Pancharatnam revisited

Erik Sjöqvist
Department of Quantum Chemistry, Uppsala University,
Box 518, S-751 20 Uppsala, Sweden

Some recent ideas concerning Pancharatnam’s prescription of relative phase between quantal states are delineated. Generalisations to mixed states and entangled two-photon states are discussed. An experimental procedure to test the geometric phase as a Pancharatnam relative phase is described. We further put forward a spatial split-beam dual to Pancharatnam’s relative phase.

1 Introduction

Consider a pair of vectors $|A\rangle$ and $|A\rangle' = e^{i\alpha}|A\rangle$ representing one and the same quantal state $A$. Let us ask: what is their relative phase? Probably everyone would agree that the answer to this question is $\alpha$. Less obvious, however, is the case when the pair of vectors represent different quantal states. Apparently no one had asked this question until Pancharatnam came up with a physical prescription for the relative phase between distinct polarisation states of light. Although his analysis concerned classical light it has been realised that Pancharatnam’s concept of relative phase has a quantal counterpart with surprisingly rich structure related to the geometry of the quantum state space. This counterpart, known as quantum parallel transport, is the origin of the geometric phase first discovered for cyclic adiabatic evolution and later generalised to nonadiabatic and noncyclic pure state evolutions, as well as to mixed quantal states.

The purpose of this report is to describe some recent ideas concerning the concept of relative phase. To do this we first describe Pancharatnam’s original idea and how it relates to the geometry of quantum state space. The basic observation in this context is the experimental fact that the maximum of the interference oscillations, obtained when a variable phase shift is introduced to one of the interfering states $A$ and $B$, is shifted when $A \neq B$. This shift constitutes the desired concept of relative phase. To develop this idea in detail, let us suppose that $|A\rangle$ and $|B\rangle$ are two normalised Hilbert space representatives of the nonorthogonal states $A$ and $B$, respectively, and assume further that $|A\rangle$ is exposed to the $U(1)$ shift $e^{i\chi}$. The resulting interference pattern is determined by

$$I = \left| e^{i\chi}|A\rangle + |B\rangle \right|^2 = 2 + 2|\langle A|B\rangle| \cos[\chi - \arg\langle A|B\rangle]$$

(1)

that attains its maximum at the Pancharatnam relative phase $\phi \equiv \arg\langle A|B\rangle$. 

1
This phase reduces naturally to the $U(1)$ case whenever $|B⟩ = e^{iα}|A⟩$ as it yields $φ = α$.

Pancharatnam’s prescription of relative phase has a peculiar property that arises when two in-phase ($φ = 0$) vectors $|A⟩$ and $|B⟩$ are sent through a polariser that projects onto a third state $C$. The resulting state vectors are $|C⟩' = |C⟩⟨C|A⟩$ and $|C⟩'' = |C⟩⟨C|B⟩$. Their relative phase becomes

$$\arg \langle C|C'⟩'' = \arg \langle A|C⟩⟨C|B⟩⟨B|A⟩ ≡ \Delta(A, B, C),$$

where we have used the assumption that $\arg ⟨A|B⟩ = 0$. The right-hand side $\Delta(A, B, C)$ of Eq. (2) is independent of the choice of Hilbert space representatives of the three states $(A, B, C)$ and is therefore a property of the quantum state space. $\Delta(A, B, C)$ further fulfills the following two important properties that makes it related to oriented area:

a) It is additive: $\Delta(A, B, C, D) = \Delta(A, B, C) + \Delta(A, C, D)$.

b) It depends upon orientation: $\Delta(A, C, B) = -\Delta(A, B, C)$.

Indeed, in the case of qubits (two-level systems) these properties lead to the following natural geometric interpretation of $\Delta(A, B, C)$ known as the geometric phase. Let us take $A$ at the north pole, $B$ with polar angles $(θ, ϕ)$, and $C = B + dB$ with polar angles $(θ + dθ, ϕ + dϕ)$ being infinitesimally close to $B$ on the Bloch sphere. It yields $\Delta(A, B, B + dB) = -\frac{1}{2}(1 - \cos θ)dϕ$, which is minus one-half the solid angle enclosed by the spherical triangle defined by the three points $A, B, B + dB$ on the Bloch sphere. Thus, for a finite spherical triangle defined by any $A, B, C$ on the Bloch sphere one may divide the geodesic line connecting $B$ and $C$ into small pieces and use the properties a) and b) to obtain

$$\Delta(A, B, C) = -\Delta(A, C, B) = -\frac{1}{2}Ω(A, B, C),$$

where $Ω(A, B, C)$ is the enclosed solid angle.

In the next section we apply the above ideas to mixed quantal states. Section 3 contains a description of Pancharatnam’s relative phase for nonclassical two-photon polarisation states. Generalising the above solid angle formula to any continuous path on the Bloch sphere raises the question how such a quantity can be tested experimentally. In section 4 a general procedure how this can be achieved is described. A spatial dual to Pancharatnam’s original idea concerning internal states is discussed in section 5. The paper ends with the conclusions.
2 Generalisation to mixed states

Suppose a mixed quantal state evolves as $\rho_A \rightarrow \rho_B = U \rho_A U^\dagger$ with $U$ unitary. How is Pancharatnam’s prescription of relative phase generalised to such states? Here, we discuss such a generalisation that was discovered in the context of interferometry.

The mixed state phase is based on the observation that the evolution governed by the unitarity $U$ of any density operator may be described as

$$\rho_A = \sum_k w_k |A_k\rangle \langle A_k| \rightarrow \rho_B = \sum_k w_k |B_k\rangle \langle B_k|,$$

where each $|B_k\rangle = U |A_k\rangle$. Evidently, each such orthonormal pure state component of the density operator contributes to the interference according to Eq. (1). Thus, the total interference profile becomes

$$I = \sum_k w_k \left| e^{i\chi} |A_k\rangle + |B_k\rangle \right|^2 = 2 + 2 \sum_k w_k |\langle A_k|B_k\rangle| \cos[\chi - \arg \langle A_k|B_k\rangle],$$

where we have used that the $w_k$’s sum up to unity. This takes a more transparent and explicitly basis independent form when writing the interference profile in terms of the quantity $\text{Tr}(U \rho_A)$ as

$$I = 2 + 2 |\text{Tr}(U \rho_A)| \cos[\chi - \arg \text{Tr}(U \rho_A)].$$

The important observation from Eq. (6) is that the interference oscillations produced by the variable $U(1)$ phase $\chi$ is shifted by $\Phi = \arg \text{Tr}(U \rho_A)$ for any mixed state $\rho_A$. This shift reduces to Pancharatnam’s original prescription when $\rho_A = |A\rangle \langle A| \rightarrow \rho_B = |B\rangle \langle B|$. These two facts are the central properties for $\Phi$ being a natural generalisation of Pancharatnam’s relative phase to mixed states. Moreover, the visibility of the interference pattern is $|\text{Tr}(U \rho_A)| \geq 0$, which reduces to the expected $|\langle A|B\rangle|$ in Eq. (1) for pure states.

The oriented area introduced in Eq. (2) may be generalised to the sequence $\rho_A \rightarrow \rho_B \rightarrow \rho_C$ of nondegenerate density operators, all with the same set of eigenvalues $\{w_k\}$, but with different sets of eigenvectors $\{|A_k\rangle\}$, $\{|B_k\rangle\}$, $\{|C_k\rangle\}$. First, let us introduce the quantities $\Delta(A_k, B_k, C_k)$ for each set of eigenstates.
pure state components corresponding to one and the same eigenvalue \( w_k \). In terms of these we may take the mixed state generalisation of Eq. (2) as

\[
\Delta(A, B, C) = \arg \left[ \sum_k w_k |\langle A_k | U | A_k \rangle|^e^{i \Delta(A_k, B_k, C_k)} \right], \tag{7}
\]

where \( U \) now takes \( \rho_A \) to \( \rho_C \) via \( \rho_B \). Note that \( \Delta(A, B, C) \) in Eq. (7) is nonadditive due to its nonlinear dependence upon the additive pure state quantities \( \Delta(A_k, B_k, C_k) \), but it depends upon the orientation of the unitary path \( U \) as it fulfils \( \Delta(A, C, B) = -\Delta(A, B, C) \).

For a mixed qubit state \( \rho_A = \frac{1}{2}(1 + r) |A_0 \rangle \langle A_0| + \frac{1}{2}(1 - r) |A_1 \rangle \langle A_1| \), where \( \langle A_k | A_l \rangle = \delta_{kl} \) and \(-1 \leq r \leq 1\), we have \( |\langle A_0 | U | A_0 \rangle| = |\langle A_1 | U | A_1 \rangle| \) and \( \Delta(A_0, B_0, C_0) = -\Delta(A_1, B_1, C_1) = -\frac{1}{2} \Omega(A_0, B_0, C_0) \equiv -\frac{1}{2} \Omega \). Thus, if the \( \rho \)'s are nondegenerate \( (r \neq 0) \) we obtain

\[
\Delta(A, B, C) = -\arctan \left[ r \tan \frac{\Omega}{2} \right], \tag{8}
\]

which is the mixed state generalisation of Eq. (7) and reduces to the pure state case when \( |r| = 1 \).

3 Two-photon relative phase

Pancharatnam’s original work applies to polarised classical light waves as well as to quantal single photon states. In this section we generalise this situation to nonclassical entangled two-photon polarisation states.

Consider first a photon pair being prepared in a product polarisation state \( |\Pi_0 \rangle = |A \rangle \otimes |A' \rangle \equiv |AA' \rangle \). Suppose each of these states undergo pure rotations around some spherical triangle on the Bloch sphere defined by the points \( A, B, C \) and \( A', B', C' \), respectively. According to the theory of geometric phases the final states pick up the phases \( -\frac{1}{2} \Omega(A, B, C) \equiv -\frac{1}{2} \Omega \) and \( -\frac{1}{2} \Omega(A', B', C') \equiv -\frac{1}{2} \Omega' \), respectively. Thus, the total state \( \Pi_0 \longrightarrow \Pi_f = \Pi_0 \) acquires the relative phase

\[
\phi = \arg(\Pi_0 | \Pi_f) = -\frac{\Omega + \Omega'}{2}, \tag{9}
\]

after traversing the pair of loops.

Now, in presence of entanglement this additive property has to be modified as follows. Any entangled two-photon polarisation state can be written on Schmidt form as

\[
|\Pi_0 \rangle = \sqrt{\lambda} |AA' \rangle + \sqrt{1 - \lambda} |A_\perp A'_\perp \rangle, \tag{10}
\]
where $\langle A|A_\perp \rangle = \langle A'|A'_\perp \rangle = 0$. The real-valued parameter $0 \leq \lambda \leq 1$ determines the degree of entanglement $\Xi \equiv |1 - 2\lambda|$ with $\Xi$ ranging from 0 (maximally entangled states) to 1 (product states). Suppose again that each of the two states $A$ and $A'$ is rotated around some spherical triangle so that they pick up the phases $-\frac{1}{2}\Omega$ and $-\frac{1}{2}\Omega'$, respectively. On the other hand, the orthogonal states $A_\perp$ and $A'_\perp$ are taken around paths with opposite orientation, but enclose the same area so that they pick up the phases $+\frac{1}{2}\Omega$ and $+\frac{1}{2}\Omega'$, respectively. Thus, the photons acquire the relative phase

$$\tilde{\phi} = \arg(\Pi_0|\Pi_f) = \arctan \left[ (1 - 2\lambda) \tan \frac{\Omega + \Omega'}{2} \right],$$

and the visibility

$$|\langle \Pi_0|\Pi_f \rangle| = \sqrt{\cos^2 \frac{\Omega + \Omega'}{2} + (1 - 2\lambda)^2 \sin^2 \frac{\Omega + \Omega'}{2}} \leq 1.$$
photon pair through a Franson type interferometer. In the longer arms local rotations are applied in such a way that $|\Pi_f\rangle$ is obtained. In one of the shorter arms the $U(1)$ shift $e^{i\chi}$ is added to $|\Pi_0\rangle$. To observe a superposition of $|\Pi_0\rangle$ and $|\Pi_f\rangle$ we require that the source produces photon pairs randomly. This is the case with the source for polarisation entangled photon pairs demonstrated experimentally in [12]. If the photons arrive simultaneously in the output detectors, they both either took the shorter path ($|\Pi_0\rangle$) or the longer path ($|\Pi_f\rangle$). Thus, the measured coincidence profile becomes

$$I = |e^{i\chi}|^2 + 2|\langle \Pi_0 | \Pi_f \rangle | \cos [\chi - \arg \langle \Pi_0 | \Pi_f \rangle],$$

which verifies the two-photon Pancharatnam phase in Eq. (11) and the reduced visibility in Eq. (12).

4 Test of geometric phase

Consider the continuous path $\eta : t \in [0, \tau] \rightarrow |A_t\rangle \langle A_t| = U_t|A_0\rangle \langle A_0|U_t^\dagger$ of normalised pure state projectors with $\langle A_0 | A_\tau \rangle \neq 0$ and $U_t$ unitary. Dividing the path into small pieces and using the additive property a) described in Sec. 1, we obtain the geometric phase associated with $\eta$ as

$$\phi_g[\eta] = \lim_{N \rightarrow \infty} \arg \left( \langle A_0 | A_\tau \rangle \langle A_\tau | A_{(N-1)\tau/N} \rangle \cdots \langle A_{\tau/N} | A_0 \rangle \right).$$

(16)

$\phi_g[\eta]$ is a property of the path $\eta$ as it is independent of the lift $\eta \rightarrow \tilde{\eta} : t \in [0, \tau] \rightarrow |A_t\rangle$ and, in the $N \rightarrow \infty$ limit, of the particular subdivision of the path. A parallel lift is defined by requiring that each $\langle A_{(j+1)\tau/N} | A_{j\tau/N} \rangle$, $j = 0, ..., N - 1$, be real and positive, which in the $N \rightarrow \infty$ limit takes the form

$$\langle A_t | \dot{A}_t \rangle = 0.$$  

(17)

For such a parallel lift, we obtain $\phi_g[\eta]$ as

$$\phi_g[\eta] = \arg \langle A_0 | A_\tau \rangle.$$  

(18)

We here argue that the geometric phase for any such path $\eta$ can be made experimentally accessible as a Pancharatnam relative phase. The idea is based on Uhlmann’s approach to the geometric phase for any quantal state $\rho_t$ using the purification $W_t$ such that $W_t W_t^\dagger = \rho_t$. For the unitary pure state path $\eta$ we may write $W_t = U_t |A_0\rangle \langle A_0|\tilde{U}_t$, where an ancilla state $|\tilde{A}_t\rangle = \tilde{U}_t|A_0\rangle$ is attached to the system. Now, a parallel purification is obtained by choosing $\tilde{U}_t$ so that

$$W_t^\dagger W_t = \text{hermitian}.$$  

(19)
For such a parallel purification, the quantity
\[ \beta = \arg \text{Tr}[W_0^\dagger W_\tau], \]
(20)
takes, in the pure state case, the form
\[ \beta = \arg \langle \tilde{A}_\tau | A_\tau \rangle = \arg \langle A_0 | \tilde{U}_\tau^\dagger U_\tau | A_0 \rangle, \]
(21)
which can be shown to fulfil \[ \beta = \phi_g[\eta]. \]

Now, for pure states the purification does not need any entanglement between the system and the ancilla. Thus, the above procedure should work as a superposition effect for the system alone. To see this, consider the lift \[ \eta \to |A_t\rangle = U_t |A_0\rangle \]
and the auxiliary path \[ |\tilde{A}_t\rangle = \tilde{U}_t |A_0\rangle, \]
both in the Hilbert space of the system. By letting the auxiliary path be exposed to a \( U(1) \) shift \( e^{i\chi} \), the interference profile at each instant of time \( \tau \) reads
\[ I = \left| e^{i\chi} |\tilde{A}_\tau\rangle + |A_\tau\rangle \right|^2 = 2 + 2 \langle \tilde{A}_\tau | A_\tau \rangle \cos[\chi - \arg \langle \tilde{A}_\tau | A_\tau \rangle] \]
(22)
that attains its maximum at the relative phase \( \phi_\tau = \arg \langle A_0 | \tilde{U}_\tau^\dagger U_\tau | A_0 \rangle \). This phase is generally related to the geometric phase as \( \phi_\tau = \phi_g[\eta] + \gamma_\tau \), where \( \gamma_\tau \) is the accumulation of local phase changes along \( \eta \). However, in analogy with the above purification approach, \( \gamma_\tau \) can be made to vanish at each instant of time by choosing the auxiliary \( \tilde{U}_t \) so that it fulfils the parallelity condition \( (19) \). In other words, if \( \tilde{U}_t \) fulfils Eq. \( (19) \) it is assured that \( \phi_\tau \) is identical to the geometric phase associated with the continuous path \( \eta \).

To illustrate this idea, consider a spin-\( \frac{1}{2} \) initially polarised in the +z direction and undergoing precession around the axis \( n = (\sin \theta, 0, \cos \theta) \) under the action of the Hamiltonian
\[ H = \frac{1}{2} \left( \sin \theta \sigma_x + \cos \theta \sigma_z \right), \]
(23)
\( \sigma_x \) and \( \sigma_z \) being the standard Pauli spin operators in the \( x \) and \( z \) direction, respectively. If we put \( \tilde{U}_t = \exp(-i\tilde{H}t) \) and insert \( |+z\rangle \langle +z| \) and \( H \) into Eq. \( (19) \), we obtain
\[ \tilde{H} = \frac{1}{2} \cos \theta \sigma_z. \]
(24)
Thus, the Pancharatnam phase difference becomes
\[ \phi_\tau = \langle +z | \tilde{U}_\tau^\dagger U_\tau | +z \rangle = - \arctan \left( \cos \theta \tan \frac{\varphi}{2} \right) + \frac{\varphi}{2} \cos \theta \]
(25)
with \( \varphi \) the precession angle of the spin. This is precisely the noncyclic geometric phase \( -\frac{1}{2} \Omega_{gc} \), where \( \Omega_{gc} \) is the solid angle enclosed by the path \( \eta \) and the shortest geodesic connecting its end-points on the Bloch sphere.
Furthermore, the state $| -z\rangle\langle -z |$ acquires the relative phase $+\frac{i}{2} \Omega_{gc}$ under the action of the unitarities $U_t$ and $\tilde{U}_t$. Thus, according to Eq. (3) we obtain
\[
\Phi_\tau = -\arctan \left[ r \tan \frac{\Omega_{gc}}{2} \right],
\] which is the mixed state generalisation of Eq. (25). An explicit neutron interferometer experiment implementing the procedure described in this section has been proposed in [4]. This experiment tests the geodesically closed solid angle in noncyclic precession of the neutron spin.

5 Dual setup

It has been demonstrated [3] that Pancharatnam’s relative phase for an internal spin degree of freedom may be tested in interferometry. In such an experiment the spin state is changed due to some local interaction in one of the beams, while the other beam is exposed to a variable $U(1)$ shift. Here, we argue that the spatial beam-pair itself may acquire an unambiguous Pancharatnam relative phase, dual to the usual phase acquired by the spin state. This dual phase could be tested polarimetrically by letting both a nontrivial spin-space interaction and a spin measurement be delocalised to both the spatial beams.

Let us first briefly review the Pancharatnam relative phase for a precessing spin-$\frac{1}{2}$ system. The evolution of the state vector reads
\[
|A_0\rangle = \cos \frac{\theta}{2} |+z\rangle + \sin \frac{\theta}{2} |-z\rangle \rightarrow
|A_f\rangle = e^{-i\varphi/2} \cos \frac{\theta}{2} |+z\rangle + e^{i\varphi/2} \sin \frac{\theta}{2} |-z\rangle,
\] where $\varphi$ is the precession angle and $| \pm z \rangle$ are eigenvectors of the Pauli spin operator $\sigma_z$ in the $z$ direction. The Pancharatnam relative phase between $|A_0\rangle$ and $|A_f\rangle$ is
\[
\phi = \arg \langle A_0 | A_f \rangle = -\arctan \left( \cos \frac{\theta}{2} \tan \frac{\varphi}{2} \right),
\] where we have assumed that $\langle A_0 | A_f \rangle \neq 0$. This phase could be tested in neutron interferometry by letting one beam be exposed to a time-independent uniform magnetic field in the $z$ direction and applying a variable phase shift $e^{i\chi}$ to the other beam. The last beam-splitter in the interferometer makes the two spin states $A_0$ and $A_f$ to interfere, yielding the output interference profile as
\[
I = |e^{i\chi} |A_0\rangle + |A_f\rangle|^2 = 2 + 2|\langle A_0 | A_f \rangle| \cos [\chi - \arg \langle A_0 | A_f \rangle]
\]
with visibility
\[ |\langle A_0|A_f\rangle| = \sqrt{1 - \sin^2 \theta \sin^2 \frac{\varphi}{2}}. \] (30)

Now, the basic setup to test the dual Pancharatnam relative phase was proposed in [3] for neutrons and is sketched in Fig. 1. It consists of three steps:

(i) A superposed spatial beam state is created by sending a beam of neutrons polarised along the +z direction through a beam-splitting crystal plate with transmission probability \( T \) and reflection probability \( R \). With 0 and 1 the two spatial beam states, this yields the total state vector
\[ |\Psi_0\rangle = |A_0\rangle + z\rangle = \left[ \cos \left( \frac{\theta}{2} \right) |0\rangle + \sin \left( \frac{\theta}{2} \right) |1\rangle \right] + z\rangle, \] (31)
where we have set \( \sqrt{T} = \cos \left( \frac{\theta}{2} \right) \).

(ii) The 0−beam is exposed to a magnetic field \( B_0 = B_0 \hat{x} \) and the 1−beam is exposed to \( B_1 = B_1 \hat{x} \), as shown in Fig. 1. \( B_0 \) and \( B_1 \) are time-independent and uniform over some finite spatial region. The change in the spin state in each beam is described by the unitary operators
\[ U(\varphi_0) = e^{-i\varphi_0 \sigma_x / 2}, \]
\[ U(\varphi_1) = e^{-i\varphi_1 \sigma_x / 2}, \] (32)
where \( \sigma_x \) is the Pauli spin operator in the \( x \) direction. Here, \( \varphi_0 \propto B_0 \) and \( \varphi_1 \propto B_1 \) are the Larmor precession angles. By introducing the spatial state vectors
\[ |A^+\rangle = e^{-i\Delta \varphi / 4} \cos \frac{\theta}{2} |0\rangle + e^{i\Delta \varphi / 4} \sin \frac{\theta}{2} |1\rangle, \]
\[ |A^-\rangle = e^{i\Delta \varphi / 4} \cos \frac{\theta}{2} |0\rangle + e^{-i\Delta \varphi / 4} \sin \frac{\theta}{2} |1\rangle \] (33)
with \( \Delta \varphi = \varphi_0 - \varphi_1 \) and \( \chi = (\varphi_0 + \varphi_1)/2 \), we may write the final total state vector as
\[ |\Psi_f\rangle = \frac{1}{2} \left[ e^{-i\chi/2} |A^+\rangle + e^{i\chi/2} |A^-\rangle \right] + z\rangle \]
\[ + \frac{1}{2} \left[ e^{-i\chi/2} |A^+\rangle - e^{i\chi/2} |A^-\rangle \right] - z\rangle. \] (34)

(iii) A spin analyser measures the spin in the \( z \) direction in each beam, as shown in Fig. 1. The interference profile \( I \) in the +z spin channel, say, is...
proportional to the sum of $+z$ detections in the two spatially separated analysers. This is dual to the interference measurement in a single spatial channel, while summing over the two spin states, in the setup designed to measure the spin-$-\frac{1}{2}$ Pancharatnam relative phase $\chi$. The interference profile in the $+z$ spin channel reads

$$I = \left| e^{i\chi/2}|A^-\rangle + e^{-i\chi/2}|A^+\rangle \right|^2 \propto 2 + 2|\langle A^-|A^+\rangle| \cos \left[ \chi - \arg \langle A^-|A^+\rangle \right].$$

(35)

Thus, by keeping $\Delta \varphi \propto B_0 - B_1$ fixed and varying $\chi \propto B_0 + B_1$ the dual Pancharatnam relative phase $\arg \langle A^-|A^+\rangle$ and visibility $|\langle A^-|A^+\rangle|$ can be measured.

6 Conclusions

We have discussed extensions of Pancharatnam’s prescription of relative phase to mixed states and entangled two-photon states. The solid angle relation for single qubit states becomes nonlinear in these extensions and we have discussed experiments to test this nonlinearity. A continuously evolving quantal state is associated with a geometric phase at each point along the path in state space. For a given lift of such a path we have proposed a general procedure to systematically cancel the accumulation of local phase changes relative a specific auxiliary path in Hilbert space. This guarantees that Pancharatnam’s relative phase between the two paths in Hilbert space is identical to the geometric phase. Finally, we have argued that a spatially delocalised state may acquire a dual Pancharatnam relative phase that could be measured polarimetrically. We hope that these ideas may lead to new experimental tests of Pancharatnam’s prescription of relative phase as well as to trigger further extensions to the theory of interference.

Acknowledgements

I wish to thank Jeeva Anandan, Artur Ekert, Marie Ericsson, Björn Hessmo, Daniel Oi, Arun Pati, and Vlatko Vedral for collaboration on Pancharatnam’s phase. This work was supported by the Swedish Research Council.
References

1. S. Pancharatnam, Proc. Indian Acad. Sci. A 44, 247 (1956).
2. M.V. Berry, Proc. R. Soc. London Ser. A 392, 45 (1984).
3. Y. Aharonov and J.S. Anandan, Phys. Rev. Lett. 58, 1593 (1987).
4. J. Samuel and R. Bhandari, Phys. Rev. Lett. 60, 2339 (1988).
5. A. Uhlmann, Rep. Math. Phys. 24, 229 (1986).
6. E. Sjöqvist, A.K. Pati, A. Ekert, J.S. Anandan, M. Ericsson, D.K.L. Oi, and V. Vedral, Phys. Rev. Lett. 85, 2845 (2000).
7. A.G. Wagh, V.C. Rakhecha, P. Fischer, and A. Ioffe, Phys. Rev. Lett. 81, 1992 (1998).
8. N. Mukunda and R. Simon, Ann. Phys. (N.Y.) 228, 205 (1993).
9. J. Brendel, W. Dultz, and W. Martienssen, Phys. Rev. A 52, 2551 (1995).
10. B. Hessmo and E. Sjöqvist, Phys. Rev. A 62, 062301 (2000).
11. J. D. Franson, Phys. Rev. Lett. 62, 2205 (1989).
12. A.G. White, D.F.V. James, P.H. Eberhard, and P.G. Kwiat, Phys. Rev. Lett. 83, 3103 (1999).
13. A. Uhlmann, in Symmetry in Physics V, Algebraic Systems, Their Representations, Realizations, and Physical Applications, Ed. B. Gruber (Plenum Press, New York, 1993);
14. E. Sjöqvist, Phys. Lett. A 286, 4 (2001).
15. E. Sjöqvist, Phys. Rev. A 63, 035602 (2001).
Figure Captions

Fig.1. Set up for testing the dual Pancharatnam relative phase. The source prepares a coherent superposition of the two spatial beam states 0 and 1, both being spin-polarised perpendicular to the plane of the set up. Each beam is exposed to a magnetic field in a common direction in the plane of the set up, but with different strength $B_0 \neq B_1$. The dual Pancharatnam relative phase is measured by the pair of spin analyzers by keeping the difference $B_0 - B_1$ fixed, while varying the sum $B_0 + B_1$. 
