Experimentally friendly formulation of quantum backflow

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(Dated: August 19, 2020)

Quantum backflow is usually understood as a quantum interference phenomenon where probability current points in the opposite direction to particle’s momentum. Here, we quantify the amount of quantum backflow for arbitrary momentum distributions, paving the way towards its experimental verification. We give examples of backflow in gravitational and harmonic potential. The former is especially appealing as the probability of finding a free falling particle above initial level grows for suitably prepared quantum state with most momentum downwards.

INTRODUCTION

A wave function of a quantum particle has physically observable characteristics that can be local or global. The probability of finding the particle in a specific region of space or the probability current are examples of local characteristics, which can be determined given access to only small part of the wave function. In contradistinction, the momentum is a property of the entire wave function, e.g. requires the determination of the de Broglie wavelength. Already at this level of generality, it is clear that the probability current and the momentum of a quantum particle may behave very differently.

Quantum backflow (QB) is an interference effect that arises from this observation [1]. In order to understand the statement better, and to simplify the subsequent analysis, let us suppose that a particle in the one-dimensional space interacts with a potential that depends only on the particle’s position. Intuitively, we may think that if the momentum distribution concentrates within the positive half-line, the probability current, too, will be non-negative. In other words, the direction of the particle’s velocity, defined by the rate of change how probable it is to find the particle in a region of space, should coincide with the direction of its momentum. It turns out that this is not the case for a suitably chosen quantum state. In a sense, the probability flows ‘backwards’. Hence the term “quantum backflow”.

QB was first studied by Allcock in his work on the arrival time in quantum mechanics [2]. Later, Bracken and Melloy [3] gave a detailed analysis of QB as an eigenvalue problem. The analysis was rigorously phrased in the mathematical language of integral operators on separable Hilbert space in [4]. In the same paper, a numerical approximation of the optimal QB state was given. See also [5, 6] for interesting case studies. QB in systems interacting with linear potential was studied in [7] and [8]. Recently, there have been attempts at analysing QB in the relativistic setting [9,11], in the setting of quantum particle decay [12, 13], as well as the attempts at describing quantum backflow in dissipative [14] and many-particle systems [15].

Among others, Palmero et al. [1] proposed an experimental scheme that “could lead to the observation of quantum backflow” in $^7$Li Bose-Einstein condensate. To the best of our knowledge, however, QB has not yet been confirmed experimentally. Nevertheless, in a recent experimental work by Eliezer et al. [16], “optical backflow” in transverse momentum of a beam of light was reported. In contradistinction, the present study focuses on backflow of individual non-relativistic quantum particles.

Our goal here is to introduce QB as a phenomenon that has no analogue in classical mechanics. We extend the customary definition of QB to states with non-zero probability of measuring negative momentum. Hence, our approach should be applicable to realistic experimental situations. We study cases of QB in gravitational field near the surface of the Earth and in harmonic potential. We also comment on possible experimental verification of QB using atomic gravimeters.
QUANTUM BACKFLOW

Here, we provide a definition of QB that can be put to test in a relatively uncomplicated experiment. The definition conveys how necessary conditions for the probability current that follow from the classical equations of motion are no longer satisfied in the quantum regime.

Let us focus on the system of a lone particle in the one-dimensional space. For future reference, we set the vertical direction with the $x$ axis pointing downwards. Suppose the particle interacts with arbitrary potential $V(x)$. We examine its dynamics from classical and quantum points of view with the same initial conditions. The quantum system at time $t$ is fully described by its wave function $\psi_t(x)$. The classical model requires simultaneous knowledge of position and momentum, whose precise estimation is famously forbidden by the uncertainty principle. We therefore propose an operational approach in which distribution of position and momentum is estimated with finite precision, and given by the probability density function

$$f_t(x, p) = |\langle \phi | W(x, p) | \psi \rangle|^2,$$

where $W(x, p)$ is the Wigner-Moyal transform of $\psi$ and $\phi$ (see e.g. Eq. (6.63) in [17]):

$$\langle \phi | W(x, p) | \psi \rangle = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{-\frac{\pi}{\hbar} p^2} \phi^*(a, \frac{1}{2} x) \psi(a, \frac{1}{2} x) da.$$

The function $\phi$ represents the finite precision of the measurement apparatus, centred at zero, with finite width $\sigma_\phi$. The marginals of $f_t$ agree with densities of position and momentum, derived from $\psi_t(x)$, "up to $\sigma_\phi^2\), i.e.:

$$P_t(x = x_0) = \int_{-\infty}^{\infty} f_t(x_0, p) dp = \langle |\psi|^2 \ast |\phi|^2 \rangle(x_0),$$

$$P_t(p = p_0) = \int_{-\infty}^{\infty} f_t(x, p_0) dx = \langle |\tilde{\psi}|^2 \ast |\tilde{\phi}|^2 \rangle(p_0),$$

where $\tilde{\psi}(p)$ and $\tilde{\phi}(p)$ are Fourier transforms of $\psi$ and $\phi$, respectively, and $\ast$ stands for convolution.

In other words, the family of operators $W(x, p)|\phi\rangle\langle\psi|W(x, p)$, defined on the phase space $(x, p) \in \mathbb{R}^2$, is a positive-operator valued measure that allows experimental estimation of the joint probability distribution $f_t(x, p)$ of position and momentum in the state $\psi$, with finite precision given by a square-integrable function $\phi$.

We now derive probability currents in the quantum and classical models corresponding to the rate of change of probability of finding the particle above the level $x = a$. This is a textbook exercise in the quantum case, leading to the familiar formula

$$j_t(a) = \frac{d}{dt} P_t(x \leq a) = \frac{\hbar m}{2} \Im \left( \frac{\partial}{\partial x} \psi_t^*(a) \frac{\partial}{\partial x} \psi_t(x) \right|_{x=a},$$

(5)

where $\Im$ stands for the imaginary part of a complex number.

Let us fix $t$ for now and consider a putative classical system, with the distribution of position and momentum given by $f_t(x, p)$. The distribution evolves for a short time $\tau$, $t \leq \tau \leq t + \Delta t$ according to the Hamilton equations of motion: $\dot{x}(\tau) = p/m$, $\dot{p}(\tau) = -\frac{\partial}{\partial x} V$. This implies that, unlike in the quantum case, the probability current of the classical system,

$$(j_{cl})_t(a) = \left. \frac{d}{d\tau} P_{cl}(x(\tau) \leq a) \right|_{\tau = t},$$

(6)

must be bounded from below:

$$(j_{cl})_t(a) \geq \frac{1}{m} \int_{-\infty}^{0} p f_t(a, p) dp$$

(7)

(see Appendix for the detailed proof). We can say that the probability of finding the classical particle above the line $x = a$ cannot grow faster than a certain quantity derived from the distribution of only negative momenta.

In particular, if $f_t(x, p) = 0$ for $p < 0$, we get that $(j_{cl})_t(a) \geq 0$, i.e. the direction of the momentum and the probability current coincide. This leads to the usual definition of QB given by the following statements about a wave function of a quantum system in one-dimensional space [18]: a) $\tilde{\psi}$ contains only positive momenta; b) there exists $a \in \mathbb{R}$, for which $j_t(a) < 0$. To facilitate the analysis of QB for arbitrary states, we say that QB is a situation where the inequality (7) no longer holds.

**Definition 1.** The quantum backflow takes place at point $x = a$ and at time $t$, if

$$j_t(a) < \frac{1}{m} \int_{-\infty}^{0} p f_t(a, p) dp,$$

(8)

where the probability current $j_t(a)$ is given by Eq. (5) and the function $f_t(x, p)$ by Eq. (1).

Note that the definition of QB depends on the choice of the “precision” function $\phi$ in Eq. (1).

**EXAMPLES**

We give two concrete examples of QB according to Definition 1. Both involve superposition of Gaussian states
FIG. 1. Quantum backflow in gravity. The initial state is a superposition of two Gaussian wave functions with amplitudes given in (11) describing Rubidium atom. Left panel: The dashed line (almost straight) gives the lower bound on the classical probability current. The solid line is the quantum probability current. QB takes place in the interval when the solid line is below the dashed line, see Eq. (8). Right panel: As a consequence, the probability $P(x < 0)$ of finding the particle above the initial level of $x = 0$ increases despite small contributions from negative momenta. Solid line gives the probability in the presence of the gravitational field. For comparison, the dashed line represents the free particle.

Gravitational potential

Suppose the particle interacts with the potential $V(x) = -mgx$, for $g \geq 0$. Recall that by our convention, the direction of the $x$ axis and the direction of the gravitational force coincide. We choose the initial level $x = 0$ above the surface. Of course, by putting $g = 0$, we also cover the case of a free particle on the real line.

Consider the initial wave function, at time $t = 0$, being a superposition of two Gaussian states centred at $x = 0$, with the same spread $\sigma$ but different mean momenta. If the corresponding quantum particle is free, the wave function at time $t$ reads:

$$\psi_{\text{free}}(x, t) = \sum_{n=1}^{2} \frac{B_n}{\sqrt{4\sigma^2 + 2i\hbar t}} \times \exp \left( \frac{i}{\hbar} p_n (x - \frac{p_n}{2m} t) - \frac{(x - \frac{p_n}{m} t)^2}{4\sigma^2 + 2i\hbar t} \right),$$

where $p_1, p_2$ are the mean values of momentum for each branch of the superposition. In the presence of the linear potential, the wave function accelerates due to the interaction with gravity and at time $t$ reads [19]:

$$\psi(x, t) = e^{-\frac{i}{\hbar}(-mgtx + \frac{1}{2}mg^2t^2)} \psi_{\text{free}}(x - \frac{1}{2}gt^2, t).$$

In order to compute the classical limit on the probability current, we now choose realistic experimental values for the parameters. Namely, we take the Rubidium atom of mass $m \approx 1.4 \times 10^{-25}$ kg, described by the wave function with $\sigma = 1\mu$m. The required superposition can be prepared with a series of Raman pulses giving rise to a coherently combined momentum $p_1 = 0$ and $p_2 = 2\hbar k$, where the wave number $k = 2\pi/\lambda$ with $\lambda = 780$ nm. These values match cold-atom gravimeters which seem to be well-suited for measurements of QB [20]. We choose superposition amplitudes to be

$$B_1 \approx 1.18 \times 10^{-3}, B_2 \approx 4.42 \times 10^{-4},$$

which numerically optimise the effect of QB. The precision function is chosen as a Gaussian with standard deviation $\sigma_\phi = 0.1\mu$m.

In Fig. 1 we show the probability current and the overall probability of finding the particle above the initial line $x = 0$ as functions of time. We set $g = 9.8 \text{ m} \cdot \text{s}^{-2}$. It is clear that for $t$ such that approximately: $23.7\mu$s $\leq t \leq 39\mu$s, the probability current satisfies the inequality [5], i.e. QB takes place. During that time, the numerical value of the integral $\int_{-\infty}^{0} |\psi(p)|^2 dp$, i.e. the contribution of “negative momenta” to the backflow state, ranges from 0.23 to 0.13. Here, our approach allows us to separate the contribution of the negative momenta from the genuine quantum backflow. Despite the particle’s free fall, the quantum probability of finding the particle above the initial level unmistakably increases.
FIG. 2. Quantum backflow in a harmonic trap. Rb atom in the initial superposition of coherent states with parameters given in the main text gives rise to the quantum probability current (solid line) which is below the classical lower bound (dashed line). This signifies QB for approxi-mately (230, 239) μs. Probability current was evaluated at $x = \sqrt{\frac{2n}{m_\omega}} \cos(0.55\pi)$. 

Harmonic potential

Our second example is a particle in quadratic potential, due to its wide applicability. For concreteness, we again consider Rb atom, but this time in a harmonic trap with frequency $\nu = 10$ kHz [21]. It is generally known that coherent states of a quantum harmonic oscillator take the form of a Gaussian packet in position representation [22]. Hence, we take as a initial state the superposition of two coherent states $|\psi\rangle = a|\alpha\rangle + a|\beta\rangle$. Numerical optimisation of QB leads to the following parameters: $a \approx 0.73$, $\alpha = e^{i(0.9\pi - \omega t)}$, $\beta = 9e^{i(0.55\pi - \omega t)}$, where $\omega = 2\pi \nu$. Fig. 2 shows the corresponding classical bound on the probability current and the quantum prediction for QB.

CONCLUSIONS

We presented an analysis of QB for states with admissible negative momentum, whereas the standard definition of QB is applicable to wave functions with positive-only support in momentum representation. Broadly speaking, our idea is to compare the evolution of the joint probability distribution of position and momentum (known with finite precision) for a given quantum state with its hypothetical classical analogue. The classical system would evolve in infinitesimal time interval according to Hamilton’s equations of motion, starting with the same initial conditions. We defined QB as a situation when the probability current exceeds what might be possible for its classical counterpart.

In particular, we showed that a relatively easy-to-prepare superposition of two Gaussian packets exhibits QB during free-fall in the uniform gravitational field near the Earth’s surface. Of course, the Ehrenfest theorem guarantees that the average position of the quantum particle follows the classical trajectory. Nevertheless, the probability of locating the particle above the initial level displays ‘antigravitational’ quantum backflow.

ACKNOWLEDGMENTS

We thank Rainer Dumke for discussions. This work is supported by Polish National Agency for Academic Exchange NAWA Project No. PPN/PPO/2018/1/00007/U/00001.

Appendix

Here, we give a proof of the inequality [7].

Once again, we consider the distribution of position and momentum given by $f_t(x, p)$ that evolves classically for a brief time $\tau$, $t \leq \tau \leq t + \Delta t$. If our system were initially at point $x$, then $x(\tau) = x + \frac{p}{m}(\tau - t)$. The probability $P(x(\tau) \leq a)$ of finding the particle in the region “above” the line $x = a$ would be

$$P(x(\tau) \leq a) = P(x \leq a - \frac{p}{m}(\tau - t))$$

$$= \int_{-\infty}^{a} \int_{-\infty}^{\frac{p}{m}(\tau - t)} f_t(x, p) \, dp \, dx$$

$$= \int_{-\infty}^{a} \int_{-\infty}^{\infty} f_t(x - \frac{p}{m}(\tau - t), p) \, dp \, dx. \tag{12}$$

Now, the probability current of the classical system,

$$\langle j_{cl}\rangle_t(a) = -\frac{d}{dt} P(x(\tau) \leq a) \bigg|_{\tau=t}, \tag{13}$$

can be expressed in terms of the probability density function $f_t(x, p)$:

$$\langle j_{cl}\rangle_t(a) = -\frac{d}{dt} \int_{-\infty}^{a} \int_{-\infty}^{\infty} f_t(x - \frac{p}{m}(\tau - t), p) \, dp \, dx \bigg|_{\tau=t}$$

$$= \int_{-\infty}^{a} \int_{-\infty}^{\infty} \frac{\partial}{\partial x} f_t(x, p) \, dp \, dx$$

$$= \frac{1}{m} \int_{-\infty}^{\infty} pf_t(a, p) \, dp \, dx, \tag{14}$$
where the last equality is a consequence of the fact that \( \lim_{x \to \pm \infty} f_t(x, p) = 0 \). This immediately yields the inequality:

\[
(jcl)_t(a) \geq \frac{1}{m} \int_{-\infty}^{0} pf_t(a, p) \, dp. \tag{15}
\]