Spinor Analysis
B.L. van der Waerden in Groningen (Holland)

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Contents

1 The representations of the Lorentz group. 2
2 Classification of world vectors and tensors. 4
3 Invariant theory of the spinors. 5
4 The Dirac wave equation 6
5 The possible Lorentz invariant wave equations. 8
A Contragredient, cogredient transformations 8
B Remarks 9

"Let us call the novel quantities which, in addition to the vectors and tensors, have appeared in the quantum mechanics of the spinning electron, and which in the case of the Lorentz group are quite differently transformed from tensors, as spinors for short. Is there no spinor analysis that every physicist can learn, such as tensor analysis, and with the aid of which all the possible spinors can be formed, and secondly, all the invariant equations in which spinors occur?" So Mr Ehrenfest asked me and the answer will be given below.

It turns out that the question can be solved without any new tools. It is necessary, in order to find all the possible spinors, to set up all the representations of the Lorentz group, and these are known. In order to arrive at an invariant theory for the spinors, only the known two-step isomorphism between the Lorentz group and the binary unimodular group has to be converted into a "transfer principle" for the

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2See, for example, H. Weyl, Group Theory and Quantum Mechanics, Leipzig, 1928, and the literature cited therein.
covariants of both groups\textsuperscript{4}, and a known theorem of invariant theory In order to see that all possible invariant equations can be written as binary tensor equations.

On the physical side, a complete overview of all possible invariant wave equations is obtained. It shows that a Lorentz invariant linear wave equation for the spinning electron with only two wave functions and of the first order necessarily leads to the Dirac equation without the mass element, that is, is not consistent with the fact of the mass. It is therefore necessary to use either differential equations of the second order or at least four wave functions. This can be set up so: there are still a lot of possibilities.

\section{The representations of the Lorentz group.}

The task of finding all the "quantities" which are linearly transformed with Lorentz transformations according to some rule, so that the corresponding transformations of the "quantities" are also composed when the second Lorentz transformations are composed, i.e., the product of two Lorentz transformations is nothing but the problem of the representation of the Lorentz group by linear transformations.

It is known that the Lorentz group has a ambiguous representation as a binary group (that is, as a group in two complex variables); we will write this down first.

We start from a transformation group in 2 complex variables, which we will subsequently relate to the Lorentz group.

The binary transformations of determinant 1\textsuperscript{4}:

\begin{equation}
\begin{aligned}
\xi'_1 &= \alpha_{11} \xi_1 + \alpha_{12} \xi_2 \\
\xi'_2 &= \alpha_{21} \xi_1 + \alpha_{22} \xi_2
\end{aligned}
\end{equation}

\begin{equation}
\bar{\xi}'_1 = \bar{\alpha}_{11} \bar{\xi}_1 + \bar{\alpha}_{12} \bar{\xi}_2 \\
\bar{\xi}'_2 = \bar{\alpha}_{21} \bar{\xi}_1 + \bar{\alpha}_{22} \bar{\xi}_2
\end{equation}

leave, when $\eta_1, \eta_2$ is transformed cogrediently to the $\xi_i$, the expression $\xi_1 \eta_2 - \xi_2 \eta_1$ invariant, and therefore transform $\eta_2, -\eta_1$ contragrediently to $\xi_1, \xi_2$. We can therefore put

\begin{equation}
\begin{aligned}
\eta^1 &= \eta_2 \\
\eta^2 &= -\eta_1
\end{aligned}
\end{equation}

and also for all binary vectors and tensors, e.g.\textsuperscript{5}

\begin{equation}
\begin{aligned}
\xi^1 &= \xi_2 \\
\xi^2 &= -\xi_1
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
a^{11} &= a^{22} = a_{22} \\
a^{21} &= a^{12} = -a_{12}
\end{aligned}
\end{equation}

We agree to write a point (dotted indices) above the index in all variables which are transformed according to the conjugate complex transformation ($\bar{\mathbf{1}}$).\textsuperscript{6} The corresponding agreement applies to tensors; For example, $a^{\lambda\mu\nu}$ is a tensor that transforms like $\tilde{\xi}_\lambda \bar{\eta}_\mu \bar{\xi}_\nu$. For the dotted indices the above terms: $\xi^1 = \xi_2; \xi^2 = -\xi_1$ are also valid.

\begin{align*}
&\xi_1 \xi_1, \; \xi_2 \xi_2, \; \bar{\xi}_1 \bar{\xi}_2 + \bar{\xi}_2 \bar{\xi}_1, \; \frac{\bar{\xi}_1 \xi_2 - \bar{\xi}_2 \xi_1}{2i}
\end{align*}

\textsuperscript{3}See E. A. Weiss, A Spatial Analogue to the Hessian Transfer Principle, Diss. Bonn, 1924.
\textsuperscript{4}TN: originally the index of the coefficients $\alpha$ in the transformations were all in lower position.
\textsuperscript{5}TN: originally the last expression reads: $\cdots = a_{12}$.
\textsuperscript{6}TN: originally (4).
are real, and are transformed by the transformations \((1), (\bar{1})\) into real ones again; So the transformation coefficients for these expressions are real. However, these same transformation coefficients also valid for a tensor \(a_{\alpha\beta}\) and the expressions

\[
a_{11}, \quad a_{22}, \quad \frac{a_{12} + a_{21}}{2}, \quad \frac{a_{12} - a_{21}}{2i}.
\]

Hence if

\[
\begin{align*}
\frac{a_{21} + a_{12}}{2} &= x \\
\frac{a_{21} - a_{12}}{2i} &= y \\
\frac{a_{11} - a_{22}}{2} &= z \\
\frac{a_{11} + a_{22}}{2c} &= t
\end{align*}
\]

are transformed, then \(x, y, z, t\) are also transformed as real. Moreover

\[
-\frac{1}{2} a^{\alpha\beta} a_{\alpha\beta} = -\frac{1}{2} (a^{12} a_{12} + a^{21} a_{21} + a^{11} a_{11} + a^{22} a_{22})
\]

\[
= a_{12} a_{21} - a_{22} a_{11} = x^2 + y^2 + z^2 - c^2 t^2
\]

is invariant, so we have a real Lorentz transformation. It is also known that any real Lorentz transformation can be obtained in such a way. If we reverse the signs of \(a_{\mu\nu}\) in \((1)\), we obtain the same transformation for the \(a_{\bar{\mu}\bar{\nu}}\), that is, the transformations \((1)\) form an double representation of the Lorentz group. Finally, every transformation \((1)\) is to be obtained steadily from the identity, so, under our world transformations, neither the spatial reflections, nor the transformations, which change the course of the time, are preserved.

In order to generate the reflections, we also add to \((1)\) the transformation with which every \(\xi_{\nu}\) is transformed into \(\bar{\xi}_{\nu}\). It takes a form \(a^{\mu\nu} \bar{\xi}_{\mu} \xi_{\nu}\) and \(a^{\mu\nu} \bar{\xi}_{\mu} \xi_{\nu}\) and therefore \(a^{\mu\nu}\) in \(a^{\mu\nu}\) and also \(a_{\bar{\mu}\bar{\nu}}\) in \(a_{\mu\nu}\) over. It follows that \(x, z, \) and \(t\) are invariant, but \(y\) is transformed into \(-y\), that is, it creates a reflection on the \(XZ\) plane.

If we leave the reflections for the time being, it is clear that all representations of the group \((1)\) are also representations of the Lorentz group, and vice versa.

However, the group \((4)\) does not allow any other irreducible representations, as is shown in the representation theory, to be obtained when binary tensors \(a_{\alpha\beta...\bar{\gamma}\bar{\delta}...}\) symmetric in \(\alpha\beta...\) and \(\bar{\gamma}\bar{\delta}...\), are used as transformation objects. Any representation is composed of such irreducible, that is, every representation is obtained by

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7TN: originally \((4), (4)\).
8Indices occurring above and below are summed up silently.
9TN: originally the last line reads: \(= a_{12} a_{21} - a_{22} a_{11} = x^2 + y^2 + z^2 - c^2 t^2\). 
10In order to see this, for example, it suffices to consider that a 6-membered set of Lorentz transformations is obtained from \((1)\) (TN: originally \((4)\)).
11TN: cf. note 5
12TN: cf. note 5
13TN: cf. note 5
14TN: originally \(\bar{\xi}_{\nu}\).
15TN: cf. note 5.
taking one or more tensors \( a_{\alpha\beta\ldots\gamma\delta\ldots} \) with or without symmetry conditions as the transformation object. Thus the "quantities" of the Lorentz group are no other than these binary tensors, transformed by the binary tensor representations of the Lorentz group. We call them 'the quantities of the Lorentz group', spin-tensors, or spinors, namely spin-tensors of the 1st, 2nd, . . . rank depending on the number of indices. Specifically, the quantities with only one index (assuming the values 1, 2) should be called spinvectors\(^{16}\).

2 Classification of world vectors and tensors.

Since the binary tensor representations provide all the representations, the ordinary quaternary world vectors and tensors must also be found under it, that is, the world vectors must be written as spinors with binary indices.

A world vector can be given as \( a^k \) (cogredient to \( x, y, z, t \)), or contragredient to it as \( a_i = g_{ik}a^k \) \((g_{00} = -c^2, g_{11} = 1 \text{ etc.})\). We introduce a new term for each world vector by setting (corresponding to (4)):

\[
\begin{align*}
-a^{12} &= a_{21} = a^1 + ia^2 = a_1 + ia_2 \\
-a^{21} &= a_{12} = a^1 - ia^2 = a_1 - ia_2 \\
-a^{22} &= a_{11} = a^3 + ca^0 = a_3 - \frac{1}{c}a_0 \\
-a^{11} &= -a_{22} = a^3 - ca^0 = a_3 + \frac{1}{c}a_0.
\end{align*}
\]

Accordingly, we designate all world tensors as binary tensors; for example, for a world tensor \( a_{ik} \) (which is to be treated initially as a product \( a_{i}b_{k} \))\(^{17}\):

\[
\begin{align*}
a_{21,21} &= a_{1,1} + ia_{2,1} + ia_{1,2} - a_{2,2} \\
-a_{21,12} &= a_{1,1} + ia_{2,1} - ia_{1,2} + a_{2,2} \\
\ldots
\end{align*}
\]

\[
a_{22,22} = a_{3,3} + \frac{1}{c}a_{3,0} + \frac{1}{c^2}a_{0,3} + \frac{1}{c^2}a_{0,0}.
\]

The differentiation symbols \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t} \) can also be renamed as they transform as \( a_k \):

\[
\begin{align*}
\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} &= \partial_{21} \\
\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} &= \partial_{12} \\
\frac{\partial}{\partial z} - \frac{1}{c} \frac{\partial}{\partial t} &= \partial_{11} \\
\frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t} &= -\partial_{22}.
\end{align*}
\]

\(^{16}\) If one is not looking for the quantities of the Lorentz group but the three-dimensional group of rotations, it is easiest to consider this group as the subgroup of the Lorentz group, which leaves the time \( t \) invariant. The quantity \( t \) transforms as \( a_{11} + a_{22} \), thus also as a Hermite form \( \xi_1\xi_1 + \xi_2\xi_2 \); the rotation group in the representation (1) is represented by the "unitary" subgroup, which leaves this form invariant. In this subgroup, obviously, \( \xi_1, \xi_2 \) are transformed contragrediently to \( \xi_1, \xi_2 \), i.e \( \xi^1, \xi^2 \). Therefore the dotted indices become superfluous: they are replaced by contragredient undotted ones. The possible "quantities" which belong to the irreducible representations are once again the symmetric tensors \( a_{\alpha\beta\ldots\delta} \) of rank \( 2j = 0, 1, 2, \ldots \).

\(^{17}\) TN: originally the last line reads: \( = a_{3,3} + \frac{1}{c}a_{3,0} + \frac{1}{c^2}a_{0,3} + \frac{1}{c^2}a_{0,0} \)
As can be seen from (5),
\[ a_k a^k = g_{kl} a^k a^l = -\frac{1}{2} a^\lambda_\mu a_\lambda^\mu, \]
and therefore also
\[ a_k b^k = a^k b^k = g_{kl} a^k b^l = -\frac{1}{2} a^\lambda_\mu b_\lambda^\mu = -\frac{1}{2} a_\lambda^\mu b_\lambda^\mu. \]

Of course, these formulas also apply if the vectors \( a, b \) are replaced by differentiation symbols; for example,
\[ \text{div } a = -\frac{1}{2} \partial_\mu a^\mu, \]
\[ \Box = \sum \frac{\partial^2}{\partial x^2} - \frac{1}{c^2 \partial t^2} = -\frac{1}{2} \partial_\lambda^\mu \partial^\lambda_\mu. \]

The following table lists the possible types of spinors of the lowest grades, indicating which species of world tensors correspond to them:

| Rank | Spinors |
|------|---------|
| 1    | \( a_\lambda, a_\dot{\lambda} \) (spin-vectors), |
| 2    | \( a_\lambda^\mu, a_\dot{\lambda}, a_\dot{\lambda}^\mu \rightarrow \text{world vector } a_k, \) |
| 3    | \( a_\lambda^\mu_\nu, a_\dot{\lambda}^\mu_\nu, a_\lambda_\mu^\nu, a_\lambda^\mu_\dot{\nu}, \) |
| 4    | \( \{ a_\lambda^\mu_\nu_\rho, a_\dot{\lambda}^\mu_\nu_\rho, a_\lambda_\mu^\nu_\dot{\rho}, a_\lambda^\mu_\dot{\nu}_\rho \rightarrow \text{world tensor } a_{kl}. \) |

It should also be noted that one can also allow the spinors of the second rank \( a_\lambda^\mu \) and \( a_\dot{\lambda}^\mu \) to correspond to world tensors, for example the "selfdual" \( F_{kl} \), by setting:
\[ a_\alpha^\alpha_{\beta\lambda\mu} = \delta_\alpha^\beta a_{\dot{\lambda}\dot{\mu}} \]
from which
\[ a_{\dot{\lambda}\dot{\mu}} = \frac{1}{2} a_\alpha^\alpha_{\dot{\lambda}\dot{\mu}} \]
follows and by forming the corresponding \( a_{\alpha\beta\dot{\lambda}\dot{\mu}} \) from \( F_{kl} \).

It should also be pointed out that one can always restrict oneself to such spin tensors, which are symmetric in all the dotted and also in all the undotted indices, since all the others are formed linearly from them with the aid of \( \epsilon \)- or \( \delta \)-symbols (see below).

3 Invariant theory of the spinors.

We consider a system of binary vectors and tensors, tentatively without dotted indices. All invariants and covariants of this system in the group (1) are obtained by a known theorem of binary invariant theory ("the first fundamental principle of

18TN: cf. note 5
the symbolic method): all the indices of the tensors are written down and form expressions like
\[ \epsilon^{\alpha\beta} \epsilon^{\gamma\delta} a_{\alpha\beta\gamma} b_{\delta\lambda\mu} \ldots, \]
where
\[ \begin{align*}
\epsilon_{12} &= 1, & \epsilon^{21} &= -1, \\
\epsilon_{11} &= 0, & \epsilon_{22} &= 0.
\end{align*} \]
The calculation rules for the symbol \( \epsilon \) are
\[ \begin{align*}
\epsilon^{\alpha\beta} & = -\epsilon^{\beta\alpha} \\
\epsilon^{\alpha\beta} \epsilon^{\gamma\delta} + \epsilon^{\beta\gamma} \epsilon^{\alpha\delta} + \epsilon^{\gamma\alpha} \epsilon^{\beta\delta} & = 0 \\
\epsilon^{\alpha\beta} u_{\alpha\beta\gamma} + \epsilon^{\alpha\beta} u_{\gamma\alpha\beta} + \epsilon^{\alpha\beta} u_{\gamma\beta\alpha} & = 0.
\end{align*} \]

Another slightly shorter type of writing for the invariants is obtained by using the rule (2) for raising the indices and eliminating the \( \epsilon \)-symbols by means of:
\[ \epsilon^{\alpha\beta} a_{\alpha\beta} = -a^\alpha_{\alpha}. \]

The calculation rules we need now are harder to remember than (6) (from which follow):
\[ \begin{align*}
u_{\alpha\alpha} & = -u_{\alpha\alpha} \\
u_{\beta\alpha} & = u_{\alpha\beta} - u_{\alpha\beta}.
\end{align*} \]

(1) and (\( \bar{1} \)) are algebraically completely independent, and the invariance of a system of equations between binary vectors is expressed by the requirement of invariance for two independent groups, one of which operates only on the dotted indices, and the other only on the undotted indices. Thus exactly the same invariant operations are permissible as before, but they must refer either to the dotted or to the undotted indices. That is, a raised, dotted index remains dotted, and is summed only by two dotted or two undotted indices.

If a system of equations is to be invariant under reflections, it must allow the replacement of all \( a^{\mu
u} \) by \( a^{\nu\mu} \), and all \( \xi_\mu \) by \( \xi_{\bar{\nu}} \) (or quantities which likewise transform).

4 The Dirac wave equation

The Dirac wave equation of the electron is multiplied by the Dirac \( \Gamma_0 \):
\[ \frac{1}{c} \left( \frac{\hbar}{i} \frac{\partial}{\partial t} + \Phi_0 \right) \psi + \sum_{1}^{3} s'_r \left( \frac{\hbar}{i} \frac{\partial}{\partial x_r} + \Phi_r \right) \psi + mc\Gamma_0 \psi = 0 \]
where the \( s'_r \) and \( \Gamma_0 \) are four-row matrices,
\[ s'_r = \begin{pmatrix} s_r & 0 \\ 0 & -s_r \end{pmatrix}, \quad \Gamma_0 = \begin{pmatrix} 0 & E' \\ E & 0 \end{pmatrix}, \]

19 The reason why the even shorter and more convenient symbolic factorization of the tensors has been avoided here consists chiefly in the difficulty of the designation of the differentiation symbols, where it is difficult to determine the magnitudes to which they are to act.

20 TN: originally \( \xi_{\bar{\mu}} \).

21 The difference from Weyl §39, p. 172 in the member with \( \Phi_0 \) is that we use \( x, y, z, t \), and not \( x, y, z, ict \) as coordinates; Our vector \( \Phi_0 \) is real and contragredient to the coordinates and our \( \Phi_0 \) is = \( ci \) times the Weyl one.
where $E$ is the two-row unit matrix and the $s_r$ are the Pauli matrices

$$s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

We denote the first two components of the Dirac wave function $\psi$ with $\psi_1, \psi_2$; the last two (transforming as the conjugate complex and contragrediently to the first) by $\chi^1, \chi^2$, and (8) accordingly split into two equations:

$$\begin{cases} \frac{1}{c} \left( \frac{\hbar}{i} \frac{\partial}{\partial t} + \Phi_0 \right) \psi + \sum_1^3 s_r \left( \frac{\hbar}{i} \frac{\partial}{\partial x_r} + \Phi_r \right) \psi + m c \chi = 0, \\ \frac{1}{c} \left( \frac{\hbar}{i} \frac{\partial}{\partial t} + \Phi_0 \right) \chi - \sum_1^3 s_r \left( \frac{\hbar}{i} \frac{\partial}{\partial x_r} + \Phi_r \right) \chi + m c \psi = 0. \end{cases}$$

If we introduce the abbreviation $a_k$ for $\frac{\hbar}{i} \frac{\partial}{\partial x_k} + \Phi_k$\textsuperscript{22}, then we have to evaluate the quantity

$$\frac{1}{c} a_0 + \sum_1^3 s_r a_r$$

in the first equation. It is the matrix

$$\begin{pmatrix} \frac{1}{c} a_0 + a_3 & a_1 - i a_2 \\ a_1 + i a_2 & \frac{1}{c} a_0 - a_3 \end{pmatrix} = \begin{pmatrix} -a_1^{11} & -a_1^{21} \\ -a_1^{12} & -a_1^{22} \end{pmatrix} = - \begin{pmatrix} a_1^{11} & a_1^{21} \\ a_1^{12} & a_1^{22} \end{pmatrix}.$$ 

Also, in the second equation, the expression

$$\frac{1}{c} a_0 - \sum_1^3 s_r a_r = \begin{pmatrix} \frac{1}{c} a_0 - a_3 & -a_1 + i a_2 \\ -a_1 - i a_2 & \frac{1}{c} a_0 + a_3 \end{pmatrix} = \begin{pmatrix} -a_2^{12} & -a_2^{21} \\ -a_2^{11} & -a_2^{11} \end{pmatrix} = - \begin{pmatrix} a_2^{11} & a_2^{12} \\ a_2^{21} & a_2^{22} \end{pmatrix}$$

is applied. Hence the Dirac equations are written as:

$$\begin{cases} -a^{\lambda \mu} \psi_\lambda + m c \chi^\mu = 0, \\ -a_{\mu \lambda} \chi^\lambda + m c \psi_\mu = 0. \end{cases}$$

or, if we lower the indices of the $\chi$ and use the meaning for the $a^{\lambda \mu}$:

$$\begin{cases} -\left( \frac{\hbar}{i} \frac{\partial}{\partial \mu} + \Phi_\mu \right) \psi_\lambda + m c \chi^\mu = 0, \\ \left( \frac{\hbar}{i} \frac{\partial}{\partial \lambda} + \Phi_\lambda \right) \chi_\mu + m c \psi_\mu = 0. \end{cases}$$

\textbf{p.108} The derivation of the second-order wave equation

$$\hbar^2 \Box \chi_\nu = -\frac{1}{2} \hbar^2 \partial^{\lambda \mu} \partial_\lambda \chi_\nu = m^2 c^2 \chi_\nu \quad \text{(in the case } \Phi_\kappa = 0)$$

(10) (and also for $\psi$) from the Dirac occurs by using the identity (7):

$$\partial^{\lambda \mu} \partial_\lambda \chi_\nu = \partial^{\lambda \mu} \partial_\lambda \chi_\mu - \partial^{\lambda \nu} \partial_\lambda \chi_\mu = -2 \partial^{\lambda \nu} \partial_\lambda \chi_\mu = \frac{2 m c^2 i^2}{\hbar^2} \chi_\nu.$$ 

\textsuperscript{22}TN: originally $\frac{\hbar}{i} \frac{\partial}{\partial x_k} + \Phi_k$.

\textsuperscript{23}TN: originally $\cdots = \begin{pmatrix} -a_1^{11} & -a_1^{12} \\ -a_2^{21} & -a_2^{22} \end{pmatrix} = - \begin{pmatrix} a_1^{11} & a_1^{12} \\ a_2^{21} & a_2^{22} \end{pmatrix}$
5 The possible Lorentz invariant wave equations.

From a wave equation for the electron we have to demand that it is linear in the occurring wave functions $\psi$ and that it expresses the derivative $\frac{\partial \psi}{\partial t}$ or $\frac{\partial^2 \psi}{\partial t^2}$ linearly on the other derivatives of the same and lower order, with coefficients which may still be appended by the field. If the wave function $\psi$ has two components $\psi_1$, $\psi_2$, then the wave equation must also consist of two components.

If one also requires that it is of the first order, then only

$$\partial_{\lambda\mu} \psi^\mu + c_{\lambda\mu} \psi^\mu = 0$$

can be considered where the $c_{\lambda\mu}$ may depend on the field. For “no field” we must have $c_{\lambda\mu} = 0$, and one obtains a pair of the Dirac equations with $m = 0$.

If we let four $\psi$-components $m$, of which the first two are transformed according to (1), the last two to $(\bar{1})$, one obtains as possibilities:

$$\begin{cases} 
\partial_{\lambda\mu} \psi^\mu + b_{\lambda\mu} \psi^\mu + c_\mu \psi_\sigma = 0 \\
\partial_{\lambda\mu} \psi^{\lambda} + e_{\lambda\mu} \psi^{\lambda} + f_\rho \psi_\rho = 0.
\end{cases}$$

For “no field”, only one multiple of the unit matrix remains for $c_\mu$ and $f_\rho$, and the Dirac equations are obtained by giving the opposite sign to these multiples. For the field dependence of $b, c, e, f$ there are, of course, more possibilities than the Dirac alone. If a differential equation of the second order is obtained, with only two components of the wave function, the number of possibilities is still much greater: the following terms in the wave equation are possible:

$$\Box \psi^\mu + b^\lambda_{\rho\mu} \partial_{\lambda\rho} \psi^\sigma + c^\mu_{\lambda} \psi^\lambda = 0.$$  

(The conceivable $\partial^\rho \partial^\mu \psi^\nu$ can be expressed by $\Box \psi^\mu$: cf. § 4, conclusion.)

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The following appendix is not present in the actual paper, but is provided here as a terminology complement.

A Contragredient, cogredient transformations

Let $A$ be the transformation matrix

$$A = \begin{pmatrix} \alpha_1^1 & \alpha_2^1 \\ \alpha_1^2 & \alpha_2^2 \end{pmatrix}$$

The vectors $X$ and $Y$ transform cogrediently if

$$X' = AX, \text{ and } Y' = AY \iff x'_\mu = \alpha_{\mu}^\nu x_\nu \text{ and } y'_\mu = \alpha_{\mu}^\nu y_\nu.$$
in which case
\[ X'^T Y' = X^T A^T A Y. \]

The vectors \( X \) and \( Y \) transform contragrediently if
\[
X = A^T X' \iff X' = (A^{-1})^T X, \quad \text{and} \quad Y' = A Y \iff x'_\mu = x'_\nu \alpha'^\nu_{\mu} \quad \text{and} \quad y'_\mu = \alpha'^\nu_{\mu}
\]
\((\alpha'^\nu_{\mu} \) is the transpose matrix of \( \alpha^\mu_{\nu} \)) in which case
\[ X'^T Y' = X A^{-1} A Y = X^T Y. \]

B Remarks

This translation has been made available, as is, for whoever is interested, even though it was made for the personal use of the author. If you find any mistake or better expressions, feel free to contact the author of the translation for an update.

The footnotes numbering in the original publication were made on a per page basis. Here, instead, it is continuous and the original footnotes are interspersed with translation comments (see below), so the original footnotes numbering is not preserved.

I’ve tried to correct the typos in the original paper. Corrections and translation remarks are preceded by ‘TN:’ in footnotes and those footnotes are displayed in blue color. All other footnotes come from the original paper.

The transformation matrix elements of equations (1) and \((\bar{1})\) are displayed with one upper index and one lower index to make notations coherent with modern implicit summation convention (one upper index and one lower equal index letter are meant to be summed on).

The appendix A has been included to make clear the meaning of these expressions.