REGULARITY OF LYAPUNOV EXPONENTS FOR
DIFFEOMORPHISMS WITH DOMINATED SPLITTING

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ABSTRACT. We consider families of diffeomorphisms with dominated splittings and preserving a Borel probability measure, and we study the regularity of the Lyapunov exponents associated to the invariant bundles with respect to the parameter. We obtain that the regularity is at least the sum of the regularities of the two invariant bundles (for regularities in $[0, 1]$), and under suitable conditions we obtain formulas for the derivatives. Similar results are obtained for families of flows, and for the case when the invariant measure depends on the map.

We also obtain several applications. Near the time one map of a geodesic flow of a surface of negative curvature the metric entropy of the volume is Lipschitz with respect to the parameter. At the time one map of a geodesic flow on a manifold of constant negative curvature the topological entropy is differentiable with respect to the parameter, and we give a formula for the derivative. Under some regularity conditions, the critical points of the Lyapunov exponent function are non-flat (the second derivative is nonzero for some families). Also, again under some regularity conditions, the criticality of the Lyapunov exponent function implies some rigidity of the map, in the sense that the volume decomposes as a product along the two complimentary foliations. In particular for area preserving Anosov diffeomorphisms, the only critical points are the maps smoothly conjugated to the linear map, corresponding to the global extrema.

1. Introduction

Let $M$ be an orientable compact Riemannian manifold without boundary of dimension $d$. Let $f : M \to M$ be a $C^r$ diffeomorphism of $M$, $r \geq 1$, which has a dominated splitting $TM = E^1 \oplus E^2 \oplus E^3$, meaning that the splitting is continuous, invariant under $Df$, and satisfies the following conditions:
\[
\sup_{x \in M} \left\| Df \big|_{E^1(x)} \right\| \cdot \left\| Df^{-1} \big|_{E^2(f(x))} \right\| < 1,
\]
\[
\sup_{x \in M} \left\| Df \big|_{E^2(x)} \right\| \cdot \left\| Df^{-1} \big|_{E^3(f(x))} \right\| < 1.
\]

We assume that each bundle \( E^i \) is orientable and \( Df \) preserves the orientation of each \( E^i \), for \( i \in \{1, 2, 3\} \). We can allow that either \( E^1 \) or \( E^3 \) is trivial, however we assume that \( E := E^2 \) and \( F := E^1 \oplus E^3 \) are not trivial, and let \( k = \dim E \geq 1 \). This means that our considerations can be applied in the context of partially hyperbolic diffeomorphisms to the stable, unstable, center, center-stable, center-unstable, or even intermediate bundles.

Assume that \( f \) preserves the Borel probability measure \( \mu \) on \( M \). Oseledets Theorem ([48], see also [47], Chapter 4.10) gives the existence of \( n \) Lyapunov exponents (counted with their multiplicity) for \( \mu \) almost every point \( p \in M \), and a corresponding Lyapunov splitting. From these \( n \) exponents, \( k \) will correspond to the invariant bundle \( E \), meaning that the corresponding bundles of the Lyapunov splitting are inside \( E \), and we denote their sum by \( \lambda(p, f, E) \). The \emph{integrated Lyapunov exponent of} \( f \) \emph{with respect to} \( \mu \) \emph{and associated to} \emph{the bundle} \( E \) \emph{will be}

\[
\lambda(f, E, \mu) = \int_M \lambda(p, f, E) d\mu.
\]

If the measure \( \mu \) is ergodic for \( f \), then of course \( \lambda(p, f, E) = \lambda(f, E, \mu) \) for \( \mu \) almost every \( p \in M \).

From the Birkhoff Ergodic Theorem one can see that an alternative definition of the integrated Lyapunov exponent is

\[
(1.1) \quad \lambda(f, E, \mu) = \int_M \log \| Df^k |_E \| d\mu = \int_M \log |J(f|_E)| d\mu,
\]

where \( J(f|_E) \) is the Jacobian of \( f \) restricted to \( E \).

Recall that if \( f \) has a dominated splitting, then for any diffeomorphism \( g \) which is \( C^1 \) close to \( f \) the dominated splitting persists, i.e. there exists a dominated splitting for \( g \), \( TM = E^1_g \oplus E^2_g \oplus E^3_g \). If furthermore \( g \) preserves the same measure \( \mu \), then we can obtain again the integrated Lyapunov exponent \( \lambda(g, E_g, \mu) \) of \( g \) with respect to \( \mu \) and associated to the bundle \( E_g := E^2_g \).

The main goal of our paper is to study the regularity of the map \( g \mapsto \lambda(g, E_g, \mu) \). This map is always continuous because of the continuous dependence of the dominated splitting with respect to the diffeomorphism. We will find sufficient conditions that guarantee better regularity of this map, we will obtain formulas for the derivatives along one-parameter families, and we will investigate the critical points in some specific situations.
1.1. **Regularity of the integrated Lyapunov exponent.** To be more specific, the standing hypothesis and notations throughout the paper are the following:

**Hypothesis (H):** Let $f_t$, $t \in I \subseteq \mathbb{R}$, be a $C^r$ family of $C^s$ diffeomorphisms with dominated splittings $TM = E^1_t \oplus E^2_t \oplus E^3_t$, where $r \geq 1$ and $I$ is an open interval containing 0, all the bundles are orientable and $Df_t$ preserves the orientations. $E_t := E^2_t$, $F_t := E^1_t \oplus E^3_t$. All the maps $f_t$ preserve the same Borel probability $\mu$, and let $\lambda(t) = \lambda(f_t, E_t, \mu)$. Let $X$ be the $C^{r-1}$ vector field on $M$ tangent to the family $h_t := f_t \circ f_0^{-1}$ in $t = 0$. If $r \geq 2$ let $\phi^X$ be the flow on $M$ generated by $X$. If $t = 0$ we will just drop the index $t$ from all the notations: $f := f_0$, $E := E_0$, $F := F_0$, etc.

Our first result relates the regularity of $\lambda$ at $t = 0$ with the regularity of the splitting $TM = E \oplus F$ for $f_0$. If the map $t \mapsto E_t$ is of class $C^\beta$ (with respect to $t$ in $t = 0$), it is easy to see that the same holds for $t \mapsto \lambda(t)$, as long as $r \geq \beta + 1$. If $E$ is $C^\beta$ (with respect to the points on $M$) for some $\beta \in [0, 1]$, then $t \mapsto E_t$ is also $C^\beta$ at $t = 0$. Dolgopyat proved in [24] this fact for the case $\beta = 1$, and we will discuss this point with more details in subsection 2.4. Thus the regularity of $\lambda$ at $t = 0$ is at least the regularity of $E$, and by the symmetry it is also at least the regularity of $F$, at least up to the $C^1$ regularity. The next result says that in fact the regularity of $\lambda$ at $t = 0$ is at least the sum of the regularity of $E$ and the regularity of $F$, if the two regularities of the bundles are in $[0, 1)$.

**Theorem A.** Assume that (H) is satisfied for $r \geq 3$, $F$ is of class $C^\alpha$ and $E$ is of class $C^\beta$ on a neighborhood of the support of $\mu$, for some $\alpha, \beta \in [0, 1)$. Then

(i) If $\alpha + \beta < 1$ then $\lambda(t) = \lambda(0) + O(t^{\alpha+\beta})$;

(ii) If $\alpha + \beta = 1$ then $\lambda(t) = \lambda(0) + O(t \log t)$;

(iii) If $\alpha + \beta > 1$ then $\lambda(t) = \lambda(0) + t \lambda'(0) + O(t^{\alpha+\beta-1})$. Furthermore

\[
\lambda'(0) = \left. \frac{\partial}{\partial t} \int_M (\phi^X_t)^* \omega_F(V_E)d\mu \right|_{t=0} = \left. \frac{\partial}{\partial t} \int_M \omega_F((\phi^X_t)^* V_E)d\mu \right|_{t=0}
\]

and

\[
|\lambda'(0)| \leq C_{\alpha,\beta,M} \|X\|_{C^0} \|\omega_F\|_{C^0} \|V_E\|_{C^0} + C_M \|X\|_{C^1} \|\omega_F\|_{C^0} \|V_E\|_{C^0}.
\]

where $\omega_F$ is a continuous $k$-form on $M$ depending of $F$ and $V_E$ is a continuous non zero $k$-multivector field depending on $E$ (See the next subsection for a precise definition and comments on $\omega_F$ and $V_E$).

As an application we have the following corollary:

**Corollary B.** Let $\phi_1$ be the time-one map of the geodesic flow on a surface of negative curvature preserving the volume $\mu$. There exists a $C^1$ neighborhood $U$ of $\phi_1$ such that for
any $C^3$ one-parameter family of $C^3$ diffeomorphisms in $\mathcal{U}$ preserving $\mu$, the metric entropy with respect to $\mu$ is Lipschitz with respect to the parameter.

In order to prove Theorem A we will use the following result which has its own interest.

Theorem C. Let $\phi_t$ be a flow on the compact manifold $M$, generated by the $C^r$ vector field $X$. Let $f, g : M \to \mathbb{R}$ be continuous observables on $M$. Let $\mu$ be an invariant measure for $\phi$, and assume that $f$ is $C^\alpha$ and $g$ is $C^\beta$ in a neighborhood of the support of $\mu$, $\alpha, \beta \geq 0$, $r \geq \max\{\alpha, \beta\} - 1$. Let $h(t) = \int_M f(x)g(\phi_t(x))d\mu$. If either $\alpha + \beta$ is not an integer, or both $\alpha$ and $\beta$ are integers, then $h$ is $C^{\alpha+\beta}$; otherwise $h$ is $C^{\alpha+\beta-1}$.

We have the bound $\|h\|_{C^{\alpha+\beta}} \leq C_{X,\alpha,\beta}\|f\|_{C^\alpha}\|g\|_{C^\beta}$, where $C_{X,\alpha,\beta}$ depends on $\alpha, \beta$, and $X$. In particular if $\alpha, \beta \in (0, 1)$, $\alpha + \beta > 1$ then $\|h\|_{C^1} \leq C_{\alpha,\beta}\|X\|_{C^\alpha}\|f\|_{C^\alpha}\|g\|_{C^\beta}$.

1.2. Formulas for the derivatives. In order to obtain formulas for the derivatives of the integrated Lyapunov exponent with respect to the parameter, we will put the problem in the context of vector fields and forms, motivated by the book [65]. Let us consider $\omega_F \in \Omega^k(M)$, a continuous nonzero $k$-form on $M$, such that $\ker \omega_F = F \wedge TM^{\wedge(k-1)}$, and eventually after shrinking the interval $I$ let us assume that $\omega_F|_{E_t(p)^\wedge k} \neq 0$, for every $p \in M$ and every $t \in I$.

This can be done because of the continuity and the orientability of $E_t$ and $F_t$. We are making an abuse of the notations, considering that the $k$-form $\omega$ acts on $k$-multivectors, however the action is well defined since a differential form is multilinear and anti-symmetric.

Next, for every $t \in I$, we can choose a continuous nonzero $k$-multivector field $V_t$, with $V_t(p) \in E_t(p)^\wedge k$ for every $p \in M$, $t \in I$, such that

$$\omega(V_t(p)) = 1. \tag{1.4}$$

By the continuity of the map $(t, p) \to E_t(p)$ we have that $V_t$ is continuous in $t$.

In fact, if $E$ is $C^\beta$ and $F$ is $C^\alpha$ for some $\alpha, \beta \geq 0$, then we can choose $\omega_F$ to be $C^\alpha$ and $V_E := V_0$ to be $C^\beta$, for more on this construction see subsection 2.1.

We will denote by $(f_t)_*$ the action induced by $Df$ on the tangent bundle (vectors, multivectors) and $(f_t)^*$ the action induced on the cotangent bundle (forms). Since the space $E_t(p)$ is $(f_t)_*$-invariant, there is a real number $\eta_t(p)$ such that

$$\omega(V_t(p)) = \eta_t(p) \cdot V_t(f_t(p)). \tag{1.5}$$

In other words, if we denote $\tilde{\eta}_t = \eta_t \circ f_t^{-1}$, we have

$$(f_t)_*V_t(p) = \tilde{\eta}_t(p) \cdot V_t(p).$$
Observe that in fact $\eta_t$ measures the volume expansion of $Df_t$ restricted to $E_t$ using a metric which gives norm one to the multivectors $V_t$. Since the Lyapunov exponent is independent of the metric this implies that

\begin{equation}
\lambda(t) = \int_M \log \eta_t(p) d\mu = \int_M \log \tilde{\eta}_t(p) d\mu.
\end{equation}

A representation of $\omega, V, V_t$ and the action of the derivatives of $f$ and $h_t$ can be seen in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Action of the derivative $Df_t$ on $V_t$.}
\end{figure}

Recall that Dolgopyat showed that when $E$ is $C^1$ then $E_t$ is differentiable with respect to $t$ at $t = 0$. In this case $V_t$ will also be differentiable with respect to $t$ at $t = 0$, and we denote its derivative $V'$. For more on this and an explicit formula for $V'$ see subsection 2.4.

Recall that the vector field $X$ on $M$ is tangent to the family $h_t := f_t \circ f_0^{-1}$ in $t = 0$, and let $Y$ be the vector field which gives the second order correction of $h_t$ for $r \geq 3$ (exact definition and more details can be founded in the subsection 2.2). Again, for simplicity we will use the notations $\eta := \eta_0, \tilde{\eta} := \tilde{\eta}_0$.

The formula (1.2) gives a (not very explicit) formula for $\lambda'(0)$ when the sum of the regularities of $E$ and $F$ is greater than one. The next result gives explicit formulas for the first derivative of $\lambda$ in 0 when either $E$ or $F$ are $C^1$, and the second derivative of $\lambda$ in 0 when both $E$ and $F$ are $C^1$. 
Theorem D. Assume that \((H)\) is satisfied for \(r \geq 3\), and \(X, Y, \omega_F, V_E, V'\) and \(\tilde{\eta}\) are defined as above.

(i) If \(F\) is \(C^1\) on a neighborhood of \(\text{supp}(\mu)\), then \(\lambda\) is differentiable in 0 and

\[
\lambda'(0) = \int \mathcal{L}_X \omega_F(V_E) d\mu. \tag{1.7}
\]

(ii) If \(E\) is \(C^1\) on a neighborhood of \(\text{supp}(\mu)\), then \(\lambda\) is differentiable in 0 and

\[
\lambda'(0) = -\int \omega_F(\mathcal{L}_X V_E) d\mu. \tag{1.8}
\]

(iii) If both \(E\) and \(F\) are \(C^1\) on a neighborhood of \(\text{supp}(\mu)\), then \(\lambda\) has expansion of order 2 at \(t = 0\):

\[
\lambda(t) = \lambda(0) + t\lambda'(0) + \frac{t^2}{2}\lambda''(0) + o(t^2),
\]

where

\[
\lambda''(0) = \int_M -\mathcal{L}_X \omega_F(\mathcal{L}_X V_E) + \mathcal{L}_Y \omega_F(V_E) - (\mathcal{L}_X \omega_F(V_E))^2 + \frac{2}{\tilde{\eta}} \mathcal{L}_X \omega_F(f_* V') d\mu. \tag{1.9}
\]

Here \(\mathcal{L}\) is the usual Lie derivative which is well defined on \(C^1\) forms and multivector fields.

It is remarkable that the first derivative does not depend explicitly on \(f\) or \(f_t\), it only depends on the both sub-bundles \(E\) and \(F\) of the invariant splitting for \(f\), on the invariant measure \(\mu\), and on the vector field \(X\) tangent to the family \(h_t\). It is not hard to see that \(\lambda'(0)\) is independent on the choice of \(\omega_F\) and \(V_E\), and it is in fact linear in the vector field \(X\) (and in \(\omega_f\) and \(V_E\)), and bounded with respect to the \(C^1\) topology. The second derivative is the sum of a bilinear form in \(X\) and a linear form in \(Y\), and the last term does depend on \(f\).

1.3. Variable measure. We remark that Theorem D can be formulated even in the setting of invariant measures \(\mu_t\) varying with the parameter \(t \in I\). Of course, we need to impose some conditions on the regularity of the family of measures.

Let us consider now the corresponding hypothesis for variable measure. It is identical with the hypothesis \((H)\), with the only difference that now the invariant measures \(\mu\) depend on \(t\).

Hypothesis \((H')\): Let \(f_t, t \in I \subset \mathbb{R}\), be a \(C^r\) family of \(C^r\) diffeomorphisms with dominated splittings \(TM = E^1_t \oplus E^2_t \oplus E^3_t\), where \(r \geq 1\) and \(I\) is an open interval containing 0, all the bundles are orientable and \(Df\) preserves the orientations. \(E_t := E^2_t, F_t := E^1_t \oplus E^3_t\). Every map \(f_t\) preserves a Borel probability measure \(\mu_t\), \(\lim_{t \to 0} \mu_t = \mu_0 := \mu\) in the weak* topology, and let \(\lambda(t) = \lambda(f_t, E_t, \mu_t)\). Let \(X\) be the \(C^{r-1}\) vector field on \(M\).
tangent to the family $h_t := f_t \circ f_0^{-1}$ in $t = 0$. If $r \geq 2$ let $\phi^X$ be the flow on $M$ generated by $X$. If $t = 0$ we will just drop the index $t$ from all the notations: $f := f_0$, $E := E_0$, $F := F_0$, etc.

We say that the family $\mu_t$ has linear response $R(\varphi)$ for the continuous function $\varphi : M \to \mathbb{R}$, if the application $t \mapsto \int_M \varphi d\mu_t$ is differentiable in $t = 0$ and the derivative is $R(\varphi) \in \mathbb{R}$. In particular, if $\varphi$ is constant, then any family of measures has linear response, and $R(\varphi) = 0$.

We obtain the following result.

**Theorem E.** Assume that (H') is satisfied for $r \geq 3$. Then:

(i) If $F$ is $C^1$, then $\lambda$ is differentiable in $0$ if and only if the family $\mu_t$ has linear response $R(\log \eta)$ for the function $\log \eta : M \to \mathbb{R}$. In this case we have

$$\lambda'(0) = R(\log \eta) + \int_M L_X \omega F(V_E) d\mu.$$ 

(ii) If $E$ is $C^1$, then $\lambda$ is differentiable in $0$ if and only if the family $\mu_t$ has linear response $R(\log \eta)$ for the function $\log \eta : M \to \mathbb{R}$. In this case we have

$$\lambda'(0) = R(\log \eta) - \int \omega F(L_X V_E) d\mu.$$ 

(iii) Suppose that $E$ is $C^1$ and $F$ is $C^2$. In addition, suppose that $\eta$ is constant and the family $\mu_t$ has linear response $R(L_X \omega F(V_E) \circ f)$ for the function $L_X \omega F(V_E) \circ f : M \to \mathbb{R}$. Then $\lambda$ has expansion of order two at $t = 0$, and

$$\lambda''(0) = 2R(L_X \omega F(V_E) \circ f) +$$ 

$$+ \int_M L_X L_X \omega F(V_E) + L_Y \omega F(V_E) - (L_X \omega F(V_E))^2 + \frac{2}{\eta} L_X \omega F(f_* V') d\mu.$$ 

As an application we can obtain a linear response formula for the topological entropy at the time one map of a geodesic flow on a manifold of negative curvature.

**Theorem F.** Let $f$ be the time one map of a geodesic flow on a manifold of constant negative curvature, and let $f_t$ be a $C^3$ family of diffeomorphisms with $f_0 = f$. Then the map $t \mapsto h_{\text{top}}(f_t)$ is differentiable at $t = 0$, and the derivative is

$$(1.10) \quad \frac{\partial}{\partial t} h_{\text{top}}(f_t) \bigg|_{t=0} = - \int_M \omega_{E^{cs}}(L_X V_{E^u}) d\mu,$$

where $\mu$ is the Liouville measure, $X$ is the vector field tangent to the perturbation $h_t = f_t \circ f_0^{-1}$ at $t = 0$, and $\omega_{E^{cs}}$ and $V_{E^u}$ are chosen as in the subsection 1.2 for the splitting $E^{cs} \oplus E^u$. 

The result also holds for the time-$t$ map of the suspension flow over a linear Anosov map, for $t$ irrational.

1.4. Families of flows. We can obtain similar results if we consider families of flows instead of diffeomorphisms. A splitting is dominated for a $C^1$ flow $\phi$ if it is dominated for the time-one map of the flow $\phi_1$. The integrated Lyapunov exponents associated to an invariant bundle and an invariant measure is equal to the exponent corresponding to the time-one map of the flow, and the same bundle and measure.

We have the following hypothesis and notations:

**Hypothesis (HF):** Let $X_t, t \in I \subset \mathbb{R}$, be a $C^r$ family of $C^r$ vector fields, such that the corresponding flows $\phi^t$ have dominated splittings $TM = E^1_t \oplus E^2_t \oplus E^3_t$, where $r \geq 1$ and $I$ is an open interval containing 0, all the bundles are orientable and $D\phi^t$ preserves the orientations. $E_t := E^2_t$, $F_t := E^1_t \oplus E^3_t$. All the flows $\phi^t$ preserve the same Borel probability $\mu$, and let $\lambda(t) = \lambda(\phi^t, E, \mu)$. Let $X' := X'(0)$ be the $C^{r-1}$ vector field which is the derivative of $X_t$ with respect to $t$ in $t = 0$. If $r \geq 2$ then let $\phi X'$ be the flow on $M$ generated by $X'$. If $t = 0$ we will just drop the index $t$ from all the notations: $\phi := \phi^0$, $E := E_0$, $F := F_0$, etc.

We have the following result:

**Theorem G.** Assume that (HF) is satisfied for $r \geq 3$, $F$ is $C^\alpha$ and $E$ is $C^\beta$ on a neighborhood of the support of $\mu$, for some $\alpha, \beta \in [0,1]$. Then

(i) If $\alpha + \beta < 1$ then $\lambda(t) = \lambda(0) + O(t^{\alpha+\beta})$;
(ii) If $\alpha + \beta = 1$ then $\lambda(t) = \lambda(0) + O(t \log t)$;
(iii) If $\alpha + \beta > 1$ then $\lambda(t) = \lambda(0) + t \lambda'(0) + O(t^{\alpha+\beta-1})$. Furthermore

$$
\lambda'(0) = \frac{\partial}{\partial t} \left. \int_M (\phi_t^X)^* \omega_F(V_E) d\mu \right|_{t=0} = \frac{\partial}{\partial t} \left. \int_M \omega_F((\phi_t^X)_* V_E) d\mu \right|_{t=0}.
$$

(iv) If $\alpha = 1$, then $\lambda(t)$ is differentiable in $t = 0$ and

$$
\lambda'(0) = \int \mathcal{L}_{X'} \omega_F(V_E) d\mu.
$$

(v) If $\beta = 1$, then $\lambda(t)$ is differentiable in $t = 0$ and

$$
\lambda'(0) = -\int \omega_F(\mathcal{L}_{X'} V_E) d\mu.
$$

(vi) If $\alpha = \beta = 1$ then $\lambda$ has expansion of order 2 at $t = 0$. 


1.5. **Critical points of the integrated Lyapunov exponent.** Once we obtain formulas for the derivative of the integrated Lyapunov exponent which can be applied for large sets of diffeomorphisms, a natural question is what can we say about the critical points of $\lambda$. We say that a diffeomorphism $f$ is critical (for the bundle $E$ which is part of a dominated splitting and the measure $\mu$) if for any smooth family $f_t$, preserving the measure $\mu$ and passing through $f_0 = f$, we have $\lambda'(0) = 0$.

The following are some examples, the invariant measure $\mu$ is the volume, and the proofs of the claims are left to the reader:

(i) Linear automorphism of the torus: is critical for any bundle which is part of a dominated splitting.

(ii) Time-one map of a volume preserving hyperbolic flow: critical for the central bundle, not critical for the stable and unstable bundles.

(iii) Skew product over a volume preserving Anosov diffeomorphism, with rotations on the center fibers which are circles: critical for the center bundle, may be critical or not for the stable and unstable bundles.

On the other hand, if for example $\mu$ is the Dirac measure at a fixed point, then there are no critical diffeomorphisms. This is why the study of the critical diffeomorphisms is more interesting when the invariant measure is the volume, or at least the bundle $E$ is not transversal to the support of $\mu$.

1.5.1. **Non-flat critical points.** Our next results says basically that if $\mu$ is the volume, and a critical diffeomorphism has a $C^1$ splitting, then the critical point is non-degenerate or non-flat (the second derivative is nonzero). Given a family of diffeomorphisms $f_t$, satisfying the hypothesis (H), we denote by $\lambda_i(t)$ the integrated Lyapunov exponent of $f_t$ corresponding to $E_i$ with respect to the volume:

$$\lambda_i(t) = \lambda(f_t, E_i, \mu) = \int_M \log \| Df_t^{\dim(E)} \|_{E_i} \, d\mu. $$

**Theorem H.** Assume that $f$ is a $C^3$ volume preserving diffeomorphism on the compact manifold $M$. Assume that $f$ has a dominated splitting $TM = E^1 \oplus E^2 \oplus E^3$ which is $C^1$, with $E^2$ and $E^3$ nontrivial.

Then there exists a family of $C^\infty$ diffeomorphisms $h_t$, with $f_t = h_t \circ f$ satisfying the hypothesis (H) for $r = 3$ and $\mu$ equal to the volume, such that

$$\lambda''_3(0) < 0 \quad \text{and} \quad \lambda''_2(0) > 0. $$

Theorem H generalizes classical results obtained previously by Shub-Wilkinson, Ruelle, Dolgopyat and Dolgopyat-Pesin (see [64, 59, 24, 25]). Like in the papers mentioned above,
it can be used in order to remove zero exponents for volume preserving partially hyperbolic diffeomorphisms by arbitrarily small $C^\infty$ perturbations. In particular one can obtain nonuniform hyperbolicity, as well as pathological center and intermediate foliations, by doing arbitrarily small $C^\infty$ perturbations of diffeomorphisms with $C^1$ dominated splittings. For example this is the case of partially hyperbolic automorphisms on nilmanifolds, or products of volume preserving codimension one Anosov maps with rotations.

It seems very probable that the result can be adapted to more general situations, for example if we assume that only $E^2$ and $E^3$ are smooth, and uses a perturbation in the direction of $E^2 \oplus E^3$. For example this could be the case of skew products over volume preserving codimension one Anosov maps, where the fibers are circles and the fiber maps are rotations. It is worth mentioning that this construction was already known in the $C^1$ topology from [7], so the novelty here is the use of $C^\infty$ small perturbations.

1.5.2. **Critical points and rigidity.** The last result says that if again $\mu$ is the volume, and the critical diffeomorphism has a sufficiently smooth splitting forming two transversal foliations, then the critical point is rigid in the sense that the volume disintegrates as a true product along the two complimentary foliations.

**Theorem I.** Assume that $f$ is a $C^3$ volume preserving diffeomorphism on the compact manifold $M$. Assume that $f$ has a dominated splitting $TM = E^1 \oplus E^2 \oplus E^3$, $E := E^2$ and $F := E^1 \oplus E^3$ integrate to complimentary foliations $\mathcal{W}^E$ and $\mathcal{W}^F$. Assume also that $F$ is $C^1$, $\dim E = 1$, and the foliation $\mathcal{W}^E$ has $C^2$ leaves, it is absolutely continuous, and the densities of the disintegrations of the volume along the $\mathcal{W}^E$-leaves are $C^1$ along the $\mathcal{W}^E$-leaves.

If $f$ is a critical diffeomorphism for $E$ and the volume, then the disintegrations of the volume along $\mathcal{W}^E$ are invariant under $\mathcal{W}^F$-holonomy.

Let us make a few remarks on this result.

**Remark 1.** The condition required on the bundle $E$ is satisfied if $E$ is $C^1$, or more generally if $E$ is the unstable (or stable) bundle of a $C^2$ diffeomorphism.

**Remark 2.** The disintegrations of the volume along $\mathcal{W}^E$ are invariant under $\mathcal{W}^F$-holonomy if and only if the disintegrations of the volume along $\mathcal{W}^F$ are invariant under $\mathcal{W}^E$-holonomy, or we say that the volume is a "true product".

**Remark 3.** The conclusion of Theorem I is similar in some sense to the "Invariance Principle"-type results, see for example [3, 2], etc. In these results zero center exponents would imply that the disintegrations along the center foliations are invariant under
the stable and unstable holonomies. It is interesting that the criticality of an exponent will also imply a similar conclusion (of course we require stronger regularity assumptions).

The stable and unstable bundles of area preserving Anosov diffeomorphisms in dimension 2 are $C^{2-}$ (that means diffeomorphisms of class $C^r$, for all $0 \leq r < 2$), so the stable and unstable Lyapunov exponents $\lambda_s$ and $\lambda_u$ (with respect to the area) are differentiable with respect to the parameter along one-parameter families. We obtain the following corollary.

**Corollary J.** The critical diffeomorphisms for the unstable (stable) Lyapunov exponent with respect to the area, in the space of $C^\infty$ area preserving Anosov diffeomorphisms of the two-torus homotopic to the linear map $L$, are $C^\infty$ conjugated to $L$ (in particular they are the global maximum).

This corollary answers a conjecture from [29]. In fact the question posed in [29] is much weaker, they asked whether local maximality of the unstable exponent implies rigidity. A local maximum of the unstable exponent is automatically a critical point.

1.6. **Some historical remarks.** The theory of characteristic exponents originated over a century ago in the study of the stability if solutions of differential equations by A. M. Lyapunov [46]. The work of Furstenberg, Kesten, Oseledets, Kingman, Ledrappier and other built the study of Lyapunov exponents into a very active research field in its own right, and one with an unusually vast array of interaction with others areas of the mathematics and physics, as stochastic processes (random matrices [26, 27], random walks on groups [30]), spectral theory (Schrödinger-type operators [20, 21]) and smooth dynamics (non uniform hyperbolicity [8]). Since then, an extensive literature has been written about it, we refer the reader to the books [10, 65] and the expository papers [70, 66] for an approach of the theme related with our work.

In the setting of smooth dynamics, Lyapunov exponents play a key role understanding the behavior of a dynamical system. On the one hand, when the Lyapunov exponents are non zero, the theory initiated by Pesin [49] provides detailed geometric information on the dynamics and several deep results have been proved: entropy formula for smooth measure [50] and its converse [40, 42], its interplay with the Hausdorff dimension [9], the existence of uniformly hyperbolic sets having many periodic orbits (in particular, the number of orbits of period $n$ grows exponentially in $n$) and carrying large entropy [35], statistical description for the orbits of a large set of points [1, 16, 23].

On the other hand, vanishing exponents is an exceptional situation that also can be exploited. In the sixties, Furstenberg [28] proved in the setting random matrices in SL(2, $\mathbb{R}$)
that if the exponent vanishes, then the matrices either leave invariant a common line or pair of lines, or they generate a precompact group (see [41] for a generalization to any dimension). Such possibilities are degenerate and they can be easily destroyed by perturbing the matrices.

The regularity of various dynamical invariants (including Lyapunov exponents) with respect to parameters was successfully investigated in the context of Anosov systems (motivated by hyperbolic geometry) in a series of works by Katok, Knieper, Pollicott, Weiss, Contreras, Ruelle, among others (see [39, 36, 37, 38, 52, 69, 19, 57, 58, 60]). Following this direction, an active field of study is the dependence of (physical) measures with respect to parameters (so-called linear response formulas), and there are many advances in particular for one-dimensional maps by Ruelle, Baladi, Smania, Dolgopyat, (see [61, 6] and the references therein for an overview panorama of linear response).

The study of the regularity of Lyapunov exponents with respect to the volume in the context of partially hyperbolic dynamics was initiated in a remarkable paper by Shub-Wilkinson [64]. The authors established the regularity of the center exponent within a family of volume preserving partially hyperbolic diffeomorphisms of the three-torus, they showed that the second derivative is nonzero, and in conclusion they constructed open sets of such diffeomorphisms which are stably ergodic, nonuniformly hyperbolic, and with pathological center foliations. This ideas were pushed further in [62], while in [59] the regularity of the exponents and formulas for the derivatives were obtained for linear automorphisms of the $n$-torus.

Another remarkable progress was obtained in [24], in the context of Anosov actions. Here, among other things, Dolgopyat established the regularity of the Lyapunov exponents at the time-one map of the geodesic flow on a surface of constant negative curvature, and the non-vanishing of the second derivative. These ideas were used also in [25] in order to construct completely hyperbolic diffeomorphisms on any manifold. In fact the two papers [64, 24] are the main source of inspiration of our work.

In the line of removing zero exponents and obtaining nonuniform hyperbolicity, Baraviera and Bonatti obtained in [7] that the Lyapunov exponents are not locally constant in the $C^1$ topology. In a parallel direction, there exists recent research relating the zero central exponents with rigidity properties of the system, and thus suggesting that the zero exponents are a highly non-generic situation. There are several works in this direction, based on the so-called ”Invariance principle” formulated more completely in a dynamical setting by [15] and further refined and applied in various works, see for example [3, 2, 4].
In the case of cocycles with dominated splitting, one has much better regularity results. In this case one assumes that the base dynamics and the invariant measure are fixed, and allows changes for the linear bundle maps. Ruelle showed in [56] that the Lyapunov exponents are analytic with respect to the parameter, for any (fixed) base dynamics.

The situation is much more complicated in the absence of the dominated splitting, and one cannot expect in general not even the continuity of the exponents. Bochi-Viana [11, 12] showed that there are situations when $C^1$ generically the Lyapunov exponents (with respect to the volume) are zero. Some recent results established the continuity of the center exponents, again with respect to the volume, for large sets of symplectic or volume preserving partially hyperbolic diffeomorphisms with center dimension two (see [43, 44, 67]).

Again for the case of cocycles there are better results. If the base dynamics is sufficiently random (hyperbolic), and the cocycle satisfies some other conditions (bunching, accessibility), continuity of the Lyapunov exponents for two-dimensional cocycles was established in [13, 5], while a generalization for higher dimensional cocycles was announced by Avila-Eskin-Viana.

1.7. Some further questions. In this subsection we will mention some further questions which we consider interesting.

(i) How optimal are our results? The results on the regularity of invariant bundles in terms of the contraction and expansion rates are in general optimal, so the $\alpha$ and $\beta$ from our hypothesis are finite, and in general small. Our method seems to be limited in the sense that the maximum regularity of the Lyapunov exponent which we can obtain is $\alpha + \beta$. But is this indeed optimal? We don’t know in fact any example of a $C^\infty$ family of $C^\infty$ diffeomorphisms with a dominated splitting such that the integrated Lyapunov exponent corresponding to a sub-bundle is not $C^\infty$. Does such an example exist?

(ii) Our formulas for the first derivative of the Lyapunov involve only the two bundles of the dominated splitting (and the measure). Thus the problem of finding and understanding the critical points translates into a purely geometric/analytic question. If the two bundles are $C^1$ and integrable, and the measure is the volume, criticality means that the volume decomposes as a “true product” along the 2 foliations. Does a similar statement hold if the two bundles are only Hölder, with the sum of the exponents bigger than 1? What about if one is Hölder and one is smooth? What happens if the bundles are not integrable?
(iii) If $f$ is partially hyperbolic, critical for the unstable bundle and the volume, and the splitting is $C^1$, is it true that $f$ is a (local) maximum for the unstable Lyapunov exponent? The proof of Theorem H suggests that this is the case. For many perturbations supported on small enough neighborhoods of non-periodic points the second derivative of the unstable Lyapunov exponent is negative (recall that the second derivative is bilinear in $X$).

(iv) One could definitely obtain better regularity results for the Lyapunov exponents if one considers special families of perturbations. Is it possible to apply this remark in order to remove zero exponents in new and interesting situations?

(v) It seems possible to obtain further results in the case of variable measures, assuming only Hölder regularity of the bundles, but assuming in exchange better regularity of the invariant measures with respect to the parameters. Can one obtain results in this direction for relevant dynamical measures, like the SRB measures, Gibbs $u$-states, or measures of maximal entropy?

1.8. Manuscript organization. In the next section we provide some definitions and some preparative results. We begin giving more details about the definition of the $k$-form $\omega_F$ and the $k$-multivector field $V_E$ and the dependence of the regularity from the smoothness of $F$ and $E$ respectively (see subsection 2.1). Then (subsection 2.2) we describe the family of “tangent” vectors fields $X$ and $Y$ for the family $h_t = f_t \circ f^{-1}$ and their the role in the first (and second) order “Taylor expansion” (on local charts) for the family $f_t$. We prove also that the flow $\phi_t^X$ (resp. $\phi_t^Y$) generated by $X_t$ (resp. by $Y_t$) preserves $\mu$. In this proof appears for the first time, the connection with the Lie derivative. Using this information we obtain a “Taylor expansion” for the maps $t \rightarrow (f_t)^*\omega_F$ and $t \rightarrow (f_t)^*V_E$ (see subsection 2.3) and finally we investigate the regularity of of the map $t \mapsto V_t$ which is equivalent to finding the regularity of the map $t \mapsto E_t$ and we obtain some formulas for their derivatives (subsection 2.4).

Theorem C is proved in Section 3. In Section 4 we prove Theorem A and Corollary B. The obtention of the formulas for the derivatives and the proof of Theorem D are presented in Section 5. In Section 6 we deal with the case when the invariant measure $\mu_t$ depends on the map $f_t$ and we prove Theorem E and Theorem F. Section 7 is devoted to flows and we present the proof for Theorem G. In section 8 we study the case of non-flat critical points, that means when non-vanishing of the second derivative, and we prove Theorem H. Section 9 is dedicated to the study of critical points and rigidity. There we prove Theorem I and Corollary J.
2. Definitions and preliminary results

2.1. The form $\omega_F$ and the multi–vector fields $V_t$. Let $TM = E_t \oplus F_t$ be continuous splittings (on $M$ and the parameter $t$) such that $E = E_0$ is $C^\beta$ and $F = F_0$ is $C^\alpha$ with $\alpha, \beta \geq 0$. Let $k := \dim E$. We assume that $M$, $E_t$ and $F_t$ are orientable. We claim that there exist a continuous $k$–form $\omega_F$ on $M$, and continuous $k$–multivector fields $V_t$ such that:

1. $\omega_F$ is $C^\alpha$ and $\ker \omega_F(p) = F(p) \wedge T_p M^{\wedge (k-1)}$,
2. $V_t(p) \in E(p)^{\wedge k}$, and $V_E = V_0$ is $C^\beta$,
3. $\omega_F(V_t) = 1$ for every $t \in I$ (eventually for a smaller interval $I$).

A sketch of the proof is the following.

Given any smooth chart $U \subset M$ where $E$ and $F$ are parallelizable, one can choose a $C^\beta$ positively oriented base of $E$ to be $\{V_1, V_2, \ldots, V_k\}$ and a $C^\alpha$ positively oriented base of $F$ to be $\{V_{k+1}, \ldots, V_d\}$. Let $V^U := V_1 \wedge V_2 \wedge \cdots \wedge V_k$ be a nonzero $C^\beta$ $k$–multivector field in $E^{\wedge k}$ inside the chart $U$. Using a finite covering of $M$ with such charts, and a smooth partition of unity, one can construct a nonzero $C^\beta$ $k$–multivector field $\tilde{V}_E$ in $E^{\wedge k}$ on the entire $M$.

In a similar way we can construct continuous nonzero $k$–multivector field $\tilde{V}_t$ in $E^{\wedge k}_t$ on the entire $M$. Since $t \mapsto E_t$ is continuous, we can choose $\tilde{V}_t$ such that $t \mapsto \tilde{V}_t$ is also continuous.

Let $\mu$ be a nonzero smooth $d$–form on $M$. Let $\tilde{\omega}_F := i_{V_{k+1}} \cdots i_{V_d} \mu$ be a nonzero $C^\alpha$ $k$–form inside the chart $U$ with the kernel $F \wedge T M^{\wedge (k-1)}$ ($i_V \mu$ is the interior product of $\mu$ with $V$). Using again a finite covering of $M$ with such charts, and a smooth partition of unity, one can construct a nonzero $C^\alpha$ $k$–form $\tilde{\omega}_F$ with the kernel $F \wedge T M^{\wedge (k-1)}$ on the entire $M$.

The transversality of $E$ and $F$ guarantees that $\tilde{\omega}_F(\tilde{V}_E)$ is nonzero. If $\alpha \geq \beta$, we just let $\omega_F = \tilde{\omega}_F$ and $V_E = \frac{1}{\tilde{\omega}_F(V_E)} \tilde{V}_E$. Otherwise we let $V_E = \tilde{V}_E$ and $\omega_F = \frac{1}{\tilde{\omega}_F(V_E)} \tilde{\omega}_F$. Thus we get $\omega_F$ to be $C^\alpha$, $V_E$ to be $C^\beta$, and $\omega_F(V_E) = 1$.

Since $t \mapsto \tilde{V}_t$ is continuous, eventually after restricting $I$ we can assume that $\omega_F(\tilde{V}_t)$ is nonzero. Let $V_t = \frac{1}{\omega_F(V_t)} \tilde{V}_t$, so $\omega_F(\tilde{V}_t) = 1$. Also from construction we have $t \mapsto V_t$ is in fact continuous.

Remark 4. Let us remark that the choice of $\omega_F$ and $V_E$ is not unique. Given any function $h : M \to (0, \infty)$ of class $C^{\max\{\alpha, \beta\}}$, we can replace $\omega_F$ and $V_t$ by $h \omega_F$ and $\frac{1}{h} V_t$.

Notations: For simplicity in the rest of the paper we will use the notations $\omega := \omega_F$ and $V := V_E$, if no confusion can be made.
2.2. The “tangent” vector fields $X$ and $Y$ for the family $h_t := f_t \circ f_0^{-1}$. Suppose that we have a $C^r$–family of diffeomorphism $(h_t)_{t \in I}$ on $M$ such that $h = h_0 = Id$ is the identity on $M$. We are interested in approximating $h_t$ by flows.

Define the $C^{r-1}$ vector field $X$ on $M$ tangent to the family $h_t$ in $t = 0$ by

$$X(p) = \left. \frac{\partial}{\partial t} h_t(p) \right|_{t=0}.$$  

If $r \geq 2$ then $X$ is $C^1$ and will generate a flow which we denote $\phi_t^X$. The flow $\phi_t^X$ is a good approximation of first order for the family $h_t$. The following lemma is straightforward.

**Lemma 5.** Under the above conditions, the following relations hold uniformly in any charts:

(i) If $r \geq 2$ then

$$h_t(p) = \phi_t^X(p) + O(t^2) \quad (= p + tX(p) + O(t^2));$$

and

$$Dh_t(p) = D\phi_t^X(p) + o(t) \quad (= Id + tDX(p) + o(t));$$

(ii) If $2 < r \leq 3$ then

$$Dh_t(p) = D\phi_t^X(p) + O(t^{r-1}) \quad (= Id + tDX(p) + O(t^{r-1})).$$

In order to obtain a better approximation of $h_t$ (up to order two), we need to introduce the vector field $Y$, which can be seen as a “second order correction of the flow”. An intrinsic way of defining $Y$ is the following.

For $r \geq 1$, define the $C^{r-1}$ vector fields $X_t$ “tangent” to each $h_t$:

$$X_t(p) = \left. \frac{\partial}{\partial s} h_t^{-1}(h_{t+s}(p)) \right|_{s=0} = Dh_t^{-1}(h_t(p)) \cdot \frac{\partial}{\partial t} h_t(p) = [Dh_t(p)]^{-1} \cdot \frac{\partial}{\partial t} h_t(p).$$

Clearly we have that $X = X_0$. If $r \geq 2$, then we can differentiate $X_t$ with respect to $t$ and we obtain the vector fields $Y_t$:

$$Y_t(p) = \lim_{s \to 0} \frac{X_{s+t}(p) - X_t(p)}{s} = \frac{\partial}{\partial t} X_t(p).$$

Let $Y := Y_0$. We can give a formula for $Y$ in local charts. Suppose that in some chart we have

$$h(p) = p, \quad \left. \frac{\partial}{\partial t} h_t(p) \right|_{t=0} = X(p), \quad \text{and} \quad \left. \frac{\partial^2}{\partial t^2} h_t(p) \right|_{t=0} = Z(p).$$
where $X, Z : \mathbb{R}^n \to \mathbb{R}^n$ are $C^{r-1}$ respectively $C^{r-2}$. This means that we can write

\begin{equation}
  h_t(p) = p + tX(p) + \frac{t^2}{2}Z(p) + R(t,p)
\end{equation}

where $R(t,p) = o(t^2)$ uniformly on $p$.

The vector field $X$ is independent of the choice of the chart, however $Z$ is not (this is why we use $Y$ and not $Z$). We claim that

\begin{equation}
  Y = Z - DX \cdot X.
\end{equation}

In order to see this, we compute $Y$:

\[
Y(p) = \frac{\partial}{\partial t}X_t(p)\big|_{t=0} = \frac{\partial}{\partial t} \left[ [Dh_t(p)]^{-1} \cdot \frac{\partial}{\partial t} h_t(p) \right] \big|_{t=0} \\
= [Dh_0(p)]^{-1} Z(p) + \frac{\partial}{\partial t} [Dh_t(p)^{-1}] \big|_{t=0} \cdot X(p) \\
= Z(p) - \left[ Dh_t(p)^{-1} \cdot \frac{\partial}{\partial t} Dh_t(p) \cdot Dh_t(p)^{-1} \right] \big|_{t=0} \cdot X(p) \\
= Z(p) - D \left( \frac{\partial}{\partial t} h_t(p) \big|_{t=0} \right) \cdot X(p) \\
= Z(p) - DX(p) \cdot X(p),
\]

since $Dh(p) = Id$, the derivative of the inverse of a matrix function satisfies $(A(t)^{-1})' = A(t)^{-1} \cdot A'(t) \cdot A(t)^{-1}$, and the partial derivatives commute, $\frac{\partial}{\partial t} Dh_t(p)\big|_{t=0} = D \left( \frac{\partial}{\partial t} h_t \big|_{t=0} \right)$ (remember that $r \geq 2$).

**Remark 6.** We remark that the vector fields $X$ and $Y$ allow to approximate the parametric family $h_t$ with a composition of flows. In fact, if $r \geq 3$, the flows $\phi^X$ and $\phi^Y$ generated by $X$ and $Y$ are well defined. Then we have

\[
h_t(p) = \phi_t^X(\phi_{1/2}^Y(p)) + o(t^2) \quad \text{if } r \geq 3,
\]

\[
Dh_t(p) = D \left[ \phi_t^X(\phi_{1/2}^Y(p)) \right] + o(t^2) \quad \text{if } r \geq 4,
\]

in any chart and uniformly in $p$.

The proof is straightforward, one just has to check that the first two derivatives (with respect to $t$) of both sides of the equations coincide in $t = 0$. One can also approximate the family $h_t$ with $\phi_{1/2}^Y \circ \phi_t^X$, for $r$ sufficiently large.

An important observation is the following.
Lemma 7. Suppose that $h_t$ preserves the Borel probability $\mu$ for all $t \in I$. If $r \geq 2$ then $\phi^{X_t}$ preserves $\mu$, for any $t \in I$. If $r \geq 3$, then $\phi^{Y_t}$ also preserves $\mu$, for any $t \in I$. In particular, if $\mu$ is the volume on $M$ and $r \geq 3$, then the vector fields $X$ and $Y$ are divergence-free.

Proof. Recall that $f$ preserves a measure $\mu$ if and only if $\int_M g d\mu = \int_M g \circ f d\mu$ for any $C^0$ function $g : M \to \mathbb{R}$. Since the $C^1$ functions are dense in the space of $C^0$ functions, this is equivalent to $\int_M g d\mu = \int_M g \circ f d\mu$ for any $C^1$ function $g : M \to \mathbb{R}$.

If a vector field $\chi$ is differentiable and generates the flow $\phi^\chi$, then $\phi^\chi$ preserves $\mu$ if and only if $\int_M g d\mu = \int_M g \circ \phi^\chi_s d\mu$ for any $C^1$ function $g : M \to \mathbb{R}$ and any $s \in \mathbb{R}$. This in turn is equivalent to

\begin{equation}
\int_M g(\phi^\chi_s(p)) d\mu \bigg|_{s=0} = \int_M \left( \frac{\partial}{\partial s} g(\phi^\chi_s(p)) \right) \bigg|_{s=0} d\mu = \int_M L\chi g d\mu = 0,
\end{equation}

for any $C^1$ function $g : M \to \mathbb{R}$ ($L\chi$ is the Lie derivative).

Take some $g : M \to \mathbb{R}$ of class $C^1$. If every $h_t$ preserves $\mu$, then $\int_M g \circ h_t d\mu = \int_M g d\mu$ is constant. Recall that by (2.5) we have that $\frac{\partial}{\partial t} h_t(p) = Dh_t(p)X_t(p)$. We have

\begin{align*}
0 &= \frac{\partial}{\partial t} \int_M g(h_t(p)) d\mu = \int_M Dg(h_t(p)) \frac{\partial}{\partial t} h_t(p) d\mu \\
&= \int_M Dg(h_t(p)) Dh_t(p)X_t(p) d\mu = \int_M D(g \circ h_t)(p) \cdot X_t(p) \\
&= \int_M L_{X_t}(g \circ h_t) d\mu.
\end{align*}

Since $h_t$ is a diffeomorphism, this means that for each $t$ and for any $C^1$ function $\tilde{g} = g \circ h_t$ we have

$\int_M L_{X_t} \tilde{g} d\mu = 0,$

so by (2.9) we have that $\phi^{X_t}$ preserves $\mu$ for every $t$.

The Lie derivative is linear with respect to the vector fields, so by (2.9), the flows generated by $(X_t - X_s)/(t - s)$ preserve $\mu$ for all $t, s \in I$. The Lie derivative is also continuous with respect to the vector field, so from the definition of $Y_t$ (recall (2.6)) we get that the flows generated by $Y_t$ also preserve $\mu$. □

2.3. Expansions for $t \mapsto h_t^* \omega$ and $t \mapsto h_t^* V$ at $t = 0$. We refer again to figure Figure 1 for an intuitive representation of $\omega_F$ and $V_t$, and the action induced on them by the derivative of $f$ and $h_t$. 

Let $\Omega^k(M)$ be the Banach space of continuous $k$-forms on $M$. If $\omega \in \Omega^k(M)$, its norm is defined by

$$\|\omega\| = \sup\{\omega_p(v_1, \ldots, v_k) : p \in M, v_i \in T_p M, \|v_i\| = 1, i \in \{1, \ldots, k\}\}.$$  

Also let $\mathcal{X}^k(M)$ be the Banach space of continuous $k$-multivector fields on $M$. If $V \in \mathcal{X}^k(M)$ then its norm is

$$\|V\| = \sup_{p \in M} \|V(p)\|,$$

where $\|V(p)\|$ is the usual norm on the exterior product of $T_p M$. Let us remark that the pairing $(\omega, V) \mapsto \omega(V)$ is bilinear and continuous with values in $C^0(M)$.

We are interested in the Frechet differentiability of the maps $h_t^* \omega : I \subset \mathbb{R} \to \Omega^k(M)$ and $h_t^* V : I \to \mathcal{X}^k(M)$. Using smooth partitions of unity, one can see that it is sufficient to check the regularity of the maps in local charts.

We have the following lemma which is fundamental to our future considerations. Recall that $r$ is the regularity of the family $h_t$, and let $X$ and $Y$ the vector fields tangent to the family $h_t$ defined in the previous subsection.

**Lemma 8.** Let $\omega \in \Omega^k(M)$ be a continuous $k$-form.

(i) If $\omega$ is $C^\alpha$ and $r \geq \alpha + 1$ then $t \mapsto h_t^* \omega$ is $C^\alpha$.

(ii) If $\omega$ is $C^1$ and $r \geq 2$, then $t \mapsto h_t^* \omega$ is Frechet differentiable and the derivative in zero is $L_X \omega$:

$$h_t^* \omega = \omega + tL_X \omega + o(t).$$

(iii) If $\omega$ is $C^2$ and $r \geq 3$, then $t \mapsto h_t^* \omega$ is twice Frechet differentiable and the second derivative in zero is $L_X L_X \omega + L_Y \omega$:

$$h_t^* \omega = \omega + tL_X \omega + \frac{t^2}{2}(L_X L_X \omega + L_Y \omega) + o(t^2).$$

**Proof.** The part (i) and the differentiability claims follow directly from the formulas of the pullback of a form in local coordinates.

For the parts (ii) and (iii) we just have to check that if $\omega$ is $C^1$ then $\frac{\partial}{\partial t} h_t^* \omega \big|_{t=0} = L_X \omega$, and if $\omega$ is $C^2$ then $\frac{\partial^2}{\partial t^2} h_t^* \omega \big|_{t=0} = L_X L_X \omega + L_Y \omega$. Let us make first the following remarks.
Observe first that if (ii), (iii) are true for the forms \( \omega_1 \) and \( \omega_2 \) then they are also true for the form \( \omega_1 + \omega_2 \), because the Lie derivative is linear:

\[
\frac{\partial}{\partial t} h_t^*(\omega_1 + \omega_2) \bigg|_{t=0} = \frac{\partial}{\partial t} (h_t^* \omega_1) \bigg|_{t=0} + \frac{\partial}{\partial t} (h_t^* \omega_2) \bigg|_{t=0} = \mathcal{L}_X \omega_1 + \mathcal{L}_X \omega_2
\]

\[
= \mathcal{L}_X (\omega_1 + \omega_2),
\]

\[
\frac{\partial^2}{\partial t^2} h_t^*(\omega_1 + \omega_2) \bigg|_{t=0} = \frac{\partial^2}{\partial t^2} (h_t^* \omega_1) \bigg|_{t=0} + \frac{\partial^2}{\partial t^2} (h_t^* \omega_2) \bigg|_{t=0} = \mathcal{L}_X \mathcal{L}_X \omega_1 + \mathcal{L}_Y \omega_1 + \mathcal{L}_X \mathcal{L}_X \omega_2 + \mathcal{L}_Y \omega_2
\]

\[
= \mathcal{L}_X \mathcal{L}_X (\omega_1 + \omega_2) + \mathcal{L}_Y (\omega_1 + \omega_2).
\]

Observe also that if (ii), (iii) are true for the forms \( \omega_1 \) and \( \omega_2 \) then they are also true for the form \( \omega_1 \wedge \omega_2 \), because the derivatives obey the Leibniz rule.

\[
\frac{\partial}{\partial t} h_t^*(\omega_1 \wedge \omega_2) \bigg|_{t=0} = \frac{\partial}{\partial t} h_t^* \omega_1 \wedge h_t^* \omega_2 \bigg|_{t=0} + h_t^* \omega_1 \wedge \frac{\partial}{\partial t} h_t^* \omega_2 \bigg|_{t=0} = \mathcal{L}_X \omega_1 \wedge \omega_2 + \omega_1 \wedge \mathcal{L}_X \omega_2 = \mathcal{L}_X (\omega_1 \wedge \omega_2),
\]

\[
\frac{\partial^2}{\partial t^2} h_t^*(\omega_1 \wedge \omega_2) \bigg|_{t=0} = \left[ \frac{\partial^2}{\partial t^2} h_t^* \omega_1 \wedge h_t^* \omega_2 + 2 \frac{\partial}{\partial t} h_t^* \omega_1 \wedge \frac{\partial}{\partial t} h_t^* \omega_2 + h_t^* \omega_1 \wedge \frac{\partial^2}{\partial t^2} h_t^* \omega_2 \right] \bigg|_{t=0} = (\mathcal{L}_X \mathcal{L}_X \omega_1 + \mathcal{L}_Y \omega_1) \wedge \omega_2 + 2 \mathcal{L}_X \omega_1 \wedge \mathcal{L}_X \omega_2 + \mathcal{L}_X (\mathcal{L}_X \mathcal{L}_X \omega_2 + \mathcal{L}_Y \omega_2) = \mathcal{L}_X \mathcal{L}_X (\omega_1 \wedge \omega_2) + \mathcal{L}_Y (\omega_1 \wedge \omega_2).
\]

The formulas in (ii), (iii) are local, and it is sufficient to verify them in a chart \( U \subset \mathbb{R}^n \), where any form can be decomposed into a sum of forms \( g dx_1 \wedge \cdots \wedge dx_k \). The two remarks above show that we only need to verify (ii) and respectively (iii) for a zero form \( g \) of class \( C^1 \) respectively \( C^2 \), and for the one-forms \( dx_i \), or more generally for a one-form \( dg \) with \( g \) of class \( C^\infty \).

Let us prove first (ii) for a map \( g : U \to \mathbb{R} \) of class \( C^1 \), and \( h_t \) of class \( C^2 \), meaning that \( X \) is \( C^1 \). Then \( h_t^* g(p) = g(h_t(p)) \) and

\[
\frac{\partial}{\partial t} h_t^* g(p) \bigg|_{t=0} = \frac{\partial}{\partial t} g(h_t(p)) \bigg|_{t=0} = Dg(p) \cdot X(p) = \mathcal{L}_X g(p),
\]

so indeed \( \frac{\partial}{\partial t} h_t^* g \bigg|_{t=0} = \mathcal{L}_X g \).
Now let us prove (iii) for a map \( g : U \to \mathbb{R} \) is of class \( C^2 \), and \( h_t \) of class \( C^3 \), meaning that \( X \) is \( C^2 \) and \( Y \) is \( C^1 \). We have that
\[
\frac{\partial^2}{\partial t^2} h_t^*(g(p)) = \frac{\partial^2}{\partial t^2} g(h_t(p)) = \frac{\partial}{\partial t} \left[ Dg(h_t(p)) \cdot \frac{\partial}{\partial t} h_t(p) \right]
\]
\[
= D^2 g(h_t(p)) \left( \frac{\partial}{\partial t} h_t(p), \frac{\partial}{\partial t} h_t(p) \right) + Dg(h_t(p)) \cdot \frac{\partial^2}{\partial t^2} h_t(p),
\]
and in \( t = 0 \), we get
\[
\frac{\partial^2}{\partial t^2} h_t^*(g(p)) \bigg|_{t=0} = D^2 g(p) (X(p), X(p)) + Dg(p) \cdot Z(p).
\]
On the other hand
\[
\mathcal{L}_X \mathcal{L}_X g(p) = \mathcal{L}_X (Dg(p) \cdot X(p)) = D (Dg(p) \cdot X(p)) \cdot X(p)
\]
\[
= D^2 g(p) (X(p), X(p)) + Dg(p) \cdot DX(p) \cdot X(p)
\]
and, using (2.8),
\[
\mathcal{L}_Y g(p) = Dg(p) \cdot Y = Dg(p) \cdot Z(p) - Dg(p) \cdot DX(p) \cdot X(p).
\]
Combining the last 3 equalities we get that
\[
\frac{\partial^2}{\partial t^2} h_t^*(g(p)) \bigg|_{t=0} = \mathcal{L}_X \mathcal{L}_X g(p) + \mathcal{L}_Y g(p).
\]
Now consider a zero form \( g \) of class \( C^\infty \). Recall that the exterior derivative commutes with the pullback \( (h_t^*(dg)) = d(h_t^*g) \), and with the Lie derivative \( (\mathcal{L}_x (dg)) = d(\mathcal{L}_x g) \).

Let us prove (ii) for \( dg \) given \( h_t \) of class \( C^2 \). The map \((t, p) \mapsto g(h_t(p))\) is \( C^1 \) in both \( t \) and \( p \), so the partial derivatives commute: \( \frac{\partial^2}{\partial t^2} g(h_t(p)) = \frac{\partial}{\partial t} d(g \circ h_t) \). Then
\[
\frac{\partial}{\partial t} h_t^*(dg) \bigg|_{t=0} = \frac{\partial}{\partial t} d(g \circ h_t) \bigg|_{t=0} = d \frac{\partial}{\partial t} (g \circ h_t) \bigg|_{t=0} = d \mathcal{L}_X g = \mathcal{L}_X dg.
\]
Now let us prove (iii) for \( dg \), given \( h_t \) of class \( C^3 \). The map \((t, p) \mapsto g(h_t((p)))\) is \( C^3 \) in \( t \) and \( p \), so the following partial derivatives commute: \( \frac{\partial^2}{\partial t^2} g(h_t(p)) = \frac{\partial}{\partial t} d(g \circ h_t) \). Then
\[
\frac{\partial^2}{\partial t^2} h_t^*(dg) = \frac{\partial^2}{\partial t^2} d(g \circ h_t) = d \frac{\partial^2}{\partial t^2} (g \circ h_t),
\]
and
\[
\frac{\partial^2}{\partial t^2} h_t^*(dg) \bigg|_{t=0} = d (\mathcal{L}_X \mathcal{L}_X g + \mathcal{L}_Y g) = \mathcal{L}_X \mathcal{L}_X dg + \mathcal{L}_Y dg.
\]
This finishes the proof of the lemma. \( \square \)
Lemma 10. Let $\omega$ be a continuous $k$-multivector field on $M$.

(i) If $V$ is $C^\beta$ and $r \geq \beta + 1$ then $t \mapsto h_t^* V$ is $C^\beta$.

(ii) If $V$ is $C^1$ and $r \geq 2$, then $t \mapsto h_t^* V$ is Frechet differentiable and the derivative in zero is $-\mathcal{L}_X V$:

$$h_t^* V = V - t \mathcal{L}_X V + o(t).$$

(iii) If $V$ is $C^2$ and $r \geq 3$, then $t \mapsto h_t^* V$ is twice Frechet differentiable and the second derivative in zero is $\mathcal{L}_X \mathcal{L}_X V - \mathcal{L}_Y V$:

$$h_t^* V = V - t \mathcal{L}_X V + \frac{t^2}{2} (\mathcal{L}_X \mathcal{L}_X V - \mathcal{L}_Y V) + o(t^2).$$

Proof. Again, like in the case of forms, the part (i) and the differentiability claims are immediate. For the parts (ii) and (iii) we have to check again that $\frac{\partial}{\partial t} h_t^* V \big|_{t=0} = -\mathcal{L}_X V$ if $V$ is $C^1$, and $\frac{\partial^2}{\partial t^2} h_t^* V \big|_{t=0} = \mathcal{L}_X \mathcal{L}_X V - \mathcal{L}_Y V$ if $V$ is $C^2$.

One can prove the claims directly for vector fields, and using the Leibniz rule extend the result for multivector fields, similar to the proof of Lemma 8. We will give a proof using Lemma 8 and the duality between forms and multivector fields. Let us remark that $\omega(h_t^* V) = h_t^* \omega(V) \circ h_t^{-1}$ (where $\omega(h_t^* V)$ and $h_t^* \omega(V)$ are seen as maps from $M$ to $\mathbb{R}$).

Assume first that $V$ is $C^1$. It is easy to see that $\frac{\partial}{\partial t} h_t^{-1} \big|_{t=0} = -X$. For any $C^1$ form $\omega$ we have:

$$\omega \left( \frac{\partial}{\partial t} h_t^* V \right) \big|_{t=0} = \frac{\partial}{\partial t} \omega(h_t^* V) \big|_{t=0} = \frac{\partial}{\partial t} \left[ h_t^* \omega(V) \circ h_t^{-1} \right] \big|_{t=0} = \frac{\partial}{\partial t} \left[ h_t^* \omega(V) \right] \circ h_t^{-1} \big|_{t=0} + d \left[ h_t^* \omega(V) \right] \left( \frac{\partial}{\partial t} h_t^{-1} \right) \big|_{t=0} = \mathcal{L}_X \omega(V) + d(\omega(V))(-X) = \mathcal{L}_X [\omega(V)] - \omega(\mathcal{L}_X V) - \mathcal{L}_X [\omega(V)] = -\omega(\mathcal{L}_X V).$$
This clearly implies that \( \left( \frac{\partial}{\partial t} h_{t_0} V \right)_{t=0} = -\mathcal{L}_X V \).

The proof of the formula (2.13) is similar, and since we do not need it in our future considerations, we omit the proof.

We also have a result estimating the approximation of \( h_{t_0} V \) by \( \phi_{t_0}^X V \), and of \( h_t^* \omega \) by \( \phi_t^X \omega \).

**Lemma 11.** Let \( \omega \) be a \( k \)-form on \( M \), and \( V \) a \( k \)-multivector field on \( M \), and \( r \geq 2 \).

(i) If \( \omega \) is \( C^\alpha \), \( \alpha \in [0, 1] \) then

\[
(2.14) \quad h_t^* \omega = \phi_t^X \omega + O(t^{\min\{2\alpha, r-1\}}).
\]

(ii) If \( V \) is \( C^\beta \), \( \beta \in [0, 1] \), then

\[
(2.15) \quad h_t^* V = \phi_t^X V + O(t^{\min\{2\beta, r-1\}}).
\]

**Proof.** Part (i). The formula can be verified locally in charts, and applying an argument similar to the one from Lemma 8, it is sufficient to verify the formula for a \( C^\alpha \) 0-form \( g \), and a \( C^\infty \) 1-form \( dg \).

So let \( g : M \to \mathbb{R} \) be \( C^\alpha \). Then applying (2.2) we get

\[
h_t^* g(p) - \phi_t^X g(p) = g(h_t(p)) - g(\phi_t^X(p)) \leq C d(h_t(p), \phi_t^X(p))^\alpha = O(t^{2\alpha}).
\]

Now let \( g : M \to \mathbb{R} \) be \( C^\infty \). Applying (2.4) and (2.2) we get

\[
(h_t^* - \phi_t^X) dg(p) = d(g \circ h_t)(p) - d(g \circ \phi_t^X)(p)
= dg(h_t(p)) Dh_t(p) - dg(\phi_t^X(p)) D\phi_t^X(p)
= dg(h_t(p))[Dh_t(p) - D\phi_t^X(p)] + [dg(h_t(p)) - dg(\phi_t^X(p))] D\phi_t^X(p)
= O(t^{r-1}) + O(t^2) = O(t^{r-1}).
\]

This finishes the proof of the first part.

Part (ii). Since locally every multivector field is a combination of exterior products of vector fields, it is sufficient to verify the formula just for vector fields. So let \( V \) be a \( C^\beta \)
vector field on $M$. Then
\[
\| (h_t - \phi_t X) V(p) \| \leq \| Dh_t(h_t^{-1}(p))V(h_t^{-1}(p)) - D\phi_t X(\phi_t^{-1}(p))V(\phi_t^{-1}(p)) \|
\]
In a similar way, one obtains that
\[
\| (h_t - \phi_t X) V(p) \| \leq C d(h_t^{-1}(p) - \phi_t^{-1}(p)) + C \| Dh_t - D\phi_t \|
\]
where we used again (2.4) and (2.2) (which implies that also $d(h_t^{-1}(p) - \phi_t^{-1}(p)) = O(t^2)$). This finishes the proof.

2.4. Regularity of $t \mapsto V_t$ and a formula for $V' = \frac{\partial}{\partial t} V_t|_{t=0}$. In this section we will investigate the regularity of the map $t \mapsto V_t$ which is equivalent to finding the regularity of the map $t \mapsto E_t$.

So let us assume that $f_t$ has a dominated splitting $TM = E_t^1 \oplus E_t^2 \oplus E_t^3$, and denote $E_t := E_t^2$ and $F_t := E_t^1 \oplus E_t^3$. Let $\lambda_{E_t}^1 < \lambda_{E_t}^2 < \lambda_{E_t}^3 < \lambda_{E_t}^2$ be expansion bounds along the three sub-bundles for $f = f_0$:
\[
\lambda_{E_t}^1 < m(Df|_{E_t^1}) \leq \| Df|_{E_t^1} \| < \lambda_{E_t}^2,
\]
\[
\lambda_{E_t}^2 < m(Df|_{E_t^2}) \leq \| Df|_{E_t^2} \| < \lambda_{E_t}^2,
\]
\[
\lambda_{E_t}^3 < m(Df|_{E_t^3}) \leq \| Df|_{E_t^3} \| < \lambda_{E_t}^3.
\]
Then the same relations will hold for $f_t$ and the corresponding decomposition $TM = E_t^1 \oplus E_t^2 \oplus E_t^3$ for $t \in I$, where $I$ is a small interval around zero. Let $g : M \times I \to M \times I$ be the $C^r$ diffeomorphism defined by
\[
g(x, t) = (f_t(x), t).
\]
The standard Invariant Section Theorem ([31], see also [53]) tells us that $(p, t) \mapsto F_t(p)$ is of class $C^{\alpha}$ in both $t \in I$ and $p \in M$, for
\[
\alpha = \min \left\{ \log \lambda_{E_t}^1 - \log \lambda_{E_t}^2, \log \lambda_{E_t}^3 - \log \lambda_{E_t}^2 \right\},
\]
In a similar way, one obtains that $(p, t) \mapsto E_t(p)$ is of class $C^{\beta}$ in both $t \in I$ and $p \in M$, for
\[
\beta = \min \left\{ \log \lambda_{E_t}^1 - \log \lambda_{E_t}^2, \log \lambda_{E_t}^3 - \log \lambda_{E_t}^2 \right\}.
\]
If we assume some further regularity of the bundles for \( t = 0 \) (with respect to the point on the manifold \( M \)), then we can also obtain better regularity with respect to the parameter \( t \) at \( t = 0 \). More specifically we have the following result.

**Proposition 12.** Assume that \( f_t \) has a dominated splitting \( TM = E_1^t \oplus E_2^t \oplus E_3^t \) (here \( E_1 \) or \( E_3 \) can be trivial). If \( E = E_0^f \) is of class \( C^\beta \) for some \( \beta \in [0,1] \), and \( r \geq 2 \), then the map \( t \mapsto V_t \) has expansion of order \( \beta \) at \( t = 0 \).

For \( \beta = 1 \) the result was obtain by Dolgopyat in [24] (see also [53] for an alternative proof), and we use their method for \( \beta \in (0, \text{Lip}) \). The difference is that we will use the action induced on multivector fields instead of the action induced on the Grassmannian.

Let us comment that it appears that the result could be improved up to \( \beta = 1 + \beta_0 \) where \( \beta_0 \) is given by the formula (2.17), and it seems improbable to obtain a similar result for larger values of \( \beta \) without further restrictions on the family \( f_t \).

**Proof.** Suppose that \( \beta \in (0, \text{Lip}) \) (for \( \beta = 0 \) there is nothing to prove). Recall that since the bundle \( E_t \) is invariant under \( f_{ts} \), there exists \( \eta_t : M \to (0, \infty) \) such that \((f_{ts})_*V_t(p) = \eta_t(p) \cdot V_t(f_t(p))\). This means that if we denote \( \tilde{\eta}_t := \eta \circ f_t^{-1} \), then we have

\[ f_{ts}V_t = h_{ts}f_*V_t = \tilde{\eta}_t V_t. \]

In fact \( \tilde{\eta}_t = \omega(h_{ts}f_*V_t) \). Furthermore

\[
0 = h_{ts}f_*V_t - \tilde{\eta}_t V_t \\
= (h_{ts}f_* - \tilde{\eta}_t \text{Id})(V_t - V) + h_{ts}f_*V - \tilde{\eta}_t V \\
= (f_* - \tilde{\eta}_t \text{Id})(V_t - V) + [h_{ts}f_* - f_* - (\tilde{\eta}_t - \tilde{\eta}) \text{Id}](V_t - V) + [h_{ts}f_*V - f_*V] \\
- (\tilde{\eta}_t - \tilde{\eta})V.
\]

Observe that \( \lim_{t \to 0} h_{ts}f_* - f_* + (\tilde{\eta}_t - \tilde{\eta}) \text{Id} = 0 \) so

\[
[h_{ts}f_* - f_* + (\tilde{\eta}_t - \tilde{\eta}) \text{Id}](V_t - V) = o(\|V_t - V\|).
\]

Also from Lemma 10 we know that \( t \mapsto h_{ts}f_*V \) is \( C^\beta \) in \( t = 0 \) so \( h_{ts}f_*V - f_*V = O(t^\beta) \). Then

\[ (f_* - \tilde{\eta} \text{Id})(V_t - V) - (\tilde{\eta}_t - \tilde{\eta})V = o(\|V_t - V\|) + O(t^\beta). \]

Let us remark that \( V_t - V \) is in the kernel of \( \omega \), while \( V \) is in the complimentary space \( E^{\wedge k} \). Let \( \mathcal{P} : TM^{\wedge k} \to \ker(\omega) = F \wedge TM^{\wedge(k-1)} \) be the canonical projection parallel to \( E^{\wedge k} \), which is given by the formula

\[ \mathcal{P}(W) = W - \omega(W)V, \quad (\forall W \in TM^{\wedge k}). \]
Applying the projection $\mathcal{P}$ to the formula (2.19) we get

$$(2.21) \quad \mathcal{P}[(f_* - \tilde{\eta}Id)(V_t - V) - (\tilde{\eta}_t - \tilde{\eta})V] = (f_* - \tilde{\eta}Id)(V_t - V) = o(\|V_t - V\|) + O(t^\beta)$$

**Claim:** $\|(f_* - \tilde{\eta}Id)(V_t - V)\| \geq C\|V_t - V\|$ for some $C > 0$ and small $t$.

If the claim is true, then combined with (2.21) it gives immediately that $V_t - V = O(t^\beta)$ as needed.

**Proof of the claim.** In general there is no need that the operator $f_* - \tilde{\eta}Id$ is invertible, not even if we restrict it to the kernel of $\omega$. However we will see that if we restrict it to $F \wedge E^{\wedge(k-1)}$ then it is indeed invertible, and this is good enough in order to obtain the claim.

Let $\mathcal{T} = \frac{\mathcal{I}}{\tilde{\eta}}|_{F \wedge E^{\wedge(k-1)}}$. Then $\mathcal{T}$ can be decomposed into the direct sum $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_3$, where $\mathcal{T}_1 = \mathcal{T}|_{E \wedge E^{\wedge(k-1)}}$, this is because the dominated splitting is invariant under $f_*$. Because of the domination property, one can see that $\mathcal{T}_1$ is a contraction, while $\mathcal{T}_3$ is an expansion, in other words the operator $\mathcal{T}$ is hyperbolic, so $\mathcal{T} - \text{Id}$ is invertible. This in turn implies that $(f_* - \tilde{\eta}Id)|_{F \wedge E^{\wedge(k-1)}} = \tilde{\eta}(\mathcal{T} - \text{Id})$ is also invertible, so there exists $C > 0$ such that

$$(2.22) \quad \|(f_* - \tilde{\eta}Id)(W)\| > C, \quad \forall W \in F \wedge E^{\wedge(k-1)}, \quad \|W\| = 1.$$  

From the continuity of the operator $f_* - \tilde{\eta}Id$, there exists a neighborhood $\mathcal{U}$ of the set \{ $W \in F \wedge E^{\wedge(k-1)} : \|W\| = 1$ \} inside $TM^k$ such that the relation (2.22) holds for every $W \in \mathcal{U}$. Then what is left to prove is that $\frac{V_i - V}{\|V_i - V\|}$ is inside $\mathcal{U}$ for small values of $t$ and $V_t - V \neq 0$ (if $V_t - V = 0$ then there is nothing to prove).

The fact that $\frac{V_i - V}{\|V_i - V\|}$ is close to $F \wedge E^{\wedge(k-1)}$ follows from the fact that $V \in E^k$, and $V$ and $V_t$ are in fact simple multivectors. Using a partition of unity one can see that it is enough to show this fact locally, and in this case we have

$$V = V_1 \wedge V_2 \cdots \wedge V_k, \quad V_i \in E, \quad i \in \{1, 2, \ldots, k\},$$

$$V_t = (V_1 + W_{t1}) \wedge (V_2 + W_{t2}) \cdots \wedge (V_k + W_{tk}), \quad W_{ti} \in F, \quad i \in \{1, 2, \ldots, k\}.$$  

We can suppose that $\|V_i \wedge \cdots \wedge V_i\|$ and $\|V_i \| \cdots \|V_i\|$ are comparable, for any $\{i_1, \ldots, i_l\}$ subset of $\{1, 2, \ldots, k\}$ (the vector fields $V_i$ can be chosen to form locally an orthogonal base of $E$ and have all constant size for example). We can decompose $V_t - V = W_t + \tilde{W}_t$, where

$$W_t = W_{t1} \wedge V_2 \cdots \wedge V_k + \cdots + V_1 \wedge V_2 \cdots \wedge V_{k-1} \wedge W_{tk} \in F \wedge E^{\wedge(k-1)}$$

and

$$\tilde{W}_t = W_{t1} \wedge W_{t2} \wedge V_3 \cdots \wedge V_k + \cdots + W_{t1} \wedge W_{t2} \wedge \cdots \wedge W_{tk} \in F^{\wedge2} \wedge TM^{\wedge(k-2)}.$$
Since $F$ and $E$ have the angle uniformly bounded away from zero, we have that $\|W_t\|$ is comparable with $\max\{\|W_{t_1}\|, \ldots, \|W_{t_k}\|\}$ uniformly in $t$, and then each term from the formula of $\tilde{W}_t$ is bounded from above by $D\|W_t\|^2$ for some $D > 0$. Then for some $\tilde{D} > 0$ we have that
\[
\|\tilde{W}_t\| \leq \tilde{D}\|W_t\|^2.
\]
Estimating the distance between $\frac{W_t}{\|W_t\|}$ and $\frac{V_t - V}{\|V_t - V\|} = \frac{W_t + \tilde{W}_t}{\|W_t + \tilde{W}_t\|}$ we get
\[
\left\| \frac{W_t}{\|W_t\|} - \frac{W_t + \tilde{W}_t}{\|W_t + \tilde{W}_t\|} \right\| \leq \|W_t\| \cdot \left( \left| \frac{1}{\|W_t\|} - \frac{1}{\|W_t + \tilde{W}_t\|} \right| + \frac{\|\tilde{W}_t\|}{\|W_t + \tilde{W}_t\|} \right)
\leq 2\|\tilde{W}_t\| \leq \frac{2\|\tilde{W}_t\|}{\|W_t + \tilde{W}_t\|}
\leq \frac{2\tilde{D}\|W_t\|}{1 - \tilde{D}\|W_t\|} \leq 4\tilde{D}\|W_t\|,
\]
which converges uniformly to zero as $t$ goes to zero, so indeed $\frac{V_t - V}{\|V_t - V\|}$ is inside $U$ for small enough $t$ and this finishes the proof of the claim. \qed

We obtained that $V_t - V = O(t^\beta)$ for $\beta \in (0, \text{Lip}]$. If $\beta = 1$ then from Lemma 10 we know that $t \rightarrow h_{t*} f_\ast V$ is $C^1$ in $t = 0$ and
\[
h_{t*} f_\ast V - f_\ast V = -t\mathcal{L}_X (f_\ast V) + o(t) = -t\mathcal{L}_X (\tilde{\eta}V) + o(t)
= -t\tilde{\eta}\mathcal{L}_X V + -t\mathcal{L}_X \tilde{\eta}V + o(t).
\]
We also have that $V_t - V = O(t)$, so the relation (2.19) becomes
\[
(f_\ast - \tilde{\eta} Id)(V_t - V) - (\tilde{\eta}_t - \tilde{\eta})V = -t\tilde{\eta}\mathcal{L}_X V + -t\mathcal{L}_X \tilde{\eta}V + o(t).
\]
projecting by $\mathcal{P}$ on the kernel of $\omega$ we get
\[
(f_\ast - \tilde{\eta} Id)(V_t - V) = -t\tilde{\eta}\mathcal{P}\mathcal{L}_X V + o(t).
\]
Dividing by $t$ and taking the limit when $t$ goes to zero we get
\[
(f_\ast - \tilde{\eta} Id) \left( \lim_{t \to 0} \frac{V_t - V}{t} \right) = -\tilde{\eta}\mathcal{P}\mathcal{L}_X (V),
\]
so $V' := \lim_{t \to 0} \frac{V_t - V}{t}$ exists and
\[
(2.23) \quad V' = \left( Id - \frac{f_\ast}{\tilde{\eta}} \right)^{-1} \mathcal{P}\mathcal{L}_X (V).
\]
Let us remark that it is easy to see in charts that in fact
\[ L_X(V) = L_X(V_1 \wedge V_2 \wedge \cdots \wedge V_k) \]
\[ = (L_X V_1) \wedge V_2 \wedge \cdots \wedge V_k + \cdots + V_1 \wedge \cdots \wedge (L_X V_k) \in TM \wedge E^{\wedge(k-1)}, \]
so \( PL_X(V) \in F \wedge E^{\wedge(k-1)} \) and the inverse of \( \text{Id} - \mathcal{T} \) is well defined since \( \mathcal{T} = \frac{f}{\eta} \) is hyperbolic.

We also obtain a formula for \( V' \) if \( \beta \geq 1 \). Since \( F = E^1 \oplus E^2 \oplus E^3 \) and \( PL_X V \in F \wedge E^{\wedge(k-1)} \), we can decompose it as
\[ PL_X V = P_1(L_X V) + P_3(L_X V), \]
where \( P_i(L_X V) \in E^i \wedge E^{\wedge(k-1)}, i = 1, 3. \)

**Proposition 13.** Assume that \( f_t \) has a dominated splitting \( TM = E^1_t \oplus E^2_t \oplus E^3_t \). If \( E = E^2_0 \) is of class \( C^1 \) and \( r \geq 2 \), then the derivative of the map \( t \mapsto V_t \) in \( t = 0 \) is
\[ V' = \left[ \left[ \text{Id} - \left( \frac{f}{\eta} \right) \right] \right]^{-1}_{F \wedge E^{\wedge(k-1)}} \mathcal{P}(L_X V) \]
\[ = \sum_{n \geq 0} \left( \frac{f}{\eta} \right)^n P_1(L_X V) - \sum_{n \geq 1} \left( \frac{f}{\eta} \right)^{-n} P_3(L_X V). \]

**Proof.** Recall that the formula (2.23) gives us that \( V' = (\text{Id} - \mathcal{T})^{-1} \mathcal{P}L_X V. \) Also the operator \( \mathcal{T} \) is hyperbolic, and it can be decomposed into the direct sum \( \mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_3, \) where \( \mathcal{T}_i = \mathcal{T}|_{E^i \wedge E^{\wedge(k-1)}}, i = 1, 3. \) Then we also have \( (\text{Id} - \mathcal{T})^{-1} = (\text{Id} - \mathcal{T}_1)^{-1} \oplus (\text{Id} - \mathcal{T}_3)^{-1}. \)
Since \( \mathcal{T}_1 \) is a contraction, we have that
\[ (\text{Id} - \mathcal{T}_1)^{-1} = \sum_{n \geq 0} \mathcal{T}_1^n, \]
and since \( \mathcal{T}_3 \) is an expansion, we have
\[ (\text{Id} - \mathcal{T}_3)^{-1} = -\sum_{n \geq 1} \mathcal{T}_3^{-n}. \]
Putting the formulas (2.25) and (2.26) together we obtain that indeed
\[ V' = (\text{Id} - \mathcal{T})^{-1} \mathcal{P}L_X V = (\text{Id} - \mathcal{T}_1)^{-1} \mathcal{P}_1(L_X V) + (\text{Id} - \mathcal{T}_3)^{-1} \mathcal{P}_3(L_X V) \]
satisfies the desired formula. \( \square \)
3. A result on regularity of averaged observables for flows.

In this section we will prove Theorem C. The proof is based on the following lemma:

Lemma 14. Let $\mathbb{T}$ be the circle $[0,2\pi]|_{0=2\pi}$, and let $f, g : \mathbb{T} \to \mathbb{R}$ be continuous functions. Suppose that $f$ is $C^\alpha$ and $g$ is $C^\beta$, $\alpha, \beta \geq 0$, and let $h$ be the convolution $f * g$, i.e. $h(t) = \int_\pi f(x)g(t-x)dx$. If either $\alpha + \beta$ is not an integer, or at least one of $\alpha$ or $\beta$ is an integer, then $h$ is $C^{\alpha+\beta}$ and $\|h\|_{C^{\alpha+\beta}} \leq C_{\alpha,\beta}\|f\|_{C^\alpha}\|g\|_{C^\beta}$. If $\alpha, \beta \notin \mathbb{N}$ and $\alpha + \beta \in \mathbb{N}$ then $h$ is $C^{\alpha+\beta-1+Zygmund}$ with the modulus of continuity $C\|f\|_{C^\alpha}\|g\|_{C^\beta}|t\log t|$.

This result seems to be known in the more general context of Besov spaces, but since we didn’t find a reference, we need the exact bounds, and the proof is fairly simple, we will include it here.

Proof. Step 1: Reduction to the case $\alpha, \beta \in (0,1)$. Let us remark first that the problem can be easily reduced to the case when $\alpha, \beta \in [0,1)$. Indeed, let $\alpha = a + \alpha'$, $\beta = b + \beta'$, $a, b \in \mathbb{N}$, $\alpha', \beta' \in (0,1)$. Then differentiating inside the integral and eventually changing the variable, we get $h^{(a+b)}(t) = \int_\pi f^{(a)}(x)g^{(b)}(t-x)dx$, with $f^{(a)}$ of class $C^{\alpha'}$ and $g^{(b)}$ of class $C^{\beta'}$. Furthermore $\|h\|_{C^{\alpha+\beta}} \leq 2\pi\|f\|_{C^\alpha}\|g\|_{C^\beta}$.

So we will assume that $\alpha, \beta \in [0,1)$. If either $\alpha$ or $\beta$ are 0 then the result is trivial, so we consider only the case when $\alpha, \beta \in (0,1)$.

Step 2: Estimate on Fourier coefficients of $h$. Let $\hat{f}(n)$, $\hat{g}(n)$ be the Fourier coefficients of $f$ and $g$. Then we know that $\hat{h}(n) = (\hat{f} \ast \hat{g})(n) = 2\pi \hat{f}(n)\hat{g}(n)$, and the Fourier series of $f, g, h$ are uniformly convergent since the functions are Hölder.

The Fourier coefficients of $f_t(x) = f(x-t)$ for some fixed $t$ are $e^{-int}\hat{f}(n)$, so the Fourier coefficients of $f_t - f$ are $(e^{-int} - 1)\hat{f}(n)$. If $C_f > 0$ is minimal such that $|f(x) - f(y)| \leq C_f|x-y|^\alpha$, then a bound for the $L^2$ norm of $f_t - f$ is $\sqrt{2\pi}C_f t^\alpha$. Choosing $t_k = 2^{-k} \cdot 2\pi/3$, we observe that for $2^k \leq |n| \leq 2^{k+1}$ then $|nt_k| \in [2\pi/3, 4\pi/3]$, so $|e^{-int_k} - 1| > \sqrt{3}$. Applying Parseval identity for $f_{tk} - f$, we get

$$\sum_{|n|=2^k}^{2^{k+1}} |\hat{f}(n)|^2 \leq \frac{1}{3} \sum_{|n|=2^k}^{2^{k+1}} |\hat{f}(n)|^2 |e^{-int_k} - 1|^2 \leq \frac{1}{3} \sum_{n \geq 2}^\infty |(f_{tk} - f)|^2 \leq \frac{2\pi}{3} C_f^2 t_k^{2\alpha} \leq CC_f^2 2^{-2k\alpha},$$

where $C$ denotes some universal constant. A similar computation will give that

$$\sum_{|n|=2^k}^{2^{k+1}} |\hat{g}(n)|^2 \leq CC_g^2 2^{-2k\beta},$$
and then by Cauchy-Schwartz
\[
\sum_{|n|=2^k} |\hat{h}(n)| = 2\pi \sum_{|n|=2^k} |\hat{f}(n)\hat{g}(n)| \leq CC_fC_g 2^{-k(\alpha+\beta)}.
\]

**Step 3:** The case \(\alpha + \beta < 1\). Consider first the case \(\alpha + \beta < 1\). Let \(t, s \in \mathbb{T}^1\) and \(k_0 \in \mathbb{N}\) such that \(2^{-(k_0+1)} \leq |t - s| < 2^{-k_0}\). Let \(c > 0\) be such that if \(|x| < 1\) then \(|1 - e^x| \leq c|x|\). We obtain
\[
|h(t) - h(s)| = 2\pi \sum_{n \in \mathbb{Z}} \hat{f}(n)\hat{g}(n)e^{i\pi/n}(1 - e^{i\pi(t-s)})
\leq \sum_{k < k_0} \sum_{|n|=2^k} |\hat{h}(n)| \cdot cn|t - s| + \sum_{k \geq k_0} \sum_{|n|=2^k} 2|\hat{h}(n)|
\leq CC_fC_g 2^{-k(\alpha+\beta)} 2^{k+1} 2^{-k_0} + \sum_{k \geq k_0} CC_fC_g 2^{-k(\alpha+\beta)}
= CC_fC_g 2^{-k_0 + 1} \sum_{k < k_0} 2^{k(1-\alpha-\beta)} + CC_fC_g \sum_{k \geq k_0} 2^{-k(\alpha+\beta)}
= CC_fC_g 2^{-k_0 + 1} \frac{2^{k_0(1-\alpha-\beta)} - 1}{1 - 2^{1-\alpha-\beta}} + CC_fC_g 2^{-k_0(\alpha+\beta)} \frac{1}{1 - 2^{-(\alpha+\beta)}}
\leq C_{\alpha,\beta} CC_fC_g 2^{-(k_0+1)(\alpha+\beta)} \leq C_{\alpha,\beta} CC_fC_g |t - s|^{\alpha+\beta}.
\]
This shows that \(h\) is \(C^{\alpha+\beta}\) and \(\|h\|_{C^{\alpha+\beta}} \leq C_{\alpha,\beta} \|f\|_{C^\alpha} \|g\|_{C^\beta}\) and completes the case \(\alpha + \beta < 1\).

**Remark 15.** One can see from the computation above that for \(\alpha + \beta \in (1/2, 1)\), one can take \(C_{\alpha,\beta} = \frac{C}{1-\alpha-\beta}\) for some universal constant \(C\).

**Step 4:** The case \(\alpha + \beta = 1\). If \(\alpha + \beta = 1\) then we get
\[
|h(t) - h(s)| = CC_fC_g 2^{-k_0 + 1} \sum_{k < k_0} 2^{0} + CC_fC_g \sum_{k \geq k_0} 2^{-k}
= CC_fC_g k_0 2^{-k_0} + CC_fC_g 2^{-k_0}
\leq CC_fC_g k_0 2^{-(k_0+1)} \leq CC_fC_g \|t - s\| \log(t - s)|.
\]
Then the modulus of continuity of \(h\) is \(C\|f\|_{C^\alpha} \|g\|_{C^\beta} |t \log t|\) so \(h\) is indeed Zygmund.

**Step 5:** The case \(\alpha + \beta > 1\). Now consider \(\alpha + \beta > 1\). We remark first that
\[
\sum_{|n|=2^k} |n\hat{h}(n)| \leq 2^{k+1} \sum_{|n|=2^k} |\hat{h}(n)| = CC_fC_g 2^{-(\alpha+\beta - 1)}.
\]
This implies that
\[ \sum_{n \in \mathbb{Z}} n |\hat{h}(n)| = \sum_{k \in \mathbb{N}} \sum_{|n| = 2^k} |n \hat{h}(n)| \leq \sum_{k \in \mathbb{N}} CC_f C_g 2^{-k(\alpha + \beta - 1)} = \frac{C}{\alpha + \beta - 1} C_f C_g \]
is absolutely convergent. This implies that \( h \) is \( C^1 \) and gives the bound on the derivative of \( h \), while the Fourier coefficients of \( h' \) will be \( \hat{h}'(n) = i n \hat{h}(n) \).

The proof that \( h' \) is \( C^{\alpha + \beta - 1} \) is similar to the proof of the Hölder continuity of \( h \) in the case \( \alpha + \beta < 1 \), one just uses the relation
\[ 2^{k+1} \sum_{|n| = 2^k} |\hat{h}'(n)| = \sum_{|n| = 2^k} |n \hat{h}(n)| \leq CC_f C_g 2^{-k(\alpha + \beta - 1)}. \]

\[ \square \]

Remark 16. We remark that the case \( \alpha + \beta \in \mathbb{N} \) is indeed special. One can see this by taking \( f(x) = g(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \sin 2^n x \). Then \( f \) and \( g \) are \( C^2 \), while \( h(t) = \pi \sum_{n=0}^{\infty} \frac{1}{2^n} \cos 2^n t \) is Zygmund but it is not Lipschitz because it has the derivative infinite in 0.

Now we will prove Theorem C.

Proof. We start with the remark that again we can reduce to the case when \( \alpha, \beta \in [0, 1) \).
If \( \alpha = a + \alpha', \beta = b + \beta', a, b \in \mathbb{N}, \alpha', \beta' \in [0, 1) \), denote \( f_a(x) = \frac{\partial x_a}{\partial s} f(\phi_s(x))|_{s=0} \) and \( g_b(x) = \frac{\partial x_b}{\partial s} g(\phi_s(x))|_{s=0} \). Differentiating inside the integral and eventually changing the variable we get again that
\[ h^{(a+b)}(t) = \int_M f_a(x) g_b(x) d\mu, \]
so it is enough to show the result for \( f_a \) being \( C^{\alpha'} \) and \( g_b \) being \( C^{\beta'} \).

If \( \alpha = 0 \) then clearly
\[ |h(t) - h(t')| \leq \int_M |f(x)| \cdot |g(\phi_t(x)) - g(\phi_{t'}(x))| d\mu \leq \|f\|_{C^\alpha} C_g \|X\|_{C^0} |t - t'|^\beta, \]
so \( h \) is clearly \( C^\beta \) with the required bound. The case \( \beta = 0 \) is similar.

Consequently we can assume from now on that \( \alpha, \beta \in (0, 1) \).

The case of ergodic \( \mu \). We assume first that \( \mu \) is ergodic. If \( x \in M \) is a generic point then from Birkhoff Ergodic Theorem we have
\[ h(t) = \int_M f(x) g(\phi_t(x)) d\mu = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(\phi_s(x)) g(\phi_{s+t}(x)) ds. \]
By Poincaré recurrence, there exist a sequence \( t_n \to \infty, n \in \mathbb{N} \), such that \( \lim_{n \to \infty} \phi_{t_n}(x) = x \). Let \( T_N = \frac{t_n + 1}{2\pi} \). We can construct smooth closed curves \( s \in [0, 2\pi T_n] \mapsto \phi_s^n(x) \in M \) obtained by keeping \( \phi_s^n(x) = \phi_s(x) \) for \( s \in [0, t_n] \), and completing with the curve \( \phi_s^n(x) \), \( s \in [t_n, t_n + 1] \). For \( n \) sufficiently large we can assume that the curve \( \phi_s^n(x) \) is in the neighborhood of the support of \( \mu \) where the regularity of \( f \) and \( g \) is satisfied, and \( \left\| \frac{\partial}{\partial s} \phi_s^n(x) \right\| \leq \|X\|_{C^0} \). Then let

\[
h_n(t) = \frac{1}{2\pi T_n} \int_0^{2\pi T_n} f(\phi_s^n(x))g(\phi_{s+t}^n(x))ds = \frac{1}{2\pi} \int_0^{2\pi} f(\phi_{T_nr}^n(x))g(\phi_{T_nr+t}^n(x))dr.
\]

Clearly \( \lim_{n \to \infty} h_n(t) = h(t) \) for any \( t \), so \( h_n \) converges pointwise to \( h \). Let \( f_n(r) = f(\phi_{T_nr}^n(x)) \) and \( g_n(r) = g(\phi_{T_nr}^n(x)) \). Then

\[
|f_n(r) - f_n(r')| = |f(\phi_{T_nr}^n(x)) - f(\phi_{T_nr'}^n(x))| \leq \|f\|_{C^0}d(\phi_{T_nr}^n(x), \phi_{T_nr'}^n(x))^{\alpha} \\
\leq \|X\|_{C^0}^{\alpha}\|f\|_{C^0}\|T_n^\alpha|r - r'|^{\alpha},
\]

so \( f_n \) is \( C^\alpha \) and

\[
\|f_n\|_{C^\alpha} \leq \|X\|_{C^0}^{\alpha}\|f\|_{C^0}T_n^\alpha.
\]

Similarly \( g_n \) is \( C^\beta \) with and

\[
\|g_n\|_{C^\beta} \leq \|X\|_{C^0}^{\beta}\|g\|_{C^\beta}T_n^\beta.
\]

We also have that

\[
h_n(t) = \frac{1}{2\pi} \int_0^{2\pi} f_n(r)g_n(r + t/T_n)dr.
\]

Observe that if \( \sigma \) is the involution \( \sigma(r) = -r \), and \( H_n := \sigma \circ (f_n \ast (g_n \circ \sigma)) \), then

\[
h_n(t) = \sigma \circ (f_n \ast (g_n \circ \sigma)) \left( \frac{t}{T_n} \right) = H_n \left( \frac{t}{T_n} \right).
\]

Since \( \sigma \) is an isometry, Lemma 14 says that \( H_n \) must be \( C^{\alpha + \beta} \) (if \( \alpha + \beta \neq 1 \)) and

\[
\|H_n\|_{C^{\alpha + \beta}} \leq C_{\alpha, \beta}\|X\|_{C^0}^{\alpha + \beta}\|T_n^{\alpha + \beta}\|f\|_{C^0}\|g\|_{C^\beta}.
\]

**Sub-case** \( \alpha + \beta < 1 \). If \( \alpha + \beta < 1 \) then

\[
|h_n(t) - h_n(t')| = \left| H_n \left( \frac{t}{T_n} \right) - H_n \left( \frac{t'}{T_n} \right) \right| \leq C_{\alpha, \beta}\|X\|_{C^0}^{\alpha + \beta}\|f\|_{C^0}\|g\|_{C^\beta}|t - t'|^{\alpha + \beta},
\]

so \( h_n \) are uniformly \( C^{\alpha + \beta} \), which implies that \( h = \lim_{n \to \infty} h_n \) must be also \( C^{\alpha + \beta} \) with the same upper bound on the \( C^{\alpha + \beta} \) norm. It also implies the the limit \( h_n \to h \) is uniform on compact sets.

**Sub-case** \( \alpha + \beta = 1 \). In this case Lemma 14 says that \( H_n \) are Zygmund, with the modulus of continuity \( C\|X\|_{C^0}\|f\|_{C^0}\|g\|_{C^\beta}T_n|t \log t| \). Then we get that \( h_n \) is also Zygmund,
however the modulus of continuity may not be uniform with respect to \( n \) so we cannot pass to the limit.

We will use instead the previous step and the Remark 15. Since for \( \alpha' < \alpha \), the \( C^{\alpha'} \)

norm is bounded from above by the \( C^\alpha \) norm (eventually multiplied by a fixed constant),

we get that for every \( s \in (1/2, 1) \) and any \( n \), \( h_n \) is \( C^s \) and

\[
|h_n(t) - h_n(t')| \leq \frac{C}{1-s} \|X\|_{C^\alpha} \|f\|_{C^\alpha} \|g\|_{C^\beta} |t - t'|^s.
\]

If \( |t - t'| \) is sufficiently small, we can take \( s = 1 + \frac{1}{\log(\|X\|_{C^\alpha} |t - t'|)} \) and we get

\[
|h_n(t) - h_n(t')| \leq -C\|f\|_{C^\alpha} \|g\|_{C^\beta} \log (\|X\|_{C^\alpha} |t - t'|) (\|X\|_{C^\alpha} |t - t'|)^{1+ \frac{1}{\log(\|X\|_{C^\alpha} |t - t'|)}} = -eC\|f\|_{C^\alpha} \|g\|_{C^\beta} \|X\|_{C^\alpha} |t - t'| (\log |t - t'| + \log \|X\|_{C^\alpha}).
\]

In conclusion, \( h_n \) is indeed uniformly Zygmund, so we can pass to the limit and conclude

that \( h \) is also Zygmund.

**Sub-case \( \alpha + \beta > 1 \).** Again we will have that \( H_n \) must be \( C^{\alpha+\beta} \) with the norm

\( C_{\alpha,\beta}\|X\|^\alpha_{C^\alpha} T_{n}^{\alpha+\beta} \|f\|_{C^\alpha} \|g\|_{C^\beta} \). In particular \( H_n \) is \( C^1 \) and the derivative \( H'_n \) is \( C^{\alpha+\beta-1} \)

with the constant \( C_{\alpha,\beta}\|X\|^\alpha_{C^\alpha} T_{n}^{\alpha+\beta} \|f\|_{C^\alpha} \|g\|_{C^\beta} \). Then \( h_n \) must be also \( C^1 \) and

\[
|h'_n(t) - h'_n(t')| = \frac{1}{T_n} \left| H'_n \left( \frac{t}{T_n} \right) - H'_n \left( \frac{t'}{T_n} \right) \right| \leq C_{\alpha,\beta}\|X\|^\alpha_{C^\alpha} \|f\|_{C^\alpha} \|g\|_{C^\beta} |t - t'|^{\alpha+\beta-1},
\]

so \( h'_n \) verifies the \((\alpha + \beta - 1)\)-Hölder condition uniformly with respect to \( n \).

We claim that \( h'_n \) are also uniformly bounded (this does not follow directly from the bounds above).

Again we know that for any \( s \in (0, 1) \), the maps \( h_n \) are uniformly \( C^s \), the \( C^s \) norms

of \( h_n \) are uniformly bounded by \( C_s\|X\|^r_{C^\alpha} \|f\|_{C^\alpha} \|g\|_{C^\beta} \), and \( h_n \) converges uniformly on compact sets to \( h \).

In particular for \( s = \frac{1}{2} \) the \( C^\frac{1}{2} \) property of \( h_n \) gives

\[
|h_n(t + a) - h_n(t)| \leq \|h_n\|_{C^\frac{1}{2}} a^{1/2},
\]

while the \( C^{\alpha+\beta-1} \) condition on \( h'_n \) gives

\[
|h'_n(t) - h'_n(s)| \leq \|h'_n\|_{C^{\alpha+\beta-1}} |t - s|^\alpha|t - s|^{\alpha+\beta-1} \leq \|h'_n\|_{C^{\alpha+\beta-1}} a^{\alpha+\beta-1}, \quad \forall s \in [t, t + a],
\]

or

\[
h'_n(t) \leq \|h'_n\|_{C^{\alpha+\beta-1}} a^{\alpha+\beta-1} + h'_n(s).
\]
Then we get
\[
\begin{align*}
  h'_n(t) & \leq \| h'_n \|_{C^{\alpha + \beta - 1}} a^{\alpha + \beta - 1} + \frac{1}{a} \int_t^{t+a} h'_n(s) ds \\
  & = \| h'_n \|_{C^{\alpha + \beta - 1}} a^{\alpha + \beta - 1} + \frac{1}{a} (h_n(t + a) - h_n(t)) \\
  & \leq \| h'_n \|_{C^{\alpha + \beta - 1}} a^{\alpha + \beta - 1} + \| h_n \|_{C^0} a^{-1/2} \\
  & \leq C_{\alpha, \beta} \| f \|_{C^\alpha} \| g \|_{C^\beta} \left[ \| X \|_{C^0} a^{\alpha + \beta - 1} + \| X \|_{C^0} a^{-1/2} \right].
\end{align*}
\]

Choosing \( a = \| X \|_{C^0}^{-1} \) we get
\[
 h'_n(t) \leq C_{\alpha, \beta} \| f \|_{C^\alpha} \| g \|_{C^\beta} \| X \|_{C^0}.
\]

A similar argument works for \(-h'_n(t)\), so we have the uniform bounds for \(|h'_n|\). Using also the uniform Hölder conditions on \( h'_n \), we can apply Arzela-Ascoli in order to obtain a subsequence \( h'_{n_k} \) convergent (uniformly on compact sets) to some \( h' \), and this will imply that \( h' \) must be equal to the derivative of \( h \). The uniform bounds on \( h'_{n_k} \) transfer to \( h' \), so
\[
\| h \|_{C^1} \leq C_{\alpha, \beta} \| f \|_{C^\alpha} \| g \|_{C^\beta} \| X \|_{C^0}.
\]

This concludes the proof for the case when \( \mu \) is ergodic.

**The case of general \( \mu \).** Now suppose that \( \mu \) is not ergodic, then it must have an ergodic decomposition
\[
\mu = \int_{\mathcal{M}_e} \nu \, dm_\mu(\nu),
\]
where \( \mathcal{M}_e \) are the ergodic invariant probabilities of \( \phi \) and \( m_\mu \) is a Borel probability measure on \( \mathcal{M}_e \). Then
\[
h(t) = \int_M f(x) g(\phi_t(x)) d\mu = \int_{\mathcal{M}_e} \int_M f(x) g(\phi_t(x)) d\nu dm_\mu(\nu) := \int_{\mathcal{M}_e} h_\nu(t) dm_\mu(\nu).
\]

Now since \( h_\nu(t) \) are \( C^{\alpha + \beta} \) in \( t \) with uniform bounds independent of \( \nu \), then the same must be true for \( h(t) \), and the bounds are preserved. This concludes the proof of the theorem.

\[\square\]

4. **Regularity of the averaged Lyapunov exponents**

Now we will prove Theorem A. We will invoke frequently the following lemma of calculus, we will omit its proof.

**Lemma 17.** Let \( \alpha : I \times M \to \mathbb{R} \) continuous in \( t \) and \( r \geq 0 \). If \( 0 < c < \alpha(t) \), then
\[
\log (\alpha(t) + o, O(t'^r, t'^r \log t)) = \log \alpha(t) + o, O(t'^r, t'^r \log t).
\]

(4.1)
In particular, if \( \alpha : I \times M \to \mathbb{R} \) continuous with respect to \( t \in I \) uniformly with respect to \( p \in M \) and \( \nu \) is a probability Borel measure on \( M \), then

\[
\int \log (\alpha(t,p) + o, O(t^r, t^r \log t)) \, d\nu(p) = \int \log \alpha(t,p) \, d\nu(p) + o, O(t^r, t^r \log t).
\]

We will use the notations introduced previously in the paper. Recall that from (1.6) we have

\[
\lambda(t) = \int_M \log \eta_\ast d\mu = \int_M \log \tilde{\eta}_t d\mu,
\]

where \( f_tV_t = \tilde{\eta}_t V_t, \tilde{\eta}_t = \eta_t \circ f_t^{-1} \). Since \( \omega(V_t) = 1 \), applying \( \omega \) we get

\[
\tilde{\eta}_t = \omega(f_tV_t) = \omega(h_t f_* V_t) = h_t^\ast \omega(f_* V_t) \circ h_t^{-1}.
\]

**Proof of Theorem A.** The strategy of the proof is to approximate \( \lambda(t) \) up to order \( t^{\alpha + \beta} \) by a simple formula involving the action of the flow \( \phi_t^X \) on \( \omega \) (or \( V \)), and then to use Theorem C in order to obtain the regularity of this new expression.

**Step 1:** The following approximations hold:

\[
\tilde{\eta}_t = \omega(h_t f_* V_t) = \tilde{\eta}_t \circ h_t^{-1} \cdot \omega(\phi_t^X V) + O(t^{\min\{\alpha, \beta, 2\beta\}}), \tag{4.3}
\]

\[
\eta_t \circ f^{-1} = h_t^\ast \omega(f_* V_t) = \tilde{\eta}_t \cdot \phi_t^X \omega(V) + O(t^{\min\{\alpha + 2\alpha\}}), \tag{4.4}
\]

\[
\lambda(t) = \lambda(0) + \int_M \log \omega(\phi_t^X V) d\mu + O(t^{\alpha + \beta}) = \lambda(0) + \int_M \log \phi_t^X \omega(V) d\mu + O(t^{\alpha + \beta}). \tag{4.5}
\]

In fact, we have:

\[
\tilde{\eta}_t = \omega(h_t f_* V_t) = \omega(h_t f_* V_t) + \omega(h_t f_* (V_t - V))
\]

\[
= \tilde{\eta}_t \circ h_t^{-1} \cdot \omega(h_t V_t) + \omega(h_t f_* (V_t - V))
\]

\[
= \tilde{\eta}_t \circ h_t^{-1} \cdot \omega(\phi_t^X V) + \tilde{\eta}_t \circ h_t^{-1} \cdot \omega((h_t - \phi_t^X) V) + \omega(h_t f_* (V_t - V))
\]

\[
= \tilde{\eta}_t \circ h_t^{-1} \cdot \omega(\phi_t^X V) + E_1(t,p) + E_2(t,p).
\]

We will evaluate \( E_1 \) and \( E_2 \) separately. Since \( V \) is \( C^\beta \) we can apply Lemma 11 and since \( r \geq \alpha + \beta + 1 \) we get

\[
E_1 = \tilde{\eta}_t \circ h_t^{-1} \cdot \omega((h_t - \phi_t^X) V) = \tilde{\eta}_t \circ h_t^{-1} \cdot \omega(O(t^{\min\{2\beta, r-1\}})) = O(t^{\min\{2\beta, \alpha + \beta\}}).
\]

Recall that \( V_t - V \) is in the kernel of \( \omega \), which is invariant by \( f_* \), so we have \( \omega(f_* (V_t - V)) = 0 \). We obtain

\[
E_2 = \omega(h_t f_* (V_t - V)) = h_t^\ast \omega(f_* (V_t - V)) \circ h_t^{-1}
\]

\[
= (h_t^\ast \omega - \omega)(f_* (V_t - V)) \circ h_t^{-1} = O(t^{\alpha + \beta})
\]
since from Lemma 8 we know that $h_t^*\omega - \omega = O(t^\alpha)$, and from Proposition 12 we have that $V_t - V = O(t^\beta)$.

Putting the above estimates together we obtain the formula (4.3). The proof of (4.4) is similar:

$$
\eta_t \circ f^{-1} = h_t^*\omega(f_*V_t) = h_t^*\omega(f_*V) + h_t^*\omega(f_*V_t - V))
= \tilde{\eta} \cdot h_t^*\omega(V) + h_t^*\omega(f_*V_t - V))
= \tilde{\eta} \cdot \phi_t^{X_*}\omega(V) + \tilde{\eta} \cdot (h_t^* - \phi_t^{X_*})\omega(V) + h_t^*\omega(f_*V_t - V))
:= \tilde{\eta} \cdot \phi_t^{X_*}\omega(V) + E_3(t, p) + E_4(t, p)
$$

Then applying again Lemma 11 we obtain

$$
E_3(t, p) = (h_t^* - \phi_t^{X_*})\omega(f_*V) = O(t^{\min(2\alpha, r-1)})(V) = O(t^{\min(2\alpha, \alpha + \beta)}).
$$

Furthermore

$$
E_4(t, p) = h_t^*\omega(f_*V_t - V)) = (h_t^*\omega - \omega)(f_*V_t - V)) = O(t^{\alpha + \beta})
$$

is similar to the estimation of $E_2$, and the proof of (4.4) follows.

In order to obtain the approximation (4.5), we use (4.3) or (4.4), depending whether $\beta \geq \alpha$ or not. For example if $\beta \geq \alpha$ we use (4.3) and we get

$$
\lambda(t) = \int_M \log \tilde{\eta}_t = \int_M \log \tilde{\eta} \cdot h_t^{-1} \cdot \omega(\phi_t^{X_*}V) + O(t^{\alpha + \beta})
= \int_M \log \omega(\phi_t^{X_*}V) + \int_M \log(\tilde{\eta} \cdot h_t^{-1})d\mu + O(t^{\alpha + \beta})
= \int_M \log \omega(\phi_t^{X_*}V)d\mu + \lambda(0) + O(t^{\alpha + \beta}).
$$

We used the fact that $\phi_t^{X_*}$ preserves $\mu$, so $\int_M \log(\tilde{\eta} \cdot \phi_t^{X_*})d\mu = \int_M \log \tilde{\eta}d\mu = \lambda(0)$.

If $\alpha > \beta$ we use (4.4) and we get

$$
\lambda(t) = \int_M \log \eta_t = \int_M \log \eta \circ f^{-1} = \int_M \log \tilde{\eta} \cdot \phi_t^{X_*}\omega(V) + O(t^{\alpha + \beta})
= \int_M \log \phi_t^{X_*}\omega(V) + \int_M \log \tilde{\eta}d\mu + O(t^{\alpha + \beta})
= \int_M \log \left[\omega(\phi_t^{X_*}V) \circ \phi_t^{X_*}\right] d\mu + \lambda(0) + O(t^{\alpha + \beta})
= \int_M \log \omega(\phi_t^{X_*}V)d\mu + \lambda(0) + O(t^{\alpha + \beta}).
$$

We used again the fact that $f$ and $\phi_t^{X_*}$ preserve $\mu$.

**Step 2:** The map $t \mapsto \int_M \omega(\phi_t^{X_*}V)d\mu$ is $C^{\alpha + \beta}$ (or is $C^{\alpha + \beta - 1 + \text{Zygund}}$).
This is an application of Theorem C, and it can be done in general for any $\alpha, \beta > 0$, as long as $X$ is $C^r$ with $r \geq \alpha + \beta + 1$.

Choose a finite open cover of $M$ with charts and a smooth partition of unity associated to it, $(U_i, \rho_i)$, $1 \leq i \leq N$. Since

$$\int_M \omega(\phi_t^X V) d\mu = \sum_{i=1}^N \int_M \rho_i \omega(\phi_t^X V) d\mu,$$

it is sufficient to study the regularity of $t \mapsto \int_{U_i} \rho_i \omega(\phi_t^X V) d\mu$. So we can assume that $\omega$ is $C^\alpha$ and supported in a small chart $U$.

We know that $D\phi_t^X(p)$ is $C^{r-1}$, with $r - 1 \geq \alpha + \beta$, so

$$(4.6) \quad D\phi_t^X(p) = Id + tD_1(p) + \frac{t^2}{2} D_2(p) + \cdots + \frac{t^\alpha}{\alpha!} D_\alpha(p) + O(t^{\alpha+\beta}),$$

where $a = [\alpha + \beta]$, $D_i : U_i \to \mathcal{M}_{d \times d}(\mathbb{R})$ is $C^{r-i-1}$, for all $1 \leq i \leq a$. Denote $D_0 = Id$, and observe that $D_1 = DX$.

We can assume that in the chart $U$ we have $V = V_1 \wedge V_2 \wedge \cdots \wedge V_k$, where $V_1, V_2, \ldots, V_k$ are $C^\beta$ vector fields. Then

$$\phi_t^X V \circ \phi_t^X = (\phi_t^X V_1 \wedge \phi_t^X V_2 \wedge \cdots \wedge \phi_t^X V_k) \circ \phi_t^X = D\phi_t^X(V_1(p)) \wedge D\phi_t^X(V_2(p)) \wedge \cdots \wedge D\phi_t^X(V_k(p)).$$

Using the expansion (4.6) of $D\phi_t^X(p)$ we get an expansion

$$\phi_t^X V \circ \phi_t^X = V + tD_1(V) + \frac{t^2}{2} D_2(V) + \cdots + \frac{t^\alpha}{\alpha!} D_\alpha(V) + O(t^{\alpha+\beta}),$$

where each $D_i(V)$ is an expression involving $D_1, D_2, \ldots, D_i$ and $V$, so it is of class $C^{\min\{r-i-1, \beta\}}$. In particular

$$D_1(V) = DX \cdot V_1 \wedge V_2 \wedge \cdots \wedge V_k + V_1 \wedge DX \cdot V_2 \wedge \cdots \wedge V_k + \cdots + V_1 \wedge V_2 \wedge \cdots \wedge DX \cdot V_k$$

and $\|D_1(V)\| \leq k\|X\|_{C^1}\|V\|_{C^\alpha}$.

Then, for $t$ small, we have

$$\int_U \omega(\phi_t^X V) d\mu = \int_U \omega(\phi_t^X V) \circ \phi_t^X d\mu = \sum_{i=0}^\alpha \frac{t^i}{i!} \int_U \omega(\phi_t^X(p)) D_i(V)(p) d\mu(p) + O(t^{\alpha+\beta}).$$

Now remember that $\omega$ is $C^\alpha$, while $D_i(V)$ is $C^{\min\{r-i-1, \beta\}}$. Furthermore, in the chart $U$ we have $\omega = \sum_{I \in \mathcal{I}} a_I dx_I$ and $D_i(V) = \sum_{I \in \mathcal{I}} b_I \frac{\partial}{\partial x_I}$, with $\mathcal{I}$ being the set of multi-indices
of size $k$, and all $a_I$ are of class $C^\alpha$ and $b_I$ of class $C^{\min\{r-i-1,\beta\}}$. Then by Theorem C we have that

$$A_1(t) := \int_U \omega(\phi_t^X(p))D_1(V)(p)d\mu(p) = \sum_{i \in I} \int_U a_I(\phi_t^X(p))b_I(p)d\mu(p)$$

is of class $C^{\alpha+\min\{r-i-1,\beta\}} = C^{\min\{\alpha+\beta,r-i-1+\alpha\}}$ as a function of $t$ (or of class $C^{\min\{\alpha+\beta-1,r-i-2+\alpha\}+\text{Zygmund}}$).

If $\alpha + \beta \leq r - i - 1 + \alpha$ then $A_i(t)$ is $C^{\alpha+\beta}$ (or $C^{\alpha+\beta-1+\text{Zygmund}}$). Otherwise we have $t \mapsto t^iA_i(t)$ has expansion in $t = 0$ of order $i + (r - i - 1 + \alpha) - \epsilon \geq 2\alpha + \beta - \epsilon > \alpha + \beta$ for small enough $\epsilon > 0$.

We conclude that for all $0 \leq i \leq a$, we have that $t^iA_i$ has expansion of order $(\alpha + \beta)$ (or $\alpha + \beta - 1 + \text{Zygmund}$) in $t = 0$, so also $t \mapsto \int_U \omega(\phi_t^XV)d\mu$ has expansion of order $(\alpha + \beta)$ (or $\alpha + \beta - 1 + \text{Zygmund}$) in $t = 0$.

Since in the above argument one can replace $V$ by $\phi_t^XV$ for any $s$, we obtain that $t \mapsto \int_U \omega(\phi_t^XV)d\mu$ is $C^{\alpha+\beta}$ (or is $C^{\alpha+\beta-1+\text{Zygmund}}$) for all $t$.

If in particular $\alpha, \beta \in (0,1)$, $\alpha + \beta > 1$, $r \geq \alpha + \beta + 1$, using the formula of $D_1(V)$ we get

$$\int_U \omega(\phi_t^XV)d\mu = \int_U \omega(\phi_t^XV(p))V(p)d\mu(p) + \int_U \omega(\phi_t^XV(p))D_1(V)(p)d\mu(p) + O(t^{\alpha+\beta})$$
$$= \int_U \omega(\phi_t^XV(p))V(p)d\mu(p) + \int_U \omega(p)D_1(V)(p)d\mu(p) + O(t^{\alpha+\beta}).$$

Using the estimate on the derivative in Theorem C and putting the charts together we get

$$(4.7) \quad \left| \frac{\partial}{\partial t} \int_M \omega(\phi_t^XV)d\mu \right|_{t=0} \leq C_{\alpha,\beta,M} \|X\|_{C^0} \|\omega\|_{C^0} \|V\|_{C^\beta} + C_M \|X\|_{C^1} \|\omega\|_{C^0} \|V\|_{C^0}.$$

**Step 3:** The map $t \mapsto \int_M \log \omega(\phi_t^XV)d\mu$ is $C^{\alpha+\beta}$ (or Zygmund if $\alpha + \beta = 1$) in $t = 0$.

Even if the map $t \mapsto \int_M \omega(\phi_t^XV)d\mu = \int_M \phi_t^XV\omega(V)$ is $C^{\alpha+\beta}$ (or $C^{\alpha+\beta-1+\text{Zygmund}}$) for all $t$, the map $t \mapsto \int_M \log \omega(\phi_t^XV)d\mu$ could be $C^{\alpha+\beta}$ (or Zygmund if $\alpha + \beta = 1$) only in $t = 0$. Let us remind first that $\phi_t^XV$ is uniformly $C^\beta$ in $t$ and $\phi_t^X\omega$ is uniformly $C^\alpha$ in $t$. Then either $\omega(\phi_t^XV)$ or $\phi_t^X\omega(V)$ will be uniformly $C^{\max\{\alpha,\beta\}}$ in $t$. Assume that $g(t,p) := \omega(\phi_t^XV)$ is uniformly $C^{\max\{\alpha,\beta\}}$ in $t$, the other case can be treated similarly.

**Lemma 18.** Given $g(t,p)$ uniformly $C^r$ in $t$, $s \in (0,1)$, with $g(0,p) = 1$ for all $p \in M$, and $\int_M g(t,p)d\mu$ of class $C^u$ in $t$, $u \leq 2s$, then $\int_M \log g(t,p)d\mu$ has expansion of order $u$ in $t$ at $t = 0$. 
Proof. We know that \( \log A = A - 1 + O ((A - 1)^2) \). Then
\[
\int_M \log g(t, p) d\mu = \int_M g(t, p) - 1 + O ((g(t, p) - 1)^2) d\mu
\]
\[
= -1 + \int_M g(t, p) d\mu + O(t^2)
\]
\[
= -1 + \int_M g(t, p) d\mu + O(t^u)
\]
and the result follows. The result also works for \( u = \text{Zygmund} \), and if \( u > 1 \) then
\[
\frac{\partial}{\partial t} \int_M \log g(t, p) d\mu \bigg|_{t=0} = \frac{\partial}{\partial t} \int_M g(t, p) d\mu \bigg|_{t=0}.
\]
□

Applying the above lemma for \( s = \max \{\alpha, \beta\} \) and \( u = \alpha + \beta \) or \( u = \text{Zygmund} \) is \( \alpha + \beta = 1 \), we get that indeed \( t \mapsto \int_M \log \omega(\phi_t^* V) d\mu \) has an expansion of order \( \alpha + \beta \) or \( \text{Zygmund} \) in \( t = 0 \).

Now putting Step 1 and Step 3 together we obtain the desired regularity for \( \lambda(t) \) in \( t = 0 \).

Furthermore, if \( \alpha + \beta > 1 \) then
\[
\lambda'(0) = \frac{\partial}{\partial t} \int_M \omega(\phi_t^* V) d\mu \bigg|_{t=0} = \frac{\partial}{\partial t} \int_M \phi_t^* \omega(V) d\mu \bigg|_{t=0} = \frac{\partial}{\partial t} \int_M \log \omega(\phi_t^* V) d\mu \bigg|_{t=0}
\]
and
\[
|\lambda'(0)| \leq C_{\alpha, \beta, M} \|X\|_{C^0} \|\omega\|_{C^0} \|V\|_{C^\beta} + C_M \|X\|_{C^1} \|\omega\|_{C^0} \|V\|_{C^0}.
\]
This finishes the proof of the theorem. □

Remark 19. In fact the condition \( r \geq \max \{\alpha + \beta + 1, 2\} \) is sufficient for the proof above.

Next we prove Corollary B.

Proof of Corollary B. The time one map of the geodesic flow on a surface of negative curvature is stably ergodic with respect to the volume, so let \( U \) be a \( C^1 \) neighborhood of \( \phi^1 \) in the space of volume preserving diffeomorphisms such that any \( C^2 \) map in \( U \) is ergodic with respect to the volume. Furthermore we can choose the neighborhood \( U \) such that the stable, center and unstable bundles of the \( C^2 \) maps in \( U \) are \( C^\alpha \) for some \( \alpha > 1/2 \), and also depend \( C^\alpha \) on parameters (this can be done with a mild condition on the contraction/expansion rates on the bundles, see subsection 2.4).

Let \( f_t \) be a \( C^3 \) family of \( C^3 \) diffeomorphisms in \( U \), and let \( TM = E^s_t \oplus E^c_t \oplus E^u_t \) be the corresponding splitting for every \( f_t \). Let \( X_t \) be the vector field tangent to the family at
each $f_t$. We have that $\|E_i^e\|_{C^\alpha}$ for $i \in \{s, c, u\}$, and $\|X_t\|_{C^1}$ are uniformly bounded. By Theorem A, for every $i \in \{s, c, u\}$, the map $t \mapsto \lambda(f_t, E_i^e, \mu)$ is differentiable everywhere, and the derivative is uniformly bounded, so the map is Lipschitz.

On the other hand, if we apply the Pesin formula we obtain that the metric entropy of $f_t$ with respect to $\mu$ is the sum of the positive Lyapunov exponents of $f_t$ with respect to $\mu$. If the center exponent is positive then this is

$$h_\mu(f_t) = \lambda(f_t, E^c_t, \mu) + \lambda(f_t, E^u_t, \mu) = -\lambda(f_t, E^s_t, \mu),$$

and if the center exponent is negative then this is

$$h_\mu(f_t) = \lambda(f_t, E^u_t, \mu).$$

Then we have

$$h_\mu(f_t) = \max\{\lambda(f_t, E^u_t, \mu), -\lambda(f_t, E^s_t, \mu)\}$$

is the maximum of two Lipschitz functions, so it is also Lipschitz.

\begin{remark}
The result can be formulated in a more general setting. The time one map of the geodesic flow on a surface with negative curvature can be replaced by any volume preserving partially hyperbolic diffeomorphism which has center dimension one, is stably ergodic (which is an open dense property by [18, 54]), and the invariant bundles satisfy some mild bunching conditions such that they are of class $C^{1/2}$ for example (again an open condition).
\end{remark}

5. Formulas for the derivatives (Proof of Theorem D)

In this section we will prove Theorem D. The computations are based on the estimates from Lemmas 8 and 10. The first two parts of the theorem could also be obtained from the formulas (4.3) and (4.4).

\textbf{Proof. Part (i).} We have that $\omega$ is $C^1$, so from (2.10) we know that $h_t^*\omega = \omega + t\mathcal{L}_X\omega + o(t)$. We also have that $V_t - V = o(1)$ and is in the kernel of $\omega$. Then, using the facts that $f_*V = \tilde{\eta}V$ and $\omega(V) = 1$, we get

$$\eta_t \circ f^{-1} = h_t^*\omega(f_*V_t) = (\omega + t\mathcal{L}_X\omega)(f_*V + f_*(V_t - V)) + o(t)$$

$$= \omega(\tilde{\eta}V) + t\mathcal{L}_X\omega(\tilde{\eta}V) + \omega(f_*(V_t - V)) + t\mathcal{L}_X\omega(f_*(V_t - V)) + o(t)$$

$$= \tilde{\eta}(1 + t\mathcal{L}_X\omega(V)) + o(t).$$
We compose with $f$ and we take the logarithm, and using the fact that \( \log(1+a) = a + o(a) \), we get that the following holds if $\omega$ is $C^1$:

\[
\log \eta_t = \log \eta + \log [1 + t \mathcal{L}_X \omega(V) \circ f] + o(t) = \log \eta + t \mathcal{L}_X \omega(V) \circ f + o(t).
\]

Integrating with respect to $\mu$ we obtain

\[
\lambda(t) = \int_M \log \eta_t \, d\mu = \int_M \log \eta + t \mathcal{L}_X \omega(V) \circ f \, d\mu + o(t)
\]

\[= \lambda(0) + t \int_M \mathcal{L}_X \omega(V) \, d\mu + o(t).
\]

This shows that $\lambda$ is differentiable in zero and

\[
\lambda'(0) = \int_M \mathcal{L}_X \omega(V) \, d\mu.
\]

**Part (ii).** Now assume that $V$ is $C^1$. By Proposition 12 we know that $V_t$ is differentiable in $t = 0$, so $V_t = V + tV' + o(t)$, with $V'$ in the kernel of $\omega$. We also know from Lemma 10 that $h_t V = V - t \mathcal{L}_X V + o(t)$. Using that $f_s$ preserves the kernel of $\omega$, $h_t - Id = o(1)$, we get

\[
\tilde{\eta}_t = \omega(h_t f_s V_t) = \omega(h_t f_s V) + t \omega(h_t f_s V') + o(t)
\]

\[= \omega(h_t \eta V) + t \omega(h_t f_s V') + o(t)
\]

\[= \eta \circ h_t^{-1} [\omega(V) - t \omega(\mathcal{L}_X V)] + t \omega(f_s V') + t \omega(h_t f_s V' - f_s V') + o(t)
\]

\[= \tilde{\eta} \circ h_t^{-1} [1 - t \omega(\mathcal{L}_X V)] + o(t).
\]

Composing with $f_t = h_t \circ f$ and using that $\omega(\mathcal{L}_X V) \circ f_t = \omega(\mathcal{L}_X V) \circ f + o(1)$ we get

\[
\eta_t = \tilde{\eta} \circ f_t = \eta [1 - t \omega(\mathcal{L}_X V) \circ f] + o(t) = \eta [1 - t \omega(\mathcal{L}_X V) \circ f] + o(t).
\]

Taking the logarithm we get that if $V$ is $C^1$ then the following holds:

\[
\log \eta_t = \log \eta + \log [1 - t \omega(\mathcal{L}_X V) \circ f] + o(t) = \log \eta - t \omega(\mathcal{L}_X V) \circ f + o(t).
\]

Integrating with respect to $\mu$ we obtain

\[
\lambda(t) = \int_M \log \eta_t \, d\mu = \int_M \log \eta - t \omega(\mathcal{L}_X V) \circ f \, d\mu + o(t)
\]

\[= \lambda(0) - t \int_M \omega(\mathcal{L}_X V) \, d\mu + o(t).
\]

This shows again that $\lambda$ is differentiable in zero and

\[
\lambda'(0) = -\int_M \omega(\mathcal{L}_X V) \, d\mu.
\]
Part (iii). Recall that \( V_t = V + tV' + o(t) \) and \( h_t^*\omega = \omega + t\mathcal{L}_X\omega + o(t) \). Let \( \tilde{\omega}_t = h_t^*\omega - \omega - t\mathcal{L}_X\omega = o(t) \).

We first evaluate \( \eta_t \circ f^{-1} \). Using that \( V_t - V \) is in the kernel of \( \omega \) which is preserved by \( f_* \), \( V_t - V = tV' + o(t) \), and \( \tilde{\omega}_t = o(t) \), we get

\[
\eta_t \circ f^{-1} = h_t^*\omega(f_*V_t) = h_t^*\omega(f_*V) + h_t^*\omega[f_*(V_t - V)]
= h_t^*\omega(\tilde{\eta}V) + \omega[f_*(V_t - V)] + t\mathcal{L}_X\omega[tV' + o(t)] + \tilde{\omega}_t[tV' + o(t)]
= \tilde{\eta}h_t^*\omega(\tilde{V}) + t^2\mathcal{L}_X\omega(f_*V') + o(t^2).
\]

Now we compose with \( f \) and we apply the logarithm. Invoking Lemma 17 and recalling the relation \( \log(1 + s) = s - \frac{1}{2}s^2 + o(s^2) \) we obtain

\[
\log \eta_t = \log \left[ \eta \left( h_t^*\omega(V) + \frac{t^2}{\tilde{\eta}}\mathcal{L}_X\omega(f_*V') \right) \circ f + o(t^2) \right]
= \log \eta + \log \left[ 1 + t\mathcal{L}_X\omega(V) + \tilde{\omega}_t(V) + \frac{t^2}{\tilde{\eta}}\mathcal{L}_X\omega(f_*V') \right] \circ f + o(t^2)
\]

so

\[
\log \eta_t = \log \eta + \left[ t\mathcal{L}_X\omega(V) + \tilde{\omega}_t(V) + \frac{t^2}{\tilde{\eta}}\mathcal{L}_X\omega(f_*V') - \frac{t^2}{2}[\mathcal{L}_X\omega(V)]^2 \right] \circ f + o(t^2)
\]

We used the fact that

\[
[t\mathcal{L}_X\omega(V) + \tilde{\omega}_t(V) + \frac{t^2}{\tilde{\eta}}\mathcal{L}_X\omega(f_*V')]^2 = t^2[\mathcal{L}_X\omega(V)]^2 + o(t^2).
\]

Integrating with respect to \( \mu \) and using the fact that \( f \) preserves \( \mu \) we get the following estimation on \( \lambda_t \):

\[
\lambda(t) = \lambda(0) + t \int_M \mathcal{L}_X\omega(V) d\mu + \frac{t^2}{2} \int_M \left( \frac{2}{\tilde{\eta}}\mathcal{L}_X\omega(f_*V') - [\mathcal{L}_X\omega(V)]^2 \right) d\mu + \int_M \tilde{\omega}_t(V) d\mu + o(t^2).
\]

We are left with the estimation of \( \int_M \tilde{\omega}_t(V) d\mu \). We will do this using approximations with \( C^2 \) forms.

Remember that since \( X \) preserves the measure \( \mu \), then for any \( C^1 \) function \( g : M \to \mathbb{R} \) we have \( \int_M \mathcal{L}_X g d\mu = 0 \).

Lemma 21. The following estimation holds:

\[
\int_M \tilde{\omega}_t(V) d\mu = \frac{t^2}{2} \int_M -\mathcal{L}_X\omega(\mathcal{L}_XV) + \mathcal{L}_Y\omega(V) d\mu + o(t^2)
\]
Proof. We have to show basically that the function $A(t) := \int_M \tilde{\omega}_t(V)d\mu$ is twice differentiable in $t = 0$, $A(0) = A'(0) = 0$, and the second derivative is

$$A''(0) = \int_M -\mathcal{L}_X \omega (\mathcal{L}_X V) + \mathcal{L}_Y \omega (V)d\mu.$$ 

Since $\tilde{\omega}_t = h_t^* \omega - \omega - t\mathcal{L}_X \omega$, Lemma 8 tells us that $A$ is $C^1$, and also $A(0) = A'(0) = 0$, because $\tilde{\omega}_0 = 0$ and $\frac{\partial}{\partial t}\tilde{\omega}_t|_{t=0} = 0$.

Also we have that $t \mapsto \omega + t\mathcal{L}_X \omega$ is $C^\infty$ in $t$, and the second derivative vanishes, so it is enough to show that the map

$$B(t) := A(t) + \int_M (\omega + t\mathcal{L}_X \omega) (V)d\mu = \int_M h_t^* \omega (V)d\mu$$

is twice differentiable and

$$B''(0) = \int_M -\mathcal{L}_X \omega (\mathcal{L}_X V) + \mathcal{L}_Y \omega (V)d\mu.$$ 

Consider a sequence of $C^2$ forms $\omega_n$, $n \geq 1$, that converges to $\omega$ in the $C^1$ topology. The $B_n(t) := \int_M h_t^* \omega_n (V)d\mu$ clearly converges uniformly to $B(t)$.

From Lemma 8, (2.10) and Remark 9, we have that $h_t^* \omega_n$ is differentiable with respect to $t$, and

$$\frac{\partial}{\partial t} h_t^* \omega_n = \mathcal{L}_{X_t} (h_t^* \omega_n),$$

which converges uniformly to

$$\frac{\partial}{\partial t} h_t^* \omega = \mathcal{L}_{X_t} (h_t^* \omega).$$

This implies that $B'_n$ converges uniformly to $B'$.

Also from Lemma 8, (2.11) and Remark 9, we have that $h_t^* \omega_n$ is twice differentiable with respect to $t$, and

$$\frac{\partial^2}{\partial t^2} h_t^* \omega_n = \mathcal{L}_{X_t} \mathcal{L}_{X_t} (h_t^* \omega_n) + \mathcal{L}_{Y_t} (h_t^* \omega_n).$$

Then

$$B''_n(t) = \int_M \mathcal{L}_{X_t} \mathcal{L}_{X_t} (h_t^* \omega_n) (V) + \mathcal{L}_{Y_t} (h_t^* \omega_n) (V)d\mu.$$ 

Since we have that

$$\mathcal{L}_{X_t} \mathcal{L}_{X_t} (h_t^* \omega_n) (V) = \mathcal{L}_{X_t} [\mathcal{L}_{X_t} (h_t^* \omega_n) (V)] - \mathcal{L}_{X_t} (h_t^* \omega_n) (\mathcal{L}_{X_t} V),$$

and $X_t$ preserves $\mu$, we get that

$$B''_n(t) = \int_M -\mathcal{L}_{X_t} (h_t^* \omega_n) (\mathcal{L}_{X_t} V) + \mathcal{L}_{Y_t} (h_t^* \omega_n) (V)d\mu.$$
Since \(\omega_n\) converges to \(\omega\) in the \(C^1\) topology, the right hand side of (5.6) converges uniformly to
\[
\int_M -\mathcal{L}_{X_t} (h_t^* \omega) (\mathcal{L}_{X_t} V) + \mathcal{L}_{Y_t} (h_t^* \omega) (V) d\mu,
\]
so \(B\) must be also twice differentiable with the second derivative given above. Replacing \(t = 0\) we obtain the claim.

\[\square\]

The proof of the part (iii) of the theorem follows now from Lemma 21 and (5.4).

\[\square\]

6. The case of variable measure (Proof of Theorem E)

In this section we will treat the case when the invariant measure \(\mu_t\) depends on the map \(f_t\).

**Proof of Theorem E. Part (i).** Since \(\omega\) is \(C^1\), the relation (5.1) holds:
\[
\log \eta_t = \log \eta + t \mathcal{L}_{X} \omega(V) \circ f + o(t).
\]
Integrating with respect to \(\mu_t\) we have,
\[
\lambda(t) = \int_M \log \eta_t \, d\mu_t = \int_M \log \eta \, d\mu_t + t \int_M \mathcal{L}_{X} \omega(V) \circ f \, d\mu_t + o(t)
\]
\[
= \int_M \log \eta \, d\mu_t + t \int_M \mathcal{L}_{X} \omega(V) \circ f \, d\mu_t + o(t)
\]
\[
= \int_M \log \eta \, d\mu_t + t \int_M \mathcal{L}_{X} \omega(V) \, d\mu + o(t).
\]
We used the facts that \(f\) preserves \(\mu_t\), and for a continuous map \(g : M \to \mathbb{R}\) we have
\[
\int_M g \, d\mu_t = \int_M g \, d\mu + o(1).
\]
The formula above shows that \(\lambda\) is differentiable in 0 if and only if the family \(\mu_t\) has linear response \(\mathcal{R}(\log \eta)\) for the function \(\log \eta : M \to \mathbb{R}\), and in this case we obtain
\[
\lambda'(0) = \mathcal{R}(\log \eta) + \int_M \mathcal{L}_{X} \omega(V) d\mu.
\]

**Part (ii).** Since now \(V\) is \(C^1\), the relation (5.2) holds:
\[
\log \eta_t = \log \eta - t \omega(\mathcal{L}_X V) \circ f + o(t).
\]
Again, integrating with respect to $\mu_t$ we obtain
\[
\lambda(t) = \int_M \log \eta_t \, d\mu_t = \int_M \log \eta \, d\mu_t - t \int_M \omega(L_X V) \circ f \, d\mu_t + o(t)
\]
\[
= \int_M \log \eta \, d\mu_t - t \int_M \omega(L_X V) \circ f \, d\mu + o(t)
\]
\[
= \int_M \log \eta \, d\mu_t - t \int_M \omega(L_X V) \, d\mu + o(t).
\]

The formula above shows again that $\lambda$ is differentiable in 0 if and only if the family $\mu_t$ has linear response $R(\log \eta)$ for the function $\log \eta : M \to \mathbb{R}$, and in this case
\[
\lambda'(0) = R(\log \eta) - \int_M \omega(L_X V) \, d\mu.
\]

**Part (iii).** Since both $\omega$ and $V$ are $C^1$, then formula (5.3) holds:
\[
\log \eta_t = \log \eta + \left[ tL_X \omega(V) + \tilde{\omega}_t(V) + \frac{t^2}{\eta}L_X \omega(f_* V') - \frac{t^2}{2} (L_X \omega(V))^2 \right] \circ f + o(t^2).
\]
Since $\omega$ is in fact $C^2$ then by Lemma 8 we have
\[
\tilde{\omega}_t = \frac{t^2}{2} (L_X L_X \omega + L_Y \omega) + o(t^2).
\]
Combining the two relations above we get
\[
\log \eta_t = \log \eta + tL_X \omega(V) \circ f + \frac{t^2}{2} \left[ L_X L_X \omega(V) + L_Y \omega(V) - (L_X \omega(V))^2 + \frac{2}{\eta}L_X \omega(f_* V') \right] \circ f + o(t^2).
\]
Integrating with respect to $\mu_t$, we have
\[
\lambda(t) = \int_M \log \eta \, d\mu_t + t \int_M L_X \omega(V) \circ f \, d\mu_t + \frac{t^2}{2} \int_M \left[ L_X L_X \omega(V) + L_Y \omega(V) - (L_X \omega(V))^2 + \frac{2}{\eta}L_X \omega(f_* V') \right] \circ f \, d\mu + o(t^2)
\]
\[
= \lambda(0) + t \int_M L_X \omega(V) \, d\mu + t^2 R(L_X \omega(V) \circ f) \, d\mu + \frac{t^2}{2} \int_M \left[ L_X L_X \omega(V) + L_Y \omega(V) - (L_X \omega(V))^2 + \frac{2}{\eta}L_X \omega(f_* V') \right] \, d\mu + o(t^2)
\]
We used that $\log \eta = \lambda(0)$ is constant, $\mu_t$ has linear response for $L_X \omega(V) \circ f$, and $\mu_t = \mu + o(1)$ in the weak* topology. This finishes the proof of the theorem. 

□
Next we will prove Theorem F. The strategy is to apply Theorem E for the family \( f_t \) and two families of corresponding invariant measures: measures of maximal entropy and Gibbs u-states. Since the Liouville measure for \( f \) is both the unique measure of maximal entropy and the unique Gibbs u-state, the hypothesis of Theorem E will be satisfied, and we obtain differentiability of the corresponding stable and unstable Lyapunov exponents. On the other hand, the topological entropy will be bounded above and below by the two Lyapunov exponents, so the conclusion follows. Part of the argument is the fact that the derivative of the center exponents vanishes.

**Proof of Theorem F.** Assume that \( f \) is the time-one map of the geodesic flow on the unit tangent bundle of a manifold with constant negative curvature, denoted \( M \). Let \( \mu \) be the Liouville measure on \( M \), this means that \( \mu \) is invariant under \( f \), and it is the unique measure of maximal entropy, and the unique Gibbs u-state for \( f \).

There exists a \( C^1 \) neighborhood \( \mathcal{U} \) of \( f \) such that all the diffeomorphisms \( g \in \mathcal{U} \) are partially hyperbolic with one-dimensional center, and are \( \epsilon \)-entropy expansive, for the same \( \epsilon > 0 \) (see for example [45]). In particular the entropy function is upper-semicontinuous, and there exist measures of maximal entropy.

Assume now that the smooth family \( f_t \) is in \( \mathcal{U} \), so each \( f_t \) has a measure of maximal entropy \( \mu_t \), which we can choose to be ergodic (see [68] for example). If, for some sequence \( t_n \to 0 \), we have

\[
\mu_0 = \lim_{n \to \infty} \mu_{t_n}
\]

in the weak* topology, then \( \mu_0 \) must be an invariant measure for \( f_0 = f \).

**Lemma 22.** The entropy function is upper semicontinuous in both \( f \) and \( \mu \), in the sense that

\[
\limsup_{n \to \infty} h_{\mu_{t_n}}(f_{t_n}) \leq h_{\mu_0}(f).
\]

**Proof.** This follows basically from Bowen (see [17]). Since all the maps \( f_{t_n} \) are \( \epsilon \)-entropy expansive for the same \( \epsilon > 0 \), then for any partition \( \mathcal{A} \) with size smaller that \( \epsilon \), we have that

\[
h_{\mu_{t_n}}(f_{t_n}) = h_{\mu_{t_n}}(f_{t_n}, \mathcal{A}) = \lim_{k \to \infty} H_{\mu_{t_n}} \left( \mathcal{A} \lor f_{t_n}^{-1} \mathcal{A} \lor \cdots \lor f_{t_n}^{-k+1} \mathcal{A} \right).
\]

Let \( \mathcal{A} \) be such that the boundaries of the elements have zero measure with respect to \( \mu_0 \). Since the right hand side limit is decreasing, then given \( \delta > 0 \), there exists \( k_0 \) such that

\[
H_{\mu_0} \left( \lor_{i=0}^{k_0} f^{-i} \mathcal{A} \right) < h_{\mu_0}(f) + \delta.
\]
For every element $A \in \bigvee_{i=0}^{k_0} f^{-i} \mathcal{A}$, we can choose $C = \overline{C} \subset \text{int}(A) \subset A \subset \overline{A} \subset U = \text{int}(U)$, with $\mu_0(U \setminus C) > \delta'$, and $\mu_0(\partial C) = \mu_0(\partial U) = 0$. This implies that $\lim_{n \to \infty} \mu_{t_n}(C) = \mu_0(C)$ and $\lim_{n \to \infty} \mu_{t_n}(U) = \mu_0(U)$. For large enough $n$, the element $A_n$ from the partition $\bigvee_{i=0}^{k_0} f^{-i} \mathcal{A}$ corresponding to $A$ will satisfy $C \subset A_n \subset U$ (this is because $f_{t_n}$ converges to $f$), so we get that

$$|\mu_{t_n}(A_n) - \mu_0(A)| < \delta'.$$

This will hold for all the elements of the partition and for all large enough $n$, so by taking $\delta'$ sufficiently small we obtain that there exists some $n_\delta > 0$ such that, for any $n > n_\delta$ we have

(6.3) \[ |H_{\mu_{t_n}} \left( \bigvee_{i=0}^{k_0} f^{-i} \mathcal{A} \right) - H_{\mu_0} \left( \bigvee_{i=0}^{k_0} f^{-i} \mathcal{A} \right) | < \delta. \]

Now putting together (6.1), (6.2) and (6.3), and remembering that the sequence which gives the entropy is decreasing, we get that

$$h_{\mu_{t_n}}(f_{t_n}) < h_{\mu_0}(f) + 2\delta$$

for any $n > n_\delta$. This finishes the proof of the upper semicontinuity.

\[ \square \]

Let us suppose by contradiction that $\mu_0$ is different from the Liouville measure $\mu$, then $h_{\mu_0}(f) < h_{\text{top}}(f)$, because $\mu$ is the unique measure of maximal entropy for $f$. This implies that

$$\lim_{n \to \infty} h_{\text{top}}(f_{t_n}) = \lim_{n \to \infty} h_{\mu_{t_n}}(f_{t_n}) \leq h_{\mu_0}(f) < h_{\text{top}}(f),$$

so $h_{\text{top}}$ is not continuous at $f$, and this contradicts the continuity of the topological entropy obtained in [32, 63].

Then we have that $\mu_t$ converges in the weak* topology to $\mu$ when $t$ goes to 0. We consider first the splitting $TM = E^{cs} \oplus E^u$, with $F := E^{cs}$ and $E := E^u$. We have that $\eta^u$, the expansion along $E^u$, is constant for $f$, since the curvature is constant, so $\mathcal{R}(\log \eta) = 0$. All the sub-bundles $E^s$, $E^c$ and $E^u$ are smooth.

Then from Theorem E part ii, we obtain that $t \mapsto \lambda^u(f_t, \mu_t)$ is differentiable in $t = 0$, and the derivative is

$$\frac{\partial}{\partial t} \lambda^u(f_t, \mu_t) \bigg|_{t=0} = -\int_M \omega_{E^s}(\mathcal{L}_X V_{E^u}) d\mu.$$
A similar argument gives the differentiability in 0 of the center Lyapunov exponent, and furthermore:

\[
\frac{\partial}{\partial t} \lambda_c(f_t, \mu_t) \bigg|_{t=0} = - \int_M \omega_{E^{cu}}(L_{X} V_{E^{s}})d\mu \\
= \int_M \omega_{E^{cu}}(L_{V_{E^{c}}} X)d\mu \\
= \int_M \mathcal{L}_{V_{E^{c}}} [\omega_{E^{cu}}(X)]d\mu - \int_M \mathcal{L}_{V_{E^{c}}} \omega_{E^{cu}}(X)d\mu \\
= 0.
\]

We used the fact that we can choose $V_{E^{c}}$ to be the vector field generating the geodesic flow, so on one hand it preserves the Liouville measure $\mu$, and $\int_M \mathcal{L}_{V_{E^{c}}} g d\mu = 0$ for any $C^1$ map $g : M \to \mathbb{R}$, and on another hand $V_{E^{c}}$ preserves the form $\omega_{E^{cu}}$, so $\mathcal{L}_{V_{E^{c}}} \omega_{E^{cu}} = 0$.

Combining both derivatives above we also get

\[
\frac{\partial}{\partial t} \lambda^{cu}(f_t, \mu_t) \bigg|_{t=0} = \frac{\partial}{\partial t} \lambda^{c}(f_t, \mu_t) \bigg|_{t=0} + \frac{\partial}{\partial t} \lambda^{u}(f_t, \mu_t) \bigg|_{t=0} = - \int_M \omega_{E^{cu}}(L_{X} V_{E^{u}})d\mu.
\]

Since the maps $f_t$ are partially hyperbolic with one dimensional center, and each $\mu_t$ is ergodic, from the Ruelle formula we obtain

\[
h_{\text{top}}(f_t) = h_{\mu_t}(f_t) \leq \max\{\lambda^{u}(f_t, \mu_t), \lambda^{cu}(f_t, \mu_t)\} =: A(t).
\]

Because both $\lambda^{u}(f_t, \mu_t)$ and $\lambda^{cu}(f_t, \mu_t)$ are differentiable in $t = 0$ with the same derivative, we have that also $A(t)$ is differentiable in $t = 0$ with the same derivative:

\[
A'(0) = - \int_M \omega_{E^{cu}}(L_{X} V_{E^{u}})d\mu.
\]

Also it is easy to see that

\[
h_{\text{top}}(f) = \lambda^{u}(f, \mu) = \lambda^{cu}(f, \mu) = A(0).
\]

Thus we obtained that the topological entropy is bounded from above by the differentiable function $A(t)$. The next step is to obtain a bound from below by another differentiable function with the same derivative.

Since the maps $f_t$ are partially hyperbolic and smooth, there exist ergodic Gibbs u-states $m_t$ for each $f_t$ (see [51] or [14]). This means that $m_t$ is an invariant probability measure for $f_t$, and the disintegrations of $m_t$ along the unstable foliations of $f_t$ are absolutely continuous with respect to the Lebesgue measure on the leaves. This in particular implies that $h_{m_t}(f_t) \geq \lambda^{u}(f_t, m_t)$ (see for example Ledrappier-Young [42]).

Since $f_t$ converges to $f$ in the $C^2$ topology, any weak limit of Gibbs u-states is a Gibbs u-state (see [51] for example), and since $\mu$ is the unique Gibbs u-state for $f$, we get that the
measures \(m_t\) converge in the weak* topology to \(\mu\) when \(t\) goes to 0. Applying Theorem F again we obtain that \(t \mapsto \lambda^u(f_t, m_t)\) is differentiable in \(t = 0\), and the derivative is
\[
\left. \frac{\partial}{\partial t} \lambda^u(f_t, m_t) \right|_{t=0} = -\int_M \omega_{E^s}(\mathcal{L}_X V_{E^u})d\mu.
\]

Using the variational principle we get
\[
h_{\text{top}}(f_t) \geq h_{m_t}(f_t) \geq \lambda^u(f_t, m_t) =: B(t),
\]
where \(B(t)\) is differentiable in \(t = 0\) and
\[
B'(0) = -\int_M \omega_{E^s}(\mathcal{L}_X V_{E^u})d\mu = A'(0).
\]

Evaluating in \(t = 0\) we obtain again
\[
h_{\text{top}}(f) = \lambda^u(f, \mu) = \lambda^{cu}(f, \mu) = B(0).
\]

In conclusion, we have that
\[
B(t) \leq h_{\text{top}}(f_t) \leq A(t), \quad B(0) = h_{\text{top}}(f_0) = A(0),
\]
and both \(A\) and \(B\) are differentiable in 0 with the same derivative. This clearly implies that \(t \mapsto h_{\text{top}}(f_t)\) is differentiable in \(t = 0\) and it has the same derivative,
\[
\left. \frac{\partial}{\partial t} h_{\text{top}}(f_t) \right|_{t=0} = -\int_M \omega_{E^s}(\mathcal{L}_X V_{E^u})d\mu.
\]

\[\square\]

7. The case of flows

In this section we will prove Theorem G. We will apply Theorem A and Theorem D to the family of time-one maps \(f_t = \phi^t\) of the flows generated by \(X_t\). The first step is to find \(\overline{X}\), the vector field tangent to the family \(h_t = f_t \circ f^{-1} = \phi^t_1 \circ \phi^{-1}\) in \(t = 0\).

**Lemma 23.** Assume hypothesis (HF) is satisfied, \(r \geq 2\). Let \(\phi\) be the flow \(\phi^0\) generated by \(X^0\). Then the following holds:
\[
\overline{X} = \left. \frac{\partial}{\partial t} \phi^t_1 \circ \phi^{-1} \right|_{t=0} = \int_0^1 \phi_{s^t} X' ds.
\]

**Proof.** Since it is enough to verify the formula locally in charts, we can assume that we are in \(\mathbb{R}^n\). We have
\[
\frac{\partial}{\partial s} \phi^t_s(\phi^{-1}(p)) = X'(\phi^t_s(\phi^{-1}(p))).
\]
Furthermore
\[ \frac{\partial}{\partial t} \frac{\partial}{\partial s} \phi_s^t(\phi_{-1}(p)) = \left( \frac{\partial}{\partial t} X^t \right) (\phi_s^t(\phi_{-1}(p))) + DX^t(\phi_s^t(\phi_{-1}(p))) \cdot \frac{\partial}{\partial t} \phi_s^t(\phi_{-1}(p)) \]

Evaluating in \( t = 0 \) and using the notation
\[ A(s) := \left. \frac{\partial}{\partial t} \phi_s^t(\phi_{-1}(p)) \right|_{t=0} \]
we obtain
\[ A'(s) = X'(\phi_{s-1}(p)) + DX(\phi_{s-1}(p))A(s). \]

For \( s = 0 \) we have
\[ A(0) = \left. \frac{\partial}{\partial t} \phi_0^t(\phi_{-1}(p)) \right|_{t=0} = \left. \frac{\partial}{\partial t} \phi_{-1}(p) \right|_{t=0} = 0. \]

We have to solve the differential equation (7.3) with the initial condition (7.4). The homogeneous matrix equation
\[ \Phi'(s) = DX(\phi_{s-1}(p))\Phi(s) \]
has the fundamental solution \( \Phi_0(s) = D\phi_{s-1}(p) \) (this can be seen differentiating the formula \( \frac{\partial}{\partial s} \phi_{s-1}(p) = X(\phi_{s-1}(p)) \)). Then the solution of our equation is
\[
\begin{align*}
\bar{X}(p) &= A(1) = \Phi_0(1)\Phi_0^{-1}(0)A(0) + \Phi_0(1) \int_0^1 \Phi_0^{-1}(s)X'(\phi_{s-1}(p))ds \\
&= 0 + Id \int_0^1 D\phi_{s-1}(p)^{-1}X'(\phi_{s-1}(p))ds \\
&= \int_0^1 D\phi_{1-s}(\phi_{s-1}(p))X'(\phi_{s-1}(p))ds = \int_0^1 [\phi_{(1-s)}X'](p)ds \\
&= \int_0^1 [\phi_sX'](p)ds.
\end{align*}
\]

This proves the formula (7.1) pointwise. The formula also holds if we see the right hand side as a Bochner integral in the Banach space \( \mathcal{X}^1(M) \) of \( C^1 \) vector fields (with the \( C^1 \) norm). This is because the map \( s \mapsto \phi_sX' \) is continuous in the \( C^1 \) topology on \( \mathcal{X}^1(M) \) since \( \phi_s \) is \( C^2 \) and \( X' \) is \( C^1 \) (in particular \( \bar{X} \) is \( C^1 \)).

\[ \square \]

**Proof of Theorem G.** All the claims about the regularity of \( \lambda \) follow from the previous results on families of diffeomorphisms, applied to the family \( f_t = \phi_t^1 \), which will satisfy the hypothesis (H) for \( r \geq 3 \). We only have to check that the formulas (1.11), (1.12) and (1.13) hold.
From (1.2) we have
\[
\lambda'(0) = \frac{\partial}{\partial t} \int_M (\phi_t^X)^* \omega(V) d\mu \bigg|_{t=0} = \frac{\partial}{\partial t} \int_M \omega((\phi_t^X)_* V) d\mu \bigg|_{t=0}
\]

Let
\[
\mathcal{D}(X) := \frac{\partial}{\partial t} \int_M (\phi_t^X)^* \omega(V) d\mu \bigg|_{t=0} = \frac{\partial}{\partial t} \int_M \omega((\phi_t^X)_* V) d\mu \bigg|_{t=0}
\]

Then \( \mathcal{D} \) is well defined for every vector field
\[
X \in \mathcal{X}^1_\mu(M) := \{ X \in \mathcal{X}^1(M) : \phi_*^X \mu = \mu \},
\]
i.e. the \( C^1 \) vector fields which preserve \( \mu \).

It is easy to see that \( X \in \mathcal{X}^1_\mu(M) \) if and only if \( X \in \mathcal{X}^1(M) \) and for every \( C^1 \) map \( g : M \to \mathbb{R} \) we have \( \int_M dg(X) d\mu = 0 \) (see the proof of Lemma 7). This in turn implies that \( \mathcal{X}^1_\mu(M) \) is a closed linear subspace of \( \mathcal{X}^1(M) \). The formula (4.7) shows then that \( \mathcal{D} : \mathcal{X}^1_\mu(M) \to \mathbb{R} \) is a bounded operator.

Although this does not follow directly from the definition, \( \mathcal{D} \) is also a linear operator on \( \mathcal{X}^1_\mu(M) \). Let \( \beta' < \beta \) such that \( \alpha + \beta > 1 \), and let \( V_n \) be a sequence of \( C^\infty \) multivector fields converging to \( V \) in the topology \( C^{\beta'} \). Let
\[
\mathcal{D}_n(X) := \frac{\partial}{\partial t} \int_M \omega((\phi_t^X)_* V_n) d\mu \bigg|_{t=0} = \int_M \omega \left( \frac{\partial}{\partial t}(\phi_t^X V_n) \bigg|_{t=0} \right) d\mu
\]

so the operator \( \mathcal{D}_n \) is linear in \( X \).

On the other hand from (4.7) we have
\[
|\mathcal{D}_n(X) - \mathcal{D}(X)| = \left| \frac{\partial}{\partial t} \int_M \omega((\phi_t^X)_* (V_n - V)) d\mu \bigg|_{t=0} \right| \leq C \| \omega \|_{C^\alpha} \| V_n - V \|_{C^{\beta'}} \| X \|_{C^1},
\]

so \( \mathcal{D}_n \) converges to \( \mathcal{D} \), which means that \( \mathcal{D} \) will be also linear.

We have that \( X_t \in \mathcal{X}^2_\mu(M) \) for all \( t \), so \( X' = \lim_{t \to 0} \frac{X_t - X}{t} \in \mathcal{X}^1_\mu(M) \). Also \( \phi_{ss} X' \in \mathcal{X}^1_\mu(M) \) for all \( s \in [0, 1] \), because for any \( C^1 \) map \( g : M \to \mathbb{R} \) we have
\[
\int_M dg(\phi_{ss} X') d\mu = \int_M \phi_{ss} dg(X') d\mu = \int_M d(g \circ \phi_s)(X') d\mu = 0.
\]

Since a bounded linear operator commutes with the Bochner integral we get
\[
\lambda'(0) = \mathcal{D}(X) = \mathcal{D} \left( \int_0^1 \phi_{ss} X' ds \right) = \int_0^1 \mathcal{D}(\phi_{ss} X') ds.
\]
Now we have to compute $\mathcal{D}(\phi_{s*}X')$. Observe that $\phi_t^{\phi_{s*}X'} = \phi_s \circ \phi_t^{X'} \circ \phi_{-s}$. Let $\eta^s : M \to \mathbb{R}$ be such that $\mathcal{D}(\phi_s(p)V(p)) = \eta^s(p)V(\phi_s(p))$. Then

$$\phi_{-s*}V(p) = D\phi_{-s}(\phi_s(p))V(\phi_s(p)) = [D\phi_s(p)]^{-1}V(\phi_s(p)) = \frac{1}{\eta^s(p)}V(p),$$

or $\phi_{-s*}V = \frac{1}{\eta}V$. Also

$$\phi^s\omega(V)(p) = \omega(\phi_{s*}V(p)) = \omega(D\phi_s(p)V(p)) = \omega(\eta^s(p)V(\phi_s(p))) = \eta^s(p),$$

so $\phi^s\omega = \eta^s\omega$. Then

$$\phi^s\omega(\phi_{ts}^{X'} \circ \phi_{-s*}V) = \eta^s\omega \left( \phi_{ts}^{X'} \frac{1}{\eta^s} \right) = \frac{\eta^s}{\eta^s \circ \phi_{-s}^t} \omega(\phi_{ts}^{X'} V)$$

and

$$\omega(\phi_{ts}^{\phi_{s*}X'}) = \omega(\phi_{s*} \circ \phi_{ts}^{X'} \circ \phi_{-s*}V) = \left[ \phi^s\omega(\phi_{ts}^{X'}) \circ \phi_{-s} \right] \circ \phi_{-s}$$

$$= \left[ \frac{\eta^s}{\eta^s \circ \phi_{-s}^t} \right] \circ \phi_{-s} \cdot \left[ \omega(\phi_{ts}^{X'}) \right] \circ \phi_{-s}.$$

If $\beta \geq \alpha$ then, using that $\phi$ and $\phi^{X'}$ preserve $\mu$, and applying twice Lemma 18, we obtain

$$\mathcal{D}(\phi_{s*}X') = \frac{\partial}{\partial t} \int_M \omega(\phi_{ts}^{\phi_{s*}X'}) d\mu \bigg|_{t=0} = \frac{\partial}{\partial t} \int_M \log \left[ \omega(\phi_{ts}^{\phi_{s*}X'}) \right] d\mu \bigg|_{t=0}$$

$$= \frac{\partial}{\partial t} \left[ \int_M \log(\eta^s \circ \phi_{-s}) d\mu - \int_M \log(\eta^s \circ \phi_{-s}^{X'} \circ \phi_{-s}) d\mu + \int_M \log \left( \omega(\phi_{ts}^{X'}) \right) \circ \phi_{-s} \right] d\mu \bigg|_{t=0}$$

$$= \frac{\partial}{\partial t} \int_M \log \left( \omega(\phi_{ts}^{X'}) \right) d\mu \bigg|_{t=0} = \mathcal{D}(X').$$

If $\alpha > \beta$ then one obtains in a similar way that $\mathcal{D}(\phi_{s*}X') = \mathcal{D}(X')$ again, this time using the estimation

$$\phi^{X'*} \omega(V) = \left[ \omega(\phi_{ts}^{\phi_{s*}X'}) \right] \circ \phi_{ts}^{\phi_{s*}X'} = \frac{\eta^s \circ \phi_{ts}^{X'}}{\eta^s \circ \phi_{-s}^t} \left[ \phi_{ts}^{X'*} \omega(V) \right] \circ \phi_{-s}.$$

Then from (7.5) we have

$$\lambda'(0) = \int_0^1 \mathcal{D}(\phi_{s*}X') ds = \int_0^1 \mathcal{D}(X') ds = \mathcal{D}(X'),$$

and the formula (1.11) follows.
For the proof of the formula (1.12) let us assume that \( \alpha = 1 \), then
\[
\lambda'(0) = \int_M \mathcal{L}_X \omega(V) d\mu.
\]
since \( X \mapsto \mathcal{L}_X \omega(V) \) is linear and bounded (in the \( C^1 \) topology), we can apply again the commutativity of the linear bounded operator with the Bochner integral and we obtain
\[
(7.6) \quad \lambda'(0) = \int_M \int_0^1 \mathcal{L}_{(\phi_s^* X)} \omega(V) dsd\mu = \int_0^1 \int_M \mathcal{L}_{(\phi_s^* X)} \omega(V) d\mu ds.
\]
(we changed the order of integration). We further have
\[
\int_M \mathcal{L}_{(\phi_s^* X)} \omega(V) d\mu = \int_M \left[ \phi_s^* \mathcal{L}_X \omega(V) \right] d\mu = \int_M \left[ \phi_s^* \mathcal{L}_X \omega(V) \right] d\mu
\]
\[
= \int_M \left[ \phi_s^* (\mathcal{L}_X (\eta^s \omega) + \eta^s \mathcal{L}_X \omega) \right] d\mu
\]
\[
= \int_M (\mathcal{L}_X \eta^s) \circ \phi_s d\mu + \int_M \eta^s \mathcal{L}_X \omega(\phi_s V) d\mu
\]
\[
= \int_M \mathcal{L}_X \eta^s d\mu + \int_M \eta^s \mathcal{L}_X \omega \left( \frac{1}{\eta^s} V \right) d\mu
\]
\[
= \int_M \mathcal{L}_X \omega(V) d\mu.
\]
Substituting in (7.6) we obtain (1.12).

The proof of (1.13) is similar and we omit it here.

\[\square\]

8. Non-vanishing of the second derivative

In this section we will prove Theorem H. The idea of the proof is to make an explicit perturbation supported on a small neighborhood of a non-periodic point, mixing the directions of \( E^2 \) and \( E^3 \). In this small neighborhood we approximate the bundles by linear ones, and we apply the formula (1.9). For the linear part of the bundles, the first three terms in (1.9) can be computed explicitly, and the error term is small. For the last term in (1.9), \( \frac{2}{3} \mathcal{L}_X \omega (f_s V') \), we use the expansion (2.24) of \( V' \), and if the return time of the support of the perturbation is large, then only the first term is significant, and this can be again computed explicitly for the linear part.

Proof. We have that \( TM = E^1 \oplus E^2 \oplus E^3 \) is a dominated splitting of class \( C^1 \) for \( f \) (\( E^1 \) can be trivial) and the measure \( \mu \) is the volume on \( M \). Let \( p_0 \in M \) be a non-periodic point for \( f_0 \) and consider a smooth chart \((U, \phi)\) from a neighborhood \( U \) of \( p_0 \) to \( B(0,1) \subset \mathbb{R}^d \) such that:
\[ \phi(p_0) = 0; \]
\[ \phi_*E^3(0) = \text{span}\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots \frac{\partial}{\partial x_i}\}; \]
\[ \phi_*E^2(0) = \text{span}\{ \frac{\partial}{\partial x_{i+1}}, \frac{\partial}{\partial x_{i+2}}, \ldots \frac{\partial}{\partial x_m}\}; \]
\[ \phi_*E^1(0) = \text{span}\{ \frac{\partial}{\partial x_{m+1}}, \frac{\partial}{\partial x_{m+2}}, \ldots \frac{\partial}{\partial x_d}\}; \]
\[ \phi_*\mu = \text{Leb}. \]

We will work mostly in this chart, and we will make some abuse using the same notations for the objects in \( M \) and their push forward in the chart. Let \( H : B(0,1) \to \mathbb{R} \) be a \( C^\infty \) function with compact support. Let \( H_r(p) = r^2 H(p) \) be the rescaling of \( H \) to the ball \( B_r := B(0,r) \). Consider the \( C^\infty \) vector field \( X_r \) with support in \( B_r \):
\[
X_r(p) = -\frac{\partial H}{\partial x_{i+1}}(p) \frac{\partial}{\partial x_1} + \frac{\partial H}{\partial x_1}(p) \frac{\partial}{\partial x_{i+1}} \quad (\text{here } p = (x_1, x_2, \ldots x_d) \in \mathbb{R}^d),
\]
or
\[
X_r(p) = -r \frac{\partial H}{\partial x_{i+1}} \left( \frac{p}{r} \right) \frac{\partial}{\partial x_1} + r \frac{\partial H}{\partial x_1} \left( \frac{p}{r} \right) \frac{\partial}{\partial x_{i+1}}
\]

We will consider the family \( h^r \) = \( \phi^r \), i.e. the flow generated by \( X_r \). The flow preserves the volume, in fact it preserves the \( x_1 x_{i+1} \)-planes, and is Hamiltonian on each plane. Observe that we have \( X_r = O(r) \) and \( DX_r = O(1) \).

We will show that for \( r \) sufficiently small, the family \( h^r \) satisfies the conclusion of the theorem. We will apply the formula (1.9) from Theorem D to the Lyapunov exponents of the bundles \( E^3 \) and \( E^2 \).

**Estimation of \( \lambda^r_{r,E^3}(0) \).** The family \( h^r \) has \( X = X_r \) and \( Y = 0 \), and in the first case we consider \( E = E^3 \) and \( F = E^1 \oplus E^2 \).

Let \( \omega^3 = \omega \) be the \( C^1 \) form corresponding to \( F = E^1 \oplus E^2 \), and \( V^3 = V \) the \( C^1 \) multivector corresponding to \( E = E^3 \) such that \( \omega_3(V_3) = 1 \). Eventually after a composition with a linear map, we can assume that (in the given chart) we have \( \omega^3(0) = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_i \), and \( V^3(0) = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \cdots \wedge \frac{\partial}{\partial x_i} \). Since \( \omega^3 \) and \( V^3 \) are \( C^1 \) we get
\[
\omega^3(p) = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_i + \alpha^3(p), \quad \alpha^3 = O(r), \quad d\alpha^3 = O(1),
\]
\[
V^3(p) = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \cdots \wedge \frac{\partial}{\partial x_i} + T^3(p), \quad T^3 = O(r), \quad DT^3 = O(1).
\]
Since \( \omega^3 \), \( d\omega^3 \), \( V^3 \) and \( DV^3 \) are all \( O(1) \), we have that \( \mathcal{L}_X \omega^3 \) and \( \mathcal{L}_X V^3 \) are \( O(1) \), while \( \mathcal{L}_X \alpha^3 \) and \( \mathcal{L}_X T^3 \) are \( O(r) \).

Let us estimate the Lie derivative of \( \omega^3 \) with respect to \( X_r \). We have:
\[
\mathcal{L}_X dx_i = d(\mathcal{L}_X x_i) = d \left[ -r \frac{\partial H}{\partial x_{i+1}} \left( \frac{p}{r} \right) \right] = -\sum_{i=1}^n \frac{\partial^2 H}{\partial x_{i+1} \partial x_i} \left( \frac{p}{r} \right) dx_i,
\]
\[ \mathcal{L}_X dx_{i+1} = d(\mathcal{L}_X, x_{i+1}) = d \left[ \frac{\partial H}{\partial x_1} \left( \frac{p}{r} \right) \right] = \sum_{i=1}^{n} \frac{\partial^2 H}{\partial x_i \partial x_1} \left( \frac{p}{r} \right) dx_i, \]

and \( \mathcal{L}_X, dx_i = 0 \) for all the other \( i \). Then

\[ \mathcal{L}_X, \omega^3 = \mathcal{L}_X, (dx_1 \wedge \cdots \wedge dx_l) + \mathcal{L}_X, \alpha^3 = (\mathcal{L}_X, dx_1) \wedge dx_2 \wedge \cdots \wedge dx_l + O(r) \]

\[ = - \left( \sum_{i=1}^{n} \frac{\partial^2 H}{\partial x_{i+1} \partial x_1} \left( \frac{p}{r} \right) dx_i \right) \wedge dx_2 \wedge \cdots \wedge dx_l + O(r) \]

\[ = - \frac{\partial^2 H}{\partial x_1 \partial x_{i+1}} \left( \frac{p}{r} \right) dx_1 \wedge \cdots \wedge dx_l - \frac{\partial^2 H}{\partial x_l^2} \left( \frac{p}{r} \right) dx_{i+1} \wedge dx_2 \wedge \cdots \wedge dx_l + \]

\[ - \sum_{i=1}^{n} \frac{\partial^2 H}{\partial x_{i+1} \partial x_i} \left( \frac{p}{r} \right) dx_i \wedge dx_2 \wedge \cdots \wedge dx_l + O(r). \]

Now let us estimate the Lie derivative of \( V^3 \). We also have:

\[ \mathcal{L}_X, \frac{\partial}{\partial x_i} = \left[ X_r, \frac{\partial}{\partial x_i} \right] = \frac{\partial X_r}{\partial x_i} = - \frac{\partial^2 H}{\partial x_{i+1} \partial x_1} \left( \frac{p}{r} \right) \frac{\partial}{\partial x_1} + \frac{\partial^2 H}{\partial x_1 \partial x_{i+1}} \left( \frac{p}{r} \right) \frac{\partial}{\partial x_{i+1}}. \]

Then we have:

\[ \mathcal{L}_X, V^3 = \mathcal{L}_X, \left( \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_l} + T^3 \right) \]

\[ = \sum_{i=1}^{l} \frac{\partial}{\partial x_1} \wedge \cdots \wedge \mathcal{L}_X, \frac{\partial}{\partial x_i} \wedge \cdots \wedge \frac{\partial}{\partial x_l} + O(r) \]

\[ = - \frac{\partial^2 H}{\partial x_1 \partial x_{i+1}} \left( \frac{p}{r} \right) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \cdots \wedge \frac{\partial}{\partial x_l} + \]

\[ + \sum_{i=1}^{l} \frac{\partial^2 H}{\partial x_{i+1} \partial x_i} \left( \frac{p}{r} \right) \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_{i+1}} \wedge \cdots \wedge \frac{\partial}{\partial x_l} + O(r). \]

Computing \( \mathcal{L}_X, \omega^3(V^3) \) we get

\[ \mathcal{L}_X, \omega^3(V^3) = - \frac{\partial^2 H}{\partial x_1 \partial x_{i+1}} \left( \frac{p}{r} \right) + O(r). \]

Computing \( \mathcal{L}_X, \omega^3(\mathcal{L}_X, V^3) \) we get

\[ \mathcal{L}_X, \omega^3(\mathcal{L}_X, V^3) = \left[ \frac{\partial^2 H}{\partial x_1 \partial x_{i+1}} \left( \frac{p}{r} \right) \right]^2 - \frac{\partial^2 H}{\partial x_{i+1}^2} \left( \frac{p}{r} \right) \frac{\partial^2 H}{\partial x_i^2} \left( \frac{p}{r} \right) + O(r). \]

Now we will estimate the second derivative of the Lyapunov exponent corresponding to \( E^3 \). We remark that \( \mathcal{L}_X, \omega^3 \) is supported in \( B_r = B(0, r) \), since here is where \( X_r \) is
Integrating by parts the second term we obtain

\[ \lambda''_{r,E^1}(0) = \int_M -\mathcal{L}_X,\omega^3(\mathcal{L}_X,V^3) - (\mathcal{L}_X,\omega^3(V^3))^2 + \frac{2}{\eta_3} \mathcal{L}_X,\omega^3(f_*V^3) d\mu \]

\[ = -\int_{B_r} \mathcal{L}_X,\omega^3(\mathcal{L}_X,V^3) + (\mathcal{L}_X,\omega^3(V^3))^2 d\mu + \int_{B_r} \frac{2}{\eta_3} \mathcal{L}_X,\omega^3(f_*V^3) d\mu \]

\[ := -I_1 + I_2. \]

Computing \( I_1 \) in the chart and using the change of variables \( p = (x_i)_{1 \leq i \leq d} = rq = (ry_i)_{1 \leq i \leq d}, \) \( dp = r^d dq, \) we get

\[ I_1 = \int_{B_r} \mathcal{L}_X,\omega^3(\mathcal{L}_X,V^3) + (\mathcal{L}_X,\omega^3(V^3))^2 d\mu \]

\[ = \int_{B_r} \left[ \frac{\partial^2 \omega}{\partial x_1 \partial x_{l+1}} \left( \frac{p}{r} \right) \right]^2 - \frac{\partial^2 \omega}{\partial x_1 ^2} \left( \frac{p}{r} \right) \frac{\partial^2 \omega}{\partial x_1 ^2} \left( \frac{p}{r} \right) + \left[ \frac{\partial^2 \omega}{\partial x_1 \partial x_{l+1}} \left( \frac{p}{r} \right) \right]^2 + O(r)d\mu \]

\[ = r^d \int_{B(0,1)} 2 \left[ \frac{\partial^2 \omega}{\partial x_1 \partial x_{l+1}} \right]^2 - \frac{\partial^2 \omega}{\partial x_1 ^2} \frac{\partial^2 \omega}{\partial x_1 ^2} d\mu + \int_{B(0,r)} O(r)d\mu \]

\[ = r^d K + O(r^{d+1}), \]

where \( K \) is independent of \( r, \) and is given by the formula

\[ K = \int_{B(0,1)} 2 \left[ \frac{\partial^2 \omega}{\partial x_1 \partial x_{l+1}} \right]^2 - \frac{\partial^2 \omega}{\partial x_1 ^2} \frac{\partial^2 \omega}{\partial x_1 ^2} d\mu. \]

Integrating by parts the second term we obtain

\[ \int_{B(0,1)} \frac{\partial^2 \omega}{\partial x_1 ^2} \frac{\partial^2 \omega}{\partial x_1 ^2} d\mu = - \int_{B(0,1)} \frac{\partial \omega}{\partial x_1 } \frac{\partial^3 \omega}{\partial x_1 \partial x_1 ^2} d\mu = \int_{B(0,1)} \left[ \frac{\partial^2 \omega}{\partial x_1 \partial x_{l+1}} \right]^2 d\mu, \]

so we have in fact that, if \( H \) is not constant zero, then

\[ K = \int_{B(0,1)} \left[ \frac{\partial^2 \omega}{\partial x_1 \partial x_{l+1}} \right]^2 d\mu > 0. \]

Now we will estimate \( I_2. \) Remember that \( \mathcal{P} = Id - \omega^3(\cdot)V^3 \) is the projection to the kernel of \( \omega_F = \omega^3. \) Denote

\[ A^3(p) := \mathcal{P}(\mathcal{L}_X,V^3)(p) = O(1). \]

We can apply formula \((2.24)\) and we get

\[ V^3 = \left[ Id - \left( \frac{f_*}{\eta_3} \right) \right]_{\mathcal{F} \cup E^{k-1}}^{-1} \mathcal{P}(\mathcal{L}_X,V^3) = \left[ Id - \left( \frac{f_*}{\eta_3} \right) \right]_{\mathcal{F} \cup E^{k-1}}^{-1} A^3. \]
In this case since \( E = E^3 \), and \( E \) dominates \( F = E^1 \oplus E^2 \), the operator \( \left( \frac{f_r}{\eta_3} \right)_F \) is a contraction by some \( \nu \in (0, 1) \), so

\[
V^3 = \sum_{k \geq 0} \left( \frac{f_r}{\eta_3} \right)^k A^3.
\]

The support of \( A^3 \) is inside \( B_r \), so the support of \( f^k A^3 \) is inside \( f^k(B_r) \). Let \( t_r = \min\{k \geq 1, B_r \cap f^k(B_r) \neq \emptyset\} \) be the return time of the set \( B_r \) to itself under \( f \). Since \( p \) is not periodic we have \( \lim_{r \to 0} t_r = \infty \). Since \( \left( \frac{f_r}{\eta_3} \right)_F \) is a contraction by \( \nu < 1 \), we have that, for \( C \) independent of \( r \)

\[
\left\| \left( \frac{f_r}{\eta_3} \right)^k A^3 \right\| \leq \nu^k \| A^3 \| \leq C \nu^k.
\]

Since \( \mathcal{L}_X, \omega^3 \) is also bounded independently of \( r \), we have the following estimation of \( I_2 \)

\[
I_2 = \int_{B_r} \frac{2}{\eta} \mathcal{L}_X, \omega^3(f_r V^3) d\mu = \int_{B_r} 2 \mathcal{L}_X, \omega^3 \left( \left( \frac{f_r}{\eta_3} \right)^k A^3 \right) d\mu = \sum_{k \geq 1} \int_{B_r} 2 \mathcal{L}_X, \omega^3 \left( \left( \frac{f_r}{\eta_3} \right)^k A^3 \right) d\mu = \sum_{k \geq 1} \int_{B_r} 2 \mathcal{L}_X, \omega^3 \left( \left( \frac{f_r}{\eta_3} \right)^k A^3 \right) d\mu.
\]

and

\[
|I_2| \leq \sum_{k=t_r}^{\infty} \int_{B_r \cap f^k(B_r)} C \nu^k d\mu \leq \sum_{k=t_r}^{\infty} \int_{B_r} C \nu^k d\mu \leq C \nu^r r^d,
\]

where \( C \) is again some constant independent on \( r \).

In conclusion,

\[
(8.1) \quad \lambda''_{r, E^3}(0) = -Kr^d + O(r^{d+1}) + O(r^{d+1} \nu^r),
\]

so \( \lim_{r \to 0} \frac{\lambda''_{r, E^3}(0)}{r^d} = -K < 0 \) and as a consequence \( \lambda''_{r, E^3}(0) < 0 \) for all \( r \) sufficiently small.

**Estimation of \( \lambda''_{r, E^2}(0) \).** We can do a similar estimation for the integrated Lyapunov exponent corresponding to the bundle \( E^2 \). In this case we will consider the corresponding \( \omega^2, V^2, V^2', \bar{\eta}_2 \).

We obtain again

\[
\lambda''_{r, E^2}(0) = \int_M -\mathcal{L}_X, \omega^2(\mathcal{L}_X, V^2) - (\mathcal{L}_X, \omega^2(V^2))^2 + \frac{2}{\bar{\eta}_2} \mathcal{L}_X, \omega^2(f_r V^2') d\mu
\]

\[
= -\int_{B_r} \mathcal{L}_X, \omega^2(\mathcal{L}_X, V^2) + (\mathcal{L}_X, \omega^2(V^2))^2 d\mu + \int_{B_r} \frac{2}{\bar{\eta}_2} \mathcal{L}_X, \omega^2(f_r V^2') d\mu
\]

\[
:= -I_3 + I_4.
\]
Because of the symmetry between $E^2$ and $E^3$ with respect to $X_r$, we obtain again the estimate

$$I_3 = r^d K + O(r^{d+1}),$$

where, if $H$ is not trivial,

$$K = \int_{B(0,1)} \left[ \frac{\partial^2 H}{\partial x_1 \partial x_{l+1}} \right]^2 d\mu > 0.$$

We will estimate $I_4$. We write again

$$\omega^2(p) = dx_{l+1} \wedge \cdots \wedge dx_m + \alpha^2(p), \quad \alpha^2 = O(r), \quad d\alpha^2 = O(1),$$

$$V^2(p) = \frac{\partial}{\partial x_{l+1}} \wedge \cdots \wedge \frac{\partial}{\partial x_m} + T^2(p), \quad T^2 = O(r), \quad DT^2 = O(1).$$

We obtain the analog formula for $L_{X_r} \omega^3$ and $L_{X_r} V^2$:

$$L_{X_r} \omega^3 = \frac{\partial^2 H}{\partial x_1 \partial x_{l+1}} \left( \frac{p}{r} \right) dx_{l+1} \wedge \cdots \wedge dx_m + \frac{\partial^2 H}{\partial x_1^2} \left( \frac{p}{r} \right) dx_1 \wedge dx_{l+2} \wedge \cdots \wedge dx_m +$$

$$+ \sum_{i \in \{2, \ldots, m+1, \ldots, n\}} \frac{\partial^2 H}{\partial x_1 \partial x_i} \left( \frac{p}{r} \right) dx_i \wedge dx_{l+2} \wedge \cdots \wedge dx_m + O(r),$$

$$L_{X_r} V^2 = \frac{\partial^2 H}{\partial x_{l+1} \partial x_{l+1}} \left( \frac{p}{r} \right) \frac{\partial}{\partial x_{l+1}} \wedge \cdots \wedge \frac{\partial}{\partial x_m} +$$

$$- \sum_{i=l+1}^m \frac{\partial^2 H}{\partial x_{l+1} \partial x_i} \left( \frac{p}{r} \right) \frac{\partial}{\partial x_{l+1}} \wedge \cdots \wedge \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_m} + O(r).$$

Consider now $P = Id - \omega^2(\cdot) V^2$ the projection to the kernel of $\omega_F = \omega^2$, and denote

$$A^2(p) := P(L_{X_r} V^2)(p) = \sum_{i=l+1}^m \frac{\partial^2 H}{\partial x_{l+1} \partial x_i} \left( \frac{p}{r} \right) \frac{\partial}{\partial x_{l+1}} \wedge \cdots \wedge \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_m} + O(r).$$

From formula (2.24) we have

$$V^2' = \left[ \left[ Id - \left( \frac{f^*}{\eta^2} \right) \right]_{F \wedge E^{k-1}} \right]^{-1} A^2.$$

This time the operator $\left( \frac{f^*}{\eta^2} \right)_{F \wedge E^{k-1}}$ is not a contraction, but it is hyperbolic and decomposes in the sum of the contraction $T_1$ and the expansion $T_3$. In our case

$$T_1 A^2 = O(r)$$

and

$$T_3 A^2 = \sum_{i=l+1}^m \frac{\partial^2 H}{\partial x_{l+1} \partial x_i} \left( \frac{p}{r} \right) \frac{\partial}{\partial x_{l+1}} \wedge \cdots \wedge \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_m} + O(r).$$
Then applying the second part of (2.24) we get

\[ V^{2'} = \sum_{k \geq 0} \left( \frac{f_s}{\eta_2} \right)^k T_1 A^2 - \sum_{k \geq 1} \left( \frac{f_s}{\eta_2} \right)^{-k} T_3 A^2 \]

Now we evaluate \( I_4 \). Let \( t_r \) be the first return time of \( B_r \) under \( f^{-1} \). Like in the previous case, since \( \left( \frac{f_s}{\eta_2} \right) \mid_{E^3 \Lambda E^{k-1}} \) is an expansion, there exist \( C > 0 \) and \( 0 < \nu < 1 \) such that

\[
\left\| \left( \frac{f_s}{\eta_2} \right)^{-k} T_3 A^2 \right\| \leq C \nu^k.
\]

We have

\[
I_4 = \int_{B_r} \frac{2}{\eta_2} L_{X_r} \omega^2(f_s V^{2'}) d\mu
\]

\[
= \int_{B_r} 2L_{X_r} \omega^2 \left[ \left( \frac{f_s}{\eta_2} \right)^k T_1 A^2 - \frac{f_s}{\eta_2} \sum_{k \geq 1} \left( \frac{f_s}{\eta_2} \right)^{-k} T_3 A^2 \right] d\mu
\]

\[
= 2 \sum_{k \geq 1} \int_{B_r} 2L_{X_r} \omega^2 \left( \frac{f_s}{\eta_2} \right)^k T_1 A^2 d\mu - 2 \sum_{k \geq 0} \int_{B_r} 2L_{X_r} \omega^2 \left( \frac{f_s}{\eta_2} \right)^{-k} T_3 A^2 d\mu + O(r^{d+1})
\]

\[
= - \int_{B_r} 2L_{X_r} \omega^2 (T_3 A^2) d\mu - 2 \sum_{k \geq 1} \int_{B_r} 2L_{X_r} \omega^2 \left( \frac{f_s}{\eta_2} \right)^{-k} T_3 A^2 d\mu + O(r^{d+1}) + O(r^d \nu^t)
\]

\[
= \int_{B_r} 2 \frac{\partial^2 H}{\partial x_{t+1}^r} \left( \frac{p}{r} \right) \frac{\partial^2 H}{\partial x_1^r} \left( \frac{p}{r} \right) d\mu + O(r^{d+1}) + O(r^d \nu^t)
\]

\[
= 2K + O(r^{d+1}) + O(r^d \nu^t).
\]

Putting \( I_3 \) and \( I_4 \) together we obtain

\[
(8.2) \quad \lambda''_{r, E^3}(0) = Kr^d + O(r^{d+1}) + O(r^d \nu^t),
\]

so \( \lim_{r \to 0} \frac{\lambda''_{r, E^3}(0)}{r^d} = K > 0 \) and as a consequence \( \lambda''_{r, E^3}(0) > 0 \) for all \( r \) sufficiently small.

\[ \square \]

9. Critical points and rigidity

In this section we will prove Theorem I and Corollary J. Let us remark that assuming better regularity of the bundles \( E \) and (or) \( F \) would simplify the proof considerably. For example if \( E \) is sufficiently smooth we can assume that \( V \) is constant \( \frac{\partial}{\partial x_1} \) in some local chart, and this simplifies considerably the expression of the derivative of the Lyapunov exponent. However we want to apply the result to dynamical foliations in order to obtain
rigidity, and assuming better regularity of the invariant bundles already implies rigidity in many situations (see [33]).

The strategy of the proof is the following. First we see that if the diffeomorphism $f$ is critical, then by moving the derivatives in the formula (1.7) away from $X$, we obtain that some specific continuous one-form on $M$ is locally exact, or the integral vanishes over every local piecewise $C^1$ closed curve. Then we choose a closed curve which is a “rectangle” formed by $W^E$ and $W^F$ pieces. The integral of the one-form over a $W^E$ piece can be written in terms of the densities of the disintegrations of the volume along $W^E$. The integral of the one-form over a $W^F$ piece can be written in terms of the Jacobian of the holonomy along $W^F$ between $W^E$ pieces. Then the vanishing of the integral over all such closed curves will imply that the disintegrations of the volume along $W^E$ are invariant under the $W^F$ holonomy.

In the rest of the section $\mu$ will be the Lebesgue measure on the manifold $M$, and $\mu_E$ is the Lebesgue measure on the leaves of the foliation $W^E$ tangent to the sub-bundle $E$. Given a foliation chart $U$ for the foliation $W^E$, following Rokhlin [55] we can define the disintegrations of $\mu$ along the local leaves of $W^E$ in $U$. We will denote by $m_E(x)$ the disintegration along the local leaf of $W^E$ passing through $x$, this is defined for $\mu$-almost every $x$ in $U$. There is also a quotient measure defined on the space of local leaves in $U$. If $T$ is a transversal to the foliation $W^E$ in $U$, the quotient measure induced on $T$ is denoted $\mu_T$.

In our case the disintegrations $m_E$ of $\mu$ along the local leaves of $W^E$ are absolutely continuous with respect to the Lebesgue measure on the leaves, $\mu_E$. We denote by $\rho_x$ the densities of the disintegrations with respect to Lebesgue, or

$$dm_E(x) = \rho_x d\mu_E(x).$$

If there is no need to specify it, we will drop the $x$ from the above notations.

**Proof of Theorem I.** We divide the proof in several steps.

**Step 1: Moving the derivatives away from $X$.**

If $f$ is critical for $E$ and the volume $\mu$, then by (1.7) we know that for every $C^3$ divergence free (volume preserving) vector field $X$ on $M$ we have

$$\int_M \mathcal{L}_X \omega(V) d\mu = 0.$$

Suppose that $X$ is supported in a foliation chart $U$ for $W^E$, with some transversal $T$. Assume that in this chart we have the quotient measure $\mu_T$ and the disintegration of the volume along the $W^E$ leaves have the density $\rho$ with respect to Lebesgue on the leaves.
µE. Since \( \frac{V}{\|V \|} \) is a unit vector field generating \( \mathcal{W}^E \), and \( \mu_E \) is the Lebesgue measure on \( \mathcal{W}^E \), the Fundamental Theorem of Calculus gives that for every piece \( \mathcal{W}^E(p) \) and every \( C^1 \) function \( g : \mathcal{W}^E(p) \to \mathbb{R} \) with compact support,

\[
\int_{\mathcal{W}^E(p)} dg \left( \frac{V}{\|V \|} \right) d\mu_E = 0.
\]

This means that we can integrate by parts on \( \mathcal{W}^E(p) \). We have

\[
0 = \int_M \mathcal{L}_X \omega(V) d\mu = \int_M i_X d\omega(V) + di_X \omega(V) d\mu \\
= \int_M -i_V d\omega(X) d\mu + \int_U d[\omega(X)](V) d\mu \\
= \int_M -i_V d\omega(X) d\mu + \int_T \int_{\mathcal{W}^E(p)} d[\omega(X)] \left( \frac{V}{\|V \|} \right) \rho\|V\| d\mu_E d\mu_T \\
= \int_M -i_V d\omega(X) d\mu - \int_T \int_{\mathcal{W}^E(p)} d(\rho\|V\|) \left( \frac{V}{\|V \|} \right) \omega(X) d\mu_E d\mu_T \\
= \int_M -i_V d\omega(X) d\mu - \int_T \int_{\mathcal{W}^E(p)} d \log(\rho\|V\|)(V) \omega(X) \rho d\mu_E d\mu_T \\
= - \int_M [i_V d\omega + d \log(\rho\|V\|)(V) \omega] (X) d\mu := - \int_M \alpha(X) d\mu,
\]

where \( \alpha = i_V d\omega + d \log(\rho\|V\|)(V) \omega \) is a continuous 1-form in \( U \). We used the fact that \( \rho \) and \( V \) are differentiable along the \( \mathcal{W}^E \) leaves, which we know from the hypothesis on \( E \).

Remark 24. If \( V \) is \( C^1 \) then \( d \log(\rho\|V\|)(V) = \text{div}(V) \) and the formula above can be deduced directly (\( \text{vol} \) is the volume d-form):

\[
\int_M di_X \omega(V) d\mu = \int_M di_X \omega \wedge i_V \text{vol} = - \int_M \omega(X) di_V \text{vol} = \int_M \omega(X) \text{div}(V) d\mu.
\]

Step 2: The form \( \alpha = i_V d\omega + d \log(\rho\|V\|)(V) \omega \) is exact.

From the previous step we know that for every \( C^3 \) divergence free vector field \( X \) supported in \( U \) we have

\[
(9.1) \quad \int_U \alpha(X) d\mu = 0.
\]

The proof of this step is given by the following lemma.

Lemma 25. Let \( U \) be an open set in \( \mathbb{R}^d \), and \( \alpha \) a continuous 1-form in \( U \). Suppose that for every \( C^\infty \) divergence free vector field \( X \) supported in \( U \) we have \( \int_U \alpha(X) d\mu = 0 \).
Then $\alpha$ is exact, i.e. for every piecewise $C^1$ curve $\gamma \subset U$ we have
\[ \int_{\gamma} \alpha = 0. \]

**Proof.** First assume that $\gamma$ is a $C^\infty$ simple closed curve in $U$. There exists a tubular neighborhood $U_0$ of $\gamma$ which is $C^\infty$ diffeomorphic to $B_{\mathbb{R}^{n-1}}(0,1) \times \mathbb{T}^1$, and thus it is $C^\infty$ foliated by $C^\infty$ closed curves corresponding to the curves $\{x\} \times \mathbb{T}^1$, $x \in B_{\mathbb{R}^{n-1}}(0,1)$. Let $T$ be a $C^\infty$ transversal to the foliation of closed curves of the tubular neighborhood of $\gamma$.

Let $X_0$ be a $C^\infty$ vector field in $U_0$ tangent to the foliation of $U_0$ by closed curves, and such that the period of all the closed curves for the flow $\phi^{X_0}$ generated by $X_0$ is one (this can be done by pulling back under the diffeomorphisms the unit vector field in $B_{\mathbb{R}^{n-1}}(0,1) \times \mathbb{T}^1$ tangent to the $\{x\} \times \mathbb{T}^1$ curves).

We claim that we can rescale $X_0$ by a $C^\infty$ nonzero scalar function $f$ such that $fX_0$ is divergence free. The map $f$ must satisfy:
\[ \text{div}(fX_0) = f \text{div}(X_0) + df(X_0) = 0, \]
or
\[ d(\log f)(X_0) = -\text{div}(X_0). \]

We can take the initial conditions $f(p) = 1$ (or $\log f(p) = 0$) for all $p \in T$, and we get that for all $t \in [0,1]$
\[ \log f(\phi^t_{X_0}(p)) = -\int_0^t \text{div}(X_0)(\phi^s_{X_0}(p)) \, ds. \]

This formula defines the $f$ on $U_0$, but it may have a discontinuity at the return of the flow $\phi^{X_0}$ at the transversal $T$. The flow $\phi^t_{X_0}$ is periodic with period one, or $\phi^1_{X_0} = Id$, and $\det[D\phi^1_{X_0}](p) = 1$. From the Liouville formula we have that
\[ \det[D\phi^t_{X_0}(p)] = \det[D\phi^0_{X_0}(p)] \exp \left( \int_0^1 \text{div}(X_0)(\phi^s_{X_0}(p)) \, ds \right). \]

This implies that $\int_0^1 \text{div}(X_0)(\phi^s_{X_0}(p)) \, ds = 0$, so $f(p) = f(\phi^1_{X_0}(p))$, and that $f$ is well defined and $C^\infty$ on $U_0$, and $fX_0$ is a $C^\infty$ rescaling of $X_0$ which is divergence free.

Let $X := fX_0$ be the divergence free vector field tangent to the foliation of $U_0$ by closed curves. Let us comment that the election of $X$ is not unique, in fact given any $C^\infty$ scalar function $g$ which is constant on the closed leaves of the foliation, we have $\text{div}(gX) = 0$. Then without loss of generality we can assume that the period of $\phi^X$ on each closed curve $\gamma_p$ is one.
Up to rescaling, we can assume that the volume \( \mu \) restricted to \( U_0 \) is a probability invariant measure for \( \phi^{X_0} \). The ergodic decomposition of \( \mu \) for the flow generated by \( X \) in \( U_0 \) consists of the measures \( \nu_p \) supported on the closed curves \( \gamma_p \), \( p \in T \):

\[
\int_{\gamma_p} gd\nu_p = \int_0^1 g(\phi_t^X(p))dt, \quad \forall g \in C^0(U_0, \mathbb{R}),
\]

and

\[
\int_{U_0} gd\mu = \int_T \int_{\gamma_p} gd\nu_p dm_T, \quad \forall g \in C^0(U_0, \mathbb{R}),
\]

where \( m_T \) is the quotient measure on \( T \), which can be identified with the set of ergodic invariant measures for \( \phi^{X_0} \). It is easy to see that \( m_T \) has full support (it is in fact smooth if \( T \) is smooth).

Also observe that we have

\[
\int_{\gamma_p} \alpha(X) d\nu_p = \int_0^1 \alpha(X)(\phi_t^X(p))dt = \int_0^1 \phi^{X*}\alpha = \int_{\gamma_p} \alpha
\]

where \( \phi^{X*}\alpha \) is the pull back of \( \alpha \) by the map \( \phi^X : [0, 1] \to U_0 \).

Now we will apply this to (9.1) with \( X \) replaced by the divergence free \( gX \), where \( g \) is constant on the closed leaves and supported in \( U_0 \). We get

\[
0 = \int_{U_0} \alpha(gX)d\mu = \int_T g(p) \int_{\gamma_p} \alpha(X) d\nu_p dm_T = \int_T g(p) \int_{\gamma_p} \alpha dm_T.
\]

Since the above relation holds for any compactly supported \( C^\infty \ g : T \to \mathbb{R} \), and the map \( p \mapsto \int_{\gamma_p} \alpha \) is continuous, we have that \( \int_{\gamma_p} \alpha = 0 \) for all \( p \in T = \text{supp}(m_T) \), and in particular for the initial smooth simple closed curve \( \gamma \).

**Remark 26.** An alternative method to the use of the ergodic decomposition would be to use the explicit formula of densities \( \rho \) of the disintegrations of the volume along the curves of a flow generated by a \( C^1 \) vector field \( X \).

We continue with the proof of Lemma 25. We obtained that \( \int_{\gamma} \alpha = 0 \) for any simple closed \( C^\infty \) curve \( \gamma \). Now given any piecewise \( C^1 \) closed curve \( \gamma \), since \( \alpha \) is \( C^0 \), we can make a standard approximation of \( \gamma \) with (a sum of) \( C^\infty \) simple closed curves \( \gamma_k \), in the sense that \( \int_{\gamma_k} \alpha \to \int_{\gamma} \alpha \), and the conclusion of the lemma follows.

**Remark 27.** The converse of the Lemma 25 is also true, and is an immediate consequence of the Birkhoff Ergodic Theorem.

**Step 3: The integral of \( \alpha \) on \( \mathcal{W}^E \) curves.**
We want to apply the previous step to a piecewise \( C^1 \) simple closed curve formed by two \( W^E \) pieces and two \( W^F \) pieces (curves inside \( W^E \) and \( W^F \)). Suppose that we have such a “rectangle” formed by the points \( a, b, c, d \) with \( \gamma_{ab} \) a \( W^E \) piece between \( a \) and \( b \), \( \gamma_{bc} \) a \( W^F \) piece between \( b \) and \( c \), \( \gamma_{cd} \) a \( W^E \) piece between \( c \) and \( d \), and \( \gamma_{da} \) a \( W^F \) piece between \( d \) and \( a \).

The restriction of \( i_V d\omega \) to \( E = T W^E \) is zero, because \( V \) is generating \( E \) and \( i_V d\omega(V) = d\omega(V, V) = 0 \). Since \( \frac{V}{\|V\|} \) gives the arc length parametrization of \( \gamma_{ab} \), we have

\[
\int_{\gamma_{ab}} \alpha = \int_{\gamma_{ab}} i_V d\omega + \int_{\gamma_{ab}} d\log(\rho\|V\|)(V)\omega
= \int_{\gamma_{ab}} d\log(\rho\|V\|)(V)\omega \left( \frac{V}{\|V\|} \right) d\mu_E
= \int_{\gamma_{ab}} d\log(\rho\|V\|) \left( \frac{V}{\|V\|} \right) d\mu_E
= \log(\rho(b)\|V(b)\|) - \log(\rho(a)\|V(a)\|).
\]

Similarly we get that

\[
\int_{\gamma_{cd}} \alpha = \log(\rho(d)\|V(d)\|) - \log(\rho(c)\|V(c)\|).
\]

(9.2)

**Step 4: The integral of \( \alpha \) on \( W^F \) curves.**

The restriction of \( \omega \) on \( F = \ker \omega = T W^F \) is zero from the definition of \( \omega \). Then

\[
\int_{\gamma_{bc}} \alpha = \int_{\gamma_{bc}} i_V d\omega.
\]

Since \( F = \ker \omega \) is integrable, we can choose \( f, g : M \to \mathbb{R} \) such that \( \omega = fdg \) (take \( g \) constant on the leaves of \( W^F \) with \( dg \neq 0 \) and rescale by some \( f \)). If \( f \) and \( dg \) would be \( C^1 \) (this would happen if \( F \) would be \( C^2 \)), then \( d\omega = df \wedge dg = d\log f \wedge \omega \), and \( i_V d\omega|_F = d\log f \).

In our case \( F \) and \( \omega \) are only \( C^1 \), so \( g \) is \( C^1 \), and \( dg \) and \( f \) could be only continuous. We claim that however \( dg \) and \( f \) are \( C^1 \) along curves in \( W^F \) leaves, and the relation \( i_V d\omega|_F = d\log f \) still holds.

We will check the differentiability of \( dg \) along \( W^F \). Let \( X \) be a \( C^1 \) vector field in \( F \), then the curves of the flow \( \phi^X_t \) are inside \( W^F \). We have

\[
dg(p) = d(g \circ \phi^X_t)(p) = dg(\phi^X_t(p)) \cdot D\phi^X_t(p),
\]

or

\[
dg(\phi^X_t(p)) = dg(p) \cdot [D\phi^X_t(p)]^{-1}.
\]
Since $t \mapsto D\phi_t^X(p)$ is $C^1$, we get that $dg$ is $C^1$ along the curve $t \mapsto \phi_t^X(p)$, and then the same must hold for $f$ (and log $f$) since $\omega$ is also $C^1$, so $d(\log f)(X)$ is well defined along $W^F$.

Now assume that the flow generated by the vector field $X$ joins $b$ with $c$ in $W^F(b)$, or $\phi_t^X(b) = c$ for some $T > 0$ and $\gamma_{bc}$ is $\phi_t^X(b)$, $t \in [0, T]$. Then

$$i_V d\omega(X)(\phi_t^X(b)) = -i_X d\omega(V)(\phi_t^X(b)) = -\mathcal{L}_X \omega(V)(\phi_t^X(b))$$

$$= -\lim_{s \to 0} \phi_s^{* \omega, s} \omega - \omega(V)(\phi_t^X(b)) = -\lim_{s \to 0} \frac{f \circ \phi_s^X d(g \circ \phi_s^X) - fdg}{s}(V)(\phi_t^X(b))$$

$$= -\lim_{s \to 0} \frac{f(\phi_{t+s}^X(p)) - f(\phi_t^X(p))}{s} dg(V)(\phi_t^X(p)) = \frac{df(X)(\phi_t^X(p))}{f(\phi_t^X(p))} fdg(X)(\phi_t^X(p))$$

$$= -d(\log f)(X)(\phi_t^X(p))\omega(V)(\phi_t^X(p)) = -\frac{\partial}{\partial t} \log(f(\phi_t^X(p))).$$

We used the facts that $\omega(V) = 1$, $\mathcal{L}_X \omega = di_X \omega + i_X d\omega$ and $di_X \omega = 0$ since $X \in \ker \omega$. Then

$$\int_{\gamma_{bc}} \alpha = \int_{\gamma_{bc}} i_V d\omega = \int_0^T i_V d\omega(X)(\phi_t^X(b)) dt$$

$$= -\int_0^T \frac{\partial}{\partial t} \log(f(\phi_t^X(p))) dt = \log f(b) - \log f(c) = \log \frac{f(b)}{f(c)}.$$ 

Let $h_{bc}: W^E(b) \to W^E(c)$ be the holonomy between $W^E(b)$ and $W^E(c)$ along the $W^F$ foliation and homemotopic to $\gamma_{bc}$. Then $h_{bc}$ is $C^1$, and the Jacobian of $h_{bc}$ is given by

$$Dh_{bc}(b) \left( \frac{V(b)}{\|V(b)\|} \right) = Jh_{bc}(b) \frac{V(c)}{\|V(c)\|}.$$ 

Since $g \circ h_{bc} = g$ we get

$$dg(b)V(b) = d(g \circ h_{bc})V(b) = \|V(b)\| dg(c) Dh_{bc}(b) \left( \frac{V(b)}{\|V(b)\|} \right)$$

$$= Jh_{bc}(b) \left( \frac{V(b)}{\|V(c)\|} \right) dg(c)V(c)$$

Because $\omega = f dg$ and $\omega(V) = 1$ we have that $dg(V) = \frac{1}{f}$. Then we obtain

$$\frac{f(b)}{f(c)} = \frac{dg(c)V(c)}{dg(b)V(b)} = \frac{1}{Jh_{bc}(b)} \frac{\|V(c)\|}{\|V(b)\|},$$

so

$$\int_{\gamma_{bc}} \alpha = \log \left( \frac{\|V(c)\|}{\|V(b)\|} \right) - \log Jh_{bc}(b).$$ 

(9.3)
Similarly we get

\[ \int_{\gamma_{da}} \alpha = \log \left( \frac{\| V(a) \|}{\| V(d) \|} \right) - \log Jh_{da}(d) = \log \left( \frac{\| V(a) \|}{\| V(d) \|} \right) + \log Jh_{bc}(a), \]

since \( h_{da} = h_{bc}^{-1} \) and \( h_{bc}(a) = d \).

**Step 5: Concluding the invariance of disintegrations under holonomies.**

Now we put together (9.2), (9.3), (9.4), and we use that \( \alpha \) is locally exact. Then

\[ 0 = \int_{\gamma_{ab}} \alpha + \int_{\gamma_{bc}} \alpha + \int_{\gamma_{cd}} \alpha + \int_{\gamma_{da}} \alpha \]

\[ = \log(\rho(b)\| V(b) \|) - \log(\rho(a)\| V(a) \|) + \log \left( \frac{\| V(c) \|}{\| V(b) \|} \right) - \log Jh_{bc}(b) + \]

\[ + \log(\rho(d)\| V(d) \|) - \log(\rho(c)\| V(c) \|) + \log \left( \frac{\| V(a) \|}{\| V(d) \|} \right) + \log Jh_{bc}(a) \]

\[ = \log \left( \frac{Jh_{bc}(a)\rho(d)}{\rho(a)} \right) - \log \left( \frac{Jh_{bc}(b)\rho(c)}{\rho(b)} \right). \]

Assume that we fix two nearby local \( W^E \) leaves, \( W^E(b) \) and \( W^E(c) \), and let \( h_{bc} \) be the local holonomy between them along \( W^F \). Then for any \( a \in W^E(b) \), the expression

\[ \frac{Jh_{bc}(a)\rho(h_{bc}(a))}{\rho(a)} = k \]

is constant.

Remember that \( m_E(b) \) is the disintegration of \( \mu \) along the local \( W^E \) leaf passing through \( b \), and \( \rho(a) \) is the density of \( m_E(b) \) with respect to the Lebesgue measure on the local leaf, \( \mu_E(b) \). On the other hand, from the formula of change of variable, we have that \( Jh_{bc}(a)\rho(h_{bc}(a)) \) is the density of the pull-back under the holonomy \( h_{bc} \) (or the push-forward under \( h_{bc}^{-1} \)) of \( m_E(c) \), the disintegration of \( \mu \) along the local \( W^E \) leaf passing through \( c \).

Then the formula (9.5) shows that the holonomy along \( W^F \) preserves the disintegrations of \( \mu \) along \( W^E \) local leaves, modulo the multiplication by a constant. In general the disintegrations are well defined modulo multiplication with constants, due to the choice of the foliation chart, so if the local foliation chart is properly chosen, then the disintegrations will be invariant under the \( W^F \) holonomies. This concludes the proof of Theorem I.

\[ \square \]

**Remark 28.** Let us comment that even though we obtain the above result for the case when one foliation is one-dimensional, it seems very probable that the result should work...
for higher dimensional foliations too, at least under the condition that the two bundles are $C^1$ and integrable.

Proof of Corollary J. The stable and unstable bundles of an area preserving $C^\infty$ diffeomorphism are of class $C^{2-}$ by [53], so we can apply Theorem I and obtain that the disintegrations of the area along the unstable foliation are invariant under the stable holonomy. Since the disintegrations are smooth, meaning that the densities are uniformly $C^\infty$ along the unstable leaves, then the Jacobian of the stable holonomy between unstable leaves must be also uniformly $C^\infty$. Because the unstable leaves are one dimensional this implies that the stable holonomies between unstable leaves are uniformly $C^\infty$, and the Journé Lemma [34] gives that the stable foliation is $C^\infty$. Once we get the smoothness of the foliations, we can use [33] or [22] and we obtain the conclusion on rigidity.

□

Remark 29. We used several dynamical results in order to simplify the proof, however one can obtain a more general result. The property of being critical is in fact a property of the two transversal bundles, without considering any dynamics. One can show that if the bundles are $C^1$ and critical in the sense above, then there exists a diffeomorphism preserving the area and taking the two foliations into two foliations of the two-torus by straight lines.

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