Jensen’s Inequality for $g$-Convex Function under $g$-Expectation

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Abstract. A real valued function defined on $\mathbb{R}$ is called $g$-convex if it satisfies the following “generalized Jensen’s inequality” under a given $g$-expectation, i.e., $h(\mathbb{E}^g[X]) \leq \mathbb{E}^g[h(X)]$, for all random variables $X$ such that both sides of the inequality are meaningful. In this paper we will give a necessary and sufficient conditions for a $C^2$-function being $g$-convex. We also studied some more general situations. We also studied $g$-concave and $g$-affine functions.

1 Introduction

Jensen’s inequality plays an important role in probability theory. It claims that for any given convex function $h$ defined on $\mathbb{R}$ we have

$$h(\mathbb{E}[X]) \leq \mathbb{E}[h(X)]$$

for each random variable $X$ such that $\mathbb{E}[X]$ and $\mathbb{E}[h(X)]$ are meaningful. Here $\mathbb{E}[\cdot]$ stands for the expectation related to a probability $P$. It is worth to mention that its converse is also true: If the above inequality holds true for all random variables $X$ such that both $\mathbb{E}[X]$ and $\mathbb{E}[h(X)]$ are meaningful, then $h$ is a convex function.

In 1997 Peng [P1997] (see also [P1995]) introduced the notion of $g$-expectation $\mathbb{E}^g[\cdot]$ defined via a backward stochastic differential equation of which the generator is a given function $g = g(t, y, z)_{(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d}$. A $g$-expectation preserves most properties of the classical expectations except that it is a nonlinear functional. Its nonlinearity is characterized by its generator $g$. It becomes a typical example of nonlinear expectations under which the time-consistency holds true

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thus a theory of nonlinear martingales can be developed. It is also a useful tool to
the nonlinear dynamic pricing as well as dynamic risk measures in finance.

A very interesting problem is whether, for a \( g \)-expectation, the following
generalized Jensen’s inequality is true:

\[
    h(\mathbb{E}^g[X]) \leq \mathbb{E}^g[h(X)],
\]

for each \( X \) s.t. \( \mathbb{E}^g[X] \) and \( \mathbb{E}^g[h(X)] \) are meaningful.

This problem was initialed in \([BCHMP, CHMP2000]\) in which a counterexample
was given to show that the above generalized Jensen’s inequality fails for a very
simple convex functions \( h \). A sufficient condition for a special situation was
also provided. Chen, Kulperger and Jiang \([CKJ, 2003]\) have obtained a very
interesting result: provided \( g \) does not depend on \( y \), the above generalised
Jensen’s inequality holds true for each convex function \( h \) if and only if \( g \) is a
super-homogeneous function, i.e., \( g(t, \lambda z) \geq \lambda g(t, z) \), \( dP \times dt - a.s. \) for \( \lambda \in \mathbb{R} \)
and \( z \in \mathbb{R}^d \). This result was improved by \([Hu, 2005]\) showing that, in fact, \( g \)
must be independent of \( y \).

In this paper we study this problem with a different point of view: For each
fixed function \( g \), to give an explicit characterization to \( h \) satisfying the above
generalised Jensen inequality. We have obtained the following result: For a
\( C^2 \)-function \( h \) the above generalised Jensen inequality holds if and only if \( h \)
satisfies:

\[
    \frac{1}{2} h''(y) |z|^2 + g(t, h(y), h'(y)z) - h'(y)g(t, y, z) \geq 0, \ dP \times dt - a.s., \ \forall (y, z) \in \mathbb{R} \times \mathbb{R}^d.
\]

The previously mentioned result of classical Jensen’s inequality just corresponds
a special case where \( g \equiv 0 \). The above mentioned results in \([CKJ]\) and \([Hu]\)
can be also obtained from our new result. For the case where \( h \) is only a continuous
function we have also obtained a similar result by using the notion of the well-
known viscosity solution in partial differential equations.

It is natural to call a \( h \) satisfying the above inequality to be a \( g \)-convex
function. In general, a continuous function \( h \) satisfying the generalized Jensen’s
inequality is called a \( g \)-convex function. In this paper we will study this type
of functions. We also investigate the related \( g \)-concave as well as \( g \)-affine func-
tions. A deep relation of \( g \)-convexity and backward stochastic viability property
introduced by Buckdahn, Quincampoix and Rascanu in \([BQR]\) is also disclosed.

This paper is organized as follows. In Section 2 we recall some facts about
\( g \)-expectation and BSDEs. The notion of \( g \)-convexity as well as the necessary
and sufficient condition for a \( g \)-convex \( C^2 \)-function will be given in Section 3. We
establish the necessary and sufficient condition for a continuous \( g \)-convex function
in Section 4. An equivalence between \( g \)-convexity and backward stochastic
viability property is given in Section 5. Finally, in Section 6 we study functional
operations preserving \( g \)-convexity and apply the results obtained in foregoing
sections to prove some properties of \( g \)-expectations.
2 Some Facts about $g$-Expectations

Let $(\Omega, \mathcal{F}, P)$ be a given probability space and let $(W_t)_{t \geq 0}$ be a $d$-dimensional Brownian motion in this space. The natural filtration generated by $W$ will be denoted by $(\mathcal{F}_t)_{t \geq 0}$.

Let $T > 0$ be a fixed real number. For any $0 \leq t \leq T$, we denote by $L^p(\mathcal{F}_t)$, the space of $\mathcal{F}_t$-measurable random variables satisfying $E[|X|^p] < \infty$, for $p \geq 1$. For a positive integer $n$ and $z \in \mathbb{R}^n$, we denote by $|z|$ the Euclidean norm of $z$. We will denote by $L^2_T(0, T; \mathbb{R}^n)$, the space of all progressively measurable $\mathbb{R}^n$-valued processes such that $E\left[\int_0^T |\psi_t|^2 \, dt\right] < \infty$; and by $S^2_T(0, T; \mathbb{R}^n)$ the elements in $L^2_T(0, T; \mathbb{R}^n)$ with continuous paths such that $E\left[\sup_{t \in [0, T]} |\psi_t|^2\right] < \infty$. And we denote by $D^2_T(0, T)$ the set of all $\mathbb{R}CLL$ (right continuous with left limit) processes $\phi$ in $L^2_T(0, T; \mathbb{R})$ such that $E[\sup_{t \in [0, T]} |\phi|^2] < \infty$.

Let us consider a function $g$, which will be in the sequel the generator of the backward stochastic differential equation (BSDE), defined on $\Omega \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$, with values in $\mathbb{R}^m$, such that the process $(g(t, y, z))_{t \in [0, T]}$ is progressively measurable for each $(y, z)$ in $\mathbb{R}^m \times \mathbb{R}^{m \times d}$.

Throughout this paper the function $g$ will satisfy the following conditions:

\[
\begin{align*}
(a) & \quad \text{There exists a constant } \mu > 0, \text{ for each } (y, z), (\bar{y}, \bar{z}) \in \mathbb{R}^m \times \mathbb{R}^{m \times d}, \\
& \quad |g(t, y, z) - g(t, \bar{y}, \bar{z})| \leq \mu(|y - \bar{y}| + |z - \bar{z}|); \\
(b) & \quad (g(t, 0, 0))_{t \in [0, T]} \in L_2(0, T; \mathbb{R}^m).
\end{align*}
\]

(2.1)

It is by now well known (see Pardoux and Peng [PP]) that under the assumptions (2.1), for any random variable $X \in L^2(\mathcal{F}_T)$, the BSDE

\[
Y_t = X + \int_t^T g(s, Y_s, Z_s) \, ds - \int_t^T Z_s \, dW_s, \quad t \in [0, T],
\]

has a unique adapted solution $(Y_t, Z_t)_{t \in [0, T]} \in S^2_T(0, T; \mathbb{R}^m) \times L^2_T(0, T; \mathbb{R}^{m \times d})$.

In the sequel we denote equation (2.2) by $(g, T, X)$.

In this paper we mainly discuss the 1-dimensional BSDE, i.e., $m = 1$. The following situations are typical:

\[
\begin{align*}
(a) & \quad g(\cdot, 0, 0) \equiv 0, \\
(b) & \quad g(\cdot, y, 0) \equiv 0, \forall y \in \mathbb{R}.
\end{align*}
\]

(2.3)

Obviously (b) implies (a). The following notion of $g$-expectation was introduced by Peng [P1997].

**Definition 2.1** Let $m = 1$. We denote by $E^0_{t,T}[X] := Y_t$:

\[
E^0_{t,T}[\cdot] : L^2(\mathcal{F}_T) \rightarrow L^2(\mathcal{F}_t), \quad 0 \leq t \leq T < \infty.
\]

(2.4)

$(E^0_{t,T}[\cdot])_{0 \leq t \leq T}$ is called $g$-expectation.
Applications of $g$-expectations in dynamic superpricing and dynamic risk measures can be found in [B-El] [CE] [CHMP] [DE] [EPQ] [EQ] [E-RC] [P2004] [P2004b] [Rosazza] [Yong].

**Remark 2.2** The $g$-expectation originally introduced in [P1997] corresponds to the case in which $g$ satisfies (2.3)-b, that is, the situation of "zero interest rate" (see next section or [P2004]). Peng [P2002], [P2005] also introduced the notion of $g$-evaluation if $g$ satisfies (2.3)-a, the situation of "self-financing". For the simplicity, we call them all $g$-expectation here whenever $g$ satisfies (2.3).

Also we have the following properties about $g$-expectation (see [P2004] Theorem 3.4):

**Proposition 2.3** Let the generator $g$ satisfies (2.3) and (2.3)-a. Then the above defined $g$-expectation $E^g[\cdot]$ satisfies, for each $t \leq T < \infty$, $X, \bar{X} \in L^2(F_T)$,

- **(A1)** $E^g_{t,T}t[X] \geq E^g_{t,T}[X]$, a.s., if $X \geq \bar{X}$;
- **(A2)** $E^g_{t,T}[X] = X$;
- **(A3)** $E^g_{s,T}[E^g_{t,T}[X]] = E^g_{s,T}[X]$, a.s., for $s \leq t$;
- **(A4)** $1_A E^g_{t,T}[X] = E^g_{t,T}[1_A X]$, $\forall A \in F_t$, where $1_A$ is the indicator function of $A$, i.e. $1_A(\omega)$ equals 1 when $\omega \in A$ and 0 otherwise.

If (2.3) does not hold, (A1)-(A3) still hold true. But (A4) is replaced by the following: for each $X_1, \cdots, X_N \in L^2(F_T)$ and for each $F_t$-partition $\{A_i\}_{i=1}^N$ of $\Omega$ (i.e. $A_i \in F_t$, $A_i \cap A_j = \emptyset$ if $i \neq j$ and $\bigcup A_i = \Omega$), we have

- **(A4')** $\sum_{i=1}^N 1_A E^g_{t,T}[X_i] = E^g_{t,T}[\sum_{i=1}^N 1_A X_i]$.

**Lemma 2.4** (See [EPQ] or Proposition 2.2 in [BCHMP]) Let $g$ satisfy (2.4) and $m = 1$, and let $X \in L^2(F_T)$. Then the solution $(Y_t, Z_t)_{t \in [0,T]}$ of BSDE (2.2) satisfies

$$E \left[ \sup_{s \in [0,T]} (e^{\beta s} |Y_s|^2) + \int_t^T e^{\beta s} |Z_s|^2 \ ds | F_t \right] \leq K E \left[ e^{\beta T} |X|^2 + \left( \int_t^T e^{(\beta/2)s} |g(s,0,0)| \ ds \right)^2 | F_t \right].$$

where $\beta = 2(\mu + \mu^2)$ and $K$ is a positive constant only depending on $\mu$.

**Remark 2.5** The above lemma implies that $E^g[\cdot]$ is continuous in $L^2$.

The decomposition theorem of $E^g$-supermartingale obtained in [P1999] (see also [P2004]) will play an important role in this paper.

**Proposition 2.6** (Decomposition theorem of $E^g$-supermartingale) We assume that $g$ satisfies (2.7) and $m = 1$. Let $Y \in D^2_T(0,T)$ be a $g$-supermartingale, namely, for each $0 \leq s \leq t \leq T$,

$$E_{s,t}^g[Y_t] \leq Y_s.$$
Then there exists a unique $\mathcal{F}_t$-adapted increasing and RCLL process $A \in D^2_F (0, T)$ (thus predictable) with $A_0 = 0$, such that, $Y$ is the solution of the following BSDE:

$$Y_t = Y_T + (A_T - A_t) + \int_t^T g(t, Y_s, Z_s) - \int_t^T Z_s dW_s, \quad t \in [0, T].$$

### 3 g-Convexity for $C^2$-functions

To begin with we give the notion of $g$-convexity.

**Definition 3.1** For a given $g$-expectation $\mathbb{E}^g[\cdot ]$, a function $h : \mathbb{R} \rightarrow \mathbb{R}$ is said to be $g$-convex (resp. $g$-concave) if for each $X \in L^2 (\mathcal{F}_T)$ such that $h(X) \in L^2 (\mathcal{F}_T)$, one has

$$h(\mathbb{E}^g_{t,T}[X]) \leq \mathbb{E}^g_{t,T}[h(X)], \quad \text{(resp. } h(\mathbb{E}^g_{t,T}[X]) \leq \mathbb{E}^g_{t,T}[h(X)]) \text{ } \mathbb{P}-\text{a.s., } t \in [0, T].$$

$h$ is called $g$-affine if it is both $g$-convex and $g$-concave.

Clearly, for each $g$ the function $h(y) = y$ is $g$-affine. Throughout this paper, we only consider the case where $h$ is continuous. In the case when $h$ is a $C^2$-function, we have the following result. For notational convenience, we denote

$$L^t,y,z_{g}\varphi := \frac{1}{2}\varphi_{yy}(y)|z|^2 + g(t, \varphi(y), \varphi_y(y)z) - \varphi_y(y) g(t, y, z), \quad \varphi \in C^2 (\mathbb{R}).$$

**Theorem 3.2** Let $g$ satisfy (2.1) and let $h \in C^2 (\mathbb{R})$. Then the following two statements are equivalent:

(i) $h$ is $g$-convex (resp. $g$-concave);

(ii) For each $y \in \mathbb{R}$, $z \in \mathbb{R}^d$,

$$L^t,y,z_{g} h \geq 0 \ (\text{resp. } \leq 0), \quad dP \times dt-a.s. \quad (3.1)$$

**Remark 3.3** If we assume furthermore $g(t, y, 0) \equiv 0 \ (\text{2.11-b})$, then we can define

$$\mathbb{E}^g [X \mid \mathcal{F}_t] = \mathbb{E}^g_{t,T}[X].$$

Thus the Jensen’s inequality becomes

$$\mathbb{E}^g [h(X) \mid \mathcal{F}_t] \geq h(\mathbb{E}^g[X \mid \mathcal{F}_t]).$$

In particular, when $t = 0, \quad \mathbb{E}^g[h(X)] \geq h(\mathbb{E}^g[X]).$

Before the proof of Theorem 3.2, we prove the following lemma.

**Lemma 3.4** Assume that $g$ satisfies (2.1) and an $C^2$-function $h$ satisfies (3.1). Then $h$ is convex in the usual sense.
Proof. For each $y_0 \in \mathbb{R}$, one has,

$$
0 \leq \frac{1}{2} h''(y_0) |z|^2 + g(t, h(y_0), h'(y_0)z) - h'(y_0)g(t, y_0, z)
= \frac{1}{2} h''(y_0) |z|^2 + (g(t, h(y_0), h'(y_0)z) - g(t, 0, 0))
+ h'(y_0)(g(t, 0, 0) - g(t, y_0, z)) + g(t, 0, 0) - h'(y_0)(g(t, 0, 0))
\leq \frac{1}{2} h''(y_0) |z|^2 + 2C_1 |h'(y_0)| |z| + C_1 + C_1 |g(t, 0, 0)|
$$

where $C_1$ only depends on $\mu$ and $y_0$. Thus

$$
0 \leq \frac{1}{2} h''(y_0) |z|^2 + 2C_2 |h'(y_0)| |z| + C_2, \quad \forall z \in \mathbb{R}^d.
$$

where $C_2$ only depends on $C_1$ and $M = E \left[ \int_0^T |g(t, 0, 0)|^2 \, dt \right]$. Thus $h''(y_0)$ must be non-negative. \(\blacksquare\)

Lemma 3.5 Let $h \in C(\mathbb{R})$ be a convex function. If for each $X \in L^\infty(\mathcal{F}_T)$ we have

$$
E^0_{t,T}[h(X)] \geq h(E^0_{t,T}[X]), \quad \text{a.s.,} \quad \forall t \in [0, T], \quad (3.2)
$$

then this relation also holds for each $X \in L^2(\mathcal{F}_T)$ such that $h(X) \in L^2(\mathcal{F}_T)$.

Proof. We need to consider two cases: (a) $h$ is a monotone function; (b) there exists a $\tilde{y} \in \mathbb{R}$ such that $h(y) \geq h(\tilde{y})$. For case (a), we have

$$
E^0_{t,T}[h((-n) \vee X \wedge m)] \geq h(E^0_{t,T}[((-n) \vee X \wedge m)]), \quad \text{a.s.} \quad m, n = 1, 2, \cdots.
$$

Since, for each fixed $n$, the sequence $\{h((-n) \vee X \wedge m)\}_{m=1}^\infty$ (resp. $\{(-n) \vee X \wedge m\}_{m=1}^\infty$) monotonically converges to $h((-n) \vee X)$ (resp. $(-n) \vee X$) in $L^2(\mathcal{F}_T)$ as $m \to \infty$. We then can pass limit on the both sides of the above inequality and obtain

$$
E^0_{t,T}[h((-n) \vee X)] \geq h(E^0_{t,T}[(-n) \vee X]), \quad \text{a.s.} \quad n = 1, 2, \cdots.
$$

Similarly, when $n \uparrow \infty$, the sequence $\{h((-n) \vee X)\}_{n=1}^\infty$ (resp. $\{(-n) \vee X\}_{n=1}^\infty$) monotonically converges to $h(X)$ (resp. $X$) in $L^2(\mathcal{F}_T)$. Thus we can pass limit on the both sides of the above inequality and obtain \(\blacksquare\). For case (b), we observe that then $h$ increases on $[\tilde{y}, \infty)$ and decreases on $(-\infty, \tilde{y}]$ thus, as $m \to \infty$, $h((-m + \tilde{y}) \vee X \wedge (m + \tilde{y}))$ increasingly converges to $h(X)$ in $L^2(\mathcal{F}_T)$. We then pass limit on both sides of

$$
E^0_{t,T}[h((-m + \tilde{y}) \vee X \wedge (m + \tilde{y}))] \geq h(E^0_{t,T}[(-m + \tilde{y}) \vee X \wedge (m + \tilde{y})]),
\quad \text{a.s.} \quad m = 1, 2, \cdots.
$$

and thus obtain \(\blacksquare\). \hfill \blacklozenge
Proof of Theorem 3.2. (ii) $\implies$ (i): We first consider the case where $X$ is bounded. The corresponding solution $Y$ of $(g, T, X)$ is also bounded since $X$ and $g(\cdot, 0, 0)$ are bounded. We now apply Itô’s formula to $h(Y_t)$:

$$-dh(Y_t) = \left[-\frac{1}{2} h''(Y_t)|Z_t|^2 + h'(Y_t)g(t, Y_t, Z_t)\right]dt - h'(Y_t)Z_t dW_t$$

$$= [g(t, h(Y_t), h'(Y_t)Z_t) + \psi_t]dt - h'(Y_t)Z_t dW_t,$$

where

$$\psi_t = \frac{1}{2} h''(Y_t)|Z_t|^2 - g(t, h(Y_t), h'(Y_t)Z_t) + h'(Y_t)g(t, Y_t, Z_t).$$

From (3.1), it follows that $\psi_t \leq 0$ and thus $h(Y_t)$ is a $g$-subsolution. By comparison theorem of BSDE it follows that

$$E^g_{t,T}[h(X)] = E^g_{t,T}[h(Y_T)] \geq h(Y_t) = h(E^g_{t,T}[X]), \ a.s., \ \forall X \in L^\infty(F_T).$$

On the other hand, from Lemma 3.4 $h$ is convex. This with Lemma 3.5 yields (i).

(i) $\implies$ (ii): We only give a proof for the situation where $g(\cdot, y, z)$ is a continuous process on $[0, T]$. For each fixed $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, we consider the following SDE

$$-dY^{t,y,z}_s = 1_{[t,T]}(s)[g(s, Y^{t,y,z}_s, z)ds - zdW_s], \ Y_0 = y.$$ 

We apply Itô’s formula on $[t, T]$:

$$-dh(Y^{t,y,z}_s) = \left[-\frac{1}{2} h''(Y^{t,y,z}_s)|z|^2 + h'(Y^{t,y,z}_s)g(s, Y^{t,y,z}_s, z)\right]ds - h'(Y^{t,y,z}_s)z dW_s$$

For a large number $m > 0$, let $\tau_m = \inf\{s \geq t : |Y^{t,y,z}_s - y| = m\}$. It is that $Y^{t,y,z}_s$ is bounded on $[t, \tau_m]$. By (i),

$$E^g_{t,T}(Y^{t,y,z}_s, Y^{t,y,z}_{s\wedge \tau_m}) \geq h(Y^{t,y,z}_{s\wedge \tau_m}), \ P-a.s., \forall r \in [t, \tau_m].$$

That is, $h(Y^{t,y,z}_s)$ is a $g$-submartingale on $[t, \tau_m]$. By the decomposition theorem of $g$-submartingale (Proposition 2.6), it follows that there exist an increasing process $(A_s)_{s \geq t}$ such that

$$h(Y^{t,y,z}_{s\wedge \tau_m}) = h(y) - \int_t^{s\wedge \tau_m} g(r, h(Y^{t,y,z}_r), Z_r) dr + A_{s\wedge \tau_m} + \int_t^{s\wedge \tau_m} Z_r dW_r.$$ 

This with

$$h(Y^{t,y,z}_{s\wedge \tau_m}) = h(y) - \int_t^{s\wedge \tau_m} \left[\frac{1}{2} h''(Y^{t,y,z}_r)|z|^2 - h'(Y^{t,y,z}_r)g(r, Y^{t,y,z}_r, z)\right]dr$$

$$+ \int_t^{s\wedge \tau_m} h'(Y^{t,y,z}_r)z dW_r.$$
yields $Z_s \equiv h'(Y^{t,y,z}_s)z$ and

$$-\frac{1}{2}h''(Y^{t,y,z}_s)z^2 + h'(Y^{t,y,z}_s)g(s,Y^{t,y,z}_s,z) \leq g(s,h(Y^{t,y,z}_s),h'(Y^{t,y,z}_s))$$

on $[t,\tau_m]$. Since $g(\cdot, y, z)$ is a continuous process (otherwise a technique in the proof of Theorem 8.1 in [P2005b] is needed), as $s = t$, we can obtain (3.1). The proof is complete.

**Example 3.6** For the case $g = \langle \xi_t, z \rangle$, the $g$-expectation corresponds to the classical linear expectation (Girsanov transformation). Theorem 3.2 becomes the classical results: $h \in C^2(R)$ is $g$-convex if and only if $h''(y) \geq 0$.

**Remark 3.7** A $g$-concave function is concave in the usual sense. Its proof is similar to that of Corollary 5.6.

From Theorem 3.2 we can also derive the following result of [CKJ] and its improved version [Hu].

**Proposition 3.8** Let $g$ satisfy (2.1). Then the following two statements are equivalent:

(i) For each convex function $h$, and each $X \in L^2(F_T)$ such that $h(X) \in L^2(F_T)$,

$$\mathbb{E}^g_{t,T}[h(X)] \geq h(\mathbb{E}^g_{t,T}[X]), \quad \forall 0 \leq t \leq T;$$

(ii) $g$ is independent of $y$, and is super-homogeneous in $z$, i.e., for any $\lambda \in \mathbb{R}$,

$$g(t, \lambda z) \geq \lambda g(t, z).$$

**Proof.** (ii) $\Rightarrow$ (i): In the case when $h \in C^2$, this can be proved by (3.1). For general situation we can apply the same technique in the proof of (3.1) (see [CKJ]).

(i) $\Rightarrow$ (ii): For each given $a, b \in \mathbb{R}$, take $h(x) = ax + b$. Obviously it is a convex function and in $C^2(\mathbb{R})$. Thus the inequality (3.1) yields $g(t, ay + bz) - ag(t, y, z) \geq 0$, $dP \times dt$-a.s. Since $a, b$ can be chosen arbitrarily, $g$ must be independent of $y$ and super-homogeneous in $z$.

**Corollary 3.9** Let $g$ satisfy (2.1) and be independent of $z$, and $h \in C^2(\mathbb{R})$. Then the following two statements equivalent:

(i) $h$ is $g$-convex;

(ii) $h$ is convex ($h''(y) \geq 0$ for each $y$) and satisfies

$$\forall y, \quad g(t, h(y)) - h'(y)g(t, y) \geq 0, \quad dP \times dt \text{-a.s.}$$

**Corollary 3.10** Let $g$ satisfy (2.1) and be independent of $y$, and let $h \in C^2(\mathbb{R})$ be $g$-convex. Moreover if there exist a set $\Gamma \in \Omega \times [0, T]$ with positive measure, in which $g(t, 0) > 0$ (resp. $g(t, 0) < 0$), then $h'(y) \leq 1$ (resp. $h'(y) \geq 1$).

A simple and fundamentally important result in stochastic analysis is that, for each martingale $X$ and for each convex function $h$ such that $h(X) \in L^1$, the process $h(X)$ is a submartingale. For $g$-expectation, we have:
Theorem 3.11 If \((Y_t)_{t \in [0,T]}\) is a \(g\)-martingale, and \(h\) is a \(g\)-convex function (resp. \(g\)-concave function, \(g\)-affine function), then \((h(Y_t))_{t \in [0,T]}\) is a \(g\)-submartingale (resp. \(g\)-supermartingale, \(g\)-martingale) provided \(h(Y_t) \in L^2(F_t), \; t \in [0,T]\).

Proof. Let \(Y_t\) be a \(g\)-martingale and \(h\) a \(g\)-convex function, then
\[
\mathbb{E}_{s,t}[h(Y_t)|F_s] \geq h(\mathbb{E}_{s,t}[Y_t|F_s]) = h(Y_s),
\]
for any \(0 \leq s < t \leq T\), as required. The proofs of other cases are similar.

Moreover its inverse also holds, namely,

Theorem 3.12 Let \(g\) satisfy (2.1). If for each \(g\)-martingale \((Y_t)_{t \in [0,T]}\), \((h(Y_t))_{t \in [0,T]}\) is a \(g\)-submartingale (resp. \(g\)-supermartingale, \(g\)-martingale), then \(h\) is a \(g\)-convex (resp. \(g\)-concave, \(g\)-affine) function.

Proof. We only prove the case of \(g\)-submartingale. Since, for each \(X \in L^2(F_T)\), \((E^g_{t,T}[X])_{t \in [0,T]}\) is a \(g\)-martingale. Define \(\tilde{Y}_t = h(E^g_{t,T}[X])\), we have, for \(0 \leq s < t \leq T\),
\[
E^g_{s,t}[h(E^g_{t,T}[X])] = E^g_{s,t}[h(E^g_{s,t}[Y])] = E^g_{s,t}[\tilde{Y}_t] \geq \tilde{Y}_s = h(E^g_{s,t}[X])
\]
In particular, as \(t = T\), it follows that \(E^g_{s,T}[h(X)] \geq h(E^g_{s,T}[X])\) for \(s \in [0,T]\). Thus \(h\) is a \(g\)-convex function.

4 \(g\)-Convexity for Continuous Functions

In this section we consider \(g\)-convex functions \(h \in C(\mathbb{R})\), i.e., without the \(C^2\)-assumption.

We now recall the definition of viscosity subsolutions.

Definition 4.1 Let \(g\) satisfy (2.1) and independent of \(\omega\). A continuous function \(u : \mathbb{R} \to \mathbb{R}\) is called a viscosity subsolution of \(L^{1,y,z}_g u = 0\) if, for any \(\varphi \in C^2(\mathbb{R})\), and \(x \in \mathbb{R}\) such that \(u - \varphi\) attains local maximum at \(x\), one has for each \((t,z) \in [0,T] \times \mathbb{R}\),
\[
L^{t,x,z}_g \varphi = \frac{1}{2} \varphi''(x)|z|^2 + g(t,u(x),\varphi'(x)z) - \varphi'(x)g(t,x,z) \geq 0
\]

Theorem 4.2 Let \(h \in C(\mathbb{R})\) be of polynomial growth. Moreover let us assume that \(g\) satisfies (2.1) and is independent of \(\omega\). The the following conditions are equivalent:
(i) \(h\) is a viscosity subsolution of \(L^{1,y,z}_g h = 0\);
(ii) \(h\) is \(g\)-convex.

Remark 4.3 For more basic definitions, results and related literature on viscosity solutions of PDE, we refer to Crandall, Ishii and Lions [CIL].

For proving this theorem, we need the following lemma.
Lemma 4.4 If $g$ satisfies (2.1) and $h$ is a continuous viscosity subsolution of $\mathcal{L}_y^{t,x,z}h = 0$ for each $(t, z) \in [0, T] \times \mathbb{R}^d$, then $h$ is convex in the usual sense.

Proof. If on the contrary $h$ is not convex, then there are constants $-\infty < a < b < \infty$ such that the relation $\psi \geq h$ fails on $[a, b]$, where 

$$
\psi(x) := \frac{h(b)(x - a)}{b - a} + \frac{h(a)(b - x)}{b - a}.
$$

We set $h_\delta(x) := \psi(x) - \delta(x - a)(x - b)$ and 

$$
\delta_0 = \inf \{ \delta > 0 : h_\delta(x) \geq h(x), \forall x \in [a, b] \}.
$$

It is easy to check that $\delta_0 > 0$, $h_{\delta_0} \geq h$ on $[a, b]$ and there exists $\bar{x} \in (a, b)$ such that $h_{\delta_0}(\bar{x}) = h(\bar{x})$. But since for each $z \in \mathbb{R}^d$, $h$ is a viscosity subsolution of $\mathcal{L}_y^{t}h = 0$, and $h_{\delta_0} - h$ attains minimum at $\bar{x}$, we have 

$$
0 \leq \mathcal{L}_y^{t} h_{\delta_0}(\bar{x}) = -\delta_0|z|^2 + g(t, [\frac{h(b) - h(a)}{b - a} - \delta_0(2\bar{x} - b + a)]z)
$$

$$
+ \frac{h(b) - h(a)}{b - a} - \delta_0(2\bar{x} - b + a)]g(t, z).
$$

Since $g$ is Lipschitz in $z$, there exists a positive constant $C$ independent of $z$, such that 

$$
-\delta_0|z|^2 + C_1|z| + C_2 \geq 0, \forall z \in \mathbb{R}^d.
$$

This contradicts to $\delta_0 > 0$. Thus $h$ must be convex. ■

Combining this Lemma with Theorem 4.2 we immediately have a more explicit characterization for a continuous $g$-convex function:

Corollary 4.5 We assume the same conditions as in the above theorem. Then the following condition is equivalent:

(i) $h$ is convex and for each $y$ such that $h''(y)$ exists, $\mathcal{L}_y^{t,y,z}h(y) \geq 0$;

(ii) $h$ is $g$-convex.

Proof. If $h$ is a viscosity subsolution of $\mathcal{L}_y^{t,y,z}h = 0$ then $h$ is convex. On the other hand, by Alvarez, Lasry and Lions [ALL], if $h$ is convex and for each $y$ such that $h''(y)$ exists, one has $\mathcal{L}_y^{t,y,z}h(y) \geq 0$, then $h$ is a viscosity subsolution of $\mathcal{L}_y^{t,y,z}h = 0$. ■

The proof of Theorem 4.2(i)⇒(ii). Given $(t, x, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, we consider the following SDE 

$$
dX^{t,x,z}_s = -g(s, X^{t,x,z}_s; z)ds + zdW_s, \quad s \in (t, T], \quad X^{t,x}_0 = x, \quad s \in [0, t].
$$

It is clear that $X^{t,x,z}$ is also a $g$-martingale on $[0, T]$: In particular $\mathbb{E}_s^{g}[X^{t,x,z}_T] = X^{t,x,z}_s$, and $\mathbb{E}_t^{g}[X^{t,x,z}_T] = \mathbb{E}[X^{t,x,z}_T] = x$. On the other hand, by nonlinear Feynman-Kac formula, the function $u(t, x) := \mathbb{E}[h(X^{t,x,z}_T)]$ defined on $[0, T] \times \mathbb{R}$ is the viscosity solution of the parabolic PDE 

$$
\partial_t u + \frac{1}{2} \partial_{xx} u(t, x)|z|^2 - \partial_x u g(t, x, z) + g(t, u, z\partial_x u) = 0, \quad u|_{z=T} = h(x).
$$
But the function defined by \( v(t, x) := h(x) \) is a viscosity subsolution of \( \partial_t v + \mathcal{L}^{i, z}_{t, v} = 0 \) with terminal condition \( v|_{t=T} = h \). It follows from the maximum principle of viscosity solution that

\[
 u(t, x) \geq h(x), \quad \forall (t, x) \in [0, T] \times \mathbb{R}.
\]

Or

\[
 \mathbb{E}^g[h(X^t_{T, x}; z)|\mathcal{F}_t] = \mathbb{E}^g_{t, T}[h(X^t_{T, x}; z)] \\
 \geq h(x) = h(\mathbb{E}^g[T^t_{X^t_{T, x}; z}]) = h(\mathbb{E}^g_{t, T}[X^t_{T, x}; z]|\mathcal{F}_t]).
\]

We now apply a technique initialed in [P1995, pp.107, Theorem 4.6; Peng1995:Xiangfan Summer School]: Let \( \{A_i\}_{i=1}^N \) be an \( \mathcal{F}_t \)-measurable partition of \( \Omega \); and \( z_i \in \mathbb{R}^n \), \( x_i \in \mathbb{R}, i = 1, \ldots, N \) be given. We set \( \eta = \sum_{i=1}^N 1_{A_i, z_i}, \zeta = \sum_{i=1}^N 1_{A_i, x_i} \). It is easy to check that \( \sum_{i=1}^N 1_{A_i, X^t_{s, x_i;z_i}} = X^t_{T, \zeta; \eta} \).

\[
 \sum_{i=1}^N 1_{A_i} \mathbb{E}^g[h(X^t_{T, x_i}; z_i)|\mathcal{F}_t] = \mathbb{E}^g[\sum_{i=1}^N 1_{A_i} h(X^t_{T, x_i}; z_i)|\mathcal{F}_t] \\
 = \mathbb{E}^g[h(\sum_{i=1}^N 1_{A_i} X^t_{T, x_i; z_i})|\mathcal{F}_t] \\
 = \mathbb{E}^g[h(X^t_{T, \zeta; \eta})|\mathcal{F}_t]
\]

Thus

\[
 \mathbb{E}^g[h(X^t_{T, \zeta; \eta})|\mathcal{F}_t] = \sum_{i=1}^N 1_{A_i} \mathbb{E}^g[h(X^t_{T, x_i; z_i})|\mathcal{F}_t] \geq \sum_{i=1}^N 1_{A_i} \mathbb{E}^g[h(X^t_{T, x_i; z_i})|\mathcal{F}_t] \\
 \geq \sum_{i=1}^N 1_{A_i} h(\mathbb{E}^g[X^t_{T, x_i; z_i}|\mathcal{F}_t]) = h(\sum_{i=1}^N 1_{A_i} \mathbb{E}^g[X^t_{T, x_i; z_i}|\mathcal{F}_t]) \\
 = h(\mathbb{E}^g[X^t_{T, \zeta; \eta}|\mathcal{F}_t]) = h(\zeta).
\]

\[
 \mathbb{E}^g_{t, T}[h(X^t_{T, \zeta; \eta})] = \sum_{i=1}^N 1_{A_i} \mathbb{E}^g_{t, T}[h(X^t_{s, x_i; z_i})] \geq \sum_{i=1}^N 1_{A_i} h(x_i) = h(\zeta),
\]

In other words, for bounded \( \mathcal{F}_t \)-measurable simple functions \( \zeta, \eta \),

\[
 \mathbb{E}^g_{t, T}[h(\zeta - \int_t^T g(s, X^t_{s, x; \zeta; \eta}) ds + \int_t^T \eta dW_s)] \geq h(\zeta) \quad (4.1)
\]

It follows that for any bounded \( \mathcal{F}_t \)-measurable random variables \( \zeta, \eta \), we also have (4.1). Moreover, for any bounded \( \mathcal{F}_t \)-adapted process \( \eta \) and bounded \( \mathcal{F}_t \)-measurable random variables \( \zeta \), we have

\[
 \mathbb{E}^g_{t, T}[h(\zeta - \int_t^T g(s, X^t_{s, x; \zeta; \eta}) ds + \int_t^T \eta dW_s)] \geq h(\zeta) \quad (4.2)
\]
Indeed we note that \[ \zeta - \int_t^T g(s, X^t_s, \zeta, \eta) ds + \int_t^T \eta_t dW_s \in L^{2m+1}(\mathcal{F}_T) \]
with the polynomial growth of \( h \) and the continuity of \( \mathbb{E}^g[\cdot] \) yields (4.2).

Now for any given bounded \( \mathcal{F}_T \)-measurable \( X \), let \((Y_t, Z_t)_{t \in [0,T]} \) be the solution of the BSDE

\[
Y_t = X + \int_t^T g(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s
\]

Let \( h \) be a function with polynomial growth \( |h(x)| \leq C(1 + |x|^m) \). We note that \( \sup_{t \in [0,T]} |Y_t(\omega)| \in L^\infty(\mathcal{F}_T) \) and,

\[
E[\left( \int_0^T |Z_s|^2 ds \right)^{2m+1}] < \infty.
\]

We can find a sequence of \( \mathcal{F}_t \)-measurable simple functions \( \{\zeta^i\}_{i=1}^\infty \) that converges to \( Y_t \) in \( L^\infty(\mathcal{F}_T) \) and a sequence of \( \mathcal{F}_t \)-progressively measurable simple processes \( \{\eta^i_t\}_{t \in [0,T]} \) such that

\[
\lim_{i \to \infty} E\left[ \int_0^T |Z_s - \eta^i_s|^2 ds \right]^{2m+1} = 0.
\]

It follows from BDG-inequality that the random variables

\[
X^i := \zeta^i - \int_t^T g(s, X^t_s, \zeta^i, \eta^i_s) ds + \int_t^T \eta^i_s dW_s
\]
converges in \( L^{2m+1}(\mathcal{F}_T) \) to \( X \). Thus \( h(X^i) \) converges to \( h(X) \) in \( L^2(\mathcal{F}_T) \). Thus

\[
\mathbb{E}^g_{t,T}[h(X)] = \lim_{i \to \infty} \mathbb{E}^g_{t,T}[h(X^i)] \geq \lim_{i \to \infty} h(\zeta^i) = h(Y_t) = h(\mathbb{E}^g_{t,T}[X]).
\]

Thus (ii) holds for the case where \( X \in L^\infty(\mathcal{F}_T) \). This with the fact that \( h \) is convex and Lemma [3.2] it follows that (ii) holds for all \( X \in L^2(\mathcal{F}_T) \) such that \( h(X) \in L^2(\mathcal{F}_T) \). The proof is complete. ■

**The proof of Theorem 4.2 (ii)⇒(i).** We will apply a technique in [P1995, pp.126]. For a fixed \( t, x, z \), let \( \varphi \) be a smooth and polynomial growth function such that \( \varphi \geq h \) and \( h(x) = \varphi(x) \). We consider

\[
X^{t,x,z}_s = x - \int_s^t g(r, X^{t,x,z}_r, z) dr + z(W_s - W_t), \quad s \in [t, t+\delta].
\]

where \( \delta \) is a small positive number such that \( t + \delta \leq T \). It is clear that \( X^{t,x,z}_s \) is a \( g \)-martingale. Since \( h \) is \( g \)-convex, we have

\[
\mathbb{E}^g_{t,t+\delta}[\varphi(X^{t,x,z}_{t+\delta})] \geq \mathbb{E}^g_{t,t+\delta}[h(X^{t,x,z}_{t+\delta})] \geq h(\mathbb{E}^g_{t,t+\delta}[X^{t,x,z}_{t+\delta}]) = h(x) = \varphi(x).
\]

Or

\[
Y_t - \varphi(x) = \mathbb{E}^g_{t,t+\delta}[\varphi(X^{t,x,z}_{t+\delta})] - \varphi(x) \geq 0.
\]

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where \((Y, Z)\) solve the BSDE
\[
\begin{aligned}
-dY_s &= g(s, Y_s, Z_s)ds - Z_s dW_s, \quad s \in [t, t + \delta], \\
Y_{t+\delta} &= \varphi(X_{t+\delta}^{t,x,z}).
\end{aligned}
\]

We consider
\[
Y_1^s := Y_s - \varphi(X_s^{t,x,z}), \quad Z_1^s := Z_s - \varphi_x(X_s^{t,x,z})z
\]
which is the solution of the BSDE
\[
\begin{aligned}
-dY_1^s &= [g(s, Y_1^s + \varphi(X_s^{t,x,z}), Z_1^s + \varphi_x(X_s^{t,x,z})z) + \tilde{L} \varphi(X_s^{t,x,z})]ds - Z_1^s dW_s, \\
Y_{t+\delta}^1 &= 0.
\end{aligned}
\]

where \(s \in [t, t + \delta]\) and \(\tilde{L} \varphi(x) = \frac{1}{2} \varphi_{xx}(x)|z|^2 - \varphi_x(x)g(s, x, z)\). We can prove that \(E[|Y_1^t - Y_2^t|] = o(\delta)\), where \((Y_2^t, Z_2^t)\) solves
\[
\begin{aligned}
-dY_2^s &= [g(s, Y_2^s + \varphi(x), Z_2^s + \varphi_x(x)z) + \tilde{L} \varphi(x)]ds - Z_2^s dW_s, \quad s \in [t, t + \delta], \\
Y_{t+\delta}^2 &= 0.
\end{aligned}
\]

But it is easy to check that \(Z^2 \equiv 0\) and
\[
\begin{aligned}
-dY_2^s &= [g(s, Y_2^s + \varphi(x), \varphi_x(x)z) + \tilde{L} \varphi(x)]ds, \\
Y_{t+\delta}^2 &= 0.
\end{aligned}
\]

Thus from the Lipschitz continuity of \(g(s, \cdot, z)\), we have
\[
Y_t - \varphi(x) = Y_1^t = Y_2^t + o(\delta)
\]
\[
= \int_t^{t+\delta} [g(s, Y_2^s + \varphi(x), \varphi_x(x)z) + \tilde{L} \varphi(x)]ds + o(\delta)
\]
\[
= \int_t^{t+\delta} [g(s, \varphi(x), \varphi_x(x)z) + \tilde{L} \varphi(x)]ds + o(\delta) \geq 0
\]

From which it follows that \(\tilde{L} \varphi(x) + g(t, \varphi(x), \varphi_x(x)z) = L_t^{t,x,z} \varphi(x) \geq 0\). Thus \(h\) is a viscosity subsolution of \(L_t^{t,x,z} u = 0\). 

5 \textit{g-Convexity and Viability}

Surprisingly to us, the notion of g-convexity has a deep relation with the notion of viability for BSDE introduced and systematically studied by Buckdahn, Quincampoix and Rascanu in [BQR]. We recall the notion and a result about the backward stochastic viability property.

\textbf{Definition 5.1 (Definition 3 in [BQR])} \textit{Let} \(K\) \textit{be a nonempty closed subset of} \(\mathbb{R}^m\).
(a) A stochastic process \((Y_t)_{t \in [0,T]}\) is viable in \(K\) if and only if
\[ Y_t \in K, \quad P\text{-a.s., } \forall t \in [0,T]. \]

(b) The closed set \(K\) enjoys the backward stochastic viability property, denoted \(g\text{-BSVP}\), for \((2.2)\) if and only if:
\[ \forall \tau \in [0,T], \forall X \in L^2(\mathcal{F}_\tau) \text{ such that } X \in K \text{ P-a.s., } \exists \text{ a solution } (Y,Z) \text{ to BSDE } (2.2) \text{ over the time interval } [0,\tau], \]
\[ Y_s = X + \int_s^\tau g(r,Y_r,Z_r) \, dr - \int_s^\tau Z_r \, dW_r, \quad s \in [0,\tau] \]
such that \((Y_s)_{s \in [0,\tau]}\) is viable in \(K\).

**Lemma 5.2 (Theorem 2.4 in [BQR])** Suppose that \(g\) satisfies condition \((2.1)\). Let \(K\) be a nonempty closed set. If \(K\) enjoys \(g\text{-BSVP}\) for \((2.2)\), then \(K\) is convex.

**Remark 5.3** In the above lemma, the authors in [BQR] assume that \(g\) also satisfies the following conditions: \(g(\omega,\cdot,y,z)\) is continuous, as a part of whole assumptions. But in their proof, we can see that this condition is needless to this lemma, condition \((2.1)\) is enough.

**Theorem 5.4** Let \(g\) satisfy \((2.1)\) and \(h: \mathbb{R} \rightarrow \mathbb{R}\) be a continuous function. Moreover assume that
\[ \bar{g}(t,y^1,y^2,z^1,z^2) = \left( \begin{array}{c} g(t,y^1,z^1) \\ g(t,y^2,z^2) \end{array} \right). \]
Then the following statements are equivalent:
(i). \(h\) is \(g\)-convex;
(ii). \(\text{epi}(h)\) enjoys \(\bar{g}\text{-BSVP}\) where
\[ \text{epi}(h) = \{(x_1,x_2) \in \mathbb{R}^2; \ h(x_1) \leq x_2\}. \]

**Proof.** (i)\(\Rightarrow\)(ii): It is obvious that \(\text{epi}(h)\) is a closed set in \(\mathbb{R}^2\).

Given any \(X = (X_1,X_2)^T \in \text{epi}(h)\) P-a.s. such that \(X \in L^2(\mathcal{F}_T,\mathbb{R}^2)\). By the definition of \(\text{epi}\), we have
\[ h(X_1) \leq X_2, \ P - a.s., \]
which implies by the comparison theorem of BSDE and the \(g\)-convexity of \(h\) that
\[ h(E^q_{t,T}[X_1]) \leq E^q_{t,T}[h(X_1)] \leq E^q_{t,T}[X_2], \ P - a.s. \]
Thus
\[ (E^q_{t,T}[X_1],E^q_{t,T}[X_2]) \in \text{epi}(h), \ P - a.s. \ t \in [0,T]. \]
It is clear that \( \bar{g} \) satisfies (2.1). Moreover, \((\mathbb{E}^g_{t,T}[X_1], \mathbb{E}^g_{t,T}[X_2])_{t \in [0,T]} \) is the unique solution of the following equation

\[
\begin{pmatrix}
Y_1^t \\
Y_2^t
\end{pmatrix}
= \begin{pmatrix}
X_1 \\
X_2
\end{pmatrix} + \int_t^T \begin{pmatrix}
g(s, Y_1^s, Z_1^s) \\
g(s, Y_2^s, Z_2^s)
\end{pmatrix} ds - \int_t^T \begin{pmatrix}
Z_1^s \\
Z_2^s
\end{pmatrix} dW_s.
\]

Then (5.1) implies that \( \text{epi}(h) \) enjoys \( \bar{g} \)-BSVP, as required.

(ii) \( \Rightarrow \) (i): Assume that \( \text{epi}(h) \) enjoys \( \bar{g} \)-BSVP, i.e., for any \( X \in L^2(F_T; \mathbb{R}^2) \) such that \( X \in \text{epi}(h) \), we have

\[
(\mathbb{E}^g_{t,T}[X_1], \mathbb{E}^g_{t,T}[X_2]) \in \text{epi}(h), \quad P - a.s., \quad t \in [0,T],
\]

and by the definition of \( \text{epi}(h) \),

\[
h(\mathbb{E}^g_{t,T}[X_1]) \leq \mathbb{E}^g_{t,T}[X_2], \quad P - a.s., \quad t \in [0,T].
\]

For any given \( X_1 \) such that \( X_1 \in L^2(F_T) \), putting \( X_2 = h(X_1) \) yields

\[
h(\mathbb{E}^g_{t,T}[X_1]) \leq \mathbb{E}^g_{t,T}[X_2] = \mathbb{E}^g_{t,T}[h(X_1)],
\]

as required. ■

Remark 5.5 In the proof of Theorem 5.4, we note that we do not need condition (b) of (2.1), \((g(t, 0, 0))_{t \in [0,T]} \in L_2^g \) is enough.

Corollary 5.6 If a continuous functions \( h \) is \( g \)-convex, then \( h \) is convex.

Proof. It is clear that \( \text{epi}(h) \) enjoys \( g \)-BSVP. By Theorem 2.4 in [BQR], \( \text{epi}(h) \) is a convex set, which implies that \( h \) is a convex function. ■

Clearly, a \( g \)-affine function must be affine in the usual sense. Then we have

Theorem 5.7 Let \( g \) satisfy (2.1). Then the following two statements are equivalent:

(i) A function \( h \) is \( g \)-affine;
(ii) \( h \) has the form: \( h(y) = ay + b \) for some \((a,b) \in \Pi_a^g\) where

\[
\Pi_a^g := \{(a,b); g(t, ay + b, az) = ag(t, y, z), \quad dP \times dt - a.s.\}
\]

6 More Properties of \( g \)-Convexity

6.1 Functional operations preserving \( g \)-convexity

It is natural to build up new \( g \)-convex functions from simpler ones, via operations preserving \( g \)-convexity, or even yielding it.

Proposition 6.1 Let \( g \) satisfy (2.1), \( \varphi \in C(\mathbb{R}) \). If \( D \) is a nonempty subset of \( g \)-convex functions dominated by \( \varphi \), then the function

\[
f(y) = \sup \{h(y) : h \in D\},
\]

is \( g \)-convex.
Proof. It is clear that \( f \) is convex. For any given \( h \in \mathcal{D} \), Jensen’s inequality for \( g \)-expectation holds, thus for any \( X \in L^\infty(\mathcal{F}_T) \), we have

\[
\mathbb{E}_{t,T}^g [h(X)] \geq h(\mathbb{E}_{t,T}^g[X]).
\]

From the definition of \( f \) and comparison theorem of BSDEs, it follows that

\[
\mathbb{E}_{t,T}^g [f(X)] \geq \mathbb{E}_{t,T}^g [h(X)] \geq h(\mathbb{E}_{t,T}^g[X]).
\]

This with the arbitrariness of \( h \) and Lemma 3.5 yields what is required. ■

Clearly, the function \( f \) in Theorem 6.1 may be only continuous instead of in \( C^2 \) and if \( h_1 \) and \( h_2 \) are \( g \)-convex, then so is \( h(y) = h_1(y) \vee h_2(y) \). In addition, for the case of \( g \)-concavity, we also have the same result, in which the "sup" is replaced by "inf".

The following result is also easy:

**Proposition 6.2** Let \( \varphi \in C^2(\mathbb{R}) \), \( g \) satisfy (2.1). If there exists at least one \( g \)-convex function that dominates \( \varphi \), then \( \varphi \) is \( g \)-convex if and only if it is represented as the supremum of all \( g \)-convex \( C^2 \)-functions that dominate \( \varphi \).

Motivated by Proposition 6.1 and the discussions about abstract convexity in [PR] or [Singer], we can find \( g \)-convex functions by another way.

For given \( g \), we define

\[
\Pi_g^v = \{(a,b) \in \mathbb{R}^2 : g(t, ay + b, az) \geq ag(t, y, z), \forall y, z, \ dP \times dt \text{-a.s.}\}
\]

It is clear that that \( \Pi_g^v \) cannot be empty, at least it contains a element \((1, 0)\), and if \( g = \langle \xi_t, z \rangle \) where \((\xi_t)_{t \in [0, T]}\) is a \( \mathbb{R}^d \)-valued progressively measurable process, \( \Pi_g^v = \mathbb{R}^2 \). For each \((a, b) \in \Pi_g^v\), \( h(y) = ay + b \) is an affine \( g \)-convex function.

**Proposition 6.3** Let \( g \) satisfy (2.1) and \( \phi \in C(\mathbb{R}) \). Then

\[
\begin{aligned}
f(y) = \sup \{ h(y) = ay + b : \forall (a, b) \in \Pi_g^v \text{ such that } h \leq \phi \}
\end{aligned}
\]

is \( g \)-convex.

**Proof.** The proof is similar to that of Proposition 6.1. ■

**Remark 6.4** From the above theorem, it follows that for each \((a, b) \in \Pi_g^v\),

\[
\mathbb{E}_{t,T}^g[aX + b] \geq a\mathbb{E}_{t,T}^g[X] + b.
\]

But we cannot change the sign "\( \geq \)" to "\( = \)" in general although \( h(y) = ay + b \) is an affine function, because \( h \) here may be not a \( g \)-affine function.

The following property is easy to be proved:
Proposition 6.5 Let \( g \) satisfy (2.1) and let \( h \) and \( \psi \) be two continuous functions. Then

(i) If \( \psi \) is \( g \)-affine and \( h \) is \( g \)-convex, then \( h \circ \psi \) is \( g \)-convex.

(ii) If \( h \) is \( g \)-convex and increasing, and \( \psi \) is \( g \)-convex, then \( h \circ \psi \) is \( g \)-convex.

We also have the following stability property for \( g \)-convex functions.

Theorem 6.6 Let \( g \) satisfy (2.1) and the \( g \)-convex (resp. concave) functions \( h_k : \mathbb{R} \to \mathbb{R} \) converge pointwise for \( k \to \infty \) to \( h : \mathbb{R} \to \mathbb{R} \). Then \( h \) is \( g \)-convex (resp. concave) and, for each compact set \( S \in \mathbb{R} \), the convergence of \( h_k \) to \( h \) is uniform on \( S \).

Proof. Convexity of \( h \) is trivial since \( h_k \) is convex. And for each compact set \( S \in \mathbb{R} \), the convergence of \( h_k \) to \( h \) is uniform on \( S \) (See [HL, pp. 177, Theorem 3.1.5]).

We now prove that \( h \) is \( g \)-convex. Given bounded \( F_T \)-measurable random variable \( X \), we assume \( |X| \leq M \). The uniform convergence means that there exists a function \( \delta_M(k) \) with \( \delta_M(k) \to 0 \) as \( k \to \infty \) such that for each \( x \in B[0,M] \) we have

\[
|h_k(x) - h(x)| \leq \delta_M(k).
\]

This implies that \( h_k(X) \to h(X) \) in \( L^2 \) as \( k \to \infty \). Therefore by the continuity of \( \mathbb{E}^g[\cdot] \), we have

\[
\mathbb{E}^g_{t,T}[h(X)] \geq h(\mathbb{E}^g_{t,T}[X]), \quad P - a.s. \text{ for } t \in [0,T].
\]

This with Lemma 3.5 it follows that for each \( X \in L^2(F_T) \) such \( h(X) \in L^2(F_T) \),

\[
\mathbb{E}^g_{t,T}[h(X)] \geq h(\mathbb{E}^g_{t,T}[X]), \quad P - a.s. \text{ for } t \in [0,T].
\]

Thus \( h \) is \( g \)-convex. \[■\]

6.2 Some interesting properties of \( g \)-convexity

As mentioned before, for given \( g \), the set of all \( g \)-convex functions is a subset of that of convex functions. From Corollary 6.5 and Hu’s result in [Hu] (see also Corollary 3.8) it follows that if \( g \) is not super-homogeneous, then this inclusion is strict.

Unlike the classical situation, in general \( h \) is \( g \)-convex does not implies that \( -h \) is \( g \)-concave. Let us consider the following example.

Example 6.7 Let \( g = |z| \), the following statements are equivalent:

(i) A function \( h \) is \( g \)-convex;

(ii) \( h \) is convex \((h''(y) \geq 0 \text{ a.e.})\).

Moreover the following statements are also equivalent:

(iii) A function \( h \) is \( g \)-concave;

(iv) \( h \) is concave \((h''(y) \leq 0 \text{ a.e.}) \) and nondecreasing \((h'(y) \geq 0 \text{ a.e.})\).
The following property implies that a convex function may not be a $g$-convex one.

**Example 6.8** In the case when $g = ay$ where $a \in \mathbb{R}$, we have $\mathbb{E}^g_{t,T}[c] = ce^{a(T-t)}$ and $\mathbb{E}^g_{t,T}[h(c)] = h(c)e^{a(T-t)}$. If $h$ is $g$-convex then

$$h(c)e^{a(T-t)} = \mathbb{E}^g_{t,T}[h(c)] \geq h(ce^{a(T-t)}).$$

*From this relation it is easy to find a convex $h$ which is not $g$-convex.*

We consider the following self-financing condition:

$$\mathbb{E}^g_{t,T}[0] \equiv 0, \quad \forall 0 \leq t \leq T.$$

**Corollary 6.9** Let $g$ satisfy (2.1). Then the following three statements are equivalent:

(i) $\mathbb{E}^g[\cdot]$ satisfies the self-financing condition;
(ii) $g$ satisfies (2.3)–a;
(iii) The constant function $h \equiv 0$ is $g$-affine.

**Proof.** The proof of the equivalence between (i) and (ii) can be found in [P2006b, Proposition 3.7]. The equivalence between (ii) and (iii) follows from Theorem 5.7 immediately.

The "zero interest rate" condition means:

$$\mathbb{E}^g_{t,T}[\eta] = \eta, \quad \forall 0 \leq t \leq T, \eta \in L^2(F_T).$$

**Corollary 6.10** Let $g$ satisfy (2.1). Then the following three statements are equivalent:

(i) $\mathbb{E}^g[\cdot]$ satisfies the zero interest rate condition;
(ii) $g$ satisfies (2.3)–b;
(iii) For each constant $c$, the functions $h(y) = c$ and $y$ are $g$-affine.

**Proposition 6.11** Let $g$ satisfy (2.1) and $c$ be a constant, then the following statements are equivalent

(i) $\mathbb{E}^g_{t,T}[X + c] = (\text{resp.} \geq, \leq) \mathbb{E}^g_{t,T}[X] + c$, for each $X \in L^2(F_T)$;
(ii) $g(t, y + c, z) = (\text{resp.} \geq, \leq) g(t, y, z)$ for each $y \in \mathbb{R}, z \in \mathbb{R}^d$.

**Remark 6.12** (ii) means that $g$ is a periodic function in $y$ with period $c$.

**Proof.** It is clear that the function $h(y) = y + c$ is $g$-affine (resp. $g$-convex, $g$-concave).}

**Corollary 6.13** The following statements are equivalent

(i) $\mathbb{E}^g_{t,T}[X + c] = \mathbb{E}^g_{t,T}[X] + c$, for each $X \in L^2(F_T), c \in \mathbb{R}$;
(ii) $g$ is independent of $y$.
(iii) $h(y) = y + c$ is $g$-affine, $c \in \mathbb{R}$.

This result is a generalization of Lemma 3.2 in [P2004]. In addition, it is clear that if, for each $c \in \mathbb{R}, h + c$ is $g$-convex implies $h$ is $g$-convex, then $g$ must be independent of $y$. 

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