Probability, Preclusion and Biological Evolution in Heisenberg-Picture Everett Quantum Mechanics

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June 28, 2021

Abstract

The fact that certain “extraordinary” probabilistic phenomena—in particular, macroscopic violations of the second law of thermodynamics—have never been observed to occur can be accounted for by taking hard preclusion as a basic physical law; i.e. precluding from existence events corresponding to very small but nonzero values of quantum-mechanical weight. This approach is not consistent with the usual ontology of the Everett interpretation, in which outcomes correspond to branches of the state vector, but can be successfully implemented using a Heisenberg-picture-based ontology in which outcomes are encoded in transformations of operators. Hard preclusion can provide an explanation for biological evolution, which can in turn explain our subjective experiences of, and reactions to, “ordinary” probabilistic phenomena, and the compatibility of those experiences and reactions with what we conventionally take to be objective probabilities arising from physical laws.

Key words: quantum mechanics, Everett interpretation, probability, preclusion, Heisenberg picture, biological evolution, Principal Principle

1 A categorization of probabilistic phenomena

The issue of how probability can either be incorporated into, or be shown to arise from, Everett quantum mechanics [1–9] has long been contentious (10–27 are but a sampling of the extensive and diverse literature on the subject). It has been argued that at least
some of the difficulty associated with this issue is related to difficulty in understanding the nature of probability *per se*. So it makes sense to focus directly on the physical phenomena that we associate with probability. These can usefully be placed into two categories:

**Extraordinary probabilistic phenomena** These are events to which we ascribe extremely low probability. Not “low probability” at the level of winning a lottery; rather, “low probability” at the level of macroscopic entropy decrease, e.g., observing an ice cube placed in water becoming colder while the water becomes warmer. The characteristics of extraordinary probabilistic phenomena relevant for the present discussion are twofold:

*Nonobservation* Although physical theory ascribes to them a nonzero probability of occurring, they have in fact never been observed to occur.

*Collective nature* The smallness of the probability of occurrence is related to the fact that the phenomena in question involve a large number (order of Avogadro’s number) of constituents (typically molecules).

**Ordinary probabilistic phenomena** All other phenomena of physics which are described using probability: passage of photons through a polarizing beam-splitter, absorption of a photon during photosynthesis, radioactive decay of a uranium nucleus...

To start, we focus on extraordinary probabilistic phenomena and, making use of the nonprobabilistic parts of the quantum formalism, attempt to account for their nonobservation. Preclusion, an approach applied to the Everett interpretation in a “soft” qualitative form by Geroch and presented in a “hard” quantitative form by Buniy, Hsu and Zee (BHZ) suggests itself. BHZ propose that components of the Schrödinger-picture

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1We are of course not claiming that there is any difficulty within the usual theory of probability in understanding why events with extremely low but nonzero probability have not been observed; after all, we have observed until now only a finite number of events. The issue is that, as detailed in the above-cited references, there are difficulties in incorporating the usual theory of probability into Everett quantum mechanics. The approach we are taking is to start with the formalism of Everett quantum mechanics, which is deterministic, modify it in a way that preserves this determinism, and show that the resulting theory successfully accounts for the various kinds of phenomena which in, non-Everettian versions of quantum mechanics, are described probabilistically.

2Having been unaware of this important work at the time of writing of my previous paper on preclusion in Everett quantum mechanics, I regrettably failed to cite it there.

3Preclusion plays a role in the non-Everett approaches due to Sorkin and Galvan. Hanson employs the related idea of “mangling” in the context of the world-counting approach to probability in Everett quantum mechanics. Cournot’s principle, essentially soft preclusion, is used in the theory of classical probability as axiomatized by Kolmogorov to relate the mathematical formalism to physical reality. “An event with very small probability is morally impossible: it will not happen.”
wavefunction with norm less than a small positive number be removed from the wavefunction. In other words, events with quantum-mechanical amplitudes smaller than a certain level simply do not occur.

This would seem to explain, by fiat, the nonobservation of extraordinary probabilistic phenomena. The probability of an event in the standard Copenhagen interpretation of quantum mechanics is given by the Born rule; that is, it is the square norm of the component of the wavefunction corresponding to that event. Removing from the wavefunction events with sufficiently small norm therefore implies the nonoccurrence of those events.

However, the fact that the events we are most interested in precluding are collective events presents a difficulty if we employ the usual ontology of the Everett interpretation in the Schrödinger picture.

In Sec. 2 below we describe this difficulty in detail. Sec. 3 reviews the Heisenberg-picture description of ideal measurement in Everett quantum mechanics and the ontology that it motivates. In Sec. 4 we show how the preclusion of extraordinary probabilistic phenomena can be successfully implemented in the Heisenberg-picture Everett interpretation. Sec. 5 details a consistency condition that preclusion should obey to provide sensible descriptions of sequences of measurements in which the outcomes of later measurements are contingent on the outcomes of earlier measurements. In Sec. 6 we show how preclusion of extraordinary probabilistic phenomena provides an explanation for biological evolution and, in particular, for the evolution of subjective judgements regarding ordinary probabilistic phenomena, present a preliminary model of biological evolution relying on preclusion rather than objective probability, and review recent experiments pointing to the evolutionary origins of subjective probability. We conclude with a summary of our results (Sec. 7) and a discussion (Sec. 8).

2 Incompatibility of preclusion and Schrödinger-picture Everett ontology

We take as a model collective phenomenon the Bernoulli process and the associated law of large numbers. The classical Bernoulli process is a sequence of $N$ binary events with probability $p$ for one of the two possible outcomes, say outcome 1, (and for outcome 2, of course, probability $q = 1 - p$). The law of large numbers, a theorem which follows from the axioms of probability theory, states that by taking $N$ sufficiently large, the probability that the relative frequency $f_1$ for outcome 1 (i.e., fraction of events with outcome 1) differs by any fixed amount from $p$ can be made as small as desired. Equivalently, the probability that $f_1$ is within a fixed amount of $p$ can be made as close to unity as desired. The probability distribution for $f_1$ thus becomes peaked around the value $p$. What is key in deriving this result is that the probability distribution is for relative frequency regardless of order; i.e.
it is the sum of the probabilities for all sequences with a given relative frequency.

The quantum analog of the Bernoulli process involves a product state $|\Psi\rangle$ of $N$ qubits or spins (we will use the two terms interchangeably), i.e., quantum systems each of which has a state space spanned by two basis vectors $|S^{(i)}; \alpha_1\rangle$, $|S^{(i)}; \alpha_2\rangle$, with the amplitudes associated with the basis vectors the same for each qubit. That is,

$$|\Psi\rangle = \prod_{i=1}^{N}(c_1|S^{(i)}; \alpha_1\rangle + c_2|S^{(i)}; \alpha_2\rangle)$$

(1)

where $\alpha_1$, $\alpha_2$ are nondegenerate eigenvalues labeling the basis vectors and

$$|c_1|^2 + |c_2|^2 = 1.$$  

(2)

This can be expanded as

$$|\Psi\rangle = \sum_{i_1=1}^{2} \ldots \sum_{i_N=1}^{2} |B_{i_1,\ldots,i_N}\rangle$$

(3)

where

$$|B_{i_1,\ldots,i_N}\rangle = c_1^{r_1(i^{(1)},\ldots,i^{(N)})} c_2^{r_2(i^{(1)},\ldots,i^{(N)})} \prod_{p=1}^{N} |S^{(p)}; \alpha_{i^{(p)}}\rangle,$$

(4)

is one of the $2^N$ branches of $|\Psi\rangle$, and where

$$r_i(i^{(1)},\ldots,i^{(N)}) = \sum_{p=1}^{N} \delta_{i_{i^{(p)}}},$$

(5)

satisfying

$$r_1(i^{(1)},\ldots,i^{(N)}) + r_2(i^{(1)},\ldots,i^{(N)}) = N,$$

(6)

is the number of factors of $c_i$ in each term in (4). Each branch of the state vector is regarded as an Everett world in which the corresponding sequence of measurement outcomes $\alpha_{i^{(1)}}, \ldots, \alpha_{i^{(N)}}$ has occurred. The number of $\alpha_1$ outcomes in the world with outcome sequence $\alpha_{i^{(1)}}, \ldots, \alpha_{i^{(N)}}$ is $r_1(i^{(1)},\ldots,i^{(N)})$ and the relative frequency of $\alpha_1$ outcomes in this world is

$$f_1(i^{(1)},\ldots,i^{(N)}) = r_1(i^{(1)},\ldots,i^{(N)})/N.$$  

(7)

For sufficiently large $N$ we would like to preclude all branches except those for which the relative frequency $f_1(i^{(1)},\ldots,i^{(N)})$ is close to the Born-rule value

$$f_{B,1} = |c_1|^2.$$  

(8)
The weight (square norm) of a branch with outcome sequence $\alpha_{i(1)}, \ldots, \alpha_{i(N)}$ is given by

$$w(i^{(1)}, \ldots, i^{(N)}) = |c_1|^{2r_1(i^{(1)}, \ldots, i^{(N)})}|c_2|^{2r_2(i^{(1)}, \ldots, i^{(N)})}.$$  (9)

The weight of a branch with the Born-rule relative-frequency of “spin up” ($\alpha_1$) is

$$w_B = |c_1|^{2N}|c_1|^2 \left(1 - |c_1|^2\right)^N(1 - |c_1|^2)$$  (10)

So, to establish the Born rule by preclusion, we have to find a real number, $\epsilon_P$, smaller than $w_B$ and larger than the weight of any other branch. But this is in general impossible. Suppose $|c_1| > |c_2|$. Then the weight (9) of a branch will be a monotonically-increasing function of $r_1(i^{(1)}, \ldots, i^{(N)})$, the number of $\alpha_1$ outcomes in that branch. In particular, the branch with $(i^{(1)}, \ldots, i^{(N)}) = (1, 1, 1, \ldots, 1)$ will have larger weight than any other, contradicting the requirement that the Born-rule weight is larger. We obtain the same type of contradiction in the case $|c_2| > |c_1|$. For $|c_1| = |c_2| = 1/\sqrt{2}$ all branches have the same weight. It is only if $|c_1| = 1$ or $|c_2| = 1$ that a preclusion approach can work.

If branches with common values of relative frequency are grouped together prior to being subject to preclusion, then indeed the law of large numbers with the Born-rule value for the peak relative frequency can be recovered using the same mathematics as in the classical-probability case. Groups of branches with relative frequencies near the Born-rule value (8) survive preclusion essentially because there are so many branches in the group, even though there are branches with non-Born values of relative frequency which individually have larger weights. For details see BHZ.

But what is the justification for grouping together branches in this manner before applying preclusion? What dictates that we should group together branches with the same relative frequency $f$ for outcome 1, even though, e.g., one branch may have the first $fN$ components with outcome 1 and another may have the last $fN$ components with outcome 1? Might we not just as well consider that, say, a branch with the first $1.01fN$ components with outcome 1 should be grouped together with a branch with the first $fN$ components with outcome 1? Keep in mind that the components correspond to measurements of distinguishable and possibly macroscopic systems.

This, then, is the issue: The usual ontology of the Everett interpretation regards individual Everett branches as worlds. But it is the sums of weights for branches with nearby relative frequencies which obey Bernoulli’s law of large numbers. If we want to make use of preclusion to give physical meaning to the branch weights in a way that will yield the law of large numbers with a peak at the Born-rule value, we require an ontology which focuses on entities which effectively correspond to the required groups of branches.

\[^4\]Of course in general there will be no branch with precisely the relative frequency (8), because $N|c_1|^2$ may not be an integer. The goal then would be preclude all branches except the one with the weight closest to $w_B$.  

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The same issue of grouping also arises in attempts to demonstrate the existence of probability and the Born rule in Everett quantum mechanics in the context of infinite ensembles \[10, 12\]. In that context I have demonstrated that using an ontology motivated by the Heisenberg-picture formulation of the Everett interpretation and taking into account the fundamental physical limitations on relative-frequency-measuring devices yields the Born rule in a straightforward manner \[54\]. This approach also works in conjunction with finite ensembles and preclusion, as will now be shown.

\[5\]

3 Ideal measurement in Everett quantum mechanics in the Heisenberg picture

We cast our description of the quantum Bernoulli process in terms of observers performing ideal measurements on the ensemble of two-state systems. In this section we will describe ideal measurement of a single system, and in the next section progress to measurements of ensembles of systems and relative frequency.

The Schrödinger-picture description of ideal measurement \[55\] of a single system \(S\) involves an observer \(O\) equipped with a measuring apparatus who measures \(S\) in a superposition of eigenstates of the observable which the apparatus measures, and in doing so evolves from a state of ignorance to a state of entanglement with \(S\):

\[
|O; \beta_0\rangle \left(c_1|S; \alpha_1\rangle + c_2|S; \alpha_2\rangle\right) \rightarrow c_1|O; \beta_1\rangle|S; \alpha_1\rangle + c_2|O; \beta_2\rangle|S; \alpha_2\rangle.
\] (11)

In the Heisenberg picture the state vector does not change, so we must look at changes in the operators to be able to characterize ideal measurement. The time evolution in (11) can be implemented by

\[
\hat{U} = \exp(-i\hat{H}(t - t_{in})),
\] (12)

where \(t\) is the time at the end of the measurement, \(t_{in}\) is the time at the beginning (which we also take to be the initial time at which the Heisenberg and Schrödinger pictures coincide), and where

\[
\hat{H} = \sum_{i=1}^{2} \hat{h}_i^O \otimes \hat{P}_i^S,
\] (13)

with

\[
\hat{h}_i^O = i\kappa(|O; \beta_i\rangle\langle O; \beta_0| - |O; \beta_0\rangle\langle O; \beta_i|), \quad i = 1, 2,
\] (14)

and

\[
\hat{P}_i^S = |S; \alpha_i\rangle\langle S; \alpha_i|.
\] (15)

The methodology used in the calculations which appear in Sec. 4 below differs somewhat from that employed in the corresponding section, 3.5, of \[54\], and allows us to extend some of the results of the latter; see footnote \[5\] p. 18.
Let \( \hat{a} \) and \( \hat{b} \) be the Schrödinger-picture operators of which \( |S; \alpha_i\rangle \), \( |O; \beta_i\rangle \) in (11) are the respective eigenvectors,

\[
\hat{a} = \sum_{i=1}^{2} \alpha_i \hat{P}^S_i, \\
\hat{b} = \sum_{i=0}^{2} \beta_i \hat{P}^O_i,
\]

(16) (17)

with

\[
\hat{P}^O_i = |O; \beta_i\rangle \langle O; \beta_i|.
\]

(18)

We will take the \( \beta_i \)'s to be nondegenerate, and take \( t \) to be equal to the time at which a measurement begun at time \( t_{in} \) is complete, i.e.,

\[
t = t_{in} + \frac{\pi}{2 \kappa}.
\]

(19)

Then the operator corresponding to the spin state of the system \( S \) is unchanged by the measurement interaction:

\[
\hat{a}(t) = \hat{a}
\]

(20)

(we distinguish Heisenberg-picture operators by explicit time arguments), while the operator \( \hat{b} \) corresponding to the state of awareness of the observer \( O \) is changed by the measurement interaction into the form

\[
\hat{b}(t) = \sum_{i=1}^{2} \hat{b}_i \otimes \hat{P}^S_i,
\]

(21)

where

\[
\hat{b}_i = \exp(i\hat{h}_i^O (t - t_{in})) \hat{b} \exp(-i\hat{h}_i^O (t - t_{in})).
\]

(22)

For the time-independent Heisenberg-picture state vector we take the before-measurement state vector on the left-hand side of (11),

\[
|\psi_{in}\rangle = |O; \beta_0\rangle (c_1|S; \alpha_1\rangle + c_2|S; \alpha_2\rangle).
\]

(23)

We find that in the Heisenberg picture this ideal measurement has the following characteristics [54, 56–58]:

**Before measurement (time \( t_{in} \)).**

The time-independent Heisenberg-picture state vector is an eigenstate of the time-dependent Heisenberg-picture operator corresponding to the observer’s state of awareness,

\[
\hat{b}(t_{in}) = \hat{b},
\]

(24)

with the eigenvalue corresponding to a state of ignorance:

\[
\hat{b}|\psi_{in}\rangle = \beta_0|\psi_{in}\rangle.
\]

(25)
After measurement (time $t$).

The operator corresponding to the observer’s state of awareness has changed from the form (24) to the form (21), a sum of operators $\hat{b}_i$ which act in the state space of the observer $O$ multiplied by projection operators into states of the observed system $S$. The time-independent Heisenberg-picture state vector is an eigenstate of the $\hat{b}_i$’s with eigenvalues corresponding to states in which the observer has observed the $S$ in the state labeled by eigenvalue $\alpha_i$:

$$\hat{b}_i |\psi_{in}\rangle = \beta_i |\psi_{in}\rangle, \quad i = 1, 2$$  \hspace{1cm} (26)

Up to this point we have only transcribed the description of ideal measurement from the Schrödinger picture to the Heisenberg picture. However, since in the Heisenberg picture it is the operators that evolve, it is natural to think in terms of an ontology reflecting this difference \[54\]. Rather than states (Everett branches) which encode properties of an entire globally-defined world at a given time, and in which properties such as spin or states of awareness of observers have definite values when the state is an eigenstate of a suitable operator, we regard the things which exist in the multiverse \[60\] as corresponding not to states but to operators, which have definite values when they satisfy “eigenoperator” conditions such as (25), (26). During the ideal-measurement process, the state of awareness of $O$ undergoes a transition from a single entity in a state of ignorance ($\beta_0$) regarding the properties of $S$ to two “Everett copies” in states of awareness $\beta_1$, $\beta_2$, i.e., respectively having seen $S$ in state $\alpha_1$ and having seen seen $S$ in state $\alpha_2$. As in the Schrödinger picture ontology, the possible values of physical properties are those of the eigenvalues of the corresponding operator; it is necessary to consider only nondegenerate eigenvalues so that there is no ambiguity regarding the nature and number of the splitting into Everett copies\[6\].

To introduce preclusion, we define the weight associated with $\hat{b}_i$ to be the matrix element of the projector\[7\] $\hat{P}^S_i$ associated with $\hat{b}_i$ between the initial Heisenberg-picture state and its adjoint:

$$w_{\hat{b},i} = \langle \psi_{in} | \hat{P}^S_i | \psi_{in} \rangle.$$  \hspace{1cm} (27)

Then an observer $O$ with state of awareness $\beta_i$ exists at time $t$ if $w_{\hat{b},i} > \epsilon_P$.

The weight (27) is of course just the usual value of the probability, in the state $|\psi_{in}\rangle$, for the outcome corresponding to the projection operator $\hat{P}^S_i$. We note that even the usual quantum formalism employs hard preclusion with $\epsilon_P = 0$, since, in any interpretation, components of the state vector with zero norm correspond to outcomes which with certainty

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\[6\] For a more extensive discussion of the ontology of Heisenberg-picture Everett quantum mechanics, see \[54\] Sec. 5.

\[7\] Termed a “label” in \[56\], since such factors are responsible for ensuring the correct matching-up of entangled systems and observers in Einstein-Podolsky-Rosen-type scenarios.
do not occur. In the unphysical limiting case of an infinitely-large ($N = \infty$) ensemble of measured systems one can show that all outcomes of the measurement of relative frequency inconsistent with the Born rule are precluded even if $\epsilon_P = 0$ [54]. Preclusion with $\epsilon_P > 0$ enables us to implement this approach for the physically far more interesting case of a finite ensemble.

### 4 A model of relative-frequency measurement in Heisenberg-picture Everett quantum mechanics with preclusion

Our model for relative-frequency measurement involves three types of quantum systems. Measured systems $S(p)$, $p = 1, \ldots, N$, constitute the ensemble of identical systems required to investigate the relative frequency of a particular outcome. The state space of each $S(p)$ is spanned by the two eigenstates of the (Schrödinger-picture) operator $\hat{a}(p)$,

$$\hat{a}(p) |S(p); \alpha_i(p)\rangle = \alpha_i(p) |S(p); \alpha_i(p)\rangle, \quad i(p) = 1, 2, \quad p = 1, \ldots, N,$$

$$\alpha_1 \neq \alpha_2.$$  

For each system $S(p)$ there is an observer/measuring apparatus $O(p)$, the state space of which is spanned by the three eigenvectors of $\hat{b}(p)$,

$$\hat{b}(p) |O(p); \beta_i(p)\rangle = \beta_i(p) |O(p); \beta_i(p)\rangle, \quad i(p) = 0, 1, 2, \quad p = 1, \ldots, N,$$

$$\beta_i \neq \beta_j, \quad i \neq j.$$  

The eigenvalues $\beta_0$, $\beta_1$ and $\beta_2$ correspond, respectively, to $O(p)$ being in a state of ignorance, having observed $\alpha_1$, and having observed $\alpha_2$. Each measurement is implemented by an interaction of the form described in Sec. 3.

The third type of system is the relative-frequency observer $F$, which queries the observers $O(p)$ and computes the relative frequency for the observation $\beta_1$. The possible values of relative frequency, for any fixed value of $N$, are $0, 1/N, 2/N, \ldots, (N - 1)/N, 1$. As $N$ increases without limit the resolution needed to distinguish between adjacent values of relative frequency becomes finer and the total number of possible values of relative frequency becomes larger. But any given device will only have a limited resolution [11] as well.

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8 Much of the analysis of the finite-ensemble case is similar or identical to the analysis of the infinite-ensemble case presented in [54], and the reader is referred to that reference for intermediate steps that have been left out of Sec. 4.

9 By including the $O(p)$’s in the model, instead of having $F$ interact with the $S(p)$’s directly, we refute claims in [61, 62] that measurements of individual systems are incompatible with obtaining the Born rule. See discussion in [54, Sec. 5].
as a limit on its information-storage capacity set by its construction but ultimately capped by gravitational considerations \[59\]. Since \( \mathcal{F} \) cannot store an unlimited number of digits, for sufficiently large \( N \) it will be unable to record precisely the value of any possible result for the relative-frequency measurement. Nor, of course, can it retain the values obtained from each of the \( \mathcal{O}^{(p)} \)'s as \( N \) increases without limit.

With this in mind, we construct a model of a relative-frequency observer/measuring device \( \mathcal{F} \) by taking the state space of \( \mathcal{F} \) to be spanned by a discrete finite set of eigenvectors of an operator \( \hat{f} \):

\[
\hat{f} |\mathcal{F}; \phi_i \rangle = \phi_i |\mathcal{F}; \phi_i \rangle, \quad i = 0, \ldots, \nu + 1,
\]

where

\[
\phi_i = (i - 1)/\nu, \quad i = 1, \ldots, \nu + 1
\]

are the possible outputs of the relative-frequency device after the completion of the relative-frequency measurement, and

\[
\phi_0 = -1/\nu
\]

corresponds to a state of ignorance.

Define the relative frequency function for the result \( \beta^{(p)}_i = \beta_1 \),

\[
f(\beta^{(1)}_1, \ldots, \beta^{(N)}_1) = (1/N) \sum_{p=1}^{N} \delta_{\beta^{(p)}_i, 1}, \quad \delta(p) \neq 0 \quad \forall \quad p.
\]

The possible values of this function are

\[
f_l = l/N, \quad l = 0, \ldots, N.
\]

Define also the finite-resolution relative frequency function \( \phi(\beta^{(1)}_1, \ldots, \beta^{(N)}_1) \) to be that \( \phi_i, i = 1, \ldots, \nu + 1 \), which is closest in value to \( f(\beta^{(1)}_1, \ldots, \beta^{(N)}_1) \):

\[
\phi(\beta^{(1)}_1, \ldots, \beta^{(N)}_1) = \arg \min_{\phi_i} |\phi_i - f(\beta^{(1)}_1, \ldots, \beta^{(N)}_1)|, \quad \delta(p) \neq 0 \quad \forall \quad p
\]

(smaller \( \phi_i \) in case of a tie)

where the \( \phi_i \)'s are as given in (33). It will also prove convenient to define

\[
\tilde{\phi}(\beta^{(1)}_1, \ldots, \beta^{(N)}_1) = \phi(\beta^{(1)}_1, \ldots, \beta^{(N)}_1), \quad \delta(p) \neq 0 \quad \forall \quad p
\]

\[
= \phi_0 \quad \text{otherwise}
\]

where \( \phi_0 \) is as given in (34).

The unitary operator corresponding to the measurement of all of the \( \mathcal{O}^{(p)} \)'s by \( \mathcal{F} \) can then be written as

\[
\hat{U}_\mathcal{F} = \sum_{k=0}^{\nu+1} \hat{u}_k^\mathcal{F} \otimes \hat{P}_k^\mathcal{O}.
\]
where
\[ \hat{u}_k^F |\mathcal{F}; \phi_0 \rangle = |\mathcal{F}; \phi_k \rangle, \quad k = 0, \ldots, \nu + 1, \] \hfill (41)
Note that \( \hat{u}_0^F \) acts as the identity on the ignorant state \( |\mathcal{F}; \phi_0 \rangle \). \( \hat{P}_k^\Omega \) is the projection operator which projects out those states corresponding to \( \phi_k \):
\[ \hat{P}_k^\Omega = \sum_{i(1)=0}^{2} \cdots \sum_{i(N)=0}^{2} \delta_{\nu \delta(\beta_{i(1)}, \ldots, \beta_{i(N)}), k-1} \otimes_{p=1}^{N} \hat{P}_{i(p)}^{O(p)} \] \hfill (42)
where
\[ \hat{P}_{i(p)}^{O(p)} = |O(p), \beta_{i(p)} \rangle \langle O(p), \beta_{i(p)}|. \] \hfill (43)
At time \( t_1 \) subsequent to the conclusion of all the measurements—i.e., after each \( O(p) \) measures its corresponding \( S(p) \) and all \( N \) of the \( O(p) \)’s are subsequently measured by \( \mathcal{F} \)—the relative-frequency operator \( \hat{f} \) has the form
\[ \hat{f}(t_1) = \sum_{k=0}^{\nu+1} \hat{f}_k \otimes \hat{L}_k, \] \hfill (44)
where
\[ \hat{f}_k = \hat{u}_k^F \dagger \hat{u}_k^F, \quad k = 0, \ldots, \nu + 1, \] \hfill (45)
and
\[ \hat{L}_k = \sum_{i(1)=1}^{2} \cdots \sum_{i(N)=1}^{2} \left( \otimes_{p=1}^{N} \hat{u}_{i(p)}^{(p)} \right) \hat{P}_k^\Omega \left( \otimes_{q=1}^{N} \hat{u}_{i(q)}^{(q)} \right) \otimes_{r=1}^{N} \hat{P}_{i(r)}^{S(r)}, \quad k = 0, \ldots, \nu + 1. \] \hfill (46)
\( \hat{f}(t_1) \) in \( (44) \) is of the post-ideal-measurement form \( (21) \), so the weight associated with the \( k \)th possible outcome is
\[ W_{f,k}(t_1) = \langle \psi(t_0) | \hat{L}_k | \psi(t_0) \rangle. \] \hfill (47)
where \( |\psi(t_0)\rangle \) is the constant Heisenberg-picture state.
We take \( |\psi(t_0)\rangle \) to be the product state in which \( \mathcal{F} \) and the \( O(p) \)’s are ignorant and each of the \( S(p) \)’s is in a superposition of \( |S(p); \alpha_1 \rangle \) and \( |S(p); \alpha_2 \rangle \) with the same coefficients:
\[ |\psi(t_0)\rangle = |\mathcal{F}; \phi_0 \rangle \prod_{p=1}^{N} |O(p); \beta_0 \rangle \prod_{q=1}^{N} |S(q); \psi_0\rangle, \] \hfill (48)
\[ |S(q); \psi_0\rangle = c_1^{(q)} |S(p); \alpha_1 \rangle + c_2^{(q)} |S(p); \alpha_2 \rangle, \quad p = 1, \ldots, N, \] \hfill (49)
with
\[ c_1^{(q)} = c_1, \quad c_2^{(q)} = c_2 \quad \forall \ q, \] \hfill (50)
\[ |c_1|^2 + |c_2|^2 = 1. \] \hfill (51)
Using (46) and (48) in (47), we find (see [54], Sec. 3)
\[ W_{f,0}(t_1) = 0 \quad \forall N. \] (52)
That is, at the completion of the relative-frequency measurement there is no Everett copy of \( F \) which is in a state of ignorance.

The weights corresponding to the other Everett copies of \( F \) are
\[ W_{f,k}(t_1) = \sum_{l \mid 0 \leq l \leq N, N\phi_k - \frac{1}{2\nu} < l \leq N\phi_k + \frac{1}{2\nu}} p_{N,l} \] (53)
where
\[ p_{N,l} = \frac{N!}{l!(N-l)!}p^lq^{N-l}, \] (54)
and where we have set
\[ p = |c_1|^2, \quad q = |c_2|^2, \] (55)
so
\[ q = 1 - p. \] (56)

The expression (53) for \( W_{f,k}(t_1) \) can be evaluated by a careful application of the DeMoivre-Laplace theorem. Let \( \phi_k' \) be that finite-resolution relative frequency which is closest to \( p \), i.e.
\[ k' = \arg \min_k |\phi_k - p|, \quad k = 1, \ldots, \nu + 1 \] (57)
For concreteness suppose that
\[ \phi_k' > p, \] (58)
so
\[ \phi_k' = p + \Delta. \] (59)
(See Fig. 1) Since the \( \phi_k' \)'s are uniformly spaced a distance \( \frac{1}{\nu} \) apart,
\[ 0 \leq \Delta < \frac{1}{2\nu}. \] (60)
We will treat separately the case \( \Delta = \frac{1}{2\nu} \), i.e., two finite-resolution relative frequencies equally close to \( p \).
Using (59) we can write (53), for the case \( k = k' \), as

\[
W_{f,k'}(t_1) = \sum_{l \mid 0 \leq l \leq N, -\sqrt{N} \left( \frac{l-\Delta}{\sqrt{pq}} \right) \leq \sqrt{N} \left( \frac{l+\Delta}{\sqrt{pq}} \right)} p_{N,l} \tag{61}
\]

Define

\[
D'(A) = \sum_{l \mid 0 \leq l \leq N, -A \leq \frac{l-Np}{\sqrt{Npq}} \leq A} p_{N,l} \tag{62}
\]

where

\[
A > 0 \tag{63}
\]

is an arbitrary positive number. Provided that \( N \) is sufficiently large compared to \( A \), the sum in (61) will contain all of the summands in (62) as well as possibly others; and since all summands are positive, the sum in (61) will then be at least as large as that in (62). Specifically,

\[
W_{f,k'}(t_1)_{[N]} \geq D'(A)_{[N]}, \quad \sqrt{N} \left( \frac{1}{2\nu} \pm \Delta \sqrt{pq} \right) > A. \tag{64}
\]

(We have added the subscript “\([N]\)” to \( W_{f,k'}(t_1)_{[N]} \) and \( D'(A)_{[N]} \) in (64) to remind the reader of the dependence of these quantities on \( N \).

Using the DeMoivre-Laplace theorem [63, eq. (4.5.2), p. 204] and the definition of the error function [64, eq. 40:3:1, p. 406],

\[
\lim_{N \to \infty} D'(A)_{[N]} = \text{erf}(A/\sqrt{2}), \tag{65}
\]

This implies that, for any positive number \( \epsilon_D \), there exists an integer \( N_D(\epsilon_D, A) \) such that

\[
D'(A)_{[N]} > \text{erf}(A/\sqrt{2}) - \epsilon_D, \quad N > N_D(\epsilon_D, A) \tag{66}
\]

From (64) and (66) we obtain

\[
W_{f,k'}(t_1)_{[N]} > \text{erf}(A/\sqrt{2}) - \epsilon_D, \quad N > \max \left( N_D(\epsilon_D, A), \frac{A^2pq}{\left( \frac{1}{2\nu} - \Delta \right)^2} \right), \quad \epsilon_D > 0 \tag{67}
\]

From the definition of the error function it follows that

\[
\lim_{A \to \infty} \text{erf}(A/\sqrt{2}) = 1. \tag{68}
\]

So for any positive number \( \epsilon_e \) there exists a number \( A_e(\epsilon_e) \) such that

\[
\text{erf}(A/\sqrt{2}) > 1 - \epsilon_e, \quad A > A_e(\epsilon_e). \tag{69}
\]
Using (69) in (67), we obtain

\[ W_{f,k}'(t_1)_N > 1 - \epsilon_e - \epsilon_D, \quad N > \max \left( N_D(\epsilon_D, A), \frac{A^2pq}{\left( \frac{1}{2\nu} - \Delta \right)^2} \right), \quad A > A_e(\epsilon_e), \epsilon_D, \epsilon_e > 0 \]

\[ (70) \]

Let \( \epsilon \) be a number satsifying

\[ \epsilon > \epsilon_D + \epsilon_e. \]

Then

\[ W_{f,k}'(t_1)_N > 1 - \epsilon, \quad N > \max \left( N_D(\epsilon_D, A), \frac{A^2pq}{\left( \frac{1}{2\nu} - \Delta \right)^2} \right), \quad A > A_e(\epsilon_e), \epsilon > \epsilon_D + \epsilon_e, \epsilon_D, \epsilon_e, \epsilon > 0 \]

\[ (72) \]

So \( W_{f,k}'(t_1) \), the weight for the finite-resolution relative frequency closest to the Born-rule relative frequency \( p \), can be brought arbitrarily close to unity by making the size \( N \) of the spin ensemble being measured sufficiently large.\(^{10}\)

Of course \( W_{f,k}'(t_1) \) will never be larger than unity. The \( W_{f,k} \)'s sum to unity \[54, eq. (100)],

\[ \sum_{k=1}^{\nu+1} W_{f,k}(t_1) = 1, \]

\[ (73) \]

and are all positive,

\[ W_{f,k}(t_1) > 0, \quad k = 1, \ldots, \nu + 1 \]

\[ (74) \]

so

\[ W_{f,k}'(t_1) < 1. \]

\[ (75) \]

Using (72) with (73) with (74) we obtain

\[ W_{f,k}(t_1) < \epsilon, \quad k \neq k', \quad N > \max \left( N_D(\epsilon_D, A), \frac{A^2pq}{\left( \frac{1}{2\nu} - \Delta \right)^2} \right), \quad A > A_e(\epsilon_e), \epsilon > \epsilon_D + \epsilon_e, \epsilon_D, \epsilon_e, \epsilon > 0 \]

\[ (76) \]

\(^{10}\)The conditions in (72) are to be read from right to left: Pick a desired positive \( \epsilon \), then choose a positive \( \epsilon_D \) and a positive \( \epsilon_e \) such that their sum is less than \( \epsilon \). Then choose a positive number \( A \) greater than \( A_e(\epsilon_e) \). Using \( A \) and \( \epsilon_D \) we can now select a value for \( N \) large enough so that \( W_{f,k}'(t_1) \) is within the desired distance \( \epsilon \) of 1.

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I.e., the same \( N \) that yields \( W_{f,k'}(t_1) \) within \( \epsilon \) of 1 will ensure that any other \( W_{f,k}(t_1), k \neq k' \), will be within \( \epsilon \) of 0. For any preclusion threshold \( \epsilon_P \) between \( \epsilon \) and \( 1 - \epsilon \), Everett copies of the relative-frequency observer \( \mathcal{F} \) perceiving the non-Born finite-resolution relative frequencies \( \phi_k, k \neq k' \), will be precluded while the Everett copy perceiving the finite-resolution relative frequency \( \phi_k \), closest to the Born value will survive. This conclusion also holds if the closest finite-resolution relative frequency to \( p \) is less than \( p \).

What about the case \( \Delta = \frac{1}{2\nu} \), i.e., \( p \) equidistant from the two closest finite-resolution relative frequencies? Denote these \( \phi_k \)'s by

\[
\phi_{k<} = p - \frac{1}{2\nu}, \quad (77)
\]

\[
\phi_{k>} = p + \frac{1}{2\nu}. \quad (78)
\]

Using (53) and (77),

\[
W_{f,k<}(t_1) = \sum_{l \mid 0 \leq l \leq N, -\frac{1}{\nu} \sqrt{\frac{N}{pq}} < l - Np \sqrt{Npq} \leq 0} p_{N,l}. \quad (79)
\]

This can be evaluated in essentially the same manner as \( W_{f,k'}(t_1) \) above.

Define

\[
D_{<}(A) = \sum_{l \mid 0 \leq l \leq N, -A \leq l - Np \sqrt{Npq} \leq 0} p_{N,l}, \quad A > 0 \quad (80)
\]

Then

\[
W_{f,k<}(t_1) \geq D_{<}(A), \quad \frac{1}{\nu} \sqrt{\frac{N}{pq}} > A \quad (81)
\]

The DeMoivre-Laplace theorem implies that, for any positive number \( \epsilon_D \), there exists a number \( N_{<}(\epsilon_D, A) \) such that

\[
D_{<}(A) > \frac{1}{2} \text{erf}(A / \sqrt{2}) - \epsilon_D, \quad N > N_{<}(\epsilon_D, A) \quad (82)
\]

Using this and (69) in (81)

\[
W_{f,k<}(t_1) > \frac{1}{2} - \frac{\epsilon_e}{2} - \epsilon_D, \quad N > \max \left( N_{<}(\epsilon_D, A), \nu^2 A^2 pq \right), \quad A > A_e(\epsilon_e), \quad \epsilon_D, \epsilon_e > 0. \quad (83)
\]

Let \( \epsilon_\leq \) be a positive number satisfying

\[
\epsilon_\leq > \frac{\epsilon_e}{2} + \epsilon_D. \quad (84)
\]
Then
\[ W_{f,k<}(t_1) > \frac{1}{2} - \epsilon, \quad N > \max \left( N_<(\epsilon_D, A), \nu^2 A^2 pq \right), \quad A > A_e(\epsilon_e), \quad \epsilon_0 > \frac{\epsilon_e}{2} + \epsilon_D, \]
\[ \epsilon_D, \epsilon_e, \epsilon_0 > 0. \]  
(85)

Using (78) in (53),
\[ W_{f,k>}(t_1) = \sum_{l \mid 0 \leq l \leq N, 0 < \frac{l - Np}{\sqrt{Npq}} < \frac{1}{\nu} \sqrt{\frac{N}{pq}}} p_{N,l}. \]  
(86)

To evaluate (86), define
\[ \tilde{W}_{f,k>}(t_1) = \sum_{l \mid 0 \leq l \leq N, 0 \leq \frac{l - Np}{\sqrt{Npq}} \leq \frac{1}{\nu} \sqrt{\frac{N}{pq}}} p_{N,l} \]  
(87)
and
\[ D_>(A) = \sum_{l \mid 0 \leq l \leq N, 0 \leq \frac{l - Np}{\sqrt{Npq}} \leq A} p_{N,l}, \quad A > 0. \]  
(88)

These satisfy
\[ \tilde{W}_{f,k>}(t_1) \geq D_>(A), \quad \frac{1}{\nu} \sqrt{\frac{N}{pq}} > A. \]  
(89)

Using (69) and (82) in (89),
\[ \tilde{W}_{f,k>}(t_1) > \frac{1}{2} - \frac{\epsilon_e}{2} - \epsilon_D, \quad N > \max \left( N_<(\epsilon_D, A), \nu^2 A^2 pq \right), \quad A > A_e(\epsilon_e), \quad \epsilon_D, \epsilon_e > 0. \]  
(90)
(Note that the conditions on \( N \) are the same as for \( W_{f,k<}(t_1) \) in (83).)

From (86) and (87),
\[ \tilde{W}_{f,k>} - W_{f,k>} = 0, \quad [Np] \neq Np, \quad p_{N,[Np]}, \quad [Np] = Np, \]  
(91)
where \([x]\) is the floor of \( x \). Therefore
\[ \tilde{W}_{f,k>} - W_{f,k>} \leq p_{N,[Np]}. \]  
(92)
Using (90) and (92),
\[ W_{f,k}(t_1) > \frac{1}{2} - \frac{\epsilon_e}{2} - \epsilon_D - p_{N,\lfloor Np \rfloor}, \quad N > \max \left( N_\prec(\epsilon_D, A), \nu^2 A^2 pq \right), \quad A > A_\epsilon(\epsilon_e), \]
where \( \epsilon_D, \epsilon_e > 0 \).

The DeMoivre-Laplace theorem [63, eq. (4.5.1), p. 204] gives an asymptotic expression for \( p_{N,l} \),
\[ p_{N,l} \sim \frac{e^{-(l-Np)^2/2Npq}}{\sqrt{2\pi Npq}}, \quad |l - Np| = o(N^{2/3}). \]

Since \( |\lfloor Np \rfloor - Np| < 1 \), we can make use of (94) with \( l = \lfloor Np \rfloor \) to conclude that, for any positive number \( \epsilon_* \), there is an integer \( N_*(\epsilon_*, p) \) such that
\[ p_{N,\lfloor Np \rfloor} < \frac{1 + \epsilon_*}{\sqrt{2\pi Npq}}, \quad N > N_*(\epsilon_*, p). \]

(The limit involved in (94) is uniform in \( l \) [63, footnote p. 204], so \( N_*(\epsilon_*, p) \) does not depend on \( l \).) From (93) and (96) we obtain
\[ W_{f,k}(t_1) > \frac{1}{2} - \frac{\epsilon_e}{2} - \epsilon_D - \frac{1 + \epsilon_*}{\sqrt{2\pi Npq}}, \quad N > \max \left( N_\prec(\epsilon_D, A), \nu^2 A^2 pq, N_*(\epsilon_*, p) \right), \]
\[ A > A_\epsilon(\epsilon_e), \quad \epsilon_D, \epsilon_e, \epsilon_* > 0. \]

Let \( \epsilon_> \) be a positive number satisfying
\[ \epsilon_> > \frac{\epsilon_e}{2} + \epsilon_D + \frac{1 + \epsilon_*}{\sqrt{2\pi Npq}}. \]

Since \( \epsilon_* \) is positive, this implies
\[ \epsilon_> > \frac{\epsilon_e}{2} + \epsilon_D \]
which, with (98), yields
\[ \sqrt{N} > \frac{1 + \epsilon_*}{\left( \epsilon_> - \frac{\epsilon_e}{2} - \epsilon_D \right) \sqrt{2\pi pq}}. \]

Conversely, (99) and (100) give (98). Using (97)-(100) we obtain
\[ W_{f,k}(t_1) > \frac{1}{2} - \epsilon_>, \quad N > \max \left( N_\prec(\epsilon_D, A), \nu^2 A^2 pq, N_*(\epsilon_*, p), \frac{(1 + \epsilon_*)^2}{(\epsilon_> - \frac{\epsilon_e}{2} - \epsilon_D)^2 2\pi pq} \right), \]
\[ A > A_e(\epsilon_\text{e}), \quad \epsilon_\text{e} > \frac{\epsilon_\text{e}}{2} + \epsilon_D, \quad \epsilon_D, \epsilon_\text{e}, \epsilon_\text{e}, \epsilon > 0. \tag{101} \]

Setting
\[ \epsilon_\text{e} = \epsilon, \tag{102} \]
(85) and (101) give
\[ W_{f,k<}(t_1) > \frac{1}{2} - \epsilon \quad \text{and} \quad W_{f,k>}(t_1) > \frac{1}{2} - \epsilon, \]
\[ N > \max \left( N_<(\epsilon_D, A), \nu^2 A^2 pq, N_*(\epsilon_*, p), \frac{(1 + \epsilon_*)^2}{(\epsilon_\text{e} - \frac{\epsilon_\text{e}}{2} - \epsilon_D)^2 2\pi pq} \right), \]
\[ A > A_e(\epsilon_\text{e}), \quad \epsilon_\text{e} > \frac{\epsilon_\text{e}}{2} + \epsilon_D, \quad \epsilon_D, \epsilon_\text{e}, \epsilon_*, \epsilon > 0. \tag{103} \]

Using (73) and (74) with (103) we obtain
\[ W_{f,k}(t_1) < 2\epsilon, \quad k \neq k_<, k_>, \]
\[ N > \max \left( N_<(\epsilon_D, A), \nu^2 A^2 pq, N_*(\epsilon_*, p), \frac{(1 + \epsilon_*)^2}{(\epsilon_\text{e} - \frac{\epsilon_\text{e}}{2} - \epsilon_D)^2 2\pi pq} \right), \]
\[ A > A_e(\epsilon_\text{e}), \quad \epsilon_\text{e} > \frac{\epsilon_\text{e}}{2} + \epsilon_D, \quad \epsilon_D, \epsilon_\text{e}, \epsilon_*, \epsilon > 0. \tag{104} \]

So, if the two finite-resolution relative frequencies \( \phi_{k_<} \) and \( \phi_{k_>} \) closest to \( p \) are equidistant from \( p \), the same \( N \) that yields \( W_{f,k<}(t_1) \) and \( W_{f,k>}(t_1) \) within \( \epsilon \) of \( 1/2 \) will ensure that any other \( W_{f,k}(t_1) \), \( k \neq k_<, k_> \), will be within \( 2\epsilon \) of 0. For any preclusion threshold \( \epsilon_P \) between \( 2\epsilon \) and \( 1/2 - \epsilon \), Everett copies of the relative-frequency observer \( \mathcal{F} \) perceiving the non-Born finite-resolution relative frequencies \( \phi_k, k \neq k_<, k_> \), will be precluded while the two Everett copies perceiving the finite-resolution relative frequencies \( \phi_{k_<}, \phi_{k_>} \) closest to the Born value will survive.\(^{11}\)

5 Precluded contingent events

Any theory of probability, classical or quantum, that employs hard preclusion must deal with apparently-paradoxical situations of the following sort. Consider a probabilistic event the occurrence of which is contingent on the occurrence of at least one of a set of \( N \) antecedent probabilistic events. Say the probability of any one of the antecedent events

\(^{11}\)By considering the \( N \to \infty \) limit of (103) and (104) we extend the results of [54, Sec. 3.5] for \( \Delta = 1/2\nu \) to include the cases in which \( \nu \) is even or \( p \) differs from 1/2.
is \( p \). Then if \( p \ll 1 \), the probability of the contingent event is approximately \( Np \). If the preclusion threshold \( \epsilon_P \) is such that \( p < \epsilon_P < Np \), the contingent event will occur despite none of the antecedent events required for its occurrence having occurred.

To avoid such situations we modify the preclusion formalism we have been employing so far by making the preclusion threshold \( \epsilon_P \) a function of the dimension \( D \) of the projection operator corresponding to the event which is (possibly) being precluded. Specifically,

\[
\epsilon_P = \epsilon_P(D) = D\epsilon_1
\]

where \( \epsilon_1 \) is the preclusion threshold for 1-dimensional projectors. Note that in the Schrödinger picture this dimension-dependence of the preclusion threshold follows automatically from the formalism, in which events are precluded based on the norms of (product) state vectors to which they correspond; see BHZ.

As an illustration, consider an “at-least-one” observer \( \mathcal{G} \) that measures whether one or more of a set of qubits in the state \( |\Psi\rangle \) of (11) is in state \( \alpha_1 \). The two states of the observer correspond, respectively, to “all qubits in state 1” and “at least one qubit in state 2:”

\[
\hat{g}|\mathcal{G};\gamma_0\rangle = \gamma_0|\mathcal{G};\gamma_0\rangle \quad \text{(all qubits in state 1)},
\]

\[
\hat{g}|\mathcal{G};\gamma_1\rangle = \gamma_1|\mathcal{G};\gamma_1\rangle \quad \text{(at least one qubit in state 2)}. \tag{107}
\]

For simplicity we do not include observers of the individual qubits or a separate ready state for \( \mathcal{G} \), the role of the latter being played by \( |\mathcal{G};\gamma_0\rangle \). I.e., \( \mathcal{G} \) is in state \( |\mathcal{G};\gamma_0\rangle \) before the measurement, at time \( t_{in} \), and and remains in that state after completion of the measurement at time \( t \) if all of the qubits are in state 1, else transitions to state \( |\mathcal{G};\gamma_1\rangle \).

The unitary transformation which effects this dynamics is

\[
\hat{U}^{\mathcal{GS}}(t) = \exp \left( -i\hat{H}^{\mathcal{GS}}(t-t_{in}) \right) \tag{108}
\]

where

\[
\hat{H}^{\mathcal{GS}} = \hat{h}^\mathcal{G} \otimes \hat{P}^S_{\mathcal{G}} \tag{109}
\]

with

\[
\hat{h}^\mathcal{G} = i\kappa (|\mathcal{G};\gamma_1\rangle\langle\mathcal{G};\gamma_0| - |\mathcal{G};\gamma_0\rangle\langle\mathcal{G};\gamma_1|) \tag{110}
\]

and, using (15),

\[
\hat{P}^S_{\mathcal{G}} = 1 - \prod_{p=1}^{N} \hat{P}_{1}^{S(p)} \tag{111}
\]

Using (107)-(111), we find the unitary transformation, at time \( t \), to be

\[
\hat{U}^{\mathcal{GS}}(t) = \hat{P}^S_{\mathcal{G}} + \left[ \cos(\kappa(t-t_{in})) + i\sin(\kappa(t-t_{in})) \frac{\hat{h}^\mathcal{G}}{\kappa} \right] \otimes \hat{P}^S_{\mathcal{G}}, \tag{112}
\]

\[12\]I thank an anonymous referee of [30] for pointing out this issue to me.
where
\[ \hat{P}_S^G = 1 - \hat{P}_S^G = \prod_{p=1}^{N} \hat{P}_1^{S(p)} \]  

(113)

The Heisenberg-picture at-least-one operator at time \( t \) is then
\[
\hat{g}(t) = \hat{U}^{G \downarrow} \hat{g} \hat{U}^{G \uparrow}
\]
\[ = \hat{g} \otimes \hat{P}_S^G + \left[ \cos^2(\kappa(t - t_{in}))\hat{g} + i \sin(\kappa(t - t_{in})) \cos(\kappa(t - t_{in})) \frac{1}{\kappa} [\hat{g}, \hat{h}^G] \right] \otimes \hat{P}_g^G 
\]  

(114)

where
\[
\hat{g}_1 = \frac{1}{\kappa^2} \hat{h}^G \hat{g} \hat{h}^G
\]
\[ = \gamma_1 |G; \gamma_0\rangle \langle G; \gamma_0| + \gamma_0 |G; \gamma_1\rangle \langle G; \gamma_1|. \]

Choosing the time \( t \) to be the time \( t_1 \) at which the measurement is complete (see eq. (19)), the Heisenberg-picture at-least-one operator is in the post-ideal-measurement form,
\[
\hat{g}(t_1) = \hat{g} \otimes \hat{P}_S^G + \hat{g}_1 \otimes \hat{P}_g^G 
\]  

(116)

The weight for \( G \) having determined that at least one of the systems \( S^{(q)} \) is in state 2 is
\[
W_S^1 = \langle \Psi | \hat{P}_S^G | \Psi \rangle. 
\]  

(117)

Using (1), (2), (55) and (111),
\[
W_S^1 = 1 - (1 - q)^N. 
\]  

(118)

Suppose now that the probability \( q \) of any individual qubit being in the state 2 is small enough to be precluded, i.e.,
\[
q < \epsilon_1 
\]  

(119)

so of course
\[
Nq < N \epsilon_1 
\]  

(120)

But
\[
1 - (1 - q)^N \leq Nq 
\]  

(121)

(induction on \( N \)). So, whenever the outcome of any qubit being in state 2 is precluded, the rule (105), with the dimension \( D \) equal to \( N \), ensures that the outcome of \( G \) having detected at least one of the qubits in state 2 is also precluded:
\[
W_S^1 < \epsilon_P(N). 
\]  

(122)
6 Ordinary probabilistic events and biological evolution

6.1 The connection between extraordinary and ordinary probabilistic events

Accounting for the nonobservation of extraordinary probabilistic events is no small thing; the laws of thermodynamics play a key role in making our world what it is. But of course there are myriad ordinary probabilistic phenomena encountered in physics and in day-to-day life. How, if at all, are these related to extraordinary probabilistic events and preclusion?

There is no shortage of events of low probability which are known to have occurred. Therefore the preclusion parameter \( \epsilon_p(N) \) for sub-Avogadro values of \( N \) commonly encountered in physics and in life must be extremely small\(^{13} \) in order not to be in conflict with experience, much smaller than the probability of a deviation of the empirical relative frequency from the theoretical probability. Preclusion, it might seem, can have no more direct connection with ordinary probabilistic phenomena than can the limiting case of an infinitely large system.

Again it makes sense to appeal directly to the empirically-observed features associated with the phenomena we are trying to understand, in this case ordinary probabilistic phenomena as defined in Sec. 1. The objective features associated with ordinary probabilistic phenomena are precisely the subjective judgements we have regarding them. In the words of de Finetti [65, p. 174], “What do we mean when we say, in ordinary language, that an event is more or less probable? We mean that we would be more or less surprised to learn that it has not happened.” These judgements, also termed “subjective probabilities” or “credences,” are subjective in that they are the beliefs of individuals; but their existence is an objective fact.

A feature of subjective judgements of probability is that, in many though by no means in all cases, they are more or less in accord with the quantities given by physical theories regarding ordinary probabilistic phenomena which we usually think of as “objective probabilities.” This connection between subjective judgement and objective-probability-as-given-by-physical-theory bears a resemblance to the Principal Principle of Lewis [66] which, as formulated by Papineau [18], states that “it it rational to set your subjective probabilities equal to the objective probabilities.” In the present context, where no notion of objective probability governing ordinary probabilistic phenomena has been introduced, what we wish to explain is what we might call the “empirical Principal Principal,” the

\(^{13}\)Estimates for the size of the preclusion parameter, based on quantum-gravitational considerations, are presented in BHZ. Note that the parameter \( \epsilon \) of BHZ is related to the parameter \( \epsilon_1 \) of the present paper by \( \epsilon = \sqrt{\epsilon_1} \).
observation that in many cases subjective judgements assigned to ordinary probabilistic phenomena are consistent with the predictions of physical theory, i.e., with the probabilities assigned by the Born rule based on the same quantum-mechanical amplitudes that, for Avogadro-number-sized ensembles, would lead to the preclusion of extraordinary probabilistic phenomena.\footnote{\textsuperscript{14}}

We have argued above that the preclusion of extraordinary probabilistic phenomena cannot be directly related to the kind of probability witnessed in ordinary probabilistic phenomena essentially because the probabilities involved in the latter are too large. However, if the phenomenon of subjective judgement of probability is one that comes about through \textit{biological evolution}, then the requisite small quantum weights can arise not through the growth in the size of an ensemble at a single time, but rather through the repetition of interactions between organisms and their environment over generations upon generations of geological time. After a sufficiently large number of generations, the existence of species with subjective judgements of probability that are suited for survival in environments that have lower quantum-mechanical amplitude will be precluded. Surviving species will be those with subjective judgements of probability suited for survival in environments with higher quantum-mechanical amplitude. The latter, it can be argued, will be environments in which frequencies are close to those required by the Born rule, so subjective judgements of probability will match Born-rule values, the objective probabilities of standard quantum theory. In the next two subsections we present a model illustrating this process.\footnote{\textsuperscript{15}}

\subsection*{6.2 Biological evolution from hard preclusion}

Existing models of biological evolution describe ordinary probabilistic phenomena, including evolutionary change from one generation to the next, in terms of objective probability; e.g., a species of predator with slightly superior eyesight will in each generation likely do somewhat better at surviving than a similar species with poorer eyesight, and over many generations will with high probability supplant the less sharp-sighted species. For us to argue that subjective probability, or for that matter any other observed phenomenon, is the result of biological evolution, we require a model of biological evolution which does not require for its description the same ordinary probabilistic phenomena the origin of which we wish to explain, but which rather makes use only of the physical concepts we have at our disposal, namely unitary quantum-mechanical time evolution and hard preclusion.

Here we present a basic proof-of-concept version of such a model. For simplicity we work in the Schrödinger picture with discrete time-steps, one per generation, and do not...
model competition between species but only natural selection. Specifically, the model demonstrates that the weight for the survival of a species which is fit to a lower-weight (less “probable”) environment eventually drops below the preclusion threshold while that for a species fit to a higher-weight (more “probable”) environment remains above that threshold.

An issue that has to be kept in mind in constructing a model of this sort is the tension between the unitarity of quantum mechanics and the irreversibility of the notion of “extinction.” This is of course not an insuperable obstacle. Following the approach of other models of irreversible processes in quantum mechanics (see, e.g., the discussion of the decay of an excited state in [69, Sec. 6.1]), we set up a unitary framework in which the transition of a species from an extinct state to an alive state is allowed, then apply the framework to a time regime small compared to the time required for a return from the extinct state to the alive state.

The objects in the model are an environment \( \mathcal{E} \) and two species of organisms \( \mathcal{O}_B \) and \( \mathcal{O}_M \). The environment is represented by a tensor product of states labeled by a discrete time parameter \( t \):

\[
|\mathcal{E}\rangle = \prod_{t=0}^{T} |\mathcal{E}_t\rangle.
\]

(123)

The state \( |\mathcal{E}_t\rangle \) represents the state of that environment that the organisms interact with at time \( t \). I.e., each \( |\mathcal{E}_t\rangle \), \( t = 0, \ldots, T \), is the state vector of a distinct entity, as indicated in (123). \( |\mathcal{E}_t\rangle \) is unchanged under unitary time evolution, although \( |\mathcal{E}_t\rangle \) affects the evolution of organisms at time \( t \) (see (129)).

For simplicity take all the \( |\mathcal{E}_t\rangle \)'s to have the same structure, the superposition of two states \( |\mathcal{E}_t; B\rangle \) and \( |\mathcal{E}_t; M\rangle \):

\[
|\mathcal{E}_t\rangle = \mu_B |\mathcal{E}_t; B\rangle + \mu_M |\mathcal{E}_t; M\rangle,
\]

(124)

where \( \mu_B \) and \( \mu_M \) are independent of \( t \), and

\[
\langle \mathcal{E}_t; B | \mathcal{E}_t; B \rangle = \langle \mathcal{E}_t; M | \mathcal{E}_t; M \rangle = 1,
\]

(125)

\[
\langle \mathcal{E}_t; B | \mathcal{E}_t; M \rangle = 0,
\]

(126)

\[
|\mu_B|^2 + |\mu_M|^2 = 1,
\]

(127)

\[
|\mu_B| > |\mu_M|.
\]

(128)

The labels \( B \) and \( M \) stand, respectively, for “Born-like” and “maverick” states. We will say more about this choice of terminology below; for now they are simply names for states satisfying (125) - (128).

The organisms \( \mathcal{O}_B \) and \( \mathcal{O}_M \) thrive only if they are in, respectively, Born-like and maverick environments, and decline otherwise. Again being guided by the desire for simplicity, we make the impact of the environment determinative and extreme. If a species is alive
at time $t$ in a favorable environment ($\mathcal{O}_B$ in $|\mathcal{E}_t; B\rangle$ or $\mathcal{O}_M$ in $|\mathcal{E}_t; M\rangle$), that species will be alive at time $t + 1$. If a species is alive at time $t$ in an unfavorable environment ($\mathcal{O}_B$ in $|\mathcal{E}_t; M\rangle$ or $\mathcal{O}_M$ in $|\mathcal{E}_t; B\rangle$), that species will be extinct at time $t + 1$.

For unitarity the dynamics must also include the possibility of transitions from states in which a species is extinct to states in which it is alive. To incorporate these transitions while allowing “extinction” to retain the meaning we want it to have, we introduce for each species a single alive state but $\Omega \gg 1$ extinct states. A species which becomes extinct progresses through a succession of extinct states until, after $\Omega$ time steps, it again becomes alive—$\Omega$ is essentially the Poincaré recurrence time. This dynamics is compatible with unitarity, while restricting attention to states with $t < \Omega$ allows us to model the desired (highly-simplified) evolutionary biology.

In detail: The state in which $B$ is alive is $|\mathcal{O}_B; A\rangle$, while the states in which $B$ is extinct are $|\mathcal{O}_B; X_j\rangle$, $j = 0, \ldots, \Omega - 1$. Similarly, $\mathcal{O}_M$ is alive in $|\mathcal{O}_M; A\rangle$ and extinct in $|\mathcal{O}_M; X_j\rangle$, $j = 0, \ldots, \Omega - 1$. Let $\hat{U}_t$ be the unitary operator that implements time evolution from time $t$ to time $t + 1$. We specify that $\hat{U}_t$ does not have any effect on the environment basis vectors $|\mathcal{E}_t; B\rangle$, $|\mathcal{E}_t; M\rangle$, while the modification to the organism basis vectors effected by $\hat{U}_t$ depends on the states of those vectors as well as on the environment at time $t$. That is, at the next time step a species remains alive if the environment is definitely the type that is favorable for it, and becomes extinct if it is definitely in an unfavorable environment. An extinct species in a favorable environment will remain extinct by transitioning to the the next extinct state, “next” being defined modulo $\Omega$. An extinct species in an unfavorable environment will likewise remain extinct and transition to the next extinct state unless it is already in the final extinct state, in which case it will transition to the alive state.

The action of $\hat{U}_t$ on a basis of states of the $\mathcal{E}_t$, $\mathcal{O}_B$ and $\mathcal{O}_M$ is thus given by

$$
\begin{align*}
\hat{U}_t|\mathcal{E}_t; B\rangle|\mathcal{O}_B; A\rangle|\mathcal{O}_M; A\rangle &= |\mathcal{E}_t; B\rangle|\mathcal{O}_B; A\rangle|\mathcal{O}_M; X_0\rangle \\
\hat{U}_t|\mathcal{E}_t; B\rangle|\mathcal{O}_B; A\rangle|\mathcal{O}_M; X_k\rangle &= |\mathcal{E}_t; B\rangle|\mathcal{O}_B; A\rangle \\
&\quad \cdot \left(|\mathcal{O}_M; X_{k+1}\rangle(1 - \delta_{k,\Omega - 1}) + |\mathcal{O}_M; A\rangle\delta_{k,\Omega - 1}\right) \\
\hat{U}_t|\mathcal{E}_t; B\rangle|\mathcal{O}_B; X_j\rangle|\mathcal{O}_M; A\rangle &= |\mathcal{E}_t; B\rangle \left(|\mathcal{O}_B; X_{j+1}\rangle(1 - \delta_{j,\Omega - 1}) + |\mathcal{O}_B; X_0\rangle\delta_{j,\Omega - 1}\right) \\
&\quad \cdot |\mathcal{O}_M; X_0\rangle \\
\hat{U}_t|\mathcal{E}_t; B\rangle|\mathcal{O}_B; X_j\rangle|\mathcal{O}_M; X_k\rangle &= |\mathcal{E}_t; B\rangle \left(|\mathcal{O}_B; X_{j+1}\rangle(1 - \delta_{j,\Omega - 1}) + |\mathcal{O}_B; X_0\rangle\delta_{j,\Omega - 1}\right) \\
&\quad \cdot \left(|\mathcal{O}_M; X_{k+1}\rangle(1 - \delta_{k,\Omega - 1}) + |\mathcal{O}_M; A\rangle\delta_{k,\Omega - 1}\right) \\
\hat{U}_t|\mathcal{E}_t; M\rangle|\mathcal{O}_B; A\rangle|\mathcal{O}_M; A\rangle &= |\mathcal{E}_t; M\rangle|\mathcal{O}_B; X_0\rangle|\mathcal{O}_M; A\rangle \\
\hat{U}_t|\mathcal{E}_t; M\rangle|\mathcal{O}_B; A\rangle|\mathcal{O}_M; X_k\rangle &= |\mathcal{E}_t; M\rangle|\mathcal{O}_B; X_0\rangle \\
&\quad \cdot \left(|\mathcal{O}_M; X_{k+1}\rangle(1 - \delta_{k,\Omega - 1}) + |\mathcal{O}_M; X_0\rangle\delta_{k,\Omega - 1}\right)
\end{align*}
$$

(129)
\[
\hat{U}_t |\mathcal{E}_t; M \rangle |\mathcal{O}_B; X_j \rangle |\mathcal{O}_M; A \rangle = |\mathcal{E}_t; M \rangle \left( |\mathcal{O}_B; X_{j+1} \rangle (1-\delta_{j,\Omega-1}) + |\mathcal{O}_B; A \rangle \delta_{j,\Omega-1} \right) \\
\cdot |\mathcal{O}_M; A \rangle \\
\hat{U}_t |\mathcal{E}_t; M \rangle |\mathcal{O}_B; X_j \rangle |\mathcal{O}_M; X_k \rangle = |\mathcal{E}_t; M \rangle \left( |\mathcal{O}_B; X_{j+1} \rangle (1-\delta_{j,\Omega-1}) + |\mathcal{O}_B; A \rangle \delta_{j,\Omega-1} \right) \\
\cdot \left( |\mathcal{O}_M; X_{k+1} \rangle (1-\delta_{k,\Omega-1}) + |\mathcal{O}_M; X_0 \rangle \delta_{k,\Omega-1} \right)
\]

Indices \( j, k \) run from 0 through \( \Omega - 1 \).

Taking the inner products of the above states with a dual basis, and keeping in mind the orthogonality of alive states to extinct states and of extinct states with different indices to each other, we obtain the matrix representation

\[
\hat{U}_t =
\begin{pmatrix}
\delta_{l,0} & \delta_{m,k+1} & \delta_{m,0} & 0 \\
\delta_{m,k+1} & \delta_{m,k+1} & \delta_{m,0} & 0 \\
\delta_{m,0} & \delta_{m,0} & \delta_{m,0} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

where \( l, m \) are row indices and \( j, k \) are column indices. Blocks with no nonzero entries are left blank. For example, the first entry in the first column is

\[
\langle \mathcal{E}_t; B | \langle \mathcal{O}_B; A | \langle \mathcal{O}_M; A | \hat{U}_t |\mathcal{E}_t; B \rangle |\mathcal{O}_B; A \rangle |\mathcal{O}_M; A \rangle = 0,
\]

and the second entry in the first column is

\[
\langle \mathcal{E}_t; B | \langle \mathcal{O}_B; A | \langle \mathcal{O}_M; X_m | \hat{U}_t |\mathcal{E}_t; B \rangle |\mathcal{O}_B; A \rangle |\mathcal{O}_M; A \rangle = \delta_{m,0}.
\]

From this representation it is straightforward to show that \( \hat{U}_t \) is unitary.

If we limit the time over which we track evolution to be smaller than the Poincaré recurrence time, i.e., we take \( T < \Omega - 1 \), then the indices \( j, k, l \) and \( m \) will never be as
large as $\Omega - 1$ and we can replace (129) with

\[
\hat{U}t\left|\mathcal{E}_t; B\right|\mathcal{O}_B; A\rangle\mathcal{O}_M; A \rangle = \left|\mathcal{E}_t; B\right|\mathcal{O}_B; A\rangle\mathcal{O}_M; X_0\rangle
t\in\mathbb{Z}.
\]

Using (123), (124), (133), (134) and (135), we find after the first timestep

\[
\hat{U}t\left|\mathcal{E}_t; B\right|\mathcal{O}_B; X_j\rangle\mathcal{O}_M; X_k \rangle = \left|\mathcal{E}_t; B\right|\mathcal{O}_B; X_{j+1}\rangle\mathcal{O}_M; X_{k+1}\rangle
\]

and we can replace (129) with

\[
\left|\mathcal{E}_t; B\right|\mathcal{O}_B; X_j\rangle\mathcal{O}_M; X_k \rangle = \left|\mathcal{E}_t; B\right|\mathcal{O}_B; X_{j+1}\rangle\mathcal{O}_M; X_{k+1}\rangle
\]

Proceeding by mathematical induction, we hypothesize that

\[
|\Phi, n\rangle = \left[\prod_{t=n}^{T} \left(\mu_B|\mathcal{E}_t; B\rangle + \mu_M|\mathcal{E}_t; M\rangle\right)\right]
\]

Take the initial state at $t = 0$, $|\Phi, 0\rangle$, to be one in which both species are alive:

\[
|\Phi, 0\rangle = |\mathcal{E}\rangle|\mathcal{O}_B; A\rangle|\mathcal{O}_M; A\rangle.
\]

Evolution from one discrete time step to the next is implemented by

\[
|\Phi, t + 1\rangle = \hat{U}_t|\Phi, t\rangle.
\]

Using (123), (124), (133), (134) and (135), we find after the first timestep

\[
|\Phi, 1\rangle = \left[\prod_{t=1}^{T} \left(\mu_B|\mathcal{E}_t; B\rangle + \mu_M|\mathcal{E}_t; M\rangle\right)\right]
\]

\[
\left(\mu_B|\mathcal{O}_B; A\rangle|\mathcal{O}_M; X_0\rangle|\mathcal{E}_0; B\rangle + \mu_M|\mathcal{O}_B; X_0\rangle|\mathcal{O}_M; A\rangle|\mathcal{E}_0; M\rangle\right).
\]

I.e., after one timestep there is a superposition of two states, in each of which only one species is alive. This of course is a result of our having chosen the two species to be incapable of surviving even a single generation in an unfavorable environment. After two timesteps,

\[
|\Phi, 2\rangle = \left[\prod_{t=2}^{T} \left(\mu_B|\mathcal{E}_t; B\rangle + \mu_M|\mathcal{E}_t; M\rangle\right)\right]
\]

\[
\cdot \left(\mu_B|\mathcal{O}_B; A\rangle|\mathcal{O}_M; X_1\rangle|\mathcal{E}_1; B\rangle|\mathcal{E}_0; B\rangle + \mu_B\mu_M|\mathcal{O}_B; X_0\rangle|\mathcal{O}_M; X_1\rangle|\mathcal{E}_1; M\rangle|\mathcal{E}_0; B\rangle\right)
\]

\[
+ \mu_M\mu_B|\mathcal{O}_B; X_1\rangle|\mathcal{O}_M; X_0\rangle|\mathcal{E}_1; B\rangle|\mathcal{E}_0; M\rangle + \left(\mu_M\right)^2|\mathcal{O}_B; X_1\rangle|\mathcal{O}_M; A\rangle|\mathcal{E}_1; M\rangle|\mathcal{E}_0; M\rangle\right)
\]

Proceeding by mathematical induction, we hypothesize that

\[
|\Phi, n\rangle = \left[\prod_{t=n}^{T} \left(\mu_B|\mathcal{E}_t; B\rangle + \mu_M|\mathcal{E}_t; M\rangle\right)\right]
\]
\[
\begin{align*}
\cdot \left( (\mu_B)^n |O_B ; A > |O_M ; X_{n-1} > |\mathcal{E}_{n-1} ; B > |\mathcal{E}_{n-2} ; B > \cdots |\mathcal{E}_0 ; B > \right) \\
+ |\zeta_n > \\
+ (\mu_M)^n |O_B ; X_{n-1} > |O_M ; A > |\mathcal{E}_{n-1} ; M > |\mathcal{E}_{n-2} ; M > \cdots |\mathcal{E}_0 ; M > \\
\end{align*}
\]

where \( |\zeta_n > \) contains no factors of alive species, i.e., neither \( |O_B ; A > \) nor \( |O_M ; A > \). If \( (138) \) is true for some \( n \), then, using \( (135) \),

\[
|\Phi, n + 1 > = \left[ \prod_{t=n+1}^T \left( \mu_B |\mathcal{E}_t ; B > + \mu_M |\mathcal{E}_t ; M > \right) \right] \tilde{U}_n \left\{ \left( |\mu_B \mathcal{E}_n ; B > + \mu_M |\mathcal{E}_n ; M > \right) \right\} (139)
\]

\[
\cdot \left( (\mu_B)^n |O_B ; A > |O_M ; X_{n-1} > |\mathcal{E}_{n-1} ; B > |\mathcal{E}_{n-2} ; B > \cdots |\mathcal{E}_0 ; B > \right) \\
+ |\zeta_n > \\
+ (\mu_M)^n |O_B ; X_{n-1} > |O_M ; A > |\mathcal{E}_{n-1} ; M > |\mathcal{E}_{n-2} ; M > \cdots |\mathcal{E}_0 ; M > \\
\right\}
\]

and, using \( (133) \),

\[
|\Phi, n + 1 > = \left[ \prod_{t=n+1}^T \left( \mu_B |\mathcal{E}_t ; B > + \mu_M |\mathcal{E}_t ; M > \right) \right]
\]

\[
\left\{ (\mu_B)^{n+1} |O_B ; A > |O_M ; X_n > |\mathcal{E}_n ; B > |\mathcal{E}_{n-1} ; B > |\mathcal{E}_{n-2} ; B > \cdots |\mathcal{E}_0 ; B > \\
+ (\mu_B)^n \mu_M |O_B ; X_0 > |O_M ; X_n > |\mathcal{E}_n ; M > |\mathcal{E}_{n-1} ; B > |\mathcal{E}_{n-2} ; B > \cdots |\mathcal{E}_0 ; B > \\
+ \tilde{U}_n \left[ \zeta_n \left( \mu_B |\mathcal{E}_n ; B > + \mu_M |\mathcal{E}_n ; M > \right) \right] \\
+ (\mu_M)^n \mu_B |O_B ; X_n > |O_M ; X_0 > |\mathcal{E}_n ; B > |\mathcal{E}_{n-1} ; M > |\mathcal{E}_{n-2} ; M > \cdots |\mathcal{E}_0 ; M > \\
+ (\mu_M)^{n+1} |O_B ; X_n > |O_M ; A > |\mathcal{E}_n ; M > |\mathcal{E}_{n-1} ; M > |\mathcal{E}_{n-2} ; M > \cdots |\mathcal{E}_0 ; M > \right\}
\]

By the induction hypothesis, \( |\zeta_n > \) contains no factors of \( |O_B ; A > \) or \( |O_B ; M > \) and, as discussed above, \( n < \Omega - 1 \), allowing us to use \( (133) \) rather than the full action \( (129) \) of \( \tilde{U}_t \). It follows that \( \tilde{U}_n [\zeta_n (|\mathcal{E}_n ; B > + |\mathcal{E}_n ; M >)] \) also contains no factors of \( |O_B ; A > \) or \( |O_B ; M > \), nor does \( |\zeta_{n+1} > \) defined as

\[
|\zeta_{n+1} > = (\mu_B)^n \mu_M |O_B ; X_0 > |O_M ; X_n > |\mathcal{E}_n ; M > |\mathcal{E}_{n-1} ; B > |\mathcal{E}_{n-2} ; B > \cdots |\mathcal{E}_0 ; B > (141)
\]

\[
+ \tilde{U}_n \left[ \zeta_n \left( \mu_B |\mathcal{E}_n ; B > + \mu_M |\mathcal{E}_n ; M > \right) \right] \\
+ (\mu_M)^n \mu_B |O_B ; X_n > |O_M ; X_0 > |\mathcal{E}_n ; B > |\mathcal{E}_{n-1} ; M > |\mathcal{E}_{n-2} ; M > \cdots |\mathcal{E}_0 ; M > .
\]
So, using (141) to write (140) as

\[ |\Phi, n + 1 \rangle = \left[ \prod_{t=n+1}^{T} \left( \mu_B |E_t; B \rangle + \mu_M |E_t; M \rangle \right) \right] \]

and comparing with (138), we see that the induction hypothesis is true for \( n + 1 \) if it is true for \( n \) (since \(|\zeta_{n+1}\rangle\) satisfies the same conditions as required for \(|\zeta_n\rangle\)) and so, with (136) and (137), is true for all \( n \).

For \( n > 0 \) there is one branch of the Schrödinger-picture state (138) in which the species \( O_B \) is alive and \( O_M \) is extinct, and another distinct branch in which \( O_M \) is alive and \( O_B \) is extinct. In all other branches neither species is alive. The weights for \( O_B \) and \( O_M \) being alive are the respective square norms of these branches:

\[ W_B = |\mu_B|^{2n}, \quad (143) \]
\[ W_M = |\mu_M|^{2n}. \quad (144) \]

Both \(|\mu_B|\) and \(|\mu_M|\) are less than unity, and we’ve specified that \(|\mu_B| > |\mu_M|\). So as the number of generations \( n \) of biological evolution increases, both \( W_B \) and \( W_M \) will decrease, with \( W_M \) dropping below the preclusion threshold first. That is, for at least some period of time the species that is fit to survive in a higher-weight ("more probable") environment will be present while the species fit to survive in a lower-weight ("less probable") environment will be extinct.

### 6.3 Born-like and maverick states

If we now base a model of the environment on the same collection of \( N \) spins described in Sec. 2 we can show that the sum of branches with relative frequencies for spin-up and spin-down near the Born-rule values will constitute a state with norm increasing towards unity as \( N \) becomes large, while the remaining "maverick" branches, i.e., those with relative frequencies far from the Born-rule values, will constitute a state with norm approaching zero for large \( N \). We can therefore identify the respective sums at time \( t \) of Born-like branches and maverick branches with \( \mu_B |E_t; B \rangle \) and \( \mu_M |E_t; M \rangle \) of Sec. 6.2 since these will satisfy the conditions \( \langle E_t; B |E_t; M \rangle = 0 \) and \(|\mu_B| > |\mu_M|\). Applying the results of Sec. 6.2 we conclude that species that thrive in a Born-like environment (for purposes of
visualization, consider an organism that eats spins and must consume up-spins and down-spins in a ratio close to \( p : q \) will persist in the presence of a preclusion threshold that would lead organisms requiring a far-from-Born-rule environment to become extinct.

Referring to (1)-(6), define a state composed of branches with relative frequencies near the Born-rule values,

\[
|\tilde{E}_t; B\rangle = \sum_{i_1=1}^{2} \cdots \sum_{i_N=1}^{2} |B_{i_1,\ldots,i_N}\rangle \\
\theta (r_1(i_1,\ldots,i_N) - N(p - \delta)) \theta (N(p + \delta) - r_1(i_1,\ldots,i_N)),
\]

(145)

and a state composed of the remaining branches with relative frequencies farther from the Born-rule values,

\[
|\tilde{E}_t; M\rangle = |\Psi\rangle - |\tilde{E}_t; B\rangle \\
= \sum_{i_1=1}^{2} \cdots \sum_{i_N=1}^{2} |B_{i_1,\ldots,i_N}\rangle \\
\left[ \bar{\theta} (N(p - \delta) - r_1(i_1,\ldots,i_N)) + \bar{\theta} (r_1(i_1,\ldots,i_N) - N(p + \delta)) \right],
\]

(146)

where

\[0 < \delta < \min(p, q),\]

(147)

and

\[
\theta(x) = \begin{cases} 
0, & x < 0 \\
1, & x = 0 \\
1, & x > 0,
\end{cases}
\]

(148)

\[
\bar{\theta}(x) = \begin{cases} 
0, & x < 0, \\
0, & x = 0, \\
1, & x > 0,
\end{cases}
\]

so

\[
(\theta(x))^2 = \theta(x),
\]

(149)

\[
(\bar{\theta}(x))^2 = \bar{\theta}(x),
\]

(150)

\[\text{We will be identifying } |\tilde{E}_t; B\rangle, |\tilde{E}_t; M\rangle, \text{ up to norm, with } |E_t; B\rangle, |E_t; M\rangle; \text{ see } (165). \text{ As discussed in Sec. 6.2, } |\tilde{E}_t\rangle \text{ represents a physical object distinct from } |\tilde{E}_t\rangle, t \neq t', \text{ so strictly speaking we should write in the rest of this section } |B_{i_1,\ldots,i_N}\rangle_t, \text{ where } |B_{i_1,\ldots,i_N}\rangle_t \text{ is defined by } (14) \text{ with } |S^{(p)}; \alpha^{(p)}\rangle \text{ replaced by } |S^{(p)}; \alpha^{(p)}\rangle_t.\]
\[ \theta(x) + \tilde{\theta}(-x) = 1, \]
\[ \theta(x)\tilde{\theta}(-x) = 0. \]  

From (4),
\[
\langle B_{j(1), \ldots, j(N)} | B_{i(1), \ldots, i(N)} \rangle = |c_1|^{2r_1(i(1), \ldots, i(N))} |c_2|^{2r_2(i(1), \ldots, i(N))} \prod_{p=1}^{N} \delta_{j(p), i(p)}
\]  

Using (145), (146), (152) and (153), we verify that
\[
\langle \tilde{E}_t; B | \tilde{E}_t; M \rangle = 0.
\]

From (5) we see that \( r_i(i^{(1)}, \ldots, i^{(N)}) \) takes integer values in the range \([0, N]\),
\[ 0 \leq r_i(i^{(1)}, \ldots, i^{(N)}) \leq N. \]

Therefore
\[
\sum_{n=0}^{N} \delta_{n, r_i(i^{(1)}, \ldots, i^{(N)})} = 1.
\]

Using (145), (149) and (153)
\[
\langle \tilde{E}_t; B | \tilde{E}_t; B \rangle = \sum_{i^{(1)}=1}^{2} \cdots \sum_{i^{(N)}=1}^{2} \delta_{n, r_i(i^{(1)}, \ldots, i^{(N)})}
\]
\[
\theta(n - N(p - \delta)) \theta(N(p + \delta) - n) |c_1|^{2n} |c_2|^{2(N-n)}
\]

using (156)
\[
= \sum_{n|N(p-\delta)\leq n\leq N(p+\delta)} \left( \frac{N}{n} \right) |c_1|^{2n} |c_2|^{2(N-n)}. \]
Using (55), rewrite this as

\[
\langle \tilde{E}_t; B | \tilde{E}_t; B \rangle = \sum_{n=|\delta \sqrt{Npq} \leq n \leq \sqrt{Npq} \leq \delta \sqrt{Npq}} \binom{N}{n} p^n q^{N-n}
\]

(161)

Following the approach of Sec. 4, we use (66), (69) and (71) in (161) to obtain

\[
\langle \tilde{E}_t; B | \tilde{E}_t; B \rangle > 1 - \epsilon, \quad N > \max \left( N_D(\epsilon_D, A), \frac{A^2 pq}{\delta^2} \right), \quad A > A_e(\epsilon_e), \quad \epsilon > \epsilon_D + \epsilon_e,
\]

\[
\epsilon_D, \epsilon_e, \epsilon > 0
\]

(162)

From (145), (146) and (154)

\[
\langle \tilde{E}_t; B | \tilde{E}_t; B \rangle + \langle \tilde{E}_t; M | \tilde{E}_t; M \rangle = \langle \Psi | \Psi \rangle = 1.
\]

(163)

Using (162) and (163),

\[
\langle \tilde{E}_t; M | \tilde{E}_t; M \rangle < \epsilon, \quad N > \max \left( N_D(\epsilon_D, A), \frac{A^2 pq}{\delta^2} \right), \quad A > A_e(\epsilon_e), \quad \epsilon > \epsilon_D + \epsilon_e,
\]

\[
\epsilon_D, \epsilon_e, \epsilon > 0
\]

(164)

So, if we make the identifications, for all t,

\[
| \tilde{E}_t; B \rangle = \mu_B | E_t; B \rangle \quad \text{and} \quad | \tilde{E}_t; M \rangle = \mu_M | E_t; M \rangle
\]

(165)

we see from (126), (162) and (164) that (126), (127) and (128) are satisfied for any \( \delta \) satisfying (147) provided \( N \) is large enough so that \( \epsilon < 1/2 \).

The results of Sec. 6.2 therefore apply, and we can conclude that species fit to survive in environments \( | E_t; B \rangle \) with relative frequencies near the Born-rule value—and one aspect of fitness to survive in an environment can be possessing a sense of subjective probability consistent with that environment—will have an advantage over species fit to survive in maverick environments \( | E_t; M \rangle \).

\[\text{17}\]

The reader may object that in focusing on the states defined in (145) and (146) we are performing the same sort of unjustified grouping that we objected to in Sec. 2. It seems clear that this objection would be dispelled by an analysis in the Heisenberg picture, where the focus would be on the operators representing \( \mathcal{O}_B \) and \( \mathcal{O}_M \). The “preferences” of \( \mathcal{O}_B \) and \( \mathcal{O}_M \) for different ranges of ratios of up-spins and down-spins, i.e., the different transformations induced by environments with different ratios, would group the branches in (145) and (146) just as in Sec. 2 the different transformations of \( \mathcal{F} \) due to different ratios of observers’ perceptions leads to a grouping, with the parameter \( \delta \) playing a role analogous to that played by \( 1/\nu \).
6.4 Experimental evidence for the biological evolution of subjective judgements of probability of ordinary probabilistic events

Is there evidence that subjective probability is in fact a product of evolution and is to some extent innate to individual organisms? A substantial body of research would seem to support this idea.

Human infants less than a year in age show surprise at the occurrence of improbable events, for example the outcomes of lottery-like games. “…12-month-olds have rational expectations about the future based on estimations of event possibilities, without the need of sampling past experiences. …Our results suggest that at the onset of human decision processes the mind contains an intuition of elementary probability that cannot be reduced to the encountered frequency of events or elementary heuristics. …We presented movies in which three identical objects and one different in color and shape bounced randomly inside a container with an open pipe at its base, as in a lottery game …[Subsequently], an occluder hid the container and one object, either one of the three identical objects (probable outcome) or else the different one (improbable outcome), exited from the pipe. …Infants had no information about frequency distributions of actual outcomes, so their reactions could not be primed by previous experience. …Despite the complexity of the task and the lack of habituation, infants looked significantly longer when they witnessed the improbable outcome. …This result suggests that infants do not need to experience outcome frequency to respond to probabilities. …These experiments show that just as infants expect that future events will respect physical constraints …they also expect that in the future the most likely outcome will occur. Because no frequency information about actual outcomes was provided, these expectations are grounded on intuitions about single event probabilities based on future possibilities. Undoubtedly, infants respond to frequencies …. However, our experiments show that the origin of the concept of probability cannot be reduced to experiencing frequencies [70].”

Results in a similar experiment “suggest that not only were infants surprised at improbable outcomes, but also that they anticipated the occurrence of a probable future event, even without any previous experience with them. …Infants can reason about single-case probabilities without previous experience with the outcomes of the scenes as early as 12 months of age. This result adds compelling further evidence to the finding that single-case probabilities are meaningful for young humans when they can represent and track possible outcomes …A well-known position in evolutionary psychology holds that humans can understand probabilities only as collections of experienced events …According to this perspective, our intuitions about the future are entirely dependent on our experience of the past. Instead, we showed that infants have no difficulty in reacting to single-case probabilities despite having no information about the past frequencies of the outcomes [71].”
Infants also show surprise at mismatch between the statistical properties of populations of items with the statistical properties of samples randomly drawn from those populations. They can use the observed statistics of populations of items to guide actions, and are sensitive to the presence or absence of random sampling. “Human learners make inductive inferences based on small amounts of data: we generalize from samples to populations and vice versa. The academic discipline of statistics formalizes these intuitive statistical inferences. What is the origin of this ability? We report six experiments investigating whether 8-month-old infants are ‘intuitive statisticians.’ Our results showed that, given a sample, the infants were able to make inferences about the population from which the sample had been drawn. Conversely, given information about the entire population of relatively small size, the infants were able to make predictions about the sample.

“One important assumption in a statistical inference task, using probability or heuristics, is the assumption of random sampling. It is only under conditions of random sampling that the inference is warranted. If the learner has evidence that she is not receiving a random sample from the population, she can no longer use the statistical information in the sample to make guesses about the overall population.

“Recent studies in our laboratory suggest that 11-month-old infants are sensitive to sampling conditions. . . . Infants were given evidence that the random sampling assumption had been violated. The experimenter first expressed a preference for one type of ping-pong balls, say red ones, by showing the infant a small container with both types of ping-pong balls (red and white) and selectively picking up only the red ones and placing them in another container. On the test trials, the experimenter looked into the box while pulling out the ping-pong balls, i.e., she had visual access to the content of the box. If infants were sensitive to the fact that the random sampling assumption had been violated, their looking times should no longer be predicted by which sample was more probable given the content of the box. Instead their looking times should be predicted by whether the sample was consistent with the experimenter’s preference or not. That was exactly what we found; infants looked longer when the experimenter pulled out a sample that was inconsistent with her expressed preference, regardless of the content of the box. In the third condition, the experimenter expressed a preference but she was blindfolded during the sampling process. The infants were able to integrate these two sources of information, and their looking times were once again predicted by the content of the box (F. Xu and S. Denison, unpublished data). These results suggest that infants engaged in a rather sophisticated form of statistical inference in this task, and their looking time patterns on the test trials were not simply a matter of matching the sample to the overall population in terms of distribution.

“The present studies provide evidence that early in development infants are able to use a powerful statistical inference mechanism for inductive learning. They can make generalizations about a population based on a sample, and conversely, they can make predictions about a sample given information about a population. This ability for performing intu-
itive statistics develops early and in the absence of schooling or explicit teaching. It may be the roots of later acquisition of statistical principles, in both the course of developing an understanding of scientific inquiry and learning about probabilistic reasoning and statistics [72].

Similar results have been obtained in studies of nonhuman primates; see, e.g., [73–76]. For a recent review of research in this area, see [77].

7 Summary

In brief: Hard preclusion, applied to the Heisenberg-picture ontology of Everett quantum mechanics, explains the nonobservation of extraordinary probabilistic phenomena, e.g., macroscopic entropy decrease. The evolution of organisms fit to maverick environments, i.e., environments with statistics far from Born-rule values, is an extraordinary probabilistic phenomenon and is precluded; nonprecluded species evolve subjective judgements of probability consistent with Born-rule values.

In more detail:

1. We show that hard preclusion cannot account for the nonobservation of extraordinary probabilistic phenomena (Sec. 1) if the ontology of the Everett interpretation implied by the Schrödinger picture is employed (Sec. 2).

2. We show that, if the Everett-interpretation ontology implied by the Heisenberg picture (Sec. 3) is employed, hard preclusion can account in a straightforward and objective manner for the nonobservation of extraordinary probabilistic phenomena. As a model of extraordinary probabilistic phenomena we analyze an ensemble of spins, demonstrating for sufficiently large ensemble size the preclusion of relative-frequency measurements differing from those given by the Born rule and, as a by-product, extending the results of [54] for the case of infinite ensemble size (Sec. 4). A model showing the consistency of preclusion when applied to observations of contingent events is also presented (Sec. 5).

3. We argue that the long time scales involved in biological evolution render the quantum weights for the survival of species with subjective judgements of probability differing greatly from those given by the Born rule small enough to be subject to preclusion, thus accounting for the perception, by surviving species, of ordinary probabilistic phenomena in a manner consistent with the Principal Principle (Sec. 6.1).

4. A proof-of-concept model of natural selection relying on hard preclusion rather than objective probability is presented (Sec. 6.2). A specialization of this model to two species existing in a spin-ensemble environment shows that species that thrive in
Born-like states can survive preclusion thresholds that drive species that thrive in maverick states to extinction (Sec. 6.3).

5. We review experimental evidence that subjective probability is innate, a product of biological evolution (Sec. 6.4).

8 Discussion

Theories of probability have been classified as either “objective” or “subjective” [78]. The present theory is an objective theory in that it is based on the objective existence of both quantum weights and the preclusion threshold, both features of the physical world. In particular it is a matter of objective fact whether or not the weight for a quantum event is so small that the event is precluded. The probabilities that are assigned to ordinary probabilistic phenomena, however, reside purely in the minds of organisms performing probabilistic judgments, e.g., the infants in the experiments described in Sec. 6.4. As already pointed out, such judgements are themselves objective phenomena. “The subjective opinion, as something known by the individual under consideration is, at least in this sense, something objective and can be a reasonable subject of a rigorous study [78, p. 5].” These subjective opinions, in turn, result from the objective process of biological evolution based on quantum mechanics with preclusion.

The idea that subjective probability is at least to some degree innate and a product of evolution has been entertained by at least one subjectivist. Koopmans writes that “probability derives directly from the intuition … probability as well as logic may be derived by race experience through the process of evolution [79].” But, in the usual description, biological evolution is a process guided by objective probability. Without objective probability—or some other objective feature of the world, such as the hard preclusion employed here—no explanation is provided as to why one species survives and another becomes extinct, or why one trait is preserved and another extinguished. This point is discussed further below.

The hard-preclusion theory as presented above has a “two-level ontology:” an underlying level of Heisenberg-picture operators which evolve continuously; and a level of events which either occur or fail to occur as the weights associated with them are respectively above or below the preclusion threshold [30]. In this it is like any other quantum theory in which Heisenberg-picture operators correspond to events [54, 56, 58, 80, 88], except that in the usual theories the preclusion threshold is (implicitly) set to a value of

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18The theory thus calls to mind Heisenberg’s “potentia” [89, 90]. The place of this notion in quantum mechanics has been explored extensively (see, e.g., [91, 99]) but, to the best of my knowledge, always in terms of a single possibility being actualized stochastically. In the present theory, all possibilities with weights above the preclusion threshold are actualized deterministically.
Combining the two levels into one may be possible by introducing changes to the Heisenberg-picture equations of motion for the operators, e.g., a Heisenberg-picture version of the Schrödinger-picture “snap-to” rule proposed in BHZ. Whether or not this is done, however, hard preclusion is not simply an interpretation of quantum mechanics but a modification to it.

Now, one goal of the Everett program has been “to take the mathematical formalism of quantum mechanics as it stands without adding anything to it.” If we find it necessary to modify the formalism, is the effort worth it? The answer to that question remains as simple and compelling as ever: The Everett interpretation, by virtue of eliminating the insistence on a single outcome to quantum measurement, avoids the nonlocality that Bell’s theorem assures us must be present in any single-outcome quantum theory.

The Everett interpretation thereby being an eminently worthwhile enterprise, it is all the more important to make as clear as possible the reasons for incorporating preclusion into Everett quantum mechanics. There are two motivations for believing that preclusion is in fact a real physical process.

The first of these motivations is the one which has been guiding our presentation throughout this paper: Everett quantum mechanics with preclusion provides an explicitly-specified, noncircular explanation for probabilistic phenomena. To recapitulate, events with quantum-mechanical weight below the preclusion threshold do not occur. In particular, events corresponding to macroscopic entropy decrease do not occur. But in addition the biological evolution of organisms fit to non-Born-rule “maverick” environments does not occur, explaining why those organisms that do evolve possess traits, including the trait of subjective probability, fit to Born-rule environments. The full gamut of probabilistic phenomena, nonobservation of extraordinary probabilistic phenomena as well as subjective judgements of ordinary probabilistic phenomena, is explained.

Deutsch has stated: “A scientific explanation is a statement of what is there in reality, and how it behaves and how that accounts for the explicanda.” We agree with this idea of the meaning of explanation, from which it follows that a necessary condition for something to be the explanation of a physical phenomenon is that the thing that provides the explanation is something that exists objectively. From this viewpoint, let us briefly examine some prominent approaches to Everettian probability.

- In the Deutsch-Wallace decision-theoretic approach, the existence of decision-making agents satisfying certain nonprobabilistic conditions of rationality is
assumed. The laws of quantum mechanics are then argued to constrain the subjective judgements of the agents to act as if they were maximizing probabilistic expectation values. But rational agents come about through biological evolution, and without objective probability or some surrogate such as preclusion the fact that the decision-making agents exist in the first place is unexplained. If we take as given the existence of the organisms and that they follow the specified rules of rationality then the values that these organisms attach to decisions are shown to be constrained to be consistent with the Born rule, so this line of reasoning may play a role in showing how the innate “ability for performing intuitive statistics,” developed through biological evolution, becomes the “roots of later acquisition of statistical principles.”

- Zurek’s envariance approach uses a symmetry in the physics of entangled quantum states of two systems to argue that a property pertaining to just one of the systems and associated with basis states of the system is constrained to have values given by the Born rule. This is of course suggestive of and consistent with the property so constrained indeed being objective probability given by the Born rule. Now objective probability plays certain roles; e.g., the smallness of the objective probability of an event is sufficient to explain why this event does not occur, or occurs rarely. But, in the envariance approach, no argument is given that the property associated with a particular basis state being small should be taken to mean that an observer can discount the possibility of the system existing in that state.

- Vaidman has introduced an approach based on the idea of ignorance-based or self-locating uncertainty, the uncertainty experienced by an experimenter after she has performed a quantum experiment that leads to Everett splitting but before she has become aware of the result. A subjective probability is associated with this uncertainty. McQueen and Vaidman present a thought experiment and symmetry arguments, distinct from those employed in either the decision-theory

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22Wallace argues that objective as well as subjective probabilities are present in his version of the decision-theoretic approach, but this claim is questioned by Brown and Ben Porath: “If agent-independent chances existed, then there would be more to Everettian ontology than just the universal wavefunction, contrary to the aims of the approach... Note that the probabilities in the argument are credences...” Even if the approach somehow gave rise to something we would consider to be objective probability, it would only do so in a circular fashion.

23This in fact seem to be the approach Deutsch is taking. “[No] analysis from within physics [could] ever decide what is rational, or what is within the scope of reason.” We would argue that rational behavior, being a type of behavior leading to the survival of the species, could come about through biological evolution guided by preclusion, along the lines of the subjective-probabilistic behavior displayed by the subjects of the experiments described in Sec. 6.4.

24Kolmogorov: “We apply the theory of probability to the actual world of experiments in the following manner... If P(A) is very small, one can be practically certain that when conditions for a repeatable experiment are realized only once, the event A would not occur at all.”
or envariance approaches, to argue that this subjective probability must have the Born-rule value.\textsuperscript{25} Again, because only subjective probability exists in the theory, it shares with the decision-theory and envariance approaches the inability to provide explanations for physical phenomena.\textsuperscript{26}

On the other hand, to the extent that preclusion is taken to be an objective physical process, Everett quantum mechanics with preclusion can provide an explanation for physical phenomena, including but not limited to the biological evolution that may be the basis of subjective probability. (The fact that preclusion involves the quantum-mechanical weight, i.e., has the Born-rule value, is of course an \textit{input}, whereas the other theories described above present arguments \textit{deriving} the relation between subjective probability and the Born-rule value.)

The theory thus resolves the interpretational issues facing probability\textsuperscript{131} (with or without any connection to quantum mechanics with any number of outcomes). It makes use of the strength of subjective interpretations of probability: regarding ordinary probabilistic phenomena, there is no question about “what kinds of things are probabilities \textsuperscript{131},” they are states in the brains of organisms. But its foundation of objective preclusion provides what purely subjective interpretations lack, the ability to explain physical phenomena including the evolution of subjective probability—in particular, what I termed above (Sec. 6.1) the “empirical Principal Principle,” the connection between subjective probability and physical law as embodied by the Born rule.

What Everett quantum mechanics with preclusion lacks is a \textit{normative} component. The consistency of the reactions of infants to the outcomes of lottery-like games with the expected statistics of those outcomes may indeed be explained by the theory. But were a precocious infant to ask “Why \textit{should} I react this way? I thought probability is supposed to be ‘applicable,’ a ‘guide to life’ \textsuperscript{131},” we could only respond by saying “You evolved that way.” However, as pointed out by Papineau\textsuperscript{18, Secs. 6-9}, other interpretations of objective probability, both propensity-based and frequency-based, are equally incapable of answering this question of “why.” It may be possible to add to the theory, from the outside, a normative element. Alternatively, it may not be the case that “the inability...to provide this sort of [normative] explanation is a deficiency. Rather it may simply be the elimination of an illusion which, like absolute simultaneity, has no objective correlate in the physical world \textsuperscript{30}.”

\textsuperscript{25}Kent\textsuperscript{130} and McQueen and Vaidman\textsuperscript{129} criticize the related approach of Sebens and Carroll\textsuperscript{27} on the grounds that the latter consider Everett copying of an observer to occur even before the observer has interacted with a device measuring a quantum system in a superposition. We note that the viewpoint taken by Sebens and Carroll is contrary to that motivated by the Heisenberg-picture Everett ontology. See e.g. the discussion of “Siamese \((\nu + 1)-\text{tuplets}\)” in\textsuperscript{54} Sec. 5.

\textsuperscript{26}And shares with them as well the potential relevance for further investigations of how more sophisticated subjective evaluations of ordinary probabilistic phenomena develop starting from their primitive innate roots.
Finally, as alluded to above, there is a second motivation for believing in the existence of preclusion, one completely unrelated to any role that may be claimed for preclusion as a desideratum for addressing issues of probability in the Everett interpretation. It has been argued for decades, using a variety of approaches, that quantum mechanics and relativity taken together imply the existence of a minimum spatial length scale at or near the Planck length (for a review see [132]). Buniy, Hsu and Zee [133] argue that the existence of a minimum spatial length leads in turn to the existence of a minimum length for state vectors in quantum-mechanical Hilbert space\(^{27}\)—that is, to preclusion of states with norm below some minimum value.

So, whether or not we find preclusion desirable for its ability to explain probability in Everett quantum mechanics, preclusion may well be present in quantum physics desired or not. Given its presence, it will play a role vis-à-vis probabilistic phenomena, as investigated in BHZ, [30], [134] and the present paper. Probability may thus be a telltale sign of quantum gravity making its effect known in our daily lives.

Acknowledgments

I would like to thank Jianbin Mao, Jacob A. Rubin and Allen J. Tino for helpful discussions, and Rainer Plaga and Michael Price for comments on an earlier paper on this subject. I am particularly grateful to Michael Weissman for pointing out to me the possibility of modifying the rule linking mathematical formalism to probability while retaining unchanged the structure of the underlying formalism.

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\(^{27}\)We note that arguments that lead to this conclusion may imply a more complicated rule for preclusion than the one employed in this paper, with a fixed \(\epsilon_1\) in all situations. For example, a minimum length of the Planck length \(l_P\) suggests that a device of size \(d\) can measure angles no smaller than \(\theta_{\text{min}} \approx l_P/d\). The change in the state vector of a qubit caused by a rotation, which change is itself a state vector, should be precluded if the magnitude of the rotation is smaller than \(\theta_{\text{min}}\). Performing a \(\theta_{\text{min}}\) rotation about the \(y\)-axis of a state \(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\) spin-up along the \(z\)-axis, and assuming \(d \gg l_P\) so \(\theta_{\text{min}} \ll 1\), the norm squared of the difference between the original and rotated states is approximately

\[
\left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \left(1 - \frac{i}{2} \theta_{\text{min}} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|^2 = \left(\frac{l_P}{2d}\right)^2.
\]
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