On a result of Pazy concerning the asymptotic behaviour of nonexpansive mappings

Heinz H. Bauschke, Graeme R. Douglas, and Walaa M. Moursi

May 15, 2015

Abstract

In 1971, Pazy presented a beautiful trichotomy result concerning the asymptotic behaviour of the iterates of a nonexpansive mapping. In this note, we analyze the fixed-point free case in more detail. Our results and examples give credence to the conjecture that the iterates always converge cosmically.

2010 Mathematics Subject Classification: Primary 47H09, Secondary 90C25.

Keywords: Cosmic convergence, firmly nonexpansive mapping, nonexpansive mapping, Poincaré metric, projection operator.

1 Introduction

Throughout, $X$ is a finite-dimensional real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$, and $T: X \to X$ is nonexpansive, i.e., $(\forall x \in X)(\forall y \in X) \|Tx - Ty\| \leq \|x - y\|$. Then, using (3), the vector

$$v := P_{\text{ran}(\text{Id} - T)}(0)$$

(1)
is well defined. The following remarkable result\(^1\) was proved by A. Pazy in 1971.

**Fact 1.1 (Pazy’s trichotomy; \([8]\)).** Let \(x \in X\). Then

\[
\lim_{n \to \infty} \frac{T^n x}{n} = -v. \tag{2}
\]

Moreover, exactly one of the following holds:

1. \(0 \in \text{ran}(\text{Id} - T)\, \text{and} \, (T^n x)_{n \in \mathbb{N}}\) is bounded for every \(x \in X\).
2. \(0 \not\in \text{ran}(\text{Id} - T) \setminus \text{ran}(\text{Id} - T), \|T^n x\| \to \infty\) and \(\frac{1}{n} T^n x \to 0\) for every \(x \in X\).
3. \(0 \not\in \text{ran}(\text{Id} - T)\, \text{and} \, \lim_{n \to \infty} \frac{1}{n} \|T^n x\| > 0\) for every \(x \in X\).

Now consider the case when \(T\) does not have a fixed point. Let \(x \in X\). In view of Fact\(^{1.1}\), \(\|T^n x\| \to \infty\) and it is natural to ask whether additional asymptotic information is available about the (eventually well defined) sequence

\[
(Q_n(x))_{n \in \mathbb{N}} := \left(\frac{T^n x}{\|T^n x\|}\right)_{n \in \mathbb{N}}. \tag{3}
\]

Since \(X\) is finite-dimensional, for every \(x \in X\), \((Q_n(x))_{n \in \mathbb{N}}\) has cluster points. If the sequence \((Q_n(x))_{n \in \mathbb{N}}\) actually converges, then we refer to this also as cosmic convergence\(^2\).

Combining (1) and (2), we obtain the following necessary condition for cosmic convergence:

\[
0 \not\in \text{ran}(\text{Id} - T) \Rightarrow v \neq 0 \text{ and } (\forall x \in X) \, Q_n(x) \to -v/\|v\|. \tag{4}
\]

*The aim of this note is to provide conditions sufficient for convergence of \((Q_n(x))_{n \in \mathbb{N}}\) in the case when \(0 \in \text{ran}(\text{Id} - T) \setminus \text{ran}(\text{Id} - T)\).*

To the best of our knowledge, nothing was previously known about the behaviour of \((Q_n(x))_{n \in \mathbb{N}}\) in this case\(^3\). The results in this note nurture the conjecture that the sequence \((Q_n(x))_{n \in \mathbb{N}}\) actually converges. Notation and notions not explicitly defined may be found in \([3, 13, 14]\).

---

\(^1\)In fact, Fact\(^{1.1}\) holds in general Hilbert space. See also \([10, 11, 12]\) and \([9]\) for even more general settings. We thank Simeon Reich for bringing these references to our attention.

\(^2\)It will become clear in Section 2.4 why we speak of cosmic convergence.

\(^3\)Let us mention in passing that the study of \((Q_n(x))_{n \in \mathbb{N}}\) in the case when \(\text{Fix} T \neq \varnothing\) seems of little interest. Indeed, if \(T: x \mapsto 0\), then \((Q_n(x))_{n \in \mathbb{N}}\) is never well defined.
2 Results

2.1 The one-dimensional case

Theorem 2.1. Suppose that $X$ is one-dimensional and that $\text{Fix } T = \emptyset$. Then $T$ admits cosmic convergence; in fact, exactly one of the following holds:

(i) $(\forall x \in X) \ T x > x$, $T^n x \to +\infty$, and $Q_n(x) \to +1$.

(ii) $(\forall x \in X) \ T x < x$, $T^n x \to -\infty$, and $Q_n(x) \to -1$.

Proof. We can and do assume that $X = \mathbb{R}$. If there existed $a$ and $b$ in $\mathbb{R}$ such that $Ta > a$ and $Tb < b$, then the Intermediate Value Theorem would provide a point $z$ between $a$ and $b$, which is absurd in view of the hypothesis. It follows that either $\text{ran}(\text{Id} - T) \subseteq \mathbb{R}^-$ or $\text{ran}(\text{Id} - T) \subseteq \mathbb{R}^+$. Let us first assume that $\text{ran}(\text{Id} - T) \subseteq \mathbb{R}^-$, i.e., $(\forall x \in \mathbb{R}) \ x < T x$. Let $x \in \mathbb{R}$. On the one hand, we have $x < T x < T(T x) = T^2 x < T^3 x < \cdots < T^n x < T^{n+1} x < \cdots$. On the other hand, by Fact 1.1(ii) & (iii), $|T^n x| \to +\infty$. Altogether, $T^n x \to +\infty$ and hence $Q_n(x) \to +1$. Finally, the case when $\text{ran}(\text{Id} - T) \subseteq \mathbb{R}^+$ is treated similarly. ■

2.2 Composition of two projectors

In this section, we assume that

$$A \text{ and } B \text{ are nonempty closed convex subsets of } X$$

with corresponding projectors (nearest point mappings) $P_A$ and $P_B$, respectively, and that

$$T = P_B P_A.$$  

(5) (6)

We begin with a few technical lemmas.

Lemma 2.2. Let $K$ be a nonempty closed convex cone. Then $\bigl(K^\ominus\bigr)^\perp = K \cap (-K)$.

Proof. We will use repeatedly the fact that (see Corollary 6.33) $(K^\ominus)^\ominus = K$. "\subseteq": Indeed, $(K^\ominus)^\perp \subseteq (K^\ominus)^\ominus = K$ and $(K^\ominus)^\perp \subseteq (K^\ominus)^\ominus = -K$; hence, $(K^\ominus)^\perp \subseteq K \cap (-K)$. "\supseteq": Let $x \in K \cap (-K)$. Then $\langle x, K^\ominus \rangle \leq 0$ and $\langle -x, K^\ominus \rangle \leq 0$ and thus $\langle x, K^\ominus \rangle = 0$, i.e., $x \in (K^\ominus)^\perp$.

We write $S^\ominus := \{x \in X \mid \text{sup } \langle x, S \rangle \leq 0\}$ and $S^\oplus := -S^\ominus$ for a subset $S$ of $X$. 

4We write $S^\ominus := \{x \in X \mid \text{sup } \langle x, S \rangle \leq 0\}$ and $S^\oplus := -S^\ominus$ for a subset $S$ of $X$. 

3
Lemma 2.3. The set of (oriented) functionals separating the sets A and B satisfies
\[ U := \{ u \in X \setminus \{0\} \mid \sup \langle A, u \rangle \leq \inf \langle B, u \rangle \} = (\overline{\text{cone}}(A - B))^\circ \setminus \{0\}. \]  
(7)

Moreover, \[(\text{rec} A) \cap (\text{rec} B) \subseteq \bigcap_{u \in U} \{ u \} = \overline{\text{cone}}(A - B) \cap \overline{\text{cone}}(B - A).\]  
(8)

Consequently, if \( A \cap B = \emptyset \), then \( U \neq \emptyset \) and \((\text{rec} A) \cap (\text{rec} B)\) is a nonempty closed convex cone that is contained in a proper hyperplane of \( X \).

Proof. Since (7) is easily checked, we turn to (8). Let us first deal with the inclusion. If \( U = \emptyset \), then the intersection is trivially equal to \( X \) and we are done. So suppose that \( u \in U \), set \( R := \text{rec} A \) and \( S := \text{rec} B \). Then \( A + R = A \) and \( B + S = B \); consequently, \( \sup \langle A + R, u \rangle \leq \inf \langle B + S, u \rangle \). Since \( R \) and \( S \) are cones, we deduce that \( R \subseteq \{ u \}^\circ \) and \( S \subseteq \{ u \}^\circ \). Therefore, \( R \cap S \subseteq \{ u \}^\perp \). This completes the proof of the inclusion. Now \( x \in \bigcap_{u \in U} \{ u \} \iff (\forall u \in U) \langle x, u \rangle = 0 \iff x \in U^\perp = (\overline{\text{cone}}(A - B))^\perp = (A - B) \cap (-\overline{\text{cone}}(A - B)) = \overline{\text{cone}}(A - B) \cap \overline{\text{cone}}(B - A) \) by Lemma 2.2. The “Consequently” part follows from the Separation Theorem (see, e.g., [7 Theorem 2.5]).

Lemma 2.4. \( 0 \in \overline{\text{ran}}(\text{Id} - T) \setminus \text{ran}(\text{Id} - T) \iff \text{Fix} T = \emptyset. \)

Proof. By [1] (see also [4] for extensions to firmly nonexpansive operators), we always have \( 0 \in \overline{\text{ran}}(\text{Id} - T) \), and this implies the result. 

We are now ready for the main result of this section.

Theorem 2.5. Suppose that \( \text{Fix} T = \emptyset \). Let \( b_0 := x \in X \) and set \((\forall n \in \mathbb{N}) a_{n+1} := P_A b_n \) and \( b_{n+1} := P_{B a_{n+1}} T b_n. \) Then the following hold:

(i) \( \|a_n\| \to +\infty, \|b_n\| \to +\infty, b_n - a_n \to g, \) and \( a_{n+1} - b_n \to -g \), where \( g := P_{B - A}(0). \)

(ii) All cluster points of \( (b_n / \|b_n\|)_{n \in \mathbb{N}} \) lie in the set
\[ ((\text{rec} A) \cap (\text{rec} B)) \cap ((\text{rec} A) \cap (\text{rec} B))^\oplus, \]
which is a closed convex cone in \( X \) that properly contains \( \{0\} \).

(iii) Neither \((\text{rec} A) \cap (\text{rec} B)\) nor \((\text{rec} A) \cap (\text{rec} B))^\oplus\) is a linear subspace of \( X \).

(iv) (Cosmic Convergence) The sequence \((Q_n(x))_{n \in \mathbb{N}} = (b_n / \|b_n\|)_{n \in \mathbb{N}}\) converges provided one of the following holds:

\footnote{We use \( \text{rec} S := \{ x \in X \mid x + S \subseteq S \} \) to denote the recession cone of a nonempty convex subset of \( X \).}
(a) \(((\text{rec } A) \cap \text{rec } B) \cap ((\text{rec } A) \cap \text{rec } B)\) is a ray.
(b) \(\text{rec } A \cap \text{rec } B\) is a ray.
(c) \(\dim X = 2\).

\textbf{Proof.} Set \(R := (\text{rec } A) \cap \text{rec } B\), which is a nonempty closed convex cone.

(i) See [2, Theorem 4.8].

(ii) Note that (i) makes the quotient sequence eventually well defined. Let \(q\) be cluster point of \((b_n/\|b_n\|)_{n \in \mathbb{N}}\), say

\[
\frac{b_{k_n}}{\|b_{k_n}\|} \to q \tag{10}
\]

for some subsequence \((b_{k_n})_{n \in \mathbb{N}}\) of \((b_n)_{n \in \mathbb{N}}\). Then [3, Proposition 6.50] implies that \(q \in \text{rec } B\). Furthermore, since \(a_{k_n} - b_{k_n} \to -g\) and \(\|b_{k_n}\| \to +\infty\), we deduce that

\[
\frac{a_{k_n}}{\|b_{k_n}\|} = \frac{a_{k_n} - b_{k_n}}{\|b_{k_n}\|} + \frac{b_{k_n}}{\|b_{k_n}\|} \to q. \tag{11}
\]

As before, this implies that \(q \in \text{rec } A\). Thus

\[
q \in (\text{rec } A) \cap (\text{rec } B) = R. \tag{12}
\]

On the other hand, using [15, Theorem 3.1] and [3, Proposition 6.34], we have

\[
b_{n+1} - b_n = (b_{n+1} - a_{n+1}) + (a_{n+1} - b_n) \in \text{ran}(P_B - \text{Id}) + \text{ran}(P_A - \text{Id}) \tag{13a}
\]
\[
\subseteq \text{ran}(P_B - \text{Id}) + \text{ran}(P_A - \text{Id}) = (\text{rec } B)^\oplus + (\text{rec } A)^\oplus \tag{13b}
\]
\[
\subseteq (\text{rec } A)^\oplus + (\text{rec } B)^\oplus = \left(\text{rec } A \cap (\text{rec } B)\right)^\oplus = R^\oplus. \tag{13c}
\]

It follows that \(b_n - b_0 = \sum_{k=0}^{n-1}(b_{k+1} - b_k) \in nR^\oplus = R^\oplus\); hence, \((b_n - b_0)/\|b_n - b_0\| \in R^\oplus\) which implies that \(q \in R^\oplus\). Altogether, \(q \in R \cap R^\oplus\). Since \(\|q\| = 1\), we deduce that \(\{0\} \nsubseteq R \cap R^\oplus\). Finally, if \(R\) was a linear subspace of \(X\), then \(R \cap R^\oplus = R \cap R^\perp = \{0\}\), which is absurd. Hence \(R\) is not a linear subspace. If \(R^\oplus\) were a linear subspace of \(X\), then so would be \(R^{\oplus\oplus} = R\), which is absurd.

(iv) In view of (ii) \(R \cap R^\oplus\) contains a ray and it suffices to show that \(R \cap R^\oplus\) is precisely a ray. Indeed, each of the listed conditions guarantees that — for (iv)(c) use (iii). \blacksquare

In Figure 1, we visualize Theorem 2.5(iv)(c) for the case when \(A\) and \(B\) are nonintersecting unbounded closed convex subsets in the Euclidean plane.
Figure 1: A GeoGebra snapshot in $\mathbb{R}^2$ for two sets $A$ (the black line) and $B$ (the blue region) illustrating Theorem 2.5(iv)(c). Shown are the first few iterates of the sequence $(T^n x)_{n \in \mathbb{N}} = (b_n)_{n \in \mathbb{N}}$ (red points) and of the sequence $(a_n)_{n \in \mathbb{N}}$ (blue points). We visually confirm cosmic convergence: the sequence $(Q_n(x))_{n \in \mathbb{N}}$ converges to $(1/\sqrt{2})(1,1)$.

2.3 Firmly nonexpansive operators

Recall that $x \in X$ belongs to the horizon cone of a nonempty subset $C$ of $X$, written $x \in C^\infty$ if there exist sequences $(c_n)_{n \in \mathbb{N}}$ in $C$ and $(\lambda_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_{++}$ such that $\lambda_n \to 0$ and $\lambda_n c_n \to x$. Note that $C^\infty = \overline{C}^\infty$; furthermore, if $C$ is closed and convex, then $C^\infty = \text{rec } C$ (see [14, Section 6.G]). The notion of the horizon cone allows us to present a superset of cluster points of the iterates of $T$.

**Theorem 2.6.** Suppose that $\text{Fix } T = \emptyset$, let $x_0 := x \in X$, and set $(\forall n \in \mathbb{N})$ $x_{n+1} := Tx_n$. Then the following hold:

(i) All cluster points of $(x_n/\|x_n\|)_{n \in \mathbb{N}}$ lie in the cone

$$R := (\text{ran } T)^\infty \cap (\text{ran}(T - \text{Id}))^\infty = (\text{ran } T)^\infty \cap \text{rec } (\overline{\text{ran }} (T - \text{Id})).$$

(ii) If $T$ is firmly nonexpansive, then $R = \text{rec } (\overline{\text{ran }} T) \cap \text{rec } (\overline{\text{ran }} (T - \text{Id}))$.

(iii) (cosmic convergence) If $R$ is a ray, then $(Q_n(x))_{n \in \mathbb{N}} = (x_n/\|x_n\|)_{n \in \mathbb{N}}$ converges.
Proof. By Fact [1.1] $\|x_n\| \to \infty$; thus, the quotient sequence is eventually well defined. (i) Let $q$ be a cluster point of $(x_n/\|x_n\|)_{n \in \mathbb{N}}$. It is clear that $q \in (\text{ran}\ T)_{\infty}$. For every $n \in \mathbb{N}$, we have $x_{n+1} - x_0 = \sum_{k=0}^{n}(x_{k+1} - x_k) \in (n+1)\text{ran}(T - \text{Id}) \subseteq \text{cone ran}(T - \text{Id})$; hence, $(x_{n+1} - x_0)/\|x_{n+1}\| \in \text{cone ran}(T - \text{Id})$ and thus $q \in (\text{ran}(T - \text{Id}))_{\infty}$. Since $T$ is nonexpansive, $\text{ran}(\text{Id} - T)$ is convex (see [3, Lemma 4] which yields the right identity. (ii) Since $T$ is firmly nonexpansive, so is $\text{Id} - T$ which implies that $\text{ran}(\text{Id} - (\text{Id} - T)) = \text{ran} T$ is convex (again by [3, Lemma 4]). The conclusion now follows because the horizon cone and recession cone coincide for closed convex sets. (iii) This is clear. 

The following result allows a reduction to lower-dimensional cases.

**Theorem 2.7.** Let $Y$ be a linear subspace of $X$, and let $B : Y \Rightarrow Y$ be maximally monotone. Set $A := BP_Y$ and suppose that $T = J_A := (\text{Id} + A)^{-1}$. Let $x \in X$. Then the following hold:

(i) $A : X \rightrightarrows X$ is maximally monotone and $T = P_{Y^\perp} + J_BP_Y$, where $J_B := (\text{Id} + B)^{-1}$.

(ii) $(\forall n \in \mathbb{N}) \ T^n x = P_{Y^\perp} x + J_B^n(P_Y x)$.

Proof. (i) This follow from [3, Proposition 23.23]. (ii) Clear from (i) and induction. 

We are now in a position to obtain a positive result for proximity operators of certain convex functions.

**Corollary 2.8.** Let $f : X \to ]-\infty, +\infty]$ be convex, lower semicontinuous, and proper on $\mathbb{R}$, let $a \in X$ such that $\|a\| = 1$, set $F : X \to ]-\infty, +\infty] : x \mapsto f(\langle a, x \rangle)$, and suppose that $T = P_F := (\text{Id} + \partial f)^{-1}$ is the associated proximity operator. Let $x \in X$. Then

$$\forall n \in \mathbb{N} \quad T^n x = P_{\{a\}^\perp}(x) + P_f^n(\langle a, x \rangle)a,$$

where $P_f := (\text{Id} + \partial f)$ is the proximity operator of $f$. Consequently, if $f$ is bounded below but without minimizers, then $T$ admits cosmic convergence and $(Q_n(x))_{n \in \mathbb{N}} = (T^n x/\|T^n x\|)_{n \in \mathbb{N}}$ converges either to $+a$ or to $-a$.

Proof. Set $Y := \mathbb{R}a$ and $\varphi : Y \to ]-\infty, +\infty] : \xi a \mapsto f(\xi)$. Then $F = \varphi \circ P_Y$ and $\partial F = (\partial \varphi) \circ P_Y$. By Theorem 2.7 $T x = P_{\{a\}^\perp}(x) + P_f(\langle a, x \rangle a) = P_{\{a\}^\perp}(x) + P_f(\langle a, x \rangle)a$. Concerning the “Consequently” part, observe that if $f$ is bounded below but without minimizers, then $0 \in \text{ran}(\text{Id} - P_f) \setminus \text{ran}(\text{Id} - P_f)$ and the result follows from Theorem 2.1.

We conclude this section with two examples: the first is covered by our analysis but the second is not.

**Example 2.9.** Suppose that $X = \mathbb{R}^2$ and set

$$F : \mathbb{R}^2 \to ]-\infty, +\infty] : (\xi_1, \xi_2) \mapsto \begin{cases} \frac{1}{\xi_1 + \xi_2}, & \text{if } \xi_1 + \xi_2 > 0; \\ +\infty, & \text{otherwise} \end{cases}$$

(16)
and suppose that $T = P_F$. Set $a = (1, 1)/\sqrt{2}$, and $f(\xi) = 1/((\sqrt{2}\xi))$, if $\xi > 0$ and $f(\xi) = +\infty$ otherwise. Then Corollary 2.8 applies and we obtain cosmic convergence; indeed,

$$Q_n(x) = \frac{T_n(x)}{\|T_n(x)\|} \rightarrow a.$$  \hfill (17)

**Example 2.10.** Suppose that $X = \mathbb{R}^2$, set

$$F: \mathbb{R}^2 \rightarrow [-\infty, +\infty] : (\xi_1, \xi_2) \mapsto \begin{cases} \exp(\xi_1), & \text{if } \xi_2 > 0; \\ +\infty, & \text{otherwise,} \end{cases}$$  \hfill (18)

and suppose that $T = P_F$. Then $F$ is not of a form that makes Corollary 2.8 applicable. Interestingly, numerical experiments suggest that

$$Q_n(x) = \frac{T_n x}{\|T_n x\|} \rightarrow (-1, 0);$$  \hfill (19)

however, we do not have a proof for this conjecture.

### 2.4 Poincaré metric and cosmic interpretation

In this section, we provide a different interpretation of our convergence results which also motivates the terminology “cosmic convergence” used above. We first observe that $X$ can be equipped with the *Poincaré metric*, which is defined by

$$\Delta: X \rightarrow X \rightarrow \mathbb{R}: (x, y) \mapsto \left\| \frac{x}{1 + \|x\|} - \frac{y}{1 + \|y\|} \right\|.$$  \hfill (20)

Note that $\Delta$ is just the standard Euclidean metric after the bijection $x \mapsto x/(1 + \|x\|)$ between $X$ and the open unit ball was applied. The metric space $(X, \Delta)$ is not complete; however, regular convergence of sequences in the Euclidean space $X$ is preserved. To complete $(X, \Delta)$, define the equivalence relation

$$x \equiv y :\iff x \in \mathbb{R}^+ y$$  \hfill (21)

on $X \setminus \{0\}$, with equivalence class

$$\text{dir } x := \mathbb{R}^+ x$$  \hfill (22)

for $x \in X \setminus \{0\}$. Following [14], we write

$$\text{hnz } X := \{ \text{dir } x \mid x \in X \setminus \{0\} \} \quad \text{and} \quad \text{csm } X := X \cup \text{hnz } X.$$

\hfill (23)
Here hzn is the *horizon* of X while csm X denotes the *cosmic closure* of X. A convenient representer of dir x is \(x/\|x\|\). These particular representers form the unit *sphere* which we can think of adjoining to the open unit ball. More precisely, we extend \(\Delta\) from \(X \times X\) to \(csm\ X \times csm\ X\) as follows:

\[
x \in X \quad \text{dir} \ y \in \text{hzn} \ X \quad \Rightarrow \quad \Delta(x, \text{dir} \ y) := \Delta(\text{dir} \ y, x) := \frac{x}{1 + \|x\|} - \frac{y}{\|y\|}.
\]

(24)

and

\[
\text{dir} \ x \in \text{hzn} \ X \quad \text{dir} \ y \in \text{hzn} \ X \quad \Rightarrow \quad \Delta(\text{dir} \ x, \text{dir} \ y) := \Delta(\text{dir} \ y, \text{dir} \ x) := \frac{x}{\|x\|} - \frac{y}{\|y\|}.
\]

(25)

Equipped with \(\Delta\), the Bolzano-Weierstrass theorem implies that the cosmic closure csm X is a (sequentially) *compact* metric space; in particular, any sequence \((x_n)_{n \in \mathbb{N}}\) in X such that \(\|x_n\| \to +\infty\) has a convergent subsequence in \((\text{csm} \ X, \Delta)\). In the previous sections, we concentrated on the case when \((x_n)_{n \in \mathbb{N}} = (T^n x)_{n \in \mathbb{N}}\) and \(\text{Fix} \ T = \emptyset\); then, of course, it may or may not be true that the entire sequence converges in \((\text{csm} \ X, \Delta)\). This provides an *a posteriori* motivation for our terminology.

### 2.5 Conclusion

We have taken a closer look at Pazy’s trichotomy theorem for nonexpansive operators. The question whether or not \((T^n x)_{n \in \mathbb{N}}\) always cosmically converges when \(T\) has no fixed points remains open; however, we have presented various partial results indicating that the answer may be affirmative. Future work may focus on analyzing larger classes of nonexpansive operators, e.g., general proximity operators or averaged operators. Another promising avenue may be to use tools from non-euclidean geometry (see [6]). Furthermore, it is presently unclear how the presented results extend to infinite-dimensional settings.

### References

[1] H.H. Bauschke, The composition of finitely many projections onto closed convex sets in Hilbert space is asymptotically regular, *Proceedings of the AMS* 131 (2003), 141–146.

[2] H.H. Bauschke and J.M. Borwein, Dykstra’s alternating projection algorithm for two sets, *Journal of Approximation Theory* 79 (1994), 418–443.
[3] H.H. Bauschke and P.L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, 2011.

[4] H.H. Bauschke, V. Martin-Marquez, S.M. Moffat, and X. Wang, Compositions and convex combinations of asymptotically regular firmly nonexpansive mappings are also asymptotically regular, *Fixed Point Theory and Applications* 2012:53. Available at http://www.fixedpointtheoryandapplications.com/content/2012/1/53

[5] GeoGebra, http://www.geogebra.org

[6] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, 1984.

[7] B.S. Mordukhovich and N.M. Nam, *An Easy Path to Convex Analysis and Applications*, Morgan & Claypool Publishers, 2014.

[8] A. Pazy, Asymptotic behavior of contractions in Hilbert space, *Israel Journal of Mathematics* 9 (1971), 235–240.

[9] A.T. Plant and S. Reich, The asymptotics of nonexpansive iterations, *Journal of Functional Analysis* 54 (1983), 308–319.

[10] S. Reich, Asymptotic behavior of contractions in Banach spaces, *Journal of Mathematical Analysis and Applications* 44 (1973), 57–50.

[11] S. Reich, On the asymptotic behavior of nonlinear semigroups and the range of accretive operators I, *Journal of Mathematical Analysis and Applications* 79 (1981), 113–126.

[12] S. Reich, On the asymptotic behavior of nonlinear semigroups and the range of accretive operators II, *Journal of Mathematical Analysis and Applications* 87 (1982), 134–146.

[13] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[14] R.T. Rockafellar and R.J-B. Wets, *Variational Analysis*, Springer-Verlag, corrected 3rd printing, 2009.

[15] E.H. Zarantonello, Projections on convex sets in Hilbert space and spectral theory, in: E.H. Zarantonello (editor), *Contributions to Nonlinear Functional Analysis*, Academic Press, pp. 237–424, 1971.