Community Detection via Facility Location

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In this paper we apply theoretical and practical results from facility location theory to the problem of community detection in networks. The result is an algorithm that computes bounds on a minimization variant of local modularity. We also define the concept of an edge support and a new measure of the goodness of community structures with respect to this concept. We present preliminary results and note that our methods are massively parallelizable.

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I. INTRODUCTION

In this paper, we apply results from facility location theory to community detection. Leveraging recent developments in both fields, we compute a weighting of the input graph that represents pertinent information for community detection algorithms. We show how to compute this weighting efficiently using techniques from facility location theory. We can interpret the weights as probabilities and randomly sample over a space of good community assignments. Computing the weights involves solving a linear program (LP) [12] that has special structure. Solvers for this special kind of LP require only linear space and linear time per iteration. Furthermore, this solution strategy is amenable to massive parallelism.

We also give new measures for evaluating the quality of community assignments and show that our algorithms provide a provable lower bound on solution quality with respect to one of these. We demonstrate empirically that another of our measures is complementary to modularity, and that optimizing based on this new measure better resolves small communities in large graphs and better matches common sense community structures in familiar datasets. Thus, we make four contributions in this work: we demonstrate a connection between community detection and facility location; we use that connection to compute lower bounds on solution quality; we show how to compute new measures for the goodness of community structure that contrast with modularity; and we apply massively parallelizable methods to compute these bounds and measures.

II. BACKGROUND

Newman and Girvan’s concept of modularity [16] is now ubiquitous in the community detection literature. There are several variations on this concept, such as [4, 7, 10, 12, 20], and many heuristics to optimize the original concept and these variations, e.g. [17, 8]. In order to compute community structures with good modularity in large network instances, researchers commonly use one of two approaches: greedy heuristics, such as [8] and [18], and metaheuristic approaches, such as simulated annealing [17]. Agarwal and Kempe [11] applied mathematical programming to the problem of maximizing modularity, resulting in an algorithm to compute upper bounds for that measure.

We present an alternative that employs results from the vast facility location literature to community detection. We model a variation of modularity as an uncapacitated facility location problem (to be defined below), and employ the simple and powerful Volume algorithm [2] to solve the problem. Mulvey and Crowder [14] used similar techniques, applying older subgradient methods, to solve p-median problems that approximately cluster points in n-dimensional space.

We first observe that specializing a minimization version of the modularity problem produces an uncapacitated facility location problem. We then discuss its solutions and the interpretation and use of its results.

III. STRONGLY-LOCAL MODULARITY (SLM)

Girvan and Newman define the modularity \( Q \) for a graph \( G \) as follows: \( Q = \sum_s (e_{ss} - a_s^2) \), where \( s \) is a community in the domain \( \{1 \ldots q\} \), \( e_{ss} \) is the fraction of \( E(G) \) (the edge set of the graph) that connects a node in community \( r \) to one in community \( s \), and \( a_s \) is the fraction of edges that have at least one endpoint in \( s \) \( (a_s = \sum_r e_{rs}) \). Squaring \( a_s \) gives the probability that an edge would have both endpoints in community \( s \) in a random graph with the same endpoint degree distribution.

Modularity is a way to measure the quality of community assignment: it rewards communities that are better connected than would be expected in a random graph reflecting the endpoint degree distribution.

Now consider a simple variation of the modularity concept: \( Q^- = \sum_s (1 - (e_{ss} - a_s^2)) \). Minimizing \( Q^- \) is similar to, though not identical to, maximizing \( Q \). Basic algebra shows that a community assignment minimizing \( Q^- \) has at most as many communities as one that maximizes \( Q \), and this is typically a strict inequality.

It is well-known that community assignments of maximum modularity fail to resolve small communities in large graphs [8]. Reflecting on this work, it would seem that the \( Q^- \) measure will compound this problem by resolving even fewer communities. However, we provide a remedy via a further modification described below, and
our switching of optimization sense will prove useful.

Muff, Rao, and Caflisch [13] define the local modularity to be the same as modularity, except that the denominator in the fractions \( e_{rs} \) are the numbers of edges in a cluster’s “neighborhood,” defined to be itself and all neighboring clusters. We use a metric that also focuses on local structure, but is even more restrictive, requiring no information about the structure of neighboring communities. We define a strongly local community to consist of a single representative node and all of its immediate neighbors, i.e., a full community of radius one. Let \( Q_s = e_{ss} - a_s^2 \). We can compute this measure for any strongly local community without knowing any community assignments other than the vertices in \( s \). Ignoring algorithmic details, we need only know the number of triangles in the strongly local community and the degree of each node.

Now we give the key definition that allows us to model the problem using facility location theory. Let

\[
\tilde{Q}_s = \begin{cases} 
Q_s & \text{if } s \text{ is a strongly local community} \\
0 & \text{otherwise}
\end{cases}
\]

We define the Strongly-Local Modularity (SLM) as follows:

\[
\tilde{Q}^- = \sum_s (1 - \tilde{Q}_s).
\]

We use SLM in combination with a relaxed notion of community assignment in which community representatives can share common neighbors within their respective communities.

IV. MODELING SLM AS A FACILITY LOCATION PROBLEM

We transform instances of the community detection problem into instances of the Uncapacitated Facility Location Problem (UFLP) [12]. Given a set of potential facility locations \( L \), a set of customers \( C \), a set of facility opening costs \( f_i \), and a set of service costs \( c_{ij} \) (the cost to serve customer \( j \) using facility \( i \) ), the objective function of UFLP is

\[
F(x) = \sum_{i \in L} f_i x_i + \sum_{i \in L, j \in C} c_{ij} y_{ij},
\]

where the variables \( x_i \) indicate whether or not location \( i \) hosts a facility, and the variables \( y_{ij} \) indicate whether or not location \( i \) serves customer \( j \). Solutions to UFLP minimize \( F(x) \) subject to the constraints that every customer must be served, and that no customer can be served by a facility that does not exist. UFLP is a well known NP-hard problem [6, 7, 11], yet it has special structure that enables efficient computation of fractional solutions.

We consider all vertices to be potential facility locations, with facility opening costs \( f_s = (1 - Q_s) \). Each vertex is a customer that must be served by a facility (and may serve itself if it hosts a facility). The service cost is zero for a node to serve a neighbor in the graph. Nodes cannot serve non-neighbors (cost is effectively infinite). The solution to the UFLP is a minimum-cost facility and service assignment in which every vertex is served.

UFLP is an integer program (IP), but we need only solve the linear programming relaxation of the IP [12]. This relaxation has special structure that obviates the need for a general linear program solver. We apply Lagrangian relaxation in conjunction with an elegant subgradient method known as the Volume algorithm (VA) [2] in the Lagrangian relaxation framework of [3]. The memory usage of this combined procedure is on the order of the problem input size. VA makes a series of linear-time passes over the data. There are no known asymptotic bounds on the number of iterations. However, in practice, the total runtime is comparable to the \( O(n \log^2 n) \) runtime of the most familiar fast modularity heuristic, the CNM greedy algorithm [5]. We have observed this experimentally on graphs with up to 100 million edges.

The volume algorithm provides a fractional solution to the UFLP that in turn provides a provable lower bound on \( Q^- \) where all communities are strongly local.

![FIG. 1: The support of Zachary’s karate club. Solid edges have stronger support than speckled edges and larger vertices are more likely to be leaders. Note the nearly-invisible edges linking portions of the club destined to split.](image)

Our community-assignment procedure selects a set of facilities to “open.” Each open facility represents a leader of a subset of a strongly-local community. That is, every community has at least one node that is adjacent to all other nodes in the community. The set of leaders, therefore, forms a dominating set, that is, a set of vertices \( D \) such that each vertex in the graph is either in \( D \) or adjacent to an element of \( D \).

In our community-finding procedure, called SNL, we set the facility-opening costs as described above and use VA to compute an optimal fractional placement of facilities. We then open each facility with probability equal to its fractional assignment value. If this does not produce a dominating set, then we repair it to make a dominating set. We then assign all the other vertices to a commu-
We define the support of an edge \((u, v)\) to be a real number between 0 and 1 that indicates the level of support/evidence for nodes \(u\) and \(v\) being in the same community. Given any randomized algorithm \(A\) for community detection, such as the metaheuristic approach of [17], we can compute a support with respect to \(A\). Given any randomized algorithm \(A\) for community detection, such as the metaheuristic approach of [17], we can compute a support with respect to \(A\) by sampling: generate many community assignments using \(A\), then compute the fraction of times each edge is intra-community. We now show how to compute a support with respect to SNL without sampling.

Given a fractional solution \(x\) to an instance of UFLP, we define the support with respect to SNL to be a set of values \(z\), where \(z_j\) is a probability that in a set of community leaders sampled from \(x\), edge \(j\) could link two vertices in the same community. Formally,

\[
z_{e=(v,w)} = 1 - [(1-x_v) \star (1-x_w) \star \Pi_{u \in N(v) \cap N(w)} (1-x_u)].
\]

An edge \(e = (v, w)\) has strong support if it is unlikely that none of the vertices capable of serving both \(v\) and \(w\) will become a server. This includes \(v, w,\) and their mutual neighbors. Figure 1 depicts the support of Zachary’s karate club dataset [19], an abstraction of a social network that famously split into two. The larger vertices and darker edges have higher \(x\) and \(z\) values, respectively. Even before community assignments have been specified, the community structure begins to emerge in fractional form. Note that some edges that are destined to become inter-community edges have very low support and are therefore almost invisible.

Given the support of a graph, we define a new measure to evaluate the effectiveness of community assignments. We define the support variance \((\text{Var}_s)\) as follows, assuming that \(\delta(v, w)\) is an indicator function with value 1 if \(v\) and \(w\) are in the same community and 0 otherwise.

\[
\text{Var}_s = \sum_{(v, w) \in E(G)} (\delta(v, w) - y_{vw})^2.
\]

VI. PRELIMINARY COMPUTATIONAL RESULTS

We applied our methods to several familiar datasets. Figure 2 shows the support of Zachary’s karate club. The colored images in Figure 2 depict the solutions of four algorithms: our facility location-based rounding heuristic (SNL); the CNM greedy algorithm; a combination of these two (SNL-CNMM), in which SNL is used to compute strongly-local communities, then CNM is allowed to merge these; and the eigenvector-based approach of Newman, augmented with a Kernighan-Lin-like postprocessing step (Newman-KL) [15]. Newman-KL gives one of the best known values for modularity.

In this case, intuition and history favor the facility location-based community assignments with low support variance over those with high modularity. For example, the latter split the topmost community despite reasonably strong support for the edges holding it together.
yielding an algorithm to compute a provable lower bound location theory to the problem of community detection, and maximizes modularity. However, as the number of ample shown is a ring of ten 5-cliques, and grouping the merging the endpoints of the edge that has the least support in their first step. The right hand instance, from [8], has been used to show that modularity optimization fails to resolve small communities in large graphs. The example shown is a ring of ten 5-cliques, and grouping the 5-cliques individually both minimizes support variance and maximizes modularity. However, as the number of 5 cliques increases, the common sense solution continues to minimize support variance, but is discarded by modularity optimization methods in favor of larger communities.

Figure 3 shows two instances that have been demonstrated in recent literature to present inherent problems for modularity algorithms. The modularity of the left hand instance, from [7], tricks greedy algorithms into merging the endpoints of the edge that has the least support in their first step. The right hand instance, from [8], has been used to show that modularity optimization fails to resolve small communities in large graphs. The example shown is a ring of ten 5-cliques, and grouping the 5-cliques individually both minimizes support variance and maximizes modularity. However, as the number of 5 cliques increases, the common sense solution continues to minimize support variance, but is discarded by modularity optimization methods in favor of larger communities.

VII. CONCLUSIONS

We have applied models and algorithms from facility location theory to the problem of community detection, yielding an algorithm to compute a provable lower bound on a minimization variant of local modularity, a support measure that can be computed without sampling, and a randomized rounding heuristic that can be generalized into a class of heuristics. We have also introduced a new measure for evaluating the quality of community structures. The effectiveness of our heuristics for large graphs remains open, but the solution techniques themselves are scalable and based upon simple traversals of the network that are massively parallelizable in a more natural way than the priority queue-based methods previously published. We will explore the scalability of our methods on supercomputers in future work.

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