Maximum efficiency of the collisional Penrose process

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We consider the collision of two particles that move in the equatorial plane near a general stationary rotating axially symmetric extremal black hole. One of the particles is critical (with fine-tuned parameters) and moves in the outward direction. The second particle (usual, not fine-tuned) comes from infinity. We examine the efficiency $\eta$ of the collisional Penrose process. There are two relevant cases here: a particle falling into a black hole after collision (i) is heavy or (ii) has a finite mass. We show that the maximum of $\eta$ in case (ii) is less than or equal to that in case (i). It is argued that for superheavy particles, the bound applies to nonequatorial motion as well. As an example, we analyze collision in the Kerr-Newman background. When the bound is the same for processes (i) and (ii), $\eta = 3$ for this metric. For the Kerr black hole, recent results in the literature are reproduced.

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I. INTRODUCTION

Investigation of high-energy collisions of particles near black holes comes back to Refs. [1] - [3]. In recent years, interest in this issue was revived after the observation made by Bañados, Silk and West (the BSW effect, after the names of its authors) that particle collision near the Kerr black hole can lead, under certain additional conditions, to the unbounded growth of the energy in their center of mass $E_{c.m.}$. Later on, in a large series of works,
this observation was generalized and extended to other objects and scenarios. The energy that appears in the BSW effect is relevant for an observer who is present just near the point of collision in the vicinity of the black hole horizon. Meanwhile, what is especially physically important is the Killing energy $E$ of debris after such a collision measured by an observer at infinity. Strong redshift ”eats” a significant part of $E_{c.m.}$, so it was not quite clear in advance, to what extent the energy $E$ may be high. If $E$ exceeds the initial energy of particles, we are faced with the energy extraction from a black hole. This is the so-called collisional Penrose process (the Penrose process that occurs due to particle collisions).

It turned out that energy extraction from the Kerr black hole is possible but it was found to be relatively modest \cite{3,6,7}. A more general situation not restricted to the Kerr metric was considered in Ref. \cite{8} where quite general upper bounds were derived that depend on the details of the metric. But, also, an indefinitely large $E$ turned out to be impossible.

The aforementioned results on the energy extraction were obtained for the standard BSW scenario: both colliding particles move towards a black hole, one of which is ”critical” with fine-tuned parameters (energy and angular momentum), and the second particle is ”usual” (not fine-tuned). However, numeric findings in Ref. \cite{9} showed that if the critical particle moves away from a black hole, the efficiency of the process significantly increases and attains 13.92 for the Kerr metric. In what follows, we call this the Schnittman process (scenario). Later, it was numerically \cite{10} and analytically \cite{11} found that if colliding particles move in opposite directions and both of them are usual, a formally infinite efficiency becomes possible (the so-called super-Penrose process). The problem is, however, that a usual particle with a finite energy cannot move away from a black hole although for the critical particle this is possible (\cite{5,8,12}). Therefore, one could think that an outgoing usual particle could be created in some preceding collision. However, more careful treatment showed that the kind of particles under discussion cannot appear as a result of preceding collisions with finite masses and angular momenta. For a divergent mass of an initial particle this is possible but this reduces the physical value of the process \cite{13,14}. One is led to the conclusion that starting from initial conditions in which usual outgoing particles near the black hole horizon are absent one cannot obtain them by means of additional collisions. As a result, the super-Penrose process near black holes is impossible. Accounting for more involved scenarios in which particles that intermediate between the critical and usual ones participate, only confirmed this conclusion \cite{12}. (There is another option when collision occurs near a white
black hole but we do not discuss these rather exotic objects here.)

Quite recently, the numerical estimates of the efficiency of the energy extraction from the Kerr black hole found in Ref. were derived analytically.

In the present work, we consider particle collisions in the Schnittman scenario near more general stationary rotating axially symmetric black holes. It turns out that this not only generalizes the aforementioned results but leads also to some new qualitative possibilities absent for the Kerr black hole. In particular, the maximum efficiency of the energy extraction becomes possible even without heavy produced particles that are necessary in the Kerr case. As an example, we consider collisions in the Kerr-Newman background. In doing so, colliding particles are assumed to be neutral. We do not discuss another mechanism of the energy extraction related to the electrically charged particles.

Throughout the paper, the fundamental constants $G = c = 1$.

II. BASIC FORMULAS

We consider the axially symmetric metric of the form

$$ds^2 = -N^2dt^2 + g_{\phi}(d\phi - \omega dt)^2 + \frac{dr^2}{A} + g_{\theta}d\theta^2,$$

where the metric coefficients do not depend on $t$ and $\phi$. Correspondingly, the energy $E = -mu_0$ and angular momentum $L = mu_\phi$ are conserved. Here, $m$ is a particle's mass, $u^\mu = \frac{dx^\mu}{d\tau}$ being the four-velocity, $\tau$ the proper time along a trajectory. We assume that the metric is symmetric with respect to the equatorial plane $\theta = \frac{\pi}{2}$ and (unless otherwise stated explicitly) confine ourselves by motion within this plane. Then, without the loss of generality, we can redefine the radial coordinate in such a way that $A = N^2$. We do not restrict ourselves by the Kerr or another concrete form of the metric, so the results are quite general. The equations of motion for a geodesic particle read

$$m\dot{t} = \frac{X}{N^2},$$

$$m\dot{\phi} = \frac{L}{g_{\phi}} + \frac{\omega X}{N^2},$$

$$m\dot{r} = \sigma Z,$$

where

$$X = E - \omega L.$$
\[ Z = \sqrt{X^2 - N^2(m^2 + \frac{L^2}{g\phi})}, \]  
(6)
dot denotes differentiation with respect to \( \tau \), the factor \( \sigma = +1 \) or \(-1\) depending on the direction of motion.

As usual, we assume the forward-in-time condition \( \dot{t} > 0 \), whence
\[ X \geq 0. \]  
(7)
The equality can be achieved on the horizon only where \( N = 0 \).

A. Classification of particles and their properties near the horizon

Particles with \( X_H > 0 \) separated from zero are called usual, particles with \( X_H = 0 \) are called critical. If \( X_H \neq 0 \) but is small (of the order \( N \)), we call a particle near-critical. Subscript "H" means that the corresponding quantity is calculated on the horizon. In what follows, we need the approximate expression for relevant quantities \( X \) and \( Z \) near the horizon, where \( N \ll 1 \). For the near-critical particle,
\[ L = \frac{E}{\omega_H} (1 + \delta), \]  
(8)
where \( \delta \ll 1 \). Near the horizon, we assume the Taylor expansion
\[ \delta = C_1 N + C_2 N^2 + O(N^3), \]  
(9)
\[ \omega = \omega_H - B_1 N + B_2 N^2 + O(N^3). \]  
(10)
In what follows, we use the notations
\[ b = B_1 \sqrt{g_H}, h = \omega_H \sqrt{g_H}, b_2 = B_2 \sqrt{g_H}. \]  
(11)
One can find from (6), (9) and (8) - (10) the expansions for \( X \) and \( Z \). For the critical particles it reads
\[ X = NE\left(\frac{b}{h} - C_1\right) + (C_1 \frac{b}{h} - \frac{b_2}{h} - C_2)EN^2 + O(N^3), \]  
(12)
\[ Z = Ns + \tau N^2 + O(N^3), s = s(E, C_1) = \sqrt{E^2[(\frac{b}{h} - C_1)^2 - \frac{1}{h^2}]} - m^2, \]  
(13)
\[ \tau = \frac{E^2(\rho + \frac{1}{2h^2}g_{tt})}{s}, \quad (14) \]

\[ \rho = -C_1 \frac{b}{h} + C_1 C_2 + C_1 \left( \frac{b^2 - 1}{h^2} + \frac{b_2}{h} \right) - C_2 \frac{b}{h} - \frac{bb_2}{h^2}. \quad (15) \]

For the critical one we can put \( C_1 = 0 \) and obtain

\[ X = NE \frac{b}{h} + O(N^2), \quad (16) \]

\[ s(E, 0) = \sqrt{E^2 \left( \frac{b^2}{h^2} - \frac{1}{h^2} \right) - m^2}, \quad (17) \]

\[ Z = N \sqrt{E^2 \left( \frac{b}{h} \right)^2 - \frac{1}{h^2}} - m^2 + N^2 E^2 \frac{1}{\sqrt{E^2 \left( \frac{b^2}{h^2} - \frac{1}{h^2} \right) - m^2}} + O(N^3). \quad (18) \]

For a usual particle,

\[ X = X_H + B_1 LN - B_2 LN^2 + O(N^3), \quad (19) \]

\[ Z = X - \frac{N^2}{2X} \left( m^2 + \frac{L^2}{g_H} \right) + O(N^3). \quad (20) \]

### III. GENERAL CONDITIONS FOR ESCAPING TO INFINITY

It is convenient to rewrite (6) as

\[ Z^2 = \frac{g_{00}L^2}{g_{\phi}} - 2\omega LE + E^2 - N^2m^2 = \frac{g_{00}}{g_{\phi}} (L - L_+)(L - L_-), \quad (21) \]

\[ L_\pm = \frac{E g_{\phi} \omega \pm N \sqrt{Y}}{g_{00}}, \quad (22) \]

\[ Y = (E^2 + m^2 g_{00}) g_{\phi}. \quad (23) \]

Near the horizon, \( g_{00} = -N^2 + g_{\phi} \omega^2 \approx (g_{\phi} \omega^2)_H > 0 \).

Then,

\[ L_\pm = L_H + N(L_H \frac{b}{h} \pm \frac{\sqrt{E^2 \frac{b^2}{h^2} + m^2}}{\omega_H}) + O(N^2). \quad (24) \]

On the horizon,

\[ L_\pm = \frac{E}{\omega_H} = L_H. \quad (25) \]

We are interested in the conditions when a particle can escape to infinity. In general, they are model-dependent and depend on the behavior of the metric not only near the horizon
but in the intermediate region between the horizon and infinity as well. However, there are some general necessary conditions which should be obeyed just near the horizon. In what follows, we make one more assumption that they are also sufficient (say, there are no additional maxima of the potential barrier). Here, there are two options.

\[ E \geq m, L < L_H, \sigma = +1, \delta < 0, \quad (26) \]

\[ E \geq m, L_H < L < L_-, \sigma = -1 \text{ or } \sigma = +1. \quad (27) \]

In the second case a particle bounces from the turning point before reaching the horizon. This requires \( \delta > 0 \).

In case (26), \( C_1 < 0 \). In case (27), it follows from (24) that

\[ 0 < C_1 \leq C_m, \quad (28) \]

\[ C_m = \frac{b}{\hbar} - \sqrt{\frac{E^2}{\hbar^2} + m^2} = \frac{b}{\hbar} - \sqrt{\frac{1}{\hbar^2} + \frac{m^2}{E^2}}. \quad (29) \]

**IV. COLLISION NEAR HORIZON**

Let two particles 1 and 2 collide to produce new particles 3 and 4. The conservation laws of the energy and angular momentum read

\[ E_1 + E_2 = E_3 + E_4, \quad (30) \]

\[ L_1 + L_2 = L_3 + L_4. \quad (31) \]

It follows from (30) and (31) that

\[ X_1 + X_2 = X_3 + X_4. \quad (32) \]

The conservation of the radial momentum gives us

\[ \sigma_1 Z_1 + \sigma_2 Z_2 = \sigma_3 Z_3 + \sigma_4 Z_4. \quad (33) \]

We concentrate on the following scenario [9]. A usual particle 2 comes from infinity, collides with an outgoing particle 1 near the horizon and produces particles 3 and 4. Particle
4 is usual, it falls down into a black hole. Particle 3 that escapes to infinity should be near-critical. This follows from the analysis carried out in [5] for collisions near the Kerr black hole and in [8] for a much more general case. (It is also worth noting that an individual particle that moves near the horizon in the outer region along an outgoing geodesics extendable indefinitely in the past, should be critical - see, e.g., discussion in Sec. IV A of [12].) It either goes to infinity immediately after collision or moves inward, bounces from the potential barrier first and only afterwards escapes to infinity. Particle 1 is produced from the previous collision and for this reason should be near-critical as well as particle 3. For simplicity, we assume that particle 1 is exactly critical.

Now, Eq. (33) reads

\[ Z_4 - Z_2 = \sigma_3 Z_3 - Z_1. \]  

(34)

Using power expansions for each type of particles and neglecting terms of the order \( N^2 \) we have

\[ \sigma_3 s(E_3, C_1) = A_1 + E_3(C_1 - \frac{b}{\hbar}) \equiv F, \]  

(35)

where \( s(E, C) \) is taken from Eq. (13),

\[ A_1 \equiv \frac{b}{\hbar} E_1 + s(E_1, 0). \]  

(36)

It is easy to find from (35) that

\[ C_1 = \frac{b}{\hbar} - \frac{A_1^2 + m_3^2 + \frac{E_3^2}{\hbar^2}}{2A_1E_3}. \]  

(37)

By substitution back into (35) we obtain

\[ F = \frac{A_1^2 - m_3^2 - \frac{E_3^2}{\hbar^2}}{2A_1E_3}. \]  

(38)

V. ENERGY EXTRACTION AND SIGN OF \( \sigma_3 \)

The energy extraction can be measured by the quantity

\[ \eta = \frac{E_3}{E_1 + E_2}. \]  

(39)
We are interested in obtaining the maximum possible value of $\eta$. For given $E_1$ and $E_2$, this requires the maximum value of $E_3$. This value obeys bounds that follow from (35) - (38).

The form of the bound depends on $\sigma_3$.

If $\sigma_3 = +1$, it follows from (35) and (38) that
\[ E_3 \leq \lambda_0 \equiv h\sqrt{A_1^2 - m_3^2} < hA_1. \]  
(40)

If $\sigma_3 = -1$, $C_1 \geq 0$ according to (28), and we have from (37) that
\[ E_3^2 - 2E_3bhA_1 + h^2(A_1^2 + m_3^2) \equiv (E_3 - \lambda_+)(E_3 - \lambda_-) \leq 0, \]  
(41)

where
\[ \lambda_+ = h(bA_1 \pm \sqrt{A_1^2(b^2 - 1) - m_3^2}), \]  
(42)
\[ C_1 = -\frac{(E_3 - \lambda_+)(E_3 - \lambda_-)}{2A_1E_3h^2}. \]  
(43)

Therefore,
\[ \lambda_- \leq E_3 \leq \lambda_+. \]  
(44)

But $\lambda_+ > hbA_1 > hA_1$ since the nonnegativity of the square root in (42) requires $b \geq 1$. Thus $\lambda_0 < \lambda_+$ and the scenario in which the maximum value of $E_3$ is equal to $\lambda_+$ is more effective than that with $E_3 = \lambda_0$. Therefore, in what follows we concentrate on the case $\sigma_3 = -1$.

It is implied that after bounce from the potential barrier particle 3 escapes to infinity, so $E_3 \geq m_3$, $\lambda_+ \geq m_3$. It is clear from (42) that
\[ m_3 \leq A_1\sqrt{b^2 - 1}. \]  
(45)

Thus
\[ m_3 \leq E_3 \leq \lambda_+. \]  
(46)

When $E_3 = \lambda_+$, inequality $E_3 \leq \lambda_+$ turns into equality that requires $C_1 = 0$ according to (43).

To make $\lambda_+$ as large as possible, we choose $m_3 = 0$. Then,
\[ (\lambda_+)_{\text{max}} = hA_1(b + \sqrt{b^2 - 1}). \]  
(47)

If also $m_1 = 0$,
\[ A_1 = E_1\frac{(b + \sqrt{b^2 - 1})}{h}, \]  
(48)
\[ (\lambda_+)_{\text{max}} = E_1(b + \sqrt{b^2 - 1})^2. \]  
(49)
VI. PRODUCTION OF HEAVY PARTICLES

The subsequent properties of scenarios depend crucially on the value of \( m_4 \). This was noticed for the Kerr metric in [17] and is generalized below. It turns out that if \( m_4 \) is not finite arbitrary quantity but is adjusted to the location of collision in a special way, one can derive some universal bounds for the efficiency of the energy extracted. Namely, we suppose in this Section that \( m_4^2 \) has the order \( N^{-1} \), so that

\[
m_4^2 = \frac{\mu}{N} + \mu_0, \tag{50}
\]

where \( \mu_{0,1} = O(1) \). Then, it follows from (6) that

\[
Z_4 \approx \sqrt{X_4^2 - \mu N}, \tag{51}
\]

where we neglected the terms \( N^2 \) inside the square root.

In the above expansion in powers of \( N \) that gave rise to (35) it was tacitly implied that \( m_4 \) was finite. For (50), it should be somewhat modified. Eq. (35) is still valid but with another expression for \( A_1 \):

\[
A_1 = \frac{b}{h} E_1 + s(E_1, 0) - \frac{\mu}{2X_2} = E_1 \left( \frac{b + \sqrt{b^2 - 1}}{h} \right) - \frac{\mu}{2X_2}. \tag{52}
\]

Correspondingly, the expression (47) also changes. Now,

\[
(\lambda_+)_{\text{max}} = E_1 (b + \sqrt{b^2 - 1})^2 - \frac{\mu h}{2X_2} (b + \sqrt{b^2 - 1}) < \lambda_{\text{max}}(\mu = 0) = E_1 (b + \sqrt{b^2 - 1})^2. \tag{53}
\]

Now, we have

\[
\eta \leq \frac{\lambda_{\text{max}}}{E_1 + E_2} \leq \frac{[b + \sqrt{(b^2 - 1)}]^2 E_1}{E_1 + E_2} \leq \eta_0, \tag{54}
\]

where

\[
\eta_0 \equiv [b + \sqrt{(b^2 - 1)}]^2 \tag{55}
\]

is the maximum possible value of \( \eta_0 \). Comparison of the terms \( N^2 \) in the momentum conservation allows us, in principle, to find \( \mu_0 \) but we omit this unimportant part.

For the Kerr metric,

\[
b = 2, \ h = 1 = b_2, \tag{56}
\]

and we obtain \( \eta_0 = (2 + \sqrt{3})^2 \approx 13.92 \) that agrees with previous results [9], [16], [17].
VII. GENERALIZATION TO NONEQUATORIAL MOTION

It is interesting that the bound on efficiency found for equatorial motion admits generalization to the nonequatorial one, provided \( m_4 \) has the form (50), so particle 4 is heavy. Now, instead of (6), we have

\[
Z = \sqrt{X^2 - N^2 (m^2 + \frac{L^2}{g\phi} + g\theta^2)}. \tag{57}
\]

Assuming that \( \theta^2 \) are finite for all particles, we see that only terms of the order \( N^2 \) in the near-horizon expansion can change. Meanwhile, the above result was obtained on the basis of the conservation of the radial momentum \( (33) \) and its expansion with respect to \( N \) in which terms of the zero and first order only were taken into account. Therefore, the bound (55) remains valid.

VIII. FINITE \( m_4 \)

A. Restriction on \( E_2 \) from terms of order \( N^2 \)

If the mass \( m_4 \) is finite, the situation becomes much more complex since one should take into account terms \( N^2 \) in \( (33) \). This is because just these terms can give some lower bound on \( E_2 \) (see below). For the same reason, generalization to the nonequatorial motion is not straightforward since it depends on terms \( N^2 \) that are themselves model-dependent. Let us denoted by \( Y_L \) and \( Y_R \) the coefficients at terms \( N^2 \) in the left and right hand sides of (34) respectively. Then, direct calculation shows that

\[
Y_L = E_3 (C_2 - C_1 \frac{b}{h}) + \frac{b_2}{h} (E_3 - E_1) + \frac{1}{2 (X_2)_H} [m_2^2 - m_4^2] + \frac{\alpha}{h^2}, \tag{58}
\]

where

\[
\alpha = E_1 - E_3 + \frac{E_2^2 - (E_1 - E_3 + E_2)^2}{2 (X_2)_H}. \tag{59}
\]

Calculating \( Y_R \) with arbitrary \( m_3 \) and \( \sigma_3 \), we obtain

\[
Y_R = (\frac{g_1}{2g_H h^2} - \frac{bb_2}{h^2}) (\frac{E_1^2}{\sqrt{E_1^2 (\theta^2 - 1) - m_2^2}} + \sigma_3 \frac{\lambda_+^2}{\sqrt{\lambda_+^2 (\theta^2 - 1) - m_3^2}} - \sigma_3 \frac{\lambda_+^2 C_2 \frac{b}{h^2}}{\sqrt{\lambda_+^2 (\theta^2 - 1) - m_3^2}}). \tag{60}
\]
We are interested in the case of the potentially maximum efficiency of extraction, so we put $E_3 = \lambda_+$. It is realized when $C_1 = 0$. Then, equation $Y_L = Y_R$ gives us

$$m_4^2 + 2 (X_2)_H S = m_2^2 + \frac{[E_2^2 - (E_1 - \lambda_+ + E_2)]}{\hbar^2} + 2 (X_2)_H \left(\frac{b_2 h - 1}{\hbar^2}\right)(\lambda_+ - E_1),$$  \hspace{1cm} \text{(61)}

$$S \equiv Y_R - E_3 C_2.$$  \hspace{1cm} \text{(62)}

We find

$$E_2 = \frac{1}{2}(\lambda_+ - E_1) - \frac{m_3^2 h^2}{2(\lambda_+ - E_1)} + Q,$$  \hspace{1cm} \text{(63)}

where

$$Q = \frac{h^2 (m_4^2 + 2 (X_2)_H S)}{2(\lambda_+ - E_1)} + (X_2)_H (1 - b_2 h).$$  \hspace{1cm} \text{(64)}

\textbf{B. Properties of } S

It is convenient to represent the quantity $S$ in the form

$$S = S_1 + S_2,$$  \hspace{1cm} \text{(65)}

where

$$S_1 = \left(\frac{bb_2}{\hbar^2} - \frac{g_1}{2g_H h^2}\right)\left(-\sigma_3\frac{\lambda_+^2}{\sqrt{\lambda_+^2 \frac{(b^2 - 1)}{h^2} - m_3^2}} + \frac{E_1^2}{\sqrt{E_1^2 \frac{(b^2 - 1)}{h^2} - m_2^2}}\right),$$  \hspace{1cm} \text{(66)}

$$S_2 = -C_2 \lambda_+ \left[\frac{\sqrt{\lambda_+^2 \frac{(b^2 - 1)}{h^2} - m_3^2}}{\sqrt{\lambda_+^2 \frac{(b^2 - 1)}{h^2} - m_3^2}} + 1\right]$$  \hspace{1cm} \text{(67)}

and we put $E_3 = \lambda_+$.

Let us remind that the case of interest (maximum efficiency of collision) is realized for $\sigma_3 = -1$. Then, if

$$\frac{bb_2}{\hbar^2} - \frac{g_1}{2g_H h^2} > 0,$$  \hspace{1cm} \text{(68)}

both $S_1 > 0$, $S_2 > 0$, so

$$S > 0.$$  \hspace{1cm} \text{(69)}

We also remind that $(X_2)_H > 0$ since particle 2 is usual and the forward-in-time condition (7) should be satisfied. For the Kerr metric, (68) is satisfied, $b_2 h - 1 = 0$ and $Q > 0$. However, in general, (68) can be violated, $1 - b_2 h$ can have any sign and one cannot exclude any sign of $Q$ in advance, so both situations should be considered separately.
C. \( Q \geq 0 \)

It follows from (63) that

\[ E_2 \geq \kappa, \]  

(70)

where

\[ \kappa = \left( \frac{y}{2} - \frac{m_2 h^2}{2y} \right), \quad y = \lambda + E_1. \]  

(71)

As particle 2 comes from infinity, \( E_2 \geq m_2 \). It makes sense to consider two subcases separately in the manner close to that in [17].

At first, let

\[ m_2 \leq \kappa. \]  

(72)

From (71) we have

\[ h^2 m_2^2 + 2m_2 y - y^2 \leq 0, \]  

whence

\[ m_2 \leq m_+ = \frac{y}{h^2} (\sqrt{1 + h^2} - 1). \]  

(74)

The efficiency of extraction

\[ \eta = \frac{\lambda_+}{E_1 + E_2} \leq \frac{\lambda_+}{E_1 + \kappa} = \frac{2\lambda_+ y}{2E_1 y + y^2 - m_2 h^2} = f(m_2). \]  

(75)

The function \( f \) increases monotonically from \( m_2 = 0 \) to \( m_2 = m_+ \). When \( m_2 = 0 \),

\[ f(0) = \frac{2\lambda_+}{E_1 + \lambda_+} = \frac{2}{1 + (b + \sqrt{b^2 - 1})^2}, \]  

(76)

where (49) was used. For the Kerr metric, \( f(0) = \frac{2(2+\sqrt{3})^2}{1+(2+\sqrt{3})^2} \approx 1.87 \). For \( b \gg 1 \), \( f(0) \approx 2 \).

When \( m_2 \) takes a maximum possible value \( m_+ \), by substitution of (74) into (75) we find

\[ f(m_+) = \frac{\lambda_+ (\sqrt{1 + h^2} + 1)}{E_1 \sqrt{1 + h^2} + \lambda_+} = g(\lambda_+). \]  

(77)

The function \( g(\lambda_+) \) achieves the maximum value at \( \lambda_+ = (\lambda_+)_{\text{max}} = E_1[(b + \sqrt{b^2 - 1})^2] \) according to (49). Then,

\[ g[(\lambda_+)_{\text{max}}] = \eta_1 = \frac{(b + \sqrt{b^2 - 1})^2(\sqrt{1 + h^2} + 1)}{\sqrt{1 + h^2} + (b + \sqrt{b^2 - 1})^2} < (b + \sqrt{b^2 - 1})^2 \eta_0, \]  

(78)
where we took into account that $b \geq 1$, $(b + \sqrt{b^2 - 1})^2 \geq 1$.

We see that under the condition (72), the efficiency of extraction for finite masses $m_4$ is always less than that for (50).

Now, let

$$m_2 > \kappa.$$  \hspace{1cm} (79)

Then,

$$\kappa < m_2 \leq E_2.$$  \hspace{1cm} (80)

Therefore, we can put $E_2 = m_2$ in (39). For $E_3 = \lambda_+$, we have

$$\eta \leq \frac{\lambda_+}{m_2 + E_1}.$$  \hspace{1cm} (81)

As this quantity is monotonically decreasing when $m_2$ grows, the maximum is still achieved if $m_2 = \kappa$, so it coincides with (78).

For the Kerr case (56), we return to the results of [17],

$$g(\lambda_+) = \frac{\lambda_+}{(2 - \sqrt{2}) E_1 + \lambda_+ (\sqrt{2} - 1)}. \hspace{1cm} (82)$$

When $\lambda_+$ takes the maximum posible value $\lambda_+ = E_1 (2 + \sqrt{3})^2$, we obtain

$$\eta_1 = \frac{(2 + \sqrt{3})^2}{2 - \sqrt{2} + (2 + \sqrt{3})^2 (\sqrt{2} - 1)} \approx 2.19.$$  \hspace{1cm} (83)

For dirty black holes with $b \gg 1$, $\eta_0 \approx 2b$ is also large.

\section{D. $Q < 0$}

This case has no analog for the Kerr metric. Now,

$$m_2 \leq E_2 < \kappa.$$  \hspace{1cm} (84)

This requires the validity of (74). Now, we can put $m_2 = 0$ safely in the expression for $\eta$ and obtain $\eta = \eta_0$ according to (55).

It is interesting that now we have not only the upper bound but also the lower one:

$$f(m_2) < \eta < \frac{\lambda_+}{E_1 + m_2} \leq \eta_0,$$  \hspace{1cm} (85)
where \( \eta_0 \) is given by Eq. (55). As \( f(m_2) \geq f(0) \),
\[
\eta > 2 \frac{(b + \sqrt{b^2 - 1})^2}{1 + (b + \sqrt{b^2 - 1})^2} = \frac{2\eta_0}{1 + (b + \sqrt{b^2 - 1})^2} \equiv \eta_2. \tag{86}
\]

Thus
\[
\eta_2 < \eta \leq \eta_0. \tag{87}
\]

It is seen that always \( \eta_2 > 1 \), so extraction does occur.

The maximum value \( \eta_0 \) coincides with \( \eta_0 \), so this maximum is the same for heavy particles and those with finite masses.

**E. Massless particles**

The case \( Q < 0 \) implies some relationship between \( C \) and other parameters. It arises when \( (68) \) is violated or \( b_2 h > 1 \) (or both). To avoid cumbersome expressions, let us consider the situation when particles 2 and 3 are massless or have negligible masses. If one put \( m_2 = m_3 = 0 \), it is seen from \( (66), (67) \) with \( \sigma_3 = -1 \) and \( \lambda_+ \) given by \( (49) \) that
\[
S_1 = (bb_2 - \frac{g_1}{2g_H}) \frac{2E_1(b + \sqrt{b^2 - 1})}{\sqrt{b^2 - 1} h}, \tag{88}
\]
\[
S_2 = C_2E_1 \frac{(b + \sqrt{b^2 - 1})}{\sqrt{b^2 - 1}}, \tag{89}
\]
where \( C_2 > 0 \). Then, according to \( (64) \), \( Q < 0 \) entails
\[
0 < C_2 < C_2^{(0)}, \tag{90}
\]
\[
C_2^{(0)} = -\frac{m_2^2 \sqrt{b^2 - 1}}{2 (X_2)_H E_1(b + \sqrt{b^2 - 1})} + D < D, \tag{91}
\]
\[
D = 2 \frac{T}{\hbar^2}, \quad T = (b^2 - 1)(b_2 h - 1) + bh(b_1 \frac{g_1}{2g_H} - bb_2). \tag{92}
\]

If \( D < 0 \), the coefficient \( C_2^{(0)} < 0 \), condition \( (90) \) cannot be obeyed, so \( Q \geq 0 \).

**IX. KERR-NEWMAN BLACK HOLE**

Here, we consider an important example of the extremal Kerr-Newman black hole. Although in astrophysical application the electric charge is quite small, investigation of the
properties of such a black hole has obvious theoretical interest. For the corresponding metric, one has the following values of the horizon coefficients relevant in our context (θ = \( \frac{\pi}{2} \)):

\[
\omega_H = \frac{a}{M^2 + a^2}, \quad b = \frac{2a}{M}, \quad b_2 = \frac{a^3}{M^3},
\]

(93)

\[
h = \frac{a}{M},
\]

(94)

\[
\frac{g_1}{2g_H} - bb_2 = 1 - \frac{a^2}{M^2} - 2\frac{a^4}{M^4},
\]

(95)

\[
b_2h - 1 = \frac{a^4}{M^4} - 1.
\]

(96)

These quantities can be obtained by straightforward calculations from the known metric coefficients. Now, \( M^2 = q^2 + a^2 \), where, \( q \) is the electric charge, \( a = \frac{J}{M} \). \( J \) is the angular momentum.

For simplicity, we assume that \( \frac{m_4^2}{E_{1(X)}^2} \ll 1 \), so in (91) \( C_2^{(0)} \approx D \).

It is convenient to introduce a variable \( y = \frac{a^2}{M^2} \). Then, after some algebra one finds

\[
T = -3y^2 - 2y + 1 = (y + 1)(1 - 3y).
\]

(97)

Here, the condition \( b^2 \geq 1 \) in (55) requires \( y \geq \frac{1}{3}, \frac{a}{M} \geq \frac{1}{2} \) where (93) is taken into account. Also, the existence of a black hole horizon entails \( y \leq 1 \). Thus

\[
\frac{1}{4} \leq y \leq 1.
\]

(98)

The function \( T(y) \) is monotonically decreasing and has one root \( y = y_0 = \frac{1}{3} \) that lies within the interval (98).

Thus for \( \frac{1}{\sqrt{3}} < \frac{a}{M} \leq 1 \) the coefficients \( T < 0, D < 0 \). As a result, \( C_2^{(0)} < 0 \) and condition (90) cannot be satisfied, so negative values of \( Q \) are forbidden. As \( Q \geq 0 \), the energy extraction \( \eta = \eta_1 \) is given by (78) that now reads

\[
\eta_1 = \eta_0 \left( \sqrt{1 + y} + 1 \right), \quad \eta_0 = \left( 2x + \sqrt{4x^2 - 1} \right)^2, \quad x = \frac{a}{M}.
\]

(99)

If \( \frac{1}{2} \leq \frac{a}{M} \leq \frac{1}{\sqrt{3}} \), \( C_2^{(0)} \geq 0 \). Then, if (90) is satisfied, we have \( Q < 0 \) and the energy extraction is the same for production of massive particles and the ones with modest mass \( m_4 \), so \( \eta = \eta_0 \) according to (55). If \( C_2 \geq C_2^{(0)} \), the negative value of \( Q \) is forbidden again, so extraction is given by (99).
For slightly positive $C_2$, when $y$ increases and passes through the value $y_0$, the maximum efficiency extraction changes abruptly from $\eta_0(y_0) = 3$ to $\eta_1(y_0) = \frac{3(2+\sqrt{3})}{2+3\sqrt{3}} \approx 1.56$.

It is also instructive to compare $\eta_1(b, h)$ for the Kerr-Newman and Kerr metrics. It follows from (93) and (94) that $b \leq 2$, $h \leq 1$, the equality being achieved for the Kerr black hole. We have from (78) that $\eta_1(b, h) \leq \eta_1(2, h)$ since $\eta_1$ is monotonically increasing function of $b$ at fixed $h$. Also, $\eta_1(2, h) \leq \eta_1(2, 1)$ since $\eta_1(2, h)$ is monotonically increasing function of $h$. Thus $\eta_1(\text{Kerr-Newman}) \leq \eta_1(\text{Kerr})$.

Remembering also the expression (55) for $\eta_0$, we see that for the Kerr-Newman black hole the extraction is less effective than for the Kerr one in both cases (for heavy particles and for particles with finite $m_4$).

X. SUMMARY AND CONCLUSIONS

We considered the Schnittman scenario. In this scenario, an ingoing a usual particle falling down from infinity collides near the with the outgoing critical one. It is this scenario in which the efficiency of the energy extraction from the Kerr black hole was found to increase as compared to the standard BSW process attaining almost 14\cite{9}. We analytically derived the upper bound on the extraction efficiency $\eta$ for such a scenario that is valid not only for the Kerr metric\cite{17}, \cite{16} but applies to any rotating stationary axially symmetric black hole. We found $\eta_0$, the absolute maximum of $\eta$. In this context, one should distinguish two situations: (i) the mass of a particle that is produced due to collision and falls into a black hole scales like $m_4 \sim N^{-1/2}$ in the point of collision, (ii) $m_4 = O(1)$. For collisions in the Kerr background, $\eta_0$ is realized in case (i) only\cite{17} whereas in case (ii) the allowed maximum value of $\eta = \eta_1 < \eta_0$. However, we saw that in general the situation is more involved. Depending on the relation between the parameters of the problem, either the maximum of $\eta$ equal to $\eta_0$ is achieved for heavy particles only or $\eta = \eta_0$ can be realized for any $m_4$.

The expressions for $\eta_0$ and $\eta_1$ contain the metric coefficients and some their first derivatives on the horizon that are combined in two parameters $b$ and $h$. The results apply to "dirty" (surrounded by matter) black holes, the Kerr-Newman one, etc. In particular, we found the intervals of the Kerr-Newman parameter in which both maxima (for heavy particles and the ones with finite $m_4$) can coincide ($\frac{b}{M} \leq \frac{1}{\sqrt{3}}$) and those where they cannot
\( \frac{1}{\sqrt{3}} < \frac{a}{M} \leq 1 \). It turned out that the energy extraction for the Kerr-Newman black hole with \( a < 1 \) is less effective than for the Kerr one \( (a = 1) \).

In the case of heavy particles, the bound obtained is valid also for nonequatorial motion.

Our approach is quite generic in that it is model-independent and can be used for further investigation of the collisional Penrose process near a wide class of black holes. It would be of interest to extend it to generic nonequatorial motion and compare to the approach developed in [16].

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