LIPSCHITZ EQUIVALENCE OF SELF-SIMILAR SETS WITH EXACT OVERLAPS

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Abstract. In this paper, we study a class \( A(\lambda, n, m) \) of self-similar sets with \( m \) exact overlaps generated by \( n \) similitudes of the same ratio \( \lambda \). We obtain a necessary condition for a self-similar set in \( A(\lambda, n, m) \) to be Lipschitz equivalent to a self-similar set satisfying the strong separation condition, i.e., there exists an integer \( k \geq 2 \) such that \( x^{2k} - mx^k + n \) is reducible, in particular, \( m \) belongs to \( \{a^n : a \in \mathbb{N} \} \) with \( i \geq 2 \).

1. Introduction

Recall that a compact subset \( K \) of Euclidean space is said to be a self-similar set [6], if \( K = \bigcup_{i=1}^N S_i(K) \) is generated by contractive similitudes \( \{S_i\} \), with ratio set \( \{r_i\} \subset (0, 1) \) satisfying \( |S_i(x) - S_j(y)| = r_i|x-y| \) for all \( x, y \). The classical dimension result under the open set condition (OSC) is

\[
\dim_H K = s \quad \text{with} \quad \sum_{i=1}^n (r_i)^s = 1. \tag{1.1}
\]

In particular, \( K \) is said to be dust-like when the strong separation condition (SSC) holds, i.e., \( S_i(K) \cap S_j(K) = \emptyset \) for all \( i \neq j \), then the open set condition holds and thus (1.1) is valid.

The self-similar sets with overlaps have complicated structures, for example, Hochman [5] studied the self-similar sets

\[ E_\theta = E_\theta/3 \cup (E_\theta/3 + \theta/3) \cup (E_\theta/3 + 2/3) \]

and obtained \( \dim_H E_\theta = 1 \) for any \( \theta \) irrational. If \( \theta \) is rational, Kenyon [8] obtained that the OSC is fulfilled for \( E_\theta \) if and only if \( \theta = p/q \in \mathbb{Q} \) with \( p \equiv q \not\equiv 0 \pmod{3} \).

Rao and Wen [11] also discussed the structure of \( E_\theta \) with \( \theta \in \mathbb{Q} \) using the key idea “graph-directed structure” introduced by Mauldin and Williams [9].

Recently, Jiang, Wang and Xi [7] investigated a class \( A(\lambda, n, m) \) of self-similar sets with exact overlaps where \( \lambda \in (0, 1) \) and \( m, n \in \mathbb{N} \) with \( 1 \leq m \leq n - 2 \). Let \( f_i(x) = \lambda x + b_i \) with \( 0 = b_1 < b_2 < \cdots < b_n = 1 - \lambda \). Write \( I = [0, 1] \) and \( I_i = f_i(I) \).

Assume that

\[
\frac{|I_i \cap I_{i+1}|}{|I_i|} \in \{0, \lambda\} \quad \text{if} \quad I_i \cap I_{i+1} \neq \emptyset, \quad \text{and} \quad \# \left\{ i : \frac{|I_i \cap I_{i+1}|}{|I_i|} = \lambda \right\} = m.
\]
We call \( E = \bigcup_{i=1}^{n} f_i(E) \) a self-similar set with exact overlap, denoted by \( E \in \mathcal{A}(\lambda, n, m) \). It is proved in [7] that \( \dim_{H} E = \frac{\log \beta}{- \log \lambda} \) where the P.V. number \( \beta > 1 \) is a root of the irreducible polynomial \( x^2 - nx + m = (x - \beta)(x - \beta') \) with \( |\beta'| < 1 < \beta \).

In this paper, we will compare self-similar sets in \( \mathcal{A}(\lambda, n, m) \) with dust-like self-similar sets in terms of Lipschitz equivalence.

Two compact subsets \( X_1, X_2 \) of Euclidean spaces are said to be Lipschitz equivalent, denoted by \( X_1 \simeq X_2 \), if there is a bijection \( f : X_1 \to X_2 \) and a constant \( C > 0 \) such that for all \( x, y \in X_1 \),
\[
C^{-1}|x - y| \leq |f(x) - f(y)| \leq C|x - y|.
\]

Cooper and Pignataro [1], Falconer and Marsh [3], David and Semmes [2] and Wen and Xi [12] showed that two self-similar sets need not be Lipschitz equivalent although they have the same Hausdorff dimension.

We concern the Lipschitz equivalence between two self-similar sets with the SSC and with overlaps respectively.

1. David and Semmes [2] posed the \{1, 3, 5\}={'}1, 4, 5\} problem. Let \( H_1 = (H_1/5) \cup (H_1 + 2/5) \cup (H_1 + 4/5) \) and \( H_2 = (H_2/5) \cup (H_2 + 3/5) \cup (H_2 + 4/5) \) be \{1, 3, 5\}, \{1, 4, 5\} self-similar sets respectively. The problem asks about the Lipschitz equivalence between \( H_1 \) (with the SSC) and \( H_2 \) (with the touched structure). Rao, Ruan and Xi [10] proved that \( H_1 \) and \( H_2 \) are Lipschitz equivalent.

2. Guo et al. [4] studied the Lipschitz equivalence for \( K_n = (\lambda K_n) \cup (\lambda K_n + \lambda^i(1 - \lambda)) \cup (\lambda K_n + 1 - \lambda) \) with overlaps and proved that \( K_n \simeq K_m \) for all \( n, m \geq 1 \). In particular, for \( n = 1 \), \( K_1 \in \mathcal{A}(\lambda, 3, 1) \) is Lipschitz equivalent to a dust-like set \( F = (\lambda F) \cup (\lambda^{1/2} F + 1 - \lambda^{1/2}) \).

We will state our main result.

**Theorem 1.** Suppose \( E \in \mathcal{A}(\lambda, n, m) \) and \( P(x) = x^2 - nx + m \). If there is a dust-like self-similar set \( F \) such that \( E \simeq F \), then there exists an integer \( k \geq 2 \) such that
\[
P(x^k) = x^{2k} - nx^k + m \quad \text{is reducible in } \mathbb{Z}[x].
\]
In particular, we have
\[
m \in \{a^i \mid a \in \mathbb{N} \text{ and } i \in \mathbb{N} \text{ with } i \geq 2\}.
\]

By this theorem, if \( m \in \{2, 3, 5, 6, 7, 10, 11, 12, 13, 14, 15, 17, \cdots \} \), then we cannot find a dust-like self-similar set to be Lipschitz equivalent to \( E \in \mathcal{A}(\lambda, n, m) \).

**Example 1.** For \( n = 3 \) and \( m = 1 \), we have \( P(x) = x^2 - 3x + 1 \) and an example \( K_1 \simeq F = (\lambda F) \cup (\lambda^{1/2} F + 1 - \lambda^{1/2}) \) in [4] as above. Now, \( P(x^2) = (x^2 - x - 1)(x^2 + x - 1) \) is reducible and \( 1 \in \{a^i \mid a \in \mathbb{N} \text{ and } i \in \mathbb{N} \text{ with } i \geq 2\} \).

The paper is organized as follows. In Section 2 we show any self-similar set in \( \mathcal{A}(\lambda, n, m) \) has graph-directed structure and obtain the logarithmic commensurability of ratios for the dust-like self-similar set by the approach of Falconer and Marsh [3]. Using the dimension polynomials and their irreducibility, we give the proof of Theorem 1 in Section 3.

### 2. Logarithmic commensurability of ratios

At first, we show that any self-similar set with exact overlaps will generate a graph-directed construction.

**Lemma 1.** There are graph-directed sets \( \{E_i\}_{i=1}^{n} \) with ratio \( \lambda \) satisfying the SSC and \( E_1 = E \).
\textbf{Proof.} Consider the set $G$ in the following form
\[ G = \bigcup_{i=1}^{k} (E + a_i) \] with \( 0 = a_1 < a_2 < \cdots < a_k \) and \( k \leq n - 1 \) such that \((I + a_i) \cap (I + a_{i+1}) \neq \emptyset\) with \( I = [0,1]\) for all \( i \leq k - 1 \) satisfying
\[ |(I + a_i) \cap (I + a_{i+1})| = 0 \text{ or } \lambda. \]

Let \( \mathcal{G} \) be the collection of all sets in the form as above. For every \( G \in \mathcal{G} \), considering the natural decomposition at the touched point \((|I + a_i) \cap (I + a_{i+1})| = 0\) or on the exact overlapping \((|(I + a_i) \cap (I + a_{i+1})| = \lambda)\), we have the decomposition
\[ G = \bigcup_{G' \in \mathcal{G}} \bigcup_i (\lambda G' + b_i G') \]
which is a disjoint union. That means we obtain a graph directed construction satisfying the SSC. In fact, we only need to choose a subgraph generated by \( E \) with \( k = 1 \).

The main result of this section is the following Proposition 1. We will use the approach by Falconer and Marsh [3]. In [3], the authors discussed the dust-like self-similar sets, now we will deal with the graph-directed sets.

**Proposition 1.** Suppose \( E \in \mathcal{A}(\lambda, n, m) \) and \( F = \bigcup_{j=1}^{l} g_j(F) \) is a dust-like self-similar set such that \( E \simeq F \). Assume \( r_j \) is the contractive ratio of \( g_j \) for any \( j \). Then there is a ratio \( r \in (0,1) \) and positive integers \( k \) and \( k_1 \leq k_2 \leq \cdots \leq k_i \) such that
\[ \lambda = r^k, \quad r_1 = r^{k_1}, \quad r_2 = r^{k_2}, \ldots, \quad r_t = r^{k_t}. \]

Without loss of generality, we only need to show that
\[ \frac{\log r_j}{\log \lambda} \in \mathbb{Q}, \]
or \( \frac{\log(r_j)'}{\log \lambda'} \in \mathbb{Q} \) with \( s = \dim_H E = \dim_H F \). Suppose \( f: F \to E \) is a bi-Lipschitz bijection and \( c \geq 1 \) is a constant satisfying
\[ c^{-1}|x - y| \leq |f(x) - f(y)| \leq c|x - y| \quad \text{for all } x, y \in F. \]

Denote \( \Sigma^* = \bigcup_{k \geq 0} \{1, \cdots, t\}^k \). For any \( j = j_1 \cdots j_k \in \Sigma^* \), we write \( F_j = g_{j_{j_1} \cdots j_k}(F) \).

Suppose \( e \) is an admissible path of length \(|e|\) in the directed graph beginning at vertex \( v = b(e) \), then
\begin{equation}
|E_e| = \lambda^{|e|}|E_v| \quad \text{and} \quad \mathcal{H}^s(E_e) = \lambda^{|e|} \mathcal{H}^s(E_v) = \lambda^{|e|} \mathcal{H}^s(E_{b(e)}).
\end{equation}

Because of the SSC on \( F \), we assume that there is a constant \( \xi > 0 \) such that
\begin{equation}
d(F_j, F \setminus F_j) \geq \xi |F_j| \quad \text{for all } j \in \Sigma^*,
\end{equation}
and
\begin{equation}
\xi |E_{e_j}| \leq |F_j| \leq \xi^{-1} |E_{e_j}| \quad \text{for all } j \in \Sigma^*,
\end{equation}
where we denote by \( E_{e_j}(\subset E) \) the smallest copy containing \( f(F_j) \).

**Lemma 2.** There is a positive integer \( N \) such that for any copy \( F_j \) of \( F \) and smallest copy \( E_{e_j}(\subset E) \) containing \( f(F_j) \), there is a set \( \Delta_j \) composed of paths \( e' \) with length \( N \) satisfying
\[ f(F_j) = \bigcup_{e' \in \Delta_j} E_{e_j + e'}. \]
Proof. Now let \( N = \left\lceil \log \frac{c^{-1}2^{(n-1)^{2}}} \right\rceil + 1 \). It suffices to show that if \( z \in E_{e_{j+e}'} \) with \( E_{e_{j+e}'} \cap f(F_j) \neq \emptyset \) then \( z \in f(F_j) \). In fact, if \( z \in f(F \setminus F_j) \) and \( z' \in E_{e_{j+e}'} \cap f(F_j) \), by (2.2)–(2.3) we have
\[
|z - z'| \geq d(f(F_j), f(F \setminus F_j)) \geq c^{-1} \xi |F_j| \geq c^{-1} \xi^2 |E_{e_j}|.
\]
On the other hand, using (2.1) and the fact that \( 1 = |E| \leq |E_{e_j}| \leq n - 1 \), we have
\[
|z - z'| \leq |E_{e_{j+e}'}| \leq \lambda^N(n - 1)|E_{e_j}| < c^{-1} \xi^2 |E_{e_j}|,
\]
this is a contradiction. \qed

For any Borel set \( B \subset F \), we let
\[
h(B) = \frac{\mathcal{H}^s(f(B))}{\mathcal{H}^s(B)}.
\]
Since \( f: F \to E \) is bi-Lipschitz, we have
\[
d = \sup_{j \in \Sigma^*} h(F_j) < \infty.
\]

Lemma 3. There is a finite set \( \Lambda \) such that
\[
\frac{h(F_{j,\j})}{h(F_j)} \in \Lambda
\]
for all \( j \in \Sigma^* \) and all \( j \in \{1, \ldots, t\} \).

Proof. We note that
\[
\frac{h(F_{j,\j})}{h(F_j)} = \frac{\mathcal{H}^s(f(F_{j,\j}))}{\mathcal{H}^s(f(F_j))} \mathcal{H}^s(F_j) / \mathcal{H}^s(F_{j,\j}) = \frac{\mathcal{H}^s(F_j)}{\mathcal{H}^s(F_{j,\j})} \frac{\lambda^{s|e_{j+\j}|}}{\lambda^{s|e_j|}} \frac{\mathcal{H}^s(f(F_{j,\j}))}{\mathcal{H}^s(f(F_j))} / \lambda^{s|e_{j+\j}|}/ \lambda^{s|e_j|}.
\]
Now, \( \frac{\mathcal{H}^s(F_j)}{\mathcal{H}^s(F_{j,\j})} \in \{(r_j)^{-s}\} \). Suppose \( M \) is a upper bound for difference of lengths of \( e_{j+\j} \) and \( e_j \), we have
\[
\frac{\lambda^{s|e_{j+\j}|}}{\lambda^{s|e_j|}} \in \{\lambda^k : k \leq M\}
\]
which is a finite set. By Lemma 2, we also obtain that
\[
\frac{\mathcal{H}^s(f(F_j))}{\lambda^{s|e_j|}} = \sum_{\epsilon' \in \Delta} \mathcal{H}^s(E_{e_{j+\j}}(\epsilon')) \lambda^{s|e_j|} = \lambda^{s|e_j|+N} \sum_{\epsilon' \in \Delta} \mathcal{H}^s(E_{\epsilon'}(\epsilon')) / \lambda^{s|e_j|} \in \lambda^{sN} \left\{ \sum_{\epsilon' \in \Delta} \mathcal{H}^s(E_{\epsilon'}(\epsilon')) : \Delta \subset \{\epsilon' : |\epsilon'| = N\} \right\}
\]
which is also a finite set. \qed

Lemma 4. There is a copy \( F_{j_1 \ldots j_k} \) of \( F \) and a constant \( \tilde{d} > 0 \) such that
\[
(2.4) \quad \frac{\mathcal{H}^s(f(B))}{\mathcal{H}^s(B)} = \tilde{d}
\]
for Borel set \( B \subset F_{j_1 \ldots j_k} \).

Proof. Suppose \( \alpha = \max_{x \in (-\infty, 1) \cap \Lambda} x < 1 \) or \( \alpha = 1/2 \) if \( (-\infty, 1) \cap \Lambda = \emptyset \). Take \( \epsilon > 0 \) such that
\[
(2.5) \quad \max_{t} \left( \alpha r_t^s + (1 + \epsilon)(1 - r_t^s) \right) < 1.
\]
Let $d = \sup_{j \in \Sigma^*} h(F_j) < \infty$ and take a sequence $j = j_1 \cdots j_k^*$ such that $\frac{d}{h(F_j)} < 1 + \epsilon$. We notice that

$$\tilde{d} \doteq h(F_j) = \sum_j \frac{\mathcal{H}^s(F_{j^*})}{\mathcal{H}^s(F_j)} h(F_{j^*}) = \sum_j \frac{\mathcal{H}^s(F_{j^*})}{\mathcal{H}^s(F_j)} = \sum_j (r_j)^s = 1,$$

i.e., we have

$$(2.6)\quad 1 = \sum_j (r_j)^s \frac{h(F_{j^*})}{h(F_j)} \text{ with } \sum_j (r_j)^s = 1.$$

We will first show that $h(F_{j^*}) \geq h(F_j)$ for all $j$. Otherwise, without loss of generality, we assume that $\frac{h(F_{j^*})}{h(F_j)} < 1$. Then

$$\frac{h(F_{j^*})}{h(F_j)} \leq \alpha \quad \text{and} \quad \frac{h(F_{j^*})}{h(F_j)} \leq \frac{d}{h(F_j)} < 1 + \epsilon \quad \text{for} \quad j \geq 2.$$

It follows from (2.5) that

$$1 = \sum_j (r_j)^s \frac{h(F_{j^*})}{h(F_j)} \leq \alpha r_1^s + (1 + \epsilon)(1 - r_1^s) < 1,$$

this is a contradiction. Now $h(F_{j^*}) \geq h(F_j)$ for all $j$, by (2.6) we obtain that

$$h(F_{j^*}) = h(F_j) = \tilde{d} \quad \text{for all} \quad j.$$

In the same way, we have

$$h(F_{j^*j_1^*j_2^*}) = h(F_j) = \tilde{d} \quad \text{for all} \quad j_1, j_2.$$

Again and again, we obtain

$$h(F_{j^*}) = \tilde{d} \quad \text{for any} \quad j' \text{ with prefix } j.$$

Then (2.4) follows. \hfill \square

**Proof of Proposition 1.** Take $j = j_1 \cdots j_{k^*}$ in Lemma 4. For any $j$, we consider the sequence $j[j]^k = j * [j]^k$ where the sequence $[j]^k$ is composed of $k$ successive digits $j$. Then

$$h(F_{j[j]^k}) = 1 \quad \text{with} \quad k > k'.$$

Hence we obtain that

$$(r_j)^{k - k'} = \frac{\mathcal{H}^s(F_{j[j]^k})}{\mathcal{H}^s(F_{j[j]^k})} = \frac{h(F_{j[j]^k})}{h(F_{j[j]^k})} \cdot \frac{\sum_{e' \in \Delta_{j[j]^k}} \mathcal{H}^s(E_b(e'))}{\sum_{e' \in \Delta_{j[j]^k'}} \mathcal{H}^s(E_b(e'))} \cdot \lambda_s(0(1 - |e_{j[j]^k}|))$$

$$= \frac{\sum_{e' \in \Delta_{j[j]^k}} \mathcal{H}^s(E_b(e'))}{\sum_{e' \in \Delta_{j[j]^k'}} \mathcal{H}^s(E_b(e'))} \cdot \lambda_s(0(1 - |e_{j[j]^k}|)).$$

From the finiteness, we can find $k \neq k'$ such that $\Delta_{j[j]^k} = \Delta_{j[j]^k'}$ then

$$(r_j)^{k - k'} = \lambda_s(0(1 - |e_{j[j]^k}|)),$$

that means $(r_j)^{k - k'} = \lambda_s(0(1 - |e_{j[j]^k}|))$, i.e.,

$$\log r_j / \log \lambda \in \mathbb{Q}$$

for all $j$. Then Proposition 1 is proved. \hfill \square
3. Proof of Theorem

3.1. Dimension polynomials. From [7] we have
\[ P(x) = x^2 - nx + m = (x - \beta)(x - \beta') \text{ with } |\beta'| < 1 < \beta. \]
Using notations in Proposition 1, we consider the following two polynomials

\[ (3.1) \quad \tilde{P}(x) = P(x^k) \quad \text{and} \quad \tilde{Q}(x) = x^{k_1} - \sum_{i=1}^{t} x^{k_i - k_1}. \]

**Proposition 2.** Let \( s = \dim_H E = \dim_H F \) and \( r \) the ratio in Proposition 1. Then
\[ \tilde{P}(r^{-s}) = \tilde{Q}(r^{-s}) = 0. \]

**Proof.** It follows from [7] that for \( s = \dim_H E \),
\[ (\lambda^{-s})^2 - n(\lambda^{-s}) + m = 0. \]
On the other hand, for \( s = \dim_H F \), by the SSC we have
\[ \sum_{i=1}^{t} (r_i)^s = 1. \]
Then the proposition follows the relations in Proposition 1. \( \square \)

3.2. Irreducibility of polynomial.

**Proposition 3.** For any \( Q \in \{ x^p - \sum_{i=0}^{p-1} b_i x^i : p \geq 1, b_i \in \mathbb{Z} \text{ and } b_i \geq 0 \} \), we have
\[ P(x^q) \nmid Q(x). \]

**Proof.** Let \( Q(x) = (\sum a_i x^i) (x^{2q} - nx^q + m) \). Suppose
\[ \sum a_i x^i = P_0 + P_1 + \cdots + P_{q-1} \]
where \( P_v = \sum_{i \equiv v (\text{mod } q)} a_i x^i \) for \( v = 0, 1, \cdots, (q - 1) \). Then we have
\[ Q(x) = P_0 P(x^q) \oplus P_1 P(x^q) \oplus \cdots \oplus P_{q-1} P(x^q), \]
where \( \oplus \) means the orthogonality of above polynomials in the basis \( \{ 1, x, x^2, \cdots \} \).

Without loss of generality, we assume that
\[ \deg \left( \sum a_i x^i \right) \equiv u \pmod{q} \quad \text{with} \quad 0 \leq u \leq q - 1. \]
Let \( c_i = a_{qi + u} \), then
\[ P_u = x^u (c_0 + c_1 x^q + c_2 x^{2q} + \cdots + c_r x^{lq}) = x^u U(x^q). \]
Since \( p \equiv 2q + \deg(\sum a_i x^i) \equiv u \pmod{q} \), we have
\[ x^u U(x^q) P(x^q) = x^p - \sum_{j \equiv u \pmod{q}} b_j x^j, \]
which implies
\[ U(x) P(x) = x^{p'} - \sum_{i=0}^{p'} b_i' x^i \quad \text{with} \quad b_i' \in \mathbb{Z} \quad \text{and} \quad b_i' \geq 0. \]
Therefore we obtain that
\[ (x^2 - nx + m)(c_0 + c_1 x + c_2 x^2 + \cdots + c_l x^l) = x^{l+2} - \sum_{i=0}^{l+1} b_i' x^i, \]
where
\begin{equation}
(3.2) \quad c_l = 1.
\end{equation}

We recall that
\[ x^2 - nx + m = (x - \beta)(x - \beta') \quad \text{with} \quad \beta > 1 > |\beta'|. \]

Now, we have the following

**Claim 1.** For any \(0 \leq i \leq l - 1\),
\begin{equation}
(3.3) \quad c_{i+1} \leq c_i \beta^{-1} \leq 0.
\end{equation}

We will verify (3.3) by induction.

1. For \(i = 0\), we have \(c_0m = -b_0' \leq 0\) and thus \(c_0 \leq 0\).
2. For \(i = 1\), we have \(-c_0n + mc_1 = -b_1' \leq 0\) and thus \(c_1 \leq \frac{n}{m}c_0 \leq \beta^{-1}c_0 \leq 0\).

Here \(\frac{\beta}{m} > 1 > \beta^{-1}\).

3. Assume that (3.3) is true for \(i - 1\), i.e., we have \(c_i \leq c_{i-1} \beta^{-1} \leq 0\). Hence
\[ mc_{i+1} - nc_i + \beta c_i \leq mc_{i+1} - nc_i + c_{i-1} = -b'_{i+1} \leq 0, \]
which implies
\[ mc_{i+1} \leq \frac{(n - \beta)}{m}c_i = \beta^{-1}c_i \leq 0 \]
due to \(\frac{(n - \beta)}{m} = \beta^{-1}\). Then (3.3) is verified. In particular, we have
\[ c_l \leq 0 \]
which contradicts to (3.2).

**Proposition 4.** Suppose \(m \not\in \{a^i \mid a \in \mathbb{N} \text{ and } i \in \mathbb{N} \text{ with } i \geq 2\}\). Then
\[ P(x^q) \text{ is irreducible in } \mathbb{Z}[x] \text{ for any } q \geq 1. \]

**Proof.** Note that \(P(x) = P(x^1)\) is irreducible (e.g. see [7]). Without loss of generality, we assume that \(q \geq 2\). Let \(\omega = e^{2\pi \sqrt{-1}/q}\). Then
\[ P(x^q) = \left( \prod_{i=0}^{q-1} (x - \omega^i \beta^{1/q}) \right) \cdot \left( \prod_{i=0}^{q-1} (x - \omega^i (\beta')^{1/q}) \right). \]
Suppose on the contrary that \(P(x^q) = Q_1(x)Q_2(x)\) and \(Q_1(x), Q_2(x) \in \mathbb{Z}[x]\) with \(\deg Q_1, \deg Q_2 \geq 1\). We note that
\[ m = |P(0)| = |Q_1(0)| \cdot |Q_2(0)|, \]
where
\[ |Q_1(0)| = |\beta^{u_1}(\beta')^{v_1}|^{1/q} \in \mathbb{N} \quad \text{and} \quad |Q_2(0)| = |\beta^{u_2}(\beta')^{v_2}|^{1/q} \in \mathbb{N} \]
with \(u_1, v_1, u_2, v_2 \geq 1\).

We will show that \(u_1 = v_1\). Otherwise by symmetry we may assume that \(u_1 > v_1\), then
\[ (\beta^{u_1-v_1}) = \frac{|Q_1(0)|^q}{|\beta^{v_1}||m^{v_1}|^q} = \frac{|Q_1(0)|^q}{(m^{v_1})^q}, \]
which implies
\[ R(\beta) = 0 \quad \text{with} \quad R(x) = m^{v_1}x^{u_1-v_1} - |Q_1(0)|^q \in \mathbb{Z}[x]. \]
By [7], we obtain that $P(x) = x^2 - nx + m$ is an irreducible polynomial satisfying $P(\beta) = 0$. Therefore, we have

$$P | R$$

but $R$ only has roots with module $\beta$.

Now $R'(\beta) = P'(\beta) = 0$ with $|\beta'| < |\beta|$. This is a contradiction.

In the same way, we have $u_2 = v_2$. Now we obtain that

$$u_1 = v_1 \text{ and } u_2 = v_2.$$ 

Let $u_1/q = j/i$ with $(i, j) = 1$ and $i \geq 2$, then $u_2/q = (i - j)/i$ since $u_1 + u_2 = q$. Hence

$$|Q_1(0)| = m^i j \in N \text{ and } |Q_2(0)| = m^{i-1} \in N$$

and thus $m^i = a \in N$ and $m = a^i$ with $i \geq 2$. This is a contradiction. $\square$

### 3.3. Proof of Theorem.

It follows from Propositions 1-2 that there are $r \in (0, 1)$ and $k, k_1 \leq k_2 \leq \cdots \leq k_t \in N$ such that

$$\bar{P}(r^{-s}) = \bar{Q}(r^{-s}) = 0,$$

where $\bar{P}$ and $\bar{Q}$ are defined in (3.1). Suppose on the contrary that $\bar{P}(x) = P(x^k) = x^{2k} - nx^k + m$ is irreducible in $\mathbb{Z}[x]$, then we have

$$P(x^k)(x^i - \sum_{i=1}^{t} x^{k_i - k_i}),$$

which contradicts to Proposition 3. Therefore $P(x^k)$ is reducible in $\mathbb{Z}[x]$, and thus $m \in \{a^i \mid a \in N \text{ and } i \in N \text{ with } i \geq 2\}$ by Proposition 4.

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