Quantum mechanics as a complete physical theory

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Abstract

We show that the principles of a “complete physical theory” and the conclusions of the standard quantum mechanics do not irreconcilably contradict each other as is commonly believed. In the algebraic approach, we formulate axioms that allow constructing a renewed mathematical scheme of quantum mechanics. This scheme involves the standard mathematical formalism of quantum mechanics. Simultaneously, it contains a mathematical object that adequately describes a single experiment. We give an example of the application of the proposed scheme.

1 Introduction

In the seminal work by Einstein, Podolsky, and Rosen [1], the main principles were formulated that must be satisfied by a complete physical theory in the authors’ opinion:

a. “each element of the physical reality must have a counterpart in the physical theory” and

b. “if we can confidently (i.e., with the probability one) predict the value of some physical quantity without perturbing the system in any way, then there exists an element of the physical reality corresponding to this physical quantity.”

The standard quantum mechanics (the theory traced back to Bohr, Heisenberg, Dirac, and von Neumann) did not accept this maxim. A single experiment has no adequate counterpart in the mathematical formalism of the standard quantum mechanics. Moreover, the belief was firmly established that such a counterpart cannot exist.

In this work, we attempt to formulate the basic points of a mathematical scheme of quantum mechanics satisfying two seemingly incompatible requirements. First, the standard mathematical formalism of quantum mechanics can be reproduced within this scheme. Second, this scheme involves a mathematical object that adequately describes a single experiment.

Most of these points are formulated in [2]. We clarify and extend them here. In particular, we essentially clarify the role of the measuring device. In this work, in variation with [2], greater attention is paid to mathematical aspects of the theory. A more detailed phenomenological justification of the axioms and the relations of the proposed approach to other versions of the theory can be found in [3].

2 Observables and physical states

The proposal is to construct the mathematical formalism of quantum mechanics based on the algebraic approach to quantum theory [3], where observable quantities correspond to elements of some algebra. The main points of this algebraic approach can be formulated more simply if in addition to the directly observable quantities, their complex combinations are also included.
in the consideration. These combinations are called dynamic quantities in what follows. We accept the following statement as the first postulate.

Elements of an involutive, associative, and (in general) noncommutative algebra \( \mathfrak{A} \) correspond to dynamic variables such that the following conditions are satisfied:

a. for each element \( \hat{R} \in \mathfrak{A} \), there exists a Hermitian element \( \hat{A} (\hat{A}^* = \hat{A}) \) such that \( \hat{R}^* \hat{R} = \hat{A}^2 \), and

b. if \( \hat{R}^* \hat{R} = 0 \), then \( \hat{R} = 0 \).

We assume that the algebra has a unit element \( \hat{I} \) and that Hermitian elements of \( \mathfrak{A} \) correspond to observable quantities. We let \( \mathfrak{A}_+ \) denote the set of these elements.

We now formulate the second postulate.

Mutually commuting elements of the set \( \mathfrak{A}_+ \) correspond to compatible (simultaneously measurable) observables.

A specific feature of compatible observables is then that they allow a system of measuring devices whereby these observables can be repeatedly measured in an arbitrary sequence. The results of a repeated measurement of observables are then unchanged. We say that the corresponding measurements are reproducible.

We let \( \Omega_\xi (\Omega \equiv \{\hat{Q}\} \in \mathfrak{A}_+) \) denote the maximal real commutative subalgebra of the algebra \( \mathfrak{A} \). This is the subalgebra of compatible observables. The subscript \( \xi (\xi \in \Xi) \) distinguishes different such subalgebras. If \( \mathfrak{A} \) is commutative (the algebra of classical dynamic variables), then the set \( \Xi \) consists of one element. If \( \mathfrak{A} \) is noncommutative (the algebra of quantum dynamic variables), then the set \( \Xi \) has the power of the continuum.

Hermitian elements of \( \mathfrak{A} \) are a latent form of observable quantities. The explicit form is a certain number, which transpires in an individual observation. We assume that there exists some physical reality that determines the result of such an individual observation. We call this physical reality the physical state of the quantum object.

In what follows, we need the following definition [4]. Let \( \mathfrak{B} \) be a real (complex) commutative algebra and \( \tilde{\varphi} \) be a linear functional on \( \mathfrak{B} \). If

\[
\tilde{\varphi}(\hat{B}_1 \hat{B}_2) = \tilde{\varphi}(\hat{B}_1)\tilde{\varphi}(\hat{B}_2) \text{ for all } \hat{B}_1 \in \mathfrak{B} \text{ and } \hat{B}_2 \in \mathfrak{B},
\]

then the functional \( \tilde{\varphi} \) is called the real (complex) homomorphism on the algebra \( \mathfrak{B} \).

We now formulate the third postulate.

The physical state of a quantum object involved in an individual observation is described by a functional \( \varphi(\hat{A}) \) (in general, multivalued), with \( \hat{A} \in \mathfrak{A}_+ \), whose restriction \( \varphi_\xi(\hat{A}) \) to each subalgebra \( \Omega_\xi \) is single-valued and is a real homomorphism \( (\varphi_\xi(\hat{A}) = \hat{A} \text{ is a real number}) \).

A functional \( \varphi \) is multivalued because the result of an observation can depend not only on the quantum object under observation but also on properties of the device used for the observation. A typical measuring device consists of an analyzer and a detector. The analyzer is a device with one input and several output channels. As an example, we consider the device measuring an observable \( \hat{A} \). For simplicity, we assume that the spectrum of this observable is discrete. Each output channel of the analyzer must then correspond to a certain point of the spectrum. The detector registers the output channel through which the quantum object leaves the analyzer. The corresponding point of the spectrum is taken to be the value of the observable \( \hat{A} \) registered by the measuring device.

In general, the value not of one observable \( \hat{A}_i \) but of an entire set of compatible observables can be registered in one experiment. All these observables must belong to a single subalgebra \( \Omega_\xi \). Each output channel of the analyzer must then correspond to a set of points of the spectra of the observables \( \hat{A}_i \), one point for each independent observable. Obviously, the analyzer must
be constructed appropriately. The analyzer constructed in this way (and the entire measuring device) is said to be compatible with the subalgebra \( \mathfrak{Q}_\xi \).

The set of observables \( \hat{A} \) need not necessarily be registered by one device. A group of measuring devices (one complicated device) can be used for this purpose. In this case, the entire group (each of its elements) must be compatible with the subalgebra \( \mathfrak{Q}_\xi \). We assume that the restriction \( \varphi_\xi(\hat{A}) \) of \( \varphi \) corresponding to a certain physical state describes the value of the observable \( \hat{A} \) that is registered in this physical state by the measuring device compatible with the subalgebra \( \mathfrak{Q}_\xi \).

We assume that the compatibility of a device with a subalgebra is determined by classical characteristics (construction, spatial position, etc.) of the device. If the device is compatible with a subalgebra \( \mathfrak{Q}_\xi \), we say that it belongs to the type \( \mathfrak{Q}_\xi \).

One observable can belong to two (and more) different subalgebras, \( \hat{A} \in \mathfrak{Q}_\xi \cap \mathfrak{Q}_\xi' \). If the functional \( \varphi \) is multivalued at a point \( \hat{A} \), it can occur that \( \varphi_\xi(\hat{A}) \neq \varphi_{\xi'}(\hat{A}) \). Physically, this means that we can register different values of the same observable in the same physical state using different measuring devices. Therefore, the functional \( \varphi \) does not describe the value of the observable \( \hat{A} \) in a certain type to the observable \( \hat{A} \). Accordingly, the physical reality is not the value of the observable \( \hat{A} \) in a given physical state but the response of the measuring device to this state.

If the functional \( \varphi \) is single-valued at a point \( \hat{A} \), we say that the corresponding physical state \( \varphi \) is stable on the observable \( \hat{A} \). In this case, we can say that the observable \( \hat{A} \) has a definite value in the physical state \( \varphi \) and this value is the physical reality.

The functionals involved in the third postulate can be shown to have the following properties [4]:

1. \( \varphi_\xi(0) = 0 \);
2. \( \varphi_\xi(\hat{I}) = 1 \);
3. \( \varphi_\xi(\hat{A}^2) \geq 0 \);
4. if \( \lambda = \varphi_\xi(\hat{A}) \), then \( \lambda \in \sigma(\hat{A}) \);
5. if \( \lambda \in \sigma(\hat{A}) \), then \( \lambda = \varphi_\xi(\hat{A}) \) for some \( \varphi_\xi(\hat{A}) \).

Here, \( \sigma(\hat{A}) \) is the spectrum of the element \( \hat{A} \) in the algebra \( \mathfrak{A} \). The corresponding properties of individual measurements are postulated in the standard quantum mechanics but are a consequence of the third postulate here.

The multivaluedness of a functional \( \varphi \) allows introducing it consistently. This can be verified by direct construction. Evidently, it suffices to construct the restriction \( \varphi_\xi \) of \( \varphi \) to each subalgebra \( \mathfrak{Q}_\xi \).

We describe several ways of constructing the functional \( \varphi \). The first is as follows. In each subalgebra \( \mathfrak{Q}_\xi \), we arbitrarily choose a system \( G(\mathfrak{Q}_\xi) \) of independent generators. We next require \( \varphi_\xi \) to be a certain mapping of \( G(\mathfrak{Q}_\xi) \) to a real number set \( S_\xi \) (allowable points of the spectra for the corresponding elements of the set \( G(\mathfrak{Q}_\xi) \)). On the other elements of \( \mathfrak{Q}_\xi \), the functional \( \varphi_\xi \) is constructed by linearity and multiplicativity.

It is clear that this procedure is always possible if each functional \( \varphi_\xi \) is constructed independently of the others. On the other hand, the functional \( \varphi \) resulting from this construction is highly ambiguous.

We can attempt constructing a single-valued functional \( \varphi \). For this, we choose some subalgebra \( \mathfrak{Q}_1 \) (of type \( \mathfrak{Q} \)) and let \( G(\mathfrak{Q}_1) \) be a set of generators of \( \mathfrak{Q}_1 \). We define the restriction \( \varphi_1 \)
of \( \varphi \) to the subalgebra \( \Omega_1 \) by requiring \( \varphi_1 \) to be some mapping of \( G(\Omega_1) \) to a real number set \( S_1 \) (points of the spectra for the corresponding elements of \( \Omega_1 \)).

We next choose another subalgebra \( \Omega_2 \). With \( \Omega_1 \cap \Omega_2 \equiv \Omega_1_2 \neq \emptyset \), we first construct a set of generators \( G_{12} \) of \( \Omega_{12} \), and then supplement it with the set \( G_{21} \) to the complete set of generators of \( \Omega_2 \). The restriction \( \varphi_2 \) is constructed as follows. If \( \hat{A} \in G_{12} \), then \( \varphi_2(\hat{A}) = \varphi_1(\hat{A}) \). If \( \hat{A} \in G_{21} \), then the functional \( \varphi_2 \) is defined such that it is a mapping of \( G_{21} \) to some allowable set of points in the spectra of the corresponding elements of the algebra \( \Omega_2 \).

The same scheme is used to construct the restrictions of the functional \( \varphi \) to other subalgebras \( \Omega_\xi \). But this scheme can become inconsistent at a certain stage because the subalgebra \( \Omega_\xi \) can have nonempty intersections with other subalgebras \( \Omega_{\xi_1}, \Omega_{\xi_2}, \ldots, \Omega_{\xi_n} \) for which the restrictions of \( \varphi \) are already fixed. The corresponding mappings can be nonallowable for elements of the subalgebra \( \Omega_\xi \).

In this case, we proceed as follows. Let \( k \) be the maximum number \((1 \leq k \leq n)\) such that the definitions \( \varphi_\xi(\hat{A}) = \varphi_{\xi_i}(\hat{A}) \) are allowable for all \( i \) \((1 \leq i \leq k)\). With these definitions, we define the functional \( \varphi_\xi(\hat{A}) \) on the subalgebra \( \Omega_k = [\Omega_{\xi_1} \cup \ldots \cup \Omega_{\xi_k}] \cap \Omega_\xi \). We next take some set \( G^{(k)} \) of generators of the subalgebra \( \Omega_k \) and supplement it with the set \( G_k \) to a complete set of generators of \( \Omega_\xi \). On the generators in \( G_k \), the functional \( \varphi_\xi(\hat{A}) \) is defined such that it realizes the mapping of \( G_k \) to the set of allowable points of the spectra of the corresponding elements of \( \Omega_\xi \). At this stage, the functional \( \varphi \) can become multivalued, and a single-valued functional therefore cannot be constructed. But the arising ambiguity is minimal in a certain sense. It is therefore impossible to avoid ambiguity in the functional \( \varphi \) in the general case. Kochen and Specker have given a specific example of this case.

But it is always possible to construct a functional \( \varphi \) that is single-valued on all observables belonging to any preset subalgebra \( \Omega_\xi \). For this, it suffices to assign the subalgebra \( \Omega_\xi \) number \( 1 \) (set \( \xi = 1 \)) and define the restriction \( \varphi_1 \) of \( \varphi \) to \( \Omega_1 \) as described above. We must next exhaust all subalgebras \( \Omega_i \) (of type \( \Omega_\xi \)) that have nonempty intersections with \( \Omega_1 \). To construct the restriction \( \varphi_i \) of \( \varphi \) to each \( \Omega_i \), it suffices to use the recipe used in the previous version in constructing the restriction \( \varphi_2 \). By construction, such a functional \( \varphi \) is single-valued on all elements belonging to \( \Omega_1 \). Different subalgebras \( \Omega_i \) can have common elements that do not belong to \( \Omega_1 \). On these elements, the functional \( \varphi \) can be multivalued.

We accept the following statement as the fourth postulate.

The equality \( \varphi_\xi(\hat{A}_1) = \varphi_\xi(\hat{A}_2) \) is satisfied for all \( \varphi_\xi \) if and only if \( \hat{A}_1 = \hat{A}_2 \).

In other words, the functional \( \varphi \) separates arbitrary two different observables. The equality \( \hat{A}_1 = \hat{A}_2 \) in particular denotes that both elements \( \hat{A}_1 \) and \( \hat{A}_2 \) simultaneously belong (or do not belong) to a domain of the functional \( \varphi_\xi \).

## 3 The quantum ensemble

The functional \( \varphi \) maps the set \( \Omega_\xi = (Q)_\xi \) into a real number set,

\[
(Q)_\xi \xrightarrow{\varphi} \{Q = \varphi(Q)\}_\xi.
\]

For different functionals \( \varphi_i \) and \( \varphi_j \), the sets \( \{\varphi_i(Q)\}_\xi \) and \( \{\varphi_j(Q)\}_\xi \) can be different or can coincide. If \( \varphi_i(Q) = \varphi_j(Q) = Q \) for all \( Q \in (Q) \), then the functionals \( \varphi_i \) and \( \varphi_j \) are said to be \( \{Q\} \)-equivalent. We let \( \{\varphi\}_{Q} \) be the set of all physical states to which there correspond \( \{Q\} \)-equivalent functionals that are stable on the observables in the subalgebra \( (Q)_\xi \). The set of the corresponding physical states is said to be a (pure) quantum state and is denoted by \( \Psi_{Q} \). The set of physical systems that are in these physical states is said to be the quantum \( \Psi_{Q} \)-ensemble.
Strictly speaking, the above definition of the quantum state is only valid for a physical system that does not contain identical particles. Describing identical particles requires some generalization of the definition of the quantum state [2].

We consider a quantum $\Psi_Q$-ensemble as a parent population (in the probability theory sense) and each experiment to measure an observable $\hat{A}$ as a trial. As the event $\hat{A}$, we consider the experiment where the measured value of the observable $\hat{A}$ is not greater than $\tilde{A}$, i.e., $\varphi(\hat{A}) = \hat{A} \leq \tilde{A}$. This event is not unconditional. By the second postulate, one trial cannot be an event for two noncommuting observables. The probability of the event $\hat{A}$ is determined by the structure of the quantum ensemble and by this condition. Let this probability be equal to $P(\hat{A})$.

We let $\{\varphi\}_{\hat{A}} Q$ (with $\{\varphi\}_{\hat{A}} Q \subset \{\varphi\}_Q$) denote the set of physical states involved in a denumerable sampling from mutually independent random trials of measuring the observable $\hat{A}$. We note that if $\hat{B}$ is an observable not commuting with $\hat{A}$, then the probability that the sets $\{\varphi\}_{\hat{A}} Q$ and $\{\varphi\}_{\hat{B}} Q$ intersect is equal to zero. Indeed, on the one hand, the observables $\hat{A}$ and $\hat{B}$ cannot be measured in one trial. On the other hand, the set $\{\varphi\}_Q$ has the power of the continuum. Therefore, the probability that the same state from $\{\varphi\}_Q$ is repeated in two random denumerable samplings is equal to zero. Therefore, the additional condition is automatically satisfied with the probability one for the described samplings.

By definition, the probability of the occurrence of an event $\hat{A}$ in each of these trials is $P(\hat{A})$. It determines the probabilistic measure $\mu(\varphi)$ ($\varphi(\hat{A}) \leq \tilde{A}$) on any such sampling. The measure $\mu(\varphi)$ in turn determines a distribution of the values $A_i = \varphi_i(\hat{A})$ of the observable $\hat{A}$ and the mathematical expectation $<A>$ in this sampling,

$$<A> = \int_{\{\varphi\}_{\hat{A}} Q} d\mu(\varphi) \varphi(\hat{A}).$$

For $1 \leq i \leq n$, let the functionals $\varphi_i \in \{\varphi\}_{\hat{A}} Q$, then by Khinchin’s theorem (the law of large numbers; see, e.g., [3]), the random quantity $A_n = (A_1 + \ldots + A_n)/n$ converges to $<A>$ in probability as $n \to \infty$. Therefore,

$$\text{P- lim}_{n \to \infty} \frac{1}{n} (\varphi_1(\hat{A}) + \ldots + \varphi_n(\hat{A})) = <A> = \equiv \Psi_Q(\hat{A}). \quad (2)$$

Formula (2) defines a functional (the quantum average) on the set $\mathfrak{A}_+$. The totality of all quantum experiments unambiguously indicates that we must accept the following linearity postulate.

The functional $\Psi_Q(\cdot)$ is linear on the set $\mathfrak{A}_+$.

This implies that

$$\Psi_Q(\hat{A} + \hat{B}) = \Psi_Q(\hat{A}) + \Psi_Q(\hat{B})$$

also in the case where $[\hat{A}, \hat{B}] \neq 0$.

Quantum experiments support one more postulate. To determine the probability $P(\hat{A})$ experimentally, we must conduct random tests. But these tests are often accompanied in practice by some condition because the quantum average of an observable is experimentally found using not a random set of devices capable of measuring this observable but a certain type of such devices. Devices of different types can be used in different series of experiments. Experience shows that the probability $P(\hat{A})$ is the same in all these cases. The representativity postulate is therefore true.

The probability $P(\hat{A})$ of detecting an event $\hat{A}$ for a system in any quantum state $\Psi_Q$ is independent of the type of the measuring device used for that purpose.
Obviously, an ideal measuring device is understood here. Any realistic device introduces a systematic error.

We now discuss how the representativity postulate requirement can be realized in the proposed approach. Let a functional \( \varphi^\mu \) be multivalued on an observable \( \hat{A} \). This is related to the fact that the result of the observation of a certain quantum object can depend not only on the internal properties of this object but also on the type of the device intended for investigating the observable \( \hat{A} \). We label different types of devices in some fixed manner. Let \( \{D_1(\hat{A}), D_2(\hat{A}), \ldots, D_n(\hat{A})\} \) be the set of different types of devices. We label the values of a multivalued functional \( \varphi^\mu \) on the elements \( \hat{A} \) in accordance with the labeling of the measuring devices,

\[
\varphi^\mu(\hat{A}) = \{A_{\mu 1}, \ldots, A_{\mu n}\},
\]

where \( A_{\mu i} \) is the indication of the device \( D_i(\hat{A}) \) in the physical state \( \varphi^\mu \). Each specific device \( D_i(\hat{A}) \) has entirely definite physical characteristics and is therefore compatible with a certain subalgebra \( \Omega_i \) (\( \hat{A} \in \Omega_i \)). In this sense, it "knows" which value of the functional to choose. For a specific physical state, some (or all) values \( A_{\mu i} \) can coincide.

We next consider another functional \( \varphi^\nu \), whose values on the observable \( \hat{A} \) are

\[
\varphi^\nu = \{A_{\nu 1}, \ldots, A_{\nu m}\},
\]

where \( \nu 1, \ldots, \nu n \) is a certain permutation of \( \mu 1, \ldots, \mu n \) and \( A_{\nu i} \) denotes the indication of the device \( D_i(\hat{A}) \) in the physical state \( \varphi^\nu \). Let the functionals \( \varphi^\mu \) and \( \varphi^\nu \) coincide on the other observables.

If a physical state \( \varphi^\mu \) belongs to the quantum state \( \Psi_Q = \{\varphi\}_Q \), then the physical state \( \varphi^\nu \) also belongs to \( \Psi_Q \). Indeed, if \( \hat{A} \in \{\hat{Q}\} \), then the functionals \( \varphi^\mu \) and \( \varphi^\nu \) are single-valued on \( \hat{A} \) and therefore coincide. If \( \hat{A} \notin \{\hat{Q}\} \), then the functionals \( \varphi^\mu \) and \( \varphi^\nu \) coincide on \( \hat{Q} \in \{\hat{Q}\} \) by construction. This argument holds if we consider the functionals obtained by any other permutation of the indices \( \mu 1, \ldots, \mu n \).

We now verify that for the representativity postulate to be satisfied, it suffices to require the relative probabilities of hitting the states \( \varphi^\mu \) and \( \varphi^\nu \) to be the same. Indeed, let the device \( D_i(\hat{A}) \) be used to find the mean values of the observable \( \hat{A} \) in a quantum state \( \Psi_Q \) experimentally and let \( \Psi_Q(\hat{A}) \) denote the result of this experiment. Theoretically, the quantity \( \Psi_Q(\hat{A}) \) is constructed as follows. We take some physical state \( \varphi^\mu \). This state contributes \( A_{\mu i} \) to \( \Psi_Q(\hat{A}) \) with the weight \( w_{\mu i} \) (the probability of hitting the state \( \varphi^\mu \)). We next take the state \( \varphi^\nu \). It contributes \( A_{\nu j} \) with the weight \( w_{\nu j} = w_{\mu i} \). In the same way, we must take all the states obtained by other permutations of the arguments in \( \Psi_Q(\hat{A}) \) into account. Following the same scheme, we must then take all the physical states that are not related to \( \varphi^\mu \) by a permutation of the arguments in \( \Psi_Q(\hat{A}) \) into account.

Now let the device \( D_j(\hat{A}) \) be used to find the mean value. The result of this experiment is denoted by \( \Psi_Q(\hat{A}) \). To calculate \( \Psi_Q(\hat{A}) \) theoretically, we use the above scheme. We again start with the states \( \varphi^\mu, \varphi^\nu \). . . . We then obtain a set of values \( A_{\mu i}, A_{\nu j} \) . . . that contribute to \( \Psi_Q(\hat{A}) \) with the weight \( w_{\mu i} \). But the sets \( (A_{\mu i}, A_{\nu i}) \) and \( (A_{\mu j}, A_{\nu j}) \) differ by only a permutation of elements. Therefore, the total contributions of the physical states \( \varphi^\mu, \varphi^\nu \) to \( \Psi_Q(\hat{A}) \) and \( \Psi_Q(\hat{A}) \) are the same. Similarly, we consider the contributions of the physical states that are not related to \( \varphi^\mu \) a permutation of arguments. It follows that \( \Psi_Q(\hat{A}) = \Psi_Q(\hat{A}) \).

For the representativity postulate to be satisfied, it now suffices to verify that for each physical state \( \varphi^\mu \), there exists the corresponding state of the type \( \varphi^\nu \). Obviously, this is indeed so if one additional point is introduced into the constructive scheme of building the set of physical
states described above. Along with each multivalued functional \( \varphi^\mu \) constructed in accordance with the proposed scheme, all the functionals whose values are obtained by all possible permutations of the arguments in Eq. (3) must be introduced into the set of functionals.

Therefore, although the devices \( D_i(\hat{A}) \) and \( D_j(\hat{A}) \) can give different results for the same physical state in individual observations, they give the same result for mean values of the observable \( \hat{A} \). The proposed model thus gives the same results for determining quantum mean values of observables as the standard quantum mechanics.

Any element \( \hat{R} \) of the algebra \( \mathfrak{A} \) is uniquely represented as \( \hat{R} = \hat{A} + i\hat{B} \), where \( \hat{A}, \hat{B} \in \mathfrak{A}_+ \). Therefore, the functional \( \Psi_Q \) can be uniquely extended to a linear functional on \( \mathfrak{A} \): \( \Psi_Q(\hat{R}) = \Psi_Q(\hat{A}) + i\Psi_Q(\hat{B}) \).

Let \( \hat{R}^* \hat{R} = \hat{A}^2 \in \{ \hat{Q} \} \) (\( \hat{A} \in \mathfrak{A}_+ \)) if \( \varphi \in \{ \varphi \}_Q \), then \( \varphi(\hat{R}^* \hat{R}) = A^2 \). Therefore,

\[
\Psi_Q(\hat{R}^* \hat{R}) = A^2 = \varphi(\hat{A}^2)_{\varphi \in \varphi}_Q.
\]

In accordance with inequality (1 /3/), we have \( A^2 \geq 0 \). In addition, it follows from condition (1 /1/) and the fourth postulate that \( \sup \varphi(\hat{R}^* \hat{R}) > 0 \), if \( \hat{R} \neq 0 \).

We define the norm of an element \( \hat{R} \) by

\[
\| \hat{R} \|^2 = \sup \varphi(\hat{R}^* \hat{R}) = \sup_Q \Psi_Q(\hat{R}^* \hat{R}).
\] (5)

Because \( \Psi_Q \) is a positive linear functional, all the axioms of a norm are indeed satisfied for \( \| \hat{R} \| \) (see [3, 4]). Because \( \varphi([\hat{A}^2]^2) = [\varphi(\hat{A}^2)]^2 \), we have \( \| \hat{R}^* \hat{R} \| = \| \hat{R} \|^2 \), and \( \mathfrak{A} \) is therefore a \( C^* \)-algebra. Thus, a necessary condition for the consistency of the linearity postulate is the following requirement: the algebra \( \mathfrak{A} \) can be endowed with the structure of a \( C^* \)-algebra.

This requirement can be formulated in purely algebraic terms because the relation [3]

\[
\rho(\hat{R}) = \| \hat{R} \| = \rho^{1/2}(\hat{R}^* \hat{R}).
\] (6)

is valid for a \( C^* \)-algebra. Here, \( \rho(\hat{R}) \) is the spectral radius of the element \( \hat{R} \), \( \rho(\hat{R}) = \sup_{\lambda \in \sigma(\hat{R})} |\lambda_R| \), where \( \lambda_R \in \sigma(\hat{R}) \).

Using Eq. (1 /5/), we can rewrite relation (6) as

\[
\| \hat{R} \| = [\sup_{\varphi} \varphi(\hat{R}^* \hat{R})]^{1/2}.
\]

which agrees with Eq. (3). Therefore, the spectral radius (a purely algebraic notion) of each element of the algebra \( \mathfrak{A} \) must satisfy the norm axioms and the condition \( \rho^2(\hat{R}) = \rho(\hat{R}^* \hat{R}) \).

In view of the above, it is useful to give a new formulation of the first postulate.

**Elements of the algebra \( \mathfrak{A} \) that has the structure of a \( C^* \)-algebra correspond to dynamic quantities.**

We did not accept this formulation of the first postulate initially because it follows directly from the experiment that the observables have algebraic properties and the quantum mean values have the linearity property. But the mathematical relations involved in the definition of a \( C^* \)-algebra are not directly related to the experiment.

Postulating that the observables belong to a normed algebra, we must consider the observables to be bounded. In any experiment, we always deal with bounded values of observables. Normalizability of the algebra is therefore not a restriction from the experimental standpoint. But many unbounded operators, which are obviously not elements of a normed algebra, occur in the quantum theory. In the algebraic approach, "unbounded observables" are conventionally
considered as elements adjoint to the algebra of bounded dynamic quantities; in other words, unbounded observables are assumed to admit a spectral representation where the spectral projectors are elements of a normed algebra. The ensuing problems are common to the algebraic approach to quantum theory in general. We do not consider them here.

4 Time evolution and the ergodicity condition

In the standard quantum mechanics, the time evolution is determined by the Heisenberg equation

\[
\frac{d\hat{A}(t)}{dt} = i\left[\hat{H}, \hat{A}(t)\right], \quad \hat{A}(0) = \hat{A}, \tag{7}
\]

where \(\hat{A}(t)\) and the Hamiltonian \(\hat{H}\) are operators in some Hilbert space. But for (7) to preserve its physical meaning, it suffices to consider \(\hat{A}(t)\) and \(\hat{H}\) as elements of some algebra (in particular, of \(\mathfrak{A}\)) or elements adjoint to the algebra.

In our case, the evolution equation can be rewritten in terms of physical states. We therefore accept the fifth postulate.

A physical state of a quantum system evolves in time as

\[
\varphi(\hat{A}) \rightarrow \varphi_e(\hat{A}) \equiv \varphi(\hat{A}(t)), \tag{8}
\]

where \(\hat{A}(t)\) is defined by Eq. (7).

Equation (8) describes time evolution of a physical state entirely unambiguously. It is a different story, though, that an observation allows determining the initial value \(\varphi(\hat{A})\) of a functional only up to its belonging to a certain quantum state \(\{\varphi\}_Q\). Most of our predictions regarding the time evolution of a quantum object are therefore probabilistic. In addition, Eqs. (7) and (8) are valid only for systems that are not exposed to first-class actions (in von Neumann’s terminology [8]), i.e., do not interact with a classical measuring device.

We now return to the linearity postulate. From the experimental standpoint, this postulate is well justified. But it is not quite clear whether it can be realized within the mathematical scheme considered here. It turns out that this postulate can be related to the time evolution of the quantum system. For this, we must impose restrictions on the Hamiltonian \(\hat{H}\).

We now accept the sixth postulate.

The Hamiltonian \(\hat{H}\) is a Hermitian spectral (possibly, adjoint) element of the algebra \(\mathfrak{A}\). The spectrum of \(\hat{H}\) contains at least one discrete nondegenerate value \(E_0\).

This implies that the Hamiltonian \(\hat{H}\) has an integral representation of the form

\[
\hat{H} = \int \hat{p}(dE) \ E, \tag{9}
\]

where \(\hat{p}(dE)\) are orthogonal projectors. Hereinafter integrations (and also limits) on algebra \(\mathfrak{A}\) are understood in sense of the weak topology of \(C^*-\)algebra.

Somewhat conventionally, we can represent \(\hat{p}(dE)\) as

\[
\hat{p}(dE) = \hat{p}_p(dE) + \hat{p}_c(dE) = \sum_n \hat{p}_n \delta(E - E_n) \ dE + \hat{p}_c(dE). \tag{10}
\]

Here \(\hat{p}_p(dE)\) and \(\hat{p}_c(dE)\) concern to point and continuous spectrums, correspondingly. Besides, \(\hat{p}_n \hat{p}_m = \hat{p}_n \hat{p}_m = 0\) for \(m \neq n\), \(\hat{p}_n \hat{p}_c(dE) = \hat{p}_c(dE) \hat{p}_n = 0\). The sum over \(n\) in (11) must necessarily involve at least one term \((n = 0)\) with a nondegenerate value \(E_0\).
In addition to this last restriction, other requirements are always assumed in considering any quantum mechanics model. Requiring a discrete point in the spectrum does not seem too restrictive either. For example, a one-particle quantum system can have a purely continuous energy spectrum. But it can be considered as a one-particle state of an extended system that can also be in the vacuum state in addition to the one-particle state. The energy spectrum of the extended system already has a discrete nondegenerate point in the spectrum.

The quantity $E_0$ need not necessarily be the lower bound of the spectrum. By the nondegeneracy of $E_0$, we assume that the projector $\hat{p}_0$ in decomposition (11) is one-dimensional. A projector $\hat{p}$ is said to be one-dimensional if it cannot be represented as

$$\hat{p} = \sum_\alpha \hat{p}_\alpha, \quad \hat{p}_\alpha \neq \hat{p}, \quad \hat{p}\hat{p}_\alpha = \hat{p}_\alpha\hat{p} = \hat{p}_\alpha.$$

Let us remark that, if two elements $\hat{A}_1$ and $A_2$ of algebra $\mathfrak{A}$ have identical spectral representation of type (10), then they obey the fourth postulate. Therefore, such elements coincide.

**Statement 1.**

*If $\hat{A} \in \mathfrak{A}_+$, then $\hat{A}_0 = \hat{p}_0\hat{A}\hat{p}_0$ has the form $\hat{A}_0 = \hat{p}_0\Psi_0(\hat{A})$, where $\Psi_0(\hat{A})$ is some functional.*

**Proof.** Because $[\hat{A}_0, \hat{p}_0] = 0$, it follows that $\hat{A}_0$ and $\hat{p}_0$ have the common spectral decomposition of unity. Because the projector $\hat{p}_0$ is one-dimensional, the spectral decomposition of $\hat{A}_0$ must have the form $\hat{A}_0 = \hat{p}_0\Psi_0(\hat{A}) + \hat{A}_0'$, where $\hat{A}_0'$ is orthogonal to $\hat{p}_0$. Therefore, $\hat{A}_0 = \hat{p}_0\hat{A}_0 = \hat{p}_0\hat{p}_0\Psi_0(\hat{A}) + \hat{p}_0\hat{A}_0' = \hat{p}_0\Psi_0(\hat{A})$. The statement is proved.

**Statement 2.**

*The functional $\Psi_0(\hat{A})$ is linear.*

**Proof.** Indeed,

$$\hat{p}_0\Psi_0(\hat{A} + \hat{B}) = \hat{p}_0(\hat{A} + \hat{B})\hat{p}_0 = \hat{p}_0\Psi_0(\hat{A}) + \hat{p}_0\Psi_0(\hat{B}).$$

Because $\hat{p}_0 \neq 0$, it follows that $\Psi_0(\hat{A} + \hat{B}) = \Psi_0(\hat{A}) + \Psi_0(\hat{B})$, which was to be proved.

A physical state $\varphi_{0\alpha}$ is said to be ground if $\varphi_{0\alpha}(\hat{p}_0) = 1$. By linearity, the functional $\Psi_0(\hat{A})$ is uniquely extended to the algebra $\mathfrak{A}$, $\Psi_0(\hat{A} + i\hat{B}) = \Psi_0(\hat{A}) + i\Psi_0(\hat{B})$, where $\hat{A}, \hat{B} \in \mathfrak{A}_+$.

**Statement 3.** *The functional $\Psi_0(\hat{A})$ is positive.*

**Proof.** In accordance with property (1/3/), there is the inequality $\varphi_{0\alpha}(\hat{p}_0\hat{R}^*\hat{R}\hat{p}_0) \geq 0$. On the other hand,

$$\varphi_{0\alpha}(\hat{p}_0\hat{R}^*\hat{R}\hat{p}_0) = \varphi_{0\alpha}(\hat{p}_0\Psi_0(\hat{R})^*) = \Psi_0(\hat{R}^*\hat{R}).$$

The statement is proved.

**Statement 4.** *The functional $\Psi_0$ satisfies the normalization condition $\Psi_0(\hat{I}) = 1$.*

**Proof.** Indeed,

$$1 = \varphi_{0\alpha}(\hat{p}_0\hat{I}\hat{p}_0) = \varphi_{0\alpha}(\hat{p}_0\Psi_0(\hat{I})) = \Psi_0(\hat{I}),$$

which was to be proved.

To find the physical meaning of the functional $\Psi_0$, we consider an element $\tilde{A}$ in the algebra $\mathfrak{A}$ that corresponds to an observable $\tilde{A}$ averaged in time,

$$\tilde{A} = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} dt \hat{A}(t) = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} dt \exp[-i\hat{H}t] \hat{A} \exp[i\hat{H}t].$$

The average is understood with respect to the weak topology of $C^*$-algebra. Substituting the spectral decomposition of $\hat{H}$ in (11), we obtain

$$\tilde{A} = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} dt \left[ \sum_n \hat{p}_n \hat{A}_n + \int \hat{p}_c(dE) \hat{A}_c(dE') \exp[i(E' - E)] + \hat{B}(t) \right],$$

9
where

\[ \hat{B}(t) = \sum n \left[ \hat{p}_n \hat{A} \int \hat{p}_c(dE) \exp[i(E_n - E)t] + \hat{p}_c(dE) \hat{A} \hat{p}_n \exp[-i(E_n - E)t] \right] + \sum_{n,m} \hat{p}_n \hat{A} \hat{p}_m \exp[i(E_n - E_m)t] \]

It can be easily shown that the occurrence of time exponentials in this expression implies the relation

\[
\lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} dt \hat{B}(t) = 0,
\]

and therefore

\[ \bar{A} = \sum n \hat{p}_n \hat{A} \hat{p}_n + \hat{D}, \]

where

\[ \hat{D} = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} dt \int \hat{p}_c(dE) \hat{A} \hat{p}_c(dE') \exp[i(E' - E)]. \]

In a right-hand side of this equality the integrand \( \hat{A} \exp[i(E' - E)] \) is majorized by magnitude \( ||\hat{A}|| \). Therefore, the integrals and the limit exist. Because \( \hat{p}_c(dE) = \hat{p}_c\hat{p}_c(dE) = \hat{p}_c(dE)\hat{p}_c \),

we have \( \hat{D} = \hat{p}_c\hat{D}\hat{p}_c \). Therefore,

\[ \bar{A} = \sum n \hat{p}_n \hat{A} \hat{p}_n + \hat{p}_c\hat{D}\hat{p}_c. \]

We now consider the value of the observable \( \bar{A} \) in the physical ground state \( \varphi_{0\alpha} \),

\[
\varphi_{0\alpha} \left( \sum n \hat{p}_n \hat{A} \hat{p}_n + \hat{p}_c\hat{D}\hat{p}_c \right) = \varphi_{0\alpha}(\hat{p}_0 \hat{A} \hat{p}_0) + \varphi_{0\alpha}(\hat{F}), \tag{12} \]

where \( \hat{F} = \sum_{n \neq 0} \hat{p}_n \hat{A} \hat{p}_n + \hat{p}_c\hat{D}\hat{p}_c \). We here use the linearity of the functional \( \varphi_{0\alpha} \) on mutually commuting elements \( \hat{p}_n \hat{A} \hat{p}_n, \hat{p}_m \hat{A} \hat{p}_m \) and \( \hat{p}_c\hat{D}\hat{p}_c \). Because \( \hat{p}_n\hat{p}_0 = \hat{p}_c\hat{p}_0 = 0 \) for \( n \neq 0 \), the right-hand side of (12) can be rewritten as

\[
\Psi_0(\hat{A}) + \varphi_{0\alpha} \left( (\hat{I} - \hat{p}_0)\hat{F}(\hat{I} - \hat{p}_0) \right) = \Psi_0(\hat{A}) + \varphi_{0\alpha}(\hat{I} - \hat{p}_0)\varphi_{0\alpha} \left( (\hat{I} - \hat{p}_0)\hat{F}(\hat{I} - \hat{p}_0) \right) = \Psi_0(\hat{A}).
\]

We finally have

\[ \varphi_{0\alpha}(\bar{A}) = \Psi_0(\bar{A}). \]

The value of the observable \( \bar{A} \) is the same in all physical ground states.

The functional \( \Psi_0 \) has all the properties that must be possessed by a functional determining quantum mean values. It is linear, is positive, and is equal to unity on the unit element. In addition, it is continuous as a linear functional on the \( C^* \)-algebra. Instead of the linearity axiom, we can therefore accept the seventh postulate (the ergodicity axiom).

The mean value of an observable \( \bar{A} \) in the ground state of a quantum ensemble is equal to the value of observable \( \bar{A} \) (time-averaged value of the observable \( \bar{A} \)) in any physical ground state.

To construct the standard mathematical formalism of quantum mechanics, we can now use the canonical construction of Gelfand-Naimark-Segal (GNS)(see, e.g., [3]).

We consider two elements \( \hat{R}, \hat{S} \in \mathfrak{A} \) equivalent if the condition \( \Psi_0 \left( \hat{K}^*(\hat{R} - \hat{S}) \right) = 0 \) is valid for any \( \hat{K} \in \mathfrak{A} \). We let \( \Phi(\hat{R}) \) denote the equivalence class of the element \( \hat{R} \) and consider the set
\( \mathfrak{A}(\Psi_0) \) of all equivalence classes in \( \mathfrak{A} \). We make \( \mathfrak{A}(\Psi_0) \) a linear space setting \( a\Phi(\hat{R}) + b\Phi(\hat{S}) = \Phi(a\hat{R} + b\hat{S}) \). The scalar product in \( \mathfrak{A}(\Psi_0) \) is defined as \( \left( \Phi(\hat{R}), \Phi(\hat{S}) \right) = \Psi_0(\hat{R}^*\hat{S}) \). This scalar product generates the norm \( \| \Phi(\hat{R}) \|^2 = \Psi_0(\hat{R}^*\hat{R}) \) in \( \mathfrak{A}(\Psi_0) \). Completion with respect to this norm makes \( \mathfrak{A}(\Psi_0) \) a Hilbert space. Each element \( \hat{S} \) of the algebra \( \mathfrak{A} \) is uniquely assigned a linear operator \( \Pi(\hat{S}) \) acting in this space as \( \Pi(\hat{S})\Phi(\hat{R}) = \Phi(\hat{S}\hat{R}) \).

We note that using the ground state \( \Psi_0 \) is not a necessary condition in the GNS construction. All our argument can be based not on the projection operator \( \hat{p}_0 \) but on any other one-dimensional projection operator \( \hat{p} \). In this case, there is also the equality

\[
\hat{p}\hat{A}\hat{p} = \hat{p}\Psi_p(\hat{A}),
\]

where \( \Psi_p(\hat{A}) \) is a functional with the linearity and positivity properties; in addition, \( \Psi_p(\hat{I}) = 1 \).

Instead of the physical ground states \( \varphi_{0a} \), we can use the basic physical states \( \varphi_{pa} \). These states satisfy the condition \( \varphi_{pa}(\hat{p}) = 1 \). The set \( \{ \varphi_{pa} \} \) of physical states constitutes the base quantum state \( \{ \varphi \} \). Instead of the ergodicity axiom, we can postulate that the quantum average in the state \( \{ \varphi \} \) is defined by the functional \( \Psi_p \) in Eq. (14). In this case, we lose the relation of the quantum average to the time average. As a compensation, the theory becomes more flexible because the assumptions regarding the spectrum of the Hamiltonian become redundant.

5 An example

To illustrate the above, we consider a quantum system whose observable quantities are described by Hermitian \( 2 \times 2 \) matrices. The elements \( \hat{H}, \hat{p}_0, \) and \( \hat{A} \) are given by

\[
\hat{H} = \begin{bmatrix} E_0 & 0 \\ 0 & -E_0 \end{bmatrix}, \quad \hat{p}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.
\]

Obviously, \( \hat{p}_0\Psi_0(\hat{A}) = \hat{p}_0\hat{A}\hat{p}_0 = \hat{p}_0 d \), i.e.,

\[
\Psi_0(\hat{A}) = d. \quad (14)
\]

In addition,

\[
\hat{A} = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} dt e^{-iE_0t\tau_3} \hat{A} e^{iE_0t\tau_3} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \quad (15)
\]

where \( \tau_i \) are Pauli matrices.

All physical states can easily be constructed. We consider a Hermitian matrix \( \hat{A} \), i.e., with \( a^* = a, d^* = d, \) and \( c = b^* \). Any such matrix can be represented as

\[
\hat{A} = r_0\hat{I} + r \hat{\tau}(\hat{n}), \quad (16)
\]

where \( \hat{n} \) is the unit three-dimensional vector, \( \hat{\tau}(\hat{n}) = (\hat{\tau} \hat{n}) \). For Eq. (16) to be valid, we must

\[
r = \left( \frac{(a-d)^2}{4} + bb^* \right)^{1/2}, \quad r_0 = \frac{a+d}{2},
\]

\[
n_1 = \frac{b+b^*}{2r}, \quad n_2 = \frac{b-b^*}{2ir}, \quad n_3 = \frac{a-d}{2r}.
\]

The commutator of the matrices \( \hat{\tau}(\hat{n}), \hat{\tau}(\hat{n}') \) is nonvanishing for \( \hat{n}' \neq \pm \hat{n} \). Therefore, each matrix \( \hat{\tau}(\hat{n}) \) (up to a sign) is a generator of a real maximal commutative subalgebra. Because \( \hat{\tau}(\hat{n})\hat{\tau}(\hat{n}) = \hat{I} \), the spectrum of \( \hat{\tau}(\hat{n}) \) consists of two points \( \pm 1 \).
Let \( \{ f(\vec{n}) \} \) be the set of all functions taking the values \( \pm 1 \) and such that \( f(-\vec{n}) = -f(\vec{n}) \).

A physical state is described by a functional whose value coincides with one of the points in the spectrum of the corresponding algebra element. For each point of the spectrum, there exists an appropriate functional. Therefore, to the set of physical states, there corresponds a set of functionals defined by

\[
\varphi(\hat{\tau}(\vec{n})) = f(\vec{n}).
\]

Taking properties (3) into account (which must be possessed by each physical state), we obtain

\[
\varphi(\hat{A}) = r_0 + r f(\vec{n}). \tag{17}
\]

The ground state is any functional \( \varphi_0 \) such that

\[
f(n_1 = 0, n_2 = 0, n_3 = 1) = -1.
\]

Substituting the element \( \vec{A} \) in (17), we obtain

\[
\varphi_0(\vec{A}) = \frac{a + d}{2} + \frac{a - d}{2} = d.
\]

This agrees with (14).

In this model, we can do without multivalued functionals. If we considered the algebra of matrices describing unit spin, multivalued functionals would inevitably arise. This was noted (in other terms) by Kochen and Specker [5].

6 Conclusions

In principle, a physical state can be considered as a special hidden parameter. But because of the multivaluedness of the functional \( \varphi \), the conditions of the Kochen and Specker no-go theorem [5] are not satisfied for this hidden parameter. It was noted in [3] that the conditions of the von Neumann no-go theorem [8] are not satisfied for \( \varphi \). In addition, it was also shown there that the conditions of Bell’s theorem [9] are not satisfied for the functional \( \varphi \). Therefore, the arguments that are usually adduced by opponents of the use of hidden parameters in quantum mechanics become inapplicable for the physical state.

Regarding the multivaluedness of the functional \( \varphi \), the physical process that is usually called a measurement should rather be called an observation. The term “observation” better expresses the fact that the indication of the device in use has two causes: the physical state of the observed object and the type of the device. The type of the device is then defined not only by the observable that it must register but also by an ignored parameter of the device. In the proposed approach, the type of the measuring device plays the role of such an ignored parameter. Different types of devices correspond to different maximal commutative subalgebras to which a given observable belongs. Unlike the values of a hidden parameter, the value of the ignored parameter can in principle be established experimentally.

The result of an observation experiment (the physical reality) is thus determined by two other physical realities: the physical state of the observed object and the type of the observing device. In an individual quantum experiment, in contrast to a classical one, the second physical reality cannot be neglected in general. But even in the quantum case, it is possible not to take the type of the measuring device into account in an experiment determining the mean values of an observable.

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