ON THE DIRICHLET BOUNDARY VALUE PROBLEM FOR THE NORMALIZED $p$-LAPLACIAN EVOLUTION

AGNID BANERJEE
Department of Mathematics, Purdue University, West Lafayette, IN 47907

NICOLA GAROFALO
Dipartimento di Ingegneria Civile, Edile e Ambientale (DICEA)
Università di Padova, 35131 Padova, ITALY

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ABSTRACT. In this paper, we study the potential theoretic aspects of the normalized $p$-Laplacian evolution, see (1.1) below. A systematic study of such equation was recently started in [1], [4] and [25]. Via the classical Perron approach, we address the question of solvability of the Cauchy-Dirichlet problem with “very weak” assumptions on the boundary of the domain. The regular boundary points for the Dirichlet problem are characterized in terms of barriers. For $p \geq 2$, in the case of space - time cylinder $G \times (0, T)$, we show that $(x, t) \in \partial G \times (0, T)$ is a regular boundary point if and only if $x \in \partial G$ is a a regular boundary point for the $p$-Laplacian. This latter operator is the steady state corresponding to the evolution (1.1) below. Consequently, when $p \geq 2$ the Cauchy-Dirichlet problem for (1.1) can be solved in cylinders whose section is regular for the $p$-Laplacian. This can be thought of as an analogue of the results obtained in [17] for the standard parabolic $p$-Laplacian
\[
\text{div}(|Du|^{p-2}Du) - u_t = 0.
\]

1. Introduction. It was observed by Sternberg [32] in 1929 that the method developed by Perron [29] for solving the Dirichlet boundary value problem for the Laplace equation can be extended to the heat equation. Such generalization, which owes to the ideas of Perron, Wiener, Brelot, Bauer, Doob, is nowadays well known for linear parabolic equations, see [8]. In this paper we study the potential theoretic aspects of the normalized $p$-Laplacian evolution
\[
u_t = |Du|^{2-p}\text{div}(|Du|^{p-2}Du), \quad 1 < p < \infty.
\]
This equation has been recently studied in [1], [4] and [25]. It is an evolution, associated with the $p$-Laplacian $\Delta_p u = \text{div}(|Du|^{p-2}Du)$, that generalizes the motion by mean curvature, which corresponds to the case $p = 1$, and also the heat equation, corresponding to the case $p = 2$. The case $p = \infty$ of the normalized $\infty$-Laplacian evolution, see (5.6), has been studied in [15]. The equation (1.1) also arises in image processing, see [4], where the Cauchy-Neumann problem was studied. In [1]
we studied the Cauchy problem for (1.1), and the Cauchy-Dirichlet problem. In the interesting paper [25], solutions to (1.1) have been characterized by asymptotic mean value properties. These properties are connected with the analysis of tug-of-war games with noise in which the number of rounds is bounded. The value functions for these games approximate a solution to the pde (1.1) when the parameter that controls the size of possible steps go to zero. Furthermore, this equation has the advantage of being 1-homogeneous but has the disadvantage of having the non-divergence structure.

In the elliptic case, Granlund, Lindqvist and Martio in a series of papers (of which we only quote [10]) initiated the study of the nonlinear potential theory for the $p$-Laplacian
\[
\Delta_p u = \text{div}(|Du|^{p-2}Du) = 0,
\]
(1.2)
An account of the elliptic nonlinear potential theory is given in the monograph [12]. Analogous to the case of Laplace equation, a sufficiency criterion for the $p$-Laplacian was proved by [26] and then generalized to more general quasilinear equations in [9]. The necessity of such a criterion was finally established in [18]. For the standard parabolic $p$-Laplacian
\[
\text{div}(|Du|^{p-2}Du) = u_t
\]
(1.3)
the study of the Perron method goes back to the work [17], in which the authors established a characterization of regular points on the lateral boundary in terms of Wiener test for the $p$-Laplacian. Later such result was generalized to more general quasilinear parabolic equations in [31]. In this context, we would like to mention that for the heat equation, characterization of regular boundary points in an arbitrary domain in space-time (not necessarily space-time cylinders), was established in [5], and subsequently generalized to divergence form parabolic PDE with $C^{1,\text{Dini}}$ coefficients in [7]. The characterization of regular points for arbitrary domains in space time remains open for the equation (1.3) as well as for the equation (1.1). We intend to come back to this question in a future study.

This paper is organized as follows. In Section 2, we introduce the relevant notions and gather some known results. We also establish a basic existence result by an application of Krylov-Safonov Hölder estimate. Moreover, we use such an existence result in the proof of elliptic type comparison principle as well. In Section 3, we introduce the notion of generalized supersolutions analogous to its elliptic counterpart. We show that such a notion is equivalent to the one in the viscosity sense. In Section 4, we establish a barrier characterization of regularity and then prove our main result which is stated below:

**Theorem 1.1.** Let $p \geq 2$ and $\Omega = G \times (0,T)$. Let $x_0 \in \partial G$ and $0 < t_0 \leq T$. Then, $\xi_0 = (x_0, t_0)$ is a regular boundary point for equation (1.1) if and only if $x_0 \in \partial G$ is a regular boundary point for the $p$-Laplacian.

In closing, we include an appendix to this paper which contains a discussion on the large time behavior of Cauchy-Dirichlet problem to (1.1). We also indicate some interesting open issues in that direction. Furthermore, we establish a Tychonoff type maximum principle for the normalized $\infty$-Laplacian evolution. This partly answers a question raised in [15]( see Remark 4.10 in that paper). We show that linear type barriers can be employed in this situation. In the case $p < \infty$, such a result has been established in [1] using an explicit solution computed in that paper.
2. Preliminaries. We would like to emphasize that for the rest of the paper, unless otherwise specified, by a neighborhood $V$ of a point $(x_0, t_0) \in \mathbb{R}^{n+1}$ we mean a parabolic neighborhood, i.e., $V = U \cap \{ t \leq t_0 \}$ for some Euclidean neighborhood $U$ of $(x_0, t_0)$ in $\mathbb{R}^{n+1}$. Local extrema in the definitions below are to be understood with respect to parabolic neighborhoods. We now introduce the relevant notions of a subsolution and supersolution.

**Definition 2.1.** An upper semicontinuous function $u \in L^\infty(\Omega \times (0, T])$, is called a *viscosity subsolution* of (1.1) provided that, if $u - \phi$ has a local maximum at $z_0 \in \Omega \times (0, T)$,

\begin{equation}
\phi_t \leq \left( \delta_{ij} + (p - 2) \frac{D_i \phi D_j \phi}{|D\phi|^2} \right) D_{ij} \phi \quad \text{at } z_0,
\end{equation}

then for every such $\phi \in C^2(\Omega \times [0, T])$, either

\begin{equation}
\left\{ \begin{array}{l}
\phi_t = \delta_{ij} + (p - 2) \frac{D_i \phi D_j \phi}{|D\phi|^2} D_{ij} \phi \\
\text{if } D\phi(z_0) \neq 0,
\end{array} \right.
\end{equation}

or

\begin{equation}
\inf_{|a|=1} \left\{ \phi_t - (\delta_{ij} + (p - 2) a_i a_j) D_{ij} \phi \right\} \leq 0 \quad \text{at } z_0,
\end{equation}

\begin{equation}
\left\{ \begin{array}{l}
\phi_t = 0 \\
\text{if } D\phi(z_0) = 0.
\end{array} \right.
\end{equation}

Analogous definitions are given for supersolutions and solutions.

In fact by arguing as in [1] proposition 2.8, we have the following equivalent definition which allows to reduce the number of test functions in the definition.

**Definition 2.2.** An upper semicontinuous function $u \in L^\infty(\Omega \times (0, T])$, is called a *viscosity subsolution* of (1.1) provided that if $\phi \in C^2(\Omega \times [0, T])$,

\begin{equation}
\phi_t \leq \left( \delta_{ij} + (p - 2) \frac{D_i \phi D_j \phi}{|D\phi|^2} \right) D_{ij} \phi \quad \text{at } z_0,
\end{equation}

then either

\begin{equation}
\left\{ \begin{array}{l}
\phi_t = \delta_{ij} + (p - 2) \frac{D_i \phi D_j \phi}{|D\phi|^2} D_{ij} \phi \\
\text{if } D\phi(z_0) \neq 0,
\end{array} \right.
\end{equation}

or

\begin{equation}
\left\{ \begin{array}{l}
\phi_t \leq 0 \\
\text{if } D\phi(z_0) = 0, D^2 \phi(z_0) = 0.
\end{array} \right.
\end{equation}

Analogous definitions are given for supersolutions and solutions.

**Remark 2.3.** In [1], although for local extrema, Euclidean neighborhoods have been considered in the definitions of subsolutions and supersolutions. However, by arguing as in Lemma 1.4 in [3], one can show that for continuous subsolutions and supersolutions, this is equivalent to consider parabolic neighborhoods instead.

**Remark 2.4.** Although Proposition 2.8 in [1] is stated for continuous functions, note that, for its validity, we just require a supersolution to be lower semicontinuous and a subsolution to be upper semicontinuous. In order to see this, one just needs to observe that in the proof of Proposition 2.8 in [1], the maximum/minimum of a upper/lower semicontinuous function is attained in a compact set. Therefore, with the same notations as in that proof (say for the case of subsolution), $\xi_\epsilon$ attains its maximum at some point $(x_\epsilon, y_\epsilon, t_\epsilon)$ for each $\epsilon$. Moreover by uppersemicontinuity, the inequality (2.14) in [1] continues to hold. The rest of the proof remains the same without any change.
Now we state the comparison theorem adapted to upper semicontinuous sub-solutions and lower semicontinuous supersolutions, see [2]. For this, we introduce the relevant notion of upper and lower semicontinuous relaxation.

Let \( h : L \to \mathbb{R} \) where \( L \) is an arbitrary set. The lower semicontinuous relaxation \( h^* \) of \( h \) with respect to \( L \) is defined as

\[
h^*(x) = \lim_{\varepsilon \to 0} \inf_{|y-x| \leq \varepsilon, y \in L} h(y)
\]  

(2.7)

Analogously, the upper semicontinuous relaxation \( h^* \) is defined. We note that when \( \Omega \) is an open set in \( \mathbb{R}^{n+1} \) and \( u : \Omega \to \mathbb{R} \) is a lower semicontinuous function, then \( u^* = u \) in \( \Omega \). Similarly, \( v^* = v \) when \( v \) is upper semicontinuous.

Henceforth in this paper, if \( D \subset \mathbb{R}^{n+1} \) we will indicate with \( \partial_p D \) its parabolic boundary. If \( \Omega \subset \mathbb{R}^n \) and \( T > 0 \), then for the space-time cylinder \( \Omega_T = \Omega \times (0, T) \) we have

\[
\partial_p \Omega_T = (\Omega \times \{0\}) \cup (\partial \Omega \times [0, T]).
\]

**Theorem 2.5.** Let \( \Omega \) be a bounded domain and \( \Omega_T = \Omega \times (0, T) \). Let \( v \) be a subsolution and \( u \) be a supersolution in \( \Omega_T \) such that \( v^* \leq u^* \) on the parabolic boundary \( \partial_p \Omega_T \). Then \( v \leq u \) in \( \Omega_T \).

**Proof.** Since \( u \) is lower semicontinuous and \( v \) is upper semicontinuous, we have that \( v = v^* \) and \( u = u^* \) in \( \Omega_T \). Theorem 4.1 in [2] now implies that \( v \leq u \) in \( \Omega_T \). \( \square \)

In what follows, \( Q \) will stand for a cube defined by

\[
Q = (a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n),
\]

and \( Q_T = Q \times (0, T) \) will be called a box in the subsequent discussion.

For domains \( \Omega \subset \mathbb{R}^n \) satisfying an uniform exterior sphere condition at the boundary, the existence of solution to the Cauchy-Dirichlet problem in \( \Omega_T = \Omega \times (0, T) \) has been obtained by probabilistic arguments in [25] as a limit of value functions of tug of war games with noise. We first show that by an application of Krylov-Safonov Hölder estimate, the exterior sphere condition can be weakened in some sense. Moreover, we shall use such a basic existence result for the proof of elliptic type comparison principle as well.

**Theorem 2.6.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain such that it satisfies an (uniform) exterior cone condition at each point on the boundary. Furthermore, we assume that there exists an exhaustion of \( \Omega \) by compact subdomains \( \Omega_k \nearrow \Omega \) such that each point of \( \partial \Omega_k \) can be touched by an exterior cone of fixed size independent of \( k \). Let \( g \) be a continuous function on \( \partial_p \Omega_T \). Then, there exists a unique solution to (1.1) in \( \Omega_T \) such that \( u = g \) on \( \Gamma_T \).

**Proof.** For every \( i, j = 1, \ldots, n \) and for a given \( \varepsilon > 0 \) we consider the regularized matrix

\[
a^\varepsilon_{ij}(\sigma) = \delta_{ij} + (p - 2)\frac{\sigma_i \sigma_j}{\varepsilon^2 + |\sigma|^2}, \quad i, j = 1, \ldots, n.
\]

For smooth boundary values \( g \) and given \( \varepsilon > 0 \), we consider the following Cauchy-Dirichlet problem

\[
\begin{cases}
u^\varepsilon = a^\varepsilon_{ij}(Du^\varepsilon)u^\varepsilon_{ij} & \text{in } \Omega \times (0, T) \\
u^\varepsilon(x, t) = g, & (x, t) \in \partial_p \Omega \times (0, T),
\end{cases}
\]

(2.8)
The existence of a solution $u^\varepsilon$ is guaranteed by Theorem 4.4 on p. 560 of [20]. Moreover, the following uniform ellipticity condition is satisfied independent of $\varepsilon > 0$,

$$\min\{1, p - 1\} \left| \xi \right|^2 \leq a_{ij}(\sigma)\xi_i\xi_j \leq \max\{1, p - 1\} \left| \xi \right|^2. \quad (2.9)$$

Now we consider the sequence of compactly contained cylinders $G_k = \Omega_k \times (1/k, T)$. The solution $u^\varepsilon$ is smooth up to $\partial G_k$. If we take any point $\xi = (x, t)$ on the lateral boundary of $\Omega_T$ (this means $t > 0$), then for $k$ large enough there exist points $\xi_k = (x_k, t) \in \partial G_k$ such that $\xi_k \to \xi$. Therefore, by applying Theorem 7.44 in [22], and by arguing as in the proof of Theorem 6.32 in [22], we conclude that there exist $R_0 > 0$ and $0 < \gamma < 1$, such that for all $r \leq R_0$,

$$\text{osc}_{Q_k,r \cap \Omega_k} u^\varepsilon \leq \gamma \text{osc}_{Q_k, 2r \cap \Omega_k} u^\varepsilon + \text{osc}_{\partial \Omega_k \cap Q_k, 2r} u^\varepsilon. \quad (2.10)$$

$R_0$ and $\gamma$ can be chosen independent of $k$ and $\varepsilon$ because of the uniform ellipticity condition (2.9) and the fact that each point of $\partial \Omega_k$ can be touched from outside by a cone of fixed size independent of $k$. We mention that in (2.10), $\text{osc} u^\varepsilon$ stands for oscillation of $u^\varepsilon$, whereas $\partial \Omega_k \cap Q_k$ is the box centered at $\xi_k$.

Theorem 7.44 in [22] is applicable on each $G_k$ since $u^\varepsilon$ is smooth up to $\partial G_k$. Moreover, since $u^\varepsilon$ is continuous up to $\partial \Omega_T$, we can let $k \to \infty$ in (2.10) and obtain

$$\text{osc}_{Q_r \cap \Omega} u^\varepsilon \leq \gamma \text{osc}_{Q_{2r} \cap \Omega} u^\varepsilon + \text{osc}_{\partial \Omega \cap Q_{2r}} g \leq \gamma \text{osc}_{Q_{2r} \cap \Omega} u^\varepsilon + Kr \quad (2.11)$$

where $Q_r$ is the box centered at $\xi$. The last inequality in (2.11) follows from the fact that $g$ is smooth and hence Lipschitz continuous on the parabolic boundary. For points on the bottom and corner of the cylinder $\Omega_T$, i.e., points $(x, 0) \in \Omega \times (t = 0)$, we can infer the oscillation estimate (2.11) with $Q_r = B(x, r) \times [0, r^2]$, and $Q_{2r} = B(x, r) \times [0, 4r^2]$, by a similar application of Theorem 7.44 in [22] and by an approximation argument as before. Uniform Hölder estimates independent of $\varepsilon$ at the parabolic boundary thus follow in a standard way from the inequality (2.11). For points away from the parabolic boundary, we have the classical interior Krylov-Safonov Hölder estimate ([19]) for non-divergence form uniformly parabolic PDE. In conclusion, the functions $u^\varepsilon$ are uniformly $\alpha$-Hölder continuous in $\Omega_T$ with an exponent $\alpha$ which is independent of $\varepsilon$.

Therefore, by Ascoli-Arzelà, for a subsequence $\varepsilon_k \to 0$ one has $u^{\varepsilon_k} \to u$ which is a solution to (1.1) and which takes the parabolic boundary values $g$. For continuous boundary values $g$, we take a sequence of smooth $g_k$’s which converges uniformly to $g$ on $\Gamma_T$. Let $u_k$ be the corresponding solutions. By the maximum modulus principle we have

$$\max_{\Omega_T} |u_k - u_j| = \max_{\Gamma_T} |g_k - g_j|. \quad (2.12)$$

Thus, $u_k \to u$. Since the equation (1.1) is stable under uniform limits, we again conclude that $u$ is a solution to (1.1) in $\Omega_T$ with parabolic boundary values $g$. \(\square\)

**Remark 2.7.** The boundary Hölder estimate for uniformly parabolic pde’s has been obtained by probabilistic arguments in [11]. The Hölder continuity at the boundary can be alternatively obtained using the barrier method, see [27].

Given $0 < \lambda \leq \Lambda$, we will denote by

$$[[\lambda, \Lambda]] = \{A \in S_\Lambda \mid \lambda I_n \leq A \leq \Lambda I_n\}.$$
We let $P^+_\lambda, \Lambda$ and $P^-_{\lambda, \Lambda}$ denote the maximal and minimal Pucci extremal operators corresponding to $\lambda, \Lambda$, i.e., for every $M \in \mathcal{S}_n$ we have
\[
P^+_\lambda, \Lambda(M) = P^+_\lambda, \Lambda(M, \lambda, \Lambda) = \lambda \sum_{e_i > 0} e_i + \sum_{e_i < 0} e_i, \\
P^-_{\lambda, \Lambda}(M) = P^-_{\lambda, \Lambda}(M, \lambda, \Lambda) = \lambda \sum_{e_i > 0} e_i + \sum_{e_i < 0} e_i,
\]
where $e_i = e_i(M), i = 1, \ldots, n$, indicate the eigenvalues of $M$. As is well known, $P^+_{\lambda, \Lambda}$ and $P^-_{\lambda, \Lambda}$ are uniformly elliptic fully nonlinear operators, and moreover
\[
P^+_{\lambda, \Lambda}(M) = \sup_{A \in \mathcal{S}(\lambda, \Lambda)} \{\text{trace}(AM)\}, \quad P^-_{\lambda, \Lambda}(M) = \inf_{A \in \mathcal{S}(\lambda, \Lambda)} \{\text{trace}(AM)\}. \tag{2.13}
\]
Consider the following differential inequalities which are to be understood in the usual viscosity sense.
\[
P^+_\lambda, \Lambda(D^2u) - u_t \geq 0 \geq P^-_{\lambda, \Lambda}(D^2u) - u_t. \tag{2.14}
\]

**Definition 2.8.** Given a bounded open set $D \subset \mathbb{R}^{n+1}$, the symbol $\mathcal{S}(\lambda, \Lambda)$ will indicate the class of functions $u \in C(D)$ which are viscosity solutions of (2.14), in the sense that $u$ is at the same time a viscosity subsolution of the fully nonlinear equation
\[
P^+_{\lambda, \Lambda} u - u_t = 0, \tag{2.15}
\]
and a viscosity supersolution of
\[
P^-_{\lambda, \Lambda} u - u_t = 0. \tag{2.16}
\]

Let $u$ be a solution to (1.1). Note that from Definition 2.1, at a maximum point of $u - \varphi$, we have that
\[
\varphi_t \leq a_{ij}(D\varphi)D_{ij}\varphi, \tag{2.17}
\]
where
\[
\min\{1, p - 1\} \mathbb{I}_n \leq [a_{ij}] \leq \max\{1, p - 1\} \mathbb{I}_n. \tag{2.18}
\]
The inequality (2.17) is seen to be true by considering the matrix $[a_{ij}]$ in both cases when $D\varphi$ vanishes and when it does not, see [1]. It is thus easily seen from Definition 2.1 and (2.18) that any viscosity solution to (1.1) belongs to the class $\mathcal{S}(\lambda, \Lambda)$, with $\lambda = \min\{1, p - 1\}$ and $\Lambda = \max\{1, p - 1\}$.

Hence by the results in [35], the following Hölder bounds for solutions in terms of their $L^\infty$ norms are a direct consequence of the discussion above and Theorem 4.19 in [35].

**Theorem 2.9.** Let $u$ be a solution to (1.1) in an open set $\Omega \subset \mathbb{R}^{n+1}$. Then, given a compact set $K$ in $\Omega$, there exists $\alpha > 0$ depending on $p, n$ and $K$, such that
\[
|u(x, t) - u(y, s)| \leq C|u||_{\infty, K, \Omega}(|x - y| + |t - s|^{1/2})^\alpha. \tag{2.19}
\]

**Remark 2.10.** Any solution of (1.1) is in fact Lipschitz continuous in space. This can be seen as follows. From the discussion above, we have that such a solution is in $C^0_{loc}$. Now, given any cylinder $B(x_0, r) \times (T_1, T_2)$ which is compactly contained in the region of the space-time where $u$ is defined, we solve the boundary value problem
\[
\begin{cases}
    u^\varepsilon_i = a_{ij}^\varepsilon(Du^\varepsilon)u^\varepsilon_{ij}, & \text{in } B(x_0, r) \times (T_1, T_2) \\
    u^\varepsilon(x, t) = u(x, t), & (x, t) \in \partial_p(B(x_0, r) \times (T_1, T_2)),
\end{cases} \tag{2.20}
\]
where $a_{ij}^\varepsilon$ has the same definition as before. For each $\varepsilon > 0$, let $u^\varepsilon$ be the solution to the Cauchy-Dirichlet problem (2.20). Since the boundary datum $u$ is Hölder continuous, by arguing as in the proof of Theorem 2.6 one obtains for $u^\varepsilon$ Hölder estimates up to the boundary independent of $\varepsilon$. Therefore by Ascoli-Arzelà and uniqueness, we have that (on a subsequence) $u^\varepsilon$ converge uniformly to $u$. The uniform spatial Lipschitz estimate independent of $\varepsilon$ for $u^\varepsilon$ follows by employing the Bernstein’s method, see for instance [4], or Theorem 11.3 in [22]. See also [15], where the Bernstein’s method has been successfully adapted to the case $p = \infty$. The Lipschitz continuity of $u$ in the space variable thus follows.

The uniform Hölder estimate (2.19) leads to the following convergence result.

**Lemma 2.11.** Given an open set $\Omega \in \mathbb{R}^{n+1}$ suppose that $u_k$ is a locally bounded sequence of solutions of (1.1) in $\Omega$. Then, there exists a subsequence that converges locally uniformly in $\Omega$ to a solution of (1.1).

**Proof.** By Theorem 2.9, $u_k$’s are equicontinuous. The conclusion then follows from Ascoli-Arzelà and from the fact that (1.1) is stable under uniform limits.

### 3. Perron process

In this section we introduce for equation (1.1) notions of generalized subsolution and supersolution analogous to those introduced by Riesz for the Laplacian. For the $p$-Laplacian $\Delta_p$, see chapter 7 in [12]. We then show that bounded generalized supersolutions are supersolutions in the viscosity sense, and vice-versa. A similar statement obviously holds for subsolutions.

**Definition 3.1.** A function $u : \Omega \to (-\infty, \infty]$ is called a generalized supersolution to (1.1) if

i) $u$ is lower semicontinuous;

ii) $u < +\infty$ in a dense set of $\Omega$;

iii) $u$ satisfies the following comparison principle: On each set of the form $G \times (t_1, t_2)$ with closure in $\Omega$, where $G \subset \mathbb{R}^n$ is an open set, if $h$ is a solution to (1.1) continuous up to the closure of $G \times (t_1, t_2)$, and $h \leq u$ on the parabolic boundary, then $h \leq u$ in $G \times (t_1, t_2)$.

Generalized subsolutions are defined analogously.

**Remark 3.2.** Note that it is quite immediate that if $u$ were a continuous supersolution in the viscosity sense to (1.1), then by the comparison principle Theorem 2.5, $u$ is also a generalized supersolution. However for the upper semicontinuous subsolutions and lower semicontinuous supersolutions in the sense of Definition 2.1, the comparison principle Theorem 2.5 doesn’t allow us to infer that the inequality “$v \leq u$” continues to be valid at $t = T$. The fact that a viscosity supersolution is a supersolution in the sense of Definition 3.1 is implied by the following proposition below.

**Proposition 3.3.** Let $u$ be a lower semicontinuous supersolution to $P_{\lambda, \Lambda} - \partial_t$ in the viscosity sense. Then,

$$u(x, t) = \lim_{r \to 0} \inf_{B(x, r) \times (t - r^2, t)} u. \quad (3.1)$$

**Proof.** Without loss of generality, we can assume that $(x, t) = (0, 0)$ and $u(0, 0) = 0$. If the conclusion is not true, we may assume that there exists $r_0 > 0$ such that

$$\inf_{B(0, r) \times (-r^2, 0)} u > \varepsilon, r \leq r_0 \quad (3.2)$$
Since \( u(0,0) = 0 \) and \( u \) is lower semicontinuous, we have that there exists a sequence \((r_n)\) such that \( r_0 > r_1 > r_2 > \ldots \) such that \( r_j < \frac{1}{2} r_{j-1} \) and
\[
\inf_{B(0,r_j) \times [0,r_j^2]} u > -2^{-j}
\] (3.3)
Denote \( Q_j = B(0,r_j) \times (-r_j^2, r_j^2) \), \( Q_j^+ = B(0,r_j) \times [0,r_j^2) \), \( Q_j^- = B(0,r_j) \times (-r_j^2, 0) \) and \( U_{j+1}^- = B(0,r_j) \times (-r_j^2, \frac{r_j^2}{2}) \). Define \( v_j = u + 2^{-j} \) which is a non-negative supersolution in \( Q_j \). The weak Harnack inequality implies
\[
\inf_{Q_{j+1}^-} u \leq \inf_{U_{j+1}^-} v_j \leq \left( \frac{1}{U_{j+1}^-} \int_{U_{j+1}^-} v_j^2 \right)^{\frac{1}{2}} \leq \inf_{Q_{j+1}^-} v_j \leq c 2^{-j}
\] (3.4)
For large enough \( j \)'s, this contradicts (3.2) and the conclusion follows.

**Remark 3.4.** The weak Harnack inequality for lower semicontinuous (and not just continuous) supersolutions can be found in the course notes of Imbert and Silvestre. (see Theorem 2.4.15 in [13])

As a corollary we have the following.

**Corollary 3.5.** Let \( u \) be a supersolution in the sense of Definition 2.1. Then \( u \) is a generalized supersolution. Analogous statement for subsolutions hold.

**Proof.** It suffices to check the property iii) of a generalized supersolution. Suppose \( h \) is a solution to (1.1) continuous up to the closure of \( G \times (t_1, t_2) \), and \( h \leq u \) on the parabolic boundary. Since \( u \) is lower-semicontinuous, the lower semicontinuous relaxation \( u_* \) of \( u \) with respect to \( G \times (t_1, t_2) \) satisfies
\[
u_* \geq u \text{ on } \partial_p(G \times (t_1, t_2))
\] (3.5)
and equals \( u \) in \( G \times (t_1, t_2) \). Therefore, by Theorem 2.5, \( h \leq u \) in \( G \times (t_1, t_2) \).

Now given any point \((x,t_2)\), by Proposition 3.3, we can take a sequence of points \((x_n,t_n) \in G \times (t_1, t_2) \) such that \((x_n,t_n) \to (x,t_2)\) and \( u(x_n,t_n) \to u(x,t_2)\). Then by the continuity of \( h \), we have that \( h(x,t_2) \leq u(x,t_2) \). The conclusion thus follows.

The proof for the converse is inspired by the ideas in [16].

**Theorem 3.6.** Let \( u \) be a bounded generalized supersolution (subsolution). Then \( u \) is a supersolution (subsolution) in the viscosity sense of Definition 2.1.

**Proof.** We only treat the case of a supersolution since that of a subsolution is analogous. Let \( u \) be a generalized supersolution, and suppose that \( u \) is not a viscosity supersolution. By the equivalent Definition 2.2, there exist \((x_0,t_0)\) and \( \phi \in C^2 \) such that \( u(x_0,t_0) = \phi(x_0,t_0) \), and \( u(x,t) > \phi(x,t) \) in \( Q_0 = B(x_0,r_0) \times [t_0 - r_0^2, t_0] \) for some \( r_0 > 0 \) such that one of the following cases occur:

Case 1: \( D\phi(x_0,t_0) \neq 0 \) and \(|D\phi|^2 \text{div}(|D\phi|^{p-2} D\phi) > \phi_t \) at \((x_0,t_0)\). In this case, for a small enough \( r_0 > 0 \), we have that \( \phi \) is a classical subsolution to (1.1) in \( Q_0 \). Since \( u \) is lower semicontinuous and \( \partial_t Q_0 \) is compact, there exists \( \delta > 0 \) such that \( \phi + \delta \leq u \) on \( \partial_t Q_0 \). Let \( h \) be the solution in \( Q_0 \) with parabolic boundary values \( \phi + \delta \). By property iii) of Definition 3.1, one has \( \phi + \delta \leq h \leq u \) in \( Q_0 \), which is a contradiction as \( u(x_0,t_0) = \phi(x_0,t_0) \). So \( u \) must be a viscosity supersolution in this case.
Case 2: $D\phi(x_0, t_0) = D^2\phi(x_0, t_0) = 0$ and $\phi_t(x_0, t_0) < 0$. As before, for small enough $t_0$, we would have that $\Lambda_{\min}(p-2)D^2\phi + \Delta\phi > \phi_t$ in $Q_0$. This implies that $\phi$ is a continuous viscosity subsolution in $Q_0$. Then, we can repeat the argument as in Case 1, and again reach a contradiction.

This implies the desired conclusion. \qed

Remark 3.7. In the above proof, $\Lambda_{\min}(A)$ denotes the smallest eigenvalue of a symmetric matrix $A$. From now on, unless otherwise specified, by a supersolution or subsolution, we will mean a generalized supersolution or subsolution.

Before presenting a version of an elliptic type comparison, we establish the following intermediate lemma.

Lemma 3.8. Suppose $K = \bigcup_{i=1}^M Q_i \times (t_{i1}, t_{i2})$ (finite union of boxes) such that for any pair $Q_i, Q_j, i \neq j$ in the collection, either they overlap in the interior or their closures are disjoint. Let $u$ be a bounded supersolution and $v$ be a bounded subsolution in a domain $\Omega$ which contains $\overline{K}$, and suppose that $v \leq u$ on $\partial K$. Then, $v \leq u$ in $K$.

Proof. Consider the sequence of times $s_1 < s_2 < s_3 < \ldots s_N$ in the increasing order such that $s_k = t_{ij}$ for some $i = 1, \ldots, M$, $j = 1, 2$. Consider the set $K \cap [s_1 < t < s_2]$ which is a disjoint union of sets of the form $U_k \times (s_1, s_2)$ where each $U_k$ is a (connected) domain (union of subcollection of cubes $Q_i$'s). Moreover, for each $k$ one has

$$v \leq u \text{ on } \partial_p(U_k \times (s_1, s_2)).$$

(3.6)

Now, each $U_k$ is a union of closed cubes such that any two cubes overlap in the interior or have disjoint closures. Therefore, one can see that each point on $\partial U_k$ can be touched by an exterior cone of aperture bounded from below. By slightly shrinking each cube $Q_i$ in the collection that constitute $U_k$, it can be seen that the same configuration can be retained. I.e., either the shrunk cubes overlap or their closures are disjoint. Therefore, $U_k$ can be exhausted by compact subdomains which satisfy the hypothesis of Theorem 2.6. For each $k$, since $v$ is upper semicontinuous and $u$ is lower semicontinuous, let $\theta$ be a continuous function such that $v \leq \theta \leq u$ on $\partial_p(U_k \times (s_1, s_2))$. The possibility of finding such a continuous function $\theta$ is guaranteed by the fact that $\partial_p(U_k \times (s_1, s_2))$ is compact. Since $U_k$ satisfies the hypothesis of Theorem 2.6, there exists a unique solution $h$ to equation (1.1) in $(U_k \times (s_1, s_2))$ with parabolic boundary values $\theta$. By the definition of supersolutions and subsolutions, one thus concludes

$$v \leq h \leq u \text{ in } U_k \times (s_1, s_2).$$

(3.7)

This gives $v \leq u$ in $K \cap [t \leq s_2]$. Now we proceed inductively to reach the same conclusion in $K \cap (s_i \leq t \leq s_{i+1}]$. The desired conclusion thus follows. \qed

Remark 3.9. We mention that in the proof of Lemma 3.8, from (3.6), Proposition 3.3 and the comparison principle Theorem 2.5, we cannot infer that $v \leq u$ at $t = s_2$ in (3.7). This depends on the fact that $u \geq v$ does not imply that $\lim\inf u \geq \lim\sup v$. That is why we solved the Cauchy-Dirichlet problem as an intermediate step in the proof.

We now state an elliptic type comparison theorem which is analogous to Lemma 4.3 in [17]. Nevertheless, in order to better clarify the arguments, we provide complete details.
Lemma 3.10. Suppose \( u \) is a supersolution and \( v \) is a subsolution in a bounded open set \( \Omega \in \mathbb{R}^{n+1} \). If \( u \) and \( v \) are bounded, and if at each point \( \xi_0 \) of \( \partial \Omega \), one has
\[
\lim_{\xi \to \xi_0} v(\xi) \leq \liminf_{\xi \to \xi_0} u(\xi),
\]
then \( v \leq u \) in \( \Omega \).

Proof. For each \( \varepsilon > 0 \), consider the set
\[
K_\varepsilon = \{ \xi \in \Omega \mid v(\xi) \geq u(\xi) + \varepsilon \},
\]
which, in view of (3.8), is a compact subset of \( \Omega \). Therefore, there is an open set \( D \), compactly contained in \( \Omega \) and containing \( K_\varepsilon \), such that \( D \) is a union of finitely many boxes. Moreover, since \( D \) is compactly contained and the number of boxes are finite, by enlarging the \( Q_j \)'s a bit if necessary, we can assume that \( D = \bigcup_{j=1}^M Q_i \times (t_1, t_2) \), which satisfies the hypothesis of Lemma 3.8. Now, we have \( v \leq u + \varepsilon \) on \( \partial D \), and so Lemma 3.8 implies \( v \leq u + \varepsilon \) in \( D \). This gives \( v \leq u + \varepsilon \) in \( \Omega \), and the desired conclusion follows by letting \( \varepsilon \to 0 \).

3.1. Parabolic modification. Let \( Q_T = Q \times (0, T) \) be a box with closure in \( \Omega \). Let \( u \) be a supersolution in \( \Omega \) and bounded on \( Q_T \). We let
\[
v = \sup \{ h \mid h \text{ is a solution to (1.1) which is continuous in } \overline{Q}_T \text{ and } h \leq u \text{ on } \partial_p Q_T \},
\]
and define the parabolic modification \( U \) of \( u \) as follows
\[
\begin{cases}
U = u & \text{in } \Omega - (Q \times (0, T)), \\
U = v & \text{in } Q \times (0, T).
\end{cases}
\]
We intend to show that \( U \) is a generalized supersolution. Since \( \overline{Q}_T \) is contained in \( \Omega \), for some \( \varepsilon > 0 \) we must have that \( Q_{T+\varepsilon} = Q \times (0, T + \varepsilon) \) is contained in \( \Omega \). Since \( u \) is a supersolution, it is thus clear that \( U \leq u \) in \( \Omega \). By taking \( \theta_j \nearrow u \) on \( \partial_p Q_{T+\varepsilon} \), we consider the corresponding solutions \( h_j \) to (1.1) such that \( h_j = \theta_j \) on \( \partial_p Q_{T+\varepsilon} \). From the comparison principle it follows that \( \{h_j\}_{j \in \mathbb{N}} \) is a bounded increasing sequence of solutions to (1.1), and we denote \( w = \lim h_j \) in \( Q_{T+\varepsilon} \).

We claim that \( w \) restricted to \( Q \times (0, T) \) equals \( v \). First, we note that by Lemma 2.11 \( w \) is a solution in \( Q_T \). Moreover, by the definition of \( v \) we have that \( w \leq v \). Conversely we have
\[
\liminf_{x \to y \in \partial_p Q_T} w(x) \geq \liminf_{x \to y \in \partial_p Q_T} h_j(x) = \theta_j(y),
\]
for all \( j \). Hence, by letting \( j \to \infty \) in the latter inequality, we find
\[
\liminf_{x \to y \in \partial_p Q_T} w(x) \geq u(y).
\]
For any \( h \) which is a solution to (1.1), such that \( h \leq u \) on \( \partial_p Q_T \), we thus have
\[
\liminf_{x \to y \in \partial_p Q_T} w(x) \geq h(y)
\]
By the comparison theorem we can thus conclude that \( w \geq h \) in \( Q_T \), and hence \( w \geq v \) which justifies the claim. Lower semicontinuity at each point on the parabolic boundary follows from (3.10). Furthermore, at each point of \( Q \times \{ t = T \} \) we have \( v \leq u \). Since \( v \) is Hölder continuous restricted to \( Q \times (0, T] \), the lower semicontinuity of \( U \) follows. In a standard way, by using the fact that \( v \) is a solution in \( Q_T \), (3.10) and Theorem 2.5, one verifies that \( U \) satisfies the comparison property iii) of a generalized supersolution. Therefore, \( U \) is a supersolution.
3.2. Perron method. In what follows, we let $\Omega$ be a bounded open set in $\mathbb{R}^{n+1}$.
Let $f : \partial \Omega \to \mathbb{R}$ be any bounded function. We now introduce the relevant notions
of upper and lower Perron solutions.
A function $u$ is said to belong to the upper class $\mathcal{U}_f$ if $u$ is bounded from below, it
is a supersolution in $\Omega$, and satisfies $\lim_{\eta \to \xi} \inf_u(\eta) \geq f(\xi)$ at each $\xi \in \partial \Omega$. The lower
class $\mathcal{L}_f$ is defined analogously. Since $f$ is bounded, large positive constants belong
to $\mathcal{U}_f$ and large negative constants belong to $\mathcal{L}_f$. Thus, the classes $\mathcal{U}_f$ and $\mathcal{L}_f$ are
non-empty.

**Definition 3.11.** We define the upper solution $\overline{\Pi}_f$ relative to $f$ as
\[
\overline{\Pi}_f = \inf \{ u \mid u \in \mathcal{U}_f \}. \tag{3.12}
\]

The lower solution $\underline{\Pi}_f$ is defined analogously. Notice that $\min \{ u, \| f \|_\infty \} \in \mathcal{U}_f$ if
$u \in \mathcal{U}_f$, and $\max \{ v, -\| f \|_\infty \} \in \mathcal{L}_f$ if $v \in \mathcal{L}_f$. Since $f$ is bounded, in (3.12) we can
take the infimum over bounded $u$’s in $\mathcal{U}_f$. It thus follows from Lemma 3.10 that
$\underline{\Pi}_f \leq \overline{\Pi}_f$. Moreover, $\overline{\Pi}_f, \underline{\Pi}_f$ are bounded by the same constants as $f$.

By using the parabolic modification, it is quite classical to show that $\overline{\Pi}_f$ and $\underline{\Pi}_f$
are solutions to (1.1). We nevertheless provide the details.

**Theorem 3.12.** The upper Perron solution $\overline{\Pi}_f$ and the lower Perron solution $\underline{\Pi}_f$
are solutions to (1.1) in $\Omega$.

**Proof.** Fix a box $Q_T$ with closure in $\Omega$. Select a countable dense subset $\{ y_j \}_{j \in \mathbb{N}}$ of
$Q_T$. For each $j \in \mathbb{N}$ we choose a sequence of functions $u_{i,j} \in \mathcal{U}_f$ such that
\[
\lim_{i \to \infty} u_{i,j}(y_j) = \overline{\Pi}_f(y_j). \tag{3.13}
\]

As previously said, we may assume that the $u_{i,j}$’s are bounded. Replacing $u_{i,j+1}$
by $\min \{ u_{i,j}, u_{i,j+1} \}$, we have that
\[
\lim_{i \to \infty} u_{i,j}(y_k) = \overline{\Pi}_f(y_k), \quad k = 1, 2, ..., j. \tag{3.14}
\]

By taking the parabolic modification $U_{i,j}$ of $u_{i,j}$, it holds that in $Q_T$
$\overline{\Pi}_f \leq U_{i,j} \leq u_{i,j}$. \tag{3.15}

From Lemma 2.11, by first letting $i \to \infty$ and then $j \to \infty$, we can assume that
after possibly passing to a subsequence, $\lim_{j \to \infty} U_{i,j} = v$, where $v$ is a solution of
(1.1) in $Q_T$. We note that from the construction,
$\overline{\Pi}_f \leq v$ in $Q_T$. \tag{3.16}

On the other hand, $v = \overline{\Pi}_f$ in the dense subset chosen above. If now $u$ is any
function in $\mathcal{U}_f$, then its parabolic modification $U$ in $Q_T$ is not greater than $u$.
Moreover, since $v = \overline{\Pi}_f$ in a dense subset, and $U \geq \overline{\Pi}_f$ by definition, by continuity
we have $u \geq U \geq v$ in $Q_T$. Since this is true for all $u \in \mathcal{U}_f$, we conclude that
$\overline{\Pi}_f \geq v$ in $Q_T$, and therefore they are equal. This proves that $\overline{\Pi}_f$ is a solution of
(1.1). The proof for $\underline{\Pi}_f$ is analogous. \qed

**Remark 3.13.** Let $\Omega = Q \times (0, T)$, and suppose that $f : \partial \Omega \to \mathbb{R}$ is continuous.
Then, the upper and lower Perron solutions coincide with the solution as in Theorem 2.6. As one would anticipate, the value of $f$ on the top cap of the cylinder does not have any influence on the solution. This is so since if $h$ is the solution to (1.1) as in Theorem 2.6 corresponding to boundary values $f$, then in a routine way one can
show that the functions \( h + \frac{\varepsilon}{T - t} \) belong to \( U_f \) for \( \varepsilon > 0 \), and to \( L_f \) for \( \varepsilon < 0 \). By letting \( \varepsilon \to 0 \), one can see that \( H_f = \overline{H_{-f}} \).

4. Barriers and boundary regularity. We begin this section by introducing the classical notion of a barrier. In what follows \( \Omega \) will denote a bounded open set in \( \mathbb{R}^{n+1} \).

**Definition 4.1.** A function \( w \) is a barrier in \( \Omega \) at the point \( \xi_0 \in \partial \Omega \) if:

i) \( w \) is a positive generalized supersolution in \( \Omega \);

ii) \( \lim_{\zeta \to \xi_0} w(\zeta) > 0 \) if \( \xi \in \partial \Omega \) and \( \xi \neq \xi_0 \);

iii) \( \lim_{\zeta \to \xi_0} w(\zeta) = 0 \).

As for heat equation, the existence of a barrier is a completely local question. Suppose there exists a neighborhood \( N \) of \( \xi_0 = (x_0, t_0) \) such that a barrier at \( \xi_0 \) can be found in \( N \cap \Omega \). Then, we can define a barrier in \( \Omega \) as follows. Let \( Q = B_r(x_0) \times (t_0 - r, t_0 + r) \) be compactly contained in \( N \). Let \( m = \inf_{N \setminus Q} w \).

Without loss of generality, we can assume that \( m > 0 \). If we define

\[
\begin{cases}
  v = \min\{w, m\} \text{ in } \Omega \cap Q, \\
  v = m \text{ in } \Omega \setminus Q,
\end{cases}
\]

(4.1)

then it is easy to see that \( v \) is a barrier in \( \Omega \) at \( \xi_0 \).

**Theorem 4.2.** Suppose \( f : \partial \Omega \to \mathbb{R} \) is bounded and continuous at \( \xi_0 \in \partial \Omega \). If there exists a barrier in \( \Omega \) at \( \xi_0 \), then

\[
\lim_{\xi \to \xi_0} H_f(\xi) = f(\xi_0) = \lim_{\xi \to \xi_0} H_f(\xi).
\]

**Proof.** Given \( \varepsilon > 0 \) there exists \( \delta = \delta(\xi_0, \varepsilon) > 0 \) such that \( |f(\xi) - f(\xi_0)| \leq \varepsilon \), when \( |\xi - \xi_0| \leq \delta \). Because of lower semicontinuity, we can assert that there exists \( M > 0 \) such that

\[
Mw(\xi) > 2\sup |f| \text{ for } |\xi - \xi_0| > \delta.
\]

Then, the function \( Mw + \varepsilon + f(\xi_0) \) belongs to the upper class \( U_f \), whereas \( -Mw - \varepsilon + f(\xi_0) \) belongs to the lower class \( L_f \). Such functions, have the respective limits \( f(\xi_0) + \varepsilon \) and \( f(\xi_0) - \varepsilon \) at \( \xi_0 \). We thus have

\[
\limsup_{\xi \to \xi_0} H_f(\xi) \leq f(\xi_0) + \varepsilon,
\]

and

\[
\liminf_{\xi \to \xi_0} H_f(\xi) \geq \liminf_{\xi \to \xi_0} H_f(\xi) \geq f(\xi_0) - \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, the conclusion follows.

**Definition 4.3.** A boundary point \( \xi_0 \) is called regular if \( \lim_{\xi \to \xi_0} H_f = f(\xi_0) \) whenever \( f : \partial \Omega \to \mathbb{R} \) is continuous.

**Remark 4.4.** Because \( \overline{H}_f = -\overline{H}_{-f} \), we could replace \( \overline{H}_f \) by \( \overline{H}_f \) in the above definition.

We now have the classical characterization of regular boundary points in terms of barriers.

**Theorem 4.5.** A boundary point \( \xi_0 \) is regular if and only if there is a barrier at \( \xi_0 \).
Proof. The sufficiency has been established in the previous theorem. For the necessity, let \( x_0, t_0 \) and define
\[
ψ(x, t) = \frac{1}{2}|x - x_0|^2 + ε(t - t_0)^2,
\]
where \( ε \) is chosen such that \( 0 < 2ε \text{ diam}(Ω) < n + p - 2 \).
Now \( D_tψ = (x - x_0)_t \), thus \( Dψ \) vanishes precisely when \( x = x_0 \), and \( D_{ij}ψ = δ_{ij} \).
Moreover,
\[
\left( δ_{ij} + (p - 2) \frac{D_iψD_jψ}{|Dψ|^2} \right) D_{ij}ψ = n + p - 2
\]
in \( Ω \setminus \{(x_0, t) \mid (x_0, t) ∈ Ω \} \). We now consider points \((x_k, t) ∈ Ω \) such that \( x_k → x_0 \).
Define
\[
a_k = \frac{Dψ(x_kt)}{|Dψ(x_kt)|}.
\]
After possibly passing to a subsequence, we have that \( a_k → a_{x_0,t} \) such that \( |a_{x_0,t}| = 1 \). Therefore, \((x_0, t) \) there exists a unit vector \( a = a_{x_0,t} \) such that
\[
(δ_{ij} + (p - 2)a_ia_j)D_{ij}ψ = n + p - 2.
\]
Moreover,
\[
|ψ_t| = 2ε|t - t_0| \leq 2ε \text{ diam}(Ω) \leq n + p - 2.
\]
By the choice of \( ε \) we can check that \( ψ \) is a continuous subsolution in \( Ω \). Since \( x_0 \) is assumed to be regular, \( w = H_ψ \) satisfies
\[
\lim_{ξ → x_0} w(ξ) = ψ(ξ₀) = 0.
\]
Since \( ψ ∈ L_ψ \), we have that \( w ≥ ψ \) by definition. In conclusion, \( w \) is a barrier at \( ξ₀ \).

Definition 4.6. A point \((x_0, t_0) ∈ ∂Ω \) is called an initial point if for each \((x, t) ∈ Ω \) one has \( t ≥ t_0 \).

We next show that an initial point is always regular.

Proposition 4.7. Let \( x_0, t_0 \) be an initial point. Then, \( x_0 \) is regular.
Proof. Let \( 2Ω \) be the sphere of radius \( R = t_0 - t_1 \) centered at \((x_0, t_1) \) which touches \( Ω \) precisely at \( ξ₀ \) since \( x_0 \) is an initial point. Let \( λ = \min\{1, p - 1\} \) and \( Λ = \max\{1, p - 1\} \). Also, let \( w \) be defined by
\[
w = e^{-R^2} - e^{-r^2},
\]
where
\[
r = r(x, t) = \left( |x - x_0|^2 + |t - t_1|^2 \right)^{1/2}.
\]
We have \( w(ξ₀) = 0 \), and \( w > 0 \) in the rest of \( Ω \). After a computation, we find at points \((x, t) \) such that \( x ≠ x_0 \),
\[
|Dw|^2 - p \text{div}(|Dw|^{p-2} Dw) ≤ 2e^{-r^2} nΛ.
\]
Moreover,
\[
w_t = 2e^{-r^2} (t - t_1) ≥ 2e^{-r^2} (t_0 - t_1).
\]
Therefore, if \( t_0 - t_1 ≥ 2Λn \), we have that \( w \) is a classical supersolution in points \((x, t) ∈ Ω \) such that \( x ≠ x_0 \). The fact that \( w \) is a supersolution in the viscosity sense in the whole of \( Ω \) follows from a compactness argument as employed in the proof of Theorem 4.5. Therefore it follows that \( w \) is a barrier at \( ξ₀ \).
Before stating the next theorem, we introduce the following notations:

\[ \Omega_- = \Omega \cap \{ t < t_0 \}, \quad \Omega_+ = \Omega \cap \{ t > t_0 \}. \]

The next result is a crucial ingredient in the proof of the main result in this paper. It states that the regularity at a point \( \xi_0 = (x_0, t_0) \in \partial \Omega \) is completely determined by times \( t < t_0 \). This is the analogue of Theorem 6.4 in [17] for the parabolic \( p \)-Laplacian.

**Theorem 4.8.** Let \( \xi_0 = (x_0, t_0) \in \partial \Omega \) such that \( \xi_0 \in \partial \Omega_- \). Then, \( \xi_0 \) is a regular point for the domain \( \Omega \) if and only if it is a regular boundary point for \( \Omega_- \).

**Proof.** The necessity of the condition is trivial: since \( \Omega_- \subset \Omega \), any barrier for \( \Omega \) is also a barrier for \( \Omega_- \).

For the sufficiency, suppose \( \xi_0 \) is a regular boundary point of \( \Omega_- \). Given the subsolution \( \psi \) defined in the proof of Theorem 4.5, consider the lower Perron solution in \( \Omega \) relative to \( \psi \), \( H = \frac{H^\Omega_{\partial \psi}}{\partial \psi} \). Clearly, it satisfies \( H \geq \psi \). One has

\[ \lim_{\xi \to \xi_0, \xi \in \Omega_-} H(\xi) = \psi(\xi_0) = 0. \]  \hspace{1cm} (4.2)

One way to see the validity of (4.2) is as follows. Let \( u \) be any bounded function in the upper class for \( \psi \) in \( \Omega_- \). For \( \varepsilon > 0 \), consider the function

\[ \left\{ \begin{array}{ll} v = \sup u & \text{if } t > t_0 - \varepsilon, \\ v = u(x, t) & \text{if } t \leq t_0 - \varepsilon. \end{array} \right. \]

Then, \( v \) is in the upper class \( \mathcal{U}_\psi \). By the arbitrariness of \( \varepsilon \), it follows that the restriction of \( H \) to \( \Omega_- \) coincides with the upper Perron solution of \( \psi \) in \( \Omega_- \). We conclude that (4.2) follows from the fact that \( \xi_0 \) is a regular boundary point of \( \Omega_- \).

Next, we claim that \( H \) is also a barrier at \( \xi_0 \) in \( \Omega \). Since \( H \geq \psi \) and \( \psi > 0 \) when \( \xi \neq \xi_0 \), we only need to show that

\[ \lim_{\zeta \to \xi_0, \zeta \in \Omega} H(\zeta) = 0. \]  \hspace{1cm} (4.3)

If \( \xi_0 \) does not belong to \( \partial \Omega_+ \), there is nothing to prove. Assume therefore that \( \xi_0 \in \partial \Omega_+ \), and let

\[ \left\{ \begin{array}{ll} \phi = \psi & \text{in } \partial \Omega, \\ \phi = H & \text{in } \Omega \cap \partial \Omega_+. \end{array} \right. \]

The restriction of \( \phi \) to \( \partial \Omega_+ \) is continuous at \( \xi_0 \). In order to see this, observe that if \( (x_k, t_k) \in \partial \Omega \to \xi_0 \), then the fact that \( \lim_{k \to \infty} \phi(x_k, t_k) = \phi(\xi_0) \) follows from the continuity of \( \psi \). When \( (x_k, t_k) \in \Omega \cap \partial \Omega_+ \), then we must have \( t_k = t_0 \) for every \( k \in \mathbb{N} \). Since \( (x_k, t_0) \in \Omega \), by the continuity of \( H \) in \( \Omega \) we see that there exist points \( (x_k, s_k) \in \Omega_- \) such that \( |s_k - t_0| \leq \frac{1}{k} \) and

\[ |H(x_k, t_0) - H(x_k, s_k)| \leq \frac{1}{k}. \]  \hspace{1cm} (4.4)

Therefore, from (4.2) it follows that \( H(x_k, s_k) \to H(\xi_0) = 0. \) This proves the claim that the restriction of \( \phi \) to \( \partial \Omega_+ \) is continuous at \( \xi_0 \).

Returning to the proof of (4.3) above, at this point we let \( h = \frac{H^\Omega_{\partial \phi}}{\partial \phi} \), i.e., \( h \) is the lower Perron solution in \( \Omega_+ \) relative to \( \phi \). We claim that \( h = H \) in \( \Omega_+ \). To see this claim we notice that if \( u \in \mathcal{L}_\phi(\Omega) \), then \( u \) restricted to \( \Omega_+ \) is in \( \mathcal{L}_\phi(\Omega_+) \). This
implies that $H \leq h$ in $\Omega_+$. For the reverse inequality, let $v \in L_\phi(\Omega_+)$. Without loss of generality we can assume that $v$ be bounded. Then, for each $\xi \in \partial \Omega_+$ one has

$$\limsup_{\zeta \to \xi} v(\zeta) \leq \phi(\xi) \leq \liminf_{\zeta \to \xi} H(\zeta). \quad (4.5)$$

We note that (4.5) holds since $H = \phi$ in $\Omega \cap \partial \Omega_+$, and $H \geq \psi$ in $\Omega$. By the comparison principle we conclude that $v \leq H$. This implies the reverse inequality and establishes $h = H$. By Proposition 4.7 we know that any initial point is regular. Therefore,

$$\lim_{\zeta \to \xi_0, \zeta \in \Omega} H(\zeta) = \phi(\xi_0) = 0. \quad (4.6)$$

We now take points $(x_k, t_0) \in \Omega$ such that $(x_k, t_0) \to (x_0, t_0) = \xi_0$. By the continuity of $H$, for each $k \in \mathbb{N}$, we can find $\zeta_k$ in $\Omega_-$ such that

$$\begin{cases} |H(x_k, t_0) - H(\zeta_k)| \leq \frac{1}{k}, \\ |\zeta_k - (x_k, t_0)| \leq \varepsilon(k) \to 0 \text{ as } k \to \infty. \end{cases} \quad (4.7)$$

Hence,

$$H(x_k, t_0) \to H(\xi_0) = 0. \quad (4.8)$$

From (4.2), (4.7) and (4.8), it follows that (4.3) holds, thus completing the proof. □

The equation (1.1) is an evolution associated with the $p$-Laplacian. Regular points for $p$-Laplacian have a geometric characterization in terms of the Wiener-Mazya criterion stated below. We recall from the potential theory for the $p$-Laplacian that, given a bounded open set $G \subset \mathbb{R}^n$, a point $x_0 \in \partial G$ is regular for the $p$-Laplacian if

$$h_f(x_0) = f(x_0),$$

whenever $f \in W^{1,p}(G) \cap C(\overline{G})$, and $h_f$ is the Perron solution for the $p$-Laplacian, see [12], [18]. Note that such a $f$ is resolutive, i.e., the upper and lower Perron solution coincide, see Corollary 9.29 in [12]. We recall that a condenser is a couple $(K, D)$, where $K$ is a compact set, $D$ is an open set, and $K \subset D \subset \mathbb{R}^n$. Given a condenser $(K, D)$ we denote by

$$\mathcal{F}_D(K) = \{ \phi \in C_0^\infty(D) \mid 0 \leq \phi \leq 1, \phi \equiv 1 \text{ in a neighborhood of } K \}. \quad (4.9)$$

The capacity of a condenser $(K, D)$ is defined as

$$\text{cap}_p(K, D) = \inf_{\phi \in \mathcal{F}_D(K)} \int_D |D\phi|^p dx.$$

The Wiener-Mazya criterion states that a boundary point $x_0$ is regular for the $p$-Laplacian if and only if

$$\int_0^1 \left[ \text{cap}_p(\overline{B(x_0, t)} \setminus G, B(x_0, 2t)) \right]^{\frac{1}{p-1}} \frac{dt}{t} = \infty, \quad (4.9)$$

see [26], [9] for the sufficiency, and [18] for the necessity.

Before stating our next lemma, we recall the notion of viscosity supersolution for the $p$-Laplacian as in [16].

**Definition 4.9.** Let $G$ be an open set in $\mathbb{R}^n$. A function $u : G \to (-\infty, \infty]$ is called a viscosity supersolution to the $p$-Laplace equation in $G$ if:

i) $u$ is lower semicontinuous;

ii) $u$ is finite in a dense subset of $G$;
iii) whenever \( x_0 \in G \) and \( \phi \in C^2(G) \) are such that \( u(x_0) = \phi(x_0) \) and \( u(x) > \phi(x) \) for \( x \neq x_0 \), and \( D\phi(x_0) \neq 0 \), we have \( \text{div}(|D\phi|^{p-2}D\phi) \leq 0 \).

In what follows we will need the notion of \( p \)-superharmonic function. For the latter we refer to Chapter 7 in [12]. The following, which is the main result in [16], is very useful in our situation.

**Theorem 4.10.** A function \( u \) is a viscosity supersolution to the \( p \)-Laplace equation if and only if \( u \) is a \( p \)-superharmonic.

**Lemma 4.11.** Let \( u \) be a bounded \( p \)-superharmonic function where \( p \geq 2 \). Then, \( v(x,t) = u(x) \) is a supersolution to (1.1).

**Proof.** Clearly, \( v \) is lower semicontinuous. Let \( \phi \) be a test function such that \( v - \phi \) has a local minimum at \((x_0,t_0)\). First we observe that, since \( v \) is time-independent, for \( t \leq t_0 \) one has

\[
v(x_0,t_0) - \phi(x_0,t_0) \leq v(x_0,t) - \phi(x_0,t),
\]

and thus \( \phi_t(x_0,t_0) \geq 0 \). Next, we show that \( v \) is a viscosity supersolution in the sense of Definition 2.2. There are 2 cases:

Case 1: \( D\phi(x_0,t_0) \neq 0 \). By arguing with respect to the \( x \) variable, from Theorem 4.10, we conclude that \( |D\phi|^{2-p}\text{div}(|D\phi|^{p-2}D\phi) \leq 0 \leq \phi_t(x_0,t_0) \).

Case 2: \( D\phi(x_0,t_0) = D^2\phi(x_0,t_0) = 0 \). In this case, since \( \phi_t(x_0,t_0) \geq 0 \), (2.6) in Definition 2.2 is satisfied.

The conclusion thus follows. \( \square \)

We are ready to prove the main result of our paper.

**Proof of Theorem 1.1.** By Theorem 4.8 we may assume that \( t_0 < T \). Suppose first that \( \xi_0 \) is regular for the equation (1.1). Let \( \phi \) be a function in \( W^{1,p}(G) \cap C(\overline{G}) \), and denote by \( h_\phi \) the \( p \)-harmonic Perron solution with boundary values \( \phi \). We define \( f \) by setting

\[
\begin{cases}
  f(x,t) = \phi(x) & \text{if } 0 < t < T, \\
  f(x,t) = h_\phi & \text{if } t = 0, \text{ or } T.
\end{cases}
\]

We thus have that \( f \) is bounded, and continuous at \( \xi_0 \). Moreover, if \( u \) is a \( p \)-superharmonic function from the elliptic upper class of \( \phi \) (without loss of generality we can assume that \( u \) be bounded), then from Lemma 4.11 we conclude that \( v(x,t) = u \in \mathcal{U}_f \). Thus,

\[
\liminf_{(x,t) \to \xi_0} \overline{f}(x,t) \leq \liminf_{x \to x_0} h_\phi(x). \tag{4.10}
\]

Since by assumption,

\[
\lim_{(x,t) \to \xi_0} \overline{f}(x,t) = f(\xi_0), \tag{4.11}
\]

we find

\[
\lim_{x \to x_0} h_\phi(x) \geq \phi(x_0). \tag{4.12}
\]

Consequently,

\[
\phi(x_0) \leq \liminf_{x \to x_0} h_\phi(x) \leq \limsup_{x \to x_0} h_\phi(x) = -\liminf_{x \to x_0} h_{-\phi}(x) \leq \phi(x_0), \tag{4.13}
\]

and therefore \( x_0 \) is regular for the \( p \)-Laplacian. Note that in (4.13), we have crucially used the resolutivity of \( \phi \).
Conversely, suppose that \( x_0 \in \partial G \) is a regular boundary point for the p-Laplacian. To prove that \( (x_0,t) \) is regular for (1.1) we construct a barrier at \( \xi_0 \) as follows: Let \( g = |x-x_0| \). Consider the following generalized Dirichlet problem

\[
\begin{aligned}
\text{div}(\varepsilon^2 + |Du|^2)^{p/2-1}Du &= -(\varepsilon^2 + |Du|^2)^{p/2-1} \\
-u + g &\in W^{1,p}_0(G).
\end{aligned}
\] (4.14)

The operator in (4.14) satisfies the structural conditions (6.1 - 6.3) in [24] uniformly in \( \varepsilon \). In order to see that the strict monotone condition of the principal part (i.e. condition 6.1 in [24]) is satisfied, we observe that

\[
\langle (\varepsilon^2 + |b|^2)^{p/2-1}b - (\varepsilon^2 + |a|^2)^{p/2-1}a, b-a \rangle \geq \frac{\langle (\varepsilon^2 + |b|^2)^{p/2-1} + (\varepsilon^2 + |a|^2)^{p/2-1} \rangle}{2} |b-a|^2
\] (4.15)

Moreover, it can be seen quite easily that (6.7) in [24] is satisfied uniformly in \( \varepsilon \) with \( q = \frac{p}{2} \). Therefore by Theorem 6.12 in [24] (theory of pseudomonotone operators), we have the existence of solution \( u^\varepsilon \) to the generalized Dirichlet problem (4.14) for each \( \varepsilon > 0 \). By Theorem 6.3 in [24], we have that \( u^\varepsilon \) are uniformly bounded in \( W^{1,p}(G) \). Then, from [30], we infer that \( u^\varepsilon \) have uniform Hölder norms in \( \varepsilon \) on any compact subset of \( G \). Consequently by Theorem 1 in [33], we conclude that \( u^\varepsilon \)'s are uniformly bounded in \( C^{1,\alpha}_{loc} \). The elliptic theory ensures that \( u^\varepsilon \) is a classical solution to the equation in (4.14) for each \( \varepsilon > 0 \) (see [21] Chapter 4). Therefore, \( u^\varepsilon \) solves

\[
\left( \delta_{ij} + (p-2) \frac{Du \cdot D_j u^\varepsilon}{|Du|^2} \right) D_{ij} u^\varepsilon = -1.
\] (4.16)

For a subsequence \( \varepsilon_k \to 0 \), we have weak convergence of \( u^\varepsilon_k \) in \( W^{1,p}(G) \) and, thanks to the uniform \( C^{1,\alpha}_{loc} \) estimates in \( \varepsilon_k \), uniform convergence of \( u^\varepsilon_k \) with its first derivatives on compact subsets of \( G \). Consequently, by passing to the limit we conclude in a standard way that the limit \( u \) is a weak solution to

\[
\text{div}(|Du|^{p-2}Du) = -|Du|^{p-2}
\] (4.17)

such that \( u - g \in W^{1,p}_0(G) \).

Theorem 2.5 in [9] implies that the Wiener-Mazya criterion (4.9) is a sufficient criterion for the regularity of a boundary point for the equation (4.17). Therefore, we conclude that

\[
\lim_{x \to x_0} u(x) = g(x_0) = 0
\] (4.18)

Since \( u \) is \( p \)-superharmonic and \( g \) is \( p \)-subharmonic (which can be checked by a direct computation), from Lemma 3.18 in [12] we see that \( u(x) \geq |x-x_0| \). Moreover, since \( u^\varepsilon_k \) converges to \( u \) uniformly on compact subsets, it follows that \( u \) is a viscosity solution in the sense of [2] to

\[
\left( \delta_{ij} + (p-2) \frac{Du \cdot D_j u}{|Du|^2} \right) D_{ij} u = -1.
\]

Define

\[ v(x,t) = u(x) + (t_0 - t). \]

Then, \( v \) is a viscosity solution to (1.1) in \( G_{t_0} = G \times (0,t_0) \). Moreover, from (4.18) it follows that \( v \) is a barrier for \( \xi_0 \) in \( G_{t_0} \). Therefore, \( \xi_0 \) is a regular boundary point for \( G_{t_0} \). Theorem 4.8 now ensures that \( \xi_0 \) is regular for \( G \times (0,T) \).
Remark 4.12. The reason for which in the proof of Theorem 1.1 we took a detour by first solving the approximating problems (4.14) is because, in general, a weak solution to
\[
\text{div}(|Du|^{p-2} Du) = -|Du|^{p-2} \tag{4.19}
\]
is not a viscosity solution to
\[
(\delta_{ij} + (p - 2) \frac{D_i h D_j h}{|D h|^2}) D_{ij} h = -1. \tag{4.20}
\]
This can be quite easily seen. For instance, in the case \( p > 2 \), any constant is a solution to (4.19) but not a viscosity solution to (4.20).

We conclude our discussion with the following corollary.

Corollary 4.13. Let \( G \) be a regular domain for the \( p \)-Laplacian, \( p \geq 2 \). Then, the Cauchy-Dirichlet problem corresponding to (1.1) is solvable in \( G \times (0, T) \) for continuous boundary values.

Proof. Note that the regularity of the points of the form \((x, 0)\) follow from Proposition 4.7. The rest follows from Theorem 1.1. Continuity at \( t = T \) can be ensured by solving the Cauchy-Dirichlet problem in \( G \times (0, T + \varepsilon) \) for some \( \varepsilon > 0 \). \( \square \)

Remark 4.14. The case \( 1 < p < 2 \) of Theorem 1.1 remains open.

5. Appendix.

5.1. Large time behavior. Let \( \Omega \) be a sufficiently smooth (for instance, \( C^2 \)) bounded open set in \( \mathbb{R}^n \), and let \( g \in C^2(\Omega) \), and consider the Cauchy-Dirichlet problem corresponding to the equation (1.1), i.e.,
\[
\begin{cases}
\text{div}(|Du|^{p-2} Du) = |Du|^{p-2} u_t \quad \text{in } \Omega \times (0, \infty), \\
u(x, t) = g(x), \quad (x, t) \in \partial \mu(\Omega \times (0, \infty)).
\end{cases} \tag{5.1}
\]
One has the following result on the large-time behavior for

Theorem 5.1. Let \( u \) be the unique solution to the Cauchy-Dirichlet problem (5.1). Then, \( u(x, t) \to v(x) \) as \( t \to \infty \) where \( v \) is the unique \( p \)-harmonic function with boundary values \( g \).

Proof. See Theorem 7.6 in [1] for the case \( 1 < p \leq 2 \) and [14] for the case \( p \geq 2 \). \( \square \)

Following the result in [14] in the case \( p \geq 2 \), we can show that even the energy converges as \( t \to \infty \).

Lemma 5.2. Let \( p \geq 2 \). With the same assumptions as in Theorem 5.1, one has
for \( t \to \infty \)
\[
\int_\Omega |Du(x, t)|^p dx \to \int_\Omega |Dv(x)|^p dx. \tag{5.2}
\]
Proof. Since the solution \( u \) is obtained as the limit of solutions \( u^\varepsilon \) of the regularized problems (2.20), from standard barrier type arguments for time-difference quotients, Lemma 10.1 and Theorem 11.1 in [22], one has that
\[
||u^\varepsilon||_{L^\infty(\Omega \times (0, \infty))} \leq K||D^2 g||_{L^\infty(\Omega)} \tag{5.3}
\]
\[
||Du^\varepsilon||_{L^\infty(\Omega \times (0, \infty))} \leq C(\Omega, n, p, ||g||_{C^2(\Omega)}). \tag{5.4}
\]
Now, at each time level \( t \), \( u^\varepsilon(\cdot, t) \) solves the following elliptic pde
\[
\text{div}(\varepsilon + |Du^\varepsilon|^{p/2-1} Du^\varepsilon) = u^\varepsilon_t(\varepsilon + |Du^\varepsilon|^2)^{p/2-1} \tag{5.5}
\]
In the case \( p \geq 2 \), Theorem 1 in [23] now implies that \( u(\cdot, t) \) has uniform \( C^{1,\alpha}(\Omega) \) bounds independent of \( t \). Since \( u(\cdot, t) \to v \), by the theorem of Ascoli-Arzelà we obtain the desired conclusion.

**Remark 5.3.** The convergence of the energy for the case \( p < 2 \) remains an open question at the moment. The rate of convergence of \( u \) and its corresponding \( p \)-energy is also an interesting question in this direction. For the case \( p \geq 2 \), it has been shown in [14] that the rate of convergence is polynomial in time. However, in the same paper it has been mentioned that an exponential rate of convergence is expected to be true.

5.2. Tychonoff type maximum principle for the normalized \( \infty \)-Laplacian evolution. The corresponding equation is

\[
\frac{u_t}{|Du|^2} D_{ij} u = \Delta_{\infty}^N u \tag{5.6}
\]

A solution to (5.6) is to be interpreted in the viscosity sense, see [15].

**Theorem 5.4.** Let \( u \) be a viscosity solution of (5.6) in \( \mathbb{R}^n \times [0, T] \). Assume further that for some \( A, a > 0 \) the following growth estimate is satisfied:

\[
u(x, t) \leq Ae^{a|x|^2}, \quad \text{for every } (x, t) \in \mathbb{R}^n \times [0, \infty).
\]

Then,

\[
\sup_{\mathbb{R}^n \times [0, T]} u = \sup_{\mathbb{R}^n} u(x, 0).
\]

**Proof.** The idea of the proof follows [8]. Let \( K = \sup_{\mathbb{R}^n} u(x, 0) \). Without loss of generality, we can assume that \( K < \infty \), otherwise there is nothing to prove. For \( \varepsilon > 0 \) define

\[
v = K + \varepsilon \exp \left( \frac{k|x|^2}{1 - \mu t} + \nu t \right) - 1.
\]

Using the boundedness of the principal part in (5.6), it follows from a simple calculation that for \( x \neq 0 \),

\[
\Delta_{\infty}^N v - v_t \leq \varepsilon \exp \left( \frac{k|x|^2}{1 - \mu t} + \nu t \right) (|x|^2(16k^2n^2 - \mu k) + 16kn - \nu)
\]

when \( t \leq \frac{1}{2\mu} \). So given any \( k > 0 \), by choosing \( \mu, \nu \) large enough depending only on \( k \), we conclude that \( v \) is a classical supersolution at the points \((x, t)\) such that \( x \neq 0 \). By a compactness argument, it follows that \( v \) is a viscosity supersolution in \( \mathbb{R}^n \times [0, \frac{1}{2\mu}] \).

Now if \( k \) is chosen large enough depending only \( a \), we have that for any given \( \varepsilon > 0 \) and \( y \in \mathbb{R}^n \), there exists \( R > 0 \) large enough depending only on \( \varepsilon \) and \( y \) such that

\[
v(x, t) \geq u(x, t) \text{ when } |x - y| = R \tag{5.7}
\]

At \( t = 0 \), the same inequality holds since \( v \geq K \). By the comparison principle in \( B(y, R) \times (0, \frac{1}{2\mu}) \) for the equation (5.6) (see Theorem 3.1 in [15]), we have that \( v \geq u \) in \( B(y, R) \times (0, \frac{1}{2\mu}) \). In particular,

\[
v(y, t) \geq u(y, t).
\]

The conclusion now follows in \( [0, \frac{1}{2\mu}] \) by letting \( \varepsilon \to 0 \) and by using the arbitrariness of \( y \). We can thus repeat the argument in \( [\frac{1}{2\mu}, \frac{1}{\mu}] \) and so on, to obtain the same conclusion in \( \mathbb{R}^n \times [0, T] \).
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E-mail address: agnidban@gmail.com
E-mail address: rembrandt54@gmail.com