INFINITE PRODUCT REPRESENTATIONS FOR MULTIPLE GAMMA FUNCTION

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Abstract. Two kinds of infinite product representations for Vignéras multiple gamma function are presented. As an application of these formulas, a multiplication formula for the function is derived.

INTRODUCTION

In a series of papers [1, 3, 5, 7], Barnes introduced multiple gamma functions associated with a certain generalization of the Hurwitz zeta function. In relevant with a special case of Barnes’ function, Vignéras [29] introduced her multiple gamma functions \(G_r(z)\) \((r \in \mathbb{Z}_{\geq 0})\) as a sequence of meromorphic functions uniquely determined by the following relations:

\[
\begin{align*}
(i) & \quad G_0(z) = z, \\
(ii) & \quad G_r(1) = 1, \\
(iii) & \quad G_r(z + 1) = G_{r-1}(z)G_r(z) \\
(iv) & \quad d_{r+1} \log G_r(z + 1) \geq 0 \quad \text{for } z \geq 0.
\end{align*}
\]

This formulation can be considered as a generalization of the Bohr-Morellup theorem. For example, \(G_1(z)\) is the celebrated Euler gamma function \(\Gamma(z)\) (cf. Artin [3], Whittaker-Watson [31]). \(G_2(z)\) is \(G\)-function introduced in Barnes [4].

Vignéras multiple gamma function has various applications to number theory, geometry and mathematical physics. It is known that Vignéras’ function appears in factors in the determinants of the Laplacians on the some compact Riemann surfaces. Sarnak [25] applied \(G_2(z)\) to the representation of the determinants of the Laplacian on spinor fields on a Riemann surface. Many researchers [11, 13, 16, 28, 30] computed the determinants in the case where the Laplacians are on the \(n\)-sphere. \(G_2(z)\) is used to represent factors of Töplitz determinants. For example, we can refer to Böttcher-Silbermann [9]. Tracy [26] and Basor-Tracy [10] applied the fact to a representation of coefficients in an asymptotic expansion for \(\tau\)-function of the Ising model. As mentioned in [1], it is expected to apply Vignéras’ function to random matrix theory. \(G_2(z)\) also appears in an asymptotic behavior of the mean value of a certain \(L\)-function [12]. Vignéras’ functions are closely related with Kurokawa’s multiple sine function [15, 19, 20, 24]. The gamma factor of the Selberg zeta function is represented as products of these functions.

As its significance is gradually recognized, studies on Vignéras’ function have become of a much interest to researchers in the theory of special functions. Ueno

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and the author [27] derived an asymptotic expansion ("higher Stirling formula") and an Weierstrass’ infinite product representation for $G_r(z)$. The author [23] gave another proof of Barnes’ generalization of the Hölder theorem [8]. It is proved that Vignéras functions satisfy no algebraic differential equation. We should note results by Ferreira and Lopez [14] and by Adamtik [1, 2].

In this paper, we present two types of infinite product representations of Vignéras’ multiple gamma function, which can be considered as a generalization of the Gauss and of the Euler product formula of Euler’s gamma function

\[
\Gamma(z + 1) = \lim_{N \to \infty} \frac{N!}{(z + 1)(z + 2) \cdots (z + N)(N + 1)^z}
\]

\[
= \prod_{n=1}^{\infty} \left[ (1 + \frac{z}{n})^{-1} \left( 1 + \frac{1}{n} \right)^z \right]
\]

(cf. Artin [3], Whittaker-Watson [31]). Our main theorem is stated as follows: If $z$ is not negative integer, the multiple gamma function $G_r(z)$ is represented as

\[
G_r(z + 1) = \lim_{N \to \infty} \prod_{n=1}^{N} \frac{G_{r-1}(n)}{G_{r-1}(z + n)} \prod_{k=0}^{r-1} G_k(N + 1)(z_k)
\]

\[
= \prod_{n=1}^{\infty} \left[ \frac{G_{r-1}(n)}{G_{r-1}(z + n)} \prod_{k=0}^{r-1} \left( \frac{G_k(n + 1)}{G_k(n)} \right)^{z_k} \right].
\]

In the case when $r = 1$, these formulas coincide with (0.2) and (0.3). We can find the representation for $G_2(z)$ in Jackson [15]. It should be noted that infinite product formula of these types for a $q$-analogue of the multiple gamma function were already obtained in [22]. However, in contrast to simplicity in $q$-case, some delicate techniques are necessary to deal with infinite products of Vignéras’ function. We verify (0.4) and (0.5) in section 1. The point is to apply an asymptotic expansion in [27] to estimations for products of Vignéras’ functions.

In section 2, as an application of infinite product representations, we derive a multiplication formula for Vignéras’ multiple gamma function, which can be regarded as a generalization of the well known formula

\[
\prod_{m=0}^{p-1} \Gamma \left( \frac{z + m}{p} \right) = \frac{(2\pi)^{p-1}}{p^{\frac{p-1}{2}}} \Gamma(z)
\]

for Euler’s gamma function (cf. Artin [3], Whittaker-Watson [31]). It is described as follows:

\[
\prod_{q_1, q_2, \cdots, q_r = 0}^{p-1} G_r \left( \frac{z + q_1 + \cdots + q_r}{p} \right) = \frac{e^{\phi_r(z)}}{p^{Q_r(z)}} G(z)
\]

It might be seem that formula of this type can be guessed easily from (0.1). However, it is not easy to determine explicit forms of $\phi_r(z)$ and $\psi_r(z)$. The reason why we can do it is usefulness of our representations (1.4).

On preparing the first draft of this paper, the author was informed about Kuribaysahi’s result [17]. He proved a multiplication formula

\[
\prod_{q_1, \cdots, q_r = 0}^{p-1} \Gamma_r \left( \frac{z + q_1 + \cdots + q_r}{p} \right) = p^{Q_r(z)} \Gamma_r(z).
\]
Infinite Product Representations for Multiple Gamma Function

for a function \( \Gamma_r(z) \) defined as

\[
\Gamma_r(z) := \exp \left[ \frac{\partial}{\partial s} \zeta_r(s, z) \right]_{s=0},
\]

where \( \zeta_r(s, z) \) is a generalization of the Hurwitz zeta function defined as the analytical continuation of the series

\[
\zeta_r(s, z) := \sum_{n_1, \ldots, n_r=0}^{\infty} (z + n_1 + \cdots + n_r)^{-s}, \quad \Re s > r.
\]

At the end of section 2, discussions about relations between his result and ours are added. We can find the following relation:

\[
Q_r(z) = (-1)^r \psi_r(z),
\]

although they look different at first sight. It may be worth noting that the relation \( 0.7 \) seems to be applicable to Kurokawa’s multiple sine function [18, 19, 20, 21]. \( 0.7 \) can be verified without use of the zeta function. We give an elementary proof in appendix.

For simplicity, we call Vignéras multiple gamma function only “multiple gamma function” in the following sections.

Notations: In this paper, we use notation \( B_r(z) \) for the Bernoulli polynomial defined by the generating function

\[
\sum_{r=0}^{\infty} B_r(z) t^r = \frac{t e^t}{1 - e^t},
\]

and \( B_r \) for the Bernoulli number defined as \( B_r := B_r(0) \). We introduce the Stirling number \( r S_j \) of the 1st kind by

\[
t(t - 1) \cdots (t - r + 1) = \sum_{j=0}^{r} r S_j t^j.
\]

The notation \( \zeta(s) \) is used to refer to the Riemann zeta function defined as the series \( \zeta(s) := \sum_{n=1}^{\infty} n^{-s} \) and its analytical continuation. \( \zeta'(s) \) is the first derivative of \( \zeta(s) \) defined by \( \zeta'(s) := \frac{d}{ds} \zeta(s) \).

1. Infinite Product Representation

As mentioned in introduction, our main theorem is described as follows:

Theorem 1.1. If \( z \) is not negative integer and is included in any finite region of complex plane, the multiple gamma function \( G_r(z) \) is represented as

\[
G_r(z + 1) = \lim_{N \to \infty} \left[ \prod_{n=1}^{N} \frac{G_{r-1}(n)}{G_{r-1}(z + n)} \prod_{k=0}^{r-1} G_k(N + 1)(z^k) \right]
\]

(1.1)

\[
= \prod_{n=1}^{\infty} \left[ \frac{G_{r-1}(n)}{G_{r-1}(z + n)} \prod_{k=0}^{r-1} \left( \frac{G_k(n + 1)}{G_k(n)} \right)^{(z^k)} \right].
\]

(1.2)
Proof. From the Gauss product representation \([1,2]\), the Euler product representation \([1,2]\) follows immediately. So, we give a proof of \([1,2]\) in this section. We apply an asymptotic expansion for \(G_r(z)\), which was firstly appeared in \([27]\).

**Theorem 1.2** (Ueno-Nishizawa). Let us put \(0 < \delta < \pi\), then, as \(|z| \to \infty\) in the sector \(\{z \in \mathbb{C} | \arg z | < \pi - \delta\} \)

\[
\log G_r(z+1) = \left\{ \left( \frac{z+1}{r} \right) + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} G_{r,j}(z) \right\} \log(z+1) - \sum_{j=0}^{r-1} \frac{G_{r,j}(z)(z+1)^{j+1}}{(j+1)^2} - \sum_{j=0}^{r-1} G_{r,j}(z) \zeta'(-j) + O(z^{-1}).
\]

(1.3)

where a polynomial \(G_{r,j}(z)\) is defined by the generating function

\[
\left( \frac{z-u}{r-1} \right) =: \sum_{j=0}^{r-1} G_{r,j}(z)u^j \quad (r=0 \ldots r-1), \quad G_{r,j}(z) = 0, \quad (j \geq r).
\]

In our proof, the following lemma is useful:

**Lemma 1.3.** For arbitrary \(x, y \in \mathbb{C}\),

(i) \[\sum_{k=0}^{r} \binom{x}{r-k} \binom{y}{k} = \binom{x+y}{r},\]

(ii) \[\sum_{k=0}^{r} \binom{x}{r-k} G_{k,j}(y) = G_{r,j}(x+y).
\]

Noting this lemma and that

\[\sum_{j=0}^{r-1} G_{r,j}(z+N-1) \left\{ \frac{(z+N)^{j+1}}{j+1} - \frac{N^{j+1}}{(j+1)^2} \right\} = \int_{N}^{z+N} \frac{dv}{v} \int_{0}^{v} \left( \frac{z+N-1-u}{r-1} \right) du,
\]

we rewrite the logarithms of terms in brackets of \([1,1]\) and have the following asymptotic behavior as \(N \to \infty\):

\[
\log \left[ \prod_{n=1}^{N} \frac{G_{r-1}(n)}{G_{r-1}(z+n)} \prod_{k=0}^{r-1} G_{k}(N+1)^{(z-k)} \right] = \\
= \log G_r(z+1) + \sum_{k=0}^{r} \binom{z}{r-k} \left\{ \binom{N}{k} + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} G_{k,j}(N-1) \right\} \log N - \\
- \sum_{j=0}^{r-1} \sum_{k=0}^{r} \binom{z}{r-k} G_{k,j}(N-1) \frac{N^{j+1}}{j+1} - \sum_{j=0}^{r-1} \sum_{k=0}^{r} \binom{z}{r-k} G_{k,j}(N-1) \zeta'(-j) - \\
- \left\{ \binom{z+N}{r} + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} G_{r,j}(z+N-1) \right\} \log(z+N) + \\
+ \sum_{j=0}^{r-1} G_{r,j}(z+N-1) \frac{(z+N)^{j+1}}{j+1} + \sum_{j=0}^{r-1} G_{r,j}(z+N-1) \zeta'(-j) + O(N^{-1}) = \\
= \log G_r(z+1) + \int_{0}^{z} \frac{du}{u+N} \left( \frac{z+N}{r} \right) + \sum_{j=0}^{r} \frac{B_{j+1}}{j+1} G_{r,j-1}(z+N-1) -
\]
Lemma 1.4 (Ueno-Nishizawa). For arbitrary \( z \in \mathbb{C} \), we have

\[
\binom{z}{r} + \sum_{j=0}^{r-1} B_{j+1} G_{r,j}(z-1) = \int_{-1}^{z-1} \binom{t}{r-1} dt + \sum_{j=0}^{r-1} B_{j+1} G_{r,j}(-1).
\]

This was already shown in [27]. Therefore, we have proved theorem 1.1. \( \Box \)

2. Multiplication formula

As an application of Gauss' product representation, we demonstrate the multiplication formula of the multiple gamma function.

Theorem 2.1.

\[
\prod_{q_1, \cdots, q_r=0}^{p-1} G_r \left( \frac{z+q_1+\cdots+q_r}{p} \right) = e^{\phi_r(z)} \left/ p^{\psi_r(z)} \right. G(z)
\]

where

\[
\phi_r(z) = \sum_{j=0}^{r-1} \left[ \sum_{q_1, \cdots, q_r=0}^{p-1} G_{r,j} \left( \frac{z+q_1+\cdots+q_r}{p} - 2 \right) - G_{r,j}(z-1) \right] \zeta(-j)
\]

\[
\psi_r(z) = \binom{z}{r} + \sum_{j=0}^{r-1} B_{j+1} G_{r,j}(z-1).
\]

Proof. From the infinite product representation, it follows that

\[
\lim_{N \to \infty} \prod_{q_1, \cdots, q_r=0}^{p-1} G_r \left( \frac{z+q_1+\cdots+q_r}{p} \right) = \prod_{k=0}^{r-1} G_r(p(N-1))^{\left( \frac{z}{r-k} \right)} \times \prod_{j=0}^{r-1} G_r(p(N-1))^{\left( \frac{z}{r-j} \right)} \times \prod_{m=0}^{pN} \psi_{r-1}(z+m)
\]

We substitute the asymptotic expansion to the logarithm of terms in the second bracket.

\[
\log \prod_{k=0}^{r} G_r(N) \prod_{q_1, \cdots, q_r}^{p(N-1)} \left( \frac{z+q_1+\cdots+q_r}{p} - 1 \right) \times \prod_{j=0}^{r-1} B_{j+1} G_{r,j} \left( \frac{z+q_1+\cdots+q_r}{p} + N - 2 \right) \bigg/ \prod_{m=0}^{pN} \psi_{r-1}(z+m)
\]

\[
= \left\{ \sum_{q_1, \cdots, q_r} \left( \binom{z+q_1+\cdots+q_r}{r} / p \right) + \sum_{j=0}^{r-1} B_{j+1} G_{r,j} \left( \frac{z+q_1+\cdots+q_r}{p} + N - 2 \right) \right\} \log N - \left\{ \binom{z+p(N-1)-1}{r} + \sum_{j=0}^{r-1} B_{j+1} G_{r,j} \left( \binom{z+p(N-1)-2}{r} \right) \right\} \log \left( N - 1 - \frac{1}{p} \right) -
\]
We show that its divergent terms vanish. First, we compute terms including log $p$.

**Proposition 2.2.** If we define $\psi_0(z) = 0$ and

$$\psi_r(z) := \left( \frac{z}{r} \right) + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} G_{r,j}(z-1),$$

then $\psi_r(z)$ satisfies $\psi_0(z) = z$ and

$$\left( \frac{p(N-1)-1}{r} \right) + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} G_{r,j}(z+p(N-1)-2) - \sum_{m=0}^{p(N-1)-1} \psi_{r-1}(z+m) = \psi_r(z).$$

$\psi_r(z)$ does not depend on $N$ and is uniquely determined as the polynomial satisfying the above recurrence relation.

**Proof.** This proposition immediately follows from the relation

$$\sum_{l=0}^{L-1} \binom{x+k}{k} = \binom{z+L}{k+1} - \binom{z}{k+1}$$

for $L \in \mathbb{Z}_{\geq 0}$. \hfill $\square$

Next, we simplify terms including $\zeta'(-j)$ and give a explicit form of $\phi_r(z)$.

**Proposition 2.3.** If we define

$$\phi_{r,j}(z) := \sum_{q_1, \cdots, q_r = 0}^{p-1} G_{r,j} \left( \frac{z + q_1 + \cdots + q_r}{p} + N - 2 \right) - G_{r,j}(z-1),$$

then $\phi_r(z) = \sum_{j=0}^{r-1} \phi_{r,j}(z) \zeta'(-j)$ is uniquely determined as a polynomial satisfying the recurrence relation $\phi_0(z) = 0$ and

$$\sum_{j=0}^{r-1} \sum_{q_1, \cdots, q_r = 0}^{p-1} G_{r,j} \left( \frac{z + q_1 + \cdots + q_r}{p} + N - 2 \right) - G_{r,j}(z + p(N-1)-1) \zeta'(-j) - \sum_{m=0}^{pN-1} \phi_{r-1}(z+m) = \phi_r(z).$$
Proof. It is sufficient to prove
\[
\phi_{r,j}(z) = \sum_{q_1, \ldots, q_r} G_{r,j} \left( \frac{z + q_1 + \cdots + q_r}{p} + N - 2 \right) - G_{r,j}(z + p(N - 1) - 1) - \\
- \sum_{m=0}^{p(N-1)-1} \left[ \sum_{q_1, \ldots, q_r} G_{r-1,j} \left( \frac{z + m + q_1 + \cdots + q_{r-1}}{p} + N - 2 \right) - G_{r-1,j}(z + m + p(N - 1) - 1) \right].
\]

However, we can conclude it from the identity
\[
\sum_{m=0}^{L} G_{r,j}(z + m) = G_{r+1,j}(z + L) - G_{r+1,j}(z), \quad (L \in \mathbb{Z}_{\geq 0}),
\]
and
\[
\sum_{m=0}^{p(N-1)-1} \sum_{q_1, \ldots, q_{r-1}=0}^{p-1} G_{r,j} \left( \frac{z + m + q_1 + \cdots + q_{r-1}}{p} - 2 \right) = \\
\sum_{q_1, \ldots, q_{r-1}=0}^{p-1} \left[ G_{r} \left( \frac{z + q_1 + \cdots + q_{r-1}}{p} + N - 2 \right) - G_{r} \left( \frac{z + q_1 + \cdots + q_{r-1}}{p} - 2 \right) \right],
\]
The uniqueness of \(\phi_{r}(z)\) follows from its polynomiality. \(\square\)

In order to finish our proof, we verify that the rest of terms vanish as \(N \to \infty\). By lemma 1.3 we can see that
\[
\sum_{k=0}^{r} \left\{ \sum_{q_1, \ldots, q_r=0}^{p-1} \left( \frac{z + q_1 + \cdots + q_r}{r - k} \right) \right\} \times \\
\sum_{j=0}^{k} \left[ \binom{N}{k} + \frac{B_{j+1} G_{k,j}(N-1)}{j+1} - \sum_{j=0}^{r} G_{k,j}(N-1) \frac{N^2}{(j+1)^2} \right] - \\
\sum_{k=0}^{r} \left( \frac{z - 1}{r - k} \right) \left\{ \binom{N}{k} + \frac{B_{j+1} G_{k,j}(N-1)}{j+1} - \sum_{j=0}^{r} G_{k,j}(N-1) \frac{N^2}{(j+1)^2} \right\} \\
= \sum_{q_1, \ldots, q_r=0}^{p-1} \left[ \left( \frac{z + q_1 + \cdots + q_r}{r} \right) + \sum_{j=0}^{r} G_{j+1,j} \left( \frac{z + q_1 + \cdots + q_r}{p} + N - 2 \right) \right] - \\
\sum_{j=0}^{r-1} G_{r,j} \left( \frac{z + q_1 + \cdots + q_r}{p} + N - 2 \right) \frac{N^2}{(j+1)^2} - \\
\left\{ \frac{z + N - 1}{r} \right\} + \sum_{j=0}^{r-1} G_{j+1,j} \left( z + N - 2 \right) - \sum_{j=0}^{r-1} G_{r,j}(z + N - 2) \frac{N^2}{(j+1)^2} \right\}.
\]
From the same argument as proof of theorem 1.4, it follows that the above terms tend to zero as \(N \to \infty\). Therefore, we have proved theorem 2.1 \(\square\)

Our result is closely related with Kuribayashi [17]. In order to explain his result, we introduce some functions. \(\zeta_r(s, z)\) is defined as a special case of Barnes’ zeta function...
function \[\zeta_r(s, z) := \sum_{n_1, \ldots, n_r=0}^{\infty} (z + n_1 + \cdots + n_r)^{-s}\]
for \(\Re s > r\). This function can be continued analytically to a meromorphic function whose poles are placed at \(s = 1, \ldots, r\). We call the analytic continuation also \(\zeta_r(s, z)\). The gamma function \(\Gamma_r(z)\) associated with \(\zeta_r(s, z)\) is introduced as
\[
\Gamma_r(z) := \exp \left[ \frac{\partial}{\partial s} \zeta_r(s, z) \bigg|_{s=0} \right].
\]
Kuribayashi exhibit the following multiplication formula:

**Theorem 2.4** (Kuribayashi). \(\Gamma_r(z)\) satisfies the following multiplication formula:
\[
\prod_{q_1, \ldots, q_r=0}^{p-1} \frac{\Gamma_r \left( \frac{z + q_1 + \cdots + q_r}{p} \right)}{\Gamma_r \left( \frac{z}{p} \right)} = p^{Q_r(z)} \Gamma_r(z),
\]
where
\[
Q_r(z) = \frac{(-1)^r}{(r-1)!} \sum_{r=1}^{r} \frac{S_l}{l} \left\{ z^l - (-1)^l B_l \right\}.
\]
As a consequence of facts in Vardi [28], a relation between \(G_r(z)\) and \(\Gamma_r(z)\) is expressed as follows:
\[
G_r(z) = R_r(z) \Gamma_r(z)^{(-1)^{r-1}}
\]
where
\[
R_r(z) := \exp \left[ \sum_{j=0}^{r-1} G_{r,j}(z-1) \zeta'(-j) \right].
\]
Thus, we have
\[
(2.2) \quad Q_r(z) = (-1)^r \psi_r(z) = (-1)^r \left[ \frac{z}{r} + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} G_{r,j}(z-1) \right].
\]
Our expression is useful in some cases of studies on related functions. For example, noting that \(G_{r,0}(z) = \binom{z}{r-1}\), we can check that the relation follows
\[
(2.3) \quad (-1)^r Q_r(r-z) = Q_r(z).
\]
from the definition of \(\psi_r(z)\) and (2.2). It plays an important role in the multiplication formula
\[
\prod_{q_1, \ldots, q_r=0}^{p-1} S_r \left( \frac{z + q_1 + \cdots + q_r}{p} \right) = S_r(z).
\]
for Kurokawa’s multiple sine function [18, 19, 20, 21] introduced as
\[
S_r(z) := \Gamma_r(r-z) \Gamma_r(z)^{(-1)^{r+1}}.
\]
In Kuribayashi’s original proof, (2.3) is verified through a rather complicated argument. He applied a relation between \(\zeta_r(-m, z)\) \((m \in \mathbb{Z}_{\geq 0})\) and the Bernoulli polynomials \(B_l(z)\). However, once (2.2) is obtained, we can check (2.3) immediately.
3. Appendix : An Elementary Proof for (2.2)

Without facts of zeta functions, we can prove (2.2) directly as follows: First, we rewrite Kuribayashi’s $Q_r(z)$ as

$$(-1)^r Q_r(z) = \frac{1}{(r-1)!} \sum_{l=0}^{r-1} S_l \left( \frac{(-1)^{l+1} B_{l+1}}{l+1} - \frac{(z-1)^{l+1}}{l+1} \right). \quad (3.1)$$

The second term can be written as follows:

$$\frac{1}{(r-1)!} \sum_{l=0}^{r-1} S_l \frac{(z-1)^{l+1}}{l+1} = \int_0^z \frac{(t-1)}{(r-1)} dt - \int_0^1 \frac{(t-1)}{(r-1)} dt.$$ 

From Lemma 1.4 and $G_{r,j}(0) = \frac{(-1)^j}{(r-1)!} r^{-1} S_j,$ it follows that

$$\int_0^z \frac{(t-1)}{(r-1)} dt - \int_0^1 \frac{(t-1)}{(r-1)} dt = \left( \frac{z}{r} \right) + \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} \frac{G_{r,j}(z-1)}{r-1} S_j - \frac{1}{(r-1)!} \sum_{j=0}^{r-1} \frac{B_{j+1}}{j+1} \frac{1}{r-1} S_j.$$

Therefore, we obtain (2.2) by substituting this to (3.1).

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