Intersection models and forbidden pattern characterizations for 2-thin and proper 2-thin graphs*

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Abstract

The thinness of a graph is a width parameter that generalizes some properties of interval graphs, which are exactly the graphs of thinness one. Graphs with thinness at most two include, for example, bipartite convex graphs. Many NP-complete problems can be solved in polynomial time for graphs with bounded thinness, given a suitable representation of the graph. Proper thinness is defined analogously, generalizing proper interval graphs, and a larger family of NP-complete problems are known to be polynomially solvable for graphs with bounded proper thinness. It is known that the thinness of a graph is at most its pathwidth plus one. In this work, we prove that the proper thinness of a graph is at most its bandwidth, for graphs with at least one edge. It is also known that boxicity is a lower bound for the thinness. The main results of this work are characterizations of 2-thin and 2-proper thin graphs as intersection graphs of rectangles in the plane with sides parallel to the Cartesian axes and other specific conditions. We also bound the bend number of graphs with low thinness as vertex intersection graphs of paths on a grid ($B_k$-VPG graphs are the graphs that have a representation in which each path has at most \(k\) bends). We show that 2-thin graphs are a subclass of $B_1$-VPG graphs and, moreover, of monotone $L$-graphs (that is, $B_1$-VPG graphs admitting a representation that uses only one of the four possible shapes $L$, $J$, $\Gamma$, $\square$, with an extra condition), and that 3-thin graphs are a subclass of $B_3$-VPG graphs. We also show that $B_0$-VPG graphs may have arbitrarily large thinness, and that not every 4-thin graph is a VPG graph. Finally, we characterize 2-thin graphs by a set of forbidden patterns for a vertex order.
1. Introduction

A large family of graph width parameters has been studied since the introduction of treewidth in the 80’s [39], some of them very recently defined, like twin-width [4] in 2020. For each width parameter, a growing family of NP-complete problems are known to be polynomial time solvable on graph classes of bounded width. Thus, it is interesting to find structural properties of a graph class that ensure bounded width. In this work we will focus on a width parameter called the thinness of a graph.

A graph $G = (V, E)$ is $k$-thin if there exist an ordering $v_1, \ldots, v_n$ of $V$ and a partition of $V$ into $k$ classes $(V^1, \ldots, V^k)$ such that, for each triple $(r, s, t)$ with $r < s < t$, if $v_r, v_s$ belong to the same class and $v_r v_s \in E$, then $v_s v_t \in E$. Such an ordering and partition are called consistent. The minimum $k$ such that $G$ is $k$-thin is called the thinness of $G$ and is denoted by thin($G$). The thinness is unbounded on the class of all graphs, and graphs with bounded thinness were introduced by Mannino, Oriolo, Ricci and Chandran in [30] as a generalization of interval graphs (intersection graphs of intervals of the real line), which are exactly the 1-thin graphs [34]. Graphs of thinness at most two include, for example, convex bipartite graphs [7].

In [5], the concept of proper thinness is defined in order to obtain an analogous generalization of proper interval graphs (intersection graphs of intervals of the real line such that no interval properly contains another). A graph $G = (V, E)$ is proper $k$-thin if there exist an ordering and a partition of $V$ into $k$ classes such that both the ordering and its reverse are consistent with the partition. Such an ordering and partition are called strongly consistent. The minimum $k$ such that $G$ is proper $k$-thin is called the proper thinness of $G$ and is denoted by pthin($G$). Proper interval graphs are exactly the proper 1-thin graphs [36], and in [5] it is proved that the proper thinness is unbounded on the class of interval graphs. Examples of thin representations of graphs are shown in Figure 1.

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In [8], the concept of (proper) independent thinness was introduced in order to bound the (proper) thinness of the lexicographical and direct products of two graphs. In this case it is required, additionally, the classes of the partition to be independent sets. These concepts are denoted by thin\text{ind}(G) and pthin\text{ind}(G), respectively.

The relation between thinness and other well known width parameters is surveyed in [5]. The pathwidth (resp. bandwidth) of a graph $G$ can be defined as one less than the maximum clique size of an interval (resp. proper interval) supergraph of $G$, chosen to minimize its maximum clique size [27, 32]. In [30] it is proved that the thinness of a graph is at most its pathwidth plus one but, indeed, the proof shows that the independent thinness of a graph is at most its pathwidth plus one. Combining the ideas behind that proof and a characterization of the bandwidth of a graph as a proper pathwidth, due to Kaplan and Shamir [27], it can be proved that the proper independent thinness of a graph is at most its bandwidth plus one. We will furthermore prove that the proper thinness of a graph with at least one edge is at most its bandwidth.

Several graph classes are defined by means of a geometrical intersection model, being the most prominent of such classes the class of interval graphs, introduced by Hajós in 1957 [21]. Moreover, many of these classes are generalizations of interval graphs, like circular-arc graphs [18], vertex and edge intersection graphs of paths on a grid [2, 20], and graphs with bounded boxicity. The boxicity of a graph, introduced by Roberts in 1969 [37], is the minimum dimension in which a given graph can be represented as an inter-
section graph of axis-parallel boxes. Chandran, Mannino, and Oriolo in [11] proved that the thinness of a graph is at least its boxicity, so 2-thin graphs are a subclass of boxicity 2 graphs, i.e., intersection graphs of axis-parallel rectangles in the plane. Their proof is constructive, and in their boxicity 2 model for 2-thin graphs, the upper-right corners of the rectangles lie in two diagonals, according to the class the corresponding vertex belongs to. They furthermore proved that boxicity 2 graphs have unbounded thinness. Indeed, while the independent set problem is NP-complete on boxicity 2 graphs, even if the rectangle model is given [17, 25], that and a wide family of NP-complete problems are polynomial-time solvable on graphs of bounded thinness, when the representation is given, and the family of problems can be enlarged for proper thinness [5, 6, 30]. So, it is of interest to study which boxicity 2 graphs have bounded (proper) thinness. We will do in Section 4 a slight modification to their model for 2-thin graphs, needed to obtain, together with the diagonals property, a full characterization in Theorem 8. Namely, we modify it to satisfy a further property that we call blocking, needed to prove the equivalence in the theorem. Notice that when restricting the upper-right corners of the rectangles to lie in one diagonal, we obtain the class of interval graphs, i.e., 1-thin graphs. The definition of p-box graphs [41] is similar, since they are the intersection graphs of rectangles such that the lower-right corners lie in a diagonal.

The models with this kind of restrictions, like endpoints, corners or sides of the geometrical objects lying on a line, are known in the literature as grounded models (see Figure 2). A graph is $B_k$-VPG if it is the vertex intersection graph of paths with at most $k$ bends in a grid [2]. VPG graphs, without bounds in the number of bends, are also known as string graphs. A subclass of $B_1$-VPG graphs is the class of L-graphs, in which all the paths have the shape L. Many classes can be characterized by different grounded L-models. For instance, circle graphs are exactly the doubly grounded L-graphs where both endpoints of the paths belong to an inverted diagonal [2], and p-box graphs are also characterized as monotone L-graphs [1]. A very nice survey on this kind of models can be found in [26]. We present here another grounded rectangle model for 2-thin graphs that gives rise to a grounded L-model for them and a grounded $B_0$-VPG model for independent 2-thin graphs. Based on these models, we can also obtain a $B_3$-VPG representation for 3-thin graphs and a $B_1$-VPG representation for independent 3-thin graphs. We furthermore show that $B_0$-VPG graphs have unbounded thinness, and that not every 4-thin graph is a VPG graph. The L-model can be
modified to prove that 2-thin graphs are monotone L-graphs. Notice that
the class of monotone L-graphs or p-box graphs contains all trees [41], which
have unbounded thinness [3], and that the octahedron $\overline{3K_2}$ is an example of
a graph of thinness 3 which is not p-box since it has boxicity 3 [37].

As a last result, we obtain forbidden ordered pattern characterizations
for 2-thin graphs, independent 2-thin graphs, and proper independent 2-thin
graphs.

1.1. Basic definitions

All graphs in this work are finite, have no loops or multiple edges, and are
undirected unless we say explicitly digraphs. For a graph $G$, denote by $V(G)$
its vertex set and by $E(G)$ its edge set. For a subsets $A$ of $V(G)$, denote by
$G[A]$ the subgraph of $G$ induced by $A$.

A digraph is a graph $D = (V, A)$ such that $A$ consists of ordered pairs of
$V$, called arcs. A cycle of a digraph $D$ is a sequence $v_1, v_2, \ldots, v_i$, $1 \leq i \leq n,$
of vertices of $V(D)$ such that $v_i = v_i$ and, for all $1 \leq j < i$, $v_jv_{j+1} \in A(D)$. A
digraph is acyclic if it has no directed cycles. A topological ordering of a
digraph $D$ is an ordering $<$ of its vertices such that for each arc $vw \in A(D)$,
$v < w$. A digraph admits a topological ordering if and only if it is acyclic,
and such an ordering can be computed in $O(|V| + |A|)$ time [28].

Denote the size of a set $S$ by $|S|$. A clique or complete set (resp. independent set) is a set of pairwise adjacent (resp. nonadjacent) vertices. The
clique number of a graph is the size of a maximum clique. Let $X, Y \subseteq V(G)$. We say that $X$ is complete to $Y$ if every vertex in $X$ is adjacent to every vertex in $Y$, and that $X$ is anticomplete to $Y$ if no vertex of $X$ is adjacent to a vertex of $Y$. A graph is complete if its vertex set is a complete set. A graph is bipartite if its vertex set can be partitioned into two independent sets, and complete bipartite if those sets are complete to each other.

Given a graph $G$ and two disjoint subsets $A, B$ of $V(G)$, the bipartite graph $G[A, B]$ is defined as the subgraph of $G$ formed by the vertices $A \cup B$ and the edges of $G$ that have one endpoint in $A$ and one in $B$. Notice that $G[A, B]$ is not necessarily an induced subgraph of $G$.

A $t$-coloring of a graph is a partition of its vertices into $t$ independent sets. The smallest $t$ such that $G$ admits a $t$-coloring is called the chromatic number of $G$. A graph is perfect if for every induced subgraph of it, the chromatic number equals the clique number.

A graph $G(V, E)$ is a comparability graph if there exists an ordering $v_1, \ldots, v_n$ of $V$ such that, for each triple $(r, s, t)$ with $r < s < t$, if $v_r v_s$ and $v_s v_t$ are edges of $G$, then so is $v_r v_t$. Such an ordering is a comparability ordering. A graph is a co-comparability graph if its complement is a comparability graph. A graph is a permutation graph if it is both a comparability and a co-comparability graph, and a bipartite permutation graph if it is, moreover, bipartite.

A bipartite graph with bipartition $(X, Y)$ is an interval bigraph if every vertex can be assigned an interval on the real line such that for all $x \in X$ and $y \in Y$, $x$ is adjacent to $y$ if and only if the corresponding intervals intersect. A proper interval bigraph is an interval bigraph admitting a representation in which the family of intervals of each of $X$, $Y$ is inclusion-free. Proper interval bigraphs are equivalent to bipartite permutation graphs [22].

2. Algorithmic aspects of thinness

The recognition problem for $k$-thin and proper $k$-thin graphs is open, even for fixed $k \geq 2$. Some related algorithmic problems were also studied: partition into a minimum number of classes (strongly) consistent with a given vertex ordering [3,6], existence of a vertex ordering (strongly) consistent with a given vertex partition [5], existence of a vertex ordering (strongly) consistent with a given vertex partition requiring additionally that the vertices on each class are consecutive in the order [9,35]. In this work, in order to prove the intersection model characterizations, we will deal with the problem of
the existence of a vertex ordering (strongly) consistent with a given vertex partition and that extends a partial order of the vertices that is a total order when restricted to each of the parts.

The problem of finding a partition into a minimum number of classes (strongly) consistent with a given vertex ordering can be solved by coloring a conflict graph, that is shown to belong to a class in which the coloring problem is polynomial time solvable. Namely, let \( G \) be a graph and \(<\) an ordering of its vertices. In [3], it was defined the graph \( G_< \) having \( V(G) \) as vertex set, and \( E(G_<) \) is such that for \( v < w, \ vw \in E(G_<) \) if and only if there is a vertex \( z \) in \( G \) such that \( w < z, \ zv \in E(G) \) and \( zw \notin E(G) \). Similarly, in [5], it was introduced the graph \( \tilde{G}_< \), which has \( V(G) \) as vertex set, and \( E(\tilde{G}_<) \) is such that for \( v < w, \ vw \in E(\tilde{G}_<) \) if and only if either \( vw \in E(G_<) \) or there is a vertex \( x \) in \( G \) such that \( x < v, \ xw \in E(G) \) and \( xv \notin E(G) \). An edge of \( G_< \) (respectively \( \tilde{G}_< \)) represents that its endpoints cannot belong to the same class in a vertex partition that is consistent (respectively strongly consistent) with the ordering \(<\), and, as it was observed in the respective works, such a partition is a coloring of the corresponding graph.

In those works it was proved that \( G_< \) and \( \tilde{G}_< \) are co-comparability graphs, thus perfect [31]. This has two main implications. The first one is that the optimum coloring can be computed in polynomial time [19], and thus the problem of finding a partition into a minimum number of classes (strongly) consistent with a given vertex ordering can be solved in polynomial time. The other one is that the chromatic number equals the clique number, and the following corollary was used to prove upper and lower bounds for the thinness and proper thinness of a graph.

**Corollary 1.** Let \( G \) be a graph, and \( k \) a positive integer. Then \( \text{thin}(G) \geq k \) (resp. \( \text{pthin}(G) \geq k \)) if and only if, for every ordering \(<\) of \( V(G) \), the graph \( G_< \) (resp. \( \tilde{G}_< \)) has a clique of size \( k \).

The problem about the existence of a vertex ordering (strongly) consistent with a given vertex partition was shown to be NP-complete [5], but the complexity remains open when the number of parts is fixed. The problem about the existence of a vertex ordering consistent with a given vertex partition requiring additionally that the vertices on each class are consecutive in the order was shown to be polynomial time solvable [9]. The same problem for strong consistency was shown NP-complete when the number of parts is arbitrary [9], and polynomial time solvable when the number of parts is
fixed [35]. Under the same additional constraints, the problem of finding a minimum size partition for a given ordering is polynomial time solvable, and the complexity of the problem of finding an ordering that minimizes the size of a minimum size partition is unknown.

Let us solve now the following problem.

(Strongly) Consistent Extending Order – (S)CEO

Instance: A graph $G$, a partition $\Pi = \{V^1, \ldots, V^k\}$ and a partial order $<$ of $V(G)$ that is total and (strongly) consistent restricted to each $V^j$, $1 \leq j \leq k$.

Question: Does there exist a total ordering of $V(G)$ extending $<$ and (strongly) consistent with $\Pi$?

Given the input of the (S)CEO problem, we define a digraph $D(G, \Pi, <)$ (resp. $\tilde{D}(G, \Pi, <)$) having $V(G)$ as vertex set and such that an ordering of $V(G)$ is a solution to (S)CEO if and only if it is a topological ordering of $D(G, \Pi, <)$ (resp. $\tilde{D}(G, \Pi, <)$). The problem then reduces to the existence of a topological order of a digraph, which is polynomial time solvable [28]. Given two vertices $v_i \in V^i$, $v_j \in V^j$, $i \neq j$, we create the arc $v_i v_j$ in $D(G, \Pi, <)$ if and only if $v_i v_j \notin E(G)$ and there exists $v'_j \in V^j$ with $v'_j < v_j$ and $v_i v'_j \in E(G)$, and in $\tilde{D}(G, \Pi, <)$ if and only if $v_i v_j \notin E(G)$ and either there exists $v'_j \in V^j$ with $v'_j < v_j$ and $v_i v'_j \in E(G)$, or there exists $v'_i \in V^i$ with $v'_i > v_i$ and $v'_i v_j \in E(G)$. Additionally, in order to ensure that a topological ordering of the digraph extends $<$, we create in both cases the arc $vv'$ for every pair of vertices $v < v'$.

Lemma 2. Let $G$ be a graph, $\Pi = \{V^1, \ldots, V^k\}$ a partition and $<$ a partial order of $V(G)$ that is total and consistent restricted to each $V^j$, $1 \leq j \leq k$. An ordering of $V(G)$ is consistent with the partition $\Pi$ and extends the partial order $<$ if and only if it is a topological ordering of $D(G, \Pi, <)$.

Proof. Suppose first $<$ is a total ordering of $V(G)$ consistent with the partition $\Pi$ and that extends the partial order $<$. Let $vv'$ be an arc such that $v < v'$. Then $v < v'$, since $<$ extends $<$. Let $v_i v_j$ be an arc with $v_i \in V^i$, $v_j \in V^j$, $i \neq j$, such that $v_i v_j \notin E(G)$ and there exists $v'_j \in V^j$ with $v'_j < v_j$ and $v_i v'_j \in E(G)$. Suppose that $v_j < v_i$. Since $<$ extends $<$, $v'_j < v_j < v_i$, and $v_i v'_j \in E(G)$ but $v_i v_j \notin E(G)$, a contradiction because $<$ is consistent with the partition $\Pi$. Thus $v_i < v_j$ for every such arc, and $<$ is a topological ordering for $D(G, \Pi, <)$.
Conversely, suppose $\prec$ is a topological ordering for $D(G, \Pi, \prec)$. Let $v'_j < v_j < v_i$ such that $v_i \in V^i$, $v_j, v'_j \in V^j$, $v'_j v_i \in E(G)$. If $i = j$, then $v'_j < v_j < v_i$ because $\prec$ is total restricted to $V^i$ and $\prec$ extends $\prec$, so $v_j v_i \in E(G)$ because $\prec$ is consistent restricted to $V^i$. If $i \neq j$, then $v'_j < v_j$ because $\prec$ is total restricted to $V^j$ and $\prec$ extends $\prec$. Since $\prec$ is a topological ordering for $D(G, \Pi, \prec)$, $v_i v_j$ is not an arc of $D(G, \Pi, \prec)$, thus $v_i v_j \in E(G)$.

\[\square\]

**Lemma 3.** Let $G$ be a graph, $\Pi = \{V^1, \ldots, V^k\}$ a partition and $\prec$ a partial order of $V(G)$ that is total and strongly consistent restricted to each $V^j$, $1 \leq j \leq k$. An ordering of $V(G)$ is strongly consistent with the partition $\Pi$ and extends the partial order $\prec$ if and only if it is a topological ordering of $\tilde{D}(G, \Pi, \prec)$.

**Proof.** Suppose first $\prec$ is a total ordering of $V(G)$ strongly consistent with the partition $\Pi$ and that extends the partial order $\prec$. Let $vv'$ be an arc such that $v < v'$. Then $v \prec v'$, since $\prec$ extends $\prec$. Let $v_i v_j$ be an arc with $v_i \in V^i$, $v_j \in V^j$, $i \neq j$, such that $v_i v_j \notin E(G)$ and there exists $v'_j \in V^j$ with $v'_j < v_j$ and $v_i v'_j \in E(G)$. Suppose that $v_j < v_i$. Since $\prec$ extends $\prec$, $v'_j \prec v_j \prec v_i$, and $v_i v'_j \in E(G)$ but $v_i v_j \notin E(G)$, a contradiction because $\prec$ is strongly consistent with the partition $\Pi$. Thus $v_i \prec v_j$ for every such arc. Analogously, let $v_i v_j$ be an arc with $v_i \in V^i$, $v_j \in V^j$, $i \neq j$, such that $v_i v_j \notin E(G)$ and there exists $v'_i \in V^i$ with $v'_i > v_i$ and $v'_i v_j \in E(G)$. Suppose that $v_j < v_i$. Since $\prec$ extends $\prec$, $v_j \prec v_i \prec v'_i$, and $v_i v'_j \in E(G)$ but $v_i v_j \notin E(G)$, a contradiction because $\prec$ is strongly consistent with the partition $\Pi$. Thus $v_i \prec v_j$ for every such arc. Therefore, $\prec$ is a topological ordering for $\tilde{D}(G, \Pi, \prec)$.

Conversely, suppose $\prec$ is a topological ordering for $\tilde{D}(G, \Pi, \prec)$. Let $v_i < v'_i < v''_i$ in $V^i$, such that $v_i v''_i \in E(G)$. Then $v_i < v'_i < v''_i$ because $\prec$ is total restricted to $V^i$ and $\prec$ extends $\prec$, so $v_i v'_i, v'_i v''_i \in E(G)$ because $\prec$ is strongly consistent restricted to $V^i$. Let $v'_j < v_j < v_i$ such that $v_i \in V^i$, $v_j, v'_j \in V^j$, $i \neq j$, and $v'_j v_i \in E(G)$. Then $v'_j < v_j$ because $\prec$ is total restricted to $V^j$ and $\prec$ extends $\prec$. Since $\prec$ is a topological ordering for $\tilde{D}(G, \Pi, \prec)$, $v_i v_j$ is not an arc of $\tilde{D}(G, \Pi, \prec)$, thus $v_i v_j \in E(G)$. Analogously, let $v_j \prec v_i \prec v'_i$ such that $v_i, v'_i \in V^i$, $v_j \in V^j$, $i \neq j$, and $v_j v'_i \in E(G)$. Then $v_i < v'_i$ because $\prec$ is total restricted to $V^i$ and $\prec$ extends $\prec$. Since $\prec$ is a topological ordering for $\tilde{D}(G, \Pi, \prec)$, $v_i v_j$ is not an arc of $\tilde{D}(G, \Pi, \prec)$, thus
Given a graph \( G \), the pathwidth \( \text{pw}(G) \) (resp. bandwidth \( \text{bw}(G) \)) may be defined as one less than the maximum clique size in an interval (resp. proper interval) supergraph of \( G \), chosen to minimize its clique size [27]. It was implicitly proved in [30] that \( \text{thin}_{\text{ind}}(G) \leq \text{pw}(G) + 1 \). We will reproduce here the proof, emphasizing the independence of the classes defined.

A characterization in [27] of the bandwidth as a proper pathwidth allows to mimic the proof in [30] and prove that \( \text{pthin}_{\text{ind}}(G) \leq \text{bw}(G) + 1 \). This bound can be improved for non-empty graphs. We use a third equivalent definition of bandwidth, namely \( \text{bw}(G) = \min_f \max \{|f(v_i) - f(v_j)| : v_iv_j \in E\} \), for \( f : V(G) \to \mathbb{Z} \) an injective labeling, and Corollary 1 to prove the following.

**Theorem 4.** Let \( G \) be a graph with at least one edge. Then \( \text{pthin}(G) \leq \text{bw}(G) \).

**Proof.** Suppose on the contrary that \( \text{pthin}(G) > \text{bw}(G) \). By Corollary 1 for every vertex order \( < \) of \( G \), there is a clique of size \( \text{bw}(G) + 1 \) in \( \tilde{G}_< \). Let \( f \) be a labeling of \( V(G) \) realizing the bandwidth and \( < \) be the order induced by \( f \). Suppose \( v_1 < v_2 < \cdots < v_b < v_{b+1} \) is a clique of \( \tilde{G}_< \), where \( b = \text{bw}(G) \). As \( v_1v_{b+1} \in E(\tilde{G}_<) \), there exists either \( v_0 \) such that \( v_0 < v_1 \), \( v_0v_{b+1} \in E(G) \), and \( v_0v_1 \notin E(G) \), or \( v_{b+2} \) such that \( v_{b+2} > v_{b+1} \), \( v_1v_{b+2} \ug E(G) \), and \( v_{b+1}v_{b+2} \notin E(G) \). In the first case, \( |f(v_{b+1}) - f(v_0)| = f(v_{b+1}) - f(v_0) \geq b + 1 \) and, in the second case, \( |f(v_{b+2}) - f(v_1)| = f(v_{b+2}) - f(v_1) \geq b + 2 - 1 = b + 1 \). In either case, it is a contradiction with the fact that \( f \) realizes the bandwidth \( b \). So, \( \text{pthin}(G) \leq \text{bw}(G) \).

This bound can be arbitrarily bad, for example, for the complete bipartite graphs \( K_{n,n} \), that are proper 2-thin and have unbounded bandwidth. However, it is tight (up to a constant factor) for grids [11, 14].

A path decomposition [38] of a graph \( G = (V,E) \) is a sequence of subsets of vertices \( (X_1, X_2, \ldots, X_r) \) such that

1. \( X_1 \cup \cdots \cup X_r = V \).
(2.) For each edge $vw \in E$, there exists an $i \in \{1, \ldots, r\}$, so that both $v$ and $w$ belong to $X_i$.

(3.) For each $v \in V$ there exist some $s(v), e(v) \in \{1, \ldots, r\}$, so that $s(v) \leq e(v)$ and $v \in X_j$ if and only if $j \in \{s(v), s(v)+1, \ldots, e(v)\}$.

The width of a path decomposition $(X_1, X_2, \ldots, X_r)$ is defined as $\max_i |X_i| - 1$. The pathwidth of a graph $G$ is the minimum possible width over all possible path decompositions of $G$.

A proper path decomposition \cite{27} of a graph $G = (V, E)$ is a path decomposition that additionally satisfies

(4.) For every $u, v \in V$, $\{s(u), s(u)+1, \ldots, e(u)\} \not\subset \{s(v), s(v)+1, \ldots, e(v)\}$.

The proper pathwidth of a graph $G$ is the minimum possible width over all possible proper path decompositions of $G$. Kaplan and Shamir \cite{27} proved that the proper pathwidth of a graph equals its bandwidth.

**Theorem 5.** For a graph $G$, $\text{thin}_{\text{ind}}(G) \leq \text{pw}(G) + 1$ and $\text{pthin}_{\text{ind}}(G) \leq \text{bw}(G) + 1$.

**Proof.** (slight modification of the one in \cite{30}) Consider an optimal (proper) path decomposition $(X_1, X_2, \ldots, X_r)$ of width $q$. Let $k = q + 1$ be the cardinality of the biggest set. We demonstrate that the graph is (proper) independent $k$-thin. We first describe an ordering and then give a description of how we can assign the vertices to $k$ classes, in order to get a partition into $k$ independent sets which is (strongly) consistent with the ordering.

We order the vertices $v$ according to $s(v)$, and breaking ties arbitrarily. Notice that, in the proper case, by (4.), if $s(v) = s(w)$ then also $e(v) = e(w)$, and if $s(v) < s(w)$ then also $e(v) < e(w)$.

We assign the vertices in each $X_i$ to pairwise distinct classes, for $1 \leq i \leq r$. We can do it for $i = 1$, since $|X_1| \leq k$. On each step, for $i > 1$, we assign the vertices of $X_i \setminus X_{i-1}$ to pairwise distinct classes that are not used by the vertices in $X_i \cap X_{i-1}$. This can be done because $|X_i| \leq k$.

No vertex is assigned to more than one class because of condition (3.). The classes are independent sets because of condition (2.).

Let $v \in V(G)$, with $s(v) = i$. Then all neighbors $u$ of $v$ smaller than $v$ are also present in $X_i$, i.e., $e(u) \geq i$. Suppose $u < z < v$, $uv \in E(G)$, $u, z$ in
the same class. Note that if \( z \) and \( u \) are in the same class, there is no subset \( X_k \) such that \( z, u \) are both in \( X_k \). Thus \( u < z \) tells us that \( s(u) < s(z) \). Also, \( z < v \) tells us that \( s(z) \leq s(v) \). But by the claim above, \( e(u) \geq s(v) \), since \( u \) is a neighbor of \( v \) smaller than \( v \). Thus, by (3.), \( u \in X_{s(z)} \), a contradiction with the fact that no set contains two vertices of the same class. Thus, the partition and the ordering of \( V(G) \) we obtained are consistent. It follows that \( \text{thin}_{\text{ind}}(G) \leq \text{pw}(G) + 1 \).

For a (proper) path decomposition, the reverse of the order defined is a decreasing order by \( e(v) \). So the same argument proves that, in that case, the partition and the ordering of \( V(G) \) we obtained are strongly consistent. It follows that \( \text{pthin}_{\text{ind}}(G) \leq \text{bw}(G) + 1 \). □

This bounds are tight, respectively, for interval and proper graphs, where both the (proper) independent thinness and the (proper) pathwidth plus one equal the clique number of the graph, but can be arbitrarily bad, for example, for complete bipartite graphs, that are proper independent 2-thin but have unbounded pathwidth.

Notice that Theorem 4 improves that bound for proper thinness of graphs with at least one edge. As a consequence of Theorem 4, we have the following.

**Corollary 6.** Let \( G \) be a connected graph. Then \( \text{pthin}(G) \leq |V(G)| - \text{diam}(G) \), where \( \text{diam}(G) \) denotes the diameter of \( G \).

**Proof.** In [12], it was proved that for a connected graph, \( \text{bw}(G) \leq |V(G)| - \text{diam}(G) \). □

### 4. Rectangle intersection models for 2-thin and proper 2-thin graphs

We call box a rectangle that is aligned with the Cartesian axes in \( \mathbb{R}^2 \), i.e., the Cartesian product of two segments \( [x_1, x_2] \times [y_1, y_2] \). We say that the box \( b \) is defined by \( x_1, x_2, y_1, y_2 \), which will be denoted by \( X_1(b), X_2(b), Y_1(b), \) and \( Y_2(b) \), respectively. The upper-right corner of \( b \) is the point \( (x_2, y_2) \). The vertical (resp. horizontal) prolongation is the Cartesian product \( P_Y(b) = [x_1, x_2] \times \mathbb{R} \) (resp. \( P_X(b) = \mathbb{R} \times [y_1, y_2] \)).

A set of boxes is 2-diagonal if their upper-right corners are pairwise distinct and each of them lies, for some constant values \( d_1 < 0 < d_2 \), either
in the intersection of the diagonal \( y = x + d_1 \) and the 4th quadrant of the Cartesian plane, or in the intersection of the diagonal \( y = x + d_2 \) and the 2nd quadrant of the Cartesian plane. A set of boxes is \textit{weakly 2-diagonal} if their upper-right corners are pairwise distinct and each of them lies, for some constant values \( d_1 < d_2 \), either in the diagonal \( y = x + d_1 \) or in the diagonal \( y = x + d_2 \). We will call \( y = x + d_1 \) the \textit{lower diagonal} and \( y = x + d_2 \) the \textit{upper diagonal}.

A 2-diagonal model is \textit{blocking} if for every two non-intersecting boxes \( b_1, b_2 \) in the upper and lower diagonal, resp., either the vertical prolongation of \( b_1 \) intersects \( b_2 \) or the horizontal prolongation of \( b_2 \) intersects \( b_1 \) (see Figure 3). A 2-diagonal model is \textit{bi-semi-proper} if for any two boxes \( b, b' \), defined by \( x_1, x_2, y_1, y_2 \) and \( x'_1, x'_2, y'_1, y'_2 \) and such that \( y_2 - x_2 = y'_2 - x'_2 \) and \( x_2 < x'_2 \), it holds \( x_1 \leq x'_1 \) and \( y_1 \leq y'_1 \) (see Figure 4).

Let \( G = (V, E) \) be a 2-thin graph, with partition \( V^1, V^2 \) consistent with an order \( < \). Let \( V^1 = v_1 < \cdots < v_{n_1}, V^2 = w_1 < \cdots < w_{n_2} \). Let \( U(1, v_i) = i \) if \( v_i \) has no neighbors smaller than itself in \( V^1 \), or \( \min \{ j : v_j < v_i, v_j v_i \in E(G) \} \), otherwise. Let \( U(2, w_i) = i \) if \( w_i \) has no neighbors smaller than itself in \( V^2 \), or \( \min \{ j : w_j < w_i, w_j w_i \in E(G) \} \), otherwise. Let \( U(2, v_i) = 0 \) if \( v_i \) is adjacent to all the vertices of \( V^2 \) which are smaller than itself, or \( \max \{ j : w_j < v_i, w_j v_i \notin E(G) \} \), otherwise. Let \( U(1, w_i) = 0 \) if \( w_i \) is adjacent to all the vertices of \( V^1 \) which are smaller than itself, or \( \max \{ j : v_j < w_i, v_j w_i \notin E(G) \} \), otherwise.

Figure 3: Upper-right corner, vertical and horizontal prolongations (first), a weakly 2-diagonal model (second), a blocking 2-diagonal model (third), a 2-diagonal model that is not blocking, where the gray boxes do not satisfy the required property (fourth).

Figure 4: The first two situations are bi-semi-proper, the last three are not.
We define the following model of $G$ as intersection of boxes in the plane (slight modification of the model in [11]), which we will denote by $M_1(G)$: the upper-right corner of $v_i$ is $(i + n_2, i)$, for $1 \leq i \leq n_1$, and the upper-right corner of $w_i$ is $(i, i + n_1)$, for $1 \leq i \leq n_2$; the lower-left corner of $v_i$ is $(U(2, v_i) + 0.5, U(1, v_i) - 0.5)$, for $1 \leq i \leq n_1$, and the lower-left corner of $w_i$ is $(U(2, w_i) - 0.5, U(1, w_i) + 0.5)$, for $1 \leq i \leq n_2$. Intuitively, the boxes having the upper right corner in the higher (resp. lower) diagonal “go down” (resp. left) and stop just to avoid the greatest non-neighbor smaller than themselves in the other class (if any), and “go left” (resp. down) enough to catch all the neighbors smaller than themselves in their own class (if any, and without intersecting a non-neighbor). An example in depicted in Figure 5.

Lemma 7. Let $G = (V, E)$ be a 2-thin graph, with partition $V^1, V^2$ consistent with an order $<$. Then $M_1(G)$ is a blocking 2-diagonal intersection model for $G$ that respects the relative order on each class. Moreover, if the order and the partition are strongly consistent, the model is bi-semi-proper.

Proof. It is straightforward that the model is 2-diagonal (after a translation of $(n_2, n_1)$ to the origin) and respects the relative order on each class. Let us prove that $M_1(G)$ is an intersection model for $G$ and that it is blocking.

• Let $v_i < v_j$ adjacent. Then $U(1, v_j) \leq i$ and $U(2, v_j) \leq n_2$, so the boxes of $v_i$ and $v_j$ intersect.

• Let $v_i < v_j$ not adjacent. Because of the consistency between order and partition, $U(1, v_j) > i$, thus the boxes of $v_i$ and $v_j$ do not intersect.

• Let $w_i < w_j$ adjacent. Then $U(2, w_j) \leq i$ and $U(1, w_j) \leq n_1$, so the boxes of $w_i$ and $w_j$ intersect.

• Let $w_i < w_j$ not adjacent. Because of the consistency between order and partition, $U(2, w_j) > i$, thus the boxes of $w_i$ and $w_j$ do not intersect.

• Let $v_i < w_j$. Then $U(2, v_i) < j$. If they are adjacent, by the consistency between order and partition, $U(1, w_j) < i$ and the boxes of $v_i$ and $w_j$ intersect. If they are not adjacent, $U(1, w_j) \geq i$ and the boxes of $v_i$ and $w_j$ do not intersect. In this case, $P_Y(w_j) \cap v_i \neq \emptyset$.

• Let $w_i < v_j$. Then $U(1, w_i) < j$. If they are adjacent, by the consistency between order and partition, $U(2, v_j) < i$ and the boxes of $v_i$ and $w_j$
intersect. If they are not adjacent, \( U(2, v_j) \geq i \) and the boxes of \( v_i \) and \( w_j \) do not intersect. In this case, \( P_X(v_j) \cap w_i \neq \emptyset \).

It remains to observe that if the order and the partition are strongly consistent and \( x_i < x_j \) are in the same class, then \( U(1, x_i) \leq U(1, x_j) \) and \( U(2, x_i) \leq U(2, x_j) \), so the model is bi-semi-proper.

\[ \square \]

**Theorem 8.** A graph is 2-thin if and only if it has a blocking 2-diagonal model. Moreover, if a graph \( G \) is 2-thin and the partition \( V^1, V^2 \) of its vertices is consistent with an order \( < \), then there exists a blocking 2-diagonal model such that on each of the diagonals lie, respectively, the upper-right corners of the vertices of \( V^1 \) and \( V^2 \), in such a way that their order corresponds to \( < \) restricted to the respective part. Conversely, if a graph \( G \) admits a blocking 2-diagonal model, then there exists an order of the vertices of \( G \) that is consistent with the partition given by the diagonals where the upper-right corners lie, and extends their order on the respective diagonals.

**Proof.**

\( \Rightarrow \) It follows from Lemma 7.

\( \Leftarrow \) Let us consider a blocking 2-diagonal model of \( G \), and let \( V^1 \) and \( V^2 \) be the vertices corresponding to boxes whose upper-right corners lie in the lower and upper diagonal, respectively. We will slightly abuse notation and use it indistinctly for a vertex and the box representing it.

Let \( \Pi = \{ V^1, V^2 \} \), \( < \) be the order of \( V^1 \) and \( V^2 \) defined by the \( X_2 \) coordinates on each of the sets, and where a vertex of \( V^1 \) and a vertex of \( V^2 \) are not comparable.

Let us first prove that \( < \) is consistent restricted to \( V^i \), \( i = 1, 2 \). Let \( x < y < z \) in \( V^1 \) with \( xz \in E(G) \) (the definitions are symmetric with respect to both classes). Then \( X_2(x) < X_2(y) < X_2(z) \) and since \( xz \in E(G) \), it holds \( X_1(z) < X_2(x) < X_2(y) \) and \( Y_1(z) < Y_2(x) = X_2(x) + d_1 < X_2(y) + d_1 = Y_2(y) \). Therefore, \( yz \in E(G) \).

Let \( D = D(G, \Pi, <) \). By the blocking property, given two vertices \( v_i \in V^i \), \( v_{3-i} \in V^{3-i} \), if \( v_iv_{3-i} \in A(D) \), then the appropriate prolongation of \( v_{3-i} \) intersects \( v_i \). As observed above, an ordering of \( V(G) \) is consistent with the partition \( V^1, V^2 \) and extends the partial order \( < \) if and only if it is a topological ordering of \( D \).
Let us prove now that $D$ is acyclic, thus it admits a topological ordering. Suppose it is not, and let us consider a shortest oriented cycle of $D$. Moreover, since the subdigraph induced by each class is complete and acyclic, the cycle has at most two vertices of each class, and necessarily an arc from $V^1$ to $V^2$ and another from $V^2$ to $V^1$.

**Case 1:** The cycle consists of two vertices, $v_1 \in V^1$ and $v_2 \in V^2$.

In this case, $v_1$ and $v_2$ do not intersect but the horizontal prolongation of $v_1$ intersects $v_2$ and the vertical prolongation of $v_2$ intersects $v_1$, which is not possible.

**Case 2:** The cycle is $v_1w_1w_2$ such that $v_1$, $w_1 \in V^1$ and $w_2 \in V^2$.

Since $v_1w_1 \in D$, we have $X_2(v_1) < X_2(w_1)$ and therefore $Y_2(v_1) < Y_2(w_1)$. The horizontal prolongation of $v_1$ intersects $v_2$, therefore $Y_1(v_2) < Y_2(v_1) < Y_2(w_1)$, and the vertical prolongation of $v_2$ intersects $w_1$, therefore $X_1(w_1) < X_2(v_2)$, contradicting that $v_2$ and $w_1$ do not intersect because they are not adjacent.

**Case 3:** The cycle is $v_2w_2v_1$ such that $v_1 \in V^1$ and $v_2, w_2 \in V^2$.

Since $v_2w_2 \in D$, we have $X_2(v_2) < X_2(w_2)$. The vertical prolongation of $v_2$ intersects $v_1$, therefore $X_1(v_1) < X_2(v_2) < X_2(w_2)$, and the horizontal prolongation of $v_1$ intersects $w_2$, therefore $Y_1(w_2) < Y_2(v_1)$, contradicting that $v_1$ and $w_2$ do not intersect because they are not adjacent.

**Case 4:** The cycle is $v_1w_1w_2v_2$ such that $v_1, w_1 \in V^1$ and $v_2, w_2 \in V^2$.

Since $v_1w_1 \in D$, for $i = 1, 2$, we have $X_2(v_i) < X_2(w_i)$ and therefore $Y_2(v_i) < Y_2(w_i)$.

The vertical prolongation of $v_2$ intersects $w_1$ and $v_2$ does not intersect $w_1$, therefore $Y_2(v_1) < Y_2(w_1) < Y_1(v_2)$. The horizontal prolongation of $v_1$ intersects $w_2$ and $v_1$ does not intersect $w_2$, therefore $X_2(v_2) < X_2(w_2) < X_1(v_1)$. This contradicts for $v_1$ and $v_2$ the fact that the model is blocking. □

Propositions 9 and 12 show that the blocking property is necessary for Theorem 8, since there are graphs having a 2-diagonal model which are not 2-thin.

Let $G$ be the graph defined in the following way: $V(G) = A \cup B$, where $A = a_1, \ldots, a_{36}$, $B = b_1, \ldots, b_{36}$, and $A = A_0 \cup A_1 \cup \cdots \cup A_5$, $B = B_0 \cup B_1 \cup \cdots \cup B_5$, where $A_5 = \{a_{33}, a_{34}, a_{35}, a_{36}\}$, $A_0 = \{a_i : 1 \leq i \leq 32, i \text{ is odd}\}$, and, for $1 \leq k \leq 4$, $A_k = \{a_i : 8(k - 1) < i \leq 8k \text{ and } i \text{ is even}\}$; $B_j = \{b_i : a_i \in A_j\}$, for $0 \leq j \leq 5$. The edges joining $A$ and $B$ are such that: for $1 \leq j \leq 4$, $A_j$ is complete to $B_j$ and $B_5$, and anticomplete to $B_0$ and $B_i$, $1 \leq i \leq 4$, $i \neq j$; $A_5$ is anticomplete to $B_0$ and complete to $B \setminus B_0$. Besides, $a_{2k-1}a_{2k} \in E(G)$.
and $b_{2k-1}b_{2k} \in E(G)$, for $1 \leq k \leq 16$, and these are the only internal edges of $A$ and $B$.

**Proposition 9.** The graph $G$ has a representation as intersection of boxes having the 2-diagonal property.

**Proof.** The representation is as follows: the upper right corner of $a_i$ is $(i, i + 36)$ for $1 \leq i \leq 36$; the lower left corners are, for $a_i$ in $A_0$, $(i - 0.5, i + 35.5)$, for $a_i$ in $A_1$, $(i - 1.5, 0)$, for $a_i$ in $A_2$, $(i - 1.5, 8.5)$, for $a_i$ in $A_3$, $(i - 1.5, 16.5)$, for $a_i$ in $A_4$, $(i - 1.5, 24.5)$, for $a_i$ in $A_5$, $(i - 0.5, 0)$. If the lower left and upper right corners of $a_i$ are $(x, y)$ and $(w, z)$, respectively, then the lower left and upper right corners of $b_i$ are $(y, x)$ and $(z, w)$, respectively. It is not hard to check that this is a representation of $G$. It is drawn in Figure A.15. □

**Lemma 10.** Let $H$ be a complete bipartite graph with bipartition $(A, B)$. In every 2-thin representation of $H$, except perhaps for the greatest vertex of $A$ and the greatest vertex of $B$ (according to the order associated with the representation), every vertex of $A$ is in one class and every vertex of $B$ is in the other class.

**Proof.** Let $\prec$ be the order associated with a 2-thin representation of $H$. Let $a_M$ (resp. $b_M$) be the greatest vertex of $A$ (resp. $B$) according to $\prec$. By
symmetry of the graph (since the sizes of $A$ and $B$ are not specified in the statement), we may assume without loss of generality $a_M > b_M$.

Claim 11. $A \setminus \{a_M\}$ is complete to $B \setminus \{b_M\}$ in $G_\prec$.

Let $a \in A \setminus \{a_M\}$, $b \in B$ such that $b < a$. Then, $b < a < a_M$, $a_M b \in E(G)$ and $a_M a \not\in E(G)$, therefore $ab \in E(G_\prec)$. Now, let $b \in B \setminus \{b_M\}$, $a \in A$ such that $a < b$. Then, $a < b < b_M$, $b_M a \in E(G)$ and $b_M b \not\in E(G)$, thus $ab \in E(G_\prec)$. Hence, $A \setminus \{a_M\}$ is complete to $B \setminus \{b_M\}$ in $G_\prec$. \hfill \Box

By the claim above, $A \setminus \{a_M\}$ and $B \setminus \{b_M\}$ are in different sets of the partition, and since there are only two sets in the partition, the statement holds.

Proposition 12. The graph $G$ has thinness 3.

Proof. Let $V^1 = A \setminus A_0$, $V^2 = B \setminus B_0$, $V^3 = A_0 \cup B_0$ and the order given by $a_1, a_2, b_1, b_2, a_3, a_4, b_3, b_4, \ldots a_{35}, a_{36}, b_{35}, b_{36}$. It is not difficult to see that the order and the partition are consistent.

Now, suppose that $G$ admits a 2-thin representation $(\prec, V^1, V^2)$. Notice that $(A_j \cup A_5, B_j \cup B_5)$ induce a complete bipartite graph, for $j = 1, \ldots, 4$. So, by Lemma 10 and transitivity, except perhaps for a few vertices, the vertices of $A \setminus A_0$ are in one of the sets of the partition, say $V^1$, and the vertices of $B \setminus B_0$ are in the other, say $V^2$. Let us call $A'$ the vertices of $A \setminus A_0$ that are in $V^2$ and $B'$ the vertices of $B \setminus B_0$ that are in $V^1$. For $j = 1, \ldots, 5$, let $A'_j = A_j \setminus A'$ and $B'_j = B_j \setminus B'$.

For $1 \leq j \leq 5$, let $a^{i}_j, a^3_j$ and $b^{i}_j, b^3_j$ be the greatest and smallest vertices of $A'_j$ and $B'_j$, respectively.

Let $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$. Then, for every vertex $a \in A'_i \cup A'_k \cup A'_\ell$, either $a > b^i_M$ or $a < a'$ for every $a' \in A'_i$, and, analogously, for every vertex $b \in B'_i \cup B'_k \cup B'_\ell$, either $b > a^i_M$ or $b < b'$ for every $b' \in B'_i$. By symmetry of the graph $G$, we may assume $a^1_M < a^2_M < a^3_M < a^4_M$ and $a^4_M > \max\{b^1_M, b^2_M, b^3_M, b^4_M\}$. By the observation above, for every $b \in B'_3 \cup B'_2 \cup B'_1$ and $b' \in B'_4$, it holds $b < b'$. In particular, $b^4_M > b^1_S > \max\{b^3_M, b^2_M, b^1_M\}$. Suppose $a^3_M < b^3_S$. Then $a^3_M > b^3_S > b^3_M$, $a^3_M b^3_M \in E(G)$ and $a^3_M b^3_S \not\in E(G)$, a contradiction. Then $a^3_M < b^3_S < b^4_M$, and hence, for every $a \in A'_3 \cup A'_2 \cup A'_1$ and $a' \in A'_4$, it holds $a < a'$, i.e., $a^4_S > a^3_M$.

Suppose $b^3_M > a^3_S$. Then $b^3_M > a^3_S > a^3_M$, $a^3_M b^3_M \in E(G)$ and $b^3_M a^3_S \not\in E(G)$, a contradiction. Then $b^3_M < a^3_S$. Suppose now $b^3_M > a^3_M$. Then
\[ b_2^a > a_3^a > a_2^a, \quad b_2^M a_M^a \in E(G) \text{ and } b_2^M a_M^a \notin E(G), \text{ a contradiction. Then } b_2^M < a_M^a, \text{ and hence, for every } b \in B'_2 \cup B'_1 \text{ and } b' \in B'_3, \text{ it holds } b < b', \text{ i.e., } b_S^2 > \max\{b_2^M, b_M^a\}. \]

Next, suppose \( a_M^3 > b_S^2 \). Then \( a_M^3 > b_S^2 > b_2^a, \quad a_M^3 \notin E(G) \text{ and } a_M^2 b_S^3 \notin E(G), \text{ a contradiction. Then } a_M^2 < b_S^3 < b_M^2, \text{ and hence, for every } a \in A'_2 \cup A'_1 \text{ and } a' \in A'_3, \text{ it holds } a < a', \text{ i.e., } a_M^2 < a_S^3. \text{ Suppose now } b_M^2 > a_S^3. \text{ Then } b_M^2 > a_S^3 > a_M^3, \quad a_M^3 \notin E(G) \text{ and } b_M^2 a_S^3 \notin E(G), \text{ a contradiction. Then } b_M^2 < a_S^3. \]

Suppose \( b_M^1 > a_M^2. \text{ Then } b_M^1 > a_M^2 > a_M^1, \quad b_M^1 a_M^1 \in E(G) \text{ and } b_M^1 a_M^2 \notin E(G), \text{ a contradiction. Then } b_M^1 < a_M^2, \text{ and hence, for every } b \in B'_1 \text{ and } b' \in B'_2, \text{ it holds } b < b', \text{ i.e., } b_S^2 > b_M^1. \text{ Suppose } b_M^1 > a_S^2. \text{ Then } b_M^1 > a_S^2 > a_M^1, \quad a_M^1 b_M^1 \in E(G) \text{ and } b_M^1 a_S^2 \notin E(G), \text{ a contradiction. Then } b_M^1 < a_S^2. \]

Finally, suppose \( a_M^1 > b_S^2. \text{ Then } a_M^1 > b_S^2 > b_M^1, \quad a_M^1 b_M^1 \in E(G) \text{ and } a_M^1 b_S^2 \notin E(G), \text{ a contradiction. Then } a_M^1 < b_S^2, \text{ and hence, for every } a \in A'_1 \text{ and } a' \in A'_2, \text{ it holds } a < a', \text{ i.e., } a_M^1 < a_S^2. \text{ Suppose now } b_M^1 > a_S^2. \text{ Then } b_M^1 > a_S^2 > a_M^1, \quad a_M^1 b_M^1 \in E(G) \text{ and } b_M^1 a_S^2 \notin E(G), \text{ a contradiction. Then } b_M^1 < a_S^2. \]

So, \( A'_1 < A'_2 < A'_3 < A'_4, \quad B'_1 < B'_2 < B'_3 < B'_4, \quad \text{ and } \max\{a_M^1, b_M^1\} < \min\{a_S^3, b_S^2\} < \max\{a_M^1, b_M^2\} < \min\{a_S^3, b_S^3\} < \max\{a_M^3, b_M^3\} < \min\{a_S^4, b_S^2\} < b_M^3 < a_M^4. \]

The vertices in \( B'_3 \) have to be greater than \( b_M^3, \) which is a non-neighbor of \( a_M^4 \) and smaller than it. Similarly, the vertices in \( A'_5 \) have to be greater than \( a_M^3. \]

Let \( a_2^a, a_3^a \) be the second and third greatest vertices of \( A'_2, \) respectively. Let \( a_0 \) be the neighbor of \( a_2^a \) in \( A_0. \]

Suppose first \( a_0 \in V'. \text{ If } a_0 > a_2^a, \text{ then } a_0 < a_2^a, \text{ because } a_0 a_2^a \notin E(G). \]

If \( a_0 < a_2^a, \text{ then } a_0 > a_3^a, \text{ because } a_2^a a_3^a \notin E(G). \]

Let \( b_5^5 \in B'_5. \text{ Then } b_5^5 > b_M^3 > a_3^a > a_2^a > a_0 > a_S^2, \text{ but } b_5^5 a_S^2 \in E(G) \text{ and } b_5^5 a_0 \notin E(G), \text{ a contradiction.} \]

Suppose now \( a_0 \in V'. \text{ If } a_0 > a_2^a, \text{ then } a_0 < a_2^a, \text{ because } a_0 a_2^a \notin E(G). \]

If \( a_0 < a_2^a, \text{ then } a_0 > b_M^1, \text{ because } a_2^a > b_M^1 \text{ and } a_2^a b_M^1 \notin E(G). \text{ Let } a_5^5 \in A'_5. \]

Then \( a_5^5 > a_3^a > a_2^a > a_0 > b_M^1, \text{ but } a_3^a b_M^1 \in E(G) \text{ and } a_5^5 a_0 \notin E(G), \text{ a contradiction.} \]

\[
\]

For proper 2-thin graphs, we can relax the 2-diagonal property and do not require the blocking property, by requiring the model to be bi-semi proper.

**Theorem 13.** Let \( G \) be a graph. The following statements are equivalent:

(i) \( G \) is a proper 2-thin graph.
(ii) \( G \) has a bi-semi-proper blocking 2-diagonal model.

(iii) \( G \) has a bi-semi-proper weakly 2-diagonal model.

Moreover, if \( G \) is proper 2-thin and the partition \( V^1, V^2 \) of its vertices is strongly consistent with an order \( < \), then there exists a bi-semi-proper blocking 2-diagonal model such that on each of the diagonals lie, respectively, the upper-right corners of the vertices of \( V^1 \) and \( V^2 \), in such a way that their order corresponds to \( < \) restricted to the respective part. Furthermore, if \( G \) admits a bi-semi-proper weakly 2-diagonal model, then there exists an order of the vertices of \( G \) that is consistent with the partition given by the diagonals where the upper-right corners lie, and extends their order on the respective diagonals.

Proof. (i) \( \Rightarrow \) (ii) It follows from Lemma 7.

(ii) \( \Rightarrow \) (iii) This is straightforward.

(iii) \( \Rightarrow \) (i) Let us consider a bi-semi-proper weakly 2-diagonal model of \( G \). We will slightly abuse notation and use it indistinctly for a vertex and the box representing it. Let \( V^i \) be the set of vertices \( v \) such that \( Y^i(v) - X^i(v) = d_i \), for \( i = 1, 2 \). We may assume without loss of generality that \( d_1 < d_2 \).

Let \( \Pi = \{V^1, V^2\}, < \) be the order of \( V^1 \cup V^2 \) defined by the \( X^i \) coordinates on each of the sets, and where a vertex of \( V^1 \) and a vertex of \( V^2 \) are not comparable.

Let us first prove that \( < \) is strongly consistent restricted to \( V^i, i = 1, 2 \). Let \( x < y < z \) in \( V^1 \) with \( xz \in E(G) \) (the definitions are symmetric with respect to both classes). Then \( X^2(x) < X^2(y) < X^2(z) \) and since \( xz \in E(G) \), it holds \( X^1(z) < X^2(x) < X^2(y) \) and \( Y^1(z) < Y^2(x) = X^2(x) + d_1 < X^2(y) + d_1 = Y^2(y) \). Therefore, \( yz \in E(G) \). By the bi-semi-proper property, \( X^1(y) \leq X^1(z) < X^2(x) \) and \( Y^1(y) \leq Y^1(z) < Y^2(x) \), and therefore \( xy \in E(G) \).

Let \( D = \tilde{D}(G, \Pi, <) \). By Lemma 4, an ordering of \( V(G) \) is strongly consistent with the partition \( V^1, V^2 \) and extends the partial order \( < \) if and only if it is a topological ordering of \( D \), thus let us prove that \( D \) is acyclic. Suppose it is not, and consider a shortest oriented cycle of \( D \). Moreover, since the subdigraph induced by each class is complete and acyclic, the cycle has at most two vertices of each class, and necessarily an arc from \( V^1 \) to \( V^2 \) and another from \( V^2 \) to \( V^1 \).

The possible types of arcs joining \( v_1 \in V^1 \) and \( v_2 \in V^2 \) are the following:
• Type $a$ if $X_2(v_1) > X_2(v_2)$ and $Y_2(v_1) < Y_2(v_2)$;
• Type $b$ if $X_2(v_1) < X_2(v_2)$ and $Y_2(v_1) < Y_2(v_2)$;
• Type $c$ if $X_2(v_1) > X_2(v_2)$ and $Y_2(v_1) > Y_2(v_2)$;
• Type $d$ if $X_2(v_1) = X_2(v_2)$ and $Y_2(v_1) < Y_2(v_2)$;
• Type $e$ if $X_2(v_1) > X_2(v_2)$ and $Y_2(v_1) = Y_2(v_2)$,

and we use the subindex $ij$ if the orientation of the arc is from $V^i$ to $V^j$.

The following properties hold because the model is bi-semi-proper.

• Type $a_{12}$: $Y_1(v_2) > Y_2(v_1)$ and $X_1(v_1) < X_2(v_2)$;
• Type $a_{21}$: $Y_1(v_2) < Y_2(v_1)$ and $X_1(v_1) > X_2(v_2)$;
• Type $b_{12}$: $Y_1(v_2) > Y_2(v_1)$ or $X_1(v_2) > X_2(v_1)$;
• Types $b_{21}$ and $c_{12}$: cannot exist;
• Type $c_{21}$: $Y_1(v_1) > Y_2(v_2)$ or $X_1(v_1) > X_2(v_2)$;
• Type $d_{12}$: $Y_1(v_2) > Y_2(v_1)$;
• Types $d_{21}$ and $e_{12}$: cannot exist;
• Type $e_{21}$: $X_1(v_1) > X_2(v_2)$.

Let us see the possible cycles.

**Case 1:** The cycle consists of two vertices, $v_1 \in V^1$ and $v_2 \in V^2$.

It cannot happen, since for every $t \in \{b, c, d, e\}$, one of $t_{12}$ or $t_{21}$ cannot exist, and $a_{12}$ and $a_{21}$ cannot occur simultaneously.

**Case 2:** The cycle is $v_1 w_1 v_2$ such that $v_1, w_1 \in V^1$ and $v_2 \in V^2$.

Since $v_1 w_1 \in D$, we have $X_2(v_1) < X_2(w_1)$ and therefore $Y_2(v_1) < Y_2(w_1)$.

As the model is bi-semi-proper, $X_1(v_1) \leq X_1(w_1)$ and $Y_1(v_1) \leq Y_1(w_1)$.

The arc $w_1 v_2$, as seen before, may be of type $a_{12}$, thus $Y_1(v_2) > Y_2(w_1)$ and $X_1(w_1) < X_2(v_2)$; or type $b_{12}$, thus $Y_1(v_2) > Y_2(w_1)$ or $X_1(v_2) > X_2(w_1)$; or type $d_{12}$, and in that case $Y_1(v_2) > Y_2(w_1)$.

Also, the arc $v_2 v_1$, as seen before, can be type $a_{21}$, thus $Y_1(v_2) < Y_2(v_1)$ and $X_1(v_1) > X_2(v_2)$; or type $c_{21}$, thus $Y_1(v_1) > Y_2(v_2)$ or $X_1(v_1) > X_2(v_2)$; or type $e_{21}$, and in that case $X_1(v_1) > X_2(v_2)$. 

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The options $a_{12}$ and $a_{21}$ imply $Y_2(w_1) < Y_1(v_2) < Y_2(v_1)$, but we know that $Y_2(v_1) < Y_2(w_1)$.

The options $a_{12}$ and $c_{21}$ imply either $Y_2(v_1) > Y_1(v_1) > Y_2(v_2) > Y_1(v_2) > Y_2(w_1)$, which contradicts $Y_2(v_1) < Y_2(w_1)$, or $X_1(v_1) > X_2(v_2) > X_1(w_1)$, which contradicts $X_1(v_1) \leq X_1(w_1)$.

The options $a_{12}$ and $e_{21}$ imply $X_1(v_1) > X_2(v_2) > X_1(w_1)$, which contradicts $X_1(v_1) \leq X_1(w_1)$.

The options $b_{12}$ and $a_{21}$ are incompatible because they imply either $Y_2(v_1) > Y_1(v_1) > Y_2(w_1)$, that contradicts $Y_2(v_1) < Y_2(w_1)$, or $X_2(v_1) > X_1(v_1) > X_2(v_2) > X_1(w_1)$, that contradicts $X_2(v_1) < X_2(w_1)$.

The options $b_{12}$ and $c_{21}$ are incompatible because $Y_2(v_1) > Y_1(v_1) > Y_2(w_1)$ contradicts $Y_2(v_1) < Y_2(w_1)$; $X_2(v_1) > X_1(v_1) > X_2(v_2)$ contradicts $X_2(v_1) < X_2(w_1)$; $X_1(v_1) > X_2(v_2) > X_1(v_2) > X_2(w_1)$ and $Y_1(v_1) > Y_2(v_2)$ implies $Y_2(v_2) = X_2(v_2) + d_2 > X_1(v_2) + d_2 > X_2(v_1) + d_2 = Y_2(v_1) + d_2 - d_1 > Y_1(v_1) + d_2 - d_1 > Y_2(v_2) + d_2 - d_1 > Y_2(v_2)$, a contradiction. Finally, suppose that $Y_1(v_2) > Y_2(w_1)$ and $X_1(w_1) \geq X_1(v_1) > X_2(v_2)$.

The existence of the arc $w_1v_2$ implies that either there exists $v'_2 \in V^2$ with $X_2(v'_2) < X_2(v_2)$ and $w_1v'_2 \in E(G)$, or there exists $v'_1 \in V^1$ with $X_2(v'_1) > X_2(w_1)$ and $v'_1v_2 \in E(G)$. But $X_2(v'_2) < X_2(v_2) < X_1(w_1)$ contradicts $w_1v'_2 \in E(G)$ and $X_2(v'_1) > X_2(v_2)$ implies $X_1(v'_1) \geq X_1(w_1) > X_2(v_2)$ which contradicts $v'_1v_2 \in E(G)$.

The options $b_{12}$ and $e_{21}$ are incompatible because they imply either $Y_2(v_1) = Y_1(v_2) > Y_2(w_1)$, a contradiction, or $X_1(v_1) > X_2(v_2) > X_1(v_2) > X_2(w_1)$, a contradiction too.

The options $d_{12}$ and $a_{21}$ imply $Y_2(v_1) > Y_1(v_2) > Y_2(w_1)$, a contradiction.

The options $d_{12}$ and $c_{21}$ imply either $Y_2(v_1) > Y_1(v_1) > Y_2(v_2) > Y_1(v_2)$, a contradiction, or $X_2(v_1) > X_1(v_1) > X_2(v_2) = X_2(w_1)$, a contradiction too.

The options $d_{12}$ and $e_{21}$ imply $Y_2(v_1) = Y_2(v_2) > Y_1(v_2) > Y_2(w_1)$, a contradiction.

**Case 3:** The cycle is $v_2w_2v_1$ such that $v_1 \in V^1$ and $v_2, w_2 \in V^2$.

Since $v_2w_2 \in D$, we have $X_2(v_2) < X_2(w_2)$ and therefore $Y_2(v_2) < Y_2(w_2)$.

As the model is bi-semi-proper, $X_1(v_2) \leq X_1(w_2)$ and $Y_1(v_2) \leq Y_1(w_2)$.

The arc $v_1v_2$, as seen before, can be type $a_{12}$, thus $Y_1(v_2) > Y_2(v_1)$ and $X_1(v_1) < X_2(v_2)$, or type $b_{12}$, thus $Y_1(v_2) > Y_2(v_1)$ or $X_1(v_2) > X_2(v_1)$; or type $d_{12}$, and in that case $Y_1(v_2) > Y_2(v_1)$.

Also, the arc $w_2v_1$, as seen before, can be type $a_{21}$, thus $Y_1(w_2) < Y_2(v_1)$ and $X_1(v_1) > X_2(w_2)$, or type $c_{21}$, thus $Y_1(v_1) > Y_2(w_2)$ or $X_1(v_1) > X_2(w_2)$; or type $e_{21}$, and in that case $X_1(v_1) > X_2(w_2)$.
The options $a_{12}$ and $a_{21}$ imply $Y_1(v_2) > Y_2(v_1) > Y_1(w_2)$, a contradiction.

The options $a_{12}$ and $c_{21}$ imply either $Y_2(v_2) > Y_1(v_2) > Y_2(v_1)$, a contradiction, or $X_2(v_2) < X_1(v_1) < X_2(v_2)$, a contradiction.

The options $a_{12}$ and $e_{21}$ imply $X_2(v_2) < X_1(v_1) < X_2(v_2)$, a contradiction.

The options $b_{12}$ and $a_{21}$ imply either $Y_1(v_2) > Y_2(v_1) > Y_1(w_2)$, a contradiction, or $X_2(v_2) > X_1(v_2) > X_2(v_1) > X_1(v_1) > X_2(w_2)$, a contradiction.

The options $b_{12}$ and $c_{21}$ are incompatible because $Y_2(v_2) > Y_1(v_2) > Y_2(v_1) > Y_2(w_2)$ contradicts $Y_2(v_2) < Y_2(v_2)$; $X_1(v_2) > X_2(v_1) > X_2(v_2)$ contradicts $X_1(v_2) < X_2(v_2)$; $X_1(v_2) > X_2(v_1)$ and $Y_1(v_1) > Y_2(w_2)$ implies $Y_2(v_2) = X_2(v_2) + d_2 > X_1(v_2) + d_2 > X_2(v_1) + d_2 = Y_2(v_1) + d_2 - d_1 > Y_1(v_1) + d_2 - d_1 > Y_2(w_2) + d_2 - d_1 > Y_2(w_2)$, a contradiction. Finally, suppose that $Y_1(v_2) > Y_2(v_1)$ and $X_1(v_1) > X_2(v_2)$. The existence of the arc $v_1v_2$ implies that either there exists $v'_2 \in V^2$ with $X_2(v'_2) < X_2(v_2)$ and $v_1v'_2 \in E(G)$, or there exists $v'_1 \in V^1$ with $X_2(v'_1) > X_2(v_1)$ and $v'_1v_2 \in E(G)$. But $X_2(v'_2) < X_2(v_2) < X_2(v_2) < X_1(v_1)$ contradicts $v_1v'_2 \in E(G)$ and $X_2(v'_1) > X_2(v_1)$ implies $X_1(v'_1) \geq X_1(v_1) > X_2(w_2) > X_2(v_2)$ which contradicts $v'_1v_2 \in E(G)$.

The options $b_{12}$ and $e_{21}$ imply $Y_2(v_2) > Y_1(v_2) > Y_2(v_1) = Y_2(w_2)$, a contradiction.

The options $d_{12}$ and $a_{21}$ imply $Y_1(v_2) > Y_2(v_1) > Y_1(w_2)$, a contradiction.

The options $d_{12}$ and $c_{21}$ imply either $Y_2(v_2) > Y_1(v_2) > Y_2(v_1)$, a contradiction, or $X_2(v_2) = X_1(v_2) > X_1(v_1) > X_2(v_2)$, a contradiction.

The options $d_{12}$ and $e_{21}$ imply $Y_2(v_2) > Y_1(v_2) > Y_2(v_1) = Y_2(w_2)$, a contradiction.

**Case 4:** The cycle is $v_1w_1v_2w_2$ such that $v_1, w_1 \in V^1$ and $v_2, w_2 \in V^2$.

Since $v_iw_i \in D$, for $i = 1, 2$, we have $X_2(v_i) < X_2(w_i)$ therefore $Y_2(v_i) < Y_2(w_i)$. As the model is bi-semi-proper, $X_1(v_i) \leq X_1(w_i)$ and $Y_1(v_i) \leq Y_1(w_i)$.

The arc $w_1v_2$, as seen before, can be type $a_{12}$, thus $Y_1(v_2) > Y_2(w_1)$ and $X_1(w_1) < X_2(v_2)$; or type $b_{12}$, thus $Y_1(v_2) > Y_2(w_1)$ or $X_1(v_2) > X_2(w_1)$; or type $d_{12}$, and in that case $Y_1(v_2) > Y_2(w_1)$.

Also, the arc $w_2v_1$, as seen before, can be type $a_{21}$, thus $Y_1(w_2) < Y_2(v_1)$ and $X_1(v_1) > X_2(w_2)$; or type $c_{21}$, thus $Y_1(v_1) > Y_2(w_2)$ or $X_1(v_1) > X_2(w_2)$; or type $e_{21}$, and in that case $X_1(v_1) > X_2(w_2)$.

The options $a_{12}$ and $a_{21}$ imply $Y_2(v_1) > Y_1(w_1) > Y_2(v_2)$, a contradiction.
The options $a_{12}$ and $c_{21}$ imply $Y_2(v_2) > Y_1(v_2) > Y_2(w_1) > Y_2(v_1) > Y_1(v_1) > Y_2(w_2)$, a contradiction.

The options $a_{12}$ and $e_{21}$ imply $X_1(w_1) < X_2(v_2) < X_2(w_2) < X_1(v_1)$, a contradiction.

The options $b_{12}$ and $a_{21}$ imply either $Y_2(v_1) > Y_1(w_1) > Y_1(v_1) > Y_2(w_1)$, a contradiction, or $X_2(v_2) > X_1(v_2) > X_2(w_1) > X_2(v_1) > X_1(v_1) > X_2(w_2)$, a contradiction.

The options $b_{12}$ and $c_{21}$ are incompatible because $Y_2(v_2) > Y_1(v_2) > Y_2(w_1) > Y_2(v_1) > Y_2(w_2)$ contradicts $Y_2(v_2) < Y_2(w_2)$; $X_2(v_2) > X_1(v_2) > X_2(w_1) > X_2(v_1) > X_2(w_2)$, contradicts $X_2(v_2) < X_2(w_2)$; $X_1(v_2) > X_2(w_1)$ and $Y_1(v_1) > Y_2(w_2)$ implies $Y_2(v_2) = X_2(v_2) + d_2 > X_1(v_2) + d_2 > X_2(w_1) + d_2 > X_2(v_1) + d_2 > Y_1(v_1) + d_2 = Y_2(v_1) + d_2 > d_1 > Y_2(w_2) + d_2 > Y_2(v_2)$, a contradiction. Finally, suppose that $Y_1(v_2) > Y_2(w_1)$ and $X_1(v_1) > X_2(w_2)$. The existence of the arc $w_1v_2$ implies that either there exists $v'_2 \in V^2$ with $X_2(v'_2) < X_2(v_2)$ and $w_1v'_2 \in E(G)$, or there exists $v'_1 \in V^1$ with $X_2(v'_1) > X_2(w_1)$ and $v'_1v_2 \in E(G)$. But $X_2(v'_2) < X_2(v_2) < X_2(w_2) < X_1(v_1) \leq X_1(w_1)$ contradicts $w_1v'_2 \in E(G)$, and $X_2(v'_1) > X_2(w_1)$ implies $X_1(v'_1) \geq X_1(w_1) \geq X_1(v_1) > X_2(w_2) > X_2(v_2)$ which contradicts $v'_1v_2 \in E(G)$.

The options $b_{12}$ and $e_{21}$ imply $X_1(v_1) > X_2(w_1) > X_2(v_2) > X_2(w_2)$, a contradiction.

The options $d_{12}$ and $a_{21}$ imply $Y_1(v_2) > Y_2(w_1) > Y_2(v_1) > Y_1(w_2)$, a contradiction.

The options $d_{12}$ and $c_{21}$ imply $Y_2(v_2) > Y_1(v_2) > Y_2(w_1) > Y_2(v_1) > Y_1(v_1) > Y_2(w_2)$, a contradiction.

The options $d_{12}$ and $e_{21}$ imply $Y_2(v_2) > Y_1(v_2) > Y_2(w_1) > Y_2(v_1) = Y_2(w_2)$, a contradiction.

\[\square\]

The bi-semi-proper requirement is necessary, otherwise we can represent any interval graph (we place the intervals on a diagonal line and make each of them the diagonal of a square box, see Figure 2), and interval graphs may have arbitrarily large proper thinness [5].

The model $\mathcal{M}_1$ can be transformed into a model where the rectangles lie within the 3rd quadrant of the Cartesian plane, and each of the rectangles has either its top side or its right side on a Cartesian axe. We call such a model 2-grounded. Since every pair of intersecting rectangles has nonempty intersection within the third quadrant, it is enough to define the model $\mathcal{M}_2$.
as the intersection of $\mathcal{M}_1$ and the 3rd quadrant of the Cartesian plane (see Figure 8). By adjusting the definitions of blocking and bi-semi-proper for 2-grounded models, we can prove characterizations analogous to those in Theorems 8 and 13 for (proper) 2-thin graphs.

5. Thin graphs as VPG graphs

We will first prove that $B_0$-VPG graphs have unbounded thinness, and that not every 4-thin graph is a VPG graph. Then, we will prove that graphs with thinness at most three have bounded bend number as VPG graphs.

Let $G$ be a graph. Let $\Delta(G)$ be the maximum degree of a vertex in $G$. A subgraph $H$ (not necessarily induced) of $G$ is a spanning subgraph if $V(H) = V(G)$. If $X \subseteq V(G)$, denote by $N(X)$ the set of vertices of $G$ having at least one neighbor in $X$. The vertex isoperimetric peak of a graph $G$, denoted as $b_v(G)$, is defined as $b_v(G) = \max \limits_{s} \min \limits_{X \subset V, |X| = s} |N(X) \cap (V(G) \setminus X)|$, i.e., the maximum over $s$ of the lower bounds for the number of boundary vertices (vertices outside the set with a neighbor in the set) in sets of size $s$.

**Theorem 14.** [11] For every graph $G$ with at least one edge, $\text{thin}(G) \geq b_v(G)/\Delta(G)$.

The following corollary is also useful.

**Corollary 15.** Let $G$ be a graph such that $\Delta(G) \leq d$, and $H$ be a (not necessarily induced) subgraph of $G$ with at least one edge and such that $b_v(H) \geq b$. Then $\text{thin}(G) \geq b/d$.

**Proof.** Let $G'$ be the subgraph of $G$ induced by $V(H)$. Then $\Delta(G') \leq \Delta(G)$. Since $H$ is a spanning subgraph of $G'$, then $b_v(G') \geq b_v(H)$, and since $G'$ is an induced subgraph of $G$, then $\text{thin}(G') \geq \text{thin}(G)$. So, by Theorem 14, $\text{thin}(G) \geq \text{thin}(G') \geq b_v(G')/\Delta(G') \geq b_v(H)/\Delta(G) \geq b/d$. □

For a positive integer $r$, the $(r \times r)$-grid $GR_r$ is the graph whose vertex set is $\{(i, j) : 1 \leq i, j \leq r\}$ and whose edge set is $\{(i, j)(k, l) : |i - k| + |j - l| = 1, \text{ where } 1 \leq i, j, k, l \leq r\}$.

The thinness of the two dimensional $r \times r$ grid $GR_r$ was lower bounded by $r/4$, by using Theorem 14 and the following lemma.

**Lemma 16.** [14] For every $r \geq 2$, $b_v(GR_r) \geq r$. 25
We use these results to prove the unboundedness of the thinness of $B_0$-VPG graphs.

**Proposition 17.** The class of $B_0$-VPG graphs has unbounded thinness.

**Proof.** Let $r \geq 2$ and let $G_r$ be the intersection graph of the following paths on a grid: $\{(i-0.1,j)-(i+1.1,j)_{0 \leq i,j \leq r}\} \cup \{(i,j-0.1)-(i,j+0.1)_{1 \leq i,j \leq r}\}$. The grid $GR_r$ is a subgraph of $G_r$, and $\Delta(G_r) = 6$ (see Figure 6). So, by Lemma 16 and Corollary 15, thin($G_r$) $\geq r/6$. □

**Proposition 18.** Not every 4-thin graph is a VPG graph.

**Proof.** The edge subdivision of the complete graph $K_5$ is 4-thin (see Figure 7 for a representation). Nevertheless, it is not a string graph [40], and string graphs are equivalent to VPG graphs [2]. □

To prove that graphs with thinness at most three have bounded bend number, we start by defining an intersection model obtained from $M_2$ by keeping from each rectangle the path formed by the top and right sides (all the paths have the shape $L$ and it can be proved that both intersection models produce the same graph). We can then reflect vertically and horizontally the model in order to obtain the L-model $M_3$ (see Figure 8). In this way, we prove that 2-thin graphs are L-graphs, thus $B_1$-VPG. When the graph is independent 2-thin it is enough to keep the horizontal part for the paths that are grounded to the y-axis and the vertical part for the paths that are grounded to the x-axis, so independent 2-thin graphs are $B_0$-VPG. Again, by adjusting the definitions of blocking, bi-semi-proper to these 2-grounded
Figure 7: A 4-thin representation of the edge subdivision of $K_5$.

Figure 8: The model $M_2$ as intersection of grounded rectangles (left) and the L-model $M_3$ (right) for the graphs in Figure 1.
models, we can prove characterizations analogous to those in Theorems 8
and 13 for (independent) (proper) 2-thin graphs.

For 3-thin graphs, we use the previous ideas for the intersections within
each class and between each pair of classes. A sketch of a \( B_3 \)-VPG model
for 3-thin graphs and a \( B_1 \)-VPG model for independent 3-thin graphs can be
found in Figure 9.

The 4-wheel \( W_4 \) (obtained from a 4-cycle by adding a universal vertex) is
2-thin and proper independent 3-thin but not \( B_0 \)-VPG (in a \( B_0 \)-VPG graph,
\( N[v] \) is interval for every \( v \)). so the bound of the bend number in those
cases is tight. We conjecture that also the 3 bends bound is tight for 3-thin
graphs, but we are missing an example.

The bend number of 2-thin graphs as edge intersection graphs of paths on
a grid (EPG graphs) is unbounded, since already proper independent 2-thin
graphs contain the class of complete bipartite graphs, that has unbounded
EPG bend number.

Further modifying the model \( M_3 \) by translating vertically the paths that
are grounded to the \( x \)-axe and horizontally the paths that are grounded to the
\( y \)-axe in order to have the bends lying in the diagonal \( y = -x \), we can obtain
a monotone L-model \( M_4 \) for every 2-thin graph. Moreover, we will prove
in the next section that 2-thin graphs are exactly the blocking monotone
L-graphs. An L-model is blocking if for every two non-intersecting L’s, either
the vertical or the horizontal prolongation of one of them intersects the other.

6. Characterization by forbidden patterns

A trigraph \( T \) is a 4-tuple \((V(T), E(T), N(T), U(T))\) where \( V(T) \) is the
vertex set and every unordered pair of vertices belongs to one of the three
disjoint sets \( E(T), N(T), \) and \( U(T) \) called respectively edges, non-edges, and
undecided edges. A graph \( G = (V(G), E(G)) \) is a realization of a trigraph \( T \) if \( V(G) = V(T) \) and \( E(G) = E(T) \cup U' \), where \( U' \subseteq U(T) \). When representing a trigraph, we will draw solid lines for edges, dotted lines for non edges, and nothing for undecided edges. As \( (E(T), N(T), U(T)) \) is a partition of the unordered pairs, it is enough to give any two of these sets to define the trigraph, and we will often define a trigraph by giving only \( E \) and \( N \).

An ordered graph is a graph given with a linear ordering of its vertices. We define the same for a trigraph, and call it a pattern. We say that an ordered graph is a realization of a pattern if they share the same set of vertices and linear ordering and the graph is a realization of the trigraph. When, in an ordered graph, no ordered subgraph is the realization of given pattern, we say that the ordered graph avoids the pattern. The mirror or reverse of a pattern is the same pattern, except the ordering, which is reversed.

Given a family of patterns \( \mathcal{F} \), the class \( \text{Ord}(\mathcal{F}) \) is the set of graphs that have the following property: there exists an ordering of the nodes, such that none of the ordered subgraphs is a realization of a pattern in \( \mathcal{F} \), i.e., the ordered graph avoids all the patterns in \( \mathcal{F} \).

Many natural graph classes can be described as \( \text{Ord}(\mathcal{F}) \) for sets \( \mathcal{F} \) of small patterns \([13, 10, 15, 16, 24]\). For instance, for \( P_1 = (\{1, 2, 3\}, \{13\}, \emptyset) \) and its mirror \( P_2 = (\{1, 2, 3\}, \{13\}, \{12\}) \), the class of interval graphs is \( \text{Ord}(\{P_1\}) \) \([34]\) and the class of proper interval graphs is \( \text{Ord}(\{P_1, P_2\}) \) \([36]\), and for \( P_3 = (\{1, 2, 3\}, \{13\}, \{12, 23\}) \) and \( P_4 = (\{1, 2, 3\}, \{12, 13, 23\}, \emptyset) \), the class of bipartite permutation graphs is \( \text{Ord}(\{P_3, P_4\}) \) \([16]\). For \( P_5 = (\{1, 2, 3, 4\}, \{13, 24\}, \{23\}) \), the class of mono-
tone L-graphs is $\text{ORD}({P_6})$ [10]. See Figure 11 for a graphical representation of the aforementioned patterns.

The class of bipartite graphs can be characterized as $\text{ORD}({P_5})$, where $P_5 = ({\{1,2,3\}, \{12,23\}, \emptyset})$ [16]. In the literature, there are two ways of defining patterns for subclasses of bipartite graphs. The first one [23] involves a total order of the vertices, that are colored black or white, a set of compulsory edges, and a set of compulsory non-edges. We will call such a structure a bicolored pattern, and we will describe it as a 4-tuple containing a set of ordered vertices, the subset of white vertices, the set of edges and the set of non-edges. For instance, the paths in Figure 12 are described as $Q_1 = ({\{1,2,3\}, \{3\}, \{13\}, \{23\}})$, $Q_2 = ({\{1,2,3\}, \{1,2\}, \{13\}, \{23\}})$, $Q_3 = ({\{1,2,3\}, \{1\}, \{13\}, \{12\}})$, $Q_4 = ({\{1,2,3\}, \{2,3\}, \{13\}, \{12\}})$.

We say that a bipartite graph $H$ belongs to $\text{BICOLORD}(\mathcal{F})$, for a fixed family of bicolored patterns $\mathcal{F}$, if $H$ admits a bipartition $V(H) = A \cup B$ and an ordering of $A \cup B$ that avoids the patterns from $\mathcal{F}$. It was proved in [23] that interval bigraphs are exactly $\text{BICOLORD}({Q_1, Q_2})$. 

Figure 11: The patterns used in the characterizations of this section. The solid lines denote compulsory edges and the dotted lines are compulsory non-edges in the pattern.

Figure 12: Examples of bicolored patterns.
The other (slightly different) way [24] is the following. A bipartite pattern is a bipartite trigraph whose vertices in each part of the bipartition are linearly ordered. We will denote such pattern as a 4-tuple containing two disjoint sets of ordered vertices, the set of edges and the set of non-edges. For instance, the paths in Figure 13 are described as 

\[
\text{Pattern } R_1 = (\{1, 2\}, \{1', 2'\}, \{12', 21'\}, \{11'\}), \text{ Pattern } R_2 = (\{1, 2\}, \{1', 2'\}, \{12', 21'\}, \{22'\}), \text{ Pattern } R_3 = (\{1, 2, 3\}, \{1', 2', 3'\}, \{13', 31', 33'\}, \{23', 32'\}), \text{ Pattern } R_4 = (\{1, 2, 3\}, \{1'\}, \{11', 31'\}, \{21'\}), \text{ Pattern } R_4' = (\{1\}, \{1', 2', 3'\}, \{11', 13'\}, \{12'\}).
\]

We say that a bipartite graph \( H \) belongs to BiOrd(\( \mathcal{F} \)), for a fixed family of bipartite patterns \( \mathcal{F} \), if \( H \) admits a bipartition \( V(H) = A \cup B \) and an ordering of \( A \) and of \( B \) so that no pattern from \( \mathcal{F} \) occurs. Several known bipartite graph classes can be characterized as BiOrd(\( \mathcal{F} \)). For instance, bipartite convex graphs are BiOrd(\( \{R_4\} \)), and proper interval bigraphs are BiOrd(\( \{R_1, R_2\} \)) [24].

We will state now a lemma that is necessary to prove the forbidden pattern characterizations of 2-thin graphs and (proper) independent 2-thin graphs.

**Lemma 19.** Let \( G \) be a graph, \( \{V^1, V^2\} \) a partition of \( V(G) \), and \( < \) a partial order of \( V(G) \) that is total when restricted to each of \( V^1, V^2 \). Then \( D(G, \{V^1, V^2\}, <) \) is acyclic if and only if \( G[V^1, V^2] \) ordered according to \( < \) avoids the bipartite patterns \( R_2 \) and \( R_3 \), and \( \tilde{D}(G, \{V^1, V^2\}, <) \) is acyclic if and only if \( G[V^1, V^2] \) ordered according to \( < \) avoids the bipartite patterns \( R_1, R_2, R_4 \) and \( R_4' \). Furthermore, if \( G[V^1, V^2] \) has no isolated vertices, then \( \tilde{D}(G, \{V^1, V^2\}, <) \) is acyclic if and only if \( G[V^1, V^2] \) ordered according to \( < \) avoids the bipartite patterns \( R_1 \) and \( R_2 \).

**Proof.** If the pattern \( R_2 \) occurs, then the vertices \( 2, 2' \) form a directed cycle both in \( D(G, \{V^1, V^2\}, <) \) and \( \tilde{D}(G, \{V^1, V^2\}, <) \). If the pattern \( R_1 \)
occurs, then the vertices 1, 1′ form a directed cycle in \( \tilde{D}(G, \{V^1, V^2\}, <) \). If the pattern \( R_3 \) occurs, then the vertices 2, 3, 2′, 3′ form a directed cycle in \( D(G, \{V^1, V^2\}, <) \). If the pattern \( R_4 \) (resp. \( R_4' \)) occurs, then the vertices 2, 1′ (resp. 1, 2′) form a directed cycle in \( \tilde{D}(G, \{V^1, V^2\}, <) \).

In order to prove the converse, suppose that there is a directed cycle in \( D(G, \{V^1, V^2\}, <) \). As in the proof of Theorem 8, we can reduce up to symmetry (since the definition of the digraph and the patterns are symmetric) to the following three cases.

**Case 1:** The cycle consists of two vertices, \( v_1 \in V^1 \) and \( v_2 \in V^2 \).

In this case, \( v_1 \) and \( v_2 \) are not adjacent and, by definition of the digraph, there exist \( v'_1 \in V^1, v'_2 \in V^2 \) such that \( v'_1 < v_1, v'_2 < v_2, \) and \( v'_1 v_2, v_1 v'_2 \in E(G) \). So, the pattern \( R_2 \) occurs in \( G[V^1, V^2] \) ordered according to \(<\).

**Case 2:** The cycle is \( v_1 w_1 v_2 \) such that \( v_1, w_1 \in V^1 \) and \( v_2 \in V^2 \).

By definition of the digraph, we have \( v_1 < w_1, v_1 v_2, w_1 v_2 \notin E(G) \), there exists in \( V^2 \) a vertex \( v'_2 < v_2 \) such that \( w_1 v'_2 \in E(G) \), and there exists in \( V^1 \) a vertex \( v'_1 < v_1 \) such that \( v'_1 v_2 \in E(G) \). So, \( \{v'_1, v_1, v'_2, v_2\} \) form the pattern \( R_2 \) in \( G[V^1, V^2] \) ordered according to \(<\).

**Case 3:** The cycle is \( v_1 w_1 v_2 v_2 \) such that \( v_1, w_1 \in V^1 \) and \( v_2, v_2 \in V^2 \).

By definition of the digraph, we have \( v_1 < w_1, v_2 < w_2, w_1 v_2, v_1 w_2 \notin E(G) \), there exists in \( V^2 \) a vertex \( v'_2 < v_2 \) such that \( w_1 v'_2 \in E(G) \), and there exists in \( V^1 \) a vertex \( v'_1 < v_1 \) such that \( v'_1 w_2 \in E(G) \). If \( w_1 w_2 \in E(G) \), then \( \{v'_1, v_1, v'_2, v_2\} \) form the pattern \( R_2 \) in \( G[V^1, V^2] \) ordered according to \(<\). If, otherwise, \( w_1 w_2 \in E(G) \), then \( \{v'_1, v_1, v'_2, v_2, v_2\} \) form the pattern \( R_3 \) in \( G[V^1, V^2] \) ordered according to \(<\).

Suppose now that there is a directed cycle in \( \tilde{D}(G, \{V^1, V^2\}, <) \). As in the proof of Theorem 8, we can reduce up to symmetry (since the definition of the digraph and the patterns are symmetric) to the following three cases.

**Case 1:** The cycle consists of two vertices, \( v_1 \in V^1 \) and \( v_2 \in V^2 \).

In this case, by definition of the digraph, \( v_1 \) and \( v_2 \) are not adjacent and, on the one hand, either there exists in \( V^2 \) a vertex \( v'_2 < v_2 \) with \( v_1 v'_2 \in E(G) \) or there exists in \( V^1 \) a vertex \( v''_1 > v_1 \) with \( v''_1 v_2 \in E(G) \) and, on the other hand, either there exists in \( V^1 \) a vertex \( v'_1 < v_1 \) with \( v'_1 v_2 \in E(G) \) or there exists in \( V^2 \) a vertex \( v''_2 > v_2 \) with \( v_1 v''_2 \in E(G) \).

If the existent vertices are \( v'_1 \) and \( v'_2 \), then \( \{v'_1, v_1, v'_2, v_2\} \) form the pattern \( R_2 \) in \( G[V^1, V^2] \) ordered according to \(<\). If this is the case for \( v''_1 \) and \( v''_2 \), then the pattern \( R_1 \) is formed by \( \{v_1, v''_1, v_2, v''_2\} \). If the existent vertices are \( v'_1 \) and \( v''_1 \), then \( \{v'_1, v_1, v''_1, v_2\} \) form the pattern \( R_4 \).

**Case 2:** The cycle is \( v_1 w_1 v_2 \) such that \( v_1, w_1 \in V^1 \) and \( v_2 \in V^2 \).
By definition of the digraph, we have $v_1 < w_1$, $v_1v_2$, $w_1v_2 \not\in E(G)$, and, on the one hand, either there exists in $V^2$ a vertex $v'_2 < v_2$ with $w_1v'_2 \in E(G)$ or there exists in $V^1$ a vertex $w'_1 > w_1$ with $w'_1v_2 \in E(G)$ and, on the other hand, either there exists in $V^1$ a vertex $v'_1 < v_1$ with $v'_1v_2 \in E(G)$ or there exists in $V^2$ a vertex $v''_2 > v_2$ with $v_1v''_2 \in E(G)$.

If the existent vertices are $v'_1$ and $v'_2$, then $\{v'_1, w_1, v'_2, v_2\}$ form the pattern $R_2$ in $G[V^1, V^2]$ ordered according to $<$. If this is the case for $v''_1$ and $v''_2$, then the pattern $R_1$ is formed by $\{v_1, w'_1, v_2, v''_2\}$. In the case of $v'_2$ and $v''_2$, if at least one of $v_1v'_2, w_1v''_2$ is not an edge, then either $R_1$ or $R_2$ is formed by $\{v_1, w_1, v'_2, v''_2\}$. If $v_1v'_2$ and $w_1v''_2$ are edges, then $\{v_1, v'_2, v_2, v''_2\}$ form $R_4'$. In the case of $v'_1$ and $w''_2$, the vertices $\{v'_1, v_1, w'_1, v_2\}$ form $R_4$. 

**Case 3:** The cycle is $v_1w_1v_2w_2$ such that $v_1, w_1 \in V^1$ and $v_2, w_2 \in V^2$.

By definition of the digraph, we have $v_1 < w_1$, $v_2 < w_2$, $w_1v_2, v_1w_2 \not\in E(G)$, and, on the one hand, either there exists in $V^2$ a vertex $v'_2 < v_2$ with $w_1v'_2 \in E(G)$ or there exists in $V^1$ a vertex $w'_1 > w_1$ with $w'_1v_2 \in E(G)$ and, on the other hand, either there exists in $V^1$ a vertex $v'_1 < v_1$ with $v'_1w_2 \in E(G)$ or there exists in $V^2$ a vertex $w''_2 > w_2$ with $v_1w''_2 \in E(G)$.

Suppose the existent vertices are $v'_1$ and $v'_2$. If $v_1w_2 \not\in E(G)$, then $R_2$ is formed by $\{v'_1, w_1, v'_2, w_2\}$, otherwise, $R_4$ is formed by $\{v'_1, v_1, w_1, w_2\}$. Similarly, suppose that the existent vertices are $w''_2$ and $v''_1$. If $v_1v'_2 \not\in E(G)$, then $R_1$ is formed by $\{v_1, w''_1, v_2, w''_2\}$, otherwise, $R_4$ is formed by $\{v_1, w_1, w''_1, v_2\}$. In the case of $v'_2$ and $w''_2$, if $v_1w''_2 \not\in E(G)$, then $R_2$ is formed by $\{v_1, v'_2, v_2, w''_2\}$, otherwise, $R_4$ is formed by $\{v_1, v'_2, v_2, v''_2\}$. The last case is symmetric.

To conclude, notice that, in a bipartite graph with no isolated vertices, the patterns $R_4$ or $R_4'$ imply either $R_1$ or $R_2$. Indeed, in the case of $R_4 = (\{v_1, v_2, v_3, v'_1\}, \{v_1v'_1, v_3v'_1\}, \{v_2v'_1\})$, since $v_2$ is not an isolated vertex, there is a vertex $v'$ with $v_2v' \in E(G)$, and either $R_1$ or $R_2$ occurs, when $v' > v'_1$ or $v' < v'_1$, respectively. The case of $R_4'$ is symmetric.

While a characterization of $k$-thin and proper $k$-thin graphs by forbidden induced subgraphs is open for $k \geq 2$, they may be defined by means of forbidden patterns, due to Corollary 1 i.e., we forbid the patterns for an order $<$ that produce a clique of size $k + 1$ in $G_<$ (resp. $\tilde{G}_<$). This approach leads to a high number of forbidden patterns (see Figures A.16 and A.17 for the patterns corresponding to 2-thin and proper 2-thin graphs, respectively).

However, the model of 2-thin graphs as monotone $L$-graphs leads to the following forbidden pattern characterization for the class, with only four
symmetric patterns.

Recall that an L-model is blocking if for every two non-intersecting L’s, either the vertical or the horizontal prolongation of one of them intersects the other.

**Theorem 20.** Let us define the patterns 

\[ P_6 = (\{1, 2, 3, 4\}, \{13, 24\}, \{23\}) \]

\[ P_7 = (\{1, 2, 3, 4, 5\}, \{13, 35\}, \{23, 34\}) \]

\[ P_8 = (\{1, 2, 3, 4, 5, 6\}, \{13, 46\}, \{23, 45\}) \]

\[ P_9 = (\{1, 2, 3, 4, 5, 6\}, \{14, 34, 36\}, \{24, 35\}) \] (see Figure 11).

Let \( G \) be a graph. The following statements are equivalent:

1. \( G \) is a 2-thin graph.
2. \( G \) has a blocking monotone L-model.
3. \( G \in \text{Ord}(\{P_6, P_7, P_8, P_9\}) \).

**Proof.** 

(i) \( \Rightarrow \) (ii) Let \( G \) be a 2-thin graph with partition \( V^1, V^2 \) and a consistent ordering \( < \). Consider the model \( \mathcal{M}_4(G) \). It is easy to see that it is an intersection model for \( G \), given that \( \mathcal{M}_1(G) \) is. We will see that it is blocking. Let us consider two non-adjacent vertices \( v_1, v_2 \). If \( v_1 \in V^1 \) and \( v_2 \in V^2 \), then either the horizontal prolongation of \( v_1 \) intersects \( v_2 \) or the vertical prolongation of \( v_2 \) intersects \( v_1 \), because the model \( \mathcal{M}_1(G) \) is blocking (Lemma 7). If \( v_1 < v_2 \) and both belong to \( V^1 \) (resp. \( V^2 \)), then the vertical (resp. horizontal) prolongation of \( v_2 \) intersects \( v_1 \) (see Figure 10).

(ii) \( \Rightarrow \) (iii) Let \( G \) be a graph admitting a blocking monotone L-model and consider the ordering of the vertices according to the L-corners along the diagonal. It is known that the pattern \( P_6 \) is not possible in an L-model for that vertex ordering [10]. Let us see that the other patterns are not possible when the model is blocking. Figure [14] shows schematic representations of patterns \( P_7, P_8, \) and \( P_9 \). The light parts are optional, according to the undecided edges of the trigraph. In each of the cases, vertices labeled as \( x \) and \( y \) violate the blocking property.

(iii) \( \Rightarrow \) (i) Let \( G \in \text{Ord}(\{P_6, P_7, P_8, P_9\}) \) and let \( v_1, \ldots, v_n \) be an ordering of the vertices avoiding the patterns \( P_6, P_7, P_8, \) and \( P_9 \). If the order avoids \( P_1 \), then \( G \) is an interval graph, in particular 2-thin. Otherwise, let \( n_1 \) be such that \( v_1, \ldots, v_{n_1} \) avoids \( P_1 \) but there exist \( 1 \leq i < j < n_1 + 1 \) such that \( v_i v_{n_1+1} \in E(G) \) and \( v_j v_{n_1+1} \notin E(G) \). Let \( V^1 = \{v_1, \ldots, v_{n_1}\} \) and
Figure 14: Schematic L-models of patterns $P_7$, $P_8$, and $P_9$. The light parts are optional, according to the undecided edges of the trigraph. In each of the cases, vertices labeled as $x$ and $y$ violate the blocking property.

$V^2 = \{v_{n+1}, \ldots, v_n\}$. Consider the ordering $<$ such that $V^1$ is ordered increasingly according to the vertex indices and $V^2$ is ordered decreasingly according to the vertex indices. The graph $G[V^2]$ ordered by $<$ avoids $P_1$, since otherwise either $P_7$ or $P_8$ occurs in the original ordering. It remains to prove that $D(G, \{V^1, V^2\}, <)$ is acyclic. By Lemma 19 this is so if and only if $G[V^1, V^2]$ with the sets ordered according to $<$ avoids the bipartite patterns $R_2$ and $R_3$. But this holds because the original ordering avoids $P_6$ and $P_9$, respectively. \qed

Notice that the blocking property is crucial, since every tree is a monotone L-graphs \cite{11} and trees may have arbitrarily large thinness \cite{5}.

By combining results from \cite{16, 22, 24}, we have the following two characterization theorems. They show, among other equivalences, that (proper) independent 2-thin graphs are equivalent to (proper) interval bigraphs, respectively.

**Theorem 21.** Let $G$ be a graph. The following statements are equivalent:

(i) $G$ is an independent 2-thin graph.

(ii) $G$ is an interval bigraph.

(iii) $G$ is bipartite and $G \in \text{BICOLORD}(\{Q_1, Q_2\})$.

(iv) $G$ is bipartite and $G \in \text{BIORD}(\{R_2, R_3\})$.

(v) $G \in \text{ORD}(\{P_5, P_6, P_9\})$. 

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Proof. \((i) \iff (iii)\) It is straightforward from the definition of independent thinness.

\((ii) \iff (iii)\) It is proved in [22].

\((i) \implies (iv)\) Let \(\{V^1, V^2\}\) be a partition of \(V(G)\) into independent sets and \(<\) an ordering of \(V(G)\) that is consistent with the partition \(\{V^1, V^2\}\). By Lemma 2, \(D(G, \{V^1, V^2\}, <)\) is acyclic. By Lemma 19, \(<\) avoids \(R_2\) and \(R_3\), so \(G \in \text{BiOrd}(\{R_2, R_3\})\).

\((iv) \implies (i)\) Let \(\{V^1, V^2\}\) be a bipartition of \(V(G)\) and \(<\) an order of \(V^1\) and of \(V^2\) that avoids \(R_2\) and \(R_3\). By Lemma 19, \(D(G, \{V^1, V^2\}, <)\) is acyclic. By Lemma 2, there is an ordering of \(V(G)\) that is consistent with the partition \(\{V^1, V^2\}\), so \(G\) is independent 2-thin.

\((iv) \implies (v)\) Let \(\{V^1, V^2\}\) be a bipartition of \(G\) and \(<\) an ordering of \(V^1\) and of \(V^2\) that avoids \(R_2\) and \(R_3\). Consider the order of \(V(G)\) such that every vertex of \(V^1\) precedes every vertex of \(V^2\), \(V^1\) is ordered according to \(<\) and \(V^2\) is ordered according to the reverse of \(<\). This order avoids \(P_5\) because every edge has an endpoint in \(V^1\) and the other in \(V^2\). It also avoids \(P_6\) and \(P_9\), because otherwise, by the way of defining the ordering of \(V(G)\) and by the edges in the patterns, the first two (resp. three) vertices of \(P_6\) (resp. \(P_9\)) belong to \(V^1\), and the last two (resp. three) to \(V^2\). Thus, with the order \(<\) of \(V^1\) and of \(V^2\) the the pattern \(R_2\) (resp. \(R_3\)) occurs, which is a contradiction.

\((v) \implies (i)\) Since a graph is independent 2-thin if and only if each of its connected components is (see, for example, [8]), we may assume \(G\) is connected and non-trivial. Let \(\{V^1, V^2\}\) be the bipartition of \(V(G)\) and let \(<\) be an order of \(V^1 \cup V^2\) that avoids \(P_5\), \(P_6\), and \(P_9\). Since the graph is connected and the order avoids \(P_5\), either every vertex of \(V^1\) precedes every vertex of \(V^2\), or every vertex of \(V^2\) precedes every vertex of \(V^1\). We may assume the first case. Consider \(V^1\) ordered according to \(<\) and \(V^2\) ordered according to the reverse of \(<\). We will call this partial order \(<'\). Since \(<\) avoids \(P_6\) and \(P_9\), \(G\) ordered according to \(<'\) avoids \(R_2\) and \(R_3\). By Lemma 19, \(D(G, \{V^1, V^2\}, <')\) is acyclic. By Lemma 2, there is an ordering of \(V(G)\) that is consistent with the partition \(\{V^1, V^2\}\), so \(G\) is independent 2-thin. \(\square\)

In particular, independent 2-thin graphs can be recognized in polynomial time [33]. Even cycles of length at least 6 are bipartite and 2-thin but not
interval bigraphs [33], so not independent 2-thin graphs.

**Theorem 22.** Let $G$ be a graph. The following statements are equivalent:

(i) $G$ is a proper independent 2-thin graph.

(ii) $G$ is a proper interval bigraph.

(iii) $G$ is a bipartite permutation graph.

(iv) $G$ is bipartite and $G \in \text{BICOLORD}(\{Q_1, Q_2, Q_3, Q_4\})$.

(v) $G$ is bipartite and $G \in \text{BIORD}(\{R_1, R_2\})$.

(vi) $G \in \text{ORD}(\{P_3, P_4\})$.

Proof. (i) $\Leftrightarrow$ (iv) It is straightforward from the definition of proper independent thinness.

(ii) $\Leftrightarrow$ (iii) It is proved in [22].

(ii) $\Leftrightarrow$ (v) It is proved in [24].

(iii) $\Leftrightarrow$ (vi) It is proved in [16].

(i) $\Rightarrow$ (v) Let $\{V^1, V^2\}$ be a partition of $V(G)$ into independent sets and $<$ an ordering of $V(G)$ that is strongly consistent with the partition $\{V^1, V^2\}$. By Lemma [3], $\hat{D}(G, \{V^1, V^2\}, <)$ is acyclic. By Lemma [19], $<\text{ avoids } R_1$ and $R_2$, so $G \in \text{BIORD}(\{R_1, R_2\})$.

(v) $\Rightarrow$ (i) Since a graph is proper independent 2-thin if and only if each of its connected components is (see, for example, [8]), we may assume $G$ is connected and non-trivial. Let $\{V^1, V^2\}$ be the bipartition of $V(G)$ and let $<$ be an order of $V^1$ and of $V^2$ that avoids $R_1$ and $R_2$. By Lemma [19], $\hat{D}(G, \{V^1, V^2\}, <)$ is acyclic. By Lemma [3], there is an ordering of $V(G)$ that is strongly consistent with the partition $\{V^1, V^2\}$, so $G$ is proper independent 2-thin.

In particular, proper independent 2-thin graphs can be recognized in linear time [23, 42]. The bipartite claw (the subdivision of $K_{1,3}$) is bipartite and
proper 2-thin but not bipartite permutation \[29\], so not proper independent 2-thin.

The results above show that, for a (proper) \(k\)-thin graph with a partition \(V^1, \ldots, V^k\) consistent with some ordering, not only \(G[V^i]\) is a (proper) interval graph for every \(1 \leq i \leq k\), but also \(G[V^i, V^j]\) is a (proper) interval bigraph for every \(1 \leq i, j \leq k\).

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Appendix A. Omitted figures

Figure A.15: The graph $G$ used to prove that the blocking property is necessary for the characterization of 2-thin graphs as boxicity 2 graphs (Proposition 9).
Figure A.16: Reference triangle (left) and forbidden patterns for thinness at most 2. Red vertices form a clique in $G_<$, and labeled vertices are the witnesses for the respective edges of the clique, according to the labeling of the first triangle.
Figure A.17: Reference triangle (top-left) and forbidden patterns for proper thinness at most 2, other than those in Figure A.16 and their reverses. Red vertices form a clique in $\tilde{G}_<$, and labeled vertices are the witnesses for the respective edges of the clique, according to the labeling of the first triangle. We draw only those in which the witness of $C$ is greater then the orange vertices. The complete set is obtained by the patterns in Figure A.16 those in the present figure, and their reverses.