Minimum-energy wavelet frames generated by the Walsh polynomials

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Abstract: Drawing inspiration from the construction of tight wavelet frames generated by the Walsh polynomials, we introduce the notion of minimum-energy wavelet frames generated by the Walsh polynomials on positive half-line \( \mathbb{R}^+ \) using unitary extension principles and present its equivalent characterizations in terms of their framelet symbols. Moreover, based on polyphase components of the Walsh polynomials, we obtain a necessary and sufficient condition for the existence of minimum-energy wavelet frames in \( L^2(\mathbb{R}^+) \). Finally, we derive the minimum-energy wavelet frame decomposition and reconstruction formulae which are quite similar to those of orthonormal wavelets on local fields of positive characteristic.

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1. Introduction

The notion of frames was first introduced by Duffin and Schaeffer (1952) in connection with some deep problems in nonharmonic Fourier series. Frames are basis-like systems that span a vector space but allow for linear dependency, which can be used to reduce noise, find sparse representations, or...
obtain other desirable features unavailable with orthonormal bases. The idea of Duffin and Schaeffer did not generate much interest outside nonharmonic Fourier series until the seminal work by Daubechies, Grossmann, and Meyer (1986). They combined the theory of continuous wavelet transforms with the theory of frames to introduce wavelet (affine) frames for $L^2(\mathbb{R})$. After their work, the theory of frames began to be studied widely and deeply. Today, the theory of frames has become an interesting and fruitful field of mathematics with abundant applications in signal processing, image processing, harmonic analysis, Banach space theory, sampling theory, wireless sensor networks, optics, filter banks, quantum computing, medicine, and so on. An introduction to the frame theory and its applications can be found in Christensen (2003), Daubechies (1992), Debnath and Shah (2015), Shah (2015), Debnath (2011b). Recent results in this direction can also be found in Farkov, Maksimov, and Stroganov (2011), Meenakshi, Manchanda, and Siddiqi (2012), Shah (2015), Sharma and Manchanda (2013) and the references therein.

An important example about frame is wavelet frame, which is obtained by translating and dilating a finite family of functions. One of the most useful methods to construct wavelet frames is through the concept of unitary extension principle (UEP) introduced by Ron and Shen (1997) and were subsequently extended by Daubechies, Han, Ron, and Shen (2003) in the form of the oblique extension principle (OEP). They give sufficient conditions for constructing tight and dual wavelet frames for any given refinable function $\phi(x)$ which generates a multiresolution analysis. The resulting wavelet frames are based on multiresolution analysis, and the generators are often called framelets. The advantages of MRA-based wavelet frames and their promising features in applications have attracted a great deal of interest and effort in recent years. To mention only a few references on wavelet frames, the reader is referred to Chui and He (2000), Dong et al. (2012), Farkov, Lebedeva, and Skopina (2015), Gao and Cao (2008), Han (2012), Huang and Cheng (2007), Huang, Li, and Li (2012), Zhu, Li, and Huang (2013) and many references therein.

The past decade has also witnessed a tremendous interest in the problem of constructing compactly supported orthonormal scaling functions and wavelets with an arbitrary dilation factor $p \geq 2, p \in \mathbb{N}$ (see Debnath & Shah, 2015). The motivation comes partly from signal processing and numerical applications, where such wavelets are useful in image compression and feature extraction because of their small support and multifractal structure. Lang (1996) constructed several examples of compactly supported wavelets for the Cantor dyadic group by following the procedure of Daubechies (1992) via scaling filters and these wavelets turn out to be certain lacunary Walsh series on the real line. Kozyrev (2002) found a compactly supported $p$-adic wavelet basis for $L^2(\mathbb{Q}_p)$ which is an analog of the Haar basis. The concept of multiresolution analysis on a positive half-line $\mathbb{R}^+$ was recently introduced by Farkov (2009). He pointed out a method for constructing compactly supported orthogonal $p$-wavelets related to the Walsh functions, and proved necessary and sufficient conditions for scaling filters with $p^n$ many terms ($p, n \geq 2$) to generate a $p$-MRA in $L^2(\mathbb{R}^+)$. Subsequently, dyadic wavelet frames on the positive half-line $\mathbb{R}^+$ were constructed by Shah and Debnath (2011a) using the machinery of Walsh–Fourier transforms. They have established a necessary and sufficient conditions for the system \[
\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) : j \in \mathbb{Z}, k \in \mathbb{Z}^+\}\] to be a frame for $L^2(\mathbb{R}^+)$. Wavelet packets and wavelet frame packets related to the Walsh polynomials were deeply investigated in a series of papers by the author in Shah (2009, 2012a, 2012b), Shah and Debnath (2011b). Recent results in this direction can also be found in Farkov, Maksimov, and Stroganov (2011), Meenakshi, Manchanda, and Siddiqi (2012), Shah (2015), Sharma and Manchanda (2013) and the references therein.
A constructive procedure for constructing tight wavelet frames generated by the Walsh polynomials using extension principles was first reported by Shah (2013). He provided a sufficient condition for finite number of functions \( \{ \psi_1, \psi_2, \ldots, \psi_p \} \) to form a tight wavelet frame for \( L^2(\mathbb{R}^+) \). Although wavelet frames have many desirable features but the computational complexity and numerical instability during the course of decomposition and reconstruction of functions always remains a debate of discussion (see Dong et al., 2012; Han, 2012). Therefore, in order to reduce the computational complexity and maintain the numerical stability, we shall introduce the concept of minimum-energy wavelet frames associated with the Walsh polynomials on \( \mathbb{R}^+ \) by extending the above-described method (Shah, 2013). More precisely, we present an equivalent characterizations of minimum-energy wavelet frames related to Walsh polynomials is also given. Finally, we derive the minimum-energy wavelet frame decomposition and reconstruction formulas which are quite similar to those of orthonormal wavelets on positive half-line \( \mathbb{R}^+ \).

The paper is structured as follows. In Section 2, we introduce some notations and preliminaries related to the operations on positive half-line \( \mathbb{R}^+ \) including the definitions of the Walsh–Fourier transform, \( p \)-multiresolution analysis and minimum-energy wavelet frame related to the Walsh polynomials. In Section 3, we construct minimum-energy wavelet frames generated by the Walsh polynomials and establish a necessary and sufficient condition for the existence of minimum-energy wavelet frames in \( L^2(\mathbb{R}^+) \). Section 4, deals with the decomposition and reconstruction algorithms of the minimum-energy wavelet frames on a half-line \( \mathbb{R}^+ \).

2. Walsh–Fourier analysis and MRA-based wavelet frames

We start this section with certain results on Walsh–Fourier analysis. We present a brief review of generalized Walsh functions, Walsh–Fourier transforms, and its various properties.

As usual, let \( \mathbb{R}^+ = [0, +\infty), \mathbb{Z}^+ = \{0, 1, 2, \ldots\} \) and \( \mathbb{N} = \mathbb{Z}^+ - \{0\} \). Denote by \( [x] \) the integer part of \( x \). Let \( p \) be a fixed natural number greater than 1. For \( x \in \mathbb{R}^+ \) and any positive integer \( j \), we set

\[
x_j = [p^j x] \pmod{p}, \quad x_{-j} = [p^{1-j} x] \pmod{p}, \quad (2.1)
\]

where \( x_j, x_{-j} \in \{0, 1, \ldots, p-1\} \). It is clear that for each \( x \in \mathbb{R}^+ \), there exist \( k = k(x) \) in \( \mathbb{N} \) such that \( x_j = 0 \) \( \forall j > k \).

Consider on \( \mathbb{R}^+ \) the addition defined as follows:

\[
x \oplus y = \sum_{j=0}^{\infty} \zeta_j p^{-j} + \sum_{j=0}^{\infty} \zeta_j p^{-j},
\]

with \( \zeta_j = x_j + y_j \pmod{p}, j \in \mathbb{Z} \setminus \{0\} \), where \( \zeta_j \in \{0, 1, \ldots, p-1\} \) and \( x_j, y_j \) are calculated by (2.1). As usual, we write \( z = x \ominus y \) if \( z \oplus y = x \), where \( \ominus \) denotes subtraction modulo \( p \) in \( \mathbb{R}^+ \).

For \( x \in [0, 1) \), let \( r_0(x) \) is given by

\[
r_0(x) = \begin{cases} 
1, & \text{if } x \in [0, 1/p) \\
\epsilon_\rho^\ell, & \text{if } x \in [\ell p^{-1}, (\ell+1)p^{-1}), \quad \ell = 1, 2, \ldots, p-1,
\end{cases}
\]

where \( \epsilon_\rho = \exp(2\pi i/p) \). The extension of the function \( r_0 \) to \( \mathbb{R}^+ \) is given by the equality \( r_0(x+1) = r_0(x), \ x \in \mathbb{R}^+ \). Then, the generalized Walsh functions \( \{w_m(x) : m \in \mathbb{Z}^+\} \) are defined by

\[
w_0(x) \equiv 1 \quad \text{and} \quad w_m(x) = \prod_{j=0}^{k} (r_0(p^j x))^\delta_j
\]
where $m = \sum_{j=0}^{m} \mu_j p^j$, $\mu_j \in \{0, 1, \ldots, p - 1\}$, $\mu_k \neq 0$. They have many properties similar to those of the Haar functions and trigonometric series, and form a complete orthogonal system. Further, by a Walsh polynomial we shall mean a finite linear combination of Walsh functions.

For $x, y \in \mathbb{R}^+$, let

$$
\chi(x, y) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x y_j + x_j y)\right),
$$

(2.2)

where $x_j, y_j$ are given by (2.1).

We observe that

$$
\chi(x, y) = \chi(x, z) \chi(y, z),
$$

and

$$
\chi(x \oplus y, z) = \chi(x, z) \chi(y, z),
$$

where $x, y, z \in \mathbb{R}^+$ and $x \oplus y$ is $p$-adic irrational. It is well known that systems $\{\chi(x, z)\}_{z=0}^{+\infty}$ and $\{\chi(x, y)\}_{y=0}^{+\infty}$ are orthonormal bases in $L^2(\mathbb{R}^+)$ (see Golubov, Efimov, & Skvortsoy, 1991).

The Walsh–Fourier transform of a function $f \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ is defined by

$$
\hat{f}(\xi) = \int_{\mathbb{R}^+} f(x) \chi(x, \xi) \, dx,
$$

(3.3)

where $\chi(x, \xi)$ is given by (2.2). The Walsh–Fourier operator $F : L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$, $F f = \hat{f}$, extends uniquely to the whole space $L^2(\mathbb{R}^+)$. The properties of the Walsh–Fourier transform are quite similar to those of the classic Fourier transform (see Golubov et al., 1991; Schipp, Wade, & Simon, 1990). In particular, if $f \in L^2(\mathbb{R}^+)$ then $\hat{f} \in L^2(\mathbb{R}^+)$

and

$$
\|\hat{f}\|_{L^2(\mathbb{R}^+)} = \|f\|_{L^2(\mathbb{R}^+)},
$$

By $p$-adic interval $I \subset \mathbb{R}^+$ of range $n$, we mean intervals of the form

$$
I = I_n^k = [kp^{-n}, (k+1)p^{-n}), \quad k \in \mathbb{Z}^+.
$$

The $p$-adic topology is generated by the collection of $p$-adic intervals and each $p$-adic interval is both open and closed under the $p$-adic topology (see Schipp et al., 1990). The family $\{0, p^j : j \in \mathbb{Z}\}$ forms a fundamental system of the $p$-adic topology on $\mathbb{R}^+$. Therefore, for each $0 \leq j, k < p^n$, the Walsh function $w_j(x)$ is piecewise constant and hence continuous. Thus $w_j(x) = 1$ for $x \in I_n^k$.

Let $\mathcal{E}_n(\mathbb{R}^+)$ be the space of $p$-adic entire functions of order $n$, that is, the set of all functions which are constant on all $p$-adic intervals of range $n$. Thus, for every $f \in \mathcal{E}_n(\mathbb{R}^+)$, we have

$$
f(x) = \sum_{k \in \mathbb{Z}^+} f(p^{-n}k) \chi_n(x), \quad x \in \mathbb{R}^+.
$$

(4.4)

Clearly each Walsh function of order $p^{n-1}$ belong to $\mathcal{E}_n(\mathbb{R}^+)$. The set $\mathcal{E}(\mathbb{R}^+)$ of $p$-adic entire functions on $\mathbb{R}^+$ is the union of all the spaces $\mathcal{E}_n(\mathbb{R}^+)$. It is clear that $\mathcal{E}(\mathbb{R}^+)$ is dense in $L^p(\mathbb{R}^+)$, $1 \leq p < \infty$ and each function in $\mathcal{E}(\mathbb{R}^+)$ is of compact support.
Next, we give a brief account of the MRA-based wavelet frames generated by the Walsh polynomials on a positive half-line \( \mathbb{R}^+ \). Following the unitary extension principle, one often starts with a refinable function or even with a refinement mask to construct desired wavelet frames. A compactly supported function \( \phi \in L^2(\mathbb{R}^+) \) is called a refinable function, if it satisfies an equation of the type

\[
\phi(x) = p \sum_{k=0}^{p^n-1} c_k \phi(px \oplus k), \quad x \in \mathbb{R}^+
\]

(2.5)

where \( c_k \) are complex coefficients. Applying the Walsh–Fourier transform, we can write this equation as

\[
\hat{\phi}(\xi) = h_0(p^{-1}\xi)\hat{\phi}(p^{-1}\xi),
\]

(2.6)

where

\[
\hat{h}_0(\xi) = \sum_{k=0}^{p^n-1} c_k \hat{w}_k(\xi),
\]

(2.7)

is a generalized Walsh polynomial, which is called the mask or symbol of the refinable function \( \phi \) and is of course a \( p \)-adic step function. Observe that \( w_k(0) = \hat{\phi}(0) = 1 \). Hence, letting \( \xi = 0 \) in (2.6) and (2.7), we obtain \( \sum_{k=0}^{p^n-1} c_k = 1 \). Since \( \phi \) is compactly supported and in fact \( \text{supp} \phi \subset [0, p^n-1] \), therefore \( \phi \in L^2_{p^{-1}\xi}((\mathbb{R}^+)) \) and hence as a result \( \hat{\phi}(\xi) = 1 \) for all \( \xi \in [0, p^{n-1}) \) as \( \hat{\phi}(0) = 1 \). Moreover, if \( b_s = h_0(sp^{-n}) \) represents the values of the mask \( h_0(\xi) \) on \( p \)-adic intervals, i.e.

\[
b_s = \sum_{k=0}^{p^n-1} c_k \hat{w}_k(s p^{-n}), \quad 0 \leq s \leq p^n-1,
\]

(2.8)

then

\[
c_k = \frac{1}{p^n} \sum_{s=0}^{p^n-1} b_s \hat{w}_k(s p^{-n}), \quad 0 \leq k \leq p^n-1.
\]

(2.9)

and, conversely, equalities (2.8) follow from (2.9). These discrete transforms can be realized by the fast Vilenkin–Chrestenson transform (see Golubov et al., 1991). Using Parseval’s relation for the discrete transforms, Equations (2.8) and (2.9) can be written as

\[
\sum_{k=0}^{p^n-1} |c_k|^2 = \frac{1}{p^n} \sum_{s=0}^{p^n-1} |b_s|^2.
\]

(2.10)

For a compactly supported refinable function \( \phi \in L^2(\mathbb{R}^+) \), let \( V_0 \) be the closed shift invariant space generated by \( \{ \phi(x \oplus k) : k \in \mathbb{Z}^+ \} \) and \( V_j = \{ \phi(px) : \phi \in V_0 \} \) \( j \in \mathbb{Z} \). Then, it is proved in Farkov (2009) that the closed subspaces \( \{ V_j : j \in \mathbb{Z} \} \) forms a \( p \)-multiresolution analysis ( \( p \)-MRA) for \( L^2(\mathbb{R}^+) \). Recall that a \( p \)-MRA is a family of closed subspaces \( \{ V_j \}_{j \in \mathbb{Z}} \) of \( L^2(\mathbb{R}^+) \) that satisfies: (i) \( V_j \subset V_{j+1}, j \in \mathbb{Z} \), (ii) \( \bigcup_{j \in \mathbb{Z}} V_j \) is dense in \( L^2(\mathbb{R}^+) \) and (iii) \( \bigcap_{j \in \mathbb{Z}} V_j = \{ 0 \} \).

Given an \( p \)-MRA generated by a compactly supported refinable function \( \phi(x) \), one can construct a set of basic tight framelets \( \Psi = \{ \psi_1, \ldots, \psi_L \} \subset V_1 \) satisfying

\[
\hat{\psi}_j(\xi) = h_j(p^{-1}\xi)\hat{\phi}(p^{-1}\xi),
\]

(2.11)

where
The wavelet system $\mathcal{X}(\Psi) = \{ \psi_{j,k}(x) = p^{j/2} \psi(\lfloor p^j x + k \rfloor), j \in \mathbb{Z}, k \in \mathbb{Z}^+, \ell = 1, 2, \ldots, L \}$ forms a tight frame of $L^2(\mathbb{R}^+)$. In this connection, Shah (2013) gave an explicit construction scheme for the construction of tight wavelet frames generated by the Walsh polynomials using unitary extension principles in the following way.

**Theorem 2.1** Let $\phi(x)$ be a compactly supported refinable function and $\hat{\phi}(0) = 1$. Then, the wavelet system $\mathcal{X}(\Psi)$ given by (2.14) constitutes a normalized tight wavelet frame in $L^2(\mathbb{R}^+)$ provided the matrix $\mathcal{M}(\xi)$ as defined in (2.13) satisfies

$$\mathcal{M}(\xi) \mathcal{M}^*(\xi) = I_p, \quad \text{for a.e. } \xi \in \sigma(\mathcal{V}_0)$$

(2.15)

where $\sigma(\mathcal{V}_0) := \{ \xi \in [0,1]: \sum_{k \in \mathbb{Z}^+} |\hat{\phi}(\xi \oplus k)|^2 \neq 0 \}$. 

Motivated and inspired by the construction of tight wavelet frames generated by the Walsh polynomials (Shah, 2013), we extend this concept to minimum-energy wavelet frames on the positive half-line $\mathbb{R}^+$ using the machinery of unitary extension principles. Note that, in this paper, we suppose that any symbol function is a Walsh polynomial, and scaling function and wavelet functions are compactly supported.

**Definition 2.1** Let $\phi \in L^2(\mathbb{R}^+)$ satisfies $\hat{\phi} \in L^\infty$ and $\phi$ is continuous at 0, and $\hat{\phi}(0) = 1$. Suppose that $\phi$ generates a sequence of nested closed subspaces $\{ \mathcal{V}_j : j \in \mathbb{Z} \}$. Then, a finite family $\Psi = \{ \psi_1, \psi_2, \ldots, \psi_L \} \subset \mathcal{V}_1$ is called a minimum-energy wavelet frame associated with $\phi(x)$ if for all $f \in L^2(\mathbb{R}^+)$

$$\sum_{k \in \mathbb{Z}^+} |\langle f, \psi_{1,k} \rangle|^2 = \sum_{k \in \mathbb{Z}^+} |\langle f, \phi_{0,k} \rangle|^2 + \sum_{\ell=1}^L \sum_{k \in \mathbb{Z}^+} |\langle f, \psi_{0,\ell,k} \rangle|^2.$$ 

(2.16)

By Parseval’s identity, minimum-energy wavelet frame $\Psi$ must be a tight frame for $L^2(\mathbb{R}^+)$ with frames bound equal to 1. At the same time, formula (2.16) is equivalent to

$$\sum_{k \in \mathbb{Z}^+} \langle f, \phi_{1,k} \rangle \phi_{1,k} = \sum_{k \in \mathbb{Z}^+} \langle f, \phi_{0,k} \rangle \phi_{0,k} + \sum_{\ell=1}^L \sum_{k \in \mathbb{Z}^+} \langle f, \psi_{0,\ell,k} \rangle \psi_{0,\ell,k}, \quad \text{for all } f \in L^2(\mathbb{R}^+)$$

(2.17)

**3. Construction of minimum-energy wavelet frames**

In this section, we give a complete characterization of minimum-energy wavelet frames associated with some given refinable functions in terms of their framelet symbols. More precisely, we present a necessary and sufficient condition for the existence of minimum-energy wavelet frames generated by Walsh polynomials.
The following theorem presents the equivalent characterizations of the minimum-energy wavelet frame associated with given compactly supported refinable function \( \phi(x) \).

**Theorem 3.1** Suppose that every element of the framelet symbols, \( h_0(\xi), h_r(\xi), r = 1, 2, \ldots, L \) in (2.7) and (2.12) is a Walsh polynomial, and the compactly supported function \( \phi(x) \) associated with \( h_0(\xi) \) generates a nested subspace \( \{ V_j : j \in \mathbb{Z} \} \). Then the following statements are equivalent:

1. \( \Psi = \{ \psi_1, \psi_2, \ldots, \psi_L \} \) is a minimum-energy wavelet frame associated with \( \phi(x) \).
2. \( \mathcal{A}(\xi)\mathcal{A}^*(\xi) = I_p \), for a.e \( \xi \in \sigma(V_0) \).
3. \[ a_{m,n} = \sum_{k \in \mathbb{Z}^+} \left( c_{m-p,k}c_{n-p,k} + \sum_{l=1}^{L} d_{m-p,k}^l d_{n-p,k}^l \right) - p\delta_{m,n} = 0, \quad \forall \ m, n \in \mathbb{Z}^+. \] (3.2)

**Proof** By using the functional Equations (2.5) and (2.11) and notation \( a_{m,n} \), Equation (2.17) can be written as

\[ \sum_{m \in \mathbb{Z}^+} \sum_{n \in \mathbb{Z}^+} a_{m,n} \left( f, \phi(px \oplus m) \right) \phi(px \oplus n) = 0, \quad \forall f \in L^2(\mathbb{R}^+). \] (3.3)

On the other hand, formula (3.1) can be reformulated as

\[ \left| h_0(p^{-1}\xi) \right|^2 + \sum_{r=1}^{L} \left| h_r(p^{-1}\xi) \right|^2 = 1, \] (3.4)

\[ h_0(p^{-1}\xi) \overline{h_0}(\xi \oplus k/p) + \sum_{r=1}^{L} h_r(p^{-1}\xi) \overline{h}_r(\xi \oplus k/p) = 0, \quad k = 1, 2, \ldots, p - 1, \]

which is equivalent to

\[ h_0(p^{-1}\xi) \sum_{k=0}^{p-1} \overline{h_0}(\xi \oplus k/p) + \sum_{r=1}^{L} h_r(p^{-1}\xi) \left( \sum_{k=0}^{p-1} \overline{h}_r(\xi \oplus k/p) \right) = 1, \]

or

\[ h_0(p^{-1}\xi) \left( \overline{h_0}(\xi) - \sum_{k=0}^{p-1} \overline{h_0}(\xi \oplus k/p) \right) + \sum_{r=1}^{L} h_r(p^{-1}\xi) \left( \overline{h}_r(\xi) - \sum_{k=0}^{p-1} \overline{h}_r(\xi \oplus k/p) \right) = 1, \]

\[ h_0(p^{-1}\xi) \left( \sum_{k=0}^{p-1} \overline{h_0}(\xi \oplus k/p) - 2 \overline{h_0}(\xi \oplus m/p) \right) \sum_{r=1}^{L} h_r(p^{-1}\xi) \]

\[ \times \left( \sum_{k=0}^{p-1} \overline{h}_r(\xi \oplus k/p) - 2 \overline{h}_r(\xi \oplus m/p) \right) = 1, \quad m = 1, 2, \ldots, p - 1. \]

The above system is equivalent to

\[
\begin{align*}
&h_0(p^{-1}\xi) \sum_{k \in \mathbb{Z}^+} c_{pk} w_{pk}(\xi) + \sum_{r=1}^{L} h_r(p^{-1}\xi) \sum_{k \in \mathbb{Z}^+} d_{pk}^r w_{pk}(\xi) = 1, \\
&h_0(p^{-1}\xi) \left( \sum_{m=1}^{L} \sum_{k \in \mathbb{Z}^+} c_{mk} w_{mk}(\xi) \right) + \sum_{r=1}^{L} h_r(p^{-1}\xi) \left( \sum_{m=1}^{L} \sum_{k \in \mathbb{Z}^+} d_{mk}^r w_{mk}(\xi) \right) = p - 1.
\end{align*}
\] (3.5)

The above system can be further expressed as

\[
\begin{align*}
&h_0(p^{-1}\xi) \sum_{k \in \mathbb{Z}^+} c_{pk} w_{pk}(\xi) + \sum_{r=1}^{L} h_r(p^{-1}\xi) \sum_{k \in \mathbb{Z}^+} d_{pk}^r w_{pk}(\xi) = 1, \\
&h_0(p^{-1}\xi) \sum_{k \in \mathbb{Z}^+} c_{1pk} w_{1pk}(\xi) + \sum_{r=1}^{L} h_r(p^{-1}\xi) \sum_{k \in \mathbb{Z}^+} d_{1pk}^r w_{1pk}(\xi) = 1, \\
& \vdots \\
&h_0(p^{-1}\xi) \sum_{k \in \mathbb{Z}^+} c_{p-1pk} w_{p-1pk}(\xi) + \sum_{r=1}^{L} h_r(p^{-1}\xi) \sum_{k \in \mathbb{Z}^+} d_{p-1pk}^r w_{p-1pk}(\xi) = 1.
\end{align*}
\]
Multiply the identities of (3.5) with \( \hat{\phi}(p^{-1}\xi)w_m(\xi) \), \( m = 0, 1, \ldots, p-1 \), we obtain

\[
\hat{\phi}(p^{-1}\xi)w_m(\xi) = \sum_{k \in \mathbb{Z}} \left( c_{m-pk} \hat{w}_{pk}(\xi) \hat{h}_0(p^{-1}\xi) \hat{\phi}(p^{-1}\xi) + \sum_{r=1}^{l} d'_{m-pk} \hat{w}_{pk}(\xi) \hat{h}_r(p^{-1}\xi) \hat{\phi}(p^{-1}\xi) \right). \tag{3.6}
\]

Therefore, the system (3.5) can be written as

\[
\begin{align*}
\hat{\phi}(p^{-1}\xi)w_0(\xi) &= \sum_{k \in \mathbb{Z}} \left( c_{-pk} \hat{w}_{pk}(\xi) \hat{\phi}(\xi) + \sum_{r=1}^{l} d'_{-pk} \hat{w}_{pk}(\xi) \hat{\phi}'(\xi) \right), \\
\hat{\phi}(p^{-1}\xi)w_1(\xi) &= \sum_{k \in \mathbb{Z}} \left( c_{-1-pk} \hat{w}_{pk}(\xi) \hat{\phi}(\xi) + \sum_{r=1}^{l} d'_{-1-pk} \hat{w}_{pk}(\xi) \hat{\phi}'(\xi) \right), \\
&\vdots \\
\hat{\phi}(p^{-1}\xi)w_{p-1}(\xi) &= \sum_{k \in \mathbb{Z}} \left( c_{p-1-pk} \hat{w}_{pk}(\xi) \hat{\phi}(\xi) + \sum_{r=1}^{l} d'_{p-1-pk} \hat{w}_{pk}(\xi) \hat{\phi}'(\xi) \right).
\end{align*}
\]

This system of equations can be written in time domain as

\[
\begin{align*}
\phi(x) &= \sum_{k \in \mathbb{Z}} \left( c_{-pk} \phi(x \ominus k/p) + \sum_{r=1}^{l} d'_{-pk} \psi'(x \ominus k/p) \right), \\
\phi(x \ominus 1/p) &= \sum_{k \in \mathbb{Z}} \left( c_{-1-pk} \phi(x \ominus k/p) + \sum_{r=1}^{l} d'_{-1-pk} \psi'(x \ominus k/p) \right), \\
&\vdots \\
\phi(x \ominus (p-1)/p) &= \sum_{k \in \mathbb{Z}} \left( c_{p-1-pk} \phi(x \ominus k/p) + \sum_{r=1}^{l} d'_{p-1-pk} \psi'(x \ominus k/p) \right).
\end{align*}
\]

On the reformulation of above system, we obtain

\[
\phi(x \ominus m/p) = \sum_{k \in \mathbb{Z}} \left( c_{m-pk} \phi(x \ominus k/p) + \sum_{r=1}^{l} d'_{m-pk} \psi'(x \ominus k/p) \right), \quad m \in \mathbb{Z}^+. \tag{3.7}
\]

Using (2.5) and its corresponding wavelet equation, we can rewrite formula (3.7) as

\[
\sum_{m \in \mathbb{Z}} a_{m,n} \phi(x \ominus m/p) = 0, \quad \forall \ n \in \mathbb{Z}^+. \tag{3.8}
\]

Thus, the UEP condition (3.1) is equivalent to (3.8). In conclusion, the proof of the theorem reduces to the proof of the equivalence of (3.2), (3.3), and (3.8).

It is obvious that (3.2) implies (3.8) which implies (3.3). In order to prove \( (3.3) \implies (3.2) \), we assume that \( f \) be a function of compact support, i.e. \( f \in \mathcal{E}(\mathbb{R}^+) \). By using the properties that for every fixed \( m \), \( a_{m,n} = 0 \) except for finitely many \( n \), the functional

\[
\beta_n(f) = \sum_{m \in \mathbb{Z}^+} a_{m,n} \left( f, \phi(\cdot \ominus m/p) \right), \quad n \in \mathbb{Z}^+,
\]

just has finite nonzero's for \( n \in \mathbb{Z}^+ \). Since \( \hat{\phi}(\xi) \) is nontrivial function, by taking the Fourier transform of (3.3), it follows that the polynomial \( \sum_{m \in \mathbb{Z}^+} \beta_n(f)w_n(\xi) \) is identically zero. Obviously, \( \beta_n(f) = 0 \), \( n \in \mathbb{Z}^+ \). In other words, we say that

\[
\left( f, \sum_{m \in \mathbb{Z}^+} a_{m,n} \phi(x \ominus m/p) \right) = 0, \quad n \in \mathbb{Z}^+.
\]

Thus, the series in the above equation is a finite sum and hence represents a compactly supported function in \( L^2(\mathbb{R}^+) \). By choosing \( f \) to be this function, it follows that
\[ \sum_{m, n} a_{mn} \varphi(x \ominus m/p) = 0, \]

which implies that the polynomial \( \sum_{m, n} a_{mn} w(\xi) \) is identically equal to 0 so that \( a_{mn} = 0, m, n \in \mathbb{Z}^+ \).

This completes the proof of the theorem. \( \square \)

Now we shall present a necessary condition for minimum-energy wavelet frames generated by the Walsh polynomials in terms of their wavelet masks.

**Theorem 3.2** Let \( \phi \in L^2(\mathbb{R}^+) \) be a compactly supported refinable function with refinement mask \( h_0(\xi) \) such that \( \hat{\phi} \) is continuous at 0 and \( \hat{\phi}(0) = 1 \). If \( \Psi = \{\psi_1, \psi_2, \ldots, \psi_L\} \) is the minimum-energy wavelet frame associated with \( \phi(x) \), then

\[ \sum_{m=0}^{p-1} \left| h_0(\xi \oplus m/p) \right|^2 \leq 1, \quad \text{for all } \xi \in \mathbb{R}^+. \] (3.9)

**Proof** Let \( Q(\xi) \) be the first column of the modulation matrix \( M(\xi) \), as defined in (2.13). Then, \( M(\xi) = (Q(\xi), R(\xi)) \), where

\[ R(\xi) = \begin{pmatrix} h_1(\xi) & h_1(\xi \oplus 1/p) & \cdots & h_1(\xi \oplus (p-1)/p) \\ h_2(\xi) & h_2(\xi \oplus 1/p) & \cdots & h_2(\xi \oplus (p-1)/p) \\ \vdots & \vdots & \ddots & \vdots \\ h_L(\xi) & h_L(\xi \oplus 1/p) & \cdots & h_L(\xi \oplus (p-1)/p) \end{pmatrix} \] (3.10)

and

\[ Q(\xi) = [h_0(\xi \oplus 1/p) \cdots h_0(\xi \oplus (p-1)/p)]. \]

Therefore, the condition (3.1) can be reformulated as

\[ Q(\xi)Q^*(\xi) + R(\xi)R^*(\xi) = I_p, \]

or equivalently,

\[ I_p - Q(\xi)Q^*(\xi) = R(\xi)R^*(\xi). \]

Since \( R(\xi)R^*(\xi) \) is a Hermitian matrix, the matrix \( I_p - Q(\xi)Q^*(\xi) \) is positive semi-definite, so that

\[ \det \left( I_p - Q(\xi)Q^*(\xi) \right) \geq 0, \]

and this gives

\[ \sum_{m=0}^{p-1} \left| h_0(\xi \oplus m/p) \right|^2 \leq 1, \quad \text{for all } \xi \in \mathbb{R}^+. \]

In fact, we have

\[ \det \begin{pmatrix} I_p & Q^*(\xi) \\ Q(\xi) & 1 \end{pmatrix} = \det \begin{pmatrix} I_p & -Q^*(\xi) \\ -Q(\xi) & 1 \end{pmatrix} = \det \begin{pmatrix} I_p - Q(\xi)Q^*(\xi) & 0 \\ 0 & 1 - Q(\xi)Q^*(\xi) \end{pmatrix}. \]

\[ \det \begin{pmatrix} I_p & Q^*(\xi) \\ Q(\xi) & 1 \end{pmatrix} = \det \begin{pmatrix} I_p & Q^*(\xi) \\ 0 & 1 - Q(\xi)Q^*(\xi) \end{pmatrix}. \]

\[ \det \begin{pmatrix} I_p & -Q^*(\xi) \\ -Q(\xi) & 1 \end{pmatrix} = \det \begin{pmatrix} I_p & -Q^*(\xi) \\ 0 & 1 - Q(\xi)Q^*(\xi) \end{pmatrix}. \]
Therefore
\[
\det \left( I_p - Q(\xi)Q'(\xi) \right) \left( 1 - Q(\xi)Q'(\xi) \right) = \left( 1 - Q(\xi)Q'(\xi) \right)^2,
\]
and it gives \( 1 - Q(\xi)Q'(\xi) \geq 0 \). The proof of the Theorem 3.2 is completed. \( \square \)

According to the Theorem 3.2, there may not exist minimum-energy wavelet frame associated with a given compactly supported refinable function \( \phi \) and in case if it exist, then the refinement mask must satisfy (3.9). In this context, we provide a sufficient condition for minimum-energy wavelet frames related to the Walsh polynomials based on the polyphase representation of the wavelet masks \( h_0(\xi), \xi' = 0, 1, \ldots, L \).

The polyphase representation of the refinement mask \( h_0(\xi) \) can be derived by using the properties of Walsh polynomials as
\[
h_0(\xi) = \sum_{k=0}^{p^n-1} c_k w_k(\xi)
= \sum_{k=0}^{p^n-1} \sum_{m=0}^{p-1} c_{pk+m} w_{pk+m}(\xi)
= \sum_{m=0}^{p-1} w_m(\xi) \sum_{k=0}^{p^n-1} c_{pk+m} w_k(p\xi)
= \frac{1}{\sqrt{p}} \sum_{m=0}^{p-1} \mu_{0,m}(p\xi) w_m(\xi),
\]
where
\[
\mu_{0,m}(\xi) = \sqrt{p} \sum_{k=0}^{p^n-1} c_{pk+m} w_k(\xi), \quad m = 0, 1, \ldots, p - 1. \tag{3.11}
\]

Similarly, the wavelet masks \( h_r(\xi), 1 \leq r \leq L \), as defined in (2.12) can be splitted into polyphase components as
\[
h_r(\xi) = \frac{1}{\sqrt{p}} \sum_{m=0}^{p^n-1} \mu_{r,m}(p\xi) w_m(\xi), \tag{3.12}
\]
where
\[
\mu_{r,m}(\xi) = \sqrt{p} \sum_{k=0}^{p^n-1} d_{pk+m}^{r} w_k(\xi), \quad m = 0, 1, \ldots, p - 1. \tag{3.13}
\]

With the polyphase components given by (3.11) and (3.13), we formulate the polyphase matrix \( \Gamma(\xi) \) as:
\[
\Gamma(\xi) = \begin{pmatrix}
\mu_{0,0}(\xi) & \mu_{1,0}(\xi) & \cdots & \mu_{L,0}(\xi) \\
\mu_{0,1}(\xi) & \mu_{1,1}(\xi) & \cdots & \mu_{L,1}(\xi) \\
\vdots & \vdots & \ddots & \vdots \\
\mu_{0,p-1}(\xi) & \mu_{1,p-1}(\xi) & \cdots & \mu_{L,p-1}(\xi)
\end{pmatrix}.
\]
Therefore, the modulation matrix \( \mathcal{M}(\xi) \) can be expressed as
\[
\mathcal{M}(\xi) = \Gamma(p\xi) \mathcal{W}^{p}(\xi), \tag{3.14}
\]
where \( \mathcal{W}(\xi) \) is the Walsh matrix given by

\[
\mathcal{W}(\xi) = \frac{1}{\sqrt{p}} \begin{pmatrix}
W_0(\xi) & W_1(\xi) & \cdots & W_{p-1}(\xi) \\
W_0(\xi \oplus 1/p) & W_1(\xi \oplus 1/p) & \cdots & W_{p-1}(\xi \oplus 1/p) \\
\vdots & \vdots & \ddots & \vdots \\
W_0(\xi \oplus (p-1)/p) & W_1(\xi \oplus (p-1)/p) & \cdots & W_{p-1}(\xi \oplus (p-1)/p)
\end{pmatrix}
\]

Thus, we have

\[
\mathcal{M}(\xi), \mathcal{M}'(\xi) = \mathcal{W}(\xi) \Gamma^*(p\xi) \Gamma(p\xi) \mathcal{W}^*(\xi),
\]

and, hence we conclude that

\[
\mathcal{M}(\xi), \mathcal{M}'(\xi) = pI_p \iff \Gamma^*(p\xi) \Gamma(p\xi) = pI_p,
\]

or equivalently, we say that

\[
\sum_{r=0}^{l} h_r(\xi \oplus m/p) h_r(\xi \oplus n/p) = \delta_{m,n} \iff \sum_{r=0}^{l} \mu_{r,m}(p\xi) \mu_{r,n}(p\xi) = \delta_{m,n}, \quad 0 \leq m, n \leq p-1.
\]

Since the Walsh matrix \( \mathcal{W}(\xi) \) is a unitary matrix, therefore, we have

\[
\mathcal{M}(\xi), \mathcal{M}'(\xi) = \Gamma^*(p\xi)(\mathcal{W}(\xi) \mathcal{W}^*(\xi)) \Gamma(p\xi) = \Gamma(p\xi) \Gamma^*(p\xi),
\]

which implies that

\[
\sum_{k=0}^{p-1} h_r(\xi \oplus k/p) h_r(\xi \oplus k/p) = \sum_{k=0}^{p-1} \mu_{r,k}(p\xi) \mu_{r,k}(p\xi), \quad 0 \leq r, r' \leq L.
\]

Therefore, it follows from (3.9) and (3.16) that

\[
\sum_{m=0}^{p-1} |\mu_{0,m}(\xi)|^2 \leq 1, \quad \xi \in \mathbb{R}^+,
\]

which further yields

\[
\sum_{m=0}^{p-1} |b_{0,m}^{s}(\xi)|^2 \leq 1, \quad s = 0, 1, \ldots, p^n - 1,
\]

where \( b_{0,m}^{s} = \mu_{0,m}(p^{1-n} \xi_{(s)}) \). Since the polynomial \( \mu_{r,m}(p\xi) \) is constant on the intervals \( I_{ns}, \quad 0 \leq s \leq p^n - 1 \), so the polyphase components \( \mu_{r,m}(\xi) \) can also be written as

\[
\mu_{r,m}(p\xi) = \sum_{s=0}^{p^{n-1}} b_{r,m}^{s}(\xi) I_{ns}(\xi), \quad m = 0, 1, \ldots, p - 1
\]

where

\[
\sum_{r=0}^{L} b_{r,m}^{s} b_{r,m'}^{s} = \delta_{mm'}, \quad 0 \leq m, m' \leq p - 1, s = 0, 1, \ldots, p^n - 1.
\]

Now, if there exists \( \mu_{0,p}(\xi) \) such that

\[
\sum_{m=0}^{p} |\mu_{0,m}(\xi)|^2 = 1.
\]
then, we have the following theorem which provides a sufficient condition for minimum-energy wavelet frames generated by the Walsh polynomials in $L^2(\mathbb{R}^+)$. 

**Theorem 3.3** Let $h_0(\xi)$ be the refinement mask of a compactly supported refinable function $\phi(x)$ and satisfy inequality (3.17). Furthermore, if there exist $\mu_{0p}(\xi)$ of the form (3.21), then there exists a minimum-energy wavelet frame associated with $\phi(x)$.

**Proof** Under the given assumptions, it is easy to verify that

$$f = \left( \mu_{00}(\xi), \mu_{01}(\xi), \ldots, \mu_{0p-1}(\xi), \mu_{0p}(\xi) \right)^T$$

is a unit vector, where $T$ stands for the transpose of a given vector. By multiplying the diagonal matrix $D_p = \text{diag}(\xi^0, \xi^1, \ldots, \xi^p)$ to the left side of (3.22), we obtain

$$f_1 = \left( \xi^0 \mu_{00}(\xi), \xi^1 \mu_{01}(\xi), \ldots, \xi^{p-1} \mu_{0p-1}(\xi), \xi^p \mu_{0p}(\xi) \right)^T = \sum_{j=0}^{p} u_j \xi^j, \quad t_0, t_1, \ldots, t_p \in \mathbb{Z}^+,$$

where $u_j \in \mathbb{R}^+$, with $u_j \neq 0$ and $u_j \neq 0$. It is also clear that $f_1$ is a unit vector as

$$f_1^* f_1 = \left( \sum_{j=0}^{p} u_j \xi^j \right)^* \left( \sum_{j=0}^{p} u_j \xi^j \right) = 1, \quad \text{for all } \xi \in L^2[0,1]$$

and consequently, $u_0 \neq 0$.

Consider the $(p + 1) \times (p + 1)$ Householder matrix

$$H_1 = I_{p+1} - \frac{2}{\|v\|^2} vv^T,$$

where $v = u_j \pm \|u_j\|e_j$ with $e_j = (1, 0, \ldots, 0)^T_{p+1}$ and the + and - signs are so chosen that $v \neq 0$. Then

$$H_1u_j = \pm \|u_j\|e_j,$$

By the orthogonal property of the Householder matrix, we have

$$(H_1u_0)^T(H_1u_j) = u_0^T H_1^* H_1u_j = u_0^T u_j = 0.$$

Using previous equation, it follows that the first component of $H_1u_0$ is 0. Since $H_1f_1 = \sum_{j=0}^{p} (H_1u_j)\xi^j$, therefore, we can construct a diagonal matrix $D_1 = \text{diag}(\xi^0, 1, \ldots, 1, t_{1j}) \in \mathbb{Z}^+$ such that

$$f_2 = D_1u_1 f_1 = D_1 \sum_{j=0}^{p} (H_1u_j)\xi^j = \sum_{j=0}^{p} u_j^{(1)} \xi^j$$

is also a unit vector and $J_1 < J$, $u_0^{(1)} \neq 0$, $u_1^{(1)} \neq 0$.

Similarly, we define the Householder matrix

$$H_2 = I_{p+1} - \frac{2}{\|\hat{v}\|^2} \hat{v}\hat{v}^T,$$

where $\hat{v} = u_j \pm \|u_j\|e_j \neq 0$, and $D_2 = \text{diag}(\xi^{p_0}, 1, \ldots, 1, t_{1j}) \in \mathbb{Z}^+$ such that

$$f_3 = D_2 H_2 f_2 = D_2 \sum_{j=0}^{p} (H_2u_j)\xi^j = \sum_{j=0}^{p} u_j^{(2)} \xi^j$$
is also a unit vector and \( J_2 < J_1 \), \( u_0^{(2)} \neq 0 \), \( u_1^{(2)} \neq 0 \). Since every component of \( f \) is a finite sum, we repeat this procedure finite times to get some unitary matrices \( D_1, H_1, D_{N-1}, H_{N-1}, \ldots, H_2, D_1, H_1 \) such that

\[
D_1 H_1 D_{N-1} H_{N-1} \cdots H_2 D_1 H_1 f = e_1. \tag{3.25}
\]

Therefore, it is clear that \( f \) is the first column of the unitary matrix

\[
H = D_1^* H_1^* D_{N-1}^* H_{N-1}^* \cdots D_2^* H_2^*. \tag{3.26}
\]

By setting,

\[
H = \begin{pmatrix}
\mu_{0,0} (\xi) & \mu_{0,1} (\xi) & \cdots & \mu_{0,\ell} (\xi) \\
\mu_{1,0} (\xi) & \mu_{1,1} (\xi) & \cdots & \mu_{1,\ell} (\xi) \\
\vdots & \vdots & \ddots & \vdots \\
\mu_{p,0} (\xi) & \mu_{p,1} (\xi) & \cdots & \mu_{p,\ell} (\xi)
\end{pmatrix},
\]

It is immediate that \( H \) satisfies the equality \( \Gamma^*(\xi) \Gamma(\xi) = J_1 \). Further, if we choose polyphase representation of wavelet masks \( h_j (\xi), \ell = 1, 2, \ldots, L \) as defined by (3.13) or even (3.19) in Equation (2.13), then we can obtain the UEP condition (2.15). Therefore, Theorem 3.1 implies that \( \Psi \) generates a minimum-energy wavelet frame for \( L^2 (\mathbb{R}^+) \). This completes the proof of the Theorem 3.3. \( \square \)

### 4. Decomposition and reconstruction algorithms

Suppose \( \Psi = \{ \psi_1, \psi_2, \ldots, \psi_l \} \) is the minimum-energy wavelet frame associated with the compactly supported refinable function \( \phi(x) \). Then, for each \( j \in \mathbb{Z} \), we consider

\[
V_j = \overline{\text{span}} \{ \phi_{jk} : k \in \mathbb{Z}^+ \} \quad \text{and} \quad W_j = \overline{\text{span}} \{ \psi_{jk}^\ell : k \in \mathbb{Z}^+, \ell = 1, 2, \ldots, L \}. \tag{4.1}
\]

Thus,

\[
V_{j+1} = V_j + W_j, \quad j \in \mathbb{Z}. \tag{4.2}
\]

Note that decomposition (4.2) is not a direct sum decomposition since in general \( V_j \cap W_j \neq \{0\} \). Thus, it follows from (4.1) and (4.2) that any \( f \in V_{j+1} \) can be expressed as

\[
f(x) = P_{j} f(x) + Q_{j} f(x), \tag{4.3}
\]

where

\[
P_{j} f(x) = \sum_{k \in \mathbb{Z}^+} \langle f, \phi_{jk} \rangle \phi_{jk}(x), \tag{4.4}
\]

\[
Q_{j} f(x) = P_{j+1} f(x) - P_{j} f(x) = \sum_{\ell=1}^{L} \sum_{k \in \mathbb{Z}^+} \langle f, \psi_{jk}^\ell \rangle \psi_{jk}^\ell(x), \tag{4.5}
\]

are the projection and detailed operators defined on \( V_j \) and \( W_j \), respectively. The importance of this frame expansion as compared to any other expansion

\[
Q_{j} f = \sum_{\ell=1}^{L} \sum_{k \in \mathbb{Z}^+} a_{jk} \psi_{jk}^\ell. \tag{4.6}
\]

of the same \( Q_{j} f \) is that the energy in (4.5) is minimum in the sense that

\[
\sum_{\ell=1}^{L} \sum_{k \in \mathbb{Z}^+} |\langle f, \psi_{jk}^\ell \rangle|^2 \leq \sum_{\ell=1}^{L} \sum_{k \in \mathbb{Z}^+} |a_{jk}|^2. \tag{4.7}
\]
Therefore, by using (4.5) and (4.6), we have

$$
\langle Q_j f, f \rangle = \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}} |\langle f, \psi_j^l \rangle|^2 = \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}} a_{j,k} \langle f, \psi_j^l \rangle,
$$

(4.8)

and this derives

$$
0 \leq \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}} |a_{j,k} - \langle f, \psi_j^l \rangle|^2
$$

$$
= \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}} |a_{j,k}|^2 - 2 \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}} a_{j,k} \langle f, \psi_j^l \rangle + \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}} |\langle f, \psi_j^l \rangle|^2
$$

$$
= \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}} |a_{j,k}|^2 - \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}} |\langle f, \psi_j^l \rangle|^2.
$$

This inequality means that the coefficients of the error term $Q_j f$ in (4.5) have minimal $l^2$-norm among all sequences $\{a_{j,k}\}$ which satisfy (4.6).

We now discuss the decomposition and reconstruction algorithms associated with minimum-energy wavelet frames on positive half-line. For any $f \in L^2(\mathbb{R}^+)$, we consider

$$
a_{j,k} = \langle f, \phi_j^l \rangle; \quad b_{j,k}^l = \langle f, \psi_j^l \rangle, \quad l = 1, 2, \ldots, L.
$$

(4.9)

Then, by two scale relations (2.5) and the corresponding wavelet equation, we obtain

$$
\phi_{j+1} = \sum_{k \in \mathbb{Z}} c_{k-p} \phi_j^k \quad \psi_{j+1} = \sum_{k \in \mathbb{Z}} d_{k-p} \psi_j^k, \quad l = 1, 2, \ldots, L, j \in \mathbb{Z}^+.
$$

(4.10)

By taking the inner products with $f$ on both sides of the two equations in (4.10), we have a tight minimum-energy wavelet frame decomposition:

$$
a_{j+1,k} = \sum_{k \in \mathbb{Z}} c_{k-p} a_{j,k} \quad b_{j+1,k}^l = \frac{1}{\sqrt{p}} \sum_{k \in \mathbb{Z}} d_{k-p} b_{j,k}^l, \quad l = 1, 2, \ldots, L, j \in \mathbb{Z}^+.
$$

(4.11)

Using the fact that $\phi_j^k \in V_j$ and relations (2.4) and wavelet equation, from (4.3) we also have

$$
\phi_{j+1,i} = \sum_{k \in \mathbb{Z}} \left( c_{i-p} \phi_{j,k} + \sum_{l=1}^{L} d_{i-p} \psi_{j,k}^l \right), \quad i \in \mathbb{Z}^+.
$$

(4.12)

By taking the inner products with $f$ on both sides of (4.12), we have a tight minimum-energy wavelet frame reconstruction:

$$
a_{j+1,i} = \sum_{k \in \mathbb{Z}} \left( c_{i-p} a_{j,k} + \sum_{l=1}^{L} d_{i-p} b_{j,k}^l \right), \quad i \in \mathbb{Z}^+.
$$

(4.13)

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