High-Dimensional High-Frequency Regression

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Abstract

In this paper, we develop a novel high-dimensional regression inference procedure for high-frequency financial data. Unlike usual high-dimensional regression for low-frequency data, we need to additionally handle the time-varying coefficient problem. To accomplish this, we employ the Dantzig selection scheme and apply a debiasing scheme, which provides well-performing unbiased instantaneous coefficient estimators. With these schemes, we estimate the integrated coefficient, and to further account for the sparsity of the beta process, we apply thresholding schemes. We call this Thresholding dEbiased Dantzig Integrated Beta (TEDI Beta). We establish asymptotic properties of the proposed TEDI Beta estimator. In the empirical analysis, we apply the TEDI Beta procedure to analyzing high-dimensional factor models using high-frequency data.

Keywords: Dantzig selection, debiased, diffusion process, factor model, sparsity

1 Introduction

Since high-frequency financial data are widely available, researchers have developed financial models that incorporate high-frequency data to account for market dynamics. For example, researchers...
have developed several well-performing non-parametric realized volatility estimators (Aït-Sahalia et al., 2010; Barndorff-Nielsen et al., 2008; Fan and Kim, 2018; Jacod et al., 2009; Xiu, 2010; Zhang, 2006; Zhang et al., 2005). With the realized volatility estimator, several time series models have been introduced. Examples include the heterogeneous auto-regressive (HAR) models (Corsi, 2009), high-frequency based volatility (HEAVY) models (Shephard and Sheppard, 2010), realized GARCH models (Hansen et al., 2012), unified GARCH-Itô models (Kim and Wang, 2016b), realized GARCH-Itô models (Song et al., 2021), and overnight GARCH-Itô models (Kim and Wang, 2021). These models are based on the auto-regressive structure of realized volatilities, with their empirical studies showing that the auto-regressive type models help explain the volatility dynamics. On the other hand, researchers have modeled regression-based financial models, such as the CAPM (Sharpe, 1964; Lintner, 1965) and factor models (Fama and French, 1992), via diffusion processes. We call this the high-frequency regression. To evaluate coefficients of the high-frequency regression, we often employ realized volatilities. For example, Barndorff-Nielsen and Shephard (2004) estimated the market beta by calculating a ratio of the integrated covariance between assets and systematic factors to the integrated variation of the systematic factors. Related literature for this topic include works by Andersen et al. (2006); Reiß et al. (2015). Mykland and Zhang (2009) further computed the market beta as the aggregation of the market betas estimated over local blocks, while Andersen et al. (2021) investigated the intra-day variation of the spot market betas. To evaluate the coefficients of multi factor models, Aït-Sahalia et al. (2020) proposed an integrated beta approach using the instantaneous market betas. These models showed that incorporating high-frequency financial data has benefit for capturing beta dynamics. For example, intra-day data provide accurate estimations with sufficient data even in relatively short time periods.

In the field of finance, there are hundreds of potential factor candidates that explain the cross section of expected stock returns (Cochrane, 2011; Harvey et al., 2016; Hou et al., 2020; McLean and Pontiff, 2016). Thus, the factor models often run into the curse of dimensionality problem. To
tackle this problem in the high-dimensional statistics, we usually assume the sparsity of factors, that is, the number of significant factors is small. To accommodate the sparsity condition, we often employ the LASSO procedure (Tibshirani, 1996), SCAD (Fan and Li, 2001), and Dantzig selector (Candes et al., 2007). The works of Belloni et al. (2014); Feng et al. (2020); Yuan and Lin (2006); Zou (2006) are useful for further reading. These estimation methods result in sparse betas, and under the sparsity condition, they are consistent estimators. We also encounter the curse of dimensionality in high-frequency regressions, so the estimation methods developed for the finite dimension fail to estimate the betas consistently. Furthermore, we often observe that the betas are time-varying, which makes it difficult to apply the high-dimensional regression methods designed for low-frequency data directly. Thus, to benefit from the utilization of high-frequency financial data in the high-dimensional regression, we need to develop methodologies that can handle both the curse of dimensionality as well as the time-varying betas.

In this paper, we introduce a novel high-dimensional high-frequency regression estimation procedure which can accommodate the sparse and time-varying beta processes. To model the high-frequency data, we employ diffusion processes whose stochastic difference equations have a time series regression structure. We also assume that the coefficient beta process $\beta_t$ follows a diffusion process. In this paper, the parameter of interest is the integrated beta, $\int_0^1 \beta_t dt$. To handle the curse of dimensionality, we assume that the beta processes are sparse, and to account for the sparsity of the time-varying beta process, we employ the Dantzig selector procedure (Candes et al., 2007). Specifically, due to the time-varying phenomena, we cannot estimate the integrated beta directly, and so we first estimate the instantaneous betas using the Dantzig selector procedure, based on the definition of $\beta_t$. Then, to mitigate the bias coming from the Dantzig selector, we propose a debiasing scheme and estimate the integrated beta with the debiased Dantzig instantaneous beta. With the debiasing scheme, we can obtain more accurate estimators in terms of the element-wise convergence rate; however, the estimated integrated beta is not sparse. Thus, to accommodate the
sparsity, we further regularize the estimated integrated beta. We call this Thresholding dEBiased
Dantzig Integrated Beta (TEDI Beta). We also establish its asymptotic properties.

The rest of paper is organized as follows. Section 2 introduces the model set-up. Section
3 proposes the TEDI Beta estimation procedure and establishes its asymptotic properties. In
Section 4 we conduct a simulation study to check the finite sample performance of the TEDI Beta
estimation procedure, and in Section 5 we apply the TEDI Beta to the high-frequency financial
data. The conclusion is presented in Section 6 and we collect all of the proofs in the Appendix.

2 The model set-up

We consider the following non-parametric time series regression diffusion models:

\[ dY_t = \beta_t^\top dX_t + dZ_t , \tag{2.1} \]

where \( Y_t \) is a dependent process, \( X_t \) is a \( p \)-dimensional multivariate covariate process, \( \beta_t \) is a
coefficient process, and \( Z_t \) is a residual process. The \( p \)-dimensional covariate process \( X_t \) and
residual process \( Z_t \) satisfy the following diffusion processes:

\[ dX_t = \mu_t dt + \sigma_t dB_t \quad \text{and} \quad dZ_t = \nu_t dW_t , \]

where \( \mu_t \) is a drift process, \( \sigma_t \) and \( \nu_t \) are instantaneous volatility processes, \( B_t \) and \( W_t \) are \( p\)-
dimensional and one-dimensional standard Brownian motions, respectively, and \( B_t \) and \( W_t \) are
independent. The processes \( \mu_t , \beta_t , \sigma_t , \) and \( \nu_t \) are predictable. We further assume that the
coefficient \( \beta_t \) satisfies the following diffusion process:

\[ d\beta_t = \mu_{\beta,t} dt + \nu_{\beta,t} dW^\beta_t , \]

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where $\mu_{\beta,t}$ and $\nu_{\beta,t}$ are predictable, and $W^\beta_t$ is $p$-dimensional standard Brownian motion. The parameter of interest is the integrated beta:

$$I_\beta = (I_{\beta_i})_{i=1,...,p} = \int_0^1 \beta_i dt.$$

In finance, hundreds of potential factor candidates have been proposed in order to explain the cross section of expected stock returns (Cochrane, 2011; Harvey et al., 2016; Hou et al., 2020; McLean and Pontiff, 2016). That is, the dimensionality, $p$, of the covariate process is large. Thus, we often run into the curse of dimensionality problem when handling financial data. However, all of them may not be significant; thus, to account for this, we assume the coefficient process $\beta_t = (\beta_{1t}, \ldots, \beta_{pt})^\top$ satisfies the following sparsity condition:

$$\sup_{0 \leq t \leq 1} \sum_{i=1}^p |\beta_{it}|^\delta \leq s_p \quad \text{and} \quad \sum_{i=1}^p |I_{\beta_i}|^\delta \leq s_p \text{ a.s.,}$$

where $\delta \in [0, 1)$ and $s_p$ is diverging slowly with respect to $p$, for example, $\log p$. We investigate asymptotic properties under this general sparsity case. However, in practice, it is harmless to assume $\delta = 0$. That is, we can assume that several factors are significant, while others do not affect on the expected returns.
3 High-dimensional high-frequency regression

3.1 Estimation procedure

In this section, we propose an estimation procedure for large integrated betas. We first fix some notations. For any given $p_1$ by $p_2$ matrix $U = (U_{ij})$, let

$$\|U\|_{\text{max}} = \max_{i,j} |U_{ij}|, \quad \|U\|_1 = \max_{1 \leq j \leq p_2} \sum_{i=1}^{p_1} |U_{ij}|, \quad \text{and} \quad \|U\|_{\infty} = \max_{1 \leq i \leq p_1} \sum_{j=1}^{p_2} |U_{ij}|.$$ 

We denote the Frobenius norm of $U$ by $\|U\|_F = \sqrt{\text{tr}(U^\top U)}$. The matrix spectral norm $\|U\|_2$ is the square root of the largest eigenvalue of $UU^\top$. $C$'s denote generic constants whose values are free of $n$ and $p$ and may change from appearance to appearance.

From the model (2.1), the instantaneous beta $\beta_t$ satisfies the following equation:

$$\frac{d}{dt}[Y,X]_t = \beta_t^\top \frac{d}{dt}[X,X]_t \text{ a.s.,}$$

where $[\cdot, \cdot]$ denotes the quadratic variation. The beta process $\beta_t$ is a function of instantaneous volatilities of $X$ and $Y$ as follows:

$$\beta_t = \Sigma_t^{-1} \Sigma_{XY,t} \text{ a.s.,} \quad (3.1)$$

where $\Sigma_t = \sigma_t \sigma_t^\top$ and $\Sigma_{XY,t} = \frac{d}{dt}[X,Y]_t$. Thus, the instantaneous beta can be estimated by the instantaneous volatility estimators. For the finite dimensional case, the instantaneous volatility-based estimation procedure works well (Aït-Sahalia et al. 2020). However, this approach cannot explain the sparse structure (2.2). Furthermore, when the dimensionality of the covariate $X$ is larger than the sample size, this approach fails to consistently estimate the instantaneous beta. Therefore, the procedure developed for the finite dimension is neither effective nor efficient. To accommodate the sparse structure of the beta process in (2.2), we employ the Dantzig selection...
method \cite{Candes2007} as follows. Let $\Delta_i^n A = A_{i\Delta_n} - A_{(i-1)\Delta_n}$ for $1 \leq i \leq 1/\Delta_n$ and $n = 1/\Delta_n$. Define

$$\mathcal{Y}_i = \begin{pmatrix} \Delta_{i+1}^n Y \\ \Delta_{i+2}^n Y \\ \vdots \\ \Delta_{i+k_n}^n Y \end{pmatrix}, \quad \mathcal{X}_i = \begin{pmatrix} \Delta_{i+1}^n X^T \\ \Delta_{i+2}^n X^T \\ \vdots \\ \Delta_{i+k_n}^n X^T \end{pmatrix}, \quad \text{and} \quad \mathcal{Z}_i = \begin{pmatrix} \Delta_{i+1}^n Z \\ \Delta_{i+2}^n Z \\ \vdots \\ \Delta_{i+k_n}^n Z \end{pmatrix},$$

where $k_n$ is the number of observations in each window to calculate the local regression. Then, we estimate the sparse instantaneous beta as follows:

$$\hat{\beta}_{i\Delta_n} = \arg \min \| \beta \|_1 \quad \text{s.t.} \quad \left\| \frac{1}{k_n\Delta_n} \mathcal{X}_i^T \mathcal{X}_i \beta - \frac{1}{k_n\Delta_n} \mathcal{X}_i^T \mathcal{Y}_i \right\|_{\max} \leq \lambda_n,$$

where $\lambda_n$ is a tuning parameter which converges to zero. We specify $\lambda_n$ in Theorem 1. With the appropriate $\lambda_n$, we can show that the proposed Dantzig instantaneous beta estimator $\hat{\beta}_{i\Delta_n}$ is a consistent estimator (see Theorem 1). To estimate the integrated beta $I\beta$ with this consistent estimator, we usually consider the sum of the instantaneous volatility estimators $\hat{\beta}_{i\Delta_n}$’s. However, the Dantzig estimator is biased, so their summation cannot enjoy the law of large number properties.

For example, the error of the sum of the Dantzig instantaneous beta estimators is dominated by the bias terms, and so it has the same convergence rate as that of $\hat{\beta}_{i\Delta_n}$. To reduce the effect of the bias, we use a debiasing scheme as follows. We first estimate the inverse matrix of the instantaneous volatility matrix $\Sigma_{i\Delta_n}$ using the constrained $\ell_1$-minimization for inverse matrix estimation (CLIME) \cite{Cai2011}. Let $\hat{\Omega}_{i\Delta_n}$ be the solution of the following optimization problem:

$$\min \| \Omega \|_1 \quad \text{s.t.} \quad \left\| \frac{1}{k_n\Delta_n} \mathcal{X}_i^T \mathcal{X}_i \Omega - I \right\|_{\max} \leq \tau_n,$$

where $\tau_n$ is a tuning parameter which converges to zero. We specify $\tau_n$ in Theorem 1. With the appropriate $\tau_n$, we can show that the proposed CLIME estimator $\hat{\Omega}_{i\Delta_n}$ is a consistent estimator (see Theorem 1).
where $\tau_n$ is the tuning parameter specified in Theorem 2. With the CLIME estimator, we adjust the Dantzig instantaneous beta estimator as follows:

$$
\tilde{\beta}_{i\Delta_n} = \hat{\beta}_{i\Delta_n} + \frac{1}{k_n\Delta_n} \hat{\Omega}_{i\Delta_n} \chi_i^\top (Y_i - \chi_i \hat{\beta}_{i\Delta_n}).
$$

(3.4)

Then, the debiased Dantzig instantaneous beta estimator satisfies

$$
\tilde{\beta}_{i\Delta_n} - \beta_{0,i\Delta_n} = \frac{1}{k_n\Delta_n} \Omega_{0,i\Delta_n} (\chi_i^\top Z_i + A_i) + R_i \text{ a.s.,}
$$

where the subscript 0 represents the true parameters, $A_i$ is a martingale difference defined in (A.8), and $R_i$ is a negligible remaining error term (see Theorem 3). The integrated beta estimator is

$$
\hat{I}\beta = \sum_{i=0}^{\lfloor 1/(k_n\Delta_n) \rfloor - 1} \tilde{\beta}_{ik_n\Delta_n} k_n\Delta_n.
$$

The debiasing scheme helps improve the element-wise convergence rate of the debiased Dantzig integrated beta estimator. However, the debiased Dantzig integrated beta estimator does not satisfy the sparsity condition (2.2) due to the bias adjustment. To accommodate the sparsity of the integrated beta, we apply the thresholding scheme as follows:

$$
\tilde{I}\beta_i = s(\tilde{I}\beta_i) \mathbf{1} (|\tilde{I}\beta_i| \geq h_n) \quad \text{and} \quad \tilde{I}\beta = (\tilde{I}\beta_i)_{i=1,...,p},
$$

where $\mathbf{1}(\cdot)$ is an indicator function, the thresholding function $s(\cdot)$ satisfies that $|s(x) - x| \leq h_n$, and $h_n$ is a thresholding level specified in Theorem 4. Examples of the thresholding function $s(x)$ include the hard thresholding function $s(x) = x$ and the soft thresholding function $s(x) = x - \text{sign}(x)h_n$. We call this the Thresholded dEbiased Dantzig Integrated Beta (TEDI Beta) estimator. We summarize the TEDI Beta estimation procedure in Algorithm 1.
Algorithm 1 TEDI Beta estimation procedure

**Step 1** Estimate the instantaneous beta:

\[
\hat{\beta}_{i\Delta n} = \arg\min \|\beta\|_1 \text{ s.t. } \left\| \frac{1}{k_n \Delta_n} \mathcal{X}_i^\top \mathcal{X}_i \beta - \frac{1}{k_n \Delta_n} \mathcal{X}_i^\top \mathcal{Y}_i \right\|_{\text{max}} \leq \lambda_n,
\]

where \( \lambda_n = C_{\lambda} s_p \sqrt{\log p} \left( \sqrt{k_n \Delta_n} + k_n^{-1/2} \right) \) and \( k_n = c_k n^{1/2} \) for some large constants \( C_{\lambda} \) and \( c_k \).

**Step 2** Estimate the inverse instantaneous volatility matrix:

\[
\hat{\Omega}_{i\Delta n} = \arg\min \|\Omega\|_1 \text{ s.t. } \left\| \frac{1}{k_n \Delta_n} \mathcal{X}_i^\top \mathcal{X}_i \Omega - I \right\|_{\text{max}} \leq \tau_n,
\]

where \( \tau_n = C_{\tau} \sqrt{\log p} \left( \sqrt{k_n \Delta_n} + k_n^{-1/2} \right) \) for some large constant \( C_{\tau} \).

**Step 3** Debias the Dantzig instantaneous beta estimator:

\[
\tilde{\beta}_{i\Delta n} = \hat{\beta}_{i\Delta n} + \frac{1}{k_n \Delta_n} \hat{\Omega}_{i\Delta n} \mathcal{X}_i (\mathcal{Y}_i - \mathcal{X}_i \hat{\beta}_{i\Delta n}).
\]

**Step 4** Estimate the integrated beta:

\[
\tilde{I}\beta = \sum_{i=0}^{[1/(k_n \Delta_n)]-1} \tilde{\beta}_{ik_i \Delta_n} k_n \Delta_n.
\]

**Step 5** Threshold the debiased Dantzig integrated beta estimator:

\[
\tilde{I}\beta_i = s(\tilde{I}\beta_i) 1_{\{|\tilde{I}\beta_i| \geq h_n\}} \quad \text{and} \quad \tilde{I}\beta = (\tilde{I}\beta_i)_{i=1,...,p},
\]

where \( 1(\cdot) \) is an indicator function, the thresholding function \( s(\cdot) \) satisfies that \( |s(x) - x| \leq h_n \), \( h_n = C_h b_n \) for some constant \( C_h \), and \( b_n \) is defined in Theorem 3.

### 3.2 Asymptotic results

In this section, we establish the asymptotic properties for the proposed TEDI Beta estimation procedure. To investigate the asymptotic properties, we need the following technical conditions.

**Assumption 1.**

(a) The volatility process \( \Sigma_t = (\Sigma_{ijt})_{i,j=1,...,p} \) satisfies the following Lipschitz condition:

\[
|\Sigma_{ijt} - \Sigma_{ij}s| \leq L_{\sigma,t,s}|t - s|^{1/2} \quad \text{a.s.,}
\]
where $L_{\sigma,t,s}$ is some bounded positive random variable.

(b) $\mu_t, \mu_{\beta,t}, \beta_t, \nu_t, \Sigma_t$, and $\Sigma_\beta,t = \nu_{\beta,t} \nu_{\beta,t}^\top$ are almost surely bounded, and $\|\Sigma_t^{-1}\|_1 \leq C$ a.s.

(c) The drift process $\mu_{\beta,t} = (\mu_{\beta,1t}, \ldots, \mu_{\beta,pt})^\top$ and the volatility process $\Sigma_{\beta,t} = (\Sigma_{\beta,ijt})_{i,j=1,\ldots,p}$ satisfy the following sparsity condition for $\delta \in [0, 1)$:

$$\sup_{0 \leq t \leq 1} \sum_{i=1}^p |\mu_{\beta,it}|^\delta \leq s_p \quad \text{and} \quad \sup_{0 \leq t \leq 1} \sum_{i=1}^p |\Sigma_{\beta,it}|^{\delta/2} \leq s_p \ a.s.$$

(d) The inverse matrix of the volatility matrix process, $\Sigma_t^{-1} = \Omega_t = (\omega_{ijt})_{i,j=1,\ldots,p}$, satisfies the following sparsity condition:

$$\sup_{0 \leq t \leq 1} \max_{1 \leq i \leq p} \sum_{j=1}^p |\omega_{ijt}|^q \leq s_{\omega,p} \ a.s.,$$

where $q \in [0, 1)$ and $s_{\omega,p}$ is diverging slowly with respect to $p$, for example, $\log p$.

(e) $n^{c_1} \leq p \leq \exp(n^{c_2})$ for some positive constants $c_1$ and $c_2 < 1/4$.

Remark 1. To investigate estimators of time-varying processes, we need continuity conditions such as Assumption 1(a) and the diffusion process structures for $X_t$, $Y_t$ and $\beta_t$ in Section 2. Even if Assumption 1(a) is replaced by the condition that $\Sigma_t$ has a continuous Itô diffusion process structure with bounded drift and instantaneous volatility processes, we can obtain the same theoretical results with up to $\sqrt{\log p}$ order. The boundedness condition Assumption 1(b) provides sub-Gaussian tails which are often required to investigate high-dimensional inferences. On the other hand, when we investigate the asymptotic behaviors of volatility estimators such as their convergence rate, the boundedness condition can be relaxed to the locally boundedness condition (see Aït-Sahalia and Xiu (2017)). Specifically, Jacod and Protter (2011) showed in Lemma 4.4.9 that if the asymptotic result, such as stable convergence in law or convergence in probability, is satisfied under the boundedness
condition, it is also satisfied under the locally boundedness condition. Thus, the asymptotic results established in this paper also hold for the locally boundedness condition. The sparsity condition for the beta process, Assumption 1(c), is the technical condition for investigating the discretization error of Dantzig instantaneous beta estimator $\hat{\beta}_{i\Delta_n}$. Finally, to investigate asymptotic properties of the CLIME estimator, we need the sparse inverse matrix condition Assumption 1(d) (Cai et al., 2011).

In Theorems 1 and 2 below, we establish asymptotic properties for the sparse instantaneous beta and inverse matrix. Note that we use subscript 0 for the true parameters.

**Theorem 1.** Under Assumption 1(a)–(c), we choose $\lambda_n = C_\lambda s_p \sqrt{\log p} \left( \sqrt{k_n \Delta_n} + k_n^{-1/2} \right)$ for some large constant $C_\lambda$ and $k_n = c_k n^c$ for some constant $c_k$ and $c \in [1/2, 3/4]$. Then, we have, for large $n$,

$$\|\hat{\beta}_{i\Delta_n} - \beta_{0,i\Delta_n}\|_{\text{max}} \leq C\lambda_n \quad \text{and} \quad \|\hat{\beta}_{i\Delta_n} - \beta_{0,i\Delta_n}\|_1 \leq C s_p \lambda_n^{1-\delta},$$

with probability greater than $1 - p^{-a}$ for any given positive constant $a$.

**Theorem 2.** Under Assumption 1, we choose $\tau_n = C_\tau \sqrt{\log p} \left( \sqrt{k_n \Delta_n} + k_n^{-1/2} \right)$ for some large constant $C_\tau$ and $k_n = c_k n^c$ for some constants $c_k$ and $c \in [1/2, 3/4]$. Then, we have, for large $n$,

$$\|\hat{\Omega}_{i\Delta_n} - \Omega_{0,i\Delta_n}\|_{\text{max}} \leq C\tau_n \quad \text{and} \quad \|\hat{\Omega}_{i\Delta_n} - \Omega_{0,i\Delta_n}\|_1 \leq C s_{w,p} \tau_n^{1-q},$$

with probability greater than $1 - p^{-a}$ for any given positive constant $a$.

**Remark 2.** Theorems 1 and 2 show that by choosing $c = 1/2$, the estimators for the instantaneous beta and inverse matrix have element-wise convergence rates of $n^{-1/4}$ and $\ell_1$ convergence rates $n^{-1(1-\delta)/4}$ and $n^{-1(q-1)/4}$, respectively, with the log order term and the sparsity level term. We note that when choosing the sub-interval length $k_n = c_k n^{1/2}$ to estimate the instantaneous processes, we have the same order convergence rates of the statistical estimation and time-varying instantaneous
process approximation errors. That is, the order $n^{-1/4}$ is optimal for estimating each element of the instantaneous process; thus, the convergence rates are optimal up to log order.

The Dantzig instantaneous beta estimator has a near-optimal convergence rate as shown in Theorem 1. However, as discussed in the previous section, it is a biased estimator, which causes some non-negligible estimation errors when estimating the integrated beta. To tackle this problem, we employ debiasing schemes with the consistent CLIME estimator as in (3.4), and in the following theorem, we investigate its asymptotic benefits.

**Theorem 3.** Under the assumptions in Theorems 1–2, we choose $k_n = c_k n^{1/2}$ for some constant $c_k$. Then, we have

$$
\tilde{\beta}_{i\Delta_n} - \beta_{0,i\Delta_n} = \frac{1}{k_n \Omega_{i\Delta_n}} (X_i^\top Z_i + A_i) + R_i,
$$

where $A_i$ is defined in (3.5) and

$$
\|R_i\|_{\max} \leq C \left\{ s_p^{2-\delta}(\log p/n^{1/2})^{(2-\delta)/2} + s_p s_{\omega,p}(\log p/n^{1/2})^{(2-q)/2} + s_p (\log p)^{3/2}/n^{1/2} \right\},
$$

with probability greater than $1 - p^{-a}$ for any given positive constant $a$. Furthermore, we have, with probability greater than $1 - p^{-a}$ for any given positive constant $a$,

$$
\|\hat{I}\beta - I\beta_0\|_{\max} \leq C b_n,
$$

where $b_n = s_p^{2-\delta}(\log p/n^{1/2})^{(2-\delta)/2} + s_p s_{\omega,p}(\log p/n^{1/2})^{(2-q)/2} + s_p (\log p)^{3/2}/n^{1/2}$.

**Remark 3.** The debiased Dantzig instantaneous beta is decomposed by the martingale difference term $X_i^\top Z_i + A_i$ and the non-martingale remaining term $R_i$. The martingale difference term can enjoy the law of large number property, so the integrated beta estimator has a faster convergence rate than the Dantzig instantaneous beta estimator. The remaining non-martingale terms have
the same order as those of the martingale terms for the integrated beta estimator. Unlike the biased Dantzig estimator, the non-martingale remaining terms do not impact on the integrated beta estimator.

**Remark 4.** Theorem 3 shows the element-wise convergence rate for the debiased Dantzig integrated beta. When we have the exact sparse beta and inverse matrix processes, that is, $\delta = q = 0$, the debiased Dantzig integrated beta estimator has the convergence rate $s_p(s_p + s_{\omega,p})(\log p)^{3/2}/n^{1/2}$. The $n^{1/2}$ term is related with the sample size, which is known as the optimal rate. The $(\log p)^{3/2}$ term comes from handling the high-dimensional error bound. Usually, in high-dimensional literature, we have $\sqrt{\log p}$, but the debiased Dantzig integrated beta estimator has $(\log p)^{3/2}$ due to the handling of the high-dimensional error bounds for estimating two betas, such as the instantaneous beta and the integrated beta, and bounding the random processes. Finally, the $s_p$ and $s_{\omega,p}$ terms represent the sparsity levels for the beta and inverse volatility matrix. High-dimensional literature commonly assumes the sparsity level to be negligible; hence, we have the convergence rate $n^{-1/2}$ with up to $\log p$ order.

Theorem 3 indicates that, using the debiasing scheme, we obtain well-performing input integrated beta estimator $\hat{\beta}$. As described in Section 3.1, with the input integrated beta estimator $\hat{\beta}$, we apply the thresholding scheme to account for the sparsity and obtain the TEDI Beta. In the following theorem, we establish the $\ell_1$ convergence rate of the TEDI Beta estimator.

**Theorem 4.** Under the assumptions in Theorems 1–2, choose $h_n = C_h b_n$ and $k_n = c_k n^{1/2}$ for some constants $C_h$ and $c_k$, where $b_n$ is defined Theorem 3. Then, we have, with probability greater than $1 - p^{-a}$ for any given positive constant $a$,

$$
\|\hat{\beta} - \beta_0\|_1 \leq C s_p b_n^{1-d}.
$$

(3.7)

Theorem 4 shows that the TEDI Beta is a consistent estimator in terms of the $\ell_1$ norm under the
sparsity condition (2.2). When estimating the integrated beta without the debiasing step, we can obtain the convergence rate \( s_p \sqrt{\log p n^{-1/4}} \). The benefit of applying the debiasing scheme is the difference between \( b_n \) and \( s_p \sqrt{\log p n^{-1/4}} \). Under the sparsity condition, \( b_n \) is \( n^{-\left(2-(\delta+\eta)\right)/4} \) with \( \log p \) order for \( \delta, \eta \in [0, 1) \), which is faster than the convergence rate of the Dantzig integrated beta estimator. Therefore, the TEDI Beta estimator has the faster convergence rate.

### 3.3 Extension to jump diffusion processes

In financial practice, we often observe jumps. To reflect this, we can extend the continuous diffusion process (2.1) to the jump diffusion process as follows:

\[
\begin{align*}
dY_t & = dY_t^c + dY_t^J, \\
dY_t^c & = \beta_t^\top dX_t^c + dZ_t, \quad \text{and} \quad dY_t^J = J_t^y d\Lambda_t^y,
\end{align*}
\]

(3.8)

where \( Y_t^c \) and \( X_t^c \) are the continuous part of \( Y_t \) and \( X_t \), respectively, \( J_t^y \) is the jump size, and \( \Lambda_t^y \) is the Poisson process with the bounded intensity. The covariate process \( X_t \) is

\[
\begin{align*}
dX_t & = dX_t^c + dX_t^J, \\
dX_t^c & = \mu_t dt + \sigma_t dB_t, \quad \text{and} \quad dX_t^J = J_t d\Lambda_t,
\end{align*}
\]

(3.9)

where \( J_t \) is a jump size process and \( \Lambda_t \) is a \( p \)-dimensional Poisson process with bounded intensities. Under this jump diffusion model, we can still use the proposed estimation procedure, but we cannot observe the continuous diffusion process. To tackle this problem, we first detect the jumps from
the observed stock log-return data. For example, we use the truncation method as follows. Define

\[
\hat{Y}_i^c = \begin{pmatrix}
\Delta_{i+1}^n Y \mathbb{1}_{\{\Delta_{i+1}^n Y \leq u_n\}} \\
\Delta_{i+2}^n Y \mathbb{1}_{\{\Delta_{i+2}^n Y \leq u_n\}} \\
\vdots \\
\Delta_{i+k_n}^n Y \mathbb{1}_{\{\Delta_{i+k_n}^n Y \leq u_n\}}
\end{pmatrix}
\]

and

\[
\hat{X}_i^c = \left( \Delta_{i+1}^n \hat{X}_i^{cT} \begin{atop} \Delta_{i+2}^n \hat{X}_i^{cT} \\
\vdots \\
\Delta_{i+k_n}^n \hat{X}_i^{cT} \end{atop} \right),
\]

(3.10)

where \( \mathbb{1}_{\{\cdot\}} \) is an indicator function, \( k_n \) is the number of observations in each window used to calculate the local regression,

\[
\Delta_i^n \hat{X}^c = \left( \Delta_i^n X_j \mathbb{1}_{\{\Delta_i^n X_j \leq v_n\}} \right)_{j=1,...,p},
\]

and \( u_n \) and \( v_n \) are the truncation levels. We choose \( u_n = C_u s_p \sqrt{\log pn^{-\varrho}} \) and \( v_n = C_v \sqrt{\log pn^{-\varrho}} \) for \( \varrho \in [5/16, 1/2) \) and some constants \( C_u \) and \( C_v \). Then, to estimate the integrated beta \( I\beta \), we employ the estimation method in Section 3.1 using \( \hat{Y}_i^c \) and \( \hat{X}_i^c \) instead of \( Y_i \) and \( X_i \). We denote the jump-adjusted TEDI Beta estimator by \( \tilde{I}\beta^c \). In the following theorem, we investigate the asymptotic property of the jump-adjusted TEDI Beta estimator.

**Theorem 5.** Under the models (3.8)–(3.9), let assumptions in Theorem 4 hold. Then, we have, with probability greater than \( 1 - p^{-a} \) for any given positive constant \( a \),

\[
\| \tilde{I}\beta^c - I\beta_0 \|_1 \leq C s_p b_n^{1-\delta}. \tag{3.11}
\]

Theorem 5 shows that the jump-adjusted TEDI Beta estimator has the same convergence rate obtained in Theorem 4. Therefore, we conclude that the jump can be detected well and that its effects can be mitigated.
3.4 Discussion on the tuning parameter selection

To implement the TEDI Beta estimation procedure, we need to choose the tuning parameters. In this section, we discuss how to select the tuning parameters for the numerical studies. For the jump adjustment, let \( \hat{\sigma}_Y = \sqrt{\sum_{i=1}^{n} (\Delta_i^n Y)^2} \) and \( \hat{\sigma}_X = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{p} (\Delta_i^n X_j)^2 / p} \). We choose

\[
\begin{align*}
  u_n &= c_u \hat{\sigma}_Y n^{-0.47} \sqrt{\log p} \quad \text{and} \quad v_n = c_v \hat{\sigma}_X n^{-0.47} \sqrt{\log p}, \\
  (3.12)
\end{align*}
\]

where \( c_u \) and \( c_v \) are tuning parameters. To employ Algorithm 1, we first need to handle the scale problem. To do this, we standardize the variables \( \Delta_i^n \hat{Y}^c \) and \( \Delta_i^n \hat{X}_j^c \), \( j = 1, \ldots, p \), to have a mean of 0 and a variance of 1. Then, we choose

\[
\begin{align*}
  k_n = [c_k n^{1/2}], \quad \lambda_n = c_\lambda n^{-1/4} \sqrt{\log p}, \quad \tau_n = c_\tau n^{-1/4} \sqrt{\log p}, \quad \text{and} \quad h_n = c_h n^{-1/2} \log p, \\
  (3.13)
\end{align*}
\]

where \( c_k, c_\lambda, c_\tau, \) and \( c_h \) are tuning parameters. In the simulation and empirical studies, we choose \( c_u = 1, c_v = 1, c_k = 1, c_\lambda = 4, c_\tau = 1/4, \) and \( c_h = 1/2. \)

4 A simulation study

In this section, we conducted simulations to check the finite sample performance of the proposed TEDI Beta estimator. We generated the data with frequency \( 1/n^{all} \) and considered the following time series regression jump diffusion model:

\[
\begin{align*}
  dY_t & = \beta^\top dX_t^c + dZ_t + J_t^y d\Lambda_t^y, \\
  dX_t & = dX_t^c + dX_t^J, \quad dX_t^c = \sigma_t dB_t, \quad dX_t^J = J_t d\Lambda_t, \quad dZ_t = \nu_t dW_t,
\end{align*}
\]
where $B_t$ and $W_t$ are $p$-dimensional and one-dimensional independent Brownian motions, respectively, $J_t = (J_{1t}, \ldots, J_{pt})^\top$ and $J'_{t}$ are jump sizes, and $\Lambda_t = (\Lambda_{1t}, \ldots, \Lambda_{pt})^\top$ and $\Lambda'_{t}$ are the Poisson processes with the intensities $(20, \ldots, 20)^\top$ and 15, respectively. The jump sizes $J_{it}$ and $J'_{it}$ were independently generated from the Gaussian distribution with a mean of 0 and standard deviation of 0.05. We set the initial values $X_{i0}$ and $Y_0$ to 0, while $\nu_t$ follows the Ornstein–Uhlenbeck process

$$
d\nu_t = 4 (0.15 - \nu_t) \, dt + 0.05 dW_t^\nu, $$

where $\nu_0 = 0.2$ and $W_t^\nu$ is one-dimensional independent Brownian motion. The instantaneous volatility process $\sigma_t$ was taken to be a Cholesky decomposition of $\Sigma_t = (\Sigma_{ijt})_{1 \leq i,j \leq p}$, where $\Sigma_{ijt} = \xi_t 0.5|\!|i-j|\!|$ and $\xi_t$ satisfies

$$
d\xi_t = 5 (0.35 - \xi_t) \, dt + 0.15 dW_t^\xi, $$

where $\xi_0 = 1$ and $W_t^\xi$ is one-dimensional independent Brownian motion. For the coefficient process $\beta_t$, we considered the time-varying beta and constant beta processes, where $[s_p]$ factors are only significant. We first generated the time-varying beta process as follows:

$$
d\beta_t = \mu_{\beta,t} \, dt + \nu_{\beta,t} dW_t^\beta, $$

where $\mu_{\beta,t} = (\mu_{1,\beta,t}, \ldots, \mu_{p,\beta,t})^\top$, $\nu_{\beta,t} = (\nu_{i,j,\beta,t})_{1 \leq i,j \leq p}$, and $W_t^\beta$ is $p$-dimensional independent Brownian motion. We set the process $(\nu_{i,j,\beta,t})_{1 \leq i,j \leq [s_p]}$ as $\zeta_t I_{[s_p]}$, where $I_{[s_p]}$ is the $[s_p]$-dimensional identity matrix and $\zeta_t$ was generated as follows:

$$
d\zeta_t = 3 (0.75 - \zeta_t) \, dt + 0.25 dW_t^\zeta, $$
where $\zeta_0 = 0.5$ and $W_t^z$ is one-dimensional independent Brownian motion. For $i = 1, \ldots, [s_p]$, we took the initial value $\beta_{i0}$ as 0.5 and $\mu_{i, \beta, t} = 0.5$ for $0 \leq t \leq 1$. We set $\beta_{it}, i = [s_p] + 1, \ldots, p$, as zero. In contrast, for the constant beta process, we set $\beta_{it} = 1$ for $i = 1, \ldots, [s_p]$ and $0 \leq t \leq 1$, while the other $\beta_{it}$’s were set to 0. We chose $p = 100$, $s_p = \log p$, $n^{all} = 4000$, and we varied $n$ from 500 to 4000. To implement the TEDI Beta estimation procedure, we used the hard thresholding function $s(x) = x$ and employed the tuning parameter selection method discussed in Section 3.4.

For the purposes of comparison, we considered the integrated beta estimator proposed by A"ıt-Sahalia et al. (2020). We note that, for small $p$, one can account for the time variation of the beta process. We call this the AIT Beta estimator. Specifically, the AIT Beta estimator is calculated as follows:

$$\hat{\beta}_{AIT}^{\Delta_n} = (\hat{X}_i^c \hat{X}_i^c)^{-1} \hat{X}_i^c \hat{Y}_i^c$$

and

$$\hat{I}_{\beta}^{AIT} = \sum_{i=0}^{[1/(K_n \Delta_n)]-1} \hat{\beta}_{iK_n \Delta_n} K_n \Delta_n,$$

where $\hat{X}_i^c$ and $\hat{Y}_i^c$ are defined in (3.10) and we used $K_n = [n^{0.47}]$ instead of $k_n = [n^{0.5}]$. For $\hat{X}_i^c \hat{X}_i^c$ in (4.1), we added $10^{-4} I_p$ to avoid the singularity coming from the ultra high-dimensionality. We also employed the LASSO estimator (Tibshirani, 1996), which is able to explain the sparsity of the high-dimensional beta process. However, the LASSO estimator is designed for the constant beta process; thus, it fails to account for the time-varying beta process. We estimated the LASSO estimator as follows:

$$\hat{I}_{\beta}^{LASSO} = \arg\min_{\beta} \left\{ \sum_{i=0}^{n-1} \left( \Delta_i^n \hat{Y}_i^c - \Delta_i^n \hat{X}_i^c \beta \right)^2 + \lambda^{LASSO} \| \beta \|_1 \right\}.$$

where $\Delta_i^n \hat{Y}_i^c = \Delta_i^n Y^1_{\{\Delta_i^n Y \leq u_n\}}$, $\Delta_i^n \hat{X}_j^c = \left( \Delta_i^n X_j 1_{\{\Delta_i^n X_j \leq v_n\}} \right)_{j=1,\ldots,p}$, and the regularization parameter $\lambda^{LASSO}$ was selected using the 10-fold cross-validation. We calculated the average estimation errors under the $L_1$ norm, $L_2$ norm, and max norm by 1000 simulation procedures.

Figure 1 plots the log $L_1$, $L_2$, and max norm errors of the TEDI Beta, AIT Beta, and LASSO estimators for the time-varying and constant beta processes with $p = 100$ and $n = \ldots$. 


Figure 1: The log $L_1$, $L_2$, and max norm error plots of the TEDI Beta (red dot), AIT Beta (blue triangle), and LASSO (green diamond) estimators for $p = 100$ and $n = 500, 1000, 2000, 4000$. From Figure 1 we find that the estimation errors of the TEDI Beta estimator are decreasing as the number of high-frequency observations increases. For the time-varying beta process, the TEDI Beta estimator outperforms other estimators. This may be because the proposed TEDI Beta estimation method can account for both time variation and the high-dimensionality of the beta process, while the AIT Beta and LASSO estimators fail to explain one of them. When comparing the AIT Beta and LASSO estimators, the LASSO estimator shows better performance. This may be because the errors from the curse of dimensionality are much more significant than those from the time-varying beta in this simulation study. For the constant beta process, the TEDI Beta and LASSO estimators perform better than the AIT Beta estimator. This is probably due
to the fact that only the AIT Beta estimator is unable to handle the curse of dimensionality. We note that even for the constant beta process, the TEDI Beta estimator outperforms the LASSO estimator. One possible explanation for this is that the bias-adjustment scheme helps to estimate the integrated beta more accurately. From these results, we can conjecture that the TEDI Beta estimator accounts for the time variation and high-dimensionality of the beta process and is robust to the beta process structure.

5 An empirical study

We applied the proposed TEDI Beta estimator to real high-frequency trading data from January 2013 to December 2019. We took stock price data from the EOD (End of Day) website, firm fundamentals from the merged Center for Research in Security Prices (CRSP) database and Compustat database, and future price data from the FirstRate Data website. We considered the log-prices of the five assets as the dependent processes. Specifically, we selected Apple Inc. (AAPL), Berkshire Hathaway Inc. (BRK.B), General Motors Company (GM), Alphabet Inc. (GOOG), and Exxon Mobil Corporation (XOM). These firms are the top market value stocks in five global industrial classification standards (GICS) sectors: information technology, financials, consumer discretionary, communication services, and energy sectors. For the covariate process, we first collected the 74 most active future data from the FirstRate Data website and excluded futures that have negative price data or started trading after January 2013. Then, we considered the log-prices of the 66 remaining futures as the covariate processes. We listed the symbols of 66 futures in Table 1. Furthermore, we considered Fama-French five factors in Fama and French (2015) and the momentum factor in Carhart (1997). We denoted market, value, size, profitability, investment, and momentum factors by MKT, HML, SMB, RMW, CMA, and MOM, respectively. We constructed these factors with high-frequency data similar to the scheme in Aït-Sahalia et al. (2020) as follows. First, we
obtained the monthly portfolio constituents for above six factors with the stocks listed on NYSE, NASDAQ, and AMEX. Specifically, the MKT is the return of a value-weighted portfolio of whole assets, while the other factors are as follows:

\[
HML = (SH + BH) / 2 - (SL + BL) / 2,
\]

\[
SMB = (SH + SM + SL) / 3 - (BH + BM + BL) / 3,
\]

\[
RMW = (SR + BR) / 2 - (SW + BW) / 2,
\]

\[
CMA = (SC + BC) / 2 - (SA + BA) / 2,
\]

\[
MOM = (SU + BU) / 2 - (SD + BD) / 2,
\]

where small (S) and big (B) portfolios were classified by the market equity, while high (H), medium (M), and low (L) portfolios were classified by their ratio of book equity to market equity. Also, we classified robust (R), neutral (N), and weak (W) portfolios according to their profitability, while conservative (C), neutral (N), and aggressive (A) portfolios were classified by their investment. Finally, we classified up (U), flat (F), and down (D) portfolios according to the momentum of the return. The details of this process can be found in Aït-Sahalia et al. (2020). Then, we calculated each portfolio return with a frequency of five minutes using the portfolio weights adjusted at a five-minute frequency. Specifically, we obtained the return of any portfolios, \( WRet_{d,i} \), for the \( d \)th day and \( i \)th time interval as follows:

\[
WRet_{d,i} = \frac{\sum_{j=1}^{N_d} w_{d,i}^j \times Ret_{d,i}^j}{\sum_{j=1}^{N_d} w_{d,i}^j},
\]
where $N_d$ is the number of stocks for the portfolio on the day $d$, the superscript $j$ represents the $j$th stock of the portfolio, and $w_{d,i}^j$ is obtained by

$$w_{d,i}^j = w_d^j \times \prod_{t=0}^{i-1} (1 + Ret_{d,t}^j) ,$$

where $w_d^j$ is the market capitalization calculated using the close price of the $j$th stock on the day $d - 1$, and $Ret_{d,0}^j$ is the overnight return from the $(d - 1)$th day to the $d$th day. In sum, we utilized the five assets and 72 factors for the dependent processes and covariate processes, respectively. For each of the five assets, we employed the TEDI Beta estimation procedure to obtain the monthly integrated beta. We selected the tuning parameters discussed in Section 3.4 and used the hard thresholding function $s(x) = x$. The integrated betas for the non-trading period were estimated to be zero.

Figure 2 depicts the monthly integrated beta estimates for the five assets and 72 factors, and Figure 3 plots the nonzero frequency of monthly integrated betas. From Figures 2–3 we find that the value of the integrated beta varies over time and the significant betas also change over time. The interesting finding here is that there are several factors that played a significant role in most periods. Thus, to investigate the beta behavior in greater details, we draw the integrated betas for the five assets and the three most frequent factors illustrated in Figure 4. For example, AAPL has NQ (E-mini Nasdaq-100), YM (E-mini Dow), and ES (E-mini S&P 500); BRK.B has MKT, ES, and YM; GM has MKT, MOM, and EW (E-mini S&P 500 Midcap); GOOG has NQ, ES, and HML; and XOM has MKT, XAE (E-mini Energy Select Sector), and MOM. The market factor or Nasdaq index is the most significant factor over the entire time period; however, their beta values vary. In contrast, other factors are significant only for some periods. From these results, we can infer that the beta processes are sparse and time-varying. Hence, incorporating these features is important to account for market dynamics.
Figure 2: The monthly integrated beta estimates for the five assets and 72 factors. Each line represents the integrated beta estimates for each month.
Figure 3: The nonzero frequency of the monthly integrated beta estimates for the five assets and 72 factors.
Figure 4: The integrated beta estimates for the three most frequent factors among the 72 factors for each of the five assets.
Figure 5: The integrated beta estimates (left) and the nonzero frequency (right) for the six factors, MKT, HML, SMB, RMW, CMA, and MOM. Each line (left) represents the integrated beta estimates for each month.
In finance practice, the six factors (Fama-French five factors and the momentum factor) are most frequently used (Asness et al., 2013; Barroso and Santa-Clara, 2015; Carhart, 1997; Fama and French, 2015, 2016). Thus, we investigate their integrated beta behaviors. Figure 5 depicts the estimates of the monthly integrated beta for MKT, HML, SMB, RMW, CMA, and MOM with their nonzero frequency. We find that the MKT factor was significant for BRK.B, GM, and XOM, which may indicate that these firms can be adequately explained by the market movements. Other market factors are also significant for some periods; thus, these factors can explain expected stock returns for BRK.B, GM, and XOM. In contrast, for technology companies such as AAPL and GOOG, the integrated beta estimates for the MKT factor are usually small. This may be because the NQ (E-mini Nasdaq-100) factor played a significant role for the technology stocks as shown in Figure 4. Furthermore, AAPL and GOOG cannot be satisfactorily explained using the common six factors. One possible explanation is that, over the last twenty years, the technology companies have led the U.S. economy, with AAPL and GOOG as the most successful companies in the same time frame. Thus, these six factors may not work well for the period when we studied them.

6 Conclusion

In this paper, we proposed a novel Thresholding dEbiased Dantzig Integrated Beta (TEDI Beta) estimation procedure which can accommodate the sparse and time-varying beta process in the high-dimensional set-up. Specifically, to account for the sparse and time-varying beta process, we applied the Dantzig procedure to the instantaneous beta estimator, which results in a biased estimator. To reduce the bias, we proposed a debiased estimation procedure. We estimated the integrated beta with this new debiased instantaneous beta estimator. We showed that the Dantzig procedure can handle the sparsity of the instantaneous beta and that the debiased scheme mitigates the errors from the bias of the instantaneous beta estimator. To accommodate the sparsity of the
integrated beta, we further regularized the beta estimator. Finally, we showed that the proposed TEDI Beta estimator can obtain the near-optimal convergence rate. In the empirical study, we found that the beta process is sparse and time-varying. It revealed that, when analyzing the high-dimensional high-frequency regression, the TEDI Beta estimator is a useful tool which can handle the curse of dimensionality and the time-varying beta.

Table 1: Descriptions for the symbols of the 66 future factors used in the empirical study.

| Symbol | Description                  | Symbol | Description                  |
|--------|------------------------------|--------|------------------------------|
| A6     | Australian Dollar            | N6     | New Zealand Dollar           |
| AD     | Canadian Dollar              | NG     | Henry Hub Natural Gas        |
| B6     | British Pound                | NQ     | E-mini Nasdaq-100            |
| BR     | Brazilian Real               | OJ     | Orange Juice                 |
| BTP    | Euro BTP Long-Bond           | PA     | Palladium                    |
| BZ     | Brent Last Day Financial     | PL     | Platinum                     |
| CA     | Cocoa Futures                | RB     | RBOB Gasoline                |
| CL     | Crude Oil WTI                | RM     | Robusta Coffee               |
| DX     | US Dollar Index Future       | RP     | Euro-British Pound           |
| DY     | DAX                          | RS     | Canola                       |
| E1     | Swiss Franc                  | RTY    | E-mini Russell 2000          |
| E6     | Euro FX                      | RU     | Russian Ruble                |
| ED     | Eurodollar                   | SI     | Silver                       |
| ES     | E-mini S&P 500               | T6     | South African Rand           |
| EW     | E-mini S&P 500 Midcap        | UB     | Ultra US Treasury Bond       |
| FX     | Euro Stoxx 50                | US     | 30 Year US Treasury Bond     |
| G      | 10-Year Long Gilt            | VX     | VIX                          |
| GC     | Gold                         | X      | FTSE 100                     |
| GF     | Feeder Cattle                | XAE    | E-mini Energy Select Sector  |
| GG     | Euro Bund                    | XAF    | E-mini Financial Select Sector |
| HE     | Lean Hog                     | XAI    | E-mini Industrial Select Sector |
| HG     | Copper                       | YM     | E-mini Dow                   |
| HH     | Natural Gas (Henry Hub) Last-day Financial | ZC | Corn |
| HO     | NY Harbor ULSD (Heating Oil) | ZF     | 5-Year Treasury Note         |
| HR     | Euro Bobl                    | ZL     | Soybean Oil                  |
| J1     | Japanese Yen                 | ZM     | Soybean Meal                 |
| L      | 3 Month Sterling             |ZN      | 10-Year Treasury Note        |
| LE     | Live Cattle                  | ZO     | Oats                         |
| ME     | E-mini Euro FX               | ZQ     | 30 Day Fed Funds Future      |
| ML     | Milling Wheat                | ZR     | Rough Rice                   |
| MME    | MSCI Emerging Markets Index  | ZS     | Soybean                      |
| MP     | Mexican Peso                 | ZT     | 2-Year Treasury Note         |
| MX     | CAC40                        | ZW     | Wheat                        |
A Appendix

A.1 Proofs of Theorems 1 and 2

Proof of Theorem 1. We denote the true instantaneous beta at time $i\Delta_n$ by $\beta_0$. We have

\[
\Delta_{i+k}^n Y = \int_{(i+k-1)\Delta_n}^{(i+k)\Delta_n} \beta_t^\top dX_t + \int_{(i+k-1)\Delta_n}^{(i+k)\Delta_n} dZ_t
\]

\[
= \beta_0^\top \Delta_{i+k}^n X + \Delta_{i+k}^n Z + \int_{(i+k-1)\Delta_n}^{(i+k)\Delta_n} (\beta_t - \beta_0)^\top dX_t.
\]

Then, we have

\[
Y_i = X_i \beta_0 + Z_i + \tilde{X}_i,
\]

where

\[
\tilde{X}_i = \begin{pmatrix}
\int_{i\Delta_n}^{(i+1)\Delta_n} (\beta_t - \beta_0)^\top dX_t \\
\int_{(i+1)\Delta_n}^{(i+2)\Delta_n} (\beta_t - \beta_0)^\top dX_t \\
\vdots \\
\int_{(i+k_n-1)\Delta_n}^{(i+k_n)\Delta_n} (\beta_t - \beta_0)^\top dX_t
\end{pmatrix}.
\]

Since the instantaneous volatility and drift processes are bounded, $\Delta_{i+k}^n Z$ and $\Delta_{i+k}^n X_j$ are sub-Gaussian. Then, similar to proofs of Theorem 1 \cite{Kim2016}, we can show, for some large $C$,

\[
P \left( \max_{1 \leq j \leq p} \left| \frac{1}{k_n \Delta_n} \sum_{k=0}^{k_n} \Delta_{i+k}^n Z \Delta_{i+k}^n X_j \right| \geq C \sqrt{\log p/k_n} \right) \leq p^{-1-a}, \tag{A.1}
\]

where $X_j$ is the $j$th element of $X$.

Consider $\frac{1}{k_n \Delta_n} X_i^\top \tilde{X}_i$. There exist standard Brownian motions, $W_{mt}^*$ and $B_{it}^*$, such that

\[
d\beta_{mt} = \mu_{\beta,mt} dt + \sqrt{\Sigma_{\beta,mm}^*} dW_{mt}^* \quad \text{and} \quad dX_{it} = \mu_{it} dt + \sqrt{\Sigma_{ii}^*} dB_{it}^*.
\]
and let $\tilde{X}_{mt} = \int_{i\Delta_n}^t (\beta_{ms} - \beta_{m0})dX_{ms}$. By Itô’s formula, we have

$$
\int_{(i+k-1)\Delta_n}^{(i+k)\Delta_n} dX_{jt} \int_{(i+k-1)\Delta_n}^{(i+k)\Delta_n} (\beta_m - \beta_{m0})dX_{mt} = \int_{(i+k-1)\Delta_n}^{(i+k)\Delta_n} (\beta_m - \beta_{m0})\Sigma_{jmt}dt + \int_{(i+k-1)\Delta_n}^{(i+k)\Delta_n} (\tilde{X}_{mt} - \tilde{X}_m(i+k-1)\Delta_n)\Sigma_{jmt}dt + \int_{(i+k-1)\Delta_n}^{(i+k)\Delta_n} \mu_{\beta,ms}d\Sigma_{jmt}dt + \int_{(i+k-1)\Delta_n}^{(i+k)\Delta_n} (X_{jt} - X_j(i+k-1)\Delta_n)\mu_{jmt}dt = M_{1km} + D_{1km} + M_{2km} + D_{2km} + M_{3km} + D_{3km} \text{ a.s.}
$$

First, consider $D_{1km}$’s. By Assumption 1(b)–(c), we have

$$
\left| \frac{1}{k_n\Delta_n} \sum_{m=1}^{p} \sum_{k=1}^{k_n} D_{1km} \right| \leq C s_p k_n \Delta_n \text{ a.s.}
$$

For $D_{2km}$, by Assumption 1(b)–(c), the process $\sum_{m=1}^{p} |\beta_m - \beta_{m0}|$ has the sub-Gaussian tail and $\sum_{m=1}^{p} \sqrt{\Sigma_{\beta,mm}} \leq C \sum_{m=1}^{p} |\Sigma_{\beta,mm}|^{1/2} \leq C s_p$. Thus, we can show

$$
P \left( \sup_{t \in [i\Delta_n,(i+k_n)\Delta_n]} \sum_{m=1}^{p} |\beta_m - \beta_{m0}| \geq C s_p \sqrt{\Delta_n k_n \log p} \right) \leq CP \left( \sum_{m=1}^{p} |\beta_{m(i+k_n)\Delta_n} - \beta_{m0}| \geq C s_p \sqrt{\Delta_n k_n \log p} \right) \leq p^{-a-c/c_1}. \tag{A.2}
$$
Let \( E = \sup_{t \in [i\Delta_n, (i+k)\Delta_n]} \sum_{m=1}^{p} |\beta_{mt} - \beta_{m0}| \geq C s_p \sqrt{\Delta_n k_n \log p} \). Then, we have

\[
P \left( \sup_{t \in [(i+k-1)\Delta_n, (i+k)\Delta_n]} \sum_{m=1}^{p} |\bar{X}_{mt} - \bar{X}_{m(i+k-1)\Delta_n} | \geq C s_p \log p \sqrt{\Delta_n^2 k_n} \right) \leq C P \left( \sum_{m=1}^{p} \int_{(i+k-1)\Delta_n}^{(i+k)\Delta_n} (\beta_{mt} - \beta_{m0}) dX_{mt} \right) \geq C s_p \log p \sqrt{\Delta_n^2 k_n, E^c} + p^{-3-a-c/c_1}.
\]

which implies

\[
P \left( \sup_{1 \leq k \leq k_n} \sup_{t \in [(i+k-1)\Delta_n, (i+k)\Delta_n]} \sum_{m=1}^{p} |\bar{X}_{mt} - \bar{X}_{m(i+k-1)\Delta_n} | \geq C s_p \log p \sqrt{\Delta_n^2 k_n} \right) \leq C p^{-3-a-c/c_1}, \tag{A.3}
\]

Thus, we have, with probability at least \( 1 - p^{-2-a} \),

\[
\left| \frac{1}{k_n \Delta_n} \sum_{m=1}^{p} \sum_{k=1}^{k_n} D_{2km} \right| \leq C s_p \log p \sqrt{\Delta_n^2 k_n}.
\]

Similarly, we can show, with probability at least \( 1 - p^{-2-a} \),

\[
\left| \frac{1}{k_n \Delta_n} \sum_{m=1}^{p} \sum_{k=1}^{k_n} D_{3km} \right| \leq C s_p \log p \sqrt{\Delta_n^2 k_n}.
\]

Consider \( M_{1km} \)'s. By Azuma-Hoeffding inequality, we have, with probability at least \( 1 - p^{-2-a} \),

\[
\left| \frac{1}{k_n \Delta_n} \sum_{m=1}^{p} \sum_{k=1}^{k_n} M_{1km} \right| \leq C s_p \sqrt{\Delta_n k_n \log p}.
\]

For \( M_{2km} \), let

\[
\eta_{mt} = (\tilde{X}_{mt} - \tilde{X}_{m(i+k-1)\Delta_n}) \sqrt{\sum_{jjt}} \quad \text{and} \quad Q_t = 1_{\{\sum_{m=1}^{p} \left| \eta_{mt} \right| \leq C s_p \log p \sqrt{\Delta_n^2 k_n}\}}.
\]

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Then, by Azuma-Hoeffding inequality, we have, with probability at least 1 \( - p^{-3-a} \),

\[
\left| \frac{1}{k_n \Delta_n} \sum_{m=1}^{p} \sum_{k=1}^{k_n} \int_{(i+k-1)\Delta_n}^{(i+k)\Delta_n} \eta_m Q_t dB_{jt}^* \right| \leq C s_p \sqrt{\Delta_n (\log p)^{3/2}}.
\]

Thus, by (A.3), we have, with probability at least 1 \( - p^{-2-a} \),

\[
\left| \frac{1}{k_n \Delta_n} \sum_{m=1}^{p} \sum_{k=1}^{k_n} M_{2km} \right| \leq C s_p \sqrt{\Delta_n (\log p)^{3/2}}.
\]

Similarly, we can show, with probability at least 1 \( - p^{-2-a} \),

\[
\left| \frac{1}{k_n \Delta_n} \sum_{m=1}^{p} \sum_{k=1}^{k_n} M_{3km} \right| \leq C s_p \sqrt{\Delta_n (\log p)^{3/2}}.
\]

Therefore, we have, with probability at least 1 \( - C p^{-1-a} \),

\[
\max_{1 \leq j \leq p} \left| \frac{1}{k_n \Delta_n} \sum_{k=1}^{k_n} \Delta_{i+k}^n X_j \int_{(i+k-1)\Delta_n}^{(i+k)\Delta_n} \left( \beta_t - \beta_0 \right)^\top dX_t \right| \leq C s_p \sqrt{\Delta_n k_n \log p}.
\] (A.4)

With probability greater than 1 \( - p^{-1-a} \), we have

\[
\left\| \frac{d}{dt} [X, X]_{i \Delta_n} - \frac{1}{k_n \Delta_n} X_i^\top X_i \right\|_{\text{max}} \leq \left\| \Sigma_{i \Delta_n} - \frac{1}{k_n \Delta_n} \int_{i \Delta_n}^{(i+k_n)\Delta_n} \Sigma_s ds \right\|_{\text{max}} + \left\| \frac{1}{k_n \Delta_n} \int_{i \Delta_n}^{(i+k_n)\Delta_n} \Sigma_s ds - \frac{1}{k_n \Delta_n} X_i^\top X_i \right\|_{\text{max}} \leq C \sqrt{k_n \Delta_n} + C \sqrt{\log p / k_n},
\] (A.5)

where \( \frac{d}{dt} [X, X]_t = \Sigma_t \) and the last inequality can be showed similar to the proofs of Theorem 1 (Kim and Wang, 2016a). Thus, by (A.1), (A.4), and (A.5), we show the statement under the
following statements:

\[
\begin{align*}
\left\| \frac{1}{k_n \Delta_n} \mathcal{X}_i^\top Z_i \right\|_{\text{max}} &\leq C \sqrt{\log p / k_n}, \\
\left\| \frac{1}{k_n \Delta_n} \mathcal{X}_i^\top \tilde{X}_i \right\|_{\text{max}} &\leq Cs_p \sqrt{k_n \Delta_n \log p}, \\
\left\| \frac{d}{dt} [\mathbf{X}^\top \mathbf{X}]_{i \Delta_n} - \frac{1}{k_n \Delta_n} \mathcal{X}_i^\top \mathcal{X}_i \right\|_{\text{max}} &\leq C(\sqrt{k_n \Delta_n} + \sqrt{\log p / k_n}). 
\end{align*}
\] (A.6)

By (A.6), we have, for some large \( C_{\lambda} \),

\[
\begin{align*}
\left\| \frac{1}{k_n \Delta_n} \mathcal{X}_i^\top \mathcal{X}_i \beta_0 - \frac{1}{k_n \Delta_n} \mathcal{X}_i^\top \mathcal{Y}_i \right\|_{\text{max}} &= \left\| \frac{1}{k_n \Delta_n} \mathcal{X}_i^\top \mathcal{X}_i \beta_0 - \frac{1}{k_n \Delta_n} \mathcal{X}_i^\top \mathcal{X}_i - \frac{1}{k_n \Delta_n} \mathcal{X}_i^\top \mathcal{Y}_i \right\|_{\text{max}} \\
&\leq \left\| \frac{1}{k_n \Delta_n} \mathcal{X}_i^\top \mathcal{Z}_i \right\|_{\text{max}} + \left\| \frac{1}{k_n \Delta_n} \mathcal{X}_i^\top \tilde{X}_i \right\|_{\text{max}} \\
&\leq \lambda_n.
\end{align*}
\]

Thus, \( \beta_0 \) satisfies the constraint in (3.2), which implies that

\[
\| \hat{\beta}_{i \Delta_n} \|_1 \leq \| \beta_0 \|_1.
\]

We have

\[
\begin{align*}
\left\| \frac{1}{k_n \Delta_n} \mathcal{X}_i^\top \mathcal{X}_i (\hat{\beta}_{i \Delta_n} - \beta_0) \right\|_{\text{max}} &\leq \left\| \frac{1}{k_n \Delta_n} \mathcal{X}_i^\top \mathcal{X}_i \hat{\beta}_{i \Delta_n} - \frac{1}{k_n \Delta_n} \mathcal{X}_i^\top \mathcal{Y}_i \right\|_{\text{max}} \\
&\quad + \left\| \frac{1}{k_n \Delta_n} \mathcal{X}_i^\top \mathcal{X}_i \beta_0 - \frac{1}{k_n \Delta_n} \mathcal{X}_i^\top \mathcal{Y}_i \right\|_{\text{max}} \\
&\leq 2 \lambda_n
\end{align*}
\]

and

\[
\| \Sigma_{i \Delta_n} (\hat{\beta}_{i \Delta_n} - \beta_0) \|_{\text{max}} \leq \left\| \frac{1}{k_n \Delta_n} \mathcal{X}_i^\top \mathcal{X}_i (\hat{\beta}_{i \Delta_n} - \beta_0) \right\|_{\text{max}}
\]

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\[
+ (\Sigma_i \Delta_n - \frac{1}{k_n \Delta_n} X_i^\top X_i)(\widehat{\beta}_i \Delta_n - \beta_0)\|_{\text{max}}
\leq 2\lambda_n + 2\|\beta_0\|_1\|\Sigma_i \Delta_n - \frac{1}{k_n \Delta_n} X_i^\top X_i\|_{\text{max}}
\leq 2\lambda_n + C s_p (\sqrt{k_n \Delta_n} + \sqrt{\log p/k_n})
\leq C\lambda_n,
\]

where the third inequality is due to (A.6) and the sparsity condition (2.2). Then, we have

\[
\|\widehat{\beta}_i \Delta_n - \beta_0\|_{\text{max}} \leq \|\Sigma_i \Delta_n^{-1}\|_1 \|\Sigma_i \Delta_n(\widehat{\beta}_i \Delta_n - \beta_0)\|_{\text{max}}
\leq C\lambda_n,
\]

where the last inequality is due to Assumption 1(b).

Now, we consider the \(\ell_1\) norm error bound. Let \(a_n = \|\widehat{\beta}_i \Delta_n - \beta_0\|_{\text{max}}\). Define

\[
A = \widehat{\beta}_i \Delta_n - \beta_0,
A_1 = \left(\widehat{\beta}_{ji} \Delta_n 1(|\widehat{\beta}_{ji} \Delta_n| \geq 2a_n); 1 \leq j \leq p\right)^\top - \beta_0.
\]

Then, we have

\[
\|\beta_0\|_1 - \|A_1\|_1 + \|A - A_1\|_1 \leq \|A_1 + \beta_0\|_1 + \|A - A_1\|_1 = \|\widehat{\beta}_i \Delta_n\|_1 \leq \|\beta_0\|_1,
\]

which implies

\[
\|A\|_1 \leq \|A - A_1\|_1 + \|A_1\|_1 \leq 2\|A_1\|_1.
\]

Therefore, it is enough to investigate the convergence rate of \(\|A_1\|_1\). We have

\[
\|A_1\|_1 = \sum_{j=1}^p |\widehat{\beta}_{ji} \Delta_n 1(|\widehat{\beta}_{ji} \Delta_n| \geq 2a_n) - \beta_{j0}|
\]

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\[
\leq \sum_{j=1}^{p} |\hat{\beta}_{ji\Delta_n}| \mathbf{1}(|\hat{\beta}_{ji\Delta_n}| \geq 2a_n) - \beta_{j0}\mathbf{1}(|\beta_{j0}| \geq 2a_n)| + \sum_{j=1}^{p} |\beta_{j0}| \mathbf{1}(|\beta_{j0}| \leq 2a_n)
\]
\[
\leq a_n \sum_{j=1}^{p} \mathbf{1}(|\hat{\beta}_{ji\Delta_n}| \geq 2a_n) + \sum_{j=1}^{p} |\beta_{j0}| \mathbf{1}(|\beta_{j0}| \geq 2a_n) - 1(|\hat{\beta}_{ji\Delta_n}| \geq 2a_n)| + Cs_p a_n^{1-\delta}
\]
\[
\leq a_n \sum_{j=1}^{p} \mathbf{1}(|\beta_{j0}| \geq a_n) + \sum_{j=1}^{p} |\beta_{j0}| \mathbf{1}(|\beta_{j0}| \geq 2a_n) - 1(|\hat{\beta}_{ji\Delta_n}| \geq 2a_n)| + Cs_p a_n^{1-\delta}
\]
\[
\leq \sum_{j=1}^{p} |\beta_{j0}| \mathbf{1}(|\beta_{j0}| \geq 2a_n) - 1(|\hat{\beta}_{ji\Delta_n}| \geq 2a_n)| + Cs_p a_n^{1-\delta}
\]
\[
\leq \sum_{j=1}^{p} |\beta_{j0}| \mathbf{1}(|\beta_{j0}| - 2a_n) \leq |\beta_{j0}| - |\hat{\beta}_{ji\Delta_n}|) + Cs_p a_n^{1-\delta}
\]
\[
\leq \sum_{j=1}^{p} |\beta_{j0}| \mathbf{1}(|\beta_{j0}| \leq 3a_n) + Cs_p a_n^{1-\delta}
\]
\[
\leq Cs_p a_n^{1-\delta}. \tag{A.7}
\]

\[\blacksquare\]

**Proof of Theorem 2.** Similar to the proofs of Theorem 1, we can show the statement. We omit the proof.  \[\blacksquare\]

### A.2 Proof of Theorem 3

**Proof of Theorem 3.** Consider (3.5). We have

\[
\bar{\beta}_{i\Delta_n} - \beta_{0,i\Delta_n} = \frac{1}{k_n\Delta_n} \Omega_{0,i\Delta_n} \chi_i^\top Z_i + \frac{1}{k_n\Delta_n} \hat{\Omega}_{i\Delta_n} \chi_i^\top \bar{X}_i
\]
\[
- \left(\frac{1}{k_n\Delta_n} \hat{\Omega}_{i\Delta_n} \chi_i^\top \chi_i - I\right) (\bar{\beta}_{i\Delta_n} - \beta_{0,i\Delta_n}) + \frac{1}{k_n\Delta_n} (\hat{\Omega}_{i\Delta_n} - \Omega_{0,i\Delta_n}) \chi_i^\top Z_i.
\]

For \(\frac{1}{k_n\Delta_n} \hat{\Omega}_{i\Delta_n} \chi_i^\top \bar{X}_i\), by the proofs of (A.4), we have

\[
\frac{1}{k_n\Delta_n} \Omega_{0,i\Delta_n} \chi_i^\top \bar{X}_i = \frac{1}{k_n\Delta_n} \Omega_{0,i\Delta_n} A_i + R_{X,i}.
\]
where \( \| R_{X,i} \|_{\text{max}} \leq C s_p (\log p)^{3/2} \sqrt{\Delta n} \) with probability greater than \( 1 - C p^{-1-a} \), and

\[
\mathcal{A}_i = \left( \sum_{m=1}^{p} \int_{i \Delta_n}^{(i+k_n) \Delta_n} \int_{i \Delta_n}^{t} \sqrt{\Sigma_{\beta,m} \int_{m}^{t} dW_{ms}^{*} \Sigma_{jmt} dt} \right)_{j=1,\ldots,p}.
\]

(A.8)

By Theorem 2 and (A.4), we have, with probability greater than \( 1 - C p^{-1-a} \),

\[
\left\| \frac{1}{k_n \Delta_n} \tilde{\Omega}_{i,0} - \tilde{\Omega}_{i,0} \right\|_{\text{max}} \leq C s_{\omega,p} s_p \left( \sqrt{\log p/n} \right)^{2-q}.
\]

Thus, we have, with probability greater than \( 1 - C p^{-1-a} \),

\[
\frac{1}{k_n \Delta_n} \tilde{\Omega}_{i,0} x_i x_i^\top = \frac{1}{k_n \Delta_n} \Omega_{0,i} A_i + R'_{X,i},
\]

(A.9)

where \( \| R'_{X,i} \|_{\text{max}} \leq C \left\{ s_{\omega,p} s_p \left( \sqrt{\log p/n} \right)^{2-q} + s_p (\log p)^{3/2} \right\} \).

For \( \left( \frac{1}{k_n \Delta_n} \tilde{\Omega}_{i,0} x_i x_i^\top - I \right) (\beta_{i,0} - \beta_{0,i}) \), we have, with probability greater than \( 1 - p^{-1-a} \),

\[
\left\| \left( \frac{1}{k_n \Delta_n} \tilde{\Omega}_{i,0} x_i x_i^\top - I \right) (\beta_{i,0} - \beta_{0,i}) \right\|_{\text{max}} \leq C s_{\omega,p} \tau_n \lambda_n \delta
\]

(A.10)

where the second inequality is by Theorem 1 and (3.3). For the last term, we have

\[
\left\| \frac{1}{k_n \Delta_n} (\tilde{\Omega}_{i,0} - \Omega_{0,i}) x_i z_i \right\|_{\text{max}} \leq \left\| \tilde{\Omega}_{i,0} - \Omega_{0,i} \right\|_{1} \left\| \frac{1}{k_n \Delta_n} x_i z_i \right\|_{\text{max}}
\]

\[
\leq C s_{\omega,p} \tau_n^{1-q} \sqrt{\log p/k_n}
\]

\[
\leq C s_{\omega,p} (\log p)^{1-q/2} (\sqrt{k_n \Delta_n} + k_n^{-1/2})^{1-q} k_n^{-1/2}.
\]

(A.11)
where the second inequality is due to Theorem 2 and (A.1), with probability greater than $1 - Cp^{-1-a}$. By (A.9), (A.10), and (A.11), we have, with probability greater than $1 - p^{-a}$,

$$\tilde{\beta}_{i\Delta_n} - \beta_{0, i\Delta_n} = \frac{1}{k_n \Delta_n} \Omega_{0, i\Delta_n} (\mathcal{X}_i^\top Z_i + A_i) + R_i, \quad (A.12)$$

where

$$\|R_i\|_{\max} \leq C \left\{ s_p^{2-\delta} (\log p / n^{1/2})^{(2-\delta)/2} + s_p s_{\omega, p} (\log p / n^{1/2})^{(2-q)/2} + s_p (\log p)^{3/2} / n^{1/2} \right\}.$$ 

Consider (3.6). We have

$$\hat{I}_{\beta} - I_{\beta_0} = k_n \Delta_n \sum_{i=0}^{[1/(k_n \Delta_n)]-1} \left( \tilde{\beta}_{ik_n \Delta_n} - \beta_{0, ik_n \Delta_n} \right) + \sum_{i=0}^{[1/(k_n \Delta_n)]-1} \int_{ik_n \Delta_n}^{(i+1)k_n \Delta_n} (\beta_{0, ik_n \Delta_n} - \beta_{0, t}) dt - \int_{[1/(k_n \Delta_n)]k_n \Delta_n}^{1} \beta_{0, t} dt.$$ 

First, we consider the discretization error terms. Since the beta process has the sub-Gaussian tail, we can show, with probability greater than $1 - p^{-1-a}$,

$$\left\| \sum_{i=0}^{[1/(k_n \Delta_n)]-1} \int_{ik_n \Delta_n}^{(i+1)k_n \Delta_n} (\beta_{0, ik_n \Delta_n} - \beta_{0, t}) dt \right\|_{\max} \leq C \sqrt{\log p / n}.$$ 

Also, by Assumption 1(b), we have

$$\left\| \int_{[1/(k_n \Delta_n)]k_n \Delta_n}^{1} \beta_{0, t} dt \right\|_{\max} \leq C n^{-1/2} \text{ a.s.}$$

Consider $\sum_{i=0}^{[1/(k_n \Delta_n)]-1} \left( \tilde{\beta}_{ik_n \Delta_n} - \beta_{0, ik_n \Delta_n} \right)$. By (3.5), we have

$$\sum_{i=0}^{[1/(k_n \Delta_n)]-1} \left( \tilde{\beta}_{ik_n \Delta_n} - \beta_{0, ik_n \Delta_n} \right) = \sum_{i=0}^{[1/(k_n \Delta_n)]-1} \frac{1}{k_n \Delta_n} \Omega_{0, ik_n \Delta_n} (\mathcal{X}_{ik_n}^\top Z_{ik_n} + A_{ik_n}) + R_{ik_n},$$

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and, similar to the proofs of (A.12), we can show, with probability greater than $1 - p^{-1-a}$,

$$\left\| \frac{[1/(k_n \Delta_n)]^{-1}}{\sum \Omega_{0,ik_n \Delta_n} A_{ik_n}} \right\|_{\max} \leq C \frac{1}{k_n \Delta_n} \left\{ s_p^2 (\log p/n^{1/2})^{(2-\delta)/2} + s_p s_{\omega,p} (\log p/n^{1/2})^{(2-q)/2} + s_p (\log p)^{3/2}/n^{1/2} \right\}.$$

Since the inverse matrix process $\Omega_{0,ik_n \Delta_n}$ is bounded and $X$ and $Z$ have sub-Gaussian tails, similar to the proofs of Theorem 1 ([Kim and Wang 2016a]), we can show, with probability greater than $1 - p^{-1-a}$,

$$\left\| \frac{[1/(k_n \Delta_n)]^{-1}}{\sum \Omega_{0,ik_n \Delta_n} X_{ik_n}^T Z_{ik_n}} \right\|_{\max} \leq C \sqrt{\log p/n}.$$

Finally, we consider $\Omega_{0,ik_n \Delta_n} A_{ik_n}$ terms. Note that $A_{ik_n}$’s are martingales with sub-Gaussian tails. Thus, by Azuma-Hoeffding inequality, we have, with probability greater than $1 - p^{-1-a}$,

$$\left\| \frac{[1/(k_n \Delta_n)]^{-1}}{\sum \Omega_{0,ik_n \Delta_n} A_{ik_n}} \right\|_{\max} \leq C n^{-1/2} s_p \log p.$$

Therefore, the statement (3.6) is showed. ■

### A.3 Proof of Theorem 4

**Proof of Theorem 4.** By (3.6), we can find $h_n$ such that, with probability greater than $1 - p^{-a}$,

$$\left\| \frac{\hat{I} \beta - I \beta_0}{h_n} \right\|_{\max} \leq h_n/2. \tag{A.13}$$

We show the statement under the event (A.13). Similar to the proof of (A.7), we can show

$$\left\| \frac{\hat{I} \beta - I \beta_0}{h_n} \right\|_{1} \leq C s_p h_n^{1-\delta}.$$
A.4 Proof of Theorem [5]

Proof of Theorem [5]. For some large constant $C > 0$, let

\[ E_1 = \{ \max_{i,j} |\Delta_i^n X_j^c| \leq C \frac{\log pn}{\sqrt{n}} \} \cap \{ \max_{i,j} |\Delta_i^n Y_j^c| \leq C s_p \frac{\log pn}{\sqrt{n}} \}, \]
\[ E_2 = \{ \max_{i,j} \sum_{k=1}^{k_n} \mathbbm{1}_{\{\Delta_{i+k}^n X_j^c > v_n\}} \leq C k_n \Delta_n \log p \} \cap \{ \max_{i,j} \int_{i \Delta_n}^{(i+k_n) \Delta_n} d\Lambda_{jt} \leq C k_n \Delta_n \log p \}. \]

By the boundedness condition Assumption 1(b) and sparsity condition (2.2), we can show

\[ P(E_1) \geq 1 - p^{-2-a}. \]

Under the event $E_1$, we have, for large $n$,

\[ \max_{i,j} \sum_{k=1}^{k_n} \mathbbm{1}_{\{\Delta_{i+k}^n X_j^c > v_n\}} \leq \max_{i,j} \int_{i \Delta_n}^{(i+k_n) \Delta_n} d\Lambda_{jt}, \]

where $\int_{i \Delta_n}^{(i+k_n) \Delta_n} d\Lambda_{jt}$ is a Poisson with the intensity $C k_n \Delta_n$ for some constant $C > 0$, and

\[ P \left( \max_{i,j} \int_{i \Delta_n}^{(i+k_n) \Delta_n} d\Lambda_{jt} \geq C k_n \Delta_n \log p \right) \leq p^{-2-a}. \]

Thus, we have

\[ P(E_1 \cap E_2) \geq 1 - p^{-1-a}. \] (A.14)

We have

\[ \sum_{k=1}^{k_n} \Delta_{i+k}^n X_j^c \mathbbm{1}_{\{\Delta_{i+k}^n X_j^c \leq v_n\}} \Delta_{i+k}^n X_j^c \mathbbm{1}_{\{\Delta_{i+k}^n X_j^c \leq v_n\}} - \Delta_{i+k}^n X_j^c \Delta_{i+k}^n X_j^c \]

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\[
\begin{align*}
&= \sum_{k=1}^{k_n} \Delta_{i+k}^n X_j^c \Delta_{i+k}^n X_i^c \left(1_{\{|\Delta_{i+k}^n x_j| \leq v_n\}} 1_{\{|\Delta_{i+k}^n x_i| \leq v_n\}} - 1\right) \\
&\quad + \sum_{k=1}^{k_n} (\Delta_{i+k}^n X_j^c \Delta_{i+k}^n X_i^c + \Delta_{i+k}^n X_j^d \Delta_{i+k}^n X_i^c + \Delta_{i+k}^n X_j^d \Delta_{i+k}^n X_i^d) 1_{\{|\Delta_{i+k}^n x_j| \leq v_n\}} 1_{\{|\Delta_{i+k}^n x_i| \leq v_n\}} \\
&= (I)_{ijl} + (II)_{ijl}.
\end{align*}
\]

Under the event \(E_1 \cap E_2\), we have

\[
\max_{i,j,l} |(I)_{ijl}| \leq C \max_{i,j} |\Delta_i^n X_j^c|^2 \times \max_{i,j} \sum_{k=1}^{k_n} 1_{\{|\Delta_{i+k}^n x_j| > v_n\}} \\
\leq C (\log p)^2 k_n / n^2
\]

and

\[
\max_{i,j,l} |(II)_{ijl}| \leq C v_n \max_{i,j} |\Delta_i^n X_j^c| \times \max_{i,j} \int_{i \Delta_n}^{(i+k_n) \Delta_n} d\Lambda_{jt} + C v_n^2 \max_{i,l} \int_{i \Delta_n}^{(i+k_n) \Delta_n} d\Lambda_{jt} \\
\leq C (\log p)^2 n^{-2\rho-1/2}.
\]

Thus, by (A.14), we have, with probability greater than \(1 - p^{-1-a}\),

\[
\max_i \frac{1}{k_n \Delta_n} \left\| \hat{X}_i^c \hat{X}_i^c - X_i^c X_i^c \right\|_\infty \leq C (\log p)^2 n^{-2\rho},
\]

and similarly, we can show, with probability greater than \(1 - p^{-1-a}\),

\[
\max_i \frac{1}{k_n \Delta_n} \left\| \hat{Y}_i^c \hat{Y}_i^c - X_i^c Y_i^c \right\|_\infty \leq C s_p (\log p)^2 n^{-2\rho}.
\]

This result indicates that the effect of jump is negligible. Therefore, the statement can be shown by Theorem 4.

\[\blacksquare\]
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