AN EXTENDED FREUDENTHAL MAGIC SQUARE IN CHARACTERISTIC 3

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Abstract. Freudenthal’s Magic Square, which in characteristic 0 contains the exceptional Lie algebras other than $G_2$, is extended over fields of characteristic 3, through the use of symmetric composition superalgebras, to a larger square that contains both Lie algebras and superalgebras. With one exception, the simple Lie superalgebras that appear have no counterpart in characteristic 0.

1. Introduction

In 1966 [Tit66], Tits gave a unified construction of the exceptional simple Lie algebras based on two ingredients: a unital composition algebra (or Hurwitz algebra) and a central simple degree 3 Jordan algebra. This construction is valid over arbitrary fields of characteristic $\neq 2, 3$. The outcome of this construction is given by Freudenthal’s Magic Square [Sch66, Fre64] (Table 1).

| dim A | 6 | 9 | 15 | 27 |
|-------|---|---|----|----|
| 1     | $A_1$ | $A_2$ | $C_3$ | $F_4$ |
| 2     | $A_2$ | $A_2 \oplus A_2$ | $A_5$ | $E_6$ |
| 4     | $C_3$ | $A_5$ | $D_6$ | $E_7$ |
| 8     | $F_4$ | $E_6$ | $E_7$ | $E_8$ |

Table 1. Freudenthal’s Magic Square

At least in the split cases, this is a construction that depends on two Hurwitz algebras, since the central simple degree 3 Jordan algebras turn out to be the algebras of hermitian $3 \times 3$ matrices over a Hurwitz algebra. Even though the construction is not symmetric on the two Hurwitz algebras involved, the result (the Magic Square) is symmetric.

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Over the years, several symmetric constructions of Freudenthal’s Magic Square based on two Hurwitz algebras have been proposed. Vinberg, in 1966 [Vin05] gave a construction based on two Hurwitz algebras and their derivation algebras. In 1993, Allison and Faulkner [AF93] gave a very general construction based on structurable algebras. In particular, the tensor product of two Hurwitz algebras is structurable and, in this case, the construction is equivalent to the one given by Vinberg. More recently, Barton and Sudbery [BS, BS03] and Landsberg and Manivel [LM02, LM04] provided a different construction based on two Hurwitz algebras and their Lie algebras of triality. (See [Bae02] for a very nice exposition.)

The triality phenomenon was described by E. Cartan [Car38], relating the natural and the two half spin representations of the orthogonal Lie algebra in dimension 8. As shown in [KMRT98], simpler formulas can be obtained if the so called symmetric composition algebras are used, instead of the classical Hurwitz algebras. This led the second author to reinterpret the construction by Barton and Sudbery in terms of a construction of a Lie algebra $\mathfrak{g}(S, S')$ out of two symmetric composition algebras $S$ and $S'$ [Eld04]. With a few exceptions in dimension 2, any symmetric composition algebra of dimension 1, 2 or 4 is a para-Hurwitz algebra, while in dimension 8, besides the para-Hurwitz algebras, there appears a new family of symmetric composition algebras, the Okubo algebras. The construction in [Eld04] using para-Hurwitz algebras reduces naturally to the construction by Barton and Sudbery (although with slightly simpler formulas). Okubo algebras provide new constructions that highlight different order 3 automorphisms and different distinguished subalgebras of the exceptional simple Lie algebras. Further descriptions of the Lie algebras that appear in Freudenthal’s Magic Square in terms of copies of the three-dimensional simple Lie algebra $\mathfrak{sl}_2$ and its two dimensional natural module are derived in [Eld].

An interesting advantage of the constructions of the Magic Square in terms of two composition algebras and their triality Lie algebras is that this construction is also valid over fields of characteristic 3. Only some attention has to be paid to the second row (or column). Thus [Eld04], for instance, instead of a simple Lie algebra of type $E_6$, a non simple Lie algebra of dimension 78 is obtained if a two dimensional and an eight dimensional composition algebras are used as ingredients, but it contains a simple codimension 1 ideal of type $E_6$ (the simple Lie algebra of type $E_6$ has dimension 77 in characteristic 3).

The characteristic 3 presents also another exceptional feature. Only over fields of this characteristic there are composition superalgebras which are nontrivial (that is, they are not composition algebras). In [EO02] both Hurwitz superalgebras and symmetric composition superalgebras are classified. The main results assert that the only nontrivial superalgebras appear in characteristic 3 and dimensions 3 and 6, in agreement with the classification of the simple alternative superalgebras by Shestakov [She97].

The symmetric composition superalgebras can thus be plugged into the construction $\mathfrak{g}(S, S')$ in [Eld04]. Then an extension in characteristic 3 is obtained of Freudenthal’s Magic Square, in which Lie superalgebras appear (see Table 2).
### Table 2. Freudenthal’s Magic Supersquare (characteristic 3)

| dim $S'$ | dim $S$ |
|----------|---------|
| 1        | $A_1$  |
| 2        | $\tilde{A}_2$ $\oplus \tilde{A}_2$ $\tilde{A}_5$ $\tilde{E}_6$ |
| 4        | $D_6$  |
| 6        | $E_8$  |
| 3        | (6, 8) (21, 14) |
| 4        | (11, 14) (35, 20) |
| 5        | (24, 26) (66, 32) |
| 6        | (55, 50) (133, 56) |
| 7        | (21, 16) (36, 40) |
| 8        | (78, 64) |

In Table 2 only the entries above the diagonal have been displayed, as the square is symmetric. The notation $\tilde{X}$ indicates that in characteristic 3 the algebra obtained is not simple, but contains a simple codimension 1 ideal of type $X$. For instance, in the split case, $\tilde{A}_5$ denotes the projective general Lie algebra $\mathfrak{pgl}_6$. Besides, only the pairs $(\dim q_0, \dim q_1)$ are displayed for the nontrivial Lie superalgebras that appear in the “supersquare”.

The aim of this paper is the description of all the Lie superalgebras that appear in Table 2. With just a single exception, which corresponds to the pair (6, 8), these Lie superalgebras have no counterpart in characteristic 0, and they are either simple or contain a simple codimension 1 ideal (this happens again only in the second row). They will be described as contragredient Lie superalgebras, and to do so, a previous description in terms similar to those used in [Eld06b] will be provided too.

In a forthcoming paper, most of these Lie superalgebras will be shown to be isomorphic to the Lie superalgebras constructed in terms of orthogonal and symplectic triple systems in [Eld06b].

The paper is organized as follows. Section 2 will be devoted to review the symmetric composition superalgebras and their Lie superalgebras of triality. Then Section 3 will deal with the construction of a Lie superalgebra in terms of two symmetric composition superalgebras, thus extending the construction in [Eld04]. Section 4 will give the definitions and characterizations of contragredient Lie superalgebras in a way suitable for our purposes. Finally, the last Section 5 will be devoted to the description of all the split Lie superalgebras in Freudenthal’s Magic Supersquare as contragredient Lie superalgebras. Their even and odd parts will be computed too.

### 2. Symmetric composition superalgebras

Recall [EO02] that a quadratic superform on a $\mathbb{Z}_2$-graded vector space $U = U_0 \oplus U_1$ over a field $k$ is a pair $q = (q_0, b)$ where

(i) $q_0 : U_0 \rightarrow k$ is a quadratic form.
(ii) \( b : U \times U \to k \) is a supersymmetric even bilinear form. That is, \( b|_{U_0 \times U_0} \) is symmetric, \( b|_{U_1 \times U_1} \) is alternating \((b(x_1, x_1) = 0 \text{ for any } x_1 \in U_1) \text{ and } b(U_0, U_1) = 0 = b(U_1, U_0)\).

(iii) \( b|_{U_0 \times U_0} \) is the polar of \( q_0 \). That is, \( b(x_0, y_0) = q_0(x_0 + y_0) - q_0(x_0) - q_0(y_0) \) for any \( x_0, y_0 \in U_0 \).

The quadratic superform \( q = (q_0, b) \) is said to be regular if \( q_0 \) is regular (definition as in [KMRT98, p. xix]) and the restriction of \( b \) to \( U_1 \) is nondegenerate.

Then a superalgebra \( C = C_0 \oplus C_1 \) over \( k \), endowed with a regular quadratic superform \( q = (q_0, b) \), called the norm, is said to be a composition superalgebra in case

\[
\begin{align*}
q_0(x_0, y_0) &= q_0(x_0)q_0(y_0), \quad (2.1a) \\
b(x_0y, x_0z) &= q_0(x_0)b(y, z) = b(y_0x_0, zx_0), \quad (2.1b) \\
b(xy, zt) &= (-1)^{|x||y|+|x||z|+|y||z|}b(zt, xy) = (-1)^{|y||z|}b(x, z)b(y, t), \quad (2.1c)
\end{align*}
\]

for any \( x_0, y_0 \in C_0 \) and homogeneous elements \( x, y, z, t \in C \) (where \(|x|\) denotes the parity of the homogeneous element \( x \)).

The unital composition superalgebras are termed Hurwitz superalgebras.

A composition superalgebra \( S = S_0 \oplus S_1 \) with norm \( q = (q_0, b) \) is said to be symmetric if

\[ b(xy, z) = b(x, yz), \]

for any \( x, y, z \in S \). That is, the associated supersymmetric bilinear form is associative.

**Example 2.2.** Let \( V \) be a two dimensional vector space over the field \( k \) with a nonzero alternating bilinear form \((|.|)\). Consider the superspace

\[ B(1, 2) = k1 \oplus V, \]

where \( B(1, 2)_0 = k1 \) and \( B(1, 2)_1 = V \), with supercommutative multiplication given by

\[ 1x = x1 = x \quad \text{and} \quad uv = \langle u|v \rangle 1 \]

for any \( x \in B(1, 2) \) and \( u, v \in V \), and with quadratic superform \( q = (q_0, b) \) given by:

\[ q_0(1) = 1, \quad b(u, v) = \langle u|v \rangle, \]

for any \( u, v \in V \).

If the characteristic of \( k \) is 3, then \( B(1, 2) \) is a Hurwitz superalgebra [EO02 Proposition 2.7].

**Example 2.3.** Let \( V \) be as in Example 2.2, then \( \text{End}_k(V) \) is equipped with the symplectic involution \( f \mapsto f \), such that

\[ \langle f(u)|v \rangle = \langle u|f(v) \rangle, \]

for any \( u, v \in V \). Consider the superspace

\[ B(4, 2) = \text{End}_k(V) \oplus V, \]

where \( B(4, 2)_0 = \text{End}_k(V) \) and \( B(4, 2)_1 = V \), with multiplication given by:

- the usual multiplication (composition of maps) in \( \text{End}_k(V) \),
\[ v \cdot f = f(v) = \bar{f} \cdot \bar{v} \text{ for any } f \in \text{End}_k(V) \text{ and } v \in V, \]
\[ u \cdot v = \langle |u|v \rangle (w \mapsto \langle |w|u|v \rangle) \in \text{End}_k(V) \text{ for any } u, v \in V, \]
and with quadratic superform \( q = (q_0, b) \) such that
\[ q_0(f) = \det f, \quad b(u, v) = \langle u|v \rangle, \]
for any \( f \in \text{End}_k(V) \) and \( u, v \in V \).

Again, if the characteristic is 3, \( B(4, 2) \) is a Hurwitz superalgebra \(^{[EO02, \text{Proposition 2.7]}}\).

The vector space \( \text{End}_k(V) \) may be identified with \( V \otimes V \) (unadorned tensor products are assumed to be tensor products over the ground field \( k \)) by means of
\[
V \otimes V \rightarrow \text{End}_k(V)
\]
\[ x \otimes y \mapsto \langle \cdot |x \rangle y : v \mapsto \langle \cdot |x|v \rangle y. \]

The symplectic involution on \( \text{End}_k(V) \) becomes now \( x \otimes y = -y \otimes x \) for any \( x, y \in V \).

Then \( B(4, 2) \) can be identified with
\[
B(4, 2) = (V \otimes V) \oplus V,
\]
with multiplication given by
\[
\begin{cases}
(x \otimes y) \cdot (z \otimes t) = \langle \cdot |x|z \rangle t \otimes y, \\
u \cdot (x \otimes y) = \langle \cdot |u|y \rangle = -(y \otimes x) \cdot u \\
u \cdot v = -u \otimes v,
\end{cases}
\]
for any \( x, y, z, t, u, v \in V \), and the supersymmetric bilinear form \( b \) is determined by
\[
b(x \otimes y, z \otimes t) = \langle \cdot |x|z \rangle \langle \cdot |y|t \rangle, \quad b(u, v) = \langle u|v \rangle, \quad (2.5)
\]
for any \( x, y, z, t, u, v \in V \) (see \(^{[Eldb, \S 2]}}\)).

Given any Hurwitz superalgebra \( C \) with norm \( q = (q_0, b) \), the linear map \( x \mapsto \bar{x} = b(x, 1)1 - x \) is a superinvolution (\( \bar{x} = x \) and \( \overline{xy} = (-1)^{|x||y|}\bar{x}\bar{y} \) for any homogeneous \( x, y \in C \)). Then \( C \), with the same norm \( q \), but with new multiplication
\[ x \bullet y = \bar{x} \bar{y} \]
becomes a symmetric composition superalgebra, which is called a \textit{para-Hurwitz superalgebra}, and denoted by \( \bar{C} \).

Also, as for the ungraded case, if \( \varphi \) is an order 3 automorphism of a symmetric composition superalgebra \( S \), with multiplication \( \bullet \) and norm \( q \)
(we will refer to this as the symmetric composition superalgebra \( (S, \bullet, q) \)), then \( \varphi \) is an isometry of \( q \), and with the new multiplication given by
\[
x \ast y = \varphi(x) \bullet \varphi^2(y), \quad (2.6)
\]
\((S, \ast, q)\) is again a symmetric composition algebra, denoted by \( S_{\varphi} \) or \((S, \bullet, q)_{\varphi} \).

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Example 2.7. Given any order 3 automorphism of $B(1,2)$ over a field $k$ of characteristic 3 (and hence this is also an automorphism of the associated para-Hurwitz superalgebra $B(1,2)$), there is a scalar $\lambda \in k$ and a symplectic basis $\{v, w\}$ of $V$ (that is, $\langle v|w \rangle = 1$) such that $\varphi(v) = v$ and $\varphi(w) = \lambda v + w$. The symmetric composition superalgebra defined on $B(1,2)$ by means of the new multiplication given by
\[ x \cdot y = \varphi(\hat{x})\varphi^2(\hat{y}) \]
is denoted by $B(1,2)_\lambda$ (see [EO02, Example 2.9]).

For $\lambda = 1$ this is just the para-Hurwitz superalgebra $B(1,2)$.

Example 2.8. In terms of the description of $B(4,2)$ as $(V \otimes V) \oplus V$, the multiplication in the associated para-Hurwitz superalgebra $B(4,2)$ is determined by:
\[
\begin{align*}
(x \otimes y) \bullet (z \otimes t) &= (y|z)t \otimes x, \\
u \bullet (x \otimes y) &= (y|u)x = -(y \otimes x) \bullet u \\
u \bullet v &= -u \otimes v,
\end{align*}
\]
for any $x, y, z, t, u, v \in V$. The associated bilinear form is given in (2.5).

Remark 2.9. This is exactly the multiplication in the para-Cayley algebra in [Eldb, eq. (2.2)], if the last copy of $V$ is suppressed there.

The classification of the symmetric composition superalgebras appears in [EO02, Theorem 4.3]. Over fields of characteristic 2, any symmetric composition superalgebra is a symmetric composition algebra suitably graded over $\mathbb{Z}_2$. In characteristic $\neq 2$, the classification is given by:

Theorem 2.10. Let $k$ be a field of characteristic $\neq 2$, and let $S$ be a symmetric composition superalgebras over $k$. Then either:

(i) $S_1 = 0$; that is, $S$ is a symmetric composition algebra, or

(ii) the characteristic of $k$ is 3 and either there is a scalar $\lambda \in k$ such that $S$ is isomorphic to $B(1,2)_\lambda$ or $S$ is isomorphic to $B(4,2)$.

Therefore, the superalgebras in Examples 2.7 and 2.8 exhaust, up to isomorphism, all the symmetric composition superalgebras, which do not appear in the ungraded setting.

In order to superize the construction of the Lie algebras $g(S, S')$ in the Introduction, the triality Lie superalgebras are needed. Given a symmetric composition superalgebra $(S, *, q)$, its triality Lie superalgebra $\text{tri}(S, *, q) = \text{tri}(S, *, q)_0 \oplus \text{tri}(S, *, q)_1$ is defined by
\[
\text{tri}(S, *, q)_i = \{(d_0, d_1, d_2) \in \mathfrak{osp}(S, q)_i^3 : d_0(x * y) = d_1(x) * y + (-1)^{|x|} x * d_2(y) \}
\]
for any homogeneous elements $x, y \in S$.

where $i = 0, 1$. For simplicity, this Lie superalgebra will be referred to simply by $\text{tri}(S)$. Here $\mathfrak{osp}(S, q)$ denotes the orthosymplectic Lie superalgebra of the superform $q$. The bracket in $\text{tri}(S)$ is given componentwise.
Note that $\tri(S)$ is endowed with a natural automorphism $\theta : (d_0, d_1, d_2) \mapsto (d_2, d_0, d_1)$, satisfying $\theta^3 = 1$.

**Theorem 2.11.** Let $(S, *, q)$ be any of the symmetric composition superalgebras $B(1, 2)$ or $B(4, 2)$. Then for any homogeneous element $d_0 \in \osp(S, q)$, there are unique $d_1, d_2 \in \osp(S, q)$ such that $(d_0, d_1, d_2) \in \tri(S)$. Besides, the map $\Phi : \osp(S, q) \rightarrow \osp(S, q)$ given by $\Phi(d_0) = d_1$ is an automorphism of $\osp(S, q)$ with $\Phi^3 = 1$. Hence $\tri(S) = \{(d, \Phi(d), \Phi^2(d)) : d \in \osp(S, q)\}$ is isomorphic to $\osp(S, q)$. Moreover, for $B(1, 2)$, this automorphism $\Phi$ is the identity.

**Proof.** This is [EO02, Theorem 5.6]. The last assertion is proved in [EO02, p. 5464]. \qed

**Corollary 2.12.** $\tri(B(1, 2)) = \{(d, d, d) : d \in \osp(B(1, 2), q)\}$.

Given a nondegenerate even supersymmetric bilinear form $b$ on a superspace $W = W_0 \oplus W_1$, its orthosymplectic Lie superalgebra $\osp(W, b)$ is spanned by the operators

$$\sigma_{x,y} : z \mapsto (-1)^{|y||z|} b(x, z) y - (-1)^{|x||y|+|z|} b(y, z) x,$$

for homogeneous $x, y, z \in W$. Besides, for homogeneous $x, y, z, t \in W$,

$$[\sigma_{x,y}, \sigma_{z,t}] = \sigma_{x,y} \sigma_{z,t} + (-1)^{|x||y|+|z|} \sigma_{z,t} \sigma_{x,y}(t). \quad (2.13)$$

Also, given a vector space $U$ endowed with a nondegenerate alternating bilinear form $\langle \cdot, \cdot \rangle$, its symplectic Lie algebra $\sp(U, \langle \cdot, \cdot \rangle)$ is spanned by the operators

$$\gamma_{x,y} : z \mapsto \langle x|z\rangle y + \langle y|z\rangle x,$$

for $x, y, z \in U$.

Over any field $k$ of characteristic 3, consider the para-Hurwitz superalgebra $B(1, 2)$ which, for simplicity, will be denoted by $S_{1,2}$. Thus, $S_{1,2} = k1 \oplus V$, where $V$ is a two dimensional vector space endowed with a nonzero (hence nondegenerate) alternating bilinear form $\langle \cdot, \cdot \rangle$. The norm $q = (q_0, b)$ satisfies $q_0(1) = 1$ (so $b(1, 1) = 2 = -1$) and $b(u, v) = \langle u|v \rangle$ for any $u, v \in V$. The corresponding orthosymplectic Lie superalgebra $\osp(S_{1,2}, b) = \sigma_{S_{1,2}, S_{1,2}}$ satisfies:

$$\osp(S_{1,2}, b) = \sigma_{S_{1,2}, S_{1,2}} (\text{note that } \sigma_{1,1} = 0).$$

But for any $u, v, w \in V$:

$$\sigma_{u,v} : \begin{cases} 1 \mapsto 0, \\ w \mapsto -b(u, w)v - b(v, w)u = -\gamma_{u,v}(w), \end{cases} \quad (2.15)$$

while

$$\sigma_{1,u} : \begin{cases} 1 \mapsto 2u = -u \ (\text{char } k = 3), \\ v \mapsto -b(u, v)1 = -\langle u|v \rangle 1. \end{cases} \quad (2.16)$$

Also, $[\sigma_{1,u}, \sigma_{1,v}] = -\sigma_{u,v}$ and $[\sigma_{u,v}, \sigma_{1,w}] = \sigma_{1,\sigma_{u,v}(w)}$ for any $u, v, w \in V$. Thus, $\sigma_{V,V}$ can be identified with the three dimensional symplectic
Lie algebra $\mathfrak{sp}(V)$, $\sigma_{1,V}$ with $V$ ($\sigma_{1,u} \leftrightarrow u$), and therefore $\mathfrak{osp}(S_{1,2}, b)$ is isomorphic to the Lie superalgebra

$$b_{0,1} = \mathfrak{sp}(V) \oplus V,$$

with even part $\mathfrak{sp}(V)$, odd part $V$, and multiplication determined by:

- the Lie algebra $\mathfrak{sp}(V)$ is a subalgebra,
- $[\gamma, v] = \gamma(v)$ for any $\gamma \in \mathfrak{sp}(V)$ and $v \in V$,
- $[u, v] = \gamma_{u,v}$ for any $u, v \in V$.

Now, consider the superalgebra $S_{4,2} = B(4,2)$ over a field $k$ of characteristic 3, as described in Example 2.3. Then [EO02, Lemma 5.7] shows that the Lie superalgebra $\mathfrak{osp}(S_{4,2}, b)$ is isomorphic to the Lie superalgebra

$$\mathfrak{d}_{2,1} = (\mathfrak{sp}(V) \oplus \mathfrak{sp}(V) \oplus \mathfrak{sp}(V)) \oplus (V \otimes V \otimes V),$$

with even part $\mathfrak{sp}(V) \oplus \mathfrak{sp}(V) \oplus \mathfrak{sp}(V)$, odd part $V \otimes V \otimes V$, and multiplication determined by:

- the Lie algebra $\mathfrak{sp}(V) \oplus \mathfrak{sp}(V) \oplus \mathfrak{sp}(V)$ is a subalgebra,
- for any $\gamma_1, \gamma_2, \gamma_3 \in \mathfrak{sp}(V)$ and $u_1, u_2, u_3 \in V$,

$$[(\gamma_1, \gamma_2, \gamma_3), u_1 \otimes u_2 \otimes u_3] = \gamma_1(u_1) \otimes u_2 \otimes u_3 + u_1 \otimes \gamma_2(u_2) \otimes u_3 + u_1 \otimes u_2 \otimes \gamma(u_3),$$

- for any $u_1, u_2, u_3, v_1, v_2, v_3 \in V$,

$$[u_1 \otimes u_2 \otimes u_3, v_1 \otimes v_2 \otimes v_3] = -\left(\langle u_2 | v_2 \rangle \langle u_3 | v_3 \rangle \gamma_{u_1, v_1}, \langle u_1 | v_1 \rangle \langle u_3 | v_3 \rangle \gamma_{u_2, v_2}, \langle u_1 | v_1 \rangle \langle u_2 | v_2 \rangle \gamma_{u_3, v_3}\right)$$

$$= -\sum_{i=1}^{3} (\prod_{j \neq i} (u_j | v_j)) \nu_i (\gamma_{u_i, v_i}),$$

where $\nu_i : \mathfrak{sp}(V) \rightarrow \mathfrak{sp}(V)^3$ denotes the inclusion on the $i$th-component.

(A word of caution is needed here: the operators $\sigma_{a,b}$ in [EO02] are changed in sign with respect to ours.)

Moreover, the action of $\mathfrak{d}_{2,1}$ on $S_{4,2} = (V \otimes V) \oplus V$ is given by the isomorphism:

$$\rho : \mathfrak{d}_{2,1} \rightarrow \mathfrak{osp}(S_{4,2}, b)$$

such that

$$\rho((\gamma_1, \gamma_2, \gamma_3))(v_1 \otimes v_2 + v_3) = (\gamma_1(v_1) \otimes v_2 + v_1 \otimes \gamma_2(v_2)) + \gamma_3(v_3),$$

$$\rho(u_1 \otimes u_2 \otimes u_3)(v_1 \otimes v_2 + v_3) = -\sigma_{u_1 \otimes u_2, u_3}(v_1 \otimes v_2 + v_3)$$

$$= \langle u_3 | v_3 \rangle u_1 \otimes u_2 - \langle u_1 | v_1 \rangle \langle u_2 | v_2 \rangle u_3,$$

for any $\gamma_1, \gamma_2, \gamma_3 \in \mathfrak{sp}(V)$ and $u_1, u_2, u_3, v_1, v_2, v_3 \in V$.

Note that $\sigma_{u_3, v_3}(w_3) = -b(w_3, v_3) v_3 - b(v_3, w_3) u_3 = -\gamma_{u_3, v_3}(w_3)$, so (see [EO02, (2.8)])

$$\sigma_{u_1 \otimes u_2, v_1 \otimes v_2} = -\rho((v_2 | v_2) \gamma_{u_1, v_1}, \langle u_1 | v_1 \rangle \gamma_{u_2, v_2}, 0)),$$

$$\sigma_{u_3, v_3} = -\rho((0, 0, \gamma_{u_3, v_3})), $$

$$\sigma_{u_1 \otimes u_2, u_3} = -\rho(u_1 \otimes u_2 \otimes u_3),$$

(2.20)
for any \( u_1, u_2, v_1, v_2, v_3 \in V \).

Consider the natural order 3 automorphism \( \theta \) of \( \mathfrak{d}_{2,1} \) such that
\[
\begin{align*}
\theta(\gamma_1, \gamma_2, \gamma_3) &= (\gamma_3, \gamma_1, \gamma_2), \\
\theta(u_1 \otimes u_2 \otimes u_3) &= u_3 \otimes u_1 \otimes u_2,
\end{align*}
\]
for any \( \gamma_1, \gamma_2, \gamma_3 \in \mathfrak{sp}(V) \) and \( u_1, u_2, u_3 \in V \).

Then (compare to [Eldb] Proposition 2.12):

**Proposition 2.22.** For any homogeneous elements \( f \in \mathfrak{d}_{2,1} \) and \( x, y \in S_{4,2} \):
\[
\rho(f)(x \circ y) = \rho(\theta^{-1}(f))(x) \circ y + (-1)^{|f||x|} x \circ \rho(\theta^{-2}(f))(y).
\]

*Proof.* It is enough to prove this for generators of \( \mathfrak{d}_{2,1} \) and a spanning set of \( S_{4,2} \), and hence for \( f = u_1 \otimes u_2 \otimes u_3 \), \( x = v_1 \otimes v_2 \) or \( x = v_3 \), and \( y = w_1 \otimes w_2 \) or \( y = w_3 \), where \( u_i, v_i, w_i \in V \), \( i = 1, 2, 3 \). This is straightforward. For instance,
\[
\begin{align*}
\rho(u_1 \otimes u_2 \otimes u_3)(v_3 \otimes (w_1 \otimes w_2)) &= \rho(u_1 \otimes u_2 \otimes u_3)((w_2|v_3)w_1) \\
&= (w_2|v_3)v_1 \otimes u_1 \otimes u_2,
\end{align*}
\]
\[
\begin{align*}
\rho(\theta^{-1}(u_1 \otimes u_2 \otimes u_3))(v_3) \otimes (w_1 \otimes w_2) &= \rho(u_2 \otimes u_3 \otimes u_1)(v_3) \otimes (w_1 \otimes w_2) \\
&= (u_1|v_3)(u_2 \otimes u_3) \otimes (w_1 \otimes w_2) \\
&= (u_1|v_3)(u_3|u_1)w_2 \otimes u_2
\end{align*}
\]
\[
- v_3 \otimes \rho(\theta^{-2}(u_1 \otimes u_2 \otimes u_3))(w_1 \otimes w_2) = - v_3 \otimes (\rho(u_3 \otimes u_1 \otimes u_2)(w_1 \otimes w_2) \\
= v_3 \otimes (w_3|w_1)u_1 \otimes u_2 \\
&= -(u_3|w_1)v_3 \otimes u_1 \otimes u_2,
\]

and now, since \( \langle u|v \rangle w + \langle v|w \rangle u + \langle w|u \rangle v = 0 \) for any \( u, v, w \in V \) (as any trilinear alternating form on a two dimensional vector space is trivially zero),

one gets
\[
\rho(u_1 \otimes u_2 \otimes u_3)(v_3 \otimes (w_1 \otimes w_2)) = \rho(\theta^{-1}(u_1 \otimes u_2 \otimes u_3))(v_3) \otimes (w_1 \otimes w_2) \\
- v_3 \otimes \rho(\theta^{-2}(u_1 \otimes u_2 \otimes u_3))(w_1 \otimes w_2),
\]
as required.

Write \( \rho_i = \rho \circ \theta^{-i} \), then Theorem 2.11 and Proposition 2.22 give, as in [Eldb] Corollary 2.13:

**Corollary 2.23.** \( \text{tri}(S_{4,2}) = \{ (\rho_0(f), \rho_1(f), \rho_2(f)) : f \in \mathfrak{d}_{2,1} \} \).

This Corollary allows us to identify \( \text{tri}(S_{4,2}) \) to \( \mathfrak{d}_{2,1} \) by means of \( f \leftrightarrow (\rho_0(f), \rho_1(f), \rho_2(f)) \).

**Remark 2.24.** The formulas for the bracket in \( \mathfrak{d}_{2,1} \) and for the representations \( \rho_i \) of \( \mathfrak{d}_{2,1} \) \( (i = 0, 1, 2) \) are exactly the ones that appear in [Eldb] §2 for \( \mathfrak{d}_4 \), but with the last copy of \( V \) suppressed.

Denote by \( V_i \) \( (i = 1, 2, 3) \) the module \( V \) for \( \mathfrak{sp}(V)^3 = \mathfrak{sp}(V) \oplus \mathfrak{sp}(V) \oplus \mathfrak{sp}(V) \) on which only the \( i \)th component of \( \mathfrak{sp}(V)^3 \) acts: \( (s_1, s_2, s_3).v_i = s_i(v_i) \), for
any \( s_j \in \mathfrak{sp}(V), \ j = 1, 2, 3, \) and \( v_i \in V_i. \) Also denote by \( \iota_i(S_{4,2}) \) the \( \mathfrak{o}_{2,1} \)-module associated to the representation \( \rho_i. \) Then, as modules for \( \mathfrak{sp}(V)^3 \) (compare to [Eldb, (2.14)]):

\[
\begin{cases}
\iota_0(S_{4,2}) = (V_1 \otimes V_2) \oplus V_3, \\
\iota_1(S_{4,2}) = (V_2 \otimes V_3) \oplus V_1, \\
\iota_2(S_{4,2}) = (V_3 \otimes V_1) \oplus V_2 \quad (\simeq (V_1 \otimes V_3) \oplus V_2).
\end{cases}
\]

(2.25)

The multiplication \( \cdot \) on \( S_{4,2} \) (see Example 2.8) becomes the bilinear \( \text{tri}(S_{4,2}) \)-invariant map

\[
\iota_0(S_{4,2}) \times \iota_1(S_{4,2}) \rightarrow \iota_2(S_{4,2})
\]

given by,

\[
((V_1 \otimes V_2) \oplus V_3) \times (V_2 \otimes V_3) \oplus V_1) \rightarrow (V_3 \otimes V_1) \oplus V_2
\]

\[
(u_1 \otimes u_2 + u_3), (v_2 \otimes v_3 + v_1) \mapsto (\langle u_2 | v_2 \rangle v_3 \otimes u_1 - u_3 \otimes v_1)
\]

\[- (\langle u_1 | v_1 \rangle u_2 + \langle u_3 | v_3 \rangle v_2)
\]

(2.26)

for any \( u_i, v_i \in V_i, \ i = 1, 2, 3; \) and cyclically. Note that it consists of contractions for repeated indices.

This is exactly the multiplication in [Eldb, 3.1] if \( V_4 \) is ignored there.

3. THE EXTENDED FREUDENTHAL’S MAGIC SQUARE

Let \( (S, \cdot, q) \) and \( (S', *, q') \) be two symmetric composition superalgebras and define \( g = g(S, S') \) to be the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-graded anticommutative superalgebra such that

\[
\begin{align*}
g_{(0,0)} &= \text{tri}(S, \cdot, q) \oplus \text{tri}(S', *, q'), \\
g_{(1,0)} &= g_{(0,1)} = g_{(1,1)} = S \otimes S'.
\end{align*}
\]

For any \( x \in S \) and \( x' \in S', \) denote by \( \iota_i(x \otimes x') \) the element \( x \otimes x' \) in \( g_{(1,0)} \) (respectively \( g_{(0,1)}, g_{(1,1)} \)) if \( i = 0 \) (respectively, \( i = 1, 2 \)). Thus

\[
g = g(S, S') = \left( \text{tri}(S, \cdot, q) \oplus \text{tri}(S', *, q') \right) \oplus \left( \odot_{i=0}^{2} \iota_i(S \otimes S') \right).
\]

(3.1)

Then \( g \) is a superalgebra where, for \( i = 0, 1, \)

\[
g_i = (g_{(0,0)})_i \oplus (g_{(1,0)})_i \oplus (g_{(0,1)})_i \oplus (g_{(1,1)})_i,
\]

with

\[
(g_{(0,0)})_i = \text{tri}(S)_i \oplus \text{tri}(S')_i,
\]

\[
\iota_j(S \otimes S')_i = \iota_j(S_0 \otimes S'_i) \oplus \iota_j(S_1 \otimes S'_{1-i}),
\]

for \( j = 0, 1, 2. \)

The superanticommutative multiplication on \( g \) is defined by means of:

- \( g_{(0,0)} \) is a Lie subsuperalgebra of \( g, \)
- \([d_0, d_1, d_2], \iota_i(x \otimes x') \] = \( \iota_i(d_i(x) \otimes x'), \)
- \([d'_0, d'_1, d'_2], \iota_i(x \otimes x') \] = \(-1)^{|d'_i||x|} \iota_i(x \otimes d'_i(x')) ,
- \([\iota_i(x \otimes x'), \iota_{i+1}(y \otimes y') \] = \(-1)^{x'\otimes y'} \iota_{i+2}(x \otimes y) \otimes (x' \otimes y') \) (indices modulo 3),
\[\begin{align*}
&\bullet \left[t_i(x \otimes x'), t_i(y \otimes y')\right] = (-1)^{|x||x'|+|x||y'|+|y||x'|}b(x', y')\theta^i(t_{x,y}) \\
&\quad + (-1)^{|y||x'|}b(x, y)\theta^i(t'_{x,y'}),
\end{align*}\]

for any \(i = 0, 1, 2\) and homogeneous \(x, y \in S, x', y' \in S'\), \((d_0, d_1, d_2) \in \text{tri}(S)\), and \((d'_0, d'_1, d'_2) \in \text{tri}(S')\). Here \(\theta\) denotes the natural automorphism \(\theta: (d_0, d_1, d_2) \mapsto (d'_2, d_0, d_1)\) in \(\text{tri}(S)\), \(\theta'\) the analogous automorphism of \(\text{tri}(S')\), and

\[t_{x,y} = (\sigma_{x,y}, \frac{1}{2}b(x, y)1 - r_x l_y, \frac{1}{2}b(x, y)1 - l_x r_y)\quad (3.2)\]

(with \(l_x(y) = x \bullet y, r_x(y) = (-1)^{|y||x'|}y \bullet x\)), while \(t'_{x',y'}\) is the analogous triple in \(\text{tri}(S')\).

Just superizing the arguments in [Eld04, Theorem 3.1]; that is, taking into account the parity signs, one gets:

**Theorem 3.3.** With this multiplication, \(\mathfrak{g} = \mathfrak{g}(S, S')\) is a Lie superalgebra.

With the same proof as in the ungraded case [Eld06a, Theorem 12.2] the following result is obtained:

**Proposition 3.4.** Let \(S\) and \(S'\) be two symmetric composition superalgebras, and let \(\varphi\) be an automorphism of \(S\) of order 3, then the Lie superalgebras \(\mathfrak{g}(S, S')\) and \(\mathfrak{g}(S_\varphi, S')\) are isomorphic.

(The superalgebra \(S_\varphi\) was defined in [Eld06].)

Therefore, over fields of characteristic 3, there is no need to deal with the symmetric composition superalgebras \(\overline{B(1,2)}_\lambda\), but just with \(S_{1,2} = \overline{B(1,2)}\) (which is three dimensional) and with \(S_{4,2} = \overline{B(4,2)}\) (whose dimension is 6). Freudenthal’s Magic Square thus extends over these fields to the larger square

\[
\begin{array}{cccc|cc}
\text{dim } S' & 1 & 2 & 4 & 8 & 3 & 6 \\
\hline
\text{dim } S & A_1 & A_2 & C_3 & E_4 & \mathfrak{g}(S_1, S_{1,2}) & \mathfrak{g}(S_1, S_{1,2}) \\
2 & \tilde{A}_2 & \tilde{A}_2 & \tilde{A}_3 & C_5 & \mathfrak{g}(S_2, S_{1,2}) & \mathfrak{g}(S_{2,2}) \\
4 & D_6 & E_7 & \tilde{E}_6 & \tilde{E}_7 & \mathfrak{g}(S_4, S_{1,2}) & \mathfrak{g}(S_{4,2}) \\
8 & E_8 & \mathfrak{g}(S_8, S_{1,2}) & \mathfrak{g}(S_{8,2}) & \mathfrak{g}(S_{8,2}) \\
3 & \mathfrak{g}(S_{1,2}, S_{1,2}) & \mathfrak{g}(S_{1,2}, S_{1,2}) & \mathfrak{g}(S_{1,2}, S_{1,2}) & \mathfrak{g}(S_{1,2}, S_{1,2}) \\
6 & \mathfrak{g}(S_{4,2}, S_{1,2}) & \mathfrak{g}(S_{4,2}, S_{4,2}) & \mathfrak{g}(S_{4,2}, S_{4,2}) & \mathfrak{g}(S_{4,2}, S_{4,2})
\end{array}
\]

(see Table 2 where the dimensions of the even and odd part of the superalgebras involved, which are easily computed from the definitions, are displayed), where \(S_n\) denotes a symmetric composition algebra of dimension \(n\) (\(n = 1, 2, 4\) or 8).

The purpose for the remaining part of the paper is the description of the Lie superalgebras \(\mathfrak{g}(S, S_{1,2})\) and \(\mathfrak{g}(S, S_{4,2})\) in the split case, that is, in case the (super)algebra \(S\) contains nontrivial idempotents (for instance, over an algebraically closed field). Proposition 3.4 or [Eld06a, Corollary 12.1] shows
that it is enough to take then for \( S_n \) the unique split para-Hurwitz algebras (dimensions 1, 2, 4 or 8), together with \( S_{1,2} \) and \( S_{1,4} \).

This description will be given in terms of contragredient Lie superalgebras.

4. Contragredient Lie superalgebras

Let \( A = (a_{ij}) \) be a square matrix of size \( n \) over a ground field \( k \), and let \( (\mathfrak{h}, \Pi, \Pi^\vee) \) be a realization of \( A \), as defined in [Kac90] \( \S 1.1 \). That is,

- \( \Pi = \{\alpha_1, \ldots, \alpha_n\} \) is a linearly independent set in \( \mathfrak{h}^* \) (the dual of the vector space \( \mathfrak{h} \)),
- \( \Pi^\vee = \{h_1, \ldots, h_n\} \) is a linearly independent set in \( \mathfrak{h} \),
- \( \alpha_j(h_i) = a_{ij} \) for any \( i, j = 1, \ldots, n \),
- \( \dim \mathfrak{h} = 2n - \text{rank} \ A \).

As in [Kac77], for any subset \( \tau \subseteq \{1, \ldots, n\} \), consider the local Lie superalgebra

\[ \tilde{\mathfrak{g}}(A, \tau) = \mathfrak{g}^{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1, \]

with \( \mathfrak{g}_0 = \mathfrak{h}, \mathfrak{g}_1 = k e_1 \oplus \cdots \oplus k e_n, \mathfrak{g}^{-1} = k f_1 \oplus \cdots \oplus k f_n \), and bracket given by:

\[ [e_i, f_j] = \delta_{ij} h_i, \quad [h, h'] = 0, \quad [h, e_i] = \alpha_i(h) e_i, \quad [h, f_i] = -\alpha_i(h) f_i, \quad (4.1) \]

for any \( i, j = 1, \ldots, n \) and \( h, h' \in \mathfrak{h} \), where \( \mathfrak{g}_0 \) is even and \( e_i, f_i \) are even if and only if \( i \not\in \tau \).

Note that changing \( f_i \) by \( c f_i, 0 \neq c \in k \), the \( i^{\text{th}} \) row of \( A \) is multiplied by \( c \). Hence, if \( a_{ii} \neq 0 \), then \( a_{ii} \) can be taken to be 2, as it is customary.

Then (see [Kac77, 1.2.2]) there exists a minimal \( \mathbb{Z} \)-graded Lie superalgebra \( \mathfrak{g}(A, \tau) \) with local part \( \tilde{\mathfrak{g}}(A, \tau) \). To define \( \mathfrak{g}(A, \tau) \) one first considers \( \tilde{\mathfrak{g}}(A, \tau) \), the free Lie superalgebra generated by \( \mathfrak{h} \) and \( \{e_1, \ldots, e_n, f_1, \ldots, f_n\} \), subject to the relations in (4.1), endowed with the \( \mathbb{Z} \)-grading induced by the grading in \( \tilde{\mathfrak{g}}(A, \tau) \) (note that the relations are homogeneous), and then considers the largest homogeneous ideal \( \mathfrak{i}(A) \) that intersects \( \mathfrak{h} \) trivially. Then

\[ \mathfrak{g}(A, \tau) = \tilde{\mathfrak{g}}(A, \tau) / \mathfrak{i}(A). \]

Remark 4.2.

- As in [Kac90, Proposition 1.6], with \( \mathfrak{g}'(A, \tau) = [\mathfrak{g}(A, \tau), \mathfrak{g}(A, \tau)] \), the center is given by

\[ \mathfrak{c} = Z(\mathfrak{g}(A, \tau)) = Z(\mathfrak{g}'(A, \tau)) = \{h \in \mathfrak{h} : \alpha_i(h) = 0 \forall i = 1, \ldots, n\}. \]

Note that \( \dim \mathfrak{c} = n - \text{rank} \ A \).

- Let \( Q \) be the free abelian group on generators \( e_1, \ldots, e_n \). Then \( \mathfrak{g}(A, \tau) \) is \( Q \)-graded by assigning \( \deg e_i = \epsilon_i = -\deg f_i, i = 1, \ldots, n \), and \( \deg h = 0 \) for any \( h \in \mathfrak{h} \). Now \( \tilde{\mathfrak{g}}(A, \tau) \) is \( Q \)-graded too, because the relation in (4.1) are \( Q \)-homogeneous. For any \( m \in \mathbb{Z} \), the \( m^{\text{th}} \) homogeneous component of \( \tilde{\mathfrak{g}}(A, \tau) \) equals

\[ \bigoplus_{q=m_1 e_1 + \cdots + m_n e_n} \tilde{\mathfrak{g}}(A, \tau)_q, \]
and the ideal $i(A)$ is $Q$-homogeneous too, since $i(A)$ is contained in $\oplus_{\not\equiv q \in Q} \pi_q(i(A))$ ($\pi_q$ denotes the projection onto the $q$th-component), which is a $Q$-homogeneous ideal that intersects trivially $\mathfrak{h}$. Hence, by maximality, $i(A) = \oplus_{\not\equiv q \in Q} \pi_q(i(A))$.

The following notation will be used:
- $g(A, \tau)$ will be called the contragredient Lie superalgebra with Cartan matrix $A$.
- $g'(A, \tau)$ will be called the derived contragredient Lie superalgebra with Cartan matrix $A$.
- $g(A, \tau)/c$ will be called the centerless contragredient Lie superalgebra with Cartan matrix $A$.
- $g'(A, \tau)/c$ will be called the centerless derived contragredient Lie superalgebra with Cartan matrix $A$.

A bit of caution is needed here, in [VK71], it is the Lie algebra $g'(A, \tau)$ the one that is called the contragredient Lie algebra.

Note that, as in [Kac90] Proposition 1.7, $g(A, \tau)$ has no $Q$-homogeneous ideal if and only if $\det A \neq 0$ and for any $i, j$ there exist indices $i_1, \ldots, i_s$ such that $a_{i_1i_2}a_{i_2i_3} \cdots a_{i_{s}j} \neq 0$. If only this last condition holds, then any $Q$-homogeneous ideal either contains $g'(A, \tau)$ or is contained in the center. Also note that if $\det A \neq 0$, then $c = 0$, $g'(A, \tau) = g(A, \tau)$ and the four Lie superalgebras considered above coincide.

**Lemma 4.3.** Under the conditions above, let $j(A)$ be the largest $\mathbb{Z}$-homogeneous ideal of $\tilde{g}(A, \tau)$ which intersects trivially $g_{-1} \oplus g_1$. Then $j(A) = i(A) \oplus c$.

**Proof.** It is clear that $i(A) \cap g_{\pm 1} = 0$, otherwise $i(A)$ would contain some $0 \neq h \in \mathfrak{h}$ (because of $[e_i, f_i] = h_i$ for any $i$). Hence $i(A)$ is the largest homogeneous ideal intersecting trivially $g_{-1} \oplus g_0 \oplus g_1$. Thus $(i(A) \oplus c) \cap (g_{-1} \oplus g_1) = 0$, so $i(A) \oplus c \subseteq j(A)$.

Conversely, $\tilde{g}(A, \tau) = \oplus_{i \in \mathbb{Z}} \tilde{g}_i$, with $\tilde{g}_i = g_i$ for $i = -1, 0, 1$, and for any $h \in j(A) \cap g_0 = j(A) \cap \mathfrak{h}$, $[h, e_i] = \alpha_i(h)e_i \in j(A) \cap g_1 = 0$ for any $i$. Thus $\alpha_i(h) = 0$ for any $i$ and $h \in c$. Therefore $j(A) \subseteq c \oplus (\oplus_{i \neq -1, 0, 1} j(A) \cap \tilde{g}_i)$. But $\oplus_{i \neq -1, 0, 1} j(A) \cap \tilde{g}_i$ is an ideal of $\tilde{g}(A, \tau)$, since it is closed under the action of $e_i, f_i, i = 1, \ldots, n$, and $\mathfrak{h}$ (note that $[f_i, j(A) \cap \tilde{g}_2] \subseteq j(A) \cap g_1 = 0$). Hence $\oplus_{i \neq -1, 0, 1} j(A) \cap \tilde{g}_i \subseteq i(A)$, and $j(A) \subseteq i(A) \oplus c$. \hfill $\Box$

Let us consider the natural characterizations of the contragredient Lie superalgebras, that will be used in the next section. The previous notations and assumptions will be kept in the following results.

**Theorem 4.4.** Let $g$ be a $\mathbb{Z}$-graded Lie superalgebra generated by $g_0$, which is contained in $g_0$, and by elements $e_i \in g_1$ and $f_i \in g_{-1}$, $i = 1, \ldots, n$, which are even (respectively odd) if and only if $i \not\in \tau$ (resp. $i \in \tau$). Assume that there are $\alpha_1, \ldots, \alpha_n \in \mathbb{g}_0^*$ and $h_1, \ldots, h_n \in g_0$ such that

$$(g_0, \{\alpha_1, \ldots, \alpha_n\}, \{h_1, \ldots, h_n\})$$

is a realization of $A$ and that relations (4.1) hold. If any nonzero homogeneous ideal of $g$ intersects $g_0$ nontrivially, then $g$ is isomorphic to the contragredient Lie superalgebra $g(A, \tau)$.
Proof. This is clear from the definitions, which give an epimorphism $\phi : g(A, \tau) \to g$, such that $\ker \phi \cap g(A, \tau)_0 = 0$, so $\ker \phi = 0$. \qed

**Theorem 4.5.** Let $g$ be a $\mathbb{Z}$-graded Lie superalgebra such that:

(i) $g_0 = \bar{h}$ is contained in $g_0$, it is abelian and its dimension is $n$.

(ii) There are linearly independent elements $\bar{\alpha}_1, \ldots, \bar{\alpha}_n \in \bar{h}^*$ and elements $\bar{h}_1, \ldots, \bar{h}_n \in \bar{h}$ such that $\bar{\alpha}_j(\bar{h}_i) = \delta_{ij}$ for any $i, j$.

(iii) There are elements $\bar{e}_1, \ldots, \bar{e}_n \in g_1$ and $\bar{f}_1, \ldots, \bar{f}_n \in g_{-1}$, where $\bar{e}_i, \bar{f}_i$ are even (resp. odd) if $i \not\in \tau$ (resp. $i \in \tau$), and such that the relations $[\bar{e}_i, \bar{f}_i] = \delta_{ij} \bar{h}_i$ are satisfied (with $\bar{e}_i$ replaced by $\bar{e}_i$, ...), and $g$ is generated by $\bar{e}_1, \ldots, \bar{e}_n, \bar{f}_1, \ldots, \bar{f}_n$ and $\bar{h}$.

(iv) Any nonzero homogeneous ideal of $g$ intersects $g_{-1} \oplus g_1$ nontrivially. Then $g$ is isomorphic to the centerless contragredient Lie superalgebra $g(A, \tau)/\hat{c}$.

Proof. The elements $\alpha_1, \ldots, \alpha_n \in \mathfrak{h}^*$ ($\mathfrak{h} = g(A, \tau)_0$) define linearly independent elements $\hat{\alpha}_i \in (\mathfrak{h}/\mathfrak{c})^*$, because $\mathfrak{c} = \{h \in \mathfrak{h} : \alpha_i(h) = 0 \ \forall i\}$. Hence there is a linear bijection $\varphi^* : \mathfrak{h}^* \to (\mathfrak{h}/\mathfrak{c})^*$ such that $\varphi^*(\hat{\alpha}_i) = \delta_{ij}$ for any $j$. Besides, $\varphi^*$ induces a linear bijection $\varphi : \mathfrak{h}/\mathfrak{c} \to \mathfrak{h}$, such that $\bar{\alpha}_j(\varphi(h + \mathfrak{c})) = \bar{\alpha}_j(h + \mathfrak{c}) = \delta_{ij}(h)$ for any $j$ and $h \in \mathfrak{h}$. By composing with the natural projection, a surjective linear map is obtained $\phi : \mathfrak{h} \to \mathfrak{h}$ such that $\bar{\alpha}_j(\phi(h)) = \delta_{ij}(h)$ for any $j$ and $h \in \mathfrak{h}$.

Now, by definition of $\tilde{g}(A, \tau)$, $\phi$ extends to a surjective homomorphism of Lie superalgebras $\hat{\phi} : \tilde{g}(A, \tau) \to g$ which takes $e_i$ (respectively $f_i$) to $\bar{e}_i$ (respectively $\bar{f}_i$), for any $i$, and $\hat{\phi}$ is homogeneous. Condition (iv) shows that $\hat{\phi}(1(A)) = 0$, so $\hat{\phi}$ induces a surjective homomorphism $\hat{\phi} : \tilde{g}(A, \tau)/j(A) \to g$. But $\hat{\phi}$ is bijective in degrees 0, 1 and $-1$ (the elements $\bar{e}_1, \ldots, \bar{e}_n$ are linearly independent, as they are eigenvectors of eigenvalues $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ for the action of $\bar{h}$). By definition of $j(A)$, one concludes that $\ker \hat{\phi} = 0$, so $g$ is isomorphic to $\tilde{g}(A, \tau)/j(A)$, which is isomorphic to $g(A, \tau)/\hat{c}$ by Lemma 4.3 \qed

**Theorem 4.6.** Let $g$ be a $\mathbb{Q}$-graded Lie superalgebra generated by nonzero elements $\bar{e}_i \in g_{i+1}, \bar{f}_i \in g_{-i}, i = 1, \ldots, n$, where $\bar{e}_i, \bar{f}_i$ are even (resp. odd) if $i \not\in \tau$ (resp. $i \in \tau$). Assume that:

(i) The elements $\bar{e}_i = [\bar{e}_i, \bar{f}_i], i = 1, \ldots, n$, are linearly independent.

(ii) $[\bar{h}_i, \bar{e}_j] = \alpha_{ij} \bar{e}_j, [\bar{h}_i, \bar{f}_j] = -\alpha_{ij} \bar{f}_j, [\bar{e}_i, \bar{f}_j] = \delta_{ij} \bar{h}_i$, and $[\bar{h}_i, \bar{h}_j] = 0$, for any $i, j = 1, \ldots, n$.

(iii) Any nonzero $Q$-homogeneous ideal of $g$ intersects nontrivially $g_{-1} \oplus g_0 \oplus g_1, \text{ where } g_0 = \oplus \{\bar{g}_q : q = m_1 e_1 + \cdots + m_n e_n, m_1 + \cdots + m_n = m\}$

for any $m \in \mathbb{Z}$.

Then $g$ is isomorphic to the derived contragredient Lie superalgebra $g'(A, \tau)$.

Proof. Since $g$ is generated by the elements $\bar{e}_1, \ldots, \bar{e}_n, \bar{f}_1, \ldots, \bar{f}_n$, it follows that $g_0 = kh_1 \oplus \cdots \oplus kh_n$. Take an even vector space $\bar{g}_0$ of dimension
Proof. Because of item (i),

\[ \hat{\alpha}_i(h_i) = a_{ij} \text{ for any } i, j, \]

\[ \hat{\alpha}_1, \ldots, \hat{\alpha}_n \text{ are linearly independent}. \]

Then \((g_0 \oplus \hat{g}_0, \{\hat{\alpha}_1, \ldots, \hat{\alpha}_n\}, \{\hat{h}_1, \ldots, \hat{h}_n\})\) is a realization of \(A\), and \(\hat{g} = g \oplus \hat{g}_0\) is a \(Q\)-graded Lie superalgebra, with \(g_0 = g_0 \oplus \hat{g}_0, \hat{g} = \hat{g}_0\) for any \(0 \neq \epsilon \in Q\), and where \([\hat{g}_0, \hat{g}_0] = 0, \hat{g}\) is an ideal of \(\hat{g}\), and for any \(h \in \hat{g}_0\) and \(x \in \hat{g}_0\), \((0 \neq \epsilon = m_1 \epsilon_1 + \cdots + m_n \epsilon_n \in Q), [h, x] = (m_1 \hat{\alpha}_1 + \cdots + m_n \hat{\alpha}_n)(h)x \).

By Theorem 4.4 \(\hat{g}\) is isomorphic to the contragredient Lie superalgebra \(g(A, \tau)\), and hence \(g = [\hat{g}, \hat{g}]\) is isomorphic to \(g'(A, \tau)\). \(\square\)

**Theorem 4.7.** Let \(g\) be a \(Q\)-graded Lie superalgebra generated by nonzero elements \(\hat{e}_i \in g_{\hat{e}_i}, \hat{f}_i \in g_{\hat{f}_i}, i = 1, \ldots, n\), where \(\hat{e}_i, \hat{f}_i\) are even (resp. odd) if \(i \notin \tau\) (resp. \(i \in \tau\)). Assume that:

(i) If \(h_i = [\hat{e}_i, \hat{f}_i], i = 1, \ldots, n\), then \([h_i, \hat{e}_j] = a_{ij}\hat{e}_j, [h_i, \hat{f}_j] = -a_{ij}\hat{f}_j, [\hat{e}_i, \hat{f}_j] = 0\), and \([\hat{e}_i, \hat{f}_j] = -a_{ij}\hat{h}_i, \) for any \(i, j\).

(ii) Any nonzero \(Q\)-homogeneous ideal intersects nontrivially \(g_{-1} \oplus g_1\), where \(g_m = \oplus \{\hat{g}_q : q = m_1 \epsilon_1 + \cdots + m_n \epsilon_n, m_1 + \cdots + m_n = m\}\) for any \(m \in \mathbb{Z}\).

Then \(g\) is isomorphic to the centerless derived contragredient Lie superalgebra \(g'(A, \tau)/\epsilon\).

Proof. Because of item (i), \(g_0 = \text{span} \{\hat{h}_1, \ldots, \hat{h}_n\}\) holds, and condition (ii) implies that the dimension of \(g_0\) equals the rank of \(A\) (any linear combination of the rows of \(A\) which gives 0 induces a linear combination of the \(h_i\)’s which is central, and hence spans an ideal with trivial intersection with \(g_{-1} \oplus g_1\)).

As in the proof of Theorem 4.6 take a vector space \(g_0\) of dimension \(n - \text{rank } A\), and elements \(\hat{\alpha}_1, \ldots, \hat{\alpha}_n \in (g_0 \oplus \hat{g}_0)^*\) such that

\[ \hat{\alpha}_j(h_i) = a_{ij} \text{ for any } i, j, \]

\[ \hat{\alpha}_1, \ldots, \hat{\alpha}_n \text{ are linearly independent}. \]

The Lie superalgebra \(\hat{g}\) constructed as in the proof of Theorem 4.6 satisfies now the hypotheses of Theorem 4.5. Hence \(\hat{g}\) is isomorphic to the centerless contragredient Lie superalgebra \(g(A, \tau)/\epsilon\), and hence \(g = [\hat{g}, \hat{g}]\) is isomorphic to the centerless derived contragredient Lie superalgebra \(g'(A, \tau)/\epsilon = [g(A, \tau)/\epsilon, g(A, \tau)/\epsilon]\). \(\square\)

In order to present the Lie superalgebras \(g(S, S')\) in the Extended Freudenthal’s Magic Square as contragredient Lie superalgebras, it will be useful first to describe them in terms similar to those used in [Eldo].

Let \(V\) be, as before, a two dimensional vector space endowed with a nonzero alternating bilinear form \(\langle \cdot, \cdot \rangle\). For any \(n \in \mathbb{N}\) consider copies \(V_1, \ldots, V_n\) of \(V\), and for any subset \(\sigma \subseteq \{1, \ldots, n\}\) take the module for \(\mathfrak{sp}(V)^n = \mathfrak{sp}(V_1) \oplus \cdots \oplus \mathfrak{sp}(V_n)\) given by:

\[ V(\sigma) = \begin{cases} \mathfrak{sp}(V_1) \oplus \cdots \oplus \mathfrak{sp}(V_n) & \text{if } \sigma = \emptyset, \\ V_{i_1} \otimes \cdots \otimes V_{i_r} & \text{if } \sigma = \{i_1, \ldots, i_r\}, 1 \leq i_1 < \cdots < i_r \leq n. \end{cases} \]  

(4.8)
\( V_i \) is the natural module for \( \mathfrak{sp}(V_i) \) annihilated by \( \mathfrak{sp}(V_j) \) for \( j \neq i \).

Identify, as in [Eldb, Section 2], any subset \( \sigma \subseteq \{1, \ldots, n\} \) with the element \( (\sigma_1, \ldots, \sigma_n) \in \mathbb{Z}_2^n \), such that \( \sigma_i = 1 \) if and only if \( i \in \sigma \). Then for any \( \sigma, \tau \in \mathbb{Z}_2^n \), consider the natural \( \mathfrak{sp}(V)^n \)-invariant map:

\[
\varphi_{\sigma, \tau} : V(\sigma) \times V(\tau) \rightarrow V(\sigma + \tau)
\]

defined as follows:

- If \( \sigma \neq \tau \) and \( \sigma \neq \emptyset \neq \tau \), then \( \varphi_{\sigma, \tau} \) is obtained by contraction, by means of \( \langle \cdot, \cdot \rangle \) in the indices \( i \in \sigma \cap \tau \) \( (\sigma_i = 1 = \tau_i) \). Thus, for instance,

\[
\varphi_{\{1,2,3\},\{1,3,4\}}(u_1 \otimes u_2 \otimes u_3, v_1 \otimes v_3 \otimes v_4) = \langle u_1 | v_1 \rangle \langle v_3 | v_3 \rangle u_2 \otimes v_4
\]

for any \( u_1, v_1 \in V_1, u_2 \in V_2, v_3 \in V_3 \) and \( v_4 \in V_4 \).

- \( \varphi_{\emptyset,\emptyset} \) is the Lie bracket in \( \mathfrak{sp}(V)^n = \mathfrak{sp}(V_1) \oplus \cdots \oplus \mathfrak{sp}(V_n) \).

- For any \( \sigma \neq \emptyset \), \( \varphi_{\emptyset,\sigma} = -\varphi_{\sigma,\emptyset} \) is given by the natural action of the Lie algebra \( \mathfrak{sp}(V)^n \) on \( V(\sigma) \). Thus, for instance,

\[
\varphi_{\emptyset,\{1,3\}}((s_1, \ldots, s_n), u_1 \otimes u_3) = s_1(u_1) \otimes u_3 + u_1 \otimes s_3(u_3),
\]

for any \( s_i \in \mathfrak{sp}(V), i = 1, \ldots, n \), and \( u_1 \in V_1, u_3 \in V_3 \).

- Finally, for any \( \sigma = \{i_1, \ldots, i_r\} \neq \emptyset \), \( \varphi_{\sigma,\sigma} \) is given by:

\[
\varphi_{\sigma,\sigma}(u_{i_1} \otimes \cdots \otimes u_{i_r}, v_{i_1} \otimes \cdots \otimes v_{i_r}) = \prod_{j=1}^{r} \left( u_{i_k} \mid v_{i_k} \right) \nu_j (\gamma_{u_{i_j}, v_{i_j}}),
\]

for any \( u_{i_j}, v_{i_j} \in V_{i_j}, j = 1, \ldots, r \), where \( \nu_i : \mathfrak{sp}(V_i) \rightarrow \mathfrak{sp}(V_i)^n \) denotes the canonical inclusion into the \( i \)-th-component, and \( \gamma_{u,v} \) has been defined in (2.14).

**Example 4.10.** The description of \( \mathfrak{d}_{2,1} \) (characteristic 3) in (2.18) shows that, with \( n = 3 \),

\[
\mathfrak{d}_{2,1} = V(\emptyset) \oplus V(\{1,2,3\}),
\]

with even part \( V(\emptyset) = \mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2) \oplus \mathfrak{sp}(V_3) \), and odd part \( V(\{1,2,3\}) = V_1 \otimes V_2 \otimes V_3 \), and where

\[
[x_{\sigma}, y_{\tau}] = \epsilon_{\mathfrak{d}_{2,1}}(\sigma, \tau) \varphi_{\sigma,\tau}(x_{\sigma}, y_{\tau}),
\]

with

\[
\epsilon_{\mathfrak{d}_{2,1}}(\emptyset, \emptyset) = \epsilon_{\mathfrak{d}_{2,1}}(\emptyset, \{1,2,3\}) = \epsilon_{\mathfrak{d}_{2,1}}(\{1,2,3\}, \emptyset) = 1,
\]

\[
\epsilon_{\mathfrak{d}_{2,1}}(\{1,2,3\}, \{1,2,3\}) = -1.
\]

(Same behavior as \( \mathfrak{d}_1 \) in [Eldb, (2.18) and (2.19)].)

Take the canonical basic elements \( \epsilon_1 = (1,0,0) \), \( \epsilon_2 = (0,1,0) \) and \( \epsilon_3 = (0,0,1) \) in \( \mathbb{Z}_3^3 \), and for any \( i = 1,2,3 \) consider a symplectic basis \( \{v_i, w_i\} \) of
$V_i$ (that is, $⟨v_i|w_i⟩ = 1$), and the basic elements in each $\mathfrak{sp}(V_i)$:

\[
\begin{align*}
    h_i &= γ_{v_i,w_i} : \begin{cases}
        v_i &\mapsto -v_i, \\
        w_i &\mapsto w_i,
    \end{cases} \\
    e_i &= γ_{w_i,v_i} : \begin{cases}
        v_i &\mapsto -2w_i = w_i, \\
        w_i &\mapsto 0,
    \end{cases} \quad \text{(4.13)} \\
    f_i &= -γ_{v_i,w_i} : \begin{cases}
        v_i &\mapsto 0, \\
        w_i &\mapsto -2v_i = v_i,
    \end{cases}
\end{align*}
\]

which satisfy

\[
[h_i,e_i] = 2e_i, \quad [h_i,f_i] = -2f_i, \quad [e_i,f_i] = h_i.
\]

The Lie superalgebra $\mathfrak{d}_{2,1}$ is $\mathbb{Z}^3$-graded by assigning $\deg w_i = ε_i = -\deg v_i$, $i = 1, 2, 3$. Hence, $\deg h_i = 0$, $\deg e_i = 2ε_i$, $\deg f_i = -2ε_i$, for any $i$.

Then,

\[
Φ_{\mathfrak{d}_{2,1}} = \{±2ε_i : i = 1, 2, 3\} \cup \{±ε_1 ± ε_2 ± ε_3\} \quad \text{(4.14)}
\]

is the set of the nonzero degrees that appear in $\mathfrak{d}_{2,1}$. Moreover, the subalgebra $\mathfrak{h} = kh_1 ⊕ kh_2 ⊕ kh_3$ is a Cartan subalgebra of $\mathfrak{d}_{2,1}$ and there is a natural homomorphism of abelian groups:

\[
R : \mathbb{Z}^3 \rightarrow \mathfrak{h}^* \\
ε_i \mapsto R(ε_i)(h_j \mapsto δ_{ij}).
\]

The image under $R$ of $Φ_{\mathfrak{d}_{2,1}}$ is precisely the set of roots of $\mathfrak{h}$ in $\mathfrak{d}_{2,1}$.

Consider the lexicographic order on $\mathbb{Z}^3$ with $ε_1 > ε_2 > ε_3 > 0$. The set of the positive elements in $Φ_{\mathfrak{d}_{2,1}}$ which are not sums of two positive elements is

\[
Π_{\mathfrak{d}_{2,1}} = \{α_1 = 2ε_2, α_2 = ε_1 - ε_2 - ε_3, α_3 = 2ε_3\} \quad \text{(4.15)}
\]

whose elements are linearly independent over $\mathbb{Z}$ and satisfy that $Φ_{\mathfrak{d}_{2,1}}$ is contained in $\mathbb{Z}Π_{\mathfrak{d}_{2,1}} = \mathbb{Z}α_1 ⊕ \mathbb{Z}α_2 ⊕ \mathbb{Z}α_3$.

Let $Q$ be the free abelian group $\mathbb{Z}Π_{\mathfrak{d}_{2,1}}$, take $τ = \{2\}$ and consider the following elements in $\mathfrak{d}_{2,1}$:

\[
\begin{align*}
    E_1 &= e_2, \quad E_2 = w_1 ⊗ v_2 ⊗ v_3, \quad E_3 = e_3, \\
    F_1 &= f_2, \quad F_2 = -v_1 ⊗ w_2 ⊗ w_3, \quad F_3 = f_3, \\
    H_1 &= h_2, \quad H_2 = h_1 - h_2 - h_3, \quad H_3 = h_3.
\end{align*}
\]

Then $(\mathfrak{h}, \{R(α_1), R(α_2), R(α_3)\}, \{H_1, H_2, H_3\})$ is a realization of the matrix:

\[
A_{\mathfrak{d}_{2,1}} = \begin{pmatrix}
2 & -1 & 0 \\
1 & 0 & 1 \\
0 & -1 & 2
\end{pmatrix},
\]

which is the Cartan matrix $D_1$ in $[Kac77]$ p. 55], and which corresponds to the Lie superalgebra $D(2, 1) ≅ D(2, 1; 1)$, that is, to the orthosymplectic Lie superalgebra $\mathfrak{osp}(4, 2)$.

The Lie superalgebra $\mathfrak{d}_{2,1}$ is $\mathbb{Z}$-graded with $(\mathfrak{d}_{2,1})_0 = \mathfrak{h}$, $(\mathfrak{d}_{2,1})_1 = kE_1 ⊕ kE_2 ⊕ kE_3$, and $(\mathfrak{d}_{2,1})_{-1} = kF_1 ⊕ kF_2 ⊕ kF_3$. It is easily checked to be simple. Now, Theorem 4.4 with the elements $H_i, E_i, F_i$ above, $i = 1, 2, 3$, gives:
Proposition 4.17. The Lie superalgebra $\mathfrak{d}_{2,1}$ is isomorphic to the contragredient Lie superalgebra $\mathfrak{g}(A_{32,1}, \{2\})$.

5. The Lie superalgebras in the Extended Freudenthal’s Magic Square

In [Eldb], most of the Lie algebras in Freudenthal’s Magic Square were described, in the split case, by means of the $V(\sigma)$’s and $\varphi_{\sigma,\tau}$’s of the previous section. This is possible too in the superalgebra setting.

This section will be devoted to get such descriptions, or a similar one, for each Lie superalgebra in the Extended Freudenthal’s Magic Square, constructed from a couple of split symmetric composition (super)algebras. This will be used to find a description of all these superalgebras as contragredient Lie superalgebras.

Throughout this section, the characteristic of the ground field $k$ will always be assumed to be 3.

5.1. $\mathfrak{g}(S_1, S_{4,2})$. The symmetric composition superalgebra $(S_1, \bullet, q)$ is just $k1$, with $1 \bullet 1 = 1$ and $q(1) = 1$ (so $b(1, 1) = 2$). Thus $\text{tri}(S_1) = 0$ and (3.1), (4.11) and (2.25) show that

$$
\mathfrak{g}(S_1, S_{4,2}) = \text{tri}(S_{4,2}) \oplus \iota_0(S_1 \otimes S_{4,2}) \oplus \iota_1(S_1 \otimes S_{1,4,2}) \oplus \iota_2(S_1 \otimes S_{4,2})
$$

$$
= \left( V(\emptyset) \oplus V(\{1, 2, 3\}) \right) \oplus \left( V(\{1, 2\}) \oplus V(\{3\}) \right)
$$

$$
\oplus \left( V(\{2, 3\}) \oplus V(\{1\}) \right) \oplus \left( V(\{1, 3\}) \oplus V(\{2\}) \right)
$$

$$
= \bigoplus_{\sigma \in 2^{1,2,3}} V(\sigma),
$$

where $2^{1,2,3}$ denotes the power set of $\{1, 2, 3\}$. Besides,

$$
\mathfrak{g}(S_1, S_{4,2})_0 = \bigoplus_{|\sigma| \text{ even}} V(\sigma) \quad \text{and} \quad \mathfrak{g}(S_1, S_{4,2})_1 = \bigoplus_{|\sigma| \text{ odd}} V(\sigma).
$$

By invariance of the Lie bracket under the action of $\mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2) \oplus \mathfrak{sp}(V_3) = V(\emptyset)$, it follows that

$$
[x_\sigma, y_\tau] = \epsilon(\sigma, \tau) \varphi_{\sigma,\tau}(x_\sigma, y_\tau),
$$

(5.1)

for any $\sigma, \tau \in 2^{1,2,3}$, for a suitable map

$$
\epsilon : 2^{1,2,3} \times 2^{1,2,3} \rightarrow k^\times.
$$

The multiplication $\iota_i(1 \otimes S_{4,2}) \times \iota_{i+1}(1 \otimes S_{4,2}) \rightarrow \iota_{i+2}(1 \otimes S_{4,2})$ is given in (2.26), and for any $x, y \in S_{4,2}$ (see (3.2)):

$$
[\iota_0(1 \otimes x), \iota_0(1 \otimes y)] = 2t_{x,y} = (2\sigma_{x,y}, b(x, y)1 - 2r_x l_y, b(x, y)1 - 2l_x r_y),
$$

as $b(1, 1) = 2$ in $S_1$. This element is identified with the element $2\rho^{-1}(\sigma_{x,y})$ in (2.19), which is given by (2.20). By cyclic symmetry, one completes the information about the map $\epsilon$ in (5.1), displayed in Table 3 which is exactly Table 2 in [Eldb], corresponding to $\mathfrak{g}(S_1, S_8)$, but with the index 4 taken out. From this description it readily follows that $\mathfrak{g}(S_1, S_{4,2})$ is simple.
In general, in order to obtain a description of the Lie superalgebra $\mathfrak{g}(S, S')$ in the Extended Freudenthal’s Magic Square as contragredient Lie superalgebras, the complete description of the map $\epsilon$ in (5.1) is not necessary and will not be given. A detailed description of these maps appears in [Cun06].

Now, with the notations and conventions in Example 4.10, the Lie superalgebra $\mathfrak{g} = \mathfrak{g}(S_1, S_4, 2)$ is $\mathbb{Z}^3$-graded, and the set of nonzero degrees that appear is

$$\Phi_{S_1, S_4, 2} = \{\pm 2\epsilon_i, \pm \epsilon_i \pm \epsilon_j, \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3, \pm \epsilon_i : 1 \leq i < j \leq 3\}.$$  

The subalgebra $\mathfrak{h} = kh_1 \oplus kh_2 \oplus kh_3$ (recall (4.13) that $h_i = \gamma_{v_i, w_i}$ belongs to $\mathfrak{sp}(V_i) \subseteq V(0)$, where $\{v_i, w_i\}$ is a fixed symplectic basis of $V_i$) is a Cartan subalgebra of $\mathfrak{g}$ and the image of $\Phi_{S_1, S_4, 2}$ under the natural homomorphism of abelian groups

$$R : \mathbb{Z}^3 \rightarrow \mathfrak{h}^*$$

$$\epsilon_i \mapsto R(\epsilon_i)(: h_j \mapsto \delta_{ij})$$

is precisely the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$.

Consider the lexicographic order on $\mathbb{Z}^3$ with $\epsilon_1 > \epsilon_2 > \epsilon_3 > 0$. The set of positive elements in $\Phi_{S_1, S_4, 2}$ which are not sums of positive elements (the irreducible degrees) is

$$\Pi = \{\alpha_1 = \epsilon_2 - \epsilon_3, \alpha_2 = \epsilon_3, \alpha_3 = \epsilon_1 - \epsilon_2 - \epsilon_3\},$$

which is a $\mathbb{Z}$-basis of $\mathbb{Z}^3$.

Then $\left(\mathfrak{h}, \{R(\alpha_1), R(\alpha_2), R(\alpha_3)\}, \{h_2 - h_3, 2h_3, -h_1 + h_2 + h_3\}\right)$ is a realization of the matrix

$$A_{S_1, S_4, 2} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

with associated Dynkin diagram (using the conventions in [Kac77, Tables IV and V])

\[ \begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{array} \]
With \( \tau = \{2, 3\} \), consider the generators
\[
H_1 = h_2 - h_3, \quad H_2 = 2h_3, \quad H_3 = -h_1 + h_2 + h_3,
E_1 = w_2 \otimes v_3, \quad E_2 = w_3, \quad E_3 = w_1 \otimes v_2 \otimes v_3,
F_1 = v_2 \otimes w_3, \quad F_2 = -v_3, \quad F_3 = -v_1 \otimes w_2 \otimes w_3.
\]
As for Proposition 4.17 Theorem 4.3 implies the following description of \( g(S_1, S_{4,2}) \):

**Proposition 5.2.** The Lie superalgebra \( g(S_1, S_{4,2}) \) is isomorphic to the contra
gredient Lie superalgebra \( g(A_{S_1, S_{4,2}}, \{2, 3\}) \).

Moreover, the set of even and odd nonzero degrees are
\[
(\Phi_{S_1, S_{4,2}})_0 = \{ \pm 2\epsilon_i, \pm \epsilon_i \pm \epsilon_j : 1 \leq i < j \leq 3 \},
(\Phi_{S_1, S_{4,2}}) = \{ \pm \epsilon_i, \pm \epsilon_i + \epsilon_j \pm \epsilon_k : 1 \leq i \leq 3 \}.
\]
The set of “irreducible” even degrees for the lexicographic order considered is
\[
\Pi_0 = \{ \beta_1 = \epsilon_1 - \epsilon_2, \beta_2 = \epsilon_2 - \epsilon_3, \beta_3 = 2\epsilon_3 \},
\]
and it can be concluded from here that the even part \( g_0 = g(S_1, S_{4,2})_0 \) is isomorphic to the contra
gredient Lie algebra with Cartan matrix
\[
\begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -2 \\
0 & -1 & 2
\end{pmatrix}.
\]
That is, \( g(S_1, S_{4,2})_0 \) is isomorphic to the symplectic Lie algebra \( sp_6 \) (type \( C_3 \)). Here the appropriate basis of \( h \) is
\[
\{ \bar{H}_1 = h_1 - h_2, \bar{H}_2 = h_2 - h_3, \bar{H}_3 = h_3 \}.
\]
Also, the set of simple roots \( R(\Pi_0) \) of \( g_0 \) induces a triangular decomposition
\( g_0 = n^- \oplus h \oplus n^+ \), where \( n^+ = \oplus_{0 < \alpha \in (\Phi_{S_1, S_{4,2}})_0} (g_0)_R(\alpha) \) is the sum of root spaces (nonzero homogeneous components in the \( \mathbb{Z}^3 \)-grading) corresponding to the positive degrees in \( (\Phi_{S_1, S_{4,2}})_0 \), and similarly for \( n^- \). The \( g_0 \)-module \( g_1 \) is \( \mathbb{Z}^3 \)-graded consistently with the action of \( g_0 \), it is easily seen to be irreducible, and to contain the highest weight vector \( w_1 \otimes w_2 \otimes w_3 \), that is, \([n^+, w_1 \otimes w_2 \otimes w_3] = 0 \). Its highest weight is \( R(\epsilon_1 + \epsilon_2 + \epsilon_3) = \omega_3 \), which satisfies \( \omega_3(\bar{H}_1) = 0 = \omega_3(\bar{H}_2), \omega_3(\bar{H}_3) = 1 \).

Since the proof of the uniqueness result in [Hum72] Theorem A in §20.3 remains valid in this setting (note that instead of grading over the \( \mathbb{Z} \)-linear combinations of weights, which are elements of \( h^* \), we grade over a true lattice \( \mathbb{Z}^3 \)), \( g_1 \) is the unique \( g_0 \)-module with such a highest weight. Denote it by \( V(\omega_3) \). Note that \( \dim g_1 = 14 \).

Let us give the most natural presentation of this module. Consider the matrix Lie algebra \( sp_6 \), and its natural six-dimensional module, which is endowed with an alternating invariant bilinear form \( \{,\} \). Let \( \{a_1, a_2, a_3, b_1, b_2, b_3\} \) be a symplectic basis (that is, \( \{a_i | b_j\} = \delta_{ij}, \{a_i | a_j\} = 0 = \{b_i | b_j\} \) for any \( i, j \)). With \( \gamma_{x,y} = \{x, y\} + \{y, x\} \), the subspace spanned by \( \gamma_{a_i, b_i}, i = 1, 2, 3 \) is a Cartan subalgebra of \( sp(W) \simeq sp_6 \) and the weights of \( W \) are \( \{\pm \delta_i : i = 1, 2, 3\} \), where \( \delta_i(\gamma_{a_i, b_i}) = \delta_{ij} \) for any \( i, j \).
Consider the $\mathfrak{sp}_6$-module $\bigwedge^3 W$ and the homomorphism
\[
\varphi : \bigwedge^3 W \to W
\]
\[
z_1 \wedge z_2 \wedge z_3 \mapsto \{z_1|z_2\}z_3 + \{z_2|z_3\}z_1 + \{z_3|z_1\}z_2.
\]
Then $\dim \ker \varphi = 14$, the weights of $\ker \varphi$ are $\pm \delta_1 \pm \delta_2 \pm \delta_3$ and $\pm \delta_i$ ($i = 1, 2, 3$), all of them of multiplicity 1. The element $b_1 \wedge b_2 \wedge b_3$ is a highest weight vector of weight $\delta_1 + \delta_2 + \delta_3 = \omega_3$ and $\ker \varphi$ is irreducible. Therefore, up to isomorphism $V(\omega_3) = \ker \varphi$.

Let us summarize the above discussion:

**Proposition 5.3.** The Lie superalgebra $\mathfrak{g}(S_1, S_{4,2})$ is simple with even part isomorphic to the symplectic Lie algebra $\mathfrak{sp}_6$ and odd part isomorphic to the irreducible module of dimension 14 above.

There is no counterpart in characteristic 0 (see [Kac77]) to this simple Lie superalgebra.

5.2. $\mathfrak{g}(S_4, S_{1,2})$. The symmetric composition superalgebra $S_4$ is the even part of $S_{1,2}$, and its triality Lie algebra is given too by the even part of the triality Lie superalgebra of $S_4$. Therefore, six copies of $V$ are needed: $V_1$, $V_2$ and $V_3$ for $S_4$, and $V_4$, $V_5$ and $V_6$ for $S_{1,2}$.

With the same sort of arguments used so far, one has:

\[
\mathfrak{g}(S_4, S_{1,2}) = (\tri S_4) \oplus \tri S_{4,2}) \oplus (\bigoplus_{i=0}^2 \iota_i(S_4 \otimes S_{4,2}))
\]

\[
= \bigoplus_{\sigma \in S_{S_4,S_{4,2}}} V(\sigma),
\]

where

\[
S = S_{S_4,S_{4,2}} = \{\emptyset, \{4,5,6\}, \{1,2,4,5\}, \{1,2,6\}, \{2,3,5,6\}, \{2,3,4\}, \{1,3,4,6\}, \{1,3,5\}\}.
\]

For instance,

\[
\iota_1(S_4 \otimes S_{1,2}) = (V_2 \otimes V_3) \otimes ((V_5 \otimes V_6) \oplus V_4) = V(\{2,3,5,6\}) \oplus V(\{2,3,4\}).
\]

Like in all the other cases, the even (respectively odd) part is the sum of the $V(\sigma)$’s with $\sigma$ containing an even (resp. odd) number of elements.

The multiplication presents the form in (5.1) for a suitable map $\varepsilon : \mathcal{S} \times \mathcal{S} \to k^\times$. Here the nonzero even and odd degrees are:

\[
\Phi_0 = \{\pm 2\varepsilon_i : 1 \leq i \leq 6\}
\]

\[
\cup \{\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_4 \pm \varepsilon_5 \pm \varepsilon_6 \pm \varepsilon_1 \pm \varepsilon_3 \pm \varepsilon_4 \pm \varepsilon_6\},
\]

\[
\Phi_1 = \{\pm \varepsilon_4 \pm \varepsilon_5 \pm \varepsilon_6 \pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_6 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 \pm \varepsilon_1 \pm \varepsilon_3 \pm \varepsilon_5\}.
\]

With the lexicographic order given by $0 < \varepsilon_1 < \cdots < \varepsilon_6$, the set of irreducible degrees is

\[
\Pi = \{\alpha_1 = 2\varepsilon_1, \alpha_2 = \varepsilon_5 - \varepsilon_4 - \varepsilon_2 - \varepsilon_1, \alpha_3 = 2\varepsilon_2, \alpha_4 = \varepsilon_4 - \varepsilon_5 - \varepsilon_2, \alpha_5 = 2\varepsilon_3, \alpha_6 = \varepsilon_6 - \varepsilon_5 - \varepsilon_4\},
\]
which is a linearly independent set over \( \mathbb{Z} \). Then \( \Phi \) is contained in \( Q = \mathbb{Z}\Pi \) (which is isomorphic to \( \mathbb{Z}^6 \)), so that any positive element in \( \Phi \) is a sum of elements in \( \Pi \). Consider the elements (same conventions as before)

\[
H_1 = h_1, \quad H_2 = -h_5 + h_4 + h_2 + h_1, \quad H_3 = h_2, \quad H_4 = h_4 - h_3 - h_2, \\
H_5 = h_3, \quad H_6 = h_6 - h_5 - h_4,
\]

which give a realization

\[
\{h, \{R(\alpha_1), R(\alpha_2), R(\alpha_3), R(\alpha_4), R(\alpha_5), R(\alpha_6)\}, \{H_1, H_2, H_3, H_4, H_5, H_6\}\}
\]

of the regular matrix (with associated Dynkin diagram):

\[
A_{s_{4,2}} = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & -1 \\
0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix},
\]

Then:

**Proposition 5.4.** The Lie superalgebra \( g(s_{4,2}) \) is isomorphic to the contragredient Lie superalgebra \( g(A_{s_{4,2}}, \{4, 6\}) \).

Also, the set of irreducible even degrees is:

\[
\Pi_0 = \{\beta_1 = 2\epsilon_3, \beta_2 = \epsilon_6 - \epsilon_5 - \epsilon_3 - \epsilon_2, \beta_3 = 2\epsilon_2, \\
\beta_4 = \epsilon_5 - \epsilon_4 - \epsilon_2 - \epsilon_1, \beta_5 = 2\epsilon_1, \beta_6 = 2\epsilon_4\},
\]

with associated Cartan matrix of type \( D_6 \). The odd part is an irreducible module for the even part, with highest weight vector \( R(\epsilon_4 + \epsilon_5 + \epsilon_6) = \omega_6 \), and the same sort of arguments used for \( g(s_{1,4,2}) \) gives:

**Proposition 5.5.** The Lie superalgebra \( g(s_{4,2}) \) is simple with even part isomorphic to the orthogonal Lie algebra \( so_{12} \), and odd part isomorphic to the spin module for \( so_{12} \).

Note that the even part is just \( g(s_4, s_4) \), which was determined in [Eldb].

**Corollary 5.6.** The Lie superalgebra \( g(s_{4,2}) \) is isomorphic to the Lie superalgebra in [Eld06b] theorem 3.2(v)] and [Elda] Theorem 4.1(ii) \( l = 6 \).

5.3. \( g(s_{8,1,2}) \). Here, four copies of \( V \) are needed for \( S_8 \) (see [Eldb] (2.14)) and three copies for \( S_{4,2} \). The indices 1, 2, 3 and 4 will be used for \( S_8 \), while 5, 6 and 7 will be reserved for \( S_{4,2} \). Then

\[
g(s_8, s_{4,2}) = (\text{tri}(S_8) \oplus \text{tri}(S_{4,2})) \oplus (\bigoplus_{i=0}^2 \lambda_i(S_8 \otimes S_{4,2}))
\]

\[
= \bigoplus_{\sigma \in S_{s_8, s_{4,2}}} V(\sigma),
\]

where

\[
S = S_{s_8, s_{4,2}} = \{\emptyset, \{1, 2, 3, 4\}, \{5, 6, 7\}, \\
\{1, 2, 5, 6\}, \{3, 4, 5, 6\}, \{1, 2, 7\}, \{3, 4, 7\}, \\
\{2, 3, 6, 7\}, \{1, 4, 6, 7\}, \{2, 3, 5\}, \{1, 4, 5\}, \\
\{1, 3, 5, 7\}, \{2, 4, 5, 7\}, \{1, 3, 6\}, \{2, 4, 6\}\}.
\]
Hence,
\[
\Phi_0 = \{ \pm 2\epsilon_i : i = 1, \ldots, 7 \} \cup \{ \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4, \\
\phantom{\Phi_0 = \{} \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_5 \pm \epsilon_6, \pm \epsilon_3 \pm \epsilon_4 \pm \epsilon_5 \pm \epsilon_6, \pm \epsilon_3 \pm \epsilon_4 \pm \epsilon_6 \pm \epsilon_7, \\
\phantom{\Phi_0 = \{} \pm \epsilon_1 \pm \epsilon_4 \pm \epsilon_5 \pm \epsilon_7, \pm \epsilon_1 \pm \epsilon_3 \pm \epsilon_5 \pm \epsilon_7, \pm \epsilon_2 \pm \epsilon_4 \pm \epsilon_5 \pm \epsilon_7 \},
\]
\[
\Phi_1 = \{ \pm \epsilon_5 \pm \epsilon_6 \pm \epsilon_7, \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_7, \pm \epsilon_3 \pm \epsilon_4 \pm \epsilon_7, \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_5, \\
\phantom{\Phi_1 = \{} \pm \epsilon_1 \pm \epsilon_4 \pm \epsilon_5, \pm \epsilon_1 \pm \epsilon_3 \pm \epsilon_6, \pm \epsilon_2 \pm \epsilon_4 \pm \epsilon_6 \},
\]
and with the lexicographic order with \(0 < \epsilon_1 < \ldots < \epsilon_7:\)
\[
\Pi = \{ \alpha_1 = \epsilon_6 - \epsilon_5 - \epsilon_4 - \epsilon_3, \alpha_2 = 2\epsilon_2, \alpha_3 = 2\epsilon_3, \\
\phantom{\Pi = \{} \alpha_4 = \epsilon_4 - \epsilon_3 - \epsilon_2 - \epsilon_1, \alpha_5 = 2\epsilon_1, \alpha_6 = \epsilon_5 - \epsilon_4 - \epsilon_1, \alpha_7 = \epsilon_7 - \epsilon_6 - \epsilon_5 \},
\]
which is a \(\mathbb{Z}\)-linearly independent set with \(\Phi \subseteq \Pi\), so that any positive element in \(\Phi\) is a sum of elements in \(\Pi\). The associated matrix and Dynkin diagram are here:
\[
A_{S_8, S_{4,2}} = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0
\end{pmatrix}, \\
\begin{array}{cccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
\]

**Proposition 5.7.** The Lie superalgebra \(g(S_8, S_{4,2})\) is isomorphic to the contragredient Lie superalgebra \(g(A_{S_8, S_{4,2}}, \{6, 7\})\).

Also, in this situation:
\[
\Pi_0 = \{ \beta_1 = \epsilon_7 - \epsilon_6 - \epsilon_4 - \epsilon_1, \beta_2 = 2\epsilon_2, \beta_3 = 2\epsilon_1, \\
\phantom{\Pi_0 = \{} \beta_4 = \epsilon_4 - \epsilon_3 - \epsilon_2 - \epsilon_1, \beta_5 = 2\epsilon_1, \beta_6 = \epsilon_6 - \epsilon_5 - \epsilon_4 - \epsilon_3, \beta_7 = 2\epsilon_5 \},
\]
whose associated Cartan matrix is of type \(E_7\) (this also follows since \(g(S_8, S_{4,2})_0\) is isomorphic to \(g(S_8, S_4)\), which was computed in [Eld8]).

The odd part, which has dimension \(7 \times 8 = 56\), is an irreducible module with highest weight \(R(\epsilon_7 + \epsilon_6 + \epsilon_5) = \omega_7\). Thus,

**Proposition 5.8.** \(g(S_8, S_{4,2})\) is a simple Lie superalgebra, whose even part is isomorphic to the split simple Lie algebra of type \(E_7\), and whose odd part is its 56-dimensional irreducible module \(V(\omega_7)\).

5.4. \(g(S_{4,2}, S_{4,2})\). For \(g(S_{4,2}, S_{4,2})\), the indices 1, 2 and 3 will refer to the copies of \(V\) associated to the first copy of \(S_{4,2}\), while the indices 4, 5 and 6 will refer to the copies of \(V\) related to the second copy of \(S_{4,2}\). Then:
\[
g(S_{4,2}, S_{4,2}) = (\text{tri}(S_{4,2}) \oplus \text{tri}(S_{4,2})) \oplus \bigoplus_{i=0}^{2}(S_{4,2} \otimes S_{4,2}) \\
= \oplus_{\sigma \in S_{S_{4,2}, S_{4,2}}} V(\sigma),
\]
where

\[ \mathcal{S}_{S_{4,2}, S_{4,2}} = \{ \emptyset, \{1, 2, 3\}, \{4, 5, 6\}, \{1, 2, 4, 5\}, \{3, 6\}, \{1, 2, 6\}, \{3, 4, 5\}, \{2, 3, 5, 6\}, \{1, 4\}, \{2, 3, 4\}, \{1, 5, 6\}, \{1, 3, 4, 6\}, \{2, 5\}, \{1, 3, 5\}, \{2, 4, 6\} \}. \]

Hence,

\[ \Phi_0 = \{ \pm 2\epsilon_i : 1 \leq i \leq 6 \} \cup \{ \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_4 \pm \epsilon_5, \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_5 \pm \epsilon_6, \pm \epsilon_1 \pm \epsilon_3 \pm \epsilon_4 \pm \epsilon_6, \pm \epsilon_3 \pm \epsilon_4 \pm \epsilon_5 \pm \epsilon_6, \pm \epsilon_1 \pm \epsilon_4 \pm \epsilon_5 \pm \epsilon_6, \pm \epsilon_1 \pm \epsilon_4 \pm \epsilon_6 \pm \epsilon_5, \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4 \pm \epsilon_5, \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4 \pm \epsilon_6 \}, \]

\[ \Phi_1 = \{ \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3, \pm \epsilon_4 \pm \epsilon_5 \pm \epsilon_6, \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_6, \pm \epsilon_1 \pm \epsilon_3 \pm \epsilon_4, \pm \epsilon_1 \pm \epsilon_3 \pm \epsilon_5, \pm \epsilon_2 \pm \epsilon_4 \pm \epsilon_5, \pm \epsilon_2 \pm \epsilon_4 \pm \epsilon_6 \}. \]

And with the lexicographic order with \( \epsilon_1 > \cdots > \epsilon_6 > 0 \),

\[ \Pi = \{ \alpha_1 = \epsilon_1 - \epsilon_2 - \epsilon_3, \alpha_2 = \epsilon_3 - \epsilon_4 - \epsilon_5, \alpha_3 = 2\epsilon_5, \alpha_4 = \epsilon_4 - \epsilon_5 - \epsilon_6, \alpha_5 = 2\epsilon_6, \alpha_6 = \epsilon_2 - \epsilon_3 - \epsilon_4 \}, \]

which is a linearly independent set with \( \Phi \subseteq \mathbb{Z}\Pi \), so that any positive element in \( \Phi \) is a sum of elements in \( \Pi \). The associated matrix and Dynkin diagram are:

\[
A_{S_{4,2}, S_{4,2}} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & -1 \\
0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}, \]

\[
\begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4 \\
\alpha_5 \\
\alpha_6
\end{array}
\]

**Proposition 5.9.** The Lie superalgebra \( \mathfrak{g}(S_{4,2}, S_{4,2}) \) is isomorphic to the contragredient Lie superalgebra \( \mathfrak{g}(A_{S_{4,2}, S_{4,2}}, \{1, 2, 4, 6\}) \).

Also,

\[ \Pi_0 = \{ \beta_1 = 2\epsilon_4, \beta_2 = \epsilon_1 - \epsilon_2 - \epsilon_4 - \epsilon_5, \beta_3 = 2\epsilon_5, \beta_4 = \epsilon_2 - \epsilon_3 - \epsilon_5 - \epsilon_6, \beta_5 = 2\epsilon_6, \beta_6 = \epsilon_3 - \epsilon_6 \}, \]

with associated Cartan matrix of type \( B_6 \). The odd part is an irreducible module for the even part with highest weight \( R(\epsilon_1 + \epsilon_2 + \epsilon_3) = \omega_6 \), so by uniqueness, it is the spin module for the even part:

**Proposition 5.10.** \( \mathfrak{g}(S_{4,2}, S_{4,2}) \) is a simple Lie superalgebra whose even part is isomorphic to the orthogonal Lie algebra \( \mathfrak{so}_{13} \) and whose odd part is the spin module for its even part.

**Corollary 5.11.** The Lie superalgebra \( \mathfrak{g}(S_{4,2}, S_{4,2}) \) is isomorphic to the Lie superalgebra in [Eldal Theorem 3.1(ii) \((l = 6)\)].
5.5. $g(S_{1,2},S_{4,2})$. Here there is just one copy of $V$ involved in $S_{1,2}$, which will carry index 1, and three copies, with indices 2, 3 and 4, in $S_{4,2}$. Also, the triality Lie superalgebra $\tri(S_{1,2})$ will be identified to the superalgebra $\mathfrak{b}_{0,1}$ in $\mathfrak{d}_{4,2}$. Then

$$g(S_{1,2},S_{4,2}) = (\tri(S_{1,2}) \oplus \tri(S_{4,2})) \oplus (\oplus_{i=0}^{2} (S_{1,2} \otimes S_{4,2}))$$

with

$$S_{S_{1,2},S_{4,2}} = \{\emptyset,\{1\},\{2,3,4\},\{1,4\},\{1,2,3\},\{4\},\{3,4\},\{1,2\},\{1,3,4\},\{2\},\{2,4\},\{1,3\},\{1,2,4\},\{3\}\},$$

and

$$\Phi_0 = \{\pm 2\epsilon_i : 1 \leq i \leq 4\} \cup \{\pm \epsilon_i \pm \epsilon_j : 1 \leq i < j \leq 4\},$$

$$\Phi_1 = \{\pm \epsilon_i \pm \epsilon_j \pm \epsilon_k : 1 \leq i < j < k \leq 4\} \cup \{\pm \epsilon_i : 1 \leq i \leq 4\}.$$

With the lexicographic order $\epsilon_1 > \epsilon_2 > \epsilon_3 > \epsilon_4 > 0$,

$$\Pi = \{\alpha_1 = \epsilon_1 - \epsilon_2 - \epsilon_3, \alpha_2 = \epsilon_3 - \epsilon_4, \alpha_3 = \epsilon_4, \alpha_4 = \epsilon_2 - \epsilon_3 - \epsilon_4\},$$

which is a linearly independent set with $\Phi \subseteq \Pi$, so that any positive element in $\Phi$ is a sum of elements in $\Pi$. The associated matrix and Dynkin diagram are:

$$A_{S_{1,2},S_{4,2}} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -2 & 2 & -2 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad \begin{array}{cccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4
\end{array}$$

and, therefore:

**Proposition 5.12.** The Lie superalgebra $g(S_{1,2},S_{4,2})$ is isomorphic to the contragredient Lie superalgebra $g(A_{S_{1,2},S_{4,2}},\{1,3,4\})$.

Also,

$$\Pi_0 = \{\beta_1 = \epsilon_1 - \epsilon_2, \beta_2 = \epsilon_2 - \epsilon_3, \beta_3 = \epsilon_3 - \epsilon_4, \beta_4 = 2\epsilon_4\},$$

with associated Cartan matrix of type $C_4$. The odd part is an irreducible module of dimension $4 \times 8 + 4 \times 2 = 40$, with highest weight $R(\epsilon_1 + \epsilon_2 + \epsilon_3) = \omega_3$.

As for the Lie superalgebra $g(S_{1,1},S_{1,2})$, the irreducible module for the symplectic Lie algebra $\mathfrak{sp}_8$ with this highest weight is obtained as follows. Let $W$ be the natural eight dimensional module for $\mathfrak{sp}_8$, and let $\varphi : \wedge^3 W \rightarrow W$ be the linear map such that $\varphi(z_1 \wedge z_2 \wedge z_3) = \{z_1|z_2\}z_3 + \{z_2|z_3\}z_1 + \{z_3|z_1\}z_2$. This time ker $\varphi$ is not irreducible but, since the characteristic is 3, contains the irreducible submodule $\tilde{W} = \{\sum_{i=1}^{4} a_i \wedge b_i \wedge z_i : z \in W\}$, which is isomorphic to $W$ (here $\{a_i,b_i : i = 1,\ldots,4\}$ is a symplectic basis of $W$). Then ker $\varphi/\tilde{W}$ is an irreducible module of dimension 40 and the class of $b_1 \wedge b_2 \wedge b_3$ modulo $\tilde{W}$ is a highest weight vector of weight $\omega_3$.

**Proposition 5.13.** $g(S_{1,2},S_{4,2})$ is a simple Lie superalgebra with even part isomorphic to the symplectic Lie algebra $\mathfrak{sp}_8$ and odd part isomorphic to the irreducible module of dimension 40 above.
5.6. $\mathfrak{g}(S_2, S_{4,2})$. The symmetric composition superalgebra $S_2$ has a basis \{e^+, e^-\} with multiplication given by
\[
e^\pm \cdot e^\mp = e^\mp, \quad e^\pm \cdot e^\mp = 0,
\]
with norm given by $q(e^\pm) = 0$, and $b(e^+, e^-) = 1$. The orthogonal Lie algebra $\mathfrak{so}(S_2, q)$ is spanned by $\phi = \sigma_{e^-, e^+}$, which satisfies $\phi(e^\pm) = \pm e^\pm$, and its triality Lie algebra is (see [Eldu, 3.4]):
\[
\text{tri}(S_2) = \{ (\mu_0 \phi, \mu_1 \phi, \mu_2 \phi) : \mu_0, \mu_1, \mu_2 \in k, \mu_0 + \mu_1 + \mu_2 = 0 \}.
\]
Besides,
\[
t_{e^-, e^+} = (\sigma_{e^-, e^+}, \frac{1}{2} b(e^-, e^+)1 - r_v e^+, \frac{1}{2} b(e^-, e^+)1 - l_v e^+)
= (\phi, \phi, \phi),
\]
(since the characteristic is 3). Consider the two dimensional abelian Lie algebra
\[t = \{ (\mu_0, \mu_1, \mu_2) \in k^3 : \mu_0 + \mu_1 + \mu_2 = 0 \},\]
with basis \{t_1 = (1, -1, 0), t_2 = (0, 1, -1)\}. Then \text{tri}(S_2) is isomorphic to $t$ and the action of \text{tri}(S_2) on each $\iota_i(S_2)$ becomes:
\[(\mu_0, \mu_1, \mu_2).\iota_i(e^\pm) = \pm \mu_i \iota_i(e^\pm) .\]
Here,
\[
\mathfrak{g} = \mathfrak{g}(S_2, S_{4,2}) = (\text{tri}(S_2) \oplus \text{tri}(S_{4,2})) \oplus (\oplus_{i=0}^2 \iota_i(S_2 \otimes S_{4,2})) ,
\]
so, with standard identifications,
\[
\mathfrak{g}_0 = (t \oplus \mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2) \oplus \mathfrak{sp}(V_3)) \oplus (S_2^0 \otimes V_1 \otimes V_2) \oplus (S_2^1 \otimes V_2 \otimes V_3) \oplus (S_2^2 \otimes V_1 \otimes V_3)
\]
\[
\mathfrak{g}_1 = (V_1 \otimes V_2 \otimes V_3) \oplus (S_2^0 \otimes V_3) \oplus (S_2^1 \otimes V_1) \oplus (S_2^2 \otimes V_2),
\]
where, for any $i = 0, 1, 2$, $S_2^i$ is just a copy of $S_2$, with basic elements $e_{2i}^\pm$.

Consider the free $\mathbb{Z}$-module $F$ with basis $\{\delta_1, \delta_2, \epsilon_1, \epsilon_2, \epsilon_3 \}$. The Lie superalgebra $\mathfrak{g}$ is graded over $F (\cong \mathbb{Z}^5)$ by assigning degree $\epsilon_i$ to $w_i, -\epsilon_i$ to $v_i, i = 1, 2, 3$, as in Example 4.10 and
\[
\deg(e_{0i}^\pm) = \pm \delta_1, \quad \deg(e_{1i}^\pm) = \pm \delta_2, \quad \deg(e_{2i}^\pm) = \mp(\delta_1 + \delta_2). \tag{5.14}
\]

In this way, the sets of even and odd nonzero degrees that appear in $\mathfrak{g}(S_2, S_{4,2})$ are:
\[
\Phi_0 = \{ \pm 2 \epsilon_i : i = 1, 2, 3 \}
\cup \{ \pm \delta_1 \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 , \pm (\delta_1 + \delta_2) \pm \epsilon_1 \pm \epsilon_3 \},
\]
\[
\Phi_1 = \{ \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3 , \pm \delta_1 \pm \epsilon_3 , \pm \delta_2 \pm \epsilon_1 , \pm (\delta_1 + \delta_2) \pm \epsilon_2 \}.
\]
The abelian subalgebra $\mathfrak{h} = kt_1 \oplus kt_2 \oplus kh_1 \oplus kh_2 \oplus kh_3$ ($h_i = \gamma_{w_i}, i = 1, 2, 3$), is a Cartan subalgebra of $\mathfrak{g}(S_2, S_{4,2})$ and the image of $\Phi_0 \cup \Phi_1$ under
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the natural homomorphism of abelian groups
\[ R : F \rightarrow h^* \]
\[ \delta_1 \mapsto R(\delta_1) (\cdot t_1 \mapsto 1, t_2 \mapsto 0, h_j \mapsto 0), \]
\[ \delta_2 \mapsto R(\delta_2) (\cdot t_1 \mapsto -1, t_2 \mapsto 1, h_j \mapsto 0), \]
\[ \epsilon_i \mapsto R(\epsilon_i) (\cdot t_1, t_2 \mapsto 0, h_j \mapsto \delta_{ij}), \]
is precisely the set of roots of \( g(S_2, S_{4,2}) \) relative to \( h \).

Consider the lexicographic order on \( F \) with \( \delta_1 > \delta_2 > \epsilon_1 > \epsilon_2 > \epsilon_3 > 0 \).

Then, \( \Phi = \{ \alpha_1 = \delta_1 - \epsilon_1 - \epsilon_2, \alpha_2 = 2\epsilon_2, \alpha_3 = \epsilon_1 - \epsilon_2 - \epsilon_3, \alpha_4 = 2\epsilon_3, \alpha_5 = \delta_2 - \epsilon_1 \} \)
is a linearly independent set with \( \Phi \subseteq Z\Pi \), so that any positive element in \( \Phi \) is a sum of elements in \( \Pi \). Consider the elements:
\[ E_1 = e_0^+ \otimes v_1 \otimes v_2, \quad H_1 = (t_1 - t_2) + h_1 + h_2, \quad F_1 = \xi_1 e_0^- \otimes w_1 \otimes w_2, \]
\[ E_2 = \gamma_{w_2, w_2}, \quad H_2 = h_2, \quad F_2 = \xi_2 \gamma_{v_2, v_2}, \]
\[ E_3 = w_1 \otimes v_2 \otimes v_3, \quad H_3 = -h_1 + h_2 + h_3, \quad F_3 = \xi_3 v_1 \otimes w_2 \otimes w_3, \]
\[ E_4 = \gamma_{w_3, w_3}, \quad H_4 = h_3, \quad F_4 = \xi_4 \gamma_{v_3, v_3}, \]
\[ E_5 = e_1^+ \otimes v_1, \quad H_5 = (t_1 - t_2) + h_1, \quad F_5 = \xi_5 e_1^- \otimes w_1, \]
where \( \xi_1, \ldots, \xi_5 \) are suitable scalars so as to have \([E_i, F_j] = \delta_{ij} H_j \) for any \( i, j \).

With these elements \( g(S_2, S_{4,2}) \) is \( \mathbb{Z} \)-graded, by assigning degree 1 to \( E_1, \ldots, E_5 \), and degree \(-1\) to \( F_1, \ldots, F_5 \), and the hypotheses of Theorem 4.5 are satisfied relative to the rank 4 matrix (and Dynkin diagram):
\[
A_{S_2, S_{4,2}} = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & 1 \\
0 & 0 & -1 & 2 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix},
\]

Therefore:

**Proposition 5.15.** The Lie superalgebra \( g(S_2, S_{4,2}) \) is isomorphic to the centerless contragredient Lie superalgebra \( g(A_{S_2, S_{4,2}}, \{3, 5\}) / c \). It is not simple, but its derived superalgebra \([g(S_2, S_{4,2}), g(S_2, S_{4,2})]\) is a simple ideal of codimension 1.

Also, \( \Pi_0 = \{ \beta_1 = 2\epsilon_1, \beta_2 = \delta_1 - \epsilon_1 - \epsilon_2, \beta_3 = 2\epsilon_2, \beta_4 = \delta_2 - \epsilon_2 - \epsilon_3, \beta_5 = 2\epsilon_3 \} \)
is a linearly independent set with \( \Phi_0 \subseteq Z\Pi_0 \), so that any positive element in \( \Phi \) is a sum of elements in \( \Pi_0 \), and with similar arguments one obtains that the associated Cartan matrix is of type \( A_5 \), \( g_0 \) thus being isomorphic to the centerless contragredient Lie algebra \( g(A_5) / c \), which is isomorphic to the projective general Lie algebra \( pgl_6 \). The highest degree in \( g_1 \) is \( \delta_1 + \delta_2 + \epsilon_3 \), so the highest weight is \( R(\delta_1 + \delta_2 + \epsilon_3) = \omega_3 \). Therefore:

**Proposition 5.16.** \( g(S_2, S_{4,2}) \) is a Lie superalgebra with even part isomorphic to \( pgl_6 \), and odd part isomorphic to the third exterior power of the natural module for \( gl_6 \) (of dimension 20).
Since the characteristic is 3, this third exterior power is indeed a module for $\mathfrak{pg} \cong \mathfrak{gl}_6 / k1$.

5.7. $\mathfrak{g}(S_{1,2}, S_{1,2})$. Now, let us consider two copies of $S_{1,2}$: $k1 \oplus V_i$, $i = 1, 2$ (Example 2.7). Then the even and odd parts of

$$\mathfrak{g} = \mathfrak{g}(S_{1,2}, S_{1,2}) = (\text{tr}(S_{1,2}) \oplus \text{tr}(S_{1,2})) \oplus (\oplus_{i=0}^2 \mathfrak{i}(S_{1,2} \otimes S_{1,2}))$$

are given, because of Corollary 2.12, by:

$$\mathfrak{g}_0 = (\mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2)) \oplus (\oplus_{i=0}^2 \mathfrak{k} \mathfrak{t}_i(1 \otimes 1)) \oplus (\oplus_{i=0}^2 \mathfrak{i}(V_1 \otimes V_2)),$$

$$\mathfrak{g}_1 = (V_1 \oplus (\oplus_{i=0}^2 \mathfrak{k} \mathfrak{t}_i(V_1 \otimes 1))) \oplus (V_2 \oplus (\oplus_{i=0}^2 \mathfrak{k} \mathfrak{t}_i(1 \otimes V_2))).$$

Consider the quaternion algebra

$$Q = k1 \oplus \bigoplus_{i=0}^2 kx_i,$$

with $x_i^2 = -1$ for any $i = 0, 1, 2$, and $x_i x_{i+1} = -x_{i+1} x_i = x_{i+2}$ for any $i$ (indices modulo 3). Its norm $N$ satisfies $N(1) = N(x_i) = 1$, $i = 0, 1, 2$, and $N(1, x_i) = N(x_i x_j) = 0$ for any $i \neq j$ ($N(a, b) = N(a + b) - N(a) - N(b)$).

Since the characteristic of $k$ is 3, $N(1 + x_1 + x_2) = 0$, hence $N$ represents 0 and $Q$ is the split quaternion algebra. That is, $Q$ is isomorphic to Mat$_2(k)$. In particular, $Q_0 = (k1)^\perp = kx_0 \oplus kx_1 \oplus kx_2$ is a Lie algebra under the commutator $[x, y] = xy - yx$, which is isomorphic to $\mathfrak{sl}_2$.

The subspace $\bigoplus_{i=0}^2 kx_i(1 \otimes 1)$ in $\mathfrak{g}_0$ is a subalgebra, and

$$[\mathfrak{i}_i(1 \otimes 1), \mathfrak{i}_{i+1}(1 \otimes 1)] = \mathfrak{i}_{i+2}(1 \otimes 1),$$

while $[x_i, x_{i+1}] = 2x_{i+2} = -x_{i+2}$. Therefore, $\bigoplus_{i=0}^2 kx_i(1 \otimes 1)$ is isomorphic to $Q_0$ under the linear map that takes $\mathfrak{i}_i(1 \otimes 1)$ to $-x_i$.

As vector spaces, there is a natural isomorphism

$$\Gamma_0 : \mathfrak{g}_0 \rightarrow (\mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2) \oplus Q_0) \oplus (V_1 \otimes V_2 \otimes Q_0),$$

which is the identity on $\mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2)$, and such that

$$\Gamma_0(\mathfrak{i}_i(1 \otimes 1)) = -x_i, \quad \Gamma_0(\mathfrak{i}_i(1 \otimes 2)) = u_1 \otimes u_2 \otimes x_i,$$

for any $i = 0, 1, 2$ and $u_1, u_2 \in V_1, u_2 \in V_2$.

The Lie bracket in $\mathfrak{g}_0$ is then transferred to the right hand side as follows:

- $\mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2) \oplus Q_0$ is a Lie subalgebra with componentwise bracket.
- $[s_1, u_1 \otimes u_2 \otimes p] = s_1(u_1) \otimes u_2 \otimes p$, for any $s_1 \in \mathfrak{sp}(V_1)$, $u_1 \in V_1$, $i = 1, 2$, and $p \in Q_0$.
- $[q, u_1 \otimes u_2 \otimes p] = u_1 \otimes u_2 \otimes [q, p]$, for any $u_i \in V_i$, $i = 1, 2$, and $p, q \in Q_0$.
- $[\mathfrak{i}_i, u_1 \otimes u_2 \otimes p] = \mathfrak{i}_i(u_1) \otimes u_2 \otimes [\mathfrak{i}_i, p]$, for any $u_i \in V_i$, $i = 1, 2$, and $p, q \in Q_0$.
- $[\mathfrak{i}_i, \mathfrak{i}_j(u_1 \otimes u_2)] = \mathfrak{i}_{i+j}(u_1 \otimes u_2)$ and $[\mathfrak{i}_i(1 \otimes 1), \mathfrak{i}_{i+2}(u_1 \otimes u_2)] = -\mathfrak{i}_{i+1}(u_1 \otimes u_2)$ in $\mathfrak{g}_0$ (see Section 3).
- For any $u_i, u'_i \in V_i$, $i = 1, 2$, and $p, p' \in Q_0$:

$$[u_1 \otimes u_2 \otimes p, u'_1 \otimes u'_2 \otimes p'] = -\langle u_2 | u'_2 \rangle N(p, p') \gamma_{u_1, u'_1} - \langle u_1 | u'_1 \rangle N(p, p') \gamma_{u_2, u'_2} - \langle u_1 | u'_1 \rangle \langle u_2 | u'_2 \rangle [p, q].$$

This is because $[\mathfrak{i}_i(u_1 \otimes u_2), \mathfrak{i}_j(u'_1 \otimes u'_2)] = \langle u_2 | u'_2 \rangle \gamma_{u_1, u'_1} + \langle u_1 | u'_1 \rangle \gamma_{u_2, u'_2}$ (note that $S_{1,2} = k1 \oplus V$, and $\sigma_{u, u'} = -\gamma_{u, u'}$ in Section 3), $N(1, 1) = 2 = -1$ and $[\mathfrak{i}_i(u_1 \otimes u_2), \mathfrak{i}_{i+1}(u'_1 \otimes u'_2)] = -\mathfrak{i}_{i+2}(u_1 \otimes u'_1) \otimes (u_2 \otimes u'_2) = -\mathfrak{i}_{i+2}(u_1 \otimes u'_1) \otimes (u_2 \otimes u'_2) = -\mathfrak{i}_{i+2}(u_1 \otimes u'_1) \otimes (u_2 \otimes u'_2)$.
\[-\langle u_1 | u'_1 \rangle \langle u_2 | u'_2 \rangle u_{i+2} (1 \otimes 1),\] which corresponds to \([u_1 \otimes u_2 \otimes x_i, u'_1 \otimes u'_2 \otimes x_{i+1}] = \langle u_1 | u'_1 \rangle \langle u_2 | u'_2 \rangle x_{i+2} (\langle x_i, x_{i+1} \rangle = -x_{i+2}).\]

Now, as vector spaces, there is also a natural isomorphism:

\[
\begin{align*}
\Gamma_1 : g_1 &\rightarrow (V_1 \oplus V_2) \otimes Q, \\
\quad u_1 &\mapsto u_1 \otimes 1, \\
\quad u_2 &\mapsto -u_2 \otimes 1, \\
\quad \iota_3 (u_1 \otimes 1) &\mapsto -u_1 \otimes x_i, \\
\quad \iota_3 (1 \otimes u_2) &\mapsto u_2 \otimes x_i,
\end{align*}
\]

for \(i = 0, 1, 2,\) \(u_1 \in V_1,\) and \(u_2 \in V_2.\) Under \(\Gamma_0\) and \(\Gamma_1,\) the Lie bracket \(g_0 \times g_1 \rightarrow g_1\) is transferred to:

- \([s_i, (u_1 + u_2) \otimes q] = s_i (u_i) \otimes q\) for any \(s_i \in \mathfrak{sp}(V_i), u_i \in V_i, i = 1, 2\) and \(q \in Q.\)
- \([p, (u_1 + u_2) \otimes q] = (u_1 + u_2) \otimes pq\) for any \(p \in Q_0, q \in Q,\) and \(u_i \in V_i, i = 1, 2.\)
- \([u_1 \otimes u_2 \otimes p, u'_1 \otimes q] = -\langle u_1 | u'_1 \rangle u_2 \otimes pq\) for any \(u_1, u'_1 \in V_1, u_2 \in V_2, p \in Q_0\) and \(q \in Q.\)
- \([u_1 \otimes u_2 \otimes p, u'_2 \otimes q] = \langle u_2 | u'_2 \rangle u_1 \otimes pq\) for any \(u_1 \in V_1, u_2, u'_2 \in V_2,\)

for any \(u_i, u'_i \in V_i, i = 1, 2,\) and \(q, q' \in Q.\) Here \(-\) denotes the canonical involution in the quaternion algebra \(Q\) (that is, the symplectic involution in Mat_{2}(k)).

Therefore, the even and odd parts of \(g = g(S_{1,2}, S_{1,2})\) can be identified to

\[
g_0 = \left( \mathfrak{sp}(V_1) \oplus \mathfrak{sp}(V_2) \oplus \mathfrak{sl}_2 \right) \otimes (V_1 \otimes V_2 \otimes \mathfrak{sl}_2),
\]

\[
g_1 = (V_1 \oplus V_2) \otimes \mathfrak{gl}_2.
\]

The action of \(\mathfrak{sl}_2\) on \(g_0\) is given by left multiplication, hence \(g_0\) decomposes, as a module for \(\mathfrak{sl}_2,\) as \(g_0 = j_1 \oplus j_{-1},\) where \(j_1\) (respectively \(j_{-1}\)) consists of the \(2 \times 2\) matrices with zero second (resp. first) column.

Let \(\{v_i, w_i\}\) be a symplectic basis of \(V_i, i = 1, 2,\) as considered so far, and let \(\{h = E_{11} - E_{22}, e = E_{12}, f = E_{21}\}\) be the standard basis of \(\mathfrak{sl}_2\) \((E_{ij}\) denotes the \(2 \times 2\) matrix with entry \((i, j)\) equal to 1 and all the other entries 0). Note that \(\{E_{11}, E_{21}\}\) is a basis of \(j_1\), while \(\{E_{12}, E_{22}\}\) is a basis of \(j_{-1} \).

Consider here the free \(\mathbb{Z}\)-module \(F\) with basis \(\epsilon_1, \epsilon_2, \epsilon, \delta.\) The Lie superalgebra \(g = g(S_{1,2}, S_{1,2})\) is \(F\)-graded by assigning degree \(\epsilon_i\) to \(w_i\) and \(-\epsilon_i\) to \(v_i\) as usual, \(i = 1, 2,\) degree \(\epsilon + \delta\) to \(E_{11}\) \((\in j_1),\) \(-\epsilon + \delta\) to \(E_{21}\) \((\in j_1),\) \(\epsilon - \delta\) to \(E_{12}\) \((\in j_{-1}),\) and \(-\epsilon - \delta\) to \(E_{22}\) \((\in j_{-1}).\) Then it follows that, for instance, the degree of \(v_1 \otimes E_{22}\) is \(-\epsilon_1 - \epsilon - \delta,\) or that the degree of \(e = E_{12} \in \mathfrak{sl}_2\) is \(2\epsilon.\)
The sets of nonzero even and odd degrees are:

\[ \Phi_0 = \{ \pm 2\epsilon_1, \pm 2\epsilon_2, \pm 2\epsilon, \pm \epsilon_1 \pm \epsilon_2, \pm \epsilon_1 \pm 2\epsilon \pm 2\epsilon \} , \]

\[ \Phi_1 = \{ \pm \epsilon_1 \pm \epsilon \pm \delta, \pm \epsilon_2 \pm \epsilon \pm \delta \} . \]

The abelian subalgebra \( h = kh_1 \oplus kh_2 \oplus kh \) (\( h_i = \gamma v_i, w_i \in \mathfrak{sp}(V_i), i = 1, 2, \) as usual) is a Cartan subalgebra of \( g \) and the hypotheses of Theorem 4.7 are satisfied relative to the rank 3 matrix homomorphism of abelian groups:

\[ R : F \rightarrow h^* \]

\[ \epsilon_1 \mapsto R(\epsilon_1) : (h_1 \mapsto 1, h_2 \mapsto 0, h \mapsto 0) , \]

\[ \epsilon_2 \mapsto R(\epsilon_2) : (h_1 \mapsto 0, h_2 \mapsto 1, h \mapsto 0) , \]

\[ \epsilon \mapsto R(\epsilon) : (h_1 \mapsto 0, h_2 \mapsto 0, h \mapsto 1) , \]

\[ \delta \mapsto 0 , \]

is precisely the set of roots of \( g \) relative to \( h \).

Consider the lexicographic order on \( F \) with \( \delta > \epsilon_1 > \epsilon_2 > \epsilon > 0 \), then

\[ \Phi = \{ \alpha_1 = 2\epsilon_2, \alpha_2 = \epsilon_1 - \epsilon_2 - 2\epsilon, \alpha_3 = 2\epsilon, \alpha_4 = \delta - \epsilon_1 - \epsilon \} \]

is a linearly independent set in \( F \) with \( \Phi = \Phi_0 \cup \Phi_1 \subseteq \mathbb{Z} \Pi \), so that any positive element in \( \Phi \) is a sum of elements in \( \Pi \). Consider the elements:

\[ E_1 = \gamma_{v_2, w_2} , \quad E_2 = w_1 \otimes v_2 \otimes E_{21} , \quad E_3 = e , \quad E_4 = v_1 \otimes E_{21} , \]

\[ F_1 = -\gamma_{v_2, w_2} , \quad F_2 = -v_1 \otimes w_2 \otimes E_{12} , \quad F_3 = f , \quad F_4 = w_1 \otimes E_{12} . \]

Then:

\[ H_1 = [E_1, F_1] = \gamma_{v_2, w_2} = h_2 , \]

\[ H_2 = [E_2, F_2] = -\gamma_{v_1, w_1} + \gamma_{v_2, w_2} + h = -h_1 + h_2 + h , \]

\[ H_3 = [E_3, F_3] = h , \]

\[ H_4 = [E_4, F_4] = \gamma_{v_1, w_1} - h = h_1 - h , \]

and the hypotheses of Theorem 4.7 are satisfied relative to the rank 3 matrix (and Dynkin diagram):

\[
\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -2 & 2 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

Therefore:

**Proposition 5.18.** The Lie superalgebra \( g(S_{1,2}, S_{1,2}) \) is isomorphic to the centerless derived contragredient Lie superalgebra \( g'(A_{S_{1,2}, S_{1,2}}, \{4\})/\epsilon. \)

Moreover, \( \{ E_i, F_i, H_i : i = 1, 2, 3 \} \) generate the even part of \( g_0 \), the associated Cartan subalgebra is of type \( B_3 \), \( g_0 \) thus being isomorphic to the orthogonal Lie algebra \( \mathfrak{s}_7 \). The grading of \( g \) by \( F \) gives, in particular, a consistent \( \mathbb{Z} \)-grading by means of the projection \( F = \mathbb{Z} \epsilon_1 \oplus \mathbb{Z} \epsilon_2 \oplus \mathbb{Z} \epsilon \oplus \mathbb{Z} \delta \rightarrow \mathbb{Z} , \)

\[ n_1 \epsilon_1 + n_2 \epsilon_2 + n_3 \epsilon + n_4 \delta \mapsto n_4 , \]

where

\[ g = g_{-1} \oplus g_0 \oplus g_1 , \]

with

\[ g_0 = g_0 , \quad g_1 = (V_1 \oplus V_2) \otimes j_1 , \quad g_{-1} = (V_1 \oplus V_2) \otimes j_{-1} . \]
As modules over $g_0 = g_0$, $g_1$ and $g_{-1}$ are isomorphic, and $g_1$ is an irreducible module with highest weight $R(\epsilon_1 + \epsilon + \delta) = \omega_3$, so $g_1$ is the spin module for $g_0$. Hence:

**Proposition 5.19.** $g(S_{1,2}, S_{1,2})$ is a simple Lie superalgebra whose even part is isomorphic to $so_7$, and its odd part is the direct sum of two copies of the spin module for the even part.

**Corollary 5.20.** The Lie superalgebra $g(S_{1,2}, S_{1,2})$ is isomorphic to the Lie superalgebra in [Eld06b, Theorem 4.23(ii)], constructed in terms of a null orthogonal triple system.

5.8. $g(S_1, S_{1,2})$. The Lie superalgebra $g(S_1, S_{1,2})$ is a subsuperalgebra of $g(S_{1,2}, S_{1,2})$. Given the description of this latter superalgebra in 5.7, more specifically in 5.17, the even and odd parts of $g = g(S_1, S_{1,2})$ can be described as

$$g_0 = sp(V_2) \oplus sl_2 \left( \subseteq g(S_{1,2}, S_{1,2})_0 \right),$$

$$g_1 = V_2 \otimes gl_2 \left( \subseteq g(S_{1,2}, S_{1,2})_1 \right),$$

with

$$\Phi_0 = \{\pm 2\epsilon_2, \pm 2\epsilon\}, \quad \Phi_1 = \{\pm \epsilon_2 \pm \epsilon \pm \delta\}.$$

With the lexicographic order in which $\delta > \epsilon_2 > \epsilon > 0$,

$$\Pi = \{\alpha_1 = 2\epsilon_2, \alpha_2 = \delta - \epsilon_2 - \epsilon, \alpha_3 = 2\epsilon\},$$

with associated matrix and Dynkin diagram:

$$A_{S_1, S_{1,2}} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}, \quad \alpha_1 \quad \alpha_2 \quad \alpha_3$$

and the same arguments as in the previous section give:

**Proposition 5.21.** The Lie superalgebra $g(S_1, S_{1,2})$ is isomorphic to the centerless derived contragredient Lie superalgebra $g'(A_{S_1, S_{1,2}}, \{2\})/c$.

**Corollary 5.22.** $g(S_1, S_{1,2})$ is isomorphic to the projective special Lie superalgebra $psl(2, 2)$.

This is the only Lie superalgebra in the Extended Freudenthal’s Magic Square with a counterpart in characteristic 0.

Note that the even part $g_0$ is the direct sum of two copies of $sl_2$, while the odd part is a direct sum of two copies of the four dimensional irreducible module for $g_0$ which consists of the tensor product of the natural modules for each copy of $sl_2$. 
5.9. \( g(S_4, S_{1,2}) \). Here a description in terms of \( V(\sigma)'s \) is again possible. Three copies of \( V \) are needed for \( S_4 \) and an extra copy for \( S_{1,2} \). The indices 1, 2 and 3 will be used for the three copies related to \( S_4 \) and the index 4 for the extra copy attached to \( S_{1,2} \). Then

\[
g(S_4, S_{1,2}) = (\text{tr}(S_4) \otimes \text{tr}(S_{1,2})) \oplus (\oplus_{i=0}^2 \epsilon_i (S_4 \otimes S_{1,2}))
\]

where

\[
S_{S_4, S_{1,2}} = \{ \emptyset, \{4\}, \{1, 2\}, \{1, 2, 4\}, \{2, 3\}, \{2, 3, 4\}, \{1, 3\}, \{1, 3, 4\} \}.
\]

The nonzero even and odd degrees are:

\[
\Phi_0 = \{ \pm 2\epsilon_i : 1 \leq i \leq 4 \} \cup \{ \pm \epsilon_1 \pm \epsilon_2, \pm \epsilon_2 \pm \epsilon_3, \pm \epsilon_1 \pm \epsilon_3 \},
\]

\[
\Phi_1 = \{ \pm \epsilon_4, \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_4, \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4 , \pm \epsilon_1 \pm \epsilon_3 \pm \epsilon_4 \}.
\]

With the lexicographic order given by \( 0 < \epsilon_1 < \epsilon_2 < \epsilon_3 < \epsilon_4 \), the set of irreducible degrees is

\[
\Pi = \{ \alpha_1 = \epsilon_3 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_1, \alpha_3 = 2\epsilon_1, \alpha_4 = \epsilon_4 - \epsilon_2 - \epsilon_3 \},
\]

which is a linearly independent set over \( \mathbb{Z} \), and \( \Phi = \Phi_0 \cup \Phi_1 \) is contained in \( \mathbb{Z} \Pi \). The associated matrix and Dynkin diagram are:

\[
A_{S_4, S_{1,2}} = \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -2 & -1 \\
0 & -1 & 2 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix},
\]

and

\[
\alpha_1 \overrightarrow{a} \alpha_2 \overrightarrow{a} \alpha_3 \overrightarrow{a} \alpha_4
\]

\textbf{Proposition 5.23.} The Lie superalgebra \( g(S_4, S_{1,2}) \) is isomorphic to the contragredient Lie superalgebra \( g(A_{S_4, S_{1,2}}, \{4\}) \).

Also, the set of irreducible even degrees is

\[
\Pi_0 = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 = 2\epsilon_4 \},
\]

with associated Cartan matrix of type \( C_3 \oplus A_1 \). (This also follows from \( g(S_4, S_{1,2})_0 = g(S_4, S_1) \oplus \mathfrak{sp}(V_4) \).)

The odd part, which has dimension 26, is irreducible with highest weight \( R(\epsilon_2 + \epsilon_3 + \epsilon_4) = R(\epsilon_2 + \epsilon_3) + R(\epsilon_4) \), so that:

\textbf{Proposition 5.24.} \( g(S_4, S_{1,2}) \) is a simple Lie superalgebra with even part isomorphic to the direct sum \( \mathfrak{sp}_6 \oplus \mathfrak{sl}_2 \), and odd part the irreducible module which is the tensor product of the irreducible module \( V(\omega_2) \) for \( \mathfrak{sp}_6 \) (of dimension 13) , and the natural two dimensional module for \( \mathfrak{sl}_2 \).

The irreducible module \( V(\omega_2) \) for \( \mathfrak{sp}_6 \) can be described as follows. Let \( W \) be the natural six-dimensional module for \( \mathfrak{sp}_6 \), which is endowed with a nondegenerate alternating form \( \{.,.\} \), and take a symplectic basis \( \{a_i, b_i : i = 1, 2, 3 \} \). Consider the linear map \( \varphi : \mathbb{A}^2 W \rightarrow k, \ z_1 \wedge z_2 \mapsto \{z_1 | z_2 \} \). Then, since the characteristic of \( k \) is 3, \( k(\sum_{i=1}^3 a_i \wedge b_i) \) is a trivial submodule of \( \ker \varphi \), and \( \ker \varphi / k(\sum_{i=1}^3 a_i \wedge b_i) \) is an irreducible module with highest weight vector the class of \( b_2 \wedge b_3 \) modulo \( k(\sum_{i=1}^3 a_i \wedge b_i) \) (conventions as
and with the lexicographic order given by $0 < \epsilon < \epsilon^5$ and Dynkin diagrams are $\mathbb{Z}$, which is a set of irreducible degrees is $\Pi$. The associated matrix and Dynkin diagrams are

$$A_{S_8,S_{1,2}} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \alpha_1 \rightarrow \alpha_2 \rightarrow \alpha_3 \rightarrow \alpha_4 \rightarrow \alpha_5$$

and then:

**Proposition 5.25.** The Lie superalgebra $g(S_8,S_{1,2})$ is isomorphic to the contragredient Lie superalgebra $g(A_{S_8,S_{1,2}}, \{5\})$.

Also,

$$\Pi_0 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 = 2\epsilon_5\},$$

with associated Cartan matrix of type $F_4 \oplus A_1$ (this also follows from $g(S_8,S_{1,2})_0 = g(S_8,S_1) \oplus sp(V_5)$). The odd part, which has dimension 50, is irreducible with highest weight $R(\epsilon_5 + \epsilon_4 + \epsilon_3) = R(\epsilon_3 + \epsilon_4) + R(\epsilon_5)$. $R(\epsilon_5)$ is the highest weight of the natural two dimensional module for $sp(V_5) \cong sl_2$, while $R(\epsilon_3 + \epsilon_4)$ is the highest weight $\omega_4$ for the direct summand of $g(S_8,S_{1,2})_0$ isomorphic to $f_4$, the exceptional simple split Lie algebra of type $F_4$ (with the ordering of the roots given above). Thus:
Proposition 5.26. \( g(S_8, S_{1,2}) \) is a simple Lie superalgebra with even part isomorphic to \( f_4 \oplus \mathfrak{sl}_2 \), and odd part the irreducible module which is the tensor product of the natural two dimensional module for \( \mathfrak{sl}_2 \) and the 25-dimensional module of highest weight \( \omega_4 \) for \( f_4 \).

In characteristic \( > 3 \), the module \( V(\omega_4) \) for \( f_4 \) is usually described as the set of trace zero elements \( J_0 \) in the split exceptional Jordan algebra \( J = H_3(C) \), where \( C \) is the split octonion algebra. However, in characteristic 3, the identity matrix belongs to \( J_0 \), and \( J_0/k1 \) is the irreducible module \( V(\omega_4) \) in this case.

5.11. \( g(S_2, S_{1,2}) \). Take the basis \( \{e^+, e^-\} \) of the symmetric composition algebra \( S_2 \) as in 5.6 and identify \( \text{tri}(S_2) \) with \( t = \text{span}\{t_1, t_2\} \) as was done there. Take also three copies \( S_2^i \) (with basis \( \{e^+_i, e^-_i\} \) of \( S_2, i = 0, 1, 2 \). Here we consider the Lie superalgebra

\[
g = g(S_2, S_{1,2}) = (\text{tri}(S_2) \circ \text{tri}(S_{1,2})) \oplus (\oplus_{i=0}^2 (S_2 \circ S_{1,2})),
\]

so, with standard identifications:

\[
g_0 = (t \oplus \mathfrak{sp}(V)) \oplus (\oplus_{i=0}^2 S_2^i),
\]

\[
g_1 = V \oplus (\oplus_{i=0}^2 (S_2^i \circ V)).
\]

Consider the free \( \mathbb{Z} \)-module \( F \) freely generated by \( \delta_1, \delta_2 \) and \( \epsilon \). With the conventions in 5.6 \( g \) is \( F \)-graded and the sets of nonzero even and odd degrees are:

\[
\Phi_0 = \{ \pm 2\epsilon, \pm \delta_1, \pm \delta_2, \pm (\delta_1 + \delta_2) \}
\]

\[
\Phi_1 = \{ \pm \epsilon, \pm \delta_1 \pm \epsilon, \pm \delta_2 \pm \epsilon, \pm (\delta_1 + \delta_2) \pm \epsilon \}.
\]

With the lexicographic order in which \( \epsilon > \delta_1 > \delta_2 > 0 \),

\[
\Pi = \{ \alpha_1 = \delta_1, \alpha_2 = \delta_2, \alpha_3 = \epsilon - (\delta_1 + \delta_2) \}
\]

is a linearly independent set with \( \Phi = \Phi_0 \cup \Phi_1 \subseteq \mathbb{Z} \Pi \). Now consider the elements \( \{v, w\} \) denotes a symplectic basis of \( V \):

\[
E_1 = e^+_0 = \iota_0(1 \otimes e^+) \quad E_2 = e^+_1 \quad E_3 = e^+_2 \otimes w = \iota_2(e^+ \otimes w),
\]

\[
H_1 = 2(t_1 - t_2) \quad H_2 = 2(t_1 - t_2) \quad H_3 = (t_1 - t_2) - \gamma_{v,w},
\]

\[
F_1 = \xi_1 e^-_0 \quad F_2 = \xi_2 e^-_1 \quad F_3 = \xi_3 e^-_2 \otimes v,
\]

where \( \xi_1, \xi_2, \xi_3 \) are suitable scalars so as to have \( [E_i, F_j] = \delta_{ij}H_i \) for any \( i, j \).

With these elements, \( g(S_2, S_{1,2}) \) is \( \mathbb{Z} \)-graded, by assigning degree 1 to \( E_1, E_2, E_3 \) and degree \( -1 \) to \( F_1, F_2, F_3 \), and the hypotheses of Theorem 5.25 are satisfied relative to the rank 2 matrix (with associated Dynkin diagram):

\[
A_{S_2, S_{1,2}} = \begin{pmatrix}
2 & -1 & -1 \\
-1 & 2 & -1 \\
1 & 1 & 0
\end{pmatrix}
\]

Therefore:

Proposition 5.27. The Lie superalgebra \( g(S_2, S_{1,2}) \) is isomorphic to the centerless contragredient Lie superalgebra \( g(A_{S_2, S_{1,2}}, \{3\})/c. \) It is not simple, but its derived superalgebra is a simple ideal of codimension 1.
Also,
\[ \Pi_0 = \{ \beta_1 = \delta_1, \beta_2 = \delta_2, \beta_3 = 2\epsilon \} \]
is a linearly independent set with \( \Phi_0 \subseteq \mathbb{Z}\Pi_0 \) and, with already used arguments, it is obtained that the associated Cartan matrix is of type \( A_2 \oplus A_1 \), \( g_0 \) thus being isomorphic to the direct sum of the centerless contragredient Lie algebra \( g(A_2)/\mathfrak{c} \), which is isomorphic to \( \mathfrak{pgl}_3 \), and of \( \mathfrak{sl}_2 \). The highest degree in \( g_1 \) is \( (\delta_1 + \delta_2) + \epsilon \), and \( g_1 \) is easily seen to be irreducible. Hence:

**Proposition 5.28.** \( g(S_2, S_{1,2}) \) is a Lie superalgebra with even part isomorphic to the direct sum of \( \mathfrak{pgl}_3 \) and \( \mathfrak{sl}_2 \), and odd part isomorphic, as a module for the even part, to the tensor product of the “adjoint module” \( \mathfrak{psl}_3 \) of \( \mathfrak{pgl}_3 \) by the natural two dimensional module for \( \mathfrak{sl}_2 \).

Table 4 summarizes the information on the even and odd parts of the Lie superalgebras in the extended Freudenthal Magic Square obtained in this section. The notation \( (n) \) will indicate a module of dimension \( n \).

|       | \( S_{1,2} \)                                      | \( S_{4,2} \)                                      |
|-------|--------------------------------------------------|--------------------------------------------------|
| \( S_1 \) | \( \mathfrak{psl}_{2,2} \)                       | \( \mathfrak{sp}_6 \oplus (14) \)                |
| \( S_2 \) | \( (\mathfrak{pgl}_3 \oplus \mathfrak{sl}_2) \oplus (\mathfrak{psl}_3 \otimes (2)) \) | \( \mathfrak{pgl}_6 \oplus (20) \)               |
| \( S_4 \) | \( (\mathfrak{sp}_6 \oplus \mathfrak{sl}_2) \oplus ((13) \otimes (2)) \) | \( \mathfrak{so}_{12} \oplus \mathfrak{spin}_{12} \) |
| \( S_8 \) | \( (\mathfrak{f}_4 \oplus \mathfrak{sl}_2) \oplus ((25) \otimes (2)) \) | \( \mathfrak{e}_7 \oplus (56) \)                 |
| \( S_{1,2} \) | \( \mathfrak{so}_7 \oplus 2\mathfrak{spin}_7 \) | \( \mathfrak{sp}_8 \oplus (40) \)                |
| \( S_{4,2} \) | \( \mathfrak{so}_{13} \oplus \mathfrak{spin}_{13} \) |                                          |

Table 4. Even and odd parts

With the exception of \( \mathfrak{psl}_{2,2} \), none of these Lie superalgebras have a counterpart in characteristic 0. Corollary 5.11 shows that \( g(S_{4,2}, S_{4,2}) \) has already appeared in [Elda, Theorem 3.1(ii)], while Corollary 5.20 shows that \( g(S_{1,2}, S_{1,2}) \) is the Lie superalgebra that appears in [Eld06b, Theorem 4.23(ii)].

In a forthcoming paper, it will be shown that the Lie superalgebras \( g(S_r, S_{1,2}) \) and \( g(S_r, S_{4,2}) \), for \( r = 1, 4 \) and 8, and their derived subalgebras for \( r = 2 \), are precisely the simple Lie superalgebras defined in [Eld06c, Theorem 4.23(ii)] in terms of symplectic and orthogonal triple systems related to simple Jordan algebras of degree 3.

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