x_3^{v_3^1} \ldots x_n^{v_n^1}. \tag{6.8}$$

One can formally write

$$x^{v_i} = x^{v_i}(1 + x_2)^{-v_i^1}(1 + x_2)^{v_i^1} = x^{v_i}(1 + x_2)^{-v_i^1} \sum_{s=0}^{\infty} \frac{(s-v_i^1)!}{s!} (-1)^s (x_2)^s \tag{6.9}$$

where the infinite series is the Taylor expansion of $(1 + x_2)^{v_i^1}$. This series is not an element of the Laurent polynomial ring $SH^0(T^*T^n)$, however finite truncations of this series belong to the ring. Fix some positive integer $N$ and denote

$$S_N(x^{v_i}) := x^{v_i}(1 + x_2)^{-v_i^1} \sum_{s=0}^{N} \frac{(s-v_i^1)!}{s!} (-1)^s (x_2)^s \in SH^0(T^*T^n). \tag{6.10}$$

The above expression makes sense as an element of $SH^0(T^*T^n)$ whenever $v_i^1 < 0$. It is also convenient to put

$$S_N(x^{v_i}) := x^{v_i} \text{ if } v_i^1 \geq 0. \tag{6.11}$$

The claim that for sufficiently large $N$, it holds that

$$\langle x^{v_1} | \ldots | x^{v_k} \rangle = \langle S_N(x^{v_1}) | \ldots | S_N(x^{v_k}) \rangle. \tag{6.12}$$

Indeed, all terms in the difference between (6.10) and (6.9) involve powers of $x_2$ that are larger than $N$, and whenever any of those terms appears in the descendant operation, it forces the result of the operation to vanish by Lemma 6.2 if $N$ is sufficiently large.

By definition of $S_N$ and a combination of (6.7) and (6.8), the elements $S_N(x^{v_i})$ lie in the image of the Viterbo map for all $i$, regardless of the sign of $v_i^1$. We denote their preimages by $r_i$:

$$Vit(r_i) = S_N(x^{v_i}).$$

(This overrides the previous notation for the $r_i$.) The explicit formula for $r_i$ can be obtained from Theorem 6.9, compare (6.7), (6.8); it is omitted. Proposition 6.8 implies that

$$\langle S_N(x^{v_1}) | \ldots | S_N(x^{v_k}) \rangle = \langle r_1 | \ldots | r_k \rangle. \tag{6.13}$$

Now apply Viterbo maps onto the torus $(T^n)\gamma \subset T^n$ which is the product of the torus in $T^r \subset T^*T^2$ appearing in Theorem 6.9 and the zero-section in $T^*T^{n-2}$. One has

$$Vit(S_N(x^{v_i})) = \begin{cases} x^{v_i}(1 + x_2)^{v_i^1}, & v_i^1 > 0 \\ x^{v_i} \sum_{s=0}^{N} \frac{(s-v_i^1)!}{s!} (-1)^s (x_2)^s, & v_i^1 < 0. \end{cases}$$

Recall that sum appearing in the bottom row is a partial sum of the Taylor series for $(1 + x_2)^{v_i^1}$ and assuming that $N$ is large enough, the remaining terms of the series do not contribute to the descendant operation appearing in the next formula. Another reference to Theorem 6.9 now applied to the Viterbo map for the new torus $T^r$ yields

$$\langle r_1 | \ldots | r_k \rangle = \langle x^{v_1}(1 + x_2)^{v_1^1} | \ldots | x^{v_k}(1 + x_2)^{v_k^1} \rangle.$$
Lemma 6.10 follows from (6.11), (6.12), (6.13) and the obvious fact that the matching invariants computed in $T^*T^n$ and $T^*(T^n)'$ coincide, because $T^*T^n$ is symplectomorphic to $T^*(T^n)'$. \[ \square \]

**Proof of Proposition 6.6 in the case when $u$ is primitive.** Assume that $u$ is the second basic vector: $u = (0, 1, 0, \ldots, 0) \in \mathbb{Z}^n$. Writing $\Omega$ as a combination of $dx_1 \wedge dx_s$, one sees that only the summands of form $dx_2 \wedge dx_i$ contribute to the identity one needs to prove. It is sufficient to prove the statement for $\Omega = dx_2 \wedge dx_1$. For these $u$ and $\Omega$, the identity from Proposition 6.6 rewrites as

\[
(6.14) \quad \sum_{i=1}^k v_i^1 \cdot \langle x^{v_1} | \ldots | x^{v_i} x_2 | x^{v_{i+1}} | \ldots | x^{v_k} \rangle = 0.
\]

The non-trivial case is when the inputs are balanced, that is,

\[
\sum_{i=1}^k v_i = (0, -1, 0, \ldots, 0).
\]

In this case (6.14) is recovered from the wall-crossing relation from Lemma 6.10 by picking out the balanced terms therein. To see why, consider the Taylor expansion

\[
x^{v_1}(1 + x_2)^{v_1^1} = x^{v_1}(1 + v_1^1 x_2 + \mathcal{O}((x_2)^2))
\]

holding regardless of the sign of $v_1^1$. After the wall-crossing transformation from Lemma 6.10 is applied to all elements of the sequence $x^{v_1}, \ldots, x^{v_k}$, the balanced terms are the ones where exactly one input comes from the linear term of the Taylor expansion above, and the rest come from the constant term. This precisely recovers (6.14). \[ \square \]

**Remark 6.5.** The case when $u$ is primitive is sufficient for the proof of Theorem 4.5.

**6.4. Proof of Theorem 4.5.** By induction on $k$ it will be shown that for all $k \geq 1$, the statement $ST_k$ below holds.

**$ST_k$: **For all $n \geq 1$, $0 \leq m \leq n$ and $v_1, \ldots, v_k \in \mathbb{Z}^n$ satisfying

\[
(6.15) \quad v_1 + \ldots + v_k + \left(1, \ldots, 1, 0, \ldots, 0\right) = 0 \in \mathbb{Z}^n,
\]

it holds that

\[
(6.16) \quad \langle x_1 | \ldots | x_m | x^{v_1} | \ldots | x^{v_k} \rangle_{T^*T^n} = (m + k - 2)!
\]

Putting $m = 0$ in $ST_k$, one gets

\[
\langle x^{v_1} | \ldots | x^{v_k} \rangle_{T^*T^n} = (k - 2)!
\]

whenever $v_1 + \ldots + v_k = 0$, which is the statement of Theorem 4.5. So once the induction is established, Theorem 4.5 follows.

Now consider the case $k = 1$ in $ST_k$, which is the base of the induction. In this situation there is a single vector $v_1 = (-1, \ldots, -1, 0, \ldots, 0)$ and $ST_1$ asserts that for all $m \leq n$,

\[
\langle x_1 | \ldots | x_m | x_1^{-1} \ldots x_m^{-1} \rangle_{T^*T^n} = (m - 1)!
\]

By Lemma 6.4 one may assume that $m = n$, reducing the equality to Lemma 6.5.

One proceeds to the step of the induction, that is, one needs to show that $ST_{k-1}$ implies $ST_k$. Fix the numbers $n, m \leq n$ and a collection of vectors $v_1, \ldots, v_k \in \mathbb{Z}^n$...
satisfying (6.15). Recall that as usual, $x_1, \ldots, x_n$ are the multiplicative generators of $SH^0(T^n)$. Consider the inclusion $T^n T^n \subset T^n T^{n+1}$ and denote the new generator of $SH^0(T^n T^{n+1})$ by $z$; this helps distinguish its special role.

Consider the following sequence of elements in $SH^0(T^n T^{n+1})$:

$$x_1z^{-1}, x_2, \ldots, x_m, x^{v_1}, \ldots, x^{v_k}.$$ 

The next step is to apply Proposition 6.6 taking the above sequence as the sequence $x^n$ from its statement. For example, it means that the first vector $v_1$ for Proposition 6.6 is taken to be $(1, 0, \ldots, 0, -1)$ corresponding to $x_1z^{-1}$, the second vector is $(0, 1, 0, \ldots, 0)$ corresponding to $x_2$, the $(m+1)$st vector for Proposition 6.6 is the vector $v_1$ appearing in this proof, etc. Next, take $u = (0, \ldots, 0, 1)$ to be the basic vector corresponding to the variable $z$, and the 2-form $\Omega$ to be $dz \wedge dx_1$. The relation from Proposition 6.6 yields

\begin{equation}
(6.17) \quad \langle x_1 | x_2 | \ldots | x_m | x^{v_1} | \ldots | x^{v_k} \rangle_{T^n T^{n+1}} \quad + \sum_{i=1}^{k} v_i^1 \cdot \langle x_1z^{-1} | x_2 | \ldots | x_m | x^{v_1} | \ldots | x^{v_i}z | x^{v_{i+1}} | \ldots | x^{v_k} \rangle_{T^n T^{n+1}} = 0;
\end{equation}

compare to (6.14) and Example 6.2. It follows from (6.15) that $\sum_{i=1}^{k} v_i^1 = -1$. Hence if one proves that

\begin{equation}
(6.18) \quad \langle x_1z^{-1} | x_2 | \ldots | x_m | x^{v_1} | \ldots | x^{v_i}z | x^{v_{i+1}} | \ldots | x^{v_k} \rangle_{T^n T^{n+1}} = (m + k - 2)! \quad \text{for every term appearing in the sum from (6.17),}
\end{equation}

then (6.16) follows immediately and establishes $ST_k$.

To show (6.18), consider first the $SL(n+1, \mathbb{Z})$ action taking $x_1 \mapsto x_1z$ and leaving all other variables intact. By Lemma 6.3

\begin{equation}
(6.19) \quad \langle x_1z^{-1} | x_2 | \ldots | x_m | x^{v_1} | \ldots | x^{v_i}z | x^{v_{i+1}} | \ldots | x^{v_k} \rangle_{T^n T^{n+1}} \quad = \langle x_1 | x_2 | \ldots | x_m | x^{v_1} x^{v_i} | \ldots | x^{v_i}z(v_i+1) | x^{v_{i+1}} z^{v_{i+1}} | \ldots | x^{v_k} z^{v_k} \rangle_{T^n T^{n+1}}
\end{equation}

Denote for convenience

$$\tilde{v}_j^1 = \begin{cases} v_j^1 & \text{if } j \neq i, \\ (v_i^1) + 1 & \text{if } j = i, \end{cases}$$

and observe that by (6.15),

\begin{equation}
(6.20) \quad \tilde{v}_1^1 + \ldots + \tilde{v}_k^1 = 0.
\end{equation}

Consider the inclusion $SH^0(T^n T^{n+1}) \subset SH^0(T^n T^{n+2})$ and denote by $w$ the new generator of $SH^0(T^n T^{n+2})$. One applies Proposition 6.6 once again, now to the sequence

\begin{equation}
(6.21) \quad x_1, x_2, \ldots, x_m, x^{v_1} z^{v_1} w^{-1}, x^{v_2} z^{v_2}, \ldots, x^{v_k} z^{v_k},
\end{equation}

the basic vector $u = (0, \ldots, 0, 1)$ corresponding to the new variable $w$, and $\Omega = dw \wedge dz$. The result is the relation below.

\begin{equation}
(6.22) \quad \tilde{v}_1^1 \cdot \langle x_1 | x_2 | \ldots | x_m | x^{v_1} z^{v_1} w^{-1} | x^{v_2} z^{v_2} | \ldots | x^{v_k} z^{v_k} \rangle_{T^n T^{n+2}} \quad + \sum_{j=2}^{k} \tilde{v}_j^1 \cdot \langle x_1 | x_2 | \ldots | x_m | x^{v_1} z^{v_1} w^{-1} | x^{v_2} z^{v_2} | \ldots | x^{v_j} z^{v_j} w | \ldots | x^{v_k} z^{v_k} \rangle_{T^n T^{n+2}} = 0.
\end{equation}
By (6.20) and (6.22), one has reduced (6.18) to the following:

\[(6.23)\]
\[
\langle x_1 | x_2 | \ldots | x_m | x^{n_1} z^{\ell_1} w^{-1} | x^{n_2} z^{\ell_2} \ldots | x^{n_k} z^{\ell_k} w \rangle_{T^* \mathbb{T}^{n+2}} = (m + k - 2)! \]

for all \(j\). To see why this latter equality is true, observe that the vector in \(\mathbb{Z}^{n+2}\) corresponding to the monomial

\[x^{n_1} z^{\ell_1} w^{-1}\]

is primitive because the last co-ordinate of that vector is \(-1\). Consider the \(GL(n + 2, \mathbb{Z})\) transformation acting on monomials as follows:

\[w \mapsto x^{n_1} z^{\ell_1} w^{-1}, \quad x_1 \mapsto x_1, \quad \ldots, \quad x_n \mapsto x_n, \quad z \mapsto z.\]

The application of this transformation to all inputs of (6.23) turns that identity into

\[(6.24)\]
\[
\langle x_1 | x_2 | \ldots | x_m | w | \underbrace{\ldots}_{k-1 \text{ inputs}} \rangle_{T^* \mathbb{T}^{n+2}} = (m + k - 2)! \]

The precise formula for the remaining \(k - 1\) inputs is immaterial. It is obvious that the inputs in (6.24) are balanced because the inputs in (6.23) are. Now, equation (6.24) is part of the inductive hypothesis \(ST_{k-1}\) up to a relabelling of the variables, because \(x_1, \ldots, x_m, w\) are the variables corresponding to the elements of a basis. This establishes the step of the induction and concludes the proof of Theorem 4.5. \(\square\)

REFERENCES

[1] A. Abbondandolo and M. Schwarz. On the Floer homology of cotangent bundles. Comm. Pure Appl. Math., 59(2):254–316, 2006.
[2] A. Abbondandolo and M. Schwarz. Floer homology of cotangent bundles and the loop product. Geom. Topol., 14:1569–1722, 2010.
[3] M. Abouzaid. Symplectic cohomology and Viterbo’s theorem. In Free loop spaces in geometry and topology: including the monograph Symplectic cohomology and Viterbo’s theorem by Mohammed Abouzaid, volume 24 of IRMA Lect. Math. Theor. Phys., 2015.
[4] M. Akhtar, T. Coates, S. Galkin, and A. M. Kasprzyk. Minkowski polynomials and mutations. SIGMA, 8, 2012.
[5] M. Akhtar and A. Kasprzyk. Mutations of fake weighted projective planes. arXiv:1302.1152, 2013.
[6] D. Auroux. Mirror symmetry and T-duality in the complement of an anticanonical divisor. J. Gökova Geom. Topol., 1:51–91, 2007.
[7] D. Auroux. Infinitely many monotone Lagrangian tori in \(\mathbb{R}^6\). Invent. Math., 201(3):909–924, 2015.
[8] D. Auroux, D. Gayet, and J.-P. Mohsen. Symplectic hypersurfaces in the complement of an isotropic submanifold. Math. Ann., 321(4):739–754, 2001.
[9] E. Bao and K. Honda. Semi-global Kuranishi charts and the definition of contact homology. arXiv:1512.00580, 2015.
[10] V. V. Batyrev. Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties. J. Algebraic Geom., 3:493–535, 1994.
[11] P. Biran. Lagrangian non-intersections. Geom. Funct. Anal., 16(2):279–326, 2006.
[12] P. Biran and K. Cieliebak. Symplectic topology on subcritical manifolds. Comment. Math. Helv., 76(4):712–753, 2001.
[13] P. Biran and M. Khanevsky. A Floer-Gysin exact sequence for Lagrangian submanifolds. Comment. Math. Helv., 88(4):899–952, 2013.
[14] M. S. Borman and N. Sheridan. In preparation. 2016.
[15] F. Bourgeois, T. Ekholm, and Y. Eliashberg. Effect of Legendrian surgery. Geom. Topol., 16(1):301–389, 2012. With an appendix by Sheel Ganatra and Maksim Maydanskiy.
[16] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder. Compactness results in Symplectic Field Theory. *Geom. Topol.*, 7:799–888, 2003.

[17] F. Bourgeois and A. Oancea. Symplectic homology, autonomous Hamiltonians, and Morse-Bott moduli spaces. *Duke Math. J.*, 146(1):71–174, 2009.

[18] F. Bourgeois and A. Oancea. An exact sequence for contact- and symplectic homology. *Invent. Math.*, 175(3):611–680, 2009.

[19] F. Bourgeois and A. Oancea. S¹-equivalent symplectic homology and linearized contact homology. *Internat. Math. Res. Notices*, 2017(13):3849–3937, 2016.

[20] M. Carl, M. Pumperla, and B. Siebert. A tropical view on Landau-Ginzburg models. *Draft*, 2011.

[21] F. Charest and C. Woodward. Floer trajectories and stabilizing divisors. *J. Fixed Point Theory Appl.*, 19(2):1165–1236, 2017.

[22] M. Chas. Minimal intersection of curves on surfaces. *Geom. Dedicata*, 144(1):25–60, 2010.

[23] M. Chas and D. Sullivan. String topology. arXiv:math/9911159, 1999.

[24] C.-H. Cho and Y.-G. Oh. Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds. *Asian J. Math.*, 10(4):773–814, 2006.

[25] K. Cieliebak, A. Floer, and H. Hofer. Symplectic homology. II. A general construction. *Math. Z.*, 218(1):103–122, 1995.

[26] K. Cieliebak, U. Frauenfelder, and A. Oancea. Rabinowitz Floer homology and symplectic homology. *Ann. Sci. Éc. Norm. Supér.*, 43(6):957–1015, 2010.

[27] K. Cieliebak and J. Latschev. The role of string topology in symplectic field theory. In *New Perspectives and Challenges in Symplectic Field Theory*, volume 49 of CRM Proc. Lecture Notes, 2009.

[28] K. Cieliebak and K. Mohnke. Compactness for punctured holomorphic curves. *J. Symplectic Geom.*, 3(4):589–654, 2005.

[29] K. Cieliebak and K. Mohnke. Symplectic hypersurfaces and transversality in Gromov-Witten theory. *J. Symplectic Geom.*, 5(3):281–356, 2007.

[30] K. Cieliebak and K. Mohnke. Punctured holomorphic curves and Lagrangian embeddings. arXiv:1411.1870, 2014.

[31] T. Coates, A. Corti, S. Galkin, V. Golyshev, and A. M. Kasprczyk. Mirror symmetry and Fano manifolds. In *European Congress of Mathematics Krakow, 2–7 July, 2012*, pages 285–300, 2014.

[32] T. Coates, A. Corti, S. Galkin, and A. M. Kasprczyk. Quantum periods for 3-dimensional Fano manifolds. *Geom. Topol.*, 20(1):103–256, 2016.

[33] T. Coates and A. Givental. Quantum Riemann-Roch, Lefschetz and Serre. *Ann. of Math. (2)*, 165(1):15–53, 2007.

[34] D. A. Cox and S. Katz. *Mirror symmetry and algebraic geometry*, volume 68 of *Math. Surveys Monogr.*, 1999.

[35] J. A. Cruz Morales and S. Galkin. Upper bounds for mutations of potentials. SIGMA, 9(005), 2013.

[36] G. Dimitroglou Rizell. The classification of Lagrangians nearby the Whitney immersion. arXiv:1712.01182, 2017.

[37] G. Dimitroglou Rizell and J. Evans. Unlinking and unknottedness of monotone Lagrangian submanifolds. *Geom. Topol.*, 18(2):997–1034, 2014.

[38] G. Dimitroglou Rizell, E. Goodman, and A. Ivrii. Lagrangian isotopy of tori in $S^2 \times S^2$ and $CP^2$. *Geom. Funct. Anal.*, 26(5):1297–1358, 2016.

[39] L. Diogo, D. Tonkonog, R. Vianna, and W. Wu. In preparation. 2017.

[40] T. Ekholm. Rational SFT, linearized Legendrian contact homology, and Lagrangian Floer cohomology. In *Perspectives in Analysis, Geometry, and Topology*, volume 296 of *Progr. Math.*, pages 109–145, 2012.

[41] T. Ekholm, G. Dimitroglou Rizell, and D. Tonkonog. In preparation. 2017.

[42] T. Ekholm, K. Honda, and T. Kálmán. Legendrian knots and exact Lagrangian cobordisms. *J. Eur. Math. Soc.*, to appear.

[43] T. Ekholm and Y. Lekili. Duality between Lagrangian and Legendrian invariants. arXiv:1701.01284, 2017.

[44] T. Ekholm and A. Oancea. Symplectic and contact differential graded algebras. *Geom. Topol.*, 21(4):2161–2230, 2017.
[45] Y. Eliashberg, A. Givental, and H. Hofer. Introduction to Symplectic Field Theory. In Visions in Mathematics: GAFA 2000 Special volume, Part II, pages 560–673, 2000.
[46] O. Fabert. Higher algebraic structures in Hamiltonian Floer theory I. arXiv:1310.6014, 2013.
[47] A. Floer and H. Hofer. Symplectic homology. I. Open sets in $C^n$. Math. Z., 215(1):37–88, 1994.
[48] K. Fukaya. Deformation theory, homological algebra and mirror symmetry. In Proceedings of the 4th SIGRAV Graduate School on Contemporary Relativity and Gravitational Physics, 2001.
[49] K. Fukaya. Application of Floer homology of Lagrangian submanifolds to symplectic topology. In Morse Theoretic Methods in Nonlinear Analysis and in Symplectic Topology, volume 217 of NATO Sci. Ser. II, pages 231–276, 2006.
[50] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. Lagrangian Floer theory on compact toric manifolds, I. Duke Math. J., 151(1):23–175, 2010.
[51] K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono. Lagrangian Intersection Floer Theory: Anomaly and Obstruction, volume 46 of Stud. Adv. Math. American Mathematical Society, International Press, 2010.
[52] S. Galkin. Toric degenerations of Fano manifolds (in Russian). PhD thesis, Steklov Math. Institute, 2008.
[53] S. Galkin and A. Usnich. Mutations of potentials. Preprint IPMU 10-0100, 2012.
[54] S. Ganatra. Symplectic cohomology and duality for the wrapped Fukaya category. arXiv:1904.7312, 2019.
[55] S. Ganatra and D. Pomerleano. Remarks on symplectic cohomology of smooth divisor complements. Draft, 2017.
[56] A. Gathmann. Absolute and relative Gromov-Witten invariants of very ample hypersurfaces. Duke Math. J., 115(2):171–203, 2002.
[57] A. B. Givental. Homological geometry and mirror symmetry. In Proceedings of the International Congress of Mathematicians, August 31–11, 1994 Zürich, Switzerland, pages 472–480, 1994.
[58] A. B. Givental. A mirror theorem for toric complete intersections. In Topological field theory, primitive forms and related topics, volume 160 of Progr. Math., pages 141–175, 1996.
[59] V. Godin. Higher string topology operations. arXiv:0711.4859, 2007.
[60] W. M. Goldman. Invariant functions on Lie groups and Hamiltonian flows of surface group representations. Invent. Math., 85:263–302, 1986.
[61] M. Goersky and N. Hingston. Loop products and closed coedecis. Duke Math. J., 150(1):117–209, 2009.
[62] T. Graber, J. Kock, and R. Pandharipande. Descendant invariants and characteristic numbers. Amer. J. Math., 124(3):611–647, 2002.
[63] M. Gross. Tropical geometry and mirror symmetry, volume 114 of CBMS Regional Conference Series in Mathematics. 2011.
[64] M. Gross and B. Siebert. From real affine geometry to complex geometry. Ann. of Math. (2), 174(3):1301–1428, 2011.
[65] M. Guest. From quantum cohomology to integrable systems, volume 15 of Oxf. Grad. Texts Math. Oxford University Press, 2008.
[66] H. Hind. Lagrangian spheres in $S^2 \times S^2$. Geom. Topol., 14(2):303–318, 2004.
[67] H. Hind and S. Lisi. Symplectic embeddings of polydisks. Selecta Math. (N.S.), pages 1–22, 2014.
[68] H. Hofer. A general Fredholm theory and applications. Current Developments in Mathematics, pages 1–72, 2004.
[69] K. Hori and C. Vafa. Mirror symmetry. arXiv:hep-th/0002222, 2000.
[70] N. Iiten, J. Lewis, and V. Przyjalkowski. Toric degenerations of Fano threefolds giving weak Landau-Ginzburg models. J. Algebra, 374:104–121, 2013.
[71] K. Irie. A chain level Batalin-Vilkovisky structure in string topology via de Rham chains. Internat. Math. Res. Notices, 2017.
[72] K. Irie. Chain level loop bracket and pseudo-holomorphic disks. arXiv:1801.04633, 2018.
[73] L. Katzarkov and V. Przyjalkowski. Landau-Ginzburg models — old and new. In Proceedings of 18th Gökova Geometry-Topology Conference, pages 97–124, 2011.
[74] T. Kimura, J. Stasheff, and A. A. Voronov. On operad structures of moduli spaces and string theory. *Comm. Math. Phys.*, 171(1):1–25, 1995.

[75] J. Latschev. Fukaya's work on Lagrangian embeddings. *arXiv:1409.6474*, 2014.

[76] Y. I. Manin. Frobenius manifolds, quantum homology and moduli spaces, volume 47 of *Amer. Math. Soc. Colloq. Publ.* 1999.

[77] D. McDuff and D. A. Salamon. *J-Holomorphic Curves and Symplectic Topology*, volume 52 of *Amer. Math. Soc. Colloq. Publ.* 2004.

[78] A. Oancea. A survey of Floer homology for manifolds with contact type boundary or symplectic homology. *Ensaio Mat.*, 7:51–91, 2004.

[79] A. Oancea. The Künneth formula in Floer homology for manifolds with restricted contact type boundary. *Math. Ann.*, 334(1):65–89, 2006.

[80] J. Pardon. Contact homology and virtual fundamental cycles. *arXiv:1508.03873*, 2015.

[81] J. Pardon. An algebraic approach to virtual fundamental cycles on moduli spaces of pseudoholomorphic curves. *Geom. Topol.*, 20(2):779–1034, 2016.

[82] J. Pascaleff. On the symplectic cohomology of log Calabi-Yau surfaces. *arXiv:1304.5298*, 2013.

[83] J. Pascaleff. Floer cohomology in the mirror of the projective plane and a binodal cubic curve. *Duke Math. J.*, 163(13):2427–2516, 2014.

[84] J. Pascaleff and Y. Lekili. Floer cohomology of $g$-equivariant Lagrangian branes. *Compositio Math.*, 152(5):1071–1110, 2016.

[85] J. Pascaleff and D. Tonkonog. The wall-crossing formula and Lagrangian mutations. *arXiv:1711.03209*, 2017.

[86] T. Prince. The tropical superpotential for $\mathbb{P}^2$. *arXiv:1703.07620*, 2017.

[87] V. Przyjalkowski. On Landau-Ginzburg models for Fano varieties. *Commun. Number Theory Phys.*, 1(4):713–728, 2007.

[88] V. Przyjalkowski. Weak Landau-Ginzburg models for smooth Fano threefolds. *Izv. Math.*, 77(4):135–160, 2013.

[89] V. Przyjalkowski. On Calabi-Yau compactifications of toric Landau-Ginzburg models for Fano complete intersections. *arXiv:1701.04762*, 2017.

[90] A. Ritter. Topological quantum field theory structure on symplectic cohomology. *J. Topol.*, 6(2):391–489, 2013.

[91] D. Salamon and J. Weber. Floer homology and the heat flow. *Geom. Funct. Anal.*, 16(5):1050–1138, 2006.

[92] P. Seidel. A biased view of symplectic cohomology. *Current Developments in Mathematics*, 2006:211–253, 2008.

[93] P. Seidel. *Lectures on categorical dynamics*. Author’s website, 2013.

[94] P. Seidel. Disjoinable Lagrangian spheres and dilations. *Invent. Math.*, 197(2):299–359, 2014.

[95] P. Seidel. Fukaya $A_{\infty}$-structures associated to Lefschetz fibrations. III. *arXiv:1608.04012*, 2016.

[96] V. Shende, D. Treumann, and H. Williams. On the combinatorics of exact Lagrangian surfaces. *arXiv:1603.07449*, 2016.

[97] D. Sullivan. String topology: background and present state. *Current Developments in Mathematics*, 2005:41–88, 2007.

[98] F. Tabing. Computations of the structure of the Goldman Lie algebra for the torus. *arXiv:1611.04676*, 2016.

[99] F. Tabing. String homology and Lie algebra structures (Ph.D. thesis). *arXiv:1610.03933*, 2016.

[100] D. Tonkonog. From symplectic cohomology to Lagrangian enumerative geometry. *arXiv:1711.03292*, 2017.

[101] R. Vakil. The enumerative geometry of rational and elliptic curves in projective space. *arXiv:alg-geom/9709007*, 1999.

[102] R. Vianna. On exotic Lagrangian tori in $\mathbb{C}P^2$. *Geom. Topol.*, 18:2419–2476, 2014.

[103] R. Vianna. Infinitely many exotic monotone Lagrangian tori in $\mathbb{C}P^2$. *J. Topol.*, 9(2):535–551, 2016.

[104] R. Vianna. Infinitely many monotone Lagrangian tori in del Pezzo surfaces. *arXiv:1602.03556*, 2016.
[105] C. Viterbo. Functors and computations in Floer homology with applications. II. Preprint, 1996.

University of California, Berkeley