Invariance of the parity conjecture for \( p \)-Selmer groups of elliptic curves in a \( D_{2p^n} \)-extension

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Abstract
In section 2, we show a \( p \)-parity result in a \( D_{2p^n} \)-extension of number fields \( L/K \) \((p \geq 5)\) for the twist \( 1 \oplus \eta \oplus \tau \):

\[ W(E/K, 1 \oplus \eta \oplus \tau) = (-1)^{(1 \oplus \eta \oplus \tau, X_p(E/L))}, \]

where \( E \) is an elliptic curve over \( K \), \( \eta \) and \( \tau \) are respectively the quadratic character and an irreductible representation of degree 2 of \( \text{Gal}(L/K) = D_{2p^n} \), and \( X_p(E/L) \) is the \( p \)-Selmer group. The main novelty is that we use a congruence result between \( \varepsilon_0 \)-factors (due to Deligne) for the determination of local root numbers in bad cases (places of additive reduction above 2 and 3). We also give applications to the \( p \)-parity conjecture (using the machinery of the Dokchitser brothers).

1 The conjecture of Birch and Swinnerton-Dyer and the parity conjecture

Let \( K \) be a number field and \( E \) an elliptic curve defined over \( K \). Denote by \( K_v \) the completion of \( K \) at a place \( v \).

We recall a few definitions:

Definition 1.1 (Tate Module):
The \( l \)-adic Tate module of \( E \) is the inverse limit of the system of multiplication by \( l \) maps \( E[l^{n+1}] \to E[l^n] \), where \( E[m] \) denotes the kernel of multiplication by \( m \) on \( E \).

Set \( T_l(E) = \lim_{\leftarrow} E[l^n] \), \( V_l(E) = \mathbb{Q}_l \otimes_{\mathbb{Z}_l} T_l(E) \) and:

\[ \sigma'_{E/K_v,l} : \text{Gal}(\overline{K_v}/K_v) \to \text{GL}(V_l(E)^*). \]

Definition 1.2 Fix an embedding, \( \iota : \mathbb{Q}_l \hookrightarrow \mathbb{C} \); we can then associate to \( \sigma'_{E/K_v,l} \) a complex representation \( \sigma'_{E/K_v,l,\iota} \) of the Weil-Deligne group (see [9] §13). One can show that the isomorphism class of \( \sigma'_{E/K_v} := \sigma'_{E/K_v,l,\iota} \) is independent of the choice of \( l \) and \( \iota \) (see [9] §13, §14, §15).
Denote by $L(E/K, s)$ the global $L$-function, product of local $L$-functions:

$$L(E/K, s) = \prod_{v \text{ finite}} L(E/K_v, s) = \prod_{v \text{ finite}} L(\sigma'_E/K_v, s),$$

defined for $\text{Re}(s) > \frac{3}{2}$ (see [9] §17 for the correspondence between the classical definition of $L(E/K_v, s)$ and the one using $\sigma'_E/K_v$) and by

$$\Lambda(E/K, s) = A(E/K)^{s/2} L(E/K, s)(2(2\pi)^{-s}\Gamma(s))^{[K: \mathbb{Q}]},$$

the "complete" $L$-function.

Recall the following classical conjectures:

**Conjecture 1.3** (Birch and Swinnerton-Dyer: BSD):

$$\text{ord}_{s=1} \Lambda(E/K, s) = \text{rk}(E/K).$$

**Conjecture 1.4** (Functional equation of $\Lambda$: FE):

$L(E/K, s)$ has a holomorphic continuation to $\mathbb{C}$ and there is a number $W(E/K) = \prod_{v} W(E/K_v) \in \{\pm 1\}$ such that:

$$\Lambda(E/K, s) = W(E/K) \Lambda(E/K, 2 - s)$$

(see [9] §12 and §19 for the definition of $W(E/K_v) := W(\sigma'_E/K_v)$ and [9] §21 p.157 for the functional equation of $\Lambda$).

This conjecture is known in a few cases:

- For elliptic curves over $\mathbb{Q}$ thanks to modularity results on elliptic curves due to Wiles, Taylor, Breuil, Diamond and Conrad
- For elliptic curves over a totally real field $K$, we only know a meromorphic continuation and the functional equation of $\Lambda$ thanks to a potential modularity result of Wintenberger (see [16]) together with an argument of Taylor.

In general, conjecture 1.4 is not known.

The conjecture of Birch and Swinnerton-Dyer implies the following weaker conjecture:

**Conjecture 1.5** (BSD (mod 2))

$$\text{rk}(E/K) \equiv \text{ord}_{s=1} \Lambda(E/K, s) \pmod{2}.$$

Combining it with the conjectural functional equation we get:

**Conjecture 1.6** (Parity conjecture)

$$(-1)^{\text{rk}(E/K)} = W(E/K).$$
Tim and Vladimir Dokchitser showed that this conjecture is true assuming that the $6^∞$-part of the Tate-Shafarevich group of $E$ over $K(E[2])$ is finite (see [5] Th 7.1 p.20).

**Definition 1.7 Selmer group:**  
Let $X_p(E/K) := \text{Hom}_{Z_p}(S(E/K,p^∞),\mathbb{Q}_p/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$,

where $S(E/K,p^∞) := \lim \limits_{\longrightarrow} S(E/K,p^n)$ is the $p^∞$-Selmer group, sitting in an exact sequence:

$$0 \longrightarrow E(K) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow S(E/K,p^∞) \longrightarrow \text{III}_{E/K}[p^∞] \longrightarrow 0$$

If we let $\text{rk}_p(E/K) := \dim_{\mathbb{Q}_p} X_p(E/K) = \text{rk}(E/K) + \text{cork}_{Z_p} \text{III}_{E/K}[p^∞]$, a more accessible form of the conjecture [1.6] is the following:

**Conjecture 1.8 (p-parity conjecture)**  
$(-1)^{\text{rk}_p(E/K)} = W(E/K)$.

If $L/K$ is a finite Galois extension and $\tau$ is a self-dual $\mathbb{Q}_p$-representation of $\text{Gal}(L/K)$ then there is an equivariant form of the conjecture [1.8]:

**Conjecture 1.9 (p-parity conjecture for (self-dual) twists)**  
$(-1)^{\tau \cdot X_p(E/K)} = W(E/K,\tau)$, where $W(E/K,\tau) = \prod_v W(\sigma'_E/K_v \otimes \text{Res}_{D_v} \tau)$ (where $D_v \subset \text{Gal}(L/K)$ is the decomposition group at $v$).

It is this last conjecture in a particular setting that will interest us for the rest of the paper.

## 2 Invariance of the parity conjecture in a $D_{2p^n}$-extension

### 2.1 Statement of the main theorem and applications to the p-parity conjecture

Let $K$ be a number field, $E/K$ an elliptic curve and $L/K$ a finite Galois extension such that $\text{Gal}(L/K) \simeq D_{2p^n}$, with $p \geq 5$ a prime number.

$D_{2p^n}$ admits the following irreducible representations over $\overline{\mathbb{Q}}_p$:

- $1$ the trivial representation
- $\eta$ the quadratic character
- $\frac{p^n-1}{2}$ irreducible representations of degree 2; they are of the form,

$$I(\chi) := \text{Ind}_{C_{p^n}}^{D_{2p^n}}(\chi) = I(\chi^{-1}),$$

where $\chi$ is a non-trivial character of $C_{p^n}$ ($I(1) = 1 \oplus \eta$ is reducible).

See for example [12] for the description of irreducible representations of $D_{2p^n}$.

Let $\tau = I(\chi)$ be such an irreducible representation of degree 2.
**Theorem 2.1** With the notation above and \( p \geq 5 \), we have the following equality:

\[
\frac{W(E/K, \tau)}{W(E/K, 1 \oplus \eta)} = \frac{(-1)^{\langle \tau, \chi_{p}(E/L) \rangle}}{(-1)^{\langle 1 \oplus \eta, \chi_{p}(E/L) \rangle}}.
\]

In other words, the \( p \)-parity conjecture for \( E/K \) tensored by \( 1 \oplus \eta \oplus \tau \) holds:

\[
W(E/K, 1 \oplus \eta \oplus \tau) = (-1)^{\langle 1 \oplus \eta \oplus \tau, \chi_{p}(E/L) \rangle}.
\]

**Remark 2.2** The Dokchitser brothers have shown that this equality holds in two different cases:

- In the case when \( p \) is any prime number but the elliptic curve \( E/K \) has a cyclic decomposition group in all additive places above 2 and 3 (see [4] Th.4.2 (1) p.43).
- In the case when \( p \equiv 3 \pmod{4} \) (without any additional assumption) using a strong global \( p \)-parity result over totally real fields (see [5] Prop.6.12 p.18).

In particular, the statement of Thm.2.1 also holds for \( p = 3 \). Furthermore, this case can be proved without using the "painful calculation" (of [4] p.30) in the case of additive reduction (see the appendix).

Here we prove the equality for all \( p \geq 5 \) (without any additional assumption).

**Corollary 2.3**

\[
\frac{W(E/K, I(\chi))}{(-1)^{\langle I(\chi), \chi_{p}(E/L) \rangle}} \text{ does not depend on } \chi : C_{p^n} \to \mathbb{C}^*.
\]

**Theorem 2.4** Let \( K \) be a number field, \( p \geq 3 \), and \( E/K \) an elliptic curve. Suppose \( F \) is a \( p \)-extension of a Galois extension \( M/K \), Galois over \( K \). If the \( p \)-parity conjecture

\[
(-1)^{\text{rk}_{\mathbb{Z}} E/L} = W(E/L)
\]

holds for all subfields \( K \subset L \subset M \), then it holds for all subfields \( K \subset L \subset F \).

**Theorem 2.5** Let \( K \) be a number field, \( p \geq 3 \), \( E/K \) an elliptic curve and \( F/K \) a Galois extension. Assume that the \( p \)-Sylow subgroup \( P \) of \( G = \text{Gal}(F/K) \) is normal and \( G/P \) is abelian. If the \( p \)-parity conjecture holds for \( E \) over \( K \) and its quadratic extensions in \( F \), then it holds for all twists of \( E \) by orthogonal representations of \( G \).
2.2 Reduction to the case of a $D_{2p}$-extension

Here we reduce the demonstration of Theorem 2.1 by an induction argument together with the Galois invariance of root numbers due to Rohrlich (see [11, Theorem 2]), to the following statement:

**Proposition 2.6** It is sufficient to prove Theorem 2.1 in the case when $n = 1$ (i.e. $\text{Gal}(L/K) \simeq D_{2p}$).

**Proof.** Suppose Theorem 2.1 is true for $n = N - 1$. We will show that theorem is true for $n = N$.

Consider $L/K$ a finite Galois extension such that $\text{Gal}(L/K) \simeq D_{2p^N}$ and $\tau$ an irreducible representation of degree 2 of $D_{2p^N}$.

- If $\chi$ is not injective, then the statement is known by the induction hypothesis.
- If $\chi$ is injective:
  
  Let $\sigma = \text{res}(I(\chi)) := \text{res}^{D_{2p^N}}_{D_{2p^{N-1}}}(I(\chi))$.

  Then $\sigma = I(\chi')$, where $\chi' := \chi|_{C_{p^{N-1}}} : C_{p^{N-1}} \to \overline{\mathbb{F}_p}$ is injective.

  We have: $\text{Ind}^{D_{2p^N}}_{D_{2p^{N-1}}}(\sigma) = \bigoplus I(\chi_0)$, where the sum is taken over the $\chi_0$ such that $\chi_0|_{C_p^{N-1}} = \chi|_{C_{p^{N-1}}}$.

  For each such $\chi_0$ there is an element of $\text{Aut}(\mathbb{C})$ sending $\chi$ into $\chi_0$ and $I(\chi)$ into $I(\chi_0)$.

  By inductivity of root numbers in Galois extension:
  
  $$W(E/K, \sigma) = W(E/K, \text{Ind}^{D_{2p^N}}_{D_{2p^{N-1}}}(\sigma)).$$

By Galois invariance of root numbers:

$$W(E/K, I(\chi')) = W(E/K, I(\chi_0)), \forall \chi_0 \text{ such that } \chi_0|_{C_{p^{N-1}}} = \chi|_{C_{p^{N-1}}}.''

So $W(E/K, \sigma) = W(E/K, \text{Ind}^{D_{2p^N}}_{D_{2p^{N-1}}}(\sigma)) = W(E/K, \tau)p = W(E/K, \tau)$.

On the other hand,

$$\langle \sigma, X_p(E/L) \rangle = \left\langle \text{Ind}^{D_{2p^N}}_{D_{2p^{N-1}}}(\sigma), X_p(E/L) \right\rangle = p. \langle \tau, X_p(E/L) \rangle,$$

because $X_p(E/L)$ is a $\mathbb{Q}_p$-representation.

So $(-1)^{(1 \oplus \eta \oplus \sigma)} X_p(E/L) = (-1)^{(1 \oplus \eta \oplus \tau)} X_p(E/L)$.

By the induction hypothesis, $(-1)^{(1 \oplus \eta \oplus \sigma)} X_p(E/L) = W(E/K, \sigma)$. As a result, $W(E/K, 1 \oplus \eta \oplus \tau) = (-1)^{(1 \oplus \eta \oplus \tau)} X_p(E/L)$. ■

2.3 The case of a $D_{2p}$-extension

We first restate Theorem 2.1 in the case of a $D_{2p}$-extension.

Let $K$ be a number field, $E/K$ an elliptic curve and $L/K$ a Galois extension such that $\text{Gal}(L/K) \simeq D_{2p} \simeq C_p \times C_2$, with $p \geq 5$ a prime number.

Recall the irreducible representations of $D_{2p}$ over $\overline{\mathbb{Q}_p}$:

- 1 the trivial representation
• \( \eta \) the quadratic character
• \( f(\chi) \) irreducible representations of degree 2, where \( \chi \) is a non-trivial character of \( C_p \).

**Theorem 2.7** With the notation above and \( p \geq 5 \), we have the following equality:

\[
\frac{W(E/K, \tau)}{W(E/K, 1 \oplus \eta)} = \frac{(-1)^{\tau, X_p(E/L)}}{(-1)^{(1 \oplus \eta, X_p(E/L))}}.
\]

In other words, the \( p \)-parity conjecture for \( E/K \) tensored by \( 1 \oplus \eta \oplus \tau \) holds:

\[
W(E/K, 1 \oplus \eta \oplus \tau) = (-1)^{(1 \oplus \eta \oplus \tau, X_p(E/L))}.
\]

The proof of Theorem 2.7 will occupy the rest of section 2.

We use the following notation:
• \( v \) a finite place of \( K \)
• \( K_v \) the completion of \( K \) at \( v \)
• \( \frak{d}_v = \frak{f}_v \) the cardinality of the residue field of \( K_v \)
• \( 2 \mid v \) a finite place of \( L \)
• \( w \mid v \) a finite place of \( L^H \) (where \( H \) is a subgroup of \( \text{Gal}(L/K) = D_{2p} \))
• \( \delta = \text{ord}_v \) (the minimal discriminant of \( E/K_v \))
• \( \delta_H = \text{ord}_w \) (the minimal discriminant of \( E/(L^H)_w \))
• \( e_H \) the ramification index of \( (L^H)_w / K_v \)
• \( f_H \) the residue degree of \( (L^H)_w / K_v \)
• \( \omega_{E/K_v} = \) a minimal invariant differential of \( E/K_v \)
• \( C_v(E/L^H) = c_w(E/L^H)\omega(H) \),
where \[
\omega(H) = \left| \frac{\omega_{E/K_v}}{E/(L^H)_w} \right|_{(L^H)_w}.
\]

Furthermore, if \( t_v > 3 \) then:

\[
\left| \frac{\omega_{E/K_v}}{E/(L^H)_w} \right|_{(L^H)_w} = q^{\frac{\delta_H - 2H}{12}} f_H
\]

\[
= q^{\frac{\delta_H - 2H}{12}} (\text{in the case of potentially good reduction})
\]

For \( D_{2p} \), there is the following equality: \( \text{Ind}_{1}^{D_{2p}} 1 - 2 \text{Ind}_{D_{2p}}^{D_{2p}} 1 - \text{Ind}_{C_p}^{D_{2p}} 1 + 2.1 = 0 \) of virtual representations of \( G \), this gives the \( G \)-relation \( \Theta : \{1\} - 2D_2 - C_p + 2G \) in the sense of \[4\] (Def 2.1 p.11).

We recall two definitions in our setting (i.e. with \( \Theta : \{1\} - 2D_2 - C_p + 2D_{2p} \), for general definitions see \[4\].

**Definition 2.8** \((\text{[3], Def.2.13 p.14})\): Let \( \rho \) be a self-dual \( \mathbb{Q}_p[G] \)-representation. Pick a \( G \)-invariant non-degenerate \( \mathbb{Q}_p \)-linear pairing \( \langle , \rangle \) on \( \rho \) and set \( C_\Theta(\rho) = \det(\langle , \rangle |_\rho^{(1)}) \det(\frac{1}{2} (\langle , \rangle |_\rho^{D_2})^{-2} \det(\frac{1}{2} (\langle , \rangle |_\rho^{C_p})^{-1} \det(\frac{1}{2} (\langle , \rangle |_\rho^{D_{2p}})^2.
\]

As an element of \( \mathbb{Q}_p^*/\mathbb{Q}_p^2 \), this does not depend on the choice of the pairing.
**Definition 2.9** \([\mathbb{I}^4],\) Def.2.50 p.23) We define:

\[
T_{\Theta,p} = \left\{ \sigma \text{ a self-dual } \overline{\mathbb{Q}_p}[G]-\text{representation} \mid \langle \sigma, \rho \rangle \equiv \text{ord}_p C_{\Theta}(\rho) \pmod{2} \right. \forall \rho \text{ a self-dual } \mathbb{Q}_p[G]-\text{representation} \}
\]

Following the approach of the Dokchitser brothers, we have the following theorem

**Theorem 2.10** (Theorem 1.14 of [\mathbb{I}^4]). Let \(L/K\) be a Galois extension of number fields with Galois group \(G = D_{2p}\), where \(p > 2\) is a prime number. Let \(\Theta : \{1\}-2D_2-C_p+2D_{2p}\). For every elliptic curve \(E/K\), the \(\mathbb{Q}_p[G]-\text{representation} X_p(E/L)\) is self-dual, and

\[\forall \sigma \in T_{\Theta,p}, \quad (-1)^{\langle \sigma, X_p(E/L) \rangle} = (-1)^{\text{ord}_p C},\]

where \(C = \prod_{v \nmid \infty} C_v\) with \(C_v = C_v(\{1\})C_v(D_2)^{-2}C_v(C_p)^{-1}C_v(G)^2\)

and \(C_v(H) = \prod_{w \mid v} C_w(E/L^H)\).

Now, since \(1 \oplus \eta \oplus \tau \in T_{\Theta,p}\) (see \([\mathbb{I}^4],\) example 2.53 p.24), we only need to prove that:

\[(1) \quad \frac{W(E/K, \tau)}{W(E/K, 1 \oplus \eta)} = (-1)^{\text{ord}_p C}.\]

Furthermore, since we are only interested in the parity of \(\text{ord}_p C\), we do not have to determine \(C_v(D_2)\) and \(C_v(G)\), because these terms only bring an even contribution (since they appear with an even exponent).

Both sides of (1) are of local nature. As \(W(E/K, \tau) = \prod_v W(E/K_v, \tau_v)\), where \(\sigma_v := \text{res}_{Gal(L_v/K_v)} \sigma\), all we need to do is to prove the following local equality:

\[(2) \quad \frac{W(E/K_v, \tau_v)}{W(E/K_v, (1 \oplus \eta)_v)} = (-1)^{\text{ord}_p C_v},\]

for each finite place \(v\) of \(K\) \((v \mid \infty\) do not contribute, since \(p \neq 2\)).

Denote by \(G_v := Gal(L_v/K_v)\) the decomposition group of \(v\). The proof of Theorem 2.7 split in several cases:

- \(G_v = \{1\}\) (there are 2p places above \(v\) in \(L\)) \(\text{ see section } 2.3.2\)
- \(G_v = D_2\) (there are \(p\) places above \(v\) in \(L\)) \(\text{ see section } 2.3.3\)
- \(G_v = C_p\) (there are 2 places above \(v\) in \(L\)) \(\text{ see section } 2.3.2\)
- \(G_v = D_{2p}\) (there is a unique place above \(v\) in \(L\)) \(\text{ see section } 2.3.4\)

We first recall a few facts about the local Tamagawa factors of elliptic curves.
2.3.1 Local Tamagawa factors of elliptic curves

The assumptions and notation from above are in force.

The local Tamagawa factor at \( v \), \( c(E/K_v) = \#(E(K_v)/E^0(K_v)) \), (where \( E^0(K_v) = \{ \text{Points of non-singular reduction} \} \)) is determined by Tate’s algorithm (see [14] IV §9):

\[
c(E/K_v) = \begin{cases} 
1 & \text{if } E \text{ has good reduction at } v \\
1, 2, 3 \text{ or } 4 & \text{if } E \text{ has additive reduction at } v \\
n & \text{if } E \text{ has split multiplicative reduction} \\
1 \text{ or } 2 & \text{if } E \text{ has non-split multiplicative reduction}
\end{cases}
\]

of type \( I_n \) at \( v \)

If \( E \) acquires semi-stable reduction over \( L_z \), then:

1. If \( E \) has split multiplicative reduction of type \( I_n \) over \( K_v \), then:
   \[
c(E/ (L^H)_{w}) = n.e_H.
\]

2. If \( E \) has non-split multiplicative reduction of type \( I_n \) over \( K_v \), then:
   \[
c(E/ (L^H)_{w}) = \begin{cases} 
n.e_H & \text{if } E \text{ has split multiplicative reduction} \\
1 \text{ or } 2 & \text{otherwise.}
\end{cases}
\]

3. If \( E \) has potentially good reduction, then \( c(E/ (L^H)_{w}) = 1, 2, 3 \text{ or } 4 \).

4. If \( E \) has additive and potentially multiplicative reduction then:
   \[
c(E/ (L^H)_{w}) = \begin{cases} 
n.e_H & \text{if } E \text{ has split multiplicative reduction} \\
1, 2, 3 \text{ or } 4 & \text{of type } I_n \text{ over } (L^H)_{w} \text{ and } l_v \neq 2.
\end{cases}
\]

The following few remarks will be used in the subsequent computations.

**Remark 2.11** If \( w_1 \) and \( w_2 \) are two places of \( L \) above the same \( v \), then:

\[
c_{w_1}(E/L) = c_{w_2}(E/L).
\]

In particular:

\[
\begin{align*}
C_v(\{1\}) &= C_w(E/L)^r \\
C_v(C_p) &= C_w(E/L^{C_p})^{r'},
\end{align*}
\]

where \( r = \text{the number of places } w \text{ of } L \text{ such that } w \mid v \) and \( r' = \text{the number of places } w' \text{ of } L^{C_p} \text{ such that } w' \mid v \).

**Remark 2.12** If \( E/K \) has potentially good reduction at \( v \), then:

\[
\forall w \ (\text{resp. } w') \text{ place of } L \ (\text{of } L^{C_p}), c_w(E/L) \ (c_{w'}(E/L^{C_p}) \in \{1,..,4\},
\]

and therefore \( \text{ord}_p (c_v) = 0 \) and \( (-1)^{\text{ord}_p(C_v)} = (-1)^{\text{ord}_p(C_{w'})} \).
In these cases, \( C \).

### 2.3.2 The cases \( \omega \)

From 2.12 and 2.14 we deduce that the only case that needs multiplicative reduction at \( \omega \).

\[
\begin{align*}
\text{Remark 2.14} & \quad \text{If } v \mid p \text{ (i.e. } p \neq l_v, p \text{ is fixed, } l_v \text{ is variable), then } \ord_p(\omega(H)) = 0 \text{ and } (-1)^{\ord_p(C_v)} = (-1)^{\ord_p(c_v)}. \\
\text{Remark 2.15} & \quad \text{By the previous two remarks, if } E/K \text{ has good reduction at } v, \text{ then: } (-1)^{\ord_p(C_v)} = 1.
\end{align*}
\]

\[
\begin{align*}
\text{Remark 2.16} & \quad \text{We have } \frac{W(E/K_v,\tau_v)}{W(E/K_v,(1\otimes\eta)_v)} = 1 = \frac{\det\tau_v(-1)}{\det(1\otimes\eta)_v(-1)} \quad \text{in the case of good reduction,} \\
& \quad \text{we have the desired equality (2) in the case of good reduction at } v.
\end{align*}
\]

\[
\begin{align*}
\text{2.3.4 The case } G_v = D_{2p} \\
\text{Denote by } w \text{ (resp } z) \text{ the unique place of } L^{C_p} \text{ (resp } L) \text{ above } v.
\end{align*}
\]

\[
\begin{align*}
\forall w' \mid v \text{ place of } L^{C_p} \text{ and } \forall w \mid w' \text{ place of } L, \quad \left(L^{C_p}ight)_{w'} \\
\quad \text{In particular, } C_v(\{1\}) = C_v(C_p)^p, \text{ therefore } C_v = C_v(C_p)^{p-1} \text{ and } \ord_p(C_v) = 0.
\end{align*}
\]

\[
\begin{align*}
\text{Finally, we get: } \frac{W(E/K_v,\tau_v)}{W(E/K_v,(1\otimes\eta)_v)} = 1 = (-1)^{\ord_p(C_v)}.
\end{align*}
\]

\[
\begin{align*}
\text{2.3.4 The case } G_v = D_{2p} \\
\text{Denote by } w (\text{resp } z) \text{ the unique place of } L^{C_p} (\text{resp } L) \text{ above } v.
\end{align*}
\]

\[
\begin{align*}
\text{In this case, there are two possibilities for the inertia group of } G_v, \text{ } I_v = C_p \text{ or } D_{2p} \text{ (because } I_v \text{ is a normal subgroup of } G_v = D_{2p} \text{ and } G_v/I_v \text{ is cyclic).} \\
\text{Furthermore, if } l_v \neq p \text{ then } I_v = C_p.
\end{align*}
\]
- For \( l_v \neq 2 \) because the inertia group of a tamely ramified extension is cyclic.
- For \( l_v = 2 \) because the case \( I_v = D_{2p}, I_v^{\text{wild}} = D_2 \) (the wild inertia group) is impossible since \( I_v^{\text{wild}} \) is normal in \( I_v \).

### 2.3.4.1 Computation of \((-1)^{\text{ord}_p(C_v)}\)

1. If \( E/K_v \) has potentially multiplicative reduction:

   (a) If \( E/K_v \) acquires split multiplicative reduction of type \( I_n \) over \( L_z \) (and therefore over \((L^C)_{\omega}\)), then:

   \[
   C_v(\{1\}) = c_w(E/L_z) = e_{\omega}E/(L^C)_{\omega} \times c_w(E/(L^C)_{\omega})
   = \frac{e_{\omega}}{C_p} \times c_v(E/K)(C_v(C_p))
   = \frac{e_{\omega}}{C_p} \times C_v(C_p)
   \]

   but \[
   \begin{cases}
   \text{if } I_v = C_p \text{ then } e_{\omega} = p \text{ and } e_{C_p} = 1 \\
   \text{if } I_v = D_{2p} \text{ then } e_{\omega} = 2p \text{ and } e_{C_p} = 2
   \end{cases}
   \]

   In both cases we get: \( C_v = p \) and \((-1)^{\text{ord}_p(C_v)} = -1\).

   (b) If \( E/K_v \) does not acquire split multiplicative reduction of type \( I_n \) over \( L_z \) (and therefore nor over \((L^C)_{\omega}\)), then:

   \[
   c_v(\{1\}), c_v(C_p) \in \{1, 2, 3, 4\} \text{ and } \text{ord}_p\left(\frac{\omega(\{1\})}{\omega(C_p)}\right) \equiv 0 \pmod{2}.
   \]

   The second claim is a consequence of Remark 2.14 in the case \( l_v \neq p \).

   In the case \( l_v = p \), we have to distinguish two cases:

   i. If \( E/K_v \) acquires non-split multiplicative reduction of type \( I_n \) over \( L_z \) (and therefore over \((L^C)_{\omega}\)), then \( \delta_{\{1\}} = \delta_{C_p} \).

   Furthermore, \( f_{C_p} = f_{\{1\}} = 1 \) or 2 and \( \frac{\omega(\{1\})}{\omega(C_p)} = \frac{\delta f_{\{1\}} - e_{C_p}}{e_{\omega}} \),

   so \( \text{ord}_p\left(\frac{\omega(\{1\})}{\omega(C_p)}\right) \equiv 0 \pmod{2} \) (because \( p - 1 \mid (e_{\omega} - e_{C_p}) \)).

   ii. If \( E/K_v, E/(L^C)_{\omega} \text{ and } E/L_z \) have additive reduction (of type \( I_n^\gamma \)):

   - if \( I_v = C_p \), then \( f_{C_p} = f_{\{1\}} = 2 \) and the result follows.
   - if \( I_v = D_{2p}, \) since \( p \geq 5 \), \( E \) becomes of type \( I_{2p}^\gamma \) over \((L^C)_{\omega}\)

   and \( I_{2p}^\gamma \) over \( L_z \) and we get: \( \text{ord}_p(\omega(\{1\})) = \text{ord}_p(\omega(C_p)) \equiv 0 \pmod{2} \).

   To sum up, in the case of potentially multiplicative reduction:

   \[
   (-1)^{\text{ord}_p(C_v)} = \begin{cases}
   -1 & \text{ if } E/(L^C) \text{ has split multiplicative reduction} \\
   1 & \text{ otherwise.}
   \end{cases}
   \]

2. If \( E/K_v \) has potentially good reduction, then:
(a) If \( I_v = C_p \) (i.e. \( e_{\{1\}} = p \) and \( e_{C_p} = 1 \)):
We get: \( f_{\{1\}} = f_{C_p} = 2 \) so \( \text{ord}_p(\omega(C_p)) = \text{ord}_p(\omega(\{1\})) \equiv 0 \pmod 2 \)
and therefore \((-1)^{\text{ord}_p(C_v)} = 1 \) (see Remark 2.12).

(b) If \( I_v = D_{2p} \) (i.e. \( e_{\{1\}} = 2p \), \( e_{C_p} = 2 \) and \( l_v = p \)):
We get: \( C_v(\{1\}) = \omega(C_v) = q_v \)
\[ \frac{\delta}{2p} \] and therefore
\[ (-1)^{\text{ord}_p(C_v)} = (-1)^{\text{ord}_p\left(\frac{\delta}{2p}\right)} = 1. \]
i. If \( q_v \) is an even power of \( p \), then
\[ (-1)^{\text{ord}_p(C_v)} = (-1)^{\text{ord}_p\left(\frac{\delta}{2p}\right)} = 1. \]
ii. If \( q_v \) is an odd power of \( p \):
A computation of \( \frac{\delta}{2p} \) and \( \frac{\delta}{2p} \) depending on \( p \) modulo 12 gives the following table:

| \( p \mod 12 \) | \( II, II^* \) (\( \epsilon = 6 \)) | \( III, III^* \) (\( \epsilon = 4 \)) | \( IV, IV^* \) (\( \epsilon = 3 \)) | \( I_0^* \) (\( \epsilon = 2 \)) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1               | 1               | 1               | 1               | 1               |
| 5               | -1              | 1               | -1              | 1               |
| 7               | 1               | -1              | 1               | 1               |
| 11              | -1              | -1              | -1              | 1               |

Table of values of \((-1)^{\text{ord}_p(C_v)}\) depending on the Kodaira symbol of the curve (and the value of \( \epsilon = \frac{12}{\text{gcd}(\delta, 12)} \)) and \( p \) modulo 12:

In relation to the above table it may be useful to recall the following fact: if the residue characteristic of \( K_v \) is \( > 3 \), then we have the following correspondence between \( \epsilon = \frac{12}{\text{gcd}(\delta, 12)} \), the valuation of the minimal discriminant \( \delta \) and the Kodaira symbols:

- \( \epsilon = 1 \) \(\Leftrightarrow\) \( \delta = 0 \) \(\Leftrightarrow\) \( E \) is of type \( I_0 \)
- \( \epsilon = 2 \) \(\Leftrightarrow\) \( \delta = 6 \) \(\Leftrightarrow\) \( E \) is of type \( I_0^* \)
- \( \epsilon = 3 \) \(\Leftrightarrow\) \( \delta = 4 \) or \( \delta = 8 \) \(\Leftrightarrow\) \( E \) is of type \( IV \) or \( IV^* \)
- \( \epsilon = 4 \) \(\Leftrightarrow\) \( \delta = 3 \) or \( \delta = 9 \) \(\Leftrightarrow\) \( E \) is of type \( III \) or \( III^* \)
- \( \epsilon = 6 \) \(\Leftrightarrow\) \( \delta = 2 \) or \( \delta = 10 \) \(\Leftrightarrow\) \( E \) is of type \( II \) or \( II^* \).

2.3.4.2 Computation of \( \frac{W(E/K_v, \tau_v)}{W(E/K_v, (1 \oplus \eta)_v)} \)

1. The case of potentially multiplicative reduction:
We have an explicit formula of Rohrlich (see [10] Th.2 (ii) p.329):
\[ W(E/K_v, \sigma) = \det \sigma(-1)^{\text{dim} \sigma(-1)^{\chi, \sigma}}, \]
where \( \chi \) is the character of \( K_v^* \) associated to the extension \( K_v(\sqrt{-c_0}) \) of \( K_v \) (\( c_0 \) is the classical factor, see [13] p.46).
Since \( \text{dim} \tau_v = \text{dim} (1 \oplus \eta) = 2 \), \( \det(\tau_v) = \det(1 \oplus \eta) \) and \( \langle \chi, \tau_v \rangle = 0 \), we get:
\[ \frac{W(E/K_v, \tau_v)}{W(E/K_v, (1 \oplus \eta)_v)} = \frac{(-1)^{\langle \chi, \tau_v \rangle}}{(-1)^{\langle \chi, (1 \oplus \eta)_v \rangle}} = \frac{1}{(-1)^{\langle \chi, (1 \oplus \eta)_v \rangle}} = (-1)^{\langle \chi, (1 \oplus \eta)_v \rangle}. \]
(a) If the reduction of $E/K_v$ is split multiplicative (i.e. $\chi = 1$):

Then $(-1)^{\langle \chi, (1 \oplus \eta_v) \rangle} = -1$.

(b) If the reduction of $E/K_v$ is non-split multiplicative (i.e. $\chi$ is an unramified quadratic character):

i. If $E$ acquires split multiplicative reduction over $L_z$ (and therefore over $(L_{C_p})_w$), then $\eta_v = \chi$, hence $(-1)^{\langle \chi, (1 \oplus \eta_v) \rangle} = -1$.

ii. If $E$ acquires non-split multiplicative reduction over $L_z$ (and therefore over $(L_{C_p})_w$), then $\eta_v \neq \chi$, hence $(-1)^{\langle \chi, (1 \oplus \eta_v) \rangle} = 1$.

(c) If the reduction of $E/K_v$ is additive (i.e. $\chi$ is a ramified quadratic character)

i. If $E$ acquires split multiplicative reduction over $L_z$ (and therefore over $(L_{C_p})_w$), then $\eta_v = \chi$, hence $(-1)^{\langle \chi, (1 \oplus \eta_v) \rangle} = -1$.

ii. If $E$ acquires non-split multiplicative reduction over $L_z$ (and therefore over $(L_{C_p})_w$), then $\eta_v \neq \chi$, hence $(-1)^{\langle \chi, (1 \oplus \eta_v) \rangle} = 1$.

To sum up, in the case of potentially multiplicative reduction:

$$W(E/K_v, \tau_v) = W(E/K_v, (1 \oplus \eta_v)_w) = \begin{cases} -1 & \text{if } E/(L_{C_p}) \text{ has split multiplicative reduction} \\ 1 & \text{otherwise.} \end{cases}$$

$$= (-1)^{ord_p(C_v)}, \text{ by 2.3.4.1.1}$$

2. The case of potentially good reduction:

Here we have to distinguish the cases $l_v = p$ and $l_v \neq p$.

(a) The case $l_v = p$.

We have again an explicit formula of Rohrlich, since $p \geq 5$ (see [10], Th.2 (iii) p.329):

We use the following notation:

• $q = pr^s$ the cardinality of the residue field residue degree of $K_v$

• $\epsilon = \frac{12}{\gcd(12, 12)}$

• $\epsilon = \begin{cases} 1 & \text{if } r \text{ is even or } \epsilon = 1 \\ \frac{-1}{p} & \text{if } r \text{ is odd and } \epsilon = 2 \text{ or } 6 \\ \frac{-2}{p} & \text{if } r \text{ is odd and } \epsilon = 3 \\ \frac{-3}{p} & \text{if } r \text{ is odd and } \epsilon = 4. \end{cases}$

Then $\forall \sigma$ a self-dual representation of $Gal(\overline{K_v}/K_v)$ with finite image:

$$W(E/K_v, \sigma) = \begin{cases} \alpha(\sigma, \epsilon) & \text{if } q \equiv 1[\epsilon] \\ \alpha(\sigma, \epsilon)(-1)^{\langle 1 + \eta_v \rangle + \hat{\sigma_v}, \sigma} & \text{if } q \equiv -1[\epsilon] \text{ and } \epsilon = 3, 4, 6, \end{cases}$$

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where $\eta_{nr}$ is the unramified quadratic character, $\hat{\sigma}_v$ is an irreducible representation of degree 2 of $D_2$, and $\alpha(\sigma, \epsilon) := (\det \sigma)(-1)^{\dim \sigma}$.

Since $\dim \tau_v = \dim (1 \oplus \eta)_v = 2$ and $\det \tau_v = \det (1 \oplus \eta)_v$, $\alpha((1 \oplus \eta)_v, \epsilon) = \alpha(\tau_v, \epsilon)$ and we get:

$$
\frac{W(E/K_v, \tau_v)}{W(E/K_v, (1 \oplus \eta)_v)} = \begin{cases} 
1 & \text{if } q \equiv 1[\epsilon] \\
(-1)^{1+\eta_{nr}+1+\eta_v} & \text{if } q \equiv -1[\epsilon]
\end{cases}
$$

if $\epsilon = 3, 4, 6,$

$$
\frac{W(E/K_v, \tau_v)}{W(E/K_v, (1 \oplus \eta)_v)} = \begin{cases} 
1 & \text{if } q \equiv 1[\epsilon] \\
(-1)^{1+\eta_{nr}+1+\eta_v} & \text{if } q \equiv -1[\epsilon]
\end{cases}
$$

and $\epsilon = 3, 4, 6,$

($\langle \hat{\sigma}_v, \tau_v \rangle = 0$ since $\epsilon = 3, 4, 6$ and $p \geq 5$).

i. If $r$ is even, then $q \equiv 1[\epsilon] \forall \epsilon \in \{2, 3, 4, 6\}$ and therefore

$$
\frac{W(E/K_v, \tau_v)}{W(E/K_v, (1 \oplus \eta)_v)} = 1 = (-1)^{\ord_p(C_v)}
$$

by 2.b.i (in section 2.3.4.1).

ii. If $r$ is odd, then $q \equiv 1[\epsilon] \iff p \equiv 1[\epsilon]$ and:

$$
\frac{W(E/K_v, \tau_v)}{W(E/K_v, (1 \oplus \eta)_v)} = \begin{cases} 
1 & \text{if } q \equiv 1[\epsilon] \\
(-1)^{1+\eta_{nr}+1+\eta_v} & \text{if } q \equiv -1[\epsilon]
\end{cases}
$$

if $\epsilon = 3, 4, 6.$

A. If $I_v = C_p$, then $\eta_{nr} = \eta_v$ and $\frac{W(E/K_v, \tau_v)}{W(E/K_v, (1 \oplus \eta)_v)} = 1.$

B. If $I_v = D_{2p}$, then $\eta_{nr} \neq \eta_v$ and:

$$
\frac{W(E/K_v, \tau_v)}{W(E/K_v, (1 \oplus \eta)_v)} = \begin{cases} 
1 & \text{if } q \equiv 1[\epsilon] \\
-1 & \text{if } q \equiv -1[\epsilon]
\end{cases}
$$

and $\epsilon = 3, 4, 6.$

In both cases, we obtain for the values of $\frac{W(E/K_v, \tau_v)}{W(E/K_v, (1 \oplus \eta)_v)}$ exactly the same table as for the values of $(-1)^{\ord_p(C_v)}$, depending on $p$ modulo 12:

| $p \mod 12$ | 1 | 5 | 7 | 11 |
|-------------|---|---|---|----|
| $III, III^*$ ($\epsilon = 6$) | 1 | -1 | 1 | -1 |
| $III, III^*$ ($\epsilon = 4$) | 1 | 1 | -1 | -1 |
| $IV, IV^*$ ($\epsilon = 3$) | 1 | -1 | 1 | -1 |
| $I_v^*$ ($\epsilon = 2$) | 1 | 1 | 1 | 1 |
(b) The case $l_v \neq p$:

In this case, the explicit formula of Rohrlich cannot be used, since $l_v$ can be 2 or 3.

Let $\sigma$ be a representation $\sigma : \text{Gal} (\overline{K_v} / K_v) \rightarrow GL(V_v)$ with finite image; let $\sigma'_{E/K_v} : WD(\overline{K_v} / K_v) \rightarrow GL(V)$ be the representation of the Weil-Deligne group associated to the elliptic curve given by $(\sigma_{E/K_v}, N) = (\sigma_{E/K_v}, 0)$ (because the reduction is potentially good), therefore this is simply a representation of the Weil group $W(\overline{K_v} / K_v)$ (because $N = 0$) and

$$\sigma'_{E/K_v} \otimes \sigma = \sigma_{E/K_v} \otimes \sigma : W(\overline{K_v} / K_v) \rightarrow GL(W),$$

where $W = V \otimes V_\sigma$, is also a representation of the Weil group.

We first recall the link between $\varepsilon$-factors and root numbers:

$$W(E/K_v, \sigma) = \frac{\varepsilon(\sigma_{E/K_v} \otimes \sigma, \psi, dx)}{\varepsilon(\sigma_{E/K_v} \otimes \sigma, \psi, dx)} = \varepsilon(\sigma'_{E/K_v} \otimes \sigma, \psi, dx_{\psi}),$$

where $dx$ is any Haar measure, $\psi$ is any additive character of $K_v$ and $dx_{\psi}$ the self-dual Haar measure with respect to $\psi$ on $K_v$.

Here, we choose an additive character $\psi$ for which the Haar measure $dx_{\psi}$ takes values (on open compact subsets of $K_v$) in $\mathbb{Z}_p[\zeta_p]$, where $\zeta_p$ is a primitive $p$-th root of unity. For example, if the conductor of $\psi$ is trivial, then the values of $dx_{\psi}$ lie in $l_v^2 \cup \{0\} \subset \mathbb{Z}_p[\zeta_p]$.

In one of his articles (2 p.548), Deligne gives a description of the $\varepsilon$-factors in terms of $\varepsilon_0$-factors; in our settings this gives:

$$\varepsilon(\sigma_{E/K_v} \otimes \sigma, \psi, dx_{\psi}) = \varepsilon_0(\sigma_{E/K_v} \otimes \sigma, \psi, dx_{\psi}) \det(-\nu(\phi) \mid W^{I(v)}),$$

where $\phi$ is the geometric Frobenius at $v$ and $I(v) = \text{Gal}(\overline{K_v} / K_v^{ur})$. Recall that, since $l_v \neq p$, the inertia group of $D_{2p}$ is $I_v = C_p$.

i. If $E$ has additive reduction, denote by $F$ the smallest Galois extension of $K_v^{ur}$ such that $E$ has good reduction over $F$ and set $\Phi = \text{Gal}(F / K_v)$; then the restriction of $\sigma_{E/K_v}$ to $I(v)$ factors through $\Phi$.

It is known that:

• For $l_v \geq 5$, $\Phi$ is cyclic of order $\varepsilon = \frac{12}{\text{pgcd}(5, 12)}$ (dividing 12).
• For $l_v = 3$, $|\Phi| \in \{2, 3, 4, 6, 12\}$.
• For $l_v = 2$, $|\Phi| \in \{2, 3, 4, 6, 8, 24\}$.

For a more precise description of $\Phi$, see, for example, [1] or [6].

The representation $\sigma_{E/K} \otimes \sigma$ ($\sigma = \tau_v$ or $(1 \otimes \eta)_v$) restricted to $I(v)$ factors through a quotient $H$ of $I(v)$ which admits $\Phi$ and $C_p$ as quotients.

We have:

$$(V \otimes V_\sigma)^{I(v)} = (V \otimes V_\sigma)^H = \text{Hom}_H(V^*, V_\sigma) = \text{Hom}((V^{\Phi})^*, V_\sigma^{C_p})$$

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This completes the proof of Theorem 2.7. ■

Remark 2.17 This proof can be adjusted to work in the case \( \text{Gal}(L/K) \simeq D_{2p^n} \), the computations are almost the same. The idea to reduce the proof to the case of a \( D_{2p} \)-extension, using Galois invariance of Rohrlich, was suggested to me by Tim Dokchitser.
3 Appendix

The purpose of this appendix is to make a small improvement on Theorem 6.7 of [5]. The interest of this improvement is that Proposition 6.12 of [5] (which is the same statement as Theorem 2.1 for $p \equiv 3 \mod 4$) will no longer rely on the "truly painful case of additive reduction" anymore (see [4] p.30). In fact, we use the passage to the global case to avoid all places of additive reduction, not just those above 2 and 3. Since we have proved the result for $p \geq 5$ (Theorem 2.1) without using any global parity results at all, for us this is of interest essentially in the case $p = 3$.

We start by recalling the definition of an elliptic curve being close to another one:

**Proposition 3.1** Let $E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$ be an elliptic curve over a non archimedean local field $K$ (with valuation $v$ and residue characteristic $p$) and $F/K$ a finite Galois extension. There exists $\varepsilon > 0$ such that every elliptic curve $E' : y^2 + a'_1 xy + a'_3 y = x^3 + a'_2 x^2 + a'_4 x + a'_6$ over $K$ satisfying:

$$\forall i |a'_i - a_i|_v < \varepsilon,$$

the curves $E$ and $E'$ have the same conductor, valuation of the minimal discriminant, local Tamagawa factors, $C(E/F', \frac{ds}{2y + a_1 x + a_3})$, root numbers and the Tate module as a $\text{Gal}(\bar{K}/K)$-module for $l \neq p$, in all intermediate fields $F'$ of $F/K$. We will say that $E'$ is close to $E/K$.

**Proof.** This is Proposition 3.3 of [5]. \[\blacksquare\]

We now state the minor improvement of Theorem 6.7 of [5]:

**Theorem 3.2** Let $K$ a local non archimedean field of characteristic 0 and $F/K$ a finite Galois extension. Let $F/K$ be a Galois extension of totally real fields and $v_0$ a place of $K$ such that:

- $v_0$ admits a unique place $\bar{v}_0$ of $F$ above it
- $K_{v_0} \simeq K$ and $F_{\bar{v}_0} \simeq F$.

Such an extension exists (see Lemma 3.1 of [5]). Let $E/K$ be an elliptic curve with additive reduction. Then there exists an elliptic curve $E/K$ such that:

- $E$ has semi-stable reduction for all $w \neq v_0$
- $j(E)$ is not an integer (i.e. $j(E) \notin \mathcal{O}_K$)
- $E/K_{v_0}$ is close to $E/K$.

**Proof.** We first choose an elliptic curve $E/K$ such that $E/K_{v_0}$ is close to $E/K$ (this is possible, by Proposition 3.1). Now the goal is to remove all places of additive reduction by changing $E/K$ to an elliptic curve satisfying the three conditions of the theorem. Let $E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$ with $a_i \in \mathcal{O}_K$. 

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If we want a place not to be of additive reduction we have to impose one of the two following conditions:

- The valuation \( w(\Delta) \) is zero (in this case \( w \) is of good reduction).
- The valuation \( w(c_4) \) is zero (in this case \( w \) is of good or multiplicative reduction depending on \( w(\Delta) = 0 \) or \( >0 \)).

Let \( v \neq v_0 \) be a place of \( K \) not above 2.

To get the condition "\( j(E) \) is not an integer" it is sufficient to make \( v \) a multiplicative place (\( v \) is multiplicative \( \iff v(j(E)) < 0 \)).

**Step 1: Make semi-stable all places \( w \neq v_0 \) above 2**

Denote by \( v_{2,1}, \ldots, v_{2,r} \) these places.

In this case: \( [v_{2,i}(a_1) = 0 \Rightarrow v_{2,i}(c_4) = 0 \ (c_4 = (a_1^2 + 4a_2)^2 - 24a_1a_3 - 48a_4)] \).

Let \( p_0 \) and \( p_{2,i} \) be the primes ideals associated to \( v_0 \) and \( v_{2,i} \).

By the Chinese remainder theorem, there exists \( d_1 \in \mathcal{O}_K \) such that:

- \( d_1 \equiv 0 \mod p_0^n \) (i.e. \( v_0(d_1) \geq n \)).
- \( d_1 \equiv 1 - a_1 \mod p_{2,i} \forall i \in \{1, \ldots, r\} \) (i.e. \( v_{2,i}(a_1 + d_1) = 0 \)).
- \( d_1 \equiv -a_1 \mod p \) (\( p \) associated to \( v \neq v_0 \)).

So, if we let \( a'_1 = a_1 + d_1 \) for \( n \) big enough we get the curve \( y^2 + a'_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \) which is close to \( E/K \), \( v_{2,i}(a'_1) = v_{2,i}(a_1 + d_1) = 0 \) \( \forall i \in \{1, \ldots, r\} \) and \( v(a'_1) > 0 \).

**Step 2: Make \( v \) semi-stable**

By the Chinese remainder theorem, there exist \( d_2, d_3, d_4 \in \mathcal{O}_K \) such that:

- \( d_2 \equiv 0 \mod p_0^n \) (i.e. \( v_0(d_2) \geq n \)).
  
  \( d_2 \equiv 1 - a_2 \mod p \) (so \( v(a_2 + d_2) = 0 \)).
- \( d_3 \equiv 0 \mod p_0^n \) (i.e. \( v_0(d_3) \geq n \)).
  
  \( d_3 \equiv -a_3 \mod p \) (so \( v(a_3 + d_3) > 0 \)).
- \( d_4 \equiv 0 \mod p_0^n \) (i.e. \( v_0(d_4) \geq n \)).
  
  \( d_4 \equiv -a_4 \mod p \) (so \( v(a_4 + d_4) > 0 \)).

So, if we let \( a'_i = a_1 + d_1 \), \( i \in \{2, 3, 4\} \), for \( n \) big enough we get:

\( E' : y^2 + a'_1xy + a'_3y = x^3 + a'_2x^2 + a'_4x + a_6 \) is close to \( E/K \) (Proposition 3.1).

Furthemore:

- \( c'_4 = (a_1^2 + 4a_2)^2 - 24a_1a_3 - 48a_4 \)
  
  \( v(a'_1) > 0 \)
  
  \( v(a'_2) > 0 \)
  
  \( v(a'_3) > 0 \)
  
  \( v(a'_4) = 0 \)
  
  \( v(c'_4) = 0 \).

The curve \( E' : y^2 + a'_1xy + a'_3y = x^3 + a'_2x^2 + a'_4x + a_6 \) is close to \( E/K \), \( \forall w \neq v_0 \) above 2 \( w(c'_4) > 0 \), and \( v(c'_4) = 0 \). Since \( c'_4 \) does not depend on \( a_6 \), we can modify \( a_6 \) to allow places \( w \neq v_0 \) such that \( w(c'_4) > 0 \) to become places of good reduction (since \( c'_4 \) will be unchanged, some places of good reduction can become of multiplicative reduction but not of additive reduction) and such that \( v \) is of multiplicative reduction (\( v(j(E)) < 0 \)).

**Step 3. Turn additive reduction places into good reduction ones and make \( v \) multiplicative.**

Let \( v_1, \ldots, v_r, v_{r+1}, \ldots, v_t \) be the places where \( v_i(c'_4) > 0 \), \( v_i \neq v_0 \) (\( \neq v \) and not
By the Chinese remainder theorem, there exists $c$ such that:

- $c \equiv 0 \mod p_i^0$ (i.e. $v_0(c) \geq n$)
- $c \equiv 0 \mod p_i \forall i \in \{1, \ldots, r\}$ (i.e. $v_i(c) > 0$)
- $16c \equiv \alpha_i - \gamma \mod p_i \forall i \in \{r+1, \ldots, t\}$ (where $\alpha_i \neq 0, \gamma \mod p_i$)

Finally, if we let $a'_0 = a_0 + c$ for $n$ big enough, we get:

$E'': y^2 + a'_1xy + a'_3y = x^3 + a'_2x^2 + a'_4x + a'_6$

and we see that with this choice:

- $v_1, \ldots, v_t$ are all places of good reduction for $E''$
- $v$ is a place of multiplicative reduction for $E''$

This completes the proof.

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