The universality class of absorbing phase transitions with a conserved field

Michela Rossiti, Romualdo Pastor-Satorras, and Alessandro Vespignani

1) International School for Advanced Studies, SISSA/ISAS Via Beirut 2-4, 34014 Trieste, Italy
2) The Abdus Salam International Centre for Theoretical Physics (ICTP), P.O. Box 586, 34100 Trieste, Italy

We investigate the critical behavior of systems exhibiting a continuous absorbing phase transition in the presence of a conserved field coupled to the order parameter. The results obtained point out the existence of a new universality class of nonequilibrium phase transitions that characterizes a vast set of systems including conserved threshold transfer processes and stochastic sandpile models.

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Absorbing phase transitions (APT) are a category of critical nonequilibrium phase transitions, widespread in condensed matter physics and population and epidemics modeling. Directed percolation (DP) has been recognized as the paradigmatic example of a system exhibiting a transition from an active to a unique absorbing phase. DP defines a precise universality class (theoretically described by the Reggeon field theory) which has proven to be very robust with respect to the introduction of microscopic modifications. The Reggeon field theory is at the heart of a strong claim of universality, summarized in the following conjecture: Continuous absorbing phase transitions to a unique absorbing state fall generically in the universality class of directed percolation. This conjecture is expected to hold for models with short range interactions that, most importantly, do not possess additional symmetries.

Many examples of APT subject to extra symmetries, and thus out of the DP class, have been identified in recent years. Among them we find systems with symmetric absorbing states, models of epidemics with perfect immunization (the so-called dynamic percolation class), and systems with an infinite number of absorbing states. Very recently, it has been pointed out that the critical point of self-organized critical (SOC) sandpile models can also be interpreted as a continuous phase transition with many absorbing states. What distinguishes sandpile models from other models with absorbing states, is that the control parameter, represented by the global density of particles, is a conserved quantity.

Given the large class of systems whose dynamics involves conserved fields, it becomes particularly interesting to explore in general the effect of conservation rules in APT. With this purpose in mind, in this Letter we report the critical behavior of several models showing absorbing transitions that strictly conserve the number of particles or energy. In particular, we introduce a conserved lattice gas (CLG) with short range stochastic microscopic dynamical rules, that undergoes a continuous phase transition to an absorbing state at a critical value of the particle density. We present extensive numerical simulations in $d = 2$ of the stationary and spreading properties of the model, and determine the full set of critical exponents. In order to prove definitively the existence of a well-defined universality class we have also performed simulations of a conserved threshold transfer process (CTTP), and several fixed energy sandpile models with stochastic rules. All models provide critical exponents compatible with a single and broad universality class that embraces all APT in stochastic models with a conserved field. This evidence leads us to conjecture that, in absence of additional symmetries, absorbing phase transitions in stochastic models with infinite absorbing states and activity coupled to a static conserved field define a unique and per se universality class. This result is relevant in the understanding of several reaction-diffusion systems, sandpile models and activated processes that could share the same theoretical description.

The CLG model is defined on a $d$-dimensional square lattice. To each site $i$ it is associated a binary variable $n_i$ that assumes the values $n_i = 1$ if the site is occupied by a particle or $n_i = 0$ if the site is empty. Double occupancy is strictly forbidden. Nearest neighbors particles repel each other via repulsive short range interactions. As a product of this interaction, at each time step particles jump into one of their empty nearest neighbor sites, selected at random. The only dynamics in the model is due to these active particles; isolated particles do not move. The dynamics can be implemented with either sequential or parallel updating. In the latter case, an exclusion principle is applied so that two particles never attempt to move into the same site. We impose periodic boundary conditions, and since the dynamics admits neither input nor loss, the total number of particles $N = \sum_i n_i(t)$ is a conserved quantity. It is clear that the model allows an infinite number (in the thermodynamic limit) of absorbing configurations, in which there are no nearest neighbor particles.

In the CLG model, the constant particle density $n = N/L^d$ acts as a tuning parameter. Initial conditions are generated by placing at random in the lattice $nL^d$ particles, generating an homogeneous and uncorrelated distribution. For small densities, the system will very likely fall into an absorbing configurations with only isolated particles. For large densities, the system reaches a stationary active state with everlasting activity (this is trivially the case for $n > 1/2$). We shall see in the following that as we vary $n$, the CLG model exhibits a continuous transition separating an absorbing phase from an active phase. The phase transition occurs for a nontrivial density $n_c$. 


(\* 1/2\). APT are characterized by the order parameter \(\rho_0\) measuring the density of dynamical entities, in our case the density of nearest neighbor particles. The order parameter is null for \(n < n_c\), and follows a power law \(\rho_0 \sim (n - n_c)^\beta\) for \(n > n_c\). The system correlation length \(\xi\) and time \(\tau\) both diverge as \(n \rightarrow n_c^+\). In the critical region the system is characterized by power law behavior, namely \(\xi \sim (n - n_c)^{-\nu}\) and \(\tau \sim (n - n_c)^{-\nu_\tau}\). The dynamical critical exponent is defined as \(\tau \sim \xi^z\), with \(z = \nu_\eta/\nu_\tau\). These exponents fully define the critical behavior of the stationary state of the model.

In order to study the critical point of the CLG model, we performed numerical simulations in \(d = 2\) for systems with size ranging from \(L = 64\) to \(L = 512\), averaging over \(10^3 - 10^5\) independent initial configurations. Very close to the critical point we have \(\xi \gg L\), so that the actual characteristic length of the system is the lattice size \(L\). Because of its finite size, the system will enter sometimes an absorbing configuration even for values of \(n\) in the supercritical region. It is then convenient to introduce averages over a set of independent trials and calculate the quasi-stationary properties in the active phase from a restricted average over surviving trials with nonzero final activity.

As shown in Fig. 1, after a transient which depends on the system size \(L\) and \(\Delta n \equiv n - n_c\), the surviving samples average of the density of active sites reaches a stationary state \(\rho_0(L, \Delta n)\). Close to the critical point, the finite size scaling ansatz tells us that all quantities depend on the system size through the ratio \(L/\xi\), and the order parameter follows the finite size scaling form

\[
\rho_0(\Delta n, L) = L^{-\beta/\nu} \mathcal{G}(L^{\nu_\perp} \Delta n),
\]

where \(\mathcal{G}\) is a scaling function with \(\mathcal{G}(x) \sim x^\beta\) for large \(x\). For \(\Delta n = 0\) the stationary density follows the pure power law behavior \(\rho_0 \sim L^{-\beta/\nu_\perp}\). On the other hand, for values of \(n\) in the supercritical regime \(\rho_0\) should be independent of \(L\) for \(L \gg \xi\), while in the subcritical regime \(\rho_0\) should decay faster than a power law. This allows us to locate the critical value \(n_c\) of the particle density as the only value of \(n\) at which we recover a nontrivial power law scaling for the density of active sites. In Fig. 1 we observe power law scaling for \(n = 0.23875\), but clearly not for 0.2387 or 0.2388, indicating that \(n_c = 0.23875(5)\) (Figures in parenthesis indicate the statistical uncertainty in the last digit). From the power law decay we find the exponent ratio \(\beta/\nu_\perp = 0.81(3)\). An independent estimate of the exponent \(\beta\) can be obtained by looking at the scaling of the active-site density with respect to \(\Delta n\) for the size \(L = 320\). The resulting power law behavior yields \(\beta = 0.63(1)\), where the error is mainly due to the uncertainty in the critical point \(n_c\). A consistency test can be performed by considering the active site density away from the critical point. In Fig. 2 we plot \(\rho_0(\Delta n, L)L^{\nu_\perp}\) versus \(\Delta nL^{1/\nu_\perp}\) for \(\nu_\perp = 0.78, \beta/\nu_\perp = 0.81\) and \(n_c = 0.23875\). As one would expect all the data collapse onto a single curve, following the scaling form Eq. (1). A further check is provided by the direct fitting of the large \(x\) behavior of the scaling function \(\mathcal{G}(x)\) that gives \(\beta = 0.63\), recovering the independent measurement at \(L = 320\).

To determine the dynamical exponents we turn our attention to the scaling properties of time dependent quantities. In particular, we can define a characteristic time by studying the decay of the probability \(P(t)\) that a random initial configuration has survived up to time \(t\). At the critical point this probability decays, at large times, as \(P(t) \sim \exp(-t/\tau)\). At \(\Delta n = 0\) the effective characteristic length is the system size \(L\), and we have that \(\tau(L) \sim L^z\). We can access the value of \(\tau(L)\) by a direct fitting of the \(P(t)\) exponential tail; \(z\) is then estimated from the behavior of \(\tau(L)\) for different \(L\). Again, a clean power law behavior is obtained for \(n_c = 0.23875\), yielding \(z = 1.52(6)\). Also in this case a consistency check can be performed by studying the time decay of the active sites density \(\rho_{a,all}(t)\), averaging over all trials, even those that have reached an absorbing state. Assuming a single characteristic time scaling as \(L^z\), we have at \(\Delta n = 0\) \(\tau\)
\[ \rho_{\alpha_{iit}}(t, L) = t^{-\theta} F(tL^{-z}) , \]

where \( F(x) \) is a constant for \( x \ll 1 \), and decays faster than any power law for \( x \gg 1 \). Data from simulations with different \( L \) can be collapsed onto a universal curve by plotting \( \rho_{\alpha_{iit}}(t, L) t^\theta \) versus \( tL^{-z} \). The best collapse is obtained for \( \theta = 0.43 \) and \( z = 1.52 \), confirming the value obtained for the dynamical critical exponent. The exponent \( \theta = 0.43(1) \) is recovered also from a direct fitting of the decay of the stationary density averaged over surviving trials (see Fig. 4). In usual APT, the latter exponent obeys \( \theta = \beta/\nu \). This relation assumes a standard scaling behavior at \( \Delta n = 0 \) for \( \rho_{\alpha}(t) \). In our model, however, the simple scaling behavior is broken by an anomalous scaling regime (visible in Fig. 4) by the sharp drop just before the stationary state) that seems to grow steeper with increasing \( L \). It follows that data collapse in time is not achievable with standard scaling forms, and that \( \theta \) violates the usual scaling relation. Albeit its origin is not clear, it is noteworthy that this anomaly is common to all APT with conserved fields inspected so far [1,2].

At this respect, it is interesting to note that the exponents of the model fulfill all scaling relation in standard APT. In what respects to hyperscaling relations, some of them, as \( D = d + z - \beta/\nu \perp \), are fulfilled, while others, like \( \eta + \delta + \theta = d/\nu \), are not. This is again due to the \( \theta \) exponent anomaly.

In APT it is possible to obtain more information on the critical properties by studying the evolution (spread) of activity in systems which start close to an absorbing configuration [3]. In each spreading simulation, a small perturbation is added to an absorbing configuration. It is then possible to measure the spatially integrated activity \( N(t) \), averaged over all runs, and the survival probability \( P(t) \) of the activity after \( t \) time steps. Only at the critical point we have power law behavior for these magnitudes.

Here, we will follow a procedure equivalent to the definition of slowly driven simulations in sandpiles, that enlightens the connections with these models. Instead of fixing the density \( n \) by working with periodic boundary conditions, and thus studying the system at a given distance below the critical point, we impose open boundary conditions and start each spreading experiment by adding a new particle. Under these conditions, the system flows to a stationary state with balance between the input of particles and the boundary dissipation. In the limit in which the particle addition is infinite slow with respect to the spreading of activity, the system reaches a critical state with density \( n_c \) (in the thermodynamic limit) [17]. The infinitely slow drive is implemented by adding a new active particle only when the system falls into an absorbing configuration. The system thus jumps between absorbing states via avalanche-like rearrangements, and we can associate each spreading experiment with an avalanche. The probability distribution \( P_a(s) \) of having a spreading event involving \( s \) sites, as well as the the quantities \( N(t) \) and \( P(t) \) can be measured. The only characteristic length is the system size \( L \), and we can write the scaling forms [3]:

\[ N(t) = t^\nu f(t/L^z), \quad P(t) = t^{-\delta} g(t/L^z), \]
\[ P_a(s) = s^{-\tau} h(s/L^D), \]

where the scaling functions \( f(x), g(x) \) and \( h(x) \) are decreasing exponentially for \( x \gg 1 \), and we have considered that the spreading characteristic time and size are scaling as \( L^z \) and \( L^D \), respectively. Simulations were performed for system of size between \( L = 64 \) and \( L = 1024 \), averaging over at least \( 5 \times 10^6 \) spreading experiments. The extrapolation of the measured densities at infinite \( L \) yields a critical density \( n_c = 0.2388(1) \), in perfect agreement with steady state simulations. The scaling exponents are measured using the now standard moment analysis technique [1,3]. The resulting exponents are summarized in Table 1. In particular, the dynamical exponent \( z = 1.53(2) \) is in excellent agreement with the stationary state simulations, confirming the presence of a single critical behavior for both cases.

| Steady state exponents | \( \beta \) | \( \beta/\nu \) | \( z \) | \( \eta \) | \( \delta \) |
|------------------------|----------|----------|------|------|------|
| CLG                    | 0.63(1)  | 0.81(3)  | 1.52(6) | 0.43(1) |
| CTTP                   | 0.64(1)  | 0.78(3)  | 1.55(5) | 0.43(1) |
| Manna                  | 0.64(1)  | 0.78(2)  | 1.57(4) | 0.42(1) |
| DP                     | 0.583(4) | 0.80(1)  | 1.766(2) | 0.451(1) |

**TABLE I.** Critical exponents for spreading and steady state experiments. Figures in parenthesis indicate the statistical uncertainty in the last digit. Steady state Manna exponents from Ref. [3].

In order to provide further evidence for the existence of a general universality class, we have simulated several other models exhibiting an APT in the presence of a conserved field. The first is a conserved threshold transfer process (CTTP). In the CTTP, the sites of a lattice can be vacant, singly occupied, or doubly occupied by particles, corresponding to a dynamic variable \( n_i = 0, 1, 2 \) respectively. Values \( n_i > 2 \) are strictly forbidden. Dynamics affects only doubly occupied sites: every site with \( n_i = 2 \) tries to transfer both its particles to randomly selected nearest neighbors with \( n_j < 2 \). Singly occupied sites are, on the other hand, inert. The total number of particles \( N = \sum \ n_i \) is thus constant in time. Results from simulations are obtained along the lines shown for the CLG and are reported in Table 1; in this case, the largest sizes used are \( L = 512 \) for the stationary exponents and \( L = 1024 \) for the spreading exponents. As an example of our simulations, in Fig. 5 we plot \( \rho_{\alpha_{iit}}(t, L) t^\theta \) as a function of \( tL^{-z} \), which shows a remarkable data collapse. We have also investigated the Manna sandpile...
model, and its variations by the inclusion of a stochastic threshold \[20\]. In this case, an absorbing phase transition is obtained by using periodic boundary conditions and a fixed number of sand grains (energy) as reported in \[11,12\]. All models lead invariably to the same universality class as the Manna model \[21\] (complete results on these models will be reported elsewhere).

Our results provide striking evidence for a common critical behavior which is incompatible with the DP universality class (see Table I) \[22\]. More noticeably, the models share also the same scaling anomaly in the exponent \(\theta\) \[12\], signalling a common behavior in the transient regime to the stationary state in the active phase. This uniformity of results confirms the hypothesis of a unique universality class for all models with the same conservation symmetry, and lead us to conjecture that, in the absence of additional symmetries, absorbing phase transitions in systems with stochastic dynamics in which the order parameter is locally coupled to a static conserved field define a single and new universality class. This conjecture is supported by noticing that the sandpile model has the same structure and basic symmetries of the present CLG, once the field \(n(x,t)\) is replaced by local energy (sand grains) field \[11,12\]. Indeed, in all models presented here, a conserved non-critical field is dynamically coupled to the non-conserved order parameter field \(\rho_a\). Very likely this basic structure will be reflected in a unique theoretical description (a field theory with the same relevant terms and symmetries) that accounts for the shared critical properties of these models.

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