Symmetries and Conservation Laws for the 2D Ricci Flow Model

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Abstract

The paper aims to study the connection between symmetries and conservation laws for the 2D Ricci flow model. The procedure starts by obtaining a set of multipliers which generates conservation laws. Then, using a general relation which connects symmetries and conservation laws for whatever dynamical system, one determines symmetries related to a chosen multiplier. On this basis, new similarity solutions of the model, not yet discussed in literature, are highlighted.

Keywords: conservation laws, Lie symmetries, invariant solutions, Ricci flow model

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1 Introduction

The concepts of symmetry, invariants and conservation laws are fundamental in the study of dynamical systems, providing a clear connection between equations of motion and their solutions. There are many reasons for computing the symmetries and the conservation laws corresponding to systems of which the evolution is described by differential equations. In the recent years, a remarkable number of mathematical models occurring in various research domains have been studied from the point of view of the symmetry groups theory [1–3]. The most suitable technique for finding classes of analytical solutions, the so-called Lie symmetry method, investigates integrability starting from the invariance of evolutionary equations under some linear transformations of the variables which define the so-called Lie group of symmetries. Conservation laws proceed from the conservation of physical quantities: linear momentum, mass, energy, electric charge, etc. Furthermore, conservation laws are applied in the study of partial differential equations, as for example in: (i) test of complete integrability and application of Inverse Scattering Transform, (ii) study of quantitative and qualitative properties for pdes (Hamiltonian structure, recursion operators) and (iii) development of numerical methods such as finite element methods [4,5].

There are many interesting results on the correspondence between symmetries and conservation laws. As examples, an identity [6] which does not depend on the use of a Lagrangian and provides a relationship between symmetries and conservation laws for self-adjoint differential equations is derived; a direct link between the components of a conserved vector of an arbitrary partial differential equation and the Lie-Bäcklund symmetry generator of the equation, which is associated with the conserved vector’s components is obtained [7].

The aim of this paper is to tackle the connection between symmetries and conservation laws using the results mentioned above and the property of the Euler differential operator to annihilate whatever expression having the mathematical form of a divergence. By applying the Euler operator on a combination of evolutionary equations, one can determine a set of multipliers generating conservation laws. We will apply this procedure for the 2D Ricci flow equation, a very interesting model coming from the gravity theory. The Lie symmetry problem and the invariant solutions of this model have been already discussed [8]. Using the procedure mentioned above, new symmetries and solutions will be highlighted.
The paper is organized as follows: after this introductive section, the problem of constructing a conserved vector will be analyzed in the second section, by means of the conservation law multipliers’ method and also by tacking into account the relations which impose an intimate connection between symmetry generators and conserved vectors. In the third section, these methods will be applied to the 2D Ricci flow model. A set of conservation laws corresponding to a multiplier \(\Lambda(x,t,y,U)\) that does not depend on the derivatives of the dependent variable \(U\), the associated Lie symmetry operators and some new invariant solutions, will be obtained. Some concluding remarks will end the paper.

2 Connection between symmetries and conservation laws

For each partial differential equation (pde) or for each pde system there is a local group of transformations (called the symmetry group) that acts on the space of its independent and dependent variables, with the property that it maps the set of all analytical solutions to itself, and so it leaves the form of the equation (or system) unchanged. The widely applicable method to find the symmetry group associated with a pde (or pde system) is called the classical Lie method. Consequently, the knowledge of Lie point symmetries allows us to construct the group-invariant solutions. Two solutions should be equivalent if there should be a symmetry transformation that could transform the one into the other. Moreover, new solutions might be obtained from the known ones: by applying the symmetry group to a known solution, a family of new solutions would be created.

Let us consider a \(n\)-th order partial differential system:

\[
\Delta^{\nu}[x,u(x),u^{(n)}(x)] = 0, \quad \nu = 1,q
\]

where \(x \equiv \{x^i, i = 1,p\} \subset R^p\) represent independent variables, while \(u \equiv \{u^\alpha, \alpha = 1,q\} \subset R^q\) dependent ones. The notation \(u^{(n)}\) designates the set of partial derivatives of \(u\) with respect to \(x\), up to the \(n\)-th order.

Let us consider the infinitesimal symmetry operator with the general form:

\[
X = \sum_{i=1}^p \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^q \phi_\alpha(x,u) \frac{\partial}{\partial u^\alpha}
\]

In order to explicit the above mentioned property of classical symmetries, the Lie method applied the criterion of infinitesimal invariance of the system to the action of the operator. More precisely, the following condition ought to be imposed:

\[
X^{(n)}(\Delta^{\nu})|_{\Delta=0} = 0
\]

where \(X^{(n)}\) represents the extension of \(n\)-th order of the Lie symmetry generator.

A conservation law for partial differential equations is a divergence expression which vanishes for the solutions of the pde system. The origin of conservation laws comes from physical principles such as the conservation of mass, momentum and energy. Although there is a well-known systematic method able to find conservation laws for variational pdes, the applicability of Noether’s method is limited by the fact that many interesting pde systems are not variational. A systematic procedure able to find conservation laws, called the direct method, has been developed. The direct method consists of two main steps:

(i) to determine a set of conservation law multipliers so that a linear combination of the pdes with the conservation law multipliers should create a divergence expression. For our given differential system one could define a set of conservation law multipliers \(\{\Lambda_\nu : M \times S^{(n)} \rightarrow R\}, \nu = 1,q\), if smooth functions \(\{P_i : M \times S^{(n)} \rightarrow R\}, i = 1,p\) should exist such that everywhere on \(M \times S^{(n)}\) we could have:

\[
\Lambda_\nu[x,U(x),U^{(n)}(x)]\Delta^{\nu}[x,U(x),U^{(n)}(x)] = \sum_{i=1}^p P_i[x,U(x),U^{(n)}(x)]
\]
It is known that conservation law multipliers could be found using the method \[12\] of the Euler operator. Thereby, we can solve the following system of pdes:

\[
E_\rho \left( \Lambda_\nu \left[ x, U(x), U^{(n)}(x) \right] \Delta^\nu [x, U(x), U^{(n)}(x)] \right) = 0, \quad \forall \rho = 1, q \tag{5}
\]

for the unknown functions \( \{ \Lambda_\nu, \nu = 1, q \} \).

(ii) ones having determined a set of conservation law multipliers, to find the corresponding fluxes in order to obtain the conservation law:

\[
D_i P_i \left[ U(x), U^{(n)}(x) \right] \mid_{U(x)=u(x)} = \Lambda_\nu \left[ x, U(x), U^{(n)}(x) \right] \Delta^\nu [x, U(x), U^{(n)}(x)] \mid_{U(x)=u(x)} = \\
\Lambda_\nu \left[ x, u(x), u^{(n)}(x) \right] \Delta^\nu [x, u(x), u^{(n)}(x)] = 0 \tag{6}
\]

The key property of this set of conservation law multipliers is that their existence implies the existence of conservation laws. Conversely, it turns out that for non-degenerate pde systems, every conservation law up to equivalence must arise from a set of conservation law multipliers. Generally, the correspondence between sets of conservation law multipliers and equivalent conservation laws could be many-to-one. However, if a pde system admits a Cauchy-Kolvalevskaya form \[13\] and the sets of conservation law multipliers satisfy some mild conditions, then there is a one-to-one correspondence between each set of conservation law multipliers and each set of equivalent conservation laws.

Further, a relation \[7\] was derived between the symmetry operator \( X \) and the components of a conserved vector \( P_i \). It has the form:

\[
X^{(r)}(P^i) + P^i D_k(\xi^k) - P^k D_k(\xi^i) = 0, \quad i = 1, p \tag{7}
\]

where \( X \) is applied in the \( r \)-th extended form.

The conditions \(7\) with \( X \) known, joined to the conservation law \( D_i P^i = 0 \), might be viewed as a system of linear partial differential equations which could be solved for the components \( P^i \), \( i = 1, p \) of the conserved vector. Yet, \(7\) could be used as well to obtain the symmetry operators \( X \) associated with a given conserved vector. In the next section, the latter way will be applied.

3 Application to the 2D Ricci flow model

3.1 Multipliers of the model

In this section, the relationship between symmetries and conservation laws will be applied to the 2D Ricci flow model. Its evolutionary equation, bearing large consequences in gravity theory, has the form:

\[
u_t = \frac{u_{xy}}{u} - \frac{u_x u_y}{u^2} \tag{8}\]

It is a well-known equation which has been studied as a continuum limit of the Toda-type equation. Among the main results concerning \(8\) one may mention: (i) it could be obtained as a particular case of the 3D Ricci flow equation which accepts a Killing vector;(ii) by linearization, it presents various classes \[14\] of solutions, these depending on the ”sector” where it is defined; (iii) an effective study of its Lie symmetries and invariant solutions was performed \[8\].

Now, the first task will be to obtain conservation laws for the previous model, using the multipliers’ method. A multiplier \( \Lambda \) of equation \(8\) has the property :

\[
\Lambda \left[ U_t - \frac{U_{xy}}{U} + \frac{U_x U_y}{U^2} \right] = D_i P^i [t, x, y, U, U^{(2)}] \tag{9}
\]
for all functions $U(t, x, y)$, not only for the solutions $u(t, x, y)$ of (8).

Let us consider multipliers of the form $\Lambda = \Lambda(t, x, y, U)$. Multipliers which depend on the first order and higher order partial derivatives of $U$ could also be considered. Yet, in two dimensions, calculations rapidly become more complicated. Therefore, computer assisted calculations may lead to further conservation laws.

The right hand side of (9) is a divergence expression. Hence, this expression should vanish identically when applying the Euler operator which, in two dimensions, takes the form:

$$E_U = \frac{\partial}{\partial U^t} - D_x \frac{\partial}{\partial U^x} - D_y \frac{\partial}{\partial U^y} + D_x^2 \frac{\partial}{\partial U^2_x} + D_y^2 \frac{\partial}{\partial U^2_y} + D_x D_y \frac{\partial}{\partial U^xy} + ...$$ (10)

The determining equation for the multiplier $\Lambda(t, x, y, U)$ is:

$$E_U \left[ \Lambda \left( U_t - \frac{U_{xy}}{U} + \frac{U_x U_y}{U^2} \right) \right] = 0$$ (11)

The expansion of (11) yields the following pde:

$$-2\Lambda \frac{U U_{xy}}{U} + \left( \frac{\Lambda U}{U^2} - \frac{\Lambda U}{U} \right) U_x U_y - \Lambda_y U_x - \Lambda_x U_y - \left( \Lambda_t + \frac{\Lambda U}{U} \right) = 0$$ (12)

Because (12) is satisfied for all functions $U(t, x, y)$ and because the multiplier $\Lambda$ is chosen as not to depend on derivatives of $U$, the coefficient functions of various derivatives of $U$ should vanish. Thereby, the determining system for $\Lambda(t, x, y, U)$ is generated:

$$\Lambda_t = \Lambda = \Lambda_{xy} = 0$$ (13)

By solving the system (13), one obtains the general multiplier:

$$\Lambda = f(x) + g(y), \ \forall f(x), \ \forall g(y)$$ (14)

### 3.2 Conservation laws and symmetries

The first aim of this section is to determine the conserved vector associated with the multiplier (14). From (9) and (14) and by performing elementary manipulations, one obtains:

$$[f(x) + g(y)] \left[ U_t - \frac{U_{xy}}{U} + \frac{U_x U_y}{U^2} \right] = D_t \left[ (f(x) + g(y)) U \right] + D_x \left[ -g(y) \frac{U_y}{U} \right] + D_y \left[ -f(x) \frac{U_x}{U} \right] = D_t P^i$$ (15)

Consequently, when $U(t, x, y)$ is a solution of (8), the previous expression leads to the following set of conservation laws:

$$D_t \left[ (f(x) + g(y)) u \right] + D_x \left[ -g(y) \frac{u_y}{u} \right] + D_y \left[ -f(x) \frac{u_x}{u} \right] = 0, \ \forall f(x), \ \forall g(y)$$ (16)

As a result, one obtains that whatever conserved vector of the Ricci flow model (8) with multiplier of the form (14), should have the components:

$$P^1 = (f(x) + g(y)) u, \ P^2 = -g(y) \frac{u_y}{u}, \ P^3 = -f(x) \frac{u_x}{u}$$ (17)

The next aim is to illustrate how we might find the point symmetries associated with the conservation law (10). The starting point of computations is clearly the relation (7) in which
we consider $P^i$ as known components of the conserved vector and $X$ as an unknown symmetry operator. The symmetry conditions for (17) are:

\[
X^{(1)}(P^1) + P^1 D_x(\xi) + P^1 D_y(\eta) - P^2 D_x(\varphi) - P^3 D_y(\varphi) = 0
\]
\[
X^{(1)}(P^2) + P^2 D_t(\varphi) + P^2 D_y(\eta) - P^1 D_t(\xi) - P^3 D_y(\xi) = 0
\]
\[
X^{(1)}(P^3) + P^3 D_t(\varphi) + P^3 D_x(\xi) - P^1 D_t(\eta) - P^2 D_x(\eta) = 0
\]

where the symmetry operator $X$ is of type (2) and has the first extension:

\[
X^{(1)}(t, x, y, u) = \varphi \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \phi \frac{\partial}{\partial u} + \phi^x \frac{\partial}{\partial u_x} + \phi^y \frac{\partial}{\partial u_y}
\]

Coefficient functions $\phi^t$, $\phi^x$, $\phi^y$ will be calculated according to the symmetry theory [9].

Let us suppose that $\Lambda = f(x) + g(y) \neq 0$. The expansion of the determining equations (18) and separation by various monomials in derivatives of $u$, do generate the following differential system:

\[
\varphi_u = 0
\]
\[
\xi_y = 0
\]
\[
\xi_u = 0
\]
\[
\eta_x = 0
\]
\[
\eta_u = 0
\]
\[
\varphi_x g(y) = 0
\]
\[
\varphi_y f(x) = 0
\]

\[
v \left[ \xi \frac{df(x)}{dx} + \eta \frac{dg(y)}{dy} + (f(x) + g(y))(\xi_x + \eta_y) \right] + (f(x) + g(y))\phi = 0
\]

\[
u \eta \frac{dg(y)}{dy} - g(y)[\phi - u\phi_u - u\varphi_u] = 0
\]
\[
u \xi \frac{df(x)}{dx} - f(x)[\phi - u\phi_u - u\varphi_u] = 0
\]
\[
u^2 (f(x) + g(y))\xi_x + g(y)\phi_y = 0
\]
\[
u^2 (f(x) + g(y))\eta_x + f(x)\phi_x = 0
\]

with unknown functions $\varphi$, $\xi$, $\eta$, $\phi$.

Looking to the system (20), we are able to distinguish two cases for solving it. More precisely, the latter five equations of the system (20) should be solved by taking into account the following situations:

**Case (I)**: $f(x) \neq 0, g(y) \neq 0 \Rightarrow \varphi(t), \xi(t, x), \eta(t, y), \phi(t, x, y, u)$

and

**Case (II)**: $f(x) = 0, g(y) \neq 0 \Rightarrow \varphi(t, y), \xi(t, x), \eta(t, y), \phi(t, x, y, u)$ or $g(y) = 0, f(x) \neq 0 \Rightarrow \varphi(t, x), \xi(t, x), \eta(t, y), \phi(t, x, y, u)$

### 3.3 Symmetries and invariant solutions for Case (I)

Under the conditions (21), through a computational method (Maple 10 program), six solutions could be obtained. They correspond to the following Lie symmetry operators:
(1) $\forall a_1 = \text{const.}, \forall f(x) \neq \text{const.}, \forall g(y) \neq \text{const.}$

\[ X^{(I)}_1 = a_1 \frac{\partial}{\partial t} \]  

(2) $\forall b_j, = \text{const.}, j = 1, 3, 4, \forall g(y) = b_2 = \text{const.} \neq 0, \forall f(x) \neq \text{const.}$

\[ X^{(I)}_2 = b_3 \frac{\partial}{\partial t} + [-b_1 y + b_4] \frac{\partial}{\partial y} + b_1 u \frac{\partial}{\partial u} \]  

(3) $\forall p_j, = \text{const.}, j = 1, 3, 4, \forall f(x) = p_2 = \text{const} \neq 0, \forall g(y) \neq \text{const.}$

\[ X^{(I)}_3 = p_3 \frac{\partial}{\partial t} + [-p_1 x + p_4] \frac{\partial}{\partial x} + p_1 u \frac{\partial}{\partial u} \]  

(4) $\forall s_k, = \text{const.}, k = 1, 7, k \neq 2, 3, \forall f(x) = s_2 = \text{const} \neq 0, \forall g(y) = s_3 = \text{const} \neq 0$

\[ X^{(I)}_4 = s_4 \frac{\partial}{\partial t} + [s_5 x + s_6] \frac{\partial}{\partial x} + \left( (-s_5 - s_1) y + s_7 \right) \frac{\partial}{\partial y} + s_1 u \frac{\partial}{\partial u} \]  

(5) $\forall m_i = \text{const.}, i = 1, 4, f(x) = (x - m_1), g(y) = (-2(y + m_2))^{(-1/2)}$

\[ X^{(I)}_5 = \left[ m_3 t + m_4 \right] \frac{\partial}{\partial t} - m_3 \left[ x - m_1 \right] \frac{\partial}{\partial x} + 2m_3 y + m_1 \frac{\partial}{\partial y} = \left[ m_3 t + m_4 \right] \frac{\partial}{\partial t} - m_3 f(x) \frac{\partial}{\partial x} - \frac{m_3}{2} g(y)^{(-2)} \frac{\partial}{\partial y} \]  

(6) $\forall n_i = \text{const.}, i = 1, 4, f(x) = (x - n_2), g(y) = (y - n_3)$

\[ X^{(I)}_6 = \left[ \frac{n_1 t + n_4}{3} \right] \frac{\partial}{\partial t} - \frac{n_1 (x - n_2)}{3} \frac{\partial}{\partial x} - \frac{n_1 (y - n_3)}{3} \frac{\partial}{\partial y} + n_1 u \frac{\partial}{\partial u} = \left[ \frac{n_1 t + n_4}{3} \right] \frac{\partial}{\partial t} - \frac{n_1 f(x)}{3} \frac{\partial}{\partial x} - \frac{n_1 g(y)}{3} \frac{\partial}{\partial y} + n_1 u \frac{\partial}{\partial u} \]  

It is important to remind some results [8]:

(i) by applying the classical symmetry approach, the general Lie operator for the Ricci flow model (8) is obtained in the form:

\[ V = \left( c_1 t + c_2 \right) \frac{\partial}{\partial t} + \xi(x) \frac{\partial}{\partial x} + \eta(y) \frac{\partial}{\partial y} + u \left[ c_1 - \frac{d \xi(x)}{dx} - \frac{d \eta(y)}{dy} \right] \frac{\partial}{\partial u} \]  

with $c_1$, $c_2$ arbitrary constants and $\xi(x)$, $\eta(y)$ arbitrary functions.

(ii) in the linear sector of invariance (when $\xi(x)$, $\eta(y)$ are considered as having linear forms), the Ricci flow model admits a 6–parameters family of Lie operators which generates the following 6 independent symmetry operators:

\[ V_1 = t \frac{\partial}{\partial t} + u \frac{\partial}{\partial u}, \ V_2 = \frac{\partial}{\partial t}, \ V_3 = \frac{\partial}{\partial x}, \ V_4 = \frac{\partial}{\partial y}, \ V_5 = x \frac{\partial}{\partial x} - u \frac{\partial}{\partial u}, \ V_6 = y \frac{\partial}{\partial y} - u \frac{\partial}{\partial u} \]  

Remark 1 : The Lie point symmetries associated to the conservation law (10), which have the expressions (23)-(28) could be expressed as a linear combination of the independent operators (30).
These expressions are:

\[
\begin{align*}
X_1^{(f)} &= a_1 V_2, \quad X_2^{(f)} = b_3 V_2 + b_4 V_4 - b_1 V_6, \quad X_3^{(f)} = p_3 V_2 + p_4 V_3 - p_1 V_5 \\
X_4^{(f)} &= s_4 V_2 + s_6 V_3 + s_7 V_4 + s_5 V_6 - (s_1 + s_5) V_6 \\
X_5^{(f)} &= m_3 V_1 + m_4 V_2 + m_1 m_3 V_3 + 2m_2 m_3 V_4 - m_3 V_5 + 2m_3 V_6 \\
X_6^{(f)} &= \frac{n_1}{3} V_1 + n_4 V_2 + n_1 n_2 V_3 + n_1 n_3 V_4 - \frac{n_1}{3} V_5 - \frac{n_1}{3} V_6
\end{align*}
\]  

The next objective of this section is to determine the invariant solutions of the analyzed model, generated by the Lie operators \(23\) - \(28\). The similarity reduction method \(15\) will be applied.

Consequently, the operator \(X_1^{(f)}\) from \(23\) has the characteristic equations:

\[
\frac{dt}{a_1} = \frac{dy}{0} = \frac{dx}{0} = \frac{du}{0}.
\]  

By integrating these equations, one obtains three invariants:

\[
I_1^{(1)} = x, \quad I_2^{(1)} = y, \quad I_3^{(1)} = u.
\]  

By designating the invariant \(I_3^{(1)} = H_1(x, y)\) as a function of the other two and then inserting it to the Ricci flow equation \(35\), the similarity reduced equation takes the form:

\[
(H_1)_x (H_1)_y - H_1 (H_1)_{xy} = 0.
\]  

The solution of this equation is:

\[
H_1(x, y) = h_1(x) h_2(y), \quad \forall \ h_1(x), \ \forall h_2(y)
\]  

Thereby, the invariant solution corresponding to the operator \(X_1^{(f)}\) has the final form:

\[
u_1^{(f)}(x, y) = h_1(x) h_2(y), \quad \forall \ h_1(x), \ \forall h_2(y)
\]  

Through a similar algorithm, the following results could be obtained:

(1) The Lie operators \(23\), \(25\), \(26\) do generate, respectively, the following similarity variables:

\[
\begin{align*}
v_2 &= x, \quad w_2 = \frac{b_1 y - b_4}{b_2} \exp\left(\frac{b_1 t}{b_3}\right), \quad v_3 = y, \quad w_3 = \frac{p_1 x - p_4}{p_1} \exp\left(\frac{p_1 t}{p_3}\right) \\
v_4 &= \frac{s_5 x + s_6}{s_5} \exp\left(-\frac{s_5 t}{s_4}\right), \quad w_5 = \frac{(s_5 + s_1) y - s_7}{s_5 + s_1} \exp\left(\frac{s_5 + s_1}{s_4} t\right)
\end{align*}
\]  

and, through similarity reduction, evolutionary equations will look like:

\[
\begin{align*}
b_1 w_2 (H_2)^2 (H_2)_{w_2} + b_1 (H_2)^3 - b_3 H_2 (H_2)_{w_2} + b_3 (H_2)_{v_2} (H_2)_{w_2} &= 0 \\
p_1 w_3 (H_3)^2 (H_3)_{w_3} + p_1 (H_3)^3 - p_3 H_3 (H_3)_{v_3} (H_3)_{w_3} &= 0 \\
s_5 v_4 (H_4)^2 (H_4)_{v_4} - (s_5 + s_1) w_4 (H_4)^2 (H_4)_{w_4} - s_1 (H_4)^3 &+ s_4 H_4 (H_4)_{v_4} - s_4 (H_4)_{v_4} (H_4)_{w_4} &= 0
\end{align*}
\]
(2) The invariant solutions of the model, associated to \( X_2^{(I)}, X_3^{(I)} \) respectively \( X_4^{(I)} \), become stationary:

\[
\begin{align*}
    u_2^{(I)}(x, y) &= \frac{b_1}{b_1 y - b_4} F_2(x), \quad u_3^{(I)}(x, y) = \frac{m_1}{m_1 x - m_3} G_3(y), \quad \forall F_2(x), \quad \forall G_3(y) \\
    u_4^{(I)}(x, y) &= q_1 \left( \frac{s_5 x + s_6}{s_5} \right)^q \frac{q_2}{q_1} \left( \frac{(s_5 + s_1) y - s_7}{s_5 + s_1} \right)^{q_2(q_2-q_1)/(s_5+s_1)}, \quad \forall q_1, q_2 = \text{const.}
\end{align*}
\]

(3) By integrating characteristic equations, the Lie operators \(27\) and \(28\) do lead to the following new variables:

\[
\begin{align*}
    v_5 &= m_4 x + m_3 (x - m_1) t, \quad w_5 = \frac{y + m_2}{(m_4 t + m_4)^2} \\
    v_6 &= x(n_1 t + 3n_4) - n_1 n_4 t, \quad w_6 = y(n_1 t + 3n_4) - n_1 n_3 t
\end{align*}
\]

Through the similarity reduction method, they generate the reduced differential equations:

\[
\begin{align*}
    m_3(v_5 - m_1 m_4)(H_5)^2(H_5)_{v_5} - 2m_3 w_5 (H_5)^2 (H_5)_{w_5} - \\
    - H_5(H_5)_{v_5 w_5} + (H_5)_{v_5} (H_5)_{w_5} = 0
\end{align*}
\]

\[
\begin{align*}
    n_1(v_6 - 3n_2 n_4)(H_6)^2(H_6)_{v_6} + n_1(w_6 - 3n_3 n_4) (H_6)^2 (H_6)_{w_6} + \\
    + 3n_1 (H_6)^3 - H_6 (H_6)_{v_6 w_6} + (H_6)_{v_6} (H_6)_{w_6} = 0
\end{align*}
\]

(4) The invariant solutions of the model, corresponding to \( X_5^{(I)} \), respectively to \( X_6^{(I)} \) are nonstationary. They have the expressions:

\[
\begin{align*}
    u_5^{(I)}(t, x, y) &= \rho_1 \left[ (m_1 m_4 - v_5) w_5^{(1/2)} \right]^{-\rho_2}, \quad \forall \rho_1, \rho_2 = \text{const.} \\
    u_6^{(I)}(t, x, y) &= \gamma_1 (v_6 - 3n_2 n_4) \gamma_2 (-w_6 + 3n_3 n_4)^{-3+\gamma_2}, \quad \forall \gamma_1, \gamma_2 = \text{const.}
\end{align*}
\]

where invariants \( v_5, w_5, v_6, w_6 \), expressed in the terms of original variables, are provided by \(46\).

### 3.4 Symmetries and invariant solutions for Case (II)

By solving the latter five equations of the determining system \(20\) under the conditions \(22\), the following Lie symmetry operators will be generated:

\[
\begin{align*}
    X_1^{(II)} &= c_2 \frac{\partial}{\partial t} + \xi(x) \frac{\partial}{\partial x} - \left( \frac{d\xi(x)}{dx} u \right) \frac{\partial}{\partial u}, \quad c_2 = \text{const.}, \quad \forall \xi(x) \\
    X_2^{(II)} &= c_2' \frac{\partial}{\partial t} + \eta(y) \frac{\partial}{\partial y} - \left( \frac{d\eta(y)}{dy} u \right) \frac{\partial}{\partial u}, \quad c_2' = \text{const.}, \quad \forall \eta(y)
\end{align*}
\]

**Remark 2:** The forms \(50\), \(51\) represent particular cases of the general operator \(29\). They could be obtained by imposing \(c_1 = 0, \eta(y) = 0\), respectively \(c_1 = 0, \xi(x) = 0\) in \(29\).

Let us obtain invariant solutions for the 2D Ricci model, associated to operator \(50\). Similar results will be derived for operator \(51\).

The function \(u = \Psi(t, x, y)\) is an invariant solution generated by \(50\), providing that:

\[
X_1^{(II)} [u - \Psi(t, x, y)] |_{u=\Psi} = 0
\]

The previous condition has the equivalent form:

\[
\xi(x)\Psi + c_2 \Psi_t + \xi(x)\Psi_x = 0
\]
The general solution of (53) is:

\[ \Psi(t, x, y) = F(v, w) \xi(x), \quad v = y, \quad w = t - \int c^2 \xi(x) \, dx \]  

(54)

with arbitrary functions \( \xi(x) \) and \( F(v, w) \).

The substitution of (54) into the 2D equation (8) yields a reduced differential equation of the form:

\[ F^2 F_w + c^2 F F_{vw} - c^2 F_v F_w = 0 \]  

(55)

which admits as a solution an arbitrary function \( F_1(v) \).

Consequently, by taking into account (54), the invariant solution associated to operator (50) becomes:

\[ u_1^{(II)}(x, y) = \frac{F_1(y)}{\xi(x)}, \quad \forall F_1(y), \quad \forall \xi(x) \]  

(56)

Due to similar reasons, the invariant solution attached to operator (51) will be:

\[ u_2^{(II)}(x, y) = \frac{F_2(x)}{\eta(y)}, \quad \forall F_2(x), \quad \forall \eta(y) \]  

(57)

4 Concluding remarks

In literature the problem of constructing a conserved vector is generally approached by means of the direct method of solving \( D_i P^i = 0 \), for a given equation with no recourse to symmetry properties. It usually involves ad hoc assumptions able to simplify the procedure. For a known symmetry operator \( X \), determining the conserved vector \( P = (P^i) \), \( i = 1, p \) would become simpler if the relation (7) which connects symmetries and conservation laws, should be added to the conservation law \( D_i P^i = 0 \). Conversely, the special relation (7) could also be used to obtain the symmetry generators \( X \) associated with a given conserved vector \( P \). The latter approach is applied in this paper for the 2D Ricci flow model.

The central objects, in the study of conservation laws, are the sets of conservation law multipliers. The key property of these multipliers is that their existence implies the existence of conservation laws. The main results of this paper consist in obtaining: (i) the set of multipliers (14) which depend upon two arbitrary functions \( f(x) \) and \( g(y) \); (ii) the set of local conservation laws (16) associated to the mentioned multipliers; (iii) two sets of symmetry operators (23)-(28) and (50), (51), by making use of the the tight relationship (7) between symmetries and conservation laws and by solving the determining differential system (20) in two different cases. In the first case, each of the Lie symmetry generators could be expressed as a linear combination of the independent operators (30), obtained [8] for the linear sector of invariance of the Ricci flow model. In the second case, both operators (50) and (51), could be identified as particular forms of the general Lie operator [8] which depends on two arbitrary functions; (iv) six stationary and two nonstationary invariant solutions of the Ricci model, associated to the mentioned symmetry operators are computed. Among, them, some new similarity solutions, not yet given in literature, were highlighted. They are listed in (45), (46), (49), (56), (57).

These results have been obtained by assuming a conservation law multiplier which does not depend on the derivatives of the variable \( U(t, x, y) \). The multipliers which depend on the first order and higher order partial derivatives of \( U \) may lead to further conservation laws. They will be studied in forthcoming works.

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