Strong edge-colorings for \( k \)-degenerate graphs

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Abstract We prove that the strong chromatic index for each \( k \)-degenerate graph with maximum degree \( \Delta \) is at most \((4k - 2)\Delta - k(2k - 1) + 1\). This confirms a conjecture of Chang and Narayanan.

Keywords Strong edge-coloring · \( k \)-degenerate graphs · Induced matching

A strong edge-coloring of a graph \( G \) is an edge-coloring so that no edge can be adjacent to two edges with the same color. So in a strong edge-coloring, every color class gives an induced matching. The strong chromatic index \( \chi'_s(G) \) is the minimum number of colors needed to color \( E(G) \) strongly. For example, the chromatic index of a 5-cycle is 3, while its strong chromatic index is 5. This notion was introduced by Fouquet and Jolivet [5]. Erdős and Nešetřil during a seminar in Prague in 1985 proposed some open problems, one of which is the following

Conjecture 1 (Erdős and Nešetřil, 1985) If \( G \) is a simple graph with maximum degree \( \Delta \), then \( \chi'_s(G) \leq 5\Delta^2/4 \) if \( \Delta \) is even, and \( \chi'_s(G) \leq (5\Delta^2 - 2\Delta + 1)/4 \) if \( \Delta \) is odd.

This conjecture is true for \( \Delta \leq 3 \) [1], [6]. Cranston [4] showed that \( \chi'_s(G) \leq 22 \) for \( \Delta = 4 \). Chung et al. [3] showed that the upper bounds are exactly the numbers of edges in \( 2K_2 \)-free graphs. Molloy and Reed [8] proved that graphs with sufficient large maximum degree \( \Delta \) has strong chromatic index at most \( 1.998\Delta^2 \). For more results see [9] (Chapter 6, problem 17).

A graph is \( k \)-degenerate if every subgraph has minimum degree at most \( k \). Chang and Narayanan [2] recently proved that a 2-degenerate graph with maximum degree
Δ has strong chromatic index at most 10Δ − 10. Luo and the author in [7] improved the upper bound to 8Δ − 4.

In [2], the following conjecture was made

**Conjecture 2** (Chang and Narayanan [2]) There exists an absolute constant c such that for any k-degenerate graphs G with maximum degree Δ, \( \chi'_c(G) \leq ck^2\Delta \). Furthermore, the \( k^2 \) may be replaced by \( k \).

In this paper, we prove a stronger form of the conjecture. Unlike the priming processes in [2] and [7], we find a special ordering of the edges and by using a greedy coloring obtain the following result.

**Theorem 1** The strong chromatic index for each k-degenerate graph with maximum degree Δ is at most \((4k−2)Δ−k(2k−1)+1\).

Thus, 2-degenerate graphs have strong chromatic index at most 6Δ − 5.

**Proof** By definition of k-degenerate graphs, after the removal of all vertices of degree at most \( k \), the remaining graph has no edges or has new vertices of degree at most \( k \), thus we have the following simple fact on k-degenerate graphs (see also [2]).

Let \( G \) be a k-degenerate graph. Then there exists \( u \in V(G) \) so that \( u \) is adjacent to at most \( k \) vertices of degree more than \( k \). Moreover, if \( Δ(G) > k \), then the vertex \( u \) can be selected with degree more than \( k \).

We call a vertex \( u \) a special vertex if \( u \) is adjacent to at most \( k \) vertices of degree more than \( k \). An edge is a special edge if it is incident to a special vertex and a vertex with degree at most \( k \). The above fact implies that every k-degenerate graph has a special edge, and if \( Δ \leq k \), then every vertex and every edge are special.

We order the edges of \( G \) as follows. First we find in \( G \) a special edge, put it at the beginning of the list, and then remove it from \( G \). Repeat the above step in the remaining graph. When the process ends, we have an ordered list of the edges in \( G \), say \( e_1, e_2, \ldots, e_m \), where \( m = |E(G)| \). So \( e_m \) is the special edge we first chose and placed in the list.

Let \( G_i \) be the graph induced by the first \( i \) edges in the list, \( i = 1, 2, \ldots, m \). Then \( e_i \) is a special edge in \( G_i \). We now count the edges of \( G_i \) within distance one to \( e_i \) in \( G \). We may call the edges in \( G_i \) blue edges and the edges in \( G − G_i \) yellow edges. Let \( u_i, v_i \) be the endpoints of \( e_i \) with \( u_i \) being a special vertex in \( G_i \).

We first count the blue edges incident to \( u_i \) and its neighbors. The vertex \( u_i \) has three kinds of neighbors: the neighbors in \( X_1 \) sharing blue edges with \( u_i \) and having degree more than \( k \), the neighbors in \( X_2 \) sharing blue edges with \( u_i \) and having degree at most \( k \) (thus \( v_i \in X_2 \)), and the neighbors in \( X_3 \) sharing yellow edges with \( u_i \). By definition, \(|X_1| \leq k\), so at most \( |X_1|Δ + k(|X_2|−1) \) blue edges are incident to \( X_1 \cup (X_2−\{v_i\}) \). For each vertex \( u \) in \( X_3 \), \( uu_i \) is a yellow edge in \( G_i \) but will be a special edge in \( G_j \) for some \( j > i \). So either \( u \) or \( u_i \) has degree at most \( k \) in \( G_j \) (thus also in \( G_i \)), and if \( u_i \) has degree at least \( k \) in \( G_m \) for some \( m \), then all yellow edges incident to \( u_i \) in \( G_m \) should have degree at most \( k−1 \) in \( G_m \), in order for the yellow edges to be special later. Then among vertices in \( X_3 \), at most \( x = \max\{0, k−|X_1|−|X_2|\} \) vertices have degree more than \( k \) in \( G_i \), and all other vertices have degree at most \( k−1 \) in \( G_i \).
Therefore at most $x\Delta + (|X_3| - x)(k - 1)$ blue edges are incident to $X_3$. Note that $d(u_i) \leq \Delta$, $|X_2| \leq \Delta$ and $|X_1| + x \leq k$, then at most

$$
|X_1|\Delta + k(|X_2| - 1) + x\Delta + (|X_3| - x)(k - 1) = (|X_1| + x)\Delta + (k - 1)(d(u_i)
- |X_1| - x - 1) + |X_2| - 1 \leq 2k\Delta - k^2
$$

blue edges are within distance one to $e_i$ from $u_i$ side (not including the edges incident to $v_i$).

We also count the blue edges incident to $v_i$ and its neighbors. Similarly, $v_i$ has two kinds of neighbors: the neighbors in $Y_1$ sharing blue edges with $v_i$, and the neighbors in $Y_2$ sharing yellows edges with $v_i$. From the fact that $e_i$ is a special edge, $|Y_1| \leq k$, so at most $(|Y_1| - 1)\Delta$ blue edges are incident to $Y_1 - \{u_i\}$. For each vertex $v$ in $Y_2$, $vv_i$ is a yellow edge in $G_i$ but will be a special edge in $G_s$ for some $s > i$. Similar to above, at most $k - |Y_1|$ vertices in $Y_2$ have degree more than $k$ in $G_i$, and all other vertices in $Y_2$ have degree at most $k - 1$ in $G_i$. So at most $(k - |Y_1|)(\Delta - 1) + (|Y_2| - (k - |Y_1|))(k - 1)$ blue edges are incident to $Y_2$. In total, at most

$$(|Y_1| - 1)\Delta + (k - |Y_1|)(\Delta - 1) + (|Y_2| - (k - |Y_1|))(k - 1) \leq (2k - 2)\Delta - k(k - 1)
$$

So in $G_i$, the number of blue edges within distance one to $e_i$ is at most

$$2k\Delta - k^2 + (2k - 2)\Delta - k(k - 1) \leq (4k - 2)\Delta - k(2k - 1)
$$

Now color the edges in the list one by one greedily. For each $i$, when it is the turn to color $e_i$, only the edges in $G_i$ (the blue edges) have been colored. Since there are at least $(4k - 2)\Delta - k(2k - 1) + 1$ colors, we are able to color the edges so that edges within distance one get different colors. \qed

We shall note that the above result is not only true for simple graphs, but also for multigraphs.

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