A stochastic Poisson structure associated to a Yang-Baxter equation

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Abstract. We consider a simple solution of a Yang-Baxter equation on loop algebra and deduce from it a Sklyanin Poisson structure which operates continuously on a Sobolev test algebra on the Wiener space of the Lie algebra.

It is very classical that the solution of the classical Yang-Baxter equation on a finite dimensional algebra gives a Poisson structure on the algebra of smooth function on the finite dimensional Lie algebra [1]. Solution of Yang-Baxter equation in infinite dimension were studied by Belavin-Drinfeld in [2]. So it should be a Poisson structure associated to the solution of this Yang-Baxter equation. We refer to the review [3]. Our goal in this communication is to define functional spaces where this Poisson structure act continuously in a simple case. For that we use the apparatus of infinite dimensional analysis.

Poisson structure in infinite dimensional analysis were studied by Dito-Léandre [4] and after by Léandre in [5], [6], [7], [8], [9], [10], [11]. Especially Léandre studied the case of hydrodynamic Poisson structure in [9], [10], [11]. We consider another type of Poisson structure on the Wiener space.

1. The deterministic case
Let $so(n)$ be the Lie algebra of the orthogonal group endowed with its biinvariant Killing form $Tr$. We consider several complexified Banach spaces:

- The Banach space $H_p$ of maps $s \rightarrow B(s)$ from $S^1$ into $so(n)$ such that

$$\|B\|_p = \left( \int_{S^1} |B(s)|^p dg(s) \right)^{1/p} < \infty$$

- The Hilbert space $H.S$ of maps $s \rightarrow B(s)$ from $S^1$ into $so(n)$ such that

$$\|B\|_S^2 = \int_{S^1} |B(s)|^2 dg(s) + \int_{S^1} |B'(s)|^2 dg(s) < \infty$$

where $dg(s)$ denotes the normalized Haar measure on $S^1$.

$H.S$ is an Hilbert space for the Hilbert norm $\|\|_S$ and $\|\|_2$. 

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If $B(s) = \sum x_n s^n$, the associated Hilbert transform $R$ is given by

$$R(B)(s) = -\sum_{n<0} x_n s^n + \sum_{n\geq 0} x_n s^n$$  \hspace{1cm} (3)

$R$ is a continuous operator in all the $H_p$ [12] and in $H.S.$

Let $F$ be a smooth Gateaux map on $H.S.$ We say that $F$ is strongly Gateaux smooth if for all $p_1, \ldots, p_r$, $\sum \frac{1}{p_i} < 1$

$$|dF(h_1, \ldots, h_r)| \leq C_{p_1, \ldots, p_r}(F)(B) \prod \|h_i\|_{p_i}$$  \hspace{1cm} (4)

for all $h_i$ in $H.S.$ The smallest quantity which satisfy the previous inequality is still denoted by $C_{p_1, \ldots, p_r}(F)(B)$. We suppose that $d^r F$ is step by step Gateaux differentiable for the norm $C_{p_1, \ldots, p_r}(F)(B)$.

We consider another pairing for $h_1, h_2 \in H_{p_1}, H_{p_2}$, $1/p_1 + 1/p_2 \leq 1$ related to the pairing of [1]:

$$< h_1, h_2 > = \int_{S^1} Tr(h_1(s)sh_2(s))dg(s)$$  \hspace{1cm} (5)

This pairing is not degenerated. $R$ is antisymmetric for $<,>$. If $F$ is strongly Gateaux smooth, we can write:

$$dF(h) = < \nabla F, h >$$  \hspace{1cm} (6)

Let us remark that $\nabla F$ is computed for $<,>$ and not for $\|,\|_2$.

**Theorem 1** Let us denote by $A$ the space of strongly Gateaux smooth functional on $H.S.$ It is an algebra.

**Definition 2** Let $F^1, F^2$ belonging to $A$. The Sklyanin bracket is given by

$$\{F^1, F^2\}(B) = < B, [\nabla F^1, R(B\nabla F^2)] + [R(\nabla F^1 B), \nabla F^2]>$$  \hspace{1cm} (7)

This object has a sense by the help of the Hoelder inequality. Namely $\nabla F^1, \nabla F^2, B, R(B\nabla F^2), R(\nabla F^1 B)$ belong to all the $H_p$.

**Theorem 3** If $F^1, F^2$ belong to $A$, $\{F^1, F^2\}$ belong still to $A$.

**Scheme of the proof:** The hypothesis that $F$ is strongly Gateaux smooth implies that for all $p_i$, $\sum \frac{1}{p_i} < 1$, $1/p_1 + 1/q_1 = 1$

$$\|d^{r-1} \nabla F(h_2, \ldots, h_r)\|_{q_1} \leq (C_{p_1, \ldots, p_r} F)(B) \prod_{i=2}^{r} \|h_i\|_{p_i}$$  \hspace{1cm} (8)

It is enough to show that some quantities of the type $\psi = < B, \nabla F^1 R(B\nabla F^2) >$ are strongly Gateaux smooth. This leads to estimate some quantities of the type

$$< h_1, d^{r-1} \nabla F^1(h_1, \ldots, h_r)R(h_2d^{r-2} \nabla F^2(h_1^2, \ldots, h_r^2)) >$$  \hspace{1cm} (9)

The result arise from the Hoelder inequality, the previous inequality and from the fact that $R$ is continuous in all the $H_p$ ([12]).

We put

$$[B_1, B_2]_R = [R(B_1), B_2] + [B_1, R(B_2)]$$  \hspace{1cm} (10)

$R$ satisfied the modified Yang-Baxter equation

$$[R(B_1), R(B_2)] - R([B_1, B_2]_R) = -[B_1, B_2]$$  \hspace{1cm} (11)

$R$ is antisymmetric for $<,>$, $\nabla F$ is computed for $<,>$ and $Tr(xyz) = Tr(yzx)$. Therefore ([1], [3]).

**Theorem 4** The Sklyanin bracket defines a Poisson structure on $A.$
2. The stochastic Sklyanin-Poisson structure
We consider the Gaussian process $s \rightarrow B(s)$ with reproducing Hilbert space $H.S$. By using
the theory of radonification ([13]), $s \rightarrow B(s)$ is in fact only continuous. $\text{sup}_s |B(s)|$ belong to all
the $L^p(dP)$. $dP$ is the law of the process $s \rightarrow B(s)$ on $C(S^1, so(n))$, the space of continuous
functions from $S^1$ into $so(n)$ endowed with the uniform topology.

If $h$ belong to $H.S$, it is classical that the law of $B + \lambda h \lambda \in R$ is absolutely continuous with
respect of the law of $B$. So if we consider a Wiener functional (almost surely defined!) $F(B)$,
the Wiener functional $F(B + \lambda h)$ is well defined. Let us recall that the Malliavin Calculus [14]
has a lot of precursors motivated by mathematical physics [15], [16], [17], [18] but we use in
this note the final version of the Malliavin Calculus [19]. We can apply the machinery of the
Malliavin Calculus and consider the Gateaux derivative (almost surely defined!) in the direction
of $h \in H.S$ of $F$. We iterate the procedure and we suppose that the iterated Gateaux derivatives
(almost surely defined) satisfy still (4). We suppose that we can take the Gateaux derivative of
$dt^F$ for the norm $C_{p_1...,p_r}(F)$.

Definition 5A Wiener functional $F$ is said to be strongly smooth in the Malliavin sense if
the previous requirements are satisfied and if all his strong Sobolev norms $\|F\|_{p_1,...,p_r,p}$, $\sum _{1 \leq i \leq r} 1/p_i < 1$,
infinity $\geq p \geq 1$ are finite $\|F\|_{p_1,...,p_r,p} = E[C_{p_1,...,p_r}(F)]^{1/p} < \infty $ (12)

Theorem 6 The space $W_{\infty -}$ is a Frechet algebra.

Definition 7 Let $F^1, F^2 \in W_{\infty -}$. The stochastic Sklyanin bracket is defined by

$$\{F^1, F^2\}(B) = \langle B, [\nabla F^1, R(B \nabla F^2)] + [R(\nabla F^1 B), \nabla F^2] \rangle$$ (13)

For the same reason than in the Definition 2, the stochastic Sklyanin bracket is defined, but
almost surely!

Theorem 8 If $F^1, F^2$ are strongly smooth in the Malliavin sense, $\{F^1, F^2\}$ is still strongly
smooth in the Malliavin sense, and its strong Sobolev norms can be estimated in terms of the
strong Sobolev norms of $F^1$ and $F^2$.

Scheme of the proof: The algebra is the same as in the proof of Theorem 3. Only estimates
are differents. Difficulties come when in expressions taking the derivatives of quantities as $\psi$ in
the proof of Theorem 3 there is no derivative of $B$ which appear. But this can be handle by
using the Hoelder inequality because

$$E[\text{sup}_s |B(s)|^{1/p}] < \infty$$ (14)

♦

Definition 9 Let $A_{\infty -}$ be a topological unital Frechet algebra. A Poisson structure $\{,\}$ on $A_{\infty -}$
is a continuous bilinear antisymmetric map from $A_{\infty -} \times A_{\infty -}$ into $A_{\infty -}$, which is a derivation
in each argument, and which satisfy the Jacobi relation: if $F^1, F^2, F^3$ are element of $A_{\infty -}$

$$\{F^1, \{F^2, F^3\}\} + \{F^2, \{F^3, F^1\}\} + \{F^3, \{F^1, F^2\}\} = 0$$ (15)

Theorem 10 The stochastic Sklyanin bracket defines a Poisson structure on $W_{\infty -}$.

Scheme of the proof: Only the estimates are different than in the algebraic proof of
Theorem 4. But they are carried out by the previous theorem.♦

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