Unilateral global bifurcation and nodal solutions for the $p$-Laplacian with sign-changing weight

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In this paper, we shall establish a Dancer-type unilateral global bifurcation result for a class of quasilinear elliptic problems with sign-changing weight. Under some natural hypotheses on perturbation function, we show that $(\mu_k^{\nu}(p), 0)$ is a bifurcation point of the above problems and there are two distinct unbounded continua, $(C_k^{\nu})^+$ and $(C_k^{\nu})^-$, consisting of the bifurcation branch $C_k^{\nu}$ from $(\mu_k^{\nu}(p), 0)$, where $\mu_k^{\nu}(p)$ is the $k$th positive or negative eigenvalue of the linear problem corresponding to the above problems, $\nu \in \{+, -\}$. As applications of the above unilateral global bifurcation result, we study the existence of nodal solutions for a class of quasilinear elliptic problems with sign-changing weight. Moreover, based on the bifurcation result of Drábek and Huang (1997) Dai G, Ma R. Unilateral global bifurcation phenomena and nodal solutions for $p$-Laplacian. J. Differ. Equ. 2012;252:2448–2468., we study the existence of one-sign solutions for a class of high-dimensional quasilinear elliptic problems with sign-changing weight.

Keywords: $p$-Laplacian; unilateral global bifurcation; nodal solutions; sign-changing weight

AMS Subject Classifications: 35B05; 35B32; 35J25

1. Introduction

In [1], Rabinowitz established Rabinowitz’s unilateral global bifurcation theory. However, as pointed out by Dancer [2,3] and López-Gómez [4], the proofs of these theorems contain gaps. Fortunately, Dancer [2] gave a corrected version of the unilateral global bifurcation theorem. In 1997, Drábek and Huang [5] proved a Dancer-type bifurcation theorem (Theorem 4.5, [5]) in which the continua bifurcated from the principle eigenvalue for a high-dimensional $p$-Laplacian problem with sign-changing weight in $\mathbb{R}^N$. However, no any information on the high eigenvalue for $p$-Laplacian problem with sign-changing weight.

For the case of definite weight, Dai and Ma [6] established a Dancer-type unilateral global bifurcation result for the one-dimensional $p$-Laplacian. In [7], Girg and Takáč proved a Dancer-type bifurcation theorem for a high-dimensional $p$-Laplacian equation.

It is the main purpose of this paper to establish a similar result to Dai and Ma’s about the continuum of radial solutions for the following $N$-dimensional $p$-Laplacian problem on the unit ball of $\mathbb{R}^N$ with $N \geq 1$ and $1 < p < +\infty$.

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where $B$ is the unit open ball of $\mathbb{R}^N$, $\varphi_p(s) = |s|^{p-2}s$, $m \in M(B)$ is a sign changing function with

$$M(B) = \{m \in C(\overline{B}) \text{ is radially symmetric } |\text{meas}\{x \in B, m(x) > 0\} \neq 0\}.$$  

$g : B \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition in the first two variables and radially symmetric with respect to $x$. Moreover, we also assume that $g$ satisfies $g(x, s; 0) \equiv 0$ for any $(x, s) \in B \times \mathbb{R}$.

It is clear that the radial solutions of (1.1) is equivalent to the solutions of the following problem

$$\begin{cases} -\text{div}(\varphi_p(\nabla u)) = \mu m(x)\varphi_p(u) + g(x, u; \mu), & \text{in } B, \\ u = 0, & \text{on } \partial B, \end{cases}$$

(1.1)

where $r = |x|$ with $x \in B$, $m \in M(I)$ is a sign changing with $I = (0, 1)$ and

$$M(I) = \{m \in C(\overline{I}) \text{ is radially symmetric } |\text{meas}\{r \in I, m(r) > 0\} \neq 0\}.$$  

We also assume that the perturbation function $g$ satisfies the following hypotheses

$$\lim_{s \to 0} \frac{g(r, s; \mu)}{|s|^{p-1}} = 0$$

(1.3)

uniformly for a.e. $r \in I$ and $\mu$ on bounded sets.

Under the conditions of $m \in M(I)$ and (1.3), we shall show that $(\mu^+_{k}(p), 0)$ is a bifurcation point of (1.2) and there are two distinct unbounded continua, $(C^+_{k})$ and $(C^-_{k})$, consisting of the bifurcation branch $C^+_{k}$ from $(\mu^+_{k}(p), 0)$, where $\mu^+_{k}(p)$ is the $k$th positive or negative eigenvalue of the linear problem corresponding to (1.2), where $v \in \{+, -\}$.

On the basis of the unilateral global bifurcation result (see Theorem 3.2), we investigate the existence of radial nodal solutions for the following $p$-Laplacian problem

$$\begin{cases} -\text{div}(\varphi_p(\nabla u)) = \gamma m(x) f(u), & \text{in } B, \\ u = 0, & \text{on } \partial B, \end{cases}$$

(1.4)

where $f \in C(\mathbb{R}, \mathbb{R})$, $\gamma$ is a parameter.

It is clear that the radial solutions of (1.4) is equivalent to the solutions of the following problem

$$\begin{cases} \left(r^{N-1}|u'|^{p-2}u'ight)' + \gamma r^{N-1}m(r) f(u) = 0, & r \in I, \\ u'(0) = u(1) = 0, \end{cases}$$

(1.5)

where $r = |x|$ with $x \in B$.

It is well known that when $m(r) \equiv 1$ and $f(r, u) = \lambda \varphi_p(u)/\gamma$, problem (1.5) has a non-trivial solution if and only if $\lambda$ is an eigenvalue of the following problem

$$\begin{cases} \left(r^{N-1}|u'|^{p-2}u'ight)' + \lambda r^{N-1}\varphi_p(u) = 0, & r \in I, \\ u'(0) = u(1) = 0. \end{cases}$$

(1.6)

In particular, when $\lambda = \lambda_{k}(p)$, there exist two solutions $u^+_k$ and $u^-_k$, such that $u^+_k$ has exactly $k - 1$ zeros in $I$ and is positive near $0$, and $u^-_k$ has exactly $k - 1$ zeros in $I$ and is negative near $0$ (see [8], Theorem 1.5.3).
When \( p = 2, N = 1 \) and \( m(r) \geq 0 \), Ma and Thompson [9] considered the interval of \( \gamma \), in which there exist nodal solutions of (1.5) under some suitable assumptions on \( f \). The results of the above have been extended to the the case of weight function changes its sign by Ma and Han [10]. The results they obtained extended some well-known theorems of the existence of positive solutions for related problems [11–13] and sign-changing solutions.[14] For the case \( p \neq 2 \) but \( N = 1, m(r) \geq 0 \), Dai and Ma [6] proved the existence of nodal solutions for (1.5).

However, few results on the existence of radial nodal solutions, even positive solutions, have been established for \( N \)-dimensional \( p \)-Laplacian problem with sign-changing weight \( m(r) \) on the unit ball of \( \mathbb{R}^N \). In this paper, we shall establish a similar result to Ma and Thompson [9] for \( N \)-dimensional \( p \)-Laplacian problem with sign-changing weight. Problem with sign-changing weight arises from the selection-migration model in population genetics. In this model, \( m(r) \) changes sign corresponding to the fact that an allele \( A_1 \) holds an advantage over a rival allele \( A_2 \) at some points and is at a disadvantage at others; the parameter \( r \) corresponds to the reciprocal of diffusion, for detail, see [15]. For the applications of nodal solutions, see Lazer and McKenna [16] and Kurth [17].

In high-dimensional general domain case, based on Drábek and Huang’s results (note their results also valid for bounded smooth domain), we shall investigate the existence of one-sign solutions for the problem (1.4) with \( 1 < p < N \) and the general smooth domain \( \Omega \subset \mathbb{R}^N \) with \( N \geq 2 \), i.e.

\[
\begin{cases}
- \text{div} (\varphi_p(\nabla u)) = \gamma m(x) f(u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\] (1.7)

By a solution of (1.7), we understand \( u \in W^{1,p}_0(\Omega) \) satisfying (1.7) in the weak sense.

The rest of this paper is arranged as follows. In Section 2, we establish the eigenvalue theory of second order \( p \)-Laplacian Dirichlet boundary value problem in the radial case with sign-changing weight. In Section 3, we establish the unilateral global bifurcation theory for (1.2). In Section 4, we prove the existence of nodal solutions for (1.5). In Section 5, we study the existence of one-sign solutions for (1.7).

2. Some preliminaries

In [18], by Prüfer transformation, Meng, Yan and Zhang established the spectrum of one-dimensional \( p \)-Laplacian with an indefinite integrable weight. When \( m \equiv 1 \), using oscillation method, Peral [8] established the eigenvalue theory of \( N \)-dimensional \( p \)-Laplacian on the unit ball. However, applying their methods to \( N \)-dimensional \( p \)-Laplacian on the unit ball with indefinite weight is very difficult, even cannot be used. In [19], using variational method, Anane, Chakrone and Monssa established the spectrum of one-dimensional \( p \)-Laplacian with an indefinite weight. While, we do not know whether or not the eigenvalue function \( \mu_k^p(p) \) is continuous with respect to \( p \), which was obtained by Anane, Chakrone and Monssa. In this section, by similar method of Anane, Chakrone and Monssa’s, we can establish the eigenvalue theory of second-order \( p \)-Laplacian Dirichlet boundary value problem in the radial case with indefinite weight. It is well known the continuity of eigenvalues with respect to \( p \) is very important in the studying of the global bifurcation phenomena for \( p \)-Laplacian problems, see [8,20–22]. In this section, we shall also show that \( \mu_k^p(p) \) is continuous with respect to \( p \). Moreover, we also establish a key lemma which will be used in Section 4.
Applying the similar method to prove [19, Theorem 1] with obvious changes, we can obtain the following theorem.

**Theorem 2.1** Assume $m \in M(I)$. The eigenvalue problem

$$
\begin{cases}
(r^{N-1} |u'|^{p-2} u')' + \mu m(r) r^{N-1} |u|^{p-2} u = 0, & r \in I, \\
u'(0) = u(1) = 0
\end{cases}
$$

(2.1)

has two infinitely many simple real eigenvalues

$$0 < \mu_1^+(p) < \mu_2^+(p) < \cdots < \mu_k^+(p) < \cdots, \quad \lim_{k \to +\infty} \mu_k^+(p) = +\infty,$$

$$0 > \mu_1^-(p) > \mu_2^-(p) > \cdots > \mu_k^-(p) > \cdots, \quad \lim_{k \to +\infty} \mu_k^-(p) = -\infty$$

and no other eigenvalues. Moreover,

1. Every eigenfunction corresponding to eigenvalue $\mu_k^+(p)$, has exactly $k - 1$ zeros.
2. For every $k$, $\mu_k^+(m)$ verifies the strict monotonicity property with respect to the weight $m$.

**Remark 2.1** Using the Gronwall inequality,[23] we can easily show that all zeros of eigenfunction corresponding to eigenvalue $\mu_1^+(p)$ is simple.

We first show that the principle eigenvalue function $\mu_1^+(1, +\infty) \to \mathbb{R}$ is continuous.

**Theorem 2.2** The eigenvalue function $\mu_1^+ : (1, +\infty) \to \mathbb{R}$ is continuous.

**Proof** The proof is similar to that of [20], but we give a rough sketch of the proof for reader’s convenience. We only show that $\mu_1^+ : (1, +\infty) \to \mathbb{R}$ is continuous since the case of $\mu_1^-$ is similar. In the following proof, we shall shorten $\mu_1^+$ to $\mu_1$.

From the variational characterization of $\mu_1(p)$, it follows that

$$
\mu_1(p) = \sup \left\{ \mu > 0 \bigg| \mu \int_B m(x) |u|^p \, dx \leq \int_B |\nabla u|^p \, dx, \text{ for all } u \in C^\infty_{r,c}(B) \right\},
$$

(2.2)

where $C^\infty_{r,c}(B) = \{ u \in C^\infty_c(B) | u \text{ is radially symmetric} \}$.

Let $\{ p_j \}_{j=1}^\infty$ be a sequence in $(1, +\infty)$ convergent to $p > 1$. We shall show that

$$
\lim_{j \to +\infty} \mu_1(p_j) = \mu_1(p).
$$

(2.3)

To do this, let $u \in C^\infty_{r,c}(I)$. Then, from (2.2),

$$
\mu_1(p_j) \int_B m(x) |u|^{p_j} \, dx \leq \int_B |\nabla u|^{p_j} \, dx.
$$

On applying the dominated convergence theorem we find that

$$
\limsup_{j \to +\infty} \mu_1(p_j) \int_B m(x) |u|^p \, dx \leq \int_B |\nabla u|^p \, dx.
$$

(2.4)

Relation (2.4), the fact that $u$ is arbitrary and (2.2) yield

$$
\limsup_{j \to +\infty} \mu_1(p_j) \leq \mu_1(p).
$$

(2.5)
Thus, to prove (2.3) it suffices to show that
\[
\liminf_{j \to +\infty} \mu_1(p_j) \geq \mu_1(p).
\]  
(2.5)

Let \( \{p_k\}_{k=1}^{\infty} \) be a subsequence of \( \{p_j\}_{j=1}^{\infty} \) such that \( \lim_{k \to +\infty} \mu_1(p_k) = \liminf_{j \to +\infty} \mu_1(p_j) \).

Let us fix \( \varepsilon_0 > 0 \) so that \( p - \varepsilon_0 > 1 \) and for each \( 0 < \varepsilon < \varepsilon_0 \), \( W^{1,p-\varepsilon}_r(B) \) is compactly embedded into \( L^{p+\varepsilon}_r(B) \), where \( W^{1,p-\varepsilon}_r(B) = \{ u \in W^{1,p-\varepsilon}_0(B) | u \text{ is radially symmetric} \} \), \( L^{p+\varepsilon}_r(B) = \{ u \in L^{p+\varepsilon}_r(B) | u \text{ is radially symmetric} \} \). For \( k \in \mathbb{N} \), let us choose \( u_k \in W^{1,p_k}_r(B) \) such that
\[
\int_B |\nabla u_k|^{p_k} \, dx = 1 \quad (2.6)
\]
and
\[
\int_B |\nabla u_k|^{p_k} \, dx = \mu_1(p_k) \int_B m(x) |u_k|^{p_k} \, dx. \quad (2.7)
\]

For \( 0 < \varepsilon < \varepsilon_0 \), there exists \( k_0 \in \mathbb{N} \) such that \( p - \varepsilon < p_k < p + \varepsilon \) for any \( k \geq k_0 \). Thus, for \( k \geq k_0 \), (2.6) and Hölder’s inequality imply that
\[
\int_B |\nabla u_k|^{p-\varepsilon} \, dx \leq |B|^{\frac{p_k-p+\varepsilon}{p_k}}, \quad (2.8)
\]
where \( |B| \) denotes the measure of \( B \). This shows that \( \{u_k\}_{k=k_0}^{\infty} \) is a bounded sequence in \( W^{1,p-\varepsilon}_r(B) \). Passing to a subsequence if necessary, we can assume that \( u_k \to u \) in \( W^{1,p-\varepsilon}_r(B) \) and hence that \( u_k \to u \) in \( L^{p+\varepsilon}_r(B) \). Furthermore, \( u \in L^p_r(B) \) and \( u_k \to u \) in \( L^{p_k}_r(B) \) for \( k \geq k_0 \). It follows that
\[
\left| \int_B |u_k|^{p_k} \, dx - \int_B |u|^{p_k} \, dx \right|
\leq \int_B p_k |u + \theta u_k|^{p_k-1} |u_k - u| \, dx
\leq (p + \varepsilon) \left( \int_B |u + \theta u_k|^{p_k} \, dx \right)^{\frac{p_k-1}{p_k}} \left( \int_B |u_k - u|^{p_k} \, dx \right)^{1/p_k}
\leq (p + \varepsilon) \left( \|u\|_{p_k} + \|u_k\|_{p_k} \right)^{p_k-1} \left( \int_B |u_k - u|^{p_k} \, dx \right)^{1/p_k} \to 0
\]
as \( k \to +\infty \). It is clear that
\[
\int_B |u|^{p_k} \, dx - \int_B |u|^p \, dx \to 0 \quad \text{as} \quad k \to +\infty.
\]

Thus,
\[
\int_B |u_k|^{p_k} \, dx \to \int_B |u|^p \, dx.
\]

Similarly, we can also obtain that
\[
\int_B m^+(x) |u_k|^{p_k} \, dx \to \int_B m^+(x)|u|^p \, dx
\]
and
\[
\int_B m^-(x) |u_k|^p \, dx \to \int_B m^-(x)|u|^p \, dx,
\]
where \(m^+(x) = \max\{m(x), 0\}\), \(m^-(x) = -\min\{m(x), 0\}\). Therefore,
\[
\int_B m(x) |u_k|^p \, dx = \int_B m^+(x) |u_k|^p \, dx - \int_B m^-(x) |u_k|^p \, dx
\]
\[
\to \int_B m^+(x)|u|^p \, dx - \int_B m^-(x)|u|^p \, dx
\]
\[
= \int_B m(x)|u|^p \, dx.
\]  
(2.9)

We note that (2.6) and (2.7) imply that
\[
\mu_1(p_k) \int_B m(x) |u_k|^p \, dx = 1
\]  
(2.10)

for all \(k \in \mathbb{N}\). Thus, letting \(k \to +\infty\) in (2.10) and using (2.9), we find that
\[
\liminf_{j \to +\infty} \mu_1(p_k) \int_B m(x)|u|^p \, dx = 1.
\]  
(2.11)

On the other hand, since \(u_k \rightharpoonup u\) in \(W^{1,p-\varepsilon}_r(B)\), from (2.8) we obtain that
\[
\|\nabla u\|_{p-\varepsilon} \leq \liminf_{k \to +\infty} \|\nabla u_k\|_{p-\varepsilon} \leq |B|^\frac{1}{p'}.
\]

Now, letting \(\varepsilon \to 0^+\) and applying Fatou’s Lemma we find that
\[
\|\nabla u\|_{p} \leq 1.
\]  
(2.12)

Hence \(u \in W^{1,p}_r(B)\), here \(W^{1,p}_r(B)\) denotes the radially symmetric subspace of \(W^{1,p}(B)\). We claim that actually \(u \in W^{1,p-\varepsilon}_r(B)\) for each \(0 < \varepsilon < \varepsilon_0\). For \(\phi \in C_0^\infty(\mathbb{R}^N)\), it is easy to see that
\[
\left| \int_B u \frac{\partial \phi}{\partial x_i} \, dx \right| \leq \|\nabla u\|_{p-\varepsilon} \|\phi\|_{(p-\varepsilon)'} = \|\phi\|_{(p-\varepsilon)'}, \ i = 1, \ldots, N.
\]

Then, letting \(\varepsilon \to 0^+\) we obtain that
\[
\left| \int_B u \frac{\partial \phi}{\partial x_i} \, dx \right| \leq \|\nabla u\|_p \|\phi\|_{(p)'} = \|\phi\|_{(p)'}, \ i = 1, \ldots, N,
\]
where \(p' = p/(p - 1)\). Since \(\phi\) is arbitrary, from Proposition IX-18 of [24] we find that \(u \in W^{1,p}_r(B)\), as desired.

Finally, combining (2.11) and (2.12) we obtain that
\[
\liminf_{j \to +\infty} \mu_1(p_k) \int_B m(x)|u|^p \, dx \geq \int_B |\nabla u|^p \, dx.
\]

This and the variational characterization of \(\mu_1(p)\) imply (2.5) and hence (2.3). This concludes the proof of the lemma.

Using Remark 2.1, Theorem 2.1 and Theorem 2.2, we shall show that all eigenvalue functions \(\mu_k^\pm : (1, +\infty) \to \mathbb{R}, 2 \leq k \in \mathbb{N}\) are continuous.
Theorem 2.3  For each $2 \leq k \in \mathbb{N}$, the eigenvalue function $\mu_k^i : (1, +\infty) \to \mathbb{R}$ is continuous.

Proof  Let $u_k^i$ be an eigenfunction corresponding to $\mu_k^i(p)$. By Theorem 2.1 and Remark 2.1, we know that $u$ has exactly $k - 1$ simple zeros in $I$, i.e. there exist $c_{k,1}, \ldots, c_{k,k-1} \in I$ such that $u(c_{k,1}) = \cdots = u(c_{k,k-1}) = 0$. For convenience, we set $c_{k,0} = 0$, $c_{k,k} = 1$, $J_i = (c_{k,i-1}, c_{k,i})$ and $B_i = \{x \in B | c_{k,i-1} < |x| < c_{k,i} \}$ for $i = 1, \ldots, k$. Let $\mu_1^i(p, m/J_i, J_i)$ denote the first positive or negative eigenvalue of the restriction of problem (2.1) on $J_i$ for $i = 1, \ldots, k$. We note that Lemma 3 of [19] also holds for (2.1). It follows that $\mu_k^i(p) = \mu_1^i(p, m/J_i, J_i)$ for $i = 1, \ldots, k$. Using similar proof as Theorem 2.2, we can show that $\mu_k^i(p, m/J_i, J_i)$ is continuous with respect to $p$ for $i = 1, \ldots, k$. Therefore, $\mu_k^i(p)$ is also continuous with respect to $p$. □

Finally, we give a key lemma that will be used in Section 4. Firstly, as an immediate consequence of Lemma 4.1 of [20], we obtain the following Sturm-type comparison theorem.

Lemma 2.1  Let $b_2(r) > b_1(r) > 0$ for $r \in (0, 1)$ and $b_i(r) \in L^\infty(0, 1)$, $i = 1, 2$. Also let $u_1, u_2$ be solutions of

$$
\left(r^{N-1} \varphi_p \left(u'\right)\right)' + b_i(r)r^{N-1} \varphi_p(u) = 0, \quad i = 1, 2,
$$

respectively. If $u_1$ has $k$ zeros in $(0, 1)$, then $u_2$ has at least $k + 1$ zeros in $(0, 1)$.

Let

$$
I^+ := \{r \in \hat{T} | m(r) > 0 \}, \quad I^- := \{r \in \hat{T} | m(r) < 0 \}.
$$

Lemma 2.2  Assume $m \in M(I)$. Let $\hat{T} = [a, b]$ be such that $\hat{T} \subset I^+$ and $\text{meas} \hat{T} > 0$.

Let $g_n : \hat{T} \to (0, +\infty)$ be continuous function and such that

$$
\lim_{n \to +\infty} g_n(r) = +\infty \quad \text{uniformly on } \hat{T}.
$$

Let $y_n \in E$ be a solution of the equation

$$
\left(r^{N-1} \varphi_p \left(y'_n\right)\right)' + r^{N-1} m(r)g_n(r)\varphi_p \left(y_n\right) = 0, \quad r \in (0, 1).
$$

Then the number of zeros of $y_n|\hat{T}$ goes to infinity as $n \to +\infty$.

Proof  After taking a subsequence if necessary, we may assume that

$$
m(r)g_n(r) \geq \lambda_j, \quad r \in \hat{T},
$$

as $j \to +\infty$, where $\lambda_j$ is the $j$th eigenvalue of the following problem

$$
\begin{cases}
\left(r^{N-1} \varphi_p \left(u'\right)\right)' + \lambda r^{N-1} \varphi_p(u(r)) = 0, \quad r \in (a, b), \\
u'(a) = u(b) = 0.
\end{cases}
$$

Let $\varphi_j$ be the corresponding eigenvalue of $\lambda_j$. It is easy to check that the number of zeros of $\varphi_j$ goes to infinity as $j \to +\infty$. By Lemma 2.1, one has that the number of zeros of $y_n|\hat{T}$ goes to infinity as $n \to +\infty$. It follows the desired results. □
3. Unilateral global bifurcation phenomena for (1.2)

If \( m(r) \equiv 1 \), Del Pino and Elgueta [21] established the global bifurcation theory for one-dimensional \( p \)-Laplacian eigenvalue problem. Peral [8] got the global bifurcation theory for \( p \)-Laplacian eigenvalue problem on the unite ball. In [20], Del Pino and Manásevich obtained the global bifurcation from the principle eigenvalue for \( p \)-Laplacian eigenvalue problem on the general domain. If \( m(r) \geq 0 \) and is singular at \( r = 0 \) or \( r = 1 \), Lee and Sim [22] also established the bifurcation theory for the one-dimensional \( p \)-Laplacian eigenvalue problem. However, if \( m(r) \) changes sign, there are a few paper involving the bifurcation theory for \( p \)-Laplacian eigenvalue problem. In this section, we shall study the unilateral global bifurcation phenomena for the \( N \)-dimensional \( p \)-Laplacian eigenvalue problem with sign-changing weight in the radial case.

Let \( Y = L^1(0, 1) \) with its usual normal \( \| \cdot \|_{L^1} \) and \( E = \{ u \in C^1(\bar{I}) \mid u'(0) = u(1) = 0 \} \) with the norm \( \| u \| = \max_{r \in I} |u(r)| + \max_{r \in I} |u'(r)| \).

Considering the following auxiliary problem
\[
\begin{cases}
-\left( r^{N-1} |u'|^{p-2} u' \right)' = r^{N-1} h(r), & \text{a.e. } r \in I, \\
u'(0) = u(1) = 0
\end{cases}
\tag{3.1}
\]
for a given \( h \in Y \). By a solution of problem (3.1), we understand a function \( u \in E \) with \( r^{N-1} \varphi_p(u') \) absolutely continuous which satisfies (3.1).

We have known that for every given \( h \in Y \), there is a unique solution \( u \) to the problem (3.1) (see [20]). Let \( G_p(h) \) denote the unique solution to (3.1) for a given \( h \in Y \). It is well known that \( G_p : Y \to E \) is continuous and compact (see [8,20]).

Define \( T^p_\mu(u) = G_p(\mu m(r) \varphi_p(u(r))) \). Let \( \Psi_{p,\mu} \) be defined in \( E \) by
\[
\Psi_{p,\mu}(u) = u - T^p_\mu(u),
\]
where \( \mu \) is a positive parameter. It is not difficult to show that \( \Psi_{p,\mu} \) is a non-linear compact perturbation of the identity. Thus, the Leray–Schauder degree \( \deg \Psi_{p,\mu}, B_r(0), 0 \) is well defined for arbitrary \( r \)-ball \( B_r(0) \) and \( \mu \neq \mu^*_k \).

Firstly, we can compute \( \deg \Psi_{2,\mu}, B_r(0), 0 \) for any \( r > 0 \) as follows.

**Lemma 3.1** For \( r > 0 \), we have
\[
\deg \Psi_{2,\mu}, B_r(0), 0 = \begin{cases}
1, & \text{if } \mu \in \left( \mu^*_1(2), \mu^*_1(2) \right), \\
(-1)^k, & \text{if } \mu \in \left( \mu^*_k(2), \mu^*_k(2) \right), k \in \mathbb{N}, \\
(-1)^k, & \text{if } \mu \in \left( \mu^*_k(2), \mu^*_k(2) \right), k \in \mathbb{N}.
\end{cases}
\]

**Proof** We divide the proof into two cases.

**Case 1** \( \mu \geq 0 \).
Since \( G_2 \) is compact and linear, by [25, Theorem 8.10] and Theorem 2.1 with \( p = 2 \),
\[
\deg \Psi_{2,\mu}, B_r(0), 0 = (-1)^{m(\mu)},
\]
where \( m(\mu) \) is the sum of algebraic multiplicity of the eigenvalues \( \mu \) of (2.1) satisfying \( \mu^{-1} \mu^*_k < 1 \). If \( \mu \in [0, \mu^*_1(2)) \), then there are no such \( \mu \) at all, then
\[
\deg \Psi_{2,\mu}, B_r(0), 0 = (-1)^{m(\mu)} = (-1)^0 = 1.
\]
If $\mu \in (\mu_k^+ (2), \mu_{k+1}^+ (2))$ for some $k \in \mathbb{N}$, then
\[
\left( \mu_j^+ (2) \right)^{\frac{1}{j}} > 1, \quad j \in \{1, \ldots, k\}.
\]

This together with Theorem 2.1 implies that
\[
\deg (\Psi_{2, \mu}, B_r (0), 0) = (-1)^k.
\]

**Case 2** $\mu < 0$.

In this case, we consider a new sign-changing eigenvalue problem
\[
\begin{cases}
(r N^{-1} u')' + \hat{\mu} \hat{m} (r) r N^{-1} u = 0, & r \in I, \\
u'(0) = u(1) = 0,
\end{cases}
\]
where $\hat{\mu} = -\mu$, $\hat{m} (r) = -m (r)$. It is easy to check that
\[
\hat{\mu}_k^+ (2) = -\mu_k^- (2), \quad k \in \mathbb{N}.
\]

Thus, we may use the result obtained in **Case 1** to deduce the desired result. $\square$

As far as the general $p$ is concerned, we can compute the extension of the Leray–Schauder degree defined in [26] by the deformation along $p$.

**Lemma 3.2**

(i) Let $\{ \mu_k^+ (p) \}_{k \in \mathbb{N}}$ be the sequence of positive eigenvalues of (2.1). Let $\mu$ be a constant with $\mu \neq \mu_k^+ (p)$ for all $k \in \mathbb{N}$. Then for arbitrary $r > 0$,
\[
\deg (\Psi_{p, \mu}, B_r (0), 0) = (-1)^\beta,
\]
where $\beta$ is the number of eigenvalues $\mu_k^+ (p)$ of problem (2.1) less than $\mu$.

(ii) Let $\{ \mu_k^- (p) \}_{k \in \mathbb{N}}$ be the sequence of negative eigenvalues of (2.1). Consider $\mu \neq \mu_k^- (p), k \in \mathbb{N}$, then
\[
\deg (\Psi_{p, \mu}, B_r (0), 0) = (-1)^\beta, \quad \forall r > 0,
\]
where $\beta$ is the number of eigenvalues $\mu_k^- (p)$ of problem (2.1) larger than $\mu$.

**Proof** We shall only prove the case $\mu > \mu_1^+ (p)$ since the proof for the other cases are similar. We also only give the proof for the case $p > 2$. Proof for the case $1 < p < 2$ is similar. Assume that $\mu_k^+ (p) < \mu < \mu_{k+1}^+ (p)$ for some $k \in \mathbb{N}$. Since the eigenvalues depend continuously on $p$, there exists a continuous function $\chi : [2, p] \to \mathbb{R}$ and $q \in [2, p]$ such that $\mu_k^+ (q) < \chi (q) < \mu_{k+1}^+ (q)$ and $\lambda = \chi (p)$. Define
\[
\Upsilon (q, u) = u - G_q \left( \chi (q) m (r) \varphi_q (u) \right).
\]

It is easy to show that $\Upsilon (q, u)$ is a compact perturbation of the identity such that for all $u \neq 0$, by definition of $\chi (q)$, $\Upsilon (q, u) \neq 0$, for all $q \in [2, p]$. Hence, the invariance of the degree under homo-topology and Lemma 3.1 imply
\[
\deg (\Psi_{p, \mu}, B_r (0), 0) = \deg (\Psi_{2, \chi (2)}, B_r (0), 0) = (-1)^k. \quad \square
\]
Define the Nemitskii operator $H : \mathbb{R} \times E \to Y$ by

$$H(\mu, u)(r) := \mu m(r)\varphi_p(u(r)) + g(r, u(r); \mu).$$

Then it is clear that $H$ is continuous (compact) operator and problem (1.2) can be equivalently written as

$$u = G_p \circ H(\mu, u) := F(\mu, u).$$

$F$ is completely continuous in $\mathbb{R} \times E \to E$ and $F(\mu, 0) = 0, \forall \mu \in \mathbb{R}$.

Using the similar method to prove [6, Theorem 2.1] with obvious changes, we may obtain the following result.

**Theorem 3.1** Assume that (1.3) holds and $m \in M(I)$, then from each $(\mu^+_{\nu_k}, 0)$ it bifurcates an unbounded continuum $C^+_{\nu_k}$ of solutions to problem (1.2), with exactly $k - 1$ simple zeros, where $\mu^+_{\nu_k}$ is the eigenvalue of problem (2.1).

In what follows, we use the terminology of Rabinowitz [27]. Let $S^+_k$ denote the set of functions in $E$ which have exactly $k - 1$ interior nodal (i.e. non-degenerate zeros) in $I$ and are positive near $t = 0$, and set $S^-_k := -S^+_k$, and $S_k = S^+_k \cup S^-_k$. They are disjoint and open in $E$. The following global bifurcation result is a generalization of Theorem 3.2 of [6]. The essential idea is similar to the proof of Theorem 3.2 of [6]

**Theorem 3.2** Assume that (1.3) holds and $m \in M(I)$, then there are two distinct unbounded continua, $(C^+_{\nu_k})^+$ and $(C^+_{\nu_k})^-$, consisting of the bifurcation branch $C^+_{\nu_k}$. Moreover, for $\sigma \in \{+, -\}$, we have that

$$(C^+_{\nu_k})^\sigma \subset \left(\{\mu^+_{\nu_k}, 0\} \cup (\mathbb{R} \times S^\sigma_k)\right).$$

### 4. Existence of nodal solutions for (1.5)

In this section, we shall investigate the existence and multiplicity of nodal solutions for problem (1.5) under the linear growth condition on $f$.

Firstly, we suppose that

(H1) $f \in C(\mathbb{R}, \mathbb{R})$ with $f(s)s > 0$ for $s \neq 0$;

(H2) there exists $f_0 \in (0, +\infty)$ such that

$$f_0 = \lim_{|s| \to 0} \frac{f(s)}{\varphi_p(s)};$$

(H3) there exists $f_{\infty} \in (0, +\infty)$ such that

$$f_{\infty} = \lim_{|s| \to +\infty} \frac{f(s)}{\varphi_p(s)}.$$

Let $\mu^\pm_k$ be the $k$th positive or negative eigenvalue of (2.1). Applying Theorem 3.2, we shall establish the existence of nodal solutions of (1.5) as follows.

**Theorem 4.1** Assume that (H1), (H2) and (H3) hold and $m \in M(I)$. Assume that for some $k \in \mathbb{N}$, either

$$\gamma \in \left(\mu^+_k(p)/f_{\infty}, \mu^+_k(p)/f_0\right) \cup \left(\mu^-_k(p)/f_0, \mu^-_k(p)/f_{\infty}\right)$$
or
\[ \gamma \in (\mu^+(p)/f_0, \mu^+(p)/f_\infty) \cup (\mu^-(p)/f_\infty, \mu^-(p)/f_0). \]

Then (1.5) has two solutions \( u_k^+ \) and \( u_k^- \) such that \( u_k^+ \) has exactly \( k-1 \) zeros in \( I \) and is positive near 0, and \( u_k^- \) has exactly \( k-1 \) zeros in \( I \) and is negative near 0.

**Proof** We only prove the case of \( \gamma > 0 \). The case of \( \gamma < 0 \) is similar. Consider the following problem
\[
\begin{cases}
(r^{N-1} \varphi_p (u'))' + \mu \gamma r^{N-1} m(r) f(u) = 0, & r \in I, \\
u'(0) = u(1) = 0.
\end{cases}
\] (4.1)

Let \( \xi \in C(\mathbb{R}, \mathbb{R}) \) be such that \( f(s) = f_0 \varphi_p(s) + \xi(s) \) with \( \lim_{|s| \to 0} \xi(s)/\varphi_p(s) = 0 \). Hence, the condition (1.3) holds. Using Theorem 3.2, we have that there are two distinct unbounded continua, \( (C_k^+)\sigma \) and \( (C_k^-)\sigma \), consisting of the bifurcation branch \( C_k^\sigma \) from \( (\mu^+(p)/\gamma f_0, 0) \), such that
\[ (C_k^\sigma) \subset \left( \{ (\mu^+/\gamma f_0, 0) \} \cup (\mathbb{R} \times S_k^\sigma) \right). \]

It is clear that any solution of (4.1) of the form \( (1, u) \) yields a solutions \( u \) of (1.5). We shall show that \( (C_k^+)\sigma \) crosses the hyperplane \( \{1\} \times E \) in \( \mathbb{R} \times E \). To this end, it will be enough to show that \( (C_k^+)\sigma \) joins \( (\mu^+(p)/\gamma f_0, 0) \) to \( (\mu^+(p)/\gamma f_\infty, +\infty) \). Let \( (\eta_n, y_n) \in (C_k^+)\sigma \) satisfy \( \eta_n + \|y_n\| \to +\infty \). We note that \( \eta_n > 0 \) for all \( n \in \mathbb{N} \) since \( (0,0) \) is the only solution of (4.1) for \( \mu = 0 \) and \( (C_k^+)\sigma \cap \{(0) \times E\} = \emptyset \).

**Case 1** \( \mu^+(p)/f_\infty < \gamma < \mu^+(p)/f_0 \). In this case, we only need to show that
\[ \left( \frac{\mu^+(p)}{\gamma f_\infty}, \frac{\mu^+(p)}{\gamma f_0} \right) \subseteq \{ \mu \in \mathbb{R} | (\mu, u) \in (C_k^+)\sigma \}. \]

We divide the proof into two steps.

**Step 1** We show that if there exists a constant \( M > 0 \) such that \( \eta_n \subset (0, M] \) for \( n \in \mathbb{N} \) large enough, then \( (C_k^+)\sigma \) joins \( (\mu^+(p)/\gamma f_0, 0) \) to \( (\mu^+(p)/\gamma f_\infty, +\infty) \).

In this case, it follows that \( \|y_n\| \to +\infty \). Let \( \tilde{\xi} \in C(\mathbb{R}, \mathbb{R}) \) be such that \( f(s) = f_\infty \varphi_p(s) + \tilde{\xi}(s) \). Then \( \lim_{|s| \to +\infty} \tilde{\xi}(s)/\varphi_p(s) = 0 \). Let \( \tilde{\xi}(u) = \max_{0 \leq |s| \leq u} |\tilde{\xi}(s)| \). Then \( \tilde{\xi} \) is non-decreasing and
\[ \lim_{u \to +\infty} \frac{\tilde{\xi}(u)}{|u|^{p-1}} = 0. \] (4.2)

We divide the equation
\[ \left( r^{N-1} \varphi_p (y_n') \right)' - \eta_n \gamma r^{N-1} m(r) f_\infty \varphi_p (y_n) = \eta_n \gamma r^{N-1} m(r) \tilde{\xi} (y_n) \]
by \( \|y_n\|^{p-1} \) and set \( \overline{y}_n = y_n / \|y_n\| \). Since \( \overline{y}_n \) is bounded in \( E \), after taking a subsequence if necessary, we have that \( \overline{y}_n \rightharpoonup \overline{y} \) for some \( \overline{y} \in E \). Moreover, from (4.2) and the fact that \( \tilde{\xi} \) is non-decreasing, we have that
\[ \lim_{n \to +\infty} \frac{\tilde{\xi}(y_n(r))}{\|y_n\|^{p-1}} = 0. \]

Therefore, by Theorem 2.1,\n\[ -\left(r^{N-1} \varphi_p \left(\tilde{y}\right)\right)' = \overline{\mu} \gamma r^{N-1} m(r) f_\infty \varphi_p \left(\tilde{y}\right), \]
where $\overline{\mu} = \lim_{n \to +\infty} \eta_n$, again choosing a subsequence and relabeling it if necessary.

We claim that $\overline{\mu} \in \left(C_k^+\right)^\sigma$.

It is clear that $\|\tilde{y}\| = 1$ and $\tilde{y} \in \left(C_k^+\right)^\sigma \subseteq \left(C_k^+\right)^\sigma$ since $\left(C_k^+\right)^\sigma$ is closed in $\mathbb{R} \times E$. Therefore, by Theorem 2.1, $\overline{\mu} \gamma f_\infty = \mu_k^+(p)$, so that $\overline{\mu} = \mu_k^+ / \gamma f_\infty$. Therefore, $\left(C_k^+\right)^\sigma$ joins $(\mu_k^+(p) / \gamma f_0, 0)$ to $(\mu_k^+(p) / \gamma f_\infty, +\infty)$.

**Step 2** We show that there exists a constant $M$ such that $\eta_n \in (0, M]$ for $n \in \mathbb{N}$ large enough.

On the contrary, we suppose that $\lim_{n \to +\infty} \eta_n = +\infty$. Since $(\eta_n, y_n) \in (C_k^+)^\sigma$, it follows that
\[ \left(r^{N-1} \varphi \left(y_n'\right)\right)' + \gamma \eta_n r^{N-1} m(r) f(y_n) = 0. \]

Let
\[ 0 < \tau(1, n) < \cdots < \tau(k, n) = 1 \]
be the zeros of $y_n$ in $I$. Then, after taking a subsequence if necessary,
\[ \lim_{n \to +\infty} \tau(l, n) := \tau(l, \infty), \quad l \in \{1, \cdots, k - 1\}. \]

It follows that either there exists at least one $l_0 \in \{1, \cdots, k - 1\}$ such that
\[ \tau(l_0, \infty) < \tau(l_0 + 1, \infty) \text{ or } \tau(1, \infty) = 1. \]

Notice that Lemma 2.2 and the fact $y_n$ has exactly $k - 1$ simple zeros in $I$ yield
\[ \left\{ \left[ \bigcup_{l=1}^{k-1} (\tau(l, \infty), \tau(l + 1, \infty)) \right] \cup (0, \tau(1, \infty)) \right\} \cap I^+ = \emptyset, \]
which implies that
\[ \left\{ \left[ \bigcup_{l=1}^{k-1} (\tau(l, \infty), \tau(l + 1, \infty)) \right] \cup (0, \tau(1, \infty)) \right\} \subseteq (I \setminus I^+). \]

Therefore,
\[ \operatorname{meas} (I \setminus I^+) \geq \operatorname{meas} \left\{ \left[ \bigcup_{l=1}^{k-1} (\tau(l, \infty), \tau(l + 1, \infty)) \right] \cup (0, \tau(1, \infty)) \right\} = 1. \]

However, this contradicts $(H_2)$: $0 < \operatorname{meas} (I \setminus I^+) < 1$.

**Case 2** $\mu_k^+(p) / f_0 < \gamma < \mu_k^+(p) / \gamma f_\infty$. In this case, we have that
\[ \frac{\mu_k^+(p)}{\gamma f_0} < 1 < \frac{\mu_k^+(p)}{\gamma f_\infty}. \]

Assume that $(\eta_n, y_n) \in (C_k^+)^\sigma$ is such that $\lim_{n \to +\infty} (\eta_n + \|y_n\|) = +\infty$. In view of **Step 2 of Case 1**, we have known that there exists $M > 0$, such that for $n \in \mathbb{N}$ sufficiently
large, $\eta_n \in (0, M]$. Applying the same method used in Step 1 of Case 1, after taking a subsequence and relabeling it if necessary, it follows that

$$(\eta_n, y_n) \rightarrow \left(\frac{\mu_k^+(p)}{\gamma f_\infty}, +\infty\right) \text{ as } n \rightarrow +\infty.$$  

Thus, $(C_k^+)\sigma$ joins $(\mu_k^+(p)/\gamma f_0, 0)$ to $(\mu_k^+(p)/\gamma f_\infty, +\infty)$.

Using the similar proof with that of Theorem 4.1, we can obtain the more general results as follows.

**Theorem 4.2** Assume that $(H_1)$, $(H_2)$ and $(H_3)$ hold and $m \in M(I)$. Assume that for some $k, n \in \mathbb{N}$ with $k \leq n$, either

$$\gamma \in \left(\mu_k^+(p)/f_\infty, \mu_k^+(p)/f_0\right) \cup \left(\mu_k^-(p)/f_0, \mu_k^-(p)/f_\infty\right)$$

or

$$\gamma \in \left(\mu_n^+(p)/f_0, \mu_n^+(p)/f_\infty\right) \cup \left(\mu_n^-(p)/f_\infty, \mu_n^-(p)/f_0\right).$$

Then (1.5) has $n - k + 1$ pairs solutions $u_j^+$ and $u_j^-$ for $j \in \{k, \ldots, n\}$ such that $u_j^+$ has exactly $j - 1$ zero in $I$ and is positive near 0, and $u_j^-$ has exactly $j - 1$ zero in $I$ and is negative near 0.

**Remark 4.1** We would like to point out that Theorem 1.1 of [9] is the corollary of Theorem 4.1 even in the case of $p = 2$ and $N = 1$.

**Remark 4.2** We also note that Theorem 4.1 and Theorem 4.2 is valid for the problems on annular domain because it can be convert the equivalent one-dimensional problems.

An open problem. When $m \geq 0$, using Lemma 2.1, we can easily get that (1.5) has no non-trivial solution if $\gamma m f(s)/\varphi_p(s)$ not cross any eigenvalue of (1.6). Therefore, we conjecture that (1.5) has no non-trivial solution if

$$\mu_k^+(p) < f(s)/\varphi_p(s) < \mu_{k+1}^+(p) \text{ or } \mu_k^-(p) > -f(s)/\varphi_p(s) > \mu_{k+1}^-(p) \text{ for } s \neq 0.$$  

5. One-sign solutions for (1.7)

In this section, based on the bifurcation result of Drábek and Huang [5], we shall study the existence of one-sign solutions for problem (1.7). From now on, for simplicity, we write $X := W_0^{1,p}(\Omega)$.

The main results of this section are the following theorem.

**Theorem 5.1** Let $(H_1)$, $(H_2)$ and $(H_3)$ hold, and $m \in M(\Omega)$. Assume that either

$$\gamma \in \left(\mu_1^+(p)/f_\infty, \mu_1^+(p)/f_0\right) \cup \left(\mu_1^-(p)/f_0, \mu_1^-(p)/f_\infty\right)$$

or

$$\gamma \in \left(\mu_1^+(p)/f_0, \mu_1^+(p)/f_\infty\right) \cup \left(\mu_1^-(p)/f_\infty, \mu_1^-(p)/f_0\right),$$

then problem (1.7) possesses at least a positive and a negative solution.
Remark 5.1  By the $C^{1,\alpha} (0 < \alpha < 1)$ regularity results for quasilinear elliptic equations with $p$-growth condition,[28] $u \in C^{1,\alpha}(\overline{\Omega})$ for any solution $u$ of (1.7) since $f$ is continuous and subcritical.

In order to prove Theorem 5.1, we consider the following eigenvalue problem

$$
\begin{align*}
-\text{div} (\varphi_p(\nabla u)) &= \mu \gamma m(x) f(u), & \text{in } \Omega, \\
u &= 0, & \text{on } \partial \Omega,
\end{align*}
$$

(5.1)

where $\mu$ is a parameter. Let $\zeta \in C(\mathbb{R}, \mathbb{R})$ be such that $f(s) = f_0 \varphi_p(s) + \zeta(s)$ with $\lim_{|s| \to 0} \zeta(s)/\varphi_p(s) = 0$. Let us consider

$$
\begin{align*}
-\text{div} (\varphi_p(\nabla u)) &= \mu \gamma m(x) f_0 \varphi_p(u) + \mu \gamma m(x) \zeta(u), & \text{in } \Omega, \\
u &= 0, & \text{on } \partial \Omega
\end{align*}
$$

(5.2)

as a bifurcation problem from the trivial solution $u \equiv 0$.

Let $S^+ = \left\{ u \in C^{1,\alpha}(\overline{\Omega}) \left|u(x) > 0 \text{ for all } x \in \Omega \right. \right\}$, $S^- = \left\{ u \in C^{1,\alpha}(\overline{\Omega}) \left|u(x) < 0 \text{ for all } x \in \Omega \right. \right\}$.

Applying Theorems 4.4 and 4.5 of [5] to (5.2), we can obtain the following unilateral global bifurcation result, which plays a fundamental role in our study.

**Lemma 5.1**  Let $\nu \in \{+, -\}$. There are two distinct unbounded continua, $C^\nu_+$ and $C^\nu_-$, consisting of the bifurcation branch $C^\nu$ from $(\mu^\nu_k(p)/\gamma f_0, 0)$. Moreover, for $\sigma \in \{+, -\}$, we have

$$
C^\nu_\sigma \subset \left(\{(\mu^\nu(p)/\gamma f_0, 0)\} \cup (\mathbb{R} \times S^\sigma)\right).
$$

We use Lemma 5.1 to prove the main results of this section.

**Proof of Theorem 5.1**  Since the proof is similar to that of Theorem 4.1, we only give a rough sketch of the proof. We only prove the case of $\gamma > 0$. The case of $\gamma < 0$ is similar. It is clear that any solution of (5.1) of the form $(1, u)$ yields a solution $u$ of (1.7). We shall show that $C^+_\sigma$ crosses the hyperplane $\{1\} \times X$ in $\mathbb{R} \times X$. To this end, it will be enough to show that $C^+_\sigma$ joins $(\mu^+_1(p)/\gamma f_0, 0)$ to $(\mu^+_1(p)/\gamma f_\infty, +\infty)$.

Let $(\mu_n, y_n) \in C^+_\sigma$ where $y_n \neq 0$ satisfies $\mu_n + \|y_n\|_X \to +\infty$. We note that $\mu_n > 0$ for all $n \in \mathbb{N}$ since $(0, 0)$ is the only solution of (5.1) for $\mu = 0$ and $C^+_\sigma \cap \{(0) \times X\} = \emptyset$.

**Case 1**  $\mu^+_1(p)/\gamma f_\infty < \gamma < \mu^+_1(p)/f_0$.

In this case, we only need to show that

$$
\left(\frac{\mu^+_1(p)}{\gamma f_\infty}, \frac{\mu^+_1(p)}{\gamma f_0}\right) \subseteq \left\{\mu \in \mathbb{R} \left| (\mu, u) \in C^+_\sigma \right. \right\}.
$$

We divide the proof into two steps.
Step 1 We show that if there exists a constant $M > 0$ such that $\eta_n \subset (0, M]$ for $n \in \mathbb{N}$ large enough.

In this case it follows that $\|y_n\|_X \to +\infty$. Similar to the proof of Theorem 4.1, we divide the equation

$$-\text{div} (\varphi_p (\nabla y_n)) - \mu_n m(x) \varphi_p (y_n) = \mu_n \gamma m(x) \xi (y_n)$$

by $\|y_n\|_X^{p-1}$ and set $\overline{y}_n = y_n / \|y_n\|_X$. Since $\overline{y}_n$ is bounded in $X$, after taking a subsequence if necessary, we have that $\overline{y}_n \to \overline{y}$ for some $\overline{y} \in X$ and $\overline{y}_n \to \overline{y}$ in $L^{p'} (\Omega)$. Obviously, for any $\epsilon > 0$, there exists a constant $C > 0$ such that

$$|\xi(s)| \leq C + \epsilon |s|^{p-1}.$$  \hspace{1cm} (5.3)

From (5.3), we can obtain that

$$\lim_{n \to +\infty} \frac{\xi (y_n(t))}{\|y_n\|_X^{p-1}} = 0 \text{ in } L^{p'} (\Omega).$$

By the compactness of $R_p : L^{p'} (\Omega) \to X$ (see [20]), we obtain

$$-\text{div} (\varphi_p (\nabla \overline{y})) - \overline{\mu} \gamma f_{\infty} m(x) \varphi_p (\overline{y}) = 0,$$

where $\overline{\mu} = \lim_{n \to +\infty} \mu_n$, again choosing a subsequence and relabeling it if necessary. The rest proof of this step is the same as the proof of Theorem 4.1.

Step 2 We show that there exists a constant $M$ such that $\mu_n \in (0, M]$ for $n \in \mathbb{N}$ large enough.

On the contrary, we suppose that $\lim_{n \to +\infty} \mu_n = +\infty$. Since $(\mu_n, y_n) \in C_{\sigma}^+$, it follows that

$$\text{div} (\varphi_p (\nabla y_n)) + \gamma \mu_n m(x) \frac{f(y_n)}{\varphi_p (y_n)} \varphi_p (y_n) = 0 \text{ in } \Omega^+,$$

where $\Omega^+ = \{ x \in \Omega | m(x) > 0 \}$. By Theorem 2.6 of [29], we have $y_n$ must change sign in $\Omega^+$, which contradicts Lemma 5.1. The rest proof of this step is similar to the proof of Theorem 4.1. $\square$

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References

[1] Rabinowitz PH. Some global results for nonlinear eigenvalue problems. J. Funct. Anal. 1971;7:487–513.
[2] Dancer EN. On the structure of solutions of non-linear eigenvalue problems. Indiana Univ. Math. J. 1974;23:1069–1076.
[3] Dancer EN. Bifurcation from simple eigenvalues and eigenvalues of geometric multiplicity one. Bull. Lond. Math. Soc. 2002;34:533–538.
[4] López-Gómez J. Spectral theory and nonlinear functional analysis. Boca Raton (FL): Chapman and Hall/CRC; 2001.
[5] Drábek P, Huang YX. Bifurcation problems for the $p$-Laplacian in $\mathbb{R}^N$. Trans. Am. Math. Soc. 1997;349:171–188.
[6] Dai G, Ma R. Unilateral global bifurcation phenomena and nodal solutions for $p$-Laplacian. J. Differ. Equ. 2012;252:2448–2468.
[7] Girg P. Bifurcations of positive and negative continua in quasilinear elliptic eigenvalue problems. Ann. Inst. H. Poincaré Anal. Non Linéaire. 2008;9:275–327.
[8] Peral I. Multiplicity of solutions for the $p$-Laplacian. ICTP SMR 990/1; 1997.
[9] Ma R, Thompson B. Nodal solutions for nonlinear eigenvalue problems. Nonlinear Anal. 2004;59:707–718.
[10] Ma R, Han X. Existence of nodal solutions of a nonlinear eigenvalue problem with indefinite weight function. Nonlinear Anal. 2009;71:2119–2125.
[11] Erbe LH. Eigenvalue criteria for existence of positive solutions to nonlinear boundary value problems. Math. Comput. Model. 2000;32:529–539.
[12] Erbe LH, Wang HY. On the existence of positive solutions of ordinary differential equations. Proc. Am. Math. Soc. 1994;120:743–748.
[13] Henderson J, Wang HY. Positive solutions for nonlinear eigenvalue problems. J. Math. Anal. Appl. 1997;208:252–259.
[14] Naito Y, Tanaka S. On the existence of multiple solutions of the boundary value problem for nonlinear second order differential equations. Nonlinear Anal. 2004;56:919–935.
[15] Fleming WH. A selection-migration model in population genetics. J. Math. Biol. 1975;2:219–233.
[16] Lazer AC, McKenna PJ. Large-amplitude periodic oscillations in suspension bridges: some new connections with nonlinear analysis. SIAM Rev. 1990;32:537–578.
[17] Kurth M. On the existence of infinitely many modes of a nonlocal nonlinear Schrödinger equation related to dispersion-managed solitons. SIAM J. Math. Anal. 2005;36:967–985.
[18] Meng G, Yan P, Zhang M. Spectrum of one-dimensional $p$-Laplacian with an indefinite integrable weight. Mediterr. J. Math. 2010;7:225–248.
[19] Anane A, Chakrone O, Monssa M. Spectrum of one dimensional $p$-Laplacian with indefinite weight. Electron. J. Qual. Theory Differ. Equ. 2002;2002:1–11.
[20] Del Pino M, Manásevich R. Global bifurcation from the eigenvalues of the $p$-Laplacian. J. Differ. Equ. 1991;92:226–251.
[21] Del Pino M, Elgueta M, Manásevich R. A homotopic deformation along $p$ of a Leray-Schauder degree result and existence for $\left(\frac{|u'|^{p-2}u'}{\gamma}+f(t,u)\right)' = 0, u(0) = u(T) = 0, p > 1$. J. Differ. Equ. 1989;80:1–13.
[22] Lee YH, Sim I. Global bifurcation phenomena for singular one-dimensional $p$-Laplacian. J. Differ. Equ. 2006;229:229–256.
[23] Evans LC. Partial differential equations. Providence (RI): AMS; 1998.
[24] Brezis H. Analyse Fonctionnelle. Théorie et applications. Paris: Masson; 1983.
[25] Deimling K. Nonlinear functional analysis. New York (NY): Springer-Verlag; 1987.
[26] Browder FE, Petryshyn WF. Approximation methods and the generalized topological degree for nonlinear mapping in Banach spaces. J. Funct. Anal. 1969;3:217–245.
[27] Rabinowitz PH. Some aspects of nonlinear eigenvalue problems. Rocky Mountain J. Math. 1973;3:161–202.
[28] Lieberman GM. Boundary regularity for solutions of degenerate elliptic equations. Nonlinear Anal. 1988;12:1203–1219.
[29] Allegretto W, Huang YX. A Pocone’s identity for the $p$-Laplacian and applications. Nonlinear Anal. 1998;32:819–830.