A POSITIVE SOLUTION FOR AN ASYMPTOTICALLY CUBIC QUASILINEAR SCHröDINGER EQUATION

XIANG-DONG FANG

School of Mathematical Sciences
Dalian University of Technology, 116024 Dalian, China

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Abstract. We consider the following quasilinear Schrödinger equation

\[-\Delta u + V(x)u - \Delta(u^2)u = q(x)g(u), \quad x \in \mathbb{R}^N,\]

where \( N \geq 1, 0 < q(x) \leq \lim_{|x| \to \infty} q(x), \) \( g \in C(\mathbb{R}^+, \mathbb{R}) \) and \( g(u)/u^3 \to 1, \) as \( u \to \infty. \) We establish the existence of a positive solution to this problem by using the method developed in Szulkin and Weth [27, 28].

1. Introduction. In this paper, we are concerned with the following quasilinear Schrödinger equation

\[-\Delta u + V(x)u - \Delta(u^2)u = q(x)g(u), \quad x \in \mathbb{R}^N.\]  (1)

This equation is related to the modified Schrödinger equation

\[i\frac{\partial \psi}{\partial t} = -\Delta \psi + W(x)\psi - q(x)h(|\psi|^2)\psi - \kappa \Delta(|\psi|^2)\psi,\]  (2)

where \( \psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}, \) \( W = W(x), x \in \mathbb{R}^N \) is a given potential, \( \kappa \) is a positive constant and \( h \) is a real function. The form of (2) has been derived as models of several physical phenomena, see e.g. [19, 22]. We are interested in the standing wave solutions of the form: \( z(t, x) = u(x)e^{i\lambda t}. \) Observe that \( z(t, x) \) solves (2) if and only if \( u(x) \) satisfies (1) with \( V(x) = W(x) - \lambda \) and \( g(u) = h(u^2)u. \)

The existence of solutions of (1) has been studied widely. A positive ground state solution of problem (1) with \( q(x)g(u) = |u|^{p-2}u, 4 \leq p < 2 \cdot 2^*, N \geq 3 \) has been obtained in [20] using a constrained minimization method. In [19], the quasilinear problem was reduced to a semilinear one by using a change of variables and an Orlicz space framework was used to prove the existence of a positive solution. This method was also employed by [9], but the usual Sobolev space \( H^1(\mathbb{R}^N) \) was used.

The semilinear equation (i.e., \( \kappa = 0 \) in (1)) with asymptotically linear nonlinearity has been extensively studied. The existence of a positive solution has been proved in [26] with assuming radial symmetry. In [10] and [16], for the non symmetric asymptotically linear case, they obtained a Mountain Pass positive solution. Later in [18], under restrictive conditions on the potential \( V, \) the existence of a

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positive solution corresponding to higher energy levels was shown, see also [17]. Subsequently, the result in [17] was extended to the quasilinear Schrödinger problem (see [5]). In a recent paper [8], they proved a positive bound state with the nonlinearity \( f \in C^3 \). The existence of sign-changing solutions was proved for an asymptotically linear Schrödinger equation with deepening potential well in [21].

We consider the problem

\[-\Delta u + V(x)u - \Delta(u^2)u = q(x)g(u), \quad u \in H^1(\mathbb{R}^N). \tag{3}\]

Setting \( G(u) := \int_0^u g(s)ds \), we suppose that \( V \), \( q \) and \( g \) satisfy the following assumptions:

1. \( V \in C(\mathbb{R}^N), \lim_{|x| \to \infty} V(x) = V_\infty > 0 \), and \( \inf_{x \in \mathbb{R}^N} V(x) > 0 \).
2. \( q \in C(\mathbb{R}^N), \lim_{|x| \to \infty} q(x) = q_\infty > 0 \), \( \inf_{x \in \mathbb{R}^N} q(x) > 0 \) and \( q(x) \leq q_\infty \), for all \( x \in \mathbb{R}^N \), with the strict inequality holding on a subset of positive Lebesgue measure in \( \mathbb{R}^N \).

3. \( V_\infty < q_\infty \).
4. \( \sup_{x \in \mathbb{R}^N} [(V(x) - V_\infty) + q_\infty - q(x)] < \mu \), where \( (V(x) - V_\infty)^+ := \max\{V(x) - V_\infty, 0\} \) and \( \mu \) is the number in the right side of the inequality in Lemma 3.15.
5. \( g \in C([0, \infty)) \) and \( g(u) = o(u) \), as \( u \to 0 \).
6. \( g(u)/u^3 \to 1 \), as \( u \to \infty \).
7. \( u \to g(u)/u^3 \) is positive for \( u \neq 0 \), nondecreasing on \( (0, \infty) \).

**Theorem 1.1.** Suppose that \( (V), (Q), (R_1), (R_2) \) and \( (g_1)\)-(\(g_3)\) hold and the following hypothesis holds

- \( (H) \) the least energy level \( c_\infty \) of \( (7) \) is an isolated critical level for \( I_\infty \). Then \( (3) \) has a positive solution.

**Remark 1.** The functions \( V(x) \equiv V_\infty \) and \( q(x) = q_\infty - \frac{d}{2} e^{-|x|^2} \), for \( x \in \mathbb{R}^N \), satisfies \( (V), (Q), (R_1) \) and \( (R_2) \), where \( V_\infty < q_\infty \) and \( \mu = 2 \cdot \frac{e^{-c_\infty}}{||u||_{L^4}^4} \) is in Lemma 3.15. Note that \( \mu \) is independent of the particular choice of \( V(x) \) and \( q(x) \) by (11) and the definition of \( c_\infty \).

In [19], it is stated that \( V(x) \leq V_\infty \) for all \( x \in \mathbb{R}^N \) (see (V4) there) and the nonlinearity is autonomous, that is, \( q(x)g(u) := \lambda |u|^{p-1}u, 4 \leq p < 2 \cdot 2^* \). In our paper \( (V) \) is weaker. It may be possible that \( V(x) > V_\infty \) in our case. But we need an additional condition \( (R_2) \) to have delicate estimates for the energy of the functional. The existence of a ground state solution was proved in [19], while the minimal energy level need not be attained in our case (see Proof of Theorem 1.1).

To our knowledge, there is only a paper [5] concerned with the existence of solutions to quasilinear Schrödinger equations when the nonlinearity is asymptotically cubic at infinity and the potential \( q(x) \leq q_\infty \) (or \( V(x) \geq V_\infty \) equivalently). The conditions on the potential in [5] (see \((a_1)-(a_4)\) there) are somewhat stronger than ours. For the nonlinearity, an example is the function \( g(u) = u^3 \) satisfying our assumptions but not the condition \((g_2)\) in [5], since there \( Q(u) = \frac{1}{4}g(u)u - G(u) \equiv 0 \), for all \( u > 0 \).

**Remark 2.** Since the solutions set of the limiting equation might be complicated, we need the hypothesis \((H)\) which comes from [3] (see also [5]) in order to make sure that our minimax value will not coincide with the critical values of \((7)\).

According to Theorems 1.2 and 1.3 in [2] (take \( p = 3 \) there), it is easy to see that the ground state solution for the limiting problem \((7)\) is unique and non-degenerate with \( g(u) = u^3 \) (see also [1, 23]).
Notation. $C, C_1, C_2, \ldots$ will denote different positive constants whose exact value is inessential. $B_{\rho}(y) := \{ x \in \mathbb{R}^N : |x - y| < \rho \}$. The usual norm in the Lebesgue measure $L^p(\mathbb{R}^N)$ is denoted by $\|u\|_p$. $E$ denotes the Sobolev space $H^1(\mathbb{R}^N)$ with the norm

$$\|u\| := \left( \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 \right)^{1/2}$$

and $S$ is the unit sphere in $E$. For $y \in \mathbb{R}^N$, let $y \ast u$ for the translate of $u \in E$, that is, $(y \ast u)(x) := u(x - y)$.

2. Preliminary results. We observe that formally the problem (3) is the Euler-Lagrange equation corresponding to the functional

$$J(u) := \frac{1}{2} \int_{\mathbb{R}^N} (1 + 2u^2)|\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x)u^2 - \int_{\mathbb{R}^N} q(x)G(u).$$

(4)

Obviously the presence of the second order nonhomogeneous term $\Delta(u^2)u$ prevents us to work directly with the functional $J$ in $E$. To overcome this difficulty, we employ an argument developed in [9]. We make a change of variables $v := f^{-1}(u)$, where $f$ is defined by

$$f'(t) = \frac{1}{(1 + 2f^2(t))^{1/2}} \text{ on } [0, +\infty) \text{ and } f(t) = -f(-t) \text{ on } (-\infty, 0].$$

Let us collect some properties of $f$, which have been proved in [9, 11, 12, 15, 30].

**Lemma 2.1.** The function $f$ satisfies the following properties:

1. $f$ is uniquely defined, $C^\infty$ and invertible;
2. $|f'(t)| \leq 1$ for all $t \in \mathbb{R}$;
3. $|f(t)| \leq |t|$ for all $t \in \mathbb{R}$;
4. $f(t)/t \to 1$ as $t \to 0$;
5. $f(t)/\sqrt{t} \to 2^{1/4}$ as $t \to +\infty$;
6. $f(t)/t \leq t f'(t) \leq f(t)$ for all $t > 0$;
7. $|f(t)| \leq 2^{1/4} |t|^{1/2}$ for all $t \in \mathbb{R}$;
8. $f^2(t) - f(t)f'(t)t \geq 0$ for all $t \in \mathbb{R}$;
9. There exists a positive constant $C$ such that $|f(t)| \geq C|t|$ for $|t| \leq 1$ and $|f(t)| \geq C|t|^{1/2}$ for $|t| \geq 1$;
10. $|f(t)f'(t)| < 1/2$ for all $t \in \mathbb{R}$;
11. the function $f(t)f'(t)t^{-1}$ is decreasing for $t > 0$;
12. the function $f^p(t)f'(t)t^{-1}$ is increasing for $p \geq 3$ and $t > 0$.

Therefore, after the change of variables, we have the following functional

$$I(v) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V(x)v^2 - \int_{\mathbb{R}^N} q(x)G(f(v)), \quad (5)$$

which is well defined on $E$ and belongs to $C^1$ under the assumptions $(V)$, $(Q)$, $(g_1)$ and $(g_2)$. Moreover, the critical points of $I$ correspond to the weak solutions of the Euler-Lagrange equation

$$-\Delta v + V(x)f(v)f'(v) = q(x)g(f(v))f'(v), \quad v \in E.$$

According to [9], if $v \in E$ is a critical point of the functional $I$, then $u = f(v) \in E$ is a solution of (3).

Define the Nehari manifold

$$\mathcal{M} := \{ v \in E \setminus \{0\} : \langle I'(v), v \rangle = 0 \}. \quad (6)$$
Set $c_0 := \inf_M I$. Let $g(u) := -g(-u)$ for $u < 0$ and then $g$ is odd.

For the limiting problem

$$ -\Delta v = \tilde{g}(v) $$

(7)

where $\tilde{g}(v) = q_\infty g(f(v))f'(v) - V_\infty f(v)f'(v)$, we define the functional and the corresponding Nehari manifold

$$ I_\infty(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V_\infty f^2(v) - \int_{\mathbb{R}^N} q_\infty G(f(v)),$$

$$ \mathcal{M}_\infty := \{v \in E \setminus \{0\} : \langle I'_\infty(v), v \rangle = 0\}. $$

(8)

and set $c_\infty := \inf_{\mathcal{M}_\infty} I_\infty$.

Note that $(R_1)$ guarantees that (1.3) in [4] holds. It is easy to see that $\tilde{g}$ satisfies all assumptions of Theorem 1 in [4], so there exists a radially symmetric positive ground state solution $u_\infty \in C^2(\mathbb{R}^N)$ associated to equation (7).

Let

$$ E := \left\{ v \in E : \int_{\mathbb{R}^N} |\nabla v|^2 < \int_{\mathbb{R}^N} q(x)v^2 \right\} $$

and

$$ E_\infty := \left\{ v \in E : \int_{\mathbb{R}^N} |\nabla v|^2 < \int_{\mathbb{R}^N} q_\infty v^2 \right\}. $$

By the condition $(Q)$, we have $E \subset E_\infty$ and $E \neq \emptyset$. Obviously, not all functions in $E_\infty$ belong to $E$, while we can prove that the modified functions do (see Lemma 3.10).

Under our assumptions, we can not make sure whether or not $M$ is of class $C^1$, so we take the methods developed by [27, 28].

3. Proof of Theorem 1.1.

Lemma 3.1.

(1) For each $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$ |g(u)| \leq \varepsilon |u| + C_\varepsilon |u|^{p-1} \text{ for all } u \in \mathbb{R}, $$

where $4 < p < 2 \cdot 2^* = 2N/(N-2)$ if $N \geq 3$, $2^* := \infty$ if $N = 1$ or 2.

(2) $G(u) \geq 0$ and $\frac{1}{2}g(u)u \geq G(u)$.

This follows easily from $(g_1)$-$\,(g_3)$. Set $h_u(t) := I(tu)$, $t > 0$.

Lemma 3.2.

(1) For every $u \in E$, there is a unique $t_u > 0$ such that $t_u u \in M$. Moreover, $I(t_u u) = \max_{t \geq 0} I(tu)$.

(2) If $u \notin E$, then $tu \notin M$ for any $t > 0$.

Proof. (1) For every $u \in E$, note that

$$ \frac{h_u(t)}{t^2} = \frac{1}{2} \int_{u \neq 0} |\nabla u|^2 + V(x) \frac{f^2(tu)}{(tu)^2} u^2 - \int_{u \neq 0} q(x)G(f(tu)) \frac{f^4(tu)}{(tu)^4} \frac{u^2}{tu}. $$

By $(g_2)$ and $(g_3)$, we have $\frac{G(s)}{s^2} \leq \frac{1}{4}$, for any $s \in \mathbb{R}$. It follows from Lemma 2.1-(4),(5) and the Lebesgue dominated convergence theorem that $\lim_{t \to 0} \frac{h_u(t)}{t^2} = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 > 0$ and $\lim_{t \to \infty} \frac{h_u(t)}{t^2} = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - q(x)u^2 < 0$. Note also that $h_u'(t) = (I'(tu), u) = 0$ is equivalent to

$$ \int_{\mathbb{R}^N} |\nabla u|^2 = \int_{u \neq 0} \left[ \frac{q(x)g(f(tu))f'(tu)}{tu} - \frac{V(x)f(tu)f'(tu)}{tu} \right] u^2. $$
Set \(Y(s) := \frac{g(x)g(f(s))f'(s) - V(x)f(s)f'(s)}{s}\). Using \((g_3)\) and Lemma 2.1-(12), we get \(g(f(s))f'(s) = \frac{g(f(s))}{f'(s)}\), \(\frac{f'(s)f'(s)}{f'(s)}\) is increasing for \(s > 0\). Then \(Y(s)\) is increasing for \(s > 0\) by Lemma 2.1-(11). This completes the proof.

(2) Arguing by contradiction, suppose \(tu \in M\) for some \(t > 0\). Since \(h_u(t) = 0\), \((g_2)\) and \((g_3)\), we obtain that

\[
\int_{\mathbb{R}^N} |\nabla u|^2 = \int_{u \neq 0} \left[ \frac{q(x)g(f(tu))f'(tu)}{tu} - \frac{V(x)f(tu)f'(tu)}{tu} \right] u^2 \leq \int_{u \neq 0} \frac{q(x)f^3(tu)f'(tu)}{tu} u^2 \leq \int_{\mathbb{R}^N} q(x)u^2.
\]

The last inequality above holds from Lemma 2.1-(5), (10), (12), hence a contradiction with \(u \notin E\).

Lemma 3.3.

(1) There exists \(\rho > 0\) such that \(\inf_M I \geq \inf_{S_\rho} I > 0\), where \(S_\rho := \{u \in E : \|u\| = \rho\}\).

(2) \(\|u\|^2 \geq \inf_M I\) for all \(u \in M\).

(3) If \(V\) is a compact subset of \(E\), there exists \(R > 0\) such that \(I \leq 0\) on \((\mathbb{R}^N)^+ \setminus B_R(0)\).

Proof. (1) For every \(u \in M\), there is \(s > 0\) such that \(su \in S_\rho\), then \(I(u) = I(tu,u) \geq I(su)\) by Lemma 3.2-(1). Hence \(\inf_M I \geq \inf_{S_\rho} I\). We claim that there exist \(C_1, \rho > 0\) such that

\[
\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)f^2(u)) \geq C_1 \|u\|^2 \text{ whenever } \|u\| \leq \rho.
\]

Arguing by contradiction, there is \(u_n \to 0\) in \(E\) such that

\[
\int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)v_n^2) + \int_{\mathbb{R}^N} V(x) \left( \frac{f^2(u_n)}{u_n^2} - 1 \right) v_n^2 \leq \frac{1}{n},
\]

where \(v_n := \frac{u_n}{\|u_n\|^r}\). Since \(u_n \to 0\) in \(L^2(\mathbb{R}^N)\), for fixed \(\varepsilon > 0\), the measure

\[
|\{x \in \mathbb{R}^N : |u_n(x)| > \varepsilon\}| \to 0
\]
as \(n \to \infty\). Using the Hölder inequality,

\[
\int_{|u_n| > \varepsilon} v_n^2 \leq \|\{x \in \mathbb{R}^N : |u_n(x)| > \varepsilon\}\|^{(r-2)/r} \|v_n\|^2 \to 0,
\]

where \(r = 2^*\) if \(N \geq 3\) and \(r > 2\) if \(N = 1\) or \(2\). So we have by Lemma 2.1-(4),

\[
\int_{\mathbb{R}^N} V(x) \left( \frac{f^2(u_n)}{u_n^2} - 1 \right) v_n^2 \to 0.
\]

Hence \(\|v_n\| = 1\) and \(v_n \to 0\) in \(E\), a contradiction.

By \((Q)\), Lemma 3.1-(1), Lemma 2.1-(3), (7) and the Sobolev inequality, we obtain that

\[
\int_{\mathbb{R}^N} q(x)G(f(u)) \leq C_2 \varepsilon \|u\|^2 + C_3 C_\varepsilon \|u\|^{p/2}.
\]
Take \( \varepsilon \) small enough, such that for small \( \rho \), \( I(u) \geq C_4\|u\|^2 - C_5\|u\|^{p/2} \) and \( \inf_{S_{\rho}} I > 0 \).

(2) By Lemma 3.1-(2) and Lemma 2.1-(3), for \( u \in \mathcal{M} \),
\[
\inf_{\mathcal{M}} I \leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)f^2(u) \leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 = \frac{1}{2}\|u\|^2.
\]

(3) Without loss of generality, we may assume that \( \mathcal{V} \subset S \). Suppose by contradiction that there exist \( u_n \in \mathcal{V} \) and \( w_n = t_n u_n \), such that \( u_n \to u \), \( t_n \to \infty \) and \( I(w_n) \geq 0 \). It follows from \((g_2)\) and Lemma 2.1-(5) that
\[
\int_{u_n \neq 0} \left( V(x) \frac{f^2(t_n u_n)}{(t_n u_n)^2} - 2q(x) \frac{G(f(t_n u_n))}{f^2(t_n u_n)} + f^4(t_n u_n) \right)^2 u_n^2 \to - \int_{\mathbb{R}^N} q(x)u^2,
\]
as \( n \to \infty \). Note that \( u \in \mathcal{E} \), so we have
\[
0 \leq \frac{2I(t_n u_n)}{t_n^2} \to \int_{\mathbb{R}^N} |\nabla u|^2 - \int_{\mathbb{R}^N} q(x)u^2 < 0,
\]
a contradiction. \( \square \)

The above follows from a similar argument as Lemmas 3.3 and 4.4 in [15], so we omit it.

**Lemma 3.4.** If \( (u_n) \subset \mathcal{M} \) is a Palais-Smale sequence, then \( (u_n) \) is bounded.

**Proof.** Arguing indirectly, suppose \( \|u_n\| \to \infty \) and \( v_n := \frac{u_n}{\|u_n\|} \). If
\[
\sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |v_n|^2 \to 0,
\]
then \( v_n \to 0 \) in \( L^{p/2}(\mathbb{R}^N) \) for \( 2 < p/2 < 2^* \) by P.L. Lions’ lemma (cf. [29], Lemma 1.21). It follows from Lemma 3.1-(1) and Lemma 2.1-(3),(7) that \( G(f(tv_n)) \to 0 \) for every \( t > 0 \). Since \( (u_n) \subset \mathcal{M} \) is a Palais-Smale sequence, there exists \( d > 0 \) such that \( d \geq I(u_n) \geq I(tv_n) \) for every \( t > 0 \) and using Lemma 2.1-(9)
\[
I(tv_n) = \frac{1}{2} \int_{\mathbb{R}^N} t^2 |\nabla v_n|^2 + V(x)f^2(tv_n) + o(1)
\]
\[
\geq \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 + \frac{C^2t^2}{2} \int_{|v_n| \leq 1} V(x)v_n^2 + o(1)
\]
\[
= \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla v_n|^2 + C^2t^2 \int_{\mathbb{R}^N} V(x)v_n^2 - \frac{C^2t^2}{2} \int_{|v_n| > 1} V(x)v_n^2 + o(1)
\]
\[
\geq \frac{t^2}{2} \min\{1, C^2\} - \frac{C^2}{2} \int_{|v_n| > 1} V(x)(tv_n)^{p/2} + o(1)
\]
\[
= \frac{t^2}{2} \min\{1, C^2\} + o(1),
\]
where \( o(1) \to \infty \), as \( n \to \infty \). We have a contradiction with \( t \) large enough. Then there is a sequence \( (y_n) \subset \mathbb{R}^N \) and \( \delta > 0 \) such that
\[
\liminf_{n \to \infty} \int_{B_1(0)} (y_n * v_n)^2 \geq \delta.
\]
If the sequence \( (y_n) \) is bounded in \( \mathbb{R}^N \), going if necessary to a subsequence, \( v_n \rightharpoonup v \neq 0 \) by the local compactness of the Sobolev embedding theorem. Note that
\[ \langle I'(u_n), \varphi \rangle \to 0, \text{ for every } \varphi \in C_0^\infty(\mathbb{R}^N). \] So \( \frac{\langle I'(u_n), \varphi \rangle}{n_{\|u_n\|}} \to 0 \), that is,

\[
\int_{u_n \neq 0} \nabla v_n \nabla \varphi + \frac{V(x)f(u_n)f'(u_n)}{u_n} v_n \varphi = \int_{u_n \neq 0} \frac{q(x)g(f(u_n))f'(u_n)}{u_n} v_n \varphi + o(1). \]

Let \( x \in \mathbb{R}^N \) be such that \( v(x) \neq 0 \). Then \( u_n(x) \to \infty \). It follows from \((g_2)\) and Lemma 2.1-(5) that

\[
\frac{g(f(u_n))f'(u_n)}{n_{\|u_n\|}} = \frac{f^3(u_n)}{f^3(u_n)} \to 1. \]

By the Lebesgue dominated convergence theorem, \( \int_{\mathbb{R}^N} \nabla v_n \nabla \varphi = \int_{\mathbb{R}^N} q(x) v_n \varphi \). Since the essential spectrum of \(-\Delta - q \) is \([-q, \infty) \) (cf. Theorem 3.15 in [25]), a contradiction.

Then \( |y_n| \to \infty \), we can assume that \( y_n \ast v_n \to w \neq 0 \). Similarly, we obtain that \( \int_{\mathbb{R}^N} \nabla w \nabla \varphi = \int_{\mathbb{R}^N} q \ast w \phi \), a contradiction. The conclusion follows.

The above lemmas also follow for the limiting functional \( I_\infty \) except that \( M \) and \( E \) should be replaced by \( M_\infty \) and \( E_\infty \). Let \( U := E \cap S \) (recall that \( S \) is the unit sphere in \( E \)) and define the mapping \( m : U \mapsto M \) by \( m(w) := t_w w \), where \( t_w \) is as in Lemma 3.2-(1). Since \( E \) is open in \( U \), \( U \) is an open subset of \( S \). Similarly, \( U_\infty := E_\infty \cap S \) and define the mapping \( m_\infty : U_\infty \mapsto M_\infty \) by \( m_\infty (w) := t_w w \).

**Lemma 3.5.** Assume \( w_n \in U, \ w_n \to w_0 \in \partial U \) and \( m(w_n) := t_n w_n \). Then \( t_n \to \infty \) and \( I(m(w_n)) \to I(w) \).

**Proof.** Taking the same argument as Lemma 4.6 in [15], we have that \( t_n \to \infty \).

Note that \( \int_{\mathbb{R}^N} |\nabla w| \geq \int_{\mathbb{R}^N} q(x) w_0^2 \). Using \((g_2)\) and \((g_3)\), we get \( \frac{G(s)}{s^2} \leq \frac{1}{2} \). Then according to Lemma 2.1-(7),

\[
I(tw_0) = \frac{1}{2} \int_{\mathbb{R}^N} t^2 |\nabla w_0|^2 + V(x) f^2(tw_0) - \int_{\mathbb{R}^N} q(x) G(f(tw_0)) = \frac{1}{2} \int_{\mathbb{R}^N} t^2 q(x) \left[ \frac{1}{2} - \frac{G(f(tw_0))}{f^2(tw_0)} \right] f^2(tw_0) \]
\[
\geq \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(tw_0). \]

So \( I(tw_0) \to \infty \), as \( t \to \infty \). Given \( C > 0 \), take \( t > 0 \) such that \( I(tw_0) \geq C \). Therefore

\[
\lim_{n \to \infty} \inf I(t_n w_n) \geq \lim_{n \to \infty} I(t_n w_n) = I(tw_0) \geq C. \]

**Lemma 3.6.** The map \( m : U \mapsto M \) is continuous.

**Proof.** Assume \( w_n \to w_0 \) with \( w_n \in U \). Let \( m(w_n) := t_n w_n \). If \( w_0 \in \partial U \), by Lemma 3.5, \( t_n \to \infty \) and \( I(m(w_n)) \to \infty \). It has a contradiction with Lemma 3.3-(3). In the following, we need to prove that \( m(w_n) \to m(w_0) \). According to Lemma 3.3-(3), \( (t_n) \) is bounded. We may assume \( t_n \to t_0 \), up to a subsequence. Note that \( 0 = \langle I'(t_n w_n), t_n w_n \rangle \to \langle I'(t_0 w_0), t_0 w_0 \rangle \), then \( t_0 w_0 = m(w_0) \). The proof is complete.

By Lemma 3.2, the mapping \( m \) is bijective and the inverse of \( m \) is given by \( m^{-1} : M \mapsto U \), \( m^{-1}(u) = \frac{\psi}{\|\psi\|} \).

**Lemma 3.7.** The mapping \( m^{-1}(u) \) is Lipschitz continuous.
Proof. Taking an argument as in [27], we have by Lemma 3.3, for all $u, v \in \mathcal{M}$,
\[
\|m^{-1}(u) - m^{-1}(v)\| = \left\| \frac{u - v}{\|u\|} + \frac{(||v| - \|u||)v}{\|u\|\|v\|} \right\| \leq C\|u - v\|.
\]

So we have the following lemma.

**Lemma 3.8.** The map $m$ is a homeomorphism between $U$ and $\mathcal{M}$.

Define the functional $\Psi : U \to \mathbb{R}$ by
\[
\Psi(w) := I(m(w)).
\]
According to Lemma 4.8 in [15], the following lemma follows from Lemmas 3.2 and 3.3.

**Lemma 3.9.**

(1) $\Psi \in C^1(U, \mathbb{R})$ and
\[
\langle \Psi'(w), z \rangle = \|m(w)\|\|I'(m(w)), z\rangle \quad \text{for all } z \in T_w(U).
\]

(2) If $(w_n)$ is a Palais-Smale sequence for $\Psi$, then $(m(w_n))$ is a Palais-Smale sequence for $I$. If $(u_n) \subset \mathcal{M}$ is a bounded Palais-Smale sequence for $I$, then $(m^{-1}(u_n))$ is a Palais-Smale sequence for $\Psi$.

(3) $w$ is a critical point of $\Psi$ if and only if $m(w)$ is a nontrivial critical point of $I$. Moreover, the corresponding values of $\Psi$ and $I$ coincide and $\inf_U \Psi = \inf_{\mathcal{M}} I$.

**Lemma 3.10.** $c_0 \leq c_\infty$.

Proof. Let $u_\infty \in \mathcal{M}_\infty$ such that $I_\infty(u_\infty) = c_\infty$. It is obvious that $u_\infty \in \mathcal{E}_\infty$. In our case we cannot make sure that $u_\infty \in \mathcal{E}$. But we prove the modified function does. Let $u_\infty^g := y * u_\infty$. Using Lemma 2.1-(5),(10), (g2) and the translation invariance, we obtain that
\[
\frac{J_\infty(ru_\infty^g)}{y^2} = \int_{\mathbb{R}^N} |\nabla u_\infty^g|^2 + \int_{\mathbb{R}^N} V_\infty \frac{f(\frac{ru_\infty^g)}{ru_\infty^g})f'(\frac{ru_\infty^g)}{ru_\infty^g} (u_\infty^g)^2}{ru_\infty^g} - \int_{\mathbb{R}^N} q_\infty \frac{g(\frac{ru_\infty^g)}{ru_\infty^g})}{f^3(\frac{ru_\infty^g)}{ru_\infty^g})} \frac{f^3(\frac{ru_\infty^g)}{ru_\infty^g})f'(\frac{ru_\infty^g)}{ru_\infty^g} (u_\infty^g)^2}{ru_\infty^g} = \int_{\mathbb{R}^N} |\nabla u_\infty|^2 - \int_{\mathbb{R}^N} q_\infty u_\infty^2 < 0,
\]
as $r \to \infty$. So there exist $\eta < 0$ and $R > 0$ such that $\frac{J_\infty(ru_\infty^g)}{r^2} \leq \eta$, for $r \geq R$. It follows from (V), Lemma 2.1-(3) and the Lebesgue dominated convergence theorem that
\[
\lim_{|y| \to \infty} \int_{\mathbb{R}^N} [V(x + y) - V_\infty] \frac{f(\frac{ru_\infty^g)}{ru_\infty^g})f'(\frac{ru_\infty^g)}{ru_\infty^g} (u_\infty^g)^2}{ru_\infty^g} = 0.
\]
Similarly,
\[
\lim_{|y| \to \infty} \int_{\mathbb{R}^N} [q_\infty - q(x + y)] \frac{g(\frac{ru_\infty^g)}{ru_\infty^g})}{f^3(\frac{ru_\infty^g)}{ru_\infty^g})} \frac{f^3(\frac{ru_\infty^g)}{ru_\infty^g})f'(\frac{ru_\infty^g)}{ru_\infty^g} (u_\infty^g)^2}{ru_\infty^g} = 0,
\]
by (Q), (g2), (g3) and Lemma 2.1-(7). Then $\frac{J(\frac{ru_\infty^g)}{r^2})}{r^2} = \frac{J_\infty(\frac{ru_\infty^g)}{r^2})}{r^2} + o(1)$, where $o(1) \to 0$, as $|y| \to \infty$. So there is $S > 0$ such that $\frac{J(\frac{ru_\infty^g)}{r^2})}{r^2} \leq \frac{\eta}{2} < 0$, for $r \geq R$ and $|y| \geq S$.

Note that $J(\frac{ru_\infty^g)}{r^2}) > 0$ for $r$ small enough, and the similar argument as in the proof of Lemma 3.2-(1) shows that $\frac{J(\frac{ru_\infty^g)}{r^2})}{r^2}$ is strictly decreasing in $r \in (0, \infty)$. 

\[\Box\]
Then, there exists a unique \( T^y \in (0, R) \) such that \( J(T^y u_\infty^y) = 0 \), i.e. \( T^y u_\infty^y \in \mathcal{M} \) for \( |y| \geq S \) (By Lemma 3.2-(2), we have \( u_\infty^y \in \mathcal{E} \) for \( |y| \geq S \)). Therefore

\[
\int_{\mathbb{R}^N} |\nabla u_\infty|^2 = \int_{\mathbb{R}^N} \frac{q(x+y)g(f(T^y u_\infty))f'(T^y u_\infty)}{T^y u_\infty} u_\infty^2 \\
- \int_{\mathbb{R}^N} \frac{v(x+y) f(T^y u_\infty) f'(T^y u_\infty)}{T^y u_\infty} u_\infty^2 \\
= \int_{\mathbb{R}^N} \frac{q_\infty g(f(T^y u_\infty)) f'(T^y u_\infty)}{T^y u_\infty} u_\infty^2 \\
- \int_{\mathbb{R}^N} \frac{V_\infty f(T^y u_\infty) f'(T^y u_\infty)}{T^y u_\infty} u_\infty^2 + o(1),
\]

where \( o(1) \to 0 \), as \( |y| \to \infty \). Since \( \frac{q_\infty g(f(s)) f'(s)}{s} - \frac{V_\infty g(f(s)) f'(s)}{s} \) is strictly increasing for \( s > 0 \), we get \( T^y \to 1 \), as \( |y| \to \infty \). Hence \( c_0 \leq I(T^y u_\infty^y) \to c_\infty \). \( \square \)

In the following, we describe a splitting lemma on \( \mathcal{M} \).

**Lemma 3.11.** If there exists \( (u_n) \subset \mathcal{M} \) such that

\[
I(u_n) \to c, \quad I'(u_n) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N),
\]

then replacing \( \{u_n\} \) if necessary by a subsequence, there exists a solution \( u^0 \in \mathcal{E} \) of

\[
-\Delta v + V(x) f(v) f'(v) = q(x) g(f(v)) f'(v),
\]

a finite sequence \( u^1, \ldots, u^k \in \mathcal{E} \) of solutions of

\[
-\Delta v + V_\infty f(v) f'(v) = q_\infty g(f(v)) f'(v),
\]

and \( k \) sequences \( (y_n^j) \subset \mathbb{R}^N \) satisfying

\[
|y_n^j| \to \infty, \quad |y_n^j - y_n^{j'}| \to \infty, \quad j \neq j', \quad n \to \infty,
\]

\[
\int_{\mathbb{R}^N} |\nabla(u_n - u^0 - \sum_{j=1}^k u^j(\cdot - y_n^j))|^2 + f^2(u_n - u^0 - \sum_{j=1}^k u^j(\cdot - y_n^j)) \to 0,
\]

\[
\int_{\mathbb{R}^N} |\nabla u_n|^2 + f^2(u_n) \to \sum_{j=0}^k \int_{\mathbb{R}^N} |\nabla u^j|^2 + f^2(u^j),
\]

\[
I(u^0) + \sum_{j=1}^k I_\infty(u^j) = c.
\]

**Proof.** By Lemma 3.4, \( (u_n) \) is bounded in \( \mathcal{E} \). Going if necessary to a subsequence, \( u_n \to u^0 \) in \( \mathcal{E} \). We claim that \( \int_{\mathbb{R}^N} f^2(u_n) - f^2(u_n - u^0) - f^2(u^0) \to 0, \)

\( \int_{\mathbb{R}^N} [f(u_n) f'(u_n) - f(u_n - u^0) f'(u_n - u^0)] \varphi \to 0 \), uniformly in \( ||\varphi|| \leq 1 \) as \( n \to \infty \).

We prove the second. Note that \( [f(s) f'(s)]' \leq [f'(s)]^2 \leq 1 \). For \( R > 0 \), using the mean value theorem and the Hölder inequality,

\[
\left| \int_{|x| > R} [f(u_n) f'(u_n) - f(u_n - u^0) f'(u_n - u^0)] \varphi \right| \leq ||\varphi|| \left( \int_{|x| > R} |u|^2 \right)^{1/2}.
\]

Thus, for every \( \varepsilon > 0 \), there exists \( R > 0 \) such that

\[
\left| \int_{|x| > R} [f(u_n) f'(u_n) - f(u_n - u^0) f'(u_n - u^0)] \varphi \right| \leq \varepsilon ||\varphi||.
\]
It follows from the Rellich theorem and Hölder inequality that
\[ \left| \int_{|x| \leq R} [f(u_n) f'(u_n) - f(u_n - u^0) f'(u_n - u^0) - f(u^0) f'(u^0)] \phi \right| \leq \varepsilon \| \phi \|. \]
The first is similar.

Note that for \( s \geq C > 0 \), using Lemma 2.1-(6),(7) and (g2),
\[ g(f(s)) f'(s) = \frac{g(f(s)) f'(s) s}{s} \leq \frac{g(f(s)) f(s)}{s} \leq C_1 f^p(s) \leq C_2 s^{p/2 - 1}, \quad (9) \]
where \( 4 < p < 2 \cdot 2^* \). Taking a similar argument as in Proposition A.1 in [13], we get by (g1), (g2), Lemma 2.1-(7) and (9), \( \int_{\mathbb{R}^N} G(f(u_n)) - G(f(u_n - u^0)) - G(f(u^0)) \, dx \rightarrow 0 \), \( \int_{\mathbb{R}^N} [g(f(u_n)) f'(u_n) - g(f(u_n - u^0)) f'(u_n - u^0) - g(f(u^0)) f'(u^0)] \phi \, dx \rightarrow 0 \), uniformly in \( \| \phi \| \leq 1 \) as \( n \rightarrow \infty \) (see also Lemmas 3.4 and 3.5 in [31]). Hence \( I(u_n - u^0) = I(u_n) - I(u^0) + o(1) \), \( I_\infty(u_n - u^0) = I_\infty(u_n) - I_\infty(u^0) + o(1) \), \( I'(u_n - u^0) = I'(u_n) - I'(u^0) + o(1) \), \( I'_\infty(u_n - u^0) = I'_\infty(u_n) - I'_\infty(u^0) + o(1) \).

Let \( u^1_n := u_n - u^0 \). Then \( u^1_n \rightarrow 0 \) in \( L^p_{\text{loc}}(\mathbb{R}^N) \) by the local compactness of the Sobolev embedding theorem. According to (V), (Q) and Lemma 2.1-(3),(7), we have \( I_\infty(u^1_n) = I(u_n) - I(u^0) + o(1) \) and \( I'_\infty(u^1_n) = I'(u_n) - I'(u^0) + o(1) \). Let
\[ \delta := \limsup_{n \rightarrow \infty} \left( \sup_{g \in \mathbb{R}^N} \int_{B_1(g)} |u^1_n|^2 \right) \]
If \( \delta = 0 \), then \( u^1_n \rightarrow 0 \) in \( L^2(\mathbb{R}^N) \) by P.L. Lions’ Lemma (cf. [29], Lemma 1.21), where \( p \) is in Lemma 3.1-(1). For every \( \varepsilon > 0 \) there is \( C_\varepsilon > 0 \) such that, using Lemma 2.1-(6),(3),(7) and Lemma 3.1-(1),
\[ \int_{\mathbb{R}^N} g(f(u^1_n)) f'(u^1_n) u^1_n \leq \int_{\mathbb{R}^N} g(f(u^1_n)) f(u^1_n) \leq \varepsilon \int_{\mathbb{R}^N} |u^1_n|^2 + C_\varepsilon \int_{\mathbb{R}^N} |u^1_n|^p/2. \]
It follows from \( (I'_\infty(u^1_n), u^1_n) = o(1) \) and Lemma 2.1-(6) that
\[ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u^1_n|^2 + V_\infty f^2(u^1_n) \leq \int_{\mathbb{R}^N} |\nabla u^1_n|^2 + V_\infty f(u^1_n) f'(u^1_n) u^1_n \rightarrow 0, \]
and we finish the proof.

If \( \delta > 0 \), there exists \( (y^1_n) \subset \mathbb{R}^N \) such that \( \int_{B_1(0)} [(-y^1_n) * u^1_n]^2 > \delta/2 \). By the local compactness of the Sobolev embedding theorem, \( (-y^1_n) * u^1_n \rightarrow u \neq 0 \). Since \( u^1_n \rightarrow 0 \) in \( E \), we have \( (y^1_n) \) is unbounded. We may assume that \( |y^1_n| \rightarrow \infty \). It is easy to see that \( I'_\infty \) is weakly sequentially continuous and invariant under translation, then \( I'_\infty(u^1_n) = 0 \in H^{-1}(\mathbb{R}^N) \).

Let \( u^2_n := u_n - (-y^1_n) * u^1_n \), we have \( u^2_n \rightarrow 0 \) in \( E \) and take the same argument as above except that \( I \) should be replaced by \( I_\infty \). Since \( I_\infty(u) \geq c_\infty > 0 \) for every nontrivial critical point \( u \) of \( I_\infty \), the iteration must terminate after a finite number of steps with \( (I(u_n)) \) is bounded.

**Lemma 3.12.** If \( (u_n) \) is bounded in \( E \), then there exist \( C_1, C_2 > 0 \) such that
\[ C_1 \| u_n \|^2 \leq \int_{\mathbb{R}^N} |\nabla u_n|^2 + \int_{\mathbb{R}^N} V(x) f^2(u_n) \leq C_2 \| u_n \|^2, \]
where \( C_1 \) and \( C_2 \) depend on the bound on \( \| u_n \| \) and \( V(x) \).

**Proof.** See Lemma 3.2 of [14] in detail. For the sake of completeness, we include a sketch of it. Obviously, we have by Lemma 2.1-(3), \( \int_{\mathbb{R}^N} |\nabla u_n|^2 + V(x) f^2(u_n) \leq \| u_n \|^2 \).
Arguing by contradiction, assume $u_n \neq 0$ such that
\[
\int_{\mathbb{R}^N} |\nabla w_n|^2 + \int_{u_n \neq 0} V(x)\frac{f^2(u_n)}{u_n^2} w_n^2 \to 0,
\]
where $w_n := \frac{u_n}{\|u_n\|}$. Since $\int_{\mathbb{R}^N} |\nabla w_n|^2 \to 0$, we get
\[
\int_{\mathbb{R}^N} V(x)w_n^2 \to 1. \quad (10)
\]
Set $A_n := \{x \in \mathbb{R}^N : |u_n(x)| \geq C \}$ and $B_n := \mathbb{R}^N \setminus A_n$, for a given $C > 0$. Note that $(u_n)$ is bounded in $E$, for $\varepsilon > 0$ there exists $C$ large enough such that $|A_n| \leq \varepsilon$. It follows from Lemma 2.1-(6) that the function $\frac{f(t)}{t}$ is decreasing for $t > 0$. Hence
\[
\frac{f^2(C)}{C^2} \int_{B_n} V(x)w_n^2 \leq \int_{B_n} V(x)\frac{f^2(u_n)}{u_n^2} w_n^2 \to 0.
\]
Take $\varepsilon$ small enough, then we have by Hölder inequality and $(V)$
\[
\int_{A_n} V(x)w_n^2 \leq C_1 \varepsilon^{(2^* - 2)/2^*} \leq \frac{1}{2},
\]
a contradiction with $(10)$.

\[\square\]

**Proposition 1.** Assume that $(V)$, $(Q)$, $(R_1)$, $(g_1)$-$(g_3)$ are satisfied. If $c_0 < c_\infty$, then $(3)$ has a nontrivial ground state solution.

**Proof.** Though Ekeland’s variational principle (cf: Theorem 4.8.1 in [7]) follows in a complete metric space, it still holds in $U$ by Lemma 3.5 (the possibility that the limit reaches the boundary of $U$ is ruled out). So there exists $(w_n) \subset U$ satisfying $\Psi(w_n) \to c_0$ and $\Psi'(w_n) \to 0$. Let $u_n := m(w_n)$. Then $(u_n)$ is a Palais-Smale sequence for the functional $I$ at level $c_0$ by Lemma 3.9, and moreover $(u_n)$ is bounded in $E$ by Lemma 3.4. According to Lemma 3.11 and $c_0 < c_\infty$, we have $\int_{\mathbb{R}^N} |\nabla(u_n - u^0)|^2 + f^2(u_n - u^0) \to 0$. Lemma 3.12 implies that $\|u_n - u^0\| \to 0$. Using $I(m(w_n)) \to I(u^0)$, $m^{-1}(u^0) \in U$ and then $u^0 \in M$ in view of Lemmas 3.5 and 3.8.

In the following, we assume that $c_0 = c_\infty$. Next we introduce the barycenter map $\beta$ of a given function $u \in E \setminus \{0\}$ which has been constructed in [3] and [6]. Set $\tau(u)(x) = \frac{1}{|B_1|} \int_{B_1(x)} |u(y)|dy$, $\tau(u) \in L^\infty$ and is continuous; $\hat{u}(x) = [\tau(u)(x) - \frac{1}{2} \max \tau(u)]^+$, $\hat{u} \in C_0(\mathbb{R}^N)$; we define the barycenter of $u$ by setting
\[
\beta(u) = \frac{1}{|\hat{u}|_{L^1}} \int_{\mathbb{R}^N} x\hat{u}(x)dx \in \mathbb{R}^N.
\]
Since $\hat{u}$ has compact support, $\beta(u)$ is well defined. The function $\beta$ has the following properties:

1. $\beta$ is continuous in $E \setminus \{0\}$.
2. If $u$ is radial, $\beta(u) = 0$.
3. For every $t > 0$, $u \in E \setminus \{0\}$, $\beta(tu) = \beta(u)$.
4. Given $y \in \mathbb{R}^N$, $\beta(y + u) = \beta(u) + y$.

Now we define $b := \inf_{u \in M} \beta(u) = 0 \quad I(u) = \inf_{w \in U} \beta(m(w)) = 0 \Psi(w)$. It is easy to see that $b \geq c_0 = c_\infty$.

**Proposition 2.** Assume that $(V)$, $(Q)$, $(R_1)$, $(g_1)$-$(g_3)$ are satisfied. If $b = c_0 = c_\infty$, then $(3)$ has a nontrivial ground state solution.
Proof. By the definition of $b$, we get a minimizing sequence $(v_n) \subset U$ of $\Phi$ with $\beta(m(v_n)) = 0$ at level $b$. According to Ekeland’s variational principle, we can find some $(w_n) \subset U$ such that $\Phi(w_n) \to b, \Phi'(w_n) \to 0, \|w_n - v_n\| \to 0$. Put $u_n := m(w_n)$, then $(u_n) \subset \mathcal{M}$ is a Palais-Smale sequence for $I$ at level $b$. Using Lemma 3.4, $(u_n)$ is bounded and $u_n \to u^0$ after passing to a subsequence. It follows from the property (1) of $\beta$ that $\beta(u_n)$ is bounded in $\mathbb{R}^N$.

We claim $u^0 \neq 0$. In fact, if $u^0 = 0$, we have in view of Lemma 3.11

$$\int_{\mathbb{R}^N} |\nabla (u_n - u^1(\cdot - y_n^1))|^2 + f^2(u_n - u^1(\cdot - y_n^1)) \to 0.$$  

Using Lemma 3.12, we obtain that $\|u_n - u^1(\cdot - y_n^1)\| \to 0$. Since $|\beta(u^1(\cdot - y_n^1))| = |y_n^1| \to \infty$ by the properties (2), (4) of $\beta$, a contradiction with the boundedness of $\beta(u_n)$.

According to the claim, we have $u_n \to u^0$ in $E$ by Lemmas 3.11 and 3.12. Taking the same argument in Proposition 1, $u^0 \in \mathcal{M}$.  

Next, we consider the case $b > c_0 = c_\infty$. By the proof of Lemma 3.10, there exist $S > 0$ and $T^y \in (0, R)$ for some fixed $R > 1$ such that $T^yu^y_\infty \in \mathcal{M}$, for $|y| \geq S$. It is clear that $T^y \to 1$, as $|y| \to \infty$ and is continuous with respect to $y$, and moreover

$$\sup_{|y| \geq S} T^y < R (11)$$

is independent of the particular choice of $V(x)$ and $q(x)$. Now define the continuous operator $\Gamma : \mathbb{R}^N \to \mathcal{M}$ as $\Gamma[y] := T^yu^y_\infty \in \mathcal{M}$, for $|y| \geq S$. It follows from the properties of $\beta$ that

$$\beta(\Gamma[y]) = y. (12)$$

Lemma 3.13. $I(\Gamma[y]) \to c_\infty$, if $|y| \to \infty$.

Proof. By the translation invariance of the limiting functional $I_\infty$, it is easy to see that $I(\Gamma[y]) \to I_\infty(u_\infty) = c_\infty$.  

Lemma 3.14. There exists some $\delta > 0$ (with $c_\infty + \delta < b$) such that $\beta(u) \neq 0$ for every $u \in \mathcal{M} \cap I^{-\infty + \delta}$, where $I^{-c} := \{u \in E : I(u) \leq c\}$.

Proof. According to the definition of $b$ and $b > c_0 = c_\infty$, it is obvious that the conclusion holds.

Let $c_* := \inf\{c > c_\infty : c$ is a critical value of $I_\infty\}$ and $\tilde{c} := \min\{c_* , 2c_\infty\}$. According to the hypothesis $(H)$, we have $c_\infty < \tilde{c} \leq 2c_\infty$.

Lemma 3.15. If $\sup_{x \in \mathbb{R}^N} [(V(x) - V_\infty)^+ + q_\infty - q(x)] < 2 \cdot \frac{\tilde{c} - c_\infty}{R^2\|u_\infty\|_2^2}$, where $R$ is in (11), then $I(\Gamma[y]) < \tilde{c}$.

Proof. In view of $(g_2)$ and $(g_3)$, $G(s) \leq \frac{1}{4}s^4$, for any $s > 0$. We have by Lemma 2.1-(3),(7),

$$I(\Gamma[y]) = I_\infty(\Gamma[y]) + \int_{\mathbb{R}^N} \frac{1}{2} (V(x) - V_\infty) f^2(T^yu^y_\infty) + (q_\infty - q(x)) G(f(T^yu^y_\infty))$$

$$\leq c_\infty + \int_{\mathbb{R}^N} \frac{1}{2} (V(x) - V_\infty)^+(T^yu^y_\infty)^2 + \frac{1}{2} (q_\infty - q(x))(T^yu^y_\infty)^2$$

$$< c_\infty + \frac{\tilde{c} - c_\infty}{R^2\|u_\infty\|_2^2} \int_{\mathbb{R}^N} (T^yu^y_\infty)^2 = \tilde{c}.$$

$\square$
Proof of Theorem 1.1. By Lemma 3.10, we have $c_0 \leq c_\infty$. If $c_0 < c_\infty$, we get a nontrivial solution of (3) in view of Proposition 1. In the case $b = c_0 = c_\infty$, Proposition 2 gives the desired conclusion. So we will consider the case $b > c_0 = c_\infty$.

Arguing by contradiction, suppose the functional $I$ does not have a critical value in $(c_\infty, \bar{c})$. By [24] P.86, there exists a decreasing flow $\eta$ with $\eta : \Psi^c \setminus K_{\bar{c}} \mapsto \Psi^{c_\infty+\delta}$ (rule out the possibility that the flow $\eta$ reaches at boundary of $U$ by Lemmas 3.5 and 3.9), where $\Psi^c := \{ w \in U : \Psi(w) \leq c \}$, $K_{\bar{c}} := \{ w \in U : \Psi'(w) = 0 \text{ and } \Psi(w) = c \}$ and $\delta$ is as in Lemma 3.14. Moreover, $\eta(u) = u$ for every $u \in \Psi^{c_\infty+\delta}$.

By Lemmas 3.13 and 3.14, there exists $\rho_1 \geq S > 0$ such that for every $\rho > \rho_1$

$$c_\infty < \max_{|y|=\rho} I(\Gamma[y]) = \max_{|y|=\rho} \Psi(m^{-1}(\Gamma[y])) < c_\infty + \delta < b.$$  

Define a continuous map $h : B_\rho(0) \mapsto \partial B_{\rho}(0)$ as

$$h(y) = \rho \cdot \frac{\beta \circ m \circ \eta \circ m^{-1}(\Gamma[y])}{|\beta \circ m \circ \eta \circ m^{-1}(\Gamma[y])|}.$$  

According to (12), $h(y) = y$ for $y \in \partial B_{\rho}(0)$, a contradiction with D.11 in [29]. Arguing as Lemma 2.4 in [8], it is easy to see that the solution $u^0$ does not change sign. The elliptic regularity theory implies that $u^0 \in C^2(\mathbb{R}^N)$ (see Lemma 1.30 in [29]). It follows from the strong maximum principle that $u^0$ is positive. \hfill $\square$

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E-mail address: fangxd0401@dlut.edu.cn