Semiclassical Quantum Gravity:
Statistics of Combinatorial Riemannian Geometries

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(Dated: 1 September 2004)

Abstract

This paper is a contribution to the development of a framework, to be used in the context of semiclassical canonical quantum gravity, in which to frame questions about the correspondence between discrete spacetime structures at “quantum scales” and continuum, classical geometries at large scales. Such a correspondence can be meaningfully established when one has a “semiclassical” state in the underlying quantum gravity theory, and the uncertainties in the correspondence arise both from quantum fluctuations in this state and from the kinematical procedure of matching a smooth geometry to a discrete one. We focus on the latter type of uncertainty, and suggest the use of statistical geometry as a way to quantify it. With a cell complex as an example of discrete structure, we discuss how to construct quantities that define a smooth geometry, and how to estimate the associated uncertainties. We also comment briefly on how to combine our results with uncertainties in the underlying quantum state, and on their use when considering phenomenological aspects of quantum gravity.

PACS numbers: 04.60.Pp, 02.40.Sf, 05.90.+m.

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I. INTRODUCTION

Ever since quantum mechanics and general relativity became two of the pillars of contemporary physics, it has been clear that a new, more general theory is needed that contains both of them in some appropriate limit. The task of finding such a quantum theory of gravity has indeed been an excruciating one and has not yet been completed, in part because of the numerous conceptual challenges that such a theory faces [1]. One of them is the question that will motivate our considerations: What is the fundamental structure of spacetime? That is, can we expect the continuum picture of a (smooth) manifold to be valid at arbitrarily small scales? Do we need to modify it into a “foamy” picture with a manifold of fluctuating topology [2], and possibly dimensionality? Or should it be entirely replaced by a different, quantum picture that is discrete, polymeric, fuzzy? Is it even appropriate to pose the basic questions in such a geometric language? It is by now generally believed that the smooth manifold picture is inadequate, but there is a wide variety of approaches to the way in which the quantum realm manifests itself, which in some cases depend on the prejudices of individual quantum gravity practitioners, for example regarding the amount of classical structures and non-dynamical elements that remain in the final formulation. In the present work we shall not pretend to be free from such prejudices; however, we will attempt to attack our main problem from a general perspective, and present a framework that could be adapted to different approaches and formulations of the dynamics.

Within the search for a consistent theory of quantum gravity, there has been over the past several years an increase in the amount of work devoted to the semiclassical sector of the theory and its physical predictions. These subjects involve several issues, such as identifying those states which have a semiclassical interpretation, and predicting phenomenological effects due to these states that can be seen as quantum gravity corrections to the dynamics of quantum fields. As a result of their different uses of the classical structures, each approach faces a different challenge when it comes to the question of the (semi-)classical limit. As loop quantum gravity [3] is the main inspiration for our framework, let us look at the semiclassical limit there.

A key aspect, underlying the very possibility of phenomenological predictions, is that of establishing a correspondence between the structure in terms of which quantum gravity is formulated, and the smooth geometry in terms of which the usual field theories are formulated and observations interpreted. One expects the two descriptions to be very close at large scales, and to start departing significantly at small length scales, since all of the field theories we use to describe the behavior of matter, and even of gravitation, are based on manifolds that become “featureless” at small scales, and are actually locally flat in an infinitesimal neighborhood of each point. We will therefore take the point of view that the manifold $M$ is only a convenient tool which gives us an effective “low-energy” picture, within a certain degree of accuracy at each scale. With this observation at the center of our discussion, we will describe a framework that will quantify this accuracy, and identify scales at which one can cross over between the quantum picture of spacetime, assumed to be discrete, and a continuum, classical geometry.

The issue of recovering a continuum theory as an appropriate limit of a discrete one is of course not new, and has been studied in various contexts for a long time. In quantum gravity the best-known approaches in which the issue arises are the ones based on piecewise linear manifolds, such as the various versions of Regge calculus [4] or dynamical triangulations [5],
graph-based approaches such as loop quantum gravity, the more recently developed spin foam models, or causal sets, but the list includes many others, both covariant and based on space+time splittings. In the causal set approach and in dynamical triangulations, the only variable used is combinatorial, while in other approaches one uses a combinatorial structure which is “dressed” with additional variables, such as edge lengths of a manifold triangulation in Regge calculus, or holonomies of connections along graph edges in loop quantum gravity. Although there may or may not be a “minimal length” in those frameworks, one may think of their discreteness as giving rise to a characteristic length scale $\ell$ (whose value in terms of fundamental constants is to be determined within the theory) above which the discrete structure $\Omega$ can be considered as kinematically well approximated by a smooth manifold.

When one faces the issue of recovering the continuum theory, the first question that arises is whether the discrete dynamics converges to that of the continuum as $\ell \to 0$. To address this question, one usually considers a fixed smooth manifold with metric $(M, g_{ab})$ (to be thought of as either the spatial or the spacetime geometry), and embeds in it an unspecified, but increasingly finer sequence of discrete structures $\Omega(\ell)$ (for example, triangulations of $M$ with edge lengths induced by $g_{ab}$ of order $\ell$), and shows that the value of the discrete version of some quantity $Q$ that governs the dynamics, such as the action in a covariant approach, approaches the value of the continuum version, calculated for $(M, g_{ab})$, as $Q(\Omega(\ell)) = Q(M, g_{ab}) + O(\ell^2)$. A number of results of this type are known. For the most part, the emphasis has been on features of the discrete dynamics itself and on showing that the discrete variables approach the continuum ones fast enough.

Here we want to explore the kinematical aspects of this correspondence in more detail. As already mentioned, we feel that, given the importance of potentially observable corrections to continuum theories, it is important to have a way to quantify the extent to which a discrete geometry corresponds to a smooth one, and the amount of uncertainty in the correspondence. In other words, if the discrete theory is fundamental and not an approximation, the limit $\ell \to 0$ is not to be taken, and we need to know exactly what the $O(\ell^2)$ terms are. This is the goal of the framework that we propose. In this work, the first in a series of papers, we intend to motivate the issues to be addressed and to provide the first steps. We start by considering only those aspects that are related to an underlying combinatorial structure; aspects related to additional variables to be assigned to the underlying discrete structure will be treated separately in a forthcoming publication. All considerations will be independent of the dynamics of the theory, and in fact we don’t need any details of the quantum theory, except for occasional references to an underlying quantum state $\Psi$, assumed to be semiclassical. This paper can be separated in two parts. In the first one we develop a way of assigning a smooth classical geometry to a given (random) graph. The second part places this procedure in the more general context of building a macroscopic geometry $(M, g)$, starting from a semiclassical state $\Psi_{sc}$, or a graph $\Omega$ together with the expectation values of observables $\langle O \rangle$, and raises the related issues of coarse-graining and combining statistical and quantum uncertainties.

Let us now comment on the way in which our work differs from previous results on the continuum limit of a discrete theory. The first key feature of our approach is related to the fact that, rather than reconstructing a given smooth geometry, we are interested in constructing one using a discrete structure $\Omega$, which may not even be embeddable in a smooth geometry. For example, in the picture of quantum geometry arising from loop quantum
gravity, the geometry at Planck length appears to be distributional, with support on the edges of a graph. In the current formulation of this approach, those graphs are embedded in a given manifold $\mathcal{M}$, but the expectation is that it should be possible to formulate it in terms of abstract graphs. The theory should then include criteria for recognizing graphs which look like manifolds at large scales, and specify how to determine the emergent geometry and evaluate the uncertainties involved in the construction. Several scales, associated with qualitatively different descriptions of the geometry, will arise in this process, and it should ultimately be possible to establish a correspondence between those provided by continuum theories, such as the Planck length $L_P$ at which classical geometry is expected to break down, and the ones provided by the discrete theory, which in the present paper are purely combinatorial and dimensionless, such as the amount of coarse-graining necessary for $\Omega$ to be embeddable.

There is a caveat, however: a random complex carries information from the metric of the underlying manifold. As will be shown in detail, curvature quantities can be related to combinatorial properties of the complex and vice versa. But this raises a potentially worrisome point: would it be consistent to use a complex that determines, in the above sense, a classical metric $g_1$, say, but dress it with semiclassical states that are peaked around a completely different, non gauge-equivalent classical metric $g_2$? To circumvent this potential problem, it would be convenient to have criteria to decide on the right random complex to use for the semiclassical situation at hand. This will imply, at this level, to use the quantum geometry $g_2$ implied by a quantum state $\Psi$ and compare it with the geometry $g_1$ consistent with the discrete structure $\Omega$.

As tools for constructing a geometry $(\mathcal{M}, g_{ab})$, we will identify examples of quantities $Q_X$ associated with appropriate regions or submanifolds $X$ of $\mathcal{M}$; the second key feature of our approach concerns the way in which uncertainties $\Delta Q_X$ are calculated using statistical techniques. Given one discrete structure $\Omega$, the $\Delta Q_X$ represent the uncertainty in the estimate of the effective geometry $(\mathcal{M}, g_{ab})$ that $\Omega$ could be considered a discretization of. Such statistical fluctuations can be, and have been, calculated in some cases, using both analytical and numerical methods. Here we will use as far as possible analytical techniques, in the spirit of the random lattice approach to gauge theories in Minkowski space pioneered by T D Lee and collaborators [9]. Most of the explicit calculations will be done in two dimensions; results in three or more dimensions in general will have to be obtained numerically.

More specifically, in Sec II we define the setting of our work, introduce and motivate the choice we make for $\Omega$, that of a cell complex, and comment on the possibility of obtaining cell complexes starting from graphs. Sec III contains the main results in this paper; after an introduction, clarifying what we mean by constructing an approximate geometry from $\Omega$, we show explicitly how to carry out the construction in 2D. In Sec IV we return to more basic questions about our discrete structure: we discuss the role of different length scales in the discrete-continuum transition, and the more general setting of structures that are not embeddable at a certain length scale. For example, discrete structures associated with candidate semiclassical states for quantum gravity will probably not be generic graphs, and it seems reasonable to start exploring the obstructions to graph embeddability by looking at cases that are not too “severe”, where the discrete structure is, in an appropriate sense, “almost embeddable”. To that end, we discuss the notion of coarse-graining for a cell complex, intended to produce in those cases an embeddable, smoothed-out version of the discrete structure.
In Sec V we discuss the consequences of the quantum fluctuations \( \langle \Delta Q \rangle \), and possible uses of our results. As stated earlier, from the point of view of the continuum, those fluctuations contribute to the total uncertainty in the geometry \((\mathcal{M}, g_{ab})\); thus, on the one hand they will ultimately allow us to quantify the goodness of \( \Psi \) as a semiclassical state, and on the other hand they will play an important role in the relationship between \( \Psi \) and continuum-based phenomenology. Note that these modifications are of a different nature than the corrections to field dispersion relations that arise purely from discretization effects in Ref [10]. Finally, we return to the motivation for this work, and discuss possible applications in loop quantum gravity phenomenology and other directions for future work.

Regarding notation, we will follow the following convention. Statistical averages and quantum expectation values will be indicated by angle brackets, as in \( \langle Q \rangle \), and means with respect to probability distributions by overbars, as in \( \overline{Q} \); as for uncertainties, \( (\Delta Q)^2 \) will denote a quantum fluctuation, while \( (\Delta Q)^2 \) (the subscript “c” being understood) or \( \sigma^2 \) will denote the statistical uncertainty or variance of a classical probability distribution.

II. MATHEMATICAL SETTING

Our framework can be seen as a “bridge” between a discrete, pre-geometrical description of spacetime, motivated by quantum gravity ideas, and the classical, continuum-based geometrical description, for situations in which the quantum theory provides us with a “semiclassical state”. In this section, we define the notions needed to translate this statement and the conceptual points discussed in the introduction into a specific program.

A. Cell Complexes and Tilings

The most basic variable in this paper will be a cell complex; given their central role in what follows, we start by recalling a few useful definitions and facts about cell complexes and their relationship with manifolds. In topology, a \( k \)-dimensional (open) cell is a space homeomorphic to the interior of a \( k \)-ball. A cell complex is a set of nonempty, pairwise disjoint cells, such that (a) The closure of each cell is homeomorphic to a ball and its boundary to a sphere in some dimension, and (b) The boundary of each cell is a union of cells; in our case, this will always be a finite union. (0-dimensional balls are pairs of vertices.)

Given a differentiable manifold \( \mathcal{M} \), a cell decomposition of \( \mathcal{M} \) is a cell complex homeomorphically embedded in it. Our assumptions then imply that the cell decomposition is locally finite, in the sense that every compact subset of \( \mathcal{M} \) intersects only a finite number of cells. For example, a finite 3-dimensional cell complex \( \Omega \) (one whose maximal cell dimensionality is 3) consists of a set of \( N_0 \) vertices \( v_I \), \( N_1 \) edges \( e_I \), \( N_2 \) 2-cells \( \omega_I \), and \( N_3 \) 3-cells \( C_I \); if \( \Omega \) is a cell decomposition of \( \mathcal{M} \), then the 3-cells together with their boundaries, \( (\cup_I C_I) \cup (\cup_I \omega_I) \cup (\cup_I e_I) \cup (\cup_I v_I) \approx \mathcal{M} \). We will say that a cell complex \( \Omega \) is embeddable if there is a differentiable manifold \( \mathcal{M} \) of which \( \Omega \) is a cell decomposition.

We will also mention the more general concept of tiling of \( \mathcal{M} \). This term often denotes a collection of pairwise disjoint open subsets \( \omega_I \) of \( \mathcal{M} \) whose closures cover \( \mathcal{M} \); here we will consider a tiling to include the union of the boundaries of the \( \omega_I \), partitioned into submanifolds of various dimensionalities. A simple example will illustrate the concept. Given a smooth loop \( \alpha \) in \( S^2 \), such that \( S^2 \setminus \alpha \) consists of two open “half-spheres” \( \omega_1 \) and \( \omega_2 \), the set
\{\omega_1, \omega_2, \alpha\} is a tiling of \(S^2\), but not a cell decomposition. However, if we pick a point \(v \in \alpha\) and call \(e\) the edge \(\alpha \setminus v\), then \(C_1 := \{\omega_1, \omega_2, e, v\}\) is a cell decomposition of \(S^2\); alternatively, if we pick two points on \(\alpha\) and join them with an extra edge that does not meet \(\alpha\) elsewhere, we divide \(S^2\) into three open wedges which, together with the elements on their boundaries, make up another cell decomposition \(C_2\). In a cell complex \(\Omega\) homeomorphic to a \(D\)-manifold \(M\) (but not in any tiling), the \(N_k(\Omega)\) satisfy

\[ \sum_{k=0}^{D} (-1)^k N_k(\Omega) = (-1)^D \chi(\Omega), \]

where \(\chi(\Omega)\) is the Euler number of the complex \(\Omega\), or of the manifold \(M\).

However, cell complexes need not be embeddable. They could, for example, have “regions” with different dimensionalities; in an embeddable cell complex every cell is, or is on the boundary of, one of maximal dimensionality. For a different type of example, consider the above cell complexes \(C_1\) and \(C_2\), and remove their 2-dimensional cells; in the first case the remaining 1-dimensional skeleton is homeomorphic to \(S^1\), while in the second case one is left with a non-embeddable complex.

A useful operation on cell decompositions is duality, which produces a new \(D\)-dimensional cell complex \(\Omega^*\) from any given \(\Omega\) of the same dimensionality. One associates with each \(k\)-dimensional cell \(\omega\) in \(\Omega\) (a \(D\)-dimensional cell complex must have cells of all dimensionalities \(0 \leq k \leq D\)) a \((D-k)\)-dimensional dual cell \(\omega^*\) in \(\Omega^*\), whose boundary consists of the duals of all cells which have \(\omega\) on their boundary. If \(\Omega\) is a cell decomposition of a manifold \(M\), then \(\Omega^*\) is also homeomorphic to \(M\); however, since the duality \(\Omega \leftrightarrow \Omega^*\) is an operation between abstract cell complexes, in general there is no natural embedding of \(\Omega^*\) in \(M\) (the mapping \(f : \Omega \to M\) does not induce a mapping \(f^* : \Omega^* \to M\), unless \(M\) is endowed with more structure). Duality is defined for general tilings, but their duals may not be homeomorphic to the original \(M\), while non-embeddable cell complexes may not have well-defined duals.

**B. Triangulations and the Voronoi Procedure**

Of all types of cell decompositions of manifolds, the most useful ones for us are triangulations, in which the cell complexes are simplicial, and their dual complexes. In triangulations, of course, all 2-cells (triangles) have 3 edges and 3 vertices, all 3-cells (tetrahedra) have 4 faces, 6 edges, and 4 vertices; in general, all \(k\)-simplices have \(k+1\) faces on their boundary, etc. Their dual cell decompositions therefore satisfy incidence properties which state that each \((D-k)\)-cell is on the boundary of (is shared by) \(k+1\) cells of dimensionality one unit higher, etc, and can be concisely written as follows: For each \(l\)-dimensional cell \(\omega \in \Omega\), the number of \(k\)-cells that have \(\omega\) in their closure (with \(0 \leq l \leq k \leq D\)) is

\[ N_{k|l}(\omega) = \binom{D+1-l}{k-l}. \]

In particular, each vertex has \(N_{1|0} = D + 1\) edges. Thus, in two dimensions all dual vertices are trivalent, while in three dimensions they are shared by four edges. This property already makes such complexes useful, since for example quantum geometry results in loop quantum gravity show that 4-valent vertices of graphs in 3 dimensions are the fundamental units of volume \(\Pi\). Also, if each edge terminates at two vertices, we find a useful relation between
the total numbers of edges and vertices (if finite),
\[ N_1(\Omega) = \frac{1}{2} (D + 1) N_0(\Omega) . \] (2.3)

However, the main reason why these two complexes are useful is more general; it lies in the fact that they can be obtained in a manifold using just a set of points as input, and in the way they encode geometrical information on \((\mathcal{M}, q_{ab})\) when the set of points they are based on is chosen at random.

Given any locally finite set of points \(p_I\) in a Riemannian manifold \((\mathcal{M}, q_{ab})\), one can obtain from it a tiling of \(\mathcal{M}\); if the points are at generic locations and sufficiently dense (with respect to both local length scales, determined by the metric, or global ones, determined by the metric and topology), the result is actually a cell decomposition, called the Voronoi complex, and its dual is called the Delaunay triangulation. We start by introducing the procedure in general, without any additional assumptions on the set of points, and then consider in the next subsection the case in which the \(\{p_I\}\) are randomly sprinkled points. (Capital latin indices \(I, J, \ldots\) will be used to denote points or elements of various dimensionalities in a complex; the type of object they refer to should be clear from the context.)

For each embedded point \(p_I\), we can define an open region \(\omega_I \subset \mathcal{M}\) as the set of all manifold points which are closer to \(p_I\), with respect to \(q_{ab}\) than to any other \(p_J\); clearly the union of the closures of such regions is \(\mathcal{M}\), so the \(\{\omega_I\}\) define a tiling. The boundaries of the \(\omega_I\) are made of manifold points that are equidistant from more than one of the \(p_I\)’s; those equidistant from \(p_I\) and \(p_J\) and closer to them than to any other \(p_K\) are the codimension-1 common boundary of the \(D\)-dimensional regions \(\omega_I\) and \(\omega_J\) around \(p_I\) and \(p_J\), respectively. Common portions of boundaries among more than two cells, if present, define lower-dimensional portions of the \(\partial \omega_I\)’s. We will always call the resulting complex a Voronoi complex, even when not all the sets just described are cells; if the \(p_I\) are close enough to each other compared to all length scales in the manifold, associated with the metric or the topology, we actually obtain a cell complex \(\Omega\) homeomorphic to \(\mathcal{M}\).

Generically, vertices of a Voronoi complex are equidistant from \(D + 1\) points, since a higher number of points at the same distance can only be obtained in degenerate situations; then the \(D + 1\) ways of picking \(D\) of those points define \(D + 1\) Voronoi edges incident on that point. In fact, when the \(p_I\) are at generic locations, all of the relationships are satisfied, regardless of whether we have a cell complex; because in our applications the points will be randomly sprinkled in \(\mathcal{M}\), we will not discuss special arrangements and will always assume that those relations hold. Given any Voronoi vertex \(v_I\), the \(D + 1\) points it is close to are on the surface of a \(D\)-sphere around \(v_I\) and define a simplex \(\omega_I^*\), dual to \(v_I\). If \(\Omega\) is a cell complex, there are enough Voronoi vertices for the dual complex \(\Omega^*\) defined by all those simplices to be a Delaunay triangulation of the manifold, which has the \(p_I\) themselves as vertices. Intuitively, \(k + 1\) sprinkled points that are “clustered closely enough” define a \(k\)-simplex in \(\Omega^*\) that lies in the \(k\)-plane through those points. Thus, the concept of Delaunay triangulation is less general than that of “Voronoi tiling”.

When does the Voronoi procedure not produce a cell complex? A simple example is that of a 2-sphere with two points \(p_1\) and \(p_2\) chosen on it, in which case the procedure gives the tiling we called \(\{\omega_1, \omega_2, \alpha\}\) in Sec [12A] and its dual consists of \(p_1\), \(p_2\) and an edge between them, which is not homeomorphic to \(S^2\). In this example, both the non-trivial topology and the high curvature (in terms of the point density) contribute to the outcome. However, one can easily modify it into other examples in which only one of the factors is present.
FIG. 1: Example of 2D Voronoi (thin lines) and Delaunay (thick lines) complexes; the points they are based on are the Delaunay vertices. Notice that in 2D edges are dual to edges, whereas vertices and 2-cells are duals of each other, and that dual edges are orthogonal to each other, although they do not necessarily meet.

(a flat cylinder, or a plane with a small region blown up into a long tube or a balloon, respectively), and it produces a similar effect. All such Voronoi tilings satisfy the incidence relations (2.2), and those same relations, imposed on an abstract complex, guarantee that it is homeomorphic to some manifold. For the time being, however, we would like to work with ordinary cell complexes. This means that for us an embeddable complex will be one such that the relations (2.2) are satisfied and in which every cell has on its boundary (the appropriate number of) cells of all lower dimensionalities.

By construction, the two types of complexes just defined are embeddable. However, since in this paper our intention is to construct geometries from discrete structures, we are interested in characterizing those structures which can arise from the Voronoi procedure in some Riemannian manifold. This condition excludes the “lower-dimensional parts” in the complex, but other possible “defects” are not excluded. Voronoi complexes provide the most convenient combinatorial set of conditions for being embeddable in a manifold. These conditions exclude the “degenerate” Voronoi complexes mentioned earlier (which is not a big loss), but they do include ones which are not locally finite, and could easily be extended to the generalized complexes that are not made of topological cells.
C. Random Voronoi Complexes

In order to understand how discrete structures embedded in a manifold $\mathcal{M}$ encode its geometry, we need to introduce a few notions related to randomly distributed points. A random point distribution on $\mathcal{M}$ with a volume element (in particular, one given by a Riemannian metric, $\sqrt{\text{g}}$) is the outcome of a uniform, binomial or Poisson, point process. If the total volume $V_{\mathcal{M}}$ is finite, a uniform point process is specified by stating that, each time a point $x$ is chosen in $\mathcal{M}$, the probability that $x$ fall in any given measurable region $X \subseteq \mathcal{M}$ is

$$P(x \in X) = V_X / V_{\mathcal{M}},$$  \hspace{1cm} (2.4)

or the infinitesimal version, that the probability density is $\tilde{P}_M(x|\sqrt{g}) = V_{\mathcal{M}}^{-1} \sqrt{g} d^D x$. If the process is repeated $N$ times, with no correlations among points (we will not keep track of the order they came in), we get a uniform sprinkling of points with density $\rho := N / V_{\mathcal{M}}$ that for our purposes we will think of as being of order $\ell_c^{-D}$. The probability density for each point to fall in an infinitesimal region $d^D x$ is

$$d\mu = \rho \sqrt{g} d^D x. \hspace{1cm} (2.5)$$

One of the most useful finite probabilities in this context is the one for exactly $k$ points out of $N$ to fall inside $X$ (without specifying which ones). It is easy to see that this probability follows a binomial distribution,

$$P(k, X | N, \mathcal{M}) = \binom{N}{k} \left( \frac{V_X}{V_{\mathcal{M}}} \right)^k \left( 1 - \frac{V_X}{V_{\mathcal{M}}} \right)^{N-k}, \hspace{1cm} (2.6)$$

which, as $V_{\mathcal{M}}$ and $N$ become very large, with $\rho = \text{constant}$, approaches a Poisson distribution,

$$P(k, X | N, \mathcal{M}) \approx \frac{e^{-\rho V_X} (\rho V_X)^k}{k!}.$$

This last equation justifies the name Poisson distribution that is often used for the sets of points used in this paper, and corresponds to the infinite volume situation.

A random Voronoi complex in a manifold $(\mathcal{M}, g_{ab})$ is the result (assumed to be a cell complex for the time being) of applying the Voronoi procedure of Sec II C to a Poisson distribution of points in $(\mathcal{M}, g_{ab})$ (see, e.g., Ref [12] and references therein). This is the discrete structure we use in this paper to encode information on a spatial geometry. Such structures have been called “random lattices” in the context of gauge theory (the subject is actually older than that, but for references with physical motivations somewhat related to ours, see Refs [9] and [13]). The randomness of the point distributions and their finite density imply that all complexes are locally finite, and all vertices are $(D+1)$-valent with probability 1. Most of the general discussion will be valid for complexes and manifolds of arbitrary dimension $D$, but actual calculations will be carried out for 2-dimensional ones.

D. Remarks on the Use of Voronoi Complexes

The discrete structures one uses in loop quantum gravity to construct spin networks, the basic states that form the usual basis for the kinematical Hilbert space $\mathcal{H}_{\text{kin}}$, are graphs
embedded in a manifold. Thus, although we don’t know yet how to formulate a manifold-independent theory, it may be useful to try to establish a correspondence between certain types of graphs and the Voronoi cell complexes that our statistical machinery is based on. Thus, suppose that we are given a graph $\gamma$, consisting only of vertices and edges, but without higher-order cells. We then pose the following question: Can we construct a full Voronoi complex just from $\gamma$? To recover $\Omega$ as an abstract complex in $D$ dimensions, what we need to do is specify which edges in $\gamma$ form (the boundary of) an elementary 2-cell, which of the latter form (the boundary of) a 3-cell, and so on. If all goes well, the resulting cell complex will satisfy the correct incidence relations for a Voronoi complex, as specified by (2.2).

Let us begin with a proposal for a construction in two dimensions. Given a graph $\gamma$ in which every vertex is trivalent, define a loop to be a chain of consecutive edges $e_1 e_2 \cdots e_K = (v_1 \leftrightarrow v_2 \leftrightarrow \cdots \leftrightarrow v_K \leftrightarrow v_{K+1})$ that closes on itself, i.e., $v_1 = v_{K+1}$. Most loops are not to be thought of as boundaries of 2-cells; we call plaquette a loop $\alpha$ such that, for any two vertices $v_I$ and $v_J \in \alpha$, the shortest path in the graph between $v_I$ and $v_J$ is part of $\alpha$. The cell complex $\Omega$ we are looking for has as 0-cells and 1-cells the same ones as $\gamma$, trivially, and its 2-cell are identified with a set of plaquettes $\alpha$ such that $\Omega$ is a Voronoi cell decomposition of a 2-manifold, i.e., such that every edge is shared by exactly two 2-cells; if such a choice is not possible, we consider the graph non-embeddable. However, we expect that if we apply this construction to the 1-skeleton of an actual Voronoi complex $\Omega$, we recover the original $\Omega$. The main questions will then be: How do we recognize from the graph whether it corresponds to a “good situation”? What can we say about cases where it does not?

Similarly, to construct a 3D Voronoi complex from a four-valent graph, we start by defining a candidate 3-cell $C$ as a finite set of plaquettes $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ such that (i) every edge is shared by exactly two 2-cells $\alpha_i$ and $\alpha_j$, as in the 2D construction above, (ii) the topology of $C$ is that of a 2-sphere, and (iii) for every two vertices $v_i \in \alpha_i$ and $v_j \in \alpha_j$, the shortest path in the graph between them is part of $C$. A collection of 3-cells defined in this way gives a good Voronoi complex if all edges in it are shared by exactly three plaquettes, and each plaquette by exactly two 3-cells. These definitions could then be generalized in an obvious way to higher dimensions, but in this paper we will not need to consider explicitly cells of higher dimensionality. We can call abstract $D$-dimensional Voronoi graph one such that all vertices have the same valence $D+1$, and the above construction gives a good $D$-dimensional Voronoi complex.

Let us conclude with two remarks on the concepts we have introduced so far. First, we are not suggesting that all semiclassical quantum gravity states are associated with Voronoi complexes, just as in ordinary quantum mechanics, not all semiclassical states are coherent states. However, the latter have properties that make them easy to work with, and they encode in a convenient, minimal set of parameters a point in classical phase space and the freedom in the (minimum) uncertainties in the canonical variables. We propose to consider states based on Voronoi complexes as playing a similar role for the discrete-to-continuum transition. We do not know yet how to phrase a minimum uncertainty condition in this context, or questions about the existence of processes which might produce such states. Filling in the first gap is one of the goals of our program; the second one will probably require a much more complete knowledge of semiclassical quantum gravity, including its dynamics. From a geometrical point of view, however, there is a strong motivation for using Voronoi complexes, that we will be exploring in this paper.

Second, it is known that in order for states based on a set of graphs to span a dense
subset of the (kinematical) Hilbert space of loop quantum gravity, one needs to consider graphs with an arbitrary number of edges and connectivities. If one restricts oneself to graphs with only four-valent vertices (in 3+1 dimensions), one does not obtain but a high-codimension subspace of the Hilbert space. We are nevertheless suggesting that by restricting our attention to states defined over such graphs we will not lose important information. Are the semiclassical states defined over such a restricted class of graphs sufficient to display the needed semi-classical features? We do not have a definite proof for this, but we can argue in favor of such states. Consider for instance the example of a simple harmonic oscillator with a finite number of degrees of freedom. In this case the usual Gaussian coherent states one defines, peaked at phase space points, span only a finite-dimensional submanifold of the infinite-dimensional Hilbert Space of the theory. In spite of this, one can regard the coherent states as ‘enough’ for describing semiclassical states in some cases. Again for the coherent states of the free Maxwell theory, one can take coherent states and they approximate very well the semiclassical properties that we are interested in. We will then by analogy assume that the states we are considering here, defined over Voronoi complexes, will be enough to describe the semiclassical sector of the theory.

III. THE INTRINSIC GEOMETRY OF VORONOI COMPLEXES

Having introduced the necessary background concepts, we can now describe in more detail what we intend to do in the rest of the paper. Our general goal is the following: given a Voronoi cell complex Ω, i.e., one satisfying the incidence relations (2.2), determine the range of classical geometries that are consistent with it. From the quantum gravity point of view, it is not clear whether it is reasonable to associate a single Ω with a semiclassical Ψ (different points of view underly for example the proposal in Ref [14], or the shadow state proposal [15], in which different discrete structures are seen as tools for probing the state Ψ). We do so here because it allows us to separate the effects of the classical uncertainty in the discrete-to-continuum transition from those of the quantum fluctuations in Ψ, and if more Ω’s need to be considered one can always combine their uncertainties later.

We emphasize that the effective geometry will only be a spatial one; the recovery of an effective spacetime geometry requires either structures that can be interpreted as discrete spacetimes and the use of Lorentzian statistical geometry (see, e.g., Ref [16]), or additional variables on Ω that can be interpreted as discrete versions of dynamical data (as in Ref [8]). Even in this context, our actual calculations will concern cases in which the topology of M is trivially determined by that of Ω, and only Sec IV will discuss a more general situation.

A. The Discrete-Continuum Transition

There is an analogy with our situation in elementary physics. When one is dealing with a fluid, one ‘knows’ that at some ‘microscopic scale’, one is dealing with molecules, individual entities with which one can associate, classically, a position and a velocity. One then considers cells at a mesoscopic (crossover) scale inside of which one averages velocities, energies and so on, and one assigns such quantities to the cell as a whole. Finally, one goes to much larger, macroscopic scales and regards those properties of the cell as being local, defined by continuum (and differentiable) fields. In a sense, this is the procedure we are envisaging:
There is a microscopic (discrete scale) $\ell_d$ where the ‘true’ discrete geometry is defined by a graph. We will then consider large sets of cells at a scale $\ell_c$, over which we will average the combinatorial quantities of the cells, to smooth out statistical fluctuations and define mesoscopic quantities that vary slowly between such groups of cells. On the larger macroscopic scale, this will allow us to view the mesoscopic quantities as the local values of continuum fields that define a geometry. This passage between the microscopic scale $\ell_d$ and the final macroscopic one is what we call the discrete-continuum transition.

To define a macroscopic geometry $(\mathcal{M}, q_{ab})$, the quantities we consider will be geometric invariants $Q_X$ associated with extended submanifolds $X \subset \mathcal{M}$, large enough to correspond to a large number of cells of $\Omega$ considered as a cell decomposition of $\mathcal{M}$, but macroscopically small so that they do not correspond to integrating or averaging over regions where the geometry varies. In this paper, the submanifolds $X$ will be simply open regions of $\mathcal{M}$ containing large numbers of $D$-cells in $D$ dimensions, although for other purposes one might consider hypersurfaces in $\mathcal{M}$, approximated by large collections of $(D - 1)$-cells, or submanifolds of higher codimension. As for the $Q_X$ themselves, the above analogy with the thermodynamic limit leads us to divide the possible invariants into extensive ones (the simplest example is the volume $V_X$, for which we obtain values by counting either Voronoi cells or Voronoi vertices contained in $X$) and intensive ones (examples of this type are curvature invariants, averaged over $X$). For the purposes of this paper, the latter are the less trivial and the more fundamental ones (relationships between most quantities are curvature-dependent), and we will concentrate on those in this paper. By constructing an effective continuum geometry here we thus mean finding the values of a sufficiently large set of curvature invariants $Q_X$, and providing a quantitative measure of the goodness of the construction, in terms of the values of the uncertainties $\Delta Q_X$, using $\Omega$.

In order to learn how to do this, we will first provisionally assume that a classical geometry with known values for all of its $Q_X$ is given, which allows us to calculate statistical distributions of Voronoi complex variables. The relationship will then be inverted to allow us to estimate the $Q_X$ and their uncertainties from a given $\Omega$. In other words the main idea is that, when one obtains a cell complex as the result of applying the Voronoi construction to a random point sprinkling of density $\rho$ in a Riemannian manifold $(\mathcal{M}, q_{ab})$, the combinatorial quantities $N_{klij}$ for the complex satisfy dimension-dependent identities, and one can also calculate (at least in principle) geometry-dependent probability distributions for values of those quantities. These two types of relationships together imply that the complex encodes enough information about the manifold $\mathcal{M}$ and metric $q_{ab}$ that we could reconstruct $(\mathcal{M}, q_{ab})$ from it, up to statistical uncertainties on volume scales at or below $\rho^{-1}$.

Thus, if we are given an abstract, embeddable Voronoi complex in the sense of Sec III, we can construct an approximate $(\mathcal{M}, q_{ab})$ that is a good continuum version of $\Omega$ on scales larger than the average embedded cell size $V_M/N_D =: \rho^{-1}$. From a practical point of view, we quickly run into the difficulty that only a few of the relevant probability distributions, for low-dimensional flat or constant-curvature spaces, are known. Therefore, we will treat in detail the two-dimensional case, where we can derive the results we need analytically, and outline the procedure in three dimensions, where analogous calculations will have to be done with computer simulations.
B. The Two-Dimensional Case

In two dimensions, any metric is conformally flat, and can be locally written as \( q_{ab} = e^{2f} g_{ab} \), where \( f \) is a scalar function, and \( g_{ab} \) a fixed flat metric on a portion of the 2-manifold; if Cartesian coordinates are used for \( g_{ab} \), the line element is then \( q_{ab} \text{d}x^a \text{d}x^b = e^{2f} (\text{d}x^2 + \text{d}y^2) \).

The function \( f \) is in turn related to the scalar curvature by \( R = -2 e^{-f} \nabla^2 f \); \( R \) is the continuum geometrical quantity we will associate with a cell complex in this section. When attempting to construct a geometry, it would seem natural to consider first finding a distance for any pair of cells (or of vertices) in the complex. However, the relationship between areas and lengths is curvature-dependent, and if we consider the cell density to be a basic parameter, before we assign distances to pairs of objects in the cell complex, we first need to find out the curvature that best fits each portion of \( \Omega \). Besides, in physical applications one is often directly interested in the curvature of a manifold, since it affects, e.g., the propagation of matter fields on it.

In order to find out how to assign a value of \( R \) to a subset of a cell complex \( \Omega \), we need to learn to recognize complexes that might arise from a point sprinkling in a manifold with curvature \( R \). We will therefore use Eqs. (2.1) and (2.3) to determine the mean and variance of the number of edges of a 2-cell on a 2-sphere \((S^2, s_{ab})\) of constant scalar curvature. If we denote by \( \Omega_R \) a Voronoi complex on such an \( S^2 \), Eq. (2.1) becomes \( N_0 - N_1 + N_2 = \chi(\Omega_R) \); then, using (2.3) and noticing that the average value of \( N_1(\omega) \) over the complex is given by \( \langle N_1(\omega) \rangle_{\omega \in \Omega_R} = 2 N_1(\Omega_R)/N_2(\Omega_R) \), where the 2 is due to the fact that each edge is shared by 2 2-cells, we obtain

\[
\langle N_1(\omega) \rangle_{\omega \in \Omega_R} = 6 \left( 1 - \frac{\chi(\Omega_R)}{\rho V} \right) = 6 \left( 1 - \frac{\int_M R \text{d}V}{4\pi \rho V} \right) = 6 \left( 1 - \frac{R}{4\pi \rho} \right). \tag{3.1}
\]

Here, we have used the Gauss-Bonnet theorem relating \( \chi(\Omega_R) \) and the scalar curvature of \((S^2, s_{ab})\). Since the average does not depend on \( \Omega_R \), it also equals the mean number \( \overline{N_1} \) of neighbors of a cell taken over all random complexes in the geometry \((S^2, s_{ab})\) with density \( \rho \). Eq (3.1) can then be easily inverted to give the scalar curvature in terms of the mean number of edges of a cell,

\[
R = 4\pi \rho \left( 1 - \frac{1}{6} \overline{N_1} \right). \tag{3.2}
\]

Finding the variance \( \sigma_1^2 = \overline{N_1}^2 - (\overline{N_1})^2 \) of the distribution of the number of neighbors is considerably more difficult. We already know \( (\overline{N_1})^2 \); for \( \overline{N_1} \), we will use a trick [17]. Notice that for any 2-cell \( \omega \), \( N_1(\omega) \) equals the number of vertices \( N_0(\omega) \), so our task can be seen as that of calculating \( \overline{N_0^2} \). The latter has been calculated (i.e., analytically reduced to an integral which is then numerically evaluated) by Brakke [17] for the flat case; we will now generalize his calculation to the case of a constant positive curvature manifold. For any \( \omega \), a quantity related to \( N_0^2 \) whose mean is much simpler to calculate directly is the number of (unordered) pairs of vertices not sharing an edge, for if we denote this quantity by \( N_{0,0'}(\omega) \), then by definition \( N_{0,0'} = \frac{1}{2} N_0 (N_0 - 3) \). Thus, \( \overline{N_0^2} = 2 \overline{N_{0,0'}} + \frac{3}{2} \overline{N_0} \) (notice that our \( \overline{N_{0,0'}} \) is Brakke’s \( I(v,v) \)), where \( \overline{N_{0,0'}} \) can be found by integrating a suitable probability density, as we now show.

Consider the 2-sphere as embedded in 3-dimensional Euclidean space, and call \( C \) the center of the sphere; the radius \( a \) of the sphere is related to the scalar curvature of \( S^2 \) by \( R = 2/a^2 \). Choose an arbitrary point \( O \) on the 2-sphere as its origin. To locate any other
point \( P \) in \( S^2 \) we will initially use the spherical coordinates \((\chi, \theta)\), where \( \chi \in [0, \pi] \) is the angle at \( C \) between the lines \( CO \) and \( CP \), and \( \theta \in [0, 2\pi] \) the azimuthal angle on \( S^2 \) around \( O \). The line element on \( S^2 \) is then given by the familiar form
\[
ds^2 = s_{ab} \, dx^a \, dx^b = a^2 \, (d\chi^2 + \sin^2 \chi \, d\theta^2) .\]  (3.3)

Consider now an arbitrary cell in a random Voronoi complex on \((S^2, s_{ab})\) of density \( \rho \), and for convenience choose the coordinates such that the sprinkled point or “seed” \( S_0 \) that defines this cell is at the origin. We would like to find the expected number of pairs of vertices \((P_1, P_2)\) of this cell which do not share an edge, i.e., which are not consecutive. Any cell vertex \( P_i \) is equidistant from three seeds in the sprinkling, in this case \( S_0 \) and two others, \((S_{i1}, S_{i2})\), where \( i = 1, 2 \) labels the vertex they are associated with. Therefore, when we count pairs of vertices we need to count pairs of configurations \((S_0, S_{i1}, S_{i2})\) such that the disk \( \hat{V}_i \) inside the circle through each triple of points is void of other seeds (so that they really define a vertex), and the four seeds \( S_{ij} \) are distinct (so that the vertices \( P_1 \) and \( P_2 \) are not consecutive). What is the probability density for all of this to happen, in terms of all possible locations for the four seeds in question?

The probability measure for a seed to be located at \( S = (\chi, \theta) \) is \( \rho \, (a^2 \sin \chi \, d\chi \, d\theta) = \rho a^2 \, d(\cos \chi) \, d\theta \); therefore, for the pair of seeds \((S_{i1}, S_{i2})\) giving the vertex \( P_i \), at locations \((\chi_{i1}, \theta_{i1})\) and \((\chi_{i2}, \theta_{i2})\), respectively, with \( i = 1 \) or 2,
\[
d\mu_i = \rho a^4 \, d(\cos \chi_{i1}) \, d\theta_{i1} \, d(\cos \chi_{i2}) \, d\theta_{i2} .\]  (3.4)

We can make sure that none of the four seeds is contained in the disk defined by \( S_0 \) and the two seeds in the other pair by specifying appropriate ranges for the allowed positions of the seeds, and we can impose that the union of the two disks \( \hat{V}_i \) contain no additional seeds by multiplying the measure by the probability \( e^{-\rho A(V_1 \cup V_2)} \).

To control more easily the ranges of integration, it is convenient to make a coordinate transformation from the eight variables \((\chi_{ij}, \theta_{ij})\) to a set of four variables \((\zeta, \phi, \omega_1, \omega_2)\) which specify the location of \( P_1 \) and \( P_2 \) (or equivalently, the size and location of the two circles), and four variables \( \alpha_{ij} \) which specify the location of the four points on the two circles. Given two circles through \( S_0 \), or two points \( P_i = (\chi_i, \theta_i) \), labeled so that \( \theta_2 - \theta_1 \leq \pi \), call \( Q = (2\zeta, \phi) \) the other point at which the circles intersect, besides \( S_0 \) (if the circles are tangent at \( S_0 \) we identify \( \phi \) with the direction of that tangent, and \( Q = S_0 \), but this happens with probability zero). Also, call \( \omega_1 \) and \( \omega_2 \) the angles \( Q S_0 P_1 \) and \( Q S_0 P_2 \), taken to be positive respectively in the clockwise and counterclockwise directions from \( \phi \). Then
\[
\theta_1 = \phi - \omega_1 , \quad \theta_2 = \phi + \omega_2 , \quad \text{and the ranges of values for the new angles are}
\]
\[
0 < \phi < 2\pi , \quad -\frac{\pi}{2} < \omega_1 < \frac{\pi}{2} , \quad -\omega_1 < \omega_2 < \frac{\pi}{2} . \]  (3.6)

The distance \( \chi_i \) of \( P_i \) from \( S_0 \) can be expressed in terms of \( \zeta \) using the spherical cosine law applied to the isosceles triangle \( S_0 Q P_i \), i.e.,
\[
\cos \chi_i = \cos \chi_i \cos 2\zeta + \sin \chi_i \sin 2\zeta \cos \omega_i , \quad \text{which implies the relationship}
\]
\[
\tan \chi_i = \frac{\tan \zeta}{\cos \omega_i} , \quad \text{with} \quad 0 < \zeta < \frac{\pi}{2} . \]  (3.7)
FIG. 2: Construction for the calculation of 2D statistical fluctuations on the two sphere; see text for explanation (only one of the two $P_i$ and one of the $S_{ij}$ are labelled).

The additional variables are defined as follows. On the circle around $P_1$ (resp, $P_2$) call $S_{i1}$ the first seed one meets going around the circle clockwise (resp, counterclockwise) from $S_0$, and $S_{i2}$ the second one. Then $\alpha_{i1}$ is the angle $P_iS_0S_{i1}$ and $\alpha_{i2}$ the angle $P_iS_0S_{i2}$ (both measured clockwise for $i = 1$ and counterclockwise for $i = 2$), i.e.,

$$\theta_{ij} = \theta_i + (-1)^i \alpha_{ij} = \phi + (-1)^i (\omega_i + \alpha_{ij}) \ ,$$

with ranges given by

$$-\omega_i < \alpha_{i1} < \frac{\pi}{2} , \quad \alpha_{i1} < \alpha_{i2} < \frac{\pi}{2} \ .$$

The Eqs (3.8) constitute half of the coordinate transformation. To find the other half, we apply the spherical cosine law to the triangle $S_0P_iS_{ij}$; this gives $\cos \chi_i = \cos \chi_i \cos \chi_{ij} + \sin \chi_i \sin \chi_{ij} \cos \alpha_{ij}$, which implies the relationship

$$\tan \frac{\chi_{ij}}{2} = \tan \chi_i \cos \alpha_{ij} \ .$$

We can then substitute (3.7) in this last equation, and express the result in the more convenient form

$$\cos \chi_{ij} = \frac{\cos^2 \omega_i - \tan^2 \zeta \cos^2 \alpha_{ij}}{\cos^2 \omega_i + \tan^2 \zeta \cos^2 \alpha_{ij}} \ .$$
With (3.8) and (3.11) we can now rewrite the full measure of integration (3.4) in terms of the new variables. After a somewhat lengthy calculation, one obtains

\[ d\mu_1 d\mu_2 = d\zeta d\phi d\omega_1 d\omega_2 d\alpha_{11} d\alpha_{12} d\alpha_{21} d\alpha_{22} \]
\[ \times 256 \rho^4 a^8 \tan^7 \zeta (1 + \tan^2 \zeta) \left[ \prod_{i,j=1,2} \frac{\cos \alpha_{ij}}{(\cos^2 \omega_i + \tan^2 \zeta \cos^2 \alpha_{ij})^2} \right] \times \]
\[ \times \cos^3 \omega_1 \cos^3 \omega_2 \sin(\omega_1 + \omega_2) \sin(\alpha_{22} - \alpha_{21}) \sin(\alpha_{12} - \alpha_{11}) . \]  

(3.12)

We now need to calculate the area \( A(V_1 \cup V_2) \) of the union of the disks around \( P_1 \) and \( P_2 \). The line \( S_0Q \) divides each of the \( V_i \) into two parts, and \( V_1 \cup V_2 \) is the disjoint union of one part from each \( V_i \) (see figure) which, for the purpose of finding its area, it is convenient to think of as \( V_i \) with the “wedge” \( S_0P_iQ \) removed and replaced by the triangle \( S_0P_iQ \). Thus we can write

\[ A(V_1 \cup V_2) = \sum_{i=1}^{2} (A_{\text{disk}} - A_{\text{wedge}} + A_{\text{triangle}})_i = \sum_{i=1}^{2} \left[ \pi + 2\omega - (\pi - \beta) \cos \chi \right] a^2 , \]  

(3.13)

where some simple spherical geometry gives

\[ A_{\text{disk}}_i = 2\pi (1 - \cos \chi_i) a^2 \]
\[ A_{\text{wedge}}_i = 2\beta_i (1 - \cos \chi_i) a^2 \]
\[ A_{\text{triangle}}_i = 2 (\beta_i + \omega_i - \frac{1}{2} \pi) a^2 . \]  

(3.14)

Here, \( \beta_i \) is half of the internal angle of the wedge or the triangle \( S_0P_iQ \) at \( P_i \). Using the sine law with half the triangle \( QS_0P_i \), we obtain that it is related to our variables by

\[ \cos \beta_i = \cos \zeta \sin \omega_i , \]  

(3.15)

and we obtain from (3.7) that

\[ \cos^2 \chi_i = \frac{\cos^2 \omega_i}{\cos^2 \omega_i + \tan^2 \zeta} . \]  

(3.16)

Putting these pieces together we therefore have

\[ A(V_1 \cup V_2) = a^2 \sum_{i=1}^{2} \left[ \pi + 2\omega_i - \frac{2 \cos \omega_i}{\sqrt{\cos^2 \omega_i + \tan^2 \zeta}} (\pi - \arccos(\cos \zeta \sin \omega_i)) \right] , \]  

(3.17)

and the expectation value we are looking for is obtained integrating the product of these probabilities over all locations of the four seeds,

\[ \overline{N_{0,0'}} = \int d\mu_1 \int d\mu_2 e^{-\rho A(V_1 \cup V_2)} , \]  

(3.18)

Finally, substituting this in the expression for the variance of \( N_0 \), we get

\[ \sigma_1^2 = 2 \overline{N_{0,0'}} + 3 \overline{N_1} - \overline{N_1^2} = 2 \int d\mu_1 \int d\mu_2 e^{-\rho A(V_1 \cup V_2)} - 18 \left( 1 - \frac{3}{4\pi} \frac{R}{\rho} + \frac{1}{8\pi^2} \frac{R^2}{\rho^2} \right) . \]  

(3.19)
This integral can be evaluated numerically for any given value of $\rho/R$.

We are now in a position to discuss the geometries we associate with a 2-dimensional Voronoi complex $\Omega$. Since in two dimensions the curvature is completely characterized by the Ricci scalar $R$, our goal is to associate a value of $R$, with suitable uncertainties, with every set $U$ in a cover of a manifold $\mathcal{M} \simeq \Omega$ that is made of sufficiently small sets. We start by selecting a larger collection of candidate sets, and then explain how to pick the appropriate ones among them. For each 2-cell $\omega_0 \in \Omega$ consider the family of sets \( \{U_{\omega_0,\lambda}\} \), $\lambda = 0, 1, 2, \ldots$, defined by

$$U_{\omega_0,0} = \omega_0, \quad U_{\omega_0,\lambda+1} = U_{\omega_0,\lambda} \cup \{\omega \in \Omega \mid \partial \omega \cap \partial U_{\omega_0,\lambda} \neq \emptyset\};$$

(3.20)

in other words, $U_{\omega_0,\lambda}$ is $\omega_0$ together with the first $\lambda$ layers of neighboring 2-cells around it. Roughly speaking, if we call $N_{\omega_0,\lambda}$ the number of 2-cells in $U_{\omega_0,\lambda}$, in an approximately flat 2-geometry we expect to have $N_{\omega_0,\lambda} \approx 1 + 6 + 12 + \ldots + 6\lambda = 3\lambda^2 + 3\lambda + 1$. A value of $R$ can be assigned to each $\omega_0$ simply by using (3.2) as an estimate,

$$R(\omega_0) = 4\pi \rho \left(1 - \frac{1}{6} N_1(\omega_0)\right).$$

(3.21)

This value however is not a good one as far as manifold geometry goes; differences between it and those obtained for neighboring 2-cells should be interpreted not as real variations of $R$ but as statistical fluctuations. What we should do instead is average the values obtained for a cluster of neighboring 2-cells, i.e., a suitably large $U_{\omega_0,\lambda}$. If we knew that the continuum geometry will turn out to be a constant curvature one, the best strategy would be to pick $\lambda$ as large as possible, in order to maximize the statistics. In practice, if we use too many 2-cells we may be combining regions that in a good fit of the geometry would have different curvatures, so we need to specify a procedure for picking an optimal $\lambda$. Pick some value of $\lambda$. Then, at scale $\lambda$, $U_{\omega_0,\lambda}$ provides a sample of size $N_{\omega_0,\lambda}$ from the ensemble of all 2-cells in a manifold of assumed constant curvature, and our best estimate for the scalar curvature around $\omega_0$ is the average

$$R(U_{\omega_0,\lambda}) = \langle R(\omega) \rangle_{\omega \in U_{\omega_0,\lambda}} = 4\pi \rho \left(1 - \frac{1}{6} \langle N_1(\omega)\rangle_{\omega \in U_{\omega_0,\lambda}}\right),$$

(3.22)

with the variance in the distribution of averages over such samples being given by

$$\sigma^2_{\langle N_1 \rangle} = N_{\omega_0,\lambda}^{-1} \sigma_1^2.$$

(3.23)

If this variance is small, we can use it to estimate a range of values of $R$ which would have produced values of $\langle N_1(\omega)\rangle$ within this tolerance, by setting $\Delta R = (dR/d\langle N_1 \rangle) \sigma_{\langle N_1 \rangle}$, or

$$\Delta R(U_{\omega_0,\lambda}) = \frac{2\pi \rho}{3} \sigma_{\langle N_1 \rangle} = \frac{2\pi \rho}{3} N_{\omega_0,\lambda}^{-1/2} \sigma_1.$$

(3.24)

Finally, the whole process is self-consistent and gives a good continuum approximation to $\Omega$ if differences between the scalar curvatures estimated from neighboring regions $U_{\omega_0,\lambda}$ are smaller than the statistical uncertainty $\Delta R(U_{\omega_0,\lambda})$ within individual regions, i.e., $R$ is slowly varying on the scales given by $\lambda$. 

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C. Considerations on the General Case

In three dimensions, there are three independent local curvature invariants. Different choices are possible; for example, they can be thought of as the eigenvalues of the Ricci tensor $R_{ab}$, or as the scalars $R = R^a_a$, $R^a_b R^b_a$, and $R^a_b R^b_c R^c_a$. To recover them from $\Omega$ we need to use three independent local combinatorial graph quantities, chosen among the $N_k(\omega)$. One may think that a possible choice for these is the set of numbers $(N_0(C), N_1(C), N_2(C))$, which count the number of vertices, edges and 2-faces of a 3-cell $C$. However, for any given $C$, only one of those three numbers is independent, since its vertices, edges and faces form a 2-dimensional complex $\partial C$ homeomorphic to $S^2$, to which we can apply Eqs. 2.1–2.3 with $\chi(\partial C) = 2$, and we get that $N_0(C) - N_1(C) + N_2(C) = 2$ and $N_1(C) = \frac{3}{2} N_0(C)$, which is easily solved to give, for example,

$$N_0(C) = 2 N_2(C) - 4, \quad N_1(C) = 3 N_2(C) - 6. \quad (3.25)$$

It seems reasonable to assume that, analogously to the 2D result, the number $N_2(C)$ of 2-faces of a 3-cell allows us to estimate the scalar curvature $R$. The other curvature invariants are associated with the anisotropies in the curvature, since knowledge of the traces of powers of $R^a_b$ can be thought of as corresponding to knowledge of its eigenvalues. We can then conjecture that this type of curvature information is encoded in direction-dependent combinatorial quantities of the 3-cells, i.e., any statistically significant differences between properties of 2-cells on different parts of the 3-cell boundaries; for example, the direction in which faces of a 3-cell $C$ have the least number of edges are those in which $\partial C$ is “most curved”, and should correspond to an eigenvector of $R^a_b$. Checking such statements is beyond the scope of this article, and will most likely require computational work with families of simple 3-geometries.

IV. SMALL-SCALE GEOMETRY AND COARSE-GRAINING

In the previous section we assumed that all complexes are embeddable in some manifold, but one of the ultimate goals of graph-based approaches to quantum gravity is to obtain a manifold-independent formulation. This section represents a tentative first step in that direction. It is somewhat more speculative than the previous ones, both for this general reason and for a more specific one. Even though our motivation comes from the semiclassical sector of quantum gravity, we will continue to work in a purely geometrical setting, i.e., we do not consider any quantum aspects, states, operators and the like. What we shall assume is that it is meaningful to consider a fixed discrete structures even at scales below the cross-over scale we have considered before, either because the quantum-to-classical transition occurs at a still smaller scale $\ell_q < \ell_d$, or because we can anyway learn something about the continuum interpretation of the theory by pushing this geometrical view as far as possible. This is the meaning we shall give to the term “small scale geometry”. We will discuss two aspects of this issue here, the different scales (“combinatorial” scales $\ell$ in the discrete structure, or metric scales $L$ in the continuum) that play a role in the construction of a Riemannian manifold, and how to deal with some of the obstructions that may come up in this process. As we will see, even in the simple setting in which we consider only the milder obstructions, different situations may arise; we will need to introduce the concept of coarse-graining, and to define other length scales in addition to the ones we have already seen in other sections.
A.  Length Scales

One general issue that needs to be addressed is that of identifying the various length scales that appear in our framework, and assigning dimensional values to combinatorial quantities in $\Omega$. Since we are assuming that the fundamental theory is formulated in terms of $\Omega$, we would want to describe those length scales using quantities intrinsic to $\Omega$, and be able to make statements such as “at scale $\ell_q$, $\Omega$ cannot be embedded in a manifold,” which implies that a continuum geometry is not available to provide a meaning for length scales.

Continuum theories include a number of length scales associated with various phenomena; results of measurements are interpreted in terms of those scales and provide values for them, so in order to provide values for scales intrinsic to $\Omega$ it is natural to go through a correspondence with continuum length scales. An obvious one to try to identify in terms of $\Omega$ is the Planck length $L_P$ (when $\Omega$ is dressed with other variables it may be possible to do this even at the kinematical level, as in the case of the calculation of the loop spacing for the heuristic weave states in loop quantum gravity [18]), but any prediction of the discrete theory that has a continuum counterpart can be used, and when more than one will be available the discrete theory can be tested.

A discrete structure of the type we are discussing also provides various scales. For example, if we start with a Voronoi cell complex, we can embed it in a geometry that has no length scales smaller than the cell size. The latter cannot be assigned a dimensional value yet, because the situation is invariant under a global rescaling of the metric, but we will say that $\Omega$ is associated with a semiclassical cross-over scale $\ell_s$ equal to its discreteness scale $\ell_d$. If we start with a $D$-dimensional Voronoi complex $\Omega$ that is not a cell complex, but is such that we can obtain a cell complex $\Omega'$ by coarse-graining $\Omega$ and removing a fraction $0 < \xi < 1$ of its $D$-dimensional tiles, we will say that $\Omega'$ corresponds to the semiclassical scale $\ell_s$, at which it and its dual $\Omega^{*}$ represent the manifold well, while $\Omega$ is characterized by the discreteness scale $\ell_d = (1 - \xi)^{1/D} \ell_s$. In either case, it may not be possible to determine the metric or curvature of the geometry at scale $\ell_s$, because at that scale we do not have enough statistics to reliably use the techniques we will describe. Instead, the classical scale $\ell_c$ is the linear size, in terms of number of cells, of the regions in $\Omega$ used as approximations to the manifold regions $X$ such that the continuum geometry can be determined with small statistical fluctuations. That geometry itself, possibly together with the global topology of $\mathcal{M}$, may also determine larger length scales $\ell_g$ (both local ones defined by the curvature and its rate of change or higher derivatives, and possibly global ones defined for example by non-trivial homotopy generators).

Summarizing, the length scales $\ell_d \leq \ell_s \leq \ell_c \leq \ell_g$ are determined by the structure of $\Omega$, if $\Omega$ is a discrete structure of a type that allows us to define $\ell_s$, given a suitable definition of a coarse-graining procedure. In quantum gravity, we will then require, as part of the definition of a semiclassical state, the condition that $\Psi$ be associated with a discrete structure for which $\ell_s$ exists. On the other hand, continuum-based physics determines a length scale $L_P$ for quantum gravity, given in terms of fundamental physical constants by dimensional arguments and back-of-the-envelope calculations of quantum gravity fluctuations [2], which appears also in the eigenvalues of the quantum geometry operators for areas and volumes; It is not clear to us whether this scale should be identified with $\ell_d$ or $\ell_c$, if different, or how this relationship might be affected by renormalization arguments. Finally, physics also determines a phenomenological, “macroscopic” scale $L_m \gg L_P$, which depends on the experimental...
techniques used but that can be assumed to be larger than $\ell_s$. A good understanding of the relationships between all of these scales is outside the scope of this paper, but our discussion of coarse-graining will be a start in this direction.

**B. Coarse-Graining**

Some complexes $\Omega$ can be directly associated with a “macroscopic” geometry, in the sense that they can be embedded homeomorphically in a manifold at a length scale $\ell_c$ and the procedures previously described give a well-defined (approximate) geometry at a length scale $\ell_s$, in which case the “semiclassical” interpretation is relatively straightforward, within the appropriate statistical uncertainties. That is, $\Omega$ gives us a classical, mesoscopic geometry. At larger scales, for which the number of points to be taken per sample cell is larger, one expects a very good approximation, according to our own set of conditions. The remaining question then is: What if $\Omega$ does not satisfy the requirements for approximating a nice geometry? Although there then is no direct association between graph and geometry, if in an appropriate sense the obstructions only occur on small scales, it may be possible to coarse-grain the graph to a larger-scale one that is associated with a continuum geometry.

The coarse-graining procedure will be one which, intuitively, takes a graph $\Omega$ at a certain scale and maps it to a different, larger-scale $\Omega'$. There are two possibilities on how to perform the coarse graining in this case. The first possibility, that we call soft coarse-graining, has the following strategy: One assumes that one has a Voronoi or Delaunay complex and one gets to a coarser one by “ignoring” the structure below a certain scale simply by averaging the quantities of interest over neighboring cells. This is what we have already done in Sec III when looking for the optimal mesoscopic scale at which to define slowly varying quantities; we are not actually defining a new complex, and it can always be done. In the second possibility, that we will refer to as hard coarse-graining, we have a procedure that connects two graphs, and one gets the larger graph $\Omega'$ by removing cells (Delaunay vertices) from $\Omega$, in a precise way.

Hard coarse graining, which implies a modification of the underlying graph, represents a new input in our considerations. This procedure is achieved by a sequence of *cellular moves* that refine the cellular decomposition [19]. These moves have been shown to provide a prescription for refining and coarsening the complex (they are in a sense more elementary than the Pachner moves). The intuitive idea for the coarse graining is that two adjacent $n$-cells of the complex that share an $(n-1)$-cell get ‘fused’ into a new $n$-cell. This refining/coarsening procedure has already been used for defining renormalization prescriptions in discrete systems [20], and the approximation to smooth manifolds by our proposed hard coarse-graining is an explicit example of some of those prescriptions.

In order to have a proper understanding of the different scenarios that one might encounter given an arbitrary graph at small scales, with the purpose of incorporating it into our geometrical formalism, let us make the following classification in decreasing order of ‘complexity’:

1. In the most exotic scenario one starts with a Voronoi complex that does not have a dual cell complex. This means that there is no dual triangulation and the corresponding graph might be disconnected. Since this happens for cases where, say, one sprinkles
points in a manifold with a density that is “too low”, and no real statistics is possible, we will exclude this case from our considerations here.

2. The next possibility is that the given cell complex $\Omega$ can not be embedded into any manifold $\mathcal{M}$. This means that the complex is not topologically a cell decomposition of a manifold. In this case one might hope that with a proper coarse graining one might be able to take this complex to a new one $\Omega'$ that does admit such interpretation and can be embedded. Physically, one could say that the original complex represented a space that on certain small scales has ‘topological’ and/or ‘dimensional’ fluctuations. This is the trickiest situation where one needs a ‘hard coarse graining’, for which we are not aware of an existing specific procedure, and it is certainly a question worth pursuing.

3. A milder scenario is to consider as the starting point a cell complex $\Omega$ that is a cell decomposition of a manifold $\mathcal{M}$ but that is not Voronoi, in the sense that the valence of the graph is not equal to $D + 1$. In this scenario, one would allow for “fluctuating valence” in the Voronoi cell, but with average $\langle N_{1|0} \rangle = 4$, in 3D. This means that the graph that defines the quantum state at Planck scale does not have the ‘Voronoi signature’, which might be the case for an arbitrary graph in loop quantum gravity. In this case we can implement the cell moves of Ref [13] to turn it into a Voronoi cell complex; if the authors’ conjecture is correct, via these cellular moves one can always transform any cell decomposition into a Voronoi one, that will serve as starting point for the statistical geometry considerations described in previous sections. One should also note that in some cases this ‘restructuring’ of the cell complex might not correspond to a course graining in the strict sense, since the moves that ‘lower’ the valence of the Voronoi graph do not remove cell complexes to it as one might expect from a coarse graining procedure.

In the next, final section we shall consider quantum issues and relate the formalisms we have been employing to the task of bridging the gap between a semiclassical state in loop quantum gravity and classical geometry.

V. PLANCK SCALE GEOMETRY AND BEYOND

In this section we make precise the connection between the previous sections and semiclassical canonical quantum gravity, i.e., the low energy limit of loop quantum gravity. This section has two parts. In the first one, we outline how one can try to use our framework to complement present approaches to semi-classical states in LQG. In the second part we outline the steps to be followed in order to estimate statistical errors in making phenomenological predictions out of the semiclassical states.

A. Semiclassical States

In order to deal with the low energy limit of the theory, we will again have to deal with the fact that there might be new length scales in the description of the quantum geometry. So
far we have considered in great detail the way in which we may approximate a macroscopic geometry starting from a graph or complex at a much smaller scale. Even though we have introduced a microscopic discrete scale $\ell_d$, at which the finest graph might be defined, this scale is not necessarily related to a particular quantum scale $\ell_q$. As discussed in Sec III, there are standard dimensional arguments which indicate that the relevant scale $\ell_q$ for a quantum state is the Planck length $L_P$. This only means that the quantum excitations of the geometry, which could be elements of area through edges of the graph or contributions to the volume from the vertices, yield eigenvalues of the order of the Planck scale for single edges/vertices (at least for the unrenormalized value of the Barbero-Immirzi parameter $\gamma$ of order one that is used nowadays). Whenever we consider a semiclassical state $\Psi_{sc}$ in loop quantum gravity, there are several steps and assumptions involved in estimating the continuum geometry it corresponds to. Let us now explore what those assumptions are.

The first assumption is that the true nature of geometry at the Planck length is described by loop quantum geometry. Second, we assume that we can define certain states (Coherent [24], Shadow [15], Gaussian [26], etc.) that will be semiclassical in the sense of approximating a geometry (and its time derivative), in the ‘correct phase’. Finally, one is assuming that a definite picture of the quantum geometry will emerge from a precise merger of both loop quantum gravity and our “discrete-statistical” approach. In this viewpoint, the final picture looks like this: The ‘true state’ is given by a properly defined state in loop quantum gravity, featuring quantum behavior at the Planck scale. The graph that we use to define this ‘shadow state’ (to give an example) will be a Voronoi graph. The semiclassical state will have the property that, when probed on a mesoscopic scale $\ell_c$ (still to be specified in practice) it will behave quasi-classically (hopefully, with respect the the coarse-grained operators that we need to define). This means that in order to construct the desired semiclassical state following the steps of Ref [21], the operators that we specify as belonging to the set to be approximated (together with the tolerances) will not be operators assigned to observables at the quantum, Planck scale.

We are interested in constructing states that approximate quantities at a scale $\ell_s$, where the quantum-classical transition takes place. As already mentioned, this new scale, yet to be identified, could be assumed to be close to the mesoscopic scale $\ell_c$, but it should be smaller. Going to our analogy model of fluids, the transition quantum-classical can be assumed to take place at a smaller scale than the one at which averages are taken and the continuum approximation emerges (for the simple reason that one adds velocities of particles contained in the region, which already presupposes classical attributes for the constituents of the fluid). The viewpoint that new, mesoscopic scales have to be considered has already been explored by other authors who have explored the semiclassical limit before [25].

The next question we would like to address is the possible utility of our statistical methods for distinguishing between good and bad candidates for semiclassical states. That is, suppose that we are given an alleged semiclassical state $\Psi_{sc}$ and we are assigned the task of testing it. The strategy for doing so is the following. Compute expectation values and fluctuations for the cell-based, observables that the maker of the state claims it approximates well. Make sure that the state is defined on a Voronoi complex. Dress the complex with the expectation values found for the observables. Now the crucial step is to compare this dressing with the geometry we expect to get from the complex itself, as in Sec III and verify that they are consistent. If they are not, return the state to the maker.
B. Phenomenology

Suppose now that the given state satisfies all criteria set by some considerations, then in our procedure we would have to pass it to the next step in the testing line: see whether it fits observations. Now this is a very intricate question, for we have not specified the phenomenological criteria that a semiclassical state should satisfy, nor the expected observable deviations from the classical realm (Lorentz violation, modified dispersion relations, non-commutativity, birefringence, etc) \[27\]. Note also that this procedure implies the knowledge of more quantities assigned to the cell complex, more than what we have up to now specified in the complex. This further ‘dressing of the complex’ will be dealt with in a forthcoming paper \[8\].

What we could attempt to answer is the following simpler question. Suppose that we expect the state to approximate the geometry \(g_{ab}\). As we have discussed before, there are now two sources of uncertainty in the approximation of this spacetime geometry (even though we were working in the canonical picture, we can talk about a spacetime geometry). The first source is a well known one, due to quantum fluctuations: observables \(O_I\) that were chosen to be approximated by the state have quantum dispersions \((\Delta \hat{O}_I)^{\Psi}\). The second source of uncertainty is the statistical nature of our reconstruction procedure. That is, as we have argued before, the macroscopic continuous geometry is only a fiction, very useful for describing phenomena at certain scales, but it is only an approximation to the true geometry. But there is not a unique macroscopic metric that can be approximated by the graph: rather, there is an ‘ensemble’ of such metrics, which then introduces a classical or cross-over uncertainty \((\Delta O_I)^{c}\) in the observables. The observational imprint that this uncertainty might bring is manifested by the fact that we normally use a fiducial classical metric to perform geometrical measurements. We expect that this geometry belongs to the set of possible geometries that are approximated by the graph, but it might turn out to be not the ‘most probable’ one, and thus one might be introducing an extra error in our interpretation of the experimental results. Note that these errors are of a different nature than those that arise from the use of a regular lattice in \[10\]. We expect that we won’t have these ‘discretization errors’ due to the use of the Voronoi construction. In order to have full control over the different sources of uncertainty, one would like to quantify them and decide which one is the dominant one. At this point we are not in the position of having a working hypothesis for this question, and we can only make an educated guess as to the way in which the two uncertainties interact with each other. Under the most naive assumption of independence, one might say that the total uncertainty is

\[
(\Delta O_I)^2 = (\Delta O_I)^{\Psi} + (\Delta O_I)^{c}.
\]

An important issue is to quantify the fluctuations that can be attributed to both effects, and know how the total fluctuation for each observable depends on them, since we would like to be able to subtract any statistical ‘spurious’ information and be able to measure the pure quantum contributions to the problem.

Assuming that the semiclassical state had these desired properties —an issue that lies outside the scope of this paper— would lead us to conclude, from a strict viewpoint that we have succeeded in our task of constructing a semiclassical geometry. This is because this state should then approximate any observations, at the mesoscopic level that we shall perform on the geometry (with external matter) and thus, if we now take the viewpoint that it is enough to describe what we can observe, we would reach our conclusion.
C. Summary and Outlook

To summarize, we have introduced a framework for establishing a bridge between the description of geometry of space in terms of semiclassical states for quantum gravity, and the one used in phenomenological calculations of corrections to field theory in curved space. At the most basic level, our proposal fits in with the general idea that the type of discreteness one encounters in a class of approaches to quantum gravity implies that the amount of information contained in a finite spacetime volume is finite, which has been explored in a variety of contexts \[28\]. Our work then suggests a specific way to translate this idea into quantitative expressions for the information a discrete structure contains about a continuum geometry, using statistical geometry. To illustrate how the framework is used, we have characterized a semiclassical quantum gravity state by a single cell complex and a single set of values for variables motivated by loop quantum gravity. Such a characterization, of course, is not intended to be complete; it might arise as part of the specification of a coherent state \[24\], or from a “shadow state” used to probe $\Psi_{sc}$ \[15\], but for our purposes it is just a tool that allows us to isolate the statistical effects we want to study. It is a tool that complements the criteria one might have in choosing the quantum state.

The general picture that emerged in this framework is one in which, from a certain length scale upward, if the quantities characterizing the state are sufficiently regular, the mesoscopic geometry is determined up to quantifiable uncertainties. Once those uncertainties will be better understood, especially in three dimensions, one can begin to look into the possibility of adding matter coupled to gravity, and how these uncertainties need to be combined with quantum fluctuations to give measurable effects on observable quantities. The answer to this last question will probably depend on which observations are carried out, and it is in principle possible that one can use observable effects that depend only on quantum uncertainties; after all, going back to the gas analogy, when we want to study the quantum properties of atoms, we do not need to combine their quantum fluctuations with the statistical mechanics description of a gas. But a gas is different, and we do not need to have information about the state of the whole gas to even be able to describe what each atom does.

In this paper we have concentrated our attention to graphs that did not have any other geometrical information other than what is already in its definition. In order to have an approach closer to the variables used in LQG one would like to have assigned certain (classical) quantities to the graph $\Omega$, that is, one would like to consider ‘dressed’ graphs. This will be dealt with in a subsequent publication \[5\].

Acknowledgements

LB and AC would like to thank Perimeter Institute for hospitality. Partial support for this work was provided by NSF grant PHY-0010061, CONACyT grant J32754-E and DGAPA-UNAM grant 112401.

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