THE SPECTRAL DENSITY FUNCTION FOR THE
LAPLACIAN ON HIGH TENSOR POWERS OF A LINE
BUNDLE.

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1. Introduction

Let $X$ be a compact $2n$-dimensional almost Kähler manifold, with symplectic form $\omega$ and almost complex structure $J$. Almost Kähler means that $\omega$ and $J$ are compatible in the sense that

$$\omega(Ju, Jv) = \omega(u, v) \quad \text{and} \quad \omega(\cdot, J\cdot) \gg 0.$$ 

The combination thus defines an associated Riemannian metric $\beta(\cdot, \cdot) = \omega(\cdot, J\cdot)$. Any symplectic manifold possesses such a structure. We will assume further that $\omega$ is ‘integral’ in the cohomological sense. This means we can find a complex hermitian line bundle $L \to X$ with hermitian connection $\nabla$ whose curvature is $-i\omega$.

Recently, beginning with Donaldson’s seminal paper, the notion of “nearly holomorphic” or “asymptotically holomorphic” sections of $L^\otimes k$ has attracted a fair amount of attention. Let us recall that one natural way to define spaces of such sections is by means of an analogue of the $\bar{\partial}$-Laplacian $\Box$.

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The hermitian structure and connection on $L$ induce corresponding structures on $L^\otimes k$. In combination with $\beta$ this defines a Laplace operator $\Delta_k$ acting on $C^\infty(X; L^\otimes k)$. Then the sequence of operators
\[
D_k = \Delta_k - nk
\]
has the same principal and subprincipal symbols as the d-bar Laplacian in the integrable case; in fact in the Kähler case $D_k$ is the $\overline{\partial}$-Laplacian. (By Kähler case we mean not only that $J$ is integrable but also that $L$ is hermitian holomorphic with $\nabla$ the induced connection.) The large $k$ behavior of the spectrum of $\Delta_k$ was studied (in somewhat greater generality) by Guillemin-Uribe [6]. For our purposes, the main results can be summarized as follows:

**Theorem 1.1.** [6] There exist constants $a > 0$ and $M$ (independent of $k$), such that for large $k$ the spectrum of $D_k$ lies in $(ak, \infty)$ except for a finite number of eigenvalues contained in $(-M, M)$. The number $n_k$ of eigenvalues in $(-M, M)$ is a polynomial in $k$ with asymptotic behavior $n_k \sim k^n \text{vol}(X)$, and can be computed exactly by a symplectic Riemann-Roch formula.

Furthermore, if the eigenvalues in $(-M, M)$ are labeled $\lambda_j^{(k)}$, then there exists a spectral density function $q \in C^\infty(X)$ such that for any $f \in C(\mathbb{R})$,
\[
\frac{1}{n_k} \sum_{j=1}^{n_k} f(\lambda_j^{(k)}) \to \frac{1}{\text{vol}(X)} \int_X (f \circ q) \frac{\omega^n}{n!},
\]
as $k \to \infty$.

The proof of Theorem 1.1 is based on the analysis of generalized Toeplitz structures developed in [4].

By the remarks above, in the Kähler case all $\lambda_j^{(k)} = 0$, corresponding to eigenfunctions which are holomorphic sections of $L^\otimes k$. Hence $q \equiv 0$ for a true Kähler structure. In general, it is therefore natural to consider sections of $L^\otimes k$ spanned by the eigenvalues of $D_k$ in $(-M, M)$ as being analogous to holomorphic sections.

The goal of the present paper is to derive a simple geometric formula for the spectral density function $q$. Our main result is:

**Theorem 1.2.** The spectral density function is given by
\[
q = -\frac{5}{24} |\nabla J|^2
\]

**Corollary 1.3.** The spectral density function is identically zero iff $(X, J, \omega)$ is Kähler.

It is natural to ask if one can choose $J$ so that $q$ is very small, i.e. if the symplectic invariant
\[
j(X, \omega) := \inf \{ \| \nabla J \|^2_\infty ; J \text{ a compatible almost complex structure} \}
\]
SPECTRAL DENSITY FUNCTION

is always zero. We have learned from Miguel Abreu that for Thurston’s manifold \( j = 0 \); it would be very interesting to find \((X, \omega)\) with \( j > 0 \).

The proof of Theorem \ref{1.2} starts with the standard and very useful observation that sections of \( L^\otimes_k \) are equivalent to equivariant functions on an associated principle bundle \( \pi : Z \to X \). We endow \( Z \) with a ‘Kaluza-Klein’ metric such that the fibers are geodesic. Then the main idea exploited in the proof is the construction of approximate eigenfunctions (quasimodes) of the Laplacian \( \Delta_Z \) concentrated on these closed geodesics. Such quasimodes are equivariant and thus naturally associated to sections of \( L^\otimes_k \). Moreover, the value of the spectral density function \( q(x) \) is encoded in the eigenvalue of the quasimode concentrated on the fiber \( \pi^{-1}(x) \subset Z \).

2. Preliminaries

The associated principle bundle to \( L \) is easily obtained as the unit circle bundle \( Z \subset L^* \). There is a 1-1 correspondence between sections of \( L^\otimes_k \) and functions on \( Z \) which are \( k \)-equivariant with respect to the \( S^1 \)-action, i.e.

\[
f(z, e^{i\theta}) = e^{ik\theta} f(z).
\]

The connection \( \nabla \) on \( L \) induces a connection 1-form \( \alpha \) on \( Z \). The curvature condition on \( \nabla \) translates to

\[
d\alpha = \pi^* \alpha,
\]

where \( \pi : Z \to X \). Together with the Riemannian metric on \( X \) and the standard metric on \( S^1 = \mathbb{R}/2\pi \mathbb{Z} \), this defines a ‘Kaluza-Klein’ metric \( g \) on \( Z \) such that the projection \( Z \to X \) is a Riemannian submersion with totally geodesic fibers. With these choices the correspondence between equivariant functions and sections extends to an isomorphism between

\[
L^2(X, L^\otimes_k) \simeq L^2(Z)_k,
\]

where \( L^2(Z)_k \) is the \( k \)-th isotype of \( L^2(Z) \) under the \( S^1 \) action.

The Laplacian on \( Z \) is denoted \( \Delta_Z \). By construction it commutes with the generator \( \partial \theta \) of the circle action, and so it also commutes with the ‘horizontal Laplacian’:

\[
\Delta_h = \Delta_Z + \partial^2 \theta.
\]

The action of \( \Delta_h \) on \( L^2(Z)_k \) is equivalent under \((2.1)\) to the action of \( \Delta_k \) on \( L^2(X, L^\otimes_k) \).

For sufficiently large \( k \), we let \( \mathcal{H}_k \subset L^2(Z)_k \) denote the span of the eigenvectors with eigenvalues in the bounded range \((-M, M)\). The corresponding orthogonal projection is denoted \( \Pi_k : L^2(Z) \to \mathcal{H}_k \). The following fact appears in the course of the proof of Theorem \ref{1.1}.

**Lemma 2.1.** \[\Box\] There is a sequence of functions \( q_j \in C^\infty(X) \) such that

\[
\| \Pi_k \left( \Delta_h - nk - \sum_{j=0}^N k^{-j} \pi^* q_j \right) \Pi_k \| = O(k^{-(N+1)}).
\]

Moreover, the spectral density function \( q \) in Theorem \ref{1.2} is equal to \( q_0 \).
3. QUASIMODES ON THE CIRCLE BUNDLE

The key to the calculation of the spectral density function at \( x_0 \in X \) is the observation that, with the Kaluza-Klein metric, the assumptions on \( X \) imply the stability of the geodesic fiber \( \Gamma = \pi^{-1}(x_0) \). Thus one should be able to construct an approximate eigenfunction, or quasimode, for \( \Delta_Z \) which is asymptotically localized on \( \Gamma \). The lowest eigenvalue of the quasimode (or rather a particular coefficient in its asymptotic expansion) will yield the spectral density function.

The computation is largely a matter of interpolating between two natural coordinate systems. From the point of view of writing down the Kaluza-Klein metric explicitly, the obvious coordinate system to use is given by first trivializing \( Z \) to identify a neighborhood of \( \Gamma \) with \( S^1 \times U_{x_0} \), where \( U_{x_0} \) is a neighborhood of \( x_0 \) in \( X \) (the base point \( x_0 \) will be fixed throughout this section). On \( U_{x_0} \) we can introduce geodesic normal coordinates centered at \( x_0 \). These coordinates will be denoted \((\theta, x^1, \ldots, x^{2n})\). The base point \( z_0 \in \Gamma \) corresponding to \( \theta = 0 \) is arbitrary. In such coordinates the connection \( \alpha \) takes the form \( \alpha = d\theta + \alpha_j dx^j \).

We will follow the quasimode construction outlined in Babich-Buldyrev [1], which is essentially based in the normal bundle \( N\Gamma \subset TZ \). Let \( \psi : N\Gamma \to Z \) be the map defined on each fiber \( N_x \Gamma \) by the restriction of the exponential map \( \exp_x : T_x Z \to Z \). Of course, \( \psi \) is only a diffeomorphism near \( \Gamma \). The Fermi coordinate system along \( \Gamma \) is defined by the combination of \( \psi \) and the choice of a parallel frame for \( N\Gamma \). Let \( \gamma(s) \) be a parametrization of \( \Gamma \) by arclength, with \( \gamma(0) = z_0, \gamma'(0) = \partial_\theta \). Let \( e_j(s) \) be the frame for \( N_{\gamma(s)} \Gamma \) defined by parallel transport from the initial value \( e_j(0) = \partial_j \). Then the Fermi coordinates are defined by

\[
(s, y^j) \mapsto \psi(y^j e_j(s)).
\]

Note that \( s = \theta \) only on \( \Gamma \).

3.1. The ansatz. Now we can formulate the construction of an asymptotic eigenfunction as a set of parabolic equations on \( N\Gamma \). Let \( \kappa \) be an asymptotic parameter (eventually to be related to \( k \)). Setting \( u = e^{iks}U \) we consider the equation

\[
(\Delta_Z - \lambda)e^{iks}U(s, y) = 0.
\]

Since we are hoping to localize near \( y = 0 \), the ansatz is to substitute \( u^j = \sqrt{\kappa} y^j \) and do a formal expansion

\[
e^{-iks} \Delta_Z e^{iks} = \kappa^2 + \kappa L_0 + \sqrt{\kappa} L_1 + L_2 + \ldots.
\]

This defines differential operators \( L_j \) on a neighborhood of the zero-section in \( N\Gamma \), but since the coefficients are polynomial in the \( y^j \) variables, they extend naturally to all of \( N\Gamma \). We also make an ansatz of formal expansions
for $\lambda$ and $U$:

$$\lambda = \kappa^2 + \sigma + \ldots$$

$$U = U_0 + \kappa^{-1}U_1 + \ldots$$

Substituting these expansions into (3.1) and reading off the orders gives the equations

$$\mathcal{L}_0 U_0 = 0$$
$$\mathcal{L}_1 U_0 = 0$$
$$\mathcal{L}_0 U_1 = -(\mathcal{L}_2 - \sigma)U_0$$

(3.3)

Since $\mathcal{L}_j$ is well-defined on $NT$, we can seek global solutions $U_j(s, y)$, subject to the boundary condition $\lim_{|y| \to \infty} U_j = 0$. It turns out that $\mathcal{L}_0$ is a very familiar parabolic operator and the second equation is satisfied as a trivial consequence of the first. Solutions of the third equation exist only for a certain value of $\sigma$, and the main goal of this section is to compute this quantity.

By pulling back by $\psi$, we can use ($\theta, x$) as an alternate coordinate system on $NT$ (near the zero section). We’ll use $\tilde{\beta}_{ij}, \tilde{\alpha}_i, \tilde{\omega}_{ij}, \tilde{J}_j$ to denote the various tensors lifted from $X$ and written in these coordinates (so all are independent of $\theta$). We let $\Gamma^\mu_{\mu\nu}$ denote the Christoffel symbols of the Kaluza-Klein metric $g$ in the ($\theta, x$) coordinates. To reduce notational complexity insofar as possible, we will adopt the convention that unbarred expressions involving $\beta_{ij}, \alpha_i, \omega_{ij}, J_j$ and their derivatives are to be evaluated at the base point $x_0 \in X$, e.g. $\beta_{ij} = \tilde{\beta}_{ij}\big|_{x=0}$. The Christoffel symbols of $\beta_{ij}$ (evaluated at $x_0$) will be denoted by $\Gamma^i_{jk}$, with the same convention for derivatives. (Thus $\Gamma^i_{jk} = 0$, but its derivatives are not zero.) The freedom in the trivialization of $Z$ may be exploited to assume that

$$\alpha_j = 0, \quad \partial_j \alpha_k = \frac{1}{2} \omega_{jk}.$$ 

We’ll use $g_{\mu\nu}$ to denote the Kaluza-Klein metric expressed in the ($\theta, x$) coordinates (with the convention that Greek indices range over $0, \ldots, 2n$ and Roman over $1, \ldots, 2n$). Let $\partial_j$ denote the vector field $\frac{\partial}{\partial x^j}$ on $X$. The horizontal lift of $\partial_j$ to $Z$ is then

$$E_j = \partial_j - \bar{\alpha}_j \partial_\theta.$$ 

(3.4)

The Kaluza-Klein metric is determined by the conditions:

$$g(E_j, \partial_\theta) = 0, \quad g(\partial_\theta, \partial_\theta) = 1, \quad g(E_j, E_k) = \bar{\beta}_{jk}.$$ 

Substituting in with (3.4) we quickly see that

$$g_{00} = 1, \quad g_{j0} = \bar{\alpha}_j, \quad g_{jk} = \bar{\beta}_{jk} + \bar{\alpha}_j \bar{\alpha}_k.$$ 

In block matrix form we can write

$$g = \begin{pmatrix} 1 & \bar{\alpha} \\ \bar{\alpha} & \bar{\beta} + \bar{\alpha} \bar{\alpha} \end{pmatrix},$$ 

(3.5)
from which
\[
g^{-1} = \begin{pmatrix}
1 + \alpha \beta^{-1} \alpha - \beta^{-1} \alpha \\
-\beta^{-1} \alpha - \beta^{-1} \alpha & \beta^{-1}
\end{pmatrix}.
\]

We’ll use \( G_{\mu\nu} \) to denote the Kaluza-Klein metric written in the Fermi coordinates \((s,y)\), i.e.
\[
G_{00} = g(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}), \quad G_{0j} = g(\frac{\partial}{\partial s}, \frac{\partial}{\partial y^j}), \quad G_{ij} = g(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}).
\]

\( G_{\mu\nu} \) is well-defined in a neighborhood of \( y = 0 \), and with the ansatz above we only need to know its Taylor series to determine \( L_j \). As noted, the heart of the calculation will be the change of coordinates from \((\theta,x)\) to \((s,y)\).

By assumption \( G_{\mu\nu} = \delta_{\mu\nu} \) to second order in \( y \). After the substitution \( u_j = \sqrt{\kappa} y_j \), we can write the Taylor expansions of various components as
\[
G_{00} = 1 + \kappa^{-1} a^{(2)} + \kappa^{-3/2} a^{(3)} + \kappa^{-2} a^{(4)} + \ldots
\]
\[
G_{0j} = \kappa^{-1} b_j^{(2)} + \kappa^{-3/2} b_j^{(3)} + \ldots
\]
\[
G_{jk} = \delta_{jk} + \kappa^{-1} c_{jk}^{(2)} + \ldots,
\]

where \((l)\) denotes the term which is a degree \( l \) polynomial in \( u \). Then using the definition
\[
\Delta Z = -\frac{1}{\sqrt{G}} \partial_\mu \left[ \sqrt{G} G^{\mu\nu} \partial_\nu \right],
\]
we can substitute the expansions \((3.7)\) into \((3.2)\) and read off the first few orders in \( \kappa \):
\[
\begin{align*}
\mathcal{L}_0 &= -2i \partial_s - a^{(2)} - \partial_u^2 \\
\mathcal{L}_1 &= -a^{(3)} + 2ib^{(2)} \frac{\partial}{\partial \omega^j} + i(\frac{\partial}{\partial \omega^j} b^{(3)}) \\
\mathcal{L}_2 &= -\partial_s^2 + 2ia^{(2)} \partial_s - a^{(4)} + (a^{(2)})^2 + (b^{(2)})^2 \\
&\quad + i \left[ -\frac{1}{2} \partial_s \text{Tr} c^{(2)} + 2b^{(3)} \frac{\partial}{\partial \omega^j} + (\frac{\partial}{\partial \omega^j} b^{(3)}) \right] \\
&\quad + c^{ik} \frac{\partial}{\partial \omega^i} \frac{\partial}{\partial \omega^k} + (\frac{\partial}{\partial \omega^j} c^{ik}) \frac{\partial}{\partial \omega^k} - \frac{1}{2} \partial_s [a^{(2)} + \text{Tr} c^{(2)}] \frac{\partial}{\partial \omega^j}
\end{align*}
\]

3.2. The metric in Fermi coordinates. For use in the calculation, let us first work out some simple implications of \( J^2 = -1 \). Using conventions as above, this means \( J^k_j J^m_j = -\delta^m_j \). Differentiating at the base point \( x_0 \) gives us
\[
(\partial_i J^k_j) J^m_k = -J^k_j (\partial_i J^m_k), \quad J^k_j (\partial_i J^i_k) = 0
\]
The other basic fact is \( d\omega = 0 \), which translates to
\[
\partial_i \omega_{jk} + \partial_j \omega_{ik} + \partial_k \omega_{ij} = 0.
\]

Lemma 3.1.
\[
\partial_i J^i_j = 0.
\]
Proof. Using the fact that \( J_l^j \omega_{jk} \beta^{kl} \) we have
\[
J_j^k (\partial_l J_k^l) = - (\partial_l J_j^k) J_k^l = - (\partial_l \omega_{jk}) \omega^{kl} = - \frac{1}{2} (\partial_l \omega_{jk} - \partial_k \omega_{jl}) \omega^{kl} = \frac{1}{2} (\partial_l \omega_{kl}) \omega^{kl} = - \frac{1}{2} (\partial_l J_k^l) J_k^j = 0
\]

A similar fact, which will also be useful, is:

**Lemma 3.2.** For any vector \( v^j \) we have
\[
(\partial_l J_j^m) v^j (\omega v)_m = 0.
\]

Proof.
\[
(\partial_l J_j^m) v^j (\omega v)_m = (\partial_l J_j^m) v^j J_m^s v_s = - (\partial_l J_m^s) v^j J_j^m v_s = - (\partial_l \omega_{ms}) (Jv)^m v_s = (\partial_l \omega_{sm}) (Jv)^m v_s = - (\partial_l J_j^m) (\omega v)_m v_s = 0
\]

To proceed, we must determine the terms in the Taylor expansion of \( G_{\mu\nu} \) in terms of the geometric data \( \beta, \omega, J, \alpha \). In terms of \( \partial_j \), let the parallel frame \( e_j(s) \) be written \( T_k^j \partial_k \) (fixing \( j \) for the moment). The parallel condition is
\[
\partial_s T_k^j = - \Gamma_{0l}^k T_j^l,
\]
where
\[
\Gamma_{0l}^k |_{x=0} = \frac{1}{2} \beta^{km} (\partial_l \alpha_m - \partial_m \alpha_l) = \frac{1}{2} \beta^{km} \omega_{lm} = \frac{1}{2} J_l^k.
\]
The solution is
\[
T_j^k = (e^{-\frac{1}{2} J^j})^k_j.
\]
Since this is the matrix relating the \( x \)-frame to the \( y \)-frame at \( x = 0 \), we have \( \frac{\partial z^k}{\partial x^j} = T_j^k \). This makes it convenient to introduce
\[
z^k = T_j^k y^j.
\]
The transformation to Fermi coordinates may now be written as
\[
\theta = s + A(s, z), \quad x^j = z^j + B^j(s, z).
\]
The functions $A$ and $B$ are determined by the condition that the ray $t \mapsto (s, ty)$ be a geodesic. Of course, we are really just interested in the Taylor expansions:

$$A = \kappa^{-1} A^{(2)} + \kappa^{-3/2} A^{(3)} + \kappa^{-2} A^{(4)} + \ldots,$$

$$B^j = \kappa^{-1} B^{j(2)} + \kappa^{-3/2} B^{j(3)} + \ldots,$$

where degrees are labeled as above.

Denoting the $t$ derivative by a dot, the geodesic equations are

$$\ddot{\theta} = -\Gamma^0_{00} \dot{\theta}^2 - 2\Gamma^0_{0l} \dot{\theta} \dot{x}_l - \Gamma^0_{jl} \dot{x}_j \dot{x}_l$$

$$\ddot{x}_k = -\Gamma^k_{00} \dot{\theta}^2 - 2\Gamma^k_{0l} \dot{\theta} \dot{x}_l - \Gamma^k_{jl} \dot{x}_j \dot{x}_l$$

(3.9)

The Christoffel symbols of $g_{ij}$ are

$$\tilde{\Gamma}^0_{00} = \tilde{\Gamma}^j_{00} = 0$$

$$\tilde{\Gamma}^0_{0j} = \frac{1}{2} (\tilde{J} \tilde{\alpha})_j$$

$$\tilde{\Gamma}^0_{jk} = \frac{1}{2} \left[ \partial_j \tilde{\alpha}_k + \partial_k \tilde{\alpha}_j + \tilde{\alpha}_j (\tilde{J} \tilde{\alpha})_k + \tilde{\alpha}_k (\tilde{J} \tilde{\alpha})_j \right] - \tilde{F}^l_{jk} \tilde{\alpha}_l$$

$$\tilde{\Gamma}^j_{0k} = -\frac{1}{2} \tilde{J}^j_k$$

$$\tilde{\Gamma}^j_{lk} = -\frac{1}{2} \tilde{J}^l_i \tilde{\alpha}_k - \frac{1}{2} \tilde{J}^l_k \tilde{\alpha}_l + \tilde{F}^j_{lk}$$

Substituting the Taylor expansion of the Christoffel symbols into (3.9) and equating coefficients, we find $A^{(2)} = 0$, $B^{(2)} = 0$,

$$A^{(3)} = - (\partial_m \partial_j \alpha_l) z^m z^j z^l$$

$$A^{(4)} = -\frac{1}{24} (\partial_k \partial_m \partial_j \alpha_l) z^k z^m z^j z^l - \frac{1}{24} (\partial_k F^i_{jl}) z^k z^j z^l (\omega z)_i,$$

$$B^{k(3)} = -\frac{1}{6} (\partial_m F^k_{jl}) z^m z^j z^l.$$

(3.10)

Using $x = z + \kappa^{-3/2} B^{(3)} + \ldots$, we can then determine the coefficients of the expansion of $\tilde{\alpha}_k$:

$$\tilde{\alpha}^{(1)}_k = -\frac{1}{2} (\omega z)_k$$

$$\tilde{\alpha}^{(2)}_k = \frac{1}{2} (\partial_l \partial_m \alpha_k) z^l z^m$$

$$\tilde{\alpha}^{(3)}_k = \frac{1}{6} (\partial_j \partial_l \partial_m \alpha_k) z^j z^l z^m + \frac{1}{12} \omega_{kl} (\partial_m F^i_{jl}) z^m z^j z^l$$

(3.11)

The Fermi coordinate vector fields are

$$\partial_s = (1 + \partial_s A) \partial_0 + (z^l + B^l) \partial_l,$$

$$\frac{\partial}{\partial y^j} = \left( \frac{\partial}{\partial y^j} A \right) \partial_0 + \left( T^j + \frac{\partial}{\partial y^j} B^l \right) \partial_l.$$
Note that $z^j = T_k^j(s)g^k$, so $z'^j = -\frac{1}{2}(Jz)^j$. To compute $a^{(l)}$, we use (3.10) and (B.11) to expand $G_{00} = g(\partial_s, \partial_s)$. The second order term is

$$a^{(2)} = 2\alpha_l^{(1)} z'^j + z'^j z'_l = -\frac{z^2}{4}$$

At third order we have

$$a^{(3)} = 2(A^{(3)})' + 2\alpha_m^{(2)} z'^m$$

$$= -\frac{1}{3} (\partial_j \partial_l \alpha_m)[2z^j z'^l z'^m + z^j z'^l z''_m] + (\partial_j \partial_l \alpha_m)z^j z'^l z'^m$$

$$= \frac{1}{3} (\partial_j \partial_l \alpha_m)(Jz)^j z'^l z'^m - \frac{1}{3} (\partial_j \partial_l \alpha_m)z^j z'^l (Jz)^m$$

$$= -\frac{1}{3} (\partial_l \alpha_m) z^j z'^l (Jz)^m$$

Thus, by Lemma 3.2 we have

$$a^{(3)} = 0.$$

The fourth order term is somewhat more complicated:

$$a^{(4)} = 2A^{(4)} + 2\alpha_m^{(3)} z'^m + 2\alpha_m^{(1)} (B'^m)^{(3)} + z^l (\beta_m^{(2)} + \alpha_m^{(1)} \alpha_m^{(1)}) z'^m + 2z'^m (B'^m)^{(3)}$$

We’ll expand the first term,

$$2A^{(4)} = \frac{1}{24} (\partial_k \partial_m \partial_j \alpha_l)[3z^j z'^m (Jz)^j z'^l + z^j z'^m z'^j (Jz)^l]$$

$$+ \frac{1}{24} (\partial_k F_{jl}^i)[(Jz)^k z'^j z'^l (\omega z)_i + 2z^k z'^j (Jz)^l (\omega z)_i + z^k z'^j z'^l _i]$$

and the second,

$$2\alpha_k^{(3)} z'^k = -\frac{1}{6} (\partial_j \partial_l \partial_m \alpha_k) z^j z'^m (Jz)^k - \frac{1}{12} \omega_{ki}(\partial_m F_{jl}^i)z^m z'^j (Jz)^l$$

The terms involving $\partial_m \alpha_k$ combine to form factors of $\omega_{mk}$:

$$2A^{(4)} + 2\alpha_k^{(3)} z'^k = -\frac{1}{8} (\partial_j \partial_l \omega_{mk}) z^j z'^m (Jz)^k + \frac{1}{24} (\partial_k F_{jl}^i)(Jz)^k z^j z'^l (\omega z)_i$$

$$+ \frac{1}{12} (\partial_k F_{jl}^i) z^k z'^j (Jz)^l (\omega z)_i + \frac{1}{8} (\partial_k F_{jl}^i) z^k z'^j z'^l$$

After noting that $2\alpha_m^{(1)} (B'^m)^{(3)} + 2z'^m (B'^m)^{(3)} = 0$, we are left with the term

$$z'^l (\beta_m^{(2)} + \alpha_m^{(1)} \alpha_m^{(1)}) z'^m = \frac{1}{8} (\partial_j \partial_l \beta_{lm})(Jz)^l z^j z^k (Jz)^m + \frac{z^4}{16}$$
So in conclusion,
\[
a^{(4)} = -\frac{1}{8} (\partial_j \partial_i \omega_{mk}) z^j z^l z^m (J z)^k + \frac{1}{24} (\partial_k F_{jl}^i) (J z)^k z^j z^l (\omega z)_i \\
+ \frac{1}{12} (\partial_k F_{jl}^i) z^k z^j (J z)_i + \frac{1}{8} (\partial_k F_{jl}^i) z^k z^j z^l \\
+ \frac{1}{8} (\partial_j \partial_l \beta_{lm}) (J z)^l z^k (J z)^m + \frac{z^4}{16}
\]

For \( b_j = g(\partial_s, \partial_p) \) the third order term will prove irrelevant, so we compute only
\[
b_j^{(2)} = \partial_j A^{(3)} + \alpha_m^{(2)} T^m_j \\
= -\frac{1}{6} (\partial_k \partial_l \alpha_m) [2 T^k_j z^l z^m + z^k z^l T^m_j] + \frac{1}{2} (\partial_k \partial_l \alpha_m) z^k z^l T^m_j \\
= \frac{1}{3} (\partial_k \partial_l \alpha_m) T^k_j z^l z^m + \frac{1}{3} (\partial_k \partial_l \alpha_m) z^k z^l T^m_j \\
= \frac{1}{3} (\partial \omega_{km}) z^k z^l T^m_j
\]

Finally, we have \( c_{lm} = g(\partial_s, \partial_p) \). It is convenient to insert factors of \( T \):
\[
T^j c_m^{(2)} T^m = \beta_{jk}^{(2)} + \alpha_j^{(1)} \alpha_k^{(1)} + (\partial_s \beta_k^{(3)}) + (\partial_k \beta_j^{(3)}) \\
= \frac{1}{2} (\partial_k \partial_l \beta_{jk}) z^l z^m + \frac{1}{4} (\omega z)_j^l (\omega z)_k^m - \frac{1}{6} (\partial_j F_{ikl}) z^j z^l \\
- \frac{1}{3} (\partial_m F_{jik}) z^m z^l - \frac{1}{6} (\partial_k F_{ijl}) z^j z^l - \frac{1}{3} (\partial_m F_{kij}) z^m z^l
\]

3.3. Parabolic equations. The first of the equations (3.3) involves the operator
\[
\mathcal{L}_0 = -2i \partial_s + \frac{u^2}{4} - \partial_u^2
\]
The equation \( \mathcal{L}_0 U_0 = 0 \) is then instantly recognizable as the Schrödinger equation for a harmonic oscillator. The “ground state” solution
\[
U_0 = e^{-i n s/2} e^{-u^2/4}.
\]
Now \( e^{i k s} U \) is supposed to be periodic, which means we must require
\[
\kappa - \frac{n}{2} = k \in \mathbb{Z}.
\]
A function on \( z \) which is \( e^{i k s} \times \) (periodic) comes from a section of \( L^k \), so this \( k \) is our usual asymptotic parameter, and
\[
k^2 = k^2 + nk + \frac{n^2}{4}
\]
By the standard analysis of the quantum harmonic oscillator, a complete set of solutions to \( \mathcal{L}_0 U = 0 \) can be generated by application of the “creation operator”
\[
\Lambda^*_j = -i e^{-i s/2} (\partial_u - \frac{u_j}{2})
\]
We will need

\[ U_{ij} = \Lambda_i^* \Lambda_j^* U_0, \quad U_{ijkl} = \Lambda_i^* \Lambda_j^* \Lambda_k^* \Lambda_l^* U_0, \]

which are easily computed explicitly:

\[ U_{ij} = (-u_j u_k + \delta_{ij}) e^{-is} U_0, \]
\[ U_{ijkl} = (u_i u_j u_k u_l - \delta_{ij} u_k u_l - \delta_{ik} u_j u_l - \delta_{il} u_j u_k - \delta_{jk} u_i u_l - \delta_{kl} u_i u_j + \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) e^{-2is} U_0. \]

Since \( a^{(3)} = 0 \) and \( \partial_{uj} b^{(2)} = 0 \), the next operator is

\[ L_1 = 2ib^{(2)} \frac{\partial}{\partial \sigma}. \]

It then follows from \( b^{(2)} w^j = 0 \) that

\[ L_1 U_0 = 0. \]

Moreover, it is easy to check, using the creation operators, that \( U_0 \) is the unique solution of \( L_0 U = 0 \) for which this is true.

Consider finally the third equation

\[ (3.19) \]
\[ L_0 U_1 = -(L_2 - \sigma) U_0, \]

from which we’ll determine \( \sigma \). Since \( L_2 U_0 \) has coefficients polynomial in \( u_j \) of order no more than four, we can expand

\[ (3.20) \]
\[ L_2 U_0 = [C^{ijkl} u_i u_j u_k u_l + C^{ij} u_i u_j + C] U_0. \]

**Proposition 3.3.** The equations (3.3) have a solution \( U_0, U_1 \in C^{\infty}(\Gamma) \) if and only if

\[ (3.21) \]
\[ \sigma = C + C_t t + 3C_{kk} t, \]

where the coefficients \( C^{ijkl} \) are assumed symmetrized.

**Proof.** In terms of the basis for the kernel of \( L_0 \) we can rewrite (3.20) as

\[ L_2 U_0 = e^{2is} D^{ijkl} U_{ijkl} + e^{is} D^{ij} U_{ij} + DU_0. \]

Observe that

\[ L_0(e^{2is} U_{ijkl}) = -4 U_{ijkl}, \quad L_0(e^{is} U_{ij}) = -2 U_{ij}. \]

So the equation \( L_0 U_1 = -(L_2 - \sigma) U_0 \) has a solution only if \( \sigma = D \), in which case we can set

\[ U_1 = \frac{1}{4} e^{2is} D^{ijkl} U_{ijkl} + \frac{1}{2} e^{is} D^{ij} U_{ij}. \]

To compute \( D \) we note

\[ C^{ij} u_i u_j U_0 = -C^{ij} e^{is} U_{ij} + C_t t U_0, \]
and (with the symmetry assumption),
\[
C_{ijkl} u_i u_j u_k u_l U_0 = C_{ijkl} e^{2i\alpha} U_{ijkl} + \left[ 6 C_{ijkl} u_k u_l - 3 C_{kk} \right] U_0
\]
\[
= C_{ijkl} e^{2i\alpha} U_{ijkl} + (\ldots) e^{i\alpha} U_{jk} + 3 C_{kk} U_0
\]
This means that
\[
D = C + C_l + 3 C_{kk} u_k.
\]

To conclude the computation, we will examine $L_2 U_0$ piece by piece and form the contractions of coefficients according to (3.21). From (3.8) we break up $L_2 U_0 = W_1 + \ldots W_6$, where
\[
W_1 = [- \partial_s^2 + 2ia^{(2)} \partial_s] U_0
\]
\[
W_2 = [-a^{(4)} + (a^{(2)})^2] U_0
\]
\[
W_3 = (b^{(2)})^2 U_0
\]
\[
W_4 = i \left[ - \frac{1}{2} \partial_s \text{Tr} e^{(2)} + 2b^{(3)} \partial_s + \left( \frac{\partial}{\partial u} b^{(3)} \right) \right] U_0
\]
\[
W_5 = \left[ e^{jk(2)} \frac{\partial}{\partial u} \frac{\partial}{\partial u} + \left( \frac{\partial}{\partial u} c^{jk(2)} \right) \frac{\partial}{\partial u} \right] U_0
\]
\[
W_6 = - \frac{1}{2} \frac{\partial}{\partial u} \left[ a^{(2)} + \text{Tr} c^{(2)} \right] \frac{\partial}{\partial u} U_0
\]
By (3.17) we compute
\[
W_1 = [- \partial_s^2 + 2ia^{(2)} \partial_s] U_0 = \left[ \frac{n^2}{4} - \frac{n z^2}{4} \right] U_0
\]
The contribution to $\sigma$ from $W_1$ is thus:
\[
(3.22) \quad - \frac{n^2}{4}
\]
For $W_2$, from the calculations of $a^{(2)}$ and $a^{(4)}$ we have
\[
-a^{(4)} + (a^{(2)})^2 = \frac{1}{8} (\partial_j \partial_k \omega_{mk}) z^j z^l z^m (Jz)^k - \frac{1}{24} (\partial_k F_{ij}^k)(Jz)^l (\omega z)_i
\]
\[
- \frac{1}{12} (\partial_k F_{ij}^k z^k z^j (Jz)^l (\omega z)_i - \frac{1}{8} (\partial_k F_{ij}^k) z^k z^j z^l i
\]
\[
- \frac{1}{8} (\partial_j \partial_k \beta_{lm})(Jz)^l z^j z^k (Jz)^m
\]
We symmetrize and take the contractions to find the contribution to $\sigma$:
\[
\frac{1}{8} (\partial^j \partial^k \omega_{mk}) \omega^{kn} + \frac{1}{4} (\partial_j \partial^l \omega_{lk}) \omega^{jk} - \frac{1}{12} (\beta_{lm} \partial_k F_{lm}^k) - \frac{1}{6} (\partial^k F_{kl})
\]
\[
- \frac{1}{4} (\partial_j \partial_k \beta_{lm}) \omega^{jl} \omega^{km} - \frac{1}{8} (\beta_{lm} \partial^k \partial_k \beta_{lm})
\]
Let us simplify this expression. By \( d\bar{\omega} = 0 \) we have

\[
(\partial_j \partial_l \omega_{lk}) \omega^{jk} = \frac{1}{2} (\partial^j \partial_j \omega_{mk}) \omega^{mk}.
\]

From \( \bar{\omega}_{mk} = -\bar{\beta}_{mr} J_k^r \) we derive

\[
(\partial^j \partial_j \omega_{mk}) \omega^{mk} = \beta^{lm} \partial^j \beta_{lm} - (\partial^j \partial_j J_m^m) J^k.
\]

Finally from \( \bar{J}^2 = -1 \) we obtain

\[
(\partial^j \partial_j J_m^m) J^k_m = -(\partial_j J_m^m)(\partial_j J^k_m) = |\nabla J|^2.
\]

Combining these facts gives

\[
\frac{1}{8} (\partial^j \partial_j \omega_{mk}) \omega^{mk} + \frac{1}{4} (\partial_j \partial_l \omega_{lk}) \omega^{jk} = \frac{1}{4} \beta^{lm} \partial^j \beta_{lm} - \frac{1}{4} |\nabla J|^2.
\]

Evaluating the Christoffel symbols gives

\[
\beta^{lm} \partial_k F_{lm}^k = \frac{1}{2} \beta^{lm} \partial^k \beta_{lm} - \frac{1}{2} \beta^{lm} \partial^k \partial^k \beta_{lm}
\]

and

\[
\partial^k F_{kl} = \frac{1}{2} \beta^{lm} \partial^k \beta_{lm}
\]

Thus the final contribution from \( W_2 \) to \( \sigma \) is

\[
(3.23) \quad -\frac{1}{4} |\nabla J|^2 - \frac{1}{4} (\partial_j \partial_k \beta_{lm}) \omega^{ji} \omega^{km} + \frac{1}{12} \beta^{lm} \partial^k \partial_k \beta_{lm} - \frac{1}{12} \partial^j \partial^l \beta_{jl}
\]

By our calculations,

\[
\frac{1}{8}(\partial^2) (\partial \omega_{km}) \omega^{jk} = \frac{1}{9} (\partial_i \omega_{km}) z^k z^j (\partial_j J^m_i) z^j,
\]

which (recalling that \( \partial^j J^m_j = 0 \)) gives a contribution from \( W_3 \) of

\[
\frac{1}{9} (\partial \omega_{km}) (\partial^k \omega^{lm}) + \frac{1}{9} |\nabla J|^2
\]

By \( d\bar{\omega} = 0 \), we have

\[
(\partial \omega_{km}) (\partial^k \omega^{lm}) = -\frac{1}{2} (\partial_k \omega_{ml}) (\partial^k \omega^{lm}) = \frac{1}{2} |\nabla J|^2.
\]

So the contribution from \( W_3 \) simplifies to

\[
(3.24) \quad -\frac{1}{6} |\nabla J|^2
\]

The terms in \( W_4 \) are purely imaginary and therefore must contribute zero because \( \sigma \) is real. This can easily be confirmed explicitly.

To compute \( W_5 \) we need to consider

\[
\omega^{j(2)} \partial_{\omega} \partial_{\omega} U_0 + (\partial_{\omega} \omega^{j(2)}) \partial_{\omega} U_0
\]

Noting that \( \partial_{\omega} U_0 = -\frac{u^k}{U_0} U_0 \), this becomes

\[
\left[ \frac{1}{4} \omega^{j(2)} \omega^j U_0 - \frac{1}{2} \beta^{jk} \omega^{j(2)} - \frac{1}{2} u_k (\partial_{\omega} \omega^{j(2)}) \right] U_0
\]
If $c^{jk}_2$ is written $E^{jk}_{lm} u^l u^m$, then under contraction the contribution is
\[
\frac{1}{4} (\beta^{lm} \beta_{jk} E^{jk}_{lm} + E^{jk}_{jk}) - \frac{1}{2} \beta^{lm} \beta_{jk} E^{jk}_{lm} - \frac{1}{2} (E^{jk}_{jk} + E^{jk}_{kj}) = -\frac{1}{4} (\beta^{lm} \beta_{jk} E^{jk}_{lm} + E^{jk}_{jk})
\]
This is the same as the contribution of
\[
-\frac{1}{4} c^{(2)}_j u^j u^k = -\frac{1}{8} (\partial_j \partial_k \beta_{lm}) z^j z^k z^l z^m + \frac{1}{4} (\partial_m F_{jk}) z^m z^j z^l,
\]
yielding
\[
-\frac{1}{8} \beta^{lm} (\partial_j \partial_k \beta_{lm}) - \frac{1}{4} (\partial^j \partial^k \beta_{jk}) + \frac{1}{4} \beta^{lm} \partial_k F_{lm} + \frac{1}{2} (\partial^m F^k_{mk}),
\]
which vanishes upon substitution of the $F$. Hence the total contribution of $W_5$ to $\sigma$ is zero.

Finally, we evaluate the expression appearing in $W_6$:
\[
\frac{1}{4} u^j \partial_j [a^{(2)} + \text{Tr} c^{(2)}] = \frac{1}{2} [a^{(2)} + \text{Tr} c^{(2)}] = \frac{1}{4} (\beta^{lm} \partial_j \partial_k \beta_{lm}) - \frac{1}{6} (\beta^j \partial_k F_{ik}) z^j z^k - \frac{1}{3} (\partial_m F^l_{ml} z^m z^i,
\]
The contribution is
\[
\frac{1}{4} (\beta^{lm} \partial^k \partial_k \beta_{lm}) - \frac{1}{6} (\beta^j \partial_k F^l_{ik}) - \frac{1}{3} (\partial^m F^l_{ml}).
\]
This contribution from $W_6$ can be reduced to
\[
(3.25) \quad \frac{1}{6} (\beta^{lm} \partial^k \partial_k \beta_{lm}) - \frac{1}{6} (\partial^j \partial^k \beta_{kl})
\]
Adding together (3.22), (3.23), (3.24), and (3.25) gives
\[
\sigma = -\frac{n^2}{4} - \frac{1}{12} |\nabla J|^2 - \frac{1}{4} (\partial_j \partial_k \beta_{lm}) \omega^{jl} \omega^{km} + \frac{1}{4} \beta^{lm} \partial^k \partial_k \beta_{lm} - \frac{1}{4} \partial^j \partial^l \beta_{jl}
\]
The last three terms on the right-hand side could be written in terms of the curvature tensors:
\[
-\frac{1}{4} (\partial_j \partial_k \beta_{lm}) \omega^{jl} \omega^{km} + \frac{1}{4} \beta^{lm} \partial^k \partial_k \beta_{lm} - \frac{1}{4} \partial^j \partial^l \beta_{jl} = \frac{1}{4} (R + \frac{1}{2} R_{ljkm} \omega^{lj} \omega^{km}).
\]
To complete the calculation we cite a lemma which can be found, for example, in [7].

**Lemma 3.4.** For an almost Kähler manifold,
\[
R + \frac{1}{2} R_{ljkm} \omega^{lj} \omega^{km} = -\frac{1}{2} |\nabla J|^2.
\]
This lemma leads us to the final result that
\[
(3.26) \quad \sigma = -\frac{n^2}{4} - \frac{5}{24} |\nabla J|^2
\]
3.4. Quasimodes. Let us introduce the function
\[ h(x) = -\frac{5}{24} |\nabla J(x)|^2 \]

**Proposition 3.5.** Fix \( x_0 \in X \) and let \( \Gamma = \pi^{-1}(x_0) \). There exists a sequence \( \psi_k \in L^2(Z)_k \) with \( \| \psi_k \| = 1 \) such that
\[ \| (\Delta h - nk - h(x_0))\psi_k \| = O(k^{-1/2}). \]
Moreover, \( \psi_k \) is asymptotically localized on \( \Gamma \) in the sense that if \( \varphi \in C^\infty(Z) \) vanishes to order \( m \) on \( \Gamma \), then
\[ \langle \psi_k, \varphi \psi_k \rangle = O(k^{-m/2}). \]

**Proof.** Let \( W \) be a neighborhood of \( \Gamma \) in which Fermi coordinates \((s, y)\) are valid, and \( \chi \in C^\infty(Z) \) a cutoff function with \( \text{supp}(\chi) \subset W \) and \( \chi = 1 \) in some neighborhood of \( \Gamma \). Then we define the sequence \( \psi_k \in L^2(Z)_k \) by
\[ \psi_k(s, y) = \Lambda_k \chi e^{iks}[U_0 + \kappa^{-1} U_1], \]
where \( U_j(s, y) \) are the solutions obtained above, \( \kappa = k + n/2 \), and \( \Lambda_k \) normalizes \( \| \psi_k \| = 1 \). This could be written as
\[ \psi_k(s, y) = \Lambda_k \chi e^{iks} [P_0 + P_2(y) + \kappa P_4(y)] e^{-\kappa y^2/4}, \]
where \( P_l \) is a polynomial of degree \( l \) (with coefficients independent of \( k \)). Since \( P_0 = 1 + O(k^{-1}) \), we have that
\[ \Lambda_k \sim \left( \frac{k}{2\pi} \right)^{n/2} \text{ as } k \to \infty. \]
The concentration of \( \psi_k \) on \( \Gamma \) described in \((3.28)\) then follows immediately from \((3.29)\).

By virtue of the factor \( e^{-\kappa y^2/4} \), we can turn the formal considerations used to obtain the operators \( \mathcal{L}_j \) into estimates. With cutoff, \( \chi \mathcal{L}_j \) could be considered an operator on \( Z \) with support in \( W \). By construction we have
\[ \chi \left[ e^{iks} \Delta Z e^{iks} - \kappa^2 - \kappa \mathcal{L}_0 - \sqrt{\mathcal{L}_1} - \mathcal{L}_2 \right] = \sum_{l, m, |\beta| \leq 2} E_{l, m, \beta}(s, y) \kappa^l \partial_s^m \partial_y^\beta, \]
where \( A_{l, m, \beta} \) is supported in \( W \) and vanishes to order \( 2l + |\beta| + 1 \) at \( y = 0 \). We also have
\[ (\kappa \mathcal{L}_0 + \sqrt{\kappa} \mathcal{L}_1 + \mathcal{L}_2 - \sigma)(U_0 + \kappa^{-1} U_1) = \kappa^{-1} (\sqrt{\kappa} \mathcal{L}_1 + \mathcal{L}_2 - \sigma) U_1 \]
Combining these facts with the definition of \( \psi_k \) we deduce that
\[ (\Delta Z - \kappa^2 - \sigma) \psi_k(s, y) = \Lambda_k \sum_{l \leq 4} k^l F_l(s, y) e^{-\kappa y^2/4}, \]
where \( F_l \) is supported in \( W \) and vanishes to order \( 2l + 1 \) at \( y = 0 \). Using this order of vanishing we estimate
\[ \left\| \Lambda_k k^l F_l e^{-\kappa y^2/4} \right\|^2 = O(k^{-1}). \]
Noting that $\Delta_Z - \kappa^2 - \sigma = \Delta_h - nk - h(x_0)$ on $L^2(Z)_k$, we obtain the estimate (3.27).

4. Spectral density function

Let $\psi_k \in L^2(Z)_k$ be the sequence produced by Proposition 3.5. As in §2, we let $\Pi_k$ denote the orthogonal projection onto the span of low-lying eigenvectors of $\Delta_h - nk$. Consider

$$\phi_k = \Pi_k \psi_k \quad \eta_k = (I - \Pi_k) \psi_k.$$ 

By Theorem 1.1 (for $k$ sufficiently large, which we’ll assume throughout),

$$\| (\Delta_h - nk) \phi_k \| < M, \quad \| (\Delta_h - nk) \eta_k \| > ak \| \eta_k \|.$$

By Proposition 3.5 we have a uniform bound

$$\| (\Delta_h - nk) \psi_k \| \leq C,$$

so these estimates imply in particular that

$$ak \| \eta_k \| < C + M.$$

Hence $\| \eta_k \| = O(k^{-1})$.

From Lemma 2.1 we know that $q$ satisfies

$$\langle \phi_k, (\Delta_h - nk - \pi^* q) \phi_k \rangle = O(1/k).$$

Let $r_k = (\Delta_h - nk + h(x_0)) \psi_k$, which by Proposition 3.5 satisfies $\| r_k \| = O(k^{-1/2})$. So

$$\langle \phi_k, (\Delta_h - nk - \pi^* q) \phi_k \rangle$$

$$= \langle \phi_k, (h(x_0) - \pi^* q) \phi_k \rangle + \langle \phi_k, (\Delta_h - nk - h(x_0)) \phi_k \rangle$$

$$= \langle \phi_k, (h(x_0) - \pi^* q) \phi_k \rangle + \langle \phi_k, r_k \rangle - \langle \phi_k, (\Delta_h - nk - h(x_0)) \eta_k \rangle.$$ 

The left-hand side is $O(1/k)$, while the second term on the right is $O(k^{-1/2})$. The third term on the right-hand side is equal to

$$\langle (\Delta_h - nk) \phi_k, \eta_k \rangle < M \| \eta_k \| = O(k^{-1}).$$

Therefore, the first term on the right-hand side of (4.1) can be estimated

$$\langle \phi_k, (h(x_0) - \pi^* q) \phi_k \rangle = O(k^{-1/2}).$$

Because $\| \eta_k \| = O(1/k)$ this implies also that

$$h(x_0) - \langle \psi_k, (\pi^* q) \psi_k \rangle = O(k^{-1/2}).$$

Since $q$ is smooth, the localization of $\psi_k$ on $\Gamma$ from Proposition 3.5 implies that

$$\langle \psi_k, (\pi^* q) \psi_k \rangle = q(x_0) + O(k^{-1/2}).$$

Thus $q(x_0) = h(x_0)$. This proves Theorem 1.2.
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