I. INTRODUCTION

The second law of thermodynamics establishes the celebrated time’s arrow by directing the spontaneous evolution of a system undergoing any thermodynamic process. It is irreversible: for example, when interacting with a thermal bath of infinite heat capacity, the system eventually adopts the thermal distribution of the same bath reaching equilibrium with it and thereby maximum thermodynamic entropy [1]. This phenomenon, generally referred as thermalization, is not limited by the macroscopic scale but also can be revealed in microscopic systems, whose dynamics is governed by the laws of quantum mechanics [2, 3]. Adapting the description of an open quantum system’s dynamics, the thermalization process with respect to a bath at temperature $T$ can be associated with a quantum operation (thermal map) represented as a completely positive trace-preserving (CPTP) map that, regardless of the input state of the system, produces a canonical thermal state of temperature $T$. Evidently, sequential interaction of the thermodynamic system with multiple thermal baths at same temperature has no distinct feature other than producing the same thermal state and so does sequential action of the corresponding thermal maps on the input state of the system.

However, quantum theory allows in principle multiple quantum operations to occur in a way not compatible with any underlying causal structure. A particular example is provided by the quantum SWITCH, a higher-order quantum process that executes the operations in a superposition of their possible relative orders making thereby the corresponding causal order indefinite. Compared with operations occurring in a well-defined order, the advent of quantum SWITCH has offered advantages in a plethora of information processing tasks like channel discrimination [4], computation [5], classical and quantum communication [6–9], and others. In a similar spirit, the relevance of the quantum SWITCH in the domain of quantum thermodynamics has recently been observed in the framework of quantum cooling [10–13], boosted quantum battery charging [14], and thermodynamic work extraction games [15, 16]. As regards the latter, quantum SWITCH of two thermal maps has been considered for extraction of the free energy amount of work from thermal states [15] as well as with respect to work extraction under a unitary cyclic evolution of a quantum system [16]. Indeed, amount of work that can be extracted from a quantum system under a unitary cycle is given by ergotropy [17] that quantifies the energy difference between the concerned state and the corresponding passive state, i.e., the equal entropic state with minimum energy [18–21]. Evidently, the energy of a single passive state cannot be lowered further, and no work can be extracted from it under unitary cyclic evolution. However, possessing multiple copies of a passive state can in principle lead to a non-zero amount of extractable work when a global unitary is performed [22]. Interestingly, thermal states are an only class of passive states (dubbed as completely passive states) that do not allow to extract a non-zero ergotropic work even from infinitely many copies of them [18].

Since a sequential action of two identical thermal maps produces a thermal and thereby completely passive state, no extractable work is stored at the output end. Interestingly, relaxing a definite causal order of the maps’ occurrence by superposing their sequential application orders via the quantum SWITCH can activate a non-zero ergotropy at the output end even if the input state is thermal. However, for a thermal input state, the non-zero ergotropy is activated only if its temperature is below a certain threshold bound $T_C$ [16]. In the present paper, we consider a generic framework for ergotropic work extraction under the sequential occurrence of thermal maps and question whether the temperature bound $T_C$ can be overcome. Firstly, we generalize the setup to occurrence of asymptotically many identical thermal maps in an indefinite causal order via the N-SWITCH: in this case, the amount of ergotropy stored in the output state can be doubled compared with the scenario of two thermal maps controlled by the quantum SWITCH (i.e., 2-SWITCH). However, increasing the number of switched thermal maps still
does not violate the temperature bound $T_C$, with a restricted classical communication channel between the control and the sender.

Thereafter, we move towards a more general structure of the quantum SWITCH and allow the locally thermal input to be correlated initially with the control qubit. Notably, the local and global ergotropic difference for locally thermal correlated quantum states have important consequences to identify the nature of shared correlation amongst them [23–25]. However, thermal maps break entanglement, and the signature of the correlation between the parts of the joint system vanishes whenever one part evolves under such operations. Interestingly, the signature of initial correlation can outlast further if these thermal interactions are controlled via the quantum SWITCH and accordingly the potential ergotropy of the final output can supersede that of the uncorrelated quantum SWITCH. Moreover, we have further strengthen the power of entangled causal order by violating the threshold temperature bound $T_C$ in an entangled 2-SWITCH settings only.

The remainder of the paper is organized as follows. In Section II, we introduce the necessary framework and recall the concepts of ergotropy and thermal maps, and the quantum SWITCH. In Section III, we discuss action of the quantum N-SWITCH on identical thermal maps and recover the threshold temperature bound $T_C$ for the activation of ergotropy. In Section IV, we generalize the quantum SWITCH setup by allowing correlations between the input thermal state and the control qubit and exhibit the possible configuration to overcome the temperature bound. Last but not least, in Section V, we draw the conclusions and possible future directions as well as applications of the results.

II. FRAMEWORK

A. Ergotropy of a quantum state

It is well known that internal energy of an isolated (quantum) system $S$ evolving under certain Hamiltonian $H$ does not change. However, it is possible to lower it by coupling the system to a macroscopic source represented by properly chosen external time-dependent potential $V(t)$ [17]. Such potential varies cyclically the Hamiltonian of the system during a certain time interval $t \in [0, \tau]$. The evolution of the system’s state $\rho(t)$ is thus governed by the effective Hamiltonian $H'(t) = H + V(t)$ with $V(0) = V(\tau) = 0$.

In particular, the lowest energy can be reached by an appropriate choice of the external potential $V(t)$ that leads to a passive state of the system at $t = \tau$. Indeed, let us assume that the initial state of the system is given by $\rho(0) = \sum_k p_k |\psi_k\rangle \langle \psi_k|$, while its initial ("isolated") Hamiltonian can be expanded as $H = \sum_i \epsilon_i |\psi_i\rangle \langle \psi_i|$ with increasing energies $\epsilon_i \leq \epsilon_{i+1}$. Then the effective Hamiltonian $H'(t)$ with the optimal choice of the external potential generates a unitary evolution under the operator $U_p$ that produces the passive state

$$\rho(\tau) = U_p \rho(0) U_p^\dagger = \sum_i p_i |\epsilon_i\rangle \langle \epsilon_i|,$$

with decreasing populations $p_i \geq p_{i+1}$. The amount of extractable work under such setting is maximum work that can be extracted from $\rho(0) \equiv \rho$ via unitary cycles and termed as ergotropy,

$$W(\rho) = \max_{U \in \mathcal{U}} \text{Tr}[\rho - U \rho U^\dagger] H] = \text{Tr}[\rho - U \rho U^\dagger] H] = \sum_{i,j} p_i \epsilon_j (|\langle \epsilon_i | \epsilon_j \rangle|^2 - \delta_{ij}),$$

where $\mathcal{U}$ is the set of all unitary transformations generated from the effective time-dependent Hamiltonian $H + V(t)$.

Note that the passive states are diagonal in the Hamiltonian eigenbasis, along with an inverse population distribution with respect to the energy levels [18]. Therefore, the ergotropy $W(\rho)$ can originate from two different consecutive contributions [26] (for the sake of simplicity, we consider a two-dimensional system that allows one to provide them with simple analytic expressions):

1) incoherent counterpart $W_i(\rho)$ which can be extracted from $\rho$ by reorientation of the diagonal elements $p_i$ (i.e., probabilities) inverse to the energy levels,

$$W_i(\rho) = \max \{0, \delta \rho\},$$

where $\delta \rho = \rho_{22} - \rho_{11}$ is the population imbalance,

2) coherent ergotropy $W_c(\rho)$ which can be extracted by the transformation of basis $\{|\psi_i\rangle\} \rightarrow \{|\epsilon_j\rangle\}$, after the probabilities have been permutated and $W_i(\rho)$ has been extracted,

$$W_c(\rho) = \frac{1}{2} \left( \eta - \sqrt{\eta^2 - 4|\rho_{12}|^2} \right).$$

where $\eta = \sqrt{2P(\rho) - 1}$ depends on the purity $P(\rho) = \text{Tr}(\rho^2)$ of the system’s state, and $\rho_{12}$ is its off-diagonal element.

Obviously, this also implies that a single copy of $\rho$ has zero ergotropy if it is passive in nature.

For the multipartite quantum systems, there are different aspects of ergotropy depending upon the restrictions imposed on the unitary operation $U$. When the constituents of a multipartite quantum states are allowed to apply only a local unitary, i.e., $\mathcal{U} := \bigotimes \mathcal{U}_k$ in Eq. (2), then the amount of extractable work is referred as local ergotropy ($W_l$). On the other hand, considering the set of all unitary acting on the state jointly will give rise to the global ergotropy ($W_g$). Obviously, $W_g \geq W_l$ and also there are quantum states for which $W_l = 0$ while $W_g \neq 0$ [23, 25]. Such states are generally termed as locally passive, or more strongly locally thermal quantum states.
However, there can be an intermediate concept of locally extractable ergotropy, namely the daemonic ergotropy \( W_d \) [24], which involves a measurement on a subsystem \( A \) of the multipartite quantum state of \( SA \) and, depending upon the outcome, the other part is used to extract work unitarily. Indeed, it is defined as an average ergotropy of post-selected states of the system with respect to the optimal projective measurement \( \{ \Pi_a \} \) on \( A \),

\[
W_d(\rho_{SA}) = \max_{\{ \Pi_a \}} \left( \sum_a p_a W(\rho_{S|a}) \right),
\]

where \( W(\rho_{S|a}) = p_a^{-1} \text{Tr}_A(\Pi_a \rho_{SA}) \) is the post-selected state of \( S \) associated to the measurement outcome \( a \) obtained with the probability \( p_a = \text{Tr}(\Pi_a \rho_{SA}) \). On a slightly different note, while the difference of \( W_g \) and \( W_f \) is a potential candidate to witness the presence of entanglement in the quantum state [25], the presence of quantum discord as well as genuinely classical-classical correlations between \( S \) and \( A \) implies a gap between \( W_d \) and \( W_f \) [24].

### B. Thermalizing maps

Any quantum operation on a quantum system \( S \) can be obtained as local dynamics of a unitary interaction between the system and environment. In particular, a general thermal operation \( T \) can be modelled by an energy-preserving unitary interaction of \( S \) with a thermal bath \( E \) at temperature \( T \) [27]. Indeed, it is given by

\[
T(\rho) = \text{Tr}_E [U_{SE}(\rho \otimes \tau_E) U_{SE}^\dagger],
\]

where \( \tau_E \) is the state of thermal bath \( E \), i.e., the thermal state following Gibbs probability distribution,

\[
\tau_E = \frac{1}{Z_\beta} e^{-\beta H} = \frac{1}{Z_\beta} \sum_j e^{-\beta E_j} |j\rangle \langle j|,
\]

where \( Z_\beta = \text{Tr}[e^{-\beta H}] = \sum_k e^{-\beta E_k} \) is the canonical partition function, \( H \) is the governing Hamiltonian, \( \beta = (k_B T)^{-1} \), and \( k_B \) is the Boltzmann constant. It is clear from that thermal states follow inverse population distribution with the energy levels and hence can be identified as passive states, which possess no ergotropy.

In turn, we identify a particular class \( \Lambda_\beta \) of thermal operations dubbed thermalizing maps which produce a thermal state \( \tau_\beta \),

\[
\Lambda_\beta[\rho] = \tau_\beta,
\]

for any input state \( \rho \) of \( S \), i.e., are pin maps. This operation can be realized if the energy-conserving unitary \( U_{SE} \) of the thermal operation \( T \) is a SWAP operator

\[
U_{SE} = \sum_{i,j} |ij\rangle \langle ji|,
\]

which interchanges the states of the system and the bath, i.e.,

\[
U_{SE}(\rho \otimes \tau_E) U_{SE}^\dagger = \tau_E \otimes \rho.
\]

For a chosen thermal state \( \tau_\beta \), the corresponding qudit thermalizing map can be realized as a convex combination of \( d \) pin maps \( E_k \), each having a fixed output, namely, one of the elements of the energy eigenbasis \( \{ |e_k\rangle \}_{k=0}^{d-1} \)

\[
\Lambda_\beta[\rho] = \sum_k e^{-\beta E_k} Z |e_k\rangle \langle e_k|.
\]

Alike every quantum CPTP operations, thermalizing maps can also be expressed in terms of Kraus operators

\[
\Lambda_\beta(\rho) = \sum_{k=0}^{d-1} E_k \rho E_k^\dagger,
\]

where \( E_k = \sqrt{e^{-\beta E_k/I}} \{(k/d) \langle k \mod d |\} \) \langle k \mod d | \), and, evidently, \( \sum_k E_k^\dagger E_k = \mathbb{I} \). In particular, assuming that the system is a qubit (hence, \( d = 2 \)) whose dynamics is governed by a Hamiltonian with \( \epsilon_0 = 0 \) and \( \epsilon_1 = \epsilon \), the Kraus operators can be given in the following form,

\[
\begin{align*}
E_0 &= \sqrt{\rho} |0\rangle \langle 0|, \\
E_1 &= \sqrt{\rho} |1\rangle \langle 1|, \\
E_2 &= \sqrt{1 - \rho} |1\rangle \langle 0|, \\
E_3 &= \sqrt{1 - \rho} |0\rangle \langle 1|,
\end{align*}
\]

where \( \rho = \frac{1}{1 + \exp(-\beta \epsilon)} \).

### C. Controlled quantum operations

Quantum SWITCH is a particular example of a higher-order operation mapping two or more quantum operations to another quantum operation that is not compatible with any well-defined order of their occurrence [5]. It can be interpreted as coherent control of the order of the operations’ occurrence by an external quantum system (control system \( C \)). For example, considering \( N \) such operations \( \{ \Lambda_1, \cdots, \Lambda_N \} \), quantum SWITCH assigns them a CPTP map \( S(\Lambda_1, \cdots, \Lambda_N) \) given by

\[
S(\Lambda_1, \cdots, \Lambda_N)(\rho_T \otimes \omega_C) = \sum_{i_1 \cdots i_N} S_{i_1 \cdots i_N} (\rho_T \otimes \omega_C) S_{i_1 \cdots i_N}^\dagger,
\]

where \( \rho_T \) is the input state of the thermodynamic system \( S \) (target), and \( \omega_C \) is the input state of control system. Note that, if all possible cyclic permutations of these \( N \) channels have been considered, then the quantum capacity of the side channel should be \( \log N \) bits. The Kraus operators \( S_{i_1 \cdots i_N} \) corresponding to Eq. (13) will take the form,

\[
S_{i_1 \cdots i_N} = \sum_{j=0}^{N-1} P_j \left( \prod_{i}^{j \leq i < j + 1} E_{i_i}^{(N)} \right) T \otimes |j\rangle \langle j|_C,
\]
where $E_i^{(k)}$ denotes the $i^{th}$-Kraus element of the operation $\Lambda_k$, while $\mathcal{P}_j$ represents the $j^{th}$ cyclic permutation of Kraus elements of different operations. In particular, the usual quantum SWITCH for two quantum operations is represented by the Kraus operators

$$S_{ij} = E_i^{(1)} E_j^{(2)} \otimes |0\rangle \langle 0|_C + E_j^{(1)} E_i^{(2)} \otimes |1\rangle \langle 1|_C. \quad (15)$$

It is further important to observe that the action of quantum SWITCH in Eq. (13) does not depend on the correlations that can be initially shared between the target and control system. This allows for a further generalization,

$$S(\Lambda_1, \cdots, \Lambda_N)[\rho_{TC}] = \sum_{i_1, \cdots, i_N} S_{i_1 \cdots i_N} (\rho_{TC}) S^T_{i_1 \cdots i_N}. \quad (16)$$

where $\rho_{TC}$ is the joint state of the target and control system, and the Kraus operators $S_{i_1 \cdots i_N}$ coincide with ones as in Eq. (14).

### III. CONTROLLED CONFIGURATION OF THERMAL MAPS

Let us first illustrate a thermodynamic framework for distant work transfer. We consider the scenario where Alice, the sender, wishes to transfer a work storage, viz., a quantum battery to Bob, at a distant location. Obviously, if they share perfect quantum channel then Alice is able to transport exact amount of work to the receivers end. But, if the channel is a thermal map, then the output, being a fixed thermal state, possesses no ergotropic work content. Moreover, if the input state at Alice’s end is initially a thermal state, then even a perfect quantum channel will not be helpful for work transfer. However, a conjunction of these two seemingly impossible scenario gives rise to a possibility of work transportation, using the complete quantumness of the associated dynamics. The scenario specifically assumes that the distant parties Alice and Bob are connected via a noisy transmission line, consists of a series of thermal maps and also there is perfect side channel, whose input and output is accessed by a third party, Charlie. While Charlie can share any prior correlation with Alice, he is only allowed to communicate a bit of classical information to Bob. Importantly, the input system at Alice’s lab contains no local ergotropy, i.e., locally thermal in nature and both her and Bob’s systems are governed by the two-level Hamiltonian $H_A (\beta) = |1\rangle \langle 1|$. On the other hand, we associate a trivial Hamiltonian for Charlie’s quantum system, i.e., $H_C = 1$.

Before presenting the results with quantum control, we will first consider the classically definite sequence of thermal maps. Note that, the thermal map $\Lambda_\beta$ acts as a pin-map which produces a fixed output $\tau_\beta$ irrespective of the input quantum state. Hence, if there are $N$-different thermal maps $\Lambda_{\beta_1}, \Lambda_{\beta_2}, \cdots, \Lambda_{\beta_N}$, then for every possible causally ordered sequence of these maps one of the thermal states $\tau_{\beta_k}$, $k \in \{1, 2, \cdots, N\}$ will be transferred at the receivers end. Now, every causally separable configuration of these thermal maps can be identified as a subjective ignorance (i.e., convex combination) of all these causally ordered sequences. As a consequence, the final state will be

$$\rho_{del} = \sum_{k=1}^N p_k \tau_{\beta_k}$$

where, $p_k$ is the probability for all possible causal configuration of these thermal maps for which the final action is $\Lambda_{\beta_k}$. Now, since the convex combination of any thermal states are always passive in nature [23, 28], the extractable ergotropy will vanish at the receivers’ end.

Also importantly, the action of a pin-map destroys any possible correlation hold by the input quantum state, which further implies that after the causally separable configuration of all these $N$ thermal channels, the output state at the receivers end becomes completely uncorrelated with the rest of the environment. Hence, it possesses no more daemonic ergotropy, i.e., a ergotropic work assisted by the classical communication from an initially correlated ancillary quantum system.

#### A. Quantum-controlled thermalization

Now let us assume that the order of maps $\Lambda_\beta$ at the same inverse temperature $\beta$ is controlled coherently by an external quantum control system within the quantum SWITCH (see Fig. 1). For example, when a qubit target system is thermalized with two baths at inverse temperature $\beta$ in an indefinite causal order via the quantum SWITCH, its final state is not necessarily the thermal state $\tau_\beta$, so that certain amount of work can be extracted from it [15, 16]. Indeed, when the target stays initially in a thermal state of inverse temperature
whose ergotropy, as the one of \((18)\) for 2-SWITCH, is incoherent. \(\sigma\)

This means that if the control qubit is measured in the basis
\[
\{ \pm \} = \{ \frac{\ket{0} \pm \ket{1}}{\sqrt{2}} \}
\]
after the system is thermalized with the baths, the resulting (un-normalized) state of the target system
\[
\hat{\rho}_\pm = \frac{1}{2Z_\beta} \begin{pmatrix} 1 + \frac{1}{Z_\beta z_{\beta in}} & 0 \\ 0 & e^{-\beta} \left( 1 + \frac{e^{-(\alpha_\beta + m_\beta)}}{Z_\beta z_{\beta in}} \right) \end{pmatrix},
\]
generally speaking, differs from the thermal state \(\tau_\beta\) obtained when the thermal baths are placed in a classical configuration and can carry some amount of ergotropy. Both states do not carry any coherence with respect to the energetic eigenbasis, hence, their ergotropy is of genuinely incoherent origin and can be extracted by permutating the energetic populations. Indeed, overall daemonic ergotropy of \((17)\) is given by \cite{16}
\[
W_d(\sigma_2) = \frac{1}{2Z_\beta^2 z_{\beta in}} \max \{ 0, e^{-2\beta} - e^{-\beta} \},
\]
where \(\sigma_2 = S(\Lambda_\beta, \Lambda_\beta) [\tau_{\beta in} \otimes (+) (+)]\) is the output of the quantum SWITCH. In this way, if a qubit system at inverse temperature \(\beta_{in}\) gets thermalized with two thermal baths at temperature \(\beta\) in an indefinite causal order via the quantum SWITCH, there exists a \textit{temperature bound}
\[
\beta_{in} > 2\beta,
\]

Hence, when a yes/no measurement of the control with respect to the state \(\ket{\gamma_+}\) is performed, we can easily get the corresponding post-measurement states
\[
\hat{\rho}_+ = \frac{1}{NZ_\beta} \begin{pmatrix} 1 + \frac{N-1}{Z_\beta z_{\beta in}} & 0 \\ 0 & e^{-\beta} \left( 1 + \frac{e^{-(\alpha_\beta + m_\beta)}}{Z_\beta z_{\beta in}} \right) \end{pmatrix},
\]
\[
\hat{\rho}_- = \frac{N-1}{NZ_\beta} \begin{pmatrix} 1 - \frac{1}{Z_\beta z_{\beta in}} & 0 \\ 0 & e^{-\beta} \left( 1 - \frac{e^{-(\alpha_\beta + m_\beta)}}{Z_\beta z_{\beta in}} \right) \end{pmatrix},
\]
whose ergotropy, as the one of \((18)\) for 2-SWITCH, is incoherent. The resulting daemonic ergotropy is given by
\[
W_d(\sigma_N) = \frac{N-1}{NZ_\beta^2 z_{\beta in}^2} \max \{ 0, e^{-2\beta} - e^{-\beta} \},
\]

\[
S(\Lambda_\beta, \Lambda_\beta) [\tau_{\beta in} \otimes (+) (+)] = \frac{1}{2} \left( (\tau_\beta + \tau_\beta \tau_{\beta in} \tau_\beta) \otimes (+) (+) + (\tau_\beta - \tau_\beta \tau_{\beta in} \tau_\beta) \otimes (-) (-) \right).
\]

This means that for the control qubit is measured in the basis
\[
\{ \pm \} = \{ \frac{\ket{0} \pm \ket{1}}{\sqrt{2}} \}
\]
when the thermal baths are placed in a classical configuration and can carry some amount of ergotropy. Both states do not carry any coherence with respect to the energetic eigenbasis, hence, their ergotropy is of genuinely incoherent origin and can be extracted by permutating the energetic populations. Indeed, overall daemonic ergotropy of \((17)\) is given by \cite{16}
\[
W_d(\sigma_2) = \frac{1}{2Z_\beta^2 z_{\beta in}} \max \{ 0, e^{-2\beta} - e^{-\beta} \},
\]
where \(\sigma_2 = S(\Lambda_\beta, \Lambda_\beta) [\tau_{\beta in} \otimes (+) (+)]\) is the output of the quantum SWITCH. In this way, if a qubit system at inverse temperature \(\beta_{in}\) gets thermalized with two thermal baths at temperature \(\beta\) in an indefinite causal order via the quantum SWITCH, there exists a \textit{temperature bound}
\[
\beta_{in} > 2\beta,
\]

A natural question arises whether the temperature bound can be shifted or overcome. Firstly, we check whether it can be changed by increasing the number of thermal baths controlled by the quantum SWITCH. The target qubit system initially remains in the same thermal state \(\tau_{\beta in}\), and we assume that is thermalized with \(N\) baths at inverse temperature \(\beta\). Their occurrence is controlled by the generalized \(N\)-SWITCH which entangles an \(N\)-dimensional system (playing the role of the control system) with the cyclic orders of the thermalizing maps. Indeed, when the control system stays initially in the state \(\ket{\gamma_+}\)
\[
\psi_{\tau_{\beta in}} = \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} \ket{i},
\]
we get the following state out of the \(N\)-SWITCH,

\[
S(\Lambda_\beta, ..., \Lambda_\beta) [\tau_{\beta in} \otimes \ket{\gamma_+} \bra{\gamma_+}] = \frac{1}{N} \left( (\tau_\beta + (N-1)\tau_\beta \tau_{\beta in} \tau_\beta) \otimes \ket{\gamma_+} \bra{\gamma_+} + (N-1)(\tau_\beta - \tau_\beta \tau_{\beta in} \tau_\beta) \otimes (\mathbb{I} - \ket{\gamma_+} \bra{\gamma_+}) \right).
\]

where \(\sigma_N = S(\Lambda_\beta, ..., \Lambda_\beta) [\tau_{\beta in} \otimes \ket{\gamma_+} \bra{\gamma_+}]\) is the output of the \(N\)-SWITCH. This means that for the initial temperatures \(\beta_{in}\) satisfying the bound \((20)\), local ergotropy extraction can be improved by increasing the number of thermal baths \(N\),

\[
W_d(\sigma_N) = 2 \left( 1 - \frac{1}{N} \right) W_d(\sigma_2),
\]

up to doubling the work which can be extracted in the usual 2-SWITCH scenario. However, the temperature bound \((20)\) remains unchanged and cannot be beaten in this manner.
Crucially, sharing a priori correlations (classical as well as quantum) between the target and control systems, as for separable $\sigma_{TC}$, does not allow one to locally extract ergotropy from the target system when the maps act consecutively or in mixture of their causal orders (for example, when the control system is discarded). However, as we will see in the following, it allows one to gain ergotropy beyond the temperature bound (20) when the application order of the thermalizing maps is put into a superposition by the quantum SWITCH.

A. Classical correlations between target and control

We start the analysis with the case of target and control systems sharing prior classical correlations. Importantly, we exclude quantum discord, which is known to positively contribute to ergotropy [29], and assume classical-classical correlations only [30, 31],

$$\rho_{TC} = \sum_i p_i \Pi_i \otimes \Theta_i,$$  \hspace{1cm} (27)

where $p_i$ are probabilities, i.e., $\sum_i p_i = 1$, and $\{\Pi_i\}$ and $\{\Theta_i\}$ are projectors onto certain orthonormal bases in the respective subsystems. The condition (26) of local thermality puts constraints on the possible expansions (27) of the joint state $\rho_{TC}$, namely,

$$\rho_{TC} = \frac{1}{Z_{\beta_m}} \left( |0\rangle \langle 0| \otimes |\psi\rangle \langle \psi| + e^{-\beta_m} |1\rangle \langle 1| \otimes |\psi^+\rangle \langle \psi^+| \right),$$  \hspace{1cm} (28)

where $\{\Theta_i\} = \{|\psi\rangle, |\psi^+\rangle\}$ constitute an arbitrary orthonormal basis for the control qubit. After the target is thermalized with the quantum-controlled baths, a joint output state $\sigma_{CC} = S(\Lambda_\beta, \Lambda_\beta) [\rho_{TC}]$ is produced by the quantum SWITCH reading

$$\sigma_{CC} = \frac{1}{2Z_{\beta_m}} \left( (\tau_\beta + \tau_\beta |0\rangle \langle 0| \otimes |\psi\rangle \langle \psi| + (\tau_\beta - \tau_\beta |0\rangle \langle 0| \otimes Z |\psi\rangle \langle \psi| Z \right.$$  

$$+ e^{-\beta_m} \left( (\tau_\beta + \tau_\beta |1\rangle \langle 1| \otimes |\psi^+\rangle \langle \psi^+| + (\tau_\beta - \tau_\beta |1\rangle \langle 1| \otimes Z |\psi^+\rangle \langle \psi^+| Z \right),$$  \hspace{1cm} (29)

where $Z$ is the Pauli Z-operator. Evidently, $\sigma_{CC}$ has block-diagonal form and, therefore, do not carry any coherence between target and control. On the other hand, the target component of each summand in (29) carries no coherence as well. Therefore, daemonic ergotropy that can be extracted from the quantum SWITCH with classically correlated input is of purely incoherent nature.

It can be demonstrated that, similar to the case of uncorrelated initial state, the maximal daemonic ergotropy can be achieved for $\Theta_i = |\pm\rangle$. Indeed, after measurement of the control in the optimal $\{|\pm\rangle\}$ basis, the (un-normalized) post-measurement states read as

$$\hat{\rho}_\pm = \frac{1}{2Z_\beta} \left( \begin{array}{cc} 1 & \frac{1}{Z_{\beta_m}} \\ \frac{1}{Z_{\beta_m}} & e^{-\beta} \left( 1 \mp \frac{1}{Z_{\beta_m}} e^{-(\beta+\beta_m)} \right) \end{array} \right),$$  \hspace{1cm} (30)

carrying the incoherent ergotropy

$$W_d(\sigma_{CC}) = \frac{1}{2Z_\beta^2 Z_{\beta_m}} \max\{0, e^{-2\beta} - e^{-\beta_m} + 2e^{-(\beta+\beta_m)} \},$$  \hspace{1cm} (31)

where $\sigma_{CC} = S(\Lambda_\beta, \Lambda_\beta) [\rho_{TC}]$ is the output of the quantum SWITCH. Hence, initial classical correlations between target
and control alters the temperature bound (20) for the non-zero ergotropy,

$$\ln(e^{\beta_{in}} + 2) > 2\beta.$$ \hspace{1cm} (32)

This means that initial classical correlations improve work extraction from the quantum SWITCH by shifting the temperature bound towards smaller $\beta_{in}$ (see Fig. 2). Moreover, it can easily seen that the left-hand side of (32) is bounded below by $\ln(3)$ corresponding to the infinite initial temperature (i.e., $\beta_{in} = 0$). Hence, the inverse temperature $\tilde{\beta} = \frac{\ln(3)}{2}$ of the maps can be seen as a critical value where the temperature bound (32) becomes trivial. Indeed, for $\beta < \tilde{\beta}$, classical correlations can activate ergotropy extraction in the entire range of initial temperatures $\beta_{in}$, so that the temperature bound is overcome completely. On the other hand, for $\beta \geq \frac{1}{2} \ln 3$, there still exists a finite range $\beta_{in} \in [0, \ln(e^\beta - 2)]$ with no possible ergotropy extraction. Therefore, we can conclude that prior classical correlations allow one to beat the temperature bound (20) only partially. Indeed, the temperature bound (32) from classically correlated target and control improves non-zero work extraction only in a tiny region of small inverse temperatures $\beta_{in}$ of initial state of the target and $\beta$ of the maps (see Fig. 3). In the following section, we assume quantum nature of the initial correlations between target and control and ask what are the advantages it can additionally offer. In particular, we examine whether the temperature bound (20) can be completely reduced in this setting.

**B. Do quantum correlations perform better?**

Now, we allow prior quantum correlations between target and control and identify maximal work that can be extracted from the output of the quantum SWITCH. Importantly, it can be shown that maximal ergotropy from target initially sharing quantum discord with control does not exceed one from target initially entangled with control. Therefore, for the sake of simplicity, we focus on entanglement in what follows. As in the case of initial classical correlations, we take into account that the target is initially in a thermal state $T_{\beta_{in}}$ due to condition (26). Hence, the joint state $\rho_{TC}$ can be seen as a purification $|T_{\alpha}(\beta_{in}) \rangle \langle T_{\alpha}(\beta_{in})|$ of $T_{\beta_{in}}$, where

$$|T_{\alpha}(\beta_{in})\rangle = \frac{1}{\sqrt{Z_{\beta_{in}}}} \left( |0\rangle + e^{-\frac{\beta_{in}}{2}} |1\rangle \right),$$ \hspace{1cm} (33)

is the initial pure joint state where the basis states of the target are entangled with the states $|\psi\rangle = \sqrt{\alpha}|0\rangle + e^{i\phi} \sqrt{1-\alpha}|1\rangle$ and $|\psi\rangle = e^{-i\phi} \sqrt{1-\alpha}|0\rangle - \sqrt{\alpha}|1\rangle$ of the control qubit that constitute an arbitrary orthonormal basis with $\alpha \in [0, 1]$ and $\phi \in [0, 2\pi]$. The output of the quantum SWITCH (see Appendix A for its precise form)

$$\sigma_{QC} = S(\Lambda_{\beta}, \Lambda_{\beta}) \left[ |T_{\alpha}(\beta_{in})\rangle \langle T_{\alpha}(\beta_{in})| \right],$$ \hspace{1cm} (34)

in contrast to (29), can carry coherence between target and control and, in turn, allows for a non-zero coherent contribution to ergotropy. This can be seen from the post-selected states of the target when the control system is measured in the $\{\pm\}$ basis,

$$\hat{\rho}_{\pm} = \frac{1}{2Z_{\beta}} \left( \pm \frac{e^{2i\phi} \sqrt{\alpha(1-\alpha)}}{Z_{\beta}^{2}Z_{\beta_{in}}} + \frac{1 + 2 \cos \phi \sqrt{\alpha(1-\alpha)}}{Z_{\beta}^{2}Z_{\beta_{in}}} e^{-\frac{\beta_{in}}{2}} e^{-\frac{\beta}{2}} \right) \left( 1 \pm \frac{2 \cos \phi \sqrt{\alpha(1-\alpha)}}{Z_{\beta}^{2}Z_{\beta_{in}}} e^{-\gamma(\beta_{in})} \right).$$ \hspace{1cm} (35)

For a given purification $(\alpha, \phi)$ of the initial thermal state $T_{\beta_{in}}$, the corresponding incoherent ergotropy reads

$$W_{d,c}(\alpha, \phi) = \frac{1}{2Z_{\beta}Z_{\beta_{in}}} \max \left\{ 0, e^{-2\beta} - e^{-\beta_{in}} + \left( 1 + 2 \cos \phi \sqrt{\alpha(1-\alpha)} \right) e^{-(2\beta + \beta_{in})} - \left( 1 - 2 \cos \phi \sqrt{\alpha(1-\alpha)} \right) \right\},$$ \hspace{1cm} (36)

while the expression for coherent ergotropy $W_{d,c}(\alpha, \phi)$ is omitted for the sake of brevity. Given the $(\beta_{in}, \beta)$-pair, we report

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Figure 3. Comparison of regions of non-zero ergotropy in the space of $(\beta_{in}, \beta)$-pairs for the initially uncorrelated and classically correlated target and control.
\begin{align}
\mathcal{W}_d(\sigma_{QC}) &= \max_{\alpha, \phi} \mathcal{W}_d(\sigma_{QC}; \alpha, \phi). \tag{37}
\end{align}

Importantly, the optimal purification that yields the maximal ergotropy is expectedly insensitive of the phase angle \( \phi \) and determined solely by \( \alpha \). Therefore, without loss of generality, we fix \( \phi = 0 \) and find that the optimal purification \( \alpha_{opt} \) depends crucially on relation between the temperatures \( \beta_{in} \) of the initial thermal state and \( \beta \) of the maps,

\[
\alpha_{opt}(\beta_{in}, \beta) = \begin{cases} 
0 \text{ or } 1, & \text{for } \beta_{in} \leq 2\beta \\
\frac{1}{2}, & \text{for } \beta_{in} > 2\beta 
\end{cases} . \tag{38}
\]

Indeed, taking into account the temperature bound (20) affecting the ergotropy for the initially uncorrelated target and control, we conclude that, given the temperature \( \beta \) of the maps, initial temperatures violating (20) demand purification in the computational basis of the control for the maximal ergotropy extraction, whereas, for ones violating the temperature bound, the optimal purification is in the \( \{|\pm\rangle\} \) basis.

For the purification on the computational basis, the resulting ergotropy

\[
\mathcal{W}_d(\sigma_{QC}) = \frac{1}{2} \tanh \left( \frac{\beta}{2} \right) \left( \sqrt{1 + \frac{1}{4} \sinh^{-2}(\beta) \cosh^{-2}\left(\frac{\beta_{in}}{2}\right)} - 1 \right) \tag{39}
\]

is non-zero for almost any \( (\beta_{in}, \beta) \)-pair, except for extreme cases of infinite inverse temperatures, and is constituted only by the coherent counterpart. This stands in stark contrast to prior classical correlations which allow one for exclusively incoherent ergotropy extraction only within a certain region of temperature pairs. Interestingly, in the range \( 0 \leq \beta_{in} \leq 2\beta \), the amount of extractable work is larger for higher input temperatures, i.e., smaller \( \beta_{in} \) values (see Fig. 4). This feature differs from behavior of ergotropy in the setups with initially uncorrelated and classically correlated target and control, where, in the regions of non-zero ergotropy, it increases while increasing \( \beta_{in} \). Indeed, coherent ergotropy varies monotonically with the entanglement between the target and the control. Since the latter depends on the inverse temperature as \( (\cosh \frac{\beta_{in}}{2})^{-1} \), i.e., is its decreasing function and, therefore, makes the extractable work decreasing while increasing \( \beta_{in} \).

On the other hand, for temperatures satisfying the temperature bound (20), we find enhancement of ergotropy under initially entangled target and control compared with uncorrelated ones. Interestingly, in this case, the maximal ergotropy and its behavior coincide with the maximal ergotropy that can be extracted when the target and control are correlated only classically. Indeed, the sharp change in the optimal configuration at \( \beta_{in} = 2\beta \) makes the type of extractable ergotropy from being coherent \( (\beta_{in} \leq 2\beta) \) to incoherent \( (\beta_{in} > 2\beta) \). This results in a non-monotonic behaviour of the maximal extractable ergotropy \( \mathcal{W}_d \) that decreases with \( \beta_{in} \) for \( \beta_{in} \leq 2\beta \) while increasing with \( \beta_{in} \) for \( \beta_{in} > 2\beta \), see Fig. 5.

![Figure 4. Daemonic ergotropy from the output of thermalizing maps at inverse temperature \( \beta = 1 \) controlled by the quantum SWITCH depending on the purification of the initial thermal state. For \( \beta_{in} < 2\beta \) we the optimal \( \alpha = 0 \) or 1. For \( \beta_{in} > 2\beta \), the optimal \( \alpha = \frac{1}{2} \).](image)

V. CONCLUSIONS

Thermal states and channels are considered to be useless for work extraction in thermodynamics. However, in the quantum domain, there is a possibility of making the causal order of two processes indefinite. The quantum SWITCH is an example of such a operation that can induce causal indefiniteness in the action of two or more maps (channels) using a control system. Using the quantum SWITCH, it was shown in [16] that a finite amount of ergotropic work extraction is possible using the apparently thermodynamically useless setting of an initial thermal state passing through two thermal maps. The quantum SWITCH puts the thermal maps in a superposition of causal orders which accounts for the positive work extraction. However, the ergotropics gains reported in [16] are conditional and occurs only when the temperature of the thermal maps should be less than twice the temperature of the input thermal state. In this work, we attempt to lift these restrictions on work extraction. In earlier works, the state of the control was taken to be in the Fourier basis (c.f. [32]) that induces superposition of causal order. Interestingly, the quantum SWITCH can support more general causal orderings when the target and the control pre-share some correlation. In this generalized setting, we consider initially correlated target control states such that the marginal target state is thermal. Note that in absence of the SWITCH, no ergotropy can be extracted from the target system when it passes through two thermal maps irrespective of the correlation that the target share with any other system.

In this work we systematically analyze all possible correlations that the target and control can share and explore its impact on local ergotropy extraction from the target system. First we examine the situation when the target and control share classical correlation. In this case we report enhanced ergotropy extraction in the domain where the product target
control configuration with the control in the superposition state gave a finite ergotropy extraction. In the region where no ergotropy extraction was possible using the product setting, we observe two distinct features depending on the temperature of the thermal maps. If the inverse temperature of the maps is less than a critical value, the prior classical correlation enable extraction of ergotropy for all input target temperatures: “complete activation”. In the other case, although the temperature bound gets shifted to include a greater domain of input temperature for ergotropy extraction but there still remains a forbidden zone: “partial activation”.

We then move on to the domain of quantum correlations and consider entangled target control states. We find out that prior entanglement can induce unconditional “activation” of ergotropy extraction in the entire forbidden region. Inside the forbidden region, the amount of ergotropy obtained is also greater compared to what can be extracted using only classical correlations. Outside the forbidden region there is no difference in the enhancement offered by prior classical correlation and discord over the initial product configuration. Surprisingly, prior entanglement do not give any more extractable ergotropy than offered by a separable discordant state.

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**Appendix A: The state after the SWITCH action**

The output state \(\sigma\) in the computational basis \(|ij\rangle\), where \(i, j \in \{0, 1\}\) is given in the matrix form by

\[
\sigma_Q = \begin{bmatrix}
\rho(1-\alpha+q(-1+2\alpha)) & e^{-i\phi}p^2q\sqrt{\alpha}(1-\alpha) & 0 & -(1-p)p\alpha\sqrt{q(1-q)} \\
\frac{e^{i\phi}p^2q\sqrt{\alpha}(1-\alpha)}{\rho(1-\alpha+q(-1+2\alpha))} & \frac{p(1-\alpha+q(-1+2\alpha))}{\rho} & 0 & 0 \\
-(1-p)p\alpha\sqrt{q(1-q)} & 0 & -(1-p)p\alpha\sqrt{q(1-q)} & 0 \\
\frac{-(1-p)p\alpha\sqrt{q(1-q)}}{e^{-i\phi}(1-p)p(1-\alpha)\sqrt{q(1-q)}} & 0 & \frac{-(1-p)p\alpha\sqrt{q(1-q)}}{e^{i\phi}(1-p)p(1-\alpha)\sqrt{q(1-q)}} & e^{i\phi}(1-p)p(1-\alpha)\sqrt{q(1-q)}
\end{bmatrix}.
\]

Here \(p\) is related to the temperature of the thermal maps via \(\beta = \ln \frac{p}{1-p}\), while the temperature of the local thermal state is given by \(\beta_{in} = \ln \frac{\beta}{1-\beta}\).

**Appendix B: The post measurement states**

The un-normalized post measurement states of the target system after the control is measured in the \(|+\rangle, |-\rangle\)-basis read as

\[
\tilde{\rho}_s = \frac{1}{2} \begin{bmatrix}
p \pm 2\cos\phi p^2q\sqrt{\alpha(1-\alpha)} & \pm p(1-p)\sqrt{q(1-q)}(e^{-2i\phi}(1-\alpha)-\alpha) \\
\pm p(1-p)\sqrt{q(1-q)}(e^{2i\phi}(1-\alpha)-\alpha) & (1-p) \pm 2\cos\phi p^2(1-p)^2(1-q)\sqrt{\alpha(1-\alpha)}
\end{bmatrix}.
\]

The corresponding normalized states are \(\rho_s = \frac{\tilde{\rho}_s}{p_s}\), where \(p_s = \text{tr}\tilde{\rho}_s\). In this case also, \(p\) is related to the temperature of the thermal maps via \(\beta = \ln \frac{p}{1-p}\), while \(\beta_{in} = \ln \frac{\beta}{1-\beta}\) is the temperature of the local thermal state.
Thermodynamic Batteries powered by generalized measurements with indefinite causal order activated by coherent-controlled thermalisation, "Quantum network boosted by entanglement with a control system," (2022), arXiv:2206.05247.