Method for obtaining a differential equation of the asphalt concrete mix behavior under railway transport load

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Abstract. The use of asphalt concrete in the railway track construction allows you to ensure high strength and stability of the railway. The supporting asphalt layer is subjected to force from the railway transport. The study of material behavior under load is of great practical importance. Rheological modeling is a popular tool for studying the behavior of a material under load. The accepted rheological model is described by the corresponding differential equation. Establishing the initial conditions of a differential equation is difficult if its order is 2 or higher. The article describes an establishing initial conditions method for solving the rheological equation of the model. The model's behavior analytical description over an infinitesimal time period allows you to set initial conditions. The proposed method for establishing the model equation initial conditions simplifies the process of material under load analytical description. This approach to the asphalt concrete layer under load study allows a more conscious approach to the choice of its parameters when designing the railway track elements.

1. Introduction

Currently, the traditional ballast tracks are being replaced by more modern ones. In such structures, the asphalt concrete mix is laid as a support layer under the ballast prism [1-3]. It replaces the crushed stone sub-ballast layer and geotextile. Asphalt concrete compensates for some of the disadvantages of the crushed stone base, increases the durability of the elements of the upper structure.

The main advantages of this technology are as follows [4-6]:
- even load distribution;
- almost complete water resistance;
- increased path shear resistance;
- ease of installation of the ballast.

The use of asphalt concrete in the railway track construction allows you to ensure high strength and stability of the railway [7]. Theoretical study of the asphalt concrete support layer behavior during the railway track operation allows you to choose its physical and mechanical parameters. Rheological modeling is a popular tool for studying the behavior of a material under load. If the model is described by the second order or higher equation, it is difficult to establish the equation initial conditions.

The purpose of this work is to establish initial conditions method for solving the rheological equation of the model.

2. Method

Various authors have proposed different asphalt concrete rheological models [8-15]. But most of them do not fully describe the stress-strain state of asphalt concrete [16]. The Boguslavsky model [17] is the
closest to asphalt concrete mechanical properties (figure 1).

Figure 1. The Boguslavsky model.

Static loading of the model by pressure $\sigma_e = \text{const}$ is considered. It is necessary to find the strain law $\varepsilon(t)$. Under the condition $\sigma_e > \sigma_0$, the total strain can be represented as the sum of elastic-viscous model part strain $\varepsilon_{e-v}$ and plastic strain $\varepsilon_{pl}$.

$$\varepsilon = \varepsilon_{e-v} + \varepsilon_{pl}. \quad (1)$$

The Boguslavsky model is a parallel connection of the Kelvin body and the Maxwell body

$$\begin{cases}
\sigma_K = E_K \varepsilon_K + \eta_K \varepsilon'_K; \\
\varepsilon'_M = \frac{1}{E_M} \sigma'_M + \frac{1}{\eta_M} \sigma_M; \\
\sigma = \sigma_K + \sigma_M; \\
\varepsilon_K = \varepsilon_M = \varepsilon_{e-v}.
\end{cases} \quad (2)$$

After transformations of the equations system (2), we obtain a differential equation for the Boguslavsky model elastic-viscous part

$$\frac{\eta_K \eta_M}{E_M} \varepsilon'' + \left(\eta_M + \frac{E_K}{E_M} \eta_M + \eta_K\right) \varepsilon' + E_K \varepsilon = \frac{\eta_M}{E_M} \sigma' + \sigma. \quad (3)$$

In the case $E_K = E_M = E$, equation (3) takes the form

$$\frac{\eta_K \eta_M}{E} \varepsilon'' + (2\eta_M + \eta_K) \varepsilon' + E \varepsilon = \frac{\eta_M}{E} \sigma' + \sigma. \quad (4)$$

We introduce the notation

$$\frac{\eta_M}{E} = \theta; \quad \frac{\eta_K}{E} = \tau, \quad (5)$$

where $\theta$ is relaxation time, $\tau$ is strain retardation time.
Taking into account the introduced notation (5), equation (4) takes the form
\[ \theta \varepsilon \sigma^* + (2\eta_0 + \varepsilon E) \varepsilon' + E \varepsilon = \theta \sigma' + \sigma. \] (6)

The differential equation for the plastic part of Boguslavsky model has the form
\[ \sigma_c - \sigma_0 = \eta_3 \varepsilon'. \] (7)

Dividing both sides of equation (6) by \( \theta \varepsilon E \), and taking into account \( \sigma(t) = \sigma_c = \text{const} \), we obtain
\[ \varepsilon^* + \frac{2\theta + \tau}{\theta \tau} \varepsilon' + \frac{1}{\theta \tau} \varepsilon = \frac{\sigma}{\theta \varepsilon E}. \] (8)

The solution to the second-order non-uniform linear differential equation with constant coefficients will be sought as the sum of the general solution of the corresponding uniform differential equation \( \xi \) and some particular solution to the non-uniform equation \( \tilde{\varepsilon} \)
\[ \varepsilon = \xi + \tilde{\varepsilon}. \] (9)

The uniform differential equation corresponding to (8) has the form
\[ \varepsilon^* + \frac{2\theta + \tau}{\theta \tau} \varepsilon' + \frac{1}{\theta \tau} \varepsilon = 0. \] (10)

For (10), the characteristic equation has the form
\[ p^2 + \frac{2\theta + \tau}{\theta \tau} p + \frac{1}{\theta \tau} = 0. \] (11)

The roots of characteristic equation (11) are equal
\[ p_1, p_2 = \frac{-\theta + 0.5 \tau}{\theta \tau} \pm \sqrt{\left(\frac{\theta + 0.5 \tau}{\theta \tau}\right)^2 - \frac{1}{\theta \tau}}. \] (12)

Then the general solution of equation (10) will be as follows
\[ \xi = C_1 e^{p_1 t} + C_2 e^{p_2 t}. \] (13)

Since the right-hand side of equation (8) is a constant, a particular solution to the non-uniform differential equation (8) will be sought in the form
\[ \tilde{\varepsilon} = A, \] (14)

where \( A \) is constant.

Taking into account (14), equation (8) takes the form
\[ \tilde{\varepsilon} = \frac{\sigma_c}{E}. \] (15)

Substituting (13) and (15) in equation (6) we have
\[ \varepsilon = C_1 e^{p_1 t} + C_2 e^{p_2 t} + \frac{\sigma_c}{E}. \] (16)

The equation (16) is a general solution of differential equation (6) for the case \( \sigma(t) = \sigma_c = \text{const} \).
To find the constants $C_1$ and $C_2$, it is necessary to set the initial conditions for $\varepsilon$ and $\varepsilon'$ at any known moment in time $t$. In this case, the model is rapidly loaded with constant force $\sigma_c$, which does not change further.

Consider the initial point in time. For this moment, the equalities

$$\varepsilon(0) = 0; \varepsilon'(0) = 0; \sigma(0) = 0.$$  

(17)

We give time an infinitesimal increment $\Delta \to 0$ and consider the moment $t_{0+} = t_0 + \Delta = \Delta$. Since $\Delta$ is infinitesimal, the strain can be considered insignificant $\varepsilon(\Delta) \to 0$, and the force at this moment already reaches its limit value, i.e.

$$\sigma(\Delta) = \sigma_c.$$  

(18)

The main difficulty is finding the value $\varepsilon'(t_{0+}) = \varepsilon'(\Delta)$, which in this case is one of the initial conditions. To find it, we integrate both sides of equation (6) in the interval from $t_0 = 0$ to $t_{0+} = \Delta$

$$\theta \varepsilon \varepsilon'(t)_{t_0}^{\Delta} + (2\eta + \xi \varepsilon) \varepsilon(t)_{t_0}^{\Delta} + \int_{0}^{\Delta} \varepsilon \sigma(t)_{t_0}^{\Delta} + \int_{0}^{\Delta} \sigma(t)_{t_0}^{\Delta} = 0.$$  

(19)

The integrands in both sides of equation (19) are bounded; therefore, the values of the corresponding integrals will be the smaller, the smaller the integration interval. Since the time interval $\Delta$ is infinitesimal, both integrals can be neglected. Then equation (19) takes the form

$$\varepsilon'(\Delta) = \frac{1}{E}\sigma_c.$$  

(20)

Thus, the initial conditions are as follows:

$$\text{for } t = 0 \quad \varepsilon = 0, \quad \varepsilon' = \sigma_c / (\xi \varepsilon).$$  

(21)

Differentiating (16) we obtain

$$\varepsilon' = C_1 p_1 \varepsilon^{p_1} + C_2 p_2 \varepsilon^{p_2}.$$  

(22)

Substituting the obtained initial conditions (21) into equations (16) and (22) we obtain the system

$$\begin{cases} 
0 = C_1 + \frac{C_2}{E} + \sigma_c \varepsilon_c; \\
\frac{\sigma_c}{E} = C_1 p_1 + C_2 p_2 \varepsilon.
\end{cases}$$  

(23)

From the system of equations (23) we find the values of the constants

$$C_1 = \frac{-\sigma_c \varepsilon_c}{E \left(1 + p_1 \varepsilon_c\right) p_1}; \quad C_2 = \frac{\sigma_c \varepsilon_c}{E \left(1 + p_2 \varepsilon_c\right) p_1};$$  

(24)

Substitute (24) into (16)

$$\varepsilon_{c+}(t) = \frac{(1 + p_1 \varepsilon_c) p_2 \sigma_c \varepsilon_c \varepsilon^{p_1}}{(p_2 - p_1)} + \frac{(1 + p_2 \varepsilon_c) p_1 \sigma_c \varepsilon_c \varepsilon^{p_2}}{(p_2 - p_1)} + \frac{\sigma_c}{E}.$$  

(25)

Next, we determine the plastic strain. For this, from equation (7) we express...
\[ \varepsilon' = \frac{\sigma_e - \sigma_0}{\eta_S}. \]  \hspace{1cm} (26)

Integrating expression (26) we obtain
\[ \varepsilon_{pl} = \frac{\sigma_e - \sigma_0}{\eta_S} t. \] \hspace{1cm} (27)

Substituting expressions (25), (27) into equation (1) we obtain the total deformation for the Boguslavsky model
\[ \varepsilon(t) = \frac{(p_1+p_2\theta)p_2}{(p_2-p_1)} E \sigma_e e^{\varepsilon_{pl}} + \frac{(1+p_2\theta)p_1}{(p_2-p_1)} E \sigma_e e^{\varepsilon_{pl}} + \frac{\sigma_e - \sigma_0}{\eta_S} t. \] \hspace{1cm} (28)

where \( \theta \) and \( \tau \) are determined by the expression (5), \( p_1, p_2 \) are determined by the expression (12).

3. Conclusion
Thus, an analysis of the rheological model behavior over an infinitesimal period of time allows us to obtain the initial conditions of the model differential equation.

The proposed method for establishing the model equation initial conditions simplifies the process of material under load analytical description. This approach to the asphalt concrete layer under load study allows a more conscious approach to the choice of its parameters when designing the railway track elements.

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