Calculus of the fractional order operators in a discrete time domain

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Abstract. The article presents the elementary theory of differential and integral operators of the fractional order in a discrete time approach. A notion of a simple proper fraction operator has been introduced. It has been done for the time equivalent by applying the Taylor series. On this basis a new theory of a certain complexity operators has been formed which includes differential operators of the fractional order. Somewhat more general approach has been presented in the later part of the article by introducing a rational power of the convolution operator. Both approaches to the fractional operators are realized by non–recursive digital filters of infinite impulse responses. The stability of such filters is also being considered. The article also contains the application to the distributed parameters electrical circuits theorem.

Key words: operational calculus, digital filters, fractional order operator, convolution operator, discrete time.

1. Introduction

The fractional order differential–integral calculus has been known for over 300 years. Eminent mathematicians of the era – Leibniz and Newton [27] (their famous calculus wars [3]) or more recent as Hadamard [6] took part in the creation of this calculus. The history of creation and development of fractional calculus can be found in literature [4]. However, the application of fractional order differential–integral calculus is a relatively recent part of this mathematical discipline in which [2,7,11–12] can be considered as groundbreaking work and [1,5,8–10,13,26,28] as significant achievement.

This publication presents the mathematical convolution method as well as digital filter impulse response method with the application for distributed parameters system analysis in the theory of electrical circuits. The authors’ previous studies in this field can be found in publications [14–23].

The rest of this work is organized as follows. Section 2 introduces the theory of a fractional order differential operator using digital filters (discrete time domain) as well as discussion on stability of such filters. Section 3 proposes the use of differential operator in the theory of electric transmission line. Section 4 extends the fractional order differential operator to the rational order differential operator.

2. Theory of a differential operator of a non–integer order

The differential operator of an integer order ( p is a natural number) may be presented in the following discrete time form:

\[ f(z) = (1-z^p)^n = \sum_{n=0}^{\infty} \frac{d^n f(z)}{dz^n} \bigg|_{z=0} z^n = \]

\[ = \sum_{n=0}^{\infty} (-1)^n \frac{p!}{m!(p-m)!} z^n = \sum_{n=0}^{\infty} (-1)^n \frac{p!}{m!(p-m)!} \sum_{m=0}^{n} h_m z^m \]

where z is a unit delay operator (a complex variable). As a matter of fact, it is a well–known Newton formula [27] and also a FIR digital filter which is always BIBO–stable.

For \( p = -1 \) this operator becomes an integral operator which can be calculated from the inverse filter:

\[ \frac{1}{a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n} = \]

\[ = h_0 + h_1 z + h_2 z^2 + \ldots + h_n z^n \]

The above implies a recursive formula for the impulse samples of the inverse filter:

\[ a_0 h_0 = 1 \]
\[ a_0 h_1 + a_1 h_0 = 0 \]
\[ a_0 h_2 + a_1 h_1 + a_2 h_0 = 0 \]
\[ a_0 h_3 + a_1 h_2 + a_2 h_1 + a_3 h_0 = 0 \]
\[ \ldots \]
\[ \sum_{n=0}^{m} a_n h_{m-n} = 0 \quad \text{for} \quad m = 1,2,\ldots \]

Applying the inverse formulas (2) for the \((1-z)^{-1}\) filter the integral filter is obtained:

\[ (1-z)^{-1} = 1 + z + z^2 + \ldots + z^n + \ldots \] (3)

In fact, the formula (3) may be calculated from a geometric sequence which is convergent for \(|z|<1\) or by applying the Taylor series:

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where:

\[
f(z) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m f(z)}{dz^m} z^n = \sum_{m=0}^{\infty} h_m z^m
\]  
(4)

In fact, the function:

\[f(z) = (1-z)^{-1}\]

used in the formula (4) gives the result:

\[
h_m = \frac{1}{m!} \left. \frac{d^m (1-z)^{-1}}{dz^m} \right|_{z=0} = \frac{(-1)^m m!}{p(p-1)(p-2)\ldots[p-(m-1)]} = \frac{(-p)^m}{m!} \frac{1}{2} \frac{3-p}{3} \ldots \frac{m-1-p}{m} = \frac{(-p)^m}{m!} K_m
\]  
(8)

for \( m \geq 0 \), or in a recursive way:

\[
h_{0} = 0 \quad \text{for} \quad n < 0
\]

\[
h_{1} = 1 \quad \text{for} \quad n = 0
\]

\[
h_{n} = \frac{n-1-p}{n} h_{n-1} \quad \text{for} \quad n > 0
\]

The integral filter is not a recursive one. It is not a FIR filter. Thus it is not a BIBO-stable (it has infinite impulse response). However, it may be a stable filter when it is assumed that the input signal is limited to a finite time period. In such a case the transient operator has a form of a finite convolution:

\[
y_a = \sum_{m=-\infty}^{\infty} h_{m-a} x^n \rightarrow \sum_{m=0}^{\infty} h_{m-a} z^m
\]  
(5)

where the input signal \( \{x_n\} \) fulfills the condition:

\[x_n = 0 \quad \text{for} \quad n \in \{0,1,2,\ldots,N-1\}\]

A detailed development of the formula (5) renders:

\[
y_0 = h_0 x_0
\]

\[
y_1 = h_1 x_0 + h_0 x_1
\]

\[
y_2 = h_2 x_0 + h_2 x_1 + h_1 x_2
\]

\[
\ldots
\]

\[
y_{N-1} = h_{N-1} x_0 + h_{N-1} x_1 + \ldots + h_1 x_{N-1}
\]

\[
y_N = h_N x_0 + h_N x_1 + \ldots + h_N x_N
\]

\[
y_{N+1} = h_{N+1} x_0 + h_{N+1} x_1 + \ldots + h_{N+1} x_{N+1}
\]

\[
\ldots
\]

\[
y_{N+n} = h_{N+n} x_0 + h_{N+n} x_1 + \ldots + h_{N+n} x_{N+n}
\]

\[
\ldots
\]

The expansion of (6) implies that in order to ensure the stability of the filter (5) it is sufficient to limit the impulse output. Absolute summation is not necessary.

The generalization of the expansion (1) as a Taylor series (4) allows for a definition of a differential operator of the fraction order \( 0 < p < 1 \):

\[
f(z) = (1-z)^{-p} = \sum_{m=0}^{\infty} \frac{1}{m!} \left. \frac{d^m f(z)}{dz^m} \right|_{z=0} z^m = \sum_{m=0}^{\infty} h_m z^m
\]  
(7)

where:

\[
h_{m} = \frac{1}{m!} \left. \frac{d^m f(z)}{dz^m} \right|_{z=0} = \frac{(-p)^m}{m!} \frac{1}{2} \frac{3-p}{3} \ldots \frac{m-1-p}{m} = \frac{(-p)^m}{m!} K_m
\]

The examination of the formula above is done through the integral criteria:

\[
K(y) = \exp \left[ \int_{y}^{\infty} \ln \left( \frac{x-p}{x} \right) dx \right] = \alpha \exp \left[ y \ln \left( \frac{y-p}{y} \right) \right] \exp \left[ -p \left( \ln \left( y-p \right) - 1 \right) \right]
\]  
(9)

where:

\[
\alpha = \exp \left[ -p \left( 1-p \right) \ln \left( 1-p \right) \right]
\]

In order to estimate the value of the coefficient the following expression is extended in a power series:

\[
\frac{1}{1-p} = 1 + p + p^2 + p^3 + \ldots
\]

and after the integration:

\[-\ln \left( 1-p \right) = p + \frac{p^2}{2} + \frac{p^3}{3} + \frac{p^4}{4} + \ldots\]

hence:

\[-p \left( 1-p \right) \ln \left( 1-p \right) = \left( \frac{1}{2} - 1 \right) p^2 + \left( \frac{1}{3} - \frac{1}{2} \right) p^3 + \left( \frac{1}{4} - \frac{1}{3} \right) p^4 + \left( \frac{1}{5} - \frac{1}{4} \right) p^5 + \ldots < 0\]
thus \( \alpha < 1 \). The remaining elements in (9) behave under the condition \( y \to \infty \) in the following way:

\[
\exp\left[y \ln \left(\frac{y-p}{y}\right)\right] \xrightarrow{y \to \infty} \exp(-p)
\]

and:

\[
\exp\left[-p \left[\ln(y-p) - 1\right]\right] \xrightarrow{y \to \infty} \exp[-p \ln(y)] = \exp[\ln(y^{-p})] = y^{-p}
\]

Thus, the integral criterion when applied for the function (9) renders:

\[
\int_{1}^{\infty} K(y) \frac{dy}{y} = \alpha' \int_{1}^{\infty} y^{-(1+p)} dy = \frac{\alpha'}{p} < \infty
\]

It means that the digital filter (7) as a discrete–time model of a differential operator of a fractional order is BIBO–stable.

By applying the Taylor expansion in the series (7) for a definition of a fraction order integral operator:

\[
0 < p \leq 1
\]

it is obtained:

\[
(1-z)^p = \sum_{n=0}^{\infty} h_n z^n
\]

where:

\[
h_0 = 1
\]

and for \( m > 0 \):

\[
h_m = \frac{p}{m} \frac{1+\frac{p}{2}}{\frac{3}{2}} \frac{p^{\frac{3}{2}}}{\frac{4}{3}} \frac{\ldots}{m} \frac{m-1+\frac{p}{m}}{m} = \frac{m-1+\frac{p}{m}}{m} h_{m-1}
\]

Each element in the product (11) belongs to the range \((0,1)\). It means that the sequence \( \{h_n\} \) is limited. However, it is not given as an absolute sum. Thus, the digital filter (10) is stable upon a condition only.

A periodical, differential filter defined by the operator \((0 < p < 1)\) is a certain generalization:

\[
(a-z)^p = a^p \left(1 - \frac{z}{a}\right)^p = \sum_{n=0}^{\infty} h_n z^n
\]

where:

\[
h_n = (-p)^{\frac{m-1-p}{m}} \frac{1}{2} \frac{2-p}{3} \frac{3-p}{4} \ldots \frac{m-1-p}{m} a^{-p-n}
\]

and an appropriate integral filter \((0 < p \leq 1)\):

\[
(a-z)^{-p} = a^{-p} \left(1 - \frac{z}{a}\right)^{-p} = \sum_{n=0}^{\infty} h_n z^n
\]

where:

\[
h_n = p^{\frac{1+p}{2}} \frac{2+p}{3} \frac{3+p}{4} \ldots \frac{m-1+p}{m} a^{p-n}
\]

Both filters (12) and (14) are BIBO–stable when:

\[
|\alpha| > 1
\]

The expressions (12)–(15) may be treated together for differential filters \((-1 \leq p \leq 1)\):

\[
(a-z)^p = \sum_{n=0}^{\infty} h_n z^n
\]

where:

\[
h_n = (-p)^{\frac{m-1-p}{m}} \frac{1}{2} \frac{2-p}{3} \frac{3-p}{4} \ldots \frac{m-1-p}{m} a^{-p-n}
\]

for \( m \geq 0 \) and:

\[
h_n = 0
\]

for \( m < 0 \).

The joined results of (1) and (7)–(8) help to find a differential operator of an improper fraction:

\[
f(z) = (1-z)^N (1-z)^p = \sum_{n=0}^{\infty} k_n z^n
\]

where \( N \) is an integer positive number (an integral part of the order of the operator) and \( p \) is a proper fraction of the operator \((0 < p < 1)\). When applying a partial decomposition:

\[
(1-z)^p = \sum_{n=0}^{\infty} (-1)^n \left(\frac{N}{m}\right) z^n = \sum_{n=0}^{\infty} g_n z^n
\]

where:

\[
g_m = \begin{cases} \frac{N!}{m!(N-m)!} & \text{for } m \in \{0,1,\ldots,N\} \\ 0 & \text{for } m \not\in \{0,1,\ldots,N\} \end{cases}
\]

and:

\[
(1-z)^p = \sum_{n=0}^{\infty} h_n z^n
\]

where:

\[
h_n = \begin{cases} 0 & \text{for } n < 0 \\ 1 & \text{for } n = 0 \\ \frac{n-1-p}{n} h_{n-1} & \text{for } n > 0 \end{cases}
\]

and after composing the filters (17) and (18) the filter (16) is obtained in which the convolution:
$k_n = 0 \quad \text{for } n < 0$

$k_0 = g_0 h_0$

$k_1 = g_0 h_1 + g_1 h_0$

$k_2 = g_0 h_2 + g_1 h_1 + g_2 h_0$

\[ \cdots \]

$k_N = g_0 h_N + g_1 h_{N-1} + \cdots + g_N h_0$

$k_{N+1} = g_0 h_{N+1} + g_1 h_N + \cdots + g_N h_1$

\[ \cdots \]

$k_{N+n} = g_0 h_{N+n} + g_1 h_{N+n-1} + \cdots + g_N h_n$

Due to the fact, that one of the filters is a FIR filter each element $k_n$ is calculated as a finite sum. Yet, the given filter is not a FIR filter but a BIBO-stable one (a composition of BIBO-stable filters is still BIBO-stable).

For example, for a differential operator of the $p = \frac{1}{2}$ order a digital filter with an impulse response is obtained:

\[ \{ h_n \} = \{ 1, -0.5000, -0.1250, -0.0625, -0.0391, -0.0273, -0.0205, -0.0161, -0.0131, -0.0109, -0.0093, -0.0080, -0.0070, -0.0062, -0.0055, \ldots \} \]

The square value of the differential operator of the $\frac{1}{2}$ order should render an operator of the first order. This means that a square value of the convolution is obtained:

\[ h_0 h_0 = 1.0000 \]

\[ h_0 h_1 + h_1 h_0 = -1.0000 \]

\[ h_0 h_2 + h_1 h_1 + h_2 h_0 = 0.0000 \]

\[ h_0 h_3 + h_1 h_2 + h_2 h_1 + h_3 h_0 = 0.0000 \]

\[ h_0 h_4 + h_1 h_3 + h_2 h_2 + h_3 h_1 + h_4 h_0 = -0.0001 \]

\[ \cdots \]

Fig. 1. Impulse responses of the operators: a) differential of the 1st order, b) differential of the $\frac{1}{2}$ order, c) integral of the 1st order, d) integral of the $\frac{1}{2}$ order.

| n | p = 0.3 | p = 0.5 | p = 0.7 |
|---|--------|--------|--------|
|   | Diff. | Int. | Diff. | Int. | Diff. | Int. |
| 0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 1 | -0.3000 | 0.3000 | -0.5000 | 0.5000 | -0.7000 | 0.7000 |
| 2 | -0.1050 | 0.1950 | -0.1250 | 0.3750 | -0.1050 | 0.5950 |
| 3 | -0.0595 | 0.1495 | -0.0625 | 0.3125 | -0.0455 | 0.5355 |
| 4 | -0.0402 | 0.1233 | -0.0391 | 0.2734 | -0.0262 | 0.4953 |
| 5 | -0.0297 | 0.1060 | -0.0274 | 0.2461 | -0.0173 | 0.4656 |
| 6 | -0.0233 | 0.0936 | -0.0205 | 0.2256 | -0.0124 | 0.4423 |
| 7 | -0.0190 | 0.0842 | -0.0161 | 0.2095 | -0.0094 | 0.4233 |
| 8 | -0.0159 | 0.0768 | -0.0131 | 0.1964 | -0.0074 | 0.4074 |
| 9 | -0.0136 | 0.0708 | -0.0109 | 0.1855 | -0.0060 | 0.3938 |
| 10 | -0.0118 | 0.0658 | -0.0093 | 0.1762 | -0.0050 | 0.3820 |
| 11 | -0.0104 | 0.0616 | -0.0080 | 0.1682 | -0.0042 | 0.3716 |
| 12 | -0.0093 | 0.0580 | -0.0070 | 0.1612 | -0.0036 | 0.3623 |
| 13 | -0.0084 | 0.0549 | -0.0062 | 0.1550 | -0.0031 | 0.3539 |
| 14 | -0.0076 | 0.0522 | -0.0055 | 0.1495 | -0.0027 | 0.3463 |

3. Linear equations with differential operators of non–integer order and their application in the analysis of distributed parameters circuits

It is difficult to find a general formula for linear equations with a differential operator of the fraction order. The reason is that the polynomial method which is used in a classical theory of differential operators does not work here. The notion of a rational function does not apply here either. For linear equations with differential operator as a combination of a product of operators (12) it means $(a - z)^{\frac{p}{n}}$. However, some generalized form of the equation can be formulated as a combination of the products of the operators of the (12) type, i.e. $(a - z)^p$ which map the digital filters of the impulse responses denoted by:

\[ h_n = \begin{cases} 0 & \text{for } n < 0 \\ a^n & \text{for } n = 0 \end{cases} \] (19)

\[ h_n = \frac{n-1-p}{na} h_{n-1} \quad \text{for } n > 0 \]

for $0 < p < 1$. This combination is made of the integral operator (14) $(a - z)^{-p}$ for which:

\[ h_n = \begin{cases} 0 & \text{for } n < 0 \\ a^{-n} & \text{for } n = 0 \end{cases} \] (20)

\[ h_n = \frac{n-1+p}{na} h_{n-1} \quad \text{for } n > 0 \]

for $0 < p \leq 1$ and the operators of the integer order, i.e. the integer (positive and negative) power of the operator $(a - z)$ for which $0 < p \leq 1$.

It may be an equation with a linear combination of the fractional operators of the type:
\[
\frac{(a_i - z)^{p_i}}{(b_i - z)^{q_i}} = f(z)
\]

(21)

where \(p_i, q_i\) are any given positive numbers. In this way, a differential equation may take the following form:

\[
\left( \sum c_i f(z) \right) y = x
\]

(22)

where \(f(z)\) are fraction functions of the \((21)\) type and \(x, y\) are the input and output signals. In order to solve equation (22) the impulse function \(h_n'\) should be found which will correspond to the fraction (21). It will be a combination of functions (19) and (20). Next, there should follow a juxtaposition of the impulse functions which create the operator on the left side of the equation (22). In order to find the solution of the differential equation (22) the inverse operator should also be determined according to the inverse formula (2).

3.1 The operator of the wave impedance. Differential equations of the transmission line:

\[
\frac{\partial u}{\partial x} = R i + L \frac{\partial i}{\partial t}
\]

\[
\frac{\partial i}{\partial x} = G u + C \frac{\partial u}{\partial t}
\]

take the operator forms:

\[
\frac{du}{dt} = (R + sL)i
\]

\[
\frac{di}{dx} = (G + sC)u
\]

(23)

or the form of one operator equation:

\[
\frac{du}{dt} = \left( \frac{R + sL}{G + sC} \right)^2 \frac{i}{u}
\]

or:

\[
u \frac{du}{dt} = \left( \frac{R + sL}{G + sC} \right)^2 i \frac{di}{dt}
\]

(24)

The integration of the differential equation (24) will render a formula with the wave impedance operator:

\[
u u = \frac{R + sL}{G + sC} i
\]

(25)

When applying the unit time delay \(z\) with a sample time \(\tau\), the wave impedance operator takes the form:

\[
\sqrt{\frac{R + sL}{G + sC}} = \sqrt{R + \frac{L}{\tau} (1 - z)} \sqrt{G + \frac{C}{\tau} (1 - z)} = \sqrt{\frac{a - z}{b - z}}
\]

where:

\[
a = 1 + \frac{\tau R}{L} > 1
\]

\[
b = 1 + \frac{\tau G}{C} > 1
\]

So, the wave impedance operator may be written with differential and integral operators of the \(\frac{1}{2}\) order:

\[
Z_f = \rho (a - z)^{\frac{1}{2}} (b - z)^{-\frac{1}{2}}
\]

where:

\[
\rho = \sqrt{\frac{L}{C}}
\]

is a wave resistance. The equation (25) takes now a form of an equation with differential operator of the \(\frac{1}{2}\) order:

\[
\rho (a - z)^{\frac{1}{2}} (b - z)^{-\frac{1}{2}} i = u
\]

A similar result for the wave impedance operator can be obtained by applying a bilinear transformation:

\[
\sqrt{\frac{R + sL}{G + sC}} = \sqrt{\frac{R + \frac{2L}{\tau} 1 - z}{G + \frac{2C}{\tau} 1 - z}} = \rho \sqrt{\frac{a - z}{b - z}}
\]

where:

\[
\rho = \sqrt{\frac{L}{C}} \sqrt{1 - \frac{\tau R}{2L}} \approx \sqrt{\frac{L}{C}}
\]

and:

\[
a = 1 + \frac{\tau R}{2L} > 1
\]

\[
b = 1 + \frac{\tau G}{2C} > 1
\]

3.2 The propagation operator. In the continuous–time domain the propagation operator, which appears in the solution of differential equations (23) has a form:
\[ \gamma l = l \sqrt{(R + sl)(G + sc)} = \]
\[ = l \sqrt{\frac{R + \frac{L}{\tau} (1 + z)}{G + \frac{C}{\tau} (1 + z)}} = (26) \]
\[ = \alpha (a - z)^{\frac{1}{2}} (b - z)^{\frac{1}{2}} \]

where:
\[ a = 1 + \frac{\tau R}{L} > 1, \quad b = 1 + \frac{\tau G}{C} > 1, \quad \alpha = \frac{l}{\sqrt{LC}} \]
in which \( \nu \) is the velocity of light and \( l \) is geometric length of transmission line.

When applying the bilinear transformation it is obtained:
\[ \gamma l = l \sqrt{\frac{R + \frac{L}{2\tau} (1 - z)}{G + \frac{C}{2\tau} (1 - z)}} = \]
\[ = \alpha (a - z)^{\frac{1}{2}} (b - z)^{\frac{1}{2}} \]

where:
\[ a = \frac{1 + 2\tau R}{1 - \frac{2\tau R}{L}} > 1, \quad b = \frac{1 + 2\tau G}{1 - \frac{2\tau G}{C}} > 1 \]
\[ \alpha = \frac{l}{\sqrt{LC}} \sqrt{\frac{1 - \frac{2\tau R}{L}}{1 - \frac{2\tau G}{C}}} = \frac{l}{2\nu} \]

The solutions of differential equations (23) with wave impedance, propagation operator and boundary conditions as signals \( u_1, u_2 \) can be presented in the following form:
\[ u_1 = \cosh \gamma l u_1 + Z_r \sinh \gamma l i_2 \]
\[ i_1 = Z_r^{-1} \sinh \gamma l u_2 + \cosh \gamma l i_2 \]

(27)

For the adjusted circuit at the end of transmission line:
\[ i_2 = Z_r' u_2 \]

(28)

Substituting (28) into (27) it is obtained:
\[ u_2 = \exp(-\gamma l) u_1 \quad \text{or} \quad u_1 = \exp(\gamma l) u_2 \]

(29)

The exponential operator (wave operator) from the equation (29) can be expanded into the following series:
\[ \exp(\gamma l) = 1 + \gamma l + \frac{1}{2} (\gamma l)^2 + \frac{1}{6} (\gamma l)^3 + \frac{1}{24} (\gamma l)^4 + \ldots \]
\[ \exp(-\gamma l) = 1 - \gamma l + \frac{1}{2} (\gamma l)^2 - \frac{1}{6} (\gamma l)^3 + \frac{1}{24} (\gamma l)^4 - \ldots \]

Thus, from (26) it is obtained:
\[ \exp(\gamma l) = 1 + \alpha (a - z)^{\frac{1}{2}} (b - z)^{\frac{1}{2}} + \frac{1}{2} \alpha^2 (a - z)(b - z) + \]
\[ + \frac{1}{6} \alpha^3 (a - z)^{\frac{3}{2}} (b - z)^{\frac{3}{2}} + \]
\[ + \frac{1}{24} \alpha^4 (a - z)^2 (b - z)^2 + \ldots \]

The wave impedance operator takes the form:
\[ (Z_n) = \rho \left[(a - z)^{\frac{1}{2}} (b - z)^{\frac{1}{2}}\right] = \rho \{a_n\} \ast \{b_n\} = \rho z_n \]

where the components (differential):
\[ \{a_n\} = \left\{(a - z)^{\frac{1}{2}}\right\}_n \]

and (integral):
\[ \{b_n\} = \left\{(b - z)^{\frac{1}{2}}\right\}_n \]

are collected in the table 2.

| \( n \) | \( \{a_n\} = \left\{(a - z)^{\frac{1}{2}}\right\}_n \) | \( \{b_n\} = \left\{(b - z)^{\frac{1}{2}}\right\}_n \) |
|-----|----------------|----------------|
| 0 | 1.0000 | 1.0000 |
| 1 | -0.5000 | 0.5000 |
| 2 | -0.1250 | 0.3750 |
| 3 | -0.0625 | 0.1250 |
| 4 | -0.0391 | 0.0781 |
| 5 | -0.0274 | 0.0454 |
| 6 | -0.0205 | 0.0293 |
| 7 | -0.0161 | 0.0209 |
| 8 | -0.0131 | 0.0164 |
| 9 | -0.0109 | 0.0135 |
| 10 | -0.0093 | 0.0120 |
| 11 | -0.0080 | 0.0110 |
| 12 | -0.0070 | 0.0104 |
| 13 | -0.0062 | 0.0100 |
| 14 | -0.0055 | 0.0100 |

\[ \{z_n\} \] operator, which is a convolution of other operators, is defined by the sequence:
\[ z_0 = \left(\frac{a}{b}\right)^{\frac{1}{2}} \]
\[ z_1 = -0.5(ab)^{\frac{1}{2}} + 0.5a^{\frac{1}{2}}b^{-\frac{1}{2}} \]
\[ z_2 = -0.125a^{-\frac{3}{2}} - 0.25a^{-\frac{1}{2}}b^{-\frac{1}{2}} + 0.375a^{\frac{1}{2}}b^{-\frac{3}{2}} \]
\[ z_3 = -0.062a^{-\frac{5}{2}}b^{-\frac{1}{2}} - 0.062a^{-\frac{3}{2}}b^{-\frac{3}{2}} \]
\[ -0.187a^{-\frac{1}{2}}b^{-\frac{5}{2}} + 0.312a^{\frac{3}{2}}b^{-\frac{5}{2}} \]

...
3.3 An infinite circuit. The Fig. 2 shows a diagram of an infinite homogenous electric circuit containing operators: horizontal \( r \) (resistance type) and vertical \( g \) (conductance type).

![Diagram of an infinite homogenous electric circuit](image)

Fig. 2. The infinite homogenous electric circuit.

The input impedance at any place of the chain fulfills the recursive equation:

\[
Z_{n+1} = r + \frac{1}{g} + \frac{1}{Z_n}
\]

where does the limit impedance of an infinite chain equation result:

\[
Z = r + \frac{1}{g} + \frac{1}{Z}
\]

or:

\[
gZ^2 - rgZ - r = 0
\]

The equation (34) has a solution:

\[
Z = \frac{1}{2} \left[ r + \frac{1}{g} \pm \sqrt{4 + rg} \right]
\]

![Diagram of an RLGC infinite homogenous electric circuit](image)

Fig. 3. The RLGC infinite homogenous electric circuit.

For the \( RLGC \) circuit operators \( r \) and \( g \) takes the form of differential operators:

\[
\begin{align*}
    r &\to R + sL \\
g &\to G + sC
\end{align*}
\]

hence:

\[
Z(s) = \frac{1}{2} L \left\{ \frac{(a + s)^{\frac{1}{2}}}{(b + s)^{\frac{1}{2}}} \left[ s^2 + (a + b)s + ab + 4\omega^2 \right] \right\}^{\frac{1}{2}}
\]

where:

\[
a = \frac{R}{L}, \quad b = \frac{G}{C}, \quad \omega^2 = \frac{1}{LC}
\]

The second–degree polynomial in expression (35) has a decomposition into two conjugated complex roots:

\[
s^2 + (a + b)s + ab + 4\omega^2 = (\sigma + s)(\sigma^* + s)
\]

where:

\[
\sigma = \frac{a + b}{2} + j\sqrt{\left(2\omega^2 - \frac{a - b}{2}\right)}
\]

so the function (35) takes the form:

\[
Z(s) = \frac{1}{2} L \left( a + s \right)^{\frac{1}{2}} \left( b + s \right)^{\frac{1}{2}} \left( \sigma + s \right)^{\frac{1}{2}} \left( \sigma^* + s \right)^{\frac{1}{2}}
\]

\[
\cdot \frac{\left( \sigma + s \right)^{\frac{1}{2}} \left( \sigma^* + s \right)^{\frac{1}{2}} + \left( a + s \right)^{\frac{1}{2}} \left( b + s \right)^{\frac{1}{2}}}{b + s}
\]

Digital transformation of the components of the function (36) gives:

\[
(a + s)^{\frac{1}{2}} \to a + \frac{1}{\tau}(1 - z)^{\frac{1}{2}} = \frac{1}{\sqrt{\tau}}(1 + \sigma \tau - z)^{\frac{1}{2}}
\]

so the function (36) takes the digital form:

\[
Z = \frac{1}{2} \frac{L}{\tau} \left( \hat{a} - z \right)^{\frac{1}{2}} \left( \hat{b} - z \right)^{\frac{1}{2}} \cdot \left( \hat{\sigma} - z \right)^{\frac{1}{2}} \left( \hat{\sigma}^* - z \right)^{\frac{1}{2}} + \left( \hat{a} - z \right)^{\frac{1}{2}} \left( \hat{b} - z \right)^{\frac{1}{2}}
\]

\[
\cdot \frac{\left( \hat{\sigma} - z \right)^{\frac{1}{2}} \left( \hat{\sigma}^* - z \right)^{\frac{1}{2}} + \left( \hat{a} - z \right)^{\frac{1}{2}} \left( \hat{b} - z \right)^{\frac{1}{2}}}{\hat{b} - z}
\]

where:

\[
\hat{a} = 1 + a\tau > 1 \\
\hat{b} = 1 + b\tau > 1 \\
\hat{\sigma} = 1 + \sigma\tau, |\sigma| > 1
\]

The result (37) is known for the presence of conjugated complex roots. Under formula (12) it occurs:

\[
(\sigma - z)^p = \sum_{n=0}^{\infty} k_n \sigma^n z^m
\]

where for \( 0 < p < 1 \):

\[
k_n = \frac{\left( \frac{2}{1} - p \right) \left( \frac{3}{2} - p \right) \ldots \left( \frac{2m - 1}{m} - p \right)}{m}
\]

thus:

\[
(\sigma - z)^p (\sigma^* - z)^p = \sum_{n=0}^{\infty} h_n z^m
\]

where:
4. Rational power of the convolution operator

Assuming that \( h \) is a deterministic signal of the discrete–time type, such as \( h_0 = 1 \):

\[
(h * h)_a = \sum_{m=0}^{n} h_{n-m}h_m = 2h_n + \sum_{m=1}^{n-1} h_{n-m}h_m
\]

The cube convolution is given by the following expression:

\[
(h * h * h)_a = \sum_{m=0}^{n} h_{n-m}(h * h)_m = h_n(h * h)_0 + h_0(h * h)_n + \sum_{m=1}^{n-1} h_{n-m}(h * h)_m = 3h_n + \sum_{m=1}^{n-1} h_{n-m}(h * h)_m
\]

By introducing a symbol of \( \mathcal{G} \)–times convolution power ( \( \mathcal{G} \) is positive integer number):

\[
h_n^\mathcal{G} = h * h * \ldots * h
\]

the following identity can be obtained:

\[
h_n^\mathcal{G} = \mathcal{G}h_n + \sum_{m=1}^{n-1} h_{n-m}h_m^2 + \sum_{m=1}^{n-1} h_{n-m}(h_m^2 + \ldots + h_m^{[\mathcal{G}-1]})
\]

which can be proved by the induction:

\[
h_n^{[\mathcal{G}+1]} = \sum_{m=0}^{n} h_{n-m}h_m^{[\mathcal{G}]} = h_n^{[\mathcal{G}]} + h_n^{[\mathcal{G}]} + \sum_{m=1}^{n-1} h_{n-m}h_m^{[\mathcal{G}]}
\]

\[
= (\mathcal{G} + 1)h_n + \sum_{m=1}^{n-1} h_{n-m}h_m + \sum_{m=1}^{n-1} h_{n-m}(h_m^2 + \ldots + h_m^{[\mathcal{G}]})
\]

Assuming that \( h_0 = 1 \) (which does not affect mathematical generalization of the equation) an inductive proof of the formula (30) is obtained. By means of the definition of the convolution power a convolution root of the operator \( \alpha \) is defined:

\[
h^\mathcal{G} = \alpha
\]

thus:

\[
h = \alpha^{1/\mathcal{G}}
\]

The identity (30) applied for the equation (31) implies the following recursive expression:

\[
h_n = \frac{1}{\mathcal{G}} \left[ a_n - \sum_{m=1}^{n-1} h_{n-m}(h_m^2 + \ldots + h_m^{[\mathcal{G}-1]}) \right]
\]

In particular for the power of the \( \mathcal{G}/2 \) order:

\[
h_n = \frac{1}{2} \left( a_n - \sum_{m=1}^{n-1} h_{n-m}h_m \right)
\]

and for the power of the \( \mathcal{G}/4 \) order:

\[
h_n = \frac{1}{4} \left( a_n - \sum_{m=1}^{n-1} h_{n-m}h_m - \sum_{m=1}^{n-2} h_{n-m} \sum_{p=0}^{m} h_{n-p}h_p \right)
\]

Table 3

| \( n \) | \( k_{n-m} \) | \( U_{n-m} \) |
|---|---|---|
| 1 | \( U_0, k_0, k_0 \) | \( U_1 \) |
| 2 | \( U_2, k_2, k_0, k_0 \) | \( U_2 \) |
| 3 | \( U_3, k_2, k_0, k_0, k_0 \) | \( U_3 \) |
| 4 | \( U_4, k_2, k_0, k_0, k_0, k_0 \) | \( U_4 \) |...
As an example the expansion of the convolution formula (32) were calculated for the differential operator \((1-z)^\frac{1}{2}\):

\[
\begin{align*}
h_0 &= 1.0000 \\
h_1 &= 0.5h_0 = 0.5000 \\
h_2 &= -0.5h_1 = -0.1250 \\
h_3 &= -0.5(h_2 + h_1) = -0.0625 \\
h_4 &= -0.5(h_3 + h_2 + h_1) = -0.0391 \\
&\vdots
\end{align*}
\]

and for the integral operator \((1-z)^\frac{1}{2}\):

\[
\begin{align*}
h_0 &= 1.0000 \\
h_1 &= 0.5h_0 = 0.5000 \\
h_2 &= 0.5(1-h_1) = 0.3750 \\
h_3 &= 0.5(1-h_2-h_1) = 0.3125 \\
h_4 &= 0.5(1-h_3+h_2+h_1) = 0.2734 \\
&\vdots
\end{align*}
\]

The expansion of the differential operator \((1-z)^\frac{1}{2}\) according to the DF impulse response formula (19) gives:

\[
\begin{align*}
h_0 &= 1.0000 \\
h_1 &= -0.3333 \\
h_2 &= -0.1111 \\
h_3 &= -0.0617 \\
h_4 &= -0.0411 \\
&\vdots
\end{align*}
\]

and according to the convolution formula (33) gives:

\[
\begin{align*}
h_0 &= 1.0000 \\
h_1 &= \frac{1}{3}a_1 = -0.3333 \\
h_2 &= \frac{1}{3}(a_2 - 3h_1) = -0.1111 \\
h_3 &= \frac{1}{3}(a_3 - 6h_2 + h_1h_2) = -0.0617 \\
h_4 &= \frac{1}{3}(a_4 - 6h_3 - 3h_2h_1 + 3h_1h_2h_3) = -0.0411 \\
&\vdots
\end{align*}
\]

As for the integral operator \((1-z)^\frac{1}{2}\) the results are presented below, according to the DF impulse response method (20) on the left and according to the convolution method (33) on the right:

\[
\begin{align*}
h_0 &= 1.0000 \\
h_1 &= 0.3333 \\
h_2 &= 0.2222 \\
h_3 &= 0.1728 \\
h_4 &= 0.1440 \\
&\vdots
\end{align*}
\]

5. Results and Discussion

Using presented in the article methods it is possible to determine the differential–integral operator of the rational order with digital filters, i.e. in the discrete time domain. The results obtained by the digital filter impulse response method coincide with the results of the convolution method. It should be noticed that the convolution method is more complex and will need more computing power but unlike the digital filter impulse response it can be used to obtain a rational order operator.

The application of differential–integral operators in electrical engineering were also presented. In the theory of electric systems with distributed parameters, the differential and integral operators of the \(\frac{1}{2}\) order were used in order to obtain the wave impedance and the propagation operator as well as the discrete impedance of the infinite electric circuit.

In further research, the authors will focus on extending the theory of rational order operators and their application in theory of electric transmission line equations in discrete time domain.

6. Conclusion

The current electric transmission line theory allows to describe the phenomena for a lossless or non–distorting line loaded with resistance at most [24–25]. The discrete–time analysis presented in the article allows to overcome this limitation. The operators appearing in the transmission line theory can be presented as a combination of differential–integral operators. The examples shown in Section 3 use the \(\frac{1}{2}\) order differential and integral operators. In future work it will be shown that it is possible to use operators of any rational order in electric transmission line theory. Perhaps this generalized theory will find application in other fields of science, in particular those involving the propagation of waves.

Abbreviations

BIBO: Bounded Input Bounded Output; FIR: Finite Impulse Response; DF: Digital Filter

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