Some recent results on singular $p$-Laplacian equations

Umberto Guarnotta, Roberto Livrea
Dipartimento di Matematica e Informatica, Università di Palermo,
Via Archirafi 34, 90123 Palermo, Italy
E-mail: umberto.guarnotta@unipa.it, roberto.livrea@unipa.it

Salvatore A. Marano
Dipartimento di Matematica e Informatica, Università di Catania,
Viale A. Doria 6, 95125 Catania, Italy
E-mail: marano@dmi.unict.it

Abstract

A short account of some recent existence, multiplicity, and uniqueness results for singular $p$-Laplacian problems either in bounded domains or in the whole space is performed, with a special attention to the case of convective reactions. An extensive bibliography is also provided.

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1 Introduction

When studying quasi-linear elliptic systems in the whole space and with singular, possibly convective, reactions, a natural preliminary step is looking
for the previous literature on equations of the same type, which we have done in the latest years.

At first, this obviously led us to investigate singular $p$-Laplacian Dirichlet problems as

$$\begin{cases}
-\Delta_p u = h(x, u, \nabla u) & \text{in } \Omega,
\end{cases}$$

$$\begin{cases}
u > 0 & \text{in } \Omega,
\end{cases}$$

$$\begin{cases}
u = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $1 < p < \infty$, the symbol $\Delta_p$ denotes the $p$-Laplace operator, namely

$$\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u),$$

$\Omega$ is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary $\partial \Omega$, while $h \in C^0(\Omega \times \mathbb{R}^+ \times \mathbb{R}^N)$ satisfies

$$\lim_{t \to 0^+} h(x, t, \xi) = \infty.$$ 

If $p = 2$ then various special (chiefly non-convective) cases of $(1.1)$ have been thoroughly studied (see Subsection 3.1). Both surveys [1, 2, 3] and a monograph [4], besides many proceeding papers, are already available. The main purpose of Section 3 below is to provide a short account on some recent existence, multiplicity or uniqueness results for $p \neq 2$ and the relevant technical approaches. Let us also point out [5, 6, 7]. The work [5] treats a singular $p(x)$-Laplacian Robin problem, while [6, 7] are devoted to singular $(p, q)$-Laplacian equations with Neumann and Robin boundary conditions, respectively; cf. [8] too.

Section 4 aims at performing the same as regards singular $p$-Laplacian problems in the whole space. So, it deals with situations like

$$\begin{cases}
-\Delta_p u = h(x, u, \nabla u) & \text{in } \mathbb{R}^N,
\end{cases}$$

$$\begin{cases}
u > 0 & \text{in } \mathbb{R}^N,
\end{cases}$$

$$\begin{cases}
u(x) \to 0 & \text{as } |x| \to \infty.
\end{cases}$$

To the best of our knowledge, except [4], even when $p = 2$ and $h$ does not depend on $\nabla u$, there are no surveys concerning $(1.2)$. Hence, this probably represents the first contribution.

Both sections are divided into four parts. The first is a historical sketch of the case $p = 2$. The next two treat existence, multiplicity, and uniqueness in the non-convective case. The fourth is devoted to singular problems with convection. Since the literature on $(1.1)-(1.2)$ is by now very wide and our knowledge is limited, significant works may have been not mentioned here, something of which we apologize in advance. Moreover, for the sake of brevity, we did not treat singular parabolic boundary-value problems and refer the reader to [2, 4, 10, 11, 12].
2 Basic notation

Let $X(\Omega)$ be a real-valued function space on a nonempty measurable set $\Omega \subseteq \mathbb{R}^N$. If $u_1, u_2 \in X(\Omega)$ and $u_1(x) < u_2(x)$ a.e. in $\Omega$ then we simply write $u_1 < u_2$. The meaning of $u_1 \leq u_2$, etc. is analogous. Put

$$X(\Omega)_+ := \{u \in X(\Omega) : u \geq 0\}.$$  

The symbol $u \in X_{loc}(\Omega)$ means that $u : \Omega \to \mathbb{R}$ and $u|_K \in X(K)$ for all nonempty compact subset $K$ of $\Omega$. Given $1 < r < N$, define

$$r' := \frac{r}{r-1}, \quad r^* := \frac{Nr}{N-r}.$$  

Let us next recall the notion and some relevant properties of the so-called Beppo Levi space $\mathcal{D}_0^{1,r}(\mathbb{R}^N)$, addressing the reader to [13, Chapter II] for a complete treatment. Set

$$\mathcal{D}^{1,r} := \{z \in L_{loc}^{1}(\mathbb{R}^N) : |\nabla z| \in L^r(\mathbb{R}^N)\}$$

and denote by $\mathcal{R}$ the equivalence relation that identifies two elements in $\mathcal{D}^{1,r}$ whose difference is a constant. The quotient set $\dot{\mathcal{D}}^{1,r}$, endowed with the norm

$$\|u\|_{1,r} := \left(\int_{\mathbb{R}^N} |\nabla u(x)|^r dx\right)^{1/r},$$

turns out complete. Write $\mathcal{D}_0^{1,r}(\mathbb{R}^N)$ for the subspace of $\dot{\mathcal{D}}^{1,r}$ defined as the closure of $C_0^\infty(\mathbb{R}^N)$ under $\| \cdot \|_{1,r}$, namely

$$\mathcal{D}_0^{1,r}(\mathbb{R}^N) := \overline{C_0^\infty(\mathbb{R}^N)}\| \cdot \|_{1,r}.$$  

$\mathcal{D}_0^{1,r}(\mathbb{R}^N)$, usually called Beppo Levi space, is reflexive and continuously embeds in $L^{r^*}(\mathbb{R}^N)$, i.e.,

$$\mathcal{D}_0^{1,r}(\mathbb{R}^N) \hookrightarrow L^{r^*}(\mathbb{R}^N).$$

Consequently, if $u \in \mathcal{D}_0^{1,r}(\mathbb{R}^N)$ then $u$ vanishes at infinity, meaning that the set $\{x \in \mathbb{R}^N : |u(x)| \geq \varepsilon\}$ has finite measure for any $\varepsilon > 0$.

3 Problems in bounded domains

3.1 The case $p = 2$

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary $\partial \Omega$, let $a : \Omega \to \mathbb{R}_+^N$ be nontrivial measurable, and let $\gamma > 0$. The simplest singular
elliptic Dirichlet problem writes as
\[
\begin{cases}
-\Delta u = a(x)u^{-\gamma} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\] (3.1)

Since the pioneering papers \[14, 15, 16, 17, 18\], a wealth of existence, uniqueness or multiplicity, and regularity results concerning (3.1) have been published. We refer the reader to the monograph \[4\] as well as the surveys \[1, 2\] for an exhaustive account. Roughly speaking, four basic questions can be identified:

- find the right conditions on the datum \(a\). Usually, \(a \in L^q(\Omega)\) with \(q \geq 1\) is enough for existence. However, starting from the works \[19, 20\], the case when \(a\) is a bounded Radon measure took interest.

- consider non-monotone singular terms. This is a difficult task, mainly when we want to guarantee uniqueness of solutions.

- insert convective terms on the right-hand side. For equations driven by the Laplacian, good references are \[4\ Section 9\] and \[21\]. Otherwise, cf. \[22, 23, 24, 25\].

- substitute the Laplacian with more general elliptic operators. Obviously, a first attempt might be considering equations driven by the \(p\)-Laplacian, and this section aims to provide a short account of the nowadays literature. However, further possibly non-homogeneous operators have been considered; see, e.g., \[14, 15, 26, 27, 28, 29, 20, 30, 31\].

Incidentally, we recall that (3.1) stems from important applied questions, as the study of heat conduction in electrically conducting materials \[32\], chemical heterogeneous catalysts \[33\], and non-Newtonian fluids \[34\].

### 3.2 Existence and multiplicity

Consider the model problem
\[
\begin{cases}
-\Delta_p u = a(x)u^{-\gamma} + \lambda f(x, u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\] (3.2)

where \(a : \Omega \to \mathbb{R}_0^+\) denotes a nonzero measurable function, \(\gamma, \lambda > 0\), while \(f : \Omega \times \mathbb{R}_0^+ \to \mathbb{R}\) satisfies Carathéodory’s conditions. Let us stress that, here, the parameter \(\lambda\) multiplies the non-singular term.
In 2006, Perera and Silva investigated (3.2) under the assumptions below, where $f$ is allowed to change sign.

(a1) There exist $\varphi_0 \in C_0^1(\Omega)_+$ and $\hat{q} > N$ such that $a\varphi_0^{-\gamma} \in L^\hat{q}(\Omega)$.

(a2) With appropriate $\delta, c_1 > 0$ one has

\[ f(x, t) \geq -c_1 a(x) \quad \text{in} \; \Omega \times [0, \delta]. \]

(a3) To every $M > 0$ there correspond $h \in L^1(\Omega)$ and $c_2 > 0$ such that

\[ -h(x) \leq f(x, t) \leq c_2 \quad \forall (x, t) \in \Omega \times [0, M]. \]

(a4) With appropriate $q \in ]1, p^*[ \text{ and } c_3 > 0$ one has

\[ f(x, t) \leq c_3 (t^{q-1} + 1) \quad \text{in} \; \Omega \times \mathbb{R}_0^+. \]

(a5) There are $t_0 > 0$ and $\mu > p$ such that

\[ 0 < \mu \int_0^t f(x, \tau)d\tau \leq tf(x, t) \quad \forall (x, t) \in \Omega \times [t_0, +\infty[. \]

They seek distributional solutions to (3.2), i.e., functions $u \in W_0^{1,p}(\Omega)$ such that $u > 0$ and

\[ \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_\Omega a u^{-\gamma} \varphi dx + \int_\Omega f(\cdot, u) \varphi dx \quad \forall \varphi \in C_0^\infty(\Omega). \]

**Theorem 3.1** ([35], Theorems 1.1–1.2). Let (a1)–(a3) be satisfied. Then Problem (3.2) admits a distributional solution for every $\lambda > 0$ small. If, in addition, (a4)–(a5) hold true then a further distributional solution exists by decreasing $\lambda$ when necessary.

Proofs employ perturbation arguments and variational methods, previously introduced in [36]. An immediate but hopefully useful consequence of Theorem 3.1 is the next

**Corollary 3.2.** Let (a1) be fulfilled. Suppose $f$ does not depend on $x$ and, moreover, $f(t) \geq 0$ in a neighborhood of zero once $\text{ess inf}_\Omega a = 0$. Then, for every $\lambda > 0$ sufficiently small, the problem

\[- \Delta_p u = a(x)u^{-\gamma} + \lambda f(u) \quad \text{in} \; \Omega, \quad u > 0 \quad \text{in} \; \Omega, \quad u = 0 \quad \text{on} \; \partial \Omega \quad (3.3)\]

possesses a distributional solution.
Further results concerning (3.3) can be found in Aranda-Godoy [37], where a continuous non-increasing function \( g(u) \) takes the place of \( u^{-\gamma} \) and, from a technical point of view, fixed point theorems for nonlinear eigenvalue problems are exploited.

The case \( \lambda = 0 \) in (3.3) was well investigated by Canino, Sciunzi, and Trombetta [38], with a special attention to uniqueness (see the next section). Here, given \( u \in W_{1, p}^{1, p}(\Omega) \),

\[
 u = 0 \text{ on } \partial \Omega \overset{\text{def}}{=} u \geq 0 \text{ and } (u - \varepsilon)^+ \in W_{0}^{1, p}(\Omega) \quad \forall \varepsilon > 0.
\]

**Theorem 3.3 ([38], Theorem 1.3).** Let \( \lambda = 0 \). If \( \gamma \geq 1 \) and \( a \in L^1(\Omega) \) then (3.3) admits a distributional solution \( u \in W_{\text{loc}}^{1, p}(\Omega) \) such that \( \text{ess inf}_K u > 0 \) for any compact set \( K \subseteq \Omega \). Moreover, \( u^{1+(\gamma-1)/p} \in W_{0}^{1, p}(\Omega) \). If \( 0 < \gamma < 1 \) then (3.3) has a solution \( u \in W_{\text{loc}}^{1, p}(\Omega) \) in each of the following cases:

- \( 1 < p < N \) and \( a \in L^m(\Omega) \), with \( m := \left( \frac{p^*}{p - \gamma} \right)^{1-\gamma} \).
- \( p = N \) and \( a \in L^m(\Omega) \) for some \( m > 1 \).
- \( p > N \) and \( a \in L^1(\Omega) \).

The proof of this result relies on a technique previously introduced in [29] for the semi-linear case. It employs truncation and regularization arguments. The work [39] contains a version of Theorem 3.3 for the so-called \( \Phi \)-Laplacian. A more general problem patterned after

\[
 -\Delta_p u = \mu u^{-\gamma} \text{ in } \Omega, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,
\]

where \( \mu \) denotes a non-negative bounded Radon measure on \( \Omega \) while \( \gamma \geq 0 \), is thoroughly studied in [10]; see also [11] and the references therein.

Finally, as regards Problem (3.2) again, the papers [42] [43] [44] [45] do not require Ambrosetti-Rabinowitz’s condition \((a_5)\), while [46] establishes the existence of at least three weak solutions. Moreover, a possibly non-homogeneous elliptic operator is considered in [44], but \( \lambda = 1 \).

The nice paper [47] investigates the problem

\[
 \begin{cases} 
 -\Delta_p u = \lambda u^{-\gamma} + u^{q-1} & \text{in } \Omega, \\
 u > 0 & \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega,
 \end{cases}
\]

where \( 0 < \gamma < 1 \) and \( 1 < p < q < p^* \). It should be noted that, here, contrary to above, the parameter \( \lambda \) multiplies the singular term. Combining known variational methods with a \( C^{1, \alpha}(\Omega) \)-regularity result [47] Theorem 2.2] for solutions to (3.5) and a strong comparison principle [47] Theorem 2.3], the authors obtain the following
Theorem 3.4 ([17], Theorem 2.1). Suppose $0 < \gamma < 1$ and $1 < p < q < p^*$. Then there is $\Lambda > 0$ such that (3.5) has:

- at least two ordered solutions in $C^1(\overline{\Omega})$ for every $\lambda \in ]0, \Lambda[$,
- at least one solution in $C^1(\overline{\Omega})$ when $\lambda = \Lambda$, and
- no solutions once $\lambda > \Lambda$.

The case $q = p^*$ is also studied and it is shown that $\gamma < 1$ is a reasonable sufficient (and likely optimal) condition to get $C^1(\overline{\Omega})$-solutions of (3.5).

If $p = 2$ and, roughly speaking, $a \equiv -1$ while $f$ does not depend on $u$ then Problem (3.2) was fruitfully studied in [48].

We end this section by pointing out two very recent works, namely [49], which deals with possibly non-monotone singular reactions (see also [50, 51], essentially based on sub-super-solution methods) and [31], devoted to singular equations driven by the $(p, q)$-Laplace operator $u \mapsto \Delta_p u + \Delta_q u$.

3.3 Uniqueness

Surprisingly enough, if $p \neq 2$, uniqueness of solutions looks a difficult matter, even for the model problem

\[
\begin{aligned}
-\Delta_p u &= a(x)u^{-\gamma} \quad \text{in } \Omega, \\
u &> 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

As observed in [38], this is mainly caused by the fact that, in general, solutions do not belong to $W^{1,p}_0(\Omega)$ once $\gamma \geq 1$. The paper [38] provides two different results. The first one (Theorem 1.4) holds in star-shaped domains, while the other is the following

Theorem 3.5 ([38], Theorem 1.5). Assume that either $\gamma \leq 1$ and $a \in L^1(\Omega)$ or $\gamma > 1$ and

- $a \in L^m(\Omega)$ for some $m > \frac{N}{p}$ if $1 < p < N$,
- $a \in L^m(\Omega)$ with $m > 1$ when $p = N$, and
- $a \in L^1(\Omega)$ if $p > N$.

Then (3.6) possesses a unique distributional solution.
We next point out that, for $\gamma \leq 1$, Theorem 3.4 of [40] establishes the uniqueness of renormalized solutions to (3.4).

The situation becomes quite clear when $p = 2$ and one seeks sufficiently regular solutions. Denote by $\varphi_1$ a positive eigenfunction corresponding to the first eigenvalue $\lambda_1$ of the problem $-\Delta u = \lambda u$ in $\Omega$, $u = 0$ on $\partial\Omega$.

**Theorem 3.6** ([17], Theorems 1–2). Let $p = 2$ and let $a \in C^{0,\alpha}(\overline{\Omega})$ be positive. Then (3.6) has a unique solution $u \in C^{2,\alpha}(\Omega) \cap C^0(\Omega)$. Moreover,

- there exist $c_1, c_2 > 0$ such that $c_1\varphi_1^{2/(1+\gamma)} \leq u \leq c_2\varphi_1^{2/(1+\gamma)}$ in $\overline{\Omega}$,
- $u \in H^1_0(\Omega) \iff \gamma < 3$, and
- $\gamma > 1 \implies u \not\in C^1(\overline{\Omega})$.

See also the nice paper [52]. As regards weak solutions, one has

**Theorem 3.7** ([53], Theorem 3.1). Suppose $p = 2$ and $a \in L^1(\Omega)$. Then (3.6) admits at most one solution belonging to $H^1_0(\Omega)$.

Another uniqueness case occurs when $\gamma > 1$.

**Theorem 3.8** ([53], Theorem 1.3). If $p = 2$, $\gamma > 1$, and $a \in L^1(\Omega)$ then (3.6) possesses at most one solution $u \in H^1_{\text{loc}}(\Omega)$ such that $u^{(\gamma+1)/2} \in H^1_0(\Omega)$.

### 3.4 Equations with convective terms

Consider the problem

\[
\begin{aligned}
-\Delta_p u &= f(x, u, \nabla u) + g(x, u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]

where $p < N$ while $f : \Omega \times \mathbb{R}_0^+ \times \mathbb{R}^N \to \mathbb{R}_0^+$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}_0^+$ satisfy Carathéodory’s conditions. In 2019, Liu, Motreanu, and Zeng established the existence of solutions $u \in W_0^{1,p}(\Omega)$ to (3.7) under the hypotheses below, where $\lambda_1$ stands for the first eigenvalue of $-\Delta_p$ in $W_0^{1,p}(\Omega)$.

(h$_1$) There exist $c_0, c_1, c_2 > 0$ such that $c_1 + c_2\lambda_1^{1-1/p} < \lambda_1$ and

\[
f(x, t, \xi) \leq c_0 + c_1t^{p-1} + c_2|\xi|^{p-1} \quad \forall (x, t, \xi) \in \Omega \times \mathbb{R}_0^+ \times \mathbb{R}^N.
\]

(h$_2$) $g(x, \cdot)$ is non-increasing on $(0, 1]$ for all $x \in \Omega$ and $g(\cdot, 1) \neq 0$.  


(h₃) With appropriate \( \theta \in \text{int}(C⁰₀(\Omega)⁺) \), \( \hat{q} > \max\{N, p'\} \), and \( \varepsilon₀ > 0 \), the map \( x \mapsto g(x, \varepsilon \theta(x)) \) belongs to \( L^\hat{q}(Ω) \) for any \( \varepsilon \in (0, \varepsilon₀) \).

Condition (h₃) was previously introduced by Faraci and Puglisi [54]. It represents a natural generalization of (a₁) in Section 3.2.

Theorem 3.9 ([55], Theorem 25). Let (h₁)–(h₃) be satisfied. Then (3.7) has a solution \( u \in \text{int}(C⁰₀(\Omega)⁺) \).

We think worthwhile to sketch the main ideas of the proof. For every fixed \( w \in C⁰₀(\Omega) \), an intermediate problem, where \( \nabla w \) replaces \( \nabla u \) in \( f(x, u, \nabla u) \) and the singular term remains unchanged, is considered. The authors construct a positive sub-solution \( \tilde{u} \in \text{int}(C⁰₀(\Omega)⁺) \) independently of \( w \) and show the existence of a solution greater than \( \tilde{u} \). If \( S(w) \) denotes the set of such solutions then, via suitable properties of the multi-function \( w \mapsto S(w) \), it is proved that the map \( \Gamma \), which assigns to each \( w \) the minimal element of \( S(w) \), is completely continuous. Now, Leray–Schauder’s alternative principle applied to \( \Gamma \) yields a solution \( u \in \text{int}(C⁰₀(\Omega)⁺) \) to (3.7).

The recent paper [24], partially patterned after [55], treats the Robin problem

\[
\begin{cases}
-\text{div} A(\nabla u) = f(x, u, \nabla u) + g(x, u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu_A} + \beta u^{p-1} = 0 & \text{on } \partial\Omega,
\end{cases}
\]

where \( A : \mathbb{R}^N \to \mathbb{R}^N \) denotes a continuous strictly monotone map having suitable properties, which basically stem from Lieberman’s nonlinear regularity theory [56] and Pucci-Serrin’s maximum principle [57]. By the way, the conditions on \( A \) include classical non-homogeneous operators as, e.g., the \((p, q)\)-Laplacian. Moreover, \( \beta \) is a positive constant while \( \frac{\partial}{\partial \nu_A} \) indicates the co-normal derivative associated with \( A \). If \( p = 2 \) then a uniqueness result is also presented; cf. [24, Theorem 4.2].

The special case \( A(\xi) := |\xi|^{p-2}\xi \), \( g(x, t) := t^{-\gamma} \) for some \( 0 < \gamma < 1 \), and \( \beta = 0 \) (which reduces (3.8) to a Neumann problem) has been investigated in [8] without imposing any global growth condition on \( t \mapsto f(x, t, \xi) \). Instead, a kind of oscillatory behavior near zero is taken on. For such an \( f \), the work [25] establishes the existence of a solution \( u \in C⁰₀(\Omega) \) to the parametric problem

\[
\begin{cases}
-\text{div} A(\nabla u) = f(x, u, \nabla u) + \lambda u^{-\gamma} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]

provided \( \lambda > 0 \) is small enough.
Finally, the very recent paper [58] treats \( \Phi \)-Laplacian equations with strongly singular reactions perturbed by gradient terms.

4 Problems on the whole space

4.1 The case \( p = 2 \)

Let \( N \geq 3 \), let \( a : \mathbb{R}^N \to \mathbb{R}_0^+ \) be nontrivial measurable, and let \( \gamma > 0 \). The simplest singular elliptic problem in the whole space writes as

\[
\begin{cases}
-\Delta u = a(x)u^{-\gamma} & \text{in } \mathbb{R}^N, \\
u > 0 & \text{in } \mathbb{R}^N.
\end{cases}
\]

Sometimes it is also required that \( u(x) \to 0 \) as \( |x| \to \infty \). Since the pioneering papers [59, 60, 61, 62], some existence and uniqueness results concerning (4.1) have been published. We refer the reader to the monograph [4] for a deep account. Roughly speaking, four basic questions can be identified:

- find the right hypotheses on \( a \). Usually, \( a \in C_{loc}^{0, \alpha}(\mathbb{R}^N)_+ \) as well as

\[
\int_1^\infty r \max_{|x|=r} a(x) \, dr < \infty
\]

(cf. condition (a8) below) guarantee both existence and uniqueness of solutions \( u \in C_{loc}^{2, \alpha}(\mathbb{R}^N) \).

- replace \( u^{-\gamma} \) with a function \( f(u) \) such that \( \lim_{t \to 0^+} f(t) = \infty \). This was done in [63, 64] for decreasing \( f \). Later on, also non-monotone singular reactions were fruitfully treated [65, 66, 67].

- put convective terms on the right-hand side. For equations driven by the Laplacian, a good reference is [4, Section 9.8]; cf. in addition [68, 69].

- generalize the left-hand side of the equation. The case of a second-order uniformly elliptic operator is treated in [70, 77], while [71] deals with \( u \mapsto -\Delta u + c(x)u \), where \( c \in L_{loc}^\infty(\mathbb{R}^N)_+ \).

The equation of Problem (4.1) arises in the boundary-layer theory of viscous fluids [72, 73, 74] and is called Lane-Emden-Fowler equation. Its importance in scientific applications has by now been widely recognized; see, e.g., [75].
4.2 Existence and multiplicity

To the best of our knowledge, the first paper treating singular $p$-Laplacian equations on the whole space is that of Goncalves and Santos [76], published in 2004. The authors consider the problem

\[
\begin{align*}
-\Delta_p u &= a(x) f(u) \quad \text{in } \mathbb{R}^N, \\
u > 0 &\quad \text{in } \mathbb{R}^N, \\
u(x) &\to 0 \quad \text{as } |x| \to \infty,
\end{align*}
\]

where $a \in C^0(\mathbb{R}^N)_+$ is radially symmetric while $f \in C^1(\mathbb{R}^+, \mathbb{R}^+)$, and assume that:

(a6) the function $t \mapsto \frac{f(t)}{t^{p-1}}$ is non-increasing on $\mathbb{R}^+$.

(a7) $\lim \inf_{t \to 0^+} f(t) > 0$ as well as $\lim_{t \to \infty} \frac{f(t)}{t^{p-1}} = 0$.

(a8) if $\Phi(r) := \max_{|x|=r} a(x)$, $r > 0$, then

\[
0 < \int_1^\infty r \Phi(r) \frac{1}{r^{p-1}} dr < \infty \quad \text{for } 1 < p \leq 2,
\]

\[
0 < \int_1^\infty r \frac{(p-2)N+1}{p-1} \Phi(r) dr < \infty \quad \text{for } p > 2.
\]

**Theorem 4.1** ([76], Theorem 1.1). Under (a6)–(a8), Problem (4.2) admits:

- a radially symmetric solution $u \in C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})$ when $p < N$.
- no radially symmetric solution in $C^1(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus \{0\})$ if $p \geq N$.

The proof exploits fixed point arguments, the shooting method, and sub-super-solution techniques.

One year later, Covei [77] did not assume $a$ radially symmetric but locally Hölder continuous and positive, replaced conditions (a6)–(a7) with those below, and obtained similar results. See also [78], where the asymptotic behavior of solutions is described.

(a6') The function $t \mapsto \frac{f(t)}{(t+\beta)^{p-1}}$ turns out decreasing on $\mathbb{R}^+$ for some $\beta > 0$.

(a7') $\lim_{t \to 0^+} \frac{f(t)}{t^{p-1}} = \infty$ and $f(t) \leq c$ for any $t$ large enough.
The work [79] treats the parametric problem

\[
\begin{aligned}
-\Delta_p u &= a(x)u^{-\gamma} + \lambda b(x)u^{q-1} \quad \text{in } \mathbb{R}^N, \\
 u &> 0 \quad \text{in } \mathbb{R}^N,
\end{aligned}
\]  

(4.3)

where \(1 < p < N\), \(0 < \gamma < 1\), \(\lambda > 0\), \(\max\{p, 2\} < q < p^*\), and the coefficients fulfill

\[
a \in L^{\frac{p}{p^* - (1 - \gamma)}}(\mathbb{R}^N), \quad a \neq 0, \quad b \in L^{\frac{p}{p^* - q}}(\mathbb{R}^N), \quad b > 0.
\]  

(4.4)

**Theorem 4.2** ([79], Theorem 1.2). If (4.4) holds then there exists \(\Lambda > 0\) such that (4.3) possesses

- at least two solutions in \(D_0^{1,p}(\mathbb{R}^N)\) for every \(\lambda \in ]0, \Lambda[\),
- at least one solution belonging to \(D_0^{1,p}(\mathbb{R}^N)\) when \(\lambda = \Lambda\), and
- no solutions once \(\lambda > \Lambda\).

It may be of interest to point out that this result is proved by combining sub-super-solution methods with the mountain pass theorem for continuous functionals.

**Remark 4.3.** If \(b \equiv 0\) then Problem (4.3) reduces to a well-known one, very important in scientific applications; cf. [80, Remark 2.2].

A meaningful case occurs when \(a, b : \mathbb{R}^N \to \mathbb{R}^+_0\) turn out nonzero locally Hölder continuous functions. In fact, define

\[
M(x) := \max\{a(x), b(x)\}, \quad x \in \mathbb{R}^N.
\]  

(4.5)

From [81, Remarks 1–2] it follows

**Lemma 4.4.** Suppose that \(p < N\), the functions \(a, b : \mathbb{R}^N \to \mathbb{R}^+_0\) are non-trivial and locally Hölder continuous, while \((a_8)\) holds with \(M\) in place of \(a\). Then the problem

\[
\begin{aligned}
-\Delta_p w &= M(x) \quad \text{in } \mathbb{R}^N, \\
w &> 0 \quad \text{in } \mathbb{R}^N, \\
w(x) &\to 0 \quad \text{as } |x| \to \infty
\end{aligned}
\]  

(4.6)

admits a solution \(w_M \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)\) for suitable \(\alpha \in ]0, 1[\).

Via sub-super-solution techniques, Lemma 4.4 gives rise to

**Theorem 4.5** ([81], Theorem 1.1). Let \(\gamma > 0\), let \(p < q\), and let \(M\) be given by (4.3). Under the assumptions of Lemma 4.4 there exists \(\lambda^* > 0\) such that (4.3) has:
• at least one solution $u \in C^1(\mathbb{R}^N)$ for every $0 \leq \lambda < \lambda^*$. Moreover, $u(x) \to 0$ as $|x| \to \infty$.

• no solution once $\lambda > \lambda^*$.

This result was next generalized under various aspects by the same author and Rezende [82]; cf. also [80].

Finally, infinite semi-positone problems, i.e., $\lim_{t \to 0^+} f(t) = -\infty$, were fruitfully investigated in [83]. Precisely, given $a \in L^\infty(\mathbb{R}^N)$ and $f \in C^0(\mathbb{R}^+)$, consider the problem

\[
\begin{cases}
-\Delta_p u = \lambda a(x) f(u) & \text{in } \mathbb{R}^N, \\
u > 0 & \text{in } \mathbb{R}^N, \\
u(x) \to 0 & \text{as } |x| \to \infty,
\end{cases}
\] (4.7)

where $\lambda > 0$, $1 < p < N$. The following conditions will be posited.

(a9) There exists $\gamma \in ]0,1[$ such that $\lim_{t \to 0^+} t^\gamma f(t) = c_0 \in \mathbb{R}^-$. 

(a10) $\lim_{t \to \infty} f(t) = \infty$ but $\lim_{t \to \infty} \frac{f(t)}{t^{p-1}} = 0$.

(a11) $\inf_{|x|=r} a(x) > 0$ for all $r > 0$ and $0 < a(x) < \frac{C_0}{|x|^p}$ in $\mathbb{R}^N \setminus \{0\}$ with suitable $C_0 > 0$, $\sigma > N + \gamma \frac{N-p}{p-1}$.

Theorem 4.6 ([83], Theorem 1.4). If (a9)-(a11) hold and $\lambda$ is sufficiently large then (4.7) has a solution in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$.

4.3 Uniqueness

As far as we know, uniqueness has been addressed only in [76, Remark 1.2] and [77, Section 2] under the key assumption (a′6) above. The arguments of both papers rely on a famous result by Diaz and Saa [84]. Theorem 1.3 of [83] contains a nice idea to achieve uniqueness for singular problems in exterior domains.

4.4 Equations with convective terms

To the best of our knowledge, there is only one paper concerning singular quasi-linear elliptic equations in the whole space and with convective terms, namely [86]. It treats the problem

\[
\begin{cases}
-\text{div } A(\nabla u) = f(x, u) + g(x, \nabla u) & \text{in } \mathbb{R}^N, \\
u > 0 & \text{in } \mathbb{R}^N,
\end{cases}
\] (4.8)
where $N \geq 2$ and $1 < p < N$. The differential operator $u \mapsto \text{div} A(\nabla u)$ is as in (3.8), while $f : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}_0^+$ and $g : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_0^+$ fulfill Carathéodory’s conditions. Moreover,

\[
\liminf_{t \to 0^+} f(x, t) > 0 \quad \text{uniformly with respect to } x \in B_\sigma(x_0),
\]

\[
f(x, t) \leq h(x)t^{-\gamma} \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \quad \text{where } h \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N),
\]

and

\[
g(x, \xi) \leq k(x)|\xi|^r \quad \text{in } \mathbb{R}^N \times \mathbb{R}^N, \quad \text{with } k \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N).
\]

Here, $x_0 \in \mathbb{R}^N$, $\sigma \in ]0, 1[$, $\gamma \geq 1$, $r \in [0, p - 1[$, as well as

\[
\eta > (p^*)', \quad \theta > \left(\frac{1}{(p^*)'} - \frac{r}{p}\right)^{-1}.
\]

**Theorem 4.7** ([86], Theorem 1.2). Under (4.9)–(4.11), there exists a distributional solution $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ to (4.8) such that $\text{ess inf}_K u > 0$ for every compact set $K \subseteq \mathbb{R}^N$.

To prove this result, the authors first solve some auxiliary problems, obtained by shifting the singular term and working in balls, via sub-super-solution techniques. A compactness result, jointly with a fine local energy estimate on super-level sets of solutions, then yields the conclusion.

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**References**

[1] J. Hernández, F.J. Mancebo, and J.M. Vega, *Nonlinear Singular Elliptic Problems: Recent Results and Open Problems*, in: Nonlinear elliptic and parabolic problems, pp. 227–242, Progr. Nonlinear Differential Equations Appl. 64, Birkhäuser, Basel, 2005.
[2] J. Hernández and F.J. Mancebo, *Singular Elliptic and Parabolic Equations*, in: M. Chipot and P. Quittner (eds), Handbook of Differential Equations, vol. 3, pp. 317–400, Elsevier, Amsterdam, 2006.

[3] V. Radulescu, *Singular phenomena in nonlinear elliptic problems: from blow-up boundary solutions to equations with singular nonlinearities*, in: Handbook of differential equations: stationary partial differential equations, Vol. IV, 485–593, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2007.

[4] M. Ghergu and V.D. Radulescu, *Singular elliptic problems: bifurcation and asymptotic analysis*, Oxford Lecture Ser. Math. Appl. 37, Oxford Univ. Press, Oxford, 2008.

[5] K. Saoudi, *The fibering map approach to a p(x)-Laplacian equation with singular nonlinearities and nonlinear Neumann boundary conditions*, Rocky Mountain J. Math. 48 (2018), 927–946.

[6] N.S. Papageorgiou, C. Vetro, and F. Vetro, *Singular Neumann (p, q)-equations*, Positivity 24 (2021), 1017–1040.

[7] N.S. Papageorgiou, V. Radulescu, and D. Repovs, *Robint double-phase problems with singular and superlinear terms*, Nonlinear Anal. Real World Appl. 58 (2021), Paper no. 103217, 20 pp.

[8] N.S. Papageorgiou, V. Radulescu, and D. Repovs, *Positive solutions for nonlinear Neumann problems with singular terms and convectons*, J. Math. Pures Appl. (9) 136 (2020), 1–21.

[9] I. De Bonis and D. Giachetti, *Nonnegative solutions for a class of singular parabolic problems involving p-Laplacian*, Asymptot. Anal. 91 (2015), 147–183.

[10] F. Oliva and F. Petitta, *A nonlinear parabolic problem with singular terms and nonregular data*, Nonlinear Anal. 194 (2020), 111472, 13 pp.

[11] J. Giacomoni, D. Kumar, and K. Sreenadh, *A qualitative study of (p, q) singular parabolic equations: local existence, Sobolev regularity and asymptotic behavior*, Adv. Nonlinear Stud. 21 (2021), 199–227.

[12] S. Ciani and U. Guarnotta, *On a non-homogeneous parabolic equation with singular and convective reaction*, preprint.

[13] G.P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems*, 2nd ed., Springer Monographs in Mathematics, Springer, New York, 2011.
[14] C.A. Stuart, *Existence and approximation of solutions of non-linear elliptic equations*, Math. Z. **147** (1976), 53–63.

[15] M.G. Crandall, P.H. Rabinowitz, and L. Tartar, *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations **2** (1977), 193–222.

[16] M.M. Coclite and G. Palmieri, *On a singular nonlinear Dirichlet problem*, Comm. Partial Differential Equations **14** (1989), 1315–1327.

[17] A.C. Lazer and P.J. McKenna, *On a singular nonlinear elliptic boundary-value problem*, Proc. Amer. Math. Soc. **111** (1991), 721–730.

[18] Y.S. Choi, A.C. Lazer, and P.J. McKenna, *Some remarks on a singular elliptic boundary value problem*, Nonlinear Anal. **32** (1998), 305–314.

[19] L. Orsina and F. Petitta, *A Lazer–McKenna type problem with measures*, Differential Integral Equations **29** (2016), 19–36.

[20] F. Oliva and F. Petitta, *Finite and infinite energy solutions of singular elliptic problems: Existence and uniqueness*, J. Differential Equations **264** (2018), 311–340.

[21] C. Aranda and E. Lami Dozo, *Multiple solutions to a singular Lane-Emden-Fowler equation with convection term*, Electron. J. Differential Equations 2007, Paper no. 05, 21 pp.

[22] L. Boccardo, *Dirichlet problems with singular and gradient quadratic lower order terms*, ESAIM Control Optim. Calc. Var. **14** (2008), 411–426.

[23] D. Arcoya, J. Carmona, T. Leonori, P.J. Martínez-Aparicio, L. Orsina, and F. Petitta, *Existence and nonexistence of solutions for singular quadratic quasilinear equations*, J. Differential Equations **246** (2009), 4006–4042.

[24] U. Guarnotta, S.A. Marano, and D. Motreanu, *On a singular Robin problem with convection terms*, Adv. Nonlinear Stud. **20** (2020), 895–909.

[25] N.S. Papageorgiou and Y. Zhang, *Nonlinear nonhomogeneous Dirichlet problems with singular and convection terms*, Bound. Value Probl. 2020, Paper no. 153, 21 pp.

[26] S.M. Gomes, *On a singular nonlinear elliptic problem*, SIAM J. Math. Anal. **17** (1986), 1359–1369.

[27] J. Chabrowski, *Existence results for singular elliptic equations*, Hokkaido Math. J. **20** (1991), 465–475.
[28] S.B. Cui, *Positive solutions for Dirichlet problems associated to semilinear elliptic equations with singular nonlinearity*, Nonlinear Anal. **21** (1993), 181–190.

[29] L. Boccardo and L. Orsina, *Semilinear elliptic equations with singular nonlinearities*, Calc. Var. Partial Differential Equations **37** (2010), 363–380.

[30] J. Giacomoni, D. Kumar, and K. Sreenadh, *Sobolev and Hölder regularity results for some singular nonhomogeneous quasilinear problems*, Calc. Var. Partial Differential Equations **60** (2021), Paper No. 121, 33 pp.

[31] N.S. Papageorgiou and P. Winkert, *Singular Dirichlet (p,q)-equations*, Mediterr. J. Math. **18** (2021), Paper no. 141, 20 pp.

[32] W. Fulks and J.S. Maybee, *A singular non-linear equation*, Osaka Math. J. **12** (1960), 1–19.

[33] W.L. Perry, *A monotone iterative technique for solution of pth order (p < 0) reaction-diffusion problems in permeable catalysis*, J. Comput. Chemistry **5** (1984), 353–357.

[34] G. Astrita and G. Marrucci, *Principles of non-Newtonian fluid mechanics*, McGraw-Hill, New York, 1974.

[35] K. Perera and E.A.B. Silva, *Existence and multiplicity of positive solutions for singular quasilinear problems*, J. Math. Anal. Appl. **323** (2006), 1238–1252.

[36] K. Perera and Z. Zhang, *Multiple positive solutions of singular p-Laplacian problems by variational methods*, Bound. Value Prob. Bound. Value Probl. 2005, 377–382.

[37] C. Aranda and T. Godoy, *Existence and multiplicity of positive solutions for a singular problem associated to the p-Laplacian operator*, Electron. J. Differential Equations 2004, Paper no. 132, 15 pp.

[38] A. Canino, B. Scìunzi, and A. Trombetta, *Existence and uniqueness for p-Laplace equations involving singular nonlinearities*, NoDEA Nonlinear Differential Equations Appl. **23** (2016), Paper no. 8, 18 pp.

[39] J.V. Gonçalves, M.L. Carvalho, and C.A. Santos, *About positive $W^{1,p}_{\text{loc}}(\Omega)$-solutions to quasilinear elliptic problems with singular semilinear term*, Topol. Methods Nonlinear Anal. **53** (2019), 491–517.

[40] L.M. De Cave, R. Durastanti, and F. Oliva, *Existence and uniqueness results for possibly singular nonlinear elliptic equations with measure data*, NoDEA Nonlinear Differential Equations Appl. **25** (2018), Paper no. 18, 35 pp.
[41] V. De Cicco, D. Giachetti, F. Oliva, and F. Petitta, *The Dirichlet problem for singular elliptic equations with general nonlinearities*, Calc. Var. Partial Differential Equations 58 (2019), Paper No. 129, 40 pp.

[42] S.T. Kyritsi and N.S. Papageorgiou, *Pairs of positive solutions for singular p-Laplacian equations with a p-superlinear potential*, Nonlinear Anal. 73 (2010), 1136–1142.

[43] N.S. Papageorgiou and G. Smyrlis, *A bifurcation-type theorem for singular nonlinear elliptic equations*, Methods Appl. Anal. 22 (2015), 147–170.

[44] N.S. Papageorgiou and G. Smyrlis, *Nonlinear elliptic equations with singular reaction*, Osaka J. Math. 53 (2016), 489–514.

[45] N.S. Papageorgiou and P. Winkert, *Solutions with sign information for nonlinear nonhomogeneous problems*, Math. Z. 292 (2019), 871-891.

[46] J.I. Diaz, J.M. Morel, and L. Oswald, *An elliptic equation with singular nonlinearity*, Comm. Partial Differential Equations 12 (1987), 1333–1344.

[47] P. Candito, U. Guarnotta, and K. Perera, *Two solutions for a parametric singular p-Laplacian problem*, J. Nonlinear Var. Anal. 4 (2020), 455–468.

[48] J.V.A. Gonçalves, M.C. Rezende, and C.A. Santos, *Positive solutions for a mixed and singular quasilinear problem*, Nonlinear Anal. 74 (2011), 132–140.

[49] D.D. Hai, *On a class of singular p-Laplacian boundary value problems*, J. Math. Anal. Appl. 383 (2011), 619–626.

[50] B. Bougherara, J. Giacomoni, and J. Hernández, *Existence and regularity of weak solutions for singular elliptic problems*, Electron. J. Differ. Equ. Conf. 22 (2015), 19–30.

[51] A. Canino and B. Sciunzi, *A uniqueness result for some singular semilinear elliptic equations*, Commun. Contemp. Math. 18 (2016), 1550084, 9 pp.

[52] F. Faraci and D. Puglisi, *A singular semilinear problem with dependence on the gradient*, J. Differential Equations 260 (2016), 3327–3349.

[53] Z. Liu, D. Motreanu, and S. Zeng, *Positive solutions for nonlinear singular elliptic equations of p-Laplacian type with dependence on the gradient*, Calc. Var. Partial Differential Equations 58 (2019), Paper no. 28, 22 pp.
[56] G.M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. 12 (1988), 1203–1219.

[57] P. Pucci and J. Serrin, *The maximum principle*, Prog. Nonlinear Differential Equations Appl. 73, Birkhäuser Verlag, Basel, 2007.

[58] M.L. Carvalho, J.V. Goncalves, E.D. Silva, and C.A.P. Santos, *A type of Brézis-Oswald problem to Φ-Laplacian operator with strongly-singular and gradient terms*, Calc. Var. Partial Differential Equations 60 (2021), Paper No. 195, 25 pp.

[59] T. Kusano and C.A. Swanson, *Entire positive solutions of singular semilinear elliptic equations*, Japan. J. Math. (N.S.) 11 (1985), 145–155.

[60] R. Dalmasso, *Solutions d’équations elliptiques semi-linéaires singulières*, Ann. Mat. Pura Appl. (4) 153 (1988), 191–201 (in French).

[61] A.L. Edelson, *Entire solutions of singular elliptic equations*, J. Math. Anal. Appl. 139 (1989), 523–532.

[62] A.V. Lair and A.W. Shaker, *Entire solution of a singular semilinear elliptic problem*, J. Math. Anal. Appl. 200 (1996), 498–505.

[63] A.V. Lair and A.W. Shaker, *Classical and weak solutions of a singular semilinear elliptic problem*, J. Math. Anal. Appl. 211 (1997), 371–385.

[64] Z. Zhang, *A remark on the existence of entire solutions of a singular semilinear elliptic problem*, J. Math. Anal. Appl. 215 (1997), 579–582.

[65] F.C.S. Cîrstea and V.D. Rădulescu, *Existence and uniqueness of positive solutions to a semilinear elliptic problem in \( \mathbb{R}^N \)*, J. Math. Anal. Appl. 229 (1999), 417–425.

[66] J.V. Gonçalves and C.A. Santos, *Existence and asymptotic behavior of non-radially symmetric ground states of semilinear singular elliptic equations*, Nonlinear Anal. 65 (2006), 719–727.

[67] J.V. Gonçalves, A.L. Melo, and C.A. Santos, *On existence of \( L^\infty \)-ground states for singular elliptic equations in the presence of a strongly nonlinear term*. Adv. Nonlinear Stud. 7 (2007), 475–490.

[68] M. Ghergu and V.D. Rădulescu, *Ground state solutions for the singular Lane-Emden-Fowler equation with sublinear convection term*, J. Math. Anal. Appl. 333 (2007), 265–273.

[69] J.V. Gonçalves and F.K. Silva, *Existence and nonexistence of ground state solutions for elliptic equations with a convection term*, Nonlinear Anal. 72 (2010), 904–915.
[70] J. Chabrowski and M. König, *On entire solutions of elliptic equations with a singular nonlinearity*, Comment. Math. Univ. Carolin. **31** (1990), 643–654.

[71] C.O. Alves, J.V. Gonçalves, and L.A. Maia, *Singular nonlinear elliptic equations in $\mathbb{R}^N$*, Abstr. Appl. Anal. **3** (1998), 411–423.

[72] A.J. Callegari and M.B. Friedman, *An analytical solution of a nonlinear, singular boundary value problem in the theory of viscous fluids*, J. Math. Anal. Appl. **21** (1968), 510–529.

[73] A. Callegari and A. Nachman, *Some singular, nonlinear differential equations arising in boundary layer theory*, J. Math. Anal. Appl. **64** (1978), 96–105.

[74] A. Callegari and A. Nachman, *A nonlinear singular boundary value problem in the theory of pseudoplastic fluids*, SIAM J. Appl. Math. **38** (1980), 275–281.

[75] A.C. Fowler, *Mathematical models in the applied sciences*, Cambridge Univ. Press, Cambridge, 1997.

[76] J.V. Gonçalves and C.A. Santos, *Positive solutions for a class of quasilinear singular equations*, Electron. J. Differential Equations 2004, Paper no. 56, 15 pp.

[77] D.-P. Covei, *Existence and uniqueness of positive solutions to a quasilinear elliptic problem in $\mathbb{R}^N$*, Electron. J. Differential Equations 2005, Paper no. 139, 15 pp.

[78] D.-P. Covei, *Existence and asymptotic behavior of positive solution to a quasilinear elliptic problem in $\mathbb{R}^N$*, Nonlinear Anal. **69** (2008), 2615–2622.

[79] X. Liu, Y. Guo, and J. Liu, *Solutions for singular $p$-Laplacian equations in $\mathbb{R}^N$*, J. Syst. Sci. Complex. **22** (2009), 597–613.

[80] S. Carl and K. Perera, *Generalized solutions of singular $p$-Laplacian problems in $\mathbb{R}^N$*, Nonlinear Stud. **18** (2011), 113–124.

[81] C.A. Santos, *Non-existence and existence of entire solutions for a quasilinear problem with singular and super-linear terms*, Nonlinear Anal. **72** (2010), 3813–3819.

[82] M.C. Rezende and C.A. Santos, *Positive solutions for a quasilinear elliptic problem involving sublinear and superlinear terms*, Tokyo J. Math. **38** (2015), 381–407.

[83] P. Drábek and L. Sankar, *Singular quasilinear elliptic problems on unbounded domains*, Nonlinear Anal. **109** (2014), 148–155.
[84] J.I. Diaz and J.E. Saa, *Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), 521–524.

[85] M. Chhetri, P. Drábek, and R. Shivaji, *Analysis of positive solutions for classes of quasilinear singular problems on exterior domains*, Adv. Nonlinear Anal. **6** (2017), 447–459.

[86] L. Gambera and U. Guarnotta, *Strongly singular convective elliptic equations in $\mathbb{R}^N$ driven by a non-homogeneous operator*, Comm. Pure Appl. Anal., doi:10.3934/cpaa.2022088.