Modular properties of Eisenstein series and statistical mechanics

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The temperature inversion properties of the internal energy, $E$, on odd spheres, and its derivatives, together with their expression in elliptic terms, as expounded in previous papers, are extended to the integrals of $E$, thence making contact with the theory of modular forms with rational period functions.

I point out that the period functions of (holomorphic) Eisenstein series computed by Zagier were already available since the time of Ramanujan and I give a rederivation by contour integration. Removing both the Planck and Casimir terms gives a fully subtracted form of the series which allows a more elegant and compact treatment. I expound the relation to Eichler cohomology cocycles and also rewrite the theory in a distributional, Green function way.

Some historical and technical developments of the Selberg–Chowla formula are presented, and it is suggested that this be renamed the Epstein–Kober formula. On another point of historical justice, the work of Koshliakov on Dirichlet series is reprised. A representation of a ‘massive’ generalised Dirichlet series due to Berndt is also reproved, applied to the Epstein series and to a derivation of the standard statistical mode sum, interpreted as a Kronecker limit formula.

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1. Introduction.

In two previous works, [1], [2], we have investigated the thermal quantities for scalar and spinor fields on the space–time $T \times S^d$, where $d$ is odd. The conformal scalar internal energy, $E$, for example, is a linear combination of ‘partial’ energies, $\epsilon_t$, $t = 1, 2, \ldots$, (see below). These quantities possess a known behaviour under modular transformations and are central to elliptic function theory. In particular, a temperature inversion symmetry can be exhibited.

From basic elliptic properties, it was shown that $E$ could be expressed as a polynomial in $\epsilon_2$ and $\epsilon_3$, while, for the specific heat, and all higher derivatives, one has to include $\epsilon_1$, preferably via a modular covariant derivative. The free energy involves an integration and things are not so simple as obstructions arise to simple modular behaviour. It is this aspect that is explored in the present work. I use this thermal angle just to motivate the introduction of various quantities, the analysis of which allows us to make contact with various topics in the theory of modular forms. Some of the procedures can be given a physical terminology, which might be suggestive.

2. Mellin transforms and the period polynomial.

It is advantageous to begin from the basic Mellin transform, used e.g. by Malurkar and Hardy, for the Eisenstein series (or partial internal energy, [2] [1]),

$$
\epsilon_t(b) = -\frac{B_{2t}}{4t} + \sum_{n=1}^{\infty} \frac{n^{2t-1}q^{2n}}{1-q^{2n}} \\
= \frac{1}{2} \zeta_R(1-2t) + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \Gamma(s) \zeta_R(s) \zeta_R(s-2t+1) (2\pi b)^{-s},
$$

(1)

where $q = e^{i\pi \tau}$ and $c$ lies above $2t$. I am using $b = -i\tau = \beta/2\pi = 1/\xi$ as yet another convenient variable. Although, physically, $b$ is real, if we wish to consider general modular transformations then we must allow it to become complex.

The Mellin transform has been used systematically in finite temperature field theory and elsewhere in discussions of asymptotic limits. The survey by Elizalde et al, [3], is useful and I further mention, as being somewhat relevant here, Cardy [4], Kutasov and Larsen, [5]. Terras, [6] pp.55,229, can be consulted for aspects of the mathematical side. In number theory see Hardy and Littlewood, [7], and Landau, [8,9].
Equation (1) can be looked upon as a continuous expansion in $b$. Various limits in $b$ are uncovered by displacing the integration contour. The pole in the integrand at $s = 2t$ corresponds to the Planck term, while the remaining one at $s = 0$ corresponds to minus the Casimir value, which is the first term in (1). The high temperature limit ($b \to 0$) follows by pushing the contour to the left, beyond the Casimir pole, the contribution from which cancels the first term in (1) leaving just the effect of the Planck pole and a correction that tends to zero exponentially. Conversely, on moving the contour all the way to the right only the zero temperature Casimir term remains.

This cosmetic asymmetry can be avoided (if desired) by also extracting the Planck term, defining,

$$
\epsilon_t^{\text{sub}}(b) = \epsilon_t(b) - \frac{1}{2} \zeta_R(1 - 2t)(1 + (ib)^{-2t}),
$$

and writing

$$
\epsilon_t^{\text{sub}}(b) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \Gamma(s) \zeta_R(s) \zeta_R(s - 2t + 1)(2\pi b)^{-s},
$$

where now $c$, at the moment, lies anywhere between 0 and $2t$.

The quantity in the integrand is the Hecke $L$–series of the normalised Eisenstein modular form,

$$
L(G_t, s) = \zeta_R(s) \zeta_R(s - 2t + 1),
$$

satisfying the (typical) functional relation

$$
(2\pi)^{-s} \Gamma(s)L(G_t, s) = (-1)^t (2\pi)^{s - 2t} \Gamma(2t - s) L(G_t, 2t - s).
$$

I now consider a multiple integration with respect to $b$ of the internal energy. Actually for the free energy only one integration is needed but the general case is mathematically important. So integrate $\epsilon_t^{\text{sub}}$, $h$ times to give, using (3),

$$
\frac{1}{\Gamma(h)} \int_x^{\infty} db \frac{(b - x)^{h-1}}{(2\pi)^s} \epsilon_t^{\text{sub}}(b)
\quad = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \frac{\Gamma(s - h)}{(2\pi)^s} \zeta_R(s) \zeta_R(s - 2t + 1)x^{-s+h},
$$

with $h < c < 2t$.

I have imposed the boundary condition that each successive integral vanishes at $x = \infty$, i.e. at zero temperature.
More poles have been introduced into the integrand from the $\Gamma$ function and the contour has to be moved to the right to avoid them but we should stop when they pass beyond the contour at $\text{Re } s = c = 2t - \delta$, $0 < \delta < 1$. Hence the maximum $h$ can be is $2t - 1$.

I now set $h$ equal to this maximum, $2t - 1$, simply for the reason that it gives the ultimately recognisable, and important, quantity,

$$
\overline{\phi}_{2t}(x) \equiv \frac{2(2\pi)^{2t}}{\Gamma(2t - 1)} \int_{\infty}^{\infty} db (b - x)^{2t - 2} \epsilon_t \epsilon_t^\text{sub}(b)
$$

$$
= \frac{2}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \frac{\Gamma(s - 2t + 1)}{(2\pi)^{s-2t}} \zeta_R(s) \zeta_R(s - 2t + 1) x^{-s+2t-1},
$$

with $2t - 1 < c < 2t$. The first line is recognised as a Weyl fractional integral.

The significance of this quantity for statistical mechanics on spheres is admittedly obscure because the internal energy is a sum of the $\epsilon_t$ for different $t$, and the integral depends on $t$. Only for the 3–sphere, where $t = 2$ and $\overline{E} = \epsilon_2$, is there a direct relation but even then $\overline{\phi}_4$ is a third integral of $E$ having no particular thermal meaning. Nevertheless, its further analysis is not without interest as it connects with some salient mathematical concepts.

According to my general programme, interest lies in the behaviour under modular transformations. My treatment is equivalent to that of Malurkar, [10], cf also Guinand [11]. In contrast to the derivatives of $E$, there are obstructions to the modular invariance of the integrals and the period polynomials or, rather, period functions, provide a measure of these obstructions, as we will see.

Firstly under translations, $b \rightarrow b - i$, and, because of the Planck subtraction,

$$
\epsilon_t \epsilon_t^\text{sub}(b - i) - \epsilon_t \epsilon_t^\text{sub}(b) = (-)^{t+1} \frac{1}{2} \zeta_R(1 - 2t) \left( \frac{1}{(b - i)^{2t}} - \frac{1}{b^{2t}} \right),
$$

which implies, through (7), that

$$
\overline{\phi}_{2t}(x - i) - \overline{\phi}_{2t}(x) = 2i \frac{\zeta_R(2t)}{x(x - i)}. \tag{9}
$$

Under inversion, $x \rightarrow 1/x$, the integrand involves

$$
\epsilon_t \epsilon_t^\text{sub}(1/b) = (-)^t \frac{b^{2t}}{2t} \epsilon_t \epsilon_t^\text{sub}(b)
$$

since the regularising subtraction, (2), maintains the inversion property,

$$
\epsilon_t(1/b) = (-)^t \frac{b^{2t}}{2t} \epsilon_t(b),
$$

4
enjoyed by the full Eisenstein series in (1),

Therefore,

$$
\overline{\phi}_{2t}(1/x) = \frac{2(2\pi)^{2t}}{\Gamma(2t-1)} \int_{1/x}^{\infty} db \ (b - 1/x)^{2t-2} \epsilon_t^{\text{sub}}(b)
= - \frac{2(2\pi)^{2t}}{\Gamma(2t-1)} \int_0^{\infty} \frac{db'}{b'^2} \ (1/b' - 1/x)^{2t-2} \epsilon_t^{\text{sub}}(1/b')
= (-1)^t x^{2-2t} \frac{2(2\pi)^{2t}}{\Gamma(2t-1)} \int_0^{x} db \ (x - b)^{2t-2} \epsilon_t^{\text{sub}}(b).
$$

(10)

The amount by which $\overline{\phi}_{2t}$ violates the pure modular inversion property is the quantity,

$$
\overline{\phi}_{2t}(x) - (ix)^{2t-2} \overline{\phi}_{2t}(1/x) \equiv \overline{P}_t(x),
$$

or, from (7) and (10)

$$
\overline{P}_t(x) = \frac{2(2\pi)^{2t}}{\Gamma(2t-1)} \int_0^{\infty} db \ (x - b)^{2t-2} \epsilon_t^{\text{sub}}(b).
$$

(12)

The calculation of $\overline{P}_t$ from the Mellin contour form goes as follows. From (7),

$$
\overline{\phi}_{2t}(1/x) = \frac{2(2\pi)^{2t}}{\Gamma(2t-1)} \int_{c-i\infty}^{c+i\infty} ds \ \Gamma(s - 2t + 1) \ \zeta_R(s) \zeta_R(s - 2t + 1) x^{s-2t+1}
= \frac{2}{x^{2t-2}} \frac{1}{2\pi i} \int_{2t-c-i\infty}^{2t-c+i\infty} ds \ \frac{\Gamma(1 - s)}{(2\pi)^{-s}} \zeta_R(2t - s) \zeta_R(1 - s) x^{2t-s-1},
$$

(13)

by setting $s \to 2t - s$.

The $\zeta$-functional equation (one can also, more easily, use (5)) leads to,

$$
\zeta_R(s) \zeta_R(s - 2t + 1)
= \frac{2\Gamma(1 - s)}{(2\pi)^{1-s}} \sin(\pi s/2) \zeta_R(1 - s) \frac{2\Gamma(2t - s)}{(2\pi)^{2t-s}} \sin(\pi (s - 2t + 1)/2) \zeta_R(2t - s)
= \frac{4\Gamma(1 - s)\Gamma(2t - s)}{(2\pi)^{2t+1-2s}} \sin(\pi s/2) \sin(\pi (s - 2t + 1)/2) \zeta_R(1 - s) \zeta_R(2t - s)
= (-1)^t \frac{2\Gamma(1 - s)\Gamma(2t - s)}{(2\pi)^{2t+1-2s}} \sin(\pi s) \zeta_R(1 - s) \zeta_R(2t - s)
= \frac{(-1)^{t-1}\Gamma(1 - s)}{(2\pi)^{2t-2s} \Gamma(s - 2t + 1)} \zeta_R(1 - s) \zeta_R(2t - s),
$$
and so (13) becomes,

$$
\phi_{2t}(1/x) = \frac{2(-1)^{t-1}}{x^{2t-2}} \frac{1}{2\pi i} \int_{2t-c-i\infty}^{2t-c+i\infty} ds \frac{\Gamma(s-2t+1)}{(2\pi)^{s-2t}} \zeta_R(s) \zeta_R(s-2t+1) x^{2t-s-1}.
$$

(14)

I then find the alternative expression for the obstruction, $\mathcal{P}_t(x)$, defined in (11) or (12),

$$
\mathcal{P}_t(x) = \frac{1}{\pi i} \left( \int_{c-i\infty}^{c+i\infty} ds \frac{\Gamma(s-2t+1)}{(2\pi)^{s-2t}} \zeta_R(s) \zeta_R(s-2t+1) x^{2t-s-1} \right) ds \frac{\Gamma(s-2t+1)}{(2\pi)^{s-2t}} \zeta_R(s) \zeta_R(s-2t+1) x^{2t-s-1}.
$$

Translating the contours so that they mutually cancel leaves the contributions of the intervening poles i.e. those between 0 and 2t. Thus

$$
\mathcal{P}_t(x) = \frac{1}{\pi i} \phi_C ds \frac{\Gamma(s-2t+1)}{(2\pi)^{s-2t}} \zeta_R(s) \zeta_R(s-2t+1) x^{2t-s-1}
$$

$$
= (-1)^{t-\frac{1}{2}} \phi_C ds \zeta_R(s) \zeta_R(2t-s) \text{cosec}(\pi s/2) x^{2t-s-1}
$$

$$
= 2\pi \zeta_R(2t-1)((ix)^{2t-2} - 1) - 4i \sum_{j=1}^{t-1} \zeta_R(2j) \zeta_R(2t-2j)(ix)^{2t-2j-1}
$$

$$
= 2\pi \zeta_R(2t-1)((ix)^{2t-2} - 1) - i(-1)^t (2\pi)^t \sum_{j=1}^{t-1} \frac{B_{2j} B_{2t-2j}}{(2j)! (2t-2j)!} (ix)^{2t-2j-1},
$$

(15)

which is a polynomial of degree $(2t-2)$ and is the final, main result of this section. The odd powers come from the poles of the cosec while the two even powers arise from the poles of the Riemann $\zeta$–functions. $\mathcal{P}_t(x)$ is real if $x$ is.

Setting $x$ to zero in (15) and (12) produces the specific formulae,

$$
\zeta_R(2t-1) = -\frac{2(2\pi)^{2t-1}}{\Gamma(2t-1)} \int_0^{\infty} db b^{2t-2} \epsilon_t^{sub}(b)
$$

$$
= -(-1)^t \frac{2(2\pi)^{2t-1}}{\Gamma(2t-1)} \int_0^{\infty} db \epsilon_t^{sub}(b),
$$

(16)

using inversion. These can be checked numerically. More generally, expanding (12) gives the other nonzero moments of $\epsilon_t^{sub}(b)$,

$$
\int_0^{\infty} db b^{2j-1} \epsilon_t^{sub}(b) = \frac{(-1)^j}{8j(t-j)} B_{2j} B_{2t-2j}, \quad 1 < j < t - 1.
$$

(17)
The periods given by Kohnen and Zagier, [12], are identical to (16) and (17) and I now have made contact with this mathematical notion.

The periods of a cusp modular form, \( F \), of weight \( 2t \) are the numbers (e.g. Lang [13]),
\[
r_n(F) = \int_0^\infty db \, F(ib) \, b^n, \quad 0 \leq n \leq 2t - 2. \tag{18}
\]

The end points are cusps where \( F \) vanishes (exponentially) to ensure convergence. The restriction on \( n \) is a polynomial one (see [13].)

It is important for the following to assemble the periods into a period polynomial, which, as usual, is a handier quantity,
\[
r_F(x) = \sum_{n=0}^{2t-2} (-1)^n \binom{2t-2}{n} r_n(F) \, x^{2t-2-n} \tag{19}
\]
\[
= \int_0^\infty db \, (x-b)^{2t-2} F(ib).
\]

In order to extend the notion to non–cusp forms (such as Eisenstein series) one course (see Zagier, [14] and Kohnen and Zagier, [12] §4), is to continue the integer, \( n \), into the complex plane by employing the Hecke \( L \)-series of \( F \) via the Mellin transform,
\[
L(F,s) = \frac{(2\pi)^s}{\Gamma(s)} \int_0^\infty dt \, (F(it) - F(i\infty)) \, t^{s-1}, \tag{20}
\]
which converges for \( s > 2t - 2 \) and, continued in \( s \), satisfies the functional relation,
\[
(2\pi)^{-s} \Gamma(s) \, L(F,s) = (-1)^t (2\pi)^{s-2t} \Gamma(2t-s) \, L(F,2t-s), \tag{21}
\]
by virtue of the (inversion) modularity of \( F \), and conversely. This is usually attributed to Hecke, [15] (but see the Appendix). Terras, [6] p.229, gives a useful pedagogical treatment of this topic.

Then the periods could be defined, in general, by, [12],[14],
\[
r_n(F) = \lim_{s \to n+1} \frac{\Gamma(s)}{(2\pi)^s} \, L(F,s) \tag{22}
\]
\[
\equiv \lim_{s \to n+1} L^* (F,s),
\]
because this coincides with the integral expression, (18), for cusp forms.

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\(^2\) I prefer the normalisation of Kohnen and Zagier, [12].
The explicit values of \( r_n \) \((n = 0, \ldots, 2t - 2)\) for Eisenstein series (which is all that is required because of a decomposition theorem for modular form space) are easily computed from (4) and are given in [12]. They agree with (16) and (17), as already noted.

However, the Dirichlet series, \( L^*(F, s) \) has poles at \( s = 0 \) and \( s = 2t \). A means of incorporating these values is given by Zagier, [14], using the period polynomial. One notes that the summation in the cusp period polynomials can be extended,

\[
 r_F(x) = \sum_{n \in \mathbb{Z}} (-1)^n \binom{2t - 2}{n} r_n(F) x^{2t-2-n}
 = \sum_{n \in \mathbb{Z}} (-1)^n \binom{2t - 2}{n} i^{n+1} L^*(F, n+1) x^{2t-2-n},
\]

because the binomial coefficient vanishes outside the range \( n = 0, \ldots, 2t - 2 \). For non–cusp forms, however, the sum extends from \(-1\) to \( n = 2t - 1 \), the nonzero end values arising from the singularities of \( L^*(F, s) \) mentioned above. The resulting expression can then be taken as a definition of the period polynomial of a non–cusp form.

Again, for the Eisenstein series, \( G_{2t} = \epsilon_t \), the computation is straightforward, [14], and (23) with (4) yields, \((n = 2j - 1)\),

\[
 r_G(x) = \frac{(2t - 2)!}{2(2\pi)^{2t}} \left( 2\pi \zeta_R(2t - 1) \left( (-1)^{t-1} x^{2t-2} - 1 \right) - \right)
 (2\pi)^t \sum_{j=0}^{t} (-1)^{t-j} \frac{B_{2j} B_{2t-2j}}{(2j)! (2t-2j)!} x^{2j-1},
\]

which is \(1/x\) times a polynomial. For comparison with [14], Zagier’s \( X \) equals my \( ix \).

I now comment on this construction. The periods, and period polynomials, for non–cusp forms are defined by analogy to those of cusp forms, the justification being that the expressions, for \( F(i\infty) \) equal to zero, reduce to those for cusp forms. The means to attain this is not unique. Instead of the ‘zeta–function’ regularisation, as in the definition of \( L \), (20), I employ the \textit{subtracted} form, as in (2),

\[
 F_{\text{sub}} \equiv F - a_0 - \frac{a_0}{\tau^{2t}}, \quad a_0 = F(i\infty),
\]

whose moments are non–infinite and provide an alternative definition of the periods of a non–cusp form which equally reduces to that for cusp forms when \( a_0 = 0 \). This
subtraction maintains inversion modularity, but violates translational, whereas the subtraction of just $a_0$ does the reverse. Making use of my earlier considerations of Eisenstein series, see (12), I define the subtracted period polynomial as an integral,

$$r^\text{sub}_F(x) = \int_0^\infty db \ (b-x)^{2t-2} F^\text{sub}(b).$$

For Eisenstein series, (12) shows that $r^\text{sub}_G(x)$ is proportional to $\bar{P}_t(x)$ whose explicit form is given in (15).

It is necessary to compare this definition with that of Zagier, (24). The difference is the absence of the ‘end point’ terms proportional to $1/x$ and $x^{2t-1}$. This can be remedied by including, in the terminology of this paper, the regularised contributions from the subtracted terms, the Casimir and Planck terms, which can be accomplished simply by expanding the closed contour in (15) to include the corresponding poles at $s = 0$ and $s = 2t$. This just extends the sum by the two end points giving the well known quantity,

$$\bar{R}_t(x) = 2\pi \zeta_R(2t-1)((ix)^{2t-2} - 1) - 4i \sum_{j=0}^{t} \zeta_R^{2j}(2t-2j)(ix)^{2t-2j-1}, \quad (25)$$

which agrees, up to a factor, with Zagier’s expression given in (24).

Furthermore, the use of the subtracted form, $F^\text{sub}$, allows one to give a cleaner integral expression for the enlarged period polynomial than that in [14], i.e.,

$$r^\text{enl}_F(x) = \int_0^\infty db \ (b-x)^{2t-2} F^\text{sub}(b) + \frac{a_0}{2t-1} (x^{2t-1} + x^{-1}).$$

While equivalent to the method in [14], I believe the contour approach is neater and is capable of being extended to the general theory of periods. Zagier, [14], shows that logistic simplifications occur in this theory when it is extended to include non-cusp forms such as the Eisenstein series.

As an historical point, it will be noticed that the above expressions have been available since the time of Ramanujan, with contour derivations by Malurkar, [10], and Guinand, [11]. In order to investigate this point further, I return to the mode sum expressions for the thermodynamic related quantities.
3. Back to summations.

I start with the Eisenstein series forms and work the development to parallel
the quantities of the previous section. This allows me to introduce the useful calculations of Smart, [16], on the Epstein function (which will arise later) using its relation with Eisenstein series.

The subtracted partial energy \( \epsilon_{t}^{\text{sub}} \) in (2) corresponds to the the ‘regularised’ Eisenstein series,

\[
G_{t}^{\text{sub}}(\tau) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(m\tau + n)^{2t}} \tag{26}
\]

where \( \tau \) is defined by, [16], eqn.(2.11),

\[
\phi_{2t}(\tau) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{m^{2t-1}} \frac{1}{m\tau + n}. \tag{29}
\]

The relation is, [2],

\[
\epsilon_{t}^{\text{sub}}(\tau) = \frac{(2t - 1)!}{2(2\pi i)^{2t}} G_{t}^{\text{sub}}(\tau). \tag{27}
\]

Now, corresponding to (7), integrate (26) with respect to \( \tau \) from \( \tau \) to \( i\infty \), \( 2t - 1 \) times to give, formally,

\[
\frac{1}{(2t - 1)!} \phi_{2t}(\tau), \tag{28}
\]

one finds easily the inversion relation, [16], eqn.(1.10b),

\[
\phi_{2t}(\tau) - \tau^{2t-2} \phi_{2t}(-1/\tau) = -4 \sum_{j=1}^{t-1} \tau^{t-2j-1} \zeta_{R}(2j) \zeta_{R}(2t - 2j). \tag{31}
\]

Also, under translations,

\[
\phi_{2t}(\tau + 1) - \phi_{2t}(\tau) = 2 \frac{\zeta_{R}(2t)}{\tau(\tau + 1)}. \tag{32}
\]
Up to a simple factor, one might expect $\phi_{2t}$ to be the same as $\bar{\phi}_{2t}$ in (7). However the cocycle functions in (31) and (15) differ by the two even powers. I note that these are related to the $j = 1$ and $j = 2t - 1$ terms in the sum in (30) which have gone out in the passage to the difference, (31), by antisymmetry. However the summation over $m$, respectively $n$, for these values of $j$ is conditionally convergent and is therefore subject to ambiguity, that is to say, a choice has to be made, as in the discussion of the Eisenstein series, $G_2$. The regularised Mellin transform approach results in a different definition of the series (29) to that used by Smart, [16], p.3. The relation is easily found by taking the even powers in (15) over to the left and then we see that one has the relation, (further remarks are given later),

$$\bar{\phi}_{2t}(x) = i\phi_{2t}(\tau) - 2\pi\zeta_R(2t - 1), \quad x = -i\tau,$$

also agreeing with (9) and (32).

Smart, [16], also gives the relation with a known Lambert series as

$$S_t(\tau) \equiv \sum_{m=1}^{\infty} \frac{1}{m^{2t-1}} \frac{q^{2m}}{1 - q^{2m}} = -\frac{1}{4\pi i} \phi_{2t}(\tau) - \frac{1}{2\pi i\tau} \zeta_R(2t) - \frac{1}{2} \zeta_R(2t - 1)$$

$$= \frac{1}{4\pi} \left( \bar{\phi}_{2t}(x) + \frac{2}{x} \zeta_R(2t) \right)$$

$$\equiv \frac{1}{4\pi} \psi_{2t}(x),$$

(34)

where $\psi_{2t}$ is the function having $\bar{R}_{2t}(x)$, (25), as its inversion cocycle function,

$$\bar{\psi}_{2t}(x) - (ix)^{2t-2} \bar{\psi}_{2t}(1/x) = \bar{R}_{2t}(x),$$

(35)

as is easily confirmed.

The introduction of $\bar{\psi}$ has restored translational invariance, (the Planck term has been put back in (34)),

$$\bar{\psi}_{2t}(x - i) - \bar{\psi}_{2t}(x) = 0.$$

(36)

It is almost convention to consider modular integrals, for example, to be periodic (e.g. Knopp, [19]). See some later comments.

Equations (35) and (34) imply an inversion relation for the series $S_t$ which has a certain history, some of which is detailed by Berndt, [20], p.153, who says that it was first written down by Ramanujan. As is clear, it is quite the same as the expression for the Eisenstein period polynomials.
The Lambert series, \(S_t(\tau)\), has been considered by Apostol, [21], Berndt, [20], Grosswald, [22], and Smart [16], for example. It can be related to statistical mechanics in the following way.

One can resum the statistical expression for the partial free energy in a standard manner, beginning, \((q = e^{\pi i \tau} = e^{-\pi/\xi})\),

\[
f_t = \epsilon_{t,0} - \frac{\xi}{2\pi} \sum_{n=1}^{\infty} n^{2t-2} \log(1 - q^{2n}) = \epsilon_{t,0} - \frac{\xi}{2\pi} \sum_{m,n=1}^{\infty} \frac{n^{2t-2}}{m} q^{2mn}.
\]

(37)

The scaled entropy is defined by,

\[
s_t(\xi) = \frac{\partial f_t(\xi)}{\partial \xi} = -\frac{1}{2\pi} \left(1 - \frac{2\pi}{\xi} D\right) \sum_{m,n=1}^{\infty} \frac{n^{2t-2}}{m} q^{2mn}.
\]

(38)

Concentrate now on the summation and rewrite it,

\[
\sum_{m,n=1}^{\infty} \frac{n^{2t-2}}{m} q^{2mn} = D^{2t-2} \sum_{m=1}^{\infty} \frac{1}{m^{2t-1}} \frac{q^{2m}}{1 - q^{2m}},
\]

(39)

in terms of \(S_t(\tau), (34)\).

This Lambert series has been discussed in connection with generalised Dedekind \(\eta\)–functions by Berndt, [20], and it is worth noting that, in a certain sense, \(\log \eta\) is a modular form of zero weight, in agreement with the results \(\epsilon_1 = -D \log \eta, [1]\), which could therefore be written, \(\epsilon_1 = -D \log \eta\), in terms of the covariant derivative, \(D\).

The question, not answered here, is whether there is an ‘expansion’ or representation theorem for modular integrals combining that for modular forms (of any real weight, see [23]). It is not likely that one can find a pure elliptic formula. As evidence I cite the particular evaluations of Smart who gives at the lemniscate point \((\tau = i, x = 1)\),

\[
\overline{\psi}_4(1) = \frac{7\pi^4}{90} - 2\pi \zeta_R(3),
\]

and, more generally, if \(2t = 0(\text{mod}4)\),

\[
S_t(i) = \frac{\zeta_R(2t)}{2\pi} - \frac{\zeta_R(2t - 1)}{2} + \frac{1}{2\pi} \sum_{j=1}^{t-1} (-1)^{j+1} \zeta_R(2t - 2j) \zeta_R(2j),
\]

a result actually due to Lerch which follows from (35) with (25).

As a final general comment in this section, it is always possible to obtain the required quantities as (infinite) \(q\)-series by direct integration, but these do not count
as closed forms. For example, one integration yields the \( q \)-series for the partial free energy in terms of the arithmetic functions, \( \sigma_k(m) \), (the sum of the \( k \)-th powers of the divisors of \( m \)),

\[
f_t = \epsilon_{t,0} + \frac{1}{\beta} \sum_{m=1}^{\infty} \sigma_{2t-1}(m) \frac{q^{2m}}{m},
\]

which amounts only to doing the \( n \)-summation in (37), or an integration of the standard relation, (1),

\[
\epsilon_t = -\frac{B_{2t}}{4t} + \sum_{m=1}^{\infty} \sigma_{2t-1}(m) q^{2m}.
\]

One also has the formula, e.g. \([24,25]\),

\[
S_t(\tau) = \sum_{m=1}^{\infty} \sigma_{-2t+1}(m) q^{2m} = \sum_{m=1}^{\infty} \frac{\sigma_{2t-1}(m)}{m^{2t-1}} q^{2m}, \quad \forall \ t,
\]

in agreement with the result of a \((2t - 1)\)-fold integration.

4. Another approach via modular integrals.

The period functions are cocycle functions associated with Eichler cohomology and modular integrals. These notions put a different slant on the preceding formulae. In effect one begins again.

The best place to start is the important (but particular) analytical result of Bol, \([26]\), which states that, for any reasonable function, \( \varphi(\tau) \),

\[
(D^{r+1}\varphi)(\gamma\tau) = (c\tau + d)^{r+2}D^{r+1}((c\tau + d)^r \varphi(\gamma\tau)),
\]

where \( r \) is a nonnegative number, and I use,

\[
(D^{r+1}\varphi)(\tau) \equiv \left( \frac{dq}{dq} \right)^{r+1} \varphi(\tau),
\]

with,

\[
\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}).
\]

This shows that, if \( \varphi \) is an automorphic form of weight \(-r\), then \( D^{r+1}\varphi \) is an automorphic form of weight \( 2 + r \), see also Petersson, \([27]\). Note that it is the ordinary derivative that enters here. The proof of (41) follows by induction.
The classic discussion of modular integrals can proceed by asking for the inverse of Bol’s result. That is, given a form of weight $2 + r$, $\rho(\tau)$, as a source, can one integrate the differential equation,

$$D^{r+1} \varphi = \rho \quad (42)$$

to obtain $\varphi$, a form of weight $-r$? Clearly this will not be so in general because of the freedom introduced by constants of integration. That is, one would expect $\varphi$ to behave as a form of weight $-r$ up to an additional piece, the general form of which can be determined from the differential equation (42) and Bol’s theorem, (41).

Generally we would write (ignoring any multiplier systems for simplicity),

$$\varphi(\gamma \tau) = (c\tau + d)^{-r}(\varphi(\tau) + P(\gamma, \tau)) \quad (43)$$

where $P(\gamma, \tau)$ has to be a polynomial in $\tau$ of order $\leq r$. The inverse to Bol’s result is that, if any $\varphi$ satisfies (43), then $\rho$, of (42), is a form of weight $(2 + r)$. This follows most simply by substituting (43) into (41).

$P$ must satisfy the 1–cocycle consistency condition, [28],

$$P(\gamma_1 \gamma_2, \tau) = (c_2 \tau + d_2)^r P(\gamma_1, \gamma_2 \tau) + P(\gamma_2, \tau) \equiv P(\gamma_1, \tau) |_{\gamma_2} + P(\gamma_2, \tau) . \quad (44)$$

$\varphi$ is known as an automorphic or Eichler integral, of weight $-r$, and $P$ is the associated period polynomial, [28,29]. I have also introduced the useful, standard ‘stroke’ operator, $|\gamma$, sometimes denoted $[\gamma]$.

The general solution (equivalently the indefinite integral) of the differential equation (42) is a particular integral plus the complementary function (a zero mode) i.e.

$$\varphi(\tau) = \frac{(2\pi i)^{r+1}}{\Gamma(r + 1)} \int_{\tau_0}^{\tau} d\sigma (\tau - \sigma)^r \rho(\sigma) + \Theta(\tau) , \quad (45)$$

where $\Theta(\tau)$ is a polynomial of degree $r$. The integral, in the upper half plane, is path independent if $\rho$ has weakish analytic properties and, if desired, the solution can be made independent of the lower limit by suitable adjustment of $\Theta(\tau)$. In elementary fashion, $\Theta(\tau)$ is determined by the first $r$ derivatives of $\varphi(\tau)$ at $\tau_0$.

All this is general analysis. In modular applications to $\text{SL}(2,\mathbb{Z})$, $\tau_0$ is chosen as the cusp $i\infty$ and the limits in (45) reversed. In terms of Fourier expansions, a holomorphic cusp form source is

$$\rho(\tau) = \sum_{m=1}^{\infty} \rho_m q^{2m} , \quad (46)$$
and the particular Eichler integral of \( \rho \), is, from (45),

\[
\varphi(\tau) = \sum_{m=1}^{\infty} \frac{\rho_m}{m^{r+1}} q^{2m},
\]

which has the same periodicity as \( \rho \), a result of choosing \( \tau_0 = i\infty \).

The problem in the present Eisenstein case has already been noted. The Eisenstein series, \( G_t(\tau) \) does not vanish at \( i\infty \) (or 0). It is a ‘non–cusp’ form and the direct integration of (42) meets obstacles at an elementary level. This can be appreciated simply from the appearance of \( 1/m \) factors during integration, and \( m \) can be zero in \( G_t \). This fits in because the \( m = 0 \) terms in \( G_t \) are those making \( G_t \) nonzero at \( i\infty \), the zero temperature value. Saying it yet again, the Fourier transform of \( G_t \) begins with an \( m = 0 \) term, the Casimir contribution, so (47) makes no immediate sense.

One way of avoiding these obstructions, is to regularise \( G_t \) by dropping the offending term(s) as in (26) and consider the differential equation,

\[
D^{2t-1} \varphi_{2t} = -4\pi \epsilon_{t_{\text{sub}}},
\]

which has the particular integral, \( \varphi_{2t} = \overline{\varphi}_{2t} \) where,

\[
\overline{\varphi}_{2t}(\tau) = -4\pi \frac{(2\pi i)^{2t-1}}{\Gamma(2t-1)} \int_{\tau}^{i\infty} d\tau' (\tau' - \tau)^{2t-2} \epsilon_{t_{\text{sub}}}^{\text{sub}}(\tau'),
\]

the constants and boundary conditions being chosen so as to reproduce (7). The upper limit is a cusp. Smart’s function, \( \phi_{2t} \), results if one adds a complementary constant (and multiplies by \( i \)). (See (33).)

Because of the subtraction, the quantity \( \epsilon_{t_{\text{sub}}}^{\text{sub}}(\tau) \) is not exactly a modular form, rather one has, expressed slightly generally, \( (r = 2t - 2) \),

\[
\epsilon(\gamma\tau) = (c\tau + d)^{r+2} (\epsilon(\tau) + E(\gamma, \tau))
\]

and also

\[
\phi(\gamma\tau) = (c\tau + d)^{-r} (\phi(\tau) + P(\gamma, \tau)),
\]

which are related by the same differential equation, (42),

\[
D^{r+1} \phi = -4\pi \epsilon.
\]

Bol’s formula, (41), relates \( P \) and \( E \),

\[
D^{r+1} P = -4\pi E,
\]

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and these relations generalise the usual ones applicable to cusp forms. The cocycle formula (44) still holds of course, as (50) is the same as (43).

I remark that $\overline{\phi}$ and $\epsilon_{\text{sub}}$, not being periodic, do not quite possess Fourier expansions, as in (46) and (47). However $\overline{\psi}$, in view of (36), does,

$$
\overline{\psi}(x) = \sum_{m=1}^{\infty} \psi_m q^{2m},
$$

Because $P$, this time, satisfies (51), it is not required to be a polynomial. Generally, since the modular group is generated by $S$ and $T$, $P(\gamma, \tau)$ is a rational period function. In the present case, however, $P(S, \tau)$ is a polynomial, given by (15),

$$
P(S, \tau) = \overline{P}_t(\tau),
$$

while the (non–zero) $P(T, \tau)$, which follows from (9),

$$
P(T, \tau) = 2 \zeta_R(2t) \left( \frac{1}{\tau} - \frac{1}{\tau + 1} \right),
$$

is a ratio of polynomials. Of course, $P(\pm 1, \tau) = 0$ and I remark that the polynomial $P(S, \tau)$ runs from $\tau^0$ to $\tau^{2t-2}$.

As a simple check of the algebra, set $\gamma_1 = \gamma_2 = S$ in the cocycle relation, (44). This gives,

$$
P(S, \tau)|_{(1+S)} \equiv P(S, \tau) + \tau^r P(S, S\tau) = P(S, \tau) + \tau^r P(S, -1/\tau) = 0
$$

which, for $r = 2t - 2$, agrees with the explicit form, (15).

Also, as examples,

$$
P(TS, \tau) = 2 \zeta_R(2t) \frac{\tau^{2t}}{1 - \tau} + \overline{P}_t(\tau)
$$

$$
P(ST, \tau) = \frac{2 \zeta_R(2t)}{\tau(1 + \tau)} + \overline{P}_t(1 + \tau).
$$

I note that the period function appears to have poles at $-1, 0, +1$ and $i\infty$.

From (49) and the form of $\epsilon_{\text{sub}}$, (2), one finds

$$
E(S, \tau) = 0, \quad E(T, \tau) = \left( \frac{1}{(\tau + 1)^{2t}} - \frac{1}{\tau^{2t}} \right)
$$

\[3\] Beware that some authors, e.g. Knopp, swap the usage of the symbols $S$ and $T$. 

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which are consistent with (53) and (52).

For completeness, and to aid comparison, this is a convenient place to outline some standard elementary things about period polynomials, mostly for \( \text{SL}(2, \mathbb{Z}) \).

Two basic ‘polynomials’ are the inversion one, \( P(S, \tau) \), and the translation one, \( P(T, \tau) \). For cusp forms, the latter vanishes because of periodicity, \( P(T, \tau) = 0 \), [19], corresponding to the periodicity of the modular integrals, cf (46), (47). In this case, \( P(S, \tau) \) satisfies the Eichler–Shimura relations

\[
\begin{align*}
P(S, \tau) \big| (1 + S) &= 0, \\
P(S, \tau) \big| (1 + TS + (TS)^2) &= 0
\end{align*}
\]

with \( S^2 = 1 \) and \((TS)^3 = 1\). A standard way of proving these uses the complex integral form for \( P \), e.g. [14], but they can also follow, algebraically from the cocycle relation as I now show. (See also Knopp, [30], for slightly different details.)

For the first identity, just set, as above, \( \gamma_1 = 1 \) and \( \gamma_2 = S \) in (44) and it falls straight out. The second identity is slightly more work. Using (44) twice, one has, dropping the \( \tau \) argument temporarily and noting \((TS)^2 = (TS)^{-1}\),

\[
P(S) \big| (1 + TS + (TS)^2) = P(S) + P(STS) - P(TS) + P(SS^{-1}T^{-1}) - P(TSTS).
\]

Further, setting \( \gamma_1 = T \) in the cocycle condition, and using the assumption that \( P(T) = 0 \), yields

\[
P(T\gamma) = P(\gamma).
\]

One then sees that the first and third terms on the right–hand side of (55) cancel, as do the second and fifth. The fourth term vanishes on its own because (56) shows that \( P(T^{-1}) = 0 \), and (54) follows.

In one development, the notion of modular integral has been somewhat divorced from that of integration. One says that a **modular integral** is any function (with appropriate analyticity properties) that obeys the quasi–modular relation (43), where the **period function** \( P \) must still, of course, satisfy the cocycle condition (44). Motivation is then provided to look for modular integrals with **rational** period functions, *i.e.* ratios of polynomials. In this case it seems to be conventional to **assume** that the translational period function, \( P(T, \tau) \), vanishes, [19], so that both Eichler–Shimura relations, (54), still hold. There is a certain interest in computing such rational period functions and investigating their zeros and poles.

The easiest introduction to this topic is by Knopp, [30]. It is, however, somewhat peripheral to our considerations which are more concerned with non–cusp forms *i.e.* the Eisenstein series, *e.g.* [14].
5. Green function distributional description.

Developing the formalism a little further, it is possible to give the discussion of Eichler integrals a distributional or Green function look. For this purpose, it is better to use the ‘real’ variable form and consider the differential equation,

$$\frac{d^{2t-1}}{dx^{2t-1}} \psi(x) = \rho(x), \quad (57)$$

for $x > 0$ with solution vanishing at $\infty$, *cf* (7),

$$\psi(x) = \int_{0}^{\infty} dx' \Phi_{2t-1}(x' - x) \rho(x') \quad (58)$$

where the generalised function, $\Phi_{\alpha}(x)$, is

$$\Phi_{\alpha}(x) = \frac{x^\alpha - 1}{\Gamma(\alpha)}. \quad (59)$$

The *generalised* function $x^\alpha_+$ is concentrated on the positive $x$-axis, *i.e.* $x^\alpha_+$ is equal to $x^\alpha$ for $x \geq 0$ and is zero for $x < 0$, *e.g.* Gelfand and Shilov, [31] I, §5.5. $\Phi_{2t-1}$ acts as a Green function satisfying,

$$\frac{d^{2t-1}}{dx^{2t-1}} \Phi_{2t-1}(x' - x) = \delta(x' - x).$$

The formal, distributional equivalent is obtained from the convolution, valid for all $\alpha$ and $\beta$,

$$\Phi_{\alpha} * \Phi_{\beta} = \Phi_{\alpha + \beta}, \quad (60)$$

by setting $\alpha = -\beta = 2t - 1$ and noting

$$\Phi_{-k}(x) = \delta^{(k)}(x), \quad k = 0, 1, 2, \ldots \quad (61)$$

An important formula is the Fourier transform of $\Phi$, [31] I.p.360,

$$\int_{0}^{\infty} dx \Phi_{\alpha}(x) e^{i\sigma x} = \frac{1}{(\sigma + i0)^\alpha} e^{i\pi \alpha/2}, \quad (62)$$

and the inverse, which is a (continuous) eigenfunction expansion,

$$\Phi_{\alpha}(x) = e^{i\pi \alpha/2} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\sigma \frac{e^{-i\sigma x}}{(\sigma + i0)^\alpha}. \quad (63)$$

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In the present case, the source $\rho$ is assumed to be periodic under $x \rightarrow x - i$ (the Planck term is not removed) and to converge as $x \rightarrow \infty$. So one has the ‘Fourier’ series,

$$\rho(x) = \sum_{m=1}^{\infty} \rho_m e^{-2m\pi x}, \quad x > 0. \quad (64)$$

Substitution of this and (63) into (58) gives,

$$\psi(x) = (-1)^t \frac{1}{2\pi} \sum_{m=1}^{\infty} \rho_m \int_{-\infty}^{\infty} d\sigma \frac{e^{i\sigma x}}{(\sigma + i0)^{2t-1}(\sigma - 2m\pi i)}. \quad (65)$$

Closing the contour in the upper half $\sigma$–plane I obtain, $x > 0$,

$$\psi(x) = \frac{1}{(2\pi)^{2t-1}} \sum_{m=1}^{\infty} \frac{\rho_m}{m^{2t-1}} e^{-2m\pi x}, \quad (66)$$

which, it is no surprise to see, is just (47) derived by a much longer route but one on which we encounter some handy information. It is seen also that the source periodicity has been transmitted to the solution.

Including an $m = 0$ term in the Fourier series, (64), still gives convergence at infinity, but of course the solution, (66), breaks down, as has already been mentioned. As with all Green function approaches, when this happens the corresponding eigenfunction has to be removed from the source. This can be done formally by redefining the Green function to exclude this function which is, in the present case, a constant zero mode. Altering the contour in the Fourier integral (63) gives a modified Green function,

$$\Phi_{2t-1}^{\text{sub}}(x) \equiv (-1)^t \frac{1}{2\pi i} \int_{-\infty}^{\infty+i\delta} d\sigma \frac{e^{-i\sigma x}}{\sigma^{2t-1}}, \quad 0 < \delta < 2, \quad (67)$$

which automatically takes out any constant part of the source. This can be seen by going back to (65) and replacing the integral by one from $-\infty + i\delta$ to $\infty + i\delta$,

$$\psi(x) = (-1)^{t+1} \frac{1}{2\pi} \sum_{m=1}^{\infty} \rho_m \int_{-\infty+i\delta}^{\infty+i\delta} d\sigma \frac{e^{i\sigma x}}{\sigma^{2t-1}(\sigma - 2m\pi i)}. \quad (68)$$

For $0 < \delta < 2$, if $x > 0$, closing the contour in the upper half plane still yields (66). However now, even if the source Fourier series, (64), is extended to the constant term, $m = 0$, the result is again (66). This should be denoted by $\psi^{\text{sub}}(x)$ and one can write

$$\psi^{\text{sub}}(x) = \int_0^{\infty} dx' \Phi_{2t-1}^{\text{sub}}(x' - x) \rho(x'). \quad (69)$$
Retaining the general structure, (63), for any \( \alpha \), one can define for any \( \alpha \),
\[
\Phi_{\alpha}^{\text{sub}}(x) \equiv e^{i\pi\alpha/2}\frac{1}{2\pi} \int_{-\infty+i\delta}^{\infty+i\delta} d\sigma \frac{e^{-i\sigma x}}{\sigma^\alpha}.
\] (70)

Then \( \Phi_{\alpha}^{\text{sub}}(x) \) is still concentrated on the positive \( x \)-axis and the convolution (60) also remains valid,
\[
\Phi_{\alpha}^{\text{sub}} \ast \Phi_{\beta}^{\text{sub}} = \Phi_{\alpha+\beta}^{\text{sub}}
\] (71)
together with (61),
\[
\Phi_{-k}^{\text{sub}} = \delta^{(k)}.
\] (72)

To check these statements, firstly, if \( x < 0 \), closing the contour in (70) in the upper half plane yields zero as the only singularity is at \( \sigma = 0 \). Then
\[
(\Phi_{\alpha}^{\text{sub}} \ast \Phi_{\beta}^{\text{sub}})(x) = \frac{e^{i\pi(\alpha+\beta)/2}}{(2\pi)^2} \int_{-\infty}^{\infty} dx' \int_{-\infty+i\delta}^{\infty+i\delta} d\sigma \int_{-\infty+i\delta}^{\infty+i\delta} d\sigma'\frac{e^{-i\sigma x - iX'(\sigma'-\sigma)}}{\sigma^\alpha \sigma'^\beta}
\]
where the lower limit on the \( x' \) integral is allowed because \( \Phi_{\beta}^{\text{sub}}(x') \) is concentrated on the positive \( x' \) axis.

Setting \( \sigma = s + i\delta \), \( \sigma' = s' + i\delta \), I get, in detail,
\[
(\Phi_{\alpha}^{\text{sub}} \ast \Phi_{\beta}^{\text{sub}})(x) = e^{i\pi(\alpha+\beta)/2}\frac{1}{2\pi} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} ds'\frac{e^{-i\sigma x - iX'(s'-s)}}{\sigma^\alpha \sigma'^\beta}
\]
\[
= e^{i\pi(\alpha+\beta)/2}\frac{1}{2\pi} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} ds'\frac{e^{-i\sigma x}}{\sigma^\alpha \sigma'^\beta} \delta(s-s')
\]
\[
= e^{i\pi(\alpha+\beta)/2}\frac{1}{2\pi} \int_{-\infty}^{\infty} ds \frac{e^{-i\sigma x}}{\sigma^\alpha \sigma'^\beta} = e^{i\pi(\alpha+\beta)/2}\frac{1}{2\pi} \int_{-\infty}^{\infty+i\delta} d\sigma \frac{e^{-i\sigma x}}{\sigma^\alpha + \beta}
\]
\[
= \Phi_{\alpha+\beta}^{\text{sub}}(x).
\]

Equation (72) follows either from (71) or (70) since the integrand has no singularity, and so one can set \( \delta = 0 \).

\emph{Eichler cohomology}.

As a side remark, the general solution to (57) consists of the particular integral, (58), plus a solution of the homogeneous equation, \textit{i.e.} a zero mode, which in this case is a polynomial, \( \Theta(x) \), of degree at most \( 2t-2 \). The contribution of this to the cocycle polynomial, \( P \) of (43), constitutes an element of the set of \textit{coboundaries} and a cohomology of polynomials can be set up (Eichler cohomology). Working
with the cohomology classes removes the constants of integration ambiguity in the solution of (57). That is, there is a one-to-one correspondence between \( \rho \) and an element of the cohomology group denoted \( H^1(\Gamma, \Pi_r) \), where \( \Pi_r \) is the vector space of polynomials of degree \( \leq r = 2t - 2 \) (for cusp forms) and \( \Gamma \) is (here) the modular group. The development of these ideas, while significant generally, is not required here.

6. Epstein approach to thermal quantities.

An ‘earlier’ form of the free energy, or effective action, follows from the thermal \( \zeta \)–function, which, in this case is related to the Epstein \( \zeta \)–function. Starting from this I will be able to make contact with the Eisenstein formulation and derive useful relations. Using the Epstein function is equivalent to the non-holomorphic Eisenstein series. As a rule the analysis is harder. In reality one is doing twice the necessary work.

In some ways we are going backwards. The use of holomorphic forms, e.g. (27), should be considered a simplification available through working with an explicit square root of the Laplacian.

Again splitting up the degeneracy, one is led to define a ‘partial’ \( \zeta \)–function,

\[
\zeta_t(s, \beta) = \frac{i}{2\beta} \sum_{m,n=-\infty}^{\infty} \frac{n^{2t-2}}{(4\pi^2m^2/\beta^2 + n^2/a^2)^s},
\]

and the related free energy,

\[
F_t = -\frac{\xi}{8\pi} \lim_{s \to 0} \frac{1}{s} \sum_{m,n=-\infty}^{\infty} \frac{n^{2t-2}}{(m^2 + n^2\xi^{-2})^s}.
\]

I have rescaled the free energy by defining, as above and as in [2], \( \bar{F}_t = aF_t \) and have also taken a preliminary limit of \( s \to 0 \) in an extracted factor of \( (2\pi/\beta)^2s \). This will be justified in a moment.

\[
\bar{F}_t = (-1)^{t-1} \frac{\xi}{8\pi} \left( \frac{d}{d(1/\xi^2)} \right)^{t-1} \lim_{s \to 0} \frac{\Gamma(s-t+1)}{\Gamma(s+1)} \sum_{m,n=-\infty}^{\infty} \frac{1}{(m^2 + n^2\xi^{-2})^{s-t+1}}.
\]

Not surprisingly, the degeneracy factor is replaced by a derivative and the resulting sum is precisely an Epstein \( \zeta \)–function. This has been used before e.g.
The point is that there is no pole at \( s = 0 \), which justifies the limit. In terms of the Epstein function, \( Z_2 \),

\[
\mathcal{F}_t = (-1)^{t-1} \frac{\xi}{8\pi} \left( \frac{d}{d(1/\xi^2)} \right)^{t-1} \lim_{s \to 0} \frac{\Gamma(s-t+1)}{\Gamma(s+1)} Z_2(2s-2t+2, A),
\]

(76)

where \( A \) is the matrix (the ‘modulus’),

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & \xi^{-2} \end{pmatrix}.
\]

The idea now is to use the standard functional relation satisfied by \( Z_2(s, A) \), repeated here for convenience,

\[
Z_2(2s, A) = \frac{\pi^{2s-1}}{\sqrt{\det A}} \frac{\Gamma(1-s)}{\Gamma(s)} Z_2(2-2s, A^{-1}),
\]

(77)

to give

\[
\mathcal{F}_t = \frac{\xi}{8\pi} \left( - \frac{d}{d(1/\xi^2)} \right)^{t-1} \xi \lim_{s \to 0} \pi^{2s-2t+1} \frac{\Gamma(t-s)}{\Gamma(s+1)} Z_2(2t-2s, A^{-1}).
\]

\( Z_2(2s, A) \) has a pole at \( s = 1 \) only, so working at \( t > 1 \) to begin with, one has

\[
\mathcal{F}_t = \xi^{(-1)^{t-1}} \frac{\Gamma(t)}{8\pi^{2t}} \left( \frac{d}{d(1/\xi^2)} \right)^{t-1} \xi Z_2(2t, A^{-1}).
\]

(78)

As the simplest case, set \( t = 2 \) giving the three-sphere. Then

\[
\mathcal{F}_3 = \frac{\xi}{8\pi^{4}} \frac{d}{d(1/\xi^2)} \xi Z_2(4, A^{-1}) = -\frac{\xi^4}{16\pi^{4}} \frac{d}{d\xi} \xi Z_2(4, A^{-1}),
\]

(79)

agreeing with the result in [32], allowing for the changes in notation.

This formula can be taken further as shown by Epstein [33] p.633 who gives,

\[
Z_2(4, A^{-1}) = \frac{\pi^4}{45} + \pi\xi^{-3}\zeta_R(3) + 2\pi\xi^{-3}\chi(q') + 4\pi^2\xi^{-2} D' \chi(q'),
\]

(80)

where \( q' = e^{-\pi\xi} \) and

\[
\chi(q') = S_2(-1/\tau) = \sum_{m=1}^{\infty} \frac{q'^{2m}}{m^3(1-q'^{2m})}.
\]
Equation (79) with (80) is appropriate for the high temperature limit since $q'$ is the ‘inverse’ temperature Boltzmann factor. The low temperature form follows by simple homogeneous scaling of the Epstein $\zeta$–function, which gives,

$$Z_2(2s, A^{-1}) = \xi^{-2s} Z_2(2s, A),$$  \hspace{1cm} (81)

and therefore the equivalent form,

$$Z_2(4, A^{-1}) = \frac{\pi^4}{45} \xi^{-4} + 2 \pi \xi^{-1} \zeta_R(3) + 2 \pi \xi^{-1} \chi(q) + 4 \pi^2 \xi^{-2} D \chi(q),$$  \hspace{1cm} (82)

where $q = e^{-\pi/\xi}$.

The low temperature form (82) yields a somewhat simpler expression for the free energy, (79), as the $\zeta_R(3)$ term goes out and there is some cancellation that produces,

$$\overline{F}_3 = \frac{1}{240} + \frac{\xi}{2\pi} D^2 \chi(q).$$

I have thus regained, at some length, the summation form (37) with (39).

All that has been done is to rederive, in a special case and in a detailed way, the standard statistical mode sum, the general form of which was obtained, in this fashion, in [34,35] and we are no further forward in finding a closed form for the free energy, which is one of my aims.

The equivalence of (80) and (82), derived here rather trivially (granted Epstein’s calculation), is a known inversion identity, and can also be derived from (31). The exact identity is written out in Katayama [36]. His derivation seems to be similar to that via the Epstein function. Clearly the higher odd sphere expressions will involve the Riemann $\zeta$–function at positive odd integers. Katayama writes out the one appropriate for the (partial) five-sphere.

Smart, [16], also derives these identities, which is not surprising since his work is concerned with the evaluation of the Epstein $\zeta$–function at integer arguments, through which he is led to the forms $\phi_{2k}$. We see that Epstein had already arrived at the same development. Consult also Bodendiek and Halbritter, [37], for a related treatment.
7. The Epstein–Kober–Selberg–Chowla formula.

Smart makes use of the, oft quoted, paper of Selberg and Chowla, [38], which is concerned, partly, with the evaluation of the Epstein function via the Kronecker limit formula \(^4\) and elegantly yields some of the finite forms for the complete elliptic integral used in [2]. The important expression for the Epstein function quoted by Selberg and Chowla, [39], and derived by them in [38], (equ.(6)), in terms of \(K\)-Bessel functions, is essentially the same as that already given by Epstein, [33], p.631, equn.(12), p.622, equn.(16). Rankin [40] obtained the same result and it also occurs in Bateman and Grosswald, [41]. These are the standard references. See also Terras, [6] p.209. Smart, [16], employs this formula to split the Epstein function into (simpler) holomorphic and anti–holomorphic parts using the explicit form of the Bessel function, \(K_{t-1/2}\), which henceforth disappears from his analysis. Other expressions were given by Deuring [42] and Mordell, [43].

An early, and little known, derivation is that by Kober, [44]. Following, and generalising, Watson, [45], he obtains, starting from the Bessel function end, the formula (I retain his notation),

\[
\frac{1}{\sqrt{u}} \Delta^{(2w+1)/4} \frac{1}{8\pi^{w+1/2}} \Gamma(w + 1/2) Z_2(a, b, c; w + 1/2) = \frac{u^{-w}}{4} \frac{\Gamma(w)}{\pi^w} \zeta_R(2w) + \frac{u^w}{4} \frac{\Gamma(w + 1/2)}{\pi^{w+1/2}} \zeta_R(2w + 1) + \sum_{n=1}^{\infty} \sigma_{2w}(n)n^{-w} \cos(2\pi vn)K_w(2\pi vn),
\]

(83)

where the quadratic form in the Epstein function, \(Z_2\), is \(am_1^2 + 2bm_1m_2 + cm_2^2\) and

\[
\Delta = ac - b^2 \equiv a^2 u^2, \quad v \equiv \frac{b}{a}.
\]

Equation (83) is identical to the corresponding one in [33] and [38] and therefore, with justice, should be called the Epstein–Kober formula. It is the Fourier expansion of the Epstein function in \(v\). For the sum of squares case, \(v = 0\), the result is rederived by Guinand [46] in the same way.

Kober also discusses behaviour under ‘reciprocal’ transformations, \(u \rightarrow 1/u\). In particular the diagonal case, \(v = 0\), when one can write \(u = x\).

\(^4\)Landau, [9], gives some interesting history of this famous formula.
In this way of doing things, the Bessel function can be eliminated using the standard formula
\[ \sum_{n=1}^{\infty} \frac{\sigma_{2w}(n)}{n^{s+w}} = \zeta_R(s+w) \zeta_R(s-w), \quad (84) \]
and the Mellin transform (Heaviside’s integral),
\[ \frac{1}{4\pi^{-s}} \Gamma\left(\frac{s+w}{2}\right) \Gamma\left(\frac{s-w}{2}\right) = \int_{0}^{\infty} dy y^{s-1} K_w(2\pi y), \quad \text{Re } s > 1 + |\text{Re } w|, \quad (85) \]
so that,
\[ \sum_{n=1}^{\infty} \frac{\sigma_{2w}(n)}{n^{w}} K_w(2\pi un) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \, u^{-s} \xi(s+w) \xi(s-w), \quad (86) \]
with \( c > 1 + w \) and where,
\[ \xi(2w) \equiv \frac{1}{2} \pi^{-w} \Gamma(w) \zeta_R(2w). \]
This is the same as Deuring’s expression, [42] p.589.

The integrand possesses a functional equation which follows most easily from that for the Riemann \( \zeta \)-function, \( \xi(2w) = \xi(1-2w) \), and is,
\[ \xi(s+w) \xi(s-w) \equiv f_w(s) = f_w(1-s). \quad (87) \]

Using (87) one could derive the functional equation for the Epstein \( \zeta \)-function in (83). The usual derivation employs the inversion relation for generalised \( \theta \)-functions, [33,47].

Furthermore, it is possible to reverse the argument and use (86) to obtain (83) (for \( v = 0 \)). The diagonal Epstein function is particularly easy to deal with being the \( \zeta \)-function on the torus, \( S^1 \times S^1 \). It can be expressed very simply in terms of the \( \zeta \)-functions on the circle factors (these are Riemann \( \zeta \)-functions ) and the result is exactly (83) with (86) without the intervention of the Bessel function. The individual Riemann \( \zeta \)-functions in (83) arise from adjusting the zero modes. All this is standard and can easily be generalised to higher dimensions.

A natural step is to proceed as with the holomorphic modular forms and ask for the corresponding period polynomials which will follow directly from (86) and (87) in the standard manner.
There are four poles in the relevant strip, \(-c \leq s \leq c\), and a straightforward contour calculation produces the result,

\[
\sum_{n=1}^{\infty} \frac{\sigma_{2w}(n)}{n^w} K_w(2\pi nu) - \frac{1}{u} \sum_{n=1}^{\infty} \frac{\sigma_{2w}(n)}{n^w} K_w(2\pi n/u) = \frac{1}{2} \xi(2w)(u^{w-1} - u^{-w}) + \frac{1}{2} \xi(-2w)(u^{-w-1} - u^w),
\]

The right hand side of (88) might be termed a period function.

This formula was obtained by Guinand, [46], but using the complete equation (83), with \(v = 0\), and properties of the Epstein function. This is unnecessary as I have just shown.

In order to make a connection with the period polynomials encountered earlier, e.g. (35), one introduces the formula for the K-Bessel function, written as,

\[
K_{t-1/2}(x) = \sqrt{\frac{2}{\pi}} x^{t-1/2} (-1)^{t-1} \left( \frac{d}{dx} \right)^{t-1} e^{-x} \frac{x}{x}, \quad t \in \mathbb{Z}.
\]

Then, somewhat similarly to [16],

\[
\sum_{n=1}^{\infty} \frac{\sigma_{2t-1}(n)}{n^{t-1/2}} K_{t-1/2}(2\pi nu) = \sqrt{\frac{2}{\pi}} \frac{(-1)^{t-1}}{(2\pi)^{t-1/2}} \left( \frac{1}{u \, du} \right)^{t-1} \sum_{n=1}^{\infty} \frac{\sigma_{2t-1}(n)}{n^{2t-1}} q^{2n}
\]

\[
= \sqrt{\frac{2}{\pi}} \frac{(-1)^{t-1}}{(2\pi)^{t-1/2}} \left( \frac{1}{u \, du} \right)^{t-1} S_t(iu),
\]

so that, for the left-hand side of (88), one finds,

\[
(-1)^{t-1} \frac{1}{\pi^t} \left( \left( \frac{d}{du^2} \right)^{t-1} S_t(iu) - \frac{1}{u} \left( \frac{d}{d(1/u^2)} \right)^{t-1} S_t(i/u) \right).
\]

It is not obvious from (35) that this reduces so nicely to (88), and I leave this as a question mark.

Further expressions involving Bessel functions appear in the Appendix.

8. Conclusion

It does not seem possible to ellipticise integrals of the internal energy, in particular the free energy (except for the circle), because the modular properties are more complicated possessing nonzero cocycle functions. These are related to a known Lambert series connected with the Eisenstein series and Zagier’s expression for the period functions of these is more neatly re-computed by a contour method. The use of the fully subtracted series \(\epsilon_t^{\text{sub}}\) allows a more compact treatment.

Sufficient historical material has been exhibited to indicate that the Selberg–Chowla formula should be renamed the Epstein–Kober formula.
 Appendix, Dirichlet Series.

It is possible to give a fairly broad context to the inversion behaviour (e.g. modular properties), without too much restriction, in terms of Dirichlet series. Although these ideas, by now classic, are associated with Hecke, and have been expounded by Ogg [48,49], for example, the simplest case occurs earlier in other places. I mention Koshliakov, [50], whose discussion I now paraphrase both for historical justice and interest.

Define two ordinary Dirichlet series (‘zeta functions’),

\[ \phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \quad (\text{Re } s > \nu_a), \quad \psi(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}, \quad (\text{Re } s > \nu_b > \nu_a), \]

for some \( \nu \)'s depending on the asymptotics of \( a_n \) and \( b_n \), and assume that the \( a_n \) and \( b_n \) are such that the corresponding ‘cylinder kernels’ satisfy the ‘formula of transformation’,

\[ \sum_{n=0}^{\infty} a_n e^{-nb\rho} = \frac{a}{\rho^n} \sum_{n=0}^{\infty} b_n e^{-nb/\rho}, \quad \nu = \nu_a, \quad \rho > 0, \quad (91) \]

for fixed \( a \) and \( b \), where the numbers (not necessarily integers) of ‘zero modes’ are taken to be,

\[ a_0 = -\phi(0) \quad b_0 = -\psi(0). \]

Then Koshliakov, [50], shows the equivalence of (91) with

\[ a \frac{\Gamma(\nu - s) \psi(\nu - s)}{b^{\nu-s}} = \frac{\Gamma(s) \phi(s)}{b^s}, \quad (92) \]

by brute force using the summation function, \( \sigma(z) \), given by the Bessel function expression,

\[ \sigma(z) = -2ab z^{(\nu-1)/2} \sum_{n=1}^{\infty} \frac{b_n}{n^{(\nu-1)/2}} K_{\nu-1}(2b\sqrt{n}z), \]

and Heaviside’s formula, (85). \( \sigma(z) \) has the important property of possessing poles at \( z = 1, 2, \ldots \) with residues \( a_1, a_2, \ldots \) and also has a cut along the negative \( x \)-axis with discontinuity \( ab^\nu 2\pi i \psi(0)(-x)^{\nu-1}/\Gamma(\nu) \), \( z = x + iy \).

Koshliakov shows that \( \phi(s) \) is a single–valued holomorphic function having a first-order pole at \( s = \nu \),

\[ \phi(s) \sim -\psi(0) \frac{ab^\nu}{\Gamma(\nu)} \frac{1}{s - \nu}, \quad (93) \]
and it is then a theorem that (92) together with (93) is equivalent to (91), as can also be established easily, and instructively, by Mellin transform. It follows, symmetrically, that \( \psi(s) \) is a single-valued holomorphic function having a first-order pole at \( s = \nu \),

\[
\psi(s) \sim -\phi(0) \frac{b^\nu}{a \Gamma(\nu)} \frac{1}{s - \nu}.
\]

Koshliakov also gives an integral form of \( \phi(s) \),

\[
\phi(s) = -\frac{\sin \pi s}{\pi} \int_0^\infty dx \frac{\sigma(x)}{x^s}, \quad \text{Re} \, s < 0, \quad (94)
\]

showing that \( \phi(s) \) vanishes at negative integers in agreement with (92).

Hecke, [15], derives the same equivalence, but only for \( a_n = b_n \), and eight years later. The more general case can also be found in the standard references Ogg [48,49] and Weil, [51], and no doubt elsewhere. A related formula involving Ramanujan’s ‘reciprocal functions’ occurs in Hardy and Littlewood, [7].

Defining cylinder–kernels, or theta–series, including the zero modes, by,

\[
\Phi(\beta) = \sum_{n=0}^{\infty} a_n e^{-n\beta}, \quad \Psi(\beta) = \sum_{n=0}^{\infty} b_n e^{-n\beta},
\]

the reciprocal relation, (91), reads,

\[
\Phi(b\rho) = \frac{a}{\rho^\nu} \Psi(b/\rho), \quad (95)
\]

(continued from the regions of convergence in \( s \)) and one has, in standard fashion (cf (20)),

\[
\phi(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\beta \beta^{s-1} \left( \Phi(\beta) - a_0 \right), \quad \psi(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\beta \beta^{s-1} \left( \Psi(\beta) - b_0 \right).
\]

As an example, the Dirichlet series corresponding to the classical Eisenstein series is, from (1) or (84),

\[
\sum_{m=1}^{\infty} \sigma_{2t-1}(m) m^{-s} = \zeta_R(s) \zeta_R(s + 1 - 2t),
\]

and this is a case discussed by Koshliakov, [50] p.19, with \( a_n = b_n = \sigma_{2t-1}(n), \nu = 2t, a = (-1)^t, b = 2\pi, \phi(0) = \psi(0) = -B_{2t}/4t. \)

Koshliakov also considers Jacobi elliptic cases that yield reciprocal identities covered earlier by Glaisher [25].
**Generalised Dirichlet series.**

I have described above the classic Dirichlet set up. A certain, and sometimes only apparent, generalisation is obtained by replacing \( n \) by arbitrary sequences, \( \lambda_m \) and \( \mu_n \), and defining new \( \phi \) and \( \psi \) by

\[
\phi(s) = \sum_{m=1}^{\infty} \frac{a_m}{\lambda_m^s}, \quad \psi(s) = \sum_{n=1}^{\infty} \frac{b_n}{\mu_n^s},
\]

which satisfy, by definition, the functional equation (one of many possible),

\[
\Gamma(\delta - s) \psi(\delta - s) = \Gamma(s) \phi(s), \quad \delta \in \mathbb{R},
\]

with corresponding heat-kernels,

\[
\Phi(\beta) = \sum_{m=1}^{\infty} a_m e^{-\lambda_m \beta}, \quad \Psi(\beta) = \sum_{n=1}^{\infty} b_n e^{-\mu_n \beta}.
\]

Bochner, [52], (see also Chandrasekharan and Narasimhan, [53], Knopp, [54]), derives, by Mellin transforms, the ‘modular relation’,

\[
\Phi(\beta) - \beta^{-\delta} \Psi\left(\frac{1}{\beta}\right) = B(\beta),
\]

where the ‘residual function’ \( B(\beta) \) is

\[
B(\beta) = \frac{1}{2\pi i} \int_{C(S)} ds \chi(s) \beta^{-s} = \sum_{s \in S} \beta^{-s} \text{Res} \chi(s).
\]

\( \chi(s) \) is the joint continuation of the two sides of (97) and is assumed to have only simple poles for singularities confined to a compact region, \( S \), of the \( s \)-plane, usually on the real axis. \( C(S) \) is a loop surrounding \( S \). I especially note that if there exist higher-order poles then integer powers of log \( \beta \) occur.

In certain situations, e.g. Koshliakov’s, \( B(\beta) \) can be naturally amalgamated with the left-hand side to give an exact modular relation like (91), e.g. Bochner, [52] p.342. Typically one would include zero modes \( \lambda_0 = 0 \) and \( \mu_0 = 0 \) in (98), but not in (96). (There seems to be a misprint on p.342 of [52] which has \( \lambda_0 = \mu_0 = 1 \).)

There is another formula connected with this analysis that is useful. It is concerned with the representation of the **modified** series,

\[
\phi(s, w) = \sum_{n=1}^{\infty} \frac{a_n}{(\lambda_n + w^2)^s} \quad \text{Re } w > 0.
\]
There are many reasons why we should wish to add the number $w^2$. For example it might correspond to a mass or simply be added for extra flexibility as when calculating heat–kernel expansion coefficients via resolvents or it might be another part of the denominator as in Epstein’s derivation of (83); see [55].

The representation alluded to is, (Berndt [56]),

$$\Gamma(s) \phi(s, w) = R(s, w) + 2 \sum_{n=1}^{\infty} b_n \left( \frac{\mu_n}{w^2} \right)^{(s-\delta)/2} K_{s-\delta}(2w\sqrt{\mu_n}),$$  \hspace{5cm} (101)

where $s$ is such that the summation converges absolutely and $R(s, w)$ is given by

$$R(s, w) = \sum_{s' \in S} \Gamma(s-s') w^{2s'-2s} \text{Res} \chi(s').$$ \hspace{5cm} (102)

Berndt’s second proof of (101) proceeds via the cylinder–kernels, $\Phi$ and $\Psi$. I prefer his first, Mellin, proof, [57]. Of course, the information employed is the same. I give this proof out of interest.

The Mellin transform relation between $\phi(s)$ and the $\phi(s, w)$ of (100), is

$$\Gamma(s) \phi(s, w) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds' w^{2s'-2s} \Gamma(s-s') \Gamma(s') \phi(s')$$

where $c$ is chosen sufficiently positive to make both $\phi(s)$ and $\psi(s)$ converge. The vertical contour is now moved to the left so as to run from $\delta-c-i\infty$ to $\delta-c+i\infty$ picking up the residues, $R(s, w)$, (102), on the way. The region, $S$, lies between these vertical lines and also the horizontal pieces contribute zero.

On the shifted vertical contour make the coordinate change $s' \to \delta-s'$ and use (97) \textit{i.e.} $\chi(\delta-s') = \Gamma(s') \psi(s')$, valid in this range, and so for this part I find,

$$\sum_n b_n \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds' \Gamma(s'-\delta+s) \Gamma(s') \mu_n^{s'-s} w^{2s'-2s} = 2 \sum_{n=1}^{\infty} b_n \left( \frac{\mu_n}{w^2} \right)^{(s-\delta)/2} K_{s-\delta}(2w\sqrt{\mu_n}),$$

on using (85). Hence (101) has been established.

In specific situations (101) is quite familiar in the $\zeta$–function area and in background and finite–temperature quantum field theory.

An illustrative example is the simplest, diagonal Epstein function,

$$\phi(s) = Z_p(s) \equiv \sum_{m=-\infty}^{\infty'} \frac{1}{(m.m)^s}, \quad m = (m_1, m_2, \ldots, m_p),$$  \hspace{5cm} (103)
which obeys the functional equation (Epstein [33]),

\[ \Gamma(s) Z_p(s) = \pi^{2s-p/2} \Gamma(p/2 - s) Z_p(p/2 - s), \]  

(104)

so that, see (97), \( \delta = p/2, a_n = 1, \lambda_m = m.m, b_n = \pi^{p/2}, \mu_n = \pi^2 m.m. \)

\( \Gamma(s) Z_p(s) \) has poles at \( s = 0 \) and \( s = p/2 \) and (101) gives for the modified, or inhomogeneous, or ‘massive’ Epstein function, \( Z_p(s, w) \),

\[
\sum_{m=-\infty}^{\infty} \frac{1}{(m.m + w^2)^s} = -w^{-2s} + \frac{\Gamma(p/2)\Gamma(s - p/2)}{\Gamma(s)} w^{p-2s} + 2\pi^s \sum_{m=-\infty}^{\infty} \left( \frac{|m|}{w} \right)^{s-p/2} K_{s-p/2}(2w\pi|m|),
\]

(105)

where the dash means to omit the \( m = 0 \) term. Note that the \( w^{-2s} \) could be included in the left–hand sum if this were extended to include \( m = 0 \).

The right–hand side can be taken as the continuation of the left–hand side showing the single pole at \( s = p/2 \), as is correct and may be proved in other ways.

Actually (105) is shown much more directly using the Jacobi inversion relation, which is, of course, how Epstein obtained (104). Equivalent to Jacobi inversion is Poisson summation.

Equations like (105) also occur in lattice summations. Some references, but none earlier than those already quoted, are given in the review by Glasser and Zucker [58] p.109.

A similar formula holds for the general Epstein function which may, or may not, have a pole at \( s = p/2 \) and may, or may not, vanish at \( s = 0 \). The works [59,3,60], for example, can be consulted for details, some further references and physical applications.

When \( p = 1, \phi(s) = 2\zeta_R(2s) \) and I recover an earlier, well–known formula, [45,44], which yields, after putting \( w = u m_2 \) and summing over \( m_2 \), the formula for the Epstein \( \zeta \)–function mentioned earlier; see (83).

It is left as an exercise to show that the non-diagonal formula, (83), can be obtained in the same way using reciprocal relations for the Hurwitz \( \zeta \)–function, or, equivalently, the Lipshitz formula (cf Epstein, [33], Kober [44] p.622).

As an application of (105) it might be helpful if I present a standard derivation of the usual statistical free energy mode sum from the thermal \( \zeta \)–function expression, for a reasonably general system. The starting point is the determinant
form,
\[ F = \frac{i}{2} \lim_{s \to 0} \frac{\zeta(s, \beta)}{s} = -\frac{1}{2\beta} \lim_{s \to 0} \frac{1}{s} \sum_{m=-\infty}^{\infty} \frac{d_n}{(\omega_n^2 + 4\pi^2m^2/\beta^2)^s}, \] (106)

where \(\omega_n^2\) and \(d_n\) are the eigenvalues and degeneracies of the appropriate operator (related to the Laplacian) on the spatial section, \(\mathcal{M}\). The \(\omega_n\) are the single-particle energies. The dash here means that the denominator should never be zero (cf (74)) but the sum includes \(m = 0\). For simplicity it is assumed that there is no spatial zero mode and so the dash can be removed.

Now consider (105) for \(p = 1\). Include the \(w^{-2s}\) with the left-hand sum, set \(w = \beta\omega_n/2\pi\), multiply by the degeneracy, \(d_n\) and sum over \(n\). The limit (106) is then easily taken since \(s\Gamma(s) = \Gamma(s + 1)\) and only the \(K_{1/2}\) Bessel appears. There is nothing deep about this calculation and the result is a standard form,

\[
F = \frac{1}{2} \zeta_{\mathcal{M}}(-1/2) + \frac{1}{\beta} \sum_n d_n \sum_{m=1}^{\infty} e^{-m\omega_n\beta} m \\
= \frac{1}{2} \zeta_{\mathcal{M}}(-1/2) + \frac{1}{\beta} \sum_n d_n \log(1 - e^{-\omega_n\beta}).
\] (107)

To avoid specifically field theoretic problems, it has been assumed that \(\zeta_{\mathcal{M}}(-1/2)\) exists. This is true for the cases discussed in this paper.

The result, (107), is an analogue of the (first) Kronecker limit–formula applied to the generalised torus \(S^1 \times \mathcal{M}\); see [61]. The original formula relates to \(\mathcal{M} = S^1\) and, in this case, there exists a functional relation, (97), so that the limit–formula is often expressed in terms of the remainder about the pole in \(\zeta(s, \beta)\) at \(s = 1\). In the general case this is not possible.

Finally, if one introduces the standard arithmetic quantity \(r_p(n)\), equal to the number of representations of the integer \(n\) as the sum of \(p\) squares, then the diagonal Epstein function, (103), is obtained by writing,

\[ \phi(s) = Z_p(s) = \sum_n \frac{r_p(n)}{n^s}, \]

and is treated as such by Koshliakov who also gives the ‘exact’, product forms of \(Z_p(s)\) for \(p = 2\) and \(p = 4\).
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