Multipoint Cauchy problem for nonlinear wave equations in vector-valued spaces
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Abstract
In this paper, regularity properties, Strichartz type estimates to solutions of multipoint Cauchy problem for linear and nonlinear abstract wave equations in vector-valued function spaces are obtained. The equation includes a linear operator $A$ defined in a Hilbert space $H$, in which by choosing $H$ and $A$ we can obtain numerous classis of nonlocal initial value problems for wave equations which occur in a wide variety of physical systems.

Key Word: Wave equations, Positive operators, Semigroups of operators, local solutions
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1. Introduction, definitions

Consider the multipoint Cauchy problem for nonlinear abstract wave equations (NLAWE)

$$\partial_t^2 u - \Delta u + Au = F(u), \; x \in \mathbb{R}^n, \; t \in [0, T],$$

(1.1)

$$u(0, x) = \varphi(x) + \sum_{k=1}^{m} \alpha_k u(\lambda_k, x), \; \text{for a.e.} \; x \in \mathbb{R}^n,$$

(1.2)

$$u_t(0, x) = \psi(x) + \sum_{k=1}^{m} \beta_k u_t(\lambda_k, x), \; \text{for a.e.} \; x \in \mathbb{R}^n,$$

(1.3)

where $m$ is a positive integer, $\alpha_k, \; \beta_k$ are complex numbers, $\lambda_k \in (0, T]$, $A$ is a linear and $F$ is a nonlinear operators in a Hilbert space $E$, $\Delta$ denotes the Laplace operator in $\mathbb{R}^n$, $u = u(t, x)$ is the $E$-valued unknown function and $\varphi, \psi$ are data functions.

If we put $F(u) = \lambda |u|^p u$ in (1.1) we get the multipoint initial value problem for the following NLAWE

$$\partial_t^2 u - \Delta u + Au = \lambda |u|^p u, \; x \in \mathbb{R}^n, \; t \in [0, T],$$

(1.4)
where \( p \in (1, \infty) \), \( \lambda \) is a real number.

Let \( \mathbb{N} \), \( \mathbb{R} \) and \( \mathbb{C} \) denote the sets of all natural, real and complex numbers, respectively. For \( E = \mathbb{C} \), \( \alpha_k = \beta_k = 0 \) and \( A = 0 \) the problem (1.4) become the classical Cauchy problem for nonlinear wave equation \((\text{NWE})\)

\[
\frac{\partial^2 u}{\partial t^2} - \Delta u = \lambda |u|^{p-1} u, \quad x \in \mathbb{R}^n, \quad t \in [0, T],
\]

\[
u(0, x) = \varphi(x), \quad u_t(0, x) = \psi(x) \quad \text{for a.e.} \quad x \in \mathbb{R}^n.
\]

The existence of solutions and regularity properties of Cauchy problem for NWE studied e.g in [4], [6], [9], [14], [17], [19–22], [24], [27], [30–31], [35, 37], [39–42] and the references therein. In contrast, to the mentioned above results we will study the regularity properties of the abstract Cauchy problem (1.1). Abstract differential equations studied e.g. in [1–3], [7–8], [11], [13], [15–16], [18], [25], [28–29], [32–34], [41] and [43]. Since the Hilbert space \( \mathcal{H} \) is arbitrary and \( A \) is a possible linear operator, by choosing \( \mathcal{H} \) and \( A \) we can obtain numerous classes of wave equations and its systems which occur in a wide variety of physical systems.

Our main goal is to obtain the existence, uniqueness and Strichartz type estimates, i.e. estimates in the form of space time integrability to solution of (1.1) – (1.3). Strichartz type estimates to solutions of Cauchy problem for wave equations studied e.g in [10], [14], [20], [22], [24], [35], [37]. If we choose \( H \) a concrete space, for example \( H = L^2(\Omega) \), \( A = L \), where \( \Omega \) is a dom in \( \mathbb{R}^d \) with sufficiently smooth boundary, in variables \( y = (y_1, y_2, \ldots, y_d) \) and \( L \) is an elliptic operator in \( L^2(\Omega) \) in (1.2), then we obtain existence, uniqueness and the regularity properties of the mixed problem for linear wave equation

\[
\frac{\partial^2 u}{\partial t^2} - \Delta u + Lu = F(t, x, y), \quad x \in \mathbb{R}^n, \quad y \in \Omega,
\]

and for the NLS equation

\[
\frac{\partial^2 u}{\partial t^2} - \Delta u + Lu = F(u), \quad x \in \mathbb{R}^n, \quad y \in \Omega,
\]

where \( u = u(t, x, y) \).

Moreover, let we choose \( E = L^2(0, 1) \) and \( A \) to be differential operator with generalized Wentzell-Robin boundary condition defined by

\[
D(A) = \{ u \in W^{2,2}(0, 1), \quad B_j u = A u(j) = 0, \quad j = 0, 1 \},
\]

\[
A u = a u^{(2)} + b u^{(1)}
\]

where \( a = a(y) \) and \( b = b(y) \) are complex-valued functions. Then, from the main our theorem we get the existence, uniqueness and regularity properties of multipoint Wentzell-Robin type mixed problem for the following wave equation

\[
\frac{\partial^2 u}{\partial t^2} - \Delta u + a \frac{\partial^2 u}{\partial y^2} + b \frac{\partial u}{\partial y} = F(t, x),
\]

\[
B_j u = 0, \quad j = 0, 1.
\]
\[
\begin{align*}
\frac{\partial^2}{\partial t^2} u - \Delta u + a \frac{\partial^2 u}{\partial y^2} + b \frac{\partial u}{\partial y} &= F(u), \\
\end{align*}
\] (1.10)

where

\[
\begin{align*}
\frac{\partial^2}{\partial t^2} u_m - \Delta u_m + \sum_{j=1}^N a_{mj} u_j &= F_m(t, x), \quad t \in [0, T], \ x \in R^n, \\
\end{align*}
\] (1.11)

and the same for infinity many system of NWE equation

\[
\begin{align*}
\frac{\partial^2}{\partial t^2} u_m - \Delta u_m + \sum_{j=1}^N a_{mj} u_j &= F_m(u_1, u_2, \ldots u_N), \quad t \in [0, T], \ x \in R^n, \\
\end{align*}
\] (1.12)

where \(a_{mj}\) are complex numbers, \(u_j = u_j(t, x)\).

2. Definitions and Background

Let \(E\) be a Banach space. \(L^p(\Omega; E)\) denotes the space of strongly measurable \(E\)-valued functions that are defined on the measurable subset \(\Omega \subset R^n\) with the norm

\[
\|f\|_p = \|f\|_{L^p(\Omega; E)} = \left( \int_{\Omega} \|f(x)\|_E^p \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.
\]
Let \( H \) be a Hilbert space. For \( p = 2 \) and \( E = H \) the space \( L^p (\Omega; E) \) become the \( H \)-valued functions space \( L^2 (\Omega; H) \) with inner product:

\[
(f, g)_{L^2(\Omega; H)} = \int_{\Omega} (f(x), g(x))_H \, dx, \text{ for any } f, g \in L^2 (\Omega; H).
\]

Let \( L^q_tL^r_x (E) = L^q_tL^r_x ((a,b) \times \Omega; E) \) denotes the space of strongly measurable \( E \)-valued functions that are defined on the measurable set \((a,b) \times \Omega\) with the norm

\[
\|f\|_{L^q_tL^r_x ((a,b) \times \Omega; E)} = \left( \int_a^b \left[ \int_{\Omega} \|f(t,x)\|_E^r \, dx \right]^{\frac{q}{r}} \, dt \right)^{\frac{1}{q}}, \quad 1 \leq q, r < \infty.
\]

Let \( C (\Omega; E) \) denote the space of \( E \)-valued, bounded strongly continuous functions on \( \Omega \) with norm

\[
\|u\|_{C(\Omega; E)} = \sup_{x \in \Omega} \|u(x)\|_E.
\]

\( C^m (\Omega; E) \) will denote the spaces of \( E \)-valued bounded strongly continuous and \( m \)-times continuously differentiable functions on \( \Omega \) with norm

\[
\|u\|_{C^m(\Omega; E)} = \max_{0 \leq |\alpha| \leq m} \sup_{x \in \Omega} \|D^\alpha u(x)\|_E.
\]

Let \( E_1 \) and \( E_2 \) be two Banach spaces. \( B (E_1, E_2) \) will denote the space of all bounded linear operators from \( E_1 \) to \( E_2 \). For \( E_1 = E_2 = E \) it will be denoted by \( B (E) \).

A closed densely defined linear operator \( A \) is said to be absolutely positive in a Banach space \( E \) (see [11], § 11.2) if \( D(A) \) is dense on \( E \), the resolvent \( (A - \lambda^2 I)^{-1} \) exists for \( \text{Re} \lambda > \omega \) and

\[
\left\| (A - \lambda^2 I)^{-1} \right\|_{B(E)} \leq M_0 |\text{Re} \lambda - \omega|^{-1}.
\]

It is known [38, §115.1] that there exist fractional powers \( A^\theta \) of a absolutely positive operator \( A \). Let \( E (A^\theta) \) denote the space \( D(A^\theta) \) with the graphical norm

\[
\|u\|_{E(A^\theta)} = \left( \|u\|^p + \|A^\theta u\|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad 0 < \theta < \infty.
\]

For case of Hilbert space \( H \) and \( p = 2 \), \( E (A^\theta) \) will be denoted by \( H (A^\theta) \).

**Remark 1.1.** It is known that if the operator \( A \) is absolutely positive in a Banach space \( E \) and \( 0 \leq \alpha < 1 \) then it is an infinitesimal generator of group of bounded linear operator \( U_A (t) \) satisfying

\[
\|U_A (t)\|_{B(E)} \leq M e^{\omega|t|}, \quad t \in (-\infty, \infty),
\]
\[ \|A^n U_A(t)\|_{B(E)} \leq M |t|^{-\alpha}, \quad t \in (-\infty, \infty) \tag{2.1} \]

(see e.g. [29, § 1.6], Theorem 6.3).

Let \( E \) be a Banach space. \( S = S(R^n; E) \) denotes \( E \)-valued Schwartz class, i.e. the space of all \( E \)-valued rapidly decreasing smooth functions on \( R^n \) equipped with its usual topology generated by seminorms. \( S(R^n; \mathbb{C}) \) denoted by \( S \).

Let \( S'(R^n; E) \) denote the space of all continuous linear operators, \( L : S \to E \), equipped with the bounded convergence topology. Recall \( S(R^n; E) \) is norm dense in \( L^p(R^n; E) \) when \( 1 < p < \infty \).

Let \( F \) denotes the Fourier trasformation, \( \hat{u} = Fu \) and

\[ s \in \mathbb{R}, \quad \xi = (\xi_1, \xi_2, ..., \xi_n) \in R^n, \quad |\xi|^2 = \sum_{k=1}^{n} \xi_k^2; \]

\[ \langle \xi \rangle := \left( 1 + |\xi|^2 \right)^{\frac{1}{2}}. \]

Consider the \( E \)-valued Sobolev space \( W^{s,p}(R^n; E) \) and homogeneous Sobolev spaces \( W^{s,p}(R^n; E) \) defined by respectively,

\[ W^{s,p}(R^n; E) = \{ u : u \in S'(R^n; E), \| u \|_{W^{s,p}(R^n; E)} = \left\| F^{-1} \left( 1 + |\xi|^2 \right)^{\frac{s}{2}} \hat{u} \right\|_{L^p(R^n; E)} < \infty \}; \]

\[ \hat{W}^{s,p}(R^n; E) = \{ u : u \in S'(R^n; E), \| u \|_{\hat{W}^{s,p}(R^n; E)} = \left\| F^{-1} |\xi|^s \hat{u} \right\|_{L^p(R^n; E)} < \infty \}. \]

For \( \Omega = R^n \times G, \quad p = (p_1, p_2), s \in \mathbb{R} \) and \( l \in \mathbb{N} \) we define the \( E \)-valued anisotropic Sobolev space \( W^{s,l,p}(\Omega; E) \) by

\[ W^{s,l,p}(\Omega; E) := \{ u \in S'(\Omega; E) \| u \|_{W^{s,l,p}(\Omega; E)} = \| u \|_{W^{s,p}(\Omega)} + \| u \|_{W^{l,p}(\Omega)} \}, \]

where

\[ \| u \|_{W^{s,l,p}(\Omega; E)} = \left\| F^{-1} \left( 1 + |\xi|^2 \right)^{\frac{s}{2}} \hat{u} \right\|_{L^p(\Omega; E)} < \infty, \]

\[ \| u \|_{W^{l,p}(\Omega; E)} = \| u \|_{L^p(\Omega; E)} + \sum_{|\beta| = l} \| D_\beta^\alpha u \|_{L^p(\Omega; E)}. \]

The similar way, we define homogeneous anisotropic Sobolev spaces \( \hat{W}^{s,l,p}(\Omega; E) \) as:

\[ \hat{W}^{s,l,p}(\Omega; E) := \{ u \in S'(\Omega; E), \| u \|_{\hat{W}^{s,l,p}(\Omega; E)} = \| u \|_{W^{s,p}(\Omega; E)} + \| u \|_{\hat{W}^{l,p}(\Omega; E)} \}, \]

where

\[ \| u \|_{\hat{W}^{s,p}(\Omega; E)} = \left\| F^{-1} |\xi|^s \hat{u} \right\|_{L^p(\Omega; E)} < \infty. \]
Let $A$ be a linear operator in a Banach space $E$. Consider Sobolev-Lions type homogeneous and inhomogeneous abstract spaces, respectively

$$\dot{W}^{s,p}(R^n; E (A), E) = \dot{W}^{s,p}(R^n; E) \cap L^p(R^n; E (A)),$$

$$\|u\|_{\dot{W}^{s,p}(R^n; E (A), E)} = \|u\|_{\dot{W}^{s,p}(R^n; E)} + \|u\|_{L^p(R^n; E (A))} < \infty,$$

$$W^{s,p}(R^n; E (A), E) = W^{s,p}(R^n; E) \cap L^p(R^n; E (A)),$$

$$\|u\|_{W^{s,p}(R^n; E (A), E)} = \|u\|_{W^{s,p}(R^n; E)} + \|u\|_{L^p(R^n; E (A))} < \infty.$$

Sometimes we use one and the same symbol $C$ without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say $\alpha$, we write $C_{\alpha}$.

**Definition 1.1. (Solution).** A function $u : [0, T] \times R^n \rightarrow H (A)$ is called a (strong) solution to problem (1.1) – (1.3) if it lies in the class

$$C^1([0, T]; \dot{W}^{2, \gamma}(R^n; H (A))) \cap C^1([0, T]; \dot{W}^{2, \gamma}(R^n; H (A))),$$

for $\gamma \geq n \left( \frac{1}{2} - \frac{1}{k} \right)$, $k > 1$ and obeys the formula

$$u (t, x) = U_{\Delta + A} (t) \left[ \varphi (x) + \sum_{k=1}^{m} \alpha_k u (\lambda_k, x) \right] +$$

$$\tilde{U}_{\Delta + A} (t) \left[ \psi (x) + \sum_{k=1}^{m} \beta_k u (\lambda_k, x) \right] + \int_{0}^{t} \tilde{U}_{\Delta + A} (t - s) F (u (s)) ds \quad (2.2)$$

for all $t \in (0, T)$, where $U_{A + \Delta} (t)$ is a cosine, $\tilde{U}_{A + \Delta} (t)$ is a sine operator-functions (see e.g. [11]) with generator of $A + \Delta$, i.e.

$$U_{A + \Delta} (t) = \frac{1}{2} \left( e^{t(\Delta - A)^{\frac{1}{2}}} + e^{-t(\Delta - A)^{\frac{1}{2}}} \right), \quad (2.3)$$

$$\tilde{U}_{A + \Delta} (t) = \frac{1}{2} A^{-\frac{1}{2}} \left( e^{t(\Delta - A)^{\frac{1}{2}}} - e^{-t(\Delta - A)^{\frac{1}{2}}} \right).$$

We say that $u$ is a global solution if $T = \infty$.

We write $a \lesssim b$ to indicate that $a \leq C b$ for some constant $C$, which is permitted to depend on some parameters.

3. The exsistence of solution to multipoint Cauchy problem for linear wave equation
Consider the abstract Schrödinger equation
\[ \partial^2_t u - \Delta u + Au = 0, \ t \in [0, T], \ x \in \mathbb{R}^n, \]
where \( A \) is a linear operator in a Hilber space \( H \).

It can be shown that the fundamental solutions of the free abstract Schr"odinger equation (3.1) can be expressed as
\[ U_{A+\Delta} (t) (x, y) = C (t, A) U_\Delta (t) (x, y), \]
\[ \hat{U}_{A+\Delta} (t) (x, y) = S (t, A) U_\Delta (t) (x, y) \]
where \( C (t, A) \) is a cosine, \( S (t, A) \) is a sine operator-functions (see e.g. [11]) with generator of \( A \), i.e.
\[ C (t, A) = \frac{1}{2} \left( e^{tA}+e^{-tA} \right), \ S (t, A) = \frac{1}{2}A^{-\frac{1}{2}}\left( e^{tA} - e^{-tA} \right). \]
and \( U_\Delta (t) (x, y) = e^{\Delta t} (x, y) \) is a fundamental solution of the free wave equation:
\[ \partial^2_t u - \Delta u = 0, \ x \in \mathbb{R}^n, \ t \in [0, T], \]
i.e.
\[ U_\Delta (t) (x, y) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}, \ t \neq 0, \]
\[ U_\Delta (t) f (x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{-|x-y|^2}{4t}} f (y) \, dy. \]

Lemma 3.1. Let \( A \) be an absolute positive operator in a Banach space \( E \) and \( 0 \leq \alpha < 1 \). Then the following dispersive inequalities holds
\[ \| A^\alpha U_{A+\Delta} (t) f \|_{L^p_x (\mathbb{R}^n; E)} \lesssim t^{-n (\frac{1}{p} - \frac{1}{2}) + \alpha} \| f \|_{L^p_x (\mathbb{R}^n; E)}, \]  
\[ \| A^\alpha U_{A+\Delta} (t-s) f \|_{L^\infty_x (\mathbb{R}^n; E)} \lesssim |t-s|^{-n (\frac{1}{p} - \frac{1}{2}) + \alpha} \| f \|_{L^1_x (\mathbb{R}^n; E)} \]
for \( t \neq 0, \ 2 \leq p \leq \infty, \ \frac{1}{p} + \frac{1}{p'} = 1 \).

Proof. By using (3.3) and Young’s integral inequality we have
\[ \| U_\Delta (t) f \|_{L^p_x (\mathbb{R}^n; E)} \lesssim |t|^{-n (\frac{1}{p} - \frac{1}{2})} \| f \|_{L^p_x (\mathbb{R}^n; E)}, \]
\[ \| U_\Delta (t) f \|_{L^\infty_x (\mathbb{R}^n; E)} \lesssim |t|^{-\frac{n}{2p}} \| f \|_{L^1_x (\mathbb{R}^n; E)}. \]
By (1.10) we get
\[ \| A^\alpha U_A (t) \|_{B(E)} \lesssim |t|^{-\alpha}, \ t \neq 0. \]
By using then the properties of \( U_{A+\Delta} (t) = U_\Delta (t) U_A (t) \), the estimates (3.7) and (3.6) we obtain (3.4) and (3.5).
In this section, we make the necessary estimates to solution of the following Cauchy problem for the linear abstract wave equation

\[ u_{tt} - \Delta u + Au = F(t,x), \quad x \in \mathbb{R}^n, \quad t \in (0, \infty), \]  \hfill (3.7)

\[ u(0,x) = \varphi(x) + \sum_{k=1}^{m} \alpha_k u(\lambda_k, x), \quad \text{for a.e.} \ x \in \mathbb{R}^n, \]  \hfill (3.8)

\[ u_t(0,x) = \psi(x) + \sum_{k=1}^{m} \beta_k u_t(\lambda_k, x), \quad \text{for a.e.} \ x \in \mathbb{R}^n, \]  \hfill (3.9)

where \( A \) is a linear operator in a Hilbert space \( H \).

**Condition 3.1.** Assume:

1. \(|\alpha_k + \beta_k| > 0, \sum_{k,j=1}^{m} \alpha_k \beta_j \neq 0;\)
2. \( E \) is a Banach space;
3. \( A \) is absolute positive operator in a Banach space \( E \) and \( \gamma \geq \frac{n}{p} \) for \( p \in [1, \infty] \).

First we need the following lemma:

**Lemma 3.2.** Suppose the Condition 3.1 hold, \( \varphi \in \tilde{W}^{\gamma,p}(\mathbb{R}^n; E(A)) \) and \( \psi \in \tilde{W}^{\gamma-1,p}(\mathbb{R}^n; E(A)) \). Then problem (2.1) – (2.2) has a unique generalized solution.

**Proof.** By using of the Fourier transform we get from (3.1):

\[ \hat{u}_{tt}(t, \xi) + A_{\xi} \hat{u}(t, \xi) = \hat{F}(t, \xi), \]  \hfill (3.10)

\[ \hat{u}(0, \xi) = \hat{\varphi}(\xi) + \sum_{k=1}^{m} \alpha_k \hat{u}(\lambda_k, \xi), \]  \hfill (3.10)

\[ \hat{u}_t(0, \xi) = \hat{\psi}(\xi) + \sum_{k=1}^{m} \beta_k \hat{u}_t(\lambda_k, \xi) \quad \text{for a.e.} \ \xi \in \mathbb{R}^n. \]

where \( \hat{u}(t, \xi) \) is a Fourier transform of \( u(t, x) \) with respect to \( x \) and

\[ A_{\xi} = A + |\xi|^2, \quad \xi \in \mathbb{R}^n. \]

Consider the problem

\[ \hat{u}_{tt}(t, \xi) + A_{\xi} \hat{u}(t, \xi) = \hat{F}(t, \xi), \]  \hfill (3.11)

\[ \hat{u}(0, \xi) = u_0(\xi), \quad \hat{u}_t(0, \xi) = u_1(\xi), \quad \xi \in \mathbb{R}^n, \quad t \in [0, T], \]

where \( u_0(\xi), u_1(\xi) \in D(A) \) and for \( \xi \in \mathbb{R}^n \). By virtue of [11, §11.2, 11.4] we obtain that \( A_{\xi} \) is a generator of a strongly continuous cosine operator function
and the Cauchy problem (3.11) has a unique solution for all $\xi \in \mathbb{R}^n$, moreover, the solution can be expressed as

$$\dot{u} (t, \xi) = C (t, \xi, A) u_0 (\xi) + S \left( t, \xi, A \right) u_1 (\xi) + \int_0^t S \left( t - \tau, \xi, A \right) \hat{F} (\tau, \xi) d\tau, \ t \in (0, T),$$

where $C (t, \xi, A)$ is a cosine and $S (t, \xi, A)$ is a sine operator-functions generated by $A\xi$, i.e.

$$C (t, \xi, A) = \frac{1}{2} \left( e^{tA_{\xi}} + e^{-tA_{\xi}} \right), \quad S (t, \xi, A) = \frac{1}{2} A^{-\frac{1}{2}} \xi \left( e^{tA_{\xi}} - e^{-tA_{\xi}} \right).$$

Using the formula (3.12) and the condition (3.10) we get

$$u_0 (\xi) = \hat{\varphi} (\xi) + \sum_{k=1}^m \alpha_k [C (\lambda_k, \xi, A) u_0 (\xi) + S (\lambda_k, \xi, A) u_1 (\xi)] + \sum_{k=1}^m \alpha_k \int_0^\lambda S (\lambda_k - \tau, \xi, A) \hat{F} (\tau, \xi) d\tau, \ \tau \in (0, T).$$

Then,

$$\left[ I - \sum_{k=1}^m \alpha_k C (\lambda_k, \xi, A) \right] u_0 (\xi) - \sum_{k=1}^m \alpha_k S (\lambda_k, \xi, A) u_1 (\xi) = \sum_{k=1}^m \frac{\lambda_k}{\alpha_k} \int_0^\lambda S (\lambda_k - \tau, \xi, A) \hat{F} (\tau, \xi) d\tau + \hat{\varphi} (\xi).$$

Differentiating both sides of formula (3.12) we obtain

$$\dot{u}_t (t, \xi) = A_{\xi} S (t, \xi, A) u_0 (\xi) + C (t, \xi, A) u_1 (\xi) + \frac{1}{2} \hat{F} (t, \xi) + \int_0^t A S (t - \tau, \xi, A) \hat{F} (\tau, \xi) d\tau, \ t \in (0, \infty).$$

Using the above formula and the condition

$$\dot{u}_t (0, \xi) = \hat{\psi} (\xi) + \sum_{k=1}^m \beta_k \dot{u}_t (\lambda_k, \xi)$$

we obtain

$$u_1 (\xi) = \hat{\psi} (\xi) + \sum_{k=1}^m \beta_k \left[ A_{\xi} S (\lambda_k, \xi, A) u_0 (\xi) + C (\lambda_k, \xi, A) u_1 (\xi) \right] + \sum_{k=1}^m \beta_k \int_0^\lambda S (\lambda_k - \tau, \xi, A) \hat{F} (\tau, \xi) d\tau, \ \tau \in (0, T).$$
\[
\sum_{k=1}^{m} \beta_k \left[ \frac{1}{2} \tilde{g}(\lambda_k, \xi) + \int_{0}^{\lambda_k} S(\lambda_k - \tau, \xi, A) \hat{F}(\tau, \xi) d\tau \right].
\]

Thus,

\[
- \sum_{k=1}^{m} \beta_k A \xi S(\lambda_k, \xi, A) u_0(\xi) + \left[ I - \sum_{k=1}^{m} \beta_k C(\lambda_k, \xi, A) \right] u_1(\xi) =
\]

\[
\sum_{k=1}^{m} \beta_k \left[ \frac{1}{2} \tilde{g}(\lambda_k, \xi) + \int_{0}^{\lambda_k} S(\lambda_k - \tau, \xi, A) \hat{F}(\tau, \xi) d\tau \right] + \hat{\psi}(\xi).
\]  

(3.14)

Now, we consider the system of equations (3.13) and (3.14) in \( u_0(\xi) \) and \( u_1(\xi) \). The determinant of this system is

\[
D(\xi) = \begin{vmatrix}
\alpha_{11}(\xi) & \alpha_{12}(\xi) \\
\alpha_{21}(\xi) & \alpha_{22}(\xi)
\end{vmatrix},
\]

where

\[
\alpha_{11}(\xi) = I - \sum_{k=1}^{m} \alpha_k C(\lambda_k, \xi, A), \quad \alpha_{12}(\xi) = - \sum_{k=1}^{m} \alpha_k S(\lambda_k, \xi, A),
\]

\[
\alpha_{21}(\xi) = - \sum_{k=1}^{m} \beta_k A \xi S(\lambda_k, \xi, A), \quad \alpha_{22}(\xi) = I - \sum_{k=1}^{m} \beta_k C(\lambda_k, \xi, A).
\]

We find the determinant of the system (3.13)-(3.14):

\[
D(\xi) = I - \sum_{k=1}^{m} (\alpha_k + \beta_k) C(\lambda_k, \xi, A) + \sum_{k,j=1}^{m} \alpha_k \beta_j [C(\lambda_k, \xi, A)C(\lambda_j, \xi, A) - A \xi S(\lambda_k, \xi, A) S(\lambda_j, \xi, A)].
\]

By properties of operator functions \( C(\lambda, \xi, A) \) and \( S(\lambda, \xi, A) \) we get \( D(\xi) \neq 0 \). Solving the system (3.13) – (3.14), by using the property of sine and cosine operator function [11, §11.2, 11.4] we get

\[
u_0(\xi) = D^{-1}(\xi) \left[ \left( I - \sum_{k=1}^{m} \beta_k C(\lambda_k, \xi, A) \right) f_1 - \sum_{k=1}^{m} \alpha_k S(\lambda_k, \xi, A) f_2 \right],
\]

(3.15)

\[
u_1(\xi) = D^{-1}(\xi) \left[ \left( I - \sum_{k=1}^{m} \alpha_k C(\lambda_k, \xi, A) \right) f_2 + \sum_{k=1}^{m} \beta_k A \xi S(\lambda_k, \xi, A) f_1 \right],
\]

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where
\[
f_1 = \sum_{k=1}^{m} \alpha_k \int_{0}^{\lambda_k} S(\lambda_k - \tau, \xi, A) \hat{F}(\tau, \xi) \, d\tau + \hat{\varphi}(\xi),
\]
\[
f_2 = \sum_{k=1}^{m} \beta_k \left[ \frac{1}{2} \hat{F}(\lambda_k, \xi) + \int_{0}^{\lambda_k} S(\lambda_k - \tau, \xi, A) \hat{F}(\tau, \xi) \, d\tau \right] + \hat{\psi}(\xi). \tag{3.16}
\]

From (3.12), (3.15) and (3.16) we get that the solution of (3.10) can be expressed as
\[
\hat{u}(t, \xi) = D^{-1}(\xi) \left\{ \left[ C(t, \xi, A) \left( I - \sum_{k=1}^{m} \beta_k C(\lambda_k, \xi, A) \right) + S(t, \xi, A) \sum_{k=1}^{m} \beta_k A \xi S(\lambda_k, \xi, A) \right] f_1 + \left[ S(t, \xi, A) \left( I - \sum_{k=1}^{m} \alpha_k C(\lambda_k, \xi, A) \right) - C(t, \xi, A) \sum_{k=1}^{m} \alpha_k S(\lambda_k, \xi, A) \right] f_2 \right\}, \quad t \in (0, T). \tag{3.17}
\]

We obtain from (3.17) that there is a generalized solution of (3.10) given by
\[
u(t, x) = S_1(t, x, A) \varphi(x) + S_2(t, x, A) \psi(x) + \Phi(t, x, A), \tag{3.18}
\]
where $S_1(t, A)$ and $S_2(t, A)$ are linear operator functions in $E$ defined by
\[
S_1(t, x, A) \varphi = F^{-1}D^{-1}(\xi) \left[ C(t, \xi, A) \left( I - \sum_{k=1}^{m} \beta_k C(\lambda_k, \xi, A) \right) + S(t, \xi, A) \sum_{k=1}^{m} \beta_k A \xi S(\lambda_k, \xi, A) \right] \hat{\varphi}(\xi),
\]
\[
S_2(t, x, A) \psi = F^{-1}D^{-1}(\xi) \left[ -C(t, \xi, A) \sum_{k=1}^{m} \alpha_k S(\lambda_k, \xi, A) + S(t, \xi, A) \left( I - \sum_{k=1}^{m} \alpha_k C(\lambda_k, \xi, A) \right) \right] \hat{\psi}(\xi), \tag{3.19}
\]
\[
\Phi(t, x, A) = F^{-1}D^{-1}(\xi) \left\{ \left[ C(t, \xi, A) \left( I - \sum_{k=1}^{m} \beta_k C(\lambda_k, \xi, A) \right) \right] g_1(t) - \sum_{k=1}^{m} \alpha_k S(\lambda_k, \xi, A) g_2(\xi) \right\} + S(t, \xi, A) \left[ \sum_{k=1}^{m} \beta_k A \xi S(\lambda_k, \xi, A) \right] g_1(\xi) +
\]
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\[
\left( I - \sum_{k=1}^{m} \alpha_k C(\lambda_k, \xi, A) \right) \sum_{k=1}^{m} \beta_k A_\xi S(\lambda_k, \xi, A) g_2(\xi),
\]
here
\[
g_1(\xi) = \sum_{k=1}^{m} \alpha_k \int_0^{\lambda_k} S(\lambda_k - \tau, \xi, A) \hat{F}(\tau, \xi) d\tau,
\]
(3.20)
\[
g_2(\xi) = \sum_{k=1}^{m} \beta_k \left[ \frac{1}{2} \hat{F}(\lambda_k, \xi) + \int_0^{\lambda_k} S(\lambda_k - \tau, \xi, A) \hat{F}(\tau, \xi) d\tau \right].
\]

4. Strichartz inequalities for linear wave equation

The proof of Strichartz type estimates involves basically two type ingredients. The first one consists of specific estimates, in particular stationary phase estimates, on evolution groups associated with homogenous equations. The second one consists of abstract arguments, not specific to the wave equations. This is mainly duality argument and were first applied in [37].

**Condition 4.1.** Assume \( n > 1 \),
\[
\frac{1}{q} + \frac{n-1}{2r} < \frac{n-1}{4}, \quad 2 \leq q, r \leq \infty \quad \text{and} \quad (n, q, r) \neq (2, 2, \infty).
\]

**Remark 4.1.** If \( \frac{1}{q} + \frac{n-1}{2r} = \frac{n-1}{4} \), then \((q, r)\) is called sharp admissible, otherwise \((q, r)\) is called nonsharp admissible. Note in particular that when \( n > 2 \) the endpoint \((2, \frac{2(2n-2)}{n-3})\) is called sharp admissible.

For a space-time slab \([0, T] \times R^n\), we define the \( E \)-valued Strichartz norm
\[
\|u\|_{S^0([0,T];E)} = \sup_{(q,r) \ \text{admissible}} \|u\|_{L^q_t L^r_x([0,T] \times R^n; E)},
\]
where \( S^0 \) \((0, T); E\) is the closure of all \( E \)-valued test functions under this norm and \( N^0 \) \((0, T); E\) denotes the dual of \( S^0 \) \((0, T); E\).

Assume \( H \) is an abstract Hilbert space and \( Q \) is a \( H \)-valued Hilbert space of function. Suppose for each \( t \in \mathbb{R} \) an operator \( U(t) \): \( Q \rightarrow L^2(\Omega; E) \) obeys the following estimates:
\[
\|U(t)f\|_{L^2_t(\Omega; H)} \lesssim \|f\|_Q \tag{4.1}
\]
for all \( t, \Omega \subset R^n \) and all \( f \in Q \). Moreover,
\[
\|U(s) U^*(t)g\|_{L^2_t(\Omega; H)} \lesssim |t-s|^{-\frac{n-1}{2}} \|g\|_{L^2_t(\Omega; H)} \tag{4.2}
\]
\[
\|U(s) U^*(t)g\|_{L^2_t(\Omega; H)} \lesssim (1 + |t-s|^{-\frac{n-1}{2}}) \|g\|_{L^2_t(\Omega; H)} \tag{4.3}
\]
for all $t \neq s$ and all $g \in L^1_x(\Omega; H)$.

For proving the main theorem of this section, we will use the following bilinear interpolation result (see [5], Section 3.13.5(b)).

**Lemma 4.1.** Assume $A_0$, $A_1$, $B_0$, $B_1$, $C_0$, $C_1$ are Banach spaces and $T$ is a bilinear operator bounded from $(A_0 \times B_0, A_0 \times B_1, A_1 \times B_0)$ into $(C_0, C_1)$, respectively. Then whenever $0 < \theta_0, \theta_1 < \theta < 1$ are such that $1 \leq \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{s}$ and $\theta = \theta_0 + \theta_1$, the operator is bounded from

$$(A_0, A_1)_{\theta_0 \theta_1} \times (B_0, B_1)_{\theta_1 \theta_1}$$

to $(C_0, C_1)_{\theta r}$.

By following [22, Theorem 1.2] we have:

**Theorem 4.1.** Assume $U(t)$ obeys (4.1)-(4.3). Let $U(t)$ generates absolute positive infinitesimal generator operator $A$ and $0 \leq \alpha < 1$. Then the following estimates are hold

$$\|U(t)f\|_{L^1_t L^\infty_x(H)} \lesssim \|f\|_Q, \tag{4.4}$$

$$\left\| \int U^*(s) F(s) ds \right\|_Q \lesssim \|F\|_{L^1_t L^\infty_x(H)}, \tag{4.5}$$

$$\int_{s < t} \|A^\alpha U(t) U^*(s) F(s) ds\|_{L^1_t L^\infty_x(H)} \lesssim \|F\|_{L^1_t L^\infty_x(H)}, \tag{4.6}$$

for all sharp admissible exponent pairs $(q, r), (\tilde{q}, \tilde{r})$. Furthermore, if the decay hypothesis is strengthened to (4.3), then (4.4) – (4.6) hold for all admissible $(q, r), (\tilde{q}, \tilde{r})$.

**Proof. The first step:** Consider the nonendpoint case, i.e. $(q, r) \neq \left(2, \frac{2n - 1}{n - 1}\right)$ and will show firstly, the estimates (4.4), (4.5). By duality, (4.4) is equivalent to (4.5). By the $TT^*$ method, (4.4) is in turn equivalent to the bilinear form estimate

$$\left| \int \int \langle (A^\alpha U(s))^* F(s), (A^\alpha U(t))^* G(t) \rangle ds dt \right| \lesssim \|F\|_{L^1_t L^\infty_x(H)} \|G\|_{L^1_t L^\infty_x(H)} \tag{4.7}$$

By symmetry it suffices to show to the retarded version of (4.7)

$$|T(F, G)| \lesssim \|F\|_{L^1_t L^\infty_x(H)} \|G\|_{L^1_t L^\infty_x(H)}, \tag{4.8}$$

where $T(F, G)$ is the bilinear form defined by

$$T(F, G) = \int_{s < t} \int \langle (A^\alpha U(s))^* F(s), (A^\alpha U(t))^* G(t) \rangle ds dt$$
By real interpolation between the bilinear form of (4.1) and due to estimate (1.10) we get
\[
|\langle (A^2 U(s))^* F(s), (A^2 U(t))^* G(t) \rangle| \lesssim \|F(s)\|_{L^2_x} \|G(t)\|_{L^2_x}.
\]
By using the bilinear form of (4.2) and (1.10) we have
\[
|\langle (A^2 U(s))^* F(s), (A^2 U(t))^* G(t) \rangle| \lesssim \|F(s)\|_{L^2_x(\Omega; H)} \|G(t)\|_{L^2_x(\Omega; H)}.
\]
\[
|t-s|^{-\frac{n}{2}} \|F(s)\|_{L^2_x(\Omega; H)} \|G(t)\|_{L^2_x(\Omega; H)}.
\]
In a similar way, we obtain
\[
|\langle (A^2 U(s))^* F(s), (A^2 U(t))^* G(t) \rangle| \lesssim (4.9)
\]
\[
|t-s|^{-1-\beta(r,\tilde{r})} \|F(s)\|_{L^{q'}(\Omega; H)} \|G(t)\|_{L^{q'}(\Omega; H)},
\]
where \(\beta(r,\tilde{r})\) is given by
\[
\beta(r,\tilde{r}) = \frac{n}{2} - 1 - \frac{n}{2} \left( \frac{1}{r} - \frac{1}{\tilde{r}} \right).
\]
It is clear that \(\beta(r, r) \leq 0\) when \(n > 2\). In the sharp admissible case we have
\[
\frac{1}{q} + \frac{1}{q'} = -\beta(r, r),
\]
and (4.8) follows from (4.10) and the Hardy-Littlewood-Sobolev inequality ([20]) when \(q > q'\).

If we are assuming the truncated decay (4.3), then (4.10) can be improved to
\[
|\langle (A^2 U(s))^* F(s), (A^2 U(t))^* G(t) \rangle| \lesssim (4.12)
\]
\[
(1 + |t-s|)^{-1-\beta(r,\tilde{r})} \|F(s)\|_{L^{q'}(\Omega; H)} \|G(t)\|_{L^{q'}(\Omega; H)}
\]
and now Young’s inequality gives (4.8) when
\[
-\beta(r, r) + \frac{1}{q} > \frac{1}{q'},
\]
i.e. \((q, r)\) is nonsharp admissible. This concludes the proof of (4.4) and (4.5) for nonendpoint case.

**The second step:** It remains to prove (4.4) and (4.5) for the endpoint case, i.e. when
\[
(q, r) = \left(2, \frac{2(n-1)}{n-3}, n > 2\right).
\]
It suffices to show (4.8). By decomposing \( T(F,G) \) dyadically as \( \sum_j T_j(F,G) \), where the summation is over the integers \( \mathbb{Z} \) and

\[
T_j(F,G) = \int_{t^{-2^{l-1}} < s \leq t^{-2^j}} \langle (A^2 U(s))^* F(s), (A^2 U(t))^* G(t) \rangle ds dt
\]

we see that it suffices to prove the estimate (4.13)

\[
\sum_j |T_j(F,G)| \lesssim \|F\|_{L^2_t L_x^r(H)} \|G\|_{L^2_t L_x^r(H)}.
\]

For this aim, before we will show the following estimate

\[
|T_j(F,G)| \lesssim 2^{-j \beta(a,b)} \|F\|_{L^2_t L_x^{r_1}(H)} \|G\|_{L^2_t L_x^{r_2}(H)}
\]

for all \( j \in \mathbb{Z} \) and all \( \left( \frac{1}{r_1}, \frac{1}{r_2} \right) \) in a neighbourhood of \( \left( \frac{1}{r}, \frac{1}{r_1} \right) \). For proving (4.15) we will use the real interpolation of \( H \)-valued Lebesgue space and sequence spaces \( l_q^t(H) \) (see e.g. [38] \S 1.18.2 and 1.18.6). Indeed, by [38, \S 1.18.4] we have

\[
\left( L^2_t L^{p_0}_x(H), L^2_t L^{p_1}_x(H) \right)_{\theta,2} = L^2_t L^{p_2}_x(H)
\]

whenever \( p_0, p_1 \in [1, \infty] \), \( p_0 \neq p_1 \) and \( \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} \) and \( (l^{p_0}_q(H), l^{p_1}_q(H))_{\theta,1} = l^{p_1}_q(H) \) for \( s_0, s_1 \in \mathbb{R}, s_0 \neq s_1 \) and

\[
\frac{1}{s} = \frac{1 - \theta}{s_0} + \frac{\theta}{s_1},
\]

where

\[
l^q_s(H) = \left\{ u = \{u_j\}_{j=1}^\infty, u_j \in H \right\},
\]

\[
\|u\|_{l^q_s(H)} = \left( \sum_{j=1}^{\infty} 2^{jsq} \|u_j\|_H^q \right)^{\frac{1}{q}} < \infty.
\]

By (4.16) the estimate (4.15) can be rewritten as

\[
T : L^2_t L^2_x(H) \times L^2_t L^2_x(H) \to l^2_\infty(H)^{\beta(a,b)}
\]

where \( T = \{ T_j \} \) is the vector-valued bilinear operator corresponding to the \( T_j \).

We apply Lemma 3.2 to (4.17) with \( r = 1, p = q = 2 \) and arbitrary exponents \( a_0, a_1, b_0, b_1 \) such that

\[
\beta(a_0, b_1) = \beta(a_1, b_0) \neq \beta(a_0, b_0).
\]

Using the real interpolation space identities we obtain

\[
T : L^2_t L^{a_1,2}_x(E^*) \times L^2_t L^{b_1,2}_x(E^*) \to l^1_1(H)^{\beta(a,b)}
\]
for all \((a, b)\) in a neighbourhood of \((r, r)\). Applying this to \(a = b = r\) and using
the fact that \(L^r(H) \subset L^{r,2}(H)\) we obtain
\[
T : L^2 L_x^{a',2}(H) \times L^2 L_x^{b',2}(H) \to l^1(H)
\]
which implies (4.15).

We are now ready to state the Strichartz estimates to solution (3.7) \(-(3.9).

**Theorem 4.2.** Assume the Conditions 3.1 and 4.1 are satisfied and
\[
\frac 1 q + \frac n r = \frac n 2 - \gamma = \frac 1 q + \frac n r - 2, \quad 0 \leq \alpha < 1.
\]
Let
\[
\varphi \in \dot{W}^{2,\gamma}(\mathbb{R}^n; H(A)), \quad \psi \in \dot{W}^{2,\gamma-1}(\mathbb{R}^n; H(A)),
\quad F \in L^{\tilde{r}} \left( [0, T]; L^{\tilde{r}'}(\mathbb{R}^n; H) \right)
\]
and let \(u : [0, T] \times \mathbb{R}^n \to H(A)\) be a solution to (3.7 \- 3.9). Then
\[
\| A^\alpha u \|_{L^q([0, T]; L^r(\mathbb{R}^n; H))} + \| A^\alpha u \|_{C([0, T]; L^{r'}(\mathbb{R}^n; H))} +
\]
\[
\| A^\alpha \partial_t u \|_{C([0, T]; \dot{W}^{2,\gamma-1}(\mathbb{R}^n; H))} \leq \| A\varphi \|_{\dot{W}^{2,\gamma}(\mathbb{R}^n; H)} + \| A\psi \|_{\dot{W}^{2,\gamma-1}(\mathbb{R}^n; H)} + \| F \|_{L^{\tilde{r}'}([0, T]; L^{\tilde{r}'}(\mathbb{R}^n; H))}.
\]

**Proof.** By (3.18) \- (3.20) the solution of (3.7 \- 3.9) can be expressed as
\[
u(t, x) = S_1(t, x, A) \varphi(x) + S_2(t, x, A) \psi(x) + \Phi(t, x, A), \tag{4.19}
\]
where
\[
S_1(t, x, A) \varphi = F^{-1} D^{-1}(\xi) B_1(t, \xi, A) \hat{\varphi}(\xi),
\]
\[
S_2(t, x, A) \psi = F^{-1} D^{-1}(\xi) B_2(t, \xi, A) \hat{\psi}(\xi), \tag{4.20}
\]
here
\[
B_1(t, \xi, A) = \left[ C(t, \xi, A) \left( I - \sum_{k=1}^m \beta_k C(\lambda_k, \xi, A) \right) + S(t, \xi, A) \sum_{k=1}^m \beta_k A_\xi S(\lambda_k, \xi, A) \right], \tag{4.21}
\]
\[
B_2(t, \xi, A) = \left[ -C(t, \xi, A) \sum_{k=1}^m \alpha_k S(\lambda_k, \xi, A) + S(t, \xi, A) \left( I - \sum_{k=1}^m \alpha_k C(\lambda_k, \xi, A) \right) \right] .
\]
By the usual reduction using Littlewood-Paley theory we may assume that the spatial Fourier transform of $\varphi$, $\psi$, $F$ and $u$ are all localized in the annulus $\{||\xi|| \sim 2^j\}$ for some $j$ in a similar way as scular case (see Corollary 1.3 in [22] and Lemma 5.1 of [30] and the subsequent discussion). The cases $r = \infty$ or $\tilde{r} = \infty$ can also be treated by this argument, but the $H$-valued Lebesgue spaces $L^r_x$, $L^\infty_x$ must be replaced by their $H$-valued Besov space counterparts. By the gap condition, the estimate is scale invariant, and so we may assume $j = 0$. Now that frequency is localized, $(-\Delta + A)^{\frac{1}{2}}$ becomes an invertible smoothing operator, and we may replace the Sobolev norms $W^{2,\gamma}(R^n; H)$, $W^{2,\gamma-1}(R^n; H)$ with the $L^2(R^n; H)$ norm. Combining these reductions with (4.19) – (4.21), we see that (4.18) will follow from the estimates

$$
\| S_{t\pm}(t, x, A) \varphi \|_{C(L^2(R^n; H))} \lesssim \| \varphi \|_{L^2(R^n; H)},
$$

$$
\| S_{t\pm}(t, x, A) \varphi \|_{L^1_t(L^\infty_x(R^n; H))} \lesssim \| \varphi \|_{L^2(R^n; H)},
$$

$$
\| S_{t\pm}(t, x, A) \psi \|_{C(L^2(R^n; H))} \lesssim \| \psi \|_{L^2(R^n; H)},
$$

$$
\| S_{t\pm}(t, x, A) \psi \|_{L^1_t(L^\infty_x(R^n; H))} \lesssim \| \psi \|_{L^2(R^n; H)},
$$

(4.22)

$$
\left\| \int_{t>s} S_{t\pm}(t, x, A) S_{s\pm}^*(s, x, A) F(s) \right\|_{C(L^2(R^n; H))} \lesssim \| F \|_{L^2_t L^\infty_x},
$$

$$
\left\| \int_{t>s} S_{t\pm}(t, x, A) S_{s\pm}^*(s, x, A) F(s) \right\|_{L^1_t L^\infty_x(H)} \lesssim \| F \|_{L^2_t L^\infty_x}, \ i = 1, 2,
$$

where the truncated wave evolution operators $S_{t\pm}(t, x, A)$ are given by

$$
\hat{S}_{t\pm}(t, \xi, A) f(\xi) = \chi_{[0, T]}(t) \beta(\xi) \hat{S}(t, \xi, A)
$$

for some Littlewood-Paley cutoff function $\beta$ supported on $\{||\xi|| \sim 1\}$. Apply Theorem 4.1 with all of the above estimates (4.22) will follow from Theorem 4.1 with $\Omega = R^n$, $Q = L^2(R^n; H)$, once we show that operator functions $S_{t\pm}(t, x, A)$ obey the energy estimate (4.1) and the truncated decay estimate (4.3). Consider first, the nonendpoint case. By the method of $TT^*$ and in view of (4.20) – (4.21) it will follow once we prove

$$
\left\| \int_{s<t} A^\alpha S_{t\pm}(t-s, x, A) F(s) ds \right\|_{L^1_t L^\infty_x(H)} \lesssim \| F \|_{L^2_t L^\infty_x(H)},
$$

(4.23)

The energy estimate (3.4):

$$
\| U_{\Delta + A}(t) f \|_{L^2_x(H)} \lesssim \| f \|_{L^2_x(H)}
$$

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follows from Plancherel’s theorem, the untruncated decay estimate
\[ \| U_\Delta (t - s) f \|_{L^\infty_x(H)} \lesssim |t - s|^{-\frac{n}{2}} \| f \|_{L^1_x(H)}, \]
from the equality
\[ U_{\Delta + A} (t) f = U_\Delta (t) U_A (t) f, \]
the explicit representation of the wave evolution operator
\[ U_\Delta (t) f(x) = (2\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{\frac{i|x - y|^2}{2t}} f(y) dy, \]
and from the estimate (3.5). Due to properties of the operator \( A \), the groups \( U_{\Delta + A} (t) \),
by (4.20) – (4.21) and by the dispersive estimate (3.4) we have
\[ \left\| A^\alpha \Phi_i \right\|_E \lesssim \int_{s < t} \left\| A^\alpha S_{i \pm} (t - s, x, A) ds \right\|_{B(H)} \| F(s) \|_H ds \lesssim \int_{\mathbb{R}} |t - s|^{-n(\frac{1}{2} - \frac{1}{p}) - \alpha} \| F(s) \|_H ds, \]
where
\[ \Phi_i = \int_{s < t} S_{i \pm} (t - s, x, A) F(s) ds, \quad i = 1, 2. \]

Moreover, from above estimate by the Hardy-Littlewood-Sobolev inequality, we get
\[ \left\| A^\alpha \Phi_i \right\|_{L^q_t L^r_x (\mathbb{R}^{n+1}; H)} \lesssim \left\| \int_{\mathbb{R}} |t - s|^{-n(\frac{1}{2} - \frac{1}{p}) - \alpha} \| F(s) \|_{L^\infty_x(\mathbb{R}^n; H)} ds \right\|_{L^q_t(\mathbb{R})} \lesssim \| F \|_{L^q_t L^r_x (\mathbb{R}^{n+1}; H)} , \]
where
\[ \frac{1}{q_1} = \frac{1}{q} + \frac{1}{p} + \frac{1}{2} - \frac{\alpha}{n}. \]

The argument just presented also covers (4.24) in the case \( q = \tilde{q}, r = \tilde{r} \). It allows to consider the estimate in dualized form:
\[ \left\| \int_{s < t} \left\langle S_{i \pm} (t - s, x, A) F(s), G(t) \right\rangle ds dt \right\| \lesssim \| F \|_{L^q_t L^r_x (H)} \| G \|_{L^q_1 L^{r'} (H)} \] (4.25)
when
\[ \frac{1}{q_1} = \frac{1}{q} + \frac{1}{p} + \frac{1}{2} - \frac{\alpha}{n}. \]
The case $\tilde{q} = \infty, \tilde{r} = 2$ follows from (4.25). Now, consider the endpoint case, i.e. $(q, r) = \left(\frac{2n}{2n-2}, 2\right)$. It is suffices to show the following estimates

\begin{align}
\|A^\alpha S_{i\pm} (t-s, x, A) \phi\|_{L^2_t L^\infty_x(H)} &\lesssim \|A\phi\|_{W^{s,2}(\mathbb{R}^n; H)}, \quad (4.26) \\
\|A^\alpha S_{i\pm} (t-s, x, A) \phi\|_{C^0(L^2_t(H))} &\lesssim \|A\phi\|_{W^{s,2}(\mathbb{R}^n; H)},
\end{align}

\begin{align}
\|A^\alpha S_{i\pm} (t-s, x, A) \psi\|_{L^2_t L^\infty_x(H)} &\lesssim \|A\psi\|_{W^{r-1,2}(\mathbb{R}^n; H)}, \quad (4.27) \\
\|A^\alpha S_{i\pm} (t-s, x, A) \psi\|_{C^0(L^2_t(H))} &\lesssim \|A\psi\|_{W^{r-1,2}(\mathbb{R}^n; H)},
\end{align}

\begin{align}
\left\| \int_{s<t} A^\alpha S_{i\pm} (t-s, x, A) F(s) \, ds \right\|_{L^2_t L^\infty_x(H)} &\lesssim \|F\|_{L^{\tilde{q}_i'} L^{\tilde{r}_i'}(H)}, \quad (4.28) \\
\left\| \int_{s<t} A^\alpha S_{i\pm} (t-s, x, A) F(s) \, ds \right\|_{C^0 L^2_t(H)} &\lesssim \|F\|_{L^{\tilde{q}_i'} L^{\tilde{r}_i'}(H)}, \quad (4.29)
\end{align}

Indeed, applying Theorem 4.1 with the energy estimate

\[ \|S_{i\pm} (t-s, x, A) f\|_{L^2(\mathbb{R}^n; H)} \lesssim \|f\|_{L^2(\mathbb{R}^n; H)} \]

which follows from Plancherel’s theorem, the untruncated decay estimate (4.3) and by using of Lemma 4.1 we obtain the estimates (4.27) and (4.28). Let us temporarily replace the $C^0_t L^2_x(H)$ norm in estimates (4.26), (4.27) by the $L^2_t L^\infty_x(H)$. Then, all of the above the estimates will follow from Theorem 4.1, since we show that $S_{i\pm} (t, x, A)$ obey the energy estimate (4.1) and the truncated decay estimate (4.2). The estimate (4.1) is obtain immediate from Plancherel’s theorem, and (4.2) follows in a similar way as in [31, p. 223-224]. To show that the quantity

\[ G_i F(t) = \int_{s<t} A^\alpha S_{i\pm} (t-s, x, A) F(s) \, ds, \quad i = 1, 2 \]

is continuous in $L^2(\mathbb{R}^n; H)$, we use the the identity

\[ G_i F(t+\varepsilon) = S_{i\pm} (\varepsilon, x, A) G_i F(t) + G_i \left( \chi_{[t,t+\varepsilon]} F \right)(t), \]

the continuity of $S_{i\pm} (\varepsilon, x, A)$ as an operator on $L^2(\mathbb{R}^n; H)$, and the fact that

\[ \left\| \chi_{[t,t+\varepsilon]} F \right\|_{L^{\tilde{q}_i'} L^{\tilde{r}_i'}(H)} \to 0 \text{ as } \varepsilon \to 0. \]

From the estimates (4.26) – (4.29) we obtain (4.18) for endpoint case.

5. Strichartz type estimates for solution to nonlinear wave equation
For the Cauchy problem for scalar wave equation
\[ \partial_t^2 u - \Delta u = F(u), \quad x \in \mathbb{R}^n, \ t \in [0, T], \]
where nonlinearity \( F \in C^1 \) satisfies
\[ F(u) = O\left(\|u\|^k\right), \quad |u| |F_u(u)| \sim |F(u)|, \quad k > 1. \]

The question of how much regularity \( \gamma = \gamma(k, n) \) is needed to insure local well-posedness of this problem was addressed for higher dimensions and nonlinearities in [20]; and then almost completely answered in [24].

Let
\[ X = L^2(\mathbb{R}^n; H), \ Y = W^{2,2}(\mathbb{R}^n; H(A), H), \ H_j = (X,Y)_{\frac{2j+2}{4}}, j = 0, 1. \]

**Remark 5.1.** By using J. Lions-I. Petree result (see e.g [38, §1.8.]) we obtain that the map \( u \rightarrow u^{(j)}(t_0), t_0 \in [0, T] \) is continuous from \( W^{2,2}(0,T; X,Y) \) onto \( H_j \) and there is a constant \( C_1 \) such that
\[ \|u^{(j)}(t_0)\|_{H_j} \leq C_1 \|u\|_{W^{2,2}(0,T; X,Y)}, \quad 1 \leq p \leq \infty. \]

Consider the multipoint initial-value problem (1.1) – (1.3). By reasoning as [22, Corollary 9.1] we prove the following result:

**Theorem 5.1.** Assume: (1) Conditions 3.1 and 4.1 are satisfied;
(2) the function \( F : H_1 \rightarrow H \) is continuously differentiable and obeys the power type estimates
\[ F(u) = O\left(\|u\|^k\right), \quad |u| \|F_u(u)\| \sim |F(u)| \]
for some \( k > 1 \), where \( F_u(u) \) denotes the derivative of operator function \( F \) with respect to \( u \in H \) and here \( \|u\| = \|u\|_H \);
(3) \( n \geq 4, \gamma = \frac{n-3}{2(n-1)}, k_0 = \frac{(n+1)^2}{(n-1)^2+4}; \)
(4) \( \varphi \in \dot{W}^{2,\gamma}(\mathbb{R}^n; H(A)), \psi \in \dot{W}^{2,\gamma-1}(\mathbb{R}^n; H(A)). \)

Then for \( k \geq k_0 \) there is a \( T > 0 \) depending only on
\[ \|\varphi\|_{\dot{W}^{2,\gamma}(\mathbb{R}^n; H(A))} + \|\psi\|_{\dot{W}^{2,\gamma-1}(\mathbb{R}^n; H(A))} \]
and a unique weak solution \( u \) to (1.1) – (1.3) with
\[ u \in L^q(0,T; L^r_0(\mathbb{R}^n; H(A))), \]
where
\[ q_0 = \frac{2(n+1)}{n-3}, \quad r_0 = \frac{2(n^2-1)}{(n^2-1)+4}, \quad 0 \leq \alpha < 1. \]
In addition, the solution satisfies
\[ u \in C \left( [0, T] ; \tilde{W}^{2, \gamma} (\mathbb{R}^n; H (A)) \right) \cap C^1 \left( [0, T] ; \tilde{W}^{2, \gamma-1} (\mathbb{R}^n; H (A)) \right) \] (5.3)
and depends continuously on the data.

**Proof.** We apply the standard fixed point argument in the space
\[ V = V (T; M) = \{ u : u \in L^q ([0, T] ; L^r (\mathbb{R}^n; H (A))) , \| Au \|_{L^q_t L^r_x (H)} \leq M \} \]
with \( T \) and \( M \) to be determined. Then we will used the estimate (4.18). By (4.19), the problem of finding a solution \( u \) of (1.1) – (1.3) is equivalent to finding a fixed point of the mapping
\[ S (u) = S_1 (t, x, A, \varphi (x)) + S_2 (t, x, A, \psi (x)) + G (F (u)) , \] (5.4)
where \( S_i (t, x, A), i = 1, 2 \) are operator function defined by (4.20) – (4.21) and \( G (F (u)) \) defined by (3.20), where
\[ g_1 (\xi) = \sum_{k=1}^{m} \alpha_k \int_0^{\lambda_k} S (\lambda_k - \tau, \xi, A) \hat{F} (u) (\tau, \xi) d\tau , \] (5.5)
\[ g_2 (\xi) = \sum_{k=1}^{m} \beta_k \left[ \frac{1}{2} \hat{F} (u) (\lambda_k, \xi) + \int_0^{\lambda_k} S (\lambda_k - \tau, \xi, A) \hat{F} (u) (\tau, \xi) d\tau \right] . \]

Accordingly, we will find \( M, T \) so that \( S \) is a contraction on \( V(T, M) \). It will suffice to show that for all \( M \) there is a \( T > 0 \) so that
\[ \| S (u) - S (v) \|_V \leq \frac{1}{2} \| u - v \|_V . \] (5.6)

By (3.20) we get \( G (F (0)) = 0 \). It implies
\[ S (0) = S_1 (t, x, A, \varphi (x)) + S_2 (t, x, A, \psi (x)) . \]
So \( S(0) \) is finite by (4.18) applied to the homogeneous problem and by using the properties of functions \( S_1, S_2 \) with relation of the operator \( A \), i.e., the fact that \( S : V \to V \) follows by picking \( M \) large enough so
\[ \| S (0) \|_V \leq \frac{M}{2} . \] (5.7)
Again in view of (4.18) we have
\[ \| S (u) - S (v) \|_V = G (F (u) - F (v)) \lesssim \| F (u) - F (v) \|_{L^q_t L^r_x (H)} . \] (5.8)
The assumptions (5.2) give

\[ \|F(u) - F(v)\|_H = \left\| \frac{1}{\lambda} \int_0^\lambda F(\lambda u + (1 - \lambda) v) \, d\lambda \right\|_H = \left\| \int_0^\lambda (u - v) \nabla F(\lambda u + (1 - \lambda) v) \, d\lambda \right\|_H \lesssim \|u - v\|_H (\|u\|_H + \|v\|_H)^{k-1}. \]

Using this in (5.8) gives

\[ \|S(u) - S(v)\|_V \lesssim \|u - v\|_H (\|u\|_H + \|v\|_H)^{k-1} \|L^{q', L^{p'}}_V\|. \quad (5.9) \]

Moreover, by the generalized Hölder inequality we have

\[ \|u - v\|_H \|u\|_H + \|v\|_H)^{k-1} \|L^{q', L^{p'}}_V \leq (5.10) \]

\[ \|u - v\|_H \|u\|_H + \|v\|_H)^{k-1} \|L^{q', L^{p'}}_V \leq (5.11) \]

where \(1 \leq p < \infty\) is chosen so that

\[ \frac{1}{q'} = \frac{1}{q} + \frac{1}{q/(k-1)} + \frac{1}{p'} = \frac{1}{r} + \frac{1}{r/(k-1)}. \]

By the assumptions on \(u, v\) the estimate (5.10) simplifies to

\[ \|u\|_H + \|v\|_H)^{k-1} \|L^{q', L^{p'}}_V \lesssim T^\frac{1}{k-1} \|u - v\|_V. \quad (5.11) \]

Thus if we choose \(T\) so that \(T^\frac{1}{k-1} < 1\), then (5.9) and (5.11) give the desired contraction (5.6).

To obtain the regularity (5.3) for \(u\) we apply (5.11) with \(v = 0\) to obtain

\[ \|S(u)\|_{L^{q', L^{p'}}_V(H)} \leq T^\frac{1}{k-1} \|u\|_V < \infty, \]

and (5.3) follows from (4.18).

Finally, we need to show uniqueness. Suppose that we have two solutions \(u, v\) to (1.1) – (1.3) for time \([0, T^*]\) such that

\[ \|Au\|_{L^q([0, T^*]; L^r(R^n, H))} \leq M, \quad \|Av\|_{L^q([0, T^*]; L^r(R^n, H))} \leq M \]

for some \(M\). Choose \(0 < T \leq T^*\) such that \(T^\frac{1}{k-1} < 1\). By the above arguments (5.6) holds, which implies that \(u = v\) for time \([0, T]\). Since \(T\) depends only on \(M\), we may iterate this argument and obtain \(u = v\) for all times \([0, T^*]\).

6. The existence and uniqueness for the system of wave equation
Consider at first, the multipoint Cauchy problem for linear system of wave equations

\[
\partial_t^2 u_m - \Delta u_m + \sum_{j=1}^{N} a_{mj} u_j = F_j (t, x), \quad t \in [0, T], \ x \in \mathbb{R}^n, \quad (6.1)
\]

\[
u_m (0, x) = \varphi_m (x) + \sum_{k=1}^{m} \alpha_k u_m (\lambda_k, x), \quad \text{for a.e.} \ x \in \mathbb{R}^n,
\]

\[
\partial_t u_m (0, x) = \psi_m (x) + \sum_{k=1}^{m} \beta_k \partial_t u_m (\lambda_k, x), \quad \text{for a.e.} \ x \in \mathbb{R}^n,
\]

where \( u = (u_1, u_2, ..., u_N), \ u_j = u_j (t, x), \ a_{mj} \) are complex numbers. Let \( l_2 = l_2 (N) \) and \( l_2^2 = l_2^2 (N), \ N \in \mathbb{N} \) (see [38, §1.18]). Let \( A \) be the operator in \( l_2 (N) \) defined by

\[
D (A) = \left\{ u = \{u_j\} \in l_2 (N), \ \|Au\|_{l_2(N)} = \left( \sum_{m=1}^{N} |a_{mj} u_j|^2 \right)^{\frac{1}{2}} < \infty \right\},
\]

\[
A = [a_{mj}], \ a_{mj} = a_{jm}, \ s > 0, \ m, j = 1, 2, ..., N, \ N \in \mathbb{N}.
\]

It is clear that for \( N = m < \infty \) the space \( l_2 (N) \) conside with the finite dimensional vector space \( \mathbb{C}^m \).

From Theorem 4.2 we obtain the following result

**Theorem 6.1.** Assume the Conditions 4.1 are hold. Let

\[
|\alpha_k + \beta_k| > 0, \ \sum_{k,j=1}^{m} \alpha_k \beta_j \neq 0, \ 0 < \alpha < 1
\]

and

\[
\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - \gamma = \frac{1}{q} + \frac{n}{r} - 2.
\]

Suppose \( n \geq 1 \) and

\[
\varphi \in \tilde{W}^{\gamma,2} (\mathbb{R}^n; D (A)), \ \psi \in \tilde{W}^{\gamma-1,2} (\mathbb{R}^n; D (A)),
\]

\[
F \in L^{q'} ([0, T]; L^r (\mathbb{R}^n; l_2 (N))).
\]

Let \( u : [0, T] \times \mathbb{R}^n \to l_2 (N) \) be a solution to (6.1). Then

\[
\|A^\alpha u\|_{L^q ([0, T]; L^r (\mathbb{R}^n; l_2 (N)))} + \|A^\alpha u\|_{C ([0, T]; L^2 (\mathbb{R}^n; l_2 (N)))} + \tag{6.2}
\]

\[
\|A^\alpha \partial_t u\|_{C ([0, T]; \tilde{W}^{2, \gamma -1} (\mathbb{R}^n; l_2 (N)))} \lesssim \|A \varphi\|_{\tilde{W}^{2, 2, \gamma} (\mathbb{R}^n; l_2 (N))} + \|A \psi\|_{\tilde{W}^{2, \gamma-1} (\mathbb{R}^n; l_2 (N))} + \|F\|_{L^{q'} ([0, T]; L^r (\mathbb{R}^n; l_2 (N)))}.
\]
Proof. It is easy to see that A is a symmetric operator in $l_2(N)$ and other conditions of Theorem 4.2 are satisfied. Hence, from Theorem 4.2 we obtain the conclusion.

Consider now, the Cauchy problem (1.10). Let $A$ be the operator in $l_2(N)$ defined by

\[
D(A) = \left\{ u = \{u_j\} \in l_2^s, s > 0 \mid \|u\|_{l_2^s(N)} = \left( \sum_{j=1}^{N} |2^{aj}u_j|^2 \right)^{\frac{1}{2}} < \infty \right\},
\]

\[
A = [a_{mj}], a_{mj} = a_{jm}, s > 0, m, j = 1, 2, ..., N, \ N \in \mathbb{N},
\]

Let

\[
X(N) = L^2(R^n; l_2(N)), \ Y(N) = W^{2,2}(R^n; l_2^s(N), l_2(N)),
\]

\[
H_j(N) = (X(N), Y(N))_{\frac{1}{1+2j}, 2}, j = 0, 1,
\]

where $H_j(N)$ denote the real interpolation spaces between $X(N)$ and $Y(N)$.

Remark 6.1. It is known that (see e.g. [38, § 1.18] the real interpolation spaces $(l_2^s(N), l_2^s(N))_{\theta, 2}, \ \theta \in (0, 1)$ between $l_2^s(N)$ and $l_2(N)$ defined as

\[
(l_2^s(N), l_2(N))_{\theta, 2} = l_2^{s(1-\theta)}(N).
\]

So, it can be shown that

\[
H_j(N) = W^{2(1-\theta), 2}(R^n; l_2^{s(1-\theta)}(N), l_2(N)).
\]

We obtain from Theorem 5.1 the following result

Theorem 6.2. Assume: (1) the function $F : l_2^{s/4}(N) \to l_2(N)$ is continuously differentiable and obeys the power type estimates

\[
F(u) = O(\|u\|^k_{l_2}), \ |u|_{l_2} \|F_u(u)\|_{l_2} \sim \|F(u)\|_{l_2}
\]

for some $k > 1$;

(2) $n \geq 4, \ \gamma = \frac{n-3}{2(n-1)}, \ k = \frac{(n+1)^2}{(n-1)^2+4}$;

(3) $\varphi \in W^{2,\gamma}(R^n; l_2^s(N)), \ \psi \in W^{2,\gamma-1}(R^n; l_2^s(N))$;

(4) assume the Conditions 4.1 are hold and

\[
|\alpha_k + \beta_j| > 0, \ \sum_{k, j=1}^{m} \alpha_k \beta_j \neq 0,
\]

\[
\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - \gamma = \frac{1}{q} + \frac{n}{r} - 2.
\]

Then there is a $T > 0$ depending only on

\[
\|\varphi\|_{W^{2,\gamma}(R^n; l_2^s(N))} + \|\psi\|_{W^{2,\gamma-1}(R^n; l_2^s(N))}
\]
and a unique weak solution $u$ to (1.12) – (1.9) with

$$u \in L^{q_0} ([0, T]; L^{r_0} (R^n; l^s_2 (N))) ,$$

where

$$q_0 = \frac{2(n+1)}{n-3}, \quad r_0 = \frac{2(n^2-1)}{(n^2-1)+4} .$$

In addition, the solution satisfies

$$u \in C \left( [0, T]; \dot{W}^{2, \gamma} (R^n; l^s_2 (N)) \right) \cap C^1 \left( [0, T]; \dot{W}^{2, \gamma-1} (R^n; l^s_2 (N)) \right)$$

and depends continuously on the data.

**Proof.** It is easy to see that $A$ is a symmetric operator in $l^2$ and other conditions of Theorem 5.1 are satisfied. Hence, from Theorem 5.1 we obtain the conclusion.

7. The existence and uniqueness of solution to anisotropic wave equation

Let $\Omega = R^n \times G, G \subset R^d, d \geq 2$ is a bounded domain with $(d-1)$-dimensional boundary $\partial G$. Consider at first, the multipoint mixed problem for the following wave equation

$$\partial_t^2 u - \Delta_x u + \sum_{|\alpha| \leq 2l} a_\alpha(y) D_\alpha u = F(t, x), \quad x \in R^n, \ y \in G, \ t \in [0, T], \ p \geq 0,$$

$$B_j u = \sum_{|\beta| \leq l_j} b_{j\beta}(y) D_\beta u = 0, \ x \in R^n, \ y \in \partial G, \ j = 1, 2, ..., m, \quad (7.2)$$

$$u(0, x, y) = \varphi(x, y) + \sum_{k=1}^m \alpha_k u(\lambda_k, x, y), \ \text{for a.e.} \ x \in R^n, \ y \in G, \quad (7.3)$$

$$\partial_t u(0, x, y) = \psi(x, y) + \sum_{k=1}^m \beta_k \partial_t u(\lambda_k, x, y), \ \text{for a.e.} \ x \in R^n, \ y \in G,$$

where $u = u(t, x, y)$ is a solution, $a_\alpha, b_{j\beta}$ are the complex valued functions, $\lambda = \pm 1, \ \alpha = (\alpha_1, \alpha_2, ..., \alpha_d), \ \beta = (\beta_1, \beta_2, ..., \beta_d), \ \mu_i < 2l$ and

$$D_x^k = \frac{\partial^k}{\partial x^k}, \ D_j = -i \frac{\partial}{\partial y_j}, \ D_y = (D_1, ..., D_d), \ y = (y_1, ..., y_d).$$

Let

$$\xi' = (\xi_1, \xi_2, ..., \xi_{d-1}) \in R^{d-1}, \ \alpha' = (\alpha_1, \alpha_2, ..., \alpha_{d-1}) \in Z^{d-1},$$
\[ A(y_0, \xi', D_y) = \sum_{|\alpha'| + |\beta| \leq 2l} a_{\alpha'}(y_0) \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_{d-1}^{\alpha_{d-1}} D_y^\beta \text{ for } y_0 \in \bar{G} \]

\[ B_j(y_0, \xi', D_y) = \sum_{|\beta'| + |\beta| \leq l_j} b_{j\beta'}(y_0) \xi_1^{\beta_1} \xi_2^{\beta_2} \cdots \xi_{d-1}^{\beta_{d-1}} D_y^\beta \text{ for } y_0 \in \partial G. \]

For \( \Omega = \mathbb{R}^n \times G, s \in \mathbb{R} \) and \( l \in \mathbb{N} \) let \( \tilde{W}^{s,l,p}(\Omega) = W^{s,l,p}(\Omega; \mathbb{C}). \)

From Theorem 4.2 we obtain the following result

**Theorem 7.1.** Assume the following conditions be satisfied:

1. \( G \in C^2, a_\alpha \in C(\bar{G}) \) for each \( |\alpha| = 2l \) and \( a_\alpha \in L_{\infty}(G) \) for each \( |\alpha| < 2l; \)
2. \( b_{j\beta} \in C^{2l-l_j}(\partial G) \) for each \( j, \beta \) and \( l_j < 2l, \sum_{j=1}^l b_{j\beta}(y') \sigma_j \neq 0, \)
   for \( |\beta| = m, y' \in \partial G, \) where \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_d) \in \mathbb{R}^d \) is a normal to \( \partial G; \)
3. \( y \in \bar{G}, \xi \in \mathbb{R}^d, \mu \in S(\varphi_0) \) for \( 0 \leq \varphi_0 < \pi, |\xi| + |\mu| \neq 0 \) let \( \mu + \sum_{|\alpha| = 2l} a_\alpha(y) \xi^\alpha \neq 0; \)
4. \( \mu + A(y_0, \xi', D_y) \vartheta(y) = 0, \)
5. \( B_j(y_0, \xi', D_y) \vartheta(0) = h_j, j = 1, 2, \ldots, l \)
has a unique solution \( \vartheta \in C_0(\mathbb{R}_+) \) for all \( h = (h_1, h_2, \ldots, h_d) \in \mathbb{C}^d \) and for \( \xi' \in \mathbb{R}^{d-1}; \)
6. assume the Conditions 4.1 are hold and

\[
\left| \alpha_k \right| + \left| \beta_k \right| > 0, \sum_{k,j=1}^m \alpha_k \beta_j \neq 0, 0 \leq \alpha < 1,
\]

\[
\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - \gamma = \frac{1}{q} + \frac{n}{r} - 2;
\]

6. \( n \geq 1 \) and

\[
\varphi \in \tilde{W}^{\gamma,2}(\mathbb{R}^n; \mathbb{W}^{2,2}(G)), \psi \in \tilde{W}^{\gamma-1,2}(\mathbb{R}^n; \mathbb{W}^{2,2}(G)),
\]

\[
F \in L^{\tilde{q}}\left( [0,T]; L^{r'}(\mathbb{R}^n; \mathbb{L}^2(G)) \right).
\]

Let \( u : [0,T] \times \mathbb{R}^n \to L^2(G) \) be a solution to (5.1). Then

\[
\|A^\alpha u\|_{L^q([0,T]; L^r(R^n; L^2(G)))} + \|A^\alpha u\|_{C([0,T]; L^2(R^n; L^2(G)))} +
\]

\[
\|A^\alpha \partial_t u\|_{C([0,T]; W^{2,\gamma-1}(R^n; L^2(G)))} \lesssim \|A\varphi\|_{\tilde{W}^{2,\gamma}(R^n; L^2(G))} +
\]

\[
\|A\psi\|_{\tilde{W}^{2,\gamma-1}(R^n; L^2(G))} + \|F\|_{L^{r'}([0,T]; L^{r'}(R^n; L^2(G)))}.
\]

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Theorem 4.2 implies the assertion.

Then the problem (7.1) – (7.3) can be rewritten as the problem (3.7) – (3.9), where \( u(x) = u(x,.,), f(x) = f(x,.) \), \( x \in \mathbb{R}^n \) are the functions with values in \( H = L^2(G) \). By virtue of [8, Theorem 8.2], operator \( A \) is absolute positive in \( L^2(G) \). Moreover, in view of (1)–(6) all conditions of Theorem 4.2 are hold. Then Theorem 4.2 implies the assertion.

Consider now, the multipoint mixed problem for nonlinear wave equation

\[
\partial_t^2 u + \Delta x u + \sum_{|\alpha| \leq 2l} a_\alpha (y) D_y^\alpha u = F(u),
\]

\( x \in \mathbb{R}^n, y \in G, t \in [0,T], p \geq 0, \) (7.4)

\[
B_j u = \sum_{|\beta| \leq l_j} b_{j\beta} (y) D_y^\beta u = 0, x \in \mathbb{R}^n, y \in \partial G, j = 1,2, ..., l, \]

(7.5)

\[
u(0,x,y) = \varphi(x,y) + \sum_{k=1}^{m} \alpha_k u(\lambda_k,x,y), \text{ for a.e. } x \in \mathbb{R}^n, y \in G \]

(7.6)

\[
\partial_t u (0,x,y) = \psi(x,y) + \sum_{k=1}^{m} \beta_k \partial_\nu u(\lambda_k,x,y), \text{ for a.e. } x \in \mathbb{R}^n, y \in G.
\]

From Theorem 5.1 we obtain

**Theorem 7.2.** Assume the following conditions be satisfied:

1. \( G \in C^2, a_\alpha \in C(\overline{G}) \) for each \( |\alpha| = 2l \) and \( a_\alpha \in L_\infty(G) \) for each \( |\alpha| < 2m; \)
2. \( b_{j\beta} \in C^{2m-2l_j}(\partial G) \) for each \( j, \beta \) and \( m_j < 2l, \sum_{j=1}^{l} b_{j\beta}(y') \sigma_j \neq 0, \) for \( |\beta| = l_j, y' \in \partial G, \) where \( \sigma = (\sigma_1, \sigma_2, ..., \sigma_d) \in \mathbb{R}^d \) is a normal to \( \partial G; \)
3. for \( y \in \overline{G}, \xi \in \mathbb{R}^d, \mu \in S(\varphi_0) \) for \( 0 \leq \varphi_0 < \pi, |\xi| + |\mu| \neq 0 \) let \( \mu + \sum_{|\alpha| = 2d} a_\alpha (y) \xi^\alpha \neq 0; \)
4. for each \( y_0 \in \partial G \) local BVP in local coordinates corresponding to \( y_0: \)

\[
\mu + A(y_0, \xi', D_{y}) \vartheta(y) = 0,
\]

\[
B_j (y_0, \xi', D_y) \vartheta(0) = h_j, j = 1,2, ..., l
\]

has a unique solution \( \vartheta \in C_0(\mathbb{R}_+) \) for all \( h = (h_1, h_2, ..., h_d) \in \mathbb{C}^d \) and for \( \xi' \in \mathbb{R}^{d-1}; \)

\[
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\]
(5) assume the Condition 4.1 are hold and and

\[ |\alpha_k + \beta_k| > 0, \sum_{k,j=1}^m \alpha_k \beta_j \neq 0, \]

\[ \frac{1}{q} + \frac{n}{r} = \frac{n}{2} - \gamma = \frac{1}{q} + \frac{n}{r} - 2; \]

(6) the function \( F : W^{\frac{n}{2},2}(G) \to L^2(G) \) is continuously differentiable and obeys the power type estimates

\[ F(u) = O \left( \|u\|_{L^2(G)}^k \right), \|u\|_{L^2(G)} \|F_u(u)\|_{L^2(G)} \sim \|F(u)\|_{L^2(G)} \]

for some \( k > 1; \)

(7) \( n \geq 4, \gamma = \frac{n-3}{2(n-1)}, k = \frac{(n+1)^2}{(n-1)^2+4}, \)

(8) \( \varphi \in W^{2,\gamma}(\mathbb{R}^n; W^{2,2}(G)) \) and \( \psi \in \tilde{W}^{2,\gamma-1}(\mathbb{R}^n; W^{2,2}(G)) \).

Then there is a \( T > 0 \) depending only on

\[ \|\varphi\|_{\tilde{W}^{2,\gamma}(\mathbb{R}^n; W^{2,2}(G))} + \|\psi\|_{\tilde{W}^{2,\gamma-1}(\mathbb{R}^n; W^{2,2}(G))} \]

and a unique weak solution \( u \) to (7.4) – (7.6) with

\[ u \in L^{q_0} \left( [0, T] ; L^{r_0} \left( \mathbb{R}^n; W^{2,2}(G) \right) \right), \]

where

\[ q_0 = \frac{2(n+1)}{n-3}, r_0 = \frac{2(n^2-1)}{(n^2+4)}. \]

In addition, the solution satisfies

\[ u \in C \left( [0, T]; \tilde{W}^{2,\gamma}(\mathbb{R}^n; W^{2,2}(G)) \right) \cap C^1 \left( [0, T]; \tilde{W}^{2,\gamma-1}(\mathbb{R}^n; W^{2,2}(G)) \right) \]

and depends continuously on the data.

**Proof.** The problem (7.4) – (7.6) can be rewritten as the problem (1.1), where \( u(x) = u(x,.) \), \( f(x) = f(x,.) \), \( x \in \mathbb{R}^n \) are the functions with values in \( H = L^2(G) \). By virtue of [8, Theorem 8.2], operator \( A + \mu \) is absolute positive in \( L^2(G) \) for sufficiently large \( \mu > 0 \). Moreover, in view of (1)-(8) all conditions of Theorem 5.1 are hold. Then Theorem 5.1 implies the assertion.

8. The Wentzell-Robin type mixed problem for wave equations

Consider at first, the linear problem (1.7) – (1.9). From Theorem 4.2 we obtain the following result

**Theorem 8.1.** Suppose the the following conditions are satisfied:
(1) $a$ is positive, $b$ is a real-valued functions on $(0, 1)$. Moreover, $a(.) \in C(0, 1)$ and
\[
\exp \left( - \int \frac{x}{b(t)} a^{-1} (t) \, dt \right) \in L_1(0, 1);
\]

(2) assume the Conditions 4.1 are hold and
\[
|\alpha_k + \beta_k| > 0, \sum_{k,j=1}^m \alpha_k \beta_j \neq 0, 0 \leq \alpha < 1,
\]
\[
\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - \gamma = \frac{1}{q} + \frac{n}{r} - 2;
\]

Let $n \geq 1$ and
\[
\varphi \in \dot{W}^{\gamma, 2}(\mathbb{R}^n; W_2^2(0, 1)), \psi \in \dot{W}^{\gamma^{-1}, 2}(\mathbb{R}^n; W_2^2(0, 1)),
\]
\[
F \in L_{\tilde{q}'}([0, T]; L_{\tilde{r}'}(\mathbb{R}^n; L_2^2(0, 1))).
\]

Let $u : [0, T] \times \mathbb{R}^n \to L_2^2(0, 1)$ be a solution to (1.7) – (1.9). Then
\[
\|A^\alpha u\|_{L^q([0, T]; L^r(\mathbb{R}^n; L_2^2(0, 1)))} + \|A^\alpha u\|_{C([0, T]; L^2(\mathbb{R}^n; L_2^2(0, 1)))} +
\]
\[
\|A^\alpha \partial_t u\|_{C([0, T]; \dot{W}^{2^{-}\gamma - 1} \mathbb{R}^n; L_2^2(0, 1))} \lesssim \|A\varphi\|_{\dot{W}^{2^{-}\gamma} \mathbb{R}^n; L_2^2(0, 1)} +
\]
\[
\|A\psi\|_{\dot{W}^{2^{-}\gamma - 1} \mathbb{R}^n; L_2^2(0, 1))} + \|F\|_{L_{\tilde{q}'}([0, T]; L_{\tilde{r}'}(\mathbb{R}^n; L_2^2(0, 1)))}.
\]

**Proof.** Let $H = L_2^2(0, 1)$ and $A$ is a operator defined by (4.1). Then the problem (1.7) – (1.9) can be rewritten as the problem (1.2). By virtue of [12, 23] the operator $A$ generates analytic semigroup in $L_2^2(0, 1)$. Hence, by virtue of (1)-(2) all conditons of Theorem 4.2 are satisfied. Then Theorem 4.2 implies the assertion.

Consider now, the problem (1.10) – (1.8) – (1.9). In this section, from Theorem 5.1 we obtain the following result:

**Theorem 8.2.** Suppose the the following conditions are satisfied:
(1) $a$ is positive, $b$ is a real-valued functions on $(0, 1)$. Moreover, $a(.) \in C(0, 1)$ and
\[
\exp \left( - \int \frac{x}{b(t)} a^{-1} (t) \, dt \right) \in L_1(0, 1);
\]

(2) the function $F : W^{k, 2}(0, 1) \to L_2^2(0, 1)$ is continuously differentiable and obeys the power type estimates
\[
F(u) = O \left( \|u\|_L^k_{L^2(0, 1)} \right), \|u\|_{L^2(0, 1)} \|F_u (u)\|_{L^2(0, 1)} \sim \|F(u)\|_{L^2(0, 1)}.
\]

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for some $k > 1$;
\begin{align*}
(3) \quad n \geq 4, \quad \gamma = \frac{n-3}{2(n-1)}, \quad k = \frac{(n+1)^2}{(n-1)^2 + 4}, \\
(4) \quad \varphi \in \dot{W}^{2,\gamma} (R^n; W^{2,2} (0,1)), \quad \psi \in \dot{W}^{2,\gamma-1} (R^n; W^{2,2} (0,1)); \\
(5) \quad \text{assume the Conditions 4.1 are hold and}
\end{align*}

\begin{align*}
|\alpha_k + \beta_k| > 0, \quad \sum_{k,j=1}^{m} \alpha_k \beta_j \neq 0,
\end{align*}

\begin{align*}
\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - \gamma = \frac{1}{q} + \frac{n}{r} - 2.
\end{align*}

Then for $k \geq k_0$ there is a $T > 0$ depending only on

\begin{align*}
\|\varphi\|_{\dot{W}^{2,\gamma} (R^n; W^{2,2} (0,1))} + \|\psi\|_{\dot{W}^{2,\gamma-1} (R^n; W^{2,2} (0,1))}
\end{align*}

and a unique weak solution $u$ to (1.10) – (1.8) – (1.9) with

\begin{align*}
u \in L^{q_0} \left([0,T]; L^{r_0} (R^n; W^{2,2} (0,1))\right),
\end{align*}

where

\begin{align*}
q_0 = \frac{2(n+1)}{n-3}, \quad r_0 = \frac{2 (n^2 - 1)}{(n^2 - 1) + 4}.
\end{align*}

In addition, the solution satisfies

\begin{align*}
u \in C \left([0,T]; \dot{W}^{2,\gamma} (R^n; W^{2,2} (0,1))\right) \cap C^1 \left([0,T]; \dot{W}^{2,\gamma-1} (R^n; W^{2,2} (0,1))\right)
\end{align*}

and depends continuously on the data.

**Proof.** Let $H = L^2 (0,1)$ and $A$ is a operator defined by (1.4). Then the problem (1.10) – (1.8) – (1.9) can be rewritten as the problem (5.1). By virtue of [12, 23] the operator $A$ generates analytic semigroup in $L^2 (0,1)$. Hence, by virtue of (1)-(5), all conditons of Theorem 5.1 are satisfied. Then Theorem 5.1 implies the assertion.

1. H. Amann, Linear and quasi-linear equations,1, Birkhauser, Basel 1995.

2. A. Arosio, Linear second order differential equations in Hilbert space. The Cauchy problem and asymptotic behaviour for large time, Arch. Rational Mech. Anal., v 86, (2), 1984, 147-180.

3. A. Ashyralyev, N. Aggez, Nonlocal boundary value hyperbolic problems Involving Integral conditions, Bound.Value Probl., 2014 V. 2014:214.

4. H. Bahouri, P. Gérard, High frequency approximation of solutions to critical nonlinear wave equations, Amer. J. Math. 121 (1999), 131–175.

5. J. Bergh and J. Lofstrom, Interpolation spaces: An introduction, Springer-Verlag, New York, 1976.
6. M. Christ and M. Weinstein, Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation. J. Funct. Anal. 100 (1991), 87-109.

7. G. Da Prato and E. Giusti, A characterization on generators of abstract cosine functions, Boll. Del. Unione Mat., (22)1967, 367–362.

8. Denk R., Hieber M., Prüss J., $\mathbb{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type, Mem. Amer. Math. Soc. 166 (2003), n.788.

9. E. B. Davies and M. M. Pang, The Cauchy problem and a generalization of the Hille-Yosida theorem, Proc. London Math. Soc. (55) 1987, 181-208.

10. D. Foschi, Inhomogeneous Strichartz estimates, Jour. Hyperbolic Diff. Eqs 2, No. 1 (2005) 1–24.

11. H. O. Fattorini, Second order linear differential equations in Banach spaces, in North Holland Mathematics Studies, V. 108, North-Holland, Amsterdam, 1985.

12. A. Favini, G. R. Goldstein, J. A. Goldstein and S. Romanelli, Degenerate second order differential operators generating analytic semigroups in $L_p$ and $W^{1,p}$, Math. Nachr. 238 (2002), 78 –102.

13. A. Favini, V. Shakhmurov, Y. Yakubov, Regular boundary value problems for complete second order elliptic differential-operator equations in UMD Banach spaces, Semigroup form, v. 79 (1), 2009.

14. J. Ginibre, G. Velo, Generalized Strichartz inequalities for the wave equation, Jour. Func. Anal., 133 (1995), 50–68.

15. J. A. Goldstein, Semigroup of linear operators and applications, Oxford, 1985.

16. V. I. Gorbachuk and M. L. Gorbachuk, Boundary value problems for differential-operator equations, Naukova Dumka, Kiev, 1984.

17. M. Grillakis, Regularity for the wave equation with a critical nonlinearity, Comm. Pure Appl. Math., 45 (1992), 749–774.

18. S. G. Krein, Linear differential equations in Banach space, Providence, 1971.

19. C. Kenig, G. Ponce, L. Vega, A bilinear estimate with applications to the KdV equation, J. Amer. Math. Soc. 9 (1996), 573–603.

20. L. Kapitanski, Weak and yet weaker solutions of semilinear wave equations, Comm. Part. Diff. Eq., 19 (1994), 1629–1676.
21. S. Klainerman, M. Machedon, On the regularity properties of a model problem related to wave maps, Duke Math. J. 87 (1997), 553–589.

22. M. Keel and T. Tao, Endpoint Strichartz estimates. Amer. J. Math. 120 (1998), 955-980.

23. V. Keyantuo, M. Warma, The wave equation with Wentzell–Robin boundary conditions on $L^p$-spaces, J. Differential Equations 229 (2006) 680–697.

24. H. Lindblad, C. D. Sogge, Restriction theorems and semilinear Klein-Gordon equations in (1+3) dimensions, Duke Math. J. 85 (1996), no 1, 227–252.

25. A. Lunardi, Analytic semigroups and optimal regularity in parabolic problems, Birkhauser, 2003.

26. M. Meyries, M. Veraar, Pointwise multiplication on vector-valued function spaces with power weights, J. Fourier Anal. Appl. 21 (2015)(1), 95–136.

27. N. Masmoudi, K. Nakanishi, From nonlinear Klein-Gordon equation to a system of coupled nonlinear Schrodinger equations, Math. Ann. 324 (2002), (2) 359–389.

28. S. Piskarev and S.-Y. Shaw, Multiplicative perturbations of semigroups and applications to step responses and cumulative outputs, J. Funct. Anal. 128 (1995), 315-340.

29. A. Pazy, Semigroups of linear operators and applications to partial differential equations. Springer, Berlin, 1983.

30. C. D. Sogge, Lectures on Nonlinear Wave Equations, International Press, Cambridge, MA, 1995.

31. C. D. Sogge, Fourier Integrals in Classical Analysis, Cambridge University Press, 1993.

32. V. B. Shakhmurov, Nonlinear abstract boundary value problems in vector-valued function spaces and applications, Nonlinear Anal-Theor., v. 67(3) 2006, 745-762.

33. R. Shahmurov, On strong solutions of a Robin problem modeling heat conduction in materials with corroded boundary, Nonlinear Anal., Real World Appl., v.13, (1), 2011, 441-451.

34. R. Shahmurov, Solution of the Dirichlet and Neumann problems for a modified Helmholtz equation in Besov spaces on an annuals, J. Differential equations, v. 249(3), 2010, 526-550.

35. I. E. Segal, Space-time decay for solutions of wave equations, Adv. Math. 22 (1976), 304–311.
36. E. M. Stein, Singular Integrals and differentiability properties of functions, Princeton Univ. Press, Princeton. NJ, 1970.

37. R. S. Strichartz, Restriction of Fourier transform to quadratic surfaces and decay of solutions of wave equations, Duke Math. J. 44 (1977), 705–774.

38. H. Triebel, Interpolation theory, Function spaces, Differential operators, North-Holland, Amsterdam, 1978.

39. T. Tao, Local and global analysis of nonlinear dispersive and wave equations, CBMS regional conference series in mathematics, 2006.

40. T. Tao, Low regularity semi-linear wave equations. Comm. Partial Differential Equations, 24 (1999), no. 3-4, 599–629.

41. C. Travis and G. F. Webb, Second order differential equations in Banach spaces, Nonlinear Equations in Abstract Spaces, (ed. by V. Lakshmikantham), Academic Press, 1978, 331-361.

42. G. B. Whitham, Linear and Nonlinear Waves, Wiley–Interscience, New York, 1975.

43. S. Yakubov and Ya. Yakubov, Differential-operator Equations. Ordinary and Partial Differential Equations, Chapman and Hall /CRC, Boca Raton, 2000.