Stationary and static cylindrically symmetric Einstein spaces of the Lewis form

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Abstract. The derivation of the general solutions for stationary and static cylindrically symmetric Einstein spaces of Lewis form is revisited and the physical and geometrical meaning of the parameters appearing in the resulting solutions are investigated. It is shown that three of the parameters (and the value of the cosmological constant) are essential, of which one characterizes the local gravitational field and appears in the Cartan scalars, while the remaining two give information about the topological identification made to produce cylindrical symmetry. Other than the cosmological constant, they can be related to the parameters of the vacuum Weyl and Lewis classes, whose interpretation was previously investigated by da Silva et al. (1995a, 1995b).

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1. Introduction

In a recent paper, Santos (1993) gave a set of solutions for stationary cylindrically symmetric spacetimes of the Lewis form (1932) with a cosmological constant. Krasinski had previously given these in a different form (Krasinski 1975a, 1994). Both use coordinates in which the static limit is hard to obtain. We shall reconsider the derivation of the complete set of stationary and static solutions. In another set of recent papers, da Silva et al. (1995a, 1995b) have given some physical identifications for the parameters of the corresponding vacuum solutions. Here we extend that investigation to the cases with cosmological constant.

We first make some remarks on the problem of definition of cylindrical symmetry. In Kramer et al (1980) a spacetime is defined to be cylindrically symmetric if it is axisymmetric about an infinite axis and translationally symmetric along that axis. The definition of axisymmetry gives some difficulties (Mars and Senovilla 1993, 1995). In the non-singular case, we can characterize axisymmetry by the requirement that some Killing vector (a) has closed spacelike orbits near the axis and (b) has length zero at the axis. In order for the axis to be non-singular we require elementary flatness
at the axis. If such an axisymmetric spacetime is also stationary, then one can prove that the stationary and axisymmetry Killing vectors commute and form surfaces with orthogonal transitivity, but these results depend on the existence of regular axis points (see chapter 17 of Kramer et al (1980)).

These ideas are not currently well-defined if the axis is not regular, or has no regular part, but to exclude metrics without regular axes would be too restrictive as it excludes many of the known solutions which one would want to call cylindrical. In particular it rules out solutions which if continued to the axis ($\rho = 0$ in the usual coordinate systems) would have strange behaviour but which can be joined at some $\rho = \rho_0 > 0$ to a perfectly regular and well-behaved interior solution. This requirement, together with the existence of two commuting spacelike Killing vectors of which one has closed orbits and the other has non-closed orbits, so that they are transitive on surfaces of topology $S^1 \times \mathbb{R}^1$, could be taken as the definition of cylindrical symmetry.

However, at present we know no way to test whether a metric could form a part of a globally well-behaved cylindrically symmetric solution in the sense of Kramer et al and it is therefore simpler to adopt the less demanding definition requiring only the existence of spacelike surfaces of symmetry with topology $S^1 \times \mathbb{R}^1$.

If there is no regular axis, one could define the presence of a singular axis by requiring the spacelike Killing vector with closed orbits to have a length which tends to zero, but the subsequent treatment of such spacetimes has some difficulties. In particular one cannot prove the orthogonal transitivity property, either in the stationary case (where it can occur with a timelike surface of transitivity), or in the more general non-stationary cylindrically symmetric case, in which Kramer et al (1980) assume it to occur with a spacelike surface of transitivity (see equation (20.1) there). In particular, if in the latter form one assumes that the metric coefficients are independent of time, one would wish to call the solution static even though the metric does not take the usual form for a stationary axisymmetric metric with orthogonal transitivity: for some such solutions see Chitre et al (1975), MacCallum (1983). Another possible problem is shown by the solution discussed by Lemos (1994) in which the “axis” $\rho = 0$ in our coordinates is interpreted as the horizon of a cylindrical black hole.

Thus the Lewis metric form used by Santos does not include all possible stationary Einstein metrics with cylindrical symmetry, defined as above. Even in the vacuum case the solutions of the Lewis form in general do not have well-defined axes (where the Killing vector with closed orbits has zero length), or if they do the axis is not in general elementarily flat. We therefore begin by simply assuming the Lewis form of the metric (Lewis, 1932),

$$ds^2 = -f dt^2 + 2k dt d\varphi + \ell d\varphi^2 + e^\nu dr^2 + e^\mu dz^2$$  \hspace{1cm} (1.1)

where $f$, $k$, $\ell$, $\nu$ and $\mu$ are functions of $r$ only, and the ranges of the coordinates $t$, $r$, $z$ and $\varphi$ are initially taken to be

$$-\infty < t < \infty \quad -\infty < r < \infty \quad -\infty < z < \infty \quad 0 \leq \varphi \leq 2\pi$$  \hspace{1cm} (1.2)

with the hypersurfaces $\varphi = 0$ and $\varphi = 2\pi$ identified to ensure the cylindrical symmetry. Note that we have not assumed $0 \leq r$, in order both to avoid assuming the existence of an axis and to allow more freedom of coordinate choice. The coordinates are numbered

$$x^0 = t \quad x^1 = r \quad x^2 = z \quad x^3 = \varphi.$$  \hspace{1cm} (1.3)

We should note immediately that the identification made on $\varphi$ makes this an improper coordinate system: it does not obey the differential geometric requirement
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of giving a homeomorphism between a region of the manifold and an open set in \( \mathbb{R}^4 \). For that reason, we should really use at least two coordinate patches to cover the whole manifold. The permissible local transformations of coordinates, on these two or more patches, which are compatible with a identification in \( \varphi \), with period \( P' \) say, are (MacCallum, 1997)

\[
\bar{t} = T_1t + T_0, \quad \bar{r} = \bar{r}(r), \quad \bar{z} = Z_1z + Z_0, \quad \bar{\varphi} = X_1t + X_2\varphi + X_0, \quad (1.4)
\]

where the upper case letters denote constants, and \( \bar{r} \) is an arbitrary function. It is clear that the changes of origin of \( t, z \) and \( \varphi \) do not alter any metric coefficient and can be ignored, so we need only consider the more restricted transformations

\[
\bar{t} = T_1t, \quad \bar{r} = \bar{r}(r), \quad \bar{z} = Zz, \quad \bar{\varphi} = X_1t + X_2\varphi. \quad (1.5)
\]

However, we could also locally allow the more general form

\[
\bar{t} = T_1t + T_2\varphi, \quad \bar{r} = \bar{r}(r), \quad \bar{z} = Zz, \quad \bar{\varphi} = X_1t + X_2\varphi. \quad (1.6)
\]

In MacCallum (1997), it is shown that of the four parameters \( T_1, T_2, X_1 \) and \( X_2 \) only two are essential; they fix the topological identification needed to reach the actual solution starting from its locally equivalent standard form. If in the above transformations the barred coordinates refer to this standard form then the essential parameters \( S \) and \( P \) of MacCallum (1997) are \(-X_2/T_2 \) and \( 2\pi X_2 \) (or we could say, more succinctly, that \( X_2 \) and \( T_2 \) are essential).

The radial coordinate can be chosen so that \( \nu = \mu \) (as Santos (1993) does initially) or so that \( \nu = 0 \), which is the more convenient choice of radial coordinate introduced later by Santos (1993), called \( r^* \) in that paper, which is in fact the natural \( r \) coordinate arising from the existence of hypersurface-homogeneity of Bianchi type I on timelike surfaces (which thus necessarily have geodesic normal unit vectors). With this choice the remaining coordinate freedom of \( r \) is \( \bar{r} = \pm r + R \) where \( R \) is a constant.

2. The Einstein space solutions

We now revisit and complete the derivation of the solutions of the Einstein equations

\[
R_{ab} = \Lambda g_{ab} \quad (2.1)
\]

for the metric form considered. Following Santos (1993) we find that if \( u = f/\ell \) and \( v = k/\ell \) then with the choice of \( r \) given by \( \nu = \mu \), and denoting \( d/dr \) by \( ' \), we have

\[
u' = \Theta'(u + v^2), \quad \varphi' = \Phi'(u + v^2) \quad (2.2)
\]

where \( (\rho\Theta)' = 0 = (\rho\Phi)' \), \( \rho \) (denoted \( D \) in Santos (1993)) being the usual radius parameter \( \rho^2 = f \ell + k^2 \) (which in vacuum and some other cases can also be taken as an isothermal coordinate in the \((r, z)\) surfaces). There are now several possibilities. If both \( \Theta' \) and \( \Phi' \) are non-zero, they must be multiples of one another, say \( \Phi' = A\Theta' \), and this leads (as in Santos (1993)) to \( v = Au + B \) where \( A \) and \( B \) are constants, i.e. to \( k = Af + B\ell \). If one of \( \Theta' \) and \( \Phi' \) is zero, so that either \( \Theta \) or \( \Phi \) is constant (possibly zero), then again \( f \), \( k \) and \( \ell \) are linearly dependent, and this is also true if \( \ell = 0 \) so that \( u \) and \( v \) are not well-defined. Thus in all cases the proposition of MacCallum (1997) applies, and all solutions are locally equivalent either to a diagonal metric, or one with a null Killing vector, or a ‘windmill’ form (which can be considered as a complex diagonal metric). Moreover, the solution for the diagonal case will give the solutions
for the remaining cases if we allow complex values for parameters and coordinates and take limits as in McIntosh (1992).

Taking the form (1.1) with \( k = 0 = \nu \) we find (cf. Linet (1985) and Santos (1993)) that if \( G = \rho e^{\mu/2} \) then

\[ G'' + 3\Lambda G = 0. \]

The distinct solutions (after choice of the origin and sign of \( r \)) are

\[ G = C \cosh \left( \sqrt{3|\Lambda|}r \right), \quad \Lambda < 0; \quad (2.3) \]
\[ = C \exp \left( \sqrt{3|\Lambda|}r \right), \quad \Lambda < 0; \quad (2.4) \]
\[ = C \sinh \left( \sqrt{3|\Lambda|}r \right), \quad \Lambda < 0; \quad (2.5) \]
\[ G = Cr, \quad \Lambda = 0; \quad (2.6) \]
\[ G = C \sin \left( \sqrt{3\Lambda}r \right), \quad \Lambda > 0; \quad (2.7) \]

where in all cases \( C > 0 \). The interpretation of \( G < 0 \) is unclear. For example, it might represent a double-covering of the spacetime or a region interior to a horizon.

The vacuum solutions are included for comparison purposes although these solutions, which can all be related to the Kasner form (see McIntosh (1992) and references therein), have been discussed in many earlier papers. By scaling the three ignorable coordinates we can arrive at \( C = 1/\sqrt{|3\Lambda|} \) if \( \Lambda \neq 0 \): for \( \Lambda = 0 \) we can normalize so that \( C = 1 \), and we denote these normalized cases by \( G_0 \). They have the property that

\[ (G_0')^2 + 3\Lambda(G_0)^2 = \eta \]

where in the five cases (2.3–2.7) \( \eta \) is respectively \(-1, 0, 1, 1 \) and \( 1 \).

The case \( \Lambda < 0 \) gives three forms of solution, but Linet (1985) considers only (2.5), and the differences are not obvious in Krasinski’s complicated form. In Santos (1993) all three are included in the general form

\[ G = C_1 \cosh \left( \sqrt{-3\Lambda}r \right) + C_2 \sinh \left( \sqrt{-3\Lambda}r \right), \quad (2.8) \]

the three cases corresponding to \( |C_1| > |C_2|, |C_1| = |C_2|, \) and \( |C_1| < |C_2| \). In the first and third cases the \( C \) above is \( \sqrt{|C_1^2 - C_2^2|} \), while in the second case \( C = |C_1| \). The origin shifts in \( r \) required to reach (2.3) and (2.5) are respectively arctanh(\( C_2/C_1 \)) and arctanh(\( C_1/C_2 \)). In the similar formulae for the case (2.6) \( C = C_2 \) and the shift is \( C_1/C_2 \) and in the case (2.7) \( C = \sqrt{C_1^2 + C_2^2} \) and \( R = \arctan(C_1/C_2) \).

With these specializations, and using allowable scalings of the remaining coordinates, we can write the resulting metric (Linet, 1985) as

\[ ds^2 = dr^2 + G_0^{2/3} \left[ \sum_{j=0,2,3} \varepsilon_j \exp(2(p_j - 1/3)U)(dx^j)^2 \right] \]

where \( \varepsilon_j \) is 1 except for \( x^0 = t \), for which it is \(-1 \), and \( dU/dr = 1/G_0 \). (No constant of integration is given in \( U \) as it can be absorbed by rescalings of coordinates, but the rescalings allowed appear explicitly in the discussion below.) The constants \( p_j \) (which are \( 1/3 + KK_j/2 \) in terms of Linet’s notation) obey

\[ \sum_j p_j = 1, \quad \sum_j p_j^2 = (2\eta + 1)/3. \]
The parametrization here has been chosen to agree with the usual forms of the Kasner vacuum metric (equations (11.50) and (11.51) in Kramer et al (1980)) and its ‘windmill’ counterpart (McIntosh, 1992) for the case Λ = 0, where $G = r = \exp U$. As in the Kasner vacuum case, a complex coordinate change gives related ‘cosmological’ solutions, i.e. solutions for which the essential variable has a timelike character. These related metrics have been given several times: Kasner himself (1925) gave the case related to (2.7), Saunders (1967) (see equation (11.52) in Kramer et al (1980)) the cases related to (2.5) and (2.7), Kale (1970) the case related to (2.3), Spindel (1979) gave the full set, and there may be other occurrences we have not found. The metric forms are included in those considered by Kellner (1975), but he seems not to give explicitly the solutions for a cosmological constant. Linet (1985) only considered the forms with (2.5) and (2.7), as he wished to have an axis $\rho = 0$ at some $r$.

Note that for $\eta = 0$ the only real solution is $p_i = 1/3$ for all $i$: other cases with $\eta = 0$, and all cases with $\eta = -1$, will have complex values of $p_i$. It is not obvious whether in general the resulting metrics, with the coordinates also considered to be complex, will have a real section, as happens in the ‘windmill’ case (McIntosh, 1992). However, there are clearly cases where this does happen: for example, with $\eta = -1$, if $p_0 = 1$, $p_2 = \bar{p}_3 = \sqrt{2i/\sqrt{3}}$ where we get a real solution by replacing $z$ and $\varphi$ by $x - iy$ and $x + iy$ respectively. Linet ignores these cases, taking his $K^2 > 0$. They are implicitly present in Santos (1993), where they correspond to complex values of $\delta$ and $\alpha\gamma/\beta$ in his equation (51), as corrected (see the erratum to the paper and below). The similar examples with $p_2 = 1$ are covered by his equation (57). If we assume that $p_2$ is real and $p_0 = a + ib = \bar{p}_3$, we find this is possible, with non-zero real $b$, for values of $a$ outside $[0, 2/3]$ if $\eta = 1$, for $a \neq 1/3$ if $\eta = 0$ and for any $a$ if $\eta = -1$. The cases with a null Killing vector can be obtained by going to coordinates in which the metrics appear stationary and then taking a suitable limit.

There are clearly two essential parameters in (2.9), one of the $p_i$, say $p_2$, and the cosmological constant (and the discrete parameter $\eta$ if $\Lambda < 0$). One could argue that $\Lambda$ should not be regarded as a parameter, since in the usual Einstein theory it would be a universal constant. However, if the term is regarded as arising as a relic of quantum processes, or if we consider that in writing down the whole family we have an infinite number of completely disjoint spacetimes, and hence no physical reason for using the same $\Lambda$, it may make sense to treat $\Lambda$ as a parameter. To make the solution cylindrical, $\varphi$ must be periodic, and the period is specified by another essential global parameter (see MacCallum (1997)): note that scaling $\varphi$ to make the period $2\pi$ alters the $g_{\varphi\varphi}$ of (2.9) by a constant factor. Thus in the form (2.9) for a cylindrical metric with a periodic $\varphi$ there are three parameters, $\Lambda$, $p_2$ and the period (plus the discrete parameter $\eta$ if $\Lambda < 0$).

For the distinct possible $G_0$ given by (2.3–2.7) the formulae for $U$ are respectively

\begin{align*}
U &= \arctan \left| \sinh \left( \sqrt{3} |\Lambda| r \right) \right|, \quad \Lambda < 0; \\
&= \exp \left( -\sqrt{3} |\Lambda| r \right), \quad \Lambda < 0; \\
&= \ln \tanh \frac{1}{2} \left( \sqrt{3} |\Lambda| r \right), \quad \Lambda < 0; \\
U &= \ln r, \quad \Lambda = 0; \quad \text{(2.14)} \\
U &= \ln \tan \frac{1}{2} \left( \sqrt{3} |\Lambda| r \right), \quad \Lambda > 0, \quad \text{(2.15)}
\end{align*}

cf. Santos (1993), equations (45)–(49).
We can now obtain the general forms of the corresponding classes of stationary cylindrically symmetric metrics by applying the transformations (1.6), taking the barred coordinates to be the ones for the standard form just discussed. This will introduce a fourth essential parameter (not counting \( \eta \)) which specifies the lines in the original \((\bar{\varphi}, \bar{t})\) surface parallel to \( t = 0 \) along which the topological identification is to be made, as well as some inessential parameters: as noted above \( X_2 \) and \( T_2 \) can be considered the essential parameters in addition to \( \Lambda \) and \( p_2 \). We do the calculations explicitly only for the diagonal form, assuming \( p_2 \) to be real, which leaves us the possibility of complex \( p_0 \) and \( p_3 \). Then

\[
\begin{align*}
\text{ds}^2 &= dr^2 + G_0^{2/3} \left( Z^2 \exp(2(p_2 - 1/3)U) \right) dz^2 \\
&\quad + \left[ X_2^2 \exp(2(p_3 - 1/3)U) - T_2^2 \exp(2(p_0 - 1/3)U) \right] d\varphi^2 \\
&\quad + 2[X_1 X_2 \exp(2(p_3 - 1/3)U) - T_1 T_2 \exp(2(p_0 - 1/3)U)] dt d\varphi \\
&\quad - \left[ T_2^2 \exp(2(p_0 - 1/3)U) - X_1^2 \exp(2(p_3 - 1/3)U) \right] dt^2
\end{align*}
\]

(2.16)

and the full coordinate freedom includes replacing \( r \) in the formulae for \( G_0 \) and \( U \) by \( \pm r + R \) where \( R \) is a constant. The geometry has 4 essential parameters \( \Lambda, p_2, X_2 \) and \( T_2 \). \( G \) is now given by

\[
G^2 = Z^2 (T_1 X_2 - T_2 X_1)^2 G_0^2.
\]

The remaining free parameters \( T_1, X_1, R \) and \( Z \) are inessential in specifying the geometry of the region covered, but may be required in order to match this metric to an interior using the Lichnerowicz conditions, which are the same as the earlier Darmois conditions but expressed in ‘admissible coordinates’ continuous across the boundary (see Bonnor and Vickers (1981) and applications in Krasinski (1975a) and Bonnor et al (1997)). This coordinate choice, and therefore the inessential coordinate parameters, are not required in the Darmois form, in which the conditions are just that the first and second fundamental forms of the surface, considered from the two sides, must agree, and thus only invariantly-defined parameters can appear. Of course, on trying to match a given solution to the solution here one may find it has more essential parameters than can in fact be matched by the solutions here, or parameters which appear in an incompatible way: there is no guarantee that a match is possible.

These solutions can now be identified with earlier forms. Krasinski (1975a) used a “radial” coordinate \( x^2 = k/f \) in the above notation: this leads to rather complicated forms, and the static limit \( k = 0 \) is awkward to handle. (This coordinate choice is made in order to fit the exteriors easily to his non-static interior solutions.) Indeed Krasinski (1975b) gives only the case \( \Lambda = 0 \) in the explicitly static form, as his equation (7.2), and says “The generalization of (7.2) to the case \( \Lambda \neq 0 \) is unexpectedly very involved, so I do not present it here”, though in fact the solutions themselves appear in stationary form. We can identify his classes, simply by looking at the coefficients in the linear dependence of \( f, k \) and \( \ell \) (cf. MacCallum (1997)) as follows. His class A is locally equivalent to the ‘windmill’ form, Class B is locally equivalent to a real diagonal form, and Classes C and D are locally equivalent to the form with a null Killing vector, the difference being that in Class D the identification is made along this Killing vector’s orbit. The identification of Class B with the (locally) static cases other than his (9.3) is stated in Krasinski (1975a), in the paragraph following Table IX, and the proof of this (which was known to Krasinski at that time (private communication, 1997)) is hinted at but not completely described in the paragraph following his equation (9.2).
The version given originally by Santos (1993) contains some typographical errors, corrected in the subsequent erratum.

(i) If $K = -\alpha^2 < 0$, we have
\[
\ell / \rho = \frac{1}{\alpha} \sinh \left( \frac{\alpha \Theta}{2\beta} \right),
\]
\[
k / \rho = -\cosh \left( \frac{\alpha \Theta}{2\beta} \right) - \frac{\beta}{\alpha} \sinh \left( \frac{\alpha \Theta}{2\beta} \right),
\]
\[
f / \rho = -2\beta \cosh \left( \frac{\alpha \Theta}{2\beta} \right) - \frac{\beta^2 + \alpha^2}{\alpha} \sinh \left( \frac{\alpha \Theta}{2\beta} \right).
\]

(ii) If $K = 0$,
\[
\ell / \rho = \frac{\Theta}{2\beta},
\]
\[
k / \rho = -1 - \frac{\Theta}{2},
\]
\[
f / \rho = -\frac{\beta}{2} \Theta - 2\beta.
\]

(iii) If $K = \alpha^2 > 0$,
\[
\ell / \rho = \frac{1}{\alpha} \sin \left( \frac{\alpha \Theta}{2\beta} \right),
\]
\[
k / \rho = -\cos \left( \frac{\alpha \Theta}{2\beta} \right) - \frac{\beta}{\alpha} \sin \left( \frac{\alpha \Theta}{2\beta} \right),
\]
\[
f / \rho = -2\beta \cos \left( \frac{\alpha \Theta}{2\beta} \right) - \frac{\beta^2 - \alpha^2}{\alpha} \sin \left( \frac{\alpha \Theta}{2\beta} \right),
\]

where
\[
\Theta = \gamma \Omega + \zeta, \quad e^{3\mu/2} = eG \exp(\delta \Omega), \quad \Omega = \int dr/G.
\]

$\delta$, $\gamma$, $\epsilon$ and $\zeta$ are constants of integration.

The 9 constant parameters, $\Lambda$, $C_1$, $C_2$, $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$ and $\zeta$, in these solutions are related, in each case, by one equation, as follows:

(i) $K < 0$
\[
3 |\Lambda| \left( C_1^2 - C_2^2 \right) + \frac{1}{4} \delta^2 + \frac{3 \alpha^2 \gamma^2}{16 \beta^2} = 0, \quad \Lambda < 0;
\]
\[
-C_2 \left( C_2 + \frac{1}{8} \delta^2 \right) + \frac{3 \alpha^2 \gamma}{16 \beta^2} = 0, \quad \Lambda = 0;
\]
\[
-3 \Lambda \left( C_2^2 + C_1^2 \right) + \frac{1}{4} \delta^2 + \frac{3 \alpha^2 \gamma^2}{16 \beta^2} = 0, \quad \Lambda > 0.
\]

(ii) $K = 0$
\[
3 |\Lambda| \left( C_1^2 - C_2^2 \right) + \frac{1}{4} \delta^2 = 0, \quad \Lambda < 0;
\]
\[
C_2 \left( C_2 + \frac{1}{8} \delta^2 \right) = 0, \quad \Lambda = 0;
\]
\[
-3 \Lambda \left( C_1^2 + C_2^2 \right) + \frac{1}{4} \delta^2 = 0, \quad \Lambda > 0.
\]
We now seek to identify the 9 parameters in the form of Santos (1993) with the parameters of the general solution (2.16). We have already identified $C_1$ and $C_2$ with $C$ and $R$ above and we see that $\Omega = U/3|\Lambda|$ if $\Lambda \neq 0$, or $U/C$ if $\Lambda = 0$. For the five possible $G_0$ we have $Z^2(T_1X_2 - T_2X_1)^x$ equal to, respectively $3|\Lambda|(C_1^2 - C_2^2)$, $3|\Lambda|C_1^2$, $3|\Lambda|(C_2^2 - C_1^2)$, $C_2^2$, and $3\Lambda(C_2^2 + C_1^2)$. We find $\delta = \sqrt{3|\Lambda|C(3p_2 - 1)}$ if $\Lambda \neq 0$ or $\delta = C(3p_2 - 1)$ if $\Lambda = 0$, and $\varepsilon = Z^2/(T_1X_2 - T_2X_1)$. In case (i) $\alpha\gamma = 6|\Lambda|C/\beta(p_3 - p_0)$ (or $\alpha\gamma = 2C/\beta(p_3 - p_0)$ if $\Lambda = 0$); in case (iii) $p_0$ and $p_3$ are complex conjugates and $\alpha\gamma = -6i|\Lambda|C/\beta(p_3 - p_0)$. The relations listed as (2.26–2.28) and (2.32–2.34) are then equivalent to the second of the equations (2.10), the first of these being already built into the forms above.

The parameters $\alpha$ and $\beta$ are most easily identified from the linear dependence of $f$, $k$, and $\ell$ which for case (i) is

$$f - 2\beta k - \beta^2 \ell = -\alpha^2 \ell$$

and can be compared with

$$f + \left(\frac{T_1}{T_2} + \frac{X_1}{X_2}\right)k - \frac{T_1X_1}{T_2X_2}\ell = 0$$

giving us that $\beta + \alpha$ and $\beta - \alpha$ are $T_1/T_2$ and $X_1/X_2$ (the correct pairing depending on the choice which keeps $t$ timelike). Finally, use of the transformation

$$\bar{t} = t - T_2\varphi/T_1, \quad \bar{\varphi} = -X_1t/X_2 + \varphi,$$

on the metric (2.16), and the same in terms of $\alpha$ and $\beta$ on that of Santos, enables us to identify $\zeta$ in terms of $\ln(X_2/T_1)$.

3. Invariant properties

As discussed above, we have 4 essential parameters in the general stationary metric, including $\Lambda$ but excluding $\eta$. From the work of MacCallum (1997) and the results of Da Silva et al (1995a), Da Silva et al (1995b) we can expect that 2 of these ($T_2$ and $X_2$) will appear in the holonomy around the curves on which only $\varphi$ changes, and the remaining two ($\Lambda$ and $p_2$) in invariants of the local curvature. Further parameters may appear from coordinate conditions on a matching to an interior solution.

We can compute the “Cartan scalars” which completely characterise the local geometry according to the procedure for invariant classification described in Karlhede (1980), Paiva et al (1993) and MacCallum and Skea (1994). For case (i) with the $G_0$ of equation (2.5) and $C = 1/\sqrt{3|\Lambda|}$, this calculation, which was carried out using the computer algebra package CLASSI built on SHEEP, gave, as the non-trivial classifying factors:

(iii) $K > 0$

$$3|\Lambda|(C_1^2 - C_2^2) + \frac{1}{4}k^2 - \frac{3\alpha^2\gamma^2}{16\beta^2} = 0, \quad \Lambda < 0;$$

$$-C_2^2 + \frac{1}{4}\delta^2 - \frac{3\alpha^2\gamma^2}{16\beta^2} = 0, \quad \Lambda = 0;$$

$$-3\Lambda(C_2^2 + C_1^2) + \frac{1}{4}\delta^2 - \frac{3\alpha^2\gamma^2}{16\beta^2} = 0, \quad \Lambda > 0.$$
invariants, the cosmological constant and
\[
\Psi_0 = \Psi_4 = 1/4\alpha\beta^{-1}\gamma|\Lambda|\sinh^{-2}(\sqrt{3}|\Lambda|r)
+ 1/4\alpha\beta^{-1}\gamma|\Lambda|\cosh(\sqrt{3}|\Lambda|r)\sinh^{-2}(\sqrt{3}|\Lambda|r)
\]
\[
\Psi_2 = -1/8\alpha^2\beta^{-2}\gamma^2|\Lambda|\sinh^{-2}(\sqrt{3}|\Lambda|r) + 1/3|\Lambda|\sinh^{-2}(\sqrt{3}|\Lambda|r)
- 1/6\delta|\Lambda|\cosh(\sqrt{3}|\Lambda|r)\sinh^{-2}(\sqrt{3}|\Lambda|r)
\]
\[
D\Psi_{01'} = D\Psi_{50'}
= \sqrt{6}\left(1/16\alpha^3\beta^{-3}\gamma^3|\Lambda|^{3/2}\sinh^{-3}(\sqrt{3}|\Lambda|r)
- 5/12\alpha\beta^{-1}\gamma|\Lambda|^{3/2}\cosh(\sqrt{3}|\Lambda|r)\sinh^{-3}(\sqrt{3}|\Lambda|r)
- 5/24\alpha\beta^{-1}\gamma|\Lambda|^{3/2}\sinh^{-1}(\sqrt{3}|\Lambda|r)
- 2/3\alpha\beta^{-1}\gamma|\Lambda|^{3/2}\sinh^{-3}(\sqrt{3}|\Lambda|r)
\right)
\]
\[
D\Psi_{10'} = D\Psi_{41'}
= \sqrt{6}\left(-1/16\alpha^3\beta^{-3}\gamma^3|\Lambda|^{3/2}\sinh^{-3}(\sqrt{3}|\Lambda|r)
- 1/12\alpha\beta^{-1}\gamma|\Lambda|^{3/2}\cosh(\sqrt{3}|\Lambda|r)\sinh^{-3}(\sqrt{3}|\Lambda|r)
- 1/24\alpha\beta^{-1}\gamma|\Lambda|^{3/2}\sinh^{-1}(\sqrt{3}|\Lambda|r)
+ 1/6\alpha\beta^{-1}\gamma|\Lambda|^{3/2}\sinh^{-3}(\sqrt{3}|\Lambda|r)
\right)
\]
\[
D\Psi_{21'} = D\Psi_{30'}
= \sqrt{6}\left(1/8\alpha^2\beta^{-2}\gamma^2|\Lambda|^{3/2}\cosh(\sqrt{3}|\Lambda|r)\sinh^{-3}(\sqrt{3}|\Lambda|r)
+ 1/12\delta|\Lambda|^{3/2}\sinh^{-1}(\sqrt{3}|\Lambda|r)
+ 1/6\delta|\Lambda|^{3/2}\sinh^{-3}(\sqrt{3}|\Lambda|r)
- 1/3|\Lambda|^{3/2}\cosh(\sqrt{3}|\Lambda|r)\sinh^{-3}(\sqrt{3}|\Lambda|r)
\right)
\]
We see that the other essential parameter that appears is \(\alpha\gamma/\beta\) which is a linear combination of the \(p_i\) and so equivalent to specifying \(p_2\) as expected. The results for the other possible cases described above are similar in form. Thus, as expected, we find that only two parameters are essential locally: all the other parameters can be locally removed by coordinate transformations.

We can also relate the Cartan scalars above to their limiting forms for the Weyl vacuum class considered by da Silva et al. (1995b). In the limit \(\Lambda = 0\) we can identify the other parameters in the Cartan scalars with the \(n\) of the Weyl form by \(\delta = 2(n^2 - 3)/(n^2 + 3)\), \(\alpha\gamma/\beta = 8n/(n^2 + 3)\), and with these relations the Cartan scalars above have the da Silva et al. values as their limits.

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