Study of shadow and parameter estimation of non-commutative Kerr-like Lorentz violating black holes

Sohan Kumar Jha
Chandernagore College, Chandernagore, Hooghly, West Bengal, India

Anisur Rahaman∗
Durgapur Govt. College, Durgapur, Bardwan - 713214, West Bengal, India

(Dated: March 1, 2022)

Abstract

The first image of supermassive black hole M87∗ and various observations made by the Event Horizon Telescope (EHT) collaboration have given us a new window to test the viability of modified theories of gravity. In this manuscript, we extensively study the non-commutative Kerr-like black hole, where the source of matter has a Lorentzian distribution. We investigate the deviation of shape and size of ergosphere and black hole shadow. We observe that with the increase in Lorentz violating parameter $l$, the size of the black hole shadow increases, and with the increase in the non-commutative parameter $b$, the size of the black hole decreases. We also demonstrate the variations of average radius $R_s$, deviation $\delta s$, and emission rate with Lorentz violation and for the use of non-commutativity. In this letter, we try to constrain parameters of a Non-commutative Kerr-like black hole using observable from EHT collaboration. Our study shows that a non-commutative Kerr-like black hole is a suitable candidate for an astrophysical black hole and a possible bound of the parameter associated with the non-commutativity is determined.

PACS numbers:

I. INTRODUCTION

The black hole is one of the important predictions of the general theory of relativity. As an important event for the observation of astrophysical black holes, the Event Horizon Telescope (EHT) has published the first image with ultra-high angular resolution of the shadow of supermassive black hole M87∗ at the center of the nearby galaxy Messier 87 [1–6]. The shadow is found to have an angular diameter $42 \pm 3 \mu\text{as}$ with the deviation from circularity $\Delta C \leq 0.1$ and axial ratio $\lesssim \frac{4}{3}$. It has opened a new window for the observation of black holes and to test modified theories of gravity in the strong gravity regime. The most important characteristic of this image is that a dark interior is surrounded by a light ring. The dark area and the light ring are referred to as the shadow and the photon ring of the black hole respectively.

A two-dimensional dark area in the celestial sphere which is known as black hole shadow is caused by the strong gravity of the black hole. It was first examined by Synge in 1966 for a Schwarzschild black hole [7], and the radius of the shadow was given by Luminet [8]. The shadow of a non-rotating black hole is a standard circle, while the shadow of a rotating black hole elongates in the direction of the rotating axis due to the dragging effects of spacetime [15, 16]. Hioki and Maeda [9] proposed two observables based on the feature that point at the boundary of the Kerr shadow to match the astronomical observations. One of which roughly describes the size of the shadow and the other describes the deformation of its shape from a reference circle. Also, using the method given in [17] one can find the deviation from circularity $\Delta C$. These various observables are very useful in testing and constraining the parameters involved in the modified theories of gravity.

The formulation of the Standard Model(SM) of particle physics and the general theory of gravity (GTR) entirely depends on the principle of Lorentz invariance. The GTR does not take into account the quantum properties of particles, and SM, on the other hand, neglects all gravitational effects of particles. At the Planck scale, one cannot neglect gravitational interactions between particles, and hence merger of SM with GTR in a single theory become essential. It is indeed available from the quantum gravity concept. At this scale, it is expected to face a violation of Lorentz symmetry [18]. Several studies related to Lorentz violation in different aspects have come in the literature [19–44]. The standard model extension (SME) is an effective field theory that couples SM to GTR [45–48] where

∗Electronic address: anisur.associates@iucaa.ac.in; manisurn@gmail.com (Corresponding Author)
Lorentz violation (LV) is introduced. One of the theories that belongs in this class is the bumblebee model, where LV is introduced through an axial-vector field $B_\mu$ which is known as the bumblebee field. Recently, in [49, 50], a Kerr-like solution was obtained from the Einstein-bumblebee theory. In [51], it has been found that Kerr-Sen-like solution is also possible from Einstein-bumblebee theory.

Non-commutative spacetime has been extensively studied in recent years [52–57]. The correction due to noncommutative spacetime has been studied in various field. A fertile field for applying the idea of noncommutative spacetime is the black hole physics. Several ways are available in the literature to implement a noncommutative spacetime in theories of gravity [58–62]. Through a modification of the matter source by a Gaussian mass distribution noncommutativity has been brought in the black hole physics in the article [63] and through a Lorentzian mass distribution it has been introduced in the article [64]. An interesting extension concerning the thermodynamic similarity between Reissner-Nordström black hole the noncommutative Schwarzschild black hole and the has been made in the article [70]. The thermodynamical aspects of noncommutative black holes have been investigated by taking on the tunneling formalism the in the articles [66–73]. In [74], by taking the mass density to be a Lorentzian smeared mass distribution the thermodynamic properties of noncommutative BTZ black holes have been studied. So far we have found non-commutativity can be implemented by modifying the source of matter, replacing the Dirac delta function either by the Gaussian distribution [63] or by the Lorentzian distribution [64]. In this manuscript, we have introduced non-commutativity into Kerr-like black hole [49] by considering Lorentzian distribution as it has been used in [64].

Lorentz violation and non-commutativity of spacetime both are supposed to carry the quantum gravity effect. In this article we carry out an investigation through a generalized spacetime metric that combine both the effect and study the black hole shadow in the light of the recently obtained information form the M87† data.

The manuscript is organized as follows. In sect. II, we briefly review the Non-commutative Kerr-like black hole. In sect. III, we study the black hole shadows, average radius $R_\star$, and deviation $\delta_\star$. Sect. IV is devoted to investigate the variation of emission rate. We try to constrain parameters using observables from EHT in sect. V. Sect. VI contains a summary and discussion of our findings.

II. NON-COMMUTATIVE KERR-LIKE BLACK HOLE

To introduce non-commutativity into Kerr-like black hole we consider that the mass density of black hole is given by a Lorentzian distribution as follows [64],

$$\rho_b = \frac{\sqrt{b}M}{\pi^{3/2}(\pi b + r^2)^2}. \quad (1)$$

Here $b$ is the strength of non-commutativity of spacetime and $M$ is the total mass diffused throughout a region with linear size $\sqrt{b}$. For the smeared matter distribution, we can further obtain [77]

$$\mathcal{M}_b = \int_0^\rho \rho_b(r)4\pi r^2 dr = \frac{2M}{\pi} \left( \tan^{-1} \left( \frac{r}{\sqrt{b}} \right) - \frac{\sqrt{b}r}{\pi b + r^2} \right) = -\frac{4\sqrt{b}M}{\pi^2} + M + \mathcal{O} \left( b^{3/2} \right). \quad (2)$$

In [50] it has been shown that Einstein-bumblebee gravity puts forward a Lorentz violating Kerr-like solution. Therefore the gained knowledge of the articles [64, 77] and [50] we can have a generalization that enables us to noncommutative Kerr-like Lorentz violation black hole. In this situation the metric is given by

$$ds^2 = -\left( 1 - \frac{2M_\rho r}{\rho^2} \right) dt^2 - \frac{4M_\rho a \sqrt{1 + \frac{l}{b}} \sin^2 \theta}{\rho^2} dtd\varphi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{A \sin^2 \theta}{\rho^2} d\varphi^2, \quad (3)$$

where

$$\rho^2 = r^2 + (1 + l)a^2 \cos^2 \theta, \Delta = \frac{r^2 - 2M_\rho r}{1 + l} + a^2, A = \left[ r^2 + (1 + l)a^2 \right]^2 - \Delta(1 + l)^2 a^2 \sin^2 \theta. \quad (4)$$

When $l \to 0$ and $b \to 0$, it recovers the usual Kerr metric. The metric (3) carries the information of two effects which are supposed occur in the vicinity Planck scale one is noncommutating effect of the spacetime and other one is the Lorentz violating effect. The parameter $b$ is connected with the noncommutativity and the parameter $l$ is associated with Lorentz violation. Equating $\Delta = 0$, we get the expressions for Event horizon and Cauchy horizon which are given by

$$r_\pm = M \pm \sqrt{-\pi l a^2 - \pi a^2 + \pi M^2 - 8\sqrt{\pi M} b \sqrt{\pi}}, \quad (5)$$
where \( \pm \) signs correspond to event horizon and Cauchy horizon respectively. The event horizon and Cauchy horizon are labelled by \( r_{eh} \) and \( r_{ch} \) respectively.

From the above plots we see that there exists critical values of \( a \), for fixed values of \( b \) and \( l \), critical values of \( b \) for fixed values of \( a, l \) and critical values of \( l \) for fixed values of \( b, a \). The critical value of \( a, b \) and \( l \) are designated by \( a_c, b_c \) and \( l_c \) respectively. In these cases \( \Delta = 0 \) has only one root. For \( a < a_c \) we have black hole and for \( a > a_c \) we have naked singularity. Similarly for \( b < b_c \) we have black hole, but for \( b > b_c \) we have naked singularity and for \( l < l_c \) signifies the black hole, however \( l > l_c \) represents the naked singularity. Numerical computation shows that we have \( a_c = 0.52747M \) for \( b = 0.02M^2 \) and \( l = 0.3 \). Similarly for \( a = 0.5M \) and \( l = 0.3 \) we have \( b_c = 0.0223654M^2 \), and for \( a = 0.6M \) and \( b = 0.01M^2 \) we find \( l_c = 0.524023 \).

There exists black hole when the following inequality is maintained

\[
- \pi a^2 - \pi b^2 + \pi M^2 - 8\sqrt{\pi M\sqrt{b}} \geq 0,
\]

(6)

However when in the equation equality is maintained it corresponds to extremal black holes, and when the equation is strictly greater than 0 we have non-extremal black holes with Cauchy and Event horizons.
FIG. 3: The upper one is the parameter space \((a/M - l)\) for various values of \(b\) and the lower one is the parameter space \((a/M - b/M^2)\) for various values of \(l\). The colored regions correspond to parameter space for which we have black hole.

From the above plots, we observe that as \(l\) increases the parameter space \((a/M - b/M^2)\) for which we have black hole gets shrunk and as \(b\) increases the parameter space \((a/M - l)\) for which we have black hole also reduces.

Let us now focus on the static limit surface (SLS). At the SLS, the asymptotic time-translational Killing vector becomes null which is mathematically given by

\[
g_{tt} = \rho^2 - 2Mr = 0. \quad (7)
\]

The real positive solutions of above equation gives radial coordinates of ergosphere:

\[
r_{\text{ergo}}^\pm = \frac{2\sqrt{\pi}M \pm \sqrt{-4\pi a^2 \cos^2(\theta) - 4\pi a^2 l \cos^2(\theta) + 4\pi M^2 - 32\sqrt{\pi} M \sqrt{b}}}{2\sqrt{\pi}}. \quad (8)
\]
The radial equation of motion can be written down in the familiar form

\[ r'' = \frac{\sigma^2}{\Sigma} \frac{\dot{r}^2}{\ell + a} \]  

where \( \Sigma = r^2 - 2Mr + Q^2 \) and \( \ell = \ell_0 + 2Mra \), and \( a \) is the angular momentum per unit mass of the black hole. From above we can conclude that the shape and size of the ergosphere depend on rotational parameter \( a \), non-commutative parameter \( b \), and LV parameter \( l \). The size of the ergosphere increases with the increase of \( b \) and \( l \).

III. PHOTON ORBIT AND BLACK-HOLE SHADOW

In this section we study the black hole shadow related to this modified theory. There are several studies related to the black hole shadow from which we will get necessary inputs for study our study \([10][13]\). In order to study the shadow we introduce two conserved parameters \( \xi \) and \( \eta \) as usual which are defined by

\[ \xi = \frac{L_z}{E} \quad \text{and} \quad \eta = \frac{Q}{E^2}, \]  

where \( E, L_z \), and \( Q \) are the energy, the axial component of the angular momentum, and Carter constant respectively. Then the null geodesics in the bumblebee rotating black hole spacetime in terms of \( \xi \) are given by

\[ \rho'^2 \frac{d\rho}{d\lambda} = \pm \sqrt{R}, \quad \rho'^2 \frac{d\theta}{d\lambda} = \pm \sqrt{S}, \]

\[ (1 + \ell) \Delta \rho'^2 \frac{dt}{d\lambda} = A - 2\sqrt{1 + \ell M} \rho \xi, \]
\[ (1 + \ell) \Delta \rho'^2 \frac{d\phi}{d\lambda} = 2\sqrt{1 + \ell M} r a + \frac{\xi}{\sin^2 \theta} (\rho^2 - 2Mr), \]

where \( \lambda \) is the affine parameter and

\[ R(r) = \left[ \frac{r^2 + (1 + \ell)a^2}{\sqrt{1 + l}} - a\xi \right]^2 - \Delta \left[ \eta + (\xi - a\sqrt{1 + l})^2 \right], \quad \Theta(\theta) = \eta + (1 + l)a^2 \cos^2 \theta - \xi^2 \cot^2 \theta. \]

The radial equation of motion can be written down in the familiar form

\[ \left( \rho'^2 \frac{dr}{d\lambda} \right)^2 + V_{eff} = 0. \]
The effective potential $V_{\text{eff}}$ here reads

$$V_{\text{eff}} = -\left[\frac{r^2 + (1 + l)a^2}{\sqrt{1 + l}} - a\xi\right]^2 + \Delta \left[\eta + (\xi - a\sqrt{1 + l})^2\right]. \quad (13)$$

The unstable spherical orbit on the equatorial plane is given by the following equations:

$$\theta = \frac{\pi}{2}, \quad R(r) = 0, \quad \frac{dR}{dr} = 0, \quad \frac{d^2 R}{dr^2} < 0, \quad \text{and} \quad \eta = 0. \quad (14)$$

We plot the potential $V_{\text{eff}}$ against $r/M$ with $\xi = \xi_c + 0.2$ where $\xi_c$ is the value of $\xi$ for equatorial spherical unstable orbit.

**FIG. 5:** The left and the right panels describe the effective potential for prograde orbits and the retrograde orbits respectively for various values of $a$ with $b = 0.01M^2$ and $l = 0.1$.

**FIG. 6:** The left and the right panels describe the effective potential for prograde orbits and the retrograde orbits respectively for various values of $b$ with $a = 0.1M$ and $l = 0.1$.

**FIG. 7:** The left panel describes the effective potential for prograde orbits and right panel describes the retrograde orbits for various values of $l$ with $a = 0.1M$ and $b = 0.03M^2$.

Above plots show that the turning points for prograde orbits shift towards left as $a$ or $b$ increases. We also plot critical radii of prograde and retrograde orbits for different scenario in the Fig. below.
It can be concluded from the above plots that critical radii, both for the prograde and retrograde orbits, decrease with an increase in the non-commutative parameter \( b \). On the other hand, the critical radius for prograde orbit decreases with the increase in \( l \) but for retrograde orbit it increases with the increase in \( l \). For more generic orbits \( \theta \neq \pi/2 \) and \( \eta \neq 0 \), the solution of Eqn. (14) \( r = r_s \), gives the \( r \)- constant orbit, which is also called spherical orbit and the conserved parameters of the spherical orbits are given by

\[
\xi_s = \frac{(a^2 (1 + l) + r^2) \Delta'(r) - 4r \Delta(r)}{a \sqrt{1 + l \Delta'(r)}},
\]

\[
\eta_s = \frac{r^2 \left( 8 \Delta(r) \left( 2a^2 (1 + l) + r \Delta'(r) \right) - r^2 \Delta'(r)^2 - 16 \Delta'(r)^2 \right)}{a^2 (1 + l) \Delta'(r)^2},
\]

where ' stands for differentiation with respect to radial coordinate. The above expressions in the limit \( l \to 0 \) and \( b \to 0 \) reduce to those for Kerr black hole. It would be useful at this point to introduce two celestial coordinates for better study of the shadow. The two celestial coordinates, which are used to describe the shape of the shadow that an observer sees in the sky, can be given by

\[
\alpha(\xi, \eta; \theta) = \lim_{r \to \infty} -\frac{rp(\phi)}{p(t)} = -\xi_s \csc \theta,
\]

\[
\beta(\xi, \eta; \theta) = \lim_{r \to \infty} \frac{rp(\theta)}{p(t)} = \sqrt{\left( \eta_s + a^2 \cos^2 \theta - \xi_s^2 \cot^2 \theta \right)},
\]

(16)

where \((p(t), p(r), p(\theta), p(\phi))\) are the tetrad components of the photon momentum with respect to locally non-rotating reference frames [13].

With these inputs, we now plot black hole shadows for various cases which are depicted in the figures below.
\[ l = -0.9 \]
\[ l = 0 \]
\[ l = 0.9 \]
\[ b = 0 \]
\[ b = 0.01 M^2 \]
\[ b = 0.02 M^2 \]

FIG. 10: The left panel gives shapes of the shadow for various values of \( l \) with \( a = 0.1 M \), \( b = 0.01 M^2 \), and \( \theta = \pi/2 \). The right panel gives shapes of the shadow for various values of \( b \) with \( a = 0.1 M \), \( l = 0.1 \), and \( \theta = \pi/2 \).

FIG. 11: The shapes of the shadow for various values of \( a \) with \( b = 0.01 M^2 \), \( l = 0.1 \), and \( \theta = \pi/2 \).

From the above plots, we observe that the size of the shadow increases with an increase in \( l \), whereas it decreases with an increase in \( b \). Besides, if we increase \( a \) then the shadow shifts toward the right.

Using the parameters which are introduced by Hioki and Maeda [9], we analyze the deviation from circular form \( \delta_s \) and the size \( R_s \) of the shadow cast by the black hole.
FIG. 12: The black hole shadow and reference circle. $ds$ is the distance between the left points of the shadow and the reference circle.

For calculating these parameters, we consider five points $(\alpha_t, \beta_t), (\alpha_b, \beta_b), (\alpha_r, 0), (\alpha_p, 0)$ and $(\bar{\alpha}_p, 0)$ which are top, bottom, rightmost, leftmost of the shadow and leftmost of the reference circle respectively. So, we have

$$R_s = \frac{(\alpha_t - \alpha_r)^2 + \beta_t^2}{2|\alpha_t - \alpha_r|}$$

and

$$\delta_s = \frac{|\bar{\alpha}_p - \alpha_p|}{R_s}.$$ 

In the following Fig. We plot $R_s$ and $\delta_s$ for various scenarios to study how $R_s$ and $\delta_s$ varies with parameters of the modified theory of gravity.

FIG. 13: The left panel shows variation of $R_s$ for various values of $a$ with $l = -0.2$ and $\theta = \pi/2$, and the right panel shows the variation of $R_s$ for various values of $l$ with $a = 0.3M$ and $\theta = \pi/2$.

FIG. 14: The left panel shows variation of $R_s$ for various values of $a$ with $b = 0.01M^2$ and $\theta = \pi/2$ and the right panel shows the variation of $R_s$ for various values of $b$ with $a = 0.3M$ and $\theta = \pi/2$. 
FIG. 15: The left panel shows variation of $\delta_s$ for various values of $a$ with $l = -0.2$ and $\theta = \pi/2$ and the right panel shows the variation of $\delta_s$ for various values of $l$ with $a = 0.1M$ and $\theta = \pi/2$.

FIG. 16: The left panel shows the variation of $\delta_s$ for various values of $a$ with $b = 0.1M^2$ and $\theta = \pi/2$ and the right panel one gives the variation of $\delta_s$ for various values of $b$ with $a = 0.2M$ and $\theta = \pi/2$.

From the above plots we observe that $R_s$ decreases with an increase in $b$ for fixed values of $a$ and $l$, whereas for fixed values of $a$ and $b$, it increases with an increase in $l$. On the other hand, $\delta_s$ increases with an increase in $l$ for fixed values of $a$ and $b$ as well as with an increase in $b$ for fixed values of $a$ and $l$.

**IV. ENERGY EMISSION RATE**

In this part, we study the possible visibility of the non-commutative Kerr-like black hole through shadow. In the vicinity of limiting constant value, the cross-section of the black hole’s absorption moderates lightly at high energy. We know that a rotating black hole can absorb electromagnetic waves, so the absorbing cross-section for a spherically symmetric black hole is [79]

$$\sigma_{\text{lim}} = \pi R_s^2.$$  \hspace{1cm} (17)

Using the above equation the energy emission rate is obtained [80]:

$$\frac{d^2E}{d\omega dt} = \frac{2\pi^3 R_s^2}{e(\frac{\omega}{\pi}) - 1} \omega^3,$$ \hspace{1cm} (18)

where $T = \frac{\sqrt{1+\ell^2} \chi'(r_+)}{4\pi[r_+^2 + (1+\ell)a^2]}$ is the Hawking temperature and $\omega$ the frequency.
\[ a = 0.1M \]
\[ a = 0.3M \]
\[ a = 0.5M \]

\[ b = 0 \]
\[ b = 0.01M^2 \]
\[ b = 0.02M^2 \]

FIG. 17: The left panel gives variation of emission rate against \( \omega \) for various values of \( a \) with \( b = 0.01M^2 \) and \( l = 0.3 \). The right panel gives variation of emission rate against \( \omega \) for various values of \( b \) with \( a = 0.1M \) and \( l = 0.3 \).

FIG. 18: It gives variation of emission rate against \( \omega \) for various values of \( l \) with \( b = 0.02M^2 \) and \( a = 0.1M \).

In the Fig. we have shown the plots of energy emission rate versus \( \omega \) for various cases. It is clear from the plots that the emission rate decreases with an increase in the value of \( b \) for any set of fixed values of \( a \) and \( l \). It also decreases with an increase in \( l \), for \( a \) and \( b \) being fixed, and with an increase in \( a \), when \( l \) and \( b \) remain fixed. Let us now consider the case when some medium is present which is a more natural one.

We now consider the situation when the black hole is veiled with dispersive medium like plasma and compute the emission rate when the black hole in this situation. In the presence of plasma the celestial coordinates are given by

\[ \alpha(\xi, \eta; \theta) = -\frac{\xi \csc \theta}{n}, \]
\[ \beta(\xi, \eta; \theta) = \sqrt{\eta + (1 + l)a^2 \cos^2 \theta - \xi^2 \cot^2 \theta - (n^2 - 1) a^2 (1 + l) \sin^2 \theta} \]

\[ \frac{d^2 E(\omega)}{d\omega dt} \]

where \( n = \sqrt{1 - \frac{k}{r}} \) is the refractive index of the plasma, \( k \) being the plasma constant. Using these expressions combined with the expression of emission rate we investigate the variation of rate of emission for various situations.
Here we observe that the rate of emission increases with the plasma constant $k$ for fixed values of $a$, $b$, and $l$. The other variations are similar to those which we have seen without plasma, though with reduced values.

V. CONSTRAIN FROM THE OBSERVED DATA FOR M87$^*$

This section is devoted to the constraining of the parameter existing in the modified theory. We compare the shadows produced from the numerical calculation by the non-commutative Kerr-like black holes with the observed one for the M87$^*$ black hole. For comparison, we consider the experimentally obtained astronomical data for the deviation from circularity $\Delta \leq 0.10$ and angular diameter $\theta_d = 42 \pm 3 \mu as$. The boundary of the shadow is described by the polar coordinate $(R(\phi), \phi)$ with the origin at the center of the shadow $(\alpha_C, \beta_C)$ where $\alpha_C = \frac{|\alpha_{\text{max}} + \alpha_{\text{min}}|}{2}$, and $\beta_C = 0$.

If a point $(\alpha, \beta)$ over the boundary of the image subtends an angle $\phi$ on the $\alpha$ axis at the geometric center, $(\alpha_C, 0)$, and $R(\phi)$ be the distance between the point $(\alpha, \beta)$ and $(\alpha_C, 0)$, then the average radius $R_{\text{avg}}$ of the image is given by [83]

$$R_{\text{avg}}^2 \equiv \frac{1}{2\pi} \int_0^{2\pi} d\phi R^2(\phi),$$

(20)

where $R(\phi) = \sqrt{(\alpha(\phi) - \alpha_C)^2 + \beta(\phi)^2}$, and $\phi = \tan^{-1} \frac{\beta(\phi)}{\alpha(\phi) - \alpha_C}$.

With the above inputs, the circularity deviation $\Delta C$ is defined by [17]

$$\Delta C \equiv 2 \sqrt{\frac{1}{2\pi} \int_0^{2\pi} d\phi (R(\phi) - R_{\text{avg}})^2}.$$  

(21)
We also consider the angular diameter of the shadow which is defined by

$$\theta_d = \frac{2}{\pi} \sqrt{\frac{A}{\pi}},$$

(22)

where $A = 2 \int_{r_+}^{r_-} \beta d\alpha$ is the area of the shadow and $d = 16.8 \text{Mpc}$ is the distance of $M87^*$ from the Earth. These relations will enable us to accomplish a comparison between the theoretical predictions for non-commutative Kerr-like black-hole shadows and the experimental findings of the EHT collaboration. In the figures below the deviation from circularity, $\Delta C$ is shown for non-commutative Kerr-like black holes for inclination angles $\theta = 90^\circ$ and $\theta = 17^\circ$ respectively.

FIG. 21: The left panel is for $l = 0.4$, and the right panel is for $b = 0.01 M^2$ where the inclination angle is $90^\circ$. The black solid lines correspond to $\Delta C = 0.1$.

FIG. 22: The left panel is for $l = 0.4$, and the right one is for $b = 0.01 M^2$ where the inclination angle is $17^\circ$.

In the figures below the angular diameter $\theta_d$ is shown for non-commutative Kerr-like black holes for inclination angles $\theta = 90^\circ$ and $\theta = 17^\circ$ respectively.
FIG. 23: The left panel is for $l = 0.4$, and the right panel is for $b = 0.01M^2$ where the inclination angle is $90^\circ$. The black solid lines correspond to $\theta_d = 39\mu$as.

FIG. 24: The left panel is for $l = 0.4$, and the right panel is for $b = 0.01M^2$ where the inclination angle is $17^\circ$. The black solid lines correspond to $\theta_d = 39\mu$as.

From the above plots, we can conclude that the constrain $\Delta C \leq 0.1$ is satisfied for finite parameter space when the inclination angle is $90^\circ$, whereas, when the inclination angle is $17^\circ$, the constrain is satisfied for the entire parameter space. For inclination angles $\theta = 90^\circ$ and $\theta = 17^\circ$, the constrain $\theta_d = 42 \pm 3\mu$ within $1\sigma$ region is satisfied for finite parameter space. The circular asymmetry in the $M87^*$ shadow can also be defined in terms of the axial ratio $D_X$ which is the ratio of major to the minor diameter of the shadow [1]. It is defined by [84]

$$D_X = \frac{\Delta Y}{\Delta X} = \frac{\beta_t - \beta_b}{\alpha_r - \alpha_p}$$

We should have $1 < D_X \lesssim 4/3$ in accordance with the EHT observations of $M87^*$ [1]. Note that $D_X$ is another way of defining $\Delta C$. Axial ratio of $4 : 3$ indeed corresponds to a $\Delta C \leq 0.1$ [1]. In the figures below axial ratio, $D_X$ is shown for non-commutative Kerr-like black holes for inclination angles $\theta = 90^\circ$ and $\theta = 17^\circ$ respectively.
From the plots above we see that the condition $1 < D_X \lesssim 4/3$ is satisfied for the entire parameter space of non-commutative Kerr-like black holes. Thus non-commutative Kerr-like black holes are remarkably consistent with EHT images of $M87^*$. Therefore, we can not rule out non-commutative Kerr-like black holes from the observational data of $M87^*$ black hole shadow.

We can find a bound of the parameter $b$ associated with the non-commutativity of the spacetime in a similar way we determined the bounds of the parameter $l$ in [81]. By modelling M87* black hole as Kerr black hole, the author of the article [85] obtained a lower limit of $a$ for the M87* black hole. Bringing this result under consideration in [81] we put the interval of interest for $a$ as $[0.50M, 0.99M]$, and the using the experimental constraints $\Delta C \leq 0.10$ and $\theta_d = 42 \pm 3\mu as$ with the information $a \in [0.50M, 0.99M]$, we observed that $l \in (-1, 0.621031)$. In a similar way taking into account the bounds $a \in [0.50M, 0.99M]$ and $l \in (-1, 0.621031)$ and the experimental constraints $\Delta C \leq 0.1$ and $\theta_d = 39 \pm 3\mu as$, we get a bound on the parameter $b$ which is linked with non-commutativity of the space time. We find that the parameter $b \in [0, 0.000505973M^2]$. The upper bound of $b$ is found out to be 0.000505973$M^2$.

VI. SUMMARY AND DISCUSSION

In this work, we have extensively studied the Non-commutative Kerr-like black hole. The spin, the mass, the LV parameter, and the non-commutative parameter involved in it determine the gravitational field. We have investigated the effect of LV parameter $l$ and non-commutative parameter $b$ on the size of black hole shadow. We have observed that the size of the black hole shadow increases with an increase in the value of the parameter $l$, and it decreases with
an increase in the value of the parameter $b$. Thus, it can be safely concluded that LV and non-commutativity, both, have a significant impact on black hole shadow.

We have also studied emission rate in the absence and presence of plasma. Our study shows that the nature of variation of emission rate remains unaltered with and without plasma, though with reduced values. We can also obtain the results for Kerr and Kerr-like black holes with suitable limits. These results have clearly established the influence of the LV parameter and non-commutative parameter on emission rate.

We have used EHT observations to constrain parameters in our modified theories. For inclination angle $\theta = 90^\circ$, the deviation from circularity $\Delta C \leq 0.1$ and angular diameter $\theta_d = 42 \pm 3\mu as$ within $1\sigma$ region are satisfied for finite parameter space ($b/\sqrt{M} - a/\sqrt{M}$) and $(l - a/\sqrt{M})$. For inclination angle $\theta = 17^\circ$, the circularity deviation $\Delta C \leq 0.1$ is satisfied for entire parameter space ($b/\sqrt{M} - a/\sqrt{M}$) and $(l - a/\sqrt{M})$. The angular diameter $\theta_d = 42 \pm 3\mu as$ within $1\sigma$ is satisfied for finite parameter space ($b/\sqrt{M} - a/\sqrt{M}$) and $(l - a/\sqrt{M})$. The axis ratio $D_X$ satisfies the constraint $1 < D_X \lesssim 4/3$ for the entire parameter space at both the inclination angles $\theta = 90^\circ$ as well as $\theta = 17^\circ$. Therefore our study enables to establish the fact that non-commutative Kerr-like black holes are remarkably consistent with EHT images of $M87^*$. It demands that ruling out non-commutative Kerr-like black holes from the observational data of $M87^*$ black hole shadow would be illogical. Thus non-commutative Kerr-like black hole may be considered as a suitable candidate for astrophysical black hole. It has also been shown that the possible upper bound of $b$ which is associated with the non-commutativity is $0.000505973M^2$.

\[1\] K. Akiyama et al.: Astrophys. J. 875, L1 (2019).
\[2\] K. Akiyama et al.: Astrophys. J. 875, L2 (2019).
\[3\] K. Akiyama et al.: Astrophys. J. 875, L3 (2019).
\[4\] K. Akiyama et al.: Astrophys. J. 875, L4 (2019).
\[5\] K. Akiyama et al.: Astrophys. J. 875, L5 (2019).
\[6\] K. Akiyama et al.: Astrophys. J. 875, L6 (2019).
\[7\] J. L. Synge: MNRAS 131 463 1966.
\[8\] J. P. Luminet: Astronomy & Astrophysics, 75, 228 (1979).
\[9\] K. Hioki and K. I. Maeda: Phys. Rev. D80, 024042 (2009).
\[10\] F. Atamurotov, B. Ahmedov, A. Abdjabbarov: Phys. Rev. D92, 084005 (2015).
\[11\] V. Perlick, O. Y. Tsupko: Phys. Rev. D 95, 104003 (2017).
\[12\] Shao-Wen Wei, Yu-Xiao Liu: JCAP 11, 063 (2013).
\[13\] G. Z. Babar, A. Z. Babar, F. Atamurotov: Euro. Phys. Jour. C80, 761 (2020).
\[14\] S. Dastan, R. Saafari, S. Soroushfar: arXiv:1610.09477
\[15\] J. M. Bardeen, W. H. Press, S. A. Teukolsky: Astro-phys. J. 178, 347 (1972).
\[16\] S. Chandrasekhar, The Mathematical Theory of Black Holes, Oxford University Press, New York, (1992).
\[17\] T. Johannsen, D. Psaltis: Astrophys. J. 718, 446 (2010).
\[18\] D. Mettingly: Living Rev. Rel. 8, 5 (2005).
\[19\] K. Bakke and H. Belich: Eur. Phys. J. Plus 129: 147 (2014).
\[20\] V. A. Kostelecky, C. D. Lane: Journal of Mathematical Physics 40, 6245 (1999).
\[21\] T. J. Yoder and G. S. Adkins: Phys. Rev. D 86, 116005 (2012).
\[22\] R. Lehnert: Phys. Rev. D 68, 085003 (2003).
\[23\] O. G. Kharlanov, V. Ch. Zhukovsky: J. Math. Phys. 48, 092302 (2007).
\[24\] V. A. Kostelecky and M. Mewes: Phys. Rev. Lett. 87, 251304 (2001).
\[25\] V. A. Kostelecky and M. Mewes: Phys. Rev. D 66, 056005 (2002).
\[26\] V. A. Kostelecky and M. Mewes: Phys. Rev. Lett. 97, 140401 (2006).
\[27\] V. A. Kostelecky and M. Mewes: Phys. Rev. Lett. 87, 251304 (2001).
\[28\] S. Carroll, G.B. Field, and R. Jackiw: Phys. Rev. D 41, 1231 (1990).
\[29\] C. Adam and F. R. Klinkhamer: Nucl. Phys. B 607, 247 (2001).
\[30\] W. F. Chen and G. Kunstatter: Phys. Rev. D 62, 105029 (2000).
\[31\] C. D. Carone, M. Sher, M. Vanderhaeghen: Phys. Rev. D74, 077901 (2006).
\[32\] F.R. Klinkhamer and M. Schreck: Nucl. Phys. B848, 90 (2011).
\[33\] M. Schreck: Phys. Rev. D86, 065038 (2012).
\[34\] M. A. Hohensee, R. Lehnert, D. F. Phillips, R. L. Walsworth: Phys. Rev. D80, 036010 (2009).
\[35\] B. Altschul and V. A. Kostelecky: Phys. Lett. B628, 106.
\[36\] R. Bluhm, N. L. Gagne, R. Potting, A. Vrublevskis: Phys. Rev. D 77, 125007 (2008).
\[37\] R. Bluhm, V. Alan Kostelecky: Phys. Rev. D71, 065005 (2005).
\[38\] R. V. Maluf, V. Santos, W. T. Cruz, C. A. S. Almeida: Phys. Rev. D88, 025005 (2013).
\[39\] R.V. Maluf, C.A.S. Almeida, R. Casana, M.M. Ferreira, Jr.: Phys. Rev. D 90 025007 (2014).
\[40\] Q. G. Bailey and V. A. Kostelecky: Phys. Rev. D74, 045001 (2006).
