A characterisation of the continuum Gaussian free field in $d \geq 2$ dimensions.

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Abstract

We prove that under certain mild moment and continuity assumptions, the $d$-dimensional Gaussian free field is the only stochastic process in $d \geq 2$ that is translation invariant, exhibits a certain scaling, and satisfies the usual domain Markov property. Our proof is based on a decomposition of the underlying functional space in terms of radial processes and spherical harmonics.

1 Introduction

The continuum Gaussian free field (GFF) is a generalisation of Brownian motion to higher dimensions, which is defined on balls as follows.

Definition 1 (Gaussian free field). Let $d \geq 1$ and $B \subseteq \mathbb{R}^d$ denote the open unit ball.

Then for $a \in \mathbb{R}^d$ and $r > 0$ the $d$-dimensional zero boundary continuum GFF in $a + rB$ is the centred Gaussian process $(h_{a + rB}, f)_{f \in C_\infty^c(\mathbb{R}^d)}$ whose covariance is given by

$$
\mathbb{E}((h_{a + rB}, f)(h_{a + rB}, g)) = \int \int_{(\mathbb{R}^d)^2} f(z)G^{a + rB}(z, w)g(w)\, dz\, dw; \quad f, g \in C_\infty^c(\mathbb{R}^d),
$$

where $G^{a + rB}$ denotes the zero boundary Green’s function for the Laplacian in $a + rB$.

In dimension $d = 1$, this corresponds to a slightly awkward definition of the Brownian bridge on intervals $(a - r, a + r)$: in this case the GFF can actually be defined as an a.s. continuous function, i.e. indexed instead by points. In $d \geq 2$ the GFF does not make sense as a pointwise defined function, thus the more general definition. However, one can still heuristically see the continuum GFF as a Gaussian height function parameterised by a domain $D \subseteq \mathbb{R}^d$ [She07]. The Gaussian free field has played many roles in probability and mathematical physics since the 1970s: it is the stationary solution of the stochastic heat equation; is used to describe both free and interactive Euclidean Quantum field theories; it appears as a corrector in stochastic homogenisation; to name just a few. Recently, the 2d GFF has played a crucial role in the probabilistic study of statistical physics models, Schramm-Loewner evolution, Liouville quantum gravity and Liouville field theory [Dub09, MS16, DS11, DKRV16]. It is both the proven and the conjectured scaling limit of several natural discrete height functions [Nad97, Ken01, RV07].

There are many characterisations of Brownian motion that single it out among $1d$ stochastic processes and it is similarly natural to also ask how the GFF in higher dimensions can be characterised. Recently, the 2d continuum GFF was characterised as the only conformally invariant field, satisfying a domain Markov property and certain minimal moment and continuity assumptions [BPR20, BPR21].

This short note provides a first characterisation of the $d$-dimensional continuum Gaussian free field for $d \geq 3$, answering Question 6.1 in [BPR20]. It also provides a new characterisation of the GFF in $d = 2$ that relaxes the assumption of conformal invariance and thereby generalises the main result of [BPR20] via a new proof. Our characterisation is based only on scaling and the domain Markov property and in particular, we do not need to assume rotational invariance.

More precisely, but still somewhat informally, we prove that, under certain mild moment and continuity assumptions, the $d$-dimensional Gaussian free field is the only stochastic process in $d \geq 2$ that is translation invariant, exhibits a certain scaling, and satisfies the following domain Markov
property - for each ball, the field inside can be written as a sum of an independent scaled copy of the original field plus the “harmonic extension” of its behavior on the boundary. We prove the theorem by identifying the covariance structure as the Green’s kernel (lighter step), and then proving Gaussianity (a bit more involved). The two arguments are quite independent of each other (although we use the knowledge of the covariance kernel to simplify some arguments later on), but both make strong use of the domain Markov property. In particular, our proof of Gaussianity via spherical harmonics stresses the nice interplay between the GFF and harmonic functions, and could be of independent interest.

This characterisation could potentially be helpful when identifying the GFF as a continuum scaling limit, since proving conformal invariance or even rotational invariance from the lattice is known to be very difficult.\footnote{In fact, let us thank here H. Duminil-Copin and V. Tassion for asking us whether only rotational invariance should suffice for the characterisation of the 2d continuum GFF. This discussion happened in a nice and friendly workshop in Fribourg, organised by I. Manolescu, who we would hereby like to thank too. We would finally like to thank N. Berestycki, J.C. Mourrat and G. Ray for many interesting and fruitful discussions on this topic.}

However, the key step in this case would be proving this concrete form of domain Markov property in the limit, which admittedly is probably not easy at all - it would in some sense already mean a certain link of the covariance kernel to random walks and Brownian motion.

**Theorem 2** (Characterisation of the $d$-dimensional GFF). Let $d \geq 2$ and $B \subseteq \mathbb{R}^d$ denote the open unit ball. Suppose that for every $a \in \mathbb{R}^d$, $r > 0$ $(h^{a+rB}, f)_{f \in C^\infty_c(\mathbb{R}^d)}$ is a centered stochastic process indexed by $C^\infty_c(\mathbb{R}^d)$.

If the collection $\{h^{a+rB} : a \in \mathbb{R}^d, r > 0\}$ satisfies the following conditions, then for some $c$ independent of $a$ and $r$, each $h^{a+rB}$ has the law of $c$ times a $d$-dimensional zero boundary Gaussian free field in $a + rB$.

**A. Linearity** $(h^B, f)$ is a.s. linear in $f$.

**B. Zero outside of $B$** For any $f$ with supp$(f) \subset \mathbb{R}^d \setminus B$ $(h^B, f) = 0$ a.s.

**C. Translation and scaling** The law of $h^{a+rB}$ is the rescaled image of $h^B$ under translation by $a$ and scaling by $r$:

$$(h^{a+rB}, f(r \cdot + a)) = r^{d/2} (h^B, f(\cdot)).$$

Note that if $h$ were a function and $(\cdot, \cdot)$ was the $L^2$ inner product, this would be equivalent to $h^{a+rB}(a + r \cdot) = r^{d/2} h^B(\cdot)$, exhibiting half the scaling factor of $G^B$.

**D. Domain Markov property** Suppose that $a + rB \subseteq B$. Then we can write

$$(h^B, f)_{f \in C^\infty_c(\mathbb{R}^d)} = (h^{a+rB}, f)_{f \in C^\infty_c(\mathbb{R}^d)} + (\varphi^{a+rB}, f)_{f \in C^\infty_c(\mathbb{R}^d)}$$

where the two summands are independent, $h^{a+rB}$ is equal in law to $h^{a+rB}$, and $\varphi^{a+rB}$ is a stochastic process that a.s. corresponds to integrating against a harmonic function when restricted to $a + rB$.

**E. Existence of fourth moments** We have $\mathbb{E}(h^B, f)^4 < \infty$ for all $f \in C^\infty_c(\mathbb{R}^d)$.

**F. Continuity of the covariance functional** The covariance $K_2(f, g) := \mathbb{E}(h^B(f)h^B(g))$, is a continuous bilinear form on $C^\infty_c(D)$.

**G. Zero boundary conditions** For any sequence $(f_n)_{n \geq 0}$ of smooth positive functions with $\int f_n$ uniformly bounded, $(d(supp(f_n)), 0) \to 1$ as $n \to \infty$ and $\sup_n \sup_{r<1} \sup_{x,y \in 0(rB)} |f_n(x)/f_n(y)| < \infty$, we have that $K_2(f_n, f_n) \to 0$.

We have not striven for most general technical assumptions, but rather have tried to keep the proofs light and self-contained. However, we believe that several assumptions can most likely be relaxed.

- First, the moment assumption can be probably relaxed using methods of [BPR21].
- Also, although using spherical harmonics is key in our proofs, one could make this work for other reasonable domains, e.g. with smooth boundary. This follows rather directly from our techniques if the domain Markov property is assumed for both the domain shape and balls, but probably variations of this are also possible.
- We use the same form of domain Markov property (DMP) as in [BPR20], but only for balls. The DMP plays a key role in our argument - it is used both in showing that the covariance kernel is harmonic off the diagonal, and in proving Gaussianity. It seems difficult to relax the harmonicity condition for the boundary data extension in the current approach, but it would be
very interesting to determine if this is possible. On the other hand, as in the case of Brownian motion, one could easily envisage replacing the condition of having an exact (scaled) copy of the field with a more relaxed, martingale type of condition.

- It might seem a bit surprising that we are not using rotational invariance, however, the reason is that the DMP already implies rather easily that the covariance is a harmonic function and this is a very strong property. The exact scaling could similarly be relaxed, at least to some extent.

- Finally, let us comment on the exact form of the zero boundary condition. In [BPR20] this convergence was only asked for rotationally symmetric functions. We need the slightly generalised form (that is implied for example by conformal invariance and the rotationally invariant form) basically in two places: to determine that the covariance is the Green’s kernel and to obtain the uniqueness of the domain Markov property. Notice that our condition is still less restricting than asking zero boundary conditions locally.

We will now present the argument in three sections. First, we discuss some immediate consequences of the assumptions (along similar lines to [BPR20], so we will keep this brief). In Section 3, using the DMP, scaling and translation invariance, we show how to deduce that the covariance kernel is the Green’s kernel (Proposition 4). Finally, the main part of the paper is Section 4, where we prove Gaussianity (Proposition 12) - we do this using solely the DMP, and a decomposition of the underlying functional space using spherical harmonics. Theorem 2 is an immediate consequence of Proposition 4 and Proposition 12. Sections 3 and 4 can be read quite independently of each other, although we use the identification of the covariance kernel to simplify some arguments in the latter.

**Definition 3** (Scaling function). In what follows we write \( s(r) \) for the function on \((0, \infty)\) defined by \(-\log r\) when \(d = 2\) and \(r^{2-d}\) when \(d \geq 3\).

## 2 Immediate consequences

Here we discuss some immediate properties of our assumptions. The section is self-contained, but we remain brief, as similar properties have been shown in detail in the \(2d\) case in [BPR20].

**The domain Markov decomposition is unique.** Indeed, suppose that for some \(a, \mathbb{R}\) we had two decompositions as in D of Theorem 2:

\[
h^B = h_{a}^{B} + \varphi_{a}^{B} = h_{a+r}^{B} + \varphi_{a+r}^{B}.
\]

Then for any \(z \in a + r \mathbb{B}\), by harmonicity, there exists a sequence \((f_n)_{n \geq 0}\) of functions satisfying the zero boundary condition assumption \(G\) from Theorem 2, and such that \((\varphi, f_n(a + r)) = \varphi(z)\) for all \(n\) and any harmonic \(\varphi\) in \(a + r \mathbb{B}\). The zero boundary condition then implies that \((h_{a}^{B} - h_{a+r}^{B}, f) \rightarrow 0\) a.s. along a subsequence as \(n \rightarrow \infty\). Thus, it must be the case that \(\varphi_{a}^{a+r} = \varphi_{a+r}^{a} \) a.s. Applying this for a dense collection of \(z\) and using harmonicity of \(\varphi_{a}^{a+r} \), \(\varphi_{a+r}^{a}\) proves the uniqueness.

We will use the following consequences of this uniqueness repeatedly:

- if \(a + r \mathbb{B} \subset a' + r' \mathbb{B} \subset \mathbb{B}\) then
  \[
  \varphi_{a}^{a+r} - \varphi_{a'}^{a'+r'} \text{ is independent of } \varphi_{a}^{a+r} \text{ and equal in law to } (r'/r) \frac{2-d}{2} \varphi_{a}^{a'+r'(a-a')}^{B}.
  \]
  \[\text{(2.1)}\]

- we can apply the domain Markov property in several balls at once. More precisely, if \(B_1, \ldots, B_n\) are \(n\) balls, and \(h^B = h^i + \varphi^i\) in each ball, then
  \[
  \varphi := h^B - \sum_{i=1}^{n} h^i \text{ is a.s. harmonic in } \bigcup_{i} B_i
  \]
  \[\text{(2.2)}\]

and \(h^i\) is independent of \(\{\varphi, (h^j)_{j \neq i}\}\) for each \(i\). Indeed, for any fixed \(i\) we can write \(\varphi = \varphi^i - \sum_{j \neq i} h^j\) and since \(h^j\) is zero in \(B_i\) for all \(j \neq i\), \(\varphi = \varphi^i\) is harmonic in \(B_i\). But also for every \(j \neq i\), \(h^j\) is zero in \(B_i\) and therefore \(h^j\) is measurable with respect to \(\varphi^i\). So the collection \(\{\varphi, (h^j)_{j \neq i}\} = \{\varphi^i - \sum_{j \neq i} h^j, (h^j)_{j \neq i}\}\) is \(\varphi^i\)-measurable and therefore independent of \(h^i\).
The 2-point function. For \( z_1, z_2 \in (1 - \varepsilon)B \) with \(|z_1 - z_2| > 2\varepsilon\), define the harmonic functions \( \varphi^{z_1+\varepsilon}_B, \varphi^{z_2+\varepsilon}_B \) according to the domain Markov decomposition of \( h^B \) in \( z_1 + \varepsilon B, z_2 + \varepsilon B \) respectively. Then by (2.1) and (2.2), the quantity

\[
  k_2(z_1, z_2) = \mathbb{E}(\varphi^{z_1+\varepsilon}_B(z_1)\varphi^{z_2+\varepsilon}_B(z_2))
\]

(2.3)
is well-defined, i.e., it does not depend on \( \varepsilon > 0 \) satisfying the above conditions. Note that, by harmonicity, we can alternatively write

\[
  k(z_1, z_2) = \mathbb{E}((h^B, \eta_i)(h^B, \eta_j))
\]

(2.4)
for any smooth functions \( \eta_i, \eta_j \) of mass one, that are supported in \( z_1 + \varepsilon B, z_2 + \varepsilon B \) and rotationally symmetric about \( z_1, z_2 \) respectively.

For every \( \delta > 0 \), the following crude upper bound for \( z_1 \neq z_2 \in (1 - \delta)B \) is rather direct with \( C = C(\delta) \):

\[
  |k_2(z_1, z_2)| \leq C(1 + s(|z_1 - z_2|)).
\]

(2.5)
To justify this, observe that by Cauchy–Schwarz and the domain Markov property, it suffices to show that \( \mathbb{E}(\varphi^{z+\varepsilon}_B(z)^2) \leq C(1 + s(\varepsilon)) \) for any \( z \in (1 - \delta)B \) and \( \varepsilon < \delta \). Indeed, as long as \( d(z_i, \partial B) > |z_i - z_j|/2 \) for \( i = 1, 2 \) we can set \( \varepsilon = |z_i - z_j|/2 \) in the definition of \( k_2(z_1, z_2) \); otherwise we crudely set \( \varepsilon = \delta/2 \) leading to the dependence of \( C(\delta) \).

Let us first consider the case \( z = 0 \). If \( 2^{-n} \leq \varepsilon \leq 2^{-(n+1)} \), then the domain Markov property implies that \( \mathbb{E}(\varphi^0_B(0)^2) \leq \mathbb{E}(\varphi^{z+\varepsilon}_B(z)^2) \). Further, by the domain Markov property and also scaling, the right-hand side can be written as the sum \( \sum_{m=0}^{n-1}2^{m(d-2)/2}X_m \), with the \( X_m \) i.i.d. each having the law of \( \varphi^0_B(0) \). Adding up the variances gives the desired bound.

When \( z \neq 0 \) with \( |z| = r < 1 \), let \( r' = 1/(1 + r) \in (1/2, 1) \), \( \varepsilon' = \varepsilon/(1 + r) \) and \( z' = -r'z \). By applying the Markov property for \( h^{z'+\varepsilon'}B \) in \( \varepsilon'B \) we can write

\[
  \varphi^{z'}_B(0) = \varphi^{z'+\varepsilon'}_B(0) + \tilde{\varphi}(0)
\]
where the summands are independent and by translation invariance and scaling, \( \tilde{\varphi}(0) \) has the law of \( (r')^{2-d} \times \varphi^{z+\varepsilon}_B(z) \). But since the variance of the summands add up, the variance of \( \tilde{\varphi}(0) \) is no greater than the variance of \( \varphi^{z'+\varepsilon'}_B(0) \). Hence the claim follows from the case \( z = 0 \).

From here, the assumption of continuity for \( K_2 \) shows that \( k_2 \) is the covariance function of the field, in the sense that for any \( f_1, f_2 \in C_c^\infty(B) \), we have

\[
  K_2(f_1, f_2) = \iint_{B^2} f_1(z_1)f_2(z_2)k_2(z_1, z_2)\,dz_1dz_2.
\]

(2.6)

This essentially just uses (2.4), associativity of convolution and the fact that smooth functions can be approximated by mollified versions. The proof is written in detail in [BPR20, Proof of Lemma 2.18] and does not depend on the dimension \( d \).

The 4-point function. Similarly, due to (2.2), for \( z_1, \ldots, z_4 \in B \) with \( \min_{j \neq i} |z_i - z_j| > 2a_i \) and \( d(z_i, \partial B) > a_i \) for each \( i \), \( k_4(z_1, \ldots, z_4) = \mathbb{E}((\prod_{i=1}^4 \varphi^{z_i+a_i}_B(z_i))) \) is well-defined, i.e., it does not depend on the choice of \( a_i \) satisfying the above conditions. Again we can write this as \( \mathbb{E}(\prod_{i=1}^4 (h^B, \eta_i)) \) for any smooth functions \( (\eta_i)_{1 \leq i \leq 4} \), with \( \eta_i \) having mass one, being rotationally symmetric about \( z_i \) and supported in \( z_i + a_i B \) for \( i = 1, 2, 3, 4 \).

Using the same argument as in the 2-point case, one can now bound \( |k_4(z_1, \ldots, z_4)| \). Indeed, by Holder’s inequality it suffices this time to obtain bounds on the fourth moments of \( \varphi^{z+\varepsilon}_B(z) \) for \( \varepsilon > 0 \). This can be done very similarly to the 2-point bound. One first treats \( z = 0 \) using the i.i.d nature of increments - the only change here is that when expanding \( \mathbb{E}((\sum_{m=0}^{n-1} \alpha_m X_m)^4) \) with independent \( X_m \)'s one must consider both \( \sum \alpha_i^4 \mathbb{E}(X_i^4) \) and \( \sum_{i \neq j} \alpha_i^2 \alpha_j^2 \mathbb{E}(X_i^2)\mathbb{E}(X_j^2) \) - and then transports this bound to general \( z \) using translation invariance and scaling. We omit the details and give the final bound: there are constants \( C(\delta) > 0 \) such that for any distinct \( z_1, z_2, z_3, z_4 \in (1 - \delta)B \)

\[
  |k_4(z_1, \ldots, z_4)|^4 \leq C(\delta) \prod_{i=1}^4 (1 + \max_{j \neq i} s(|z_i - z_j|)^2).
\]

(2.7)
Spherical averages. For \( z \in \mathbb{B} \) and \( \varepsilon < d(z, \partial \mathbb{B}) \) we define \( h_\varepsilon(z) = \varphi_\varepsilon^{z+\varepsilon\mathbb{B}}(z) \) to be the \( \varepsilon \)-spherical average of \( h^\mathbb{B} \) around \( z \). Then we have that
\[
E(h_\varepsilon(z)^2) = \int k_2(x, y)\rho_\varepsilon^2(dy), \tag{2.8}
\]
where \( \rho_\varepsilon \) is uniform measure on \( \partial(z + \varepsilon \mathbb{B}) \). Indeed, let us write \( \rho_n \) for a sequence of smooth test functions with total mass one that are rotationally symmetric about \( z \) and supported in the annular region \( z + \{(1 - 2^{-2n})\varepsilon \mathbb{B} \setminus (1 - 2^{-n})\varepsilon \mathbb{B}\} \) for each \( n \). Then \( E((h^\mathbb{B}, \rho_n)^2) = E(h_\varepsilon(z)^2) + E((h_\varepsilon^{z+\varepsilon\mathbb{B}}, \rho_n)^2) \) for each \( n \), where the second term on the right-hand side goes to 0 as \( n \to \infty \) by the zero boundary condition assumption. It therefore follows that
\[
E(h_\varepsilon(z)^2) = \lim_{n \to \infty} E((h^\mathbb{B}, \rho_n)^2) = \lim_{n \to \infty} \iint k_2(x, y)\rho_n(x)\rho_n(y) \, dx \, dy.
\]
This limit in \( n \) is equal to the right-hand side of (2.8), since by (2.5), \( \iint k_2(x, y)(\rho_n - \rho)(dx)(\rho_n - \rho)(dy) \to 0 \) as \( n \to \infty \).

Using the same construction as above with \( z = 0 \) and \( \varepsilon = \varepsilon_n \) converging to 1 as \( n \to \infty \), we also see that
\[
E(h_\varepsilon(0)^2) \to 0 \text{ as } r \uparrow 1. \tag{2.9}
\]

3 Covariance is the Green function

Let us start by showing that scaling and translation invariance together with the domain Markov property already imply that the covariance kernel is the Green’s function:

**Proposition 4.** The function \( k_2(x, y) \) (defined for \( x \neq y \)) is a positive multiple of the zero boundary Green’s function.

**Proof.** Write \( G^\mathbb{B} \) for the zero boundary Green’s function in \( \mathbb{B} \). We are going to use the following characterisation of \( G^\mathbb{B} \) (see for example [WP20, Lemma 3.7]).

* Suppose that for \( y \in \mathbb{B} \), \( k_y(x) \) is a harmonic function in \( \mathbb{B} \setminus \{y\} \), such that \( k_y(x) - bs(|x - y|) \) is bounded in a neighbourhood of \( y \) for some \( b > 0 \) and such that \( (k_y, f_n) \to 0 \) as \( n \to \infty \) for any sequence of functions \( f_n \) as in our zero boundary condition \( G \) of Theorem 2.² Then \( k_y(x) = bG^\mathbb{B}(x, y) \).

First, let us see that \( k_2(y, x) \) for fixed \( y \in \mathbb{B} \) is harmonic off the diagonal. Indeed, by the harmonicity of the function appearing in the domain Markov decomposition it is easy to see that for any \( \eta > 0 \) and \( x \in \mathbb{B} \) such that \( |x - y| \land d(x, \partial \mathbb{B}) > \eta \):
\[
k_2(x, y) = \int k_2(w, y)\rho_\varepsilon^2(dw) \tag{3.1}
\]
where \( \rho_\varepsilon \) is uniform measure \( \partial(x + \eta \mathbb{B}) \). The proof of this is given in [BPR20, Lemma 2.9], which applies directly to all \( d \geq 2 \). This implies that \( x \mapsto k_2(x, y) \) is harmonic in \( \mathbb{B} \setminus \{y\} \). Note that in particular we have continuity in this region, which we did not assume a priori.

We now check the boundary condition for \( k_y := k_2(y, \cdot) \): if \( f_n \) are a sequence of functions as in \( G \) of Theorem 2, we have \( (k_y, f_n) = \int k_2(y, x)f_n(x) \, dx = \int k_2(w, w)f_n(x) \, dx \rho_\varepsilon^2(dw) \) for some \( \delta(y) > 0 \) and all \( n \) large enough. By Cauchy–Schwarz we see that the right-hand side is bounded by the square root of \( E(h_\varepsilon(y)^2)E((h^\mathbb{B}, f_n)^2) \) which converges to 0 by the zero boundary condition \( G \).

Now, notice that by (2.5), for any \( \delta > 0 \) there are some constants \( c_1, c_2 > 0 \) such that \( k_2(0, w) + c_1s(|w|) + c_2 \) is a positive harmonic function in \( (1 - \delta)\mathbb{B} \setminus \{0\} \). Thus by Böcher’s theorem [ABR01, Chapter III], we conclude that \( k_2(0, w) + c_1s(|w|) + c_2 \) is of the form \( c_3 s(|w|) + \nu(w) \), where \( \nu(w) \) is harmonic in the whole of \( (1 - \delta)\mathbb{B} \). In particular, there is some \( b > 0 \) such that \( k_2(0, w) - bs(|w|) \) is harmonic and bounded in \( (1 - 2\delta)\mathbb{B} \). Note that \( b \) must be positive, since \( \int k_2(0, z)\rho_\varepsilon^2(dz) = E(h_\varepsilon(0)^2) \) is positive and increasing to \( \infty \) as \( w \downarrow 0 \) by the domain Markov property.

²The proof in [WP20] works exactly the same if we use this “zero boundary condition” for \( k_y \) rather than a pointwise zero boundary condition.
From here, we return to a fixed $y \in \mathbb{B}$. We choose $\varepsilon > 0$ such that $d(y, \partial \mathbb{B}) > 2\varepsilon$ and write $h_{\mathbb{B}}|_{y + 2\varepsilon \mathbb{B}} = h_{\mathbb{B}}|_{y + 2\varepsilon \mathbb{B}} + \varphi_{y}^{\mathbb{B} + 2\varepsilon \mathbb{B}}$. Then for $x \in y + \varepsilon \mathbb{B}$ we have $k_2(x, y) = \mathbb{E}(h_{\mathbb{B}}(\eta_x)(h_{\mathbb{B}}(\eta_y))$ where $\eta_x$ and $\eta_y$ are smooth functions with mass one, radially symmetric around $x, y$ and supported in small non-intersecting balls around $x$ and $y$ respectively. Using the decomposition and harmonicity of $\varphi_{y}^{\mathbb{B} + 2\varepsilon \mathbb{B}}$ we have

$$k_2(x, y) = \mathbb{E}(h_{\mathbb{B}}^{\varepsilon}(\eta_x)(h_{\mathbb{B}}^{\varepsilon}(\eta_y)) + \mathbb{E}(\varphi_{y}^{\mathbb{B} + 2\varepsilon \mathbb{B}}(x)\varphi_{y}^{\mathbb{B} + 2\varepsilon \mathbb{B}}(y))$$

where the second term on the right-hand side is bounded by Cauchy-Schwarz and (2.5). The first term is equal to $bs(|x - y|) + O(1)$ in some neighbourhood of $y$ by the assumption $C$ on the distribution of $h_{\mathbb{B}}^{\varepsilon}$ and the previous paragraph.

We have therefore shown that $x \mapsto k_2(x, y)$ satisfies the condition $\star$ and is therefore equal to a multiple (which must actually be $b$) of $G^\mathbb{B}(x, y)$. Since $y \in \mathbb{B}$ was arbitrary we are done. \qed

**Corollary 5.** For any $f \in H^{-1}(\mathbb{B})$, take a sequence of smooth functions $(f_n)_{n \geq 0}$ such that $G^\mathbb{B}(f - f_n, f - f_n) \to 0$. Then we may define $(h^\mathbb{B}, f)$ to be the $L^2$ limit of $(h^\mathbb{B}, f_n)$ as $n \to \infty$. The limit does not depend on the approximation.

In what follows, we will therefore use the notation $(h^\mathbb{B}, f)$ for $f \in H^{-1}(\mathbb{B})$ without further justification.

## 4 Gaussianity

It now remains to argue that the field is Gaussian. We do this via a decomposition of the field into a sequence of radial processes. We show using the domain Markov property that each of these processes is Gaussian, and that moreover, the whole sequence is jointly Gaussian.

First, we will see a bound for the 4-point function that is reminiscent of Wick’s theorem: this will help us deduce continuity of our processes. In the second subsection, we do the basic case - the case of spherical averages. Finally, we extend this to a wider range of processes, obtained from so called spherical harmonics. The Gaussianity of the field $(h^\mathbb{B}, f)_{f \in C_c(\mathbb{R}^d)}$ is proved in this final subsection.

### 4.1 Weak Gaussianity in terms of the 4-point function

To prove continuity of our radial processes, we need the following bound that is implied by our assumption on existence of fourth moments. Notice that this can be seen as establishing a very weak form of Wick’s theorem, i.e. getting us closer to Gaussianity. Recall that for $r > 0$, $h_r(0) = \varphi_r^\mathbb{B}(0)$ is the spherical average at radius $r$ around the origin, and converges to 0 in probability as $r \uparrow 1$.

**Lemma 6.** For $r \in (0, 1)$ we have that

$$\mathbb{E}(h_r(0)^4) = \int_{\partial(\mathbb{R}\mathbb{B})^4} k_4(z_1, z_2, z_3, z_4) \prod_{i=1}^4 \rho_0^*(dz_i).$$

Moreover, for some constant $c(d)$ and some $\eta \in [0, 1)$:

$$|k_4(z_1, z_2, z_3, z_4)| \leq c(d)\delta^{-\eta}g(z_1, z_2, z_3, z_4)$$

for all $\delta \in (0, 1]$ and $z_1, \ldots, z_4 \in \partial(1 - \delta)\mathbb{B}$, where

$$g(z_1, z_2, z_3, z_4) = G^\mathbb{B}(z_1, z_2)G^\mathbb{B}(z_3, z_4) + G^\mathbb{B}(z_1, z_3)G^\mathbb{B}(z_2, z_4) + G^\mathbb{B}(z_1, z_4)G^\mathbb{B}(z_2, z_3)$$

is the four-point function for the $d$-dimensional zero boundary GFF in $\mathbb{B}$.

Note that for any $0 < r < 1 - \eta$, our upper bound (2.7) gives that, for some constant $C(\eta)$ depending only on $\eta$,

$$\int_{\partial(\mathbb{R}\mathbb{B})^4} |k_4(z_1, z_2, z_3, z_4)| \prod_{i=1}^4 \rho_0^*(dz_i) \leq C(\eta)s(r)^2$$
and in particular, the integral is finite. Indeed, it suffices to bound the integral over the region where \( \min(|z_i - z_j|) = |z_1 - z_2| \). However, on this region, by (2.7), we can bound \( |k_4| \) above by an \( \eta \)-dependent constant times
\[
(1 + s(|z_1 - z_2|)) \sum_{i,j \neq k} \sqrt{1 + s(|z_i - z_k|)} \sum_{j \neq k} \sqrt{1 + s(|z_j - z_k|)}.
\]

Then expanding the product of the two sums, we can show the desired bound for each term separately, using Cauchy–Schwarz in the integral over \( z_3 \) and \( z_4 \) for all terms, except \( s(|z_3 - z_4|) = s(|z_1 - z_4|) \) which can be integrated directly. Here we are using the fact that for any fixed \( z \in \partial(r\mathbb{B}) \), we have that \( \int_{\partial(r\mathbb{B})} s(|z - z|) \rho_i'(dz) = O(s(r)) \). The expression for the 4-point function now follows from dominated convergence:

**Proof of (4.1).** Fix \( r \in (0,1) \) and for each \( n \geq 1 \) partition the sphere \( \partial((r - 2^{-n})\mathbb{B}) \) into regions each having diameter no larger than \( 2^{-n} \). For each of these regions we can then choose a ball of maximal radius centred at a point in the region, so that the ball intersected with the sphere lies inside the region, but within distance \( 2^{-2n} \) from the boundary of the region. This produces a sequence \( \{z_{i,n}\}_{1 \leq i \leq m_n} \) and radii \( \{r_{i,n}\}_{1 \leq i \leq m_n} \) (all less than \( 2^{-(n+1)} \)) such that the balls \( z_{i,n} + r_{i,n}\mathbb{B} \) do not intersect.

We can now set \( \nu_{i,n} \) for each \( i \), \( n \) to be a smooth mollifier supported on \( z_{i,n} + r_{i,n}\mathbb{B} \), that is radially symmetric about \( z_{i,n} \) and has total mass one. Using that \( k_2 = bG^8 \), we see that \( m_n^{-1} \sum_{i=1}^{m_n} (h^8, \nu_{i,n}) \to h_r(0) \) in \( L^2 \) as \( n \to \infty \). On the other hand, the definition of \( k_4 \) gives that \( k_4(z_{i,n}, z_{j,n}, z_{k,n}, z_{l,n}) = \mathbb{E}(h^8, \nu_{i,n})(h^8, \nu_{j,n})(h^8, \nu_{k,n})(h^8, \nu_{l,n}) \) for any distinct \( 1 \leq i, j, k, l \leq m_n \) and every \( n \). Dominated convergence using (4.3) then allows us to conclude.

The Wick-type of bound is slightly trickier:

**Proof of (4.2).** Fix \( z_1, z_2, z_3, z_4 \) distinct and for each \( j \) write \( a_j = \min_{i \neq j} d(z_i, z_j)/2 \).

Denote \( B_j := (z_j + a_j\mathbb{B}) \cap \mathbb{B} \) for each \( j \) so that the \( B_j \) do not intersect. Write \( \rho_j \) for uniform measure on \( \partial(z_j + a_j\mathbb{B}) \) and \( \tilde{\rho}_j \) for harmonic measure on \( \partial B_j \setminus \partial \mathbb{B} \), seen from \( z_j \).

**Claim 7.** We can write
\[
k_4(z_1, z_2, z_3, z_4) = \mathbb{E}(h^8, \tilde{\rho}_1)(h^8, \tilde{\rho}_2)(h^8, \tilde{\rho}_3)(h^8, \tilde{\rho}_4)). \tag{4.4}
\]

This claim could be checked straightforwardly if we could apply the domain Markov property inside the regions \( B_j \) to \( B_4 \), so it should not be too surprising. However, since the Markov property has only been assumed for balls, we will have to do a little bit of work to show this. This will be carried out shortly, but let us first see how (4.2) follows.

First, notice that using the domain Markov property we can write
\[
\mathbb{E}(h^{28}, \tilde{\rho}_j)^4 = \mathbb{E}(h^{28}, \rho_j)^4 + 6 \mathbb{E}(h^{28}, \rho_j)^2 \mathbb{E}(h^{28}, \tilde{\rho}_j)^2 + \mathbb{E}(h^{28}, \tilde{\rho}_j)^4)
\]

and hence bound \( \mathbb{E}(h^{28}, \tilde{\rho}_j)^4 \leq \mathbb{E}(h^{28}, \rho_j)^4 \) for each \( j \). By the same dominated convergence argument as for (4.2), and writing \( k^n_{28} \) for the four-point function of \( h^{28} \), we see that the latter is equal to \( \int_{\partial(z_j + a_j\mathbb{B})} |k^n_{28}(x, y, z, w)| \rho_j(dx) \ldots \rho_4(dw) \). But then by (4.3) and using that each \( z_j + a_j\mathbb{B} \) is far from the boundary of \( 2\mathbb{B} \) we see that this is less than some constant, not depending on \( \delta \), times
\[
\sup_{\partial B_j \setminus \partial \mathbb{B}} \left\| \frac{d\tilde{\rho}_j}{d\rho_j} \right\| s(a_j)^2. \tag{4.5}
\]

In the case that \( a_j > \delta \), \( \sup_{\partial B_j \setminus \partial \mathbb{B}} \|d\tilde{\rho}_j/d\rho_j\| \) can be bounded above by the probability that \( d \)-dimensional Brownian motion on the half-space \( \{(x_1, \ldots, x_d) : x_1 > 0\} \), started from \( (\delta/a_j) \) reaches the boundary of the unit sphere before hitting the hyperplane \( \{x_1 = 0\} \). This is bounded above by a constant times \( (\delta/a_j) \) [Bur86, Theorem 4.4]. When \( \delta > a_j \) we have \( \sup_{\partial B_j \setminus \partial \mathbb{B}} \|d\tilde{\rho}_j/d\rho_j\| = 1 \). So overall, we obtain the bound (where from now on \( a \leq b \) means \( a \leq Cb \) for some constant \( C \) depending on the dimension):
\[
\mathbb{E}(h^{28}, \tilde{\rho}_j)^4 \lesssim (\delta/a_j \wedge 1) \sqrt{s(a_j)}.
\]
Thus applying Cauchy–Schwarz to (4.4) we see that
\[ k_4(z_1, z_2, z_3, z_4) \lesssim \prod_{j=1}^{4} \left( \frac{\delta}{a_j} \wedge 1 \right) \sqrt{s(a_j)}. \]

On the other hand, we can lower bound \( g(z_1, z_2, z_3, z_4) \) using the explicit expression
\[ G^B(x, y) = s(|x - y|) - s(|x||y - \tilde{x}|) \quad ; \quad \tilde{x} = |x|^{-2} x. \]

In particular, when \( |x| = |y| = 1 - \delta = r \), we have that \( |x|^2 |y - \tilde{x}|^2 |x - y|^{-2} = (1 + |x - y|^{-2} \delta^2 (2 - \delta)^2) \). This implies that, on the region \( |x - y| > d \), we have \( G^B(x, y) \gtrsim \delta^2 |x - y|^{-2} \). On the region \( |x - y| < \delta \) we have \( G^B(x, y) \gtrsim s(|x - y|) \).

Without loss of generality we can now assume that \( a_1 \geq a_2 \geq a_3 \geq a_4 \). Combining the above lower bounds for \( G^B \) and the definition of \( g \) with the upper bounds for \( k_4 \), we obtain (4.2) under the condition that \( a_2 > a_1/10 \). Note that the \( \delta^{-\eta} \) correction is only actually needed when the dimension \( d = 2 \).

When \( a_2 \leq a_1/10 \), i.e., one point is considerably further than the rest, our bounds do not suffice - this is because we haven’t taken properly care of cancellations occurring in the third moment, when three points are together. Let us do that now. Notice that when \( a_2 \leq a_1/10 \) we have \( B_2, B_3, B_4 \subset z_2 + a_1 B \); we write \( \hat{\rho}_1 \) for harmonic measure seen from \( z_2, z_3, z_4 \) on \( \partial(z_2 + a_1 B) \setminus \partial B \). We need an extension of Claim 7, that first separates the three points, and looks at the occurring cancellations:

**Claim 8.** In the case that \( a_2 < a_1/10 \) we can further write \( k_4(z_1, z_2, z_3, z_4) \) as

\[ \mathbb{E}(h^B_1, \hat{\rho}_1) \prod_{i \neq 1} (h^B_i, \hat{\rho}_i) + \sum_{\sigma \in S(2,3,4)} \mathbb{E}(h^B_1, \hat{\rho}_1)(h^B_1, \hat{\rho}_1(2)) \mathbb{E}(h^B_3, \hat{\rho}_3(3)) \mathbb{E}(h^B_4, \hat{\rho}_4(4) - \hat{\rho}_3(4)). \]  

(4.6)

Again, we postpone the proof of the claim and first see how it implies (4.2). From the same Cauchy–Schwarz argument and bounds as in \( a_2 > a_1/10 \) case (noting also that \( d(z_2, \partial(z_2 + a_1 B)) > a_1/2 \) for \( j = 3, 4 \), we can deduce that the first term in (4.6) is \( \lesssim \delta^{-\eta} g(z_1, z_2, z_3, z_4) \) for some \( \eta \in [0, 1) \). Again the \( \delta^{-\eta} \) correction is only needed when \( d = 2 \).

To deal with the latter terms in (4.6), we use the fact that the covariance of \( h^B_1 \) is equal to \( bG^B \). Indeed, using the explicit expression for \( G^B \) we see that the latter term is given by

\[ 4b^2 \sum_{\sigma \in S(2,3,4)} G^B(z_1, z_\sigma(2)) G^B(z_2 + a_1 B) \cap B(z_3, z_\sigma(4)), \]

and since \( G^{D'} \leq G^D \) for \( D' \subset D \), (4.2) then follows in the case \( a_2 \leq a_1/10 \) too.

It now remains to prove the claims.

**Proof of Claims:** We start with (4.4). As mentioned above, we cannot apply the Markov property directly inside the regions \( B_i \). Instead we will work in \( 2B \) and use the equivalent of (4.4) for the field \( h^{2B} \), together with the Markov decomposition \( h^{2B} = h^{2B}_1 + \varphi^{2B} \). In this decomposition, the two summands are independent, \( h^{2B}_1 \) has the law of \( h^B \) and \( \varphi^{2B} \) is harmonic inside \( B \).

For each \( i = 1, \ldots, 4 \), let \( \eta_i \) be a smooth function supported in \( B_i \), rotationally symmetric about \( z_i \), such that \( k_4(z_1, z_2, z_3, z_4) = \mathbb{E}(\prod_{i=1}^{4} (h^B_1, \eta_i)) \). Let us also denote by \( \rho'_i \) the harmonic measure on \( \partial B_i \) (now including the part on \( \partial B \)) seen from \( z_i \).

By the Markov decomposition for the field \( h^{2B} \) in \( \{z_1 + a_1 B, \ldots, z_4 + a_4 B\} \), as in (2.2) with \( n = 4 \), we have that \( \mathbb{E}(\prod_{i=1}^{4} (h^{2B}_i, \eta_i)) = \mathbb{E}(\prod_{i=1}^{4} (h^{2B}_1, \rho'_i)) \). We use the domain Markov decomposition \( h^{2B} = h^{2B}_1 + \varphi^{2B} \) to rewrite this as

\[ \mathbb{E}(\prod_{i=1}^{4} (h^{2B}_1 + \varphi^{2B}, \eta_i)) = \mathbb{E}(\prod_{i=1}^{4} (h^{2B}_1 + \varphi^{2B}, \rho'_i)). \]

Opening the brackets and using the independence of \( h^{2B}_1 \) and \( \varphi^{2B} \), we can write this further as

\[ \sum_{S \cup S' = \{1,2,3,4\}} \mathbb{E}(\prod_{i \in S} (h^{2B}_1, \eta_i)) \mathbb{E}(\prod_{j \in S'} (\varphi^{2B}, \eta_j)) = \sum_{S \cup S' = \{1,2,3,4\}} \mathbb{E}(\prod_{i \in S} (h^{2B}_1, \rho'_i)) \mathbb{E}(\prod_{j \in S'} (\varphi^{2B}, \rho'_j)). \]
As \( h^{S}_{28} \) has the law of \( h^{S} \), to prove that \( E(\prod_{i=1}^{4}(h^{S}, \eta_{i})) = E(\prod_{i=1}^{4}(h^{S}, \rho_{i}')) \) it suffices to show that all terms with \( |S| \neq 4 \) cancel out. Since \((h^{S}, \tilde{\rho}_{j}) = (h^{S}, \rho_{j}')\) a.s. for each \( i \), this proves the claim.

To show the cancellation, first observe that as \( \psi_{28}^{S} \) is harmonic inside each \( B_{j} \), we have that \( (\varphi_{28}^{S}, \eta_{j}) = (\varphi_{28}^{S}, \rho_{j}') \) for every \( j = 1 \ldots 4 \) and thus the terms with \( |S'| = 4 \) cancel out. Also, both \( h^{S}_{28} \) and \( \varphi_{28}^{S} \) are mean zero, so all terms with \( |S| = 1 \) or \( |S'| = 1 \) also cancel out. We are left to consider the cases when \( |S| = 2 \) and \( |S'| = 2 \). But we already know that

\[
E(\prod_{j \in S'}(\varphi_{28}^{S}, \eta_{j})) = E(\prod_{j \in S'}(\varphi_{28}^{S}, \rho_{j}')), 
\]

and thus it remains to just verify that for \( i \neq j \), we have that \( E((h^{S}_{28}, \eta_{i})(h^{S}_{28}, \eta_{j})) = E((h^{S}_{28}, \rho_{i}')(h^{S}_{28}, \rho_{j}')) \). This can be however verified directly via the fact that the covariance is a multiple of the Green’s function and all \( \eta_{i}, \eta_{j}, \rho_{i}', \rho_{j}' \) have disjoint support.

The proof of (4.6) given (4.4) follows from the same argument, using the domain Markov decomposition of \( h^{S} \) inside \( z_{1} + a_{1}B \) and \( z_{2} + a_{1}B \). We omit the details.

\[ \square \]

### 4.2 Gaussianity of spherical averages

We now show that the \( r \)-spherical average process around 0 with a varying radius is a Gaussian process.

**Lemma 9.** \((h_{s}(0))_{r \in (0,1)}\) is a Gaussian process.

To prove this, we will use the fact that any continuous stochastic process (indexed by positive time) with independent increments, is Gaussian. The fact that \( h_{s}(0) \) has a continuous modification in \( r \) comes from Lemma 6 of the previous subsection.

**Proof of Lemma 9.** Since the variance of this process increases as \( r \downarrow 0 \), it is natural to parameterise the time so that the process starts at \( r = 1 \). Thus we set \( X_{t} = h_{1-t}(0) \). Then \( X_{t} \to 0 \) in probability as \( t \downarrow 0 \) and \( X \) has independent increments by the domain Markov property (more precisely (2.1)). Thus, to show that it is a Gaussian process, by [Kal97, Theorem 11.4], it suffices to prove that it admits a continuous modification in \( t \).

We apply Kolmogorov’s continuity criterion to show this. By the domain Markov property and scaling assumption it suffices to control the behavior at time 0, i.e. it is enough to show that for some \( C \in (0, \infty) \), some \( \eta < 1 \) and all \( \delta \in (0,1) \):

\[
E(X_{\delta}^{4}) \leq C\delta^{2-\eta}. \quad (4.7)
\]

For this we use (4.2). We obtain that

\[
E(X_{\delta}^{4}) = \int_{\partial((1-\delta)B)^{d}} k_{4}(z_{1}, z_{2}, z_{3}, z_{4}) \prod_{i=1}^{4} \rho_{0}^{1-\delta}(dz_{i}) \lesssim \delta^{-\eta} \int_{\partial((1-\delta)B)^{d}} g(z_{1}, z_{2}, z_{3}, z_{4}) \prod_{i=1}^{4} \rho_{0}^{1-\delta}(dz_{i}).
\]

Now recalling that \( g(z_{1}, z_{2}, z_{3}, z_{4}) \) is the four-point function for the zero boundary GFF in \( B \), we see that the integral on the right hand side is the fourth moment of the spherical average of the GFF at radius \( 1 - \delta \). But this spherical average is a centred Gaussian with variance \( -\log(1 - \delta) \) when \( d = 2 \) and \( 1 - (1 - \delta)^{4-d} \) when \( d > 2 \) - see [WP20, equation (13)]. Since these are both of order \( \delta \) as \( \delta \to 0 \) the fourth moment is of order \( \delta^{2} \) by Gaussianity of the GFF itself.

\[ \square \]

### 4.3 Gaussianity in the general case

In what follows, we will often use the co-ordinates \( z = (r, \overline{\theta}) = (|z|, z/|z|) \) for a point in \( \mathbb{R}^{d} \), \( d \geq 2 \).

We will generalise the case of spherical averages to a wider class of processes, stemming from so called spherical harmonics. The interest comes from the following classical theorem (see e.g. [SW71, Chapter IV]).

**Theorem 10** (Expansion using spherical harmonics). In each \( d \geq 2 \), there is a collection of smooth functions \( \psi_{n,j}(\overline{\theta}) : \partial B \to \mathbb{R} \) with \( n \in \mathbb{N}_{0} \), \( M_{n} \in \mathbb{N} \) and \( j \in \{1, \ldots, M_{n}\} \) such that
A. The functions $\psi_{n,j}(\overline{\theta})$ form an orthonormal basis of $L^2(\partial \mathbb{B})$.

B. For every $n \in \mathbb{N}_0$ and $j \in \{1, \ldots, M_n\}$, we have that $(r, \overline{\theta}) \mapsto r^n \psi_{n,j}(\overline{\theta})$ is harmonic in $\mathbb{B}$.

C. For each $n \in \mathbb{N}_0$, one can find radially symmetric functions $f_{n,i}(r) : [0, 1] \to \mathbb{R}$ with $i \in \mathbb{N}_0$ such that $\{e_{n,j,i} = f_{n,i}(\psi_{n,j})\}_{n \in \mathbb{N}_0, j \in \{1, \ldots, M_n\}, i \in \mathbb{N}}$ form an orthonormal basis of $L^2(\partial \mathbb{B})$.

**Remark 11.** In fact one can write out a specific collection of such functions using Legendre polynomials and Bessel functions and choose $e_{n,j,i}$ to be eigenfunctions of $\Delta$. This is not necessary here.

For example, in $d = 2$, one has $M_0 = 1$ and $M_n = 2$ for $n \geq 1$, with the $\psi$'s given by the usual Fourier series on the circle. That is,

$$\{\psi_{n,1,k} = \{\psi_{n,1,k} \}_{k \geq 1}, \{\psi_{n,2,k} \}_{k \geq 1, 1} := \{J_0(\alpha_{n,k} r) \sin(n\theta), J_1(\alpha_{n,k} r) \cos(n\theta)\}_{n \geq 1, k \geq 1}\}$$

form an orthonormal basis of $L^2(\mathbb{B})$, where $(J_n)_{n \geq 0}$ are the Bessel functions and $\alpha_{n,k}$ are the zeroes of $J_n$ for each $n$.

Using these notations, the main result of this section can be stated as follows:

**Proposition 12** (Gaussianity). The random variables $(h^B, e_{n,j,i})_{n \in \mathbb{N}, j \in \{1, \ldots, M_n\}, i \in \mathbb{N}}$ are jointly Gaussian. In particular $(h^B, f)_{f \in C_c^\infty(\mathbb{B})}$ is a Gaussian process.

To prove Proposition 12 we will choose appropriate radial functions $g_{n,j}$ for which we can verify that $h^B$ tested against $g_{n,j}(r)\psi_{n,j}(\overline{\theta})$ on each sphere at radius $r$ is a Gaussian process in $r \in (0, 1)$. The key observation is the following. For $r \in (0, 1)$ and a smooth function $\psi : \partial \mathbb{B} \to \mathbb{R}$ let $\nu^\psi_r$ be the signed measure defined by the condition that

$$\nu^\psi_r(\phi) = \int_{\partial \mathbb{B}} \psi(\overline{\theta}) \phi(r, \overline{\theta}) \rho^0_r(d\overline{\theta})$$

for all functions $\phi$ such that $\phi(r, \overline{\theta}) \in L^1(\partial \mathbb{B}, \rho^0_r(d\overline{\theta}))$, where as before $\rho^0_r$ is uniform measure on $\partial \mathbb{B}$.

**Lemma 13.** Suppose that $\varphi$ is a harmonic function in $\mathbb{B}$. Suppose also that $\psi(\overline{\theta})$ is a smooth function defined on $\partial \mathbb{B}$ such that $(r, \overline{\theta}) \mapsto \psi(\overline{\theta}) r^n$ is harmonic in $\mathbb{B}$. Then $r^{-n} \nu^\psi_r(\varphi)$ is constant as a function of $r$ on $(0, 1)$.

**Proof.** Let us fix $r_0 \in (0, 1)$: we will show that $\left(\frac{d}{dr}\right)|_{r = r_0} r^{-n} \nu^\psi_r(\varphi) = 0$, which implies the result. By the second Green’s identity applied in $r_0 \mathbb{B}$ to the harmonic functions $\varphi$ and $(r, \overline{\theta}) \mapsto r^n \psi$, we can write

$$\int_{\partial \mathbb{B}} (\varphi \frac{d}{dr} [r^n \psi] - r^n \psi \frac{d}{dr} \varphi) = \int_{r_0 \mathbb{B}} (\varphi \Delta [r^n \psi] - r^n \psi \Delta \varphi),$$

where we are integrating against the standard volume measure on $r_0 \mathbb{B}$ on the right-hand side, and the induced measure on its boundary on the left, which is a multiple of uniform measure. Now the right-hand side is zero as both $r^n \psi$ and $\varphi$ are harmonic by assumption. Thus we deduce that

$$n \int_{\partial \mathbb{B}} \varphi(r_0, \overline{\theta}) \psi(\overline{\theta}) \rho^0_r(d\overline{\theta}) = r_0 \int_{\partial \mathbb{B}} \psi(\overline{\theta}) \varphi(r, \overline{\theta}) \rho^0_r(d\overline{\theta}).$$

It follows that

$$\frac{d}{dr} [r^{-n} \nu^\psi_r(\varphi)]|_{r = r_0} = -nr_0^{-n-1} \int_{\partial \mathbb{B}} \varphi(\overline{\theta}) \psi(r_0, \overline{\theta}) + r_0^{-n} \int_{\partial \mathbb{B}} \psi(\overline{\theta}) \frac{d}{dr} \varphi(r, \overline{\theta})|_{r = r_0} = 0.$$
Proof. First let us fix \( a_0, \ldots, a_k \) in \( \mathbb{R} \). We again parametrise the process from large radii towards small radii, i.e. let’s set \( X_t = A_{1-t} \). We first argue using Lemma 13 that \( X_t \) has independent increments.

For clarity, let us first show this for \( \tilde{X}_t = (h^\beta, (1-t)^{-n_i}\nu^\psi_{ni,i}) \). Indeed, for each \( i \in \{1, \ldots, k\} \) and each \( s < r < 1 \) we can write by domain Markov property
\[
(h^\beta, s^{-n_i}\nu^\psi_{ni,i}) = (h^\beta, s^{-n_i}\nu^\psi_{ni,i}) + (\varphi^\beta, s^{-n_i}\nu^\psi_{ni,i}).
\]
As already mentioned, \( \psi_{ni,j} \) satisfies the conditions of Lemma 13, and thus
\[
(h^\beta, s^{-n_i}\nu^\psi_{ni,i}) \to \lim_{u \uparrow r} (h^\beta, u^{-n_i}\nu^\psi_{ni,i}) = (h^\beta, r^{-n_i}\nu^\psi_{ni,i})
\]
where the last equality follows since \((h^\beta, u^{-n_i}\nu^\psi_{ni,i})\) converges to 0 as \( u \uparrow r \) in \( L^2 \), by a direct calculation using the Green’s function.

As \( h^\beta \) and \( \varphi^\beta \) are independent, we conclude that \( \tilde{X}_t = (h^\beta, (1-t)^{-n_i}\nu^\psi_{ni,i}) \) has independent increments. The same argument can be directly applied to get the claim for \( X_t \).

Now, \( X_t \) is also centred and square integrable, converging to 0 in probability as \( t \downarrow 0 \), by definition and assumptions on \( h^\beta \). Moreover, using the fact that each \( \psi_{n,i} \) is bounded on \( \partial \mathbb{B} \), we can use the 4th moment bound (4.2) to apply Kolmogorov’s criteria similarly to the case of the spherical average and obtain that \( X_t \) possesses an a.s. continuous modification. This implies that the process is Gaussian by [Kal97, Theorem 11.4].

For the final claim, let us fix \( r \in (0,1) \). Then \((h^\beta, \nu^\psi_{n,i,j})_{n \geq 0, j \in \{1, \ldots, M_n\}}\) is jointly Gaussian, since for any \( k \geq 1, b_1, \ldots, b_k \in \mathbb{R}, n_i \in \mathbb{N}_0 \) and \( j_i \in \{1, \ldots, M_{n_i}\} \), we can apply the above with \( a_i = r^{n_i}b_i \) for each \( i \) to see that \( \sum b_i(h^\beta, \nu^\psi_{n,i,j}) \) is a Gaussian random variable.

There is one more step required to deduce Proposition 12: to extend the joint Gaussianity to varying radii.

**Lemma 15.** For any \( k \geq 1 \), and any \((r_i, n_i, j_i)_{1 \leq i \leq k}\) with \( r_1 \in (0,1), n_i \in \mathbb{N}_0 \) and \( j_i \in \{1, \ldots, M_{n_i}\} \) for each \( i \), the vector
\[
\left( (h^\beta, \nu^\psi_{r_1, j_1}), \ldots, (h^\beta, \nu^\psi_{r_k, j_k}) \right)
\]
is Gaussian.

In particular, we have that \((h^\beta, \nu^\psi_{r, j})_{r \in (0,1), n \geq 0, j \in \{1, \ldots, M_n\}}\) is jointly Gaussian.

**Proof.** The second statement is an immediate consequence of the first. So, let us fix \( k, (r_i, n_i, j_i)_{1 \leq i \leq k} \) as in the statement. For now, let us assume that \( r_1 > \cdots > r_k > 0 \) (we come back to the general case at the end of the proof). We iterate the domain Markov property inside each ball \( r_i \mathbb{B} \) as follows: we write \( h^\beta = h^\beta \), then \( h^\beta = h^\beta + \varphi^1 \) by applying the Markov property inside \( r_1 \mathbb{B} \), then \( h^\beta = h^\beta + \varphi^2 \) inside \( r_2 \mathbb{B} \) etc. so that \((\varphi^1, \ldots, \varphi^k)\) are mutually independent by (2.1). Shortening the notation \( \nu^\psi_{n,i,j} \) to \( \nu^i \) for each \( i = 1, \ldots, k \), it suffices to show that
\[
(h^\beta, \nu^i) = (0, \nu^i) = (0, \nu^1) (0, \nu^2) \cdots (0, \nu^{k-1}) (0, \nu^{k}),
\]
is a Gaussian vector - indeed, we can write each \((h^\beta, \nu^i) = \sum_{j=1}^i (\varphi^j, \nu^i) \). However, as in the proof Lemma 14, it follows from Lemma 13 that
\[
(h^{k-1}, \nu^i) = (\varphi^k, \nu^i) = \left( \frac{r_{k-1} \nu^k}{r_k} \right) (\varphi^k, \nu^k).
\]
for any \( \psi_{n,i} \) and any \( j \). Hence, (4.10) can be rewritten as
\[
((\varphi^1, \nu^1), \left( \frac{r_2}{r_1} \right)^{n_2} (\varphi^2, \nu^2), \left( \frac{r_3}{r_2} \right)^{n_3} (\varphi^3, \nu^3), \ldots, (\varphi^{k-1}, \nu^{k-1}), \left( \frac{r_k}{r_{k-1}} \right)^{n_k} (\varphi^k, \nu^k)).
\]
But now \((\varphi^k, \nu^k)\) is Gaussian, and each pair \((\varphi^i, \nu^i), \left( \frac{r_{i+1}}{r_i} \right)^{n_{i+1}} (\varphi^{i+1}, \nu^{i+1})\) with \( i = 1, \ldots, k-1 \) is jointly Gaussian by Lemma 14. Moreover, this singleton, and each pair in the list are independent of one another by construction. Thus we indeed have a Gaussian vector.

Finally, if the \( r_i \)'s are not distinct, the same argument still works: the “pairs” just mentioned will simply be larger tuples, concluding the proof of the lemma.
We are now ready to prove Proposition 12.

Proof of Proposition 12. Consider the basis \((e_{n,j,i} = f_{n,i}(\psi_{n,j}))_{n \in \mathbb{N}_0, j \in \{1, \ldots, M_n\}, i \in \mathbb{N}}\) of \(L^2(\mathbb{B})\) given by Theorem 10. The previous lemma implies that
\[
\{(h^B, f_{n,i}(r)\psi_{n,j})\}_{r \in (0,1), n \in \mathbb{N}_0, j \in \{1, \ldots, M_n\}, i \in \mathbb{N}}
\]
is a Gaussian process. In particular,
\[
(h^B, f_{n,i}(\psi_{n,j}))_{n \in \mathbb{N}_0, j \in \{1, \ldots, M_n\}, i \in \mathbb{N}}
\]
is also a Gaussian process. Indeed, the random variables in (4.12) exist by Corollary 5, and we obtain the Gaussianity since they can be defined as almost sure limits of weighted sums of elements in the collection (4.11).

The fact that \(e_{n,j,i}\) form a basis of \(L^2(\mathbb{B})\), together with linearity of \((h^B, f)\) and that \(h^B\) is zero outside of \(\mathbb{B}\), now implies that \((h^B, f)_{f \in C^\infty_c(\mathbb{R}^d)}\) is a Gaussian process. \(\square\)

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