Fractional dimensional Fock space and Haldane’s exclusion statistics. q/p case.

K.N. Ilinski $^{1,2}$* J.M.F. Gunn$^{1,}$†

$^1$ School of Physics and Space Research, University of Birmingham, Birmingham B15 2TT, United Kingdom

$^2$ Institute of Spectroscopy, Russian Academy of Sciences, Troitsk, Moscow region, 142092, Russian Federation

Abstract

The discussion of Fractional dimensional Hilbert spaces in the context of Haldane exclusion statistics is extended from the case [10] of $g = 1/p$ for the statistical parameter to the case of rational $g = q/p$ with $q,p$-coprime positive integers. The corresponding statistical mechanics for a gas of such particles is constructed. This procedure is used to define the statistical mechanics for particles with irrational $g$. Applications to strongly correlated systems such as the Hubbard and $t − J$ models are discussed.
1 Introduction

Last year a series of papers devoted to Haldane exclusion statistics and a generalized Pauli principle have appeared. Two new concepts were introduced in Haldane’s original paper [1]. Let us briefly review them.

1. The first concept is the Haldane dimension, $d(N)$, which is the dimension of the one-particle Hilbert space associated with the $N$-th particle, keeping the coordinates of the other $N-1$ particles fixed. The statistical parameter, $g$, of a particle (or ‘$g$-on’) is then defined by the equation

$$g = - \frac{d(N + m) - d(N)}{m}$$

and conditions of homogeneity on $N$ and $m$ are imposed. In addition the system is confined to a finite region where the number $K$ of independent single-particle states is finite and fixed. Here the usual Bose and Fermi ideal gases have $g = 0$ for Bose case (i.e. $d(N)$ does not depend on $N$) and $g = 1$ for Fermi case – that is the dimension is reduced by unity for each added fermion, which is the usual Pauli principle.

2. The second concept is the Haldane-Wu state-counting procedure, which is a combinatorial expression for the number of ways, $W$, to place $N$ $g$-ons into $K$ single-particle states.

$$W = \frac{(d(N) + N - 1)!}{(d(N) - 1)!N!}$$

$$d(N) = K - g(N - 1),$$

This expression was used by many authors [2, 3, 4, 5] to describe the thermodynamical properties of $g$-ons. In particular, in Refs. [3, 4, 7, 8] it was shown that excitations in the Calogero-Sutherland model obey the form of statistical mechanics defined by Wu [2] for $g$-ons, with fractional $g$, in general. (Possible generalizations, which depend linearly on $g$, of the Haldane-Wu state-counting procedure were considered in Ref. [9]).

In our previous paper [10] we argued that these two ideas introduced by Haldane are distinct, in that the most natural combinatorial expression for $W$, using the quantity $d(N)$, is not (2). This led us to a definition of the dimension of a Hilbert space which is fractional and is related in a natural manner to particles having a fractional $g = 1/m$. We constructed the statistical mechanics for such particles and showed that many features agree with Wu’s statistical mechanics, however the agreement is not complete. Since these arguments are relevant to this letter, let us briefly summarise them.

The first argument for the expression for $W$ in (2) is motivated by initially remembering the fermionic case, writing (2) in the form

$$W_F = K(K - 1)(K - 2)(K - 3)\cdots(K - N + 1)/N!$$

where the $i$th bracket in the numerator is the number of ways to insert the $i$th particle into the system when all the particles are distinguishable. The answer is then corrected, for indistinguishability, by the $N!$ in the denominator. By analogy, expression (2) can be rewritten replacing the factor of the initial number of available states, $K$, in fermionic expression by an effective number of allowed states, $K + (1 - g)(N - 1)$. Thus the space of available single-particle
states *swells* before the particles are added. This contradicts the assumption of fixing the size of the system which leads to a fixed number of single-particle states. From the definition of the Haldane dimension, \( d(N) \), we would have naively expected a result of the form

\[
W = K(K - g) \cdots (K + (1 - g)(N - 1) - N + 1)/N!
\]

where the number of available states decreases in a manner proportional to \( g \) as the particles are added. This expression has the disadvantage that it does not even give us the correct interpolation to the Bose limit.

Another argument (which is perhaps more relevant to the subject of this paper) arises from comparing the prediction of (2) for the case \( g = 2 \) and a straightforward calculation for \( K \) single-particle states in the \( N \)-particle sector. For example with \( K = 10 \) and \( N = 3 \) the Haldane-Wu procedure gives \( W = 8!/(3!5!) = 56 \) and straightforward calculation gives \( W_0 = (10 \cdot 8 \cdot 6)/3! = 80 \). It is easy to see that, for general large \( K \) and \( N \), the deviation of \( \ln W \) from straightforward counting is the following:

\[
\ln W_0 - \ln W = \frac{1}{2} \frac{N^2}{K} + O \left( \frac{N^3}{K^2} \right).
\]

This deviation is important for cases where the occupation numbers are not small. So, the above discussion may be summarised by stating that Haldane’s definition of the fractional dimension and the Haldane-Wu state-counting procedure are not consistent. Moreover, *we cannot expect the agreement between predictions of theories based on \( W \) and state-counting procedures, which fit \( W_0 \), for \( g = 2 \).*

Certainly one may ask why one should consider state-counting procedures such as \( W_0 \). Many applications of \( W \) are known, so it might be said that one should regard the Haldane dimension \( d(N) \) as a less important variable. However, our main motivation in considering the original Haldane definition of exclusion statistics is to note their realization in strongly correlated systems such as the Hubbard model with infinite \( U \) (with statistical parameter \( g = 2 \)) or the \( t - J \) model. There the charge degree of freedom on one site exactly obeys the \( W_0 \) state-counting procedure with \( g = 2 \). Examples for the other integeral statistical parameters are readily constructed using fermions with more spin states or ‘flavours’. Another possibility is to model finite (as against infinite) interaction effects by varying the statistical parameter, \( g \), from 2 to 1.

These problems require an appropriate definition of: \( d(N) \), a fractional dimensional Hilbert space and the state-counting procedure corresponding to \( W_0 \) for general rational \( g = q/p \) (In the paper \([10]\) only the simplest case \( g = 1/m \) was considered). These issues will be addressed in this paper. We will see that the agreement with statistical mechanics based on \( W \) (2) is decreases with increasing \( q \) but this is not surprising because of the aforementioned disagreement of the state-counting procedures for integer \( g \).

In the next sections we generalise the considerations of Ref. \([11]\) for \( g = q/p \). We will give the description of the corresponding Hilbert space, a calculation of dimension which reproduces the correct Haldane dimension \( d(N) \) and discuss the corresponding statistical mechanics.
2 Hilbert space definition of Haldane’s dimension

In this section we review the Hilbert space definition of Haldane dimension for fractional values of statistical parameter \( g \). To do this we follow Ref. [10].

First of all let us return to the motivation for the introduction of Haldane’s dimension, \( d(N) \), and discuss an alternative view of this quantity. In the original definition the dimension \( d(N) \) reflects the number of independent possibilities to add the \( N \)-th particle to the system. Intuitively it is clearest to consider a system on a lattice. The decrease in the number of allowed states \( d(N) \) with the increase in the number of particles is a consequence of the exclusion of the particles and gives us a Generalized Pauli exclusion principle. Haldane’s suggestion, to complete such a statistical description, was to impose linearity of the variation in \( d \) as a function of \( N \). The coefficient of proportionality is \( g \). This immediately implies an obvious condition on the statistical constant \( g \): the allowed number of states for the first particle is equal to the number of independent degrees of freedom of the system, \( K' \), and to keep the particle homogeneity condition we have to assume that \( g = K'/M \) where \( M \) is the maximum allowed number of particles (i.e. \( g \) should be rational). Moreover, if we assume that the statistical parameter does not depend on the number \( K' \) (and statistics can be realized on single site with \( K' = 1 \)), we must only consider the set of statistical parameters

\[
g = \frac{1}{m} \quad m = 1, 2, 3, \ldots \in N .
\]

This case was considered in Ref. [10]. To consider general fractional values (or indeed \( g \) integral and greater than unity) of statistical parameter, \( g = q/p \) (where \( q \) and \( p \) are coprime positive integers), we must relax the condition of \( g \) being independent of \( K' \), and allow use of \( K' > 1 \). To describe \( g = p/q \), we require at least \( q \) of them (for example for the \( t - J \) model we have two single-particle levels per each site). Generally, the set of states should be divided into blocks with \( q \) states in each of them and the number of states \( K' = qK \).

Our main idea now is to consider the process of inserting of the \( N \)-th particle into the system as a probabilistic process (in Gibbs spirit), i.e. we assume that the probability of such an insertion plays the role of the Haldane’s measure of the probability to add the \( N \)-th particle to the system. Let us illustrate the idea for the case of a single block (with \( q \) levels) and give an interpretation of \( d(N) \) in that case.

First of all we have the vacuum state (empty block states) to which we can add the first particle. We assume that the statistics reveals itself at the level of two particles and it is irrelevant for the case \( N = 1 \), so \( d(1) = q \) (because we can put the particle in any of \( q \) levels in the block). Now let us assume that we cannot add the second particle to the system at each attempt and the process is a probabilistic one with the probability \( (1 - 1/p) \) of success per each level. Then result probability to add the second particle to the block is \( q(1 - 1/p) \). Then if we consider a large number, \( Q \), of copies of the system and perform trials with each, we will find approximately \( q(1 - 1/p)Q \) double-occupied blocks so that on average only a \( q(1 - 1/p) \) part of the particle is in each block. We interpret this as a fractional dimension of the subspace with double occupation of the block and \( d(2) = q(1 - 1/p) \). This implies that the statistical parameter will be \( g = d(1) - d(2) = q/p \).

Let us continue the procedure and consider a third particle. We can repeat the previous argument with only a small correction: now we consider the conditional probability to add
a third particle to the systems with the condition that there are already two particles in the system (whose coordinates we have to fix) before the trials. This conditional probability is, by assumption, $(1 - 2/p)$ per level which determines the average probability for the third particle in each block and the equality $d(3) = q(1 - 2/p)$. Then the full probability to find the block with $N = 3$ is $q^3(1 - 1/p)(1 - 2/p)$.

We see that the probability to find $N > p$ particles in the block is equal to zero and this avoids difficulties with a nonpositive number of $N$-particle states which occurs in the approach based on the expression (2) [4, 11]. This is achieved by eliminating of the probability on the $m + 1$-th step. In this context the original Pauli principle ‘There are no double-occupied states’ can be reformulated as ‘The probability to find a double-occupied state is equal zero’.

Moreover we can give a ‘geometrical’ definition of the fractional dimension by drawing parallels with the notion of a noninteger dimension in the framework of dimensional regularization. Indeed, usually tin the calculation of thermodynamical quantities of an ideal gas, such as the partition function or the mean value of an arbitrary physical variable $\hat{O}$, we have to compute the following traces:

$$Z = Tr(Id \cdot e^{-\beta H}) \quad \text{or} \quad \langle \hat{O} \rangle = Tr(Id \cdot e^{-\beta H} \cdot \hat{O}) \quad (4)$$

where the Hamiltonian $H$:

$$H = \sum_{i,j=1}^{K,q} \epsilon_{n_{i,j}} \quad (5)$$

is of the usual ideal gas form, and does not depend on the statistics but depends only on the occupation numbers of the particles on the $i$-th block and an ‘unit operator’ $Id$; these completely define the exclusion statistics of the particles:

$$Id = \sum_{\{n_{i,j} \geq 0\}} \alpha_{\{n_{i,j}\}_{i,j=1}^{K,q}} \{\{n_{i,j}\}_{i,j=1}^{K,q}\} \langle\{n_{i,j}\}_{i,j=1}^{K,q}\rangle \quad (6)$$

Here $\{\{n_{i,j}\}_{i,j=1}^{K,q}\}$ is the state with $n_{i,j}$ particles in the state $i, j$, and $\alpha_{\{n_{i,j}\}_{i,j=1}^{K,q}}$ is the probability to find this state. It is obvious that the expression (6) gives correct answers for fermions and bosons and there is no contradiction in using the expression for intermediate statistics. Moreover we can say that the Hilbert space of the theory is constructed if we define the operator $Id$ and use the equalities (6) (the scalar product can be defined in the same way). We will not go more into the mathematical details here and will discuss them elsewhere.

We can now interpret $\alpha_{\{n_{i,j}\}_{i,j=1}^{K,q}}$ as the dimension of the subspace spanned by the vector $\{\{n_{i,j}\}_{i,j=1}^{K,q}\}$. Indeed, usually the dimension of a subspace $S$ can be defined as the trace of unit operator on the subspace

$$\dim S = Tr(Id|_S)$$

(this definition was used in dimensional regularization where $d - \epsilon = \sum_{j} \delta_{j}$). The full dimension of the $N$-particle subspace of the space of states is then given by the formula:

$$W_0 = \sum_{\{n_{i,j} \geq 0\}, \sum_{n_{i,j}=N}} \alpha_{\{n_{i,j}\}_{i,j=1}^{K,q}} = Tr(Id|\sum_{n_{i,j}=N}) \quad (7)$$

which we will use later for the state-counting procedure in the next section. Haldane’s dimension $d(N)$ of the $N$-particle subspace with an arbitrary fixed $N - 1$-particle subspace
\[ |\{n_{i,j}\}_{i,j=1}^{K,q}\rangle, \sum n_{i,j} = N - 1 \]

is then described by the relation:

\[
d(N) = \sum_{i,j} \alpha_{\{n_{i,j}\}_{i,j=1}^{K,q}} \frac{\sum_{n_{i,j}}^{N-1}}{\sum_{n_{i,j}=N-1}}.
\]  \hspace{1cm} (8)

(This is a sum of conventional probabilities to add \(N\)-th particle to the system with the condition that before the addition the system is in the state \(|\{n_{i,j}\}_{i,j=1}^{K,q}\rangle, \sum n_{i,j} = N - 1\).

The statements made so far are general and did not require any concrete choice of the probabilities \(\alpha_{\{n_{i,j}\}_{i,j=1}^{K,q}}\). Moreover when we discussed these ideas for the single block we defined probabilities with only a single index and so we have a choice to define probabilities with several indices.

In fact, there is single self-consistent way to define \(\alpha_{\{n_{i,j}\}_{i,j=1}^{K,q}}\) such that

1. the definition of the \(N\)-th particle dimension \(d(N)\) \hspace{1cm} (8) actually gives Haldane’s conjecture \(K - g(N - 1)\) for \(d(N)\);

2. The Hilbert space of the system with \(K\) blocks is factorized into the product of Hilbert spaces corresponding to each block. This property together with the Hamiltonian of the form \[5\] is characteristic of an ideal gas of the particles with any statistics and will lead to the factorization of partition function and other physical quantities.

To prove this let us use the assumption about the tensor product nature of the full Hilbert space which immediately implies the operator \(Id\) for the complete system is a tensor product of the operators \(\{Id_i\}_{i=1}^{K}\) for each state:

\[
Id = Id_1 \otimes Id_2 \otimes \cdots \otimes Id_K.
\]  \hspace{1cm} (9)

The last relation is equivalent to the following expression for probabilities \(\alpha_{\{n_{i,j}\}_{i,j=1}^{K,q}}\):

\[
\alpha_{\{n_{i,j}\}_{i,j=1}^{K,q}} = \prod_{i=1}^{K} (1 - 1/p)(1 - 2/p)...(1 - (\sum_j n_{i,j} - 1)1/p).
\]  \hspace{1cm} (10)

Combined with \(\text{(8)}\) we obtain the required equality for \(d(N)\):

\[
d(N) = \sum_{l=1,j=1}^{K,q} (1 - 1/p \sum_{j'} n_{l,j'}) = q(K - 1/p(N - 1)) \hspace{1cm} (11)
\]

Moreover, we can say that \(W_0\) in \(\text{(7)}\) together with Haldane’s dimension \(d(N)\) \hspace{1cm} (11) and the corresponding \(\alpha_{\{n_{i,j}\}_{i,j=1}^{K,q}}\) \hspace{1cm} (10) replaces expression \(\text{(3)}\) in the theory. These are used in the investigation of thermodynamical properties in the rest of the paper.

We conclude this section with a note about the connection between the subject of section 2 and Ref\[11\], where the microscopic origin of the Haldane-Wu state-counting procedure was examined. As in this work, the notion of statistics was considered in a probabilistic spirit. The author assumed that a single level may be occupied by any number of particles, and each occupancy is associated with an a priori probability. These probabilities are determined by enforcing consistency with \textit{the Haldane-Wu state-counting procedure} and not with Haldane’s
definition of exclusion statistics. In detail, the distinction with the current work are: the
definition of the Haldane dimension was not used, and a Hilbert space was not constructed
and the a priori probabilities in Ref. [11] may be negative. As we have shown [10], the Haldane
dimension and the Haldane-Wu state-counting procedure are distinct; the formulations of this
work and Ref. [11] are correspondingly distinct.

3 Statistical mechanics of $q/p$-ons

Let us consider initially the second virial coefficient, $B_2$, which reflects the statistical interactions
between the particles and has been calculated for the other representations of particles obeying
exclusion statistics [3, 4]

$$B_2 = -Z_1(2Z_2/Z_1^2 - 1).$$

where $Z_i$ is the $i$-particle’s partition function. For the case of $g = q/p$, a straightforward
 calculation gives:

$$Z_2 = e^{-2\beta\epsilon}(q^2C_K^2 + K\frac{q(q + 1)}{2}(1 - 1/p)), \quad Z_1 = e^{-\beta\epsilon}qK$$

which results in

$$B_2 = q/p + 1/p - 1$$

coinciding with Wu’s expression [2, 3]. $B_2 = 2q/p - 1$, only for the case $q = 1$, but yields the
 expressions which might be anticipated, on physical grounds, for integer values of $g$ where Wu’s
 result is more difficult to interpret.

To calculate the partition function of an ideal $g$-on gas, we return to the formulae (4),(5) of
the previous section and use our self-consistent choice of the operator $Id$ (6) with the coefficients
(10). As was noted above, such a choice allows us to factorize the statistical operator $Id$ into
a tensor product of the operators for single states (9) which leads to the factorization of the
partition function of the system:

$$Z = \prod_{i=1}^{K} Z_i.$$ (13)

Here the function $Z_i$ is the partition function associated with a single block labelled by index
$i$:

$$Z_i = Tr(Id_i \cdot e^{-\beta(\epsilon_i - \mu)n_i}).$$

For the case of $g = q/p$ the last expression can be rewritten as

$$Z_i = 1 + qe^{-\beta(\epsilon_i - \mu)} + \frac{q(q - 1)(q + 1)}{2p}e^{-2\beta(\epsilon_i - \mu)} \cdots + \frac{p(p - 1)(p - 2)\cdots}{p^{p-1}} C_{q+p-1}^{p} e^{-p\beta(\epsilon_i - \mu)}$$

or in a more closed form

$$Z_i = \sum_{n=0}^{p} C_{p}^{n} C_{q+n-1}^{n} \frac{e^{-n\beta(\epsilon_i - \mu)}}{n!} \frac{1}{p^{n}}.$$ (14)

It is obvious that the formulae (13), (14) interpolate between integer $g$ (in particular, Fermi
($g = p = 1$)) and Bose ($g = 0, p = \infty$) cases and hence lead to the an interpolation for all
statistical quantities.
The most interesting object for comparison (for different \( p \) and \( q \)) is the distribution function which in our approach may be calculated as (we once more put all energies equal to the single \( \epsilon \) to make the expression more compact)

\[
 n(\beta, \mu) \equiv \langle N/qK \rangle = \frac{1}{\beta} \frac{\partial \ln Z_i}{\partial \mu}
\]

which results in the next equality:

\[
 n(\beta, \mu) = \frac{1}{q} \left( \sum_{n=1}^{m} C_p^m C_{q+n-1}^m n! \frac{e^{-n\beta(\epsilon_i - \mu)}}{p^n} \right) \cdot \left( \sum_{n=0}^{m} C_p^n C_{q+n-1}^n n! \frac{e^{-n\beta(\epsilon_i - \mu)}}{p^n} \right)^{-1}. \tag{15}
\]

Let us briefly discuss the behaviour of this distribution function at low temperature (in the high temperature limit all statistical effects disappear). We can easily see from (15) that in the low temperature limit the function \( n(\beta, \mu) \) for our \( q/p \)-particles behaves exactly like the standard distribution function \[2, 13\], i.e. at any positive value of \( g \) we have a ‘Fermi level’:

\[
 n(\beta, \mu) = \frac{p}{q} \quad \text{for} \quad \epsilon < \mu \\
 n(\beta, \mu) = 0 \quad \text{for} \quad \epsilon > \mu \tag{16}
\]

At low, but finite, temperatures we will have a discrepancy with the standard result. To simplify the comparison we remember that the distribution function of \( g \)-ons in the Haldane-Wu approach can be expressed as

\[
 n(\beta, \mu) = \frac{1}{w(\beta, \mu) + g} \tag{17}
\]

and the function \( w(\beta, \mu) \equiv w(\xi) \) is a solution of equation \[2, 13\]:

\[
 w^g(\xi)(w(\xi) + 1)^{1-g} = e^{\beta(\epsilon - \mu)} \equiv \xi. \tag{18}
\]

At low enough, but finite, temperatures and energy \( \epsilon \) above the Fermi level (i.e. \( \xi \) is very big) we find the following expansion for \( n \) as a function of variable \( \xi \):

\[
 n(\beta, \mu) = \frac{1}{\xi} \left( 1 + \frac{1}{\xi} \left( 1 - 2g \right) + \ldots \right). \tag{19}
\]

The first term on the RHS is a pure Boltzmann distribution and this is a common feature for any statistics in this limit. But the next term reflects the statistics of the particles. There is no difficulty in showing that in the same limit expression \[13\] gives us similar asymptotics for the distribution function:

\[
 n(\beta, \mu) = \frac{1}{\xi} \left( 1 + \frac{1}{\xi} \left( 1 - q/p - 1/p \right) + \cdots \right), \tag{20}
\]

that is consistent with comparison of second virial coefficients and coincides with \[19\] for the case \( q = 1 \).

Below the Fermi level at low, but finite, temperatures \( \xi \) is a small, but finite, parameter. From eq.\[13\] we immediately obtain the following approximation for \( n(\xi) \):

\[
 n(\beta, \mu) = \frac{p}{q} - \frac{p^2}{q^2} \epsilon p/q + \cdots .
\]
In contrast with this the asymptotic behaviour for the distribution function \( n(\beta, \mu) \) contains terms which are linear in \( \xi \):

\[
n(\beta, \mu) = \frac{p}{q} \left( 1 - \xi \frac{p}{q + p - 1} + \cdots \right).
\] (21)

In conclusion of this section let us return to the state-counting procedure for probabilistic \( g \)-ons. For convenience set all energies to be identical \( \epsilon_i = \epsilon \) and rewrite the formulae for the dimension of the \( N \)-particle subspace of the full Hilbert space \( \left( \right) \) in the form without constrained summation:

\[
W_0 = \sum_{n_{i,j}=0}^{\infty} \alpha_{\{n(i,j)\}} K_{i,j} \cdot \delta \sum_{i,j} n_{i,j} N
\]

which then will be treated as a Fourier integral:

\[
W_0 = \frac{1}{2\pi} \int_{0}^{2\pi} d\phi \sum_{n_{i,j}=0}^{\infty} \alpha_{\{n(i,j)\}} K_{i,j} \cdot e^{\frac{1}{3} \sum_{i,j=1}^{n(i,j)} n(i,j) - N\phi}.
\] (22)

This last equality and the factorization property \( (\) for the coefficients \( \alpha \) allow us to perform the summation over \( n_{i,j} \) and obtain an expression for \( W_0 \) with only a single integral over the auxiliary variable \( \phi \):

\[
W_0 = \frac{1}{2\pi} \int_{0}^{2\pi} d\phi \cdot z(e^{i\phi}) K \cdot e^{-iN\phi}.
\] (22)

where the function \( z(e^{i\phi}) \) coincides with the single state partition function for imaginary energy and chemical potential \( (\beta(\mu - \epsilon) \rightarrow i\phi) \):

\[
z(e^{i\phi}) = \sum_{n=0}^{p} C_p^n \cdot \frac{C_q^m}{C_{q+n-1}^m} \cdot \frac{e^{in\phi}}{p^n}.
\]

Eq.(22) replaces the Haldane-Wu state-counting expression \( (\) in the theory of probabilistic \( g \)-ons.

It is interesting to note that in some sense our consideration is complementary to the Haldane-Wu case: we can easily find the distribution function analytically but to find the entropy in the large \( N \) limit an algebraic equation of the \( p \)-th order must be solved, whilst in the Haldane-Wu approach the form of the entropy is obvious but an algebraic equation of the \( p \)-th order is encountered when deriving the distribution function.

4 Discussion

Let us discuss a few of the results and their consequences in more detail. Perhaps the least attractive aspect of our formulation is that the results depend on both of the parameters \( p \) and \( q \). However this apparent drawback is not as significant as one might initially presume: this is because as \( p \) and \( q \) both become large, with their ratio fixed, the results only depend on \( p/q \). This may be seen by examining expressions \( (\) (\).

An interesting extension of this logic is that we may define the statistical mechanics of irrational \( g \) by considering a sequence of rational numbers, \( p/q \), which approach the required irrational number.
In conclusion, the discussion of Fractional dimensional Hilbert spaces in the context of Haldane exclusion statistics has been extended from the case of $g = 1/p$ for the statistical parameter to the case of rational $g = q/p$ with $q, p$-coprime positive integers. The corresponding statistical mechanics for a gas of such particles is constructed.

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