SPECTRAL FLOW AND ITERATION
OF CLOSED SEMI-RIEMANNIAN GEODESICS

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ABSTRACT. We introduce the notion of spectral flow along a periodic semi-Riemannian geodesic, as a suitable substitute of the Morse index in the Riemannian case. We study the growth of the spectral flow along a closed geodesic under iteration, determining its asymptotic behavior.

1. INTRODUCTION

Closed geodesics are critical points of the geodesic action functional in the free loop space of a semi-Riemannian manifold \((M, g)\); a very classical problem in Geometry is to establish multiplicity of closed geodesics (see [17]). By “multiplicity of closed geodesics”, it is always meant multiplicity of “prime closed geodesics”, i.e., those geodesics that are not obtained by iteration of another closed geodesic. One of the difficult aspects in the variational theory of geodesics is precisely the question of distinguishing iterates. In a celebrated paper by R. Bott [9] it is studied the Morse index of a closed Riemannian geodesic; the main result is a formula establishing the growth of the index under iteration. This formula shows that, given a closed geodesic \(\gamma\), its iterates \(\gamma^{(N)}\) either have Morse index that grows linearly with \(N\), or they have all null index. This has been used by Gromoll and Meyer in another celebrated paper [15], where the authors develop an equivariant Morse theory to prove the existence of infinitely many prime closed geodesics in compact Riemannian manifolds whose free loop space has unbounded Betti numbers. Roughly speaking, the (uniform) linear growth of the Morse index of an iterate implies that if there were only a finite number of prime closed geodesics, then the homology generated by their iterates would not suffice to produce the homology of the entire free loop space. Refinements of this kind of results have appeared in subsequent literature (see [6, 17]).

More recently, an increasing interest has arisen around the question of existence of periodic solutions of more general variational problems, and especially in the context of semi-Riemannian geometry. Recall that a semi-Riemannian manifold is a manifold \(M\) endowed with a nondegenerate, but possibly non positive definite, metric tensor \(g\). In this context, the geodesic variational theory is extremely more involved, even in the fixed endpoint case (see [1, 3]), due to the strongly indefinite character of the action functional. When the metric tensor is Lorentzian, i.e., it has index equal to 1, and the metric is stationary, i.e., time invariant, then it is possible to perform a certain reduction of the geodesic variational problem that yields existence results similar to the positive definite case (see [8, 10, 13, 14, 19]). For instance, it is proven in [8] that any stationary Lorentzian manifold having a compact Cauchy surface and whose free loop space has unbounded Betti numbers has infinitely many distinct prime closed geodesics. A Bott type result on the Morse index of an iterate has been proven in [16] for stationary Lorentzian metrics.

Dropping the stationarity assumption is at this stage a quite challenging task. The first problem that one encounters is the fact that the critical points of the geodesic action functional have truly infinite Morse index, so that standard Morse theory fails. In order to...
develop Morse theory for strongly indefinite functionals (see [2]), one computes the dimension of the intersection between the stable and the unstable manifolds as the difference of a sort of generalized index function defined at each critical point. In the fixed endpoint geodesic case, such generalized index function can be described explicitly as a kind of algebraic count of the degeneracies of the index form along the geodesic. More precisely, this is the so-called spectral flow of the path of index forms along the geodesic. Several extensions of the Morse index theorem (see [13, 22]) show that this number is related to a symplectic invariant associated to a fixed endpoint geodesic, called the Maslov index. The Maslov index is the natural substitute for the number of conjugate points along a geodesic, which may be infinite when the metric is non positive definite.

As to the periodic case, the notion of spectral flow \( sf(\gamma) \) of a closed geodesic \( \gamma \) has been introduced only recently (see [11]): this is a generalization of the Morse index of the geodesic action functional in the Riemannian case. For the reader’s convenience, in Section 3 we will review briefly this definition, that is given in terms of the choice of a periodic frame along the geodesic. An explicit computation shows that the periodic spectral flow equals the fixed endpoint spectral flow plus a concavity index, as in the original paper by M. Morse [20], plus a certain degeneracy term (see Theorem 5.2).

The main purpose of the present paper is to establish the growth of the spectral flow under iteration of the closed geodesic, along the lines of [9]. Given a closed semi-Riemannian geodesic \( \gamma \), we will show the existence of a function \( \lambda_\gamma \) defined on the unit circle \( S^1 \) and taking values in \( \mathbb{Z} \) (Definition 4.1), with the property that the spectral flow of the \( N \)-th iterate \( \gamma^{(N)} \) of \( \gamma \) equals the sum of the values that \( \lambda_\gamma \) takes at the \( N \)-th roots of unity (Theorem 5.3). This function is continuous, i.e., locally constant, at the points of \( S^1 \setminus \{1\} \) that are not eigenvalues of the linearized Poincaré map \( P_\gamma \) of \( \gamma \) (Proposition 4.4); the jump of \( \lambda_\gamma \) at an eigenvalue of \( P_\gamma \) is bounded by the dimension of the corresponding eigenspace (Corollary 4.11). As in the Riemannian case, knowing the exact value of the jumps of \( \lambda_\gamma \) at each discontinuity point would determine entirely the function \( \lambda_\gamma \). It should be observed that these discontinuities correspond to isolated degeneracy instants of an analytic path of self-adjoint Fredholm operators, and the value of the jump equals the contribution of the degeneracy instant to the spectral flow of the path. In principle, these jumps can be computed using higher order methods (see [11]), involving a finite number of derivatives of the path. As to the point \( z = 1 \), there is always a discontinuity of \( \lambda_\gamma \) when \( g \) is not positive definite (see Corollary 4.8): the value of the jump at \( z = 1 \) equals the index of the metric tensor \( g \). Concerning the nullity of the iterates \( \gamma^{(N)} \), the semi-Riemannian case is totally analogous to the Riemannian case, where the question is reduced to studying the spectrum of the linearized Poincaré map.

Using these properties of the spectral flow function \( \lambda_\gamma \), we then study the asymptotic behavior of the sequence \( N \to sf(\gamma^{(N)}) \in \mathbb{Z} \), by first showing that the limit \( \lim_{N \to \infty} \frac{1}{N} sf(\gamma^{(N)}) \) exists and is finite (Proposition 6.6). More precisely, using a certain finite dimensional reduction, we show that \( sf(\gamma^{(N)}) \) is the sum of a linear term in \( N \), a uniformly bounded term, and the term of a sequence which is either bounded or it satisfies a sort of uniform linear growth in \( N \) (Proposition 6.1, Lemma 6.2 and Proposition 6.8). When \( \gamma \) is a hyperbolic geodesic, i.e., when \( P_\gamma \) has no eigenvalue on the unit circle, then \( |sf(\gamma^{(N)})| \) either grows linearly with \( N \), or it is constant equal to the index of the metric tensor \( g \) (Proposition 6.12). In view to the development of a full-fledged Morse theory for semi-Riemannian closed geodesics, the most important result is that the spectral flow of an iterate \( \gamma^{(N)} \) is either bounded, or it has a uniform linear growth (Proposition 6.7, Corollary 6.10). This implies, in particular, that if a semi-Riemannian manifold has only a finite number of distinct prime closed geodesics, then for \( k \in \mathbb{Z} \) with \( |k| \) sufficiently large, the total number of geometrically distinct closed geodesics whose spectral flow is equal to \( k \) has to be uniformly bounded (Proposition 6.11). This is the key point of Gromoll and Meyer celebrated Riemannian multiplicity result.
The results are obtained mostly by functional analytical techniques. Using periodic frames along the geodesic (Section 3), the problem is cast into the language of differential systems in $\mathbb{R}^n$ and studied in the appropriate Sobolev space setting. Following Bott’s ideas, the spectral flow function $\lambda_\gamma$ is then obtained by considering a suitable complexification of the index form and of the space of infinitesimal variations of the geodesic (Subsections 4.1 and 4.2). The central property of $\lambda_\gamma$, that gives $s(\gamma^{(N)})$ as a sum of the values of $\lambda_\gamma$ at the $N$-th roots of unity, is proved in Section 5 using Bott’s suggestive terminology, this is called the Fourier theorem. Its proof in the non positive definite case relies heavily on a very special property of the index form, which is that of being represented by a compact perturbation of a fixed symmetry of the Hilbert space of variations of $\gamma$. By a symmetry of a Hilbert space it is meant a self-adjoint operator $\mathcal{J}$ whose square $\mathcal{J}^2$ is the identity. For paths of the form symmetry plus compact, the spectral flow only depends on the endpoints of the path, which is used in the proof of the Fourier theorem. The question of continuity of $\lambda_\gamma$, which is quite straightforward in the positive definite case, is more involved in the general semi-Riemannian case. At points $z \in S^1 \setminus \{1\}$, it is obtained by showing a perturbation result for the spectral flow of paths of self-adjoint Fredholm operators restricted to continuous families of closed subspaces of a fixed Hilbert space (Corollary 2.3). The definition and a few basic properties of spectral flow on varying domains are discussed preliminarily in Section 2. As to the point $z = 1$, where Corollary 2.3 does not apply, we use a certain finite dimensional reduction formula for the spectral flow (Proposition 4.5), which was proved recently in [17] to show that the spectral flow function has in $z = 1$ a sort of artificial discontinuity when $\gamma$ is nondegenerate. The reduction formula is used also in last section, where we obtain the iteration formula for the spectral flow (Proposition 6.1) and we prove estimates on its growth. For simplicity, in this paper we will only consider orientation preserving closed geodesics; however, in Subsection 4.5 we discuss briefly how to deal with the non-orientation preserving case.

Future developments of the theory of periodic semi-Riemannian geodesics should include an equivariant version of the strongly indefinite Morse theory, along the lines of [23]. A preliminary important step would deal with the case of nondegenerate critical orbits; in the context of periodic geodesics, this would apply to the so-called bumpy metrics. Recall that a metric is bumpy if all its closed geodesics are nondegenerate. Bumpy metrics are generic in the Riemannian setting (see [4, 5, 18, 24]; nothing is known with this respect in the nonpositive definite case.

2. Spectral flow on varying domains

Let $H$ be a real or complex Hilbert space; we will denote by $B(H)$ the Banach algebra of all bounded operators on $H$, by $GL(H)$ the open subset of $B(H)$ consisting of all isomorphisms, by $O(H)$ the subgroup of $GL(H)$ consisting of all isometries, and by $F_{sa}(H)$ the set of all self-adjoint Fredholm operators on $H$. The adjoint of an operator $T$ on $H$ will be denoted by $T^*$. Let us recall that the spectral flow is a integer invariant associated to a continuous path $T : [a, b] \to F_{sa}(H)$, which is:

(i) fixed endpoint homotopy invariant;
(ii) additive by concatenation;
(iii) invariant by cogredience, i.e., given two Hilbert spaces $H_1, H_2$, a continuous curve $T : [a, b] \to F_{sa}(H_1)$ and a continuous curve $M : [a, b] \to Iso(H_1, H_2)$ of isomorphisms from $H_1$ to $H_2$, then the spectral flow of $T$ on $H_1$ coincides with the spectral flow of $[a, b] \ni t \mapsto M_t T_t M_t^*$ on $H_2$.

We will denote by $s(T, [a, b])$ the spectral flow of the curve $T$; recall that $s(T, [a, b])$ is a sort of algebraic count of the degeneracy instants of the path $T_t$ as $t$ runs from $a$ to $b$. Details on the definition and the basic properties of the spectral flow can be found, for instance, in refs. [7][11][21]. There are several conventions on how to compute the contribution of the
endpoints of the path, in case of degenerate endpoints; although making a specific choice is irrelevant in the context of the present paper, we will follow the convention in [21].

Property (i) above holds in fact in a slightly more general form, as follows. If \( h : [a, b] \times [c, d] \to \mathcal{F}_{sa}(H) \) is a continuous map such that \( \dim(\text{Ker}(h_{a,s})) \) and \( \dim(\text{Ker}(h_{b,s})) \) are constant for all \( s \in [c, d] \), then the spectral flow of the path \([a, b] \ni t \mapsto h_{t,c}\) equals the spectral flow of the curve \([a, b] \ni t \mapsto h_{t,d}\) (see Corollary 2.3).

We will need to consider paths of Fredholm operators defined on varying domains. Let us consider the following setup: let \([a, b] \ni t \mapsto H_t\) be a continuous path of closed subspaces of \( H \). Recall that this means that, denoting by \( P_t : H \to H \) the orthogonal projection onto \( H_t \), then the curve \( t \mapsto P_t \) is continuous relatively to the operator norm topology of \( B(H) \). For instance, the kernels of a continuous family \( t \mapsto F_t \) of surjective bounded linear maps from \( H \) to some other Hilbert space \( H' \) form a continuous family of closed subspaces of \( H \) ([13] Lemma 2.9). A simple lifting argument in fiber bundles shows that there exists a continuous curve \( t \mapsto \Phi_t \in \text{GL}(H) \) and a closed subspace \( H_s \) of \( H \) such that \( \Phi_t(H_s) = H_t \) for all \( t \). We will call the pair \((\Phi, H_s)\) a \textit{trivialization} of the path \( t \mapsto H_t \). Assume now that \([a, b] \ni t \mapsto T_t \in B(H)\) is a continuous curve with the property that \( P_t T_t|_{H_s} : H_s \to H_s \) belongs to \( \mathcal{F}_{sa}(H_s) \) for all \( t \). Then, given a trivialization \((\Phi, H_s)\) for \((H_t)_{t \in [a,b]}\), for all \( t \in [a,b] \) the operator \( P_s\Phi_t^*P_tT_t\Phi_t|_{H_s} : H_s \to H_s \) belongs to \( \mathcal{F}_{sa}(H_s) \), where \( P_s \) is the orthogonal projection onto \( H_s \). We can therefore give the following definition:

**Definition 2.1.** The spectral flow of the path \( T \) over the varying domains \((H_t)_{t \in [a,b]}\), denoted by \( \text{sf}(T:(H_t)_{t \in [a,b]}) \), is defined as the spectral flow of the continuous path \([a, b] \ni t \mapsto \Phi_t^*P_tT_t\Phi_t|_{H_s} \) of self-adjoint Fredholm operators on \( H_s \).

Invariance by cogredience shows easily that the above definition does not depend on the choice of the trivialization \((\Phi, H_s)\) of \((H_t)_{t \in [a,b]}\). Namely, assume that \((\Phi, H_s)\) is another trivialization of \((H_t)_{t \in [a,b]}\). Denoting by \( P_s \) (resp., \( P_s \)) the orthogonal projection onto \( H_s \) (resp., onto \( H_s \)), and setting \( B_t = P_s\Phi_t^*P_tT_t\Phi_t|_{H_s} \), \( \tilde{B}_t = P_s\Phi_t^*P_t\tilde{T}_t\Phi_t|_{H_s} \), and \( \Psi_t = \Phi_t^*\Phi_t \), one has:

\[
\tilde{B}_t = (\Psi_t|_{H_s})^*B_t(\Psi_t|_{H_s})
\]

for all \( t \), hence \( \text{sf}(B, [a, b]) = \text{sf}(\tilde{B}, [a, b]) \).

Let us study how the spectral flow varies with respect to the domain.

**Lemma 2.2.** Let \([a, b] \ni t \mapsto T_t \in B(H)\) be a continuous map and \([c, d] \ni s \mapsto H_s\) be a continuous path of closed subspaces of \( H \), with the property that \( P_s T_s|_{H_s} \in \mathcal{F}_{sa}(H_s) \) for all \( s \) and \( t \). For all \( s \in [c, d] \), denote by \( h_s \) the spectral flow of the path \([a, b] \ni t \mapsto P_s T_s|_{H_s}\) of Fredholm operators on \( H_s \). Similarly, for all \( t \in [a, b] \) denote by \( v_s \) the spectral flow of the (constant) path of Fredholm operators \( T_t \) on the varying domains \((H_s)_{s \in [c,d]}\). Then:

\[
(2.1) \quad h_c - h_d = v_a - v_b.
\]

**Proof.** Choose a trivialization \((\Phi, H_s)\) for \((H_s)_{s \in [c,d]}\), and define a continuous map \( B : [a, b] \times [c, d] \to \mathcal{F}_{sa}(H_s) \) by:

\[
B_s,t = P_s\Phi_t^*P_s T_t\Phi_t|_{H_s}.
\]

By definition, \( v_s = \text{sf}(s \mapsto B_s,t, [c,d]) \) and \( h_s = \text{sf}(t \mapsto B_s,t, [a,b]) \); formula (2.1) follows immediately from the homotopy invariance and the concatenation additivity of the spectral flow. \( \square \)

\[1\text{In fact, one can find the curve } \Phi_t \text{ taking values in } O(H), \text{ see [7].} \]
Corollary 2.3. Under the assumptions of Lemma 2.2 if \( \text{Ker}(P_sT_a|_{H_s}) \) and \( \text{Ker}(P_sT_b|_{H_s}) \) have constant dimension for all \( s \in [a, b] \), then the spectral flow of \( t \mapsto P_sT_t|_{H_s} \) on \( H_s \) does not depend on \( s \).

Proof. This follows easily from the fact that curves of self-adjoint Fredholm operators with kernel of constant dimension have null spectral flow. Thus, under our assumptions both terms \( s_a \) and \( s_b \) in (2.1) vanish. \( \square \)

3. Spectral flow along a closed geodesic

3.1. Periodic geodesics. We will consider throughout an \( n \)-dimensional semi-Riemannian manifold \((M, g)\), denoting by \( \nabla \) the covariant derivative of its Levi–Civita connection, and by \( R \) its curvature tensor, chosen with the sign convention \( R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \). Let \( \gamma : [0, 1] \to M \) be a periodic geodesic in \( M \), i.e., \( \gamma(0) = \gamma(1) \). We will assume that \( \gamma \) is orientation preserving, which means that the parallel transport along \( \gamma \) is orientation preserving; the non orientation preserving case can be studied similarly, as explained in Subsection 4.5. If \( M \) is orientable, then every closed geodesic is orientation preserving. Moreover, given any closed geodesic \( \gamma \), its two-fold iteration \( \gamma^{(2)} \), defined by \( \gamma^{(2)}(t) = \gamma(2t) \), is always orientation preserving.

Let us denote by \( T \gamma : T\gamma(0)M \oplus T\gamma(0)M \to T\gamma(0)M \oplus T\gamma(0)M \) the linearized Poincaré map of \( \gamma \), defined by:

\[
I_\gamma(V, W) = \int_0^1 g(D_tV, D_tW) + g(RV, W) \, dt,
\]

where we set \( R = R(\dot{\gamma}, \cdot)\). Closed geodesics in \( M \) are the critical points of the geodesic action functional \( f(\gamma) = \frac{1}{2} \int_0^1 g(\dot{\gamma}, \dot{\gamma}) \, dt \) defined in the free loop space \( \Omega M \) of \( M \); \( \Omega M \) is the Hilbert manifold of all closed curves in \( M \) of Sobolev class \( H^1 \). The index form \( I_\gamma \) is the second variation of \( f \) at the critical point \( \gamma \); unless \( g \) is positive definite, the Morse index of \( f \) at each non constant critical point is infinite. The notion of Morse index is replaced by the notion of spectral flow.

Let us denote by \( \mathcal{P}_\gamma : T\gamma(0)M \oplus T\gamma(0)M \to T\gamma(0)M \oplus T\gamma(0)M \) the linearized Poincaré map of \( \gamma \), defined by:

\[
\mathcal{P}_\gamma(v, v') = (V(1), \frac{d}{dt}V(1)),
\]

where \( V \) is the unique Jacobi field along \( \gamma \) such that \( V(0) = v \) and \( \frac{d}{dt}V(0) = v' \). Fixed points of \( \mathcal{P}_\gamma \) correspond to periodic Jacobi fields along \( \gamma \). Moreover, \( \mathcal{P}_\gamma \) preserves the symplectic form \( \omega \) of \( T\gamma(0)M \oplus T\gamma(0)M \) defined by:

\[
\omega((v, v'), (w, w')) = g(v, w') - g(v', w).
\]

3.2. Periodic frames and trivializations. Consider a smooth periodic orthonormal frame \( T \) along \( \gamma \), i.e., a smooth family \([0, 1] \ni t \mapsto T_t \) of isomorphisms:

\[
T_t : \mathbb{R}^n \to T\gamma(t)M
\]

with \( T_0 = T_1 \), and

\[
g(T_te_i, T_te_j) = \epsilon_i \delta_{ij},
\]

where \( \{e_i\}_{i=1,...,n} \) is the canonical basis of \( \mathbb{R}^n \), \( \epsilon_i \in \{-1, 1\} \) and \( \delta_{ij} \) is the Kronecker symbol. The existence of such frame is guaranteed by the orientability assumption on the closed geodesic. The pull-back by \( T_t \) of the metric \( g \) gives a symmetric nondegenerate bilinear form \( G \) on \( \mathbb{R}^n \), whose index is the same as the index of \( g \); note that this pull-back does not depend on \( t \), by the orthogonality assumption on the frame \( T \). In the sequel, we
will also denote by $G : \mathbb{R}^n \to \mathbb{R}^n$ the symmetric linear operator defined by $(Gv) \cdot w$; by (3.3), $G$ satisfies:

\begin{equation}
G^2 = \text{Id}.
\end{equation}

Moreover, the pull-back of the linearized Poincaré map $\mathfrak{P}_\gamma$ by the isomorphism $T_0 \oplus T_0 : \mathbb{R}^n \oplus \mathbb{R}^n \to T_{\gamma(0)}M \oplus T_{\gamma(0)}M$ gives a linear endomorphism of $\mathbb{R}^n \oplus \mathbb{R}^n$ that will be denoted by $\mathfrak{P}$.

For all $t \in [0, 1]$, define by $\mathcal{H}^1_{\per}$ the Hilbert space of all $H^1$-vector fields $V$ along $\gamma|_{[0, t]}$ satisfying:

\[ T_0^{-1}V(0) = T_t^{-1}V(t). \]

Observe that the definition of $\mathcal{H}^1_{\per}$ depends on the choice of the periodic frame $T$, however, $\mathcal{H}^1_{\per}$, which is the space of all periodic vector fields along $\gamma$, does not depend on $T$. Although in principle there is no necessity of fixing a specific Hilbert space inner product, it will be useful to have one at disposal, and this will be chosen as follows. For all $t \in [0, 1]$, consider the Hilbert space:

\[ H^1_{\per}([0, t], \mathbb{R}^n) = \left\{ V \in H^1([0, t], \mathbb{R}^n) : \bar{V}(0) = \bar{V}(t) \right\}; \]

a natural Hilbert space inner product in $H^1_{\per}([0, t], \mathbb{R}^n)$ is given by:

\begin{equation}
\langle \bar{V}, \bar{W} \rangle = \bar{V}(0) \cdot \bar{W}(0) + \int_0^t \bar{V}(s) \cdot \bar{W}(s) \, ds,
\end{equation}

where $\cdot$ is the Euclidean inner product in $\mathbb{R}^n$. The map $\Psi_t : \mathcal{H}_{\per}^1 \to H^1_{\per}([0, t], \mathbb{R}^n)$ defined by $\Psi_t(V) = \bar{V}$, where $\bar{V}(s) = T_t^{-1}(V(s))$ is a linear isomorphism; the space $\mathcal{H}_{\per}^1$ will be endowed with the pull-back of the inner product (3.5) by the isomorphism $\Psi_t$. Denote by $\bar{T}_t \in \text{End}(\mathbb{R}^n)$ the pull-back by $T_t$ of the endomorphism $R_{\gamma(t)} = R(\gamma, \cdot)\gamma$ of $T_{\gamma(t)}M$:

\[ \bar{T}_t = T_t^{-1} \circ R_{\gamma(t)} \circ T_t; \]

observe that $t \mapsto \bar{T}_t$ is a smooth map of $G$-symmetric endomorphisms of $\mathbb{R}^n$. Finally, denote by $\Gamma_t \in \text{End}(\mathbb{R}^n)$ the Christoffel symbol of the frame $T$, defined by:

\[ \Gamma_t(v) = T_t^{-1}(\frac{d}{dt}V) - \frac{d}{dt}V(t), \]

where $V$ is any vector field satisfying $V(t) = v$, and $V = \Psi_t^{-1}(\bar{V})$. A straightforward computation shows that $\Gamma_t$ is $G$-anti-symmetric for all $t$.

The push-forward by $\Psi_t$ of the index form $I_s$ on $\mathcal{H}_{\per}^1$ is given by the bounded symmetric bilinear form $\bar{T}_t$ on $H^1_{\per}([0, t], \mathbb{R}^n)$ defined by:

\begin{equation}
\bar{T}_t(\bar{V}, \bar{W}) = \int_0^t G(\bar{V}(s), \bar{W}(s)) + G(\Gamma_s V(s), \bar{W}(s)) + G(\bar{V}(s), \Gamma_s \bar{W}(s))
\end{equation}

\[ + G(\Gamma_s \bar{V}(s), \Gamma_s \bar{W}(s)) + G(\bar{W}(s), \bar{W}(s)) \, ds. \]

Finally, for $t \in [0, 1]$, we will consider the isomorphism

\[ \Phi_t : H^1_{\per}([0, t], \mathbb{R}^n) \to H^1_{\per}([0, 1], \mathbb{R}^n), \]

defined by $\bar{V} \mapsto \bar{V}$, where $\bar{V}(s) = \bar{V}(st)$, $s \in [0, 1]$. The push-forward by $\Phi_t$ of the bilinear form $\bar{T}_t$ is given by the bounded symmetric bilinear form $\bar{I}_t$ on $H^1_{\per}([0, 1], \mathbb{R}^n)$ defined by:

\begin{equation}
\bar{I}_t(\bar{V}, \bar{W}) = \frac{1}{t^2} \int_0^1 G(\bar{V}(r), \bar{W}(r)) + tG(\Gamma_{tr} \bar{V}(r), \bar{W}(r)) + tG(\bar{V}(r), \Gamma_{tr} \bar{W}(r))
\end{equation}

\[ + t^2 G(\Gamma_{tr} \bar{V}(r), \Gamma_{tr} \bar{W}(r)) + t^2 G(\bar{W}(r), \bar{W}(r)) \, dr. \]
3. Spectral flow of a periodic geodesic. For \( t \in [0, 1] \), define the Fredholm bilinear form \( B_t \) on the Hilbert space \( H^1_{\text{per}}([0, 1], \mathbb{R}^n) \) by setting:
\[
B_t = t^2 \cdot \tilde{I}_t.
\]
From (3.7), one sees immediately that the map \([0, 1] \ni t \mapsto B_t\) can be extended continuously to \( t = 0 \) by setting:
\[
B_0(\tilde{V}, \tilde{W}) = \int_0^1 G(\tilde{V}'(r), \tilde{W}'(r)) \, dr.
\]
Observe that \( \text{Ker}(B_0) \) is \( n \)-dimensional, and it consists of all constant vector fields. For all \( t \in [0, 1] \), the bilinear form \( \tilde{I}_t \) on \( H^2_{\text{per}}([0, 1], \mathbb{R}^n) \) is represented with respect to the inner product (3.3) by a compact perturbation of the symmetry \( \mathcal{J} \) of \( H^1_{\text{per}}([0, 1], \mathbb{R}^n) \) given by \( \tilde{V} \mapsto G\tilde{V} \).

**Definition 3.1.** The spectral flow \( \text{sf}(\gamma) \) of the closed geodesic \( \gamma \) is defined as the spectral flow of the continuous path of Fredholm bilinear forms \([0, 1] \ni t \mapsto B_t\) on the Hilbert space \( H^1_{\text{per}}([0, 1], \mathbb{R}^n) \).

It is a non trivial fact that the definition of spectral flow along a closed geodesic does not depend on the choice of a periodic orthonormal frame along the geodesic. This result is obtained in [7] by determining an explicit formula giving the spectral flow in terms of some other integers associated to the geodesic, such as the Maslov index and the concavity number. The Maslov index is a symplectic invariant, which is computed as an intersection number in the Lagrangian Grassmannian of a symplectic vector space; it will be denoted by \( i_{\text{Maslov}}(\gamma) \).

Recall that a Jacobi field along \( \gamma \) is a smooth vector field \( J \) along \( \gamma \) that satisfies the second order linear equation:
\[
\frac{D^2}{dt^2} J(t) = R(\gamma(t), J(t)) \gamma(t), \quad t \in [0, 1];
\]
let us denote by \( \mathcal{J}_\gamma \) the \( 2n \)-dimensional real vector space of all Jacobi fields along \( \gamma \). Let us introduce the following spaces:
\[
\mathcal{J}_\gamma^{\text{per}} = \{ J \in \mathcal{J}_\gamma : J(0) = J(1), \ \frac{D}{dt} J(0) = \frac{D}{dt} J(1) \},
\]
\[
\mathcal{J}_\gamma^0 = \{ J \in \mathcal{J}_\gamma : J(0) = J(1) = 0 \}, \quad \text{and}
\]
\[
\mathcal{J}_\gamma^* = \{ J \in \mathcal{J}_\gamma : J(0) = J(1) \}.
\]
It is well known that \( \mathcal{J}_\gamma^{\text{per}} \) is the kernel of the index form \( I_\gamma \) defined in (3.1), while \( \mathcal{J}_\gamma^0 \) is the kernel of the restriction of the index form to the space of vector fields along \( \gamma \) vanishing at the endpoints. We denote by \( n_{\text{per}}(\gamma) \) and \( n_0(\gamma) \) the dimensions of \( \mathcal{J}_\gamma^{\text{per}} \) and \( \mathcal{J}_\gamma^0 \), respectively. The nonnegative integer \( n_{\text{per}}(\gamma) \) is the nullity of \( \gamma \) as a periodic geodesic, i.e., the nullity of the Hessian of the geodesic action functional at \( \gamma \) in the space of closed curves. Observe that \( n_{\text{per}}(\gamma) \geq 1 \), as \( \mathcal{J}_\gamma^{\text{per}} \) contains the one-dimensional space spanned by the tangent field \( J = \dot{\gamma} \). Similarly, \( n_0(\gamma) \) is the nullity of \( \gamma \) as a fixed endpoint geodesic, i.e., it is the nullity of the Hessian of the geodesic action functional at \( \gamma \) in the space of fixed endpoints curves in \( M \). In this case, \( n_0(\gamma) > 0 \) if and only if \( \gamma(1) \) is conjugate to \( \gamma(0) \) along \( \gamma \). The index of concavity of \( \gamma \), that will be denoted by \( i_{\text{conc}}(\gamma) \) is a nonnegative integer invariant associated to periodic solutions of Hamiltonian systems. In our notations, \( i_{\text{conc}}(\gamma) \) is equal to the index of the symmetric bilinear form:
\[
(J_1, J_2) \mapsto g(\frac{D}{dt} J_1(1) - \frac{D}{dt} J_1(0), J_2(0))
\]
defined on the vector space \( \mathcal{J}_\gamma^* \). It is not hard to show that this bilinear form is symmetric, in fact, it is given by the restriction of the index form \( I_\gamma \) to \( \mathcal{J}_\gamma^* \).
Theorem 3.2. Let \((M, g)\) be a semi-Riemannian manifold and let \(\gamma : [0, 1] \rightarrow M\) be a closed oriented geodesic in \(M\). Then, the spectral flow \(\text{sf}(\gamma)\) is given by the following formula:

\[
\text{sf}(\gamma) = \dim(\mathcal{J}^p_{\gamma} \cap \mathcal{J}^q_{\gamma}) - \text{iMaslov}(\gamma) - \text{icone}(\gamma) - n_-(g),
\]

where \(n_-(g)\) is the index of the metric tensor \(g\).

**Proof.** See [7, Theorem 5.6]. \(\square\)

Formula (3.9) proves in particular that the definition of spectral flow for a periodic geodesic \(\gamma\) does not depend on the choice of an orthonormal frame along \(\gamma\).

4. THE SPECTRAL FLOW FUNCTION

4.1. The basic data. As described in Section 3, the choice of a smooth periodic orthonormal frame along a closed orientation preserving semi-Riemannian geodesic produces the following objects:

- a non degenerate symmetric bilinear form \(G\) on \(\mathbb{R}^n\) and a symplectic form \(\varpi\) on \(\mathbb{R}^n \oplus \mathbb{R}^n\) defined by \(\varpi((v, v'), (w, w')) = G(v, w') - G(v', w)\);
- a smooth 1-periodic curve \(\Gamma : \mathbb{R} \rightarrow \text{End}(\mathbb{R}^n)\) of \(G\)-anti-symmetric linear endomorphisms of \(\mathbb{R}^n\);
- a smooth 1-periodic curve \(\overline{\Gamma} : \mathbb{R} \rightarrow \text{End}_G(\mathbb{R}^n)\) of \(G\)-symmetric linear endomorphisms of \(\mathbb{R}^n\);
- a linear endomorphism \(\mathcal{P} : \mathbb{R}^n \oplus \mathbb{R}^n \rightarrow \mathbb{R}^n \oplus \mathbb{R}^n\) that preserves the symplectic form \(\varpi\).

We will use a complexification of these data. More precisely, \(G\) will be extended by sesquilinearity on \(\mathbb{C}^n\), \(\Gamma\) and \(\overline{\Gamma}\) will be extended to \(\mathbb{C}\)-linear endomorphisms of \(\mathbb{C}^n\), and \(\mathcal{P}\) will be extended to \(\mathbb{C}\)-linear endomorphisms of \(\mathbb{C}^{2n}\). Let \(\mathcal{H}\) be the complex Hilbert space \(H^1([0, 1], \mathbb{C}^n)\) endowed with inner product:

\[
\langle \tilde{V}, \tilde{W} \rangle = \tilde{V}(0) \cdot \tilde{W}(0) + \int_0^1 \tilde{V}'(r) \cdot \tilde{W}'(r) \, dr,
\]

where \(\cdot\) denotes the canonical Hermitian product in \(\mathbb{C}^n\): \(v \cdot w = \sum_{j=1}^n v_j \bar{w}_j\). Given a unit complex number \(z\), let \(\mathcal{H}_z\) denote the Hilbert subspace of \(\mathcal{H}\) defined by:

\[
\mathcal{H}_z = \{ \tilde{V} \in \mathcal{H} : \tilde{V}(1) = z \tilde{V}(0) \};
\]

moreover, we will denote by \(\mathcal{H}_0\) the subspace:

\[
\mathcal{H}_0 = \{ \tilde{V} \in \mathcal{H} : \tilde{V}(0) = \tilde{V}(1) = 0 \}.
\]

For \(t \in [0, 1]\), let \(B_t : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}\) denote the bounded Hermitian form defined by:

\[
B_t(\tilde{V}, \tilde{W}) = \int_0^1 G(\tilde{V}'(r), \tilde{W}'(r)) + tG(\Gamma_r \tilde{V}(r), \tilde{W}'(r)) + tG(\tilde{V}'(r), \Gamma_r \tilde{W}(r))
\]

\[
+ t^2G(\Gamma_r \tilde{V}(r), \Gamma_r \tilde{W}(r)) + t^2G(\overline{\Gamma_r} \tilde{V}(r), \tilde{W}(r)) + t^2G(\overline{\Gamma_r} \tilde{W}(r), \tilde{V}(r)) \, dr.
\]

With the above data, the Jacobi equation along \(\gamma\) gives the following second order linear homogeneous equation for vector fields in \(\mathbb{C}^n\):

\[
\tilde{V}''(r) + 2\Gamma_r \tilde{V}'(r) + (\Gamma_r^2 + \Gamma_r^2 - \overline{\Gamma}_r) \tilde{V}(r) = 0.
\]

The linear map \(\mathcal{P} : \mathbb{C}^n \oplus \mathbb{C}^n \rightarrow \mathbb{C}^n \oplus \mathbb{C}^n\) is given by:

\[
\mathcal{P}(v, v') = (\tilde{V}(1), \tilde{V}'(1) + \Gamma_0 \tilde{V}(1)),
\]

where \(\tilde{V} \in C^2([0, 1], \mathbb{C}^n)\) is the unique solution of (4.4) satisfying \(\tilde{V}(0) = v\) and \(\tilde{V}'(0) = v' - \Gamma_0 v\). We observe that \(t \mapsto B_t\) can extended in the obvious way for \(t \in [0, +\infty)\), and \(B_N\) corresponds with the index form associated to the \(N\)-th iterate \(\gamma^{(N)}\).
4.2. The spectral flow function on the circle. As in Section 3 it is easy to see that for all \( t \in [0, 1] \) and \( z \in S^1 \), the restriction of the Hermitian form \( B_t \) to \( \mathcal{H}_z \times \mathcal{H}_z \) is represented by a compact perturbation of the symmetry \( \mathcal{J} \) of \( \mathcal{H} \) defined by \( V \mapsto GV \); observe that each \( \mathcal{H}_z \) is invariant by \( \mathcal{J} \).

**Definition 4.1.** The spectral flow function \( \lambda_{\gamma} : S^1 \to \mathbb{N} \) is defined by:

\[
\lambda_{\gamma}(z) = \text{spectral flow of } [0, 1] \ni t \mapsto B_t \text{ on the space } \mathcal{H}_z \times \mathcal{H}_z.
\]

Let us recall that the spectral flow of a continuous path of symmetric bilinear forms \( B \) on a real Hilbert space \( H \) equals the spectral flow of the continuous path of Fredholm Hermitian forms obtained by taking the sesquilinear extension of \( B \) on the complexified Hilbert space \( H^C \). In particular, \( \lambda_{\gamma}(1) = \mathfrak{s}(\gamma) \).

Moreover, it is easy to see that \( \mathcal{J} \) is invariant by \( \lambda_{\gamma} \).

**Lemma 4.2.** It is shown in [12] that the spectral flow function \( \mathcal{J} \) is invariant by \( \lambda_{\gamma} \).

The map \( \lambda_{\gamma} \) is defined by:

\[
\lambda_{\gamma}(1, N) = \mathfrak{s}(\gamma^{(N)}),
\]

where \( \gamma^{(N)} \) is the \( N \)-th iterate of \( \gamma \).

4.3. Continuity and jumps of the spectral flow function. The reader will easily check by an immediate partial integration argument that equation (4.4) characterizes the elements in the kernel of the Hermitian form \( B_1 \) in (4.3); more precisely:

**Lemma 4.2.** \( \tilde{V} \in \mathcal{H} \) is in the kernel of \( B_1 : \mathcal{H}_z \times \mathcal{H}_z \to \mathbb{C} \) if and only if it satisfies (4.4) and the boundary conditions:

\[
\tilde{V}(1) = z \cdot \tilde{V}(0), \quad \text{and} \quad \tilde{V}'(1) = z \cdot \tilde{V}'(0).
\]

Thus, \( B_1 \) is degenerate on \( \mathcal{H}_z \) if and only if \( z \) is in the spectrum of the linearized Poincaré map \( \mathfrak{P}_\gamma \), and \( \dim(\ker(B_1|_{\mathcal{H}_z \times \mathcal{H}_z})) = \dim(\ker(\mathfrak{P}_\gamma - z \cdot \text{Id})) \).

**Lemma 4.3.** The map \( S^1 \ni z \mapsto \mathcal{H}_z \subset \mathcal{H} \) is a continuous map of closed subspaces of \( \mathcal{H} \).

**Proof.** \( \mathcal{H}_z \) is the kernel of a continuous family \( F_z : \mathcal{H} \to \mathbb{C}^n \) of surjective bounded linear maps, defined by \( F_z(V) = \tilde{V}(1) - z \tilde{V}(0) \) (see [13 Lemma 2.9]).

**Proposition 4.4.** The spectral flow function \( \lambda_{\gamma} \) is constant on every connected subset \( A \) of \( S^1 \setminus \{1\} \) that does not contain elements in the spectrum of the linearized Poincaré map \( \mathfrak{P}_\gamma \).

**Proof.** This follows easily from Corollary 2.3. By Lemma 4.2, the assumption on the spectrum of the Poincaré map says that \( B_1 \) does not degenerate on \( \mathcal{H}_z \), for all \( z \in A \). Moreover, it is easy to see that \( B_0 \) is nondegenerate on \( \mathcal{H}_z \) for all \( z \neq 1 \) (while the kernel of \( B_0|_{\mathcal{H}_z \times \mathcal{H}_z} \) consists of all constant maps, and it has dimension \( n \)). This concludes the proof.

We conclude that \( \lambda_{\gamma} \) has a finite number of jumps on \( S^1 \), that can occur only at those points in the spectrum of \( \mathfrak{P}_\gamma \) that lie in \( S^1 \) or at \( z = 1 \).

Let us study now the behavior of \( \lambda_{\gamma} \) around \( z = 1 \); the result of Proposition 4.4 cannot be extended to \( z = 1 \), because \( B_0 \) is always degenerate on \( \mathcal{H}_1 \). We will show that, unless the metric tensor \( g \) is positive definite, then \( \lambda_{\gamma} \) is indeed discontinuous at \( z = 1 \). To this

\[\text{In fact, the very same argument shows that } z \mapsto \mathcal{H}_z \text{ is a real analytic map. The same conclusion holds in Lemma 4.7.}\]
aim, we will need to determine an alternative description of the function $\lambda_\gamma$ based on a finite dimensional reduction for the computation of the spectral flow.

4.4. A finite dimensional reduction. By a symmetry of a Hilbert space $H$ we mean a bounded self-adjoint operator $\mathfrak{z}$ on $H$ satisfying $\mathfrak{z}^2 = 1$. For paths $[a, b] \ni t \mapsto T_t \in \mathcal{F}_{sa}(H)$ of Fredholm self-adjoint operators of the form $T_t = \mathfrak{z} + K_t$, where $\mathfrak{z}$ is a fixed symmetry of $H$ and $K_t$ is a compact self-adjoint operator on $H$ for all $t$, the spectral flow $\text{sf}(T, [a, b])$ depends only on the endpoints $T_a$ and $T_b$. More precisely, $\text{sf}(T, [a, b])$ is the relative dimension of the generalized negative spaces of $T_t$ at $t = a$ and at $t = b$. We want to compare the spectral flow of a path $T$ on $H$ with the spectral flow of its restriction to a finite codimensional subspace of $H$. Let us recall the following result from [7]:

**Proposition 4.5.** Let $T : [a, b] \to \mathcal{F}_{sa}(H)$ be a continuous curve where each $T_t$ is a compact perturbation of a fixed symmetry $\mathfrak{z}$ of $H$, and let $\mathcal{V} \subset H$ be a closed subspace of finite dimension in $H$. Set $B_t = \langle T_t \cdot, \cdot \rangle$ and $\mathcal{V}_t = (T_t \mathcal{V})^\perp = \mathcal{V}^\perp \cap \mathcal{V}$; then:

\begin{equation}
\text{sf}(T, [a, b]) - \text{sf}(P_{\mathcal{V}} T |_{\mathcal{V}}, [a, b]) = n_-\left( (B_a|_{\mathcal{V}_a \cap \mathcal{V}_a}) + \dim(\mathcal{V} \cap \mathcal{V}_a) \right) - \dim(\mathcal{V} \cap \mathcal{V}_a) - n_-\left( (B_b|_{\mathcal{V}_b \cap \mathcal{V}_b}) + \dim(\mathcal{V} \cap \mathcal{V}_b) \right),
\end{equation}

where $n_-$ denotes the index of a Hermitian (or symmetric in the real case) bilinear form.

**Proof.** See [7, Theorem 4.3]. □

We apply this result to the path of bilinear forms $[0, 1] \ni t \mapsto B_t$ given in (4.3), to the Hilbert space $H = \mathcal{H}_\gamma$ in (4.1) and to the closed finite codimensional subspace $\mathcal{V} = \mathcal{H}_a$ defined in (4.2), obtaining:

**Proposition 4.6.** For all $z \in S^1$, the following equality holds:

\begin{equation}
\lambda_\gamma(z) - \lambda_a^0 = (1 - \delta_{z, 1}) \cdot n_-(g) - n_0(\gamma) + \dim(J_\gamma^{(1)}(z)) - n_-(b_z),
\end{equation}

where $J_\gamma^{(1)}(z)$ is the finite dimensional vector space:

$\mathcal{J}_\gamma^{(1)}(z) = \mathcal{H}_a \cap \text{Ker}(B_1|_{\mathcal{H}_\gamma \times \mathcal{H}_\gamma})$

$= \{ \tilde{V} \in C^2([0, 1], \mathbb{C}^n) : \tilde{V}(0) = \tilde{V}(1) = 0, \tilde{V}^\prime(1) = z\tilde{V}^\prime(0) \}$

and $b_z$ is the Hermitian form on the finite dimensional space:

$\mathcal{J}^{(2)}(z) = \{ \tilde{V} \in C^2([0, 1], \mathbb{C}^n) : \tilde{V}(1) = z\tilde{V}(0) \}$,

given by the restriction of $B_1$, or, more explicitly:

$b_z(\tilde{V}, \tilde{W}) = G(z\tilde{V}(1) - \tilde{V}^\prime(0), \tilde{W}(0))$.

**Proof.** Formula (4.9) follows directly by applying Proposition 4.3 to the above setup. It is easily obtained after checking the following identities:

- $\mathcal{H}^{\perp a_0}_a \cap \mathcal{H}_x = \{ \tilde{V} : \tilde{V}(t) = A(z - 1)t + A, \text{ for some } A \in \mathbb{C}^n \}$, thus the index of the restriction of $B_0$ to such space is equal to 0 if $z = 1$ and is equal to $n_-(G) = n_-(g)$ if $z \neq 1$;
- $\mathcal{H}_a \cap (\mathcal{H}^{\perp a_0}_a \cap \mathcal{H}_x) = \{ 0 \}$;
- $\text{Ker}(B_0|_{\mathcal{H}_\gamma \times \mathcal{H}_\gamma}) = \{ \tilde{V} : \tilde{V}^\prime = 0, (\tilde{V}(1), \tilde{V}^\prime(1)) = z \cdot (\tilde{V}(0), \tilde{V}^\prime(0)) \}$, such space is $\{ 0 \}$ if $z \neq 1$, and it consists of constant functions if $z = 1$;
- $\mathcal{H}_a \cap (\text{Ker}(B_0|_{\mathcal{H}_\gamma \times \mathcal{H}_\gamma})) = \{ 0 \}$;
- $\text{Ker}(B_1|_{\mathcal{H}_\gamma \times \mathcal{H}_\gamma}) = \{ \tilde{V} \text{ solution of (4.4)} : (\tilde{V}(1), \tilde{V}^\prime(1)) = z \cdot (\tilde{V}(0), \tilde{V}^\prime(0)) \}$.

\[\text{Here } \delta_{z, 1} \text{ is the Kronecker symbol, equal to 1 if } z = 1 \text{ and to 0 otherwise.}\]
\( \mathcal{H}_0 \cap \text{Ker}(B_1|_{\mathcal{H}_x \times \mathcal{H}_x}) = J_\gamma(1)(z); \)

\( \mathcal{H}_x^{\perp |_x} \cap \mathcal{H}_x = J_\gamma(2)(z); \)

\( \mathcal{H}_x^{\perp |_x} = \text{Ker}(B_1|_{\mathcal{H}_x \times \mathcal{H}_x}) = \{ \bar{V} \text{ solution of (4.4)}, \bar{V}(0) = \bar{V}(1) = 0 \}. \)

We recall that the term \( n_0(\gamma) \) in (4.9) is the nullity of \( \gamma \) as a \textit{fixed endpoints} geodesic, i.e., the multiplicity of \( \gamma(1) \) as conjugate point to \( \gamma(0) \) along \( \gamma \). In particular, it coincides with the dimension of the space:

\[
(4.10) \quad \{ \bar{V} \in C^2([0,1],\mathbb{C}^n) : \bar{V} \text{ solution of (4.4)}, \bar{V}(0) = \bar{V}(1) = 0 \}.
\]

Observe that for all \( z \in S^1 \), the space \( J_\gamma(1)(z) \) is contained in (4.10); it follows that

\[
(4.11) \quad - n_0(\gamma) + \dim(J_\gamma(1)(z)) \leq 0, \quad \forall \ z \in S^1.
\]

**Lemma 4.7.** The map \( z \mapsto J_\gamma(2)(z) \subset \mathcal{H} \) is continuous at those points \( z \in S^1 \) that are not in the spectrum of \( \mathcal{P} \).

**Proof.** It suffices to show that \( z \mapsto J_\gamma(2)(z) \) is a continuous family of closed subspaces of the finite dimensional closed subspace \( S = \{ \bar{V} \in C^2([0,1],\mathbb{C}^n) : \bar{V} \text{ is solution of (4.4)} \} \) of \( \mathcal{H} \). If we identify \( S = \mathbb{C}^n \oplus \mathbb{C}^n \) via the map \( \bar{V} \mapsto (\bar{V}(0), \bar{V}(1)) \), then \( J_\gamma(2)(z) \) is the kernel of the linear map \( L_z = \pi_1 \circ (\mathcal{P} - z \cdot I) : \mathbb{C}^n \oplus \mathbb{C}^n \to \mathbb{C}^n \), where \( \pi_1 : \mathbb{C}^n \oplus \mathbb{C}^n \to \mathbb{C}^n \) is the projection on the first summand. Clearly, \( z \mapsto L_z \) is continuous. If \( z \in S^1 \) is not in the spectrum of \( \mathcal{P} \), then \( \mathcal{P} - z \cdot I \) is an isomorphism, and thus \( L_z \) is surjective, which concludes the proof (see [13] Lemma 2.9).

**Corollary 4.8.** Let \( \mathcal{A} \) be a connected subset of \( S^1 \) that contains \( z = 1 \), and that does not contain any eigenvalue of the linearized Poincaré map \( \mathcal{P}_\gamma \). Then, \( \lambda_\gamma \) is constant equal to \( \lambda_\gamma(1) + n_-(g) \) on \( \mathcal{A} \setminus \{1\} \).

**Proof.** Using formula (4.9), the reader will easily convince himself that the statement is equivalent to proving that the quantity \( \dim(J_\gamma(1)(z)) - n_-(b_z) \) is constant on every connected subset \( \mathcal{A} \) of \( S^1 \) that does not contain elements in the spectrum of \( \mathcal{P}_\gamma \). This follows immediately from the continuity of the subspaces \( \mathcal{A} \ni z \mapsto J_\gamma(2)(z) \) proved in Lemma 4.7.

and from the following observations:

(a) \( \text{Ker}(b_z) = \{ \tilde{W} \text{ solution of (4.4)}, \tilde{W}(0) = \tilde{W}(1) = 0 \} \), thus \( \text{Ker}(b_z) \) does not depend on \( z \in \mathcal{A} \), and \( n_-(b_z) \) is constant on \( \mathcal{A} \);

(b) \( J_\gamma(1)(z) \subset \text{Ker}(B_1|_{\mathcal{H}_x \times \mathcal{H}_x}) \), hence \( \dim(J_\gamma(1)(z)) = 0 \) for all \( z \in \mathcal{A} \).

In order to prove (a), note that if \( z \) is not in the spectrum of \( \mathcal{P}_\gamma \), then the map \( J_\gamma(2)(z) \ni \bar{V} \mapsto \bar{V}'(1) - \bar{V}'(0) \in \mathbb{C}^n \) is an isomorphism. It follows immediately from the definition of \( b_z \) that \( \text{Ker}(b_z) = \{ \bar{W} \in J_\gamma(2)(z), \bar{W}(0) = 0 \} \), which gives the desired conclusion.

**Remark 4.9.** Note that, when \( 1 \) is not in the spectrum of \( \mathcal{P}_\gamma \), the discontinuity at \( z = 1 \) of \( \lambda_\gamma \) occurs only in the non Riemannian case.

**4.5. Non orientation preserving closed geodesics.** When the geodesic \( \gamma \) is non orientation preserving, the definition of the spectral flow given in Definition 3.1 does not apply because there is no periodic orthonormal frame along the geodesic. Let us now indicate briefly how to modify the construction in order to get a well defined notion of spectral flow also in this case. One can consider an arbitrary smooth orthonormal frame \( \mathbf{T} = (T_t)_{t \in [0,1]} \) along the geodesic as in (3.2), which will not satisfy \( T_0 = T_1 \); denote by \( S \in \text{GL}(\mathbb{C}^n) \) the complexification of the isomorphism \( T_1^{-1}T_0 \). Then, it would be natural to define the spectral flow \( \text{sf}(\gamma) \) as the spectral flow of the path of Fredholm bilinear forms \([0,1] \ni t \mapsto B_t \).
given in (3.8) on the space:

\[ H^1_0([0, 1], \mathbb{C}^n) = \{ \tilde{V} \in H^1([0, 1], \mathbb{C}^n) : \tilde{V}(1) = S\tilde{V}(0) \}. \]

(compare with Definition 3.1). However, with this definition, using Proposition 4.5 one checks easily that formula (3.9) will not hold in general. More precisely, the right hand side of (3.9) will contain an extra term, given by the index of the restriction of the metric tensor \( g \) on the image of the operator \( S - \text{Id} \). This is proved easily with the help of Proposition 4.5 as in the proof of Proposition 4.6. Namely, in this case, the \( B_0 \)-orthogonal space to \( H^1_0([0, 1], \mathbb{C}^n) \) in \( H^1_S([0, 1], \mathbb{C}^n) \) is given by the \( n \)-dimensional space of all affine maps \( \tilde{V}(t) = (S - \text{Id})Bt + B \), with \( B \in \mathbb{C}^n \). The restriction of \( B_0 \) to such space has index equal to the index of the restriction of \( g \) to the image of \( S - \text{Id} \). Note that \( S = \text{Id} \) in the orientation preserving case. Thus, one way to make the definition independent on the orthogonal frame would be to restrict to frames \( T \) for which the operator \( S = T_{-1}^{-1}T_0 \) is such that \( g \) is positive semi-definite on the image of \( S - \text{Id} \). This is always possible when \( g \) is not negative definite, by a simple linear algebra argument. Or, more simply, one could define \( \sf(B) \) as the difference between the spectral flow of the path \( t \mapsto \tilde{B}_t \) on \( H^1_0([0, 1], \mathbb{C}^n) \) minus the index of the restriction of \( g \) to \( \text{Im}(S - \text{Id}) \), which by what has been observed, is a quantity independent of the frame. With such definition, formula (3.9) holds also in the non orientation preserving case, and the entire paper carries over to the non orientable case.

### 4.6. On the jumps of the spectral flow function

By the results of Proposition 4.4 and Corollary 4.8 we know that \( \lambda_z \) is a piecewise constant function on \( S^1 \), whose jumps occur at the eigenvalues of the linearized Poincaré map and at 1. Thus, \( \lambda_z \) is uniquely determined once we know its value at some given point, say at \( z = 1 \), and the value of \( \lambda_z \) at each discontinuity point. When 1 is not an eigenvalue of \( \mathcal{P}_z \), then the jump of \( \lambda_z \) at 1 is \( n_-(g) \), as proved in Corollary 4.8 but \( \lim_{\theta \to +0} \lambda_z(e^{i\theta}) = \lim_{\theta \to -0} \lambda_z(e^{i\theta}) \). In order to give an upper estimate for the jumps at the eigenvalues of \( \mathcal{P}_z \), we need the following:

**Lemma 4.10.** Let \( I \ni t \mapsto B_t \) be a continuous curve of Hermitian forms on a Hilbert space \( H \), and let \( I \ni t \mapsto H_t \) be a continuous family of closed subspaces of \( H \), such that \( B_t|_{H_t \times H_t} \) is Fredholm for all \( t \). Assume that a point \( t_0 \) in the interior of \( I \) is an isolated degeneracy instant for \( B_t|_{H_t \times H_t} \). Then, for \( \varepsilon > 0 \) small enough, \( \mathcal{S}(\{(H_t)_{t \in [t_0-\varepsilon, t_0+\varepsilon]\}}) \leq \dim(\ker(B_{t_0}|_{H_{t_0} \times H_{t_0}})) \).

**Proof.** Using a trivialization for the path \( I \ni t \mapsto H_t \), it suffices to prove the result for the spectral flow of a continuous path of Fredholm Hermitian forms \( t \mapsto B_t \) on a fixed Hilbert space \( H \). When \( H \) is finite dimensional, in which case the spectral flow is the variation of the index function, the result is elementary and well known. The infinite dimensional case can be reduced to the finite dimensional one by an argument of functional calculus. More precisely, for all \( t \), let \( T_t \) be the self-adjoint Fredholm operator such that \( B_t = \langle T_t, \cdot \rangle \); let \( \varepsilon > 0 \) be small enough so that the spectrum of \( T_{t_0} \) has empty intersection with \([-2\varepsilon, 2\varepsilon] \setminus \{0\} \). Then, for \( t \) sufficiently close to \( t_0 \), the spectrum of \( T_t \) has empty intersection with \([-\varepsilon, \varepsilon] \setminus \{0\} \). Thus, denoting by \( \chi_{[-\varepsilon, \varepsilon]} \) the characteristic function of \([-\varepsilon, \varepsilon] \), the map \( t \mapsto P_t = \chi_{[-\varepsilon, \varepsilon]}(T_t) \) is a map of finite rank projections which is continuous near \( t_0 \); the image \( H_t \) of \( P_t \) is \( T_t \)-invariant, so that \( \ker(B_t) = \ker(B_t|_{H_t \times H_t}) \). By definition (see [21]), the spectral flow through \( t_0 \) of the path \( T \) is the variation of index of the restriction of \( B_t \) to \( H_t \times H_t \), which reduces the general case to a finite dimensional one.

With this, we are now able to prove that the jump of \( \lambda_z \) at \( z \in S^1 \) can be estimated with the dimension of the \( z \)-eigenspace of \( \mathcal{P}_z \).

**Corollary 4.11.** Let \( e^{i\theta_0} \) be an eigenvalue of \( \mathcal{P}_z \). Then:

\[
\left| \lim_{\theta \to +0} \lambda_z(e^{i(\theta_0+\theta)}) - \lim_{\theta \to -0} \lambda_z(e^{i(\theta_0+\theta)}) \right| \leq \dim(\ker(\mathcal{P}_z - e^{i\theta_0} \cdot 1)).
\]
Proof. By the concatenation additivity of the spectral flow and as \( B_0|_{\mathcal{H}_z \times \mathcal{H}_z} \) is non-degenerate for \( z \neq 1 \), for \( \theta > 0 \) small enough the difference \( \lambda_{\gamma}(e^{i(\theta_0+\theta)}) - \lambda_{\gamma}(e^{i\theta_0}) \) is equal to the spectral flow of the constant Fredholm Hermitian form \( B_1 \) on the continuous curve of closed subspaces \([ -\theta, \theta ] \ni \rho \mapsto \mathcal{H}_{e^{i(\theta_0+\rho)}} \). By assumption, there is an isolated degeneracy instant of \( B_1 \) at \( \rho = 0 \). By Lemma 4.10 the jump of \( \lambda_{\gamma} \) at \( e^{i\theta_0} \) is less than or equal to the dimension of \( \text{Ker}(B_1|_{\mathbb{H}_{e^{i\theta_0}} \times \mathbb{H}_{e^{i\theta_0}}}) \); this is equal to \( \text{dim(}\text{Ker}(\mathfrak{P}_{\gamma} - e^{i\theta_0} \cdot 1)) \) by Lemma 4.2.

\[ \square \]

5. The Fourier Theorem

We will now fix an integer \( N \geq 1 \) and we set:

\[ \omega = e^{2\pi i/N}. \]

For all \( k \geq 0 \), given \( \tilde{V}_k \in \mathcal{H}_{\omega^k} \) we will assume that \( \tilde{V}_k \) is extended to a continuous map \( \tilde{V}_k : \mathbb{R} \to \mathbb{C}^n \) by setting:

\[ \tilde{V}_k(t + m) = \omega^km \tilde{V}_k(t), \quad m \in \mathbb{Z}, \quad t \in [0, 1[. \]

Lemma 5.1. The map \( \Phi_N : \mathcal{H}_1 \to \bigoplus_{k=0}^{N-1} \mathcal{H}_{\omega^k} \) defined by:

\[ \Phi_N(\tilde{V}) = (\tilde{V}_0, \ldots, \tilde{V}_{N-1}), \]

where:

\[ \tilde{V}_k(t) = \frac{1}{N} \sum_{j=0}^{N-1} \omega^{-jk} \tilde{V} \left( \frac{t + j}{N} \right), \quad t \in [0, 1[, \]

is a linear isomorphism, whose inverse \( \Psi_N : \bigoplus_{k=0}^{N-1} \mathcal{H}_{\omega^k} \to \mathcal{H}_1 \) is given by:

\[ \Psi_N(\tilde{V}_0, \ldots, \tilde{V}_{N-1}) = \tilde{V}, \]

where

\[ \tilde{V}(t) = \sum_{k=0}^{N-1} \tilde{V}_k(tN). \]

Proof. A matter of straightforward calculations, based on the identity:

\[ \sum_{j=0}^{N-1} \omega^{jk} = \begin{cases} N, & \text{if } k \equiv 0 \pmod{N} \\ 0, & \text{otherwise.} \end{cases} \]

First, one needs to prove that \( \Phi_N \) is well defined, i.e., that the map \( \tilde{V}_k \) in (5.1) belongs to \( \mathcal{H}_{\omega^k} \):

\[ \tilde{V}_k(1) = \frac{1}{N} \sum_{j=0}^{N-1} \omega^{-jk} \tilde{V} \left( \frac{1+j}{N} \right) = \frac{1}{N} \sum_{j=1}^{N} \omega^{-k(j-1)} \tilde{V} \left( \frac{j}{N} \right) = \omega^k \left[ \frac{1}{N} \sum_{j=1}^{N} \omega^{-kj} \tilde{V} \left( \frac{j}{N} \right) \right] = \omega^k \tilde{V}(1), \]

i.e., \( \tilde{V}_k \in \mathcal{H}_{\omega^k} \). Similarly, \( \Psi_N \) is well defined. Clearly, \( \Phi_N \) and \( \Psi_N \) are linear and bounded.
In order to check that $\Psi_N$ is a right inverse for $\Phi_N$, set $(\tilde{W}_0, \ldots, \tilde{W}_{N-1}) = \Phi_N \circ \Psi_N(\tilde{V}_0, \ldots, \tilde{V}_{N-1})$. Then, for all $k = 0, \ldots, N-1$ and all $t \in [0, 1]$:

$$
\tilde{W}_k(t) = \frac{1}{N} \sum_{j=0}^{N-1} \sum_{l=0}^{N-1} \omega^{-kj} \tilde{V}_l(t + j) = \frac{1}{N} \sum_{j=0}^{N-1} \sum_{l=0}^{N-1} \omega^{-kj} \omega^l \tilde{V}_l(t) \\
= \frac{1}{N} \sum_{k=0}^{N-1} \left( \sum_{j=0}^{N-1} \omega^{j(l-k)} \right) \tilde{V}_l(t) = \sum_{k=0}^{N-1} \delta_{l,k} \tilde{V}_l(t) = \tilde{V}_k(t),
$$

i.e., $\Phi_N \circ \Psi_N$ is the identity of $\bigoplus_{k=0}^{N-1} \mathcal{H}_k$.

Finally, to check that $\Psi_N$ is a left inverse for $\Phi_N$, set $\tilde{V} = \Psi_N \circ \Phi_N(\tilde{W})$ and compute:

$$
\tilde{V}(t) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \omega^{-kj} \tilde{W}(t + j) \\
= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \omega^{-kj} \tilde{W}(t) = \sum_{k=0}^{N-1} \delta_{k,0} \tilde{W}(t) = \tilde{W}(t),
$$

for all $t \in [0, 1]$. Thus, $\Psi_N \circ \Phi_N$ is the identity of $\mathcal{H}_1$, which concludes the proof. \hfill \Box

**Lemma 5.2.** Given $\tilde{V}, \tilde{W} \in \mathcal{H}_1$, set:

$$
\Phi_N(\tilde{V}) = (\tilde{V}_0, \ldots, \tilde{V}_{N-1}), \quad \text{and} \quad \Phi_N(\tilde{W}) = (\tilde{W}_0, \ldots, \tilde{W}_{N-1}),
$$

with $\tilde{V}_k, \tilde{W}_k \in \mathcal{H}_k$ for all $k = 0, \ldots, N - 1$. Then, the following identities hold:

$$
B_0(\tilde{V}, \tilde{W}) = N^2 \sum_{k=0}^{N-1} B_0(\tilde{V}_k, \tilde{W}_k), \quad \text{and} \quad B_N(\tilde{V}, \tilde{W}) = N^2 \sum_{k=0}^{N-1} B_1(\tilde{V}_k, \tilde{W}_k).
$$

**Proof.** Again, a matter of direct calculations, as follows:

$$
B_0(\tilde{V}, \tilde{W}) = N^2 \sum_{k,j} \int_0^1 G(\tilde{V}_k'(rN), \tilde{W}_j'(rN)) \, dr \\
= N \sum_{k,j} \int_0^N G(\tilde{V}_k'(s), \tilde{W}_j'(s)) \, ds = N \sum_{k,j,l} \int_0^{l+1} G(\tilde{V}_k'(s), \tilde{W}_j'(s)) \, ds \\
= N \sum_{k,j,l} \int_0^1 G(\tilde{V}_k'(s+l), \tilde{W}_j'(s+l)) \, ds = N \sum_{k,j,l} \omega(k-j) \int_0^1 G(\tilde{V}_k'(s), \tilde{W}_j'(s)) \, ds \\
= N^2 \sum_k \int_0^1 G(\tilde{V}_k'(s), \tilde{W}_k'(s)) \, ds = N^2 \sum_k B_0(\tilde{V}_k, \tilde{W}_k).
$$
Similarly,

\[ B_N(\tilde{V}, \tilde{W}) = N^2 \int_0^1 \sum_{k,j} \left[ G(\tilde{V}_k^r(rN), \tilde{W}_j^r(rN)) + G(\Gamma_N r \tilde{V}_k(rN), \tilde{W}_k^r(rN)) \right. \]

\[ + G(\tilde{V}_k^r(rN), \Gamma_N, \tilde{W}_j(rN)) + G(\Gamma_N, \tilde{V}_k(rN), \Gamma_N, \tilde{W}_j(rN)) \]

\[ + G(\tilde{R}_N r \tilde{V}_k(rN), \tilde{W}_j(rN)) \right) \, \mathrm{d}r \]

\[ = N^2 \sum_{k,j} \int_0^{l+1} \left[ G(\tilde{V}_k^r(s), \tilde{W}_j^r(s)) + G(\Gamma_r \tilde{V}_k(s), \tilde{W}_j(s)) \right. \]

\[ + G(\tilde{V}_k^r(s), \Gamma_s \tilde{W}_j(s)) + G(\Gamma_s \tilde{V}_k(s), \Gamma_s \tilde{W}_j(s)) \]

\[ + G(\tilde{R}_s \tilde{V}_k(s), \tilde{W}_j(s)) \right) \, \mathrm{d}s \]

\[ = N \sum_{k,j,l} \omega^{(k-j)} \int_0^1 \left[ G(\tilde{V}_k^r(s), \tilde{W}_j^r(s)) + G(\Gamma_r \tilde{V}_k(s), \tilde{W}_j(s)) \right. \]

\[ + G(\tilde{V}_k^r(s), \Gamma_s \tilde{W}_j(s)) + G(\Gamma_s \tilde{V}_k(s), \Gamma_s \tilde{W}_j(s)) \]

\[ + G(\tilde{R}_s \tilde{V}_k(s), \tilde{W}_j(s)) \right) \, \mathrm{d}s \]

\[ = N^2 \sum_k \int_0^1 \left[ G(\tilde{V}_k^r(s), \tilde{W}_k^r(s)) + G(\Gamma_r \tilde{V}_k(s), \tilde{W}_k(s)) \right. \]

\[ + G(\tilde{V}_k^r(s), \Gamma_s \tilde{W}_k(s)) + G(\Gamma_s \tilde{V}_k(s), \Gamma_s \tilde{W}_k(s)) \]

\[ + G(\tilde{R}_s \tilde{V}_k(s), \tilde{W}_k(s)) \right) \, \mathrm{d}s \]

\[ = N^2 \sum_k B_1(\tilde{V}_k, \tilde{W}_k). \]

**Theorem 5.3** (Fourier theorem).

\[ \text{sf}(\gamma^{(N)}) = \sum_{k=0}^{N-1} \lambda_{\gamma}(\omega^k). \]

**Proof.** This follows immediately from [4.1, Lemma 5.1, 5.2] and the following two observations:

1. the spectral flow of a path of compact perturbations of a fixed symmetry only depends on the endpoints of the path;
2. the spectral flow is additive by direct sums. \( \square \)

6. A SUPERLINEAR ESTIMATE FOR THE SPECTRAL FLOW OF AN ITERATE

We will now use the Fourier theorem and formula (4.9) in order to establish estimates on the growth of the spectral flow for the \( N \)-th iterate of a closed geodesic \( \gamma \). We will use the notations in Subsection 4.4, an immediate application of Proposition 4.6, Theorem 5.3, and Eq. (4.7) gives the following:
Proposition 6.1. Given a closed geodesic and an integer $N \geq 1$:
\begin{equation}
\sigma(\gamma^{(N)}) = -N(i_{\text{Maslov}}(\gamma)) - n_-(g) + \sum_{k=0}^{N-1} \dim(\mathcal{J}_\gamma^{(1)}(\omega^k)) - \sum_{k=0}^{N-1} n_-(b_{\omega^k}),
\end{equation}
where $\omega = e^{2\pi i/N}$.

Let us estimate the last two terms in formula (6.1).

Lemma 6.2. The quantity $\sum_{k=0}^{N-1} \dim(\mathcal{J}_\gamma^{(1)}(\omega^k))$ is uniformly bounded:
\begin{equation}
0 \leq \sum_{k=0}^{N-1} \dim(\mathcal{J}_\gamma^{(1)}(\omega^k)) \leq 2 \dim(M).
\end{equation}

Proof. If we identify the space $S = \{ \tilde{V} \in C^2([0,1],\mathbb{C}^n) : \tilde{V} \text{ is solution of (4.4)} \}$ with $\mathbb{C}^n \oplus \mathbb{C}^n$ via the map $\tilde{V} \mapsto (\tilde{V}(0), \tilde{V}'(0))$, then for all $z \in S^1$, $\mathcal{J}_\gamma^{(1)}(z)$ is identified with a subspace of Ker$(\Psi_{\gamma} - z \cdot I)$. The conclusion follows from the fact that $\sum_{\gamma \in \mathcal{C}} \dim(\text{Ker}(\Psi_{\gamma} - z \cdot I)) \leq 2 \dim(M)$. □

A rough estimate for the term containing the index of the Hermitian forms $b_z$ is given in the following:

Lemma 6.3. For all $z \in S^1$, $n_-(b_z) \leq 2 \dim(M) - n_0(\gamma)$; if $z$ is not in the spectrum of $\Psi_{\gamma}$, then $n_-(b_z) \leq \dim(M) - n_0(\gamma)$. It follows that
\begin{equation}
0 \leq \sum_{k=0}^{N-1} n_-(b_{\omega^k}) \leq N \cdot \left[ \dim(M) - n_0(\gamma) \right] + 4 \dim(M)^2 - 2 n_0(\gamma) \dim(M).
\end{equation}

Proof. Arguing as in the proof of Corollary 4.8, one proves easily that $\dim(\mathcal{J}_\gamma^{(2)}(z)) \leq 2 \dim(M)$, and that $\dim(\text{Ker}(b_z)) \geq n_0(\gamma)$. This last inequality follows from the fact that the space $\{ \tilde{W} \text{ solution of (4.4)} : \tilde{W}(0) = \tilde{W}(1) = 0 \}$ is always contained in the kernel of $b_z$. This proves that $n_-(b_z) \leq 2 \dim(M) - n_0(\gamma)$. When $z \in S^1$ is not in the spectrum of $\Psi_{\gamma}$, then it is shown in the proof of Corollary 4.8 that $\dim(\mathcal{J}_\gamma^{(2)}(z)) = \dim(M)$, which gives the improved inequality $n_-(b_z) \leq \dim(M) - n_0(\gamma)$. Inequality (6.3) follows now easily, observing that there are at most $2 \dim(M)$ eigenvalues of $\Psi_{\gamma}$ (on the unit circle). □

Corollary 6.4. Set $C_\gamma = 4 \dim(M)^2 - 2 n_0(\gamma) \dim(M) + n_-(g)$; then:
\begin{align}
(6.4) \quad \sigma(\gamma^{(N)}) & \geq \left[ -i_{\text{Maslov}}(\gamma) + n_0(\gamma) - \dim(M) \right] \cdot N - C_\gamma, \\
(6.5) \quad \sigma(\gamma^{(N)}) & \leq -i_{\text{Maslov}}(\gamma) \cdot N - n_-(g) + 2 \dim(M),
\end{align}
for all $N \geq 1$.

Proof. Follows easily from (6.1), (6.2) and (6.3). □

Inequality (6.4) becomes interesting when $i_{\text{Maslov}}(\gamma) < n_0(\gamma) - \dim(M)$, while (6.5) when $i_{\text{Maslov}}(\gamma) > 0$. Thus, the question is understanding the asymptotic behavior of $\sigma(\gamma^{(N)})$ when
\begin{equation}
n_0(\gamma) - \dim(M) \leq i_{\text{Maslov}}(\gamma) \leq 0;
\end{equation}

note that $n_0(\gamma) - \dim(M) \leq -1$, and that $C_\gamma$ is bounded uniformly on $\gamma$:
\[ 2 \dim(M)^2 + 2 \dim(M) + n_-(g) \leq C_\gamma \leq 4 \dim(M)^2 + n_-(g). \]

By (3.9), (6.6) is equivalent to:
\begin{align}
(6.7) \quad \sigma(\gamma) & \geq \dim(\mathcal{J}_{\gamma}^{\text{per}} \cap \mathcal{J}_{\gamma}^0) - i_{\text{conc}}(\gamma) - n_-(g) \\
& \geq \dim(\mathcal{J}_{\gamma}^{\text{per}} \cap \mathcal{J}_{\gamma}^0) - i_{\text{conc}}(\gamma) - n_-(g) + \dim(M) - n_0(\gamma).
\end{align}
Lemma 6.5. If for some \( k \geq 1 \), \( |\sf(\gamma(\mathcal{L}))| > 2 \dim(M) + n_-(g) \), then the sequence \( N \mapsto |\sf(\gamma(kN))| \) has superlinear growth.

Proof. The inequality \( |\sf(\gamma(\mathcal{L}))| > 2 \dim(M) + n_-(g) \) implies that (6.7) is not satisfied by the iterate \( \gamma(\mathcal{L}) \); this follows easily considering the trivial inequalities:

\[
\dim(\mathcal{J}_{\gamma(k)}^0) \leq \dim(\mathcal{J}_{\gamma(k)}^0) \leq \dim(M) - 1
\]

and

\[
i_{\text{conc}}(\gamma(\mathcal{L})) \leq \dim(\mathcal{J}_{\gamma(k)}^0) \leq \dim(\mathcal{J}_{\gamma(k)}^0) = 2 \dim(M).
\]

By Corollary 6.4: \( N \mapsto |\sf(\gamma(kN))| \) has superlinear growth.

The result of Lemma 6.5 is not yet satisfactory; we want to prove that if \( \sf(\gamma(\mathcal{L})) \) is not bounded, then the entire sequence \( N \mapsto |\sf(\gamma(\mathcal{L}))| \) has superlinear growth. Let us study more precisely the behavior of \( \sf(\gamma(\mathcal{L})) \) as \( N \to \infty \):

Proposition 6.6. The limit:

\[
L_\gamma = \lim_{N \to \infty} \frac{1}{N} \sf(\gamma(\mathcal{L}))
\]

exists, and it is finite.

Proof. Using (6.1), it suffices to show that the limit:

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} n_-(b_{e,2\pi k/N})
\]

exists and is finite. By Lemma 4.7, the map \( S^1 \ni z \mapsto n_-(b_z) \in \mathbb{N} \) is constant on every connected component of \( S^1 \) that does not contain elements in the spectrum of \( \mathcal{P}_\gamma \); thus, this function is Riemann integrable on \( S^1 \), and:

\[
0 \leq \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} n_-(b_{e,2\pi k/N}) = \frac{1}{2\pi} \int_{S^1} n_-(b_{e,\theta}) \, d\theta < +\infty.
\]

Clearly, the following inequality holds:

\[-i_{\text{Maslov}}(\gamma) + n_0(\gamma) - \dim(M) \leq L_\gamma \leq -i_{\text{Maslov}}(\gamma);\]

moreover, in the second inequality, the equality holds if and only if \( b_z \) is positive semi-definite at each point of \( S^1 \) that does not belong to the spectrum of \( \mathcal{P}_\gamma \). With this, we can finally prove the following:

Proposition 6.7. The sequence \( |\sf(\gamma(\mathcal{L}))| \) is either bounded or it has superlinear growth.

Proof. The thesis is equivalent to proving that \( L_\gamma = 0 \) if and only if the sequence \( \sf(\gamma(\mathcal{L})) \) is bounded. The “if” part is trivial. Now, assume by contradiction that \( L_\gamma = 0 \) and that the sequence \( \sf(\gamma(\mathcal{L})) \) is unbounded. By Lemma 6.5 there exists \( k \geq 1 \) such that the subsequence \( N \mapsto |\sf(\gamma(kN))| \) has superlinear growth, i.e., \( \lim_{N \to \infty} \frac{1}{N} |\sf(\gamma(kN))| = kL_\gamma \neq 0 \), which is a contradiction.

Our goal will now be to prove the occurrence of a uniform superlinear growth for the spectral flow of an iterate. To this aim, we need to study the sequence:

\[
B_N = \sum_{k=0}^{N-1} n_-(b_{e,2\pi k/N}),
\]

which is in a sense, the non-trivial part in formula (6.1). A substantial improvement to the result of Lemma 6.5 can be obtained as follows.
Proposition 6.8. The limit

\( K_\gamma = \lim_{N \to \infty} \frac{1}{N} B_N \)

exists, and it is a nonnegative real number. This number is zero if and only if \( B_N \) is bounded, which occurs if and only if \( n_-(b_z) \) vanishes almost everywhere on \( S^1 \). If \( B_N \) is not bounded, then its superlinear growth is uniform in the following sense: there exist a constant \( \alpha \in \mathbb{R} \), such that for all \( N, P \in \mathbb{N} \):

\[
K_\gamma P - \alpha \leq B_{N+P} - B_N \leq K_\gamma P + \alpha,
\]

Proof. The existence of the limit has already been established in the proof of Proposition 6.6. From the equality in (6.9) one obtains easily that the limit is zero if and only if \( n_-(b_z) \) vanishes almost everywhere on \( S^1 \), and as it has always a finite number of discontinuities (the eigenvalues of \( \mathbb{P}_\gamma \)), one deduces that this occurs precisely when \( B_N \) is bounded. As to the last statement, assume that \( e^{i\theta_1}, \ldots, e^{i\theta_k} \) are all the eigenvalues of \( P_\gamma \) in \( S^1 \setminus \{1\} \), with \( 0 < \theta_1 < \ldots < \theta_k < 2\pi \); set \( \theta_0 = 0 \) and \( \theta_{k+1} = 2\pi \). For \( j = 0, \ldots, k \), define \( d_j \) as

\[
d_j = \lim_{\theta \to 0^+} n_-(b_z e^{i(\theta_j + \theta_0)}) \geq 0;
\]

recalling (a) in the proof of Corollary 4.8, \( d_j \) is the constant value of the map \( n_-(b_z) \) in the arc \( \mathcal{A}_j = \{ e^{i\theta} : \theta \in [\theta_j, \theta_{j+1}] \} \). With these notations, we have:

\[
K_\gamma = \frac{1}{2\pi} \sum_{j=0}^k d_j (\theta_{j+1} - \theta_j);
\]

by Lemma 6.3:

\[
d_j \leq 2 \dim(M) - n_0(\gamma).
\]

By the first part of the proof, the assumption that \( B_N \) is unbounded is equivalent to the fact that at least one of the term in the sum (6.12) is positive (i.e., \( K_\gamma > 0 \)). Finally, define constants \( a_N \) and \( C_{N,j} \), for \( N \geq 1 \) and \( j \in \{0, 1, \ldots, k\} \) by:

\[
a_N = \text{cardinality of } \{ j : N\theta_j \equiv 0 \mod 2\pi \},
\]

\[
C_{N,j} = \left\lfloor \frac{N(\theta_{j+1} - \theta_j)}{2\pi} \right\rfloor,
\]

where \( \lfloor \cdot \rfloor \) denotes the integer part function: \( \lfloor x \rfloor = \max\{m \in \mathbb{Z} : m \leq x\} \). Clearly, \( 0 \leq a_N \leq k + 1 \); moreover, for all \( N \) and \( j \), the arc \( \mathcal{A}_j \) contains a number of \( N \)-th roots of unity which is at most \( C_{N,j} + 1 \) and at least \( C_{N,j} - 1 \). With this in mind, we proceed to
the final calculation giving the desired uniform superlinear growth, as follows:

\[ \mathcal{B}_{N+P} \mathcal{B}_N = \sum_{l=0}^{N+P-1} n_-(b_{e^{2\pi i l/(N+P)}}) - \sum_{l=0}^{N-1} n_-(b_{e^{2\pi i l/N}}) \]

\[ = \sum_{l=1}^{N+P-1} n_-(b_{e^{2\pi i l/(N+P)}}) - \sum_{l=1}^{N-1} n_-(b_{e^{2\pi i l/N}}) \]

\[ \geq \sum_{j=0}^{k} d_j(C_{N+P,j} - 1) - \sum_{j=0}^{k} d_j(C_{N,j} + 1) - a_N \max_{z \in S^1} [n_-(b_z)] \]

by Lemma 6.3

\[ \geq \sum_{j=0}^{k} d_j(C_{N+P,j} - C_{N,j} - 2) - (k + 1)[2 \dim(M) - n_0(\gamma)] \]

\[ \geq \sum_{j=0}^{k} d_j(C_{N+P,j} - C_{N,j}) - 2 \left( \sum_{j=0}^{k} d_j \right) - (k + 1)[2 \dim(M) - n_0(\gamma)] \]

\[ \geq \sum_{j=0}^{k} \frac{d_j(\theta_{j+1} - \theta_j)}{2\pi} - 4 \left( \sum_{j=0}^{k} d_j \right) - (k + 1)[2 \dim(M) - n_0(\gamma)] \]

by \( 6.12 \)

\[ \geq K \gamma P - 5(k + 1)[2 \dim(M) - n_0(\gamma)]. \]

This concludes the proof of the first inequality in (6.11). The second inequality in (6.11) is obtained similarly:

\[ \mathcal{B}_{N+P} \mathcal{B}_N \leq \sum_{j=0}^{k} d_j(C_{N+P,j} + 1) - \sum_{j=0}^{k} d_j(C_{N,j} - 1) + a_N \max_{z \in S^1} [n_-(b_z)] \]

by Lemma 6.3

\[ \leq \sum_{j=0}^{k} d_j(C_{N+P,j} - C_{N,j} + 2) + (k + 1)[2 \dim(M) - n_0(\gamma)] \]

\[ \leq \sum_{j=0}^{k} d_j(C_{N+P,j} - C_{N,j}) + 2 \left( \sum_{j=0}^{k} d_j \right) + (k + 1)[2 \dim(M) - n_0(\gamma)] \]

\[ \leq \sum_{j=0}^{k} \frac{d_j(\theta_{j+1} - \theta_j)}{2\pi} - 4 \left( \sum_{j=0}^{k} d_j \right) + (k + 1)[2 \dim(M) - n_0(\gamma)] \]

by \( 6.13 \)

\[ \leq K \gamma P + 5(k + 1)[2 \dim(M) - n_0(\gamma)]. \]

From (6.1), (6.8), and (6.10) one obtains immediately:

\[ L_\gamma = -K_\gamma - i_{\text{Maslov}}(\gamma); \]

moreover, we can finally prove the uniform superlinear growth of \( \mathfrak{s}(\gamma^{(N)}) \):

**Proposition 6.9.** With the notations of Corollary 6.4 and Proposition 6.8 the following inequalities hold:

\[ (6.13) \quad L_\gamma \cdot P - 2 \dim(M) - \alpha \leq \mathfrak{s}(\gamma^{(N+P)}) - \mathfrak{s}(\gamma^{(N)}) \leq L_\gamma \cdot P + 2 \dim(M) + \alpha. \]

**Proof.** Immediate from Propositions 6.1, 6.8 and Lemma 6.2 \( \square \)

**Corollary 6.10.** The sequence \( \mathfrak{s}(\gamma^{(N)}) \) is either bounded or it has uniform linear growth.

**Proof.** We have seen in the proof of Proposition 6.7 that the sequence \( \mathfrak{s}(\gamma^{(N)}) \) is bounded if and only if \( L_\gamma = 0 \), so that the thesis follows from Proposition 6.9 \( \square \)
Denote by $\Lambda M$ the free loop space of $M$, and by $f : \Lambda M \to \mathbb{R}$ the geodesic action functional of $(M, g)$, whose critical points are well known to be closed geodesics. There is an equivariant action of the orthogonal group $O(2)$ on $\Lambda M$, obtained from the natural action of $O(2)$ on the parameter space $S^1$. A critical $O(2)$-orbit of $f$ consists of all closed geodesics that are obtained by rotation and inversion of a given closed geodesic in $(M, g)$; it is immediate that all the closed geodesics in the same critical orbit have equal spectral flow. Using equivariant Morse theory applied to the geodesic action functional, Gromoll and Meyer have proved that, in the Riemannian case, the contribution to the homology of a critical orbit of $f$ in the free loop space $\Lambda M$ in a fixed dimension $k$ is given only by those closed orbits whose Morse index is an integer between $k - \dim(M)$ and $k$. A key point of their multiplicity result is that, assuming the existence of only a finite number of distinct closed prime geodesics, one has a uniformly bounded number of distinct orbits with a fixed Morse index ([15, Corollary 2]). Aiming at the development of an equivariant Morse theory for strongly indefinite functionals, we prove an extension of their result, replacing the Morse index with the spectral flow.

Proposition 6.11. Let $(M, g)$ be a semi-Riemannian manifold that has only a finite number of distinct prime closed geodesics. Then, for $k \in \mathbb{Z}$ with $|k|$ sufficiently large, the total number of critical orbits of the geodesic action functional $f$ in the free loop space $\Lambda M$ having spectral flow equal to $k$ is bounded uniformly in $k$.

Proof. Let $\gamma_1, \ldots, \gamma_r$ be the family of all distinct prime closed geodesics in $M$ and fix some integer $k$ with $|k| > 2 \dim(M) + n_-(g)$ (recall Lemma 6.5). We can remove from the family those geodesics whose spectral flow is not unbounded by iteration, and assume that all these geodesics have iterates with unbounded spectral flow; in particular, $L_{\gamma_i} \neq 0$ for all $i$. For $i = 1, \ldots, r$, let $N_i \geq 1$ be the first integer such that $sf(\gamma_i^{(N_i)}) = k$; if no such integer $N_i$ exists, we can remove also $\gamma_i$ from the family. From (6.13) we obtain easily that $|sf(\gamma_i^{(N_i+P)}) - k| > 1$ when

$$P > \frac{1 + \alpha + 2 \dim(M)}{|L_{\gamma_i}|}.$$  

Thus, there are at most:

$$r + \sum_{i=1}^r \left\lfloor \frac{1 + \alpha + 2 \dim(M)}{|L_{\gamma_i}|} \right\rfloor$$

critical orbits of $f$ with spectral flow equal to $k$. \hfill \square

Finally, let us recall that $\gamma$ is said to be hyperbolic if the linearized Poincaré map $\mathcal{P}_\gamma$ does not have eigenvalues on the unit circle. For hyperbolic geodesics, the iteration formula for the spectral flow has a simple expression:

Proposition 6.12. If $\gamma$ is hyperbolic, then:

$$sf(\gamma^{(N)}) = Nsf(\gamma) + (N - 1) n_-(g).$$

Proof. This follows easily from Corollary 4.8 and Theorem 5.3 \hfill \square

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\footnote{i.e., the Hilbert manifold of all curves $\gamma : S^1 \to M$ having Sobolev regularity $H^1$.}
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