The Energy-Momentum Tensor in Field Theory II *

Hidenori SONODA†

Department of Physics and Astronomy, UCLA, Los Angeles, CA 90095-1547, USA

In a previous paper, field theory in curved space was considered, and a formula that expresses the first order variation of correlation functions with respect to the external metric was postulated. The formula is given as an integral of the energy-momentum tensor over space, where the short distance singularities of the product of the energy-momentum tensor and an arbitrary composite field must be subtracted, and finite counterterms must be added. These finite counterterms have been interpreted geometrically as a connection for the linear space of composite fields over theory space. In this paper we will study a second order consistency condition for the variational formula and determine the torsion of the connection. A non-vanishing torsion results from the integrability of the variational formula, and it is related to the Bose symmetry of the product of two energy-momentum tensors. The massive Ising model on a curved two-dimensional surface is discussed as an example, and the short-distance singularities of the product of two energy-momentum tensors are calculated explicitly.

September 1995

* This work was supported in part by the U.S. Department of Energy, under Contract DE-AT03-88ER 40384 Mod A006 Task C.
† sonoda@physics.ucla.edu
1. Introduction and review

In a previous paper [1] we have studied field theory in curved space and introduced the energy-momentum tensor as a composite field that generates infinitesimal deformations of the external metric. It is convenient to study field theory in an arbitrary curved background because the energy-momentum tensor is most naturally defined as the field which is conjugate to the external metric [2]. The properties of the energy-momentum tensor in flat space can be obtained by taking the flat metric limit. We have introduced a formula, called the variational formula, which expresses the first order change of correlation functions of arbitrary composite fields under an infinitesimal change of the external metric. The variational formula treats the short-distance singularities in the product of the energy-momentum tensor and an arbitrary composite field more carefully than earlier studies\[1\]. The singularities must be subtracted, and finite counterterms, denoted by $K$, must be added in the variational formula. The finite counterterms $K$ can be interpreted geometrically as a connection for the linear space of composite fields over theory space. (The theory space is parameterized by the external metric $h_{\mu\nu}$ and spatially constant parameters $g^i$.)

We have studied the consistency of the variational formula with the renormalization group (RG), the variational formula with respect to the spatially constant parameters, and diffeomorphism. We have obtained two main results: First, we have obtained an expression of the short distance singularities of the product of the energy-momentum tensor and an arbitrary composite field in terms of the connection $K$ or finite counterterms. Second, we have found that the connection $K$ also gives the Schwinger terms in the euclidean version of the equal-time commutator between the energy-momentum tensor and an arbitrary composite field.

In the present paper we wish to check further consistency of the variational formula with respect to the external metric: we will study the second order variation of the vacuum energy and impose Maxwell’s integrability condition. This integrability condition is related to the symmetry of the operator product expansions (OPE’s) under interchange of two Bosonic fields. We will see that the integrability condition gives rise to a non-vanishing torsion of the connection $K$.

There are two kinds of variational formula for field theory in D-dimensional curved space with metric $h_{\mu\nu}$ and spatially constant parameters $g^i(i = 1, \ldots, N)$. The first kind,
introduced in ref. [1], expresses the first order variation of correlation functions under an arbitrary change of the external metric $h_{\mu\nu}$:

$$\langle \Phi_{a_1}(P_1)\ldots\Phi_{a_n}(P_n)\rangle_{h,g} - \langle \Phi_{a_1}(P_1)\ldots\Phi_{a_n}(P_n)\rangle_{h+\delta h,g}$$

$$= \lim_{\epsilon \to 0} \left[ \int_{\rho(r,P_k) \geq \epsilon} d^D r \sqrt{h} \frac{1}{2} \delta h_{\mu\nu}(r) \times \left\langle \left( \Theta^{\mu\nu}(r) - \langle \Theta^{\mu\nu}(r)\rangle_{h,g} \right) \Phi_{a_1}(P_1)\ldots\Phi_{a_n}(P_n) \right\rangle_{h,g} \right]$$

$$+ \sum_{k=1}^{n} \left\{ \delta h \cdot K(h(P_k), g) - \int_{\epsilon}^{1} d\rho \delta h \cdot C(\rho; h(P_k), g) \right\}^{b}_{a_k} \times \langle \Phi_{a_1}(P_1)\ldots\Phi_{b}(P_k)\ldots\Phi_{a_n}(P_n)\rangle_{h,g} ,$$

where $\rho(r, P)$ is the geodesic distance between the two points $r$ and $P$. The OPE coefficient $C$ is defined by

$$\int_{\rho(r,P)=\rho} d^{D-1}\Omega \frac{1}{2} \delta h_{\mu\nu} \Theta^{\mu\nu}(r) \Phi_{a}(P) = \left[ \delta h \cdot C(\rho; g, h) \right]^{b}_{a} \Phi_{b}(P) + o\left( \frac{1}{\rho} \right) ,$$

where $d^{D-1}\Omega$ is the angular volume element:

$$d^D r \sqrt{h} = d\rho \ d^{D-1}\Omega ,$$

and $\delta h \cdot C$ is a short-hand notation for

$$\left[ \delta h \cdot C(\rho; g, h) \right]^{b}_{a} \equiv \sum_{m=0}^{\infty} \frac{1}{m!} \nabla_{\mu_1} \ldots \nabla_{\mu_m} \frac{1}{2} \delta h_{\mu\nu} \cdot (C^{\mu\nu,\mu_1\ldots\mu_m}(\rho; g, h))^{b}_{a} .$$

In eq. (1.2) we only keep the terms which cannot be integrated over $\rho$ all the way to zero. The definition (1.2) implies that the OPE coefficients satisfy the algebraic constraints

$$C^{\mu\nu,\mu_1\ldots\mu_m}(\rho; h, g) = \frac{1}{\rho^2} h_{\mu_{m+1}\mu_{m+2}} C^{\mu\nu,\mu_1\ldots\mu_{m+1}\mu_{m+2}}(\rho; h, g) + o\left( \frac{1}{\rho^3} \right) .$$

In ref. [1] we have imposed the consistency of the variational formula (1.1) with the RG, and found the following expression of the OPE coefficient $\delta h \cdot C$ in terms of the connection $K$:

$$\delta h \cdot C(\rho; h, g) = \frac{\partial}{\partial \rho} \delta h \cdot S(\rho; h, g) ,$$
where the matrix $S(\rho; h, g)$ is defined by

$$\delta h \cdot S(\rho; h, g) \equiv \left[ G(\rho; h, g) \cdot \left\{ \frac{\delta h}{\rho^2} \cdot \mathcal{K} \left( h/\rho^2, g/(\ln \rho) \right) \right\} \right.$$ \hspace{1cm} (1.7)

$$+ G(\rho; h, g) - G(\rho; h + \delta h, g) \right] \cdot G^{-1}(\rho; h, g).$$

The matrix $G$ is defined by the following RG equation and the initial condition:

$$\frac{d}{dt} G(\rho; h, g) \equiv \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ G(e^{-\Delta t \rho}; e^{-2\Delta t h} g + \Delta t \beta) - G(\rho; h, g) \right]$$ \hspace{1cm} (1.8)

$$= \gamma(h, g) G(\rho; h, g), \quad G(1; h, g) = 1,$$

where $\gamma(h, g)$ is the matrix of scale dimension in the basis $\{ \Phi_a \}$, and the running parameter $g^i(t)$ is defined by

$$\frac{\partial}{\partial t} g^i(t) = \beta^i(g(t)), \quad g^i(0) = g^i,$$ \hspace{1cm} (1.9)

where $\beta^i(g)$ is the beta function of the $i$-th parameter $g^i$. Note that the initial condition for $G$ implies

$$S(\rho = 1; h, g) = \mathcal{K}(h, g).$$ \hspace{1cm} (1.10)

Similarly, the consistency with diffeomorphism gives rise to the following expression of the euclidean version of the equal-time commutator between the energy-momentum tensor and an arbitrary composite field $\Phi_a$:

$$(u \cdot \tilde{\mathcal{C}}(\rho; h, g)) \Phi_a(P) = \mathcal{L}_u \Phi_a + (\mathcal{L}_u h \cdot S(\rho; h, g)) \Phi_a,$$ \hspace{1cm} (1.11)

where $u$ is an arbitrary vector field, $\mathcal{L}_u$ is the Lie derivative in the direction of $u$, and $\tilde{\mathcal{C}}$ is defined by

$$\int_{\rho(r, P) = \rho} d^{D-1} \Omega(r) \ N_\mu(r) u_\nu(r) \Theta^{\mu\nu}(r) \Phi_a(P) = (u \cdot \tilde{\mathcal{C}}(\rho; h, g)) \Phi_a(P) + o(\rho^0),$$ \hspace{1cm} (1.12)

where $N^\mu(r)$ is the unit outward normal vector at $r$, and we only keep the terms non-vanishing as $\rho \to 0$. Eq. (1.11) shows that the anomalous part of the commutator is determined by the connection $\mathcal{K}$ through $S$ defined by eq. (1.7). In components, $\tilde{\mathcal{C}}$ is written as

$$(u \cdot \tilde{\mathcal{C}})_a^b(\rho; h, g) = \sum_{m=0}^{\infty} \frac{1}{m!} \nabla_{\mu_1} \cdots \nabla_{\mu_m} u_\nu \cdot (\tilde{\mathcal{C}}^{\mu\nu, \mu_1 \cdots \mu_m})_a^b(\rho; h, g),$$ \hspace{1cm} (1.13)
where
\[
\tilde{\mathcal{C}}_{\mu_\nu, \mu_1...\mu_m}(\rho; h, g) = h_{\mu_\nu \mu_{m+1}} \tilde{\mathcal{C}}_{\mu_\nu, \mu_{m+1}...\mu_m}(\rho; h, g),
\]  
\(1.14\)
and the unintegrable part of \(\rho \tilde{\mathcal{C}}\)'s coincide with \(\mathcal{C}\)'s:
\[
\rho \tilde{\mathcal{C}}_{\mu_\nu, \mu_1...\mu_m}(\rho; h, g) = \mathcal{C}_{\mu_\nu, \mu_1...\mu_m}(\rho; h, g) + o\left(\frac{1}{\rho}\right).
\]  
\(1.15\)

There are four sources of algebraic constraints on the connection \(\mathcal{K}\). One is the constraint (1.5). Since Eq. (1.11) determines \(\tilde{\mathcal{C}}_{\mu_\nu, \mu_1...\mu_m}\), which is a trace over \(\mu\), the existence of \(\tilde{\mathcal{C}}_{\mu_\nu, \mu_1...\mu_m}\) which gives the correct trace (1.14) can constrain the connection \(\mathcal{K}\). This is the second constraint. The relation (1.15) can give the third constraint, since both \(\mathcal{C}\)'s and \(\tilde{\mathcal{C}}\)'s are given in terms of the same connection \(\mathcal{K}\). Finally, algebraic constraints analogous to (1.5) exist also for \(\tilde{\mathcal{C}}\)’s:
\[
\tilde{\mathcal{C}}_{\mu_\nu, \mu_1...\mu_m}(\rho) = \frac{1}{\rho^2} h_{\mu_{m+1} \mu_{m+2}} \tilde{\mathcal{C}}_{\mu_\nu, \mu_1...\mu_m}(\rho) + o\left(\frac{1}{\rho^2}\right).
\]  
\(1.16\)

These constrains are very useful when we determine the connection \(\mathcal{K}\) in practice.

The second kind of variational formula gives the first order change of correlation functions under an arbitrary change of the parameter \(g^i\) of the theory [4]:
\[
- \frac{\partial}{\partial g^i} \langle \Phi_{a_1}(P_1)...\Phi_{a_n}(P_n) \rangle_{h,g}
\]
\[
= \lim_{\epsilon \to 0} \left[ \int_{\rho(r, P_k) \geq \epsilon} d^D r \sqrt{h} \left( \langle O_i(r) - \langle O_i(r) \rangle_{h,g} \rangle_{h,g} \Phi_{a_1}(P_1)...\Phi_{a_n}(P_n) \right)_{h,g}
\right.
\]
\[
+ \sum_{k=1}^{n} \left\{ (c_i)(h(P_k), g) - \int_{\epsilon}^1 \frac{1}{\rho} (C_i)(\rho; h(P_k), g) \right\}_{a_k}^{b_k}
\]
\[
\times \left\langle \Phi_{a_1}(P_1)...\Phi_{b}(P_k)...\Phi_{a_n}(P_n) \rangle_{h,g} \right]_{a_k},
\]
\(1.17\)

where
\[
\int_{\rho(r, P) = \rho} d^{D-1} \Omega O_i(r) \Phi_a(P) = [C_i(\rho; g, h(P))]_{a}^{b} \Phi_b(P) + o\left(\frac{1}{\rho}\right),
\]  
\(1.18\)

and the finite counterterm \(c_i\) can be interpreted as the \(g^i\) component of a connection over theory space.

The consistency between the variational formula (1.17) and the RG gives the following relation between the OPE coefficient \(\mathcal{C}_i\) and the connection \(c_i\) [4]:
\[
\mathcal{C}_i(\rho; h, g) = \frac{\partial}{\partial \rho} S_i(\rho; h, g),
\]  
\(1.19\)
where
\[ S_i(\rho; h, g) \equiv \left[ G(\rho; h, g) \frac{\partial g^j(\ln \rho)}{\partial g^i} c_j(h, g) - \frac{\partial}{\partial g^i} G(\rho; h, g) \right] \cdot G^{-1}(\rho; h, g) . \] (1.20)

It has also been found that the consistency among the two kinds of variational formula and the RG gives the trace condition [1]:
\[ 2h \cdot \mathcal{K}(h, g) = \gamma(h, g) + \beta^i(g)c_i(h, g) . \] (1.21)

The purpose of the present paper is to check further the consistency of the variational formula with respect to the metric (1.1). In particular we will study the second order variation of the vacuum energy using the first order variational formula (1.1) recursively. We will find that the symmetry of the second order variation results in a non-vanishing torsion of the connection \( \mathcal{K} \). It will be shown that part of this result is equivalent to the Bose symmetry between two energy-momentum tensors.

The paper is organized as follows. In sect. 2 we will calculate the second order variation of the vacuum energy using the variational formula (1.1). By imposing an integrability condition (or Maxwell’s relation) to the second order variation, we will derive the torsion of the connection \( \mathcal{K} \). In sect. 3 we will study an implication of the Bose symmetry among the composite fields with integer spins, and show that part of the result of sect. 2 can be also obtained from the Bose symmetry between two energy-momentum tensors. In sect. 4 we will find a relation between the particular matrix elements of the connections \( c_i \) and \( \mathcal{K} \), i.e., \( (c_i)\Theta^{\mu\nu} \) and \( \mathcal{K}O_i \). In sect. 5 we will study the example of the massive Ising model on a curved two-dimensional surface, and compute explicitly \( \mathcal{K}\Theta^{\mu\nu} \), i.e., the elements of the connection \( \mathcal{K} \) for the energy-momentum tensor. In the zero mass limit, the result is shown to agree with conformal field theory. We conclude the paper in sect. 6.

2. Second order variation of the vacuum energy

We consider a field theory in \( D \)-dimensional curved space with an external metric \( h_{\mu\nu} \) and spatially constant parameters \( g^i (i = 1, \ldots, N) \). Let \( F[h, g] \) be the total vacuum energy (or free energy). Under an infinitesimal change of the metric and the parameters, the free energy changes by
\[
F[h + \delta h, g + \delta g] - F[h, g] = \int d^DP\sqrt{h(P)} \left[ \frac{1}{2} \delta h_{\mu\nu}(P) \langle \Theta^{\mu\nu}(P) \rangle_{h,g} + \delta g^i \langle O_i(P) \rangle_{h,g} \right] . \] (2.1)
We wish to calculate the terms of order $\delta h_1 \delta h_2$ in

$$\Delta[\delta h_1, \delta h_2] \equiv F[h + \delta h_1 + \delta h_2, g] - F[h + \delta h_1, g] - F[h + \delta h_2, g] + F[h, g]. \quad (2.2)$$

(We ignore the terms of order $(\delta h_1)^2$ and $(\delta h_2)^2$.) The integrability of the first order variational formula (2.1) demands that the second order variation (2.2) must be symmetric with respect to $\delta h_1$ and $\delta h_2$.

We can calculate the second order change (2.2) by applying the variational formula (1.1) to eq. (2.1). There are two ways of doing this depending on the order of changing the metric, as indicated in Fig. 1. Integrability of the vacuum energy demands that the two paths in the figure give the same result.

$$\begin{align*}
\Delta[\delta h_1, \delta h_2] &= (F[(h + \delta h_1) + \delta h_2, g] - F[h + \delta h_1, g]) \\
&\quad - (F[h + \delta h_2, g] - F[h, g]) \\
&= \int d^D P \sqrt{h + \delta h_1} \frac{1}{2} (\delta h_2)_{\mu\nu}(P) \langle \Theta^{\mu\nu}(P) \rangle_{h + \delta h_1, g} \\
&\quad - (F[h + \delta h_2, g] - F[h, g]). \quad (2.3)
\end{align*}$$

Along the first path, we obtain

$$\Delta[\delta h_1, \delta h_2] = \left( F[(h + \delta h_1) + \delta h_2, g] - F[h + \delta h_1, g] \right) \left( F[h + \delta h_2, g] - F[h, g] \right)$$

Applying the variational formula (1.1) to the integrand, we obtain

$$\Delta[\delta h_1, \delta h_2] = - \int_{\rho\geq\epsilon} d^D P \sqrt{h(P)} \, d^D r \sqrt{h(r)}$$

$$\times \frac{1}{2} (\delta h_2)_{\mu\nu}(P) \frac{1}{2} (\delta h_1)_{\alpha\beta}(r) \langle \Theta^{\alpha\beta}(r) \Theta^{\mu\nu}(P) \rangle_{h, g}^{c}$$
\[
+ \int d^D P \sqrt{h(P)} \frac{1}{2} (\delta h_2)_{\mu\nu}(P) \left\{ h^{\alpha\beta}(P) \frac{1}{2} (\delta h_1(P))_{\alpha\beta} \langle \Theta^{\mu\nu}(P) \rangle_{h,g} \right\} 
\]

(2.4)

\[ + \delta h_1(P) \cdot \left( \int_1^\epsilon d\rho \ C(\rho; h(P), g) - \mathcal{K}(h(P), g) \right)^{\mu\nu \ a} \langle \Phi_a(P) \rangle_{h,g} \right\}.
\]

The second path gives the same result as above except that \(\delta h_1\) and \(\delta h_2\) are interchanged.

Since the energy-momentum tensor satisfies the canonical RG equation

\[
\frac{d}{dt} \Theta^{\mu\nu} = (D + 2) \Theta^{\mu\nu},
\]

(2.5)

the general formulas (1.6), (1.7) imply

\[
\delta h \cdot C(\rho; h, g) \Theta^{\mu\nu} = \frac{\partial}{\partial \rho} \left( \delta h \cdot S(\rho; h, g) \right) \Theta^{\mu\nu},
\]

(2.6)

where

\[
\delta h \cdot S(\rho; h, g) \Theta^{\mu\nu} = \frac{1}{\rho^{D+2}} \frac{\delta h}{\rho^2} \cdot \mathcal{K} \left( h/\rho^2, g(\ln \rho) \right) \cdot G^{-1}(h; h, g) \Theta^{\mu\nu}.
\]

(2.7)

Using the component notation of eq. (1.4), this can be written as

\[
[C^\alpha\beta,\mu_1...\mu_m(\rho; h, g)]^{\mu\nu \ a} = \frac{\partial}{\partial \rho} [S^\alpha\beta,\mu_1...\mu_m(\rho; h, g)]^{\mu\nu \ a},
\]

(2.8)

where

\[
[S^\alpha\beta,\mu_1...\mu_m(\rho; h, g)]^{\mu\nu \ a} = \frac{1}{\rho^{D+4}} \left( \mathcal{K}^\alpha\beta,\mu_1...\mu_m \left( h/\rho^2, g(\ln \rho) \right) \right)^{\mu\nu \ b} \cdot (G^{-1})_b^a(\rho; h, g).
\]

(2.9)

Therefore, the invariance of eq. (2.4) under the interchange of \(\delta h_1\) and \(\delta h_2\) implies that

\[
I_{12}(\epsilon) = I_{21}(\epsilon),
\]

(2.10)

where

\[
I_{12}(\rho) \equiv \int d^D P \sqrt{h(P)} \frac{1}{2} (\delta h_2)_{\mu\nu}(P) \left( -\frac{1}{2} (\delta h_1)_{\alpha\beta} h^{\alpha\beta} \Theta^{\mu\nu}(P) 
\right.
\]

\[
+ \sum_{m=0}^{\infty} \frac{1}{m!} \nabla_{\mu_1}...\nabla_{\mu_m} \frac{1}{2} (\delta h_1)_{\alpha\beta}(P) \cdot S^\alpha\beta,\mu_1...\mu_m(\rho; h, g) \Theta^{\mu\nu} \right).
\]

(2.11)

We recall that the connection \(\mathcal{K}\) was originally introduced in ref. [1] as finite counterterms in the variational formula (1.4). This means that in the definition of \(I_{12}(\rho)\) above, \(S(\rho)\) only has those terms which give non-vanishing contributions for an infinitesimal \(\rho\). This implies
that eq. (2.10) is actually valid for an arbitrary $\epsilon$ which is not necessarily infinitesimal as long as $I_{12}(\epsilon)$ is well-defined.

Therefore, from eq. (2.10), we find that

$$f(\rho) \equiv (\delta h_1)_{\mu\nu} \left[ h_{\mu\nu} (\delta h_2)_{\alpha\beta} \Theta^{\alpha\beta} \right. $$

$$+ \left. \sum_{m=0}^{\infty} \frac{1}{m!} \nabla_{\mu_1} \cdots \nabla_{\mu_m} (\delta h_2)_{\alpha\beta} \cdot S_{\mu\nu, \mu_1 \cdots \mu_m}^{\alpha\beta} (\rho; h, g) \Theta^{\mu\nu} \right] - (\delta h_1 \leftrightarrow \delta h_2)$$

is a total divergence with respect to space. Since $\delta h_1, \delta h_2$ are arbitrary, this means that for each integer $m$ we must find

$$S_{\alpha\beta, \mu_1 \cdots \mu_m}^{\alpha\beta} (h/\rho^2, g(ln\rho)) \Theta^{\mu\nu} = \delta_{m,0} \left( h_{\alpha\beta} \Theta^{\mu\nu} - h^{\mu\nu} \Theta_{\alpha\beta} \right)$$

$$+ (-)^m \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \nabla_{\nu_1} \cdots \nabla_{\nu_n} \left\{ S_{\mu\nu, \mu_1 \cdots \mu_m, \nu_1 \cdots \nu_n}^{\alpha\beta} (h/\rho^2, g(ln\rho)) \Theta^{\alpha\beta} \right\} .$$

This is the main result of this section. The first term on the right-hand side is independent of the geodesic distance $\rho$. In the next section we will see that the $\rho$ dependence of eq. (2.13) is a consequence of the Bose statistics of the energy-momentum tensor.

Finally we observe that, if we take $\rho = 1$, eq. (2.13) gives the torsion of the connection $K$:

$$\tau(\delta h_1, \delta h_2; h, g) \equiv (\delta h_1 \cdot K)(\delta h_2)_{\alpha\beta} \Theta^{\alpha\beta} - (\delta h_2 \cdot K)(\delta h_1)_{\alpha\beta} \Theta^{\alpha\beta}$$

$$= \frac{1}{2} \left\{ (\delta h_1)^{\mu}_{\alpha\beta} (\delta h_2)_{\rho\sigma} \Theta^{\sigma\rho} - (\delta h_2)^{\mu}_{\alpha\beta} (\delta h_1)_{\rho\sigma} \Theta^{\sigma\rho} \right\}$$

$$+ \sum_{m=1}^{\infty} \frac{1}{m!} \left[ (\delta h_2)_{\alpha\beta} (-)^m \nabla_{\mu_1} \cdots \nabla_{\mu_m} \left( \frac{1}{2} (\delta h_1)_{\mu\nu} \cdot (K_{\alpha\beta, \mu_1 \cdots \mu_m}) \Theta^{\mu\nu} \right) \right.$$ 

$$\left. - (\nabla_{\mu_1} \cdots \nabla_{\mu_m} (\delta h_2)_{\alpha\beta}) \cdot \frac{1}{2} (\delta h_1)_{\mu\nu} (K_{\alpha\beta, \mu_1 \cdots \mu_m}) \Theta^{\mu\nu} \right].$$

The sum over the positive integer $m$ is a total derivative in space.

To summarize, the condition of integrability of the vacuum energy gives eq. (2.13). This determines the torsion of the connection $K$ up to total derivatives in space.

### 3. Bose symmetry of the OPE coefficients in curved space

To improve our understanding of the integrability condition (2.13) in the previous section, let us study the property of OPE’s in curved space under interchange of two fields.
In curved space we consider an OPE of two composite fields

\[ A(r)B(P) = C_{AB}^a(P, v)\Phi_a(P), \quad (3.1) \]

where \( v \) is the geodesic coordinate of \( r \) with respect to \( P \) (i.e., \( r = \text{Exp}_v(P) \)). For simplicity, we take \( A \) and \( B \) to be scalar fields. We define the angular integral

\[ C_{AB}^{\mu_1...\mu_m,a}(\rho; h(P)) \equiv \int_{\|v\| = \rho} d^{D-1}\Omega(r) v^{\mu_1}...v^{\mu_m} C_{AB}^a(P, v), \quad (3.2) \]

where \( d^D r \sqrt{h} = d\rho d^{D-1}\Omega(r) \), and \( \|v\| \) is the norm. Our goal is to find the symmetry of the integrated OPE coefficient under interchange of \( A \) and \( B \): we wish to know the relation between \( C_{AB}^{\mu_1...\mu_m,a} \) and \( C_{BA}^{\mu_1...\mu_m,a} \).

Let us introduce \( V^a_b(P, r) \) which parallel transports a tensor \( \Phi_a \) at \( P \) to a tensor at \( r = \text{Exp}_v(P) \) along the geodesic. The transported tensor is given by \( \sum_b \Phi_b(P)V^b_c(P, r) \). The inverse of \( V \) parallel transports a tensor from \( r \) to \( P \). So, we can write

\[ (V(P, r))^{-1} = V(r, P). \quad (3.3) \]

Now, the Bose symmetry

\[ B(r)A(P) = A(P)B(r) \quad (3.4) \]

implies that

\[ B(r)A(P) = \sum_a C_{AB}^a(r, w)\Phi_a(r), \quad (3.5) \]

where \( w \) is the tangent vector at \( r \) such that \( \text{Exp}_w(r) = P \). (See Fig. 2.)

![Fig. 2 two nearby points P, r](image.png)
Using the Taylor expansion

\[ \Phi_a(r) = \sum_{n=0}^{\infty} \frac{1}{n!} v_{\nu_1} \cdots v_{\nu_n} \nabla_{\nu_1} \cdots \nabla_{\nu_n} \Phi_b(P) \cdot V^b_a(P, r), \quad (3.6) \]

we obtain

\[ B(r)A(P) = (C_{AB})^a(r, w) \sum_{n=0}^{\infty} \frac{1}{n!} v_{\nu_1} \cdots v_{\nu_n} \nabla_{\nu_1} \cdots \nabla_{\nu_n} \Phi_b(P) \cdot V^b_a(P, r) . \quad (3.7) \]

Therefore, using the definition (3.2), we find

\[ C_{\mu_1 \cdots \mu_m, a}(r, w; h, \mathcal{g}) \Phi_a(P) \]
\[ = \sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{n!} \int d^{D-1}\Omega_p(-v)^{\mu_1} \cdots (-v)^{\mu_m} (-v)^{\nu_1} \cdots (-v)^{\nu_n} \]
\[ \times V^b_a(P, r) C^a_{AB}(r, w) \nabla_{\nu_1} \cdots \nabla_{\nu_n} \Phi_b(P) . \quad (3.8) \]

In order to proceed further we need three equations. The first equation is the Taylor expansion

\[ V^b_a(P, r) C^a_{AB}(r, w) = \sum_{m=0}^{\infty} \frac{1}{m!} v^\mu_1 \cdots v^\mu_m C^a_{AB, \mu_1 \cdots \mu_m}(P; ||v||) , \quad (3.9) \]

This is valid since we can formally expand the OPE coefficients as

\[ C^a_{AB}(P, v) = \sum_{m=0}^{\infty} \frac{1}{m!} v^\mu_1 \cdots v^\mu_m C^a_{AB, \mu_1 \cdots \mu_m}(P; ||v||) , \quad (3.10) \]

where \( C^a_{AB, \mu_1 \cdots \mu_m}(P; ||v||) \), except for its dependence on the norm \( ||v|| \), is an ordinary tensor field and admits the Taylor expansion

\[ V^b_a(P, r) C^b_{AB, \mu_1 \cdots \mu_m}(r; ||v||) V^\mu_1^a (r, P) \cdots V^\mu_m^a (r, P) \]
\[ = \sum_{n=0}^{\infty} \frac{1}{n!} v^\nu_1 \cdots v^\nu_n \nabla_{\nu_1} \cdots \nabla_{\nu_n} C^a_{AB, \mu_1 \cdots \mu_m}(P; ||v||) , \quad (3.11) \]

just as eq. (3.6).

To describe the other two equations, we first introduce \( W^\mu_a(P, v) \) which parallel transports tangent vectors from \( r = \text{Exp}_v(P) \) to \( P \) in the geodesic coordinate system around \( P \). The two parallel transports \( V \) and \( W \) are related to each other by the following coordinate transformation:

\[ W^\mu_a(P, v) = V^\mu_a(P, r) \frac{\partial v^\alpha}{\partial v^\mu_r} \bigg|_{v_r=0} , \quad (3.12) \]

10
where the vector $v_P$ at $P$ and the vector $v_r$ at $r$ correspond to the same point: $\text{Exp}_{v_P}(P) = \text{Exp}_{v_r}(r)$. In the geodesic coordinate $v_P$ around $P$, the volume element at the point $\text{Exp}_{v_P}(P)$ is given by $\text{det}W(P, v_P)$. The second equation we need is the following expansion of the volume element:

$$\text{det}W(P, v) = \sum_{k=0}^{\infty} \frac{1}{k!} v^{\kappa_1} \ldots v^{\kappa_k} \nabla^{\kappa_1} \ldots \nabla^{\kappa_k} \text{det}W(P, -v). \quad (3.13)$$

The third equation we need is the one about integrals of total derivatives:

$$\nabla_{\mu_1} \ldots \nabla_{\mu_m} \int d^{D-1}\Omega_\rho(v) v^{\mu_1} \ldots v^{\mu_m} t(P, v) \quad (3.14)$$

$$= \int \frac{d^{D-1}\Omega_\rho(v)}{\text{det}W(P, v)} v^{\mu_1} \ldots v^{\mu_m} \nabla_{\mu_1} \ldots \nabla_{\mu_m} (\text{det}W(P, v) t(P, v)),$$

where $t(P, v)$ is an arbitrary tensor at $P$ which depends also on a vector $v$ at $P$. The tensor can be formally expanded as

$$t(P, v) = \sum_{n=0}^{\infty} \frac{1}{n!} v^{\nu_1} \ldots v^{\nu_n} t_{\nu_1 \ldots \nu_n}(P, \|v\|), \quad (3.15)$$

where $t_{\nu_1 \ldots \nu_n}(P, \|v\|)$ is an ordinary tensor field with an additional dependence on the norm of the vector $v$. The covariant derivative of $t(P, v)$ is defined by

$$\nabla_\mu t(P, v) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} v^{\nu_1} \ldots v^{\nu_n} \nabla_\mu t_{\nu_1 \ldots \nu_n}(P, \|v\|), \quad (3.16)$$

where we have an ordinary covariant derivative on the right-hand side. ($\|v\|$ is fixed under the derivative.) We will not prove the second and third equations $(3.13)$, $(3.14)$ in this paper. For a mathematical background we refer the reader to refs. [5] and [6]. ($W$ here is denoted as $V$ in these references.)

Using eqs. $(3.9)$ and $(3.13)$, eq. $(3.8)$ gives

$$C_{BA}^{\mu_1 \ldots \mu_m, a}(\rho; h, g) \Phi_a(P)$$

$$= (-)^m \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \int \frac{d^{D-1}\Omega_\rho(v)}{\text{det}W(P, v)} v^{\nu_1} \ldots v^{\nu_n}$$

$$\times \nabla_{\nu_1} \ldots \nabla_{\nu_n} (\text{det}W(P, v) v^{\mu_1} \ldots v^{\mu_m} C_{AB}^{\rho}(P, v) \Phi_a(P))$$

after a change of coordinates $v \rightarrow -v$. Then, by using eq. $(3.14)$, we obtain the desired relation:

$$C_{BA}^{\mu_1 \ldots \mu_m, a}(\rho; h, g) \Phi_a(P)$$

$$= (-)^m \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \nabla_{\nu_1} \ldots \nabla_{\nu_n} [C_{AB}^{\mu_1 \ldots \mu_m, \nu_1 \ldots \nu_n, a}(\rho; h, g) \Phi_a(P)]. \quad (3.18)$$
This implies that the difference

\[ C_{BA}^\mu_1...\mu_m,a(\rho; h, g)\Phi_a(P) - (-)^m C_{AB}^\mu_1...\mu_m,a(\rho; h, g)\Phi_a(P) \]  

is a total derivative in space. Eq. (3.18) is the main result of this section; it is a direct consequence of the Bose symmetry (3.4).

Now for the energy-momentum tensor, the only difference is that it is a tensor, and the integrated OPE must be defined with the parallel transport operator as

\[
\int_{\rho(r,P)=\rho} d^{D-1}\Omega(r) v^{\mu_1}...v^{\mu_m} V^\alpha(\rho, r) V^\beta(\rho, r) \Theta^\gamma(\rho, r) \Theta^\mu(\rho, r) \\
= \sum_a [C^\alpha,\mu_1...\mu_m(\rho; h(P), g)]^{\mu\nu} a \Phi_a(P).
\]  

(3.20)

If we only keep the terms which cannot be integrated over \( \rho \) to zero, this \( C \) coincides with the \( C \) in the previous sections. The relation analogous to eq. (3.18) is given by

\[
(C^{\alpha,\mu_1...\mu_m}(\rho; h, g))^{\mu\nu} a \Phi_a(P) \\
= (-)^m \sum_{n=0}^\infty \frac{(-)^n}{n!} \nabla_{\nu_1}...\nabla_{\nu_n} [(Q^{\alpha,\mu_1...\mu_m,\nu_1...\nu_n}(\rho; h, g))^{\alpha\beta} a \Phi_a(P)].
\]  

(3.21)

This is precisely what we obtain by differentiating eq. (2.13) with respect to \( \rho \), thanks to the relation (1.6) between \( C \) and \( S \). Hence, the \( \rho \) dependence of the integrability condition (2.13) is consistent with the invariance of the OPE under interchange of two energy-momentum tensors.

Eq. (3.18) generalizes easily to a product of any two bosonic composite fields.

4. A relation between \( c_i \) and \( K \)

In the same way as in sect. 2, the symmetry of the second order variation

\[
\Delta[\delta h, \delta g] \equiv F[h + \delta h, g + \delta g] - F[h, g] - F[h + \delta h, g] + F[h, g]
\]  

(4.1)

gives a relation among the mixed matrix elements of the connection \( (c_i, K) \).
By demanding that the two paths shown in Fig. 3 give the same second order variation (4.1), we find the following relation (we skip the derivation, since it is analogous to the derivation in sect. 2):

\[(c_i)\Theta^\mu\nu - (K^\mu\nu)\mathcal{O}_i = -h^{\mu\nu}\mathcal{O}_i + \sum_{m=1}^{\infty} \frac{(-)^m}{m!} \nabla_{\mu_1}...\nabla_{\mu_m} [(K^{\mu\nu, \mu_1...\mu_m}) \mathcal{O}_i] . \tag{4.2}\]

This also gives torsion of the connection \((c_i, K)\), which is related to the Bose symmetry of the operator product \(\mathcal{O}_i(r)\Theta^{\mu\nu}(P)\) under interchange of the two fields.

5. Example: Massive Ising model on a curved surface

We have obtained two constraints (2.13) (or equivalently (2.14)) and (4.2). As an example of a practical use of these constraints, we take the massive Ising model on a curved two-dimensional surface with metric \(h_{\mu\nu}\), and apply the integrability constraints and the other algebraic constraints discussed in sect. 1 to determine the connection \(K\) explicitly.

The model has three parameters: \(g_1, m,\) and \(\kappa\), where \(g_1\) is the cosmological constant, \(m\) is the mass, and \(\kappa\) is the coefficient of the Ricci curvature \(R\) in the vacuum energy density. The metric and the parameters satisfy the following RG equations:

\[
\frac{dt}{dt} h_{\mu\nu} = 2h_{\mu\nu} , \quad \frac{dt}{dt} g_1 = 2g_1 + \frac{m^2}{2} \beta_1 , \quad \frac{dt}{dt} m = m , \quad \frac{dt}{dt} \kappa = c' , \tag{5.1}\]
where \( c' \) is a constant to be determined later. The constant \( \beta_1 \) depends on how we normalize \( m \). If we choose \( m \) such that it gives the physical mass of the free fermion in flat space, then
\[
\beta_1 = -\frac{1}{2\pi} .
\] (5.2)

The above RG equations for the parameters imply the following RG equation for the field \( O_m \) conjugate to \( m \):
\[
\frac{d}{dt} O_m = O_m - m\beta_1 .
\] (5.3)

Eqs. (5.1) also imply the following trace of the energy-momentum tensor:
\[
\Theta = 2g_1 + \frac{m^2}{2} \beta_1 + mO_m + c'R .
\] (5.4)

By taking the limit \( m = 0 \), we find that the constant \( c' \) is related to the central charge \( c = \frac{1}{2} \) of the conformal Ising model as
\[
c' = -\frac{c}{24\pi} .
\] (5.5)

Our goal is to calculate the singularities in the product of two energy-momentum tensors \( \Theta^{\mu\nu}(r)\Theta^{\alpha\beta}(P) \) explicitly. The answer is well-known in the massless limit \( m = 0 \), but in the following we will be able to calculate the correction due to the non-vanishing mass \( m \).

As a preparation we first calculate the singularities in the product \( \Theta^{\mu\nu}(r)O_m(P) \). The most general form of the connection is given by
\[
(\delta h \cdot K)O_m = \delta h(aO_m + b m) ,
\] (5.6)
where \( \delta h \equiv \delta h^\mu_\mu \) is the trace, and \( a, b \) are constants. Using the trace condition (1.21), we find
\[
a = \frac{1}{4} , \quad b = \frac{1}{4} \left( (c_m)_m^1 - \beta_1 \right) ,
\] (5.7)
where the constant \( (c_m)_m^1 \) is a matrix element of the connection \( c_m \) for the conjugate field \( O_m \). Under a redefinition of the cosmological constant
\[
g_1 \rightarrow g_1 + \frac{k}{2} m^2 ,
\] (5.8)
the conjugate field \( O_m \) and the connection \( (c_m)_m^1 \) change by
\[
O_m \rightarrow O_m - km , \quad (c_m)_m^1 \rightarrow (c_m)_m^1 + k ,
\] (5.9)
so that the linear combinations

\[ O_m + m(c_m)_m^1, \quad mO_m + 2g_1 \]  

are invariant under the redefinition (5.8). In particular, the trace (5.4) is invariant.

Taking the column vector \((O_m, 1)^T\) as the basis, the general formula (1.7) gives

\[ S(\rho; m) = \begin{pmatrix} 0 & 0 \\ m[-\beta_1 (\ln \rho + 1) + (c_m)_m^1] & 1 \end{pmatrix}. \]

Hence, (1.6) and (1.11) give

\[ \delta h \cdot C(\rho; h, m)O_m = \frac{1}{2} \delta h \frac{-m\beta_1}{\rho} \]  

(5.12)

\[ u \cdot \tilde{C}(\rho; h, m)O_m = u^\mu \partial_\mu (O_m + m(c_m)_m^1 - m\beta_1 (1 + \ln \rho)) \]  

(5.13)

Now we consider the product of two energy-momentum tensors. The most general form of the connection, allowed by covariance and the \(Z_2\) invariance under \(m \rightarrow -m\), is given as follows:

\[ \delta h \cdot K(h, m, g_1)\Theta^{\alpha\beta} \]

\[ = \frac{1}{2} h^{\alpha\beta} \delta h \left( A_1 + g_1 A_3 + m^2 A_5 + RA_7 \right) + \delta h^{\alpha\beta} \left( A_2 + g_1 A_4 + m^2 A_6 + RA_8 \right) \]

\[ + \frac{C_1}{2} \delta h h^{\alpha\beta} \Theta + C_2 \delta h \Theta^{\alpha\beta} + C_3 h^{\alpha\beta} h_{\mu\nu} \Theta^{\mu\nu} + C_4 \delta h^{\alpha\beta} \Theta \]

(5.14)

\[ + \frac{B_1}{4} h^{\alpha\beta} \nabla^2 \delta h + \frac{B_2}{4} (\nabla^\alpha \nabla^\beta + \nabla^\beta \nabla^\alpha) \delta h \]

\[ + \frac{B_3}{2} h^{\alpha\beta} \nabla^\mu \nabla^\nu \delta h_{\mu\nu} + \frac{B_4}{2} \nabla^2 \delta h^{\alpha\beta}, \]

where \(A\)'s, \(B\)'s, and \(C\)'s are all constants.

The constants \(A, B, C\)'s will be determined in four steps. First we use the integrability constraints obtained in sect. 2 to relate some of the unknown constants. Second we will impose consistency with the previous results (5.12), (5.13) on the trace of the energy-momentum tensor. Third, we will write down the OPE coefficients \(C, \tilde{C}\) in terms of the connection \(K\) and impose the algebraic constraints (1.5), (1.16). At this point we will still have some undetermined constants. Finally we will determine the remaining constants by imposing the algebraic constraints (1.14) and (1.15).
5.1. torsion constraint

The integrability condition (2.13) (or equivalently (2.14)) gives, for \( \rho = 1 \), the following conditions:

\[
\begin{align*}
(K^{\mu\nu}) \Theta^{\alpha\beta} &= h^{\mu\nu} \Theta^{\alpha\beta} - h^{\alpha\beta} \Theta^{\mu\nu} + (K^{\alpha\beta}) \Theta^{\mu\nu} \\
(K^{\mu\nu,\gamma\delta}) \Theta^{\alpha\beta} &= (K^{\alpha\beta,\gamma\delta}) \Theta^{\mu\nu}.
\end{align*}
\]

Eq. (5.15) gives

\[
C_3 = C_2 - \frac{1}{2},
\]

and eq. (5.16) gives

\[
B_2 = B_3.
\]

5.2. trace condition

The trace anomaly (5.4) provides further constraints. Using the explicit form of the trace anomaly, the variational formula gives

\[
- \langle (h + \delta h)_{\alpha\beta} \Theta^{\alpha\beta}(P) \rangle_{h+\delta h,g} + \langle h_{\alpha\beta} \Theta^{\alpha\beta}(P) \rangle_{h,g} \\
= m \left[ - \langle O_m \rangle_{h+\delta h,g} + \langle O_m \rangle_{h,g} \right] + c' (-R(h + \delta h) + R(h)) \\
= m \int_{\rho \geq \epsilon} d^D r \sqrt{h} \frac{1}{2} \delta h_{\mu\nu}(r) \langle \Theta^{\mu\nu}(r) O_m(P) \rangle_{h,g}^c \\
+ m \left[ \delta h \cdot \left\{ \mathcal{K} - \int_{\epsilon}^1 d\rho \mathcal{C}(\rho) \right\} \right]_m^a \langle \Phi_a(P) \rangle_{h,g} + c' (-R(h + \delta h) + R(h)) .
\]

On the other hand, the variational formula for the energy-momentum tensor gives

\[
- \langle (h + \delta h)_{\alpha\beta} \Theta^{\alpha\beta}(P) \rangle_{h+\delta h,g} + \langle h_{\alpha\beta} \Theta^{\alpha\beta}(P) \rangle_{h,g} \\
= - \delta h_{\alpha\beta} \langle \Theta^{\alpha\beta} \rangle_{h,g} + m \int_{\rho \geq \epsilon} d^D r \sqrt{h} \frac{1}{2} \delta h_{\mu\nu}(r) \langle \Theta^{\mu\nu}(r) O_m(P) \rangle_{h,g}^c \\
+ h_{\alpha\beta} \langle (\delta h \cdot \mathcal{K}) \Theta^{\alpha\beta} \rangle_{h,g} - \int_{\epsilon}^1 d\rho \ m \ (\delta h \cdot \mathcal{C}(\rho))_m^a \langle \Phi_a \rangle_{h,g} .
\]

We demand consistency between the two expressions. Using the variation of the Ricci curvature

\[
-R(h + \delta h) + R(h) = \delta h_{\mu\nu} R^{\mu\nu} + \nabla^2 \delta h - \nabla^\mu \nabla^\nu \delta h_{\mu\nu},
\]

\[
R(h + \delta h) = R(h) + \delta R(h) = R(h) + \delta h_{\mu\nu} R^{\mu\nu} + \nabla^2 \delta h - \nabla^\mu \nabla^\nu \delta h_{\mu\nu},
\]

\[
\int_{\rho \geq \epsilon} d^D r \sqrt{h} \frac{1}{2} \delta h_{\mu\nu}(r) \langle \Theta^{\mu\nu}(r) O_m(P) \rangle_{h,g}^c \\
+ m \left[ \delta h \cdot \left\{ \mathcal{K} - \int_{\epsilon}^1 d\rho \mathcal{C}(\rho) \right\} \right]_m^a \langle \Phi_a(P) \rangle_{h,g} + c' (-R(h + \delta h) + R(h)) .
\]

\[
\int_{\rho \geq \epsilon} d^D r \sqrt{h} \frac{1}{2} \delta h_{\mu\nu}(r) \langle \Theta^{\mu\nu}(r) O_m(P) \rangle_{h,g}^c \\
+ m \left[ \delta h \cdot \left\{ \mathcal{K} - \int_{\epsilon}^1 d\rho \mathcal{C}(\rho) \right\} \right]_m^a \langle \Phi_a(P) \rangle_{h,g} + c' (-R(h + \delta h) + R(h)) .
\]
we obtain

\[ h_{\alpha\beta}(K^{\mu\nu})\Theta^{\alpha\beta} = m(K^{\mu\nu})O_m + 2c'R^{\mu\nu} + 2\Theta^{\mu\nu} \]  
(5.22a)

\[ h_{\alpha\beta}(K^{\mu\nu,\gamma\delta})\Theta^{\alpha\beta} = c'(4h^{\gamma\delta}h^{\mu\nu} - 2(h^{\mu\gamma}h^{\nu\delta} + h^{\mu\delta}h^{\nu\gamma})) . \]  
(5.22b)

We can obtain (5.22a) also from eq. (4.2) of the previous section. With the connection (5.4), (5.7) for the conjugate field \( O_m \), eq. (5.22a) gives

\[ A_1 + A_2 = 0 , \quad A_3 + A_4 = -\frac{1}{2} , \]
\[ A_5 + A_6 = \frac{1}{4}(c_m) \frac{1}{m} - \frac{3}{8} \beta_1 , \quad A_7 + A_8 = \frac{c'}{4} , \]  
(5.23)

\[ C_1 + C_4 = -\frac{3}{4} , \quad C_2 = 1 , \quad C_3 = \frac{1}{2} , \]  
(5.24)

and eq. (5.22b) gives

\[ B_1 + B_4 = 3c' , \quad B_2 = B_3 = -c' . \]  
(5.25)

We have used eqs. (5.17) and (5.18).

5.3. OPE coefficients \( C, \tilde{C} \) in terms of the connection \( K \)

Using eqs. (2.6) and (2.7), we can write down the OPE coefficients \( C \) in terms of the connection (5.14) as follows:

\[(C^{\mu\nu}(\rho))\Theta^{\alpha\beta} = -\frac{2}{\rho^2} [h^{\mu\nu}h^{\alpha\beta}A_1 + (h^{\mu\alpha}h^{\nu\beta} + h^{\mu\beta}h^{\nu\alpha})A_2] \]

\[ + \frac{\beta_1 m^2}{2\rho} [h^{\mu\nu}h^{\alpha\beta}A_3 + (h^{\mu\alpha}h^{\nu\beta} + h^{\mu\beta}h^{\nu\alpha})A_4] \]  
(5.26a)

\[(C^{\mu,\gamma\delta}(\rho))\Theta^{\alpha\beta} = 0 . \]  
(5.26b)

Using the algebraic constraint (1.5), we find that eq. (5.26a) implies the vanishing of \( A_1, A_2 \):

\[ A_1 = A_2 = 0 . \]  
(5.27)

This is consistent with (5.23).
Using the general formula (1.11), we can calculate the coefficients $\tilde{C}$. The result is as follows:

$$(\tilde{C}^{\mu\nu}_{\mu}(\rho))\Theta^{\alpha\beta}$$
$$= \frac{1}{6} \left[ (-B_1 - c' - 2B_4)h^{\alpha\beta}\partial^\nu R + (c' + B_4)(h^{\nu\beta}\partial^{\alpha} R + h^{\nu\alpha}\partial^{\beta} R) \right]$$
$$+ \nabla^\nu \Theta^{\alpha\beta},$$
(5.28a)

$$(\tilde{C}^{\mu\nu}_{\mu}(\gamma\rho))\Theta^{\alpha\beta}$$
$$= h^{\gamma\nu}h^{\alpha\beta} \left[ A_3(g_1 + \frac{1}{2}\beta_1 m^2 \ln \rho) + m^2 A_5 ight.$$
$$+ R(A_7 + \frac{1}{6}(-B_1 - 2c' - 3B_4)) + C_1 \Theta]$$
$$+ (h^{\nu\alpha}h^{\gamma\beta} + h^{\nu\beta}h^{\gamma\alpha}) \left[ A_4(g_1 + \frac{1}{2}\beta_1 m^2 \ln \rho) + m^2 A_6 ight.$$
$$+ R(A_8 + \frac{1}{3}(c' + B_4)) + C_4 \Theta]$$
$$- h^{\nu\alpha}h^{\gamma\beta} - h^{\nu\beta}h^{\gamma\alpha} + 2h^{\gamma\nu}h^{\alpha\beta} + h^{\alpha\beta}h^{\nu\gamma},$$
(5.28b)

$$(\tilde{C}^{\mu\nu}_{\mu}(\gamma\delta\epsilon(\rho)))\Theta^{\alpha\beta} = (K^{\nu\gamma,\delta\epsilon} + K^{\nu\delta,\epsilon\gamma} + K^{\nu\epsilon,\gamma\delta})\Theta^{\alpha\beta}$$
$$= (B_1 - c')g^{\alpha\beta}(h^{\nu\gamma}h^{\delta\epsilon} + h^{\nu\delta}h^{\gamma\epsilon} + h^{\nu\epsilon}h^{\gamma\delta})$$
$$- c'(h^{\nu\gamma}(h^{\delta\alpha}h^{\epsilon\beta} + h^{\delta\beta}h^{\epsilon\alpha}) + ...)$$
$$+ B_4 \left( h^{\gamma\delta}(h^{\epsilon\alpha}h^{\nu\beta} + h^{\epsilon\beta}h^{\nu\alpha}) + ... \right),$$
(5.28c)

where we have used eqs. (5.24) and (5.25), and the last omitted terms are obtained by symmetrizing with respect to $\gamma$, $\delta$, and $\epsilon$.

Since eq. (5.28b) has no $1/\rho^2$ singularity, the algebraic constraint (1.16) implies that

$$h^{\delta\epsilon}(\tilde{C}^{\mu\nu}_{\mu}(\gamma\delta\epsilon))\Theta^{\alpha\beta} = 0.$$  (5.29)

Using (5.25), this gives

$$B_1 = \frac{5c'}{2}, \quad B_4 = \frac{c'}{2}.$$  (5.30)

We also notice that the right-hand side of eq. (5.28b) should not depend on the cosmological constant $g_1$, since the short-distance singularities are independent of the additive constant in the vacuum energy. Using the trace anomaly (5.4), the absence of $g_1$ in eq. (5.28b) gives

$$A_3 = -3 - 2C_1, \quad A_4 = \frac{5}{2} + 2C_1.$$  (5.31)

To summarize so far, eqs. (5.27), (5.31), (5.25), (5.30), and (5.24) give all the constants except for $A_6$, $A_8$, and $C_1$. To determine these three remaining constants, we must use the algebraic constraints (1.14) and (1.15).
5.4. Further algebraic constraints

To determine the remaining unknown constants, we try to construct coefficients \( \tilde{C}^{\mu
u, \mu_1...\mu_m} \) which are related to eqs. (5.28) by eqs. (1.14). At the same time we must satisfy the constraints (1.15). We have found it convenient to do this construction in a complex coordinate system \( z \) in which the metric has only \( h_{zz} \) non-vanishing. While constructing the coefficients \( \tilde{C}^{\mu
u, \mu_1...\mu_m} \), it is also important to satisfy the Bose symmetry (3.21). We will omit the detail here. The final results are as follows:

\[
\begin{align*}
A_3 &= A_4 = -\frac{1}{4}, & A_5 &= A_6 = \frac{1}{8} \left( (c_m)^1_m - \frac{3}{2} \beta_1 \right), \\
A_7 &= \frac{7c'}{8}, & A_8 &= -\frac{5c'}{8}, & C_1 &= -\frac{11}{8}, & C_4 &= \frac{5}{8}.
\end{align*}
\]

Written in a more transparent form, our final results are given by

\[
\begin{align*}
\frac{1}{\rho} \int_{\rho(z,P) = \rho} d\Omega_{\rho(z,P) = \rho} \left( \delta h^{\mu\nu} \Theta^{\mu\nu}(z,\bar{z})(P) \Theta^{zz}(P) \right) \\
&= -\frac{c'}{4} (\nabla^2)^4 \delta h_{zz}(P) + \nabla^2 \delta h_{zz} \cdot \Theta^{zz}(P) + \nabla^2 \Theta^{zz}(P) \\
&\quad - m^2 \frac{\beta_1}{16} \nabla^2 \delta h_{zz}(P) \\
&\quad + m^2 \left( -\nabla^2 \delta h \cdot \nabla^2 + \nabla^2 \delta h^{zz} \cdot \nabla^2 + \nabla^2 \Theta^{zz}(P) \right) O_m(P) + o(\rho^0), \tag{5.33a}
\end{align*}
\]

\[
\begin{align*}
\frac{1}{\rho} \int_{\rho(z,P) = \rho} d\Omega_{\rho(z,P) = \rho} \delta h^{\mu\nu} \Theta^{\mu\nu}(z,\bar{z}) O_m(P) \\
&= \frac{1}{2} (\nabla^2 \Theta^{zz})(P) \left[ \frac{1}{2} (O_m(P) + m(c_m)^1_m) - m \beta_1 \left( \frac{1}{4} + \frac{1}{2} \ln \rho \right) \right] \\
&\quad - m \frac{\beta_1}{8} \nabla^2 \delta h(P) \\
&\quad + (\nabla^2 \delta h_{zz} \cdot \nabla^2 + \nabla^2 \delta h^{zz} \cdot \nabla^2) O_m(P) + o(\rho^0), \tag{5.33b}
\end{align*}
\]

where \( \delta h^{\mu\nu} \) is an arbitrary symmetric tensor which vanishes at \( P \). In the \( m = 0 \) limit, this agrees with conformal field theory [7].

6. Conclusion

In this paper we have continued the study of the energy-momentum tensor initiated in ref. [1]. In particular we have examined integrability of the first order variational formula (2.1) for the vacuum energy, and derived the condition (2.13). We have observed that this
integrability condition \((2.13)\) gives the torsion of the connection \(K\) up to total derivatives as in eq. \((2.14)\). We have also interpreted the dependence of eq. \((2.13)\) on the geodesic distance \(\rho\) as the Bose symmetry between two energy-momentum tensors. Finally, using the integrability condition and other algebraic constraints, we have determined the short-distance singularities of the product of two energy-momentum tensors in the massive Ising model in two dimensions.

The massive Ising model may be too simple. After all, it is a theory of the free massive Majorana fermion, and the OPE of two energy-momentum tensors can be calculated directly using the elementary spinor fields (at least in flat space). The aim of the example is to show that we understand the general properties of the energy-momentum tensor well enough to determine the OPE explicitly without explicit calculations.

The goal of ref. \([1]\) and the present paper is understanding the first order variational formula \((1.1)\), which actually defines the energy-momentum tensor through its integral over space. We have applied only a limited integrability check in this paper. In a separate paper \([8]\) we plan to address the full integrability condition for the variational formula applied to arbitrary correlation functions instead of the vacuum energy.
References

[1] H. Sonoda, “The Energy-Momentum Tensor in Field Theory I,” UCLA preprint (April, 1995), UCLA/95/TEP/10, hep-th 9504133
[2] J. Schwinger, Phys. Rev. 127 (1962) 324; 130 (1962) 406, 800
[3] K. G. Wilson, Phys. Rev. D2 (1970) 1478
[4] H. Sonoda, in Proceedings of the Conference Strings ’93, eds. M. B. Halpern, G. Rivlis, and A. Sevrin (World Scientific, 1995), and references therein
[5] D. Friedan, Ann. Phys. 163 (1985) 318
[6] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry I (Interscience, 1963)
[7] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. B241 (1984) 333
[8] H. Sonoda, “The Energy-Momentum Tensor in Field Theory III,” paper in preparation