Abstract. In this paper, under suitable settings, we can obtain the existence of solutions to a class of prescribed Weingarten curvature equations in warped product manifolds of special type by the standard degree theory based on the a priori estimates for the solutions. This is to say that the existence of closed hypersurface (which is graphic with respect to the base manifold and whose k-th Weingarten curvature satisfies some constraint) in a given warped product manifold of special type can be assured.

Keywords: Prescribed Weingarten curvature equations, k-convex, starshaped, warped product manifolds.

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1. Introduction

Throughout this paper, let \((M^n, g)\) be a compact Riemannian \(n\)-manifold with the metric \(g\), and let \(I\) be an (unbounded or bounded) interval in \(\mathbb{R}\). Clearly, \(\bar{M} := I \times f M^n\) is actually the \((n+1)\)-dimensional warped product manifold (sometimes, for simplicity, just say warped product) endowed with the following metric
\[
\bar{g} = dt^2 + f^2(t)g,
\]
where \(f : I \rightarrow \mathbb{R}^+\) is a positive differential function defined on \(I\). Given a differentiable function \(u : M^n \rightarrow I\), its graph actually corresponds to the following graphic hypersurface
\[
\mathcal{G} = \{X(x) = (u(x), x)| x \in M^n\}
\]
in \(\bar{M}\). Equivalently, we can say that \(\mathcal{G}\) is graphic w.r.t. the base manifold \(M^n\). Denote by \(\bar{\nabla}, D\) the Riemannian connections on \(\bar{M}\) and \(M^n\), respectively. Let \(\{e_i\}_{i=1,2,...,n}\) be an orthonormal frame field in \(M^n\). Then one can find an orthonormal frame field \(\{\bar{e}_\alpha\}_{\alpha=0,1,...,n}\) in \(\bar{M}\) such that \(\bar{e}_i = \frac{1}{f}e_i\), \(1 \leq \alpha = i \leq n\) and \(\bar{e}_0 = \partial/\partial t\). The existence of the frame fields can always be assured in the tangent space of a prescribed point. Denote by \(\bar{\nabla} u_i := D_i u, u_{ij} := D_j D_i u\) and \(u_{ijk} := D_k D_j D_i u\) the covariant derivatives of \(u\) w.r.t. the metric \(g\). Clearly, the tangent vectors of \(\mathcal{G}\) are given by
\[
X_i = (Du, 1) = e_i + u_i \partial/\partial t = f \bar{e}_i + u_i \bar{e}_0, \quad i = 1,2,...,n.
\]
Let \(\langle\cdot, \cdot\rangle\) be the inner product w.r.t. the metric \(\bar{g}\). Then the induced metric \(\bar{g}\) on \(\mathcal{G}\) has the form
\[
\bar{g}_{ij} = \langle X_i, X_j \rangle = f^2 \delta_{ij} + u_i u_j,
\]
its inverse is given by
\[
\bar{g}^{ij} = \frac{1}{f^2} \left( \delta^{ij} - \frac{u^i u^j}{f^2 + |Du|^2} \right),
\]
where \( u^i = g^{ij}u_j = \delta^i_ju_j \) and \(|Du|^2 = u^iu_i\). Of course, in this paper we use the Einstein summation convention – repeated superscripts and subscripts should be made summation\(^2\).

The outward unit normal vector field of \( G \) is given by
\[
\nu = \frac{1}{\sqrt{f^2 + |Du|^2}} \left( f \frac{\partial}{\partial t} - u^if^{-1}e_i \right) = \frac{1}{\sqrt{f^2 + |Du|^2}} (f\bar{e}_0 - u^i\bar{e}_i),
\]
and the component \( h_{ij} \) of the second fundamental form \( A \) of \( G \) is computed as follows
\[
(1.3) \quad h_{ij} = -\langle \nabla X_j X_i, \nu \rangle = \frac{1}{\sqrt{f^2 + |Du|^2}} (-fu_{ij} + 2fu_{ij} + f^2f'\delta_{ij}).
\]

One can also see [3, Subsection 2.2] for the computations of the above geometric quantities. Denote by \( \lambda_1, \lambda_2, \ldots, \lambda_n \) the principal curvatures of \( G \), which are actually the eigenvalues of the matrix \( (h_{ij})_{n \times n} \) w.r.t. the metric \( \bar{g} \). The so-called \( k \)-th Weingarten curvature at \( X(x) = (u(x), x) \in G \) is defined as
\[
\sigma_k(\lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}.
\]

\( V = f(u)\frac{\partial}{\partial t} \) is the position vector field\(^3\) of hypersurface \( G \) in \( \bar{M} \), and clearly, for any \( x \in M^n \), \( V|_x \) is a one-to-one correspondence with \( X(x) \). Let \( \nu(V) \) be the outward unit normal vector field along the hypersurface \( G \) and \( \lambda(V) = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) be the principal curvatures of \( G \) at \( V \). Define the annulus domain \( \bar{M}^+ := \{(t, x) \in \bar{M} | r_1 \leq t \leq r_2 \} \) with \( r_1 < r_2 \). In this paper, we consider the following Weingarten curvature equation
\[
(1.5) \quad \sigma_k(\lambda(V)) = \sum_{l=0}^{k-1} \alpha_l(u(x), x)\sigma_l(\lambda(V)), \quad \forall V \in G, \quad 2 \leq k \leq n,
\]
where \( \{\alpha_l(u(x), x)\}_{l=0}^{k-1} \) are given smooth functions defined on \( G \). The \( k \)-th Weingarten curvature \( \sigma_k(\lambda(V)) \) is also called \( k \)-th mean curvature. Besides, when \( k = 1, 2 \) and \( n \), \( \sigma_k(\lambda(V)) \) corresponds to the mean curvature, the scalar curvature and the Gaussian curvature of \( G \) at \( V \).

We also need the following conception:

**Definition 1.1.** For \( 1 \leq k \leq n \), let \( \Gamma_k \) be a cone in \( \mathbb{R}^n \) determined by
\[
\Gamma_k = \{ \lambda \in \mathbb{R}^n | \sigma_l(\lambda) > 0, \ l = 1, 2, \ldots, k \}.
\]

A smooth graphic hypersurface \( G \subset \bar{M} \) is called \( k \)-admissible if at every position vector \( V \in G \), \( (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Gamma_k \).

For the Eq. (1.5), we can prove the following:

**Theorem 1.2.** Let \( M^n \) be a compact Riemannian \( n \)-manifold \( (n \geq 3) \) and \( \bar{M} = I \times_f M^n \), with the metric \( (1.4) \), be the warped product manifold defined as before. Assume that the warping function \( f \) is positive differential, \( f' > 0 \), and \( \alpha_l(u(x), x) \in C^\infty(I \times M^n) \) are positive functions for all \( 0 \leq l \leq k - 1 \). Suppose that
\[
(1.6) \quad \sigma_k(e) \left( f' \right)^k \geq \sum_{l=0}^{k-1} \alpha_l(u(x), x)\sigma_l(e) \left( f' \right)^l \quad \text{for} \ u \geq r_2,
\]

\(^2\)In this setting, repeated Latin letters should be made summation from 1 to \( n \).

\(^3\)In \( \mathbb{R}^{n+1} \) or the hyperbolic \( (n + 1) \)-space \( \mathbb{H}^{n+1} \), there is no need to define the vector field \( V \) since these two spaces are two-points homogeneous and global coordinate system can be set up, and then \( X(x) \) can be seen as the position vector directly.
Remark 1.1. (1) The Eq. (1.5) is actually a combination of elementary symmetric functions of eigenvalues of a given (0, 2)-type tensor. Equations of this type are important not only in the study of PDEs but also in the study of many important geometric problems. For instance, if \( \lambda(V) \) in the Eq. (1.5) were replaced by eigenvalues of the Hessian \( D^2 u \) of a graphic function \( u \) defined over a bounded \((k-1)\)-convex domain \( \Omega \subset \mathbb{R}^n \), Krylov [14] studied the corresponding PDE

\[
\sigma_k(D^2 u(x)) = \sum_{l=0}^{k-1} \alpha_l(x) \sigma_l(D^2 u(x)), \quad \forall x \in \Omega,
\]

with a prescribed Dirichlet boundary condition (DBC for short) and coefficients \( \alpha_l(x) \geq 0 \) for all \( 0 \leq l \leq k-1 \), and observed that the natural admissible cone to make equation elliptic is \( \Gamma_k \); recently, Guan-Zhang [11] showed that comparing with Krylov’s this observation, for the admissible solution of Eq. (1.10) with prescribed DBC in the sense that \( \lambda(D^2 u) \in \Gamma_{k-1} \), there is no sign requirement for the coefficient function of \( \alpha_{k-1}(x) \). Moreover, they also investigated the solvability of the following fully nonlinear elliptic equation

\[
\sigma_k(D^2 u + uI) \leq \sum_{l=0}^{k-1} \alpha_l(x) \sigma_l(D^2 u + uI), \quad \forall x \in \mathbb{S}^n,
\]

for some unknown function \( u : \mathbb{S}^n \to \mathbb{R} \) defined over \( \mathbb{S}^n \), where \( \alpha_l(x) \), \( 0 \leq l \leq k-2 \), are positive functions; Fu-Yau [6,7] proposed an equation of this type in the study of the Hull-Strominger system in theoretical physics; Phong-Picard-Zhang investigated the Fu-Yau equation and its generalization in series works [24, 25, 26]. Recently, inspired by Krylov’s and Guan-Zhang’s works [11, 14], Chen-Shang-Tu [2] considered the following equation

\[
\sigma_k(\kappa(X)) = \sum_{l=0}^{k-1} \alpha_l(X) \sigma_l(\kappa(X)), \quad \forall X \in \mathcal{M} \subset \mathbb{R}^{n+1}, \quad 2 \leq k \leq n
\]

Remark 1.1. (1) The \( k \)-admissible and the graphic properties of the hypersurface \( \mathcal{G} \) make sure that the Eq. (1.5) is a single scalar second-order elliptic PDE of the graphic function \( u \), which is the cornerstone of the a prior estimates given below. If furthermore \( M^n \) is convex, then \( M^n \) is diffeomorphic to \( \mathbb{S}^n \) (i.e., the Euclidean unit \( n \)-sphere), \( \mathcal{G} \) is also a graphic hypersurface over \( \mathbb{S}^n \) and should be starshaped. In this setting, Theorem 1.2 degenerates into the following:

- **FACT 1.** Under the assumptions of Theorem 1.2, if furthermore \( M^n \) is convex, then there exists a smooth \( k \)-admissible, starshaped closed graphic hypersurface \( \mathcal{G} \) contained in the interior of the annulus \( M^+ \) and satisfying the Eq. (1.5).

(2) We refer readers to, e.g., [21 Appendix A], [22] pp. 204-211 and Chapter 7 for an introduction to the notion and properties of warped product manifolds. Submanifolds in warped product manifolds have nice geometric properties and interesting results can be expected – see, e.g., several nice eigenvalue estimates for the drifting Laplacian and the nonlinear \( p \)-Laplacian on minimal submanifolds in warped product manifolds of prescribed type have been shown in [17] Sections 3-5.
on an embedded, closed starshaped \(n\)-hypersurface \(\tilde{M}\), \(n \geq 3\), where \(\kappa(X)\) are principal curvatures of \(\tilde{M}\) at \(X\), and \(\alpha_l(x)\), \(0 \leq l \leq k - 1\), are positive functions defined over \(\tilde{M}\). Under the \(k\)-convexity for \(\tilde{M}\) and several other growth assumptions (see [2] Theorem 1.1), they can show the existence of solutions to Eq. (1.10). This result has already been generalized by Shang-Tu [27] to the situation that the ambient space \(\mathbb{R}^{n+1}\) was replaced by the hyperbolic space \(\mathbb{H}^{n+1}\).

If \(M^n = S^n\), \(I = (0, \ell)\) with \(0 < \ell \leq \infty\), putting a one-pint compactification topology by identifying all pairs \(\{0\} \times S^n\) with a single point \(p^*\) to \(\tilde{M}\) (see, e.g., [5] page 705 for this notion) and requiring that \(f(0) = 0\), \(f'(0) = 1\), then the warped product manifold \(\tilde{M}\) becomes the spherically symmetric manifold \(\tilde{M} := [0, \ell) \times f S^n\). The single point \(p^*\) is called the base point of \(\tilde{M}\). Applying FACT 1 in Remark 1.1 directly, one has:

**Corollary 1.3.** Under the assumptions of Theorem 1.2 with additionally \(M^n = S^n\), \(I = (0, \ell)\) with \(0 < \ell \leq \infty\), one-pint compactification topology imposed, \(f(0) = 0\) and \(f'(0) = 1\), then there exists a smooth \(k\)-admissible, starshaped (w.r.t. the base point \(p^*\)), closed hypersurface \(\mathcal{G}\) contained in the interior of the annulus \(\tilde{M}^+ \subset \tilde{M}\) and satisfying the Eq. (1.5).

**Remark 1.2.** (1) If furthermore the warping function \(f\) satisfies \(f''(t) + Kf(t) = 0\) for some constant \(K\), i.e. the Jacobi equation, then

\[
 f(t) = \begin{cases} 
 \sin(\sqrt{K}t)/\sqrt{K}, & K > 0, \ \ell = \pi/\sqrt{K}, \\
 t, & K = 0, \ \ell = \infty, \\
 \sinh(\sqrt{-K}t)/\sqrt{-K}, & K < 0, \ \ell = \infty,
\end{cases}
\]

and moreover, in this setting, \(\tilde{M}\) corresponds to \(S^{n+1}(1/\sqrt{K})\) (i.e., the Euclidean \((n + 1)\)-sphere with radius \(1/\sqrt{K}\)) with the antipodal point of \(p^*\) missed, \(\mathbb{R}^{n+1}\) and \(\mathbb{H}^{n+1}(K)\) (i.e., the hyperbolic \((n + 1)\)-space with constant curvature \(K < 0\)), respectively. From this, one can see that spherically symmetric manifolds cover space forms as a special case and actually they were called *generalized space forms* by Katz and Kondo [12].

(2) Clearly, our Corollary 1.3 covers Chen-Shang-Tu’s and Shang-Tu’s main results in [2, 27] (mentioned in (3) of Remark 1.1) as special cases.

(3) Spherically symmetric manifolds have nice symmetry in non-radial direction, which leads to the fact that one can use this kind of manifolds as model space in the study of comparison theorems. In fact, Prof. J. Mao and his collaborators have used spherically symmetric manifolds as model space to successfully obtain Cheng-type eigenvalue comparison theorems for the first Dirichlet eigenvalue of the Laplacian on complete manifolds with radial (Ricci and sectional) curvatures bounded, Escobar-type eigenvalue comparison theorem for the first nonzero Steklov eigenvalue of the Laplacian on complete manifolds with radial sectional curvature bounded from above, heat kernel and volume comparison theorems for complete manifolds with suitable curvature constraints, and so on – see [5, 18, 19, 21, 29] for details.

This paper is organized as follows. In Section 2 we will list some useful formulae including several basic properties of \(\sigma_k\), structure equations for hypersurfaces in warped product manifolds. A priori estimates (including \(C^0\), \(C^1\) and \(C^2\) estimates) for solutions to the Eq. (1.5) will be shown continuously in Sections 3-5. In Section 6 by applying the degree theory, together with the a priori estimates obtained, we can prove the existence of solutions to prescribed Weingarten curvature equations of type (1.5).

### 2. Some useful formulae

Except the setting of notations in Section 1 denote by \(\bar{\nabla}, \nabla\) the Riemannian connections on \(\tilde{M}\) and \(\mathcal{G}\), respectively. The curvature tensors in \(\tilde{M}\) and \(\mathcal{G}\) will be denoted by \(\bar{R}\) and \(R\).
respectively. Let \( \{E_0 = \nu, E_1, \cdots, E_n\} \) be an orthonormal frame field in \( \mathcal{G} \) and \( \{\omega_0, \omega_1, \cdots, \omega_n\} \) is its associated dual frame field. The connections forms \( \{\omega_{ij}\} \) and curvature forms \( \{\Omega_{ij}\} \) in \( \mathcal{G} \) satisfy the structure equations

\[
d\omega_i - \sum_i \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \omega_{ji} = 0,
\]

\[
d\omega_{ij} - \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij} = -\frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l.
\]

The coefficients \( h_{ij}, 1 \leq i, j \leq n, \) of the second fundamental form are given by Weingarten equation

\[
\omega_{i0} = \sum_j h_{ij} \omega_j.
\]

The covariant derivatives of the second fundamental form \( h_{ij} \) in \( \mathcal{G} \) are given by

\[
\sum_k h_{ijk} \omega_k = dh_{ij} + \sum_l h_{il} \omega_{lj} + \sum_l h_{lj} \omega_{li},
\]

\[
\sum_l h_{ijk}\omega_l = dh_{ijk} + \sum_l h_{ijl} \omega_l + \sum_l h_{ilk} \omega_l + \sum_l h_{ljk} \omega_l.
\]

The Codazzi equation is

\[
h_{ijk} - h_{ikj} = -\bar{R}_{0ijk},
\]

and the Ricci identity can be obtained as follows:

**Lemma 2.1.** (see also [3] Lemma 2.2) Let \( X(x) \) be a point of \( \mathcal{G} \) and \( \{E_0 = \nu, E_1, \cdots, E_n\} \) be an adapted frame field such that each \( E_i \) is a principal direction and \( \omega_i^k = 0 \) at \( X(x) \). Let \( (h_{ij}) \) be the second quadratic form of \( \mathcal{G} \). Then, at the point \( X(x) \), we have

\[
h_{liti} = h_{litl} - h_{tim}(h_{mi}h_{li} - h_{ml}h_{ti}) - h_{mi}(h_{ml}h_{li} - h_{ml}h_{ti}) + \bar{R}_{0liti} - 2h_{ml}\bar{R}_{milt} + h_{il}\bar{R}_{0i0t} + h_{lt}\bar{R}_{0i0l} + \bar{R}_{0lijk}.
\]

As mentioned in Section [1], one can suitably choose local coordinates such that \( \{e_i\}_{i=1,2,\cdots,n} \) is an orthonormal frame field in \( M^n \), and then one can find an orthonormal frame field \( \{\bar{e}_i\}_{\alpha=0,1,\cdots,n} \) in \( \bar{M} \) such that \( \bar{e}_i = (1/f)e_i, 1 \leq \alpha = i \leq n \) and \( \bar{e}_0 = \partial/\partial t \). Correspondingly, the associated dual frame field of \( \{\bar{e}_i\}_{\alpha=0,1,\cdots,n} \) should be \( \{\theta_\alpha\}_{\alpha=0,1,\cdots,n} \) with \( \theta_\alpha = f\theta_i, 1 \leq i \leq n, \) and \( \theta_0 = dt \). Clearly, \( \{\theta_i\}_{i=1,2,\cdots,n} \) is the dual frame field of the orthonormal frame field \( \{e_i\}_{i=1,2,\cdots,n} \). We have the following fact:

**Lemma 2.2.** (see [3]) On the leaf \( M_t \) of the warped product manifold \( \bar{M} = I \times_f M^n \), the curvature satisfies

\[
\bar{R}_{ijk0} = 0
\]

and the principal curvature is given by

\[
\kappa(t) = \frac{f'(t)}{f(t)}
\]

where the outward unit normal vector \( \bar{e}_0 = \frac{\partial}{\partial t} \) is chosen for each leaf \( M_t \).
Remark 2.1. In fact, the leaf $M_t$ can also be seen as a closed graphic hypersurface in $\bar{M}$, which corresponds to the graph of some constant function, i.e. $u = \text{const}$. Besides, we refer readers to [3] Section 2 or [23] for the geometry of hypersurfaces in warped product manifolds if necessary.

Consider two functions $\tau : G \to \mathbb{R}$ and $\Lambda : G \to \mathbb{R}$ given by
\begin{align}
\tau &= f(\nu, e_0) = \langle V, \nu \rangle, \\
\Lambda &= \int_0^u f(s) ds,
\end{align}
where $V = fe_0 = f\frac{\partial}{\partial t}$ is the position vector field and $\nu$ is the outward unit normal vector field. Then we have:

**Lemma 2.3.** (see [1]) The gradient vector fields of the functions $\tau$ and $\Lambda$ are
\begin{align}
\nabla E_i \Lambda &= f \langle e_0, E_i \rangle, \\
\nabla E_i \tau &= \sum_j \nabla E_j \Lambda h_{ij},
\end{align}
and the second order derivatives of $\tau$ and $\Lambda$ are given by
\begin{align}
\nabla^2 E_i \Lambda &= -\tau h_{ij} + f' g_{ij}, \\
\nabla^2 E_i \tau &= -\tau \sum_k h_{ik} h_{kj} + f' h_{ij} + \sum_k (h_{ijk} + \bar{R}_{0ijk}) \nabla E_k \Lambda.
\end{align}

The following Newton-Maclaurin inequality will be used frequently (see, e.g., [16, 28]).

**Lemma 2.4.** Let $\lambda \in \mathbb{R}^n$. For $0 \leq l \leq k \leq n$, $r > s \geq 0$, $k \geq r$, $l \geq s$, we have
\[k(n - l + 1)\sigma_{l-1}(\lambda) \sigma_k(\lambda) \leq l(n - k + 1)\sigma_l(\lambda) \sigma_{k-1} \]
and
\[
\left[ \frac{\sigma_k(\lambda)}{C_n^k} \right] \frac{1}{\sigma_l(\lambda)/C_n^l} \leq \left[ \frac{\sigma_r(\lambda)}{C_n^r} \right] \frac{1}{\sigma_s(\lambda)/C_n^s}, \quad \text{for } \lambda \in \Gamma_k.
\]

At end, we also need the following truth to ensure the ellipticity of the Eq. (3.1).

**Lemma 2.5.** Let $G = \{(u(x), x) | x \in M^n \}$ be a smooth $(k - 1)$-admissible closed hypersurface in $\bar{M}$ and $\alpha_l(u, x) \geq 0$ for any $x \in M^n$ and $0 \leq l \leq k - 2$. Then the operator
\[
G(h_{ij}(V), u, x) := \frac{\sigma_k(\lambda(V))}{\sigma_{k-1}(\lambda(V))} - \sum_{l=0}^{k-2} \alpha_l(u, x) \frac{\sigma_l(\lambda(V))}{\sigma_{k-1}(\lambda(V))}
\]
is elliptic and concave with respect to $h_{ij}(V)$.

**Proof.** The proof is almost the same with the one of [11 Proposition 2.2], and we prefer to omit here. ∎
3. $C^0$ Estimate

We consider the family of equations for $0 \leq t \leq 1$,

$$
(3.1) \quad \frac{\sigma_k(\lambda(V))}{\sigma_{k-1}(\lambda(V))} - k-2 \sum_{l=0}^{k-2} t\alpha_l(u, x) \frac{\sigma_l(\lambda(V))}{\sigma_{k-1}(\lambda(V))} - \alpha_{k-1}(u, x, t) = 0,
$$

where

$$
\alpha_{k-1}(u, x, t) := t\alpha_{k-1}(u, x) + (1-t)\varphi(u) \frac{\sigma_k(e)}{\sigma_{k-1}(e)} f',
$$

and $\varphi$ is a positive function defined on $I$ and satisfying the following conditions:

(a) $\varphi(u) > 0$;
(b) $\varphi(u) > 1$ for $u \leq r_1$;
(c) $\varphi(u) < 1$ for $u \geq r_2$;
(d) $\varphi'(u) < 0$.

**Lemma 3.1 ($C^0$ estimate).** Assume that $0 \leq \alpha_l(u, x) \in C^\infty(I \times M^n)$. Under the assumptions (1.0) and (1.7) mentioned in Theorem 1.2, if $G = \{(u(x), x) | x \in M^n\} \subset \bar{M}$ is a smooth $(k-1)$-admissible, closed graphic hypersurface satisfied the curvature equation (3.1) for a given $t \in [0, 1]$, then

$$
r_1 \leq u(x) \leq r_2, \quad \forall x \in M^n.
$$

**Proof.** Assume that $u(x)$ attains its maximum at $x_0 \in M^n$ and $u(x_0) \geq r_2$. Then from (1.3), one has

$$
h_j^i = \frac{1}{v} \left[ f'^i \delta_j^i + \frac{1}{v^2} (f'u_j u^i - f u_j^i) \right],
$$

where $v = \sqrt{f'^2 + |Du|^2}$, which implies

$$
h_j^i(x_0) = \frac{1}{f} \left( f'^i \delta_j^i - \frac{u_j^i}{f} \right) \geq \frac{f'}{f} \delta_j^i.
$$

Note that $\frac{\sigma_k}{\sigma_{k-1}}$ and $\frac{\sigma_{k-1}}{\sigma_l}$ with $0 \leq l \leq k-2$ is concave in $\Gamma_{k-1}$. Thus,

$$
\frac{\sigma_k}{\sigma_{k-1}}(h_j^i) \geq \frac{\sigma_k}{\sigma_{k-1}} \left( \frac{f'^i}{f} \delta_j^i \right) + \frac{\sigma_k}{\sigma_{k-1}} \left( - \frac{1}{f^2} u_j^i \right) \geq \frac{\sigma_k}{\sigma_{k-1}} \left( \frac{f'}{f} \delta_j^i \right).
$$

Therefore, it follows that

$$
\frac{\sigma_k(\lambda(V))}{\sigma_{k-1}(\lambda(V))} \geq \frac{\sigma_k(e)}{\sigma_{k-1}(e)} \frac{f'}{f}.
$$

Similarly, one can get

$$
\frac{\sigma_l(\lambda(V))}{\sigma_{k-1}(\lambda(V))} \leq \frac{\sigma_l(e)}{\sigma_{k-1}(e)} \left( \frac{f}{f'} \right)^{k-l-1}.
$$

Combining with the above two inequalities, we have

$$
\frac{\sigma_k(e)}{\sigma_{k-1}(e)} \frac{f'}{f} - k-2 \sum_{l=0}^{k-2} t\alpha_l(u, x) \frac{\sigma_l(e)}{\sigma_{k-1}(e)} \left( \frac{f}{f'} \right)^{k-l-1} \leq \alpha_{k-1}(u, x, t).
$$
Clearly, if \( t = 0 \), the above inequality is contradict with (3.11). When \( 0 < t \leq 1 \), we can obtain

\[
\alpha_{k-1}(u, x) = \left(1 - \frac{1}{t}\right) \frac{f'}{f} \frac{\sigma_k(e)}{\sigma_{k-1}(e)} + \frac{1}{t} \alpha_{k-1}(x, u, t)
\]

\[
\geq \left(\frac{1}{t} \frac{f'}{f} - \left(1 - \frac{1}{t}\right) \frac{f'}{f}\right) \frac{\sigma_k(e)}{\sigma_{k-1}(e)} - \sum_{l=0}^{k-2} \alpha_l(u, x) \frac{\sigma_l(e)}{\sigma_{k-1}(e)} \left(\frac{f}{f'}\right)^{k-l-1}
\]

\[
> \frac{f'}{f} \frac{\sigma_k(e)}{\sigma_{k-1}(e)} - \sum_{l=0}^{k-2} \alpha_l(u, x) \frac{\sigma_l(e)}{\sigma_{k-1}(e)} \left(\frac{f}{f'}\right)^{k-l-1},
\]

which is contradict with

\[
\frac{f'}{f} \frac{\sigma_k(e)}{\sigma_{k-1}(e)} - \sum_{l=0}^{k-2} \alpha_l(u, x) \frac{\sigma_l(e)}{\sigma_{k-1}(e)} \left(\frac{f}{f'}\right)^{k-l-1} \geq \alpha_{k-1}(u, x)
\]

in view of (1.6) and the condition \( \varphi(u) < 1 \) for \( u \geq r_2 \). This shows \( \sup u \leq r_2 \). Similarly, we can obtain \( \inf u \geq r_1 \) in view of (1.7) and the condition \( \varphi(u) > 1 \) for \( u \leq r_1 \). Our proof is finished. \( \square \)

Now, we can prove the following uniqueness result.

**Lemma 3.2.** For \( t = 0 \), there exists a unique admissible solution of the Eq. (3.1), namely \( G_0 = \{ (u(x), x) \in M | u(x) = u_0 \} \), where \( u_0 \) is the unique solution of \( \varphi(u_0) = 1 \).

**Proof.** Let \( G_0 \) be a solution of (3.1), and then for \( t = 0 \),

\[
\frac{\sigma_k(\lambda(V))}{\sigma_{k-1}(\lambda(V))} - \varphi(u) \frac{\sigma_k(e)}{\sigma_{k-1}(e)} \frac{f'}{f} = 0.
\]

Assume that \( u(x) \) attains its maximum at \( x_0 \in M^n \). Then one has

\[
\frac{\sigma_k(\lambda(V))}{\sigma_{k-1}(\lambda(V))} \geq \frac{\sigma_k(e)}{\sigma_{k-1}(e)} \frac{f'}{f},
\]

which implies

\[
\varphi(u_{\text{max}}) \geq 1.
\]

Similarly, the minimum \( u_{\text{min}} \) of \( u(x) \) satisfies

\[
\varphi(u_{\text{min}}) \leq 1.
\]

Since \( \varphi \) is a decreasing function, we obtain

\[
\varphi(u_{\text{max}}) = \varphi(u_{\text{min}}) = 1,
\]

which implies that \( u(x_0) = u_0 \) for any \( (u(x), x) \in G_0 \), with \( u_0 \) the unique solution of \( \varphi(u_0) = 1 \). \( \square \)

4. \( C^1 \) Estimate

We can rewritten the Eq. (3.1) as follows:

\[
G(h_{ij}(V), u, x, t) = \frac{\sigma_k(\kappa(V))}{\sigma_{k-1}(\kappa(V))} - \sum_{l=0}^{k-2} t\alpha_l(u, x) \frac{\sigma_l(\kappa(V))}{\sigma_{k-1}(\kappa(V))} = \alpha_{k-1}(u, x, t).
\]

For convenience, we will simplify notations as follows:

\[
G_k(h_{ij}(V)) := \frac{\sigma_k(\lambda(V))}{\sigma_{k-1}(\lambda(V))}, \quad G_l(h_{ij}(V)) := -\frac{\sigma_l(\lambda(V))}{\sigma_{k-1}(\lambda(V))},
\]

\[
G(h_{ij}(V), u, x, t) = \alpha_{k-1}(u, x, t).
\]
and
\[ G^{ij}(\lambda(V)) := \frac{\partial G}{\partial h_{ij}}, \quad G^{ij,rs}(\lambda(V)) := \frac{\partial^2 G}{\partial h_{ij} \partial h_{rs}}. \]

**Lemma 4.1 (C^1 estimate).** Assume that \( k \geq 2 \) and
\[ \alpha_i(u, x) \geq c_t > 0, \quad \forall x \in M^n \]
for \( 0 \leq l \leq k - 1 \). Under the assumption (1.3), if the smooth \((k - 1)\)-admissible, closed graphic hypersurface \( G \) satisfies the Eq. (1.5) and \( u \) has positive upper and lower bounds, then there exists a constant \( C \) depending on \( n, k, c_t, |\alpha_t|_{C^1}, \) the \( C^0 \) bound of \( f \) and the curvature tensor \( \bar{R} \), the minimum and maximum values of \( u \) such that
\[ |\nabla u(x)| \leq C, \quad \forall x \in M^n. \]

**Proof.** First, we know from (1.3) and (2.8) that
\[ \tau = \frac{f_2(u)}{\sqrt{f_2(u) + |Du|^2}}. \]
It is sufficient to obtain a positive lower bound of \( \tau \). Define
\[ \psi = -\log \tau + \gamma(\Lambda), \]
where \( \gamma(t) \) is a function chosen later. Assume that \( x_0 \) is the maximum value point of \( \psi \). If \( V \) is parallel to the normal direction \( \nu \) of \( G \) at \( x_0 \), our result holds since \( \langle V, \nu \rangle = |V| \). So, we assume that \( V \) is not parallel to the normal direction \( \nu \) at \( x_0 \), we may choose the local orthonormal frame field \( \{E_1, \cdots, E_n\} \) on \( G \) satisfying
\[ \langle V, E_1 \rangle \neq 0 \quad \text{and} \quad \langle V, E_i \rangle = 0, \quad \forall i \geq 2. \]
Then, we arrive at \( x_0 \),
\[ (4.1) \quad \tau_i = \tau \gamma' \Lambda_i \]
and
\[ \psi_{ii} = \frac{-\tau_{ii}}{\tau} + \frac{(\tau_i)^2}{\tau^2} + \gamma'' \Lambda_i^2 + \gamma' \Lambda_{ii} \]
\[ = \frac{1}{\tau} \left( \sum_k (\bar{R}_{ii} + \bar{R}_{00}) \Lambda_k + f' h_{ii} - \tau h_{ii}^2 \right) \]
\[ + ((\gamma')^2 + \gamma'') \Lambda_i^2 + \gamma'(f' - \tau h_{ii}) \]
in view of
\[ \tau_{ii} = \sum_k (h_{ii} + \bar{R}_{00}) \langle V, E_k \rangle + f' h_{ii} - \tau \sum_k h_{ik} h_{ki}. \]
By (2.7), (2.8) and (4.1), we have at \( x_0 \)
\[ (4.2) \quad h_{11} = \tau \gamma', \quad h_{ii} = 0, \quad \forall i \geq 2. \]
Therefore, we can rotate the coordinate system such that \( \{E_i\}_{i=1}^n \) are the principal curvature directions of the second fundamental form \( h_{ij} \), i.e. \( h_{ij} = h_{ii} \delta_{ij} \). Since \( \Lambda_1 = \langle V, E_1 \rangle, \Lambda_i = \langle V, E_i \rangle \) for any \( i \geq 2 \). So, we can get
\[ G^{ii}_{\psi_{ii}} = -\frac{f'}{\tau} G^{ii}_{\psi_{ii}} = -\frac{1}{\tau} G^{ii}_{\bar{R}_{00}} + \frac{1}{\tau} G^{ii}_{\bar{R}_{ii}} \Lambda_1 + G^{ii}_{h_{ii}} \]
\[ + ((\gamma')^2 + \gamma'') G^{11} \Lambda_i^2 + \gamma' G^{ii}(f' - \tau h_{ii}). \]
Noting that

\[ G^{ij} h_{ij} = G - \sum_{l=0}^{k-2} (k-l)\alpha_l G_l = \alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2} (k-l)\alpha_l G_l \]

and

\[ G^{ij} h_{ij} = \nabla_1 \alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2} t_1 \alpha_l G_l, \]

we conclude

\[
G^{ii} \psi_{ii} = \frac{\Lambda_1}{\tau} \left( -\nabla_1 \alpha_{k-1}(u,x,t) + \sum_{l=0}^{k-2} t_1 \alpha_l G_l \right) \\
+ \frac{f'}{\tau} \left( -\alpha_{k-1}(u,x,t) + \sum_{l=0}^{k-2} (k-l)\alpha_l G_l \right) + G^{ii} h^2_{ii} \\
- \frac{1}{\tau} G^{ii} R_{0ii1} \Lambda_1 + \left( (\gamma')^2 + \gamma'' \right) G^{11} \Lambda^2_1 + \gamma' G^{ii} (f' - \tau h_{ii}) \\
= \frac{1}{\tau} \left( -\Lambda_1 \nabla_1 \alpha_{k-1}(u,x,t) - f'\alpha_{k-1}(u,x,t) \right) \\
+ \frac{1}{\tau} \sum_{l=0}^{k-2} t G_l \left( \Lambda_1 \nabla_1 \alpha_l + f'(k-l)\alpha_l \right) + G^{ii} h^2_{ii} \\
- \frac{1}{\tau} G^{ii} R_{0ii1} \Lambda_1 + \left( (\gamma')^2 + \gamma'' \right) G^{11} \Lambda^2_1 + \gamma' G^{ii} (f' - \tau h_{ii}).
\]

(4.3)

Since \( \langle V, E_i \rangle = 0 \) for \( i = 2, \cdots, n \), we obtain

\[ V = \langle V, E_1 \rangle E_1 + \langle V, \nu \rangle \nu = \Lambda_1 E_1 + \tau \nu, \]

which results in

\[ \Lambda_1 \nabla_1 \alpha_t(u,x) + (k-l)f'\alpha_t(u,x) = \nabla V \alpha_t(u,x) + (k-l)f'\alpha_t(u,x) - \tau \nabla V \alpha_t(u,x). \]

We know from the assumption (1.8) that

\[ [(k-l)f'\alpha_t(u,x) + \nabla V \alpha_t(u,x)] = \left[ (k-l)f'\alpha_t(u,x) + f \frac{\partial \alpha_t(u,x)}{\partial u} \right] \leq 0. \]

Thus,

(4.4) \[ \Lambda_1 \nabla_1 \alpha_t(u,x) + (k-l)f'\alpha_t(u,x) \leq -\tau \nabla V \alpha_t(u,x) \]

and

(4.5) \[ \Lambda_1 \nabla_1 \alpha_{k-1}(u,x,t) + f'\alpha_{k-1}(u,x,t) \leq (1-t)\varphi' \frac{\sigma_k(e)}{\sigma_{k-1}(e)} - \tau \nabla V \alpha_{k-1}(u,x,t). \]

Taking (4.4) and (4.5) into (4.3), we have at \( x_0 \)
Choosing 

\[ 0 \geq G^{ii} \varphi_{ii} \]

\[ \geq G^{ii} h_{ii}^2 + \left( (\gamma')^2 + \gamma'' \right) G^{11} \Lambda_1^2 + \gamma' G^{ii} (f' - \tau h_{ii}) - \frac{1}{\tau} G^{ii} R_{0ii1} \Lambda_1 \]

\[ - t \sum_{l=0}^{k-2} G_l \bar{\nabla}_\nu \alpha_l(u, x) - \frac{(1 - t)}{\tau} \varphi' \frac{\sigma_k(e)}{\sigma_{k-1}(e)} + \bar{\nabla}_\nu \alpha_{k-1}(u, x, t) \]

(4.6)

\[ = G^{ii} \left( h_{ii} - \frac{1}{2} \gamma' \tau \right)^2 + \left( (\gamma')^2 + \gamma'' \right) G^{11} \Lambda_1^2 + G^{ii} \left( \gamma' f' - \frac{1}{4} (\gamma')^2 \tau^2 \right) \]

\[ - \frac{1}{\tau} G^{ii} R_{0ii1} \Lambda_1 - t \sum_{l=0}^{k-2} G_l \bar{\nabla}_\nu \alpha_l(u, x) - \frac{(1 - t)}{\tau} \varphi' \frac{\sigma_k(e)}{\sigma_{k-1}(e)} + \bar{\nabla}_\nu \alpha_{k-1}(u, x, t). \]

Choosing

\[ \gamma(t) = - \frac{\alpha}{t} \]

for sufficiently large positive constant \( \alpha \), we have

\[ \gamma'(t) = \frac{\alpha}{t^2}, \quad \gamma''(t) = - \frac{2\alpha}{t^3}. \]

Therefore, (4.6) becomes

(4.7)

\[ 0 \geq G^{ii} \left( \gamma' f' - \frac{1}{4} (\gamma')^2 \tau^2 \right) - c_1 \left( \sum_{l=0}^{k-2} |G_l| + 1 \right) - \frac{1}{\tau} G^{ii} R_{0ii1} \Lambda_1 \]

in view of

\[ (\gamma')^2 + \gamma'' \geq 0, \]

where \( c_1 \) is a positive constant depending on \( |\alpha_l| \). Since \( V = \langle V, E_1 \rangle E_1 + \langle V, \nu \rangle \nu \), we can find that \( V \perp \text{Span}(E_2, \ldots, E_n) \), i.e., \( V \) is orthogonal with the subspace spanned by \( E_2, \ldots, E_n \). On the other hand, \( E_1, \nu \) are orthogonal with \( \text{Span}(E_2, \ldots, E_n) \). It is possible to choose suitable coordinate system such that \( E_1 \perp \text{Span}(E_2, \ldots, E_n) \), which implies that the pairs \( \{V, E_1\} \) and \( \{\nu, E_1\} \) lie in the same plane and

\[ \text{Span}(E_2, \ldots, E_n) = \text{Span}(E_2, \ldots, E_n), \]

where of course \( \{E_0 = \bar{e}_0, E_1, \ldots, E_n\} \) is a local orthonormal frame field in \( \bar{M} \). Therefore, we can choose \( E_2 = \bar{E}_2, \ldots, E_n = \bar{E}_n \), and then vectors \( \nu \) and \( E_1 \) can be decomposed into

\[ \nu = \langle \nu, \bar{e}_0 \rangle \bar{e}_0 + \langle \nu, \bar{E}_1 \rangle \bar{E}_1 = \frac{\tau}{f} \bar{e}_0 + \langle \nu, \bar{E}_1 \rangle \bar{E}_1, \]

\[ E_1 = \langle E_1, \bar{e}_0 \rangle \bar{e}_0 + \langle E_1, \bar{E}_1 \rangle \bar{E}_1. \]

By (2.4) and the fact \( V = \Lambda_1 E_1 + \tau \nu \), we can obtain

\[ R_{0ii1} = \bar{R}(\nu, E_i, E_i, E_1) \]

\[ = \frac{\tau}{f} \langle E_1, \bar{e}_0 \rangle \bar{R}(\bar{e}_0, \bar{E}_1, \bar{E}_1, \bar{e}_0) + \frac{\nu, \bar{E}_1}{\Lambda_1} \bar{R}(\bar{E}_1, \bar{E}_1, \bar{E}_1, \bar{E}_1) \]

\[ = \tau \frac{\nu, \bar{E}_1}{\Lambda_1} \bar{R}(\bar{e}_0, \bar{E}_1, \bar{E}_1, \bar{e}_0) - \frac{\nu, \bar{E}_1}{\Lambda_1} \bar{R}(\bar{E}_1, \bar{E}_1, \bar{E}_1, \bar{E}_1) \]

(4.8)

\[ = \tau \left( \frac{1}{f} \langle E_1, \bar{e}_0 \rangle \bar{R}(\bar{e}_0, \bar{E}_1, \bar{E}_1, \bar{e}_0) - \frac{\nu, \bar{E}_1}{\Lambda_1} \bar{R}(\bar{E}_1, \bar{E}_1, \bar{E}_1, \bar{E}_1) \right) , \]
where the third equality comes from \( \langle V, \tilde{E}_1 \rangle = 0 \). Substituting (4.8) into (4.7) yields
\[
0 \geq G^{ii} \left( \gamma' f' - \frac{1}{4} (\gamma')^2 \tau^2 \right) - c_1 \left( \sum_{l=0}^{k-2} |G_l| + 1 \right) - c_2 \sum_i G^{ii}
\]

where \( c_2 > 0 \) depends on the \( C^0 \) bound of \( f \) and the curvature tensor \( \bar{R} \). To continue our proof, we need to estimate \( G_l \) for \( 0 \leq l \leq k - 2 \). Let \( P \in \mathbb{R} \) be a fixed positive number.

(I) If \( \frac{\sigma_k}{\sigma_{k-1}} \leq P \), then we get from \( \alpha_l \geq c_l \) that
\[
|G_l| = \frac{\sigma_l}{\sigma_{k-1}} \leq \frac{1}{\alpha_l} \left( \frac{\sigma_k}{\sigma_{k-1}} + \alpha_l(u, x, t) \right) \leq c_3 (P + 1),
\]

where the constant \( c_3 > 0 \) depends on \( c_l, |\alpha_l|_{C^0} \).

(II) If \( \frac{\sigma_k}{\sigma_{k-1}} > P \), then by Lemma 2.4 one has
\[
|G_l| = \frac{\sigma_l}{\sigma_{k-1}} \leq \frac{\sigma_l}{\sigma_{l+1}} \cdots \frac{\sigma_{l+1}}{\sigma_{l+2}} \cdots \frac{\sigma_{k-2}}{\sigma_{k-1}} \leq c_4 \left( \frac{\sigma_{k-1}}{\sigma_k} \right)^{k-1-l} \leq P^{-(k-1-l)},
\]

where the positive constant \( c_4 \) depends on \( k \).

So, \( |G_l| \) can be bounded for any \( 0 \leq l \leq k - 2 \). By the definition of operator \( G \) and a direct computation, we have \( \Sigma_{1} G^{ii} \geq \frac{n-k+1}{k} \), and so we can choose sufficiently large \( \alpha \) such that
\[
0 \geq G^{ii} \left[ \gamma' f' - (\gamma')^2 \tau^2 \right].
\]

Thus,
\[
\gamma' f' \leq (\gamma')^2 \tau^2,
\]

which means
\[
\tau \geq c_5
\]

for some positive constant \( c_5 \) depending on \( n, k, \alpha_l, |\alpha_l|_{C^1} \), the \( C^0 \) bound of \( f \) and the curvature tensor \( \bar{R} \). The conclusion of Lemma 4.1 follows directly.

\[\square\]

Remark 4.1. After several careful revisions to the manuscript of this paper, we prefer to number (by subscripts) nearly all the constants in the \( C^1 \) and \( C^2 \) estimates, and we believe that this way can reveal the relations among constants clearly to readers.

5. \( C^2 \) Estimates

This section devotes to the \( C^2 \) estimates. However, before that, we need to make some preparations. First, we need the following fact:

Lemma 5.1. Let \( \mathcal{G} = \{(u(x), x) | x \in M^n\} \) be a \((k-1)\)-admissible solution of the Eq. (3.1) and assume that \( \alpha_l(u, x) \geq 0 \) for \( 0 \leq l \leq k - 1 \). Then, we have the following inequality
\[
G^{ij} h_{ijp} \geq \nabla_p \nabla_p \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} \frac{1}{1 + \frac{1}{k+1-l}} \frac{t(\nabla_p \alpha_l)^2}{\alpha_l} G_l - \sum_{l=0}^{k-2} t \nabla_p \nabla_p \alpha_l G_l.
\]

Proof. Differentiating the Eq. (3.1) once, we have
\[
\nabla_p \alpha_{k-1}(u, x, t) = G^{ij} h_{ijp} + \sum_{l=0}^{k-2} t \nabla_p \alpha_l G_l.
\]
Differentiating the Eq. (3.11) twice, we obtain

\[ \nabla_p \nabla_p \alpha_{k-1}(u, x, t) = G^{ij,rs} h_{ijp} h_{rsp} + G^{ij} h_{ijpp} + 2 \sum_{l=0}^{k-2} t \nabla_p \alpha_l G^{ij}_l h_{ijp} + \sum_{l=0}^{k-2} t \nabla_p \nabla_p \alpha_l G_l. \]

Moreover, since the operator \( \left( \frac{\alpha_l}{\alpha_{k-1}} \right) \) is concave for \( 0 \leq l \leq k-2 \), we have (see also (3.10) in [11])

\[ G^{ij,rs} h_{ijp} h_{rsp} \leq \left( 1 + \frac{1}{k-1-l} \right) G^{-1}_l G^{ij}_l G^{rs}_{ijp} h_{rsp}. \]

Thus, in view that \( G_k \) is concave in \( \Gamma_{k-1} \), we have

\[
\begin{align*}
\nabla_p \nabla_p \alpha_{k-1}(u, x, t) & \leq \sum_{l=0}^{k-2} t \alpha_l G^{ij,rs}_{ijp} h_{rsp} + G^{ij} h_{ijpp} + 2 \sum_{l=0}^{k-2} t \nabla_p \alpha_l G^{ij}_l h_{ijp} + \sum_{l=0}^{k-2} t \nabla_p \nabla_p \alpha_l G_l \\
& \leq \sum_{l=0}^{k-2} t \alpha_l G^{-1}_l \left( 1 + \frac{1}{k-1-l} \right) (G^{ij}_l h_{ijp})^2 + G^{ij} h_{ijpp} + 2 \sum_{l=0}^{k-2} t \nabla_p \alpha_l G^{ij}_l h_{ijp} + \sum_{l=0}^{k-2} t \nabla_p \nabla_p \alpha_l G_l \\
& = \frac{k-l}{k-1-l} \sum_{l=0}^{k-2} t \alpha_l G^{-1}_l \left( G^{ij}_l h_{ijp} + \frac{1}{1 + k-l \alpha_l} \nabla_p \alpha_l G_l \right)^2 + \sum_{l=0}^{k-2} \frac{1}{1 + k-l \alpha_l} t \left( \nabla_p \alpha_l \right)^2 G_l \\
& + G^{ij} h_{ijpp} + \sum_{l=0}^{k-2} t \nabla_p \nabla_p \alpha_l G_l \\
& \leq \sum_{l=0}^{k-2} \frac{1}{1 + k-l \alpha_l} t \left( \nabla_p \alpha_l \right)^2 G_l + G^{ij} h_{ijpp} + \sum_{l=0}^{k-2} t \nabla_p \nabla_p \alpha_l G_l,
\end{align*}
\]

which completes the proof of Lemma 5.1 \( \square \)

We also need the following truth:

**Lemma 5.2.** Let \( G = \{(u(x), x) | x \in M^n\} \) be a \((k-1)\)-admissible solution of the Eq. (3.1) with the position vector \( V \) in \( M \). We have the following equality

\[ G^{ij} \tau_{ij} + \sum_k \tau G^{ij}_{ik} h_{kj} \]

\[ = \left( \nabla_p \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} t \nabla_p \alpha_l G_l + \sum_p G^{ij} \tilde{R}_{0ijp} \right) (V, E_p) + f' \left( \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} (k-l) \alpha_l G_l \right). \]

**Proof.** By Lemma 2.3, we have

\[ \tau_{ij} = -\tau \sum_k h_{ik} h_{kj} + f' h_{ij} + \sum_k (h_{ijk} + \tilde{R}_{0ijk}) (V, E_p), \]

which results in

\[ G^{ij} \tau_{ij} = -\tau G^{ij} \sum_k h_{ik} h_{kj} + f' G^{ij} h_{ij} + \sum_k G^{ij} (h_{ijk} + \tilde{R}_{0ijk}) (V, E_p). \]
Note that

\[ G_{ij} h_{ij} = G - \sum_{l=0}^{k-2} (k-l) \alpha_l G_l = \alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2} (k-l) \alpha_l G_l \]

and

\[ G_{ij} h_{ijp} = \nabla_p \alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2} t \nabla_p \alpha_l G_l. \]

Thus,

\[ G_{ij} \tau_{ij} = \left( \nabla_p \alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2} t \nabla_p \alpha_l G_l + \sum_p G_{ij} \bar{R}_0 ijp \right) \langle V, E_p \rangle \]

\[ + f' \left( \alpha_{k-1}(u,x,t) - \sum_{l=0}^{k-2} (k-l) t \alpha_l G_l \right) - \sum_k \tau G_{ij} h_{ik} h_{kj}. \]

Therefore, we complete the proof. \( \square \)

Now we begin to estimate the second fundamental form.

**Lemma 5.3 (C² estimates).** Assume that \( k \geq 2 \) and \( \alpha_l(u,x) \geq c_l > 0, \forall x \in \mathbb{M}^n \)

for \( 0 \leq l \leq k-1 \). If the \( k \)-admissible, closed graphic hypersurface \( G = \{(u(x),x) \mid x \in \mathbb{M}^n\} \) satisfies the Eq. (3.1) with the position vector \( V \) in \( \bar{M} \), then there exists a constant \( C \) depending on \( n, k, c_l, |\alpha_l|_{C^2}, |\nabla u|_{C^0} \), the \( C^0, C^1 \) bounds of \( f \) and the curvature tensor \( \bar{R} \) such that for \( 1 \leq i \leq n \), the principal curvatures of \( G \) at \( V \) satisfy

\[ |\lambda_i(V)| \leq C, \forall x \in \mathbb{M}^n. \]

**Proof.** Since \( k \geq 2 \), \( G \) is 2-admissible, for sufficiently large \( c_6 \), one has

\[ |\lambda_i| \leq c_6 H, \]

where the positive constant \( c_6 \) depends on \( n, k \). So, we only need to estimate the mean curvature \( H \) of \( G \). Taking the auxiliary function

\[ W(x) = \log H - \log \tau. \]

Assume that \( x_0 \) is the maximum point of \( W \). Then at \( x_0 \), one has

\[ 0 = W_i = \frac{H_i}{H} - \frac{\tau_i}{\tau} \]

and

\[ 0 \geq W_{ij}(x_0) = \frac{H_{ij}}{H} - \frac{\tau_{ij}}{\tau}. \]

Choosing a suitable coordinate system \( \{x^1, x^2, \ldots, x^n\} \) in the neighborhood of \( X_0 = (u(x_0), x_0) \in G \) such that the matrix \( (h_{ij})_{n \times n} \) is diagonal at \( X_0 \), i.e., \( h_{ij} = h_{ii} \delta_{ij} \). This implies at \( x_0 \),

\[ 0 \geq G^{ij} W_{ij}(x_0) = \sum_{p=1}^{n} \frac{1}{H} G^{ii} h_{pp} - \frac{G^{ii} \tau_{ii}}{\tau}. \]
By (2.3), we can obtain
\[
\begin{align*}
  h_{ppii} &= h_{iipp} + h_{pp}^2 h_{ii} - h_{ii}^2 h_{pp} + \bar{R}_{0ip;i} + \bar{R}_{0pii;i} - 2h_{pp}\bar{R}_{piip} \\
  &
  + h_{ii}\bar{R}_{00i} + h_{pp}\bar{R}_{00i} + h_{ii}\bar{R}_{00p} + h_{ii}\bar{R}_{00i} - 2h_{ii}\bar{R}_{piip}.
\end{align*}
\]

Note that
\[
G^{ij} h_{ij} = G - \sum_{l=0}^{k-2} (k-l)\alpha_l G_l = \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} (k-l)\alpha_l G_l.
\]

So, we have
\[
\begin{align*}
  \sum_p G^{ii} h_{ppii} &= \sum_p G^{ii} (h_{iipp} + \bar{R}_{0iip;i} + \bar{R}_{0pii;i}) - \sum_p h_{pp} G^{ii} (h_{ii}^2 + 2\bar{R}_{piip} - \bar{R}_{0ii}) \\
  &+ \sum_p G^{ii} h_{ii} (h_{pp}^2 - 2\bar{R}_{piip} + \bar{R}_{0ii} + \bar{R}_{0pi} + \bar{R}_{0ii}) \\
  \geq \sum_p G^{ii} h_{iipp} + (|A|^2 - c_8) \left( \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} (k-l)\alpha_l G_l \right) - c_7 \\
  &- HG^{ii} (h_{ii}^2 + c_9),
\end{align*}
\]

where the positive constant \( c_7 \) depends on the \( C^1 \) bound of the curvature tensor \( \bar{R} \), the positive constants \( c_8, c_9 \) depend on the \( C^0 \) bound of the curvature tensor \( \bar{R} \). Together with Lemma 5.1, we know that (5.3) becomes
\[
\begin{align*}
  0 \geq \frac{1}{H} \sum_{p=1}^{n} G^{ii} h_{iipp} + \frac{|A|^2 - c_8}{H} \left( \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} (k-l)\alpha_l G_l \right) - \frac{G^{ii} \bar{R}_{ii}}{\tau} \\
  - \frac{c_7}{H} \sum_i G^{ii} - G^{ii} (h_{ii}^2 + c_9) \\
  \geq \frac{1}{H} \sum_{p=1}^{n} \left( \nabla_p \nabla_p \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} \frac{1}{1 + \frac{1}{k+1-l}} t (\nabla_p \alpha_l)^2 \alpha_l G_l - \sum_{l=0}^{k-2} t \nabla_p \nabla_p \alpha_l G_l \right) - \frac{G^{ii} \bar{R}_{ii}}{\tau} \\
  + \frac{|A|^2 - c_8}{H} \left( \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} (k-l)\alpha_l G_l \right) - \frac{c_7}{H} \sum_i G^{ii} - G^{ii} (h_{ii}^2 + c_9).
\end{align*}
\]

By Lemma 5.2, the above inequality becomes
\[
\begin{align*}
  0 \geq \frac{1}{H} \sum_{p=1}^{n} \left( \nabla_p \nabla_p \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} \frac{1}{1 + \frac{1}{k+1-l}} t (\nabla_p \alpha_l)^2 \alpha_l G_l - \sum_{l=0}^{k-2} t \nabla_p \nabla_p \alpha_l G_l \right) \\
  + \frac{|A|^2 - c_8}{H} \left( \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} (k-l)\alpha_l G_l \right) - \frac{c_7}{H} \sum_i G^{ii} - G^{ii} (h_{ii}^2 + c_9) \\
  - \frac{1}{\tau} \left( \nabla_p \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} t \nabla_p \alpha_l G_l + \sum_p G^{ii} \bar{R}_{0iip} \right) \langle V, E_p \rangle \\
  - \frac{f'}{\tau} \left( \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} (k-l)\alpha_l G_l \right) + G^{ii} h_{ii}^2.
\end{align*}
\]
Hence, we have
\[ 0 \geq \frac{|A|^2}{H} \left( \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} (k - l) t \alpha_l G_l \right) - \left( \frac{c_7}{H} + c_9 \right) \sum_i G^{ii} - \frac{\langle V, E_p \rangle}{\tau} \sum_p G^{ii} \tilde{R}_{0iip} \]
\[ + \frac{1}{H} \sum_{p=1}^{n} \left( \nabla_p \nabla_p \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} t \nabla_p \nabla_p \alpha_l G_l \right) - \frac{\langle V, E_p \rangle}{\tau} \left( \nabla_p \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} t \nabla_p \alpha_l G_l \right) \]
\[ - \left( \frac{c_8}{H} + \frac{f'}{\tau} \right) \left( \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} (k - l) t \alpha_l G_l \right). \]

A direction calculation implies
\[ \text{(5.4)} \quad |\nabla_p \alpha_{k-1}(u, x, t)| \leq c_{10}, \quad |\nabla_p \nabla_p \alpha_{k-1}(u, x, t)| \leq c_{11}(1 + H), \]
where the positive constant $c_{10}$ depends on $|\alpha_l|_{C^1}$, and the positive constant $c_{11}$ depends on $|\alpha_l|_{C^2}$. So
\[ - \frac{1}{H} c_{12} \left( \sum_{l=0}^{k-2} |G_l| + 1 \right) (H + 1) - c_{13} \left( \sum_{l=0}^{k-2} |G_l| + 1 \right) \]
\[ \leq \frac{1}{H} \sum_{p=1}^{n} \left( \nabla_p \nabla_p \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} t \nabla_p \nabla_p \alpha_l G_l \right) - \frac{\langle V, E_p \rangle}{\tau} \left( \nabla_p \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} t \nabla_p \alpha_l G_l \right) \]
\[ - \left( \frac{c_8}{H} + \frac{f'}{\tau} \right) \left( \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} (k - l) t \alpha_l G_l \right) \]
where the positive constant $c_{12}$ depends on $c_8, c_{10}$, the $C^1$ bound of $f$, and the positive constant $c_{13}$ depends on $c_8, c_{11}$, the $C^1$ bound of $f$. Then, together with the fact $|A|^2 \geq \frac{1}{n} H^2$, we have
\[ \frac{1}{n} H \alpha_{k-1}(u, x, t) - \left( \frac{c_7}{H} + c_{14} \right) \sum_i G^{ii} \]
\[ \leq \frac{|A|^2}{H} \left( \alpha_{k-1}(u, x, t) - \sum_{l=0}^{k-2} (k - l) t \alpha_l G_l \right) - \left( \frac{c_7}{H} + c_9 \right) \sum_i G^{ii} - \frac{\langle V, E_p \rangle}{\tau} \sum_p G^{ii} \tilde{R}_{0iip}, \]
where the positive constant $c_{14}$ depends on $c_9$, the $C^0$ bound of the curvature tensor $\tilde{R}$. Combining the fact $\sum_i G^{ii} \geq \frac{n-k+1}{k}$ with the above two inequalities, we have
\[ 0 \geq \frac{1}{n} H \alpha_{k-1}(u, x, t) - \left( \frac{c_7}{H} + c_{14} \right) - \frac{1}{H} c_{12} \left( \sum_{l=0}^{k-2} |G_l| + 1 \right) (H + 1) - c_{13} \left( \sum_{l=0}^{k-2} |G_l| + 1 \right). \]
Let us divide the rest of the proof into two cases:

Case I. If $\frac{\sigma_k}{\sigma_{k-1}} \leq H^\frac{k}{2}$, then we get from $\alpha_l \geq c_l$ that
\[ |G_l| = \frac{\sigma_l}{\sigma_{k-1}} \leq \frac{1}{\alpha_l} \left( \frac{\sigma_k}{\sigma_{k-1}} + \alpha_l(u, x, t) \right) \leq c_{15}(H^\frac{k}{2} + 1), \]
where the positive constant $c_{15}$ depends on $c_l, |\alpha_l|_{C^0}$. Thus, we have a contradiction when $H$ is large enough, which implies $H \leq C$. 
Case II. If $\frac{\sigma_k}{\sigma_{k-1}} > H^\frac{k}{k-1}$, then by Lemma 2.4, one has

$$|G_t| = \frac{\sigma_t}{\sigma_{k-1}} \leq \frac{\sigma_t \cdot \sigma_{l+1}}{\sigma_{l+2} \cdot \sigma_{k-2}} \leq c_{16} \left( \frac{\sigma_{k-1}}{\sigma_k} \right)^{k-1-t} \leq H^{\frac{k-1-t}{k}},$$

where the constant $c_{16} > 0$ depends on $k$. In this case, we can also derive $H \leq C$ easily.

In sum, the conclusion of Lemma 5.3 follows directly by using the fact $|\lambda_i| \leq c_0 H$.

\[\square\]

6. Existence

In this section, we use the degree theory for nonlinear elliptic equations developed in [15] to prove Theorem 1.2.

After establishing a priori estimates (see Lemmas 3.1, 4.1 and 5.3), we know that the Eq. (3.1) is uniformly elliptic. By [4], [13] and Schauder estimates, we have

$$|u|_{C^{4, \alpha}(M^n)} \leq C$$

for any $k$-convex solution $G$ to the equation (3.1). Define

$$C^4_{0, \alpha}(M^n) = \{ u \in C^4(M^n) : G = \{(u(x), x) | x \in M^n \} is k-convex \}.$$ Let us consider the function

$$F(\cdot; t) : C^4_{0, \alpha}(M^n) \to C^2(\alpha(M^n),$$

which is defined by

$$F(u, x, t) = \frac{\sigma_k(\kappa(V))}{\sigma_{k-1}(\kappa(V))} - \sum_{l=0}^{k-2} t\alpha_l(u, x) \frac{\sigma_l(\kappa(V))}{\sigma_{k-1}(\kappa(V))} - \alpha_{k-1}(u, x).$$

Set

$$O_R = \{ u \in C^4_{0, \alpha}(M^n) : |u|_{C^4, \alpha}(M^n) < R \},$$

which clearly is an open set in $C^4_{0, \alpha}(M^n)$. Moreover, if $R$ is sufficiently large, $F(u, x, t) = 0$ does not have solution on $\partial O_R$ by the priori estimate established in (6.1). Therefore, the degree $\deg(F(\cdot; t), O_R, 0)$ is well-defined for $0 \leq t \leq 1$. Using the homotopic invariance of the degree, we have

$$\deg(F(\cdot; 1), O_R, 0) = \deg(F(\cdot; 0), O_R, 0).$$

Lemma 3.2 shows that $u = u_0$ is the unique solution to the above equation for $t = 0$. By direct calculation, one has

$$F(su_0, x; 0) = [1 - \varphi(su_0)] \frac{\sigma_k(e)}{\sigma_{k-1}(e)} \frac{f'(su_0)}{f(su_0)}.$$

Using the fact $\varphi(u_0) = 1$, we have

$$\delta_{u_0} F(u_0, x; 0) = \frac{d}{ds} \bigg|_{s=1} F(su_0, x; 0) = -\varphi'(u_0) \frac{\sigma_k(e)}{\sigma_{k-1}(e)} \frac{f'(u_0)}{f(u_0)} > 0,$$

where $\delta F(u_0, x; 0)$ is the linearized operator of $F$ at $u_0$. Clearly, $\delta F(u_0, x; 0)$ has the form

$$\delta_u F(u_0, x; 0) = -a^{ij} \omega_{ij} + b^i \omega_i - \varphi'(u_0) \frac{\sigma_k(e)}{\sigma_{k-1}(e)} \frac{f'(u_0)}{f(u_0)} \omega,$$
where \((a^{ij})_{n \times n}\) is a positive definite matrix. Since
\[-\varphi'(u_0) \frac{\sigma_k(e)}{\sigma_{k-1}(e)} f'(u_0) > 0,\]
then \(\delta F(u_0, x; 0)\) is an invertible operator. Therefore,\[\deg(F(\cdot; 1), O_R, 0) = \deg(F(\cdot; 0), O_R, 0) = \pm 1,\]
which implies that we can obtain a solution at \(t = 1\). This finishes the proof of Theorem 1.2.

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