The Wideband Slope of Interference Channels: The Small Bandwidth Case

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Abstract

This paper studies the low-SNR regime performance of a scalar complex $K$-user interference channel with Gaussian noise. The finite bandwidth case is considered, where the low-SNR regime is approached by letting the input power go to zero while bandwidth is small and fixed. We show that for all $\delta > 0$ there exists a set with non-zero measure (probability) in which the wideband slope per user satisfies $S_0 < \frac{2}{K} + \delta$. This is quite contrary to the large bandwidth case [1], where a slope of 1 per user is achievable with probability 1. We also develop an interference alignment scheme for the finite bandwidth case that shows some gain.

I. INTRODUCTION

This paper and the companion paper [1] study the bandwidth-power trade-off of a $K$-user interference channel in the low-SNR (signal-to-noise) regime, where explicitly

$$\text{SNR} \triangleq \frac{P}{BN_0}. \quad (1)$$

Bandwidth and input power, two important design parameters, are related by the function $R\left(\frac{E_b}{N_0}\right)$, where $\frac{E_b}{N_0}$ is the transmitted energy per bit, and $R$ is the spectral efficiency. The concept of the low-SNR regime was introduced by S. Verdú in the 2002 paper [2]. A system working in this regime is characterized by very small spectral efficiency, so that the $R\left(\frac{E_b}{N_0}\right)$ curve can be closely approximated by its first-order approximation, which is determined by two measures: the minimum energy per bit $\frac{E_b}{N_0}_{\text{min}}$ and the wideband slope $S_0$. $\frac{E_b}{N_0}_{\text{min}}$ is the minimum transmitted energy per bit required by reliable communication, which is generally achieved at zero spectral efficiency; and $S_0$ is the first-order slope of $R\left(\frac{E_b}{N_0}\right)$ as $\frac{E_b}{N_0}$ approaches $\frac{E_b}{N_0}_{\text{min}}$. These two measures are defined by

$$\frac{E_b}{N_0}_{\text{min}} = \lim_{\text{SNR} \to 0} \frac{\text{SNR}}{R(\text{SNR})}. \quad (2)$$

$$S_0 \triangleq \lim_{\frac{E_b}{N_0} \to \frac{E_b}{N_0}_{\text{min}}} \frac{R\left(\frac{E_b}{N_0}\right)}{10 \log_{10} \frac{\frac{E_b}{N_0}}{\frac{E_b}{N_0}_{\text{min}}}} - 10 \log_{10} 2, \quad (3)$$

Further manipulations in [2] show that $\frac{E_b}{N_0}_{\text{min}}$ and $S_0$ can be determined by the first and second order derivative of $R(\text{SNR})$ at zero SNR:

$$\frac{E_b}{N_0}_{\text{min}} = \frac{\log_e 2}{R(0)}, \quad (4)$$

$$S_0 = -\frac{2}{R(0)} \left(\frac{\dot{R}(0)}{R(0)}\right)^2, \quad (5)$$

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where \( \hat{R}(0) \) and \( \ddot{R}(0) \) are the first-order and the second-order Taylor expansion coefficients for \( SNR \to 0 \). \( \hat{R}(0) = \frac{dR(SNR)}{dSNR} \bigg|_{SNR=0} \) and \( \ddot{R}(0) = \frac{d^2R(SNR)}{dSNR^2} \bigg|_{SNR=0} \) if \( R(SNR) \) is differentiable.

What is interesting is that there are two distinct ways to approach the low-SNR regime, which have very different impacts on the performance of the interference channel. Although approaching the low-SNR regime by letting \( B \to \infty \) is emphasized in previous papers (hence the term “wideband slope”), it is not the only way. As can be noted from the definition of \( SNR(1) \), \( SNR \) approaches zero if either \( B \to \infty \) or \( P \to 0 \). Consider a point-to-point AWGN channel with spectral efficiency

\[
R = \log \left(1 + \frac{P}{BN_0}\right).
\]

The low-SNR results are based on a Taylor series of \( \log(1 + x) \), as also seen by (4-5), therefore as long as \( SNR = \frac{P}{BN_0} \to 0 \) in any manner, low-SNR results such as minimum energy per bit and wideband slope are unchanged. The key is that the spectral efficiency \( R \to 0 \), not that \( B \to \infty \). For the interference channel, on the other hand, different results are obtained depending on how the low-SNR regime is approached.

In the first approach, let \( B \to \infty \) while \( P \) is fixed and finite. We call this the large bandwidth regime. In (1) we proved that in this case a wideband slope of \( K \) was achievable with probability one by using channel delays.

In the second approach, let \( P \to 0 \) while \( B \) is fixed and finite. In this case, the rate \( BR \) in bits/s must necessarily approach 0 as well, and we therefore call this the low-rate regime. This is the case considered in this paper, and as will be seen the results are quite different than the the case in (1).

To put the results of this paper in context, consider the completely symmetric channel: the channel between receiver pairs \((i, j)\) is the same for all \( 1 \leq i, j \leq K \), both \( i = j \) and \( i \neq j \). We call this channel the \( K \)-channel. The capacity of this channel is fully known: because of the symmetry all receivers must be able to decode all messages, and the capacity is therefore given by the MAC (multiple access channel) bound into one of the nodes. For this channel, FDMA (frequency division multiple access) or TDMA (time division multiple access) is optimum, and the degrees of freedom \( K \) is 1 (1 per user) while the wideband slope is 2 \((2/K \) per user). A key question is if this channel is typical. For degrees of freedom the answer is no: the results in (4) and (3) show that the degrees of freedom is \( K/2 \) \((\frac{1}{2} \) per user) almost everywhere for a scalar channel. Thus, the degrees of freedom is discontinuous in \( K \), and in fact almost everywhere. Similarly, (3) shows that for time-varying channels, the degrees of freedom is \( K/2 \) with probability one. In (1) we proved analogously that in the large bandwidth regime the wideband slope is \( K(1 \) per user) with probability one for a line-of-sight channel. Thus, also the wideband slope is discontinuous in \( K \) and again in fact discontinuous with probability one.

The main result of this paper is that in the low-rate regime the wideband slope is upper semi-continuous in \( K \). That is, for any \( \delta > 0 \) there exists an open set \( \mathcal{C}_\delta \) of channels so that \( \mathcal{C}_\delta \in \text{cl}(\mathcal{C}_0) \) (cl means closure) and \( S_0 \leq 2 + \delta \) in \( \mathcal{C}_0 \). While this does not give a complete characterization of the wideband slope as in (1), it does show that interference alignment in the low-rate regime does not give the same dramatic gain in performance as in the large bandwidth and high SNR regimes. We still show that interference alignment can outperform TDMA, but in line with the outer bound, not by much.

\[ y_j[n] = C_{jj}x_j[n] + \sum_{i \neq j} C_{ji}\tilde{x}_i[n - n_{ji}] + z_j[n] \]

where

\[
\tilde{x}_i[n] = \sum_{m=-\infty}^{\infty} x_i[m] \text{sinc}(n - m - \delta_{ji}).
\]

and

\[
n_{ji} = \left\lceil \tau_{ji}B + \frac{1}{2} \right\rceil \quad (7)
\]

\[
\delta_{ji} = \tau_{ji}B - \left\lceil \tau_{ji}B + \frac{1}{2} \right\rceil \quad (8)
\]

II. System Model and Preliminaries

In (1) we derived the following baseband model for the interference channel (in a line-of-sight model):

\[
y_j[n] = C_{jj}x_j[n] + \sum_{i \neq j} C_{ji}\tilde{x}_i[n - n_{ji}] + z_j[n] \]

where

\[
\tilde{x}_i[n] = \sum_{m=-\infty}^{\infty} x_i[m] \text{sinc}(n - m - \delta_{ji}).
\]
are the symbol and fractional delays, respectively. It was these delays that allowed interference alignment in \( \text{III} \) as \( B \to \infty \).

In the present paper we keep \( B \) fixed; we will further assume that \( B \) is so small that the delays are insignificant, \( n_{ji} = 0, \delta_{ji} \approx 0 \), and we therefore arrive at the usual model for the interference channel,

\[
y_j[n] = C_{jj}x_i[n] + \sum_{i \neq j} C_{ji}x_i[n] + Z_j[n],
\]

where \( C_{ji} \) is a complex scalar and the noise \( Z_j \) is i.i.d. (independent, identically distributed) circularly symmetric complex random variable with distribution \( \mathcal{CN}(0, B N_0) \); since \( B \) does not play any role in the rest of the paper we will put \( B = 1 \) and omit it from future formulas. Notice that the model (9) is valid also for a non line-of-sight model, as long as delays along all paths are insignificant.

### A. Circularly Asymmetric Signaling

To characterize the Shannon capacity region of the model (9), most research restricts the inputs to be circularly symmetric, i.e., the the real part of the input \( \text{Re}\{x_j\} \) and the imaginary part of the input \( \text{Im}\{x_j\} \) are i.i.d.. However, [6] shows that circularly asymmetric signaling achieves higher degree of freedom in the high-SNR regime. Although the specific interference alignment technique they proposed is not applicable to the low-SNR regime, that work still has inspired our interference alignment for the low-SNR regime. In section [IV] we will see that circularly asymmetric signaling indeed benefits system performance.

In circularly asymmetric signaling, the transmitters are allowed to allocate power on real and imaginary dimensions, and the real part of the input \( \text{Re}\{x_j\} \) is allowed to be correlated with the imaginary part of the input \( \text{Im}\{x_j\} \), while in circularly symmetric signaling, \( \text{Re}\{x_j\} \) and \( \text{Im}\{x_j\} \) are required to be i.i.d.. To characterize such transmission schemes, it is more convenient to consider the scalar complex channel as a two-dimensional vector real channel.

Following [6], we extend (9) into an equivalent two-dimensional real channel,

\[
Y_j = |C_{jj}|X_j + \sum_{i=1, i \neq j}^{K} |C_{ji}|U_j\xi_i + Z_j \tag{10}
\]

where \( U_{ji} =\left( \begin{array}{cc} \cos(\phi_{ji}) & -\sin(\phi_{ji}) \\ \sin(\phi_{ji}) & \cos(\phi_{ji}) \end{array} \right) \) is the rotation matrix with angle \( \phi_{ji} \), and the 2 \times 1 vector white Gaussian noise is \( Z_j \sim \mathcal{N}(0, BN_0) \). Notice that we let receiver \( j \) be phase-synchronized with the received \( x_j \) so that \( \phi_{ji} = 0 \). Without loss of generality, we can assume \( N_0 = 1 \) whenever convenient.

The input signal \( X_j \) is related to the scalar complex model by: \( X_j = \left( \begin{array}{c} \text{Re}\{x_j\} \\ \text{Im}\{x_j\} \end{array} \right) \). We assume that an \( (2^nR_j, n) \) code is used at receiver \( j \), for \( j = 1, \cdots, K \). At the transmitter \( j \), the input message \( W_j \) is drawn uniformly randomly from the index set \( \{1, \cdots, 2^{nR_j}\} \), and a deterministic function yields the length \( n \) transmitted codeword \( X^n_j(W_j) \). The codebook of user \( j \) is composed by the set of codewords \( X^n_j(1), \cdots, X^n_j(2^{nR_j}) \). We require each user to satisfy power constraint \( P_j/B \) per second per Hz. Recall that we may assume \( B = 1 \). Denote the \( i \)th entry of \( X^n_j \) by \( X^{(i)}_j \). Therefore the input must satisfy constraint

\[
\frac{1}{n} \sum_{i=1}^{n} E \left[ X^{(i)}_j \left( X^{(i)}_j \right)^T \right] \preceq V_j, \tag{11}
\]

where \( \text{Tr}(V_j) = P_j, j = 1, \cdots, K \). For any two given matrices \( A \) and \( B \), the notation \( A \preceq B \) means that the matrix \( B - A \) is positive semi-definite. Notice that given the assumption \( B = N_0 = 1 \), we have

\[
\text{SNR}_j = \frac{P_j}{BN_0} = P_j \tag{12}
\]

Corresponding to the \( X^n_j(W_j) \) codebook, we also define four Gaussian random variables \( X^n_{jG}, X_{jG}, Y^n_{jG}, Y_{jG} \), and \( Y^n_{jG} \) as follows for later use. Let \( X^n_{jG} \) be i.i.d. vector Gaussian random variable, \( X^n_{jG} \sim \mathcal{CN}(0, V_j) \), where \( V_j = \frac{1}{N} \sum_{n=1}^{N} X^n_j X^n_j^H \),
$V_j \preceq V_j$ given power constraint (11). Let $X_{ijG}$ be i.i.d. vector Gaussian random variable, $X_{ijG} \sim \mathcal{N}(0, V_j)$. $Y_{ijG}$ and $Y_{ijG}$ are defined as

$$Y_{ijG} = |C_{jj}|X_{ijG} + \sum_{i=1,i\neq j}^{K} |C_{ij}|U_{ji}X_{ijG} + Z_j,$$

$$Y_{ijG} = |C_{jj}|X_{ijG} + \sum_{i=1,i\neq j}^{K} |C_{ij}|U_{ji}X_{ijG} + Z_j.$$

**B. Performance Criterion And Performance Measures**

For more than two users it is complicated to compare complete slope regions, and we are therefore looking at a single quantity—the sum slope $S_0$, to characterize performance. The formal definitions are as follows.

**Definition 1.** (Sum slope). $S_0$ is defined as the first-order slope of the $R_{\text{sum}} \left( \frac{E_b}{N_0 \text{sum}} \right)$ curve, where $R_{\text{sum}} \triangleq \sum_{j=1}^{K} R_j$ and $\frac{E_b}{N_0 \text{sum}} \triangleq \frac{\sum_{j=1}^{K} P_j}{N_0 B \sum_{j=1}^{K} R_j}$. It characterizes the wideband slope of $R_{\text{sum}}$ as $\frac{E_b}{N_0 \text{sum}}$ approaches its minimum value $\frac{E_b}{N_0 \text{min}}$:

$$\frac{E_b}{N_0 \text{min}} = \lim_{P_{\text{sum}} \to 0} \frac{\sum_{j=1}^{K} P_j}{\sum_{j=1}^{K} R_j \cdot N_0 B} \cdot R_{\text{sum}} \left( \frac{E_b}{N_0} \right) 10 \log_{10} 2 \tag{13}$$

$$S_0 \triangleq \lim_{P_{\text{sum}} \to 0} \frac{R_{\text{sum}} \left( \frac{E_b}{N_0} \right) 10 \log_{10} 2}{10 \log_{10} \frac{E_b}{N_0 \text{sum}} - 10 \log_{10} \frac{E_b}{N_0 \text{min}}} \tag{14}$$

Denote the sum power constraint by $P_{\text{sum}} = \sum_{j=1}^{K} P_j$. Under the assumption that $N_0 B = 1$, $\frac{E_b}{N_0 \text{min}}$ and $S_0$ can be obtained from the first and second order derivatives of $R_{\text{sum}} (P_{\text{sum}})$:

$$\frac{E_b}{N_0 \text{min}} = \frac{\log_e 2}{R_{\text{sum}} (0)}; \quad S_0 = \frac{2 \left( \hat{R}_{\text{sum}} (0) \right)^2}{R_{\text{sum}} (0)} \tag{15} \tag{16}$$

Notice that constraints on $P_j$ or $R_j$ are required for a well-posed problem; otherwise the best low-SNR performance is achieved by allocating all power to the user with largest direct link gain so that $\frac{E_b}{N_0 \text{min}}$ is minimized. Such a solution is just a single user solution and gives no insight into the interference channel. To fix this insufficiency while keeping our problem relatively simple to analyze, we require the interference channel to work under the equal-power constraint, which is defined as

**Definition 2.** Equal power constraint is the case where the sum rate $R_{\text{sum}}$ is maximized under the constraint $P_1 = P_2 = \cdots = P_K$.

Given (4), we can see that if two systems achieve equal $\frac{E_b}{N_0 \text{min}}$ value, the $\frac{E_b}{N_0}$ value of the system with higher wideband slope approaches its minimum value faster, and the system is therefore more spectrally efficient. On the other hand, we should notice that the priority in the low-SNR regime is to minimize $\frac{E_b}{N_0 \text{min}}$. Based on this observation, we make the following statement:

**Remark 3.** To make fair comparison of the wideband slopes between different systems, they must have equal $\frac{E_b}{N_0 \text{min}}$ in the first place.

The results in (7) reveal that the optimal achievable minimum energy per bit $\frac{E_b}{N_0 \text{min}}$ of an interference channel is equal to that of its corresponding interference-free channel. The first-order optimality criterion under the equal power constraint is stated in the following lemma.
Lemma 4. The optimal minimum energy per bit of the interference channel defined by (9) is
\[ \frac{E_b}{N_{0 \min}} = \frac{K \log_e 2}{\sum_{j=1}^{K} |C_{jj}|^2} \]  
under the equal power constraint.

Given Remark 3 any achievable scheme or capacity outer bound gives valid bound on the sum slope only if it has correct \( \frac{E_b}{N_{0 \min}} \) values, stated in Theorem 4.

For performance measure we use
\[ \Delta S_0 = \frac{S_0}{S_{0,\text{no interference}}} \]
The quantity \( S_{0,\text{no interference}} \) is the wideband slope of the corresponding interference-free channel:
\[ R_j = \log \left( 1 + |C_{jj}|^2 P_j \right) . \]
We can interpret \( \Delta S_0 \) as the loss in wideband slope due to interference.

Under the equal power constraint, \( S_{0,\text{no interference}} \) the sum slope of the interference-free channel, and \( S_{0,\text{TDMA}} \) and \( S_{0,\text{TIN}} \) the sum slope achieved by TDMA and treating interference as noise (TIN) respectively, are listed as follows for comparison purposes; they can be obtained directly obtained from (15)
\[ S_{0,\text{no interference}} = 2 \left( \sum_{j} |C_{jj}|^2 \right)^2 \sum_{j} |C_{jj}|^4 \]  
(18)

The \( R_{\text{sum}} (P_{\text{sum}}) \) achieved by TIN is
\[ R_{\text{sum}} (P_{\text{sum}}) = \sum_{j=1}^{K} \log \left( 1 + \frac{|C_{jj}|^2 P_{\text{sum}}}{K + P_{\text{sum}} \sum_{i \neq j} |C_{ji}|^2} \right) , \]
which gives
\[ S_{0,\text{TIN}} = \frac{2 \left( \sum_{j} |C_{jj}|^2 \right)^2}{\sum_{j=1}^{K} \left( |C_{jj}|^4 + 2 \sum_{i \neq j} |C_{ji}|^2 |C_{jj}|^2 \right)} ; \]  
(19)
\[ \Delta S_0 = \frac{\sum_{j=1}^{K} |C_{jj}|^4}{\sum_{j=1}^{K} \left( |C_{jj}|^4 + 2 \sum_{i \neq j} |C_{ji}|^2 |C_{jj}|^2 \right)} \]  
(20)

The \( R_{\text{sum}} (P_{\text{sum}}) \) achieved by TDMA is
\[ R_{\text{sum}} (P_{\text{sum}}) = \frac{1}{K} \sum_{j=1}^{K} \log \left( 1 + |C_{jj}|^2 P_{\text{sum}} \right) , \]
which gives
\[ S_0 = \frac{2 \left( \sum_{j} |C_{jj}|^2 \right)^2}{K \sum_{j} |C_{jj}|^4} \]  
(21)
\[ \Delta S_0 = \frac{1}{K} \]  
(22)

III. GENERALIZED Z-CHANNEL OUTER BOUND

In this section, we develop a new outer bound on the wideband slope for a set of the 2-dimensional vector channels defined by (10), under the equal power constraint. The outer bound is specific to the low-rate regime.

The outer bound is derived from the sum Shannon capacity of a type of generalized Z-channel, which is constructed by elimination of a subset of the interference links. In Section III-A we show that for a subset of channels \( \mathcal{C} \), the optimal sum
capacity of their corresponding Z-channels can be achieved by i.i.d. 2-dimensional vector Gaussian inputs. Further, assuming that channel coefficients \( C_{ji} \) is drawn from i.i.d. continuous distribution, the set \( C \) has non-zero probability. In Section III-B, the Z-channel outer bound is used to derive an outer bound on the wideband slope.

A. Generalized Z-Channel And Its Sum Capacity

We define the generalized Z-channel corresponding to the interference channel (10) as

\[
\hat{Y}_j = |C_{jj}|X_j + \sum_{i=j+1}^{K} |C_{ji}| U_j X_i + Z_j.
\]  

Figure 1. Generalized Z-channel

Eliminating a subset of interference links will not reduce channel capacity and therefore, the sum capacity outer bound for the generalized Z-channel is also a sum capacity outer bound for the interference channel.

To derive the Z-channel sum capacity, we provide receiver \( j, j = 2, \cdots, K \) with side information \( S^n_j = \left( S^n_{j1}, \cdots, S^n_{j(j-1)} \right)^T \), where

\[
S^n_{jp} = |C_{pj}| U_{pj} X^n_p + \sum_{i=j+1}^{K} |C_{pi}| U_{pi} X^n_i + W_{jp}^n
\]  

\( p = 1, \cdots, j - 1 \). The entries in the length \( n \) noise vector \( W_{jp}^n \) are i.i.d 2 \( \times \) 1 vector Gaussian noise with the same marginal distribution as \( Z_j \). Further, they satisfy the following properties

- \( W_{j(p-1)}, \cdots, W_{j1} \) are independent of all input length \( n \) codewords \( X^n_i, i = 1, \cdots, K \);
- \( (Z_j, W_{j(j-1)}, \cdots, W_{j1}) \) are jointly Gaussian random variables, with zero mean and covariance matrix

\[
K_{S_j} = \begin{pmatrix}
I & A_{j(j-1)} & \cdots & A_{j1} & A_{j1} \\
A_{j(j-1)}^T & I & A_{(j-1)(j-2)} & \cdots & A_{(j-1)1} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
A_{j1}^T & A_{j1}^T & I & A_{21} \\
A_{j1}^T & A_{j1}^T & A_{j1}^T & I & \vdots
\end{pmatrix}
\]  

To guarantee such multivariate Gaussian random variable exists, \( A_{jk} \) should be chosen such that for all \( j = 1, \cdots, K \)

\[
K_{S_j} \succeq 0
\]

We emphasize the following property of \( K_{S_j} \), which will play a key role in the proof of the main result.

**Lemma 5.** The distributions of \( S^n_{(j-1)p} \), \( S^n_{(j-1)(p-1)}, \cdots, S^n_{j(j-1)}, X^n_j \) and \( S^n_{jp} | S^n_{j(p-1)}, \cdots, S^n_{j1} \) are equal.
Proof of Lemma 5 is in Appendix A.

Lemma 6. The distributions of $S_{n}^{j-1}$, $S_{j}^{n} - 1$, $S_{n}^{j-1}$, and $S_{j}^{n}$ are equal.

The proof of Lemma 6 is almost identical to the proof of Lemma 5 and will therefore be omitted.

Define the average covariance matrix of the input at transmitter as

$$
\tilde{V}_j \triangleq \frac{1}{n} \sum_{i=1}^{n} E \left[ X_j^{(i)} (X_j^{(i)})^T \right]
$$

for any length $n$ input sequence $X_n^j$. It must satisfy the power constraint defined in (11), i.e., $\tilde{V}_j \preceq V_j$. The next lemma states how to choose $A_{jk}$.

Lemma 7. Let $A_{jp}$, $j = 2, \cdots, K$ and $p = 1, \cdots, j - 1$ be

$$
A_{jp} = \frac{|C_{pj}|^2}{|C_{jj}|^2} U(-\phi_{pj}) + \frac{|C_{pj}|^2}{|C_{jj}|^2} \sum_{i=j+1}^{K} |C_{ji}|^2 U(\phi_{ji}) V_i U(-\phi_{pj} - \phi_{ji}) - \sum_{i=j+1}^{K} |C_{ji}|^2 |C_{pi}|^2 U(\phi_{ji}) V_i U(-\phi_{pi})
$$

(27)

If $A_{jp}$ defined by (27) satisfy $K S_j \succeq 0$, then

$$
X_{jG} \to \hat{Y}_{jG} \to (S_{j1G}, \cdots, S_{j(j-1)G})^T
$$

(28)

forms a Markov chain for all $j = 2, \cdots, K$.

Here $X_{jG}$ and $\hat{Y}_{jG}$ are defined in section II-A; the proof of Lemma 7 is in Appendix B.

For a channel realization, denote its channel coefficients by $C \triangleq \{C_{ij}; i, j = 1, \cdots, K\}$. In the following lemma, we state a sufficient condition on $C$ so that $K S_j \succeq 0$ if $A_{jp}$ is chosen according to (27).

Lemma 8. For any $0 < \alpha < 1$ there exist some $\epsilon_\alpha, \epsilon'_\alpha > 0$ and $\epsilon''(C) > 0$ so that if

$$
C \in C_{\alpha} \triangleq \left\{ C_{ij} : \frac{|C_{ij}|^2}{|C_{jj}|^2} - \alpha, |\phi_{ji}| < \epsilon'_\alpha \right\}
$$

(29)

then $K S_j \succeq 0$ for $A_{jp}$ chosen according to (27).

Proof of Lemma 8 is in Appendix C.

Our main result of this section is stated in the following theorem.

Theorem 9. For every interference channel realization $C \in C = \bigcup_{\alpha \in (0,1)} C_{\alpha}$ defined by (29) there exists an $\epsilon''(C) > 0$ so that if $P_j < \epsilon''(C)$ the sum capacity of its corresponding Z-channel is given by

$$
\sum_{j=1}^{K} R_j \leq C_{\text{sum}} = \max_{\text{Tr}(V_j) \leq P_j} \sum_{j=1}^{K} I(X_{jG}; \hat{Y}_{jG})
$$

(31)

$V_j \succeq 0, j = 1, \cdots, K$
Because the sum capacity of the interference channel is outer bounded by the sum capacity of the generalized Z-channel, (31) is an outer bound for the sum capacity of the interference channel.

Proof of Theorem 9 is in Appendix D.

Note that the bound in Theorem 9 is valid for 

\[ P_j < \epsilon''(\mathcal{C}) \],

and it therefore bounds the actual capacity for suitably low

SNR. However, we will mainly use it to bound the wideband slope, a weaker result.

B. Sum Slope Outer Bound for the Interference Channel

Given the capacity in Theorem 9, we have following result on the low-rate performance of the interference channel.

Theorem 10. For the interference channel (9), the sum capacity is outer bounded by (31) for low

SNR. Under the equal

power constraint, the minimum energy per bit of this upper bound satisfy the requirement imposed by Remark 5, which is

\[ \frac{E_b}{N_0} \mid_{\text{min}} = \frac{K \log 2}{\sum_{j=1}^{K} |C_{jj}|^2} \] (33)

For channel realizations \( \mathcal{C} \in \mathcal{C} = \bigcup_{\alpha \in (0,1)} \mathcal{C}_\alpha \) defined as (29) it therefore gives the following valid upper bound on the sum slope:

\[ S_0 \leq \left( \sum_{j=1}^{K} |C_{jj}|^2 \right)^2 \]

\[ \times \max_{\text{Tr} (\hat{V}_j) \leq 1} \left( \sum_{j=1}^{K} |C_{jj}|^2 \text{Tr} (\hat{V}_j^2) \right) \]

\[ + 2 \sum_{j=1}^{K-1} \sum_{i=j+1}^{K} |C_{jj}|^2 |C_{ji}|^2 \text{Tr} \left( \hat{V}_j U_{ji} \hat{V}_i U_{ji}^\dagger \right)^{-1} \] (36)

Proof of Theorem 10 is in Appendix E.

Theorem 11. For the symmetric channel where \( C_{jj} = 1, C_{ji} = \alpha \in (0,1) \), the sum slope is bounded by

\[ S_0 \leq \frac{2K}{\alpha K + (1 - \alpha)} \]

Proof of Theorem 11 is in Appendix E.

As discussed in the introduction, the wideband slope in the point \( \mathcal{C} = 1 \) is \( \frac{2}{K} \) per user, achievable by TDMA. Theorem 11 shows that the point \( \mathcal{C} = 1 \) is not exceptional in the low-rate regime: for \( \alpha \) close to 1 (from below) the channel with \( C_{jj} = 1, C_{ji} = \alpha \) has slope close to \( \frac{2}{K} \). However, the set of channels \( C_{jj} = 1, C_{ji} = \alpha \) still has Lebesgue measure zero, i.e., if the channel coefficients are drawn from a continuous distribution, this set has probability zero. The main result of the paper is the following theorem that shows that the set of channels with slope close to \( \frac{2}{K} \) can be extended to a set of non-zero measure.

Theorem 12. For all \( \sigma > 0 \), there exists an open set \( \tilde{\mathcal{C}}_\sigma \subset \mathcal{C}^{K(K-1)} \) with \( 1 \in \text{cl} \left( \tilde{\mathcal{C}}_\sigma \right) \), so that for \( \mathcal{C} \in \tilde{\mathcal{C}}_\sigma \)

\[ S_0 \leq 2 + \sigma \] (37)

If the magnitude and phase of the channel coefficients are drawn from continuous random distribution, \( \Pr \left( \tilde{\mathcal{C}}_\sigma \right) > 0 \).
And as $\sigma \to 0$,
\[
\lim_{\sigma \to 0} \Delta S_0 = \frac{1}{K}
\]
Because $\Delta S_0$ achieved by TDMA is $\frac{1}{K}$, when $\sigma$ is small, TDMA transmission scheme is almost optimal for channels in $\tilde{C}_\sigma$.

Proof of Theorem 12 is in Appendix G.

IV. SUM SLOPE ACHIEVABLE SCHEME

In the previous section, we have shown that there exist a set of channels $C_\sigma$, $\Pr(\tilde{C}_\sigma) > 0$, for which TDMA is almost optimal. However, we also notice that the probability that a channel realization is not in $C_\sigma$ is likewise greater than zero.

Therefore, it is natural to ask the question: for channels not in $\tilde{C}_\sigma$, can we find achievable schemes better than TDMA or Treating Interference as Noise (TIN)?

In section IV-A, we propose a circularly asymmetric transmission scheme and analyze its theoretical performance. Simulation results are shown in section IV-B. We will also discuss possible improvements of this scheme.

A. One-Dimensional Gaussian Signaling

In this section, we use the complex scalar channel model defined in (9). We define a one-dimensional Gaussian signaling transmission scheme and analyze its performance. The idea is to align interference as much as possible.

Definition 13. One-dimensional Gaussian signaling transmission scheme

- At transmitter $j$, let input sequence be $x_j[n] = w_j[n] e^{j \theta_j}$, where $w_j[n]$ is drawn from i.i.d real Gaussian random variable with distribution $\mathcal{N}(0, \text{SNR}_j)$, and the phase $\theta_j$ is a prior chosen design parameter, unchanged for all $n$ during the transmission.
- At receiver $j$, interference is treated as noise.

We call this one-dimensional because every transmitter only transmits along $e^{j \theta_j}$, therefore only one dimension is used out of the two-dimensional signal space.

Our objective is to find the set of phases $\theta = \{\theta_1, \cdots, \theta_K\}$ that maximize the achievable wideband slope $S_0$.

The achievable $S_0$ for any $\theta$ is stated in the next lemma. For computational convenience, we return to the equivalent two-dimensional real channel model. In the equivalent 2-dimensional real channel model, the input $X_j$ has covariance matrix

\[
V_j = P_j \begin{pmatrix}
\cos^2 \theta & \frac{\sin 2 \theta}{2} \\
\frac{\sin 2 \theta}{2} & \sin^2 \theta
\end{pmatrix},
\]

$\text{rank}(V_j) = 1$. We denote the normalized covariance matrix by $\hat{V}_j = \frac{V_j}{P_j}$.

Lemma 14. For the equivalent 2-dimensional real channel model defined by (10), the sum slope achieved by the one-dimensional Gaussian signaling is

\[
S_0 = \frac{\left(\sum_{j=1}^{K} |C_{jj}|^2\right)^2}{\sum_{j=1}^{K} |C_{jj}|^2 + \sum_{j=1}^{K} \sum_{i \neq j} |C_{jj}|^2 |C_{ji}|^2 + f(\theta)},
\]

where

\[
f(\theta) \triangleq \sum_{j=1}^{K} \sum_{i \neq j} |C_{jj}|^2 |C_{ji}|^2 \cos 2 (\phi_{ji} - \theta_j + \theta_i).
\]
Proof: Treating interference as noise at the receiver, the achievable sum rate \( R_{\text{sum}} \) is

\[
R_{\text{sum}} = \sum_{j=1}^{K} \left( \frac{1}{2} \log \left| \mathbf{I}_2 + \frac{2}{K} P_{\text{sum}} \left( |C_{jj}|^2 \mathbf{U}(\phi_{jj}) \tilde{\mathbf{V}}_j \mathbf{U}_2 (-\phi_{jj}) \right. \right. \right. \\
\left. \left. \left. + \sum_{i=1, i \neq j}^{K} |C_{ji}|^2 \mathbf{U}(\phi_{ji}) \tilde{\mathbf{V}}_i \mathbf{U}_2 (-\phi_{ji}) \right. \right. \right. \right) \\
\left. \left. \left. - \frac{1}{2} \log \left| \mathbf{I}_2 + \frac{2}{K} P_{\text{sum}} \sum_{i=1, i \neq j}^{K} |C_{ji}|^2 \mathbf{U}(\phi_{ji}) \tilde{\mathbf{V}}_i \mathbf{U}_2 (-\phi_{ji}) \right. \right. \right) \right) \right) \right)
\]

(40)

under the equal power constraint where \( \text{SNR}_j = \frac{\text{SNR}}{K} \). Combining (4), (5) and (40), we have

\[
\begin{align*}
\dot{R}_s (0) &= \sum_{j=1}^{K} |C_{jj}|^2 \\
\ddot{R}_s (0) &= \frac{2 \sum_{j=1}^{K} |C_{jj}|^4}{K} + \frac{2 \sum_{j=1}^{K} \sum_{i \neq j}^{K} |C_{jj}|^2 |C_{ji}|^2}{K^2} \\
&\quad + \frac{2}{K^2} \sum_{j=1}^{K} \sum_{i \neq j}^{K} (|C_{jj}|^2 |C_{ji}|^2 \cos 2 (\phi_{ji} - \theta_j + \theta_i)).
\end{align*}
\]

(43)

Given \( S_0 = \frac{2 \dot{R}_s (0) - \ddot{R}_s (0)}{R_s (0)} \) (38) follows.

Given (38), maximizing \( S_0 \) is equivalent to finding the set of \( \theta_j \) that minimizes \( f (\theta) \).

Denote this optimization problem by \( P (\theta) \), which is defined as

\[
\begin{align*}
\min \quad & f (\theta) \\
\text{subject to} \quad & \theta_j \in [-\pi, \pi].
\end{align*}
\]

Notice that \( \theta_j \mod 2\pi \) will not affect the value of \( f (\theta) \). Given the definition of the objective function in (39), the constraint \( \theta_j \in [-\pi, \pi] \) can be discarded. Therefore, \( P (\theta) \) can be solved using standard numerical methods for unconstrained optimization problems.

B. Simulation Results and Discussions

In this section, we simulate the performance of the one-dimensional signaling scheme in a 10-user interference channel with unit direct link gains and symmetric weak interference link gains, i.e., \( |C_{jj}|^2 = 1 \) and \( |C_{ji}|^2 = a < 1 \) for all \( i, j = 1, \cdots, 10 \); the phases \( \phi_{ji} \) is drawn from \( U [-\pi, \pi] \) in each channel realization. This performance will be compared with existing achievable schemes: treating interference as noise and TDMA.

The simulation results are presented below. We can see that when \( \alpha \), the ratio between the direct link gain and the interference link gain, is close to 1, then with non-zero probability the one dimensional Gaussian signaling transmission scheme performs better than TDMA.

Fig. 2 illustrates the empirical cumulative distribution functions of the sum slope achieved by the one-dimensional interference alignment scheme at different \( a \) values. For comparison, \( S_0 \) achieved by treating interference as noise are also shown, and TDMA always achieves \( S_0 = 2 \) for all \( a \) value.
In Figure 2 we compare the empirical cumulative distribution functions of $S_0$ achieved by treating interference as noise (TIN), interference alignment (INTA) and TDMA under different $\alpha$ values.

In Figure 3 we compare the median value of $S_0$ achieved by one-dimensional interference alignment scheme with the performance of treating interference as noise and TDMA.

V. Conclusion

The main result of this paper can be summarized as follows. In the low rate regime, the wideband slope is (upper semi-) continuous in the point $\frac{1}{2}$, the point where all channels are identical, and where the wideband slope (per user) is $\frac{2}{R}$. This does not give a full characterization of the wideband slope. However, it is a stark contrast to the large bandwidth regime $\frac{1}{2}$, where a wideband slope of 1 is achievable almost everywhere, implying discontinuity in the point $\frac{1}{2}$. It is also a contrast to the high SNR regime, where $\frac{1}{2}$ DoF per user is achievable almost everywhere $\frac{3}{2}$, $\frac{4}{2}$, $\frac{5}{2}$, and where the DoF is discontinuous almost everywhere. The results in $\frac{1}{2}$ and $\frac{3}{2}$, $\frac{4}{2}$, $\frac{5}{2}$ were obtained by using interference alignment, and the result in this paper implies that interference alignment does not give the dramatic gains in the low rate regime seen elsewhere. Yet, we show that interference alignment can still give some gain.

One implication of the result is that in networks, as opposed to point-to-point channels, it is important how the low SNR regime is approached. This may effect how networks are designed and operates for maximum energy efficiency.
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**APPENDIX A**

**PROOF OF LEMMA 5**

Given \( (44) \), we have

\[
\begin{align*}
S_{(j-1)p}^n &= |C_{(j-1)j}|U_{p(j-1)}X_{(j-1)}^n + \sum_{i=j}^{K} |C_{pi}|U_{pi}X_{(j-1)}^n + W_{(j-1)p}^n \\
S_{(j-1)(p-1)}^n &= |C_{(j-1)j}|U_{(p-1)(j-1)}X_{(j-1)}^n + \sum_{i=j}^{K} |C_{(p-1)i}|U_{(p-1)i}X_{(j-1)}^n + W_{(j-1)(p-1)}^n \\
&\vdots \\
S_{(j-1)1}^n &= |C_{(j-1)j}|U_{1(j-1)}X_{(j-1)}^n + \sum_{i=j}^{K} |C_{1i}|U_{1i}X_{(j-1)}^n + W_{(j-1)1}^n
\end{align*}
\]

When \( X_{(j-1)}^n \) is given, it can be subtracted from \( S_{(j-1)p}^n, S_{(j-1)(p-1)}^n, \ldots, S_{(j-1)1}^n \) to give

\[
\begin{align*}
\hat{S}_{(j-1)p}^n &= S_{(j-1)p}^n - |C_{(j-1)j}|U_{p(j-1)}X_{(j-1)}^n \\
&= \sum_{i=j}^{K} |C_{pi}|U_{pi}X_{(j-1)}^n + W_{(j-1)p}^n \quad (44) \\
\hat{S}_{(j-1)(p-1)}^n &= S_{(j-1)(p-1)}^n - |C_{(j-1)j}|U_{(p-1)(j-1)}X_{(j-1)}^n \\
&= \sum_{i=j}^{K} |C_{(p-1)i}|U_{(p-1)i}X_{(j-1)}^n + W_{(j-1)(p-1)}^n \quad (45) \\
&\vdots \\
\hat{S}_{(j-1)1}^n &= S_{(j-1)1}^n - |C_{(j-1)j}|U_{1(j-1)}X_{(j-1)}^n \\
&= \sum_{i=j}^{K} |C_{1i}|U_{1i}X_{(j-1)}^n + W_{(j-1)1}^n \quad (46)
\end{align*}
\]
while

$$S_{jp}^n = \sum_{i=j}^K |C_{pi}| U_{pi} X_n^i + W_{jp}^n$$ (47)

$$S_{j(p-1)}^n = \sum_{i=j}^K |C_{(p-1)i}| U_{(p-1)i} X_n^i + W_{j(p-1)}^n$$ (48)

$$\vdots$$

$$S_{j1}^n = \sum_{i=j}^K |C_{1i}| U_{1i} X_n^i + W_{j1}^n.$$ (49)

We know that \((Z_j, W_{j(j-1)}, \ldots, W_{j1})\) are jointly Gaussian random variables, with zero mean and covariance matrix \(K_{S_j}\) equal to:

$$
\begin{pmatrix}
I & A_{j(1)} & \cdots & A_{j2} & A_{j1} \\
A_{j(1)}^T & I & A_{(j-1)(j-2)} & \cdots & A_{(j-1)1} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
A_{j1}^T & & I & A_{21} \\
A_{j1}^T & A_{j1}^T & & & I
\end{pmatrix}
$$

which is defined in (27). It is clear that the covariance matrices of the jointly Gaussian random variables \((W_{(j-1)p}, W_{(j-1)(p-1)}, \ldots, W_{j1})\) and \((W_{jp}, W_{j(j-1)}, \ldots, W_{j1})\) are the same:

$$\begin{align*}
\text{cov} & \left( W_{(j-1)p}, W_{(j-1)(p-1)}, \ldots, W_{j1} \right) \\
= & \text{cov} \left( W_{jp}, W_{j(j-1)}, \ldots, W_{j1} \right) \\
= & \begin{pmatrix}
I & A_{p(1)} & \cdots & A_{p2} & A_{p1} \\
A_{p(1)}^T & I & A_{(p-1)(p-2)} & \cdots & A_{(p-1)1} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
A_{p2}^T & & I & A_{21} \\
A_{p1}^T & A_{p1}^T & & & I
\end{pmatrix}
\end{align*}$$ (50)

Comparing (44)–(46) and (47)–(49), we can see that distribution of \(S_{j(1)p}^n \mid S_{j(j-1)(p-1)}^n, \ldots, S_{j1}^n\) and \(S_{j(p-1)p}^n \mid S_{j(p-1)(p-2)}^n, \ldots, S_{j1}^n\) are equal as long as \(W_{(j-1)p}^n \mid W_{(j-1)(p-1)}^n, \ldots, W_{j1}^n\) and \(W_{jp}^n \mid W_{j(j-1)p}^n, \ldots, W_{j1}^n\) have the same distribution. Recall that \(W_{ji}\) is i.i.d. Gaussian random variables which is independent from the input signals \(X_n\). Therefore given (50), Lemma 5 is proved.

**APPENDIX B**

**PROOF OF LEMMA 7**

Lemma 7 is proved using the following lemma from [8].

**Lemma 15** ([8] Lemma 4, p5037). Let \(X, Y, Z\) be jointly Gaussian vectors. If \(\text{cov} (Y)\) is invertible, then \(X \rightarrow Y \rightarrow Z\) forms a Markov chain if and only if

$$\text{cov} (X, Z) = \text{cov} (X, Y) \text{cov} (Y)^{-1} \text{cov} (Y, Z)$$

Given Lemma 15 and the fact that \(\text{cov} (\hat{Y}_{jG})\) is invertible, \(X_{jG} \rightarrow \hat{Y}_{jG} \rightarrow S_j\) forms a Markov chain if and only if

$$\text{cov} (X_{jG}, S_j)$$
\[
\begin{align*}
\text{LHS} & = \begin{pmatrix}
|C_{1j}|^2 V_j U (-\phi_{1j}) \\
|C_{2j}|^2 V_j U (-\phi_{2j}) \\
\vdots \\
|C_{(j-1)j}|^2 V_j U (-\phi_{(j-1)j})
\end{pmatrix}^T
\end{align*}
\]

and the right hand side is

\[
\begin{align*}
\text{RHS} & = \left|C_{pj}\right|^2 V_j U (-\phi_{pj}) \\
& = \left|C_{jj}\right|^2 V_j U (-\phi_{jj}) \left( \sum_{i=j}^{K} \left|C_{ji}\right|^2 U (\phi_{ji}) V_i U (-\phi_{ji}) + I \right)^{-1} \\
& \quad \left( \sum_{i=j}^{K} \left|C_{ji}\right|^2 U (\phi_{ji}) V_i U (-\phi_{ji}) + \lambda_{pj} I \right) \\
& \quad \left( \sum_{i=j+1}^{K} \left|C_{pi}\right|^2 U (\phi_{ji}) V_i U (-\phi_{pi}) \right)
\end{align*}
\]

Solving the equation above, we have

\[
A_{jp} = \frac{|C_{pj}|^2}{|C_{jj}|^2} U (\phi_{pp} - \phi_{pj})
\]

\[
+ \frac{|C_{pj}|^2}{|C_{jj}|^2} \sum_{i=j+1}^{K} \left|C_{ji}\right|^2 U (\phi_{ji}) V_i U (\phi_{pj} - \phi_{ji})
\]

\[
- \sum_{i=j+1}^{K} \left|C_{pi}\right|^2 U (\phi_{ji}) V_i U (-\phi_{pi})
\]

APPENDIX C

PROOF OF LEMMA 8

First, consider the simple case where \( \frac{|C_{pj}|^2}{|C_{jj}|^2} = \alpha, \phi_{ji} = 0 \) and \( P_j = 0 \), that is, \( K_x_j = 0 \). For this case, given (27) we have

\[
A_{ji} = B = \begin{pmatrix}
\alpha & 0 \\
0 & \alpha
\end{pmatrix}
\]

for all \( i, j \). It is easy to check that the eigenvalues of \( K_{S_j} \) are \( \lambda_1 = 1 - \alpha \) and \( \lambda_2 = 1 + (j - 1) \alpha \), with multiplicity 2 \( (j - 1) \) and 2 respectively. Therefore, \( K_{S_j} \) is positive definite if \( 0 < \alpha < 1 \).

Now let us consider the case where \( \phi_{ji} \) and \( P_j \) are small but non-zero, and \( \frac{|C_{pj}|^2}{|C_{jj}|^2} \) are not necessarily equal to \( \alpha \). Denote the \( (p, q) \)th element of \( B \) by \( b_{pq} \). It is well known that the eigenvalues of symmetric matrix are locally (Lipschitz) continuous with respect to its elements. Therefore, corresponding to every \( \alpha \in (0, 1) \), for any \( \epsilon > 0 \), there exist some strictly positive real numbers \( \epsilon_\alpha, \epsilon'_\alpha \) and \( \epsilon''_\alpha \) such that if \( \left| \frac{|C_{pj}|^2}{|C_{jj}|^2} - \alpha \right| < \epsilon_\alpha, |\phi_{ji}| < \epsilon'_\alpha \), and \( P_j < \epsilon''_\alpha \) then every eigenvalues \( \lambda_\alpha \) of \( K_{S_j} \)
satisfies $|\lambda_s - \lambda_1| < \hat{\epsilon}$ or $|\lambda_s - \lambda_2| < \hat{\epsilon}$. The bound on $P_j$ may depend $C$ to ensure that the two last terms in (27) are of bounded variation. For any $0 < \alpha < 1$ we can always find some $\hat{\epsilon} > 0$ that guarantees $\lambda_s > 0$, and $K_{S_j}$ is positive definite as a result.

APPENDIX D

PROOF OF THEOREM

**Lemma 16.** ([8] Lemma 2) Let $X^n = (X_1, \ldots, X_n)$ and $Y^n = (Y_1, \ldots, Y_n)$ be two sequences of random vectors, and let $X'_G$, $X'_G$; $Y'_G$, and $Y'_G$ be Gaussian vectors with covariance matrices satisfying

$$\text{cov} \left( \frac{X_G'}{Y_G'} \right) = \frac{1}{n} \sum_{i=1}^{n} \text{cov} \left( \frac{X_i}{Y_i} \right) \leq \text{cov} \left( \frac{X'_G}{Y'_G} \right)$$

then we have

$$h \left( \frac{X^n}{Y^n} \right) \leq nh \left( \frac{X'_G}{Y'_G} \right) \leq nh \left( \frac{X_G'}{Y_G'} \right)$$

$$h \left( \frac{Y^n}{X^n} \right) \leq nh \left( \frac{Y'_G}{X'_G} \right) \leq nh \left( \frac{Y_G'}{X_G'} \right)$$

By Fano’s inequality, the sum capacity of the generalized Z-channel (23) must satisfy

$$n \sum_{j=1}^{K} R_j - n\epsilon$$

\((a)\) $I \left( \frac{X^n}{Y^n}; \frac{Y^n}{X^n} \right) + \sum_{j=2}^{K} I \left( \frac{X^n_j}{Y^n_j}; \frac{Y^n_j}{X^n_j} \right) \leq h \left( \frac{\hat{Y}_n}{X^n} \right) - h \left( \frac{\hat{Y}_n}{Y^n} \right) + \sum_{j=2}^{K} I \left( \frac{X^n_j}{\hat{Y}_n}; \frac{\hat{Y}_n}{X^n_j} \right) \leq h \left( \frac{\hat{Y}_n}{X^n} \right) - h \left( \frac{\hat{Y}_n}{X^n} \right) + \sum_{j=2}^{K} I \left( \frac{X^n_j}{\hat{Y}_n}; \frac{\hat{Y}_n}{X^n_j} \right) \leq h \left( \frac{\hat{Y}_n}{X^n} \right) - h \left( \frac{\hat{Y}_n}{X^n} \right) + \sum_{j=2}^{K} \sum_{p=1}^{j-1} \left( h \left( \frac{\hat{Y}_n}{X^n_j}; \frac{\hat{Y}_n}{X^n_j} \right) - h \left( \frac{S_{j,p}^n}{X^n_j}; \frac{S_{j,p}^n}{X^n_j} \right) \right) + \sum_{j=2}^{K} \sum_{p=1}^{j-1} \left( h \left( \frac{S_{j,p}^n}{X^n_j}; \frac{S_{j,p}^n}{X^n_j} \right) \right) - h \left( \frac{S_{j,p}^n}{X^n_j}; \frac{S_{j,p}^n}{X^n_j} \right) + \sum_{j=2}^{K} \left( h \left( \frac{\hat{Y}_n}{X^n_j}; \frac{\hat{Y}_n}{X^n_j} \right) - h \left( \frac{S_{j,p}^n}{X^n_j}; \frac{S_{j,p}^n}{X^n_j} \right) \right) - h \left( \frac{S_{j,p}^n}{X^n_j}; \frac{S_{j,p}^n}{X^n_j} \right) + \sum_{j=3}^{K} \sum_{p=j-1}^{j-1} h \left( \frac{S_{j,p}^n}{X^n_j}; \frac{S_{j,p}^n}{X^n_j} \right)\right)
\[ + \sum_{j=3}^{K-1} \sum_{p=1}^{j-2} h \left( S^n_{jp}, S^n_{j(p-1)}, \ldots, S^n_{j1} \right) \\
- \sum_{j=3}^{K-1} \sum_{p=1}^{j-1} h \left( S^n_{jp}, S^n_{j(p-1)}, \ldots, S^n_{j1}, X^n_j \right) \\
- \sum_{j=K}^{j-1} \sum_{p=1}^{j-1} h \left( S^n_{jp}, S^n_{j(p-1)}, \ldots, S^n_{j1}, X^n_j \right) \\
+ \sum_{j=2}^{K} h \left( \hat{Y}^n_j \mid S^n_{j(j-1)}, S^n_{j(j-2)}, \ldots, S^n_{j1} \right) \\
- \sum_{j=2}^{K} h \left( \hat{Y}^n_j \mid S^n_{j(j-1)}, S^n_{j(j-2)}, \ldots, S^n_{j1}, X^n_j \right) \]

\[ (f) \quad nh \left( \hat{Y}_{1G} \right) - h \left( \hat{Y}^n_1 \mid X^n \right) + h \left( S^n_{21} \right) + \sum_{j=3}^{K} h \left( S^n_{j(j-1)} \mid S^n_{j(j-2)}, \ldots, S^n_{j1} \right) \\
+ \sum_{j=3}^{K-1} \sum_{p=1}^{j-2} h \left( S^n_{jp}, S^n_{j(p-1)}, \ldots, S^n_{j1} \right) \\
- h \left( S^n_{j(j-1)p}, S^n_{j(j-1)(p-1)}, \ldots, S^n_{j(j-1)1}, X^n_{j(j-1)} \right) \\
- \sum_{p=1}^{K-1} h \left( S^n_{-Kp}, S^n_{K(p-1)}, \ldots, S^n_{K1}, X^n_K \right) \\
+ \sum_{j=2}^{K} h \left( \hat{Y}^n_j \mid S^n_{j(j-1)}, S^n_{j(j-2)}, \ldots, S^n_{j1} \right) \\
- \sum_{j=2}^{K} h \left( \hat{Y}^n_j \mid S^n_{j(j-1)}, S^n_{j(j-2)}, \ldots, S^n_{j1}, X^n_j \right) \]

\[ (g) \quad nh \left( \hat{Y}_{1G} \right) - h \left( \hat{Y}^n_1 \mid X^n \right) + h \left( S^n_{21} \right) + \sum_{j=3}^{K} h \left( S^n_{j(j-1)} \mid S^n_{j(j-2)}, \ldots, S^n_{j1} \right) \\
- \sum_{p=1}^{K-1} h \left( S^n_{Kp}, S^n_{K(p-1)}, \ldots, S^n_{K1}, X^n_K \right) \\
+ \sum_{j=2}^{K} h \left( \hat{Y}^n_j \mid S^n_{j(j-1)}, S^n_{j(j-2)}, \ldots, S^n_{j1} \right) \\
- \sum_{j=2}^{K} h \left( \hat{Y}^n_j \mid S^n_{j(j-1)}, S^n_{j(j-2)}, \ldots, S^n_{j1}, X^n_j \right) \]

\[ (h) \quad nh \left( \hat{Y}_{1G} \right) + \sum_{j=3}^{K} h \left( S^n_{j(j-1)} \mid S^n_{j(j-2)}, \ldots, S^n_{j1} \right) \\
- \sum_{j=3}^{K} h \left( \hat{Y}^n_j \mid S^n_{j(j-1)}, S^n_{j(j-2)}, \ldots, S^n_{j1} \right) \\
+ \sum_{j=3}^{K} h \left( \hat{Y}^n_j \mid S^n_{j(j-1)}, S^n_{j(j-2)}, \ldots, S^n_{j1}, X^n_j \right) \\
- nh \left( W_{K(K-1)}, \ldots, W_{K1} \right) \]
\[-\sum_{j=2}^{K} h \left( \mathcal{Y}_{j,G}^{n} \mid \mathcal{S}_{j(j-1)}^{n}, \mathcal{S}_{j(j-2)}^{n}, \ldots, \mathcal{S}_{j1}^{n}, \mathcal{X}_{j}^{n} \right)\]

\(\overset{(i)}{=} nh \left( \mathcal{Y}_{1,G} \right) - nh \left( W_{K(K-1)}, \ldots, W_{K1} \right) + \sum_{j=2}^{K} h \left( \mathcal{Y}_{j,G}^{n} \mid \mathcal{S}_{j(j-1)}^{n}, \mathcal{S}_{j(j-2)}^{n}, \ldots, \mathcal{S}_{j1}^{n} \right) - nh \left( \mathcal{N}_{K}^{1}, W_{K(K-1)}, \ldots, W_{K1} \right)\]

\(\overset{(j)}{=} nh \left( \mathcal{Y}_{1,G} \right) - nh \left( \mathcal{N}_{K}, W_{K(K-1)}, \ldots, W_{K1} \right) + \sum_{j=2}^{K} h \left( \mathcal{Y}_{j,G}^{n} \mid \mathcal{S}_{j(j-1)}^{n}, \mathcal{S}_{j(j-2)}^{n}, \ldots, \mathcal{S}_{j1}^{n} \right)\]

\(\overset{(k)}{\leq} nh \left( \mathcal{Y}_{1,G} \right) - nh \left( \mathcal{N}_{K}, W_{K(K-1)}, \ldots, W_{K1} \right) + n \sum_{j=2}^{K} h \left( \mathcal{Y}_{j,G}^{n} \mid \mathcal{S}_{j(j-1)}^{n}, \mathcal{S}_{j(j-2)}^{n}, \ldots, \mathcal{S}_{j1G} \right)\]

\(\overset{(l)}{=} nh \left( \mathcal{Y}_{1,G} \right) - nh \left( \mathcal{N}_{K}, W_{K(K-1)}, \ldots, W_{K1} \right) + n \sum_{j=2}^{K} h \left( \mathcal{Y}_{j,G} \right) - n \sum_{j=2}^{K-1} h \left( \mathcal{S}_{j(j-1)G}, \mathcal{S}_{j(j-2)G}, \ldots, \mathcal{S}_{j1G} \right) + n \sum_{j=2}^{K} h \left( \mathcal{S}_{j(j-1)G}, \mathcal{S}_{j(j-2)G}, \ldots, \mathcal{S}_{j1G} \mid \mathcal{Y}_{j,G} \right)\]

\(\overset{(m)}{=} nh \left( \mathcal{Y}_{1,G} \right) - nh \left( \mathcal{N}_{K}, W_{K(K-1)}, \ldots, W_{K1} \right) + n \sum_{j=2}^{K} h \left( \mathcal{Y}_{j,G} \right) - n \sum_{j=2}^{K} h \left( \mathcal{S}_{j(j-1)G}, \mathcal{S}_{j(j-2)G}, \ldots, \mathcal{S}_{j1G} \right) + n \sum_{j=2}^{K} h \left( \mathcal{S}_{j(j-1)G}, \mathcal{S}_{j(j-2)G}, \ldots, \mathcal{S}_{j1G} \mid \mathcal{Y}_{j,G} \right) + nh \left( \mathcal{S}_{K(K-1)G}, \mathcal{S}_{K(K-2)G}, \ldots, \mathcal{S}_{K1G} \mid \mathcal{Y}_{K,G} \right)\]

\(\overset{(n)}{=} nh \left( \mathcal{Y}_{1,G} \right) - nh \left( \mathcal{N}_{K}, W_{K(K-1)}, \ldots, W_{K1} \right) + n \sum_{j=2}^{K} h \left( \mathcal{Y}_{j,G} \right) - n \sum_{j=2}^{K} h \left( \mathcal{S}_{j(j-1)G}, \mathcal{S}_{j(j-2)G}, \ldots, \mathcal{S}_{j1G} \right) + n \sum_{j=2}^{K} h \left( \mathcal{S}_{j(j-1)G}, \mathcal{S}_{j(j-2)G}, \ldots, \mathcal{S}_{j1G} \mid \mathcal{Y}_{j,G} \right) + nh \left( W_{K(K-1)}, W_{K(K-2)}, \ldots, W_{K1} \mid \mathcal{X}_{K} \right)\)
\[
\begin{align*}
\hat{X}_{1G} & = nh(\hat{Y}_{1G}) - nh(N_K, W_{K(K-1)}, \ldots, W_K) \\
& + n \sum_{j=2}^{K} h(\hat{Y}_{jG}) \\
& - n \sum_{j=2}^{K} h(S_{j(j-1)G} | \hat{S}_{j(j-2)G}, \ldots, \hat{S}_{j1G}) \\
& + n \sum_{j=3}^{K} h(S_{j(j-2)G} | S_{j(j-3)G}, \ldots, S_{j1G}, S_{j(j-1)G}) \\
& + nh(W_{K(K-1)}, W_{K(K-2)}, \ldots, W_K | N_K) \\
\end{align*}
\]

= \[
\begin{align*}
\hat{Y}_{1G} & = nh(\hat{Y}_{1G}) - nh(N_K, W_{K(K-1)}, \ldots, W_K) \\
& + n \sum_{j=2}^{K} h(\hat{Y}_{jG}) - h(S_{21G}) \\
& - n \sum_{j=3}^{K} h(S_{j(j-1)G}) \\
& + nh(W_{K(K-1)}, W_{K(K-2)}, \ldots, W_K | N_K) \\
\end{align*}
\]

(a) is from Fano’s inequality.

(b) is from the expansion of mutual information: \( I(X^n; \hat{Y}_1^n) = h(\hat{Y}_1^n) - h(\hat{Y}_1^n | X^n) \), and the chain rule which gives 

\[
I(X^n; \hat{Y}_j^n, \hat{S}_j) = I(X^n; \hat{Y}_j^n) + I(X^n; \hat{S}_j | \hat{Y}_j^n). 
\]

(c) is from the chain rule, which gives 

\[
I(X^n; \hat{S}_j) = \sum_{p=1}^{j-1} I(X^n; \hat{S}_{jp} | \hat{S}_{p-1}^n, \ldots, \hat{S}_{1}^n). 
\]

(d) is from the expansion of mutual information.

(e) is from the inequality \( h(\hat{Y}_j^n) \leq nh(\hat{Y}_{1G}) \). It holds because Gaussian random variable maximize entropy under given
power constraint, and line 2 to line 6 in (e) is equivalent to line 2 and line 3 in (d).

(f) is from the following equation:

\[-h \left( S^n_1 | X^n_1 \right) - \sum_{j=3}^{K-1} \sum_{p=1}^{j-1} h \left( S^n_{jp} | S^n_{(p-1)}, \cdots, S^n_{j-1}, X^n_j \right) \]

\[= - \sum_{j=3}^{K} \sum_{p=1}^{j-1} h \left( S^n_{(j-1)p} | S^n_{(j-1)(p-1)}, \cdots, S^n_{j-1}, X^n_{j-1} \right). \tag{52} \]

(g) is from Lemma 5. Because random variables \( S^n_{j-1} \mid S^n_{(j-1)(p-1)}, \cdots, S^n_{j-1}, X^n_{j-1} \) and \( S^n_p \mid S^n_{(p-1)}, \cdots, S^n_1 \) have the same marginal distribution, \( h \left( S^n_{j-1} \mid S^n_{(j-1)(p-1)}, \cdots, S^n_{j-1}, X^n_{j-1} \right) \) and \( h \left( S^n_p \mid S^n_{(p-1)}, \cdots, S^n_1 \right) \) are equal, which gives

\[ \sum_{j=3}^{K} \sum_{p=1}^{j-2} h \left( S^n_{jp} \mid S^n_{p(p-1)}, \cdots, S^n_{j-1}, X^n_{j-1} \right) = 0. \]

(h) Given \( S^n_{Kp} = |C_p| U_{pK} X^n_K + W^n_{Kp} \), the summation in the third line after (g) gives

\[ \sum_{p=1}^{K-1} h \left( S^n_{Kp} \mid S^n_{K(p-1)}, \cdots, X^n_K \right) \]

\[= \sum_{p=1}^{K-1} h \left( W^n_{Kp} \mid W^n_{K(p-1)}, \cdots, W^n_{K1} \right) \]

\[= h \left( W^n_{K(K-1)}, \cdots, W^n_{K1} \right) \]

\[= nh \left( W^n_{K(K-1)}, \cdots, W^n_{K1} \right) \tag{55} \]

It is also easy to see that \( S^n_{21} \) and \( \hat{Y}^n_1 \mid X^n_1 \) have same marginal distribution, therefore

\[ h \left( S^n_{21} \right) - h \left( \hat{Y}^n_1 \mid X^n_1 \right) = 0 \tag{56} \]

(i) Now combine the second and the last terms after (h):

\[ \sum_{j=3}^{K} h \left( S^n_{j(j-1)} \mid S^n_{j(j-2)}, \cdots, S^n_{j1} \right) - \sum_{j=2}^{K} h \left( \hat{Y}^n_j \mid S^n_{j(j-1)}, S^n_{j(j-2)}, \cdots, S^n_{j1}, X^n_j \right) \]

\[= \sum_{j=3}^{K} h \left( S^n_{j(j-1)} \mid S^n_{j(j-2)}, \cdots, S^n_{j1} \right) - \sum_{j=3}^{K} h \left( \hat{Y}^n_j \mid S^n_{j(j-1)(j-2)}, \cdots, S^n_{j(j-1)1}, X^n_{j-1} \right) \]

\[-h \left( \hat{Y}^n_K \mid S^n_{K(K-1)}, \cdots, S^n_{K1}, X^n_K \right) \tag{57} \]

\[= -h \left( \hat{Y}^n_K \mid S^n_{K(K-1)}, \cdots, S^n_{K1}, X^n_K \right) \tag{58} \]

\[= nh \left( N^n_K \mid W^n_{K(K-1)}, \cdots, W^n_{K1} \right) \tag{59} \]

(h-1) is from Lemma 5. Given that random variables \( \hat{Y}^n_{j-1} \mid S^n_{(j-1)(j-2)}, \cdots, S^n_{(j-1)1}, X^n_{j-1} \) and \( S^n_{j-1} \mid S^n_{j-1}, X^n_{j-1} \) have the same marginal distribution, we have

\[ h \left( S^n_{j(j-1)} \mid S^n_{j(j-2)}, \cdots, S^n_{j1} \right) = h \left( \hat{Y}^n_{j-1} \mid S^n_{(j-1)(j-2)}, \cdots, S^n_{(j-1)1}, X^n_{j-1} \right), \]

(j) From chain rule of entropy, we know that

\[ h \left( W^n_{K(K-1)}, \cdots, W^n_{K1} \right) = h \left( N^n_K, W^n_{K(K-1)}, \cdots, W^n_{K1} \right), \tag{60} \]
form a Markov chain, and the following equality holds:
\[
\sum_{j=1}^{K} h \left( \mathbf{S}_{j(j-1)G}, \mathbf{S}_{j(j-2)G}, \ldots, \mathbf{S}_{j1G} \mid \mathbf{Y}_{jG} \right)
= + \sum_{j=2}^{K-1} h \left( \mathbf{S}_{j(j-1)G}, \mathbf{S}_{j(j-2)G}, \ldots, \mathbf{S}_{j1G} \mid \mathbf{Y}_{jG} \right)
+ h \left( \mathbf{S}_{jK-1G}, \mathbf{S}_{jK-2G}, \ldots, \mathbf{S}_{j1G} \mid \mathbf{Y}_{jG} \right)
\]

(n) Combining Lemma 7 and Lemma 8 we know that for channels in \( \mathcal{C}_\alpha \), if the power constraint \( P_j \) satisfies \( P_j \leq \epsilon''_\alpha \), then
\[
\mathbf{X}_{jG} \rightarrow \hat{\mathbf{Y}}_{jG} \rightarrow \mathbf{S}_{j}
\]  
(61)

We can conclude that the achievable sum capacity of the generalized Z-channel must satisfy
\[
\sum_{j=1}^{K} R_j \leq \max_{\mathbf{V}_j \geq 0, j = 1, \cdots, K} \frac{\Tr (\mathbf{V}_j) \leq \SNR_j}{\sum_{j=1}^{K} I \left( \mathbf{X}_{jG} ; \hat{\mathbf{Y}}_{jG} \right)}
\]  
(63)

Notice that for Z-channel, this sum capacity outer bound is achievable because the expression above is identical to the sum capacity achieved by treating interference as noise. Since the generalized Z-channel is obtained by eliminating some of the interference links from the interference channel, (63) is an outer bound for the sum capacity of the interference channel. Theorem 9 is proved.

APPENDIX E
PROOF OF THEOREM 10

In Theorem 9, we have proved that the sum capacity \( (51) \) of the generalized Z-channel is achieved by i.i.d. Gaussian input,

\[
\sum_{j=1}^{K} R_j \leq \max_{\mathbf{V}_j \geq 0, j = 1, \cdots, K} \frac{\Tr (\mathbf{V}_j) \leq P_j}{\sum_{j=1}^{K} I \left( \mathbf{X}_{jG} ; \hat{\mathbf{Y}}_{jG} \right)}
\]  
(65)

Define the normalized covariance matrix \( \hat{\mathbf{V}}_j = \frac{\mathbf{V}_j}{P_j} \), \( \Tr \left( \hat{\mathbf{V}}_j \right) = 1 \). Consider the equal power constraint where \( P_j = P_{\text{sum}} / K \) for all users.
For an expression of the form \( \log |I + xA| \), let the eigenvalue of matrix \( A \) be \( 0 \leq \lambda_i (A) < \infty \). Then

\[
\log |I + xA| = \sum_{i=1}^{n} \log (1 + x\lambda_i (A))
\]
\[
= \sum_{i=1}^{n} \left( x\lambda_i (A) - \frac{1}{2} x^2 \lambda_i^2 (A) + o(x^2) \right)
\]
\[
= x \text{Tr}(A) - \frac{1}{2} x^2 \text{Tr}(A^2) + o(x^2)
\] (67)

The second equation uses Taylor’s theorem for several variables at \( \hat{\lambda}_i (A) = x\lambda_i (A) \), since when \( x \to 0 \), \( x\lambda_i (A) \to 0 \) as well.

Combining (67), (4), (5) and (31), we find (33) and (34).

APPENDIX F

PROOF OF THEOREM 11

To maximize the right hand side of (34), we need to solve the following optimization problem

\[
\min_{\hat{V}_1, \cdots, \hat{V}_K} \sum_{j=1}^{K} |C_{jj}|^4 \text{Tr} \left( \hat{V}_j^2 \right)
\]
\[
+ 2 \sum_{j=1}^{K-1} \sum_{i=j+1}^{K} |C_{jj}|^2 |C_{ji}|^2 \text{Tr} \left( \hat{V}_j U_{ji} \hat{V}_i \hat{U}_{ji}^\dagger \right)
\] (68)

s.t.
\[
\text{Tr} \left( \hat{V}_j \right) = 1
\]
\[
\hat{V}_j \succeq 0.
\] (69)

First, consider a simple case where the channel is strictly symmetric: \( \phi_{ji} = 0 \), \( |C_{jj}|^2 = 1 \) and \( |C_{ji}|^2 = \alpha < 1 \) for all \( i, j \). (68) becomes

\[
\min_{\hat{V}_1, \cdots, \hat{V}_K} \sum_{j=1}^{K} \text{Tr} \left( \hat{V}_j^2 \right) + 2\alpha \sum_{j=1}^{K-1} \sum_{i=j+1}^{K} \text{Tr} \left( \hat{V}_j \hat{V}_i \right)
\] (70)

s.t.
\[
\text{Tr} \left( \hat{V}_j \right) = 1
\]
\[
\hat{V}_j \succeq 0.
\] (71)

Let the \( 2 \times 2 \) real positive definite matrix \( \hat{V}_j \) be

\[
\hat{V}_j = \begin{pmatrix} k_{j1} & k_{j3} \\ k_{j3} & k_{j2} \end{pmatrix}.
\] (72)

Substituting (72) into (70), we construct a non-linear optimization problem from (70) on standard form:

\[
\min_{k_{j1}, k_{j2}, k_{j3}, \cdots, k_{K1}, k_{K2}, k_{K3}} \sum_{j=1}^{K} \left( k_{j1}^2 + k_{j2}^2 + 2k_{j3}^2 \right)
\]
\[
+ 2\alpha \sum_{j=1}^{K-1} \sum_{i=j+1}^{K} (k_{j1} k_{i1} + k_{j2} k_{i2} + 2k_{j3} k_{i3})
\] (73)

s.t.
\[
-k_{j1} \leq 0
\]
\[
-k_{j2} \leq 0
\]
\[
k_{j3} - k_{j1} k_{j2} \leq 0
\]
\[
k_{j1} + k_{j2} = 1
\] (74)
\[
(75)
\]
\[
(76)
\]
\[
(77)
\]
The optimal solution of the problem defined by (73)–(77) is also the optimal solution of the problem defined by (70). Denote the optimization problem defined by (73)–(77) as \( (P_k) \), where \( k = (k_{11}, k_{12}, k_{13}, \ldots, k_{K1}, k_{K2}, k_{K3}) \) represents the set of feasible solutions. Notice that while any positive \( k_{j1}, k_{j2} \) with \( k_{j1} + k_{j2} \leq 1 \) satisfies the power constraint, we require constraint (77) to be an equality. Because only when it is satisfied with equality, the system can achieve correct \( \frac{P_k}{N_0} \) mino.

Denote the objective function in (73) by \( f(k) \). Construct the Lagrangian function for problem (73) as

\[
F(k, u_1, u_2, u_3, v) = f(k) - \sum_{j=1}^{K} u_{j1}k_{j1} - \sum_{j=1}^{K} u_{j2}k_{j2} + \sum_{j=1}^{K} u_{j3}(k_{j3}^2 - k_{j1}k_{j2}) + \sum_{j=1}^{K} v_j(k_{j1} + k_{j2} - 1).
\] (78)

To find a optimal solution for this problem, we use Karush-Kuhn-Tucker (KKT) sufficient condition. It is stated as followed.

**Theorem 17. (KKT Sufficient Condition)** Consider an optimization problem \( (P) \) defined as

\[
\begin{align*}
\min_{x} & \quad f(x) \\
\text{subject to} & \quad g_k(x) \leq 0, \quad k = 1, \ldots, m \\
& \quad h_l(x) = 0, \quad l = 1, \ldots, n,
\end{align*}
\]

with Lagrangian function

\[
L(x, u, v) = f(x) + g(x)^T u + h(x)^T v
\]

Let \( x \) be a feasible solution of \( (P) \), and suppose \( (x, u, v) \) satisfy

\[
\begin{align*}
\nabla_x L(x, u, v) &= 0 \\
u &\geq 0 \\
u_k, g_k(x) &= 0
\end{align*}
\]

Then if \( f(x) \) is a pseudoconvex function, \( g_k(x), \ k = 1, \ldots, m \) are quasiconvex functions, and \( h_l(x), \ l = 1, \ldots, n \) are linear functions, then \( x \) is a global optimal solution.

Given \( (P_k) \), it is clear that the objective function \( f(k) \) is a convex function, the equality constraints (77) are linear, and the sets of inequality constraints (74), (75), and (76) are convex. Notice that a convex function is a special case of pseudoconvex and quasiconvex. Comparing the standard problem \( (P) \) in Theorem 17 with our optimization problem \( (P_k) \), we can conclude that any feasible \( k \) satisfying

\[
\begin{align*}
\nabla_k F(k, u_1, u_2, u_3, v) &= 0 \\
u_1, u_2 \ and \ u_3 &\geq 0 \\
u_{j1}k_{j1} &= 0 \\
u_{j2}k_{j2} &= 0 \\
u_{j3}(k_{j3}^2 - k_{j1}k_{j2}) &= 0
\end{align*}
\]

is a global optimal for \( (P_k) \). Solving \( \nabla_k F(k, u_1, u_2, u_3, v) \) we have

\[
\begin{align*}
\frac{\nabla F}{\nabla k_{j1}} &= 2k_{j1} + 2\alpha \sum_{i=1, i \neq j}^{K} k_{i1} - u_{j1} - u_{j3}k_{j2} + v_j = 0 \\
\frac{\nabla F}{\nabla k_{j2}} &= 2k_{j2} + 2\alpha \sum_{i=1, i \neq j}^{K} k_{i2} - u_{j2} - u_{j3}k_{j1} + v_j = 0
\end{align*}
\]
\[ \nabla F = 4k_{j3} + 4\alpha \sum_{i=1,i\neq j}^{K} k_{i3} + 2u_{j3}k_{j3} = 0. \]

It is easy to check that \( k_{j1} = k_{j2} = \frac{1}{2}, k_{j3} = 0 \) while the Lagrange multipliers \( u_{j1} = u_{j2} = u_{j3} = 0 \), and \( v_j = -1 - \alpha (K - 1) \) satisfy KKT condition.

Therefore, \( k_{j1} = k_{j2} = \frac{1}{2}, k_{j3} = 0 \), i.e. \( \tilde{\nu}_x = \left( \begin{array}{ccc} 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{array} \right) \) is a global optimal solution. Substitute this optimal solution into the formula of sum slope (34), the sum slope has upper bound

\[ S_0 \leq \frac{2K}{\alpha K + (1 - \alpha)} \]

**APPENDIX G**

**PROOF OF COROLLARY 12**

Before proving this result, we state existing results for general parametric optimization problems. A general parametric optimization problem \( P (\lambda) \) depending on parameters \( \lambda \in \mathbb{R}^r \) is defined by

\[ \min_{x} f (x, \lambda) \]

subject to \( x \in \mathbb{R}^n \)

\[ g_i (x, \lambda) \leq 0, \; i = 1, \ldots , s \]

\[ g_i (x, \lambda) = 0, \; i = s + 1, \ldots , m \]

where \( f \) and \( g_i \) are real functions. Denote the parametric feasible region by

\[ A (\lambda) \triangleq \{ x \in \mathbb{R}^n : g_i (x, \lambda) \leq 0 \; \text{if} \; i = 1, \ldots , s; \]

\[ g_i (x, \lambda) = 0 \; \text{if} \; i = s + 1, \ldots , m \}. \]

And denote the parametric optimal value function by \( \nu (\lambda) \triangleq \inf_{x \in A (\lambda)} f (x, \lambda) \). The following theorem gives the sufficient condition under which \( \nu (\lambda) \) is a continuous function of \( \lambda \).

**Theorem 18** (Theorem 3, p.70, [11]). Suppose that

1) the function \( f \) is continuous on \( x \times \lambda \);

2) the correspondence \( A \) is continuous on \( \lambda \);

3) the subsets \( A (\lambda) \) are non empty and compact

Then the optimal value function \( \nu (\lambda) \) is continuous and the correspondence optimal solution set is upper semi-continuous.

Let \( C \) correspond to \( \lambda \), and let the \( \lambda \) as that defined in Appendix A correspond to \( \lambda \) of Theorem 18. It is easy to see that the objective function of (58) is continuous on \( \lambda \times C \) while the feasible region \( A (C) \) is non empty, compact, and independent of \( C \). Therefore, all three conditions in Theorem 18 are satisfied and the optimal value function \( f (\lambda, C) \) is continuous on \( C \).

Further, in Theorem 11 we have shown that when \( C_{\alpha} = \{ C : \phi_{ji} = 0, |C_{jj}|^2 = 1, |C_{ji}|^2 = \alpha \} \), the optimal value of the objective function of the optimization problem \( P_{\lambda} (C_{\alpha}) \) is

\[ f (\lambda, C_{\alpha}) = \frac{2K}{\alpha K + (1 - \alpha)}. \]

Given the continuity of \( f (\lambda, C_{\alpha}) \) provided by Theorem 18, for any \( \sigma \), there exist \( \sigma_1, \sigma_2, \sigma_3 \) such that for the channels \( C \in \mathcal{C}_{\alpha} \), where the set \( \mathcal{C}_{\alpha} \) is defined as

\[ \mathcal{C}_{\alpha} = \{ C : |\phi_{ji}| < \sigma_1 \]

\[ |C_{jj}|^2 - 1 | < \sigma_2 \]

\[ \sqrt{|C_{ji}|^2 - \alpha} < \sigma_3 \]
\frac{|c_{ij}|^2}{|c_{ji}|^2} < 1
\quad \mathcal{C} \in C_\alpha \}

the optimal value of the objective function of the optimization problem \( P_k(\mathcal{C}) \) satisfies

\left| f (k, \mathcal{C}) - f (k, \mathcal{C}_o) \right| < \sigma.

Notice that \( C_\alpha \) is defined in Theorem 9.

Because \( 1 \in \text{cl} \left( \tilde{C}_\sigma \right) \), as \( \alpha \rightarrow 1 \), for any positive \( \sigma \), there exists \( \tilde{C}_\sigma \), such that for \( \mathcal{C} \in \tilde{C}_\sigma \) its sum slope satisfies

\begin{equation}
S_0 \leq 2 + \sigma,
\end{equation}

(79)

If the magnitude and phase of the channel coefficients are drawn from continuous random distribution, \( Pr \left( \tilde{C}_\sigma \right) > 0 \).

And as \( \sigma \rightarrow 0 \),

$$
\lim_{\sigma \rightarrow 0} \Delta S_0 = \frac{1}{K}
$$