New Classes of Regular Symmetric Fractals

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Abstract
The paper introduces new fractal families many of which approach optimal information dimension for annular and checkerboard structures and include the Sierpinski carpet and the Menger sponge as special cases. The complementary mapping is defined, and a notation to represent the families is proposed. The new classes represent an enhanced set that goes beyond the recently published results on optimal information dimensionality and they can be expected to have applications in natural and engineered self-similar systems.

Keywords Noninteger dimensions · Fractal data · Information theory · Optimal dimensionality

1 Introduction

The importance of fractals to the study of natural, social and engineered systems is well established [7, 18]. They belong to the larger field of scale-invariant and self-similar systems [2, 3, 10, 19, 23]. In a recent paper [11], novel fractals with dimension that is close to the optimal value from the perspective of information efficiency [12–14] were presented. Designs were found for up to ten-way branching at each iteration, and they included symmetric and asymmetric ones, as well as those where in addition to holes, a few layers of other regions are peeled. Only representative fractals were described with the hope that the method can be used to find many other designs that will provide insight in transformations that occur in natural and engineered systems. The notion of dimension is applicable at all conceivable scales; therefore, this research is relevant for investigation of anomalous mechanical properties that emerge from compressing three-dimensional volumes into lower dimensions [1, 4] and also in investigations of metamaterials [6].
In more advanced investigations, the mathematical basis of noninteger spaces [21, 22] will have to be brought into the analysis. Some progress in that direction has been made in applications to the information dimension of physical space [15], which takes us to the view that space is not to be seen as a three-dimensional container. The evolutionary stages of a noninteger dimensional physical space were considered [16], and this has applications to astronomy as well as other physical systems [9, 17].

This paper lists new fractals that are motivated by examples of physical systems where influences flow out from the center in a manner that is captured by annular mapping as well as those given by checkerboard patterns. This work has many parallels with constructions in [11]. Since natural and engineered systems are likely to have a range of characteristics arising from the underlying physics associated with different values of information efficiency, the examples are of more than just a theoretical interest.

2 Iterated Function Systems

Iterated function system (IFS) fractals are created on the basis of simple plane transformations that include scaling, dislocation and the rotation of the plane axes [5, 8]. Creating an IFS fractal consists of following steps:

1. Draw an initial pattern on the plane
2. Transform the initial pattern using appropriate contracting transformation
3. Combine initial and transformed patterns
4. Repeat as many times as desired

In many cases, each iteration consists of one or more affine transformations of the type (with suitable constants (a, b, c, e, f, g):

\[
\begin{pmatrix}
x_{n+1} \\ y_{n+1}
\end{pmatrix} = \begin{pmatrix} a & b \\ c & e \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} f \\ g \end{pmatrix}
\]

(1)

For example, the binary tree fractal of Fig. 1 that has been drawn to 6 iterations requires the specification of the scaling factor r and the angle \( \theta \). Here \( a = e = r \cos \theta \); \( b = r \sin \theta \); \( c = -r \); \( f = 0 \), \( g = 1 \).

We can also see the algorithm as one where each “cube” is replaced by the number \( N(\epsilon) \) at each iterative step, with \( \frac{1}{\epsilon} = n \), and dimension \( d \) equal to [7, 11]:

\[
d = \frac{\ln N(\epsilon)}{\ln n}
\]

(2)

Thus if, on a surface, the nine sub-squares of each square, obtained upon a three-way division along both axes, are replaced by 8 squares in an iterative scheme, the dimension will be \( \frac{\ln 8}{\ln 3} \approx 1.893 \). This provides an easy procedure to compute the dimension for regular mappings as will be shown in the examples below.
3 Annular Planar Maps

We will first consider planar fractals (dimensions $1 < d < 2$), which have annular square forms.

**Definition** An annular planar fractal where at each iteration a square is replaced by $n \times n$ sub-squares with alternating bands of sides $n, n-l, n-m\ldots$ will be called $A_{2n}^2(n, n-l, n-m, \ldots)$, where the sub-squares at the layers of $n, n-l, n-m, \ldots$ are retained and the others are removed.

Note that the layer counts of sub-squares from outside with count of $n$ in decrement by 2.

**Examples** The first iteration of two mappings based on 49- and 64-cell subdivision is presented in Fig. 2.

The left mapping shows the original square replaced by $7 \times 7$ cells of which all the ones in the outer parameter are occupied and the ones inside one sub-square removed in a square ring. Likewise, the right mapping based on $8 \times 8$ cells is $A_{8}^2(8, 4)$. 
For each of these cases, we need to count the number of sub-squares that are retained in each iteration and use formula (2) to compute the dimension. In (a), the basic $7 \times 7$ plane is replaced by 25 smaller squares $= 7^2 - 5^2 + 3^2 - 1 = 32$, and this substitution is repeated. In the $8 \times 8$ example of (b), each iteration gives us $8^2 - 6^2 + 4^2 - 2^2 = (8 - 6)(8 + 6) + (4 - 2)(4 + 2) = 2(14 + 6) = 40$ sub-squares. Therefore,

$$d(\mathcal{A}_7^2(7, 3)) = \frac{\ln 32}{\ln 7} \approx 1.781$$
$$d(\mathcal{A}_8^2(8, 4)) = \frac{\ln 40}{\ln 8} \approx 1.774$$

**Definition** \(\mathcal{A}_n\)'s complementary fractal mapping where the light and dark sub-squares are exchanged will be called \(\mathcal{B}_n\).

**Definition** Let \(\alpha\) represent the number of sub-squares (out of \(n^2\)) in the iterative map that are retained.

**Corollary**

$$d(A_n) = \frac{\ln \alpha}{\ln n}, \text{ and } d(B_n) = \frac{\ln (n^2 - \alpha)}{\ln n} \quad (3)$$

**Example** The complementary mapping for \(A_7^2\) (7, 3) will be \(B_7^2(7, 3) = A_7^2(5, 1)\).

$$d(B_7^2(7, 3)) = d(A_7^2(5, 1)) = \frac{\ln 17}{\ln 7} \approx 1.456$$

In Fig. 3, the iteration maps each square into 81 sub-squares of which 49 are retained (shown in dark).

The dimensionality of this mapping will be:

$$d(A_9^2(9, 5, 1)) = \frac{\ln 49}{\ln 9} \approx 1.771$$

![Fig. 3 A_9 (9, 5, 1) fractal](image-url)
For a more general case, consider the alternating annular square rings of $11 \times 11$ given in Fig. 4.

The sub-squares that are retained will have the count $11^2 - 9^2 + 7^2 - 5^2 + 3^2 - 1 = 2(20 + 12 + 4) = 72$. Therefore,

$$d[A^2_{11}(11, 7, 3)] = \frac{\ln 72}{\ln 11} \approx 1.784$$

In general, for the $n \times n$ case, we can compute the retained sub-squares as:

$$4(n-1 + n - 5 + \cdots + n - l)$$

where $n-l$ is the least positive term in the sequence. Simplifying, we find that for even $n$,

$$d = \frac{\ln 0.5(n^2 + 2n)}{\ln n}, \text{even}$$

For odd $k$,

$$d = \begin{cases} \frac{\ln 0.5(n^2 + 2n + 1)}{\ln n} ; & n - 1 \mod 4 = 0 \\ \frac{\ln 0.5(n^2 + 2n - 1)}{\ln n} ; & n - 1 \mod 4 \neq 0 \end{cases}$$

These may be simplified further.

For example, (3) may be rewritten as:

$$d = 1 + \frac{\ln (0.5n + 1)}{\ln n}, \text{even}$$

When $n = 8$ (Fig. 2b), $d = 1 + \frac{\ln 5}{\ln 8} \approx 1.774$, as computed.

**Theorem** The dimensionality of the map $A$ is $s$ if $\alpha = n^s$. 
Proof This follows from the fact \( d(A_n) = \frac{\ln \alpha}{\ln n} \). It also follows that \( d(A_n) \) will be less than 1, if \( \alpha < n \), for the logarithm function is monotonically increasing.

4 Basic Planar Maps

We now consider basic annular maps for \( n = 3, 4 \) and 5 as in Fig. 5. The dimensionality of these maps is.

\[
\begin{align*}
A_3^2(3), \text{ also known as Sierpinski (Figs. 5a and 6)} & = \frac{\ln 8}{\ln 3} \approx 1.893 \\
A_4^2(4) \text{ (Figs. 5b and 7)} & = \frac{\ln 12}{\ln 4} \approx 1.792 \\
A_5^2(5,1) \text{ (Fig. 5c)} & = \frac{\ln 17}{\ln 5} \approx 1.760 \\
\end{align*}
\]

Their complementary maps, also shown in Fig. 5, have dimensionality of \( \frac{\ln 1}{\ln 3} = 0 \); \( \frac{\ln 4}{\ln 4} = 1 \); and \( \frac{\ln 8}{\ln 5} \approx 1.292 \).

Clearly, the complementary maps of \( A_3^2(3) \) and \( A_4^2(4) \) are not interesting with values of 0 and 1 for the first one telescopes down to nothingness, whereas the second is a stable point.

As mentioned before, \( A_3^2(3) \) is the Sierpinski map [20], which is shown in Fig. 6 for its second, third, and fourth iterations: The next figure shows the \( A_4^2(4) \) map of Fig. 5(b) after the second iteration.

![Fig. 5](image1)

(a) (b) (c)

**Fig. 5** a \( A_3^2(3) \), b \( A_4^2(4) \), c \( A_5^2(5,1) \), together with their complementary mappings

![Fig. 6](image2)

**Fig. 6** \( A_3^2(3) \) (Sierpinski carpet) in its second, third, and fourth iterations
This fractal is basically similar to the Sierpinski carpet excepting that the whites are bigger.

5 Related Symmetric Checkerboard Maps

Another interesting regular fractal is given by the mapping of Fig. 8 that represents the part of the checkerboard.

\[
d[C(3)] = \frac{\ln 4}{\ln 3} \approx 1.262
\]

\[
d[\overline{C}] = \frac{\ln 5}{\ln 3} \approx 1.465
\]

A mapping where \( C \) is rotated will naturally not change the dimensionality. The next order symmetric checkerboard map will be \( 5 \times 5 \) as shown in Fig. 9:
Fig. 9 $5 \times 5$ Checkerboard map

Its dimensionality will be given by:

$$d[C(5)] = \frac{\ln 13}{\ln 5} \approx 1.594$$

Its complement mapping (with three white squares at the top that is not shown here) will have dimensionality of $\frac{\ln 12}{\ln 5} \approx 1.544$. Since the difference in the counts of black and white squares will only be one, the two dimensionality figures will be quite close to each other.

One can also modify the checkerboard map in a variety of ways to get new fractals.

6 Annular Fractals in Three Dimensions

For clarity, the generalizations of the annular mappings into three dimensions will be shown with the superscript 3, as in $A_n^{3}(k)$.

The Menger sponge, the 3-dimensional generalization of $A_2^{3}(3)$, is very well known [20].

Let us consider the generalization of $A_4^{3}(4)$. In its first iteration, it will look like Fig. 10.

The number of sub-cubes generated from a cube at each iteration is 32 and therefore

$$d[A_4^{3}(4)] = \frac{\ln 32}{\ln 4} \approx 2.5$$

The first iteration of $A_5^{3}(5)$ is shown in Fig. 11.

Its dimensionality is:

$$d[A_5^{3}(5)] = \frac{\ln 44}{\ln 5} \approx 2.351$$
This dimensionality value is less than that for $A^3_4(4)$ for it is less dense. A much more dense fractal $A^3_5(1)$ is given in Fig. 12.

$$d[A^3_5(1)] = \frac{\ln 112}{\ln 5} \approx 2.932$$

Finally, we present $A^3_5(5, 1)$ mapping in Fig. 13. It is more dense than $A^3_5(5)$ and less dense than $A^3_5(1)$ mapping.

$$d[A^3_5(5, 1)] = \frac{\ln 57}{\ln 5} \approx 2.512$$
7 Conclusions

The paper introduced new fractal families many of which approach optimal information dimension for annular and checkerboard structures and include the Sierpinski carpet and the Menger sponge as special cases. The complementary mappings were defined, and a notation to represent the families was proposed. The new classes represent an enhanced set that goes beyond the recently published results on optimal information dimensionality.

This work may be extended by defining other classes that are either larger in the number of sub-squares at each iteration or are according to some other simple definition for the basic planar forms. It would be worthwhile to look for members of these families in engineered and natural physical systems.
Data Availability  The paper has no associated data.

References

1. K. Bertoldi, V. Vitelli, J. Christensen et al., Flexible Mech. Metamater. Nat. Rev. Mater. 2, 17066 (2017)
2. A. Bunde, S. Havlin, Fractals in science (Springer, Heidelberg, 2013)
3. T. Burns, R. Rajan, A mathematical approach to correlating objective spectro-temporal features of non-linguistic sounds with their subjective perceptions in humans. Front. Neurosci. 13, 794 (2019)
4. C. Coulais, C. Kettenis, M. van Hecke, A characteristic length scale causes anomalous size effects and boundary programmability in mechanical metamaterials. Nature Phys 14, 40–44 (2018)
5. G. Edgar, Measure, topology, and fractal geometry (Springer-Verlag, New York, 2008)
6. G. Failla, M. Zingales, Advanced materials modelling via fractional calculus: challenges and perspectives. Philos. Trans. A Math. Phys. Eng. Sci. 378(2172), 20200050 (2020)
7. K.J. Falconer, Fractal geometry: mathematical foundations and applications (Wiley, Hoboken, 2003)
8. T. Iwaniec, G. Martin, Geometric function theory and non-linear analysis (Oxford Mathematical Monographs, Oxford, 2001)
9. L. Jun, M. Ostoja-Starzewski, Edges of Saturn’s rings are fractal. Springerplus 4, 158 (2015)
10. S. Kak, Power series models of self-similarity in social networks. Inf. Sci. 376, 31–38 (2017)
11. S. Kak, Fractals with optimum information dimension. Circuit Syst. Signal Process 40(11), 5733–5743 (2021)
12. S. Kak, The base-e representation of numbers and the power law. Circuits Syst. Signal Process. 40, 490–500 (2021)
13. S. Kak, Information, representation, and structure. International Conference on Recent Trends in Mathematics and Its Applications to Graphs, Networks and Petri Nets, New Delhi, India (2020). https://doi.org/10.36227/techrxiv.12722549.v1
14. S. Kak, The intrinsic dimensionality of data. Circuits Syst. Signal Process. 40, 2599–2607 (2021)
15. S. Kak, Information theory and dimensionality of space. Sci. Rep. 10, 20733 (2020)
16. S. Kak, Asymptotic freedom in noninteger spaces. Sci. Rep. 11, 1–5 (2021)
17. S. Kak, Information-theoretic view of the variation of the gravitational constant. TechRxiv (2021). https://doi.org/10.36227/techrxiv.14527104.v1
18. B.B. Mandelbrot, The fractal geometry of nature (W. H. Freeman, New York, 1983)
19. S.J. Miller (ed.), Benford’s law: theory and applications (Princeton University Press, Princeton, 2015)
20. S. Semmes, Some novel types of fractal geometry (Oxford Mathematical Monographs, Oxford, 2001)
21. F.H. Stillinger, Axiomatic basis for spaces with noninteger dimensions. J. Math. Phys. 18, 1224–1234 (1977)
22. V.E. Tarasov, Vector calculus in non-integer dimensional space and its applications to fractal media. Commun. Nonlinear Sci. Numer. Simul. 20, 360–374 (2015)
23. T. Vicsek, Fluctuations and scaling in biology (Oxford University Press, Oxford, 2001)

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