Thermoelastic Damping in Fully Clamped Circular Plate Resonators Based on Nonlocal Thermoelasticity

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Abstract. Thermoelastic damping (TED) is the main source of energy dissipation in resonators. The modelling of TED is of great importance for the design of micro-electro-mechanical systems (MEMS) and the improvement of quality factor. Most of the previous TED models are based on classical continuum mechanics and Fourier heat conductive model. However, small-scale effects are widely found for structures at nano and sub-micron level and the classical theories cannot describe their size-dependent behaviors. Based on the nonlocal elasticity theory and the nonlocal heat conductive model, the thermoelastic coupled governing equations of a circular plate with out-of-plane vibration under the influence of nonlocal effect are derived in this paper. The influences of nonlocal thermoelasticity on TED in circular plates are discussed by numerical examples under different boundary conditions, frequencies, thicknesses and modes.

1. Introduction
Micro-electro-mechanical systems are widely applied in the fields of automobiles, home appliances, national defense, and aerospace for their advantages of low cost and high sensitivity. Microresonators are typical components of MEMS and can be used in high-precision sensors. To improve the sensitivity of resonators and reduce energy dissipation, the quality factor of the resonator needs to be improved as much as possible. For resonators that are well-supported and work in vacuum, TED is the main source of energy loss, which has a decisive influence on the quality factor and cannot be eliminated [1]. Hence, modeling the TED in typical components can help design high-quality resonators, which has always been a challenging task.

The first analytical model of TED was proposed by Zener [2, 3], with a homogeneous isotropic beam as the study object. The temperature field function in the form of an infinite series was derived. The approximate TED series solution is obtained by the energy method. Lifshitz and Roukes (L-R) [4] gave an accurate temperature field solution for the beam and obtained the analytical solution of TED in slender beams through the complex-frequency method. Subsequent studies on TED are mostly based on the established framework by Zener and L-R.

The circular plate is a common structure of resonators. Researchers have conducted in-depth research on TED in circular plates [5-8]. Sun et al. [5, 6] gave the analytical solutions of TED in circular plates under axisymmetric and arbitrary out-of-plane vibration. Li et al. [7] deduced the general expression of TED in rectangular/circular plates. Fang et al. [8] derived a more accurate circular plate TED model considering two-dimensional heat conduction.
The size-dependent behaviors of the microstructures can be observed in experiments, which cannot be explained by the classic continuum mechanics and Fourier’s model. The nonlocal elasticity theory proposed by Eringen [9] is a size-dependent theory. The theory is based on the classical continuum mechanics and further considers the long-range force among atoms, taking. Moreover, when the external characteristic length decreases to close to the mean-free path of phonons, the nonlocal effect is also observed in heat conduction process. Guyer and Krumhansl [10] gave a nonlocal heat conductive model, which considers both the nonlocal effect and the thermal relaxation of heat conduction.

Based on nonlocal effects, researchers have proposed some new TED models. However, these researches on the nonlocal effect of TED are all focused on microbeam structures, and there lacks a TED model investigating the influence of nonlocal thermoelasticity on microplates. In this paper, the equation of motion for the nonlocal circular plate is derived by using Hamilton principle. Based on Guyer-Krumhansl model, the nonlocal heat conduction equation in the circular plate is established. On this basis, an analytical TED model in circular plates considering nonlocal thermoelasticity is derived, and the validity of the model is proved. The effects of nonlocal thermoelasticity on TED in plate are discussed by numerical examples under different thicknesses, frequencies and small-scale parameters.

2. Model formulation

2.1. Equation of motion

Illustrated in Figure 1 is an isotropic circular thin plate of radius \(a\), thickness \(h\). The origin is fixed at the center of the plate. The \(x-y\) plane is located at the mid-plane of the plate and the \(z\)-axis perpendicular to the mid-plane. \(u\), \(v\) and \(w\) denote the displacements of the plate along the \(x\), \(y\) and \(z\)-axes.

![Figure 1. A schematic of a circular plate.](image)

Considering thermoelastic coupling, the strain displacement relation is

\[
\varepsilon_{xx} = -z \frac{\partial^2 w_0}{\partial x^2}, \varepsilon_{yy} = -z \frac{\partial^2 w_0}{\partial y^2}, \varepsilon_{zz} = \frac{1}{1-\nu} \left[-v (\varepsilon_{xx} + \varepsilon_{yy}) + (1+\nu) \alpha_T \theta \right], \gamma_{xy} = -2z \frac{\partial^2 w_0}{\partial x \partial y}
\]

(1)

where \(w_0\) is the transverse displacement of mid-plane \(v\) and \(\alpha_T\) the Poisson’s ratio and the thermal expansion coefficient respectively. The temperature variation \(\theta = T - T_0\).

The cubic dilatation is the internal heat source during vibration and is given as

\[
e = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = -\frac{2 \nu}{1-\nu} z \nabla^2 w_0 + \frac{1+\nu}{1-\nu} \alpha_T \theta
\]

(2)

According to nonlocal elasticity theory, the stress-strain relation can be written as follows [9]

\[
\begin{align*}
\sigma_{xx} - \mu \nabla^2 \sigma_{xx} &= \frac{E}{1-\nu^2} \left[ (\varepsilon_{xx} + \nu \varepsilon_{yy}) - (1+\nu) \alpha_T \theta \right] \\
\sigma_{yy} - \mu \nabla^2 \sigma_{yy} &= \frac{E}{1-\nu^2} \left[ (\varepsilon_{yy} + \nu \varepsilon_{xx}) - (1+\nu) \alpha_T \theta \right] \\
\sigma_{xy} - \mu \nabla^2 \sigma_{xy} &= G \gamma_{xy}
\end{align*}
\]

(3)
where $\mu$ the nonlocal parameter, $E$ the Young’s modulus and $G$ the shear modulus.

According to equation (3), the moment and torsion components are expressed as

$$
\begin{align*}
M_{xx} - \mu^2 \nabla^2 M_{xx} &= -D \left( \frac{\partial^2 w_0}{\partial x^2} + \nu \frac{\partial^2 w_0}{\partial y^2} \right) - D(1+\nu)\alpha_y M_T \\
M_{yy} - \mu^2 \nabla^2 M_{yy} &= -D \left( \frac{\partial^2 w_0}{\partial y^2} + \nu \frac{\partial^2 w_0}{\partial x^2} \right) - D(1+\nu)\alpha_y M_T \\
M_{xy} - \mu^2 \nabla^2 M_{xy} &= -D(1-\nu) \frac{\partial^2 w_0}{\partial x \partial y}
\end{align*}
$$

(4)

where the flexural rigidity $D = Eh^3/12(1-\nu^2)$ and the thermal moment $M_T = (12/\rho) \int_{W/2}^{h/2} \theta \mathrm{d}z$.

The variations of strain energy $\delta U$ and the kinetic energy $\delta T$ are

$$
\delta U = \delta \iint_V \left[ (\sigma_{xx} \varepsilon_{xx} + \sigma_{yy} \varepsilon_{yy} + \sigma_{xy} \varepsilon_{xy}) \right] \mathrm{d}V = \iint_A \left[ -M_{xx} \delta \frac{\partial^2 w_0}{\partial x^2} - M_{yy} \delta \frac{\partial^2 w_0}{\partial y^2} - 2M_{xy} \delta \frac{\partial^2 w_0}{\partial x \partial y} \right] \mathrm{d}A
$$

(5)

$$
\delta T = \int_A \rho h \frac{\partial w_0}{\partial t} \frac{\partial \delta w_0}{\partial t} \mathrm{d}A
$$

(6)

Based on Hamilton principle, combining equations (4), (5) and (6), the equation of motion for circular plates considering thermal effect is

$$
D \nabla^4 w_0 + D(1+\nu) \alpha_y \nabla^2 M_T + \left[ (1-\mu^2) \nabla^2 \right] \rho h \frac{\partial^2 w_0}{\partial t^2} = 0
$$

(7)

To adapt to the circular boundary conditions, here the Cartesian coordinate system can be transformed into the polar coordinate system where Laplace operator $\nabla^2 = (\partial^2 / \partial r^2) + (1/r)(\partial / \partial r) + (1/r^2)(\partial^2 / \partial \theta^2)$.

2.2. Temperature field

The nonlocal heat conductive model proposed by Guyer and Krumhansl is written as [10]

$$
q + \tau_0 \frac{\partial q}{\partial t} = -k \nabla T + l^2 \left[ \nabla^2 q + 2\nabla(\nabla \cdot q) \right]
$$

(8)

where $q$ is the heat flux, $k$ the heat conductivity of material, $T$ the temperature function, $\tau_0$ the relaxation time, $l$ the mean-free path of phonon. The relaxation time $\tau_0$ is set to 0 for simplicity.

Assume the transverse displacement and the relative temperature change to be

$$
w_0(r, \theta, t) = W(r, \theta) e^{\imath \omega t}, \theta(r, \theta, z, t) = \theta_0(r, \theta, z) e^{\imath \omega t}
$$

(9)

Taking the cubic dilatation as the heat source in circular plates, the equation of energy conservation is

$$
-\nabla \cdot q = \rho C_v \frac{\partial \theta}{\partial t} + \frac{E \alpha_T \theta_0}{1-2\nu} \frac{\partial \varepsilon}{\partial t}
$$

(10)

where $C_v$ is the heat capacity coefficient at constant volume.

Substituting equation (8) into equation (10), the heat conduction equation is
where $\Delta_E = \frac{E_0 T_0}{C_v}$ is Zener modulus and $\chi = \frac{k}{C_v}$ the heat diffusivity. $(1+\nu)\Delta_E \left(1 - (1-2\nu)\right) \ll 1$ is neglected.

For plate resonators operating in vacuum, the adiabatic boundary conditions for the upper and lower surface are assumed, which is $\partial_\eta [\partial_\eta u]/[\partial \eta^2] \rightarrow -\infty$. Hence, we can obtain the temperature field as follows

$$\partial_\phi (r, \theta, z) = \frac{\Delta_E}{(1-\nu)\partial_r} \nabla^2 W_0 \left[ z - \frac{\sin(\zeta)}{m \cos(\chi/2)} \right]$$

where $c = \sqrt{\frac{i\omega}{\chi - 3i \omega}} = \frac{\omega^2}{h} \sqrt{a_i, a_i} = \frac{\xi}{\eta} \left( \eta - i \alpha \right) , \ \xi = h \sqrt{\omega^2/2 \chi } , \ \alpha = \frac{3 \omega^2}{\chi} \left( 1 + 9 \omega^2 \right)^2 \chi , \ \alpha_0 = \frac{1}{1 + 9 \omega^2 \chi} \chi^2$

and $\eta = \sqrt{a_i^2 + a_i^2 + a_i}$.  

2.3. TED solution
Substituting the thermal moment into equation (7), we have

$$\nabla^4 W + \mu^2 \lambda \nabla^2 W - \lambda W = 0$$

where $\lambda = \rho h^2 / D_n$ , $D_n = D [1 + \Delta_p (1 + f(\omega)) \right] , \Delta_p = \Delta_E (1 + \nu)/(1-\nu)$ and $f(\omega) = \frac{24 \omega^2}{c^3 h} [\chi/2 \tan(\chi/2)].$

Assuming the transverse deflection $W(r, \theta) = R(r) \phi(\theta)$, equation (13) can be transformed into

$$\frac{\partial^2 \phi}{\partial \theta^2} + \gamma^2 \phi = 0$$

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} + \left( \frac{\mu^2 \lambda^2 + \lambda^2 \left[ \mu^2 \lambda^2 + 4 \right] / 2}{r^2} - \gamma^2 \right) R = 0$$

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} + \left( \frac{\mu^2 \lambda^2 - \lambda^2 \left[ \mu^2 \lambda^2 + 4 \right] / 2}{r^2} - \gamma^2 \right) R = 0$$

where $n$ is the eigenvalue.

Equation (14) can be easily solved. For simplicity, $\phi(\theta)$ is assumed to be

$$\phi(\theta) = c_i \cos(n \theta) = e^{i n \theta}$$

The solutions of the two Bessel equations in equations (15) and (16) are

$$R_n (r) = A_n J_n (\alpha r) + B_n Y_n (\alpha r) , R_i (r) = C_i I_n (\beta r) + D_i K_n (\beta r)$$

where $A_n, B_n, C_n$ and $D_n$ are four constants depending on the boundary conditions and $\alpha = \lambda \sqrt{\left[ \mu^2 \lambda^2 + 4 \right] / 2}$ and $\beta = \lambda \sqrt{\left[ \mu^2 \lambda^2 + 4 - \mu^2 \lambda^2 \right] / 2}$

Together, the transverse deflection of circular plate is written as

$$w_0 (r, \theta, t) = [R_i (r) + R_n (r)] \phi(\theta) e^{i n \theta} = [A_n J_n (\alpha r) + B_n Y_n (\alpha r) + C_i I_n (\beta r) + D_i K_n (\beta r)] e^{i n \theta + \alpha r}$$

Due to the displacement at the center cannot be infinite, we have $B_n = D_n = 0$.  


For fully clamped circular plates, the boundary conditions are given as

$$W(r, \theta)_{r=0} = \frac{\partial W(r, \theta)}{\partial r} \bigg|_{r=a} = 0$$  \hspace{1cm} (20)

Substituting equation (19) into equation (20), we have

$$\beta a I_{n+1}(\beta a) J'_\varepsilon(\alpha a) - \alpha a J_{n+1}(\alpha a) I'_\varepsilon(\beta a) = 0$$  \hspace{1cm} (21)

From equation (21), the isothermal natural frequency of plate is obtained as

$$\omega_0 = \lambda^2 \sqrt{\frac{D}{\rho h}}$$  \hspace{1cm} (22)

For $\Delta_\varepsilon << 1$, the natural frequency considering thermal effect is approximated as

$$\omega = \sqrt{\frac{D\{1 + \Delta_\varepsilon (1 + f(\omega_0))\}}{\rho h}} \lambda^2 = \omega_0 \sqrt{1 + \Delta_\varepsilon (1 + f(\omega_0))} \approx \omega_0 \left[1 + \frac{\Delta_\varepsilon}{2} (1 + f(\omega_0))\right]$$  \hspace{1cm} (23)

where $\omega_0$ can be replaced with $\omega$ without reducing accuracy.

The complex-frequency approach is utilized to acquire the TED in circular plates, which is

$$Q_{TED}^1 = \frac{\text{Im}(\omega)}{\text{Re}(\omega)} = \frac{\Delta_\varepsilon \text{Im}(f(\omega_0))}{1 + \frac{\Delta_\varepsilon}{2} (1 + \text{Re}(f(\omega_0)))} \approx \Delta_\varepsilon \text{Im}(f(\omega_0))$$  \hspace{1cm} (24)

where the imaginary part of $f(\omega_0)$ is

$$\text{Im}(f(\omega_0)) = 24 \frac{a_2}{\xi^2 \left(\eta^2 + \frac{a_2^2}{\eta^2}\right)} + 24 \left(\frac{\eta^3 - 3\frac{a_2^2}{\eta}}{\xi^3} \sinh \left(\frac{\xi a_2}{\eta}\right) + \frac{\frac{a_2^2}{\eta^3} - 3a_2 \eta}{\xi^3} \sin \left(\xi \eta\right)\right)$$  \hspace{1cm} (25)

3. Validation and discussions

When the internal characteristic length of the material is negligible compared with the external characteristic length, the nonlocal elasticity theory will degenerate into the classical continuum mechanics. The same is true for nonlocal heat conductive model. To verify our proposed model, the nonlocal parameters $\mu$ and $l$ are set to zero, then Eq.(24) will degenerate into L-R classical expression[4].

In this section, the TED values under different thicknesses, frequencies and small-scale parameters are calculated by MATLAB to investigate the influence of nonlocal effects. Silicon is a common material for microresonators with its thermal mechanical properties as follows: $E = 160\text{GPa}; \quad v = 0.22; \quad \alpha_T = 2.6 \times 10^6 \text{K}^{-1}; \quad C_v = 1.63 \times 10^6 \text{J} \cdot \text{m}^3 \cdot \text{K}^{-1}; \quad k = 150 \text{W} \cdot \text{m}^1 \cdot \text{K}^{-1}; \quad \rho = 2330 \text{kg} \cdot \text{m}^{-3}; \quad T_0 = 300\text{K}.$
Figure 2. Comparison of the TED predictions between the proposed nonlocal model and the classical L-R model. Figure 3. Variation of thermoelastic damping versus normalized frequency for different phonon mean-free paths.

Figure 2 shows a comparison of the proposed nonlocal TED model with the classical L-R model under fully clamped boundary conditions, where the plate radius \( a = 20\mu m \). The thickness where the TED reaches its peak is called the critical thickness. Nonlocal elasticity makes the critical thickness increase without changing the TED peak value. By contrast, nonlocal heat conduction makes TED increase, especially near the critical thickness. Moreover, the nonlocal elasticity and nonlocal heat conduction are considered at the same time. When the thickness of the circular plate is small, the TED prediction of the nonlocal model is slightly lower than that of the classical model. When the thickness increases to close to or exceed the critical thickness, the nonlocal model has a larger prediction than the classical one.

Figure 3 shows the variation of TED in fully clamped plate versus normalized frequency \( \xi = \sqrt{\omega/2\chi} \) with different phonon mean-free paths. Apparently, the influence of nonlocal heat conduction on TED depends on its driven frequency. The frequency where TED reaches its peak is called critical frequency. When the driven frequency is much lower than its critical frequency, there is little difference between the TED predictions of nonlocal TED model and the classical L-R model. However, when the driven frequency approaches or exceeds the critical frequency, nonlocal heat conduction increases TED significantly. At the same time, the critical frequency of TED peak increases.

Figure 4 shows the variation TED versus nonlocal elasticity parameters \( \mu/a \) and phonon mean-free path \( l \) for clamped circular plates of different thicknesses. The radius of the circular plate is \( a = 20\mu m \), and the thickness is \( h = 1, 2.4 \) and \( 2.7\mu m \) respectively. The range of \( \mu/a \) is 0 to 0.5, and the mean-free path of phonon \( l \) is 0 to 200nm. It can be seen that for a circular plate of \( h = 1\mu m \), TED monotonously decreases with the increase of nonlocal elasticity parameter, while the increase of phonon mean-free path has little impact on TED value. For circular plates of \( h = 2.4 \) and \( 2.7\mu m \), TED increases with the increase of phonon mean-free path and the effects of nonlocal elasticity on plates of different thicknesses are quite different. For the plate of \( h = 2.4\mu m \), TED decreases with the increase of nonlocal elasticity parameter. For the plate of \( h = 2.7\mu m \), TED increases first and then decreases with the increase of nonlocal elasticity parameter. As can be seen from Figure 2, \( h = 2.4\mu m \) and \( h = 2.7\mu m \) are located on the left and right side of the critical thickness, respectively. This means that the influence of the nonlocal parameters on TED is determined by the relative position of the plate thickness to the critical thickness.
4. Conclusions

The small-scale TED model for circular plate resonators is presented in this paper based on nonlocal elasticity theory and nonlocal heat conductive model. Through the complex-frequency approach, the analytical TED expression is derived. The numerical results show that nonlocal elasticity leads the critical thickness to increase without changing the TED peak value. Nonlocal heat conduction makes TED increase especially at high frequency and large thickness. Moreover, the effect of nonlocal elasticity on TED value depends on the relation between the plate thickness and critical thickness.

5. References

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