A class of Locally Nilpotent Commutative Algebras

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Abstract

This paper deals with the variety of commutative nonassociative algebras satisfying the identity $L_x^3 + \gamma L_x = 0$, $\gamma \in K$. In [2] it is proved that if $\gamma = 0, 1$ then any finitely generated algebra is nilpotent. Here we generalize this result by proving that if $\gamma \neq -1$, then any such algebra is locally nilpotent. Our results require characteristic $\neq 2, 3$.

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1 Introduction

Let $\mathcal{A}$ be the free commutative (nonassociative) algebra on a finite set $\{t_1, \ldots, t_k\}$ of generators. Then $\mathcal{A}$ is spanned by the commutative monomials in the $t_i$'s and, therefore, it is naturally graded $\mathcal{A} = \oplus_{n \geq 1} \mathcal{A}_n$, with $\mathcal{A}_1 = \text{span} \{t_1, \ldots, t_k\}$. In other words $\deg(t_i) = 1$ for $i = 1, \ldots, k$. (Actually, $\mathcal{A}$ is $\mathbb{Z}^k$-graded too, with $t_i$ being homogeneous of degree the $k$-tuple with 1 in the $i$-th position and 0's elsewhere).

Let $L_{x_1}L_{x_2} \cdots L_{x_n}$ be a string of left multiplications by monomials $x_i$. The length of the string is $n$. The total degree of the string is $\sum_{i=1}^n \deg(x_i)$. The max degree is the maximum of $\{\deg(x_1), \deg(x_2), \ldots, \deg(x_n)\}$.

This paper studies the variety of commutative (nonassociative) algebras $\mathcal{A}$ satisfying the identity

$$L_xL_xL_x + \gamma L_x^3 = 0, \quad \gamma \in K$$

whose linearizations are

$$y(x(xa)) + x(y(xa)) + x(x(ya)) + 2\gamma((xy)x)a + \gamma((xx)y)a = 0$$

and $f(x, y, z, a) = z(y(xa)) + y(z(xa)) + z(x(ya)) + x(z(ya)) + y(x(za))$

$$+ x(y(za)) + 2\gamma[(xy)z)a + ((yz)x)a + ((zx)y)a] = 0 \quad (1)$$

In [2] it is proved that if $\gamma = 0, 1$ then any finitely generated algebra is nilpotent. Here we generalize this result by proving that if $\gamma \neq -1$, then any such algebra is locally nilpotent.

In terms of the operators $L_x$'s and using that $\mathcal{A}$ is commutative we obtain the following identities

$$L_xL_xL_y + L_xL_yL_x + L_yL_xL_x + \gamma[2L_{(xy)x} + L_{(xx)y}] = 0 \quad (2)$$

$$L_xL_yL_z + L_xL_zL_y + L_yL_xL_z + L_yL_zL_x$$

$$+ L_zL_xL_y + L_zL_yL_x + 2\gamma[L_{(xy)z} + L_{(yz)x} + L_{(zx)y}] = 0 \quad (3)$$
\[L_x(xa) + L_x L_x L_a + L_x L_x L_a + \gamma [2L_a L_x L_x + L_a L_{xx}] = 0 \quad (4)\]

\[L_y(ax) + L_y L_x ax + L_y L_x ay + L_y L_x L_a + L_x L_y L_a + 2\gamma [L_a L_x y + L_a L_x L_y + L_a L_y L_x] = 0 \quad (5)\]

A string or a linear combinations of strings of the same length is called *reducible* if it is expressible as a linear combination of strings each of which has the same total degree but of shorter lengths. If \(X\) and \(Y\) are strings or linear combinations of strings of the same length we will say that \(X \equiv Y\) if and only if \(X - Y\) is reducible. That is, \(X - Y\) is expressible as a linear combination of strings each of which has the same total degree but length less than the common length of \(X\) and \(Y\). When \(X \equiv Y\) we say \(X\) is equivalent to \(Y\).

Using the above definition, identities (2), (3), (4) and (5) become

\[L_x L_y L_x + L_x L_y L_x + L_x L_y L_x \equiv 0 \quad (6)\]
\[L_x L_y L_z + L_x L_y L_z + L_y L_x L_z + L_y L_x L_z + L_x L_y L_x + L_x L_y L_x \equiv 0 \quad (7)\]
\[L_x L_x L_a \equiv -2\gamma L_a L_x L_x \quad (8)\]
\[L_y L_x L_a \equiv -(L_x L_y L_a + 2\gamma [L_a L_x L_y + L_a L_y L_x]) \quad (9)\]

## 2 Reducing string

**Lemma 1.** \(L_x L_{x_1} \cdots L_{x_n} L_x\) is equivalent to a linear combination of strings of the same total length where the \(L_x\)'s are adjacent.

**Proof.** We proceed by induction on \(n\): For \(n = 1\) we get the string \(L_x L_y L_x\).

Using (6) we have \(L_x L_y L_x \equiv -L_x L_x L_y - L_y L_x L_x\).

Assume that the result is true if the distance that the \(L_x\)'s are apart is less than \(k\). Using (8).

\[L_x L_{x_1} L_x L_{x_2} L_{x_3} \cdots L_{x_k} L_x \equiv -(L_{x_1} L_{x_1} L_{x_2} + 2\gamma (L_x L_x L_{x_1} L_x + L_x L_x L_{x_1})) L_{x_3} \cdots L_{x_k} L_x\]

Altogether, there are three strings represented in the above expression. In each of these three strings, the \(L_x\)'s are less than \(k\) apart.

By induction each is equivalent to a linear combination of strings where the \(L_x\)'s are adjacent. \(\Box\)
Lemma 2. The string \( L_x L_x L_y L_y \) is reducible if \( \gamma \neq \pm \frac{1}{2} \).

Proof. \[
(L_x L_x L_y) L_y \equiv -2\gamma (L_y L_x L_x) L_y \quad \text{using (8)} \\
\equiv 2\gamma (L_y L_y L_x + L_x L_x L_y) L_y \quad \text{using (6)} \\
\equiv 2\gamma L_x (L_y L_x L_y + L_x L_y L_y) \\
\equiv -2\gamma L_x (L_y L_y L_x) \quad \text{using (6)} \\
\equiv 4\gamma^2 L_x (L_y L_y L_y) \quad \text{using (8)} \text{ with } x \text{ and } y \text{ interchanged.}
\]

Therefore \((4\gamma^2 - 1)L_x L_x L_y L_y\) is reducible, and since \( \gamma \neq \pm \frac{1}{2} \), \( L_x L_x L_y L_y \) is reducible.

\[\square\]

Lemma 3. \( L_x L_x L_x \) is reducible.

Proof. \( L_x L_x L_x = -\gamma L_x^3 \equiv 0 \) \[\square\]

Lemma 4. \( L_x L_x L_x L_x_1 L_x_2 \cdots L_x_k L_y L_y \) is reducible if \( \gamma \neq \pm \frac{1}{2} \).

Proof. We use induction on \( k \). If \( k = 0 \), \( L_x L_x L_y L_y \) is reducible using Lemma 2. Now in general, using (6),
\[
L_x L_x L_x_1 L_x_2 \cdots L_x_k L_y L_y \equiv -2\gamma L_{x_1} L_x L_x_2 L_x_2 \cdots L_x_k L_y L_y \\
\equiv 0 \text{ by induction.}
\]

\[\square\]

Let \( A \) be the free commutative (but not associative) algebra with \( k \) generators. Let \( \text{dim}[n, k] \) be the dimension of the subspace of \( A \) which is spanned by terms of degree less than \( n \). Thus \( \text{dim}[n, k] \) is the number of distinct monomials of \( A \) with degree less than \( n \).

The proof of the following results depends on the above Lemmas and it is similar to the one given in [2].

Lemma 5. Let \( A \) be the free commutative (but not associative) algebra with \( k \) generators satisfying the identity \( L_x^3 + \gamma L_x^3 = 0 \) with \( \gamma \neq \pm \frac{1}{2} \). Then any string of total degree \( \geq n \text{dim}[n, k] \) is reducible to strings whose max degree is \( \geq n \) or that have an adjacent pair of identical \( L_i \)s.
Theorem 1. Let $A$ be the free commutative (but not associative) algebra with $k$ generators satisfying identity $L^3_x + \gamma L^3_x = 0$ with $\gamma \not= \pm \frac{1}{2}$. Then any string of total degree $\geq 2n \dim[n,k] + (n - 2)$ is reducible to a linear combination of strings of maximal degree greater than or equal to $n$.

3 Nilpotency

In this section $A$ will be a commutative algebra satisfying the identity $L^3_x + \gamma L^3_x = 0$.

We define the function $J(x, y, z)$ by $J(x, y, z) = (xy)z + (yz)x + (zx)y$.

Lemma 6. Let $A$ be a commutative algebra over a field of characteristic $\not= 2$ that satisfies the identity $L^3_x + \gamma L^3_x = 0$ with $\gamma \not= 0, -1$. Let $W$ be the linear subspace of $A$ generated by the elements of the form $x^3$ with $x \in A$. Then $W$ is an ideal of $A$.

Proof. Recall the first linearization of $L^3_x + \gamma L^3_x = 0$:

$$y(x(ax)) + x(y(xa)) + x(x(ya)) + 2\gamma((xy)x)a + \gamma((xx)y)a = 0$$

Replacing $a$ by $x$ and $y$ by $a$ we obtain $ax^3 + x(ax^2) + x(x(ax)) + 2\gamma x(x(ax)) + \gamma x(x^2a) = 0$. That is,

$$(1 + 2\gamma)x(x(ax)) + (1 + \gamma)x(x^2a) + x^3a = 0. \quad \text{(10)}$$

On the other hand, we have

$$J(xa, x, x) = 2x(x(ax)) + x^2(xa), \quad J(x^2, x, a) = x^3a + x^2(xa) + x(x^2a).$$

Subtracting both identities we have

$$x(x^2a) = J(x^2, x, a) - J(xa, x, x) + 2x(x(ax)) - x^3a.$$  

Replacing this value in (10) and reordering we have

$$(3 + 4\gamma)x(x(ax)) + (1 + \gamma)[J(x^2, x, a) - J(xa, x, x)] - \gamma x^3a = 0 \quad \text{(**) }$$

Using that $x(x(ax)) + \gamma x^3a = 0$, we obtain that

$$-4\gamma(1 + \gamma)x^3a + (1 + \gamma)[J(x^2, x, a) - J(xa, x, x)] = 0.$$
Since $\gamma \neq 0, -1$, we obtain that
\[ x^3a = \frac{1}{4\gamma}[J(x^2, x, a) - J(xa, x, x)] \subseteq W \]

since $J(x, a, a) = \frac{1}{2}[(x + a)^3 - (x - a)^3] - a^3 \subseteq W$ and $J(x, y, z) = \frac{1}{6}(J(x + z, x + z, y) - J(x - z, x - z, y)) \subseteq W$. Therefore $W$ is an ideal of $A$.

\[ \square \]

If $A$ satisfies the identity $L^3_x + \gamma L_{x^3} = 0$ and $\gamma \neq -1$, then $A$ satisfies the identity
\[ x(x(xx)) = 0 \] (11)

Linearizing completely (11) we get:

\[ 0 = g(x, y, z, a) = (a(xyz) + a(yzx) + a(zyx) + x(ayz)) + y(xaz) + y(zax) + z(axy) + z(yax). \] (12)

**Theorem 2.** Let $A$ be a commutative algebra over a field of characteristic $\neq 2, 3$ that satisfies the identity $L^3_x + \gamma L_{x^3} = 0$ with $\gamma(\gamma^2 - 1)(4\gamma^2 - 1) \neq 0$. Then the ideal $W$ of $A$ in Lemma 6 satisfies $W^2 = 0$.

**Proof.** Consider the free commutative algebra $A$ on two variables $x$ and $y$. This is $\mathbb{Z}^2$-graded: $A = \oplus_{n,m \geq 1} A_{n,m}$. Also, there is a natural order 2 automorphism $\phi$ which permutes these two variables. The automorphism $\phi$ satisfies $\phi(A_{n,m}) = A_{m,n}$ for any $n, m$. In particular it restricts to a linear order 2 automorphism of $A_{3,3}$. The natural basis for the subspace of fixed elements by $\phi$ (that is, the subspace of elements which are symmetric in the two variables) in $A_{3,3}$ is the following set $B$ consisting of 27 elements:

\[ B = \{(x(x(x(yy)))) + (y(y(y(x(xx))))) + (x(x(y(x(yy))))) + (y(y(x(yy)))),(x(x(y(x(yy))))) + (y(y(x(x(yy))))) + (x(x((xy)(yy)))) + (y(y((xx)(xy)))),(x(y(x(yy)))) + (y(x(x(yy)))),(x((xy)(yy)))) + (y((xx)(xy))),((xx)(yy)) + (y((yy)(xx))),((xy)(xx)) + (y((yy)(xx))),((yy)(xx)) + (y((yy)(xx))),((xx)((yy)(xx))) + (y((yy)((xx)(yy)))) \} \]
A subspace of symmetric elements in $\mathcal{A}_{3,3}$ obtained from specializations of the full linearization $f(x, y, z, a)$ of the identity $L_2^3 + \gamma L_3^3 = 0$ as in [3]:

$$
\mathcal{L} = \{ f(y, y, (xx)x) + f(x, x, (yy)y), f(x, y, (xy)x) + f(y, x, (yx)y), f(yy, x, xx) + f(xx, y, yy), f(x, y, x, y, xy), f(xy, y, x, x, y), f((yy)x, x, x, y), f((xy)x, y, y, x) + f((yx)y, x, x, y), f(yy, x, y, x, y) + f(yy, xx, x, y), f(xy, xy, x, y) + f(xy, xx, y, y), f(yx, y, x, y) + f(x, x, y, yy)y, f(x, y, x, y, y, y) f(x, x, y, yy) + f(y, x, x, y), f(xy, y, x, x, y) + f(xx, x, x, y) + f(yy, x, x, y), f(yy, x, x, x) + f(xx, y, x, y) + f(yy, x, x, y), f(yy, x, x, y, x) + f(xx, y, x, x) + f(yy, x, x, x), f(yy, x, x, x) + f(xx, y, x, x), f(yy, x, x, x, y) + f(xx, y, x, x, y) + f(yy, x, x, x, y)
\}.
$$

On the other hand, consider the following family $\mathcal{L}$ of 27 elements in the subspace of symmetric elements in $\mathcal{A}_{3,3}$ obtained from specializations of the identity $L_2^3 + \gamma L_3^3 = 0$ as in [1]:

$$
\mathcal{L} = \{ f(x, y, y, (xx)x) + f(x, x, (yy)y), f(x, y, y, (xy)x) + f(y, x, (yx)y), f(yy, x, xx) + f(xx, y, yy), f(x, y, x, y, xy), f(yy, x, y, x, y) + f(yy, xx, x, y), f(xy, xy, x, y) + f(xy, xx, y, y), f(yx, y, x, y) + f(x, x, y, yy)y, f(x, y, x, y, y, y) f(x, x, y, yy) + f(y, x, x, y), f(xy, y, x, x, y) + f(xx, x, x, y) + f(yy, x, x, y), f(yy, x, x, x) + f(xx, y, x, y) + f(yy, x, x, x), f(yy, x, x, x) + f(xx, y, x, x), f(yy, x, x, x, y) + f(xx, y, x, x, y) + f(yy, x, x, x, y)
\}.
$$

Note that the specializations of all the elements in $\mathcal{L}$ by means of elements in a commutative algebra $A$ over a field of characteristic $\neq 2, 3$ satisfying the identity $L_2^3 + \gamma L_3^3 = 0$ is 0.

Let $M$ be the matrix with rows representing the elements of $\mathcal{L}$ in the basis $\mathcal{B}$.

Using SAGE ([3]) and MAGMA ([1]) we compute the determinant of this matrix:

$$
\det(M) = -2^{33}3^4\gamma^5(\gamma - 1)(\gamma + 1)^{10}(2\gamma - 1)^3(2\gamma + 1)
$$

For those values of $\gamma$ for which the determinant is not zero, we can invert the matrix and therefore we can write any element of $\mathcal{B}$ as a linear combination of relations in $\mathcal{L}$. In particular, when $\det(M) \neq 0$, $x^3y^3 = 0$ is an identity in our algebra $A$, so we conclude that $W^2 = 0$. \(\square\)
Theorem 3. Any commutative algebra over a field of characteristic \( \neq 2, 3 \), satisfying the identity \( L^3_x + \gamma L x^3 = 0 \) with \( \gamma \neq 0, \pm 1, \pm \frac{1}{2} \) is locally nilpotent.

More precisely, any commutative algebra generated by \( k \) elements over a field of characteristic \( \neq 2, 3 \), satisfying the identity \( L^3_x + \gamma L x^3 = 0 \) with \( \gamma \neq 0, \pm 1, \pm \frac{1}{2} \) is nilpotent of index at most \( 2^{4n \dim[n,k] + 2(n-2)} \), where \( n \) is the index of nilpotency of the free commutative algebra on \( k \) generators satisfying the identity \( x^3 = 0 \).

Proof. Any product of total degree \( \geq 2^{4n \dim[n,k] + 2(n-2)} \) is expressible as a string of length greater than \( 4n \dim[n,k] + 2(n-2) \).

By Theorem 1, any string of total degree \( \geq 2n \dim[n,k] + (n-2) \) in the finitely generated commutative algebra is reducible to a linear combination of strings in which one of the factors is of degree greater than or equal to \( n \). Passing to the homomorphic image satisfying the identity \( L^3_x + \gamma L x^3 = 0 \) with \( \gamma \neq 0, \pm 1, \pm \frac{1}{2} \), this factor of degree greater than \( n \) must lie in \( W \).

If we let the length of the string be twice as long, then there will be two factors from \( W \). On multiplying these strings out, the result will be zero because \( W^2 = 0 \). This finishes the proof of Theorem 3. \( \square \)

4 Exceptional Cases

We now look at the five cases which arose as exceptions in Theorem 3.

Case \( \gamma = 0 \) or 1. In [2] it was proved that in these cases every finitely generated commutative algebra \( A \) satisfying the identity \( L^3_x = 0 \), or \( L^3_x + L x^3 = 0 \) is nilpotent.

Case \( \gamma = -1 \). The identity becomes \( L^3_x - L x^3 = 0 \) or \( x(x(xy)) = x^3 y \). We observe that any associative algebra satisfies this identity. In particular the algebra of polynomials in a finite set of variables satisfies \( L^3_x - L x^3 = 0 \), it is finitely generated but not nilpotent. Therefore, Theorem 3 cannot be extended to this case.

Case \( \gamma = \frac{1}{2} \). The identity becomes \( L^3_x + \frac{1}{2} L x^3 = 0 \).

In this case, replacing \( a \) by \( x \) in identity (4) with \( \gamma = \frac{1}{2} \) we obtain

\[
L x^3 + L x L x^2 + L x^3 + \frac{1}{2}[2L x^3 + L x L x^2] = 0
\]
That is,

\[ L_{x^3} + 2L_{x^3} + \frac{3}{2}L_xL_{x^2} = 0 \]

Therefore, \( \frac{3}{2}L_xL_{x^2} = 0 \) and characteristic not 3 implies that

\[ L_xL_{x^2} = 0. \] (13)

Now, replacing \( a \) by \( x^2 \) in identity (4) with \( \gamma = \frac{1}{2} \) we obtain the identity

\[ L_{x^4} + L_xL_{x^3} + L_x^2L_x^2 + L_xL_xL_x + \frac{1}{2}L_{x^2}L_x^2 = 0 \]

where \( x^4 = x^3x \).

Using that \( x^4 = 0, L_{x^3} = -2L_{x^3} \) and \( L_xL_{x^2} = 0 \) we have that

\[-2L_x^4 + L_x^2L_x^2 + L_xL_xL_x = 0. \] (14)

Multiplying (14) to the left by \( L_x \) and using (13) we obtain

\[ L_x^5 = 0, \] (15)

so \( L_x \) is nilpotent, and by linearization of \( L_x^5 \), using (13), we get

\[ L_x^2L_x^4 = 0. \]

Linearize (13) to obtain \( 2L_xL_{xy} + L_yL_{x^2} = 0 \) which, with \( y = x^2 \) gives

\[ (L_{x^2})^2 = 4L_x^4, \]

which transforms, using (14), into

\[ L_x^2L_x^2 = 0. \] (16)

Finally, with \( y = x^2 \) in (6), equations (13) and (16) give

\[ L_{x^2x^2} = 0. \] (17)

That is, for every \( x \in A \), \( x^2x^2 \in \text{ann}(A) = \{ z \in A : zA = 0 \} \).

**Theorem 4.** Any commutative algebra over a field of characteristic \( \neq 2, 3 \), satisfying the identity \( L_x^3 + \frac{1}{2}L_{x^3} = 0 \), is locally nilpotent.
Proof. Let $A$ be a commutative algebra over a field of characteristic $\neq 2, 3$, satisfying the identity $L_x^3 + \frac{1}{2}L_x^3 = 0$. It is enough to prove that $\tilde{A} = A/\text{ann}(A)$ is locally nilpotent.

But according to (17), $\tilde{A}$ satisfies the equation $x^2 x^2 = 0$, and hence, by linearization, the equation $(xy)x^2 = 0$. On the other hand, equation (13) shows that $x(yx^2) = 0$. In particular, $\tilde{A}$ is a Jordan algebra. Let $W$ be the ideal spanned by its cubes (Lemma 5). Since $(x^3)^2 = L_x^3(x^3) = 2x(x(x^3)) = 0$, and $L_x^2 = 4L_y^2 = 0 = L(x^2)^2$, it turns out that the cubes are absolute zero divisors of the Jordan algebra $\tilde{A}$. Thus $W$ is a locally nilpotent ideal of $\tilde{A}$, because of Zel’manov’s Local Nilpotence Theorem (see [3, p. 1004] and [5]). Also the algebra $\tilde{A}/W$ satisfies the identity $x^3 = 0$, so it is locally nilpotent (see [6, p. 114]). Therefore the Jordan algebra $\tilde{A}$ is locally nilpotent ([6, Chapter 4, Lemma 7]), and so is $A$, as required.

Case $\gamma = -\frac{1}{2}$. The identity becomes $L_x^3 = \frac{1}{2}L_x^3$.

In this case, our algebra $A$ satisfies $x^3 x = 0$ too. Also, $x^3 x^2 = 2x(x(x^2)) = 0$, and with $y = x^2$ in (2) it follows that $L_y^5 = 0$. Now, identities (5), (8) and (9) show that

\begin{align*}
L_x^2 L_y & \equiv L_y L_x^2, \tag{18} \\
L_x L_y L_x & \equiv -2L_y L_x^2, \tag{19}
\end{align*}

and these, together with Lemma 3, immediately imply the following result

Lemma 7. Any string of the form $L_x L_{x_1} \cdots L_{x_n} L_x$ is equivalent to a linear combination of strings of the same total length which end up in $L_x^2$. Also, any string containing three equal elements $L_x$ is reducible.

There is the following counterpart to Theorem 1

Theorem 5. Let $A$ be the free commutative (but not associative) algebra with $k$ generators satisfying identity $L_x^3 - \frac{1}{2}L_x^3 = 0$. Then any string of total degree $\geq 2n \dim[n, k] + (n - 2)$ is reducible to a linear combination of strings of maximal degree greater than or equal to $n$.

Proof. If the total degree of a string $L_{x_1} \cdots L_{x_m}$ is $\geq 2n \dim[n, k]$ and its maximal degree is $< n$, then its length $m$ is greater than $2 \dim[n, k]$, and therefore there are three monomials $x_i$ which are equal. By the previous Lemma this is reducible. The process can be continued until the maximal degree be $\geq n$, as required. \qed
And there is too the counterpart to Theorem 2.

**Theorem 6.** Let $A$ be a commutative algebra over a field of characteristic $\neq 2, 3$ that satisfies the identity $L^3 = \frac{1}{2} L^3$. Then the ideal $W$ of $A$ in Lemma 2 satisfies $W^2 = 0$.

**Proof.** Consider again the free commutative algebra $A$ on two variables $x$ and $y$ and the subspace of symmetric elements in $A_{3,3}$ spanned by the linearly independent subset (consisting of 19 elements):

$$
\mathcal{B}' = \{ x(x(x(yy)))+y(y(x(xy))) , x(x(y(xyy)))+y(y(x(xxy))) , \\
x(x(y(xy)))+y(y(x(xy)))+y(y(x(xyy))) , \\
x(x(y(xxy)))+y(y(x(yxy)))+y(y(x(yyxy))) , \\
x(x(xxy)))+y(y(x(xyy)))+y(y(x(yyxy))) , \\
x((x(xxy)))+y((x(x)x(yy)))+y((x(x)y)) , \\
x((yy)(x(y)))+y((y)(x)) ) (x(yy))+(y(x))(y(y)) , \\
(x(x)(y))(y)(xx) , (x)(x))(y)(xx) , (x(x))(y)(y(y)) \}.
$$

On the other hand, consider the following family $\mathcal{L}'$ of 19 elements in the subspace of symmetric elements in $A_{3,3}$ obtained from specializations of the full linearization $f(x, y, z, a)$ of the identity $L^3 = \frac{1}{2} L^3$ as in (11):

$$
\mathcal{L}' = \{ f(y, y, y, (xx)x)+f(x, x, x, (yy)y) , f(x, y, y, (xx)y)+f(y, x, x, (yy)x) , \\
f(x, y, y, (xy)x)+f(y, x, x, (yx)y) , f((xx)y)(y, y, y) , f((xx)y(y, y, y)+f((xx)y, y, y) , \\
f((xx)y)(x, x, y)+f((x)(x)y)(x, x, y) , f((yy)y)(x, x, y)+f((xx)x)(y, y, y) , \\
f(y, y, (xx)x)+f(x, x, x, yy)y) , f(x, y, y, x, y)y) , f(y, y, x, y) \} (x+y, y, y, x) y) , \\
(f(y, y, x, y) x)+f(x, x, y, y)y) , f(x, y, x, y) y)+f(y, y, x, x, y) , \\
f(x, x, y, x) x)+f(x, x, x, yy)y) , (f(x, x, y, y) x)+f(y, y, x, x, y) y) , \\
(f, y, y, x, y) x)+f(x, x, y, y) y)+f(x, x, y) x)+f(y, y, x) x)+f(y, y, x, x) y}.$$  

Let $M'$ be the matrix with rows representing the elements of $\mathcal{L}'$ in the basis $\mathcal{B}'$. Using again SAGE (11) and MAGMA (11) we compute the determinant of this matrix:

$$\det(M') = 2^{14}3^4.$$  

In particular, this shows that in any commutative algebra satisfying the identity $L^3 = \frac{1}{2} L^3$, $x^3y^3 = 0$ for any $x, y$, so that $W^2 = 0$.  

$\square$
Finally the same arguments as for Theorem 3 settle this exceptional case:

**Theorem 7.** Any commutative algebra over a field of characteristic \( \neq 2, 3 \), satisfying the identity \( L_x^3 = \frac{1}{2} L_x^3 \) is locally nilpotent.

More precisely, any commutative algebra generated by \( k \) elements over a field of characteristic \( \neq 2, 3 \), satisfying the identity \( L_x^3 = \frac{1}{2} L_x^3 \) is nilpotent of index at most \( 2^{4n \dim[n,k]} \), where \( n \) is the index of nilpotency of the free commutative algebra on \( k \) generators satisfying the identity \( x^3 = 0 \).

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