Invited Comment

Microscopic nuclear structure models and methods: chiral symmetry, wobbling motion and $\gamma$–bands

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Abstract

A systematic investigation of the nuclear observables related to the triaxial degree of freedom is presented using the multi-quasiparticle triaxial projected shell model (TPSM) approach. These properties correspond to the observation of $\gamma$-bands, chiral doublet bands and the wobbling mode. In the TPSM approach, $\gamma$-bands are built on each quasiparticle configuration and it is demonstrated that some observations in high-spin spectroscopy that have remained unresolved for quite some time could be explained by considering $\gamma$-bands based on two-quasiparticle configurations. It is shown in some Ce-, Nd- and Ge-isotopes that the two observed aligned or s-bands originate from the same intrinsic configuration with one of them as the $\gamma$-band based on a two-quasiparticle configuration. In the present work, we have also performed a detailed study of $\gamma$-bands observed up to the highest spin in dysposium, hafnium, mercury and uranium isotopes. Furthermore, several measurements related to chiral symmetry breaking and wobbling motion have been reported recently. These phenomena, which are possible only for triaxial nuclei, have been investigated using the TPSM approach. It is shown that doublet bands observed in lighter odd–odd Cs-isotopes can be considered as candidates for chiral symmetry breaking. Transverse wobbling motion recently observed in $^{135}$Pr has also been investigated and it is shown that TPSM approach provides a reasonable description of the measured properties.

Keywords: nuclear structure, high spin, $\gamma$ bands, chiral symmetry, wobbling motion

(Some figures may appear in colour only in the online journal)

1. Introduction

Atomic nucleus is one of the most fascinating quantum many-body systems that depicts a rich variety of shapes and structures. At the same time, it is also one of the most challenging problems in physics to investigate theoretically. *Ab-initio* approaches can be applied only to few-body systems and many-body exact Schrodinger wavefunction remains inaccessible for majority of nuclei across the periodic table. The phenomenal progress in understanding the properties of nuclei has been achieved using phenomenological models and methods. These phenomenological models are primarily based on empirical observations and have played a pivotal role in unraveling the intrinsic structures of atomic nuclei [1–6].

The pioneering work of Bohr, Mottelson and Rainwater laid the foundation of the phenomenological models in nuclear physics. It was demonstrated that properties of atomic nuclei can be elucidated by considering rotational, vibrational and single-particle motion as three basic degrees of freedom.
and led to the development of the collective model in sixties and seventies [1, 7]. This model is even being used today with the parameters determined through microscopic approaches rather than following the empirical route.

The nuclear physics research is going through a renaissance with the state-of-the-art tools and techniques being developed to probe the wealth of nuclear properties. The availability of the leadership computing facilities has made it possible to apply \textit{ab initio} methods to lighter mass region with remarkable success. On the other hand, the density functional approach is now widely used to explore the ground-state properties all across the nuclear landscape [8–10]. In recent years, the progress achieved in applying these modern tools is quite remarkable and it is expected that it would be possible to apply these techniques to investigate a broad spectrum of nuclear properties all across the nuclear periodic table in a foreseeable future [11, 12]. However, at the moment, these models have limitations and cannot be employed to investigate, for instance, the rich-band structures observed in deformed nuclei.

In the absence of a fully microscopic theory, semi-microscopic models have been developed to study the properties of band structures in medium and heavy mass nuclei. In this class of models, the triaxial projected shell model (TPSM) approach has been demonstrated to correlate the high-spin data of well deformed and transitional data with remarkable success [13]. The purpose of the present work is to provide an overview of the recent applications of the TPSM approach to a wide range of nuclear properties [14–28]. We also report new results on the observation of $\gamma$-bands based on excited configurations. Furthermore, we shall present a systematic investigation of $\gamma$-bands observed up to the highest spin in dysprosium, hafnium, mercury and uranium isotopes. The manuscript is organized in the following manner. In the next section, a few details of the TPSM approach are provided and some technical aspects of the model are included in the appendix. In section 3, the results of the calculations for $\gamma$-, chiral- and wobbling-bands are displayed and discussed. Finally, the work presented in this manuscript is summarized and concluded in section 4.

2. Outline of the TPSM

The basic strategy of the TPSM approach is similar to the spherical shell model with the only difference that deformed basis are employed for diagonalizing the shell model Hamiltonian rather than the spherical one. The deformed basis are constructed by solving the triaxial Nilsson potential with optimum quadrupole deformation parameters of $\varepsilon$ and $\varepsilon'$. In principle, the deformed basis can be constructed with arbitrary deformation parameters, however, the basis are constructed with expected or known deformation parameters (so called optimum) for a given system under consideration. These deformation values lead to an accurate Fermi surface and it is possible to choose a minimal subset of the basis states around the Fermi surface for a realistic description of a given system. The Nilsson basis states are then transformed to the quasiparticle space using the simple Bardeen–Cooper–Schriefer ansatz for treating the pairing interaction.

As the deformed basis are defined in the intrinsic frame of reference and do not have well defined angular-momentum, in the second stage these basis are projected onto states with well defined angular-momentum using the angular-momentum projection technique [29–31]. The three-dimensional angular-momentum projection operator is given by

$$\hat{P}_{MK}^I = \frac{2I + 1}{8\pi^2} \int d\Omega \, D_{MK}^I(\Omega) \hat{R}(\Omega),$$

with the rotation operator

$$\hat{R}(\Omega) = e^{-i\alpha J} e^{-i\beta J} e^{-i\gamma K}.$$

Here, ‘$\Omega$’ represents a set of Euler angles ($\alpha$, $\gamma = [0, 2\pi]$, $\beta = [0, \pi]$) and the $J$s are angular-momentum operators. The projected basis states considered in the present work for the even–even system are composed of vacuum, two-proton, two-neutron and two-proton plus two-neutron quasiparticle configurations, i.e.

$$\{ \hat{P}_{MK}^I | \Phi \}, \quad \hat{P}_{MK}^I a_{p_1} a_{p_2} | \Phi \}, \quad \hat{P}_{MK}^I a_{n_1} a_{n_2} | \Phi \}, \quad \hat{P}_{MK}^I a_{p_1} a_{p_2} a_{n_1} a_{n_2} | \Phi \} \}.$$

where $| \Phi \}$ in (3) represents the triaxial qp vacuum state. In majority of the nuclei, near-yраст spectroscopy up to $I = 20$ is well described using the above basis space of two-neutron, two-proton and two-neutron plus two-proton configurations as one expects two-protons to align after two-neutrons rather than four-neutrons considering the blocking argument. However, we would like to add that this may not be the case for all the nuclei and there are indications that four-neutron states may become important in the description of high-spin states in some rare-Earth region nuclei [32]. For odd-proton (neutron) systems, the basis space is composed of one-quasiproton (quasineutron) and two-quasineutrons (quasiprotons). In the case of odd–odd nuclei, the basis space is simply one-quasiproton coupled to one-quasineutron. This basis space for odd and odd–odd nuclei is also quite limited and needs to be extended for describing the higher spin states more accurately.

The advantage of the TPSM approach is that not only the yrast band, but also the rich excited band structures can be investigated. As a matter of fact, the major focus of the present work is to study the $\gamma$-bands which form the first excited band in many transitional nuclei. The Nilsson triaxial quasiparticle states do not have well defined projection along the symmetry axis, $\Omega$ and are a superposition of these states. For instance, the triaxial self-conjugate vacuum state is a superposition of $K = 0, 2, 4, \ldots$ states—only even-states are possible due to symmetry requirement [33]. For the symmetry operator, $\hat{S} = e^{-i\kappa K}$, we have

$$\hat{P}_{MK}^I \hat{S}^{-1} = e^{i\kappa(K-\kappa)} \hat{P}_{MK}^I \hat{S}^{-1} | \Phi \},$$

where, $\hat{S} | \Phi \} = e^{-i\kappa K} | \Phi \}$, and $\kappa$ characterizes the intrinsic states. For the self-conjugate vacuum state $\kappa = 0$ and, therefore, it follows from the above equation that only $K = \text{even}$, values are permitted for this state. For 2-qp states, the
possible values for \( K \)-quantum number are both even and odd depending on the structure of the qp state. For the 2-qp state formed from the combination of the normal and the time-reversed states, \( \kappa = 0 \) and again only \( K = \) even values are permitted. For the combination of the two normal states, \( \kappa = 1 \), and only \( K = \) odd states are allowed.

The projected states for a given configuration that constitute a rotational band are obtained by specifying the corresponding \( K \)-value in the angular-momentum projection operator. The projected states from \( K = 0, 2 \) and 4 correspond to ground-, \( \gamma \)- and \( \gamma' \)-bands, respectively. As stated earlier, for two-quasiparticle states, both even- and odd-\( K \) values are permitted, depending on the signature of the two quasiparticle states. In this description, the aligning states that cross the ground-state band and lead to upbend or backbend phenomenon have low-\( K \) configurations. These states are close to the rotational axis as compared to the deformation axis and can be easily aligned. The projection from the same quasiparticle intrinsic state with \( K' = K + 2 \) is the \( \gamma \)-band built on these quasiparticle state. The \( \gamma' \)-bands built on the two-quasiparticle states shall form one of the major focal issues of the present work and shall be discussed in detail in the next section.

In the third and the final stage of the TPSM analysis, the projected basis are employed to diagonalize the shell model Hamiltonian. The model Hamiltonian consists of pairing and quadrupole–quadrupole interaction terms, i.e.,

\[
\hat{H} = \hat{H}_0 - \frac{1}{2} \chi \sum_{\mu} \hat{Q}_\mu^\dagger \hat{Q}_\mu - G_M \hat{P}^\dagger \hat{P} - G_Q \sum_{\mu} \hat{P}_\mu^\dagger \hat{P}_\mu. \tag{5}
\]

In the above equation, \( \hat{H}_0 \) is the single-particle Nilsson Hamiltonian [34]. The parameters of the Nilsson potential are fitted to a broad range of nuclear properties and is quite appropriate to employ it as a mean-field potential. The QQ-force strength \( \chi \) in equation (5) is related to the quadrupole deformation \( \varepsilon \) as a result of the self-consistent HFB condition and the relation is given by [35]:

\[
\chi_{\tau \tau'} = \frac{2}{3} \varepsilon \omega_\tau \omega_{\tau'} \frac{\omega_\tau}{\omega_\tau^0} \frac{\omega_{\tau'}}{\omega_{\tau'}^0}, \tag{6}
\]

where \( \omega_\tau = \omega_0 a_\tau \), with \( \omega_0 = 41.4678A^{-1/2} \) MeV, and the isospin-dependence factor \( a_\tau \) is defined as

\[
a_\tau = \left[ 1 \pm \frac{N - Z}{A} \right]^{1/2}, \tag{7}
\]

with \( + \) (–) for \( \tau = \) neutron (proton). The harmonic oscillation parameter is given by \( b_\tau^2 = b_0^2/a_\tau \) with \( b_0^2 = \hbar / (m \omega_0) = A^{1/2} \) fm\(^2\). It needs to be added that the strengths in the TPSM are fixed as in the original projected shell model approach with axial symmetry. The corrections are expected from the non-axial component of the quadrupole field, which have been neglected in the present work. The monopole pairing strength \( G_M \) (in MeV) is of the standard form

\[
G_M = \frac{G_1 \mp G_2 \frac{N - Z}{A}}{A}, \tag{8}
\]

where the minus (plus) sign applies to neutrons (protons). In the present calculation, we choose \( G_1 \) and \( G_2 \) such that the calculated gap parameters reproduce the experimental mass differences. The values \( G_1 \) and \( G_2 \) vary depending on the mass region and shall be listed in the discussion of the results. The above choice of \( G_M \) is appropriate for the single-particle space employed in the model, where three major shells are used for each type of nucleon. For most of the nuclei, equation (8) has been employed. However, it has been found that for protons, \( G_M = \frac{G_1}{A} \) reproduces the pairing gaps for \( A \sim 130 \) region [24, 36] slightly better and has been employed in this region. The quadrupole pairing strength \( G_Q \) is considered to be proportional to \( G_M \) and the proportionality constant being fixed as 0.18. These interaction strengths are consistent with those used in our earlier studies [14–17, 19, 33, 35, 37–41].

It is shown in the appendix that the projection formalism outlined above can be transformed into a diagonalization problem following the Hill–Wheeler prescription. The Hamiltonian in equation (5) is diagonalized using the projected basis of equation (3). The obtained wavefunction can be written as

\[
\psi_{\kappa \nu}^m = \sum_{K, \kappa} a_{K \nu}^m | \Phi_K \rangle. \tag{9}
\]

Here, the index ‘\( \nu \)’ labels the states with same angular momentum and ‘\( \kappa \)’ the basis states. In equation (9), \( a_{K \nu}^m \) are the amplitudes of the basis states \( \kappa \). These wavefunctions are used to calculate the electromagnetic transition probabilities. The reduced electric quadrupole transition probability \( B(E2) \) from an initial state \((\sigma_i, I_i)\) to a final state \((\sigma_f, I_f)\) is given by [42]

\[
B(E2, I_i \rightarrow I_f) = \frac{e^2}{2I_f + 1} | \langle \sigma_i, I_i | \hat{Q}_0^2 | \sigma_f, I_f \rangle |^2. \tag{10}
\]

As in our earlier publications [20–24], we have used the effective charges of 1.6e for protons and 0.6e for neutrons. The effective charges are employed instead of the bare charges as core is used in the TPSM approach and valance particles occupy only three major oscillator shells. In \textit{ab initio} approaches with no core, only bare charges need to be used. The reduced magnetic dipole transition probability \( B(M1) \) is computed through

\[
B(M1, I_i \rightarrow I_f) = \frac{\mu_N^2}{2I_f + 1} | \langle \sigma_i, I_i | \hat{A}_M | \sigma_f, I_f \rangle |^2, \tag{11}
\]

where the magnetic dipole operator is defined as

\[
\hat{A}_M = g_\nu \hat{S} \times \hat{S} + (g_\nu^2 - g_\sigma^2) \hat{S}. \tag{12}
\]

Here, \( \nu \) is either \( \nu \) or \( \pi \), and \( g_\nu \) and \( g_\sigma \) are the orbital and the spin gyromagnetic factors, respectively. In the calculations we use for \( g_\nu \) the free values and for \( g_\sigma \) the free values damped by
a 0.85 factor, i.e.,
\[ g_i^\gamma = 1, \quad g_i^\nu = 0, \]
\[ g_i^\gamma = 5.586 \times 0.85, \]
\[ g_i^\nu = -3.826 \times 0.85. \] (13)

The reduced matrix element of an operator \( \hat{\mathcal{O}} \) (\( \hat{\mathcal{O}} \) is either \( \hat{Q} \) or \( \hat{M} \)) can be expressed as [42]

\[
\langle \sigma_1, I \| \hat{\mathcal{O}}_L \| \sigma_1, I \rangle = \sum_{\gamma_i \gamma_k} a_{\gamma_i}^{\gamma} a_{\gamma_k}^{\gamma} \sum_{M_i M_k} (-1)^{h - M_i} 
\times \left( \begin{array}{ccc} I \mspace{-10mu} & L \mspace{-10mu} & I \mspace{-10mu} \\ -M_i \mspace{-10mu} & M \mspace{-10mu} & M_i \end{array} \right)
\times (\Phi_{\gamma_i}) \hat{\mathcal{O}}_L \mathcal{P}_{\gamma_i} \mathcal{P}_{\gamma_k} \Phi_{\gamma_k} (\Omega)
\times (\Phi_{\gamma_i}) \hat{\mathcal{O}}_L \mathcal{P}_{\gamma_i} \Phi_{\gamma_i}. \] (14)

In the above expression, the symbol ( ) denotes a 3j-coefficient.

3. Results and discussions

In the past few years, TPSM approach has been used quite extensively to shed light on some of the outstanding issues related to the triaxility in atomic nuclei and in the present section we shall provide a brief overview of these investigations and some new results, obtained recently, shall also be presented and discussed. This section is divided into subsections, discussing various aspects and implications on the presence of triaxial deformations in atomic nuclei.

3.1. Observation of the \( \gamma \)-bands based on two-quasiparticle states

\( \gamma \)-bands are observed in most of the transitional nuclei all across the nuclear periodic table and have been studied using various theoretical approaches and methods [44–59]. In phenomenological and also in some microscopic models, these bands are interpreted as emerging due to vibrational motion in the \( \gamma \)-degree of freedom of the nuclear deformation [59–71].

In TPSM approach, \( \gamma \)-band structure results from the projection of the \( K = 2 \) component of the triaxial vacuum state. \( \gamma \)-bands are also possible based on the multi-quasiparticle triaxial states, apart from the vacuum configuration, and have not been studied as most of the models in this class consider vacuum configuration only. The basis of TPSM approach has been enlarged to include multi-quasiparticle states and it is now possible to investigate \( \gamma \)-bands built on quasiparticle structures. It is demonstrated in the present work that excited band structures observed in some nuclei that have remained abstruse for many years are, in fact, the \( \gamma \)-bands based on two-quasiparticle states. In the following, we shall provide results of the TPSM calculations for \( 132,134 \)Ce-, \( 134,136,138 \)Nd- and \( 68 \)Ge-isotopes where some excited bands are proposed as the \( \gamma \)-bands based on two-quasiparticle configurations.

In \( 134 \)Ce, the ground-state band is observed to fork into two \( s \)-bands and the band heads of both these bands are known to have negative g-factors, indicating that both of them have neutron configuration [72]. In many rotational nuclei, the ground-state band is crossed by a two quasiparticle state having pair of particles with angular-momentum aligned along the rotational axis [73]. These two-quasiparticle bands become favoured at some spin, depending on the region, are referred to as the \( s \)-bands. For a class of nuclei in \( A \approx 130 \) region, both protons and neutrons occupy same aligning configuration, and therefore, the two \( s \)-bands are expected to have similar g-factors.

TPSM study for \( 132,134 \)Ce and \( 134,136,138 \)Nd isotopes has been performed with both neutrons and protons in \( N = 3, 4 \) and 5 shells and with pairing strength parameters of \( G_1 = 20.82 \) and \( G_2 = 13.58 \). The calculations have been performed with the deformation parameters displayed in table 1. The band diagrams of \( 134 \)Ce and \( 138 \)Nd are provided in figure 1 as representative examples. A band diagram is a set of projected energies for various intrinsic configurations, i.e., the diagonal matrix elements before configuration mixing. The projection formalism not only projects out the angular momentum, but also the ‘K’ value which is specified in the projection matrix and, therefore, projected bands are also specified by this quantum number. When projected energies are plotted as a function of angular momentum for a given configuration and \( K \)-value, it forms a band. These band diagrams are quite

| \( A \) | \( \varepsilon \) | \( \varepsilon' \) | \( \gamma \) |
|---|---|---|---|
| \( 132 \)Ce | 0.183 | 0.100 | 29 |
| \( 134 \)Ce | 0.150 | 0.100 | 34 |
| \( 134 \)Nd | 0.200 | 0.120 | 31 |
| \( 136 \)Nd | 0.158 | 0.110 | 35 |
| \( 138 \)Nd | 0.170 | 0.110 | 33 |
instructive as they provide information on the underlying intrinsic structures of various bands. The bands in figure 1 and in other band diagrams, presented later in this article, are labeled as $(K, nqp)$, where $nqp$ is the number of quasiparticles in a given configuration. For instance, $(0, 0)$ is the projection from the vacuum configuration with $K = 0$ and corresponds to the ground-state band. The normal $\gamma$-band with configuration of $(2, 0)$ is the first excited band and is noted to depict quite large odd-even staggering for both the systems. For $^{134}$Ce, the ground-state band is crossed by two-quasineutron configuration, $(1, 2n)$, at $I = 8$ and above this spin value the yrast states originate from this quasiparticle configuration.

What is most interesting to note from figure 1 is that the configuration $(3, 2n)$, which is the $\gamma$-band built on the two-neutron configuration $(1, 2n)$, crosses the ground-state band at $I = 10$. It is evident that two-proton aligned structure, $(1, 2p)$, crosses the ground-state at $I = 10$, but is higher in energy than the configuration $(3, 2n)$. Although, the final placement of the band structures vary after considering the configuration mixing, but it is expected that lowest two s-bands in $^{134}$Ce to emerge from the same neutron configuration as revealed through the g-factor measurements. The band diagram for $^{138}$Nd, shown in the lowest panel of figure 1, depict a completely different behaviour as compared to $^{134}$Ce with two two-proton aligned bands crossing the ground-state band at $I = 10$. The first two-proton band that crosses has the configuration $(1, 2p)$ and the second has the configuration $(3, 2p)$, which is the $\gamma$-band based on the parent two-proton configuration. It is, therefore, expected that two s-bands observed in $^{138}$Nd should both have positive g-factors.

The resulting s-band structures obtained after diagonalization of the shell model Hamiltonian, equation (5), are shown in figure 2 with the available experimental data for all the studied Ce- and Nd-isotopes. In this figure four s-bands are plotted with dominant components from the configurations, $(1, 2n)$, $(1, 2p)$, $(3, 2n)$, and $(3, 2p)$. For $^{132}$Ce, the lowest two s-bands originate from the configurations, $(1, 2n)$ and $(1, 2p)$, which are normal neutron and proton s-bands. The $\gamma$-bands built on these two-quasiparticle structures are quite high in energy. Two observed s-bands for $^{132}$Ce, also shown in the figure, are noted to be reproduced quite well by TPSM calculations and g-factors for the two s-bands are predicted to have opposite signs as one corresponds to the proton and the other to the neutron configuration. In the case of $^{134}$Ce, the lowest two calculated s-bands have configurations of $(1, 2n)$ and $(3, 2n)$ and, therefore, both the observed s-bands are predicted to have neutron configuration as the
second s-band is the $\gamma$-band built on the two-neutron configuration. As already mentioned, the g-factor measurements have been carried out for the band head states, $I = 10$, for the two s-bands and both have negative g-factors, confirming that both states are based on the neutron configuration [83].

For $^{134}\text{Nd}$, the two s-bands are predicted to have (1, 2p) and (1, 2n) configurations and, therefore, it is predicted that two observed s-bands for this system should correspond to normal proton and neutron structures. In $^{138}\text{Nd}$, a completely different scenario is predicted for the lowest two s-bands with both of them having proton intrinsic structure. The lowest s-band is predicted to have (1, 2p) configuration and the second s-band has (3, 2p) configuration which is the $\gamma$-band based on the preceding configuration. It would be quite interesting to perform the g-factor measurements for this system as both the s-bands are expected to have positive g-factors.

It is expected that $\gamma$-bands based on the multi-quasiparticle states should be more widespread in nuclear periodic table. In the following we shall report a recent study for $^{68}\text{Ge}$, where multiple s-bands are also observed and it is shown that one of the observed s-band in $^{68}\text{Ge}$ has the structure of the $\gamma$-band built on two-quasineutron configuration. TPSM calculations for $^{68}\text{Ge}$ have been performed with the deformation parameters, $\varepsilon = 0.22$ and $\varepsilon' = 0.16$ using the oscillator space of $N = 3, 4$ and $5$ (both for protons and neutrons) and the pairing strengths of $G_1 = 20.82$ and $G_2 = 13.58$. Figure 3 is the obtained band diagram for $^{68}\text{Ge}$ and it is seen that $\gamma$-band depicts quite large odd-even staggering with the even-spin members closely following the ground-state band. It is further noted that ground-state band is crossed by two-quasiparticle bands having (1, 2n) and (3, 2n) configurations at $I = 8$. These two bands are two-neutron aligned and the $\gamma$-bands built on this two-neutron state. The odd-spin members from the two-neutron $\gamma$-band form the yrast states from $I = 9$ to 13. It is also evident from figure 3 that two-aligned band having (1, 2p) configuration also crosses the ground-state band at $I = 10$ and from $I = 14$ the four-quasiparticle state formed from two-neutron plus two-proton configuration becomes lowest in energy.

The band structures obtained after diagonalization are displayed in figure 4 along with the experimental data which depicts multiple s-bands above the ground-state band. It is evident from the figure that TPSM approach reproduces the observed band structures remarkably well. TPSM also predicts many new states for the $\gamma$-band and also a few lower states for the bands labelled as (1, 2p) and odd-spin member of (3, 2n). These levels have been assigned based on the dominant components in the calculated wavefunctions plotted in figure 5. The ground-state band, shown in the top panel of figure 5, has the dominant component, as expected, of (0, 0) for $I = 0$, 2 and 4. For $I = 6$, the contribution of the two-neutron aligned state having (1, 2n) configuration becomes equally important. It is noted from the wavefunction analysis that two observed s-bands beginning with $I = 8$ have dominant components of (1, 2n) and (3, 2n) configurations and, therefore, both these bands have neutron configuration and one of them is the $\gamma$-band based on the two-quasineutron configuration. The third observed s-band beginning with $I = 12$ has the (1, 2p) configuration. It needs to be clarified that it is somewhat erroneous to label these bands with a specific two-quasiparticle configuration as there is a substantial mixing and further for high-spin states four-quasiparticle states become lower in energy.

It is quite clear from above discussion that the lowest two observed s-bands have both neutron structure and the third s-band has the proton structure. The g-factor measurements have been performed for the lowest two s-bands and both are confirmed to have neutron configuration and, therefore, validating the TPSM predictions [83] that the second s-band is the $\gamma$-band built on first s-band. We have also evaluated the intra-band BE2 transition probabilities for various band

![Figure 3](image-url)  
Figure 3. Band diagram for $^{68}\text{Ge}$ isotope. The labels ($K$, n-qp) indicate the $K$-value and the quasiparticle character of the configuration, for instance, (3, 2n) corresponds to the 2n-aligned $\gamma$-band built on 2n-aligned state $K = 1$ quasiparticle configuration.

![Figure 4](image-url)  
Figure 4. Comparison of the measured energy levels of $^{68}\text{Ge}$ nucleus and the results of TPSM calculations. Data has been taken from [81].
structures discussed above and are plotted in figure 6. Measured BE2 values are also available for some transitions, which have been depicted in the figure. For the g-band, the experimental values are well reproduced by the TPSM calculations, however, disagreement is clearly seen for the I = 10 transition in the two-quasineutron band. More experimental data is needed to test the predictions of the TPSM in detail.

3.2. Systematic investigation of γ-band structures in Dy, Hf, Hg and U nuclei

A systematic investigation of the band structures observed in 154–164Dy, 180Hf, 180Hg and 238U nuclei have been performed. These nuclei have been chosen in the present study as γ-bands are known up to highest spin and it is possible to test the predictions of the TPSM approach in the limit of largest angular momentum. TPSM calculations for these nuclei have been performed with the deformations listed in table 2. For 154–164Dy, 180Hf and 180Hg nuclei, calculations have been performed in the configuration space with N = 4, 5, 6 for neutrons and N = 3, 4, 5 for protons; and the pairing strength parameters of G1 = 21.24 and G2 = 13.86. As a representative example, the band diagram for 154Dy is displayed in figure 7. In the band diagram, the ground-state band having (0, 0) configuration is crossed at I = 14 by two-neutron aligned configuration, (1, 2n). Further, it is noted that four-quasiparticle configuration, (2, 2n2p), crosses the two-neutron state at I = 26 and above this spin value, it is expected that four-quasiparticle configurations will dominate the yrast band.

The band diagram for 238U is displayed in figure 8. The TPSM calculations for this system were carried out within the space of N = 5, 6, 7 for neutrons and N = 4, 5, 6 for protons and with the pairing strengths of G1 = 16.80 and G2 = 12.80. It is noted from the figure that the first crossing due to alignment of two-neutrons occurs at I = 20 and the second due to the alignment of two-neutrons plus two-protons occurs at I = 30. The two-proton aligned band is seen to remain always higher in energy as compared to the neutron-aligned configuration.

The projected energies for yrast-, γ- and γγ-bands, obtained after diagonalization of the shell model Hamiltonian, are plotted in figures 9–10,11 along with the available experimental data. To have a better comparison between theoretical and experimental energies, these quantities have been subtracted by the rigid-body reference energy, \( \kappa I(I + 1) \). It is evident from the three figures that observed yrast bands in all the studied nuclei are reproduced fairly well by the TPSM calculations. For the γ-bands, the agreement between observed and calculated energies is also quite good, except that some deviations are noted in 156Dy and 166Dy for high-spin states above I = 20. These
deviations are also observed in some of the yrast bands and are expected since for spin above \( I = 14 \) as due to the alignment of two-quasineutrons, \((1, 2n)\), and the second

Table 2. Axial and triaxial quadrupole deformation parameters \( \varepsilon \) and \( \varepsilon' \) employed in the TPSM calculation. Axial deformations are taken from [84] and non-axial deformations are chosen in such a way that band heads of the \( \gamma \)-bands are reproduced. The \( \gamma \) deformation is related to the above two parameters through \( \gamma = \tan^{-1}(\varepsilon'/\varepsilon) \).

|         | \(^{154}\)Dy | \(^{158}\)Dy | \(^{160}\)Dy | \(^{162}\)Dy | \(^{164}\)Dy | \(^{180}\)Hf | \(^{180}\)Hg | \(^{238}\)U |
|---------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| \( \varepsilon \) | 0.262       | 0.260       | 0.270       | 0.280       | 0.252       | 0.215       | 0.220       | 0.210       |
| \( \varepsilon' \) | 0.100       | 0.110       | 0.110       | 0.120       | 0.130       | 0.100       | 0.090       | 0.085       |
| \( \gamma \) | 20          | 23          | 22          | 23          | 27          | 24          | 22          | 22          |

Figure 7. Band diagram for \(^{154}\)Dy nucleus. The labels \((0, 0), (2, 0), (4, 0), (1, 2n), (3, 2n), (1, 2p), (3, 2p), (0, 2n2p), (2, 2n2p)\) and \((4, 2n2p)\) correspond to ground, \( \gamma \), two neutron-aligned, \( \gamma \)-band based this two neutron-aligned state, two proton-aligned, \( \gamma \)-band built on this two proton-aligned state, two neutron plus two proton aligned, \( \gamma \) and \( \gamma \gamma \)-band built on this four-quasiparticle state.

![Figure 7](image1.jpg)

Figure 8. Band diagram for \(^{238}\)U nucleus. The labels \((0, 0), (2, 0), (4, 0), (1, 2n), (3, 2n), (1, 2p), (3, 2p), (0, 2n2p), (2, 2n2p)\) and \((4, 2n2p)\) correspond to ground, \( \gamma \), two neutron-aligned, \( \gamma \)-band on this two neutron-aligned state, two proton-aligned, \( \gamma \)-band on two this proton-aligned state, two neutron plus two proton aligned, \( \gamma \) and \( \gamma \gamma \)-band built on this four-quasiparticle state.

![Figure 8](image2.jpg)

Figure 9. Comparison of the measured energy levels of yrast-, \( \gamma \)-, and \( \gamma \gamma \)-bands for \(^{154,156,158}\)Dy nuclei and the results of TPSM calculations. The scaling factor \( \kappa \) appearing in the \( \gamma \)-axis is defined as \( \kappa = 32.32\text{A}^{-5/3} \). Data is taken from [32, 85, 86].

\( \gamma \gamma \)-bands have not been observed in any of the studied nuclei and it is noted from figures 9–11 that predicted band heads of these bands are quite high in energy (more than 2 MeV), but it is observed that for high-spin states these bands become quite close to the \( \gamma \)-bands and it should be possible to populate them at higher spin.

To probe the crossing features in the studied nuclei, the alignments of the \(^{154–164}\)Dy, \(^{160}\)Hf, \(^{162}\)Hg and \(^{238}\)U nuclei are displayed in figures 12 and 13 as a function of the rotational frequency. In the experimental alignment plot of \(^{154}\)Dy, two upbends are clearly observed, whereas in the theoretical plot only a broad upbend is noted. These upbends are due to the crossing of the aligned quasiparticle configurations along the yrast line. In order to investigate the structural changes as a function of angular momentum, the wavefunctions are plotted in figure 14. In the top panel of this figure, the yrast wavefunction depicts the first crossing at \( I = 14 \) as due to the alignment of two-quasineutrons, \((1, 2n)\), and the second
crossing at $I = 18$ arises from the alignment of four-quasiparticle configuration, $(2, 2n2p)$. The broad upbend noted in the alignment figure is due to combination of these crossings. In the experimental plot, the two crossings are clearly evident and this indicates that interaction among the crossing bands is overpredicted by the TPSM calculations. Figure 14 also depicts the wavefunctions for the $\gamma$- and $\gamma\gamma$-bands and $\gamma$-band up to $I = 12$ has the expected dominant component of $(2, 0)$. However, above this spin value, $\gamma$-band is a mixture of many different configurations. $\gamma\gamma$-band has the expected structure of $(4, 0)$ configuration up to $I = 9$ and above this spin value it also has mixture of many configurations.

TPSM calculated alignment for $^{156}$Dy, shown in the second panel of figure 12, again shows a smoother behaviour as compared to the experimental alignment. For $^{154}$–$^{164}$Dy, shown in figure 12, and for $^{180}$Hf, $^{162}$Hg and $^{238}$U, depicted in figure 13, the agreement between the calculated and the experimental values is better than the previous two cases.

The wavefunctions for the yrast-, $\gamma$- and $\gamma\gamma$-bands are displayed in figure 15 for $^{238}$U. The cross over between the ground-state configuration and the two-neutron aligned band is noted at $I = 20$ for the yrast band. Above $I = 26$, two-proton aligned configuration is also observed to become important and above $I = 30$, the four-quasiparticle configuration becomes dominant. For the $\gamma$-band, $(2, 0)$ is the dominant configuration up to $I = 16$, but above this spin value it is a highly mixed band. For $\gamma\gamma$-band, $(4, 0)$, is the dominant configuration up to $I = 10$ and above this spin, it is again a mixed band.

The measured BE2 transition probabilities are available along the yrast band for most of the nuclei studied in this section and are depicted in figures 16 and 17 along with the evaluated BE2 transitions using the TPSM wavefunctions and the projected expression given in equation (14). It is evident from the figure that calculated BE2s reproduce the known transitions remarkably well. It is also noted from the figure that TPSM results depict two drops in $^{154}$Dy and $^{156}$Dy isotopes and are due to two crossings evident from the wavefunction analysis. These crossings are not apparent in the alignment plot of figure 12, but are noted in the BE2 transitions.

3.3. Chiral symmetry and doublet band structures

The spontaneous symmetry breaking mechanism has played a central role in elucidating the intrinsic structures of quantum many-body system [106]. What all is possible from the experimental analysis of a bound quantum many-body system is a set of energy levels that are labelled by the quantum numbers related to the symmetries preserved by the system. For instance, for the case of atomic nucleus the energy levels are labelled by angular-momentum and parity quantum numbers that are related to the rotational and reflection symmetries. It is through breaking of these symmetries that
provides an insight into the excitation modes of the atomic nucleus. The most celebrated model that employs the symmetry breaking mechanism is the Nilsson model. This model breaks the rotational symmetry and has provided invaluable insights into the structure of atomic nuclei.

Figure 12. Comparison of the aligned angular momentum, \( i = I_\text{ref} - I_{\text{rot}} \) as a function of rotational frequency, \( \hbar \omega = \frac{K^2}{2} \) for 154–164 Dy nuclei. \( I_\text{ref} = \sqrt{I(I+1)} - K^2 \) and \( I_{\text{rot}} = \omega (J_0 + \omega^2 J_1) \). The reference band Harris parameters used are \( J_0 = 23 \) and \( J_1 = 90 \).

Figure 13. Comparison of the aligned angular momentum, \( i = I_\text{ref} - I_{\text{rot}} \) as a function of rotational frequency, \( \hbar \omega = \frac{K^2}{2} \) for 180 Hf, 180 Hg and 238 U nuclei. \( I_\text{ref} = \sqrt{I(I+1)} - K^2 \) and \( I_{\text{rot}} = \omega (J_0 + \omega^2 J_1) \). The reference band Harris parameters used are \( J_0 = 23 \) and \( J_1 = 90 \).

Figure 14. Probability of various projected K-configurations in the wavefunctions of the yrast-, \( \gamma \)- and \( \gamma\gamma \)-bands in 154 Dy.

Figure 15. Probability of various projected K-configurations in the wavefunctions of the yrast-, \( \gamma \)- and \( \gamma\gamma \)-bands in 238 U.

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nuclei may be a manifestation of the breaking of the chiral symmetry in the intrinsic frame of reference [108–114]. The chiral symmetry is possible for nuclei having triaxial shapes with total angular momentum having components along all the three mutually perpendicular axis. In odd–odd nuclei, the three angular-momentum vectors that form the chiral geometry are that of core; and of valence-neutron and -proton. The three angular-momentum vectors can either form left- or right-handed system and can be transformed into each other with the chiral symmetry transformation operator, \( \hat{T} \hat{R}_y \), where \( \hat{T} \) is the time-reversal operator and \( \hat{R}_y \) is the rotation by 180° about y-axis [115]. The chiral dynamical variable so called ‘handedness’ (\( \sigma \)) is not a measurable quantity and what is measurable in the laboratory frame is the set of states of chiral doublet bands which have a well defined value of complementary variable, the chirality (\( \Sigma \)). In the strong chiral symmetry breaking limit with the three angular-momentum vectors perpendicular to each other, \( \sigma \) assumes the values of \( \pm 1 \). The left- and right-handed states are well separated and there is no possibility of tunneling between the two states. This results into two degenerate doublet bands in the laboratory frame of reference.

For the weak chiral symmetry breaking, the three angular-momentum vectors are not orthogonal which, as a matter of fact, is true in most of the physical situations. The handedness variable then takes the value between \(+1\) and \(-1\) (the value of 0 corresponds to the planar situation). For this case, the tunneling between the two states takes place and in the laboratory frame this corresponds to the mixing of the two solutions with the result that two bands tend to be non-degenerate.

It is evident from the above discussion that a fingerprint of the chiral symmetry is the energy difference between the states of the doublet bands. In figure 18, this energy difference, \( \delta(E) \), is plotted for odd–odd Cs-isotopes [24] using both observed and TPSM calculated energies. These calculations have been performed with the parameter set given in [24]. The doublet band observed in these nuclei have been proposed to arise from the breaking of the chiral symmetry mechanism. It is apparent from the figure that \( \delta(E) \) varies from 0.2 to 0.4 for most of the cases. As already pointed out above, for the case of strong chiral symmetry breaking this value should be close to zero and also should be constant as a function of angular-momentum [116]. It is noted from figure 18 that \( \delta(E) \) varies with angular-momentum, in particular, for heavier Cs-isotopes and, therefore, it is difficult to make a definitive statement on the chiral nature of the observed doublet bands for heavier Cs-isotopes.

Further, it has been demonstrated in [116] that similar analysis as that for the energies can also be performed for the transition probabilities as chiral symmetry operator commutes, not only with the Hamiltonian, but also with the electromagnetic transition operators. The deviation between the yrast and the partner bands for the transition probabilities, \( B(\lambda \mu; I_l \rightarrow I) \), is defined as [116]

\[
\varepsilon(\lambda \mu; I_l) = \frac{A - B}{A + B},
\]

In recent years, it has also been demonstrated that doublet band structures observed in some odd–odd and odd-mass nuclei may be a manifestation of the breaking of the chiral symmetry in the intrinsic frame of reference [107]. The observation of the rotational band structures built on each intrinsic state is a manifestation of the breaking of this symmetry.

Figure 16. Comparison of experimental and calculated B(E2) for \(^{154-166}\text{Dy}\). Data is taken from [85, 94–98].

Figure 17. Comparison of experimental and calculated B(E2) for \(^{180}\text{Hf}, ^{180}\text{Hg}\) and \(^{238}\text{U}\). Data is taken from [92, 99].
In the strong chiral limit, the above defined $\varepsilon$ parameter is expected to tend to zero and deviations from it correspond to the tunneling between the two solutions. The above quantity is displayed in figures 19 and 20 for the E2 and M1 transitions using the TPSM expressions and wavefunctions. The figures also display the experimental values, wherever available. It is evident from figure 19 that calculated values for E2 transitions are close to the zero line for most of the cases and the experimental values available for $^{126}$Cs and $^{128}$Cs are well reproduced by the TPSM approach. The deviations for the M1 transitions, depicted in figure 20, are again noted to be close to zero line for the light Cs-isotopes. For the heavier Cs-isotopes, in particular, for $^{128}$Cs, $\varepsilon$(M1) is somewhat larger and varies with spin.

For odd-mass nuclei, the three angular-momentum vectors that form chiral geometry are that of core, odd-proton (neutron) and a pair of neutrons (protons). It has been proposed that in odd-proton $^{105}$Rh and $^{107}$Rh isotopes the first excited band is the normal $\gamma$-band up to spin, $I = 21/2$, but above this spin value this band forms the chiral partner of the yrast band. In order to investigate how $\gamma$-bands in these isotopes are transformed into the conjectured chiral bands, we have carried out TPSM study for these isotopes [23] with the deformation parameters, $\varepsilon = 0.260$ for $^{105}$Rh and $\varepsilon = 0.224$ for $^{107}$Rh within the configuration space of $N = (3, 4, 5)$ major shells for neutrons and $(2, 3, 4)$ for protons, and pairing strengths of $G_1 = 20.25$ and $G_2 = 16.20$.

As an illustrative example, the band diagram for $^{105}$Rh is shown in figure 21 and in order to have a better visualization of the bands, the energies have been subtracted by a reference contribution. The ground-state band has the intrinsic configuration of one quasiparticle with $K = 7/2$. For odd-mass nuclei, the $\gamma$-band head has two possible configurations with $K = \pm \frac{3}{2}$, resulting into $K = 11/2$ and $3/2$. Both these $\gamma$-bands are shown in figure 21 with $K = 11/2$ being favoured in energy. It is noted from the figure that the ground-state band is crossed at $I = 19/2$ by another band having $K = 5/2$, which is a three-quasiparticle state. It is seen that the $\gamma$-band built on this three-quasiparticle state having $K = 9/2$ also crosses the ground-state configuration at $I = 23/2$. There is a further simultaneous crossing by two three-quasiparticle bands having $K = 9/2$ and $13/2$ at $I = 33/2$.

The projected bands depicted in figure 21 and many more, around forty for each angular-momentum, are employed to diagonalize the shell model Hamiltonian as explained earlier. The results of yrast and the first excited band are only shown above $I = 23/2$ as
they have been proposed to originate from the chiral symmetry breaking above this spin value. It is evident from the two figures that only for the optimum triaxial deformation of 

\[ \varepsilon' = 0.15 \text{ for } ^{103}\text{Rh and } \gamma = 33^\circ \text{ for } ^{105}\text{Rh}, \]

and the figures that only for the optimum triaxial deformation of 

\[ \varepsilon' = 0.15 \text{ for } ^{103}\text{Rh and } \gamma = 33^\circ \text{ for } ^{105}\text{Rh}, \]

a good agreement is obtained between theoretical and the experimental energies. In file 24, \( \delta(E) \) is displayed as a function of angular momentum for the two isotopes and it is quite evident 

\[ \varepsilon' = 0.15 \text{ for } ^{103}\text{Rh and } \gamma = 33^\circ \text{ for } ^{105}\text{Rh}, \]
that deviations are quite large and, therefore, it is difficult to classify these doublet bands as chiral partners.

In order to probe further the evolution of the yrast and the first excited band structures with spin, the wavefunctions of the two bands are plotted in figure 25 for $^{105}$Rh as a representative case. It is noted from the upper panel that yrast band up to $I = \frac{11}{2}$ has the dominant $K = \frac{7}{2}$ configuration and above this spin value it is $K = \frac{5}{2}$ three-quasiparticle configuration that is important. It is also seen that above $I = \frac{11}{2}$, the three quasiparticle configuration with $K = \frac{9}{2}$ dominates the yrast states. The first excited band from $I = \frac{11}{2}$ up to $I = \frac{17}{2}$ has the expected dominant component of $K = \frac{11}{2}$, which is the $\gamma$-band built on the ground-state configuration. However, above $I = \frac{17}{2}$, it is a highly mixed state.

### 3.4. Wobbling motion

Wobbling motion like chiral symmetry, discussed above, is possible only for triaxial nuclei. This mode was predicted by Bohr and Mottelson in late sixties [1] for even–even nuclei and it was shown that the frequency of the wobbling motion is proportional to angular momentum, $I$ with enhanced $\Delta I = 1$, E2 inter-connecting transitions. It has been recently shown that for odd-mass nuclei, the wobbling mode is modified depending on the direction of the angular-momentum vector of the odd-particle. For systems with angular-momentum of the odd-particle aligned along the medium axis of the core, which has largest moment of inertia, referred to as transverse mode, the wobbling frequency decreases with spin. This frequency increases if angular-momentum of the last particle is anti-parallel to the medium axis and has been called as the longitudinal mode [117, 118]. The wobbling frequency is given by [119]

$$E_{\text{wob}}(I) = E(I, n_\omega = 1) - \frac{[E(I - 1, n_\omega = 0) + E(I + 1, n_\omega = 0)]}{2}$$  \hspace{1cm} (17)

In the present work we have performed TPSM analysis to describe the observed band structures and the transverse wobbling mode in $^{135}$Pr nucleus. These calculations have been performed with the deformation parameters, $\varepsilon = 0.16$ and $\varepsilon' = 0.11$, within the configuration space of $N = 3,4,5$ major shells for neutrons and $N = 2, 3, 4$ for protons; and with the pairing interaction parameters, $G_1 = 20.12$ and $G_2 = 13.13$. The results obtained after diagonalization of the shell model Hamiltonain are presented in figure 26. In the top two panels of the figure, the results are compared for the yrast and the $n = 1$ wobbling bands. In the bottom panel, the calculated wobbling frequency, calculated from the TPSM energies, is compared with the frequency obtained from the measured energies. It is noted from the figure that wobbling frequency from TPSM depicts less drop with spin compared to the frequency obtained from the measured energies. It is noted from the figure that wobbling frequency from TPSM depicts less drop with spin compared to the frequency obtained from the measured energies.

The wavefunctions for the yrast and the $n = 1$ wobbling bands are depicted in figure 27 and it is evident that three-quasiparticle band crosses the one-quasiparticle at $I = \frac{25}{2}$ for both the bands. It turns out that this crossing occurs much earlier in $^{135}$La and plays a vital role in understanding the longitudinal wobbling mode observed recently for this system [36]. The calculations reported here are preliminary and we are in the process of performing a detailed analysis of the wobbling motion for all the nuclei where this mode has been observed.
4. Summary and future prospects

During the last decade, research in nuclear theory has witnessed a discernable progress in the development of state-of-the-art models and techniques to elucidate the rich variety of shapes and structures in nuclei. There is a great optimism that in the coming years it should be possible to apply \textit{ab initio} methods of Green’s function \cite{120,121}, shell model \cite{122} and density functional theory \cite{8–10}, to investigate majority of the properties all across the nuclear periodic table with the availability of more powerful computing facilities. However, at the moment these methods have limited applicability and are used to describe nuclei in lighter mass regions or ground-state properties only. To study, for instance, the rich band structures observed in medium and heavy mass regions, alternative methods with moderate computational requirements ought to be explored.

Recently, TPSM approach has been developed to describe the rich band structures observed in well deformed and transitional nuclei. This model employs the basis that are solutions of the triaxial Nilsson potential and then three dimensional projection is performed to project out the states with well defined angular momentum quantum number. The advantage of this approach is that systematic studies of a large class of nuclei can be performed with a minimal computational effort. As a matter of fact, already a number of systematic investigations have been undertaken using this model and it has been demonstrated to reproduce the known experimental data remarkably well. This model has been applied to investigate a broad range of properties related to the triaxial degree of freedom of the nuclear deformation.

It is known that most of the deformed nuclei are axially symmetric with well defined angular momentum projection along the symmetry axis so called the ‘\(K\)’ quantum number. Band structures in well deformed nuclei are labelled with this quantum number and transitions in these nuclei are known to follow the selection rules based on this quantum number. However, there are also many regions, so called transitional regions, where the axial symmetry is known to be broken and a triaxial degree of freedom plays an important role.

In most of these triaxial nuclei, \(\gamma\)-bands are observed which traditionally have been considered as vibrations in the non-axisial degree of freedom. In the TPSM interpretation, \(\gamma\)-bands emerge from the projection of the \(K = 2\) component of the triaxial vacuum configuration. \(\gamma\)-bands are known in some Dy, Hf, Hg and U nuclei up to very high angular momentum and in the present work we have performed a systematic investigation of these nuclei using the TPSM approach. The observed yrast and \(\gamma\)-bands in all these studied nuclei have been demonstrated to be well described by the TPSM calculations. Deviations have also been noted above \(I = 20\) and one possible reason for this discrepancy could be due to neglect of the four-quasineutron and -quasiproton configurations in the present work.

Obviously, the question arises on the connection between the \(\gamma\)-bands obtained in the TPSM approach and the traditional \(\gamma\)-vibrational interpretation of these bands. In order to address this question, the phase of the odd–even staggering of the \(\gamma\)-band, which is considered to provide clues on the \(\gamma\)-vibrational or \(\gamma\)-deformation character of a nucleus \cite{123}, has been investigated systematically for Se- and Ge-isotopes \cite{22}. It has been noted in this work that staggering phase of the \(\gamma\)-bands obtained in TPSM approach without band mixing is same as that calculated from the Davydyov–Filippov model with odd-spin states lower than the even-spin states and, therefore, corresponds to the triaxial deformation. However, when band mixing is performed, the staggering phase of the
The possibility of observing $\gamma$-bands based on the quasiparticle configurations has also been explored in the present work. It is quite evident from the very construction of the basis states in TPSM approach that $\gamma$-bands are possible based on each intrinsic state. $\gamma$-band based on the ground-state are quite well established and have been studied using many different approaches and methods. However, $\gamma$-bands built on the excited quasiparticle configurations have remained rather abstruse as most of the earlier models didn’t consider the quasiparticle excitations. It has been demonstrated that some of the excited band structures in Ce- and Nd-isotopes are, as a matter of fact, the $\gamma$-bands based on two-quasiparticle states. In some of these isotopes, two $s$-bands are observed with similar $g$-factors and this was an unsolved problem for several years.

In the conventional approach, two $s$-bands are expected to be based on neutron and proton aligned structures and, therefore, $g$-factors of the two $s$-bands should have opposite signs. Measurements of $g$-factors of two $s$-bands provide same signs for both the $s$-bands and, thereby, indicating that both $s$-bands have either proton or neutron structures. What has been shown in the TPSM studies that second $s$-band in many of these nuclei is the $\gamma$-band based on the two-quasineutron or -quasiproton states. As the parent band and the $\gamma$-band based on it, originate from the same intrinsic configurations, two $s$-bands in the TPSM approach are expected to have similar $g$-factors. TPSM has also provided an explanation on the observation of three $s$-bands in $^{86}$Ge. It has been shown that one of the $s$-band is the $\gamma$-band based on the two-neutron quasiparticle state.

Further, recently new observations have been made in a set of nuclei that are considered as fingerprints of the triaxial deformation. These new observations include the occurrence of doublet bands and excited bands with dominant $\Delta I = 1$ $E2$ transitions and have been regarded as manifestations of chiral and wobbling motion in rotational nuclei. These observations are entirely as a consequence of the triaxial shape of system and are not possible in the axial limit. In the present work, the appearance of the doublet bands have been investigated for odd–odd Cs-isotopes from $A = 124$ to $132$ and also for the odd-proton $^{105}$Rh and $^{107}$Rh isotopes. It is expected that the doublet band should be degenerate in the limit of strong chiral symmetry breaking. It has been shown that the energy difference of the Cs-isotopes for lighter isotopes is small and, therefore, may be regarded as originating from the chiral symmetry breaking mechanism. However, for heavier isotopes, the energy difference is large and also they have a considerable variation with spin and, therefore, the doublet bands cannot be considered as chiral partners. We have also evaluated the differences in the $BE2$ and $M1$ transitions and similar conclusions have been drawn from these quantities as from the energy considerations.

The high-spin doublet band structures in $^{105}$Rh and $^{107}$Rh have also been proposed to originate from chiral symmetry breaking. In the low-spin regime, these nuclei have regular $\gamma$-bands and in the high-spin region, the observed energy difference between the $\gamma$-band and the yrast sequence decrease with spin and based on this inference, the high spin band structures have been regarded as chiral partners. (For the normal $\gamma$-band, this difference remains almost constant.) It has been shown that energy difference, both experimental and theoretical, is too large as compared to the Cs-isotopes and for some angular momentum value it is more than 1 MeV. It has been also demonstrated in our earlier study [37] that quadrupole moment varies with spin quite appreciably and, therefore, the doublet bands in $^{105}$Rh and $^{107}$Rh cannot be regarded as candidates for chiral symmetry breaking.

In order to probe the chiral symmetry and wobbling motion, what has been done in the present work is to calculate the energies and transition probabilities in the TPSM approach and check whether they are consistent with those expected for chiral symmetry and wobbling motion. It is certainly more interesting to directly investigate these phenomena using the TPSM wavefunctions. For chiral symmetry, this amounts to evaluation of the angular-momentum projections along the three orthogonal axis. The problem is that projected basis are defined in the laboratory frame and also form a non-orthogonal set. We are presently working on this problem and hopefully in the near future, we shall be in a position to provide some answers.

We would like to add that one of the the major problem in the TPSM model is that first of all the pairing plus quadrupole interaction used is quite rudimentary and needs to be generalized for a better description of the nuclear properties. Secondly, the coupling constants of the interaction are adjusted through self-consistency conditions with the input deformation parameters. This means that optimum deformation value for a system under investigation should be known prior to performing the TPSM calculations. Although, the final results should be independent of the input deformation in case a very large basis set is employed in the study, however, in practice a limited basis space is employed and the results become basis dependent. In order to circumvent this problem, we are considering to fix the coupling constant of the interaction through a mapping procedure as has been done in other approaches [125, 126]. The energy surface, for instance, obtained from the density functional theory can be mapped to the surface obtained using the model interaction employed in the TPSM approach with adjustable coupling constants.

There are several other possible ways of improving the predictive power of the TPSM calculations. For instance, the diagonalization of the shell model Hamiltonian is performed for a fixed value of the deformation parameter and for a more accurate description, a set of deformation values should be considered by employing the generator coordinate method (GCM). This will allow the possibility to have coupling between the $\gamma$-deformation and $\gamma$-vibrational degrees of freedom. One of the major discrepancy that has surfaced in the application of the TPSM approach is the band head energy of the $\gamma\gamma$-band [14–16]. It has been observed that for most of
known cases, TPSM calculations underpredicts band head energy of this band by more than 1 MeV. It is expected that $\gamma\gamma$-band to have significant vibrational component because of mixing with the quasiparticle states which are close in energy. We hope that by performing GCM with both $\beta$ and $\gamma$ as generator coordinates, the $\gamma\gamma$-band and other properties shall be described more accurately. It is also quite important to include higher multi-quasiparticle states in the basis space. It has been noted in the present investigation that TPSM results tend to disagree above $I = 20$ and the reason for this discrepancy is the neglect of the four-proton and -neutron quasiparticle states. We are presently working to improve the TPSM approach along the lines discussed above and the results shall be presented in future publications.

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**Appendix**

The matrix elements of the Hamiltonian and transition operators in TPSM approach are evaluated using the Wick theorem. For instance, for a two-body operator of the form $\hat{O}\hat{O}$, we have

$$\langle\hat{O}\hat{O} \rangle = \langle\hat{O}\rangle^2 + \langle\hat{O}\rangle \langle \hat{O} : \hat{O} \rangle + \langle \hat{O} : \hat{O} \rangle^2,$$

(18)

where $\langle \hat{O} \rangle$ is the contraction and $\langle \hat{O} : \hat{O} \rangle$ is the normal ordered form of the operator. The rotated matrix elements of one-body and two-body operators are given by

$$\langle \hat{H}^{(1)}(\Omega) \rangle = \langle \hat{O} \rangle \{ \langle \hat{O} : [\Omega] \rangle + \langle \hat{O} : [\Omega] \rangle \},$$

$$(19a)$$

$$\langle \hat{H}^{(2)}(\Omega) \rangle = \langle \hat{O} \rangle \{ \langle \hat{O} : [\Omega] \rangle + \langle \hat{O} : [\Omega] \rangle \},$$

$$(19b)$$

where $[\Omega] = \hat{R}(\Omega) / \hat{R}(\Omega)$. The detailed expressions of the above rotated matrix elements are given in [35]. In TPSM approach, diagonalization of the shell model Hamiltonian follows from the Hill–Wheeler method. In this method, the following ansatz is used for wavefunction

$$|\psi\rangle = \int d\Omega \, F(\Omega) \hat{R}(\Omega) \, |\Phi\rangle.$$  

(20)

For the projection operator, the above wavefunction is written as

$$|\psi_{IM}\rangle = \sum_{s,K} F^{I}_{s,K} \hat{P}^{I}_{s,K} |\Phi_{i}\rangle,$$

(21)

where

$$\hat{P}^{I}_{s,K} = \frac{(2I + 1)}{8\pi^2} \int d\Omega \, D^{I}_{s,K}(\Omega) \hat{R}(\Omega).$$

(22)

Using the variational ansatz

$$\delta \left[ \langle \psi_{IM} | H | \psi_{IM} \rangle \right] = 0$$

and substituting $|\psi_{IM}\rangle$ from equation (21), we obtain

$$\sum_{s,K} \{ \hat{H}^{I}_{s,K} | \Phi_{i}\rangle - E \cdot \hat{H}^{I}_{s,K} | \Phi_{i}\rangle \} = 0,$$

(24)

where the Hamiltonian and norm kernels are given by

$$\hat{H}^{I}_{s,K} = \langle \Phi_{i} | \hat{H}^{I}_{s,K} | \Phi_{i}\rangle,$$

$$(25a)$$

$$\langle \Phi_{i} | \hat{P}^{I}_{s,K} | \Phi_{i}\rangle = \langle \Phi_{i} | \hat{P}^{I}_{s,K} | \Phi_{i}\rangle.$$  

(25b)

In the TPSM model, we work in a representation in which the norm matrix is diagonal, i.e.

$$\sum_{s,K} \hat{N}^{I}_{s,K} U_{s,K} = n \cdot U_{s,K}^{I}$$

(26)

and solve equation (24) with the eigenstates of the above norm equation as the basis states.

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