On the form of potential spherical classes in $H_*Q_0S^0$

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Abstract

This note is about spherical classes in $H_*Q_0S^0$. A conjecture, due to Ed. Curtis, predicts that only Hopf invariant one and Kervaire invariant one elements will give rise to spherical classes in $H_*Q_0S^0$. Yet, there has been no proof of this conjecture around. Assuming that this conjecture fails, there must exist some other spherical classes in $H_*Q_0S^0$. This note determines the form of these potential spherical classes, and sets the target for someone who wishes to prove the conjecture, in the sense that correctness of the Curtis conjecture will be the same as failure of any classes predicted in this paper being spherical.

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1 Introduction and statement of results

We start by considering the problem of calculating the image of the Hurewicz homomorphism

\[ h : 2\pi_*QX \rightarrow H_*QX \]

for ant path connected CW-complex \( X \) of finite type, where \( QX \) is given by \( QX = \text{colim} \Omega^4 \Sigma^4 X \) and \( H_* \) denotes \( H_*(-; \mathbb{Z}/2) \). We then restrict our attention to the cases \( X = S^n \) with \( n > 0 \). This in return turns out to be fruitful, and we derive some interesting results about spherical classes in \( H_*Q_0S^0 \) where \( Q_0S^0 \) denotes the base point component of \( QS^0 = \text{colim} \Omega^4 S^4 \). Considering the collection of spaces \( \{ QS^n : n \geq 0 \} \) helps to eliminates some of these potential classes from being spherical in \( H_*Q_0S^0 \). Here \( QS^{-n} = \Omega Q_0S^{-n+1} \) with \( Q_0S^{-n} \) standing for the base component of \( QS^{-n} \).

We start by explaining our approach, and state our results together with some notes and explanations comparing our results to some other results that previously have been known.

Recall that given any space \( Y \), spherical classes in \( H_*Y \) are those element which belong to the image of the Hurewicz homomorphism. This makes it straightforward to see that if \( y \in H_*Y \) is spherical, then it has two basic properties:

- \( y \) is primitive;
- \( y \) is \( A \)-annihilated.

Here primitive is understood to be primitive with respect to the co-product induced by the diagonal map \( Y \rightarrow Y \times Y \). By \( y \in H_*Y \) being \( A \)-annihilated we mean that

\[ Sq_i^*y = 0 \text{ for any } i > 0 \]

where \( Sq_i^* : H_*Y \rightarrow H_{*-1}Y \) is the dual to the \( i \)-th Steenrod operation \( Sq_i : H^*Y \rightarrow H^{*+i}Y \). One notes that not every class in \( H_*Y \) may have both properties. Hence the above properties give an upper bound on the set of all spherical classes in \( H_*Y \), although they do not in general characterise such classes.

Our first result describes all \( A \)-annihilated classes in \( H_*QX \) which are of the form \( Q^Ix \) for some \( x \in H_*X \). We need the following definition to state this result. Let \( n \) be a positive integer with \( n = \sum n_i2^i \) with \( n_i \in \{0, 1\} \). Define the function \( \rho : \mathbb{N} \rightarrow \mathbb{N} \) by \( \rho(n) = \min\{i : n_i = 0\} \). We then have the following.

**Theorem 1.** Suppose \( Q^Ix \in H_*QX \) is given, with \( I = (i_1, \ldots, i_r) \) admissible, and \( \text{excess}(Q^Ix) > 0 \). Such a class is \( A \)-annihilated if and only if the following conditions are satisfied:

1. \( x \in \overline{H}_*X \) is \( A \)-annihilated;
2- excess($Q^I x$) $< 2^{\rho(i_1)}$;
3- $0 \leq 2i_{j+1} - i_j < 2^{\rho(i_{j+1})}$;
where excess($Q^I x$) = $i_1 - (i_2 + \cdots + i_r + \dim x)$. If $l(I) = 1$, then the first two conditions characterize all $A$-annihilated class of the form $Q^I x$ of positive excess. Notice that the fact that excess($Q^I x$) $> 0$ means that $Q^I x$ is not a square, i.e. it is an indecomposable.

Here $\Pi_*$ stands for the reduced homology; $I = (i_1, \ldots, i_r)$ is called admissible if $i_j \leq 2i_{j+1}$ for all $1 \leq j \leq r - 1$; and the length of $I$ is defined by $l(I) = r$.

**Remark 2.** Notice that if $Q^I x$ is an $A$-annihilated class with $Q^I x$ being of positive excess, then $I$ cannot have any even entry. This is easy to see, once we observe that having $i_1$ even implies that $\rho(i_1) = 0$. This together with condition 2 of the above theorem implies that excess($Q^I x$) $< 2^0 = 1$, i.e. excess($Q^I x$) $= 0$ which is a contradiction. Moreover, if there exists $j > 1$ with $i_j$ even, then condition 3 implies that $i_{j-1}$ is even. Iterated application of this will imply that $i_1$ is even which leads to a contradiction.

Conditions (2) – (3) were used in Curtis’s work [C75, Theorem 6.3] to describe the 0-line of the $E_2$-term of the unstable Adams spectral sequence converging to the 2-primary component of $\pi_* \Omega^n S^{n+k}$ were his condition (1), corresponding to our condition (2), is adapted to work for any space $\Omega^n S^{n+k}$. Curtis claims that these conditions describe a basis for the 0-line of the $E_2$-term of the unstable Adams spectral sequence, whereas Wellington [WS2, Remark 11.26] shows that this claim is not valid for the case $k = 0$. In fact Curtis’s claim is correct in odd dimensions, and can fail only in even dimensions. This later is a rather nontrivial corollary of Theorem 1 which is an outcome of its proof.

**Corollary 3.** Let $x \in \Pi_* X$ be fixed. Suppose

$$\xi = \sum Q^I x$$

is $A$-annihilated, with excess($Q^I x$) $> 0$, and $I$ runs over certain admissible sequences. Then each term $Q^I x$ is $A$-annihilated. In particular any odd dimensional class of the above form has this property.

**Note 4.** We note that the above corollary reflect an important fact about the Dyer-Lashof algebra described as following. Assume that $Q^I + Q^J \in R$ is $A$-annihilated, where $R$ is the Dyer-Lashof algebra. Assume that $l(I) = l(J)$, excess($Q^I$) $> 0$ and excess($Q^J$) $> 0$. Then $Q^I$ and $Q^J$ both are $A$-annihilated. This claim is not true if we remove the condition on the excess, i.e. there are counter examples if we allow some terms of trivial excess. In the next section, we will analyse an example due to Wellington [WS2, Remark 11.26]. This fixes previous theorem of Curtis [C75, Theorem 6.3].
A class of important examples satisfying the conditions of Corollary 3 is provided by cases $X = S^n$ with $n > 0$. Notice that $H_\ast S^n$ is given by $E_{\mathbb{Z} / 2}(g_n)$, the exterior algebra on generator $g_n \in H_n S^n$. Then we have

$$H_\ast QS^n \simeq \mathbb{Z} / 2 [Q^I g_n : I \text{ admissible}, \text{excess}(Q^I g_n) > 0].$$

Now suppose $\xi_n \in H_\ast QS^n$ is an odd dimensional $A$-annihilated class, e.g. an odd dimensional spherical class spherical. We then may write

$$\xi_n = \sum Q^I g_n$$

with $\text{excess}(Q^I g_n) > 0$. Combining this with Corollary 3 we obtain the following.

**Lemma 5.** Let $\xi_n = \sum Q^I g_n \in H_\ast QS^n$ be an odd dimensional $A$-annihilated class. Then each $Q^I g_n$ is $A$-annihilated, i.e. it satisfies conditions of Theorem 1.

The advantage of working with spherical classes is that they do pull back. In this case, we can obtain a much stronger result as following.

**Corollary 6.** Suppose $n > 0$ and $\xi_n \in H_\ast QS^n$ is spherical given by

$$\xi_n = \sum Q^I g_n$$

modulo decomposable terms, with $\text{excess}(Q^I g_n) > 0$. Then each $Q^I g_n$ is $A$-annihilated.

The proof is obvious when $n > 1$. If $\xi_n$ is odd dimensional, then this is just the statement of Lemma 5. If $\xi_n$ is even dimensional, then it pulls back to an odd dimensional spherical class $\xi_{n-1} \in H_\ast QS^{n-1}$. Now applying Lemma 5 to $\xi_{n-1}$ proves the lemma (and even a bit more!). But, the proof for the case $n = 1$ depends on some observations about $A$-annihilated primitive classes in $H_\ast Q_0 S^0$. To state our next result we need to recall some facts about $H_\ast Q_0 S^0$ and its submodule of primitive classes. Recall that $\pi_0 QS^0 \simeq \mathbb{Z}$. Given $n : S^0 \to QS^0$ we let $[n] \in H_0 Q_n S^0$ be the image of image of $1 \in H_0 S^0$, the generator of the non-base-point component in $H_0 QS^0$ under the Hurewicz map $\pi_0 QS^0 \to H_0 QS^0$. One then has $[n] * [m] = [n + m]$.

The homology ring $H_\ast Q_0 S^0$ is given by [CLM76, Part I, Lemma 4.10]

$$H_\ast Q_0 S^0 \simeq \mathbb{Z} / 2 [Q^I x_i : \text{excess}(Q^I x_i) > 0, (I, i) \text{ admissible}]$$

where $x_i = Q^i[1] * [-2]$ with $*$ being the loop sum in $H_\ast QS^0$. Regarding the submodule of primitives classes in this ring, there are different ways to describe this submodule. First, we note that the odd dimensional class $x_{2n+1}$ gives rise to a unique primitive class, say $p_{2n+1}$, i.e. modulo decomposable terms we have

$$p_{2n+1} = x_{2n+1}.$$
We also may define $p_{2n} = p_n^2$. Then it is well known [Z09, Page 92, First Description] that any primitive class in $H_\ast Q_0 S^0$ can be written as a linear combination of terms of the form $Q^i p_{2n+1}$ with $I$ admissible, and not necessarily $(I, 2n+1)$. Second, we note that $x_1$ is primitive. It is well known that applying Kudo-Araki operations $Q^i$ to any primitive class, results in another primitive class. In particular, we obtain primitive classes $Q^{2n} x_1$. Now, set

$$p'_{2n+1} = p_{2n+1} + Q^{2n} x_1.$$  

Similarly, we may define $p'_{2n} = (p_n')^2$. From the above description it is obvious that any primitive class in $H_\ast Q_0 S^0$ can be written in terms of $Q^i p'_{2n+1}$ with $I$ admissible, and not necessarily $(I, 2n+1)$. This is the basis described by Madsen [M70 Proposition 6.7]. This is of course our favorite basis, as $x_{2i+1}$ and $p'_{2i+1}$ show similar behavior under the action of the Steenrod algebra. Regarding the problem of spherical classes in $H_\ast Q_0 S^0$ it is well known that there is Hopf invariant one element in $2\pi Q_0 S^0$ if and only if $p'_{2i+1}$ is spherical which will happen only if $2n+1 = 2^s - 1$ for some $s > 0$. Moreover, it is well known there is Kervaire invariant one element in $2\pi Q_0 S^0$ if and only if $Q^{2n+1} p'_{2n+1} = (p'_{2n+1})^2$ is spherical which can happen only if $2n+1 = 2^s - 1$ for some $s > 0$ [M70 Proposition 7.3]. This latter result has also been proved by others in various equivalent form, see for example [E81 Proposition 4.1], [ST82, Theorem A]. It is also known that $Q^k p'_{2n+1}$ can not be spherical if $k \neq 2n+1$ [Z09, Chapter 2, Remark 5]. Our next result, which is the main goal of this paper settles down the rest of potential spherical classes in $H_\ast Q_0 S^0$ and describes their form.

**Main Theorem.** Let $\theta \in H_\ast Q_0 S^0$ be a spherical class which is not a Hopf invariant one class, neither a Kervaire invariant one class. Then $\theta$ satisfies one of the following cases.

1. If $\sigma_\ast \theta \neq 0$ and $\theta$ is an odd dimensional class, then

$$\theta = \sum Q^i p'_{2i+1},$$  

with $l(I) > 1$ such that each of terms $Q^i p'_{2i+1}$ in the above sum is $A$-annihilated.

2. If $\sigma_\ast \theta \neq 0$ and $\theta$ is an even dimensional class, then

$$\theta = \sum Q^i p'_{2i+1} + P^2,$$

with $l(I) > 1$ where I has only has odd entries. In this case $(I, 2i+1)$ satisfies condition 3 of Theorem 2, i.e. $0 < 2i_{j+1} - i_j < 2^{\delta(i_{j+1})}$ for $1 \leq j \leq r$ with $i_{r+1} = 2i+1$. Moreover, excess($Q^i p'_{2i+1}$) $- 1 < 2^{\delta(i)}$ for every $Q^i p'_{2i+1}$ involved in the above sum. Here $P$ is a primitive term. If $P \neq 0$, then it is of odd dimension. If $P = 0$, then each term in the above expression for $\theta$ is $A$-annihilated.

3. If $\sigma_\ast \theta = 0$, then $\theta = \xi^2$, with $\xi$ an odd dimensional $A$-annihilated primitive class, i.e.

$$\theta = (\sum Q^i p'_{2i+1})^2,$$
with \( l(I) > 0 \) such that each of terms \( Q^I p^{2i+1}_{2i+1} \) in the above sum is \( A \)-annihilated. Moreover, assume that \( f \in 2\pi_sQ_0S^0 \) be a class with \( hf = \theta \). Then \( f \) maps nontrivially under the projection

\[
2\pi_sQ_0S^0 \rightarrow 2\pi_s\text{Coker} J
\]

where \( J \) is the \( J \)-homomorphism. In all of the above cases \((I, 2i + 1)\) is supposed to be admissible.

Along the line, we establish a notation for \( H^*_QX \) which previous was known to be an exterior algebra [CP89, Theorem 1.1]. This will involve an observation on the image of homology of the complex transfer, viewed as a map

\[
Q\Sigma\mathbb{C}P \rightarrow Q_0S^0.
\]

Moreover, a calculation for the submodule of primitive classes in \( H_\ast QCP \) will be presented, and will be applied in proving a part of our main theorem. This paper is organised following. We prove Theorem 1, and corollaries implies by it, in Section 1. We then move to prove our main theorem. We prove our main theorem in several steps, as the proof is quite long. We have use single numbering in this section, where as in other sections, the results are labeled based on the number of section.

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2 Proof of Theorem 1

We start by recall some fact about \( H_\ast QX \) where \( X \) is an arbitrary path connected space. Recall that having fixed an additive basis \( \{x_\alpha\} \) for \( \Pi_\ast X \), with \( X \) path connected, then \( H_\ast QX \) is a polynomial algebra with generators given by the symbols \( Q^I x_\alpha \), with \( I \) admissible. Allowing the empty sequence \( \phi \) to be an admissible sequence, with \( Q^\phi x = x \), and \( \text{excess}(\phi) = +\infty \), then we can see that \( Q^I x \) is a decomposable if and only if \( \text{excess}(Q^I x_\alpha) = 0 > 0 \) where in this case \( Q^I x = (Q^{i_2} \cdots Q^{i_r} x)^2 \) with \( I(i_1, \ldots, i_r) \). Hence, Theorem 2 determines all \( A \)-annihilated classes in \( H_\ast QX \) classes of the form \( Q^I x \) with are not square. Notice that in general, any class in \( H_\ast QX \) involving at least one term \( Q^I x \) of positive excess with \( I \) determines a nonzero class in \( QH_\ast QX \), the module of indecomposables of \( H_\ast QX \), i.e. it gives rise to an indecomposable element.

We only use the Nishida relations. The Nishida relation is given as following [CLM76 Part I, Theorem 1.1(9)],

\[
Sq_a^b Q^I = \sum_{r \geq 0} \binom{b-a}{a-2r} Q^{b-a+r} Sq_r^I.
\]
Notice that $Sq^a I$ with $l(I) > 1$ may be computed by iterated use of the Nishida relations. One observes that the Nishida relations respect the length, i.e. if

$$Sq^a I = \sum Q^K Sq^a K,$$

then $l(I) = l(K)$.

Let $R$ denote the Dyer-Lashof algebra. Then according to [M75, Equation 3.2] the Nishida relations maybe used to define an action $N : A \otimes R \to R$ as following

$$N(Sq^a, Q^b) = \left( b - a \right) Q^{b-a},$$

$$N(Sq^a, Q^{i_1} \cdots Q^{i_r}) = \sum \left( i_1 - a \right) Q^{i_1-a} \cdots N(Sq^a, Q^{i_r}).$$

In other words suppose $Sq^a I = \sum Q^K Sq^a K$ where $aK \in \mathbb{Z}$. Then we have

$$N(Sq^a, I) = \sum_{aK=0} Q^K.$$

We note that if a sequence $I$ is admissible, then it is not clear whether or not after applying $Sq^a$ we will get a sum of admissible terms, i.e we may need to use the Adem relations to rewrite terms in admissible form. This means that we may decide about vanishing or non-vanishing of a homology class $Sq^a I x$ after rewriting it in admissible form.

**Example 2.1.** Consider $Q^9 Q^5 g_1$ which is an admissible term. One has

$$Sq^4 Q^9 Q^5 g_1 = Q^7 Q^3 g_1,$$

where $Q^7 Q^3$ is not admissible. Although it may look nontrivial, however the Adem relation $Q^7 Q^3 = 0$ implies that $Q^7 Q^3 g_1 = 0$. Indeed the class $Q^9 Q^5 g_1$ is not $A_*$-annihilated, which can be seen by applying $Sq^2$ as we have

$$Sq^2 Q^9 Q^5 g_1 = Q^7 Q^5 g_1 \neq 0.$$

Notice that the right hand side of the above equation is an admissible term.

According to the above example, part of the job in distinguishing between $A$-annihilated and not-$A$-annihilated classes $Q^I x$ is to choose the right operation $Sq^a$ in a way that the outcome is admissible and there is no need to use the Adem relations after the Kudo-Araki operation. The reason being that it is practically impossible to use the Adem relations when $l(I)$ is big. The following lemma [C75, Lemma 6.2] tells us when it is not possible to choose the “right” operation and provides us with the main tool towards the proof of Theorem 1.
Lemma 2.2. Suppose $I$ is an admissible sequence such that $2i_{j+1} - i_j < 2^{\rho(i_{j+1})}$ for all $1 \leq j \leq r - 1$. Let

$$N(Sq_*^a, Q^I) = \sum_{K\text{ admissible}} Q^K.$$ 

Then

$$\text{excess}(K) \leq \text{excess}(I) - 2^{\rho(i_1)}.$$ 

The above lemma can also be obtained by combining [W82, Theorem 7.11], [W82, Theorem 7.12] and [W82, Lemma 12.5]. Now we are ready to prove Theorem 1. We break it into little lemmata. The following proves Theorem 1 in one direction.

Lemma 2.3. Let $x \in H_\ast X$ be $A$-annihilated, and $I$ an admissible sequence with $\text{excess}(Q^I x) > 0$ such that

1. $\text{excess}(Q^I x) < 2^{\rho(i_1)}$;
2. $2i_{j+1} - i_j < 2^{\rho(i_{j+1})}$ for all $1 \leq j \leq r - 1$.

Then $Q^I x$ is $A$-annihilated.

Proof. Let $r > 0$. Then we have the following

$$Sq_*^a Q^I x = \sum Q^K Sq_*^a Q^K x = \sum_{a^K = 0} Q^K x.$$ 

But notice that according to Lemma 2.2

$$\text{excess}(Q^K x) \leq \text{excess}(Q^I x) - 2^{\rho(i_1)} < 0.$$ 

Hence the above sum is trivial, and we are done.

This proves the Theorem 2 in one direction. Now we have to show that the reverse direction holds as well. That is we have to show if either of conditions (1)-(3) of Theorem 2 does not hold then $Q^I x$ will be not-$A$-annihilated.

Remark 2.4. Before proceeding, we recall a basic property of the function $\rho$ defined before Theorem 2 which is as following. Notice that given a positive integer $n$, then $\rho(n)$ is the least integer $t$ such that

$$\binom{n - 2^t}{2^t} \equiv 1 \mod 2.$$ 

Notice that if $n = \sum n_i 2^i$ and $m = \sum m_i 2^i$ are given with $n_i, m_i \in \{0, 1\}$ then $\binom{n_i}{m_i} = 1 \mod 2$ if and only if $n_i \geq m_i$ for all $i$. This makes it easy to verify the above property for $\rho$.

The next three lemmata show that if any of conditions (1), (2) or (3) doesn’t hold, then $Q^I x$ will not be $A$-annihilated.

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Lemma 2.5. Let $X$ be path connected. Suppose $I = (i_1, \ldots, i_r)$ is an admissible sequence, such that $\text{excess}(Q^I x) \geq 2^{\rho(i_1)}$. Then such a class is not $A$-annihilated.

Proof. This is quite straightforward. We may use $Sq^\rho_\ast$ with $\rho = \rho(i_1)$, which gives

$$Sq^\rho_\ast Q^I x = Q^{i_1-2\rho} Q^{i_2} \cdots Q^{i_r} x + O$$

where $O$ denotes other terms given by

$$O = \sum_{t > 0} \left( \frac{i_1 - 2^\rho}{2^\rho - 2t} \right) Q^{i_1-2^\rho+t} Sq^t_\ast Q^{i_2} \cdots Q^{i_r} x.$$ 

Notice that $\text{excess}(Q^I x) \geq 2^{\rho(i_1)}$ ensures that $i_1$ is not of the form $2^\rho$. Looking at the binary expression implies that all coefficients in $O$ are nontrivial, and $O$ will depend on the action of $Sq^\rho_\ast$ on terms $Q^{i_2} \cdots Q^{i_r} x$. However, all of these terms are terms of lower excess, and they will not cancel the first term in the right hand side of the above relation.

Notice that at the right hand side of the term $Q^{i_1-2\rho} Q^{i_2} \cdots Q^{i_r} x$ is obviously admissible. Moreover,

$$\text{excess}(Q^{i_1-2\rho} Q^{i_2} \cdots Q^{i_r} x) = \text{excess}(Q^I x) - 2^\rho \geq 0.$$ 

This proves that $Sq^\rho_\ast Q^I x \neq 0$. Notice that if $\text{excess}(Sq^\rho_\ast Q^I x) = 0$, then

$$Sq^\rho_\ast Q^I x = (Q^{i_2} \cdots Q^{i_r} x)^2 \neq 0.$$ 

This completes the proof. \[\square\]

The above lemma shows that if (2) of Theorem 2 does not hold, then we will have a class which is not $A$-annihilated. Next, we move on to the case when condition (3) does not hold.

Lemma 2.6. Let $X$ be path connected. Suppose $I = (i_1, \ldots, i_r)$ is an admissible sequence, and let $Q^I x$ be given with $\text{excess}(Q^I x) > 0$ such that $2i_{j+1} - i_j \geq 2^{\rho(i_{j+1})}$ for some $1 \leq j \leq r-1$. Then such a class is not $A$-annihilated.

Proof. Assume that $Q^I x$ satisfies the condition above. We may write this condition as

$$i_j - 2^\rho \leq 2i_{j+1} - 2^{\rho+1} = 2(i_{j+1} - 2^\rho),$$

where $\rho = \rho(i_{j+1})$. This is the same as admissibility condition for the pair $(i_j - 2^\rho, i_{j+1} - 2^\rho)$. In this case we use $Sq^{\rho+j}_\ast$ where we get

$$Sq^{\rho+j}_\ast Q^I x = Q^{i_1-2^{\rho+j-1}} Q^{i_2-2^{\rho+j-2}} \cdots Q^{i_{j-2^\rho}} Q^{i_{j+1}-2^\rho} Q^{i_{j+2}} \cdots Q^{i_r} x + O$$

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where \( O \) denotes other terms, and similar to previous lemma will be a sum of terms of lower excess. The first term in right hand side of the of the above equality is admissible. Moreover,

\[
\text{excess}(Sq_x^{2^j} Q^I x) = (i_1 - 2^{j-1}) - (i_2 - 2^{j-2}) - (i_j - 2^j - (i_{j+1} - 2^j) - (i_{j+2} + \cdots + i_r + \dim x) \\
= i_1 - (i_2 + \cdots + i_r + \dim x) \\
= \text{excess}(Q^I x) > 0,
\]

where by abuse of notation we have written \( \text{excess}(Sq_x^{2^j} Q^I x) \) to denote the excess of the first term in the above equality. This implies that

\[ Sq_x^{2^j} Q^I x \neq 0, \]

and hence completes the proof.

**Remark 2.7.** According to the proof in this case we always have

\[ \text{excess}(Sq_x^{2^j} Q^I x) > 0 \]

which means that we always end up with an indecomposable term after applying the “right” operation, i.e. the outcome will not be a square. This little observation will be useful.

Now we show that the condition (1) is also necessary in the proof of the main theorem.

**Lemma 2.8.** Let \( X \) be path connected, and \( x \in \overline{H}_* X \) be not \( A \)-annihilated. Then \( Q^I x \) is not \( A \)-annihilated.

**Proof.** Let \( t \) be the least number that

\[ Sq_x^{2^t} x \neq 0. \]

If \( I = (i_1, \ldots, i_r) \), we apply \( Sq_x^{2^t + r} \) to \( Q^I x \), where we get

\[
Sq_x^{2^t + r} Q^I x = Q^{i_1-2^t} \cdots Q^{i_r-2^t} S q_x^{2^t} x + O,
\]

where \( O \) denotes sum of other terms which are of the form \( Q^I y \) with \( \dim y > \dim Sq_x^{2^t} x \). This means that the first term in the above equality will not cancel with any of other terms.

By abuse of notation we write \( \text{excess}(Q^{i_1-2^t} \cdots Q^{i_r-2^t} S q_x^{2^t} x) \) to denote the excess of the first term in the above equality. We have \( \text{excess}(Q^{i_1-2^t} \cdots Q^{i_r-2^t} S q_x^{2^t} x) = \text{excess}(Q^I x) > 0. \) Moreover,

\[ Q^{i_1-2^t} \cdots Q^{i_r-2^t} \]

is admissible. Hence \( Sq_x^{2^t+r} Q^I x \neq 0. \) Note that similar to the previous lemma we end up with an indecomposable term. \( \square \)
This completes the final step in the proof of Theorem 1. Our next task is to prove Corollary 3, which is very important for us. We recall that according to Corollary 3 if we have an \(A\)-annihilated sum of terms of the form \(Q^l x\) with \(\text{excess}(Q^l x) > 0\), then each of these terms must be \(A\)-annihilated.

### 2.1 Separating not-\(A\)-annihilated classes

Here we give a sketch of the proof for Corollary 3, and refer the reader to [Z09, Subsection 3.2] for more details. We show that if \(Q^l x\) and \(Q^J x\) are two terms which are not \(A\)-annihilated, both of positive excess, then their sum is not \(A\)-annihilated. This then will prove Corollary 3, as well as the general claim Note 4.

Theorem 1 provides us with a complete description of \(A\)-annihilated classes in \(H_\ast QX\) of the form \(Q^I x\) of positive excess, with \(X\) being path connected. In return, this also classifies all such classes that are not \(A\)-annihilated. For instance given a class \(Q^l x \in H_\ast QX\), this class will not be \(A\)-annihilated, if at least one of conditions in Theorem 2 does not hold; i.e.

- \(x\) is not \(A\)-annihilated,
- \(\text{excess}(Q^l x) \geq 2^{\rho(i_1)}\),
- There exists \(j\) such that \(2i_{j+1} - i_j \geq 2^{\rho(i_{j+1})}\).

Therefore, proving Corollary 3 comes down to analyse these cases and showing that if we have two terms \(Q^l x\) and \(Q^J x\) with \(l(I) = l(J)\), both of positive excess, and each being not-\(A\)-annihilated for one of the above reasons the \(Q^l x + Q^J x\) is not \(A\)-annihilated as well. It is obvious, that if the two terms are not \(A\)-annihilated for the same reason, then their sum is not \(A\)-annihilated. For instance we have the following example.

**Lemma 2.9.** Suppose \(Q^l x, Q^J x \in H_\ast QX\) are given, \(l(I) = l(J) = r\), with \(x \in H_\ast X\) not being \(A\)-annihilated. Then \(Q^l x + Q^J x\) is not \(A\)-annihilated.

**Proof.** We need to find an integer \(k\) such that \(S^k \ast Q^l x \neq S^k \ast Q^J x\). We do the same as we did in the proof of Lemma 2.8. Let \(t\) be the least number such that \(S^t 2^r x \neq 0\). We then observe that

\[
S^t 2^{t+r} Q^l x \neq S^t 2^{t+r} Q^J x.
\]

This completes the proof. \(\square\)

As another example consider the following case.

**Lemma 2.10.** Let \(Q^l x\) and \(Q^J x\) be two classes that are not \(A\)-annihilated such that \(\text{excess}(Q^l x) \geq 2^{\rho(i_1)}\), and \(\text{excess}(Q^J x) \geq 2^{\rho(j_1)}\). Then there exists \(k\) such that

\[
S^k \ast Q^l \neq S^k \ast Q^J x.
\]
Proof. If we choose \( \rho = \min\{\rho(i_1), \rho(j_1)\} \), then it is clear that
\[
Sq^2 \rho^* Q^I x \neq Sq^2 \rho^* Q^J x.
\]

The key ingredient is that if \( Sq^k Q^I x \neq 0 \) with \( k \) being least such number, and \( Sq^{k'} Q^J x \neq 0 \) with \( k' \) being least such number. We then may use \( Sq^{\min(k,k')} \) to show that \( Q^I x + Q^J x \) is not \( A \)-annihilated. We refer the reader to [Z09, Subsection 3.2] for more details.

Finally, we give an example showing that the conditions \( \text{excess}(Q^I) > 0 \), \( \text{excess}(Q^J) > 0 \) are necessary. The following example is due to Wellinton [W82, Remark 11.26].

Example 2.11. One may check that the following class is an \( A \)-annihilated element in the Dyer-Lashof algebra \( R \),
\[
(Q^{2062} Q^{1031} Q^{519} Q^{263} Q^{135} Q^{71} Q^{39})^2 + Q^{4120} Q^{2062} Q^{1031} Q^{519} Q^{263} Q^{135} Q^{71} Q^{39}
\]
Notice that according to Theorem 1 \( Q^{1031} Q^{519} Q^{263} Q^{135} Q^{71} Q^{39} \) which is involved in both terms is \( A \)-annihilated. This then makes it easy to see that the above sum is \( A \)-annihilated, as there are a few operations available at this stage which may map the above sum nontrivially. However, each term is not \( A \)-annihilated under the action of \( Sq^2 \). Notice that the first term is a square, and of trivial excess, and so does not satisfy conditions of Corollary 3.

Lemma 5 is just a special case of Corollary 3, and Corollary 6 is implied by Corollary 3 and Lemma 5. Recall that according to Corollary 6 if \( \xi_n = \sum Q^I g_n \in H_* QS^n \) is spherical, then every term in the above sum is \( A \)-annihilated. This is the statement of Lemma 5 if \( \xi_n \) is odd dimensional. Let \( n > 1 \), and let \( \xi_n \) be even dimensional. As \( \xi_n \) is spherical, then it pulls back to an odd dimensional spherical class \( \xi_{n-1} \in H_* QS^{n-1} \). According to Lemma 5
\[
\xi_{n-1} = \sum Q^I g_{n-1}
\]
with each \( Q^I g_{n-1} \) being \( A \)-annihilated. Applying suspension to the above class we obtain,
\[
\xi_n = \sum Q^I g_n
\]
with each \( Q^I g_n \) being \( A \)-annihilated. This completes the proof of Corollary 6.

3 Proof of The Main Theorem

Our main theorem is a combination of Corollary 3, and facts implies by it in Remark 4, together with a comparison between different bases for the submodule of primitives
in $H_*Q_0S^0$. As stated, our theorem result, couples this results with the behavior of the potential spherical classes $\theta \in H_*Q_0S^0$ under the homology suspension

$$\sigma_* : H_*Q_0S^0 \rightarrow H_{*+1}QS^1.$$  

If a spherical class does survive under the suspension homomorphism, then our main theorem reduces to a form of Remark 4. But if our spherical class $\theta \in H_*Q_0S^0$ dies under the homology suspension, it then tells us that $\theta$ is a decomposable primitive, i.e. a square. This then implies that $\theta = \zeta^{2t}$ for some $t > 0$. We shall show that it is not possible to have a spherical class $\theta = \zeta^{2t} \in H_*Q_0S^n$ with $t > 1$. This will be achieved by a preparation on the behavior of the $S^1$-transfer, together with some observations on $H_*Q_0S^{-1}$, $H_*Q_0S^{-2}$ and $H_*Q\Sigma^{-1}CP_+$. Finally, we note that it is in general true that if $\theta = \zeta^{2t} \in H_*QS^n$ is spherical with $n > 1$, then $t < 2$. The proof of this fact is a simplified version of its proof for the case when $n = 0$ where one will need to desuspend only once and apply $S_4^{-1}$. We leave the details to reader. A proof of this is to be found in [Z09, Page 60].

### 3.1 The homology rings $H_*Q_0S^{-1}$, $H_*Q\Sigma^{-1}CP$

We start by recalling some fact about the Eilenberg-MacLane spectral sequence. This spectral sequence is one of the main tools in calculating the homology of loop spaces is the Eilenberg-Moore spectral sequence. We recall the following [G04, Proposition 7.3].

**Proposition 3.1.** Let $X$ be simply connected, with $H^*X$ polynomial. Then $H_*\Omega X$ is an exterior algebra, and the suspension

$$\sigma_* : QH_*\Omega X \rightarrow PH_*X$$

is an isomorphism, and the Eilenberg-Moore spectral sequence

$$E^2 = \text{Cotor}^{H_*X}(\mathbb{Z}/2, \mathbb{Z}/2) \Rightarrow H_*\Omega X$$

collapses. In particular,

$$H_*\Omega X \simeq E_{\mathbb{Z}/2}(\sigma^{-1}_*PH_*X),$$

where $E_{\mathbb{Z}/2}(\sigma^{-1}_*PH_*X)$ denotes the exterior algebra over $\mathbb{Z}/2$ generated by $\sigma^{-1}_*PH_*X$.

The next result identifies some cases where $H^*Q_0X$ is a polynomial algebra. Recall that given any space, one may define the Frobenius homomorphism $s : H^*X \rightarrow H^*X$ as before, i.e. $s(x) = x^2$. One then has the following [G04, Lemma 7.2].

**Lemma 3.2.** The cohomology algebra $H^*Q_0X$ is a polynomial algebra if $s : H^*X \rightarrow H^*X$ is injective. Here $Q_0X$ denotes the base point component of $QX$. 

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The above theorems provide the main tool to calculate the homology rings $H_*Q_0S^{-1}$ and $H_*\Omega_0QP$, where one chooses $X = \widetilde{Q_0S^0}/Q\overline{P}$. Here $\overline{Y}$ denotes the universal cover of a given space $Y$. We refer the reader to [G04] for the proof of the machinery provided above. We recall the calculation of $H_*Q_0S^{-1}$.

**Example 3.3.** First, notice that the squaring map $H^*S^0 \to H^*S^0$ is injective. This implies that $H^*Q_0S^0$ is polynomial. Recall from Appendix D that $Q_0S^0 = P \times Q\overline{S^0}$. Hence $H^*Q\overline{S^0}$ is polynomial as well. On the other hand notice that $Q\overline{S}^{-1} = \Omega Q_0S^0$, which implies that $Q_0S^{-1} = \Omega Q_0S^0$. Now putting $X = \tilde{Q_0S^0}$ in Proposition 5.4 implies that $H_*Q_0S^{-1}$ is an exterior algebra, with $\sigma_* : QH_*Q_0S^{-1} \to PH_*Q_0S^0$ an isomorphism, i.e.

$$H_*Q_0S^{-1} = E_{\mathbb{Z}/2}(\sigma_*^{-1}PH_*Q_0S^0).$$

This is due to Cohen-Peterson [CP89, Theorem 1.1].

We shall combine the information by this example with the $S^1$-example to establish our notation for $H_*Q_0S^{-1}$ which will provide us with a “geometric description” of its generators. But, first we look into another example.

**Example 3.4.** Let $X = \mathbb{C}P, \mathbb{C}P_+$. Then the Frobenius homomorphisms $H^*X \to H^*X$ is injective. We then obtain

$$H_*Q\Sigma^{-1}\mathbb{C}P \simeq E_{\mathbb{Z}/2}(\sigma_*^{-1}PH_*Q\mathbb{C}P).$$

The above example calculates $H_*Q\Sigma^{-1}\mathbb{C}P$. We only need to describe the submodule of primitives in this algebra. However, unlike the case of $H_*Q_0S^{-1}$, we do not have a natural and geometric way of identifying the generators of $H_*Q\Sigma^{-1}\mathbb{C}P$.

First, we deal with $H_*Q_0S^{-1}$ and determine its structure as a module over the Steenrod algebra $A$, and the Dyer-Lashof algebra $R$.

The $S^1$-transfer is a map $\lambda_\mathbb{C} : Q\Sigma\mathbb{C}P_+ \to Q\mathbb{S}^0$. The homology of this map is known based on the work of Mann-Miller-Miller [MMM86, Lemma 7.4]. It factors through the complex $J$-homomorphism

$$J_\mathbb{C} : U \to Q_1\mathbb{S}^0.$$

Using the translation map $*[-1]$ we then will land in $Q_0\mathbb{S}^0$. The map $\lambda_\mathbb{C}$ is an infinite loop map, obtained as the infinite loop extension of the composite

$$\Sigma\mathbb{C}P_+ \xrightarrow{} U \xrightarrow{} Q_1\mathbb{S}^0 \xrightarrow{} Q_0\mathbb{S}^0.$$
x_{2i+1} + O_{2i+1} where O_{2i+1} denotes the other terms. On the other hand, notice that 
\Sigma c_i is primitive. Also, the image must have the same behavior under the action of the 
Steenrod algebra as \Sigma c_{2i}. Hence we obtain,

\((\lambda_C)_* \Sigma c_{2i} = p_{2i+1} + Q^{2i}x_1 = p'_{2i+1}.\)

Moreover, notice that \Sigma c_0 maps to \(x_1 = p_1 = p'_1\) where \(c_0\) is the generator coming from the disjoint base point. This then allows one to calculate \((\lambda_C)_* : H_*Q\Sigma CP_+ \to H_*Q_0S^0\). Notice that this in particular implies that \((\lambda_C)_* : PH_*Q\Sigma CP_+ \to PH_*Q_0S^0\) is an epimorphism. We are now ready to complete the calculation of \(H_*Q_0S^{-1}\).

**Theorem 3.5.** The homology algebra \(H_*Q_0S^{-1}\) as an \(R\)-module is given by

\[ E_{\mathbb{Z}/2}(Q^I w_{2i}': I \text{ admissible, dim } I > 2i), \]

with \(w_{2i}' = (\Omega \lambda_C)_* c_{2i} \) which satisfies \(\sigma_* w_{2i}' = p_{2i+1}'\). Two generators \(Q^I w_{2i}'\) and \(Q^I w_{2j}'\) may be identified if and only if they map to the same element in \(H_*Q_0S^0\) under the homology suspension \(\sigma_* : H_{*+1}Q_0S^{-1} \to H_*Q_0S^0\). The behavior of generators \(w_{2i}'\) under the Steenrod operation is very much like \(c_{2i} \in H_2\mathbb{C}P_\), i.e.

\[ Sq_*^{2k} w_{2i}' = \binom{i-k}{k} w_{2i-2k}. \]

This together with the Nishida relations completely determines the \(A\)-module structure of \(H_*Q_0S^{-1}\).

Moreover, the the mapping

\[ (\Omega \lambda_C)_* : H_*Q\Sigma CP_+ \to H_*Q_0S^{-1} \]

is an epimorphism.

**Proof.** The fact that \((\lambda_C)_* \Sigma c_{2i} = p_{2i+1}\) allows us to define unique elements \(w_{2i}' \in H_{2i}Q_0S^{-1}\) by

\[ (\Omega \lambda_C)_* c_{2i} = w_{2i}'. \]

Notice that the space \(Q_0S^{-1}\) is an infinite loop space, and hence we may consider terms of the form \(Q^I w_{2i}' \in H_*Q_0S^{-1}\). These classes have the property that

\[ \sigma_* Q^I w_{2i}' = Q^I p'_{2i+1} \]

where \(\sigma_*\) denotes the homology suspension. The fact that elements of the form \(Q^I p'_{2i+1}\) generate all primitives in \(H_*Q_0S^0\) implies that elements of the form \(Q^I w_{2i}'\) generate \(QH_*Q_0S^{-1}\), and therefore \(H_*Q_0S^{-1}\) is the exterior algebra generated by \(Q^I w_{2i}'\) with \(I\) admissible. This also determines the action of the Dyer-Lashof algebra on the homology.
ring $H_\ast Q_0 S^{-1}$. Moreover, our definition of the generators $w'_2i$ allows us to derive the action of the Steenrod operation on these classes, namely we have

$$Sq^2_i w'_{2i} = \binom{i-k}{k} w'_{2i-2k}.$$  

This together with the Nishida relations describes the action of the Steenrod algebra on the generators $Q^I w'_{2i}$, and hence completely determines the action of the Steenrod algebra on the homology ring $H_\ast Q_0 S^{-1}$.

Finally notice that although we have identified generators of $H_\ast Q_0 S^{-1}$, however there are some relations among these generators. For example consider $Q^3x_1 = x_1^4 = Q^2Q^1x_1 \in H_4 Q_0 S^0$. Hence in $H_\ast Q_0 S^{-1}$ we have

$$Q^3 w_0 = Q^2 Q^1 w_0.$$  

This then shows that two generators in this algebra maybe identified if they map to the same primitive class under the homology suspension. Finally, the definition of the generators $w'_2i$ shows that the looped transfer induces an epimorphism in homology.

This completes the proof.  

It is possible to choose a different set of generators for the submodule of primitive in $H_\ast Q_0 S^0$ in order to give a presentation of $H_\ast Q_0 S^{-1}$ with no relation among its generators [Z09, Proposition 5.37]. However, this description is not as much easy to work with as the above presentation. As the above presentation is adequate for our purpose, we then choose to work with this description. Indeed, it is possible to identify a subalgebra of $H_\ast Q_0 S^{-1}$ with no relations among its generators.

**Lemma 3.6.** There is no relation among the generators of the subalgebra of $H_\ast Q_0 S^{-1}$ given by

$$E_{\mathbb{Z}/2}(Q^I w'_{2i} : (I, 2i + 1) \text{ admissible, } \text{excess}(I, 2i + 1) > 0),$$

with $\text{excess}(I, 2i + 1) = i_1 - (i_2 + \cdots + i_r + 2i + 1)$ where $I = (i_1, \ldots, i_r)$.

We leave the proof to the reader, as it is an obvious outcome of applying the homology suspension.

Next, we determine the submodule of primitives in $H_\ast QCP$ and $H_\ast Q_0 S^{-1}$. This will imply that $(\Omega \lambda_\mathcal{C})_\ast$ is an epimorphism when restricted to the submodules. We quote the result on this and refer the reader to [Z09, Section 5.8] for proofs and details of the calculations.

**Proposition 3.7.** Any primitive class $H_\ast Q_0 S^{-1}$ maybe written as linear combination of classes of the form $Q^L \mathcal{P}^{S^{-1}}_{4n+2}$ and $Q^K \mathcal{P}^{S^{-1}}_{i,j}$ where $L$ and $K$ are chosen to be admissible.
The primitive classes \( p_{4n+2}^{S^{-1}} \), \( Q^K p_{i,j}^{S^{-1}} \) are defined by
\[
\begin{align*}
p_{4n+2}^{S^{-1}} &= w_{4n+2}' \in H_{4n+2}Q_0S^{-1} \\
p_{i,j}^{S^{-1}} &= Q^{2i+1}w_{2j}' \in H_{2i+2j+1}Q_0S^{-1},
\end{align*}
\]
modulo decomposable terms, with \( j \) being even.
Similarly, any primitive class in \( H_*Q\mathbb{C}P \) may be written as linear combination of classes of the form \( Q^L p_{4n+2}^{CP} \) and \( Q^K p_{i,j}^{CP} \) where \( L \) and \( K \) are chosen to be admissible. The primitive classes \( p_{4n+2}^{CP} \), \( Q^K p_{i,j}^{CP} \) are defined by
\[
\begin{align*}
p_{4n+2}^{CP} &= c_{4n+2} \in H_{4n+2}Q\mathbb{C}P \\
p_{i,j}^{CP} &= Q^{2i+1}c_{2j} \in H_{2i+2j+1}Q\mathbb{C}P,
\end{align*}
\]
modulo decomposable terms, with \( j \) being even. Moreover,
\[
\begin{align*}
(\Omega\lambda_{\mathbb{C}})\ast p_{4n+2}^{CP} &= p_{4n+2}^{S^{-1}}, \\
(\Omega\lambda_{\mathbb{C}})\ast p_{i,j}^{CP} &= p_{i,j}^{S^{-1}}.
\end{align*}
\]

Recall from Example 3.4, that we may apply Proposition 3.4 to calculate \( H_*Q\Sigma^{-1}\mathbb{C}P \) using
\[
H_*Q\Sigma^{-1}\mathbb{C}P \simeq E_{Z/2}(\sigma_*^{-1}PH_*Q\mathbb{C}P),
\]
and our calculation of the primitive classes. We then have the following observation.

**Proposition 3.8.** As an \( R \)-module \( H_*Q\Sigma^{-1}\mathbb{C}P \) is given by the exterior algebra over the generators \( Q^I v_{4n+1}^{CP} \) and \( Q^L v_{i,j-1}^{CP} \) with \( I \) and \( L \) admissible, and \( \dim I > 4n+1 \) and \( \dim L > 2i+2j \). Here
\[
\begin{align*}
\sigma_* v_{4n+1}^{CP} &= p_{4n+2}^{CP}, \\
\sigma_* v_{i,j-1}^{CP} &= p_{i,j}^{CP},
\end{align*}
\]
with \( v_{4n+1}^{CP} \in QH_{4n+1}Q\Sigma^{-1}\mathbb{C}P \) and \( v_{i,j-1}^{CP} \in QH_{2i+2j}Q\Sigma^{-1}\mathbb{C}P \) where \( j \) is even. The generators \( Q^I v_{4n+1}^{CP} \) are independent from each other for different choices of admissible \( I \). Two generators of the form \( Q^L v_{i,j-1}^{CP} \) are identified if they map to the same class in \( H_*Q\mathbb{C}P \) under the homology suspension.

Notice that the above presentation does not give a geometric meaning for the generators of \( H_*Q\Sigma^{-1}\mathbb{C}P \), i.e. we do not know of a natural way to define the generators \( v_{4n+1}^{CP} \) and \( v_{i,j-1}^{CP} \). However, this description is enough for our purpose. Notice that any spherical class \( H_*Q_0S^{-1} \) is primitive, and hence lies in the image of \( (\Omega\lambda_{\mathbb{C}})\ast \). Moreover, the description of \( H_*Q\Sigma^{-1}\mathbb{C}P \) tell us that any primitive class in \( H_*Q\mathbb{C}P \) pulls back through the homology suspension. The following observation is then clear.
Lemma 3.9. Suppose \( \xi \in H_*Q_0S^{-1} \) is a spherical class. This pulls back to a spherical class in \( \xi_2 \in H_*QS^{-2} \) which lives in an exterior subalgebra of \( H_*Q_0S^{-2} \) given by

\[
E_{Z/2}(Q^I v_{4n+1}, Q^Lv_{i,j-1} : I, L \text{ admissible}),
\]

where \( v_{4n+1} \in H_{4n+1}QS^{-2} \) and \( v_{i,j-1} \in H_{2i+2j}QS^{-2} \) are defined by

\[
v_{4n+1} = (\Omega^2 \lambda_\mathbb{C})_* v^{CP}_{4n+1},
\]
\[
v_{i,j-1} = (\Omega^2 \lambda_\mathbb{C})_* v^{CP}_{i,j-1}.
\]

Moreover, the definition of these classes imply that

\[
\sigma_* v_{4n+1} = p^{S^{-1}}_{4n+2},
\]
\[
\sigma_* v_{i,j-1} = p^{S^{-1}}_{i,j}.
\]

Kudo's transgression theorem also implies that

\[
\sigma_* Q^I v_{4n+1} = Q^I p^{S^{-1}}_{4n+2},
\]
\[
\sigma_* Q^L v_{i,j-1} = Q^L p^{S^{-1}}_{i,j}.
\]

We conclude this section with the following theorem.

Theorem 3.10. Let \( \xi = \zeta^2 \in H_*Q_0S^0 \) be a spherical class with \( \sigma_* \xi \neq 0 \). Then it is impossible to have \( t > 1 \).

Remark 3.11. We would like to recall some facts about Hopf algebras \cite{MM65} Proposition 4.23] which will be use in the proof of the above theorem. Suppose \( H \) is a connected bicommutative Hopf algebra of finite type over \( k = \mathbb{Z}/2 \). Then there is an exact sequence of the following form

\[
0 \to P(sH) \to PH \to QH \to Q(rH) \to 0.
\]

Here Frobenius homomorphism \( s = s_H : H \to H \) is given by \( s_H(h) = h^2 \). The map \( QH \to Qk(rH) \) is the square root map. In particular, examples \( H = H_*Q_0S^n \) with \( n \in \mathbb{Z} \) satisfy the conditions stated above. The square root map \( r : H \to H \) in our examples has the property that

\[
r Q^{2n} = Q^n r,
\]
\[
r Q^{2n+1} = 0.
\]

The above theorem then tells us that if \( \xi \in H \) is a primitive which is not a square, then the image of the \( \xi \) in \( QH \) must belong to the kernel of the square root map. For example, in \( H_*Q_0S^0 \) we have that

\[
r x_{2i} = x_i,
\]
\[
r x_{2i+1} = 0.
\]
Hence, if $Q^I x_m + D$, with $D$ being a sum of decomposable terms, is a primitive class which is not a square then $Q^I x_m$ must belong to the kernel of the square root map $r : QH_*Q_0S^0 \to QH_*Q_0S^0$. This then implies that either $I$ must have an odd entry, or $m$ is odd.

**Proof.** We do the proof for $t = 2$, i.e. $\xi = \xi^4 = Q^{2d} Q^d \zeta$ where $d = \dim \zeta$. Other cases are similar. Since $\xi$ is an $A$-annihilated primitive, then $\zeta$ also must be an $A$-annihilated primitive class. As we recall at the beginning, we may write $\zeta$ as a sum, $Q^J p_{2j+1}'$ where $J$ is admissible but $(J, 2j + 1)$ is not necessarily admissible. Hence we may write

$$\xi = \sum Q^{2d} Q^d L_{p_{2l+1}'}^d,$$

with $L$ admissible, taking all of the above terms into one big sum, where some $(L, 2l + 1)$ are admissible and some are not. Such a class pulls back to a $4d - 1$ dimensional class $\xi_{-1} \in H_{4d-1}Q_0S^{-1}$ given by

$$\xi_{-1} = \sum Q^{2d} Q^d Q^L w_{2l} + D_{-1},$$

where $D_{-1}$ denotes the decomposable part. This is an odd dimensional primitive class, i.e. its indecomposable part must belong to the kernel of the square root map. Hence either $d$ is odd, $L$ has at least one odd entry, or $l$ is odd. Hence we may rewrite the above class as

$$\xi_{-1} = \sum_{l \text{ odd}} Q^{2d} Q^d Q^L w_{2l} + \sum_{l \text{ even}} Q^{2d} Q^d Q^L w_{2l} + D_{-1}.$$

We have already calculated the set of primitive classes in $H_*Q S^{-1}$, hence we write

$$\xi_{-1} = \sum_{l \text{ odd}} Q^{2d} Q^d Q^L p_{2l}^{S^{-1}} + \sum_{l \text{ even}} Q^{2d} Q^d Q^K p_{k,l}^{S^{-1}} + D_{-1},$$

where $D_{-1}$ is an odd dimensional decomposable primitive class. Hence $D_{-1} = 0$, i.e.

$$\xi_{-1} = \sum_{l \text{ odd}} Q^{2d} Q^d Q^L p_{2l}^{S^{-1}} + \sum_{l \text{ even}} Q^{2d} Q^d Q^K p_{k,l}^{S^{-1}}.$$

Notice that $Q^d Q^L p_{2l}^{S^{-1}}$ and $Q^K p_{k,l}^{S^{-1}}$ are of dimension $2d - 1$. We plan make use of $Sq_*^1$, but not here as in this exterior algebra $Sq_*^1 \xi_{-1}$ will be a square which is trivial in the exterior algebra. Instead we desuspend once more. The class $\xi_{-1}$ pulls back to a spherical class $\xi_{-2} \in H_*Q_0S^{-2}$, according to Lemma implies that $\xi_{-2}$ is in the exterior subalgebra generated by $\text{im}(\Omega^2 \lambda_{C})_*$. Hence, we may write

$$\xi_{-2} = \sum_{l \text{ odd}} Q^{2d} Q^d Q^L v_{2l-1} + \sum_{l \text{ even}} Q^{2d} Q^K v_{k,l-1} + D_{-2},$$
where $D_{-2}$ denotes the decomposable part. The classes $Q^dQ^L v_{2l-1}$ and $Q^K v_{k,l-1}$ are of dimension $2d - 2$. Although one may decide to rewrite this sum in terms of primitives, however this form is enough for us to get a contradiction. Observe that

$$ Sq^1 \xi_{-2} = \sum_{l \text{ odd}} Q^{2d-1}Q^L v_{2l-1} + \sum_{l \text{ even}} Q^{2d-1}Q^K v_{k,l-1} + Sq^1 D_{-2}. $$

Notice that $Sq^1 D_{-2}$ is a decomposable. On the other hand terms $Q^{2d-1}Q^L v_{2l-1}$ and $Q^{2d-1}Q^K v_{k,l-1}$ are separated under this action as the map to distinct terms under the homology suspension, which also shows that these classes do not belong to ker $\sigma_*$. This shows that $Sq^1 \xi_{-2} \neq 0$. But this is a contradiction to the fact that $\xi_{-2}$ is $A$-annihilated. Hence we have completed the proof.

A similar approach maybe taken to prove the following.

**Lemma 3.12.** Let $\theta = \zeta^2 \in H_\ast Q_0 S^0$ be a spherical class. Then $\zeta$ must be an odd dimensional class.

We leave the proof to the reader, and only note that a similar claim is valid if we replace $Q_0 S^0$ by $Q S^n$ with $n > 0$ in the above theorem.

### 3.2 Completing the proof of the Main Theorem

Our main theorem is stated in terms of primitive classes in $H_{2i+1}Q_0 S^0$. Recall that any spherical class in $H_\ast Q_0 S^0$ may be written as a linear combination of terms of the form $Q^I p_{2i+1}$ with $I$, and not necessarily $(I, 2i + 1)$, being admissible. The class $p_{2i+1} \in H_{2i+1}Q_0 S^0$ is the unique primitive such that

$$ p_{2i+1} = x_{2i+1} + D_{2i+1} $$

with $D_{2i+1}$ being is sum of decomposable terms. This then makes the following observation evident.

**Lemma 3.13.** If $(I, 2n+1)$ is admissible, then modulo decomposable terms

$$ Q^I p_{2n+1} = Q^I x_{2n+1}. $$

Notice that a spherical class is $A$-annihilated and primitive. Combining this with Remark 3.11 enables us to have the following observation.

**Lemma 3.14.** Suppose $\xi_0 \in H_\ast Q_0 S^0$ is an $A$-annihilated primitive class with $\sigma_\ast \xi_0 \neq 0$. Then

$$ \xi_0 = \sum Q^I x_{2i+1} $$

modulo decomposable terms, where $(I, 2i + 1)$ runs over certain admissible sequences of positive excess.
Proof. The fact that \( \sigma_* \xi_0 \neq 0 \) implies that modulo decomposable terms

\[
\xi_0 = \sum Q^I x_n
\]

where \((I, n)\) is admissible with \(\text{excess}(Q^I x_n) > 0\). The fact that \(\xi_0\) is an indecomposable primitive implies that indecomposable part of \(\xi_0\) belongs to the kernel of the square root map \(r : H_*Q_0S^0 \to H_*Q_0S^0\). Notice that if we have two distinct admissible sequences \((J, j)\) and \((K, k)\) with only even entries, then \(rQ^J x_j \neq rQ^K x_k\). Hence the indecomposable part of \(\xi_0\) belongs to the kernel of \(r\) if and only if every \(Q^I x_n\) belong to the kernel. We show that assuming \(n \neq 2i + 1\) leads to a contradiction.

Assume that \(n\) is even. Since \(Q^I x_n\) belong to \(\ker r\), then \(I = (i_1, \ldots, i_t)\) must have at least one odd entry. Let \(s_0 = \max(s : 1 \leq s \leq t, i_s \text{ is odd})\). Then \(i_{s_0} + 1\) is even. Notice that if \(s_0 = t\), then we have \(x_n\) next to it which is even. In this case one applies \(Sq^{2s_0}_{2s_0}\) to \(\xi_0\). Notice that according to Theorem 1 a class \(Q^I x_n\) with \((I, n)\) having at least one even entry is not \(A\)-annihilated. Moreover, according to Note 4, all terms of the form \(Q^I x_n\) with \(\text{excess}(Q^I x_n) > 0\) are separated under the action of this operation from each other. Moreover, notice that

\[
\text{excess}(Sq^{2s_0}_{2s_0} Q^I x_n) = \text{excess}(Q^I x_n) > 0,
\]

which implies that the outcome is not a decomposable, and hence is separated from any other decomposable term. This implies that \(Sq^{2s_0}_{2s_0} \xi_0 \neq 0\) which contradicts the fact that \(\xi_0\) must be \(A\)-annihilated. Hence \(n\) must be odd. This implies that modulo decomposable terms

\[
\xi = \sum Q^I x_{2i+1},
\]

with \((I, 2i+1)\) admissible. \(\Box\)

The previous lemma together with the above observation, leads us to a better and more clear expression for a potential spherical classes in \(H_*Q_0S^0\). We have the following.

Corollary 3.15. Let \(\xi_0 \in H_*Q_0S^0\) be \(A\)-annihilated primitive class with \(\sigma_* \xi_0 \neq 0\). Then

\[
\xi_0 = \sum Q^I p_{2i+1}
\]

with \((I, 2i+1)\) admissible modulo decomposable terms. If \(\xi_0\) is odd dimensional, then the decomposable part is trivial. If \(\xi_0\) is even dimensional, then the decomposable part is either trivial or square of a primitive.

Moreover, assume that \(l(I) > 1\) for every \(I\) involved in the above expression for \(\xi_0\). Then \(I\) will have only odd entries.
Proof. Notice that \( \xi_0 = \sum Q^I x_{2i+1} \) modulo decomposable terms. Previous lemma allows us to replace \( Q^I x_{2i+1} \) with \( Q^I p_{2i+1} \) modulo decomposable terms. Therefore \( \xi_0 = \sum Q^I p_{2i+1} \) modulo decomposable terms. However this decomposable part is primitive, hence it must be square. If \( \xi_0 \) is an odd dimensional class, then the decomposable part is trivial. If \( \xi_0 \) is even dimensional then it is either square or trivial.

If \( \xi_0 \) is even dimensional, then we may write

\[
\xi_0 = \sum Q^I p_{2i+1} + P^2
\]

with \( (I, 2i+1) \) admissible, and \( P \) a primitive class. The fact that decomposable terms die under suspension, together with the Lemma 3.13 show that

\[
\sigma_* Q^I p_{2i+1} = \sigma_* Q^I x_{2i+1} = Q^I Q^{2i+1} g_1 \in H_* QS^1.
\]

Hence, we have

\[
\sigma_* \xi_0 = \sum Q^I Q^{2i+1} g_1
\]

being an odd dimensional \( A \)-annihilated class. Corollary 3, then implies that each term \( Q^I Q^{2i+1} g_1 \) must be \( A \)-annihilated which in particular means that \( I \) has only odd entries, and \( (I, 2i + 1) \) satisfies condition 3 of Theorem 1.

If \( \xi_0 \) is odd dimensional, then

\[
\xi_0 = \sum Q^I p_{2i+1}
\]

with \( (I, 2i+1) \) being admissible. If we have \( I = (i_1, \ldots, i_r) \), with \( i_s \) being even, then applying \( Sq_{2i-1}^* \) will show that \( \xi_0 \) is not \( A \)-annihilated which is a contradiction. This then shows that \( I \) must have only odd entries. This completes the proof.

To complete the proof, we need to translate the above results and express our results in terms of \( Q^I p_{2i+1} \) whereas the above results are expressed in terms of primitive classes \( Q^I p_{2i+1} \). The main reason for this, is the difference that \( p_{2i+1} \) and \( p'_{2i+1} \) show under the action of the Steenrod algebra. The following remark explains this more.

Remark 3.16. The action of the Steenrod algebra on \( H_* SO \) is given by

\[
Sq_*^k p_{2n+1} = \binom{2n+1-k}{k} p_{2n+1-k}^{SO}.
\]

This makes it easy to see that the primitive classes \( p'_{2n+1} \) behave similar to \( x_{2n+1} \) under the action of the Steenrod algebra, i.e.

\[
Sq_*^k p'_{2n+1} = \binom{2n+1-k}{k} p'_{2n+1-k}.
\]

Note that the above action is trivial when \( k \) is odd.

We refer the reader to [W82, Lemma 5.2] to see that

\[
Sq_*^k p_n = \binom{n-k-1}{k} p_{n-k}.
\]

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Notice that mod 2, we have

\[
\left( \begin{array}{c}
2i - 2k \\
2k
\end{array} \right) = \left( \begin{array}{c}
2i - 2k + 1 \\
2k
\end{array} \right).
\]

This implies that \( p_{2n+1} \) and \( p'_{2n+1} \) behave in the same way under the operations \( Sq^{2k} \). However, the primitive classes \( p'_{2n+1} \) are annihilated under the operations \( Sq^{2k+1} \) whereas the primitive classes \( p_{2n+1} \) have chance to survive under these operation, e.g. \( Sq^1p_3 = x_1^2 \).

Recall that the action of \( Sq^k \) on \( x_{2i+1} \) is given by

\[
Sq^k x_{2i+1} = \left( \begin{array}{c}
2n + 1 - k \\
k
\end{array} \right)x_{2n+1-k}.
\]

This shows that the primitive classes \( p'_{2i+1} \) behaves like \( x_{2i+1} \) under the action of the Steenrod operation. On the other hand, the class \( Q^l x_{2i+1} \) with \( (I, 2i+1) \) admissible behaves like \( Q^l Q^{2i+1} \) under the action of the Steenrod algebra. This then shows that if \( (I, 2i+1) \) is admissible and \( l(I) > 0 \), then \( Q^l p'_{2i+1} \) behaves like \( Q^l Q^{2i+1} \) under the action of the Steenrod algebra. Notice that this claim is not true about the classes \( Q^l p_{2i+1} \).

The above remark combined with Note 4 implies the following.

**Theorem 3.17.** Suppose \( \sum Q^l p'_{2i+1} \) is an \( A \)-annihilated sum, with \( l(I) > 0 \). Then each term \( Q^l p'_{2i+1} \) must be \( A \)-annihilated.

Next, we compare the behavior of primitive classes \( Q^l p'_{2i+1} \) and \( Q^l p_{2i+1} \) under the homology suspension.

**Remark 3.18.** We like to draw the reader’s attention to the behavior of \( Q^l p_{2n+1} \), and \( Q^l p'_{2n+1} \) under the homology suspension. First let \( I = \phi \). Recall that modulo decomposable terms,

\[
p_{2n+1} = x_{2n+1}, \quad p'_{2n+1} = x_{2n+1} + Q^{2n}x_1.
\]

We then obtain,

\[
\sigma_* p_{2n+1} = Q^{2n+1}g_1, \quad \sigma_* p'_{2n+1} = Q^{2n+1}g_1 + (Q^n g_1)^2.
\]

Now suppose \( I = (i_1, \ldots, i_r) \) is a sequence with \( i_r \) odd such that \( (I, 2n+1) \) is admissible. We then have

\[
\sigma_* Q^l Q^{2n}x_1 = Q^l Q^{2n}g_1^2 = Q^l (Q^n g_1)^2 = 0.
\]

In fact we don’t need to restrict to \( i_r \), similar statement holds if we assume only \( I \) has at least one odd entry. Notice that \( Q^l Q^{2n}x_1 \) is a primitive class, which can be written in terms of \( Q^l x_j \) modulo
decomposable terms, where \((J, j)\) is admissible. Any class \(Q^I x_j\) with \((J, j)\) dies under suspension, if and only if \(\text{excess}(Q^I x_j) = 0\), i.e. \(Q^I x_j\) is decomposable. Hence \(Q^I Q^{2n} x_1\) is a decomposable primitive, and hence a square term. We then observe that if we choose \(I\) to be even dimensional, then \(Q^I Q^{2n} x_1\) is odd dimensional which makes it impossible to be a square, hence \(Q^I Q^{2n} x_1 = 0\). In this case we have

\[
\sigma_* Q^I p_{2n+1} = Q^I Q^{2n+1} g_1 = \sigma_* Q^I p_{2n+1},
\]

as well as

\[
Q^I p_{2n+1} = Q^I p_{2n+1}.
\]

Now, we restrict our attention to the classes \(Q^I p_{2i+1}\) with \(l(I) > 0\). The result reads as following.

**Lemma 3.19.** Suppose \(\xi_0 \in H_* Q_0 S^0\) is an odd dimensional \(A\)-annihilated primitive class. Then

\[
\xi_0 = \sum Q^I p_{2i+1},
\]

with \((I, 2i + 1)\) admissible, and each \(Q^I p_{2i+1}\) being \(A\)-annihilated.

The above theorem is just a combination of Corollary 3.15, Theorem 3.17, and Remark 3.18.

Our main theorem now follows from combining Theorem 3.10, Lemma 3.11, Corollary 3.15 and Lemma 3.19.

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