On the Decomposability of 1-Parameter Matrix Flows

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Abstract

For general complex or real 1-parameter matrix flow \( A(t)_{n,n} \) this paper considers ways to decompose flows globally via one constant matrix \( C_{n,n} \) as \( A(t) = C^{-1} \cdot \text{diag}(A_1(t), ..., A_\ell(t)) \cdot C \) with each diagonal block \( A_k(t) \) square and the number of blocks \( \ell > 1 \) if possible. The theory behind our algorithm is elementary and uses the concept of invariant subspaces for the Matlab \texttt{eig} computed ‘eigenvectors’ of one flow matrix \( A(t) \) to find the coarsest simultaneous block structure for all flow matrices \( A(t) \). The method works very efficiently for all matrix flows, be they differentiable, continuous or discontinuous in \( t \), and for all types of square matrix flows such as hermitean, real symmetric, normal or general complex and real flows \( A(t) \), with or without Jordan block structures and with or without repeated eigenvalues. Our intended aim is to discover decomposable flows as they originate in sensor given outputs for time-varying matrix problems and thereby reduce the complexities of their numerical treatment.

Keywords: time-varying matrices, decomposable matrix flow, numerical algorithm, block diagonal matrix flow

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1 Introduction

This paper deals with time-varying, i.e., 1-parameter varying matrix flows \( A(t) \in \mathbb{C}^{n,n} \) where \( t_0 \leq t \leq t_f \in \mathbb{R} \) or \( t \) follows a finite section of a curve in \( \mathbb{C} \). In many applications it is of interest to learn whether a dense given matrix flow \( A(t) \) can actually be decomposed into an array of block flows \( A_k(t) (k = 1, ..., \ell) \) of smaller dimensions

\[
A(t)_{n,n} = C^{-1} \cdot \begin{pmatrix} A_1(t) & O & O & \cdots & O \\ O & A_2(t) & O & \cdots & O \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O & O & \cdots & O & A_\ell(t) \end{pmatrix} \cdot C.
\]

Here \( C \in \mathbb{C}^{n,n} \) is an invertible matrix that is constant for all parameters \( t \). If \( A(t) \) can be decomposed in this fashion, the numerical problem at hand with \( A(t) \) may be ‘divided and conquered’ into \( \ell \) smaller subproblems for the individual blocks \( A_k(t) \) and these subproblems can usually be solved more quickly. The matrix flow itself may derive over time from given equations or it may be generated from sensor data that arrive at a constant discrete sampling gap rate \( \tau \).

Decomposable matrix flows have been intimately linked to eigencurve crossings of matrix flows \( A(t) \) for over 90 years. In 1927 and 1929, Hund [7] and von Neumann and Wigner [9] proved that hermitean matrix flows \( A(t) = (A(t))^* \) whose eigencurves crossed each other must be decomposable in the above sense. Eigencurve crossing is sufficient for matrix flow decomposability but the converse implication is not true. In [12], the author studied the eigencurves of hermitean and general matrix flows and developed an algorithm to deduct the coarsest block diagonalisation dimensions of hermitean matrix flows from their eigencurve crossing data. The biggest drawback of that method is the fact that decomposable matrix flows need not show eigencurve crossings at all. This paper introduces a different algorithm that uses standard matrix invariant subspace theory to decompose matrix flows into block-diagonal flows – if possible – for both hermitean and general complex matrix flows, or it decides that such decompositions are impossible for \( A(t) \).

Details and test follow.

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2 Theory

To start we consider a ‘proper’ $n$ by $n$ hermitean time-varying matrix flow $A(t)_{n,n}$ that can be block diagonalized uniformly as in (1) for a nonsingular matrix $C_{n,n}$ and all $t_o \leq t \leq t_f$ with $\ell > 1$ diagonal blocks for ‘properness’.

Any hermitean matrix flow $A(t)$ allows us to diagonalize the flow matrix $A(t_a)$ for any $t_a \in [t_o, t_f]$ via a unitary similarity transformation $V(t_a)$, i.e., $A(t_a) \cdot V(t_a) = V(t_a) \cdot D(t_a)$ with $D(t_a)$ real diagonal. The transforming $V(t_a)$ contains the eigenvectors of $A(t_a)$ in its columns and the eigenvalues of $A(t_a)$ appear on the diagonal of $D(t_a)$. On the other hand, we have assumed that $A(t_a) = C^{-1} \cdot \text{blockdiag}\{A_1(t_a), ..., A_{\ell}(t_a)\} \cdot C$. Each eigenvector in $V(t_a)$ is associated with one of the eigenvalues of $A(t_a)$. In fact the eigenvector columns of $V(t_a)$ that are associated with one diagonal block $A_i(t_a)$ form an orthonormal basis for an invariant subspace of $A(t)$ of which there are $\ell$ by assumption.

As all matrices of the decomposable hermitean flow $A(t)$ share the same invariant subspace structure expressed in (1), then for any $t_b \neq t_a \in [t_o, t_f]$ the matrix

$$\langle \tilde{V}(t_a) \rangle^* \cdot A(t_b) \cdot \tilde{V}(t_a)$$

must be block diagonal and have that same common block structure as soon as we re-sort the eigenvectors in $V(t_a)$ into $\ell$ vector groups according to their associated diagonal blocks $A_i(t_b)$ in the column reshuffled unitary matrix $\tilde{V}(t_a)$.

**Theorem 1**: If a hermitean time-varying matrix flow $A(t)$ can be properly and uniformly diagonalized by a constant unitary similarity $V^*...V$, then the eigenvectors of any flow matrix $A(t_a)$ will block diagonalize every other flow matrix $A(t_b)$ upon re-sorting the columns of $V$ and the similarity performed by $V^*...V$ on $A(t_b)$ results in the identical block diagonalization for any $t_b$, albeit maybe a finer one as $V$ diagonalizes $A(t_a)$ by design.

For general complex matrix flows $A(t)_{n,n}$ that are diagonalizable throughout their single parameter range, the same invariant subspace argument holds except that the unitary eigenvector matrix $\tilde{V}(t_a)$ needs to be replaced by a general similarity via a nonsingular matrix $\tilde{W}(t_a)$ so that the inverse $\tilde{V}(t_a)^*$ of $\tilde{V}(t_a)$ in formula (2) becomes $\tilde{W}(t_a)^{-1}$.

**Theorem 2**: If a diagonalizable general complex time-varying matrix flow $A(t)$ can be properly and uniformly diagonalized by a constant matrix similarity $C^{-1}...C$, then the eigenvectors of any flow matrix $A(t_a)$ will block diagonalize – upon re-sorting – every other flow matrix $A(t_b)$ by similarity into block diagonal form which may be finer than the coarsest possible block diagonal form of that flow.

Here the term ‘coarsest block diagonal form’ refers to one with the minimal possible block number $\ell$ in (1). Note for example, that $D(t_a)$ in formula (2) represents the finest, i.e., a block diagonalization with $\ell = n$ one by one diagonal blocks for $A(t_a)$.

The next section will deal with computing the minimal number $\ell$ of invariant subspaces of a properly decomposable matrix flow $A(t)$ and with re-sorting the columns of their respective eigenvector matrices $V(t_a)$ or $\tilde{W}(t_a)$, so that the coarsest simultaneous diagonal block reduction or a finer one can be achieved for any flow matrix $A(t_b)$ effectively.

3 The Algorithm and Computed Results

As the theory tells us, to solve the matrix flow decomposability problem it suffices to compute

- **(A)** the eigenvector matrix $X(t_a)$ of one specific flow matrix $A(t_a)$ and apply the similarity $(X(t_a))^{-1} \cdot A(t_b) \cdot X(t_a)$ to any other flow matrix $A(t_b)$ in order to learn about the coarsest (or a finer) block-diagonalization of the respective matrix flow. Theory predicts perfect zeros in the updated $(X(...)^{-1} \cdot A(...) \cdot X...$ flow matrix, but numerical rounding errors and conditioning problems always create relatively small magnitude entries in some entry positions of the computed matrix $\tilde{A}(t_b) = (X(t_a))^{-1} \cdot A(t_b) \cdot X(t_a)$ that theoretically ought to be zero. These tiny magnitude entries must be replaced by zeros in order to be able to view the block structure of $A(t_b)$ properly. For this purpose we form

- **(B)** the logical 0-1 matrix for our computed $\tilde{A}(t_b)$ and then
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We rearrange its rows and columns by collecting equal 0-1 pattern row vectors therein into groups in order to exhibit the block diagonal structure of the studied flow. This process works equally well for all time-varying matrix flows and offers a great improvement over what could be gleaned geometrically from eigencurve crossings in [12]. Besides, in [12] the general complex matrix flow case was generally found to be intractable via coalescing eigencurves. And here this problem does not even appear.

Figure 1 shows nine Matlab spy graphs for a dense non-normal complex 17 by 17 matrix flow. Reading this figure row by row, the first row of graphs shows the Matlab spy 0-1 pattern transition from \(A(t_a)\) to \(A(t_b)\); the second row shows the 0-1 pattern transition from \(A(t_b)\) to \(A(t_c)\), and the third row the one from \(A(t_c)\) to \(A(t_{rd})\) for a randomly chosen parameter \(t_{rd} \in \mathbb{C}\). The middle column (B) of graphs displays varying 0-1 patterns due to the varying starting matrices \(A(t_a), ..., A(t_c)\). However, these middle column spy graphs are all equivalent to a 7 4 3 2 1 block diagonalization for the flow \(A(t)\). The third column (C) of 0-1 spy graphs is computed from the respective data in column (B) and the same non-zero diagonal blocks do appear, but in differing orders, upon re-sorting the 0-1 rows of column (B) into equal row groups.

The general complex matrix flow \(A(t)_{17,17}\) had originated from a flow with the same block-diagonal structure 7 4 3 2 1 that was transformed into a dense general flow by a fixed dense unitary similarity for testing and validating our MATLAB algorithm `deccomplflow9.m`.

![Figure 1](image-url)
larity was relatively easy since the dense $A(t)_{17,17}$ flow example was constructed from a diagonalizable complex general matrix flow. Our algorithms work equally well for matrix flows that are built from proper Jordan blocks such as the next 9 by 9 complex flow example with Jordan blocks of sizes 4 and 5 shows in Figure 2 below. Note the ‘holes’ in the respective diagonal 0-1 blocks that seem to only occur for Jordan blocks. We know not why.

By depending only on elementary invariant subspace theory, our algorithm and code works well with real time parameters $t \in \mathbb{R}$, as well as with more general complex parameters $t \in \mathbb{C}$ as shown in all our figures 1 through 4.

The web depository [13] also contains a simpler algorithm (deccompl.m) for finding the block diagonal dimensions of a general matrix flow for just one time $t_b$ from the Matlab eig diagonalization of $A(t_a)$ with $t_a \neq t_b$. Besides, there is a different 9-graph Matlab m-file (deccomplflow9a.m) in [15] that computes the pattern transitions not along the chain from $t_a$ to $t_b$, then to $t_c$ and then from $t_c$ to $t_{rd}$ as deccomplflow9.m does, but instead computes the transitions starting always from $t_a$ to each of $t_b$, $t_c$ and $t_{rd}$ in turn.

The ‘Matrixflow Decomp’ folder at [13] contains the Matlab m-files for general 1-parameter matrix flows in the subfolder ‘general flows’. The subfolder ‘hermitean flows’ at [13] deals with hermitean or symmetric single parameter matrix flows. The hermitean flow methods decherm.m, dechermflow9.m and dechermflow9a.m are made simpler by the fact that they do not have to deal with the Matlab eig.m m-file output for derogatory non-normal matrices with proper Jordan block structures.

![Figure 2 (Dense complex matrix flow $A(t)$, formed from two Jordan blocks of sizes 4 and 5)](image)

The occurrence of Jordan blocks and their treatment in eig.m may also create bands of 0-1 entry rows with 1s in all positions when computing $\tilde{A}(t_b) = (X(t_a))^{-1} \cdot A(t_b) \cdot X(t_a)$. Those all 1s rows need to be taken care of differently than in the hermitean matrix flow case, where such cannot occur.
Figure 3 below shows such a banded 0-1 pattern matrix with all 1s rows for a dense example flow that was built from a general matrix flow containing two Jordan blocks. Grouping identical 0-1 rows in graph column (B) together into one diagonal block in column (C) for Jordan block containing general flows – as is sufficient for hermitean flows – would result in all rows of the 0-1 pattern matrix becoming indistinguishable here, indicating falsely that the general flow is indecomposable.

The actual re-sorting from (B) to (C) spy 0-1 matrices for general flows uses both the zero and the non-zero patterns of each non-all-1s row of the ‘spy’ graphs in column (B) to arrive at the 0-1 graph column (C). This helps us detect the block-diagonal dimensions correctly while also allowing us to determine the total sum of all Jordan block dimensions for such flows.

Each of our Matlab codes provides on-screen interpretations of the computed outputs and describes the resulting block dimension sizes, for both hermitean and general flows. For the latter, the on-screen block dimensions refer to the summed dimensions of all Jordan blocks in the listed flow dimensions if followed by a (J). On-screen there are warnings when the norm of an intermediate matrix \( A(t...) \) becomes excessively large in which case the computed block dimension results may be erroneous or unreliable.

Note that the intermediate second column graphical output (B) in Figures 1 through 3 may differ from row to row, but that the final graphs in column (C) are identical in each test run, except for permutations of their diagonal block order.

The use and success of our decomposition algorithm does not depend on or require any smoothness conditions on the 1-parameter matrix flows. Real-time or discontinuous data from sensor inputs is quite admissible. Here is the output of a general complex matrix flow, constructed with time-varying entries and random entry blocks of the
same sizes throughout.

\[
A(t) \quad B(t) \quad C(t)
\]

Finally [13] includes Matlab codes for constructing over a dozen example flows in both, the hermitean and the general cases, that can be implemented inside our respective flow decomposing routines by entering their matrix dimension number \(n\). Plotting can be turned off by setting the input parameter zeich unequal to 1 and the block dimension information will still appear on-screen. Without graphing, the CPU times of the algorithms were in the thousands of a second in all tested dimensions and, with graphing included, the computations and all visual ‘spy’ displays would appear in a fraction of a second in all our tests.

4 An Outlook and Adjacent Areas of Research

It might be of interest to size each occurring Jordan block in a general matrix flow \(A(t)\) individually rather than summarily, but we have not done so. Regarding Jordan structures of fixed or static matrices \(A_{n,n}\) it appears to be nearly impossible in general and at least very expensive to try and determine the Jordan structure of even small dimensioned static matrices \(A\) reliably by numerical means such as `eig` in Matlab. More involved studies of computing the Jordan normal form of static matrices \(A_{n,n}\) have appeared in [11] and likewise for the Kronecker normal form of singular matrix pencils in [6]. Yet the problem of block diagonalizing time-varying general matrix flows \(A(t)\) in the presence of Jordan structures or repeated eigenvalues has been easily answered computationally here.
by using elementary invariant subspace theory. This again shows that time- or parameter-varying matrix flows $A(t)$ follow different fundamental concepts than classic static matrix theory and analysis. Could one and how could one alter the Francis multishift implicit QR method somehow, for example, to account for repeated eigenvalues and higher dimensional principal subspaces of static matrices, we wonder.

An application of our matrix flow decompositions helps with the matrix field of values problem for decomposing and general complex matrices in [14].

Separately Loisel and Maxwell [8, Thm 2.5, Sect 5, 6.2, and 7.1] have developed an ODE solver to find eigencrossing points of hermitean block-diagonal matrix flows for the field of values boundary computation problem, while Dieci et al [12,3,4,5] have studied multi-parameter flows and their eigencrossings and singular value crossings using geometric localization and zoom-in optimization methods. Maybe our invariant subspace based idea can be extended and adapted to help with such problems.

Finally, Sabuya [10] has dealt with a related problem to classify all matrix flows $A(x)$ that are block-diagonalizable under time varying similarities $T^{-1}(x) \cdot A(x) \cdot T(x)$ in contrast to our unified fixed $C \cdot A(t) \cdot C$ block-diagonal similarities.

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image files:
Decgen9pics17.png
Decgen9apics9.png
Decgen9pics14.png
Decgen9pics13.png