CLASS NUMBER RELATIONS ARISING FROM INTERSECTIONS OF
SHIMURA CURVES AND HUMBERT SURFACES

JIA-WEI GUO AND YIFAN YANG

ABSTRACT. By considering the intersections of Shimura curves and Humbert surfaces
on the Siegel modular threefold, we obtain new class number relations. The result is a
higher-dimensional analogue of the classical Hurwitz-Kronecker class number relation.

1. INTRODUCTION

As arithmetic and algebraic objects, modular forms and their generalizations, such
as harmonic Maass forms, weakly holomorphic modular forms, etc., play a significant
role in number theory. In many instances, class numbers emerge naturally as Fourier
coefficients of modular forms. For example, suppose that \( \theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi inz} \) is the
classical Jacobi theta function. Then the coefficients of the weight 3/2 modular form
\( \theta(z)^3 = \sum_{n} r_3(n) e^{2\pi inz} \) can be represented by certain sums of Hurwitz class numbers
\( H(m) \), the counting function of \( \text{SL}(2,\mathbb{Z}) \)-equivalence class of positive definite binary qua-
dratic forms of discriminant \( -m \), weighted by the sizes of automorphism groups of qua-
dratic forms. Moreover, Zagier \cite{17} showed that with \( H(0) \) defined to be \( -1/12 \), the
function \( \sum_{n \in \mathbb{Z}} H(n) e^{2\pi inz} \) is the holomorphic part of a harmonic Maass form of weight
3/2. Around the same time, motivated by Dirichlet’s class number formula, Cohen \cite{1}
generalized the definition of the Hurwitz class numbers and define \( H(r,n) \) via values of
Dirichlet L-function for quadratic character \( (n \cdot \cdot) \). He showed that each power series of the
form \( \sum_{n=0}^\infty H(r,n) e^{2\pi inz} \) is a modular form of weight \( r + 1/2 \) on \( \Gamma_0(4) \), which is referred
to as Cohen’s Eisenstein series of weight \( r + 1/2 \).

On the other hand, because CM-points on a modular curve or a Shimura curve \( X \) cor-
respond to abelian varieties with extra endomorphisms, when we embed \( X \) into a modular
variety of higher dimension, CM-points often lie on intersections of \( X \) with other special
cycles of the modular variety. By analyzing what CM-point lie on the intersection and
comparing with the intersection number, one obtain relations among class numbers. For
example, the classical Hurwitz-Kronecker class number relation

\[
\sum_{x \in \mathbb{Z}} H(4n - x^2) + \sum_{dd'=n, d, d'>0} \min(d, d') = \begin{cases} 
-1/12, & \text{if } n = 0, \\
2\sum_{d|n} d, & \text{if } n > 0,
\end{cases}
\]

(1)
can be interpreted as a relation arising from the intersection of two certain curves on
\( X_0(1) \times X_0(1) \) defined by modular correspondence (see \( 3 \)). Note that the generating
function for the right-hand side of (1) is the Eisenstein series \( -E_2(z)/12 \) of weight 2,
which is the holomorphic part of a harmonic Maass form. As another important example,
Hirzebruch and Zagier \cite{7} considered a certain series of curves \( T_1, T_2, \ldots \) that are images

The authors would like to thank Professor Fernando Rodriguez Villegas for bringing this problem to their
attention and for sharing his private notes with them. The authors would also like to thank Professor Takao
Yamazaki and Professor Takuya Yamauchi for many fruitful discussions. The authors are partially supported by
Grant 102-2115-M-009-001-MY4 of the Ministry of Science and Technology, Taiwan (R.O.C.).
of modular curves or Shimura curves on the Hilbert modular surface $SL(2, \mathfrak{o}) \backslash \mathbb{H}^2$, where $\mathfrak{o}$ is the ring of integers in $\mathbb{Q}(\sqrt{D})$, $p \equiv 1 \mod 4$, and obtained formulas involving Hurwitz class numbers for the intersection numbers of $T_m$ and $T_n$. They also showed that if we fix $m$, then the generating series for the intersection numbers is a modular form of weight 2 and Nebentype. This results in identities relating sums of class numbers to Fourier coefficients of a modular form of weight 2 and Nebentype.

Note that in the two aforementioned examples, the generating series for intersection numbers are a modular form and the holomorphic part of a harmonic Maass form, respectively. These arithmetic and geometric phenomenon of modular forms has been further explored and generalized by Kudla and Milson, from the aspect of automorphic representations, as correspondence between spaces of higher genus modular forms and intersections of special cycles on the quotients of symmetric spaces for orthogonal groups and the unitary groups $\mathfrak{U}(1)$.

In this paper, we shall consider the case of Shimura curves on the Siegel modular threefold $\mathcal{A}_2 := Sp(4, \mathbb{Z}) \backslash \mathbb{H}_2$. By considering the intersections of a Shimura curve with Humbert surfaces on $\mathcal{A}_2$, we will obtain identities relating sums of Hurwitz numbers to the Fourier coefficients of Cohen’s Eisenstein series of weight $5/2$. To state our theorem, let us first recall some definitions about quadratic forms. Let $D_0$ be a squarefree integer, and $D_0 = p_1 \ldots p_k$ be its prime factorization. Here we are concerned with positive definite binary quadratic forms $Q$ of discriminants $-16D_0$ such that all integers represented by $Q$ are congruent to 0 or 1 modulo 4. Such a quadratic form is either primitive or of the form $4Q’$ for some quadratic form $Q’$ of discriminant $-D_0$, and the latter case occurs only when $D_0 \equiv 3 \mod 4$. Let $C_{-16D_0}$ and $C_{-D_0}$ (in the case $D_0 \equiv 3 \mod 4$) be the class group of primitive binary quadratic forms of discriminant $-16D_0$ and $-D_0$, respectively. Define characters $\chi_{-4}$ and $\chi_{p_j}, j = 1, \ldots, k$, of $C_{-16D_0}$ by

$$\chi_{-4} : Q \mapsto \left( \frac{-4}{a} \right), \quad \chi_{p_j} : Q \mapsto \begin{cases} \left( \frac{a}{p_j} \right), & \text{if } p_j \text{ is odd}, \\ \left( \frac{8}{a} \right), & \text{if } p_j = 2, \end{cases}$$

where $a$ is any positive integer represented by $Q$ satisfying $(a, 2D_0) = 1$. These characters generate the group of genus characters of $C_{-16D_0}$ with a single relation that either $\chi_{-4}\chi_{p_1} \ldots \chi_{p_k}$ or $\chi_{p_1} \ldots \chi_{p_k}$ is the trivial character depending on whether the odd part of $D_0$ is congruent to 1 or 3 modulo 4 (see [2, Theorem 3.15]). The condition that all integers represented by $Q$ are congruent to 0 or 1 modulo 4 means that $\chi_{-4}(Q) = 1$ so that the number of $p_j$ such that $\chi_{p_j}(Q) = -1$ is even. Thus, we may associate to $Q$ an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant

$$D = D_Q = \prod_{p_j : \chi_{p_j}(Q) = -1} p_j,$$

Set also

$$N = N_Q = \prod_{p_j : \chi_{p_j}(Q) = 1} p_j.$$

Let $\mathcal{O}$ be an Eichler order of level $N$ in the quaternion algebra of discriminant $D$ over $\mathbb{Q}$, and let $X_0^D(N)$ be the Shimura curve associated to $\mathcal{O}$.

Similarly, we may define $\chi_{p_j}$ on $C_{-D_0}$ and the product $\chi_{p_1} \ldots \chi_{p_k}$ is the trivial character. Thus, for a quadratic form $Q$ of discriminant $-16D_0$ of the form $4Q’$, we may also associate to $Q$ a Shimura curve $X_0^D(N)$ with $D$ and $N$ given by (3) and (4), respectively.
Now for a negative discriminant $d$, let $h_{D,N}(d)$ be the number of CM-points of discriminant $d$ on $X_D^0(N)$. Write $d$ as $d = f^2d_0$, where $d_0$ is a fundamental discriminant, and define

$$H_{D,N}([d]) = \frac{1}{2\omega_{D_0}(d)} \sum_{r|n} \frac{1}{e_{r^2d_0}} h_{D,N}(r^2d_0),$$

where

$$\omega_{D_0}(d) = \#\{p \text{ prime} : p|D_0, \ p \nmid d\}$$

and

$$e_{r^2d_0} = \begin{cases} 3, & \text{if } r^2d_0 = -3, \\ 2, & \text{if } r^2d_0 = -4, \\ 1, & \text{else.}
\end{cases}$$

(In accordance with the usual notation for Hurwitz class numbers, $H_{D,N}$ are defined for nonnegative integers.) Note that we have the formula

$$h_{D,N}(r^2d_0) = h(r^2d_0) \prod_{p|D,N} m(\mathcal{O}, r^2d_0, p),$$

where

$$m(\mathcal{O}, r^2d_0, p) = \begin{cases} 1 - \left(\frac{d_0}{p}\right), & \text{if } p|D \text{ and } p \nmid r, \\ 0, & \text{if } p|D \text{ and } p|r, \\ 1 + \left(\frac{d_0}{p}\right), & \text{if } p|N \text{ and } p \nmid r, \\ 2, & \text{if } p|N \text{ and } p|r,
\end{cases}$$

is the number of local optimal embeddings of discriminant $r^2d_0$ into $\mathcal{O}_p := \mathcal{O} \otimes \mathbb{Z}_p$. Define also that

$$H_{D,N}(0) = \frac{1}{2} \text{Vol}(X_D^0(N)) = -\frac{D_0}{12} \prod_{p|D} \left(1 - \frac{1}{p}\right) \prod_{p|N} \left(1 + \frac{1}{p}\right)$$

(noticing that when $D = N = 1$, $H_{1,1}(0) = -1/12 = H(0)$), and $H_{D,N}(x) = 0$ if $-x$ is not 0 or a negative discriminant. With notations given as above, our main result can now be stated as follows.

**Theorem 1.** Assume that $D_0$ is a positive squarefree integer. Let $Q(x, y) = ax^2 + 2bxy + cy^2$ be a positive definite quadratic form of discriminant $-16D_0$ such that every integer represented by $Q$ is congruent to 0 or 1 modulo 4. Let $D$ and $N$ be defined by (3) and (4), respectively. Assume that $D > 1$. Then for all positive integers $n$ congruent to 0 or 1 modulo 4, we have the class number relations

$$\sum_{u, v \in \mathbb{Z}, \ u \equiv n \mod 2, \ v \equiv cn \mod 2} H_{D,N} \left(D_{0n} - \frac{Q(r, u)}{4}\right) = a_n H_{D,N}(0),$$

where $a_n$ are the coefficients in

$$\sum_{n=0}^{\infty} a_n q^n = \mathcal{H}(z) := \theta(z)^5 - 20\theta(z)\frac{\eta(4z)^8}{\eta(2z)^4} = 1 - 10q - 70q^4 - 48q^5 + \cdots,$$

and $H_{D,N}(m)$ and $H_{D,N}(0)$ are defined by (5) and (6), respectively.
Remark 1. We remark that \( H(z)/120 \) is Cohen’s Eisenstein series of weight 5/2 defined in [1].

Note that when \( D_0 \) is congruent to 3 modulo 4 and \( Q \) is of the form \( 4Q' \), the conditions \( u \equiv an, v \equiv cn \mod 2 \) imply that \( u \) and \( v \) are both even. Thus, we may write the identity as

\[
\sum_{u,v \in \mathbb{Z}} H_{D,N}(D_0u - 4Q'(u,v)) = a_n H_{D,N}(0).
\]

We now explain the geometric meaning of these identities in more details. Let \( O \) be an Eichler order of level \( N \) in the indefinite quaternion algebra \( B \) of discriminant \( D \) over \( \mathbb{Q} \). Let \( Q_{D,N} \) be the set of all points in \( \mathcal{A}_2 \) whose corresponding principally polarized abelian surfaces have quaternionic multiplication by \( O \) such that the Rosati involutions coincide with a positive involution of the form \( \alpha \mapsto \mu^{-1} \overline{\alpha} \mu \) on \( O \) for some \( \mu \in O \) with \( \mu^2 + DN = 0 \), where \( \overline{\alpha} \) denotes the conjugate of \( \alpha \) in \( O \). It can be shown that \( Q_{D,N} \) consists of a finite number \( r_{D,N} \) of irreducible components, each of which is the image of the Shimura curve \( X_0^D(N) \) under some natural map (see [14] for the case \( N = 1 \)). In [12, 13], Rotger gave upper and lower bounds for the number \( r_{D,1} \) and in some cases an exact formula for \( r_{D,1} \). To give an exact formula for \( r_{D,1} \), Lin and Yang [10] showed that each irreducible component in \( Q_{D,1} \) can be associated with a positive definite quadratic form \( Q \) of discriminant \(-16D\) such that every positive integer \( a \) represented by \( Q \) satisfies

1. \( a \equiv 0, 1 \mod 4 \), and
2. \( \left( -\frac{D,a}{Q} \right) \simeq B \).

Furthermore, they showed that the irreducible components in \( Q_{D,1} \) are in one-to-one correspondence with \( \text{GL}(2, \mathbb{Z}) \)-equivalence classes of such quadratic forms. Hence the problem of determining \( r_{D,1} \) reduces to that of counting certain \( \text{GL}(2, \mathbb{Z}) \)-equivalence classes of quadratic forms. Note that the work of Rotger [12, 13, 14] and Lin and Yang [10] can be easily extended to the case \( N > 1 \). Moreover, the condition \( \left( -\frac{D,a}{Q} \right) \simeq B \) is equivalent to that \( \chi_{p_j}(Q) = -1 \) for \( p_j | D \) and \( \chi_{p_j}(Q) = 1 \) for \( p_j | N \). Thus, each quadratic form in Theorem 1 corresponds to a unique Shimura curve \( \mathcal{X} \) in \( Q_{D,N} \).

Now recall that the Humbert surface \( H_n \) of discriminant \( n \) is defined to be the locus in \( \mathcal{A}_2 \) of principally polarized abelian surfaces having a symmetric (Rosati invariant) endomorphism of discriminant \( n \). The Humbert surfaces are hypersurfaces in Siegel’s modular threefold. For each positive integer \( n \) congruent to 0 or 1 modulo 4, we defined the Humbert divisor by

\[
G_n := \sum_{k^2 | n} v_{n/k^2} H_{n/k^2},
\]

where the sum runs over any positive integer \( k \) such that \( n/k^2 \) is an integer congruent to 0 or 1 modulo 4, \( v_4 = 1/2 \), and \( v_{n/k^2} = 1 \) for other cases. Denote by \( \mathcal{A}_2 \) the Satake compatification of \( \mathcal{A}_2 \). A fundamental result of van der Geer [16] Theorem 8.1 and Corollary 8.2] showed that the classes \( G_n \) span a 1-dimensional subspace of the 4th homology group of \( \mathcal{A}_2 \) with coefficients in \( \mathbb{Q} \), and \( G_n \) is the divisor of some Siegel modular form of weight \(-a_n/2\), where \( a_n \) are the integers given in Theorem 1. Thus, if we let \( I(\mathcal{X}, G_n) \) denote the intersection number of a Shimura curve \( \mathcal{X} \) in \( Q_{D,N} \) with \( G_n \), then there exists a constant \( c \) such that

\[
I(\mathcal{X}, G_n) = ca_n
\]
for all \( n \). In other words, the generating series
\[
\sum_{n=0}^{\infty} I(\mathfrak{X}, G_n)e^{2\pi i nz}
\]
is a modular form of weight 5/2 on \( \Gamma_0(4) \), agreeing with the general results of Kudla and Millson [9]. (Here the constant \( I(\mathfrak{X}, G_0) \) of the generating series should be interpreted as the volume of \( \mathfrak{X} \).) Using the fact that the restriction of a Siegel modular form of weight \( k \) to \( \mathfrak{X} \) is a modular form of weight \( 2k \) on \( X_0^D(N) \), we can determine \( c \) in terms of the volume of \( X_0^D(N) \).

On the other hand, the 0-dimensional components of the intersection of \( \mathfrak{X} \) and \( G_n \), being moduli points corresponding to abelian surfaces with endomorphism algebras strictly larger than the quaternion algebra of discriminant \( D \) over \( \mathbb{Q} \), must be CM-points on \( \mathfrak{X} \). By carefully analyzing which CM-points lie on the intersections and determining their multiplicities, we obtain the class number relations in our main theorem.

**Remark 2.** Note that the relations (7) hold only for the case \( D > 1 \). When \( D = 1 \), the Shimura curve \( X_0^1(N) \) is the usual modular curve and we also need to take the contribution from the cusps into account. In [6, Satz 4], Hermann obtained a formula for the contribution from the cusps in the case the Humbert surface is \( H_1 \) and the quadratic form is of the form \( 4Q' \) for some quadratic form \( Q' \) of discriminant \(-p\), where \( p \) is a prime congruent to 3 modulo 4. His proof used the fact that the Siegel modular form with divisor \( H_1 \) is the product of the ten theta functions of even characteristics. However, because we do not have such knowledge about Siegel modular forms with divisor \( H_1 \) for general \( n \) (except for some small \( n \), see [4]), it is not easy to extend Hermann’s formula to the case of general \( n \) and \( D_0 \).

2. Shimura curves on Siegel’s modular threefold

In this section we shall introduce and review some properties of Shimura curves on Siegel’s modular threefold. The materials are taken from [10, 12, 13, 14].

2.1. Abelian surfaces with quaternionic multiplication. Let \( \mathcal{O} \) be an Eichler order of squarefree level \( N \) in an indefinite quaternion algebra \( B \) of discriminant \( D \) over \( \mathbb{Q} \). Fix an embedding
\[
\phi : B \hookrightarrow M(2, \mathbb{R}).
\]
As a complex torus, an abelian surface with quaternionic multiplication by \( \mathcal{O} \) is necessarily isomorphic to \( A_z = \mathbb{C}^2/\Lambda_z \) for some \( z \in \mathbb{H} \), where \( \Lambda_z = \phi(\mathcal{O})v_z \) with \( v_z = (\frac{z}{1}) \). Such a complex torus \( A_z \) can always be principally polarized. Namely, choose a pure quaternion \( \mu \) in \( \mathcal{O} \) such that \( \mu^2 + DN = 0 \) and the \((2,1)\)-entry of \( \phi(\mu) \) is positive. We define \( E_\mu : \Lambda_z \times \Lambda_z \to \mathbb{Z} \) by
\[
E_\mu(\phi(\alpha)v_z, \phi(\beta)v_z) = \text{tr}(\mu^{-1}\alpha \overline{\beta})
\]
and extend the definition of \( E_\mu \) \( \mathbb{R} \)-bilinearly to \( \mathbb{C}^2 \). It can be shown that \( E_\mu \) is a Riemann form on \( \Lambda_z \) and the corresponding polarization \( \rho_\mu \) of \( A_z \) is principal. Conversely, if \( A_z \) is a simple abelian surface, then all principal polarizations of \( A_z \) arise in this way.

Note that the polarization \( \rho_\mu \) has the property that the restriction of the Rosati involution with respect to \( \rho_\mu \) to \( \mathcal{O} \) coincides with the involution \( \alpha \mapsto \mu^{-1}\overline{\alpha} \mu \), where \( \overline{\alpha} \) denotes the quaternionic conjugate of \( \alpha \). Thus, if we let \( Q_{D,N} \) denote the set of moduli points on \( \mathfrak{A}_2 \)

---

1Fernando Villegas has some partial result in his private notes.
whose corresponding principally polarized abelian surfaces have quaternionic multiplication by \( \mathcal{O} \) such that the Rosati involution restricted to \( \mathcal{O} \) coincides with \( \alpha \mapsto \eta \phi(\eta)^{-1} \) for some \( \eta \in \mathcal{O} \) satisfying \( \eta^2 + DN = 0 \), then

\[
\mathcal{Q}_{D,N} = \bigcup_{\mu} \mathcal{X}_\mu,
\]

where

\[
\mathcal{X}_\mu = \{(A_\mu, \rho_\mu) : z \in \mathcal{X}_0^D(N)\},
\]

and \( \mu \) are elements of \( \mathcal{O} \) such that \( \mu^2 + DN = 0 \) and the \((2, 1)\)-entry of \( \phi(\mu) \) is positive.

In [10], in order to determine the exact number of Shimura curves in \( \mathcal{Q}_{D,1} \), Lin and Yang showed that there is a one-to-one correspondence between Shimura curves in \( \mathcal{Q}_{D,1} \) and \( \text{GL}(2, \mathbb{Z}) \)-equivalence classes of positive definite binary quadratic forms of discriminant \(-16D \) such that all integers \( a \) represented by the quadratic form satisfy \( a \equiv 0, 1 \mod 4 \) and \( \left(\frac{-D,a}{\mathbb{Q}}\right) \simeq B \). These quadratic forms arise from singular relations satisfied by Shimura curve \( \mathcal{X}_\mu \), which we now review in the next section.

2.2. Quadratic forms associated to Shimura curves. In this section, we shall review the definition of quadratic forms associated to Shimura curves in \( \mathcal{Q}_{D,N} \). These quadratic forms appeared in the works of Hashimoto [5] and Runge [15] and were studied in more details in [10].

Let \((A, \rho)\) be a principally polarized abelian surface over \( \mathbb{C} \). The polarization \( \rho \) induces an involution \( f \mapsto f^\dagger \), called the Rosati involution, on the endomorphism ring of \( A \). In terms of the Riemann form \( E \) associated to \( \rho \), this involution is characterized by the property that

\[
E(f \omega_1, \omega_2) = E(\omega_1, f^\dagger \omega_2)
\]

for all periods \( \omega_1 \) and \( \omega_2 \) of \( A \). An endomorphism \( f \) of \( A \) is called symmetric or Rosati invariant if \( f = f^\dagger \). The symmetric endomorphisms can be described in terms of the period matrix of \( A \) as follows.

Let \( \tau = (\tau_1, \tau_2) \in \mathbb{H}_2 \) be a normalized period matrix for \((A, \rho)\). An endomorphism \( f \) of \( A \) can be represented by a matrix \( R_f \in M(2, \mathbb{C}) \) such that

\[
R_f(\tau_1, \tau_2) = (\tau_1, \tau_2)M_f
\]

for some \( M_f \in M(4, \mathbb{Z}) \). Then \( f \) is symmetric if and only if \( M_f \) satisfies

\[
M_f \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix} M_f,
\]

i.e., if and only if \( M_f \) is of the form

\[
M_f = \begin{pmatrix} a_1 & a_2 & 0 & b \\ a_3 & a_4 & -b & 0 \\ 0 & c & a_1 & a_3 \\ -c & 0 & a_2 & a_4 \end{pmatrix}
\]

for some integers \( a_j, b \) and \( c \). Then (9) implies that

\[
a_2\tau_1 + (a_4 - a_1)\tau_2 - a_3\tau_3 + b(\tau_2^2 - \tau_1\tau_3) + c = 0.
\]

Note that since \( f \in \mathbb{Z} \) if and only if \( a_2 = a_3 = b = c = 0 \) and \( a_1 = a_4 \), the relation is nontrivial if and only if \( f \notin \mathbb{Z} \).

Conversely, if \( \tau \in \mathbb{H}_2 \) satisfies

\[
a\tau_1 + b\tau_2 + c\tau_3 + d(\tau_2^2 - \tau_1\tau_3) + e = 0, \quad a, b, c, d, e \in \mathbb{Z},
\]

then (10) implies that

\[
a_2\tau_1 + (a_4 - a_1)\tau_2 - a_3\tau_3 + b(\tau_2^2 - \tau_1\tau_3) + c = 0.
\]

Therefore, the Rosati involution on the endomorphism ring of \( A \) is characterized by the property that

\[
E(f \omega_1, \omega_2) = E(\omega_1, f^\dagger \omega_2)
\]

for all periods \( \omega_1 \) and \( \omega_2 \) of \( A \).
then its associated abelian surface has an endomorphism \( f \) with

\[
M_f = \begin{pmatrix}
0 & a & 0 & d \\
-c & b & -d & 0 \\
0 & e & 0 & -c \\
-e & 0 & a & b
\end{pmatrix}.
\]

A relation of this type \( (10) \) for \( \ell = (a, b, c, d, e) \in \mathbb{Z}^5 \) is called a singular relation satisfied by \( \tau \) (or by its associated abelian surface \( (A, \rho) \)). A singular relation is called primitive if \( \gcd(a, b, c, d, e) = 1 \). Note that the matrix \( M_f \) above satisfies \( x^2 - bx + ac + de = 0 \). Thus, the discriminant for a singular relation \( \ell = (a, b, c, d, e) \) is defined naturally as the discriminant of the polynomial, i.e., as

\[
\Delta(\ell) = b^2 - 4(ac + de).
\]

Then the property that \( \text{Im} \, \tau \) is positive definite implies that \( \Delta(\ell) > 0 \). We also remark that Humbert proved that under the natural projection \( \mathbb{H}_2 \to \mathbb{A}_2 \), all primitive singular relations with the same discriminant \( n \) of the form \( (10) \) defines the same zero locus in \( \mathbb{A}_2 \), which is called the Humbert surface \( H_n \) of discriminant \( n \).

Observe that for a given \( \tau \in \mathbb{H}_2 \), if we let \( \mathcal{L} \) be the set of singular relations satisfied by \( \tau \), then \( \Delta \) defines a positive definite quadratic form on \( \mathcal{L} \) so that \( \mathcal{L} \) becomes a positive definite lattice. Let \( \langle \cdot, \cdot \rangle_\Delta \) denote the bilinear form associated to \( \Delta \), i.e.,

\[
\langle \ell_1, \ell_2 \rangle_\Delta = \frac{1}{2}(\Delta(\ell_1) + \Delta(\ell_2) - \Delta(\ell_1 + \ell_2)).
\]

By a direct computation, we find that

\[
\langle \ell_1, \ell_2 \rangle_\Delta^2 \equiv \Delta(\ell_1)\Delta(\ell_2) \mod 4
\]

for all singular relations \( \ell_1, \ell_2 \) satisfied by a given \( \tau \in \mathbb{H}_2 \).

We now consider the case \( (A, \rho) = (A_2, \rho_\mu) \in \mathcal{X}_\mu \), where \( \mathcal{X}_\mu \) is a Shimura curve in \( \mathbb{Q}_{D,N} \). Since the Rosati involution restricted to the Eichler order \( \mathcal{O} \) coincides with the involution \( \alpha \mapsto \mu^{-1} \overline{\alpha} \mu \), an element \( \alpha \) of \( \mathcal{O} \) is Rosati invariant if and only if

\[
\alpha \in \mu^\perp = \{ \beta \in \mathcal{O} : \text{tr}(\beta \overline{\mu}) = 0 \}.
\]

Therefore, for all \( (A, \rho) \) in \( \mathcal{X}_\mu \), the lattice of singular relations satisfied by \( (A, \rho) \) contains a sublattice isomorphic to \( \mu^\perp / \mathbb{Z} \), and there is a natural quadratic form on \( \mu^\perp / \mathbb{Z} \) defined by \( \alpha + \mathbb{Z} \mapsto \text{disc}(\alpha) = \text{tr}(\alpha)^2 - 4 \text{nr}(\alpha) \). Consequently, we may associate to \( \mathcal{X}_\mu \) a \( \text{GL}(2, \mathbb{Z}) \)-equivalence class of quadratic forms \( Q_\mu(x, y) \) defined by

\[
Q_\mu(x, y) = \text{disc}(ax + \beta y) = \text{tr}(ax + \beta y)^2 - 4 \text{nr}(ax + \beta y)
\]

where \( \{ \alpha, \beta \} \) is a \( \mathbb{Z} \)-basis for \( \mu^\perp / \mathbb{Z} \).

**Theorem A** ([10] Theorem 3 and Proposition 27]).

1. The quadratic form \( Q_\mu \) above has the property that all positive integers represented by \( Q_\mu \) satisfy
   (a) \( a \equiv 0, 1 \mod 4 \), and
   (b) that the quaternion algebra \( \left( \frac{-DNa}{Q} \right) \) has discriminant \( D \).

2. Conversely, assume that \( N \) is squarefree. Then each positive definite binary quadratic forms \( Q \) of discriminant \( -16DN \) having the two properties in Part (1) is \( \text{GL}(2, \mathbb{Z}) \)-equivalent to \( Q_\mu \) for some \( \mu \in \mathcal{O} \).

3. Assume that \( N \) is squarefree. Then the correspondence \( \mathcal{X}_\mu \leftrightarrow Q_\mu \) between Shimura curves in \( \mathbb{Q}_{D,N} \) and \( \text{GL}(2, \mathbb{Z}) \)-equivalence classes of quadratic forms of discriminant \( -16DN \) having the properties in Part (1) is one-to-one and onto.
(4) If \( Q_\mu(x, y) \) is not ambiguous, i.e., if \( Q_\mu(-x, y) \) is not \( \text{SL}(2, \mathbb{Z}) \)-equivalent to \( Q_\mu(x, y) \), then \( X_\mu \cong X_0^D(N)/\langle w_{DN} \rangle \). If \( Q_\mu \) is ambiguous, then \( Q_\mu \) primitively represents \( m \) or \( 4m \) for some positive divisor \( m \) of \( DN \) and we have \( X_\mu \cong X_0^D(N)/\langle w_{m}, w_{DN} \rangle \). Here \( w_n \) denotes the Atkin-Lehner operator on \( X_0^D(N) \) induced by an element of norm \( n \) in \( \mathcal{O} \).

Note that in [10], we only proved the case \( N = 1 \), but it is easy to see that the proof also works for the case of squarefree \( N \) with \( (D, N) = 1 \).

**Remark 3.** Note that if \( D_0 \) is not squarefree, then there may exist a quadratic form of discriminant \( -16D_0 \) that is neither primitive nor of the form \( 4Q' \) for some primitive \( Q' \), but all integers represented by it are congruent to \( 0 \) or \( 4 \mod 4 \). In such a case, there may exist different and nonisomorphic Shimura curves in \( \mathcal{A}_2 \) with this quadratic form. See [15] Example 13] for an example.

### 3. Singular relations satisfied by points on Shimura curves

As before, we assume that \( D_0 \) is a positive squarefree integer. For a positive definite binary quadratic form \( Q \) of discriminant \( -16D_0 \) such that all integers represented by \( Q \) are congruent to \( 0 \) or \( 1 \mod 4 \), define \( D \) and \( N \) by (3) and (4), respectively. Let \( X \) be the Shimura curve in \( \mathcal{Q}_{D,N} \) associated to \( Q \) as described in Theorem A. To facilitate our discussion on the intersection of \( X \) and Humbert surfaces, in this section, we shall make singular relations satisfied by points on \( X \) explicitly. We recall that \( Q \) is either primitive or is \( 4Q' \) for some quadratic form \( Q' \) of discriminant \( -D_0 \).

#### 3.1. Singular relations satisfied by Shimura curves.

**Lemma 4 ([10] Lemma 38).**

1. Assume that \( Q \) is primitive. Let \( p \) be a prime represented by \( Q \) such that \( p \nmid DN \) (so that \( p \equiv 1 \mod 4 \) and the quaternion algebra \( \left( \frac{-DN,p}{Q} \right) \) has discriminant \( D \)). Let \( B = \left( \frac{-DN,p}{Q} \right) \) be the quaternion algebra generated by \( I \) and \( J \) with \( I^2 = -DN \), \( J^2 = p \), and \( IJ = -JI \). Choose an even integer \( s \) such that \( s^2DN + 1 \equiv 0 \mod p \). Then the \( \mathbb{Z} \)-module \( \mathcal{O} \) spanned by

\[
\begin{align*}
    e_1 &= 1, &
    e_2 &= \frac{1 + J}{2}, &
    e_3 &= \frac{I + J}{2}, &
    e_4 &= \frac{sDNJ + IJ}{p},
\end{align*}
\]

is an Eichler order of level \( N \) in \( B \). Moreover, with the choice \( \mu = I \), the \( \mathbb{Z} \)-module \( \mu^\perp/\mathbb{Z} \) is spanned by \( e_2 \) and \( e_4 \) and hence the quadratic form \( Q_\mu \) defined by \( Q_\mu(x, y) = \text{disc}(e_2x + e_4y) \) is

\[
px^2 + 4sDNxy + 4tDNy^2
\]

(which is necessarily \( \text{GL}(2, \mathbb{Z}) \)-equivalent to \( Q \) since \( p \) is a prime), where \( t = (s^2DN + 1)/p \).

2. Assume that \( Q = 4Q' \) for some quadratic form \( Q' \) of discriminant \( -DN \). Let \( p \) be a prime represented by \( Q' \) and \( B = \left( \frac{-DN,p}{Q} \right) \). Let \( s \) be an odd integer such that \( s^2DN + 1 \equiv 0 \mod 4p \). Then the \( \mathbb{Z} \)-module \( \mathcal{O} \) spanned by

\[
\begin{align*}
    e_1 &= 1, &
    e_2 &= \frac{1 + I}{2}, &
    e_3 &= J, &
    e_4 &= \frac{sDNJ + IJ}{2p},
\end{align*}
\]

is an Eichler order of level \( N \) in \( B \). Moreover, with the choice \( \mu = I \), the \( \mathbb{Z} \)-module \( \mu^\perp/\mathbb{Z} \) is spanned by \( e_3 \) and \( e_4 \) and hence the quadratic form \( Q_\mu \) is

\[
4px^2 + 4sDNxy + 4tDNy^2
\]
(which is $GL(2, \mathbb{Z})$-equivalent to $Q$), where $t = (s^2DN + 1)/4p$.

Note again that Lemma 38 of [10] considered only the case $N = 1$, but it is easy to see that it can be generalized as above. Note also that the construction of Eichler orders of the form given in the lemma first appeared in [8] and was used in [5].

Let $B = \left( -\frac{DN, p}{\mathbb{Q}} \right)$ be the quaternion algebra of discriminant $D$ over $\mathbb{Q}$ and $\mathcal{O}$ be the Eichler order of level $N$ given in the lemma. Choose the embedding $\phi : B \hookrightarrow M(2, \mathbb{R})$ to be

$$\phi(I) = \begin{pmatrix} 0 & -1 \\ DN & 0 \end{pmatrix}, \quad \phi(J) = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & -\sqrt{p} \end{pmatrix}.$$  

Let $\alpha_1, \ldots, \alpha_4$ be a symplectic basis with respect to the element $\mu = I$ in the lemma, i.e.,

$$(\text{tr}(\mu^{-1} \alpha_i \tau_j))_{i,j=1,\ldots,4} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. $$

For $z \in \mathbb{H}$, let $v_z = (\bar{z})$, $\Lambda_z = \phi(\mathcal{O})v_z$, and $A_z = \mathbb{C}^2/\Lambda_z$. Write

$$\phi(\alpha_1)v_z, \ldots, \phi(\alpha_4)v_z = (\tau_1, \tau_2),$$

where $\tau_j \in M(2, \mathbb{C})$. Then the normalized period matrix for the principally polarized abelian surface $(A_z, \rho_\mu)$ is $\tau_z := \tau_2^{-1} \tau_1$. The quaternion modular embedding

$$\iota_{\mu} : \mathbb{H} \to \mathbb{H}_2, \quad \iota_{\mu} : z \mapsto \tau_z$$

defines an explicit map from $X_0^D(N)$ to $X_{\mu}$ in $Q_{D,N}$. Moreover, for $\gamma \in \mathcal{O}$, the matrix for $\gamma$, as an endomorphism of $A_z$, with respect to the basis $\phi(\alpha_1)v_z, \ldots, \phi(\alpha_4)v_z$ is the matrix $M_\gamma$ such that

$$(\gamma \alpha_1, \ldots, \gamma \alpha_4) = (\alpha_1, \ldots, \alpha_4)M_\gamma.$$ 

From this, we obtain the following description of singular relations satisfied by the Shimura curve associated to $Q$.

**Lemma 5.** We retain the notations above and those in Lemma 2.

1. Assume that $Q$ is primitive. A symplectic basis for $\mathcal{O}$ with respect to $\mu = I$ is

$$\alpha_1 = e_3 - \frac{p-1}{2} e_4, \quad \alpha_2 = -sDN e_1 - e_4, \quad \alpha_3 = e_1, \quad \alpha_4 = e_2.$$ 

The normalized period matrix of $(A_z, \rho_\mu)$ with respect to this symplectic basis is

$$\tau_z = \frac{1}{\mu z} \begin{pmatrix} -\epsilon^2 + (p-1)sDNz/2 + DN\epsilon^2z^2 & -1 - 2sDNz + DN\epsilon z^2 \\ -\epsilon - (p-1)sDNz - DN\epsilon z^2 & -\epsilon^2 + (p-1)sDNz/2 + DN\epsilon^2z^2 \end{pmatrix},$$

where $\epsilon = (1 + \sqrt{p})/2$ and $\tau = (1 - \sqrt{p})/2$. Moreover, the singular relations corresponding to $e_2$ and $e_4$ are

$$\ell_1 = (1, 1, (1-p)/4, 0, 0), \quad \ell_2 = (0, 2sDN, 0, 1, DN(s^2DN - t)).$$

respectively.

2. Assume that $Q = 4Q'$ for some quadratic form $Q'$ of discriminant $-DN$. A symplectic basis for $\mathcal{O}$ with respect to $\mu = I$ is

$$\alpha_1 = e_2, \quad \alpha_2 = -e_4, \quad \alpha_3 = e_1, \quad \alpha_4 = e_3.$$ 

The normalized period matrix of $(A_z, \rho_\mu)$ with respect to this symplectic basis is

$$\tau_z = \frac{1}{4z} \begin{pmatrix} DNz^2 + 2z - 1 & -DNz^2/\sqrt{p} - 1/\sqrt{p} \\ -DNz^2/\sqrt{p} - 1/\sqrt{p} & DNz^2 - 2sDNz/p - 1/p \end{pmatrix}.$$
Moreover, the singular relations corresponding to \( e_3 \) and \( e_4 \) are
\[
\ell_1 = (1, 0, -p, 0, -(1 + sD)N)/2, \quad \ell_2 = (0, 0, (1 - sD)N)/2, 1, -tDN),
\]
respectively.

Essentially, everything (in the case \( N = 1 \)) in the lemma is contained in the proof of Theorem 5.1 of [5] and Lemma 40 of [10], although in the proofs therein the singular relations in (17) and (19) were obtained by direct computation using the description of \( \tau_z \) above. Here we point out that these singular relations correspond to the basis for \( \mu^+ / \mathbb{Z} \) given in Lemma 4.

Note that the explicit description of \( \ell_1 \) and \( \ell_2 \) in (17) and (19) shows that for \( m_1, m_2 \in \mathbb{Z} \), the singular relation \( m_1\ell_1 + m_2\ell_2 \) is primitive if and only if \((m_1, m_2) = 1\). Consequently, we have the following result.

**Corollary 6.** A Shimura curve \( X \) in \( \mathbb{Q}_{D, N} \) is contained in the Humbert surface of discriminant \( n \) if and only if its associated quadratic form primitively represents \( n \).

Note that if \( z \) is not a CM point, then the lattice of singular relation satisfied by \( \tau_z \) has rank 2 and is spanned by \( \ell_1 \) and \( \ell_2 \) in the lemma. In the next section, we will consider the case of CM-points.

### 3.2. Singular relations satisfied by CM-points.

We retain all the notations in the previous section. In this section, we shall describe singular relations satisfied by CM-points on a Shimura curve \( X \) associated to a quadratic form \( Q \) of discriminant \(-16DN\).

Recall that a CM-point \( z \) on \( X_{0}^{0}(N) \) is defined to be the common fixed point of \( \phi(\psi(K)) \) in the upper half-plane for some embedding \( \psi \) from an imaginary quadratic field \( K \) into \( \mathbb{B} \). Its discriminant is defined to be the discriminant of the quadratic order \( R \) in \( K \) such that \( \psi(K) \cap \mathcal{O} = \psi(R) \). To describe singular relations satisfied by the corresponding point \( \tau_z \) on \( X \), we let
\[
\beta_1 = 2e_2 - e_1 = J, \quad \beta_2 = e_3 = \frac{I + IJ}{2}, \quad \beta_3 = e_4 = \frac{sDNJ + IJ}{p}
\]
in the case \( Q \) is primitive, and
\[
\beta_1 = e_3 = J, \quad \beta_2 = 2e_2 - e_1 = I, \quad \beta_3 = e_4 = \frac{sDNJ + IJ}{2p}
\]
in the case \( Q = 4Q' \). Then they form a basis for the \( \mathbb{Z} \)-module of elements of trace zero in \( \mathcal{O} \).

**Lemma 7 ([10] Lemma 40).** Let \( z \) be a CM-point of discriminant \( d \) on \( X_{0}^{0}(N) \), \( \psi \) be its corresponding optimal embedding of discriminant \( d \), \( \tau_z \) be its corresponding point on \( X \), and \( L_z \) be the lattice of singular relations satisfied by \( \tau_z \). Write
\[
\psi(\sqrt{d}) = b_1\beta_1 + b_2\beta_2 + b_3\beta_3.
\]

1. Assume that \( Q \) is primitive. Let \( \ell_1 \) and \( \ell_2 \) be the singular relations satisfied by \( X \) given in (17). Then in addition to \( \ell_1 \) and \( \ell_2 \), \( \tau_z \) also satisfies
\[
\ell_4 = (0, b_2, b_3 + b_2(1 - p)/4, 0, b_1).
\]

These three singular relations form a \( \mathbb{Z} \)-basis for \( L_z \). Moreover, the Gram matrix of \( L_z \) with respect to this basis is
\[
\begin{pmatrix}
p & 2sDN & -b_2p/2 - b_3 \\
2sDN & 4tDN & 2b_1 - b_2sDN \\
-b_2p/2 - b_3 & 2b_1 - b_2sDN & b_2/4
\end{pmatrix}.
\]
(Note that \(b_2\) and \(b_3\) are necessarily even, while \(b_1\) is even or odd depending on whether \(d\) is even or odd.)

(2) Assume that \(Q = 4Q'\) for some quadratic form \(Q'\) of discriminant \(-DN\). Let \(\ell_1\) and \(\ell_2\) be the singular relations satisfied by \(X\) given in (19). Then in addition to \(\ell_1\) and \(\ell_2\), \(\tau_3\) also satisfies

\[
\ell_3 = (0, -b_2, b_3/2, 0, -b_1/2).
\]

These three singular relations form a \(\mathbb{Z}\)-basis for \(L_z\). Moreover, the Gram matrix of \(L_z\) with respect to this basis is

\[
(23) \begin{pmatrix}
4p & 2sDN & -b_3 \\
2sDN & 4tDN & b_1 \\
-b_3 & b_1 & b_2^2
\end{pmatrix}.
\]

(Note that \(b_1\) and \(b_3\) are necessarily even, while \(b_2\) is even or odd depending on whether \(d\) is even or odd.)

In both cases, the discriminant of \(L_z\) is \(4|d|\).

Proof. The case where \(Q\) is primitive was proved (in the case \(N = 1\)) in Lemma 40 of [10]. Here we sketch a proof for the case \(Q = 4Q'\).

Recall from Lemma 5 that \(\tau_3\) is given by

\[
\tau_3 = \frac{1}{4z} \begin{pmatrix}
DNz^2 + 2z - 1 \\
-DNz^2 - \sqrt{\delta} - 1/\sqrt{\delta} \\
DNz^2 - 2sDNz/p - 1/p
\end{pmatrix}.
\]

Write \(\tau_3\) as \((\tau_1' \tau_2' \tau_3')\). We have

\[
\begin{pmatrix}
\tau_1 \\
\tau_2 \\
\tau_3 \\
\tau_2 - \tau_1 \tau_3 \\
1
\end{pmatrix} = \frac{1}{4} \begin{pmatrix}
1 \\
1/\sqrt{\delta} \\
0 \\
(sDN - 1)/2p \\
1
\end{pmatrix} \begin{pmatrix}
1 & 2 & 1 \\
0 & -2sDN/p & 1/p \\
DN(1 + s)/p & (sDN - 1)/2p & 0 \\
DNz
\end{pmatrix} \begin{pmatrix}
-1/z \\
1 \\
0 \\
1 \\
DNz
\end{pmatrix}.
\]

Write \(\psi(\sqrt{\delta}) = c_1I + c_2J + c_3IJ\). Changing \(\psi(\sqrt{\delta})\) to \(-\psi(\sqrt{\delta})\) if necessary, we may assume that \(c_1 + c_3\sqrt{\delta} > 0\). Since \(z\) is fixed by \(\psi(\sqrt{\delta})\), we have

\[
z = \frac{c_2\sqrt{\delta} + \sqrt{\delta}}{DN(c_1 + c_3\sqrt{\delta})}, \quad -\frac{1}{z} = \frac{-c_2\sqrt{\delta} + \sqrt{\delta}}{c_1 - c_3\sqrt{\delta}},
\]

and

\[
\begin{pmatrix}
-1/z \\
1 \\
DNz
\end{pmatrix} = \begin{pmatrix}
\gamma & 0 \\
0 & 1 \\
\overline{\gamma} & 0
\end{pmatrix} \begin{pmatrix}
\sqrt{\delta} \\
1
\end{pmatrix}, \quad \gamma = \frac{1}{c_1 - c_3\sqrt{\delta}}, \quad \delta = \frac{-c_2\sqrt{\delta}}{c_1 - c_3\sqrt{\delta}}
\]

where \(\overline{\gamma}\) and \(\delta\) are Galois conjugates of \(\gamma\) and \(\delta\), respectively. Hence, \((a_1, \ldots, a_5)\) is a singular relation for \(\tau_3\) if and only if \((a_1, \ldots, a_5)\) is in the nullspace of

\[
\begin{pmatrix}
1/\sqrt{\delta} & 0 & 1 \\
1/p & -2sDN/p & 1/p \\
(sDN - 1)/2p & DN(1 + s)/p & (sDN - 1)/2p
\end{pmatrix} \begin{pmatrix}
\gamma & 0 \\
0 & 1 \\
\overline{\gamma} & 0
\end{pmatrix}.
\]
Using the relations

\[ c_1 = b_2, \quad c_2 = b_1 + \frac{b_3 s DN}{2p}, \quad c_3 = \frac{b_3}{2p}, \]

we check directly that \( \ell_3 = (0, -b_2, b_3/2, 0, -b_1/2) \) is a singular relation satisfied by \( \tau_\zeta \).

Finally, we observe that the assumption that \( \psi \) is an optimal embedding of discriminant \( d \) implies that

\[ \gcd(b_1, b_2, b_3) = \begin{cases} 
1, & \text{if } d \text{ is odd,} \\
2, & \text{if } d \text{ is even,}
\end{cases} \]

which implies that if \( \ell \) is a singular relation satisfied by \( \tau_\zeta \) and \( r_1, r_2, r_3 \) are the coefficients in \( \ell = r_1 \ell_1 + r_2 \ell_2 + r_3 \ell_3 \), then \( r_1, r_2, r_3 \) are all integers. We conclude that \( \ell_1, \ell_2, \ell_3 \) form a \( \mathbb{Z} \)-basis for \( L_\zeta \). \( \square \)

### 3.3. Galois orbits of CM-points.

Observe that if \( \psi : K \to B \) is an embedding from an imaginary quadratic field \( K \) into \( B \), then so is \( \overline{\psi} : a \mapsto \overline{\psi}(a) \), where \( \overline{\psi}(a) \) denotes the quaternionic conjugate of \( \psi(a) \), and \( \phi(\psi(K)) \) and \( \phi(\overline{\psi}(K)) \) have a common fixed point \( z_\psi \) in the upper half-plane. To remove the ambiguity when we talk about the embedding corresponding to \( z_\psi \), we say an embedding \( \psi \) is normalized (with respect to \( \phi \)) if

\[ \phi(\psi(a)) \left( \begin{array}{c} z_\psi \\ 1 \end{array} \right) = a \left( \begin{array}{c} z_\psi \\ 1 \end{array} \right) \]

for all \( a \in K \) (as opposed to \( \phi(\psi(a)) \left( \begin{array}{c} z_\psi \\ 1 \end{array} \right) = \overline{\tau}(z_\psi) \), where \( \overline{\tau} \) is the complex conjugate of \( \tau \)). It is clear that \( \psi \) is normalized with respect to \( \phi \) if and only if the \((2,1)\)-entry of \( \phi(\psi(a)) \) has the same sign as the imaginary part of \( a \) for any \( a \in K \). From now on, whenever we mention the embedding corresponding to a CM-point, we refer to the one that is normalized.

Denote by \( \text{CM}(d) \) the set of CM-points of discriminant \( d \) on \( X^D_0(N) \). By local consideration, we have

\[
|\text{CM}(d)| = h(d) \prod_{\text{primes } q} m(\mathcal{O}, d, q),
\]

where \( h(d) \) is the class number of \( R \) and \( m(\mathcal{O}, d, q) \) is the number of local optimal embeddings of \( R \) into \( \mathcal{O} \otimes \mathbb{Z}_q \). The formula for \( m(\mathcal{O}, d, q) \) is given by

\[
m(\mathcal{O}, d, q) = \begin{cases} 
1, & \text{if } q \nmid DN, \\
1 - \left( \frac{4}{q} \right), & \text{if } q|D \text{ and } q \nmid f, \\
0, & \text{if } q|D \text{ and } q|f, \\
1 + \left( \frac{4}{q} \right), & \text{if } q|N \text{ and } q \nmid f, \\
2, & \text{if } q|N \text{ and } p|f,
\end{cases}
\]

assuming \( N \) is squarefree, where \( f \) is the positive integer such that \( d = f^2 d_0 \) for a fundamental discriminant \( d_0 \).

Let \( L \) be the ring class field of \( R \). Then the class group of \( R \) acts on \( \text{CM}(d) \) and two CM-points of discriminant \( d \) are in the same orbit under this action if and only if their corresponding (normalized) optimal embeddings are locally equivalent at every place. In view of the Shimura reciprocity law, this amounts to the property that two CM-points of discriminant \( d \) are in the same \( \text{Gal}(L/K) \)-orbit if and only if their corresponding optimal embeddings are locally equivalent at every place. This leads to the following explicit criterion when two CM-points are \( \text{Gal}(L/K) \)-conjugates.
Lemma 8. Let \( z \) and \( z' \) be two CM-points on \( X_1^D(N) \) and \( \psi \) and \( \psi' \) be their respective normalized optimal embeddings into \( \mathcal{O} \). Write \( \psi(\sqrt{d}) = b_1\beta_1 + b_2\beta_2 + b_3\beta_3 \) and \( \psi'(\sqrt{d}) = b'_1\beta_1 + b'_2\beta_2 + b'_3\beta_3 \). Write \( d = f^2d_0 \) for a fundamental discriminant \( d_0 \). We have the following properties.

1. We have
\[
 b_1^2p \equiv d \mod \begin{cases} 4DN & \text{in the case } Q \text{ is primitive,} \\ DN & \text{in the case } Q = 4Q'. \end{cases}
\]

2. Let \( q \) be a prime divisor of \( DN \). If \( \psi \) and \( \psi' \) are equivalent locally at \( q \), then
\[
 b_1 \equiv b_1' \mod \begin{cases} 4, & \text{if } q = 2, \\ q, & \text{if } q \neq 2. \end{cases}
\]
Moreover, if \( q \) does not divide \( (f, N) \), then this is necessary and sufficient condition for \( \psi \) and \( \psi' \) to be equivalent at \( q \).

3. If \( z \) and \( z' \) lie in the same \( \text{Gal}(L/K) \)-orbit, then
\[
 b_1 \equiv b_1' \mod \begin{cases} 2DN & \text{in the case } Q \text{ is primitive,} \\ DN & \text{in the case } Q = 4Q'. \end{cases}
\]
Moreover, if \( (N, f) = 1 \), then this is a necessary and sufficient condition for \( z \) and \( z' \) to be in the same \( \text{Gal}(L/K) \)-orbit. In such a case, the set of \( \text{Gal}(L/K) \)-orbits of CM-points of discriminant \( d \) on \( X_1^D(N) \) are in one-to-one correspondence with the set
\[
 \{ r \mod 2DN : pr^2 \equiv d \mod 4DN \}
\]
in the case \( Q \) is primitive, and with the set
\[
 \{ r \mod DN : pr^2 \equiv d \mod DN \}
\]
in the case \( Q = 4Q' \).

Proof. The case of primitive \( Q \) was proved in details in Lemma 46 of [10]. Here we sketch the proof of the case \( Q = 4Q' \).

We have
\[
d = -n\text{r}(\psi(\sqrt{d})) = b_1^2p + DN(b_1b_3s - b_2^2 + b_3^2t), \quad t = \frac{s^2DN + 1}{4p}.
\]

Therefore, we have \( b_1^2p \equiv d \mod DN \).

Now let \( q \) be a prime divisor of \( DN \). Since \( q \) is odd and different from \( p, 1, I, J \), and \( IJ \) form a \( \mathbb{Z}_q \)-basis for \( \mathcal{O}_q := \mathcal{O} \otimes \mathbb{Z}_q \). A direct computation shows that if \( \gamma = d_1 + d_2I + d_3J + d_4IJ \in \mathcal{O}_q \) and \( c'I + c'_2J + c'_3IJ = \gamma(c_1I + c_2J + c_3IJ)\gamma^{-1} \), then
\[
 c'_2 - c_2 = \frac{2DN(c_1d_0d_3 - c_3d_0d_1 + c_1d_1d_2 - c_2d_1^2 - c_3d_2d_3p + c_2d_3^2p)}{\text{nr}(\gamma)}.
\]

Therefore, if \( \gamma \in \mathcal{O}_q^\times \), then \( c'_2 \equiv c_2 \mod q \), which implies that if \( \psi \) and \( \psi' \) are locally equivalent at \( q \), then \( b_1' \equiv b_1 \mod q \). In the case \( q \nmid (N, f) \), the number of solutions of the congruence equation \( pr^2 \equiv d \mod q \) coincides with the number of local optimal embeddings discriminant \( d \) at \( q \). Thus, in the case \( q \nmid (N, f) \), local optimal embeddings of discriminant \( d \) at the place \( q \) are in one-to-one correspondence with the solutions of \( pr^2 \equiv d \mod q \). Then the lemma follows. \( \square \)
Remark 9. Note that when \( q(\langle N, f \rangle) \), the number of local optimal embeddings of discriminant \( d \) is 2 by (25), but the congruence equation \( b_1^2p \equiv d \mod q \) has only one solution, so the condition for local equivalence of \( \psi \) and \( \psi' \) at \( q \) given in the lemma does not work. It is possible to work out a condition in such a case, but it is not needed for the purpose of this paper.

Corollary 10. Let \( X \) be a Shimura curve in \( Q_{D, N} \). Let \( d \) be a negative discriminant such that CM-points of discriminant \( d \) exist on the Shimura curve \( X \). Let \( K = Q(\sqrt{d}) \), \( R \) be the quadratic order of discriminant \( d \), and \( L \) be the ring class field of \( R \). If \( z \) and \( z' \) are two CM-points of discriminant \( d \) lying in the same \( \text{Gal}(L/K) \)-orbit, then their lattices of singular relations \( L_z \) and \( L_{z'} \) are isomorphic.

Proof. Let \( Q \) be the quadratic form associated to \( X \). The case \( Q \) is primitive was proved in [10] Proposition 41] (for the case \( N = 1 \)). Here we provide a proof for the case \( Q = 4Q' \) (so that \( DN \) is assumed to be congruent to 3 modulo 4).

Let \( B, O, \) and \( \mu \) be given as in Part (2) of Lemma (4) and \( \beta_1, \beta_2, \beta_3 \) be given by (21). Assume that \( z \) and \( z' \) are CM-points of discriminant \( d \) and \( \psi \) and \( \psi' \) are their respective optimal embeddings. Write \( \psi(\sqrt{d}) = b_1\beta_1 + b_2\beta_2 + b_3\beta_3 \) and \( \psi'(\sqrt{d}) = b'_1\beta_1 + b'_2\beta_2 + b'_3\beta_3 \). Recall from (23) that the Gram matrices for \( z \) and \( z' \) are

\[
A = \begin{pmatrix} 4p & 2sDN & -b_3 \\ 2sDN & 4tDN & b_1 \\ -b_3 & b_1 & b_2^2 \end{pmatrix}, \quad A' = \begin{pmatrix} 4p & 2sDN & -b'_3 \\ 2sDN & 4tDN & b'_1 \\ -b'_3 & b'_1 & (b'_2)^2 \end{pmatrix}.
\]

It is clear that \( A' = UAU^t \), where

\[
U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u & v & 1 \end{pmatrix}
\]

with

\[
u = t(b_3 - b'_3) + \frac{s}{2}(b_1 - b'_1), \quad v = -\frac{s}{2}(b_3 - b'_3) - \frac{p}{DN}(b_1 - b'_1).
\]

Since \( \psi(\sqrt{d})/2 \in O \) or \( (1 + \psi(\sqrt{d}))/2 \in O \) depending on whether \( d \) is even or odd, the integers \( b_1 \) and \( b_3 \) are necessarily even, and so are \( b'_1 \) and \( b'_3 \). Furthermore, by Lemma 8 one has \( b_1 \equiv b'_1 \mod DN \). Consequently, the numbers \( u \) and \( v \) are both integers and \( L_z \approx L_{z'} \). This proves the assertion. \( \square \)

4. Intersection of Shimura curves with Humbert surfaces

As in the previous sections, we assume that \( D_0 \) is a positive squarefree integer. Let \( Q \) be a quadratic form of discriminant \( -16D_0 \) appearing in the statement of Theorem (1) \( D \) and \( N \) be defined by (3) and (4), respectively, and \( X \) be the Shimura curve corresponding to \( Q \) in \( Q_{D, N} \) as described in Theorem [A]

In this section, we study the intersection of the Shimura curve \( X \) with Humbert divisors \( G_n \) defined by (8). We let \( I(X, G_n) \) be the intersection number of \( X \) with \( G_n \), and let \( I_0(X, G_n) \) and \( I_1(X, G_n) \) be the contributions to \( I(X, G_n) \) from the 0-dimensional components and the 1-dimensional components of the intersection, respectively. We also recall from Theorem [A] that \( X \approx X_0^D(N)/W \), where

\[
W = W_Q := \begin{cases} \langle w_{DN} \rangle, & \text{if } Q \text{ is not ambiguous}, \\
\langle w_m, w_{DN} \rangle, & \text{if } Q \text{ is ambiguous, representing } m \text{ or } 4m, m|DN. \end{cases}
\]
Proposition 11. For all positive integer \(n\) such that \(n \equiv 0 \text{ or } 1 \mod 4\), we have

\[
I(\mathcal{X}, G_n) = \frac{1}{|W|} a_n H_{D,N}(0),
\]

where \(H_{D,N}(0)\) is defined by (6). Consequently, one has

\[
\frac{1}{|W|} H_{D,N}(0) + \sum_{n=1}^{\infty} I(\mathcal{X}, G_n) e^{2\pi i n z} = \frac{1}{|W|} H_{D,N}(0) \mathcal{H}(z).
\]

Proof. By Theorem 8.1 of [16], there is a constant \(c\) (depending on \(X\)) such that \(I(\mathcal{X}, G_n) = c a_n\) for all \(n\). Thus, one only needs to know the value of one particular \(I(\mathcal{X}, G_n)\). Here we let \(n\) be a fundamental discriminant not represented by \(Q\) so that \(\mathcal{X}\) does not lie on \(G_n\). By Theorem B, \(G_n\) is the divisor of a Siegel modular form \(F\) of weight \(-a_n/2\) on \(\text{Sp}(4, \mathbb{Z})\).

Since, in general, the restriction of a Siegel modular form \(F\) of weight \(2k\) on \(X_0^D(N)/W\), the restriction of \(F\) along \(\mathcal{X}\) has zeros. Hence \(c = H_{D,N}(0)/|W|\) and \(I(\mathcal{X}, G_n) = a_n H_{D,N}(0)/|W|\). Then the proposition follows. \(\square\)

We next determine \(I_0(\mathcal{X}, G_n)\) in terms of class numbers. This is the most complicated part of the proof.

4.2. Determination of \(I_0(\mathcal{X}, G_n)\). By Lemma 4 we may assume that the quadratic form \(Q\) is \(p x^2 + 4 s D N x y + 4 t D N y^2\) from Part (1) of the lemma or \(4 p x^2 + 4 s D N x y + 4 t D N y^2\) from Part (2) of the lemma, depending on whether \(Q\) is primitive or of the form \(4 Q'\). Let \(\mathcal{O}\) be the Eichler order of level \(N\) in \(\mathbb{H}^0\) spanned by \([D N,0]\) or \([1,0]\). Let \(\mathcal{F}\) be the image of a fixed fundamental domain of \(X_0^D(N)\) under \(\iota\). To avoid the cumbersome factor \(1/|W|\) and the complexity relating to fixed points of Atkin-Lehner involutions, here we will work on \(\mathcal{F}\) instead of \(\mathcal{X}\).

For a singular relation \(\iota = (c_1, \ldots, c_5) \in \mathbb{Z}^5\), we let

\[
\mathcal{H}_\iota = \left\{ \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix} \in \mathbb{H}_2 : c_1 \tau_1 + c_2 \tau_2 + c_3 \tau_3 + c_4 (\tau_2^2 - \tau_1 \tau_3) + c_5 = 0 \right\}
\]

be the locus of points in \(\mathbb{H}_2\) satisfying \(\iota\). We first recall that if \(\mathcal{F} \not\subseteq \mathcal{H}_\iota\), then each point \(\tau_2\) in \(\mathcal{H}_\iota \cap \mathcal{F}\) corresponds to an abelian surface with endomorphism algebra strictly larger
than quaternion algebra over \( \mathbb{Q} \), and hence \( z \) is a CM-point on the Shimura curve and \( \tau_2 \in M(2, K) \) for some imaginary quadratic field \( K \).

**Lemma 12.** Let \( \ell \) be a singular relation such that \( F \not\subset H_\ell \). Then \( H_\ell \) intersects \( F \) transversally at each point of intersection.

**Proof.** Here we only prove the case \( Q \) is of the form \( 4Q' \).

Let \( \ell = (c_1, \ldots, c_5) \) be a singular relation such that \( F \not\subset H_\ell \) and assume that \( H_\ell \) intersects \( F \) at \( \tau_z, z \in \mathbb{H} \). For convenience, we express \( (\tau^1, \tau^2, \tau^3) \) as a row vector \( (\tau_1, \tau_2, \tau_3) \). Using the explicit expression in (18), we find that the tangent space of \( F \) at \( \tau_2 \) is spanned by

\[
(26) \quad \left( DN + z^{-2}, -DN + z^{-2}, 0 \right) = \frac{1}{\sqrt{p}} (-p\tau_2, -\tau_1 + 1/2, -\tau_2).
\]

We next compute the tangent space of \( H_\ell \). If \( c_3 \neq 0 \) or \( c_4 \neq 0 \), we may express \( \tau_3 \) as a function of \( \tau_1 \) and \( \tau_2 \). Then the tangent space of \( H_\ell \) at \( (\tau_1, \tau_2, \tau_3) \) is spanned by

\[
(27) \quad (c_4 \tau_1 - c_3, 0, 0, 0), \quad (0, c_4 \tau_1 - c_3, 2c_4 \tau_2 + c_2).
\]

The three vectors in (26) and (27) are coplanar if and only if

\[
2c_2 \tau_1 + 2(p c_1 + c_3 - c_4) \tau_2 + 2c_4 (\tau_1 - p \tau_3) \tau_2 - c_2 = 0.
\]

Since for \( \tau_z \), we have \( \tau_1 - p \tau_3 = (1 + sDN)/2 \), the condition reduces to

\[
(28) \quad 2c_2 \tau_1 + (2pc_1 + c_3 + (sDN - 1)c_4) \tau_2 - c_2 = 0.
\]

Now since \( \tau_2 \in M(2, K) \) for some imaginary quadratic field \( K \), considering the real part and the imaginary part separately, we see that the \( \mathbb{Z} \)-module of quintuple of integers \( (c_1, \ldots, c_5) \) satisfying both (28) and \( c_1 \tau_1 + c_2 \tau_2 + c_3 \tau_3 + c_4 (\tau_2^2 - \tau_1 \tau_3) + c_5 = 0 \) has rank 2. However, as \( F \subset H_\ell, H_{\ell'}, \) this \( \mathbb{Z} \)-module has to be spanned precisely by \( \ell_1 \) and \( \ell_2 \), where \( \ell_1 \) and \( \ell_2 \) are the singular relations in (19). Therefore, (28) cannot hold for any singular relation with \( c_3, c_4 \neq 0 \) not in the span of \( \ell_1 \) and \( \ell_2 \), i.e., \( H_\ell \) intersects \( F \) transversally at \( \tau_2 \).

If \( c_1, c_4 \neq 0 \), we may express \( \tau_1 \) as a function of \( \tau_2 \) and \( \tau_3 \). Then by a similar computation, we find that \( H_\ell \) intersects \( F \) transversally at \( \tau_2 \).

Finally, if \( c_1 = c_3 = c_4 = 0 \), then the singular relation is \( c_2 \tau_2 + c_5 = 0 \) and the tangent space of \( H_\ell \) at \( \tau_2 \) is spanned by \( (1, 0, 0) \) and \( (0, 0, 1) \). These two vectors cannot be coplanar with the vector in (26) since \( -\tau_1 + 1/2 \) is never 0 in \( \mathbb{H} \). This completes the proof of the lemma. \( \square \)

**Corollary 13.** We have

\[
(29) \quad I_0(\mathcal{X}_\mu, G_n) = \frac{1}{2|W|} \sum_{\ell: \Delta(\ell) = n, F \not\subset H_\ell} |H_\ell \cap F|',
\]

where \( |H_\ell \cap F|' \) denotes the weighted cardinality of the set \( H_\ell \cap F \) with the weight of CM-points of discriminant \( -4 \) given by 1/2, the weight of CM-points of discriminant \( -3 \) given by 1/3, and the weight of CM-points of other discriminant given by 1.

Note that the appearance of the factor 1/2 in (29) is due to the fact that \( \ell \) and \( -\ell \) define the same surface \( H_\ell \).

Let \( \ell_1, \ell_2 \) be the singular relations given by (17) or (19) satisfied by all points \( \tau_2 \) in \( F \). Observe that if \( \ell \) is a singular relation of discriminant \( n \) such that \( F \not\subset H_\ell \), then the Gram
matrix for the lattice spanned by $\ell_1$, $\ell_2$, and $\ell$ is of the form
\begin{equation}
M_\ell := \begin{pmatrix}
a & b & \langle \ell_1, \ell \rangle \\
b & c & \langle \ell_2, \ell \rangle \\
\langle \ell_1, \ell \rangle & \langle \ell_2, \ell \rangle & n
\end{pmatrix},
\end{equation}
where $\langle \cdot, \cdot \rangle_\Delta$ is the inner product defined in (12) and $(a \ b \ c)$ is the Gram matrix of $\langle \cdot, \cdot \rangle_\Delta$ with respect to $\ell_1$ and $\ell_2$. Here we shall partition the sum in (23) according to the values of $\langle \ell_1, \ell \rangle_\Delta$ and $\langle \ell_2, \ell \rangle_\Delta$.

For convenience, given arbitrary integers $u$ and $v$, we let
\begin{equation}
M_{u,v} = \begin{pmatrix}
a & b & u \\
b & c & v \\
u & v & n
\end{pmatrix}.
\end{equation}

We have
\begin{equation}
I_0(X_\mu, G_n) = \frac{1}{2|W|} \sum_{u,v \in \mathbb{Z}} \sum_{F \not\subset \mathcal{H}_\ell, M_\ell = M_{u,v}} |\mathcal{H}_\ell \cap F|'.
\end{equation}

In order for $\mathcal{H}_\ell \cap F$ to be nonempty, the integer $u$ and $v$ need to satisfy
1. $u \equiv an \ mod \ 2$ and $v \equiv cn \ mod \ 2$, due to (13),
2. $\det M_{u,v} = 4|d|$ for some negative discriminant $d$ such that $\mathbb{Q}(\sqrt{d})$ can be embedded in the quaternion algebra $B$, due to Lemma [7].

Now assume that $u$ and $v$ are integers satisfying these conditions. Write $d = f^2d_0$, where $d_0$ is a negative fundamental discriminant. We need to study for each divisor $r$ of $f$, how many CM-points $\tau_z$ of discriminant $r^2d_0$ in $F$ satisfy some singular relations $\ell$ with $M_\ell = M_{u,v}$, and how many such $\ell$ they satisfy. We first mention a rather trivial but yet important observation.

**Lemma 14.** Let the notations be as above. For a CM-point $\tau_z$ of discriminant $r^2d_0$, let $M_z$ be the Gram matrix for the lattice of singular relations satisfied by $\tau_z$ given in (22) or (23). Then $\tau_z \in \mathcal{H}_\ell$ for some $\ell$ with $M_\ell = M_{u,v}$ if and only if the equation
\begin{equation}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
x & y & w
\end{pmatrix}
M_z
\begin{pmatrix}
1 & 0 & x \\
0 & 1 & y \\
0 & 0 & w
\end{pmatrix}
= M_{u,v}
\end{equation}
in $x$, $y$, and $w$ is solvable in integers.

Moreover, in the case the equation is solvable in integers, there is precisely one singular relation $\ell$ with $\tau_z \in \mathcal{H}_\ell$ and $M_\ell = M_{u,v}$.

**Proof.** The first statement is obvious. Here the correspondence between a solution $(x, y, w)$ and a singular relation is $(x, y, w) \leftrightarrow x\ell_1 + y\ell_2 + w\ell_3$, where $\ell_3$ is the singular relation for $\tau_z$ described in Lemma [7]. The uniqueness of $\ell$ is due to the fact that the quadratic form on the lattice of singular relations satisfied by $\tau_z$ is nondegenerate.

**Proposition 15.** Let the notations be as above. (In particular, $d$ is the negative discriminant such that $\det M_{u,v} = 4|d|$.) The number of CM-points of discriminant $r^2d_0$ satisfying some singular relation $\ell$ with $M_\ell = M_{u,v}$ is
\begin{equation}
2 \cdot \frac{h_{D,N}(r^2d_0)}{2^{\omega_{D,N}(d)}},
\end{equation}
where $h_{D,N}(r^2d_0)$ is the number of CM-points of discriminant $r^2d_0$ on $X^D_0(N)$ and
\[\omega_{D,N}(d) = \# \{q \ prime : q|DN, \ q \nmid d \} .\]
Proof. We only prove the case $Q$ is primitive. The proof of other case $Q = 4Q$ is similar.

Let $b_1 \beta_1 + b_2 \beta_2 + b_3 \beta_3$ be the image of $\sqrt{r^2 d_0}$ under the normalized optimal embedding associated to a CM-point $z$ of discriminant $r^2 d_0$, where $\beta_1, \beta_2, \beta_3$ are given in (20). We recall from Lemma 7 that $\ell_1, \ell_2, \ell_3$ are integers. Since (34) holds at $\ell_1, \ell_2, \ell_3$ and

$$\ell_3 = (0, b_2/2, b_3/2 + b_2(1 - p)/4, 0, b_1)$$

form a basis of singular relations for $\tau_z$, the Gram matrix $M_z$ with respect to $\ell_1, \ell_2, \ell_3$ is

$$M_z = \begin{pmatrix}
p & 2sDN & -b_2p/2 - b_3 \\
2sDN & 4tDN & 2b_1 - b_2sDN \\
-b_2p/2 - b_3 & 2b_1 - b_2sDN & b_3^2/4
\end{pmatrix},$$

and $\det M_z = 4 |r^2 d_0|$. By Lemma 14 we need to determine whether the equation in (33) is solvable in integers.

By considering the determinants of the two sides of (33), we see that $w = r'$ or $w = -r'$, where $r' = f/r$. Then (33) is solvable in integers if and only if

$$x = t(u + w(b_2p/2 + b_3)) - \frac{s}{2}(v - w(2b_1 - b_2sDN)),$$

$$y = -\frac{s}{2}(u + w(b_2p/2 + b_3)) + \frac{p}{4DN}(v - w(2b_1 - b_2sDN))$$

are integers. Since $s$ is assumed to be even (see the assumptions in Lemma 4), $x$ is always an integer, while the condition for $y$ being an integer reduces to that

$$w(2b_1 - b_2sDN) \equiv v \mod 4DN, \quad w = \pm r'.$$

Note that $b_2$ is even. Thus, the congruence equation further reduces to

(34) \hspace{1cm} b_1w \equiv v/2 \mod 2DN, \quad w = \pm r'.

We observe that

$$-4d = \det M_{u,v} = -pv^2 + 4sDNuv - 4tDNu^2 + 4DNn.$$

Since $t = (s^2 DN + 1)/p \equiv 1 \mod 4$ and $u^2 \equiv n \mod 4$ by (13), we have

(35) \hspace{1cm} p(v/2)^2 \equiv d \mod 4DN.

Also, we recall from Part (1) of Lemma 8 that

(36) \hspace{1cm} pb_1^2 \equiv r^2 d_0 \mod 4DN.

Naturally, we will consider the congruence equation (34) locally.

For a prime divisor $q$ of $DN$, let $S(O, r^2 d_0, q)$ be the set of $O_q$-inequivalent optimal embeddings of discriminant $r^2 d_0$ into $O_q := O \otimes \mathbb{Z}_q$. Its cardinality $m(O, r^2 d_0, q)$ is given by (25). For an optimal embedding in $S(O, r^2 d_0, q)$, write again the image of $\sqrt{r^2 d_0}$ under this embedding as $b_1 \beta_1 + b_2 \beta_2 + b_3 \beta_3$, $b_j \in \mathbb{Z}_q$.

Consider the case $w = r'$ first. Let $T(O, r^2 d_0, q)$ be the subset of $S(O, r^2 d_0, q)$ such that (34) holds at $q$, i.e.,

(37) \hspace{1cm} b_1r' \equiv v/2 \mod \begin{cases} 4, & \text{if } q = 2, \\ q, & \text{if } q \neq 2. \end{cases}

For the case $q$ is a prime dividing $d$ (and also $DN$), we have by (35) and (36),

$$v \equiv 0 \mod \begin{cases} 4, & \text{if } q = 2, \\ q, & \text{if } q \neq 2, \end{cases}$$
and
\[ p(b'_1 r')^2 \equiv (r r')^2 d_0 = d \equiv \begin{cases} 8, & \text{if } q = 2, \\ q, & \text{if } q \neq 2. \end{cases} \]

Thus, (37) holds for every element in \( S(\mathcal{O}, r^2 d_0, q) \). In other words, for the case \( q|d \), we have \( |\mathcal{T}(\mathcal{O}, r^2 d_0, q)| = m(\mathcal{O}, r^2 d_0, q) \).

For the case \( q \) does not divide \( d \) (but divides \( DN \)), we have \( m(\mathcal{O}, r^2 d_0, q) = 2 \) by (25), and according to Part (2) of Lemma 8 the equivalence of local optimal embeddings is completely determined by the residue class of \( b_1 \) modulo 4 if \( q = 2 \), or of \( b_1 \) modulo \( q \) if \( q \) is odd. The computation (38) above and (35) show that both elements in \( S(\mathcal{O}, r^2 d_0, q) \) satisfy
\[ p(b_1 r')^2 \equiv p(v/2)^2 \mod 8, \quad \text{if } q = 2, \]
\[ q, \quad \text{if } q \neq 2. \]

However, for the given \( v \), exactly one of them will satisfy (37). Therefore, in the case \( q \) does not divide \( d \), we have \( |\mathcal{T}(\mathcal{O}, r^2 d_0, q)| = m(\mathcal{O}, r^2 d_0, q)/2 \).

In summary, we find that the total number of CM-points of discriminant \( r^2 d_0 \) such that their corresponding \( b_1 \) satisfy (34) with \( w = r' \) is
\[ h(r^2 d_0) \prod_{q|(DN, d)} m(\mathcal{O}, r^2 d_0, q) \prod_{q|DN, q|d} m(\mathcal{O}, r^2 d_0, q) = \frac{h_{D,N}(r^2 d_0)}{2^{r^2 d_0}}, \]

where \( h_{D,N}(r^2 d_0) \) denotes the number of CM-points of discriminant \( r^2 d_0 \) on \( X_0^D(N) \). It is clear that the case \( w = -r' \) give the same number of CM-points, and the proof of the proposition is complete.

Putting (32), Lemma [4] and the above proposition together, and summing over all positive divisors \( r \) of \( f \), we obtain
\[ I_0(\mathcal{X}, G_n) = \frac{1}{|W|} \sum_{u \equiv r \bmod{2}, v \equiv c \bmod{2}, \det M_{u,v} > 0} H_{D,N}(\det M_{u,v}/4). \]

Since \( \det M_{u,v} = 4DNn - Q(v, -u) \), we arrive at the following formula for \( I_0(\mathcal{X}, G_n) \).

**Proposition 16.** We have
\[ I_0(\mathcal{X}, G_n) = \frac{1}{|W|} \sum_{u \equiv a \bmod{2}, v \equiv c \bmod{2}, Q(v, u) < 4DNn} H_{D,N}(DNn - \frac{Q(v, u)}{4}). \]

**4.3. Determination of \( I_1(\mathcal{X}, G_n) \).** As above, we let \( \mathcal{F} \) be the image of a fundamental domain of \( X_0^D(N) \) under the quaternion modular embedding given by (16) or (18), according to whether \( Q \) is primitive or is of the form \( 4Q' \). We have
\[ I_1(\mathcal{X}, G_n) = \frac{1}{2|W|} \sum_{\ell \colon \Delta(\ell) = n, \mathcal{F} \subset \mathcal{H}_{\ell}} 1. \]

Here again, the appearance of the factor \( 1/2 \) is due to the fact that \( \mathcal{H}_{\ell} \) and \( \mathcal{H}_{-\ell} \) define the same surface in \( \mathbb{H}_2 \). Note that \( \mathcal{F} \subset \mathcal{H}_{\ell} \) if and only if \( \ell \) is in the span of \( \ell_1 \) and \( \ell_2 \), i.e., if and only if the matrix \( M_0 \) in (20) has determinant 0. Also, such a singular relation is completely determined by the values of \( \langle \ell_1, \ell \rangle_\Delta \) and \( \langle \ell_2, \ell \rangle_\Delta \). It follows that
\[ I_1(\mathcal{X}, G_n) = \frac{1}{2|W|} \sum_{u \equiv a \bmod{2}, v \equiv c \bmod{2}, \det M_{u,v} = 0} 1. \]
where $M_{u,v}$ is the matrix in (31). Since $\det M_{u,v} = 4DNn - Q(v,-u)$ and
\[
\frac{1}{2} \text{Vol}(\mathcal{F}) = \frac{DN}{12} \prod_{q \mid D} \left( 1 - \frac{1}{q} \right) \prod_{q \mid N} \left( 1 + \frac{1}{q} \right) = H_{D,N}(0),
\]
we may write the formula as follows.

**Proposition 17.** We have
\[
\sum_{u \equiv a \mod 2, \ v \equiv c \mod 2} H_{D,N} \left( DNn - \frac{Q(v,u)}{4} \right).
\]

4.4. **Proof of Theorem 1** Combining Propositions 11, 16, and 17, we obtain the class number relations (7).

**REFERENCES**

[1] Henri Cohen. Sums involving the values at negative integers of $L$-functions of quadratic characters. *Math. Ann.*, 217(3):271–285, 1975.

[2] David A. Cox. *Primes of the form $x^2 + ny^2$*. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1989. Fermat, class field theory and complex multiplication.

[3] Benedict H. Gross and Kevin Keating. On the intersection of modular correspondences. *Invent. Math.*, 112(2):225–245, 1993.

[4] David Gruenewald. *Explicit algorithms for Humbert surfaces*. 2008. Thesis (Ph.D.)–University of Sydney.

[5] Ki-ichiro Hashimoto. Explicit form of quaternion modular embeddings. *Osaka J. Math.*, 32(3):533–546, 1995.

[6] Carl Friedrich Hermann. Die Nullstellen der Thetareihen zu positiven, binär-quadratischen Formen. *Manuscripta Math.*, 74(1):107–115, 1992.

[7] Friedrich Hirzebruch and Don Zagier. Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus. *Invent. Math.*, 36:57–113, 1976.

[8] Tomoyoshi Ibukiyama. On maximal orders of division quaternion algebras over the rational number field with certain optimal embeddings. *Nagoya Math. J.*, 88:181–195, 1982.

[9] Stephen S. Kudla and John J. Millson. Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables. *Inst. Hautes Études Sci. Publ. Math.*, (71):121–172, 1990.

[10] Yi-Hsuan Lin and Yifan Yang. Quaternionic loci in Siegel’s modular threefold. *ArXiv e-prints*, July 2018.

[11] Takayuki Oda. On modular forms associated with indefinite quadratic forms of signature $(2, n - 2)$. *Math. Ann.*, 231(2):97–144, 1977/78.

[12] Victor Rotger. Quaternions, polarization and class numbers. *J. Reine Angew. Math.*, 561:177–197, 2003.

[13] Victor Rotger. Modular Shimura varieties and forgetful maps. *Trans. Amer. Math. Soc.*, 356(4):1535–1550, 2004.

[14] Victor Rotger. Shimura curves embedded in Igusa’s threefold. In *Modular curves and abelian varieties*, volume 224 of *Progr. Math.*, pages 263–276. Birkhäuser, Basel, 2004.

[15] Runge. Endomorphism rings of abelian surfaces and projective models of their moduli spaces. *Tohoku Math. J.* (2), 51(3):283–303, 1999.

[16] Gerard van der Geer. On the geometry of a Siegel modular threefold. *Math. Ann.*, 260(3):317–350, 1982.

[17] Don Zagier. Nombres de classes et formes modulaires de poids 3/2. *C. R. Acad. Sci. Paris Sér. A-B*, 281(21):A1, A883–A886, 1975.

Department of Mathematics, National Taiwan University, Taipei, Taiwan 10617

E-mail address: jiaweigo312@gmail.com

Department of Mathematics, National Taiwan University and National Center for Theoretical Sciences, Taipei, Taiwan 10617

E-mail address: yangyifan@ntu.edu.tw