Large-$x$ structure of physical evolution kernels in Deep Inelastic Scattering

G. Grunberg

Centre de Physique Théorique, École Polytechnique
91128 Palaiseau Cedex, France
E-mail: georges.grunberg@cpht.polytechnique.fr

Abstract: The modified evolution equation for parton distributions of Dokshitzer, Marchesini and Salam is extended to non-singlet Deep Inelastic Scattering coefficient functions and the physical evolution kernels which govern their scaling violation. Considering the $x \to 1$ limit, it is found that the leading next-to-eikonal logarithmic contributions to the physical kernels at any loop order can be expressed in term of the one-loop cusp anomalous dimension, a result which can presumably be extended to all orders in $(1-x)$, and has eluded so far threshold resummation. Similar results are shown to hold for fragmentation functions in semi-inclusive $e^+e^-$ annihilation. Gribov-Lipatov relation is found to be satisfied by the leading logarithmic part of the modified physical evolution kernels.
1. Introduction

There has been recently renewed interest [1–9] in threshold resummation of “next-to-eikonal” logarithmically enhanced terms which are suppressed by some power of the gluon energy \((1 - x)\) for \(x \to 1\) in momentum space (or by some power of \(1/N, N \to \infty\) in momentum space). In particular, in [2, 5–9] this question has been investigated at the level of “physical evolution kernels” which control the scaling violation of (non-singlet) structure functions. The scale–dependence of the Deep Inelastic Scattering (DIS) coefficient function \(C_2(x, Q^2, \mu_F^2)\) corresponding to the flavor non-singlet \(F_2(x, Q^2)\) structure function \((F_2(x, Q^2)/x = C_2(x, Q^2, \mu_F^2) \otimes q_{2,ns}(x, \mu_F^2))\), where \(q_{2,ns}(x, \mu_F^2)\) is the corresponding quark distribution) can be expressed in terms of \(C_2(x, Q^2, \mu_F^2)\) itself, yielding the following “physical” evolution equation (see e.g. Refs. [10–14]):

\[
\frac{\partial C_2(x, Q^2, \mu_F^2)}{\partial \ln Q^2} = \int_x^1 \frac{dz}{z} K(z, a_s(Q^2)) C_2(x/z, Q^2, \mu_F^2) \equiv K(x, a_s(Q^2)) \otimes C_2(x, Q^2, \mu_F^2),
\]

(1.1)

where \(\mu_F\) is the factorization scale (I assume for definitness the \(\overline{MS}\) factorization scheme is used). \(K(x, a_s(Q^2))\) is the momentum space physical evolution kernel, or physical anomalous dimension; it is independent of the factorization scale and renormalization–scheme invariant.
In [15], the result for the leading contribution to this quantity in the $x \to 1$ limit was derived, which resums all logarithms at the leading eikonal level, and nicely summarizes analytically in momentum space the standard results [16,17] of threshold resummation:

$$K(x, a_s(Q^2)) \sim \mathcal{J}\left(\frac{rQ^2}{r}\right) + B_{DIS}^{D}(a_s(Q^2)) \delta(1-x),$$  \hspace{1cm} (1.2)

where $r = \frac{1-x}{x}$ (with $rQ^2 \equiv W^2$ the final state “jet” mass), $B_{DIS}^{D}(a_s)$ is related to the the quark form factor, and $\mathcal{J}(Q^2)$, the “physical Sudakov anomalous dimension” (a renormalization scheme invariant quantity), is given by:

$$\mathcal{J}(Q^2) = A(a_s(Q^2)) + \frac{dB(a_s(Q^2))}{d\ln Q^2} = A(a_s(Q^2)) + \beta(a_s(Q^2)) \frac{dB(a_s(Q^2))}{da_s} \equiv \sum_{i=1}^{\infty} j_i a_s^i(Q^2).$$  \hspace{1cm} (1.3)

In eq.(1.3),

$$A(a_s) = \sum_{i=1}^{\infty} A_i a_s^i$$  \hspace{1cm} (1.4)

is the universal “cusp” anomalous dimension [18] (see also [19]), with $a_s \equiv \frac{\alpha_s}{4\pi}$ the $\overline{MS}$ coupling,

$$\beta(a_s) = \frac{da_s}{d\ln Q^2} = -\beta_0 a_s^2 - \beta_1 a_s^3 - \beta_2 a_s^4 + \ldots$$  \hspace{1cm} (1.5)

is the beta function (with $\beta_0 = \frac{4}{3} C_A - \frac{2}{3} n_f$) and

$$B(a_s) = \sum_{i=1}^{\infty} B_i a_s^i$$  \hspace{1cm} (1.6)

is the usual final state “jet function” anomalous dimension. It should be noted that $j_1 = A_1$ (the one loop cusp anomalous dimension), and also that both $A(a_s)$ and $B(a_s)$ (in contrast to $\mathcal{J}(Q^2)$) are renormalization scheme-dependent quantities. The renormalization group invariance of $\mathcal{J}(Q^2)$ yields the standard relation:

$$\mathcal{J}((1-x)Q^2) = j_1 a_s + a_s^2[-j_1 \beta_0 L_x + j_2] + a_s^3[j_1 \beta_0^2 L_x^2 - (j_1 \beta_1 + 2j_2 \beta_0) L_x + j_3] + a_s^4[-j_1 \beta_0^3 L_x^3 + \left(\frac{5}{2}j_1 \beta_1 \beta_0 + 3j_2 \beta_0^2\right) L_x^2 - (j_1 \beta_2 + 2j_2 \beta_1 + 3j_3 \beta_0) L_x + j_4] + \ldots,$$  \hspace{1cm} (1.7)

where $L_x \equiv \ln(1-x)$ and $a_s = a_s(Q^2)$, from which the structure of all the eikonal logarithms in $K(x, a_s(Q^2))$ can be derived. A term like $\frac{L_x^p}{x}$ arising from $\frac{\mathcal{J}(rQ^2)}{r}$ in eq.(1.2) must be interpreted as usual as a standard $++$-distribution. All the eikonal logarithms are thus absorbed into the single scale $(1-x)Q^2$ (see also [20–22] and section VI-E in [11]).
However, no analogous result holds [6] at the next-to-eikonal level (except [2] at large-$\beta_0$). In this note, I show that the leading next-to-eikonal logarithmic contributions to the physical evolution kernel at a given order in $a_s$ can actually be determined in term of lower order leading eikonal coefficients, representing the first step towards threshold resummation at the next-to-eikonal level. This result is obtained by extending the approach of [23, 24] (which deals with parton distributions) to the DIS coefficient functions themselves.

2. The modified physical kernel

I consider the class of modified physical evolution equations:

$$
\frac{\partial C_2(x, Q^2, \mu_F^2)}{\partial \ln Q^2} = \int_1^1 \frac{dz}{z} \ K(z, a_s(Q^2), \lambda) \ C_2(x/z, Q^2/z^\lambda, \mu_F^2),
$$

where for book-keeping purposes I introduced the parameter $\lambda$, which shall eventually be set to its physically meaningful value $\lambda = 1$, in straightforward analogy to the modified evolution equation for parton distributions of [24]. I note that $K(x, a_s, \lambda = 0) \equiv K(x, a_s)$, the ‘standard’ physical evolution kernel. Eq.(2.1) allows to determine $K(x, a_s, \lambda)$ given $K(x, a_s)$ (or vice-versa). Indeed, expanding $C_2(y, Q^2/z^\lambda, \mu_F^2)$ around $z = 1$, keeping the other two variables fixed, and reporting into eq.(2.1), one easily derives the following relation between $K(x, a_s, \lambda)$ and $K(x, a_s)$:

$$
K(x, a_s) = K(x, a_s, \lambda) - \lambda [\ln x \ K(x, a_s, \lambda)] \otimes K(x, a_s)
$$

$$
+ \frac{\lambda^2}{2} [\ln^2 x \ K(x, a_s, \lambda)] \otimes [\beta(a_s) \frac{\partial K(x, a_s)}{\partial a_s} + K(x, a_s) \otimes K(x, a_s)] + ... ,
$$

where only terms with a single overall factor of $\lambda$ need actually to be kept up to next-to-eikonal order, since one can check terms with more factors of $\lambda$, which are associated to more factors of $\ln x$, are not relevant to determine the next-to-eikonal logarithms in the physical kernel. In the rest of the paper (except section 4) I shall therefore simply use:

$$
K(x, a_s) = K(x, a_s, \lambda) - \lambda [\ln x \ K(x, a_s, \lambda)] \otimes K(x, a_s) + ... .
$$

Eq.(2.3) can be solved perturbatively. Setting:

$$
K(x, a_s, \lambda) = K_0(x, \lambda) a_s + K_1(x, \lambda) a_s^2 + K_2(x, \lambda) a_s^3 + K_3(x, \lambda) a_s^4 + ... \ (2.4)
$$

(and similarly for $K(x, a_s)$), one gets:

$$
K_0(x, \lambda) = K_0(x)
$$

$$
K_1(x, \lambda) = K_1(x) + \lambda [\ln x \ K_0(x)] \otimes K_0(x)
$$

$$
K_2(x, \lambda) = K_2(x) + \lambda [\ln x \ K_1(x)] \otimes K_0(x) + [\ln x \ K_0(x)] \otimes K_1(x) + ...
$$

$$
K_3(x, \lambda) = K_3(x) + \lambda [\ln x \ K_2(x)] \otimes K_0(x) + [\ln x \ K_1(x)] \otimes K_1(x)
$$

$$
+ [\ln x \ K_0(x)] \otimes K_2(x) + ... .
$$
The $K_i(x)$’s are determined in terms of splitting functions and coefficient functions as follows [14]:

\begin{align}
K_0(x) &= P_0(x) \\
K_1(x) &= P_1(x) - \beta_0 c_1(x) \\
K_2(x) &= P_2(x) - \beta_1 c_1(x) - \beta_0(2c_2(x) - c_1^{(2)}(x)) \\
K_3(x) &= P_3(x) - \beta_2 c_1(x) - \beta_1(2c_2(x) - c_1^{(2)}(x)) - \beta_0(3c_3(x) - 3c_2(x) \otimes c_1(x) + c_1^{(3)}(x)) ,
\end{align}

where $P_i(x)$ are the standard $(i+1)$-loop splitting functions, $c_i(x)$ are the $i$-loop coefficient functions, and $c_1^{(2)}(x) \equiv c_1(x) \otimes c_1(x)$, etc.

Consider now the $x \rightarrow 1$ limit. The one-loop splitting function is given by [25]:

\begin{equation}
P_0(x) = A_1 p_{qq}(x) + B_1^s \delta(1-x) ,
\end{equation}

with $A_1 = 4C_F$, and$^1$

\begin{equation}
p_{qq}(x) = \frac{1}{1-x} - 1 + \frac{1}{2}(1-x) = \frac{x}{1-x} + \frac{1}{2}(1-x) = \frac{1}{r} + \frac{1}{2}(1-x) .
\end{equation}

Moreover, at the next-to-eikonal level we have, dropping from now on $\delta$ function contributions:

\begin{equation}
P_1(x) = \frac{A_2}{r} + C_2 L_x + D_2 + ... ,
\end{equation}

with [26]:

\begin{equation}
C_2 = A_1^2 .
\end{equation}

Also:

\begin{equation}
c_1(x) = \frac{c_{11} L_x + c_{10}}{r} + b_{11} L_x + b_{10} + ... 
\end{equation}

with $c_{11} = A_1 = 4C_F$, $b_{11} = 0$. From eq.(2.6) one can derive [6,7] the following expansions for $x \rightarrow 1$:

\begin{align}
K_0(x) &= P_0(x) = \frac{k_{10}}{r} + h_{10} + ... \\
K_1(x) &= \frac{k_{21} L_x + k_{20}}{r} + h_{21} L_x + h_{20} + ... \\
K_2(x) &= \frac{k_{32} L_x^2 + k_{31} L_x + k_{30}}{r} + h_{32} L_x^2 + h_{31} L_x + h_{30} + ... \\
K_3(x) &= \frac{k_{43} L_x^3 + k_{42} L_x^2 + k_{41} L_x + k_{40}}{r} + h_{43} L_x^3 + h_{42} L_x^2 + h_{41} L_x + h_{40} + ... .
\end{align}

$^1$p_{qq}(x) is defined to be 1/2 the corresponding function in [7].
3. Leading next-to-eikonal logarithms

3.1 Two loop kernel

From eq.(2.6), (2.7), (2.9) and (2.11) one deduces: \( k_{10} = A_1, h_{10} = 0, \) and \( k_{21} = -\beta_0 A_1, h_{21} = C_2. \) Then eq.(2.5) yields for \( x \to 1: \)

\[
K_0(x, \lambda) = P_0(x) \quad (3.1)
\]

\[
K_1(x, \lambda) = \frac{k_{21} L_x + k_{20}}{r} + (h_{21} - \lambda k_{10}^2) L_x + \mathcal{O}(L_x^0) .
\]

Now

\[
h_{21}(\lambda) = h_{21} - \lambda k_{10}^2 = C_2 - \lambda A_1^2 = (1 - \lambda) A_1^2 . \quad (3.2)
\]

Thus, setting \( \lambda = 1, \) one finds that the leading next-to-eikonal logarithm in \( K_1(x, \lambda = 1) \) vanishes, yielding the relation:

\[
h_{21} = k_{10}^2 = 16C_F^2 , \quad (3.3)
\]

which is correct [6, 7]. This finding is not surprising: up to two loop, the leading next-to-eikonal logarithm is contributed only by the splitting function, since \( b_{11} = 0 \) (e.g. \( h_{21} = C_2 \)), and one effectively recovers the result (eq.(2.10)) holding [24] for the two loop splitting function. The situation however changes drastically at three loop, where the leading next-to-eikonal logarithm is contributed by the coefficient function rather then the splitting function, and the crucial question is whether the leading next-to-eikonal logarithm still vanishes for \( \lambda = 1. \)

3.2 Three loop kernel

Eq.(2.5) yields for \( x \to 1: \)

\[
K_2(x, \lambda) = \frac{k_{32} L_x^2 + k_{31} L_x + k_{30}}{r} + (h_{32} - \lambda \frac{3}{2} k_{21} k_{10}) L_x^2 + \mathcal{O}(L_x) . \quad (3.4)
\]

Requiring \( h_{32}(\lambda), \) the coefficient of the \( \mathcal{O}(L_x^2) \) term, to vanish for \( \lambda = 1 \) predicts:

\[
h_{32} = \frac{3}{2} k_{21} k_{10} = -\frac{3}{2} \beta_0 A_1^2 = -24\beta_0 C_F^2 , \quad (3.5)
\]

which is indeed the correct [6, 7] value. I stress that this result is not a consequence of the relation [24, 27, 28] \( C_3 = 2A_1 A_2 \) for \( P_2(x). \) Indeed it is well-known [29] that the \( P_i(x) \)'s, and in particular \( P_2(x), \) have only a single next-to-eikonal logarithm:

\[
P_2(x) = \frac{A_3}{r} + C_3 L_x + D_3 + ... , \quad (3.6)
\]

and thus \( P_2(x) \) cannot contribute to the double logarithm in \( K_2(x). \) Rather, \( h_{32} \) is contributed by the coefficient functions in eq.(2.6), and eq.(3.5) yields a prediction for the \( \mathcal{O}(L_x^2) \) term in \( c_2(x). \)
3.3 Four loop kernel

Eq. (2.5) yields for $x \to 1$:

$$K_3(x, \lambda) = \frac{k_{43}L_x^3 + k_{42}L_x^2 + k_{41}L_x + k_{40}}{r}$$

$$+ [h_{43} - \lambda (\frac{4}{3}k_{10}k_{32} + \frac{1}{2}k_{21}^2)]L_x^3 + \mathcal{O}(L_x^2), \quad (3.7)$$

where $k_{32} = A_1\beta_0^3$ (consistently with eq.(1.7)). Requiring $h_{43}(\lambda)$, the coefficient of the $\mathcal{O}(L_x^3)$ term, to vanish for $\lambda = 1$ predicts:

$$h_{43} = \frac{4}{3}k_{10}k_{32} + \frac{1}{2}k_{21}^2 = \frac{11}{6}\beta_0^2 A_1^2 = \frac{88}{3}\beta_0^3 C_F^2, \quad (3.8)$$

which is again the correct [6,7] value.

3.4 Five loop kernel

One can similarly predict the leading next-to-eikonal logarithm in the five loop physical kernel (which depends on the four loop coefficient function). Using eq.(2.3), the coefficient of the $\mathcal{O}(L_x^4)$ term in $K_4(x, \lambda)$ is found to be given by:

$$h_{54}(\lambda) = h_{54} - \lambda (\frac{5}{4}k_{10}k_{43} + \frac{5}{6}k_{21}k_{32}), \quad (3.9)$$

where $k_{43} = -A_1\beta_0^3$ (again consistent with eq.(1.7)). Requiring this coefficient to vanish for $\lambda = 1$ predicts:

$$h_{54} = \frac{5}{4}k_{10}k_{43} + \frac{5}{6}k_{21}k_{32} = \frac{25}{12}\beta_0^3 A_1^2 = -\frac{100}{3}\beta_0^3 C_F^2. \quad (3.10)$$

3.5 All-order relations

Defining moments by

$$K(N, a_s) = \int_0^1 dx x^{N-1}K(x, a_s), \quad (3.11)$$

eq.(2.3) yields in moment space:

$$K(N, a_s) = \frac{K(N, a_s, \lambda)}{1 + \lambda K(N, a_s, \lambda)}, \quad (3.12)$$

where $\dot{f} \equiv \partial f/\partial N$. Assuming the leading next-to-eikonal logarithms vanish to all orders (in an expansion in $1/r$) for $\lambda = 1$, i.e. that $h_{i+1,i}(\lambda = 1) = 0$ for $i \geq 0$, one can derive [30] from eq.(3.12) the resummation formula:

$$\sum_{i=0}^{\infty} h_{i+1,i}L_x^i a_s^{i+1} = \frac{A_1}{\beta_0} \frac{A_1 a_s}{1 + a_s\beta_0 L_x} \ln(1 + a_s\beta_0 L_x), \quad (3.13)$$

which correctly reproduces the results in the previous subsections.
One can further show [30] that the moment space functional relation which accounts for leading logarithms at all orders in \((1 - x)\) is:

\[
K(N, a_s) = K[N - \lambda K(N, a_s), a_s, \lambda] .
\]  

(3.14)

Eq.(3.12) results from expanding the right hand side of eq.(3.14) to first order in \(\Delta N \equiv \lambda K(N, a_s)\). It is interesting that eq.(3.14) is identical to the functional relation\(^3\) obtained [28, 32] for the splitting functions in the conformal limit (where the splitting functions coincide with the \(K_i\)’s).

4. Leading next-to-next-to-eikonal logarithms

It can be checked [30] that similar methods allow to predict using eq.(2.2) the leading logarithmic contributions at the next-to-next-to-eikonal level, i.e. the coefficient of the \((1 - x)L_x^i\) term in \(K_i(x)\). The crucial new point, however, is that the leading term in the eikonal expansion has to be defined in term of the one-loop splitting function prefactor \(p_{qq}(x)\) (eq.(2.8)), instead of \(1/r\) as in eq.(2.12). Namely, keeping only leading logarithms at each eikonal order, the predicted \(f^{c}_{ji}\) coefficients \((j = i + 1, i \geq 0)\) are defined\(^4\) by:

\[
K_i(x)|_{\text{LL}} = L_x^i[p_{qq}(x) k_{ji} + h_{ji} + (1 - x)f^{c}_{ji} + (1 - x)^2 g_{ji} + O((1 - x)^3)] .
\]  

(4.1)

Eq.(2.2) yields the corresponding \(f^{c}_{ji}(\lambda)\) coefficients in \(K_i(x, \lambda)\):

\[
f^{c}_{21}(\lambda) = f^{c}_{21} + \lambda \frac{1}{2} k_{10}^2
\]

(4.2)

\[
f^{c}_{32}(\lambda) = f^{c}_{32} - \lambda\left(-\frac{3}{4} k_{10} k_{21} + k_{10} h_{21}\right) + \lambda^2 \frac{1}{2} k_{10}^3
\]

\[
f^{c}_{43}(\lambda) = f^{c}_{43} - \lambda\left(-\frac{2}{3} k_{10} k_{32} + \frac{1}{2} (h_{21} - \frac{1}{2} k_{21}) k_{21} + k_{10} h_{32}\right) + \lambda^2 k_{10} k_{21} ,
\]

where I used that \(h_{10} = 0\), and one should note the presence of contributions quadratic in \(\lambda\). Assuming the \(f^{c}_{ji}(\lambda)\)’s vanish for \(\lambda = 1\), one thus derives the relations (with \(f^{c}_{10} = 0\)):

\[
f^{c}_{21} = -\frac{1}{2} k_{10}^2 = -8C_F^2
\]

(4.3)

\[
f^{c}_{32} = -\frac{3}{4} k_{10} k_{21} + k_{10} h_{21} + \frac{1}{2} k_{10}^3 = 12C_F^2 \beta_0 + 32C_F^3
\]

\[
f^{c}_{43} = -\frac{2}{3} k_{10} k_{32} + \frac{1}{2} (h_{21} - \frac{1}{2} k_{21}) k_{21} + k_{10} h_{32} - k_{10}^2 k_{21} = -\frac{44}{3} C_F^2 \beta_0^2 - 64C_F^3 \beta_0 ,
\]

which are seen to be correct using eq.(3.26) in [7]. The latter equation also makes it likely that similar leading logarithmic predictions can be obtained to any order in \((1 - x)\), using

\(^3\)A similar functional relation has been obtained in a different context in [31].

\(^4\)The motivation for the superscript “c” (for “classical”) shall be clarified in the Conclusion section.
the same prefactor $p_{qq}(x)$ as in eq.(4.1) to define the leading term in the eikonal expansion. Indeed, one derives for instance [30] the $O((1-x)^2)$ coefficients in eq.(4.1) (with $g_{10}=0$):

\begin{align}
g_{21} &= \frac{1}{3}k_{10}^2 = \frac{16}{3}c_F^2 \\
g_{32} &= \frac{1}{6}k_{10}k_{21} + (f_{21} + \frac{1}{2}h_{21} + \frac{1}{3}k_{21})k_{10} = \frac{1}{2}k_{10}k_{21} = -8c_F^2\beta_0 \, ,
\end{align}

which are correct [7]. I note that $f_{21}$ and $g_{21}$ coincide (like $h_{21}$) with the splitting functions contributions.

5. Fragmentation functions in $e^+e^-$ annihilation

Similar results hold for physical evolution kernels associated to fragmentation functions in semi-inclusive $e^+e^-$ annihilation (SIA), provided one sets $\lambda = -1$ in the analogue of eq.(2.1):

\begin{equation}
\frac{\partial C_T(x,Q^2,\mu_F^2)}{\partial \ln Q^2} = \int_x^1 \frac{dz}{z} K_T(z,a_s(Q^2),\lambda) C_T(x/z,Q^2/z^\lambda,\mu_F^2) \, ,
\end{equation}

where $C_T$ denotes a generic non-singlet SIA coefficient function. I first note that threshold resummation in this case [33] leads at the leading eikonal level to an equation similar to eq.(1.2):

\begin{equation}
K_T(x,a_s(Q^2)) \sim \frac{J ((1-x)Q^2)}{1-x} + B_{\beta}^{SIA}(a_s(Q^2)) \delta(1-x) \, ,
\end{equation}

where $x$ should now be identified to Feynman-$x$ rather then Bjorken-$x$, and I used the results of [34] which imply that the “physical Sudakov anomalous dimension” $J(Q^2)$ is the same for structure and fragmentation functions. The statement above eq.(5.1) then follows from the following two observations:

i) The predictions in eq.(3.3), (3.5), (3.8) and (3.10) depend only upon coefficients of leading eikonal logarithms in the physical evolution kernels.

ii) Eq.(3.26) in [7] shows that the latter coefficients are identical for deep-inelastic structure functions and for $e^+e^-$ fragmentation functions (consistently with the remark below eq.(5.2)), but that the coefficients of the leading next-to-eikonal logarithms are equal only up to a sign change (in an expansion in $1/r$) between deep-inelastic structure functions and fragmentation functions.

One deduces the resummation formula ($j = i + 1$):

\begin{equation}
\sum_{i=0}^{\infty} k_{ij}^{SIA} L_x a_s^{i+1} = - \frac{A_1}{\beta_0} \frac{A_1 a_s}{1 + a_s \beta_0 L_x} \ln(1 + a_s \beta_0 L_x) \, .
\end{equation}
6. Conclusion

A modified evolution equation for DIS non-singlet structure functions, analogous to the one used in [24] for parton distributions, but which deals with the physical scaling violation and coefficient functions, has been proposed. It allows to relate the leading next-to-eikonal logarithmic contributions in the momentum space physical evolution kernel to coefficients of leading eikonal logarithms at lower loop order (depending only upon the one-loop cusp anomalous dimension $A_1$), which represents the first step towards threshold resummation at the next-to-eikonal level. This result also explains the observed [6, 7] universality of the leading next-to-eikonal logarithmic contributions to the physical kernels of the various non-singlet structure functions, linking them to the known [35] universality of the eikonal contributions. Similar results hold at the next-to-next-to-eikonal level with a proper definition of the leading eikonal piece, and can presumably be extended to leading logarithmic contributions at all orders in $(1 - x)$. Analogous results are obtained for fragmentation functions in semi-inclusive $e^+e^-$ annihilation.

One may ask to what extent the success of the present approach may be attributed, as suggested in [24, 32] for the splitting functions case, to the classical nature [36] of soft radiation. In fact, the main result of this paper for the (modified) DIS physical evolution kernel can be summarized (barring the $\delta$-function contribution) by the following equation:

$$K(x, a_s, \lambda = 1) \sim \left[ \frac{x}{1 - x} + \frac{1}{2} (1 - x) \right] J( (1 - x) Q^2 ) + \text{subleading logarithms} , \quad (6.1)$$

where the second term (the “subleading logarithms”) is contributed by all powers in $(1 - x)$ except the leading eikonal one. The first term in eq.(6.1) accounts for the leading logarithmic contributions to the modified kernel (together with some subleading logarithms) to all powers in $(1 - x)$ at any given loop order, and implies leading logarithmic contributions are actually absent beyond $\mathcal{O}(1 - x)$ power. This term has the remarkable effective one-loop splitting function form $4C_F a_{phys} \left( (1 - x) Q^2 \right) p_{qq}(x)$, with the “physical coupling” $a_{phys}(Q^2) \equiv \frac{1}{4C_F} J(Q^2)$.

As pointed out in [32], the $\frac{1}{1 - x}$ part of the one-loop prefactor (eq.(2.8)) should be interpreted as corresponding to universal classical radiation, a QCD manifestation of the Low-Burnett-Kroll theorem [36], while the $1 - x$ part represents a genuine quantum contribution. Now, it is clear that at the next-to-eikonal level, the $1 - x$ part of the prefactor is irrelevant: only the “classical” $1/r$ part is required to separate those leading logarithms in the standard ($\lambda = 0$) physical evolution kernel which are correctly predicted in the present approach (the $h_{ji}$ in eq.(4.1)), hence “inherited” in the sense of [32], from the “primordial” ones (those which at each loop order carry the same color factors as the leading $\mathcal{O}(1/(1 - x))$ eikonal logarithms, and can thus be absorbed into the definition of the leading term). However, it appears from the results of section 4 that, at next-to-next-to-eikonal level, the full one-loop prefactor has to be used into the definition of the leading term.
to properly isolate the “inherited” next-to-next-to-eikonal logarithms (the \( f_{ji}^q \) in eq.(4.1)). Moreover, although the “inherited” \( f_{ji}^q \) are purely “classical” (like the \( h_{ji} \)), the “inherited” \( g_{ji} \) at the \( \mathcal{O}((1-x)^2) \) level are a mixture of “quantum” and “classical”. Indeed, setting \( f_{q10}^q = \frac{1}{2}k_{10} \) and \( f_{q21}^q = \frac{1}{2}k_{21} \) (the “quantum parts” of the \( \mathcal{O}(1-x) \) coefficients), one finds \( g_{q21}^q = k_{10}f_{q10}^q = \frac{1}{2}k_{10}^2 \) and \( g_{21} = g_{q21} + g_{c21} \), with \( g_{c21}^q = \frac{1}{2}k_{21}f_{q10}^q + v_{10}\), and \( g_{21}^q = -\frac{1}{3}k_{10}k_{21} \). In both cases \( g_{cji}^q = -\frac{1}{3}g_{qji}^q \), which shows the “inherited” \( g_{ji} \) coefficients are actually dominantly “quantum”.

It can be further checked [30] that the very same first term in eq.(6.1) also accounts for the leading logarithmic contributions to the \( \lambda = -1 \) modified SIA physical evolution kernel to all powers in \( (1-x) \), which implies that the leading logarithmic parts of the modified DIS and SIA physical evolution kernels satisfy Gribov-Lipatov relation [37], namely we have:

\[
K(x,a_s,\lambda=1)|_{LL} = K_T(x,a_s,\lambda=-1)|_{LL} = p_{qq}(x) J ((1-x)Q^2) |_{LL} ,
\]

where \( J ((1-x)Q^2) |_{LL} = A (a_s((1-x)Q^2)) |_{LL} = \frac{A_{1a_s}(Q^2)}{1+a_s(Q^2)\beta_0 L_x} \) is the leading logarithmic contribution to eq.(1.7). Indeed, once transformed back to the standard \( (\lambda = 0) \) physical kernels, eq.(6.2) is consistent with eq.(3.26) in [7] at least to next-to-next-to-eikonal order, and is probably correct to all orders in \( (1-x) \) (with identically vanishing contributions beyond \( \mathcal{O}(1-x) \) order). On the other hand, contrary to the splitting functions case where it has been checked up to three loops [32, 38], a full Gribov-Lipatov relation \( K(x,a_s,\lambda=1) = K_T(x,a_s,\lambda=-1) \) does not seem to hold for subleading logarithms beyond the leading eikonal level.

The resummation of the subleading logarithmic contributions at next-to-eikonal order in eq.(6.1), not addressed here, remains an open issue: the present method does not work for them, except in the conformal limit, where one recovers the results of [24].

Acknowledgements I thank G. Marchesini for stimulating discussions and hospitality at the University of Milano-Bicocca where part of this paper has been written. I also wish to thank the referees for constructive suggestions.

References

[1] M. Kramer, E. Laenen and M. Spira, Nucl. Phys. B 511 (1998) 523 [arXiv:hep-ph/9611272].
[2] G. Grunberg, arXiv:0710.5693 [hep-ph].
[3] E. Laenen, L. Magnea and G. Stavenga, Phys. Lett. B 669 (2008) 173 [arXiv:0807.4412 [hep-ph]].
[4] E. Laenen, G. Stavenga and C. D. White, JHEP 0903 (2009) 054 [arXiv:0811.2067 [hep-ph]].
[5] S. Moch and A. Vogt, JHEP 0904 (2009) 081 [arXiv:0902.2342 [hep-ph]].
[6] G. Grunberg and V. Ravindran, JHEP 0910 (2009) 055 [arXiv:0902.2702 [hep-ph]].
[7] S. Moch and A. Vogt, JHEP 0911 (2009) 099 [arXiv:0909.2124 [hep-ph]].
[8] G. Grunberg, arXiv:0910.3894 [hep-ph].
[9] G. Soar, A. Vogt, S. Moch and J. Vermaseren, arXiv:0912.0369 [hep-ph].
[10] W. Furmanski and R. Petronzio, Z. Phys. C 11 (1982) 293.
[11] G. Grunberg, Phys. Rev. D 29 (1984) 2315.
[12] S. Catani, Z. Phys. C 75 (1997) 665 [hep-ph/9609263].
[13] J. Blumlein, V. Ravindran and W. L. van Neerven, Nucl. Phys. B 586 (2000) 349 [arXiv:hep-ph/0004172].
[14] W. L. van Neerven and A. Vogt, Nucl. Phys. B 603 (2001) 42 [arXiv:hep-ph/0103123].
[15] E. Gardi and G. Grunberg, Nucl. Phys. B 794 (2008) 61 [arXiv:0709.2877 [hep-ph]].
[16] G. Sterman, Nucl. Phys. B 281 (1987) 310.
[17] S. Catani and L. Trentadue, Nucl. Phys. B 327 (1989) 323.
[18] G. P. Korchemsky and G. Marchesini, Phys. Lett. B 313 (1993) 433.
[19] S. Catani, B. R. Webber and G. Marchesini, Nucl. Phys. B 349 (1991) 635.
[20] D. Amati, A. Bassetto, M. Ciafaloni, G. Marchesini and G. Veneziano, Nucl. Phys. B 173 (1980) 429.
[21] M. Ciafaloni, Phys. Lett. B 95 (1980) 113;
[22] M. Ciafaloni and G. Curci, Phys. Lett. B 102 (1981) 352.
[23] Y. L. Dokshitzer, V. A. Khoze and S. I. Troian, Phys. Rev. D 53 (1996) 89 [arXiv:hep-ph/960425].
[24] Yu. L. Dokshitzer, G. Marchesini and G. P. Salam, Phys. Lett. B 634 (2006) 504 [arXiv:hep-ph/0511302].
[25] G. Altarelli and G. Parisi, Nucl. Phys. B 126 (1977) 298.
[26] G. Curci, W. Furmanski and R. Petronzio, Nucl. Phys. B 175 (1980) 27.
[27] S. Moch, J. A. M. Vermaseren and A. Vogt, Nucl. Phys. B 688 (2004) 101 [hep-ph/0403192].
[28] B. Basso and G. P. Korchemsky, Nucl. Phys. B 775 (2007) 1 [arXiv:hep-th/0612247].
[29] G. P. Korchemsky, Mod. Phys. Lett. A 4 (1989) 1257.
[30] G. Grunberg, “On the large-\(x\) structure of physical evolution evolution kernels”, to be published.
[31] A. H. Mueller, Nucl. Phys. B 228 (1983) 351.
[32] Yu. L. Dokshitzer and G. Marchesini, Phys. Lett. B 646 (2007) 189 [arXiv:hep-th/0612248].
[33] M. Cacciari and S. Catani, Nucl. Phys. B 617 (2001) 253 [arXiv:hep-ph/0107138].
[34] S. Moch and A. Vogt, Phys. Lett. B 680 (2009) 239 [arXiv:0908.2746 [hep-ph]].
[35] S. Moch, J. A. M. Vermaseren and A. Vogt, Nucl. Phys. B 813 (2009) 220 [arXiv:0812.4168 [hep-ph]].
[36] F. E. Low, Phys. Rev. 110 (1958) 974; T. H. Burnett and N. M. Kroll, Phys. Rev. Lett. 20 (1968) 86.
[37] V. N. Gribov and L. N. Lipatov, Sov. J. Nucl. Phys. 15 (1972) 438, ibid. 15 (1972) 675.
[38] A. Mitov, S. Moch and A. Vogt, Phys. Lett. B 638 (2006) 61 [arXiv:hep-ph/0604053].