Indices of inseparability for elementary abelian 
\(p\)-extensions

Kevin Keating  
Department of Mathematics  
University of Florida  
Gainesville, FL 32611  
USA  
keating@ufl.edu

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Abstract

Let \(K\) be a local field whose residue field \(\overline{K}\) is a finite field of characteristic \(p\) and let \(L/K\) be a finite totally ramified Galois extension. Fried [3] and Heiermann [5] defined the “indices of inseparability” of \(L/K\), a refinement of the ramification data of \(L/K\). We give a method for computing the indices of inseparability of the extension \(L/K\) in terms of the norm group \(N_{L/K}(L^\times)\) in the case where \(K\) has characteristic \(p\) and \(\text{Gal}(L/K)\) is an elementary abelian \(p\)-group with a single ramification break. In some cases our methods lead to simple formulas for the indices of inseparability.

1 Introduction

Let \(K\) be a local field with finite residue field \(\overline{K}\) and let \(L/K\) be a finite separable extension. The \textit{indices of inseparability} of \(L/K\) were defined by Fried [3] for fields of characteristic \(p\), and by Heiermann [5] for fields of characteristic 0. The indices of inseparability of \(L/K\) constitute a refinement of the usual ramification data for \(L/K\), as described for instance in Serre [7, IV], in that the indices of inseparability determine the ramification data, but the ramification data does not always determine the indices of inseparability. More precisely, suppose \(L/K\) is Galois, with Galois group \(G = \text{Gal}(L/K)\). Then \(G\) has a filtration by the lower ramification subgroups \(G \supseteq G_0 \supseteq G_1 \supseteq \ldots\), as defined in [7, IV §1], and the ramification data of \(L/K\) is determined by the sequence \((|G_i|)_{i\geq 0}\). Each ramification subgroup \(G_i\) is normal in \(G\), and the quotients \(G_i/G_{i+1}\) for \(i \geq 1\) are elementary abelian \(p\)-groups. If these quotients are all cyclic then the ramification data of \(L/K\) determines the indices of inseparability, but if there is some
Let $K^{alg}$ be an algebraic closure of $K$. Class field theory gives a one-to-one correspondence between the set of finite abelian subextensions $L/K$ of $K^{alg}/K$ and the set of closed finite-index subgroups $H$ of $K^\times$. This correspondence maps the extension $L/K$ onto the group $N_{L/K}(L^\times)$, where $N_{L/K} : L \to K$ is the norm map. In principle it should be possible to determine the indices of inseparability of $L/K$ in terms of the norm group $N_{L/K}(L^\times)$. In fact it is straightforward to compute the ramification data of $L/K$ in terms of $N_{L/K}(L^\times)$ (see for instance Theorem 2 in [7, XV §2]), but extracting the additional information contained in the indices of inseparability seems to be considerably more difficult. Perhaps this should not be surprising, since the ramification data of $L/K$ can be defined in terms of the norm $N_{L/K}$. This is the approach taken by Fesenko and Vostokov in [2, III §3], but their definition does not extend in any obvious way to give a general norm-based definition of the indices of inseparability.

In this paper we consider a special case of the problem of determining the indices of inseparability of an abelian extension in terms of its norm group. In order to isolate the difficulties involved in passing from the usual ramification data to the refinement given by the indices of inseparability we restrict our attention to the case of an extension with a single ramification break. Let $L/K$ be a finite totally ramified abelian extension whose Galois group $G = \text{Gal}(L/K)$ has a single ramification break $b \geq 1$. Then $G \cong G_b/G_{b+1}$ is an elementary abelian $p$-group. Thus if $[L : K] > p$ we are in a situation where knowledge of the ramification data does not determine the indices of inseparability.

In section 2 we define the indices of inseparability of a finite totally ramified extension of local fields $L/K$. In section 3 we give an explicit computation of the norm group $N_{L/K}(L^\times)$ in the case where $L/K$ is an elementary abelian $p$-extension with a single ramification break. In section 4 we assume $\text{char}(K) = p$ and interpret the results of section 3 in terms of Artin-Schreier theory. This leads to an algorithm for computing the indices of inseparability in terms of the Artin-Schreier equations which define $L$. In section 5 we use the results of sections 3 and 4 to give explicit formulas for the indices of inseparability of certain elementary abelian $p$-extensions in characteristic $p$. In section 6 we show how the results of section 5 can be extended to apply to some extensions of local fields of characteristic 0.

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2 Ramification and indices of inseparability

In this section we define the ramification data and indices of inseparability for a finite totally ramified Galois extension of local fields. We then describe how the indices of inseparability are computed in terms of the ramification data.

Let $K$ be a local field with finite residue field $\overline{K}$ of characteristic $p$ and let $v_K$ be the normalized valuation on $K$. Let $e = v_K(p)$ be the absolute ramification index of $K$; thus $e = \infty$ if and only if $\text{char}(K) = p$. Let $\mathcal{O}_K = \{\alpha \in K : v_K(\alpha) \geq 0\}$ be the ring of
integers of $K$, let $\mathcal{M}_K$ be the maximal ideal of $\mathcal{O}_K$, and for $r \geq 1$ let $U^r_K = 1 + \mathcal{M}^r_K$. Let $L/K$ be a finite separable totally ramified Galois subextension of $K^{alg}/K$ of degree $n = ap^r$, with $p \nmid a$. Also let $v_p$ denote the $p$-adic valuation on $\mathbb{Z}$.

We begin by recalling the definition of the ramification data of $L/K$. Set $G = \text{Gal}(L/K)$ and let $\pi_L$ be a uniformizer for $L$. For a nonnegative real number $x$ we define the $x$th ramification group of $G$ (with respect to the lower numbering) by

$$G_x = \{\sigma \in G : v_L(\sigma(\pi_L) - \pi_L) \geq x + 1\}.$$  

(2.1)

We easily deduce the following: $G_x$ is a normal subgroup of $G$ which is independent of the choice of $\pi_L$; $G_x \leq G_y$ for $y \leq x$; $G_0 = G$; $G_x = \{1\}$ for all sufficiently large $x$; and $G_x = G_i$, where $i = \lfloor x \rfloor$ is the smallest integer such that $i \geq x$.

Let $b \geq 0$. If there is $\sigma \in G$ such that $v_L(\sigma(\pi_L) - \pi_L) = b + 1$ then we say that $b$ is a lower ramification break for $G$; in this case $b$ must be an integer. This is equivalent to having $b \in \mathbb{Z}$ and $G_{b+1} \neq G_b$. If $b \geq 1$ then there is a group embedding of $G_b/G_{b+1}$ into $\mathcal{M}_L^b/\mathcal{M}_L^{b+1}$ which carries $\sigma G_{b+1}$ onto $\frac{\sigma(\pi_L) - \pi_L}{\pi_L} + \mathcal{M}_L^{b+1}$. Hence $G_b/G_{b+1}$ is an elementary abelian $\mathbb{Z}$-group.

We now recall the definition of the indices of inseparability $i_0, i_1, \ldots, i_\nu$ of $L/K$ as formulated by Fried in the case char($K$) = $p$ [3 pp. 232–233] (see also [4, §2]), and by Heiermann in the case char($K$) = 0 [5 §3]. Set $q = |\overline{K}|$ and let $\mu_{q-1}$ denote the group of $q - 1$ roots of unity in $K$. Then $R = \mu_{q-1} \cup \{0\}$ is the set of Teichmüller representatives for $K$; if char($K$) = $p$ then $R$ is a subfield of $K$ which can be identified with the residue field of $K$. Given uniformizers $\pi_K$, $\pi_L$ for $K$, $L$ there are unique $c_h \in R$ such that $\pi_K = \sum_{h=0}^{\infty} c_h \pi_L^{h+n}$. Let $0 \leq j \leq \nu$ and set

$$i_j = \min\{h \geq 0 : c_h \neq 0, v_p(h+n) \leq j\};$$

(2.2)

if $c_h = 0$ for all $h \geq 0$ such that $v_p(h+n) \leq j$ let $i_j = \infty$. Note that since $v_p(n) = \nu$ we have $i_\nu = 0$ and

$$i_j = \min\{h \geq 0 : c_h \neq 0, v_p(h) \leq j\}$$

(2.3)

for $0 \leq j < \nu$. The indices of inseparability of $L/K$ are now defined recursively by $i_0 = i_\nu = 0$ and $i_j = \min\{i_j, i_{j+1} + ne\}$ for $j = \nu - 1, \ldots, 1, 0$. Equivalently,

$$i_j = \min\{i_{j'} + ne(j' - j) : j \leq j' \leq \nu\}.$$  

(2.4)

If char($K$) = 0 then $i_j$ may depend on the choice of $\pi_L$ (but not on the choice of $\pi_K$). Nevertheless, $i_j$ is independent of the choice of uniformizers [5 Th. 7.1]. If char($K$) = $p$ then it is easily seen that $i_j = \tilde{i}_j$ does not depend on the choice of uniformizers. It follows immediately from the definitions that $0 = i_\nu \leq i_{\nu-1} \leq \cdots \leq i_0$. Furthermore, if $v_p(i_j) = j' < j$ then $i_j = i_{j-1} = \cdots = i_{j'}$.

The connection between the ramification data of the extension $L/K$ and the indices of inseparability of $L/K$ can be described most easily in terms of the Hasse-Herbrand
function of $L/K$, which is defined by

$$\phi_{L/K}(x) = \int_0^x \frac{dt}{|G_0 : G_t|} \tag{2.5}$$

for $x \geq 0$. Thus $\phi_{L/K} : [0, \infty) \to [0, \infty)$ is a continuous increasing piecewise linear function which is differentiable at all positive values except for the lower ramification breaks of $L/K$. If $x > 0$ is not a ramification break then the derivative of $\phi_{L/K}$ at $x$ is $\phi'_{L/K}(x) = |G_x|/n$, where $n = ap^\nu = [L : K] = |G_0|$. Hence the ramification data of $L/K$ is encoded in the function $\phi_{L/K}$.

One can recover the Hasse-Herbrand function $\phi_{L/K}$ (and hence the orders of the ramification subgroups) from the indices of inseparability of $L/K$ by the formula

$$\phi_{L/K}(x) = \frac{1}{n} \cdot \min\{i_j + p^j x : 0 \leq j \leq \nu\} \tag{2.6}$$

(see [5, Cor.6.11]). Let $S$ be one of the segments that make up the graph of $\phi_{L/K}$. It follows from (2.6) that $S$ can be extended to meet the $y$-axis at $(0, i_j/n)$ for some $0 \leq j \leq \nu$. It follows that if $L/K$ has $k$ positive ramification breaks then the graph of $\phi_{L/K}$ determines $k + 1$ indices of inseparability for $L/K$. In particular, if $k = \nu$ then the ramification data of $L/K$ determines all $\nu + 1$ indices of inseparability. The difficult cases where $L/K$ has fewer than $\nu$ positive ramification breaks occur when $L/K$ has a positive ramification break $b$ such that the elementary abelian $p$-group $G_b/G_{b+1}$ has rank greater than 1.

3 Abelian extensions with one break

Let $K$ be a local field whose residue field $\overline{K}$ is a finite field of characteristic $p$, and let $L/K$ be a nontrivial totally ramified Galois extension with a single ramification break $b \geq 1$. We assume for simplicity that if char$(K) = 0$ then $p > 2$ and $p \nmid b$. (The condition $p \nmid b$ is automatic if either char$(K) = p$ or $[L : K] > p$.) The aim of this section is to give an explicit computation of the norm group $H = N_{L/K}(L^\times)$. In section 4 we will use this computation in the case char$(K) = p$ to get information about the subgroup of the additive group of $K$ which corresponds to $L/K$ under Artin-Schreier theory. We note that Monge uses the same methods in Chapter 6 of [6] to compute the norm groups of certain extensions of local fields of characteristic 0.

Set $G = \text{Gal}(L/K)$. Then $G_b = G$ and $G_{b+1} = \{1\}$, so $G \cong G_b/G_{b+1}$ is an elementary abelian $p$-group, say $G \cong (\mathbb{Z}/p\mathbb{Z})^\nu$ with $\nu \geq 1$. It follows from (2.5) that the graph of the function $\phi_{L/K}(x)$ consists of a line segment of slope 1 from $(0, 0)$ to $(b, b)$ and a ray of slope $p^{-\nu}$ starting at $(0, 0)$. Hence by (2.6) we have $i_j \geq bp^\nu - bp^j$ for $0 \leq j \leq \nu$, with equality for $j = 0$ and $j = \nu$.

Let $E_K(X) = \prod_{\mu \mathbb{Z}} (1 - X^c)^{-\mu(c)/c} \in \mathcal{O}_K[[X]]$ denote the Artin-Hasse exponential series (cf. [11, III §1]), where $\mu$ is the Möbius function. Of course, $E_K(X)$ depends only on $p$ and on the characteristic of $K$: If char$(K) = p$ then $E_K(X) = E_p(X) \in \mathbb{F}_p[[X]]$, 

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while if char($K$) = 0 then $E_K(X) = E_0(X) \in \mathbb{Z}_p[[X]]$. Since $E_K(X) = 1 + X + \ldots$, the norm group of $L/K$ can be described in terms of this series:

**Proposition 3.1** Let $\pi_L$ be a uniformizer for $L$. The subgroup $H = N_{L/K}(L^x)$ of $K^x$ is generated by $U_{b+1}^b$, $(K^x)^p$, $N_{L/K}(\pi_L)$, and the set

$$\{N_{L/K}(E_K(r\pi_L^k)) : r \in R, 1 \leq k \leq b, \ p \nmid k\}.$$  \hspace{1cm} (3.1)

*Proof:* Since $b$ is the only lower ramification break of $L/K$, it follows from [7, V §6, Cor. 3] that $N_{L/K}(U_{b+1}^b) = U_{b+1}^b$. Since $K^x/H \cong G$ is killed by $p$ we have $H \supset (K^x)^p$, so $H$ contains all the listed elements. Since $p \nmid b$ the group $L^x$ is generated by $U_{b+1}^b$, $(L^x)^p$, $\pi_L$, and the set

$$\{E_K(r\pi_L^k) : r \in R, 1 \leq k \leq b, \ p \nmid k\}.$$ \hspace{1cm} (3.2)

Therefore $H$ is generated by the listed elements. \hfill \Box

Let $\pi_L$ be a uniformizer of $L$ and let

$$g(X) = X^{p^e} + a_1X^{p^e-1} + \cdots + a_{p^e-1}X + a_{p^e}$$ \hspace{1cm} (3.3)

be the minimum polynomial for $\pi_L$ over $K$. For $0 \leq j \leq \nu$ define

$$\widehat{i}_j = \min\{p^\nu v_K(a_k) - k : 1 \leq k \leq p^\nu, \ v_p(k) \leq j\}.$$ \hspace{1cm} (3.4)

Then by [5, Prop. 3.12] we have

$$i_j = \min\{\widehat{i}_{j'} + p^\nu e(j' - j) : j' \leq j' \leq \nu\}.$$ \hspace{1cm} (3.5)

(cf. (2.4)). In particular, if char($K$) = $p$ then $i_j = \widehat{i}_j$. Set $t = N_{L/K}(\pi_L)$; then $t$ is a uniformizer for $K$, and it follows from our assumptions that $t = (-1)^{p^e}a_{p^e} = -a_{p^e}$. For $k \geq 1$ write $k = k_0 + k_1p^e$ with $1 \leq k_0 \leq p^e$ and define $a_k = t^{k_1}a_{k_0}$. Since $i_0 = p^\nu b - b$ and $p \nmid b$, by (3.5) we have $i_0 = \widehat{i}_0$. By (3.3) we get $v_K(a_{k_0}) = b - b_1$, and hence $v_K(a_b) = b$.

Recall that $R = \mu_{q-1} \cup \{0\}$ is the set of Teichmüller representatives of $K$. There are unique $c_{i,h} \in R$ such that $a_i = \sum_{h=1}^{\infty} c_{i,h}t^h$. It follows from (3.4) that for $0 \leq j \leq \nu$ we have

$$\widehat{i}_j = \min\{hp^\nu - i : 1 \leq i \leq p^\nu, \ v_p(i) \leq j, \ h \geq 1, \ c_{i,h} \neq 0\}.$$ \hspace{1cm} (3.6)

Since $a_{i+p^\nu} = ta_i$ we see that $c_{i,h}$ depends only on the value of $hp^\nu - i$. It follows that

$$\widehat{i}_j = \min\{hp^\nu - i : i \geq 1, \ v_p(i) \leq j, \ h \geq 1, \ c_{i,h} \neq 0\}.$$ \hspace{1cm} (3.7)

Using the fact that $bp^\nu - bp^j \leq i_j \leq i_0 = bp^\nu - b$ we get

$$\widehat{i}_j = \min\{bp^\nu - i : b \leq i \leq bp^j, \ v_p(i) \leq j, \ c_{i,b} \neq 0\}.$$ \hspace{1cm} (3.8)
For $k \geq 1$ such that $p \nmid k$ set

$$g_k(X) = (-1)^{k+1} \prod_{\zeta^b = 1} g(\zeta X^{1/k}), \quad (3.9)$$

where the product is taken over the $k$th roots of unity $\zeta \in K^{alg}$. Then $g_k(X)$ is a monic polynomial of degree $p^\nu$ with coefficients in $O_K$ whose roots are the $k$th powers of the roots of $g(X)$. For $r \in R$ set $h_k^r(X) = -r^{p^\nu} \cdot g_k(-r^{-1}(X-1))$. Then $h_k^r(X)$ is a polynomial of degree $p^\nu$ with coefficients in $O_K$ whose multiset of roots is $\{1 - r \omega^k : g(\omega) = 0\}$. Since we are assuming either char$(K) = p$ or $p > 2$ we see that $h_k^r(X)$ is monic and

$$N_{L/K}(1 - r \pi_L^k) = -h_k^r(0) = r^{p^\nu} \cdot g_k(r^{-1}). \quad (3.10)$$

**Lemma 3.2** For $i \geq 1$ we have $v_K(a_i) \geq f_i$, where $f_i = \left[ \frac{b - \frac{p^{v_p(i)}b - i}{p^\nu}}{p^\nu} \right]$.

**Proof:** Let $j = \nu_p(i)$. If $j \geq \nu$ then $a_i = -t^{i/p^\nu}$ and the claim is obvious. If $j < \nu$ then we may assume $1 \leq i \leq p^\nu$. It follows from (3.1) and the inequality $\hat{i}_j \geq i_j \geq b p^\nu - b p^j$ that

$$v_K(a_i) \geq p^{-\nu}(\hat{i}_j + i) \geq b - b p^{i-\nu} + p^{-\nu}i. \quad (3.11)$$

Hence $v_K(a_i) \geq f_i$ in this case as well. \qed

Let $\tilde{g}(X) = a_1 X^{\nu-1} + \cdots + a_{p^\nu-1} X$; then $g(X) = X^{p^\nu} + \tilde{g}(X) - t$. Choose $s \in K^{alg}$ such that $s^k = r$. It follows from (3.9) and (3.10) that

$$N_{L/K}(1 - r \pi_L^k) = (-1)^{k+1} r^{p^\nu} \cdot \left( \prod_{\zeta^b = 1} (\zeta^{p^\nu} s^{-p^\nu} - t + \tilde{g}(\zeta s^{-1})) \right). \quad (3.12)$$

By Lemma 3.2 we have $v_K(a_i) \geq \lceil (1 - p^{-1})b \rceil$ for $1 \leq i \leq p^\nu - 1$. Since $2 \lceil (1 - p^{-1})b \rceil \geq b + 1$ we get the following congruences modulo $\mathcal{M}_{K,b+1}$.

$$N_{L/K}(1 - r \pi_L^k) \equiv (-1)^{k+1} r^{p^\nu} \cdot \left( \prod_{\zeta^b = 1} (\zeta^{p^\nu} s^{-p^\nu} - t) \right) \cdot \left( 1 + \sum_{\zeta^b = 1} \frac{\tilde{g}(\zeta s^{-1})}{\zeta^{p^\nu} s^{-p^\nu} - t} \right) \quad (3.13)$$

$$\equiv (1 - r^{p^\nu} t^k) \cdot \left( 1 + \sum_{\zeta^b = 1} \frac{\zeta^{-p^\nu} s^{p^\nu} \tilde{g}(\zeta s^{-1})}{1 - t s^{p^\nu} \zeta^{-p^\nu}} \right) \quad (3.14)$$

$$\equiv (1 - r^{p^\nu} t^k) \cdot \left( 1 + \sum_{\zeta^b = 1} \sum_{i=1}^{p^\nu-1} \sum_{j=0}^{\infty} t^i a_i \zeta^{-jp^\nu-i} s^{jp^\nu+i} \right). \quad (3.15)$$
Since \( \sum_{k=1}^{\infty} \zeta^{-jp^\nu i} \) equals \( k \) if \( k \mid jp^\nu + i \), and 0 otherwise, we get the following congruences modulo \( \mathcal{M}_K^{b+1} \):

\[
N_{L/K}(1 - r\pi^k_L) \equiv (1 - r^\nu t^k) \cdot \left( 1 + k \sum_{j=0}^{\infty} \sum_{0<i<p^\nu \atop k\mid jp^\nu + i} t^j a_s s^{jp^\nu+i} \right) \tag{3.16}
\]

\[
\equiv (1 - r^\nu t^k) \cdot \left( 1 + k \sum_{k\mid m \atop p^\nu \nmid m} a_m r^{m/k} \right) \tag{3.17}
\]

\[
\equiv (1 - r^\nu t^k) \cdot \left( 1 + k \sum_{p^\nu \nmid l} a_{lk} r^{l} \right) \tag{3.18}
\]

In order to apply Proposition 3.1, we express our norm computations in terms of the Artin-Hasse exponential:

\[
N_{L/K}(E_K(r\pi^k_L)) = \prod_{\nu | c} N_{L/K}(1 - r^e \pi^{ck}_L)^{-\mu(c)/c} \tag{3.19}
\]

\[
\equiv \prod_{\nu | c} (1 - r^e \pi^{ck}_L)^{-\mu(c)/c} \cdot \prod_{\nu | c} \left( 1 + ck \cdot \sum_{p^\nu \nmid l} a_{lck} r^{lc} \right) \tag{3.20}
\]

\[
\equiv E_K(r^\nu t^k) \cdot \left( 1 - k \sum_{\nu | c} \sum_{p^\nu \nmid l} \mu(c) a_{lck} r^{lc} \right), \tag{3.21}
\]

with all congruences modulo \( \mathcal{M}_K^{b+1} \). By setting \( m = lc \) we get

\[
N_{L/K}(E_K(r\pi^k_L)) \equiv E_K(r^\nu t^k) \cdot \left( 1 - k \sum_{\nu | c} \sum_{\nu | m \atop p^\nu \nmid m} \mu(c) a_{mk} r^{m} \right) \tag{3.22}
\]

\[
\equiv E_K(r^\nu t^k) \cdot \left( 1 - k \sum_{\nu | c} a_{mk} r^{m} \sum_{\nu | m \atop p^\nu \nmid m} \mu(c) \right) \tag{3.23}
\]

\[
\equiv E_K(r^\nu t^k) \cdot \left( 1 - k \sum_{j=0}^{\nu-1} a_{kjp} r^{jp} \right) \tag{3.24}
\]

\[
\equiv E_K(r^\nu t^k) \cdot E_K \left( -k \sum_{j=0}^{\nu-1} a_{kp} r^{jp} \right), \tag{3.25}
\]

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For $1 \leq k \leq b$ and $0 \leq j \leq \nu - 1$ define
\[ S_{kj} = \{ h \in \mathbb{Z} : f_{k_{p^j}} \leq h \leq b \}. \] (3.26)

Using the expansion $a_i = \sum_{h=1}^{\infty} c_{i,h} t^h$ we can rewrite (3.25) in the form
\[ N_{L/K}(E_K(r_{\pi_L^k})) \equiv E_K(r_{p^r t^k}) \cdot E_K \left( -k \cdot \sum_{j=0}^{\nu-1} \sum_{h \in S_{kj}} c_{k_{p^j},h} r_{p^j t^h} \right) \] (3.27)
\[ \equiv E_K(r_{p^r t^k}) \cdot \prod_{j=0}^{\nu-1} \prod_{h \in S_{kj}} E_K(-k c_{k_{p^j},h} r_{p^j t^h}). \] (3.28)

As before all congruences are taken modulo $M_{b+1}^b$.

4 Artin-Schreier theory

Let $K$ be a local field of characteristic $p$ with finite residue field $\overline{K}$, and let $L/K$ be a finite totally ramified Galois extension with a single ramification break $b \geq 1$. As above $\pi_L$ is a uniformizer for $L$ and $t = N_{L/K}(\pi_L)$ is a uniformizer for $K$. In this section we interpret the norm computations from section 3 in terms of Artin-Schreier theory.

Let $K^{ab}/K$ denote the maximum abelian subextension of $K^{al}/K$. For $\eta \in K^\times$ let $\sigma_{\eta}$ denote the element of Gal($K^{ab}/K$) that corresponds to $\eta$ under class field theory. For each $\beta \in K$ let $\rho_\beta \in K^{al}$ be a root of the polynomial $X^p - X - \beta$. Then $\rho_\beta \in K^{ab}$, so we can set $[\beta, \eta)_K = \sigma_{\eta}(\rho_\beta) - \rho_\beta$. This construction is independent of the choice of $\rho_\beta$ and defines a bilinear pairing
\[ [ , )_K : K \times K^\times \to \mathbb{F}_p. \] (4.1)

Let $\varphi : K \to K$ denote the Artin-Schreier operator $\varphi(x) = x^p - x$. Then the kernel of $[ , )_K$ on the left is $\varphi K$, and the kernel of $[ , )_K$ on the right is $(K^\times)^p$. Let $\Omega_K = K dt$ denote the module of Kähler differentials of $K$, and for $\omega \in \Omega_K$ let Res($\omega$) denote the residue of $\omega$ at 0. By Schmid’s theorem [4 XIV §5, Prop. 15] the pairing (4.1) can be computed in terms of the trace map $\text{Tr}_{\overline{K}/\mathbb{F}_p} : \overline{K} \to \mathbb{F}_p$ using the formula
\[ [\beta, \eta)_K = \text{Tr}_{\overline{K}/\mathbb{F}_p} \left( \text{Res} \left( \frac{d\eta}{\eta} \right) \right). \] (4.2)

Let $B$ denote the subgroup of the additive group of $K$ which corresponds under Artin-Schreier theory to the $(\mathbb{Z}/p\mathbb{Z})^\nu$-extension $L/K$: $B = \{ \beta \in K : X^p - X - \beta \text{ splits over } L \}$. (4.3)
Recall that $H = N_{L/K}(L^\times)$ is the subgroup of $K^\times$ that corresponds to $L/K$ under class field theory. The groups $B$ and $H$ are orthogonal complements of each other with respect to $[\ , \ )_K$:

\[
H = \{ \eta \in K^\times : [\beta, \eta]_K = 0 \text{ for all } \beta \in B \} \\
B = \{ \beta \in K : [\beta, \eta]_K = 0 \text{ for all } \eta \in H \}.
\] (4.4) (4.5)

We need the following lemma regarding the Artin-Hasse exponential series $E_p(X) \in F_p[[X]]$:

**Lemma 4.1** We have $E'_p(X) = E_p(X) \cdot \frac{\lambda(X)}{X}$, with $\lambda(X) = X + X^p + X^{p^2} + \cdots$.

**Proof:** Recall that $E_0(X) \in \mathbb{Z}_p[[X]]$ is the Artin-Hasse series in characteristic 0. By [1, p. 52] we have

\[
E_0(X) = \exp \left( X + \frac{1}{p}X^p + \frac{1}{p^2}X^{p^2} + \cdots \right),
\] (4.6)

where $\exp(X)$ is the usual exponential series. By taking the formal derivative of this equation we get

\[
E'_p(X) = E_0(X) \cdot \frac{\lambda(X)}{X}.
\] (4.7)

Since $E_p(X)$ is the image of $E_0(X)$ under the canonical map $\gamma : \mathbb{Z}_p[[X]] \to F_p[[X]]$, the lemma follows by applying $\gamma$ to (4.7). \qed

Using Lemma 4.1 we see that for $\alpha \in \mathcal{M}_K$ we have

\[
\frac{dE_p(\alpha)}{E_p(\alpha)} = \frac{E'_p(\alpha)\alpha' dt}{E_p(\alpha)} = \frac{\lambda(\alpha)\alpha' dt}{\alpha},
\] (4.8)

where $\alpha'$ denotes the formal derivative of $\alpha$ with respect to $t$. By applying this formula to (3.28) we get

\[
\frac{dN_{L/K}(E_p(r\pi_L^k))}{N_{L/K}(E_p(r\pi_L^k))} \equiv k\lambda(r^pt^k)\frac{dt}{t} + \sum_{j=0}^{\nu-1} \sum_{h \in S_h} h\lambda(-kc_{kpj}, h)r^pt^h)\frac{dt}{t} \pmod {\mathcal{M}_K^b dt},
\] (4.9)

Let $\overline{K}_0$ be a subgroup of the additive group of $\overline{K}$ which is a complement of $\varphi \overline{K}$; then $\overline{K}_0$ is cyclic of order $p$. Define

\[
K_0 = \{ x_0 + x_1t^{-1} + \cdots + x_st^{-s} \in K : s \geq 0, x_0 \in \overline{K}_0, x_i \in \overline{K}, x_{pi} = 0 \text{ for } i \geq 1 \}.
\] (4.10)

Also define $B_0 = B \cap K_0$. Then $K = K_0 \oplus \varphi K$ and $B = B_0 \oplus \varphi K$, so $B/\varphi K \cong B_0$. Let $\xi = x_0 + x_1t^{-1} + \cdots + x_bt^{-b}$ be an element of $K_0 \cap \mathcal{M}_K^{-b}$. We wish to determine the conditions that $x_0, x_1, \ldots, x_b$ must satisfy for $\xi$ to lie in $B_0$. Since $B_0 \subset K_0 \cap \mathcal{M}_K^{-b}$, this will give a characterization of $B_0$, and hence of $B$.

Since $B$ is the orthogonal complement of $H$ with respect to $[\ , \ )_K$ we have $\xi \in B_0$ if and only if $[\xi, \eta)_K = 0$ for all $\eta \in H$. Since $v_K(\xi) \geq -b$, by (4.2) we have $[\xi, \eta)_K = 0$ for
all \( \eta \in U_K^{b+1} \). Hence by Proposition 3.1 we see that \( \xi \in B_0 \) if and only if \( [\xi, t]_K = 0 \) and \( [\xi, N_{L/K}(E_p(r\pi_L^k))]_K = 0 \) for all \( r \in K \) and all \( k \) such that \( 1 \leq k \leq b \) and \( p \nmid k \). Using (4.2) we deduce that \( \xi \in B_0 \) if and only if

\[
\text{Tr}_{K/F_p} \left( \text{Res} \left( \xi \cdot \frac{dN_{L/K}(E_p(r\pi_L^k))}{N_{L/K}(E_p(r\pi_L^k))} \right) \right) = 0
\]

(4.11) for all \( r \in K \) and all \( 1 \leq k \leq b \) such that \( p \nmid k \). Since \( x_0 \in K_0 \) and \( \wp_K = \ker(\text{Tr}_{K/F_p}) \), we have \( \text{Tr}_{K/F_p}(x_0) = 0 \) if and only if \( x_0 = 0 \). Using (4.9) we see that (4.11) is equivalent to

\[
\text{Tr}_{K/F_p} \left( x_k r^{p^i} - \sum_{j=0}^{\nu-1} \sum_{h \in S_{kj}} h c_{kp^j, h} x_k r^{p^j} \right) = 0.
\]

(4.12)

Set \( m = [K : F_p] \) and let \( \tau \) be the Frobenius automorphism of \( K \). Then

\[
\text{Gal}(K/F_p) = \{ \tau^0, \tau^1, \ldots, \tau^{m-1} \},
\]

(4.13)

and we can rewrite (4.12) in the form

\[
\sum_{i=0}^{m-1} x_k^{p^i} \tau^{i+v}(r) = \sum_{i=0}^{m-1} \sum_{j=0}^{\nu-1} \sum_{h \in S_{kj}} h c_{kp^j, h} x_k^{p^i} \tau^{i+j}(r).
\]

(4.14)

Since (4.14) holds for all \( r \in K \) we get

\[
\sum_{i=0}^{m-1} x_k^{p^i} \tau^{i+v} = \sum_{i=0}^{m-1} \sum_{j=0}^{\nu-1} \sum_{h \in S_{kj}} h c_{kp^j, h} x_k^{p^i} \tau^{i+j}.
\]

(4.15)

Since \( \tau^u = \tau^v \) if and only if \( u \equiv v \mod m \) this implies

\[
\sum_{i=0}^{m-1} x_k^{p^i-v} \tau^i = \sum_{i=0}^{m-1} \sum_{j=0}^{\nu-1} \sum_{h \in S_{kj}} h c_{kp^j, h} x_k^{p^i-j} \tau^i.
\]

(4.16)

It follows from the \( K \)-independence of \( \tau^0, \tau^1, \ldots, \tau^{m-1} \) that (4.16) holds if and only if

\[
x_k = \sum_{j=0}^{\nu-1} \sum_{h \in S_{kj}} h c_{kp^j, h} x_k^{p^{j-i}}.
\]

(4.17)

Let \( V \) denote the subgroup of \( K \) such that \( B_0 + \mathcal{M}_K^{-b+1} = Vt^{-b} + \mathcal{M}_K^{-b+1} \). Since \( f_{bp^j} = b \) we have \( S_{b_3} = \{ b \} \) for \( 0 \leq j \leq \nu - 1 \). Hence by (4.17) every element of \( V \) is a root of the polynomial

\[
q(X) = X - \sum_{j=0}^{\nu-1} b c_{bp^j, b} X^{p^{j-i}}.
\]

(4.18)
Since the only ramification break of $L/K$ is $b$ we have $V \cong B_0 \cong (\mathbb{Z}/p\mathbb{Z})^\nu$. Hence $V$ is equal to the set of roots of $q(X)$. Let $\psi : V \to B_0$ be the inverse of the projection of $B_0$ onto $V$. Then for each $x_b \in V$ there are uniquely determined $x_j \in \overline{K}$ such that 

$$
\psi(x_b) = x_1t^{-1} + x_2t^{-2} + \cdots + x_b t^{-b}, \text{ with } x_i = 0 \text{ for } p \mid i.
$$

For each $k$ the map $\psi_k : V \to \overline{K}$ which takes $x_b$ to $x_k$ is a group homomorphism.

Since the maps $\phi_i : V \to \overline{K}$ defined by $\phi_i(x) = x^{p^i}$ for $1 \leq i \leq \nu$ are linearly independent over $\overline{K}$, they form a basis for the $\overline{K}$-vector space of homomorphisms from $V$ to $\overline{K}$. It follows that for $0 \leq k \leq b$ there are uniquely determined $v_{jk} \in \overline{K}$ such that

$$
\psi_k(x_b) = w_0x_b^{p^0} + w_1x_b^{p^0-1} + \cdots + w_{\nu-1}x_b^{p^0}.
$$

By setting $w_j = w_{j1}t^{-1} + w_{j2}t^{-2} + \cdots + w_{jb}t^{-b}$ we get

$$
\psi(x_b) = w_0x_b^{p^0} + w_1x_b^{p^0-1} + \cdots + w_{\nu-1}x_b^{p^0}
$$

for all $x_b \in V$. Hence

$$
B_0 = \{w_0x_b^{p^0} + w_1x_b^{p^0-1} + \cdots + w_{\nu-1}x_b^{p^0} : x_b \in V\}.
$$

Of course we have $v_K(w_j) \geq -b$, and $v_K(w_j) \leq -1$ if $w_j \neq 0$. In addition, since $\hat{v}_0 = i_0 = bp^\nu - b$, it follows from (3.8) that $c_{b,b} \neq 0$. Hence $v_K(w_0) = -b$.

The maps $\psi_k$ can be determined inductively using (4.17). In fact the computations in this section can be used to derive an algorithm for computing the indices of inseparability of the extension $L/K$ in terms of the norm group $H = N_{L/K}(L^\times)$:

**Algorithm 4.2** Given $H$, use (4.2) and (3.5) to determine $B$ and $B_0 = B \cap K_0$. For each $k$ such that $1 \leq k \leq b$ we get the map $\psi_k : V \to \overline{K}$, from which we obtain $v_{jk} \in \overline{K}$ satisfying (4.19). Use reverse induction on $k$ to determine $c_{kp^i,b}$ for all $k,j$ such that $0 \leq k \leq b$, $p \nmid k$, and $1 \leq j \leq \nu - 1$: For the base case note that since $V$ is equal to the set of roots of $q(X)$, equation (4.18) determines $c_{bp^i,b}$ for all $j$. Now let $1 \leq k \leq b$ be such that $p \nmid k$ and assume that we have computed $c_{kp^j,b}$ for all $j,r$ such that $k < r \leq b$ and $p \nmid r$. If $h \in S_{kj}$ and $h < b$ then $kp^j < kp^j + (b-h)p^\nu < bp^j$, so $c_{kp^j,h} = c_{kp^j+(b-h)p^\nu,b}$ has been determined. Hence (4.17) can be used to compute $c_{kp^j,b}$ for $0 \leq j \leq \nu - 1$. Once all the $c_{kp^j,b}$ have been found use (3.8) to compute the indices of inseparability of $L/K$; since $i_j \geq bp^\nu - bp^j$ this computation does not depend on $c_{kp^j,b}$ for $k > b$.

### 5 Some explicit formulas

In this section we continue to assume that $K$ is a local field of characteristic $p$ with finite residue field $\overline{K}$, and that $L/K$ is a totally ramified elementary abelian $p$-extension of degree $p^\nu$ with a single ramification break $b \geq 1$. We use the results of sections 3 and 4 to prove theorems which relate the indices of inseparability of $L/K$ to the description of $L/K$ in terms of Artin-Schreier theory. Our approach is to use (4.17), (4.21), and (3.8) to get relations between the $K$-valuations of $w_0, w_1, \ldots, w_{\nu-1}$ and the indices of inseparability $i_0, i_1, \ldots, i_\nu$. In some cases we are able to derive explicit formulas for $i_j$. 

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in terms of the valuations of the $w_i$. We then use Schmid’s formula (4.2) to translate theorems expressed in terms of Artin-Schreier theory into theorems expressed in terms of local class field theory. We remark that although $B_0$ and $w_0, w_1, \ldots, w_{\nu-1}$ depend on $t$, and hence on the choice of $\pi_L$, the results of this section do not, of course, depend on this choice.

**Theorem 5.1** Let $L/K$ be a totally ramified $(\mathbb{Z}/p\mathbb{Z})^\nu$-extension with a single ramification break $b$ satisfying $1 \leq b \leq p - 1$. Then for $0 \leq j \leq \nu - 1$ we have

$$i_j = bp' + \min\{p^{j'}v_K(w_j') : 0 \leq j' \leq j\}. \quad (5.1)$$

**Proof:** Since $b < p$ we have $S_{kj} = \{b\}$ for $1 \leq k \leq b$ and $0 \leq j \leq \nu - 1$. Hence for $1 \leq k \leq b$ formula (4.17) simplifies to

$$\psi_k(x_b) = \sum_{j=0}^{\nu-1} b \nu^{j'} c^{b x_b}_{k p' l} . \quad (5.2)$$

Comparing this with (4.19) and (4.20) we get

$$w_j = \sum_{k=1}^{b} b c^{j'} c^{b}_{k p' l} . \quad (5.3)$$

for $0 \leq j \leq \nu - 1$.

We now prove our claim by induction on $j$. Since $i_0 = bp' - b$ and $v_K(w_0) = -b$ the claim holds for $j = 0$. Let $1 \leq j \leq \nu - 1$ and assume that the claim holds for $j - 1$. Suppose $w_j = 0$. Then $c_{kp' l} = 0$ for $1 \leq k \leq b$. Since $b < p$ and $i_{j-1} \geq bp' - bp' - 1$, by (3.8) we get $i_j = i_{j-1}$. We also have

$$\min\{p^{j'}v_K(w_j') : 0 \leq j' \leq j\} = \min\{p^{j'}v_K(w_j') : 0 \leq j' \leq j - 1\}. \quad (5.4)$$

Therefore the claim for $j$ follows from the claim for $j - 1$ in this case. If $w_j \neq 0$ then $v_K(w_j) = -k$ for some $k$ such that $1 \leq k \leq b$. Since $b < p$ this implies that the right side of (5.1) is equal to $bp' - kp'$. By (5.3) we have $c_{kp' l} = 0$ and $c_{kp' l} = 0$ for all $l$ such that $k < l < b$. Since $i_{j-1} \geq bp' - bp' - 1$ it follows from (3.8) that $i_j = bp' - kp'$. Hence the claim holds for $j$. $\square$

Although there does not seem to be a simple formula for the indices of inseparability similar to Theorem 5.1 which is valid for all $b$, it is possible to get some information about $i_j$ in the general case. Let $\mathbb{F}_p^{\nu}$ denote the unique subfield of $\overline{K}^{alg}$ with $p^{\nu}$ elements.

**Lemma 5.2** Let $L/K$ be a totally ramified $(\mathbb{Z}/p\mathbb{Z})^\nu$-extension with a single ramification break $b \geq 1$, and let $0 \leq k \leq b - 1$. Then the following are equivalent:

1. $\mathbb{F}_p^{\nu} \subset \overline{K}$ and $B_0 + M_{\overline{K}}^k$ is an $\mathbb{F}_p^{\nu}$-subspace of $M_{\overline{K}}^b$.
2. $w_j \in M_{\overline{K}}^k$ for $1 \leq j \leq \nu - 1$. 

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Proof: Suppose condition 2 holds for \( k \). Then \( w_{jb} = 0 \) for \( 1 \leq j \leq \nu - 1 \), so we have \( x_b = \psi_b(x_b) = w_{0b}x_b^{p^\nu} \). Since \( V \subset \overline{K} \) is the set of solutions to this equation we deduce that \( \mathbb{F}_{p^\nu} \subset \overline{K} \) and \( V \) is a vector space over \( \mathbb{F}_{p^\nu} \). Furthermore, by (4.21) we have

\[
B_0 + \mathcal{M}_K^{-k} = \{w_{0b}x_b^{p^\nu} : x_b \in V\} + \mathcal{M}_K^{-k},
\]

so \( B_0 + \mathcal{M}_K^{-k} \) is an \( \mathbb{F}_{p^\nu} \)-subspace of \( \mathcal{M}_K^b \). Conversely, if condition 1 holds for \( k \) then \( V \) is a vector space over \( \mathbb{F}_{p^\nu} \), and for every \( c \in \mathbb{F}_{p^\nu} \) and \( x_b \in V \) we have

\[
\psi(cx_b) \equiv c\psi(x_b) \pmod{\mathcal{M}_K^{b+1}}.
\]

Since \( (B_0 + \mathcal{M}_K^{-k})/\mathcal{M}_K^{-k} \) is a one-dimensional vector space over \( \mathbb{F}_{p^\nu} \) this implies

\[
\psi(cx_b) \equiv c\psi(x_b) \pmod{\mathcal{M}_K^{-k}}.
\]

It follows that \( \psi_i(cx_b) = c\psi_i(x_b) \) for \( k < i \leq b \), so we have \( w_{ji} = 0 \) for \( 1 \leq j \leq \nu - 1 \) and \( k < i \leq b \). Therefore condition 2 holds for \( k \).

\[
\square
\]

Theorem 5.3 Let \( L/K \) be a totally ramified \( (\mathbb{Z}/p\mathbb{Z})^{\nu} \)-extension with a single ramification break \( b \geq 1 \), and let \( k \) be an integer such that \( [b/p] \leq k \leq b - 1 \). Then the following are equivalent:

1. \( i_j \geq bp^\nu - kp^j \) for \( 1 \leq j \leq \nu - 1 \).
2. \( \mathbb{F}_{p^\nu} \subset \overline{K} \) and \( B_0 + \mathcal{M}_K^{-k} \) is an \( \mathbb{F}_{p^\nu} \)-subspace of \( \mathcal{M}_K^b \).
3. \( w_j \in \mathcal{M}_K^{-k} \) for \( 1 \leq j \leq \nu - 1 \).

Proof: The equivalence of conditions 2 and 3 follows from Lemma 5.2. To prove the equivalence of conditions 1 and 3 we use reverse induction on \( k \). By (3.8) we see that condition 1 holds for \( k = b - 1 \) if and only if \( c_{bp^j,b} = 0 \) for \( 1 \leq j \leq \nu - 1 \). Using (4.18) we see that this is equivalent to \( q(X) = X - c_{bp^j,b}X^{p^\nu} \), which holds if and only if condition 3 holds for \( k = b - 1 \). This proves the base case. Now let \( [b/p] < r \leq b - 1 \) and assume that the theorem holds for \( k = r \). If \( p \mid r \) then condition 3 holds for \( k = r \) if and only if condition 3 holds for \( k = r - 1 \), and condition 1 holds for \( k = r \) if and only if condition 1 holds for \( k = r - 1 \) (since \( p^{j+1} \nmid i_j \)). Hence the theorem holds for \( k = r - 1 \) in this case. From now on we assume that the theorem holds for \( k = r \) with \( [b/p] < r \leq b - 1 \) and \( p \nmid r \).

Suppose condition 1 holds for \( k = r - 1 \). Then by (5.7) we see that \( c_{rp^j,h} = 0 \) for all \( (j,h) \) such that \( 1 \leq j \leq \nu - 1 \) and \( 1 \leq h \leq b \). Hence by (4.17) we have

\[
\psi_r(x_b) = \sum_{h \in S_0} h c_{r,h}^{p^\nu} \psi_h(x_b)^{p^\nu}.
\]

Since condition 1 holds for \( k = r - 1 \) it also holds for \( k = r \). It follows from the inductive assumption that condition 3 holds for \( k = r \). Hence for every \( h \) such that \( r < h \leq b \) we have \( \psi_h(x_b) = w_{0h}x_b^{p^\nu} \) for all \( x_b \in V \). In particular, \( x_b = w_{0b}x_b^{p^\nu} \) with \( w_{0b} \neq 0 \). It follows
that $\psi_h(x_b) = w_{0h}w_{0b}^{-1}x_b$ for $r < h \leq b$. Since $r < b$, every $h \in S_{r0}$ satisfies $r < h \leq b$. Hence by (5.8) we have

$$\psi_r(x_b) = \sum_{h \in S_{r0}} hc_{r,h}^{p^\nu} u_{0h}^{p^\nu} w_{0b}^{-p^\nu} x_b^{p^\nu}. \quad (5.9)$$

Since $v_K(w_j) \geq -r$ for $1 \leq j \leq \nu - 1$, and (5.9) expresses $\psi_r(x_b)$ as a $\mathbb{K}$-multiple of $x_b^{p^\nu}$, we deduce that $v_K(w_j) \geq -r + 1$ for $1 \leq j \leq \nu - 1$. Therefore condition 3 holds for $k = r - 1$.

Assume conversely that condition 3 holds for $k = r - 1$. Then condition 3 also holds for $k = r$, so by the inductive assumption, condition 1 holds for $k = r$. It follows from (3.7) that $c_{r,p^j,h} = 0$ for all $(j,h)$ such that $1 \leq j \leq \nu - 1$ and $1 \leq h < b$. Therefore by (4.17) we get

$$\psi_r(x_b) = \sum_{j=1}^{\nu-1} b c_{r,p^j,b}^{p^\nu-j} x_b^{p^\nu-j} + \sum_{h \in S_{r0}} hc_{r,h}^{p^\nu} \psi_h(x_b)^{p^\nu}. \quad (5.10)$$

As in the preceding paragraph we have $\psi_h(x_b) = w_{0h}w_{0b}^{-1}x_b$ for every $h \in S_{r0}$. Hence

$$\psi_r(x_b) = \sum_{j=1}^{\nu-1} b c_{r,p^j,b}^{p^\nu-j} x_b^{p^\nu-j} + \sum_{h \in S_{r0}} hc_{r,h}^{p^\nu} u_{0h}^{p^\nu} w_{0b}^{-p^\nu} x_b^{p^\nu}. \quad (5.11)$$

It follows from (4.19) that $w_{jr} = b c_{r,p^j,b}^{p^\nu-j}$ for $1 \leq j \leq \nu - 1$. Hence by condition 3 for $k = r - 1$ we get $c_{r,p^j,b} = 0$ for $1 \leq j \leq \nu - 1$. Suppose condition 1 does not hold for $k = r - 1$, and let $1 \leq j \leq \nu - 1$ be minimum such that $i_j < bp^\nu - (r - 1)p^j$. Since $r - 1 \geq b/p$ we have $i_1 < i_0$, and for $2 \leq j \leq \nu - 1$ we have $i_j < i_{j-1}$ by the minimality of $j$. Hence $p^j \mid i_j$ for $1 \leq j \leq \nu - 1$. Since condition 1 holds for $r$ it follows that $i_j = bp^\nu - rp^j$. By (3.8) this implies $c_{r,p^j,b} \neq 0$, a contradiction. Hence condition 1 holds for $k = r - 1$. \hfill \Box

Corollary 5.4 Let $L/K$ be a totally ramified $(\mathbb{Z}/p\mathbb{Z})^\nu$-extension with a single ramification break $b$ such that $b \geq 2$ and $\mathbb{F}_{p^\nu}$ is not contained in $\mathbb{K}$. Then there is at least one $1 \leq j \leq \nu - 1$ such that $i_j = bp^\nu - bp^j$.

Proof: Since $\mathbb{F}_{p^\nu}$ is not contained in $\mathbb{K}$, condition 2 in Theorem 5.3 does not hold for $k = b - 1$. Since $b \geq 2$ we have $b - 1 \geq [b/p]$, so condition 1 also fails for $k = b - 1$. Let $1 \leq j \leq \nu - 1$ be minimum such that $i_j < bp^\nu - (b - 1)p^j$. As in the proof of Theorem 5.3 we get $p^j \mid i_j$. Since we also have $i_j \geq bp^\nu - bp^j$ we conclude that $i_j = bp^\nu - bp^j$. \hfill \Box

Corollary 5.5 Let $L/K$ be a totally ramified $(\mathbb{Z}/p\mathbb{Z})^2$-extension with a single ramification break $b \geq 1$. Then

$$i_1 = bp^2 + \min\{v_K(w_0), pv_K(w_1)\} = bp^2 + \min\{-b, pv_K(w_1)\}. \quad (5.12)$$
Proof: Set \( l = -v_K(w_1) \) and \( d = \lfloor b/p \rfloor \). If \( d + 1 \leq l \leq b \) then by Theorem 5.3 we have

\[
bp^2 - lp \leq i_1 < bp^2 - (l - 1)p < bp^2 - b = i_0.
\]  

(5.14)

Since \( i_1 < i_0 \) we have \( p \mid i_1 \), and hence \( i_1 = bp^2 - lp \). If \( l \leq d \) then the conditions of Theorem 5.3 hold for \( k = d \), so we have \( i_1 \geq bp^2 - dp \). Furthermore, (5.11) is valid with \( \nu = 2 \) and \( r = d \), so \( w_{1d} = be_{dp,b}^0 \). If \( l = d \) we get \( c_{dp,b} \neq 0 \), and hence \( i_1 = bp^2 - dp \). We conclude that \( i_1 = bp^2 + \min \{ -b, pv_K(w_1) \} \) in every case.

The power series \( E_p(X) \) induces a bijection from \( \mathcal{M}_K \) onto \( U^1_K \). Let \( \Lambda_K : U^1_K \rightarrow \mathcal{M}_K \) denote the inverse of this bijection. Recall that \( H = N_{L/K}(L^\times) \) is the subgroup of \( K^\times \) which corresponds to \( L/K \) under class field theory.

Lemma 5.6 Suppose \( i \geq \lfloor b/p \rfloor \). Then \( \Lambda_K(H \cap U^i_K) \) is a subgroup of the additive group of \( \mathcal{M}_K \).

Proof: By equation (6) in [1, p. 52], for \( \alpha_1, \alpha_2 \in \mathcal{M}_K^i \) we have

\[
E_p(\alpha_1 + \alpha_2) \equiv E_p(\alpha_1)E_p(\alpha_2) \pmod{\mathcal{M}_K^{pi}}.
\]  

(5.15)

Since \( E_p(\mathcal{M}_K^i) = U^i_K \), it follows that

\[
\Lambda_K(u_1u_2) \equiv \Lambda_K(u_1) + \Lambda_K(u_2) \pmod{\mathcal{M}_K^{pi}}
\]  

(5.16)

for \( u_1, u_2 \in U^i_K \). Since \( pi \geq b + 1 \) we have \( U^{pi}_K \subset U^{b+1}_K \subset H \), and hence \( \mathcal{M}_K^{pi} \subset \Lambda_K(H \cap U^i_K) \). Thus \( \Lambda_K(H \cap U^i_K) \) is a subgroup of \( \mathcal{M}_K \). \( \square \)

Lemma 5.7 Let \( \alpha \in \mathcal{M}_K^i \) with \( i \geq \lfloor b/p \rfloor \). Then for every \( \beta \in \mathcal{M}_K^{-b} \) and \( \zeta \in \overline{K} \) we have

\[
[\zeta \beta, E_p(\alpha)]_K = [\beta, E_p(\zeta \alpha)]_K.
\]  

(5.17)

Proof: Since \( pi - 1 \geq b \) it follows from (4.8) that

\[
\frac{dE_p(\alpha)}{E_p(\alpha)} \equiv \alpha' dt \pmod{\mathcal{M}_K^i dt}
\]  

(5.18)

\[
\frac{dE_p(\zeta \alpha)}{E_p(\zeta \alpha)} \equiv \zeta \alpha' dt \pmod{\mathcal{M}_K^i dt}.
\]  

(5.19)

The lemma now follows from Schmid’s formula [12]. \( \square \)

Theorem 5.8 Let \( k \geq \lfloor b/p \rfloor - 1 \). Then \( \Lambda_K(H \cap U^{k+1}_K) \) is an \( \mathbb{F}_{p^r} \)-subspace of \( \mathcal{M}_K \) if and only if \( B_0 + \mathcal{M}_K^{-k} \) is an \( \mathbb{F}_{p^r} \)-subspace of \( K \).
Proof: If \( \mathbb{F}_{p^v} \) is not contained in \( \overline{K} \) then neither \( \Lambda_K(H \cap U_K^{k+1}) \) nor \( B_0 + \mathcal{M}^{-k}_K \) is a vector space over \( \mathbb{F}_{p^v} \). If \( \mathbb{F}_{p^v} \subset \overline{K} \) and \( k \geq b \) then \( \Lambda_K(H \cap U_K^{k+1}) = \mathcal{M}^{k+1}_K \) and \( B_0 + \mathcal{M}^{k}_K = \mathcal{M}^{-k}_K \) are both vector spaces over \( \mathbb{F}_{p^v} \). Hence we may assume \( \mathbb{F}_{p^v} \subset \overline{K} \) and \( k \leq b - 1 \).

Suppose \( B_0 + \mathcal{M}^{k}_K \) is an \( \mathbb{F}_{p^v} \)-subspace of \( K \). By Lemma 5.6 we see that \( \Lambda_K(H \cap U_K^{k+1}) \) is a subgroup of \( \mathcal{M}_K \). We need to show that \( \Lambda_K(H \cap U_K^{k+1}) \) is stable under multiplication by elements of \( \mathbb{F}_{p^v} \). Let \( \beta \in B_0 + \mathcal{M}^{k}_K \) and \( \eta \in H \cap U_K^{k+1} \). Then \( \eta = E_p(\alpha) \) for a uniquely determined \( \alpha \in \mathcal{M}^{k+1}_K \). For \( \zeta \in \mathbb{F}_{p^v} \) we have \( \zeta \beta \in B_0 + \mathcal{M}^{k}_K \). Since we also have \( \Lambda_K(H \cap U_K^{k+1}) \) is an \( \mathbb{F}_{p^v} \)-subspace of \( \mathcal{M}_K \),

\[
[B_0, H)_K = [\mathcal{M}^{k-1}_K, U_K^{k+1})_K = 0,
\]

(5.20)

by Lemma 5.7 we get \( \langle \beta, E_p(\zeta \alpha) \rangle_K = 0 \). Hence \( E_p(\zeta \alpha) \) lies in the orthogonal complement of \( B_0 + \mathcal{M}^{k}_K \) with respect to \( \langle \ , \rangle_K \). Since \( B = B_0 \oplus \wp K \), the orthogonal complement of \( B_0 \) is equal to the orthogonal complement of \( B \), which is \( H \). Since we also have \( E_p(\zeta \alpha) \in U_K^{k+1} \) we get \( E_p(\zeta \alpha) \in H \cap U_K^{k+1} \). Hence \( \zeta \alpha \in \Lambda_K(H \cap U_K^{k+1}) \) for all \( \zeta \in \overline{K} \), so \( \Lambda_K(H \cap U_K^{k+1}) \) is an \( \mathbb{F}_{p^v} \)-subspace of \( \mathcal{M}_K \).

Conversely, suppose that \( \Lambda_K(H \cap U_K^{k+1}) \) is an \( \mathbb{F}_{p^v} \)-subspace of \( \mathcal{M}_K \). Then for every \( \eta = E_p(\alpha) \in H \cap U_K^{k+1} \) and every \( \zeta \in \mathbb{F}_{p^v} \) we have \( E_p(\zeta \alpha) \in H \cap U_K^{k+1} \). Therefore by Lemma 5.7 we have

\[
\langle \zeta \beta, E_p(\zeta \alpha) \rangle_K = \langle \beta, E_p(\zeta \alpha) \rangle_K = 0
\]

(5.21)

for every \( \beta \in B_0 \). Hence \( \zeta \beta \) lies in the orthogonal complement of \( H \cap U_K^{k+1} \). Since \( H \) and \( U_K^{k+1}(K^x)^p \) are closed finite-index subgroups of \( K^x \) we have

\[
(H \cap U_K^{k+1})^\perp = \left( (H \cap U_K^{k+1})(K^x)^p \right)^\perp
\]

(5.22)

\[
= (H \cap (U_K^{k+1}(K^x)^p))^\perp
\]

(5.23)

\[
= H^\perp + (U_K^{k+1}(K^x)^p)^\perp
\]

(5.24)

\[
= B_0 + \mathcal{M}^{k}_K + \wp K.
\]

(5.25)

Since we also have \( \zeta \beta \in (K_0 \cap \mathcal{M}^b_K) + \overline{K} \) we get

\[
\zeta \beta \in (B_0 + \mathcal{M}^{k}_K + \wp K) \cap ((K_0 \cap \mathcal{M}^b_K) + \overline{K}) \subset B_0 + \mathcal{M}^{-k}_K.
\]

(5.26)

It follows that \( B_0 + \mathcal{M}^{k}_K \) is an \( \mathbb{F}_{p^v} \)-subspace of \( K \).

By combining Theorem 5.8 with Theorem 5.3 we get the following result:

**Corollary 5.9** Let \( L/K \) be a totally ramified \((\mathbb{Z}/p\mathbb{Z})^{v'}\)-extension with a single ramification break \( b \geq 1 \) and let \( k \) be an integer such that \( [b/p] \leq k \leq b - 1 \). Then the following are equivalent:

1. \( i_j \geq bp^v - kp^j \) for \( 1 \leq j \leq v - 1 \).

2. \( \mathbb{F}_{p^v} \subset \overline{K} \) and \( \Lambda_K(H \cap U_K^{k+1}) \) is an \( \mathbb{F}_{p^v} \)-subspace of \( \mathcal{M}_K \).

We also have the following reformulation of Corollary 5.5.
Corollary 5.10  Let $L/K$ be a totally ramified $(\mathbb{Z}/p\mathbb{Z})^2$-extension with a single ramification break $b \geq 1$. If $\mathbb{F}_p \not\subseteq \overline{K}$, let $k = b$; otherwise let $k$ be the smallest nonnegative integer such that $\Lambda_K(H \cap U^{k+1}_K)$ is an $\mathbb{F}_p$-subspace of $\mathcal{M}_K$. Then $i_1 = p^2b - \max\{b, pk\}$.

Proof: If $\mathbb{F}_p \not\subseteq \overline{K}$ then $B_0 + \mathcal{M}_K^{b+1}$ is not a vector space over $\mathbb{F}_p$, so $v_K(w_1) = -b$ by Lemma 5.2. Hence by Corollary 5.3 we have $i_1 = bp^2 - bp$. If $\mathbb{F}_p \subseteq \overline{K}$ we have $i_1 = p^2b - bp$. If $k \geq d$ then by Theorem 5.8 we see that $B_0 + \mathcal{M}_K^k$ is an $\mathbb{F}_p$-subspace of $\mathcal{M}_K^b$. Hence by Lemma 5.2 we have $v_K(w_1) = -k$, so by Corollary 5.3 we get $i_1 = p^2b - pk$. If $k < d$ then $\Lambda_K(H \cap U^d_K)$ is an $\mathbb{F}_p$-subspace of $\mathcal{M}_K$. Hence by Theorem 5.8 we see that $B_0 + \mathcal{M}_K^d$ is an $\mathbb{F}_p$-subspace of $\mathcal{M}_K^b$. Hence $v_K(w_1) \geq -d + 1$, so by Corollary 5.3 we have $i_1 = p^2b - b$. We conclude that $i_1 = p^2b - \max\{b, pk\}$ in every case. \qed

6  The case $\text{char}(K) = 0$

In this section we show how to extend the results of the previous section to apply to certain extensions of local fields of characteristic 0. Let $K$ be a finite extension of the $p$-adic field $\mathbb{Q}_p$ and set $e = v_K(p)$. Let $L/K$ be a totally ramified Galois extension with a single ramification break $b$ such that $1 \leq b < e$. Then $\text{Gal}(L/K) \cong (\mathbb{Z}/p\mathbb{Z})^\nu$ for some $\nu \geq 1$.

Let $\pi_K, \pi_L$ be uniformizers for $K, L$ and recall that $R$ denotes the set of Teichmüller representatives for $K$. Choose $c_h \in R$ such that $\pi_K = \sum_{h=0}^{\infty} c_h \pi_L^{h+e}$ and let $\overline{c}_h$ denote the image of $c_h$ in $\overline{K}$. Since $K' = \overline{K}((T))$ is a local field of characteristic $p$ with residue field $\mathbb{F}_p$, by $[5, \text{Th. 2.2}]$ there is a totally ramified extension $L'/K'$ and a uniformizer $\pi_{L'}$ for $L'$ such that $T = \sum_{h=0}^{\infty} \overline{c}_h \pi_{L'}^{h+e}$.

For $0 \leq j \leq \nu - 1$ the inequalities $i_{j+1} \geq bp'^j - bp'^{j+1}$ and $i_j \leq i_0 = bp' - b$ imply that $i_j - i_{j+1} \leq bp'^{j+1} - b < ep'^j$. Hence $i_j = i_0$ for $0 \leq j \leq \nu$. Since $c_h = 0$ if and only if $\overline{c}_h = 0$, this implies that the extension $L'/K'$ has the same indices of inseparability as $L/K$. It follows that $L'/K'$ has the same ramification data as $L/K$, so $L'/K'$ has a single ramification break $b$. It can be shown using $[5, \text{Prop. 6.3b}]$ that $L'/K'$ is Galois. Hence $\text{Gal}(L'/K') \cong (\mathbb{Z}/p\mathbb{Z})^\nu$, so the results of section 5 apply to $L'/K'$. Since $b + 1 \leq e$ there is an isomorphism

$$\rho : \mathcal{O}_L/\mathcal{M}_L^{(b+1)p'^\nu} \rightarrow \mathcal{O}_{L'}/\mathcal{M}_{L'}^{(b+1)p'^\nu}$$

(6.1)

of $\overline{K}$-algebras such that $\rho(\pi_L + \mathcal{M}_L^{(b+1)p'^\nu}) = \pi_{L'} + \mathcal{M}_{L'}^{(b+1)p'^\nu}$. It follows from the series expressions for $\pi_K$ and $T$ that $\rho(\pi_K + \mathcal{M}_L^{(b+1)p'^\nu}) = T + \mathcal{M}_L^{(b+1)p'^\nu}$. Set $H = N_{L/K}(L^\times)$ and $H' = N_{L'/K'}((L')^\times)$. Let $\overline{H}$ denote the image of $N_{L/K}(\mathcal{O}_L^\times)$ in

$$(\mathcal{O}_K/\mathcal{M}_K^{(b+1)p^\nu}) \cong \mathcal{O}_K^{\times}/U_K^{b+1}$$

(6.2)
and let $\tilde{H}'$ denote the image of $N_{L'/K'}(O_{L'}^\times)$ in

$$(O_{K'}/\mathcal{M}_{K'}^{b+1})^\times \cong O_{K'}^\times/U_{K'}^{b+1}. \quad (6.3)$$

Since the norm of $u \in O_L^\times$ is the determinant of the $O_K$-linear map from $O_L$ to $O_L$ defined by $x \mapsto xu$, we have $\rho(H) = \tilde{H}'$.

As before we let $E_0(X) \in \mathbb{Z}_p[[X]]$ denote the Artin-Hasse exponential series in characteristic 0, and let $\Lambda_K : U_K^{1} \to \mathcal{M}_K$ denote the inverse of the bijection from $\mathcal{M}_K$ to $U_K^1$ induced by $E_0(X)$. Similarly, we have $E_p(X) \in \mathbb{F}_p[[X]]$ and $\Lambda_{K'} : U_{K'}^{1} \to \mathcal{M}_{K'}$. Let $u \in U_K^1$ and $u' \in U_{K'}^1$ be such that

$$\rho(u + \mathcal{M}_L^{(b+1)p^r}) = u' + \mathcal{M}_{L'}^{(b+1)p^r}. \quad (6.4)$$

Since $E_p(X)$ is the image of $E_0(X) \in \mathbb{Z}_p[[X]]$ in $\mathbb{F}_p[[X]]$ we have

$$\rho(\Lambda_K(u) + \mathcal{M}_L^{(b+1)p^r}) = \Lambda_{K'}(u') + \mathcal{M}_{K'}^{(b+1)p^r}. \quad (6.5)$$

Since $\rho(\tilde{H}) = \tilde{H}'$, it follows that for $1 \leq k \leq b$, $\rho$ induces a $\overline{K}$-isomorphism from $\Lambda_K(H \cap U_K^{k+1})/\mathcal{M}_K^{b+1}$ onto $\Lambda_{K'}(H' \cap U_{K'}^{k+1})/\mathcal{M}_{K'}^{b+1}$. Therefore $\Lambda_K(H \cap U_K^{k+1})/\mathcal{M}_K^{b+1}$ is an $\mathbb{F}_p^\nu$-subspace of $\mathcal{M}_K/\mathcal{M}_K^{b+1}$ if and only if $\Lambda_{K'}(H' \cap U_{K'}^{k+1})/\mathcal{M}_{K'}^{b+1}$ is an $\mathbb{F}_p^\nu$-subspace of $\mathcal{M}_{K'}/\mathcal{M}_{K'}^{b+1}$.

Assume $\mathbb{F}_p^\nu \subset \overline{K}$ and let $\mathbb{Z}_p^\nu$ denote the ring of integers of the unique unramified subextension of degree $\nu$ of $K/\mathbb{Q}_p$. Then $\Lambda_K(H \cap U_K^{k+1})/\mathcal{M}_K^{b+1}$ is an $\mathbb{F}_p^\nu$-subspace of $\mathcal{M}_K/\mathcal{M}_K^{b+1}$ if and only if $\Lambda_{K'}(H' \cap U_{K'}^{k+1})$ is a $\mathbb{Z}_p^\nu$-submodule of $\mathcal{M}_{K'}$. Hence we have the following analogs of Corollary 5.9 and Corollary 5.10:

**Corollary 6.1** Let $K$ be a finite extension of $\mathbb{Q}_p$ and let $e = v_K(p)$ be the absolute ramification index of $K$. Let $L/K$ be a totally ramified $(\mathbb{Z}/p\mathbb{Z})^\nu$-extension with a single ramification break $b$ such that $1 \leq b < e$, and set $H = N_{L/K}(L^\times)$. Let $k$ be an integer such that $[b/p] \leq k \leq b - 1$. Then the following are equivalent:

1. $i_j \geq bp^r - kp^j$ for $1 \leq j \leq \nu - 1$.
2. $\mathbb{F}_p^\nu \subset \overline{K}$ and $\Lambda_K(H \cap U_K^{k+1})$ is a $\mathbb{Z}_p^\nu$-submodule of $\mathcal{M}_K$.

**Corollary 6.2** Let $K$, $L$, $e$, $b$ be as in Corollary 6.1 and assume that $\nu = 2$. If $\overline{K}$ does not have a subfield isomorphic to $\mathbb{F}_p^2$ let $k = b$; otherwise let $k$ be the smallest nonnegative integer such that $\Lambda_K(H \cap U_K^{k+1})$ is a $\mathbb{Z}_p^2$-submodule of $\mathcal{M}_K$. Then $i_1 = p^2b - \max\{b, pk\}$.

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