Classification of sesquilinear forms with the first argument on a subspace or a factor space

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Dedicated to R. A. Horn on the occasion of his 65th birthday

Abstract

Let $V$ be a vector space over a field or skew field $F$, and let $U$ be its subspace. We study the canonical form problem for bilinear or sesquilinear forms

$$U \times V \to F, \quad (V/U) \times V \to F$$

and linear mappings $U \to V, V \to U, V/U \to V, V \to V/U$. We solve it over $F = \mathbb{C}$ and reduce it over all $F$ to the canonical form problem for ordinary linear mappings $W \to W$ and bilinear or sesquilinear forms $W \times W \to F$. Moreover, we give an algorithm that realizes this reduction. The algorithm uses only unitary transformations if $F = \mathbb{C}$, which improves its numerical stability. For linear mapping this algorithm can be derived from the algorithm by L. A. Nazarova,

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1 Introduction

In this paper, we give canonical matrices of bilinear or sesquilinear forms

\[ U \times V \to \mathbb{C}, \quad (V/U) \times V \to \mathbb{C}, \]

where \( V \) is a complex vector space and \( U \) is its subspace.

We use the following canonical matrices of bilinear or sesquilinear forms on a complex vector space given in \([\text{1}]\) (see also \([\text{2}, \text{3}]\)). Two square complex matrices \( A \) and \( B \) are said to be congruent or *congruent if there is a nonsingular \( S \) such that \( S^T AS = B \) or, respectively, \( S^* AS = B \), where \( S^* := \overline{S}^T \) denotes the complex conjugate transpose of \( S \). Define the \( n \)-by-\( n \) matrices

\[
\Gamma_n = \begin{bmatrix}
0 & -1 & & \\
1 & 1 & & \\
-1 & -1 & 1 & \\
1 & 1 & & 0
\end{bmatrix}, \quad \Delta_n = \begin{bmatrix}
0 & 1 \\
1 & i \\
i & 0
\end{bmatrix},
\]

\[
J_n(\lambda) = \begin{bmatrix}
\lambda & 1 & 0 \\
& \ddots & \ddots \\
& & \ddots & 1 \\
0 & & & \lambda
\end{bmatrix}.
\]

Theorem 1 (\([\text{1}, \text{p. 351}]\)). (a) Every square complex matrix is congruent to a direct sum, determined uniquely up to permutation of summands, of matrices of the form

\[
J_n(0), \quad \Gamma_n, \quad \begin{bmatrix}
0 & I_n \\
J_n(\lambda) & 0
\end{bmatrix},
\]

in which \( \lambda \neq 0, \lambda \neq (-1)^{n+1} \), and \( \lambda \) is determined up to replacement by \( \lambda^{-1} \).
(b) Every square complex matrix is *congruent to a direct sum, determined uniquely up to permutation of summands, of matrices of the form

$$ J_n(0), \quad \lambda \Gamma_n, \quad \begin{bmatrix} 0 & I_n \\ J_n(\mu) & 0 \end{bmatrix}, $$

in which $|\lambda| = 1$ and $|\mu| > 1$. Alternatively, one may use the symmetric matrix $\Delta_n$ instead of $\Gamma_n$. \[\square\]

A canonical form of a square matrix for congruence/*congruence over any field $\mathbb{F}$ of characteristic different from 2 was given in [5] up to classification of Hermitian forms over finite extensions of $\mathbb{F}$.

Let us formulate the main result. For generality, we will consider matrices over any field or skew field $\mathbb{F}$ with involution $\alpha \mapsto \bar{\alpha}$, that is, a bijection on $\mathbb{F}$ such that

$$ \bar{\alpha + \beta} = \bar{\alpha} + \bar{\beta}, \quad \bar{\alpha \beta} = \bar{\beta} \bar{\alpha}, \quad \bar{\bar{\alpha}} = \alpha $$

for all $\alpha, \beta \in \mathbb{F}$.

We denote the $m$-by-$n$ zero matrix by $0_{mn}$, or by $0_m$ if $m = n$. It is agreed that there exists exactly one matrix of size $n \times 0$ and there exists exactly one matrix of size $0 \times n$ for every nonnegative integer $n$; they represent the linear mappings $0 \to \mathbb{F}^n$ and $\mathbb{F}^n \to 0$ and are considered as the zero matrices $0_{n0}$ and $0_{0n}$. For every $p \times q$ matrix $M_{pq}$ we have

$$ M_{pq} \oplus 0_{m0} = \begin{bmatrix} M_{pq} & 0 \\ 0 & 0_{m0} \end{bmatrix} = \begin{bmatrix} M_{pq} \quad 0_{p0} \\ 0_{mq} \quad 0_{m0} \end{bmatrix} = \begin{bmatrix} M_{pq} \\ 0_{mq} \end{bmatrix} $$

and

$$ M_{pq} \oplus 0_{0n} = \begin{bmatrix} M_{pq} & 0 \\ 0 & 0_{0n} \end{bmatrix} = \begin{bmatrix} M_{pq} \quad 0_{pn} \\ 0_{0q} \quad 0_{0n} \end{bmatrix} = \begin{bmatrix} M_{pq} \\ 0_{pn} \end{bmatrix} .$$

In particular,

$$ 0_{p0} \oplus 0_{0q} = 0_{pq} .$$

For each matrix $A = [a_{ij}]$ over $\mathbb{F}$, we define its conjugate transpose

$$ A^* = \overline{A^T} = [\bar{a}_{ji}] .$$

If $S^*AS = B$ for some nonsingular matrix $S$, then $A$ and $B$ are said to be *congruent (or congruent if $\mathbb{F}$ is a field and the involution on $\mathbb{F}$ is the identity—in what follows we consider congruence as a special case of *congruence).
A sesquilinear form on right vector spaces $U$ and $V$ over $\mathbb{F}$ is a map $\mathcal{G}: U \times V \to \mathbb{F}$ satisfying
\[
\mathcal{G}(u\alpha + u'\beta, v) = \bar{\alpha}\mathcal{G}(u, v) + \bar{\beta}\mathcal{G}(u', v), \\
\mathcal{G}(u, v\alpha + v'\beta) = \mathcal{G}(u, v)\alpha + \mathcal{G}(u, v')\beta,
\]
for all $u, u' \in U$, $v, v' \in V$, and $\alpha, \beta \in \mathbb{F}$. If $\mathbb{F}$ is a field and the involution on $\mathbb{F}$ is the identity, then a sesquilinear form becomes bilinear—we consider bilinear forms as a special case of sesquilinear forms.

If $e_1, \ldots, e_m$ and $f_1, \ldots, f_n$ are bases of $U$ and $V$, then
\[
G_{ef} = [\alpha_{ij}], \quad \alpha_{ij} := \mathcal{G}(e_i, f_j),
\]
is the matrix of $\mathcal{G}$ in these bases. Its matrix in other bases $e'_1, \ldots, e'_m$ and $f'_1, \ldots, f'_n$ can be found by the formula
\[
G_{e'f'} = S^* G_{ef} R,
\]
where $S$ and $R$ are the change of basis matrices.

For every $u \in U$ and $v \in V$,
\[
\mathcal{G}(u, v) = [u]_e G_{ef} [v]_f,
\]
where $[u]_e$ and $[v]_f$ are the coordinate column-vectors of $u$ and $v$.

In this paper, we study sesquilinear forms
\[
U \times V \to \mathbb{F}, \quad (V/U) \times V \to \mathbb{F},
\]
in which $U$ is a subspace of $V$, so we always consider their matrices (1) in those bases of $U$ and $V$ that are concordant as follows.

**Definition 2.** Let $\mathcal{G}$ be one of sesquilinear forms (3), in which $V$ is a right space over $\mathbb{F}$, and $U$ is its subspace. Choose a basis $e_1, \ldots, e_n$ of $V$ such that
\[
\begin{cases}
e_1, \ldots, e_m \text{ is a basis of } U & \text{if } \mathcal{G}: U \times V \to \mathbb{F}, \\
e_m+1, \ldots, e_n \text{ is a basis of } U & \text{if } \mathcal{G}: (V/U) \times V \to \mathbb{F}.
\end{cases}
\]
By the *matrix of* $G$ *in the basis* $e_1, \ldots, e_n$, we mean the block matrix

$$[A|B] = \begin{bmatrix}
\alpha_{11} & \cdots & \alpha_{1m} & \alpha_{1,m+1} & \cdots & \alpha_{1n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\alpha_{m1} & \cdots & \alpha_{mm} & \alpha_{m,m+1} & \cdots & \alpha_{mn}
\end{bmatrix},$$

(5)

in which

$$\alpha_{ij} = \begin{cases} 
G(e_i, e_j) & \text{if } G: U \times V \to \mathbb{F}, \\
G(e_i + U, e_j) & \text{if } G: (V/U) \times V \to \mathbb{F}.
\end{cases}$$

By the *block-direct sum* of block matrices $[A_1|B_1]$ and $[A_2|B_2]$, we mean the block matrix

$$[A_1|B_1] \uplus [A_2|B_2] := \begin{bmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \end{bmatrix}.$$

In Section 5 we will prove the following theorem (a stronger statement was proved in [2, Theorem 1] in the case $U = V$).

**Theorem 3.** Let $\mathbb{F}$ be a field or skew field with involution (possibly, the identity if $F$ is a field), $V$ be a right vector space over $\mathbb{F}$, and $U$ be its subspace. Let $G$ be one of sesquilinear forms

$$U \times V \to \mathbb{F}, \quad (V/U) \times V \to \mathbb{F}. \quad (6)$$

(a) There exists a basis $e_1, \ldots, e_n$ of $V$ satisfying (4), in which the matrix of $G$ is a block-direct sum of a $p$-by-$p$ matrix

$$[K|0_p], \quad K \text{ is nonsingular},$$

and matrices of the form

$$[J_q(0)|0_0] \quad (q \geq 1), \quad [J_q(0)|E_q] \quad (q \geq 0),$$

(8)

in which

$$E_q := \begin{bmatrix} 0 \\
\vdots \\
0_1 \end{bmatrix} \quad \text{if } q \geq 1, \quad E_0 := 0_{01}$$

(9)
(the summands (7) or (8) may be absent). The block $K$ is determined by $G$ uniquely up to *congruence, and the summands of the form (8) are determined by $G$ uniquely up to permutation.

(b) If $\mathbb{F} = \mathbb{C}$, then one can replace in this direct sum the summand (7) by

$$[K_1|0_{p_1,0}] \oplus \cdots \oplus [K_s|0_{p_s,0}],$$

where $K_1 \oplus \cdots \oplus K_s$ is the canonical form of $K$ defined in Theorem 4 and each $K_i$ is $p_i$-by-$p_i$. The obtained block-direct sum is determined by $G$ uniquely up to permutation of summands, and so it is a canonical matrix of the sesquilinear (in particular, bilinear) form $G$.

Let us formulate an analogous statement for matrices of linear mappings.

**Definition 4.** Let $\mathbb{F}$ be a field or skew field, $V$ be a right vector space over $\mathbb{F}$, and $U$ be its subspace. Let $\mathcal{A}$ be one of linear mappings

$$U \to V, \quad V \to U, \quad V/U \to V, \quad V \to V/U.$$  

Choose a basis $e_1, \ldots, e_n$ of $V$ such that

$$\begin{cases} 
  e_1, \ldots, e_m \text{ is a basis of } U, & \text{if } U \to V \text{ or } V \to U, \\
  e_{m+1}, \ldots, e_n \text{ is a basis of } U, & \text{if } V/U \to V \text{ or } V \to V/U.
\end{cases} \quad (10)$$

By the matrix $A_e$ of $\mathcal{A}$ in the basis $e_1, \ldots, e_n$, we mean its matrix in the bases

$$\begin{cases} 
  e_1, \ldots, e_m \text{ of } U, & \text{if } U \to V \text{ or } V \to U, \\
  e_1 + U, \ldots, e_m + U \text{ of } V/U, & \text{if } V/U \to V \text{ or } V \to V/U,
\end{cases}$$

and $e_1, \ldots, e_n$ of $V$. We divide $A_e$ into two blocks

$$A_e = \begin{cases} 
  \begin{bmatrix} A \\ B \end{bmatrix}, & \text{if } U \to V \text{ or } V/U \to V, \\
  [A|B], & \text{if } V \to U \text{ or } V \to V/U,
\end{cases} \quad (11)$$

where $A$ is $m$-by-$m$.

The following theorem will be proved in Section 5.
Theorem 5. Let $F$ be a field or skew field, $V$ be a right vector space over $F$, and $U$ be its subspace. Let $A$ be one of linear mappings \[ U \to V, \quad V \to U, \quad V/U \to V, \quad V \to V/U. \] (12)

(a) There exists a basis $e_1, \ldots, e_n$ of $V$ satisfying (10), in which for the matrix $A_e$ of $A$ we have:

\[ \begin{cases} 
A^T_e, & \text{if } U \to V \text{ or } V/U \to V, \\
A_e, & \text{if } V \to U \text{ or } V \to V/U 
\end{cases} \]

is a block-direct sum of a $p$-by-$p$ matrix

\[ [K|0_p], \quad K \text{ is nonsingular}, \] (13)

and matrices of the form

\[ [J_q(0)|0_q], \quad [J_q(0)|E_q], \] (14)

where $E_q$ was defined in (9) (the summands (13) or (14) may be absent). The block $K$ is determined by $A$ uniquely up to similarity, and the summands of the form (14) are determined by $A$ uniquely up to permutation.

(b) If $F = \mathbb{C}$, then one can replace the summand (13) by a block-direct sum of square matrices of the form

\[ [J_q(\lambda)|0_q]. \]

The obtained matrix is determined by $A$ uniquely up to permutation of summands, and so it is a canonical matrix of the linear mapping $A$.

We do not rate Theorem 5 as new; it is readily available from the canonical form problem solved in \[4, \S 2\]. We include it in our paper since the singular indecomposable summands of the canonical forms in Theorems 3 and 5 coincide, and our proofs of Theorems 3 and 5 are similar and are based on regularization algorithms that decompose the matrix of each form (3) and each mapping (12) into a block-direct sum of

- its regular part $[K|0_p]$ with nonsingular $K$ (see (7) and (13)), which is determined by (3) or (12) up to *congruence or similarity, and of

- its singular summands of the form $[J_q(0)|0_q]$ and $[J_q(0)|E_q]$ (see (8) and (14)), which are determined uniquely.
If $\mathbb{F} = \mathbb{C}$, then these algorithms can use only unitary transformations, which improves their numerical stability. These algorithms extend the regularization algorithm [2] for a bilinear/sesquilinear form, which decomposes its matrix into a direct sum of a nonsingular matrix and several singular Jordan blocks. An analogous regularization algorithm was given by Van Dooren [7] for matrix pencils and was extended to matrices of cycles of linear mappings in [6].

The canonical form problems for matrices of forms (3) and mappings (12) are special cases of the canonical form problem for block matrices, whose form resembles

\[
A = \begin{bmatrix} A_1 & \cdots & A_{k-1} & \boxed{A_k} & A_{k+1} & \cdots & A_t \end{bmatrix}
\]  

(15)

over $\mathbb{F}$, partitioned into vertical strips, among which one strip $A_k$ is square and boxed. The number $n_k$ of rows of $A$ and the number $n_i$ of columns of each strip $A_i$ are nonnegative integers. Let

\[
B = \begin{bmatrix} B_1 & \cdots & B_{k-1} & \boxed{B_k} & B_{k+1} & \cdots & B_t \end{bmatrix}
\]  

(16)

be another bangle with the same sizes of strips and the same $k$ and $t$. We say that the bangles $A$ and $B$ are *congruent* or, respectively, *similar* and write

\[
A \sim B \quad \text{or} \quad A \simeq B
\]  

(17)

if there exists a nonsingular upper block-triangular matrix

\[
S = \begin{bmatrix}
S_{11} & \cdots & S_{1t} \\
& \ddots & \\
0 & \cdots & S_{tt}
\end{bmatrix} \quad (S_{ii} \text{ is } n_i \times n_i)
\]

over $\mathbb{F}$ such that

\[
B = S_{kk}^* AS \quad \text{or} \quad B = S_{kk}^{-1} AS.
\]
Then

\[ B_k = S_{kk}^* A_k S_{kk} \quad \text{or} \quad B_k = S_{kk}^{-1} A_k S_{kk}, \]

this means that the boxed strips of *congruent/similar bangles are *congruent/similar. The following lemma is obvious.

**Lemma 7.** Two bangles are *congruent/similar if and only if one reduces to the other by a sequence of the following transformations:

(a) Any transformation with rows of the whole matrix, and then the *congruent/similar transformation with columns of the boxed strip (this transformation reduces \((15)\) to

\[ [EA_1|\ldots|EA_{k-1} \boxed{EA_k E^*} EA_{k+1}|\ldots|EA_t] \]

or, respectively,

\[ [EA_1|\ldots|EA_{k-1} \boxed{EA_k E^{-1}} EA_{k+1}|\ldots|EA_t] \]

with a nonsingular \( E \).

(b) Any transformation with columns of an unboxed strip.

(c) Addition of a linear combination of columns of the \( i^{th} \) strip to a column of the \( j^{th} \) strip if \( i < j \).

Note that the canonical form problem for matrices of forms \((3)\) and mappings \((12)\) is the canonical form problem for bangles \((15)\) with two strips. But applying our algorithm to bangles with two strips we can produce bangles with three strips (see Section 3.2); so we consider bangles with an arbitrary number of strips.

The paper is organized as follows. In Section 2 we formulate our main theorem about the existence of a regularizing decomposition of a bangle. In Sections 3 and 4 we construct regularizing decompositions of bangles with respect to *congruence and similarity. In Section 5 we use these decompositions to prove the main theorem and Theorems 3 and 5.
2 Bangles

In this section, we formulate our main theorem, which reduces the canonical form problem for bangles up to *congruence/similarity to the canonical form problem for nonsingular matrices up to *congruence/similarity, and solves it for complex bangles.

By the block-direct sum of two bangles (15) and (16) with the same number of strips and the same position of the boxed strip, we mean the bangle

\[
A \uplus B := \begin{bmatrix}
A_1 & 0 & \cdots & A_k & 0 & \cdots & A_t \\
0 & B_1 & \cdots & 0 & B_k & \cdots & 0 & B_t
\end{bmatrix}.
\]

**Definition 8.** A regularizing decomposition of a bangle

\[
A = [A_1 \ldots | A_{k-1} | A_k | A_{k+1} | \ldots | A_t]
\]

over a field or skew field \(F\) with respect to *congruence/similarity is a bangle \(\Sigma_A\) satisfying two conditions:

(i) \(\Sigma_A\) is *congruent/similar to \(A\), and

(ii) \(\Sigma_A\) is the block-direct sum of

- its regular part

\[
[0_{p_0} | \ldots | 0_{p_0} | K | 0_{p_0} | \ldots | 0_{p_0}], \quad K \text{ is nonsingular},
\]

(18)

- and its singular part being a block-direct sum of matrices of the form

\[
[0_{q_0} | \ldots | 0_{q_0} | J_q(0) | 0_{q_0} | \ldots | 0_{q_0}],
\]

\[
[\ldots | E_q | \ldots | J_q(0) | \ldots], \quad [\ldots | J_q(0) | \ldots | E_q | \ldots],
\]

(19) (20)

in which \(E_q\) is defined in (9) and the dots denote sequences of strips \(0_{q_0}\).

Both the regular and the singular parts may have size 0-by-0.

The following theorem generalizes Theorems 3 and 5.
Theorem 9. (a) Over a field or skew field $F$, any bangle $A$ possesses regularizing decompositions for $*$congruence and for similarity, their regular parts are determined by $A$ uniquely up to $*$congruence and, respectively, similarity, and their singular parts are determined by $A$ uniquely up to permutation of summands.

(b) If $F = \mathbb{C}$ and $\Sigma_A$ is a regularizing decomposition of a bangle $A$ for to $*$congruence, then its regular part (18) is $*$congruent to the block-direct sum

$$\sum_i [0_{p_i,0} \cdots 0_{p_i,0} | K_i ] 0_{p_i,0} \cdots 0_{p_i,0},$$

in which $K_1 \oplus \cdots \oplus K_s$ is the canonical form of $K$ defined in Theorem 7 and each $K_i$ is $p_i$-by-$p_i$. Replacing in $\Sigma_A$ the regular part by this block-direct sum, we obtain a canonical form of $A$ for $*$congruence (in particular, for congruence) since the obtained bangle is $*$congruent to $A$ and is determined by $A$ uniquely up to permutation of summands.

(c) If $F = \mathbb{C}$ and $\Sigma_A$ is a regularizing decomposition of a bangle $A$ for similarity, then its regular part is similar to a block-direct sum of matrices of the form

$$[0_{q,0} \cdots 0_{q,0} | J_q(\lambda) | 0_{q,0} \cdots 0_{q,0}], \quad \lambda \neq 0.$$

Replacing in $\Sigma_A$ the regular part by this block-direct sum, we obtain a canonical form of $A$ for similarity since the obtained bangle is similar to $A$ and is determined by $A$ uniquely up to permutation of summands.

Note that for bangles with respect to similarity this theorem can be deduced from the canonical form problem solved in \[4, \S 2\].

3 Regularization for *congruence

We give an algorithm that for every bangle over a field or skew field $F$ constructs its regularizing decomposition for $*$congruence. If $F = \mathbb{C}$, then we can improve the numerical stability of this algorithm using only unitary transformations. The algorithm is the alternating sequence of left-hand and right-hand reductions, which we define in Sections 3.1 and 3.2.
3.1 Left-hand reduction for *congruence

Let

\[ A = \begin{bmatrix} A_1 & \hdots & A_{k-1} & A_k & A_{k+1} & \hdots & A_t \end{bmatrix} \]  

(21)

be a bangle over \( \mathbb{F} \). Producing *congruence transformations (a)–(c) from Lemma 7 with \( A \), we can reduce its submatrix \([ A_1 | A_2 | \hdots | A_{k-1} ]\) by the following transformations:

(a') arbitrary transformations of rows;

(b') arbitrary transformations of columns within any vertical strip \( A_i \);

(c') addition of a linear combination of columns of the \( i^{th} \) strip to a column of the \( j^{th} \) strip if \( i < j \).

First we reduce \([ A_1 | A_2 | \hdots | A_{k-1} ]\) to the form

\[ \begin{bmatrix} 0 & 0 & A_{12} & \hdots & A_{1,k-1} \\ 0 & I & A_{22} & \hdots & A_{2,k-1} \end{bmatrix} \]  

(22)

using transformations (b') with \( A_1 \) and (a'), then make zero \( A_{22}, \hdots, A_{2,k-1} \) by transformations (c'). Transforming analogously the submatrix \([ A_{12} | \hdots | A_{1,k-1} ]\), we reduce (22) to the form

\[ \begin{bmatrix} 0 & 0 & 0 & 0 & B_3 & \hdots & B_{k-1} \\ 0 & 0 & 0 & I & 0 & \hdots & 0 \\ 0 & I & 0 & 0 & 0 & \hdots & 0 \end{bmatrix}; \]

and so on. Repeat this process until obtain

\[ \begin{bmatrix} 0 & 0 & 0 & 0 & \hdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \hdots & 0 & I_{r_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & I_{r_{k-1}} & \hdots & 0 & 0 \\ 0 & I_{r_k} & 0 & 0 & \hdots & 0 & 0 \end{bmatrix}, \quad r_2 \geq 0, \hdots, r_k \geq 0, \]  

(23)

and extend the obtained partition into horizontal strips to the whole bangle (21). Make zero all horizontal strips of the blocks \( A_k, \hdots, A_t \) except for the
first strip and obtain

\[ L_k(M) := \begin{bmatrix}
0 & 0 & \ldots & 0 & 0 & M_1 & M_2 & \ldots & M_k & M_{k+1} & \ldots & M_t \\
0 & 0 & \ldots & 0 & I_{r_2} & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & I_{r_k} & \ldots & 0 & 0 & 0 & \ldots & 0 & r_k & 0 & \ldots & 0 \\
\end{bmatrix} \quad (24) \]

(we have divided the boxed block \( A_k \) into \( k \) vertical strips conformally to its partition into horizontal strips) for some

\[ M = \begin{bmatrix} M_1 & M_2 & \ldots & M_t \end{bmatrix} =: L(A). \quad (25) \]

Clearly, \( r_2, \ldots, r_k \) are uniquely determined by \( A \).

**Definition 10.** We say that a bangle \( A \) reduces to a bangle \( B \) by *admissible permutations* and write

\[ A \not\sim B \]

if \( A \) reduces to \( B \) by a sequence of the following transformations:

- permutation of rows of the whole matrix and then the same permutation of columns of the boxed strip,

- permutation of columns in an unboxed strip.

Clearly,

\[ A \not\sim B \quad \implies \quad A \not\approx B \text{ and } A \not\approx B \]

(in the notation (17)).

**Lemma 11.** (a) The equivalence

\[ L_k(M) \not\sim L_k(N) \iff M \not\sim N \quad (26) \]

holds for all

\[ M = \begin{bmatrix} M_1 & M_2 & \ldots & M_t \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & N_2 & \ldots & N_t \end{bmatrix}, \]

and each \( k \leq t \).

(b) If \( F = \mathbb{C} \), then for every bangle \( A \) we can find (24) using only unitary transformations.
Proof. (a) The equivalence (26) is trivial if \( k = 1 \). Let \( k \geq 2 \). Reasoning by induction on \( k \), we assume that

\[
\mathcal{L}_{k-1}(M) \sim \mathcal{L}_{k-1}(N) \iff M \sim N
\]  

(27)

and prove the equivalence (26) as follows.

(\( \Rightarrow \)) Suppose \( \mathcal{L}_k(M) \sim \mathcal{L}_k(N) \), that is,

\[
S_{kk}^* \mathcal{L}_k(M) S = \mathcal{L}_k(N)
\]  

(28)

for some nonsingular

\[
S = \begin{bmatrix}
S_{11} & \cdots & S_{1t} \\
& \ddots & \\
0 & \cdots & S_{tt}
\end{bmatrix}
\]

(29)

Since both \( \mathcal{L}_k(M) \) and \( \mathcal{L}_k(N) \) have the same first vertical strip

\[
\begin{bmatrix}
0 & 0 \\
0 & I_{r_k}
\end{bmatrix}
\]

(we join its zero horizontal strips), by (28) we have

\[
S_{kk}^* \begin{bmatrix}
0 & 0 \\
0 & I_{r_k}
\end{bmatrix} S_{11} = \begin{bmatrix}
0 & 0 \\
0 & I_{r_k}
\end{bmatrix}
\]

and so \( S_{kk} \) has the form

\[
S_{kk} = \begin{bmatrix}
P_1 & P_2 \\
0 & P_3
\end{bmatrix}
\]

(30)

Let

\[
R := \begin{bmatrix}
S_{22} & \cdots & S_{2t} \\
& \ddots & \\
0 & \cdots & S_{tt}
\end{bmatrix}
\]

be a submatrix of (29) with \( S_{kk} \) of the form (30). Due to (28),

\[
P_1^* \mathcal{L}_{k-1}(M) R = \mathcal{L}_{k-1}(N).
\]

(31)

So \( \mathcal{L}_{k-1}(M) \sim \mathcal{L}_{k-1}(N) \), and by (27) \( M \sim N \).
Suppose $M \sim N$. By (27), $L_{k-1}(M) \sim L_{k-1}(N)$, this ensures

$$P_{kk}^* L_{k-1}(M) P = L_{k-1}(N)$$

for some nonsingular

$$P = \begin{bmatrix} P_{11} & \cdots & P_{1t} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & P_{tt} \end{bmatrix}.$$

Denote by $B_i$ and $C_i$ the strips of $L_{k-1}(M)$ and $L_{k-1}(N)$:

$$L_{k-1}(M) = [B_1 | \ldots | B_{k-1} | B_k | B_{k+1} | \ldots | B_t],$$

$$L_{k-1}(N) = [C_1 | \ldots | C_{k-1} | C_k | C_{k+1} | \ldots | C_t].$$

Then

$$L_k(M) = \begin{bmatrix} 0 & 0 & B_1 & 0 & \ldots & B_{k-1} & B_k & B_{k+1} & B_{k+2} & \ldots & B_t \\ 0 & I_{r_k} & 0 & \ldots & 0 & B_k & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

and by (31)

$$L_k(M) \sim \begin{bmatrix} P_{kk} & P_{k,k+1}^* \\ 0 & P_{k+1,k+1} \end{bmatrix}^* L_k(M) \begin{bmatrix} I & 0 \\ 0 & (P_{k+1,k+1})^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ P \end{bmatrix} = \begin{bmatrix} 0 & 0 & C_1 & \ldots & C_t \\ 0 & I_{r_k} & C_1' & \ldots & C_t' \end{bmatrix} \sim L_k(N),$$

where $C_1', \ldots, C_t'$ are some matrices.

This proves (26). Let us give an alternative proof of (26) using *congruence transformations (a)–(c) from Lemma 7. Due to that lemma, it suffices to show that those transformations (a)–(c) with (24) that preserve all of its blocks except for $M_1, \ldots, M_t$ produce all transformations (a)–(c) with (25).

- We can add a column of $M_i$ to a column of $M_j$ if $i < j$. Indeed, in the case $j \leq k$ this is a column-transformation within the boxed block of $L_k(M)$, and so we must produce the *congruent row-transformation—add the corresponding row of the $i^{th}$ horizontal strip of (24) to the row of the $j^{th}$ horizontal strip. This spoils zero blocks of the $j^{th}$ horizontal strip, but they are repaired by additions of columns of $I_{r_j}$.
• We can also make arbitrary elementary transformations with columns of $M_i$ if $i \neq 1$: in the case $i \leq k$ these transformations spoil $I_{r_i}$ but it is restored by transformations with its columns.

(b) Let $\mathbb{F} = \mathbb{C}$. We must prove that if

$$A = \begin{bmatrix} A_1 & \ldots & A_{k-1} & [A_k] & A_{k+1} & \ldots & A_t \end{bmatrix}$$

is reduced to (25) by the algorithm from this section, then $r_2, \ldots, r_k$ and $M_1, \ldots, M_t$ can be found using only unitary transformations with $A$. By unitary column-transformations within vertical strips $A_1, \ldots, A_{k-1}$ of $A$ and by unitary row-transformations, we sequentially reduce its submatrix $[A_1|A_2|\ldots|A_{k-1}]$ to the form

$$\begin{bmatrix}
0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & H_{r_2} \\
0 & 0 & 0 & \ldots & * & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & H_{r_{k-1}} & \ldots & * & * \\
0 & H_{r_k} & * & * & \ldots & *
\end{bmatrix},$$

where each $H_{r_i}$ is a nonsingular $r_i$-by-$r_i$ block and all *’s are unspecified blocks (this reduction was studied thoroughly in [6, Section 4]). The matrix $A$ takes the form

$$\begin{bmatrix}
0 & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & H_{r_2} \\
0 & \ldots & \vdots & \vdots & \vdots \\
0 & H_{r_k} & \ldots & * & * \\
M_1 & M_2 & \ldots & M_k & M_{k+1} & \ldots & M_t
\end{bmatrix}, \quad (32)$$

in which $*_{r_2}, \ldots, *_{r_k}$ are $r_2 \times r_2, \ldots, r_k \times r_k$ matrices. Replacing $H_{r_2}, \ldots, H_{r_k}$ by the identity matrices of the same sizes and all *’s by the zero matrices, we obtain (25) because (32) can be reduced to (25) by those transformations (a)–(c) from Lemma 7 that preserve $r_2, \ldots, r_k$ and $M_1, \ldots, M_t$.

3.2 Right-hand reduction for *congruence

Let

$$A = \begin{bmatrix} [A_1] & A_2 & \ldots & A_t \end{bmatrix} \quad (33)$$
be a bangle over a field or skew field $\mathbb{F}$.

First we reduce $A$ by congruence transformations

$$\begin{bmatrix} S A_1 S^* & S A_2 & \ldots & S A_t \end{bmatrix}, \quad S \text{ is nonsingular},$$

(34)

to the form

$$\begin{bmatrix} 0_d & 0 & B_3 & \ldots & B_{t+1} \\ B'_1 & B'_2 & B'_3 & \ldots & B'_{t+1} \end{bmatrix},$$

(35)

in which the rows of $[B'_1 \ B'_2]$ are linearly independent and $B'_2$ is square.

Then we make zero $B'_3, \ldots, B'_{t+1}$ adding columns of $B'_1$ and $B'_2$, and as in **23** sequentially reduce $[B_3 \ B_4 | \ldots | B_{t+1}]$ to the form

$$\begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & I_{r_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & I_{r_{t-1}} & 0 \\ 0 & I_{r_t} & 0 & \ldots & 0 & 0 \end{bmatrix},$$

obtaining a partition of the first horizontal strip of (35) into $t$ substrips.

Conformally divide the first vertical strip of the boxed block into $t$ substrips and obtain

$$R(M) =
\begin{bmatrix}
0_{r_1} & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0_{r_2} & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & I_{r_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0_{r_{t-1}} & 0 & 0 & 0 & 0 & I_{r_{t-1}} & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0_{r_t} & 0 & 0 & I_{r_t} & 0 & \ldots & 0 \\
M_1 & M_2 & \ldots & M_{t-1} & M_t & M_{t+1} & 0 & 0 & 0 & \ldots & 0 \\
\end{bmatrix},$$

(36)

for some

$$M = \begin{bmatrix} M_1 & \ldots & M_t & M_{t+1} \end{bmatrix} =: R(A)$$

(37)

with $M_{t+1} = B_2$.

**Lemma 12.** (a) The equivalence

$$R(M) \sim R(N) \iff M \sim N$$

(38)
holds for all

\[ M = [ M_1 | \ldots | M_t | M_{t+1} ], \quad N = [ N_1 | \ldots | N_t | N_{t+1} ]. \]

(b) If \( \mathbb{F} = \mathbb{C} \), then for every bangle \( A \) of the form \((33)\) we can find \((36)\) using only unitary transformations.

Proof. (a) Let us prove the equivalence \((38)\) using *congruence transformations (a)–(c) from Lemma 7 (alternatively, one could use induction on \( t \) as in the proof of Lemma 11(a)). Due to Lemma 7, it suffices to show that those transformations (a)–(c) with \((36)\) that preserve all of its blocks except for \( M_1, \ldots, M_{t+1} \) produce all transformations (a)–(c) with \((37)\).

• We can add a column of \( M_i \) to a column of \( M_j \) if \( i < j \); by the definition of *congruence transformations we must add the corresponding row of the \( i^{\text{th}} \) horizontal strip of \((36)\) to the row of the \( j^{\text{th}} \) horizontal strip; although this spoils a zero block of the \( j^{\text{th}} \) horizontal strip if \( i \neq 1 \), but it can be repaired by additions of columns of \( I_{r_j} \).

• We can also make arbitrary elementary transformations with columns of \( M_i \) if \( i \leq t \); these transformations spoil \( I_{r_i} \) if \( i \neq 1 \), but it is restored by transformations with its columns.

(b) Let \( \mathbb{F} = \mathbb{C} \). First we reduce the bangle \((33)\) by transformations \((34)\) with unitary \( S \) to the form \((35)\), in which the rows of \( [B'_1 \ B'_2] \) are linearly independent and \( B'_2 \) is square.

Then we sequentially reduce \( [B_3 | B_4 | \ldots | B_{t+1}] \) by unitary column-transformations within vertical strips and by unitary row-transformations to the form

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & H_{r_2} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & H_{r_{t-1}} & \ldots & * & * \\
0 & H_{r_1} & * & * & \ldots & * & * \\
\end{bmatrix},
\]

where each \( H_{r_i} \) is a nonsingular \( r_i \)-by-\( r_i \) block and the *’s are unspecified
blocks. The matrix $A$ takes the form

$$
\begin{bmatrix}
0_{r_1} & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0_{r_2} & \ldots & 0 & 0 & 0 & \ldots & 0 & H_{r_2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0_{r_t} & 0 & 0 & H_{r_t} & \ldots & * & * \\
M_1 & M_2 & \ldots & M_t & M_{t+1} & \ldots & * & \ldots & * & * \\
\end{bmatrix},
$$

where $M_{t+1} = B_2$. Replacing $H_{r_2}, \ldots, H_{r_k}$ by the identity matrices of the same sizes and all *'s by the zero matrices, we obtain (36) because (39) can be reduced to (36) by those transformations (a)–(c) from Lemma 7 that preserve $r_1, \ldots, r_t$ and $M_1, \ldots, M_{t+1}$.

3.3 Regularization algorithm for *congruence

For any bangle

$$
A = [A_1 | \ldots | A_{k-1} \begin{bmatrix} A_k \end{bmatrix} A_{k+1} | \ldots | A_t]
$$

(40)

over $\mathbb{F}$, its regularizing decomposition for *congruence can be constructed as follows.

Alternating the left-hand and the right-hand reductions for *congruence, we construct the sequence of bangles

$$
A' := L(A), \quad A'' := R(A'), \quad A''' := L(A''), \quad A'''' := R(A''') , \ldots
$$

until obtain

$$
A^{(n)} = \begin{bmatrix} K \end{bmatrix} 0_{p\times0} | \ldots | 0_{p\times0} \quad \text{or} \quad A^{(n)} = 0_{p\times0} | \ldots | 0_{p\times0} \begin{bmatrix} K \end{bmatrix}
$$

(41)

with a nonsingular $K$.

Producing this reduction, we in each step have deleted the reduced parts of $A$; say, in step 1 we reduced $A$ to the form (24) and took only its unreduced part $A' = L(A)$. Let us repeat the reduction of (40) preserving all the reduced parts of $A$:

- In step 1 we transform $A$ to $\mathcal{L}_k(A')$ of the form (24).
- In step 2 we reduce its subbangle $A'$ to $\mathcal{R}(A'')$ preserving the other blocks of $\mathcal{L}_k(A')$, and so on.
After $n$ steps, instead of (41) we obtain some bangle $\hat{A}$, which is *congruent to $A$. Due to the next theorem, $\hat{A}$ is a regularizing decomposition of $A$ up to admissible permutations of rows and columns.

**Theorem 13.** If $A$ is a bangle over a field or skew field $\mathbb{F}$, then $\hat{A}$ reduces by admissible permutations of rows and columns to a regularizing decomposition of $A$ for *congruence.

**Proof.** We give a constructive proof of this theorem.

By admissible permutations of rows and columns, $\hat{A}$ reduces to a block-direct sum of the bangle (18) in which $K$ is the same as in (11), and a bangle $D$ in which each row and each column contains at most one 1 and its other entries are zero. We obtain a regularizing decomposition of $A$ for *congruence replacing $D$ in this block-direct sum by $\Sigma_D$ from the follows statement.

Let $D$ be a bangle in which each row and each column contains at most one 1 and the other entries are zero. Then $D$ reduces by admissible permutations of rows and columns to a block-direct sum $\Sigma_D$ of bangles of the form (19) and (20).

Let us prove (42). By admissible permutations of rows and columns of $D$, we reduce its boxed strip $D_k$ to a direct sum of singular Jordan blocks. Then we rearrange columns in each unboxed strip such that if its $(i, j)$ and $(i', j')$ entries are 1 and $i < i'$, then $j < j'$. It is easy to see that the obtained bangle $\Sigma_D$ is a block-direct sum of bangles of the form (19) and (20): each singular Jordan block $J_p(0)$ in the decomposition of $D_k$ gives the summand (19) if those row of $D$ that contains the last (zero) row of $J_p(0)$ is zero, and the summand (20) otherwise. The summands (20) with $p = 0$ give zero columns in unboxed strips of $D$.

### 4 Regularization for similarity

We give an algorithm that for every bangle over a field or skew field $\mathbb{F}$ constructs its regularizing decomposition for similarity. If $\mathbb{F} = \mathbb{C}$, then we can improve the numerical stability of this algorithm using only unitary transformations.
4.1 Left-hand reduction for similarity

Let 
\[ A = \begin{bmatrix} A_1 & \ldots & A_{k-1} & A_k & A_{k+1} & \ldots & A_t \end{bmatrix} \]
be a bangle over \( \mathbb{F} \). Using similarity transformations with \( A \), we can reduce its submatrix \( [A_1|A_2|\ldots|A_{k-1}] \) by transformations (a’–(c’)) from Section 3.1. We reduce this submatrix to the form
\[
\begin{bmatrix}
0 & I_{r_1} & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & I_{r_{k-1}} \\
0 & 0 & \ldots & 0 & 0
\end{bmatrix}, \quad r_1 \geq 0, \ldots, r_{k-1} \geq 0,
\]
and obtain a partition of the bangle \( A \) into \( k \) horizontal strips. Then we divide the boxed block \( A_k \) into \( k \) vertical substrips of the same sizes, make zero all horizontal strips in the blocks \( A_k, \ldots, A_t \) except for the last strip, and obtain
\[
\mathcal{L}_k(M) =
\begin{bmatrix}
0 & I_{r_1} & \ldots & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & I_{r_{k-1}} & 0 & \ldots & 0
\end{bmatrix}
\begin{bmatrix}
0_{r_1} & \ldots & 0 & 0 \\
0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0_{r_{k-1}} & \ldots & 0 & 0
\end{bmatrix}
\begin{bmatrix}
M_1 & \ldots & M_{k-1} & M_k & M_{k+1} & \ldots & M_t
\end{bmatrix}
\]
for some
\[
M = \begin{bmatrix} M_1 | \ldots | M_{k-1} & M_k & M_{k+1} & \ldots & M_t \end{bmatrix} =: L(A).
\]

Lemma 14. (a) The equivalence
\[
\mathcal{L}_k(M) \overset{\sim}{\cong} \mathcal{L}_k(N) \iff M \overset{\sim}{\cong} N
\]
holds for all
\[
M = \begin{bmatrix} M_1 | \ldots | M_{k-1} & M_k & M_{k+1} & \ldots & M_t \end{bmatrix},
N = \begin{bmatrix} N_1 | \ldots | N_{k-1} & N_k & N_{k+1} & \ldots & N_t \end{bmatrix}.
\]

(b) If \( \mathbb{F} = \mathbb{C} \), then for every bangle \( A \) we can find (43) using only unitary transformations.
Proof. (a) This statement follows from Lemma 7 since those transformations (a)–(c) with (43) that preserve all of its blocks except for $M_1, \ldots, M_t$ produce all transformations (a)–(c) with (44). For example, we can add a column of $M_i$ to a column of $M_j$ if $i < j$: although in the case $j \leq k$ we must subtract the corresponding row of the $j$th horizontal strip of (43) from the row of the $i$th horizontal strip, and this may spoil zero blocks of the $i$th horizontal strip, but they are repaired by additions of columns of $I_{r_i}$.

(b) Let $\mathbb{F} = \mathbb{C}$. By unitary column-transformations within vertical strips of $A$ and by unitary row-transformations, we sequentially reduce its submatrix $[A_1|A_2|\ldots|A_{t-1}]$ to the form

$$
\begin{bmatrix}
0 & H_{r_1} & * & * & \ldots & * & * \\
0 & 0 & 0 & H_{r_2} & \ldots & * & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & H_{r_{k-1}} \\
0 & 0 & 0 & 0 & \ldots & 0 & 0
\end{bmatrix}
$$

where each $H_{r_i}$ is a nonsingular $r_i$-by-$r_i$ block and all *'s are unspecified blocks. The matrix $A$ takes the form

$$
\begin{bmatrix}
0 & H_{r_1} & \ldots & * & * & *_{r_1} & \ldots & * & * & \ldots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & H_{r_{k-1}} & \ldots & *_{r_{k-1}} & * & \ldots & * & \ldots & * \\
0 & 0 & \ldots & 0 & 0 & M_1 & \ldots & M_{k-1} & M_k & M_{k+1} & \ldots & M_t
\end{bmatrix}
$$

in which $*_{r_1}, \ldots, *_{r_{k-1}}$ are $r_1 \times r_1, \ldots, r_{k-1} \times r_{k-1}$ matrices. Replacing $H_{r_1}, \ldots, H_{r_{k-1}}$ by the identity matrices of the same sizes and all *'s by the zero matrices, we obtain (43) since (45) reduces to (43) by those transformations (a)–(c) from Lemma 7 that preserve $r_1, \ldots, r_{k-1}, M_1, \ldots, M_t$. □

4.2 Right-hand reduction for similarity

Let

$$A = \begin{bmatrix} A_1 | A_2 | \ldots | A_t \end{bmatrix}$$

be a bangle over $\mathbb{F}$.

First we reduce $A$ by similarity transformations

$$\begin{bmatrix} S A_1 S^{-1} | S A_2 | \ldots | S A_t \end{bmatrix}, \quad S \text{ is nonsingular},$$

(47)
to the form
\[
\begin{bmatrix}
B_1 & B_2 & B_3 & \ldots & B_{t+1} \\
0 & 0 & B'_3 & \ldots & B'_{t+1}
\end{bmatrix},
\] (48)
in which the rows of \([B_1 \, B_2]\) are linearly independent and \(B_1\) is square.

Then we make zero \(B_3, \ldots, B_{t+1}\) adding columns of \(B_1\) and \(B_2\), and sequentially reduce \([B'_3 | \ldots | B'_{t+1}]\) to the form
\[
\begin{bmatrix}
0 & I_{r_2} & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & I_{r_3} & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & I_{r_t} \\
0 & 0 & 0 & 0 & \ldots & 0 & 0
\end{bmatrix}.
\]

The matrix \(A\) transforms to
\[
\mathcal{R}(M) =
\begin{bmatrix}
M_1 & M_2 & M_3 & \ldots & M_t & M_{t+1} \\
0 & 0 & 0 & \ldots & 0 & 0 & I_{r_2} & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & I_{r_3} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & I_{r_t} \\
0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0
\end{bmatrix},
\] (49)
for some
\[
M = \begin{bmatrix} M_1 & M_2 & \ldots & M_{t+1} \end{bmatrix} =: R(A)
\] (50)
with \(M_1 = B_1\).

**Lemma 15.** (a) The equivalence
\[
\mathcal{R}(M) \sim \mathcal{R}(N) \iff M \sim N
\]
holds for all
\[
M = \begin{bmatrix} M_1 & M_2 & \ldots & M_{t+1} \end{bmatrix}, \quad N = \begin{bmatrix} N_1 & N_2 & \ldots & N_{t+1} \end{bmatrix}.
\]

(b) If \(\mathbb{F} = \mathbb{C}\), then for every bangle \(A\) of the form (46) we can find (49) using only unitary transformations.
\textbf{Proof.} (a) It is easy to show that those transformations (a)–(c) from Lemma 7 with (49) that preserve all of its blocks except for $M_1, \ldots, M_{t+1}$ produce all transformations (a)–(c) with (50). Say, we can add a column of $M_i$ to a column of $M_j$ if $i < j$: although we must subtract the corresponding row of the $j^\text{th}$ horizontal strip of (49) from the row of the $i^\text{th}$ horizontal strip, and this spoils zero blocks of the $i^\text{th}$ horizontal strip if $j \neq t + 1$, but they are repaired by additions of columns of $I_{r_i}$.

(b) Let $\mathbb{F} = \mathbb{C}$. First we reduce $A$ by transformations (47) with unitary $S$ to the form (48), in which the rows of $\begin{bmatrix} B_1 & B_2 \end{bmatrix}$ are linearly independent and $B_1$ is square. Then we sequentially reduce its submatrix $\begin{bmatrix} B_3' & B_4' & \cdots & B_{t+1}' \end{bmatrix}$ by unitary column-transformations within vertical strips and by unitary row-transformations to the form

\[
\begin{bmatrix}
0 & H_{r_2} & * & * & \cdots & * & * \\
0 & 0 & 0 & H_{r_3} & * & \cdots & * \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & H_{r_t} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix},
\]

where each $H_{r_i}$ is a nonsingular $r_i$-by-$r_i$ matrix. The matrix $A$ takes the form

\[
\begin{bmatrix}
M_1 & M_2 & \ldots & M_t & M_{t+1} & * & * & \cdots & * & * \\
0 & 0 & \ldots & 0 & 0 & H_{r_2} & * & \cdots & * & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 0_{r_t} & 0 & 0 & 0 & \cdots & 0 & H_{r_t} \\
0 & 0 & \ldots & 0_{r_{t+1}} & 0 & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}. \tag{51}
\]

Replacing $H_{r_2}, \ldots, H_{r_t}$ by the identity matrices of the same sizes and all $*$’s by the zero matrices, we obtain (49) since (51) reduces to (49) by those transformations (a)–(c) from Lemma 7 that preserve $r_2, \ldots, r_{t+1}, M_1, \ldots, M_t$. \hfill $\blacksquare$

\subsection{4.3 Regularization algorithm for similarity}

For any bangle

\[
A = \begin{bmatrix} A_1 & \ldots & A_{k-1} \begin{bmatrix} A_k \end{bmatrix} & A_{k+1} & \ldots & A_t \end{bmatrix}
\]

over $\mathbb{F}$, its regularizing decomposition for similarity can be constructed as follows.
• First we apply subsequently the left-hand reduction for similarity to $A$
until obtain

$$L(L \ldots (L(A)) \ldots) = \begin{bmatrix} 0_{m0} & \ldots & 0_{m0} & B_k & B_{k+1} & \ldots & B_t \end{bmatrix},$$
in which the first $k - 1$ strips have no columns.

• Then we apply subsequently the right-hand reduction for similarity to

$$B = \begin{bmatrix} B_k & B_{k+1} & \ldots & B_t \end{bmatrix}$$
until obtain

$$R_s(R_s \ldots (R_s(B)) \ldots) = \begin{bmatrix} K & 0_{n0} & \ldots & 0_{n0} \end{bmatrix}$$ (52)
with a nonsingular $K$.

Producing this reduction, we in each step have deleted the reduced parts of $A$. Let us repeat the reduction preserving all the reduced parts of $A$ and denote the obtained bangle by $\tilde{A}$. Clearly, $\tilde{A}$ is similar to $A$. Due to the next theorem, $\tilde{A}$ is a regularizing decomposition of $A$ up to admissible permutations of rows and columns.

**Theorem 16.** If $A$ is a bangle over a field or skew field $\mathbb{F}$, then $\tilde{A}$ reduces by admissible permutations of rows and columns to a regularizing decomposition of $A$ for similarity.

**Proof.** We give a constructive proof of this theorem. By admissible permutations of rows and columns, $\tilde{A}$ is reduced to a block-direct sum of the bangle (18) with $K$ from (52) and a bangle $D$ in which each row and each column contains at most one 1 and the other entries are zero. Replacing $D$ in this block-direct sum by $\Sigma_D$ from (12), we obtain a regularizing decomposition of $A$ for similarity.

5 Proofs of Theorems 9, 3, and 5

**Proof of Theorem 9.** (a) Let us prove the statement (a) for *congruence; its proof for similarity is analogous.

Let $A$ be a bangle over $\mathbb{F}$. In view of Theorem 13 $A$ possesses a regularizing decomposition for *congruence, which is obtained from $\tilde{A}$ by admissible permutations of rows and columns.
Let $\Sigma_1$ and $\Sigma_2$ be two regularizing decompositions of $A$. Then $\Sigma_1 \sim \Sigma_2$. We need to prove that
\[ \Sigma_1^{\text{reg}} \sim \Sigma_2^{\text{reg}} \quad \text{and} \quad \Sigma_1^{\text{sing}} \sim L \Sigma_2^{\text{sing}}, \]
where $\Sigma_i^{\text{reg}}$ and $\Sigma_i^{\text{sing}}$ are the regular and the singular parts of $\Sigma_i$ ($i = 1, 2$).

If
\[ L(\Sigma_1) = R(\Sigma_1) = \Sigma_1, \]
then $\Sigma_1 = \Sigma_1^{\text{reg}}$ and (53) holds.

Let $L(\Sigma_1) \neq \Sigma_1$ or $R(\Sigma_1) \neq \Sigma_1$. Suppose for definiteness that
\[ L(\Sigma_1) \neq \Sigma_1. \]

Each row and each column of $\Sigma_i^{\text{sing}}$ ($i = 1, 2$) contains at most one 1, the other entries are zero. Due to this property, the reduction of $\Sigma_i$ to
\[ \Omega_i := L_k(L(\Sigma_i)). \]
of the form (24) can be realized by admissible permutations:
\[ \Sigma_i \sim \Omega_i; \]
moreover, $L(\Sigma_i)$ is a block-direct sum of a bangle of the form (18) and a bangle, in which each row and each column contains at most one 1, the other entries are zero. By (42), $L(\Sigma_i)$ reduces by admissible permutations of rows and columns to its regularizing decomposition, so we may take $\Omega_i$ such that $L(\Sigma_i)$ is a regularizing decomposition.

Since $\Sigma_1 \sim \Sigma_2$, we have $\Omega_1 \sim \Omega_2$, and so by (26) and (55)
\[ L(\Sigma_1) \sim L(\Sigma_2). \]
Due to (54), the size of $L(\Sigma_1)$ is less than the size of $\Sigma_1$, reasoning by induction we may assume that (53) holds for $L(\Sigma_i)$; that is,
\[ L(\Sigma_1)^{\text{reg}} \sim L(\Sigma_2)^{\text{reg}} \quad \text{and} \quad L(\Sigma_1)^{\text{sing}} \sim L(\Sigma_2)^{\text{sing}}. \]
Then
\[ \Omega_1^{\text{reg}} \sim \Omega_2^{\text{reg}} \quad \text{and} \quad \Omega_1^{\text{sing}} \sim \Omega_2^{\text{sing}} \]
since $\Omega_1$ and $\Omega_2$ have the form (24). This proves (53) due to (56).

(b) This statement follows from (a) and Theorem 1.

(c) This statement follows from (a) and the uniqueness of the Jordan Canonical Form. \qed
Proof of Theorem 3. Let $G$ be one of sesquilinear forms

$$U \times V \rightarrow \mathbb{F}, \quad (V/U) \times V \rightarrow \mathbb{F}.$$ 

Let us prove that the canonical form problem for its matrix $[A|B]$ (defined in (5)) is the canonical form problem under *congruence for the bangle

$$\begin{bmatrix} A & B \\ \end{bmatrix} \text{ or } \begin{bmatrix} B & A \\ \end{bmatrix},$$

respectively, and so Theorem 3 follows from Theorem 9.

It suffices to prove that a change of the basis of $V$ reduces $[A|B]$ by transformations

$$[A B] \rightarrow \begin{cases} S^*[A B] \begin{bmatrix} S & P \\ 0 & Q \\ \end{bmatrix} & \text{if } G: U \times V \rightarrow \mathbb{F}, \\ S^*[A B] \begin{bmatrix} S & 0 \\ P & Q \\ \end{bmatrix} & \text{if } G: (V/U) \times V \rightarrow \mathbb{F}, \end{cases} \quad (57)$$

in which $S$ and $Q$ are nonsingular matrices and $P$ is arbitrary.

Case 1: $[A|B]$ is the matrix of

$$G: U \times V \rightarrow \mathbb{F}, \quad U \subset V,$$

in a basis $e_1, \ldots, e_n$ of $V$ satisfying (4). If

$$f_j = e_1 \rho_{1j} + \cdots + e_n \rho_{nj}, \quad j = 1, \ldots, n, \quad (58)$$

is another basis of $V$ such that $f_1, \ldots, f_m$ is a basis of $U$, then the change matrix from $e_1, \ldots, e_n$ to $f_1, \ldots, f_n$ has the form

$$R = [\rho_{ij}] = \begin{bmatrix} S & P \\ 0 & Q \\ \end{bmatrix},$$

where $S$ is the change matrix from $e_1, \ldots, e_m$ to $f_1, \ldots, f_m$ in $U$. Due to (2), the matrix $[A|B]$ reduces by transformations (57).

Case 2: $[A|B]$ is the matrix of

$$G: (V/U) \times V \rightarrow \mathbb{F}, \quad U \subset V,$$
in a basis \( e_1, \ldots, e_n \) of \( V \) satisfying (4). If (58) is another basis of \( V \) such that \( f_{m+1}, \ldots, f_n \) is a basis of \( U \), then the change matrix from \( e_1, \ldots, e_n \) to \( f_1, \ldots, f_n \) has the form

\[
R = [\rho_{ij}] = \begin{bmatrix} S & 0 \\ P & Q \end{bmatrix},
\]

where \( S \) is the change matrix from \( e_1 + U, \ldots, e_m + U \) to \( f_1 + U, \ldots, f_m + U \) in \( V/U \). Hence, the matrix \([A|B]\) reduces by transformations (57).

**Proof of Theorem 5.** Let \( A \) be one of linear mappings

\[
U \rightarrow V, \quad V \rightarrow U, \quad V/U \rightarrow V, \quad V \rightarrow V/U.
\]

Let us prove that the canonical form problem for its matrix

\[
A_e = \begin{cases} 
\begin{bmatrix} A \\ B \end{bmatrix} & \text{if } U \rightarrow V \text{ or } V/U \rightarrow V, \\
[A|B] & \text{if } V \rightarrow U \text{ or } V \rightarrow V/U,
\end{cases}
\]

(see (11)) is the canonical form problem under similarity for the bangle

\[
\begin{bmatrix} B^T & A^T \end{bmatrix}, \quad \begin{bmatrix} A & B \end{bmatrix}, \quad \begin{bmatrix} A^T & B^T \end{bmatrix}, \quad \text{or } \begin{bmatrix} B & A \end{bmatrix},
\]

respectively, and so Theorem 5 follows from Theorem 9.

It suffices to prove that a change of the basis of \( V \) reduces \( A_e \) by transformations

\[
\begin{align*}
\begin{bmatrix} A \\ B \end{bmatrix} & \mapsto \begin{bmatrix} S^{-1} & * \\ 0 & Q^{-1} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} S & \text{if } A: U \rightarrow V, \\
\begin{bmatrix} A & B \end{bmatrix} & \mapsto S^{-1} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} S & 0 \\ * & Q \end{bmatrix} & \text{if } A: V \rightarrow U, \\
\begin{bmatrix} A & B \end{bmatrix} & \mapsto S^{-1} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} S & 0 \\ * & Q \end{bmatrix} & \text{if } A: V \rightarrow V/U,
\end{align*}
\]

in which \( S \) and \( Q \) are nonsingular matrices and the *’s denote arbitrary matrices.
Case 1: $A_e$ is the matrix of

$$A: U \to V \quad \text{or} \quad A: V \to U, \quad U \subset V,$$

in a basis $e_1, \ldots, e_n$ of $V$ satisfying (10). If

$$f_j = e_1 \rho_{1j} + \cdots + e_n \rho_{nj}, \quad j = 1, \ldots, n, \quad (63)$$

is another basis of $V$ such that $f_1, \ldots, f_m$ is a basis of $U$, then the change matrix from $e_1, \ldots, e_n$ to $f_1, \ldots, f_n$ has the form

$$R = [\rho_{ij}] = \begin{bmatrix} S & P \\ 0 & Q \end{bmatrix},$$

where $S$ is the change matrix from $e_1, \ldots, e_m$ to $f_1, \ldots, f_m$ in $U$. So the matrix $A_e$ reduces by transformations (59) or (60).

Case 2: $A_e$ is the matrix of

$$A: V/U \to V \quad \text{or} \quad A: V \to V/U, \quad U \subset V,$$

in a basis $e_1, \ldots, e_n$ of $V$ satisfying (10). If (63) is another basis of $V$ such that $f_{m+1}, \ldots, f_n$ is a basis of $U$, then the change matrix from $e_1, \ldots, e_n$ to $f_1, \ldots, f_n$ has the form

$$R = [\rho_{ij}] = \begin{bmatrix} S & 0 \\ P & Q \end{bmatrix},$$

where $S$ is the change matrix from $e_1 + U, \ldots, e_m + U$ to $f_1 + U, \ldots, f_m + U$ in $V/U$. Hence, the matrix $A_e$ reduces by transformations (61) or (62).

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