A Phenomenological Model of the Growth of Two–Species Atomic Bose–Einstein Condensates

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Abstract. We introduce a phenomenological mean-field model to describe the growth of immiscible two-species atomic Bose–Einstein condensates towards some equilibrium. Our model is based on the coupled Gross–Pitaevskii equations with the addition of dissipative terms to account for growth. While our model may be applied generally, we take a recent Rb-Cs experiment [McCarron et al., Phys. Rev. A 84 011603(R) (2011)] as a case study. As the condensates grow, they can pass through ranging transient density structures which can be distinct from the equilibrium states, although such a model always predicts the predominance of one condensate species over longer evolution times.

1. Introduction
First achieved experimentally in 1997 [1], binary mixtures of Bose–Einstein condensates (BECs) have been a topic of intense research both experimentally and theoretically. Experimental realisations have been achieved with two hyperfine states of the same atomic species [1–11], different isotopes of the same atomic species [12] and with different atomic species [13–15, 73]. Many aspects of these mixtures have been studied theoretically, including steady state solutions [16–28], dark–bright solitons [30–34], vortices [6, 35–47] and finite temperature models [34, 48–56, 76, 78]. One of the defining aspects of these condensates is their ability to interact with each other. Depending on the relative strength of self–interactions (interactions between two particles of the same species) and inter–species interactions, the condensates can exhibit either miscible or immiscible behaviour [17]. This introduces new physics not present in the single species condensate, such as coupled stability criteria [62–66] and modulational instabilities [67–72]. A recent experimental study of a immiscible two–species condensate [73], involving the sympathetic cooling of one condensate by the other, suggests that the growth of the condensates towards equilibrium plays an important role in determining their structure [29].

Finite temperature models have been a topic of intense theoretical research in BECs [79–82] and here we give a brief summary of their application to condensate mixtures. The Hartree–Fock model [48–51] and Popov approximation [52] have been extended from one to two–species condensates in order to study the impact of thermal clouds on the density distributions in harmonic traps, while the Hartree–Fock–Bogoliubov–Popov approximation has been applied to a homogeneous two-species system [53]. C–field methods have also been used to model the finite-temperature behaviour of miscible condensates in periodic boxes [55, 56]. Simpler finite temperature models have been employed through the addition of a dissipative term to the coupled Gross-Pitaevskii equations such that the number of particles in the system is no longer conserved (and the thermal cloud is not itself physically modelled): this led to simulations of...
growth and ensuing modulational instability in Ref. [54] and the dissipative motion dark–bright solitons in Ref. [34].

The aim of this work is to investigate the coupled dynamics of a two-species condensate during its growth towards equilibrium through a phenomenological mean-field model based on the dissipative coupled Gross–Pitaevskii equations. Our focus is on the recent Rb–Cs mixture experiment [73]. There, three distinct density profiles were observed depending on relative condensed particle numbers in each species. We first describe the stationary mean-field solutions of the system, showing that the observed structures can be qualitatively reproduced with the inclusion of a weak additional linear potential to the harmonic trapping potentials [29]. Next we apply our phenomenological growth model to study the coupled dynamics during evolution of the system towards equilibrium, and the transient and final states obtained.

2. Theoretical Framework
2.1. Coupled Gross–Pitaevskii Equations
In the limit of zero temperature and weak interactions, a two-species BEC can be described through the mean-field model of the coupled Gross-Pitaevskii equations (CGPEs),

\[ i\hbar \frac{\partial \psi_1}{\partial t} = \left( -\frac{\hbar^2}{2m_1} \nabla^2 + V_1 + g_{11} |\psi_1|^2 + g_{12} |\psi_2|^2 - \mu_1 \right) \psi_1 \]  (1)

\[ i\hbar \frac{\partial \psi_2}{\partial t} = \left( -\frac{\hbar^2}{2m_2} \nabla^2 + V_2 + g_{22} |\psi_2|^2 + g_{12} |\psi_1|^2 - \mu_2 \right) \psi_2, \]  (2)

where \( \psi_j \) are the condensate wavefunctions, \( m_j \) the atomic masses, and \( g_{jk} \) the strength of interactions. The latter are related to the s-wave scattering lengths \( a_{jj} \) through \( g_{jj} = 4\pi \hbar^2 a_{jj}/m \) and \( g_{jk} = 2\pi \hbar^2 (m_1 + m_2) a_{12}/m_1 m_2 \). We only consider repulsive interactions throughout this study, i.e. \( g_{jj} > 0 \) and \( g_{jk} > 0 \).

The trapping potentials \( V_j \) are typically harmonic and cylindrically symmetric, with the form,

\[ V_j = \frac{1}{2} m_j \left[ \omega_{j,\perp}^2 (x^2 + y^2) + \omega_{j,z}^2 z^2 \right], \]  (3)

where \( \omega_{j,\perp} \) denotes the harmonic trap frequency in the perpendicular \((xy)\) direction and \( \omega_{j,z} \) is the harmonic trap frequency in the axial \((z)\) direction.

2.2. Phenomenological Growth Model
We describe the growth of the two-species BEC through the following dissipative coupled Gross–Pitaevskii equations,

\[ i\hbar \frac{\partial \psi_1}{\partial t} = (1 - i\gamma_1) \left( -\frac{\hbar^2}{2m_1} \nabla^2 + V_1 + g_{11} |\psi_1|^2 + g_{12} |\psi_2|^2 - \mu_1 \right) \psi_1 \]  (4)

\[ i\hbar \frac{\partial \psi_2}{\partial t} = (1 - i\gamma_2) \left( -\frac{\hbar^2}{2m_2} \nabla^2 + V_2 + g_{22} |\psi_2|^2 + g_{12} |\psi_1|^2 - \mu_2 \right) \psi_2. \]  (5)

The damping term \( \gamma_j \) is introduced here phenomenologically to qualitatively account for the dissipative role of finite temperature. It allows for the loss or gain of particles in the system, and the amplitude of \( \gamma_j \) determines the growth/decay rate of each species. For simplicity, we consider \( \gamma_1 = \gamma_2 \). The inclusion of such a term was originally proposed, using general arguments, by Pitaevskii [57, 58]. First implemented to trapped Bose gases in [59], the phenomenological damped GPE has been to study, e.g. vortex lattice formation [60,61] and dark soliton decay [75] in single species condensates, and dissipative dark–bright soliton dynamics in two–species condensates [34]. A constant value of \( \gamma_i \) is typically chosen phenomenologically such that the dissipative processes match those observed experimentally.

Because of the way that the dissipative terms appear in Eqs. 4 and 5, i.e. as a global factor to the right-hand side of the CGPEs, the dynamics evolve towards the equilibrium state of the
Figure 1. (a) Schematic showing the band in the \( N_{\text{Rb}} - N_{\text{Cs}} \) plane where the experimentally observed profiles [73] fell. The three coloured regions represent the regimes associated with different observed density structures (as described in the main text). (b) and (c) show representative steady state solutions of the CGPEs for each of the three structural regimes. These three cases correspond to (i) \( N_{\text{Rb}} = 840 \) and \( N_{\text{Cs}} = 8570 \), (ii) \( N_{\text{Rb}} = 3680 \) and \( N_{\text{Cs}} = 8510 \), and (iii) \( N_{\text{Rb}} = 15100 \) and \( N_{\text{Cs}} = 6470 \). (b) shows these solutions through integrated 1D density profiles and (c) through integrated 2D density profiles. Rb ((Solid) red curve) and Cs ((Dashed) blue curve)

system (set by the chemical potentials). Depending on the initial state, this may be either through decay or growth towards the equilibrium state (we are interested in the latter). A closely related model was introduced in [54], although there the dissipative terms \( i\hbar \gamma_j \) were introduced into the Hamiltonian. There, in contrast, they acted as gain terms, causing the (unlimited) exponential growth of both condensates.

To create growth in our simulations, we take as an initial condition for \( \psi_j \) the steady state solutions of the coupled Gross-Pitaevskii equations (1) and (2) with low particle numbers \( N_1 \) and \( N_2 \). The corresponding chemical potentials are \( \mu_1^{(0)} \) and \( \mu_2^{(0)} \). For \( t > 0 \), the chemical potentials are suddenly increased to higher values, i.e. \( \mu_i \geq \mu_i^{(0)} \). Since we are considering only repulsive interactions, a larger chemical potential corresponds to a larger atom number, and so the dissipative CGPEs evolve towards this new chemical potential through the growth of the system.

3. Case Study: the Durham Rb–Cs Experiment [73]

We direct our model towards the recent \(^{87}\text{Rb} - ^{133}\text{Cs}\) two-species condensate experiment reported in Ref. [73]. The intra– and inter–species s-wave scattering lengths for Rb (labelled 1) and Cs (labelled 2) are \( a_{11} = 100a_0 \), \( a_{22} = 280a_0 \) and \( a_{12} = 650a_0 \) where \( a_0 \) is the Bohr radius. Two-species condensates are immiscible for \( g_{12} > g_{11}g_{22} \) [16]; these scattering lengths well satisfy this criterion such that the Rb-Cs condensate is highly immiscible.

The condensates experience harmonic confinement through a magnetic trap. Due to the difference in dipole moments of the species, each species experiences different trap frequencies, \( \omega_{1(2),z} = 2\pi \times 3.89 \ (2\pi \times 4.55) \text{Hz} \) in the axial direction and \( \omega_{1(2),\perp} = 2\pi \times 32.2 \ (2\pi \times 40.2) \text{Hz} \) in the transverse directions.

In this experiment, following a period of sympathetic cooling between the species, the density profiles were imaged and the atom numbers measured. These final atom numbers varied from experimental run to run but fell consistently along a band region in the \( N_{\text{Rb}} - N_{\text{Cs}} \) plane, as depicted in Fig. 1 (a). Moreover, the integrated axial density profiles largely revealed structures, depending on where they fell on the band: for large \( N_{\text{Cs}} \) and small \( N_{\text{Rb}} \) (blue region), the Rb cloud sat in the middle of the Cs cloud; for intermediate \( N_{\text{Cs}} \) and \( N_{\text{Rb}} \) (purple region), the Rb and Cs clouds sat side-by-side; for small \( N_{\text{Cs}} \) and large \( N_{\text{Rb}} \) (red region), the Cs cloud sat in
the middle of the Rb cloud.

In previous work [29], we examined the stationary solutions of the system according to the CGPEs (1) and (2) and their comparison to the experimentally observed density profiles. Using bare harmonic trapping potentials, the three qualitative density structures outlined above could not be recovered by this approach. However, the experimental trapping potential also included small asymmetrical perturbations due to an applied magnetic tilt to enhance evaporative cooling and other effects such as gravitational sag and minute beam misalignments. To first order, these effects can be modelled by introducing a linear potential to the trapping potential in each direction for just one species (which we take to be Rb),

\[ V_1 = \frac{1}{2} m_1 \left[ \omega_{1,\perp(1)}^2 (x^2 + y^2) + \omega_{1,z}^2 z^2 \right] + \alpha_x x + \alpha_z z. \]  

Even small linear potentials were found to have a significant modification to the stationary solutions of the system and promote a greater range of structural forms. In particular, for linear potentials \( \alpha_z = 1.5(\hbar \omega_1/\ell_1) \) and \( \alpha_x = 0.02(\hbar \omega_1/\ell_1) \), where \( \omega_1 = (\omega_{1,\perp} \omega_{1,z})^{1/3} \) and \( \ell_1 = \sqrt{\hbar/m\omega_1} \), a good qualitative agreement with the experimentally observed density structures was recovered in all three regimes [29]. These values are consistent with the experimentally anticipated shifts in trapping potential. Example stationary solutions from each structural regime are presented in Fig. 1 through (b) integrated 1D density profiles and (c) integrated 2D density profiles. For (i) high \( N_{\text{Cs}} \) and low \( N_{\text{Rb}} \), the integrated axial density profile shows the Rb cloud to exist within the Cs cloud, as observed experimentally. The integrated 2D profile reveals that this structure arises from the condensates lying side-by-side in the \( x \)-direction. For (ii) intermediate \( N_{\text{Rb}} \) and \( N_{\text{Cs}} \) the condensates lie side by side in the axial direction, as observed experimentally. For (iii) low \( N_{\text{Cs}} \) and high \( N_{\text{Rb}} \), the integrated axial density reveals the Cs cloud to lie within the Rb cloud, again consistent with the experimentally observed structure. In 2D this appears as the Rb cloud enclosing the Cs cloud asymmetrically.

The above results highlight the importance of including the small linear potentials in modelling this experiment. As such, we include these additional linear potentials \( (\alpha_z = 1.5(\hbar \omega_1/\ell_1) \) and \( \alpha_x = 0.02(\hbar \omega_1/\ell_1) \)) in all subsequent results.

4. Numerical Results

4.1. Typical Growth Simulation

In Fig. 2 (a) and (b), we show a typical evolution of the condensate particle numbers in each species and the combined total particles against time based on our dissipative model of Eqs. (4) and (4). Here, we start with a large number of Cs atoms in comparison to Rb. Cs decreases until it vanishes from the system while the growth of Rb continues until a maximum is reached. The total number decreases drastically at the start but then grows again slowly over time. This suggests that a low critical number of Cs atoms is required before Rb can start to grow at a quicker pace. We also present four corresponding density profiles at \( t(\bar{\omega}) = 0, 16, 40 \) and 80. The initial density distribution is a symmetric density distribution where Rb and Cs sit side by side in a transverse direction. As the condensate particles numbers evolve over time, the condensate clouds become asymmetric until a side by side formation in the axial direction emerges. Finally, a one species condensate forms for long times as no Cs particles remain in the condensate.

4.2. Growth Trajectories Through \( N_{\text{Cs}}-N_{\text{Rb}} \) Plane

Figure 3 shows the trajectories of multiple condensate particle growth curves in the \( N_{\text{Rb}}-N_{\text{Cs}} \) plane. The regions indicated by the dotted black lines have been defined in accordance to the experimental results in [73] where the density profiles observed depend on condensate particle numbers. These experimental results shall be described in more detail in the following section.

The trajectory taken in each of these simulations is determined by the final chemical potentials rather than \( \gamma_i \). For constant values of \( \gamma_1 = \gamma_2 \), the order of magnitude of \( \gamma_i \) determines solely the timescale over which growth/decay occurs although as this is already sensitive competing dynamical growth process, unequal \( \gamma_i \)’s for the two species could largely
modify the dynamics [78]. In our simulations, we first note that all of the growth curves finish either on the horizontal or vertical axes i.e. with a one component condensate where the other species has vanished from the system. One component will always vanish from the system as the growth rates do not change over time, this feature is inherent in the model. Physically, this represents all of the particles being in the thermal cloud for that component (although this is only implicit within our simplified model). Nevertheless, this is still broadly consistent with the experimental findings as there were numerous unpublished results showing images where only one of the species was condensed [84]. The influence of the final chemical potentials on the path taken is clear when comparing the green and orange growth curves in Fig. 3 respectively labelled by D and E which start from the same steady state initial condition and evolve to different final single species condensates. The nonlinear dynamics and competing processes do not give us a direct handle on the precise trajectories in the $N_{\text{Rb}} - N_{\text{Cs}}$ plane although extensive simulations have enabled us to probe the most common types of trajectories, as discussed in detail below and shown in Fig. 3.

A number of our simulations (Figure 3 A, C and D) do not lead to any changes in the integrated density profiles while growth/decay occurs and each species occupies the same overall position until one vanishes (to the thermal cloud) leaving a single species condensate. In A, the initial density profile is asymmetric with Rb and Cs sit side by side in the axial direction. Once growth/decay begins, Rb vanishes rapidly from the system leaving a condensate with only Cs present. The growth curve for B have similar initial particle numbers and similar density profile changes during phenomenological growth. In this case, Rb decays while Cs grows in the centre of the trapping potentials. This growth of Cs splits the Rb into two distinct parts either side of Cs. Over longer time evolutions, Rb vanishes leaving a Cs condensate. For D, we start with a symmetric density profile and Rb will decay over time. The asymmetric intermediate density profiles in D are due to the asymmetry present in the initial steady state solution where, as Rb slowly decreases, the right peak of Rb vanishes before the left peak giving a side–by–side density profile. We obtain a Cs only condensate over longer time scales. In E, we start with

![Figure 2](image-url)
Figure 3. Condensate particle growth curves in $N_{\text{Rb}}$ (horizontal) and $N_{\text{Cs}}$ (vertical) plane. Initial (steady state), intermediate and final density distributions depicted by stars circles and crosses respectively. Dotted lines — boundaries of experimental regions. Each set of three density profiles corresponds to initial, intermediate and final 1D integrated density profiles for Rb ((Solid) red curve) and Cs ((Dashed) blue curve).

The same initial condition as D but choose different final chemical potentials for each of these simulations resulting in different growth curves. In the intermediate plot for the green growth curve E, we observe a dark–bright soliton which oscillates in the trapping potential until all the Cs bright component no longer exists in the system. Finally in B, we observe similar density evolution to D but over a different growth path in the $N_{\text{Rb}}$–$N_{\text{Cs}}$ plane.

5. Conclusions, Discussion
We have investigated the role of growth in a two–component immiscible system modelled via a phenomenologically damped coupled GPEs. Starting with a range of steady state density distributions, the addition of a non–zero damping term simulates growth. Perturbing the chemical potential of each condensate induces competing dynamical evolution with the number of particles in the system eventually leaving one of the condensates to vanish entirely, thus leaving only a single species condensate. In a few cases, we have seen the density profiles to change from a symmetric density profile to an asymmetric one during growth provided additional linear potentials are present.

We stress that the asymmetries caused by harmonic trap offsets between the two trap centres in the axial and transverse directions are small relative to the size of the condensate clouds which explains why asymmetric density profiles can be observed in two–species experiments with different atomic species as two magnetic traps are required which are extremely hard to align perfectly. Our work shows a large number of distinct density profiles are possible during the coupled condensates dynamics. When comparing to a recent two–species experiment [73] on which our parameters are based, we observed some qualitative agreement between condensate density profiles while considering growth. However not only is the model used here a toy model with static growth, there is no explicit consideration of the thermal cloud dynamics but moreover the coupled experimental results available did not analyse the non–equilibrium
structures in detail to be able to investigate a more detailed comparison.

These results can be improved by using more accurate models for finite–temperature non–equilibrium Bose gases [80]. In 1D, this system has already been looked at with the Stochastic Projected GPE [76]. However, even our simple model is enough to reveal that modelling the two transverse directions is essential in the distribution of the condensate densities. Work is currently under way between the group of [76] and ourselves, whilst there are also other efforts for modelling multicomponent condensates with the ZNG formalism [83] to look at this system in 3D with Stochastic Projected GPE.

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References
[1] Myatt C J, Burt E A, Ghrist R W, Cornell E A and Wieman C E 1997 Phys. Rev. Lett. 78(4) 586–589
[2] Hall D S, Matthews M R, Ensher J R, Wieman C E and Cornell E A 1998 Phys. Rev. Lett. 81(8) 1539–1542
[3] Matthews M R, Anderson B P, Haljan P C, Hall D S, Wieman C E and Cornell E A 1999 Phys. Rev. Lett. 83(13) 2498–2501
[4] Maddaloni P, Modugno M, Fort C, Minardi F and Inguscio M 2000 Phys. Rev. Lett. 85(12) 2413–2417
[5] Delannoy G, McDougle G, Boyer V, Josse V, Bouyer P and Aspect A 2001 Phys. Rev. A 63(5) 051602
[6] Schweikhard V, Coddington I, Engels P, Tung S and Cornell E A 2004 Phys. Rev. Lett. 93(21) 210403
[7] Mertes K M, Merrill J W, Carretero-González R, Frantzeskakis D J, Kevrekidis P G and Hall D S 2007 Phys. Rev. Lett. 99(19) 190402
[8] Anderson R P, Ticknor C, Sidorov A I and Hall B V 2009 Phys. Rev. A 80(2) 023603
[9] Tojo T, Taguchi Y, Masuyama Y, Hayashi T, Saito H and Hirano T 2010 Phys. Rev. A 82(3) 033609
[10] Busch T, Cirac J I, Pérez-Garcia V M and Zoller P 1997 Phys. Rev. A 56(4) 2978–2983
[11] Miesner H J, Stamper-Kurn D M, Stenger J, Inouye S, Chikkatur A P and Ketterle W 1999 Phys. Rev. Lett. 82(11) 2228–2231
[12] Papp S B, Pino J M and Wieman C E 2008 Phys. Rev. Lett. 101(4) 040402
[13] Ferrari G, Inguscio M, Jastrzebski W, Modugno G, Roati G and Simoni A 2002 Phys. Rev. Lett. 89(5) 053202
[14] Modugno G, Modugno M, Riboli F, Roati G and Inguscio M 2002 Phys. Rev. Lett. 89(19) 190404
[15] Thalhammer G, Barontini G, De Sarlo L, Catani J, Minardi F and Inguscio M 2008 Phys. Rev. Lett. 100(21) 210402
[16] Ho T L and Shenoy V B 1996 Phys. Rev. Lett. 77(16) 3276–3279
[17] Pu H and Bigelow N P 1998 Phys. Rev. Lett. 80(6) 1130–1133
[18] Timmermans E 1998 Phys. Rev. Lett. 81(26) 5718–5721
[19] Ao P and Chui S T 1998 Phys. Rev. A 58(6) 4836–4840
[20] Trippenbach M, Gral K, Rzazewski K, Malomed B and Band Y B 2000 Journal of Physics B: Atomic, Molecular and Optical Physics 33 4017
[21] Barankov R A 2002 Phys. Rev. A 66(1) 013612
[22] Van Schaebroeck B 2008 Phys. Rev. A 78(2) 023624
[23] Gautam S and Angom D 2010 Journal of Physics B: Atomic, Molecular and Optical Physics 43 095302
[24] Jezeck D M and Capuzzi P 2002 Phys. Rev. A 66(1) 015602
[25] Gordon D and Savage C M 1998 Phys. Rev. A 58(2) 1440–1444
[26] Tanafsh B and Erkan K 2000 Phys. Rev. A 62(5) 053601
[27] Kim J G and Lee E K 2002 Phys. Rev. E 65(6) 066201
[28] Baláz A and Nicolin A I 2012 Phys. Rev. A 85(2) 023613
[29] Pattinson R W, Billam T P, Gardiner S A, McCarron D J, Cho H W, Cornish S L, Parker N G and Proukakis N P 2013 Phys. Rev. A 87(1) 013625
[30] Busch T and Anglin J R 2001 Phys. Rev. Lett. 87(1) 010401
[31] Kevrekidis P G, Frantzeskakis D J, Malomed B A, Bishop A R and Kevrekidis I G 2003 New Journal of Physics 5 64
[32] Becker C, Stellmer S, Soltan-Panahi P, Döröscher S, Baunert M, Richter E M, Kronjäger J, Bongs K and Sengstek K 2008 Nature Physics 4 9
[33] Hamner C, Chang J J, Engels P and Hoefer M A 2011 Phys. Rev. Lett. 106(6) 065302
