Aspects of determining $f_{B_s}$: scaling and power-law divergences

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We present preliminary results for the decay constant of the $B_s$ meson, $f_{B_s}$, at three values of $\beta = 5.7, 6.0$ and 6.2 using NRQCD and clover fermions for the heavy and light quarks respectively. As a consistency check the decay constant has also been extracted from the axial-vector current at finite momentum. In addition, we discuss the cancellation of $O(\alpha/(aM_0))$ terms and the remaining uncertainty in $f_{B_s}$ from higher order divergences.

1. Simulation details

For the heavy quark we use an NRQCD action consistent to $O(1/M_0^3)$:

$$S = Q \cdot (\Delta_t + H_0 + \delta H) Q$$

where,

$$H_0 = -\frac{\Delta^{(2)}}{M_0^2}$$

$$\delta H = -c_1 \frac{\sigma \cdot B}{2M_0} + c_2 \frac{(\Delta \cdot E - E \cdot \Delta)}{8M_0^2} - c_3 \frac{\sigma \cdot (\Delta \times E - E \times \Delta)}{8M_0^2} - c_4 \frac{2 a (\Delta^{(2)})^2}{16nM_0^2}$$

The $O(1/M_0^3)$ correction to the kinetic energy (expected to be the largest contribution from this order) and the first two discretisation corrections are also included. We implement tadpole improvement throughout, using the plaquette definition of $u_0$, and set $c_1 = 1$.

For the light quark we use the clover action with the tadpole improved value of $c_{SW}$ at $\beta = 5.7$ and 6.0 and the non-perturbatively determined value at $\beta = 6.2$; $c_{SW} = 1.61$ as determined by the Alpha collaboration, compared to $c_{SW} = 1.48$ using tadpole-improvement. The configurations at $\beta = 5.7$ and the configurations and light quark propagators at $\beta = 6.2$ were generously provided by the UKQCD collaboration.

| $\beta$ | $V$ | $N$ | $a^{-1}(m_\rho)$ (GeV) | $aM_0^3$ |
|---------|-----|-----|------------------------|----------|
| 5.7     | $12^4 \times 24$ | 278 | 1.14(3)                | 4.2      |
| 6.0     | $16^3 \times 48$ | 102 | 1.92(7)                | 2.2      |
| 6.2     | $24^3 \times 48$ | 144 | 2.63(9)                | 1.6      |

Table 1

The simulation details. The errors on $a^{-1}$ include statistical errors and those due to the chiral extrapolation of $m_\rho$. $N$ denotes the number of configurations.

The pseudoscalar decay constant is defined as

$$\langle 0 | A_\mu | PS \rangle_{QCD} = p_\mu f_{PS}$$

in Euclidean space. On the lattice matching factors $C_i$ relate the lattice operators to the current in full QCD. For the zeroth component of the current to $O(\alpha/M)$:

$$\langle A_0 \rangle_{QCD} = \sum_j C_j(\alpha, aM_0)(J_{L}^{(0)})$$

where,

$$O(1): J_{L}^{(0)} = \bar{q} \gamma_5 \gamma_0 \gamma Q$$

$$O(\frac{1}{M}): J_{L}^{(1)} = -\frac{1}{2M_0} \bar{q} \gamma_5 \gamma_0 (\gamma \cdot D)Q$$

$$O(\frac{\alpha}{M}): J_{L}^{(2)} = \frac{1}{2M_0} \bar{q} (\gamma \cdot D) \gamma_5 \gamma_0 Q.$$
Figure 1. Preliminary results for $f_{B_s}$ as a function of the lattice spacing. Our results (circles) are compared to those from JLQCD (squares) [1]. All errors include statistical and systematic uncertainties added in quadrature.

The $O(\alpha \alpha)$ discretisation error in the current is removed by defining [3],

\[
J_{L,imp}^{0} = J_{L}^{(0)} + C_{\alpha} J_{L}^{(disc)}
\]
\[
J_{L}^{(disc)} = a\bar{q}(\gamma \cdot D)\gamma_{5}\gamma_{0}Q
\]

The $C_{\alpha}$s have been calculated to 1-loop [3], however, the $q^*$ (which also depends on $aM_0$) at which the strong coupling, $\alpha$, is computed is not yet known. Thus, we average the results obtained using $aq^* = 1.0$ and $\pi$. Note that in the static limit $aq^* \sim 2$ [3] for Wilson light fermions.

The results at $\beta = 5.7$ and 6.0 have appeared previously in [4] and [5] respectively. Further details of our methods and analysis can be found in these references.

2. Scaling of $f_{B_s}$

Our results for $f_{B_s}$, calculated to $O(\alpha/M)$, are presented in figure 2 as a function of $a^{-1}$. The results are consistent with scaling, and $f_{B_s} \sim 200$ MeV. Nice agreement is found with the results of the JLQCD collaboration [3], also shown in the figure. This group uses the string tension to set $a$, and an $O(1/M_0)$ NRQCD action. However, this is unlikely to affect the comparison significantly.

Table 2 details how the errors are estimated.

| Source | $\beta = 5.7$ | 6.0 | 6.2 |
|--------|--------------|-----|-----|
| statistical | 3 | 3 | 2 |
| disc. $O((a\Lambda_{QCD})^2)$ | 13 | 4 | 2 |
| pert. $O(\alpha^2, \alpha/(aM)^2)$ | 13 | 8 | 9 |
| NRQCD $O(1/M^2)$ | 1 | 1 | 1 |
| $\kappa_s$ | +4 | +4 | +4 |
| $a^{-1}(m_\rho)$ | 3 | 4 | 3 |
| **Total** | **19** | **11** | **11** |

Table 2

Estimates of the statistical and main systematic errors, in percent, in our values for $f_{B_s}$.

Note that $a^{-1} \sim 1-2.6$ GeV is the range in which the NRQCD action and clover light fermions can be applied to the $B$ meson. For coarser values of $a$, the discretisation errors from the light quark action (and gauge action) increase rapidly, as do the $O(\alpha^2)$ perturbative errors. Conversely, if $a$ becomes too fine, the discretisation errors are under control but the $O(\alpha/(aM)^2)$ perturbative errors increase dramatically as $aM_0$ drops below 1. Overall, the perturbative errors are the main source of uncertainty, although at $\beta = 5.7$ the discretisation errors are of equal magnitude.

3. $f_{B_s}$ extracted at finite momentum

In order to investigate momentum dependent discretisation errors in the decay constant we computed the ratio

\[
\langle J_{L,imp}^{(0)} \rangle_{\vec{p}} / \langle J_{L}^{(0)} \rangle_{\vec{p}} = \sqrt{E(\vec{p}) / M}
\]

where $\langle J_{L,imp}^{(i)} \rangle_{\vec{p}} = f_1 \sqrt{E(\vec{p})}$ (without renormalisation). The RHS of equation 11 is a slowly varying function of $|\vec{p}|$, which is close to 1 for the range of momenta we studied (up to 1.5 GeV at $\beta = 6.2$). Our results, presented in table 2, are in agreement with this expectation at $\beta = 6.0$ and 6.2, i.e. the momentum dependent discretisation errors in $f_{B_s}$ are not significant. However, at $\beta = 5.7$ a 10–20% deviation from 1, is seen as $|\vec{p}|$ increases, although this is within the magnitude expected for $O((a \rho)^2)$ discretisation errors.

4. Power-law divergences

Matrix elements in NRQCD beyond zeroth order diverge as $aM_0 \to 0$. In the case of $f_{B_s}$
there are unphysical, ultra-violet, contributions to $\langle J^{(i)}_L \rangle$, $i > 0$, which are cancelled order by order in perturbation theory by terms appearing in the perturbative coefficients. In general, simulations are performed using $aM_0 > 1$ and so the unphysical contributions are not expected to be large and their cancellation should be under control.

Considering equation 10 in more detail (see 12 for definitions),

$$
\langle A_0 \rangle_{QCD} = (1 + \alpha\rho_0)\langle J^{(0)}_L \rangle + \langle J^{(1)}_L \rangle + \alpha\rho_1\langle J^{(1)}_L \rangle + \alpha\rho_2\langle J^{(2)}_L \rangle + \alpha\rho_{disc}\langle J^{(disc)}_L \rangle
$$

(12)

The explicit contributions to $\rho_0$ are

$$
\rho_0 = [B_0 - \frac{1}{2}(C_q + C_Q) - \zeta_{10} - \zeta_{10}].
$$

(13)

The lowest order divergent contribution to the current is $O(\alpha/(aM))$, which appears through the tree-level term $\langle J^{(1)}_L \rangle$ and is cancelled by the mixing term $\alpha\zeta_{10}\langle J^{(0)}_L \rangle$, where $\zeta_{10}$ is the renormalisation due to the mixing between $J^{(0)}_L$ and $J^{(1)}_L$. The remaining divergent contributions, $O(\alpha/(aM)^2, \alpha^2/(aM))$ etc, appearing in equation 12 are cancelled at higher orders; the uncertainty in $f_{B_s}$ due to these remaining terms is well within our estimates of the systematic uncertainties in table 3.

In fact a large part of $\langle J^{(1)}_L \rangle$ is unphysical, $\alpha\zeta_{10}\langle J^{(0)}_L \rangle/\langle J^{(1)}_L \rangle \sim 0.5 - 0.6$ for all $\beta_3$ and $aM_0$ for $aq^* = 2.0$. However, once the $O(\alpha/(aM))$ contribution to $\langle J^{(1)}_L \rangle$ is cancelled, the scaling behaviour of this term improves, as shown in figure 2 suggesting the remainder is physical.

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**REFERENCES**

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