QUARTIC 3-FOLD: PFAFFIANS, INSTANTONS AND HALF-CANONICAL CURVES

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Abstract. A generic quartic 3-fold $X$ admits a 7-dimensional family of representations as the Pfaffian of an 8 by 8 skew-symmetric matrix of linear forms. This provides a 7-dimensional moduli space $M$ of rank 2 vector bundles on $X$. A precise geometric description of a 14-dimensional family of half-canonical curves $C$ of genus 15 in $X$ such that the above vector bundles are obtained by Serre’s construction from $C$ is given. It is proved that the Abel–Jacobi map of this family factors through $M$, and the resulting map from $M$ to the intermediate Jacobian is quasi-finite. In particular, every component of $M$ has non-negative Kodaira dimension. Some other constructions of rank 2 vector bundles with small Chern classes are discussed; it is proved that the smallest possible charge of an instanton on $X$ is 4.

Introduction

This paper is a part of the study of moduli spaces of vector bundles with small Chern classes on certain Fano threefolds. It provides some non-existence results and constructions of a few moduli components of vector bundles on a quartic threefold. One of them is the component of kernel bundles, defined similarly to that of [MT], [IM] for the case of a cubic threefold. Our work received a strong pulse with the publication of the paper of Beauville [B], which allowed to simplify some arguments used in [MT] and at the same time put our results in a more general framework of Pfaffian hypersurfaces. We also prove that there are no normalized rank 2 stable vector bundles on $X$ with $c_2 < 4$ and exhibit two constructions of stable vector bundles with $c_1 = 0$ and $c_2 = 4$. Only one of them provides instantons; we show in fact that all the instantons of charge 4 are obtained by this construction.

In [MT], it was proved that the Abel–Jacobi map of the family of normal elliptic quintics lying on a general cubic threefold $V$ factors through a moduli component of stable rank 2 vector bundles on $V$ with Chern numbers $c_1 = 0, c_2 = 2$, whose general point represents a

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vector bundle obtained by Serre’s construction from an elliptic quintic. The elliptic quintics mapped to a point of the moduli component vary in a 5-dimensional projective space inside the Hilbert scheme of curves, and the map from the moduli component to the intermediate Jacobian is quasi-finite. Later, in [IM], this modular component was identified with the variety of representations of $V$ as a linear section of the Pfaffian cubic in $\mathbb{P}^{14}$ and it was proved that the degree of the quasi-finite map is 1, so the moduli component is birational to the intermediate Jacobian $J^2(X)$. Beauville mentions in [B] a recent work of Druel, yet unpublished, which proves that the moduli space $M_V(2; 0, 2)$ is irreducible, so its unique component is the one described above.

In the present paper, we prove that a generic quartic threefold $X$ admits a 7-dimensional family of essentially different representations as the Pfaffian of an $8 \times 8$ skew-symmetric matrix of linear forms. Thanks to [B], this provides a 7-dimensional family of arithmetically Cohen–Macaulay (ACM for short) vector bundles on $X$, obtained as the bundles of kernels of the $8 \times 8$ skew-symmetric matrices of rank 6 representing points of $X$. We show that this family is a smooth open set $M_X$ in the moduli space of stable vector bundles $M_X(2; 3, 14) \simeq M_X(2; -1, 6)$. The ACM property means the vanishing of the intermediate cohomology $H^i(X, E(j))$ for all $i = 1, 2, j \in \mathbb{Z}$.

We give also a precise geometric characterization of the ACM curves arising as schemes of zeros of sections of the above kernel vector bundles. According to Beauville, they are half-canonical ACM curves of degree 14 in $\mathbb{P}^4$; we show that they are linear sections of the rank 4 locus $Z \subset \mathbb{P}(\wedge^2 \mathcal{O}_7)$ in the projectivized space of the $7 \times 7$ skew-symmetric matrices. Linear sections of $Z$ arose already in the literature: Rødland [R] studied the sections $\mathbb{P}^6 \cap Z$, which are Calabi–Yau threefolds. We show that such curves fill out open sets of smooth points of the Hilbert schemes of $X$ (of dimension 14) and of $\mathbb{P}^4$ (of dimension 56), and that the isomorphism classes of smooth members of this family fill out a 32-dimensional moduli component $\mathcal{M}_{15}$ of curves of genus 15 with a theta-characteristic linear series of dimension 4.

Next we study the Abel–Jacobi map of the ACM half-canonical curves of genus 15 in $X$. It factors through $M_X$ via Serre’s construction: the fibers over points of $M_X$ are $\mathbb{P}^7$, and the resulting map from $M_X$ to $J^2(X)$ is étale quasi-finite, hence its image is 7-dimensional. The role of the above half-canonical curves is similar to that of normal elliptic quintics in the case of the cubic threefold $V$, where the Abel–Jacobi map factors through the instanton moduli space with fibers $\mathbb{P}^5$ and with a 5-dimensional image; as $\dim J^2(V) = 5$, the image is an open subset of $J^2(V)$. The result for a quartic threefold is somewhat weaker: here we
do not know whether the degree of the quasi-finite map is 1 and whether $M_X$ is irreducible. Moreover, as $7 = \dim M_X < 30 = \dim J^2(X)$, we cannot conclude, as in the case of a cubic threefold, that the image of $M_X$ is an open subset of an Abelian variety; we can only state that every component of it, and hence of $M_X$ itself, has a non-negative Kodaira dimension.

In Section 1, we gather preliminary results: a criterion for stability of rank 2 sheaves, Bogomolov’s inequality, and prove the non-existence of normalized rank 2 stable vector bundles with small second Chern classes. We provide two constructions of such vector bundles for $c_1 = 0$, $c_2 = 4$ and discuss the geometry of the instanton component(s).

In Section 2, we prove that a generic quartic 3-fold is Pfaffian, in using the same method as was used by Adler in his Appendix to [AR] for a cubic threefold: take a particular quartic which is Pfaffian and prove that the differential of the Pfaffian map from the family of all the $8 \times 8$ skew-symmetric matrices of linear forms to the family of quinary quartics is of maximal rank. We prove also basic facts about $M_X$: stability, dimension 7, smoothness.

Section 3 treats half-canonical ACM curves of genus 15 on $X$ and in $\mathbb{P}^4$.

Section 4 applies the technique of the Tangent Bundle Sequence of Clemens–Griffiths [CG] and Welters [W] to the calculation of the differential of the Abel–Jacobi map for the family of the above half-canonical curves $C$. It identifies the kernel of the differential with $H^1(N_{C/\mathbb{P}^4}(-1))^\vee$, and we prove that it has dimension 7.

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1. Generalities and the case $c_1 = 0$

Let $X$ be a smooth quartic threefold. It is well known that $\text{Pic}(X)$ is isomorphic to $\mathbb{Z}$, generated by the class of the hyperplane section $H$, and the group of algebraic 1-cycles modulo topological equivalence is also isomorphic to $\mathbb{Z}$, generated by the class of a line $l \subset X$. For two integers $k, \alpha$, we will denote by $M_X(2; k, \alpha)$ the moduli space of stable rank 2 vector bundles $\mathcal{E}$ on $X$ with Chern classes $c_1 = k[H]$ and $c_2 = \alpha[l]$. We will often identify the Chern classes with integers in using the generators $[H], [l]$ of the corresponding groups of algebraic cycles. We have $[H]^2 = 4[l]$.

By the definition of the Chern classes and by Riemann–Roch–Hirzebruch, we have for $\mathcal{E} \in M_X(2; k, \alpha)$:
\[ c_1(\mathcal{E}(n)) = c_1(\mathcal{E}) + 2n[H] = (k + 2n)[H], \quad c_2(\mathcal{E}(n)) = c_2(\mathcal{E}) + n[H]c_1(\mathcal{E}) + n^2[H]^2 = (\alpha + 4kn + 4n^2)[l], \quad \chi(\mathcal{E}) = \frac{2}{3}k^3 - \frac{1}{3}k\alpha + k^2 - \frac{1}{2}\alpha + \frac{7}{3}k + 2. \]

A rank 2 torsion free sheaf \( \mathcal{E} \) on \( X \) is normalized, if \( c_1(\mathcal{E}) = k[H] \) with \( k = 0 \) or \( k = -1 \). We can make \( \mathcal{E} \) normalized in replacing it by a suitable twist \( \mathcal{E}(n) \). If \( \mathcal{E} \) is locally free, we have \( \mathcal{E}^\vee \cong \mathcal{E} \otimes (\det \mathcal{E})^{-1} \), so that \( \mathcal{E} \) is self-dual when \( k = 0 \).

The following lemmas are well known:

**Lemma 1.1.** Let \( \mathcal{E} \) be a normalized rank 2 reflexive sheaf on a non-singular projective variety \( X \) with \( \text{Pic}(X) \cong \mathbb{Z} \). Then it is stable if and only if \( h^0(\mathcal{E}) = 0 \).

**Proof.** Any saturated torsion free rank 1 subsheaf of \( \mathcal{E} \) is invertible of the form \( \mathcal{O}_X(m) \) and gives an exact triple
\[
0 \rightarrow \mathcal{O}_X(m) \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Z(k - m) \rightarrow 0 ,
\]
where \( Z \) is a subscheme of \( X \) of codimension 2. The triple breaks the Gieseker stability of \( \mathcal{E} \) if and only if \( m \geq 0 \).

If we assume that \( \mathcal{E} \) has global sections, then there exists a triple (\( ) \) with \( m = 0 \), hence \( \mathcal{E} \) is not stable. If we assume that \( \mathcal{E} \) has no global sections, then in any triple (\( ) \) for \( \mathcal{E} \), we have \( m < 0 \), because \( h^0(\mathcal{E}) \geq h^0(\mathcal{O}_X(m)) \). Hence \( \mathcal{E} \) is stable. \( \square \)

**Lemma 1.2.** Let \( \mathcal{F} \) be a rank \( r \) semistable reflexive sheaf on \( X \). Then \( (2rc_2(\mathcal{F}) - (r - 1)c_1^2(\mathcal{F})) \cdot H \geq 0 \). If \( r = 2 \) and \( \mathcal{F} \) is stable, then the inequality is strict.

**Proof.** This is Bogomolov's Theorem, proved by him for \( T \)-(semi)stable sheaves \( \mathbb{B}^0 \). For another approach to the proof and for relations between different notions of (semi)stability, see e. g. \( \mathbb{K} \). \( \square \)

**Proposition 1.3.** Let \( X \) be a smooth quartic threefold. Then the following statements hold:

(i) \( M_X(2; 0, \alpha) = \emptyset \) for all odd \( \alpha \) and for \( \alpha \leq 2 \).

(ii) \( M_X(2; -1, \alpha) = \emptyset \) for \( \alpha \leq 3 \).

**Proof.** The case of odd \( \alpha \) in (i) follows trivially from Riemann–Roch–Hirzebruch: we have \( \chi(\mathcal{E}) = 2 - \frac{1}{2}\alpha \). For the remaining cases, the proof goes exactly as in \( \mathbb{B} \). The first step is to show that if \( M_X(2; \epsilon, \alpha) \neq \emptyset \) (\( \epsilon = 0, -1 \)), then \( h^0(\mathcal{E}(1)) \neq 0 \) for all \( \mathcal{E} \in M_X(2; \epsilon, \alpha) \). The second step is to verify that there are no curves on \( X \) that might be zero loci of sections of \( \mathcal{E}(1) \).

So, let \( \mathcal{E} \in M_X(2; \epsilon, \alpha) \), \( \epsilon = 0, -1, \alpha \leq 2 - \epsilon \). Assume that \( h^0(\mathcal{E}(1)) = 0 \). By Serre duality, \( h^3(\mathcal{E}(1)) = h^0(\mathcal{E}(\epsilon - 2)) = 0 \). Hence \( h^2(\mathcal{E}(1)) \geq 0 \).
\( \chi(\mathcal{E}(1)) \). We have \( \chi(\mathcal{E}(1)) = 10 - \frac{3}{2} \alpha \) if \( \epsilon = 0 \) and \( \chi(\mathcal{E}(1)) = 6 - \alpha \) if \( \epsilon = -1 \). Hence \( \dim \text{Ext}^1(\mathcal{E}, \mathcal{O}(-2)) = h^1(\mathcal{E}(\epsilon(-2))) > 0 \), and there exists a non-trivial extension of vector bundles

\[
0 \longrightarrow \mathcal{O}_X(-2) \overset{\alpha}{\longrightarrow} \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0 .
\]  

We have \( \Delta(\mathcal{F}) = (c_1(\mathcal{F})^2 - 3c_2(\mathcal{F}))H = 16 - 3\alpha > 0 \) if \( \epsilon = 0 \) and \( \Delta(\mathcal{F}) = 12 - 3\alpha \) if \( \epsilon = -1 \), so \( \mathcal{F} \) is unstable by Lemma 1.2. To fix ideas, restrict ourselves to the case \( \epsilon = 0 \), the other case being completely similar.

The unstability of \( \mathcal{F} \) can manifest itself in two ways: either \( \mathcal{F} \) contains a rank 1 saturated subsheaf \( \mathcal{O}_X(n) \) with \( n > c_1(\mathcal{F})/\text{rk} \mathcal{F} = -2/3 \), or there exists a non-trivial morphism of sheaves \( \mathcal{F} \overset{\phi}{\longrightarrow} \mathcal{O}_X(n) \) with \( n < c_1(\mathcal{F})/\text{rk} \mathcal{F} \). In the first case, \( n \geq 0 \), hence \( h^0(\mathcal{F}) \neq 0 \), hence \( h^0(\mathcal{E}) \neq 0 \), and this contradicts the stability of \( \mathcal{E} \).

In the second case, \( n \leq -1 \). If \( n < -2 \), then \( \phi \sigma = 0 \) and \( \phi \) descends to a non-trivial morphism \( \mathcal{E} \longrightarrow \mathcal{O}_X(n) \), which contradicts the stability of \( \mathcal{E} \). Hence \( n = -1 \) or \(-2 \). It cannot be \(-2 \), because otherwise the extension (4) would be split. So \( n = -1 \) and we obtain the exact triple

\[
0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}_Z(-1) \longrightarrow 0 ,
\]  
in which \( \mathcal{F}' \) is a reflexive rank 2 sheaf. We have \( c_1(\mathcal{F}') = -[H], c_2(\mathcal{F}') \leq c_2(\mathcal{F}) - c_1(\mathcal{F}')c_1(\mathcal{O}_X(1)) = (\alpha - 4)[l] < 0 \), hence \( \mathcal{F}' \) is unstable by Bogomolov’s inequality. By Lemma 1.1, \( h^0(\mathcal{F}') \neq 0 \), which implies \( h^0(\mathcal{F}) \neq 0 \) and hence \( h^0(\mathcal{E}) \neq 0 \). This contradicts the stability of \( \mathcal{E} \).

Thus we have proved \( h^0(\mathcal{E}(1)) \neq 0 \). As \( \text{Pic} \, X \cong \mathbb{Z} \), the scheme \( C = C_s \) of zeros of a non-trivial section \( s \) of \( \mathcal{E}(1) \) is a l. c. i. of pure codimension 2. Hence \( \mathcal{E}(1) \) fits into the following exact triple

\[
0 \longrightarrow \mathcal{O}_X \overset{s}{\longrightarrow} \mathcal{E}(1) \longrightarrow \mathcal{I}_C(2) \longrightarrow 0 .
\]  

We have \( [C] = (4 + \alpha)[l] \), \( \omega_C = \mathcal{O}_C(1) \), \( 2p_a(C) - 2 = 4 + \alpha \). It remains only to verify the case of \( \alpha = 2 \). We have \( p_a(C) = 4 \) and \( C \) is embedded into \( \mathbb{P}^4 \) by a subsystem of the canonical system. The exact triple (\( 3 \)), twisted by \( \mathcal{O}_X(-1) \), implies that \( C \) is not contained in a hyperplane. Hence \( C \) is not connected. It cannot be a union of more than one connected components either, because at least one of them should be of degree \( \leq 3 \) and hence \( \omega_C \) cannot be ample.

The proof is completed in a similar way in the case \( k = -1 \).

For \( k = 0 \), we have proved that there are no stable rank 2 vector bundles with \( \alpha < 4 \). However, they do exist for \( \alpha = 4 \). Indeed, assuming that \( h^0(\mathcal{E}(1)) \neq 0 \), we find only two possibilities for the zero locus \( C \) of a non-trivial section of \( \mathcal{E}(1) \): either \( C \) is a canonical curve...
8-dimensional family of such curves $C$. Only in this case the vector bundle $E$ has natural cohomology, that is, for every $t$, $h^1(E(t)) \neq 0$ for at most one value of $i$. It is reasonable to call instantons the stable vector bundles with natural cohomology, such that $c_1(E) = 0$ and the instanton condition $h^1(E(-2)) = 0$ is verified. Thus we have the following statement:

**Proposition 1.4.** (i) On any smooth quartic threefold $X$, there is an irreducible component $M^0_X$ of $M_X(2; 0, 4)$ which parametrizes the vector bundles obtained by Serre’s construction from the curves $C = C_1^4 \sqcup C_2^4$, where $C_1^4$ are plane sections of $X$. These vector bundles satisfy $h^1(E(1)) = 1$, hence they are not instantons.

(ii) Let $C$ be a smooth complete intersection of 3 quadrics in $\mathbb{P}^4$. Then there exists a smooth quartic threefold $X$ containing $C$, and the vector bundles on $X$ obtained by Serre’s construction (3) from the curve $C$ and from its generic deformations in $X$ sweep out a component $M^1_X(C)$ of $M_X(2; 0, 4)$, different from $M^0_X$. The vector bundles $E \in M^1_X(C)$ are instantons.

(iii) Let $X$ be a smooth quartic threefold. Then any component $M$ of $M_X(2; 0, 4)$, such that $h^0(E(1)) \neq 0$ for some $E \in M$, is one of the above components $M^0_X$, $M^1_X(C)$.

**Remark 1.5.** Let $C^5_8(X)$ be the 8-dimensional family of curves $C$ on the general $X = X_4$ as in Proposition 1.4 (ii), and let $\nu_2 : X \to \mathbb{P}^{14}$ be the Veronese map. For $C \in C^5_8(X)$, one can see that the 11-space $\mathbb{P}^{11}(C) = \langle \nu_2(C) \rangle \subset \mathbb{P}^{14}$ lies in a unique rank 6 quadric $Q = Q_C \supset \nu_2(X)$. Indeed, if $C$ is the intersection of three quadrics $q_i = 0$ ($i = 1, 2, 3$), then the equation of $X$ can be written in the form $q_1(x)\tilde{q}_1(x) + q_2(x)\tilde{q}_2(x) + q_3(x)\tilde{q}_3(x) = 0$, which provides the rank 6 quadric $Q = \{l_1l_1' + l_2l_2' + l_3l_3' = 0\}$, where $l_i, l_i'$ are the linear forms in the Veronese embedding corresponding to the quadratic forms $q_i, \tilde{q}_i$. It is a degenerate cone whose ridge is $\mathbb{P}^8$, the kernel of the quadratic form, and whose base is $G$, a non-degenerate 4-dimensional quadric in $\mathbb{P}^5$. The curves $C_s$ of zeros of sections $s \in H^0(E(1))$ form a projective space $\mathbb{P}^3$, naturally identified with one of the two $\mathbb{P}^{31}$’s parametrizing projective 2-planes $\{l'_1 = l'_2 = l'_3 = 0\}$ in $G$, the equations of $C_s$ being of the form $q'_1 = q'_2 = q'_3 = 0$, where $q'_i$ correspond to $l'_i$ under the Veronese map. The above $\mathbb{P}^{11}$ is the cone over this $\mathbb{P}^3$ with “vertex” $\mathbb{P}^8$. 
Let \( \Lambda_C = \{ \mathbb{P}_t^{11} : t \in \mathbb{P}^3 \} \) be the ruling of \( Q_C \) defined by \( \mathbb{P}^{11}(C) \in \Lambda \), \( \mathbb{P}^{11}(C) = \mathbb{P}_t^{11} \). Then the sections of \( E_C(1) \) are exactly the curves \( C_t = \mathbb{P}_t^{11} \cap v_2(X) \subset v_2(X) \cong X, \ t \in \mathbb{P}^3, \ C = C_0 \).

Thus we can represent \( M_{1X}^1 \), the union of the 5-dimensional components \( M_{1X}^1(C) \) of Proposition 1.4, as the variety \( \tilde{D}_6(v_2(X)) := \{ \) the \( \mathbb{P}_t^{11}'s \) contained in some rank 6 quadric \( Q \supset v_2(X) \} \), which is, in its turn, the double cover of the family \( D_6(v_2(X)) := \{ \) the rank 6 quadrics \( Q \supset v_2(X) \} \).

**Remark 1.6.** According to [Tyu], (3.1.45), the virtual dimension of \( M_X(2; 0, 2) \) is 1, so one could expect a curve of isomorphism classes of stable vector bundles with \( c_1 = 0, c_2 = 2 \). But we have proved that \( M_X(2; 0, 2) \) is empty, providing thus one more example of a situation when dimension is different from the virtual one.

**Remark 1.7.** For \( k = -1 \), we leave open the cases of \( \alpha = 4, 5 \). We construct in what follows a 7-dimensional component, or a union of 7-dimensional components of \( M_X(2; -1, 6) \) (we do not approach the question on the number of these components).

### 2. Generic quartic 3-fold is Pfaffian

Let \( E \) be an 8-dimensional vector space over \( \mathbb{C} \). Fix a basis \( e_0, \ldots, e_7 \) for \( E \), then \( e_{ij} = e_i \wedge e_j \) for \( 0 \leq i < j \leq 7 \) form a basis for the Plücker space \( \wedge^2 E \). Let \( x_{ij} \) be the corresponding (Plücker) coordinates. The embedding of the Grassmannian \( G = G(2, E) \) in \( \mathbb{P}^{27} = \mathbb{P}(\wedge^2 E) \) is precisely the locus of rank 2 skew symmetric \( 8 \times 8 \) matrices \( M \) with elements \( x_{ij} \) above the diagonal. Let \( G \subset \Omega \subset \Xi \subset \mathbb{P}^{27} \) be the filtration of \( \mathbb{P}^{27} \) by the rank of \( M \), that is \( \Omega = \{ M \mid \text{rk} M \leq 4 \}, \ Xi = \{ M \mid \text{rk} M \leq 6 \} \). Then \( G, \ \Omega \setminus G, \ \Xi \setminus \Omega \) and \( \mathbb{P}^{27} \setminus \Xi \) are orbits of \( PGL(8) \), acting via \( \wedge^2 \) of its standard representation (see e. g. [SK]), and we have \( G = \text{Sing} \Omega, \ \text{dim} G = 12, \ \Omega = \text{Sing} \Xi, \ \text{dim} \Omega = 21 \). \( \Xi \) is defined by the quartic equation \( \text{Pf}(M) = 0 \), where \( \text{Pf} \) stands for the Pfaffian of a skew-symmetric matrix. We will call \( \Xi \) the Pfaffian hypersurface of \( \mathbb{P}^{27} \).

Let \( H \subset \mathbb{P}^{27} \) be a 4-dimensional linear subspace which is not contained in \( \Xi \). Then the intersection \( H \cap \Xi \) will be called a Pfaffian quartic 3-fold. As \( \text{codim}_2 \Omega = 5 \), the linear section \( H \cap \Xi \) is nonsingular for general \( H \). Suppose that a quartic 3-fold \( X \subset \mathbb{P}^4 \) has two different representations \( \phi_1 : X \cong H_1 \cap \Xi, \phi_2 : X \cong H_2 \cap \Xi \) as linear sections of \( \Xi \). We will call them equivalent if \( \phi_2 \circ \phi_1^{-1} \) is the restriction of a transformation from \( PSL(8) \).
Proposition 2.1. A generic quartic 3-fold admits a 7-parameter family of non-equivalent representations as linear sections of the Pfaffian hypersurface in \( \mathbb{P}^{27} \).

Proof. We are using the same argument as that of [AR], Theorem (47.3). The family of quartic 3-folds in \( \mathbb{P}^4 \) is parametrized by \( \mathbb{P}^{69} \), and that of the Pfaffian representations of quartic 3-folds by an open set in the variety \( \text{Lin}(\mathbb{P}^4, \mathbb{P}^{27}) \) of linear morphisms between the two projective spaces. So, we are going to specify one particular quartic 3-fold \( X_0 = \{ F_0 = 0 \} \) which admits a Pfaffian representation \( F_0 = \text{Pf}(M_0) \), then we will show that the differential of the map \( f : \text{Lin}(\mathbb{P}^4, \mathbb{P}^{27}) \rightarrow \mathbb{P}^{69} \) at \( M_0 \) is surjective, and this will imply that \( f \) is dominant.

Let

\[
M_0 = \begin{bmatrix}
0 & x_1 & x_2 & x_3 & x_4 & x_5 & x_1 & 0 \\
-x_1 & 0 & x_5 & 0 & 0 & -x_3 & -x_1 & 0 \\
-x_2 & 0 & 0 & x_1 & x_1 & 0 & 0 & -x_4 \\
-x_3 & -x_5 & -x_1 & 0 & x_2 & 0 & 0 & 0 \\
-x_4 & 0 & -x_1 & -x_2 & 0 & x_3 & x_1 & 0 \\
-x_5 & 0 & 0 & 0 & -x_3 & 0 & x_4 & x_2 \\
-x_1 & x_3 & 0 & 0 & -x_1 & -x_4 & 0 & x_5 \\
0 & x_1 & x_4 & 0 & 0 & -x_2 & -x_5 & 0 
\end{bmatrix},
\]

\[
F_0 = \text{Pf}(M_0) = x_1^3 x_2 - x_1^3 x_3 + x_2^3 x_3 - x_1 x_2 x_3^2 - x_1^2 x_2^2 x_3 + x_1 x_2 x_3 x_4 + x_3^3 x_4 + x_2^3 x_4 - x_1 x_2 x_4^2 + x_1 x_2 x_4 x_5 + x_1^2 x_2 x_5 - x_1^2 x_3 x_5 + x_1 x_3 x_4 x_5 + x_2 x_3 x_4 x_5 + x_3^2 x_5 + x_2 x_3 x_5^2 - x_1 x_4 x_5^2 + x_1 x_3^2.
\]

A point \( M \in \text{Lin}(\mathbb{P}^4, \mathbb{P}^{27}) \) is the proportionality class of an \( 8 \times 8 \) skew-symmetric matrix of linear forms \( l_{ij} \) and is given by its \( 5 \cdot 28 = 140 \) homogeneous coordinates \( (a_{ijk}) \) such that \( l_{ij} = \sum_k a_{ijk} x_k \) (0 ≤ \( i < j \) ≤ 8, 1 ≤ \( k \) ≤ 5). We have \( \partial f(M)/\partial a_{ijk} = x_k \text{Pf}_{ij}(M) \), where \( \text{Pf}_{ij}(M) \) denotes the Pfaffian of the \( 6 \times 6 \) matrix obtained by deleting the \( i \)-th and the \( j \)-th rows and the \( i \)-th and the \( j \)-th columns of \( M \).

Computation by the Macaulay 2 program [CS] shows that, for the above matrix \( M_0 \), the 140 quartic forms \( x_k \text{Pf}_{ij}(M_0) \) generate the whole 70-dimensional space of quinary quartic forms, hence \( f \) is of maximal rank at \( M_0 \). One can also easily make Macaulay 2 to verify that \( X_0 \) is in fact nonsingular, though this is not essential for the above proof.

It remains to verify that the generic fiber of the induced map \( \tilde{f} : PGL(5) \backslash \text{Lin}(\mathbb{P}^4, \mathbb{P}^{27}) / PGL(8) \rightarrow PGL(5) \backslash \mathbb{P}^{69} \) is 7-dimensional. By counting dimensions, one sees that this is equivalent to the fact that the stabilizer of a generic point of the Grassmannian \( G(5, 28) = PGL(5) \backslash \text{Lin}(\mathbb{P}^4, \mathbb{P}^{27}) \) in \( PGL(8) \) is 0-dimensional.
Take a generic 4-dimensional linear subspace \( H \subset \mathbb{P}^{27} \). Then the quartic 3-fold \( X = H \cap \Xi \) is generic, and hence \( \text{Aut}(X) \) is trivial. Thus the stabilizer \( G_H \) of \( H \) in \( PGL(8) \) acts trivially on \( X \), and hence on \( H \). This implies the triviality of \( G_H \) by (5.3) of [3].

Let now \( \mathcal{K} \) be the kernel bundle on \( \Xi \) whose fiber at \( x \in \Xi \) is \( \ker x \). Thus \( \mathcal{K} \) is a rank 2 vector subbundle of the trivial rank 8 vector bundle \( \mathcal{E}_\Xi = E \otimes \mathcal{O}_\Xi \) over \( \Xi_0 = \Xi \setminus \Omega \). Let \( \phi : X \rightarrow H \cap \Xi \) be a representation of a nonsingular quartic 3-fold \( X \subset \mathbb{P}^4 \) as a linear section of \( \Xi \). Giving \( \phi \) is equivalent to specifying a skew-symmetric \( 8 \times 8 \) matrix \( M \) of linear forms such that the equation of \( X \) is \( \text{Pf}(M) = 0 \). Such a representation yields a rank 2 vector bundle \( \mathcal{E} = \mathcal{E}_\phi \) on \( X \), defined by \( \mathcal{E} = \phi^* \mathcal{K} \).

According to [3], Proposition 8.2, the scheme of zeros of any section \( s \neq 0 \) of \( \mathcal{E} \) is an arithmetically Cohen–Macaulay (ACM) 1-dimensional scheme \( C \) of degree 14, not contained in any quadric hypersurface and such that its canonical bundle \( \omega_C \simeq \mathcal{O}_C(2) \). Varieties satisfying the last condition are usually called half-canonical. Moreover, \( \mathcal{E} \) is also ACM and has a resolution of the form

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^4}(-1)^8 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^4}^8 \rightarrow \mathcal{E} \rightarrow 0 \tag{4}
\]

This implies in particular that two Pfaffian representations \( \phi_1, \phi_2 \) are equivalent if and only if the corresponding vector bundles \( \mathcal{E}_1, \mathcal{E}_2 \) are isomorphic. By (8.1) ibid., \( \mathcal{E} \) can be given also by Serre’s construction as the middle term of the extension

\[
0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow \mathcal{I}_{C,X}(3) \rightarrow 0 \tag{5}
\]

where \( \mathcal{I}_{C,X} \) denotes the ideal sheaf of \( C \) in \( X \). Thus, the following assertion holds.

**Corollary 2.2.** A generic quartic 3-fold \( X \subset \mathbb{P}^4 \) has a 7-dimensional family of isomorphism classes of rank 2 ACM vector bundles \( \mathcal{E} \) with \( \det \mathcal{E} \simeq \mathcal{O}(3) \) and \( h^0(\mathcal{E}) = 8 \) which are characterized by either one of the following equivalent properties:

(i) \( \mathcal{E} \) as a sheaf on \( \mathbb{P}^4 \) possesses a resolution of the form (4) with a skew-symmetric matrix of linear forms \( M \);

(ii) the scheme of zeros of any section \( s \neq 0 \) of \( \mathcal{E} \) is an ACM half-canonical curve \( C \) of degree 14 and arithmetic genus 15, not contained in any quadric hypersurface in \( \mathbb{P}^4 \);

(iii) \( \mathcal{E} \) can be obtained by Serre’s construction from a curve \( C \subset X \) as in (ii).

In fact the vector bundles under consideration are stable, so the above 7-parameter family is a part of the moduli space of vector bundles.
Theorem 2.3. Let $X$ be a generic quartic 3-fold and $M_X(2; -1, 6)$ the moduli space of stable rank 2 classes $P$ with Chern classes $c_1 = -[H], c_2 = 6[l]$, where $[l] \in H^2(X, \mathbb{Z})$ is the class of a line. Then the isomorphism classes of the ACM vector bundles of the form $G = E(-2)$, where $E$ are vector bundles introduced in Corollary 2.2, form an irreducible open subset $M_X$ of dimension 7 in the nonsingular locus of $M_X(2; -1, 6)$.

Proof. Stability. If $E$ is given by the extension (5), then twisting by $O_X(-2)$ and using $h^0(I_{C,X}(k)) = 0$ for $k \leq 2$ ((ii) of Lemma 2.2), we see that $h^0(E(-2)) = 0$. The stability follows from Lemma 1.1.

Smoothness and dimension. The stability implies that $E$ is simple, that is $h^0(E^\vee \otimes E) = 1$. Hence the tangent space $T_{[E]}M_X(2; 0, 2[l])$ at $[E]$ is identified with $\text{Ext}^1(E, E) = H^1(X, E^\vee \otimes E)$, and if $H^2(X, E^\vee \otimes E) = 0$, then $M_X(2; 0, 2[l])$ is smooth at $[E]$ of local dimension $\dim_{[E]} M_X(2; 0, 2[l]) = h^1(E^\vee \otimes E)$. 

As $\text{rk} E = 2$, we have $E^\vee = E \otimes (\det E)^{-1} \simeq E(-3)$. By Serre duality, $h^3(E^\vee \otimes E) = h^0(E^\vee \otimes E(-1)) = 0$. By (4), $h^0(E(-3)) = \chi(E(-3)) = 0$. Together with the ACM property for $E$ this gives $h^i(E(-3)) = 0$ for all $i \in \mathbb{Z}$. Now, from (5) tensored by $E(-3)$, we obtain the isomorphisms
\[ H^i(E^\vee \otimes E) = H^i(E \otimes E(-3)) = H^i(E \otimes I_C) \quad \forall i \in \mathbb{Z}. \] 

Further, the restriction sequence
\[ 0 \rightarrow E \otimes I_C \rightarrow E \rightarrow E|_C \rightarrow 0 \] 

yields $\chi(E \otimes I_C) = \chi(E) - \chi(E|_C) = 8 - 14 = -6$, so to finish the proof, it remains to prove the vanishing of $h^2(E \otimes I_C)$. By (3), (8.9), the vanishing of $h^2(E \otimes I_C)$ follows from the fact that the map $f$, introduced in the proof of Proposition 2.2, is dominant. As $h^2(O_X) = 0$, we have $0 = h^2(E \otimes I_C) = h^2(E \otimes I_C) = h^2(E \otimes I_C)$, and we are done. 

3. Curves of degree 14 and genus 15 in $\mathbb{P}^4$

Let $X = \{ F = 0 \}$ be a generic quartic 3-fold in $\mathbb{P}^4$, and $X = H \cap \Xi$ (so, the $\mathbb{P}^4$ is identified with $H$) a Pfaffian representation for $X$. For the sake of functoriality, we should have defined $\Xi$ as embedded in $\mathbb{P}(\wedge^2(E^\vee))$, so that the points $x \in X$ be interpreted as alternating bilinear forms of rank 6 on $E$, whilst $G = G(2, 8) \subset \mathbb{P}(\wedge^2E)$; to avoid this dichotomy we will work in coordinates, identifying $E$ with $E^\vee$. Let $E$ be the corresponding rank 2 vector bundle and $C$ the scheme of zeros of a section $s \neq 0$ of $E$. Let $H_{14,15}^X$, resp. $H_{14,15}^X$ denote the union of the components of the Hibert scheme of curves in $\mathbb{P}^4$, resp. $X$ whose
Proof. (i) The restriction sequence (7) yields generic points represent a curve $C$ as above. For generic $s$, the curve $C$ is nonsingular.

Similarly to the previous section, introduce the rank filtration on the $7 \times 7$ skew-symmetric matrices: $G' = G(2,7) \subset Z \subset \mathbb{P}^{20} = \mathbb{P}(\bigwedge^{2}(\mathbb{C}^{7}))$. According to [R], we have $\dim G' = 10$, $\deg G' = 42$, $\omega_{G'} = \mathcal{O}_{G'}(-7)$, $\dim Z = 17$, $\deg Z = 14$, $\omega_{Z} = \mathcal{O}_{Z}(-14)$. $G'$ will be identified with a subvariety of $G$ for the standard inclusion $\mathbb{C}^{7} \subset \mathbb{C}^{8}$.

**Proposition 3.1.** The following assertions hold:

(i) $\mathcal{H}^{0}(\mathcal{N}_{C/X}) = 14$, $\mathcal{H}^{1}(\mathcal{N}_{C/X}) = 0$. Hence $\mathcal{H}_{14,15}^{1}$ is smooth at $[C]$ of local dimension 14.

(ii) $\mathcal{H}^{0}(\mathcal{N}_{C/P^{4}}) = 56$, $\mathcal{H}^{1}(\mathcal{N}_{C/P^{4}}) = 0$. Hence $\mathcal{H}_{14,15}^{1}$ is smooth at $[C]$ of local dimension 56.

(iii) $C$ can be identified with a section of the rank 4 locus $Z$ of $7 \times 7$ skew-symmetric matrices by a $4$-dimensional linear subspace $L \subset \mathbb{P}^{20}$.

**Proof.** (i) The restriction sequence (7) yields $\mathcal{H}^{2}(\mathcal{E} \otimes \mathcal{I}_{C}) = \mathcal{H}^{1}(\mathcal{E}|_{C})$. We proved in Theorem 2.3 the vanishing of $\mathcal{H}^{2}(\mathcal{E} \otimes \mathcal{I}_{C})$. As $C$ is the scheme of zeros of a section of $\mathcal{E}$, we have $\mathcal{E}|_{C} \cong \mathcal{N}_{C/X}$. So, we obtain $\mathcal{H}^{1}(\mathcal{N}_{C/X}) = 0$. By Riemann–Roch, $\mathcal{H}^{0}(\mathcal{N}_{C/X}) = 14$ and we are done.

(ii) We have $\mathcal{H}^{1}(\mathcal{N}_{C/X}) = 0$. We are going to show that this implies $\mathcal{H}^{1}(\mathcal{N}_{C/P^{4}}) = 0$. First, by Serre duality, $0 = \mathcal{H}^{1}(\mathcal{N}_{C/X}) = \mathcal{H}^{0}(\mathcal{N}_{C/X}^{\vee}(2))$. From the restriction sequence

$$0 \to \mathcal{I}_{C,X}(2) \to \mathcal{O}_{X}(2) \to \mathcal{O}_{C}(2) \to 0$$

and from the fact that $\omega_{C} = \mathcal{O}_{C}(2)$, we deduce that $\mathcal{H}^{1}(\mathcal{I}_{C,X}(2)) = 0$. Now, the exact triple

$$0 \to \mathcal{I}_{C,X}^{2}(2) \to \mathcal{I}_{C,X}(2) \to \mathcal{N}_{C/X}^{\vee}(2) \to 0$$

yields $\mathcal{H}^{1}(\mathcal{I}_{C,X}^{2}(2)) = 0$. The ACM property for $C$ and

$$0 \to \mathcal{I}_{C,P^{4}}(-2) \to \mathcal{I}_{C,P^{4}}^{2}(2) \to \mathcal{I}_{C,X}^{2}(2) \to 0$$

imply $\mathcal{H}^{1}(\mathcal{I}_{C,P^{4}}^{2}(2)) = 0$. Now, the triple

$$0 \to \mathcal{I}_{C,P^{4}}^{2}(2) \to \mathcal{I}_{C,P^{4}}^{2}(2) \to \mathcal{N}_{C,P^{4}}^{\vee}(2) \to 0$$

and the Serre duality give $\mathcal{H}^{0}(\mathcal{N}_{C,P^{4}}^{\vee}(2)) = \mathcal{H}^{1}(\mathcal{N}_{C,P^{4}}) = 0$. By Riemann–Roch, $\mathcal{H}^{0}(\mathcal{N}_{C/P^{4}}) = 56$.

(iii) The sections of $\mathcal{E}$ are naturally identified with elements of $E^{\vee}$ via the embedding of $\mathcal{E}$ into the trivial rank 8 vector bundle $E_{X} = E \otimes \mathcal{O}_{X}$. Let $\text{Cl} : \Xi \setminus \Omega \to G = G(2,8)$ be the classifying map, sending each $x \in \Xi \setminus \Omega$ to the projectivized kernel of $x$, considered as a point of $G$, and $\text{Cl}_{X}$ the restriction of $\text{Cl}$ to $X$. We can choose the coordinates in $E$ in such a way that $s = x_{7}$. Hence $C = \text{Cl}_{X}^{-1}(\sigma_{11}(\mathbb{P}^{6}))$, where $\mathbb{P}^{6}$ is
the hyperplane \( \{x_7 = 0\} \) in \( \mathbb{P}^7 = \mathbb{P}(E) \), and \( \sigma_{11}(\mathbb{P}^6) = G' \subset G \) is the Schubert subvariety of all the lines contained in the hyperplane. We can also write \( C = \text{Cl}^{-1}(G') \cap H \). The closure of the 24-fold \( \text{Cl}^{-1}(G') \) in \( \Xi \) is defined by the 7 cubic Pfaffians \( \text{Pf}_{7}(x) \), \( 0 \leq r \leq 6 \).

As cubic forms, the Pfaffians \( \text{Pf}_{7}(x) \), \( 0 \leq r \leq 6 \) do not depend on the variables \( x_p \), \( 0 \leq p \leq 7 \). Therefore \( \text{Cl}^{-1}(G') \) is isomorphic to the cone \( C(Z) \) with vertex (or ridge) \( \mathbb{P}^6 \cong \langle e_{67}, \ldots, e_{67} \rangle \) and base

\[
Z = \{ z' : \text{Pf}_{67} z' = \ldots = \text{Pf}_{67} z' = 0 \} \subset \mathbb{P}(\wedge^2 < e_0, \ldots, e_6 >);
\]

here \( z' = (x_{pq})_{0 \leq p, q \leq 6} \) is the 8-th principal adjoint matrix of the matrix \( x \), i.e. \( z' \) is obtained from \( x \) by deleting its last column and row. It is well known that the vanishing of the principal minors of order \( 2n \) of a skew-symmetric \((2n+1) \times (2n+1)\) matrix is equivalent to the vanishing of all its minors of order \( 2n \), so \( Z \) is the locus of \( 7 \times 7 \) skew-symmetric matrices of rank 4. The projection \( \pi : \mathbb{P}^{27} \rightarrow \mathbb{P}^{20} \) with center \( \mathbb{P}^6 \) maps isomorphically (for generic \( H \)) the intersection \( H \cap C(Z) \) to \( L \cap Z \), where \( L = \pi(H) \). This ends the proof. \( \square \)

Let \( \mathcal{M}_g \) denote the moduli space of smooth curves of genus \( g \) and \( \mathcal{M}_g^* \) the subvariety of \( \mathcal{M}_g \) parametrizing half-canonical curves with a theta-characteristic \( D \) such that \( \dim |D| = r \).

**Corollary 3.2.** The following assertions hold:

(i) \( H_{14,15} \) is irreducible of dimension 56.

(ii) For generic \( \mathcal{L} \in \text{Lin}(\mathbb{P}^4, \mathbb{P}^{20}) \), the stabilizer of \( \mathcal{L} \) in \( \text{PGL}(7) \), acting on the right, is finite, and the natural map \( \text{Lin}(\mathbb{P}^4, \mathbb{P}^{20})/\text{PGL}(7) \rightarrow H_{14,15} \) is generically finite.

(iii) The natural map \( g : \text{PGL}(5) \backslash \text{Lin}(\mathbb{P}^4, \mathbb{P}^{20})/\text{PGL}(7) \rightarrow \mathcal{M}_{15} \) is generically finite and its image is a 32-dimensional irreducible component \( \hat{\mathcal{M}}_{15} \) of \( \mathcal{M}_{15}^* \).

**Proof.** (i) Indeed, \( H_{14,15} \) is the image of \( \text{Lin}(\mathbb{P}^4, \mathbb{P}^{20}) \).

(ii) This follows from the count of dimensions: \( \dim \text{Lin}(\mathbb{P}^4, \mathbb{P}^{20}) - \dim \text{PGL}(7) = (5 \cdot 21 - 1) - (7^2 - 1) = 56 = \dim H_{14,15} \).

(iii) According to Harris [1], if \( r \leq \frac{1}{2}(g - 1) \), then the codimension of any component of \( \mathcal{M}_g^* \) in \( \mathcal{M}_g \) is at most \( \frac{1}{2}r(r + 1) \). Applying this to our case, we see that the dimension of every component of \( \mathcal{M}_{15}^* \) is at least 32. Hence the component \( \hat{\mathcal{M}}_{15} \), containing the image of \( H_{14,15} \), is of dimension \( \geq 32 \). The dimension of \( \text{PGL}(5) \backslash \text{Lin}(\mathbb{P}^4, \mathbb{P}^{20})/\text{PGL}(7) \) is 32, so it remains to show that \( g \) is dominant over \( \hat{\mathcal{M}}_{15} \).

Take a generic \( C \) from the image of \( g \). \( C \) is a smooth ACM curve in \( \mathbb{P}^4 \). By the definition of \( \mathcal{M}_g^* \), every small (analytic or étale) deformation of \( C \) is accompanied by a deformation of the theta-characteristic \( D \).
embedding $C$ into $\mathbb{P}^4$. The ACM property being generic, any generic small deformation of $C$ is again in the image of $g$, and we are done. 

**Remark 3.3.** In (ii) of the lemma, the stabilizer $G_L$ of $L$ might act by non-trivial automorphisms of $C$. As $\text{Aut}(C)$ is finite, the subgroup $H_L$ fixing pointwise $C$, and hence $L = L(\mathbb{P}^4)$, is of finite index in $G_L$. So, the first assertion of (ii) is equivalent to saying that $H_L$ is finite. One can strengthen this assertion: the subgroup of $\text{PGL}(2n+1)$ fixing pointwise a generic linear $\mathbb{P}^2 \subset \mathbb{P}(\wedge^2 \mathbb{C}^{2n+1})$ for $n \geq 2$ is finite. This is easily reduced to the $2n$-dimensional case, stated in [B], (5.3).

**Proposition 3.4.** Let $\hat{H}_{14,15}^X \subset H_{14,15}^X$ be the locus of ACM half-canonical curves $C \subset X$ of degree 14 and arithmetic genus 15, not contained in any quadric hypersurface in $\mathbb{P}^4$, and $M_X \subset M_X(2;0,2[l])$ the open set defined in Theorem 2.3. Then the Serre construction defines a morphism $\phi : \hat{H}_{14,15}^X \to M_X$ with fiber $\mathbb{P}^7$. Moreover, $\hat{H}_{14,15}^X$ is isomorphic locally in the étale topology over $M_X$ to a projectivized rank 8 vector bundle on $M_X$.

**Proof.** It is easily seen that $\text{dim} \text{Ext}^1(I_C(3), \mathcal{O}_X) = 1$, so, given $C$, the Serre construction determines $E$ uniquely. This yields $\phi$ as a set theoretical map. An obvious relativization of the Serre construction shows that it is indeed a morphism.

Further, we have $h^0(E \otimes I_C) = 1$ by stability of $E$ and (B), so the projective space $\mathbb{P}^7 = \mathbb{P}(H^0(E))$ is injected into $H_{14,15}^X$ by sending each section $s \neq 0$ of $E$ to its scheme of zeros. Hence the fibers of $\phi$ are set-theoretically 7-dimensional projective spaces. The proof of the last assertion of the proposition is completely similar to that of Lemma 5.3 in [MT]. 

4. **Abel–Jacobi map**

We are going to remind briefly the Clemens–Griffiths technique for the calculation of the differential of the Abel–Jacobi map, following Welters [W], Sect. 2. Let $X$ be a nonsingular projective 3-fold with $h^{03} = 0$, and $X \subset W$ an embedding in a nonsingular possibly non-complete 4-fold. Let $\Phi : B \to \mathcal{J}_2(X)$ be the Abel–Jacobi map, where $B$ is the base of a certain family of curves on $X$. The differential $d\Phi_{[Z]}$ at a point $[Z] \in B$, representing a curve $Z$, factors into the composition of the infinitesimal classifying map $T_{B,b} \to H^0(Z, \mathcal{N}_{Z/X})$ and of the universal “infinitesimal Abel–Jacobi map” $\psi_Z : H^0(Z, \mathcal{N}_{Z/X}) \to H^1(X, \Omega_X^2)^\vee = T_{J_1(X),0}$. The adjoint $\psi^*_Z$ is identified by the following commutative square:
Here \( r_Z \) is the map of restriction to \( Z \), and the whole square (upon natural identifications) is the \( H^0 \to H^1 \) part of the commutative diagram of the long exact cohomology sequences associated to the following commutative diagram of sheaves:

\[
\begin{array}{ccc}
0 & \to & \Omega^2_X \\
\downarrow & & \downarrow \\
0 & \to & \Omega^3_X \otimes \mathcal{N}_X/W \\
\end{array}
\]

Specifying all this to the case when \( X \) is a generic quartic 3-fold, \( Z = C \subset X \) a generic curve from \( H^4_X, 14, 15, W = \mathbb{P}^4 \), we see that the dimensions in (8) form the array \( \begin{pmatrix} 35 & 30 \\ 28 & 14 \end{pmatrix} \), that \( R, r_C \) are surjective and that \( \text{corank } \beta_C = \text{corank } \psi_C^\vee = h^1(\mathcal{N}_{C/\mathbb{P}^4}(-1)). \) Dualizing, we obtain:

**Lemma 4.1.** For \( C, X \) as above, \( \dim \ker \psi_C = h^1(\mathcal{N}_{C/\mathbb{P}^4}(-1)), \) \( \dim \text{im } \psi_C = 14 - h^1(\mathcal{N}_{C/\mathbb{P}^4}(-1)). \)

We have \( \chi(\mathcal{N}_{C/\mathbb{P}^4}(-1)) = 14, \) hence \( h^0(\mathcal{N}_{C/\mathbb{P}^4}(-1)) = 14 + h^1(\mathcal{N}_{C/\mathbb{P}^4}(-1)). \)

**Lemma 4.2.** \( h^0(\mathcal{N}_{C/\mathbb{P}^4}(-1)) = 21. \)

**Proof.** Twisting the 4 exact triples in the proof of Proposition 3.1 by \( \mathcal{O}(1) \), one can see that the assertion is equivalent to

\[
h^2(I^2_{C, \mathbb{P}^4}(3)) = 21, \quad h^i(I^2_{C, \mathbb{P}^4}(3)) = 0 \quad \forall \ i \neq 2.
\]

The last equalities follow immediately from the resolution for \( I^2_{C, \mathbb{P}^4}(3) \), obtained from (4) of \( \text{[R]} \) by restriction to \( L = \mathbb{P}^4 \subset \mathbb{P}^6 \) and twisting by \( \mathcal{O}(3): \)

\[
0 \to 21\mathcal{O}_{\mathbb{P}^4}(-5) \to 48\mathcal{O}_{\mathbb{P}^4}(-4) \to 28\mathcal{O}_{\mathbb{P}^4}(-3) \to I^2_{C, \mathbb{P}^4}(3) \to 0.
\]

\[ \square \]

**Remark 4.3.** One can interprete the elements of \( H^0(\mathcal{N}_{C/\mathbb{P}^4}(-1)) \) as infinitesimal deformations of \( C \) preserving 14 points of some hyperplane section \( S = C \cap h \) of \( C \). It is easy to understand this value geometrically in constructing explicitly a 21-dimensional family of global,
non-infinitesimal deformations of $C$ which preserve $S$; one can show that every 1-parameter infinitesimal deformation lifts to a global one at list for generic $C, h$.

Indeed, lift $C$ to an element $A \in \text{Lin}(\mathbb{P}^4, \mathbb{P}^{20})$, $C = A^{-1}(L \cap Z)$, $L = A(\mathbb{P}^4)$. The ACM property of $C$ implies that $S$ spans $h$, so the set $U$ of global deformations $A'$ of $A$ such that $S \subset C' = A'^{-1}(A'(\mathbb{P}^4) \cap Z)$ are exactly the elements $A'$ with the property $A|_h = A'|_h$. Identify $\text{Lin}(\mathbb{P}^4, \mathbb{P}^{20})$ with the open subset of the Grassmannian $G(5, 26)$ parametrizing the graphs of the linear injective maps from $\mathbb{C}^5$ to $\mathbb{C}^{21}$. The graphs of the above elements $A'$ correspond to those 4-dimensional planes in $\mathbb{P}^{25}$ which contain a fixed $\mathbb{P}^3$, the graph of $A|_h$. Thus $U$ is identified with an open subset of $\mathbb{P}^{21} \subset G(5, 26)$.

We can assume $A, C, h$ generic, so that $A(h)$ is a generic linear $\mathbb{P}^3$ in $\mathbb{P}^{20}$. The ACM property for $C$ implies that the 14 points $S$ are in a sufficiently general position, so that the stabilizer of $S$ in $PGL(4) = \text{Aut}(h)$ is finite, and the subgroup fixing $S$ pointwise is trivial. This observation and Remark 3.3 imply that the stabilizer of $A(h)$ in $PGL(7)$ is finite. Hence the orbits of $PGL(7)$ have only finite intersections with $U$. Hence the map of $U$ to the quotient by $PGL(7)$ is quasi-finite, as well as that to $H_{14,15}$, and its differential is injective at (a generic) $A$.

We have obtained a 21-dimensional family of global deformations of $C$ preserving $S$. Now we want to show that any 1-parameter infinitesimal deformation of $C$ can be lifted to a global 1-parameter one in the image of $U$. Indeed, the injectivity of the differential allows to lift the infinitesimal deformation to $U$. An element of $U$ is a proportionality class of a $5 \times 21$ matrix, and $U$ is an open subset in a linear $\mathbb{P}^{21}$ inside the projective space of the proportionality classes of $5 \times 21$ matrices, so any infinitesimal deformation in $U$ is obviously lifted to a linear pencil.

Lemmas 4.1, 4.2 imply that the Abel–Jacobi map $\Phi$ has a 7-dimensional image in the 30-dimensional intermediate Jacobian $J^2(X)$ and 7-dimensional fibers. We can easily identify the irreducible components of the fiber. Indeed, by Proposition 3.3, each $C$ is contained in a $\mathbb{P}^7 = \mathbb{P}(H^0(\mathcal{E})) \subset H^X_{14,15}$. Any rationally connected variety is contracted by the Abel–Jacobi map, so each one of its fibers is a union of these $\mathbb{P}^7$'s. As the dimension of the fiber is 7, the $\mathbb{P}^7$'s are irreducible components of the fiber. Being fibers of $\phi$, the irreducible components do not meet each other, so they are in fact connected components. Thus we have proved the following theorem.

**Theorem 4.4.** Let $X$ be a generic quartic 3-fold. Let $\hat{H}^X_{14,15} \subset H^X_{14,15}$ be defined as in Proposition 3.4, and $\Phi : \hat{H}^X_{14,15} \rightarrow J^2(X)$ the Abel–Jacobi map. Then the dimension of any component of $\Phi(\hat{H}^X_{14,15})$ is
equal to 7 and the fibers of $\Phi$ are the unions of finitely many disjoint 7-dimensional projective spaces. The natural map $\psi : M_X \to J^2(X)$, defined by $\Phi = \psi \circ \phi$, is quasi-finite and étale on $M_X$.

We get immediately the following obvious corollary:

**Corollary 4.5.** Every component of $M_X$ has non-negative Kodaira dimension.

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