Serre dimension of monoid algebras

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Abstract. Let $R$ be a commutative Noetherian ring of dimension $d$, $M$ a commutative cancellative torsion-free monoid of rank $r$ and $P$ a finitely generated projective $R[M]$-module of rank $t$. Assume $M$ is $\Phi$-simplicial seminormal. If $M \in C(\Phi)$, then Serre dim $R[M] \leq d$. If $r \leq 3$, then Serre dim $R[\text{int}(M)] \leq d$. If $M \subseteq \mathbb{Z}_+^2$ is a normal monoid of rank 2, then Serre dim $R[M] \leq d$. Assume $M$ is $c$-divisible, $d = 1$ and $t \geq 3$. Then $P \cong \otimes^t P \oplus R[M]^{t-1}$. Assume $R$ is a uni-branched affine algebra over an algebraically closed field and $d = 1$. Then $P \cong \otimes^t P \oplus R[M]^{t-1}$.

Keywords. Projective modules; Serre dimension; $\Phi$-simplicial monoid.

Mathematics Subject Classification. 13C10, 13D15.

1. Introduction

Throughout, rings are commutative Noetherian with 1; projective modules are finitely generated and of constant rank; monoids are commutative cancellative torsion-free; $\mathbb{Z}_+$ denotes the additive monoid of non-negative integers.

Let $A$ be a ring and $P$ a projective $A$-module. An element $p \in P$ is called unimodular, if there exists $\phi \in \text{Hom}(P, A)$ such that $\phi(p) = 1$. We say Serre dimension of $A$ (denoted as Serre dim $A$) is \leq $t$, if every projective $A$-module of rank $\geq t + 1$ has a unimodular element. Serre dimension of $A$ measures the surjective stabilization of the Grothendieck group $K_0(A)$. Serre’s problem on the freeness of projective $k[X_1, \ldots, X_n]$-modules, $k$ a field, is equivalent to Serre dim $k[X_1, \ldots, X_n] = 0$.

After the solution of Serre’s problem by Quillen [16] and Suslin [21], many people worked on surjective stabilization of polynomial extension of a ring. Serre [20] proved Serre dim $A \leq \dim A$, Plumstead [14] proved Serre dim $A[X] \leq \dim A$, Bhatwadekar-Roy [4] proved Serre dim $A[X_1, \ldots, X_n] \leq \dim A$ and Bhatwadekar et al. [3] proved Serre dim $A[X_1, \ldots, X_n, Y_1^{\pm 1}, \ldots, Y_m^{\pm 1}] \leq \dim A$.

Anderson conjectured an analogue of Quillen–Suslin theorem for monoid algebras over a field which was answered by Gubeladze [8] as follows.

Theorem 1.1. Let $k$ be a field and $M$ a monoid. Then $M$ is seminormal if and only if all projective $k[M]$-modules are free.
Gubeladze [11] asked the following:

**Question 1.2.** Let $M \subset \mathbb{Z}_+^r$ be a monoid of rank $r$ with $M \subset \mathbb{Z}_+^r$ an integral extension. Let $R$ be a ring of dimension $d$. Is Serre dim $R[M] \leq d$?

We answer Question 1.2 for some class of monoids. Recall that a finitely generated monoid $M$ of rank $r$ is called $\Phi$-simplicial if $M$ can be embedded in $\mathbb{Z}_+^r$ and the extension $M \subset \mathbb{Z}_+^r$ is integral (see [10]). A $\Phi$-simplicial monoid is commutative, cancellative and torsion-free.

**DEFINITION 1.3.**

Let $C(\Phi)$ denote the class of seminormal $\Phi$-simplicial monoids $M \subset \mathbb{Z}_+^r$ of rank $r$ such that if $\mathbb{Z}_+^r = \{t_1^{i_1} \cdots t_r^{i_r} | s_i \geq 0\}$, then for $1 \leq m \leq r$, $M_m = M \cap \{t_1^{i_1} \cdots t_r^{i_r} | s_i \geq 0\}$ satisfies the following properties: Given a positive integer $c$, there exist integers $c_i > c$ for $i = 1, \ldots, m - 1$ such that for any ring $R$, the automorphism $\eta \in \text{Aut}_R[M_m](R[t_1, \ldots, t_m])$ defined by $\eta(t_i) = t_i + c_i^m$ for $i = 1, \ldots, m - 1$, restricts to an $R$-automorphism of $R[M_m]$. It is easy to see that $M_m \in C(\Phi)$ and rank $M_m = m$ for $1 \leq m \leq r$.

Theorem 3.4 and Proposition 3.8 answer Question 1.2 for monoids in $C(\Phi)$.

**Theorem 1.4.** Let $M \subset \mathbb{Z}_+^r$ be a seminormal $\Phi$-simplicial monoid of rank $r$ and $R$ a ring of dimension $d$.

1. If $M \in C(\Phi)$, then Serre dim $R[M] \leq d$.
2. If $r \leq 3$, then Serre dim $R[\text{int}(M)] \leq d$, where $\text{int}(M) = \text{int}(\mathbb{R}_+M) \cap \mathbb{Z}_+^r$ and $\text{int}(\mathbb{R}_+M)$ is the interior of the cone $\mathbb{R}_+M \subset \mathbb{R}_+^r$ with respect to Euclidean topology.

Corollary 3.6 follows from Theorem 1.4(1). This result is due to Anderson [1] when $R$ is a field.

**Theorem 1.5.** Let $R$ be a ring of dimension $d$ and $M \subset \mathbb{Z}_+^2$ a normal monoid of rank 2. Then Serre dim $R[M] \leq d$.

The next result answers Question 1.2 partially for 1-dimensional rings (see Theorems 3.13 and 3.16). The proof uses the technique of Kang [12], Roy [17] and Gubeladze [9]. Let us recall two definitions. (i) A monoid $M$ is called $c$-divisible, where $c > 1$ is an integer, if $cX = m$ has a solution in $M$ for all $m \in M$. All $c$-divisible monoids are seminormal. (ii) Let $R$ be a ring, $\bar{R}$ the integral closure of $R$ and $C$ the conductor ideal of $R \subset \bar{R}$. Then $R$ is called uni-branched if for any $p \in \text{Spec } R$ containing $C$, there is a unique $q \in \text{Spec } \bar{R}$ such that $q \cap R = p$.

**Theorem 1.6.** Let $R$ be a ring of dimension 1, $M$ a monoid and $P$ a projective $R[M]$-module of rank $r$.

1. If $M$ is $c$-divisible and $r \geq 3$, then $P \cong \wedge^r P \oplus R[M]^{r-1}$.
2. If $R$ is a uni-branched affine algebra over an algebraically closed field, then $P \cong \wedge^r P \oplus R[M]^{r-1}$.
If $R$ is a 1-dimensional anodal ring with finite seminormalization, then Theorem 1.6(ii) is due to Theorem 1.2 of [18]. At the end, we give some applications to minimum number of generators of projective modules.

2. Preliminaries

Let $A$ be a ring and $Q$ an $A$-module. We say $p \in Q$ is unimodular if the order ideal $O_Q(p) = \{ \phi(p) \mid \phi \in \text{Hom}(Q, A) \}$ equals $A$. The set of all unimodular elements in $Q$ is denoted by $\text{Um}(Q)$. We write $E_n(A)$ for the group generated by the set of all $n \times n$ elementary matrices over $A$ and $\text{Um}_n(A)$ for $\text{Um}(A^n)$. We denote by $\text{Aut}_A(Q)$, the group of all $A$-automorphisms of $Q$.

For an ideal $J$ of $A$, we denote by $E(A \oplus Q, J)$, the subgroup of $\text{Aut}_A(A \oplus Q)$ generated by all the automorphisms $\Delta_{a\phi} = (\begin{smallmatrix} 1 & a \phi \\ 0 & \text{id}_Q \end{smallmatrix})$ and $\Gamma_q = (\begin{smallmatrix} q & 0 \\ 0 & q \end{smallmatrix})$ with $a \in J$, $\phi \in Q^*$ and $q \in Q$. Further, we shall write $E(A \oplus Q, J)$ for $E(A \oplus Q, A)$. We denote by $\text{Um}(A \oplus Q, J)$ the set of all $(a, q) \in \text{Um}(A \oplus Q)$ with $a \in 1 + J$ and $q \in JQ$.

We state some results of Lindel [13] for later use.

**PROPOSITION 2.1** (Lemma 1.1 of [13])

Let $A$ be a ring and $Q$ an $A$-module. Let $Q_s$ be free of rank $r$ for some $s \in A$. Then there exist $p_1, \ldots, p_r \in Q \cdot \phi_1, \ldots, \phi_r \in Q^*$ and $t \geq 1$ such that following hold:

(i) $0 \simeq_A s' A = 0 \simeq_A s'^2 A$, where $s' = s^t$.

(ii) $s' Q \subset F$ and $s' Q^* \subset G$, where $F = \sum_{i=1}^r Ap_i \subset Q$ and $G = \sum_{i=1}^r A\phi_i \subset Q^*$.

(iii) The matrix $(\phi_i(p_j))_{1 \leq i, j \leq r} = \text{diagonal}(s', \ldots, s')$. We say $F$ and $G$ are $s'$-dual submodules of $Q$ and $Q^*$ respectively.

**PROPOSITION 2.2** (Lemma 1.2 and Corollary 1.3 of [13])

Let $A$ be a ring and $Q$ an $A$-module. Assume $Q_s$ is free of rank $r$ for some $s \in A$. Let $F$ and $G$ be $s$-dual submodules of $Q$ and $Q^*$ respectively. Then

(i) for $p \in Q$, there exists $q \in F$ such that $\text{ht}(O_Q(p + sq)A_s) \geq r$.

(ii) If $Q$ is projective $A$-module and $\bar{p} \in \text{Um}(Q/sQ)$, then there exists $q \in F$ such that $\text{ht}(O_Q(p + sq)) \geq r$.

**PROPOSITION 2.3** (Proposition 1.6 of [13])

Let $Q$ be a module over a positively graded ring $A = \bigoplus_{i \geq 0} A_i$ and $Q_s$ be free for some $s \in R = A_0$. Let $T \subseteq A$ be a multiplicatively closed set of homogeneous elements. Let $p \in Q$ be such that $p_{T(1+sR)} \in \text{Um}(Q_{T(1+sR)})$ and $s \in \text{rad}(O_Q(p) + A_+)$, where $A_+ = \bigoplus_{i \geq 1} A_i$. Then there exists $p' \in p + sA_+Q$ such that $p'_{T} \in \text{Um}(Q_T)$.

**PROPOSITION 2.4** (Proposition 1.8 of [13])

Under the assumptions of Proposition 2.3, let $p \in Q$ be such that $O_Q(p) + sA_+ = A$ and $A/O_Q(p)$ is an integral extension of $R/(R \cap O_Q(p))$. Then there exists $p' \in \text{Um}(Q)$ with $p' - p \in sA_+Q$. 

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The following result is due to Proposition 3.4 of [17].

PROPOSITION 2.5

Let $A, B$ be two rings with $f : A \to B$ a ring homomorphism. Let $s \in A$ be non-zero divisor such that $f(s)$ is a non-zero divisor in $B$. Assume that we have the following cartesian square:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
A_s & \xrightarrow{f(s)} & B_f(s)
\end{array}
$$

Further assume that $SL_r(B_f(s)) = E_r(B_f(s))$ for some $r > 0$. Let $P$ and $Q$ be two projective $A$-modules of rank $r$ such that (i) $\wedge^r P \cong \wedge^r Q$, (ii) $P_s$ and $Q_s$ are free over $A_s$, (iii) $P \otimes_A B \cong Q \otimes_A B$ and $Q \otimes_A B$ has a unimodular element. Then $P \cong Q$.

DEFINITION 2.6 (§ 6 of [10])

Let $R$ be a ring and $M$ a $\Phi$-simplicial monoid of rank $r$. Fix an integral extension $M \hookrightarrow \mathbb{Z}_+^r$. Let $\{t_1, \ldots, t_r\}$ be a free basis of $\mathbb{Z}_+^r$. Then $M$ can be thought of as a monoid consisting of monomials in $t_1, \ldots, t_r$.

For $x = t_1^{a_1} \cdots t_r^{a_r}$ and $y = t_1^{b_1} \cdots t_r^{b_r}$ in $\mathbb{Z}_+^r$, define $x$ is lower than $y$ if $a_i < b_i$ for some $i$ and $a_j = b_j$ for $j > i$. In particular, $t_i$ is lower than $t_j$ if and only if $i < j$.

For $f \in R[M]$, define the highest member $H(f)$ of $f$ as $am$, where $f = am + a_1m_1 + \ldots + a_km_k$ with $m, m_i \in M$, $a \in R \setminus \{0\}$, $a_i \in R$ and each $m_j$ is strictly lower than $m$ for $1 \leq i \leq k$.

An element $f \in R[\mathbb{Z}_+^r]$ is called monic if $H(f) = at^s$, where $a \in R$ is a unit and $s > 0$. An element $f \in R[M]$ is said to be monic if $f$ is monic in $R[\mathbb{Z}_+^r]$ via the embedding $R[M] \hookrightarrow R[\mathbb{Z}_+^r]$.

Define $M_0$ to be the submonoid $\{t_1^{s_1} \cdots t_r^{s_r-1} | s_i \geq 0\} \cap M$ of $M$. Clearly $M_0$ is finitely generated as $M$ is finitely generated. Also $M_0 \hookrightarrow \mathbb{Z}_+^{r-1}$ is integral. Hence $M_0$ is $\Phi$-simplicial. Further, if $M$ is seminormal, then $M_0$ is seminormal.

Grade $R[M]$ as $R[M] = R[M_0] \oplus A_1 \oplus A_2 \oplus \ldots$, where $A_i$ is the $R[M_0]$-module generated by the monomials $t_1^{s_1} \cdots t_r^{s_r-1} t_i \in M$. For an ideal $I$ in $R[M]$, define its leading coefficient ideal $\lambda(I)$ as $\{a \in R | \exists f \in I \text{ with } H(f) = am \text{ for some } m \in M\}$. □

Lemma 2.7 (Lemma 6.5 of [10]). Let $R$ be a ring and $M \subset \mathbb{Z}_+^r$ a $\Phi$-simplicial monoid. If $I \subseteq R[M]$ is an ideal, then $ht(\lambda(I)) \geq ht(I)$, where $\lambda(I)$ is defined in Definition 2.6.

3. Main theorem

This section contains main results stated in the Introduction. We also give some examples of monoids in $C(\Phi)$.
3.1 Over $C(\Phi)$ class of monoids

Lemma 3.1. Let $R$ be a ring and $M \subset \mathbb{Z}_+^r$ a monoid in $C(\Phi)$ of rank $r$. Let $f \in R[M] \subset R[\mathbb{Z}_+^r] = R[t_1, \ldots, t_r]$ with $H(f) = ut_1^{s_1} \cdots t_r^{s_r}$ for some unit $u \in R$. Then there exists $\eta \in \text{Aut}_R(R[M])$ such that $\eta(f)$ is a monic polynomial in $t_r$.

Proof. Since $M \in C(\Phi)$, we can choose positive integers $c_1, \ldots, c_{r-1}$ such that the automorphism $\eta \in \text{Aut}_{R[t_r]}[t_1, \ldots, t_r]$ defined by $\eta(t_i) = t_i + t_r^{c_i}$ for $i = 1, \ldots, r-1$, restricts to an automorphism of $R[M]$ and such that $\eta(f)$ is a monic polynomial in $t_r$. □

Lemma 3.2. Let $R$ be a ring of dimension $d$ and $M \subset \mathbb{Z}_+^r$ a monoid in $C(\Phi)$ of rank $r$. Let $P$ be a projective $R[M]$-module of rank $> d$. Write $R[M] = R[M_0] \oplus A_1 \oplus A_2 \cdots$, as defined in (2.6) and $A_+ = A_1 \oplus A_2 \oplus \cdots$ an ideal of $R[M]$. Assume that $P_s$ is free for some $s \in R$ and $P/sA_+P$ has a unimodular element. Then the natural map $\text{Um}(P) \to \text{Um}(P/sA_+P)$ is surjective. In particular, $P$ has a unimodular element.

Proof. Write $A = R[M]$. Since every unimodular element of $P/sA_+P$ can be lifted to a unimodular element of $P_1+sA_+$, if $s$ is nilpotent, then elements of $1+sA_+$ are units in $A$ and we are done. Therefore, assume that $s$ is not nilpotent.

Let $p \in P$ be such that $\tilde{p} \in \text{Um}(P/sA_+P)$. Then $O_P(p) + sA_+ = A$. Hence $O_P(p)$ contains an element of $1+sA_+$. Choose $g \in A_+$ such that $1+sg \in O_P(p)$. Applying Proposition 2.2 with $sg$ in place of $s$, we get $q \in F \subset P$ such that $\text{ht}(O_P(p+sgq)) > d$. Since $p+sgq$ is a lift of $\tilde{p}$, replacing $p$ by $p+sgq$, we may assume that $\text{ht}(O_P(p)) > d$.

By Lemma 2.7, we get $\text{ht}(\lambda(O_P(p))) \geq \text{ht}(O_P(p)) > d$. Since $\lambda(O_P(p))$ is an ideal of $R$, we get $1 \in \lambda(O_P(p))$. Hence there exists $f \in O_P(p)$ such that the coefficient of $H(f)$ (highest member of $\lambda$) is a unit.

Suppose $H(f) = ut_1^{s_1} \cdots t_r^{s_r}$ with $u$ a unit in $R$. Since $M \in C(\Phi)$, by Lemma 3.1, there exists $\alpha \in \text{Aut}_R(R[M])$ such that $\alpha(f)$ is monic in $t_r$. Thus we may assume that $O_P(p)$ contains a monic polynomial in $t_r$. Hence $A/O_P(p)$ is an integral extension of $R[M_0]/(O_P(p) \cap R[M_0])$ and $\tilde{p} \in \text{Um}(P/sA_+P)$. By Proposition 2.4, there exists $p' \in \text{Um}(P)$ such that $p' - p \in sA_+P$. This means $p' \in \text{Um}(P)$ is a lift of $\tilde{p}$. This proves the result. □

Remark 3.3. In Lemma 3.2, we do not need the monoid $M$ to be seminormal. The next result proves Theorem 1.4(1).

Theorem 3.4. Let $R$ be a ring of dimension $d$ and $M$ a monoid in $C(\Phi)$ of rank $r$. If $P$ is a projective $R[M]$-module of rank $r' \geq d + 1$, then $P$ has a unimodular element. In other words, Serre $\dim R[M] \leq d$.

Proof. We can assume that the ring is reduced with connected spectrum. If $d = 0$, then $R$ is a field. Since $M$ is seminormal, projective $R[M]$-modules are free, by Theorem 1.1. If $r = 0$, then $M = 0$ and we are done [20]. Assume $d, r \geq 1$ and use induction on $d$ and $r$ simultaneously.

If $S$ is the set of all non-zero divisor of $R$, then $\dim S^{-1}R = 0$ and so $S^{-1}P$ is free in $S^{-1}R[M]$-module ($d = 0$ case). Choose $s \in S$ such that $P_s$ is free. Consider the ring
\[ R[M]/(sR[M]) = (R/sR)[M]. \] Since \( \dim R/sR = d - 1 \), by induction on \( d \), \( \text{Um}(P/sP) \) is non-empty.

Write \( R[M] = R[M_0] \oplus A_1 \oplus A_2 \cdots \), as defined in Definition 2.6 and \( A_+ = A_1 \oplus A_2 \oplus \cdots \) an ideal of \( R[M] \). Note that \( M_0 \in \mathcal{C}(\Phi) \) and \( \text{rank } M_0 = r - 1 \). Since \( R[M]/A_+ = R[M_0] \), by induction on \( r \), \( \text{Um}(P/A_+) \) is non-empty. Write \( A = R[M] \) and consider the following fiber product diagram:

\[
\begin{array}{ccc}
A/(sA \cap A_+) & \longrightarrow & A/sA \\
\downarrow & & \downarrow \\
A/A_+ & \longrightarrow & A/(s, A_+) \\
\end{array}
\]

If \( B = R/sR \), then \( A/(s, A_+) = B[M_0] \). Let \( u \in \text{Um}(P/A_+) \) and \( v \in \text{Um}(P/sP) \). Let \( \bar{u} \) and \( \bar{v} \) denote the images of \( u \) and \( v \) in \( P/(s, A_+) \) \( P \). Write \( P/(s, A_+)P = B[M_0] \oplus P_0 \), where \( P_0 \) is some projective \( B[M_0] \)-module of rank \( = r' - 1 \). Note that \( \dim(B) = d - 1 \) and \( \bar{u}, \bar{v} \) are two unimodular elements in \( B[M_0] \oplus P_0 \).

**Case 1.** Assume \( \text{rank}(P_0) \geq \max \{2, d\} \). Then by Theorem 4.5 of [6], there exists \( \sigma \in \text{E}(B[M_0] \oplus P_0) \) such that \( \sigma(\bar{u}) = \bar{v} \). Lift \( \sigma \) to an element \( \sigma_1 \in \text{E}(P/A_+) \) and write \( \sigma_1(u) = u_1 \in \text{Um}(P/A_+) \). Then images of \( u_1 \) and \( v \) are same in \( P/(s, A_+)P \). Patching \( u_1 \) and \( v \) over \( P/(s, A_+)P \) in the above fiber product diagram, we get an element \( p \in \text{Um}(P/(sA \cap A_+)P) \).

Note that \( sA \cap A_+ = sA_+ \). We have \( P_0 \) is free and \( P/sA_+P \) has a unimodular element. Use (3.2) to conclude that \( P \) has a unimodular element.

**Case 2.** Now we consider the remaining case, namely \( d = 1 \) and \( \text{rank}(P) = 2 \). Since \( B = R/sR \) is 0 dimensional, projective modules over \( B[M_0] \) and \( B[M] \) are free, by Theorem 1.1. In particular, \( P/sP \) and \( P/(s, A_+)P \) are free modules of rank 2 over the rings \( B[M] \) and \( B[M_0] \) respectively. Consider the same fiber product diagram as above.

Any two unimodular elements in \( \text{Um}_2(B[M_0]) \) are connected by an element of \( \text{GL}_2(B[M_0]) \). Further \( B[M_0] \) is a subring of \( B[M] = A/sA \). Hence the natural map \( \text{GL}_2(B[M]) \rightarrow \text{GL}_2(B[M_0]) \) is surjective. Hence any automorphism of \( P/(s, A_+)P \) can be lifted to an automorphism of \( P/sP \). By the same argument as above, patching unimodular elements of \( P/sP \) and \( P/A_+P \), we get a unimodular element in \( P/(sA \cap A_+)P \). Since \( sA \cap A_+ = sA_+ \) and \( P/sA_+P \) has a unimodular element, by Lemma 3.2, \( P \) has a unimodular element. This completes the proof. \( \square \)

**Example 3.5.**

1. If \( M \) is a \( \Phi \)-simplicial normal monoid of rank 2, then \( M \in \mathcal{C}(\Phi) \). To see this, by Lemma 1.3 of [10], \( M \cong (a_1, a_2) \cap \mathbb{Z}_+^2 \), where \( a_1 = (a, b) \), \( a_2 = (0, c) \) and \( (a_1, a_2) \) is the group generated by \( a_1 \) and \( a_2 \). It is easy to see that \( M \cong ((1, a_1), (0, a_2)) \cap \mathbb{Z}_+^2 \), where \( \text{gcd}(b, c) = g \) and \( a_1 = b/g \), \( a_2 = c/g \). Hence \( M \in \mathcal{C}(\Phi) \).
2. If \( M \subset \mathbb{Z}_+^2 \) is a finitely generated rank 2 normal monoid, then it is easy to see that \( M \) is \( \Phi \)-simplicial. Hence \( M \in \mathcal{C}(\Phi) \) by (1).
(3) If $M$ is a rank 3 normal quasi-truncated or truncated monoid (see Definition 5.1 of [10]), then $M \in \mathcal{C}(\Phi)$. To see this, by Lemma 6.6 of [10], $M$ satisfies properties of Definition 1.3. Further, $M_0$ is a $\Phi$-simplicial normal monoid of rank 2. By (1), $M_0 \in \mathcal{C}(\Phi)$.

□

COROLLARY 3.6

Let $R$ be a ring of dimension $d$ and $M \subset \mathbb{Z}_2^r$ a normal monoid of rank 2. Then $\text{Serre dim } R[M] \leq d$.

Proof. If $M$ is finitely generated, then the result follows from Example 3.5(2) and Theorem 3.4.

If $M$ is not finitely generated, then write $M$ as a filtered union of finitely generated submonoids, say $M = \bigcup_{\lambda \in I} M_{\lambda}$. Since $M$ is normal, the integral closure $\bar{M}_{\lambda}$ of $M_{\lambda}$ is contained in $M$. Hence $M = \bigcup_{\lambda \in I} \bar{M}_{\lambda}$. By Proposition 2.22 of [5], $\bar{M}_{\lambda}$ is finitely generated. If $P$ is a projective $R[M]$-module, then $P$ is defined over $R[\bar{M}_{\lambda}]$ for some $\lambda \in I$ as $P$ is finitely generated. Now the result follows from Example 3.5(2) and Theorem 3.4. □

The following result follows from Example 3.5(3) and Theorem 3.4.

COROLLARY 3.7

Let $R$ be a ring of dimension $d$ and $M$ a truncated or normal quasi-truncated monoid of rank $\leq 3$. Then $\text{Serre dim } R[M] \leq d$.

Now we prove Theorem 1.4(2).

PROPOSITION 3.8

Let $R$ be a ring of dimension $d$ and $M$ a $\Phi$-simplicial seminormal monoid of rank $r \leq 3$. Then $\text{Serre dim } R[\text{int}(M)] \leq d$.

Proof. Recall that $\text{int}(M) = \text{int}(\mathbb{R}_+^r) \cap \mathbb{Z}_+^r$. Let $P$ be a projective $R[\text{int}(M)]$-module of rank $\geq d + 1$. Since $M$ is seminormal, by Proposition 2.40 of [5], $\text{int}(M) = \text{int}(\tilde{M})$, where $\tilde{M}$ is the normalization of $M$. Since normalization of a finitely generated monoid is finitely generated (see Proposition 2.22 of [5]), $\tilde{M}$ is a $\Phi$-simplicial normal monoid. By Theorem 3.1 of [10], $\text{int}(M) = \text{int}(\tilde{M})$ is a filtered union of truncated (normal) monoids (see Definition 2.2 of [10]). Since $P$ is finitely generated, we get that $P$ is defined over $R[N]$, where $N \subset \text{int}(M)$ is a truncated monoid. By Corollary 3.7, $\text{Serre dim } R[N] \leq d$. Hence $P$ has a unimodular element. Therefore $\text{Serre dim } R[\text{int}(M)] \leq d$. □

Assumptions. In the following examples, $R$ is a ring of dimension $d$, Monoid operations are written multiplicatively and $K(M)$ denotes the group of fractions of monoid $M$.

Example 3.9. For $n > 0$, consider the monoid $M \subset \mathbb{Z}_+^r$ generated by $\{i_1 i_2 \ldots i_r | \sum i_j = n\}$. Then $M$ is a $\Phi$-simplicial normal monoid. For integers $c_i = nk_i + 1$, $k_i > 0$
and \( i = 1, \ldots, r - 1 \), consider \( \eta \in \text{Aut}_{R[t_r]}(R[t_1, \ldots, t_r]) \) defined by \( t_i \mapsto t_i + t_i ^{c_j} \) for \( i = 1, \ldots, r - 1 \).

A typical monomial in the expansion of \( \eta(t_1^{i_1} \cdots t_{r-1}^{i_{r-1}}) = (t_1 + t_1 ^{c_1})^{i_1} \cdots (t_{r-1} + t_{r-1} ^{c_{r-1}})^{i_{r-1}} \) will look like \((t_1^{i_1-l_1}t_1^{c_1}) \cdots (t_{r-1}^{i_{r-1}-l_{r-1}}t_{r-1} ^{c_{r-1}})^{i_{r-1}}\) which belong to \( M \). So \( \eta(R[M]) \subset R[M] \). Similarly, \( \eta^{-1}(R[M]) \subset R[M] \). Hence \( \eta \) restricts to an \( R \)-automorphism of \( R[M] \). Therefore \( \eta \) satisfies the property of Definition 1.3 for \( M \). It is easy to see that \( M_m = M \cap \{t_1^{s_1} \cdots t_m^{s_m} \mid s_i \geq 0 \} \) for \( 1 \leq m \leq r - 1 \) also satisfies this property. Hence \( M \in \mathcal{C}(\Phi) \). By (3.4), Serre dim \( R[M] \leq d \).

**Example 3.10.** Let \( M \) be a \( \Phi \)-simplicial monoid generated by monomials \( t_1^2, t_2^2, t_3^2, t_1 t_2, t_2 t_3 \). For integers \( c_j = 2k_j - 1 \) with \( k_j > 1 \), consider the automorphism \( \eta \in \text{Aut}_{R[t_1]}(R[t_1, t_2, t_3]) \) defined by \( t_j \mapsto t_j + t_3 ^{c_j} \) for \( j = 1, 2 \). Then it is easy to see that \( \eta \) restricts to an automorphism of \( R[M] \).

We claim that \( M \) is seminormal but not normal. For this, let

\[
z = (t_2^2)^{-1}(t_1 t_3)(t_2 t_3) = t_1 t_2 \in K(M) \setminus M, \text{ but } z^2 \in M,
\]

showing that \( M \) is not normal. For seminormality, let

\[
z = (t_2^2)^{\alpha_1}(t_2^2)^{\alpha_2}(t_3^2)^{\alpha_3}(t_1 t_3)^{\alpha_4}(t_2 t_3)^{\alpha_5} \in K(M)
\]

with \( \alpha_i \in \mathbb{Z} \) and \( z^2, z^3 \in M \).

We may assume that \( 0 \leq \alpha_4, \alpha_5 \leq 1 \). Now \( z^2 \in M \Rightarrow \alpha_1, \alpha_2 \geq 0 \) and \( 2\alpha_3 + \alpha_4 + \alpha_5 \geq 0 \). If \( \alpha_3 < 0 \), then \( \alpha_4 = \alpha_5 = 1 \) and \( \alpha_3 = -1 \). In this case, \( z^3 = (t_1^{2\alpha_1+1}t_2^{2\alpha_2+1})^3 \notin M \), a contradiction. Therefore \( \alpha_3 \geq 0 \) and \( z \in M \). Hence \( M \) is seminormal. It is easy to see that \( M \in \mathcal{C}(\Phi) \). By Theorem 3.4, Serre dim \( R[t_1^2, t_2^2, t_3^2, t_1 t_3, t_2 t_3] \leq d \).

**Remark 3.11.**

1. Let \( R \) be a ring and \( P \) a projective \( R \)-module of rank \( \geq 2 \). Let \( \bar{R} \) be the seminormalization of \( R \). It follows from arguments in (Lemma 3.1 of [2]) that \( P \otimes_R \bar{R} \) has a unimodular element if and only if \( P \) has a unimodular element.

2. Assume \( R \) is a ring of dimension \( d \) and \( M \in \mathcal{C}(\Phi) \). Let \( \bar{M} \) be the seminormalization of \( M \). If \( \bar{M} \) is in \( \mathcal{C}(\Phi) \), then Serre dim \( R[M] \leq \max\{1, d\} \), using [2] and Theorem 3.4.

3. Let \( (R, m, K) \) be a regular local ring of dimension \( d \) containing a field \( k \) such that either char \( k = 0 \) or char \( k = p \) and tr-deg \( K/F_p \geq 1 \). Let \( M \) be a seminormal monoid. Then, using Theorem 1 of [15] and Theorem 1.2 of [23], we get Serre dim \( R[M] = 0 \). If \( M \) is not seminormal, then Serre dim \( R[M] = 1 \) using [11], [2] and [23].

**Example 3.12.** For a monoid \( M, \bar{M} \) denotes the seminormalization of \( M \).

1. Let \( M \subset \mathbb{Z}^2_+ \) be a \( \Phi \)-simplicial monoid generated by \( t_1^n, t_1 t_2, t_2^2 \), where \( n \in \mathbb{N} \). We claim that \( M \) is normal. To see this, let \( z = t_1^n t_2^2 = (t_1^n)^p (t_1 t_2)^q (t_2^2)^r \in K(M) \) with \( p, q, r \in \mathbb{Z} \) such that \( z^t \in M \) for some \( t > 0 \). Then \( i, j \geq 0 \). We need to show that
Let \( z \in M \). We may assume that \( 0 \leq q < n \). Since \( i, j \geq 0 \), we get \( p, r \geq 0 \). Thus \( z \in M \) and \( M \) is normal. Hence, by (3.6), Serre dim \( R[t_1^q, t_1 t_2, t_2^q] \leq d \).

(2) The monoid \( M \subset \mathbb{Z}_+^3 \) generated by \( t_1^2, t_1 t_2, t_2^2 \) is seminormal but not normal. For this, let \( z = (t_1 t_2^2)(t_2^2)^{-1} = t_1 \in K(M) \setminus M \). Then \( z^2 \in M \) shows that \( M \) is not normal. For seminormality, let \( z = (t_1^2)^\alpha (t_1 t_2^2)^\beta (t_2^2)^\gamma \in K(M) \) with \( \alpha, \beta, \gamma \in \mathbb{Z} \) be such that \( z^2, z^3 \in M \). We may assume \( 0 \leq \beta \leq 1 \). If \( \beta = 0 \), then \( \alpha, \gamma \geq 0 \) and hence \( z \in M \). If \( \beta = 1 \), then \( z^2 \in M \) implies \( \alpha \geq 0 \) and \( \gamma + 1 \geq 0 \). If \( \gamma = -1 \), then \( z^3 = (t_1)^{6\alpha+3} \notin M \), a contradiction. Hence \( \gamma \geq 0 \), proving that \( z \in M \) and \( M \) is seminormal. It is easy to see that \( M \in \mathcal{C}(\Phi) \). Therefore, by (3.4), Serre dim \( R[t_1^2, t_1 t_2, t_2^2] \leq d \).

(3) Let \( M \) be a monoid generated by \( (t_1^2, t_1 t_2, t_2^2) \), where \( j \geq 3 \). Then \( M \) is not seminormal. For this, if \( z = (t_1 t_2^2)(t_2^2)^{-1} = t_1 t_2^{j-2} \in K(M) \setminus M \), then \( z^2 = t_1^2 t_2^{2(j-2)} \) and \( z^3 = (t_1^2)(t_1 t_2^2)(t_2^2)^{j-6} \) are in \( M \), which shows that \( M \) is not seminormal.

If \( j = 3 \), then observe that \( t_1 t_2 \) belongs to \( \bar{M} \). Since the monoid generated by \( t_1^2, t_1 t_2, t_2^2 \) is normal, we get that \( \bar{M} \) is generated by \( t_1^2, t_1 t_2, t_2^2 \). Hence Serre dim \( R[\bar{M}] \leq d \) by (1) above. Observe that if \( j \) is odd, then \( \bar{M} = (t_1^2, t_1 t_2, t_2^2) \) and if \( j \) is even, then \( \bar{M} = (t_1^2, t_1 t_2, t_2^2) \). So Serre dim \( R[\bar{M}] \leq d \) by Example 3.12(1), (2).

In both cases, applying Remark 3.11(1), we get Serre dim \( R[M] \leq \max \{1, d\} \).

(4) Let \( M \) be a monoid generated by \( (t_1^3, t_1 t_2^2, t_2^3) \). Then \( M \) is not seminormal. For this, let \( z = (t_1 t_2^2)^2 t_2^{-3} \in K(M) \setminus M \). Then \( z^2 = t_1^2 (t_1 t_2^2) \in M \) and \( z^3 = t_1^3 t_2^3 \in M \). Hence seminormalization of \( M \) is \( \bar{M} = (t_1^3, t_1^2 t_2, t_2^3, t_1^2 t_2^3) \). By (3.9), Serre dim \( R[\bar{M}] \leq d \). Therefore, applying Remark 3.11(1), we get Serre dim \( R[M] \leq \max \{1, d\} \). \( \square \)

### 3.2 Monoid algebras over 1-dimensional rings

The following result proves Theorem 1.6(i).

**Theorem 3.13.** Let \( R \) be a ring of dimension 1 and \( M \) a c-divisible monoid. If \( P \) is a projective \( R[M] \)-module of rank \( r \geq 3 \), then \( P \cong \wedge^r P \oplus R[M] \wedge^{r-1} \).

**Proof.** If \( R \) is normal, then we are done [23]. Assume that \( R \) is not normal.

**Case 1.** Assume \( R \) has finite normalization. Let \( \tilde{R} \) be the normalization of \( R \) and \( C \) the conductor ideal of the extension \( R \subset \tilde{R} \). Then \( hC = 1 \). Hence \( R/C \) and \( \tilde{R}/C \) are zero dimensional rings. Consider the following fiber product diagram

\[
\begin{array}{ccc}
R[M] & \rightarrow & \tilde{R}[M] \\
\downarrow & & \downarrow \\
(R/C)[M] & \rightarrow & (\tilde{R}/C)[M]
\end{array}
\]

If \( P' = \wedge^r P \oplus R[M] \wedge^{r-1} \), then by [23], \( P \otimes \tilde{R}[M] \cong \wedge^r (P \otimes \tilde{R}[M]) \oplus \tilde{R}[M] \wedge^{r-1} \cong P' \oplus \tilde{R}[M] \). By Gubeladze [8], \( P/C P \) and \( P'/C P' \) are free \( (R/C)[M] \)-modules. Further, \( SL_r((\tilde{R}/C))[M] = E_r((\tilde{R}/C))[M] \) for \( r \geq 3 \) [9]. Now using standard arguments of fiber product diagram, we get \( P \cong P' \).
Case 2. Now \( R \) need not have finite normalization. We may assume that \( R \) is a reduced ring with connected spectrum. Let \( S \) be the set of all non-zero divisors of \( R \). By [8], \( S^{-1} P \) is a free \( S^{-1} R[M] \)-module. Choose \( s \in S \) such that \( P_s \) is a free \( R_s[M] \)-module.

Now we follow the arguments of Theorem 4.1 of [17]. Let \( \hat{R} \) denote the \( s \)-adic completion \( R \). Then \( \hat{R}_{\text{red}} \) has a finite normalization. Consider the following fiber product diagram

\[
\begin{array}{c}
R[M] \longrightarrow \hat{R}[M] \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
R_s[M] \longrightarrow \hat{R}_s[M]
\end{array}
\]

Since \( \hat{R}_s \) is a zero dimensional ring, by [9], \( \text{SL}_r(\hat{R}_s[M]) = E_r(\hat{R}_s[M]) \) for \( r \geq 3 \). If \( P' = \wedge^r P \oplus R[M]'^{-1} \), then \( P_s \) and \( P'_s \) are free \( R_s[M] \)-modules and by Case 1, \( P \otimes \hat{R}[M] \cong P' \otimes \hat{R}[M] \). By (2.5), \( P \cong P' \). This completes the proof. 

The following result is due to Lemma 7.1 and Remark of [12].

Lemma 3.14. Let \( R \) be a 1-dimensional uni-branched affine algebra over an algebraically closed field, \( \hat{R} \) the normalization of \( R \) and \( C \) the conductor ideal of the extension \( R \subset \hat{R} \). Then \( \hat{R}/C = R/C + a_1 R/C + \cdots + a_m R/C \), where \( a_i \in \sqrt{C} \) the radical ideal of \( C \) in \( \hat{R} \).

Lemma 3.15. Let \( R \) be a 1-dimensional ring, \( \hat{R} \) the normalization of \( R \) and \( C \) the conductor ideal of the extension \( R \subset \hat{R} \). Assume \( \hat{R}/C = R/C + a_1 R/C + \cdots + a_m R/C \), where \( a_i \in \sqrt{C} \) the radical ideal of \( C \) in \( \hat{R} \). Let \( M \) be a monoid and write \( A = \hat{R}/C \).

(i) If \( \sigma \in \text{SL}_n(A[M]) \), then \( \sigma = \sigma_1 \sigma_2 \), where \( \sigma_1 \in \text{SL}_n((R/C)[M]) \) and \( \sigma_2 \in E_n(A[M]) \).

(ii) If \( P \) is a projective \( R[M] \)-module of rank \( r \), then \( P \cong \wedge^r P \oplus R[M]'^{-1} \).

Proof.

(i) Let \( \sigma = (b_{ij}) \in \text{SL}_n(A[M]) \). Write \( b_{ij} = (b_{ij})_0 + (b_{ij})_1 a_1 + \cdots + (b_{ij})_m a_m \), where \( (b_{ij})_l \in (R/C)[M] \). If \( \alpha = ((b_{ij})_0) \), then \( \det(\sigma) = 1 = \det(\alpha) + c \), where \( c \in (\sqrt{C}/C)[M] \). Since \( c \in (R/C)[M] \) is nilpotent, \( \det(\alpha) \) is a unit in \( (R/C)[M] \). Let \( \beta = \text{diagonal}(1/(1-c), 1, \ldots, 1) \in \text{GL}_n((R/C)[M]) \) and \( \sigma_1 = \alpha \beta \in \text{SL}_n((R/C)[M]) \).

Note that \( \sigma_1^{-1} \sigma = \beta^{-1} \alpha^{-1} \sigma = \beta^{-1} 1/(1-c) \tilde{\alpha} \sigma \), where \( \tilde{\alpha} = ((\tilde{b}_{ij})_0) \), \( (\tilde{b}_{ij})_0 \) are minors of \( (b_{ij})_0 \).

\[
\sigma_2 := \sigma_1^{-1} \sigma = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & \frac{1}{1-c} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{1-c}
\end{bmatrix}
\begin{bmatrix}
1 + c_{11} & c_{12} & \cdots & c_{1n} \\
c_{21} & 1 + c_{22} & \cdots & c_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
c_{n1} & c_{n2} & \cdots & 1 + c_{nn}
\end{bmatrix},
\]

where \( c_{ij} \in (\sqrt{C}/C)[M] \).
Note that $\sigma_2 \in \text{SL}_n(A[M])$ and $\sigma_2 = \text{Id}$ modulo the nilpotent ideal of $A[M]$. Hence $\sigma_2 \in E_n(A[M])$. Thus we get $\sigma = \sigma_1 \sigma_2$ with the desired properties.

(ii) Follow the proof of Theorem 3.13 and use Lemma 3.15(i) to get the result. $\square$

Now we prove Theorem 1.6(ii) which follows from Lemmas 3.14 and 3.15.

**Theorem 3.16.** Let $R$ be a 1-dimensional uni-branched affine algebra over an algebraically closed field and $M$ a monoid. If $P$ is a projective $R[M]$-module of rank $r$, then $P \cong \wedge^r P \oplus R[M]^{-1}$.

4. Applications

Let $R$ be a ring of dimension $d$ and $Q$ a finitely generated $R$-module. Let $\mu(Q)$ denote the minimum number of generators of $Q$. By Forster [7] and Swan [22], $\mu(Q) \leq \max \{ \mu(Q_p) + \dim(R/p) \mid p \in \text{Spec}(R), Q_p \neq 0 \}$. In particular, if $P$ is a projective $R$-module of rank $r$, then $\mu(P) \leq r + d$.

The following result is well known.

**Theorem 4.1.** Let $A$ be a ring such that Serre dim $A \leq d$. Assume $A^m$ is cancellative for $m \geq d + 1$. If $P$ is a projective $A$-module of rank $r \geq d + 1$, then $\mu(P) \leq r + d$.

**Proof.** Assume $\mu(P) = n > r + d$. Consider a surjection $\phi : A^n \twoheadrightarrow P$ with $Q = \ker(\phi)$. Then $A^n \cong P \oplus Q$. Since $Q$ is a projective $A$-module of rank $\geq d + 1$, $Q$ has a unimodular element $q$. Since $\phi(q) = 0$, $\phi$ induces a surjection $\bar{\phi} : A^n/qA^n \twoheadrightarrow P$. Since $n - 1 > d$, $A^{n-1}$ is cancellative. Hence $A^{n-1} \cong A^n/qA$ and $P$ is generated by $n - 1$ elements, a contradiction. $\square$

The following result is immediate from Theorem 4.1, Theorem 3.4, Corollary 3.6 and [6].

**COROLLARY 4.2**

Let $R$ be a ring of dimension $d$, $M$ a monoid and $P$ a projective $R[M]$-module of rank $r > d$. Then

(i) If $M \in \mathcal{C}(\Phi)$, then $\mu(P) \leq r + d$.

(ii) If $M \subset \mathbb{Z}_+^2$ is a normal monoid of rank 2, then $\mu(P) \leq r + d$.

Let $M$ be a $c$-divisible monoid, $R$ a ring of dimension $d$ and $n \geq \max\{2, d + 1\}$. Then Schaubhüser [19] proved that $E_{n+1}(R[M])$ acts transitively on $\text{Um}_{n+1}(R[M])$. Using Schaubhüser’s result and arguments of Theorem 4.4 of [6], we get that if $P$ is a projective $R[M]$-module of rank $n$, then $E(R[M] \oplus P)$ acts transitively on $\text{Um}(R[M] \oplus P)$. Therefore the following result is immediate from Theorems 4.1 and 3.13.

**COROLLARY 4.3**

Let $R$ be a ring of dimension 1, $M$ a $c$-divisible monoid and $P$ a projective $R[M]$-module of rank $r \geq 3$. Then $\mu(P) \leq r + 1$. 
Acknowledgement

The authors would like to thank the referee for his/her critical remark. The second author would like to thank CSIR, India for a fellowship.

References

[1] Anderson D F, Projective modules over subrings of $k[X, Y]$ generated by monomials, Pacific J. Math. 79 (1978) 5–17
[2] Bhatwadekar S M, Inversion of monic polynomials and existence of unimodular elements (II), Math. Z. 200 (1989) 233–238
[3] Bhatwadekar S M, Lindel H and Rao R A, The Bass-Murthy question: Serre dimension of Laurent polynomial extensions, Invent. Math. 81 (1985) 189–203
[4] Bhatwadekar S M and Roy A, Some theorems about projective modules over polynomial rings, J. Algebra 86 (1984) 150–158
[5] Bruns W and Gubeladze J, Polytopes, Rings and $K$-Theory, Springer Monographs in Mathematics (2009)
[6] Dhorajia A M and Keshari M K, A note on cancellation of projective modules, J. Pure Appl. Algebra 216 (2012) 126–129
[7] Forster O, Über die Anzahl der Erzeugenden eines Ideals in einem Noetherschen Ring, Math. Z. 84 (1964) 80–87
[8] Gubeladze J, Anderson’s conjecture and the maximal class of monoid over which projective modules are free, Math. USSR-Sb 63 (1988) 165–188
[9] Gubeladze J, Classical algebraic $K$-theory of monoid algebras, Lect. Notes Math. 1437 (1990) (Springer) pp. 36–94
[10] Gubeladze J, The elementary action on unimodular rows over a monoid ring, J. Algebra 148 (1992) 135–161
[11] Gubeladze J, $K$-theory of affine toric varieties, Homology Homotopy Appl. 1 (1999) 135–145
[12] Kang M C, Projective modules over some polynomial rings, J. Algebra 59 (1979) 65–76
[13] Lindel H, Unimodular elements in projective modules, J. Algebra 172 (1995) no-2, 301–319
[14] Plumstead B, The conjectures of Eisenbud and Evans, Amer. J. Math 105 (1983) 1417–1433
[15] Popescu D, On a question of Quillen, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) 45(93) (2002) no. 3–4, 209–212
[16] Quillen D, Projective modules over polynomial rings, Invent. Math. 36 (1976) 167–171
[17] Roy A, Application of patching diagrams to some questions about projective modules, J. Pure Appl. Algebra 24 (1982) no. 3, 313–319
[18] Sarwar H P, Some results about projective modules over monoid algebras, Comm. Algebra 44 (2016) 2256–2263
[19] Schabhüser G, Cancellation properties of projective modules over monoid rings (1991) (Münster: Universität Münster, Mathematisches Institut) iv+86 pp
[20] Serre J P, Sur les modules projectifs, Sem. Dubreil-Pisot 14 (1960–61) 1–16
[21] Suslin A A, Projective modules over polynomial rings are free, Sov. Math. Dokl. 17 (1976) 1160–1164
[22] Swan R G, The number of generators of a module, Math. Z. 102 (1967) 318–322
[23] Swan R G, Gubeladze proof of Anderson’s conjecture, Contemp. Math. 124 (1992) 215–250

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