ON RESISTANCE DISTANCE OF MARKOV CHAIN AND ITS SUM RULES

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ABSTRACT. Motivated by the notion of resistance distance on graph, we define a new resistance distance between two states on a given finite ergodic Markov chain based on its fundamental matrix. We prove a few equivalent formulations and discuss its relation with other parameters of the Markov chain such as its group inverse, stationary distribution, eigenvalues or hitting time. In addition, building upon existing sum rules for the hitting time of Markov chain, we give sum rules of this new resistance distance of Markov chains that resembles the sum rules of the resistance distance on graph. This yields Markov chain counterparts of various classical formulae such as Foster’s first formula or the Kirchhoff index formulae.

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1. Introduction and main results

On a simple connected graph $G = (V, E)$, the resistance distance $\Omega_{i,j}^{G}$ between two vertices $i, j \in V$ is defined to be the voltage when a unit current enters $i$ and leaves $j$, see e.g. Tetali (1991). Equivalently, it can be defined via the notion of the generalized inverse $L^# = (L^#_{i,j})_{i,j \in V}$ of the Laplacian $L := D - A$, where $D$ is the diagonal matrix of vertex degrees, $A$ is the adjacency matrix of $G$ and $LL^#L = L$. More precisely, according to Bapat (2010); Klein (2002), we have

$$\Omega_{i,j}^{G} := L^#_{i,i} + L^#_{j,j} - L^#_{i,j} - L^#_{j,i}.$$  

(1.1)

Motivated by this definition of resistance distance on graph, we would like to define an analogous notion of resistance distance that would play a similar role between two states of a discrete-time homogeneous finite Markov chain $X = (X_n)_{n \in \mathbb{N}_0}$, where we denote $\mathbb{N}_0$ to be the set of non-negative integers. Throughout this article, we consider an ergodic (i.e. irreducible and aperiodic) Markov chain $X$ on a finite state space $\mathcal{X}$ with transition matrix $P = (P_{i,j})_{i,j \in \mathcal{X}}$ and stationary distribution $\pi = (\pi_i)_{i \in \mathcal{X}}$, which is considered to be a row vector of size $|\mathcal{X}|$. Writing $\Pi$ to be the matrix where each row is $\pi$, the fundamental matrix $F = (F_{i,j})_{i,j \in \mathcal{X}}$ associated with the Markov chain $X$, first proposed in the work of Kemeny and Snell (1976), is given by

$$F := (I - P + \Pi)^{-1},$$

where $I$ is the identity matrix of size $|\mathcal{X}| \times |\mathcal{X}|$. Note that the above inverse always exists. In view of (1.1), we now define a new resistance distance $\Omega = (\Omega_{i,j})_{i,j \in \mathcal{X}}$ of Markov chain by simply replacing $L^#$ by $F$, that is,

**Definition 1.1** (Resistance distance of Markov chain). Given an ergodic Markov chain $X$ with fundamental matrix $F = (F_{i,j})_{i,j \in \mathcal{X}}$, we define the resistance distance $\Omega_{i,j}$ between two states $i, j \in \mathcal{X}$ to be

$$\Omega_{i,j} := F_{i,i} + F_{j,j} - F_{i,j} - F_{j,i}.$$
It turns out that this definition of resistance distance admits a few equivalent formulations in terms of other important quantities and parameters of Markov chain, such as the group inverse of \(I - P\) as well as the mean hitting time of \(X\), see Proposition 1.1 below. To this end, let us proceed by briefly recalling these notions. The group inverse \(D = (D_{i,j})_{i,j \in X}\) of \(I - P\), first studied by Meyer (1975) in a Markov chain setting, is defined to be the matrix that satisfies
\[
(I - P)D(I - P) = I - P, \quad D(I - P)D = D, \quad (I - P)D = D(I - P).
\]
In this paper, as we only discuss the case where the Markov chain \(X\) is ergodic, \(D\) can be conveniently expressed as
\[
D = \sum_{n \geq 0} (P^n - \Pi),
\]
see e.g. (Meyer, 1975, Theorem 2.4). The group inverse \(D\) also appears under different names in the literature, ranging from deviation matrix Coolen-Schrijner and van Doorn (2002), ergodic potential Syski (1978) to centered resolvent Miclo (2016). We remark that the notion of group inverse is first introduced in the work of Erdélyi (1967), and group inverse is the special case of Drazin inverse when the index of the matrix is either 1 or 0. We now move on to discuss a few probabilistic parameters of interest. For \(j \in X\), we write \(\tau_j := \inf\{n \geq 0; X_n = j\}\) to be the first time that the Markov chain \(X\) hits the state \(j\), and the usual convention of \(\inf\emptyset = \infty\) applies. We also denote \(E_i\) to be the expectation under \(X_0 = i\). For example, \(E_i(\tau_j)\) is the mean hitting time of \(j\) starting from \(i\). Among various hitting time parameters as studied in Aldous and Fill (2002), we are interested in the following three: 

- **Commute time** \(t_{i,j}^c\) between \(i\) and \(j\):
  
  \[
  t_{i,j}^c := E_i(\tau_j) + E_j(\tau_i).
  \]
  Note that the commute time defines a metric on \(X\).

- **Average hitting time** \(t^{av}\):
  
  \[
  t^{av} := \sum_{i,j \in X} E_i(\tau_j)\pi_i\pi_j.
  \]
  \(t^{av}\) represents the average hitting time from \(i\) to \(j\) of \(X\) when we sample these two states \(i,j\) independently from \(\pi\). Note that the average hitting time is equal to Kemeny’s constant, that is,
  
  \[
  t^{av} = \sum_{j \in X} E_i(\tau_j)\pi_j,
  \]
  where the right hand side is the Kemeny’s constant which is independent of the starting state \(i\). We refer interested readers to Cui and Mao (2010); Kirkland (2014); Levene and Loizou (2002); Mao (2004); Pitman and Tang (2018) for further references on this parameter.

- **Forest representation of mean hitting time**: Let \(G(P)\) be the weighted direct graph on vertices \(X\) and arc weights to be the corresponding transition probabilities. The weight of a weighted direct graph is the product of its arc weights, and the weight of a set of weighted direct graphs is the sum of the weights of its members. Define \(f_{i,j}\) to be the total weight of 2-tree in-forests of \(G(P)\) that have one tree containing \(i\) and the other rooted at \(j\), where we recall an in-forest is a spanning subdigraph of \(G(P)\) all of whose weak components are converging trees (also known as in-arborescences). Let \(q_j\) to be total weight of in-trees rooted at \(j\) and \(q := \sum_{j \in X} q_j\). According to the Markov chain tree theorem Anantharam and Tsoucas (1989) and recent results in Chebotarev (2007);
Chebotarev and Deza (2018), one can express the stationary distribution and mean hitting time via these graph-theoretic parameters as, for \( i, j \in X \),

\[
\pi_j = \frac{q_j}{q}, \quad \mathbb{E}_i(\tau_j) = \frac{f_{i,j}}{q_j}. \tag{1.2}
\]

With the above notations in mind, we are now ready to present our first result that gives a few equivalent formulations of \( \Omega_{i,j} \). These formulations are particularly useful when it comes to proving various properties and sum rules of \( \Omega \).

**Proposition 1.1.** The resistance distance \( \Omega_{i,j} \) of \( X \), as defined in Definition 1.1, can be written as, for \( i, j \in X \),

1. **(Group inverse representation)**
   \[
   \Omega_{i,j} = D_{i,i} + D_{j,j} - D_{i,j} - D_{j,i}.
   \]

2. **(Mean hitting time representation)**
   \[
   \Omega_{i,j} = \pi_j \mathbb{E}_i(\tau_j) + \pi_i \mathbb{E}_j(\tau_i).
   \]

3. **(Forest representation)**
   \[
   \Omega_{i,j} = \frac{f_{i,j} + f_{j,i}}{q}.
   \]

4. **(Commute time representation for doubly stochastic \( P \))** When \( P \) is doubly stochastic, that is both the row sums and column sums of \( P \) are 1, then we have
   \[
   \Omega_{i,j} = \frac{1}{|X|} t^{c}_{i,j},
   \]
   In other words, the resistance distance is a scaled version of commute time in the doubly stochastic case.

**Remark 1.1** (Connections with existing notions of resistance distance on weighted direct graph). In this Remark, we would like to point out to readers on possible connections with existing notions of resistance distance on weighted direct graph. As \( L(P) := I - P \) can be interpreted as the Laplacian matrix of the weighted direct graph corresponding to the Markov chain \( X \) (see e.g. (Chebotarev and Agaev, 2002, Section 2.2)), existing notions of effective resistance on directed graph are thus closely related to the proposed resistance distance \( \Omega \).

In Young et al. (2016a,b), the authors propose a notion of effective resistance \( R = (R_{i,j})_{i,j \in X} \) on weighted direct graph via the reduced Laplacian. Precisely, let \( 1_N \) be the all-ones vector of length \( N := |X| \) and let \( I_N \) be the identity matrix of size \( N \). Let \( Q \in \mathbb{R}^{(N-1) \times N} \) be any matrix that satisfies
\[
Q1_N = 0, \quad QQ^T = I_{N-1}, \quad Q^TQ = I_N - \frac{1}{N} 1_N 1_N^T.
\]
Reduced Laplacian \( \overline{L(P)} \) of \( L(P) \) is then defined to be
\[
\overline{L(P)} := QL(P)Q^T.
\]
Let \( \Sigma \) be the unique solution to the Lyapunov equation
\[
\overline{L(P)}\Sigma + \Sigma \overline{L(P)}^T = I_{N-1},
\]
and \( X \) be
\[
Y := 2Q^T\Sigma Q.
\]
The authors in Young et al. (2016a,b) define the effective resistance to be
$$R_{i,j} := Y_{i,i} + Y_{j,j} - 2Y_{i,j}.$$ Note that according to (Young et al., 2016a, Section II) $L(P)$ is not unique and depends on the choice of $Q$, while $R$ is independent of the choice of $Q$. To compare $R$ with our proposed resistance distance $\Omega$, it boils down to a comparison between $Y$ and the fundamental matrix $F$ or the group inverse $D$. While both $R$ and $\Omega$ does not define a metric in general, in Proposition 1.2 below we show that $\Omega$ does define a metric when $P$ is doubly stochastic while it is unclear whether $R$ also defines a metric in this setting. On the other hand however, $\sqrt{R} = (\sqrt{R_{i,j}})$ defines a metric (see (Young et al., 2016a, Theorem 3)) while it is not clear whether $\sqrt{\Omega} = (\sqrt{\Omega_{i,j}})$ defines a metric.

In Section V of Young et al. (2016a), the authors motivate their definition $R$ by outlining a few drawbacks in defining resistance distance via Moore-Penrose generalized inverse of the directed graph’s Laplacian. Here in our proposed resistance distance $\Omega$ for Markov chain, it is defined in terms of the group inverse of $I - P$, which according to Meyer (1975) is the “correct generalized inverse to use in connection with finite Markov chains”. We leave these open questions above as future work for further comparison between $R$ and $\Omega$.

In (Albin et al., 2015, Section 4.2), the authors propose to view classical effective resistance on undirected graph via a variational formula that depends on the modulus. It is unclear to the author whether similar variational formula holds for our proposed effective resistance $\Omega$. The asymmetric nature of $P$ maybe a possible obstacle in generalizing this result to our setting.

Another related work is Boley et al. (2011). In Section 4 therein, the authors introduce a few variants of fundamental matrices and express the mean hitting time, commute time as well as the Moore-Penrose inverse of $\Pi(I-P)$ in terms of these fundamental matrices. One can easily express our proposed $\Omega$ in terms of these quantities as well by utilizing Definition 1.1 and Proposition 1.1. Finally, we mention the work Chebotarev and Agaev (2002). In Section 9 therein, the authors obtain a few interesting relationship between the fundamental matrix $F$ and the group inverse $D$. They can be applied to gain additional insights on these quantities.

We defer the proof of this Proposition to Section 2.1. We proceed to investigate whether $\Omega_{i,j}$ defines a metric on $\mathcal{X}$. Recall that in the graph setting its resistance distance $\Omega^G_{i,j}$ defines a metric as its Laplacian $L$ is symmetric. For a proof of this fact one can consult (Bapat, 2010, Section 9.1). This resembles the setting when $P$ is doubly stochastic in which $\Omega_{i,j}$ defines a metric on $\mathcal{X}$, as we shall see in the next Proposition. In general however, $\Omega$ is a semi-metric since it does not satisfy the triangle inequality.

**Proposition 1.2.** The resistance distance $\Omega_{i,j}$ of $\mathcal{X}$, as defined in Definition 1.1, satisfies, for $i, j, k \in \mathcal{X}$,

1. (non-negativity) $\Omega_{i,j} \geq 0$ and equality holds if and only if $i = j$.
2. (symmetry) $\Omega_{i,j} = \Omega_{j,i}$.
3. (triangle inequality) When $P$ is doubly stochastic, then $\Omega_{i,j} \leq \Omega_{i,k} + \Omega_{k,j}$.

In other words, $\Omega = (\Omega_{i,j})_{i,j \in \mathcal{X}}$ defines a semi-metric on $\mathcal{X}$ in general, and is a metric when $P$ is doubly stochastic.

**Remark 1.2** (Triangle inequality need not hold for reversible Markov chain). In this Remark, to demonstrate that the triangle inequality (item (3) in Proposition (1.2)) need not hold for reversible finite Markov chain, we provide a simple counterexample by looking at the three-state birth-death Markov chain. Recall that a Markov chain is reversible if and only if it satisfies the detailed
balance condition \( \pi_i P_{i,j} = \pi_j P_{j,i} \) for all \( i, j \in \mathcal{X} \). In this counterexample, suppose that the state space consists of three states with \( \mathcal{X} = \{1, 2, 3\} \), and we consider an ergodic birth-death Markov chain \( X \) on \( \mathcal{X} \) with birth probability \( P_{1,i+1} > 0 \) for \( i = 1, 2 \) and death probability \( P_{j,i-1} > 0 \) for \( j = 2, 3 \). Note that \( P_{1,3} = P_{3,1} = 0 \) as \( X \) is a birth-death chain. It is well-known that birth-death chain is reversible. Now, using Proposition 1.1 we compute

\[
\Omega_{1,3} = \pi_3 E_1(\tau_3) + \pi_1 E_3(\tau_1) = \pi_3 E_1(\tau_2) + \pi_1 E_3(\tau_2) + \pi_1 E_2(\tau_1),
\]
\[
\Omega_{1,2} + \Omega_{2,3} = \pi_2 E_1(\tau_2) + \pi_1 E_2(\tau_1) + \pi_3 E_2(\tau_3) + \pi_2 E_3(\tau_2),
\]
\[
\Omega_{1,3} - \Omega_{1,2} - \Omega_{2,3} = (\pi_3 - \pi_2) E_1(\tau_2) + (\pi_1 - \pi_2) E_3(\tau_2),
\]

where we utilize the birth-death property in the second equality of (1.3), and (1.5) follows from (1.3) and (1.4). For three-state birth-death chain with \( \pi_1 > \pi_2 \) and \( \pi_3 > \pi_2 \), by (1.5) we then have \( \Omega_{1,3} > \Omega_{1,2} + \Omega_{2,3} \). A concrete numerical example is the following birth-death chain

\[
P = \begin{pmatrix}
0.9 & 0.1 & 0 \\
0.5 & 0 & 0.5 \\
0 & 0.1 & 0.9
\end{pmatrix}.
\]

Clearly, \( P \) is ergodic with \( \pi_1 = \pi_3 = 5/11 > 1/11 = \pi_2 \). Moreover, we check that \( \Omega_{1,3} = 20 > 140/11 = \Omega_{1,2} + \Omega_{2,3} \).

The proof of the above Proposition can be found in Section 2.2. In the following, we present a generalized sum rule of \( \Omega \) as one of our major results of this article. The crux of the proof relies on the sum rule of hitting time of Markov chains Palacios and Renom (2010) and is deferred to Section 2.3.

**Lemma 1.1.** Given an ergodic Markov chain \( X \) with fundamental matrix \( F \) on \( \mathcal{X} \), for any square matrices \( M, K \) on \( \mathcal{X} \) such that

1. \( K \mathbb{1}_{|\mathcal{X}|} = \mathbb{1}_{|\mathcal{X}|} \), where \( \mathbb{1}_{|\mathcal{X}|} \) is the all-ones vector of length \( |\mathcal{X}| \),
2. \( M(K - I) \) is symmetric,

then we have

\[
\sum_{i,j} (M(K - I))_{i,j} \Omega_{i,j} = 2 \text{Tr}(M(I - K)F),
\]

where \( \text{Tr}(\cdot) \) is the trace operation.

At first glance, this theorem may seem to be restrictive due to the assumptions on the row sum of \( K \) as well as the symmetry of \( M(K - I) \). Nonetheless, in many cases these assumptions are fulfilled and we apply the above Lemma 1.1 which yields the following Corollary on the Markov chain counterpart of Kirchhoff indices:

**Corollary 1.1.** For a given ergodic Markov chain \( X \) with non-unit eigenvalues of \( P \) given by \( (\lambda_i)_{i=2}^{|\mathcal{X}|} \), we have

1. (Kirchhoff index)

\[
\sum_{i,j} \Omega_{i,j} = 2|\mathcal{X}| t^{av} = 2|\mathcal{X}| \sum_{i=2}^{|\mathcal{X}|} \frac{1}{1 - \lambda_i}.
\]
(2) (Multiplicative Kirchhoff index) Writing $M$ to be the diagonal matrix with $M_{i,i} = \pi_i$ for all $i \in \mathcal{X}$,

$$\sum_{i,j} \pi_i \pi_j \Omega_{i,j} = 2 \text{Tr}(MF - M\Pi).$$

(3) (Additive Kirchhoff index)

$$2t^{av} \leq \sum_{i,j} (\pi_i + \pi_j) \Omega_{i,j} \leq 2t^{av}(|\mathcal{X}| + 1).$$

The above formulae of Markov chain Kirchhoff indices share a striking similarity with their counterparts on graph. For instance, writing $$(\lambda_i L^G)_{i=2}^{|V|}$$ to be the non-zero eigenvalues of the Laplacian $L$, the graph counterpart of Kirchhoff index (see e.g. (Palacios and Renom, 2010, Corollary 2)) can be calculated as

$$\sum_{i,j} \Omega_{i,j}^G = 2|V| \sum_{i=2}^{|V|} \frac{1}{\lambda_i^L},$$

which resembles the corresponding formula in Corollary 1.1 item (1). For recent progress in the study of Kirchhoff indices on graph, we refer interested readers to Palacios (2016). As our second application of the main result of Lemma 1.1, we establish a Markov chain counterpart of Foster’s first formula of electrical network under a doubly stochastic setting:

**Corollary 1.2.** Suppose that $X$ is a reversible Markov chain. By writing $M$ to be the diagonal matrix with $M_{i,i} = \pi_i$ for all $i \in \mathcal{X}$, we then have, for $m \in \mathbb{N}$,

$$\sum_{i,j} \pi_j P_{j,i}^m \Omega_{i,j} = 2 \text{Tr} \left( M \left( \sum_{j=0}^{m-1} (P^j - \Pi) \right) \right).$$

In particular, when $P$ is doubly stochastic, the above gives a Markov chain analogue of the Foster’s first formula:

$$\sum_{i,j} P_{i,j} \Omega_{i,j} = 2(|\mathcal{X}| - 1).$$

The above result can be compared to its classical counterpart result in graph theory (see e.g. Palacios and Renom (2010)), which gives

$$\sum_{(i,j) \in E} \Omega_{i,j}^G = |V| - 1.$$

In this vein, we mention the work of Tetali (1994) who also gives related results in the direction of Foster’s network theorem and reversible Markov chains.

The rest of the paper is devoted to the proof of the main results. We prove Proposition 1.1 in Section 2.1, Proposition 1.2 in Section 2.2, Lemma 1.1 in Section 2.3, Corollary 1.1 in Section 2.4 and finally Corollary 1.2 in Section 2.5.

### 2. Proofs of the main results

#### 2.1. Proof of Proposition 1.1

We first prove item (1). It is well-known that $F, \Pi, D$ are connected by the formula $F = \Pi + D$, see e.g. (Meyer, 1975, Theorem 3.1). Desired result follows since

$$\Omega_{i,j} = F_{i,i} + F_{j,j} - F_{i,j} - F_{j,i} = \pi_i + D_{i,i} + \pi_j + D_{j,j} - \pi_j - D_{i,j} - \pi_i - D_{j,i} = D_{i,i} + D_{j,j} - D_{i,j} - D_{j,i}.$$
Next, we prove item (2). Using the relationship $\Omega_i \tau_j = \frac{E_i \tau_j - F_j \tau_i}{\pi_j}$ gives
\[
\Omega_{i,j} = F_{j,i} - F_{i,j} + F_{i,i} = \pi_j E_i \tau_j + \pi_i E_j \tau_i.
\]
For item (3), we only prove the case of $i \neq j$ as the case of $i = j$ is trivial. Using item (2) and (1.2), we see that
\[
\Omega_{i,j} = \frac{q_j f_{i,j}}{q_j} + \frac{q_i f_{j,i}}{q_i} = \frac{f_{i,j} + f_{j,i}}{q}.
\]
Finally, we prove item (4). In the doubly stochastic case, $\pi_i = 1/|\mathcal{X}|$ for all $i$, and so by item (2) we write
\[
\Omega_{i,j} = \frac{1}{|\mathcal{X}|} E_i \tau_j + \frac{1}{|\mathcal{X}|} E_j \tau_i = \frac{1}{|\mathcal{X}|} t^c_{i,j}.
\]

2.2. Proof of Proposition 1.2. We first prove item (1). According to Proposition 1.1 item (2),
\[
\Omega_{i,j} = \pi_j E_i \tau_j + \pi_i E_j \tau_i \geq 0.
\]
Equality holds if and only if $E_i \tau_j = E_j \tau_i = 0$ if and only if $i = j$. Next, we prove item (2). Using Proposition 1.1 item (2) again, we have
\[
\Omega_{i,j} = \pi_j E_i \tau_j + \pi_i E_j \tau_i = \pi_i E_j \tau_i + \pi_j E_i \tau_i = \Omega_{j,i}.
\]
Finally, we prove item (3) under doubly stochastic $P$. By Proposition 1.1 item (4), we see that
\[
\Omega_{i,j} = \frac{1}{|\mathcal{X}|} t^c_{i,j} \leq \frac{1}{|\mathcal{X}|} (t^c_{i,k} + t^c_{k,j}) = \Omega_{i,k} + \Omega_{k,j},
\]
where we use the triangle inequality for commute time, see e.g. (Aldous and Fill, 2002, Chapter 2, Lemma 9).

2.3. Proof of Lemma 1.1. Using Proposition 1.1 item (2), we write
\[
\sum_{i,j} (M(K-I))_{i,j} \Omega_{i,j} = \sum_{i,j} (M(K-I))_{i,j} \pi_j E_i \tau_j + \sum_{j,i} \pi_i E_j \tau_i = 2\text{Tr}(M(I-K)F),
\]
where we use the symmetry of $M(K-I)$ in the first equality, and the second equality follows from the sum rule of the hitting time of Markov chains (Palacios and Renom, 2010, Proposition 2).

2.4. Proof of Corollary 1.1. We first prove item (1). It follows from the random target lemma (see e.g. (Levin et al., 2009, Lemma 10.1)) and Proposition 1.1 item (2) that
\[
\sum_{i,j} \Omega_{i,j} = \sum_i \sum_j \pi_j E_i \tau_j + \sum_{j,i} \pi_i E_j \tau_i = \sum_i t^\text{av} + \sum_j t^\text{av} = 2|\mathcal{X}| t^\text{av} = 2|\mathcal{X}| \sum_{i=2}^{|\mathcal{X}|} \frac{1}{1 - \lambda_i},
\]
where the last equality follows from eigentime identity of ergodic Markov chain Cui and Mao (2010). We proceed to prove item (2). In Lemma 1.1, by taking $M$ to be the diagonal matrix of the row vector $\pi$ and $K = \Pi$, we readily check that $K 1_{|\mathcal{X}|} = 1_{|\mathcal{X}|}$ and $(M(K-I))_{i,j} = \pi_i \pi_j = (M(K-I))_{j,i}$, and so Lemma 1.1 gives
\[
\sum_{i,j} \pi_i \pi_j \Omega_{i,j} = 2\text{Tr}(MF - MKF) = 2\text{Tr}(MF - M\Pi),
\]
where we use $\Pi F = \Pi$ in the last equality. Finally, we prove item (3). For the lower bound, applying Proposition 1.1 item (2) again we see that
\[
\sum_{i,j} (\pi_i + \pi_j) \Omega_{i,j} = \sum_{i,j} (\pi_i + \pi_j)(\pi_j E_i(\tau_j) + \pi_i E_j(\tau_i)) \geq \sum_{i,j} \pi_i \pi_j E_i(\tau_j) + \pi_j \pi_i E_j(\tau_i) = 2t^{av}.
\]
On the other hand, for the upper bound, we have
\[
\sum_{i,j} (\pi_i + \pi_j) \Omega_{i,j} = \sum_{i,j} \pi_i^2 E_j(\tau_i) + \pi_j^2 E_i(\tau_j) + \pi_i \pi_j E_i(\tau_j) + \pi_j \pi_i E_j(\tau_i) \leq \left( \sum_{i,j} \pi_i E_j(\tau_i) + \pi_j E_i(\tau_j) \right) + 2t^{av} = 2|\mathcal{X}|t^{av} + 2t^{av},
\]
where the last equality follows again from the random target lemma.

2.5. Proof of Corollary 1.2. We first consider
\[
(2.1) \quad P^m F = P^m (\Pi + D) = \Pi + \sum_{n=0}^{\infty} (P^{m+n} - \Pi) = \Pi + D - \sum_{j=0}^{m-1} (P^j - \Pi) = F - \sum_{j=0}^{m-1} (P^j - \Pi),
\]
where we use $F = \Pi + D$ in the first and last equality. Writing $M$ to be the diagonal matrix of $\pi$ and $K = P^m$, we check that $K 1_{|\mathcal{X}|} = 1_{|\mathcal{X}|}$ and for $i \neq j$ we use the reversibility assumption on $P$ to note that $(M(K-I))_{i,j} = \pi_i P^m_{i,j} = \pi_j P^m_{j,i} = (M(K-I))_{j,i}$. By Lemma 1.1, we have
\[
\sum_{i,j} \pi_j P^m_{j,i} \Omega_{i,j} = 2\text{Tr}(MF - MP^m F) = 2\text{Tr} \left( M \left( \sum_{j=0}^{m-1} (P^j - \Pi) \right) \right),
\]
where the last equality follows from (2.1). In particular, when $P$ is doubly stochastic (and reversible by assumption), its stationary distribution is given by the discrete uniform. As a result, we take $m = 1$ and $\pi_i = 1/|\mathcal{X}|$ to see
\[
\frac{1}{|\mathcal{X}|} \sum_{i,j} P_{i,j} \Omega_{i,j} = 2\text{Tr}(M - MP) = 2 \left( 1 - \sum_i \pi_i^2 \right) = 2 \left( 1 - \frac{1}{|\mathcal{X}|} \right),
\]
from which the desired result follows.

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