Interlacing properties of zeros of multiple orthogonal polynomials

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Abstract

It is well known that the zeros of orthogonal polynomials interlace. In this paper we study the case of multiple orthogonal polynomials. We recall known results and some recursion relations for multiple orthogonal polynomials. Our main result gives a sufficient condition, based on the coefficients in the recurrence relations, for the interlacing of the zeros of neighboring multiple orthogonal polynomials. We give several examples illustrating our result.

1 Preliminaries

An important and useful relation between zeros of consecutive polynomials is the following separation of zeros

\[ y_1 < x_1 < y_2 < \ldots < x_n < y_{n+1} \]

where \( x_1, \ldots, x_n \) and \( y_1, \ldots, y_{n+1} \) are the zeros of the polynomials \( p_n \) and \( p_{n+1} \) respectively. The most important consequence of this property is that it simplifies estimates of the ratio of polynomials, needed for proving the asymptotic behavior. Among other things, the interlacing property of zeros guarantees convergence of the approximation of zeros by zeros of certain special functions using a fixed point method (see [7]) and it is equivalent to the positivity of weights in quadrature formulas (e.g., [6]).

The zeros of orthogonal polynomials interlace as a consequence of the Christoffel-Darboux formula or the recurrence relation (e.g., [8]). It holds...
even more general for a Sturm sequence of polynomials. We are interested in finding the interlacing property for multiple orthogonal polynomials from their recurrence relations.

In the first subsections 1.1 – 1.3 we briefly introduce multiple orthogonal polynomials and their recursion. In Section 2 we summarize known results, which are for the case of an AT system, a Nikishin system and a mixed type system with two Nikishin systems. We present our main result in Subsection 2.1 which states that the positivity of the coefficients $a_{\vec{n},j}$ in the nearest neighbor recurrence relations ensures the interlacing of zeros of neighboring multiple orthogonal polynomials. It is worth noting that this condition does not depend directly on whether the measures $(\mu_1, \ldots, \mu_r)$, with respect to which the polynomials satisfy orthogonality conditions, form a certain special system. Section 3 contains several examples of our result. It gives examples of multiple Hermite polynomials (see 3.1), multiple Charlier polynomials (see 3.2), both kinds of multiple Meixner polynomials (see 3.3 and 3.7), multiple Krawtchouk polynomials (see 3.4), and both kinds of multiple Laguerre polynomials (see 3.5 and 3.6).

In the following we will use the standard notation, i.e., by $\vec{n}$ we denote the multi-index $(n_1, \ldots, n_r)$, where $n_j \in \mathbb{N}$. By $|\vec{n}|$ we denote the length of the multi-index, i.e., $|\vec{n}| = \sum_{i=1}^r n_i$.

### 1.1 Multiple orthogonal polynomials

As is well known, there are two types of multiple orthogonal polynomials, type I and type II (e.g., [14, 9]). In this paper we will mainly investigate multiple orthogonal polynomials of type II, namely the unique monic polynomial of degree $|\vec{n}|$ satisfying the orthogonality conditions

$$
\int_{\mathbb{R}} x^k P_{\vec{n}}(x) d\mu_j(x) = 0, \quad \text{for } 0 \leq k \leq n_j - 1, \quad \text{and } j = 1, \ldots, r, \quad (2)
$$

with respect to $r$ different positive Borel measures $(\mu_1, \ldots, \mu_r)$ that are absolutely continuous with respect to a measure $d\mu$, i.e., for all $j = 1, \ldots, r$ we have $d\mu_j(x) = w_j(x) d\mu(x)$. We say that the multi-index $\vec{n}$ is normal when this monic polynomial $P_{\vec{n}}$ is unique.

Type I multiple orthogonal polynomials are polynomials $(A_{\vec{n},1}, \ldots, A_{\vec{n},r})$,
where $A_{\vec{n},j}$ has degree $\leq n_j - 1$ such that
\[
\sum_{j=1}^{r} \int_{\mathbb{R}} x^k A_{\vec{n},j}(x) \, d\mu_j(x) = 0, \quad \text{for } 0 \leq k \leq \vert \vec{n} \vert - 2, \text{ and } j = 1, \ldots, r, \quad (3)
\]
and
\[
\sum_{j=1}^{r} \int_{\mathbb{R}} x^{\vert \vec{n} \vert - 1} A_{\vec{n},j}(x) \, d\mu_j(x) = 1, \quad \text{for } j = 1, \ldots, r. \quad (4)
\]
These polynomials are uniquely defined if and only if the multi-index $\vec{n}$ is normal. We will use the notation \( Q_{\vec{n}}(x) = \sum_{j=1}^{r} A_{\vec{n},j}(x)w_j(x) \).

Multiple orthogonal polynomials are intimately related to Hermite-Padé approximants and often they are also called Hermite-Padé polynomials.

\section*{1.2 Recurrence relations for multiple orthogonal polynomials}

Recall that orthogonal polynomials satisfy a three term recurrence relation
\[
xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x), \quad n \geq 0,
\]
with initial values \( p_0 = 1 \) and \( p_{-1} = 0 \) (e.g., \cite{[8]}). Multiple orthogonal polynomials also satisfy finite order recurrence relations. There are two types of recurrence relations for multiple orthogonal polynomials. Often only the multiple orthogonal polynomials are considered with the following sequences of indices

- diagonal: $\vec{n} = (n, \ldots, n)$, $\vert \vec{n} \vert = rn$,
- step-line: $\vec{n} = (m+1, \ldots, m+1, m, \ldots, m)$, $\vert \vec{n} \vert = rm + s, 0 \leq s \leq r - 1$.

Any multi-index $\vec{n}$ on the step-line may be identified by the value of the length of $\vec{n}$. Observe that every natural number $n$ may be written as $n = mr + s$ with $0 \leq s \leq r - 1$ and then the corresponding multi-index $\vec{n}$ is the one above where the value $m + 1$ is repeated $s$ times. If all multi-indices are normal, then as a consequence of the orthogonality conditions we have the following finite order recurrence relation for the polynomials on the step-line:
\[
\begin{align*}
xP_n(x) &= P_{n+1}(x) + a_{n,n}P_n(x) + a_{n,n-1}P_{n-1}(x) + a_{n,n-2}P_{n-2}(x) \\
&\quad + \ldots + a_{n,n-r}P_{n-r}(x), \quad n \geq 0,
\end{align*}
\]
with initial conditions \( P_{-r} = \ldots = P_{-1} = 0 \) and \( P_0 = 1 \).

There are several ways to increase the degree of multiple orthogonal polynomials, since we are working with multi-indices. Another recurrence relation only uses the nearest neighbors of \( P_{\vec{n}} \). This will be described in the next subsection.

### 1.3 Nearest neighbor recurrence relations

Here we briefly introduce a second recursion for multiple orthogonal polynomials. For more details we refer to [9]. The nearest neighbor recurrence relation (5) connects type II multiple orthogonal polynomial \( P_{\vec{n}} \) with the polynomial of degree one higher \( P_{\vec{n} + \vec{e}_k} \) and all the neighbors of degree one lower \( P_{\vec{n} - \vec{e}_j} \) for \( j = 1, \ldots, r \). To derive it, we proceed as follows. Since both \( P_{\vec{n}} \) and \( P_{\vec{n} + \vec{e}_k} \) are monic polynomials, the difference

\[
xP_{\vec{n}}(x) - P_{\vec{n} + \vec{e}_k}(x)
\]

is a polynomial of degree \( \leq |\vec{n}| \). Choosing \( b_{\vec{n},k} \) appropriately we can also cancel the term containing \( x|\vec{n}| \). Hence

\[
xP_{\vec{n}}(x) - P_{\vec{n} + \vec{e}_k}(x) - b_{\vec{n},k}P_{\vec{n}}(x)
\]

is a polynomial of degree \( \leq |\vec{n}| - 1 \). It is easy to see that this polynomial is orthogonal to all polynomials of degree \( n_j - 2 \) with respect to \( \mu_j \), for \( j = 1, \ldots, r \). Finally, the polynomials \( P_{\vec{n} - \vec{e}_1}, \ldots, P_{\vec{n} - \vec{e}_r} \) form a basis for the linear space of all polynomials of degree \( \leq |\vec{n}| - 1 \) that satisfy the orthogonality conditions

\[
\int_{\mathbb{R}} x^k P_{\vec{n}}(x) d\mu_j(x) = 0, \quad \text{for } k \leq n_j - 2, \text{ and } j = 1, \ldots, r.
\]

Hence we can write the polynomial \( xP_{\vec{n}}(x) - P_{\vec{n} + \vec{e}_k}(x) - b_{\vec{n},k}P_{\vec{n}}(x) \) as a linear combination of the polynomials \( P_{\vec{n} - \vec{e}_1}, \ldots, P_{\vec{n} - \vec{e}_r} \), with some coefficients \( a_{\vec{n},j} \). This forms the so-called nearest neighbor recurrence relation

\[
xP_{\vec{n}}(x) = P_{\vec{n} + \vec{e}_k}(x) + b_{\vec{n},k}P_{\vec{n}}(x) + \sum_{j=1}^{r} a_{\vec{n},j}P_{\vec{n} - \vec{e}_j}.
\]  

There are \( r \) such relations for each \( k = 1, \ldots, r \). Note that the coefficients \( a_{\vec{n},j} \) do not depend on \( k \). Indeed, the coefficients \( a_{\vec{n},j} \) can be computed directly
from this recursion using the orthogonality conditions (2), giving

\[
a_{\vec{n},j} = \frac{\int_{\mathbb{R}} x^{n_j} P_{\vec{n}}(x) \, d\mu_j(x)}{\int_{\mathbb{R}} x^{n_j-1} P_{\vec{n}}(x) \, d\mu_j(x)},
\]

while the coefficients \(b_{\vec{n},k}\) can be computed by multiplying both sides of the equation by the multiple orthogonal polynomials of type I and applying the bi-orthogonality (see [4] Chapter 23, Theorem 23.1.6), giving

\[
b_{\vec{n},k} = \int_{\mathbb{R}} x P_{\vec{n}}(x) Q_{\vec{n}+\vec{e}_k}(x) \, d\mu(x). \tag{7}
\]

Similar recursion relations hold for type I multiple orthogonal polynomials, see [4, 9].

## 2 Interlacing properties

D. Kershaw proved in [5] the interlacing property for the zeros of polynomials orthogonal with respect to a Markov system. If the sequence \(\phi_1(x), \phi_2(x), \ldots\) forms an integrable Markov system on \((a, b)\) (in particular a Chebyshev system) then there exists a unique polynomial \(q_n\) of degree equal to \(n\) with real, simple zeros in \([a, b]\), such that

\[
\int_a^b q_n(x)\phi_i(x) \, dx = 0, \quad \text{for } i = 1, 2, \ldots, n,
\]

and if \(q_{n+1}\) is a polynomial of degree exactly \(n + 1\) with real, simple zeros satisfying the same orthogonality condition, then the zeros of \(q_n\) and \(q_{n+1}\) interlace.

Later, in [2], this was generalized in the sense that the Lebesgue measure was replaced by an arbitrary Borel measure, which was needed to prove interlacing in the case of multiple orthogonal polynomials for a Nikishin system. The authors showed that if \(N(\sigma_1, \ldots, \sigma_r)\) is an arbitrary Nikishin system of \(r\) measures, then the zeros of the type II multiple orthogonal polynomials \(P_{\vec{n}}\) and \(P_{\vec{n}+\vec{e}_k}\) interlace and the zeros of \(P_{\vec{n}}\) and \(Q_{\vec{n},j}\) interlace, where

\[
Q_{\vec{n},j}(x) = \int_{\mathbb{R}} \frac{P_{\vec{n}}(x) - P_{\vec{n}}(t)}{x - t} \, ds_j(t),
\]
and \( s_j \) (for \( 1 \leq j \leq r \)) are measures defined by induction using
\[
\langle \sigma_i, \sigma_j \rangle(x) = \int \frac{d\sigma_j(t)}{x-t} d\sigma_i(x) \quad \langle \sigma_1, \sigma_2, \ldots, \sigma_n \rangle = \langle \sigma_1, \langle \sigma_2, \ldots, \sigma_n \rangle \rangle,
\]
i.e., \( s_1 = \langle \sigma_1 \rangle = \sigma_1, s_2 = \langle \sigma_1, \sigma_2 \rangle, \ldots, s_r = \langle \sigma_1, \ldots, \sigma_r \rangle \).

In [1] the authors generalized the above result, proving that the zeros of \( P_{\bar{n},k} \) and \( P_{\bar{n}+\bar{e}_k,k} \) interlace, where the sequence of polynomials \( P_{\bar{n},k} \) is defined as the sequence of polynomials of degree \( n_k + \ldots + n_r \) with \( P_{\bar{n},0} = P_{\bar{n},r+1} \equiv 1 \), that is orthogonal to the measures which are given by
\[
d\mu_k(x) = \frac{|H_{\bar{n},k}(x)|}{|P_{\bar{n},k-1}(x)P_{\bar{n},k+1}(x)|} d\sigma_k(x),
\]
where \( H_{\bar{n},k} = \frac{P_{\bar{n},k-1} \Psi_{\bar{n},k-1}}{P_{\bar{n},k}} \) and \( \Psi_{\bar{n},k} \) are defined recursively as follows, for each \( \bar{n} \) we set \( \Psi_{\bar{n},0}(x) = P_{\bar{n}}(x) \) and
\[
\Psi_{\bar{n},k}(x) = \int \frac{\Psi_{\bar{n},k-1}(t)}{x-t} d\sigma_k(t), \quad \text{for } k = 1, \ldots, r.
\]

In this notation the case \( k = 1 \) corresponds to result in [2], i.e., the type II multiple orthogonal polynomials \( P_{\bar{n}} \) are \( P_{\bar{n},1} \).

Using arguments as in [3, Property (P)], one can obtain the interlacing property for the type II multiple orthogonal polynomials with respect to measures that form an AT system. Recall that a system of measures \( (\mu_1, \ldots, \mu_r) \) forms an AT system on \([a, b]\) if the measures \( \mu_j \) are absolutely continuous with respect to a measure \( \mu \) on \([a, b]\), with \( d\mu_j(x) = w_j(x) d\mu(x) \), and
\[
\{w_1, xw_1, \ldots, x^{n_1-1}w_1, w_2, \ldots, x^{n_r-1}w_r\}
\]
is a Chebyshev system on \([a, b]\). The zeros of type II multiple orthogonal polynomials for a AT system are real and simple (e.g., [4]). The results in [5] and [2] can then be summarized as

**Theorem 2.1.** Suppose that the measures \( (\mu_1, \ldots, \mu_r) \) form an AT system on \([a, b]\). Then the zeros of type II multiple orthogonal polynomials \( P_{\bar{n}} \) and \( P_{\bar{n}+\bar{e}_k} \) interlace.

**Proof.** Let \( A \) and \( B \) be two constants such that \(|A| + |B| \neq 0\). Consider \( AP_{\bar{n}} + BP_{\bar{n}+\bar{e}_k} \). Suppose that this polynomial has a multiple real zero at \( c \in \mathbb{R} \), i.e., at least a double zero
\[
AP_{\bar{n}}(x) + BP_{\bar{n}+\bar{e}_k}(x) = (x - c)^2 Q(x),
\]
for some polynomial $Q$ of degree $|n| - 1$. Denote

$$W_n(x_1, \ldots, x_{|n|}) = \det \begin{bmatrix} w_1(x_1) & \cdots & x_1^{n_1-1}w_1(x_1) & w_2(x_1) & \cdots & x_1^{n_r-1}w_r(x_1) \\ w_1(x_2) & \cdots & x_2^{n_1-1}w_1(x_2) & w_2(x_2) & \cdots & x_2^{n_r-1}w_r(x_2) \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ w_1(x_{|n|}) & \cdots & x_{|n|}^{n_1-1}w_1(x_{|n|}) & w_2(x_{|n|}) & \cdots & x_{|n|}^{n_r-1}w_r(x_{|n|}) \end{bmatrix}$$

for points $x_1, \ldots, x_{|n|}$ in $[a, b]$. This determinant vanishes if and only if $x_i = x_j$ for some $i, j$. We consider $W_n(x_1, \ldots, x_{|n|})$ as a function of $x_i$ and it changes sign only if $x$ passes through the points $x_1, \ldots, x_{|n|}$. If $Q$ is a polynomial of degree $|n| - 1$ with real coefficients, having precisely $r$ real zeros $x_1, \ldots, x_r$ on $[a, b]$ at which $Q$ changes sign then $Q(x)W_{r+1}(x, x_1, \ldots, x_r)$ does not change sign on $[a, b]$ (this is a Property (P) in [5]). In particular

$$\int_a^b (x - c)^2Q(x)W_{r+1}(x, x_1, \ldots, x_r) \, d\mu(x) \neq 0.$$

Using the orthogonality conditions

$$\int_a^b x^k(x - c)^2Q(x)w_j(x) \, d\mu(x) = 0, \quad k = 0, \ldots, n_j - 1, \quad j = 1, \ldots, r,$$

we get

$$\int_a^b (x - c)^2Q(x)W_{r+1}(x, x_1, \ldots, x_r) \, d\mu(x) = 0,$$

which forces a contradiction. This proves that $AP_{\bar{n}} + BP_{\bar{n}+\bar{e}_k}$ has only simple zeros on the real line. Hence the linear system of equations

$$\begin{bmatrix} P_{\bar{n}}(x) & P_{\bar{n}+\bar{e}_k}(x) \\ P'_{\bar{n}}(x) & P'_{\bar{n}+\bar{e}_k}(x) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

has only the trivial solution $A = B = 0$ for every $x \in \mathbb{R}$. Therefore the matrix on the left hand side has non-zero determinant and by continuity and the behavior for large $x$ we conclude that for all $x \in \mathbb{R}$

$$P_{\bar{n}}(x)P'_{\bar{n}+\bar{e}_k}(x) - P'_{\bar{n}}(x)P_{\bar{n}+\bar{e}_k}(x) > 0. \quad (9)$$
If we now fix two consecutive zeros $x_k$, $x_{k+1}$ of $P_{\vec{n}}$, then $P_{\vec{n}}'(x_k)P'_{\vec{n}}(x_{k+1}) < 0$. From the inequality (9) we get

$$P_{\vec{n}}'(x_k)P_{\vec{n}+\vec{e}_k}(x_k) < 0 \quad \text{and} \quad P_{\vec{n}}'(x_{k+1})P_{\vec{n}+\vec{e}_k}(x_{k+1}) < 0.$$ 

Hence

$$P_{\vec{n}+\vec{e}_k}(x_k)P_{\vec{n}+\vec{e}_k}(x_{k+1}) < 0,$$

which means that there is at least one zero $y_k$ of $P_{\vec{n}+\vec{e}_k}$ between $x_k$ and $x_{k+1}$. If $x_{\vec{|n|}}$ is the greatest zero of $P_{\vec{n}}$ then $P_{\vec{n}}'(x_{\vec{|n|}}) > 0$ and from inequality (9) we have $P_{\vec{n}+\vec{e}_k}(x_{\vec{|n|}}) < 0$. But since $P_{\vec{n}+\vec{e}_k}$ is a monic polynomial, the sign of $P_{\vec{n}+\vec{e}_k}(x)$ for large enough $x$ is positive, which means that there is at least one zero of $P_{\vec{n}+\vec{e}_k}$ on the right side of $x_{\vec{|n|}}$. Similarly there is at least one zero $y_1$ of $P_{\vec{n}+\vec{e}_k}$ on the left side of the least zero $x_1$ of $P_{\vec{n}}$. Since the zeros of both polynomials are real and simple we have

$$y_j < x_j < y_{j+1} \quad \text{for } j = 1, \ldots, \vec{|n|}.$$

There is an alternative definition of an AT system: instead of zeros one may consider sign changes. In this case instead of using Property (P) from [5] in the crucial step, one uses the normality of the index $\vec{n}$, which holds because $(x-c)^2(\mu_1, \ldots, \mu_r)$ is an AT system. Then the polynomial $Q$ of degree $\vec{|n|} - 1$ satisfies the orthogonality conditions of the multiple orthogonal polynomial with multi-index $\vec{n}$ for the AT-system $(x-c)^2(\mu_1, \ldots, \mu_r)$, which is not possible because of the normality.

In [3] the authors investigate a slightly more general notion of multiple orthogonal polynomials, i.e., the case of mixed type multiple orthogonal polynomials. They investigate them for two Nikishin systems. Let $N(\sigma_1^1, \ldots, \sigma_{r_1}^1)$ and $N(\sigma_1^2, \ldots, \sigma_{r_2}^2)$ be two Nikishin systems of $r_1$ and $r_2$ measures respectively, which come from the same basis measure $\sigma_0^1 = \sigma_0^2$. The polynomials $a_{n,0}, \ldots, a_{n,r_1}$, where $n = (n_1, n_2)$, $n_i = (n_{i,0}, n_{i,1}, \ldots, n_{i,r_1}) \in \mathbb{Z}^{r_i+1}$ for $i = 1, 2$, such that

1. the degree of $a_{n,j} \leq n_{1,j} - 1$, $j = 0, \ldots, r_1$, not all identically equal to zero,
2. for $i = 0, \ldots, r_2$ they satisfy

$$\int x^k \left( a_{n,0}(x) + \sum_{j=1}^{r_1} a_{n,j}(x) \hat{s}_{1,j}^1(x) \right) ds_{0,k}^2(x) = 0, \quad k = 0, \ldots, n_{2,i} - 1,$$

where $s_{j,k}^1$ are measures (and $\hat{s}_{j,k}^1$ its Cauchy transform) defined by induction using (8), i.e., $s_{0,0} = \langle \sigma_0 \rangle = \sigma_0$ and $s_{j,k} = \langle \sigma_j, \ldots, \sigma_k \rangle$,

are called mixed type multiple orthogonal polynomials. They show ([3, Theorem 3.5]) that the zeros of

$$A_{n,j} = a_{n,0} + \sum_{i=j}^{r_1} a_{n,i}(x) \hat{s}_{j+1,i}^1(x)$$

interlace, where $n^\ell = (n_1 + \vec{e}_{\ell_1}, n_2 + \vec{e}_{\ell_2}), 0 \leq \ell_1 \leq r_1$ and $0 \leq \ell_2 \leq r_2$. Note that some special cases of this theorem are

1. if $r_1 = 0$ then the zeros of $a_{n,0}$ and $a_{n^\ell,0}$ interlace, i.e., in our notation, that the zeros of type II multiple orthogonal polynomials $P_{\vec{n}}$ and $P_{\vec{n^\ell} + \vec{e}_k}$ interlace,

2. if $r_2 = 0$ and $j = 0$ then the zeros of

$$\sum_{i=0}^{r_1} a_{n,i}(x) \hat{s}_{1,i}^1(x) \quad \text{and} \quad \sum_{i=0}^{r_1} a_{n^\ell,i}(x) \hat{s}_{1,i}^1(x)$$

interlace, i.e., in our notation, that the zeros of the type I multiple orthogonal polynomials $Q_{\vec{n}}$ and $Q_{\vec{n^\ell} + \vec{e}_k}$ interlace.

### 2.1 Interlacing properties from the recurrence relation

We are interested in finding the interlacing property from the nearest neighbor recurrence relations. Note that $a_{n,j} \neq 0$ in ([5]) whenever all multi-indices are normal and $n_j > 0$. Indeed these coefficients are defined as a ratio of two integrals ([6]). Both integrals are non-zero due to normality of $\vec{n}$ and $\vec{n} + \vec{e}_k$. If

$$\int_{\mathbb{R}} x^{n_j} P_{\vec{n}}(x) d\mu_j(x) = 0$$
then \( P_n \) would satisfy the same orthogonality conditions as \( P_{\vec{n} + \vec{e}_j} \), which shows that there exist a polynomial of degree \( \leq |\vec{n}| \) satisfying the orthogonality conditions for a multi-index of length \( |\vec{n}| + 1 \) which contradicts the normality of \( \vec{n} + \vec{e}_k \).

We assume that \( P_{\vec{n}} = 0 \) if \( n_j < 0 \) for at least one \( j \) and for \( |\vec{n}| = 0 \) we assume that \( P_{\vec{n}} = 1 \). Our main result in this paper is

**Theorem 2.2.** Suppose that the zeros of \( P_{\vec{n}} \) are real and simple, and all multi-indices are normal. If for all \( 1 \leq j \leq r \) and for all multi-indices \( \vec{n} \) one has \( a_{\vec{n},j} > 0 \) whenever \( n_j > 0 \), then the zeros of \( P_{\vec{n}} \) and \( P_{\vec{n} + \vec{e}_k} \) interlace for every \( k \) with \( 1 \leq k \leq r \).

**Proof.** We will use a proof by induction on the length of \( \vec{n} \). For \( |\vec{n}| = 1 \) we have one zero \( x_1 \) of \( P_{\vec{n}} \). Evaluating the nearest neighbor recurrence relation (5) for type II multiple orthogonal polynomials at \( x_1 \), we get

\[
P_{\vec{n} + \vec{e}_k}(x_1) = -\sum_{j=1}^{r} a_{\vec{n},j} P_{\vec{n} - \vec{e}_j}(x_1).
\]

From our assumption we know that \( P_{\vec{n} - \vec{e}_j} = 1 \) only for one \( j = j_0 \) and for the other \( j \) it is zero. It follows that the sign of \( P_{\vec{n} + \vec{e}_k} \) at \( x_1 \) is minus the sign of \( a_{\vec{n},j_0} \), hence it is negative. On the other hand, since \( P_{\vec{n} + \vec{e}_k} \) is monic and of degree 2, the sign of \( P_{\vec{n} + \vec{e}_k} \) for large enough and small enough \( x \) is necessarily positive. This means that the zero of \( P_{\vec{n}} \) is between two zeros of \( P_{\vec{n} + \vec{e}_k} \).

Suppose that for \( |\vec{n}| = m - 1 \) the zeros of \( P_{\vec{n}} \) and \( P_{\vec{n} + \vec{e}_k} \) interlace for all \( k = 1, \ldots, r \). This means that \( P_{\vec{n} + \vec{e}_k} \) at the zeros \( x_i \) of \( P_{\vec{n}} \) has alternating signs, i.e.,

\[
\text{sgn } P_{\vec{n} + \vec{e}_k}(x_i) = \begin{cases} 
(-1)^{i+1} & \text{if } |\vec{n}| \text{ is even} \\
(-1)^i & \text{if } |\vec{n}| \text{ is odd}
\end{cases}
\]

Consider the case \( |\vec{n}| = m \). Let \( x_1, \ldots x_m \) be the zeros of \( P_{\vec{n}} \). Evaluating the nearest neighbor recurrence relation (5) at \( x_i \) \((1 \leq i \leq m)\) we get

\[
P_{\vec{n} + \vec{e}_k}(x_i) = -\sum_{j=1}^{r} a_{\vec{n},j} P_{\vec{n} - \vec{e}_j}(x_i).
\]

Denote by \( y_1, \ldots y_{m+1} \) the zeros of \( P_{\vec{n} + \vec{e}_k} \). From the induction assumption we know that the zeros of \( P_{\vec{n} - \vec{e}_j} \) and \( P_{\vec{n}} \) interlace, i.e.,

\[
\text{sgn } P_{\vec{n} - \vec{e}_j}(x_i) = \begin{cases} 
(-1)^i & \text{if } |\vec{n}| \text{ is even} \\
(-1)^{i+1} & \text{if } |\vec{n}| \text{ is odd}
\end{cases}
\]
Therefore, since $a_{\vec{n},j} \geq 0$ and at least for one $j$ with $1 \leq j \leq r$ we have $a_{\vec{n},j} > 0$

$$
\text{sgn } P_{\vec{n}+\vec{e}_k}(x_i) = \begin{cases} 
(-1)^{i+1} & \text{if } |\vec{n}| \text{ is even} \\
(-1)^{i+2} & \text{if } |\vec{n}| \text{ is odd}
\end{cases}.
$$

Both polynomials $P_{\vec{n}}$ and $P_{\vec{n}+\vec{e}_k}$ are monic and therefore for large enough $x$ their sign is positive. But at the point $x_m$ (the largest zero of $P_{\vec{n}}$), the polynomial $P_{\vec{n}+\vec{e}_k}$ has negative sign. Therefore there is at least one zero $y_{m+1}$ of $P_{\vec{n}+\vec{e}_k}$ such that $x_m < y_{m+1}$. Similarly there is at least one zero of $P_{\vec{n}+\vec{e}_k}$ on the left hand side of the smallest zero of $P_{\vec{n}}$, i.e., $y_1 < x_1$. Consequently, since the zeros of $P_{\vec{n}+\vec{e}_k}$ are real and simple, we have exactly one zero of $P_{\vec{n}}$ between two consecutive zeros of $P_{\vec{n}+\vec{e}_k}$

$$y_j < x_j < y_{j+1} \quad \text{for } j = 1, \ldots, m.$$ 

3 Examples

Theorem 2.2 proves interlacing of zeros of many families of multiple orthogonal polynomials, such as multiple Hermite, Charlier, Meixner of the first kind, Krawtchouk, and Laguerre polynomials of the second kind. However, there are also examples where the positivity condition ($a_{\vec{n},j} > 0$ for all $\vec{n}$ with $n_j > 0$) is too strong, showing that the condition is sufficient but not necessary. For instance in the case of multiple Laguerre polynomials of the first kind, the coefficients $a_{\vec{n},j}$ are not all positive. A similar situation occurs in the case of multiple Meixner polynomials of the second kind, since in some sense Meixner polynomials are the discrete analogs of the Laguerre polynomials. Nevertheless both examples have the interlacing property, which follows from the fact that the measures, with respect to which these polynomials satisfy orthogonality conditions, form an AT system.

3.1 Multiple Hermite polynomials

Multiple Hermite polynomials (see [4]) are given by the Rodrigues formula

$$H_{\vec{n}}^\vec{c}(x) = (-1)^{|\vec{n}|} 2^{-|\vec{n}|} e^{x^2} \prod_{j=1}^{r} (e^{-c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{c_j x}) e^{-x^2}.$$
where \(c_1, \ldots, c_r\) are distinct real numbers. These polynomials are orthogonal with respect to measures \((\mu_1, \ldots, \mu_r)\) which are given by \(d\mu_j(x) = e^{-x^2+c_j x} \, dx\) on \((-\infty, \infty)\). These measures form an AT system. An explicit expression for multiple Hermite polynomials is

\[
H_{\vec{n}}^c(x) = \frac{(-1)^{|\vec{n}|}}{2^{|\vec{n}|}} \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} \prod_{j=1}^{r} c_j^{n_j-k_j} (-1)^{|\vec{k}|} H_{|\vec{k}|}(x),
\]

where \(H_{|\vec{k}|}\) is the usual Hermite polynomial. From both formulas for \(H_{\vec{n}}^c\) we get the coefficients in the nearest neighbor recurrence relation, and the recurrence relation is

\[
xH_{\vec{n}}^c(x) = H_{\vec{n}+\vec{e}_k}^c(x) + \frac{c_k}{2} H_{\vec{n}}^c(x) + \sum_{j=1}^{r} \frac{n_j}{2} H_{\vec{n}-\vec{e}_j}^c(x),
\]

so that \(b_{\vec{n}, k} = c_k / 2\) and \(a_{\vec{n}, j} = n_j / 2\). Clearly \(a_{\vec{n}, j} > 0\) whenever \(n_j > 0\) and therefore Theorem 2.2 gives the interlacing property for the zeros of multiple Hermite polynomials.

### 3.2 Multiple Charlier polynomials

Multiple Charlier polynomials (see [4]) are given by the Rodrigues formula

\[
C_{\vec{n}}^\vec{a}(x) = \prod_{j=1}^{r} (-a_j)^{n_j} \Gamma(x+1) \prod_{j=1}^{r} (a_j^{-x} \nabla a_j^x) \frac{1}{\Gamma(x+1)},
\]

where \(a_1, \ldots, a_r > 0\) and \(a_i \neq a_j\) whenever \(i \neq j\), and \(\nabla\) is the backward difference operator, i.e.,

\[
\nabla f(x) = f(x) - f(x-1).
\]

These polynomials are orthogonal with respect to discrete measures \((\mu_1, \ldots, \mu_r)\) which are given by Poisson measures

\[
\mu_j = \sum_{k=0}^{\infty} \frac{a_j^k}{k!} \delta_k.
\]

These measures form an AT system. An explicit expression for multiple Charlier polynomials is

\[
C_{\vec{n}}^\vec{a}(x) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} \prod_{j=1}^{r} (-a_j)^{n_j-k_j} (-1)^{|\vec{k}|} (-x)_{|\vec{k}|}.
\]
From both formulas for $C_{\vec{n}}$ we get the coefficients in the nearest neighbor recurrence relation, and the recurrence relation is

$$xC_{\vec{n}}(x) = C_{\vec{n}+\vec{e}_k}(x) + (a_k + |\vec{n}|)C_{\vec{n}}(x) + \sum_{j=1}^{r} a_j n_j C_{\vec{n}-\vec{e}_j}(x),$$

so that $b_{\vec{n},k} = a_k + |\vec{n}|$ and $a_{\vec{n},j} = a_j n_j$. Since $a_j > 0$ we have $a_{\vec{n},j} > 0$ whenever $n_j > 0$ and therefore by Theorem 2.2 we get the interlacing property for the zeros of multiple Charlier polynomials.

### 3.3 Multiple Meixner polynomials of the first kind

Multiple Meixner polynomials of the first kind (see [4]) are given by the Rodrigues formula

$$M_{\vec{n}}^{\beta,\vec{c}}(x) = \frac{(\beta)_{|\vec{n}|} \prod_{j=1}^{r} \left( \frac{c_j}{c_j - 1} \right)^{n_j} \Gamma(\beta)\Gamma(x+1) \prod_{j=1}^{r} \left( c_j^{-x} n_j c_j^2 \right)}{\Gamma(|\vec{n}| + \beta + x) \Gamma(|\vec{n}| + \beta)\Gamma(x+1)},$$

where $\beta > 0$, $0 < c_i < 1$ for all $i$ and $c_i \neq c_j$ whenever $i \neq j$. These polynomials are orthogonal with respect to discrete measures $(\mu_1, \ldots, \mu_r)$ which are given by

$$\mu_j = \sum_{k=0}^{\infty} \frac{(\beta)_{k} c_j^{k} e_k}{k!} \delta_k.$$

These measures form an AT system. We can compute an explicit expression for multiple Meixner polynomials of the first kind

$$M_{\vec{n}}^{\beta,\vec{c}}(x) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} \prod_{j=1}^{r} \frac{c_j^{n_j-k_j}}{(c_j - 1)^{n_j}} (-x)^{-k} (\beta + x)_{|\vec{n}| - |\vec{k}|}.$$

From both formulas for $M_{\vec{n}}^{\beta,\vec{c}}$ we can compute the coefficients in the nearest neighbor recurrence relation, and the recurrence relation is

$$xM_{\vec{n}}^{\beta,\vec{c}}(x) = M_{\vec{n}+\vec{e}_k}^{\beta,\vec{c}}(x) + (|\vec{n}| + \beta) \frac{c_k}{1 - c_k} + \sum_{i=1}^{r} \frac{n_i}{1 - c_i} M_{\vec{n}}^{\beta,\vec{c}}(x)$$

$$+ \sum_{j=1}^{r} c_j n_j \frac{(\beta + |\vec{n}| - 1)}{(1 - c_j)^2} M_{\vec{n}-\vec{e}_j}^{\beta,\vec{c}}(x).$$
so that

\[ b_{\tilde{n},k} = (|\tilde{n}| + \beta) \frac{c_k}{1 - c_k} + \sum_{i=1}^{r} \frac{n_i}{1 - c_i}, \quad a_{\tilde{n},j} = c_j n_j \frac{(\beta + |\tilde{n}| - 1)}{(1 - c_j)^2}. \]

These recurrence coefficients were not computed earlier and appear here for the first time. Since \( 0 < c_j < 1 \) and \( \beta > 0 \) we see that \( a_{\tilde{n},j} > 0 \) whenever \( n_j > 0 \) and therefore by Theorem 2.2 we get the interlacing property of the zeros of multiple Meixner polynomials of the first kind.

### 3.4 Multiple Krawtchouk polynomials

Multiple Krawtchouk polynomials (see [4]) are type II multiple orthogonal polynomials that are orthogonal with respect to binomial measures

\[ \mu_j = \sum_{k=0}^{N} \binom{N}{k} p_k^j (1 - p_i)^{N-k} \delta_k. \]

For \( |\tilde{n}| \leq N \) they are multiple Meixner polynomials of the first kind with \( \beta = -N \) and \( c_i = \frac{p_i}{p_i-1} \), for \( 0 < p_i < 1 \) for all \( i \). Hence in this case we can immediately write the nearest neighbor recurrence relation using 3.3, i.e.,

\[ x K_{\tilde{n},N}^{\tilde{p},N}(x) = K_{\tilde{n} + \tilde{e}_k}^{\tilde{p},N}(x) + \left( (N - |\tilde{n}|)p_k + \sum_{i=1}^{r} n_i(1 - p_i) \right) K_{\tilde{n}}^{\tilde{p},N}(x) \]

\[ + \sum_{j=1}^{r} \frac{p_j}{p_j - 1} n_j \frac{(|\tilde{n}| - N - 1)}{(p_j - 1)^2} K_{\tilde{n} - \tilde{e}_j}^{\tilde{p},N}(x), \]

so that

\[ b_{\tilde{n},k} = (N - |\tilde{n}|)p_k + \sum_{i=1}^{r} n_i(1 - p_i), \quad a_{\tilde{n},j} = \frac{p_j}{p_j - 1} n_j \frac{(|\tilde{n}| - N - 1)}{(p_j - 1)^2}. \]

Since \( 0 < p_j < 1 \) and \( N > 0 \) we see that \( a_{\tilde{n},j} > 0 \) whenever \( n_j > 0 \) and therefore Theorem 2.2 gives the interlacing property of the zeros of the Multiple Krawtchouk polynomials for which \(|\tilde{n}| \leq N\).
3.5 Multiple Laguerre polynomials of the second kind

Multiple Laguerre polynomials of the second kind (see [4]) are given by the Rodrigues formula

\[ L_{\vec{c}n}(x) = (-1)^{|\vec{n}|} \prod_{i=1}^{r} c_i^{-n_i} x^{-\alpha} \prod_{j=1}^{r} \left( e^{c_j x} \frac{d^{n_j}}{dx^{n_j}} e^{-c_j x} \right) x^{|\vec{n}|+\alpha}, \]

where \( \alpha > -1, \ c_j > 0 \) and \( c_i \neq c_j \) whenever \( i \neq j \). These polynomials are orthogonal with respect to measures \((\mu_1, \ldots, \mu_r)\) which are given by

\[ d\mu_j(x) = x^\alpha e^{-c_j x} \, dx \text{ on } [0, \infty). \]

These measures form an AT system. One can compute an explicit expression for the multiple Laguerre polynomials of the second kind using Leibniz' rule:

\[ L_{\vec{c}n}(x) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} \left( |\vec{n}| + \alpha \right)^{|\vec{k}|} \prod_{j=1}^{r} c_j^{k_j} x^{|\vec{n}| - |\vec{k}|}. \]

From both formulas for \( L_{\vec{c}n}(x) \) we get the coefficients in the nearest neighbor recurrence relation, and the recurrence relation is

\[ xL_{\vec{n}}(x) = L_{\vec{n}+\vec{e}_k}(x) + \left( \frac{|\vec{n}| + 1 + \alpha}{c_k} + \sum_{j=1}^{r} \frac{n_j}{c_j} \right) L_{\vec{n}}(x) \]

\[ + \sum_{j=1}^{r} \frac{n_j}{c_j^2} (|\vec{n}| + \alpha) L_{\vec{n}-\vec{e}_j}(x), \]

so that

\[ b_{\vec{n},k} = \left( \frac{|\vec{n}| + 1 + \alpha}{c_k} + \sum_{j=1}^{r} \frac{n_j}{c_j} \right), \quad a_{\vec{n},j} = \frac{n_j}{c_j^2} (|\vec{n}| + \alpha). \]

Since \( \alpha > -1 \) we see that \( a_{\vec{n},j} > 0 \) whenever \( n_j > 0 \) and from Theorem 2.2 we then get the interlacing property for the zeros of multiple Laguerre of the second kind.

3.6 Multiple Laguerre polynomials of the first kind

Multiple Laguerre polynomials of the first kind (see [4]) are given by the Rodrigues formula

\[ L_{\vec{\alpha}n}(x) = (-1)^{|\vec{n}|} \prod_{j=1}^{r} \left( x^{-\alpha_j} \frac{d^{n_j}}{dx^{n_j}} e^{-\alpha_j x} \right) x^{-\alpha} e^{\alpha x}, \]
where \( \alpha_i - \alpha_j \not\in \mathbb{Z} \) whenever \( i \neq j \), \( \alpha_j > -1 \). These polynomials are orthogonal with respect to measures \( (\mu_1, \ldots, \mu_r) \) which are given by \( d\mu_j(x) = x^{\alpha_j}e^{-x} \, dx \) on \([0, \infty)\). These measures form an AT system. Again one can compute an explicit expression for multiple Laguerre polynomials of the first kind using Leibniz’ rule:

\[
L_{\vec{n}}(x) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \binom{n_1}{k_1} \cdots \binom{n_r}{k_r} \binom{n_r + \alpha_r}{k_r} \cdots \times \binom{|\vec{n}| - |\vec{k}| + k_1 + \alpha_1}{k_1} \prod_{i=1}^{r} k_i! (-1)^{|\vec{k}|} x^{|\vec{n}| - |\vec{k}|}.
\]

From both formulas for \( L_{\vec{n}} \) we get the coefficients in the nearest neighbor recurrence relation, and the recurrence relation is

\[
xL_{\vec{n}}^{\vec{\alpha}}(x) = L_{\vec{n} + \vec{e}_k}^{\vec{\alpha}}(x) + (|\vec{n}| + 1 + n_k + \alpha_k) L_{\vec{n}}^{\vec{\alpha}}(x) + \sum_{j=1}^{r} n_j (n_j + \alpha_j) \prod_{i \neq j}^{r} \frac{\alpha_i - n_j - \alpha_j}{n_i + \alpha_i - n_j - \alpha_j} L_{\vec{n} - \vec{e}_j}^{\vec{\alpha}}(x),
\]

so that

\[
b_{\vec{n},k} = |\vec{n}| + 1 + n_k + \alpha_k, \quad a_{\vec{n},j} = n_j (n_j + \alpha_j) \prod_{i \neq j}^{r} \frac{\alpha_i - n_j - \alpha_j}{n_i + \alpha_i - n_j - \alpha_j}.
\]

Observe that \( a_{\vec{n},j} \neq 0 \) and therefore we cannot apply Theorem 2.2 but these polynomials have the interlacing property since the measures \( (\mu_1, \ldots, \mu_r) \) form an AT system so that Theorem 2.1 can be applied. It is worth noting that in [9] it was shown that \( \sum_{j=1}^{r} a_{\vec{n},j} > 0 \). It is not clear whether such a condition is sufficient to prove the interlacing property.

### 3.7 Multiple Mexiner polynomials of the second kind

Multiple Mexiner polynomials of the second kind (see [4]) are given by the Rodrigues formula

\[
M_{\vec{n}}^{\vec{\beta},c}(x) = \left( \frac{c}{c-1} \right)^{|\vec{n}|} \prod_{j=1}^{r} (\beta_j)_{n_j} \frac{\Gamma(x + 1)}{c^x} \prod_{j=1}^{r} \frac{\Gamma(\beta_j)}{\Gamma(\beta_j + x)} \times \nabla^{n_j} \frac{\Gamma(\beta_j + n_j + x)}{\Gamma(\beta_j + n_j)} \frac{c^x}{\Gamma(x + 1)},
\]
where $0 < c < 1$ and $\beta_i - \beta_j \notin \mathbb{Z}$ whenever $i \neq j$, $\beta_j > 0$. These polynomials are orthogonal with respect to discrete measures $(\mu_1, \ldots, \mu_r)$ which are given by

$$
\mu_j = \sum_{k=0}^{\infty} \frac{(\beta_j)_k c^k}{k!} \delta_k.
$$

These measures form an AT system. An explicit expression for the multiple Meixner polynomials of the first kind is

$$
M_{\vec{n}}^{\vec{\beta},c}(x) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_r=0}^{n_r} \left( \begin{array}{c} n_1 \\ k_1 \\ \vdots \\ n_r \\ k_r \end{array} \right) \frac{c^{|\vec{n}| - |\vec{k}|}}{(c-1)^{|\vec{n}|}} \times \prod_{j=1}^{r} (\beta_j + x - \sum_{i=1}^{j-1} k_i)n_j-k_j (-x)^{|\vec{k}|}.
$$

From both formulas for $M_{\vec{n}}^{\vec{\beta},c}$ we get the coefficients in the nearest neighbor recurrence relation, and the recurrence relation is

$$
xM_{\vec{n}}^{\vec{\beta},c}(x) = M_{\vec{n}+\vec{e}_k}^{\vec{\beta},c}(x) + \left( \frac{|\vec{n}|}{1-c} + (n_k + \beta_k) \frac{c}{1-c} \right) M_{\vec{n}}^{\vec{\beta},c}(x)
$$

$$
+ \sum_{j=1}^{r} cn_j \frac{\beta_j + n_j - 1}{(1-c)^2} \prod_{i \neq j}^{r} \frac{\beta_i - n_j - \beta_j}{n_i + \beta_i - n_j - \beta_j} M_{\vec{n}-\vec{e}_j}^{\vec{\beta},c}(x),
$$

so that

$$
b_{\vec{n},k} = \frac{|\vec{n}|}{1-c} + (n_k + \beta_k) \frac{c}{1-c},$$

$$
a_{\vec{n},j} = cn_j \frac{\beta_j + n_j - 1}{(1-c)^2} \prod_{i \neq j}^{r} \frac{\beta_i - n_j - \beta_j}{n_i + \beta_i - n_j - \beta_j}.
$$

These recurrence coefficients were not computed earlier and appear here for the first time. Again we see that $a_{\vec{n},j} \neq 0$ for all $j$. However the zeros of multiple Meixner of the second kind do interlace because of Theorem [2.1]. For these multiple orthogonal polynomials one can also show that $\sum_{j=1}^{r} a_{\vec{n},j} > 0$.

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