Spatially Differential Forms of Lenz Law

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Two sets of spatially differential formulas of Lenz law on electromagnetic inductance are presented. They are a cut-magnetic flux induced voltage, which instantaneously results from cutting magnetic flux as a conductor moving with respect to an external magnetic field, and a wave-induced voltage due to the interaction of a conductor or charged particles with the arriving electromagnetic wave, which is originated from a changing magnetic flux source. Upon the Lenz differential forms, the induced electrical field strength, relevant properties and their application are discussed.

Lenz law, as an empirical rule, sums production and property of inductive electromotive force in a conductor loop when an interaction occurs between the circle loop and a steady magnetic field or changeable magnetic field. It is expressed in an integral formula of induced electromotive voltage on whole conductor loop and can be applied for a totally induced electrical voltage calculation in steady-symmetric circumstances [1],

\[ \varepsilon(t) = \oint E_c(t) \cdot dl = -\oint \frac{d\Phi}{dt} . \]  

However in physical practice there are variety of induction phenomena which occur in a non-steady status with asymmetric circumstances; or the induced electrical strength are needed to be calculated upon an interestingly particular point, segment of conductor on differently spatial condition as such where the integral form Eq.(1) can not be in use [2][3]. Moreover the integrated form will cover some explicit details of physical properties that can be revealed and applied through its spatially differential forms. Like the important role of Biot-Savart theorem (or differential form of Ampere theorem) on magnetic field calculation[1][4], therefore, the presentation of the differential form of Lenz law will be significant to time-dependent calculation of induced electrical field.

Actually electromagnetic inductances occur on a spatial conductor element or charged particles much more often than on a whole conductor loop. Mathematically the integral form is merely an accumulated sum of spatial differential inductance results along the whole loop. And in viewpoint of physics, the causes of all electromagnetic inductance phenomena could be divided into two categories: cut-magnetic flux induced electrical field and wave-induced electrical field. The integral form is given as

\[ \varepsilon(t) = \int E_c(t) \cdot dl - \int E_w(t) \cdot dl = -\left[ \int \delta \dot{\Phi}_{dec}(t) + \int \delta \dot{\Phi}_{degw}(t) \right] \]  

Upon Eq.(2), for having differently physical properties the cut-magnetic flux induced field and wave-induced field on the conductor are classified into two categories. The former is related to energy conversion from mechanical kinetic energy to electrical energy, and the latter related to the conversion from an electromagnetic wave kinetic energy to an associated energy carried or possessed by a conductor or charged particles.

The spatial differential Form of the cut-magnetic field induced voltage can be given below,

\[ d\varepsilon_r(t) = - \left[ v_{de,b}(t) \times B(t) \right] \cdot dl \]

\[ = - E_c(t) \cdot dl \]

\[ = - \delta \dot{\Phi}_{dec}(t) . \]  

Where: \(-\frac{\pi}{2} \leq \alpha_c = \frac{|n_{de}(t)| n_{dl}(t) \cdot \hat{B}(t)}{2}, 0 \leq \delta \dot{\Phi}_{dec}(t) = E_c(t) \cdot n_{dl}(t) dl \).

\[ v_{de,b}(t) \] — relative velocity of conductor element \( dl \) with respect to \( B(t) \), \( v_{de,b} = v_{de} - v_b \);

\( E_c(t) \) — induced electrical field strength in conductor element by cutting magnetic flux; or induced electrical force per unit length along \( v_{de,b}(t) \times B(t) \) direction by cutting \( B(t) \) field.

As conductor element \( dl \) moves in external magnetic field the Lorentz action will cause an energy conversion from a non-electrical energy (such as kinetic energy) to an electrical energy. It is the action that results in an induced electrical field strength \( E_c(t) \) inside of the conductor, or an elemental induced electromotive force or voltage \( d\varepsilon_c(t) \) in conductor, its value depends upon induced electrical field strength, and size and orientation of the conductor. The derivation is following:
Here $\theta_e$: the angle between the direction of the magnetic flux density at the conductor location and that of $bsv_{dt\cdot b}(t)$ or $\theta_e = n(v_{ dt \cdot b} \times B)$ and

$$E_c(t) = v_{dt\cdot b}(t) \times B(t) = v_{dt\cdot b} B \left[ v_{dt\cdot b} \times B \right].$$ \hfill (5)

Take an integrated form of Eq. (4), there exists

$$\varepsilon_c(t) = \int_t^r \!d\varepsilon_{dtc}(t)$$

$$= - \int_t^r \! [v_{dt\cdot b}(t) \times B(t)] \cdot dl$$

$$= - \int_t^r \! E_c(t) \cdot dl$$

$$= - \int_t^r \! \delta \dot{\Phi}_{dtc}(t).$$ \hfill (6)

Eq. (6) reveal that at spatially different points of the conductor, Lorentz action exerted on the positive and negative charges within the conductor, the separately electromotive force in the opposite directions, is the only cause to form the cut-induced electrical field within a conductor.

Another type of electromagnetic inductance results from the interaction of the conductor or charged particles as electromagnetic load with the arriving electromagnetic wave. The energy of the electromagnetic radiation, originated from a changing magnetic flux source or a source of magnetic flux rate, will induce an associated electrical field at the conductor location and be transferred into an associated energy of the conductor or charged particles in their overlapped space.

$$E_{sw}(t') = E_{sw}(t' = t + T)$$

$$= \frac{n(r_{sd\cdot t}) \times n_b \dot{B}[r_s(t)] \cdot ds_{\perp b}dL}{4\pi r_{sd\cdot t}^2}$$

$$= \frac{n(r_{sd\cdot t}) \times n_b \dot{\Phi}_{sw}(t')dL}{4\pi r_{sd\cdot t}^2}$$

$$= \frac{n(r_{sd\cdot t}) \times \delta M(t)}{4\pi r_{sd\cdot t}^2}$$

$$= \frac{\dot{B}[r_s(t)] \cdot dV \sin\theta}{4\pi r_{sd\cdot t}^2} n [n(r_{sd\cdot t}) \times n_b]$$

$$= E_w(t') n_{E_w}(t'),$$

where, $E_{sw}(t')$ — the induced electrical field strength, or induced electrical force per unit length on a conductor along $n(r_{sd\cdot t}) \times \delta M(t)$ direction by electromagnetic wave originated from a micro magnetic flux rate element $\delta M(t)$ in an air and vacuum space;

$\delta M = B [r_s(t)] dV = n_b \dot{\Phi}_{sw}(t')dL = n_b \dot{B}[r_s(t)] ds_{\perp b}dL$ — a spatial element of magnetic flux rate’s; $n_b$ — the unit vector of magnetic flux density $B [r_s(t)]$ at $t$ instant and $r_s(t)$ position; $\theta_w$ — the angle between the direction of $B [r_s(t)]$ and that of the relative position vector $r_{sd\cdot t}$;

$n [n(r_{sd\cdot t}) \times n_b] = n [n(r_{sd\cdot t}) \times \delta M(t)] = n_{E_w}(t')$ in an air and vacuum space;

$$0 \leq \alpha_w = n_{E_w}(t') n_{dt\cdot w}(t') \leq \pi,$$ \hfill (7)

$$0 \leq \delta \dot{\Phi}_{dt\cdot w}(t') = E_{w}(t') \cdot n_{dt\cdot w}(t')dL;$$

$$r_{sd\cdot t} = r_{sd\cdot t} n(r_{sd\cdot t})$$ — relative position vector between a micro magnetic flux’s rate spatial element $\delta M = B [r_s(t)] dV$ and a micro conductor element $dl$; $t, t'$ — emitting instant and arriving instant respectively of a micro electromagnetic wave element originated from the magnetic flux’s rate source, here $t' - t = T(t) = T(t')$. 


So the generally spatial differential form of the wave-induced voltage will be given below,

\[ d\varepsilon_w(t') = - \sum n(r_{jsw}(t')) \cdot dl \]

\[ = - \sum \frac{n(r_{jsw}) \times n_0 \delta \hat{\Phi}(t_j) dL_j}{4\pi r_{jsw}^2} \cdot dl \]

\[ = - \sum \frac{n(r_{jsw}) \times \hat{B}(r_{jsw}) dV_j}{4\pi r_{jsw}^2} \cdot dl \]

\[ = - \sum \frac{n(r_{jsw}) \times \delta M_j(t_j)}{4\pi r_{jsw}^2} \cdot dl \]

\[ = -E_w(t') \cdot dl \]

\[ = -\delta \hat{\Phi}_{dtw} [r_{dt}(t')] \).

Eq. (7,8) manifest a general explicitly formula of purely wave-induced electrical field strength at a detector position where the charged particles or a conductor serve as a electromagnetic load to interact with an arriving electromagnetic wave. In the process an energy exchange will occur among the electromagnetic energy possessed by the load and wave respectively. It is known that the associated induced electrical strength is related to spatial propagating distribution, attenuation and time delay of the wave-energy.

For a steady wave-induction circumstance, the induced voltage element can be expressed:

\[ d\varepsilon_w(t') = -E_w(t') \cdot dl \]

\[ = - \sum \frac{n(r_{sdt}) \times \delta \hat{M}(t)}{4\pi r_{sdt}^2} \cdot dl \]

\[ = - \sum \frac{n(r_{sdt}) \times \hat{B}(t) dV}{4\pi r_{sdt}^2} \cdot dl \] .

Or when the spatial shape of the magnetic flux’s rate source is long tube like shape with small cross section area \( ds_{ijb} \) usually taken in practical cases, then the induced electrical force can be expressed as:

\[ E_w(t') = \sum \frac{n(r_{sdt}) \times \delta \hat{M}(t)}{4\pi r_{sdt}^2} \]

\[ = \sum \frac{n(r_{sdt}) \times \hat{B}(t) dV}{4\pi r_{sdt}^2} \]

\[ = \sum \frac{n(r_{sdt}) \times n_0 \hat{B}(t) ds_{jb} n_0 dL}{4\pi r_{sdt}^2} \]

\[ = \sum \frac{n(r_{sdt}) \times \delta \Phi(t)}{4\pi r_{sdt}^2} dL .

Following is an illustrating example of calculating wave-induced electrical field strength in steady-symmetry circumstance (Ref. Fig. 1).

![FIG. 1: Diagram of wave-induced electrical field by a micro magnetic flux rate element](image)

Take an integral form of Eq. (11), there then exists

\[ E_w(t') = \sum \frac{n(r_{sdt}) \times \delta \hat{M}(t')} \]

\[ = \sum \frac{n(r_{sdt}) \times \hat{B}(t') dV}{4\pi r_{sdt}^2} \]

\[ = \sum \frac{n(r_{sdt}) \times \hat{B}(t') ds_{jb} n_0 dL}{4\pi r_{sdt}^2} \]

\[ = \sum \frac{2n_0 \delta \hat{\Phi}(t') n_{k_w} dL}{4\pi r_{sdt}^2} \]

\[ = \frac{\delta \hat{\Phi}(t')}{2\pi r_0} n_{k_w} = E_w(t') n_{k_w} \]

it is \( |E_w(t')| = \frac{\delta \hat{\Phi}(t')}{2\pi r_0} \). So

\[ \varepsilon_w(t') = \sum \frac{\delta \hat{\Phi}(t')}{2\pi r_0} n_{k_w} \cdot dl = \int \frac{\delta \hat{\Phi}(t')}{2\pi r_0} n_{k_w} \cdot dl \]

\[ = - \int \frac{\delta \hat{\Phi}(t')}{2\pi r_0} d\theta = - \int \frac{\delta \hat{\Phi}(t')}{2\pi} d\theta = -\delta \hat{\Phi}(t') \].

Meanwhile by applying Eq. (11) in the case, the following equations can be derived.

\[ \varepsilon_w(t') = \oint d\varepsilon_w(t') = \oint E_w(t') \cdot dl = 2\pi r_0 E_w(t') = -\delta \hat{\Phi}(t') \]

and

\[ |E_w(t')| = \frac{\delta \hat{\Phi}(t')}{2\pi r_0} \]

Eq. (11, 12) testifies that the calculated results are consistent by using any one among the differential form of wave-induced electrical field and Lenz integrated form along a closed circular loop in a steady-symmetry induction situation with even-radial electromagnetic radiant wave from center of the loop.

Combining the two types of inductance together, there are the consequences on the total inductance forms as
two aspects.

The spatial differential forms can be applied mainly to any particular point, segment or non-closed loop of conductor with asymmetry circumstance where Lenz integral form can not offer the relevant explicit solution as its mathematical limit.

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1. Calculation of induced electrical field strength will be conducted on any particular point, segment or non-closed loop of conductor with asymmetry circumstance where Lenz integral form can not offer the relevant explicit solution as its mathematical limit.

2. Explicit and quantitative analysis on the energy conversion, spatial orientation or polarization of induced electrical field strength on a conductor element and their time rate in a dynamic process of electromagnetic inductance. In addition, the time delay of the wave-induced signal, due to the wave propagating from the source of magnetic flux rate to the conductor element, can be analyzed upon the spatial differential form.

Lenz experimental law can be reduced to two fundamentally spatial differential forms: cut-magnetic field induced electrical field and wave-induced electrical field on a conductor element. In both cases a relative velocity between the conductor and a contact magnetic field or magnetic flux rate field is necessary preposition for inductance occurrence. The spatial differential forms not only manifest explicitly the physical essence of electromagnetic inductance but also can be applied as the calculating formulas in variety of calculation and analysis of electromagnetic inductance phenomena where Lenz integral form can not offer the relevant explicit solution as its mathematical limit.

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