A new predictive model to estimate the frequencies for beams with branched cracks

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Abstract. Detecting damage by using vibration signals is popular because it permits evaluating the structural integrity without being necessary scanning of the whole structure. The effect of transverse cracks is presented in detail in the literature, but in reality, the cracks can shift the direction of propagation and even split, resulting in the so-called branched crack. The effect of this type of crack is less investigated due to its complexity. We herein propose a simple model to predict frequency changes that occur due to branched cracks. Initially, we present the effect of stiffness reduction along the damaged section on the structure’s natural frequency. Next, we show that the predicted frequency drop is smaller that happens in reality. This is caused by the sudden cross-section reduction in the slice on which the transverse crack branch is. The phenomenon is similar to the stress concentration for static loads. We propose for dynamic systems a factor that considers the energy stored at the delamination ends. Considering this factor and the stiffness reduction on the damaged segment, we obtain accurate frequency changes due to any type of crack that extends in the longitudinal direction. The model is implemented in Python and tested successfully against simulation with dedicated software.

1. Introduction

As technology advances, specialists from different fields, such as mechanical, civil or aerospace engineering, have developed methods of detecting certain faults that may occur in mechanical systems or structures by measuring its modal parameters [1-4]. The possibility to analyze a structure in real-time by using the modal parameters represent an economically advantageous method that makes it likely to highlight possible damages developed during operation [5]. These so-called global damage detection methods have proven to be reliable for providing real-time information regarding the integrity of a structure. However, an accurate estimation of the modal parameters is essential for an early assessment of damage [6-8].

The methods developed so far are based on collecting the modal parameters from the field and processing them to determine whether or not the structure is suffering from damages [9]. These imply using periodically sampled response data from an array of sensors to determine the state of health and the capacity of the structure to perform its intended function. The core of the global methods consists in the negative impact of damages on the parameters of structures such as rigidity, natural frequencies and mode shapes. The parameter alteration directly depends on the type, size and position of the damage [10]. So, the exact modeling of the damage is a very important factor in determining the dynamic properties of the structure. While for transverse cracks numerous methods are proposed, for branched cracks the literature is not so abundant. Some relevant papers treating complex-shaped cracks are [11-14]. We also approached this type of crack in earlier publications [15-16].
Here, we present a method to assess the influence of a T-shaped crack on the natural frequencies of a cantilever beam. Firstly, we present a way to determine the severity of branched cracks with different penetration and length of delamination based on the stiffness decay of the affected beam segment. Then, taking into consideration the supplementary slope at the delamination ends of the T-shaped crack, we propose correction coefficients and find an accurate model to predict the natural frequencies of cantilever beams having complex-shaped cracks with different depths and placed on different location along the beam. The developed algorithm using the three features is embedded in software with the help of the Python programming language, and the depicted natural frequencies are compared with the values obtained by modal FEM simulations using the ANSYS software.

2. Materials and methods

2.1. The analyzed beam

The current study aims to define the precision of the developed algorithm to determine the natural frequency values for a cantilever beam fixed at the left end and having a T-shaped crack of different depth and position along the beam, as presented in figure 1. The transverse component of the crack is positioned in the middle of the delamination, and has the depth taken from the top surface. The crack’s delamination length for all cases is \( l = 50 \) mm, and its position \( x \) is taken from the fixed end of the beam to the left end of the longitudinal branch. The beam’s dimensions as well as material properties are presented in table 1.

![Figure 1. Cantilever beam with a T-shaped crack.](image)

Table 1. The beams main dimensions and material properties

| Length \( L [/mm] \) | Width \( B [/mm] \) | Thickness \( H [/mm] \) | Mass density \( [kg/m^3] \) | Young modulus \( [N/m^2] \) | Moment of inertia \( I [/mm^4] \) |
|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| 1000                | 20                  | 5                   | 7850                | \( 2 \cdot 10^{11} \) | 520,8333            |

First, we have performed with the help of the ANSYS simulation software modal analysis, in order to determine the undamaged and damaged beam’s natural frequencies. The 3D model of the beam is generated according to the dimensions presented in figure 1 and table 1. The study was carried out by using hexahedral mesh elements of 2 mm maximum edge size, as presented in figure 2.

![Figure 2. A zoom on the left end of the meshed cantilever.](image)
The study was made for several damage scenarios, all implying the T-shaped crack at different positions along the beam and different depths, as presented in table 2.

| Scenario | Crack location x [mm] | Crack depth h [mm] |
|----------|-----------------------|-------------------|
| 1        | 350                   | 0.2 0.4 0.6 0.8 1 1.2 1.5 |
| 2        | 400                   |                   |
| 3        | 450                   |                   |

As it can be observed from table 2, the crack is positioned in three locations, having a depth $h=0.2$ to 1.5 mm. For all damage locations and depths, the natural frequencies have been found analytically and involving the FEM.

2.2. The calculus of the natural frequencies for the decrease of the stiffness on the damaged segment

The first step for determining the natural frequencies of a damaged beam is to consider a model with decreased stiffness, as illustrated in figure 3. The current method intends to predict the impact of the stiffness alteration on the natural frequencies of the cantilever beam.

![Figure 3. The reduced section model.](image_url)

For the intact cantilever beam the dimensionless wave numbers are given by the relation:

$$\cos(\alpha L)\cosh(\alpha L) + 1 = 0$$  \hspace{1cm} (1)

where $\alpha$ is the dimensionless wave number. Knowing all $\alpha L = \lambda$ of interest, the natural frequencies of the undamaged beam are calculated by using the following relation:

$$f_i = \frac{\lambda_i^2}{2\pi \sqrt{EI/mL^2}}$$  \hspace{1cm} (2)

The normalized bending moment of the cantilever beam is determined with the relation:

$$\bar{\phi}(x) = 0.5\left[\frac{-\cos\alpha_i L + \cosh\alpha_i L}{\sin\alpha_i L + \sinh\alpha_i L} \left[\sin(\alpha_i x) + \sinh(\alpha_i x)\right] + \cos(\alpha_i x) + \cosh(\alpha_i x)\right]$$  \hspace{1cm} (3)

The strain energy is expressed with the well-known relation:

$$U_i = \frac{1}{2EI} \int_0^L \left[\bar{M}_i(x)\right]^2 dx$$  \hspace{1cm} (4)

From relation (4), we can derive for any vibration mode the strain energy coefficient:

$$\bar{S}^{L-1} = \frac{1}{L} \int_0^L \left[\bar{\phi}(x)\right]^2 dx = 0.25$$  \hspace{1cm} (5)

For a segment a-b, the stiffness decrease coefficient becomes:
Knowing the relation between the natural frequencies and strain energy $f_i = \sqrt{\mathcal{U}_i}$, the frequency for the beam with reduced cross-section becomes:

$$f_{C_i} = f_i \sqrt{1 - 4\zeta_i^a - b \frac{L - L_C}{I} \left( \frac{\zeta_i^0 - a - b}{\zeta_i^0 - L} + \zeta_i^{a - b - L} \right)}$$

(7)

After solving the previous equations and defining the coefficient $c_i^{a - b}$, it is possible to plot the square of the normalized bending moment (square of the normalized modal curvature) for the beam with reduced section a-b, as illustrated in figure 4.

**Figure 4.** Square of the normalized modal curvature for the beam with locally reduced stiffness.

To test the reliability of the described method to predict the natural frequencies of a cantilever beam having a T-shaped crack, we compare the calculated results for the cantilever beam affected by a T-shaped crack with the natural frequencies obtained through the ANSYS simulation. The percent differences between the developed analytical model and FEM simulations are shown in table 3 for a damage position $x=300$ and depths $h=0.2$ and 0.6 mm. Also, in table 4 the percent differences are shown for the damage position $x=400$ and depths $h=0.8$ and 1.2 mm. The results shown in tables 3 and 4 are determined with the described frequency estimation coefficient calculated using relation 7.

**Table 3.** Comparison between frequencies achieved with FEM and calculated from the relation considering the stiffness reduction for the damaged beam at segment between 300-350 mm.

| Mode number | Undamaged beam frequency (Hz) | Damaged beam Crack depth $h=0.2$ mm | Damaged beam Crack depth $h=0.6$ mm |
|-------------|-------------------------------|--------------------------------------|--------------------------------------|
|             | $F_D$ FEM (Hz) | $F_D$ calc. (Hz) | Error      | $F_D$ FEM (Hz) | $F_D$ calc. (Hz) | Error      |
| 1           | 4.0812           | 4.0647                    | 0.16% | 4.022           | 4.05239                    | 0.75%          |
| 2           | 25.573           | 25.528                    | 0.05% | 25.3942         | 25.4808                    | 0.34%          |
| 3           | 71.595           | 71.2355                   | 0.17% | 70.2873         | 70.9139                    | 0.89%          |
| 4           | 140.27           | 140.1569                  | 0.03% | 139.8735        | 140.069                    | 0.14%          |
| 5           | 231.81           | 231.092                   | 0.15% | 229.7252        | 230.752                    | 0.45%          |
Table 4. Comparison between frequencies achieved with FEM and calculated from the relation considering the stiffness reduction for the damaged beam at segment between 400-450 mm.

| Mode number | Undamaged beam frequency (Hz) | Damaged beam Crack depth $h=0.8$ mm | Damaged beam Crack depth $h=1.2$ mm |
|-------------|-------------------------------|-------------------------------------|-------------------------------------|
|             | FD FEM (Hz)                   | FD calc. (Hz)                       | Error (Hz)                          | FD FEM (Hz) | FD calc. (Hz) | Error (Hz) |
| 1           | 4.0812                        | 4.0291                             | 0.74%                               | 3.9853      | 4.0491       | 1.60%       |
| 2           | 25.573                        | 24.904                             | 1.46%                               | 24.3795     | 25.1347      | 3.10%       |
| 3           | 71.595                        | 70.95                              | 0.49%                               | 70.4628     | 71.1729      | 1.01%       |
| 4           | 140.27                        | 138.158                            | 0.82%                               | 136.5976    | 138.868      | 1.66%       |
| 5           | 231.81                        | 226.5                              | 1.21%                               | 222.9871    | 228.1344     | 2.31%       |

From tables 3 and 4 it can be observed that the maximum obtained error is 3.1% for the crack having a depth of $h=1.2$ mm with the location in the interval 400-450 mm.

The described method takes into account the stiffness loss on the affected beam segment, but by studying the behavior of beams with branched cracks, we observed a supplementary slope at the crack ends that led to the conclusion that we need to include a supplementary bending moment at each damage end.

2.3. Method to determine the damage severity at the ends of the damaged segment

In prior work [16] we present a simple algorithm for determining the severity $\gamma(h)$ for different transversal crack depths $h$, that consider only the undamaged and damaged beam deflections. Following mathematical relation applies:

$$\gamma(0,h) = \frac{\sqrt{\delta_D(0,h)} - \sqrt{\delta_U}}{\sqrt{\delta_D(0,h)}}$$  \hspace{1cm} (8)

The severity regression curve obtained for a beam with a thickness of $H=5$ mm and a length of $L=1$ m is illustrated in figure 5. For other lengths or thickness the curve is simply adjusted by involving the radius of gyration.

Figure 5. Curve representing the severity of a transverse crack.
To obtain more accurate results, we include in the model of the beam with a branched crack a supplementary bending moment at each damage end, see figure 6. These moments are proportional to the squared normalized modal curvature at those locations and consequently with the pseudo-severity of the transverse crack positioned at the branched crack ends, corrected with a coefficient.

\[
c_i^a = 1 - s(h) \left[ \phi_i^a(a) \right]^2 = 1 - c(h) \cdot \gamma(h) \left[ \phi_i^a(a) \right]^2
\]
\[
c_i^b = 1 - s(h) \left[ \phi_i^b(b) \right]^2 = 1 - c(h) \cdot \gamma(h) \left[ \phi_i^b(b) \right]^2
\]

where: \( \gamma(h) \) is the severity of the transverse crack with a given depth \( h \) and \( \phi_i^a(a) \) and \( \phi_i^b(b) \) are the normalized squares of the modal curvatures at locations \( a \) and \( b \). We deduced empirically \( c_i^a \) and \( c_i^b \). These coefficients are used to multiply the obtained frequency correction coefficient \( c_{i}^{a-b} \) presented in relation 7 to give more accurate frequency estimation. Hence, the severity coefficient \( c_s \) proposed for the beam with a branched crack is:

\[
c_s = c_i^a \cdot c_i^{a-b} \cdot c_i^b
\]

which considers the stiffness loss due to cross-section reduction between points \( a \) and \( b \) and the supplementary bending moments at the damage extremities. By applying the severity coefficient for the beam with a branched crack, the frequency of the cracked beam becomes:

\[
f_{Ci} = c_s \cdot f_i = c_i^a \cdot c_i^{a-b} \cdot c_i^b \cdot f_i
\]

The correction coefficients \( c(h) \) were determined based on frequencies obtained as FEM simulation results. The coefficients are suitable for any position of the T-shaped crack if its depth and delamination extent remain constant. The empirical defined coefficient curve is presented in figure 7.
2.4. The developed PyDAM application

The algorithm proposed herein can be used to predict the impact of complex-shaped cracks on the natural frequencies of beams, by involving the damage severity coefficient presented in relation 12. It works for beams that can be modeled with the Euler-Bernoulli theory irrespective of the boundary conditions.

The algorithm is computerized by using the Python programming language resulting in the software named PyDAM and the logical steps are presented below:

1. Define the boundary conditions and mode number
2. Introduce the damage limits, left and right
3. Input reduced section height ratio

The algorithm works by implementing relations that permit:

1. The calculation of the modal curvature for any position along the beam
2. Damage severity calculation for the defined depth of the transversal crack component
3. The determination of the correction coefficient for any damage depth
4. The determination of the frequency estimation coefficient for the beam with a T-shaped crack of known depth and position

The PyDAM application interface used for determining the frequency estimation coefficients of the damaged beam is presented in figure 8.

![Figure 8. The PyDAM software interface developed Python.](image)

The boundary conditions for the beam are selected from the drop-down menu, meaning: cantilever, double-clamped, free-free and simple-supported.

The transversal vibration mode number is easily selected from the drop-down menu from mode number 1 to 10.

The position of the section with a reduced stiffness is introduced as an interval value with the start of the damage location set up as the first value and the end as the second value. The values are normalized with the beam length. The third value in the input box is the normalized beam length, which obviously is one.

The stiffness reduction value is set up as the ratio between the reduced section value \( H-h \) and the thickness \( H \) of the integer beam.

Note that PyDAM permits calculating the frequencies for more branched cracks. In this case in the input boxes for the damage ends and cross-section height ratio must be inserted all crack ends.

The execution of the algorithm is done by pressing the button `Integrare`, which results in a separate window that indicates the normalized mode shape for the given mode number and boundary conditions, as well as the coefficients \( c_i^{a-b} \), \( c_i^a \) and \( c_i^b \) as well as the global severity coefficient \( c_s \).
3. Results and discussions

To obtain the natural frequencies of a cantilever beam having a T-shaped damage, we applied the developed algorithm in interactive software, named PyDAM developed with the help of the Python programming language. We aim to show in the current study that the applied method leads to more accurate results, by taking into consideration the the stiffness reduction coefficient with supplementary slope at the cross-section changes (at the delamination ends), caused by the sudden cross-section reduction in the slice on which the transverse crack branch is.

We empirically deduced the supplementary severity $s(h)$ which is integrated in the correction coefficients $c^a_i$ and $c^b_i$, which are used to multiply the obtained frequency estimation coefficient $c^{i-b}$ to give more accurate frequency values. The results obtained for the cantilever having a T-shaped damage at position $x=300$, for several crack depths, are presented in figure 9.

![Figure 9. Frequency values obtained for damage location $x=300$ mm.](image)

In table 5 we present the comparison of the frequencies obtained using the FEM, the analytical relation 7 and PyDAM for a cantilever beam affected by a T-shaped crack of depth $a=0.6$ mm, positioned at $x=300$ mm. The maximum error we obtained for all analysed cases is less than 1%. Here, we also present the severity of the transverse crack, the correction coefficient and the severity calculated with relation 11 for the T-shaped crack. One can observe that the errors are significantly smaller for the case where we use the additional bending moments at the cracks extremities.
Table 5. Results obtained for damage position $x=300$ and depth $h=0.6$ mm.

| $F_D$ (Hz) | $F_D$ FEM (Hz) | $F_D$ calc. (Hz) | Error | $F_D$ PyDAM (Hz) | Error | Severity $\gamma(h)$ | Coef. $c(h)$ | Severity $c_S$ |
|------------|----------------|------------------|-------|-----------------|-------|----------------------|-------------|-------------|
| 4.0812     | 4.0220         | 4.052            | 0.75% | 4.0226          | 0.01% |                       |             |             |
| 25.573     | 25.3942        | 25.480           | 0.34% | 25.383          | 0.04% |                       |             |             |
| 71.595     | 70.2873        | 70.913           | 0.89% | 70.236          | 0.07% | 0.001191             | 9.9         | 0.98821     |
| 140.27     | 139.8735       | 140.069          | 0.14% | 139.812         | 0.04% |                       |             |             |
| 231.81     | 229.7253       | 230.752          | 0.45% | 229.593         | 0.06% |                       |             |             |

The frequencies obtained with PyDAM can be used to calculate the relative frequency shifts (RFS), for the damaged beam in order to obtain the damage signature. In figure 10 we present the RFS’s for a T-shaped crack of depth $h=0.8$ mm in different locations along the beam.

Figure 10. RFSs for a T-shaped crack of depth $h=0.8$ mm on different locations.

Beside the stiffness reduction coefficients, by applying the supplementary bending moments for the reduced section we have obtained an accurate prediction of the natural frequencies for the beam with a T-shaped crack, the results obtained for different depths also being less than 1%.

4. Conclusions
In this paper we propose a model to predict the behavior of beams with branched cracks the can be used to calculate the natural frequencies if the crack depth, extend and location are known. The model takes into consideration beside the stiffness reduction in the affected segment the supplementary rotation at ends of the longitudinal component of the crack. In this way, the frequencies can be predicted with an error smaller than 1%, which qualifies the model to be used to create a database containing damage patterns. It was also found that the effect on the ends of the longitudinal component of the crack can be estimated with the severity of the transverse crack located at those positions, adjusted with a correction coefficient. This coefficient depends on the crack depth alone.
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