GEOMETRY OF ASYMPTOTICALLY HARMONIC MANIFOLDS WITH MINIMAL HOROSPHERES

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Abstract. \((M^n, g)\) be a complete Riemannian manifold without conjugate points. In this paper, we show that if \(M\) is also simply connected, then \(M\) is flat, provided that \(M\) is also asymptotically harmonic manifold with minimal horospheres (AHM). The (first order) flatness of \(M\) is shown by using the strongest criterion: \(\{e_i\}\) be an orthonormal basis of \(T_pM\) and \(\{b_{e_i}\}\) be the corresponding Busemann functions on \(M\).

Then, (1) The vector space \(V = \text{span}\{b_v | v \in T_pM\}\) is finite dimensional and \(\dim V = \dim M = n\). (2) \(\{\nabla b_{e_i}(p)\}\) is a global parallel orthonormal basis of \(T_pM\) for any \(p \in M\). Thus, \(M\) is a parallizable manifold. And (3) \(F : M \rightarrow \mathbb{R}^n\) defined by \(F(x) = (b_{e_1}(x), b_{e_2}(x), \cdots, b_{e_n}(x))\), is an isometry and therefore, \(M\) is flat. Consequently, AH manifolds can have either polynomial or exponential volume growth, generalizing the corresponding result of [23] for harmonic manifolds. In case of harmonic manifold with minimal horospheres (HM), the (second order) flatness was proved in [28] by showing that \(\text{span}\{b_{e_i}^2 | v \in T_pM\}\) is finite dimensional. We conclude that, the results obtained in this paper are the strongest and wider in comparison to harmonic manifolds, which are known to be AH.

In fact, our proof shows the more generalized result, viz.: If \((M, g)\) is a non-compact, complete, connected Riemannian manifold of infinite injectivity radius and of subexponential volume growth, then \(M\) is a first order flat manifold.

Contents

1. Introduction and Preliminaries 2
2. Busemann functions on AHM’s 6
3. Strong Liouville Type Property 10
   3.1. Known integral formulae for the derivative of harmonic and subharmonic functions 11
   3.2. Integral formula for the derivative of mean value of \(C^1\) function 13
4. Existence of Killing Vector Fields on AHM 18
   4.1. Killing Vector Fields 18
   4.2. Killing Vector Fields on AHM 18

2010 Mathematics Subject Classification. Primary 53C20; Secondary 53C25, 53C35.

Key words and phrases. Asymptotically harmonic manifold, harmonic manifold, Busemann function, mean curvature, horospheres, minimal horospheres, volume growth entropy, Killing vector field.

*The author would like to thank University of Extremadura, Spain, for its hospitality and support while this work was initiated.
4.3. Non Existence of Killing Vector Field of Constant Length on AH Manifolds with \(h > 0\).

5. First order flatness of AHM

5.1. Comparison of first order and second order flatness of HM

6. Consequences of Flatness of AHM

6.1. Final conclusion

7. Appendix

Asymptotically Harmonic Manifolds With Minimal Horospheres Admitting Compact Quotient

References

1. Introduction and Preliminaries

One of the important problems in the geometry of Hadamard manifold (manifold of nonpositive curvature) \(M\) is: To what extent horospheres (geodesic sphere of infinite radius) of \(M\) determine the geometry of \(M\)? This is a very popular topic of current research as can be seen from the very recent paper [10], where the connection between hessian of Busemann functions and rank of Hadamard manifold is established.

Horospheres, geodesic spheres of infinite radius, of \(M\), are level sets of Busemann functions, which are defined below.

Let \(M\) be a complete, simply connected Riemannian manifold without conjugate points. Then by Cartan-Hadamard theorem, every geodesic of \(M\) is a line. Let \(SM\) be the unit tangent bundle of \(M\). For \(v \in SM\), let \(\gamma_v\) be the geodesic line in \(M\) with \(\gamma'_v(0) = v\). Then \(b_v^+ : M \to \mathbb{R}\) and \(b_v^- : M \to \mathbb{R}\) denote the two Busemann functions associated to \(\gamma_v\), respectively, towards \(+\infty\) and \(-\infty\), and are defined as:

\[
\begin{align*}
  b^+_v(x) & = \lim_{t \to +\infty} d(x, \gamma_v(t)) - t, \\
  b^-_v(x) & = \lim_{t \to -\infty} d(x, \gamma_v(t)) + t.
\end{align*}
\]

Note that \(b^+_v(\gamma_v(r)) = +r\). The level sets, \(b_v^{\pm-1}(t)\), are called horospheres of \(M\). Thus, the two Busemann function can be interpreted as distance function from \(\pm\infty\), can be defined for any line in \(M\). Refer [24] for details on Busemann functions.

A complete Riemannian manifold is called harmonic, if all the geodesic spheres of sufficiently small radii are of constant mean curvature. The known examples of harmonic manifolds include flat spaces and locally rank one symmetric spaces. In 1944, Lichnerowicz conjectured that any simply connected harmonic manifold is either flat or a rank one symmetric space. This conjecture is true in compact case and false in non-compact one. However, the other questions in the non-compact case remain open. For the development of the conjecture see the references in [28]. In particular, if an harmonic manifold does not have conjugate points, then the geodesic spheres of any radii are of constant mean curvature and is also AH (cf. [8] Remark 2.2), cf.
Moreover, by Allamigeon’s Theorem [2], it follows that any complete, simply connected and non-compact harmonic manifold has no conjugate points. For definition and more details about harmonic manifolds see [2]. In [8] J. Heber obtained the complete classification of simply connected harmonic manifolds in the homogeneous case.

It is very well known that, eg., real hyperbolic space, $\mathbb{H}^n$, real space form, is characterized by its horospheres. Equivalently, $\mathbb{H}^n$, which is a harmonic manifold, is characterized by its volume density function, denoted by $\Theta(r)$, which in polar co-ordinates can be expressed as $\Theta(r) = r^{n-1}\sqrt{\det g_{ij}(r)}$. In this case, $\Theta'(r) = (n-1)\Delta b_{r^\pm}$.

Szabo asked: To what extend the density function of a harmonic manifold $M$, determine the geometry of $M$? The affirmative answer to his question can be found in [25], in case of harmonic symmetric spaces of rank one.

Asymptotically harmonic manifolds are asymptotic generalization of harmonic manifolds. They were originally introduced by Ledrappier [20, Theorem 1], in connection with rigidity of measures related to the Dirichlet problem (harmonic measure) and the dynamics of the geodesic flow (Bowen-Margulis measure).

Definition 1.1. A complete, simply connected Riemannian manifold without conjugate points is called AH, if the mean curvature of its horospheres is a universal constant. Equivalently, if its Busemann functions satisfy $\Delta b_{v^\pm} \equiv h$, $\forall v \in SM$, where $h$ is a nonnegative constant.

Then by regularity of elliptic partial differential equations, $b_{v^\pm}$ is a smooth function on $M$ for all $v$ and all horospheres of $M$ are smooth, simply connected hypersurfaces in $M$ with constant mean curvature $h \geq 0$. For example, every simply connected, complete harmonic manifold without conjugate points is AH. See [20] for a proof.

Till to-date the only known examples of AH manifolds are harmonic manifolds. Intuitively one thinks that the class of harmonic and AH manifolds coincide. And the Lichnerowicz conjecture should be true for AH manifolds as well. In this paper we answer this question partially that these two classes coincide in case when $h = 0$. For more details on questions related to Lichnerowicz conjecture for AH manifolds see [33], [34] and [14].

Note that study of AH manifolds starts from dimension 2. In dimension 2, the only AH manifolds are symmetric viz., $\mathbb{H}^2$ and $\mathbb{R}^2$. In dimension 3, AH manifolds have been classified in [7, 17, 29, 30] and in non-compact, Einstein and homogeneous case in [8]. For more details on AH spaces, we refer to the discussion and to the references in [7]. Important results in this context are contained in [3] and [17].
Thus, AH manifolds are defined via the property that all of its horospheres, a parallel family of hypersurfaces, have the same constant mean curvature. In view of the above classification the natural analogue of Szabo’s question is: whether the asymptotic growth of volume density function or equivalently whether the geometry of horospheres of an AH manifold influence geometry of its ambient space?

For $v \in SM$, the corresponding horosphere is \emph{totally umbilical} with constant principal curvature $h \in \mathbb{R}$ if and only if $u^+(t) = h \, \text{id}_{\gamma_v(t)^\perp}$. Thus, in this case $M$ is Einstein of constant non-positive curvature $-h^2$, as $R(t) = -h^2 \text{id}_{\gamma_v(t)^\perp}$ from \cite{3}. Thus, if $h = 0$, then $u^+(t) \equiv 0$, i.e. if every horosphere of an AH manifold is totally geodesic, then the ambient manifold is flat. See \cite{11} for more rigidity results about AH manifolds.

Moreover, the classification of 3-dimensional AH manifolds follows from \cite{29} and the recent paper \cite{30}. In particular, in \cite{30} the following result was proved:

\textbf{Theorem 1.2.} $(M, g)$ be a complete and simply connected Riemannian manifold of dimension 3 without conjugate points. If $M$ is an AH of constant $h = 0$, then $M$ is flat.

In case of harmonic manifolds flatness follows in all dimensions from \cite{28}.

\textbf{Theorem 1.3.} \cite{28} $(M, g)$ be a complete and simply connected Riemannian manifold. If $M$ is a non-compact harmonic manifold of polynomial volume growth, then $M$ is flat.

\textbf{Corollary 1.4.} \cite{23} A non-compact harmonic manifold either has polynomial volume growth or of exponential volume growth. Consequently, if $M$ is HM, then $M$ is flat.

In case of HM, it was shown that \cite{28}, the vector spaces $V = \text{span}\{b_v \mid v \in S_p M\}$ and $W = \text{span}\{b_v^2 \mid v \in S_p M\}$ are finite dimensional, where $b_v = b_v^+$ is Busemann function for $v$. Then averaging $b_v^2$ (idea which can be employed only for harmonic manifolds), it was shown that HM is Ricci flat and hence flat. Thus, the flatness Theorem \cite{28} of \cite{28} was proved by using finite dimensionality of $W$. Hence, we term this as \emph{second order flatness}. See Theorem 5.9 of §6 for a sketch of proof of Theorem 1.3 of \cite{28}. Note that to prove flatness of HM, in \cite{28}, the natural fact that $V$ is finite dimensional was not explored.

In this paper, we show that in an AHM $(M^n, g)$, dim $V = n$ and consequently, $M$ is flat. Thus, proving the flatness of AHM by the strongest flatness criterion and also strengthening the flatness result, Theorem 1.3 of \cite{28}.

We term this as \emph{first order flatness} of $M$.

The main result of our paper is Theorem 1.5.

\textbf{Theorem 1.5.} $(M^n, g)$ be an AHM with $\{e_i\}$ an orthonormal basis of $T_p M$ and $\{b_{e_i}\}$, the corresponding Busemann functions on $M$. Then

1. The vector space $V = \text{span}\{b_v \mid v \in T_p M\}$ is finite dimensional and dim $V = \text{dim} \, M = n$. 

\( \{ \nabla b_v(p) \} \) is a global parallel orthonormal basis of \( T_p M \) for any \( p \in M \). Thus, \( M \) is a parallizable manifold.

(3) \( F : M \to \mathbb{R}^n \) defined by \( F(x) = (b_{e_1}(x), b_{e_2}(x), \cdots, b_{e_n}(x)) \), is an isometry and therefore, \( M \) is flat.

One of the important steps in proving main Theorem 1.5 is, Strong Liouville Type Property, that is any subharmonic function bounded from above is constant on an AHM, is proved in §3. In fact, in §3 we show the more generalized result that, if \((M,g)\) is a non-compact, complete, connected manifold of infinite injectivity radius and of subexponential volume growth, then \((M,g)\) satisfies Strong Liouville Type Property, and hence Liouville Property. We also show that it satisfies \( L^1 \) Liouville Property too. To prove Strong Liouville Type Property, we derive an integral formula for the derivative of mean value of a \( C^1 \) function. See §3 for more details.

Using results of §3, the strongest result of the paper, that every non-trivial Killing field on AHM is parallel, is proved in §4. In §5, it is established that any parallel unit vector field on AHM is Killing of the form \( \nabla b_v, \; v \in SM \). This in turn proves Theorem 1.5 in §5. §6 is the concluding section of the paper, where we note all consequences of flatness. We also conclude the final more generalized result of this paper viz.:

**Theorem 1.6.** If \((M,g)\) is a non-compact, complete, connected manifold of infinite injectivity radius and subexponential volume growth, then \((M,g)\) is first order flat.

The proof of the result that, if \( M \) is an AHM admitting compact quotient, then \( M \) must be flat, is included in Appendix, §7, the last section of this paper. We begin by analyzing Busemann functions on AHM’s in §2.

Finally, we note that as any non-compact harmonic manifold is AH \[20\], we recover Theorem 1.3 from our main Theorem 1.5.

We now describe general preliminaries on an AHM which will be used throughout the paper.

In the sequel, the term AH manifold \( M \) tacitly means simply connected, complete Riemannian manifold without conjugate points. Therefore, \( b_v \) are smooth functions on \( M \) for all \( v \) and all horospheres of \( M \) are smooth, simply connected hypersurfaces in \( M \) with constant mean curvature \( h \geq 0 \). Hence, on an AH manifold \( M \) we can define \((1,1)\) tensor fields \( u^+ \) and \( u^- \) as follows: For \( v \in SM \) and \( x \in u^\perp \), let

\[
    u^+(v)(x) = \nabla_x \nabla b^+_{-v} \quad \text{and} \quad u^-(v)(x) = -\nabla_x \nabla b^+_{v}.
\]

Thus, \( u^\pm(v) \in \text{End}(u^\perp) \) and \( \text{tr} \; u^+(v) = \Delta b^+_{-v} = h \), \( \text{tr} \; u^-(v) = -\Delta b^+_{v} = -h \).

\( u^+(v), u^-(v) \), respectively, is the second fundamental form of the *unstable and stable horospheres* \( b_{v}^{-1}(-c), b_{v}^{+1}(c), \; c \in \mathbb{R} \), respectively, at \( p \).

Moreover, the endomorphism fields \( u^\pm \) satisfy the Riccati equation along the orbits of the geodesic flow \( \varphi^t : SM \to SM \). Thus, if \( u^\pm(t) := u^\pm(\varphi^t) \) and the Jacobi operator, \( R(t) := R(\cdot, \gamma'_v(t))\gamma'_v(t) \in \text{End}(\gamma'_v(t)^{u^\perp}) \), then

\[
    (u^\pm)'(t) + (u^\pm)^2(t) + R(t) = 0.
\]
$u^+(t)$ and $u^-(t)$ are called as unstable and stable Riccati solution respectively.

Let $(M, g)$ be a complete Riemannian manifold. Let $\Theta_x(u, r)$ be the density of the volume form of $M$ in normal coordinates centered at some point $x$; so the volume form reads $dV_M = \Theta_x(u, r) dV_{S_x M} dr$ where $dV_{S_x M}$ is the volume form of $S_x M$, where $dV_{S_x M}$ is the volume form of $S_x M$.

The function $\Theta_x$ is related to the mean curvature $h_x$ of spheres centered at $x$ and radius $r$ by the formula

\[ \frac{\Theta'_x(u, r)}{\Theta_x(u, r)} = h_x(\exp_x(ru)), \]

where $\Theta'_x(u, r)$ denotes the derivative of $\Theta_x(u, r)$ with respect to $r$. In what follows, we shall often write for short the point $\exp_x(ru)$ as $(u, r)$ to avoid cumbersome notations; moreover, we will regard $\Theta_x(u, r)$ as a function on $SM \times \mathbb{R}$.

If $\gamma_v$ is a geodesic in $M$ with $\gamma_v(0) = p, \gamma'_v(0) = v$, in an AH, then it follows that $\Delta b^\pm = \lim_{r \to \infty} \frac{\Theta'_p(v, r)}{\Theta_p(v, r)} = \text{const} = h \geq 0$, mean curvature of horospheres of $M$, viz., $b^{\pm-1}(t), t \in \mathbb{R}$.

**Proposition 1.7.** If $(M, g)$ is an AH, then Ricci$_M \leq 0$.

**Proof.** From Riccati equation we get,

\[ (u^\pm)'(t) + (u^\pm)^2(t) + R(t) = 0. \]

Hence, $\text{tr}(u^+) + \text{tr} R = 0$. This implies that Ricci$_v(v) \leq 0$, for any $v \in SM$. $\square$

### 2. Busemann functions on AHM’s

In this section we describe the behaviour of Busemann functions on AHM’s. We show that on an AHM, asymptotic geodesics are bi-asymptotic. Using Liouville Property (Corollary proved in §3), we conclude that any two geodesics of AHM having finite Hausdorff distance towards $+\infty$ are bi-asymptotic. Using standard techniques, we prove that in an AHM there exists bounded strip in all directions. We also describe ideal boundary of an AH manifolds.

If $(M, g)$ is an AH, then by Definition all of its Busemann functions satisfy $\Delta b^\pm_v \equiv h$, $\forall v \in SM$, where $h$ is a nonnegative constant.

**Proposition 2.1.** If $(M, g)$ is an AH manifold, then all of its Busemann functions are smooth. In particular, if $(M, g)$ is a real analytic manifold, then Busemann functions are real analytic. Consequently, if $(M, g)$ is a harmonic manifold, then all of its Busemann functions are real analytic.

**Proof.** We have by definition $\Delta b^\pm_v \equiv h$, $\forall v \in SM$. And Laplacian on $M$ is an elliptic operator with smooth coefficients. Then by regularity of elliptic
ASYMPTOTICALLY HARMONIC MANIFOLDS WITH $h = 0$

partial differential equations, viz., Hörmander’s Theorem 7.5.1 of [11], pg. 178, we conclude that $b_v^\pm$ is a smooth function on $M$ for all $v$.

The conclusion for harmonic manifold follows, as they are analytic and AH [20]. \hfill \square

Now we study ideal boundary of an AH manifold. We begin with the definition of asymptote on AH manifolds.

**Definition 2.2.** 1) Asymptote: Let $\gamma_v$ be a ray in a complete, non-compact Riemannian manifold $(M,g)$. A vector $w \in S_p M, p \in M$ is called an asymptotic to $v$, if $\nabla b_v, s_i(t) \to w$ for some sequence $s_i \to \infty$. Clearly, $\gamma_w$ is also a ray. We also say that $\gamma_w$ is asymptotic ray to $\gamma_v$ from $p$ towards $+\infty$.

2) Bi-asymptote: We say that two geodesics are bi-asymptotic, if they are asymptotic to each other in both positive and negative directions.

3) If all the asymptotic geodesics of $M$ are bi-asymptotic, then we say that $M$ has bi-asymptotic property.

The next proposition is an important result proved in [24].

**Proposition 2.3.** $(M,g)$ be a complete, non-compact Riemannian manifold. If $\gamma_w$ is an asymptote to $\gamma_v$ from $p$, then their Busemann functions are related by

\begin{align}
(4) \quad b_v(x) - b_w(x) & \leq b_v(p). \\
(5) \quad b_v(\gamma_w(t)) & = b_v(p) + b_w(\gamma_w(t)), \\
(6) \quad = b_v(p) - t.
\end{align}

Proof of Corollary 2.4, 2.5 and 2.8 which follows for AH manifolds, is on the same lines as the proof in case of harmonic manifolds [26].

**Corollary 2.4.** If $\gamma_v$ is a geodesic, then asymptote to $\gamma_v$ through $p \in M$, AH, is unique.

**Proof.** Let $\gamma_w$ be an asymptote to $\gamma_v$ from $p$.

Then,

$$\langle \nabla b_v(\gamma_w(t)), \gamma_w'(t) \rangle = \frac{d}{dt}(b_v(\gamma_w(t))) = -1 \quad \text{by (6)}.$$

Since $\nabla b_v$ and $\gamma_w'(t)$ both are unit tangent vectors, we have

$$\nabla b_v(\gamma_w(t)) = -\gamma_w'(t).$$

In particular,

$$\nabla b_v(p) = -\gamma_w'(0).$$

This implies that there is only one asymptotic geodesic to $\gamma_v$ starting from $p$, namely in the direction of $-\nabla b_v(p)$. \hfill \square

**Corollary 2.5.** $(M,g)$ be an AH manifold. Being asymptotic is an equivalence relation on $SM$ as well as on the space of all oriented geodesics of $M$. Moreover, two oriented geodesics are asymptotic if and only if the corresponding Busemann functions agree up to an additive constant.
Proof. If $\gamma_w$ is the asymptotic geodesic asymptotic to $\gamma_v$ starting from $p$, in a non-compact AH then, clearly $b_v - b_w$ is a harmonic function. By (4), it attains maximum at $p$, and by maximum principle, it is a constant function. This implies that $b_v(x) - b_w(x) = b_v(p)$. Conversely, suppose that $b_v(x) - b_w(x) = b_v(p)$. Therefore,
$$\nabla b_v(\gamma_w(t)) = \nabla b_w(\gamma_w(t)) = -\gamma_w'(t).$$
Thus, $\gamma_w$ is an integral curve of $-\nabla b_v$. But, from the Corollary 2.4 asymptotes are the integral curves of $-\nabla b_v$. Hence, we conclude that $\gamma_w$ is asymptotic to $\gamma_v$. Consequently, it follows that being asymptotic is an equivalence relation on $SM$.

**Lemma 2.6.** If $\gamma_v$ is a geodesic line in an AHM $M$, then $b_v^+ + b_v^- = 0$.

**Proof.** As $(M, g)$ is an AHM, $\Delta b_v^+ = \Delta b_v^- = h = 0$. Also,
$$b_v^+(x) + b_v^-(x) = \lim_{t \to \infty} d(x, \gamma_v(t)) + d(x, \gamma_v(-t)) - 2t.$$ Hence, $b_v^+(x) + b_v^-(x) \geq 0$ for all $x$, by triangle inequality, and $(b_v^+ + b_v^-)(\gamma_v(t)) = 0$, since $\gamma_v$ is a line. Thus, the minimum principle shows that $b_v^+ + b_v^- = 0$.

**Corollary 2.7.** The stable horospheres, $(b_v^+)^{-1}(c)$, and unstable horospheres, $(b_v^-)^{-1}(-c)$, of an AHM coincide like flat spaces.

**Corollary 2.8.** Any two asymptotic geodesics of an AHM, $M$, are bi-asymptotic.

**Proof.** From Lemma 2.6 it follows that
$$b_v^+ + b_v^- = 0.$$ Let $\gamma_w^\pm$ be unique asymptote to $\gamma_v$ starting from $p$ in positive and negative direction respectively with $\gamma_w^\pm(0) = p$. Therefore, from the above equation,
$$(\nabla b_v^+)(\gamma_w^+(0)) = - (\nabla b_v^-)(\gamma_w^-(0)).$$ It follows that
$$\frac{d}{dt}(\gamma_w^+(t))|_{t=0} = - \frac{d}{dt}(\gamma_w^-(t))|_{t=0}.$$ Thus, two asymptotes $\gamma_w^\pm$ fit together to form a smooth geodesic $\gamma_w$. Hence, $\gamma_w$ is bi-asymptotic to $\gamma_v$.

**Corollary 2.9.** If $c_1$ and $c_2$ are two geodesics of AHM, $M$ such that $d(c_1(t), c_2(t)) \leq C_1$, as $t \to \infty$, then $c_1$ is bi-asymptotic to $c_2$. We obtain,
$$d(c_1(t), c_2(t)) \leq C_2,$$ as $t \to -\infty$, consequently $d_H(c_1, c_2)$, Hausdorff distance between two lines is bounded. We conclude, in an AHM, there exists a bounded strip in all directions.

**Proof.** If $c_1(0) = v$ and $c_2(0) = w$, then by hypothesis $b_v^+ - b_w^+ \leq C_1$. Clearly, $b_v^+ - b_w^-$ is a bounded harmonic function. We will prove Liouville property for AHM (Corollary 3.10) in §3. Hence, $b_v^+ - b_w^-$ is constant by Liouville property on an AHM $M$. Therefore, $b_v^+ - b_w^- = constant$ implies that $c_1$ and $c_2$ are asymptotic towards $+\infty$ (from Corollary 2.5). Since $b_v^+ = -b_w^-$ from
we also have that $b_v^i - b_w^i$ is also constant, and in turn $c_1$ and $c_2$ are asymptotic towards $-\infty$ also. Consequently, using $\omega$, $d(c_1(t), c_2(t)) \leq C_2$ as $t \to -\infty$.

\section*{Ideal Boundary:} On $SM$ define an equivalence relation $v \sim w$, if $v$ is asymptotic to $w$. The equivalence classes of this relation are called points at infinity. If $M(\infty) = SM/ \sim$, then from Corollary 2.4 and 20, we conclude that $M(\infty)$ is called as equivalent classes of Busemann functions on $M$. And $\bar{M} = M \cup M(\infty)$ is Busemann compactification of an AH manifold. $M(\infty)$ is called as Busemann boundary or ideal boundary of $M$.

\begin{theorem} Let $(M, g)$ be an AH manifold. Let $J_1, J_2, \cdots, J_{n-1}$ be the Jacobi fields along a geodesic ray $\gamma_v$, arising from variations of asymptotic geodesics. If $\{e_i\}$ is an orthonormal basis of $v^\perp$ and $J_i(0) = e_i$, then
\begin{equation}
||J_1 \wedge J_2 \cdots J_{n-1}|| = e^{-ht}.
\end{equation}
\end{theorem}

\begin{proof} Let $\varphi_t$ be the flow of $\nabla b_v$ i.e., it is a one parameter family of diffeomorphism of $M$. Since, the integral curves of $\nabla b_v$ are asymptotic geodesics by Corollary 2.4, $\varphi_t$ is a variation of $\gamma_v$ through asymptotic geodesics. Therefore, $J_i(t) = d\varphi_t(e_i)$. If $\omega$ denotes the volume form on $M$, then
\begin{equation}
||J_1 \wedge J_2 \wedge \cdots J_{n-1}|| = |\det d\varphi_t| = |d\varphi_t(e_1) \wedge d\varphi_t(e_2) \wedge \cdots d\varphi_t(e_{n-1})| = |\varphi_t^* \omega|.
\end{equation}
Using properties of Lie derivative we get,
\begin{equation}
-\frac{d}{dt} \varphi_t^* \omega = L_{\nabla b_v} \omega = (\text{div}\nabla b_v) \varphi_t^* \omega = (\Delta b_v) \varphi_t^* \omega.
\end{equation}
This implies that $-\frac{d}{dt} \varphi_t^* \omega = h \varphi_t^* \omega$. But, as $\varphi_0^* \omega = \omega$, we obtain that $\varphi_t^* \omega = e^{-ht} \omega$. This gives
\begin{equation}
||J_1 \wedge J_2 \wedge \cdots J_{n-1}|| = |\det d\varphi_t| = e^{-ht}.
\end{equation}
\end{proof}

\begin{corollary} If $(M, g)$ is an AHM, then there exists a bounded strip in all directions.
\end{corollary}

\begin{proof} By Theorem 2.10 for an AHM, $||J_1 \wedge J_2 \wedge \cdots J_{n-1}|| = |\det d\varphi_t| = 1$. Hence, in an AHM there exists at least one Jacobi field $J$ along a geodesic $\gamma$ which is bounded. If $\tilde{\gamma}$ is a geodesic in the corresponding geodesic variation of $\gamma$, then $d(\gamma(t), \tilde{\gamma}(t)) \leq \int_0^t ||J(s)|| \, ds \leq C_1 l$. We conclude that $d(\gamma(t), \tilde{\gamma}(t)) \leq C$ for all $t > 0$. By Corollary 2.9, $\gamma, \tilde{\gamma}$ are bi-asymptotic geodesics and $d(\gamma(t), \tilde{\gamma}(t))$ is bounded for all $t \in \mathbb{R}$. Hence, there exists a bounded strip in all directions.
\end{proof}

\begin{remark} Let $J$ be a Jacobi field arising from a variation of asymptotic geodesics. If $J'(r) = 0$ for some $r$, then $J'(r) = u^- (J(r)) = 0$. Hence, $J \equiv 0$. Therefore, any Jacobi field arising from the variation of asymptotic geodesics is non vanishing.
\end{remark}

\begin{corollary} Let $(M, g)$ be an AH manifold and let $\varphi_t$ denote the geodesic flow of $M$. Let $H_0 = b_v^{-1}(0)$, be the horosphere of $M$. If $D$ is any domain in $H_0$ and if $\varphi_t(D) = D_t$, then $A(D) = e^{ht} A(D_t)$. In particular, for
AHM’s $A(D) = A(D_t)$ and for an AHM $\varphi_t$ is an area preserving diffeomorphism of $M$.

Proof. Since $\varphi_t(D) = D_t$, by Theorem 2.10, we obtain:

$$A(D_t) = \int_{D_t} \omega = \int_D |\det d\varphi_t| \omega = \int_D |\varphi_t^* \omega| = e^{-ht} A(D).$$

Proposition 5.1 of [28] shows that HM’s satisfy the absolute area minimizing property of horospheres. The proof also holds for AHM’s and we have that AHM’s satisfy the absolute area minimizing property of horospheres.

Proposition 2.14. Let $M$ be an AHM and $D$ be any compact subdomain of $H_0$ with $\partial D = \Gamma \subset H_0$. Let $\Sigma$ be any compact hypersurface of $M$ with $\partial \Sigma = \Gamma \subset H_0$. Then, $A(D) \leq A\Sigma$.

3. Strong Liouville Type Property

Let $(M,g)$ be a complete, connected non-compact Riemannian manifold of infinite injectivity radius and of subexponential volume growth. In this section, we show that $(M,g)$ satisfies Strong Liouville Type Property and hence Liouville Property. We also show that it also satisfies $L^1$ Liouville Property.

As AHM’s have subexponential volume growth, we infer that they satisfy all the aforementioned Liouville Type Properties in comparison with an HM [28] (which are now known to be flat spaces). Strong Liouville Type Property of AHM is an essential tool in proving the existence of Killing vector field on AHM, a major step for proving flatness of AHM.

To prove the main result of this section, we use techniques of [31] and [28]. We recall that in [28] the integral formula for the derivative of harmonic function on harmonic manifolds was proved.

Let $(M,g)$ denote a complete, connected and non-compact Riemannian manifold of infinite injectivity radius. In this section, we obtain an integral formula for the derivative of mean value of a $C^1$ function on $(M,g)$, viz., Theorem 3.12 of this section. In particular, Theorem 3.12 also holds for AH manifolds. In fact, Theorem 3.12 is the strengthening of the main result of [31] viz., Theorem 3.10 as well as strengthening of the integral formula of [28] viz., Theorem 3.8 quoted in subsection 3.1.

Definition 3.1. $(M,g)$ be a non-compact Riemannian manifold

1) $(M,g)$ is said to satisfy Strong Liouville Type Property if there are no non-trivial subharmonic functions on $(M,g)$ which are bounded from above.
2) $(M,g)$ is said to satisfy Liouville Property if there are no non-trivial bounded harmonic functions on $(M,g)$.
3) $(M,g)$ is said to satisfy $L^1$-Liouville Property if there are no non-trivial non-positive subharmonic functions in $L^1(M)$. 

3.1. Known integral formulae for the derivative of harmonic and subharmonic functions. In this subsection we recall known integral formulae for the derivative of harmonic and subharmonic functions. We also recall definitions and results needed to prove Strong Liouville Type Property for manifolds of subexponential volume growth.

**Definition : Volume growth type**

(i) If volume growth of a manifold $M$ satisfies that $\text{Vol}(B(p, r)) \geq c^r$ for large $r$, and for some $c > 1$, then it is said to have exponential volume growth.

(ii) If volume growth of a manifold $M$ is not exponential, then $M$ is said to have subexponential volume growth. Equivalently, volume growth of a manifold is subexponential, if $\lim_{r \to \infty} \frac{\log \text{Vol}(B(p, r))}{r} = 0$.

(iii) If $\text{Vol}(B(p, r)) \leq Cr^n$ for large $r$ and for some $C$ and $n > 0$, then $M$ is said to have polynomial volume growth.

Note that from the above Definition (ii) an AHM has apriori subexponential volume growth.

In the sequel, $(M, g)$ tacitly denotes a complete, non-compact Riemannian manifold of infinite injectivity radius. Let $V(p, r)$, $A(p, r)$ and $S$, respectively, denote the volume of the ball $B(p, r)$, area of the sphere $S(p, r)$ and unit sphere in $T_p M$, respectively.

**Definition 3.2.** Let $u$ be a continuous (or measurable) function on $M$. Fix $p \in M$ and $r > 0$. Then the mean value of $u$ at $p$ is its average over the ball centred at $p$ and radius $r > 0$, denoted by $A_{u, r}(p)$. We have:

$$
A_{u, r}(p) = \frac{1}{V(p, r)} \int_{B(p, r)} u \, d\mu.
$$

**Definition 3.3.** A manifold $(M, g)$ is said to satisfy weak mean value inequality, $\mathcal{WM}_R(\lambda, b)$, if there exists a constant $\lambda > 0, b > 1$ such that for any $r \leq R/b$ and $f(x) \geq 0$ satisfying $\Delta f \geq 0$ on $B_p(rb)$, then

$$
f(p) \leq \frac{\lambda}{\text{Vol}(B_p(r))} \int_{B_p(br)} f(x) \, dx.
$$

Note : If $b = 1$ in $\mathcal{WM}_R(\lambda, b)$, then it is called as Mean Value Inequality $\mathcal{M}_R(\lambda)$. It is well known that all harmonic manifolds satisfy Mean Value Inequality $\mathcal{M}_R(\lambda)$ with $\lambda = 1$ for all $R$. See [32].

We will use the following **stronger version of the maximum principle** for the subharmonic functions. Lemma 3.4 of [12] states that:

**Theorem 3.4.** At the point on the boundary where the maximum is attained, the subharmonic function has a positive outward normal derivative.

Now we recall the definition of the stability vector field as defined in [31].

**Definition 3.5.** For any real number $r > 0$, the stability vector field $H(\cdot, r)$ on $(M, g)$ is defined by

$$
H(p, r) = \int_{B(p, r)} \exp^{-1}(q) \, d\mu(q), \quad \forall p \in M,
$$

where $\exp$ is the exponential map.
where \( \exp_p^{-1} \) denotes the inverse of exponential map.

It was proved in [31] that the volume function \( V \) and the stability vector field \( H \) are related by the following differential equation.

**Lemma 3.6.** Let \( \nabla \) denote the gradient operator on \((M, g)\). Then for any \( r > 0 \) and \( p \in M \), it holds:

\[
\nabla V(p, r) - \frac{1}{r} \frac{\partial}{\partial r} H(p, r) = 0.
\]

We recall definition of angle as defined in [28].

**Definition 3.7.** Let \( p \in M \) and \( x \in S_p M \). Define angle as in [28], the function:

\[
\theta_x : M \setminus p \to \mathbb{R}, \quad q \to \theta_x(q) = \angle_p(x, v),
\]

where \( \gamma_v \) is the unique geodesic joining \( p \) to \( q \).

In [28] the integral formula for the derivative of harmonic functions was proved which led to the proof of Liouville Theorem for HM.

**Theorem 3.8.** Let \( u : M \to \mathbb{R} \) be a harmonic function on a harmonic manifold \( M \). Then for \( x \in S_p M \),

\[
(xu(p)) = \frac{1}{V(p, r)} \int_{S(p, r)} u \cos \theta_x \, d\sigma,
\]

where \( d\sigma \) denotes the volume of \( S(p, r) \).

The above integral formula was generalized in [15] to obtain the integral formula for the derivative of subharmonic functions on a harmonic manifold \( M \).

**Theorem 3.9.** Let \( u : M \to \mathbb{R} \) be a subharmonic function on a harmonic manifold \( M \). Then for \( x \in S_p M \),

\[
(\langle \nabla u(p), x \rangle) \leq \frac{1}{V(p, r)} \int_{S(p, r)} u \cos \theta_x \, d\sigma.
\]

The integral formula in Theorem 3.8 above was generalized further in [31].

**Theorem 3.10.** Let \((M, g)\) be a non-compact, connected, complete Riemannian manifold with infinite injectivity radius. If \( u \) is a function on \( M \) satisfying the mean-value property, then it holds:

For any real number \( r > 0 \),

\[
(xu(p)) = \frac{1}{V(p, r)} \int_{S(p, r)} u \cos \theta_x \, d\sigma - \frac{1}{r} \frac{u(p)}{V(p, r)} (\nabla V(p, r), x).
\]

**Corollary 3.11.** Let \((M, g)\) be a non-compact, connected and complete Riemannian manifold with minimal horospheres and infinite injectivity radius. Any bounded function on \((M, g)\) satisfying mean value property is constant.
3.2. Integral formula for the derivative of mean value of $C^1$ function. In this subsection, we prove our main result viz., the Strong Liouville Type Property, for a complete, connected, non-compact Riemannian manifold of infinite injectivity radius and of subexponential volume growth. In particular, we obtain the desired result for AHM.

First we prove the integral formula for the derivative of mean value of a $C^1$ function on $(M, g)$, a complete, connected, non-compact Riemannian manifold of infinite injectivity radius. The proof uses techniques of the proof of Theorem 3.10 of [31] and proof of Theorem 3.8 of [28].

**Theorem 3.12.** Let $(M, g)$ be a complete, connected, non-compact Riemannian manifold of infinite injectivity radius. Let $u$ be a $C^1$ differentiable function on $M$. Then the derivative of mean value of $u$ satisfies the integral formula:

$$x.A_{u,r}(p) = -\frac{A_{u,r}(p)}{V(p,r)} \langle \nabla V(p,r), x \rangle + \frac{1}{V(p,r)} \int_{S(p,r)} u \cos \theta_x \, d\sigma.$$  

**Proof.** Let $u$ be a $C^1$ function on $M$. Fix $p \in M$ and $r > 0$. We have:

$$A_{u,r}(p) = \frac{1}{V(p,r)} \int_{B(p,r)} u \, d\mu.$$  

Let $c$ be the geodesic with $c(0) = p$ and $c'(0) = x$. Then,

$$x.A_{u,r}(p) = \frac{d}{dt} A_{u,r}(c(t))_{|t=0}$$

$$= \frac{d}{dt} \left( \frac{1}{V(c(t), r)} \int_{B(c(t), r)} u \, d\mu \right)_{|t=0}.$$  

Consequently,

$$x.A_{u,r}(p) = -\frac{1}{V(p,r)^2} \langle \nabla V(p,r), x \rangle \int_{B(p,r)} u \, d\mu$$

$$+ \frac{1}{V(p,r)} \frac{d}{dt} \left( \int_{B(c(t), r)} u \, d\mu \right)_{|t=0}.$$  

From (15) we obtain

$$\frac{1}{V(p,r)^2} \langle \nabla V(p,r), x \rangle \int_{B(p,r)} u \, d\mu = \frac{A_{u,r}(p)}{V(p,r)} \langle \nabla V(p,r), x \rangle.$$  

Using techniques of proof of Theorem 2.1 of [28] we will show that

$$\frac{d}{dt} \left( \int_{B(c(t), r)} u \, d\mu \right)_{|t=0} = \int_{S(p,r)} u \cos \theta_x \, d\sigma.$$  

Finally, substituting (17) and (18) in (16) we obtain the required integral formula (14).

Now it remains to prove (18).
Consider one parameter family of diffeomorphisms of $M$, as in [28], given by

$$f_t = \exp_{c(t)} \circ P_t \circ \exp^{-1}_p : M \to M.$$ 

Let $J_x = J$ be the vector field induced by $f_t$ at time $t = 0$. Since $f_t$ maps radial geodesic starting from $p$ to geodesics starting from $c(t)$, $J$ satisfies the following property which we shall make use of:

$J$ is the unique Jacobi field satisfying $J(0) = x, J'(0) = 0$, when restricted to any geodesic starting from $p$.

As $f_t(B(p, r)) = B(c(t), r)$, we obtain

$$\left( \int_{B(c(t), r)} u d\mu \right) = \left( \int_{B(p, r)} f_t^*(u) d\mu \right) = \left( \int_{B(p, r)} f_t^* (u) f_t^*(d\mu) \right).$$

We have $f_t^*(d\mu) = |\det df_t| d\mu$. But as $J$ satisfies $J(0) = x, J'(0) = 0$, when restricted to any geodesic starting from $p$; the Jacobian of $f_t$ at $t = 0$ viz., $|\det df_t|_{t=0} = 1$, and $\frac{d}{dt} |\det df_t|_{t=0} = 0$. Therefore,

$$\frac{d}{dt} \left( \left( \int_{B(c(t), r)} u d\mu \right) \right)_{t=0} = \frac{d}{dt} \left( \left( \int_{B(p, r)} (f_t^* (u) d\mu) \right) \right)_{t=0} = \int_{B(p, r)} \frac{d}{dt} (f_t^* (u))_{t=0} d\mu.$$

Consequently,

$$\frac{d}{dt} \left( \int_{B(c(t), r)} u d\mu \right)_{t=0} = \int_{B(p, r)} L_{J_t} u d\mu = \int_{B(p, r)} L_J (u d\mu) = \int_{S(p, r)} i_J (u d\mu),$$

where $i_J$ denotes the interior product with the field $J$. However, $\omega = dr \wedge d\sigma$ and $dr|_{S(p, r)} = 0$. Therefore, $i_J (u \omega) = u i_J (\omega)$. Also $i_J dr = < J, \frac{\partial}{\partial r} >$. Hence, $i_J (u \omega) = u < J, \frac{\partial}{\partial r} > d\sigma$ and we obtain,

$$\frac{d}{dt} \left( \int_{B(c(t), r)} u d\mu \right)_{t=0} = \int_{S(p, r)} u < J, \frac{\partial}{\partial r} > d\sigma.$$

Since $J$ is a Jacobi field with $J(0) = x, J'(0) = 0$, its parallel component is

$$< J, \frac{\partial}{\partial r} > = < J(0), v > + < v, J'(0) > r = \cos \theta_x.$$

Finally we obtain,

$$\frac{d}{dt} \left( \int_{B(c(t), r)} u d\mu \right)_{t=0} = \int_{S(p, r)} u \cos \theta_x d\sigma.$$

□
Corollary 3.13. If \( u \) is a harmonic function on a harmonic manifold \((M, g)\), then we recover Theorem 3.8.

Proof. By a characterization of harmonic manifolds, any harmonic function on a harmonic manifold satisfies mean value property. And the volume density function \( \Theta_p \) is independent of point \( p \in M \). Therefore, in this case \( \nabla V(p, r) = 0 \) and \( A_{u,r}(p) = u(p) \). Therefore, the conclusion follows from the above Theorem 3.12. \(\square\)

Now we can prove our main theorem of this section.

Theorem 3.14. If \((M, g)\) is a complete, connected, non-compact Riemannian manifold of infinite injectivity radius and of subexponential volume growth, then there are no non-trivial bounded subharmonic functions on \((M, g)\). In particular, an AHM also has no non-trivial bounded subharmonic functions.

Proof. Let \( u \) be a bounded subharmonic function on \( M \). Then there exists a constant \( \alpha > 0 \) such that \( |u| \leq \alpha \). From the above integral formula, \( \forall p \in M \) and \( r > 0 \) and Lemma 3.6 we obtain,

\[
||x.A_{u,r}(p)|| \leq \frac{\alpha}{V(p,r)}||\frac{1}{r} \frac{\partial}{\partial r}H(p,r)|| + \frac{A(p,r)}{V(p,r)}.
\]

But:

\[
||\frac{1}{r} \frac{\partial}{\partial r}H(p,r)|| = ||\frac{1}{r} \frac{\partial}{\partial r} \int_{B(p,r)} exp^{-1} (q)d\mu(q)||
\]

\[
= ||\frac{1}{r} \frac{\partial}{\partial r} \int_{S(p,r)} exp^{-1} (q)d\sigma(q)||
\]

\[
\leq \frac{1}{r} \int_{S(p,r)} ||exp^{-1} (q)||d\sigma(q)
\]

\[
= A(p,r), \text{ since } ||exp^{-1} (q)|| = r, \forall q \in S(p,r).
\]

Consequently,

\[
||x.A_{u,r}(p)|| \leq 2\alpha \frac{A(p,r)}{V(p,r)}.
\]

We have,

\[
\lim_{r \to \infty} \frac{A(p,r)}{V(p,r)} \leq \lim_{r \to \infty} \frac{\int_{S} \Theta_p(r,u)d\sigma}{\int_{0}^{r} \int_{S} \Theta_p(r,u)drd\sigma} = \lim_{r \to \infty} \frac{\Theta'_p(r,u)}{\Theta_p(r,u)} = 0.
\]

Hence, for any \( p \in M \) and for any \( x \in S_p M \),

\[
||x.A_{u,r}(p)|| = 0 \text{ as } r \to \infty.
\]

We have,

\[
V(p,r)A_{u,r}(p) = \int_{B(p,r)} u \, d\mu = \int_{0}^{r} \int_{S} u(r,\varphi)\Theta_p(r,\varphi) \, drd\varphi.
\]

Therefore, differentiating (23) with respect to \( r \),

\[
V'(p,r)A_{u,r}(p) + A'_{u,r}(p)V(p,r) = \int_{S} u(r,\varphi)\Theta'_p(r,\varphi) \, drd\varphi.
\]
Therefore,

\[ A'_{u,r}(p) = \frac{1}{V(p,r)} \int_S u(r, \varphi) \Theta_p(r, \varphi) \, drd\varphi - \frac{V'(p,r)}{V(p,r)} A_{u,r}(p) \]  

(25)

As

\[ V(p,r) = \int_0^r \int_S \Theta_p(r, \varphi) \, drd\varphi, \]

we have

\[ V'(p,r) = \int_S \Theta_p(r, \varphi) \, drd\varphi = A(S(p,r)). \]

Consequently,

\[ \left| A'_{u,r}(p) \right| \leq 2 \alpha \frac{A(p,r)}{V(p,r)}. \]

(26)

From (21) it follows that

\[ \lim_{r \to \infty} \left| A'_{u,r}(p) \right| = 0. \]

(27)

Thus from (22) and (27), \( A_{u,r}(p) \) is independent of \( p \in M \) and also independent of \( r \) for sufficiently large \( r \). Hence, this clearly implies that \( u \) is constant on all balls of sufficiently large radii.

We obtain that \( u(x) = C = \text{constant}, \ \forall x \in B(p,r), r \geq N, \) for \( N \) sufficiently large. By the maximum principle of subharmonic functions,

\[ \max_{x \in B(p,r)} u(x) = \max_{y \in \partial B(p,r)} u(y) = u(z), \text{ for some } z \text{ with } r = d(p,z). \]

Now by the stronger version of the maximum principle for the subharmonic functions, Theorem [3.4] we obtain:

\[ \max_{x \in B(p,r), r < N} u(x) = \max_{x \in \partial B(p,r), r < N} u(x) \leq \max_{x \in \partial B(p,r), r \geq N} u(x) = C. \]

Therefore, for all \( x \in M \), we have \( u(x) \leq C \) i.e. \( u \) attains global maximum on \( M \). Therefore, \( u \) is a constant function on \( M \), again by the maximum principle for subharmonic functions. \( \square \)

We observe that Liouville Property holds on an AHM, in comparison with an HM.

**Corollary 3.15.** \((M, g)\) be a complete, connected, non-compact Riemannian manifold of infinite injectivity radius and of subexponential volume growth. Then \((M, g)\) satisfies Liouville Property on \( M \). In particular, an AHM also satisfies Liouville Property.

**Corollary 3.16.** \((M, g)\) be a complete, connected, non-compact Riemannian manifold of infinite injectivity radius and of subexponential volume growth. Then \((M, g)\) satisfies Strong Liouville Type Property. In particular, an AHM also satisfies Strong Liouville Type Property.

**Proof.** If \( u \) is a subharmonic function on \( M \) bounded from above, then \( g(x) = e^{u(x)} \) is a bounded non-negative subharmonic function on \( M \), as \( \Delta g = e^{u(x)}(\Delta u + ||\nabla u||^2) \geq 0. \) Therefore, by Theorem [3.14] \( g \) is a constant function and in turn, \( u \) is a constant function. \( \square \)
Corollary 3.17. \((M,g)\) be a complete, connected, non-compact Riemannian manifold of infinite injectivity radius and of exponential volume growth. If \(u\) is a bounded function on \(M\) satisfying mean value property, then the derivative of \(u\) is also bounded. In particular, harmonic functions on harmonic manifolds of exponential volume growth satisfy this property.

Proof. As \((M,g)\) is of exponential volume growth, we have

\[
\lim_{r \to \infty} \frac{A(p,r)}{V(p,r)} \leq \lim_{r \to \infty} \frac{\int_S \Theta_p(r,u) d\sigma}{\int_0^r \int_S \Theta_p(r,u) dr d\sigma} = \lim_{r \to \infty} \frac{\Theta'_p(r,u)}{\Theta_p(r,u)} = h > 0.
\]

Therefore, from the proof of Theorem 3.14, \(\|x A_{u,p}(r)\| = |xu(p)| \leq 2\alpha h\), for all \(p \in M\). \(\square\)

Now we compare results of this section with some similar type of results on Liouville Property.

Li and Wang \[21\] have proved the Liouville property in the setup of the theorem given below.

Theorem 3.18. \((M,g)\) be a complete manifold satisfying the mean value inequality \(M(\lambda)\) i.e. \(M_R(\lambda), \forall R\). If \(\lambda < 2\) and \(M\) has subexponential volume growth, then bounded harmonic functions are constants.

In fact, a slight modification of their proof of Theorem 3.18 in \[21\] shows:

Theorem 3.19. \((M,g)\) be a complete manifold of subexponential volume growth, then \((M,g)\) satisfies Strong Liouville Type Property.

Theorem 3.20. Let \((M,g)\) be a manifold of subexponential volume growth, then any non-negative superharmonic function \(u \in L^1(M)\) is constant.

Proof. The proof follows from Theorem 13.2 of \[6\]. \(\square\)

1) Note that for manifolds of subexponential volume growth, we obtain Liouville property as an application of Theorem 3.19. On the other hand Liouville property, Corollary 3.17 obtained here by direct method and also without using the mean value property, as opposed to the usual practice.

2) Note that the Strong Liouville Type Property for harmonic manifolds follows from Theorem 3.19 directly. But, the proof of Theorem 3.9 \[15\] uses harmonicity heavily. On the other hand Theorem 3.12 is quite general and suffices to prove Strong Liouville Property on manifolds of subexponential volume growth.

3) Note that conclusion of Corollary 3.17 need not imply that \(u\) is a constant function. In fact, Strong Liouville Property does not hold on AH manifolds with constant \(h > 0\). \[28\] shows that on harmonic manifolds of \(h > 0\), for each \(v \in S_p M\),

\[
h_v(x) = -n \int_0^r \Theta(r) \cos \theta_v(x),
\]

is a family of non-constant bounded harmonic functions.
4. Existence of Killing Vector Fields on AHM

In this section, we show that any Killing vector field on an AHM, $M$, is non-trivial and parallel. And thus, there exists a Killing vector field on an AHM. Further, we also show that on an AH manifold of $h > 0$, there does not exist any Killing vector field of constant length.

First we describe some preliminaries on Killing field.

4.1. Killing Vector Fields.

Definition 4.1. A vector field $X$ on a Riemannian manifold $(M, g)$ is called a Killing field, if the local flows generated by $X$ acts by isometries. Equivalently, $X$ is a Killing field if and only if $L_X g = 0$, where $L$ denotes the Lie derivative of metric $g$ with respect to $X$.

Now we recall the following results about Killing vector fields. Proofs of the results can be found in [24].

Proposition 4.2. $X$ is Killing field if and only if $v \to \nabla_v X$ is a skew symmetric $(1, 1)$-tensor.

Proposition 4.3. For a given $p \in M$, a Killing vector field $X$ is uniquely determined by $X(p)$ and $(\nabla X)(p)$.

Proposition 4.4. The set of Killing fields $\text{iso}(M, g)$ is a Lie algebra of dimension $\leq \frac{n(n+1)}{2}$. Furthermore, if $M$ is complete, then $\text{iso}(M, g)$ is the Lie algebra of $\text{Iso}(M, g)$.

Proposition 4.5. If $X$ is a Killing field on $(M, g)$ and consider the function $f = \frac{1}{2}g(X, X) = \frac{1}{2}||X||^2$, then

(1) $\nabla f = -\nabla_X X$.

(2) $\text{Hess } f(V, V) = g(\nabla_V X, \nabla_V X) - R(V, X, X, V)$.

(3) $\Delta f = ||\nabla X||^2 - \text{Ric}(X, X)$.

Lemma 4.6. If $K$ is a Killing field on a Riemannian manifold $(M, g)$, then

(28) $\nabla^2_{X,Y} K = -R(K, X)Y$.

We refer [1] for more geometric exposition on Killing vector fields. The following Proposition 4.7 can be found in [1].

Proposition 4.7. A Killing vector field $X$ on a Riemannian manifold $(M, g)$ has constant length if and only if every integral curve of the field $X$ is geodesic.

Proposition 4.8. If $X$ is a Killing vector field on a Riemannian manifold $(M, g)$, then $X$ is a Jacobi field along every geodesic $\gamma(t)$, for all $t \in \mathbb{R}$.

4.2. Killing Vector Fields on AHM. In this subsection, we prove the main result of this paper that, in an AHM every Killing vector field on $M$ is parallel.

Theorem 4.9. If $X$ is a Killing vector field on an AHM $M$, then there exists a constant $C > 0$ such that $||X|| \leq C$, and consequently, $X$ is parallel on $M$. 

Proof. Let $X$ be a non-trivial Killing vector field on an AHM $M$. From Proposition 1.7 in an AHM $M$, $\text{Ricci}_M \leq 0$. By Proposition 4.5, it follows that $f = \frac{1}{2}g(X, X) = \frac{1}{2}\|X\|^2$ is a non-negative, non-zero subharmonic function on $M$.

Case (i) Suppose that $\|X\| < C$, for some $C > 0$, then $f$ is a bounded non-negative subharmonic function on $M$. By Theorem 3.14 $f$ is a constant function on $M$. Hence,

$$\|\nabla X\|^2 = \text{Ricci}(X, X) \leq 0.$$ 

Consequently, $\|\nabla X\|^2 = 0$ and thus $X$ is a parallel vector field.

Case (ii) Fix $p \in M$. Suppose that there exists $x \in M$ such that $\|X\|(x) \to \infty$ as $d(p, x) \to \infty$ on $M$. Consider $g(x) = e^{-f(x)}$. Then $g$ is a non-negative bounded function on $M$ with $0 \leq g \leq 1$. We have,

$$\Delta g(x) = e^{-f(x)}(-\Delta f + \|\nabla f\|^2).$$

(29)

If $h(x) = -\Delta f + \|\nabla f\|^2$, then $\Delta g(x) = e^{-f(x)}h(x)$.

Note that $g$ is a smooth non-negative function, which converges to zero outside a compact set $K$ containing $p$.

(a) Suppose that $h \equiv 0$, then $g$ is a bounded harmonic function on AHM. Then $g$ is constant function on $M$ by Corollary 3.15 Liouville Property. If $g(q) = 0$ for some $q \in K$, then $g \equiv 0$. This implies that $f$ converges to $\infty$ everywhere. This is a contradiction to smoothness of $f$.

If $g(q) = c > 0$, for some $q \in K$, then $g \equiv c$, which again contradicts to smoothness of $g$, as $g$ converges to zero outside $K$. Therefore, this case can’t occur.

(b) Suppose that there exists a point $q \in K$ such that $h(q) > 0$. Then by continuity we can find a neighbourhood of $q$ say $N \subset K$ in which $h > 0$. Then $g$ is bounded subharmonic function on $N$. Again by Theorem 3.15 $g$ is constant function on $N$. Note that by argument of case (a), $g = c > 0$ on $N$. Therefore, $f$ is a positive constant on $N$ and consequently, $X$ is a non-trivial parallel vector field on $N$. Therefore, we can assume $\|X\| = 1$ on $N$. But from Proposition 4.3 a Killing vector field on $M$ is uniquely determined by $X(q) = v, \nabla X(q) = 0$. Thus, $X$ is parallel vector field on all of $M$. But, this is a contradiction to $\|X\|(x) \to \infty$. Thus, this case also can’t occur.

We conclude that, $X$ must be a bounded vector field and by case (i) $X$ is parallel.

Now we show that there exists a non-trivial parallel vector field on an AHM. We need the following important Lemma.

Lemma 4.10. If $(M, g)$ is an AHM, then $\text{Iso}(M, g)$ is diffeomorphic to $\text{Iso}(\mathbb{R}^n, \text{can})$. Consequently, $\text{Iso}(M, g)$ is diffeomorphic to semidirect product of $O(n)$ and $\mathbb{R}^n$. Hence, in particular, $\text{Iso}(M, g)$ can’t be a discrete group.
Proof. As \((M, g)\) is an AHM, then by geometry of \(M\), it is diffeomorphic to \(\mathbb{R}^n\). Let \(\varphi : M \rightarrow \mathbb{R}^n\) be diffeomorphism. Define

\[ Iso(M, g) \rightarrow Iso(\mathbb{R}^n, can) \]

\[ \psi \rightarrow \varphi \circ \psi \circ \varphi^{-1} \]

Note that as \(\psi\) is an isometry of \(M\) and \(\varphi\) is a diffeomorphism, \(\varphi \circ \psi \circ \varphi^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is a distance preserving surjective map, hence is an isometry, by Myer-Steenrod theorem. Clearly, the above map defines diffeomorphism between the two isometry groups. \(\square\)

**Corollary 4.11.** If \(X\) is a Killing vector field on an AHM, then \(X\) is a non-trivial parallel vector field on \(M\).

Proof. If \(X\) is a Killing vector field on an AHM, then \(X\) is a parallel vector field on \(M\) by Theorem 4.9. Thus, either a Killing vector field on an AHM is identically zero or is non-trivial and parallel. Suppose that there doesn’t exist any non-trivial Killing vector field on \(M\). Then, as \(Iso(M, g)\) is a Lie group, its connected component of identity is trivial. Hence, \(Iso(M, g)\) must be a discrete group. This is a contradiction, in view of Lemma 4.10, proves the required result. \(\square\)

**Corollary 4.12.** In an AHM \(M\), we have \(1 \leq \dim iso(M, g) \leq \dim M = n\).

Proof. Since any Killing field \(X\) on \(M\) is a non-trivial parallel vector field, the linear map, \(X \rightarrow X(p)\) is injective from \(iso(M, g)\) to \(T_pM\). And therefore, \(1 \leq \dim iso(M, g) \leq \dim M = n\). \(\square\)

**Corollary 4.13.** In an AHM \(M\), there exists a non-trivial Jacobi field on \(M\) which is parallel, and which arises out of variation of bi-asymptotic geodesics.

Proof. Note that from Corollary 4.11 a Killing field \(X\) on AHM is non-trivial parallel and by Proposition 4.13 \(X\) is Jacobi field along every geodesic. We may assume that \(\|X\| = 1\). Let \(X(p) = v\). Consider geodesic \(\gamma_w\), passing through \(p\) with initial velocity vector \(w \perp v\). Then \(X\) is parallel perpendicular Jacobi field along \(\gamma_w\). From [4], it follows that \(X\) arises out of variation of bi-asymptotic geodesics, bi-asymptotic to \(\gamma_w\). \(\square\)

Thus, we have shown:

**Corollary 4.14.** In an AHM \(M\), the following properties of a Killing vector field are equivalent:

1) The field \(X\) has bounded length.
2) The field \(X\) is non-trivial and parallel on \((M, g)\).
3) The field \(X\) is non-trivial parallel central Jacobi field on \(M\).
4) \(X\) has constant length.

Proof. Clearly, 1) \(\Rightarrow\) 2) from Theorem 4.9. 2) \(\Rightarrow\) 3) follows from Corollary 4.13 and trivially, 3) \(\Rightarrow\) 4) \(\Rightarrow\) 1). Thus, all the assertions 1), 2), 3) and 4) are equivalent. \(\square\)

Remark 4.15. Assertions 1), 2), 4) of Corollary 4.14 are equivalent, for a Killing vector field in any manifold of non-positive sectional curvature [4]. In case of AHM, we obtain the same conclusion without any assumption on sectional curvatures.
4.3. Non Existence of Killing Vector Field of Constant Length on AH Manifolds with $h > 0$. In this subsection, we find estimate on Ricci curvature of AH manifold $M$ of $h > 0$. Then using this estimate we show that there does not exist any Killing vector field on $M$ of constant length.

**Proposition 4.16.** If $(M, g)$ is an AH manifold of constant $h > 0$. Then $\text{Ricci} (v, v) \leq \frac{-h^2}{(n-1)} < 0$, for any $v \in SM$, where Ricci denotes the Ricci curvature of $M$.

**Proof.** Taking the trace of (3), we obtain that on any asymptotically harmonic manifold $M$,

$$- \text{Ricci}(\gamma'_v(t), \gamma'_v(t)) = \text{tr}(u^+)^2(t), \forall v \in SM.$$  

By Cauchy-Schwartz inequality,

$$\text{tr}(u^+)^2(t) \geq \left( \text{tr} u^+ \right)^2 = \frac{h^2}{n-1}.$$  

Thus, $\text{Ricci}(v, v) \leq -\frac{h^2}{(n-1)} < 0$. \square

We also recover Proposition 1.7.

**Corollary 4.17.** If $(M, g)$ is an AHM, then $\text{Ricci} (v, v) \leq 0, \forall v \in SM$.

**Lemma 4.18.** If $(M, g)$ is any Riemannian manifold of negative Ricci curvature, then $M$ has no non-trivial Killing vector fields of constant length.

**Proof.** $(M, g)$ be a Riemannian manifold with $\text{Ricci}_M < 0$. If $X$ is a Killing vector field of constant length, then $f$ as in Proposition 4.5, is a constant function and hence harmonic function. Therefore, from Proposition 4.5 (3), $\Delta f = 0 = ||\nabla X||^2 - \text{Ric}(X, X)$. Therefore,

$$0 \leq ||\nabla X||^2 = \text{Ric}(X, X) < 0.$$  

This implies that $X = 0$. Thus, $\text{Ricci} (X, X) = 0$ if and only if $X = 0$. \square

**Corollary 4.19.** If $(M, g)$ is an AH manifold with $h > 0$, then $M$ has no non-trivial Killing vector field of constant length.

**Proof.** If $(M, g)$ is an AH manifold of constant $h > 0$, then by Proposition 4.16 $\text{Ricci}_M < 0$. Hence, from Lemma 4.18 the conclusion follows. \square

5. First order flatness of AHM

In case of harmonic manifolds with polynomial volume growth, the vector spaces $V = \text{span}\{b_v|v \in T_pM\}$ and $W = \text{span}\{b^2_v|v \in T_pM\}$ are finite dimensional; where for a non-unit tangent vector $v \in T_pM$, the corresponding Busemann function is defined by $b^+_v = b_v = ||v||b_v/||v||$. The proof of flatness of HM relies heavily on finite dimensionality of $W$ (28). We term this as second order flatness of $M$. However, it turns out that HM is flat if and only if $\dim V = \text{span}\{b_v|v \in T_pM\} = n = \dim M$. This is the most natural and the strongest criterion for flatness of HM. We term this as first order flatness of complete, simply connected Riemannian manifold without conjugate points.

In this section, we prove the first order flatness of AHM; viz., that dim
Therefore, \( DF \) isometry and hence is distance preserving and we have from Theorem 5.1, that homoeomorphism which maps geodesics to geodesics, we conclude that 

\[
(33)
\]

Consequently, 

\[
\text{This implies that } ||DF(q)(w)||^2 = \sum_{i=1}^{n} \langle \nabla b_{e_i}(q), w \rangle = ||w||^2.
\]

Therefore, \( DF(q) : T_qM \rightarrow \mathbb{R}^n \) is a linear \( C^1 \) isometry. We obtain, \( F \) is an homeomorphism which maps geodesics to geodesics, We conclude that \( F \) is distance preserving and we have from Theorem [5.1] that \( F \) is a smooth isometry and hence \( M \) is flat. \( \square \)
Corollary 5.3. If $(M^n, g)$ is a complete, simply connected Riemannian manifold without conjugate points. Let $\{e_i\}$ be an orthonormal basis of $T_p M$ and let $\{b_{e_i}\}$ be the corresponding Busemann functions on $M$. Then the following are equivalent:

(i) The vector space $V = \text{span}\{b_{e_i}|v \in T_p M\}$ is finite dimensional and $\dim V = \dim M = n$.

(ii) $F : M \to \mathbb{R}^n$ defined by $F(x) = (b_{e_1}(x), b_{e_2}(x), \ldots, b_{e_n}(x))$, is an isometry and therefore, $M$ is flat.

(iii) $\{\nabla b_{e_i}(p)\}$ is a global parallel orthonormal basis of $T_p M$ for any $p \in M$. Thus, $M$ is a parallizable manifold.

Theorem 5.4. If in an AHM, $X$ is a parallel vector field with $\|X\| = 1$, then $X = \nabla b_{v}$, for some $v \in SM$. Consequently, a parallel Killing vector field in AHM with $\|X\| = 1$ is of the form $\nabla b_{v}$.

Proof. It is well known that a parallel vector field in a simply connected manifold is the gradient field for a distance function (cf. [24], pg. 192). We obtain a $f \in C^\infty(M)$ such that $\|\nabla f\| = 1$ and $X = \nabla f$. Therefore, integral curves of $X$ are unit speed geodesics of $M$ and level sets of $f$ are parallel family of hypersurfaces in $M$ (refer [26] and [24]). We have that every geodesic of an AHM $M$ is a line. Let $\gamma_v$ be a geodesic of $M$ with $\gamma_v(0) = p$, then $X(\gamma_v(t)) = \gamma_v'(t) = \nabla f(\gamma_v(t))$. We may assume that $f(p) = 0$. Thus, $\nabla f(p) = \gamma_v'(0) = v$ implies that $f(\gamma_v(t)) = -t$. If $x \in f^{-1}(-c)$, then $d(x, \gamma_v(t)) \geq d(f^{-1}(-t), f^{-1}(-c)) = |t + c| = t - c$. Equivalently, $b^+_v(x) \geq f(x)$. But, as $\nabla X = \nabla^2 f = 0$, $f$ is a harmonic function. Therefore, $b^+_v(x) - f(x)$ is a nonnegative harmonic function which attains its minimum at $p$ and hence must be a constant function. Therefore, $f = b^+_v$. \hfill $\Box$

Now we can prove our main Theorem 1.5 as stated in the introduction.

Theorem 1.5 : $(M^n, g)$ be an AHM with $\{e_i\}$ an orthonormal basis of $T_p M$ and $\{b_{e_i}\}$, the corresponding Busemann functions on $M$. Then,

(1) The vector space $V = \text{span}\{b_{e_i}|v \in T_p M\}$ is finite dimensional and $\dim V = \dim M = n$.

(2) $\{\nabla b_{e_i}(p)\}$ is a global parallel orthonormal basis of $T_p M$ for any $p \in M$. Thus, $M$ is a parallizable manifold.

And

(3) $F : M \to \mathbb{R}^n$ defined by $F(x) = (b_{e_1}(x), b_{e_2}(x), \ldots, b_{e_n}(x))$, is an isometry and therefore, $M$ is flat.

Proof. If $(M^n, g)$ is an AHM, then by Theorem 5.4 there exists a parallel Killing vector field $X$ of the form $X = \nabla b_{v}$. Since, $X$ is parallel, the vector distribution orthogonal to $X$ is also parallel and involute on $M$. Thus, by Theorem 5.4, if $\{e_i\}$ is an orthonormal basis of $T_p M$ for any $p \in M$, then $\{\nabla b_{e_i}\} \text{ is a global parallel basis of } T_p M \text{ for any } p \in M$. Now consider the vector space $V = \text{span}\{b_{e_i}|v \in T_p M\}$. Let $b = \sum a_{e_i} b_{e_i}$. But, $\nabla b = \sum a_{e_i} \nabla b_{e_i}$. This implies that $a_{e_i} = a_i$ and $\nabla b_{e_i} = \nabla b_{e_i}$. We obtain, $V = \text{span}\{b_{e_i}|v \in T_p M\} = \text{span}\{b_{e_i}\}$, is of dimension $n$. We conclude from Theorem 6.2 that $M$ is flat. \hfill $\Box$
Second order flatness of HM is already known from [28]. Now from the proof of Theorem 1.5 we also have first order flatness of HM, as per our expectations.

**Corollary 5.5.** If \((M^n, g)\) is an HM, then the vector space \(V = \text{span}\{b_\nu \mid v \in T_p M\}\) is finite dimensional and \(\dim V = \dim M = n\) and hence, \(M\) is flat of order one.

5.1. **Comparison of first order and second order flatness of HM.**

In this subsection, we sketch the proof of second order flatness of HM \((\[28\]\) and observe the merits of first order flatness of AHM, described in this paper.

\((M, g)\) be a complete manifold. Fix \(p \in M\), \(r(x) = r(p, x)\), denotes distance of \(x \in M\) from fixed point. \(WM_R(\lambda, b)\), denotes the weak mean value inequality as defined in §3. \(\mathcal{H}_l(M)\) be the linear space of all harmonic functions \(f\) on \(M\) with a polynomial growth of order at most \(l\).

\[
\mathcal{H}_l(M) = \left\{ f : M \to \mathbb{R}, \Delta f = 0 \text{ and } |f(x)| \leq O\left(r^l(x)\right) \right\}.
\]

Theorem 4.2 of Li-Wang [21] is important in our discussion.

**Theorem 5.6.** \([21]\) \((M, g)\) be a complete manifold satisfying the weak mean value inequality \(WM_R(\lambda, b)\). Suppose that the volume growth of \(M\) satisfies \(V_b(r) = O(r^\nu)\) for some point \(p \in M\) and any \(\nu > 0\). Then \(\mathcal{H}_l(M)\) is finite dimensional for all \(l \geq 0\) and \(\dim \mathcal{H}_l(M) \leq \lambda(2b + 1)^{(2l + \nu)}\).

**Corollary 5.7.** Under the conditions of Theorem 5.6, the vector space
\[
\mathcal{F} = \left\{ f : M \to \mathbb{R}, \Delta f = c \text{ and } |f(x)| \leq O\left(r^2(x)\right) \right\}
\]
is also a finite dimensional space and \(\dim \mathcal{F} = 1 + \dim \mathcal{H}_2(M)\).

It is well known that all harmonic manifolds satisfy Mean Value Inequality \(\mathcal{M}_R(\lambda)\) with \(\lambda = 1\) for all \(R\). See [32]. Thus, they also satisfy \(WM_R(\lambda, b)\), for \(\lambda = 1\) and any \(b > 1\).

**Corollary 5.8.** \((M^n, g)\) be a harmonic manifold of polynomial volume growth, \(V_b(r) = O(r^\nu)\). Then \(V = \text{span}\{b_\nu \mid v \in T_p M\}\) and \(W = \text{span}\{b_\nu^2 \mid v \in T_p M\}\) are finite dimensional vector spaces of dimension \(\leq 3^{(n+2)}\) and \(\leq 1 + 3^{(4+n)}\), respectively.

**Proof.** If \((M, g)\) is a harmonic manifold of polynomial volume growth, \(V_b(r) = O(r^nu)\) implies that \(V_b(r) = O(r^n)\) and \(\Delta b_\nu = 0\), for all \(v \in SM\). Note that as \(b_\nu\) is a Lipschitz function, \(|b_\nu(x)| \leq O(r(x))\) and hence \(V = \text{span}\{b_\nu \mid v \in S_p M\}\) is finite dimensional (Theorem 5.6) and \(\dim V \leq (2b + 1)^{(2l + \nu)} = 3^{(2+n)}\).

Also, \(\Delta b_\nu^2 = 2\) for all \(v \in SM\) and \(|b_\nu^2(x)| \leq O(r^2(x))\). Consequently, \(W = \text{span}\{b_\nu^2 \mid v \in T_p M\}\) (Corollary 5.7) is also finite dimensional vector space and \(\dim W \leq 1 + (2b + 1)^{(4+n)} = 1 + 3^{(4+n)}\). \(\square\)

We shortly describe the proof of theorem in [28], that harmonic manifolds with polynomial volume growth are flat.

**Theorem 5.9.** Harmonic manifolds with polynomial volume growth are flat [28].
Proof. The original proof of Theorem 1.3 given in [28] is based on the idea of the Szabo’s proof of the Lichnerowicz’s conjecture in compact case. In the proof, $b_v^2$ was averaged (idea which can be employed only for harmonic manifolds), and a parallel displaced family, $g_\gamma$, of real valued functions on $\mathbb{R}$, for every geodesic $\gamma$ was obtained. As $\text{span}\{b_v^2\}$ is a finite dimensional vector space (Corollary 5.8), $\text{span}\{g_\gamma\}$ is also a finite dimensional vector space and therefore, the generator function $g$ is a trigonometric polynomial. By using properties of $g$, $g$ was written in the simpler form. Then another family of radial functions $\mu_\gamma$ was introduced. The generator function, $\mu$ was obtained by generalizing co-ordinate function $r \cos \theta$ on a harmonic manifold. Then using properties of $\mu$, the two families were related. It was observed that $g$ and $\mu$ are almost periodic functions. Finally, using the Characteristic Property of an almost periodic function, it was proved that $M$ is Ricci flat and hence flat.

Conclusion:

1) Note that above sketch of Theorem 5.9 of [28] shows that the flatness of HM was proven by using finite dimensionality of, $W = \text{span}b_v^2$ completely.
2) The sketch also shows that the proof uses harmonicity of $M$ heavily, while the proof obtained in this paper for AHM is wider and strongest in comparison.
3) Also from Corollary 5.8 it is not clear whether $\dim V \leq n$. In fact, the bound $3\left(2+n\right) \gg n$ and dimension estimate of [21] is not optimal, in this case.

6. Consequences of Flatness of AHM

In this section, we describe importance of our main result that an AHM is flat. We need the following definitions.

Definition 6.1. If $(M, g)$ is a connected, non-compact Riemannian manifold, then the volume entropy $h_{vol}(M)$ of $M$ is defined as :

$$h_{vol}(M, g) = \lim_{R \to \infty} \sup \frac{\log(\text{Vol}(B_p(R)))}{R},$$

where $B_r(p) \subset M$ is the open ball of radius $R$ around $p \in M$.

Note that (34) doesn’t depend on the choice of reference point $p$ and $h_{vol}(M, g)$ is therefore well defined.

Corollary 6.2. If $(M, g)$ is an AH manifold of constant $h \geq 0$, then $M$ has either polynomial volume growth or exponential volume growth.

Proof. $(M, g)$ be an AH manifold of constant $h \geq 0$. If $h = 0$, then from our main Theorem 1.3, $M$ is flat and hence of polynomial volume growth. If $h > 0$, then as $\lim_{r \to \infty} \frac{\Theta_q(r, v)}{\Theta_q(r, v)} = h > 0$, for any $q \in M$ and $v \in S_q M$, volume growth is exponential.

Thus, we have generalized the result of [23], Corollary 1.4 stated in §1:
Corollary 1.4: If \((M, g)\) is a harmonic manifold of constant \(h \geq 0\), then \(M\) has either polynomial volume growth or exponential volume growth.

We have obtained:

**Theorem 6.3.** If \((M, g)\) is an AH manifold of constant \(h \geq 0\), then the following properties are equivalent:

1. \(M\) has subexponential volume growth.
2. \(h = 0\).
3. \(M\) is flat.
4. \(M\) is of polynomial volume growth.
5. Volume entropy of \(M\), \(h_{\text{vol}}(M) = 0 = h\).
6. \(M\) has rank \(n\).

In [11] among AH Hadamard manifolds, a rigidity theorem with respect to volume entropy for real, complex and quaternionic hyperbolic spaces were obtained. In particular, the following result was proved:

**Theorem 6.4.** \((M, g)\) be an \(n\)-dimensional Hadamard manifold of Ricci curvature \(\text{Ricci} \geq -(n - 1)\). If \((M, g)\) is AH, then \(h_{\text{vol}} = (n - 1)\) and equality holds if and only if \((M, g)\) is isometric to the real hyperbolic space \(\mathbb{H}^n\) of constant curvature \(-1\).

The following result was proved in [13]:

**Theorem 6.5.** Let \((M, g)\) be an AH manifold of constant \(h \geq 0\), such that \(||R|| \leq R_0\) and \(||\nabla R\|| \leq R'_0\) with suitable constants \(R_0, R'_0 > 0\). Then the following properties are equivalent:

1. \(M\) has rank 1.
2. \(M\) has Anosov geodesic flow \(\varphi^t : SM \to SM\).
3. \(M\) is Gromov hyperbolic.
4. \(M\) has purely exponential volume growth with growth rate \(h_{\text{vol}} = h > 0\).

**Remark 6.6.** 1) Our Theorem 6.3 is complementary to Theorem 6.4 and Theorem 6.5.

2) In view of Corollary 6.2 (Corollary 1.4), harmonic manifolds and AH manifolds share the common property viz. that they have the same volume growth. This is an important observation towards proving that class of AH and harmonic manifolds coincide.

6.1. **Final conclusion.** We showed in §3 that the Strong Liouville Type Property holds for more general type of spaces viz., for non-compact, complete, connected manifold of infinite injectivity radius and subexponential volume growth. Similarly, the results in the other sections of this paper, like the existence of Killing vector fields on AHM, can be proved in the more generality. Hence, we can conclude first order flatness even for these general type of spaces.

**Theorem 1.6 :**

If \((M, g)\) is a non-compact, complete, connected manifold of infinite injectivity radius and subexponential volume growth, then \((M, g)\) is first order flat.
7. Appendix

Asymptotically Harmonic Manifolds With Minimal Horospheres Admitting Compact Quotient

In this Appendix, we prove:

**Theorem 7.1.** If \((M, g)\) is an an AHM, admitting compact quotient, then \(M\) is flat

Note that:
1) Theorem 7.1 should be known, but the author couldn’t find the reference, where it is explicitly proved and hence she includes it here.
2) In view of Theorem 7.1, the present paper is devoted to proving flatness of non-compact AHM.

We recall here definition of complete AH manifold, that of volume growth and volume growth entropy.

**Definition 7.2.**
1) \((M, g)\) be a complete Riemannian manifold without conjugate points and \((\tilde{M}, g)\) be its universal Riemannian covering space. Then \((M, g)\) is called AH, if there exists a constant \(h \geq 0\) such that the mean curvature of all horospheres in \(\tilde{M}\) is \(h\).

2) Let \((M, g)\) and \((\tilde{M}, g)\) be as in 1) and choose \(p \in \tilde{M}\). Then the volume entropy (or asymptotic volume growth) \(h_v = h_v(M, g)\) is defined as:

\[
h_v = h_v(M, g) = \lim_{R \to \infty} \frac{\log(\text{Vol}_{\tilde{M}}(B_p(R)))}{R}.
\]

Manning [22] showed that the limit above always exists and is independent of \(p \in \tilde{M}\).

In [34] the following Proposition was proved.

**Proposition 7.3.** If \((M, g)\) is a compact AHM without focal points, then \(M\) is flat.

We prove the above mentioned result without no focal point condition, mainly by using the following result of [34] and Tits Theorem.

**Proposition 7.4.** If \((M, g)\) is a compact AH manifold with \(\tilde{M}\) as universal covering space of \(M\) and \(\Delta b_v = h\), for all \(v \in S\tilde{M}\), then \(h = h_v\).

**Theorem 7.5.** Tits Theorem: A finitely generated subgroup of a connected Lie group has either exponential growth or is almost nilpotent and hence has polynomial growth.

**Theorem 7.6.** If \((M, g)\) is compact AHM, then \((M, g)\) is flat.

**Proof.** If \(\tilde{M}\) is the universal covering space of \(M\), then \(\Delta b_v = 0\), for all \(v \in S\tilde{M}\). And hence the volume growth entropy \(h_v = 0\), by Proposition 7.4. We obtain, volume growth of \(\tilde{M}\) is subexponential and consequently, \(\pi_1(M)\) is of subexponential growth. But by Tits’ Theorem, \(\pi_1(M)\) has polynomial growth, as it is a subgroup of connected Lie group \(\text{Isom}(\tilde{M})\), of subexponential growth. By [19] compact Riemannian manifold without
conjugate points and with polynomial growth fundamental group are flat. We conclude that \((M, g)\) is flat.

\[\square\]

As any harmonic manifold is AH we conclude:

**Corollary 7.7.** If \((M, g)\) be a compact HM, then \(M\) is flat.

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