Categorification of Lagrangian intersections on complex symplectic manifolds using perverse sheaves of vanishing cycles

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Abstract

We study intersections of complex Lagrangian in complex symplectic manifolds, proving two main results.

First, we construct global canonical perverse sheaves on complex Lagrangian intersections in complex symplectic manifolds for any pair of oriented Lagrangian submanifolds in the complex analytic topology. Our method uses classical results in complex symplectic geometry and some results from [Joyc1]. The algebraic version of our result has already been obtained by the author et al. in [BBDJS] using different methods, where we used, in particular, the recent new theory of algebraic d-critical loci introduced by Joyce in [Joyc1]. This resolves a long-standing question in the categorification of Lagrangian intersection numbers and it may have important consequences in symplectic geometry and topological field theory.

Our second main result proves that (oriented) complex Lagrangian intersections in complex symplectic manifolds for any pair of Lagrangian submanifolds in the complex analytic topology carry the structure of (oriented) analytic d-critical loci in the sense of [Joyc1].

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Introduction

Let \((S, \omega)\) be a complex symplectic manifold, i.e., a complex manifold \(S\) endowed with a closed non-degenerate holomorphic 2-form \(\omega \in \Omega^2_S\). Denote the complex dimension of \(S\) by \(2n\). A complex submanifold \(M \subset S\) is Lagrangian if the restriction of \(\omega\) to a 2-form on \(M\) vanishes and \(\dim M = n\). Let \(X = L \cap M\) be the intersection as a complex analytic space. Then \(X\) carries a canonical symmetric obstruction theory \(\varphi : E^* \to \mathbb{L}_X\) in the sense of [BeFa], which can be represented by the complex \(E^* \simeq T^*S|_X \to T^*L|_X \oplus T^*M|_X\) with \(T^*S|_X\) in degree \(-1\) and \(T^*L|_X \oplus T^*M|_X\) in degree zero. Hence \(\det(E^*) \cong K_L|_X \otimes K_M|_X\). Inspired by [KoSo] \S 3.2 in primis and then by [BBDJS] \S 2.4 and close to [Joyc] \S 5.2, we will say that if we are given square roots \(K^{1/2}_L, K^{1/2}_M\) for \(K_L, K_M\), then \(X\) has orientation data. In this case we will also say that \(L, M\) are oriented Lagrangians, see Remark \ref{rem:orientation}.

We start from well known facts from complex symplectic geometry. It is well established that every complex symplectic manifold \(S\) is locally isomorphic to the cotangent bundle \(T^*N\) of a complex manifold \(N\). The fibres of the induced vector bundle structure on \(S\) are Lagrangian submanifolds, so complex analytically locally defining on \(S\) a foliation by Lagrangian submanifolds, i.e., a polarization. The data of a polarization for us will be used as a way to describe locally in the complex analytic topology the Lagrangian intersection \(X\) as a critical locus for us will be used as a way to describe locally in the complex analytic topology the Lagrangian intersection \(X\). Then \(X\) will be a critical locus for \(U, f\) is a holomorphic function on a complex manifold \(U\).

One moral of this approach is that every polarization defines a set of data for \(PV_{U,f}\) structures on them and consequently get a global object on \(X\) by gluing. This will become more clear later. In conclusion, on each chart defined by the choice of a polarization, there is naturally associated a perverse sheaf of vanishing cycles \(PV^{*}_{U,f}\) in the notation of \[\]!

Now, a natural problem to investigate is the following. Given analytic open \(R_i, R_j \subseteq X\) with isomorphisms \(R_i \cong \text{Crit}(f_i), R_j \cong \text{Crit}(f_j)\) for holomorphic \(f_i : U_i \to \mathbb{C}\) and \(f_j : U_j \to \mathbb{C}\), we have to understand whether the perverse sheaves \(P^{*}_{R_i} = PV_{U_i,f_i}\) on \(R_i\) and \(P^{*}_{R_j} = PV_{U_j,f_j}\) on \(R_j\) are isomorphic over \(R_i \cap R_j\), and if so, whether the isomorphism is canonical, for only then can we hope to glue the \(P^{*}_{R_i}\) for \(i \in I\) to make \(P^{*}_{L,M}\).

Our approach was inspired by a work of Behrend and Fantechi [BeFa]. They also investigated Lagrangian intersections in complex symplectic manifolds, but their project is probably more ambitious, as they show the existence of deeply interesting structures carried by the intersection. Unfortunately, their construction has some crucial mistakes. Our project started exactly with the aim to fix them and develop then an independent theory. In the meantime, the author worked with other collaborators on a large project [BBDJS] involving Lagrangian intersections too, but our methods here want to be self contained and independent from that. In particular, the analogue of our Theorem below for algebraic symplectic manifolds and algebraic manifolds follows from [BBJ, BBDJS,PTVV], but the complex analytic case is not available in [BBJ,PTVV].

In \[\] we will state and prove the following result:

**Theorem** Let \((S, \omega)\) be a complex symplectic manifold and \(L, M\) oriented complex Lagrangian submanifolds in \(S\), and write \(X = L \cap M\), as a complex analytic subspace of \(S\). Then we may define \(P^{*}_{L,M} \in \text{Perv}(X)\), uniquely up to canonical isomorphism, and isomorphisms \(\Sigma_{L,M} : P^{*}_{L,M} \to \mathbb{D}_X(P^{*}_{L,M})\), \(T_{L,M} : P^{*}_{L,M} \to P^{*}_{L,M}\), respectively the Verdier duality and the monodromy isomorphisms. These \(P^{*}_{L,M} \in \text{Perv}(X)\), \(\Sigma_{L,M}, T_{L,M}\) are characterized by the following property.

Given a choice of local Darboux coordinates \((x_1, \ldots, x_n, y_1, \ldots, y_n)\) in the sense of Definition \[\] such that \(L\) is locally identified in coordinates with the graph \(\Gamma_{df(x_1, \ldots, x_n)}\) of \(df\) for \(f\) a holomorphic function defined locally on an open \(U \subseteq \mathbb{C}^n\), and \(M\) is locally identified in coordinates with the graph \(\Gamma_{dg(x_1, \ldots, x_n)}\) of \(dg\) for \(g\) a holomorphic function defined locally on \(U\), and the orientations \(K^{1/2}_L, K^{1/2}_M\) are the trivial square roots of \(K_L \cong \langle dx_1 \wedge \cdots \wedge dx_n \rangle \cong K_M\), then \(P^{*}_{L,M} \cong PV^{*}_{U,g-f}\), where \(PV^{*}_{U,g-f}\) is the perverse sheaf of vanishing cycles of \(g-f\), and \(\Sigma_{L,M}\) and \(T_{L,M}\) are respectively the Verdier duality \(\sigma_{U,g-f}\) and the monodromy \(\tau_{U,g-f}\) introduced in \[\].

The same applies for \(\mathcal{D}\)-modules and mixed Hodge modules on \(X\).

Here is a sketch of the method of proof, given in detail in \[\] 2.3.
Given \((S, \omega)\) a complex symplectic manifold we want to construct a global perverse sheaf \(P_{L,M}^* \in \text{Perv}(X)\), by gluing together local data coming from choices of polarizations by isomorphisms. We consider an open cover \(\{S_i\}_{i \in I}\) of \(S\) and polarizations \(\pi_i : S_i \rightarrow E_i\), always assumed to be transverse to both the Lagrangians \(L\) and \(M\). We use the following method:

(i) For each polarization \(\pi_i : S_i \rightarrow E_i\) transverse to both the Lagrangian submanifolds \(L\) and \(M\), we define a perverse sheaf of vanishing cycle \(\PV_{ij}^*\), naturally defined on the chart induced by the choice of a polarization.

(ii) For two such polarizations \(E_i\) and \(E_j\), transverse to each other, and to both the Lagrangians, we have a way to define two perverse sheaves of vanishing cycles, \(\PV_{ij}^*\) and \(\PV_{ji}^*\), again with principal \(\mathbb{Z}_2\)-bundles, each of them parametrizing choices of square roots of the canonical bundles of \(L \cong \Gamma_{df_i}\) and \(M \cong \Gamma_{df_j}\). In this case we find an isomorphism \(\Psi_{ij}\) on double overlap \(S_i \cap S_j\) between \(\PV_{ji}^* \otimes_{\mathbb{Z}_2} Q_{f_i}\) and \(\PV_{ij}^* \otimes_{\mathbb{Z}_2} Q_{f_j}\).

(iii) For four such polarizations \(E_i, E_j, E_k\) and \(E_l\) with \(E_i\) not necessarily transverse to \(E_k\), we obtain equality between \(\Psi_{ij} \circ \Psi_{jk}\) and \(\Psi_{kl} \circ \Psi_{jk}\) on \(S_i \cap S_j \cap S_k \cap S_l\).

As perverse sheaves form a stack in the sense of Theorem \(\mathbf{1.4}\) there exists \(P_{L,M}^*\) on \(X\), unique up to canonical isomorphism, with \(P_{L,M}^*|_{S_i} \cong \PV_{ji}^* \otimes_{\mathbb{Z}_2} Q_{f_i}\), for all \(i \in I\).

Our perverse sheaf \(P_{L,M}^*\) categorifies Lagrangian intersection numbers, in the sense that the constructible function

\[
p \mapsto \sum_i (-1)^i \dim_{\mathbb{C}} H^i_{\{p\}}(X, P_{L,M}^*),
\]

is equal to the well known Behrend function \(\nu_X\) in \([\text{Behr}]\) by construction, using the expression of the Behrend function of a critical locus in terms of the Milnor fibre, as in \([\text{Behr}]\), and so

\[
\chi(X, \nu_X) = \sum_i (-1)^i \dim_{\mathbb{C}} H^i(X, P_{L,M}^*).
\]

This resolves a long-standing question in the categorification of Lagrangian intersection number, and it may have exciting far reaching consequences in symplectic geometry and topological field theory.

In \([\text{KR2}]\), Kapustin and Rozansky study boundary conditions and defects in a three-dimensional topological sigma-model with a complex symplectic target space, the Rozansky-Witten model. They conjecture the existence of an interesting 2-category, the 2-category of boundary conditions. Their toy model for symplectic manifold is a cotangent bundle of some manifold. In this case, this category is related to the category of matrix factorizations \([\text{Orlov}]\). Thus, we strongly believe that constructing a sheaf of \(\mathbb{Z}_2\)-periodic triangulated categories on Lagrangian intersection would yield an answer to their conjecture. In the language of categorification, this would give a second categorification of the intersection numbers, the first being given by the hypercohomology of the perverse sheaf constructed in the present work. Also, this construction should be compatible with the Gerstenhaber and Batalin–Vilkovisky structures in the sense of \([\text{BaGi}]\) Conj. 1.3.1.1.

Our second main result proved in \([\text{BBJ}]\) constitutes another bridge between our work and \([\text{Joyc1}]\) \([\text{BBDS}]\) \([\text{BBJ}]\). Panet
tet al. \([\text{PTVV}]\) show that derived intersections \(L \cap M\) of algebraic Lagrangians \(L, M\) in an algebraic symplectic manifold \((S, \omega)\) have \(-1\)-shifted symplectic structures, so that Theorem 6.6 in \([\text{BBDS}]\) gives them the structure of algebraic \(d\)-critical loci in the sense of \([\text{Joyc1}]\). Our second main result shows a complex analytic version of this, which is not available from \([\text{BBJ}]\) \([\text{PTVV}]\), that is, the classical intersection \(L \cap M\) of complex Lagrangians \(L, M\) in a complex symplectic manifold \((S, \omega)\) has the structure of an (oriented) complex analytic \(d\)-critical locus.

**Theorem** Suppose \((S, \omega)\) is a complex symplectic manifold, and \(L, M\) are (oriented) complex Lagrangian submanifolds in \(S\). Then the intersection \(X = L \cap M\), as a complex analytic subspace of \(S\), extends naturally to a (oriented) complex analytic \(d\)-critical locus \((X, s)\). The canonical bundle \(K_{X,s}\) in the sense of \([\text{Joyc1}]\) §2.4 is naturally isomorphic to \(K_{L}|_{X} \otimes K_{M}|_{X} \).
It would be interesting to prove an analogous version of this also for a class of ‘derived Lagrangians’ in \((S, \omega)\). Some of the authors of [BBDJS] are working on defining a ‘Fukaya category’ of (derived) complex Lagrangians in a complex symplectic manifold, using \(\mathcal{H}^*(P^\bullet_{L,M})\) as morphisms.

Outline of the paper

The paper begins with a section of background material on perverse sheaves in the complex analytic topology. Then we review basic notions in symplectic geometry. In §2 we state and prove our first main result on the construction of a canonical global perverse sheaf on complex Lagrangian intersections. In §3 we prove our second main result on the d-critical locus structure carried by Lagrangian intersections. Finally, the last section sketches some implications of the theory and proposes new ideas for further research.

Notations and conventions

Throughout we will work in the complex analytic topology over \(\mathbb{C}\). We will denote by \((S, \omega)\) a complex symplectic manifold endowed with a symplectic form \(\omega\), and its Lagrangian submanifolds will be always assumed to be nonsingular. Note that all complex analytic spaces in this paper are locally of finite type, which is necessary for the existence of embeddings \(i : X \to U\) for \(U\) a complex manifold. Fix a well-behaved commutative base ring \(A\) (where ‘well-behaved’ means that we need assumptions on \(A\) such as \(A\) is regular noetherian, of finite global dimension or finite Krull dimension, a principal ideal domain, or a Dedekind domain, at various points in the theory), to study sheaves of \(A\)-modules. For some results \(A\) must be a field. Usually we take \(A = \mathbb{Z}, \mathbb{Q}\) or \(\mathbb{C}\).

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1 Background material

In this introductory section we first recall general definitions and conventions about perverse sheaves on complex analytic spaces, and some results very well known in literature, which will be used in the sequel. This first part has a substantial overlap with [BBDJS]. Secondly, we recall some definitions and results from [Joyc1], crucially used to prove one of the main result of [BBDJS] about behavior of perverse sheaves of vanishing cycles under stabilization, stated in [2]. Finally, we establish the basic notation for symplectic manifolds, their Lagrangian submanifolds and polarizations, and we recall results from complex symplectic geometry, used in §2.

1.1 Perverse sheaves on complex analytic spaces

We discuss perverse sheaves on complex analytic spaces, as in Dimca [Dimc].

For the whole section, let \(A\) be a well-behaved commutative base ring (usually we take \(A = \mathbb{Z}, \mathbb{Q}\) or \(\mathbb{C}\)). Usually \(X\) will be a complex analytic space, always assumed locally of finite type. We start by discussing constructible complexes, following Dimca [Dimc, §2–§4].

**Definition 1.1.** A sheaf \(\mathcal{S}\) of \(A\)-modules on \(X\) is called *constructible* if all the stalks \(\mathcal{S}_x\) for \(x \in X\) are finite type \(A\)-modules, and there is a locally finite stratification \(X = \bigsqcup_{j \in J} X_j\) of \(X\), where \(X_j \subseteq X\) for \(j \in J\) are complex analytic subspaces of \(X\), such that \(\mathcal{S}|_{X_j}\) is an \(A\)-local system for all \(j \in J\).

Write \(D(X)\) for the derived category of complexes \(\mathcal{C}^\bullet\) of sheaves of \(A\)-modules on \(X\). Write \(D^b_m(X)\) for the full subcategory of bounded complexes \(\mathcal{C}^\bullet\) in \(D(X)\) whose cohomology sheaves \(\mathcal{H}^m(\mathcal{C}^\bullet)\) are constructible
for all \( m \in \mathbb{Z} \). Then \( D(X), D^b_c(X) \) are triangulated categories. An example of a constructible complex on \( X \) is the constant sheaf \( A_X \) on \( X \) with fibre \( A \) at each point.

Grothendieck’s “six operations on sheaves” \( f^*, f^!, Rf_!, Rf_*, \mathcal{R}Hom, \otimes \) act on \( D(X) \) preserving the subcategory \( D^b_c(X) \) in case \( f \) is proper.

For \( B^*, C^* \) in \( D^b_c(X) \), we may form their derived Hom \( \mathcal{R}Hom(B^*, C^*) \) \cite{Dimca §2.1}, and left derived tensor product \( B^* \mathcal{O} C^* \) in \( D^b_c(X) \), \cite{Dimca §2.2}. Given \( B^* \in D^b_c(X) \) and \( C^* \in D^b_c(Y) \), we define \( B^* \mathcal{O} C^* = \pi_X(B^*) \mathcal{O} \pi_Y(C^*) \) in \( D^b_c(X \times Y) \), where \( \pi_X : X \times Y \to X, \pi_Y : X \times Y \to Y \) are the projections.

If \( X \) is a complex analytic space, there is a functor \( \mathbb{D}_X : D^b_c(X) \to D^b_c(X) \) preserving the subcategory \( D^b_c(X) \) in \( D^b_c(X) \) and \( k \in \mathbb{Z} \).

We will use the following property: if \( f : X \to Y \) is a morphism then
\[
Rf_! \cong \mathbb{D}_Y \circ Rf_* \circ \mathbb{D}_X \quad \text{and} \quad f^! \cong \mathbb{D}_X \circ f^* \circ \mathbb{D}_Y. \tag{1.1}
\]

If \( X \) is a complex analytic space, and \( C^* \in D^b_c(X) \), the hypercohomology \( \mathbb{H}^* \) and compactly-supported hypercohomology \( \mathbb{H}^*_{cs}(C^*) \), both graded \( A \)-modules, are
\[
\mathbb{H}^k(C^*) = H^k(R\pi_*(C^*)) \quad \text{and} \quad \mathbb{H}^k_{cs}(C^*) = H^k(R\pi_!(C^*)) \quad \text{for} \ k \in \mathbb{Z},
\]
where \( \pi : X \to \ast \) is projection to a point. If \( X \) is proper then \( \mathbb{H}^*(C^*) \cong \mathbb{H}^*_{cs}(C^*) \). They are related to usual cohomology by \( H^k(A_X) \cong H^k(X; A) \) and \( H^k_{cs}(A_X) \cong H^k_{cs}(X; A) \). If \( A \) is a field then under Verdier duality \( \mathbb{H}^k(C^*) \cong \mathbb{H}^*_{cs}(\mathbb{D}_X(C^*))^* \).

Next we review perverse sheaves, following Dimca \cite{Dimca §5}.

**Definition 1.2.** Let \( X \) be a complex analytic space, and for each \( x \in X \), let \( i_x : \ast \to X \) map \( i_x : \ast \to x \). If \( C^* \in D^b_c(X) \), then the support \( \text{supp}^m C^* \) and cosupport \( \text{cosupp}^m C^* \) of \( H^m(C^*) \) for \( m \in \mathbb{Z} \) are
\[
\text{supp}^m C^* = \{ x \in X : H^m(i_x^*(C^*)) \neq 0 \} \quad \text{and} \quad \text{cosupp}^m C^* = \{ x \in X : H^m(i_x!(C^*)) \neq 0 \},
\]
where \( \{ \cdots \} \) means the closure in \( X^\text{an} \). If \( A \) is a field then \( \text{cosupp}^m C^* = \text{supp}^{-m} \mathbb{D}_X(C^*) \). We call \( C^* \) a perverse sheaf if
\[
\text{dim}_c \text{supp}^{-m} C^* \leq m \quad \text{and} \quad \text{dim}_c \text{cosupp}^m C^* \leq m
\]
for all \( m \in \mathbb{Z} \), where by convention \( \text{dim}_c 0 = -\infty \). Write \( \text{Perv}(X) \) for the full subcategory of perverse sheaves in \( D^b_c(X) \). Then \( \text{Perv}(X) \) is an abelian category, the heart of a t-structure on \( D^b_c(X) \).

Perverse sheaves have the following properties:

**Theorem 1.3.** (a) If \( A \) is a field then \( \text{Perv}(X) \) is noetherian and artinian.

(b) If \( A \) is a field then \( \mathbb{D}_X : D^b_c(X) \to D^b_c(X) \) maps \( \text{Perv}(X) \to \text{Perv}(X) \).

(c) If \( i : X \to Y \) is inclusion of a closed complex analytic subspaces, then \( R_i^* \), \( R_i! \) (which are naturally isomorphic) map \( \text{Perv}(X) \to \text{Perv}(Y) \).

Write \( \text{Perv}(Y)_X \) for the full subcategory of objects in \( \text{Perv}(Y) \) supported on \( X \). Then \( R_i^* \cong R_i! \) are equivalences of categories \( \text{Perv}(X) \to \text{Perv}(Y)_X \). The restrictions \( i^* \mid_{\text{Perv}(Y)_X} \), \( i^! \mid_{\text{Perv}(Y)_X} \) map \( \text{Perv}(X) \to \text{Perv}(Y)_X \), are naturally isomorphic, and are quasi-inverses for \( R_i^* \), \( R_i! : \text{Perv}(X) \to \text{Perv}(Y)_X \).

(d) If \( f : X \to Y \) is étale then \( f^* \) and \( f^! \) (which are naturally isomorphic) map \( \text{Perv}(Y) \to \text{Perv}(X) \). More generally, if \( f : X \to Y \) is smooth of relative dimension \( d \), then \( f^*[d] \cong f^![-d] \) map \( \text{Perv}(Y) \to \text{Perv}(X) \).

(e) \( \mathcal{O} : D^b_c(X) \to D^b_c(Y) \to D^b_c(X \times Y) \) maps \( \text{Perv}(X) \to \text{Perv}(Y) \) to \( \text{Perv}(X \times Y) \).

(f) Let \( U \) be a complex manifold. Then \( \mathcal{O}_U \mid_{\dim U} \) is perverse, where \( \mathcal{O}_U \) is the constant sheaf on \( U \) with fibre \( A \), and \( \dim U \) means shift by \( \dim U \) in the triangulated category \( D^b_c(X) \). Moreover, there is a canonical isomorphism \( \mathbb{D}_U \mid_{\dim U} \cong A_U \mid_{\dim U} \).

The next theorem is proved in \cite[Th. 10.2.9]{KashiwaraSchapira} see also \cite[Prop. 8.1.26]{HTT}. The analogue for \( D^b_c(X) \) or \( D(X) \) rather than \( \text{Perv}(X) \) is false. One moral is that perverse sheaves behave like sheaves, rather than like complexes.
Theorem 1.4. Perverse sheaves on a complex analytic space $X$ form a stack on $X$ in the complex analytic topology. Explicitly, this means the following. Let $\{U_i\}_{i \in I}$ be an analytic open cover for $X$, and write $U_{ij} = U_i \cap U_j$ for $i, j \in I$. Similarly, write $U_{ijk} = U_i \cap U_j \cap U_k$ for $i, j, k \in I$. With this notation:

(i) Suppose $\mathcal{P}^*, \mathcal{Q}^* \in \text{Perv}(X)$, and we are given $\alpha_i : \mathcal{P}^*|_{U_i} \to \mathcal{Q}^*|_{U_i}$ in $\text{Perv}(U_i)$ for all $i \in I$ such that for all $i, j \in I$ we have

$$\alpha_i|_{U_{ij}} = \alpha_j|_{U_{ij}},$$

then exists unique $\alpha : \mathcal{P}^* \to \mathcal{Q}^* \in \text{Perv}(X)$ with $\alpha|_{U_i} = \alpha_i$ for all $i \in I$.

(ii) Suppose we are given objects $\mathcal{P}^*_i \in \text{Perv}(U_i)$ for all $i \in I$ and isomorphisms $\alpha_{ij} : \mathcal{P}^*_i|_{U_{ij}} \to \mathcal{P}^*_j|_{U_{ij}}$ in $\text{Perv}(U_{ij})$ for all $i, j \in I$ with $\alpha_{ii} = \text{id}$ and

$$\alpha_{jk}|_{U_{ijk}} \circ \alpha_{ij}|_{U_{ijk}} = \alpha_{ik}|_{U_{ijk}} : \mathcal{P}^*_i|_{U_{ijk}} \to \mathcal{P}^*_k|_{U_{ijk}}$$

in $\text{Perv}(U_{ijk})$ for all $i, j, k \in I$. Then there exists $\mathcal{P}^* \in \text{Perv}(X)$, unique up to canonical isomorphism, with isomorphisms $\beta_i : \mathcal{P}^*|_{U_i} \to \mathcal{P}^*_i$ for each $i \in I$, satisfying $\alpha_{ij} \circ \beta_i|_{U_{ij}} = \beta_j|_{U_{ij}} : \mathcal{P}^*|_{U_{ij}} \to \mathcal{P}^*_j|_{U_{ij}}$ for all $i, j \in I$.

If $P \to X$ is a principal $\mathbb{Z}_2$-bundle on a complex manifold $X$, and $\mathcal{Q}^* \in \text{Perv}(X)$, we will define a perverse sheaf $\mathcal{Q}^* \otimes_{\mathbb{Z}_2} P$.

Definition 1.5. A principal $\mathbb{Z}_2$-bundle $P \to X$ on a complex analytic space $X$, is a proper, étale, surjective, complex analytic morphism of complex analytic spaces $\pi : P \to X$ together with a free involution $\sigma : P \to P$, such that the orbits of $\mathbb{Z}_2 = \{1, \sigma\}$ are the fibres of $\pi$.

Let $P \to X$ be a principal $\mathbb{Z}_2$-bundle. Write $L_P \in D^b_c(X)$ for the rank one $A$-local system on $X$ induced from $P$ by the nontrivial representation of $\mathbb{Z}_2 \cong \{1, \sigma\}$ on $A$. It is characterized by $\pi_*(A_P) \cong A_X \oplus L_P$. For each $\mathcal{Q}^* \in D^b_c(X)$, write $\mathcal{Q}^* \otimes_{\mathbb{Z}_2} P \in D^b_c(X)$ for $\mathcal{Q}^* \otimes L_P$, and call it $\mathcal{Q}^*$ twisted by $P$. If $\mathcal{Q}^*$ is perverse then $\mathcal{Q}^* \otimes_{\mathbb{Z}_2} P$ is perverse. Perverse sheaves and complexes twisted by principal $\mathbb{Z}_2$-bundles have the obvious functorial behavior.

We explain nearby cycles and vanishing cycles, as in Dimca [Dimc] §4.2.

Definition 1.6. Let $X$ be a complex analytic space, and $f : X \to \mathbb{C}$ a holomorphic function. Define $X_0 = f^{-1}(0)$, as a complex analytic subspace of $X$, and $X_* = X \setminus X_0$. Consider the commutative diagram of complex analytic spaces:

$$
\begin{array}{cccc}
X_0 & \xrightarrow{i} & X & \xrightarrow{\pi} & X_* \\
\downarrow{f} & & \downarrow{f} & & \downarrow{f} \\
\{0\} & \xrightarrow{} & \mathbb{C} & \xleftarrow{\rho} & \mathbb{C}^*.
\end{array}
$$

(1.3)

Here $i : X_0 \hookrightarrow X$, $j : X_* \hookrightarrow X$ are the inclusions, $\rho : \mathbb{C}^* \to \mathbb{C}^*$ is the universal cover of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and $X_* = X_* \times_f \mathbb{C}^* \to \mathbb{C}^*$ the corresponding cover of $X_*$, with covering map $p : X_* \to X_*$, and $\pi = j \circ p$. We define the nearby cycle functor $\psi_f : D^b_c(X) \to D^b_c(X_0)$ to be $\psi_f = i^* \circ R\pi_* \circ \pi^*$.

There is a natural transformation $\Xi : i^* \Rightarrow \psi_f$ between the functors $i^*, \psi_f : D^b_c(X) \to D^b_c(X_0)$. The vanishing cycle functor $\phi_f : D^b_c(X) \to D^b_c(X_0)$ is a functor such that for every $\mathcal{C}^*$ in $D^b_c(X)$ we have a distinguished triangle

$$
i^*(\mathcal{C}^*) \xrightarrow{\Xi(\mathcal{C}^*)} \psi_f(\mathcal{C}^*) \xrightarrow{} \phi_f(\mathcal{C}^*) \xrightarrow{[+1]} i^*(\mathcal{C}^*)
$$

in $D^b_c(X)$. Following Dimca [Dimc] p. 108, we write $\psi^p_f, \phi^p_f$ for the shifted functors $\psi_f[-1], \phi_f[-1] : D^b_c(X) \to D^b_c(X_0)$.

The generator of $\mathbb{Z} = \pi_1(\mathbb{C}^*)$ on $\mathbb{C}^*$ induces a deck transformation $\delta_{\mathbb{C}^*} : \mathbb{C}^* \to \mathbb{C}^*$ which lifts to a deck transformation $\delta_{X_*} : X_* \to X^*$ with $p \circ \delta_{X_*} = p$ and $f \circ \delta_{X_*} = \delta_{\mathbb{C}^*} \circ f$. As in Dimca [Dimc] p. 103, p. 105], we can use $\delta_{X_*}$ to define natural transformations $M_{X,f} : \psi^p_f \Rightarrow \psi^p_f$ and $M_{X,f} : \phi^p_f \Rightarrow \phi^p_f$, called monodromy.

By Dimca [Dimc] Th. 5.2.21, if $X$ is a complex analytic space and $f : X \to \mathbb{C}$ is holomorphic, then $\psi^p_f, \phi^p_f : D^b_c(X) \to D^b_c(X_0)$ both map $\text{Perv}(X) \to \text{Perv}(X_0)$. 
We will use the following property, proved by Massey [Mass]. If \( X \) is a complex manifold and \( f : X \to \mathbb{C} \) is regular, then there are natural isomorphisms
\[
\psi_f^p \circ \mathbb{D}_X \cong \mathbb{D}_{X^o} \circ \psi_f^p \quad \text{and} \quad \phi_f^p \circ \mathbb{D}_X \cong \mathbb{D}_{X^o} \circ \phi_f^p.
\] (1.4)

We can now define perverse sheaf of vanishing cycles \( PV^\bullet_{U,f} \) for a holomorphic function \( f : U \to \mathbb{C} \).

**Definition 1.7.** Let \( U \) be a complex analytic space, and \( f : U \to \mathbb{C} \) a holomorphic function. Write \( X = \text{Crit}(f) \), as a closed complex analytic subspace of \( U \). Then as a map of topological spaces, \( f|_X : X \to \mathbb{C} \) is locally constant, with finite image \( f(X) \), so we have a decomposition \( X = \bigsqcup_{c \in f(X)} X_c, \) for \( X_c \subseteq X \) the open and closed complex analytic subspace with \( f(x) = c \) for each \( x \in X_c \).

For each \( c \in \mathbb{C} \), write \( U_c = f^{-1}(c) \subseteq U \). Then we have a vanishing cycle functor \( \phi^f_{-c} : \text{Perv}(U) \to \text{Perv}(U_c) \). So we may form \( \phi^f_{-c}(A_U[\dim U]) \) in \( \text{Perv}(U_c) \), since \( A_U[\dim U] \in \text{Perv}(U) \) by Theorem 1.3(f). One can show \( \phi^f_{-c}(A_U[\dim U]) \) is supported on the closed subset \( X_c = \text{Crit}(f) \cap U_c \) in \( U_c \), where \( X_c = \emptyset \) unless \( c \in f(X) \). That is, \( \phi^f_{-c}(A_U[\dim U]) \) lies in \( \text{Perv}(U_c) \). But Theorem 1.3(c) says \( \text{Perv}(U_c) \) and \( \text{Perv}(X_c) \) are equivalent categories, so we may regard \( \phi^f_{-c}(A_U[\dim U]) \) as a perverse sheaf on \( X_c \). That is, we can consider \( \phi^f_{-c}(A_U[\dim U])|_{X_c} = i_{X_c,U_c}^* \phi^f_{-c}(A_U[\dim U]) \) in \( \text{Perv}(X_c) \), where \( i_{X_c,U_c} : X_c \to U_c \) is the inclusion morphism. Also, \( \text{Perv}(X) = \bigsqcup_{c \in f(X)} \text{Perv}(X_c) \).

Define the **perverse sheaf of vanishing cycles** \( PV^\bullet_{U,f} \) of \( U, f \) in \( \text{Perv}(X) \) to be
\[
PV^\bullet_{U,f} = \bigoplus_{c \in f(X)} \phi^f_{-c}(A_U[\dim U])|_{X_c}.
\]
That is, \( PV^\bullet_{U,f} \) is the unique perverse sheaf on \( X = \text{Crit}(f) \) with \( PV^\bullet_{U,f}|_{X_c} = \phi^f_{-c}(A_U[\dim U])|_{X_c} \) for all \( c \in f(X) \).

Under Verdier duality, we have \( A_U[\dim U] \cong \mathbb{D}_U(A_U[\dim U]) \) by Theorem 1.3(f), so \( \phi^f_{-c}(A_U[\dim U]) \cong \mathbb{D}_U \phi^f_{-c}(A_U[\dim U]) \) by (1.4). Applying \( i_{X_c,U_c}^* \) and using \( \mathbb{D}_{X_c} \circ i_{X_c,U_c}^* \equiv i_{X_c,U_c}^* \circ \mathbb{D}_{U_c} \) by (1.1) and \( i_{X_c,U_c}^* \circ i_{X_c,U_c}^* \) on \( \text{Perv}(U_c) \) by Theorem 1.3(c) also gives
\[
\phi^f_{-c}(A_U[\dim U])|_{X_c} \cong \mathbb{D}_{X_c} (\phi^f_{-c}(A_U[\dim U])|_{X_c}).
\]

Summing over all \( c \in f(X) \) yields a canonical isomorphism
\[
\sigma_{U,f} : PV^\bullet_{U,f} \xrightarrow{\cong} \mathbb{D}_X (PV^\bullet_{U,f}).
\] (1.5)

For \( c \in f(X) \), we have a monodromy operator \( M_{U,f-c} : \phi^f_{-c}(A_U[\dim U]) \to \phi^f_{-c}(A_U[\dim U]) \), which restricts to \( \phi^f_{-c}(A_U[\dim U])|_{X_c} \). Define the **twisted monodromy operator** \( \tau_{U,f} : PV^\bullet_{U,f} \to PV^\bullet_{U,f} \) by
\[
\tau_{U,f}|_{X_c} = (-1)^{\dim U} M_{U,f-c}|_{X_c} : \phi^f_{-c}(A_U[\dim U])|_{X_c} \to \phi^f_{-c}(A_U[\dim U])|_{X_c},
\] (1.6)
for each \( c \in f(X) \). Here ‘twisted’ refers to the sign \((-1)^{\dim U}\) in (1.3). We include this sign change as it makes monodromy act naturally under transformations which change dimension — without it, equation (1.28) below would only commute up to a sign \((-1)^{\dim U - \dim \mathcal{U}}\), not commute — and it normalizes the monodromy of any nondegenerate quadratic form to be the identity. The sign \((-1)^{\dim U}\) also corresponds to the twist \(\left(\frac{1}{2}(\dim U)\right)\) in the definition of the mixed Hodge module of vanishing cycles \(\mathcal{H}^\bullet_{U,f}\).

The (compatibly-supported) hypercohomology \( H^*(PV^\bullet_{U,f}), H^*_h(PV^\bullet_{U,f}) \) from (1.2) is an important invariant of \( U, f \). If \( A \) is a field then the isomorphism \( \sigma_{U,f} \) in (1.5) implies that \( H^*(PV^\bullet_{U,f}) \cong H^*_h(PV^\bullet_{U,f})^* \), a form of Poincaré duality.

We defined \( PV^\bullet_{U,f} \) in perverse sheaves over a base ring \( A \). Writing \( PV^\bullet_{U,f}(A) \) to denote the base ring, one can show that \( PV^\bullet_{U,f}(A) \equiv PV^\bullet_{U,f}(\mathbb{Z}) \otimes_\mathbb{Z} A \). Thus, we may as well take \( A = \mathbb{Z} \), or \( A = \mathbb{Q} \) if we want \( A \) to be a field, since the case of general \( A \) contains no more information.

There is a ‘Thom–Sebastiani Theorem for perverse sheaves’, due to Massey [Mass1] and Schürmann [Schu Cor. 1.3.4]. Applied to \( PV^\bullet_{U,f} \), it yields:
Theorem 1.8. Let $U, V$ be complex manifolds and $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be holomorphic, so that $f \boxplus g : U \times V \to \mathbb{C}$ is regular with $(f \boxplus g)(u,v) := f(u) + g(v)$. Set $X = \text{Crit}(f)$ and $Y = \text{Crit}(g)$ as complex analytic spaces of $U, V$, so that $\text{Crit}(f \boxplus g) = X \times Y$. Then there is a natural isomorphism

\[
\mathcal{T}S_{U,f,V,g} : \mathcal{P}V_{U \times V,f} \boxplus \mathcal{P}V_{V,g} \to \mathcal{P}V_{U,f} \boxplus \mathcal{P}V_{V,g}
\]

in $\text{Perv}(X \times Y)$, such that the following diagrams commute:

\[
\begin{array}{c}
\mathcal{P}V_{U \times V,f} \boxplus \mathcal{P}V_{V,g} \\
\mathcal{T}S_{U,f,V,g} \\
\mathcal{P}V_{U,f} \boxplus \mathcal{P}V_{V,g}
\end{array}
\]

Finally, we introduce some notation for pullbacks of $\mathcal{P}V_{V,g}$ by local biholomorphisms.

Definition 1.9. Let $U, V$ be complex manifolds, $\Phi : U \to V$ an étale morphism, and $g : V \to \mathbb{C}$ an analytic function. Write $f = g \circ \Phi : U \to \mathbb{C}$, and $X = \text{Crit}(f)$, $Y = \text{Crit}(g)$ as $\mathbb{C}$-submanifolds of $U, V$. Then $\Phi|_X : X \to Y$ is a local biholomorphism. Define an isomorphism

\[
\mathcal{P}V_{\Phi} : \mathcal{P}V_{U,f} \to \Phi^\ast(\mathcal{P}V_{V,g}) \text{ in } \text{Perv}(X)
\]

by the commutative diagram for each $c \in f(X) \subseteq g(Y)$:

\[
\begin{array}{ccc}
\mathcal{P}V_{U,f}|_{x_c} &=& \phi_{f-c}^\circ(A_U[\dim U]|_{x_c} \\
\mathcal{P}V_{V,g}|_{x_c} &=& \phi_{g-c}^\circ(A_V[\dim V]|_{x_c})
\end{array}
\]

Here $\alpha$ is $\phi_{f-c}$ applied to the canonical isomorphism $A_U \to \Phi^\ast(A_V)$, noting that $\dim U = \dim V$ as $\Phi$ is a local biholomorphism. By naturality of the isomorphisms $\alpha, \beta$ in (1.11) we find the following commute, where $\sigma_{U,f}, \tau_{U,f}$ are as in (1.5)–(1.6):

\[
\begin{array}{cccc}
\mathcal{P}V_{U,f} & \xrightarrow{\sigma_{U,f}} & \mathbb{D}(\mathcal{P}V_{U,f}) & \\
\mathcal{P}V_{V,g} & \xrightarrow{\Phi|_X} & \Phi|_X(\mathbb{D}(\mathcal{P}V_{V,g})) & \xrightarrow{\Phi|_X} \mathbb{D}(\Phi^\ast(\mathcal{P}V_{V,g}))
\end{array}
\]

If $U = V$, $f = g$ and $\Phi = \text{id}_U$ then $\mathcal{P}V_{U,f} = \text{id}_{\mathcal{P}V_{U,f}}$. If $W$ is another complex manifold, $\Psi : V \to W$ is a local biholomorphism, and $h : W \to \mathbb{C}$ is analytic with $g = h \circ \Psi : V \to \mathbb{C}$, then composing (1.11) for $\Phi$ with $\Phi|_{x_c}$ of (1.11) for $\Psi$ shows that

\[
\begin{array}{c}
\mathcal{P}V_{\Psi \Phi} = \Phi|_X(\mathcal{P}V_{\Psi}) \circ \mathcal{P}V_{\Phi} : \mathcal{P}V_{U,f} \to (\Psi \circ \Phi)|_X^\ast(\mathcal{P}V_{W,h}).
\end{array}
\]

That is, the isomorphisms $\mathcal{P}V_{\Phi}$ are functorial.
We conclude by saying that because of the Riemann–Hilbert correspondence, all our results on perverse sheaves of vanishing cycles on complex analytic spaces over a well-behaved base ring $\mathcal{A}$, translate immediately when $\mathcal{A} = \mathbb{C}$ to the corresponding results for $\mathcal{D}$-modules of vanishing cycles, with no extra work. and also to mixed Hodge modules on complex analytic spaces, see [BBDJ §2.9-2.10].

1.2 Stabilizing perverse sheaves of vanishing cycles

To set up notation for Theorem 1.13 below, we need the following theorem, which is proved in Joyce [Joyc] Prop.s 2.15, 2.16 & 2.18.

**Theorem 1.10 (Joyce [Joyc]).** Let $U, V$ be complex manifolds, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be holomorphic, and $X = \text{Crit}(f)$, $Y = \text{Crit}(g)$ as complex analytic subspaces of $U, V$. Let $\Phi : U \hookrightarrow V$ be a closed embedding of complex manifolds with $f = g \circ \Phi : U \to \mathbb{C}$, and suppose $\Phi|_X : X \to V \ni Y$ is an isomorphism $\Phi|_X : X \to Y$. Then:

(i) For each $x \in X \subseteq U$ there exist open $U' \subseteq U$ and $V' \subseteq V$ with $x \in U'$ and $\Phi(U') \subseteq V'$, an open neighbourhood $T$ of $0 \in \mathbb{C}^n$ where $n = \dim V - \dim U$, and a biholomorphism $\alpha \times \beta : V' \to U' \times T$, such that

$$(\alpha \times \beta) \circ \Phi|_{U'} = \text{id}_{U'} \times 0 : U' \to U' \times T,$$

and $g|_{V'} = f \circ \alpha + (z_1^2 + \cdots + z_n^2) \circ \beta : V' \to \mathbb{C}$. Thus, setting $f' = f|_{U'} : U' \to \mathbb{C}$, $g' = g|_{V'} : V' \to \mathbb{C}$, $\Phi' = \Phi|_{U'} : U' \to V'$, $X' = \text{Crit}(f') \subseteq U'$, and $Y' = \text{Crit}(g') \subseteq V'$, then $f' = g' \circ \Phi' : U' \to \mathbb{C}$, and $\Phi'|_{X'} : X' \to Y'$, $\alpha|_{V'} : Y' \to X'$ are biholomorphisms.

(ii) Write $N_{uv}$ for the normal bundle of $\Phi(U)$ in $V$, regarded as a vector bundle on $U$ in the exact sequence of vector bundles on $U$:

$$0 \longrightarrow TU \xrightarrow{\Phi} \Phi^*(TU) \xrightarrow{\Pi_{UV}} N_{UV} \longrightarrow 0. \quad (1.15)$$

Then there exists a unique $q_{uv} \in H^0(S^2N_{UV}^*, |_{X})$ which is a nondegenerate quadratic form on $N_{UV}|_{X}$, such that whenever $U', V', \Phi', \beta, n, X'$ are as in (i), writing $(dz_1, \ldots, dz_n)|_{U'}$ for the trivial vector bundle on $U'$ with basis $dz_1, \ldots, dz_n$, there is a natural isomorphism $\beta : (dz_1, \ldots, dz_n)|_{U'} \to N_{uv}^*|_{U'}$ making the following diagram commute:

$$N_{uv}^{|_{U'}} \xrightarrow{\Pi_{UV}|_{U'}} \Phi^*(T^*U)|_{U'} = \Phi'^*(T^*V|_{V'})$$

and $\beta = (S^2\beta)|_{X'} : (dz_1 \otimes dz_1 + \cdots + dz_n \otimes dz_n)$. \quad (1.16)

(iii) Now suppose $W$ is another complex manifold, $h : W \to \mathbb{C}$ is holomorphic, $Z = \text{Crit}(h)$ as a complex analytic subspace of $W$, and $\Psi : V \hookrightarrow W$ is a closed embedding of complex analytic subspaces with $g = h \circ \Psi : V \to \mathbb{C}$ and $\Psi|_{Y} : Y \to Z$ an isomorphism. Define $N_{uw}, q_{uw}$ and $N_{uw}, q_{uw}$ using $\Psi$ and $\Psi \circ \Phi : U \to W$ as in (ii) above. Then there are unique morphisms $\gamma_{uvw}, \delta_{uvw}$ which make the following diagram of vector bundles on $U$ commute, with straight lines exact:

$$0 \longrightarrow N_{uv} \xrightarrow{\gamma_{uvw}} N_{uw} \xrightarrow{\delta_{uvw}} N_{uv} \longrightarrow 0. \quad (1.18)$$
Restricting to $X$ gives an exact sequence of vector bundles:

$$0 \to N_{U|X} \xrightarrow{\gamma_{U|W|X}} N_{U|X} \xrightarrow{\delta_{U|W|X}} \Phi_X^*(N_{W|X}) \to 0. \quad (1.19)$$

Then there is a natural isomorphism of vector bundles on $X$

$$N_{U|X} \cong N_{U|X} \oplus \Phi_X^*(N_{W|X}), \quad (1.20)$$

compatible with the exact sequence (1.19), which identifies

$$q_{UW} \cong q_U \oplus \Phi_X^*(q_{W|X}) \oplus 0 \quad \text{under the splitting}$$

$$S^2 N_{U|X} \cong S^2 N_{U|X} \oplus \Phi_X^*(S^2 N_{W|X}) \oplus (N_{U|X} \otimes \Phi_X^*(N_{W|X})). \quad (1.21)$$

Following [Joyc1] Defs 2.19 & 2.24, we define:

**Definition 1.11.** Let $U, V$ be complex manifolds, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be holomorphic, and $X = \text{Crit}(f)$, $Y = \text{Crit}(g)$ as complex manifolds of $U, V$. Suppose $\Phi : U \to V$ is a closed embedding of complex manifolds with $f = g \circ \Phi : U \to \mathbb{C}$ and $\Phi_X : X \to Y$ an isomorphism. Then Theorem 1.10(ii) defines the normal bundle $N_{U|V}$ of $U$ in $V$, a vector bundle on $U$ of rank $n = \dim V - \dim U$, and a nondegenerate quadratic form $q_{U|V} \in H^0(S^2 N_{U|V}^*|X)$. Taking top exterior powers in the dual of (1.19) gives an isomorphism of line bundles on $U$

$$\rho_{UV} : K_U \otimes \Lambda^2 N_{U|V}^* \xrightarrow{\cong} \Phi^*(K_V),$$

where $K_U, K_V$ are the canonical bundles of $U, V$.

Write $X^{\text{red}}$ for the reduced subspace of $X$. As $q_{U|V}$ is a nondegenerate quadratic form on $N_{U|V}$, its determinant $\det(q_{U|V})$ is a nonzero section of $(\Lambda^2 N_{U|V}^*)^\otimes 2$. Define an isomorphism of line bundles on $X^{\text{red}}$:

$$J_{\Phi} = \rho_{UV}^\otimes \circ (\text{id}_{K_U^{\text{red}}} \otimes \det(q_{U|V})|_{X^{\text{red}}}) : K_U^{\text{red}}|_{X^{\text{red}}} \xrightarrow{\cong} \Phi_X^*(K_V^{\text{red}}). \quad (1.22)$$

Since principal $\mathbb{Z}_2$-bundles $\pi : P \to X$ in the sense of Definition 1.5 are complex analytic topological notions, and $X^{\text{red}}$ and $X$ have the same topological space, principal $\mathbb{Z}_2$-bundles on $X^{\text{red}}$ and on $X$ are equivalent. Define $\pi_{\Phi} : P_{\Phi} \to X$ to be the principal $\mathbb{Z}_2$-bundle which parametrizes square roots of $J_{\Phi}$ on $X^{\text{red}}$. That is, complex analytic local sections $s_{\Phi} : X \to P_{\Phi}$ of $P_{\Phi}$ correspond to local isomorphisms $\alpha : K_{U|X^{\text{red}}} \to \Phi_X^*(K_V^{\text{red}})$ on $X^{\text{red}}$ with $\alpha \circ \alpha = J_{\Phi}$.

Now suppose $W$ is another complex manifold, $h : W \to \mathbb{C}$ is holomorphic, $Z = \text{Crit}(h)$ as a complex analytic subspace of $W$, and $\Psi : V \to W$ is a closed embedding of complex manifolds with $g = h \circ \Psi : V \to \mathbb{C}$ and $\Psi|_Y : Y \to Z$ an isomorphism. Then Theorem 1.10(iii) applies, and from (1.20)–(1.21) we can deduce that

$$J_{\Phi|_{X^{\text{red}}}}(J_{\Psi}) \circ J_{\Phi} : K_U^{\text{red}}|_{X^{\text{red}}} \xrightarrow{\cong} (\Psi \circ \Phi)^*(K_V^{\text{red}}) = \Phi_X^*(\Psi|_{Y^{\text{red}}}^*(K_V^{\text{red}})). \quad (1.23)$$

For the principal $\mathbb{Z}_2$-bundles $\pi_{\Phi} : P_{\Phi} \to X$, $\pi_{\Psi} : P_{\Psi} \to Y$, $\pi_{\Phi|_{Y^{\text{red}}}} : P_{\Phi|_{Y^{\text{red}}}} \to X$, equation (1.23) implies that there is a canonical isomorphism

$$\Xi_{\Phi, \Psi} : P_{\Phi|_{Y^{\text{red}}}} \xrightarrow{\cong} \Phi_X^*(P_{\Psi}) \otimes_{\mathbb{Z}_2} P_{\Phi}. \quad (1.24)$$

It is also easy to see that these $\Xi_{\Phi, \Psi}$ have an associativity property under triple compositions, that is, given another complex manifold $T$, holomorphic $e : T \to \mathbb{C}$ with $Q := \text{Crit}(e)$, and $\Upsilon : T \to U$ a closed embedding with $e = f \circ T : T \to \mathbb{C}$ and $\Upsilon|_Q : Q \to X$ an isomorphism, then

$$(\text{id}_{\Upsilon|_Q} \otimes P_{\Psi}) \circ \Xi_{\Phi, \Upsilon} = (\Upsilon|_Q^* \circ \Xi_{\Phi, \Psi}) \circ \text{id}_{P_{\Upsilon}} \circ \Xi_{\Phi, \Psi, \Upsilon} : P_{\Phi|_{Y^{\text{red}}}} \to (\Phi \circ T)|_Q^*(P_{\Psi}) \otimes_{\mathbb{Z}_2} \Upsilon|_Q^*(P_{\Phi}) \otimes_{\mathbb{Z}_2} P_{\Upsilon}. \quad (1.25)$$
The reason for restricting to $X^{\text{red}}$ above is the following [Joyc1 Prop. 2.20], whose proof uses the fact that $X^{\text{red}}$ is reduced in an essential way.

**Lemma 1.12.** In Definition 1.11 the isomorphism $J_\Phi$ in (1.22) and the principal $\mathbb{Z}_2$-bundle $\pi_\Phi : P_\Phi \to X$ depend only on $U, V, X, Y, f, g$ and $\Phi|_X : X \to Y$. That is, they do not depend on $\Phi : U \to V$ apart from $\Phi|_X : X \to Y$.

Using the notation of Definition 1.11 we can restate Theorem 5.4 in [BBDJS]:

**Theorem 1.13.** (a) Let $U, V$ be complex manifolds, $f : U \to \mathbb{C}$, $g : V \to \mathbb{C}$ be holomorphic, and $X = \text{Crit}(f), Y = \text{Crit}(g)$ as complex analytic subspaces of $U, V$. Let $\Phi : U \to V$ be a closed embedding of complex analytic subspaces with $f = g \circ \Phi : U \to \mathbb{C}$, and suppose $\Phi|_X : X \to V \supseteq Y$ is an isomorphism $\Phi|_X : X \to Y$. Then there is a natural isomorphism of perverse sheaves on $X$:

$$\Theta_\Phi : \mathcal{P}V^*_{U,f} \to \Phi|_X^* (\mathcal{P}V^*_{V,g}) \otimes_{\mathbb{Z}_2} P_\Phi,$$

(1.26)

where $\mathcal{P}V^*_{U,f}, \mathcal{P}V^*_{V,g}$ are the perverse sheaves of vanishing cycles from 1.11 and $P_\Phi$ the principal $\mathbb{Z}_2$-bundle from Definition 1.11. and if $Q^*$ is a perverse sheaf on $X$ then $Q^* \otimes_{\mathbb{Z}_2} P_\Phi$ is as in Definition 1.5. Also the following diagrams commute, where $\sigma_{U,f}, \sigma_{V,g}, \tau_{U,f}, \tau_{V,g}$ are as in (1.27)-(1.28):\[
\begin{array}{ccc}
\mathcal{P}V^*_{U,f} & \xrightarrow{\Theta_\Phi} & \Phi|_X^* (\mathcal{P}V^*_{V,g}) \otimes_{\mathbb{Z}_2} P_\Phi \\
\sigma_{U,f} & \xrightarrow{\Phi|_X^* (\sigma_{V,g}) \circ \text{id}} & \Phi|_X^* (\mathcal{P}V^*_{V,g}) \otimes_{\mathbb{Z}_2} P_\Phi \\
\end{array}
\]

(1.27)

\[
\begin{array}{ccc}
\mathcal{P}V^*_{U,f} & \xrightarrow{\Theta_\Phi} & \Phi|_X^* (\mathcal{P}V^*_{V,g}) \otimes_{\mathbb{Z}_2} P_\Phi \\
\tau_{U,f} & \xrightarrow{\Phi|_X^* (\tau_{V,g}) \circ \text{id}} & \Phi|_X^* (\mathcal{P}V^*_{V,g}) \otimes_{\mathbb{Z}_2} P_\Phi \\
\end{array}
\]

(1.28)

If $U = V, f = g, \Phi = \text{id}_U$, then $\pi_\Phi : P_\Phi \to X$ is trivial, and $\Theta_\Phi$ corresponds to $\text{id}_{\mathcal{P}V^*_{U,f}}$ under the natural isomorphism $\text{id}_X^*(\mathcal{P}V^*_{U,f}) \otimes_{\mathbb{Z}_2} P_\Phi \cong \mathcal{P}V^*_{U,f}$.

(b) The isomorphism $\Theta_\Phi$ in (1.26) depends only on $U, V, X, Y, f, g$ and $\Phi|_X : X \to Y$. That is, if $\tilde{\Phi} : U \to V$ is an alternative choice for $\Phi$ with $\tilde{\Phi}|_X = \Phi|_X : X \to Y$, then $\Theta_\Phi = \Theta_{\tilde{\Phi}}$ by Lemma 1.12.

(c) Now suppose $W$ is another complex manifold, $h : W \to \mathbb{C}$ is holomorphic, $Z = \text{Crit}(h)$, and $\Psi : V \to W$ is a closed embedding with $g = h \circ \Psi : V \to \mathbb{C}$ and $\Psi|_Y : Y \to Z$ an isomorphism. Then Definition 1.11 defines principal $\mathbb{Z}_2$-bundles $\pi_\Phi : P_\Phi \to X$, $\pi_\Psi : P_\Psi \to Y$, $\pi_{\Psi \circ \Phi} : P_{\Psi \circ \Phi} \to X$ and an isomorphism $\Xi_{\Psi, \Phi}$ in (1.24), and part (a) defines isomorphisms of perverse sheaves $\Theta_{\Phi, \Psi : \Psi \circ \Phi}$ on $X$ and $\Theta_{\Psi, \Phi}$ on $Y$. Then the following commutes in $\text{Perv}(X)$:

$$\begin{array}{ccc}
\mathcal{P}V^*_{U,f} & \xrightarrow{\Theta_{\Psi \circ \Phi}} & (\Psi \circ \Phi)|_X^* (\mathcal{P}V^*_{W,h}) \otimes_{\mathbb{Z}_2} P_{\Psi \circ \Phi} \\
\Phi|_X^*(\Theta_\Phi) \circ \text{id} & \xrightarrow{\Phi|_X^*(\Psi) \circ \Psi|_Y^*(\mathcal{P}V^*_{W,h}) \otimes_{\mathbb{Z}_2} P_{\Psi \circ \Phi}} & \Phi|_X^*(\Psi) \circ \Psi|_Y^*(\mathcal{P}V^*_{W,h}) \otimes_{\mathbb{Z}_2} P_{\Psi \circ \Phi}. \\
\end{array}$$

(1.29)

1.3 Complex Lagrangian intersections in complex symplectic manifolds

We will start with a basic definition to fix the notation:

**Definition 1.14.** Let $(S, \omega)$ be a symplectic manifold, i.e., a complex manifold $S$ endowed with a closed non-degenerate holomorphic 2-form $\omega \in \Omega^2_S$. Denote the complex dimension of $S$ by $2n$.  

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A complex submanifold \( M \subset S \) is \textit{Lagrangian} if the restriction of the symplectic form \( \omega \) on \( S \) to a 2-form on \( M \) vanishes and \( \dim M = n \).

Holomorphic coordinates, \( x_1, \ldots, x_n, y_1, \ldots, y_n \) on an open subset \( S' \subset S \) in the complex analytic topology, are called \textit{Darboux coordinates} if \( \omega = \sum_{i=1}^n dy_i \wedge dx_i \).

**Definition 1.15.** Given an \( n \)-dimensional manifold \( N \), let us denote by \( S = T^*N \) its cotangent bundle. For any chosen point \( p \in U \subset N \), for \( U \) an open subset of \( N \) containing \( x \), let us denote by \( (x_1, \ldots, x_n) \) a set of coordinates. Then for any \( x \in U \), the differentials \( (dx_1)_x, \ldots, (dx_n)_x \) form a basis of \( T^*_xN \).

Namely, if \( y \in T^*_xN \) then \( y = \sum_{i=1}^n y_i(dx_i)_x \) for some complex coefficients \( y_1, \ldots, y_n \). This induces a set of coordinates \( (x_1, \ldots, x_n, y_1, \ldots, y_n) \) on \( T^*U \), so a coordinate chart for \( T^*N \), induced by \( (x_1, \ldots, x_n) \) on \( U \). It is well known that transition functions on the overlaps are holomorphic and this gives the structure of a complex manifold of dimension \( 2n \) to \( T^*N \).

Next, one can define a 2-form on \( T^*U \) by \( \omega = \sum_{i=1}^n dx_i \wedge dy_i \). It is easy to check that the definition is coordinate-independent. Define the 1-form \( \alpha = \sum_{i=1}^n y_i \wedge dx_i \). Clearly \( \omega = -d\alpha \), and \( \alpha \) is intrinsically defined. The 1-form \( \alpha \) is called in literature the \textit{Liouville form}, and the 2-form \( \omega \) is the \textit{canonical symplectic form}.

Next, we will review symmetric obstruction theories on Lagrangian intersections from [BeFa], and we state a crucial definition for our program.

Let \( (S, \omega) \) be a complex symplectic manifold as above, and \( L, M \subseteq S \) be Lagrangian submanifolds. Let \( X = L \cap M \) be the intersection as a complex analytic space. Then \( X \) carries a canonical symmetric obstruction theory \( \varphi : E^* \to i_X \) in the sense of [BeFa], which can be represented by the complex \( E^* \cong \left( T^*S|_X \to T^*L|_X \oplus T^*M|_X \right) \) with \( T^*S|_X \) in degree \(-1\) and \( T^*L|_X \oplus T^*M|_X \) in degree zero. Hence

\[
\det(E^*) \cong K_S|_X^{-1} \otimes K_L|_X \otimes K_M|_X \cong K_L|_X \otimes K_M|_X,
\]

(1.30)
since \( K_S \cong \mathcal{O}_S \). This motivates the following:

**Definition 1.16.** We define an \textit{orientation} of a complex Lagrangian submanifold \( L \) to be a choice of square root line bundle \( KL^{1/2} \) for \( K_L \).

**Remark 1.17.** The previous definition is inspired by [BBDJS] and close to ‘orientation data’ in Kontsevich and Soibelman [KoSo]. We point out that spin structure could have been a better choice of name than orientation, but we use orientations for consistency with [BBDJS, BBJ, BLM, Joyc1]. Also, for \textit{real} Lagrangians, a square root \( K_{L}^{1/2} \) induces an orientation on \( L \) in the usual sense.

Now, we recall well known established results in complex symplectic geometry which will be used to prove our main results. We start with the complex Darboux theorem.

**Theorem 1.18.** Let \( (S, \omega) \) be a complex symplectic manifold. Then locally in the complex analytic topology around a point \( p \in S \) is always possible to choose holomorphic Darboux coordinates.

So, basically, every symplectic manifold \( S \) is locally isomorphic to the cotangent bundle \( T^*N \) of a manifold \( N \). The fibres of the induced vector bundle structure on \( S \) are Lagrangian submanifolds, so complex analytically locally defining on \( S \) a foliation by Lagrangian submanifolds, i.e., a \textit{polarization}.

**Definition 1.19.** A \textit{polarization} of a symplectic manifold \( (S, \omega) \) is a holomorphic Lagrangian fibration \( \pi : S' \to E \), where \( S' \subseteq S \) is open.

Note that it is always possible to choose locally near a point \( p \in S \) in the complex analytic topology Darboux coordinates \( (x_1, \ldots, x_n, y_1, \ldots, y_n) \) and compatible coordinates \( x_i \) on \( E \) such that \( \pi \) can be identified with the projection \( (x_1, \ldots, x_n, y_1, \ldots, y_n) \to (x_1, \ldots, x_n) \).

We will usually consider polarizations which are \textit{transverse} to the Lagrangians whose intersection we wish to study. The point of the transversality condition is that we have a canonical one-to-one correspondence between sections \( s \) of the polarization \( E \) for the symplectic manifold \( (S, \omega) \), such that \( s^*(\omega) = 0 \) and Lagrangian submanifolds of \( S \) transverse to \( E \). Moreover, in terms of coordinates, near every point of a Lagrangian \( M \subset S \)
transverse to the polarization $E$ there exists a set of Darboux coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ such that $M = \{y_1 = \cdots = y_n = 0\}$, $E = (dx_1, \ldots, dx_n)$ and the Euler form $s$ of $M$ inside $E$ is given by $s = \sum_i y_i dx_i$.

If $L, M$ are complex Lagrangian submanifolds in $(S, \omega)$, and we consider the projection

$$\pi : (x_1, \ldots, x_n, y_1, \ldots, yn) \to (x_1, \ldots, x_n)$$

defining local coordinates on $L$, then we can always assume to choose such coordinates $x_i, y_i$ transverse to $L, M$ at a point, and transverse to other coordinate systems too. In other words, we are using the projection $\pi$ as a polarization, and we assume that the leaves are transverse to the two Lagrangians, so that $L$ and $M$ turn into the graphs of 1-forms on $N$. The Lagrangian condition implies that these 1-forms on $N$ are closed.

Recall now the Lagrangian neighbourhood theorem:

**Theorem 1.20.** If $M \subset (S, \omega)$ is a complex Lagrangian submanifold, then there exists a complex analytic neighbourhood $V \subset M$ of a point $p \in M$ isomorphic as a complex symplectic manifold to a neighbourhood $U$ of $p$ in $T^*M$, and $M$ is identified with the zero section in $T^*M$.

Note that $(S, \omega)$ need not be isomorphic to $T^*M$ in a neighbourhood of $M$, but just in a neighbourhood of a point $p \in M$.

So, we may assume that one of these 1-forms is the zero section of $T^*N$, hence identify locally $M$ with $N$. By making $L$ smaller if necessary, we may assume that the closed 1-form defined by $L$ is exact, and $L$ is the graph of the 1-form $df$, for a holomorphic function $f$ defined locally on some open submanifold $M' \subset M$, as the following lemma states:

**Lemma 1.21.** Choose locally near a point $p \in S$ in the complex analytic topology Darboux coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and compatible coordinates $x_i$ on $E$ such that $\pi : S \to E$ can be identified with the projection $(x_1, \ldots, x_n, y_1, \ldots, y_n) \to (x_1, \ldots, x_n)$. Now, given a polarization $(x_1, \ldots, x_n, y_1, \ldots, y_n) \to (x_1, \ldots, x_n)$ defining local coordinates on $L$, then $L$ is given by

$$\left\{ \left( x_1, \ldots, x_n, \frac{df}{dx_1}, \ldots, \frac{df}{dx_n} \right) : x_1, \ldots, x_n \in M' \right\}$$

for a holomorphic function $f(x_1, \ldots, x_n)$ defined locally on an open $M' \subset M \subset S$, where $M$ is the Lagrangian submanifold identified with the zero section, and the polarization $\pi$ with projection $T^*M \to M$.

So, in conclusion, if $\pi : S' \to E$ is a polarization, and $M$ a Lagrangian submanifold with $\pi : M \to E$ transverse near $x$ in $M$, then locally there is a unique isomorphism $S' \cong T^*M$ identifying $M$ with zero section and $\pi$ with projection $T^*M \to M$. Then any other Lagrangian $L$ in $S$ transverse to $\pi$ is locally described by the graph of $df$, for a holomorphic function $f$ locally defined on $M$. It is now straightforward to deduce that, as $M$ is graph of 0, and $L$ is graph of $df$, then the intersection $X = L \cap M$ is the critical locus $(df)^{-1}(0)$.

We can summarize in this way. Let $(S, \omega)$ be a complex symplectic manifold, and $L, M \subseteq S$ be Lagrangian submanifolds. Let $X = L \cap M$ be the intersection as a complex analytic space. Then $X$ is complex analytically locally modeled on the zero locus of the 1-form $df$, that is on the critical locus $\text{Crit}(f : U \to \mathbb{C})$ for a holomorphic function $f$ on a smooth manifold $U$. So, $X$ carries a natural perverse sheaf of vanishing cycles $\mathcal{P}^\bullet_{\omega, f}$ in the notation of [1.11] and a natural problem to investigate is the following. Given open $R_i, R_j \subseteq S$ with isomorphisms $R_i \cong \text{Crit}(f_i)$, $R_j \cong \text{Crit}(f_j)$ for holomorphic $f_i : U_i \to \mathbb{C}$ and $f_j : U_j \to \mathbb{C}$, we have to understand whether the perverse sheaves $\mathcal{P}^\bullet_{R_i} = \mathcal{P}^\bullet_{U_i, f_i}$ on $R_i$ and $\mathcal{P}^\bullet_{R_j} = \mathcal{P}^\bullet_{U_j, f_j}$ on $R_j$ are isomorphic over $R_i \cap R_j$, and if so, whether the isomorphism is canonical, for only then can we hope to glue the $\mathcal{P}^\bullet_{R_i}$ for $i \in I$ to make $\mathcal{P}^\bullet_{L,M}$.

We will develop this program in [2].
2 Constructing canonical global perverse sheaves on Lagrangian intersections

We can state our main result.

**Theorem 2.1.** Let \((S, \omega)\) be a complex symplectic manifold and \(L, M\) oriented complex Lagrangian submanifolds in \(S\), and write \(X = L \cap M\), as a complex analytic subspace of \(S\). Then we may define \(P_{L,M}^* \in \text{Perv}(X)\), uniquely up to canonical isomorphism, and isomorphisms

\[
\Sigma_{L,M}: P_{L,M}^* \to \mathbb{D}_X(P_{L,M}^*), \quad T_{L,M}: P_{L,M}^* \to P_{L,M}^*.
\]

respectively the Verdier duality and the monodromy isomorphisms. These \(P_{L,M}^* \in \text{Perv}(X)\) are locally characterized by the following property.

Given a choice of local Darboux coordinates \((x_1, \ldots, x_n, y_1, \ldots, y_n)\) in the sense of Definition 1.14 such that \(L\) is locally identified in coordinates with the graph \(\Gamma_{df(x_1, \ldots, x_n)}\) of \(df\) for \(f\) a holomorphic function defined locally on an open \(U \subset \mathbb{C}^n\), and \(M\) is locally identified in coordinates with the graph \(\Gamma_{dg(x_1, \ldots, x_n)}\) of \(dg\) for \(g\) a holomorphic function defined locally on \(U\), and the orientations \(K_L^{1/2}, K_M^{1/2}\) are the trivial square roots of \(K_L \equiv (\text{d}x_1 \wedge \cdots \wedge \text{d}x_n) \cong K_M\), then there is a canonical isomorphism \(P_{L,M}^* \cong \mathcal{PV}_{U,g-f}\), where \(\mathcal{PV}_{U,g-f}\) is the perverse sheaf of vanishing cycles of \(g-f\), and \(\Sigma_{L,M}\) and \(T_{L,M}\) are respectively the Verdier duality \(\mathcal{PV}_{U,g-f}\) and the monodromy \(\tau_{U,g-f}\) introduced in [41].

The same applies for \(\mathcal{O}\)-modules and mixed Hodge modules on \(X\).

A convenient way to express this is in terms of charts, by which we mean a set of data locally defined by the choice of a polarization \(\pi: S \to E\). Charts are analogous to critical charts defined by [Joyc1] 42, as in [3.1]. We will show in [3.2] that they are actually critical charts and they define the structure of a d-critical locus on the Lagrangian intersection, but for this section we will not use it.

We explained in [1.3] that the local choice of a polarization on \((S, \omega)\) yields a local description of the Lagrangian intersection as a critical locus \(P \cong \text{Crit}(f)\) for a closed embedding \(i: P \to U\), \(P\) open in \(X\), and a holomorphic function \(f: U \subset L \to \mathbb{C}\), where \(U\) is an open submanifold of \(L\). We have a local symplectic identification \(S \cong T^*U \subset T^*L\), which identifies \(L \subset S\) with the zero section in \(T^*L\), and \(M \subset S\) with the graph \(\Gamma_{df}\) of \(df\), and \(\pi: S \to E \cong L\) with the projection \(T^*L \to L\). So, for each polarization \(\pi: S \to E\) we have naturally induced a set of data \((P, U, f, i)\), which we will call an \(L\)-chart. We will also consider \(M\)-charts, namely charts coming from polarizations that identify the other Lagrangian \(M\) with the zero section, that is, charts like \((Q, V, g, j)\) where \(Q \cong \text{Crit}(g)\) for a closed embedding \(j: Q \to V\), \(Q\) open in \(X\), and a holomorphic function \(g: V \subset M \to \mathbb{C}\), where \(V\) is an open submanifold of \(M\). We have a local symplectic identification \(S \cong T^*V \subset T^*S\), which identifies \(M \subset S\) with the zero section in \(T^*S\), and \(L \subset S\) with the graph \(\Gamma_{dg}\) of \(dg\), and \(\pi: S \to E \cong M\) with the projection \(T^*S \to M\). We will use also \(LM\)-charts.

Using this general technique, let us fix the following notation we will use for the rest of the paper. We will consider mainly three kinds of charts, where by charts we basically mean a set of data associated to a choice of one or two polarizations for our symplectic complex manifold \((S, \omega)\):

(a) \(L\)-charts \((P, U, f, i)\) are induced by a polarizations \(\pi: S' \to E\) transverse to \(L, M\) with \(S' \subset S\) open, \(P \subset X\) open, and \(U \subset L\) open, and \(f: U \to \mathbb{C}\) holomorphic, and \(i: P \to U \subset L\) the inclusion, with \(i: P \to \text{Crit}(f)\) an isomorphism, so that we have local identifications

- \((S, \omega) = T^*U\);
- \(L = \text{zero section};\)
- \(M = \Gamma_{df};\)
- \(E = L = U;\)
- \(\pi: S \to E\) with \(\pi: T^*U \to U.\)

(b) \(M\)-charts \((Q, V, g, j)\) are induced by a polarization \(\tilde{\pi}: \tilde{S} \to F\) transverse to \(L, M\), with \(\tilde{S} \subset S\) open, \(Q \subset X\) open, and \(V \subset M\) open, and \(g: V \to \mathbb{C}\) holomorphic, and \(j: Q \to V \subset M\) the inclusion, with \(j: Q \to \text{Crit}(g)\) an isomorphism, so that we have local identifications

\[14\]
(c) $LM$-charts $(R, W, h, k)$ are induced by polarizations $\pi : S' \to E$, $\tilde{\pi} : \tilde{S} \to F$ transverse to $L, M$ and to each other on $\tilde{S} = S' \cap \tilde{S}$. We have $W \subset L \times M$ open, $h : W \to \mathbb{C}$ holomorphic, $k : R \to W \subset L \times M$ the diagonal map, with $k : R \to \text{Crit}(h)$ an isomorphism, and local identifications

- $(S', \omega) = T^*V$
- $M = \text{zero section}$
- $L = \Gamma_{dg}$
- $F = M = V$
- $\tilde{\pi} : S \to F$ with $\pi : T^*V \to V$.

Moreover, we will explain in [2.21] that the choice of a polarization $\pi : S \to E$ naturally induces local biholomorphisms $L \cong M$ or $M \cong L$, and thus isomorphisms

$$\Theta : K_L|_X \cong K_M|_X \quad \text{or} \quad \Xi : K_M|_X \cong K_L|_X$$

(2.2)

between the canonical bundles of the Lagrangian submanifolds. We define $\pi_{P,U,f,i} : Q_{P,U,f,i} \to P$ to be the principal $\mathbb{Z}_2$-bundle parametrizing local isomorphisms

$$\vartheta : K_L^{1/2}|_X \cong K_M^{1/2}|_X \quad \text{or} \quad \xi : K_M^{1/2}|_X \cong K_L^{1/2}|_X$$

such that $\vartheta \otimes \vartheta = \Theta$ or $\xi \otimes \xi = \Xi$.

Also, on each $L$-chart, $M$-chart, $LM$-chart, we have a natural perverse sheaf of vanishing cycles associated to the local description of the Lagrangian intersection as a critical locus. So we get a perverse sheaf of vanishing cycles $i^*(\mathcal{P}V_{U,f}^*)$ on $P$, $j^*(\mathcal{P}V_{V,g}^*)$ on $Q$, and $k^*(\mathcal{P}V_{W,h}^*)$ on $R$. These perverse sheaves together with principal $\mathbb{Z}_2$-bundles parametrizing square roots of isomorphisms (2.2) are the objects we want to glue.

Then $P^*_{L,M} \in \text{Perv}(X)$ is characterized by the following properties:

(i) If $(P, U, f, i)$ is an $L$-chart, $M$-chart, or $LM$-chart, there is a natural isomorphism

$$\omega_{P,U,f,i} : P^*_{L,M}|_P \to i^*(\mathcal{P}V_{U,f}^*) \otimes_{\mathbb{Z}_2} Q_{P,U,f,i},$$

(2.4)

Furthermore the following commute in Perv($P$):

$$\begin{align*}
P^*_{L,M}|_P \quad &\xrightarrow{\omega_{P,U,f,i}} \quad i^*(\mathcal{P}V_{U,f}^*) \otimes_{\mathbb{Z}_2} Q_{P,U,f,i} \\
\Sigma_{L,M}|_P \quad &\xrightarrow{\omega_{P,U,f,i}} \quad i^*(\Sigma_{U,f} \otimes 1_{Q_{P,U,f,i}}) \\
\mathbb{D}_P(P^*_{L,M}|_P) \quad &\xrightarrow{\mathbb{D}_P(\omega_{P,U,f,i})} \quad i^*(\mathcal{D}_{\text{Crit}(f)}(\mathcal{P}V_{U,f}^*)) \otimes_{\mathbb{Z}_2} Q_{P,U,f,i} \\
T_{L,M}|_P \quad &\xrightarrow{\omega_{P,U,f,i}} \quad i^*(\mathcal{P}V_{U,f}^*) \otimes_{\mathbb{Z}_2} Q_{P,U,f,i} \\
P^*_{L,M}|_R \quad &\xrightarrow{\omega_{P,U,f,i}} \quad i^*(\mathcal{P}V_{U,f}^*) \otimes_{\mathbb{Z}_2} Q_{P,U,f,i}.
\end{align*}$$

(2.5)
(ii) Let $\pi : S' \to E$ and $\tilde{\pi} : \tilde{S} \to F$ be polarizations transverse to $L, M$, and transverse to each other on $S' \cap \tilde{S}$. Then from $\pi$ we get an $L$-chart $(P, U, f, i)$, from $\tilde{\pi}$ we get an $M$-chart $(Q, V, g, j)$, and from $\pi$ and $\tilde{\pi}$ together we get an $LM$-chart $(R, W, h, k)$. Write $(P', U', f', i')$ for the $L$-chart determined by $\pi|_{S' \cap \tilde{S}} : S' \cap \tilde{S} \to E$, and $(Q', V', g', j')$ for the $M$-chart determined by $\tilde{\pi}|_{S' \cap \tilde{S}} : S' \cap \tilde{S} \to F$. Then $P' \subseteq P, U' \subseteq U$ are open and $f' = f|_{U'}, i' = i|_{P'}$, so $(P', U', f', i')$ is a subchart of $(P, U, f, i)$ in the sense of [3,1]. We write this as $(P', U', f', i') \subseteq (P, U, f, i)$. Similarly, $(Q', V', g', j') \subseteq (Q, V, g, j)$. Also $P' = Q' = R = X \cap S' \cap \tilde{S}$.

In this situation, Proposition 2.2 will show that there exist closed embeddings $\Phi : U' \to W$ and $\Psi : V' \to W$ such that so that $h \circ \Phi = f : U' \to C$ and $h \circ \Psi = g : V' \to C$. Moreover Crit($f$) $\simeq$ Crit($g$) as complex analytic spaces, and $f, h$ and $g, h$ are pairs of stably equivalent functions, as explained in [1.2]. Inspired by [Joyc1, Def. 2.18], we will say that $\Phi : (P', U', f', i') \to (R, W, h, k)$ is an embedding of charts if $\Phi$ is a locally closed embedding $U' \to W$ of complex manifolds such that $\Phi \circ i' = k|_{P'} : P' \to W$ and $f = h \circ \Phi : U' \to C$. As a shorthand we write $\Phi : (P', U', f', i') \to (R, W, h, k)$ to mean $\Phi$ is an embedding of $(P', U', f', i')$ in $(R, W, h, k)$. In brief, Proposition 2.2 in [4.1] will define two embeddings of charts $\Phi : (P', U', f', i') \to (R, W, h, k)$ and $\Psi : (Q', V', g', j') \to (R, W, h, k)$.

Given the embedding of charts $\Phi : (P', U', f', i') \to (R, W, h, k)$, there is a natural isomorphism of principal $\mathbb{Z}_2$-bundles

$$\Lambda_{\Phi} : Q_{R,W,h,k}|_{P'} \cong i^*(P_{\Phi}) \otimes_{\mathbb{Z}_2} Q_{P',U',f',j'}$$

(2.7)

on $P'$, for $P_{\Phi}$ defined as follows: local isomorphisms

$$\alpha : K^1_X|_{P'} \to i^*(K_{U'})|_{P'}, \quad \beta : K^1_X|_{P'} \to j^*(K_W)|_{P'}, \quad \gamma : i^*(K_{U'})|_{P'} \to j^*(K_W)|_{P'}$$

(2.8)

with $\alpha \otimes \beta = i_{P',U',f',i'}$ and $\gamma \otimes \gamma = i_{P',U',f',i'}$. $\Phi$ and $\Psi$ correspond to local sections $s_\alpha : P' \to Q_{P',U',f',i'}$, $s_\beta : P' \to Q_{R,W,h,k}|_{P'}$, and $s_\gamma : P' \to i^*(P_{\Phi})$. For $\Theta_{\Phi}$, as in Definition 1.11 and for isomorphisms $i_{R,W,h,k} : K_X \to i^*(K_{W,h,k})|_{P'}$ induced by the polarization $E_1 \times E_2$.

Then for each embedding of charts, the following diagram commutes in $\text{Perv}(P')$, for $\Theta_{\Phi}$ as in (1.26):

$$\begin{array}{ccc}
P_{L,M}|_{P'} & \xrightarrow{\psi_{P',U',f',i'}} & i^*(P_{U',f',j'}) \otimes_{\mathbb{Z}_2} Q_{P',U',f',j'} \\
\omega_{R,W,h,k}|_{P'} & \downarrow & i^*(\Theta_{\Phi}) \otimes \text{id}|_{P',U',f',j'} \\
j^*(P_{V,W,h})|_{P'} \otimes_{\mathbb{Z}_2} Q_{R,W,h,k}|_{P'} & \xrightarrow{i^*(\Phi)|_{X_0} \otimes \Lambda_{\Phi}} & i^*(\Psi_0)^*(P_{V,W,h}) \otimes_{\mathbb{Z}_2} P_{\Phi} \otimes_{\mathbb{Z}_2} Q_{P',U',f',j'}.
\end{array}$$

(2.9)

We will have an analogous commutative diagram induced by $\Psi$ on $\text{Perv}(Q')$:

$$\begin{array}{ccc}
P_{L,M}|_{Q'} & \xrightarrow{\psi_{Q',V',g',j'}} & i^*(P_{V,g'}) \otimes_{\mathbb{Z}_2} Q_{Q',V',g',j'} \\
\omega_{R,W,h,k}|_{Q'} & \downarrow & j^*(\Theta_{\Phi}) \otimes \text{id}|_{Q',V',g',j'} \\
j^*(P_{V,W,h})|_{Q'} \otimes_{\mathbb{Z}_2} Q_{R,W,h,k}|_{Q'} & \xrightarrow{j^*(\Psi)|_{X_0} \otimes \Lambda_{\Phi}} & i^*(\Phi_0)^*(P_{V,W,h}) \otimes_{\mathbb{Z}_2} P_{\Phi} \otimes_{\mathbb{Z}_2} Q_{Q',V',g',j'}.
\end{array}$$

(2.10)

Using Theorem 1.13, we get isomorphisms

$$\alpha : (i^*(P_{U',f,i}) \otimes Q_{P',U',f',i'})|_R \cong (k^*(P_{V,W,h}) \otimes Q_{R,W,h,k}),$$

$$\beta : (j^*(P_{V,g}) \otimes Q_{Q',V',g',j'})|_R \cong (k^*(P_{V,W,h}) \otimes Q_{R,W,h,k}).$$

Combining these, we get an isomorphism

$$\beta^{-1} \circ \alpha : (i^*(P_{U',f,i}) \otimes Q_{P',U',f',i'})|_R \cong (j^*(P_{V,g}) \otimes Q_{Q',V',g',j'})|_R,$$

(2.11)
that is, an isomorphism of perverse sheaves from $L$-charts and $M$-charts in $\text{Perv}(P' \cap Q')$. Later, in §2.2 we will involve also two other polarizations for an associativity result. More precisely, following notation of §2.2 we want that if we have two $L$-charts $(P_1, U_1, f_1, i_1)$ and $(P_3, U_3, f_3, i_3)$ and two $M$-charts $(Q_2, V_2, g_2, j_2)$ and $(Q_4, V_4, g_4, j_4)$ then

$$
\alpha_{2|Y}^{-1} \circ \beta_{2|Y} \circ \beta_{1|Y}^{-1} \circ \alpha_{1|Y} = \alpha_{3|Y}^{-1} \circ \beta_{3|Y} \circ \beta_{1|Y}^{-1} \circ \alpha_{1|Y} \cdot (PV^*_{U_1, f_1} \otimes_{Z_2} Q_{P_1, U_1, f_1, i_1})|_Y \rightarrow (PV^*_{U_3, f_3} \otimes_{Z_2} Q_{P_3, U_3, f_3, i_3})|_Y.
$$

(2.12)

Theorem 2.1 will be proved in §2.1–2.3. In §2.3 we will provide a descent argument, which is the most technical part of the paper. We find useful to outline here our method of the proof.

Let $\{U_a\}_{a \in I}$ be an analytic open cover for $X = L \cap M$, induced by polarizations $\pi_a : S_a \rightarrow E_a$ for $a \in I$, transverse to both $L$ and $M$, and write $U_{ab} = U_a \cap U_b$ for $a, b \in I$. Similarly, write $U_{abc} = U_a \cap U_b \cap U_c$ for $a, b, c \in I$. Define $\mathcal{P}_a$ to be $i^*_a((PV^*_{U_a, f_a}) \otimes_{Z_2} Q_{P_a, U_a, f_a, i_a})$ from the discussion above, and isomorphisms $\gamma_{ab} : \mathcal{P}_a|_{U_{ab}} \rightarrow \mathcal{P}_b|_{U_{ab}}$ in $\text{Perv}(U_{ab})$ for all $a, b \in I$ with $\beta_{ab} = \text{id}$ and

$$
\gamma_{bc}|_{U_{abc}} \circ \gamma_{ab}|_{U_{abc}} = \gamma_{ac}|_{U_{abc}} : \mathcal{P}_a|_{U_{abc}} \rightarrow \mathcal{P}_c|_{U_{abc}}
$$
in $\text{Perv}(U_{abc})$ for all $a, b, c \in I$.

The construction is independent of the choice of $\{U_a\}_{a \in I}$ above. Then by Theorem 1.4 there exists $\mathcal{P}^* \in \text{Perv}(X)$, unique up to canonical isomorphism, with isomorphisms $\omega_a : \mathcal{P}^*|_{U_a} \rightarrow \mathcal{P}_a$ for each $a \in I$, satisfying $\gamma_{ab} \circ \omega_{a|U_{ab}} = \omega_{b|U_{ab}} : \mathcal{P}^*|_{U_{ab}} \rightarrow \mathcal{P}^*|_{U_{ab}}$ for all $a, b \in I$, which concludes the proof of Theorem 2.1.

We will carry out this program in §2.1–2.3.

Theorem 2.1 resolves a long-standing question in the categorification of Lagrangian intersection number: our perverse sheaf $P^*_{L,M}$ categorifies Lagrangian intersection numbers, in the sense that the constructible function

$$
p \rightarrow \sum_i (-1)^{i-2n \dim_{C} H^i_{\{p\}}(X, P^*_{L,M}),}
$$
is equal to the well known Behrend function $\chi_X$ in Behr by construction, using the expression of the Behrend function of a critical locus in terms of the Milnor fibre, as in Behr, and so

$$
\chi(X, \nu_X) = \sum_i (-1)^{i-2n \dim_{A} H^i(X, P^*_{L,M}),}
$$

for a base ring $A$, which in this case is a field. If the intersection $X$ is compact, then $[L] \cap [M]$ is given by (2.13), where $[L], [M]$ are the homology classes of $L$ and $M$ in $S$.

Moreover, our construction may have exciting far reaching applications in symplectic geometry and topological field theory, as discussed in §4.

### 2.1 Canonical isomorphism of perverse sheaves on double overlaps

Given a complex symplectic manifold $(S, \omega)$ and Lagrangian submanifolds $L, M$ in $S$, define $X$ to be their intersection as a complex analytic set. Using results in §1.3 locally in the complex analytic topology near a point $x \in X$, we can choose an open set $S' \subset S$ and a polarization transverse to both $L$ and $M$ such that $S' \cong T^*L$ and $M \cong \Gamma_{L,f}$ so that $X = \text{Crit}(f)$ for a holomorphic function $f$ defined on $U = L \cap S'$. Thus we get a perverse sheaf of vanishing cycle $PV^*_{U,f}$. In this section we will investigate how two such local descriptions are related.

Consider $\pi_1, \pi_2 : S_1, S_2 \rightarrow E_1, E_2$ two polarizations transverse to each other and both transverse to both $L$ and $M$. Choose open neighbourhoods $U_1$ of $X \cap S_1$ in $L \cap S_1$ with $\pi_1(U_1) \subset \pi_1(M \cap S_1)$ and $V_2$ of $X \cap S_2$ in $M \cap S_2$ with $\pi_2(V_2) \subset \pi_2(L \cap S_2)$. Then we get respectively the local identifications

$$
U_1 \cong \pi_1(U_1) \subset E_1, S_1 \supset \pi_1^{-1}(\pi_1(U_1)) \cong T^*U_1, L \cap \pi_1^{-1}(\pi_1(U_1)) \cong \Gamma_0,
$$

$$
M \cap \pi_1^{-1}(\pi_1(U_1)) \cong \Gamma_{d,f}, f_1 : U_1 \rightarrow \mathbb{C},
$$
Choose an open neighbourhood $W_{12}$ of $\{ (x,x) : x \in X \cap S_1 \cap S_2 \}$ in $U_1 \times V_2$ with $(\pi_1 \times \pi_2)(W_{12}) \subset (\pi_1 \times \pi_2)(\Delta S \cap (S_1 \times S_2))$. Choose open neighbourhoods $U'_1$ of $X \cap S_1 \cap S_2$ in $U_1$ with $\{ (l, \pi_2^{-1} \circ \pi_2(l)) : l \in U'_1 \} \subset W_{12}$ and $V'_2$ of $X \cap S_1 \cap S_2$ in $V_2$ with $\{ (\pi_2^{-1} \circ \pi_1(m)) : m \in V'_2 \} \subset W_{12}$. Then we have:

**Proposition 2.2.** In the situation above, starting from polarizations $\pi_1, \pi_2 : S_1, S_2 \to E_1, E_2$ and defining $f_1 : U_1 \to \mathbb{C}$ using $\pi_1$ and $g_2 : V_2 \to \mathbb{C}$ using $\pi_2$ and using the given notation, there exists locally a holomorphic function $h_{12} : W_{12} \to \mathbb{C}$ such that the following diagram

\[
\begin{array}{ccc}
U'_1 & \xrightarrow{i_{12}} & W_{12} \\
\downarrow{f_1|_{U'_1}} & & \downarrow{h_{12}} \\
C & & C \\
\downarrow{g_2|_{V'_2}} & & \downarrow{\pi_2|_{V'_2}} \\
V'_2 & \xleftarrow{i_{21}} & W_{12}
\end{array}
\]

(2.14) is commutative, that is

\[
h_{12}(l, \pi_1|_{V'_2}^{-1}(l)) = f_1(l), \quad \text{and} \quad h_{12}(\pi_2|_{U'_1}^{-1}(m), m) = g_2(m),
\]

(2.15)

for every $l \in U'_1, m \in V'_2$. Moreover, $\Phi_{12} = id_{U'_1} \times \pi_1|_{V'_2}^{-1}$ and $\Psi_{12} = \pi_2|_{U'_1}^{-1} \times id_{V'_2}$ induce isomorphisms $\text{Crit}(h_{12}) \cong \text{Crit}(f_1) \cong \text{Crit}(g_2)$ as complex analytic spaces locally in the complex analytic topology. In particular, from Theorem 1.10 we can choose $(z_1, \ldots, z_n)$ coordinates normal to $(id_L \times \pi_1|_{V'_2}^{-1})(U'_1)$ in $W_{12}$, and $(w_1, \ldots, w_n)$ coordinates normal to $(\pi_2|_{U'_1}^{-1} \times id_{V'_2})(V'_2)$ in $W_{12}$, under which we can write $h_{12} \cong f_1 \oplus z_1^2 + \ldots + z_n^2$ and $h_{12} \cong g_2 \oplus w_1^2 + \ldots + w_n^2$. Following 1.13 we will say that $f_1$ and $g_2$ are both stably equivalent to $h_{12}$.

**Proof.** Consider the product symplectic manifold $(\mathbb{S} \times \bar{\mathbb{S}}, \omega \oplus (-\omega))$, where $\bar{S}$ denotes the symplectic manifold $S$ corresponding to the symplectic form with the opposite sign. In $\mathbb{S} \times \bar{S}$ consider the Lagrangian submanifolds $N_1 := L \times M$ and $N_2 := \Delta S$, the diagonal. As explained in 1.13, identify locally $(\mathbb{S} \times \bar{S}, \omega \oplus -\omega)$ with $(T^*(L \times M), \omega_{L \times M})$, where $\omega_{L \times M}$ is the symplectic form on $T^*(L \times M)$, and thus $\pi_1 \times \pi_2$ is identified with the projection $\pi : T^*(L \times M) \to L \times M$, that is $N_1 = \pi_1^{-1}|_L$ with the zero section, and $N_2$ with the graph $\Gamma_{dh_{12}}$ for a holomorphic function $h_{12} : L \times M \to \mathbb{C}$ normalized by $\pi_{12}(\pi_1 \times \pi_2)((L \times M) \cap \Delta S) = 0$. Consider the submanifold $P := S \times M \subset \mathbb{S} \times \bar{S}$ and intersect the Lagrangians $N_1$ and $N_2$ with this submanifold, yielding respectively $N_1 \cap P = N_1$ and $N_2 \cap P = \Delta_M$, which both lie in $S \times M$. Observe that

\[
\omega \oplus (-\omega)|_P = \pi_1^{-1}\omega,
\]

(2.16)

where $p_i : S \times S \to S$ are the projections to the $i$-th factor. Consider the following diagram of inclusions and projections in $S \times \bar{S}$ and $S$:

\[
\begin{array}{cccc}
N_1 \cap P = N_1 & \subset & P = S \times M & \subset \mathbb{S} \times \bar{S} \\
\downarrow{p_1} & & \downarrow{p_1} & \\
L & \subset & S & \subset \mathbb{S} \\
\downarrow{p_1} & & \downarrow{p_1} & \\
M & \subset & S & \subset \mathbb{S}.
\end{array}
\]

(2.17)

Under the local symplectomorphisms $S \cong T^*U_1$ and $S \times \bar{S} \cong T^*(U_1 \times V_2)$, equation 2.17 is identified with the diagram:

\[
\begin{array}{cccc}
\Gamma_{dh_{12}|_{(id_L \times \pi_1|_{V'_2}^{-1})(U'_1)}} & \subset & T^*U_1 \times \pi_{T^*U_1}(V'_2) & \subset T^*(U_1 \times V_2) \\
\downarrow{\pi_{T^*U_1}} & & \downarrow{\pi_{T^*U_1}} & \\
z(U_1) \times \pi_{T^*U_1}(V'_2) & \subset & T^*U_1 & \subset T^*U_1.
\end{array}
\]

(2.18)
Here \( z : U_1 \to T^*U_1, z : V_2 \to T^*V_2 \) are the zero section maps. To see that \( N_2 \cap P = \Delta V_2 \) is identified with \( \Gamma_{dh_{12}}|_{(id_{U_1} \times \pi_1|_{V_2}^{-1})(U_1)} \), note that \( N_2 \) is identified with \( \Gamma_{dh_{12}} \), and

\[
(\pi_1 \times \pi_2)(\Delta V_2) = (\pi_1|_{V_2} \times \pi_2|_{V_2})(V_2) = (id_{U_1} \times \pi_1|_{V_2}^{-1})(U_1) \subset U_1 \times V_2,
\]

so that \( \Delta V_2 \) is identified with a subset of \( T^*(U_1 \times V_2)|_{(id_{U_1} \times \pi_1|_{V_2}^{-1})(U_1)} \subset T^*(U_1 \times V_2) \).

Equation (2.18) shows that \( \pi_{T^*U_1} \) maps \( \Gamma_{dh_{12}}|_{(id_{U_1} \times \pi_1|_{V_2}^{-1})(U_1)} \to \Gamma_{df_1} \). Writing points of \( T^*U_1 \) as \( (x, \alpha) \) for \( x \in U_1 \) and \( \alpha \in T_x^*U_1 \), and points of \( T^*V_2 \) as \( (y, \beta) \) for \( y \in V_2 \) and \( \beta \in T_y^*V_2 \), we have

\[
\Gamma_{df_1} = \{(x, df_1(x)) : x \in U_1\}, \quad \Gamma_{dh_{12}}|_{(id_{U_1} \times \pi_1|_{V_2}^{-1})(U_1)} = \{(x, d_x h_{12}(x, \pi_1|_{V_2}^{-1}(x)), \pi_1|_{V_2}^{-1}(x), 0) : x \in U_1\},
\]

where the final term \( \beta = 0 \) as \( \Gamma_{dh_{12}}|_{(id_{U_1} \times \pi_1|_{V_2}^{-1})(U_1)} \subset T^*U_1 \times z(V_2) \). The projection \( \pi_{T^*U_1} : T^*U_1 \times T^*V_2 \to T^*V_2 \) maps \( (x, \alpha, y, \beta) \to (x, \alpha) \). So from (2.18) we see that \( d_x h_{12}(x, \pi_1|_{V_2}^{-1}(x)) = d_x f_1(x) \) for \( x \in U_1 \), that is, \( d(h_{12} \circ (id_{U_1} \times \pi_1|_{V_2}^{-1})) = df_1 \) in \( 1 \)-forms on \( U_1 \). Therefore \( h_{12} \circ (id_{U_1} \times \pi_1|_{V_2}^{-1})) = f_1 + c \) for some \( c \in \mathbb{R} \). But \( f_1 \) and \( h_{12} \) are normalized by \( f_1|_{U_1 \cap V_2} = 0 \) and \( h_{12}|_{N_1 \cap N_2} = 0 \), so as \( p_1|_{N_1 \cap N_2} \subset U_1 \cap V_2 \) we see that \( c = 0 \). Hence \( h_{12} \circ (id_{U_1} \times \pi_1|_{V_2}^{-1})) = f_1 \), and the left hand triangle of (2.14) commutes.

Using an analogous argument replacing (2.17) by the equations:

\[
\begin{align*}
N_1 \cap Q &= N_1 \\
N_2 \cap Q &= \Delta_L \\
S \times S &\quad (z(U_1) \times z(V_2)) \\
\end{align*}
\]

we see that the right hand triangle of (2.14) commutes.

Finally, the last part of Proposition 2.2 follows directly from Theorem 1.10(i).

As sketched already in [1], note that the local biholomorphisms \( \pi_1|_{V_2}^{-1}, \pi_2|_{U_1}^{-1} \) coming from polarizations \( \pi_1, \pi_2 \), induce isomorphisms (2.2) between the canonical bundles of the Lagrangian submanifolds. In terms of charts, we have an \( L \)-chart \( (P_1, U_1, f_1, i_1) \), an \( M \)-chart \( (Q_2, V_2, g_2, j_2) \) and an \( LM \)-chart \( (R_{12}, W_{12}, h_{12}, k_{12}) \) induced by \( E_1, E_2, E_3 \times E_3 \) respectively, where \( P_1 = X \cap U_1, Q_2 = X \cap V_2, R_{12} = \{x \in X : (x, x) \in W_{12}\} \). Let us denote the corresponding principal \( Z_2 \)-bundles \( Q_{P_1, U_1, f_1, i_1}, Q_{P_2, V_2, g_2, j_2} \) and \( Q_{R_{12}, W_{12}, h_{12}, k_{12}} \) parametrizing square roots of these isomorphisms of canonical bundles as explained in the introduction of [2].

Note that Proposition 2.2 defined two embeddings \( \Phi_{12} : U_1' \hookrightarrow W_{12} \) and \( \Psi_{12} : V_2' \hookrightarrow W_{12} \) which satisfy all the properties of Definition 1.11 giving embeddings of charts \( \Phi_{12} : (P_1', U_1', f_1', i_1') \hookrightarrow (R_{12}, W_{12}, h_{12}, k_{12}) \) and \( \Psi_{12} : (Q_2', V_2', g_2', j_2') \hookrightarrow (R_{12}, W_{12}, h_{12}, k_{12}) \), where \( (P_1', U_1', f_1', i_1') \) is a subchart of \( (P_1, U_1, f_1, i_1) \), and \( (Q_2', V_2', g_2', j_2') \) is a subchart of \( (Q_2, V_2, g_2, j_2) \) with \( P_1' = Q_2' = R_{12} \).

Thus Definition 1.11 gives isomorphisms of line bundles on \( P_{red} \):

\[
J_{\Phi_{12}} : K_{P_1'}|_{P_{red}} \overset{\cong}{\longrightarrow} \Phi_{12}|_{P_{red}}(K_{W_{12}}^\otimes),
\]

induced by \( \Phi_{12} : (P_1', U_1', f_1', i_1') \hookrightarrow (R_{12}, W_{12}, h_{12}, k_{12}) \), and

\[
J_{\Psi_{12}} : K_{Q_2'}|_{Q_{2, red}} \overset{\cong}{\longrightarrow} \Psi_{12}|_{Q_{2, red}}(K_{W_{12}}^\otimes),
\]
induced by $\Psi_{12} : (Q_2, V'_2, g'_2, j'_2) \hookrightarrow (R_{12}, W_{12}, h_{12}, k_{12})$.

Following Definition 1.11 define $\pi\Phi_{12} : P_{\Phi_{12}} \rightarrow P'_1$, $\pi\Psi_{12} : P_{\Psi_{12}} \rightarrow Q'_2$ to be the principal $\mathbb{Z}_2$-bundles parametrizing square roots of $J_{\Phi_{12}}, J_{\Psi_{12}}$ on $R_{12}$. Then we naturally get isomorphisms of principal $\mathbb{Z}_2$-bundles $\Lambda_{\Phi}$ and $\Lambda_{\Psi}$

$$
\Lambda_{\Phi_{12}} : Q_{R_{12}, W_{12}, h_{12}, k_{12}} \cong P_{\Phi_{12}} \otimes \mathbb{Z}_2 Q_{P_{1}, U_{1,1}, f_{1,1}, i_{1}}|_{R_{12}},
$$

(2.21)

$$
\Lambda_{\Psi_{12}} : Q_{R_{12}, W_{12}, h_{12}, k_{12}} \cong P_{\Psi_{12}} \otimes \mathbb{Z}_2 Q_{Q_{2}, V_{2,2}, g_{2}, j_{2}}|_{R_{12}}.
$$

(2.22)

Thus, we can apply Theorem 1.13 which yields natural isomorphisms of perverse sheaves on $X$:

$$
\Theta_{\Phi_{12}} : \mathcal{PV}_{U_{1,1}, f_{1}} \longrightarrow \Phi_{12}|_{R_{12}}(\mathcal{PV}_{W_{12}, h_{12}}) \otimes \mathbb{Z}_2 P_{\Phi_{12}},
$$

(2.23)

$$
\Theta_{\Psi} : \mathcal{PV}_{V_{2,2}, g_{2}} \longrightarrow \Psi_{12}(\mathcal{PV}_{W_{12}, h_{12}}) \otimes \mathbb{Z}_2 P_{\Psi_{12}},
$$

(2.24)

where $\mathcal{PV}_{U_{1,1}, f_{1}}, \mathcal{PV}_{V_{2,2}, g_{2}}, \mathcal{PV}_{W_{12}, h_{12}}$ are the perverse sheaves of vanishing cycles from 1.11 and if $Q^\bullet$ is a perverse sheaf on $X$ then $Q^\bullet \otimes \mathbb{Z}_2 P_{\Phi_{12}}$ is as in Definition 1.5. Also diagrams 1.21 and 1.28 commute. Now, combining the isomorphisms (2.21)-(2.24) we get isomorphisms

$$
\alpha_{12} = \Theta_{\Phi_{12}} \otimes \Lambda_{\Phi_{12}}^{-1} : (\mathcal{PV}_{U_{1,1}, f_{1}} \otimes \mathbb{Z}_2 Q_{P_{1}, U_{1,1}, f_{1}, i_{1}})|_{R_{12}} \longrightarrow \mathcal{PV}_{W_{12}, h_{12}} \otimes \mathbb{Z}_2 Q_{R_{12}, W_{12}, h_{12}, k_{12}},
$$

(2.25)

$$
\beta_{12} = \Theta_{\Psi_{12}} \otimes \Lambda_{\Psi_{12}}^{-1} : (\mathcal{PV}_{V_{2,2}, g_{2}} \otimes \mathbb{Z}_2 Q_{Q_{2}, V_{2,2}, g_{2}, j_{2}})|_{R_{12}} \longrightarrow \mathcal{PV}_{W_{12}, h_{12}} \otimes \mathbb{Z}_2 Q_{R_{12}, W_{12}, h_{12}, k_{12}},
$$

(2.26)

$$
\beta_{12}^{-1} \circ \alpha_{12} : (\mathcal{PV}_{U_{1,1}, f_{1}} \otimes \mathbb{Z}_2 Q_{P_{1}, U_{1,1}, f_{1}, i_{1}})|_{R_{12}} \longrightarrow (\mathcal{PV}_{V_{2,2}, g_{2}} \otimes \mathbb{Z}_2 Q_{Q_{2}, V_{2,2}, g_{2}, j_{2}})|_{R_{12}}.
$$

(2.27)

### 2.2 Comparing perverse sheaves of vanishing cycles associated to polarizations

Given a complex symplectic manifold $(S, \omega)$ and $L, M$ Lagrangian submanifolds in $S$, define $X$ to be their intersection. From 1.3 locally in the complex analytic topology near a point $x \in X$, we can choose an open set $S' \subset S$ and we can choose a polarization transverse to both $L$ and $M$ such that $S' \cong T^* L$ and $M \cong \Gamma_{df}$ so that $X \cap S' = \text{Crit}(f)$ for a holomorphic function $f$ defined on $U \subseteq L \cap S'$. Thus we get a perverse sheaf of vanishing cycle $\mathcal{PV}_{V_{1}}$. In 2.2 we compared perverse sheaves of vanishing cycles associated to two transverse polarizations. In this section we will investigate about how they behave if we consider four polarizations, pairwise transverse in a 4-cycle. This result will be used in 2.3 to prove Theorem 2.1.

We choose four polarizations $\pi_i : S \rightarrow E_i$ for $i = 1, \ldots, 4$ all transverse to each other except perhaps for the pairs $E_1, E_3$ and $E_2, E_4$. Define $L$-charts $(P_1, U_1, f_1, i_1), (P_3, U_3, f_3, i_3)$ from $\pi_1, \pi_3$ and $M$-charts $(Q_2, V_2, g_2, j_2), (Q_4, V_4, g_4, j_4)$ from $\pi_2, \pi_4$, as in the beginning of 2.2. Define $LM$-charts $(R_{12}, W_{12}, h_{12}, k_{12})$ from $\pi_1, \pi_2$, $(R_{32}, W_{32}, h_{32}, k_{32})$ from $\pi_3, \pi_2$, $(R_{34}, W_{34}, h_{34}, k_{34})$ from $\pi_3, \pi_4$, $(R_{14}, W_{14}, h_{14}, k_{14})$ from $\pi_1, \pi_4$ as in 2.1 with embeddings of charts $\Phi_{12}, \Psi_{12}, \ldots, \Phi_{14}, \Psi_{14}$ from subcharts of $(P_a, U_a, f_a, i_a), (Q_b, V_b, g_b, j_b)$ to $(R_{ab}, W_{ab}, h_{ab}, k_{ab})$.

Similarly to Proposition 2.2 we have the following result:

**Proposition 2.3.** Given a complex symplectic manifold $(S, \omega)$ and $L, M$ Lagrangian submanifolds in $S$, define $X$ to be their intersection. Suppose we have four polarizations $\pi_1, \ldots, \pi_4$, and choose data $(P_a, U_a, f_a, i_a)$ for $a = 1, 3$, $(Q_b, V_b, g_b, j_b)$ for $a = 2, 4$ and $(R_{ab}, W_{ab}, h_{ab}, k_{ab}), \Phi_{ab}, \Psi_{ab}$ for $ab = 12, 32, 34, 14$ as above.

Then there exist an open set $Z$ in $U_1 \times V_2 \times U_3 \times V_4$, a holomorphic function $F : Z \rightarrow \mathbb{C}$, and open neighbourhoods $U'_a$ of $X \cap S_1 \cap S_2 \cap S_3 \cap S_4$ in $U_a$, and $V'_b$ of $X \cap S_1 \cap S_2 \cap S_3 \cap S_4$ in $V_b$, and $W_{ab}$ of $\{(x, z) : x \in X \cap S_1 \cap S_2 \cap S_3 \cap S_4\}$ in $W_{ab}$ for all $a = 1, 3, b = 2, 4$ such that the following diagram commutes:
Moreover, locally in the complex analytic topology $\text{Crit}(F) \cong \text{Crit}(h_{ij}) \cong \text{Crit}(f_i) \cong \text{Crit}(g_j)$ as complex analytic spaces for all $i,j$. In particular, we can choose appropriate coordinate systems under which we can write $F$ as the sum of functions $h_{ij}$ or $f_i$ or $g_j$ and non degenerate quadratic forms, that is they are all stably equivalent to each other in the sense of Theorem 1.10.

Proof. In equation (2.28) there are three kinds of small triangles:

(i) Eight triangles with vertices $U'_a, W'_{ab}, Z$ or $V'_a, W'_{ab}, Z$.

(ii) Eight triangles with vertices $U'_a, W'_{ab}, C$ or $V'_b, W_{ab}, C$.

(iii) Four triangles with vertices $W'_{ab}, Z, C$. 

To show that (2.28) commutes, we must show all these triangles commute. For the triangles of type (i) we can just check this by hand in an elementary way. The triangles of type (ii) commute by a similar proof to Proposition 2.2.

Consider the product symplectic manifold $(S \times \tilde{S} \times S \times \tilde{S}, \omega - \omega + \omega + - \omega)$, where $\tilde{S}$ denotes the symplectic manifold $S$ corresponding to the symplectic form with the opposite sign. Write $p_i : S \times S \times S \rightarrow S$ for the projection to the $i$-th factor. In $S \times \tilde{S} \times S \times \tilde{S}$ consider the Lagrangian submanifolds $N_1 := L \times \Delta_S \times M$ and $N_2 := \Delta_S \times L \times M$. Identify it with $T^*(L \times M \times L \times M)$, and thus $\pi_1 \times \pi_2 \times \pi_3 \times \pi_4$ with $\pi : T^*(L \times M \times L \times M) \rightarrow L \times M \times L \times M$, that is $N_1$ with the zero section, and $N_2 := \Gamma_{dF}$ for a holomorphic function $F : Z \rightarrow C$ for open $Z \subseteq L \times M \times L \times M$ normalized by $F|_{(\pi_1 \times \pi_2 \times \pi_3 \times \pi_4)(N_1 \cap N_2)} = 0$. Consider the submanifolds

$$P_{12} := S \times S \times \Delta^3_S, \quad P_{32} := L \times S \times S \times M, \quad P_{34} := \Delta^2_S \times S \times S, \quad P_{14} := S \times \Delta^3_S \times S.$$

In the same style as the proof of Proposition 2.14 intersect the Lagrangians with these submanifolds. We can either identify $N_1$ with the zero section and $N_2 = \Gamma_{dF}$, or $N_1 = \Gamma_{-dF}$ and $N_2$ with the zero section. We will use both the options. Let us start with the submanifold $P_{12}$, for which we use the second identification.
Consider the following diagram of inclusions and projections in $S \times \tilde{S} \times S \times \tilde{S}$ and $S \times \tilde{S}$:

\[
\begin{array}{c}
N_1 \cap P_{12} = L \times \Delta_M^{34} \\
N_2 \cap P_{12} = N_2 \\
\Delta_S \\
\end{array}
\begin{array}{c}
P_{12} = S \times \tilde{S} \times \Delta_S^{34} \\
L \times M \\
\end{array}
\begin{array}{c}
P_1 = S \times \tilde{S} \times \tilde{S} \\
S \times \tilde{S} \\
\end{array}
\]

(2.29)

Under the local symplectomorphisms $S \times \tilde{S} \cong T^*(L \times M)$ and $S \times \tilde{S} \times S \times \tilde{S} \cong T^*(L \times M \times L \times M)$, equation (2.29) is identified with the diagram:

\[
\begin{array}{c}
z(L) \times z(M) \times z(L) \times z(M) \\
\Gamma_{-\Phi} \end{array}
\begin{array}{c}
\pi_{12} \\
\end{array}
\begin{array}{c}
z(L) \times z(L) \times z(M) \times T^*L \times T^*M \\
T^*L \times T^*M \\
\end{array}
\]

Here $z : L \rightarrow T^*L$, $M \rightarrow T^*M$ are the zero section maps. To understand this, note that $\pi_1 \times \pi_2 \times \pi_2 \times \pi_4$ maps $N_1 \cap P_{12} = L \times \Delta_M^{34}$ to the submanifold $(\pi_L \times \pi_M \times (\pi_3 \circ \pi_M)(L \times M))$ in $L \times M \times L \times M$. Our identification $S \times \tilde{S} \times S \times \tilde{S} \cong T^*(L \times M \times L \times M)$ maps $N_1 \rightarrow \Gamma_{-\Phi}$. Hence the top term $N_1 \cap P_{12}$ in (2.29) is identified with the top term $\Gamma_{-\Phi}|_{(\pi_L \times \pi_M \times (\pi_3 \circ \pi_M))}$ in (2.30). As for (2.17)–(2.18), we see from (2.29)–(2.30) that the triangle of type (iii) in (2.28) with vertices the top centre $L \times M$, and $L \times M \times L \times M$, and $\mathbb{C}$, commutes.

Similarly, taking intersections with the submanifold $P_{14}$ gives a diagram analogous to (2.29):

\[
\begin{array}{c}
N_1 \cap P_{14} = N_1 \\
\Delta_S^{1234} \\
\end{array}
\begin{array}{c}
P_{14} = S \times \Delta_S^{23} \times S \\
L \times M \\
\end{array}
\begin{array}{c}
P_1 = S \times \tilde{S} \times S \\
S \times \tilde{S} \\
\end{array}
\]

Using the first identification, this is identified with the diagram

\[
\begin{array}{c}
z(L) \times z(M) \times z(L) \times z(M) \\
\Gamma_{\Phi} |_{(\pi_L \times (\pi_2 \circ \pi_4) \times (\pi_1 \circ \pi_3))^{\Phi}(L \times M)} \\
\end{array}
\begin{array}{c}
\pi_{14} \\
\end{array}
\begin{array}{c}
z(L) \times T^*M \times T^*L \times z(M) \times T^*(L \times M \times L \times M) \\
T^*(L \times M) \\
\end{array}
\]

Here $\pi_1 \times \pi_2 \times \pi_2 \times \pi_4$ maps $N_2 \cap P_{14} = \Delta_S^{1234}$ to $(\pi_L \times (\pi_2 \circ \pi_4) \times (\pi_1 \circ \pi_3))^{\Phi}(L \times M)$, and we identify $N_2$ with $\Gamma_{\Phi}$, which is how we get the first term on the middle line. From this we see that the triangle of type (iii) in (2.28) with vertices the left hand $L \times M$, and $L \times M \times L \times M$, and $\mathbb{C}$, commutes. The remaining two type (iii) triangles can be shown to commute in a similar way. Hence (2.28) commutes. Finally, the last part of Proposition 2.3 follows directly from Theorem 1.10(i).
In the situation of Proposition 2.3 set $Y = X \cap S_1 \cap S_2 \cap S_3 \cap S_4$. Then following the reasoning of (2.19)–(2.27) which defined the isomorphisms of perverse sheaves $\alpha_{12}, \beta_{12}$ in (2.25)–(2.26), from (2.28) we get a commutative diagram of isomorphisms of perverse sheaves:

Since (2.31) commutes, we deduce that

$$\alpha_{32}^{-1} \circ \beta_{32}^{-1} \circ \alpha_{12}^{-1} = \alpha_{34}^{-1} \circ \beta_{34}^{-1} \circ \alpha_{14}^{-1} : \left( \mathcal{P}V_{U_1, f_1} \otimes \mathbb{Z}_2 Q_{P_3, U_1, f_1, i_1} \right) \rightarrow \left( \mathcal{P}V_{U_3, f_3} \otimes \mathbb{Z}_2 Q_{P_3, U_3, f_3, i_3} \right).$$

Equation (2.32) tells us something important. Suppose we start with polarizations $\pi_1 : S_1 \to E_1$ and $\pi_3 : S_3 \to E_3$ transverse to $L, M$, and use them to define $L$-charts $(P_1, U_1, f_1, i_1)$ and $(P_3, U_3, f_3, i_3)$, and hence perverse sheaves $\mathcal{P}V_{U_1, f_1} \otimes \mathbb{Z}_2 Q_{P_3, U_1, f_1, i_1}$ on $P_1 = X \cap S_1$ and $\mathcal{P}V_{U_3, f_3} \otimes \mathbb{Z}_2 Q_{P_3, U_3, f_3, i_3}$ on $P_3 = X \cap S_3$. We wish to relate these perverse sheaves on the overlap $X \cap S_1 \cap S_3$. To do this, we choose another polarization $\pi_2 : S_2 \to E_2$ transverse to $L, M, \pi_1, \pi_3$, and define an $M$-chart $(Q_2, V_2, g_2, j_2)$ and $LM$-charts $(R_{12}, W_{12}, h_{12}, k_{12})$ and $(R_{32}, W_{32}, h_{32}, k_{32})$. Then as in (2.27), $\alpha_{32}^{-1} \circ \beta_{32}^{-1} \circ \alpha_{12}^{-1}$ provides the isomorphism $\mathcal{P}V_{U_1, f_1} \otimes \mathbb{Z}_2 Q_{P_3, U_1, f_1, i_1} \cong \mathcal{P}V_{U_3, f_3} \otimes \mathbb{Z}_2 Q_{P_3, U_3, f_3, i_3}$ we want on $X \cap S_1 \cap S_2 \cap S_3$. Equation (2.32) shows that this isomorphism is independent of the choice of polarization $\pi_2 : S_2 \to E_2$.

### 2.3 Descent for perverse sheaves

To conclude the proof of Theorem 2.1, we use Theorem 1.4 so in particular a descent argument to glue and get a global perverse sheaf. In this section we adopt the point of view of charts induced by polarizations. This proof follows similar ideas to [BBDJ18, §6.3].

Let $(S, \omega)$ be a complex symplectic manifold and $L, M$ complex Lagrangian submanifolds in $S$, and write $X = L \cap M$, as a complex analytic subspace of $S$. Suppose we are given square roots $K_M^{1/2}, K_M^{1/2}$ for $K_L, K_M$. We may choose a family of polarizations $\pi_a : S_a \to E_a$ which defines a family $\{(R_a, U_a, f_a, i_a) : a \in A\}$ of $L$-charts $(P_a, U_a, f_a, i_a)$ on $X$ such that $\{P_a : a \in A\}$ is an analytic open cover of the analytic space $X$, so that $P_a \equiv \text{Crit}(f_a)$ for holomorphic functions $f_a : U_a \to \mathbb{C}$, and $U_a$ complex manifolds (Lagrangians), and $i_a : P_a \hookrightarrow U_a$ closed embeddings.

Then for each $a \in A$ we have a perverse sheaf

$$i_a^*(\mathcal{P}V_{U_a, f_a}) \otimes \mathbb{Z}_2 Q_{P_a, U_a, f_a, i_a} \in \text{Perv}(R_a),$$

for $Q_{P_a, U_a, f_a, i_a}$ the principal $\mathbb{Z}_2$ bundle defined in (2) point (i) parametrizing choices of square roots of canonical bundles $K_L^{1/2} \cong K_M^{1/2}$ which square to isomorphisms (2.2). As explained already in the introduction of
the proof is to use Theorem 1.4(ii) to glue the perverse sheaves (2.33) on the analytic open cover \( \{ P_a : a \in A \} \) to get a global perverse sheaf \( P_{L,M}^* \) on \( X \).

We already know from Proposition 2.22 that, given an \( L \)-chart \((P,U,f,i)\) and an \( M \)-chart \((Q,V,g,j)\) we have the isomorphism (2.11), which we recall here:

\[
\beta^{-1} \circ \alpha : (i^*(\mathcal{PV}_{U,f}^*) \otimes Q_{P,U,f,i})|_R \xrightarrow{\sim} (j^*(\mathcal{PV}_{V,g}^*) \otimes Q_{Q,V,g,j})|_R,
\]

that is, an isomorphism of perverse sheaves from \( L \)-charts and \( M \)-charts in \( \text{Perv}(P \cap Q) \).

Now, to develop our program, we have to show that if \((P_a,U_a,f_a,i_a)\) and \((P_b,U_b,f_b,i_b)\) are \( L \)-charts, then we have a canonical isomorphism

\[
\delta_{ab} : (i^*_a(\mathcal{PV}_{U_a,f_a}^*) \otimes Q_{P_a,U_a,f_a,i_a})|_{P_a \cap P_b} \xrightarrow{\sim} (i^*_b(\mathcal{PV}_{U_b,f_b}^*) \otimes Q_{P_b,U_b,f_b,i_b})|_{P_a \cap P_b}
\]

with the property that for any \( M \)-chart \((Q,V,g,j)\) coming from \( \tilde{\pi} : \tilde{S} \rightarrow F \) transverse to \( \pi_a \) and \( \pi_b \), we have

\[
\delta_{ab}|_{P_a \cap P_b \cap Q} = \alpha^{-1}_{U_b,f_b,W',h'}|_{P_a \cap P_b \cap Q} \circ \beta_{W',h',V,g}|_{P_a \cap P_b \cap Q} \circ \beta^{-1}_{W,h,V,g}|_{P_a \cap P_b \cap Q} \circ \alpha_{U_a,f_a,W,h}|_{P_a \cap P_b \cap Q}.
\]

To prove this, we first use Proposition 2.22 which provides an associativity result as in (2.23) or (2.22). In particular, it shows that if \((Q',V',g',j')\) is another such \( M \)-chart, then

\[
\alpha^{-1}_{U,b,f,b,W',h'}|_{P_a \cap P_b \cap Q} \circ \beta_{W',h',V,g}|_{P_a \cap P_b \cap Q} \circ \beta^{-1}_{W,h,V,g}|_{P_a \cap P_b \cap Q} \circ \alpha_{U,a,f,a,W,h}|_{P_a \cap P_b \cap Q} = \alpha^{-1}_{U_a,f_a,W',h'}|_{P_a \cap P_b \cap Q} \circ \beta_{W',h',V,g}|_{P_a \cap P_b \cap Q} \circ \beta^{-1}_{W,h,V,g}|_{P_a \cap P_b \cap Q} \circ \alpha_{U_b,f_b,W,h}|_{P_a \cap P_b \cap Q}.
\]

Fix two charts \((P_a,U_a,f_a,i_a)\) and \((P_b,U_b,f_b,i_b)\), and choose a family \( \{(Q_c,V_c,g_c,j_c) : c \in I\} \) of \( M \)-charts \((Q_c,V_c,g_c,j_c)\) on \( X \) transverse to both \((P_a,U_a,f_a,i_a)\) and \((P_b,U_b,f_b,i_b)\), such that \( \{Q_c : c \in I\} \) is an analytic open cover of \( P_a \cap P_b \). Then, we can use the sheaf property of morphisms of perverse sheaves in the sense of Theorem 1.4 to get \( \delta_{ab} \) as in (2.35) by gluing

\[
\alpha^{-1}_{U_b,f_b,W',h'}|_{P_a \cap P_b \cap Q} \circ \beta_{W',h',V,g}|_{P_a \cap P_b \cap Q} \circ \beta^{-1}_{W,h,V,g}|_{P_a \cap P_b \cap Q} \circ \alpha_{U_a,f_a,W,h}|_{P_a \cap P_b \cap Q}
\]

on the open cover \( \{Q_c : c \in I\} \). Also, \( \delta_{ab} \) satisfy (2.33) for all \((Q,V,g,j)\), and this is independent of choice of \((Q,V,g,j)\). This is because we can run the construction above with the family \( \{(P_a,U_a,f_a,i_a) : a \in A\} \cup \{(Q,V,g,j)\} \), yielding the same result.

Moreover, on \( P_a \cap P_b \cap P_c \), we have \( \delta_{bc} \circ \delta_{ab} = \delta_{ac} \). This is because, given locally a polarization \( \tilde{\pi} : \tilde{S} \rightarrow F \) transverse to all of \( \pi_a \), \( \pi_b \), \( \pi_c \), then on \( P_a \cap P_b \cap P_c \cap Q \), we can easily check that

\[
\gamma_{bc} \circ \gamma_{ab}|_{P_a \cap P_b \cap P_c \cap Q} = (\alpha^{-1}_{U_b,f_b,W',h'}|_{P_a \cap P_b \cap P_c \cap Q} \circ \beta_{W',h',V,g}|_{P_a \cap P_b \cap P_c \cap Q} \circ \beta^{-1}_{W,h,V,g}|_{P_a \cap P_b \cap P_c \cap Q} \circ \alpha_{U_a,f_a,W,h}|_{P_a \cap P_b \cap P_c \cap Q})
\]

\[
= (\alpha^{-1}_{U_b,f_b,W',h'}|_{P_a \cap P_b \cap P_c \cap Q}) \circ \beta_{W',h',V,g}|_{P_a \cap P_b \cap P_c \cap Q} \circ \beta^{-1}_{W,h,V,g}|_{P_a \cap P_b \cap P_c \cap Q} \circ \alpha_{U_a,f_a,W,h}|_{P_a \cap P_b \cap P_c \cap Q}
\]

As we can cover \( P_a \cap P_b \cap P_c \) by such open \( P_a \cap P_b \cap P_c \cap Q \), we deduce that \( \gamma_{bc} \circ \gamma_{ab} = \gamma_{ac} \) by the sheaf property of morphisms of perverse sheaves in the sense of Theorem 1.4.

In conclusion, we have an open cover of \( X \) by \( L \)-charts \((P_a,U_a,f_a,i_a)\), and isomorphisms (2.34), satisfying \( \gamma_{bc} \circ \gamma_{ab} = \gamma_{ac} \). By stack property of perverse sheaves in the sense of Theorem 1.4(ii), we get that there exists \( P_{L,M}^* \) in \( \text{Perv}(X) \), unique up to canonical isomorphism, with isomorphisms

\[
\omega_{P_a,U_a,f_a,i_a} : P_{L,M}^*|_{P_a} \xrightarrow{\sim} i^*_a(\mathcal{PV}_{U_a,f_a}^*) \otimes \mathbb{Z}_2 Q_{P_a,U_a,f_a,i_a}
\]

as in (2.24) for each \( a \in A \), with \( \gamma_{ab} \circ \omega_{P_a,U_a,f_a,i_a}|_{P_a \cap P_b} = \omega_{P_b,U_b,f_b,i_b}|_{P_a \cap P_b} \) for all \( a,b \in A \). Also, \( \omega_{P_a,U_a,f_a,i_a} \) with \((P_a,U_a,f_a,i_a)\) in place of \((P,U,f,i)\) define isomorphisms \( \Sigma_{L,M}|_{P_a} \), \( T_{L,M}|_{P_a} \), for each \( a \in A \). The prescribed values for \( \Sigma_{L,M}|_{P_a} \), \( T_{L,M}|_{P_a} \) agree when restricted to \( P_a \cap P_b \) for all \( a,b \in A \).
Hence, Theorem 1.4(i) gives unique isomorphisms $\Sigma_{L,M}, T_{L,M}$ in 2.1 such that (2.5)–(2.6) commute with $(P_a, U_a, f_a, i_a)$ in place of $(P, U, f, i)$ for all $a \in A$.

Also, the whole construction is independent of the choice of the family of $L$-charts and polarizations. This is because we can suppose $\{(P_a, U_a, f_a, i_a) : a \in A\}$ and $\{(\tilde{P}_a, \tilde{U}_a, \tilde{f}_a, \tilde{i}_a) : a \in A\}$ are alternative choices above, yielding $P^*_L, T^*_L, \Sigma^*_L, T^*_L, \tilde{P}_L, \tilde{T}_L, \tilde{\Sigma}_L, \tilde{T}_L$. Then applying the same construction to the family $\{(P_a, U_a, f_a, i_a) : a \in A\}$ we get $\tilde{P}^*_L, \tilde{P}^*_M$, we have canonical isomorphisms $P^*_L \cong \tilde{P}^*_L \cong P^*_M$, which identify $\Sigma^*_L, T^*_L$ with $\tilde{\Sigma}_L, \tilde{T}_L$. Thus $P^*_L, \Sigma^*_L, T^*_L$ are independent of choices up to canonical isomorphism.

3 Analytic d-critical locus structure on complex Lagrangian intersections

Pantev et al. [PTVV] show that derived intersections $L \cap M$ of algebraic Lagrangians $L, M$ in an algebraic symplectic manifold $(S, \omega)$ have $(-1)$-shifted symplectic structures, so that Theorem 6.6 in [BBDJS] gives them the structure of algebraic d-critical loci. Here, we will prove a complex analytic version of this. Theorem 3.1 states that the Lagrangian intersection $L \cap M$ of (oriented) complex Lagrangians $L, M$ has the structure of an (oriented) complex analytic d-critical locus. Notice at this point that we could have then used [BBDJS Thm 6.9] to define a perverse sheaf $P^*_L \cap M$ on $L \cap M$, instead of going through Theorem 2.1 in §2 but we wanted to provide a clear and direct proof about how to glue perverse sheaves on complex Lagrangian intersections in a complex analytic setup, and using only classical and symplectic geometry. Note also that we cannot prove Theorem 3.1 by going via [PTVV], as they do not do a complex analytic version.

Here is the result of the section.

**Theorem 3.1.** Suppose $(S, \omega)$ is a complex symplectic manifold, and $L, M$ are (oriented) complex Lagrangian submanifolds in $S$. Then the intersection $X = L \cap M$, as a complex analytic subspace of $S$, extends naturally to a (oriented) complex analytic d-critical locus $(X, s)$. The canonical bundle $K_{X,s}$ in the sense of Theorem 3.5 in [PTVV] is naturally isomorphic to $K_{L|X|s} \otimes K_{M|X|s}$.

Theorem 3.1 will be proved in §3.2 while in §3.3 we recall some material from Joyc1.

3.1 Background material on d-critical loci

Here are some of the main definitions and results on d-critical loci, from Joyce [Joyc1 Ths 2.1, 2.13, 2.21 & Defs 2.3, 2.11, 2.23].

The key idea of this section, d-critical loci, is explained in Definition 3.3 below. As a preliminary, we need to associate a sheaf $S_X$ to each complex analytic space $X$, such that (very roughly) sections of $S_X$ parametrize different ways of writing $X$ as Crit$(f)$ for $U$ a complex manifold and $f : U \to \mathbb{C}$ holomorphic.

**Theorem 3.2.** Let $X$ be a complex analytic space. Then there exists a sheaf $S_X$ of $\mathbb{C}$-vector spaces on $X$, unique up to canonical isomorphism, which is uniquely characterized by the following two properties:

(i) Suppose $U$ is a complex manifold, $R$ is an open subset in $X$, and $i : R \hookrightarrow U$ is an embedding of $R$ as a closed complex analytic subspace of $U$. Then we have an exact sequence of sheaves of $\mathbb{C}$-vector spaces on $R$:

$$0 \to I_{R,U} \to i^{-1}(O_U) \to O_X|_R \to 0,$$

where $O_X, O_U$ are the sheaves of holomorphic functions on $X, U$, and $i^*$ is the morphism of sheaves of $\mathbb{C}$-algebras on $R$ induced by $i$, which is surjective as $i$ is an embedding, and $I_{R,U} = \text{Ker}(i^*)$ is the sheaf of ideals in $i^{-1}(O_U)$ of functions on $U$ near $i(R)$ which vanish on $i(R)$.

There is an exact sequence of sheaves of $\mathbb{C}$-vector spaces on $R$:

$$0 \to S_X|_R \to i^{-1}(O_U) \to I_{R,U} \to i^{-1}(T^*U) \to 0,$$

where $d$ maps $f + I_{R,U} \mapsto df + I_{R,U} \cdot i^{-1}(T^*U)$.
(ii) Let $R, U, i, t_{R, U}$ and $S, V, j, t_{S, V}$ be as in (i) with $R \subseteq S \subseteq X$, and suppose $\Phi : U \to V$ is holomorphic with $\Phi \circ i = j|_R$ as a morphism of complex analytic spaces $R \to V$. Then the following diagram of sheaves on $R$ commutes:

\[
\begin{array}{c}
0 \to S_X|_R \xrightarrow{i_{S, V}|_R} \frac{j^{-1}(\mathcal{O}_V)}{I^2_{S, V}}|_R \xrightarrow{d} \frac{j^{-1}(T^*V)}{I_{S, V} \cdot j^{-1}(T^*V)}|_R \\
0 \to S_X|_R \xrightarrow{i_{R, U}} \frac{i^{-1}(\mathcal{O}_U)}{I^2_{R, U}} \xrightarrow{d} \frac{i^{-1}(T^*U)}{I_{R, U} \cdot i^{-1}(T^*U)}.
\end{array}
\]

Here $\Phi : U \to V$ induces $\Phi^\sharp : \Phi^{-1}(\mathcal{O}_V) \to \mathcal{O}_U$ on $U$, so we have

\[i^{-1}(\Phi^\sharp) : j^{-1}(\mathcal{O}_V)|_R = i^{-1} \circ \Phi^{-1}(\mathcal{O}_V) \to i^{-1}(\mathcal{O}_U),\]

a morphism of sheaves of $\mathbb{C}$-algebras on $R$. As $\Phi \circ i = j|_R$, equation (3.4) maps $I_{S, V}|_R \to I_{R, U}$, and so maps $I^2_{S, V}|_R \to I^2_{R, U}$. Thus (3.4) induces the morphism in the second column of (3.3). Similarly, $d\Phi : \Phi^{-1}(T^*V) \to T^*U$ induces the third column of (3.3).

There is a natural decomposition $S_X = S_X^0 \oplus \mathbb{C}_X$, where $\mathbb{C}_X$ is the constant sheaf on $X$ with fibre $\mathbb{C}$, and the subsheaf $S_X^0 \subset S_X$ is the kernel of the composition

\[
S_X \xrightarrow{\beta_X} O_X \xrightarrow{\alpha_X} O_X^{\text{red}},
\]

with $X^{\text{red}}$ the reduced complex analytic subspace of $X$, and $i_X : X^{\text{red}} \to X$ the inclusion.

Thus, if we can write $X = \text{Crit}(f)$ for $f : U \to \mathbb{C}$ holomorphic, then we obtain a natural section $s \in H^0(S_X)$. Essentially $s = f + I^2_{A_f}$, where $I_{A_f} \subset O_U$ is the ideal generated by $df$. Note that $f|_X = f + I_{A_f}$, so $s$ determines $f|_X$. Basically, $s$ remembers all of the information about $f$ which makes sense intrinsically on $X$, rather than on the ambient space $U$.

We can now define $d$-critical loci:

**Definition 3.3.** A complex analytic $d$-critical locus is a pair $(X, s)$, where $X$ is a complex analytic space, and $s \in H^0(S_X)$ for $S_X^0$ as in Theorem 3.2 satisfying the condition that for each $x \in X$, there exists an open neighbourhood $R$ of $x$ in $X$, a complex manifold $U$, a holomorphic function $f : U \to \mathbb{C}$, and an embedding $i : R \to U$ of $R$ as a closed analytic subdomain of $U$, such that $i(R) = \text{Crit}(f)$ as complex analytic subspaces of $U$, and $i_{R, U}(s|_R) = i^{-1}(f) + I^2_{R, U}$. We call the quadruple $(R, U, i, f)$ a critical chart on $(X, s)$.

Let $(X, s)$ be a complex analytic $d$-critical locus, and $(R, U, f, i)$ a critical chart on $(X, s)$. Let $U' \subseteq U$ be open, and set $R' = i^{-1}(U') \subseteq R$, $i' = i|_{R'} : R' \to U'$, and $f' = f|_{U'}$. Then $(R', U', f', i')$ is a critical chart on $(X, s)$, and we call it a subchart of $(R, U, f, i)$. As a shorthand we write $(R', U', f', i') \subseteq (R, U, f, i)$.

Let $(R, U, f, i), (S, V, g, j)$ be critical charts on $(X, s)$, with $R \subseteq S \subseteq X$. An embedding of $(R, U, f, i)$ in $(S, V, g, j)$ is a locally closed embedding $\Phi : U \to V$ such that $\Phi \circ i = j|_R$ and $f = g \circ \Phi$. As a shorthand we write $\Phi : (R, U, f, i) \to (S, V, g, j)$. If $\Phi : (R, U, f, i) \to (S, V, g, j)$ and $\Psi : (S, V, g, j) \to (T, W, h, k)$ are embeddings, then $\Psi \circ \Phi : (R, U, i, e) \to (T, W, h, k)$ is also an embedding.

**Theorem 3.4.** Let $(X, s)$ be a $d$-critical locus, and $(R, U, f, i), (S, V, g, j)$ be critical charts on $(X, s)$. Then for each $x \in R \cap S \subseteq X$ there exist subcharts $(R', U', f', i') \subseteq (R, U, f, i), (S', V', g', j') \subseteq (S, V, g, j)$ with $x \in R' \cap S' \subseteq X$, a critical chart $(T, W, h, k)$ on $(X, s)$, and embeddings $\Phi : (R', U', f', i') \to (T, W, h, k)$, $\Psi : (S', V', g', j') \to (T, W, h, k)$.

**Theorem 3.5.** Let $(X, s)$ be a complex analytic $d$-critical locus, and $X^{\text{red}} \subseteq X$ the associated reduced complex analytic space. Then there exists a holomorphic line bundle $K_{X, s}$ on $X^{\text{red}}$ which we call the canonical bundle of $(X, s)$, which is natural up to canonical isomorphism, and is characterized by the following properties:

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(i) If \((R, U, f, i)\) is a critical chart on \((X, s)\), there is a natural isomorphism
\[
\iota_{R, U, f, i} : K_{X,s}|_{R^{\text{red}}} \rightarrow i^*(K_U^{\otimes 2})|_{R^{\text{red}}},
\]
where \(K_U = \Lambda^{\dim U} T^* U\) is the canonical bundle of \(U\) in the usual sense.

(ii) Let \(\Phi : (R, U, f, i) \rightarrow (S, V, g, j)\) be an embedding of critical charts on \((X, s)\), and \(N_{UV}, q_{UV}\) be as in Proposition \(\ref{prop:critical-chart} \iota\) and set \(n = \dim V - \dim U\). Taking top exterior powers in the dual of \(\iota^*\) and pulling back to \(R^{\text{red}}\) using \(i^*\) gives an isomorphism of line bundles on \(R^{\text{red}}\)
\[
\rho_{UV} : i^*(K_U^\vee) \otimes i^*(\Lambda^n N_{UV}^\vee)|_{R^{\text{red}}} \xrightarrow{\sim} j^*(K_V)|_{R^{\text{red}}},
\] (3.6)
As \(q_{UV}\) is a nondegenerate quadratic form on \(i^*(N_{UV})\), its determinant \(\det(q_{UV})\) is a nonvanishing section of \(i^*(\Lambda^n N_{UV}^\vee)^{\otimes 2}\). Then the following diagram of isomorphisms of line bundles on \(R^{\text{red}}\) commutes:
\[
\begin{array}{ccc}
K_{X,s}|_{R^{\text{red}}} & \xrightarrow{\iota_{R, U, f, i}} & i^*(K_U^{\otimes 2})|_{R^{\text{red}}} \\
\iota_{S, U, g, j}|_{R^{\text{red}}} & \downarrow \rho_{UV} & \downarrow \text{id}_{i^*(K_U^{\otimes 2})} \otimes \det(q_{UV})|_{R^{\text{red}}} \\
& & j^*(K_V^\otimes)|_{R^{\text{red}}} \\
\end{array}
\]

Definition 3.6. Let \((X, s)\) be a complex analytic \(d\)-critical locus, and \(K_{X,s}\) its canonical bundle from Theorem \(\ref{thm:critical-chart} \iota\). An orientation on \((X, s)\) is a choice of square root line bundle \(K_{X,s}^{1/2}\) for \(K_{X,s}\) on \(X^{\text{red}}\). That is, an orientation is a line bundle \(L\) on \(X^{\text{red}}\), together with an isomorphism \(L^{\otimes 2} = L \otimes L \cong K_{X,s}\). A \(d\)-critical locus with an orientation will be called an oriented \(d\)-critical locus.

3.2 Proof of Theorem 3.1

Let \((S, \omega)\) be a complex symplectic manifold, and \(L, M \subset S\) two complex Lagrangian submanifolds of \(S\). Given the complex analytic space \(X = L \cap M\), we must construct a section \(s \in H^0(S_X^{\alpha})\) such that \((X, s)\) is a complex analytic \(d\)-critical locus. We use notation from \(\ref{notation:critical-chart}\) and in particular the notions of \(L\)-chart, \(M\)-chart, and \(LM\)-chart.

We claim that there is a unique \(d\)-critical structure \(s\) on \(X\), such that

1. every \(L\)-chart \((P, U, f, i)\) from a polarization \(\pi_1 : S_1 \rightarrow E_1\) transverse to \(L, M\) is a critical chart on \((X, s)\);
2. every \(M\)-chart \((Q, V, g, j)\) from a polarization \(\pi_2 : S_2 \rightarrow E_2\) transverse to \(L, M\) is a critical chart on \((X, s)\).

where \(L\)-charts and \(M\)-charts are defined using transverse polarizations. To show this we note that the \(L\)-chart \((P, U, f, i)\) determines a \(d\)-critical structure \(s_P\) on \(P\), and similarly the \(M\)-chart \((Q, V, g, j)\) determines a \(d\)-critical structure \(s_Q\) on \(Q\).

Next, for given \(L\)-charts and \(M\)-charts, we use the \(LM\)-charts \((R, W, h, k)\) and Proposition \(\ref{prop:critical-chart} \iota\) to show that \(s_P|_{P \cap Q} = s_Q|_{P \cap Q}\).

Then, we choose a locally finite cover of \(L\)-charts \((P_a, U_a, f_a, i_a)\) for \(a \in A\) covering \(X\), from polarizations transverse to \(L, M\). We choose \(M\)-charts \((Q_b, U_b, f_b, i_b)\) for \(b \in B\) covering \(X\), from polarizations transverse to \(L, M\) and all polarizations used to define the \((P_a, U_a, f_a, i_a)\). Then we get:
\[
s_{P_a}|_{P_a \cap Q_b} = s_{Q_b}|_{P_a \cap Q_b}
\]
for all \(a, b\). Hence \(s_{P_a}|_{P_a \cap P_{a'} \cap Q_b} = s_{P_{a'}}|_{P_a \cap P_{a'} \cap Q_b}\) for all \(a, a' \in A\), \(b \in B\). As the \(Q_b\) cover \(X\), we have \(s_{P_a}|_{P_a \cap P_{a'}} = s_{P_{a'}}|_{P_a \cap P_{a'}}\) for all \(a, a' \in A\).

So there exists a unique section \(s\) with \(s|_{P_a} = s_{P_a}\) for all \(a \in A\), as \(S_X^{\alpha}\) is a sheaf. Finally, following the same technique of \(\ref{notation:critical-chart}\) the construction is independence of choices.

For the second part of the theorem, let \((P, U, f, i)\), be a critical chart on \((X, s)\). Then Theorem 3.5(i) gives a natural isomorphism
\[
\iota_{P, U, f, i} : K_{X,s}|_{P^{\text{red}}} \rightarrow i^*(K_U^{\otimes 2})|_{P^{\text{red}}}.
\] (3.7)
Using (22), note that $K^p_L \cong K_L \otimes K_M$, as the polarization $\pi$ identifies both $L, M$ with $U$ locally, giving isomorphisms $K_U|_X \cong K_L|_X \cong K_M|_X$. Now comparing with (1.30), we get $K_X,s|_{\text{red}} \cong \det(\mathcal{L}_X)|_{\text{red}}$ for each $(P, U, f, i)$, critical chart on $(X, s)$. Comparing two critical charts, one can show that the canonical isomorphisms constructed above from two such charts are equal on the overlap. Therefore the isomorphisms glue to give a global canonical isomorphism $K_{X,s} \cong \det(\mathcal{L}_X)|_{X,\text{red}}$. This completes the proof of Theorem 3.1.

Note that we did not use $\mathcal{L}MLM$ charts and Proposition 2.3 in (22). That is because we are constructing a section $s$ of a sheaf, (effectively, a morphism in a category), rather than a (perversive) sheaf (an object in a category), so basically we only have to go up to double overlaps, not triple overlaps.

4 Relation with other works and further research

In this section we briefly discuss related work in the literature, and outline some ideas for future investigation.

The work of Behrend and Fantechi [BeFa]

The main inspiration for the present work was a result by Behrend and Fantechi [BeFa] in 2006. Their project aims to construct and study Gerstenhaber and Batalin–Vilkovisky structures on Lagrangian intersections. They consider a pair $\mathcal{P}(U, f, i)$, critical chart, so basically we only have to go up to double overlaps, not triple overlaps.

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The work of Kashiwara and Schapira [KaSc2]

Kashiwara and Schapira [KaSc3] develop a theory of deformation quantization modules, or DQ-modules, on a complex symplectic manifold \((S, \omega)\), which roughly may be regarded as symplectic versions of \(\mathcal{D}\)-modules. Holonomic DQ-modules \(\mathcal{D}^\bullet\) are supported on (possibly singular) complex Lagrangians \(L\) in \(S\). If \(L\) is a smooth, closed, complex Lagrangian in \(S\) and \(K_L^{1/2}\) a square root of \(K_L\), D’Agnolo and Schapira [DASc] show that there exists a simple holonomic DQ-module \(\mathcal{D}^\bullet\) supported on \(L\).

If \(\mathcal{D}^\bullet, \mathcal{E}^\bullet\) are simple holonomic DQ-modules on \(S\) supported on smooth Lagrangians \(L, M\), then Kashiwara and Schapira [KaSc2] show that \(R \mathcal{H}om(\mathcal{D}^\bullet, \mathcal{E}^\bullet)[n]\) is a perverse sheaf on \(S\) over the field \(\mathbb{C}(\!(\hbar)\!)\), supported on \(X = L \cap M\). Pierre Schapira explained to the authors of [BBDJS] how to prove that \(R \mathcal{H}om(\mathcal{D}^\bullet, \mathcal{E}^\bullet)[n] \cong \mathcal{P}_{L,M}\), when \(\mathcal{P}_{L,M}\) is defined over the base ring \(A = \mathbb{C}(\!(\hbar)\!)\).

The work of Baranovsky and Ginzburg [BaGi]

Apart from the mistake in the proof, Behrend and Fantechi’s work [BeFa] gives a new important understanding of a rich structure on Lagrangian intersection, investigated also by Baranovsky and Ginzburg [BaGi], who obtained analogous results for any pair of smooth coisotropic submanifolds \(L, M\) of arbitrary smooth Poisson algebraic varieties \(S\) considering first order deformations of the structure sheaf \(\mathcal{O}_S\) to a sheaf of non-commutative algebras and of the structure sheaves \(\mathcal{O}_L\) and \(\mathcal{O}_M\) to sheaves of right and left modules over the deformed algebra. The construction is canonically defined and it is independent of the choices of deformations involved.

The proof of their main result, Theorem 4.3.1 in [BaGi], shows that sometimes the Gerstenhaber and Batalin–Vilkovisky structures on Tor or Ext are well-defined globally. In their construction, this is the case, for instance, whenever in the setting of the proof of [BaGi] Thm 4.3.1, some cocycles are defined globally.

The work of Kapustin and Rozansky [KR2]

In [KR2], Kapustin and Rozansky study boundary conditions and defects in a three-dimensional topological sigma-model with a complex symplectic target space, the Rozansky-Witten model. It turns out that this model has a deep relation with the problem of deformation quantization of the derived category of coherent sheaves on a complex manifold, regarded as a symmetric monoidal category, and in particular with categorified algebraic geometry in the sense of [BFN, TV]. Namely, in the case when the target space of the Rozansky-Witten model has the form of the cotangent bundle \(T^*Y\), where \(Y\) is a complex manifold, the 2-category of boundary conditions is very similar to the 2-category of derived categorical sheaves on \(Y\).

More precisely, given a complex symplectic manifold \((S, \omega)\), Kapustin and Rozansky conjecture the existence of an interesting 2-category, with objects complex Lagrangians \(L\) with \(K_L^{1/2}\), such that \(\text{Hom}(L, M)\) is a \(\mathbb{Z}_2\)-periodic triangulated category, and if \(L \cap M\) is locally modeled on \(\text{Crit}(f : U \to \mathbb{C})\) for \(f : U \to \mathbb{C}\) a holomorphic function on a manifold \(U\), then \(\text{Hom}(L, M)\) is locally modeled on the matrix factorization category \(MF(U, f)\) as in [Orlov].

Matrix factorization and second categorification

It would be interesting to construct a sheaf of \(\mathbb{Z}_2\)-periodic triangulated categories on Lagrangian intersection, which, in the language of categorification, would yield a second categorification of the intersection numbers, the first being given by the hypercohomology of the perverse sheaf constructed in the present work.

Also, this construction should be compatible with the Gerstenhaber and Batalin–Vilkovisky structures in the sense of [BaGi] Conj. 1.3.1.
Fukaya category for derived Lagrangian and d-critical loci

It would be interesting to extend Theorem 3.1 to a class of ‘derived Lagrangians’ in \((S, \omega)\).

Given a pair \(L, M\), of derived complex Lagrangian submanifolds in the sense of \([PTVV]\) in a complex symplectic manifold \((S, \omega)\), with \(\dim \mathbb{C} S = 2n\), Joyce conjectures that there should be some kind of approximate comparison

\[
H^k(P^*_{L,M}) \approx HF^{k+n}(L, M),
\]

where \(HF^*(L, M)\) is the Lagrangian Floer cohomology of Fukaya, Oh, Ohta and Ono \([FOOO]\). Some of the authors of \([BBDJS]\) are working on defining a ‘Fukaya category’ of (derived) complex Lagrangians in a complex symplectic manifold, using \(H^*(P^*_{L,M})\) as morphisms. See \([BBDJS]\) for a more detailed discussion.

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