Spin dependent potentials from SU(2) gauge theory

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We present results on spin dependent potentials from lattice simulations of SU(2) gauge theory. The Coulomb like short range part of the central potential is identified as a mixed vector-scalar exchange while the linear long range part is pure scalar.

1. INTRODUCTION

Potential models have been found to be surprisingly successful in describing the meson spectrum, even at rather small quark masses, \( m \). To derive the semi-relativistic Hamiltonian one usually starts from the Bethe-Salpeter amplitude of a two-quark bound state with quark separation \( r \), takes the static solution \( (m \rightarrow \infty) \) of the kernel, and expands it in inverse powers of the quark mass around this limit. The result can be decomposed into the five relativistic invariants \( (S,V,T,A,\mathbf{L}) \) with \( r \) dependent (unknown) coefficient functions, the spin dependent potentials (SDPs).

So far, SDPs have been studied in the mid eighties \cite{2,3} on the lattice. The applicability of the potential approach to heavy quark physics can be tested on the lattice by comparing the spectrum and wave functions, obtained from the semi-relativistic Hamiltonian with SDPs, to NRQCD results. In addition we aim at replacing phenomenologically adjusted potentials by the ”correct” QCD potentials.

The order \( 1/m^2 \) Hamiltonian, obtained in the manner described above, is a generalisation of the familiar Breit-Fermi Hamiltonian. Apart from a purely kinetic term, it reads as follows (for equal masses), \( V_{SD} = V_0 + (V_{LS} + V_{SS} + V_{od})/m^2 \), with

\[
V_{LS}(r) = \frac{\vec{L}_1 \vec{s}_1 - \vec{L}_2 \vec{s}_2}{r} \left( \frac{V_0'(r)}{2} + V_1'(r) \right)
\]

where \( \vec{L}_i = \vec{r} \times \vec{p}_i \). \( V_{od} \) are velocity dependent corrections.

Explicit expectation values have been associated to the spin-orbit and spin-spin “potentials” \( V_1', V_2' \) and \( V_3, V_4 \), which can be computed on the lattice \cite{4}. These potentials are related to scalar \( (S) \), vector \( (V) \) and pseudo-scalar \( (P) \) exchange contributions in the following way:

\[
V_0(r) = S(r) + V(r)
\]

\[
V_3(r) = \frac{V'(r) - P'(r)}{r} - (V''(r) - P''(r))
\]

\[
V_4(r) = 2\nabla^2 V(r) + \nabla^2 P(r).
\]

In addition, relativistic invariance yields the Gromes relation \cite{6}, \( V_0'(r) = V_2'(r) - V_1'(r) \).

2. LATTICE TECHNIQUES

From the Cornell form of the central potential, \( V_0(R) = V_0 + KR - e/R \), and tree level perturbation theory one expects

\[
V_1' = -K, \ V_2' = \frac{e}{R^2}, \ V_3 = \frac{3e}{R^3}, \ V_4 = 8\pi e\delta(R)(6)
\]

Since all these potentials (up to \( V_3' \)) are predicted to be short ranged, we have chosen small lattice spacings to be sensitive to the physically interesting region: we operate at the values \( \beta = 2.74 \)
and \( \beta = 2.96 \) that correspond to resolutions \( a = 0.0408(2) \text{ fm} \) and \( a = 0.0183(6) \text{ fm} \), respectively. The physical scale has been obtained from the relation \( \sqrt{K}a = 440 \text{ MeV} \). More than 200 statistically independent configurations of lattice volume \( 32^4 \) have been collected at each coupling.

In order to reduce statistical fluctuations, temporal links are integrated analytically wherever possible \([\text{8}]\). The SDPs have been defined in a way analogous to the computation of local masses to exclude corrections proportional to \( 1/T \).

The lattice correlation functions, \( C(T) \), to be measured are Wilson loops with two colour field insertions (ears) within the temporal transporters, divided by the corresponding loop without ears (\( T \) denotes the ear-ear temporal distance). One ear excites the gluon field between the two charges while the second returns the field into its ground state. To obtain the SDPs, an integration over all possible interaction times (temporal positions of the ears) has to be performed. The minimal distance, \( \Delta \), of an ear to an “end” of the Wilson loop, occurring within the integration, represents the time the gluon field has to travel from the beginning, allowing us to reduce \( \Delta \): plateaus in the ear-ear correlators have been found from \( \Delta = 2 \) onwards (in some cases even \( \Delta = 1 \)).

In our implementation the positions of the ears in respect to the Wilson loop ends are kept fixed at \( \Delta = 2 \). The interaction time is varied by increasing the temporal Wilson loop extent. This results in smaller statistical fluctuations, compared to the standard definition \([\text{4}]\) in which the temporal extent of the Wilson loop is kept fixed and the positions of the ears are varied.

Previous authors replaced the integration over \( T \) by a discrete sum. The resulting discretization error adds to other lattice artefacts. We take into account that \( C(T) \) depends on \( T \) in a multi-exponential way and approximate the integral from \( T \) to \( T + 1 \) by the expression \(( C(T) - C(T+1))/\log(C(T)/C(T+1)) \), instead. For \( V_0' \) and \( V_2' \) where \( C(T) \) is weighted by an additional factor \( T \), an analogous interpolating formula is used. The \( T \to \infty \) tail of the correlator is estimated from an exponential fit. An integration cut-off \( T_{\text{max}} \) is chosen such that the statistical error exceeds the estimated contribution from the asymptotic tail by a factor four.

The lattice SDPs undergo multiplicative renormalizations in respect to their continuum counterparts, while \( V_0 \) does not. A non-perturbative renormalization of these quantities in the manner suggested in Ref. \([\text{3}]\) is performed. In fig. 1 the quality of this normalization is checked by comparing the Gromes combination \( V_2' - V_1' \) (in units of the string tension, \( K \)) to the force (solid curve), derived from a fit to the central potential. We find good scaling behaviour among the two data sets (\( \beta = 2.74 \) and 2.96) and validity of the Gromes relation outside of the region of lattice artifacts. A more detailed description of the improvement tricks and data analysis can be found in Ref. \([\text{8}]\).

![Figure 1. Test of the Gromes relation.](image)

3. RESULTS

We find the central potential, \( V_0 \), to contain a very clear linear confining part up to large distances which must be of scalar exchange type as a vector exchange can grow at most logarithmically with \( r \). We confirm the second spin-orbit potential \( V_2' \) to be definitely of short range nature, leaving little room for a scalar contribution to this potential. The first spin-orbit potential, \( V_1' \), is displayed in fig. 2. Our lattice resolution enables us to establish an attractive short range contribution that can be well fitted to a Coulomb \((1/R^2)\) form, in addition to the constant long-range term, which is in agreement with the string tension, \( K \).
In principle, the potentials $V_0$, $V_3$, and $V_4$ allow for a determination of $S$, $V$, and $P$ (eqs. (3)–(5)). At this stage, we will assume $P$ to vanish and $V_1$ to be pure scalar (which induces the equalities $V_2 = V$ and $V_1 = -S$). This leads to a prediction for $V_3$, according to eq. (4), which can be checked in fig. 3: we find reasonable agreement between the data sets and the predicted curve, $V''_2/R - V''_3$, the deviations being qualitatively understood from tree level lattice perturbation theory.

The remaining spin-spin potential, $V_4$, exhibits oscillatory behaviour as a lattice artifact (fig. 4) and can largely be understood as a $\delta$-contribution, according to $V_4 = 2\nabla^2 V_2$. The tree level lattice perturbative expectation $8\pi c_\delta L(R)$ is indicated by crosses in the figure. The normalization has been obtained from a $c/R^2$ fit to the $V''_2$ data points. The error ranges without symbols are obtained by using single lattice gluon exchange with an infra-red protected two-loop running coupling. The range corresponds to different choices of the $\Lambda$-parameter. Apart from the dominant $\delta$-like contribution, another very short ranged contribution seems to exist.

4. CONCLUSIONS

We find a consistent picture when identifying $-V_1$ ($V_2$) with the scalar (vector) exchange contribution, other exchange types being negligible to order $1/m^2$. Apart from the linear large distance part, we observe a short distance Coulomb like scalar contribution from the first spin-orbit potential. It appears that the Coulomb part of the central potential splits up into a vector/scalar ratio in between $3/1$ and $4/1$. The short range potentials can be qualitatively understood in terms of perturbation theory.

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