On the numerical solution of the nonlinear Korteweg–de Vries equation

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In this paper, we present a new method for solving a nonlinear third-order Korteweg–de Vries equation. This method is based on the multiquadric (MQ) quasi-interpolation operator $L_{\text{MQ}}$ and an integrated radial basis function networks scheme. In the present scheme, the second-order central divided difference of the spatial derivative is used to approximate the third-order spatial derivative, and the Taylors series expansion to discretize the temporal derivative. Then, the spatial derivative is approximated by the MQ quasi-interpolation operator $L_{\text{MQ}}$. This method is applied on some test experiments and the numerical results have been compared with the exact solutions and the solutions of other numerical methods. The $L_{\infty}, L_2$ and root-mean-square errors of the solutions show the efficiency and the accuracy of the method. Furthermore, the stability analysis of the method is surveyed.

**Keywords:** nonlinear KdV equation; multiquadric quasi-interpolation scheme; integrated radial basis function networks scheme; Taylors series expansion

1. Introduction

In this paper, we concentrate on the numerical solution of one of the well-known equation named as Korteweg–de Vries (KdV) equation:

$$u_t + \varepsilon uu_x + \mu u_{xxx} = 0, \quad x \in \Omega = [a, b], \quad t \geq t_0,$$

(1)

where $\varepsilon$ and $\mu$ are positive parameters. The KdV equation expresses a balance between dispersion form from its third derivative term $u_{xxx}$ and the shock forming tendency of its nonlinear term $uu_x$.

The most important property of Equation (1) is that solutions may exhibit solitons. Solitons are localized waves that propagate without change of its shape and velocity and are stable in mutual interaction just like the phenomenon of totally elastic collision in kinetics. Over the years, the KdV equation has found wide applications in many fields such as waves in inharmonic crystals, bubble liquid mixtures, ion acoustic wave and magneto-hydrodynamic waves in a warm plasma as well as shallow water waves (Dodd, Eilbeck, Gibbon, & Morris, 1982; Gardner & Marikawa, 1965; Korteweg–de Vries & de Vries, 1895; Washimi & Taniuti, 1966; Wijngaarden, 1968).

The KdV equation is a completely integrable Hamiltonian system which can be solved explicitly. Thus, some analytical solutions of it are found, and for appropriate initial conditions, their existence and uniqueness have been shown by Gardner, Grrne, and Kruskal (1967). The numerical solutions of the KdV equation can be found by using the techniques known as finite difference schemes, finite element schemes, and Fourier spectral methods (Geyikli & Kaya, 2005; Helal & Mehanna, 2006; Li & Wang, 2007; Yan, 2006). The difficulty of mesh generation, especially in two or more dimensions, makes these methods hard to implement. In the past two decades, the radial basis function (RBF) meshless method has been considered for the numerical solution of the various types of partial differential equations (PDEs). These methods do not require the structured grid, that is, that are truly meshless methods. Kansa (1990a, 1990b) was the first researcher to approximate PDEs using RBFs. He has directly collocated the RBFs, particularly the multiquadric (MQ), for the approximated solutions of the PDEs. The MQ–RBF was first developed by Hardy (1971) as a multi-dimensional scattered interpolation method. Franke (1982) published a detailed comparison of 29 different scattered data schemes for analytic problems. Of all the techniques tested, he concluded that the MQ approximation scheme performed the best in accuracy, visual appeal, and ease of implementation.

In most of the known methods of solving PDEs using the MQ approximation scheme, one must resolve a linear system of equations at each time step. Hon and Wu (2000) and Wu (2004, 2005) and others have provided some successful examples using the MQ quasi-interpolation scheme for solving differential equations.
Beaton and Powell (1992) proposed three univariate MQ quasi-interpolations, namely, $L_A$, $L_B$ and $L_C$. Wu and Schaback (1994) presented the univariate MQ quasi-interpolation $L_D$ and proved that the scheme is shape preserving and convergent. Chen and Wu (2006, 2007) used MQ quasi-interpolation to solve Burgers’ equation and hyperbolic conservation laws. Also, Xiao, Wang, and Zhu (2011) presented the numerical method based on Chen and Wu’s method for solving the third-order KdV equation. Recently, Jiang, Wang, Zhu, and Xu (2011) have introduced a new multi-level univariate MQ quasi-interpolation approach with high approximation order compared with the initial MQ quasi-interpolation scheme. This approach is based on inverse multiquadric (IMQ) RBF interpolation, and Wu and Schaback’s MQ quasi-interpolation operator $L_D$ that have the advantages of high approximation order.

In the Kansa’s RBF method and the other conventional RBF method, the RBF approximation is directly applied to the solution function. As such, any derivative of the solution function can be obtained by differentiating the RBF expression. But recently, Mai-Duy, Mai-Cao, and Tran-Cong (2007) and Mai-Cao and Tran-Cong (2005) have introduced an integrated radial basis function networks (IRBFNs) scheme for the approximation of the solution function and its derivatives. In the IRBFNs approach, the RBF approximation is applied to a targeted derivative (first-order or second-order) of the solution function. Consequently, the solution function is obtained by integrating the derivative with the RBF expression. Numerical experiments and theoretical analysis indicate that the IRBF scheme is more accurate than the direct radial basis function (DRBF) for solving PDEs (Aminiataei & Mazarei, 2008).

The purpose of this paper is to present a new numerical scheme to solve the nonlinear third-order KdV equation, based on the IRBF scheme and MQ quasi-interpolation scheme. In our scheme, we use the MQ quasi-interpolation scheme and the integration of it to approximate the first-order spatial derivative and the solution function, respectively, similar to the work that we did in Sarboland and Aminiataei (2014). Besides, we use the second-order central divided difference of the first-order spatial derivative to approximate the third-order spatial derivative. Also, we employ Taylor series expansion to approximate the temporal derivative as Da˘g, Conlvar, and Şahin (2011) did for Burgers’ equation.

The organization of this paper is as follows. Section 2 gives brief information about RBFs and the MQ quasi-interpolation scheme. In Section 3, an IRBFNs scheme is presented. In Section 4, we apply the method on the nonlinear third-order KdV equation. The stability analysis of the methods is discussed in Section 5. The results of three numerical experiments are presented in Section 6 and are compared with the analytical solutions and the results in Xiao et al. (2011) and Siraj, Khattak, and Tirmizi (2008).

Finally, a brief discussion and conclusion are presented in Section 7.

2. The RBFs and the MQ quasi-interpolation scheme

In this section, we present some elementary knowledge about RBF interpolation and introduce three univariate MQ quasi-interpolation schemes, namely, $L_D$, $L_{YV}$ and $L_{YV_2}$. For more details about MQ quasi-interpolation operators, one can see Beaton and Powell (1992), Wu and Schaback (1994), Chen and Wu (2006, 2007) and Powell (1992).

For a given region $\Omega = [a, b]$ and a finite set of distinct points

$$ a = x_0 < x_1 < \cdots < x_N = b, \quad h = \max_{1 \leq i \leq N} (x_i - x_{i-1}), $$

the MQ quasi-interpolation of a univariate function $f : [a, b] \rightarrow \mathbb{R}$ has the form

$$ L(f) = \sum_{i=0}^{N} f(x_i) \psi_i(x), $$

where each function $\psi_i(x)$ is a linear combination of the MQs

$$ \psi_i(x) = \sqrt{c^2 + (x-x_i)^2}, $$

and $c \in \mathbb{R}^+$ is a shape parameter. Wu and Schaback (1994) presented the univariate MQ quasi-interpolation operator $L_D$ that is defined as

$$ L_Df(x) = \sum_{i=0}^{N} f(x_i) \tilde{\psi}_i(x), $$

where

$$ \tilde{\psi}_0(x) = \frac{1}{2} + \frac{\psi_1(x) - (x-x_0)}{2(x_1-x_0)}, $$

$$ \tilde{\psi}_1(x) = \frac{\psi_2(x) - \psi_1(x)}{2(x_2-x_1)} = \frac{\psi_1(x) - (x-x_0)}{2(x_1-x_0)}, $$

$$ \tilde{\psi}_i(x) = \frac{\psi_{i+1}(x) - \psi_i(x)}{2(x_{i+1}-x_i)} = \frac{\psi_i(x) - \psi_{i-1}(x)}{2(x_i-x_{i-1})}, \quad 2 \leq i \leq N-2, $$

$$ \tilde{\psi}_{N-1}(x) = \frac{(x_N-x) - \psi_{N-1}(x)}{2(x_N-x_{N-1})} - \frac{\psi_{N-1}(x) - \psi_{N-2}(x)}{2(x_{N-1}-x_{N-2})}, $$

and

$$ \tilde{\psi}_N(x) = \frac{1}{2} + \frac{\psi_{N-1}(x) - (x_N-x)}{2(x_N-x_{N-1})}. $$

In RBFs interpolation, we can get high approximation order by increasing the number of interpolation centers but we have to solve unstable linear system of equations. By using the MQ quasi-interpolation scheme, we can avoid this problem, but the approximation order is not good. Hence, Jiang et al. (2011) have defined two
MQ quasi-interpolation operators denoted as $\mathcal{L}_W$ and $\mathcal{L}_{W_2}$, which pose the advantages of RBFs interpolation and MQ quasi-interpolation schemes. The process of MQ quasi-interpolation of $\mathcal{L}_W$ and $\mathcal{L}_{W_2}$ is as follows and is described in Jiang et al. (2011).

Suppose that $\{x_k\}_{i=1}^{\tilde{N}}$ is a smaller set from the given points $\{x_i\}_{i=1}^{N}$, where $\tilde{N}$ is a positive integer satisfying $\tilde{N} < N$ and $0 = k_0 < k_1 < \cdots < k_{\tilde{N}+1} = N$. Using the IMQ-RBF, the second derivative of $f(x)$ can be approximated by the RBF interpolant $S_f$ as

$$S_f(x) = \sum_{j=1}^{\tilde{N}} \alpha_j \tilde{\phi}(x - x_j), \quad (4)$$

where

$$\tilde{\phi}(r) = \frac{s^2}{(s^2 + r^2)^{3/2}},$$

and $s \in \mathbb{R}^+$ is a shape parameter. The coefficients $\{\alpha_j\}_{j=1}^{\tilde{N}}$ are uniquely determined by the interpolation condition

$$S_f(x_j) = \sum_{j=1}^{\tilde{N}} \alpha_j \tilde{\phi}(x_j - x_j) = f^{\prime\prime}(x_j), \quad 1 \leq i \leq \tilde{N}. \quad (5)$$

Since, Equation (5) is solvable (Madych & Nelson, 1990), so

$$\alpha = A_X^{-1} f_X^{\prime\prime}, \quad (6)$$

where

$$X = \{x_1, \ldots, x_{\tilde{N}}\}, \quad \alpha = [\alpha_1, \ldots, \alpha_{\tilde{N}}]^T,$$

$$A_X = [\tilde{\phi}(x_{j} - x_{j})], \quad f_X^{\prime\prime} = [f^{\prime\prime}(x_1), \ldots, f^{\prime\prime}(x_{\tilde{N}})]^T.$$ 

By using $f$ and the coefficient $\alpha$ defined in Equation (6), a function $e(x)$ is constructed in the following form:

$$e(x) = f(x) - \sum_{i=1}^{\tilde{N}} \alpha_i \sqrt{s^2 + (x - x_i)^2}. \quad (7)$$

Then the MQ quasi-interpolation operator $\mathcal{L}_{W}$ by using $\mathcal{L}_D$ defined by Equations (2) and (3) on the data $(x_i, e(x_i))_{0 \leq i \leq N}$ with the shape parameter $c$ is defined as follows:

$$\mathcal{L}_W f(x) = \sum_{i=1}^{\tilde{N}} \alpha_i \sqrt{s^2 + (x - x_i)^2} + \mathcal{L}_D e(x). \quad (8)$$

The shape parameters $c$ and $s$ should not be the same constant as in Equation (8).

In Equation (5), $f_X^{\prime\prime}$ can be replaced by

$$f_X^{\prime\prime} = \frac{2f(x_{k+1})}{(x_{k+1} - x_k)(x_{k+1} - x_{k-1})} - \frac{2f(x_k)}{(x_k - x_{k-1})(x_{k+1} - x_k)} + \frac{2f(x_{k-1})}{(x_k - x_{k-1})(x_{k-1} - x_{k-1})},$$

when the data $(x_i, f(x_i))_{0 \leq i \leq \tilde{N}}$ are given. So, if $f_X^{\prime\prime}$ in Equation (6) can be replaced by

$$F_X^{\prime\prime} = (f_X^{\prime\prime}, \ldots, f_X^{\prime\prime})^T, \quad (9)$$

then the quasi-interpolation operator defined by Equations (7) and (8) is denoted by $\mathcal{L}_{W_2}$. For more details about the properties and accuracy of $\mathcal{L}_W$ and $\mathcal{L}_{W_2}$, one can see Jiang et al. (2011). In this paper, we use the MQ quasi-interpolation operator $\mathcal{L}_{W_2}$ and $N = 4\tilde{N}$.

The operator $\mathcal{L}_{W_2}$ can be written in the compact form

$$\mathcal{L}_{W_2} f(x) = \sum_{i=0}^{N} f(x_i) \hat{\psi}_i(x),$$

where the basis functions $\hat{\psi}_i(x)$ are obtained by substituting Equations (6), (7) and (9) into Equation (8), see Sarboland and Aminataei (2014).

3. The IRBFNs scheme

In the past two decades, there has been some developments in applying the RBFs for the numerical solutions of various types of PDEs. In the RBF-based methods or DRBF methods, a RBF approximation is directly applied to the solution function. As such, any derivative of the solution function can be obtained by differentiating the RBF expression. Although the method has the ability to represent any continuous function to a prescribed degree of accuracy, the process of differentiation magnifies any errors that might arise from approximating the original function and thus result in inaccurate derivatives. But the IRBF method starts with the decomposition of the highest derivative (one-order in this paper) of the solution function into a linear combination of RBFs. Then, the solution function is obtained by symbolic integration. In contrast to the process of differentiation where any errors associated with the function approximation might be amplified, the integration has the effect of averaging out such errors. The IRBF method, therefore, results in a better approximation accuracy than the usual approach as in the DRBF method with the same numerical configuration. The IRBF method has been described completely in Mai-Duy et al. (2007), Mai-Cao and Tran-Cong (2005) and Aminataei and Mazarei (2008).
Due to the aforementioned advantages of the IRBF method over the DRBF method, it is decided to apply the MQ quasi-interpolation method in the indirect form. Hence, the MQ quasi-interpolation operator $L_{MQ}$ is used to approximate the highest derivative of the solution function that is described in Section 2. This method is as follows.

Let $u(x)$ be an unknown function defined on a bounded domain $[a, b] \subset \mathbb{R}$. Then

$$\frac{\partial u}{\partial x} = \sum_{i=0}^{N} \frac{\partial u}{\partial x}(x_i) \phi_i(x),$$  \hspace{1cm} (10)

and

$$u(x) = \sum_{i=0}^{N} \frac{\partial u}{\partial x}(x_i) \int \hat{\phi}_i(x) \, dx + C_1.$$  \hspace{1cm} (11)

Equations (10) and (11) can be rewritten in the compact forms as follows:

$$\frac{\partial u}{\partial x} = \sum_{i=0}^{N+1} w_i \phi_i(x),$$  \hspace{1cm} (12)

and

$$u(x) = \sum_{i=0}^{N+1} w_i \phi_i(x),$$  \hspace{1cm} (13)

where

$$\phi_i(x) = \hat{\phi}_i(x), \quad 0 \leq i \leq N, \quad \phi_{N+1}(x) = 0,$$

$$\hat{\phi}_i(x) = \int \hat{\phi}_i(x) \, dx, \quad 0 \leq i \leq N, \quad \hat{\phi}_{N+1}(x) = 1,$$

and

$$w_i = \frac{\partial u}{\partial x}(x_i), \quad 0 \leq i \leq N, \quad w_{N+1} = C_1.$$  \hspace{1cm} (14)

4. The numerical method on the nonlinear third-order KdV equation

Consider the nonlinear third-order KdV equation:

$$\frac{\partial u(x, t)}{\partial t} + \varepsilon u(x, t) \frac{\partial u(x, t)}{\partial x} + \mu \frac{\partial^3 u(x, t)}{\partial x^3} = 0,$$  \hspace{1cm} (15)

with the initial condition

$$u(x, t) = u_0(x), \quad t = 0,$$  \hspace{1cm} (16)

and the boundary conditions

$$u(x, t) = f(t), \quad x \in \partial \Omega, \quad t > 0,$$  \hspace{1cm} (17)

$$u_a(t) = g(t), \quad t > 0,$$  \hspace{1cm} (18)

where $\Omega = [a, b] \subset \mathbb{R}$, $\varepsilon$ and $\mu$ are positive parameters, $u_0(x), f(t)$ and $g(t)$ are known functions.

At first, Equation (15) is discretized in time with time step $\Delta t$ by using Taylors series expansion. In this approach, the term $u^n_i = u_i(x,t_n)$ can be arranged with the help of Taylors series expansion as follows:

$$u^n_i = \frac{u^{n+1}_i - u^n_i}{\Delta t} - \frac{\Delta t}{2} u_{xx}^n + O(\Delta t^2).$$  \hspace{1cm} (19)

Differentiating Equation (15) with respect to time, $u^n_i$ may be written as follows:

$$u^n_i = (\varepsilon u^n_i u^n_i - \mu u^n_{xx}),$$

$$= -\varepsilon u^n_i u^n_i - \varepsilon u^n_i (u^n_{xx})_x - \mu (u^n_{xxx}).$$  \hspace{1cm} (20)

Using the forward difference formula for the time derivative $u^n_i$ in Equation (20), $u^n_i$ can be rewritten as

$$\Delta t u^n_i = -\varepsilon (u^n_i (u^{n+1}_i - u^n_i) - \varepsilon u^n_i (u^{n+1}_{xx} - u^n_{xx}) - \mu (u^n_{xxx} - u^n_{xx}),$$  \hspace{1cm} (21)

Substituting Equation (21) into Equation (19) and using the resulted expression in Equation (15) lead to the following time discretized form of the KdV equation

$$u^{n+1}_i + \frac{\Delta t}{2} (u^n_i u^{n+1}_i + u^n_i u^{n+1}_i) + \frac{\Delta t}{2} u^n_{xx}$$

$$= u^n - \mu \Delta t u^n_{xx},$$  \hspace{1cm} (22)

Now, we apply the collocation method by using Equations (12) and (13). Also, we use the second-order central divided difference of the first-order spatial derivative to approximate the third-order spatial derivative

$$(u_{xxx})^n_i = \frac{2(u^n_{x+1})_i - 2(u^n_{x})_i + (u^n_{x-1})_i}{(x_{i+1} - x_i)(x_{i+1} - x_{i-1})},$$

$$(u^{n+1}_{xxx})_i = \frac{2(u^{n+1}_{x+1})_i - 2(u^{n+1}_{x})_i + (u^{n+1}_{x-1})_i}{(x_{i+1} - x_i)(x_{i+1} - x_{i-1})},$$  \hspace{1cm} (23)

Therefore, we obtain the following equations:

$$\sum_{j=0}^{N+1} w^{n+1}_j \hat{\psi}_j(x_i) + \frac{\Delta t}{2} \sum_{j=0}^{N+1} w^n_j \hat{\psi}_j(x_i) \sum_{j=0}^{N+1} w^{n+1}_j \hat{\psi}_j(x_i)$$

$$+ \sum_{j=0}^{N+1} w^n_j \hat{\psi}_j(x_i) \sum_{j=0}^{N+1} w^{n+1}_j \hat{\psi}_j(x_i)$$

$$+ \frac{\mu \Delta t}{2} \left( \alpha_i \sum_{j=0}^{N+1} w^{n+1}_j \hat{\psi}_j(x_{i-1}) - \beta_i \sum_{j=0}^{N+1} w^{n+1}_j \hat{\psi}_j(x_{i+1}) + \gamma_i \sum_{j=0}^{N+1} w^{n+1}_j \hat{\psi}_j(x_{i-1}) \right)$$

$$+ \sum_{j=0}^{N+1} w^{n+1}_j \phi_j(x_{i+1}) - \beta_i \sum_{j=0}^{N+1} w^{n+1}_j \phi_j(x_{i+1})$$

$$+ \gamma_i \sum_{j=0}^{N+1} w^{n+1}_j \phi_j(x_{i-1}) \right)$$
\[
\begin{align*}
&\sum_{j=0}^{N+1} w_j^p \tilde{\phi}_j(x_i) - \frac{\mu \Delta t}{2} \left( \sum_{j=0}^{N+1} \alpha_j \sum_{j=0}^{N+1} w_j^p \phi_j(x_{i+1}) \right) \\
&- \beta_i \sum_{j=0}^{N+1} w_j^p \psi_j(x_i) + \gamma_i \sum_{j=0}^{N+1} w_j^p \phi_j(x_{i-1}) \right), \\
&\text{for } 1 \leq j \leq N - 1 \text{ and } \sum_{j=0}^{N+1} w_j^{p+1} \tilde{\phi}_j(x_i) = f^{p+1}, \quad i = 0, N, \\
&\sum_{j=0}^{N+1} w_j^{p+1} \psi_j(x_N) = g^{p+1},
\end{align*}
\]

for \( i = 0, N, N + 1; j = 0, 1, \ldots, N + 1 \) and
\[
A_{b(1)(j+1)} = \tilde{\phi}_j(x_0), \quad A_{b(N+1)(j+1)} = \tilde{\phi}_j(x_N),
\]
\[
A_{b(N+2)(j+1)} = \psi_j(x_N),
\]
and
\[
G^{p+1} = [f^{p+1}(x_0), 0, \ldots, 0, f^{p+1}(x_N), g^{p+1}]^T.
\]

Subsequently, Equation (27) can be written as
\[
w^{p+1} = M^{-1}Nw^n + M^{-1}G^{p+1},
\]
where
\[
M = A + \frac{\varepsilon \Delta t}{2} (u^n * D + u^n_1 * A_d)
\]
\[
+ \frac{\mu \Delta t}{2} (\alpha * D_1 - \beta * D_2 + \gamma * D_3),
\]
\[
N = A_d - \frac{\mu \Delta t}{2} (\alpha * D_1 - \beta * D_2 + \gamma * D_3).
\]

and \( A = A_d + A_b \). From Equations (13) and (26), it can be written as
\[
\tilde{u}'' = Aw^n,
\]
where \( \tilde{u}'' = [u''(x_0), \ldots, u''(x_N), g'']^T \). Hence, the combination of Equations (29) and (30) is given as
\[
\tilde{u}''^{p+1} = AM^{-1}NA^{-1}u'' + AM^{-1}G^{p+1}.
\]

It is evident that to have a unique solution for the unknown vector \( u''^{p+1} \) in each time step, the matrices \( M \) and \( A \) must be non-singular and invertible. The non-singularity of these matrices depends on the situation of RBFs and cannot be proven in general. For many of the RBFs, the interpolation matrices are non-singular or specially positive-definite matrices. Notice that in many of the practical problems the singularities are rare. In the mentioned mesh-free method, the accuracy and stability of the method depend on the shape parameters \( c \) and \( s \) and also on the distance between any two collocation points. The effects of this parameter on stability of the method will be investigated in the next section.

5. The stability analysis

In this section, the stability analysis from integrated quasi-interpolation scheme is presented by using spectral radius of the amplification matrix similar to work that ul-Islam, Haq, and Uddin (2009) did. Let \( u \) be the exact, \( u' \) the numerical solution and \( e''^{p+1} = u''^{p+1} - u''^{p+1} \) the error vector of Equation (1). Also, let \( \tilde{e} = [e_0, \ldots, e_N, 0] \), then it can be written as
\[
\tilde{e}''^{p+1} = \tilde{u}''^{p+1} - \tilde{u}''^{p+1} = AM^{-1}NA^{-1} \tilde{e}''^{n} = E \tilde{e}''^{n},
\]
where \( E = AM^{-1}NA^{-1} \). For the stability of the numerical scheme, we must have \( e'' \rightarrow 0 \) as \( n \rightarrow \infty \), that is,
\(\rho(\mathbf{E}) < 1\) which is the necessary and sufficient condition for the numerical scheme to be stable, where \(\rho(\mathbf{E})\) denotes the spectral radius of the amplification matrix \(\mathbf{E}\). Equation (32) can be written as

\[
\mathbf{M} \mathbf{A}^{-1} \hat{\mathbf{e}}^{n+1} = \mathbf{N} \mathbf{A}^{-1} \hat{\mathbf{e}}^n,
\]

Equation (33) can be written in the following form by using the values of \(\mathbf{M}\) and \(\mathbf{N}\) defined in Equation (29):

\[
\left[ \mathbf{I} + \left( \frac{\Delta t}{2} \right) \mathbf{R}_1 \right] \hat{\mathbf{e}}^{n+1} = \left[ \mathbf{K} + \left( \frac{\Delta t}{2} \right) \mathbf{R}_2 \right] \hat{\mathbf{e}}^n,
\]

where

\[
\mathbf{R}_1 = \left[ \varepsilon (\mathbf{u}^n \ast \mathbf{D} + \mathbf{u}_e^n \ast \mathbf{A}_d) + \mu (\alpha \ast \mathbf{D}_1 - \beta \ast \mathbf{D}_2 + \gamma \ast \mathbf{D}_3) \right] \mathbf{A}^{-1},
\]

\[
\mathbf{R}_2 = -\mu (\alpha \ast \mathbf{D}_1 - 2 \beta \ast \mathbf{D}_2 + \gamma \ast \mathbf{D}_3) \mathbf{A}^{-1},
\]

and

\[
\mathbf{K} = \mathbf{A}^{-1} \mathbf{A}_d.
\]

The condition of stability will be satisfied if the maximum eigenvalue of the matrix \(\mathbf{E} = \left[ \mathbf{I} + (\Delta t/2) \mathbf{R}_1 \right] \left[ \mathbf{K} + (\Delta t/2) \mathbf{R}_2 \right]\) is less than unity, that is,

\[
\frac{\eta_K + (\Delta t/2) \eta_{R_2}}{1 + (\Delta t/2) \eta_{R_1}} \leq 1.
\]

where \(\eta_{R_1}, \eta_{R_2}\) and \(\eta_K\) denote the eigenvalues of the matrices \(\mathbf{R}_1, \mathbf{R}_2\) and \(\mathbf{K}\), respectively.

It is clear from Equation (34) that the stability of the method depends on the time step \(\Delta t\) and eigenvalues of the matrices \(\eta_{R_1}, \eta_{R_2}\) and \(\eta_K\). The condition numbers and magnitude of the eigenvalues of the matrices \(\mathbf{R}_1, \mathbf{R}_2\) and \(\mathbf{K}\) depend on the shape parameter and the number of collocation points. Hence, the condition number and the spectral radius of the matrix \(\mathbf{E}\) are dependent on the shape parameter and the number of collocation points. It is shown that when the shape parameter \(c\) is very large, the RBFs system error is of exponential order. But there is a certain limit for the value of \(c\) after which the solution breaks down. For the limiting value of \(c\) , the condition number of the RBFs system becomes so large that the system leads to ill-conditioning. In case of an ill-conditioned system, the spectral radius of the matrix \(\mathbf{E}\) becomes bigger than 1 so the numerical solution thus produced is not stable. Since it is not possible to find an explicit relationship among the spectral radius of the matrix \(\mathbf{E}\) and the shape parameter, this dependency is approximated numerically by keeping the number of collocation points fixed.

6. The numerical experiments

Three experiments are studied to investigate the robustness and the accuracy of the proposed method. We compare the numerical results of the KdV equation by using this scheme with the analytical solutions and solutions in Xiao et al. (2011) and Siraj et al. (2008). These methods include the MQ quasi-interpolation method (MQQI) (Xiao et al., 2011) and RBFs methods (MQ, IMQ) (Siraj et al., 2008). We denote our scheme by IMQQI. The \(L_2, L_\infty\) and root-mean-square (RMS) error norms defined by

\[
L_2 = \| u - u^* \|_2 = \sum_{j=0}^{N} (u(x_j) - u^*(x_j))^2 / (N + 1),
\]

\[
L_\infty = \| u - u^* \|_\infty = \max_{0 \leq j \leq N} |u(x_j) - u^*(x_j)|,
\]

\[
\text{RMS} = \sqrt{ \frac{1}{N} \sum_{j=0}^{N} (u(x_j) - u^*(x_j))^2 },
\]

are used to measure the accuracy of our scheme. Also, the stability analysis of the methods is considered for the first experiment. In all experiments, the shape parameter \(s\) is considered fourfold of the shape parameter \(c\). Moreover, the centers and the collocation points have been chosen as the same and equidistant. The rate of convergence in space is calculated by using the following formulae:

\[
\log(\| u - u_{h_1}^* \|/\| u - u_{h_2}^* \|) / \log(h_1/h_2),
\]

\[
\log(\| u - u_{t_1}^* \|/\| u - u_{t_2}^* \|) / \log(t_1/t_2),
\]

whereas \(u_{h_1}^*\) and \(u_{t_1}^*\) are the numerical solutions with the spatial step size \(h_1\) and time step size \(t_1\), respectively.

We perform the computations associated with our experiments in Maple 16 on a PC with a CPU of 2.4 GHZ.

**Experiment 1** Propagation of the single solitary wave (Dehghan & Shokri, 2007). In this experiment, we consider a single solitary wave propagation of nonlinear third-order KdV Equation (15) with \(\varepsilon = 6, \mu = 1\) and \([a, b] = [0, 40]\). The initial condition at \(t = 0\) is given by

\[
u_0(x) = \frac{r}{2} \text{sech}^2 \left( \frac{\sqrt{r}}{2} x - 7 \right), \quad r = 0.5.
\]

The exact solution is given by

\[
u(x, t) = \frac{r}{2} \text{sech}^2 \left( \frac{\sqrt{r}}{2} (x - rt) - 7 \right), \quad r = 0.5.
\]

The boundary functions \(f(t)\) and \(g(t)\) are extracted from the exact solution. The \(L_\infty, L_2\) and RMS errors are listed in Table 1 at different times by taking time step \(\Delta t = 0.01\) and 0.1 with \(N = 200\) and \(c = 4.075 \times 10^{-3}\). It can be observed from Table 2 that the \(L_2, L_\infty\) and RMS error norms increase slightly by increasing time step \(\Delta t\). Also, the \(L_\infty\) error is compared with the results in Xiao et al. (2011) and Siraj et al. (2008) in Table 2 and the numerical solution is compared with the exact solution and the result in Xiao et al. (2011) in Table 3 with
Table 1. $L_\infty$, $L_2$ and RMS error norms at different times with $\Delta t = 0.01$, $\Delta t = 0.1$, $N = 200$ and $c = 4.075 \times 10^{-3}$ of Experiment 1.

| $t$ | $\Delta t = 0.01$ | $\Delta t = 0.1$ |
|-----|------------------|------------------|
|     | $L_\infty$ | $L_2$ | RMS | $L_\infty$ | $L_2$ | RMS |
| 1 | $1.6728 \times 10^{-4}$ | $5.9993 \times 10^{-4}$ | $4.2316 \times 10^{-5}$ | $1.7111 \times 10^{-4}$ | $6.2137 \times 10^{-4}$ | $4.3828 \times 10^{-5}$ |
| 2 | $2.3758 \times 10^{-4}$ | $9.2242 \times 10^{-4}$ | $6.5063 \times 10^{-5}$ | $2.3897 \times 10^{-4}$ | $9.4334 \times 10^{-4}$ | $6.6538 \times 10^{-5}$ |
| 3 | $2.3758 \times 10^{-4}$ | $1.1337 \times 10^{-3}$ | $7.9697 \times 10^{-5}$ | $2.7857 \times 10^{-4}$ | $1.1457 \times 10^{-3}$ | $8.0811 \times 10^{-5}$ |
| 4 | $3.1384 \times 10^{-4}$ | $1.2931 \times 10^{-3}$ | $9.1213 \times 10^{-5}$ | $3.0797 \times 10^{-4}$ | $1.2920 \times 10^{-3}$ | $9.1133 \times 10^{-5}$ |
| 5 | $3.4136 \times 10^{-4}$ | $1.4216 \times 10^{-3}$ | $1.0027 \times 10^{-5}$ | $3.3183 \times 10^{-4}$ | $1.4046 \times 10^{-3}$ | $9.9075 \times 10^{-5}$ |

Table 2. The comparison of $L_\infty$ error between the numerical solution using our method and the solutions in Xiao et al. (2011) and Siraj et al. (2008) with $N = 200$ and $c = 4.075 \times 10^{-3}$ of Experiment 1.

| $t$ | $\Delta t = 0.1$ | $\Delta t = 0.001$ |
|-----|-----------------|-----------------|
|     | IMQI | MQQI | MQ | IMQ |
| 1   | $1.71116 \times 10^{-4}$ | $1.5259 \times 10^{-3}$ | $1.7923 \times 10^{-5}$ | $6.9584 \times 10^{-5}$ |
| 2   | $2.38973 \times 10^{-4}$ | $2.8672 \times 10^{-3}$ | $3.0151 \times 10^{-5}$ | $1.9553 \times 10^{-4}$ |
| 3   | $2.78578 \times 10^{-4}$ | $4.1428 \times 10^{-3}$ | $3.9839 \times 10^{-5}$ | $3.8286 \times 10^{-3}$ |
| 4   | $3.07970 \times 10^{-4}$ | $5.3859 \times 10^{-3}$ | $4.7835 \times 10^{-5}$ | $5.9098 \times 10^{-3}$ |
| 5   | $3.31838 \times 10^{-4}$ | $6.8141 \times 10^{-3}$ | $5.4599 \times 10^{-5}$ | $8.3667 \times 10^{-3}$ |

Table 3. The comparison between the numerical solutions using our method, the MQQI method and the exact solution with $\Delta t = 0.1$, $N = 200$ and $c = 4.075 \times 10^{-3}$ of Experiment 1.

| $x$ | $t = 1$ | $t = 3$ | $t = 5$ |
|-----|--------|--------|--------|
|     | IMQI | MQQI | Exact | IMQI | MQQI | Exact | IMQI | MQQI | Exact |
| 17  | 0.080645 | 0.081817 | 0.080625 | 0.043585 | 0.045284 | 0.043537 | 0.022560 | 0.024489 | 0.022515 |
| 18  | 0.137257 | 0.137899 | 0.137393 | 0.080469 | 0.082636 | 0.080625 | 0.043493 | 0.045640 | 0.043573 |
| 19  | 0.203742 | 0.204288 | 0.203886 | 0.137116 | 0.139860 | 0.137393 | 0.080386 | 0.084426 | 0.080625 |
| 20  | 0.247270 | 0.247960 | 0.247227 | 0.203713 | 0.206787 | 0.203886 | 0.137061 | 0.142601 | 0.137393 |
| 21  | 0.253375 | 0.253194 | 0.253251 | 0.247339 | 0.249011 | 0.247227 | 0.203699 | 0.208712 | 0.203886 |
| 22  | 0.177661 | 0.176306 | 0.177627 | 0.235452 | 0.233662 | 0.235251 | 0.247364 | 0.249883 | 0.247227 |
| 23  | 0.112325 | 0.110928 | 0.112353 | 0.177692 | 0.173616 | 0.177627 | 0.235492 | 0.231660 | 0.235251 |
| 24  | 0.063395 | 0.062591 | 0.063421 | 0.112322 | 0.108716 | 0.112353 | 0.177721 | 0.171589 | 0.177627 |
| 25  | 0.033535 | 0.033223 | 0.033545 | 0.063384 | 0.061304 | 0.063420 | 0.112335 | 0.106221 | 0.112353 |

$N = 200$ and $c = 4.075 \times 10^{-3}$ at $t = 1, 3$ and 5. In Xiao et al. (2011) and Siraj et al. (2008), time step $\Delta t = 0.001$ was used, whereas in IMQI $\Delta t = 0.1$ is used. Table 2 shows that the accuracy of the IMQI method is better than MQQI and IMQ methods when the time goes ahead. Moreover, Table 4 presents the errors and computation orders obtained using our scheme with $\Delta t = 0.1$ and different values of $N$ for $t = 2$. It can be concluded from Table 4 that the convergence rate decreases with the smaller spatial step size, but the error norms decrease slightly by increasing $N$. Table 5 shows the time rate of convergence obtained for Experiment 1 with $N = 100$, $c = 8.15 \times 10^{-3}$ and different values of $\Delta t$. It can be noted from Table 5 that the rate of convergence decreases with a smaller time step size. We also plot the graphs of absolute error and the estimated and analytical functions at $t = 5$ in Figure 1.

Table 4. The spatial rate of convergence at $t = 2$ with $\Delta t = 0.1$ of Experiment 1.

| $N$ | $L_\infty$ | Order | $L_2$ | Order |
|-----|-----------|-------|-------|-------|
| 40  | $1.72007 \times 10^{-3}$ | -- | $1.10138 \times 10^{-2}$ | -- |
| 80  | $1.48476 \times 10^{-3}$ | $2.15182$ | $3.70414 \times 10^{-3}$ | $1.57211$ |
| 120 | $6.59904 \times 10^{-4}$ | $1.99997$ | $2.00839 \times 10^{-3}$ | $1.50967$ |
| 160 | $3.73295 \times 10^{-4}$ | $1.98039$ | $1.30908 \times 10^{-3}$ | $1.48778$ |
| 200 | $2.38973 \times 10^{-4}$ | $1.99881$ | $9.43347 \times 10^{-4}$ | $1.46831$ |
Table 5. The time rate of convergence at $t = 2$ with $N = 100$ of Experiment 1.

| $\Delta t$ | $L_\infty$        | Order | $L_2$           | Order |
|----------|------------------|-------|-----------------|-------|
| 1        | $1.77413 \times 10^{-3}$ | –     | $1.10138 \times 10^{-2}$ | –     |
| 0.5      | $1.00571 \times 10^{-3}$ | 0.81890 | $3.70414 \times 10^{-3}$ | 0.80321 |
| 0.25     | $9.70341 \times 10^{-4}$ | 0.05164 | $2.00839 \times 10^{-3}$ | 0.19195 |
| 0.125    | $9.63649 \times 10^{-4}$ | 0.00998 | $1.30908 \times 10^{-3}$ | 0.04014 |
| 0.0625   | $9.61999 \times 10^{-4}$ | 0.00247 | $4.83991 \times 10^{-4}$ | 0.00944 |

Figure 1. Absolute error (a) and the analytical and estimated functions (b) at $t = 5$, with $\Delta t = 0.1$ and $N = 200$, of Experiment 1.

Table 6. The spectral radius and $L_\infty$ and $L_2$ error norms versus shape parameter $c$ when $\Delta t = 0.01$ and $N = 100$ at $t = 5$ of Experiment 1.

| $c$       | $\rho(E)$       | $L_\infty$        | $L_2$           | RMS           |
|----------|-----------------|------------------|-----------------|---------------|
| 0.0001   | 0.99994         | 2.15966 $\times 10^{-3}$ | 6.49144 $\times 10^{-3}$ | 6.45923 $\times 10^{-4}$ |
| 0.001    | 0.99994         | 2.14819 $\times 10^{-3}$ | 4.50147 $\times 10^{-3}$ | 6.41813 $\times 10^{-4}$ |
| 0.01     | 0.99994         | 2.04674 $\times 10^{-3}$ | 6.08820 $\times 10^{-3}$ | 6.05799 $\times 10^{-4}$ |
| 0.05     | 0.99994         | 1.70590 $\times 10^{-3}$ | 4.95110 $\times 10^{-3}$ | 4.92652 $\times 10^{-4}$ |
| 0.10     | 0.99994         | 1.50597 $\times 10^{-3}$ | 4.37502 $\times 10^{-3}$ | 4.35331 $\times 10^{-4}$ |
| 0.50     | 0.99994         | 1.37120 $\times 10^{-3}$ | 4.04434 $\times 10^{-3}$ | 4.02427 $\times 10^{-4}$ |
| 1.00     | 0.99994         | 1.37095 $\times 10^{-3}$ | 4.04385 $\times 10^{-3}$ | 4.02378 $\times 10^{-4}$ |
| 1.20     | 0.99994         | 1.37095 $\times 10^{-3}$ | 4.04384 $\times 10^{-3}$ | 4.02377 $\times 10^{-4}$ |
| 1.30     | 1.01481         | 1.37095 $\times 10^{-3}$ | 4.04383 $\times 10^{-3}$ | 4.02376 $\times 10^{-4}$ |
| 1.40     | 1.06072         | 1.37096 $\times 10^{-3}$ | 4.04381 $\times 10^{-3}$ | 4.02374 $\times 10^{-4}$ |
| 1.50     | 5.33193         | 2.16534 $\times 10^{0}$ | 5.65118 $\times 10^{0}$ | 5.62314 $\times 10^{-1}$ |

The relation between the spectral radius of the matrices $E$ and the different values of the shape parameter $c$ is shown in Table 6 by keeping the number collocation points fixed. It can be seen from Table 6 that the accuracy of the IMQQI method is reasonably good by keeping the shape parameter $c$ in the interval $(0, 1.4)$. But, it is clear that if the values of shape parameter $c$ are greater than the critical value $c = 1.2$, then the spectral radius of the matrix $E$ becomes bigger than 1 and hence the IMQQI method becomes unstable. Therefore, the interval stability of the IMQQI scheme is $(0, 1.2)$.

Experiment 2 Propagation of two solitary waves (Dehghan & Shokri, 2007). In this experiment, we consider two solitary waves propagation of nonlinear third-order
KdV Equation (15) with $\varepsilon = 6$ and $\mu = 1$. The initial condition at $t = 0$ is given by

$$u_0(x) = 12 \frac{\{3 + 4 \cosh(2x) + \cosh(4x)\}}{\{3 \cosh(x) + \cosh(3x)\}^2}.$$

The exact solution is given by

$$u(x, t) = 12 \frac{\{3 + 4 \cosh(2x - 8t) + \cosh(4x - 64t)\}}{\{3 \cosh(x - 28t) + \cosh(3x - 36t)\}^2},$$

and the boundary functions $f(t)$ and $g(t)$ can be obtained from the exact solution. In this experiment, we consider $\Delta t = 0.0001, 0.001$, $N = 200$ and $[a, b] = [-5, 15]$. We compare the numerical solutions using our scheme and the MQQI scheme (Xiao et al., 2011) with the exact solution in Table 7 and compare the $L_\infty$-error with the results in Xiao et al. (2011) and Siraj et al. (2008) in Table 8 for $t = 0.01, 0.05$ and 0.10. In Xiao et al. (2011) and Siraj et al. (2008), $\Delta t = 0.00001$ was used but we use $\Delta t = 0.001$. It is observable that the proposed method is more accurate in comparison with MQQI and IMQ methods.

The norms $L_\infty, L_2$ and RMS of errors are obtained in Table 9 for $t = 0.01, 0.05, 0.10, 0.15$ and 0.20. The graphs of analytical and estimated functions at $t = 0.1$ and 0.2 and their absolute error are given in Figure 2.

Table 7. The comparison between the numerical solutions using our method, the MQQI method and the exact solution with $\Delta t = 0.001$ and $N = 200$ of Experiment 2.

| $t$  | IMQQI | MQQI | Exact | IMQQI | MQQI | Exact | IMQQI | MQQI | Exact |
|------|-------|------|-------|-------|------|-------|-------|------|-------|
| 0    | 5.634246 | 5.640733 | 5.628245 | 2.593411 | 2.597785 | 2.574829 | 2.030865 | 1.990040 | 2.000572 |
| 1    | 3.192235 | 3.186663 | 3.192964 | 6.893042 | 6.922214 | 6.881609 | 1.700870 | 1.764999 | 1.717101 |
| 2    | 0.478502 | 0.478633 | 0.478495 | 1.204642 | 1.191924 | 1.207024 | 7.230998 | 7.139714 | 7.171392 |
| 3    | 0.064535 | 0.064570 | 0.064520 | 0.101147 | 0.101458 | 0.100955 | 0.465253 | 0.452621 | 0.464299 |
| 4    | 0.008725 | 0.008730 | 0.008723 | 0.012259 | 0.012227 | 0.012239 | 0.024487 | 0.024443 | 0.024308 |
| 5    | 0.001181 | 0.001181 | 0.001180 | 0.001632 | 0.001559 | 0.001630 | 0.002553 | 0.002261 | 0.002542 |

Table 8. The comparison of $L_\infty$-error between the numerical solution using our method and the solutions in Xiao et al. (2011) and Siraj et al. (2008) of Experiment 2.

| $\Delta t$ | IMQQI | MQQI | MQ–RBF | IMQ–RBF |
|------------|-------|------|--------|--------|
| 0.01       | $3.99855 \times 10^{-3}$ | $7.7405 \times 10^{-3}$ | $9.2114 \times 10^{-4}$ | $2.2071 \times 10^{-2}$ |
| 0.05       | $3.98427 \times 10^{-2}$ | $6.3762 \times 10^{-2}$ | $2.9608 \times 10^{-3}$ | $7.2316 \times 10^{-2}$ |
| 0.10       | $8.79034 \times 10^{-2}$ | $1.6196 \times 10^{-1}$ | $1.2806 \times 10^{-2}$ | $1.0121 \times 10^{-1}$ |

Figure 2. Analytical and estimated functions at $t = 0.1, 0.2$ with $\Delta t = 0.001$ and $N = 200$ of Experiment 2.
Table 9. $L_\infty, L_2$ and RMS error norms at different times with $\Delta t = 0.0001, 0.001$ and $N = 200$ of Experiment 2.

| t    | $L_\infty$ | $L_2$ | RMS     | $L_\infty$ | $L_2$ | RMS     |
|------|-------------|-------|---------|-------------|-------|---------|
| 0.01 | $4.0579 \times 10^{-3}$ | $9.105 \times 10^{-3}$ | $6.9903 \times 10^{-4}$ | $3.9985 \times 10^{-3}$ | $9.7257 \times 10^{-3}$ | $6.8600 \times 10^{-4}$ |
| 0.05 | $4.1003 \times 10^{-2}$ | $1.0295 \times 10^{-1}$ | $7.2619 \times 10^{-3}$ | $3.9842 \times 10^{-2}$ | $1.072 \times 10^{-2}$ | $7.1048 \times 10^{-3}$ |
| 0.10 | $9.1691 \times 10^{-2}$ | $2.3373 \times 10^{-1}$ | $1.6486 \times 10^{-2}$ | $8.7903 \times 10^{-2}$ | $2.2546 \times 10^{-1}$ | $1.5903 \times 10^{-2}$ |
| 0.15 | $1.3257 \times 10^{-1}$ | $3.4201 \times 10^{-1}$ | $2.4124 \times 10^{-2}$ | $1.2588 \times 10^{-1}$ | $3.2553 \times 10^{-1}$ | $2.2961 \times 10^{-2}$ |
| 0.20 | $1.6644 \times 10^{-1}$ | $4.3607 \times 10^{-1}$ | $3.0758 \times 10^{-2}$ | $1.5682 \times 10^{-1}$ | $4.1073 \times 10^{-1}$ | $2.8970 \times 10^{-2}$ |

Table 10. The percentage error using different schemes at time $t = 0.005$ with $N = 160$ and $\Delta t = 0.001$ of Experiment 3.

| x    | IMQI | MQQI | MQ | FEM | ANS | HBIM |
|------|------|------|----|-----|-----|------|
| 0.1  | 0.0000 | 0.1082 | 0.0000 | 0.0156 | 0.0078 | 3.8033 |
| 0.2  | 0.0001 | 0.1019 | 0.0003 | 0.0107 | 0.0272 | 3.7984 |
| 0.3  | 0.0005 | 0.0597 | 0.0003 | 0.0105 | 0.0634 | 3.7243 |
| 0.4  | 0.0144 | 0.1095 | 0.0103 | 0.0104 | 0.1105 | 2.9326 |
| 0.5  | 0.0374 | 0.0691 | 0.0060 | 0.0076 | 0.0603 | 0.7865 |
| 0.6  | 0.0050 | 0.0821 | 0.0101 | 0.0104 | 0.0276 | 3.2960 |
| 0.7  | 0.0003 | 0.0534 | 0.0015 | 0.0101 | 0.0223 | 6.6331 |
| 0.8  | 0.0000 | 0.0748 | 0.0007 | 0.0096 | 0.0470 | 3.6626 |
| 0.9  | 0.0000 | 0.0767 | 0.0000 | 0.0088 | 0.0442 | 3.6656 |
| 1.0  | 0.0000 | 0.0769 | 0.0000 | 0.0000 | 0.1067 | 3.7353 |

Figure 3. Absolute error (a) and the analytical and estimated functions (b) at $t = 1s$ with $\Delta t = 0.001$ and $N = 200$ of Experiment 3.

Similar to Experiment 1, we observe that by an increase in $\Delta t$ from 0.0001 to 0.001, the $L_\infty, L_2$ and RMS error norms do not increase as $t$ moves ahead from 0.01 to 0.20, in the monitored time at Table 9.

Experiment 3 A special model problem of the KdV equation was investigated in Xiao et al. (2011), Siraj et al. (2008), Aksan and Özdés (2006), Özer and Kutluay (2005) and Kutluay, Bahadir, and Özer (2000). Consider KdV Equation (15) with the initial condition

$$u_0(x) = 3C \, \text{sech}(Ax + D), \quad 0 \leq x \leq 2,$$

and boundary conditions

$$u(0, x) = u(2, x) = u_x(2, x) = 0, \quad t > 0.$$
The exact solution of this problem is taken from Alexander and Morris (1979) and is given by

\[ u(x, t) = 3C \sech^2(Ax - Bt + D), \quad 0 \leq x \leq 2, \]

where \( C, D \) are real constants, \( A = \frac{1}{2} \sqrt{\varepsilon C / \mu} \) and \( B = \varepsilon AC \).

The new method based on the MQ quasi-interpolation scheme is applied to Experiment 3 and the results of percent error are compared with those given in Xiao et al. (2011), Siraj et al. (2008), Aksan & Özdes (2006), Özer and Kutluay (2005) and Kutluay et al. (2000) in Table 10. These methods include the MQQI method (Xiao et al., 2011), RBF(MQ) method (Siraj et al., 2008), finite element method (FEM) (Aksan and Özdes, 2006), analytical–numerical solution (ANS) (Özer & Kutluay, 2005) and HBIM (Kutluay et al., 2000). In this experiment, we consider \( \varepsilon = 1, \mu = 4.84 \times 10^{-4}, C = 0.3, D = -6, N = 160 \) and \( \Delta t = 0.001 \). Moreover, we plot the graphs of absolute error and estimated and analytical functions at \( t = 1 \) s in Figure 3.

7. Conclusion

In this paper, we have presented a new numerical scheme based on the high accuracy MQ quasi-interpolation scheme and IRBF approximation scheme for solving the nonlinear third-order KdV equation.

The numerical results which are given in the previous section demonstrate the good accuracy of the present scheme. Also, the tables show that this scheme performs better than the MQQI method and IMQ–RBF method in more cases and Table 8 shows that it even performs better than the MQ–RBF method. Moreover, we have used a bigger time step \( \Delta t \), in comparison with Xiao et al. (2011) and Siraj et al. (2008).

The difficulty involved in using this approach is solving a system of algebraic equations at each time step which the dimension of the coefficient matrix is equal to the number of interpolant points and finding the appropriate value for the shape parameters \( c \) and \( s \).

Thereewith, we would like to emphasize that the scheme introduced in this paper can be well studied for any other nonlinear PDEs.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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