A Generalized Osgood Condition for Viscosity Solutions to Fully Nonlinear Parabolic Degenerate Equations

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ABSTRACT. - Using a generalized assumption of Osgood type, we prove a new comparison result for viscosity sub and supersolutions of fully nonlinear, possibly degenerate, parabolic equations. Our result allows to deal with Hamiltonian functions with a quadratic growth in the spatial gradient, under special compatibility conditions with the diffusive terms. It applies in particular to a financial differential model for pricing Mortgage-Backed Securities.

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1 Introduction

In this paper we prove a new comparison result between viscosity sub and supersolution for a nonlinear second-order parabolic, possibly strongly degenerate, equation of the following general form

$$\partial_t u + F(x, t, u, \nabla u, \nabla^2 u) = 0,$$

in $\mathbb{R}^N \times [0, T)$ $T > 0$. The unknown $u$ will always be a real-valued function on $\mathbb{R}^N \times [0, T)$; $\partial_t u$, $\nabla u$, $\nabla^2 u$ denote respectively the time derivative of $u$, the gradient of $u$ and the Hessian of $u$ in the space variables. $F$ is a real-valued function defined on $\mathbb{R}^N \times [0, T) \times (a, b) \times \mathbb{R}^N \times S^N$, where $S^N$ is the space of $N \times N$ symmetric matrices endowed with the usual ordering, while $(a, b)$ is an open, possibly unbounded interval.

Our comparison principle is based on a special compatibility condition between the diffusive terms and the quadratic dependence of the Hamiltonian on the
spatial gradient, see formula (3.10) in Theorem 3.6. This condition is somehow similar to the celebrated “null condition”, used to prove global existence for hyperbolic equations with a quadratic dependence on the gradient, see for instance [14], [21] and references therein. Notice however that with our comparison result, the dependence of the Hamiltonian on the unknown \( u \) is allowed. Moreover, it includes not only the quadratic growth with respect to the spatial gradient, but also the absence of monotonicity with respect to \( u \) and a lack of regularity in the Hamiltonian.

Actually one important motivation which has brought us to state this principle, comes from the following quasilinear equation:

\[
\partial_t U - \frac{1}{2} tr(\sigma \sigma^{\top} \nabla^2 U) - \langle \mu, \nabla U \rangle + \rho \frac{\sigma^\top \nabla U^2}{U + h + \xi(t)} + r(U + h) - \tau h = 0, \quad (1.2)
\]

in \( \mathbb{R}^N \times [0, T) \), \( \rho > 0 \), \( \tau, T > 0 \), where \( tr, \langle \rangle, | \cdot | \) denote the trace of a square matrix, the Euclidean norm and inner product, respectively, and with a continuous initial datum \( U_0 \). Moreover \( \mu : \mathbb{R}^N \times [0, T) \to \mathbb{R}^N \), \( \sigma : [0, T) \to M_{N \times d}(\mathbb{R}) \), \( \xi, r : [0, T) \to [0, \infty) \), \( h : \mathbb{R}^N \times [0, T) \to \mathbb{R} \) are continuous, where \( M_{N \times d}(\mathbb{R}) \) denotes the space of real \( N \times d \) matrices, with \( N \geq d \).

Equation (1.2) has been proposed in [27], following the probabilistic financial model by X. Gabaix and O. Vigneron [11], as a differential model for pricing some widely used American financial instruments, the Mortgage-Backed Securities (MBS). Although we shall not study here the financial issues coming from model (1.2), in section 2 we quickly analyse the particular structure of equation (1.2).

Quasilinear parabolic and elliptic equations have been extensively studied in the literature in the uniformly or strictly elliptic case. We can quote some classical books for a wide presentation of the results which are known in this direction, [13], [22]. For real applications we need to consider \( N > d \), but the case of strong degenerate equations, like (1.2), is far less classical, since we cannot expect the global existence of strong solutions. It turns out that viscosity solutions can be used to treat this kind of problem, see [7]. For stationary degenerate elliptic, but also parabolic equations, when we want to prove a comparison result by using viscosity solution’s method, the coercivity of the equation in the variable \( u \) is a standard assumption, namely for any \( R > 0 \), there exists \( \gamma_R \), which is positive in the stationary case and possibly negative in the parabolic case, such that for any \( -R \leq v \leq u \leq R, p \in \mathbb{R}^N \) and \( X \), symmetric matrix with real coefficients, it holds

\[
F(u, p, X) - F(v, p, X) \geq \gamma_R(u - v). \quad (1.3)
\]

This assumption can be more complicated to be verified if there is dependence by the spatial variable and the domain of \( F \) with respect to \( u \) is not all of \( \mathbb{R} \). Condition (1.3) is then used to state comparison results also in unbounded domains such as in [12]. There are many ways to relax condition (1.3); for instance in [17] a Osgood condition on \( F \) is used (see Definition 3.8), but also this condition is not enough for equation (1.2), because the dependence on the term \( (U + h + \xi)^{-1} \). Assumptions like (1.3) can be removed, if it is possible to prove that the Hopf’s Lemma holds in the strong form. The problem with this approach is that, for quasilinear equations, the Hopf’s Lemma can be obtained only for classical solutions. In a recent work of G. Barles and J. Busca [4], a
weak version of the Hopf’s Lemma is used to show the comparison in bounded
domain for a stationary fully nonlinear equations, but because of the degeneracy
of equation (1.2) it is not possible to follow here the same technique. Our proof
of the comparison principle for (1.1), proved in section 3, follows a different idea,
based on the structural condition (3.10) on the function $F$.

Finally, let us notice that a similar problem has been recently considered by
P.L. Lions and P.E. Souganidis in [24]. Starting from a fully nonlinear stochastic
partial differential equation, they reduce their problem to a deterministic
second order fully nonlinear pde and introduce some conditions for the original
Hamiltonian to obtain the comparison principle.

In section 3, we give some examples to show that the conditions proposed in
that paper are strictly contained in our approach. Moreover the present results
on equation (1.2) do not follow by a simple combination of the ideas in [24], see
Remark 3.14.

In a work in preparation [29], following the general theory of stochastic partial
differential equations of [23], [25], [24] we use the same kind of conditions on the
Hamiltonian, here proposed in our Theorem 3.6, to proving an existence result
for stochastic pde, which depend also on $(x,t,u)$.

## 2 The MBS Model

In the last years, the theory about financial markets, the mathematical frame-
works for modelling them, and arbitrage theory have reached an high develop-
ment and have taken a prominent position in the mathematical literature (we
refer the interested reader to some reference books about these arguments like
[20], [8]). However some financial derivatives, like MBSs, still need of specific
models which are conformed on their peculiarities. Moreover when one introduces
the incompleteness in the market, then the choice of a new measure to evalu-
te the derivatives is necessary. Via the Girsanov Theorem, [19], this choice
corresponds to assign a market price of risk. Essentially when the market is
not complete, the market price of risk is not unique and it must be chosen by
statistical analysis; this approach is also followed by X.Gabaix and O.Vigneron
in [11], but the mathematical toolbox that would required to rigorously derive
their results is still largely to be developed. In [27], we used arbitrage arguments
and the form of the market price of risk proposed in [11] to develop our differential
model (1.2). The appeal to differential equations in financial modelling has
become a standard approach, and in many cases it represents the best way for
valuing derivatives. In some instances, the expectations that result from apply-
ing the arbitrage pricing principle can be characterized as solutions to partial
differential equations. Therefore the study of solutions of the pricing equations
of more complex financial instruments and their numerical solutions have be-
come important techniques available to practitioners of modern quantitative
finance. However up to now the PDE approach for valuation of MBSs with the
additional specification of a nonlinear form for the market price of risk, was not
followed. Although the equation (1.2) represents a reduced version of a more
general situation which includes a model of the issuance of future securities, the
model outlines a new manner for treating MBSs derivatives. Not only the possi-
bility to represents the value of an MBS as the solution of a PDE is important,
but also the existence and uniqueness problem, related to that equation holds
an essential role: actually it is well known that an arbitrary specification of the market price of risk, may lead to arbitrage opportunities, hence the existence of a solution for (1.2) is the proof that exists a new risk-neutral market measure through evaluating these securities. This equation describes a first model in this direction, actually it is considered a deterministic interest rate, while in a future work, at the moment in preparation, [30], we study a model which takes into account different stochastic models for interest rate movements. However the model receives a reasonable empirical support.

Mortgage-Backed derivatives are the products of a securitization of pools of mortgages. Every mortgage-holder in a pool holds the right to prepay her debt at every time between 0 and the maturity $T$. This American-style option determines the stochasticity in the valuation of the price of a Mortgage-Backed from that pool. Many factors affect prepayments and this fact creates a remarkable complexity for generating forecasts.

The arbitrage pricing principle applies to the financial instruments whose cash flows are related to the values of economic factors, such as the interest rates $(r = r(t))$. It implies that the price $(V_t)$ of such derivative assets can be expressed as a conditional expectation with respect to a particular measure $(Q)$, over the probability space of the underlying factors that affect its value. Under suitable conditions, computing this expectation it reduces to the solution of a partial differential equation: in fact $V_t = U(X_t, T - t) + h(X_t, T - t)$, where $h$ contains information about prepayments and $X_t$ describes economic factors. In [11] the measure $Q$ depends in a nonlinear way by the price $V$ and its volatility. This aspect produces the nonlinear quadratic term in the equation (1.2). Another characteristic of problem (1.2) is the strongly degeneracy of the equation. In fact, also this aspect comes from the dependence of the payoff of the security on the trajectory followed by one or more of the underlying markovian processes. We illustrate this point with a simple consideration. One possible index $(b_t)$ of the incentive to prepay a mortgage is the amount by which a particular interest rate $r_s$ is below some given level $\tau$. The factor is represented by $b_t = \int_0^t (\tau - \tau_s)^+ ds$, and this form gives the absence of diffusion in the $b$ direction.

3 Comparison Principle and Existence

We begin with some notation and recall about viscosity solutions. Denote $Lip(\mathbb{R}^N), C^{2,1}(\mathbb{R}^N \times [0, T]), C^k(I)$, respectively, the space of the globally Lipschitz functions over $\mathbb{R}^N$, the space of all functions which have two continuous derivatives in the space variable and one continuous in the time over $\mathbb{R}^N \times [0, T)$, the space of all functions with $k = 0, 1, 2$ continuous derivatives over the interval $I$.

Definition 3.1. Given an upper semicontinuos function $u : \mathbb{R}^N \times [0, T) \rightarrow \mathbb{R}$,
the parabolic super 2-jet of $u$ at the point $(x,t)$ is,

$$
\mathcal{P}^{2,+}u(x,t):=\left\{(\partial_t \varphi(x,t), \nabla \varphi(x,t), \nabla^2 \varphi(x,t)) : \varphi \in C^{2,1}(\mathbb{R}^N \times [0,T]), u-\varphi \right\}
$$

has a local maximum at $(x,t)$ if it is, at the same time, a viscosity sub- and a super-solution of (1.1).

Remark 3.2. We shall say that a function $\varphi$, as in the previous definition, is a test function for $\mathcal{P}^{2,+}u$ at $(x,t)$.

In the definition (3.1) it is possible to replace local by global, or local strict, or global strict. In the case of a global strict maximum with $(u-\varphi)(x,t) = 0$ we will say that $\varphi$ is a good test function for $\mathcal{P}^{2,+}u$ at $(x,t)$.

Denote with $u_*, u^*$ respectively the lower and upper semicontinuous envelope of $u$. Moreover we consider as domain of $F$ with respect to $u$, in the equation (1.1), a (possibly bounded) open interval $(a,b)$, and as initial datum $u_0 \in C^0((a,b))$.

Definition 3.3. A function $u : \mathbb{R}^N \times [0,T) \rightarrow \mathbb{R}$, is called a viscosity sub (resp. super) solution of (1.1) if $a < u \leq u^* < b$ (resp. $a < u_* \leq u < b$), and

1. for every $(x,t) \in \mathbb{R}^N \times [0,T)$ and a test function $\varphi$ for $\mathcal{P}^{2,+}u^*$ (resp. $\mathcal{P}^{2,-}u^*$) at $(x,t)$,

$$
\partial_t \varphi(x,t) + F(x,t, u^*(x,t), \nabla \varphi(x,t), \nabla^2 \varphi(x,t)) \leq 0
$$

(resp. $\partial_t \varphi(x,t) + F(x,t, u_*(x,t), \nabla \varphi(x,t), \nabla^2 \varphi(x,t)) \geq 0$),

2. $u(x,0) \leq u_0(x)$, (resp. $u(x,0) \geq u_0(x)$) for $x \in \mathbb{R}^N$.

Definition 3.4. A function $u : \mathbb{R}^N \times [0,T) \rightarrow \mathbb{R}$ is called a viscosity solution of (1.1) if it is, at the same time, a viscosity sub- and a super-solution of (1.1).

In this paper we call a continuous function $\nu : [0, \infty) \rightarrow [0, \infty)$ a modulus, if $\nu(0) = 0$ and it is nondecreasing; given $p \in \mathbb{R}^N$, we denote $p \otimes p$ the $N \times N$ matrix whose entries are $p_i p_j$ for every $i, j = 1, \ldots, N$; if $X \in S^N$, then $\|X\|$ denotes the operator norm of $X$ as a self-adjoint operator on $\mathbb{R}^N$; if $f$ is a real-valued bounded function in a domain $D$, then we denote with $\|f\|_\infty$ the supremum of $f$ over $D$. Moreover we need of the following definition in which we recall a property already used in [17].
**Definition 3.5.** We say that a continuous function $\Gamma : [0,l] \to \mathbb{R}$, $l > 0$ satisfies the Osgood Condition, if the following conditions hold:

(i) $\Gamma$ is an increasing function and $\Gamma(0) = 0$;

(ii) 

$$\int_0^l \frac{dr}{\Gamma(r)} = +\infty.$$  

We are now in position to state our main comparison Theorem.

**Theorem 3.6.** Consider the following differential equation:

$$\partial_t u + F(x,t,u,\nabla u, \nabla^2 u) = 0, \quad (x,t) \in \mathbb{R}^N \times (0,T)$$  

(3.4)

and assume that,

$$F \in C(\mathbb{R}^N \times [0,T) \times (a - \epsilon_0, b + \epsilon_0) \times \mathbb{R}^N \times S^N), \quad \epsilon_0 > 0;$$  

(3.5)

- $F$ is degenerate elliptic, i.e.,

$$F(x,t,u,p,X + Y) \leq F(x,t,u,p,X),$$  

(3.6)

for every $(x,t,u,p) \in \mathbb{R}^N \times [0,T) \times (a - \epsilon_0, b + \epsilon_0) \times \mathbb{R}^N, X, Y \in S^N, Y \geq 0$.

- For every $R > 0$ there is a modulus $\nu_{1,R}$ such that

$$|F(x,t,u,p,X) - F(x,t,u,q,X)| \leq \nu_{1,R}(|p - q|)$$  

(3.7)

for all $(x,t,u,p,X) \in \mathbb{R}^N \times [0,T) \times [a,b] \times \mathbb{R}^N \times S^N$, with $|p|, |q|, \|X\| \leq R$.

- Suppose that,

$$-\epsilon_1 I_{2N} \leq \begin{pmatrix} X & O \\ O & Y \end{pmatrix} \leq \epsilon_2 \begin{pmatrix} I_N & -I_N \\ -I_N & I_N \end{pmatrix} + \epsilon_3 I_{2N}$$  

(3.8)

with $X, Y \in S^N, \epsilon_1, \epsilon_2, \epsilon_3 \geq 0$. Let $R$ be taken so that $R \geq \max(\epsilon_1, 2\epsilon_2 + \epsilon_3) + 2\epsilon_3$. Then it holds:

$$F(x,t,u,p,X + Z) - F(y,t,u,p,-Y + Z) \geq -\nu_2(\|x - y\|(|p| + \epsilon_1) + \epsilon_2(\|x - y\|^2) - \nu_{2,R}(2\epsilon_3)$$  

(3.9)

for every $(x,t,u) \in \mathbb{R}^N \times [0,T] \times [a,b], |p|, \|Z\| \leq R$, and with some moduli $\nu_2, \nu_{2,R}$ independent of the other variables and $\epsilon_1, \epsilon_2, \epsilon_3$; where $\nu_{2,R}$ possibly dependent of $R$.

- There are functions $\Gamma$ which satisfy the Osgood Condition 3.5 over

$$[0, \sqrt{\lambda_0}(b - a)], \text{ and } z \in C^1((a - \epsilon_0, b + \epsilon_0); (0, +\infty)), \text{ with } z([a,b]) \subset [\lambda_0, \Lambda_0],$$  

$\Lambda_0 > \lambda_0 > 0$, such that for every $R > 0$ there is a modulus $\nu_{\Gamma, R}$, such that,

$$\frac{1}{\lambda} F(x,t,u,\lambda q, \lambda X + \lambda q \otimes q) - \frac{1}{\lambda} F(x,t,v,\lambda q, \lambda X + \lambda q \otimes q) \geq -\Gamma(u-v)\nu_{\Gamma, R}((\lambda^2 - |z(u)|) + (\lambda^2 - |z(v)|)(1 + |q| + \|X\|))$$  

(3.10)

for $(x,t) \in \mathbb{R}^N \times [0,T), |q|, \|X\| \leq R, a \leq v \leq u \leq b, \inf_{[a,b]} z^\frac{1}{2} \leq \lambda, \lambda \leq \sup_{[a,b]} z^\frac{1}{2}, 2\kappa \leq z'(u), 2\hat{\kappa} \geq z'(v).$
If $u, \pi : \mathbb{R}^N \times [0, T) \to [a, b]$, are respectively viscosity subsolution and supersolution of the equation (3.4) and moreover there is a modulus $\omega$ such that
\[
\omega^*(x, 0) - \pi^*(y, 0) \leq \omega(|x - y|)
\]
for all $x, y \in \mathbb{R}^N$, then
\[
u^* \leq \pi^*,
\]
over $\mathbb{R}^N \times [0, T)$.

**Remark 3.7.** As regards the bounds of the solution, by financial purposes we look for bounded solution, so we limit us to state the comparison only for bounded sub/supersolution which take values only in subsets of the domain of $F$.

Conditions (3.6)-(3.9) were already used in [12], but assumption (3.10), is a relaxation of “monotonicity” assumption with respect $u$, about $F$.

We shall prove Theorem 3.6 in some steps. We start by describing some technical results which will be useful in the proof of Theorem 3.6. For the proof of these we refer the reader to the works which contain them.

For parabolic problems the monotonicity assumption with respect to $u$ can be relaxed requiring the Osgood type condition on $F$. Actually the proof of Theorem is based on the following Lemma.

**Lemma 3.8.** ([17]) Let $\Gamma$ be an Osgood type function over $[0, l]$, in the sense of Definition 3.5. Let $f$ be an upper semicontinuous function over $[0, T)$, valued in $[0, l]$. Assume that $f$ satisfies in the viscosity sense the following problem,
\[
\begin{aligned}
f'(t) &\leq \Gamma(f(t)), & t \in (0, T) \\
\min(f'(0) - \Gamma(f(0)), f(0)) &= 0.
\end{aligned}
\]
Then $f \equiv 0$.

The last Lemma is an important Technical Lemma, and is Proposition 4 in [17].

**Definition 3.9.** The super 1-jet of an upper semicontinuous function $u : [0, T) \to \mathbb{R}$ at the point $t_0$ is
\[
J^{1,+}u(t_0) := \{\varphi'(t_0) : \varphi \in C^1([0, T)), \ u - \varphi \text{ has a local maximum at } t_0\}
\]

**Lemma 3.10.** ([17], Proposition 4) Let $\psi, \varphi$ be respectively an upper/lower semicontinuous function, such that
\[
\sup \left\{ (\psi(x, t) - \varphi(y, t))^+ : |x - y| \leq 1 \right\} \leq K
\]
Set the function,
\[
\vartheta(t) = \lim_{r \to 0} \sup \left\{ (\psi(x, t) - \varphi(y, t))^+ : |x - y| \leq r \right\}.
\]
Denote with $\vartheta^*$ the upper semicontinuous envelope of $\vartheta$, and let $\varphi$ be a good test function for $J^{1,+}\vartheta^*(t_0)$.

Consider the function (which depends on two positive parameters $\alpha, \delta$)
\[
\Phi_{\alpha, \delta}(x, y, t) = \psi(x, t) - \varphi(y, t) - \alpha|x - y|^2 - \delta|x|^2 - \varphi(t).
\]
Set \( \Delta = \{(x, y, t) : |x - y| \leq 1, t \in [0, t_0 + T]\} \), and \( \Phi_{\alpha, \delta} = \sup_{\Delta} \Phi_{\alpha, \delta} \).

Then

(i) for every \( \alpha, \delta \) exists \((x_{\alpha, \delta}, y_{\alpha, \delta}, t_{\alpha, \delta}) \in \Delta \) which is a maximum point of \( \Phi_{\alpha, \delta} \);

(ii) \( \lim_{\delta \to 0} \Phi_{\alpha, \delta} = \sup_{\Delta} \{\Phi_{\alpha} \} = \Phi_{\alpha} \);

(iii) \( \lim_{\alpha \to +\infty} \Phi_{\alpha} = \vartheta(t_0) - \phi(t_0) = 0 \);

(iv) \( \lim_{\alpha \to +\infty} \lim_{\delta \to 0} \alpha |x_{\alpha, \delta} - y_{\alpha, \delta}|^2 + \delta |x_{\alpha, \delta}|^2 = 0 \).

Moreover, for a subsequence we can obtain,

(v) \( \lim_{\alpha \to +\infty} \lim_{\delta \to 0} t_{\alpha, \delta} = t_0 \);

(vi) \( \lim_{\alpha \to +\infty} \lim_{\delta \to 0} \left( \nu(x_{\alpha, \delta}, t_{\alpha, \delta}) - \nu(y_{\alpha, \delta}, t_{\alpha, \delta}) \right) = \vartheta(t_0) = \varphi(t_0) \).

Before to give the proof of Theorem 3.6, we recall the well known property about the conservation of the notion of viscosity solution with respect to a change of variable of the unknown, with respect to an increasing smooth transformation.

**Proof of Theorem 3.6.** Consider the application, \( \Psi : (a - \frac{\varepsilon_0}{2}, b + \frac{\varepsilon_0}{2}) \to \mathbb{R} \), defined by

\[
\Psi(u) = \int_{a - \frac{\varepsilon_0}{2}}^{u} \frac{1}{\sqrt{z(\tau)}} d\tau, \quad u \in \left( a - \frac{\varepsilon_0}{2}, b + \frac{\varepsilon_0}{2} \right)
\]

since that \( z > 0 \). \( \Psi \) is well defined and \( \Psi' > 0 \), so \( \Psi \) has a \( C^2 \) inverse which we denote with \( I \) defined in \((\Psi(a - \frac{\varepsilon_0}{2}), \Psi(b + \frac{\varepsilon_0}{2}))\). If we consider the functions

\( v = \Psi \circ u, \quad \nu = \Psi \circ \nu, \quad \tau = \Psi \circ \tau, \)

then by the increasing property of \( \Psi \), \( v, \nu, \tau \) are respectively subsolution and supersolution of the following equation,

\[
\partial_t v + \tilde{F}(x, t, v, \nabla v, \nabla^2 v) = 0, \quad (x, t) \in \mathbb{R}^N \times (0, T)
\]  

(3.15)

\[
\tilde{F}(x, t, v, p, X) = \frac{1}{\tilde{F}(v)} F(x, t, I(v), I'(v)p, I'(v)X + I''(v)p \otimes p),
\]

(3.16)

for every \((x, t, v, p, X) \in \mathbb{R}^m \times [0, T) \times (\Psi(a - \frac{\varepsilon_0}{2}), \Psi(b + \frac{\varepsilon_0}{2})) \times \mathbb{R}^N \times S^N \) and, by (3.11),

\[
\nu^*(x, 0) - \nu_*(y, 0) \leq \frac{1}{\sqrt{\lambda_0}} \omega(|x - y|), \quad x, y \in \mathbb{R}^N.
\]

(3.17)

we will prove the comparison between \( \nu^*, \nu_* \). Set

\[
\theta(t) := \lim_{r \to 0^+} \sup \left\{ (\nu^*(x, t) - \nu_*(y, t))^+ : |x - y| \leq r \right\}
\]

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and denote $\vartheta^* \leq \frac{b}{\sqrt{\lambda_0}}$ its upper-semicontinuous envelope. Our aim is to show that $\vartheta^* \equiv 0$.

We will obtain this assertion, proving that

\[
\begin{cases}
(\vartheta^*)(t) \leq \Gamma_0(\vartheta^*(t)) & t \in (0, T) \\
\min((\vartheta^*)(0) - \Gamma_0(\vartheta^*(0)), \vartheta^*(0)) = 0
\end{cases}
\]

where, $\Gamma_0(\theta) := \Gamma(\sqrt{\lambda_0}\theta)$ and $\Gamma$ is given by (3.10), it holds in a viscosity sense, and then using Lemma 3.8, in fact it easy to see that $\Gamma_0$ is an Osgood type function over $[0, \frac{b}{\sqrt{\lambda_0}}]$.

Let $t_0 \in [0, T)$:

- if $\vartheta^*(t_0) = 0$ and $t_0 = 0$, we have nothing to say.
- if $\vartheta^*(t_0) = 0$ and $t_0 > 0$, $\vartheta^*$ has an interior minimum at $t_0$ and this is the same for any test function $\varphi$ for $J^+\vartheta^*(t_0)$. Since that $\varphi$ is regular, follow that,

\[
\varphi'(t_0) = 0 = \Gamma_0(0) = \Gamma_0(\vartheta^*(t_0))
\]

- if $\vartheta^*(t_0) > 0$, let $\varphi$ be a good test function for $J^{1+}\vartheta^*(t_0)$. By the boundness of $\vartheta$ and $\varphi$, we can apply Lemma 3.10. Let $\Phi_{\alpha, \delta}$ be the function defined in (3.14); for fixed $\alpha, \delta$, this function obtain a maximum in a point which we denote with $(z_1, z_2, s)$ (leaving out the dipendence from the parameters, in order to simplify the notations).

Moreover, since that $\vartheta^*(t_0) > 0$ and by the conditions (iv), (v), (vi) of Lemma 3.10 and (3.17), definitively, for large $\alpha$ and small $\delta$, we can assume $\vartheta^*(z_1, s) > \varphi_*(z_2, s), \delta z_1 \leq \sqrt{\delta}, s > 0$.

In order to obtain information about the “derivatives” of $\vartheta^*$, and noting that $s \in (0, T)$, we can apply the classical maximum principle for semicontinuous functions, which is Theorem 6 in [6],

\[
u_1(x, t) = \varphi^*(x, t) - \delta|x|^2, \quad u_2(y, t) = -\varphi_*(y, t),
\]

\[
w(x, y, t) = u_1(x, t) + u_2(y, t),
\]

for every $(x, y, t) \in \mathbb{R}^N \times \mathbb{R}^N \times (0, T)$. Set

\[
b = \varphi'(s), \quad p = \left(\begin{array}{c}
\alpha(z_1 - z_2) \\
-\alpha(z_1 - z_2)
\end{array}\right), \quad A = \alpha \left(\begin{array}{cc}
I_N & -I_N \\
-I_N & I_N
\end{array}\right)
\]

where $I_N$ is the $N \times N$ identity matrix. Then $(b, p, A) \in \mathcal{P}^2, w(z_1, z_2, s)$. There exists $(b_1, X_1), (b_2, X_2) \in \mathbb{R} \times \mathcal{S}^N$ such that

\[
(b_1, \alpha(z_1 - z_2), X_1) \in \mathcal{P}^{2+}(\varphi^*(\delta|\cdot|^2))(z_1, s)
\]

and

\[
(b_2, -\alpha(z_1 - z_2), X_2) \in \mathcal{P}^{2+}(-\varphi_*(z_2, s), s),
\]
so,

$$(b_1, \alpha(z_1 - z_2) + 2\delta z_1, X_1 + 2\delta I_N) \in \overline{P^{2,+}}(z_1, s)$$

$$(-b_2, \alpha(z_1 - z_2), -X_2) \in \overline{P^{2,-}}(z_2, s).$$

Moreover, since that $A^2 = 2\alpha A$, and $\|A\| = 2\alpha$, the following relations hold

$$-3\alpha I_{2N} \leq \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq 3\alpha \begin{pmatrix} I_N & -I_N \\ -I_N & I_N \end{pmatrix} \quad (3.18)$$

$$b_1 + b_2 = \varphi'(s) \quad (3.19)$$

where $I_{2N}$ denotes the $2N \times 2N$ identity matrix. We observe that, for continuity reasons, the equation (3.15) is also satisfied, by $\overline{u}, \overline{v}$, over the closure of the parabolic-semijets. So we have,

$$b_1 \leq -\tilde{F}(z_1, s, \overline{u}^*(z_1, s), \alpha(z_1 - z_2) + 2\delta z_1, X_1 + 2\delta I_m),$$

$$b_2 \leq +\tilde{F}(z_2, s, \overline{v}_s(z_2, s), \alpha(z_1 - z_2), -X_1).$$

Set $p := \alpha(z_1 - z_2)$, then these two relations and (3.19) imply,

$$\varphi'(s) \leq \tilde{F}(z_2, s, \overline{v}_s(z_2, s), p, -X_2) + -\tilde{F}(z_1, s, \overline{u}^*(z_1, s), p + 2\delta z_1, X_1 + 2\delta I_N). \quad (3.20)$$
\[ \begin{aligned}
&\hat{F}(z_2, s, \nu_s(z_2, s), p, -X_2) - \hat{F}(z_1, s, \nu^*(z_1, s), p + 2\delta z_1, X_1 + 2\delta I_N) = \\
&\left[ \hat{F}(z_2, s, \nu_s(z_2, s), p, -X_2) - \frac{1}{I'(\nu_s(z_2, s))} F(z_2, s, I(\nu_s(z_2, s)), I'(\nu_s(z_2, s))) \times \right.
\left. (p + 2\delta z_1), -I'(\nu_s(z_2, s)) X_2 + I''(\nu_s(z_2, s)) p \otimes p \right]
\times \frac{1}{I'(\nu_s(z_2, s))} \times
\left[ F(z_2, s, I(\nu_s(z_2, s)), I'(\nu_s(z_2, s))(p + 2\delta z_1), -I'(\nu_s(z_2, s)) \times \\
X_2 + I''(\nu_s(z_2, s))(p + 2\delta z_1) \otimes (p + 2\delta z_1)) \right]
\end{aligned} \]

Now we estimate the single terms in the brackets \([\cdot]\) in the right hand side of this equality. For sufficiently large \(\alpha\), and \(\delta < 1\), set \(R_\alpha := (\alpha + 2)^2 \max(\sqrt{\Lambda_0}, \|\nu\|_\infty) + 3\alpha \sqrt{\Lambda_0}\) in (3.7), then by (3.16), the inequalities (3.18), Lemma 3.10 and the assumptions on the function \(z\), we infer,

\[ \begin{aligned}
&\hat{F}(z_2, s, \nu_s(z_2, s), p, -X_2) - \frac{1}{I'(\nu_s(z_2, s))} F(z_2, s, I(\nu_s(z_2, s)), I'(\nu_s(z_2, s))) \times \\
& (p + 2\delta z_1), -I'(\nu_s(z_2, s)) X_2 + I''(\nu_s(z_2, s)) p \otimes p \\
& \leq \frac{1}{\sqrt{\Lambda_0}} \nu_{1, R_\alpha} (|I'(\nu_s(z_2, s))(p + 2\delta z_1)| - I'(\nu_s(z_2, s))|p|) \\
& \leq \frac{1}{\sqrt{\Lambda_0}} \nu_{1, R_\alpha} (2 \sqrt{\delta \Lambda_0}),
\end{aligned} \]

since that

\[ |I'(\nu_s(z_2, s))(p + 2\delta z_1)|, \quad |I'(\nu_s(z_2, s))p| \leq R_\alpha \]

and

\[ | - I'(\nu_s(z_2, s)) X_2 + I''(\nu_s(z_2, s)) p \otimes p| \leq R_\alpha \]

The inequality (3.8) is satisfied for

\[ X = \begin{aligned}
&I'(\nu_s(z_2, s))(X_1 + 2\delta I_N) + 2\delta I''(\nu_s(z_2, s))(p \otimes z_1 + z_1 \otimes p) \\
&+ 4\delta^2 I''(\nu_s(z_1, s)) z_1 \otimes z_1
\end{aligned} \]
\( Y = \mathcal{I}(\pi_*(z_2, s))X_2. \)

\( Z = \mathcal{I}''(\pi_*(z_2, s))p \otimes p \)

In fact, (3.18) implies

\[
-\varepsilon_1 I_{2N} \leq \begin{pmatrix} X & O \\ O & Y \end{pmatrix} = \mathcal{I}'(\pi_*(z_2, s)) \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} + \begin{pmatrix} 2\delta \mathcal{I}'(\pi_*(z_2, s))I_N \\ 0 \end{pmatrix} \\
+ 2\delta \alpha \mathcal{I}''(\pi_*(z_2, s)) \begin{pmatrix} (z_1 - z_2) \otimes z_1 + z_1 \otimes (z_1 - z_2) \\ 0 \end{pmatrix} \\
+ 4\delta^2 \mathcal{I}''(\pi_*(z_2, s)) \begin{pmatrix} z_1 \otimes z_1 \\ 0 \end{pmatrix} \\
\leq 3\alpha \sqrt{\Lambda_0} \begin{pmatrix} I_N \\ -I_N \end{pmatrix} \quad + \\
+ 2(\delta \sqrt{\Lambda_0} + (\alpha \sqrt{\delta} + \delta)\|z\|_\infty) \begin{pmatrix} I_N \\ 0 \end{pmatrix}.
\]

So the relation (3.8) holds if we choose \( \varepsilon_2 = 3\alpha \sqrt{\Lambda_0}, \varepsilon_3 = 2(\delta \sqrt{\Lambda_0} + (\alpha \sqrt{\delta} + \delta)\|z\|_\infty), \varepsilon_1 = \varepsilon_2 + \varepsilon_3. \) If we choose \( R \) as in (3.8), independent of \( \delta, \) and \( R \geq R_0, \) then holds,

\[
\frac{1}{\mathcal{I}'(\pi_*(z_2, s))} \left[ F(z_2, s, \mathcal{I}(\pi_*(z_2, s)), \mathcal{I}'(\pi_*(z_2, s))(p + 2\delta z_1), -\mathcal{I}'(\pi_*(z_2, s))X_2 + \mathcal{I}''(\pi_*(z_2, s))p \otimes p)\right. \\
- F(z_1, s, \mathcal{I}(\pi_*(z_2, s)), \mathcal{I}'(\pi_*(z_2, s))(p + 2\delta z_1), \mathcal{I}'(\pi_*(z_2, s))(X_1 + 2\delta I_N) + \mathcal{I}''(\pi_*(z_2, s))(p + 2\delta z_1) \otimes (p + 2\delta z_1)] \]

\[
\leq \frac{1}{\sqrt{\Lambda_0}} \nu_2(|z_1 - z_2|(1 + \sqrt{\Lambda_0} |z_1 - z_2| + 2\sqrt{\Lambda_0 \delta}) + \varepsilon_2 |z_1 - z_2| \end{equation}

+ \frac{1}{\sqrt{\Lambda_0}} \nu_2(R(2\delta)).
\]

If we take, in (3.10) \( \lambda = \mathcal{I}'(\pi^*(z_1, s)), \lambda = \mathcal{I}'(\pi_*(z_2, s)), \kappa = \mathcal{I}''(\pi_*(z_1, s)), \)

\( \tilde{\lambda} = \mathcal{I}''(\pi_*(z_2, s)), q = p + 2\delta z_1, X = X_1 + 2\delta I_N \) and

\[
\frac{1}{\mathcal{I}'(\pi_*(z_2, s))} F(z_1, s, \mathcal{I}(\pi_*(z_2, s)), \mathcal{I}'(\pi_*(z_2, s))(p + 2\delta z_1), \mathcal{I}'(\pi_*(z_2, s))X_1 + 2\delta I_N + \mathcal{I}''(\pi_*(z_2, s))(p + 2\delta z_1) \otimes (p + 2\delta z_1) \]

\[
\leq \frac{1}{\mathcal{I}'(\pi^*(z_1, s))} F(z_1, s, \mathcal{I}(\pi^*(z_1, s)), \mathcal{I}'(\pi^*(z_1, s))(p + 2\delta z_1), \mathcal{I}'(\pi^*(z_1, s))X_1 + 2\delta I_N + \mathcal{I}''(\pi^*(z_1, s))(p + 2\delta z_1) \otimes (p + 2\delta z_1)) \leq \Gamma_0(z_1, s) - \pi_*(z_2, s))
\]
In fact, \( I'(v) = \sqrt{z(I(v))} \in [\sqrt{\lambda_0}, \sqrt{\Lambda_0}] \) and \( I''(v) = \frac{z'(I(v))}{2} \). Replacing the obtained estimates in the inequality (3.20), yields

\[
\varphi'(s) \leq \nu_{1,R_0}(2\sqrt{\delta \Lambda_0}) + \\
\frac{1}{\sqrt{\lambda_0}} \nu_2(|z_1 - z_2|(1 + \sqrt{\Lambda_0} |z_1 - z_2| + 2 \sqrt{\lambda_0} \delta + \varepsilon_2 |z_1 - z_2|^2) + \\
+ \nu_{2,R}(2\varepsilon_3) + \Gamma_0(\varphi^*(z_1, s) - \psi_*(z_2, s)).
\]

Letting \( \delta \to 0 \) (without moving \( \alpha \)), and considering the assertion (iv) of Lemma 3.10, yields,

\[
\varphi'(\lim_{\delta \to 0} s) \leq \Gamma_0(\lim_{\delta \to 0} \nu^*(z_1, s) - \psi_*(z_1, s)) \\
+ \frac{1}{\sqrt{\lambda_0}} \nu_2(\lim_{\delta \to 0} |z_1 - z_2| + 4 \sqrt{\lambda_0} |z_1 - z_2|^2)
\]

then, again with condition (iv) and (vi) of Lemma 3.10, letting \( \alpha \to \infty \) in the above inequality, yields,

\[
\varphi'(t_0) \leq \Gamma_0(\theta^*(t_0))
\]

By Lemma 3.8, and the monotonicity of the application \( \Psi \), the comparison assertion (3.12) holds.

---

In order to show why our conditions on the Hamiltonian \( F \) are more general with respect to the hypothesis up to now proposed in the viscosity theory to state the comparison principle, we build some examples which do not satisfy usual assumptions for the standard comparison result, with particular regard to the recent paper of P.L Lions and P.E. Souganidis [24].

In the example 3.11, we consider an Hamiltonian which has not the Lipschitz regularity with respect to \( u \). In that case the appeal to the Osgood property becomes a necessary requirement to obtain the comparison. For seeing that our structural approach is quite different than the technical assumptions proposed in [24] and following the same kind of setting also used in [24] for solving a stochastic pde, in the example 3.13 we give an Hamiltonian \( F_\Phi \), which comes from a function \( F = F(\nabla u, \nabla^2 u) \), through a regular transformation \( \Phi \). The first function \( F \), which we propose in such example, is not Lipschitz continuous, as instead require the authors of [24] in order to apply their techniques. Our method not only, seems to be efficient when there is a lack of regularity in the Hamiltonian, but also when we lose the hypothesis (1.12), pag. 621 in [24], see (3.28) below. Actually, in example (3.13) we consider another function \( F \) which is Lipschitz continuous, but does not satisfy that condition. In both cases our Theorem 3.6 can be successfully applied. Finally, before to begin the study of the financial model (1.2), in the observation 3.14, we remark that it seems difficult to directly combine the results of [24] for deducing the comparison principle for the equation (1.2).

**Example 3.11.** Let \( F : (-1, \infty) \times \mathbb{R}^N \times S^N \) be the Hamiltonian defined by

\[
F(u, p, X) = -\text{tr}(X) + \frac{1}{u+1} |p|^2 + \varphi(u),
\]

\[
\varphi(u) = \begin{cases} 
(u^2 + u) \log(u) & \text{if } u > 0, \\
0 & \text{if } u \leq 0.
\end{cases}
\]
This function does not get back in the class of classical problem, for which the comparison is already stated (see [7], [3]); actually we have a quadratic gradient bounds and also a lack of regularity in $u$. Of course $F$ satisfies conditions (3.5), (3.6) of Theorem 3.6 in its domain of definition, while in the interval $I \equiv [-\frac{1}{2}, \frac{1}{e}]$ verifies (3.7)-(3.9). By the regularity of $F$ with respect to $p$ and $X$, to verifying condition (3.10), it suffices to prove the existence of a function $z = z(u) > 0$ such that,

$$F_z(u, p, X) = \frac{1}{\sqrt{z(u)}}F(u, \sqrt{z(u)}p, \sqrt{z(u)}X + \frac{z'(u)}{2}p \otimes p)$$

$$= -tr(X) + \left(\frac{\sqrt{z(u)}}{u + 1} - \frac{z'(u)}{2\sqrt{z(u)}}\right)|p|^2 + \frac{\varphi(u)}{\sqrt{z(u)}}, \quad (3.23)$$

satisfies

$$F_z(u, p, X) - F_z(v, p, X) \geq -\Gamma(u - v),$$

$$\forall \frac{1}{2} \leq v < u \leq \frac{1}{e}, \quad (p, X) \in \mathbb{R}^N \times \mathcal{S}^N,$$  \quad (3.24)

with $\Gamma$ an Osgood-type function. By the previous consideration about the dependence on $u$, we cannot turn to the $u$-partial derivative of $F_z$, to obtain estimate (3.24). Consider the function

$$\Gamma(h) = \begin{cases} h \log\left(\frac{1}{h}\right) & \text{if } 0 < h < \frac{1}{e} \\ \frac{1}{e} & \text{if } \frac{1}{e} \leq h \leq \frac{1}{e} + \frac{1}{2} \end{cases}, \quad (3.25)$$

for $h > 0$; then $\Gamma$ is an Osgood-type function in the sense of definition 3.5, while it is a simple exercise to show that $\Gamma(h) = \sup_{x \in I} [x \log(x) - (x + h) \log(x + h)]$. Therefore, choosing $z(u) = (u + 1)^2$, the inequality (3.24), becomes

$$u \log(u) - v \log(v) \geq -\Gamma(u - v),$$

which by the definition of $\Gamma$ it holds. Hence, applying Theorem 3.6, we have the comparison between sub/super solutions which take values in $I$, for (3.22).

**Remark 3.12.** In [24], it was proposed a comparison principle for a class of problems which do not satisfy the usual assumptions. To treat a stochastic differential problem the authors have to prove a comparison between bounded solutions, for a deterministic problem. They consider the following equation,

$$\partial_t v = F_\Phi(v, \nabla v, \nabla^2 v),$$

$$F_\Phi(u, p, X) = \frac{1}{\Phi'(u)}F(\Phi'(u)p, \Phi'(u)X + \Phi''(u)p \otimes p),$$

$$\forall \quad (u, p, X) \in \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N,$$  \quad (3.27)

in $\mathbb{R}^N \times (0, T)$, where $\Phi$ is a smooth function, $F$ is independent of the unknown and $F(\cdot, X) \leq F(\cdot, Y)$, if $X \leq Y$. Moreover the function $F$ is a globally Lipschitz continuous function of their variables, and satisfies a structural condition (see (1.12), pag. 621 in [24]),

$$
\begin{cases}
X \cdot \nabla_X F + p \cdot \nabla_p F - F \leq C & \text{for a.e. } (X, p) \\
X \cdot \nabla_X F + p \cdot \nabla_p F - F \geq C & \text{for a.e. } (X, p)
\end{cases}
\quad (3.28)$$
for some constant $C > 0$. Under these assumptions, making a global change of the unknown $v = \phi(w)$, by a transformation $\phi$, they prove that the partial derivative of the new Hamiltonian with respect to the unknown $w$ is bounded from above. This fact, implies that the comparison result follows from the classical theory.

If we consider the problem (3.27), coming from a Lipschitz continuous function $F$ which satisfies (3.28), we can define $z(v) = [\phi'(\phi^{-1}(v))]^2$ to verify condition (3.10) of Theorem 3.6; while the other properties required by Theorem 3.6 are easily derived by the degenerate ellipticity, and the regularity assumptions on $F$, and $\Phi$. Considering the previous setting we build two Hamiltonians $F_\theta$, which satisfy our conditions but such that it is not possible to use the methods of [24].

**Example 3.13.** In many situations the Hamiltonian $F_\theta$, is not Lipschitz continuous, so the conditions (3.28), cannot be verified. Let $\gamma$ be a number in $(0,1)$, and consider the 1-dimensional Hamiltonian

$$
\tilde{F}(v, q, Y) = Y - \frac{1}{v} q^2 - v^{1-\gamma} |q|^\gamma, \quad \forall (v, q, Y) \in (0, \infty) \times \mathbb{R} \times \mathbb{R}.
$$

(3.29)

Of course $\partial_q \tilde{F}$ is not bounded from above. Moreover if $\Phi(v) = \log(v)$, and $F(p, X) = X - |p|^\gamma$, then

$$
\tilde{F} = F_\Phi.
$$

(3.30)

The assumptions (3.6), (3.7), (3.9) follow by the linearity of $F$ with respect to $X$ and the holder continuity of $F$ with respect to $p$. Now it is an exercise to prove that choosing $z(v) = e^{-2v}$, we obtain also condition (3.10). Hence although the lack of regularity for $F$, we could apply Theorem 3.6.

As a second example consider,

$$
F(X, p) = X + g(p), \quad (p, X) \in \mathbb{R}^2
$$

$$
g(p) = \begin{cases} 
\log(1 + p) & \text{if } p > 0 \\
- \log(1 - p) & \text{if } p \leq 0 
\end{cases}
$$

$$
\Phi(u) = \arctan(u).
$$

(3.31)

Then $F$ is a Lipschitz continuous function in $\mathbb{R}^2$, and $\Phi$, has the regularity required in [24]. Nevertheless $p \cdot g'(p) - g(p)$, is neither bounded from above nor from below, so (3.28) does not hold. To applying our procedure, as in example 3.11 it suffices to prove that the Hamiltonian

$$
(F_\Phi)z(u, p, X) = X + l(u)p^2 + \frac{g(w(u)p)}{w(u)},
$$

$$
l = (\sqrt{z})' + \frac{\Phi''}{\Phi'} \sqrt{z},
$$

$$
w = \Phi' \sqrt{z},
$$

(3.32)

satisfies $\Delta \equiv (F_\Phi)z(u, p, X) - (F_\Phi)z(v, p, X) \leq \gamma \cdot (u - v)$, for an appropriate function $z$, $u \geq v$ in a bounded interval, and $\gamma > 0$.

Taking $z(u) = (u^2 + 1)^2 \cdot (\beta - (1/2) \arctan^2(u))^2$, with $8\beta > \pi^2$, we have $l(u) = - \arctan(u)$, $w(u) = \beta - (1/2) \arctan^2(u)$. Then by the definition of $g$ and (3.32),

$$
\Delta \leq \frac{1}{u^2} [ (l(u) - l(v)) |\bar{w}p|^2 + \alpha (1 + \log(1 + |\bar{w}p|))],
$$

(3.33)
where \( \tilde{w} > 0 \), is a point between \( w(u) \) and \( w(v) \), \( \alpha > 0 \) depends on the bounds of \( u \). Therefore, by \( l' < 0 \), it follows the assertion.

The previous observation and the last two examples show that our comparison principle really extend the result of [24].

**Remark 3.14.** As we have already said in the introduction of the present paper, we can neither apply the existing comparison results nor the same technique used in [24], to obtain the result for the financial model. In the classical theory was not considered a quadratic dependence in the variable \( p \). In [24] the extension to problems with this kind of growth was examined, but even if we eliminate the dependence by the spatial variable \( x \) in (1.2), changing in this way the structure of the model, we can reduce the equation to a pde, whose Hamiltonian has the form (3.27), where \( F(p, X) = -(1/2)\text{tr}(\sigma\sigma^\top X) + C|\sigma^\top p + f|^2 \), for some constants \( C \) and \( f = (f_1, \ldots, f_4) \). Nevertheless this Hamiltonian is not globally Lipschitz continuous.

## 4 Application to the Financial Model

In this section we give an application of Theorem 3.6, for treating a real problem which comes from the mathematical finance, and which is described by the equation (1.2). We shall make the following assumptions:

1. **(P₁)** \( \mu \) is bounded, and \( \mu(\cdot, t) \in \text{Lip}(\mathbb{R}^N) \), for every \( t \in [0, T) \), with a Lipschitz constant independent of the time.

2. **(P₂)** \( \rho, \tau > 0 \) are known real parameters, \( h \in C^{2,1}(\mathbb{R}^N \times [0, T]) \), is bounded from above and nonnegative. Moreover \( \nabla h \) is bounded, and, \( \partial_t h(\cdot, t), \text{tr}(\sigma\sigma^\top(t))\nabla^2 h(\cdot, t), \nabla h(\cdot, t) \in \text{Lip}(\mathbb{R}^N) \) with Lipschitz constants independent of the time \( t \in [0, T) \). \( \xi \in C^{1}([0, T]) \), is positive.

3. **(P₃)** \( U_0 \in \text{Lip}(\mathbb{R}^N) \), with \( U_0 \geq 0 \), and bounded from above.

**Remark 4.1.** The assumption about the sign of \( h \) and \( \xi \), reflect the financial modelization: \( h \) describes the cash-flow from the pool of mortgages, while \( \xi \) represents the value of some Bank-account, which moves with the interest rate \( r \).

For more generality, the initial datum \( U_0 \) is assumed nonnegative and regular because, in a real framework, it is constant and equal to zero. Actually at the maturity date \( T \), the mortgage-holders have paid off their debt.

The constants \( \rho, \tau > 0 \) have a financial interpretation, in fact they represent, respectively, the risk-aversion coefficient and the coupon rate paid by the mortgage-holder.

Using Theorem 3.6, it is now easily to show the comparison for equation (1.2).

**Theorem 4.2.** Assume conditions (P₁), (P₂), (P₃). Set the functions

\[
\bar{k}(t) := e^{-\int_0^t r(s)ds} \left( \inf_{\mathbb{R}^N} U_0 + \int_0^t e^{\int_0^s r(x)dx} \inf_{\mathbb{R}^N}[(\tau - r(s))h(x, s)]ds \right)
\]

\[
\underline{k}(t) := K_0 t + c_0, \quad c_0 := \max \left( \sup_{\mathbb{R}^N} U_0, \sup_{[0,T]} \bar{k} \right),
\]

\[
K_0 := \sup_{\mathbb{R}^N \times [0, T]} \left( \frac{(\tau - r(t))h(x, t) - c_0 r(t)}{1 + t \cdot r(t)} \right)^+, \quad t \in [0, T).
\]
If exists a modulus $\nu$ such that

$$|h(x,t) - h(x,s)| \leq \nu(|t-s|), \quad (4.35)$$

and

$$\xi(t) + h(x,t) + \underline{\kappa}(t) > 0, \quad (4.36)$$

for every $t, s \in [0,T)$, $x \in \mathbb{R}^N$, then $\underline{k} \leq \underline{\kappa}$ are, respectively, $C^1([0,T))$ sub/supersolution of (1.2), and for every sub/supersolution $\underline{U}$, $\overline{U}$ of (1.2), such that $\underline{k} \leq \underline{U}$, $\overline{U} \leq \overline{\kappa}$, then $\underline{U} \leq \overline{U}$ holds in $\mathbb{R}^N \times [0,T)$.

**Proof.** Since the inequalities (4.35), (4.36), the function $k$, defined by (4.34), is a regular subsolution of (1.2): actually (4.35) implies continuity of $\inf_{x \in \mathbb{R}^N} [(\tau - r(t))h(x,t)]$ as a function of the time. It is also easy to see, by the definition of $K_0$, that $\overline{\kappa}$ is a supersolution of the same problem. By assumptions $(P_1)$-$(P_3)$, we can consider the change $u = U + h + \xi$. Then if $U$ is a viscosity sub/supersolution of (1.2) with $k \leq U \leq \overline{\kappa}$, then by (4.36), $u$ is a positive viscosity sub/supersolution of

$$\partial_t u - \frac{1}{2}tr(\sigma^\top \nabla^2 u) - (\mu, \nabla u) + \rho \frac{|\sigma^\top \nabla u - \sigma^\top \nabla h|^2}{u} + r(t)u + g(x,t) = 0, \quad (4.37)$$

in $\mathbb{R}^N \times (0,T)$, where $g := -\partial_t h + \frac{1}{2}tr(\sigma^\top \nabla^2 h) + (\mu, \nabla h) - \tau h - (\xi' + r(t)\xi)$. Moreover by the conditions on $h$, the new initial datum $u_0 = U_0 + h(x,0) + \xi(0)$, is globally Lipschitz continuous. For proving the comparison through Theorem 3.6 it suffices to show that conditions (3.5)-(3.10) hold for equation (4.37), when we consider the interval $[a,b] = [\inf_{x \in \mathbb{R}^N} (\underline{k} + h + \xi), K_0T + c_0 + \sup_{h+\xi}]$. Conditions (3.5)-(3.7) follow immediately, while conditions (3.8)-(3.9) follow by the linearity of the second order part in the equation (4.37) and by the Lipschitz regularity for $g$ and assumptions $(P_1)$, $(P_2)$. Set $m_0 := \inf_{x \in \mathbb{R}^N \times [0,T]} (\underline{k} + h + \xi)$, $M_0 := K_0T + c_0 + \sup(h+\xi)$ and consider the function $z(u) = (\lambda_1 u - \lambda_2)^2$, $u \in [m_0, M_0]$, for constants $\lambda_1$, $\lambda_2$ such that $\lambda_1 m_0 > \lambda_2 > 0$. Now we verify (3.10) with this choice. Let $\lambda$, $\bar{\lambda}$, $\kappa$, $\bar{\kappa}$ be as in condition (3.10) of Theorem 3.6, $(x,t,X) \in \mathbb{R}^N \times [0,T) \times S^N$, $R > 0$ and $|q| \leq R$, then for $M_0 \geq u > v \geq m_0$, 

```
are positive constants. By (4.38) we see that \( \Gamma(x) \) is a function as in Definition 3.5, and conditions of Theorem 3.6 are satisfied for problem (4.37). The proof of comparison is now a direct consequence of Theorem 3.6 applied to (4.37).

\[ \Gamma(x) = \rho \left( \frac{(C_2 M_0)^2}{4 \lambda_2} \right) + C_2 \left( \| \sigma^T \|_\infty, \| \nabla h \|_\infty \right) \]

where,

\[ C_1 = C_1(R, \lambda_1, \lambda_2, m_0, M_0, \| \sigma^T \|_\infty, \| \nabla h \|_\infty) \]

\[ C_2 = C_2(\lambda_1, \lambda_2, m_0, M_0, \| \sigma^T \|_\infty, \| \nabla h \|_\infty) \]

are positive constants. By (4.38) we see that \( \Gamma(x) = \rho \left( \frac{(C_2 M_0)^2}{4 \lambda_2} \right) + C_2 \) is a direct consequence of Theorem 3.6 applied to (4.37).

**Remark 4.3.** Also the functions \( k, \overline{k} \) defined in (4.34), have a financial interpretation. Actually in the real differential problem for MBS, where \( U_0 \equiv 0 \), and after the usual time-change of variable \( T - t = s \), they represent, respectively, the evolution of a Zero-Coupon-Bond with maturity \( T \), which follows the movements of \( r \), and a linear above estimate in the time for the evolution of the remaining principal in the pool.

**Corollary 4.4.** Under the same assumptions of Theorem 4.2, the problem (1.2) has a unique continuous viscosity solution \( U \), such that \( k \leq U \leq \overline{k} \) over \( \mathbb{R}^n \times [0, T) \).
Proof. By Theorem 4.2, we have proved the comparison between viscosity sub-supersolution of problem (4.2), then the assertion follows adapting a version of Perron’s Method for the elliptic case by H. Ishii in [16] to the parabolic case.

Even though, in (4.37), we have a term which depends on the unknown \( u \) which multiplies a one-order term in a decreasing form with respect to the variable \( u \), as is showed in the following Proposition 4.5 is possible to prove a regularity result of the viscosity solution. As we prove in the following Theorem, we can consider a general class of quasilinear equations for which is possible to obtain a Lipschitz regularity for the viscosity solution.

**Theorem 4.5.** Let \( v \) be a continuous viscosity solution of the following problem

\[
\partial_t v - \frac{1}{2} \text{tr}(\sigma \sigma^T \nabla^2 v) - \langle \mu, \nabla v \rangle + \lambda_1(v)\sigma^T \nabla v |^2 + \lambda_2(v)\langle \sigma^T \nabla v, w \rangle + f(x, t, v) = 0, \quad (x, t) \in \mathbb{R}^N \times (0, T).
\]

With an initial datum \( v_0 \in \text{Lip}(\mathbb{R}^N) \). Suppose that \( v \) takes values in the interval \([c, d]\), \( \lambda_1, \lambda_2 \in C^1([c, d]) \), \( \lambda_1' > 0 \) on \([c, d]\), \( \sigma, \mu \) are as in problem (1.2), with \( \mu, w \) satisfying assumption (P1), \( f \in C(\mathbb{R}^N \times [0, T) \times [c, d]) \) and \( f(\cdot, t, \cdot) \in \text{Lip}(\mathbb{R}^N \times [c, d]) \) with a Lipschitz constant which is independent of the time. If \( M > \frac{\text{Lip}(v_0)}{2} \), and

\[
C = 2\text{Lip}(\mu) + \frac{\|\lambda_1'\|^2_{\infty} \|w\|^2_{\infty}}{4\min_{[c, d]} \lambda_1'} + 2\|\lambda_2\|_{\infty} \|\sigma^T\|_{\infty} \text{Lip}(w) + \text{Lip}(f)(\frac{1}{2M} + 1)
\]

then

\[
|v(x, t) - v(y, t)| \leq 2Me^{Ct}|x - y|
\]

holds for every \( x, y \in \mathbb{R}^N \), \( t \in [0, T) \). In particular for every \( t \in [0, T) \), \( v(\cdot, t) \in \text{Lip}(\mathbb{R}^N) \).

As consequence of Theorem 4.5, we have a regularity result for our differential model.

**Proposition 4.6.** Assume the same assumptions of Theorem 4.2. If \( U \) is the viscosity solution of (1.2), then \( U(\cdot, t) \in \text{Lip}(\mathbb{R}^N) \), with a Lipschitz constant which is independent of the time.

First use Theorem 4.5 to prove this Proposition, then we shall give the proof of the Theorem.

Proof. Consider \( m_0, M_0 \) and the change \( U + h + \xi = u \), already used in the proof of Theorem 4.2. Then \( u \) solves, in a viscosity sense the equation (4.37), with an initial datum \( u_0 \in \text{Lip}(\mathbb{R}^N) \); moreover, by assumption (P3), \( u \in [m_0, M_0] \). By the regularity of \( h \), it suffices to prove the assertion for the function \( u \). For a regular trasformation \( I \), defined in a open neighbourhood of
some interval \([c, d]\), with \(I' > 0\), \(I([c, d]) = [m_0, M_0]\), the function \(v := I^{-1}(u)\), is a solution in a viscosity sense of a Cauchy problem which has the same structure of problem (4.39) with,

\[
\lambda_1(v) = \frac{d}{dv} \log \left( \frac{I'(v)}{I'(v)^2} \right), \quad \lambda_2(v) = -\frac{2\rho}{I(v)}, \quad (4.42)
\]

\[
w(x, t) = \sigma^\top(t)\nabla h(x, t),
\]

\[
f(x, t, v) = \frac{|\sigma^\top(t)\nabla h(x, t)|^2}{I(v)I'(v)} + \frac{g(x, t)}{I'(v)} + r(t)\frac{I(v)}{I'(v)}.
\]

for every \((x, t, v) \in \mathbb{R}^N \times [0, T] \times [c, d]\). Moreover the initial datum is \(v_0 = I^{-1}(u_0)\). For defining the transformation \(I\), we take the same kind of function \(z\), which we have used for proving Theorem 4.2,

\[
I(v) = m_0 \frac{e^{\frac{2m_0}{m_0}}} {2}, \quad \forall \ v \in \left[0, \frac{m_0}{2} \log \left( \frac{2M_0}{m_0} - 1 \right) \right].
\]

This choice, yields,

\[
\lambda_1'(v) = \frac{4\rho}{m_0^2} \frac{e^{\frac{2m_0}{m_0}}}{(e^{\frac{2m_0}{m_0}} + 1)^2} > 0.
\]

By the assumption \((P_1)\), \((P_2)\) it follows that \(w, f\) satisfy the conditions of Theorem 4.5, so applying this Theorem we infer that exists a constant \(C > 0\), such that

\[
|v(x, t) - v(y, t)| \leq M e^{Ct}|x - y|, \quad \forall \ x, y \in \mathbb{R}^N, \ t \in [0, T),
\]

with \(M > \text{Lip}(v_0)\), and \(C = C(M)\); hence the definition (4.45), yields

\[
|u(x, t) - u(y, t)| \leq M \left( \frac{2M_0}{m_0} - 1 \right) e^{Ct}|x - y|, \quad \forall \ x, y \in \mathbb{R}^N, \ t \in [0, T). \quad (4.48)
\]

**Proof of Theorem 4.5.** Consider the function \(Q\) defined by \(Q := ve^{-Ct}\), where \(C \geq 0\) is a positive constant given by (4.40) the function \(Q\) is a continuous viscosity solution of the equation

\[
\partial_t Q - \frac{1}{2} tr(\sigma \sigma^\top \nabla^2 Q) - \langle \mu, \nabla Q \rangle + e^{Ct} \lambda_1(Qe^{Ct})|\sigma^\top \nabla Q|^2
\]

\[
+ \lambda_2(Qe^{Ct})\langle \sigma^\top \nabla Q, w \rangle + e^{-Ct} f(x, y, t, Qe^{Ct}) + CQ = 0.
\]

With the same initial datum \(v_0\). For \(\gamma, \delta, \varepsilon > 0\), \(2M > \text{Lip}(v_0)\) we set

\[
H(x, y, t) := Q(x, t) - Q(y, t) - \varepsilon|x|^2 - K(x, y, t),
\]

\[
K(x, y, t) := M \left( \frac{|x - y|^2}{\delta} + \frac{\gamma}{T - t} \right), \quad (x, y, t) \in \mathbb{R}^N \times [0, T).
\]

We will show that for every \(\delta, \gamma > 0\) there is \(\varepsilon_0 = \varepsilon_0(\delta, \gamma) > 0\) such that for \(0 < \varepsilon < \varepsilon_0\) we have,

\[
H(x, y, t) \leq 0, \quad \forall \ (x, y, t) \in \mathbb{R}^N \times [0, T).
\]

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If the inequality (4.51) holds, then by the inequality $\frac{|x-y|^2}{\delta} + \delta \geq 2|x-y|$, for every $\delta > 0$, considering $(\overline{x}, \overline{y}, \overline{t}) \in \mathbb{R}^N \times (0, T)$, with $\overline{x} \neq \overline{y}$, and choosing $\delta = |\overline{x} - \overline{y}|$, by (4.51) it follows,

$$Q(\overline{x}, \overline{t}) - Q(\overline{y}, \overline{t}) \leq 2M|\overline{x} - \overline{y}| + \varepsilon|\overline{x}|^2 + \frac{\gamma}{T - \overline{t}}.$$  \hspace{1cm} (4.52)

Then letting $\gamma, \varepsilon \to 0$, we have the assertion of Theorem 4.5. Therefore it suffices to prove (4.51). Suppose that (4.51) were false. Then there would exist $\delta_0, \gamma_0 > 0$, such that,

$$\sup_{\mathbb{R}^N \times (0, T)} H > 0, \quad \delta = \delta_0, \quad \gamma = \gamma_0.$$  \hspace{1cm} (4.53)

holds for a subsequence $\varepsilon = \varepsilon_n \to 0$. Since that $Q$ is bounded, we see $H < 0$ for sufficiently large $\|(x, y)\|$. Since that $2M > Lip(v_0)$ we also see $H \leq 0$ at $t = 0$. Clearly $H \to -\infty$ at $t = T$; so (4.53) now implies that $H$ takes its positive maximum over $\mathbb{R}^N \times [0, T)$ at a point $(\hat{x}, \hat{y}, \hat{t})$, with $\hat{t} \in (0, T)$. First we study the behavior of the maximum point as $n \to \infty$. Since $H(\hat{x}, \hat{y}, \hat{t}) > 0$, by (4.53), it follows from this observation,

$$\varepsilon|\hat{x}| = \sqrt{\varepsilon^2 |\hat{x}|^2} = \sqrt{\varepsilon} \sqrt{\varepsilon|\hat{x}|^2} \leq \sqrt{\varepsilon(d - c)} = \sqrt{\varepsilon_n(d - c)} \to 0, \quad n \to \infty.$$  \hspace{1cm} (4.54)

Since $H$ attains its maximum over $\mathbb{R}^N \times [0, T)$ at $(\hat{x}, \hat{y}, \hat{t})$, if $w(x, y, t) := [Q(x, t) - \varepsilon|x|^2 - Q(y, t)]$, we infer that,

$$(\partial_t K(\hat{x}, \hat{y}, \hat{t}), \nabla K(\hat{x}, \hat{y}, \hat{t}), \nabla^2 K(\hat{x}, \hat{y}, \hat{t})) = \mathcal{P}^{2+} w(\hat{x}, \hat{y}, \hat{t}).$$  \hspace{1cm} (4.55)

Now we apply the usual Theorem of M.G. Crandall, and H. Ishii in [6], with $u_1(x, t) := Q(x, t) - \varepsilon|x|^2$, $u_2(y, t) = -Q(y, t)$; for $\varepsilon = \frac{2b_0}{M}$, there are $(b_1, X_1)$, $(b_2, X_2) \in \mathbb{R} \times \mathcal{S}^N$, such that,

$$(b_1, p + 2\varepsilon \hat{x}, X_1 + 2\varepsilon I_N) \in \mathcal{P}^{2+} Q(\hat{x}, \hat{t}),$$

$$(b_2, p, -X_2) \in \mathcal{P}^{2-} Q(\hat{y}, \hat{t}).$$  \hspace{1cm} (4.56)

where $p := \frac{2M}{\delta_0}(\hat{x} - \hat{y})$. Moreover

$$\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq \frac{6M}{b_0} \begin{pmatrix} I_N & -I_N \\ -I_N & I_N \end{pmatrix}, \quad b_1 + b_2 = \frac{\gamma_0}{(T - \overline{t})^2}.$$  \hspace{1cm} (4.57)

To simplify notations we respectively denote with an over-bar, and an under-bar, the value of every function which at $(\hat{x}, \hat{t})$ and $(\hat{y}, \hat{t})$, while for every function which depends only on the time we omit that dependence. By (4.49), (4.56), we can infer,

$$b_1 \leq \frac{1}{2} tr (\sigma^T (X_1 + 2\varepsilon I_N)) + \langle \overline{\sigma}, p + 2\varepsilon \hat{x} \rangle - \frac{e^{C\hat{t}} \lambda_1 (Qe^{C\hat{t}}) |\sigma^T (p + 2\varepsilon \hat{x})|^2}{\lambda_2 (Qe^{C\hat{t}}) (\sigma^T (p + 2\varepsilon \hat{x}), w) - e^{C\hat{t}} f - CQ},$$

$$b_2 \leq \frac{1}{2} tr (\sigma^T X_2) - \langle \overline{\mu}, p \rangle + \frac{e^{C\hat{t}} \lambda_1 (Qe^{C\hat{t}}) |\sigma^T p|^2}{\lambda_2 (Qe^{C\hat{t}}) (\sigma^T p, w) + e^{C\hat{t}} f + CQ}.$$  \hspace{1cm} (4.59)
Adding the inequalities (4.59) and using (4.58), yields,

\[
\frac{\gamma_0}{T^2} \leq \frac{1}{2} tr(\sigma\sigma^T (X_1 + X_2 + 2\varepsilon I_N)) + \left[ (\overline{\mu} - \underline{\mu}, p) + 2(\overline{\mu}, \varepsilon \overline{\varepsilon}) \right] \\
+ \left[ N(Q; p, \overline{x}, \overline{\theta}) - N(Q; p, \overline{x}, \overline{\theta}) \right] + \left[ -4e^{C\hat{t}}\lambda_1(\lambda e^{C\hat{t}})(\sigma^T p, \sigma^T \varepsilon \overline{\varepsilon}) \right] \\
- 4e^{C\hat{t}}\lambda_1(\lambda e^{C\hat{t}})[\sigma^T \varepsilon \overline{\varepsilon}]^2 - 2\lambda_2(\lambda e^{C\hat{t}})(\sigma^T \varepsilon \overline{\varepsilon}, \overline{\varpi}) \\
+ \left[ \lambda_2(Qe^{C\hat{t}})(\sigma^T p, \overline{w} - \overline{\varpi}) \right] + \left[ f - \hat{f} \right] e^{-C\hat{t}} - C(\overline{Q} - Q),
\]

where,

\[
N(Q; p, x, t) := e^{C\hat{t}}\lambda_1(Q e^{C\hat{t}})[\sigma^T (t)p]^2 + \lambda_2(Q e^{C\hat{t}})(\sigma^T (t)p, w(x, t)).
\]

Now, we estimate the single terms in the brackets [\( \cdot \)] for the inequality (4.60). Since \( \lambda'_1 > 0 \) over \([c, d]\), by (4.61), we have,

\[
\frac{dN}{dQ} = e^{C\hat{t}}\lambda'_1(Q e^{C\hat{t}})[\sigma^T (t)p]^2 + e^{C\hat{t}}\lambda'_2(Q e^{C\hat{t}})(\sigma^T (t)p, w(x, t)) \\
\geq - \frac{\|\lambda'_2\|^2\|w\|^2}{4\lambda'_1(Q e^{C\hat{t}})} \geq - \frac{\|\lambda'_2\|^2\|w\|^2}{4\min_{[c, d]} \lambda'_1}, \quad \forall Q, p, x, t.
\]

So by \( \overline{Q} - Q > 0 \)

\[
N(Q; p, \overline{x}, \overline{\theta}) - N(Q; p, \overline{x}, \overline{\theta}) \leq \frac{\|\lambda'_2\|^2\|w\|^2}{4\min_{[c,d]} \lambda'_1}(\overline{Q} - Q).
\]

From inequality (4.58), we have,

\[
\frac{1}{2} tr(\sigma\sigma^T (X_1 + X_2 + 2\varepsilon I_N)) \leq \varepsilon tr(\sigma\sigma^T).
\]

Conditions (4.53) and (4.54) yield,

\[
(\overline{\mu} - \underline{\mu}, p) + 2(\overline{\mu}, \varepsilon \overline{\varepsilon}) \leq Lip(\mu)\frac{2M}{\delta_0}|\overline{x} - \overline{\gamma}|^2 + 2\|\mu\|\sqrt{\varepsilon(d - c)} \\
\leq 2Lip(\mu)(\overline{Q} - Q) + 2\|\mu\|\sqrt{\varepsilon(d - c)},
\]

\[
\lambda_2(Q e^{C\hat{t}})(\sigma^T p, \overline{w} - \overline{\varpi}) \leq 2\|\lambda_2\|\|\sigma\|\|\sigma\|\|\mu\|\sqrt{\varepsilon(d - c)}Lip(\mu)(\overline{Q} - Q).\]

\[
\left[ f - \hat{f} \right] e^{-C\hat{t}} \leq Lip(f) \left[ |\overline{x} - \overline{\gamma}| + (\overline{Q} - Q)e^{C\hat{t}} \right] e^{-C\hat{t}} \\
\leq Lip(f) \left[ \frac{1}{2} \left( |\overline{x} - \overline{\gamma}|^2 + \delta_0 \right) + (\overline{Q} - Q)e^{C\hat{t}} \right] e^{-C\hat{t}} \\
\leq Lip(f) \left[ \frac{\overline{Q} - Q}{2M} + (\overline{Q} - Q)e^{C\hat{t}} \right] e^{-C\hat{t}} \\
= Lip(f)(\overline{Q} - Q)\frac{e^{-C\hat{t}}}{2M} + 1 \\
\leq Lip(f)\left( \frac{1}{2M} + 1 \right)(\overline{Q} - Q).
\]
Being $\delta_0$ fixed, $p$ is bounded for $\varepsilon \to 0$; then by (4.54), the term $[-4e^{Ct} \times \lambda_1(Qe^{Ct})(\sigma^T p, \sigma^T \varepsilon \tilde{x}) - 4e^{Ct} \lambda_1(Qe^{Ct})\sigma^T \varepsilon \tilde{x}]^2 - 2\lambda_2(Qe^{Ct})(\sigma^T \varepsilon \tilde{x}, \tilde{y})$ in (4.60), is of order $\sqrt{\varepsilon}$, for $\varepsilon \to 0$. Using that observation and introducing the estimates (4.63)-(4.67), in the inequality (4.60), yields
\[
\frac{\gamma_0}{T^2} \leq \text{str} (\sigma \sigma^T) + 2\text{Lip}(\mu)(Q - Q) + 2\|\mu\|_{\infty} \sqrt{\varepsilon (d - c)}
\]
\[+ \frac{\|\lambda'_1\|_{\infty}^2 \|w\|_{\infty}^2}{4 \min_{[c, d]} \lambda'_1} (Q - Q) + O(\sqrt{\varepsilon}) + 2\|\lambda_2\|_{\infty} \|\sigma^T\|_{\infty} \text{Lip}(w)(Q - Q)
\]
\[+ \text{Lip}(f) \left( \frac{1}{2M} + 1 \right) (Q - Q) - C(Q - Q)
\]
\[+ \text{Lip}(f) \left( \frac{1}{2M} + 1 \right) - C \right](Q - Q) = 0.
\] (4.69)

Letting $n \to \infty$ in (4.68), we see
\[
\frac{\gamma_0}{T^2} \leq \left[ 2\text{Lip}(\mu) + \frac{\|\lambda'_1\|_{\infty}^2 \|w\|_{\infty}^2}{4 \min_{[c, d]} \lambda'_1} + 2\|\lambda_2\|_{\infty} \|\sigma^T\|_{\infty} \text{Lip}(w) + \right.
\]
\[\text{Lip}(f) \left( \frac{1}{2M} + 1 \right) - C \right] (Q - Q) = 0.
\] (4.69)

In the last passage we have used the definition (4.40). The inequality (4.69) contradicts $\gamma_0 > 0$. We thus prove (4.51).

\[\square\]

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