We continue to explore the possibility that the graviton in two dimensions is related to a quadratic differential that appears in the anomalous contribution of the gravitational effective action for chiral fermions. A higher dimensional analogue of this field might exist as well. We improve the defining action for this diffeomorphism tensor field and establish a principle for how it interacts with other fields and with point particles in any dimension. All interactions are related to the action of the diffeomorphism group. We discuss possible interpretations of this field.
I. INTRODUCTION TO THE DIFFEOMORPHISM TENSOR

Recently we introduced a covariant field theory for a rank two tensor that has its roots in the coadjoint representation of the Virasoro algebra [1]. We will call this tensor the diffeomorphism tensor. In two dimensions one of the components of this rank two symmetric tensor is the quadratic differential that appears in the gravitational effective action for chiral fermions. For this reason we have been compelled to explore how the field theory for this tensor is related to two dimensional gravitation. In [2] this analysis was restricted to the cylinder but a covariant approach was lacking, while in [1] a covariant field theory was constructed emphasizing the use of differential expressions which are insensitive to a choice of connection. Although that approach gives the correct equations of motion, it is difficult to relate it to diffeomorphisms. Furthermore the conjugate momentum did not transform as an element of the adjoint representation.

In this paper we will investigate the relationship between Lie derivatives and the Lagrangian for the diffeomorphism field $D_{\mu\nu}$ as well as how the diff field interacts with matter fields such as point particles, fermions and spin one fields. The action we find that describes the diff tensor exists in two as well as higher dimensions, just as Yang-Mills. This suggests that there might be some aspect of gravitation in two dimensions that may have a non-trivial link to higher dimensions after all. Here we write $D_{\mu\nu}$ instead of $T_{\mu\nu}$ as has been done in the past [1,2], so as to avoid confusion with the energy-momentum tensor. Our guide to constructing a covariant field theory that includes interactions will be the two dimensional gauge fixed geometric actions and the isotropy algebras on the coadjoint orbits. From here we can establish a principle that one can use to identify covariant interactions. As an example, we find the point particle interaction. In the presence of a gauge field we find that the interaction of the diff field explicitly breaks the gauge invariance, which can be recovered by introducing a new group valued scalar field.

II. REVIEW: COADJOINT ORBITS, ISOTROPY GROUPS, AND CONSTRAINTS

In this section we will give a short review of previous work and then outline the procedure for this work. The relationship between coadjoint orbits of the Virasoro and the affine Lie algebras on one hand, and anomalous contributions to two dimensional effective actions on the other, has been developed in [3,4,9–11,16]. There one is able to construct geometric actions that correspond to the WZW [2] model and Polyakov [8] gravity. Physically the coadjoint vector (here shown without the central extension), $B=(A,D)$, provides the background classical fields $A$ and $D$ that couple to the bosonized chiral fermions. The field, $A$, can be associated with a background gauge field of a WZW model [9] while the field $D$ is typically set to a specific constant to study a fixed Virasoro orbit. Much of the focus in the literature on two dimensional gravity has been on the fact that the geometric action of the Virasoro group is the Polyakov action. Other authors have recognized the importance of the coadjoint orbits and considered model spaces to treat the different coadjoint orbits as elements of a Hilbert space in geometric quantization of the Virasoro algebra [23]. Our approach is to carry the coadjoint representation of the Virasoro algebra beyond conformal field theory. The property of the Virasoro orbits that we focus on is the isotropy algebra that defines each orbit. The invariance of the coadjoint vector due to this subalgebra generates an isotropy equation that we interpret as a constraint equation among conjugate variables. We will explain this point below. Then we simply ask whether we can
construct a covariant field theory reproducing this constraint for the background fields associated with the coadjoint vector \((A,D)\). This approach allows one to extend our work to a field theory in higher dimensions. This does not imply, however, that we now have a higher dimensional representation of the Virasoro algebra, but rather that there exists a field theory which, when dimensionally reduced to a 2D theory, admits constraints on the fields which agree with the isotropy algebra associated with the coadjoint representation of the Virasoro algebra. In what follows we will make these remarks concrete.

By looking at the semi-direct product of Virasoro and the affine Lie algebra on a circle, one gains insight into the structure of two dimensional gravitation through comparison with Yang-Mills. We have already seen how the anomalous structures of gravitational and gauge theories manifest themselves through geometric actions [9]. In [1,2] the focus was not on the geometric actions, but instead on the field theories which describe the dynamics for \(A\) and \(D\) separately. We call these \(\text{transverse}\) field theories as their symplectic structure is transverse to the coadjoint orbits.

The fields \(A\) and \(D\) serve as \(\text{background}\) fields in the geometric actions and are subjected to constraints that respect the isotropy algebra associated with each orbit. Different orbits correspond directly to different isotropy algebras. Our point of view is that these constraint equations are field equations that survive for a particular choice of static configurations for a gauge field \(A_\mu\) and a diff field \(D_{\mu\nu}\).

In [1] we were interested in the field theory of the diff field \(D_{\mu\nu}\) alone. \(\text{The main interest of this paper is to determine how the field } D_{\mu\nu} \text{ interacts with other fields.}\) As we shall see, the interactions of this rank two symmetric tensor are governed by diffeomorphisms and Lie derivatives.

In order to extract any information about the interaction Lagrangian, we must relate the geometry of the coadjoint orbits to constraint equations. Below is a brief description of this relationship. Using the notation of [4] for the adjoint and coadjoint representations of the centrally extended semi-direct product of the Virasoro algebra with an affine Lie algebra, one can define the algebraic action of a typical adjoint element, say \(F = (\xi(\theta), \Lambda(\theta), a)\), on an arbitrary coadjoint element \(B = (D(\theta), A(\theta), \mu)\) through

\[
B_F = (\xi(\theta), \Lambda(\theta), a) \ast (D(\theta), A(\theta), \mu) = (D(\theta)_{\text{new}}, A(\theta)_{\text{new}}, 0).
\]

Here the components of the new coadjoint elements are

\[
D(\theta)_{\text{new}} = \frac{1}{24\pi} \xi'' D + \xi' D' - \frac{g\mu}{24\pi} \xi''' - \text{Tr}(A\Lambda')
\]

and

\[
A(\theta)_{\text{new}} = \Lambda'\xi + \xi' A - [\Lambda A - A \Lambda] + k \mu \Lambda'.
\]

From these transformations one may define the coadjoint orbit of the the coadjoint element \(B = (D(\theta), A(\theta), \mu)\), as the space of all coadjoint vectors that can be reached by group transformations on \(B\). Each orbit is equipped with a non-degenerate bilinear two-form that defines a symplectic structure \(\Omega_B\):

\[
\Omega_B \left( \tilde{B}_1, \tilde{B}_2 \right) = \left\langle \tilde{B}_1 | [a_1, a_2] \right\rangle,
\]
where the coadjoint elements are given by \( \tilde{B}_i = a_i \ast \tilde{B} = \delta_{a_i} \tilde{B} \). This symplectic structure is then integrated over two-dimensional submanifolds to extract a geometric action that yields the anomalous contribution to the effective action of chiral fermions in the background fields defined by \( B \).

Using methods from [17], one obtains the action [17]

\[
S = \frac{1}{2\pi} \int d\lambda \, d\theta \, d\tau \, D(\theta) \left( \frac{\partial s}{\partial \theta} \partial_\theta s - \frac{\partial s}{\partial \theta} \partial_\theta s \right) \left( \frac{\partial \lambda s}{\partial \theta} \partial_\theta \lambda s - \frac{\partial \lambda s}{\partial \theta} \partial_\theta \lambda s \right)
\]

coupling to background diffeomorphism field

\[
+ \frac{1}{2\pi} \int d\lambda \, d\theta \, d\tau \, \mathrm{Tr} \left( A(\theta) \left( \frac{\partial \lambda s}{\partial \theta} \partial_\theta \left( g^{-1} \partial_\tau g \right) - \frac{\partial s}{\partial \theta} \partial_\theta \left( g^{-1} \partial_\tau g \right) + \left[ g^{-1} \partial_\lambda g, g^{-1} \partial_\tau g \right] \right) \right)
\]

coupling to background gauge field

\[
- \frac{\beta c}{48\pi} \int d\tau \, d\theta \left( \frac{\partial^2 s}{\partial \theta^2} \partial_\theta \partial_\theta s - \left( \frac{\partial^2 s}{\partial \theta^2} \right)^2 \partial_\theta s \right)
\]

Polyakov gravity

\[
- \frac{\beta k}{4\pi} \int d\tau \, d\theta \, \mathrm{Tr} \left( g^{-1} \partial_\theta g g^{-1} \partial_\tau g \right) + \frac{\beta k}{4\pi} \int d\lambda \, d\tau \, d\theta \, \mathrm{Tr} \left( \left[ g^{-1} \partial_\theta g, g^{-1} \partial_\lambda g \right] g^{-1} \partial_\tau g \right),
\]

Wess-Zumino-Witten

where \( s(\theta) \) and \( g(\theta) \) are diffeomorphism and loop group elements respectively, representing the bosonized fermion degrees of freedom. Each orbit yields a distinct system of classical equations of motion.

Since \( D(\theta) \) is a function of \( \theta \) only, one may integrate by parts in the first line of the above action and find that the interaction term for the background field is

\[
S_I = \int d\tau \, d\theta \, D \left( \frac{\partial \lambda s}{\partial \theta} \partial_\theta s \right).
\]

Thus \( D \) appears to be a background field that couples to the induced metric \( (\partial \lambda s/\partial \theta) \). \( D \) is the \( D_{\theta\theta} \) component of a static background rank two tensor field. All other components of \( D_{\mu\nu} \) are set to zero as they do not couple to dynamical components of the induced metric.

In a similar way we know that in the WZW model the background gauge field interacts with the bosonized fermions through the Lagrangian [18, 20]

\[
I(g, A) = \frac{1}{4\pi} \mathrm{Tr} \left( \int d\tau \, d\theta \left( A_\theta \partial_\tau g^{-1} g + A_\theta g A_\tau g^{-1} - A_\tau A_\theta \right) \right).
\]

By setting \( s(\theta) = \theta \) and \( A_\tau = 0 \), Eq. (4) reduces to the WZW model and the interaction Lagrangian

\[
\frac{1}{2\pi} \int d\lambda \, d\theta \, d\tau \, A_\theta \left( g^{-1} \partial_\lambda g, g^{-1} \partial_\tau g \right) = \frac{1}{4\pi} \mathrm{Tr} \left( A_\theta \partial_\tau g^{-1} g \right).
\]

One sees that the geometric action includes this interaction term for a particular background field configuration. By general anomaly calculations [18, 19], we readily identify the coadjoint vector \( A \) with a gauge field. Just as the field \( A_\theta \) is subject to Gauss law constraints, the same will hold for the field \( D_{\theta\theta} \); i.e. there exist constraints that must be respected by the background field. These constraints manifest themselves in a subtle but simple way from the point of view of coadjoint orbits, as they appear through the isotropy algebra of the coadjoint element.

The isotropy group of a coadjoint element is the group that leaves that coadjoint element invariant. Algebraically we have for each coadjoint element, \( B = (D(\theta), A(\theta), \mu) \);
\[ \delta D = 2\zeta_B D + D'\xi_B + \frac{e\mu}{24\pi} \zeta_B'' - \text{Tr} (A \Lambda_B) = 0 \]  

\[ \text{and} \]

\[ \delta A = A'\xi_B + \xi_B'A - [\Lambda_B A - A\Lambda_B] + k \mu A_B' = 0, \]

where \( \xi_B \) and \( \Lambda_B \) are elements of the isotropy algebra. Notice that for a given orbit, these elements are identified with the identity and do not represent bosonized chiral fermion degrees of freedom. **We can however argue that this isotropy algebra originates from constraints on the canonical variables \( A \) and \( D \) and their respective conjugate momenta.** This would illustrate the transverse relationship between the symplectic structure of an orbit, which influences the chiral fermions in the presence of the background field \( B \), and the symplectic structure that determines the dynamics for the fields \( A \) and \( D \).

### III. MAIN RESULTS: THE ORIGIN OF THE DIFFEOMORPHISM LAGRANGIAN AND THE INTERACTION LAGRANGIAN

As mentioned above, the geometric actions associated with the coadjoint orbits have a physical interpretation in terms of chiral fermions coupled to gauge and diffeomorphism fields in two space-time dimensions. In this section we discuss the main point of this work, namely the determination of the interaction Lagrangian. We will exploit the coadjoint representation to aid in determining the interaction Lagrangian for spin one fields, and borrow from the effective action to extract the interaction Lagrangian for fermions. We will then postulate the action for the point particle from the point of view of diffeomorphisms.

**A. Gauge Transformation Laws and Yang-Mills as a Guide**

We will use the structure of Yang-Mills theory as a guide to the construction of the interaction Lagrangian for the diff field. In Yang-Mills the Gauss law constraint appears as the field equation arising from varying the Lagrangian with respect to the field \( A_0 \) and evaluating at \( A_0 = 0 \). In the Yang-Mills case \( A_0 \) can always be sent to zero (i.e., we may reduce to the temporal gauge) since its conjugate momentum vanishes. In the case of the bosonized fermions, the fields \( D \) and \( A \) are the components of a spin two and spin one tensor respectively for a particular background field configuration that couples to the bosonized fermions. Nevertheless we will study the general covariance for the diff field and show that the isotropy equations will be analogous to the Gauss law constraints.

To get an appreciation of the relationship between Gauss’ law and the isotropy equations, consider the field \( A \) separately from \( D \) and set \( \xi = 0 \). Equation (6) represents the residual time-independent gauge transformations on \( A_\theta \), where \( A = (A_\theta, A_\tau) \). This is a symmetry of the Cauchy data.

Furthermore by considering the transport due to \( \xi \) and setting \( \Lambda = 0 \), we see that \( A \) comes from a tensor of rank one. That is, the isotropy equation for \( A \) is the space component of a coordinate transformation on \( A_\mu \),

\[ \delta_I A_\alpha = \xi^\beta \partial_\beta A_\alpha + A_\beta \partial_\alpha \xi^\beta. \]

Under a general coordinate transformation, we see that the \( \xi \) transformation in Equation (6) is the transformation due to time independent spatial translations. It is well known that the Yang-Mills action is the requisite action to
covariantly describe $A_\mu$ as a dynamical field in any dimension. We have utilized this fact in order to understand the pure diffeomorphism sector, and we were able to deduce the properties and the action for $D_{\mu \nu}$ in \[1,2\]. We expect then that we will recover the isotropy equations as constraint equations for these background fields.

With this in mind we deduce the properties of $D$, and extend to higher dimensions. The analogous constraint equation is

$$2\xi' D + D' \xi + \frac{\epsilon_{\mu}}{24\pi} \epsilon''' = 0.$$ 

Up to the inhomogeneous term, $D$ transforms under diffeomorphisms as $\delta D = 2\xi' D + D' \xi$, corresponding to a rank two tensor, i.e.

$$\xi^a \partial_a D_{lm} + D_{am} \partial_l \xi^a + D_{la} \partial_m \xi^a = \delta \xi D_{lm}.$$ 

In an analogous way we can use the general coordinate transformations to produce a “temporal” gauge for diffeomorphisms. In $n$ dimensions one may fix a coordinate system so that $\partial_0 D_{\mu 0} = 0$, leaving the fields $D_{\mu 0}$ without conjugate momenta. One can solve the $n$ second order equations

$$\frac{\partial x''}{\partial x^0} \left( \frac{\partial^2 x'^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} \frac{\partial^2 x'^{\beta}}{\partial x^{\nu} \partial x^{0}} D_{\alpha \beta} + \frac{\partial^2 x'^{\beta}}{\partial x^{\mu} \partial x^{0}} \frac{\partial x'^{\alpha}}{\partial x^{\nu}} D_{\alpha \beta} + \frac{\partial x'^{\alpha}}{\partial x^{0}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} \partial_\mu D_{\alpha \beta} \right) = 0,$$

for the coordinate choice. As one can see from an infinitesimal coordinate transformation of $\partial_0 D_{0
u}$, there is a residual “gauge” symmetry due to the time independent coordinate transformations. In particular, under the time independent spatial translations $D_{00}$ transforms as a scalar, $D_{0i}$ transforms as a vector and $D_{ij}$ transforms as a rank two tensor. Here the Latin indices correspond to spatial coordinates. Separating the time independent spatial translations from the time independent temporal transformations on $D_{\mu 0}$, we have

$$\delta D_{0i} = \xi^j \partial_j D_{0i} + D_{0j} \partial_i \xi^j + D_{00} \partial_i \xi^0,$$

while

$$\delta D_{00} = \xi^i \partial_i D_{00}.$$ 

So we may think of the residual symmetries as time independent spatial translations along with an inhomogeneous transformation for $D_{0i}$. $D_{00}$ transforms as a scalar. We can use $\xi^0$ from the inhomogeneous term to bring $D_{01} = 0$. This leaves another reduced residual symmetry for the remaining fields as for $i, j \neq 1$, corresponding to $x^0$ and $x^1$ independent transformations,

$$\delta D_{0i} = \xi^j \partial_j D_{0i} + D_{0j} \partial_i \xi^j + \xi^1 \partial_1 D_{0i}.$$ 

We can use $\xi^1$ to bring $D_{02} = 0$.

Again this leaves a reduced symmetry and one proceeds to bring all $D_{0i} = 0$. The $D_{0i}'s$ then serve as Lagrange multipliers for a set of constraints. The constraint hypersurface where $D_{0i} = 0$ and $\partial_0 D_{00} = 0$ is consistent with this analysis. $D$ then is the one remaining dynamical field component of a rank two symmetric tensor in two dimensions.
B. Gauss Law Constraints and Field Equations

From the point of view of two-dimensional geometric actions, the isotropy group defines the topology of the orbit through \((\Omega(G) \otimes \text{Diff} S^1)/H_{D,A}\) where \(G\) is the gauge group, \(\Omega(G)\) is the loop group of \(G\), and \(H_{D,A}\) is the isotropy group of the fields \(D\) and \(A\). The geometric actions then describe the anomalous two-dimensional fermionic vacuum in the presence of background gauge and diffeomorphism fields. As stated earlier, we are not at all interested in orbits as they necessarily contain anomalous information. Instead we are interested in making \(A\) and \(D\) dynamical variables which would certainly move us away from any orbit. In fact, if we are to preserve gauge covariance of the initial data, we must guarantee that we do not incorporate gauge variations into the dynamics. In field theories the Gauss law constraints guarantee that the initial data for the dynamical field and its associated conjugate momentum will not evolve in any (residual) gauge directions. The Gauss law constraints are the generators of the time independent gauge transformations and spatial translations. Thus the dynamical theory of \(A\) and \(D\) must be “transverse” to the coadjoint orbits. Since the isotropy condition is an equivariant relation between coadjoint elements and the adjoint representation, it is precisely the condition that defines the Gauss law. One replaces the coadjoint element that will serve as the initial data with the canonical coordinate and the adjoint element with the conjugate momentum. This follows since the conjugate momentum transforms like the adjoint elements.

In 2D the field equations of the \(D_{01}\) component become constraints on the initial data. By using the arguments of \([1]\) one can construct an action such that the variation with respect to \(D_{01}\) in two dimensions leads to the isotropy equation for the \(D_{11}\) Cauchy data.

Consider the action

\[
S_{\text{diff}} = \int d^n x \sqrt{g} \left( X^{\lambda \mu \rho} D_{\alpha}^{\rho} X_{\mu \lambda \alpha} + 2X^{\lambda \mu \rho} D_{\alpha}^{\rho} X_{\rho \mu \alpha} - \frac{q}{4} X^{\alpha \beta \rho} \nabla_{\lambda} \nabla_{\rho} X_{\lambda \alpha} - \frac{1}{2} X^{\beta \gamma \alpha} X_{\beta \gamma \alpha} \right). \tag{8}
\]

In the above \(X^{\mu \nu \rho} = \nabla^{\nu} D^{\mu \rho}\), so we may write

\[
S_{\text{diff}} = \int d^n x \sqrt{g} \left( (\nabla^{\nu} D^{\lambda} \mu) D_{\rho}^{\alpha} \nabla_{\alpha} D_{\mu \lambda} + 2(\nabla^{\nu} D^{\lambda} \mu) D_{\rho}^{\alpha} \nabla_{\mu} D_{\rho \alpha} - \frac{q}{4} (\nabla_{\alpha} \nabla_{\beta} D^{\gamma \beta}) \nabla_{\lambda} \nabla_{\rho} D^{\lambda \rho} - \frac{1}{2} (\nabla^{\alpha} D^{\beta \gamma}) \nabla_{\alpha} D_{\beta \gamma} \right).
\]

This allows us to easily see the three point function, and the fact that \(q\) will set a new length scale. Varying with respect to \(D_{i0}\) and setting \(D_{\nu 0} = 0\), we are led to the equation

\[
X^{lm0} \partial_{i} D^{lm} - \partial_{m} (X^{ml0} D_{li}) - \partial_{i} (X^{ml0} D_{mi}) - q \partial_{i} \partial_{m} \partial_{l} X^{lm0} = 0.
\]

In 1 + 1 dimensions, this corresponds to the isotropy equation on the coadjoint orbit specified by \(D\), viz. \(\xi D' + 2\xi' D + q \xi'' = 0\) where \(\xi\) takes the role of \(X^{10}\); this is in turn the conjugate momentum for \(D = D_{11}\) after setting \(D_{i0} = 0\).

C. Matter Fields Interactions

The self interaction of the diffeomorphism field suggests that interactions with other matter should have the semblance of diffeomorphisms. In \([1]\) it was suggested that the interaction Lagrangian be built up from Gauss law
constraints that are associated with the isotropy algebras in one dimension. Recall that the interaction Lagrangian of the diffeomorphism field had a structure like

$$L_{\text{int}} = X^{\lambda\mu\rho} Y_{\lambda\mu\rho},$$

where $X^{\lambda\mu\rho}$ acts as the “covariantized” conjugate momentum and $Y_{\lambda\mu\rho}$ is the “covariantized” Lie derivative of the diff field $D_{ij}$. The construction of the covariant interaction Lagrangian proceeded in the following stages:

- contract the conjugate momentum with a diff variation of the matter field;
- replace the diff fields $\xi^\alpha$ with $D^\alpha_0$ and extend to the Lie derivative;
- covariantize the interaction.

The interaction Lagrangian is then built up the following stages:

$$L^{0\text{th \ stage}} = X^{ij}_0 (\xi^l \partial_l D_{ij} + D_{lj} \partial_i \xi^l + D_{il} \partial_j \xi^l) \to L^{1\text{st \ stage}} = X^{\lambda\mu\rho}_0 (D^\alpha_0 \nabla_{\alpha} D_{\lambda\mu} + D_{\alpha\mu} \nabla_{\lambda} D^\alpha_0 + D_{\lambda\alpha} \nabla_{\mu} D^\alpha_0) \to L_{\text{int}} = X^{\lambda\mu\rho} (D^\alpha_{\rho} \nabla_{\alpha} D_{\lambda\mu} + D_{\alpha\mu} \nabla_{\lambda} D^\rho_{\alpha} + D_{\lambda\alpha} \nabla_{\rho} D^\alpha_0).$$

With this we have a principle by which we can write the interaction Lagrangian with other matter. Notice that even though $D_{\mu\nu}$ is a tensor, one does not simply use tensoriality to find the interactions of this field with other matter. One must require that the one dimensional theory yield constraint equations that serve as the Gauss law constraints.

### I. Fermions

We will use this principle to build the interaction Lagrangian for fermions coupled to the field $D_{\mu\nu}$. First write the “covariant” conjugate momentum for the fermion fields as $\sqrt{g} \bar{\Psi} \gamma^\beta$. The effect of an infinitesimal diffeomorphism $\xi$ on a spinor $\Psi$ is

$$\xi^\alpha \nabla_{\alpha} \Psi - \frac{1}{4} \nabla_{[\alpha} \xi^\beta \gamma_{\beta]} \gamma^\alpha \gamma^\beta \Psi$$

by, e.g. [24], where $\nabla$ is the spin connection. Thus we expect to see

$$D^\alpha_{\lambda} \nabla_{\alpha} \Psi - \frac{1}{4} \nabla_{[\alpha} D^\lambda_{\beta]} \gamma_{\alpha} \gamma^\beta \Psi.$$

With this we write the interaction Lagrangian density as

$$\sqrt{g} \bar{\Psi} \gamma^\lambda \left( D^\alpha_{\lambda} \nabla_{\alpha} \Psi - \frac{1}{4} \nabla_{[\alpha} D^\lambda_{\beta]} \gamma_{\alpha} \gamma^\beta \Psi \right).$$

In order to check to see whether this coupling to fermions is consistent with two dimensions we can use the geometric action to see how the bosonized fermions interact with the background field $D$. Recall the 2D result:

$$S = \int d^2 x \; D(\theta) \frac{\partial s}{\partial \theta} \int \left[ \frac{\partial^2 s}{(\partial s)^2} \partial s \partial \theta s - \frac{(\partial s)^2}{(\partial s)^3} \right] d\theta \; d\tau. \quad (9)$$

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In the bosonization of the fermions, \((\bar{\Psi} \gamma^\beta \partial_\alpha \Psi) \to \partial_\tau s/\partial \theta s\). In two dimensions the coupling to fermions is consistent with our interaction principle. This is distinct from the coupling suggested in [1], which could also yield the correct two dimensional limit. However that form of the coupling lacks any consistency as to how the diff field interacts with spin one fields. One sees that the metric has been replaced with \(g_{\mu\nu} + D_{\mu\nu}\), suggesting that the field \(D_{\mu\nu}\) is playing the role of a classical graviton field. We will see that this persists throughout the other interactions.

2. Spin One Coupling

The spin one coupling is a good test of this principle as it is should have non-trivial contributions to the isotropy equations for both the A and the D fields. Recall that we wish to reproduce the isotropy equation from our field theory that is reduced to two dimensions. The “covariant” conjugate momentum for \(A_\mu\) is

\[
F^{\rho\lambda} = \sqrt{g}(\partial^\rho A^\lambda - \partial^\lambda A^\rho + [A^\rho, A^\lambda]).
\]

The covariantized Lie derivative,

\[
D^\alpha \rho \partial_\alpha A_\lambda + A_\alpha \partial_\lambda D^\alpha \rho - \partial_\rho (D^\alpha \lambda A_\alpha).
\]

is independent of the choice of affine connection. Therefore the interaction Lagrangian is

\[
\sqrt{g}F^{\rho\lambda}(D^\alpha \rho \partial_\alpha A_\lambda + A_\alpha \partial_\lambda D^\alpha \rho - \partial_\rho (D^\alpha \lambda A_\alpha)), \tag{10}
\]

lending no new direct couplings to the metric. Notice that when \(D_{i0} = 0\), \(A_0\) has no conjugate momentum even in the presence of gravity. This interaction term, Eq.(10), is not gauge invariant. One may preserve gauge invariance by introducing a group valued scalar-field \(V\), transforming under right multiplication by a group element \(h\) as \(V \to V h\). The interaction Lagrangian is

\[
\sqrt{g}F^{\rho\lambda}(D^\alpha \rho \partial_\alpha \tilde{A}_\lambda + \tilde{A}_\alpha \partial_\lambda D^\alpha \rho - \partial_\rho (D^\alpha \lambda \tilde{A}_\alpha)), \tag{11}
\]

where

\[
\tilde{A}_\mu = A_\mu - V^{-1} \partial_\mu V.
\]

The \(V\) field could have a Lagrangian

\[
\mathcal{L} = m_A^2 \int (V^{-1} \partial_\mu V - A_\mu)(V^{-1} \partial_\nu V - A_\nu)(g^{\mu\nu} + D^{\mu\nu})d^nx. \tag{12}
\]

The \(V\) field has a coupling constant proportional to the mass of the gauge fields. Variation of the full Lagrangian with respect to \(D_{i0}\), followed by evaluation at our background fields with \(V = 1\), \(A_0 = 0\), and \(D_{0\nu} = 0\), gives the expected additional contribution to the constraint equations predicted by the 2D constraint equation

\[
2X' D + D' X + \frac{\epsilon\mu}{24\pi} X''' - Tr(\mathcal{A}E') = 0, \tag{13}
\]

where \(X\) is \(X^{10}\) and \(D\) is \(D_{11}\). Furthermore, variation with respect to \(A_0\) yields the correct addition to the constraint via the \(X\) dependent terms

9
\[ A'X + X'A - [E A - A E] + k \mu E' = 0. \]  

These equations are precisely the isotropy equations for the background A and D fields, where X replaces \( \xi \), the non-interacting conjugate momentum of D, and E replaces \( \Lambda \), the non-interacting conjugate momentum of A, for the background specified by A and D. This one dimensional system was studied both classically and quantum mechanically on a cylinder in ref. [2]. The new covariant picture described here provides a way in which one may reevaluate the Hamiltonian of the system from first principles.

3. Point Particle Couplings

The coupling of \( D_{\mu \nu} \) to the point particle helps to illuminate the role of the diff field as the graviton. Here we proceed in exactly the same way. We first identify the conjugate momentum and multiply it by the diffeomorphism shift to find that

\[
\int p_i \delta x^i \gamma \ d\tau \rightarrow \int p_i D^i_0 \gamma \ d\tau \rightarrow S_{pp} = \int \rho^\mu \rho^\nu D_{\mu \nu} d\tau.
\]

This action, together with the action in eq. (8), suggests a theory with a Newtonian potential at low energy for any dimension. After including the metric, one has

\[
m \frac{d^2 x^\beta}{d\tau^2} (\delta^\alpha_\beta + \mu^2 D_\beta \mu g^{\mu \alpha}) + \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} \left( \Gamma^\alpha_{\nu \lambda} + \frac{1}{2} \mu^2 g^{\gamma \gamma} (\partial_\gamma D_{\nu \lambda} - \partial_\gamma D_{\gamma \lambda} - \partial_\lambda D_{\gamma \nu}) \right) = 0,
\]

where we have introduced the coupling constant \( \mu \). It is clear from here that we may interpret the diff field as a disturbance in the gravitational field established by the metric, i.e. a graviton. The diff field is related to time independent diffeomorphisms in the same way that the gauge fields are related to time independent gauge transformations.

IV. CONCLUSION

Throughout this paper we have tried to illuminate the role of the diff field D in gravitation through its interactions with a variety of matter fields. We emphasized the use of isotropy algebras associated to coadjoint orbits, in order to get constraints that are related to a field theory. Throughout this work we have stayed close to Yang-Mills as it has a natural extension to higher dimensions. The way in which D appears in the geometric actions suggests that it has a classical origin quite distinct from the anomaly. Different values of D yield different background symmetries for chiral fermions, suggesting that there is a semi-classical vacuum structure that is not anomalous for two dimensional gravity and which is characterized by different coadjoint orbits. Furthermore the \( D_0^i \) components act as vector fields generating time independent coordinate transformations, and guide us to a working principle for the interaction of gravity with matter. This principle can be carried over to many other matter fields. To state this more plainly, string symmetries have taught us that:

- two dimensional classical gravity is not trivial,
- but instead has a rich structure that is related to diffeomorphisms, and
which may be related to higher dimensional theories.

In two dimensions we find that the diff field behaves as a “classical” disturbance of the metric or a classical graviton. This behavior is easily seen both in the case of fermions, and that of a point particle, where the metric appears along with the diff field as $g_{\mu\nu} + D_{\mu\nu}$. What is novel here is that apart from scalars and point particles, the diff field does NOT act as a linear shift of a fixed metric, but interacts in such a way that the $D^0_0$ components of the diff field act as elements of the algebra of diffeomorphisms. This yields unexpected results for interactions with matter; for example the spin one field, but we are comforted by the fact that these results agree with the interpretation of the constraints that arise on the coadjoint orbits in 2D. We thus use the freedom due to generalized coordinate invariance to restrict degrees of freedom on the diff field (as we discussed using the temporal gauge) instead of the metric. The role of the diff field, as opposed to that of the metric, could be that the diff field provides a description of the graviton as a fluctuation about a metric. This is distinct from the usual construction of the graviton as a linearization about a fixed metric. In our case, the graviton exhibits dynamics that could not have arisen from linearization about the metric in the Einstein-Hilbert action. Both the metric and the diff field are present from this point of view, as the metric is necessary to give a covariant meaning to the conjugate momentum of the fields and to define the covariant derivative. The D field adds further disturbances to the gravitational field in a way related to diffeomorphisms, and which is parallel to the role of the vector potential in gauge theories. In fact, in this discussion the vector potential and the diff field are simply two different components of the three-tuple that defines the centrally extended coadjoint vector. The question of geometry vs. diffeomorphisms arises in the theory of gravitation.

The idea of using the Gauss law constraints via the Virasoro algebras in order to understand gravitation have also been employed in string field theories and in lineal gravity [25]. In both of these cases the quantum states of the system carry the adjoint representation of the Virasoro algebra. In string field theory one constructs highest weight states using negative moded elements of the Virasoro algebra as gauge fixing conditions, and the other half as generators of gauge transformations. From this procedure one recovers the physical states and ghosts that are needed for the square of the BRST charge to vanish. In lineal gravity one uses a dilaton model of gravity to generate a set of constraints from the commutation relations of the energy-momentum operators. This algebra is equivalent to the Virasoro algebra, and one requires that the vacuum state be annihilated by the full Virasoro algebra.

In our case we do not use the adjoint representation, but instead focus on the states as carriers of the coadjoint representation. The vacuum state of the pure diff case will satisfy the operator constraint equation that arises from classical constraint

$$2X'D + D'X + qX''' = 0.$$  \hspace{1cm} (16)

We expect then that the vacuum state will satisfy

$$\left(2D \frac{d}{dx} \frac{\delta}{\delta D} + D \frac{d}{dx} \frac{\delta}{\delta D} + q \frac{d^3}{dx^3} \frac{\delta}{\delta D}\right)|\Psi> = 0.$$  \hspace{1cm} (17)

There is a natural extension of this constraint equation and the Hamiltonian to higher dimensions giving us direct access to higher dimensional gravity. It is not clear how the adjoint representation can ever have meaning in higher dimensions. The one feature that may be dimensionally dependent is that $q$ may be zero in all dimensions other
than two. After all, the existence of quadratic differentials implies having the ability to shift the differentials by an inhomogeneous term in one space dimension. (Here the data on the Cauchy slice acts as the one dimensional quadratic differential.) However the quantum theory for the higher dimensional theories will certainly need a quartic term in the propagator, as the three point function contains two factors of momentum. This work is being analyzed presently \[26\]. Furthermore, we will extend these ideas to the super diffeomorphism case to see whether the principle used to construct the interactions still holds \[27\].

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