ON THE CHARACTER RING OF A QUASIREDUCTIVE LIE SUPERALGEBRA

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Abstract. We study character rings of quasireductive Lie superalgebras and give a new proof of the Sergeev-Veselov theorem describing the character rings of finite-dimensional Kac-Moody superalgebras.

Order is the key to all problems.
Alexandre Dumas,
The Count of Monte Cristo

0. Introduction

One of the classical results of the representation theory of complex semisimple Lie algebras can be formulated as follows (see e.g. [14]):

The character map induces an isomorphism between the representation ring of finite-dimensional modules of a complex semisimple Lie algebra \( \mathfrak{g} \) and \( W \)-invariants in the group ring \( \mathbb{Z}[P_0] \) where \( W \) is the Weyl group and \( P_0 \) is the weight lattice of \( \mathfrak{g} \).

The “representation ring” \( \mathcal{K}(\mathfrak{g}) \) of a Lie algebra \( \mathfrak{g} \) is, by definition, the Grothendieck group of the category of finite-dimensional modules, that is the quotient of a free abelian group with generators given by all isomorphism classes of finite-dimensional \( \mathfrak{g} \)-modules by the subgroup generated by \( [N] - [M] - [N/M] \) for each module \( N \) and its submodule \( M \). We will use the same definition for Lie superalgebras and denote by \( \mathcal{K}_\pm(\mathfrak{g}) \) the quotient of \( \mathcal{K}(\mathfrak{g}) \) by the relations \( [N] = \pm [\Pi N] \), where \( \Pi \) stands for the parity change functor. In [14] Sergeev and Veselov described the ring \( \mathcal{K}_-(\mathfrak{g}) \) for the finite-dimensional Kac-Moody superalgebras and some related algebras. For \( Q \)-type superalgebras Reif described the subring corresponding to integral and half-integral weights of \( \mathcal{K}_+(\mathfrak{g}) \) in [11].

Let \( \mathfrak{g} \) be a quasireductive Lie superalgebra (see [13, 8]): this is a finite-dimensional Lie superalgebra with a reductive even part \( \mathfrak{g}_{\mathfrak{ev}} \) which acts semisimply on the odd part \( \mathfrak{g}_{\mathfrak{od}} \). Let \( \mathfrak{t} \) be a Cartan subalgebra of \( \mathfrak{g}_{\mathfrak{ev}} \) and \( \mathfrak{h} := \mathfrak{g}_{\mathfrak{od}} \) be a Cartan subalgebra of \( \mathfrak{g} \). If \( \mathfrak{g}' \) is a subalgebra of \( \mathfrak{g} \), the restriction functor \( \text{Res}_{\mathfrak{g}'}^\mathfrak{g} \) induces the ring homomorphisms \( \text{res}_{\mathfrak{g}'}^\mathfrak{g} \) from \( \mathcal{K}(\mathfrak{g}) \) to \( \mathcal{K}(\mathfrak{g}') \) and from \( \mathcal{K}_\pm(\mathfrak{g}) \) to \( \mathcal{K}_\pm(\mathfrak{g}') \). All three homomorphisms \( \text{res}_{\mathfrak{g}'}^\mathfrak{g} \) are injective as well as the homomorphism \( \text{res}_{\mathfrak{g}'}^\mathfrak{g} : \mathcal{K}_+(\mathfrak{g}) \to \mathcal{K}_+(\mathfrak{t}) \). The ring \( \mathcal{K}(\mathfrak{t}) \) can be naturally

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identified with the ring $\mathbb{Z}[t^*] \otimes \mathbb{Z}[\xi]$ where $\mathbb{Z}[t^*]$ is the group ring of $t^*$ and $\xi$ a formal variable satisfying $\xi^2 = 1$. In this paper we study the image of $\text{res}_g^h : \mathcal{K}(g) \to \mathcal{K}(t)$, which we denote by $\text{Ch}(g)$ and the images of $\text{res}_g^h : \mathcal{K}_\pm(g) \to \mathcal{K}(t)$, which we denote by $\text{Ch}_\pm(g)$. The image of $\mathcal{K}(g)$ in $\mathcal{K}(h)$ is studied in [13]. By above, $\mathcal{K}_+(g) \cong \text{Ch}_+(g)$; if $h = t$, then $\mathcal{K}_-(g) \cong \text{Ch}_-(g)$ (this holds in the cases studied by Sergeev and Veselov in [13]).

We fix a triangular decomposition of $g_\mathbb{T}$ and denote by $\pi$ the set of simple roots, by $P^+(\pi)$ the set of $g_\mathbb{T}$-dominant weights and by $P_0$ the $\mathbb{Z}$-span of $P^+(\pi)$. Let $W \subset GL(t^*)$ be the Weyl group. We set $\mathbb{Z}[P_0; \xi] := \mathbb{Z}[P_0] \otimes \mathbb{Z}[\xi]$. Since all weights of finite-dimensional $g$-module lie in $P_0$, $\text{Ch}(g)$ is a subring in the ring $R(P_0) := \text{Ch}(h) \cap \mathbb{Z}[P_0; \xi]$ (one has $R(P_0) = \mathbb{Z}[P_0; \xi]$ if $h = t$).

The $W$-action $we^\lambda := e^{w\lambda}$ naturally extends to the $W$-action on $R(P_0)$.

We denote by $\Delta$ the root system of $g$ and by $\Delta_{\text{iso}}$ the set of roots $\beta$ such that for some $e_\pm \in g_{\pm\beta}$ the subalgebra spanned by $e_+, h_\beta := [e_+, e_-]$ is isomorphic to $sl(1|1)$. Recall that any element in $R(P_0) \subset \mathbb{Z}[P_0; \xi]$ takes the form $\sum_\nu m_\nu e^\nu$ with $m_\nu \in \mathbb{Z}[[\xi]]$. We set

$$A(g) := \bigcap_{\nu \in \Delta_{\text{iso}}} \{ \sum_\nu m_\nu e^\nu \in R(P_0)_W \mid \langle \nu, h_\beta \rangle \neq 0 \implies \sum_{i \in \mathbb{Z}} (-\xi)^im_{\nu+i\beta} = 0 \}.$$ 

(1)

Considering the homomorphisms $\text{res}_g^{\mathfrak{gl}(1|1)} : \mathcal{K}(g) \to \mathcal{K}(\mathfrak{gl}(1|1))$ we obtain

$$\text{Ch}(g) \subset A(g).$$

In this paper we show that $\text{Ch}(g) = A(g)$ if $g$ is a finite-dimensional Kac-Moody superalgebra or a $Q$-type superalgebra. From [13] it follows that $\text{Ch}(g) \neq A(g)$ for $g = \mathfrak{psl}(2|2)$. In addition, we show that if $g$ is a $Q$-type superalgebra or a finite-dimensional Kac-Moody superalgebra $g \neq \mathfrak{gl}(1|1)$, then the ring $\text{Ch}(g)$ admits a “short” $\mathbb{Z}$-basis, see [14,1] for definition. If $g$ is semisimple, this short basis is $\{ b_\lambda, \xi b_\lambda \}_{\lambda \in P^+(\pi)}$ where $b_\lambda = \sum_\nu \omega_{\nu \lambda} e^\nu$.

The formula $\text{Ch}(g) = A(g)$ implies that $\text{Ch}_\pm(g) = A_\pm(g)$, where $A_\pm(g)$ are the images of $A(g)$ under the evaluations $\xi \mapsto \mp 1$. The formula $\text{Ch}_-(g) = A_- (g)$ is equivalent to the Sergeev-Veselov formula.

The formula $\text{Ch}(g) = A(g)$ for the finite-dimensional Kac-Moody superalgebras (resp., $Q$-type) can be deduced from the results of [13] (resp., [11]), but we give an alternative proof of these results. By contrast to the proof in [13], our proof does not require knowledge of any characters, but is based on the existence of “short bases”. Our main tool is the standard partial order on $t^*$. Instead of the distinguished base of simple roots used in [13] we use so-called “mixed bases”. The distinguished bases contain at most one odd root whereas the mixed bases contain the maximal possible number of odd roots. The mixed bases are useful for Kac-Wakimoto character formulae. For the $\mathfrak{osp}$-case the mixed bases were used for the description of characters in [3]. For $g \neq \mathfrak{gl}(n|n)$ our proof is based on the following simple lemma.

**Lemma.** Let $g$ be a quasireductive Lie superalgebra with a base $\Sigma$ satisfying the following properties:
Then $\text{Ch}(g)$ admits a “short basis” and $\text{Ch}(g) = \mathcal{A}(g)$.

It is easy to see that (Pr1) holds if $\Delta^+$ and $-\Delta^+$ are $W$-conjugated. The property (Pr2) looks rather technical, but, in fact, admits a nice interpretation in terms of odd reflections; each finite-dimensional Kac-Moody superalgebra admits bases satisfying

\[
\langle \eta, h_\beta \rangle \neq 0 \quad \text{and} \quad \eta - \beta \not\in P^+(\pi).
\]

which obviously implies (Pr2). The mixed triangular decompositions satisfies both (Pr1) and (Pr2) for $g \neq \mathfrak{gl}(n|n)$. For the $Q$-type superalgebras all triangular decompositions satisfy (Pr1), (Pr2). This establishes $\text{Ch}(g) = \mathcal{A}(g)$ for the $Q$-type superalgebras and the finite-dimensional Kac-Moody superalgebras $g \neq \mathfrak{gl}(n|n)$. For $\mathfrak{gl}(n|n)$ we take a triangular decompostion satisfying (Pr1) and a weaker version of (Pr2).

Contents of the paper. In Section 2 we collect some general results about the Grothendieck rings $K(g)$ and the rings $\text{Ch}(g)$ in the case when $g$ is quasireductive. In particular, we show that $\text{Ch}(g)$ is a subring in $\mathcal{A}(g)$.

In Section 3 we define “short bases”, prove the above lemma and deduce the formula $\text{Ch}(g) = \mathcal{A}(g)$ for the $Q$-type superalgebras and the finite-dimensional Kac-Moody superalgebras.

In Section 4 we describe the rings $\mathcal{A}_\pm(g)$ in the fashions of [14] and of [11]. We deduce the formula $\text{Ch}(p_n) = \mathcal{A}(p_n)$ from the results of [5].

In Appendix we recall the criteria of dominance for the finite-dimensional Kac-Moody superalgebras and obtain the formula (2) in Corollary 5.3.1.

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Index of definitions and notation Throughout the paper the ground field is $\mathbb{C}$; $\mathbb{N}$ stands for the set of non-negative integers. We denote by $\Pi$ the parity change functor.
Throughout the paper the subset $\Delta^+_\pi$ is a quasireductive Lie superalgebra.

1.1. Triangular decompositions. Recall that $g_\pi$ is a reductive Lie algebra. We fix a Cartan subalgebra $t$ in $g_\pi$. We denote by $\Delta_\pi$ the set of roots of $g_\pi$ and by $\Delta$ the set of roots of $g$. We set $h := g^t$. Let $W \subset GL(t^*)$ be the Weyl group.

Given $h \in t$ with the property $\Re \langle \alpha, h \rangle \neq 0$ for each $\alpha \in \Delta$ we have the corresponding subsets of positive roots $\Delta^+_h$ and $\Delta^+_\overline{\alpha}$ (on which $\Re \langle h, \alpha \rangle > 0$). For different choices of $h$ the subsets $\Delta^+_h$ may be different, but they can be transformed to each other by the Weyl group. We will fix one of them, $\Delta^+_h$, and consider only the subsets of positive roots $\Delta^+_h$ in $\Delta$, which contain $\Delta^+_\overline{\alpha}$. This choice fixes a triangular decomposition of $g$, compatible with the triangular decomposition of $g_\pi$, corresponding to $\Delta^+_\overline{\alpha}$.

We denote by $\pi$ the set of simple roots for $\Delta^+_h$. For each $\alpha \in \pi$ we denote by $s_\alpha$ the corresponding reflection in $W$. Recall that $P^+(\pi)$ stands for the set of the highest weights of the finite-dimensional simple $g_\pi$-modules and $P_0$ is the $Z$-span of $P^+(\pi)$ ($P_0$ is a lattice if $g_\pi$ is semisimple).

Our main tool is the standard partial order $> \subset t^*$ given by $\lambda \geq \nu$ if $\lambda - \nu \in N \Delta^+$. For each $U \subset t^*$ we denote by $\max U$ the set of maximal elements in $U$.

A set $\Sigma \subset \Delta^+$ is called a base if the elements of $\Sigma$ are linearly independent and $\Delta^+ \cap \Sigma = \emptyset$ (where $N \Sigma$ is the set of non-negative integral linear combinations of $\Sigma$; in this case we write $\Delta^+ = \Delta^+(\Sigma)$). The root systems of Kac-Moody superalgebras, $q_n$, $p_n$ and $\mathfrak{gl}(m|n)$ admit a base. The compatibility condition ($\mathfrak{n}^+ \subset \mathfrak{n}^+$) means that $\pi \subset N \Sigma$.

1.2. Set $\Delta_{iso}$. We denote by $\Delta_{iso}$ the set of roots $\alpha$ such that for some $e_\pm \in g_{\pm \alpha}$ the subalgebra spanned by $e_\pm, h_{\alpha} := [e_+, e_-]$ is isomorphic to $\mathfrak{sl}(1|1)$. Clearly, $\Delta_{iso} = -\Delta_{iso}$.

Take $\beta \in \Delta_{iso}$. Since $\beta \neq 0$, there exists $h' \in t$ satisfying $\langle \beta, h' \rangle \neq 0$. Taking $\mathfrak{sl}(1|1)$ as above we obtain $\mathfrak{sl}(1|1) + \mathbb{C} h' \cong \mathfrak{gl}(1|1)$.

Observe that for $\mathfrak{sl}(1|1)$ we have $g_\pi = t$ and $h = g$, so $\Delta = \Delta_{iso} = \emptyset$. 

\[ t, \pi, P^+(\pi), P_0, h, \Delta, W, s_\alpha, \partial \]
1.3. **Lemma.** For all \( \lambda \in \mathbb{R}_{\geq 0}P^+(\pi) \), \( w \in W \) we have \( \lambda - w\lambda \in \mathbb{R}_{\geq 0}\pi \). Moreover,

\[
\{ \lambda \in P_0 \mid \forall \alpha \in \pi \; s_\alpha \lambda \leq \lambda \} = \{ \lambda \in P_0 \mid \forall w \in W \; w\lambda \leq \lambda \} = P^+(\pi).
\]

**Proof.** Since \( g_{\mathcal{O}} \) is reductive we have

\[
P_0 = \{ \mu \in t^* \mid \forall \alpha \in \pi \; \mu - s_\alpha \mu \in \mathbb{Z}\alpha \},
\]

\[
P^+(\pi) = \{ \mu \in t^* \mid \forall \alpha \in \pi \; \mu - s_\alpha \mu \in \mathbb{N}\alpha \} = \{ \mu \in t^* \mid \forall w \in W \; \lambda - w\lambda \in \mathbb{N}\pi \}.
\]

This implies \( \lambda - w\lambda \in \mathbb{R}_{\geq 0}\pi \) for \( \lambda \in \mathbb{R}_{\geq 0}P^+(\pi) \), \( w \in W \) and gives

\[
P^+(\pi) \subset \{ \lambda \in P_0 \mid \forall \alpha \in \pi \; s_\alpha \lambda \leq \lambda \} \subset \{ \lambda \in P_0 \mid \forall w \in W \; w\lambda \leq \lambda \}.
\]

Take \( \lambda \in P_0 \) such that \( w\lambda \leq \lambda \) for all \( w \in W \). Then \( s_\alpha \lambda \leq \lambda \) for each \( \alpha \in \pi \). Since \( \lambda \in P_0 \) this gives \( \lambda - s_\alpha \lambda \in \mathbb{N}\pi \). Thus \( \lambda \in P^+(\pi) \). Hence

\[
\{ \lambda \in P_0 \mid \forall w \in W \; w\lambda \leq \lambda \} \subset P^+(\pi).
\]

This completes the proof. \( \square \)

1.4. **Modules** \( L(\lambda) \). Let \( C_\lambda \) be a simple \( \mathfrak{h} \)-module where \( t \) acts by \( \lambda \) and \( s\dim C_\lambda \geq 0 \) (such module is unique up to a parity change \( \Pi \)). We view \( C_\lambda \) as a \( \mathfrak{b} \)-module with the zero action of \( \mathfrak{n} \) and denote by \( L(\lambda) \) a simple quotient of \( M(\lambda) := \text{Ind}_\mathfrak{n}^\mathfrak{b} C_\lambda \) (such simple quotient is unique). Note that \( \dim C_\lambda = 1 \) if \( \mathfrak{h} = t \). We set

\[
P^+(\Delta^+) := \{ \lambda \mid \dim L(\lambda) < \infty \}.
\]

Clearly, \( P^+(\Delta^+) \subset P^+(\pi) \).

1.5. **\( Q \)-type superalgebras.** By \( Q \)-type superalgebras we mean one of the Lie superalgebra \( \mathfrak{q}_n \) and their subquotients \( \mathfrak{sq}_n \), \( \mathfrak{psq}_n \), \( \mathfrak{pq}_n \). These are quasireductive Lie superalgebras: \( g_{\mathcal{O}} = \mathfrak{gl}_n \) for \( \mathfrak{q}_n \), \( \mathfrak{sq}_n \) and \( g_{\mathcal{O}} = \mathfrak{sl}_n \) for \( \mathfrak{psq}_n \), \( \mathfrak{pq}_n \). One has \( \mathfrak{sq}_1 = \mathbb{C} \) and \( \mathfrak{psq}_1 = 0 \); for all other cases \( \mathfrak{h} \neq t \). In this case \( \Delta = \Delta_{\mathcal{O}} = \Delta_{\text{iso}} \) (this set is empty if \( n = 1 \)).

2. **Grothendieck rings**

In this section we collect some general results about the Grothendieck rings \( \mathcal{K}(g) \) and the rings \( \text{Ch}(g) \), \( \mathcal{A}(g) \) in the case when \( g \) is quasireductive. In Proposition 2.7 we show that \( \text{Ch}(g) \) is a subring in \( \mathcal{A}(g) \).

2.1. **Notation.** If \( \mathcal{C} \) is a category of \( g \)-modules, the Grothendieck group \( \mathcal{K}(\mathcal{C}) \) of \( \mathcal{C} \) is a free \( \mathbb{Z} \)-module spanned by \([N]\) with \( N \in \mathcal{C} \) subject to the relations \([M] + [N/M] = [N]\) for each pair of modules \( M \subset N \); we denote by \( \mathcal{K}_+(\mathcal{C}) \) the quotient of \( \mathcal{K}(\mathcal{C}) \) by the relations \([N] = \pm [\Pi N]\). If \( \mathcal{C} \) is closed under the tensor product, \( \mathcal{K}(\mathcal{C}) \) has a ring structure given by \([M] \cdot [N] := [M \otimes N]\).

We denote by \( \mathcal{F}\text{in}(g) \) the full subcategory of finite-dimensional \( g \)-modules and by \( \mathcal{K}(g) \) the Grothendieck ring of this category.
Recall that $\xi$ is a formal variable satisfying $\xi^2 = 1$. We identify $\xi$ with the image of $\Pi_{\text{triv}}$ and denote by $\psi_{\pm}(g) : K(g) \to K(g)/(\xi \mp 1)$ the canonical homomorphism. We set $K_{\pm}(g) := \psi_{\pm}(g)(K(g))$.

One readily sees that $\psi_{+} \times \psi_{-}$ gives an embedding $K(g) \hookrightarrow K_{+}(g) \times K_{-}(g)$; this embedding is strict (for instance, $(0; 1)$ does not belong to the image) and induces a bijection $K(g)_Q \overset{\sim}{\longrightarrow} K_{+}(g)_Q \times K_{-}(g)_Q$ (where for a $\mathbb{Z}$-module $M$ we set $M_Q := M \otimes_{\mathbb{Z}} \mathbb{Q}$).

2.1.1. An exact functor $F : C \to C'$ induces a group homomorphism $K(g) \to K(g')$ which is a ring homomorphism if $F$ is a tensor functor. For the restriction functor $\text{Res}_{g'}^g$ we denote the corresponding homomorphism by $\text{res}_{g'}^g$. It is easy to check that we have the following commutative diagrams

$$
\begin{array}{ccc}
K(g) & \xrightarrow{\text{res}_{g'}^g} & K(g') \\
\downarrow{\psi_{\pm}(g)} & & \downarrow{\psi_{\pm}} \\
K_{\pm}(g) & \xrightarrow{\text{res}_{g'}^g} & K_{\pm}(g')
\end{array}
$$

2.2. Structure of $K(g)$. We set $I_1 := \{ \lambda \in t^* | \Pi C_{\lambda} \cong C_{\lambda} \}$. We have

$$I_1 = \{ \lambda \in t^* | \Pi L(\lambda) \cong L(\lambda) \}.$$ 

Up to a parity shift $\Pi$ each finite-dimensional simple module is isomorphic to $L(\lambda)$ for some $\lambda \in P^+(\Delta^+)$. Thus the ring $K(g)$ is a free $\mathbb{Z}$-module with a basis

$$\{[L(\lambda)]\}_{\lambda \in P^+(\Delta^+)} \prod_{\lambda \in P^+(\Delta^+) \setminus I_1} \{\xi[L(\lambda)]\}_{\lambda \in P^+(\Delta^+) \setminus I_1}$$

where $\xi[L(\lambda)] = [L(\lambda)]$ for $\lambda \in P^+(\Delta^+) \cap I_1$. The ring $K_{\pm}(g)$ is a free $\mathbb{Z}$-module with a basis $\{\psi_{\pm}([L(\lambda)])\}_{\lambda \in P^+(\Delta^+)}$.

For each $b = \sum_\nu m_{\nu} e^\nu$ with $m_{\nu} \in \mathbb{Z}[\xi]$ we set $\text{supp}(b) := \{ \nu | m_{\nu} \neq 0 \}$.

2.2.1. Corollary. The maps $\text{res}_{h}^g : K(g) \to K(h)$, $\text{res}_{t}^g : K_{+}(g) \to K_{+}(t)$ are injective.

Proof. Take $a = \sum_{\lambda \in P^+(\Delta^+)} (k_{\lambda} + \xi n_{\lambda})[L(\lambda)] \in K(g)$ with $k_{\lambda}, n_{\lambda} \in \mathbb{Z}$ and $n_{\lambda} = 0$ for $\lambda \in I_1$. It is easy to see that each maximal element in the set $\{ \lambda | k_{\lambda} + \xi n_{\lambda} \neq 0 \}$ is a maximal element in $\text{supp}(\text{res}_{h}^g(a))$; this implies the injectivity of $\text{res}_{h}^g : K(g) \to K(h)$. The injectivity of the second map can be checked similarly. \qed
2.3. **Ring** $\text{Ch}(\mathfrak{g})$. Recall that the rings $\text{Ch}(\mathfrak{g})$, $\text{Ch}_+ (\mathfrak{g})$ are the images of the homomorphisms $\text{res}^g : \mathcal{K}(\mathfrak{g}) \to \mathcal{K}(t)$ and $\text{res}^g_+ : \mathcal{K}_+ (\mathfrak{g}) \to \mathcal{K}_+ (t)$. We set

$$Z[P_0; \xi] := Z[P_0] \otimes \mathbb{Z}[\xi], \quad Z[t^*; \xi] := Z[t^*] \otimes \mathbb{Z}[\xi];$$

we identify $\mathcal{K}(t)$ with $Z[t^*; \xi]$ and $\mathcal{K}_+(t)$ with $Z[t^*]$. We denote the homomorphisms $\psi_\pm (t)$ by $\psi_\pm$: these maps are given by $\psi_\pm (\xi) = \pm 1$, $\psi_\pm (e^\nu) = e^\nu$.

2.3.1. Let $N$ be a finite-dimensional module and $[N]$ be its image in $\mathcal{K}(\mathfrak{g})$. The map $\text{res}^g_+ : \mathcal{K}(\mathfrak{g}) \to \mathcal{K}(t)$ is given by $[N] \mapsto \text{ch}_\xi N$, where

$$\text{ch}_\xi N = \sum_\nu (\dim N_\nu \cap N_\psi) e^\nu + \xi \sum_\nu (\dim N_\nu \cap N_\tau) e^\nu$$

and $N_\nu$ is the (generalized) weight space of weight $\nu$. We set

$$\text{ch} N := \sum_{\nu \in t^*} \dim N_\nu e^\nu = \psi_+ (\text{ch}_\xi N) \quad \text{sch} N := \sum_{\nu \in t^*} \text{sdim} N_\nu e^\nu = \psi_- (\text{ch}_\xi N)$$

and view $\text{ch} N, \text{sch} N$ as elements of $Z[P_0]$. By 2.1.1 we have the following commutative diagrams

\[
\begin{array}{ccc}
\mathcal{K}(\mathfrak{g}) & \xrightarrow{\text{ch}} & \text{Ch}(\mathfrak{g}) \\
\downarrow \psi_+ & & \downarrow \psi_+ \\
\mathcal{K}_+(\mathfrak{g}) & \xrightarrow{\text{ch}} & \text{Ch}_+(\mathfrak{g})
\end{array}
\hspace{1cm}
\begin{array}{ccc}
\mathcal{K}(\mathfrak{g}) & \xrightarrow{\text{ch}} & \text{Ch}(\mathfrak{g}) \\
\downarrow \psi_- & & \downarrow \psi_- \\
\mathcal{K}_-(\mathfrak{g}) & \xrightarrow{\text{ch}} & \text{Ch}_-(\mathfrak{g})
\end{array}
\]

All maps in the above diagrams are surjective.

By Corollary 2.2.1 one has $\mathcal{K}_+(\mathfrak{g}) \xrightarrow{\sim} \text{Ch}_+(\mathfrak{g})$ and $\mathcal{K}(\mathfrak{g}) \xrightarrow{\sim} \text{Ch}(\mathfrak{g})$ if $\mathfrak{h} = t$. In particular, $\text{Ch}_+(\mathfrak{g})$ (resp., $\text{Ch}_-(\mathfrak{g})$) is a $Z$-span of $\text{ch} N$ (resp., $\text{sch} N$) with $N \in \mathcal{F}_{\text{Fin}}(\mathfrak{g})$.

2.3.2. **Remark.** If $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$ and $t \subset \mathfrak{g}'$ we have $\text{res}^{\mathfrak{g}'} \circ \text{res}^\mathfrak{g} = \text{res}^\mathfrak{g}_+$ which gives $\text{Ch}(\mathfrak{g}) \subset \text{Ch}(\mathfrak{g}')$.

2.3.3. By Corollary 2.2.1 the ring $\text{Ch}_+(\mathfrak{g})$ is a free $Z$-module with a basis $\text{ch} L(\lambda)$ for $\lambda \in P^+(\Delta^+)$; if $\mathfrak{h} = t$, then $\text{Ch}_+(\mathfrak{g})$ is a free $Z$-module with a basis $\text{sch} L(\lambda)$ for $\lambda \in P^+(\Delta^+)$. In general, the ring $\text{Ch}_-(\mathfrak{g})$ is spanned by $B_1 := \{\text{sch} L(\lambda) | \lambda \in P^+(\Delta^+) \setminus I_1\}$ (since $\text{sch} L(\lambda) = 0$ if $\lambda \in I_1$). The argument used in the proof of Corollary 2.2.1 shows that the set $B_2 := \{\text{sch} L(\lambda) | \lambda \in P^+(\Delta^+) : \text{sdim} C_\lambda \neq 0\}$ is linearly independent. Clearly, $B_2 \subset B_1$ and $B_1 = B_2 = \{\text{sch} L(\lambda) | \lambda \in P^+(\Delta^+)\}$ if $\mathfrak{h} = t$.

2.4. **The ring** $R(P_0)$. Recall that $R(P_0) := \text{Ch}(\mathfrak{h}) \cap Z[P_0; \xi]$. We set

$$R_\pm (P_0) := \psi_\pm (R(P_0)) = \text{Ch}_\pm (\mathfrak{h}) \cap Z[P_0].$$

Observe that $R(P_0) = \text{res}^\mathfrak{h}_- \mathcal{K}(\mathcal{C})$, where $\mathcal{C}$ is the full subcategory of the category $\mathcal{F}_{\text{Fin}}(\mathfrak{h})$ consisting of the modules with the $t$-eigenvalues lying in $P_0$; since $P_0$ is a subgroup of
t\ast, C is closed under the tensor product, so \( R(P_0) \) (resp., \( R_\pm(P_0) \)) is a subring of \( \mathbb{Z}[P_0; \xi] \) (resp., of \( \mathbb{Z}[P_0] \)). If \( h = t \), then \( R(P_0) = \mathbb{Z}[P_0; \xi] \) and \( R_\pm(P_0) = \mathbb{Z}[P_0] \).

2.4.1. Retain notation of 1.4. The ring \( R(P_0) \) is a free \( \mathbb{Z} \)-module with a basis \( \{ \text{ch}_\xi C_\lambda \}_{\lambda \in P_0} \) if \( \sum_{\lambda \in P_0} m_\lambda \text{ch}_\xi C_\lambda \) with \( m_\lambda \) divisible by \( \dim C_\lambda \) and \( \text{sdim} C_\lambda \) respectively (for \( R_-(P_0) \) we have \( m_\lambda = 0 \) if \( \text{sdim} C_\lambda = 0 \)).

2.4.2. W-action. The group \( W \) acts on the ring \( \mathbb{Z}[P_0; \xi] \) by \( w(e^\lambda) = e^{\omega w \lambda} \) and \( w \xi = \xi \). Recall that each \( w \in W \) acts on \( h \) by an automorphism \( \iota_w \); the module \( C_w \lambda \) is isomorphic to the \( \iota_w^{-1} \)-twist of \( C_\lambda \), so \( \dim C_\lambda = \dim C_w \lambda \) and \( \text{sdim} C_\lambda = \text{sdim} C_w \lambda \). Therefore \( R(P_0), R_\pm(P_0) \) are \( W \)-stable.

2.4.3. Remark. For \( g = q \) one has \( \text{sdim} C_\lambda = 0 \) for each \( \lambda \neq 0 \); this gives \( \text{Ch}_-(h) = \mathbb{Z} \). Moreover, by [1], \( \text{sch} L(\lambda) = 0 \) for each \( \lambda \neq 0 \).

2.4.4. Since \( \xi^2 = 1 \) we have the embedding
\[
\psi_+ \times \psi_- : R(P_0) \to R_+(P_0) \times R_-(P_0).
\]

We will use the following construction: for any subsets \( A_\pm \subset R_\pm(P_0) \) we introduce
\[
A_+ \times_{R(P_0)} A_- := \{ a \in R(P_0) | \psi_\pm(a) \in A_\pm \}.
\]

Note that \( A_+ \times_{R(P_0)} A_- \) is a subring of \( R(P_0) \) if \( A_\pm \) are rings.

2.4.5. Lemma. Let \( A \) be a \( \mathbb{Z}[\xi] \)-submodule of \( R(P_0) \) with the following property: if \( a \in R(P_0) \) and \( 2a \in A \), then \( a \in A \). Then
\[
A = \psi_+(A) \times_{R(P_0)} \psi_-(A).
\]

Proof. Take \( a \in R(P_0) \) such that \( \psi_\pm(a) \in \psi_\pm(A) \). Then \( A \) contains \( a - c(1 - \xi) \) for some \( c \in R(P_0) \). Since \( A \) is a \( \mathbb{Z}[\xi] \)-submodule, \( A \) contains \( (1 + \xi)(a - c(1 - \xi)) = (1 + \xi)a \). Similarly, \( A \) contains \( (1 - \xi)a \), so \( 2a \in A \). Then the assumption gives \( a \in A \) as required. \( \square \)
2.5. Rings $\mathcal{A}(g)$ and $\text{Ch}(g)$. Recall that $\text{Ch}_\pm(g) = \psi_\pm(\text{Ch}(g))$ and that

$$\mathcal{A}(g) := \bigcap_{\beta \in \Delta_{iso}} \{ \sum_{\nu} m_{\nu} e^\nu \in R(P_0)^W \mid \langle \nu, h_\beta \rangle \neq 0 \implies \sum_{i \in \mathbb{Z}} (-\xi)^i m_{\nu + i\beta} = 0 \}. $$

Setting $\mathcal{A}_\pm(g) := \psi_\pm(\mathcal{A}(g))$ we have

$$(3) \quad \mathcal{A}_\pm(g) := \bigcap_{\beta \in \Delta_{iso}} \{ \sum_{\nu} m_{\nu} e^\nu \in R_{\pm}(P_0)^W \mid \langle \nu, h_\beta \rangle \neq 0 \implies \sum_{i \in \mathbb{Z}} (\mp 1)^i m_{\nu + i\beta} = 0 \}$$

(recall that $R_{\pm}(P_0) = \mathbb{Z}[P_0]$ if $\mathfrak{h} = \mathfrak{t}$).

2.5.1. Corollary. $\mathcal{A}(g) = \mathcal{A}_+(g) \times \mathcal{A}_-(g)$ and $\text{Ch}(g) = \text{Ch}_+(g) \times \text{Ch}_-(g)$.

\textbf{Proof.} For $\mathcal{A}(g)$ the assumption of Lemma 2.4.5 hold. The ring $\text{Ch}(g)$ is a $\mathbb{Z}[\xi]$-submodule of $R(P_0)$. It remains to verify that for $a \in R(P_0)$ with $2a \in \text{Ch}(g)$ one has $a \in \text{Ch}(g)$. Write $2a = \sum_{\lambda \in P^+(\Delta_+)} m_\lambda ch_\xi L(\lambda)$ where $m_\lambda \in \mathbb{Z}[\xi]$; without loss of generality we may assume that $m_\lambda \not\in 2\mathbb{Z}[\xi]$ for all $\lambda$. Let $\lambda$ be maximal such that $m_\lambda \neq 0$. The coefficient of $e^\lambda$ in $2a$ is equal to $m_\lambda ch_\xi C_\lambda$. Using 2.4.1 we obtain $a \not\in R(P_0)$, a contradiction. \hfill $\square$

2.5.2. Remark. The algebras $\mathcal{A}_+(g)$ and $\mathcal{A}_-(g)$ can be very different: for example, for $\mathfrak{sl}_n$, $\mathcal{A}_-(g) = \mathbb{Z}$ whereas $\mathcal{A}_+(g)$ is a free $\mathbb{Z}$-module of infinite rank. For Kac-Moody superalgebras these algebras are not so different, see 4.1.6 below.

2.5.3. Remark. If $\Delta_{iso}$ is empty, then $\mathcal{A}(g) = R(P_0)^W$. If $\Delta_{iso}$ is non-empty and the group generated by $W$ and $-\text{Id}$ acts transitively on $\Delta_{iso}$, then for any $\beta \in \Delta_{iso}$ we have

$$\mathcal{A}(g) = \{ \sum_{\nu} m_{\nu} e^\nu \in R(g)^W \mid \langle \nu, h_\beta \rangle \neq 0 \implies \sum_{i \in \mathbb{Z}} (-\xi)^i m_{\nu + i\beta} = 0 \}. $$

Note that the above condition ($W$ acts transitively on $\Delta_{iso}/\{\pm 1\}$) holds for the Kac-Moody, $Q$-type and $P$-type superalgebras.

2.6. Case $\mathfrak{gl}(1|1)$. For the Lie superalgebra $\mathfrak{gl}(1|1)$ the algebra $\mathfrak{g}_{\mathfrak{tr}}$ is a two-dimensional commutative Lie algebra and $\mathfrak{g}_{\mathfrak{tr}} = \mathfrak{h} = \mathfrak{t}$. One has $\Delta = \Delta_{iso} = \{ \pm \beta \}$. Since $\beta \in \Delta_{iso}$ and $\dim \mathfrak{t} = 2$ we have $\{ \nu \mid \langle \nu, h_\beta \rangle = 0 \} = \mathbb{C} \beta$ A module $L(\lambda)$ is one-dimensional if $\lambda \in \mathbb{C} \beta$ and is two-dimensional with $\text{ch}_\xi L(\lambda) = e^{\lambda}(1 + \xi e^{-\beta})$ if $\lambda \not\in \mathbb{C} \beta$. Using 2.2 we obtain

$$\text{Ch}(g) = \{ \sum m_{\nu} e^\nu \mid \sum_{i \in \mathbb{Z}} (-\xi)^i m_{\nu + i\beta} = 0 \} \quad \text{for} \quad \nu \not\in \mathbb{C} \beta = \mathcal{A}(g).$$
2.7. Proposition. For a quasidefinite Lie superalgebra \( g \) we have \( \text{Ch}(g) \subset A(g) \).

Proof. Since \( g_{\text{re}} \) is a reductive Lie algebra one has \( \text{Ch}(g_{\text{re}}) = Z[P_0; \xi] \). The embeddings \( g_{\text{re}} \subset g, h \subset g \) give \( \text{Ch}(g) \subset \text{Ch}(g_{\text{re}}) \) and \( \text{Ch}(g) \subset \text{Ch}(h) \); thus

\[
\text{Ch}(g) \subset \text{Ch}(g_{\text{re}}) \cap \text{Ch}(h) = Z[P_0; \xi] \cap \text{Ch}(h) = R(P_0)^W.
\]

Fix \( \beta \in \Delta_{\text{iso}} \). By 1.2 the space \( g_{\beta} + t + g_{-\beta} \) contains a quasidefinite subalgebra \( I := gl(1|1) \times t' \) and \( t \subset I \) is the Cartan subalgebra of \( I \). View \( \text{Ch}(I) \) as a subring in \( Z[t^*; \xi] \). By 2.6 \( \sum m_{\nu}e_{\nu} \in \text{Ch}(I) \) implies \( \sum_{\xi}(-\xi)^{i}m_{\nu+i\beta} = 0 \) if \( \langle \nu, h_{\beta} \rangle \neq 0 \). Since \( \text{Ch}(g) \subset \text{Ch}(I) \) we obtain \( \text{Ch}(g) \subset A(g) \). \( \square \)

3. Short basis

In this section we show that \( A(g) = \text{Ch}(g) \) and this ring admits a “short basis” if \( g \) is a Kac-Moody superalgebras \( g \neq gl(1|1) \) or a \( Q \)-type superalgebra (see Corollaries 3.7,3.8.4). Recall that \( A(gl(1|1)) = \text{Ch}(gl(1|1)) \); it is easy to see that this ring does not admit a “short basis”.

3.1. Definition. We call a \( Z \)-basis \( B \bigcup B' \) of \( \text{Ch}(g) \) short with respect to \( \Delta^+ \) if

\[
\begin{align*}
(a) & \quad B : = \{ b_\lambda \mid \lambda \in P^+(\Delta^+) \}, \quad B' : = \{ b_\lambda \mid \text{sdim } C_\lambda \neq 0 \}; \\
(b) & \quad \text{supp}(b_\lambda - ch_\xi C_\lambda) \cap P^+(\Delta^+) = \emptyset; \\
(c) & \quad \xi b_\lambda = b_\lambda \quad \text{if sdim } C_\lambda = 0; \\
(d) & \quad \text{supp}(b_\lambda) \subset \{ \nu \in t^* \mid \nu \leq \lambda \}.
\end{align*}
\]

3.2. Properties of short bases. Assume that \( \text{Ch}(g) \) admits a short basis.

3.2.1. It is not hard to show (see, for example, [3]): that \( \dim C_\lambda = 1 \) or \( \text{sdim } C_\lambda = 0 \). Therefore \( B' : = \{ b_\lambda \mid \dim C_\lambda = 1 \} \).

3.2.2. Since \( ch_\xi L(\lambda) \in \text{Ch}(h) \cap Z[P_0; \xi] = R[P_0] \), for each \( \nu \) we have \( ch_\xi L(\lambda) \nu = m_{\lambda, \nu} ch_\xi C_\nu \) for some \( m_{\lambda, \nu} \in Z[\xi] \) with \( m_{\lambda, \nu} \in Z \) if \( \text{sdim } C_\nu = 0 \). The property (b) gives

\[
ch_\xi L(\lambda) = b_\lambda + \sum_{\nu \in P^+(\Delta^+)} m_{\lambda, \nu} b_\nu.
\]

Set \( b_\nu^\pm : = \psi_\pm(b_\nu) \); then \( b_\nu^\pm \in Z[P_0] \). By (b), (c) we get \( b_\nu^- \neq 0 \) if and only if \( b_\nu \in B' \). Moreover, the sets \( \{ \psi_+(b) \mid b \in B \} \) and \( \{ \psi_-(b) \mid b \in B' \} \) form \( Z \)-bases of \( \text{Ch}_+(g) \) and

\[
\begin{align*}
\text{ch } L(\lambda) & = \sum_{\nu} \frac{\dim L(\lambda)_\nu}{\dim C_\nu} b_\nu^+, \\
\text{sch } L(\lambda) & = \sum_{\nu : \dim C_\nu = 1} \text{sdim } L(\lambda)_\nu b_\nu^-.
\end{align*}
\]

(for the second formula we used \( \text{sdim } C_\lambda \in \{ 0, 1 \} \)).
3.3. Set $Y_\lambda$. For each $\lambda$ we set

$$Y_\lambda := \{ \mu \in P^+(\pi) \mid \mu < \lambda \}.$$ 

If the sets $Y_\lambda$ are finite, the elements $b_\lambda$ satisfying the property (b) can be constructed by the induction on the cardinality of $Y_\lambda$ via the formula $b_\lambda := \text{ch} L(\lambda) - \sum_{\nu \in Y_\lambda \cap P^+(\Delta^+)} m_{\lambda,\nu} b_\nu$ where $m_{\lambda,\nu}$ are as in $[3.2.2]$. Note that $b_\lambda$ satisfies the property (d).

3.3.1. Lemma. If $(-\mathbb{R}_{\geq 0}\Delta^+) \cap \mathbb{R}_{\geq 0}P^+(\pi) = \{0\}$, then $Y_\lambda$ is finite for each $\lambda$.

Proof. View $\lambda - \mathbb{R}\Delta^+$ as an affine space. Let us show that the set

$$M := (\lambda - \mathbb{R}_{\geq 0}\Delta^+) \cap \mathbb{R}_{\geq 0}P^+(\pi)$$

is bounded. Both $\lambda - \mathbb{R}_{\geq 0}\Delta^+$ and $\mathbb{R}_{\geq 0}P^+(\pi)$ are polyhedra (i.e., the intersection of finitely many half-spaces), so $M$ is a polyhedron as well. Assume that $M$ is unbounded. Then $M$ contains a ray, that is for some $x \in M$ and a non-zero $\gamma \in \Delta^+$ one has $\nu - x\gamma \in M$ for all $x \geq 0$. Since $\nu - x\gamma \subset \lambda - \mathbb{R}_{\geq 0}\Delta^+$ we get

$$\gamma \in \frac{\nu - \lambda}{x} + \mathbb{R}_{\geq 0}\Delta^+ \quad \text{for all} \quad x > 0.$$

Since $\mathbb{R}_{\geq 0}\Delta^+$ is closed, we obtain $\gamma \in \mathbb{R}_{\geq 0}\Delta^+$. Similarly, $\nu - x\gamma \in \mathbb{R}_{\geq 0}P^+(\pi)$ implies $\gamma \in \mathbb{R}_{\geq 0}\Delta^+$ implies $\gamma \in \mathbb{R}_{\geq 0}\Delta^+$ and the assumption gives $\gamma = 0$, a contradiction. Hence $M$ is bounded.

Let $h \in \mathfrak{t}$ be a “defining element for $\Delta^+$”, i.e. $\text{Re}\langle \alpha, h \rangle \neq 0$ for all $\alpha \in \Delta$ and $\alpha \in \Delta^+$ if and only if $\text{Re}\langle \alpha, h \rangle > 0$. Set

$$a_- := \min_{\mu \in M} \text{Re}\langle \mu, h \rangle, \quad a_+ := \max_{\mu \in M} \text{Re}\langle \mu, h \rangle, \quad u := \min_{\alpha \in \Delta^+} \text{Re}\langle \alpha, h \rangle.$$ 

Any $\mu \in Y_\lambda$ is of the form $\mu = \sum_{\alpha \in \Delta^+} \lambda - k_\alpha \alpha$ for some $k_\alpha \in \mathbb{N}$. Since $\mu \in Y_\lambda \subset M$ one has $a_- \leq u \sum_{\alpha \in \Delta^+} k_\alpha \leq a_+$. Thus each $k_\alpha$ has finitely many possible values, so $Y_\lambda$ is finite. □

3.3.2. Remark. Since $\mathfrak{g}_{\mathfrak{r}}$ is reductive, one has $-w_0\pi = \pi$ for some $w_0 \in W$. We claim that

$$-w_0\Delta^+ = \Delta^+ \implies (-\mathbb{R}_{\geq 0}\Delta^+) \cap \mathbb{R}_{\geq 0}P^+(\pi) = \{0\}. (5)$$

Indeed, assume that $-w_0\Delta^+ = \Delta^+$. If $\nu$ is a non-zero element of $(-\mathbb{R}_{\geq 0}\Delta^+)$, then $w_0\nu \in \mathbb{R}_{\geq 0}\Delta^+$ and so $w_0\nu - \nu$ is a non-zero element in $\mathbb{R}_{\geq 0}\Delta^+$. By Lemma 3.3 we obtain $\nu \not\in \mathbb{R}_{\geq 0}P^+(\pi)$.

3.4. Lemma. Assume that

- $\mathbb{R}_{\geq 0}P^+(\pi) \cap (-\mathbb{R}_{\geq 0}\Delta^+) = \{0\}$;
- for any non-zero $y \in \mathcal{A}(\mathfrak{g})$ one has $\text{supp}(y) \cap P^+(\Delta^+) \neq \emptyset$.

Then $\mathcal{A}(\mathfrak{g}) = \text{Ch}(\mathfrak{g})$ and $\text{Ch}(\mathfrak{g})$ admits a unique short $\mathbb{Z}$-basis with respect to $\Delta^+$. 

Proof. By Lemma 3.3.3 the sets $Y_\lambda$ are finite. By 3.3.3 the elements $b_\lambda$ satisfying the properties (b), (d) of (1) can be uniquely defined. Take $\lambda$ such that $\text{sdim} C_\lambda = 0$ and set $y := \xi b_\lambda - b_\lambda$. One has

$$\text{supp}(y) \cap P^+(\Delta^+) \subset \text{supp}(b_\lambda) \cap P^+(\Delta^+) = \{ \lambda \}$$

and $\lambda \not\in \text{supp}(y)$ (since $\text{sdim} C_\lambda = 0$). Hence $\text{supp}(y) \cap P^+(\Delta^+) = \emptyset$. Since $y \in \text{Ch}(g)$ and $\text{Ch}(g) \subset \mathcal{A}(g)$, the second assumption gives $y = 0$, that is $\xi b_\lambda = b_\lambda$. This gives the property (c) of (4). Hence the elements $b_\lambda$ satisfies the properties (b), (c), (d) of (1). Let us show that $B \coprod B'$ forms a $Z$-basis of $\mathcal{A}(g)$.

Let us verify that the elements of $B \coprod B'$ are linearly independent over $Z$. Let

$$\sum_{\lambda \in P^+(\Delta^+)} n_\lambda b_\lambda = 0$$

for some $n_\lambda \in Z[\xi]$ with $n_\lambda \in Z$ if $\text{sdim} C_\lambda = 0$. Observe that for each $\lambda \in P^+(\Delta^+)$ the coefficient of $e^\lambda$ in the above expression is $n_\lambda \chi C_\lambda$, so $n_\lambda \chi C_\lambda = 0$. If $\text{sdim} C_\lambda = 0$, this gives $n_\lambda = 0$ (because $n_\lambda \in Z$). If $\text{sdim} C_\lambda \neq 0$, then the formulae $\psi_\pm(n_\lambda \chi C_\lambda) = 0$ give $\psi_\pm(n_\lambda) = 0$, so $n_\lambda = 0$. Hence $B \coprod B'$ are linearly independent.

Let us check that $\mathcal{A}(g)$ lies in $Z[\xi]$-span of $B$. Take $a \in \mathcal{A}(g)$. Since $\mathcal{A}(g) \subset R(P_0)$ we have $a = \sum_{\nu} n_\nu \chi C_{\nu}$ for some $n_\nu \in Z[\xi]$. Since $b_\lambda \in \text{Ch}(g) \subset \mathcal{A}(g)$ we obtain

$$a' := a - \sum_{\lambda \in P^+(\Delta^+)} n_\lambda b_\lambda \in \mathcal{A}(g).$$

By above, $\text{supp}(a') \cap P^+(\Delta^+) = \emptyset$, so the second assumption gives $a' = 0$. Therefore $\mathcal{A}(g)$ lies in $Z[\xi]$-span of $B$.

By above, $B \coprod B'$ is a $Z$-basis of $\mathcal{A}(g)$. Since $B \subset \text{Ch}(g)$ and $\text{Ch}(g) \subset \mathcal{A}(g)$, we get $\mathcal{A}(g) = \text{Ch}(g)$.

3.5. Lemma. Assume that $\Delta^+$ admits a base $\Sigma$ and that

$$P^+(\Delta^+) \subset \{ \lambda \in P^+(\pi) \mid \forall \beta \in \Sigma \cap \Delta_{\text{iso}} \langle \lambda, h_\beta \rangle \neq 0 \implies \lambda - \beta \in P^+(\pi) \}. \tag{6}$$

Then for any non-zero $y \in \mathcal{A}(g)$ one has $\text{maxsupp}(y) \subset P^+(\Delta^+)$.

Proof. Set $S := \text{supp}(y)$. Since $y \in \mathcal{A}(g)$ the set $S$ has the following properties:

(i) $WS = S$ and $S \subset P_0$;
(ii) if $\langle \eta, h_\beta \rangle \neq 0$ for some $\beta \in \Delta_{\text{iso}}$, then $S \cap (\eta + Z \beta) \neq \{ \eta \}$.

Let $\eta$ be a maximal element in $\text{supp}(y)$. Assume that $\eta \not\in P^+(\Delta^+)$. Combining (i) and Lemma 3.3 we have $\eta \in P^+(\pi)$, so $\eta \in P^+(\pi) \setminus P^+(\Delta^+)$. By (6) there exists $\beta \in \Sigma \cap \Delta_{\text{iso}}$ such that

$$\langle \eta, h_\beta \rangle \neq 0, \quad \eta - \beta \not\in P^+(\pi).$$

By (ii) $\eta - j\beta \in S$ for some non-zero integral $j$; the maximality of $\eta$ gives $j \geq 1$. 
Since $\eta \in P^+(\pi)$ and $\eta - \beta \not\in P^+(\pi)$ there exists $\alpha \in \pi$ such that $s_\alpha \eta \leq \eta$ and $s_\alpha (\eta - \beta) > \eta - \beta$. In particular, $s_\alpha \beta < \beta$. Viewing $g$ as an $\mathfrak{sl}_2$-module for the $\mathfrak{sl}_2 \subset \mathfrak{g}_0$ corresponding to $\alpha$, we get $\beta - \alpha \in \Delta$. Since $\beta$ is a simple root, both $s_\alpha \beta$ and $\beta - \alpha$ are negative roots. Since $s_\alpha (\eta - \beta) = \eta - \beta + i \alpha$ for some $i \in \mathbb{Z}_{>0}$, this gives $s_\alpha (\eta - \beta) > \eta$. Combining with $-s_\alpha \beta > 0$ we obtain $s_\alpha (\eta - j \beta) > \eta$. Since $\eta - j \beta \in S$, we have $s_\alpha (\eta - j \beta) \in S$ which contradicts to the maximality of $\eta$. \hfill \Box

### 3.6. Corollary
Let $\mathfrak{g}$ be a quasisemisimple Lie superalgebra with a base $\Sigma$ satisfying (6) and such that $-\mathbb{R}_{\geq 0} \Delta^+ \cap \mathbb{R}_{>0} P^+(\pi) = \{0\}$. Then $\text{Ch}(\mathfrak{g}) = \mathcal{A}(\mathfrak{g})$ and this ring admits a short basis with respect to $\Delta^+$.

**Proof.** The assertion follows from Lemmata 3.4 and 3.5. \hfill \Box

### 3.7. Corollary
Let $\mathfrak{g}$ be a $Q$-type superalgebra or a finite-dimensional Kac-Moody superalgebra and $\mathfrak{g} \neq \mathfrak{gl}(n|n)$. Then $\text{Ch}(\mathfrak{g}) = \mathcal{A}(\mathfrak{g})$ and this ring admits a short basis with respect to the mixed triangular decomposition.

**Proof.** For $Q$-type case one has $\Delta^+ = \Delta^+_w = \Delta_{so}$. In particular, $\Delta^+$ and $-\Delta^+$ are $W$-conjugated. By [10],

$$P^+(\Delta^+) = \{ \lambda \in P^+(\pi) | \forall \alpha \in \pi \ s_\alpha \lambda = \lambda \implies \langle \lambda, h_\alpha \rangle = 0 \}.$$

Take $\eta \in P^+(\pi) \setminus P^+(\Delta^+)$. By above, there exists $\alpha \in \pi$ such that $\langle \eta, h_\alpha \rangle \neq 0$ and $s_\alpha \eta = \eta$. Since $s_\alpha (\eta - \alpha) = \eta + \alpha > \eta - \alpha$ we have $\eta - \alpha \not\in P^+(\pi)$. Therefore (6) holds. Combining (5) and Corollary 3.6, we obtain the assertion for $Q$-type superalgebras.

Now let $\mathfrak{g}$ be one of the algebras $\mathfrak{gl}(m|n)$ for $m \neq n$, $\mathfrak{osp}(m|n)$, $D(2|1; a)$, $F(4)$ or $G(3)$. The mixed bases for these superalgebras satisfy (6), see Appendix. It remains to verify the formula $(-\mathbb{R}_{\geq 0} \Delta^+) \cap \mathbb{R}_{>0} P^+(\pi) = \{0\}$. For $D(2|1; a)$, $F(4)$, $G(3)$ and $\mathfrak{osp}(2m + 1|n)$ the Weyl group $W$ contains $-\text{Id}$; for $\mathfrak{osp}(2m|2n)$ the sets $-\Delta^+$ and $\Delta^+$ are $W$-conjugated if $\Sigma$ is a mixed base, see Appendix. Thus (5) implies the required formula for these cases. For the remaining case $\mathfrak{gl}(m|n)$ with $m \neq n$ the formula is verified in Lemma 3.8.1 below. \hfill \Box

### 3.8. Case $\mathfrak{gl}(m|n), \mathfrak{sl}(m|n)$
One has $\mathfrak{gl}(m|n)_{\mathfrak{g}} = \mathfrak{gl}_m \times \mathfrak{gl}_n$. Let $\varepsilon_1, \ldots, \varepsilon_m, \delta_1, \ldots, \delta_n$ be the weights of the standard representation of $\mathfrak{gl}(m|n)$ and

$$\pi = \{ \varepsilon_i - \varepsilon_{i+1} \}_{i=1}^{m-1} \cup \{ \delta_j - \delta_{j+1} \}_{j=1}^{n-1}.$$

Take $\mathfrak{g} = \mathfrak{gl}(m|n)$ or $\mathfrak{g} = \mathfrak{sl}(m|n)$ for $m \neq n$. The bases compatible with the triangular decomposition of $\mathfrak{g}$ can be encoded by words in $m$ letters $\varepsilon$ and $n$ letters $\delta$: for instance, the word $\varepsilon^2 \delta$ correspond to the distinguished base $\{ \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta_1 \}$ and the word $(\varepsilon \delta)^n \varepsilon^{m-n}$ corresponds to the mixed base

$$\Sigma_{m|n} := \{ \varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \ldots, \varepsilon_n - \delta_n, \delta_n - \varepsilon_{n+1}, \varepsilon_{n+1} - \varepsilon_{n+2}, \ldots, \varepsilon_{m-1} - \varepsilon_m \}.$$

The distinguished bases correspond to the words $\varepsilon^n \delta^n$ and $\delta^n \varepsilon^m$. The bases $\Sigma$ and $-w_0 \Sigma$ correspond to the inverse words: for example, $-w_0 \Sigma_{m|n}$ corresponds to $\varepsilon^m (\delta \varepsilon)^n$. 

5
3.8.1. **Lemma.** Take $\mathfrak{g} = \mathfrak{gl}(m|n)$ or $\mathfrak{g} = \mathfrak{sl}(m|n)$ for $m \neq n$. If the first and the last letter in the word corresponding to $\Sigma$ are the same, then $(-\mathbb{R}_{\geq 0} \Delta^+) \cap \mathbb{R}_{\geq 0} P^+(\pi) = \{0\}$.

**Proof.** Take $\nu \in (-\mathbb{R}_{\geq 0} \Delta^+) \cap \mathbb{R}_{\geq 0} P^+(\pi)$ and write $\nu = \sum x_i \epsilon_i + \sum y_i \delta_i$ for $x_i, y_i \in \mathbb{R}$. The condition $\nu \in \mathbb{R}_{\geq 0} P^+(\pi)$ gives

$$x_1 \geq x_2 \ldots \geq x_m, \quad y_1 \geq y_2 \ldots \geq y_n.$$

Let the first and the last letter in the word corresponding to $\Sigma$ be $\epsilon$. Then the condition $\nu \in -\mathbb{R}_{\geq 0} \Delta^+$ gives $x_1 \leq 0 \leq x_m$, so $x_i = 0$ for all $i$. Now the condition $\nu \in -\mathbb{R}_{\geq 0} \Delta^+$ gives $y_1 \leq 0 \leq y_m$, so $y_j = 0$ for all $j$. □

3.8.2. **Case** $\mathfrak{gl}(n+1|n+1)$. We fix the base corresponding to $(\varepsilon \delta)^n \delta \varepsilon$:

$$\Sigma = \{\varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \ldots, \varepsilon_n - \delta_n, \delta_n - \delta_{n+1}, \delta_{n+1} - \varepsilon_{n+1}\}.$$

We will show that $\text{Ch}(\mathfrak{g}) = \mathcal{A}(\mathfrak{g})$ and $\text{Ch}(\mathfrak{g})$ admits a short basis with respect to $\Delta^+(\Sigma)$.

3.8.3. **Lemma.** For any non-zero $y \in \mathcal{A}(\mathfrak{g})$ one has $\text{supp}(y) \cap P^+(\Delta^+) \neq \emptyset$.

**Proof.** We retain notation of [5.2]. We denote by $(-|-)$ a standard non-degenerate bilinear form on $t^*$; for $\beta \in \Delta_{iso}$ one has $(\mu|\beta) = 0$ if and only if only if $(\mu, h_\beta) = 0$.

Set $S := \text{supp}(y)$. Recall that

(i) $WS = S$ and $S \subset P_0$;
(ii) if $(\eta, \beta) \neq 0$ for some $\beta \in \Delta_{iso}$, then $S \cap (\eta + \mathbb{Z} \beta) \neq \{\eta\}$.

We fix a total order $\succ$ on $\Sigma$: $\Sigma = \{\alpha_1 \succ \alpha_2 \succ \ldots \succ \alpha_t\}$ and extend it to the lexicographic order on $\mathbb{R} \Delta$ by setting $\sum k_i \alpha_i > 0$ if $k_1 > 0$ or $k_1 = 0$, $k_2 > 0$ and so on. For $\lambda, \nu \in P_0$ we set $\lambda \succ \nu$ if $\lambda - \nu > 0$. Note that

$$\lambda \succ \nu \implies \lambda \succ \nu.$$

Let $\eta$ be a maximal element in $\text{supp}(y)$ with respect to the order $\succ$.

Combining (i) and Lemma 5.3 we have $\eta \in P^+(\pi)$. Assume that $\eta \notin P^+(\Delta^+)$. Set $L := L(\eta)$. By Proposition 5.3 there exists $\alpha \in \pi$ such that for any $\Sigma'$ containing $\alpha$ one has $\langle w_{\Sigma'}, L, \alpha \rangle \notin \mathbb{N}$. Fix such $\alpha$. Note that $\langle \eta, \alpha \rangle \in \mathbb{N}$ since $\eta \in P^+(\pi)$. In particular, $w_{\Sigma'} L \neq \eta$ and $\alpha \notin \Sigma$. One readily sees that for $\alpha \in \pi \setminus \Sigma$ one has $\alpha \in \Sigma'$, where either

$$\Sigma' = r_\beta \Sigma \quad \text{for} \quad \beta \in \Sigma_{iso}, \quad s_\alpha \beta < 0$$

or, for $\alpha = \varepsilon_n - \varepsilon_{n+1}$, one has

$$\Sigma' = r_\beta r_\gamma \Sigma = r_\beta r_\beta \Sigma \quad \text{for} \quad \beta, \beta' \in \Sigma_{iso}, \quad (\beta|\beta') = 0, \quad s_\alpha \beta, s_\alpha \beta' < 0.$$

(The property $(\beta|\beta') = 0$ gives $\beta' \in r_\beta \Sigma_{iso}$, so $r_\beta r_\beta \Sigma$ is well-defined). Set

$$\gamma := \eta - w_{\Sigma'} L.$$
By above, $\gamma \neq 0$, so $\gamma = \beta$ if $\Sigma' = r_\beta \Sigma$ and $\gamma \in \{\beta, \beta', \beta + \beta'\}$ if $\Sigma' = r_\beta r_\beta \Sigma$. Observe that $\gamma < \alpha$ in all cases. Combining $\eta \in P^+(\pi)$ and $\langle \eta - \gamma, \alpha' \rangle \notin \mathbb{N}$ we get

$$s_\alpha(\eta - \gamma) - (\eta - \gamma) = (k + 1)\alpha \quad \text{for } k \in \mathbb{N}.$$ 

Therefore $s_\alpha(\eta - \gamma) - (\eta - \gamma) \geq \alpha > \gamma$ which implies

$$s_\alpha(\eta - \gamma) > \eta.$$

Recall that $-s_\alpha \beta > 0$ and $-s_\alpha \beta' > 0$ if $\Sigma' = r_\beta r_\beta \Sigma$.

If $\gamma = \beta$, then $\langle \eta | \beta \rangle \neq 0$ and the property (ii) gives $\eta - j\beta \in S$ for some $j \geq 1$. Then $S$ contains

$$s_\alpha(\eta - j\beta) = s_\alpha(\eta - \gamma) - (j - 1)s_\alpha \beta > \eta,$$

a contradiction. The case $\gamma = \beta'$ is similar. Consider the remaining case $\gamma = \beta + \beta'$. Then $\langle \eta | \beta \rangle, \langle \eta | \beta' \rangle \neq 0$. Let $\beta' > \beta$. By (ii) $\eta - j\beta \in S$ for some $j \in \mathbb{N}$ and then $\eta - j\beta - i\beta' \in S$ for some non-zero integral $i$. If $i < 0$, then $\eta - j\beta - i\beta' \succ \eta$ which contradicts to the maximality of $\eta$. Hence $i \geq 1$. Then

$$s_\alpha(\eta - j\beta - i\beta') = s_\alpha(\eta - \gamma) - (j - 1)s_\alpha \beta - (i - 1)s_\alpha \beta' > \eta,$$

a contradiction. Hence $\eta \in P^+(\Delta^+) \text{ as required}$. 

Combing Lemmatta 3.4, 3.8.1, 3.8.3 we obtain the

3.8.4. **Corollary.** For $\mathfrak{gl}(n|n)$ one has $\text{Ch}(\mathfrak{g}) = A(\mathfrak{g})$; this ring admits a short basis with respect to $\Delta^+(\Sigma)$.

4. **Grothendieck rings for subcategories of $\mathcal{F}\text{in}(\mathfrak{g})$**

In this section we show that our description of $\mathcal{A}_-(\mathfrak{g})$ is equivalent to Sergeev-Veselov description. In 4.3 we deduce the formula $\text{Ch}(\mathfrak{p}_n) = A(\mathfrak{p}_n)$ from the results of [5].

4.1. **Category $\mathcal{F}(P)$**. The group $\mathbb{Z}\Delta$ acts on $P_0$ by the shifts $t_\gamma(\mu) := \mu + \gamma$ for $\gamma \in \mathbb{Z}\Delta$ and $\mu \in P_0$. Let $P \subseteq P_0$ be an invariant subset (i.e., $\mu + \alpha \in P$ for all $\mu \in P$ and $\alpha \in \Delta$). We denote by $\mathcal{F}(P)$ (resp., $\mathcal{F}_h(P)$) the full subcategory of $\mathcal{F}\text{in}(\mathfrak{g})$ (resp., of $\mathcal{F}\text{in}(\mathfrak{h})$) with the weights in $P$.

4.1.1. We set $R(P) := \text{res}_h^\mathfrak{g} \mathcal{K}(\mathcal{F}_h(P))$. In other words, $R(P)$ is a $\mathbb{Z}[\xi]$-span of $\text{ch}_\mu C_\nu$ for $\nu \in P$. The ring $R(P_0)$ corresponds to the case $P = P_0$. Note that $\text{ch}_\lambda L(\lambda) \in R(P)$ for each $\lambda \in P$. We denote by $\text{Ch}(\mathcal{F}(P))$ the image $\text{res}_h^\mathfrak{g} \mathcal{K}(\mathcal{F}(P))$.

Since $P + \mathbb{Z}\Delta \subseteq P$, we have $WP = P$ and so $W(R(P)) = R(P)$. We set

$$\mathcal{A}(P) := \mathcal{A}(\mathfrak{g}) \cap R(P).$$
4.1.2. Lemma. One has \( \text{Ch}(\mathcal{F}(P)) = R(P) \cap \text{Ch}(g) \). In particular, \( \text{Ch}(g) = \mathcal{A}(g) \) implies \( \text{Ch}(\mathcal{F}(P)) = \mathcal{A}(P) \) for any \( P \).

Proof. The inclusion \( \subset \) is straightforward. For the inverse inclusion take \( a \in R(P) \cap \text{Ch}(g) \). Write \( a = \sum_{i=1}^{s} m_i \text{ch}_i L(\lambda_i) \) where \( m_i \in \mathbb{Z}[\xi] \). If \( \lambda_s \) is maximal among \( \lambda_1, \ldots, \lambda_s \), then \( \lambda_s \in \text{supp}(a) \), so \( \lambda_s \in P \) and \( \text{ch}_i L(\lambda_s) \subset R(P) \). Hence \( a' := \sum_{i=1}^{s-1} m_i \text{ch}_i L(\lambda_i) \) lies in \( R(P) \cap \text{Ch}(g) \). Using the induction on \( s \) we get \( a' \in \text{Ch}(\mathcal{F}(P)) \), so \( a \in \text{Ch}(\mathcal{F}(P)) \) as required. \( \square \)

4.1.3. Remark. Using the same argument for \( \text{ch} L(\lambda_i) \) we obtain

\[
\text{Ch}_+(\mathcal{F}(P)) = R(P) \cap \text{Ch}_+(g).
\]

For \( \text{Ch}_- \) a similar argument works if \( h = t \) or if \( \mathcal{A}_-(g) \) admits a short basis defined in 3.1.

4.1.4. Examples. Let \( g \) be one of the algebras \( \mathfrak{gl}(m|n), \mathfrak{osp}(m|n), q_n \). Let \( P_{\text{int}} \) be the integral lattice which is the lattice spanned by the weights of the standard representation. By Lemma 4.1.2 the formula \( \text{Ch}(g) = \mathcal{A}(g) \) implies \( \text{Ch}(\mathcal{F}(P_{\text{int}})) = R(P_{\text{int}}) \cap \mathcal{A}(g) \).

Let \( g := q_n \) and \( P \) be the set of “half-integral weights”. The formula \( \text{Ch}(g) = \mathcal{A}(g) \) implies \( \text{Ch}(\mathcal{F}(P)) = R(P) \cap \mathcal{A}(g) \).

4.1.5. Similarly to Corollary 2.5.1 we have

\[
\mathcal{A}(P) = \mathcal{A}_+(P) \times_{R(P_0)} \mathcal{A}_-(P) \quad \text{Ch}(P) = \text{Ch}_+(P) \times_{R(P_0)} \text{Ch}_-(P).
\]

4.1.6. Let \( g \) be such that \( h = t \) and that for some group homomorphism \( p : \mathbb{Z}\Delta \rightarrow \mathbb{Z}_2 \) one has \( p(\Delta_\tau) = 7 \). Assume, in addition, that \( P \) is a subgroup of \( P_0 \) which contains \( \mathbb{Z}\Delta \) and that \( p \) can be extended to the group homomorphism \( p : \mathbb{Z}\Delta \rightarrow \mathbb{Z}_2 \). In this case \( \mathbb{Z}[P] = R_{\Delta}(P) \) admits an automorphism \( \iota : e^\nu \mapsto (-1)^{p(\nu)}e^\nu \). We claim that this automorphism induces ring isomorphisms

\[
\mathcal{A}_-(P) \xrightarrow{\sim} \mathcal{A}_+(P), \quad \text{Ch}_-(P) \xrightarrow{\sim} \text{Ch}_+(P).
\]

Indeed, since \( \iota \) is an automorphism, it is enough to check that \( \iota(\mathcal{A}_-(P)) = \mathcal{A}_+(P) \) and \( \iota(\text{Ch}_-(P)) = \text{Ch}_+(P) \). The first formula immediately follows from [8]. For the second formula note that the condition \( h = t \) implies that \( \text{Ch}_+(P) \) are free \( \mathbb{Z} \)-modules with bases \( \{\text{ch} L(\lambda)\}_{\lambda \in P \cap P_0(\Delta_+) \cap P_0(\Delta_+) \text{ and } \{\text{sch} L(\lambda)\}_{\lambda \in P \cap P_0(\Delta_+) \text{ respectively. One readily sees } \iota(\text{ch} L(\lambda)) = \text{sch} L(\lambda); \text{ this gives the required formula.}

4.1.7. The above assumptions hold, for example, if \( g \) is a finite-dimensional Kac-Moody superalgebra and \( P = \mathbb{Z}\Delta \) or if \( g = \mathfrak{gl}(m|n), \mathfrak{osp}(m|n), p_n \) and \( P = P_{\text{int}} \) (see 4.1.4).
4.2. Sergeev-Veselov description of $A_{\pm}(g)$. Fix $\beta \in \Delta_{iso}$. Let $t = t' \times t''$ be a decomposition with $\dim t'' = 2$, $\beta \in (t'')^*$ and $(t')^* \subset \{ \mu \setminus \langle \mu, h_\beta \rangle = 0 \}$. Fix $\omega \in (t'')^*$ satisfying $\langle \omega, h_\beta \rangle \neq 0$. Since $\langle \beta, h_\beta \rangle = 0$, the elements $\omega, \beta$ form a basis of $(t'')^*$.

We set $x := e^\omega$, $y := e^\beta$ and write $f \in \mathbb{Z}[t^*]$ in the form

$$f = \sum_{a,b \in \mathbb{C}} f_{a,b} x^a y^b$$

where $f_{a,b} \in \mathbb{Z}[(t')^*].$

4.2.1. **Lemma.** Assume that $\Delta_{iso} = W_\beta \cup (-W_\beta)$. Then

$$(8) \quad A_{\pm}(g) = \{ f \in R_\pm(g)^W \mid \frac{\partial f}{\partial x} \in \mathbb{Z}[t^*](y \pm 1) \}.$$ 

**Proof.** Take $f \in \mathbb{Z}[t^*]$ and write $f = \sum m_\nu e^\nu = \sum_{a,b \in \mathbb{C}} f_{a,b} x^a y^b$, where

$$f_{a,b} = \sum_{\nu' \in (t')^*} m_{\nu' + a\omega + b\beta} e^{\nu'}.$$ 

Then $\frac{\partial f}{\partial x} = \sum_{a,b \in \mathbb{C}} a f_{a,b} x^{a-1} y^b$. The condition $\frac{\partial f}{\partial x} \in \mathbb{Z}[t^*](y - 1)$ is equivalent to

$$\forall b \quad \sum_i a f_{a,b+i} x^{a-1} = 0$$

that is for each $a, b$ one has

$$0 = a \sum_i f_{a,b+i} = a \sum_{\nu' \in (t')^*} m_{\nu'+a\omega+b\beta} e^{\nu'}.$$ 

Since $\langle \nu' + a\omega, h_\beta \rangle \neq 0$ if and only if $a = 0$ we get

$$\{ f \in \mathbb{Z}[t^*] \mid \frac{\partial f}{\partial x} \in \mathbb{Z}[t^*](y - 1) \} = \{ \sum_{\nu} m_{\nu} e^\nu \mid \langle \nu, h_\beta \rangle \neq 0 \implies \sum_i m_{\nu+i\beta} = 0 \}.$$ 

Similarly

$$\{ f \in \mathbb{Z}[t^*] \mid \frac{\partial f}{\partial x} \in \mathbb{Z}[t^*](y + 1) \} = \{ \sum_{\nu} m_{\nu} e^\nu \mid \langle \nu, h_\beta \rangle \neq 0 \implies \sum_i (-1)^i m_{\nu+i\beta} = 0 \}.$$ 

$\Box$
4.2.2. Example. Take \( g = \mathfrak{gl}(m|n) \), \( \beta := \delta_n - \varepsilon_m \) and \( \omega := \varepsilon_m \). Since \( h = t \) we have \( R(g) = Z[P_0] \) and (3) gives

\[
A_\pm(g) = \{ f \in Z[P_0]^W \mid \frac{\partial f}{\partial x} \in Z[t^*](y \pm 1) \}
\]

where \( x := e^{\varepsilon_m} \) and \( y := e^{\delta_n - \varepsilon_m} \). Set \( x_m := e^{\varepsilon_m} = x \) and \( y_n := e^{\delta_n} = y \). Then

\[
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x_m} + y_n \frac{\partial f}{\partial y_n}.
\]

Since \( x_m \) is invertible in \( Z[t^*] \), the condition \( \frac{\partial f}{\partial y} \in Z[t^*](y - 1) \) can be rewritten as

\[
x_m \frac{\partial f}{\partial x_m} + y_n \frac{\partial f}{\partial y_n} \in Z[t^*](x_m - y_n)
\]

which is the Sergeev-Veselov condition for \( A_-(\mathfrak{gl}(m|n)) \).

4.2.3. Hoyt-Reif description of \( A_-(\mathcal{P}) \). Assume that \( \mathcal{P} \subset (t^*)^+ + Z\beta + Z\omega \). Then

\[
Z[P; \xi] \subset Z[(t^*)^+; \xi] \otimes Z[x^{\pm 1}, y^{\pm 1}].
\]

For each \( c \in C^* \) we denote by \( ev_c \) the evaluation map \( ev_c : Z[P; \xi] \rightarrow Z[(t^*)^+; \xi] \) given by \( x \mapsto c \) and \( y \mapsto -1 \).

4.2.4. Corollary. Assume that \( \Delta_{iso} = W\beta \cup (-W\beta) \). If \( \mathcal{P} \subset (t^*)^+ + Z\beta + Z\omega \), then \( A_-(\mathcal{P}) = \{ f \in R[P]^W \mid ev_c(f) \text{ does not depend on } c \} \).

Proof. Take \( f = \sum m_{\nu'}e^{\nu'} \in Z[P] \). For each \( \nu' \in (t^*)^+ \) and \( j \in \mathbb{Z} \) set

\[
n_{\nu',c} := \sum_{i} m_{\nu'+j\omega - i\beta}.
\]

The condition \( \sum_{i \in \mathbb{Z}} m_{\nu'+i\beta} = 0 \) for \( \langle \nu, h_\beta \rangle \neq 0 \) means that \( n_{\nu',j} = 0 \) if \( j \neq 0 \). Since \( ev_c(f) = \sum_{\nu',j} n_{\nu',j} c^j \), the above condition holds if and only if \( ev_c(f) \) does not depend on \( c \). □

4.3. Example: \( A(\mathfrak{p}_n) \). In this section we will deduce the formula \( A(\mathfrak{p}_n) = \text{Ch}(\mathfrak{p}_n) \) form the results of [5]. It would be interesting to obtain this formula by the method used in Section [3]

4.3.1. Set \( g := \mathfrak{p}_n \). One has \( \mathfrak{g}_\mathfrak{g} = \mathfrak{gl}_n \) and \( h = t \), so \( R(P_0) = Z[P_0; \xi] \). We have

\[
\Delta_+ = \{ \varepsilon_i - \varepsilon_j \}_{1 \leq i < j \leq \mathbb{N}}, \quad \Delta_{iso} = \{ \pm (\varepsilon_i + \varepsilon_j) \}_{1 \leq i < j \leq \mathbb{N}}, \quad \Delta_T = \{ 2\varepsilon_i \}_{i=1}^n \coprod \Delta_{iso}.
\]

The group \( W = S_n \) acts on \( t^* \) by permuting \( \Delta^+ \). We set

\[
\text{str} := \sum_{i=1}^n \varepsilon_i, \quad \beta := \varepsilon_{n-1} + \varepsilon_n.
\]
One has $P_0 = \mathbb{C} \text{str} + P_{\text{int}}$. Since $\Delta_{\text{iso}} = W(\beta) \prod W(-\beta)$ we have

$$A(g) = \{ \sum \nu m\nu e^\nu \in \mathbb{Z}[P_0; \xi]^W \mid \langle \nu, h_\beta \rangle \neq 0 \implies \sum_{i \in \mathbb{Z}} (-\xi)^i m_{\nu+i\beta} = 0 \}.$$ 

4.3.2. Lemma. Denote by $\mathbb{Z}[\mathbb{C} \text{str}]$ the group ring of $\mathbb{C} \text{str}$. Then

$$\text{Ch}(g) = \mathbb{Z}[\mathbb{C} \text{str}] \otimes \text{Ch}(P_{\text{int}}), \quad A(g) = \mathbb{Z}[\mathbb{C} \text{str}] \otimes A(P_{\text{int}}).$$

Proof. The first formula follows from the fact that every finite-dimensional module $M$ can be obtained from $M \in \mathcal{F}(P_{\text{int}})$ by tensoring with one dimensional module $L(c \text{str})$. For the second formula write $a \in \mathbb{Z}[P_0; \xi]$ in the form

$$a = \sum \nu m\nu e^\nu = \sum_{c \in \mathbb{C}/\mathbb{Z}} e^{c \text{str}} a_c \quad \text{with} \quad a_c \in \mathbb{Z}[P_{\text{int}}, \xi].$$

One readily sees that $a \in A(g)$ if and only if $a_c \in A(g)$ for all $c$. This gives the second formula. \hfill \square

4.3.3. By [5] we have $\text{Ch}_-(P_{\text{int}}) = A_-(P_{\text{int}})$. Using [4.1.6, (7)] and Lemma 4.3.2 we get $\text{Ch}_+(P_{\text{int}}) = A_+(P_{\text{int}})$, $\text{Ch}(P_{\text{int}}) = A(P_{\text{int}})$ and, finally, $\text{Ch}(g) = A(g)$.

5. Appendix

In this section we recall a description of the set of dominants weights for a Kac-Moody superalgebra. The results of this section can be found in [2], [4], [6], [9] and other sources.

5.1. General results. Let $\mathfrak{s}$ be a Lie superalgebra and let $V$ be a $\mathfrak{s}$-module $V$.

5.1.1. Notation. For $a \in \mathfrak{s}$ we set $V^a := \{ v \in V \mid av = 0 \}$. We say that $a$ acts locally finitely (resp., locally nilpotently) on $V$ if any $v \in V$ lies in a finite-dimensional $a$-invariant subspace of $V$ (resp., $a^s v = 0$ for $s \gg 0$). We say that $V$ is $\mathfrak{s}$-locally finite if any $v \in V$ lies in a finite-dimensional $\mathfrak{s}$-submodule of $V$. The following important proposition holds for Lie algebras, but does not hold for Lie superalgebras.

5.1.2. Proposition ([1], Prop. 3.8). Let $\mathfrak{s}$ be a finite-dimensional Lie algebra and $F_V \subset \mathfrak{s}$ be the set of elements in $\mathfrak{s}$ which act locally finitely on a $\mathfrak{s}$-module $V$. If $F_V$ generates $\mathfrak{s}$ (as a Lie algebra), then

(i) $F_V$ spans $\mathfrak{s}$;
(ii) $V$ is $\mathfrak{s}$-locally finite.
5.1.3. Example. The Lie superalgebra $\mathfrak{osp}(2n\vert 2n)$ admit a base $\Sigma$ consisting of isotropic roots and is generated by $e_{\pm\alpha}$ with $\alpha \in \Sigma$. Since $\alpha$ is isotropic, $e_{\pm\alpha}^2 = 0$, so each $e_{\pm\alpha}$ acts nilpotently on any $\mathfrak{osp}(2n\vert 2n)$-module. However, if $V$ is an infinite-dimensional simple module, $V$ is not locally finite.

5.1.4. Corollary. Let $\mathfrak{s}$ be a finite-dimensional Lie superalgebra and $t \subset \mathfrak{s}$ be a commutative subalgebra. Let $V$ be a simple $\mathfrak{s}$-module where $t$ acts diagonally. Set

$$F := \{ a \in \mathfrak{s}^0 \mid a \text{ is ad-nilpotent and } V^a \neq 0 \}.$$

If $F + t$ generates $\mathfrak{s}^0$, then $V$ is finite-dimensional.

Proof. Take $a \in F$ and a non-zero $v \in V^a$. Since $a$ acts locally nilpotently on $U(\mathfrak{s})$, $a$ acts locally nilpotently on $U(\mathfrak{s})v$. Since $V$ is simple, $a$ acts locally nilpotently on $V$. By (ii), any cyclic $\mathfrak{s}^0$-submodule of $V$ is finite-dimensional. By PBW Theorem, $V$ is finitely generated over $\mathfrak{s}^0$; hence $V$ is finite-dimensional. \hfill \Box

5.2. Applications to Kac-Moody case. From now on $\mathfrak{g}$ is a finite-dimensional Kac-Moody superalgebra or $\mathfrak{gl}(m\vert n) = \mathfrak{sl}(m\vert n) \times \mathbb{C}$. The algebra $\mathfrak{g}^0$ is reductive.

5.2.1. Notation. We retain notation of 1.1. In our case $\mathfrak{h} = t$ and $\mathfrak{g}$ admits a non-degenerate invariant bilinear form; we denote by $(-|-)$ the corresponding form on $t^*$. Recall that $\Delta_{\text{iso}}$ stands for the set of roots $\beta$ such that for some $e_{\pm} \in \mathfrak{g}_{\beta}$ the subalgebra spanned by $e_{\pm}, h_{\beta} := [e_+, e_-]$ is isomorphic to $\mathfrak{sl}(1|1)$. One has $\Delta_{\text{iso}} = \{ \beta \in \Delta \mid (\beta|\beta) = 0 \}$.

In our case $\Delta_{\mathfrak{g}} \cap \Delta_{\text{iso}} = \emptyset$. Moreover, for each $\alpha \in \Delta_{\mathfrak{g}}$ the algebra generated by $\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}$ is isomorphic to $\mathfrak{sl}_2$; we denote by $\alpha^\vee$ the coroot corresponding to $\alpha$.

A set $\Sigma \subset \Delta^+$ is called a base if the elements of $\Sigma$ are linearly independent and $\Delta^+$ lies in $\pm N\Sigma$ (where $N\Sigma$ is the set of non-negative integral linear combinations of $\Sigma$); in this case we write $\Delta^+ = \Delta^+(\Sigma)$. For the superalgebras which we consider each set of positive roots admits a base. We denote by $\mathcal{B}$ the set of bases compatible with the fixed triangular decomposition of $\mathfrak{g}_{\mathfrak{g}}$, i.e., $\Sigma \in \mathcal{B}$ if $\Delta_{\mathfrak{g}}^+ \subset \Delta^+(\Sigma)$.

For each $\Sigma \in \mathcal{B}$ we set $\Sigma_{\text{iso}} = \Sigma \cap \Delta_{\text{iso}}$. For each $\alpha \in \pi$ we set

$$\mathcal{B}_\alpha := \{ \Sigma \in \mathcal{B} \mid \alpha \in \Sigma \text{ or } \frac{\alpha}{2} \in \Sigma \}.$$

5.2.2. Odd reflections. Given a subset of positive roots $\Delta^+$ (containing $\Delta_{\mathfrak{g}}^+$) and $\beta \in \Sigma_{\text{iso}}$, we can construct a new subset of positive roots (containing $\Delta_{\mathfrak{g}}^+$) by an odd reflection $r_\beta$:

$$r_\beta(\Delta^+) = (\Delta^+ \setminus \{ \beta \}) \cup \{-\beta\}.$$

We denote by $r_\beta \Sigma$ the base of $r_\beta \Delta^+(\Sigma)$. 
5.2.3. **Proposition** [12].

(i) Any two subsets of positive roots in $\Delta$ (containing $\Delta^+_0$) can be obtained from each other by a finite sequence of odd reflections.
(ii) The set $B_\alpha$ is non-empty for each $\alpha \in \pi$.
(iii) $r_\beta \Sigma$ contains $-\beta$, the roots $\alpha \in \Sigma \setminus \{\beta\}$, which are orthogonal to $\beta$, and the roots $\alpha + \beta$ for $\alpha \in \Sigma$ which are not orthogonal to $\beta$.

5.2.4. **Highest weight modules.** For any $\Sigma \in B$ we denote by $L(\lambda; \Sigma)$ the irreducible module of highest weight $\lambda$ with respect to the Borel subalgebra corresponding to $\Sigma$. For an odd root $\beta \in \Sigma$ one has

$$
L(\lambda; \Sigma) = \begin{cases} 
L(\lambda; r_\beta \Sigma) & \text{if } \langle \lambda, h_\beta \rangle = 0 \\
L(\lambda - \beta; r_\beta \Sigma) & \text{if } \langle \lambda, h_\beta \rangle \neq 0.
\end{cases}
$$

In particular, if $L$ is a simple highest weight module with respect to some $\Sigma \in B$, then $L$ is a simple highest weight module with respect to any $\Sigma' \in B$. We denote by $\text{wt}_\Sigma(L)$ the highest weight of $L$ with respect to $\Sigma$.

5.3. **Proposition.** For a simple highest weight $\mathfrak{g}$-module $L$ the following conditions are equivalent

(a) $L$ is finite-dimensional;  
(b) for each $\alpha \in \pi$ there exists $\Sigma \in B_\alpha$ such that $\text{wt}_\Sigma(L) \in P^+(\pi)$;  
(c) for each $\alpha \in \pi$ there exists $\Sigma \in B_\alpha$ such that $\langle \text{wt}_\Sigma(L), \alpha^\vee \rangle \in \mathbb{N}$.

**Proof.** If $L$ is finite-dimensional, then for each base $\Sigma \in B$ one has $\text{wt}_\Sigma(L) \in P^+(\pi)$ and so $\langle \text{wt}_\Sigma(L), \alpha^\vee \rangle \in \mathbb{N}$ for each $\alpha \in \pi$. This gives the implications (a) $\implies$ (b) $\implies$ (c).

For the implication (c) $\implies$ (a) we assume that for each $\alpha \in \pi$ there exists $\Sigma \in B_\alpha$ such that $\langle \text{wt}_\Sigma(L), \alpha^\vee \rangle \in \mathbb{N}$. By Corollary 5.1.4, it is enough to show that the set

$$
F := \{a \in \mathfrak{g}_\pi \mid a \text{ is ad-nilpotent and } L^a \neq 0\}.
$$

contains $\mathfrak{g}_{\pm \alpha}$ for each $\alpha \in \pi$. Fix $\alpha \in \pi$ and $\Sigma \in B_\alpha$ such that for $\nu := \text{wt}_\Sigma(L)$ one has $\langle \nu, \alpha^\vee \rangle \in \mathbb{N}$.

Since $L$ is a highest weight module, $F$ contains $\mathfrak{g}_\alpha$. Fix a highest weight vector $v \in L_\nu$. Set $\alpha' := \alpha$ if $\alpha \in \Sigma$ and $\alpha' := \alpha/2$ if $\alpha/2 \in \Sigma$. Denote by $I$ the subalgebra generated by $\mathfrak{g}_{\pm \alpha'}$. One has $I \cong \mathfrak{sl}_2$ if $\alpha' = \alpha$ and $I \cong \mathfrak{osp}(1|2)$ if $\alpha' = \alpha/2$; it is well-known that a simple highest weight $I$-module $L_I(\mu)$ is finite-dimensional if and only if $\langle \mu, \alpha^\vee \rangle \in \mathbb{N}$. Fix non-zero $e_+ \in \mathfrak{g}_{\alpha'}$. Since $\langle \nu, \alpha^\vee \rangle \in \mathbb{N}$ one has $\dim L_I(\nu) < \infty$, so $e_+ e_+^k v = 0$ for some $k > 0$. For each $\beta \in \Sigma \setminus \{\alpha'\}$ one has $[\mathfrak{g}_\beta, e_+] = 0$, so $\mathfrak{g}_\beta(e_+^k v) = 0$. Hence $e_+^k v$ is a primitive vector. Since $L$ is simple, $e_+ v = 0$. Since $\mathfrak{g}_{-\alpha}$ is spanned either by $e_-$ or by $e_+^2$, $\mathfrak{g}_{-\alpha}$ acts nilpotently on $v$, so $\mathfrak{g}_{-\alpha} \subset F$ as required. □
5.3.1. **Corollary.** Let $\Sigma \in \mathcal{B}$ be such that for each $\alpha \in \pi \setminus \Sigma$ one has $r_\beta \Sigma \in \mathcal{B}_\alpha$ for some $\beta \in \Sigma_{iso}$. Then

$$P^+(\Delta^+) = \{ \lambda \in P^+(\pi) | \forall \beta \in \Sigma \cap \Delta_{iso} \langle \lambda, h_\beta \rangle \neq 0 \implies \lambda - \beta \in P^+(\pi) \}. \tag{10}$$

**Proof.** The assertion follows from Proposition 5.3 and (9). \qed

5.3.2. **Corollary.** If $\Sigma$ satisfies the property

$$\text{each } \alpha \in \pi \text{ is the sum of at most two elements in } \Sigma$$

then the assumption of Corollary 5.3.1 holds for $\Sigma$.

**Proof.** It is enough to check that if $\alpha \in \pi$ is the sum of two different roots from $\Sigma$, then $\alpha \in r_\beta \Sigma$ for some $\beta \in \Sigma_{iso}$. Write $\alpha = \beta + \beta'$ for $\beta \neq \beta' \in \Sigma$. Since $\alpha \in \pi$, the roots $\beta$, $\beta'$ are odd.

Assume that $\beta \in \Delta_{iso}$. If $(\beta|\beta') \neq 0$, then $\alpha \in r_\beta \Sigma$ as required. If $(\beta|\beta') = 0$, then $-\beta, \beta' \in r_\beta \Sigma$, so $\alpha \not\in \Delta^+(r_\beta \Sigma)$ which contradicts to $\Delta^+(r_\beta \Sigma) = \Delta^+(\Sigma) \setminus \{\beta\} \cup \{-\beta\}$.

If $\beta, \beta' \not\in \Delta_{iso}$ are non-isotropic, then $2\beta, 2\beta' \in \Delta^+_0$ (since $\beta, \beta'$ are odd) and $2\beta - 2\beta' = 2\alpha$ which contradicts to $\alpha \in \pi$. \qed

5.4. **Dynkin diagrams.** The root systems corresponding to $g$ are called: $A(m|n)$ (for $g = \mathfrak{gl}(m|n)$ or $g = \mathfrak{sl}(m|n)$ with $m \neq n$), $B(m|n)$ (for $\mathfrak{osp}(2m+1|2n)$), $D(m|n)$ (for $\mathfrak{osp}(2m|2n)$), $G(3)$ and $F(4)$. The root system of $D(2|1)$ coincides with the root system $D(2|1)$. The root systems $D(1|n)$ is also called $C(n+1)$. The root systems $A(m|n), D(1|n)$ are called type I systems; the rest of roots systems are called type II systems. The root system $\Delta$ is of type II if and only if $g_5$ is semisimple.

The Dynkin diagram of $\Sigma \in \mathcal{B}$ was introduced in [6]. Except for $A(m|n)$ case there is a one-to-one correspondence between $\mathcal{B}$ and the sets of Dynkin diagram. For $A(m|n)$ case the set $\mathcal{B}$ admits an involution $-w_0$ and $\Sigma$, $-w_0 \Sigma$ have the same Dynkin diagram ($w_0$ is the longest element in the Weyl group).

5.4.1. **Mixed bases.** We say that a Dynkin diagram is *mixed* if this diagram contains

— as many “•”s as possible,
— as less “◦”s as possible,
— as many cycles containing the edge $⊗ − −⊗$ as possible.

We say that a base $\Sigma \in \mathcal{B}$ is *mixed* if its Dynkin diagram is mixed. The mixed Dynkin diagram is unique; a mixed base is unique if $\Delta$ is not of type $A(m|n)$ or has type $A(n+1|n)$. For $A(m|n)$ with $m \neq n \pm 1$ there are two mixed bases: $\Sigma$ and $-w_0 \Sigma$.

For $A(n|n), A(n + 1|n), B(n|n), D(n + 1|n), D(n|n)$ the mixed bases consists of odd roots.
The mixed bases satisfy the property (11). Moreover for type II systems the mixed bases are the only bases satisfying (11), see 5.5 below.

5.5. Property (11). Consider a simplified version of the Dynkin diagram: we assign to each simple root $\alpha$ a node $\bullet$ if $\alpha$ is even, $\circ$ if $\alpha$ is isotropic and $\cdot$ if $\alpha$ is odd and non-isotropic; we connect two nodes $\alpha, \beta$ if $\alpha + \beta \in \Delta$ (it is easy to see that $\alpha + \beta \in \Delta$ if and only if $(\alpha | \beta) \neq 0$). Denote the diagram by Dyn and denote by Dyn_{odd} the subdiagram which contains only $\circ$, $\bullet$ and the edges between them.

5.5.1. Remark. For $F(4)$ there are two bases in $\mathcal{B}$ which correspond to the same simplified Dynkin diagram $\circ - \circ - \circ$. 

5.5.2. Below we list the simplified Dynkin diagrams for mixed bases.

For $A (m | n)$ the diagram is $\circ - \circ - \cdots - \circ - \circ$ with $m + n - 1$ nodes where the number $\#(\circ)$ of $\circ$ nodes is 0 for $m = n, n \pm 1$ and is $m - n - 1$ if $m > n$.

For $B (m | n)$ the diagram is $\circ - \cdots - \circ - \circ - \cdots - \circ$, where $\#(\circ) = |m - n|$ and the total number of nodes is $m + n$. For $G (3)$ we have a similar simplified Dynkin diagram with three nodes and $\#(\circ) = 1$.

For $D (1 | n) = C (n + 1)$ the diagram contains $1 + n$ nodes and takes the form

\[
\begin{array}{c}
\circ \quad \circ \quad \cdots \quad \circ \quad \circ \\
\end{array}
\]

For $D (m | n)$ with $m > 1$ the diagram contains $m + n$ nodes and takes the form

\[
\begin{array}{c}
\circ \quad \circ \quad \cdots \quad \circ \quad \circ \quad \cdots \quad \circ \\
\end{array}
\]

where $\#(\circ) = 0$ for $m = n, n + 1$, $\#(\circ) = m - n - 1$ for $m > n$ and $\#(\circ) = n - m$ for $n > m$. For $F(4)$ we have a similar diagram with four nodes and $\#(\circ) = 1$.

5.5.3. Let $\ell$ (resp., $\ell'$) be the cardinality of $\pi$ (resp., of $\Sigma$).

It is easy to see that for two odd simple roots $\beta, \beta'$ with $\beta + \beta' \in \Delta$ one has $\beta + \beta' \in \pi$. As a result the number of roots in $\pi$ which can be written as $\beta + \beta'$ for $\beta, \beta' \in \Sigma$ is equal...
to the number of edges in $\text{Dyn}_{\text{odd}}$ which we denote by $\#(\text{odd edge})$ plus the number of $\bullet$s which we denote by $\#(\bullet)$. Since $\ell' = \#(\diamond) + \#(\bullet) + \#(\otimes)$ the property (11) can be rewritten as

\begin{equation}
\ell' - \ell = \#(\otimes) - \#(\text{odd edge}).
\end{equation}

For type I we have $\ell = \ell' - 1$ and $\#(\bullet) = 0$. The formula (12) means that the number of edges in $\text{Dyn}_{\text{odd}}'$ is less by one than the number of vertices. The diagrams $\text{Dyn}_{\text{odd}}$ do not have cycles, so this condition holds if and only if $\text{Dyn}_{\text{odd}}$ is connected. For $C(n + 1)$ the diagram $\text{Dyn}_{\text{odd}}$ is always connected and (12) holds.

For type II we have $\ell = \ell'$.

For $B(m|n)$ and $G(3)$ one has $\#(\bullet) \leq 1$ and the diagrams $\text{Dyn}_{\text{odd}}$ do not have cycles. Therefore (12) holds if and only if $\text{Dyn}_{\text{odd}}$ is connected and $\#(\bullet) = 1$. The mixed Dynkin diagram is the only diagram satisfying these conditions.

For the remaining cases $D(m|n)$ and $F(4)$ one has $\Delta_1 = \Delta_{\text{iso}}$, so $\#(\bullet) = 0$. From the classification it follows that Dyn may contain at most one cycle. Therefore (12) holds if and only if $\text{Dyn}_{\text{odd}}$ is connected and contains a cycle. The mixed Dynkin diagram is the only diagram satisfying these conditions.

5.5.4. Remark. If all odd roots are isotropic, then the assumption of Corollary 5.3.1 is equivalent to (11). For $B(m|n)$ the assumption of Corollary 5.3.1 holds for the mixed diagram and for $\Sigma'$ corresponding to $\diamond - \cdots - \diamond - \diamond - \cdots - \diamond$, where $\#(\diamond) = |m - n + 1|$; for instance, $B(1|1)$ admits two diagrams $\diamond - \bullet$ and $\diamond - \diamond$; the formula (10) hold in both cases. For $G(3)$ the assumption of Corollary 5.3.1 holds for the mixed diagram and for the cyclic diagram.

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