In this paper we address the question of coexistence of superconductivity and ferromagnetism in the high temperature superconductor RuSr$_2$GdCu$_2$O$_{8-δ}$. Using a field theoretical approach we study a one-fermion effective model of a ferromagnetic superconductor in which the quasiparticles responsible for the fermagnetism form the Cooper pairs as well. We discuss the physical features which are different in this model and the standard BCS model and consider their experimental consequences.

Recently an itinerant ferromagnet undergoing a high temperature superconducting transition was discovered in the copper oxide compound RuSr$_2$GdCu$_2$O$_{8-δ}$ (Ru-1212) and the experimental studies have revealed that the ferromagnetic state exists even below the superconducting transition. This prompts the interesting question of the possible many-body itinerant fermionic systems supporting both types of broken symmetry.

The search for ferromagnetic superconductors goes back to the sixties when superconducting materials with magnetic impurities were studied. The research in this direction has led to the works of Larkin and Ovchinnikov, and Fulde and Ferrell, who studied this hybridization one obtains two new fermionic fields. The tunneling between the two bands. As a result of transformation to diagonalize the kinetic term, including the change term and a complex scalar field, the superconducting fluctuations. Performing the Gaussian integration we obtain two new fermionic fields. Both fermions have ferromagnetic and pairing interactions (either strong/weak or weak/strong) and there are cross terms. The approximation of this paper involves neglecting the cross terms. Then one has two decoupled one-fermion problems. The one of interest in this paper is for the mostly superconducting fermions, because their exchange coupling is much smaller than the exchange coupling in the Ru layer, but could be comparable to Tc.

In this paper for the first time the self consistent equations for the superconducting gap and the magnetization are solved simultaneously in the mean field limit. We study the one fermion model of a ferromagnetic superconductor leaving the two fermion model with the cross terms for further investigation. This case is relevant to the doped RuO$_2$ layers in Ru-1212, where the magnetism becomes itinerant and the RuO$_2$ layers participate in the transport properties of the material.

Our model Hamiltonian is

$$H = \mu N = \int d^3r c^\dagger_\sigma(\vec{r}) \left( -\frac{1}{2m^*} \vec{\nabla}^2 - \mu \right) c_\sigma(\vec{r})$$

$$-\frac{J}{2} \int d^3r \vec{S}(\vec{r}) \cdot \vec{S}(\vec{r}) - \int d^3rc^\dagger_\uparrow(\vec{r})c^\dagger_\downarrow(\vec{r})c_\downarrow(\vec{r})c_\uparrow(\vec{r}),$$

where $c_\sigma(\vec{r})$ are the spin $\sigma$ fermion fields, $\vec{S} = \frac{1}{2}c^\dagger_\uparrow \vec{\tau}_{\sigma\uparrow}c_\uparrow - c^\dagger_\downarrow \vec{\tau}_{\sigma\downarrow}c_\downarrow$ is the spin field, $\vec{\tau}_{\sigma\tau}$ are the Pauli matrices, and $\mu$ is the chemical potential. The exchange interaction is ferromagnetic ($J > 0$) and the four fermion interaction is attractive ($g > 0$). This is the simplest model which leads to the coexistence of ferromagnetism and superconductivity.

The partition function of the model can be written as a functional integral over the Grassmann fields $c(\vec{r},\tau)$ and $\bar{c}(\vec{r},\tau)$ using a Hubbard-Stratonovich transformation of the exchange term and a complex scalar field $f(\vec{r},\tau)$ using a Hubbard-Stratonovich transformation of the second term in Eq.(1). The vector field describes the fluctuations of the magnetization, while the complex scalar field describes the superconducting fluctuations. Performing the Gaussian integral over the fermionic fields we obtain the partition function of the model as an integral over $\vec{M}$, $\vec{f}$ and $\bar{f}$ which we calculate using the steepest descent around the mean field solutions $\vec{M} = (0,0,M)$ and $\Delta = g < f >$. Here $M = -<\vec{S}^2>$ defines the magnetization of the system. The mean field equations are

$$JM + \frac{\delta F_{eff}}{\delta M} = 0, \quad 2|\Delta| \frac{\delta F_{eff}}{\delta |\Delta|} = 0,$$

where $F_{eff}$ is the Free energy of a theory with the effective Hamiltonian

$$H_{eff} = \sum_{\vec{p}} \left[ c^\dagger_{\uparrow p}\bar{c}_{\uparrow p} + c^\dagger_{\downarrow p}\bar{c}_{\downarrow p} + \Delta \bar{c}_{-\bar{p}\uparrow}c_{\bar{p}\uparrow} + \Delta \bar{c}_{-\bar{p}\downarrow}c_{\bar{p}\downarrow} \right]$$
\[\epsilon_p^\uparrow = \frac{p^2}{2m^*} - \mu + \frac{JM}{2}, \quad \epsilon_p^\downarrow = \frac{p^2}{2m^*} - \mu - \frac{JM}{2}. \tag{3}\]

Here the fermionic effective Hamiltonian \( H_{eff} \) is obtained after the Hubbard-Stratonovich transformations and setting the fields at their mean field values.

Next we diagonalize the effective Hamiltonian using a Bogoliubov transformation. After the transformation the new dispersion relations are

\[E_p^\alpha = \frac{JM}{2} + \sqrt{\epsilon_p^\alpha + |\Delta|^2}, \quad E_p^\beta = \frac{JM}{2} - \sqrt{\epsilon_p^\beta + |\Delta|^2}, \tag{4}\]

where \( \epsilon_p^\alpha = \frac{p^2}{2m^*} - \mu \). Then the mean field equations take the form

\[M = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left( 1 - n_p^\alpha - n_p^\beta \right), \tag{5}\]

\[1 = \frac{g}{2} \int \frac{d^3p}{(2\pi)^3} \frac{n_p^\beta - n_p^\alpha}{\sqrt{\epsilon_p^\alpha + |\Delta|^2}} \tag{6}\]

where \( n_p^\alpha \) and \( n_p^\beta \) are the momentum distribution function of the Bogoliubov fermions and we have assumed that \(|\Delta| \neq 0\).

From Eqn.(4) one sees that for \( M \geq 0 \) (the convention that we will use here) \( E_p^\alpha > 0 \) for all momenta \( p \) and therefore for \( T = 0 \), \( n_p^\alpha = 0 \). For \( E_p^\beta \) there are two possibilities. When \( JM < 2|\Delta| \), \( E_p^\beta < 0 \) for all \( p \) and therefore \( n_p^\beta = 1 \) for all \( p \). Substitution of this in Eqn.(5) leads to \( M = 0 \). Therefore the only solution of the mean field equations which allows for the coexistence of ferromagnetism and superconductivity is in the case when \( JM > 2|\Delta| \) which we will assume. Then the equation \( E_p^\beta = 0 \) has two solutions:

\[p_F^\pm = \sqrt{2m^*\mu \pm m^*\sqrt{(JM)^2 - 4|\Delta|^2}} \tag{7}\]

The dispersion of the \( \beta \) fermion is positive when \( p_F^- < p < p_F^+ \) and is negative in the complementary interval. The \( p \) dependence of \( E_p^\beta \) is depicted in Fig.1. With this in mind the Eqn.(5) and (6) at \( T = 0 \) have the form

\[M = \frac{1}{12\pi^2} \left[ (p_F^+)^3 - (p_F^-)^3 \right], \tag{8}\]

\[1 = \frac{g}{(2\pi)^2} \left( \int_0^\infty dp \frac{p^2}{\sqrt{\epsilon_p^\alpha + |\Delta|^2}} - \int_{p_F^-}^{p_F^+} dp \frac{p^2}{\sqrt{\epsilon_p^\alpha + |\Delta|^2}} \right). \tag{9}\]

It is difficult to solve analytically these equations, however when \( JM \) is greater, but close to \( 2|\Delta| \), \( p_F^\pm \) is approximately equal to \( p_F \) and therefore \( M \) is small as follows from Eqn.(8). In this case one can expand the r.h.s. of Eqn.(7) in the small parameter \( \sqrt{(JM)^2 - 4|\Delta|^2} \) obtaining

\[p_F^\pm = p_F \pm \frac{m^*}{2p_F} \sqrt{(JM)^2 - 4|\Delta|^2}, \tag{10}\]

where \( p_F = \sqrt{2m^*\mu} \). Substitution of these expressions in Eqn.(8) shows that in this approximation the magnetization is linear in \(|\Delta|\), namely

\[M = \frac{2}{J} \frac{r}{\sqrt{r^2 - 1}}|\Delta|, \tag{11}\]

where \( r = JM^*p_F/4\pi^2 \) and this expression is valid for large \( r \) (i.e. \( JM - 2|\Delta| \to 0^+ \)).

As in the standard BCS theory of superconductivity, the pairing of the quasiparticles occurs in the vicinity of \( p_F \), which must include the interval between \( p_F^- \) and \( p_F^+ \). Then the integration in the first integral on the r.h.s. of Eqn.(9) is limited to a shell of width \( 2\Lambda \), i.e.

\[1 = \frac{g}{(2\pi)^2} \left( \int_{p_F^-}^{p_F^+} dp \frac{p^2}{\sqrt{\epsilon_p^\alpha + |\Delta|^2}} - \int_{p_F^-}^{p_F^+} dp \frac{p^2}{\sqrt{\epsilon_p^\alpha + |\Delta|^2}} \right). \tag{12}\]

Here \( p_F + \Delta > p_F^+ \) and \( p_F - \Delta < p_F^- \).

\[\begin{tikzpicture}
  \node at (0,0) {\includegraphics[width=\textwidth]{fig1.png}};
\end{tikzpicture}\]

FIG. 1. \( E_p^\beta \) as a function of \( p \).

Substitution of the approximate expressions for \( p_F^\pm \) from Eqn.(10) in the second term in the r.h.s. of Eqn.(12) and using the expression for the magnetization from Eqn.(11) we see that this term is independent of \(|\Delta|\) and only leads to a renormalization of the cutoff \( \Lambda \). In that approximation the solution is

\[|\Delta| = \sqrt{\frac{r - 1}{r + 1}} \Lambda e^{-\frac{2\pi^2}{sm^*p_F}}, \tag{13}\]

\[M = \frac{2}{J} \frac{r}{\sqrt{r^2 + 1}} \Lambda e^{-\frac{2\pi^2}{sm^*p_F}}. \tag{14}\]
When the magnetization increases the domain of integration in the second integral on the r.h.s. of Eqn.(12) can exceed the size of the domain around \( p_F \) where the pairing occurs and which is the integration domain in the first integral of the same equation. In that case the second integral dominates and this leads to the absence of solutions with a finite gap. Taking the limiting case when the two integration domains are equal, i.e. \( p_F + \Lambda = p_F' \) and \( p_F - \Lambda = p_F' \) where \( p_F' \) are the values of the momenta from Eqn.(10) with \( \Delta = 0 \), we obtain the critical value of the magnetization

\[
M_c = \frac{\Lambda}{m^*J}(2p_F + \Lambda) \tag{15}
\]

above which the superconductivity disappears besides the existence of an attractive four fermion interaction.

Next we calculate the distribution functions \( n^\uparrow_p \) and \( n^\downarrow_p \) of the spin up and spin down quasiparticles. In terms of the distribution functions of the Bogoliubov fermions these momentum distribution functions are

\[
\begin{align*}
n^\uparrow_p &= u_p^2n^\alpha_p + v_p^2n^\beta_p, \\
n^\downarrow_p &= u_p^2(1 - n^\beta_p) + v_p^2(1 - n^\alpha_p),
\end{align*}
\tag{16}
\]

where \( u_p^2 \) and \( v_p^2 \) are the coefficients in the Bogoliubov transformation. They are independent of the magnetization and have the same form as in the BCS theory.

At zero temperature \( n^\alpha_p \) is zero and \( n^\beta_p = \theta(p_F - p) + \theta(p - p_F) \). Then the spin up and spin down quasiparticles have the following momentum distribution functions

\[
\begin{align*}
n^\uparrow_p &= \theta(p_F - p) + \theta(p - p_F') \\
n^\downarrow_p &= \theta(p_F' - p) + \theta(p_F' - p) + v_p^2[\theta(p_F - p) + \theta(p - p_F')].
\end{align*}
\tag{17}
\]

The functions are depicted on Fig.2.

The appearance of the Fermi surfaces of the Bogoliubov fermion \( \beta \) is unexpected in the superconducting phase, but it is a necessary condition for the existence of itinerant ferromagnetism. Therefore in the case of coexistence of superconductivity and ferromagnetism caused by the same quasiparticles the existence of the two Fermi surfaces is a generic property of this state. These Fermi surfaces are reflected in the spin up and spin down momentum distribution functions as well as in the anomalous Green’s functions. It is easy to show that the anomalous Green’s function,

\[
\mathcal{F}(\tau - \tau', \vec{p}) = -\langle T_{c\downarrow}(\tau, -\vec{p})c_{\uparrow}(\tau', \vec{p}) \rangle,
\tag{19}
\]

in the case \( \tau = \tau' \) is

\[
\mathcal{F}(0, \vec{p}) = \frac{|\Delta|}{2\sqrt{\epsilon_p^2 + |\Delta|^2}} \tag{20}
\]

when \( 0 < p < p_F' \) and \( p > p_F' \) and is zero when the momentum \( p \) is between the two Fermi surfaces \( p_F' \) and \( p > p_F' \).

\[
\begin{align*}
\text{FIG. 2. The zero temperature momentum distribution functions for spin up and spin down fermion.}
\end{align*}
\]

The existence of the Fermi surfaces, leads to different thermodynamic properties of the system, compared to the standard BCS theory. The specific heat has a linear temperature dependence at low temperatures as opposed to the exponential decrease of the specific heat in the BCS theory

\[
C = \frac{2\pi^2}{3} N(0) T. \tag{21}
\]

Here

\[
N(0) = \frac{m^*J}{4\pi^2} \frac{p_F^+ + p_F^-}{\sqrt{1 - \frac{4|\Delta|^2}{J^2m^*}}} = N^+(0) + N^-(0) \tag{22}
\]

is the sum of the density of states on the two Fermi surfaces of the Bogoliubov fermion \( \beta \). When the magnetization is small, from Eqns.(10) and (11) follow that the density of states increases with \( r \) as \( N(0) \to \frac{m^*J}{4\pi^2}r p_F \), as opposed to the case of ordinary weak ferromagnets, where the density of states is \( N(0) = \frac{m^*}{2\pi^2}p_F \) in this limit. Hence, the specific heat is large even at very low temperatures. In the case of a superconductor in an external magnetic field \( \vec{B} \) although there are gapless fermionic excitations the specific heat is not linear as opposed to our case. This can also be contrasted with some of the unconventional superconductors which have power law dependence of the specific heat on the temperature, depending on the nodal structure of the gap function.
Another consequence of the existence of the Fermi surfaces is the existence of paramagnons which describes the longitudinal spin fluctuations \([11]\). They exist in ferromagnetic normal metals and in our theory they survive even in the ferromagnetic superconducting phase. Their propagator is given by

\[
D_l(\omega, p) = \frac{1}{\delta + a \frac{|\omega|}{p} + b p^2},
\]

where \(a, b, \) and \(\delta\) are constants. The constant

\[
a = \frac{J \pi}{4} \left(1 - \frac{4|\Delta|^2}{f^2 M^2}\right)^{-1/2} \left(\frac{N^+(0)}{v_F^+} + \frac{N^-(0)}{v_F^-}\right)
\]

defines the analytical properties of the paramagnon and is different from zero because of the existence of the Fermi surfaces. The constant \(\delta\) is

\[
\delta = 1 - \frac{J}{2} N(0)
\]

and \(b\) is a positive constant. As we mentioned earlier, the density of states, Eqn.(22), increases as the magnetization, \(M\) decreases and therefore, at small, but finite value of the magnetization, \(M = M_0\), \(\delta\) becomes zero, as opposed to the weak ferromagnetic metals where \(\delta\) becomes zero at zero magnetization. In the case of coexistence of the superconductivity and ferromagnetism the superconductivity prevents the magnetization from becoming arbitrarilly small, because when the magnetization is smaller than the critical value \(M_0\), \(\delta\) is negative and the paramagnon fluctuations lead to an instability of that phase. The superconducting phase, with zero magnetization (BCS like regime) the spin fluctuations of the paramagnon type are absent.

Recently, a band structure calculation was performed by Pickett et al. \([5]\) and they have studied the origin of the superconducting state in the Ru-1212 compound. In our paper we considered the possibility of the coexistence of ferromagnetism and superconductivity and the physical features of such a system. We arrived at a system of self consistent equations for the magnetization and the superconducting gap, and solved analytically these equations at small magnetizations. The solutions with coexistence of superconductivity and ferromagnetism describe Bogoliubov fermions one of which has two Fermi surfaces. Therefore the spin up and spin down quasiparticles have two Fermi surfaces each. The thermodynamic properties of the coexistence phase are different from the standard BCS theory. The specific heat has a linear temperature dependence as in normal ferromagnetic metals, but increases anomalously at small magnetizations. These results are obtained in a mean field approximation, but they are generic for the coexistence state and can be used as a starting point for a calculations beyond mean field. In our model the quantum critical point is dressed, i.e. the superconducting state occurs at zero magnetization, because the superconducting gap is generated not by the spin fluctuations, but by some other means. This is to be contrasted with the theory of spin fluctuations mediated pairing in weak ferromagnetic metals \([12]\) where the quantum critical point is naked and the superconducting ferromagnetic critical temperatures go to zero at the quantum critical point. In this paper we have considered only uniform states. However, periodic solutions (like the LOFF state in the magnetic impurity case) will likely exist for certain regions of parameters, but possibly involving periodic magnetic structures as well as modulated superconducting order parameter.

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