Reasoning in Bayesian Opinion Exchange Networks Is PSPACE-Hard

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Abstract

We study the Bayesian model of opinion exchange of fully rational agents arranged on a network. In this model, the agents receive private signals that are indicative of an unknown state of the world. Then, they repeatedly announce the state of the world they consider most likely to their neighbors, at the same time updating their beliefs based on their neighbors’ announcements.

This model is extensively studied in economics since the work of Aumann (1976) and Geanakoplos and Polemarchakis (1982). It is known that the agents eventually agree with high probability on any network. It is often argued that the computations needed by agents in this model are difficult, but prior to our results there was no rigorous work showing this hardness.

We show that it is PSPACE-hard for the agents to compute their actions in this model. Furthermore, we show that it is equally difficult even to approximate an agent’s posterior: It is PSPACE-hard to distinguish between the posterior being almost entirely concentrated on one state of the world or another.

1 Introduction

Background

The problem of dynamic opinion exchange is an important field of study in economics, with its roots reaching as far as the Condorcet’s jury theorem and, in the Bayesian context, Aumann’s agreement theorem. Economists use different opinion exchange models as inspiration for explaining interactions and decisions of market participants. More generally, there is extensive interest in studying how social agents exchange information, form opinions and use them as a basis to make decisions. For a more comprehensive introduction to the subject we refer to surveys addressed to economists [AO11] and mathematicians [MT17].

Many models have been proposed and researched, with the properties studied including, among others, if the agents converge to the same opinion, the rate of such convergence, and if the consensus decision is optimal with high probability (this is called learning). Two interesting aspects of the differences between models are rules for updating agents’ opinions (e.g., fully rational or heuristic) and presence of network structure.

For example, in settings where the updates are assumed to be rational (Bayesian), there is extensive study of models where the agents act in sequence (see, e.g., [Ban92, BHW92, SS00, ADLO11] for a non-exhaustive selection of works that consider phenomena of herding and information cascades), as well as models with agents arranged in a network and repeatedly exchanging opinions as time progresses (see some references below). In this work we are interested in the network models, arguably becoming more and more relevant given the ubiquity of networks in modern society.

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On the other hand, similar questions are studied for models with so-called bounded rationality, where the Bayesian updates are replaced with simpler, heuristic rules. Some well-known examples include DeGroot model [DeG74 GJ10], the voter model [CS73 HL75] and other related variants [BC08 AOP10].

One commonly accepted reason for studying bounded rationality is that, especially in the network case, Bayesian updates become so complicated as to make fully rational behavior intractable, and therefore unrealistic. However, we are not aware of previous theoretical evidence or formalization of that assertion. Together with another paper of the same authors addressed to economists [HJMR18], we consider this work as a development in that direction.

More precisely, we show that computing an agent’s opinion in one of the most important and studied Bayesian network models is PSPACE-hard. Furthermore, it is PSPACE-hard even to approximate the rational opinion in any meaningful way. This improves on our NP-hardness result for the same problem shown in [HJMR18].

Our model and results We are concerned with a certain Bayesian model of opinion exchange and reaching agreement on a network. We are going to call it the (Bayesian) binary action model. The idea is that there is a network of honest, fully rational agents trying to learn a binary piece of information, e.g., will the price of an asset go up or down, or which political party’s policies are more beneficial to the society. We call this information the state of the world. Initially, each agent receives an independent piece of information (a private signal) that is correlated with the state of the world. According to the principle that “actions speak louder than words”, at every time step the agents reveal to their neighbors which of the two possible states they consider more likely. On the other hand, we assume that the agents are honest truth-seekers and always truthfully reveal their preferred state: According to economic terminology they act myopically rather than strategically.

More specifically, we assume that the state of the world is encoded in a random variable \( \theta \in \{T,F\} \) (standing for True and False), distributed according to a uniform prior, shared by all agents. A set of Bayesian agents arranged on a directed graph \( G = (V,E) \) performs a sequence of actions at discrete times \( t = 0,1,2,\ldots \). Before the process starts, each agent \( u \) receives a random private signal \( S(u) \in \{0,1\} \). The collection of random variables \( \{S(u) : u \in V\} \) is independent conditioned on \( \theta \). The idea is that \( S(u) = 1 \) indicates a piece of evidence for \( \theta = T \) and \( S(u) = 0 \) is evidence favoring \( \theta = F \).

At each time \( t \geq 0 \), the agents simultaneously broadcast actions to their neighbors in \( G \). The action \( A(u,t) \in \{T,F\} \) is the best guess for the state of the world by agent \( u \) at time \( t \): Letting \( \mu(u,t) \) be the respective Bayesian posterior probability that \( \theta = T \), the action \( A(u,t) = T \) if and only if \( \mu(u,t) > 1/2 \). In subsequent steps, agents update their posterior based on their neighbors’ actions (we assume that everyone is rational, and that this fact and the description of the model are common knowledge) and broadcast updated actions. The process continues indefinitely.

We are interested in computational resources required for the agents to participate in the process described above. That is, we consider complexity of computing the action \( A(u,t) \) given the private signal \( S(u) \) and history of observations \( \{A(v,t') : v \in N(u), t' < t\} \), where \( N(u) \) denotes the set of neighbors of \( u \) in \( G \). Our main result is that it is worst-case PSPACE-hard for an agent to distinguish between cases where \( \mu(u,t) \geq 1 - \exp(-\Theta(N)) \) and \( \mu(u,t) \leq \exp(-\Theta(N)) \), where \( N \) is a naturally defined size of the problem. As a consequence, it is PSPACE-hard to compute the action \( A(u,t) \).

Note a hardness of approximation aspect of our result: A priori one can imagine a reduction where it is difficult to compute the action \( A(u,t) \) when the Bayesian posterior is close to the threshold \( \mu(u,t) \approx 1/2 \). However, we demonstrate that it is already hard to distinguish between situations where the posterior is concentrated on one of the extreme values \( \mu(u,t) \approx 0 \) (and therefore almost certainly \( \theta = F \)) and \( \mu(u,t) \approx 1 \) (and therefore \( \theta = T \)).

Our hardness results carry over to other models. In particular, they extend to the case where the signals are continuous, where the prior state of the world is not uniform etc. We also note that we may assume that the agents are never tied or close to tied in their posteriors, see Remark 13.
for more details.

A good deal is known about the model we are considering. From a paper by Gale and Kariv [GK03] (with an error corrected by Rosenberg, Solan and Vieille [RSV09], see also similar analysis of earlier, related models in [BV82, TA84]) it follows that if the network $G$ is strongly connected, then the agents eventually converge to the same action (or they become indifferent). The work of Geanakoplos [Gea94] implies that this agreement is reached in at most $|V| \cdot 2^{|V|}$ time steps. Furthermore, Mossel, Sly and Tamuz [MST14] showed that in large undirected networks with non-atomic signals, learning occurs: The common agreed action is equal to the state of the world $\theta$, except with probability that goes to zero with the number of agents. A good deal remains open, too. For example, it is not known if the $|V| \cdot 2^{|V|}$ bound on the agreement speed can be improved. In this context is also interesting to note the results of [MOT16] who consider a special model with Gaussian structure and revealed beliefs. In contrast to the results presented here, it is shown that in this case, agents’ computations are efficient (polynomial time) and convergence time is $O(|V| \cdot \text{diam}(G))$.

**Proof idea** Our proof is by direct reduction from the canonical PSPACE-complete language of true quantified Boolean formulas. It maps true formulas onto networks where one of the agents’ posteriors is almost entirely concentrated on $\theta = T$ and false formulas onto networks where the posterior is concentrated on $\theta = F$. The reduction and the proof are by induction on the number of quantifier alternations in the Boolean formula. The base case of the induction corresponds to such mapping for satisfiable and unsatisfiable 3-SAT instances.

The basic idea of the reduction is to map variables and clauses of the Boolean formula onto agents or small sub-networks of agents (gadgets) in the Bayesian network. We use other gadgets to implement some useful procedures, like counting or logical operations. One challenging aspect of the reduction is that, since each such operation is implemented by Bayesian agents by broadcasting their opinions, these “measurements” themselves might shift the posterior belief of the “observer” agent. Therefore, we need to carefully compensate those unintended effects at every step.

Another interesting technical aspect of the proof is related to its recursive nature. When we establish hardness of approximation for $k$ quantifier alternations, it means that we can place an agent in our network such that the agent will be solving a “$k$-hard” computational problem. We then use this agent, together with another gadget that modifies relative likelihoods of different private signal configurations, to amplify hardness to $k + 1$ alternations.

**Related literature** One intriguing aspect of our result is a connection to the Aumann’s agreement theorem. There is a well-known discrepancy (see [CH02] for a distinctive take) between reality, where we commonly observe (presumably) honest, well-meaning people “agreeing to disagree”, and the Aumann’s theorem, stating that this cannot happen for Bayesian agents with common priors and knowledge, i.e., the agents will always end up with the same estimate of the state of the world after exchanging all relevant information. Our result hints at a computational explanation, suggesting that reasonable agreement protocols might be intractable in the presence of network structure. This is notwithstanding some positive computational results by Hanson [Han03] and Aaronson [Aar05], which focus on two agents and come with their own (perhaps unavoidable) caveats.

We find it interesting that the agents’ computations in the binary action model turn out to be not just hard, but PSPACE-hard. PSPACE-hardness of partially observed Markov decision processes (PMODPs) established by Papadimitriou and Tsitsiklis [PT87] seems to be a result of similar kind. On the other hand, there are clear differences: We do not see how to implement our model as PMODP, and embedding a TQBF instance in a PMODP looks more straightforward than what happens in our reduction. Furthermore, and contrary to [PT87], we establish hardness of approximation. We are not aware of many other PSPACE-hardness of approximation proofs, especially in recent years. Exceptions are results obtained via PSPACE versions of the PCP theorem [CFLS95, CFLS97] and a few other reductions [MHR93, HMS94, Jon97, Jon99] that concern, among others, some problems on hierarchically generated graphs and an AI-motivated problem of
planning in propositional logic.

We note that there are some results on hardness of Bayesian reasoning in static networks in the AI and cognitive science context (see [Kwi18] and its references), but this setting seems quite different from dynamic opinion exchange models.

Finally, we observe that a natural exhaustive search algorithm for computing the action \( A(u,t) \) in the binary action model requires exponential space (see [HJMR18] for a description) and that we are not aware of a faster, general method (but again, see [MOT16] for a polynomial time algorithm in a variant with Gaussian signals).

**Organization of the paper** In Section 2, we give a full description of our model, state the results precisely and give some remarks about the proofs. Section 3 contains the proof of NP-hardness, which is then used in Section 4 in the proof of PSPACE-hardness. Section 5 modifies the proof to use only a fixed number of private signal distributions. Section 6 provides a proof of \#P-hardness in a related revealed belief model. Finally, Section 7 contains some suggestions for future work.

## 2 The model and our results

In Section 2.1 we restate the binary action model in more precise terms and introduce some notation. In Section 2.2 we discuss our results in this model. In Section 2.3 we define the revealed belief model and state a \#P-hardness result for it. Finally, in Section 2.4 we explain our main proof ideas.

### 2.1 Binary action model

We consider the binary action model of Bayesian opinion exchange on a network. There is a directed graph \( G = (V,E) \), the vertices of which we call agents. The world has a hidden binary state \( \theta \in \{T,F\} \) with uniform prior distribution. We will analyze a process with discrete moments \( t = 0,1,2,\ldots \). At time \( t = 0 \) each agent \( u \) receives a private signal \( S(u) \in \{0,1\} \). The signals \( S(u) \) are random variables with distributions that are independent across agents after conditioning on \( \theta \). Accordingly, the distribution of \( S(u) \) is determined by its signal probabilities

\[
p_{0u}(u) := \Pr[S(u) = 1 \mid \theta = \theta_0], \theta_0 \in \{T,F\}.
\]

Equivalently, it is determined by its (log)-likelihoods

\[
\ell_b(u) = \log \frac{\Pr[S(u) = b \mid \theta = T]}{\Pr[S(u) = b \mid \theta = F]} = \log \frac{\Pr[\theta = T \mid S(u) = b]}{\Pr[\theta = F \mid S(u) = b]}, b \in \{0,1\}.
\]

Note that there is a one-to-one correspondence between probabilities \( p_T \) and \( p_F \) with \( p_T \neq p_F \), and likelihoods \( \ell_0, \ell_1 \) with \( \ell_0 \cdot \ell_1 < 0 \). We will always assume that a signal \( S(u) = 1 \) is evidence towards \( \theta = T \) and vice versa. This is equivalent to saying that \( p_T > p_F \) or \( \ell_1 > 0 \) and \( \ell_0 < 0 \). We allow some agents to not receive private signals: This can be “simulated” by giving them non-informative signals with \( p_T(u) = p_F(u) \). We will refer to all signal probabilities taken together as the signal structure of the Bayesian network. A specific pattern of signals \( s \in \{0,1\}^{|V'|} \) (where \( V' \) denotes the subset of agents that receive informative signals) will be called a signal configuration.

We assume that all this structural information is publicly known, but the agents do not have direct access to \( \theta \) or others’ private signals. Agents are presumed to be rational, to know that everyone else is rational, to know that everyone knows, etc. (common knowledge of rationality). At each time \( t \geq 0 \), we define \( \mu(u,t) \) to be the belief of agent \( u \): The conditional probability that \( \theta = T \) given everything that \( u \) observed at times \( t' < t \). More precisely, letting \( \mathcal{N}(u) \) be the (out)neighbors of \( u \) in \( G \) and defining

\[
H(u,t) := \{ A(v,t') : t' < t, \ v \in \mathcal{N}(u) \}.
\]
as the observation history of agent $u$ we let

$$
\mu(u, t) := \Pr[\theta = T \mid S(u), H(u, t)].
$$

Accordingly, if $(u, v) \in E(G)$ we will say that agent $u$ observes agent $v$.

Agent $u$ broadcasts to its in-neighbors the action $A(u, t) \in \{T, F\}$, which is the state of the world that $u$ considers more likely according to $\mu(u, t)$ (assume that ties are broken in an arbitrary deterministic manner, say, in favor of $F$). Then, the protocol proceeds to time step $t + 1$ and the agents update their beliefs and broadcast updated actions. The process continues indefinitely. Note that the beliefs and actions become deterministic once the private signals are fixed.

The first two time steps of the process are relatively easy to understand: At time $t = 0$ an agent broadcasts $A(u, 0) = T$ if and only if $S(u) = 1$ and the belief $\mu(u, 1)$ can be easily computed from the likelihood $\ell_{S(u)}(u)$. At time $t = 1$, an agent broadcasts $A(u, 1) = T$ if and only if

$$
\ell_{S(u)}(u) + \sum_{v \in N(u)} \ell_{S(v)}(v) > 0,
$$

where the private signals $S(v)$ can be inferred from observed actions $A(v, 0)$. The sum determines the likelihood associated with belief $\mu(u, 1)$. However, at later times the actions of different neighbors are not independent anymore and accounting for those dependencies seems difficult.

Let $\Pi$ be a Bayesian network, i.e., a directed graph $G = (V, E)$ together with the signal structure. We do not commit to any particular representation of probabilities of private signals. Our reduction remains valid for any reasonable choice. We are interested in hardness of computing the actions that the agents need to broadcast. More precisely, we consider complexity of computing the function

$$
\text{BINARY-ACTION}(\Pi, t, u, S(u), H(u, t)) := A(u, t)
$$

that computes the action $A(u, t)$ given the Bayesian network, time $t$, agent $u$, its private signal $S(u)$ and observation history $H(u, t)$. Relatedly, we will consider computing belief

$$
\text{BINARY-BELIEF}(\Pi, t, u, S(u), H(u, t)) := \mu(u, t).
$$

Note that computing BINARY-ACTION is equivalent to distinguishing between BINARY-BELIEF $> 1/2$ and BINARY-BELIEF $\leq 1/2$.

### 2.2 Our results

Our first result implies that computing BINARY-ACTION at time $t = 2$ is NP-hard. We present it as a standalone theorem, since the NP reduction and its analysis are used as a building block in the more complicated PSPACE reduction.

**Theorem 1.** There exists an efficient reduction from a 3-SAT formula $\phi$ with $N$ variables and $M$ clauses to an input of BINARY-ACTION($\Pi, t, u, H(u, t)$) such that:

- The size (number of agents and edges) of Bayesian graph $G$ is $O(N + M)$, time is set to $t = 2$ and agent $u$ does not receive a private signal.
- All probabilities of private signals are efficiently computable real numbers satisfying

$$
\exp(-(O(N))) \leq p_{\theta_0}(v) \leq 1 - \exp(-(O(N))), \ v \in V, \ \theta_0 \in \{T, F\}.
$$

- If $\phi$ is satisfiable, then the posterior $\mu(u, 2)$ satisfies

$$
\mu(u, 2) = 1 - \exp(-\Theta(N)).
$$
If $\phi$ is not satisfiable, then we have

$$\mu(u, 2) = \exp(-\Theta(N)).$$

**Corollary 2.** Both distinguishing between $\text{BINARY-BELIEF} > 1 - \exp(-O(N))$ and $\text{BINARY-BELIEF} < \exp(-O(N))$ and computing $\text{BINARY-ACTION}$ are NP-hard.

Our main result improves Theorem 1 to PSPACE-hardness. It is a direct reduction from the canonical PSPACE-complete language of true quantified Boolean formulas $\text{TQBF}$.

**Theorem 3.** There exists an efficient reduction from a $\text{TQBF}$ formula $\Phi$

$$\Phi = Q_{K}x_{K}\cdots\exists x_{1}\phi(x_{K}, \ldots, x_{1}),$$

where $\phi$ is a 3-CNF formula with $N$ variables and $M$ clauses, there are $K$ variable blocks with alternating quantifiers and the last quantifier is existential, to an input of $\text{BINARY-ACTION}(\pi, t, u, H(u, t))$ such that:

- The number of agents in Bayesian graph $G$ is $O(N^2(N + M))$, time is set to $t = 2K$ and agent $u$ does not receive a private signal.
- All probabilities of private signals are efficiently computable real numbers satisfying

$$\exp(-O(N)) \leq p_{\theta}(v) \leq 1 - \exp(-O(N)), \quad v \in V, \quad \theta \in \{T, F\}.$$ (2)

- If $\Phi$ is true, then $\mu(u, 2K) = 1 - \exp(-\Theta(N))$. If $\Phi$ is false, then $\mu(u, 2K) = \exp(-\Theta(N))$.

**Corollary 4.** Both distinguishing between $\text{BINARY-BELIEF} > 1 - \exp(-O(N))$ and $\text{BINARY-BELIEF} < \exp(-O(N))$ and computing $\text{BINARY-ACTION}$ are PSPACE-hard.

**Remark 5.** Note that the statement of Theorem 3 immediately gives $\Sigma^P_K$- and $\Pi^P_K$-hardness of approximating $\text{BINARY-BELIEF}$ at time $t = 2K$. ◊

**Remark 6.** For ease of exposition we define networks in the reductions to be directed, but due to additional structure that we impose (see paragraph “Network structure and significant times” in Section 3) it is easy to see that they can be assumed to be undirected. This is relevant insofar as a strong form of learning occurs only on undirected graphs (see [MST14] for details). ◊

One possible objection to Theorem 3 is that it uses signal distributions with probabilities exponentially close to zero and one. We do not think this is a significant issue, and it helps avoid some technicalities. Nevertheless, in Section 5 we prove a version of Theorem 3 where all private signals come from a fixed family of say, at most fifty distributions. This is at the cost of (non-asymptotic) increase in the size of the graph.

**Theorem 7.** The reduction from Theorem 3 can be modified such that all private signals come from a fixed family of at most fifty distributions.

**Remark 8.** It is possible to modify our proofs to give hardness of distinguishing between $\mu(u, t) \leq \exp(-O(N^K))$ and $\mu(u, t) \geq 1 - \exp(-O(N^K))$ for any constant $K$ (recall that $N$ is the number of variables in the formula $\phi$). This is at the cost of allowing signal probabilities in the range

$$\exp(-O(N^K)) \leq p_{\theta}(v) \leq 1 - \exp(-O(N^K))$$
or, in the bounded signal case, increasing the network size to $O(N^{K+2})$. Consequently, in the latter case we get hardness of approximation up to $\exp(-O(|V|^\alpha))$ factor for any constant $\alpha < 1$, where $|V|$ is the number of agents. ◊
2.3 Revealed beliefs

In a natural variant of our model the agents act in exactly the same manner, except that they reveal their full beliefs $A(u, t) = \mu(u, t)$ rather than just estimates of the state $\theta$. Accordingly, we call it the revealed belief model. We suspect that binary action and revealed belief models have similar computational powers. Furthermore, we conjecture that if the agents broadcast their beliefs rounded to a (fixed in advance) polynomial number of significant digits, then our techniques can be extended to establish a similar PSPACE-hardness result.

However, if one instead assumes that the beliefs are broadcast up to an arbitrary precision, our proof fails for a rather annoying reason: When implementing alternation from NP to $\Pi_2$ in the binary action model, if a formula $\phi$ has no satisfying assignments, we can exactly compute the belief of the NP observer agent. However, in case $\phi$ has a satisfying assignment, we can compute the belief only with high, but imperfect precision. The reason is that the exact value of the belief depends on the number of satisfying assignments of $\phi$. This imperfection can be “rounded away” if the agents output a discrete guess for $\theta$, but we do not know how to handle it if the beliefs are broadcast exactly.

Nevertheless, in Section 6 we present a $\#P$-hardness proof in the revealed belief model. The proof is by reduction from counting satisfying assignments in a 2-SAT formula. However, since the differences in the posterior corresponding to different numbers of satisfying assignments are small, it is not clear if they can be amplified, and consequently we do not demonstrate hardness of approximation (similar as in [PT87]). For ease of exposition we introduce an additional relaxation to the model by allowing some agents to receive ternary private signals.

Theorem 9. Assume the revealed belief model with beliefs transmitted up to arbitrary precision and call the respective computational problem BINARY-BELIEF. Additionally, assume that some agents receive ternary signals $S(u) \in \{0, 1, 2\}$.

There exists an efficient reduction that maps a 2-SAT formula $\phi$ with $N$ variables, $M$ clauses and $A$ satisfying assignments to an instance of BINARY-BELIEF($\Pi, t, u, H(u, t)$) such that:

- Bayesian network $G$ has size $O(N + M)$, time is set to $t = 2$ and agent $u$ does not receive a private signal.
- All private signal probabilities come from a fixed family of at most ten distributions.
- The likelihood of $u$ at time $t = 2$ satisfies

$$\frac{A}{2^N} \left(1 - \frac{1}{4^N}\right) \leq \frac{\mu(u, 2)}{1 - \mu(u, 2)} \leq \frac{A}{2^N} \left(1 + \frac{1}{4^N}\right).$$

In particular, rounding this likelihood to the nearest multiple of $2^{-N}$ yields $A \cdot 2^{-N}$ and allows to recover $A$.

2.4 Main proof ideas

The NP-hardness proof (in Section 3) uses an analysis of a composition of several gadgets. We will think of the agent $u$ from input to BINARY-ACTION as “observer” and accordingly call it OBS. The Bayesian network features gadgets that represent variables and clauses. The private signals in variable gadgets correspond to assignments $x$ to the formula $\phi$. Furthermore, there is an “evaluation agent” EVAL that interacts with all clause gadgets. We use more gadgets that “implement” counting to ensure that what OBS observes is consistent with one of two possible kinds of signal configurations:

- $S$(EVAL) = 1 and the signals of variable agents correspond to an arbitrary assignment $x$.
- $S$(EVAL) = 0 and the signals of variable agents correspond to a satisfying assignment $x$.  

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Then, we use another gadget to “amplify” the information that is conveyed about the state of the world by the signal $S(\text{EVAL})$. If $\phi$ has no satisfying assignment, then $S(\text{EVAL}) = 1$ and this becomes amplified to a near-certainty that $\theta = F$ (for technical reasons this is the opposite conclusion than suggested by $S(\text{EVAL}) = 1$). On the other hand, we design the signal structure such that even a single satisfying assignment tips the scales and amplifies to $\theta = T$ with high probability (whp).

We note that one technical challenge in executing this plan is that some of our gadgets are designed to “measure” (e.g., count) certain properties of the network, but these measurements use auxiliary agents with their own private signals, affecting Bayesian posteriors. We need to be careful to cancel out these unintended effects at every step.

The high-level idea to improve on the NP-hardness proof is that once we know that agents can solve hard problems, we can use them to help the observer agent solve an even harder problem. Of course this has to be done in a careful way, since the answer to a partial problem cannot be directly revealed to the observer (the whole point is that we do not know a priori what this answer is).

The PSPACE reduction is defined and Theorem 3 proved by induction. The base case is the Bayesian network from Theorem 1 but with observer agent directly observing private signals in the first $K - 1$ variable blocks. Then, we proceed to add “intermediate observers”, each of them observing one variable block less, and interacting via a gadget with the previous observer to implement the quantifier alternation by adjusting likelihoods of different assignments to variables in $\Phi$.

It is useful to view $\Phi$ as a game where two players set quantified variables (proceeding left-to-right) in $\Phi$. One player sets variables under existential quantifiers with the objective of evaluating the 3-CNF formula $\phi$ true. The other player sets variables under universal quantifiers with the objective of evaluating $\phi$ false. Under that interpretation, our reduction has the following property: The final observer agent OBS concludes that the assignment $x$ formed by private signals with high probability corresponds to a “game transcript” in the game played according to a winning strategy. Depending on which player has the winning strategy, the state of the world is either $T$ or $F$ whp.

### 3 NP-hardness: Proof of Theorem 1

We start with the NP-hardness result by reduction from 3-SAT. The reduction is used as a building block in the PSPACE-hardness proof, but it is also useful in terms of developing intuition for the more technical proof of Theorem 3. We proceed by explaining gadgets that we use, describing how to put them together in the full reduction and proving the correctness.

**Threshold gadget** Say there are agents $v_1, \ldots, v_K$ that do not observe anyone and receive private signals $S(v_i)$ with respective likelihoods $\ell_0(v_i)$ and $\ell_1(v_i)$. Additionally, there is an observer agent OBS and we would like to reveal to it, at time $t = 1$, that the sum of likelihoods of agents $v_1, \ldots, v_K$ exceeds some threshold $\delta$:

$$L := \sum_{i=1}^{K} \ell_{S(v_i)}(v_i) > \delta,$$

without disclosing anything else about the private signals. This is achieved by the gadget in Figure 1.

We describe the gadget for $\delta > 0$. Agent $B$ receives private signal with $\ell_0(B) = -\delta$ (and arbitrary $\ell_1(B)$) and agent $C$ with $\ell_1(C) = \delta$. Agents $A$, $D_1$ and $D_2$ (we will call the latter two the “dummy” agents) do not receive private signals. Our overall reduction will demonstrate the hardness of computation for agent OBS. Therefore, we need to specify the observation history of OBS. By our tie-breaking convention, it must be $A(A, 0) = A(D_1, 0) = A(D_2, 0) = F$. Furthermore, we specify $A(A, 1) = A(D_2, 1) = T$ and $A(D_1, 1) = F$.

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1 We assume that $\delta$ is chosen such that $L = \delta$ never happens.
Based on that information, agent OBS can infer that $S(B) = 0$, $S(C) = 1$ and, since the action $A(A, 2)$ is determined by the sign of $L - \delta$, that $L > \delta$. The purpose of agent $C$ is to counteract the effect of this “measurement” on the estimate of the state of the world by OBS. More precisely, let

$$P(s_1, \ldots, s_K, \theta_0) := \Pr \left[ \bigwedge_{i=1}^{K} S(v_i) = s_i \land \theta = \theta_0 \right], \quad (3)$$

$$P(s_1, \ldots, s_K, s_B, s_C, \theta_0) := \Pr \left[ \bigwedge_{i=1}^{K} S(v_i) = s_i \land S(B) = s_B \land S(C) = s_C \land \theta = \theta_0 \right]. \quad (4)$$

Based on the discussion above, we have the following:

**Claim 10.** Let $s_1, \ldots, s_K$ be private signals of $v_1, \ldots, v_K$. Similarly, let $(s_B, s_C)$ be private signals of $B$ and $C$. Then:

- If $\sum_{i=1}^{K} \ell_s(v_i) < \alpha$, then there are no $(s_B, s_C)$ that make $(s_1, \ldots, s_K, s_B, s_C)$ consistent with observations of OBS.

- If $\sum_{i=1}^{K} \ell_s(v_i) > \alpha$, then there exists unique configuration $(s_B, s_C)$ consistent with observations of OBS and the (prior) probability of this configuration when the state is $\theta_0$ is

$$P(s_1, \ldots, s_K, s_B, s_C, \theta_0) = P(s_1, \ldots, s_K, \theta_0) \cdot \alpha, \quad (5)$$

where $\alpha := (1 - p_T(B))p_T(C) = e^{\ell_s(B) + \ell_s(C)}(1 - p_F(B))p_F(C) = (1 - p_F(B))p_F(C)$ does not depend on $\theta_0$.

Similar reasoning can be made for the case when $\delta < 0$ and/or checking the opposite inequality $L < \delta$. We will say that an agent OBS observes a threshold gadget if it observes agents $A$, $D_1$ and $D_2$ and denote it as shown in Figure 2. Note that in our diagrams we use circles to denote agents and boxes to denote gadgets. The latter typically contain several auxiliary agents.

**Network structure and significant times** It might appear that the threshold gadget is more complicated than needed. The reason for this is that we will impose certain additional structure on the graph to facilitate its analysis, and later use it in the proof of Theorem 3. Specifically, we will always make sure that the graph is a DAG, with only the observer agent having in-degree zero. All agents that receive private signals will have out-degree zero, and all agents with non-zero out-degree will not receive private signals (recall a directed edge $A \rightarrow B$ indicates that $A$ observes $B$).
Furthermore, we will arrange the graph such that each agent will learn new information at a single, fixed time step. That is, for every agent $A$ there will exist a significant time $t(A)$ such that $\mu(A, t) = \frac{1}{2}$ for $t' < t(A)$ and $\mu(A, t') = \mu(A, t(A))$ for $t' > t(A)$. If $A$ receives a private signal, then $t(A) = 0$. Otherwise, $t(A)$ is determined by the (unique) path length from $A$ to an out-degree zero agent. For example, in Figure 1 significant times are $t(A) = t(D_1) = t(D_2) = 1$ and $t(OBS) = 2$.

Accordingly, we will use notation $\mu(A)$ and $A(A)$ to denote agent beliefs and actions at the significant time. Let $A$ and $B$ be agents with $t(A) < t(B) - 1$. In the following, we will sometimes say that $B$ observes $A$, even though that would contradict the significant time requirement (a direct edge $B \rightarrow A$ implies that $t(A) = t(B) - 1$). Whenever we do so, it should be understood that there is a path of “dummy” nodes of appropriate length between $A$ and $B$ (cf. $D_1$ and $D_2$ in Figure 1). For clarity, we will omit dummy nodes from the figures.

**Counting gadget** Assume now that the agents $v_1, \ldots, v_K$ receive private signals with identical likelihoods $\ell_0 < 0$ and $\ell_1 > 0$ and that a number $k$, $1 \leq k \leq K$ is given. Then, building on the threshold gadget, it is easy to convey the information that exactly $k$ out of $K$ agents received private signal 1. Letting $\delta := K\ell_0 + (k - 0.5)(\ell_1 - \ell_0)$ and $\delta' := \delta + \ell_1 - \ell_0$, we compose two threshold gadgets as shown in Figure 3.

Agent $A$ is optional: Depending on our needs we will use the counting gadget with or without it. It is used to preserve the original belief of $OBS$ after learning the count of private signals of agents $v_i$. It receives a private signal with $\ell_b(A) := \ell := -K\ell_0 - k(\ell_1 - \ell_0)$ for appropriate $b$ (depending on the sign of $\ell$) and broadcasts the corresponding state $\theta_0$. By similar analysis as for the threshold gadget and using the $P(\cdot)$ notation as in (3–5) we have:

![Figure 2: Notation for the threshold gadget.](image)

![Figure 3: Counting gadget.](image)
Claim 11. Let $s_1, \ldots, s_K$ be private signals of agents $v_1, \ldots, v_K$. Let $s$ represent private signals of all auxiliary agents in the threshold gadgets and $s_A$ a private signal of agent $A$.

Then, the only configurations $s_1, \ldots, s_K$ consistent with observations of $OBS$ are those for which $\sum_{i=1}^K s_i = k$. Furthermore, for any such configuration there exists a unique configuration $s$ (and $s_A$, if agent $A$ is present) such that (depending on the presence of $A$):

$$P(s_1, \ldots, s_K, s, \theta_0) = P(s_1, \ldots, s_K, \theta_0) \cdot \alpha = P(s_1, \ldots, s_K, \theta_0) \cdot \alpha,$$

where $\alpha := \alpha(k, K, \ell_0, \ell_1) > 0$ is easily computable and does not depend on $s_1, \ldots, s_K$ or $\theta_0$, but the value of the other term $P(\theta_0)$ is in general dependent on $\theta_0$. On the other hand, if $A$ is present, then $\beta := \beta(k, K, \ell_0, \ell_1) > 0$ does not depend at all on private signals or state of the world.

If agent $A$ is omitted, the same technique can be used to obtain inequalities (e.g., checking that at least $k$ out of $K$ private signals are ones). We will say that an agent $OBS$ observes the counting gadget if it observes both respective threshold gadgets (and $A$, if present). We will denote counting gadgets as in Figure 4.

Figure 4: Two counting gadgets illustrating the notation. The left-hand gadget ensures that at least $k$ agents received ones. The right-hand gadget ensures that exactly $k$ agents received ones. Furthermore, the equivalence symbol on the right-hand gadget denotes presence of the optional agent $A$.

Not-equal gadget

Another related gadget that we will use reveals to the observer that two agents $u, v$ with likelihoods $\ell_0, \ell_1$ and $m_0, m_1$, respectively, receive opposite signals $S(u) \neq S(v)$. Since $\ell_0 < \ell_1$ and $m_0 < m_1$, this is achieved by using two threshold gadgets to check that

$$\ell_0 + m_0 < \ell_{S(u)} + m_{S(v)} < \ell_1 + m_1,$$

where we set the thresholds in the threshold gadgets as $\ell_0 + m_0 + \varepsilon$ and $\ell_1 + m_1 - \varepsilon$ for an appropriately small $\varepsilon > 0$. We will denote the not-equal gadget as in Figure 5.

Variable and clause gadgets

Our reduction is from the standard form of 3-SAT, where we are given a CNF formula on $N$ variables $x_1, \ldots, x_N$. The formula is a conjunction of $M$ clauses $C_1, \ldots, C_M$, where each clause is a disjunction of exactly three literals on distinct variables.

We introduce two global agents. One of them is called $OBS$ and we mean it as an “observer agent”. This is the agent for which we establish hardness of computation. We will follow the rule that $OBS$ observes all gadgets that are present in the network. Second, we place an “evaluation agent” $EVAL$ with private signals $p_T := 0.9$ and $p_F := 0.4$.

Furthermore, for each variable in the CNF formula, we introduce two agents $x_i$ and $\neg x_i$ that receive private signals given by $p_T$ and $p_F$. Then, we encompass those two agents in a counting gadget as shown in Figure 6.
Then, for each clause $C_i$, we introduce a counting gadget on four agents: Three agents corresponding to the literals in the clause (note that they are observed directly and not through the variable gadgets), and the \text{EVAL} agent. The gadget ensures that at least one of those agents received signal 1. Illustration is provided in Figure 7.

The reduction We put the agents \text{EVAL} and \text{OBS} and the variable and clause gadgets together, as explained in previous paragraphs. Finally, we add two more agents $A$ and $B$. We will choose a natural number $b := C \cdot N$ for an absolute big enough constant $C > 0$. Agent $A$ receives private signals with $p_T(A) = 1 - \alpha_1 b$ and $p_F(A) = \alpha_2 b$ and agent $B$ with $p_T(B) = 1 - \alpha_3 b$ and $p_F(B) = \alpha_4 b$ for some $\alpha_1, \ldots, \alpha_4$ that we will choose shortly. Let the corresponding likelihoods be $o_1, o_2, o_3, o_4$ (note that $o_1, o_3 > 0$ and $o_2, o_4 < 0$). We also insert two not-equal gadgets observed by \text{OBS}: One of them is put between \text{EVAL} and $A$ and the other one between $A$ and $B$. The overall construction is illustrated in Figure 8.

We are reducing to the problem of computing the action of agent \text{OBS} at its significant time $t = 2$. Note that \text{OBS} observes all gadgets in the graph, and only gadgets. In particular, \text{OBS} directly infers the signals of all auxiliary agents in the gadgets, but the same cannot be said about the private signals of variable agents. The observation history $H(\text{OBS}, 2)$ is naturally determined by specifications of the gadgets.

Analysis As a preliminary matter, the reduction indeed produces an instance of polynomial size: The size of the graph is $O(N + M)$ and the probabilities of private signals satisfy

$$\exp(-O(N)) \leq p_{0\theta}(u) \leq 1 - \exp(-O(N)).$$

We inspect the construction to understand which private signal configurations are consistent with the observation history of agent \text{OBS}. First, the signals of all auxiliary agents in the gadgets can be directly inferred by \text{OBS}. With that in mind, fix a sequence of private signals to variable agents $S(x_1), \ldots, S(x_n)$. Abusing notation, we identify such sequence with an assignment.
\( C_i = \neg x_1 \lor x_3 \lor \neg x_N \)

\( x_1, \ldots, x_n \) in a natural way. Variable gadgets ensure that each “negation agent” received the opposite signal \( S(\neg x_i) = 1 - S(x_i) \). Moreover, due to clause and not-equal gadgets we have the following:

**Claim 12.**

- For every assignment \( x = (x_1, \ldots, x_n) \), there exists exactly one consistent configuration of private signals with \( S(\text{EVAL}) = 1, S(A) = 0, S(B) = 1 \).
- For every satisfying assignment \( x \), there exists exactly one consistent configuration of private signals with \( S(\text{EVAL}) = 0, S(A) = 1, S(B) = 0 \).
- There are no other consistent configurations.

As a next step, we compare the likelihoods of configurations corresponding to different assignments. To this end, we let the quantity \( P(x, \theta_0) \) be the a priori probability that private signals are in the consistent configuration corresponding to assignment \( x \), \( S(\text{EVAL}) = S_0 \) and \( \theta = \theta_0 \) (note that this is a different definition than given in (3)). Furthermore, we set \( P(x, \theta_0) := P(x, 0, \theta_0) + P(x, 1, \theta_0) \).
By inspecting the construction in a similar way as in Claims 10 and 11 we observe that, for any assignment \( x \):

\[
P(x, 1, T) = q \cdot 0.9 \cdot \alpha_1^b \cdot (1 - \alpha_3^b),
\]

\[
P(x, 1, F) = q \cdot 0.4 \cdot (1 - \alpha_2^b) \cdot \alpha_4^b.
\]

for some \( q(N, M) > 0 \) that does not depend on a specific assignment \( x \). On the other hand, for any satisfying assignment \( x \) we additionally have

\[
P(x, 0, T) = q \cdot 0.1 \cdot (1 - \alpha_1^b) \cdot \alpha_3^b,
\]

\[
P(x, 0, F) = q \cdot 0.6 \cdot \alpha_2^b \cdot (1 - \alpha_4^b).
\]

Each of those expressions is a product of four terms. The value \( q \) corresponds to the probabilities of signals in variable agents and auxiliary agents in the gadgets. The other terms arise from private signals of, respectively, \( \text{EVAL}, A \) and \( B \).

We choose \( \alpha_3 := 0.9, \alpha_2 := \alpha_4 := 0.6, \alpha_1 := 0.4 \) and note that our choice of \( b = CN \) for large enough \( C \) ensures that we can estimate\(^2\)

\[
P(x, 1, T) \in q \cdot 0.4^b \cdot \left(1 \pm \frac{1}{200N}\right)^b,
\]

\[
P(x, 1, F) \in q \cdot 0.6^b \cdot \left(1 \pm \frac{1}{200N}\right)^b,
\]

and, for satisfying assignments,

\[
P(x, 0, T) \in q \cdot 0.9^b \cdot \left(1 \pm \frac{1}{200N}\right)^b.
\]

\[
P(x, 0, F) \in q \cdot 0.6^b \cdot \left(1 \pm \frac{1}{200N}\right)^b.
\]

This in turn implies that for a satisfying assignment we have

\[
P(x, T) \in q \cdot 0.9^b \cdot \left(1 \pm \frac{1}{100N}\right)^b, P(x, F) \in q \cdot 0.6^b \cdot \left(1 \pm \frac{1}{100N}\right)^b,
\]

and for an unsatisfying one

\[
P(x, T) \in q \cdot 0.4^b \cdot \left(1 \pm \frac{1}{100N}\right)^b, P(x, F) \in q \cdot 0.6^b \cdot \left(1 \pm \frac{1}{100N}\right)^b.
\]

Accordingly, if the formula \( \phi \) has a satisfying assignment \( x^* \), it must be that the belief of agent \( \text{OBS} \) at the significant time \( t = 2 \) can be bounded by

\[
1 - \mu(\text{OBS}) = \frac{\sum_{x \in \{0, 1\}^N} P(x, F)}{\sum_{x \in \{0, 1\}^N} P(x, F) + P(x, T)} \leq \frac{\sum_{x \in \{0, 1\}^N} P(x, F)}{P(x^*, T)} \leq \frac{2^N \cdot 0.61^b}{0.89^b} \leq 0.69^b. \tag{12}
\]

At the same time, this probability can be lower bounded as

\[
1 - \mu(\text{OBS}) \geq \frac{P(x^*, F)}{\sum_{x \in \{0, 1\}^N} P(x, T) + P(x, F)} \geq \frac{0.59^b}{2^N+1 \cdot 0.91^b} \geq 0.64^b. \tag{13}
\]

If the formula \( \phi \) is not satisfiable, a simpler computation taking into account only equation \( \tag{11} \)
gives

\[
\mu(\text{OBS}) \in [0.64^b, 0.69^b]. \tag{14}
\]

Hence, \( \mu(\text{OBS}) = 1 - \exp(-\Theta(N)) \) if \( \phi \) is satisfiable and \( \mu(\text{OBS}) = \exp(-\Theta(N)) \) otherwise. \( \square \)

\(^2\) The bounds below are slightly better than needed in order to facilitate the proof of Theorem 3.
Remark 13. There are some results and proofs about opinion exchange models that are sensitive to the tie-breaking rule chosen (see, e.g., Example 3.46 in [MT17]). We claim that the reduction described above (as well as other reductions in this paper) does not suffer from this problem.

Ideally, we would like to say that ties never arise in signal configurations that are consistent with inputs to the reduction. This is seen to be true by inspection, with the following exception: Agents that do not receive private signals are indifferent about the state of the world until their significant time. We made this choice to simplify the exposition. Since significant times are common knowledge, no agent places any weight on others’ actions before their significant time (regardless of the tie-breaking rule used), and the analysis of the reduction is not affected in any way by this fact.

That being said, the ties could be avoided altogether. For example, we could introduce an agent $\text{EPS}$ that is observed by everyone else at time $t = 0$, indicating the action $A(\text{EPS}) = T$ and private signal $S(\text{EPS}) = 1$ corresponding to the likelihood $l_1(\text{EPS}) = \varepsilon$ for a small constant $\varepsilon > 0$. Since likelihoods arising in the analysis of our reduction are always bounded away from zero, $\varepsilon$ can be made small enough so that the agent $\text{EPS}$ does not affect other agents’ actions at their significant times. This almost takes care of the problem, except for the agents without private signals at time $t = 0$ (since they will acquire information from $\text{EPS}$ only at time $t = 1$). This can be solved by giving each such agent $u$ an informative private signal with likelihoods, say

$$l_1(u) = -l_0(u) = \frac{\varepsilon}{100|V|}.$$  

In that case $u$ will output an action corresponding to its private signal at time $t = 0$, but its belief due to private signal (and signals of all other non-informative agents that $u$ observes) will become dominated by belief of $\text{EPS}$ at time $t = 1$.

\[4\] PSPACE-hardness: Proof of Theorem 3

TQBF and the high-level idea Recall that we will show PSPACE-hardness by reduction from the canonical PSPACE-complete language TQBF. More precisely, we use a representation of quantified Boolean formulas

$$\Phi = Q_K x_K \cdots Q_1 x_1 : \phi(x_K, \ldots, x_1),$$

where:

- $Q_i$ is a quantifier such that $Q_i \in \{\exists, \forall\}$, $Q_i \neq Q_{i+1}$ and $Q_1 = \exists$.
- $x_K, \ldots, x_1$ are blocks of variables such that their total count is $|x_K| + \ldots + |x_1| = N$.
- $\phi$ is a propositional logical formula given in the 3-CNF form with $M$ clauses.

The language TQBF consists of all formulas $\Phi$ that are true. It is common and useful to think of $\Phi$ as defining a “position” in a game, where “Player 1” chooses values of variables under existential quantifiers, “Player 0” chooses values of variables under universal quantifiers, and the objective of Player $s$ is to evaluate $\phi$ to the value $s$. Under that interpretation, $\Phi \in \text{TQBF}$ if and only if Player 1 has a winning strategy in the given position.

Keeping that in mind, we can give an intuition for the proof: In the 3-SAT reduction, if the formula had a satisfying assignment, then agent $\text{OBS}$ could conclude whp. that the “hidden” assignment is satisfying, and $\theta = T$. Otherwise, the hidden assignment is not satisfying and $\theta = F$ whp. In the PSPACE reduction, the hidden assignment will correspond (whp.) to a “transcript” of the game played according to a winning strategy for one of the players, and $\theta$ will be determined by the winning player. This will be achieved by implementing a sequence of observer agents $\text{OBS}_1, \ldots, \text{OBS}_K$, where:

- Ultimately, the hardness will be shown for the computation of agent $\text{OBS}_K$. 

\[15\]
• Agent $OBS_i$ directly observes variable agents in blocks $x_K, \ldots, x_{i+1}$.

• For each $i$, there is a (slightly more complicated) gadget similar to the “($EVAL, A, B$)-gadget” employed in the 3-SAT reduction. This gadget involves $OBS_{i-1}$ as well as two new agents $A_i$ and $B_i$ and is observed by $OBS_i$. Its purpose is to “flip” relative likelihoods of different types of variable assignments to implement a quantifier switch.

**The reduction** Recall our formula

$$\Phi = Q_K x_K \cdots \exists x_1 : \phi(x_K, \ldots, x_1).$$

The reduction is defined inductively, with the overall structure illustrated in Figure [10]. First, we make a network identical to the one used used in 3-SAT reduction for the formula $\phi(x_K, \ldots, x_1)$ (i.e., as if all variables were existential). We call the observer agent $OBS_1$ and introduce one difference: $OBS_1$ additionally directly observes all variable agents in variable blocks $x_K, \ldots, x_2$.

Next, for each $1 < i \leq k$ we place two agents $A_i$ and $B_i$ with private signals according to probabilities $p_T(A_i) := 1 - \alpha_i, p_T(A_i) := \alpha_i, p_T(B_i) := 1 - \delta_i, p_T(B_i) := \delta_i$. The parameter $b$ is the same as in the 3-SAT reduction, i.e., $b = C \cdot N$ for some absolute $C$ big enough. The $\alpha_i$ values depend on the parity of $i$ and are provided in Table [1].

| $\alpha$   | even $i$ | odd $i$ |
|------------|----------|---------|
| $\alpha_1$ | $\frac{2}{7} \cdot 0.9$ | 0.9     |
| $\alpha_2$ | 0.9      | $\frac{4}{7} \cdot 0.9$ |
| $\alpha_3$ | 0.9      | $\frac{4}{7} \cdot 0.9$ |
| $\alpha_4$ | $\frac{4}{7} \cdot 0.9$ | 0.9     |

We place a not-equal gadget between $A_i$ and $B_i$. This agent will be observed by $OBS_i$. We would also like to place a not-equal gadget between $OBS_{i-1}$ and $A_i$. More precisely, we want a gadget that will reveal that relevant actions are different: $A(OBS_{i-1}) \neq A(A_i)$. We cannot use the standard not-equal gadget directly, since $OBS_{i-1}$ receives more complicated information than a single private signal. We now describe how to overcome this difficulty, with an illustration in Figure [9].

We put in place a gadget with a structure analogous to the not-equal gadget between $OBS_{i-1}$ and $A_i$. We will call it a **modified not-equal gadget**. It consists of two modified threshold gadgets. One of those gadgets ensures that $A(OBS_{i-1}) \neq T$ or $A(A_i) \neq T$ (we will call it a “large” threshold), and the other one ensures that $A(OBS_{i-1}) \neq F$ or $A(A_i) \neq F$ (this is a “small” threshold). Of course the conjunction of those two guarantees is equivalent to $A(OBS_{i-1}) \neq A(A_i)$. Since the analysis of two threshold gadgets is symmetric, we describe only the large threshold.

We call the main, “summing” agent of this threshold gadget $T_i$ (it is an equivalent of $A$ in Figure [1]). The agent $T_i$:

- Observes agents $OBS_{i-1}$ and $A_i$.

- Additionally observes all agents that $OBS_{i-1}$ observes.

- *Except* that it does not observe variable agents in variable block $x_i$.

The significant time of agent $OBS_{i-1}$ is $t = 2i - 2$ and we set significant time of $T_i$ to $t = 2i - 1$.

Furthermore, we place two more agents $T'_i$ and $T''_i$ corresponding to agents $B$ and $C$ in Figure [1]. Agent $T'_i$ is observed by $OBS_i$ and $T_i$, and broadcasts likelihood $-\delta$. Agent $T''_i$ is observed only
Figure 9: Modified threshold agent illustrated on case $i = K = 2$ and formula $\Phi = \forall y \exists x : \phi(y, x) = 1$. The gadget consists of nodes $T_1, T'_1, \text{ and } T''_1$. These three agents serve the role of $A, B$ and $C$ from Figure 1 and are all observed by agent $\text{OBS}_2$ (see Figure 10). The gadget implements “not-equal” behavior between agents $\text{OBS}_1$ and $A_2$. Red arrow emphasizes that agent $\text{OBS}_1$ directly observes variable agents associated with $y$. Some significant times and likelihoods are shown.

by $\text{OBS}_i$ and broadcasts likelihood $\delta$. We still need to define the threshold value $\delta$. This is not immediate, since we only have bounds $[12]-[14]$ on beliefs of agent $\text{OBS}_{i-1}$, but it can be done. Precise values for both large and small thresholds are in Table 1.

Finally, we place an agent $\text{OBS}_i$ that observes the same agents as $\text{OBS}_{i-1}$, except for variable agents in variable block $x_i$. Note that $\text{OBS}_i$ does not directly observe $\text{OBS}_{i-1}$. Additionally, $\text{OBS}_i$ observes the two not-equal gadgets defined above. It does not receive a private signal, and its significant time is $t = 2i$.

This concludes the definition of the reduction. We show hardness for the computation of agent $\text{OBS}_K$ at time $t = 2K$. Again, since this agent observes only gadgets, its observation history is naturally determined by the semantics of the gadgets. We will show that the truth value of formula $\Phi$ reduces to distinguishing between $\mu(\text{OBS}_K) \approx 1$ and $\mu(\text{OBS}_K) \approx 0$ and, by implication, $A(\text{OBS}_K) = T$ and $A(\text{OBS}_K) = F$.

Analysis: Preliminaries To start with, we note that the $i$-th stage of the inductive definition adds $O(i(N + M))$ new agents (remembering that there are dummy agents that are not shown in the figures). Consequently, the total number of agents is $O(K^2(N + M)) \leq O(N^2(N + M))$. Furthermore, the signal probabilities satisfy [2] by design.

To analyse the belief of agent $\text{OBS}_K$, we need to start with more notation and definitions. For
Figure 10: Schematic representation of the network in case $K = 2$ for formula $\Phi = \forall y \exists x : \phi(y, x) = 1$. The agents and gadgets added in the inductive definition for $i = 2$ are marked in blue. For clarity, edges from the modified not-equal gadget (cf. Figure 9, here marked with exclamation point) are not shown.

Let $G_1$ be the part of the network consisting of all agents created up to the $i$-th step of our inductive definition. Therefore, $G = G_K \supseteq \ldots \supseteq G_1$. The network was defined so that all actions of agents in $G_i$ depend only on private signals of agents in $G_i$. Furthermore, the belief $\mu(OBS_i)$ depends only on private signals of variable agents in $x_K, \ldots, x_{i+1}$ and observations of gadgets by $OBS_i$ (with the latter determined by the reduction, since $OBS_K$ observes all those gadgets as well).

We now need a careful definition in similar vein to $P(x, \theta_0)$ from the 3-SAT reduction. Given $i, 1 \leq i \leq K$, an assignment $(y, x) := (y_K, \ldots, y_{i+1}, x_i, \ldots, x_1)$, as well as $\theta_0 \in \{T, F\}$ we let $P_i(y, x, \theta_0)$ as the probability that all of the following hold:

1. For all gadgets observed by the agent $OBS_i$, $OBS_i$ observed the actions given by the reduction.
2. The assignment determined by private signals of variable agents is equal to $(y, x)$.
3. State of the world is $\theta = \theta_0$.

One checks that $P_i(y, x, \theta_0)$ depends only on private signals in $G_i$. To gain intuition, the reader is invited to convince themselves that, provided that the modified not-equal gadget ensures $A(OBS_{i-1}) \neq A(A_i)$ (we still need to prove that), $P_i(y, x, \theta_0)$ is always a sum over probabilities of one (if $\phi(y, x) = 0$) or two (in case $\phi(y, x) = 1$) signal configurations on $G_i$.

Finally, given $y$ and $\alpha, \varepsilon \in (0, 1)$ we will say that state of the world $\theta_0$ is $\alpha$-likely with error $\varepsilon$ if both

$$\exists x : P_i(y, x, \theta_0) \geq \alpha^b \cdot (1 - \varepsilon)^b,$$

$$\forall x : P_i(y, x, \theta_0) \leq \alpha^b \cdot (1 + \varepsilon)^b.$$
The analysis proceeds by induction on the block number $i$, with a two-part invariant we need to maintain. The first part says that, letting $\varepsilon := \frac{\varepsilon}{\Theta(N)}$, there exists some $\beta := \beta(i) \in (0, 1)$ such that for every partial assignment $y := (y_K, \ldots, y_{i+1})$:

1. If $i$ is odd and $\Phi_y$ is true, then $T$ is $\beta$-likely with error $\varepsilon$ and $F$ is $\frac{2}{3}\beta$-likely with error $\varepsilon$.
2. If $i$ is odd and $\Phi_y$ is false, then $T$ is $\frac{1}{3}\beta$-likely with error $\varepsilon$ and $F$ is $\frac{2}{3}\beta$-likely with error $\varepsilon$.
3. Symmetrically, if $i$ is even and $\Phi_y$ is true, then $T$ is $\frac{2}{3}\beta$-likely with error $\varepsilon$ and $F$ is $\frac{4}{3}\beta$-likely with error $\varepsilon$.
4. If $i$ is even and $\Phi_y$ is false, then $T$ is $\frac{2}{3}\beta$-likely with error $\varepsilon$ and $F$ is $\beta$-likely with error $\varepsilon$.

The second part of the invariant states that whenever $\Phi_y$ is true, the belief of agent OBS$_i$ satisfies $1 - \mu(\text{OBS}_i) \in [0.64^b, 0.69^b]$. Similarly, if $\Phi_y$ is false, then this belief satisfies $\mu(\text{OBS}_i) \in [0.64^b, 0.69^b]$. Note that this part applied to $i = K$ implies the last bullet point in the statement of Theorem 3 with $\mu(\text{OBS}_K)$ being within $\exp(-\Theta(N))$ distance to either zero or one.

**Base case** To establish the base case $i = 1$ one has to go through the proof in Section 3 and convince themselves that the analysis stays valid even when the agent OBS directly observes variable agents $y_K, \ldots, y_2$. Then, the first invariant is established with

$$\beta(1) := q^{1/b} \cdot 0.9,$$

where $q$ is the value featured in equations (10)-(11). For example, $\Phi_1$ being true means that the respective 3-CNF formula $\phi_y(x)$ is satisfiable. Taking a satisfying assignment $x$, we get by (10)

$$P_1(y, x, T) = P(x, T) \geq q \cdot 0.9^b \cdot (1 - \varepsilon)^b = \beta^b \cdot (1 - \varepsilon)^b,$$

$$P_1(y, x, F) = P(x, F) \geq q \cdot 0.6^b \cdot (1 - \varepsilon)^b = \left(\frac{2}{3}\beta\right)^b \cdot (1 - \varepsilon)^b.$$

On the other hand, by (10) and (11), for every assignment, satisfying or not, we have

$$P_1(y, x, T) \leq \max\left(q \cdot 0.9^b \cdot (1 + \varepsilon)^b, q \cdot 0.4^b \cdot (1 + \varepsilon)^b\right) \leq \beta^b \cdot (1 + \varepsilon)^b,$$

$$P_1(y, x, F) \leq q \cdot 0.6^b \cdot (1 + \varepsilon)^b = \left(\frac{2}{3}\beta\right)^b \cdot (1 + \varepsilon)^b.$$

Similar computation gives the first invariant in case $\Phi_y$ is false, this time using only (11). The second invariant is a direct consequence of equations (12)-(14).

**Induction step** We will analyze only even $i$, since the other case is analogous. Fix some $y = (y_K, \ldots, y_{i+1})$. In the following we assume that all actions observed in gadgets are as given by the reduction and that private signals for the initial blocks of variables are given by $y$. Let us call private signal configurations on $G_{i-1}$ that satisfy those conditions consistent.

In this setting, every assignment of private signals in the block $y_i$ determines the action $A(\text{OBS}_{i-1})$ and, by the second invariant, $A(\text{OBS}_{i-1}) = T$ if and only if $\Phi_y, y_i$ is true. Accordingly, we divide consistent configurations into “T-configurations” and “F-configurations”.

Our first objective is to show that the modified not-equal gadget (cf. Figure 10) ensures that $A(\text{OBS}_{i-1}) \neq A(A_i)$. Let $T_i$ be the main agent in a modified threshold gadget between $\text{OBS}_{i-1}$ and $A_i$ (cf. Figure 6). At its significant time $t = 2i - 1$, agent $T_i$ observed everything that agent $\text{OBS}_{i-1}$ observed except for the assignment $y_i$. It also observed the action $A(\text{OBS}_{i-1}) = \theta_0$. Therefore, the private signal configurations on $G_{i-1}$ consistent with observations of $T_i$ are exactly the $\theta_0$-configurations. We let

$$\mu_{\text{OBS}}(\theta_0) := E[\mu(\text{OBS}_{i-1})].$$
where the expectation is over all \( \theta_0 \)-configurations. By the second invariant, \( p_{\text{obs}}(\theta_0) \) is at a distance between 0.64\( b \) and 0.69\( b \) to 1 or 0, depending on the value of \( \theta_0 \). Let \( m(\theta_0) := \ln \frac{p_{\text{obs}}(\theta_0)}{1 - p_{\text{obs}}(\theta_0)} \).

We check that

\[
m(T) \in [0.37b, 0.45b], \quad m(F) \in [-0.45b, -0.37b].
\]

(15)

Recall that, outside of \( G_{i-1} \), agent \( T_i \) observes actions (and infers private signals) of agents \( A_i \) and \( T'^i_i \). The likelihoods of \( A_i \) are given by

\[
\ell_T(A_i) = \ln \frac{1 - \alpha^b_i}{\alpha^b_2} = \ln \frac{1 - (\frac{4}{9} \cdot 0.9)^b}{0.9^b} \in [0.16b, 0.11b]
\]

\[
\ell_F(A_i) = \ln \frac{\alpha^b_i}{1 - \alpha^b_2} = \ln \frac{(\frac{4}{9} \cdot 0.9)^b}{1 - 0.9^b} \in [-0.92b, -0.91b].
\]

The likelihood \(-\delta \) of \( T'_i \) is given by Table[4]. Let \( \theta_1 := A(OBS_{i-1}) \) and \( \theta_2 := A(A_i) \). Since private signals in \( G_{i-1}, A_i \) and \( T'_i \) are independent, it must be that \( A(T_i) = T \) if and only if

\[
m(\theta_1) + \ell_{\theta_2}(A_i) - \delta > 0.
\]

(16)

A calculation shows that the values in Table[4] ensure \( A(OBS_{i-1}) \neq A(A_i) \). For example, for the “large” threshold we have \( \delta = 0.2b \), so \( \theta_1 = \theta_2 = T \) gives

\[
m(T) + \ell_T(A_i) - \delta > (0.37 + 0.1 - 0.2) \cdot b > 0.
\]

On the other hand, if \( \theta_1 \neq T \) or \( \theta_2 \neq T \), then

\[
m(\theta_1) + \ell_{\theta_2}(A_i) - \delta < (-0.26 - 0.2) \cdot b < 0.
\]

Performing a similar reasoning for the “small” threshold, we can conclude that in every consistent configuration \( A(OBS_{i-1}) \neq A(A_i) \), as well as \( A(A_i) \neq A(B_i) \). Therefore, every consistent signal configuration on \( G_{i-1} \) can be extended to a unique configuration on \( G_i \) that is consistent with observations of \( OBS_i \). Consulting Table[4] again, we compute:

\[
\Phi_{y,y_i} \in \text{TQBF} \implies P_i(y, y_i, x; T) = P_{i-1}(y, y_i, x; T) \cdot q \cdot (1 - \alpha^b_2)
\]

(17)

\[
eq P_{i-1}(y, y_i, x; T) \cdot q \cdot \left(0.9 \cdot \frac{4}{9}\right)^b \cdot \left(1 \pm \frac{1}{200N}\right)^b
\]

\[
P_i(y, y_i, x; F) = P_{i-1}(y, y_i, x; F) \cdot q \cdot \left(0.9 \cdot \frac{4}{9}\right)^b \cdot \left(1 \pm \frac{1}{200N}\right)^b
\]

\[
\Phi_{y,y_i} \notin \text{TQBF} \implies P_i(y, y_i, x; \theta_0) \in P_{i-1}(y, y_i, x; \theta_0) \cdot q \cdot 0.9^b \cdot \left(1 \pm \frac{1}{200N}\right)^b
\]

(18)

where \( q \) is a universal factor coming from private signals in auxiliary agents in the not-equal gadgets.

Recall that the first invariant tells us that for some \( \beta' = \beta(i-1) \in (0, 1) \), if \( \Phi_{y,y_i} \) is true, then \( T \) is \( \beta' \)-likely and \( F \) is \( \frac{2}{3} \beta' \)-likely, and if \( \Phi_{y,y_i} \) is false, then \( T \) is \( \frac{2}{3} \beta' \)-likely and \( F \) is \( \frac{2}{3} \beta' \)-likely, all with error \( \frac{1}{100N} \).

Note that \( \Phi_y \) is true if and only if \( \Phi_{y,y_i} \) is true for all \( y_i \). Equivalently, \( \Phi_y \) is false if and only if there exists \( y_i \) such that \( \Phi_{y,y_i} \) is false. Take \( \beta := \beta(i) := \beta' \cdot q^{1/b} \cdot 0.9 \cdot \frac{2}{3} \). Then, the first invariant implies that if \( \Phi_y \) is true, then \( T \) is \( \frac{2}{3} \beta \)-likely and \( F \) is \( \frac{2}{3} \beta \)-likely, with error \( \frac{i}{100N} \). To see that, note
that, by \[(18)\] and induction, we have that for every \(y, y, x\):

\[
P_i(y, y, x, T) \leq P_{i-1}(y, y, x, T) \cdot q \cdot \left(0.9 \cdot \frac{4}{9}\right)^b \left(1 + \frac{1}{200N}\right)^b
\]

\[
\leq (\beta')^b \left(1 + \frac{i-1}{100N}\right)^b \cdot q \cdot \left(0.9 \cdot \frac{4}{9}\right)^b \left(1 + \frac{1}{200N}\right)^b
\]

\[
\leq \left(\frac{2}{3} \cdot \beta\right)^b (1 + \epsilon)^b.
\]

\[
P_i(y, y, x, F) \leq \left(\frac{2}{3} \cdot \beta\right)^b \left(1 + \frac{i-1}{100N}\right)^b \cdot q \cdot \left(0.9 \cdot \frac{4}{9}\right)^b \left(1 + \frac{1}{200N}\right)^b
\]

\[
\leq \left(\frac{4}{9} \cdot \beta\right)^b (1 + \epsilon)^b.
\]

At the same time, symmetric computations give also that for every \(y\) there exist \(y_i\) and \(x\) (just take arbitrary \(y_i\) and \(x\) that exists for \(\Phi^y, y_i\) by the first invariant) such that

\[
P_i(y, y_i, x, T) \geq \left(\frac{2}{3} \cdot \beta\right)^b (1 - \epsilon)^b,
\]

\[
P_i(y, y_i, x, F) \geq \left(\frac{4}{9} \cdot \beta\right)^b (1 - \epsilon)^b.
\]

On the other hand, if \(\Phi^y\) is false, then we need to consider both \[(17)\] and \[(18)\] to conclude that \(T\) is \(\frac{2}{3}\beta\)-likely and \(F\) is \(\beta\)-likely with error \(\epsilon\). However, the computation is very similar to the previous ones and we skip it. Therefore, we implemented the quantifier switch and reestablished the first induction invariant.

Finally, we need to use a computation similar as in \[(12)\] and \[(13)\] to check the second invariant. If \(\Phi^y\) is true, then, since \(T\) is \(\frac{2}{3}\beta\)-likely and \(F\) is \(\beta\)-likely with error \(\frac{i}{100N}\),

\[
1 - \mu(OBS_i) \leq \frac{2N\beta^b(4/9)^b(1 + i/100N)^b}{\beta^b(2/3)^b(1 - i/100N)^b} \leq \frac{2N(2/3)^b1.01^b}{0.99^b} \leq (2/3)^b \cdot 1.03^b \cdot 2N \leq 0.69^b,
\]

\[
1 - \mu(OBS_i) \geq \frac{\beta^b(4/9)^b(1 - i/100N)^b}{2N\beta^b(2/3)^b(1 + i/100N)^b} \geq \frac{2N(2/3)^b0.99^b}{1.01^b} \geq (2/3)^b \cdot 0.97^b \cdot 2N \geq 0.64^b.
\]

A symmetric computation confirms that the second invariant is preserved also when \(\Phi^y\) is false. \(\square\)

5 Bounded signals: Proof of Theorem \[7\]

One could object that our reduction uses private signal distributions with probabilities that are exponentially close to zero and one. Given that it is a worst-case reduction, with relevant configurations arising with exponentially small probability, we do not think this is a significant issue. In any case, in this section we explain how to modify the proof of Theorem \[3\] so that it uses only a fixed collection of (say, at most fifty) private signal distributions.

Note that the only agents we need to replace are \(A_i, B_i\) from the 3-SAT reduction, and \(A_i, B_i\) from the induction step in the \(PSPACE\) reduction, as well as their associated not-equal gadgets. We sketch the modifications on one example, since other cases are analogous. To this end, take even \(i\) and consider \(A_i, B_i\) and their not-equal gadgets (cf. Figures \[9\] and \[10\]).

Going back to the proof of Theorem \[3\], in particular equations \[(17)\]–\[(18)\], what we would like to have is that for every consistent configuration on \(G_{i-1}\), there should be a unique way of extending
it to a consistent configuration on $G_i$ such that for an assignment $(y, x)$ and $\theta_0 \in \{F, T\}$,

$$P_i(y, x, \theta_0) = P_{i-1}(y, x, \theta_0) \cdot r \cdot \begin{cases} 
\alpha_1^b & \text{if } A(OBS_{i-1}) = T \text{ and } \theta_0 = T, \\
\alpha_2^b & \text{if } A(OBS_{i-1}) = T \text{ and } \theta_0 = F, \\
\alpha_3^b & \text{if } A(OBS_{i-1}) = F \text{ and } \theta_0 = T, \\
\alpha_4^b & \text{if } A(OBS_{i-1}) = F \text{ and } \theta_0 = F,
\end{cases}$$

for some $r \in (0, 1)$ independent of $(y, x, \theta_0)$. We are going to achieve this using two independent gadgets corresponding to $A_i$ and $B_i$. Again, we only sketch the construction for $A_i$. What we need, then, is to create a gadget that extends every consistent configuration on $G_{i-1}$ to a unique consistent configuration on $G_i$ such that

$$P_i(y, x, \theta_0) = P_{i-1}(y, x, \theta_0) \cdot r \cdot \begin{cases} 
\alpha_1^b & \text{if } A(OBS_{i-1}) = T = \theta_0, \\
\alpha_2^b & \text{if } A(OBS_{i-1}) = F = \theta_0, \\
1 & \text{otherwise.}
\end{cases} \quad (19)$$

This is achieved as shown in Figure 11. We create an agent $F$ with fixed, arbitrary distribution, say $p_F(F) = 1/4$ and $p_F(F') = 3/4$. Then, we add agents $C_j, D_j, E_j$ for $j = 1, \ldots, b$ with private signal distributions

$$p_{\theta_0}(D_j) := p_{\theta_0}, \quad p_{\theta_0}(C_j) := p_{\theta_0}(E_j) := q_{\theta_0},$$

for some $(\alpha_i$-dependent) constants $p_F, p_T, q_F, q_T$ that we will specify shortly.

For each triple $C_j, D_j, E_j$ we also place three not-equal gadgets observed by $OBS_i$: Respectively, between $F$ and $C_j$, $C_j$ and $D_j$, and $D_j$ and $E_j$. We also create an agent $F'$ with the same signal distribution as $F$, and a counting gadget with equivalence observed by $OBS_i$, making sure that $S(F) + S(F') = 1$ (this is to get rid of a small distortion in (19) due to signal of agent $F$; we will not worry about it from now on). Finally, we place a gadget between $OBS_{i-1}$ and $F$ generalizing the modified not-equal gadget from Theorem 3. This gadget will be observed by $OBS_i$ and we will fill in its details later.

Figure 11: Bounded signals gadget. One out of $b$ parts is shown. The details of the modified not-equal gadget between $OBS_{i-1}$ and $F$ are not shown, and the counting gadget between $F$ and $F'$ is not included.

Let us assume for now that the modified not-equal gadget ensures that $A(OBS_{i-1}) \neq A(F)$ in every consistent configuration. Then, since not-equal gadgets guarantee $S(F) = S(D_j) \neq S(C_j) = S(E_j)$ for every $j$, we claim that it is not difficult to see that every consistent configuration on...
$G_{i-1}$ can be uniquely extended to consistent configuration on $G_i$ such that

$$P_i(y, x, \theta_0) = P_{i-1}(y, x, \theta_0) \cdot r \cdot \begin{cases} 
(\frac{q_T^2(1-p_T)}{\alpha_1})^b & \text{if } A(OBS_{i-1}) = T \text{ and } \theta_0 = T, \\
(\frac{q_F^2(1-p_F)}{\alpha_1})^b & \text{if } A(OBS_{i-1}) = T \text{ and } \theta_0 = F, \\
((1-q_T)^2p_T)^b & \text{if } A(OBS_{i-1}) = F \text{ and } \theta_0 = T, \\
\left(\frac{(1-q_F)^2p_F}{\alpha_2}\right)^b & \text{if } A(OBS_{i-1}) = F \text{ and } \theta_0 = F.
\end{cases}$$

Comparing with (19), we need to find $p_F, p_T, q_F, q_T$ satisfying

$$\frac{q_T^2(1-p_T)}{\alpha_1} = q_F^2(1-p_F) = (1-q_T)^2p_T = \frac{(1-q_F)^2p_F}{\alpha_2}.$$  (20)

Separately comparing and transforming the terms in (20): second with fourth, and then first with third, we get

$$p_F = \frac{\alpha_2 q_F^2}{\alpha_2 q_F^2 + (1-q_F)^2} , \quad p_T = \frac{q_T^2}{q_T^2 + (1-q_T)^2},$$

which can be substituted into comparison of the first and second term, yielding

$$\frac{q_T^2(1-q_T)^2}{q_T^2 + \alpha_1(1-q_T)^2} = \frac{q_F^2(1-q_F)^2}{\alpha_2 q_F^2 + (1-q_F)^2}.$$

Taking $q_T := 1-\varepsilon$ for small enough $\varepsilon > 0$, this can be checked to have a solution with $q_F = \varepsilon + O(\varepsilon^2)$, $p_F = \alpha_2 \varepsilon^2 + O(\varepsilon^3)$ and $p_T = 1 - \alpha_1 \varepsilon^2 + O(\varepsilon^3)$.

We still need to explain how to construct the modified not-equal gadget ensuring that $A(OBS_{i-1}) \neq A(F)$. This is a generalization of the construction in Figure 9 and is shown in Figure 12.

Figure 12: Implementation of the modified not-equal gadget (marked in blue in Figure 11).

Yet again, it is achieved by combining two threshold gadgets and we focus on one of them. Recall from Table 1 that this threshold was set at $\delta = 0.2b$. The objective is to ensure that $A(T) = A(F) = T$ if and only if an inequality like (16) holds.

The threshold gadget will have a “counting agent” $T$ and auxiliary agents $T'_1, \ldots, T'_b$ and $T''_1, \ldots, T''_b$. Auxiliary agents receive private signals with likelihoods $\ell_0(T'_j) := -0.2$ and $\ell_1(T''_j) := 0.2$. Agent $T$ observes $OBS_{i-1}$, as well as other gadgets and agents in the network $G_{i-1}$, in the
same way as agent $T_1$ in Figure [9]. Additionally, it directly observes all agents with private signals in the counting gadget between $F$ and $F'$, as well as all of $C_j, D_j$ and $E_j$. Finally, it observes $T_1', ..., T_k'$. Agent $OBS_j$ observes $T_1', ..., T_k', T_1''', ..., T_k'''$, and $T$. As expected, we specify that at the significant time $OBS_j$ observes actions $A(T_j') = F, A(T_j'') = T$ and $A(T) = F$. Note that we do not need to change the significant time of $OBS_j$.

Assuming that $A(OBS_{j-1}) = \theta_1$ and $A(F) = \theta_2$, we use the same reasoning as in (16) to compute the likelihood $m(\theta_1)$ that agent $T$ can infer from looking at $G_{j-1}$, another likelihood $\ell(\theta_2)$ that can be inferred from looking at $F, C_j, D_j, E_j$ and the likelihood $-\delta = -0.2b$ arising from looking at $T_1', ..., T_k'$. The bounds on $m(\theta_1)$ are the same as in (15), and as for $\ell(\theta_2)$, from (20) we get, as expected $\ell(F) = -b \ln \frac{1}{\alpha_1}$ and $\ell(T) = b \ln \frac{1}{\alpha_2}$.

Since the private signals in these three parts of the graph are conditionally independent, these likelihoods can be added up to ensure that $A(T) = F$ if and only if

$$m(\theta_1) + \ell(\theta_2) < \delta,$$

which implies, the same as in the proof of Theorem [3] that in a consistent configuration either $\theta_1 = F$ or $\theta_2 = F$.

As mentioned, other cases proceed in a similar manner. One difference is that for agents $A$ and $B$ in the base case (3-SAT reduction), EVAL is a simple agent with bounded signal (as opposed to $OBS_{j-1}$). However, this is only good news for us: We do not need to implement the modified not-equal gadget, since a simple not-equal gadget between $B$ and EVAL suffices.

\[\square\]

6 #P-hardness of revealed beliefs: Proof of Theorem [9]

**Reduction** Our reduction uses the DAG structure and the concept of significant time as explained in Section [3]. The general idea is as in Theorem [1] with some adaptations to the counting setting and revealed beliefs. We assume that the agents broadcast beliefs in the form of likelihoods.

We define a common signal distribution with $\rho_T := 3/4$ and $\rho_F := 1/4$ and respective likelihoods $\ell_1$ and $\ell_0$. The graph we construct contains an observer agent $OBS$ with no private signal and the “evaluation” agent EVAL with $(\rho_T, \rho_F)$ private signal. Given a 2-SAT formula $\phi$ with variables $x_1, ..., x_N$ and clauses $C_1, ..., C_M$, respective variable and clause gadgets are designed as follows:

For a variable $x_i$, we create two agents $x_i$ and $\neg x_i$, receiving $(\rho_T, \rho_F)$ private signals. Those two agents are observed by an auxiliary agent, which in turn is observed by agent $OBS$. The observation history of $OBS$ indicates that the auxiliary agent broadcast likelihood $A = \ell_0 + \ell_1$. At the same time, $OBS$ observes another auxiliary agent with informative private signal, broadcasting likelihood $A = -\ell_0 - \ell_1$. See Figure [13] for illustration. Since the likelihood broadcast by the agent observing $x_i$ and $\neg x_i$ is the sum of their likelihoods, we can perform an analysis similar to the threshold gadget in the binary action model. The result is that the variable gadget ensures that $S(x_i) \neq S(\neg x_i)$ and that each consistent signal configuration gives equal likelihoods of $\theta = T$ and $\theta = F$.

In the clause gadget (see Figure [14]) for a clause $C_j$ there is an auxiliary agent observing four agents:

- Two agents corresponding to literals occurring in $C_j$.
- Agent EVAL.
- Agent $E_j$ that receives a private signal $S(E_j) \in \{0, 1, 2\}$. For simplicity we relax our model a bit and allow a private signal with ternary value. Its signal distribution is such that the respective likelihoods satisfy

$$m_0 := \ell_0(E_j) = \ln \frac{\Pr[S(E_j) = 0 \mid \Theta = F]}{\Pr[S(E_j) = 0 \mid \Theta = T]},$$

$$m_1 := \ell_1(E_j) = m_0 + D,$$

$$m_2 := \ell_2(E_j) = m_0 + 2D,$$

(21)
where $D := \ell_1 - \ell_0 = 2 \ln 3$. Furthermore, the probabilities $q(\theta_0, b) := \Pr[S(E_j) = b | \theta = \theta_0]$ for $\theta_0 \in \{T, F\}$ and $b \in \{0, 1, 2\}$ are chosen such that
\[
q(T, 2)q(T, 0) = q(T, 1)^2 = q(F, 2)q(F, 0) = q(F, 1)^2.
\]
(22)
One checks that (21) and (22) are achieved (with $m_0 = -D$) by setting $q(T, 2) = q(F, 0) = q''$, $q(T, 1) = q(F, 1) = q'$, $q(T, 0) = q(F, 2) = 1 - q' - q''$, where $(q'', q')$ is the unique positive solution of
\[
\begin{align*}
q'' - q' - q'' &= 9, \\
(q')^2 &= q'(1 - q' - q'').
\end{align*}
\]
which turns out to be $q'' = 9/13$ and $q' = 3/13$.

The auxiliary agent is observed by OBS, broadcasting belief $A = 3\ell_0 + m_0 + 3D$. Since we want to be somewhat more precise in counting likelihoods induced by different assignments, we introduce additional gadgets “neutralizing” likelihoods induced by signals of agents $E_j$ and $EVAL$, illustrated in Figure 15. Their principle is basically the same as for the variable agents.
For example, for each agent $E_j$ we introduce another agent $F_j$ with the same signal distribution, an agent observing both $E_j$ and $F_j$ and broadcasting $A = 2m_0 + 2D$ to OBS and yet another agent broadcasting opposite belief $A = -2m_0 - 2D$ to OBS. In all, these agents ensure that any private signals to $E_j$ and EVAL do not affect the likelihood of state of the world $\theta$.

Figure 15: Gadgets for $E_j$ and EVAL agents.

Finally, we let $b := 2N$ and introduce agents $A_1, \ldots, A_b$ and $B_1, \ldots, B_b$. Each agent $A_i$ receives a $(p_T, p_F)$ private signal. Agent $B_i$ observes agents EVAL and $A_i$ and broadcasts $\ell_0 + \ell_1$ to agent OBS (see Figure 16). This concludes the description of the reduction.

Figure 16: One of $K$ parts of the “amplification” mechanism.

Analysis  The analysis proceeds analogous to the proof of Theorem 1. First, the network is clearly of size $O(N + M)$ and has the required DAG structure with significant time $t = 2$ for agent OBS. Next, we convince ourselves that the private signals consistent with observations of OBS can be characterized as:

- For each assignment $x$ there exists exactly one consistent configuration of private signals such that $S($EVAL$) = 1$ and $S(A_i) = 0$ for each $i \in \{1, \ldots, b\}$. 

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• For each satisfying assignment \( x \) there is exactly one consistent configuration such that \( S(\text{EVAL}) = 0 \) and \( S(A_i) = 1 \) for each \( i \in \{1, \ldots, b\} \).

• There are no other consistent signal configurations.

Then, we define \( P(x, s, \theta_0) \) as the probability that \( \theta = \theta_0 \) and that there arises the unique signal configuration consistent with assignment \( x \) and \( S(\text{EVAL}) = s \). The gadgets (recall the relation (22) for agents \( E_j \) and \( F_j \)) ensure that \( P(\cdot) \) is equal to

\[
P(x, 1, T) = q \cdot \left( \frac{1}{4} \right)^b, \quad P(x, 1, F) = q \cdot \left( \frac{3}{4} \right)^b,
\]

and, for each assignment \( x \) that is satisfying, additionally

\[
P(x, 0, T) = q \cdot \left( \frac{3}{4} \right)^b, \quad P(x, 0, F) = q \cdot \left( \frac{1}{4} \right)^b,
\]

where \( q \) is a universal common factor that depends only on \( N \) and \( M \). Recalling that \( A \) denotes the number of satisfying assignments in \( \phi \), we can conclude that the likelihood of agent \( \text{OBS} \) at its significant time \( t = 2 \) is given by

\[
\frac{\mu(\text{OBS})}{1 - \mu(\text{OBS})} = \frac{A \cdot (3/4)^b + 2^N \cdot (1/4)^b}{2^N \cdot (3/4)^b + A \cdot (1/4)^b} = \frac{A \cdot (3/4)^b + 2^N \cdot (1/4)^b}{2^N \cdot (3/4)^b + A \cdot (1/4)^b} = \frac{A + 2^N}{2^N + A} \leq \frac{A}{2^N} \left[ 1 \pm \frac{1}{4^N} \right].
\]

In particular,

\[
\left| \frac{\mu(\text{OBS})}{1 - \mu(\text{OBS})} - \frac{A}{2^N} \right| \leq \frac{A}{8^N} < \frac{1}{2^{N+1}},
\]

so rounding the likelihood to the nearest multiple of \( 2^{-N} \) successfully recovers the number of satisfying assignments \( A \).

\[\square\]

7 Conclusion

A natural open question is to make progress on the approximate-case hardness in one of the models. For example, one could try to establish \( \text{NP} \)-hardness for a worst-case network, but holding for signal configurations arising with non-negligible probability. In a different direction, as was mentioned in the introduction, there remains a gap between our \( \text{PSPACE} \)-hardness result and exponential space required by the best known algorithm.

Another interesting problem arises from trying to extend our results to the revealed beliefs model, as discussed in Sections 2.3 and 6. Thinking in terms of two-player games, consider a class of “no-mistakes-allowed” games: Games where the player with winning strategy always has exactly one winning move, with all alternative moves in a given position leading to a losing position (and this property holding recursively in all positions reachable from the initial one).

Certainly deciding if a position is winning for the first player in such games is in \( \text{PSPACE} \). On the other hand, since such a game with just the existential player corresponds to a \( \text{SAT} \) formula with zero or one satisfying assignments, by the Valiant-Vazirani theorem \( [VV86] \) it is also (morally) \( \text{NP} \)-hard. This leaves a large gap between \( \text{NP} \) and \( \text{PSPACE} \).

For example, suppose we want to prove \( \Pi_2 \)-hardness in the revealed belief model. Then it is natural to consider formulas of the form

\[
\forall x \exists y : \phi(x, y),
\]

and the question becomes: How hard is it to distinguish between the cases.
• YES: For all $x$, there exists unique $y$ such that $\phi(x, y) = 1$.
• NO: There exists unique $x_0$ such that no $y$ satisfies $\phi(x_0, \cdot)$. For all other $x$, there exists a unique $y$ such that $\phi(x, y) = 1$.

How hard is this problem? In particular, can it be shown to be harder than NP (in some sense)? Hardness of such games can be thought of as a generalization of the Valiant-Vazirani theorem.

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