ON SUCCESSIVE MINIMA-TYPE INEQUALITIES FOR THE POLAR OF A CONVEX BODY

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Abstract. Motivated by conjectures of Mahler and Makai Jr., we study bounds on the volume of a convex body in terms of the successive minima of its polar body.

1. Introduction

Let $\mathcal{K}^n$ be the set of all convex bodies, i.e., compact convex sets, in the $n$-dimensional Euclidean space $\mathbb{R}^n$ with non-empty interior. Let $\langle \cdot , \cdot \rangle$ and $\| \cdot \|$ be the standard inner product and the Euclidean norm in $\mathbb{R}^n$, respectively. We denote by $\mathcal{K}^n_0 \subset \mathcal{K}^n$ the set of all convex bodies, having the origin as an interior point, i.e., $0 \in \text{int} \, (K)$, and by $\mathcal{K}^n_s \subset \mathcal{K}^n_0$ those bodies which are symmetric with respect to $0$, i.e., $K = -K$. The volume of a set $S \subset \mathbb{R}^n$ is its $n$-dimensional Lebesgue measure and it is denoted by $\text{vol} \, (S)$.

For $K \in \mathcal{K}^n_0$ and $1 \leq i \leq n$ let

$$\lambda_i(K) = \min \{ \lambda > 0 : \dim(\lambda K \cap \mathbb{Z}^n) \geq i \}$$

be its $i$th successive minimum, which is the smallest positive dilation factor $\lambda$ such that $\lambda K$ contains $i$ linearly independent lattice points of the lattice $\mathbb{Z}^n$.

The so-called second theorem of Minkowski on successive minima provides optimal upper and lower bounds on the volume of a symmetric convex body $K \in \mathcal{K}^n_s$ in terms of its successive minima. These bounds can be easily generalized to the class $K \in \mathcal{K}^n$ as follows

$$\frac{2^n}{n!} \prod_{i=1}^{n} \frac{1}{\lambda_i(\text{cs} \, (K))} \leq \text{vol} \, (K) \leq \frac{2^n}{n!} \prod_{i=1}^{n} \frac{1}{\lambda_i(\text{cs} \, (K))}, \tag{1.1}$$

where $\text{cs} \, (K) = \frac{1}{2} (K - K) \in \mathcal{K}^n_s$ is the central symmetral of $K$. The $n$-dimensional unit cube $C_n$ shows that the upper bound is optimal, and its polar body $C_n^*$, the $n$-dimensional cross-polytope, attains the lower bound.

Here, in general, the polar body of $K \in \mathcal{K}^n$ is defined as

$$K^* = \{ y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in K \}.$$

Mahler [15] studied for $K \in \mathcal{K}^n_0$ the volume product $M(K) = \text{vol} \, (K) \text{vol} \, (K^*)$ and conjectured

$$M(K) \geq M(C_n) = \frac{4^n}{n!} \tag{1.2}$$

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Mahler [14] verified the conjecture in dimension 2, and there was a recent announcement of its proof in dimension 3 by [9]. In the general case, it is conjectured that for \( K \in \mathcal{K}^n \)

\[
M(K) \geq M(S_n) = \frac{(n+1)^{n+1}}{(n!)^2},
\]

where \( S_n \) is a simplex with centroid at the origin. This is only known to be true in the plane [14].

Combining the upper bound in (1.1) with the conjectured lower bound (1.2) leads for \( K \in \mathcal{K}^n_{(s)} \) to the inequality

\[
\text{vol}(K) \geq \frac{2^n}{n!} \prod_{i=1}^{n} \lambda_i(K^*). \tag{1.3}
\]

This inequality, which would be best possible, for instance, for the cross-polytope \( C_n^* \), was also conjectured by Mahler [16], and the previous mentioned results on the volume product \( M(K) \) implies that it is true for \( n = 2 \) and (probably for \( n = 3 \)). Even the weaker inequality,

\[
\text{vol}(K) \geq \frac{2^n}{n!} \lambda_1(K^*)^n, \tag{1.4}
\]

which has also been studied by Mahler, is open for \( n \geq 4 \).

For not necessarily symmetric bodies the same problem was studied by Makai Jr., and he conjectured for \( K \in \mathcal{K}^n \)

\[
\text{vol}(K) \geq \frac{n+1}{n!} \lambda_1(\text{cs}(K)^*)^n, \tag{1.5}
\]

and proved it for \( n = 2 \) ([3, 11]). In view of (1.3), one might conjecture the stronger inequality

\[
\text{vol}(K) \geq \frac{n+1}{n!} \prod_{i=1}^{n} \lambda_i(\text{cs}(K)^*)^n, \tag{1.6}
\]

which would be possible as the simplex \( S_n = \text{conv}\{e_1, \ldots, e_n, -1\} \) shows, where \( e_i \) is the \( i \)th unit vector and \( 1 \) is the all 1-vector. For \( n = 2 \) this is an immediate consequence of the upper bound in (1.1) and Eggeleston [2] inequality for planar convex bodies

\[
\text{vol}(K) \text{vol}(\text{cs}(K)^*) \geq 6. \tag{1.6}
\]

Actually, we believe that taking into account all successive minima one should get a stronger lower bound as in (1.5), and here we show in the plane.

**Theorem 1.1.** Let \( K \subset \mathbb{K}^2 \). Then

\[
\text{vol}(K) \geq 3 \lambda_1(\text{cs}(K)^*) \lambda_2(\text{cs}(K)^*) + \frac{1}{2} \lambda_1(\text{cs}(K)^*) \left( \lambda_2(\text{cs}(K)^*) - \lambda_1(\text{cs}(K)^*) \right)
\]

and equality holds if and only if \( K \) is up to translations and unimodular transformations equal to the triangle \( T_{s,t} = \text{conv}\{(-s, t-s), (s, t), (0, t)\} \) with \( t \geq s \in \mathbb{R}_{>0} \).
We note that for the triangle $T_{s,t}$ (see Figure 1) we have

\[(1.7) \quad \lambda_1 (cs (K)^*) = s, \quad \lambda_2 (cs (K)^*) = t, \quad \text{and} \quad \text{vol} (T_{s,t}) = 2ts - \frac{1}{2}s^2.\]

We remark that Makai-Martini\[12, \text{Proposition 3.1}\] (see also Makai\[11, \text{Proposition 1}\]) verified for simplices $S \in K^n$ the conjectured higher-dimensional analogue of (1.6), namely

\[
\text{vol} (S) \text{vol} (cs (S)^*) \geq 2^n \frac{n+1}{n!}.
\]

Application of Minkowski’s upper bound (1.1) shows inequality (1.5) for simplices. For arbitrary convex bodies $K \in K^n$ one may write (cf. \[12\])

\[
\text{vol} (K) \text{vol} (cs (K)^*) = \frac{\text{vol} (K)}{\text{vol} (cs (K))} M (cs (K)) \geq \frac{2^n}{(2n)} \frac{\pi^n}{n!} \geq \frac{(\pi/2)^n}{n!},
\]

where the lower bound on the volume product is Kuperberg’s bound\[10\], and the lower bound on the ratio $\frac{\text{vol} (K)}{\text{vol} (cs (K))}$ is the Rogers-Shephard bound (cf., e.g., \[17, \text{Theorem 10.4.1}\]. Hence, in general, we have the bound

\[
\text{vol} (K) \geq \frac{(\pi/4)^n}{n!} \prod_{i=1}^{n} \lambda_i (cs (K)^*)_i^n.
\]

In contrast to the lower bounds, in the case of upper bounds we have a complete picture.

**Theorem 1.2.** Let $K \in K^n$.

i) Then

\[
\text{vol} (K) \leq 2^n \prod_{i=1}^{n} \lambda_i (cs (K)^*).
\]

The inequality is best possible.

ii) If the centroid of $K$ is at the origin, then

\[
\text{vol} (K) \leq \frac{(n+1)^n}{n!} \prod_{i=1}^{n} \lambda_i (K^*).
\]

The inequality is best possible.
iii) For arbitrary $K \in \mathcal{K}_n^o$, the volume is in general not bounded from above by the product of $\lambda_i(K^*)$.

Observe, that $\lambda_i(K^*) \leq \lambda_i(\text{cs}(K)^*)$, $1 \leq i \leq n$, cf. Proposition 2.1.

Finally, we would like to mention that a weaker inequality than (1.4) was recently studied by Alavarez et al.\[1\]. They conjecture for $K \in \mathcal{K}_n^o$:

$$\text{vol}(K) \geq \frac{n+1}{n!} \lambda_1(K^*)^n$$

with equality if and only if $K$ is a simplex whose vertices are the only non-trivial lattice points. By the discussion above we know that it is true in the plane, for simplices and with $(\pi/4)^n/n!$ instead of $(n+1)^n/n!$ (cf.\[1, Theorem II\]). Moreover, according to Theorem 1.2 iii), there is no upper bound on the volume of this type. For an optimal lower bound on the volume of centered convex body $K$, i.e., the centroid if $K$ is at the origin, in terms of $\lambda_i(K)$ we refer to \[8\]. Instead of extending Makai’s conjecture (1.4) via higher successive minima (cf. (1.4)), Gonzales Merion & Schymura \[4\] studied possible extensions via the so called covering minima.

The paper is organized as follows: First, i.e., in Section 2, we will verify the upper bounds of Theorem 1.2. Then as preparation for the proof of Theorem 1.1 we will study gauge functions in Section 3. Finally, the content of Section 4 is the proof of Theorem 1.1.

For a general background and information on Convex Geometry and Geometry of Numbers we refer to the books \[5, 6, 17\].

2. Proof of Theorem 1.2

In order to deal with the polar of a convex body $L \in \mathcal{K}^n$, say, it is convenient to look at its support function $h_L : \mathbb{R}^n \to \mathbb{R}$ given by

$$h_L(u) = \max \{ \langle u, x \rangle : x \in L \}$$

for $u \in \mathbb{R}^n$. Then for $\lambda \in \mathbb{R}_{\geq 0}$ we have

$$y \in \lambda L^* \text{ if and only if } h_L(y) \leq \lambda.$$ (2.1)

First we observe a simple relation between the successive minin of $K \in \mathcal{K}_n^o$ and $\text{cs}(K)$ which, for $i = 1$ was already pointed out by by Alvarez et al. [1].

**Proposition 2.1.** Let $K \in \mathcal{K}_n^o$. Then, for $1 \leq i \leq n$,

$$\lambda_i(K^*) \leq \lambda_i(\text{cs}(K)^*).$$

**Proof.** Let $\lambda^*_i = \lambda_i(\text{cs}(K)^*)$ and let $z_1, \ldots, z_i \in \mathbb{Z}^n$ be linearly independent lattice points with $z_j \in \lambda^*_j \text{cs}(K)^*$, $1 \leq j \leq i$. Then, for $1 \leq j \leq i$, by the linearity of the support function

$$\lambda^*_i \geq h_L(K-K)(z_j) = \frac{1}{2} (h_K(z_j) + h_K(-z_j)) \geq \min\{h_K(z_j), h_K(-z_j)\}.$$ 

Hence, either $z_j$ or $-z_j$ belongs to $\lambda^*_j K^*$ for $1 \leq j \leq i$, and thus $\lambda_i(K^*) \leq \lambda^*_i = \lambda_i(\text{cs}(K)^*)$.

$\square$
For the proof of Theorem 1.2 ii) we will also need a classical result of Grünbaum [7], saying that for \( K \in \mathcal{K}^n_0 \) and for any halfspace \( H^+ = \{ x \in \mathbb{R}^n : \langle a, x \rangle \geq 0 \} \) containing the centroid of \( K \) it holds

\[
(2.2) \quad \text{vol} (K \cap H^+) \geq \left( \frac{n+1}{n} \right)^n \text{vol} (K).
\]

**Proof of Theorem 1.2.** For i), let \( z_1, \ldots, z_n \in \mathbb{Z}^n \) be linearly independent lattice points with \( z_i \in \lambda_i (cs (K)^*) \), \( 1 \leq i \leq n \). Then we certainly have

\[
K \subseteq P = \{ x \in \mathbb{R}^n : -h_K (-z_i) \leq \langle z_i, x \rangle \leq h_K (z_i), 1 \leq i \leq n \}.
\]

In order to estimate the volume of the parallelepiped on the right hand side we observe, that in view of \((2.1)\), \( 2 \lambda_i (cs (K)^*) = h_K (z_i) + h_K (-z_i), 1 \leq i \leq n \), and thus

\[
\text{vol} (K) \leq \text{vol} (P) = \frac{1}{|\text{det} (z_1, \ldots, z_n)|} \prod_{i=1}^n 2 \lambda_i (cs (K)^*) \leq 2^n \prod_{i=1}^n \lambda_i (cs (K)^*),
\]

where in the last inequality we used \( \text{det} (z_1, \ldots, z_n) \in \mathbb{Z} \setminus \{0\} \). The cube \( C_n \) with its polar body \( C_n^* = \text{conv} \{ \pm e_1, \ldots, \pm e_n \} \) shows that the equality is best possible.

Now assume that the centroid of \( K \) is at the origin. Let \( \lambda_i^* = \lambda_i (K)^* \), \( 1 \leq i \leq n \), and let \( z_1, \ldots, z_n \in \mathbb{Z}^n \) be linearly independent lattice points with \( z_i \in \lambda_i^* K^* \). Then, for \( 1 \leq i \leq n \),

\[
(2.3) \quad h_K (z_i) \leq \lambda_i^*.
\]

Moreover, we consider the halfspace

\[
H^+ = \{ x \in \mathbb{R}^n : \left\langle \frac{1}{\lambda_1^*} z_1 + \cdots + \frac{1}{\lambda_n^*} z_n, x \right\rangle \geq 0 \}.
\]

Then we conclude from \((2.3)\)

\[
(2.4) \quad K \cap H^+ \subseteq S = \{ x \in \mathbb{R}^n : \langle z_i, x \rangle \leq \lambda_i^*, 1 \leq i \leq n \} \cap H^+.
\]

In order to calculate the volume of the simplex \( S \) we observe that it is the image of the simplex

\[
\overline{S} = \{ x \in \mathbb{R}^n : \langle e_i, x \rangle \leq 1, 1 \leq i \leq n, \langle 1, x \rangle \geq 0 \}
\]

with respect to the linear map \( A = \left( \frac{1}{\lambda_1^*} z_1, \ldots, \frac{1}{\lambda_n^*} z_n \right) \). Hence,

\[
\text{vol} (S) = \frac{1}{|\text{det} (z_1, \ldots, z_n)|} \prod_{i=1}^n \lambda_i^* \text{vol} (\overline{S}) = \frac{1}{|\text{det} (z_1, \ldots, z_n)|} \prod_{i=1}^n \lambda_i^* \frac{n^n}{n!},
\]

and together with Grünbaum’s bound \((2.2)\) and \((2.4)\) we conclude

\[
\text{vol} (K) \leq \left( \frac{n+1}{n} \right)^n \text{vol} (K \cap H^+)
\]

\[
\leq \left( \frac{n+1}{n} \right)^n \text{vol} (S) = \frac{1}{|\text{det} (z_1, \ldots, z_n)|} \frac{(n+1)^n}{n!} \prod_{i=1}^n \lambda_i^*.
\]

Again, since \( \text{det} (z_1, \ldots, z_n) \in \mathbb{Z} \setminus \{0\} \) we get the desired bound. The simplex \( T_n = \{ x \in \mathbb{R}^n : \langle e_i, x \rangle \leq 1, 1 \leq i \leq n, \langle 1, x \rangle \geq -1 \} \) with volume \( (n+1)^n/n! \) and \( T_n^* = \text{conv} \{ e_1, \ldots, e_n, -1 \} \) shows the bound is best possible.
Finally, we point out that the assumption on the centroid is crucial for ii).
To this end, for $s \geq 1$ we consider the simplices $T(s) = \{ \mathbf{x} \in \mathbb{R}^n : \langle e_i, \mathbf{x} \rangle \leq 1, 1 \leq i \leq n, \langle \mathbf{s}, \mathbf{x} \rangle \geq -1 \}$. Then $T(s)^* = \text{conv} \{ -\frac{1}{s} \mathbf{e}_1, \ldots, \mathbf{e}_n \}$ and thus $\lambda_i(T(s)^*) = 1, 1 \leq i \leq n$. On the other hand, $\text{vol}(T(s)) \to \infty$ as $s$ approaches $\infty$. This verifies iii).

\[ \square \]

3. Gauge function

Here we collect some basic facts about gauge functions $\| \mathbf{x} \|_K$ associated to a $K \in \mathcal{K}_n^{(o)}$ which are defined by
\[
\| \mathbf{x} \|_K : \mathbb{R}^n \to [0, \infty)
\]
defined by
\[
\| \mathbf{x} \|_K = \min \{ t \geq 0 : \mathbf{x} \in t K \}.
\]
As it is well known, $\| \cdot \|_K$ satisfies the following properties:

i) $\| \mathbf{x} \|_K \geq 0$ with equality if and only if $\mathbf{x} = \mathbf{0}$,

ii) $\| \lambda \mathbf{x} \|_K = \lambda \| \mathbf{x} \|_K$ for $\lambda \in \mathbb{R}_{\geq 0}$,

iii) $\| \mathbf{x} + \mathbf{y} \|_K \leq \| \mathbf{x} \|_K + \| \mathbf{y} \|_K$.

Conversely, if $\| \cdot \|$ is a function satisfying these three properties, then its unit ball $B = \{ \mathbf{x} \in \mathbb{R}^n : \| \mathbf{x} \| \leq 1 \}$ is a convex body in $\mathcal{K}_n^{(o)}$ and $\| \cdot \| = \| \cdot \|_B$.

We also note that if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation, then $\| \mathbf{x} \|_{T(B)} = \| T^{-1} \mathbf{x} \|_B$ for all $\mathbf{x} \in \mathbb{R}^n$. From the definition of the gauge function it is evident that for $K \in \mathcal{K}_n^{(o)}$,
\[
\| \mathbf{x} \|_K = h_{K^*}(\mathbf{x})..
\]

Hence, from the linearity of the support function we immediately obtain
\[
\| \mathbf{x} \|_{cs(K)^*} = h_{cs(K)}(\mathbf{x})
\]
\[
= \frac{1}{2} (h_K(\mathbf{x}) + h_K(-\mathbf{x})) = \frac{1}{2} (\| \mathbf{x} \|_{K^*} + \| -\mathbf{x} \|_{K^*}).
\]

Combining this with the triangle inequality we conclude for $K \in \mathcal{K}_n^{(o)}$
\[
\| \mathbf{x} + \mathbf{y} \|_{cs(K)^*} = \| \mathbf{x} \|_{cs(K)^*} + \| \mathbf{y} \|_{cs(K)^*} \text{ if and only if }
\]
\[
\| \mathbf{x} + \mathbf{y} \|_{K^*} = \| \mathbf{x} \|_{K^*} + \| \mathbf{y} \|_{K^*} \text{ and } \| -(\mathbf{x} + \mathbf{y}) \|_{K^*} = \| -\mathbf{x} \|_{K^*} + \| -\mathbf{y} \|_{K^*}.
\]

4. Proof of Theorem 1.1

Since the inequality of Theorem 1.1, i.e.,
\[
\text{vol}(K) \geq \frac{3}{2} \lambda_1(cs(K)^*)\lambda_2(cs(K)^*)
\]
\[
+ \frac{1}{2} \lambda_1(cs(K)^*) \left( \lambda_2(cs(K)^*) - \lambda_1(cs(K)^*) \right)
\]
\[
= 2\lambda_1(cs(K)^*)\lambda_2(cs(K)^*) - \frac{1}{2} \lambda_1(cs(K)^*)^2
\]
is invariant with respect to translations and unimodular transformations, we may assume that $K \in \mathcal{K}_n^{(o)}$, $\lambda_2(cs(K)^*) = 1$ and the successive minima $\lambda_i(cs(K)^*)$ are obtained in direction of the unit vectors, i.e., $e_i(cs(K)^*) \in \lambda_i(cs(K)^*)cs(K)^*$, $i = 1, 2$. The latter is due to the fact that in the plane
we can always find \( z_i \in \lambda_i(\text{cs}(K)^*) \cap \mathbb{Z}^2 \) building a basis of \( \mathbb{Z}^2 \) [6, Theorem 4, p.20].

Hence, for a fixed \( t \geq 1 \) we are interested in the minimal volume among all convex bodies in the set

\[
\mathcal{A}(t) = \left\{ K \in K^2 \, : \, \lambda_1(\text{cs}(K)^*) = \frac{1}{t}, \lambda_2(\text{cs}(K)^*) = 1, \ e_i \in \lambda_i(\text{cs}(K)^*) \cap \mathbb{Z}^2, \ i = 1, 2 \right\}.
\]

Observe, that all bodies in \( \mathcal{A}(t) \) are contained in the rectangle \([-1/t, 1/t] \times [-1, 1]\) and since the volume of all these bodies is lower bounded by \( 3/2 \cdot 1/t \) (cf. (1.5), which is true for \( n = 2 \)), Blaschke’s selection theorem (cf., e.g., [5, Theorem 6.3]) ensures the existence of a convex bodies in \( \mathcal{A}(t) \) having minimal positive volume. We denote these bodies by \( \mathcal{M}(t) \), i.e.,

\[
\mathcal{M}(t) = \{ M \in \mathcal{A}(t) : \text{vol} (M) = \min \{ \text{vol} (K) : K \in \mathcal{A}(t) \} \}.
\]

Observe, that due to the triangle (cf. Theorem 1.1)

\[
T_{1,1/t} = \text{conv} \left\{ (-1/t, 1 - 1/t), (1/t, 1), (0, -1) \right\},
\]

we know that for \( K \in \mathcal{M}(t) \) (cf. (1.7))

\[
(4.1) \quad \text{vol} (K) \leq \text{vol} (T_{1,1/t}) = 2 \frac{1}{t} - \frac{1}{2} \frac{1}{t^2}
\]

and Theorem 1.1 claims that this is indeed the minimum.

In the following we will prove different geometric properties of bodies \( S \in \mathcal{M}(t) \) (or better of \( S^* \)) and at the end in Proposition 4.8 we conclude that \( \mathcal{M}(t) \) contains only — up to translations and unimodular transformations — the triangle \( T_{1,1/t} \). This proves Theorem 1.1.

Due to the definition of the successive minima, all the lattice points of \( \text{cs}(K)^* \) for \( K \in \mathcal{A}(t) \) are either contained in the boundary of \( \text{cs}(K)^* \) or lie on the line \( \text{lin} \{ e_1 \} \). For such a \( K \in \mathcal{A}(t) \) we set

\[
C_o(K) = \left\{ z \in \mathbb{Z}^2 : \|z\|_{\text{cs}(K)^*} = 1 \right\} \cup \{ \pm e_1 \},
\]

\[
C(K) = \left\{ z/\|z\|_{K^*} : z \in C_o(K) \right\}.
\]

The points in \( C(K) \) are our main objective by which we will show geometric properties of bodies in \( \mathcal{M}(t) \).

**Proposition 4.1.** Let \( K \in \mathcal{M}(t) \). Then \( K \) is a polygon and the relative interior of each edge of \( K^* \) contains a point of \( C(K) \).

**Proof.** First, we prove that \( K^* \) and thus \( K \) is a polygon. Since \( \text{cs}(K)^*, K^* \) are bounded, both are strictly contained in a square \( C_N = [-N,N]^2 \) for some large \( N \in \mathbb{R}_{>0} \). For any non-zero lattice point \( z \in C_N \), there is a supporting hyperplane in the boundary point \( \frac{z}{\|z\|_{K^*}} \) with respect to \( K^* \). Let \( C \) be the intersection of the corresponding halfspaces containing \( K^* \) together with the halfspaces bounding \( C_N \).

Obviously, \( C^* \subseteq K \) is a polygon and we claim that \( C^* \in \mathcal{A}(t) \). In order to avoid confusion, we set \( P = C^* \) and so \( C = P^* \) and we want to show \( P \in \mathcal{A}(t) \). To this end, we observe that for all \( z \in C_N \) we have by construction

\[
\|z\|_{P^*} = \|z\|_{K^*}.
\]
and hence, in view of (3.1)
\[ \|z\|_{cs(P)^*} = \|z\|_{cs(K)^*}. \]
For \( z \in \mathbb{Z}^2 \setminus C_N \) we know by construction
\[ \|z\|_{P^*} \geq \|z\|_{K^*} \]
and so
\[ \|z\|_{cs(P)^*} \geq \|z\|_{cs(K)^*} > 1. \]
Hence, \( P \in \mathcal{A}(t), P \subseteq K \) and since \( K \in \mathcal{M}(t) \), we must have \( K = P \).

Next assume that there is an edge of \( K^* \) which does not contain in its relative interior a point of \( C(K) \). Then we may move the edge a bit outward so that for this new polygon \( K_\epsilon^* \), considered as the polar of a polygon \( K_\epsilon \), it holds
\[ \|z\|_{K_\epsilon^*} = \|z\|_{K^*} \quad \text{and thus} \quad \|z\|_{cs(K_\epsilon^*)} = \|z\|_{cs(K)^*}, \]
for all \( z \in C(K) \). For all other lattice points \( z \) (which are not contained in \( \text{lin}\{e_1\} \)), we know \( \|z\|_{cs(K)^*} > 1 \) and hence, by moving just a little bit we still have \( \|z\|_{cs(K_\epsilon^*)} > 1 \) for these points.

Thus \( K_\epsilon \in \mathcal{A}(t) \) but \( K_\epsilon \) is strictly contained in \( K \), contradicting its minimality with respect to the volume. \( \square \)

In order to give a bound on the size of \( C(K) \), \( K \in \mathcal{M}(t) \), we need the next lemma.

**Lemma 4.2.** Let \( K \in \mathcal{M}(t) \), and let \((m, n) \in C_0(K)\). Then \( n \in \{-1, 0, 1\} \).

**Proof.** Assume that \((m, n) \in C_0(K)\) with \( n \geq 2 \), which is trivially a primitive lattice point.

Since \((0, t), (0, -t) \in \text{cs}(K)^* \) and \((m, n) \in \text{cs}(K)^* \), the intersection of
\[ \text{conv}\{(0, t), (0, -t), (m, n)\} \]
with the line \( \{x \in \mathbb{R}^2 : x_2 = 1\} \) has length greater or equal than \( t \geq 1 \). If the length is strictly greater than \( 1 \), this intersection contains a lattice point \( v \in \mathbb{Z}^2 \) with \( \|v\|_{cs(K)^*} < 1 \).

The only remaining case is \( n = 2 \) and \( t = 1 \), and since then \( e_1, e_2 \) are in the boundary, \( \text{cs}(K)^* = \text{conv}\{\pm e_1, \pm(1, 2)\} \). Hence up to translations we can assume that \( K \) is the parallelogram \( \text{conv}\{\pm e_1, \pm(1, -1)\} \) of volume \( 2 \). Hence, \( K \notin \mathcal{M}(t) \) (cf.(4.1)). \( \square \)

**Remark 4.3.** Let \( K \in \mathcal{M}(t) \). By Lemma 4.2 we get
\begin{itemize}
  \item[i)] if \( |C(K)| = 4 \) then \( C_0(K) = \{\pm e_1, \pm e_2\} \),
  \item[ii)] if \( |C(K)| = 6 \), then
    \[ C_0(K) = \{\pm e_1, \pm e_2, \pm(e_1 + e_2)\} \text{ or } \{\pm e_1, \pm e_2, \pm(e_2 - e_1)\}. \]
\end{itemize}

Observe that both configurations are unimodular equivalent.

Next we show that \( C(K) \) can not have more than 6 points.

**Proposition 4.4.** Let \( K \in \mathcal{M}(t) \). Then \( |C(K)| \leq 6 \), i.e., \( |C(K)| = 4 \) or \( 6 \).
Proof. Let \( K \in \mathcal{M}(t) \) and assume \(|C(K)| > 6\). Then in view of Lemma 4.2 there are at least three points in \( C_0(K) \) with last coordinate 1, and at least three points with last coordinate \(-1\). All these points lie in the boundary of \( \text{cs} (K)^* \) and hence, \( \text{cs} (K)^* \) has an edge contained in the line \( \{ x : x_2 = 1 \} \) and one contained in \( \{ x : x_2 = -1 \} \). Hence, \( \text{cs} (K) \) has the vertices \( \pm e_2 \), which shows that \( K \) has two vertices \( x, y \) with \( x - y = 2e_2 \).

On the other hand we have \( ||e_1||_{\text{cs}(K)^*} = \frac{1}{2} \) and thus \( h_{\text{cs}(K)}(e_1)^* = \frac{1}{2} \). Hence, \( K \) contains also two vertices differing in the first coordinate by \( \frac{1}{2} \). Altogether, this shows that the volume of \( K \) is at least \( 2/t \) and hence, \( K \notin \mathcal{M}(t) \) (cf. (4.1)).

Now we study the number of points of \( C(K) \) in each edge of \( K^* \). The following lemma shows that, under some translation of \( K \), the relation between the points of \( C(K) \) and the edges of \( K^* \) does not change.

**Lemma 4.5.** Let \( K \in \mathbb{R}^2 \) and \(-u \in \text{int} (K) \). Let \( v \in \mathbb{R}^2 \), such that \( \frac{v}{||v||_{K^*}} \) lies in the relative interior of the edge \( E = \{ x \in K : \langle x,f \rangle = 1 \} \) of \( K^* \).

Then \( \frac{v}{||v||_{(K+u)^*}} \) lies in the relative interior of the edge \( E' = \{ x \in (K+u)^* : \langle x,f+u \rangle = 1 \} \) of \( (K+u)^* \).

**Proof.** By assumption \( f \) is a vertex of \( K \) and so is \( f + u \) a vertex of \( K + u \). Hence \( E' \) is an edge of \( (K+u)^* \). Next, since \( \langle v,f \rangle = ||v||_{K^*} \), and

\[
||v||_{(K+u)^*} = h_{K+u}(v) = h_K(v) + \langle v,u \rangle = ||v||_{K^*} + \langle v,u \rangle,
\]

we find

\[
\langle v,f+u \rangle = \langle v,f \rangle + \langle v,u \rangle = ||v||_{K^*} + \langle v,u \rangle = ||v||_{(K+u)^*}.
\]

Thus \( \frac{v}{||v||_{(K+u)^*}} \in E' \), and since \( v/||v||_{K^*} \) was only contained in the edge \( E \), \( \frac{v}{||v||_{(K+u)^*}} \) also belongs to the relative interior of \( E' \). \[ \square \]

Next we describe in more detail the relation of the points of \( C(K) \) and the edges of \( K^* \).

**Proposition 4.6.** \( \mathcal{M}(t) \) contains a polygon \( K \) such that the relative interior of each edge of \( K^* \) contains

\[ (P) \]

1. at least two points of \( C(K) \), or
2. one point of \( C(K) \), while \( \frac{e_1}{||e_1||_{K^*}} \) or \( \frac{-e_1}{||-e_1||_{K^*}} \)

is a vertex of this edge.

Moreover, each \( K \in \mathcal{M}(t) \) has at most 4 edges, and if \( K \in \mathcal{M}(t) \) is a triangle, then \( K \) satisfies property \((P)\).

**Proof.** In the following we show that for each \( K \in \mathcal{M}(t) \) there exists another polygon \( K' \in \mathcal{M}(t) \) with the same number of edges as \( K \) satisfying property \((P)\). Together with Proposition 4.4 this implies that each \( K \in \mathcal{M}(t) \) has at most 4 edges.
So let $K \in \mathcal{M}(t)$ be a polygon which does not fulfill (P). Then, in view Proposition 4.1 we may assume that $K^*$ has an edge $E = \{ \mathbf{x} \in \mathbb{R}^2 : \langle \mathbf{f}, \mathbf{x} \rangle = 1 \} \cap K^*$ containing only one point $\mathbf{u} = (x_0, y_0) \in C(K)$ in its relative interior, and $\|x_0\|_{K^*}$ is not a vertex of $E$.

Let $\{ \mathbf{f}, \mathbf{f}_1, \ldots, \mathbf{f}_k \}$ be the vertices of $K$, and $E_i = \{ \mathbf{x} \in \mathbb{R}^2 : \langle \mathbf{f}_i, \mathbf{x} \rangle = 1 \}$, $1 \leq i \leq k$, be the supporting lines of the other edges of $K^*$. For the lines $E, E_i$ we denote by $\overline{E}, \overline{E}_i$ the corresponding halfspaces containing $K^*$, i.e.,

$$K^* = \overline{E} \cap \bigcap_{i=1}^{k} \overline{E}_i.$$ 

Let us parametrize $E$ by the angle $\theta_0 \in [0, 2\pi)$ such that

$$E = \{ (x, y) \in \mathbb{R}^2 : (\cos \theta_0)(x - x_0) + (\sin \theta_0)(y - y_0) = 0 \}.$$ 

Thus, if $\epsilon > 0$ and $\theta \in (\theta_0 - \epsilon, \theta_0 + \epsilon)$ we consider the line

$$E(\theta) = \{ (x, y) \in \mathbb{R}^2 : (\cos \theta)(x - x_0) + (\sin \theta)(y - y_0) = 0 \},$$

i.e., we rotate $E$ around $\mathbf{u}$, and the new polygon

$$K_{\theta} = E(\theta) \cap \bigcap_{i=1}^{k} \overline{E}_i.$$ 

Observe, that

$$(K_{\theta})^* = \text{conv} \{ \mathbf{f}_0, \mathbf{f}_1, \ldots, \mathbf{f}_k \} =: K_{\theta}$$

with

$$\mathbf{f}_0 = \left( \frac{\cos \theta}{\cos \theta x_0 + \sin \theta y_0}, \frac{\sin \theta}{\cos \theta x_0 + \sin \theta y_0} \right).$$

For $\epsilon$ we always assume that it is so small, that the possible rotations do not change the number of edges. Since

$$(4.2) \quad \mathbf{f}_0 \in \{ \mathbf{x} \in \mathbb{R}^2 : \langle \mathbf{u}, \mathbf{x} \rangle = 1 \}.$$ 

the volume of $K_{\theta}$, as a function in $\theta$, is monotonic in $[\theta_0 - \epsilon, \theta_0 + \epsilon]$. For each $\mathbf{v} = (v_1, v_2) \notin C_0(K)$ with $v_2 \neq 0$, we have $\|\mathbf{v}\|_{\text{cs}(K^*)} > 1$. Therefore, there exists $s > 1$ such that $\|\mathbf{v}\|_{\text{cs}(K_{\theta})^*} > 1$ for each $\mathbf{v} \notin C_0(K)$ with $v_2 \neq 0$. Thus, there exists $0 < \epsilon' < \epsilon$, such that for $\theta \in [\theta_0 - \epsilon', \theta_0 + \epsilon']$, it holds $\|\mathbf{v}\|_{\text{cs}(K_{\theta})^*} > 1$ for $\mathbf{v} \notin C_0(K)$, $v_2 \neq 0$. Since all the points $\mathbf{v} \in C(K) \setminus \{ \mathbf{u} \}$ are (also) contained in an edge of $K^*$ different from $E$, we have $\|\mathbf{v}\|_{K^*} > \|\mathbf{v}\|_{K^*}$ for $|\theta - \theta_0|$ small and so $\|\mathbf{v}\|_{\text{cs}(K_{\theta})^*} > 1$ for all $\mathbf{v} \in C_0(K)$. Therefore, after a possible unimodular transformation, we still have $K_{\theta} \in \mathcal{A}(t)$. Since $\text{vol}(K_{\theta})$ is monotonic for $|\theta - \theta_0|$ being small and $K \in \mathcal{M}(t)$, we conclude $\text{vol}(K_{\theta}) = \text{vol}(K)$, and thus $K_{\theta} \in \mathcal{M}(t)$ for $|\theta - \theta_0|$ small.

If $K$ is a triangle, i.e., let $K^*$ has the edges $E, E_1, E_2$ and so $K$ has the vertices $\mathbf{f}, \mathbf{f}_1, \mathbf{f}_2$. Since $\text{vol}(K_{\theta}) = \text{vol}(K)$, $(4.2)$ shows that the line $\{ \mathbf{x} \in \mathbb{R}^2 : \langle \mathbf{u}, \mathbf{x} \rangle = 1 \}$ must be parallel to the edge $[\mathbf{f}_1, \mathbf{f}_2]$ of $K$.

Let $\mathbf{u}' \in C_0(K)$ such that $\mathbf{u} = \frac{\mathbf{u}'}{\|\mathbf{u}'\|_{K^*}}$. If $\mathbf{u}' \neq \pm e_1$ then its last coordinate is 1 (cf. Lemma 4.2 ) and hence, after an unimodular transformation we may always assume $\mathbf{u}' \in \{ \pm e_1, \pm e_2 \}$. 


If $u' \in \{\pm e_1\}$ then the edge $[f_1, f_2]$ has normal vector $e_1$, and in view of (3.1) we get that the length of the edge $[f_1, f_2]$ has length $2$, and the height of $f$ with respect to $[f_1, f_2]$ is $2/t$. Hence, its volume is $2/t$ which is not minimal (cf. (4.1)) and so we are violating $K \in \mathcal{M}(t)$. Analogously, if $u' \in \{\pm e_2\}$ then the edge $[f_1, f_2]$ has normal vector $e_2$, and then the length of the edge $[f_1, f_2]$ is $2/t$ and the height of $f$ with respect to $[f_1, f_2]$ is $2$. Again, the volume of the triangle is contradicting $K \in \mathcal{M}(t)$.

Hence, $K$ is not a triangle, and, in particular, all triangles in $\mathcal{M}(t)$ have property (P).

So let $K$ be not a triangle. By Lemma 4.5, we may apply a translation to $K$ such that the origin is contained in the relative interior of the vertices adjacent to $f$. For convenience we denote these two vertices by $f_+, f_-$, such that $f_-, f, f_+$ are in clockwise order. Let $E_-, E_{\theta_0}(= E(\theta_0)), E_+$ be the corresponding supporting lines of $K^*$ and

$$E_+ \cap K^* = [w_1, w_2], \quad E(\theta_0) \cap K^* = [w_2, w_3], \quad E_- \cap K^* = [w_3, w_4]$$

be the associated edges of $K^*$, where $w_i$, $1 \leq i \leq 4$, are the vertices of these edges.

---

**Figure 2.** The non-triangle case

Since the origin $0$ can only be in at most one of the triangles $\text{conv} \{u, w_2, w_1\}$ and $\text{conv} \{u, w_3, w_1\}$, we assume $0 \notin \text{conv} \{u, w_2, w_1\}$. Let $\theta_1 \in [\theta_0 - \pi, \theta_0 + \pi]$ such $E(\theta_1) \cap K^* = [u, w_1]$. If $\{t w_3 : t \in \mathbb{R}\} \cap [w_1, w_2] = w'$, then let $\theta_2 \in [\theta_0 - \pi, \theta_0 + \pi]$ such that $E(\theta_2) \cap K^* = [u, w_3]$.

For $x, y \in \mathbb{R}^2$ we denote by cone $\{x, y\} = \{\lambda x + \mu y : \lambda, \mu \geq 0\}$ the cone generated by $x$ and $y$. Now we start to rotate $E(\theta)$ clockwise around $u$ and we denote the so created bodies by $K^*_{\theta}$. Then, for each point

$$x \in C_1 = \text{cone} \{u, w_1\} \cap \text{cone} \{u, \lambda - w_3\}$$

its norm $\|x\|_{K^*_{\theta}}$ is non-decreasing and $\|\lambda - x\|_{K^*_{\theta}}$ does not change; and for each point

$$x \in C_2 = \text{cone} \{u, w_3\} \cap \text{cone} \{u, -w_1\},$$
\[ \|x\|_{K_\theta} \text{ is non-increasing while } \|-x\|_{K_\theta} \text{ does not change. Therefore,} \]
\[ x \in C_1 \Rightarrow \|x\|_{cs(K_\theta)}^* \geq \|x\|_{cs(K_{\theta_0})}^*, \]
\[ (Q) \]
\[ x \in C_2 \Rightarrow \|x\|_{cs(K_\theta)}^* \leq \|x\|_{cs(K_{\theta_0})}^*. \]

Now let \( \epsilon_0 \) be maximal, such that \( K_\theta \in \mathcal{M}(t) \), for all \( \theta \in [\theta_0 - \epsilon_0, \theta_0] \).

If \( \epsilon_0 \geq \theta - \theta_1 \), then \( K_{\theta_1} \in \mathcal{M}(t) \), and for each small positive number \( r \), \( K_{\theta_1 + r} \in \mathcal{M}(t) \). Hence, for small enough \( r \), the corresponding edge \( E_+ \cap K_{\theta_1 + r}^* \) of \( K_{\theta_1 + r}^* \) has no point of \( C(K_{\theta_1 + r}) \) in its relative interior. According to Proposition 4.1 this contradicts \( K_{\theta_1 + r} \in \mathcal{M}(t) \).

Hence, we know \( \epsilon_0 < \theta - \theta_1 \). If \( 0 \in \text{conv} \{w_1, u, w_3\} \) and \( \epsilon_0 \geq \theta - \theta_2 \), then it still holds \( 0 \notin \{u, w', w_1\} \), and we replace \( K \) by \( K_{\theta_2} \) and start the rotating process again.

Hence, we may assume \( \epsilon_0 < \theta - \theta_1 \) and if \( 0 \in \text{conv} \{w_1, u, w_3\} \) we also may assume \( \epsilon_0 < \theta - \theta_2 \). Since \( K_{\theta_0 - \epsilon_0} \in \mathcal{M}(t) \) and \( \epsilon_0 \) is maximal, for each small positive number \( s \) we know \( K_{\theta_0 - \epsilon_0 - s} \notin \mathcal{M}(t) \). Then there are five cases:

(1) There exists a \( v' \in \mathbb{Z}^2 \setminus \{\pm e_1, \pm e_2\} \) such that \( \|v'\|_{cs(K_{\theta_0 - \epsilon_0})^*} = 1 \) and \( \|v'\|_{cs(K_{\theta_0 - \epsilon_0})^*} < 4 \) for some small \( s > 0 \). Then we have \( v' \in C_1 \cup C_2 \) (the norms of other points are not changed). By \( (Q) \) we conclude \( v' \in \text{cone} \{u, w_3\} \), which means that \( \frac{v'}{\|v'\|_{cs(K_{\theta_0 - \epsilon_0})^*}} \) is a new point of \( C(K_{\theta_0 - \epsilon_0}) \) that lies in the relative interior of the edge \( E(\theta_0 - \epsilon_0) \). In this case, \( E(\theta_0 - \epsilon_0) \) has two points of \( C(K_{\theta_0 - \epsilon_0}) \) in the relative interior and hence, \( K_{\theta_0 - \epsilon_0} \) fulfills property \( (P) \).

(2) \( \|e_1\|_{cs(K_{\theta_0 - \epsilon_0})^*} = \frac{1}{s} \) and \( \|e_1\|_{cs(K_{\theta_0 - \epsilon_0 - s})^*} < \frac{1}{s} \) for some small \( s > 0 \). Again, we may assume \( e_1 \in C_1 \cup C_2 \) and by \( (Q) \) \( e_1 \in \text{cone} \{u, w_3\} \). Then \( \frac{e_1}{\|e_1\|_{cs(K_{\theta_0 - \epsilon_0})^*}} \) is a new point of \( C(K_{\theta_0 - \epsilon_0}) \) in the relative interior of the edge \( E(\theta_0 - \epsilon_0) \). But then we have \( \|e_1\|_{cs(K_{\theta_0})^*} > \frac{1}{2}, \) implying \( \lambda_1(cs(K_{\theta_0})^*) > \frac{1}{2}, \) contradicting \( K_{\theta_0} \in \mathcal{M}(t) \).

(3) \( \|e_2\|_{cs(K_{\theta_0 - \epsilon_0})^*} = 1 \) and \( \|e_2\|_{cs(K_{\theta_0 - \epsilon_0 - s})^*} < 1 \) for some small \( s > 0 \). Then \( e_2 \in C_1 \cup C_2 \) and by \( (Q) \) \( e_2 \in \text{cone} \{u, w_3\} \). Then \( \frac{e_2}{\|e_2\|_{cs(K_{\theta_0 - \epsilon_0})^*}} \) is a new point of \( C(K_{\theta_0 - \epsilon_0}) \) in the relative interior of the edge \( E(\theta_0 - \epsilon_0) \). This implies \( \|e_2\|_{cs(K_{\theta_0})^*} > 1 \), contradicting \( K_{\theta_0} \in \mathcal{M}(t) \).

(4) \( \|e_1\|_{cs(K_{\theta_0 - \epsilon_0})^*} = \frac{1}{s} \) and \( \|e_1\|_{cs(K_{\theta_0 - \epsilon_0 - s})^*} > \frac{1}{s} \) for some small \( s > 0 \). Then \( e_1 \in C_1 \cup C_2 \) and in view of \( (Q) \) we get \( e_1 \in \text{cone} \{u, w_1\} \). The intersection of \( E_+ \) and \( E_{\theta_0 - \epsilon_0} \) is actually \( \frac{e_1}{\|e_1\|_{cs(K_{\theta_0 - \epsilon_0})^*}} \). In this case, \( E(\theta_0 - \epsilon_0) \) has \( u \) of \( C(K_{\theta_0 - \epsilon_0}) \) in the relative interior and \( \frac{e_1}{\|e_1\|_{cs(K_{\theta_0 - \epsilon_0})^*}} \) as a vertex. Hence, \( K_{\theta_0 - \epsilon_0} \) satisfies property \( (P) \).

(5) There exists a \( v' \in \mathbb{Z}^2 \setminus \{\pm e_1\} \) such that \( \|v'\|_{cs(K_{\theta_0 - \epsilon_0})^*} = 1 \) and \( \|v'\|_{cs(K_{\theta_0 - \epsilon_0 - s})^*} > 1 \) for some small \( s > 0 \). Then \( v' \in \text{cone} \{u, w_1\} \) (cf. \( (Q) \)), and the intersection of \( E_+ \) and \( E_{\theta_0 - \epsilon_0} \) is actually the point \( \frac{v'}{\|v'\|_{cs(K_{\theta_0 - \epsilon_0})^*}} \). If \( v' \neq \pm e_2 \) then rotation would not stop here. Hence, let \( v' = e_2 \). Since \( K_{\theta_0 - \epsilon_0} \) has at least 4 edges, and the relative interior of each edge of \( K_{\theta_0 - \epsilon_0}^* \)
contains a point of \( C(K_{0-e_0}) \) (cf. Proposition 4.1), and since now \( e_2 \) is also a vertex of \( C(K_{0-e_0}) \) we find \( |C(K_{0-e_0})| = 6 \) (cf. Remark 4.3). Hence, there exists a unimodular transformation of \( K_{0-e_0} \) mapping \( e_2 \) to a point \( C(K_{0-e_0}) \setminus \{ \pm e_1, \pm e_2 \} \) and we start the rotating process with this new body. □

Next we exclude the quadrilateral case.

**Proposition 4.7.** There are no quadrilaterals in \( \mathcal{M}(t) \).

**Proof.** Let \( K \) be a quadrilateral in \( \mathcal{M}(t) \). According to the proof of Proposition 4.6 we may assume that \( K \) satisfies property \((P)\).

Together with Proposition 4.4 we conclude that \( e_1 \parallel e_1 \parallel K^{\star} \) and \( -e_1 \parallel -e_1 \parallel K^{\star} \) are two opposite vertices of \( K^{\star} \) and each of the four edges of \( K^{\star} \) has a point of \( C(K) \) in the relative interior. In view of Remark 4.3, we may assume with \( u_1 = e_1 + e_2 \) that \( C_0(K) = \{ \pm e_1, \pm e_2, \pm u_1 \} \).

We translate \( K \) into a position, such that \( \|u_1\|_{K^{\star}} = \|-u_1\|_{K^{\star}} = 1 \) and \( \|e_2\|_{K^{\star}} = \|-e_2\|_{K^{\star}} = 1 \). In order to do so, we first find the four supporting hyperplanes of \( K \) with normal vectors \( \pm u_1, \pm e_2 \) and find the center of this parallelogram. The center of this parallelogram is in the interior of \( K \), and thus we can translate the origin point to the center of this parallelogram.

Let \( t_1 = \|e_1\|_{K^{\star}} \) and \( t_2 = \|-e_1\|_{K^{\star}} \). Then \( t_1 + t_2 = \frac{2}{t} \).

![Figure 3. The polar body of a quadrilateral satisfying the condition (P)](image)

Next we consider all the linear equations describing the edges of \( K^{\star} \) and so we get the vertices of \( K \).

i) The affine hull of the edge of \( K^{\star} \) containing \( \left( \frac{1}{t_1}, 0 \right) \) and \( e_1 + e_2 \) is given by the equation \( \{(x, y) : t_1 x + (1 - t_1) y = 1 \} \), and so \( K \) has the vertex \( (t_1, 1 - t_1) \).

ii) The affine hull of the edge of \( K^{\star} \) containing \( e_2 \) and \( \left( -\frac{1}{t_2}, 0 \right) \) is given by the equation \( \{(x, y) : -t_2 x + y = 1 \} \), and so \( K \) has the vertex \( (-t_2, 1) \).

iii) The affine hull of the edge of \( K^{\star} \) containing \( \left( -\frac{1}{t_2}, 0 \right) \) and \( -(e_1 + e_2) \) is given by the equation \( \{(x, y) : -t_2 x - (1 - t_2) y = 1 \} \), and so \( K \) has the vertex \( (-t_2, -1 + t_2) \).
iv) The affine hull of the edge of $K^*$ containing $-e_2$ and $(\frac{1}{t_1}, 0)$ is given by the equation $\{(x, y) : t_1x - y = 1\}$, and so $K$ has the vertex $(t_1, -1)$.

Therefore $\text{vol}(K) = \frac{4}{t} - \frac{2}{t^2}$, and hence $K \notin \mathcal{M}(t)$ (cf. (4.1)).

![Figure 4. A quadrilateral satisfying the condition (P)](image)

Finally, we consider the triangles in $\mathcal{M}(t)$.

**Proposition 4.8.** Up to translations and unimodular transformations, $\mathcal{M}(t)$ contains only the triangle $T_{1,1/t} = \text{conv}\{(-1/t, 1 - 1/t), (1/t, 1), (0, -1)\}$ of volume $\frac{2}{t} - \frac{1}{2t^2}$.

**Proof.** Let $K \in \mathcal{M}(t)$. According to Proposition 4.6 and Proposition 4.7, $K$ is a triangle satisfying property (P). Thus we know

1. only one edge of $K^*$ contains two points of $C(K)$ while the other two edges share a vertex in $C(K)$ and separately have one point of $C(K)$ in the relative interior of each edge, or
2. each edge of $K^*$ contains two points of $C(K)$ in the relative interior.

Therefore $|C(K)|$ has to be 6 (cf. Proposition 4.4). According to Remark 4.3, we may assume that up to an unimodular transformation $C_0(K) = \{\pm e_1, \pm e_2, \pm (e_2 - e_1)\}$.

Next we discuss the above two different cases.

1. Here we may assume that $\frac{e_1}{\|e_1\|_{K^*}}$ is a vertex of $K^*$. Then $\frac{-e_1}{\|-e_1\|_{K^*}}$ has to be in the edge opposite to this vertex. Since the two edges of $K^*$ sharing the vertex $\frac{e_1}{\|e_1\|_{K^*}}$ must contain $\frac{e_2}{\|e_2\|_{K^*}}$ and $\frac{e_1 - e_2}{\|e_1 - e_2\|_{K^*}}$ in their relative interior, respectively, the remaining edge contains either both points $\frac{-e_2}{\|-e_2\|_{K^*}}$, $\frac{e_2 - e_1}{\|e_2 - e_1\|_{K^*}}$, or only one of these points. Here we just consider the case that this edge contains both points, because otherwise each edge contains two points of $C(K)$ and this will be discussed in the next case.

Since $\|e_2\|_{K^*} + \|-e_2\|_{K^*} = 2$ and $\|e_2 - e_1\|_{K^*} + \|e_1 - e_2\|_{K^*} = 2$, we choose a translation of $K$, such that $\|e_2\|_{K^*} = \|-e_2\|_{K^*} = 1$ and $\|e_2 - e_1\|_{K^*} = \|e_1 - e_2\|_{K^*} = 1$. In order to do so, we first find the four supporting hyperplanes of $K$ with normal vectors $\pm e_2, \pm (e_2 - e_1)$ and find the center of this
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parallelogram. Then the center of this parallelogram has to be an interior point of $K$, and we can translate the origin to this center.

Then one edge of $K^\ast$ contains $\frac{e_1}{\|e_1\|_{K^\ast}}$ and $e_2$, one edge contains $e_2 - e_1$ and $-e_2$, and one edge contains $e_1 - e_2$ and $\frac{e_1}{\|e_1\|_{K^\ast}}$. From this we get $\|e_1\|_{K^\ast} = 2$ and thus $\|e_2\|_{cs(K)}^\ast = \|e_1\|_{K^\ast} - \|e_1\|_{K^\ast} = 2 - 2 < 0$, which is impossible.

**Figure 5.** First (impossible) triangle case satisfying the condition (P)

Hence, it remains only to consider the second case, and here we just assume that

2. each edge of $K^\ast$ contains two points of $C(K)$. Up to a rotation by $\pi$, i.e., up to an unimodular transformation, we may assume that the three edges $U_i$, $1 \leq i \leq 3$, are given

- $U_1$ contains $\frac{e_1}{\|e_1\|_{K^\ast}}$ and $\frac{e_2}{\|e_2\|_{K^\ast}}$,
- $U_2$ contains $\frac{e_2 - e_1}{\|e_2 - e_1\|_{K^\ast}}$ and $\frac{-e_1}{\|e_1\|_{K^\ast}}$,
- $U_3$ contains $\frac{-e_2}{\|e_2\|_{K^\ast}}$ and $\frac{e_1 - e_2}{\|e_1 - e_2\|_{K^\ast}}$.

Since $\|e_2\|_{K^\ast} + \|-e_2\|_{K^\ast} = 2$ and $\|e_2 - e_1\|_{K^\ast} + \|e_1 - e_2\|_{K^\ast} = 2$, we choose a translation of $K$, such that $\|e_2\|_{K^\ast} = \|-e_2\|_{K^\ast} = 1$ and $\|e_2 - e_1\|_{K^\ast} = \|e_1 - e_2\|_{K^\ast} = 1$. In order to do so, we proceed as in case 1., i.e., first we find the four supporting hyperplanes of $K$ with normal vectors $\pm e_2, \pm (e_2 - e_1)$ and find the center of this parallelogram. Then the center of this parallelogram has to be an interior point of $K$, and we can translate the origin to this center.

Let $t_1 = \|e_1\|_{K^\ast}, t_2 = \|-e_1\|_{K^\ast}$.
Figure 6. Second triangle case satisfying the condition (P)

Since $\|e_1\|_{cs(K)^*} = \frac{1}{t}$, we have

\[ t_1 + t_2 = \frac{2}{t}. \tag{4.3} \]

The affine hull of the edge $U_1$ containing $(\frac{1}{t_1}, 0)$ and $e_2$ is given by

\[ \{ (x, y) : t_1x + y = 1 \}. \]

The affine hull of the edge $U_2$ containing $e_2 - e_1$ and $(-\frac{1}{t_2}, 0)$ is given by

\[ \{ (x, y) : -t_2x + (1 - t_2)y = 1 \}. \]

The affine hull of the edge $U_3$ containing $-e_2$ and $e_1 - e_2$ is given by

\[ \{ (x, y) : -y = 1 \}. \]

Therefore,

\[ K = \text{conv} \{ (t_1, 1), (-t_2, 1 - t_2), (0, -1) \}, \]

with volume

\[
\text{vol}(K) = \frac{1}{2} (t_1(1 - t_2) + t_2) + \frac{1}{2} t_2 + \frac{1}{2} t_1 \\
= \frac{2}{t} - \frac{1}{2} t_1 t_2 \\
\geq \frac{2}{t} - \frac{1}{2} \frac{1}{t^2}.
\]

In the last inequality we have used (4.3) and the arithmetic-geometric mean inequality. Hence, we have equality if and only if

\[ t_1 = t_2 = \frac{1}{t}. \]

Since $K$ is supposed to have minimal volume (cf. (4.1)) we have equality.

Therefore, in this case, $K$ is a translation of $\text{conv} \{ (\frac{1}{t_1}, 1), (-\frac{1}{t_2}, 1 - \frac{1}{t_2}), (0, -1) \}$. \qed
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