We consider the quantum Non-linear Schrödinger equation
\[ i \partial_t \Psi = -\partial_x^2 \Psi + 2c \Psi^\dagger \Psi^2 \]
with positive coupling constant \( c \) varying from zero to infinity. We study quantum correlation functions of this model using the determinant representation of these correlation functions. We consider the case of a finite density ground state and evaluate, using the Riemann Hilbert problem, the asymptotics of the probability that no particles are present in the space interval \([0, x]\) in the large \( x \) limit. We call this the probability of phase separation or alternately the emptiness formation probability.

1. Introduction

The one-dimensional Bose Gas is described by quantum Bose fields \( \Psi(x, t) \) with canonical equal time commutation relations \([\Psi(x), \Psi^\dagger(y)] = \delta(x - y)\). The field \( \Psi \) annihilates the Fock vacuum \( \Psi|0\rangle = 0 \) and the Hamiltonian of the model is

\[ H = \int dx \left\{ \partial_x \Psi^\dagger \partial_x \Psi + c \Psi^\dagger \Psi^2 \right\}. \] (1)

The equation of motion is the celebrated nonlinear Schrödinger equation
\[ i \partial_t \Psi = -\partial_x^2 \Psi + 2c \Psi^\dagger \Psi^2. \] The Hamiltonian and the number operator \( Q = \int dx \Psi^\dagger \Psi \) commute
whereby the model for a given number of particles $N$ is equivalent to the quantum mechanical Hamiltonian with delta interactions

$$
\mathcal{H}_N = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1 \leq j < k \leq N} \delta(x_j - x_k). \tag{2}
$$

The Bethe Ansatz for this model was constructed by Lieb and Liniger.[1]

The ground state for the system in a box of length $L$ containing $N$ particles in the thermodynamic limit ($N \to \infty$, $L \to \infty$, $N/L = D = \text{const.}$) is a Fermi "sphere" of radius $q$. Introducing the density of particles in momentum space $\rho(\lambda)$ so that

$$
D = \frac{N}{L} = \int_{-q}^{q} d\lambda \rho(\lambda),
$$

the ground state at zero temperature is then described by the linear integral equation

$$
\frac{1}{2\pi} = \rho(\lambda) - \frac{1}{2\pi} \int_{-q}^{q} d\mu K(\lambda, \mu) \rho(\mu) = \rho(\lambda) - \frac{1}{2\pi} (\hat{K}\rho)(\lambda), \tag{3}
$$

where the kernel of the integral operator $\hat{K}$ is

$$
K(\lambda, \mu) = \frac{2c}{c^2 + (\lambda - \mu)^2}. \tag{4}
$$

Let us now lay out the structure of this paper. We begin in section 2 by defining the probability of phase separation correlation function $P(x)$ which we express in terms of the determinant of a particular Fredholm integral operator. In section 3 we use the identity $ic/(\lambda - \mu + ic) = c \int_{0}^{\infty} dse^{i(\lambda - \mu + ic)}$ to write this integral operator as a special Fredholm integral operator related to a Riemann–Hilbert problem (RP). The introduction of the $s$–integration means that this RP is infinite-dimensional, or more precisely, integral operator-valued. In section 4 we present a new representation of $Gl(2, \mathbb{C})$ in terms of such integral operators which reduces the jump condition (24) of the operator RP to the jump condition (46) of a $2 \times 2$ matrix-valued RP. Finally in section 5 we solve the $2 \times 2$ RP for large $x$. Indeed, we find that this $2 \times 2$ RP is related to that encountered if one studies the case $c = \infty$ for which the $s$–integration is unnecessary. The results obtained in section 5 lead us to a conjecture (see 16) for the asymptotic behaviour of the phase separation probability correlation function. Presently we are unable to make this conjecture completely rigorous.

### 2. Determinant Representation for Probability of Phase Separation

Although the ground state is homogeneous and translationally invariant, due to quantum fluctuations there is a non-vanishing probability that no particles will be found in the space interval $[0, x]$ which we denote

$$
P(x) = \text{Probability that no particles are found in the interval } [0, x]. \tag{5}
$$
In the thermodynamic limit the probability of an empty interval may be expressed as the ratio of two Fredholm determinants:

\[ P(x) = \frac{\langle 0 | \det(I + \hat{V}) | 0 \rangle}{\det(I - \hat{K} / 2\pi)}. \]  

(6)

The integral operator \( \hat{K} \) appearing in the denominator is given above in (4) and is \( x \) independent. Therefore when calculating the large \( x \) asymptotics we need consider only the numerator \( \langle 0 | \det(I + \hat{V}) | 0 \rangle \). The definition of the integral operator \( \hat{V} \) is more involved. \( \hat{V} \) acts on the interval \( C = [-q, q] \) in the same fashion as \( \hat{K} \), namely

\[ (\hat{V} f)(\lambda) = \int_{-q}^{q} d\mu V(\lambda, \mu) f(\mu). \]  

(7)

However in addition to depending on the parameters \( x \) and \( c \) (the length of the empty interval and coupling respectively) \( \hat{V} \) and in turn \( \det(I + \hat{V}) \) are functionals of a dual quantum field \( \hat{\varphi} (\lambda) \). Let us give the kernel of \( \hat{V} \)

\[ V(\lambda, \mu) = -\frac{c}{2\pi} \left[ \frac{\exp{i\lambda x/2 + \hat{\varphi}(\lambda)/2} \exp{-i\mu x/2 - \hat{\varphi}(\mu)/2}}{(\lambda - \mu)(\lambda - \mu + ic)} \right. 
\[ \left. + \frac{\exp{i\mu x/2 + \hat{\varphi}(\mu)/2} \exp{-i\lambda x/2 - \hat{\varphi}(\lambda)/2}}{(\mu - \lambda)(\mu - \lambda + ic)} \right]. \]  

(8)

The dual field acts in a Hilbert space with “dual” Fock vacuum \( |0\rangle \) so that \( \langle 0 | \det(I + \hat{V}) | 0 \rangle \) denotes the dual vacuum expectation value of the dual field valued determinant \( \det(I + \hat{V}) \) which may be computed from the definition of the dual field \( \hat{\varphi} (\lambda) \) given below:

\[ \hat{\varphi} (\lambda) = \hat{p} (\lambda) + \hat{q} (\lambda), \]  

(9)

where \( \forall \lambda, \mu \in C \)

\[ [\hat{p} (\lambda), \hat{q} (\mu)] = \log \left( \frac{c^2}{c^2 + (\lambda - \mu)^2} \right); \]  

(10)

\[ [\hat{p} (\lambda), \hat{p} (\mu)] = 0 = [\hat{q} (\lambda), \hat{q} (\mu)] ; \hat{p} (\lambda) |0\rangle = 0 = (0 | \hat{q} (\lambda). \]  

(11)

From the symmetry of \( \log(c^2/(c^2 + (\lambda - \mu)^2)) \) in \( \lambda \) and \( \mu \) we have the crucial property

\[ [\hat{\varphi} (\lambda), \hat{\varphi} (\mu)] = 0 \quad \forall \lambda, \mu \in C. \]  

(12)

This allows us to treat \( \hat{\varphi} (\lambda) \) simply as some function on \( C \) in our calculation of \( \det(I + \hat{V}) \). Only at the end of this calculation do we take the dual vacuum expectation value.
Obtained using the Riemann Hilbert problem in the large \( x \) limit, our conjecture for the determinant as a functional of \( \hat{\varphi}(\lambda) \) is\(^a\):

\[
\det(I + \hat{V}) \xrightarrow{x \to \infty} (\text{const}) \exp \left\{ -(xq)^2/8 - \frac{x}{2\pi} \int_{-q}^{q} \frac{\hat{\varphi}(\mu) \mu d\mu}{\sqrt{\mu^2 - q^2}} + o(x) \right\}.
\] (13)

Let us now analyse the dual vacuum expectation value of (13). Using (9)-(11) and the identity \( e^{A+B} = e^{A}e^{B}e^{-\frac{1}{2}[A,B]} \) valid \( \forall A, B \) which commute with \( [A,B] \) we obtain

\[
(0|\det(I + \hat{V})|0) \sim \exp \left\{ -(xq)^2/8 \frac{\mu d\mu}{\sqrt{\mu^2 - q^2}} \int_{-q}^{q} \frac{\nu d\nu}{\sqrt{\nu^2 - q^2}} \log \left( \frac{c^2 + (\mu - \nu)^2}{c^2} \right) \right\}
\] (14)

where we denote the dimensionless integral

\[
I(c^2/q^2) = \frac{2}{\pi^2} \int_{-1}^{1} dy \frac{ydz}{\sqrt{1 - y^2}} \int_{-1}^{1} \frac{zdz}{\sqrt{1 - z^2}} \log \left( \frac{c^2/q^2 + (y + z)^2}{c^2/q^2 + (y - z)^2} \right).
\] (15)

\( I(c^2/q^2) \) has the following properties: \( I(c^2/q^2) > 0 \), \( \frac{\partial I}{\partial (c^2/q^2)} < 0 \), \( I(c^2/q^2 \to \infty) = 0 \) and \( I(c^2/q^2 \to 0) = 1 \). Therefore we find as our conjecture for the phase separation probability correlation function

\[
P(x) \xrightarrow{x \to \infty} (\text{const}) \exp \left\{ -(xq)^2/8 \left[ 1 + I(c^2/q^2) \right] + o(x^2) \right\}.
\] (16)

The probability of phase separation is Gaussian in the length \( x \) and the rate of decay in \( x \) decreases monotonically as the coupling \( c \) varies from zero to infinity.

3. Integrable Linear Integral Operators and \( s \)–Integration

Integral operators \( I + \hat{V} \) (acting on some interval \( C \) say) where the kernel of \( \hat{V} \) has the form

\[
V(\lambda, \mu) = \frac{1}{\lambda - \mu} \sum_{j=1}^{N} e_j(\lambda)E_j(\mu),
\] (17)

with \( 0 = \sum_{j=1}^{N} e_j(\lambda)E_j(\lambda) \), belong to a special class which we will call “integrable” operators (the functions \( e_i(\lambda) \) are \( N \) linearly independent functions, continuous and integrable on the interval \( C \), similarly for \( E_j \)). Indeed such integrable integral operators form a group (where multiplication is defined in the usual way for integral

\(^a\sqrt{\lambda^2 - q^2} \) denotes the limit approaching the contour \( C \) from above. Further, we remind the reader that the Hardy small “\( o \)” symbol \( o(x) \) denotes any function of \( x \) decreasing faster than \( x \) as \( x \to \infty \).
operators) and are closely related to a certain $N \times N$ matrix Riemann–Hilbert problem.

Therefore to employ the RP for finite $c$ we must express the kernel (8) in the standard form above. To this end we rewrite the second factor in the denominators using an old identity made famous by Feynman

$$
\frac{1}{\lambda - \mu + ic} = -i \int_{0}^{\infty} ds e^{is(\lambda - \mu + ic)}.
$$

The kernel (8) becomes

$$
V(\lambda, \mu) = \int_{0}^{\infty} \frac{e_{+}(\lambda|s)e_{-}(\mu|s) - e_{-}(\lambda|s)e_{+}(\mu|s)}{\lambda - \mu} ds
$$

$$
= \int_{0}^{\infty} \frac{e_{1}(\lambda|s)E_{1}(\mu|s) + e_{2}(\lambda|s)E_{2}(\mu|s)}{\lambda - \mu} ds
$$

where

$$
e_{\pm}(\lambda|s) = \sqrt{\frac{i}{2\pi}} \exp\{\pm(i\pi \lambda/2 + \phi(\lambda)/2)\} \sqrt{c} \exp\{\pm is\lambda - cs/2\},
$$

and

$$
(e_{1}(\lambda|s), e_{2}(\lambda|s)) = (e_{+}(\lambda|s), e_{-}(\lambda|s)),
$$

$$
(E_{1}(\lambda|s), E_{2}(\lambda|s)) = (e_{-}(\lambda|s), -e_{+}(\lambda|s)),
$$

which is in the form (17) with the sum $\sum_{j=1}^{N}$ generalized to an integral $\int_{0}^{\infty} ds \sum_{j=1}^{2}$.

We must, therefore study a $2 \times 2$ operator valued RP which we now define.

The operator valued RP is to find the $2 \times 2$ matrix $\hat{\chi}(\lambda)$ whose entries are integral operators acting on the interval $[0, \infty)$ and functions of the complex plane $C$ with kernels

$$
\chi(\lambda|s, t) = \begin{pmatrix}
\chi_{11}(\lambda|s, t) & \chi_{12}(\lambda|s, t) \\
\chi_{21}(\lambda|s, t) & \chi_{22}(\lambda|s, t)
\end{pmatrix}, \quad \lambda \in C; \ s, t \in [0, \infty),
$$

such that:

1. $\chi(\lambda|s, t)$ is analytic $\forall \lambda \in C \setminus C$.

2. $\chi(\lambda|s, t)$ is, however, not analytic across the conjugation contour $C$, rather the limits $\chi_{+}(\lambda|s, t)$ and $\chi_{-}(\lambda|s, t)$, approaching $C$ from above and below respectively, satisfy the jump condition

$$
\chi_{-}^{ik}(\lambda|s, t) = \int_{0}^{\infty} \sum_{j=1}^{2} \chi_{+}^{ij}(\lambda|r, t)G_{jk}(\lambda|r, t) dr, \quad \forall \lambda \in C, \ (i, k = 1, 2),
$$

where the conjugation matrix $G_{ik}(\lambda|s, t)$ is given by

$$
G_{ik}(\lambda|s, t) = \delta_{ik}\delta(s - t) - 2\pi i e_{i}(\lambda|s)E_{k}(\lambda|t), \quad (i, k = 1, 2).
$$

---

$^b$We consider $C = [-q, q]$ as an oriented contour so that $\int_{C} = \int^{-q}_{-q}$. In general for any oriented contour we will define “+” to be the limit from the left side of the direction of travel.
3. \( \chi(\lambda|s, t) \) is canonically normalized:

\[
\chi(\infty|s, t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(s - t).
\]

(26)

In fact, if \( \hat{\chi}(\lambda) \) solves the RP defined above, then at \( \lambda = \infty \) it has the whole Laurent series expansion \( \chi(\lambda|s, t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(s - t) + \sum_{n=1}^{\infty} \frac{M_n(s, t)}{\lambda^n} \) where the integral operators \( \hat{M}_n \) are independent of \( \lambda \).

In order to extract the determinant \( \det(I + \hat{V}) \) from the operator RP we consider the formal solution to the above RP

\[
\chi_{ik}(\lambda|s, t) = \delta_{ik} \delta(s - t) + \int_C \frac{f_i(\mu|s)E_k(\mu|t)d\mu}{\mu - \lambda}
\]

(27)

where \( e_i(\lambda|s) = [(I + \hat{V})f_i](\lambda|s) = f_i(\lambda|s) + \int_C V(\lambda, \mu)f_i(\mu|s)d\mu. \)

(28)

For \( \lambda \) large we have

\[
\chi_{ik}(\lambda|s, t) \xrightarrow{\lambda \to \infty} \delta_{ik} \delta(s - t) - \frac{1}{\lambda} \int_C f_i(\mu|s)E_k(\mu|t)d\mu + O(\lambda^{-2})
\]

(29)

\[
= \delta_{ik} \delta(s - t) + \frac{M_{11}^k(s, t)}{\lambda} + O(\lambda^{-2}).
\]

(30)

But now (see 3) calculate the logarithmic derivative w.r.t. \( x \) of \( \det(I + \hat{V}) \) from (19)

\[
\frac{\partial}{\partial x} \log \det(I + \hat{V}) = \frac{i}{2} \int_C \int_0^\infty (f_1(\mu|s)e_2(\mu|s) + f_2(\mu|s)e_1(\mu|s))dsd\mu
\]

\[
= -\frac{i}{2} \int_0^\infty (M_{11}^{11}(s, s) - M_{12}^{22}(s, s))ds = -i\text{tr} \hat{M}_{11}^{11},
\]

(31)

where “tr” denotes the trace operation in the space of integral operators on \([0, \infty)\). Therefore \( \det(I + \hat{V}) \) can be obtained from the large \( \lambda \) asymptotics of the solution to the above RP.

At this point the prospect of solving an operator valued RP should seem daunting since even the solution for the matrix case is not in general known. However we now construct a representation of \( GL(2, \mathbb{C}) \) which will allow us to consider a (known) \( 2 \times 2 \) RP.

4. \( GL(2, \mathbb{C}) \) Representation by Integral Operators

Let us consider some \( 2 \times 2 \) matrix of integral operators \( \hat{O} \) with kernels

\[
O(s, t) = \begin{pmatrix} O_{11}(s, t) & O_{12}(s, t) \\ O_{21}(s, t) & O_{22}(s, t) \end{pmatrix}, \quad s, t \in [0, \infty),
\]

(32)
and multiplication defined by:

\[ \mathcal{O}_{\Pi}(s, t) = \left( \hat{O}_1 \hat{O}_\Pi \right)(s, t) = \left( \begin{array}{cc} \mathcal{O}_{11}^{\Pi}(s, t) & \mathcal{O}_{12}^{\Pi}(s, t) \\ \mathcal{O}_{21}^{\Pi}(s, t) & \mathcal{O}_{22}^{\Pi}(s, t) \end{array} \right), \quad s, t \in [0, \infty), \tag{33} \]

where

\[
\begin{align*}
\mathcal{O}_{11}^{\Pi}(s, t) &= \int_0^\infty dr \left( \mathcal{O}_{11}(s, r) \mathcal{O}_{11}^I(r, t) + \mathcal{O}_{12}(s, r) \mathcal{O}_{21}^I(r, t) \right) \\
\mathcal{O}_{12}^{\Pi}(s, t) &= \int_0^\infty dr \left( \mathcal{O}_{11}(s, r) \mathcal{O}_{12}^I(r, t) + \mathcal{O}_{12}(s, r) \mathcal{O}_{22}^I(r, t) \right) \\
\mathcal{O}_{21}^{\Pi}(s, t) &= \int_0^\infty dr \left( \mathcal{O}_{21}(s, r) \mathcal{O}_{11}^I(r, t) + \mathcal{O}_{22}(s, r) \mathcal{O}_{21}^I(r, t) \right) \\
\mathcal{O}_{22}^{\Pi}(s, t) &= \int_0^\infty dr \left( \mathcal{O}_{21}(s, r) \mathcal{O}_{12}^I(r, t) + \mathcal{O}_{22}(s, r) \mathcal{O}_{22}^I(r, t) \right). \tag{34} \end{align*}
\]

Let us now construct a special class of such operators \( \hat{O} \) which form a representation of \( \text{Gl}(2, \mathbb{C}) \) (the generalization to \( \text{Gl}(N, \mathbb{C}) \) is straightforward). Consider a pair of functions \( \alpha(s) \) and \( \beta(s) \) on \( [0, \infty) \) which we write in Dirac notation as \( \alpha \equiv \langle 1 \rangle \) and \( \beta \equiv \langle 2 \rangle \). In this notation we may write left multiplication by \( \hat{O}_{ik} \) as

\[ \hat{O}_{ik}|1\rangle = \int_0^\infty dt \mathcal{O}_{ik}(s, t) \alpha(t). \tag{35} \]

Further suppose the functions \( A(s) \equiv \langle 1 \rangle \) and \( B(s) \equiv \langle 2 \rangle \) (right multiplication by \( \hat{O}_{ik} \) is defined integrating over the first argument of \( \mathcal{O}_{ik}(s, t) \)) satisfy

\[ \langle 1 | 1 \rangle \equiv \int_0^\infty ds A(s) \alpha(s) = 1 = \langle 2 | 2 \rangle \equiv \int_0^\infty ds B(s) \beta(s). \tag{36} \]

Observe now that one may define a representation \( \hat{\mathcal{A}} \) of \( \text{Gl}(2, \mathbb{C}) \) via

\[ M \in \text{Gl}(2, \mathbb{C}) \mapsto \hat{\mathcal{A}}(M) = \left( \begin{array}{cc} I - |1\rangle \langle 1| & 0 \\ 0 & I - |2\rangle \langle 2| \end{array} \right) + \left( \begin{array}{cc} M_{11}|1\rangle \langle 1| & M_{12}|1\rangle \langle 2| \\ M_{21}|2\rangle \langle 1| & M_{22}|2\rangle \langle 2| \end{array} \right), \tag{37} \]

\( (M_{11}, M_{12}, M_{21} \text{ and } M_{22} \text{ are complex numbers and } I \text{ is the identity operator in the space of integral operators on } [0, \infty)) \). Multiplication by the integral operators (projectors) \( |1\rangle \langle 1|, |1\rangle \langle 2|, |2\rangle \langle 1| \text{ and } |2\rangle \langle 2| \) is given, for example, by

\[ |1\rangle \langle 2| f(s) = \left( \int_0^\infty ds B(s) f(s) \right) |1\rangle. \tag{38} \]

\(^c\)We assume that the integral operators \( \hat{O}_{ik} \) act on the interval \( [0, \infty) \) but this is of course not necessary.
In particular, we have \([I - |1\rangle\langle 1|][1\rangle\langle 1|] = 0\). Indeed for any \(M, N \in \text{GL}(2, \mathbb{C})\) the representation \(\hat{A}\) has the following properties:

\[
\hat{A}(MN) = \hat{A}(M)\hat{A}(N) ; \quad \hat{A}(I) = I ; \quad \hat{A}(M^{-1}) = \hat{A}^{-1}(M) \tag{39}
\]

\[
\det \hat{A}(M) = \det M = M_{11}M_{22} - M_{12}M_{21} \tag{40}
\]

\[
\text{tr} \left( \hat{A}(M) - \left( \begin{array}{cc} I - |1\rangle\langle 1| & 0 \\ 0 & I - |2\rangle\langle 2| \end{array} \right) \right) = \text{tr} M = M_{11} + M_{22} \tag{41}
\]

Notice that \((40)\) expresses an infinite dimensional determinant as the determinant of the \(2 \times 2\) matrix \(M\). Let us conclude this section by rewriting the conjugation matrix of our RP using the representation \(\hat{A}\). We begin by defining functions

\[
\begin{align*}
|1\rangle &= \sqrt{c}e^{is\lambda-cs/2} = \alpha(s), \quad \langle 1| = \sqrt{c}e^{-is\lambda-cs/2} = A(s) \\
|2\rangle &= \sqrt{c}e^{-is\lambda-cs/2} = \beta(s), \quad \langle 2| = \sqrt{c}e^{is\lambda-cs/2} = B(s) \tag{42}
\end{align*}
\]

The vectors \(|1\rangle\) and \(|2\rangle\) are normalized but not orthogonal, so that their inner products are \(\langle 1|1\rangle = c \int_0^\infty e^{-cs}ds = 1 = \langle 2|2\rangle\) and \(\langle 1|2\rangle = \langle 2|1\rangle^* = c \int_0^\infty e^{-2is\lambda-cs}ds = \frac{c}{cs-2i\lambda}\) (for \(\lambda\) real the bra and ket notation denote complex conjugation in the usual fashion although this is not necessary). Their outer products are of course integral operators in terms of which we will write the conjugation matrix. We now have

\[
\begin{align*}
e_1(\lambda|s) &= \sqrt{\frac{i}{2\pi}}e^{i(x\lambda/2+\tilde{\varphi}(\lambda)/2)}|1\rangle \equiv e_1^0(\lambda)|1\rangle, \\
E_1(\lambda|t) &= \sqrt{\frac{i}{2\pi}}e^{(-ix\lambda/2-\tilde{\varphi}(\lambda)/2)}\langle 1| \equiv E_1^0(\lambda)\langle 1|, \\
e_2(\lambda|s) &= \sqrt{\frac{i}{2\pi}}e^{-(ix\lambda/2-\tilde{\varphi}(\lambda)/2)}|2\rangle \equiv e_2^0(\lambda)|2\rangle, \\
E_2(\lambda|t) &= -\sqrt{\frac{i}{2\pi}}e^{(ix\lambda/2+\tilde{\varphi}(\lambda)/2)}\langle 2| \equiv E_2^0(\lambda)\langle 2|. \tag{43}
\end{align*}
\]

Hence, from \((25)\) we have for the conjugation matrix (where from now on the \(s\) and \(t\) dependence will be taken as understood)

\[
\hat{G}(\lambda) = \begin{pmatrix}
I - 2\pi ie_1^0(\lambda)E_1^0(\lambda)|1\rangle\langle 1| & -2\pi ie_1^0(\lambda)E_2^0(\lambda)|1\rangle\langle 2| \\
-2\pi ie_2^0(\lambda)E_1^0(\lambda)|2\rangle\langle 1| & I - 2\pi ie_2^0(\lambda)E_2^0(\lambda)|2\rangle\langle 2|
\end{pmatrix} = \hat{A}(G^0(\lambda)) \tag{44}
\]

where the \(2 \times 2\) matrix \(G^0(\lambda)\) which here plays the rôle of \(M\) is readily computed to be

\[
G^0(\lambda) = \begin{pmatrix}
2 & -e^{ix\lambda+\tilde{\varphi}(\lambda)} \\
e^{-ix\lambda-\tilde{\varphi}(\lambda)} & 0
\end{pmatrix}. \tag{45}
\]
Except for the additional exponents $\pm \tilde{\varphi}(\lambda)$, $G^\circ(\lambda)$ is the conjugation matrix one finds for the $2 \times 2$ RP encountered in the case $c = \infty$.

5. Asymptotic Analysis of the Riemann Hilbert Problem

Firstly let us sketch the derivation of the asymptotic solution to the $2 \times 2$ RP with conjugation matrix $G^\circ(\lambda)$ given above in (45) (we follow the scheme developed in [4] for the case $\varphi(\lambda) \equiv 0$). I.e. we are searching for the $2 \times 2$ matrix function $\chi^\circ(\lambda)$ normalized to $\int_C \chi^\circ G \chi^\circ d\lambda = \frac{1}{2\pi i} \int_C \frac{\varphi(\mu)}{\sqrt{\mu^2 - q^2}} d\mu$ at infinity and analytic everywhere in the complex plane $C$ save for a discontinuity across the contour $C = [-q, q]$ where $\chi^\circ(\lambda)$ satisfies

$$\chi^\circ(\lambda) = \chi^\circ_+(\lambda) G^\circ(\lambda), \quad \forall \lambda \in C.$$  \hspace{1cm} (46)

The idea is to make a "gauge transformation" under which the above RP limits for large $x$ to a new RP with conjugation matrix $\tilde{G} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Define, therefore

$$\tilde{\chi}^\circ(\lambda) = \exp \{ \psi_\infty \sigma_3 \} \chi^\circ(\lambda) \exp \{ [(ix/2)(\lambda - g(\lambda)) - \psi(\lambda)] \sigma_3 \},$$  \hspace{1cm} (47)

$(\sigma_3 = \text{diag}(1, -1))$. The function $g(\lambda) = \sqrt{\lambda^2 - q^2}$ satisfies the scalar RP $g_-(\lambda) = -g_+(\lambda)$ across $C$ with normalization $g(\lambda) \xrightarrow{\lambda \rightarrow \infty} \lambda - q^2/2\lambda + O(\lambda^{-3})$ and

$$\psi(\lambda) = -\frac{\sqrt{\lambda^2 - q^2}}{2\pi i} \int_C \frac{\varphi(\mu)}{\sqrt{\mu^2 - q^2}} d\mu.$$  \hspace{1cm} (48)

satisfies the affine scalar RP $\psi_+ + \psi_- + \varphi(\lambda) = 0$ across the contour $C$ with normalization $\psi(\lambda) \xrightarrow{\lambda \rightarrow \infty} \psi_\infty = \frac{1}{2\pi i} \int_C \frac{\varphi(\mu)}{\sqrt{\mu^2 - q^2}} d\mu$. Notice that $\psi(\lambda)$ is well defined on all of $C$ even though it depends on the dual quantum field $\varphi(\lambda)$. One may verify that $\tilde{\chi}(\lambda)$ satisfies a canonically normalized RP across $C$ with conjugation matrix

$$\tilde{G}^\circ(\lambda) = \exp \{ -(ix/2)(\lambda - g_+(\lambda)) - \psi_+(\lambda) \sigma_3 \} G^\circ(\lambda) \exp \{ [(ix/2)(\lambda - g_-(\lambda)) - \psi_-(\lambda)] \sigma_3 \}$$

$$= \left( \begin{array}{cc} 2e^{\Delta \psi(\lambda) + ixg_+(\lambda)} & -1 \\ 1 & 0 \end{array} \right) \xrightarrow{x \rightarrow \infty} \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right),$$  \hspace{1cm} (49)

where $\Delta \psi(\lambda) = \psi_+(\lambda) - \psi_-(\lambda)$. The canonically normalized RP for $\tilde{\chi}(\lambda)$ is easily solved in the large $x$ limit, we find

$$\tilde{\chi}^\circ(\lambda) \xrightarrow{x \rightarrow \infty} \left( \begin{array}{cc} \frac{1}{2}(a(\lambda) + a^{-1}(\lambda)) & \frac{i}{2}(a(\lambda) - a^{-1}(\lambda)) \\ -\frac{i}{2}(a(\lambda) - a^{-1}(\lambda)) & \frac{1}{2}(a(\lambda) + a^{-1}(\lambda)) \end{array} \right).$$  \hspace{1cm} (50)
where \(a(\lambda) = (\frac{1+q}{c-q})^{1/4}\) satisfies the canonically normalized RP \(a_-(\lambda) = ia_+(\lambda)\) (\(\forall \lambda \in C\)). Inverting the transformation \([17]\) we obtain the large \(x\) solution to the original RP,

\[
\chi^o(\lambda) \xrightarrow{x \to \infty} \begin{pmatrix} \frac{i}{2}a(\lambda) + a^{-1}(\lambda) e^{i(\lambda - \psi_\infty - ix(\lambda - g(\lambda))/2} \\
-\frac{i}{2}a(\lambda) - a^{-1}(\lambda) e^{i(\lambda + \psi_\infty - ix(\lambda - g(\lambda))/2}
\end{pmatrix} = I + \frac{1}{\lambda} \begin{pmatrix} \psi_1 - ixq^2/4 & \frac{iq}{2}e^{-2\psi_\infty} \\
-\frac{iq}{2}e^{2\psi_\infty} & -\psi_1 + ixq^2/4
\end{pmatrix} + O(\lambda^{-2}),
\]

where \(\psi_1 = \frac{1}{2\pi i} \int_C \frac{\phi^*(\mu)d\mu}{\sqrt{\mu^2 - q^2}}\).

We may now try to utilise our \(Gl(2, C)\) representation to write the solution to the finite \(c\) operator valued RP as

\[
\hat{\chi}(\lambda) = \hat{A}(\chi^o(\lambda)),
\]

since by property \([39]\)

\[
\hat{\chi}_-(\lambda) = \hat{A}(\hat{\chi}_-(\lambda)) = \hat{A}(\chi_+^o(\lambda)G^o(\lambda)) = \hat{A}(\chi_+^o(\lambda))\hat{A}(G^o(\lambda)) = \hat{\chi}_+(\lambda)\hat{G}(\lambda).
\]

The observant reader may notice that this solution does not obey the normalization at infinity since the projectors \(|1\rangle\langle 1|, |1\rangle\langle 2|, |2\rangle\langle 1|\) and \(|2\rangle\langle 2|\) are essentially singular at \(\lambda = \infty\). Moreover, as bounded operator valued functions they are analytic in the strip \(|\text{Im}\lambda| < c/2\) only. Therefore strictly speaking equation \([52]\) is not correct. The correct version is

\[
\hat{\chi}(\lambda) = \hat{\Phi}(\lambda)\hat{\chi}_+(\lambda),
\]

where \(\hat{\Phi}(\lambda)\) is an operator valued function analytic in the strip \(\mathcal{S} : |\text{Im}\lambda| < c/2\). The last equation itself may be treated as a Riemann Hilbert Problem. Indeed, let \(\Gamma\) be a simple contour belonging to the strip \(\mathcal{S}\) encircling the interval \(C\) in the counterclockwise direction. Introducing

\[
\hat{Y}(\lambda) = \begin{cases} \hat{\chi}(\lambda) & \text{if } \lambda \text{ is outside } \Gamma \\ \hat{\Phi}(\lambda) & \text{if } \lambda \text{ is inside } \Gamma, \end{cases}
\]

we can rewrite our initial operator RP 1-3 as the following new RP\(^d\):

1\(^\Gamma\). \(\hat{Y}(\lambda)\) is analytic \(\forall \lambda \in C \setminus \Gamma\).

2\(^\Gamma\). \(\hat{Y}_-(\lambda) = \hat{Y}_+(\lambda)\hat{A}(\chi^o(\lambda)), \lambda \in \Gamma\).

\(^d\) To avoid confusion, in agreement with our conventions let us reiterate that + is the limit from inside the anticlockwise contour \(\Gamma\).
From (31) and the equation
\[
\det \hat{Y}_{11}(\lambda) = 1 + \text{tr} \hat{M}_{11}^1 / \lambda + O(\lambda^{-2}),
\]
we see that to calculate \( \det (I + \hat{V}) \) we need \( \det \hat{Y}_{11}(\lambda) \) for large \( \lambda \). It follows from 2Γ that
\[
\text{Det} \hat{Y}^+_{11}(\lambda) = \det \hat{Y}_{11}(\lambda) (\chi^\circ_{22}(\lambda) - \chi^\circ_{21}(\lambda)c(\lambda)),
\]
where “det” indicates the determinant of integral operators on the space \([0, \infty)\).

Note that the product \([\hat{Y}_{11}(\lambda)]^{-1}\hat{Y}_{12}(\lambda) (= [\hat{\chi}_{11}(\lambda)]^{-1}\hat{\chi}_{12}(\lambda))\) has these properties.

Statements (a) and (b) constitute our main hypothesis concerning the solution \( \chi(\lambda) \) of the RP 1-3. By virtue of this hypothesis, equation (57) implies that
\[
\det \hat{Y}_{11}(\lambda) = (\chi^\circ_{22}(\lambda) - \chi^\circ_{21}(\lambda)c(\lambda))^{-1},
\]
for all \( \lambda \) outside of the contour \( \Gamma \). In addition property (b) leads to the equation
\[
\lim_{\lambda \to \infty} \lambda(\det \hat{Y}_{11}(\lambda) - 1) = - \lim_{\lambda \to \infty} \lambda(\chi^\circ_{22}(\lambda) - 1).
\]

Taking (57) and (51) into account we obtain
\[
\text{tr} \hat{M}_{11}^{x=\infty} \psi_1 - ixq^2/4 + o(1),
\]
which yields our conjecture for the determinant
\[
\det (I + \hat{V}) \xrightarrow{x=\infty} \text{const \ exp} \left\{ -\frac{(xq)^2/8}{2\pi} \int_{\mathbb{C}} \frac{\varphi(\mu)d\mu}{\sqrt{\mu^2 - q^2}} + o(x) \right\},
\]
as quoted in section 2. Finally let us observe that for \( c = \infty \), \( \hat{p}(\lambda), \hat{q}(\mu) = 0 \) so that for any functional \( F[\varphi(\lambda)] \) the dual vacuum expectation value \( (0|F[\varphi(\lambda)]|0) = F|\varphi=0 \). Hence \( (0|\det (I + \hat{V})|0)|_{c=\infty} = e^{-\frac{1}{8}(xq)^2/8} \). An old result first obtained in and then extended to the complete asymptotic expansion in 6.

6. Acknowledgements

This work was supported by NSF grant PHY-9321165.
7. References

1. E. H. Lieb and W. Liniger, *Phys. Rev.* 130 (1963) 1605. E. H. Lieb, *Phys. Rev.* 130 (1963) 1616.
2. V. E. Korepin, N. M. Bogoliubov and A. G. Izergin, *Quantum Inverse Scattering Method* (Cambridge University Press, 1993).
3. P. A. Deift, A. R. Its and X. Zhou (to be published), *A Riemann–Hilbert approach to Asymptotic Problems Arising in the Theory of Random Matrix Models, and in the Theory of Integrable Statistical Mechanics*.
4. J. des Cloizeaux and M. L. Mehta, *J. Math. Phys.* 14 (1973) 1648.
5. F. Dyson, *Commun. Math. Phys.* 47 (1976) 171.