Maximum Bounded Rooted-Tree Packing Problem

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Abstract

Given a graph and a root, the Maximum Bounded Rooted-Tree Packing (MBRTP) problem aims at finding \( K \) rooted-trees that span the largest subset of vertices, when each vertex has a limited outdegree. This problem is motivated by peer-to-peer streaming overlays in under-provisioned systems. We prove that the MBRTP problem is NP-complete. We present two polynomial-time algorithms that computes an optimal solution on complete graphs and trees respectively.

Keywords: Combinatorial problem, analysis of algorithms, computational complexity, distributed systems

1. Introduction

Internet is now used to transmit high-definition video streams to connected TVs and to share user-generated videos captured from high-quality cameras. Unfortunately, while the demand for transmitting videos with large bit-rate increases, the available bandwidth for a connected device has not grown that much (more devices by access point, awaited technology shift, etc.). Consequences of this lag between content and infrastructure progresses include that peer-to-peer (P2P, for short) live streaming systems, as they have been designed for years, face a new issue: the average upload capacity of peers is below the stream bit-rate and thus it is physically impossible to deliver a full-quality service to every peer. The system is said under-provisionned.

To address this issue, the use of multiple description coding technique is a promising approach. A video stream is divided into several independent
sub-streams, hereafter called *stripes*, the reception of a subset of stripes being enough to play the video. The more stripes are received, the better is the video quality. The main idea developed in [5] is that peers receive more or less degraded data video but the stream continuity is ensured.

In the current letter, we neglect the practical aspects of P2P systems (in particular peer churn and incentives to contribute) and we focus on obtaining a theoretical bound to the problem of delivering multiple stripes in an under-provisioned overlay network. Each stripe is served on an independent delivery tree from the source to a subset of peers. Our objective is to maximize the number of peers spanned in the multiple trees subject to the upload capacities of peers. The video quality experienced by a peer depends on the number of trees it belongs to.

We model the P2P network by an undirected and connected graph $G = (V, E)$ with $n = |V|$ vertices (or peers) and $m = |E|$ edges. The upload capacity of vertex $v$, denoted by $c_v$, is the number of stripes that $v$ can forward. It is a positive integer. The number of distinct stripes is represented by $K$, which is generally far smaller than $n$. A specific vertex $r$ in $V$ is given; it is called the *root* and represents the source peer of the video stream. A *rooted tree* (or *r-tree* for short) $T_k = (V_k, E_k)$ is an acyclic connected subgraph of $G$ with $r \in V_k$. Note that $(\{r\}, \emptyset)$ is the *null* r-tree. Given a rooted tree $T$ and a vertex $v$, let $C_T(v)$ be the number of children of $v$ in $T$. Throughout the paper, we use the shorthand notation $[n]$ to denote the set $\{1, 2, \ldots, n\}$. A family of $K$ rooted-trees $(T_1, T_2, \ldots, T_K)$ of $G$ is called a *bounded rooted-tree packing* if the following vertex-capacity requirements are satisfied:

$$\sum_{k=1}^{K} C_{T_k}(v) \leq c_v, \text{ for all } v \in V.$$  \hspace{1cm} (1)

The construction of multiple-tree overlay in P2P networks can be generalized as the *Maximum Bounded Rooted-Tree Packing (MBRTP)* problem, which consists of finding a bounded rooted-tree packing $(T_k = (V_k, E_k) : k \in [K])$ that spans the maximum number of vertices, that is, $\sum_{k\in[K]} |V_k|$ is maximized. Whenever only one r-tree is sought (i.e., $K = 1$), the considered problem will
be called the \textit{Maximum Bounded Rooted-Tree (MBRT)} problem.

To our knowledge, the MBRTP problem is a new optimization problem. It is loosely related to the Minimum Bounded Degree Spanning Tree problem, which tries to determine a minimum-cost spanning tree wherein any vertex has its degree at most a given value $k$ \[2\]. Variants with non-uniform degree bounds have also been studied \[3\]. Differently, the MBRT problem considers the case where not all vertices can be spanned, the MBRTP problem extending this formulation to a forest. There is no cost to minimize here.

In this paper, we prove that both MBRT and MBRTP problems are NP-hard. We present two polynomial-time algorithms that determine the optimal solutions for the MBRTP problem on complete graphs and trees, respectively.

\section{NP-Completeness of the MBRTP problem}

We prove the NP-completeness of the MBRTP problem using a reduction to 3-SAT problem. The decision problem related with MBRTP problem is:

\textbf{Question :} Does there exist a bounded $r$-tree packing of size $K$ in which the total number of vertices is greater than or equal to a positive integer $\Gamma$, i.e. $\sum_{k=1}^{K} |V_k| \geq \Gamma$ ? \hspace{0.5cm} (2)

\textbf{Theorem 1.} The MBRTP decision problem is NP-complete.

\textit{Proof.} We consider the MBRT problem ($K = 1$). Verifying that a $r$-tree solves a MBRT instance is polynomial in the size of the problem. Hence the MBRT decision problem belongs to NP.

Given an instance of the 3-SAT problem comprising a set of variables $W = \{x_i : 1 \leq i \leq n\}$ and a set of clauses $C = \{C_j : 1 \leq j \leq m\}$ on $W$ where $C_j = x_i^1 \lor x_i^2 \lor x_i^3$, we define an instance of the MBRT problem as follows (see Figure 1). Let $V' = \{r\} \cup \{i, \overline{i}, \overline{x_i} : 1 \leq i \leq n\} \cup \{C_j : 1 \leq j \leq m\}$ and let $E' = \{ri : 1 \leq i \leq n\} \cup \{ix_i, \overline{x_i} : 1 \leq i \leq n\} \cup \{x_i^lC_j : 1 \leq j \leq m, 1 \leq l \leq 3\}$. For $1 \leq i \leq n$ and for $1 \leq j \leq m$, the capacity function is defined as $c_r = n$, \hspace{0.5cm} (1)
$c_i = 1$, $c_{x_i} = c_{\overline{x_i}} = m$ and $c_{C_j} = 0$. Let $\Gamma$ be $1 + 2n + m$. This MBRT instance can be constructed in polynomial time in the size of the 3-SAT instance.

There exists a solution for our MBRT instance if and only if there exists a truth assignment for $C$.

For the forward implication, assume that there exists a $r$-tree $T = (U, F)$ of $G' = (V', E')$ satisfying \(1\) and \(2\). Inequalities \(1\) enforce that $U$ cannot contain both $x_i$ and $\overline{x_i}$ for $1 \leq i \leq n$, because $c_i = 1$ and $c_{C_j} = 0$ for $1 \leq j \leq m$. Since $|U| \geq \Gamma = 1 + 2n + m$ by \(2\), it follows that exactly one of $x_i$ and $\overline{x_i}$ belongs to $U$. Since $|U| \geq 1 + 2n + m$ and $|\{x_i, \overline{x_i}\} \cap U| = 1$, every $C_j$ for $1 \leq j \leq m$ is in $U$, and $C_j$ is adjacent to exactly one vertex among $\{x_j^1, x_j^2, x_j^3\}$ in $T$. We define the assignment function $\varphi$ as follows: $\varphi(x_i)$ is set to True if $x_i \in U$ and False if $\overline{x_i} \in U$. We directly obtain that each clause in $C$ has a true value, therefore $\varphi$ is a truth assignment for $C$.

For the backward implication, assume that we have a truth assignment $\varphi'$ for $C$. We define $W'$ the set of true literals for $\varphi'$, that is $W' = \{x_i : x_i \in W, \varphi'(x_i) = True\} \cup \{\overline{x_i} : x_i \in W, \varphi'(x_i) = False\}$. We construct the $r$-tree $T' = (U', F')$ of $G' = (V', E')$ as follows. The set of vertices $U'$ is $\{r\} \cup \{i : 1 \leq i \leq n\} \cup W' \cup \{C_j : 1 \leq j \leq m\}$. We clearly have $|U'| = 1 + 2n + m$, which makes $T'$ satisfy \(2\). We leverage on the truth assignment $\varphi'$ to associate with every clause $C_j$ one literal $x_j^i$ among $\{x_j^1, x_j^2, x_j^3\} \cap W'$. The set of edges $F'$ is defined as $\{ri : 1 \leq i \leq n\} \cup \{ix_i : 1 \leq i \leq n, x_i \in W'\} \cup \{i\overline{x_i} : 1 \leq i \leq n, \overline{x_i} \in W'\} \cup \{x_j^iC_j : 1 \leq j \leq m\}$. This construction guarantees that $T'$ is connected and without loss of generality, we can assume that $T'$ is acyclic. Since $r$ belongs to $U'$, $T'$ is a $r$-tree. Moreover, we have that $C_{T'}(r) = n$ and $C_{T'}(C_j) = 0$ for $1 \leq j \leq m$. Since $C_{T'}(i) = 1$, $C_{T'}(x_i) \leq m$, and $C_{T'}(\overline{x_i}) \leq m$ for $1 \leq i \leq n$, we have that $T'$ satisfies \(1\).
Hence MBRT is NP-complete. Consequently, MBRTP is NP-complete.

3. MBRTP on complete graphs

We present now a polynomial-time algorithm for complete graphs. Pseudocode is in Algorithm 1. We prove that it computes an optimal solution.

This algorithm performs in two stages. The first stage contains $K$ iterative steps. At each step $k \in [K]$, we compute a path rooted at $r$, noted $s_k$. Let $G_k$ be the complete graph for the set of vertices having a nonzero capacity at the beginning of the $k$th step. If the capacity $c_r$ of the root is zero, the path $s_k$ is empty, otherwise we construct $s_k$ such that it is a Hamiltonian path on $G_k$. Recall that a Hamiltonian path visits each vertex of the graph exactly once, and that computing a Hamiltonian path on a complete graph is trivially in linear-time. Then, we decrease by one the capacity of every vertex in the path, except the termination vertex. The second stage also contains $K$ steps. At each step $k \in [K]$, we compute a tree $T_k$ for the solution. The tree $T_k$ is initialized with the path $s_k$, so it is rooted in $r$ and it contains one branch containing all vertices in $s_k$. Then, every vertex $v$ in this branch having a nonzero remaining capacity attaches to $T_k$ vertices that are not yet in $T_k$ until either all vertices in $V$ are in $T_k$, or $v$ has no more capacity.

Theorem 2. Given that $G$ is a complete graph, the MBRTP problem can be solved in polynomial-time $O(nK)$.

Proof. We prove that Algorithm 1 provides the optimal solution. If $c_r = 0$, Algorithm 1 results in $K$ null $r$-trees, which is trivially optimal. Here we focus on the case where $c_r > 0$. Since graph $G$ is complete, a complete subgraph can be built from any subset of vertices of $G$, therefore it is possible to find a Hamiltonian path for each pruned subgraph $G_k$. The capacity of the root $r$ decreases by one for every Hamiltonian path $s_k$ unless either $k$ equals $K$ or $c_r$ is zero. Therefore $\min\{c_r, K\}$ non-null $r$-trees are produced at the end of the first stage. In the second stage, for each non-null $r$-tree $T_k$, the capacity checking process does not finish until any one of the following conditions is satisfied:
Algorithm 1: Algorithm for the MBRTP problem in complete graphs

**Input**: Complete graph $G$, root $r$, integer $K$, capacity $c_v$ for all $v \in V$

**Output**: $K$ $r$-trees $(T_1, T_2, \ldots, T_K)$

1. for $k \leftarrow 1$ to $K$
2. $G_k \leftarrow$ the complete graph of vertex set $\{ v | c_v > 0, v \in V \} \cup \{ r \}$
3. if $c_r$ is zero then $s_i \leftarrow (\{ r \}, \emptyset)$ else $s_k \leftarrow$ a Hamiltonian path in $G_k$ rooted in $r$;
4. for every vertex $v$ in $s_k$ except the termination vertex $c_v \leftarrow c_v - 1$
5. for $k \leftarrow 1$ to $K$
6. initialize the $r$-tree $T_k = (V_k, E_k)$ from $s_k$;
7. $V_k \leftarrow V \setminus V_k$;
8. for $v \in V_k$ do
9. while $V_k \neq \emptyset$ and $c_v > 0$ do
10. attach a vertex $v' \in V_k$ to $r$-tree $T_k$ with $v$ as parent;
11. $V_k \leftarrow V_k \setminus \{ v' \} ; c_v \leftarrow c_v - 1$

- $V_k = \emptyset$, which means that $T_k$ already includes all vertices in $G$,
- $c_v = 0, \forall v \in T_k$, that is, all vertices in $T_k$ have exhausted their capacity.

Consequently, the number of spanned vertices in the $\overline{k} = \min\{ c_r, K \}$ non-null trees is equal to $\min\{ \overline{k} + \sum_{v \in V} c_v, Kn \}$. In both cases, this number reaches the maximum imposed by (1) and (2).

The first stage takes $O(nK)$ time while the latter one terminates in time $O(nK)$ too. Considering $K \leq n$, Algorithm finishes in polynomial time.

4. MBRTP on trees

We now consider the case where $G$ is a tree. Designating vertex $r$ as the root, $G$ becomes a $r$-tree. Parameter $K$ is still the number of trees in the bounded $r$-tree packing. Given a peer $v \in V$ and an integer $k \in [K]$, let MBRTP$^v_k$ be a sub-instance of the MBRTP problem so that the underlying tree is the subtree of $G$ rooted at $v$ and the number of bounded $r$-trees to compute is $k$.

First, every vertex $v$ computes the number of spanned vertices in the optimal solution for every sub-instance MBRTP$^v_k$, for all $k \in [K]$. Each vertex $v$ stores
the results in a $K$-dimensional vector denoted by $g(v)$. The $k$th component of $g(v)$, which is noted $g(v)_k$, corresponds to the number of spanned vertices counting $v$ itself in the optimal solution of MBRTP$^v_k$. Obviously, $g(v)_k$ is monotonically increasing with respect to $k$ for any vertex $v$. For a leaf $v$ of $G$, the solution of MBRTP$^v_k$ is $k$ times the vertex $v$ itself (formally $g(v)_k = k$).

A non-leaf vertex $u$ leverages on the computations that have been made by its children to compute its own vector $g(u)$. We define $C_G(u)$ as the set of children of $u$ in $G$, and $n_G(u) = |C_G(u)|$. Let $x^i_{uv}, i \in [k], u \in V, v \in C_G(u)$ be a set of binary variables where $x^i_{uv}$ equals 1 if $u$ allocates exactly $i$ capacities to its child $v$ (i.e., $v$ is the child of $u$ in $i$ of the $k$ bounded $r$-trees), otherwise it is 0. If the variable $x^i_{uv}$ is 1, then $v$ is served with $i$ stripes that $v$ is able to relay in the sub-trees rooted at itself, spanning exactly $g(v)_i$ peers counting $v$ itself. When all variables $x^i_{uv}, i \in [k]$ are zero for $v$, vertex $v$ will receive zero stripe and neither $v$ nor its children will be spanned in the $k$ bounded $r$-trees.

Given an integer $k \in [K]$, the capacity $c_u$ of vertex $u$, and a vector $g(v)$ of every vertex $v \in C_G(u)$, the value of $g(u)_k$ can be obtained by solving the following Non-Standard Multiple-Choice Knapsack Problem (NS-MCKP):

$$\text{max } k + \sum_{v \in C_G(u)} \sum_{i=1}^k g(v)_i \times x^i_{uv} \quad \text{(NS-MCKP)}$$

$$\text{s.t. } \sum_{v \in C_G(u)} \sum_{i=1}^k i \times x^i_{uv} \leq c_u, \quad \text{(3)}$$

$$\sum_{i=1}^k x^i_{uv} \leq 1 \quad \forall v \in C_G(u), \quad \text{(4)}$$

$$x^i_{uv} \in \{0, 1\} \quad \forall v \in C_G(u) \quad \forall i \in [k].$$

Constraints (3) ensure that the capacity constraint of $u$ should not be violated by the sum of the capacity allocated to its children. Constraints (4) mean that the number of stripes sent by $u$ to $v$ is in $\{0, 1, 2, ..., k\}$.

**Lemma 3.** The above NS-MCKP can be solved in time $O(k^2 \times n_G^2(u))$.

**Proof.** We use dynamic programming as follows. Without loss of generality, we label the children of $u$ from $v_1$ to $v_{n_G(u)}$. Given two integers $d \in [n_G(u)]$
and $c \in \{0\} \cup [c_u]$, let NS-MCKP$_d^k(c)$ be the sub-instance of NS-MCKP where the set of children of $u$ is restricted to $\{v_1, ..., v_d\}$, $k$ is the number of received stripes, and the capacity is $c$. We denote by $f^k_d(c)$ the optimal solution value for the NS-MCKP$_d^k(c)$. When $d = 1$ we have

$$f^k_1(c) = \begin{cases} k & \text{if } c = 0, \\ k + g(v_1)c & \text{if } 1 \leq c < k, \\ k + g(v_1)k & \text{if } k \leq c. \end{cases}$$

When $2 \leq d \leq n_G(u)$, $f^k_d(0)$ is $k$, and whenever $c$ is greater or equal to $k \times d$, then $f^k_d(c) = k + \sum_{i \in \{d\}} g(v_i)k$, that is, the optimal solution consists of assigning $k$ capacities to every child in $\{v_1, ..., v_d\}$. For any value of $c$ ranging from 1 to $k \times d$, the solution $f^k_d(c)$ is computed by comparing $f^k_{d-1}(c)$ with what can be obtained if $u$ decides to allocate $i \in [k]$, capacities to the vertex $v_d$. Formally,

$$f^k_d(c) = \begin{cases} k & \text{if } c = 0, \\ \max \{f^k_{d-1}(c), \max \{f^k_{d-1}(c-i) + g(v_d) : i \in [k]\}\} & \text{if } 1 \leq c < k \times d, \\ k + \sum_{i \in \{d\}} g(v_i)k & \text{if } k \times d \leq c. \end{cases}$$

The solution of the NS-MCKP problem is $f^k_{n_G(u)}(c_u)$. For each value of $d$, the computation requires at most $O(k \times \min\{c_u, k \times d\})$ comparisons, as a result the overall time complexity of solving the NS-MCKP problem is $O(k^2 \times n^2_G(u))$.

**Theorem 4.** Given that $G$ is a tree, the MBRTP problem can be solved in polynomial-time $O(n^3 K^3)$.

**Proof.**

A vertex $u$ computes its vector $g(u)$ with $g(u)_k = f^k_{n_G(u)}(c_u)$ for any $k \in [K]$. This requires solving the NS-MCKP problem $K$ times. The value of $g(r)_K = f^K_{n_G(r)}(c_r)$ corresponds to the optimal solution of the MBRTP problem as it corresponds to the optimal solution of MBRTP$_K^r$. It should be noted that $f^K_{n_G(r)}(c_r)$ cannot be computed before knowing all vectors $g(.)$ of the root’s children. Consequently, the computation of vector $g(.)$ should be done from the leaves to the root $r$ in a breadth-first manner, which requires solving the

\[ \text{Theorem 4.} \]
NS-MCKP problem $nK$ times in total. We have $n_G(u) < n$, thus the overall time complexity of the proposed algorithm is $O(n^3K^3)$ provided that $G$ is a tree. As $K \leq n$, it is polynomial. \hfill \Box

5. Conclusion

In this paper we investigate the Maximum Bounded Rooted-Tree Packing Problem in under-provisioned P2P networks, which aims at maximizing the number of peers that are spanned in the multiple video delivery trees under the capacity constraint of peers. We prove that the MBRTP problem is NP-hard, while it can be polynomially solved on both complete graphs and trees.

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