ON TOPOLOGICAL COMPLEXITY OF TWISTED PRODUCTS

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Abstract. We provide an upper bound on the topological complexity of twisted products. We use it to give an estimate
\[ TC(X) \leq TC(\pi_1(X)) + \dim X \]
of the topological complexity of a space in terms of its dimension and the complexity of its fundamental group.

1. Introduction

The topological complexity \( TC(X) \) of a space \( X \) was defined by Farber \cite{F1} as the minimum integer \( k \) such that \( X \times X \) admits an open cover \( U_0, \ldots, U_k \) by \( k + 1 \) sets such that for each \( U_i \) there is a continuous motion planning algorithm \( s_i \), i.e., a continuous map \( s_i : U_i \to \mathbb{P}X \) to the path space \( \mathbb{P}X = X^{[0,1]} \) such that \( s_i(x,y)(0) = x \) and \( s_i(x,y)(1) = y \) for all \( (x,y) \in U_i \). We note that here we defined the reduced topological complexity. In the original definition the enumeration of \( U_i \) goes from 1 to \( k \). Thus, it defines the nonreduced topological complexity which is by one larger.

The topological complexity is homotopy invariant. Therefore one can define the topological complexity \( TC(\pi) \) of a discrete group \( \pi \) as \( TC(K(\pi,1)) \) where \( K(\mathbb{Z},1) \) is an Eilenberg-Maclane complex. It is known that
\[ cd(\pi) \leq TC(\pi) \leq 2cd(\pi) \]
where \( cd(\pi) \) is the cohomological dimension of a group \( \pi \) \cite{Br}. M. Farber proposed a natural question:

1.1. Problem (Farber). What kind of a discrete group invariant is \( TC(\pi) \)?

Note that for groups with infinite cohomological dimension \( TC(\pi) = \infty \). Thus, Farber’s question makes sense only for groups admitting a finite dimensional Eilenberg-MacLane complex, in particular, for torsion free groups. It is known that the reduced Lusternik-Schnirelman category of a group \( \pi \) agrees with the cohomological dimension,
\[ \text{cat}(K(\pi,1)) = cd(\pi). \]
In view of the equality \( TC(G) = \text{cat} G \) \cite{F3} for all topological groups \( G \) and the fact that \( K(\mathbb{Z},1) \) is homotopy equivalent to a topological group for abelian \( \pi \), we obtain that \( TC(\pi) = cd(\pi) \) for abelian groups. For free nonabelian groups the other bound is taken, \( TC(\pi) = 2cd(\pi) \). Using the wedge formula for the topological complexity \cite{Dr3} Yu. Rudyak noticed that for all values \( k \) and \( n \) with \( n \leq k \leq 2n \) there is a group \( \pi \) with \( cd(\pi) = n \) and \( TC(\pi) = k \) \cite{Ru}.
In this paper we investigate how the topological complexity of the fundamental group could help to estimate the topological complexity of a space. A. Costa and M. Farber [CF] obtained the following upper bound

$$TC(X) \leq 2 \text{cd}(\pi) + \dim X$$

for the topological complexity of a space $X$ with the fundamental group $\pi$. Their result is parallel to the estimate for the Lusternik-Schnirelmann category [Dr1](see also [Dr2]):

$$\text{cat} X \leq \text{cd}(\pi) + \frac{1}{2} \dim X.$$  

In this paper we improve the Costa-Farber inequality to the following

$$TC(X) \leq TC(\pi) + \dim X.$$

This result was obtained as a corollary of an upper bound formula for the topological complexity of a twisted product. It is known that in the case of the Cartesian product $X = B \times F$ there is a formula

$$TC(X) \leq TC(B) + TC(F).$$

In the case of a twisted product $X = \tilde{B} \times F$ over $B$ with the fiber $F$ and the structure group $G$ we prove the formula

$$TC(X) \leq TC(B) + TC^*_G(F)$$

where $TC^*_G(F)$ is a version of the equivariant topological complexity introduced in this paper. We note that our version of the equivariant topological complexity differs from those defined by Colman-Grant [CG] and Lubawski-Marzantowicz [LM]. We call it the strongly equivariant topological complexity.

2. Preliminaries

Inspired by the work of Kolmogorov [K] on Hilbert’s 13th problem, Ostrand [Os] gave a characterization of dimension in terms of $k$-covers. In this paper we apply his technique to the topological complexity.

A family $\mathcal{U}$ of subsets of $X$ is called a $k$-cover, $k \in \mathbb{N}$ if every its subfamily of $k$ elements forms a cover of $X$. The order $\text{Ord}_x \mathcal{U}$ of a cover $\mathcal{U}$ at a point $x \in X$ is the number of elements in $\mathcal{U}$ that contain $x$. The following is obvious.

2.1. Proposition. A family $\mathcal{U}$ that consists of $m$ subsets of $X$ is an $(n+1)$-cover of $X$ if and only if $\text{Ord}_x \mathcal{U} \geq m - n$ for all $x \in X$.

Proof. If $\text{Ord}_x \mathcal{U} < m - n$ for some $x \in X$, then $n+1 = m - (m - n) + 1$ elements of $\mathcal{U}$ do not cover $x$.

If $n+1$ elements of $\mathcal{U}$ do not cover some $x$, then $\text{Ord}_x \mathcal{U} \leq m - (n+1) < m-n$. 

2.2. Definition. Given $\Delta \subset Z$, a subset $U \subset Z$ is called deformable to $\Delta$ if there is a homotopy $h_t : U \to Z$ with $h_0 : U \to Z$ the inclusion and $h_1(U) \subset \Delta$. An open cover $\mathcal{U} = \{U_0, U_1, \ldots, U_n\}$ of $Z$ is called $\Delta$-deformable if each $U_i$ is deformable to $\Delta$. If a group $G$ acts on $Z$ and $\Delta$ is an invariant set, we call a subset $U$ equivariantly deformable to $\Delta$ if the above homotopy $h_t$ is an equivariant map for each $t \in [0,1]$. The following is well-known:
2.3. Proposition. The topological complexity $TC(X)$ of a space $X$ is the minimum number $k$ such that $X \times X$ admits an open cover $U_0, \ldots, U_k$ by $\Delta(X)$-deformable sets where $\Delta(X) = \{(x, x) \in X \times X \mid x \in X\}$ is the diagonal in $X \times X$.

We call an action of a topological group $G$ on a locally compact metric space $X$ proper if for every compact $C \subset X$ the set $\{g \in G \mid g(C) \cap C \neq \emptyset\} \subset G$ is compact. We recall that the orbit space $X/G$ of such action is always completely regular (Proposition 1.28 [Pa]).

2.4. Theorem. Let $\{U_0', \ldots, U_n'\}$ be an open cover by $G$-invariant sets of a locally compact metric space $Z$ with a proper action of a topological group $G$ on it. Then for any $m = n, n + 1, \ldots, \infty$ there is an open $(n+1)$-cover of $Z$ by $G$-invariant sets $\{U_k\}_{k=0}^m$ such that $U_k = U'_k$ for $k \leq n$ and $U_k = \cup_{i=0}^n V_i$ is a disjoint union with $V_i \subset U_i$ for $k > n$.

In particular, if $\{U_0', \ldots, U_n'\}$ is $\Delta$-deformable for some $\Delta \subset Z$, then the cover $\{U_k\}_{k=0}^m$ is $\Delta$-deformable.

Moreover, if $\{U_0', \ldots, U_n'\}$ is equivariantly $\Delta$-deformable for some subgroup $H \subset G$, then the cover $\{U_k\}_{k=0}^m$ is equivariantly $\Delta$-deformable.

Proof. We construct the family $\{U_i\}_{i=0}^m$ by induction on $m$. For $m = n$ we take $U_i = U'_i$.

Let $\mathcal{U} = \{U_0, \ldots, U_{m-1}\}$ be the corresponding family for $m > n$. By Proposition 2.1, $\text{Ord}_p \mathcal{U} \geq m - n$. Consider the set $Y = \{y \in Z \mid \text{Ord}_y \mathcal{U} = m - n\}$. Clearly, it is a closed $G$-invariant subset of $Z$. If $Y = \emptyset$, then by Proposition 2.1 $\mathcal{U}$ is an $n$-cover and we can add $U_m = U_0$ to obtain a desired $(n+1)$-cover. Assume that $Y \neq \emptyset$. We show that for every $i \leq n$ the set $Y \cap U_i$ is closed in $Z$. Let $x$ be a limit point of $Y \cap U_i$ that does not belong to $U_i$. Let $U_{i_1}, \ldots, U_{i_{m-n}}$ be the sets of the cover $\mathcal{U}$ that contain $x \in Y$. The limit point condition implies that

$$(U_{i_1} \cap \cdots \cap U_{i_{m-n}}) \cap (Y \cap U_i) \neq \emptyset.$$ 

Then $\text{Ord}_y \mathcal{U} = m - n + 1$ for all $y \in Y \cap U_i \cap U_{i_0} \cap \cdots \cap U_{i_{m-n}}$. Contradiction.

We define recursively $F_0 = Y \cap U_0$ and

$$F_{i+1} = Y \cap U_{i+1} \setminus (\cup_{k=0}^i U_k).$$

Since the sets $U_0, \ldots, U_n$ cover $Z$, this process stops at $i = n$. Note that $\{F_i\}_{i=0}^n$ is a disjoint finite family of closed $G$-invariant subsets with $\cup_{i=0}^n F_i = Y$. Let $q : Z \to Z/G$ be the projection to the orbit space. Since the action is proper, by the combination of the Urysohn Metrization theorem and Palais result mentioned before this theorem the orbit space $Z/G$ is metrizable. Note that the sets $q(F_i)$ are disjoint. Since $q$ is a quotient map and $F_i$ and $U_i$ are $G$-invariant, the sets $q(F_i)$ and $q(U_i)$ are closed and open respectively. By taking disjoint open neighborhoods of $q(F_i)$ lying in $q(U_i)$ we fix open $G$-invariant disjoint neighborhoods $V_i$ of $F_i$ with $V_i \subset U_i$. We define $U_m = \cup_{i=0}^n V_i$. In view of Proposition 2.1, $U_0, \ldots, U_{m-1}, U_m$ is an $(n+1)$-cover.

Clearly, the deformations of $U_i$ to $\Delta$, $i \leq n$, define a deformation of $U_m$ to $\Delta$. If the deformations of $U_i$, $i \leq n$, to $\Delta$ are equivariant for some subgroup $H \subset G$, then the deformation of $U_m$ is equivariant. □

The following is well known:
2.5. Proposition. Let \( X \) be a metric space, \( A \) a subset of \( X \) and \( \mathcal{V}' = \{ V'_i \}_{i \in J} \) a cover of \( A \) by sets open in \( A \). Then \( \mathcal{V}' \) can be extended to a cover \( \mathcal{V} = \{ V_i \}_{i \in J} \) of \( A \) by sets open in \( X \) with the same nerve and such that \( V_i \cap A = V'_i \) for all \( i \in J \).

Proof. The required extension \( \mathcal{V} = \{ V_i \} \) can be defined by the formula

\[
V_i = \bigcup_{a \in V'_i} B(a, \frac{d(a, A - V'_i)}{2}), \quad i \in J
\]

(see Proposition 3.1 [Sr]). \( \square \)

3. Strongly equivariant topological complexity

Let \( G \) act on \( Y \), we define the strongly equivariant topological complexity \( TC^*_G(Y) \) to be the minimal integer \( k \) such that there is an open cover of \( Y \times Y \) by \( (G \times G) \)-invariant sets \( U_0, \ldots, U_k \) such that for each \( i \) there is a \( G \)-equivariant map \( \phi_i : U_i \to Y^I \) for the diagonal action on \( U_i \) such that \( \phi(y_0, y_1)(0) = y_0 \) and \( \phi(y_0, y_1)(1) = y_1 \).

Note that in the definition of the equivariant topological complexity \( TC_G(Y) \) Colman and Grant [CG] ask the sets \( U_i \) to be \( G \)-invariant. It follows from the definitions that \( TC_G(Y) \leq TC^*_G(Y) \). Another version of an equivariant topological complexity called the symmetric topological complexity \( STC_G(Y) \) was defined by Lubawski and Marzantowicz [LM]. They require the existence of \( (G \times G) \)-equivariant deformations of \( U_i \) to the \( (G \times G) \)-saturation of the diagonal. We note that for free actions on simply connected spaces, \( TC^*_G(Y) \leq STC_G(Y) \).

Let a group \( G \) act on \( F \) and let \( p : EG \to BG \) be the projection to the orbit space of the universal \( G \)-space. Then the universal \( F \)-bundle \( p_F : F \times_G EG \to BG \) is obtained by taking the orbit space \( F \times_G EG \) of the diagonal action on \( F \times EG \).

We say that an \( F \)-bundle \( Y \to X \) is a \( G \)-equivariant if there is a map \( g : Y \to BG \) such that \( f \) is the pull-back \( g^*(p_F) \) of the universal \( F \)-bundle.

In that case we call \( X \) the twisted product over \( Y \) with the fiber \( F \) and the structure group \( G \).

3.1. Theorem. Suppose that \( p : X \to B \) is a \( F \)-bundle between locally compact metric ANR-spaces with the structure group \( G \) acting properly on \( F \). Then

\[
TC(X) \leq TC(B) + TC^*_G(F).
\]

Proof. Let \( TC(B) = n \) and \( TC^*_G(F) = m \). By Theorem 2.4 applied for the trivial group there is an \((n + 1)\)-cover \( U_0, \ldots, U_{m+n} \) of \( B \times B \) by \( \Delta(B) \)-deformable open sets. By Theorem 2.4 applied for the \((G \times G)\)-action on \( F \times F \) with the diagonal subgroup \( G = \Delta(G) \subset G \times G \) there is an \((m + 1)\)-cover \( V_0, \ldots, V_{m+n} \) of \( F \times F \) by \((G \times G)\)-invariant \( G \)-equivariantly deformable to \( \Delta(F) \) open sets.

The bundle \( p \times p : X \times X \to B \times B \) is the pull back in the diagram

\[
\begin{array}{ccc}
X \times X & \xrightarrow{g'} & (F \times F) \times_{(G \times G)} E(G \times G) \\
\downarrow {p \times p} & & \downarrow q \\
B \times B & \xrightarrow{g} & B(G \times G)
\end{array}
\]

Note that the family

\[
O_k = V_k \times_{(G \times G)} E(G \times G), \quad k = 0, \ldots, m + n.
\]

is an \((m + 1)\)-cover of the space \((F \times F) \times_{(G \times G)} E(G \times G)\).
Let $O_k' = (g')^{-1}(O_k)$. We define
\[ W_k = (p \times p)^{-1}(U_k) \cap O_k', \quad k = 0, \ldots, m + n. \]

We claim that the sets $W_k$, $k = 0, \ldots, n + m$, cover the space $X \times X$.

Let $(x, y) \in X \times X$. By Proposition 3.2.1 the point $(p(x), p(y)) \in B \times B$ is covered at least by $m + 1$ elements from $\{U_i\}$. Let $(p(x), p(y)) \in U_{i_0} \cap \cdots \cap U_{i_m}$.

By the assumption, the family $V_{i_0}, \ldots, V_{i_m}$ covers $F \times F$. Therefore, the family $O_{i_0}, \ldots, O_{i_m}$ covers $F \times F \times (G \times G) E(G \times G)$ and hence the family
\[ (g')^{-1}(O_{i_0}), \ldots, (g')^{-1}(O_{i_m}) \]
covers $X \times X$. Thus, $(x, y) \in (g')^{-1}(O_{i_s})$ for some $s$. Then $(x, y) \in W_{i_s}$.

Next we show that each set $W_k$ is deformable to $\Delta(X)$. The homotopy lifting property of the fiber bundle $f: O_k' \to B \times B$ where $f = (p \times p)|_{O_k'}$ and the fact that the inclusion $j: U_k \to B \times B$ is deformable to the diagonal $\Delta(B)$ imply that the set $W_k$ can be deformed in $O_k' \subset X \times X$ to the preimage $f^{-1}(\Delta(B))$ of the diagonal.

We note that the bundle $q$ over the diagonal $\Delta(BG) \subset \Delta(BG) \times \Delta(BG)$ is isomorphic to the twisted product bundle $q_0: (F \times F) \times G EG \to BG$ for the diagonal action of $G$ on $F \times F$. Let $\phi_0^k: V_k \to F$ be a $G$-equivariant homotopy with $\phi_0^k(x_1, x_2) = x_1$ and $\phi_0^k(x_1, x_2) = x_2$. We define a deformation $\Phi^k: V_k \to F \times F$ by the formula $\Phi^k(x_1, x_2) = \phi^k(x_1, x_2) \times x_2$. Note that it is $G$-equivariant for the diagonal action, $\Phi^k_0 = \text{id}$, and $\Phi^k(V_k) \subset \Delta(F)$. The deformation $\Phi^k$ defines a fiberwise deformation of $V_k \times G EG$ in $(F \times F) \times G EG$ to the space $\Delta(F) \times G EG$.

Thus, it defines a fiberwise deformation of $O_k$ over the diagonal $\Delta(BG)$ to
\[ \Delta(F \times G EG) \subset (F \times G EG) \times (F \times G EG) = (F \times F) \times (G \times G) E(G \times G). \]

In the pull-back diagram this defines a fiberwise deformation to $\Delta(X)$ of the set $O_k'$ over $\Delta(B)$ which is $f^{-1}(\Delta(B))$.

Thus, the concatenation of the above two deformations define a continuous deformation of $W_k$ to the diagonal $\Delta(X)$.

\[ \square \]

3.2. Proposition. Suppose that a discrete group $\pi$ acts freely and properly on a simply connected locally compact ANR space $Y$. Then $TC^*_{\pi}(Y) \leq \dim Y$.

Proof. Let $X = Y/\pi$ and let $\dim Y = k$. Since $\dim X = k$, $\dim(X \times X) \leq 2k$.

It follows from the classical dimension theory that there are 1-dimensional sets $S_0, \ldots, S_k$ that cover $X \times X$. In view of Proposition 3.2.1 and 1-dimensionality of $S_i$ there is an arbitrary small cover $U_i$ of $S_i$ of order $\leq 2$ by open in $X \times X$ sets with compact closure. Let $W_i = \bigcup_{U \in U_i} U$. Since $X \times X$ is an ANR, we may assume that for each $i$ there is a projection to the nerve $\phi_i: W_i \to K_i = N(U_i)$ and a map $\xi_i: K_i \to X \times X$ such that the composition $\xi_i \circ \phi_i: W_i \to X \times X$ is homotopic to the inclusion $W_i \subset X \times X$.

Let $q: Y \to X$ be the projection onto the orbit space of the action of $\pi \times \pi$ and let $q: Y \times Y \to Y \times Y$ be projection onto the orbit space of the diagonal subgroup action. Then there is a connecting projection $p: Y \times \pi Y \to X \times X$, $\tilde{q} = p \circ q$. Since the actions are free, $p$ is a covering map. We may assume that $U_0 \cup \cdots \cup U_k$ is a cover of $X \times X$ by even for $p$ sets. Thus, for $U \in U_i$,
\[ p^{-1}(U) = \bigsqcap_{\gamma \in J} \tilde{U}_\gamma \]
and the restriction of $p|_{U_{\gamma}} : \hat{U}_{\gamma} \to U$ is a homeomorphism for all $\gamma \in J$. Moreover, we may assume that there is an even cover $V$ of $X \times X$ and a homotopy $h^1_i$ between the inclusion $\hat{W}_i \to X \times X$ and $\xi_i \circ \phi_i$ such that for each $i$ and $U \in \mathcal{U}_i$, there is $V \in \mathcal{V}$ such that the image of $U \times [0,1]$ under that homotopy is contained in $V$.

Then $p$ induces a simplicial covering map $p'_i : K'_i \to K_i$ of the nerve $K'_i$ of the cover $\mathcal{U}'_i = \{U_{\gamma} \mid U \in \mathcal{U}_i, \gamma \in J\}$ onto the nerve $K_i$. In the above notations $p'_i$ takes a vertex corresponding to $\hat{U}_{\gamma}$ to the vertex defined by $U$.

Let $W'_i = p^{-1}(W_i)$. Then we claim that there are maps $\phi'_i : W'_i \to K'_i$ and $\xi'_i : K'_i \to Y \times_\pi Y$ that cover $\phi_i$ and $\xi_i$ with $\xi'_i \circ \phi'_i : W'_i \to Y \times_\pi Y$ homotopic to the inclusion $W'_i \subset Y \times \pi Y$. Indeed, the projection to the nerve $\phi_i : W_i \to K_i$ is defined by means of a partition of unity $\{f_j\}$ subordinated to the cover $\mathcal{U}_i = \{U_j\}$ of $W_i$ as $\phi_i(x) = (f_j(x)) \in E\pi(K_i^{(0)})$. Here we realize $K_i$ in the standard simplex in the Hilbert space spanned by the vertices $K_i^{(0)}$. Let $f_{j,\gamma}$ be the extension to $W'_i$ of the composition $f_j \circ p|_{\hat{U}_{\gamma}} : \hat{U}_{\gamma} \to [0,1]$ by $0$. Then $\{f_{j,\gamma}\}$ is the partition of unity on $W'_i$ subordinated to $\mathcal{U}'_i$ that defines a projection $\phi'_i$ and the corresponding diagram is the pull-back diagram.

The homotopy $q^1_i = h^1_i \circ p : W'_i \to X \times X$ admits a lift $\tilde{g}^1_i : W'_i \to Y \times \pi Y$ with $\tilde{g}^1_i : W'_i \to Y \times \pi Y$ the inclusion. Since for every $x' \in K'_i$ the map $\tilde{g}^1_i$ coincides with $(p|_{V_{\alpha}})^{-1} \circ h^1_i \circ p$ on $\phi_i^{-1}(x')$ where $p|_{V_{\alpha}} : V_{\alpha} \to V$ is a homeomorphism for some $V \in \mathcal{V}$ and $\alpha$, the set $\tilde{g}^1_i((\phi'_i)^{-1}(x')) = (p|_{V_{\alpha}})^{-1}(\phi_i(p'_i(x')))$ consists of one point. Hence $\tilde{g}^1_i$ factors through $\phi'_i$.

Note that $X = \Delta(Y)/\pi$ is naturally embedded in $Y \times_\pi Y$. We show that the sets $W'_i = p^{-1}(W_i)$ are deformable to $X$. The construction of a deformation of $\xi'_i$ to a map $X$ goes as follows. First we do it on all vertices $v \in (K'_i)^{(0)}$ by fixing a path $p_v : [0,1] \to Y \times \pi Y$ with $p_v(0) = \xi'_i(v)$ and $p_v(1) \in X \subset Y \times \pi Y$. Note that the inclusion $X \to Y \times \pi Y$ induces an isomorphism of the fundamental groups. Then the homotopy exact sequence of pair implies that $\pi_1(Y \times \pi Y, X) = 0$. Therefore, for every edge $[u,v] \subset K'_i$ the product of the paths $p_v \circ \xi'_i|[u,v] \circ p_u$, is path homotopic to a path in $X$. Here $p_v$ denotes the inverse path for $p_v$. This defines a deformation of $\xi'_i|[u,v]$ to $X$ that agrees with $p_u$ and $p_v$. All such deformations of edges define a deformation of $\xi'_i$ to $X$. This deformation together with a homotopy of the inclusion $W'_i \subset Y \times \pi Y$ to $\xi'_i \circ \phi'_i$ defines a deformation of $W'_i$ to $X$.

Let $V_i = q^{-1}(W'_i)$. Then each $V_i$ is $(\pi \times \pi)$-invariant since $V_i = \tilde{q}^{-1}(W_i)$. A deformation of $W'_i$ to $\Delta(Y)/\pi$ defines a $G$-equivariant deformation of $V_i$ to $\Delta(Y)$. Thus, an open cover $V_i = q^{-1}(W'_i)$, $i = 0, \ldots, k$, of $Y \times Y$ satisfies all conditions from the definition of $TC_\pi^1(Y)$.

3.3. Theorem. For a CW complex $X$ with the fundamental group $\pi$ there is the inequality

$$TC(X) \leq TC(\pi) + \dim X.$$  

Proof. We apply Theorem 3.1 to the bundle $p : \hat{X} \times_\pi E\pi \to B\pi$ with the structure group $\pi$ and the fiber $\hat{X}$, the universal cover of $X$. Note that since $E\pi$ is contractible, the map $\hat{X} \times_\pi E\pi \to X$ induced by the projection $pr_1 : \hat{X} \times E\pi \to \hat{X}$ to
the first factor is a homotopy equivalence. We apply Proposition 3.2 to complete the proof.

□

References

[CF] Costa, Armindo; Farber, Michael, Motion planning in spaces with small fundamental groups. Commun. Contemp. Math. 12 (2010), no. 1, 107-119.

[CG] Colman, Hellen; Grant, Mark Equivariant topological complexity. Algebr. Geom. Topol. 12 (2012), no. 4, 2299-2316.

[Br] K. Brown, Cohomology of groups, Springer 1982.

[Dr1] A. Dranishnikov, On the Lusternik-Schnirelman category of spaces with 2-dimensional fundamental group. Proc. Amer. Math. Soc. 137 (2009), no. 4, 1489-1497.

[Dr2] A. Dranishnikov, The Lusternik-Schnirelmann category and the fundamental group. Algebr. Geom. Topol. 10 (2010), no. 2, 917924.

[Dr3] A. Dranishnikov, Topological complexity of wedges and covering maps, Proc. AMS, to appear.

[En] R. Engelking, Theory of Dimensions Finite and Infinite, Heldermann Verlag, 1995.

[F1] Farber, Michael, Topological complexity of motion planning. Discrete Comput. Geom. 29 (2003), no. 2, 211 -221.

[F2] Farber, Michael, Invitation to topological robotics. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zrich, 2008.

[F3] M. Farber, Instability of robot motion. Top. Appl. 140 (2004), 245-266.

[K] A. N. Kolmogorov, Representation of functions of many variables, Dokl. Akad. Nauk 114 (1957), 953-956; English transl., Amer. Math. Soc. Transl. (2) 17 (1961), 369-373.

[LM] W. Lubawski and W. Marzantowicz, A new approach to the equivariant topological complexity, arXiv: 1303.0171v2 [math.AT].

[Os] Ostrand, Ph.: Dimension of metric spaces and Hilbert’s problem 13, Bull. Amer. Math. Soc. 71 1965, 619-622.

[Pa] R. Palais, On the existence of sclices for actions of non-compact Lie groups, Ann. Math., vol 73 No 2, 295-323

[Ru] Yu. Rudyak, On topological complexity of Eilenberg-MacLane spaces, arXiv:1302.1238

[Sr] T. Srinivasan, On the Lusternik-Schnirelman category of Peano continua, Topology Appl. 160 (2013), no. 13, 1742-1749. math arXiv:1212.0899

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