SEIDEL ELEMENTS AND MIRROR TRANSFORMATIONS FOR TORIC STACKS

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Abstract. We give a precise relation between the mirror transformation and the Seidel elements for weak Fano toric Deligne-Mumford stacks. Our result generalizes the corresponding result for toric varieties proved by González and Iritani in [5].

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1. Introduction

In [5], González and Iritani gave a precise relation between the mirror map and the Seidel elements for a smooth projective weak Fano toric variety $X$. The goal of this paper is to generalize the main theorem of [5] to a smooth projective weak Fano toric Deligne-Mumford stack $X'$.

Let $X'$ be a smooth projective weak Fano toric Deligne-Mumford stack, the mirror theorem can be stated as an equality between the $I$-function and the $J$-function via a change of coordinates, called mirror map (or mirror transformation). We refer to [3] and section 4.1 of [6] for further discussions.

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Let $Y$ be a monotone symplectic manifold. For a loop $\lambda$ in the group of Hamiltonian symplectomorphisms on $Y$, Seidel [10] constructed an invertible element $S(\lambda)$ in (small) quantum cohomology counting sections of the associated Hamiltonian $Y$-bundle $E_\lambda \to \mathbb{P}^1$. The Seidel element $S(\lambda)$ defines an element in $\text{Aut}(QH(Y))$ via quantum multiplication and the map $\lambda \mapsto S(\lambda)$ gives a representation of $\pi_1(\text{Ham}(Y))$ on $QH(Y)$. The construction was extended to all symplectic manifolds by McDuff and Tolman in [9]. Let $D_1, \ldots, D_m$ be the classes in $H^2(X)$ Poincaré dual to the toric divisors. When the loop $\lambda$ is a circle action, McDuff and Tolman [9] considered the Seidel element $\tilde{S}_j$ associated to an action $\lambda_j$ that fixes the toric divisor $D_j$. The definition of Seidel representation and Seidel element were extended to symplectic orbifolds by Tseng-Wang in [11].

Given a circle action on $X$ (resp. $X'$), the Seidel element in [5] (resp. [11]) is defined using the small quantum cohomology ring. In this paper, we need to define it, for smooth projective Deligne-Mumford stack, with deformed quantum cohomology to include the bulk deformations. For weak Fano toric Deligne-Mumford stack, the mirror theorem in [6] shows that the mirror map $\tau(y) \in H^{\geq 2}_{\text{orb}}(X)$, therefore, we will only need bulk deformations with $\tau \in H^{\geq 2}_{\text{orb}}(X)$.

We consider the Seidel element $\tilde{S}_j$ associated to the toric divisor $D_j$ as well as the Seidel element $\tilde{S}_{m+j}$ corresponding to the box element $s_j$. The Seidel element in definition 2.2 shows that $S = q_0 \tilde{S}$ is a pull-back of a coefficient of the $J$-function $J_{E_j}$ of the associated orbifiber bundle $E_j$, hence we can use the mirror theorem for $E_j$ to calculate $\tilde{S}_j$ when $E_j$ is weak Fano.

We extend the definition of the Batyrev element $\tilde{D}_j$ to weak Fano toric Deligne-Mumford stacks via partial derivatives of the mirror map $\tau(y)$. As analogues of the Seidel elements in B-model, the Batyrev elements can be explicitly computed from the $I$-function of $X$. The following theorem states that the Seidel elements and the Batyrev elements only differ by a multiplication of a correction function.

**Theorem 1.1.** Let $X$ be a smooth projective toric Deligne-Mumford stack with $\rho^S \in \text{cl}(C^S(X))$.

(i) the Seidel element $\tilde{S}_j$ associated to the toric divisor $D_j$ is given by

$$\tilde{S}_j(\tau(y)) = \exp\left( -q_0^{(j)}(y) \right) \tilde{D}_j(y)$$

where $\tau(y)$ is the mirror map of $X$ and the function $g_0^{(j)}$ is given explicitly in (70);

(ii) the Seidel element $\tilde{S}_{m+j}$ corresponding to the box element $s_j$ is given by

$$\tilde{S}_{m+j}(\tau(y)) = \exp\left( -g_0^{(m+j)} \right) y^{-D_{m+j}^S} \tilde{D}_{m+j}(y),$$

where $\tau(y)$ is the mirror map of $X$ and the function $g_0^{(m+j)}$ is given explicitly in (52).

It appears that the correction coefficients in the above theorem coincide with the instanton corrections in theorem 1.4 in [2]. This phenomenon also indicates the deformed quantum cohomology of the toric Deligne-Mumford stack $X$ is isomorphic to the Batyrev ring given in [6].
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2. Seidel Elements and J-functions

2.1. Generalities. In this section, we will fix our notation and construct the Seidel elements of smooth projective Deligne-Mumford stacks using $\tau$-deformed quantum cohomology.

Let $X$ be a smooth projective Deligne-Mumford stack, equipped with a $C^\times$ action.

Definition 2.1. The associated orbifiber bundle of the $C^\times$-action is the $X$-bundle over $\mathbb{P}^1$:

$$E := X \times (\mathbb{C}^2 \setminus \{0\}) / C^\times \rightarrow \mathbb{P}^1,$$

where $C^\times$ acts on $\mathbb{C}^2 \setminus \{0\}$ via the standard diagonal action.

Let $\phi_1, \ldots, \phi_N$ be a basis for the orbifold cohomology ring $H^*_\text{orb}(X) := H^*(\mathcal{I}X; \mathbb{Q})$ of $X$, where $\mathcal{I}X$ is the inertia stack of $X$. Let $\phi^1, \ldots, \phi^N$ be the dual basis of $\phi_1, \ldots, \phi_N$ with respect to the orbifold Poincaré pairing. Furthermore, let $\hat{\phi}_1, \ldots, \hat{\phi}_M$ denote a basis for the orbifold cohomology $H^*_{\text{orb}}(E) := H^*(IE; \mathbb{Q})$ of $E$. Let $\hat{\phi}^1, \ldots, \hat{\phi}^M$ be the dual basis of $\hat{\phi}_1, \ldots, \hat{\phi}_M$ with respect to the orbifold Poincaré pairing.

We will use $X$ to denote the coarse moduli space of $X$ and use $E$ to denote the coarse moduli space of $E$. Then the $C^\times$ action on $X$ descends to the $C^\times$ action on $X$ with $E$ being the associated bundle. Following [5] and [6], there is a (non-canonical) splitting

$$H^*(E; \mathbb{Q}) \cong H^*(E; \mathbb{Q}) \cong H^*(X; \mathbb{Q}) \otimes H^*(\mathbb{P}^1; \mathbb{Q}) \cong H^*(X; \mathbb{Q}) \otimes H^*(\mathbb{P}^1; \mathbb{Q}).$$

According to [3], there is a unique $C^\times$-fixed component $F_{\text{max}} \subset X^\text{orb}$ such that the normal bundle of $F_{\text{max}}$ has only negative $C^\times$-weights. Let $\sigma_0$ be the section associated to a fixed point in $F_{\text{max}}$. Following [5], there is a splitting defined by this maximal section.

$$(1) \quad H_2(E; \mathbb{Z})/\text{tors} \cong H_2(E; \mathbb{Z})/\text{tors} \cong \mathbb{Z}[\sigma_0] \oplus (H_2(X; \mathbb{Z})/\text{tors}) \cong \mathbb{Z}[\sigma_0] \oplus (H_2(X; \mathbb{Z})/\text{tors}).$$

Let $NE(X) \subset H_2(X; \mathbb{R})$ denote the Mori cone, i.e. the cone generated by effective curves and set

$$NE(X)_\mathbb{Z} := NE(X) \cap (H_2(X; \mathbb{Z})/\text{tors}).$$

Then, by lemma 2.2 of [5], we have

$$(2) \quad NE(E)_\mathbb{Z} = \mathbb{Z}_{\geq 0}[\sigma_0] + NE(X)_\mathbb{Z}.$$

Let $H_2^\text{sec}(E; \mathbb{Z})$ be the affine subspace of $H_2(E; \mathbb{Z})/\text{tors}$ which consists of the classes that project to the positive generator of $H_2(\mathbb{P}^1; \mathbb{Z})$, we set

$$NE(E)^{\text{sec}} := NE(E)_\mathbb{Z} \cap H_2^\text{sec}(E; \mathbb{Z}),$$

then we obtain

$$(3) \quad NE(E)^{\text{sec}}_\mathbb{Z} = [\sigma_0] + NE(X)_\mathbb{Z}.$$
We choose a nef integral basis \(\{p_1, \ldots, p_r\}\) of \(H^2(\mathcal{X}; \mathbb{Q})\), then there are unique lifts of \(p_1, \ldots, p_r\) in \(H^2(\mathcal{E}; \mathbb{Q})\) which vanish on \([\sigma_0]\). By abuse of notation, we also denote these lifts as \(p_1, \ldots, p_r\), these lifts are also nef. Let \(p_0\) be the pullback of the positive generator of \(H^2(\mathbb{P}^1; \mathbb{Z})\) in \(H^2(\mathcal{E}; \mathbb{Q})\). Therefore, \(\{p_0, p_1, \ldots, p_r\}\) is an integral basis of \(H^2(\mathcal{E}; \mathbb{Q})\).

Let \(q_0, q_1, \ldots, q_r\) be the Novikov variables of \(\mathcal{E}\) dual to \(p_0, p_1, \ldots, p_r\) and \(q_1, \ldots, q_r\) be the Novikov variables of \(\mathcal{X}\) dual to \(p_1, \ldots, p_r\). We denote the Novikov ring of \(\mathcal{X}\) and the Novikov ring of \(\mathcal{E}\) by

\[
\Lambda_{\mathcal{X}} := \mathbb{Q}[q_0, q_1, \ldots, q_r] \quad \text{and} \quad \Lambda_{\mathcal{E}} := \mathbb{Q}[q_0, \ldots, q_r],
\]

respectively. For each \(d \in \text{NE}(X)_\mathbb{Z}\), we write

\[
q^d := q_0^{(p_0, \beta)} q_1^{(p_1, \beta)} \cdots q_r^{(p_r, \beta)} \in \Lambda_{\mathcal{X}};
\]

and for each \(\beta \in \text{NE}(E)_\mathbb{Z}\), we write

\[
q^\beta := q_0^{(p_0, \beta)} q_1^{(p_1, \beta)} \cdots q_r^{(p_r, \beta)} \in \Lambda_{\mathcal{E}}.
\]

The \(\tau\)-deformed orbifold quantum product is defined as follows:

\[
\alpha \bullet_{\tau} \beta = \sum_{d \in \text{NE}(X)_\mathbb{Z}} \sum_{l \geq 0} \frac{1}{l!} \langle \alpha, \beta, \tau, \ldots, \tau, \phi_k \rangle_{0, l+3, d} q^d \phi^k,
\]

the associated quantum cohomology ring is denoted by

\[
QHG_{\tau}(\mathcal{X}) := (H(\mathcal{X}) \otimes \Lambda_{\mathcal{X}})_{\bullet_{\tau}}.
\]

**Definition 2.2.** The Seidel element of \(\mathcal{X}\) is the class

\[
S(\hat{\tau}) := \sum_{\alpha} \sum_{\beta \in \text{NE}(E)} \sum_{l \geq 0} \frac{1}{l!} \langle 1, \hat{\tau}_w, \ldots, \hat{\tau}_w, \iota_\tau \phi_\alpha \psi \rangle_{\mathcal{E}}^E (\hat{\tau}_{0, 2}),
\]

in \(QHG_{\tau}(\mathcal{X}) \otimes \Lambda_{\mathcal{X}}\). Here \(\iota : \mathcal{X} \rightarrow \mathcal{E}\) is the inclusion of a fiber, and \(\iota_* : H^*(\mathcal{X}; \mathbb{Q}) \rightarrow H^{*+2}(\mathcal{E}; \mathbb{Q})\)

is the Gysin map. Moreover,

\[
e^{(\hat{\tau}_{0, 2})} = q^\beta = q_0^{(p_0, \beta)} \cdots q_r^{(p_r, \beta)},
\]

where

\[
\hat{\tau}_{0, 2} = \sum_{a=0}^r p_a \log q_a \in H^2(\mathcal{E}) \quad \text{and} \quad \hat{\tau} = \hat{\tau}_{0, 2} + \hat{\tau}_w \in H^2_{\text{orb}}(\mathcal{E}).
\]

The Seidel element can be factorized as

\[
S(\hat{\tau}) = q_0 \hat{S}(\hat{\tau}), \quad \text{with} \quad \hat{S}(\hat{\tau}) \in QHG_{\tau}(\mathcal{X}).
\]

**2.2. \(J\)-functions.** We will explain the relation between the Seidel element and the \(J\)-function of the associated bundle \(\mathcal{E}\).

**Definition 2.3.** The \(J\)-function of \(\mathcal{E}\) is the cohomology valued function

\[
J_{\mathcal{E}}(\hat{\tau}, z) = e^{\hat{\tau}_{0, 2}z} \left( 1 + \sum_{\alpha} \sum_{(\beta, l) \neq (0, 0), \beta \in \text{NE}(E)} \frac{e^{(\hat{\tau}_{0, 2})}}{l!} \langle (1, \hat{\tau}_w, \ldots, \hat{\tau}_w, \phi_\alpha \psi)_{\mathcal{E}} \rangle_{0, l+2, \beta} \hat{\tau}_{w} \right),
\]

where

\[
\hat{\tau}_{n, \psi} = \sum_{n \geq 0} z^{-1-n} \phi_\alpha \psi^n.
\]
Note that when \( n = 0 \), we will have
\[
\sum_{\alpha} \langle 1, \hat{\tau}_{tw}, \cdots, \hat{\tau}_{tw}, \hat{\phi}_{\alpha}, 0, l+2, \beta \hat{\phi}^\alpha \rangle = 0, \quad \text{for} \quad (l, \beta) \neq (1, 0);
\]
\[
\sum_{\alpha} \langle 1, \hat{\tau}_{tw}, \cdots, \hat{\tau}_{tw}, \hat{\phi}_{\alpha}, 0, l+2, \beta \hat{\phi}^\alpha \rangle = \hat{\tau}_{tw}, \quad \text{for} \quad (l, \beta) = (1, 0).
\]

The \( J \)-function can be expanded in terms of powers of \( z^{-1} \) as follows:

\[
J_{E}(\hat{\tau}, z) = e^{r \sum_{a=0}^{p} \log q/a \cdot \left( 1 + z^{-1} \hat{\tau}_{tw} + z^{-2} \sum_{n=0}^{\infty} F_{n}(q_{1}, \ldots, q_{r}; \hat{\tau})q_{0}^{n} + O(z^{-3}) \right)},
\]

where

\[
F_{n}(q_{1}, \ldots, q_{r}; \hat{\tau}) = M \sum_{\alpha=1}^{M} \sum_{d \in NE(X)} \sum_{l \geq 0} \frac{1}{l!} \langle 1, \hat{\tau}_{tw}, \cdots, \hat{\tau}_{tw}, 1, \hat{\phi}_{\alpha} \psi \rangle_{0, l+2, d, \sigma_{0}} q^{d} \hat{\phi}^\alpha
\]

Proposition 2.4. The Seidel element corresponding to the \( C^\times \) action on \( X \) is given by

\[
S(\hat{\tau}) = i^*(F_{1}(q_{1}, \ldots, q_{r}; \hat{\tau})q_{0}).
\]

Proof. The proof in here is identical to the proof given in proposition 2.5 of [5] for smooth projective varieties:

Using the duality identity

\[
\sum_{\alpha=1}^{M} \hat{\phi}_{\alpha} \otimes i^* \hat{\phi}^\alpha = \sum_{\alpha=1}^{N} 1_\ast \hat{\phi}_{\alpha} \otimes \hat{\phi}^\alpha
\]

we can see that

\[
i^* F_{1}(q_{1}, \ldots, q_{r}; \hat{\tau}) = \sum_{\alpha=1}^{N} \sum_{d \in NE(X)} \sum_{l \geq 0} \frac{1}{l!} \langle 1, \hat{\tau}_{tw}, \cdots, \hat{\tau}_{tw}, 1_\ast \hat{\phi}_{\alpha} \psi \rangle_{0, l+2, d, \sigma_{0}} q^{d} \hat{\phi}^\alpha.
\]

Hence, the conclusion follows, i.e.

\[
S(\hat{\tau}) = i^*(F_{1}(q_{1}, \ldots, q_{r}; \hat{\tau})q_{0}).
\]

\[\square\]

3. SEIDEL ELEMENTS CORRESPONDING TO TORIC DIVISORS

3.1. A Review of Toric Deligne-Mumford stacks. In this section, we will define toric Deligne-Mumford stacks following the construction of [4] and [5].

A toric Deligne-Mumford stack is defined by a stacky fan \( \Sigma = (N, \Sigma, \beta) \), where \( N \) is a finitely generated abelian group, \( \Sigma \subset N_{Q} = N \otimes \mathbb{Q} \) is a rational simplicial fan, and \( \beta : Z^m \to N \) is a homomorphism. We assume \( \beta \) has finite cokernel and the rank of \( N \) is \( n \). The canonical map \( N \to N_{Q} \) generates the 1-skeleton of the fan \( \Sigma \). Let \( \bar{b}_{i} \) be the image of \( b_{i} \) under this canonical map, where \( b_{i} \) is the image \( \beta \) of the standard basis of \( Z^m \). Let \( \mathbb{L} \subset Z^m \) be the kernel of \( \beta \). Then the fan sequence is the following exact sequence

\[
0 \to \mathbb{L} \to Z^m \to N.
\]
Let $\beta^\vee : (\mathbb{Z}^*)^m \to \mathbb{L}^\vee$ be the Gale dual of $\beta$ in \cite{1}, where $\mathbb{L}^\vee := H^1(Cone(\beta^*))$ is an extension of $\mathbb{L}^* = Hom(\mathbb{L}, \mathbb{Z})$ by a torsion subgroup. The divisor sequence is the following exact sequence

\begin{equation}
0 \to N^* \xrightarrow{\beta^*} (\mathbb{Z}^*)^m \xrightarrow{\beta^\vee} L^\vee.
\end{equation}

By applying $Hom_{\mathbb{Z}}(-, \mathbb{C}^\times)$ to the dual map $\beta^\vee$, we have a homomorphism $\alpha : G \to (\mathbb{C}^\times)^m$, where $G := Hom_{\mathbb{Z}}(\mathbb{L}^\vee, \mathbb{C}^\times)$,

and we let $G$ act on $\mathbb{C}^m$ via this homomorphism.

The collection of anti-cones $\mathcal{A}$ is defined as follows:

$$\mathcal{A} := \left\{ I : \sum_{i \in I} \mathbb{R}_{\geq 0} \bar{b}_i \in \Sigma \right\}.$$

Let $U$ denote the open subset of $\mathbb{C}^m$ defined by $U := \mathbb{C}^m \setminus \bigcup_{I \notin \mathcal{A}} \mathbb{C}^I$, where

$$\mathbb{C}^I = \{(z_1, \ldots, z_m) : z_i = 0 \text{ for } i \notin I\}.$$

**Definition 3.1.** Following \cite{6}, the toric Deligne-Mumford stack $X$ is defined as the quotient stack $X := [U/G]$.

**Remark 3.2.** The toric variety $X$ associated to the fan $\Sigma$ is the coarse moduli space of $X$ \cite{1}.

**Definition 3.3** (\cite{6}). Given a stacky fan $\Sigma = (N, \Sigma, \beta)$, we define the set of box elements $Box(\Sigma)$ as follows

$$Box(\Sigma) := \left\{ v \in N : \bar{v} = \sum_{k \in I} c_k \bar{b}_k \text{ for some } 0 \leq c_k < 1, I \in \mathcal{A} \right\}.$$

We assume that $\Sigma$ is complete, then the connected components of the inertia stack $I\mathcal{X}$ are indexed by the elements of $Box(\Sigma)$ (see \cite{1}). Moreover, given $v \in Box(\Sigma)$, the age of the corresponding connected component of $I\mathcal{X}$ is defined by $\text{age}(v) := \sum_{k \notin I} c_k$.

The Picard group $Pic(\mathcal{X})$ of $\mathcal{X}$ can be identified with the character group $Hom(G, \mathbb{C}^\times)$. Hence

\begin{equation}
\mathbb{L}^\vee = Hom(G, \mathbb{C}^\times) \cong Pic(\mathcal{X}) \cong H^2(\mathcal{X}; \mathbb{Z}).
\end{equation}

We can also use the extended stacky fans introduced by Jiang \cite{7} to define the toric Deligne-Mumford stacks. Given a stacky fan $\Sigma = (N, \Sigma, \beta)$ and a finite set

$$S = \{s_1, \ldots, s_l\} \subset N_\Sigma := \{c \in N : \tilde{c} \in |\Sigma|\}.$$

The $S$-extended stacky fan is given by $(N, \Sigma, \beta^S)$, where $\beta^S : \mathbb{Z}^{m+l} \to N$ is defined by:

\begin{equation}
\beta^S(e_i) = \begin{cases}
  b_i & 1 \leq i \leq m; \\
  s_{i-m} & m+1 \leq i \leq m+l.
\end{cases}
\end{equation}
Let $L^S$ be the kernel of $\beta^S : Z^{m+l} \to N$. Then we have the following $S$-extended fan sequence

$$0 \longrightarrow L^S \longrightarrow Z^{m+l} \overset{\beta^S}{\longrightarrow} N. \tag{15}$$

By the Gale duality, we have the $S$-extended divisor sequence

$$0 \longrightarrow N^* \overset{\beta^*}{\longrightarrow} (Z^*)^{m+l} \overset{\beta^{S^*}}{\longrightarrow} L^{S^*}, \tag{16}$$

where $L^{S^*} := H^1(\text{Cone}(\beta^S)^*)$.

Assumption 3.4. In the rest of the paper, we will assume the set

$$\{v \in \text{Box}(\Sigma); \text{age}(v) \leq 1\} \cup \{b_1, \ldots, b_m\}$$

generates $N$ over $\mathbb{Z}$. And we choose the set

$$S = \{s_1, \ldots, s_l\} \subset \text{Box}(\Sigma)$$

such that the set $\{b_1, \ldots, b_m, s_1, \ldots, s_l\}$ generates $N$ over $\mathbb{Z}$ and $\text{age}(s_j) \leq 1$ for $1 \leq j \leq l$.

Let $D^S_i$ be the image of the standard basis of $(Z^*)^{m+l}$ under the map $\beta^{S^*}$, then there is a canonical isomorphism

$$L^{S^*} \otimes Q \cong (L^\vee \otimes Q) \bigoplus_{i=m+1}^{m+l} Q D^S_i, \tag{17}$$

which can be constructed as follows (6):

Since $\Sigma$ is complete, for $m < j \leq m + l$, the box element $s_{j-m}$ is contained in some cone in $\Sigma$. Namely,

$$s_{j-m} = \sum_{i \notin I^S_j} c_{ji} b_i \quad \text{in} \quad N \otimes \mathbb{Q}, \quad c_{ji} \geq 0, \quad \exists I^S_j \in A^S,$$

where $I^S_j$ is the "anticone" of the cone containing $s_{j-m}$.

By the $S$-extended fan sequence [15] tensored with $\mathbb{Q}$, we have the following short exact sequence

$$0 \longrightarrow L^S \otimes \mathbb{Q} \longrightarrow \mathbb{Q}^{m+l} \overset{\beta^S}{\longrightarrow} N \otimes \mathbb{Q} \longrightarrow 0.$$

Hence, there exists a unique $D^S_j \in L^S \otimes Q$ such that

$$\langle D^S_i, D^S_j \rangle = \begin{cases} 1 & i = j; \\ -c_{ji} & i \notin I^S_j; \\ 0 & i \in I^S_j \setminus \{j\}. \end{cases} \tag{18}$$

These vectors $D^S_j$ define a decomposition

$$L^{S^*} \otimes \mathbb{Q} = \text{Ker} \left((D^S_{m+1}, \ldots, D^S_{m+l}) : L^{S^*} \otimes \mathbb{Q} \to \mathbb{Q}^l \right) \oplus \bigoplus_{j=m+1}^{m+l} Q D^S_j.$$

We identify the first factor $\text{Ker}(D^S_{m+1}, \ldots, D^S_{m+l})$ with $L^\vee \otimes \mathbb{Q}$. Via this decomposition, we can regard $H^2(X, \mathbb{Q}) \cong L^\vee \otimes \mathbb{Q}$ as a subspace of $L^{S^*} \otimes \mathbb{Q}$.

Let $D_i$ be the image of $D^S_i$ in $L^\vee \otimes \mathbb{Q}$ under this decomposition. Then

$$D_i = 0, \quad \text{for} \quad m + 1 \leq i \leq m + l.$$
Let $\mathcal{A}^S$ be the collection of $S$-extended anti-cones, i.e.
\[
\mathcal{A}^S := \left\{ I^S : \sum_{i \notin I^S} R_{\geq 0}^{\mathcal{A}^S}(e_i) \in \Sigma \right\}.
\]

Note that
\[
\{s_1, \ldots, s_l\} \subset I^S, \quad \forall I^S \in \mathcal{A}^S.
\]
By applying $Hom\mathbb{Z}(-, \mathbb{C}^\times)$ to the $S$-extended dual map $\beta^\vee$, we have a homomorphism
\[
\alpha^S : \mathcal{G}^S \rightarrow (\mathbb{C}^\times)^{m+l}, \quad \text{where} \quad G^S := Hom\mathbb{Z}(L^S, \mathbb{C}^\times).
\]
We define $U$ to be the open subset of $\mathbb{C}^m \cup \{z_i = 0\}$ for $i \notin I^S$.

Let $G^S$ act on $U^S$ via $\alpha^S$. Then we obtain the quotient stack $[U^S/G^S]$. Jiang \cite{Jiang} showed that
\[
[U^S/G^S] \cong [U/G] = X.
\]

3.2. Mirror theorem for toric stacks. In \cite{Coates}, Coates-Corti-Iritani-Tseng defined the $S$-extended $I$-function of a smooth toric Deligne-Mumford stack $X$ with projective coarse moduli space and proved that this $I$-function is a point of Givental’s Lagrangian cone $\mathcal{L}$ for the Gromov-Witten theory of $X$. In this paper, we will only need this theorem for the weak Fano case. In this case, the mirror theorem will take a particularly simple form which can be stated as an equality of $I$-function and $J$-function via a change of variables, called mirror map.

To state the mirror theorem for weak Fano toric Deligne-Mumford stack, we need the following definitions.

We define the $S$-extended Kähler cone $C_X^S$ as
\[
C_X^S := \cap_{I^S \in \mathcal{A}^S} \sum_{I^S} R_{\geq 0} D_i^S
\]
and the Kähler cone $C_X$ as
\[
C_X := \cap_{I \in \mathcal{A}} \sum_{i \in I} R_{\geq 0} D_i.
\]
Let $p_1^S, \ldots, p_{r+l}$ be an integral basis of $L^S$, where $r = m - n$, such that $p_i^S$ is in the closure $cl(C_X^S)$ of the $S$-extended Kähler cone $C_X^S$ for all $1 \leq i \leq r + l$ and $p_{r+1}^S, \ldots, p_{m+l}$ are in $i \sum_{i=m+1}^{m+l} \mathbb{R}_{\geq 0} D_i^S$. We denote the image of $p_i^S$ in $\mathbb{L} \otimes \mathbb{R}$ by $p_i$, therefore $p_1, \ldots, p_r$ are nef and $p_{r+1}, \ldots, p_{r+l}$ are zero. We define a matrix $(m_{ia})$ by
\[
D_i^S = \sum_{a=1}^{r+l} m_{ia} p_a^S, \quad m_{ia} \in \mathbb{Z}.
\]
Then the class $D_i$ of toric divisor is given by
\[
D_i = \sum_{a=1}^{r} m_{ia} p_a.
\]
Definition 3.5 (6, Section 3.1.4). A toric Deligne-Mumford stack $\mathcal{X}$ is called weak Fano if the first Chern class $\rho$ satisfies
\[ \rho = c_1(T\mathcal{X}) = \sum_{i=1}^{m} D_i \in cl(C_{X}), \]
where $C_{X}$ is the Kähler cone of $\mathcal{X}$.

We will need a slightly stronger condition:
\[ \rho^S := D_1^S + \ldots + D_{m+1}^S \in cl(C_X^S), \]
where $C_X^S$ is the $S$-extended Kähler cone. By lemma 3.3 of [6], we can see that $\rho^S \in cl(C_X^S)$ implies $\rho \in cl(C_X)$. Moreover, under assumption 3.4, we will have $\rho^S \in cl(C_X^S)$ if and only if $\rho \in cl(C_X)$.

For a real number $r$, let $\lfloor r \rfloor$, $\lceil r \rceil$ and $\{r\}$ be the ceiling, floor and fractional part of $r$ respectively.

Definition 3.6. We define two subsets $K$ and $K_{\text{eff}}$ of $\mathbb{L}^S \otimes \mathbb{Q}$ as follows:
- $K := \{ d \in \mathbb{L}^S \otimes \mathbb{Q}; \{ i \in \{1, \ldots, m+1\}; \langle D_i^S, d \rangle \in \mathbb{Z} \} \} \cup \mathbb{A}^S$,
- $K_{\text{eff}} := \{ d \in \mathbb{L}^S \otimes \mathbb{Q}; \{ i \in \{1, \ldots, m+1\}; \langle D_i^S, d \rangle \in \mathbb{Z}_{\geq 0} \} \} \cup \mathbb{A}^S$.

Remark 3.7. We will use $K_{\mathcal{E}}$ and $K_{\text{eff,} \mathcal{E}}$ to denote the corresponding sets for the associated bundle $\mathcal{E}_j$, and use $K_{\mathcal{X}}$ and $K_{\text{eff,} \mathcal{X}}$ to denote the corresponding sets for $\mathcal{X}$.

Definition 3.8 (6, Section 3.1.3). The reduction function $v$ is defined as follows:
\[ v : K \longrightarrow \text{Box}(\Sigma) \]
\[ d \longmapsto \sum_{i=1}^{m} [\langle D_i^S, d \rangle] b_i + \sum_{j=1}^{l} [\langle D_{m+j}^S, d \rangle] s_j \]

By the $S$-extended fan exact sequence, we have
\[ \sum_{i=1}^{m} \langle D_i^S, d \rangle b_i + \sum_{j=1}^{l} \langle D_{m+j}^S, d \rangle s_j = 0 \in \mathbb{N} \otimes \mathbb{Q}. \]

Moreover, by the definition of $K$, we have $\langle D_{m+j}^S, d \rangle \in \mathbb{Z}$, for all $d \in K$ and $1 \leq j \leq l$.

Hence,
\[ v(d) = \sum_{i=1}^{m} \{ -\langle D_i^S, d \rangle \} b_i + \sum_{j=1}^{l} \{ -\langle D_{m+j}^S, d \rangle \} s_j = \sum_{i=1}^{m} \{ -\langle D_i^S, d \rangle \} b_i. \]

By abuse of notation, we use $D_i$ to denote the divisor $\{ z_i = 0 \} \subset \mathcal{X}$ and the cohomology class in $H^2(\mathcal{X}; \mathbb{Z}) \cong \mathbb{L}^S$, for $1 \leq i \leq m$.

We consider the $\mathbb{C}^\times$-action fixing a toric divisor $D_j$, $1 \leq j \leq m$, the action of $\mathbb{C}^\times$ on $\mathbb{C}^m$ is given by
\[ (z_1, \ldots, z_m) \mapsto (z_1, \ldots, t^{-1} z_j, \ldots, z_m), \quad t \in \mathbb{C}^\times. \]

We can extend this to the diagonal $\mathbb{C}^\times$-action on $U \times (\mathbb{C}^2 \setminus \{0\})$ by
\[ (z_1, \ldots, z_m, u, v) \mapsto (z_1, \ldots, t^{-1} z_j, \ldots, z_m, tu, tv), \quad t \in \mathbb{C}^\times. \]
The associated bundle $\mathcal{E}_j$ of the $\mathbb{C}^\times$-action on $\mathcal{X}$ is given by

$$\mathcal{E}_j = U \times (\mathbb{C}^2 \setminus \{0\}) / G \times \mathbb{C}^\times.$$  

We can also use the $S$-extended stacky fan of $\mathcal{X}$ to define $\mathcal{E}_j$:

$$\mathcal{E}_j = US \times (\mathbb{C}^2 \setminus \{0\}) / G_S \times \mathbb{C}^\times.$$  

Therefore $\mathcal{E}_j$ is also a toric Deligne-Mumford stack. We can identify $H^2(\mathcal{E}_j; \mathbb{Z})$ with the lattice of the characters of $G \times \mathbb{C}^\times$:

(19)  

$$H^2(\mathcal{E}_j; \mathbb{Z}) \cong \mathbb{L}^\vee \oplus \mathbb{Z} \cong H^2(\mathcal{X}; \mathbb{Z}) \oplus \mathbb{Z}.$$  

Moreover, we have the divisor sequence

$$0 \rightarrow \mathbb{N}^* \oplus \mathbb{Z} \rightarrow (\mathbb{Z}^*)^{m+2} \rightarrow \mathbb{L}^\vee \oplus \mathbb{Z}.$$  

And the $S$-extended divisor sequence

$$0 \rightarrow \mathbb{N}^* \oplus \mathbb{Z} \rightarrow (\mathbb{Z}^*)^{m+l+2} \rightarrow \mathbb{L}^\vee \oplus \mathbb{Z}.$$  

Let $\hat{D}_i^S$ be the image of the standard basis of $(\mathbb{Z}^*)^{m+l+2}$ in $\mathbb{L}^\vee \oplus \mathbb{Z}$. Then

(20)  

$$\hat{D}_i^S = (D_i^S, 0), \text{ for } i \neq j; \quad \hat{D}_j^S = (D_j^S, -1); \quad \hat{D}_{m+l+1}^S = \hat{D}_{m+l+2}^S = (0, 1).$$  

And,

(21)  

$$\hat{D}_i = (D_i, 0), \text{ for } i \neq j; \quad \hat{D}_j = (D_j, -1); \quad \hat{D}_{m+1} = \hat{D}_{m+2} = (0, 1).$$  

The fan $\Sigma_j$ of $\mathcal{E}_j$ is a rational simplicial fan contained in $N_\mathbb{Q} \oplus \mathbb{Q}$. The 1-skeleton is given by

(22)  

$$\hat{b}_i = (b_i, 0), \text{ for } 1 \leq i \leq m; \quad \hat{b}_{m+1} = (0, 1); \quad \hat{b}_{m+2} = (b_j, -1).$$  

We set

$$p_0 := (0, 1) = \hat{D}_{m+1} = \hat{D}_{m+2} \in H^2(\mathcal{E}_j; \mathbb{Q}),$$  

then a nef integral basis $\{p_1, \ldots, p_r\}$ of $H^2(\mathcal{X}; \mathbb{Q})$ can be lifted to a nef integral basis $\{p_0, p_1, \ldots, p_r\}$ of $H^2(\mathcal{E}_j; \mathbb{Q})$, under the splitting [19]. Let $p_1^S, \ldots, p_{r+t}^S$ be an integral basis of $\mathbb{L}^\vee$, such that $p_i$ is the image of $p_i^S$ in $\mathbb{L}^\vee \otimes \mathbb{R}$. Let $p_0^S, p_1^S, \ldots, p_{r+t}^S$ be an integral basis of $\mathbb{L}^\vee \oplus \mathbb{Z}$ and $p_0$ is the image of

$$p_0^S = \hat{D}_{m+l+1}^S = \hat{D}_{m+l+2}^S$$  

in $(\mathbb{L}^\vee \oplus \mathbb{Z}) \otimes \mathbb{R}$. Note that $p_{r+1}, \ldots, p_{r+t}$ are zero. We have

$$C_{\mathcal{E}_j}^S = C_\mathcal{X}^S + \mathbb{R}_{>0} p_0^S, \quad \rho_{\mathcal{E}_j}^S = \rho_{\mathcal{X}}^S + p_0^S.$$  

The following result is straightforward.

**Lemma 3.9.** If $\rho_{\mathcal{X}}^S \in cl(C_\mathcal{X}^S)$, then $\rho_{\mathcal{E}_j}^S \in cl(C_{\mathcal{E}_j}^S)$, for $1 \leq j \leq m$.

**Definition 3.10.** The $I$-function of $\mathcal{X}$ is the $H^*_{orb}(\mathcal{X})$-valued function:

(23)  

$$I_\mathcal{X}(y, z) = e^y \sum_{d \in \mathbb{R}_{\text{eff}} \mathcal{X}} \prod_{i=1}^{m+t} \left( \frac{\prod_{k=0}^{\infty} \left( D_i + ((D_i^S, d) - k) z \right)}{\prod_{k=0}^{\infty} \left( D_i + ((D_i^S, d) - k) z \right)} \right)^{y^d} 1_{v(d)},$$
where \( y^d = y_1^{(p_1^S, d)} \cdots y_{r+1}^{(p_{r+1}^S, d)} \). Similarly, The \( I \)-function of \( \mathcal{E} \) is the \( H_{\text{orb}}^* (\mathcal{E}) \)-valued function:

\[
I_{\mathcal{E}} (y, z) = \sum_{p_i \text{ logy}_i / z} p_i \prod_{i=1}^{m+l+2} \left( \prod_{k=0}^{\infty} \left( D_i + \left( (D_i^S, \beta) - k \right) z \right) \right) y^d 1_{v(\beta)},
\]

where \( y^d = y_0^{(p_0^S, \beta)} y_1^{(p_1^S, \beta)} \cdots y_{r+1}^{(p_{r+1}^S, \beta)} \).

Following section 4.1 of \([6]\), The \( I \)-functions of \( \mathcal{X} \) and \( \mathcal{E}_j \) can be rewritten in the form:

\[
I_{\mathcal{X}} (y, z) = \sum_{d \in \mathbb{K}, l \in \mathcal{K}} \prod_{i=1}^{m+l} \left( \prod_{k=0}^{\infty} \left( D_i + \left( (D_i^S, d) - k \right) z \right) \right) y^d 1_{v(d)},
\]

and

\[
I_{\mathcal{E}_j} (y, z) = \sum_{\beta \in \mathbb{K}, \mathcal{E}_j} \prod_{i=1}^{m+l+2} \left( \prod_{k=0}^{\infty} \left( D_i + \left( (D_i^S, \beta) - k \right) z \right) \right) y^d 1_{v(\beta)},
\]

respectively, because the summand with \( d \in \mathbb{K} \setminus \mathcal{K}_{\text{eff}} \) vanishes. We refer to \([6]\) for more details.

**Theorem 3.11** \([6]\), Conjecture 4.3. Assume that \( \rho^S \in cl(C_X^S) \). Then the \( I \)-function and the \( J \)-function satisfy the following relation:

\[
I_{\mathcal{X}} (y, z) = J_{\mathcal{X}} (\tau(y), z)
\]

where

\[
\tau(y) = \tau_{0,2}(y) + \tau_{tw}(y) = \sum_{i=1}^{r} (\text{logy}_i) p_i + \sum_{j=m+1}^{m+l} y^{D_j^S \mathcal{D}_j} + \text{h.o.t.} \in H_{\text{orb}}^{\leq 2} (\mathcal{X}),
\]

with

\[
\tau_{0,2}(y) \in H^2 (\mathcal{X}), \quad \tau_{tw}(y) \in H_{\text{orb}}^{\leq 2} (\mathcal{X}) \setminus H^2 (\mathcal{X}),
\]

\[
\mathcal{D}_j = \prod_{i \in \mathcal{I}_j} D_i^{(S \mathcal{D}_j)} 1_{v(D_i^S \mathcal{D}_j)} \in H_{\text{orb}}^* (\mathcal{X}).
\]

and h.o.t. stands for higher order terms in \( z^{-1} \). Furthermore, \( \tau(y) \) is called the mirror map and takes values in \( H_{\text{orb}}^{\leq 2} (\mathcal{X}) \).

For \( \tau_{0,2}(y) = \sum_{a=1}^{r} p_a \text{logy}_a \in H^2 (\mathcal{X}) \), we have

\[
\text{logy}_i = \text{logy}_i + g_i(y_1, \ldots, y_{r+1}), \quad \text{for} \quad i = 1, \ldots, r,
\]

where \( g_i \) is a (fractional) power series in \( y_1, \ldots, y_{r+1} \) which is homogeneous of degree zero with respect to the degree \( \text{deg} y_i = 2(p_i^S, d) \).

By lemma 3.39 under the assumption of theorem 3.11 we can also apply the mirror theorem to the associated bundle \( \mathcal{E}_j \), hence we have

\[
I_{\mathcal{E}_j} (y, z) = J_{\mathcal{E}_j} (\tau^{(j)}(y), z),
\]
Proposition 3.12. The function $g_i^{(j)}$ does not depend on $y_0$ and we have

$$g_i^{(j)}(y_0, \ldots, y_{r+1}) = g_i(y_1, \ldots, y_{r+1}), \quad \text{for} \quad i = 1, \ldots, r.$$  

Proof. The functions $g_i$ is the coefficients of $z^{-1}p_i$ in the expansion of $I_X$:

$$I_X(y, z) = \sum_{i=0}^{r} p_i \log y_i / z \left( 1 + z^{-1} \sum_{i=1}^{r} g_i(y)p_i + \tau_{tw} \right) + O(z^{-2}).$$

The functions $g_i^{(j)}$ is the coefficients of $z^{-1}p_i$ in the expansion of $I_{E_j}$:

$$I_{E_j}(y, z) = \sum_{i=0}^{r} p_i \log y_i / z \left( 1 + z^{-1} \sum_{i=0}^{r} g_i^{(j)}(y)p_i + \tau_{tw}^{(j)} \right) + O(z^{-2}).$$

Following the proof of lemma 3.5 of [5], we obtain the conclusion of this proposition.  

We will prove $\tau_{tw}^{(j)}$ is also independent from $y_0$. To begin with, the following lemma implies that $\tau_{tw}^{(j)}(y)$ is an (integer) power series in $y_0$.

Lemma 3.13. For any $\beta \in \mathbb{K}_{E_j}$, we have $\langle p_0^S, \beta \rangle \in \mathbb{Z}$. Furthermore, for any $\beta \in \mathbb{K}_{\text{eff}, E_j}$, we have $\langle p_0^S, \beta \rangle \in \mathbb{Z}_{\geq 0}$.

Proof. Any cone $\sigma \in \Sigma_j$ containing both $b_{m+1}$ and $b_{m+2}$ should also contain $b_j$, this is impossible since the fan $\Sigma_j$ is simplicial and $b_{m+1}$, $b_{m+2}$ and $b_j$ lie in the same plane. Hence, by the definition of $\mathbb{K}_{E_j}$ (resp. $\mathbb{K}_{\text{eff}, E_j}$), at least one of $\langle \hat{D}_{m+1}^S, \beta \rangle$ and $\langle \hat{D}_{m+2}^S, \beta \rangle$ has to be integer (resp. non-negative integer), for any $\beta \in \mathbb{K}_{E_j}$ (resp. $\beta \in \mathbb{K}_{\text{eff}, E_j}$). On the other hand, we have,

$$\langle p_0^S, \beta \rangle = \langle \hat{D}_{m+1}^S, \beta \rangle = \langle \hat{D}_{m+2}^S, \beta \rangle.$$  

Therefore, we must have $\langle p_0^S, \beta \rangle \in \mathbb{Z}$ (resp. $\langle p_0^S, \beta \rangle \in \mathbb{Z}_{\geq 0}$).

As a direct consequence of the above lemma, $\tau_{tw}^{(j)}(y)$ can only contain non-negative integer power of $y_0$.

Proposition 3.14. Let $\tau_{tw}^{(j)}(y) = \sum_{n=0}^{\infty} H_n^{(j)}(y) y_0^n$, where $H_n^{(j)}(y)$ is a (fractional) power series in $y_1, \ldots, y_n$. Then

$$H_n^{(j)}(y) = 0 \quad \text{for} \quad n \geq 1,$$

i.e. $\tau_{tw}^{(j)}(y)$ is independent from $y_0$. Moreover, we have

$$\tau_{tw}^{(j)}(y) = \tau_{tw}(y).$$
Proof. Recall $\tau^{(j)}_{\alpha}(y)$ is the coefficient of $z^{-1}$ in
\begin{equation}
\sum_{\beta} p^{[\log_2 z]} \prod_{i=1}^{m+l+2} \left( \prod_{k=0}^{\infty} \left( \hat{D}_i + \left( \langle \hat{D}_i^S, \beta \rangle - k \right) z \right) \right) y^{\beta} 1_{v(\beta)},
\end{equation}
valued in $H^{\leq 2}_{\text{ord}}(E_j) \setminus H^2(E_j)$. Hence, we only need to consider terms with $v(\beta) \neq 0$, or, equivalently, $v(d) \neq 0$, where $d$ is the natural projection of $\beta$ on to $\mathbb{K}_{\text{eff},X}$.

Therefore, it remains to examine the product factor:
\begin{align}
&\prod_{i=1}^{m+l+2} \left( \prod_{k=0}^{\infty} \left( \hat{D}_i + \left( \langle \hat{D}_i^S, \beta \rangle - k \right) z \right) \right) \\
&= \prod_{i:(\hat{D}_i^S, \beta) < 0} \prod_{0 \leq k < \langle \hat{D}_i^S, \beta \rangle} \left( \hat{D}_i + \left( \langle \hat{D}_i^S, \beta \rangle - k \right) z \right) \\
&= C_{\beta} z^{-\left( \sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle \right) + \# \{ i : (\hat{D}_i^S, \beta) \in \mathbb{Z}_{<0} \} \prod_{i:(\hat{D}_i^S, \beta) \in \mathbb{Z}_{<0}} \hat{D}_i + \text{h.o.t.},
\end{align}
where
\begin{equation}
C_{\beta} = \prod_{i:(\hat{D}_i^S, \beta) < 0} \prod_{0 \leq k < \langle \hat{D}_i^S, \beta \rangle} \left( \langle \hat{D}_i^S, \beta \rangle - k \right) \prod_{i:(\hat{D}_i^S, \beta) > 0} \prod_{0 \leq k < \langle \hat{D}_i^S, \beta \rangle} \left( \langle \hat{D}_i^S, \beta \rangle - k \right)^{-1}.
\end{equation}

By assumption, we need to have
\begin{equation}
\sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle \geq \sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle \geq 0.
\end{equation}
The equality holds if and only if
\begin{equation}
\langle \hat{D}_i^S, \beta \rangle \in \mathbb{Z}, \quad \text{for all} \quad 1 \leq i \leq m + l + 2; \quad \text{and} \quad \sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle = 0.
\end{equation}
However, this would imply $v(\beta) = 0$, hence we cannot have $\sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle = 0$. Therefore, the expansion (30) would contribute to $H^0_{\alpha}(\beta)$ only when
\begin{equation}
\sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle = 1 \quad \text{and} \quad \# \{ i : (\hat{D}_i^S, \beta) \in \mathbb{Z}_{<0} \} = 0.
\end{equation}
In this case, if $\langle p_0^S, \beta \rangle \geq 1$, then
\begin{equation}
\sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle \geq \sum_{i=1}^{m+l} \langle D_i^S, d_i \rangle + 1,
\end{equation}
therefore, we have
\begin{equation}
0 \geq \sum_{i=1}^{m+l} \langle D_i^S, d_i \rangle \geq \sum_{i=1}^{m+l} \langle D_i^S, d_i \rangle = 0.
\end{equation}
This implies, when $\langle p_0^S, \beta \rangle \geq 1$, we must have
\begin{equation}
\langle D_i^S, d_i \rangle \in \mathbb{Z}, \quad \text{for} \quad 1 \leq i \leq m + l.
\end{equation}
It is a contradiction, since \( \tau_{tw} \in H^{2}_{orb}(E_j) \setminus H^2(E_j) \) implies \( v(d) \neq 0 \). Hence \( H^{(j)}_n = 0 \) for all \( n > 0 \) and \( \tau^{(j)}_{tw}(y) \) is independent from \( y_0 \). Moreover, by the expression of \( I \)-functions and the identity

\[
\tau^{(j)}_{tw} \big|_{y_0=0} = I_X,
\]

we have \( \tau^{(j)}_{tw}(y) = \tau_{tw}(y) \).

As a direct consequence of the above lemma, we can use the following notation for the Seidel element

\[
(32) \quad S_j(\tau(y)) := S_j(\tau^{(j)}(y)),
\]

since \( S_j(\tau^{(j)}(y)) \) does not depend on \( y_0 \) or \( q_0 \).

### 3.4. Seidel Elements in terms of \( I \)-functions.

We can rewrite the \( I \)-function of the associated bundle \( E_j \) as follows:

\[
(33) \quad \sum_{i=0}^{r} p_i \log y_i / z \left( 1 + z^{-1} \left( \sum_{i=0}^{r} g_i^{(j)}(y)p_i + \tau^{(j)}_{tw}(y) \right) + z^{-2} \left( \sum_{n=0}^{2} G_n^{(j)}(y)y_0^n \right) + O(z^{-3}) \right).
\]

Then, \( \log q_i = \log g_i + g_i^{(j)}(y) \) implies

\[
(34) \quad I_{E_j}(y, z) = \sum_{i=0}^{r} p_i \log y_i / z \left( 1 + z^{-1} \tau^{(j)}_{tw}(y) + z^{-2} \left( \sum_{n=0}^{2} G_n^{(j)}(y)y_0^n \right) + O(z^{-3}) \right),
\]

where \( G_n^{(j)}(y) \) is a (fractional) power series in \( y_1, \ldots, y_{r+l} \) taking values in \( H^*_{orb}(E_j) \).

By proposition \( (2.4) \), the Seidel element \( \tilde{S}_j(\tau^{(j)}(y)) \) is the coefficient of \( q_0 / z^2 \) in

\[
\exp \left( - \sum_{i=0}^{r} p_i \log q_i / z \right) J_{E_j}(\tau^{(j)}(y), z),
\]

hence \( J_{E_j}(\tau^{(j)}(y), z) = I_{E_j}(y, z) \) and \( \log q_0 = \log y_0 + g_0^{(j)}(y) \) imply the following result:

**Theorem 3.15.** The Seidel element \( S_j \) associated to the toric divisor \( D_j \) is given by

\[
(35) \quad S_j(\tau^{(j)}(y)) = \tau(\tau^{(j)}(y)y_0).
\]

Furthermore, we have

\[
(36) \quad \tilde{S}_j(\tau(y)) = \tilde{S}_j(\tau^{(j)}(y)) = \exp(-g_0^{(j)}(y)) \tau(\tau^{(j)}(y)).
\]
3.5. Computation of \( g_0^{(i)} \). The computation is essentially the same as the proof of lemma 3.16 of [5]. Consider the product factors in \( I_{r_j} \):

\[
\prod_{i=1}^{m+l+2} \left( \frac{\prod_{k=1}^{\infty} (\hat{D}_i^S, \beta) \left( \hat{D}_i + (\hat{D}_i^S, \beta) - k \right) z}{\prod_{k=0}^{\infty} (\hat{D}_i + (\hat{D}_i^S, \beta) - k) z} \right) y^\beta 1_{v(\beta)},
\]

these factors contribute to \( g_i^{(j)} \) if

\[
v(\beta) = \sum_{i=1}^{m+l+2} \{ -\langle \hat{D}_i^S, \beta \rangle \}, \]

then, by the definition of \( k_{s_{\text{eff}}} \), we must have

\[
\langle \hat{D}_i^S, \beta \rangle \in \mathbb{Z}, \text{ for all } 1 \leq i \leq m + l + 2.
\]

In this case, the product factors can be rewritten as

\[
\prod_{i=1}^{m+l+2} \left( \frac{\prod_{k=1}^{\infty} (\hat{D}_i^S, \beta) \left( \hat{D}_i + (\hat{D}_i^S, \beta) - k \right) z}{\prod_{k=0}^{\infty} (\hat{D}_i + (\hat{D}_i^S, \beta) - k) z} \right) y^\beta 1_{v(\beta)} = \prod_{i=1}^{m+l+2} \frac{\prod_{k=-\infty}^{0} (\hat{D}_i + k z) y^\beta}{\prod_{k=-\infty}^{0} (\hat{D}_i + k z) y^\beta} \quad (37)
\]

where \( h.o.t. \) stands for higher order terms in \( z^{-1} \) and

\[
(38) \quad C_\beta = \prod_{i: \langle \hat{D}_i^S, \beta \rangle > 0} (-1)^{-\langle \hat{D}_i^S, \beta \rangle - 1} \langle \hat{D}_i^S, \beta \rangle! \prod_{i: \langle \hat{D}_i^S, \beta \rangle \geq 0} (\langle \hat{D}_i^S, \beta \rangle)!^{-1}.
\]

They contribute to the \( z^{-1} \) term if

\[
\sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle + \# \{ i : \langle \hat{D}_i^S, \beta \rangle < 0 \} \leq 1.
\]

Since we assume \( \rho_\chi^S \in \text{cl}(C_\chi^S) \), hence \( \rho_{r_j}^S \in \text{cl}(C_{r_j}^S) \). So it has to be the following three cases:

- \( \sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle = 0 \)
- \( \sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle = 1 \)
- \( \sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle = 0 \)

In the first case, we have \( \langle \hat{D}_i^S, \beta \rangle = 0 \) for all \( i \), hence \( \beta = 0 \); the second case can not happen, since \( \beta \) has to satisfy \( \langle \hat{D}_i^S, \beta \rangle = 0 \) except for one \( i \) and this implies \( \beta = 0 \).
Therefore, the coefficient of $z^{-1}$ is from the third case, where

$$
\sum_{i=1}^{m+l+2} \langle \hat{D}_i^S, \beta \rangle = 0 \quad \text{and} \quad \# \{i : \langle \hat{D}_i^S, \beta \rangle < 0 \} = 1.
$$

By the assumption $\rho^X_\mathcal{A} \in cl(S^2_\mathcal{A})$, we must have $\sum_{i=1}^{m+l} \langle D_i^S, d \rangle = 0$ and $\langle p_0^S, \beta \rangle = 0$. Moreover, $\langle D_i^S, d \rangle < 0$ for exactly one $i$ in $\{1, \ldots, m\}$. (Note that $\langle D_i^S, d \rangle \geq 0$ for $i \in \{m+1, \ldots, m+l\}$.)

Now $g_0^{(j)}$ is the coefficient corresponding to $p_0$ and $\hat{D}_j = \langle D_j, -1 \rangle = D_j - p_0$ is the only one, among $D_1, \ldots, D_m$, which contains $p_0$. By expression (37), we must have $\langle D_j^S, d \rangle < 0$ and $\langle D_i^S, d \rangle \geq 0$ for $i \neq j$. Hence we have

**Lemma 3.16.** The coefficient $g_0^{(j)}$ is given by

$$
g_0^{(j)}(y_1, \ldots, y_{r+l}) = \sum_{\langle D_i^S, d \rangle \in \mathbb{Z}, 1 \leq i \leq m+l, \langle \rho^X_\mathcal{A}, d \rangle = 0, \langle D_j^S, d \rangle < 0, \langle D_i^S, d \rangle \geq 0, \forall i \neq j} \frac{(-1)^{\langle D_j^S, d \rangle} (\langle D_i^S, d \rangle - 1)!}{\prod_{i \neq j} \langle D_i^S, d \rangle !} y^d.
$$

### 4. Batyrev Elements

In this section, we will extend the definition of the Batyrev elements in [5] to toric Deligne-Mumford stacks and explore their relationships with the Seidel elements.

#### 4.1. Batyrev Elements

Following [6], consider the mirror coordinates $y_1, \ldots, y_{r+l}$ of the toric Deligne-Mumford stacks $\mathcal{X}$ with $\rho^X_\mathcal{A} \in cl(S^2_\mathcal{A})$. Set $\mathbb{C}[y^\pm] = \mathbb{C}[y_1^\pm, \ldots, y_{r+l}^\pm]$.

**Definition 4.1.** The Batyrev ring $B(\mathcal{X})$ of $\mathcal{X}$ is a $\mathbb{C}[y^\pm]$-algebra generated by the variables $\lambda_1, \ldots, \lambda_{r+l}$ with the following two relations:

- (multiplicative): $y^d \prod_{i : \langle D_i^S, d \rangle < 0} \omega_i^{-\langle D_i^S, d \rangle} = \prod_{i : \langle D_i^S, d \rangle > 0} \omega_i^{\langle D_i^S, d \rangle}$, $d \in \mathbb{L}^S$;
- (linear): $\omega_i = \sum_{a=1}^{r+l} m_{ai} \lambda_a$, where $\omega_i$ is invertible in $B(\mathcal{X})$.

**Definition 4.2.** We define the element $\hat{p}_i^S \in H^2_{orb}(\mathcal{X}) \otimes \mathbb{Q}[[y_1, \ldots, y_{r+l}]]$ as

$$
\hat{p}_i^S = \frac{\partial \tau(y)}{\partial \log y_i}, \quad i = 1, \ldots, r+l.
$$

Recall that

$$
D_j^S = \sum_{i=1}^{r+l} m_{ij} \hat{p}_i^S, \quad \text{for} \ 1 \leq j \leq m+l,
$$

Then, the Batyrev element associated to $D_j^S$ is defined by

$$
\hat{D}_j^S = \sum_{i=1}^{r+l} m_{ij} \hat{p}_i^S, \quad \text{for} \ 1 \leq j \leq m+l.
$$
The Batyrev elements $\tilde{D}_1^S, \ldots, \tilde{D}_{m+1}^S$ satisfy the multiplicative and linear Batyrev relations for $\omega_j = \tilde{D}_j^S$.

Proof. We consider the differential operator $P_d \in \mathbb{C}[z, y, zy(\partial / \partial y)]$ for $d \in \mathbb{L}^S$, introduced by Iritani in [6], section 4.2:

$$P_d := y^d \prod_{i: (D_i^S, d) > 0} (D_i - k) \prod_{k=0}^{(D_i^S, d) - 1} (D_i - k),$$

where $D_i := \sum_{j=1}^{r+1} m_{ij} zy_j \partial / \partial y_j$.

By [6] lemma 4.6, we have

$$P_d I(y, z) = 0, \quad d \in \mathbb{L}^S.$$

Hence

$$0 = P_d (z, y, zy \partial / \partial y) I(y, z) = P_d (z, y, zy \partial / \partial y) J(\tau(y), z).$$

This implies that

$$P_d (z, y, z \tau^* \nabla 1) = 0,$$

where $\tau^* \nabla : = \nabla_{\tau, (y(\partial / \partial y_i))}$. Since

$$\tau(y) = \sum_{i=1}^r p_i \log y_i + \tau_{tw}(y)$$

and

$$\nabla_{\tau, (y(\partial / \partial y_i))} = \tau_{*}(y_i(\partial / \partial y_i)) + \frac{1}{z} y_i \frac{\partial \tau(y)}{\partial y_i} \circ \tau,$$

by setting $z = 0$, we proved that the Batyrev elements satisfy the multiplicative relation.

It is straightforward from the definition that the Batyrev elements satisfy the linear relation. $\square$

Consider the $I$-function for the bundle $E_j$ associated to the toric divisor $D_j^S$, for $1 \leq j \leq m$.

$$I_{E_j}(y, z) = \sum_{\beta \in \mathbb{I}_{E_j}} p_{\log y_i/z} \prod_{i=1}^{m+1 \ldots 2} (\prod_{k=0}^{\infty} (\tilde{D}_i^{S} + (\tilde{D}_i^{S}, \beta) - k)) \prod_{k=0}^{\infty} (\tilde{D}_i + (\tilde{D}_i^{S}, \beta) - k) y^D 1_{v(\beta)}.$$

where $y^D = y_0^{(p_0^S, \beta)} y_1^{(p_1^S, \beta)} \ldots y_{r+1}^{(p_{r+1}^S, \beta)}$. The following lemma is a generalization of lemma 3.11 in [6].

**Lemma 4.4.** The $I$-function $I_{E_j}$ of the bundle $E_j$, associated to the toric divisor $D_j^S$, satisfies the following partial differential equation:

$$z \frac{\partial}{\partial y_0} \left( y_0 \frac{\partial}{\partial y_0} \right) I_{E_j} = \left( \sum_{i=1}^{m+1} m_{ij} \left( y_i \frac{\partial}{\partial y_i} - y_0 \frac{\partial}{\partial y_0} \right) \right) I_{E_j}.$$

Proof. Consider the left hand side of the equation (43),

$$z \frac{\partial}{\partial y_0} \left( y_0 \frac{\partial}{\partial y_0} \right) I_{E_j}$$

$$= \sum_{\beta \in \mathbb{I}_{E_j}} p_{\log y_i/z} \prod_{i=1}^{m+1 \ldots 2} (\prod_{k=0}^{\infty} (\tilde{D}_i^{S} + (\tilde{D}_i^{S}, \beta) - k)) \prod_{k=0}^{\infty} (\tilde{D}_i + (\tilde{D}_i^{S}, \beta) - k) (2p_0 (p_0^S, \beta) + (p_0^S, \beta)^2 z) (y^D / y_0) 1_{v(\beta)},$$
and the right hand side of the equation \[\text{(18)}\]

\[
\left( \sum_{i=1}^{r+l} m_{ij} \left( y_i \frac{\partial}{\partial y_i} - y_0 \frac{\partial}{\partial y_0} \right) \right) I_{\mathcal{E}_j}
= \sum_{e^{i=0}} \sum_{y/z} \prod_{i=1}^{m+1+2} \left( \frac{\prod_{k=0}^{\infty} (\hat{D}_i + (\hat{D}_i^S, \beta - k) z)}{\prod_{k=0}^{\infty} (\hat{D}_i + (\hat{D}_i^S, \beta - k) z)} \right) \left( \hat{D}_j + (\hat{D}_j^S, \beta) \right) y^\beta 1_{\mathcal{E}_j}.
\]

It is suffice to prove the coefficients of $y^\beta 1_{\mathcal{E}_j}$ in them are the same, for all $\beta \in \mathbb{K}_{\mathcal{E}_j}$.
Note that, we can rewrite the product factor

\[
\frac{\prod_{k=0}^{\infty} (\hat{D}_i + (\hat{D}_i^S, \beta - k) z)}{\prod_{k=0}^{\infty} (\hat{D}_i + (\hat{D}_i^S, \beta - k) z)} = \frac{\prod_{k \leq 0, \{k\} = \{\hat{D}_i^S, \beta\}} (\hat{D}_i + kz)}{\prod_{k \leq (\hat{D}_i^S, \beta), \{k\} = \{\hat{D}_i^S, \beta\}} (\hat{D}_i + kz)}.
\]

Let $\beta' = \beta + \{\sigma_0\}$, hence we have

\[
\langle \hat{D}_j^S, \beta' \rangle = \langle \hat{D}_j^S, \beta \rangle - 1; \quad \langle \hat{D}_i^S, \beta' \rangle = \langle \hat{D}_i^S, \beta \rangle \text{ for } 1 \leq i \leq m + l \text{ and } i \neq j;
\]

\[
\langle \hat{D}_{m+l+1}^S, \beta' \rangle = \langle \hat{D}_{m+l+1}^S, \beta \rangle + 1; \quad \langle \hat{D}_{m+l+2}^S, \beta' \rangle = \langle \hat{D}_{m+l+2}^S, \beta \rangle + 1.
\]

Note that $\beta \in \mathbb{K}_{\mathcal{E}_j}$ if and only if $\beta' \in \mathbb{K}_{\mathcal{E}_j}$. Moreover,

\[
\left( y^\beta / y_0 \right) 1_{\mathcal{E}_j(\beta')} = y^\beta 1_{\mathcal{E}_j(\beta)}.
\]

Hence the coefficient of $y^\beta 1_{\mathcal{E}_j}$ in $z \frac{\partial}{\partial y_0}(y_0 \frac{\partial}{\partial y_0}) I_{\mathcal{E}_j}$ is

\[
\sum_{e^{i=0}} \sum_{y/z} \prod_{i=1}^{m+1+2} \left( \frac{\prod_{k=0}^{\infty} (\hat{D}_i + (\hat{D}_i^S, \beta - k) z)}{\prod_{k=0}^{\infty} (\hat{D}_i + (\hat{D}_i^S, \beta - k) z)} \right) \frac{\hat{D}_j + (\hat{D}_j^S, \beta) z}{(p_0 + (p_0^S, \beta + 1)z)^2}.
\]

\[
\text{•} \quad (2p_0 (p_0^S, \beta + 1) + (p_0^S, \beta + 1)^2 z)
\]

\[
= \sum_{e^{i=0}} \sum_{y/z} \prod_{i=1}^{m+1+2} \left( \frac{\prod_{k=0}^{\infty} (\hat{D}_i + (\hat{D}_i^S, \beta - k) z)}{\prod_{k=0}^{\infty} (\hat{D}_i + (\hat{D}_i^S, \beta - k) z)} \right) \frac{\hat{D}_j + (\hat{D}_j^S, \beta) z}{z} \quad \text{(since } p_0^2 = 0).\]

This is exactly the coefficient of $y^\beta 1_{\mathcal{E}_j}$ in $\left( \sum_{i=1}^{r+l} m_{ij} \left( y_i \frac{\partial}{\partial y_i} - y_0 \frac{\partial}{\partial y_0} \right) \right) I_{\mathcal{E}_j}$,

Hence the I-function of $\mathcal{E}_j$ satisfies the differential equation

\[
z \frac{\partial}{\partial y_0} \left( y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j} = \left( \sum_{i=1}^{r+l} m_{ij} \left( y_i \frac{\partial}{\partial y_i} - y_0 \frac{\partial}{\partial y_0} \right) \right) I_{\mathcal{E}_j}.
\]

\[
\square
\]

Using the expansion of $I_{\mathcal{E}_j}$, we have

\[
I_{\mathcal{E}_j}(y, z) = \sum_{e^{i=0}} \sum_{y/z} \left( 1 + z^{-1} \left( \sum_{i=0}^{r} q_i^{(j)}(y)p_i + r_i^{(j)} \right) + z^{-2} \left( \sum_{n=0}^{2} g_i^{(j)}(y_0)n \right) + O(z^{-3}) \right),
\]
Therefore, the left hand side of equation (43) is

\[ y_0 \frac{\partial}{\partial y_0} I_{E_j} = \frac{p_0}{z} \sum_{\ell \in \mathbb{Z}^r} p_{\ell \log y_i / z} \left( 1 + z^{-1} \left( \sum_{i=0}^{r} g^{(j)}_i(y) p_i + \tau^{(j)}_w \right) + z^{-2} \left( \sum_{n=0}^{2} G^{(j)}_n(y)y_0^n \right) + O(z^{-3}) \right) \]

Therefore, we obtain

\[ z \frac{\partial}{\partial y_0} \left( y_0 \frac{\partial}{\partial y_0} I_{E_j} \right) = \frac{\partial}{\partial y_0} \left( \frac{p_0}{y_0 z} \sum_{\ell \in \mathbb{Z}^r} p_{\ell \log y_i / z} \left( z^{-1} \left( \sum_{n=1}^{2} G^{(j)}_n(y)y_0^n \right) + O(z^{-2}) \right) \right) \]

On the other hand, the pull-back of the right hand side of equation (43) by \( \iota^* \) is

\[ \iota^* \left( \sum_{i=1}^{r+l} m_{ij} \left( y_i \frac{\partial}{\partial y_i} \right) - y_0 \frac{\partial}{\partial y_0} \right) I_{E_j} \]

\[ = \left( \sum_{i=1}^{r+l} m_{ij} \left( y_i \frac{\partial}{\partial y_i} \right) - y_0 \frac{\partial}{\partial y_0} \right) \iota^* I_{E_j} \]

\[ = \left( \sum_{i=1}^{r+l} m_{ij} \left( y_i \frac{\partial}{\partial y_i} \right) \right) (I_X + O(y_0)) \]

\[ = z^{-1} \left( \sum_{i=1}^{r+l} m_{ij} \left( y_i \frac{\partial}{\partial y_i} \right) \tau(y) \right) + O(z^{-2}) + O(y_0). \]

Hence we conclude the following lemma.

**Lemma 4.5.** The Batyrev element \( \hat{D}_j(y) \) is given by

\[ \hat{D}_j(y) = \iota^* G^{(j)}_i(y), \quad \text{for} \quad 1 \leq j \leq m + l. \]

Hence, the following theorem is a direct consequence of the above lemma and theorem 3.16.
Theorem 4.6. The Seidel element $\tilde{S}_j$ corresponding to the toric divisor $D_j$ is given by
\begin{equation}
\tilde{S}_j(\tau(y)) = \exp(-g^S_0(y))\tilde{D}_j(y).
\end{equation}

4.2. The computation of $\tilde{D}_j$. Using the expansion
\begin{equation}
\left( \sum_{i=1}^{r+l} m_{ij} \left( y_i \frac{\partial}{\partial y_i} \right) \right) I_X = e^{\sum_{i=1}^r p_i \log y_i/z} \left( z^{-1} \tilde{D}_j + O(z^{-2}) \right),
\end{equation}
we see that $\tilde{D}_j$ is the coefficient of $z^{-1}$ in the expansion of
\begin{equation}
e^{-\sum_{i=1}^r p_i \log y_i/z} \left( \sum_{i=1}^{r+l} m_{ij} \left( y_i \frac{\partial}{\partial y_i} \right) \right) I_X.
\end{equation}

And, by direct computation
\begin{equation}
\left( \sum_{i=1}^{r+l} m_{ij} \left( y_i \frac{\partial}{\partial y_i} \right) \right) I_X =
\sum_{\varepsilon=1}^{\infty} \sum_{d \in \mathbb{K}_{\text{eff},X}} \prod_{i=1}^{m+l} \left( \frac{\prod_{k=\varepsilon}^\infty (D_i + ((D^S_i,d) - k) z)}{\prod_{k=0}^\infty (D_i + ((D^S_i,d) - k) z)} \right) \left( \frac{D_j}{z} + \langle D^S_j, d \rangle \right) y^d 1_{\varepsilon(d)}.
\end{equation}

Hence, to compute the Batyrev element $\tilde{D}_j$, it remains to examine the expansion of the product factor
\begin{equation}
\frac{\prod_{k=\varepsilon}^\infty (D_i + ((D^S_i,d) - k) z)}{\prod_{k=0}^\infty (D_i + ((D^S_i,d) - k) z)} = C_d z^{-\sum_{i=1}^{m+l} [\langle D^S_i,d \rangle] + \# \{i : (D^S_i,d) \in \mathbb{Z}_{<0} \}} \prod_{i : (D^S_i,d) \in \mathbb{Z}_{<0}} D_i + \text{h.o.t.},
\end{equation}
where
\begin{equation}
C_d = \prod_{i : (D^S_i,d) < 0} \prod_{i : (D^S_i,d) < k < 0} (\langle D^S_i,d \rangle - k) \prod_{i : (D^S_i,d) > 0} \prod_{0 \leq k < \langle D^S_i,d \rangle} (\langle D^S_i,d \rangle - k)^{-1}
\end{equation}
The summand indexed by $d \in \mathbb{K}_{\text{eff},X}$ contributes to the coefficient of $z^{-1}$ if and only if
\begin{equation}
\sum_{i=1}^{m+l} [\langle D^S_i,d \rangle] + \# \{i : (D^S_i,d) \in \mathbb{Z}_{<0} \} \leq 1.
\end{equation}

It happens only in the following three cases:
1. \begin{align*}
&\sum_{i=1}^{m+l} [\langle D^S_i,d \rangle] + \# \{i : (D^S_i,d) \in \mathbb{Z}_{<0} \} = 0 \\
&\sum_{i=1}^{m+l} [\langle D^S_i,d \rangle] = 0 \\
&\# \{i : (D^S_i,d) \in \mathbb{Z}_{<0} \} = 1 \\
&\sum_{i=1}^{m+l} [\langle D^S_i,d \rangle] = 1 \\
&\# \{i : (D^S_i,d) \in \mathbb{Z}_{<0} \} = 0
\end{align*}

The first case happens if and only if $d = 0$. If the second case happens, then
\begin{equation}
\sum_{i=1}^{m+l} [\langle D^S_i,d \rangle] = \sum_{i=1}^{m+l} \langle D^S_i,d \rangle = \langle p^S_X, d \rangle = 0.
\end{equation}
In particular,
\[ \langle D^S_i, d \rangle \in \mathbb{Z}, 1 \leq i \leq m + l. \]
Hence we obtain the following lemma:

**Lemma 4.7.** For \( 1 \leq j \leq m + l \), the Batyrev element \( \hat{D}_j \) is given by
\[ (47) \]
\[ \hat{D}_j = D_j + \sum_{i=1}^{m} D_i \sum_{\langle \rho^S_i, d \rangle = 0} C_d \langle D^S_i, d \rangle y^d + \sum_{\langle \rho^S_i, d \rangle \in \mathbb{Z}_{<0}} C_d \langle D^S_i, d \rangle y^d 1_{v(d)}, \]
where \( C_d \) is given by equation (46).

5. **Seidel elements corresponding to Box elements**

Consider the box element \( s_j \in \text{Box}(\Sigma) \), such that
\[ s_j = \sum_{i=1}^{m} c_{ji} b_i \in \mathbb{N}_Q, \text{ for some } 0 \leq c_{ji} < 1. \]

Let \( n_j \) be the least common denominator of \( \{c_{ji}\}_{i=1}^{m} \), we define a \( \mathbb{C}^\times \)-action on \( \mathcal{U}^S \times (\mathbb{C}^2 \setminus \{0\}) \) by
\[ (z_1, \ldots, z_{m+l}, u, v) \mapsto (t^{-c_{1}n_j}z_1, \ldots, t^{-c_{m}n_j}z_m, z_{m+1}, \ldots, z_{m+l}, t^{n_j}u, t^{n_j}v), \quad t \in \mathbb{C}^\times. \]
Hence we have an associated bundle
\[ \mathcal{E}_{m+j} = \mathcal{U}^S \times (\mathbb{C}^2 \setminus \{0\})/G^S \times \mathbb{C}^\times \]
on \( \mathbb{CP}^1 \times B\mu_{n_j} \) with \( X \) being the fiber. Furthermore, \( \mathcal{E}_{m+j} \) can also be considered as a bundle over \( \mathbb{CP}^1 \), since there is a natural projection
\[ \mathbb{CP}^1 \times B\mu_{n_j} \rightarrow \mathbb{CP}^1. \]

We can identify \( H^2(\mathcal{E}_{m+j}; \mathbb{Z}) \) with \( H^2(X; \mathbb{Z}) \oplus \mathbb{Z} \), where the second summand \( \mathbb{Z} \cong \text{Pic}(\mathbb{CP}^1 \times B\mu_{n_j}) \),
and we have the following short exact sequence from remark 5.5 of [4]:
\[ (48) \]
\[ 0 \rightarrow \text{Pic}(\mathbb{CP}^1) \rightarrow \text{Pic}(\mathbb{CP}^1 \times B\mu_{n_j}) \rightarrow \mathbb{Z}/n_j \mathbb{Z} \rightarrow 0 \]
We identify an element of \( \text{Pic}(\mathbb{CP}^1) \) with its image in \( \text{Pic}(\mathbb{CP}^1 \times B\mu_{n_j}) \) under the above map. Then the weights of \( G^S \times \mathbb{C}^\times \) defining \( \mathcal{E}_{m+j} \) are given by
\[ \hat{D}_i^S = (D_i^S, -c_{ji}n_j), \quad \text{for } 1 \leq i \leq m; \quad \hat{D}_{m+j}^S = (D_{m+j}^S, 0), \quad \text{for } 1 \leq j \leq l; \]
\[ \hat{D}_{m+i+1}^S = \hat{D}_{m+i+2}^S = (0, n_j). \]
The fan of \( \mathcal{E}_{m+j} \) is contained in \( N_\mathbb{Q} \oplus \mathbb{Q} \). The 1-skeleton is given by
\[ (49) \]
\[ b_i = (b_i, 0), \text{ for } 1 \leq i \leq m; \quad b_{m+i} = (0, 1); \quad b_{m+2} = (s_j, -1). \]
Let \( E_{m+j} \) be the coarse moduli space of \( \mathcal{E}_{m+j} \). Then \( E_{m+j} \) is an X-bundle over \( \mathbb{CP}^1 \). The Seidel element is defined as in equation (5).
We set
\[ p_0 := (0, 1) \in H^2(E_{m+j}) \cong H^2(X) \oplus \text{Pic}(\mathbb{CP}^1), \]
Theorem 5.1. We will also obtain the following theorem.

Moreover, for each correction coefficient \( \tilde{\tau} \) by Lemma 5.2.

and use the same argument as in lemma 3.12 and lemma 3.14, we can show that under the canonical splitting of (17). Let \( p^S_0, p^S_1, \ldots, p^S_{r+l} \) be an integral basis of \( L^S \otimes \mathbb{Z} \) and \( p_0 \) be the image of \( p^S_0 = \hat{D}^S_{m+l+1} = \hat{D}^S_{m+l+2} \)
in \( (L^V \otimes \mathbb{Z}) \otimes \mathbb{R} \). Therefore \( p_{r+1}, \ldots, p_{r+l} \) are zero.

As in the toric divisor case, we have the following expansion of the \( I \)-function:

\[
I_{e_{m+j}}(y, z) = \sum_{m=0}^{r} p_{m} \log y/z \left( 1 + z^{-1} \left( \sum_{i=0}^{m} g_i^{(m+j)}(y) p_i + \tau_{tw}^{(m+j)}(y) \right) + z^{-2} \left( \sum_{m=0}^{2} C_m^{(m+j)}(y) y_0^n + O(z^{-3}) \right) \right),
\]

and use the same argument as in lemma 3.12 and lemma 3.14 we can show that \( g_i^{(m+j)}(y) \) and \( \tau_{tw}^{(m+j)}(y) \) are independent from \( y_0 \), for \( 1 \leq i \leq r \) and \( 1 \leq j \leq l \).

Moreover, for each \( j \in \{1, \ldots, l\} \), we have

\[
g_i^{(m+j)}(y_0, \ldots, y_{r+l}) = g_i(y_1, \ldots, y_{r+l}) \quad \text{for} \quad i = 1, \ldots, r.
\]

And

\[
\tau_{tw}^{(m+j)}(y) = \tau_{tw}(y).
\]

We will also obtain the following theorem.

**Theorem 5.1.** The Seidel element \( \tilde{S}_{m+j} \) associated to the box element \( s_j \) is given by

\[
\tilde{S}_{m+j}(\tau(y)) := \tilde{S}_{m+j}(\tau^{(m+j)}(y)) = \exp \left( -g_0^{(m+j)}(y) \right) \tau^* (g_1^{(m+j)}(y)).
\]

Using the same computation as in the toric divisor case, we can compute the correction coefficient \( g_0^{(m+j)} \):

**Lemma 5.2.** The function \( g_0^{(m+j)} \) is given by

\[
g_0^{(m+j)}(y_1, \ldots, y_{r+l}) = \sum_{1 \leq k \leq m, k \notin I_j^S} \sum_{(D^S_{k}, d) \in \mathbb{Z}, 1 \leq i \leq m+l} c_{jk} \frac{(-1)^{-(D^S_{k}, d)} - (D^S_{k}, d) - 1)!}{\prod_{i \neq k} (D^S_{i}, d)!} y^d,
\]

where \( I_j^S \) is the "anticone" of the cone containing \( s_j \).

**Proof.** The argument is almost the same as the argument in section 3.3. The only change we need to make is the paragraph above lemma 3.16.

In this case, \( g_0^{(m+j)} \) is the coefficient corresponding to \( p_0 \) and elements in \( \{\hat{D}_1, \ldots, \hat{D}_m\} \) that contain \( p_0 \) are precisely these elements:

\[
\hat{D}_k = (D_k, -c_{jk} n_j) = D_k - c_{jk} p_0, \quad \text{for} \quad 1 \leq k \leq m \quad \text{and} \quad k \notin I_j^S.
\]
Therefore, by expression \((37)\) and \((39)\), we must have \(\langle D^S_k, d \rangle < 0\) for exactly one \(k \in \{k \in \mathbb{Z} \mid 1 \leq k \leq m \text{ and } k \notin I_j^S\}\). \(\square\)

Moreover, by mimicking the computation in lemma \(4.4\) we have

**Lemma 5.3.** The \(I\)-function of \(\mathcal{E}_{m+j}\) satisfies the following differential equation:

\[
(53) \quad z \frac{\partial}{\partial y_0} \left( y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j} = y^{-D_{m+j}^{S'}} \left( \sum_{i=1}^{r} m_{ij} \left( y_i \frac{\partial}{\partial y_i} \right) - y_0 \frac{\partial}{\partial y_0} \right) I_{\mathcal{E}_j},
\]

where \(D_{m+j}^{S'} \in \mathbb{L}^S \otimes \mathbb{Q}\) is defined by \((18)\).

**Proof.** The proof is almost identical to the proof of lemma \(4.4\) except, this time, we will need to choose \(\beta' = \beta + [\sigma_0] - D_{m+j}^{S'}\). Then everything else follows. \(\square\)

Using this lemma, following the argument in the toric divisor case, we conclude

**Theorem 5.4.** The Seidel element \(\tilde{S}_{m+j}\) corresponding to the box element \(s_j\), with

\[
\tilde{s}_j = \sum_{i=1}^{m} c_{ji} \tilde{b}_i, \quad \text{for some} \quad 0 < c_{ji} < 1,
\]

is given by

\[
(54) \quad \tilde{S}_{m+j}(\tau^{(m+j)}(y)) = \exp \left( -g_0^{(m+j)} \right) y^{-D_{m+j}^{S'}} \tilde{D}_{m+j}(y),
\]

where \(\tilde{D}_{m+j}(y)\) is the corresponding Batyrev element. Moreover,

\[
(55) \quad \tilde{D}_{m+j} = \sum_{i=1}^{m} D_i \sum_{\langle D^S_i, d \rangle \in \mathbb{Z}_{>0}, \forall k \neq i} C_d(\langle D^S_{m+j}, d \rangle) y^d + \sum_{\langle D^S_i, d \rangle \in \mathbb{Z}_{<0}} C_d(\langle D^S_{m+j}, d \rangle) y^d 1_v(d),
\]

and

\[
(56) \quad C_d = \prod_{i: \langle D^S_i, d \rangle < 0} \prod_{\langle D^S_i, d \rangle < k > 0} \left( \langle D^S_i, d \rangle - k \right) \prod_{i: \langle D^S_i, d \rangle > 0 \leq k < \langle D^S_i, d \rangle} \left( \langle D^S_i, d \rangle - k \right)^{-1}.
\]

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