NORMAL APPROXIMATION IN LARGE NETWORK MODELS*

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ABSTRACT. We develop a methodology for proving central limit theorems in network models with strategic interactions and homophilous agents. Since data often consists of observations on a single large network, we consider an asymptotic framework in which the network size tends to infinity. In the presence of strategic interactions, network moments are generally complex functions of components, where a node’s component consists of all alters to which it is directly or indirectly connected. We find that a modification of “exponential stabilization” conditions from the stochastic geometry literature provides a useful formulation of weak dependence for moments of this type. We establish a CLT for a network moments satisfying stabilization and provide a methodology for deriving primitive sufficient conditions for stabilization using results in branching process theory. We apply the methodology to static and dynamic models of network formation.

JEL Codes: C31, C57, D85
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1 Introduction

Network models have attracted considerable attention in economics as tractable representations of non-market interactions, such as peer effects and social learning, and formal economic relations, such as financial and trade networks. The economic perspective on networks emphasizes the importance of strategic interactions or externalities (Jackson et al., 2016). One strand of the literature studies social interactions, how an individual’s behavior interacts with those of her social contacts. A second strand studies network formation, why an individual chooses particular social contacts in the first place, and how her choices affect others’. In both cases, externalities create a wedge between choices that are optimal from the individual’s perspective and those that are efficient for society, which has important consequences for policymaking. For instance, they motivate associational redistribution, policies that intervene on the network structure (Durlauf, 1996). However, when preferences are misaligned with the policy objective, the endogenous response to the intervention may diverge from the intended outcome (Carrell et al., 2013). It is therefore of interest to develop econometric methodologies for measuring externalities in network models.

Network data typically consists of observations on a single large network. The challenge is that strategic interactions generate autocorrelation between observations, which is a non-standard form of cross-sectional dependence. A large-sample theory requires conditions under which the amount of “independent information” grows with the number of nodes or agents in the network, despite network autocorrelation.

The main contribution of this paper is a methodology for proving CLTs for large network models with strategic interactions. These results are important for developing formally justified inference procedures. We focus on static and dynamic models of network formation as the main application of our methodology but discuss how the ideas can be applied more broadly.1 Empirical models of network formation are useful for understanding incentives for developing social connections, for example the formation of risk-sharing networks in the rural Philippines (Fafchamps and Gubert, 2007) and research partnerships in the biotechnology industry (Powell et al., 2005). They can be used distinguish between different mechanisms for link formation proposed in the theory literature, including preferential attachment (Barabási and Albert, 1999),

1A previous version of the paper also considers applications to network regressions and treatment effects with network spillovers (Leung and Moon, 2019), and a subsequent paper by Leung (2019a) applies the methodology to discrete games on networks.
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strategic transitivity (Graham, 2016), and homophily (McPherson et al., 2001). Finally, these models are useful for forecasting the effects of counterfactual interventions (Mele, 2019) and to account dependence between the network and unobservables of a social interactions model (Badev, 2013; Goldsmith-Pinkham and Imbens, 2013).

Our CLT applies to a broad class of network moments computed from static or dynamic network data. These moments are averages of node-level statistics \( n^{-1} \sum_{i=1}^{n} \psi_i \), where \( \psi_i \) is some function of the network (or network time series) and node types. In the static case, a simple example is the degree of node \( i \), which is the number of links involving \( i \). More generally, the class of moments includes the average clustering coefficient, regression estimators, and subnetwork counts, the latter of which is important for inference in strategic models of network formation (Sheng, 2020). For dynamic network data, we show that our CLT applies to the conditional likelihood estimator of Graham (2016), which is a semiparametric estimator of a dynamic network formation model. We also consider nonparametric inference, in particular, bounds on an average structural function that nonparametrically measures the impact of network lagged dependent variables on link formation.

A key observation motivating our methodology is that, in the presence of strategic interactions, \( \psi_i \) is generally a complicated function of components of a certain latent network, where a node’s component consists of all alters to which it is connected directly or indirectly (by any number of links). We therefore require a framework applicable to complex moments of this type.

We emphasize two main technical contributions. The first is an abstract CLT that holds under high-level conditions, which is an extension of limit theorems from the stochastic geometry literature. The key condition is a modification of “stabilization” assumptions from that literature that provide a useful formulation of weak dependence for our purposes. Stabilization essentially requires \( \psi_i \) to only be a function of a random subset of nodes whose size has exponential tails. However, in our applications, this random subset is generally a union of network components, which are difficult combinatorial objects to analyze. Our second – and primary – contribution is to develop a methodology and set of useful lemmas for deriving primitive sufficient conditions for stabilization and apply them to network formation.

The basic idea of the methodology is that, in the presence of strategic interactions, the “optimality” of a link may depend on the existence of “neighboring” links, which in turn may depend on other links, and so on. The longer the lengths of these
chains of dependent links, the stronger the extent of autocorrelation. We adopt a well-known technique in random graph theory and bound the lengths of these chains by subcritical branching processes whose sizes we prove have exponential tails. As discussed in Leung (2019b), primitive conditions for subcriticality can be interpreted as restrictions on the strength of strategic interactions, analogous to standard weak dependence conditions for spatial and temporal autoregressive models that bound the magnitude of the autoregressive parameter below one.

A growing literature studies frequentist inference in network formation models when the econometrician observes a single network. Leung (2019b) and Menzel (2017) develop laws of large numbers for static models with strategic interactions. The former paper modifies a weaker stabilization condition due to Penrose and Yukich (2003) and uses branching processes to derive primitive sufficient conditions. Our paper tackles the more difficult problem of obtaining a normal approximation, which naturally requires a stronger stabilization condition, and also demonstrates the broader applicability of the method by considering a more general model that nests static and dynamic network formation. We prove the CLT by extending results in Penrose and Yukich (2005) and Penrose (2007); a more detailed discussion of our contributions relative to this literature can be found at the end of §5.3. Kuersteiner (2019) takes a different approach, using a novel conditional mixingale type assumption defined in terms of a random metric of distance.

Leung (2015b) and Ridder and Sheng (2016) consider static models of strategic network formation under incomplete information. In this setting, links are independent conditional on observables. In contrast, the models studied here can be micro-founded as games of complete information and allow for unobserved heterogeneity, which generates dependence between potential links even conditional on observables.

Several papers estimate dynamic network models. Kuersteiner and Prucha (2020) analyze dynamic spatial panels, which can be interpreted as social interactions models with lagged dependent variables. Graham (2016) studies point identification and estimation of a parametric model of dynamic network formation.

Dzemski (2019) and Graham (2017) consider dyadic link formation models with fixed effects but without strategic interactions. A large statistics literature studies models without strategic interactions, for example stochastic block models (Bickel et al., 2011) and latent-space models (Hoff et al., 2002). These are useful for their parsimony and for tasks such as community detection but not for measuring exter-
normalities. More general statistical models allowing for interdependence between links are studied in Chandrasekhar and Jackson (2015) and Boucher and Mourifié (2017).

The next section presents a simplified model of network formation and examples of network moments covered by our results. Then §3 introduces key concepts used to define stabilization and discusses their intuition. In §4, we formally state our high-level conditions and CLT. We present a general model of network formation in §5 and provide primitive conditions for stabilization. Although the result is specific to network formation, its argument follows a general methodology outlined in §4 that is applicable to other models. In §6, we discuss practical inference procedures, and §7 concludes. All proofs are given in the supplemental appendix.

We introduce standard notation and terminology for networks. We represent a network on a finite set of nodes $J \subseteq \mathbb{N}$ as a $|J| \times |J|$ adjacency matrix, where the $ij$th entry $A_{ij}$, termed the potential link, is an indicator for whether nodes $i, j \in J$ are connected. Following the usual convention, we require that $A_{ii} = 0$ for all nodes $i$, meaning that there are no self links. We focus on undirected networks, so $A_{ij} = A_{ji}$.

The degree of a node $i$ in the network $A$ is $\sum_{j \neq i, j \in J} A_{ij}$. For two networks $A, A'$, we say that $A$ is a subnetwork of $A'$ if every link in $A$ is a link in $A'$. A path in a network from node $i$ to $j$ is a sequence of distinct nodes starting with $i$ and ending with $j$ such that for each consecutive node pair $k, k'$ in this sequence, $A_{kk'} = 1$. Its length is the number of links it involves. The path distance between two nodes $i, j$ in $A$ is the length of the shortest path that connects them if a path exists and $\infty$ otherwise.

The $K$-neighborhood of a node $i$ in $A$, denoted by $N_A(i, K)$, is the set of all nodes $j$ of path distance at most $K$ from $i$ (including $i$). Finally the component of a node $i$ with respect to a network $A$ is the set of all nodes $j$ at finite path distance from $i$.

## 2 Network Formation

Let $\mathcal{N}_n = \{1, \ldots, n\}$ be a set of nodes. The econometrician observes the evolution of a network on $\mathcal{N}_n$ over a finite number of time periods $t = 0, 1, \ldots, T$. Let $A_t$ be the network in period $t$ with $ij$th entry $A_{ij,t}$, an indicator for the presence of a link between nodes $i$ and $j$ in that period. In this section, we consider $T > 0$. We defer to §5 the general case that allows $T = 0$, in which case the data consists solely of $A_0$, and the setup corresponds to a static model of network formation.

Each node $i$ is endowed with a type $(X_i, Z_i)$, i.i.d. across nodes, where $X_i \in \mathbb{R}^d$
is a continuously distributed, time-invariant vector of characteristics that we call the position of node $i$. The component $Z_i = (Z_{it})_{t=0}^\infty$ is $i$’s vector of potentially time-varying attributes $Z_{it} \in \mathbb{R}^{d_z}$. Each node pair is endowed with a real-valued random utility shock $\zeta_{ij,t} = \zeta_{ji,t}$, which is i.i.d. across $i, j, t$ and independent of types.

Given an initial condition $A_0$ (defined later), we assume that the network evolves in subsequent periods according to the following dynamics: there exists a $\mathbb{R}$-valued function $V(\cdot)$ such that for any $n, i \neq j$, and $t > 0$,

$$A_{ij,t} = 1 \left\{ V \left( r^{-1} \| X_i - X_j \|, S_{ij,t}, Z_{it}, Z_{jt}, \zeta_{ij,t} \right) > 0 \right\},$$

(1)

where $\|\cdot\|$ is a norm on $\mathbb{R}^d$ and $S_{ij,t}$ is a vector of statistics that can depend on $A_{t-1}$. As an illustration, in this section, we consider the specification

$$S_{ij,t} = \left( A_{ij,t-1}, \max_k A_{ik,t-1} A_{jk,t-1} \right),$$

(2)

which will be generalized in §5. The sparsity parameter $r > 0$ is required to decay to zero with the network size $n$ when we discuss asymptotics in §4.2

We highlight three features of the model. First, we will assume that $V(\cdot)$ is eventually decreasing in its first component (Assumption 4). This captures “homophily” in positions, meaning that nodes with dissimilar positions are less likely to form links. Positions may be unobserved to the econometrician and potentially correlated with $Z_{it}$ and $S_{ij,t}$, which allows for unobserved homophily, a well-known obstacle to identification in observational network data. Positions can abstractly represent nodes’ locations in an underlying “social space,” as in latent-space models (Breza et al., 2020; Hoff et al., 2002), where “socially dissimilar” nodes are less likely to form connections. They can also represent attributes such as income and geographic location.

Second, to capture strategic interactions, specification (2) allows the latent index $V(\cdot)$ to depend on the previous state of the network $A_{t-1}$ through two sufficient statistics. The first is $A_{ij,t-1}$, which captures state dependence. The second is $\max_k A_{ik,t-1} A_{jk,t-1}$, an indicator for whether $i$ and $j$ shared a common friend last period. This generates network clustering, the well-known stylized fact that nodes with common friends are more likely to become friends (Jackson, 2010).

Because $V(\cdot)$ is decreasing in its first argument and $r$ will be taken to zero with

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2We use the notation $r^{-1} \| X_i - X_j \|$ with $r \to 0$ rather than, say, $r \| X_i - X_j \|$ with $r \to \infty$ to maintain consistency with Leung (2019b).
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the network size, our model generates clustering in the limit both due to homophily in positions and “strategic transitivity” induced by $S_{ij,t}$. Graham (2016) discusses the policy-relevance of distinguishing between these mechanisms. In contrast, commonly used sparse models that require $V(\cdot)$ to be uniformly bounded with respect to types (Bickel et al., 2011; Menzel, 2017) rule out homophily as a potential mechanism in the limit because the networks they generate are “bounded” by Erdős-Rényi graphs, which are well known to feature trivial clustering (Galhotra et al., 2018). To be clear, our framework requires homophily, but this may often be the preferred alternative, being consistent with a large literature in sociology that documents the pervasiveness of homophily (McPherson et al., 2001).

Third, most social networks are sparse, meaning that the number of connections formed by the typical node is of smaller order than the number of nodes (Barabási, 2015; Chandrasekhar, 2016). This is often accomplished by scaling the sequence of models such that the expected degree of any node is asymptotically bounded:

$$\frac{1}{n} \sum_{i=1}^{n} \sum_{j \neq i} E[A_{ij,t}] = O(1).$$

(3)

Under regularity conditions in §5, this holds if $r$ tends to zero at a certain rate with $n$. Intuitively, as $n$ increases, the number of opportunities to form links grows, which promotes density, whereas as $r$ decreases, the number of attractive partners decreases due to homophily, which promotes sparsity. Our choice for $r$ in §4 will balance these two forces so that expected degree remains asymptotically bounded but nonzero. See Remark 1.

**Network Moments.** Our objective is to prove a CLT for network moments, which are averages $n^{-1} \sum_{i=1}^{n} \psi_i$ of node statistics $\psi_i$. Node statistics can be quite general functions of the network and model primitives. We define in §5.2 a broad class of such statistics to which our results apply. Here we provide illustrative examples.

**Example 1 (Subnetwork Counts).** A simple example of a network moment is the period-$t$ average degree, where $\psi_i$ is node $i$’s period-$t$ degree $\sum_{j \neq i} A_{ij,t}$. By varying $t$, this moment measures fluctuations in network density over time, which is a useful summary statistic. Note that the average degree is proportional to the dyad count (number of links). More generally, we can count any other connected subnetwork, such
as the number of triangles, \( k \)-stars, or complete networks on \( k \)-tuples. Subnetwork counts are useful for inference in static models of network formation (Sheng, 2020), and we describe in §6 how our results can be applied to inference in this setting. Finally, we can allow \( \psi_i \) can be a complex function of the local network neighborhood of \( i \), for instance the number of units at most path distance \( K \) from \( i \).

**Example 2 (Parametric Inference).** Consider the following parametric version of (1):

\[
V( r^{-1}\|X_i - X_j\|, S_{ij,t}, Z_{it}, Z_{jt}, \zeta_{ij,t} ) = S'_{ij,t}\theta_0 + \alpha_{ij} + \zeta_{ij,t},
\]

where \( \alpha_{ij} = -r^{-1}\|X_i - X_j\| \) is a fixed effect and \( S_{ij,t} = (A_{ij,t-1}, \sum_k A_{ik,t-1}A_{jk,t-1}) \). The latter is similar to (2), except the second component of \( S_{ij,t} \) is the number of prior friends in common. When \( \alpha_{ij} \) is instead left unrestricted, this setup corresponds to that of Graham (2016), who shows \( \theta_0 \) is point identified when \( T = 3 \) and \( \zeta_{ij,t} \) is logistic. For estimation, he proposes to maximize the conditional log-likelihood

\[
\frac{1}{n} \sum_{i=1}^{n} \sum_{j \neq i} \ell_{ij}(\theta), \quad \text{where} \quad \ell_{ij}(\theta) = I_{ij} \left( G_{ij}H'_{ij}\theta - \log \left( 1 + \exp \left\{ G_{ij}H'_{ij}\theta \right\} \right) \right),
\]

\[
G_{ij} = A_{ij,2} - A_{ij,1}, \quad H_{ij} = \left( A_{ij,3} - A_{ij,0}, \sum_k A_{ik,1}A_{jk,1} - \sum_k A_{ik,0}A_{jk,0} \right),
\]

and \( I_{ij} \) is an indicator for whether \((i, j)\) is a “stable dyad” (see his Definition 2). Asymptotic normality requires a CLT for the average of the scores

\[
\frac{1}{n} \sum_{i=1}^{n} \psi_i, \quad \text{where} \quad \psi_i = \sum_{j \neq i} \nabla_{\theta} \ell_{ij}(\theta).
\]

**Example 3 (Nonparametric Inference).** Consider a nonparametric version of the setup in Example 2 where \( V(\cdot) \) is instead unknown. Let \( s = (s, z, z') \) be a vector of constants with the same dimension as \( S_{ij,t} = (S_{ij,t}, Z_{it}, Z_{jt}) \). Define the conditional average structural function (ASF)

\[
\mu(s) = \mathbb{P}(V(\alpha_{ij}, s, \zeta_{ij,t}) > 0 \mid A_{ij,0} = 1),
\]

which has the same value for all \( t > 0 \). This is the linking probability for an initially linked pair in the counterfactual world where \( S_{ij,t} = s \). Varying \( s \) reveals a
conditional average causal effect on link formation. For instance, if \( Z_{it} \) includes a treatment, then varying the associated component of \( s \) yields an average treatment effect on link formation. Alternatively, varying the component of \( s \) corresponding to \( \max_k A_{ik,t-1} A_{jk,t-1} \) in (2) yields a nonparametric measure of “strategic transitivity.” We condition on an initial link \((A_{ij,0} = 1)\) because the unconditional analog of \( \mu(s) \) is \( O(n^{-1}) \) in a sparse network, and only a trivial upper bound can be obtained for this, as we elaborate in footnote 3.

In general, the ASF is not point-identified due to the fixed effect (Chamberlain, 2010), but we can construct bounds as follows. Partition the support of \( S(s, s_0(s), \ldots, S_T(s)) \), where for \( S_{ij} = (S_{ij,0}, \ldots, S_{ij,t}) \),

\[
S_t(s) = \{ S_{ij} : S_{ij,t} = s, S_{ij,r} \neq s \ \forall r < t, r = 1, \ldots, T \},
\]

so \( S_t(s) \) is the set of values \( S_{ij} \) whose \( t \)th component first equals \( s \) at time \( t \). Define

\[
\hat{A}_{ij}(s) = \sum_{t=1}^{T} 1\{ S_{ij} \in S_t(s) \} A_{ij,t} A_{ij,0} \quad \text{and} \quad P_{ij}(s) = 1\{ S_{ij} \in \hat{S}(s) \} A_{ij,0}.
\]

By the argument in Chernozhukov et al. (2013), \( \mu_\ell(s) \leq \mu(s) \leq \mu_u(s) \) for \( \mu_\ell(s) = E[\hat{A}_{ij}(s)] / E[A_{ij,0}] \) and \( \mu_u(s) = \mu_\ell(s) + E[P_{ij}(s)] / E[A_{ij,0}] \). We can estimate the bounds using their sample analogs

\[
(\hat{\mu}_\ell(s), \hat{\mu}_u(s)) = \left( \frac{n^{-1} \sum_{i=1}^{n} \sum_{j \neq i} \hat{A}_{ij}(s)}{n^{-1} \sum_{i=1}^{n} \sum_{j \neq i} \hat{A}_{ij,0}}, \quad \frac{n^{-1} \sum_{i=1}^{n} \sum_{j \neq i} (\hat{A}_{ij}(s) + P_{ij}(s))}{n^{-1} \sum_{i=1}^{n} \sum_{j \neq i} \hat{A}_{ij,0}} \right).
\]

Inference on the ASF thus requires a joint CLT for \( n^{-1} \sum_{i=1}^{n} \psi_i \), where

\[
\psi_i = \left( \sum_{j \neq i} \hat{A}_{ij}(s), \quad \sum_{j \neq i} P_{ij}(s), \quad \sum_{j \neq i} A_{ij,0} \right).
\]

\(^3\)For the unconditional ASF (removing the conditioning on \( A_{ij,0} = 1 \)), the same bounds hold if we do not divide by \( E[A_{ij,0}] \) in the bounds formulas and remove \( A_{ij,0} \) from the definitions of \( \hat{A}_{ij}(s) \) and \( P_{ij}(s) \). The trouble is that, under specification (2), if we choose, for example, \( s = (0, 0) \) in \( s = (s, z, z') \), then \( P_{ij}(s) = 1 \) when \( i, j \) are more than two links away in every period. In a sparse network, the share of such pairs tends to one, so \( E[P_{ij}(s)] \to 1 \). On the other hand, \( E[\hat{A}_{ij}(s)] \to 0 \) under sparsity, so the bounds are trivial. For this reason, we consider the conditional ASF instead.
3 Weak Dependence

To establish a CLT, we provide conditions under which \( \{\psi_i\}_{i=1}^n \) satisfies a form of weak dependence. In §4.1, we state the formal definition. The goal of this section is to present important concepts crucial for the definition and provide intuition.

First, we need to complete the model by specifying the data-generating process for \( A_0 \). In this section, we focus on a simple dyadic model for illustration, which we later substantially generalize in §5.1. For every \( i \neq j \), we assume

\[
A_{ij,0} = 1 \left\{ V(r^{-1}\|X_i - X_j\|, 0, Z_{i0}, Z_{j0}, \zeta_{ij,0}) > 0 \right\}. \tag{4}
\]

By this we mean that in period 0, there are no strategic interactions in link formation, so we set the strategic component \( S_{ij,t} \) to zero. We may interpret (4) as a representation of the initial “meeting process” prior to the creation of social connections.

3.1 Counterfactual Subnetwork

A key concept for what follows is the distinction between the observed network and counterfactual subnetworks. (We simply write “network” in place of “network time series” in what follows.) We first require some notation. For a set of nodes \( J \subseteq \mathbb{N} \) and sparsity parameter \( r > 0 \), let \( r^{-1}T_j = ((r^{-1}X_j, Z_j))_{j \in J} \) denote the array of types (with positions scaled by \( r^{-1} \)) and \( \zeta_j = (\zeta_{ij})_{i,j \in J} \) the array of random-utility shocks, where \( \zeta_{ij} = (\zeta_{ij,t})_{t=0}^T \) for all \( i \neq j \) and \( \zeta_{ii} = 0 \) for all \( i \). For the special case where \( J = \mathcal{N}_n \), we abbreviate these as \( r^{-1}T_n \) and \( \zeta_n \).

Given a network size \( n \), define a network formation model as a function

\[ A: (r^{-1}T_J, \zeta_J) \mapsto (A_t)_{t=0}^T, \]

where \( J \) is any subset of \( \mathbb{N} \) with cardinality of \( |J| = n \) and \( A_t \) the period-\( t \) network on \( J \). Thus, \( A(\cdot) \) is the reduced-form mapping from the structural primitives to a network time series. Denote by \( A^{dy}(\cdot) \) the network formation model induced by the

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4 Strictly speaking, \( A(\cdot) \) must be indexed by the network size \( n \), i.e. \( A(\cdot) = A_n(\cdot) \), since its domain concerns objects whose dimensions depend on \( |J| = n \). We therefore have in mind a sequence of models \( \{A_n(\cdot)\}_{n \in \mathbb{N}} \), one for each network size, which under Assumption 1 will be related across \( n \). We suppress the subscript in the notation since it will be clear from the inputs of \( A(\cdot) \).
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structural model defined by (1) and (4). Given this is the true model, we call

\[ A^{\text{dy}}(r^{-1}T_n, \zeta_n). \]  

(5)

the observed network (time series). For any subset of nodes \( J \subset \mathcal{N}_n \), we call

\[ A^{\text{dy}}(r^{-1}T_J, \zeta_J) \]  

(6)

the counterfactual \( J \)-subnetwork (with respect to \( \mathcal{N}_n \)). The latter is a crucial object for what follows. *It is the result of the thought experiment in which we rewind time prior to the realization of the initial condition of the observed network, remove all nodes in \( \mathcal{N}_n \setminus J \) from the model, and then start the network formation process anew only with nodes in \( J \), all the while fixing the primitives at their original realizations.*

### 3.2 Stabilization

We next informally introduce the concept of stabilization, which is our key weak dependence condition, and provide an example to motivate and verify it. The formal definition and primitive sufficient conditions are provided in §4.1 and §5, respectively.

It is important to understand that the subnetwork of \( J \) under (5) and the sub-network under (6) generally do not coincide due to strategic interactions since \( V(\cdot) \) depends on other nodes in the network, many of which have been removed in (6). We will see this concretely in an example below. However, it is often possible to construct, for each node \( i \), a small subset of nodes \( J_i \subset \mathcal{N}_n \) such that \( \psi_i \) has a.s. the same value under the observed network and counterfactual \( J_i \)-subnetwork. Let

\[ \psi_i(\mathcal{N}_n) \quad \text{and} \quad \psi_i(J_i) \]

respectively denote the node statistic of \( i \) obtained under the observed network and counterfactual \( J_i \)-subnetwork. The following informal definition conveys the main idea behind our weak dependence condition in §4.

**Concept 1.** For any \( n \in \mathbb{N} \) and \( i \in \mathcal{N}_n \), \( \psi_i(\cdot) \) is stabilizing if there exists a set of nodes \( J_i \subset \mathcal{N}_n \) (potentially a function of \((r^{-1}T_n, \zeta_n)\) and hence random) such that

(a) \( \mathbb{P}(\psi_i(\mathcal{N}_n) = \psi_i(J_i)) = 1 \), and (b) \( \sup_n \mathbb{P}(|J_i| > t) \) decays exponentially with \( t \).
Part (a) states that \( i \)'s node statistic has the same value in almost every state of the world under both the observed network and the counterfactual \( J_i \)-subnetwork, so that the removal of nodes outside of \( J_i \) has no effect on \( i \)'s statistic. Part (b) controls the size of \( J_i \) and implies it is asymptotically bounded. Intuitively, (a) and (b) ensure node statistics are weakly dependent since each \( \psi_i(N_n) \) only depends on a relatively small “neighborhood” of alters \( J_i \). Asymptotic boundedness of \( J_i \) is sufficient for a LLN (Leung, 2019b), but for a CLT, we require exponential tails.

Contrast Concept 1 with the more familiar concept of \( M \)-dependence, which states that an observation is only correlated with neighbors in some non-random neighborhood. Stabilization is similar in spirit, except \( J_i \) is node-specific, random, and typically a complicated function of the primitives, as shown below and in §5.3. In general, bounding the size of this set is nontrivial, and one of our main contributions is to demonstrate that branching processes can be used for this purpose.

### 3.3 Verifying Stabilization

This section walks through an example illustrating the previous concepts. Let \( A_t(n) \) denote the period-\( t \) network under (5) and \( A_{ij,t}(n) \) the potential link between \( i, j \). For the counterfactual \( J_t \)-subnetwork (6), we instead write \( A_t(J_i) \) and \( A_{ij,t}(J_i) \), respectively. Suppose the network moment of interest is the period-1 average degree:

\[
\psi_i(N^{n}_A) = \sum_{j=1}^{n} A_{ij,1}(n).
\]

Recall that \( N_A(i, 1) \) is the 1-neighborhood of \( i \) in \( A_1(n) \), that is, the set of nodes at most one link away from \( i \) in period 1 (including \( i \)).

To verify stabilization, an initial idea is to take \( J_i = N_{A_1(n)}(i, 1) \) since \( \psi_i(N_n) \) is just the number of elements in this set. Then Concept 1(a) requires that \( i \)'s degree has the same value under both the observed network and the counterfactual subnetwork generated using only nodes in \( N_{A_1(n)}(i, 1) \), a.s. Unfortunately, this generally fails to hold. To see this, consider the top two panels of Figure 1, which depict an example with \( n = 4 \). Here we imagine a model for which the following events occur with positive probability. First, the primitives \( (X_i, Z_{j0}, X_j, Z_{j0}, \zeta_{ij0}) \) are realized such that only node pairs \( (i, j) \in \{(1,3), (2,3)\} \) form period-0 links, as depicted in the upper left panel. Second, in period 1,

\[
V(r^{-1}||X_1 - X_2||, (0,1), Z_{11}, Z_{21}, \zeta_{12,1}) > 0 \geq V(r^{-1}||X_1 - X_2||, (0,0), Z_{11}, Z_{21}, \zeta_{12,1}) .
\]

That is, the primitives of (1,2) are realized such that their latent index is positive if
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and only if they have a common neighbor in period 0. Since they do, they form a link in period 1. Finally, the primitives of the remaining pairs are realized such that no others form links in period 1. The upper right panel depicts this.

![Observed network, period 0](image1)

![Observed network, period 1](image2)

![Counterfactual J1-subnetwork, period 0](image3)

![Counterfactual J1-subnetwork, period 1](image4)

Figure 1: $J_i = \mathcal{N}_{A_1(n)}(i, 1)$

Now consider the bottom two panels of Figure 1, which depict the counterfactual $J_1$-subnetwork since $J_1 = \mathcal{N}_{A_1(n)}(1, 1) = \{1, 2\}$ by the upper-right panel. Recall that the counterfactual subnetwork is obtained by fixing the primitives at their original realizations, removing nodes 3 and 4 from the model, and then regrowing the network time series from the start. Since the period-0 network follows a dyadic model, nodes 1 and 2 still do not link, following their behavior in the upper-left panel. However, by (7), in period 1, their latent index is non-positive since they have no common neighbors in period 0, so they form no link in period 1, as depicted in the bottom-right panel. Hence, $\psi_1(J_1) = 0$, whereas $\psi_1(N_n) = 1$. Since this situation occurs with positive probability, Concept 1(a) fails to hold.

We can fix the problem by enlarging $J_i$ as follows:

$$J_i = \mathcal{N}_{A_i(n)}(i, 1) \cup \bigcup_{j \in \mathcal{N}_{A_1(n)}(i, 1)} \mathcal{N}_{A_0(n)}(j, 1).$$

(8)

This properly accounts for period-0 links ignored in the previous definition by including the period-0 1-neighborhoods of the period-1 1-neighbors of $i$. Under this definition, now $J_1 = \{1, 2, 3\}$ because 3 is a 1-neighbor of 1 in period 0. Figure 2 depicts the counterfactual $J_1$-subnetwork. Although node 4 is removed from the model, the left panel shows that, in period 0, the remaining nodes still form the same network as in the upper-left panel of Figure 1. This is due to the dyadic model (4). Then in period 1, nodes 1 and 2 form a link because they have a common neighbor in period
0. The remaining pairs do not form links, as in Figure 1, because their primitives are kept the same. Therefore, $\psi_1(J_1) = \psi_1(N_n)$.

![Counterfactual $J_1$-subnetworks](image)

Figure 2: $J_i = (8)$

The previous argument is only suggestive, being specific to node 1 and a particular realization of the network, but the general argument for verifying Concept 1 using (8) is as follows. By (2), for any $j \in N_n$, $A_{ij,1}(n)$ is a function of $A_0(n)$ only through $A_{ij,0}(n)$ and $\max_k A_{ik,0}(n)A_{jk,0}(n)$. These statistics, in turn, are functions of $A_0(n)$ only through the 1-neighborhoods of $i$ and $j$ in period 0, which are the sets newly added in (8). By (4), $A_{kl,0}(n)$ is a deterministic function of $(r^{-1}X_k, r^{-1}X_l, Z_{k0}, Z_{l0}, \zeta_{kl,0})$ for any $k, l \in N_n$. Hence, the 1-neighborhoods of $i, j$ in period 0 are the same in the observed network and the counterfactual $J_i$-subnetwork for (8), and therefore $A_{ij,1}(n) = A_{ij,1}(J_i)$. Because $j$ is arbitrary, we have established that $\psi_1(N_n) = \psi_1(J_i)$ a.s.

This takes care of Concept 1(a), but equally important is Concept 1(b). Under assumptions stated later, $A_t(n)$ is sparse in each period $t$ in the sense of (3). This implies that the sizes of all 1-neighborhoods in (8) are $O_p(1)$, so we can expect $|J_i| = O_p(1)$. We will be able to further strengthen this to an exponential tail bound under additional regularity conditions.

**Generalizing the Initial Condition.** Building on this intuition, we construct $J_1$ for a large class of network moments in a generalized version of model (4) that allows for strategic interactions (see Lemma SA.2.1). Such a model may better represent the long-run outcome of a dynamic process, whereas (4) is reasonable for a short-run model. However, enriching the model in this fashion significantly complicates the argument above. For example, $A_{kl,0}$ can now be a function of the 1-neighbors of $k, l$ in period 0, which in turn may be functions of links formed by their 1-neighbors, and so on. Thus, with contemporaneous strategic interactions in the initial condition, $A_{kl,0}$ can depend on the *components* of $k$ and $l$ in period 0, which are the set of nodes
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at any finite path distance from \( k \) and \( l \).

Therefore, even a node statistic as simple as the average degree is generally a function of components, not just 1-neighborhoods, as in (8). It follows that, under (8), generally \( \psi_i(N_n) \neq \psi_i(J_i) \) in the strategic model. In §5.3, we show how to modify \( J_i \) to restore this property. This discussion indicates the usefulness of the stabilization concept, as it applies to complicated statistics that are functions of components. To verify stabilization in our application, we require exponential tail bounds on component sizes. We show that the latter can be stochastically bounded by the sizes of certain branching processes for which tail bounds can be more easily obtained; see §5.3.

4 General CLT

This section presents high-level conditions for a CLT under asymptotics sending \( n \to \infty \) while keeping \( T \) fixed. We construct on a common probability space the model primitives: i.i.d. types \( \{ (X_i, Z_i) \}_{i \in \mathbb{N}} \) and i.i.d. random-utility shocks \( \{ \zeta_{ij,t} : i \neq j, \{i,j\} \subseteq \mathbb{N}, t \in \{0, \ldots, T\} \} \), with the two arrays mutually independent. Let \( f \) denote the density of \( X_i \), which we assume is continuous and bounded away from zero and infinity. From these primitives, we construct \( r^{-1}T_n \) and \( \zeta_n \) for any \( n \).

Recalling §3.1, let \( \mathcal{A}(\cdot) \) be a network formation model, for example \( \mathcal{A}^{dy} (\cdot) \) or \( \mathcal{A}^{ps} (\cdot) \) in §5. For any set of nodes \( J \subset \mathbb{N} \) and \( i \in J \), define the \( \mathbb{R}^{d_y} \)-valued node statistic under network \( \mathcal{A}(r^{-1}T_j, \zeta_j) \)

\[
\psi_i(J) \equiv \psi(i, r^{-1}T_j, \zeta_j, \mathcal{A}) \equiv \psi(i, r^{-1}T_j, \zeta_j, \mathcal{A}(r^{-1}T_j, \zeta_j)).
\]

Examples of such statistics are given in §2. As usual, we assume \( \psi(\cdot) \) is invariant to permutations of node labels, which implies \( \{ \psi_i(J) \}_{i=1}^n \) is identically distributed.\(^5\) We

\[^5\]Recalling footnote 4, we really have in mind a sequence of functions \( \{ \psi_i^n(\cdot) \}_{n \in \mathbb{N}} \), one for each network size \( |J| = n \), but we suppress the superscript in the notation. The assumptions in this section are imposed on this sequence and the sequence \( \{ \mathcal{A}_n(\cdot) \}_{n \in \mathbb{N}} \) from the referenced footnote.

\[^6\]Formally, for any bijection \( \pi : J \to J \), let \( \pi(r^{-1}T_j) = ((r^{-1}X_{\pi(i)}, Z_{\pi(i)}))_{i \in J} \) and similarly define \( \pi(\zeta_j) \) and \( \pi(\mathcal{A}(r^{-1}T_j, \zeta_j)) \) by permuting node labels according to \( \pi \). Invariance of \( \psi(\cdot) \) to permutations means \( \psi(\pi(i), \pi(r^{-1}T_j), \pi(\zeta_j), \pi(\mathcal{A}(r^{-1}T_j, \zeta_j))) = \psi(i, r^{-1}T_j, \zeta_j, \mathcal{A}(r^{-1}T_j, \zeta_j)) \).
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aim to prove a CLT for network moments

$$\frac{1}{n} \sum_{i=1}^{n} \psi_i(N_n),$$

since $\psi_i(N_n)$ is $i$'s node statistic under the observed network $A(r^{-1}T_n, \zeta_n)$. The notation allows for an extremely broad class of moments, but for a CLT, we require $\psi(\cdot)$ to satisfy two high-level conditions introduced next.

4.1 Assumptions

We first state the formal version of Concept 1. Denote by $B(x, R) \subseteq \mathbb{R}^d$ the Euclidean ball centered at $x$ with radius $R$. For any $i \in N_n$ and $B_i \supseteq B(r^{-1}X_i, R)$, define $N_n(B_i) = \{j \in N_n: r^{-1}X_j \in B_i\}$, the set of nodes with scaled positions in $B_i$. We call

$$A(r^{-1}T_{N_n(B_i)}, \zeta_{N_n(B_i)})$$

the counterfactual $N_n(B_i)$-subnetwork (with respect to $N_n$), which generalizes the counterfactual $J_i$-subnetwork in §3.1 since $A(\cdot)$ is now any network formation model rather than $A^{dy}(\cdot)$ specifically. Then $\psi_i(N_n(B_i))$ corresponds to node $i$'s statistic under the counterfactual. As stated above, we call $A(r^{-1}T_n, \zeta_n)$ the observed network, which generalizes $A^{dy}(r^{-1}T_n, \zeta_n)$ in §3.1. In general, $\psi_i(N_n(B_i))$ and $\psi_i(N_n)$ will differ due to strategic interactions, as discussed in §3.3. However, it is often possible to construct a small set $B_i$ such that the two coincide a.s. for any $i$.

**Definition 1.** For any $n \in \mathbb{N}$, $r > 0$, $i \in N_n$, and network formation model $A(\cdot)$, the radius of stabilization

$$R_i(n) \equiv R(i, r^{-1}T_n, \zeta_n, A)$$

is the smallest random variable $R \geq 0$ such that $\psi_i(N_n) = \psi_i(N_n(B_i))$ a.s. for all $B_i \supseteq B(r^{-1}X_i, R)$.

The radius of stabilization is the smallest random $R$ such that $\psi_i$ is a.s. the same under any counterfactual $N_n(B_i)$-subnetwork such that $B_i$ contains $B(r^{-1}X_i, R)$. This is closely related to Concept 1, and the exact relation will be clarified following Assumption 1. The radius $R_i(n)$ defines the sets $N_n(B_i)$, which are analogous to $J_i$. 

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To construct $R_i(n)$, we could trivially choose $R$ so large that $B(r^{-1}X_i, R)$ contains all nodes’ positions. However, this choice of radius will generally diverge with $n$. To establish a LLN, Leung (2019b) requires the radius to be $O_p(1)$, which restricts the size of the largest possible counterfactual subnetwork. The key for a CLT is stronger control on the asymptotic tail behavior of the radius. This is the purpose of the first assumption below, which constitutes our main weak dependence condition.

**Limit Sequence.** Before stating the assumption, we need to define the sequence of models along which we take limits. For technical reasons, our assumptions need to consider a sequence indexed by $n$ in which the network size is not actually $n$ but rather of asymptotic order $n$. Let $\{N_n\}_{n \in \mathbb{N}}$ with $N_n \sim \text{Poisson}(n)$ be independent of the other model primitives. The de-Poissonization argument discussed below requires us to consider two different network sizes: (a) a random size $N_n + k$, where $k$ is a constant, and (b) a non-random size $m + k$, where $m = m_n$ with $m_n/n \to c \in (0, \infty)$. The special case of $m = n$ and $k = 0$ corresponds to our original setup, and no intuition is lost to the reader who focuses on this case in what follows.

We also need to define the rate at which $r$ degenerates along the sequence. In the rest of the paper, we assume that, for some $\kappa > 0$ and all $n \in \mathbb{N}$,

$$r \equiv r_n \equiv (\kappa/n)^{1/d},$$

where $d$ is the dimension of $X_1$. Recall that $f$ is the density of $X_i$, and define $X_k = (X_1, \ldots, X_k)$, $\text{supp}(f)$ as the support of $f$, and $\text{supp}(f)^k = \times_{i=1}^{k}\text{supp}(f)$.

**Assumption 1 (Exponential Stabilization).** For any $k \in \{1, 2, 3\}$, there exist $n_0, \epsilon > 0$ such that $\limsup_{w \to \infty} w^{-1} \max\{\log \tau_{b, \epsilon}(w), \log \tau_p(w)\} < 0$, where

$$\tau_{b, \epsilon}(w) = \sup_{n > n_0} \sup_{m \in ((1-\epsilon)n, (1+\epsilon)n)} \sup_{x_k \in \text{supp}(f)^k} \mathbb{P}(R_1(m + k) > w \mid X_k = x_k),$$

$$\tau_p(w) = \sup_{n > n_0} \sup_{x \in \text{supp}(f)} \mathbb{P}(R_1(N_n + 1) > w \mid X_1 = x).$$

The values of $k$ and quantities we take supremums over are technical in nature, but the essence of the assumption is that the radius of stabilization has exponential tails in a uniform sense. The choice of node 1 is arbitrary since nodes are identically distributed. We propose the following methodology to verify this assumption.
Methodology. First, for any \( n \) and all \( i \in \mathcal{N}_n \), construct \( J_i \subseteq \mathcal{N}_n \) such that Concept 1(a) holds. Typically, \( J_i \) will be some union of components and \( K \)-neighborhoods, as in (8) and (22) below. Given \( J_i \), the radius of stabilization must satisfy
\[
\mathbf{R}_i(n) \leq \hat{\mathbf{R}}_i(n) = \max_{j \in J_i} r^{-1} \| X_i - X_j \|,
\]
(11)
since \( J_i \) is the set of nodes on which \( \psi_i \) depends and \( B(r^{-1}X_i, \hat{\mathbf{R}}_i(n)) \) is the smallest ball containing the scaled positions of these nodes. Second, show that \( |J_i| \) has exponential tails. Lemmas in §SA.2.1 can be used to establish that this quantity is stochastically bounded by the size of a certain branching process when \( J_i \) consists of components and \( K \)-neighborhoods. Then lemmas in §SA.3 can be applied to show the size of the branching process has exponential tails. Finally, since we will require nodes to be homophilous in positions, each \( j \in J_i \) will typically be close to \( i \) in terms of distance \( r^{-1} \| X_i - X_j \| \). We can then translate the exponential tail bound for \( |J_i| \) to one for \( \hat{\mathbf{R}}_i(n) \), as in Lemma SA.2.4. See §5.3 for an application of this methodology.

The next assumption is a regularity condition requiring uniformly bounded moments. Let \( \| x \|_\infty \) denote the component-wise maximum of a vector \( x \).

Assumption 2 (Bounded Moments). There exist \( p > 2, \epsilon > 0, M < \infty, \) and \( n_0 \in \mathbb{N} \) such that for any \( k \in \{1, 2, 3\}, \)
\[
\max \{ \mathbb{E}[\| \psi_1(\mathcal{N}_{m+k}) \|^p \mid \mathbf{X}_k = \mathbf{x}_k], \mathbb{E}[\| \psi_1(\mathcal{N}_{n+1}) \|^p \mid X_1 = x] \} < M
\]
for all \( n > n_0, m \in ((1-\epsilon)n, (1+\epsilon)n), \mathbf{x}_k \in \text{supp}(f)^k, \) and \( x \in \text{supp}(f) \).

This assumption needs to be verified on a case-by-case basis. Section I.3 of Leung and Moon (2019) does so for leading examples of interest under assumptions in §5.

Finally, we assume that for any \( n \in \mathbb{N}, r > 0, i \in \mathcal{N}_n, \) and \( x \in \mathbb{R}^d, \)
\[
\psi(i, r^{-1} \mathbf{T}_n, \zeta_n, \mathcal{A}) = \psi(i, ((r^{-1}X_j + x, Z_j))_{j=1}^n, \zeta_n, \mathcal{A}).
\]
(12)
This is an innocuous restriction on the node statistic \( \psi(\cdot) \) and network formation model \( \mathcal{A}(\cdot) \). It says that scaled positions \( \{r^{-1}X_j\}_{i=1}^n \) only enter these two functions through distances \( r^{-1} \| X_i - X_j \| \) (which are used to capture homophily in (1)) since
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distances are invariant to shifting positions by \( x \). Note that, since \( X_i \) and \( Z_i \) may be arbitrarily dependent, \( X_i \) can still directly enter the model as a subvector of \( Z_i \), so this does not appear to impose an economically meaningful restriction.

4.2 Main Result

Let \( \lambda_{\min}(M) \) and \( \|M\| \) be respectively the smallest eigenvalue and entrywise maximum of a matrix \( M \), \( \Sigma_n = n^{-1}\text{Var}(\sum_{i=1}^n \psi_i(N_n)) \), and \( I \) be the identity matrix with the dimensions of \( \Sigma_n \).

**Theorem 1.** Under Assumptions 1 and 2, \( \sup_n \|\Sigma_n\| < \infty \). If additionally \( \liminf_{n \to \infty} \lambda_{\min}(\Sigma_n) > 0 \), then

\[
\Sigma_n^{-1/2} \frac{1}{\sqrt{n}} \sum_{i=1}^n (\psi_i(N_n) - E[\psi_i(N_n)]) \xrightarrow{d} \mathcal{N}(0, I).
\]

A previous draft of this paper provided lower-level conditions for \( \liminf_{n \to \infty} \lambda_{\min}(\Sigma_n) > 0 \) (Leung and Moon, 2019, Theorem 5.1).

The proof proceeds in two broad steps. The first step is to establish a CLT for the “Poissonized” model in which the number of nodes is \( N_n \), so-called because \( \{X_i\}_{i=1}^{N_n} \) has the same distribution as a Poisson point process with intensity function \( n f(\cdot) \) (Penrose, 2003, Proposition 1.5). Specifically, existing results can be adapted to show

\[
\tilde{\Sigma}_n^{-1/2} \frac{1}{\sqrt{n}} \left( \sum_{i=1}^{N_n} \psi_i(N_n) - E \left[ \sum_{i=1}^{N_n} \psi_i(N_n) \right] \right) \xrightarrow{d} \mathcal{N}(0, I), \tag{13}
\]

where \( \tilde{\Sigma}_n = n^{-1}\text{Var}(\sum_{i=1}^{N_n} \psi_i(N_n)) \). This is simpler to prove than Theorem 1 because, unlike \( \{X_i\}_{i=1}^n \), the Poisson process \( \{X_i\}_{i=1}^{N_n} \) possesses a well-known spatial independence property: for any disjoint subsets \( S_1, S_2 \subseteq \mathbb{R}^d \), \( |\{X_i\}_{i=1}^{N_n} \cap S_1| \perp |\{X_i\}_{i=1}^{N_n} \cap S_2| \).

We prove (13) by adapting a theorem in Penrose and Yukich (2005) based on Stein’s method (see our Theorem SA.1.1).

Since \( N_n/n \to^p 1 \), \( \{X_i\}_{i=1}^n \) and \( \{X_i\}_{i=1}^{N_n} \) should be “similar,” so given (13), we expect a similar result for the original model. The second “de-Poissonization” step of the proof, which follows Penrose (2007), shows that this intuition is correct, provided we properly adjust \( \tilde{\Sigma}_n \) downward to obtain the correct variance \( \Sigma_n \) since \( N_n \) contributes extra randomness (see Theorem SA.1.2).
More specifically, define the add-one cost 

$$\Xi_n = \psi_{n+1}(\mathcal{N}_{n+1}) + \sum_{i=1}^{n} \left( \psi_i(\mathcal{N}_{n+1}) - \psi_i(\mathcal{N}_n) \right).$$

This is the aggregate counterfactual impact on the total \(\sum_{i=1}^{n} \psi_i(\mathcal{N}_n)\) from adding a new node labeled \(n + 1\) to the model. The first term is the direct effect of adding \(n + 1\), which is its own node statistic. The second term is the indirect effect, which is the new node’s impact on the statistics of all other nodes. A key step of the proof establishes that

$$n^{-1/2} \left( \sum_{i=1}^{N_n} \psi_i(\mathcal{N}_{N_n}) - \mathbb{E} \left[ \sum_{i=1}^{N_n} \psi_i(\mathcal{N}_{N_n}) \right] \right)$$

$$= n^{-1/2} \left( \sum_{i=1}^{n} \psi_i(\mathcal{N}_n) - \mathbb{E} \left[ \sum_{i=1}^{n} \psi_i(\mathcal{N}_n) \right] \right)$$

$$+ n^{-1/2}(N_n - n)\mathbb{E}[\Xi_{N_n}] + o_p(1).$$

This can be viewed as a first-order expansion with respect to the number of nodes. The “derivative” is \(\mathbb{E}[\Xi_{N_n}]\) since it captures the change in moments as a result of a unit increment in the number of nodes. By (13), the left-hand side is asymptotically normal, and by the Poisson CLT, so is \(n^{-1/2}(N_n - n)\mathbb{E}[\Xi_{N_n}]\). Since \(N_n\) is independent of all other primitives, it follows that the first term on the right-hand side is asymptotically normal.

Related Literature. The proof is closely based on arguments in Penrose and Yukich (2005) and Penrose (2007), whose results pertain to spatial graphs without strategic interactions. The innovation in Theorem 1 is primarily conceptual, namely, the recognition that an appropriate modification of stabilization allows us to formalize the intuition in Concept 1 and extend existing results to econometric models of interest. Our main technical innovation is the use of branching process theory to derive primitive conditions for stabilization for models with strategic interactions (see §5.3). Leung (2019b) uses branching process results to establish a law of large numbers for static models of network formation. We tackle the more difficult task of proving a central limit theorem for a general setup that includes both static and dynamic models. This requires a different estimate of the radius of stabilization (Lemma SA.2.1)
as well as new exponential tail bounds for this estimate (Lemma SA.2.4).

The setup and assumptions used in Penrose’s work are not directly applicable to our setting, so work is needed to translate the results. There are three main differences between our setups. The first is the definition of the radius of stabilization. In addition to the different formulation in terms of counterfactual subnetworks, we require invariance of i’s node statistic to the removal of nodes outside of \( \mathcal{N}_n(B_i) \), whereas the literature’s definitions demand invariance to the removal and addition of new nodes. Invariance to addition is typically violated in the models we study, which, unlike the applications in Penrose’s work, allow for strategic interactions and dynamic link formation. Second, \( X_i \) and \( Z_i \) may be dependent in our setup, whereas the literature requires independence. Third, our model includes pair-specific shocks \( \zeta_{ij} \).

5 Primitive Conditions for Stabilization

This section verifies exponential stabilization for a model of strategic network formation using the methodology presented after Assumption 1. The model generalizes that of §2 in the following ways. First, we consider general classes of network moments and network-dependent regressors \( S_{ij,t} \) that respectively include Examples 1–3 and (2) as special cases. Second, we generalize (4) to allow for strategic interactions. Finally, we allow \( T = 0 \), which corresponds to a static model of network formation. Example 1 and §6 discuss network moments useful for inference in the static model.

Let \( r^{-1}T_{n,t} = ((r^{-1}X_i, Z_{it}))_{i=1}^n \). In place of (2), we consider more generally

\[
S_{ij,t} = S(i, j, r^{-1}T_{n,t-1}, A_{t-1}),
\]

where \( S(\cdot) \) is invariant to permutations of node labels (see footnote 6).\(^7\) The next assumption requires \( S(\cdot) \) to only depend on its arguments through the 1-neighborhoods of \( i \) and \( j \) in \( A_{t-1} \). For any \( J \subseteq \mathcal{N}_n \), let \( A_{t,J} = (A_{kl,t} : k \neq l, \{k, l\} \subseteq J) \), the subnetwork of \( A_t \) on \( J \). Recall from §1 that \( \mathcal{N}_{A_t}(i, 1) \) is \( i \)'s period-\( t \) 1-neighborhood.

**Assumption 3 (Local Externalities).** For any \( n \in \mathbb{N} \), period \( t, r > 0, i, j \in \mathcal{N}_n \) with

\(^7\)Recalling footnote 4, \( S(\cdot) \) implicitly depends on \( n \), so we really have in mind a sequence of functions \( \{S_n(\cdot)\}_{n \in \mathbb{N}} \), one for each network size, but we suppress the subscript in the notation.
$i \neq j$, network $A_t$ on $N_n$, and $J = N_{A_i}(i, 1) \cup N_{A_i}(j, 1)$,

$$S(i, j, r^{-1}T_{n,t}, A_t) = S(i, j, ((r^{-1}X_i, Z_{it}))_{i \in J}, A_{jt}).$$

This states that $S_{ij,t}$ is only a function of nodes linked to either $i$ or $j$. Most models in the literature satisfy this assumption, including (2), because they use variants of statistics that capture strategic transitivity (e.g. $\max_k A_{ik,t-1}A_{jk,t-1}$ and $\sum_k A_{ik,t-1}$ and popularity (e.g. $\sum_k A_{ik,t-1}$) (Christakis et al., 2020; Goldsmith-Pinkham and Imbens, 2013; Graham, 2016; Leung, 2019b; Mele, 2017; Sheng, 2020). Dependence of $S(\cdot)$ on attributes also allows for covariate-weighted versions of these examples, such as $\sum_k A_{ik,t-1}Z_{kt-1}$.

Let $\Phi_\zeta(z) = P(\zeta_{ij,t} \geq z)$. Define $\bar{V}(r^{-1}\|X_i - X_j\|, \zeta_{ij,t}) = \sup_{s, z, z'} V(r^{-1}\|X_i - X_j\|, s, z, z', \zeta_{ij,t})$, where the supremum is taken with respect to the range of $S(\cdot)$ and support of $(Z_{it}, Z_{jt})$ over all $t$.

**Assumption 4** (Homophily). For any $\delta \in \mathbb{R}_+$, $V(\delta, \cdot)$ is invertible in its second argument, and its inverse $V^{-1}(\delta, \cdot)$ satisfies $\lim \sup_{\delta \to \infty} \delta^{-1} \log \Phi_\zeta(V^{-1}(\delta, 0)) < 0$.

Note that $\Phi_\zeta(V^{-1}(\delta, 0))$ is an upper bound on the probability that two nodes at distance $\delta$ form a link. The assumption requires this to decay exponentially with $\delta$, which restricts the tails of the random-utility shock and the nonlinearity of $V(\cdot)$ in its first argument. Leung (2019b) establishes a LLN under a weaker version of the assumption that only requires polynomial decay, which is sufficient to show asymptotic boundedness of the radius of stabilization. However, exponential stabilization seems to require exponential decay.

**Example 4.** Consider the linear latent-index model

$$V(r^{-1}\|X_i - X_j\|, S_{ij,t}, Z_{it}, Z_{jt}, \zeta_{ij,t}) = (S_{ij,t}, Z_{it}, Z_{jt})'\theta - r^{-1}\|X_i - X_j\| + \zeta_{ij,t}.$$

Suppose $S_{ij,t}$ and $Z_{it}$ have uniformly bounded support, which is typically assumed in the literature (Boucher and Mourifié, 2017; Graham, 2016; Menzel, 2017). Then for

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8Recalling footnote 7, more precisely we take the supremum over the range of $S_n(\cdot)$ and $n$. 

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some constant \( M \),

\[
P(V^{-1}|X_i - X_j|, S_{ij,t}, Z_{it}, Z_{jt}, \zeta_{ij,t}) > 0 \mid r^{-1}|X_i - X_j| = \delta) \\
\leq P(M - \delta + \zeta_{ij,t} > 0) = \Phi_\zeta(\delta - M),
\]

which decays to zero exponentially quickly with \( \delta \) if \( \zeta_{ij,t} \) has exponential tails.

Remark 1. Assumption 4 and (10) imply sparsity (3). Notice

\[
\frac{1}{n} \sum_{i=1}^{n} \sum_{j \neq i} E[A_{ij,t}] = (n - 1)P(V^{-1}|X_i - X_j|, S_{ij,t}, Z_{it}, Z_{jt}, \zeta_{ij,t}) > 0) \\
\leq (n - 1)P(\zeta_{ij,t} > V^{-1}(r^{-1}|X_i - X_j|, 0)) \\
= (n - 1)r^d \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} P(\zeta_{ij,t} > V^{-1}(|x - x'|, 0)) f(x)f(x + r(x' - x)) \, dx \, dx' \quad (16)
\]

by a change of variables. By (10), since \( f \) is continuous, this converges to

\[
k \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Phi_\zeta(V^{-1}(|x - x'|, 0)) f(x)^2 \, dx' \, dx,
\]

which is finite because the integrand decays exponentially with \( |x - x'| \) by Assumption 4.

5.1 Initial Condition

We next state a general model for \( A_0 \) that allows for strategic interactions. We require three assumptions, which essentially ensure that potential links in \( A_0 \) are weakly dependent. The first is a restriction on the strength of strategic interactions, the second a regularity condition, and the third a restriction on the selection mechanism.

Recall that \( r^{-1}T_{n,0} = ((r^{-1}X_i, Z_{il}))_{i=1}^n \) and \( S_{ij,1} = S(i, j, r^{-1}T_{n,0}, A_0) \). We assume that, for some function \( V_0(\cdot) \) possibly different from \( V(\cdot) \), for any \( n \) and \( i \neq j \),

\[
A_{ij,0} = \mathbf{1}\{V_0(r^{-1}|X_i - X_j|, S_{ij,1}, Z_{i0}, Z_{j0}, \zeta_{ij,0}) > 0\}. \quad (17)
\]

This is analogous to (1), except we include the contemporaneous network \( A_0 \) on the right-hand side through \( S_{ij,1} \). If we interpret \( V_0(\cdot) \) as the joint surplus agents \( i \) and \( j \)
enjoy from forming a link, then this corresponds to the well-known solution concept of pairwise stability under transferable utility (Jackson, 2010).

Strength of Interactions. Our first condition restricts the strength of interactions, which is measured by

\[
p_r(X_i, Z_{i0}, X_j, Z_{j0}) = P\left( \sup_s V_0(r^{-1}\|X_i - X_j\|, s, Z_{i0}, Z_{j0}, \zeta_{ij,0}) > 0 \mid X_i, Z_{i0}, X_j, Z_{j0} \right)
- P\left( \inf_s V_0(r^{-1}\|X_i - X_j\|, s, Z_{i0}, Z_{j0}, \zeta_{ij,0}) > 0 \mid X_i, Z_{i0}, X_j, Z_{j0} \right)
\]  

(assuming measurability). This is the effect on period-0 link formation of increasing the value of \(S_{ij,1}\) from its “lowest” to its “highest” possible value, conditional on types. Note that in the case of no strategic interactions (4), \(p_r(X_i, Z_{i0}, X_j, Z_{j0}) = 0 \) a.s.

Recall that \(d_z\) is the dimension of \(Z_{i0}\), and let \(\Phi_{z,0}(\cdot \mid x)\) be the conditional distribution of \(Z_{i0}\) given \(X_i = x\). For any \(h: \mathbb{R}^d \times \mathbb{R}^{d_z} \rightarrow \mathbb{R}\), define the mixed norm

\[
\|h\|_m = \sup_{x \in \mathbb{R}^d} \left( \int_{\mathbb{R}^{d_z}} h(x, z)^2 d\Phi^*(z) \right)^{1/2},
\]

where \(\Phi^*(\cdot)\) is a distribution on \(\mathbb{R}^{d_z}\) given in the next assumption.

Assumption 5 (Subcriticality). There exists a distribution \(\Phi^*(\cdot)\) on \(\mathbb{R}^{d_z}\) such that, for any \((x, z) \in \text{supp}(X_i, Z_{i0})\) and \(x' \in \text{supp}(f)\),

\[
\int_{\mathbb{R}^{d_z}} p_1(x, z; x', z') d\Phi_{z,0}(z' \mid x') \leq \int_{\mathbb{R}^{d_z}} p_1(x, z; x', z') d\Phi^*(z').
\]

Furthermore, for \(\bar{f} = \sup_{x \in \mathbb{R}^d} f(x)\) and \(\kappa\) in (10),

\[
\|h_D\|_m < 1 \quad \text{where} \quad h_D(x, z) = \kappa \bar{f} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^{d_z}} p_1(x, z; x', z')^2 d\Phi^*(z') \right)^{1/2} dx'.
\]

This is a slightly stronger version of Assumption 6 of Leung (2019b). As discussed extensively in that paper, it is analogous to standard weak dependence conditions for linear spatial and temporal autoregressive models that bound the magnitude of the autoregressive parameter below one. In our case, the assumption bounds a functional of the measure of strategic interactions (18) below one. The widely used linear-in-means model of peer effects requires a similar bound on the endogenous effect
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(Bramoullé et al., 2009; Leung, 2020b). Our condition only looks more complicated due to the nonlinearity of the model.

Assumption 5 is a restriction on the partial equilibrium effect of strategic interactions since (18) only captures the direct effect of changing $S_{ij,1}$ on $A_{ij,0}$ but not subsequent adjustments in the network that lead to a new equilibrium. Simulation results in Leung (2019b) show that the general equilibrium effect under subcriticality can still be substantive.

We next impose a mild regularity condition on (18).

**Assumption 6 (Regularity).** Either $p_r(X_i, Z_{i0}; X_j, Z_{j0}) = 0$ a.s. for any $r > 0$, or
\[
\inf_{(x,z) \in \text{supp}(X_i, Z_{i0})} \lim_{n \to \infty} n E[p_r(X_i, Z_{i0}; X_j, Z_{j0}) \mid X_i = x, Z_{i0} = z] > 0.
\]

Additionally, Assumption 4 holds with $V_0(\cdot)$ in place of $V(\cdot)$.

The case $p_r(X_i, Z_{i0}; X_j, Z_{j0}) = 0$ corresponds to a model without strategic interactions. For the more interesting case, the assumption essentially requires that strategic interactions are sufficiently nontrivial for all nodes in the sense that $p_r(X_i, Z_{i0}; X_j, Z_{j0})$ is at least order $n^{-1}$, which is a very mild requirement. It is typically satisfied because the two probabilities in (18) are upper and lower bounds on the probability of link formation, and that probability is order $n^{-1}$ under sparsity by (16).

**Selection Mechanism.** Multiple pairwise stable networks $A_0$ may satisfy (17), so the model is incomplete. Let $\mathcal{E}(r^{-1}T_{n,0}, \zeta_{n,0})$ denote the set of such networks, where $\zeta_{n,0}$ is the $n \times n$ matrix with $ij$th entry $\zeta_{ij,0}$ for $i \neq j$ and $ii$th entry equal to zero. To complete the model, we introduce a selection mechanism
\[
\lambda : (r^{-1}T_{n,0}, \zeta_{n,0}) \mapsto A_0 \in \mathcal{E}(r^{-1}T_{n,0}, \zeta_{n,0}).^9
\]

As discussed in Leung (2019b), weak dependence requires restrictions on the selection mechanism since it is possible to construct pathological cases such that the perturbation of a single node’s type shifts the whole network to an entirely different

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^9It is without loss of generality to assume that $\lambda(\cdot)$ is only a function of $(r^{-1}T_{n,0}, \zeta_{n,0})$ and not any other additional sources of randomness. This is because we can always include in each $Z_{i0}$ additional random variables that are payoff-irrelevant in the sense that they do not enter $V_0(\cdot)$ and $S(\cdot)$. Also, recalling footnote 4, we really have in mind a sequence of functions $\{\lambda_n(\cdot)\}_{n \in \mathbb{N}}$, one for each network size, but we suppress the subscript in the notation.
pairwise stable network. The next assumption rules out such cases. We state the assumption and required definitions with less commentary and refer the reader to §3.3 of Leung (2019b) for motivation and discussion. As discussed there, variants of myopic best-response dynamics are leading examples of selection mechanisms that satisfy the assumption. These are analogs of (1) and are the most widely used selection mechanisms in the theoretical and econometric literature on dynamic network formation (Graham, 2016; Jackson, 2010; Mele, 2017).

Following Leung (2019b), define the “non-robustness” indicator

$$D_{ij} = \mathbf{1}\left\{ \sup_s V_0(r^{-1}\|X_i - X_j\|, s, Z_{i0}, Z_{j0}, \zeta_{ij,0}) > 0 \right. \cap \inf_s V_0(r^{-1}\|X_i - X_j\|, s, Z_{i0}, Z_{j0}, \zeta_{ij,0}) \leq 0 \right\}. $$

Note (18) is the conditional expectation of $D_{ij}$. If $\inf_s V_0(r^{-1}\|X_i - X_j\|, s, Z_{i0}, Z_{j0}, \zeta_{ij,0}) > 0$, then the link between $i$ and $j$ in period 0 is “robust” in the sense that the pairwise stability condition (17) holds regardless of the state of the network since the latter enters $V_0(\cdot)$ only through $S(\cdot)$. Likewise, if $\sup_s V_0(r^{-1}\|X_i - X_j\|, s, Z_{i0}, Z_{j0}, \zeta_{ij,0}) \leq 0$, the potential link is “robustly absent” in the sense that no link forms, regardless of the network state. In either case, we have $D_{ij} = 0$. If instead $D_{ij} = 1$, then $A_{ij,0}$ is “non-robust” in the sense that changes to the ambient network may well affect the pairwise stability of $A_{ij,0}$.

Let $D = (D_{ij})_{i,j \in \mathcal{N}_n}$ be the network of non-robustness indicators and $C_i$ denote $i$’s component in $D$ (defined at the end of §1). Define $i$’s strategic neighborhood

$$C_i^+ = C_i \cup \left\{ j \in \mathcal{N}_n : \inf_s V_0(r^{-1}\|X_j - X_k\|, s, Z_{j0}, Z_{k0}, \zeta_{jk,0}) > 0 \text{ for some } k \in C_i \right\}, $$

which adds to $C_i$ the set of all nodes with a robust link to some node in $C_i$. The significance of this definition is that, by Proposition 1 of Leung (2019b), the subnetwork of $A_0$ on $C_i^+$ remains pairwise stable in the sense of (17) under the counterfactual model in which the set of nodes is $C_i^+$ rather than $\mathcal{N}_n$. That is, the pairwise stability of this subnetwork is invariant to the removal of nodes $\mathcal{N}_n \setminus C_i^+$ from the model. Leung (2020b) exploits this property to compute $E(r^{-1}\mathbf{T}_{n,0}, \zeta_{n,0})$ in polynomial time; we instead use this to obtain a CLT.

For any $J \subseteq \mathcal{N}_n$, define $r^{-1}\mathbf{T}_{J,0} = ((r^{-1}X_i, Z_{i0}))_{i \in J}$, and likewise define $\zeta_{J,0}$, the submatrix of $\zeta_{n,0}$ restricted to nodes in $J$. Let $\lambda(r^{-1}\mathbf{T}_{n,0}, \zeta_{n,0})$ be the restriction
of the range of $\lambda(\cdot)$ to networks on $J$. That is, if $\lambda^{-1}(T_{n,0}, \zeta_{n,0})$ outputs a network $A_0 \in \mathcal{E}(r^{-1}T_{n,0}, \zeta_{n,0})$, then $\lambda^{-1}(T_{n,0}, \zeta_{n,0})|_J$ is $A_{J,0}$.

**Assumption 7** (Selection Mechanism). The following hold for any $n \in \mathbb{N}$ and $r > 0$ in a neighborhood of zero. (a) An equilibrium exists: $\mathcal{E}(r^{-1}T_{n,0}, \zeta_{n,0})$ is nonempty a.s. (b) The selection mechanism $\lambda(\cdot)$ generates the initial condition: $A_0 = \lambda(r^{-1}T_{n,0}, \zeta_{n,0})$. (c) The selection mechanism is decentralized: for all $i \in \mathcal{N}_n$ and $J = C^+_i$, we have $\lambda(r^{-1}T_{n,0}, \zeta_{n,0})|_J = \lambda(r^{-1}T_{J,0}, \zeta_{J,0})$.

Parts (a) and (b) complete the model, with (b) specifying the model for the initial condition, which generalizes (4). The substantive restriction is part (c), which corresponds to Assumption 7 of Leung (2019b). It implies that the subnetwork of $A_0$ on any strategic neighborhood $C^+_i$ coincides with its counterfactual $C^+_i$-subnetwork (recall the definition from §3.1). As previously discussed, the set of pairwise stable subnetworks on $C^+_i$ coincide in both models, so the assumption additionally requires the selected equilibrium of both to coincide. This means the selection mechanism is sufficiently decentralized such that nodes in disjoint strategic neighborhoods coordinate separately on their pairwise stable subnetworks. It rules out pathological examples where all nodes coordinate on the basis of a common signal, such as a single node’s type. Leung (2019b) discusses how variants of myopic best-response dynamics satisfy the assumption. In the special case where the equilibrium is unique, as in (4), the assumption holds trivially.

### 5.2 Network Moments

We next define a broad class of node statistics that includes the examples in §2. Recall from §1 that $\mathcal{N}_A(i, K)$ is $i$’s $K$-neighborhood. For any network time series $\mathbf{A} = (A_t)_{t=0}^T$, define $i$’s *dynamic $K$-neighborhood* $\mathcal{N}_A^T(i, K) = \bigcup_{t=0}^T \bigcup_{j \in \mathcal{N}_A(i, K)} \bigcup_{t'=0}^T \mathcal{N}_{A_{t'}}(j, K)$. To construct (20), for each period $t$, start with $i$’s $K$-neighborhood under $A_t$. For each $j$ in this neighborhood, add to this set $j$’s $K$-neighborhood in $A_{t'}$, for all periods $t'$. While the expression appears somewhat complicated, it allows us to define a
rather general class of dynamic network moments. The next assumption states that \( \psi_i \) only depends on its arguments through \( i \)'s dynamic \( K \)-neighborhood. Recall that for \( J \subset \mathbb{N} \), \( A_{J,t} \) is the period-\( t \) subnetwork on \( J \), and define \( A_J = (A_{J,t})_{t=0}^T \).

**Assumption 8 (K-Locality).** For some integer \( K \) and any \( n \in \mathbb{N} \), \( r > 0 \), \( i \in \mathcal{N}_n \), network time series \( A = (A_t)_{t=0}^T \) on \( \mathcal{N}_n \), and \( J = \mathcal{N}_A^T(i, K) \),

\[
\psi(i, r^{-1}T_n, \zeta_n, A) = \psi(i, r^{-1}T_J, \zeta_J, A_J).
\]

**Example 5 (Subnetwork Counts).** The counts for connected subnetworks in Example 1 satisfy Assumption 8. For example, the period-0 dyad count corresponds to \( \psi_i(\mathcal{N}_n) = \sum_{j=1}^n A_{ij,0} \). This satisfies Assumption 8 for \( K = 1 \) since it is only a function of the 1-neighbors of \( i \) in period 0, which is a subset of \( \mathcal{N}_A^0(i, 1) = \cup_{j \in \mathcal{N}_A(i,1)} \mathcal{N}_A(j, 1) \).

The triangle count is proportional to \( \sum_{i \neq j \neq k} A_{ij,0}A_{jk,0}A_{ik,0} \), with corresponding node statistic \( \sum_{j,k: i \neq j \neq k} A_{ij,0}A_{jk,0}A_{ik,0} \), which also satisfies Assumption 8 for \( K = 1 \) since it is also only a function of \( i \)'s 1-neighbors.

**Example 6 (Parametric Inference).** In Example 2, the node statistic is \( \sum_j \nabla_{\theta} \ell_{ij}(\theta) \), where the score is a function of \( I_{ij}, G_{ij}, H_{ij} \) defined in the example. By definition, \( I_{ij}, G_{ij}, H_{ij} \) are only functions of the network time series through the 1-neighborhoods of \( i \) and \( j \) in periods \( t = 0, \ldots, 3 \). Furthermore, \( I_{ij} = 1 \) implies \( i, j \) are linked in either period 1 or 2 (Graham, 2016, Definition 2), so \( j \in \mathcal{N}_A(i, 1) \) for some \( t \). Therefore, the node statistic satisfies Assumption 8 for \( K = 1 \).

**Example 7 (Nonparametric Inference).** Consider the first component of \( \psi_i(\mathcal{N}_n) \) in Example 3, \( \sum_{j \neq i} \hat{A}_{ij}(s)A_{ij,0} \). By definition, \( \hat{A}_{ij}(s) \) is a function of \( S_{ij} \) for all \( j \in \mathcal{N}_A(i, 1) \). By Assumption 3, \( S_{ij} \) is a function of the network time series only through the 1-neighborhoods of \( i \) and \( j \) in \( A_t \) for all \( t \). Hence, for any \( s \), this statistic satisfies Assumption 8 for \( K = 1 \). The argument is similar for \( \sum_{j \neq i} P_{ij}(s) \).

### 5.3 Main Result

**Theorem 2.** Under Assumptions 3 and 4 (dynamic model), 5–7 (initial condition), and 8 (moments), Assumption 1 (exponential stabilization) holds.
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Verifying Assumptions. By the discussion in the previous subsection, a direct corollary is that CLTs for Examples 1–3 hold under Assumptions 2–7. At this level of generality, these conditions are likely as primitive as possible and need to be verified directly for examples of interest. For Example 1, §SA.5.1 of Leung (2019b) verifies Assumption 2, and the same argument applies to Example 3. For Example 2, Assumption 2 holds if \( S_{ij,t} \) is uniformly bounded (Assumption 2 of Graham, 2016). Assumption 3 is straightforward to verify for a given specification of \( S(\cdot) \). Verification of Assumption 4 is also straightforward; see Example 4. Assumptions 5–7 are general nonparametric restrictions on the initial condition, but for specific parametric models, they can be verified under lower-level conditions. For instance, they hold trivially in the dyadic model (4). For models with strategic interactions, Assumption 7 holds for myopic best-response dynamics, as previously discussed. Verifying Assumption 6 requires calculating the limit of \( n \mathbb{E}[D_{ij} \mid X_i, Z_i] \), which can be done by following (16). Finally, verifying Assumption 5 involves computing \( \|h_D\| \), which can be simulated. See the simulation study of Leung (2019b) and also his §A, which shows how it may be computed in closed form for normal random-utility shocks.

Proof Sketch. We prove Theorem 2 using the methodology stated after Assumption 1. Here we sketch the argument for the period-1 average degree example in §3.3. The discussion there was limited to the network formation model \( \mathcal{A}^{dy}(\cdot) \) defined from (1) and (4). We now consider the more general model \( \mathcal{A}^{ps}(\cdot) \) induced by (1) and the strategic initial conditions model from Assumption 7, with

\[
\mathcal{A}^{ps}(r^{-1}T_n, \zeta_n)
\]

(21)
denoting the resulting observed network time series. In what follows, \( A_i \) denotes the period-\( t \) network under (21) and \( A_{ij,t} \) its \( ij \)th entry.

Step 1 is to construct \( J_i \). By definition, \( \psi_i(\mathcal{N}_n) = |\mathcal{N}_{A_1}(i, 1)| \). By Assumption 3, \( i \)'s period-1 links are functions of the network through \( \mathcal{N}_{A_0}(j, 1) \) for any \( j \in \mathcal{N}_{A_1}(i, 1) \), which previously motivated the definition of \( J_i \) in (8). However, as noted at the end of §3.3, this choice of \( J_i \) does not work when we allow for strategic interactions in the initial condition because \( \mathcal{N}_{A_0}(j, 1) \) can depend on the entirety of \( j \)'s network component in period 0. Components in \( A_0 \) are typically asymptotically unbounded in size, so this fact does not help us establish Concept 1(b). However, we can refine the argument to show that \( \mathcal{N}_{A_0}(j, 1) \) actually only depends on a much smaller set.
Recall the definition of strategic neighborhoods \( C_i^+ \) from (19). As shown in Proposition 1 of Leung (2019b), \( \mathcal{N}_{A_0}(j, 1) \) only depends on \( C_j^+ \) in the sense that this 1-neighborhood is the same under both \( \mathcal{A}_T^{ps}(r^{-1} T_n, \zeta_n) \) and \( \mathcal{A}_T^{ps}(r^{-1} T_{C_i^+}, \zeta_{C_i^+}) \). This is because, for any \( k \notin C_j^+ \), the potential link between \( k \) and any \( \ell \in C_j^+ \) is robustly absent, so by Assumption 7, the absence of \( k \) under the \( C_j^+ \)-subnetwork has no impact on \( C_j^+ \). Therefore, replacing \( \mathcal{N}_{A_0}(j, 1) \) with \( C_j^+ \) in (8) results

\[
J_i = \mathcal{N}_{A_1}(i, 1) \cup \bigcup_{j \in \mathcal{N}_{A_1}(i, 1)} C_j^+, \tag{22}
\]

which satisfies Concept 1(a). Lemma SA.2.1 is the formal proof for the general case.

**Step 2** is to obtain exponential tail bounds for \( |J_i| \). To first bound \( |C_j^+| \), we use a breadth-first search: begin at \( j \), and explore its component in \( D \) by first revealing nodes in \( j \)'s 1-neighborhood, then nodes in the 1-neighbors of \( j \)'s neighbors, and so on, terminating the search once no new nodes are found. The set of nodes thus found is \( C_j \). To obtain \( C_j^+ \), add one more step in this search: for each node \( k \in C_j \), explore the set of nodes \( \ell \) such that \( \inf_{s} V(r^{-1} \|X_k - X_\ell\|, s, Z_{k0}, Z_{\ell0}, \zeta_{k0,\ell0}) > 0 \).

To bound the size of this process, we argue as follows. First, we bound the number of nodes found in the breadth-first search by a branching process, which, under Assumption 5, is subcritical in the sense that the expected number of neighbors of any node in \( D \) is asymptotically less than one. This is because \( \|h_D\|_m \) in the assumption is an upper bound on \( \mathbb{E}[\sum_j D_{ij}] \) by Jensen’s inequality. Consequently, each node \( i \) is superseded in the next step of the search by less than one node on average, so the process is below the replacement rate, and the search terminates in an asymptotically bounded number of steps. Using Assumption 6, we can strengthen this result to prove that the size of the branching process has exponential tails. Second, under Assumption 4, the number of robust links emanating from any node \( j \) has exponential tails. Combined with the first fact, this results in an exponential tail bound for \( |C_j^+| \). Formally, §SA.2.1 establishes that the breadth-first search is dominated by a branching process, and Lemmas SA.3.1 and SA.3.2 provide tail bounds.

Finally, we translate tail bounds on \( |J_i| \) to bounds on (11). By homophily, nodes \( j \in J_i \) will typically be near \( i \) in the sense that the maximum distance \( r^{-1} \|X_i - X_j\| \) over all such \( j \) is asymptotically bounded so long as \( |J_i| = O_p(1) \). In fact we can show the stronger result that (11) has exponential tails if \( |J_i| \) does. This result relies on Assumption 4, which says that link formation probabilities are exponentially decaying
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in distance. Lemma SA.2.4 provides the formal argument.

6 Inference

We next discuss three methods for inference on $E[\psi_i]$ and then turn to moment inequalities. Inference on the mean is directly relevant for quantifying the uncertainty of summary statistics in Example 1. It is also relevant for the conditional likelihood estimator in Example 2 since standard errors for the mean of the score can be converted to standard errors for the estimator using the sandwich formula.

First, we can use the generalized spatial HAC estimator for $\Sigma$ proposed in a previous draft of this paper (Leung and Moon, 2019). It consistently estimates the variance when the sample consists of only a single large network. On the other hand, it requires positions to be observed in order to construct distances $\|X_i - X_j\|$.

Second, if positions are not observed, we can use a procedure proposed by Leung (2020a). This consists of resampling $\{\psi_i\}_{i=1}^n$ to construct a test statistic and forming a confidence region for $E[\psi_i]$ using critical values constructed from a normal approximation. The procedure works so long as $n^{-1} \sum_{i=1}^n \psi_i$ is $\sqrt{n}$-consistent for its expectation, which can be established using Theorem 1. The drawback of this procedure is that it is inefficient, having a rate of convergence slower than $\sqrt{n}$.

Third, if positions are unobserved but the sample consists of multiple independent large networks, then more powerful methods are available. If $\psi_i$ is a scalar, then we can implement the robust $t$-test proposed by Ibragimov and Müller (2010). If it is a vector, we can use the randomization test proposed by Canay et al. (2017). The latter consists of computing $n^{-1} \sum_{i=1}^n \psi_i$ separately for each network and multiplying the results by $\pm 1$ to generate a permutation distribution and construct critical values. The procedure works if the limit distribution of the vector of network moments is asymptotically normal, which can be established using Theorem 1.

Moment Inequalities. The subnetwork counts in Example 1 are useful for inference in static models of network formation. Sheng (2020) defines an identified set of structural parameters in terms of moment inequalities of the form $E[n^{-1} \sum_{i=1}^n (G_i - H_i(\theta))] \leq 0$. In our context, $\sum_{i=1}^n G_i$ corresponds to a connected subnetwork count, such as the number of transitive triples (3 nodes forming a complete subnetwork). The second part $H_i(\cdot)$ is a deterministic function of the structural parameters and
subvector of types observed by the econometrician for nodes 1, …, k, where k is the size of the subnetwork being counted (e.g. 3 for transitive triples). The assumptions in Sheng (2020) imply $H_i(\cdot)$ can be computed by simulation.

Since $n^{-1} \sum_{i=1}^{n} H_i(\theta)$ is a U-statistic of order k, by the Hoeffding decomposition, 

$$n^{-1} \sum_{i=1}^{n} H_i(\theta) = n^{-1} \sum_{i=1}^{n} \tilde{H}_i(\theta) + o_p(n^{-1/2}),$$

where $\tilde{H}_i(\theta)$ only depends on $i$’s type (e.g. Leung, 2015a, Proposition B.2). Since $G_i$ satisfies Assumption 8, as discussed in Example 5, so does $G_i - \tilde{H}_i(\theta)$, and our CLT is applicable. Using the methods above, we obtain a confidence region for $E[n^{-1} \sum_{i=1}^{n} (G_i - \tilde{H}_i(\theta))]$, which can be converted to a confidence region for the structural parameters by a Bonferroni argument (Romano et al., 2014). A confidence interval for the ASF in Example 3 can be obtained similarly since the methods above deliver confidence intervals for the upper and lower bounds.

7 Conclusion

This paper develops a large-sample theory for network models with strategic interactions when the data consists of a small sample of large networks or possibly a single network. We prove a general CLT under a high-level weak dependence condition and provide a general methodology for its verification. We apply the methodology to obtain primitive conditions for CLTs for static and dynamic models of network formation with strategic interactions. In related work, we also apply the methodology to network regressions and treatment effects with spillovers (Leung and Moon, 2019) and models of discrete choice with social interactions (Leung, 2019a). There are several important directions for future work. Concentration inequalities for stabilization would be useful, for example, for deriving lower-level conditions for uniform convergence of nonparametric or high-dimensional estimators using network data. Also, it would be of interest to develop alternatives to the inference procedures in §6 that may be used when positions are unobserved.

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Supplemental Appendix

SA.1 Proof of Theorem 1

Proof of Theorem 1. First suppose $d_\psi = 1$, meaning $\psi(\cdot)$ is 1-dimensional. By Theorem SA.1.1, (SA.1.1) holds, so we may then directly apply Theorem SA.1.2 to obtain the result. For the case $d_\psi > 1$, let $t \in \mathbb{R}^{d_\psi}\setminus\{0\}$. The 1-dimensional statistic $\tilde{\psi}(i, r_n^{-1}T_n, \zeta_n, A) = t'\psi(i, r_n^{-1}T_n, \zeta_n, A)$ satisfies Assumptions 1 and 2 since $\psi(\cdot)$ satisfies them, so the result follows from the Cramér-Wold device.

SA.1.1 Poissonization

This subsection proves a CLT for the Poissonized model in which the set of nodes is $N_{N_n}$ rather than $N_n$. Throughout, we assume $d_\psi$, the dimension of $\psi(\cdot)$, is 1. Define $\tilde{\sigma}_n^2 = n^{-1}\text{Var}(\sum_{i=1}^{N_n} \psi_i(N_{N_n}))$.

Theorem SA.1.1. Suppose $d_\psi = 1$. Under Assumptions 1 and 2, $\sup_n \tilde{\sigma}_n^2 < \infty$. If additionally $\liminf_{n \to \infty} \tilde{\sigma}_n^2 > 0$, then

$$\tilde{\sigma}_n^{-1} \frac{1}{\sqrt{n}} \left( \sum_{i=1}^{N_n} \psi_i(N_{N_n}) - \mathbb{E} \left[ \sum_{i=1}^{N_n} \psi_i(N_{N_n}) \right] \right) \xrightarrow{d} \mathcal{N}(0, 1). \quad \text{(SA.1.1)}$$

The proof follows that of Theorem 2.1 in Penrose and Yukich (2005) (PY). For completeness, we next restate its key elements using our notation.

Remark SA.1.1 (Related Literature). The fact that the proof essentially carries over to our setting is not obvious because there are a number of differences between our settings. First, our definition of exponential stabilization differs because, as discussed at the end of §4.2, we only require invariance of node statistics to the removal of nodes outside the radius of stabilization, whereas PY demand invariance to removal and addition. However, it turns out the argument only requires invariance to removal; see (SA.1.9) in the proof below. (In contrast, the de-Poissonization argument in
§SA.1.2 will require more modification of their arguments. Second, PY’s model has no random-utility shocks $\zeta_{ij}$, but adding these to the model turns out to have no effect on the argument. Third, they define $T_{N_n}$ as a marked Poisson point process, which is equivalent to our representation because (1) our set of positions $X_{N_n} = (X_i)_{i=1}^{N_n}$ has the same distribution as $\mathcal{P}_{nf}$, the Poisson point process on $\mathbb{R}^d$ with intensity function $nf(\cdot)$ (Penrose, 2003, Proposition 1.5), and (2) the “marks” $Z_i$ associated with each $X_i$ are independent across nodes, each only potentially correlated with its own position $X_i$. Fourth, PY consider the simpler case where $X_i$ is a deterministic function of $Z_i$ for each $i$ since this is sufficient for their applications. We allow for dependence, but this turns out not to affect the argument. Finally, PY consider moments of the form

$$\frac{1}{n} \sum_{i=1}^{N_n} \xi((X_i, Z_i), r_n^{-1} T_{N_n}).$$

This differs from $\psi(\cdot)$ because $\mathcal{A}(\cdot)$ is not an argument (nor is $\zeta_{N_n}$ for reasons previously discussed). However, since $\mathcal{A}(\cdot)$ is a deterministic function of $r_n^{-1} T_{N_n}$ and $\zeta_{N_n}$, we can define

$$\xi(i, r_n^{-1} T_{N_n}, \zeta_{N_n}) = \psi(i, r_n^{-1} T_{N_n}, \zeta_{N_n}, \mathcal{A}),$$

in which case this coincides with PY’s setup except for the presence of $\zeta_{N_n}$.

The basis of the proof is a dependency graph CLT due to Chen and Shao (2004). The technique is to approximate $\sum_{i=1}^{N_n} \psi_i(N_{N_n})$ by a sum $\sum_{i=1}^{n} W_i$, where the dependence structure of the summands can be characterized in terms of a network (“dependency graph”) $G$ in the sense that observations unlinked in $G$ are independent.

**Definition 2.** A network $G$ over $n$ nodes (with self-links) is a dependency graph for data $\{W_i\}_{i=1}^{n}$ if for any $S_1, S_2 \subseteq N_n$ such that $G_{ij} = 0$ for all $i \in S_1$ and $j \in S_2$, we have $\{W_i: i \in S_1\} \perp \perp \{W_j: j \in S_2\}$.

Let $\|W\|_p$ be the $L_p$-norm of $W$ and $\Phi(\cdot)$ the standard normal CDF.

**Lemma SA.1.1** (Dependency Graph CLT). Let $q \in (2, 3)$ and $W = \sum_{i=1}^{n} W_i$. Suppose $G$ is a dependency graph for $\{W_i\}_{i=1}^{n}$, and let $\Gamma = \max_i \sum_{j \neq i} G_{ij}$. Further suppose

\footnote{See e.g. Last and Penrose (2017) for the definition of a marked Poisson point process.}
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\[ \mathbf{E}[W^2] = 1, \quad \mathbf{E}[W_i] = 0, \quad \text{and} \quad \|W_i\|_q \leq \theta \quad \text{for all} \ i \ \text{and some} \ \theta > 0. \]

Then

\[ \sup_t |\mathbf{P}(W \leq t) - \Phi(t)| \leq 75 \theta^q \Gamma^{5(q-1)} n. \]

**Proof.** This is Theorem 2.7 of Chen and Shao (2004).

The next two lemmas provide useful bounds on various moments and are taken from Penrose and Yukich (2005), translated to our notation. The dictionary for translation is as follows: replace \( \lambda \) (their notation) with \( n \) (our notation), \( \lambda^{-1/d} \) with \( r_n \), \( \kappa \) with \( f \), \( \Lambda_{\lambda} \) with \( \supp(f) \), \( f \) with the function mapping any real number to 1, \( \mathcal{P}_\lambda \) with \( X_{N_n} \) where

\[ X_m = (X_1, \ldots, X_m) \quad \forall m \in \mathbb{N}, \]

and \( \xi_{\lambda}(X_{ij}, \mathcal{P}_\lambda) \) with \( \psi_{ij}(\mathcal{N}_{N_n}) \) (we define the \( ij \) notation below).

As discussed above, \( X_{N_n} \) has the same distribution as the Poisson point process \( \mathcal{P}_{nf} \). It can then be represented as a different collection of independent random variables that is more useful for the argument that follows. Fix a sequence \( \{\rho_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}_+ \) to be determined later. Let \( s_n = r_n \rho_n \) and \( V_n \) be the number of cubes of the form \( Q = \prod_{i=1}^d [j_i s_n, (j_i + 1) s_n) \), where \( j_i \in \mathbb{Z} \) for all \( i \), such that \( Q \) has positive density under \( f \). Label these cubes \( Q_1, \ldots, Q_{V_n} \). Fix any \( 1 \leq i \leq V_n \). Note that \( |X_{N_n} \cap Q_i| \) has the same distribution as \( N_i \sim \text{Poisson}(\nu_i) \), where \( \nu_i = n \int_{Q_i} f(x) \, dx \). Label the points of \( X_{N_n} \cap Q_i \) as \( X_1, \ldots, X_{N_i} \), which, conditional on \( N_i \), are i.i.d. draws from a density \( f(\cdot)/\int_{Q_i} f(x) \, dx \). Under this representation,

\[ X_{N_n} = \bigcup_{i=1}^{V_n} \{X_{ij}\}_{j=1}^{N_i}. \quad (SA.1.2) \]

In the remainder of this section, we will often use this double-indexed labeling for nodes under this representation, where we label a node \( k \) by \( ij \) if its position \( X_k \) corresponds to \( X_{ij} \). We denote node \( ij \)'s node statistic by

\[ \psi_{ij}(\mathcal{N}_{N_n}). \]

The next two lemmas are restatements of Lemmas 4.2 and 4.3 of Penrose and Yukich (2005) with our translated notation. Let \( \rho_n = \alpha \log n \) for some \( \alpha \) sufficiently
large that, for $p$ in Assumption 2, some $C > 0$, all $n \geq 1$, and $\tau_p(\cdot)$ in Assumption 1,

$$
\rho_n^{d/p}(n\tau_p(\rho_n))^{(q-2)/(2q)} < Cn^{-4}, \quad \tau_p(\rho_n) < Cn^{-3}, \quad \rho_n^d < Cn^{p/(p+2)}. \tag{SA.1.3}
$$

**Lemma SA.1.2.** Under Assumption 2, for $p$ given in the assumption, there exists $C > 0$ such that $E[|\psi_{ij}(N_{N_n})|^p1\{j \leq N_i\}] \leq C\rho_n^d$, $\forall \ n > 1, j \geq 1$, and $1 \leq i \leq V_n$.

**Lemma SA.1.3.** Under Assumption 2, for $p$ given in the assumption and $1 < q < p$, there exists $C > 0$ such that $\|\sum_{j=1}^{\infty} |\psi_{ij}(N_{N_n})1\{j \leq N_i\}|_q \leq C\rho_n^d(\rho_{n+1}/p) \forall 1 \leq i \leq V_n$.

Since these lemmas are fairly straightforward consequences of the regularity condition in Assumption 2, we omit their proofs. To demonstrate that PY’s proof carries over to our setting, we focus on reproducing the proof of their Theorem 2.1 in our notation, reorganizing parts of the argument for clarity.

**Proof of Theorem SA.1.1.** Lemma SA.1.7 establishes $\sup_n \tilde{\sigma}_n^2 < \infty$. Abbreviate

$$
\Lambda_n = \sum_{i=1}^{N_n} \psi_i(N_{N_n}).
$$

We prove the following stronger result: for $q \in (2, 3]$ with $q < p$, there exists $C > 0$ such that for all $n$ sufficiently large,

$$
\sup_t |P(\text{Var}(\Lambda_n)^{-1/2}(\Lambda_n - E[\Lambda_n]) \leq t)| - \Phi(t) | \leq C(\log n)^{dq} n \text{Var}(\Lambda_n)^{-q/2}. \tag{SA.1.4}
$$

Since $\text{Var}(\Lambda_n)^{-1} = O(n^{-1})$ by assumption, the right-hand side of (SA.1.4) tends to zero, which proves the theorem.

**Setup.** Under representation (SA.1.2),

$$
\Lambda_n = \sum_{i=1}^{V_n} \sum_{j=1}^{N_i} \psi_{ij}(N_{N_n}).
$$

The key trick is to approximate this with a similar sum to which the dependency graph CLT (Lemma SA.1.1) more usefully applies.

Recall $R_i(n)$ from Definition 1. Since our setup now indexes nodes by $ij$, $1 \leq i \leq V_n$, $1 \leq j \leq N_i$, we instead write $R_{ij}(n)$ for a given network size $n$. Let $E_{ij}$ be the
event \( \{ R_{ij}(N_n) \leq \rho_n \} \) for \( \rho_n \) defined in (SA.1.3),
\[
\Lambda_n' = \sum_{i=1}^{V_n} \sum_{j=1}^{N_i} \psi_{ij}(N_n) 1 \{ \mathcal{E}_{ij} \}, \quad \text{and} \quad \Psi_i = \text{Var}(\Lambda_n')^{-1/2} \sum_{j=1}^{N_i} \psi_{ij}(N_n) 1 \{ \mathcal{E}_{ij} \},
\]
so that \( \text{Var}(\Lambda_n')^{1/2} \Lambda_n' = \sum_{i=1}^{V_n} \Psi_i. \)

We next derive a bound on the difference between \( \Lambda_n \) and \( \Lambda_n' \) that will be useful for later. By Lemma SA.1.3 and Minkowski’s inequality,
\[
\left\| \sum_{i=1}^{V_n} \sum_{j=1}^{N_i} |\psi_{ij}(N_n)| \right\|_q \leq CV_n \rho_n^{d(p+1)/p} \leq Cn \rho_n^{d/p}. \quad \text{(SA.1.5)}
\]

By Definition 1, under the event \( \mathcal{E}_n = \bigcap_{i=1}^{V_n} \bigcap_{j=1}^{\infty} \mathcal{E}_{ij} \), we have \( \Lambda_n' = \Lambda_n \), so
\[
\|\Lambda_n - \Lambda_n'\|_2 \leq \|\Lambda_n - \Lambda_n'\|_q \mathbb{P}(\mathcal{E}_n^c)^{0.5-1/q} \leq (\|\Lambda_n\|_q - \|\Lambda_n'\|_q) \mathbb{P}(\mathcal{E}_n^c)^{(q-2)/(2q)} \quad \text{(SA.1.6)}
\]
by the Hölder and Minkowski inequalities. Applying Theorem 1.6 of Penrose (2003) to expand the expectation of a sum of \( N_i \) i.i.d. random variables,
\[
\mathbb{P}(\mathcal{E}_n^c) \leq \mathbb{E}\left[ \sum_{i=1}^{V_n} \sum_{j=1}^{N_i} 1 \{ \mathcal{E}_{ij}^c \} \right] = n \int_{\mathbb{R}^d} \mathbb{P}(R_{ij}(N_n) > \rho_n \mid X_{ij} = x) f(x) \, dx \leq n \tau_p(\rho_n)
\]
for \( n \) sufficiently large. By (SA.1.5), (SA.1.6), (SA.1.7), our choice of \( \rho_n \) (SA.1.3), and Assumption 1,
\[
\|\Lambda_n - \Lambda_n'\|_2 \leq Cn^{-3}. \quad \text{(SA.1.8)}
\]

Dependency Graph. We construct a dependency graph \( G \) for \( \{ \Psi_i \}_{i=1}^{V_n} \) by setting \( G_{ij} = 1 \{ \inf \{ \|x-y\| : x \in Q_i, y \in Q_j \} \leq 2r_n \rho_n \} \). To see that this is a dependency graph, observe that under \( \mathcal{E}_{ij} \), \( B(X_{ij}, R_{ij}(N_n)r_n) \subseteq Q_i^+ \), where \( Q_i^+ = \{ x \in \mathbb{R}^d : \|x-y\| \leq r_n \rho_n \forall y \in Q_i \} \) is the \( r_n \rho_n \)-neighborhood of \( Q_i \). Thus by Definition 1,
\[
\Psi_i = \text{Var}(\Lambda_n')^{-1/2} \sum_{j=1}^{N_i} \psi_{ij}(N_n(Q_i^+)) 1 \{ \mathcal{E}_{ij} \}, \quad \text{(SA.1.9)}
\]
where \( N_n(Q_i^+)(S) = \{ j \in N_n : X_j \in S \} \) for any \( S \subseteq \mathbb{R}^d \). This means the value of \( \Psi_i \) is determined solely by nodes with positions in \( X_{N_n} \cap Q_i^+ \). Furthermore, by the spatial
independence property of Poisson point processes (Last and Penrose, 2017, Theorem 5.2), for any two disjoint sets $S_1, S_2 \subseteq \mathbb{R}^d$,

$$
\{(X_i, X_j, Z_i, Z_j, \zeta_{ij}): i, j \in N'_{n_a}(S_1)\}
\perp \{(X_i, X_j, Z_i, Z_j, \zeta_{ij}): i, j \in N'_{n_a}(S_2)\}. \quad (SA.1.10)
$$

Therefore, we have $\{\Psi_k: k \in Q_i\} \perp \{\Psi_{\ell}: \ell \in Q_j\}$ whenever $\inf\{\|x - y\|: x \in Q_i, y \in Q_j\} > 2r_n\rho_n$. Hence, $G$ is a dependency graph.

Note that (SA.1.4) holds trivially when $\text{Var}(\Lambda_n) < 1$ for $n$ sufficiently large, so we henceforth assume it is at least 1. We apply Lemma SA.1.1 to $\{W_i\}_{i=1}^{V_n}$ for $W_i = \Psi_i - \mathbb{E}[\Psi_i]$. To figure out a good choice of $\theta$, notice by Lemma SA.1.3 that

$$
\|\Psi_i\|_q \leq C \text{Var}(\Lambda_n')^{-1/2} \left\| \sum_{j=1}^{\infty} |\xi_{ij}| \right\|_q \leq C \text{Var}(\Lambda_n')^{-1/2} \rho_n^{d(p+1)/p}.
$$

Therefore, we set

$$
\theta = C \text{Var}(\Lambda_n')^{-1/2} \rho_n^{d(p+1)/p}.
$$

Let $W = \sum_{i=1}^{V_n} W_i$. Since $f$ has bounded support,

$$
V_n = O(n\rho_n^{-d}). \quad (SA.1.11)
$$

Also, there are at most $5^d$ other cubes at most $2r_n\rho_n$ from cube $Q_i$, so $\Gamma \leq 5^d$. By Lemma SA.1.1,

$$
\sup_{t} |\mathbb{P}(W \leq t) - \Phi(t)| \leq C n\rho_n^{-d} \text{Var}(\Lambda_n')^{-\eta/2} \rho_n^{d(p+1)/p} \quad (SA.1.12)
$$

**Obtaining Bound** (SA.1.4). We next replace the right-hand side of (SA.1.12) with the bound on the right-hand side of (SA.1.4). We do so by selecting $\rho_n = \alpha \log n$ and then showing we can replace $\text{Var}(\Lambda_n')$ in (SA.1.16) with $\text{Var}(\Lambda_n)$. We next show that these two variances are close for $n$ large. Trivially,

$$
\text{Var}(\Lambda_n) = \text{Var}(\Lambda_n') + \text{Var}(\Lambda_n - \Lambda_n') + 2\text{Cov}(\Lambda_n', \Lambda_n - \Lambda_n'). \quad (SA.1.13)
$$

We can bound the first term on the right-hand side as follows. By construction, $\Lambda_n'$ is the sum of $V_n$ random variables that have second moments bounded by a constant
times $\rho_n^{2d(p+1)/p}$ due to Lemma SA.1.3 for case $q = 2$. Also, as previously established, the covariance between any pair of these variables is zero when their indices $i, j$ correspond to non-adjacent cubes $Q_i, Q_j$. For adjacent cubes, the covariance is bounded by a constant times $\rho_n^{2d(p+1)/p}$ by Cauchy-Schwarz. Consequently, by (SA.1.11),

$$\text{Var}(\Lambda'_n) = O(n\rho_n^{d(p+2)/p}). \quad \text{(SA.1.14)}$$

Combining (SA.1.8), (SA.1.13), and (SA.1.14), by our choice of $\rho_n$ (SA.1.3) and Cauchy-Schwarz,

$$|\text{Var}(\Lambda_n) - \text{Var}(\Lambda'_n)| \leq Cn^{-2}. \quad \text{(SA.1.15)}$$

Given $\text{Var}(\Lambda_n) \geq 1$ from above, this implies that, for $n$ sufficiently large, $\text{Var}(\Lambda'_n) \geq \text{Var}(\Lambda_n)/2$. Hence,

$$\text{(SA.1.12)} \leq Cn\text{Var}(\Lambda_n)^{-q/2}\rho_n^{dq}, \quad \text{(SA.1.16)}$$

which delivers the bound on the right-hand side of (SA.1.4), given $\rho_n = \alpha \log n$.

**Proving (SA.1.4).** However, (SA.1.12) and (SA.1.16) do not immediately deliver (SA.1.4) because they apply to $W = \text{Var}(\Lambda'_n)^{-1/2}(\Lambda'_n - E[\Lambda'_n])$ rather than the target $\text{Var}(\Lambda_n)^{-1/2}(\Lambda_n - E[\Lambda_n])$. Let $\tilde{W} = \text{Var}(\Lambda'_n)^{-1/2}(\Lambda_n - E[\Lambda_n])$. Using the Lipschitz property of $\Phi(\cdot)$ and (SA.1.16), elementary calculations yield

$$\sup_t |P(\tilde{W} \leq t) - \Phi(t)| \leq C\beta + Cn\text{Var}(\Lambda_n)^{-q/2}\rho_n^{dq} + P(|\tilde{W} - W| \geq \beta) \quad \text{(SA.1.17)}$$

for any $\beta > 0$ (Penrose and Yukich, 2005, eq. (4.19)). Also,

$$|\tilde{W} - W| \leq \text{Var}(\Lambda'_n)^{-1/2}(|\Lambda_n - \Lambda'_n| + E[|\Lambda_n - \Lambda'_n|]). \quad \text{(SA.1.18)}$$

Note that (SA.1.8) implies $E[|\Lambda_n - \Lambda'_n|] \leq Cn^{-3}$, so under $\mathcal{E}_n$, (SA.1.18) $\leq Cn^{-3}$. On the other hand, $P(\mathcal{E}'_n) \leq Cn^{-2}$ by (SA.1.7) and (SA.1.3). So choosing $\beta = Cn^{-3}$ in (SA.1.17), we obtain

$$\sup_t |P(\tilde{W} \leq t) - \Phi(t)| \leq Cn\text{Var}(\Lambda_n)^{-q/2}\rho_n^{dq} + Cn^{-2}. \quad \text{(SA.1.19)}$$
Finally, we turn to bounding the left-hand side of (SA.1.4). We have

\[
\sup_t \left| P(\text{Var}(\Lambda_n)^{-1/2}(\Lambda_n - E[\Lambda_n]) \leq t) - \Phi(t) \right|
\]

\[
\leq \sup_t \left| P\left( \tilde{W} \leq t \left( \frac{\text{Var}(\Lambda_n)}{\text{Var}(\Lambda'_n)} \right)^{1/2} \right) - \Phi\left( t \left( \frac{\text{Var}(\Lambda_n)}{\text{Var}(\Lambda'_n)} \right)^{1/2} \right) \right|
\]

\[
+ \sup_t \left| \Phi\left( t \left( \frac{\text{Var}(\Lambda_n)}{\text{Var}(\Lambda'_n)} \right)^{1/2} \right) - \Phi(t) \right|. \tag{SA.1.20}
\]

As previously discussed, for \( n \) large, \( \text{Var}(\Lambda'_n) \geq \text{Var}(\Lambda_n)/2 \geq 0.5 \), so by (SA.1.15), there exists \( C' > 0 \) such that for all \( n > 0 \),

\[
\left| t \left( \frac{\text{Var}(\Lambda_n)}{\text{Var}(\Lambda'_n)} \right)^{1/2} - t \right| \leq |t| \left| \frac{\text{Var}(\Lambda_n)}{\text{Var}(\Lambda'_n)} - 1 \right| \leq C'n^{-2}|t|.
\]

Then since \( \Phi(\cdot) \) is Lipschitz, there exists \( C'' > 0 \) such that

\[
\sup_t \left| \Phi\left( t \left( \frac{\text{Var}(\Lambda_n)}{\text{Var}(\Lambda'_n)} \right)^{1/2} \right) - \Phi(t) \right| \leq C''n^{-2}.
\]

Combined with (SA.1.19) and (SA.1.20), we obtain

\[
\sup_t \left| P(\text{Var}(\Lambda_n)^{-1/2}(\Lambda_n - E[\Lambda_n]) \leq t) - \Phi(t) \right| \leq C(\log n)^{d_n} n \text{Var}(\Lambda_n)^{-q/2} + O(n^{-2}).
\]

By (SA.1.14) and (SA.1.15), \( \text{Var}(\Lambda_n) = O(n^{d_n(p+2)/p}) \), so the first term on the right-hand side dominates the second term, and (SA.1.4) follows. ■

**SA.1.2 de-Poissonization**

As in the previous subsection, we assume \( d_\psi = 1 \). The next result shows that, given a CLT for the Poissonized model with node set \( \mathcal{N}_{n,n} \), a CLT also holds for the original “binomial” model of interest with node set \( \mathcal{N}_n \), with a correction to the asymptotic variance. The proof follows the argument in Penrose (2007), but unlike in §SA.1.1, we need to modify some parts of the argument to account for differences between our settings, most notably the definition of stabilization.

Define \( \sigma^2_n = n^{-1} \text{Var}(\sum_{i=1}^n \psi_i(\mathcal{N}_n)) \).
Theorem SA.1.2. Suppose $d_{\psi} = 1$. Under Assumptions 1 and 2, $\sup_n \sigma_n^2 < \infty$. If additionally $\liminf_{n \to \infty} \sigma_n^2 > 0$ and (SA.1.1) holds, then

$$
\sigma_n^{-1} \frac{1}{\sqrt{n}} \left( \sum_{i=1}^{n} \psi_i(N_n) - \mathbb{E} \left[ \sum_{i=1}^{n} \psi_i(N_n) \right] \right) \xrightarrow{d} \mathcal{N}(0, 1).
$$

The proof is stated at the end of this subsection and uses the next four lemmas. Lemma SA.1.4 establishes an asymptotic equivalence between $X_n$ and $X_{N_n}$ and is taken directly from Penrose and Yukich (2003). Lemma SA.1.5 proves that node statistics of two nodes whose positions are a fixed distance apart are asymptotically uncorrelated. The third result, Lemma SA.1.6, draws on Lemma SA.1.5 to show that add-one costs of two nodes at a fixed distance apart are asymptotically uncorrelated. Finally, Lemma SA.1.7 shows that various moments are uniformly bounded.

Remark SA.1.2 (Related Literature). All of these results are analogs of lemmas in Penrose (2007), so we compare and contrast our setup with his. First, the differences discussed in Remark SA.1.1 persist here. Second, the proofs of Lemmas SA.1.5 and SA.1.6 need some modification because our setups differ, most notably the definition of stabilization. Third, to prove the lemmas, we define a coupling that follows the ideas in Penrose (2007) and Penrose and Yukich (2003). The details are different because we need to accommodate random-utility shocks $\zeta_{ij}$ and differences in notation. Fourth, Penrose considers a generalization of $\psi(\cdot)$ that is a measure in that it takes as an argument a subset of $\mathbb{R}^d$. This generalization is unnecessary for our purposes.

The lemmas utilize the following coupling. The essential idea follows Penrose and Yukich (2003), p. 298, but the formal details are different. As in §4, define i.i.d. types \(\{(X_i,Z_i)\}_{i \in \mathbb{N}}\) and i.i.d. random-utility shocks \(\{\zeta_{ij,t}: i \neq j, \{i,j\} \subseteq \mathbb{N}, t \in \{0,\ldots,T\}\}\), with the two sets mutually independent. For any $n \in \mathbb{N}$, let $N'_n$ be an independent copy of $N_n$ (defined in §4), independent of all other primitives.

- Abusing notation, for any $m \in \mathbb{N}$, we redefine $X_m = (X_5, X_7, X_9, \ldots, X_{2m+3})$. This clearly has the same distribution as the more conventional-looking array $(X_1, X_2, \ldots, X_m)$. We will see shortly below why we use only odd labels. We discuss in Remark SA.1.4 why we skip nodes 1, 4 (for now).

- Fix any $x, y \in \mathbb{R}^d$. Let $F_x$ be the set of points in $\mathbb{R}^d$ closer to $x$ than $y$ and $F_y$
the remaining set of points (break ties arbitrarily). Let $P_{nf}^x$ be the restriction of $\tilde{X}_{N_n} \equiv (X_5, X_7, X_9, \ldots, X_{2N_n+3})$ to $F_x$. Let $Q_{nf}^x$ be the restriction of $\tilde{X}_{N_n'} \equiv (X_6, X_8, X_{10}, \ldots, X_{2N_n'+3})$ to $F_y$. Construct $P_{nf}$ by concatenating the two vectors $P_{nf}^x$ and $Q_{nf}^x$. As the notation suggests, for any $n$, the set consisting of the components of $P_{nf}^x$ has the same distribution as an inhomogeneous Poisson point process on $\mathbb{R}^d$ with intensity function $nf(\cdot)$. This follows from Poisson superposition (Last and Penrose, 2017, Theorem 3.3), since $F_x$ and $F_y$ partition $\mathbb{R}^d$. Hence, $P_{nf}^x$ has the same distribution as $X_{N_n}$ used in §4.

- Similarly, let $P_{nf}^y$ be the restriction of $\tilde{X}_{N_n}$ to $F_y$ and $Q_{nf}^y$ the restriction of $\tilde{X}_{N_n'}$ to $F_x$. Construct $P_{nf}^y$ by concatenating the two vectors $P_{nf}^y$ and $Q_{nf}^y$. Again, $P_{nf}^y$ has the same distribution as $X_{N_n}$ used in §4, but note that by taking odds and evens differently on the two half-sets $F_x$ and $F_y$, we have that $P_{nf}^x$ is independent of $P_{nf}^y$.

The point of this construction is the following independence property. For any $S_1, S_2 \subseteq \mathbb{R}^d$ with $S_1 \cap S_2 = \emptyset$, define $N_x(S_1) = \{k \in \mathbb{N} : X_k \in P_{nf}^x \cap S_1\}$ and $N_y(S_2)$ similarly. By spatial independence of the Poisson point process (Last and Penrose, 2017, Theorem 5.2),

$$\{ (X_i, X_j, Z_i, Z_j, \zeta_{ij}) : i, j \in N_x(S_1)\} \perp \{ (X_i, X_j, Z_i, Z_j, \zeta_{ij}) : i, j \in N_y(S_2)\}.$$  

(SA.1.21)

This is similar to (SA.1.10) but instead uses two “interleaved” Poisson processes $P_{nf}^x$ and $P_{nf}^y$. The interleaving idea of taking two different “halves” on $F_x$ and $F_y$ of two independent Poisson processes (in our case, the odds and the evens) is taken from Penrose and Yukich (2003). Our construction only differs by building the two processes from the odd and even elements of the same countable set of random vectors.

**Remark SA.1.3.** This construction will be used to show that the node statistic of a node positioned at $x$ is asymptotically uncorrelated with that of a node positioned at $y$, for any fixed $x, y \in \mathbb{R}^d$ (Lemma SA.1.5). Intuitively, by stabilization, $x$’s statistic is primarily determined by nodes near $x$ and likewise for $y$, and by Lemma SA.1.4 and stabilization, nodes near $x$ will be given by $P_{nf}^x$ and nodes near $y$ by $P_{nf}^y$, which are independent. Note this does not mean node statistics are essentially i.i.d. because in the continuum limit, there are many nodes near $x$ and many near $y$.  

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For any \( x, y \in \mathbb{R}^d \), \( r > 0 \), and \( X \subseteq \mathbb{R}^d \), let \( \tau_{x,r} y = x + r^{-1}(y - x) \) and \( \tau_{x,r} X = \{ x + r^{-1}(y - x) : y \in X \} \). This operator dilates the point set \( X \) relative to point \( x \), whereas in the main text, we instead use \( r^{-1}X = \tau_{0,r}X \), which only dilates \( X \) relative to the origin. For technical reasons, it will be convenient to consider dilations relative to other locations. The next coupling lemma shows that the “binomial process” \( \tau_{x,r}X_n \) and Poisson process \( \tau_{x,r}P_n \) coincide on neighborhoods of \( x \) (and likewise for \( y \)).

**Lemma SA.1.4.** For any \( R > 0 \), \( x, y \in \mathbb{R}^d \), and sequence \( \{\ell_n\}_{n \in \mathbb{N}} \) with \( \ell_n/n \to 1 \),

\[
\mathbb{P}(\tau_{x,r_n}X_{\ell_n} \cap B(x, R) = \tau_{x,r_n}P_n \cap B(x, R)) \to 1.
\]

**Proof.** This is Lemma 3.1 of Penrose and Yukich (2003). ■

The next lemma utilizes nodes \( 1, \ldots, 4 \). To state the result and its proof succinctly, we define more compact notation. For \( m \in \mathbb{N} \), redefine \( \mathcal{N}_m = \{5, 7, 9 \ldots, 2m + 3\} \), mirroring the definition of \( X_m \) from the coupling. For \( x \in \mathbb{R}^d \), \( J \subseteq \mathbb{N} \), \( i \in J \), let

\[
\psi_{i,x}(J) = \psi(i, \tau_{x,r_n} T_J, \zeta_J, \mathcal{A})
\]

where \( \tau_{x,r_n} T_J = ((\tau_{x,r_n} X_i, Z_i))_{i \in J} \) and \( \zeta_J = (\zeta_{ij})_{i \neq j; i,j \in J} \). Thus, for \( x = 0 \), \( \psi_{i,x}(J) = \psi_i(J) \), the latter familiar from (9).

For any \( S \subseteq \mathbb{N} \), \( \psi_{i,x}(\mathcal{N}_m \cup S) \) corresponds to \( i \)'s node statistic in the model with node set \( \mathcal{N}_m \cup S \) and positions dilated by \( r_n^{-1} \) relative to \( x \). Recall from (SA.1.21) the definition of \( \mathcal{N}_x(S_1) \), and abbreviate \( \mathcal{N}_x \equiv \mathcal{N}_x(\mathbb{R}^d) \). Then for \( m \in \{x, y\} \), \( \psi_{i,x}(\mathcal{N}_m \cup S) \) corresponds to \( i \)'s statistic under a Poissonized model with a random number of nodes, whereas when \( m \in \mathbb{N} \), it corresponds to the usual “binomial” model with \( m \) nodes. Accordingly, in the next lemma, the vector \( V_1(x, y) \) only involves the binomial model, whereas \( V_2(x, y) \) only involves Poissonized models.

**Lemma SA.1.5.** Let \( \{\ell_n\}_{n \in \mathbb{N}} \) and \( \{m_n\}_{n \in \mathbb{N}} \) be sequences of natural numbers such that
\( \ell_n/n \) and \( m_n/n \) tend to one. For any \( x, y \in \text{supp}(f) \), define

\[
V_1(x, y) = (\psi_{1,x}(\mathcal{N}_n \cup \{1\}), \psi_{1,x}(\mathcal{N}_n \cup \{1, 2\}), \psi_{1,x}(\mathcal{N}_n \cup \{1, 3\}), \\
\psi_{1,x}(\mathcal{N}_n \cup \{1, 2, 3\}), \psi_{1,x}(\mathcal{N}_n \cup \{1\}), \psi_{1,x}(\mathcal{N}_n \cup \{1, 2\}), \\
\psi_{1,x}(\mathcal{N}_m \cup \{1, 2, 3\}), \psi_{2,y}(\mathcal{N}_m \cup \{2\}), \psi_{2,y}(\mathcal{N}_m \cup \{1, 2\}), \\
\psi_{2,y}(\mathcal{N}_m \cup \{1, 2, 3\}), \psi_{2,y}(\mathcal{N}_m \cup \{1, \ldots, 4\})) \quad \text{and} \\
V_2(x, y) = (\psi_{1,x}(\mathcal{N}_x \cup \{1\}), \psi_{1,x}(\mathcal{N}_x \cup \{1\}), \psi_{1,x}(\mathcal{N}_x \cup \{1, 3\}), \\
\psi_{1,x}(\mathcal{N}_x \cup \{1, 3\}), \psi_{1,x}(\mathcal{N}_x \cup \{1\}), \psi_{1,x}(\mathcal{N}_x \cup \{1\}), \\
\psi_{1,x}(\mathcal{N}_x \cup \{1, 3\}), \psi_{2,y}(\mathcal{N}_y \cup \{2\}), \psi_{2,y}(\mathcal{N}_y \cup \{2\}), \\
\psi_{2,y}(\mathcal{N}_y \cup \{2\}), \psi_{2,y}(\mathcal{N}_y \cup \{2, 4\})).
\]

Also define \( \mathcal{F}(w, x, y, z) = \{X_1 = x, X_2 = y, X_3 = x + r_n z, X_4 = y + r_n w\} \) for \( w, z \in \text{supp}(f) \). Under Assumption 1, for any \( \epsilon > 0 \),

\[
P\left( \|V_1(x, y) - V_2(x, y)\| > \epsilon \mid \mathcal{F}(w, x, y, z) \right) \to 0,
\]

where \( \|\cdot\| \) denotes the component-wise maximum. The terms in \( V_2(x, y) \) involving \( x \) are independent of the terms involving \( y \) by (SA.1.21).

**Remark SA.1.4.** In this lemma, we have four nodes of interest: two “focal” nodes 1, 2 and two “ancillary” nodes 3, 4. The conditioning event \( \mathcal{F}(w, x, y, z) \) says that 3 is local to 1 in terms of position (since \( r_n \to 0 \)), 4 is local to 2, and 1 and 2 are a fixed distance \( \|x - y\| \) apart and hence non-local. The lemma shows that, in various models with different node sets, node 1’s statistic is asymptotically uncorrelated with node 2’s statistic since all terms in \( V_2(x, y) \) involving node 1 are independent of those involving node 2 by the coupling construction. Node 1’s statistic only asymptotically depends on 1, 3 and not 2, 4 (and vice versa for node 2). Note that the labels of these nodes are entirely arbitrary and chosen for technical convenience since node statistics are identically distributed.

The purpose of initially holding out nodes 1, \ldots, 4 in the construction of \( X_m \) and \( \mathcal{N}_m \) is that, in the Poissonized model, the number of nodes \( N_n \) is zero with positive probability. Hence, if we had instead defined \( \mathcal{N}_m \equiv \{1, 3, 5, \ldots, 2m - 1\} \), then \( \mathcal{N}_{N_n} \) could be empty with positive probability, in which case the conditioning event is null. Our construction ensures \( \mathcal{N}_m \cup \mathcal{S} \) is always nonempty for \( m \in \{x, y\} \) and \( \mathcal{S} \) a
nonempty subset of \(\{1, \ldots, 4\}\).

**Proof of Lemma SA.1.5.** This lemma is analogous to Lemma 3.7 of Penrose (2007) and Lemma 3.2 of Penrose and Yukich (2003), but the arguments need modification for reasons discussed in Remark SA.1.2. The key step is establishing an analog of (3.4) in the latter reference, which is accomplished in (SA.1.25) below.

We will only prove convergence of arguably the most complicated of the eleven components in \(V_1(x, y)\) and \(V_2(x, y)\), namely that for any \(\epsilon > 0\),

\[
P\left(\|\psi_{2,y}(N_m \cup \{1, \ldots, 4\}) - \psi_{2,y}(N_y \cup \{2, 4\})\| > \epsilon \mid F(w, x, y, z)\right) \to 0. \tag{SA.1.22}
\]

Convergence of the other components follows from the same arguments.

Condition throughout on \(F(w, x, y, z)\). Define

\[
E_n(R) = \\{\tau_{y,r_n}X_{m_n} \cap B(y, R) = \tau_{y,r_n}B_{r_n} \cap B(y, R)\} \quad \text{for any } R > 0, \quad \text{and} \\
R^*_2 = \max\{R(2, \tau_{y,r_n}T_{N_m \cup \{1, \ldots, 4\}, \xi_{N_m \cup \{1, \ldots, 4\}}}, A), R(2, \tau_{y,r_n}T_{N_y \cup \{2, 4\}, \xi_{N_y \cup \{2, 4\}}}, A)\}.
\]

The latter is the larger of the radii of stabilization (Definition 1) corresponding to the node statistics \(\psi_{2,y}(N_m \cup \{1, \ldots, 4\})\) and \(\psi_{2,y}(N_y \cup \{2, 4\})\), respectively.

For any \(R > 0\), define \(N(y, R) = \{j \in \mathbb{N} : \tau_{y,r_n}X_j \in B(y, R)\}\), the set of nodes whose dilated positions lie within the \(R\)-ball of \(y\). This is analogous to the definition of \(N_n(B_i)\) in §4.1. By Definition 1, if \(y = 0\), then under the event \(\{R > R^*_2\}\),

\[
\psi_{2,y}(N_{m_n \cup \{1, \ldots, 4\}}) = \psi_{2,y}(N_{m_n \cup \{1, \ldots, 4\}} \cap N(y, R)). \tag{SA.1.23}
\]

By (12), this also holds for \(y \neq 0\) (also see (SA.1.26) below).

Recall that under \(F(w, x, y, z)\), the positions of nodes 1, 3, 4 are \(x, x + r_n z, y + r_n w\). Since \(\tau_{y,r_n}(\{x, x + r_n z, y + r_n w\}) = \{y + r_n^{-1}(x - y), y + r_n^{-1}(x + r_n z - y), w + y\}\), the intersection of this set and \(B(y, R)\) is \(\{w + y\}\) for \(n, R\) sufficiently large. Then

\[
(SA.1.23) = \psi_{2,y}(N_{m_n \cup \{2, 4\}} \cap N(y, R)) \quad \text{(SA.1.24)}
\]

for such \(n, R\) under \(\{R > R^*_2\}\) (and the conditioning event). For the previous choice of \(n, R\), under the event \(E_n(R) \cap \{R > R^*_2\}\),

\[
(SA.1.24) = \psi_{2,y}(N_y \cup \{2, 4\}) \cap N(y, R) = \psi_{2,y}(N_y \cup \{2, 4\}),
\]

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where the first equality uses the event $E(R_n)$, which allows us to replace the “binomial” model with the “Poissonized” model, and the second equality follows from the argument for (SA.1.23). Therefore, by the law of total probability,

$$P(\|\psi_{2,y}(N_{m_n} \cup \{1, \ldots, 4\}) - \psi_{2,y}(N_y \cup \{2, 4\})\| > \epsilon | \mathcal{F}(w, x, y, z)) \leq P(E(R_n)^c) + P(R^*_2 > R), \quad (SA.1.25)$$

where $E(R_n)^c$ is the complement of $E_n(R)$. It remains to show that the right-hand side can be made arbitrarily small by choosing $n, R$ large enough.

We first claim that by Assumption 1, $R^*_2 = O_p(1)$. This is not quite immediate from the assumption because, for instance, $R^*_2$ is defined in terms of the point sets $\tau_{y, r, n} X_{m_n}$ and $\tau_{y, r, n} P_{n_f}^y$ rather than $r_n^{-1} X_{m_n} = \tau_{0, r, n} X_{m_n}$ and $r_n^{-1} P_{n_f}^y = \tau_{0, r, n} P_{n_f}^y$ as in the assumption. However, by (12),

$$\psi(i, \tau_{z, r, n} T_m, \zeta_m, \mathcal{A}) = \psi(i, \tau_{z', r, n} T_m, \zeta_m, \mathcal{A}) \quad \forall z, z' \in \mathbb{R}^d, i \in N_m, m \in \mathbb{N}. \quad (SA.1.26)$$

Hence, picking $z = y$ and $z' = 0$, we have

$$\psi(2, \tau_{y, r, n} T_{N_{m_n} \cup \{1, \ldots, 4\}}, \zeta_{N_{m_n} \cup \{1, \ldots, 4\}}, \mathcal{A}) = \psi(2, \tau_{0, r, n} T_{N_{m_n} \cup \{1, \ldots, 4\}}, \zeta_{N_{m_n} \cup \{1, \ldots, 4\}}, \mathcal{A}).$$

Consequently,

$$R(2, \tau_{y, r, n} T_{N_{m_n} \cup \{1, \ldots, 4\}}, \zeta_{N_{m_n} \cup \{1, \ldots, 4\}}, \mathcal{A}) = R(2, \tau_{0, r, n} T_{N_{m_n} \cup \{1, \ldots, 4\}}, \zeta_{N_{m_n} \cup \{1, \ldots, 4\}}, \mathcal{A}).$$

Since $r_n^{-1} z = \tau_{0, r, n} z$, the right-hand side of the previous display is $O_p(1)$ by Assumption 1. By the same argument,

$$R(2, \tau_{y, r, n} T_{N_y \cup \{2, 4\}}, \zeta_{N_y \cup \{2, 4\}}, \mathcal{A}) = R(2, \tau_{0, r, n} T_{N_y \cup \{2, 4\}}, \zeta_{N_y \cup \{2, 4\}}, \mathcal{A}) = O_p(1).$$

Hence, $R^*_2 = O_p(1)$, as desired.

It follows that, for any $\epsilon > 0$, we can choose $R$ large enough such that for all $n$ sufficiently large, $P(R^*_2 > R) < \epsilon/2$. Furthermore, by Lemma SA.1.4, for any such $R$, we can choose $n$ large enough such that $P(E(R_n)^c) < \epsilon/2$. Combining these facts with (SA.1.25), we obtain (SA.1.22).

The next lemma shows that the add-one costs $\Xi_\ell$ and $\Xi_m$ defined in (14) are
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asymptotically uncorrelated for $\ell, m$ large.

**Lemma SA.1.6.** Let $\{\nu_n\}_{n \in \mathbb{N}} \subseteq \mathbb{N}$ be a diverging sequence such that $\nu_n/n \to 0$. Under Assumptions 1 and 2,

$$\sup_{n-\nu_n \leq \ell < m \leq n+\nu_n} \left| \mathbb{E}[\Xi_{\ell}\Xi_m] - \mathbb{E}[\Xi_{N_n}]^2 \right| \to 0.$$

**Proof.** We follow the proof of Lemma 5.1 of Penrose (2007). The main difference is that we use our Lemma SA.1.5 in place of their Lemma 3.7, but there are some additional technical details involved due to differences in our setup. For $i \leq m$, let

$$\psi_m = \psi(m, r_n^{-1}T_m, \zeta_m, A) \quad \text{and} \quad \Delta_{i,m} = \psi(i, r_n^{-1}T_{m+1}, \zeta_{m+1}, A) - \psi(i, r_n^{-1}T_m, \zeta_m, A).$$

Then

$$\mathbb{E}[\Xi_{\ell}\Xi_m] = \mathbb{E} \left[ \left( \psi_{\ell+1} + \sum_{i=1}^{\ell} \Delta_{i,\ell} \right) \left( \psi_{m+1} + \sum_{j=1}^{j \leq \ell} \Delta_{j,m} + \Delta_{\ell+1,m} + \sum_{k=\ell+2}^{m} \Delta_{k,m} \right) \right]$$

$$= \mathbb{E}[\psi_{\ell+1}\psi_{m+1}] + \ell \mathbb{E}[\Delta_{1,\ell}\psi_{m+1}] + \ell \mathbb{E}[\psi_{\ell+1}\Delta_{1,m}] + \ell(\ell - 1)\mathbb{E}[\Delta_{1,\ell}\Delta_{2,m}]$$

$$+ \ell \mathbb{E}[\Delta_{1,\ell}\Delta_{1,m}] + \ell \mathbb{E}[\psi_{\ell+1}\Delta_{\ell+1,m}] + \ell \mathbb{E}[\Delta_{1,\ell}\Delta_{\ell+1,m}]$$

$$+ (m - \ell - 1)\mathbb{E}[\psi_{\ell+1}\Delta_{\ell+2,m}] + \ell(m - \ell - 1)\mathbb{E}[\Delta_{1,\ell}\Delta_{\ell+2,m}].$$

(SA.1.27)

By Corollary 3.2.3 of Schneider and Weil (2008) and a change of variables (recalling the notation from Lemma SA.1.5),

$$\mathbb{E}[\Xi_{N_n}] = \int_{\mathbb{R}^d} \mathbb{E} \left[ \psi_{1,x}(\mathcal{N}_x \cup \{1\}) \mid X_1 = x \right] f(x) \, dx + \kappa \times$$

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{E} \left[ \psi_{1,x}(\mathcal{N}_x \cup \{1, 3\}) - \psi_{1,x}(\mathcal{N}_x \cup \{1\}) \mid X_1 = x, X_3 = x + r_n z \right] f(x) f(x + r_n z) \, dx \, dz.$$

(SA.1.28)

We need to derive the “limits” of each of the terms in (SA.1.27) and show they match with the corresponding term of $\mathbb{E}[\Xi_{N_n}]^2$. We only illustrate the first two terms; the argument for the others is similar.
Term 1. Using (SA.1.26) and \( m > \ell \geq 1 \), the first term \( E[\psi_{\ell+1}\psi_{m+1}] \) equals,

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} E[\psi_{1,x}(\mathcal{N}_\ell \cup \{1\})\psi_{2,y}(\mathcal{N}_{m-1} \cup \{1, 2\}) \mid X_1 = x, X_2 = y] f(x) f(y) \, dx \, dy.
\]

Since \( \ell, m \) are order \( n \), by Lemma SA.1.5,

\[
|\psi_{1,x}(\mathcal{N}_\ell \cup \{1\})\psi_{2,y}(\mathcal{N}_{m-1} \cup \{1, 2\}) - \psi_{1,x}(\mathcal{N}_x \cup \{1\})\psi_{2,y}(\mathcal{N}_y \cup \{2\})| \xrightarrow{P} 0 \quad (SA.1.29)
\]

conditional on \( X_1 = x, X_2 = y \), and \( \psi_{1,x}(\mathcal{N}_x \cup \{1\}) \perp \psi_{2,y}(\mathcal{N}_y \cup \{2\}) \). We then wish to use Assumption 2 and the Vitali convergence theorem to conclude that

\[
|E[\psi_{1,x}(\mathcal{N}_\ell \cup \{1\})\psi_{2,y}(\mathcal{N}_{m-1} \cup \{1, 2\}) \mid X_1 = x, X_2 = y] - E[\psi_{1,x}(\mathcal{N}_x \cup \{1\}) \mid X_1 = x] E[\psi_{2,y}(\mathcal{N}_y \cup \{2\}) \mid X_2 = y]| \to 0. \quad (SA.1.30)
\]

Supposing this were true, we would have

\[
\left| E[\psi_{\ell+1}\psi_{m+1}] - \left( \int_{\mathbb{R}^d} E[\psi_{1,x}(\mathcal{N}_x \cup \{1\}) \mid X_1 = x] f(x) \, dx \right)^2 \right| \to 0,
\]

since \( E[\psi_{1,x}(\mathcal{N}_x \cup \{1\}) \mid X_1 = x] = E[\psi_{2,x}(\mathcal{N}_x \cup \{2\}) \mid X_2 = x] \).

A technical issue with (SA.1.30) is that it involves conditional expectations, whereas the Vitali convergence theorem applies to unconditional expectations. However, it is straightforward to rewrite them as unconditional expectations by modifying the coupling construction at the start of §SA.1.2 as follows. First, let \( X_1 \) and \( X_2 \) be non-random with respective values \( x \) and \( y \). Second, let \( Z_1 \) be independently drawn from the conditional distribution of \( Z_1 \) given \( X_1 = x \) and \( Z_2 \) likewise independently drawn from the conditional distribution of \( Z_1 \) given \( X_1 = y \), both independent of all other primitives. All other elements of the coupling remain the same. Then letting \( E_*[\cdot] \) denote the expectation with respect to this new coupling and \( E[\cdot] \) the expectation under the original coupling, we have

\[
E[\psi_{1,x}(\mathcal{N}_\ell \cup \{1\})\psi_{2,y}(\mathcal{N}_{m-1} \cup \{1, 2\}) \mid X_1 = x, X_2 = y] = E_*[\psi_{1,x}(\mathcal{N}_\ell \cup \{1\})\psi_{2,y}(\mathcal{N}_{m-1} \cup \{1, 2\})],
\]

\[
E[\psi_{1,x}(\mathcal{N}_x \cup \{1\}) \mid X_1 = x] = E_*[\psi_{1,x}(\mathcal{N}_x \cup \{1\})], \quad \text{and}
\]

\[
E[\psi_{2,y}(\mathcal{N}_y \cup \{2\}) \mid X_2 = y] = E_*[\psi_{2,y}(\mathcal{N}_y \cup \{2\})]. \quad (SA.1.31)
\]
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By (SA.1.29), Assumption 2, and the Vitali convergence theorem,

$$|E \{ \psi_{1,x}(N_\ell \cup \{1\}) \psi_{2,y}(N_{m-1} \cup \{1,2\}) \} - E \{ \psi_{1,x}(N_x \cup \{1\}) \} E \{ \psi_{2,y}(N_y \cup \{2\}) \} | \to 0,$$

which implies (SA.1.30) by (SA.1.31).

**Term 2.** Next consider the second term on the right-hand side of (SA.1.27). Recalling the notation from Lemma SA.1.5, for $m > \ell \geq 1$,

$$\ell E[\Delta_{1,\ell} \psi_{m+1}]$$

$$= \ell E \left[ \left( \psi(1, r_n^{-1} T_{\ell+1}, \zeta_{\ell+1}, A) - \psi(1, r_n^{-1} T_{\ell}, \zeta_{\ell}, A) \right) \psi(m + 1, r_n^{-1} T_{m+1}, \zeta_{m+1}, A) \right]$$

$$= \ell E \left[ (\psi_{1,0}(N_{\ell-1} \cup \{1,3\}) - \psi_{1,0}(N_{\ell-1} \cup \{1\})) \psi_{2,0}(N_{m-2} \cup \{1,2,3\}) \right]$$

$$= \ell E \left[ (\psi_{1,1}(N_{\ell-1} \cup \{1,3\}) - \psi_{1,1}(N_{\ell-1} \cup \{1\})) \psi_{2,1}(N_{m-2} \cup \{1,2,3\}) \right]$$

$$= \ell \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} E \left[ (\psi_{1,x}(N_{\ell-1} \cup \{1,3\}) - \psi_{1,x}(N_{\ell-1} \cup \{1\})) \right]$$

$$\times \psi_{2,y}(N_{m-2} \cup \{1,2,3\}) \mid X_1 = x, X_2 = y, X_3 = w] f(x)f(y)f(w) \, dx \, dy \, dw,$$

where the second equality follows because node statistics are identically distributed and the third follows from (SA.1.26). By a change of variables $z = r_n^{-1}(w - x)$, the last line equals

$$\ell \frac{d}{n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} E \left[ (\psi_{1,x}(N_{\ell-1} \cup \{1,3\}) - \psi_{1,x}(N_{\ell-1} \cup \{1\})) \right]$$

$$\times \psi_{2,y}(N_{m-2} \cup \{1,2,3\}) \mid X_1 = x, X_2 = y, X_3 = x + r_nz] f(x)f(y)f(x + r_nz) \, dx \, dy \, dz. \quad (SA.1.32)$$

Recall the new coupling construction for Term 1 above. We modify this coupling by additionally defining $X_3$ as non-random and equal to $x + r_nz$ and $Z_3$ as an independent draw from the conditional distribution of $Z_1$ given $X_1 = x + r_nz$. Then following the argument for Term 1, by Lemma SA.1.5, Assumption 2, and the Vitali convergence theorem, the absolute difference between the integrand of (SA.1.32) and

$$E \left[ \psi_{1,x}(N_x \cup \{1,3\}) - \psi_{1,x}(N_x \cup \{1\}) \mid X_1 = x, X_3 = x + r_nz \right]$$

$$\times E \left[ \psi_{2,y}(N_y \cup \{2\}) \mid X_2 = y \right] f(x)f(y)f(x + r_nz) \quad (SA.1.33)$$

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is \( o(1) \).

To apply the dominated convergence theorem, we show that the integrand of (SA.1.32) is uniformly bounded by an integrable function. We continue to employ the new coupling construction used to establish (SA.1.33). Observe that if \( \|z\| \geq R(1, \tau_{x,r_n} T_{N_{\ell-1} \cup \{1,3\}}, \mathcal{A}) \), then since \( X_1 = x \) and \( X_3 = x + r_n z \), we have \( r_n^{-1} \|X_1 - X_3\| \geq R(1, \tau_{x,r_n} T_{N_{\ell-1} \cup \{1,3\}}, \mathcal{A}) \). Hence, by Definition 1

\[
\psi_{1,x}(N_{\ell-1} \cup \{1,3\}) = \psi_{1,x}(N_{\ell-1} \cup \{1\}).
\]

By Assumption 2 and the Hölder and Minkowski inequalities, there exist \( C, \epsilon > 0 \) such that for \( n \) large, the integrand of (SA.1.32) is bounded in absolute value by

\[
\left( \|\psi_{1,x}(N_{\ell-1} \cup \{1,3\})\|_p + \|\psi_{1,x}(N_{\ell-1} \cup \{1\})\|_p \right) \cdot \|\psi_{2,y}(N_{m-2} \cup \{1,2,3\})\|_p 
\times P\left( R(1, \tau_{x,r_n} T_{N_{\ell-1} \cup \{1,3\}}, \mathcal{A}) : \|z\| \right)^{1-2/p} \leq C \tau_{b,\epsilon}(\|z\|)^{1-2/p},
\]

where \( p \) is defined in Assumption 2 and, for any random variable \( W \), \( \|W\|_p = \sup_{x,y,z} E[|W|^p | X_1 = x, X_2 = y, X_3 = z]^{1/p} \). By Assumption 1,

\[
C \ell r_n^d \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tau_{b,\epsilon}(\|z\|)^{1-2/p} f(x) f(y) f(x + r_n z) \, dx \, dy \, dz
\leq C \kappa \sup_w f(w) \int_{\mathbb{R}^d} \tau_{b,\epsilon}(\|z\|)^{1-2/p} \, dz < \infty,
\]

which verifies the dominance condition. Therefore, by (SA.1.32), (SA.1.33) and dominated convergence, the following is \( o(1) \):

\[
\left| \ell \mathbb{E}[\Delta_1, \xi_{m+1}] - \kappa \int_{\mathbb{R}^d} \mathbb{E}[\psi_{2,y}(N_y \cup \{2\}) | X_2 = y] f(y) \, dy \right| \times
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{E}[\psi_{1,x}(N_x \cup \{1,3\}) - \psi_{1,x}(N_x \cup \{1\}) | X_1 = x, X_3 = x + r_n z] f(x) f(x + r_n z) \, dx \, dz.
\]

The last lemma shows that add-one costs have bounded moments.

**Lemma SA.1.7.** Let \( \{\nu_n\}_{n \in \mathbb{N}} \subseteq \mathbb{N} \) be a diverging sequence such that \( \nu_n/n \to 0 \). Under
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Assumptions 1 and 2, for $\sigma_n^2$ defined in §SA.1.1,

$$\limsup_{n \to \infty} \sup_{n - \nu_n \leq m \leq n + \nu_n} \mathbb{E}[\Xi_m^2] < \infty \quad \text{and} \quad \sup_{n} \max \{ \mathbb{E}[\Xi_n], \sigma_n^2 \} < \infty.$$ 

PROOF. The result for $\mathbb{E}[\Xi_m^2]$ follows from the proof of Lemma 5.2 of Penrose (2007). The effort of translating the proof to our notation is the same as what was done for the previous two lemmas.

For $\mathbb{E}[\Xi_n]$, recall (SA.1.28). The first term on the right-hand side is uniformly bounded in $n$ by Assumption 2. For the second term on the right-hand side, note that if $\|z\| \geq R(1, \tau_r, \mathcal{T}_n \cup \{1, 3\}, \mathcal{X}_n \cup \{1, 3\}, \mathcal{A})$, then since we are conditioning on $X_1 = x$ and $X_3 = x + r_n z$, we have $r_n^{-1} \|X_1 - X_3\| \geq R(1, \tau_r, \mathcal{T}_n \cup \{1, 3\}, \mathcal{X}_n \cup \{1, 3\}, \mathcal{A})$. Hence, by definition of the radius of stabilization, $\psi_{1,x}(\mathcal{N}_x \cup \{1, 3\}) = \psi_{1,x}(\mathcal{N}_x \cup \{1\})$. Therefore, by Assumption 2 and the Hölder and Minkowski inequalities, there exist $C, \epsilon > 0$ such that for $n$ sufficiently large, the integrand of the second term on the right-hand side is bounded above in absolute value by

$$\left( \|\psi_{1,x}(\mathcal{N}_x \cup \{1, 3\})\|_p + \|\psi_{1,x}(\mathcal{N}_x \cup \{1\})\|_p \right) f(x) f(x + r_n z) \times \mathbb{P} \left( R(1, \tau_r, \mathcal{T}_n \cup \{1, 3\}, \mathcal{X}_n \cup \{1, 3\}, \mathcal{A}) \geq \|z\| \right) 1^{-1/p} \leq C f(x) \sup_{w} f(w) \tau_p(\|z\|) 1^{-1/p},$$

(SA.1.34)

where for any random variable $W$, $\|W\|_p = \sup_{x,z} \mathbb{E}[\|W\|^p | X_1 = x, X_3 = z]^{1/p}$. The right-hand side does not depend on $n$ and has a finite integral.

For $\sigma_n^2$, by the argument for (SA.1.28),

$$\hat{\sigma}_n^2 = \int_{\mathbb{R}^d} \mathbb{E} \left[ \psi_{1,x}(\mathcal{N}_x \cup \{1\})^2 | X_1 = x \right] f(x) dx + \kappa \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( \mathbb{E} \left[ \psi_{1,x}(\mathcal{N}_x \cup \{1, 3\}) \psi_{1,x}(\mathcal{N}_x \cup \{1\}) | X_1 = x, X_3 = x + r_n z \right] - \mathbb{E} \left[ \psi_{1,x}(\mathcal{N}_x \cup \{1\}) | X_1 = x \right] \mathbb{E} \left[ \psi_{3,x}(\mathcal{N}_x+r_n z \cup \{3\}) | X_3 = x + r_n z \right] \right) f(x) f(x + r_n z) dx dz.$$

The first term is uniformly bounded by Assumption 2. The second term is also uniformly bounded by an argument similar to (SA.1.34).\footnote{For more detail, see steps 1 and 3 of the proof of Lemma H.4 in Leung and Moon (2019).}
Proof of Theorem SA.1.2. We largely follow the proof of Theorem 2.12 of Penrose (2003) and Theorem 2.3 in Penrose (2007), although the characteristic function argument at the end is different since we do not derive the limit variance. Abbreviate $\alpha_n = E[\Xi_{N_n}]$ and $\Lambda_m = \sum_{i=1}^{m} \psi_i(N_m)$. We first prove that
\[
E\left[n^{-1} (\Lambda_{N_n} - \Lambda_n - (N_n - n)\alpha_n)^2\right] \to 0. \tag{SA.1.35}
\]
The left-hand side equals
\[
\sum_{m: |m-n| \leq n^{3/4}} E\left[n^{-1} (\Lambda_m - \Lambda_n - (m-n)\alpha_n)^2\right] P(N_n = m)
+ E\left[n^{-1} (\Lambda_{N_n} - \Lambda_n - (N_n - n)\alpha_n)^2 1\{|N_n - n| > n^{3/4}\}\right]. \tag{SA.1.36}
\]
Let $\varepsilon > 0$. For $n$ sufficiently large and $m \in [n, n + n^{3/4}]$,
\[
E\left[(\Lambda_m - \Lambda_n - (m-n)\alpha_n)^2\right] = E\left[\left(\sum_{\ell=n}^{m-1} (\Xi_{\ell} - \alpha_n)\right)^2\right] \leq \varepsilon(m-n)^2 + \varepsilon n^{1/2}(m-n),
\]
where the inequality follows from expanding the square and applying Lemmas SA.1.6 and SA.1.7 to the summands. A similar argument also applies to $m \in [n - n^{3/4}, n]$. Then the first term of (SA.1.36) is bounded by
\[
n^{-1} E\left[\varepsilon(N_n - n)^2 + \varepsilon n^{1/2}|N_n - n|\right] \leq n^{-1} \varepsilon \left(n + n^{1/2} (E|N_n - n|^2)^{1/2}\right) = 2\varepsilon.
\]
Following the argument in the proof of Theorem 2.3 in Penrose (2007), the second term of (SA.1.36) can be bounded by a constant times
\[
n^{-1} n^{(2p+1)/p} P\left(|N_n - n| > n^{3/4}\right)^{1-2/p'}
\]
for some $p' \in (2, p)$ with $p$ defined in Assumption 2. This argument uses the latter assumption and Lemma SA.1.3. The previous display tends to zero since the probability decays exponentially with $n$ by Lemma 1.4 of Penrose (2003). This establishes (SA.1.35).

To calculate $n^{-1} \Var(\Lambda_n)$, consider the identity
\[
n^{-1/2} \Lambda_{N_n} = n^{-1/2} \Lambda_n + n^{-1/2} (N_n - n)\alpha_n + n^{-1/2} (\Lambda_{N_n} - \Lambda_n - (N_n - n)\alpha_n). \tag{SA.1.37}
\]
The variance of the last term tends to zero by (SA.1.35). The second term on the right-hand side has variance $\alpha_n^2$ and is independent of the first term by the coupling construction. Therefore,

$$\frac{1}{n} \text{Var} (\Lambda_n) = \frac{1}{n} \text{Var} (\Lambda_n) + \alpha_n^2 + o(1) \quad \Rightarrow \quad \sigma_n^2 = \sigma_n^2 - \alpha_n^2 + o(1) \quad \text{(SA.1.38)}$$

since $\sup_n \max \{\alpha_n, \sigma_n^2\} < \infty$ by Lemma SA.1.7.

Since $\sigma_n^2$ has strictly positive limit infimum by assumption, so does $\tilde{\sigma}_n^2$. We can then rewrite (SA.1.37) as

$$n^{-1/2} \tilde{\sigma}_n^{-1} (\Lambda_n - E[\Lambda_n]) - o_p(1) = \frac{\sigma_n}{\tilde{\sigma}_n} n^{-1/2} \sigma_n^{-1} (\Lambda_n - E[\Lambda_n]) + \frac{\alpha_n}{\tilde{\sigma}_n} n^{-1/2} (N_n - n). \quad \text{(SA.1.39)}$$

Let $\tilde{\varphi}_n(\cdot)$ be the characteristic function of the left-hand side, $\varphi_n(\cdot)$ that of $n^{-1/2} \sigma_n^{-1} (\Lambda_n - E[\Lambda_n])$, and $\varphi_p(\cdot)$ that of $n^{-1/2} (N_n - n)$. By independence of $N_n$, due to the coupling construction, the characteristic function of the right-hand side of (SA.1.39) is

$$\varphi_n \left( \frac{\sigma_n}{\tilde{\sigma}_n} x \right) \varphi \left( \frac{\alpha_n}{\tilde{\sigma}_n} x \right), \quad \text{implying} \quad \varphi_n(y) = \tilde{\varphi}_n \left( \frac{\sigma_n}{\tilde{\sigma}_n} y \right) \varphi \left( \frac{\alpha_n}{\tilde{\sigma}_n} y \right)^{-1}.$$

for $y = \sigma_n \tilde{\sigma}_n^{-1} x$. By (SA.1.1) and the Poisson CLT,

$$\tilde{\varphi}_n(x) \to e^{-x^2/2} \quad \text{and} \quad \varphi_p(x) \to e^{-x^2/2}.$$

Furthermore, this convergence is uniform over $x$ in a compact set (Klenke, 2013, Theorem 15.23). As previously discussed, $\sup_n \max \{\alpha_n, \tilde{\sigma}_n\} < \infty$ and $\lim \inf_{n \to \infty} \min \{\sigma_n, \tilde{\sigma}_n\} > 0$. Thus, there exists a compact set $K$ such that for $n$ sufficiently large, $y \tilde{\sigma}_n / \sigma_n$ and $y \alpha_n / \sigma_n$ lie in $K$. Combining these facts,

$$\varphi_n(y) = \frac{\exp \left\{ - \frac{\tilde{\sigma}_n^2 y^2}{2} \right\} + o(1)}{\exp \left\{ - \frac{\sigma_n^2 y^2}{2} \right\} + o(1)} \to e^{-y^2/2}$$

for any $y \in \mathbb{R}$, where convergence follows from (SA.1.38). Therefore,

$$n^{-1/2} \sigma_n^{-1} (\Lambda_n - E[\Lambda_n]) \xrightarrow{d} \mathcal{N}(0, 1).$$

Finally, in the previous expression, we can replace $E[\Lambda_n]$ with $E[\Lambda]$ due to (SA.1.35).
The results in this section make use of the following notation. Because we are verifying Assumption 1, we must consider a setting with \( m \) nodes and sparsity parameter \( r_n \) with \( m \) potentially different from \( n \). For \( m, n \in \mathbb{N} \), recall the definition of the network time series \( \mathcal{A}^{ps}(r_n^{-1} \mathbf{T}_m, \zeta_m) \) from (21). Let \( \mathbf{A}_t \) denote the period-\( t \) network of \( \mathcal{A}^{ps}(r_n^{-1} \mathbf{T}_m, \zeta_m) \) and \( A_{ij,t} \) the \( ij \)th entry of \( \mathbf{A}_t \). For any subset of nodes \( J \subseteq \mathcal{N}_m \), the period-\( t \) subnetwork on \( J \) of \( \mathcal{A}^{ps}(r_n^{-1} \mathbf{T}_m, \zeta_m) \) is \( \mathbf{A}_{J,t} \).

For primitives \( (r_n^{-1} \mathbf{T}_m, \zeta_m) \), let \( \mathbf{D} = \mathbf{D}(m) \) be the network on \( \mathcal{N}_m \) with \( ij \)th entry \( D_{ij} \) given in (18). Recall the definitions of \( C_i \) and \( C_i^+ \) from §5.1; throughout this section, we assume they are constructed from \( \mathbf{D}(m) \). Define \( \mathbf{M}_t = \mathbf{M}_t(m) \) as the network on \( \mathcal{N}_m \) with \( ij \)th entry

\[
M_{ij,t} = \mathbf{1} \{ \sup_s V_t(r_n^{-1} \| X_i - X_j \|, s, Z_{it}, Z_{jt}, \zeta_{ij,t}) > 0 \},
\]

where for \( t > 0 \), \( V_t(\cdot) \equiv V(\cdot) \), and \( V_0(\cdot) \) is defined in (17). Finally, let \( \mathbf{M} = \mathbf{M}(m) \) be the network with \( ij \)th entry \( M_{ij} = \max_{t=0, \ldots, T} M_{ij,t} \), the “union” of \( \mathbf{M}_0, \ldots, \mathbf{M}_T \).

**PROOF OF THEOREM 2.** This is a consequence of (11) and Lemmas SA.2.1 and SA.2.4 below. Specifically, Lemma SA.2.1 constructs, for any \( i \in \mathcal{N}_m \), a set \( J_i \) satisfying Concept 1(a). Defining \( \tilde{R}_i(m) \) as in (11) gives us an upper bound on the radius of stabilization. In §SA.2.1, we construct a branching process \( \mathbf{X}^H_{r_n}(X_i, Z_i; K) \) (SA.2.7) whose size stochastically dominates \( |J_i| \) (Lemma SA.2.2). Lemma SA.3.3 derives an exponential tail bound on the size of the branching process, which is used to establish a corresponding tail bound for \( |J_i| \) in Lemma SA.2.3. Finally, Lemma SA.2.4 translates this to a uniform exponential tail bound on \( \tilde{R}_i(m) \). Since this is an upper bound on the radius of stabilization, we have verified Assumption 1.

Our first lemma constructs \( J_i \) satisfying Concept 1(a).

**Lemma SA.2.1.** Let \( J_i \equiv J_i(m) \) be defined in (SA.2.2) in the proof below.

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Assumptions 3, 7, and 8, for any \( m, n \in \mathbb{N} \) and \( i \in \mathcal{N}_m \), with probability one,

\[
\psi(i, r_n^{-1} T_m, \zeta_m, \mathcal{A}^{ps}) = \psi(i, r_n^{-1} T_{j_t(m)}, \zeta_{j_t(m)}, \mathcal{A}^{ps}).
\]  

(SA.2.1)

PROOF. We first show that, for any \( i, t \), there exists a subset of nodes \( S_{it} \supseteq \mathcal{N}_{M_t}(i, K) \) such that the period-\( t \) subnetwork on \( \mathcal{N}_{M_t}(i, K) \) is a.s. equivalent under the observed network \( \mathcal{A}^{ps}(r_n^{-1} T_m, \zeta_m) \) and the counterfactual \( S_{it} \)-subnetwork \( \mathcal{A}^{ps}(r_n^{-1} T_{S_{it}}, \zeta_{S_{it}}) \) defined in §4.1. In what follows, for any network \( G \) on \( \mathcal{N}_m \) and \( i, j \in \mathcal{N}_m \), define

\[
\mathcal{N}_G\{i, j\} = \mathcal{N}_G(i, 1) \cup \mathcal{N}_G(j, 1).
\]

Fix \( i, t \), and choose any \( j_1, k_1 \in \mathcal{N}_{M_t}(i, K) \). By (1), \( A_{j_1, k_1, t} \) is a function of the primitives \( (X_{j_1}, X_{k_1}, Z_{j_1 t}, Z_{k_1 t}, \zeta_{j_1, k_1, t}) \) and \( S_{j_1, k_1, t} \). By Assumption 3, \( S_{j_1, k_1, t} \) is a function of the types and potential links involving nodes in \( \mathcal{N}_{M_{t-1}}(\{j_1, k_1\}) \subseteq \mathcal{N}_{M_{t-1}}(\{j_1, k_1\}) \). Therefore, \( A_{j_1, k_1, t} \) is a deterministic function of the types and potential links of nodes in \( \mathcal{N}_{M_{t-1}}(\{j_1, k_1\}) \), namely \( (r_n^{-1} T_{j_1}, \zeta_{j_1}) \) and \( A_{j_1, t-1} \) for \( J = \mathcal{N}_{M_{t-1}}(\{j_1, k_1\}) \). Now going back one period to \( t-1 \), by the same argument, for any \( j_2, k_2 \in \mathcal{N}_{M_{t-1}}(\{j_1, k_1\}) \), \( A_{j_2, k_2, t-1} \) is a deterministic function of the types and potential links involving nodes in \( \mathcal{N}_{M_{t-2}}(\{j_2, k_2\}) \). More generally, we may go back up to \( t \) periods to establish the same result: for any \( s = 2, \ldots, t \) and \( j_s, k_s \in \mathcal{N}_{M_{t-(s-1)}}(\{j_{s-1}, k_{s-1}\}) \), \( A_{j_s, k_s, t-(s-1)} \) is a deterministic function of the types and potential links involving nodes in \( \mathcal{N}_{M_{t-s}}(\{j_s, k_s\}) \).

However, things change when we go back \( s = t + 1 \) periods and reach the initial condition. At this point, we are choosing any \( j_{t+1}, k_{t+1} \in \mathcal{N}_{M_0}(\{j_t, k_t\}) \). Recall the definition of strategic neighborhoods \( C^+_i \) from §5.1. We claim that \( A_{j_{t+1}, k_{t+1}, 0} \) is a.s. equivalent under both the observed network \( \mathcal{A}^{ps}(r_n^{-1} T_m, \zeta_m) \) and the counterfactual \( J \)-subnetwork \( \mathcal{A}^{ps}(r_n^{-1} T_J, \zeta_J) \) for \( J = C^+_t \cup C^+_k \). This follows because (i) by Proposition 1 of Leung (2019b), which uses Assumption 3, the set of pairwise stable networks on \( C^+_j \cup C^+_k \) is the same under both the model that gives rise to the observed network (the one with node set \( \mathcal{N}_m \) and the model that gives rise to the counterfactual subnetwork (the one with node set \( J \)), and (ii) under Assumption 7, the equilibrium selection mechanism chooses the same pairwise stable network on \( C^+_j \cup C^+_k \) for both models.

We have therefore implicitly, recursively constructed a set \( S_{it} \) such that the subnetwork of \( A_i \) on \( \mathcal{N}_{M_t}(i, K) \) is a.s. equivalent under \( \mathcal{A}^{ps}(r_n^{-1} T_m, \zeta_m) \) and \( \mathcal{A}^{ps}(r_n^{-1} T_{S_{it}}, \zeta_{S_{it}}) \). Under Assumption 8, \( \psi(i, r_n^{-1} T_m, \zeta_m, \mathcal{A}^{ps}) \) is a deterministic function of nodes in (20),
which a deterministic function of $K$-neighborhoods in $A_t$ for all $t$. Accordingly, we define $J_i(m)$ recursively as follows. Let

$$
\mathcal{M}_{i,t} = \mathcal{R}_{i,t}(t) \cup \mathcal{R}_{i,t}(t-1) \cup \cdots \cup \mathcal{R}_{i,t}(0),
$$

where for $s \leq t$,

$$
\mathcal{R}_{i,t}(s) = \begin{cases} 
\mathcal{N}_{\mathcal{M}_t}(i,K) & \text{if } s = t, t \neq 0 \\
\bigcup_{j \in \mathcal{R}_{i,t}(s+1)} \mathcal{N}_{\mathcal{M}_t}(j,1) & \text{if } 0 < s < t \\
\bigcup_{j \in \mathcal{R}_{i,t}(1)} C_j^+ & \text{if } s = 0.
\end{cases}
$$

Define

$$
J_i \equiv J_i(m) \equiv \bigcup_{t=0}^{T} \bigcup_{j \in \mathcal{N}_{\mathcal{M}_t}(i,K)}^{T} \bigcup_{t'=0}^{T} \mathcal{M}_{j,t'}.
$$

This is similar to (20), which might be considered the “naive guess” of $J_i(m)$. This guess does not work due to strategic interactions since nodes outside of (20) influence the subnetwork time series formed on (20) (see the discussion in §5.3). In (SA.2.2), we crucially replace $\mathcal{N}_{A_t}(j,K)$ in (20) with $\mathcal{M}_{j,t'}$ to account for strategic interactions. (We also replace $\mathcal{N}_{A_t}(i,K)$ with the larger set $\mathcal{N}_{\mathcal{M}_t}(i,K)$, which is more technically convenient for the analysis that follows.) Thus, (SA.2.2) clearly contains (20), so Assumption 8 and the previous arguments establish (SA.2.1) for $J_i(m)$ given by (SA.2.2).

SA.2.1 Branching Process Bound

We next define branching processes used to stochastically bound the size of $C_i$, node $i$’s component in $D$, and that of $\mathcal{N}_{\mathcal{M}_t}(i,K)$, its $K$-neighborhood in $\mathcal{M}_t$. These are used to construct a process whose size stochastically dominates $|J_i|$ defined in (SA.2.2). The use of branching processes to bound component and neighborhood sizes is a well-known technique in random graph theory (e.g. Bollobás and Riordan, 2012).

Let $x \in \mathbb{R}^d$, $z \in \mathbb{R}^{d_z}$, $r > 0$, and $\bar{f} = \sup_{x \in \mathbb{R}^d} f(x)$. Recall the definition of $\Phi^*(\cdot)$ from Assumption 5 and $p_1(x,z;x',z')$ from (18). Denote by $\mathcal{X}_r^D(x,z)$ the multi-type Galton-Walton branching process with type space $\mathbb{R}^d \times \mathbb{R}^{d_z}$ starting at a particle of type $(x,z)$ with the following two properties (see e.g. Bollobás et al., 2007, §2.1). First,
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each particle of type \((x', z')\) is replaced in the next generation by a set of particles (its “offspring”) distributed as a Poisson point process on \(\mathbb{R}^d \times \mathbb{R}^{dz}\) with intensity

\[
d\varphi_r(x', z'; x'', z'') = \kappa f(1 + r)p_1(x', z'; x'', z'') \, d\Phi^*(z'') \, dx''. \tag{SA.2.3}
\]

Second, conditional on the types of generation \(t-1\), the point processes that constitute generation \(t\) are independent. We can interpret \(\mathcal{X}^D_r(x, z)\) as a tree network. The root of the tree, corresponding to the first generation of the process, consists of a single node (particle) of type \((x, z)\). Its neighbors are its offspring, which constitute the second generation and are nodes with types distributed according to the Poisson point process described above. Then conditional on the types of the second generation, each node in the second generation of type \((x', z')\) independently generates neighbors with types distributed as the Poisson point process above. These constitute the third generation. The process is repeated indefinitely. Let \(|\mathcal{X}_r^D(x, z)|\) denote the number of particles ultimately generated, which may be infinite. We write \((x', z') \in \mathcal{X}_r^D(x, z)\) to mean that \((x', z')\) is a type of a particle generated at some point in the process, and taking the union over all such \((x', z')\) means we collect all particle types into a set.

We next define a “fixed-depth” branching process, which is required to terminate after a given number of generations. Recalling the definitions in Assumption 4, let

\[
\bar{p}_r(x; x') = \mathbb{E} \left[ \max_{t=0, \ldots, T} 1\{\bar{V}_t(r^{-1}\|x - x'\|, \zeta_{ij,t}) > 0\} \right], \tag{SA.2.4}
\]

where \(\bar{V}_t(\delta, \zeta) = \sup_{s,z,z'} V_t(\delta, s, z, z', \zeta)\) (as in Assumptions 4 and 6) and \(V_t(\cdot) \equiv V(\cdot)\) for all \(t > 0\). Let \(\mathcal{X}_r^M(x, z; K)\) denote the branching process on \(\mathbb{R}^d \times \mathbb{R}^{dz}\) that terminates after \(K + 1\) generations, starting at a particle of type \((x, z)\), with the following two properties. First, any particle of type \((x', z')\) is replaced in the next generation by offspring with types distributed as a Poisson point process on \(\mathbb{R}^d \times \mathbb{R}^{dz}\) with intensity

\[
d\bar{\varphi}_r(x', z'; x'', z'') = \kappa f(1 + r)\bar{p}_1(x', z'; x'', z'') \, d\Phi^*(z'') \, dx''. \tag{SA.2.5}
\]

Second, conditional on the types of generation \(t - 1\), the point processes that constitute generation \(t\) are independent. Thus, the process is generated the same way as \(\mathcal{X}_r^D(x, z)\), except the intensity measure is different, and once the \((K+1)\)-th generation is born, no further offspring are generated. Define \(|\mathcal{X}_r^M(x, z)|\) and \((x', z') \in \mathcal{X}_r^M(x, z)\)
as we did for $X^D_r(x, z)$.

We use $|X^D_r(X_i, Z_{i0})|$ and $|X^M_r(X_i, Z_{i0}; K)|$ to stochastically bound $|C_i|$ and $|N^M_M(i, K)|$, respectively. Intuitively, this holds because the number of offspring of any particle stochastically dominates the degree of a node in the associated network $D$ or $M$.

**Lemma SA.2.2.** Under Assumptions 4–6, for $m, n$ sufficiently large and any $i \in N_m$, $|C_i|$ and $|N^M_M(m)(i, K)|$ are respectively stochastically dominated by $|X^D_r(X_i, Z_{i0})|$ and $|X^M_r(X_i, Z_{i0}; K)|$ conditional on $(X_i, Z_{i0})$. The same result holds for $m \leq N_n$.

**Proof.** The result for $C_i$ with non-random network size $m$ follows from Lemma SA.3.1 of Leung (2020b). The key argument in the lemma is that we can stochastically bound the degree distribution of any node $j$ in $D(m)$, which is conditionally binomial, with the Poisson point process with intensity (SA.2.3). A similar argument establishes the result for $m = N_n$, which is even simpler since the degree distribution in this case is exactly Poisson. The corresponding results for $N^M_M(m)(i, K)$ follow from the same arguments. Indeed, the proof is even simpler because we terminate the branching process after $K + 1$ generations rather than continuing indefinitely.

With this result in hand, we may stochastically bound $|J_i|$ as follows. Let $C_i^* \equiv C_i^*(m) = C_i \cup \bigcup_{j \in C_i} N^M_M(j, 1)$. Then $C_i^+ \subseteq C_i^*$ because $M_t$ is a subnetwork of $M$ for any $t$. Also let

$$J_i^* = J_i^*(m) = \bigcup_{j \in N^M_M(i, 2K + T + 1)} C_i^*.$$  \hspace{1cm} (SA.2.6)

Then $J_i \subseteq J_i^*$. We next construct a hybrid branching process

$$X^H_r(X_i, Z_{i0}; K, T)$$ \hspace{1cm} (SA.2.7)

whose size dominates $|J_i^*|$. First, for any $(x, z) \in \text{supp}(X_i, Z_{i0})$, define a smaller hybrid branching process

$$X^+_r(x, z),$$

constructed by generating $X^D_r(x, z)$, and then for each particle (in any generation) of type $(x', z')$ in $X^D_r(x, z)$, generating a branching process $X^M_r(x', z'; 1)$ independently across particles conditional on $X^D_r(x, z)$. Using Lemma SA.2.2, $|X^+_r(x, z)|$ stochastically dominates $|C_i^*|$ conditional on $(X_i, Z_{i0}) = (x, z)$. Now we can construct (SA.2.7)
by generating the fixed-depth process $\mathfrak{X}_n^M(X_i, Z_{i0}; 2K + T + 1)$ (whose size dominates that of $\mathcal{N}_M(i, 2K + T + 1)$ in (SA.2.6)) and then for each particle (in any generation) of type $(x', z')$ in $\mathfrak{X}_n^M(X_i, Z_{i0}; 2K + T + 1)$, generating the smaller hybrid process $\mathfrak{X}_n^+(x', z')$ independently across particles conditional on $\mathfrak{X}_n^M(X_i, Z_{i0}; 2K + T + 1)$. Using Lemma SA.2.2, $|\mathfrak{X}_n^H(X_i, Z_{i0}; K)|$ stochastically dominates $|J^*_i|$ and therefore $|J_i|$.

### SA.2.2 Tail Bounds

Recall the definition of $J_i(m)$ from (SA.2.2).

**Lemma SA.2.3.** Under Assumptions 4, 5, and 6, for any $k \in \{1, 2, 3\}$, there exist $n_0, \epsilon > 0$ such that $\limsup_{w \to \infty} w^{-1} \max\{\log \bar{\tau}_{b, \epsilon}(w), \log \bar{\tau}_p(w)\} < 0$, where

$$
\bar{\tau}_{b, \epsilon}(w) = \sup_n \sup_{m \in (1-\epsilon)n, (1+\epsilon)n} \sup_{x \in \text{supp}(f)} \mathbf{P}(|J_1(m + k)| > w \mid X_k = x),
$$

$$
\bar{\tau}_p(w) = \sup_n \sup_{x \in \text{supp}(f)} \mathbf{P}(|J_1(N_n + 1)| > w \mid X_1 = x).
$$

**Proof.** We prove the result for $\bar{\tau}_{b, \epsilon}(w)$ and $k = 3$, which is the most involved case. The argument for the others is similar. Fix $m \in \mathbb{N}$, and recall the definition of $J^*_i(m)$ from (SA.2.6). For any $S \subseteq \mathcal{N}_m$ with $i \notin S$, let $J^*_i(m, S) = J^*_i(m) \setminus S$. We first prove

$$
J^*_i(m) \subseteq J^*_i(m, \{2\}) \cup J^*_i(m, \{1\}). \tag{SA.2.8}
$$

Let $M^-$ and $D^-$ be the networks obtained from $M$ and $D$ respectively by deleting node 1 and its links. Let $\bar{K} = 2K + T + 1$ and $i \in J^*_i(m)$. If $i = 2$, then clearly $i \in J^*_i(m, \{1\})$, so suppose $i \neq 2$. Then either (a) $i \in \mathcal{N}_M(1, \bar{K})$ or (b) $i \in C^*_j$ for some $j \in \mathcal{N}_M(1, \bar{K})$, where $C^*_j$ is defined prior to (SA.2.6). For case (a), if $i \notin J^*_i(m, \{2\})$, then it means that node 2 lies on a path of length at most $\bar{K}$ in $M$ connecting $i$ and node 1. Therefore $i \in \mathcal{N}_M^-(2, \bar{K})$, so $i \in J^*_i(m, \{1\})$. For case (b), if $i \notin J^*_i(m, \{2\})$, then there are two possibilities. The first is that $M_{2i} = 1$, in which case $i \in \mathcal{N}_M^-(2, \bar{K})$. The second is that node 2 lies on some path in $D$ connecting $i$ and some node $j$, in which case nodes 2 and $i$ are connected through a path in $D^-$. Thus, in both cases, either $i \in J^*_i(m, \{2\})$ or $i \in J^*_i(m, \{1\})$. 

Applying (SA.2.8) several times yields

\[ J_1^*(m) \subseteq J_1^*(m, \{3\}) \cup J_3^*(m, \{1\}) \]
\[ \subseteq (J_1^*(m, \{2\}) \cup J_2^*(m, \{1\})) \cup (J_3^*(m, \{1\}) \cup J_2^*(m, \{1\})). \]

Since \(|J_1(m)| \leq |J_1^*(m)|\), by the previous equation, it suffices to prove the result replacing \(J_1(m) = 3\) and \(J_1(N_n + 3)\) in the definitions of \(\bar{\tau}_{b,\epsilon}(t)\) and \(\bar{\tau}_{p}(t)\) with \(J_1^*(m, \{1\}) = 3\) and \(J_1^*(N_n + 3, \{1\})\), respectively. The sizes of these sets are both stochastically dominated by \(|X_{r_n}^H(X_3, Z_{3,0}; K, T)|\) defined in (SA.2.7), as shown in §SA.2.1, and \(|X_{r_n}^H(X_3, Z_{3,0}; K, T)|\) satisfies the desired exponential tail bound by Lemma SA.3.3. ■

**Lemma SA.2.4.** Suppose Assumption 4 and the conclusion of Lemma SA.2.3 hold. For \(n, m \in \mathbb{N}\) sufficiently large and any \(k \in \{1, 2, 3\}\), there exist \(n_0, \epsilon > 0\) such that \(\limsup_{w \to \infty} w^{-1} \max\{\log \tau_{b,\epsilon}^*(w), \log \tau_{p}^*(w)\} < 0\), where

\[
\tau_{b,\epsilon}^*(w) = \sup_{n > n_0} \sup_{m \in ((1-\epsilon)n, (1+\epsilon)n)} \sup_{x_k \in \text{supp}(f)^k} \mathbb{P}\left( \max_{i \in J_1(n+k)} r_n^{-1} \|X_1 - X_i\| > w \ \middle| \ X_k = x_k \right),
\]
\[
\tau_{p}^*(w) = \sup_{n > n_0} \sup_{x \in \text{supp}(f)} \mathbb{P}\left( \max_{i \in J_1(n+1)} r_n^{-1} \|X_1 - X_i\| > w \ \middle| \ X_1 = x \right).
\]

**PROOF.** Since \(J_1(m) \subseteq J_1^*(m)\) (SA.2.6), it suffices to prove the result substituting \(J_1^*(m)\) and \(J_1^*(N_n + k)\) for \(J_1(m + k)\) and \(J_1(N_n + k)\), respectively. We only prove the result for \(J_1^*(m + k)\), as the argument is the same for \(J_1^*(N_n + 1)\). By the law of total probability,

\[
\mathbb{P}\left( \max_{i \in J_1^*(m+k)} r_n^{-1} \|X_1 - X_i\| > w \ \middle| \ X_k = x_k \right) \leq \mathbb{P}\left( |J_1^*(m+k)| > w' \ \middle| \ X_k = x_k \right)
\]
\[
+ \mathbb{P}\left( \max_{i \in J_1^*(m+k)} r_n^{-1} \|X_1 - X_i\| > w \ \cap \ |J_1^*(m+k)| \leq w' \ \middle| \ X_k = x_k \right).
\]

Below we choose \(w'\) to be a linear, increasing function of \(w\), so the first term on the right-hand side obeys the required exponential tail bound in \(w\) by Lemma SA.2.3.

Consider the second term. Let \(D_{ij}\) be the \(ij\)th entry of \(D(m+k)\) and \(M_{ij}\) that of \(M(m+k)\). Notice \(D_{ij} \leq M_{ij}\) for any \(i, j\). Then under the event that \(|J_1^*(m+k)| \leq w'\), for \(C_i^*(m+k)\) defined prior to (SA.2.6), we have \(C_i^*(m+k) \subseteq \mathcal{N}_{M(m+k)}(i, w')\), which
implies $J_t^*(m + k) \subseteq N_{M(m+k)}(1, \tilde{K} + w')$ for $\tilde{K} = 2K + T + 1$. Therefore,

$$
P\left( \max_{i \in J_t^*(m+k)} r_n^{-1} \|X_1 - X_i\| > w \cap |J_t^*(m+k)| \leq w' \mid X_k = x_k \right) \leq P\left( \max_{i \in N_{M(m+k)}(1,\tilde{K}+w')} r_n^{-1} \|X_1 - X_i\| > w \mid X_k = x_k \right). \quad \text{(SA.2.9)}$$

By Lemma SA.3.4, there exist $C, a_1, a_2 > 0$ such that for $n$ sufficiently large,

$$
\text{(SA.2.9)} \leq C(\tilde{K} + w')(a_1 \kappa \bar{f}(w/(\tilde{K} + w'))^d)(\tilde{K} + w')e^{-a_2 w},
$$

where $\bar{f} = \sup_x f(x)$. Choose $w' = \max\{c w - \tilde{K}, 0\}$ for some small $c > 0$. Then for $w$ large, the right-hand side is of order $w \exp\{c w \log(a_1 \kappa \bar{f} e^{-d}) - a_2 w\}$. For $c$ sufficiently small, this is order $\delta(w) = we^{-a_3 w}e^{-d \log c^c w}$ for large $w$ and some $a_3 > 0$. Since $\lim_{c \to 0} c^c = 1$, for $c$ sufficiently small we have $\limsup_{w \to \infty} w^{-1} \log \delta(w) < 0$, as desired. 

## SA.3 Auxiliary Lemmas

The lemmas in this section are used to prove results in §SA.2.2. The first two show that $|X^P_{r_n}(x, z)|$ and $|X^M_{r_n}(x, z; K)|$ defined in §SA.2.1 have exponential tails. Let $\mathcal{T} = \text{supp}(X_1, Z_{i0})$. For $r > 0$ and $\bar{f} = \sup_x f(x)$, define $g_r^\alpha(x, z) = E[\alpha^{|X^P_{r_n}(x, z)|}]$ and

$$
\psi_r(x, z) = \kappa \bar{f}(1 + r) \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} p_1(x, z; x', z')^2 \, d\Phi^*(z') \right)^{1/2} \, dx'.
$$

**Lemma SA.3.1.** Under Assumptions 4, 5, and 6, there exists $\alpha > 1$ such that, for $r$ sufficiently small, $\sup_{(x, z) \in \mathcal{T}} g_r^\alpha(x, z) < \infty$.

**Proof.** By Lemma SA.3.3 of Leung (2020b), the conclusion holds if $\inf_{(x, z) \in \mathcal{T}} \psi_0(x, z) > 0$ and $\sup_{(x, z) \in \mathcal{T}} \psi_0(x, z) < \infty$. Assumption 4 implies the second inequality, and Assumption 6 implies the first because

$$
\lim_{n \to \infty} \psi_{r_n}(x, z) \geq \lim_{n \to \infty} nE[p_{r_n}(X_i; Z_{i0}; X_j, Z_{j0}) \mid X_i = x, Z_{i0} = z].
$$

This lemma was original to an earlier version of this paper (Leung and Moon, 2019, Lemma I.4) but was subsequently generalized by Leung (2020b) and cut from the current paper for length.
Lemma SA.3.2. Define $\bar{g}_\alpha(x, z; K) = E[\alpha^{|X_M(x, z; K)|}]$. Under Assumptions 4 and 6, for any $\alpha > 1$ and $K \in \mathbb{N}$, $\sup_{r \leq \kappa} \sup_{(x, z) \in T} E[\bar{g}_\alpha(x, z; K)] < \infty$.

Proof. Let $\bar{T}_r$ be the functional that maps any $h: \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ to

$$(\bar{T}_r h)(x, z) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h(x', z') \bar{\varphi}_r(x, z; x', z').$$

A standard branching process argument yields the recursion

$$\bar{g}_\alpha(x, z; K) = \alpha \exp \left\{ \kappa (1 + r) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\bar{g}_\alpha(x', z'; K - 1) - 1) \bar{p}_r(x; x') \, d\Phi^*(z') \, dx' \right\}$$

$$= \alpha \exp \{ \bar{T}_r(\bar{g}_\alpha(x, z; K - 1) - 1) \}. \tag{SA.3.1}$$

See for example the proof of Lemma I.2 of Leung and Moon (2019). Let $\Gamma: g \mapsto \alpha \exp\{\bar{T}_r(g - 1)\}$. Then for any $(x, z) \in T$,

$$\bar{g}_\alpha(x, z; K) = \Gamma^K \alpha, \tag{SA.3.2}$$

by (SA.3.1) and the fact that $|X_M(x, z; 0)| = 1$. Now, $\Gamma \alpha$ equals

$$\alpha e^{\alpha - 1} \exp \left\{ \kappa (1 + r) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \bar{p}_r(x; x') \, d\Phi^*(z') \, dx' \right\}. \tag{SA.3.3}$$

Recall the definition of $\bar{V}_r(\delta, \cdot)$ after (SA.2.4), and let $\bar{V}_r^{-1}(\delta, \cdot)$ be its inverse, which exists by Assumptions 4 and 6. Under these two assumptions, $\sup_x \int_{\mathbb{R}^d} \bar{p}_1(x; x') \, dx' < \infty$ since

$$\bar{p}_1(x; x') \leq \sum_{t=0}^{T} \Phi_\zeta(\bar{V}_r^{-1} ||x - x'||, 0))$$

decays exponentially in $||x - x'||$. Consequently, there exists $C < \infty$ such that, for any $r \leq \kappa$, (SA.3.3) is bounded by $\alpha e^{\alpha - 1} C$. Likewise, $\Gamma^2 \alpha < \alpha e^{\alpha - 1} C - 1 C < \infty$, and repeating this argument, we obtain $\sup_{(x, z) \in T} \Gamma^K \alpha < \infty$. Combined with (SA.3.2), this proves the claim.

\[\blacksquare\]
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The next lemma establishes an exponential tail bound for $|\mathbf{x}_{r_n}^H(X_i, Z_{i0}; K, T)|$ defined in (SA.2.7).

**Lemma SA.3.3.** Under Assumptions 4, 5, and 6, there exists $n_0 \in \mathbb{N}$ such that

$$\limsup_{w \to \infty} w^{-1} \log \beta(w) < 0$$

for

$$\beta(w) = \sup_{n > n_0} \sup_{(x,z) \in T} P\left(|\mathbf{x}_{r_n}^H(X_i, Z_{i0}; K, T)| > w \mid X_i = x, Z_{i0} = z\right).$$

**Proof.** Let $\tilde{K} = 2K + T + 1$ and $\Gamma_n(x, z) = \bigcup_{(x', z')} \mathbf{x}_{r_n}^M(X_i, Z_{i0}; \tilde{K})$, where conditional on particle types in $\mathbf{x}_{r_n}^M(X_i, Z_{i0}; \tilde{K})$, $\{\mathbf{x}_{r_n}^D(x', z') : (x', z') \in \mathbf{x}_{r_n}^M(X_i, Z_{i0}; \tilde{K})\}$ are independent branching processes with intensity (SA.2.3). Then $\mathbf{x}_{r_n}^H(x, z; K, T) = \bigcup_{(x', z') \in \Gamma_n(x, z)} \mathbf{x}_{r_n}^M(x', z'; 1)$, where, conditional on $\Gamma_n(x, z)$, $\{\mathbf{x}_{r_n}^M(x', z'; 1) : (x', z') \in \Gamma_n(x, z)\}$ are independent branching processes with intensity (SA.2.5).

By the law of total probability,

$$P\left(|\mathbf{x}_{r_n}^H(x, z; K, T)| > w\right) \leq P\left(|\Gamma_n(x, z)| > w\right)$$

$$+ P\left(\sum_{(x', z') \in \Gamma_n(x, z)} |\mathbf{x}_{r_n}^M(x', z'; 1)| > w \cap |\Gamma_n(x, z)| \leq w\right). \quad (SA.3.4)$$

For any $\alpha > 1$, the second term on the right-hand side equals

$$E\left[P\left(\sum_{(x', z') \in \Gamma_n(x, z)} |\mathbf{x}_{r_n}^M(x', z'; 1)| > w \mid \Gamma_n(x, z)\right) 1\{|\Gamma_n(x, z)| \leq w\}\right]$$

$$\leq \alpha^{-w} E\left[\prod_{(x', z') \in \Gamma_n(x, z)} E\left[\alpha^{\mathbf{x}_{r_n}^M(x'', z''; 1)} \mid \Gamma_n(x, z)\right] 1\{|\Gamma_n(x, z)| \leq w\}\right]$$

$$\leq \alpha^{-w} \left(\sup_{n > n_0} \sup_{(x'', z'') \in \mathcal{T}} E\left[\alpha^{\mathbf{x}_{r_n}^M(x'', z''; 1)}\right]\right)^{w'}, \quad (SA.3.5)$$

where the second line uses Markov’s inequality and conditional independence of the branching processes, and the third line also uses conditional independence.

By Lemma SA.3.2, we can choose $\alpha$ and $n_0$ such that the supremum term in (SA.3.5) is bounded above by some $C > 0$. Then for any $c > 0$ and $w' = cw$,

$$(SA.3.5) < \exp \{-w \log \alpha + cw \log C\}.$$
Choosing \( c \) such that \( \log \alpha > c \log C \), the second element on the right-hand side of (SA.3.4) is bounded above by \( e^{-\beta w} \) for some \( \beta > 0 \). By a similar argument,

\[
P(|\Gamma_n(x, z)| > w') \leq \alpha^{-w'} \left( \sup_{n > n_0} \sup_{(x', z') \in T} \mathbb{E} \left[ \alpha^{x_{n_0}^{(x', z')}(x', z')} \right] \right)^{w''} + P \left( x_{\tilde{r}_n}^M(x, z; \tilde{K}) > w'' \right). \tag{SA.3.6}
\]

By Lemma SA.3.1, we can choose \( \alpha, n_0 \) such that the supremum term is bounded by a finite constant. Then choosing \( w'' = c'w' \) for some small enough \( c' > 0 \), the first term on the right-hand side can be bounded above by \( e^{-\beta'w'} = e^{-\beta'cw} \) for some \( \beta' > 0 \), recalling \( w' = cw \). By Lemma SA.3.2, the second term on the right-hand side of (SA.3.6) is \( O(e^{-\beta''w''}) \) for some \( \beta'' > 0 \), uniformly over all \( n \) sufficiently large. We have therefore established that (SA.3.4) is \( O(e^{-\gamma w}) \) for some \( \gamma > 0 \).

The last lemma bounds the maximal distance between \( i \) and \( N_M(i, K) \).

**Lemma SA.3.4.** Under Assumptions 4 and 6, for any \( \epsilon > 0 \), \( k \in \{1, 2, 3\} \), and \( K \in \mathbb{N} \), there exist constants \( n_0, a_1, a_2, C > 0 \) such that for \( \bar{f} = \sup_x f(x) \),

\[
\max \left\{ \sup_{n > n_0} \sup_{m \in ((1-\epsilon)n, (1+\epsilon)n)} \sup_{x_k \in \text{supp}(f)^k} P \left( \max_{i \in N_M((n+k)(1, K))} r_n^{-1} \|X_1 - X_i\| > w \mid X_k = x_k \right), \right. \\
\sup_{n > n_0} \sup_{x \in \text{supp}(f)} P \left( \max_{i \in N_M(N_n+1)(1, K)} r_n^{-1} \|X_1 - X_i\| > w \mid X_1 = x \right) \}
\leq CK(a_1\bar{f}w/K)^d e^{-a_2w}.
\]

**Proof.** We only prove the bound for the probability involving \( N_M(N_n+1)(1, K) \). The argument for the other term is similar. Abbreviate \( M = M(N_n + 1) \), and let \( M_{ij,t} \) be the \( ij \)th entry of \( M_{ij} \). Recall the definition of \( \tilde{V}_t(\delta, \cdot) \) after (SA.2.4), and let \( \tilde{V}_t^{-1}(\delta, \cdot) \) be its inverse, which exists by Assumptions 4 and 6.

Let \( N_M^*(1, K) \) be the set of nodes at path distance \( K \) from node 1 in \( M \). Then

\[
P \left( \max_{i \in N_M(1, K)} r_n^{-1} \|X_1 - X_i\| > w \right) \leq \sum_{t=1}^K P \left( \max_{i \in N_M^*(1, t)} r_n^{-1} \|X_1 - X_i\| > w \right). \tag{SA.3.7}
\]

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Note that if \( \max_{i \in \mathcal{N}_M(i,1,K)} r_n^{-1} \|X_i - X_{j_0}\| > w \), then there must exist distinct nodes \( j_0 \neq \cdots \neq j_K \) with \( j_0 = 1 \) such that for all \( \ell = 1, \ldots, K \), we have \( r_n^{-1} \|X_{j_{\ell-1}} - X_{j_\ell}\| > w/K \) and \( M_{j_{\ell-1}j_\ell, t} = 1 \) for some \( t \). This implies \( \zeta_{j_{\ell-1}j_\ell} > V_t^{-1}(w/K, 0) \) for some \( t \) and all \( \ell \) by Assumptions 4 and 6, so

\[
P \left( \max_{i \in \mathcal{N}_M(i,1,K)} r_n^{-1} \|X_i - X_i\| > w \right) \\
\leq P \left( \bigcup_{j_0 \neq \cdots \neq j_K \in \mathcal{N}_{\mathcal{N}_{n+1}} j_0=1} \bigcup_{t=1}^T \left\{ \zeta_{j_{\ell-1}j_\ell} > V_t^{-1}(w/K, 0) \cap M_{j_{\ell-1}j_\ell, t} = 1 \right\} \right) \\
\leq E \left[ \sum_{j_0 \neq \cdots \neq j_K \in \mathcal{N}_{\mathcal{N}_{n+1}} j_0=1} \prod_{\ell=1}^K \rho_n(X_{j_{\ell-1}}, X_{j_\ell}; w/K) \right], \quad \text{(SA.3.8)}
\]

where

\[
\rho_n(X_{j_{\ell-1}}, X_{j_\ell}; w/K) = \sum_{t=0}^T P \left( \zeta_{j_{\ell-1}j_\ell} > \max \left\{ V_t^{-1}(r_n^{-1} \|X_{j_{\ell-1}} - X_{j_\ell}\|, 0), V_t^{-1}(w/K, 0) \right\} \mid X_{j_{\ell-1}}, X_{j_\ell} \right).
\]

By Corollary 3.2.3 of Schneider and Weil (2008), (SA.3.8) equals

\[
\int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \rho_n(x_1, x_{j_1}; w/K) \left( \prod_{\ell=1}^{K-1} \rho_n(x_{j_\ell}, x_{j_{\ell+1}}; w/K) n f(x_{j_\ell}) \, dx_{j_\ell} \right) n f(x_{j_K}) \, dx_{j_K} f(x_1) \, dx_1.
\]

The previous expression is bounded above by

\[
\left( \sup_x n \int_{\mathbb{R}^d} \rho_n(x, y; w/K) f(y) \, dy \right)^K \\
\leq \sum_{t=0}^T \left( k \bar{f} \left( \int_{w/K}^{\infty} u^{d-1} \Phi_\zeta(V_t^{-1}(u, 0)) du + \frac{(w/K)^d}{d} \Phi_\zeta(V_t^{-1}(w/K, 0)) \right) \right)^K
\]

by a change of variables to polar coordinates. Since \( \Phi_\zeta(V_t^{-1}(u, 0)) \leq c_1 e^{-c_2 u} \) for some \( c_1, c_2 > 0 \) by Assumption 4, the second term in the inner parentheses dominates, and the claim follows from integration by parts and (SA.3.7). \( \blacksquare \)