On the compactness of Hamiltonian stationary Lagrangian surfaces in Kähler surfaces

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Abstract
We prove a bubble tree convergence theorem for a sequence of closed Hamiltonian stationary Lagrangian surfaces with bounded areas and Willmore energies in a complete Kähler surface. We also prove two strong compactness theorems on the space of Hamiltonian stationary Lagrangian tori in $\mathbb{C}^2$ and $\mathbb{C}P^2$ respectively.

Mathematics Subject Classification 53D12

1 Introduction

This paper concerns with compactness of a sequence of closed Hamiltonian stationary Lagrangian surfaces in a complete Kähler surface.

Let $(M, \omega, \bar{g}, J)$ be a $2n$-dimensional symplectic manifold with a symplectic 2-form $\omega$, an almost complex structure $J$ and a compatible metric $\bar{g}$. An immersion $F : \Sigma \rightarrow M$ is called Hamiltonian stationary Lagrangian (HSL) if it is Lagrangian and is a critical point of the volume functional among all Hamiltonian variations [31]. When a Lagrangian immersion in a Kähler-Einstein manifold is stationary with respect to all Lagrangian variations, the immersion is minimal (i.e. special Lagrangian if the ambient space is a Calabi-Yau manifold).

As a natural generalization of the minimal Lagrangians, HSLs exist in abundance: the totally geodesic $\mathbb{RP}^n$ in $\mathbb{CP}^n$ [31], the flat tori $S^1 (a_1) \times \cdots \times S^1 (a_n)$ in $\mathbb{C}^n$ [32], explicit examples in various Kähler ambient manifolds $[1,2,4,5,27,29]$; a complete classification of HSL tori in $\mathbb{C}^2$ via techniques in integrable systems and in $\mathbb{CP}^2$ [18] and other homogeneous Kähler surfaces $[19,22,26,28,30]$, and construction via the perturbation and gluing techniques $[3,23,25]$.

A regularity theory is developed in $[10]$, in particular, it is shown that a $C^1$-regular Hamiltonian stationary Lagrangian submanifold in $\mathbb{C}^n$ is smooth; the methods are further applied...
to obtain the smoothness estimates and small Willmore energy regularity in [11], which is
essential in proving a compactness theorem for HSL submanifolds in $\mathbb{C}^n$ with uniformly 
bounded areas and total extrinsic curvatures in $\mathbb{C}^n$. The regularity and compactness results 
in [10,11] rely on the assumption that the ambient space is $\mathbb{C}^n$ since it is used, in an essential 
way, that the Lagrangian phase angle $\Theta$ can be written as $\arctan \lambda_1 + \cdots + \arctan \lambda_n$ for the 
graphic representation $(x, Du)$, where $\lambda_i$’s are the eigenvalues of $D^2u$. Therefore, a boot-
strapping between $u$ and $\Theta$ becomes effective for $\Theta$ is a fully nonlinear second order elliptic 
operator and satisfies the Hamiltonian stationary equation 
$$\Delta_g \Theta = 0.$$ 

Unlike minimal submanifolds, the Simons’ identity for the Laplacian of the second funda-
mental form is not as useful for HSLs.

Our main result is

**Theorem 1.1** Let $(M, \omega, J, \bar{g})$ be a complete Kähler surface and $\Sigma$ be a closed ori-
entable surface. Assume that $h_n$ is a Riemannian metric of constant curvature on $\Sigma$ and 
$F_n : (\Sigma, h_n) \to (M, \bar{g})$ is a smooth branched conformal HSL immersion, and the areas and 
Willmore energies of $F_n$ are uniformly bounded above and $F_n(\Sigma)$ lie in a fixed compact set $K$ in $M$, for all $n \in \mathbb{N}$.

Then either $\{F_n\}$ converges to a point, or there is a stratified surface $\Sigma_\infty$ and a continuous 
mapping $F_\infty : \Sigma_\infty \to M$ so that a subsequence of $\{F_n\}$ converges to $F_\infty$ in the sense of 
bubble tree, and on each component of $\Sigma_\infty$, $F_\infty$ is a smooth branched conformal HSL 
immersion. Moreover, the area identity holds:

$$\lim_{n \to \infty} \text{Area}(F_n) = \text{Area}(F_\infty). \quad (1.1)$$

The measure $d\mu_L := (F_\infty)_* d\mu_\infty$ on $L = F_\infty(\Sigma_\infty)$ admits the structure of a varifold with 
$L^2$ generalized mean curvature $\bar{H}_\infty$ which satisfies

$$\int \bar{g}(\bar{H}_\infty, J\nabla f) d\mu_L = 0, \quad \text{for all } f \in C^\infty_c(M). \quad (1.2)$$

This generalizes the compactness theorem in [11], by allowing a general Kähler surface as 
the ambient space. Our approach is different from [10,11] due to the fact that the Lagrangian 
phase angle of a Lagrangian submanifold in a Calabi-Yau manifold does not necessarily 
admit an expression as a sum of the arctans, even in a local Darboux coordinates.

In light of the two-dimensional structure of the variation problem of the area functional, a strong compactness theorem [34, Proposition 4.7], among other important results, is proved 
for weakly conformal, minimizing Lagrangian maps with a uniform area bound, i.e. a sub-
sequence converges in the $W^{1,2}_{loc}$-topology to a minimizing Lagrangian map; this is applied 
to develop a deep theory of existence and regularity for minimizing Lagrangian maps [34].

We employ the bubble tree convergence of conformal mappings that parametrize the HSLs 
and use the construction in [6], while the bubble tree convergence for harmonic maps is first 
constructed in [33] since the seminal work [36]. Theorem 1.1 describes the singular points 
in the limit as branch points, and excludes the conical singularities in [34] since they have 
infinite Willmore energy. Without a uniform bound on the Willmore energies, Theorem 1.1 
fails; the sequence of HSL tori $S^1(1) \times S^1(1/n)$ in $\mathbb{C}^2$ has uniform bound on areas but not 
on the Willmore energies, and the limit is not a branched immersion.

A key ingredient in the proof of Theorem 1.1 is a removable singularity result (see The-
orem 3.2). First, it describes the point singularity at the limit as branched points, which is 
new even when the ambient space is Euclidean [11]; second, it is needed in our derivation.
of higher order estimates. Moreover, the removable singularity result implies the nonexistence of branched non-minimal HSL 2-spheres in a Kähler-Einstein surface, and this rigidity strengthens Theorem 1.1, as we will demonstrate in the next two theorems.

First, we state a strong compactness theorem for branched conformal HSL tori in $\mathbb{C}^2$.

**Theorem 1.2** Let \( \{F_n : (\mathbb{T}^2, h_n) \to \mathbb{C}^2\} \) be a sequence of smooth branched conformal HSL immersions with uniformly bounded areas and Willmore energies. Assume that \( 0 \in F_n(\mathbb{T}^2) \) for all \( n \in \mathbb{N} \). Then either \( \{F_n\} \) converges to a point, or a subsequence of \( \{F_n\} \) converges smoothly to a smooth branched conformal HSL immersion \( F_\infty : (\mathbb{T}^2, h_\infty) \to \mathbb{C}^2 \), and the corresponding conformal structures of \( h_n \) converge to the conformal structure of \( h_\infty \).

It is well-known that there is no immersed HSL 2-sphere in $\mathbb{C}^2$. We extend this to the case of branched conformal HSL 2-sphere in Corollary 3.1. This extension is essential in proving Theorem 1.2, since it implies that in the bubble tree convergence in Theorem 1.1: (i) non-trivial bubble cannot develop in the limiting process, and (ii) the sequence conformal structures of \( \{h_n\} \) does not degenerate as \( n \to \infty \). A similar argument is used in [8] for Lagrangian self-shrinking tori in $\mathbb{C}^2$.

Next, when \( M = \mathbb{CP}^2 \) with the Fubini-Study metric, Corollary 3.1 and a result of Yau [37] assert that the only HSL 2-sphere is the double cover of a totally geodesic $\mathbb{RP}^2$ in $\mathbb{CP}^2$. So bubbles as in (i) and (ii) above cannot rise when area is small and strong convergence holds.

**Theorem 1.3** Let \( \{F_n : (\mathbb{T}^2, h_n) \to \mathbb{CP}^2\} \) be a sequence of smooth branched conformal HSL immersions. Assume that there are positive constants \( C_1 < 2 \text{Area}(\mathbb{RP}^2) \) and \( C_2 \) so that

\[
\text{Area}(F_n) \leq C_1, \quad W(F_n) \leq C_2 \tag{1.3}
\]

for all \( n \in \mathbb{N} \) where \( W(F_n) \) is the Willmore energy of \( F_n \). Then either \( \{F_n\} \) converges to a point, or a subsequence of \( \{F_n\} \) converges smoothly to a smooth branched conformal immersion \( F_\infty : (\mathbb{T}^2, h_\infty) \to \mathbb{CP}^2 \), and the corresponding conformal structures of \( h_n \) converge to the conformal structure of \( h_\infty \).

The paper is organized as follows. In Sect. 2, we discuss some background in Lagrangian submanifolds, surface theory and the bubble tree convergence. In Sect. 3, we prove a $C^k$ estimates and a removable singularity theorem for branched conformal HSL immersions. We prove Theorem 1.1 in Sect. 4, where the bubble tree is constructed. In the last section, we derive Theorems 1.2, 1.3.

## 2 Background

### 2.1 Lagrangian immersions

Let \((M, \omega, \bar{g}, J)\) be a smooth Kähler manifold of complex dimension \( n \) where \( J \) is an integrable complex structure, \( \omega \) is a closed 2-form and \( \omega, \bar{g}, J \) satisfy

\[
\bar{g}(X, Y) = \bar{g}(JX, JY) \tag{2.1}
\]

and

\[
\omega(X, Y) = \bar{g}(JX, Y) \tag{2.2}
\]

for all tangent vectors \( X, Y \).
Let $S$ be an orientable real $n$ dimensional manifold. An immersion $F : S \to M$ is Lagrangian if $F^* \omega = 0$. By (2.2), this is equivalent to that $J$ maps the tangent space of $F$ to its normal space. A smooth vector field $X$ on $M$ is Lagrangian (resp. Hamiltonian) if the 1-form
\begin{equation}
\alpha_X := \iota_X \omega
\end{equation}
is closed (resp. exact). It follows from Cartan’s formula that if $X$ is Lagrangian and $\{ \psi_t : t \in (-\varepsilon, \varepsilon) \}$ is the one-parameter group of diffeomorphisms generated by $X$, then $\psi_t \circ F$ is also a Lagrangian immersion for each $t$. Using (2.2) we can verify that a vector field $X$ on $M$ is Hamiltonian if and only if
\begin{equation}
X = J \nabla f
\end{equation}
for some smooth function $f : M \to \mathbb{R}$.

A Lagrangian immersion $F : S \to M$ is called Hamiltonian stationary, or HSL for simplicity, if it is a critical point of the volume functional among all compactly supported Hamiltonian variations. By the first variation formula for volume and (2.4), that $S$ is HSL is equivalent to
\begin{equation}
\int_S \bar{g}(\vec{H}, J\nabla f) \, d\mu = 0, \quad \forall f \in C^\infty_c(M),
\end{equation}
where $\vec{H}$ is the mean curvature vector of the immersion and $d\mu$ is the volume element in the metric $g = F^* \bar{g}$, as demonstrated in [31].

For an immersion $F : S \to M$, define the mean curvature 1-form $\alpha := \alpha_{\vec{H}}$ on $S$ by
\begin{equation}
\alpha(Y) = \omega(\vec{H}, F_* Y)
\end{equation}
for all tangent vector $Y$ of $S$. Using (2.1), (2.2) and (2.6), when $F$ is a Lagrangian immersion, we have
\begin{align}
\bar{g}(\vec{H}, J\nabla f) &= -\bar{g}(J \vec{H}, \nabla f) \\
&= -\omega(\vec{H}, (\nabla f)^T) \\
&= -\alpha(\nabla f|_S) \\
&= -\langle \alpha, df|_S \rangle_g,
\end{align}
where $\nabla$ is the pullback connection $F^* \nabla$ and $(\nabla f)^T$ is the tangential part of $\nabla f$ along $F(S)$. Thus (2.5) is equivalent to
\begin{equation}
\int_S \langle \alpha, df \rangle_g \, d\mu = 0, \quad \forall f \in C^\infty_c(S).
\end{equation}
Thus the mean curvature 1-form $\alpha$ satisfies $d^* \alpha = 0$ when $F$ is a HSL immersion. On the other hand, it is proved in [13] that any Lagrangian immersion in a Kähler manifold $M$ satisfies $d\alpha = F^* \text{Rc}$, where $\text{Rc}$ is the Ricci 2-form of $(M, \omega, J, \bar{g})$. Hence the mean curvature 1-form of an HSL immersion satisfies an elliptic system
\begin{equation}
\begin{cases}
d\alpha = F^* \text{Rc}, \\
d^* \alpha = 0.
\end{cases}
\end{equation}
When $(M, \omega, \bar{g}, J)$ is Kähler-Einstein, it follows from (2.9) that $\alpha$ is a harmonic 1-form on $S$ since $F^* \text{Rc} = F^*(c\omega)$ vanishes on $S$ for $F$ is Lagrangian.
2.2 Basic surface theory

Let \((S, g)\) be a closed orientable real 2-dimensional Riemannian surface. Let \(g_S\) denote the genus of \(S\). By the uniformization theorem, there is a conformal diffeomorphism \(\phi : (\Sigma, h) \rightarrow (S, g)\), where

(a) when \(g_S = 0\), \(\Sigma\) is the two sphere \(S^2\) with the round metric \(h\),
(b) when \(g_S = 1\), \(\Sigma\) is the torus \(\mathbb{T}^2 := S^1 \times S^1\) and \(h\) is given by
\[
h = \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix},
\]
where \(\tau = \tau_1 + \sqrt{-1}\tau_2\) satisfies
\[-\frac{1}{2} \leq \tau_1 \leq \frac{1}{2}, \tau_2 > 0, \tau_1^2 + \tau_2^2 \geq 1 \text{ and } \tau_1 \geq 0 \text{ whenever } \tau_1^2 + \tau_2^2 = 1. \tag{2.11}\]
(c) when \(g_S \geq 2\), \(\Sigma\) is a closed orientable surface of genus \(g_S\) and \(h\) is a metric on \(\Sigma\) with constant Gauss curvature \(-1\).

Each of the metric described above will be called a model metric.

Given any immersion \(F : S \rightarrow M\), using the induced metric \(g\), there is a conformal diffeomorphism \(\phi : (\Sigma, h) \rightarrow (S, g)\). By taking \(F \circ \phi\), from now on we assume that \(F : (\Sigma, h) \rightarrow (M, \bar{g})\) is a conformal immersion from \(\Sigma\) with a model metric.

When studying the compactness of the space of HSL immersions, we will need to consider objects with singularities.

**Definition 2.1** Let \((\Sigma, h)\) be a Riemann surface. A smooth mapping \(F : \Sigma \rightarrow (M, \bar{g})\) is called a branched conformal immersion, if \(F^*\bar{g} = \lambda h\), where \(\lambda \geq 0\) and is zero only at finitely many points. The points in \(\Sigma\) where \(\lambda = 0\) are called the branch points of \(F\). The set of branch points is denoted \(\mathcal{B}\).

**Definition 2.2** A branched conformal immersion \(F\) from a Riemann surface \(\Sigma\) to a Kähler surface \((M, \omega, \bar{g}, J)\) is Lagrangian if \(F^*\omega = 0\). A branched conformal Lagrangian immersion is HSL if its mean curvature 1-form \(\alpha\) satisfies \(d^*\alpha = 0\) on \(\Sigma \setminus \mathcal{B}\) where \(g = F^*\bar{g}\).

Let \(\mathbb{D}(r) = \{z \in \mathbb{C} : |z| < r\}\) and \(\mathbb{D} = \mathbb{D}(1)\). Let \(\delta = dx^2 + dy^2\) be the standard metric on \(\mathbb{D}(r)\) and \(F : (\mathbb{D}(r), \delta) \rightarrow (M, \bar{g})\) be a branched conformal immersion. By the conformality,
\[
\bar{g} \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \right) = \bar{g} \left( \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y} \right), \quad \bar{g} \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right) = 0. \tag{2.12}\]
This implies that
\[
\lambda = \frac{1}{2} |\nabla F|_{\bar{g}}^2 := \frac{1}{2} \left( |\frac{\partial F}{\partial x}|_{\bar{g}}^2 + |\frac{\partial F}{\partial y}|_{\bar{g}}^2 \right) = |\frac{\partial F}{\partial x}|_{\bar{g}}^2 = |\frac{\partial F}{\partial y}|_{\bar{g}}^2
\]
and
\[
g := F^*\bar{g} = \frac{1}{2} |\nabla F|_{\bar{g}}^2 \delta, \quad g^{-1} = 2|\nabla F|_{\bar{g}}^{-2} \delta. \tag{2.13}\]

**Remark 1** For any branched conformal immersion \(F : (\Sigma, h) \rightarrow (M, \bar{g})\) and conformal diffeomorphism \(\theta : \mathbb{D}(r) \rightarrow \theta(\mathbb{D}(r)) \subset \Sigma\), (2.13) is applicable to the branched conformal immersion \(F \circ \theta\).
For any branched conformal immersion $F : \Sigma \to (M, \tilde{g})$, its Willmore energy is defined as

$$W(F) = \frac{1}{4} \int_{\Sigma} |H|^2 d\mu_{\tilde{g}}. \quad (2.14)$$

When the branched conformal immersion $F$ is Lagrangian, we also have

$$W(F) = \frac{1}{4} \int_{\Sigma} |\alpha|^2 d\mu_{\tilde{g}}. \quad (2.15)$$

### 2.3 Bubble tree convergence

In this subsection, we recall the definition of bubble tree convergence. First we recollect the definition of stratified surface ([6,9]).

**Definition 2.3** Let $(\Sigma, d)$ be a connected compact metric space. We call $\Sigma$ a stratified surface with singular set $P$ if $P \subset \Sigma$ is a finite set such that (i) $(\Sigma \setminus P, d)$ is a smooth Riemann surface without boundary (possibly disconnected) and $d$ is given by a smooth Riemannian metric $h$ on $\Sigma \setminus P$, and (ii) For each $p \in P$, there is $\delta > 0$ so that $B_{\delta}(p) \cap P = \{p\}$ and $B_{\delta}(0) \setminus \{p\}$ is a union of $m(p)$ topological disks with their centers deleted, where $1 < m(p) < \infty$, and on each punctured disk, the metric $h$ can be extended smoothly to the whole disk.

Next we recall the definition of bubble tree convergence ([6], see also [33]).

**Definition 2.4** Let $\{F_n : \Omega \to M\}$ be a sequence of smooth mappings to $M$. Let $\Sigma_{\infty}$ be a stratified surface. We say that $\{F_n\}$ converges to $F_{\infty} : \Sigma_{\infty} \to M$ in the sense of bubble tree if for each $n \in \mathbb{N}$, there are open sets $U_n \subset \Sigma$ and $V_n \subset \Sigma_{\infty}$ so that

1. $\Sigma_{\infty} \setminus \bigcup_n V_n = P$, and $\Sigma_{\infty} \setminus V_n$ is a union of topological disks with finitely many small disks removed.
2. Each $\Sigma \setminus U_n$ is a smooth surface with boundary, possibly disconnected. Moreover, $\{F_n(\Sigma \setminus U_n)\}$ converges to $F_{\infty}(P)$ in Hausdorff distance.
3. There is a sequence of diffeomorphisms $\varphi_n : U_n \to V_n$, such that $\{F_n \circ \varphi_n^{-1}\}$ converges to $F_{\infty}|_{\Omega}$ smoothly in any $\Omega \subset \subset \Sigma_{\infty} \setminus P$.

### 3 Small energy regularity and removable singularity

Let $(M, \omega, \tilde{g}, J)$ be a complete Kähler surface. By the Nash embedding theorem, we may assume that $(M, \tilde{g})$ is isometrically embedded into an Euclidean space $\mathbb{R}^N$. For any immersion $F : V \to M \subset \mathbb{R}^N$ defined in a local coordinates $(x^1, \ldots, x^n)$ and $g = (g_{ij})$, the equation $\tilde{H} = \text{tr} \nabla dF$ is locally given by

$$\Delta_{\tilde{g}} F - g^{ij} A^M \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \tilde{H}, \quad (3.1)$$

where $\Delta_{\tilde{g}} F = (\Delta_{\tilde{g}} F^1, \ldots, \Delta_{\tilde{g}} F^N)$ and $A^M$ is the second fundamental form of $M$ in $\mathbb{R}^N$.

For a branched conformal immersion $F : \mathbb{D}(r) \to M$, by (2.13) we have

$$\Delta_{\tilde{g}} F = 2 |\nabla F|^{-2} \Delta F, \quad (3.2)$$

Diagram
where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. When $F$ is also Lagrangian, the normal bundle is spanned by $\left\{ J \frac{\partial F}{\partial x}, J \frac{\partial F}{\partial y} \right\}$. Hence we can write

$$\vec{H} = H^x J(F) \frac{\partial F}{\partial x} + H^y J(F) \frac{\partial F}{\partial y}$$

for some functions $H^x, H^y : \mathbb{D}(r) \to \mathbb{R}$. If we write $\alpha = \alpha_x dx + \alpha_y dy$, then by (2.13)

$$\alpha_x = \frac{1}{2} |\nabla F|^2_{\bar{g}} H^x, \quad \alpha_y = \frac{1}{2} |\nabla F|^2_{\bar{g}} H^y.$$  

(3.3)

Together with (3.1) and (3.2), we have

$$\Delta F = \alpha_x J(F) \frac{\partial F}{\partial x} + \alpha_y J(F) \frac{\partial F}{\partial y} + A^M \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial x} \right) + A^M \left( \frac{\partial F}{\partial y}, \frac{\partial F}{\partial y} \right).$$  

(3.4)

Next, we recall the $\varepsilon$-regularity result in [6, Proposition 2.1]). For any subset $U \subset \mathbb{D}(r)$, denote

$$W_{\mathbb{R}^N}(F, U) = \frac{1}{4} \int_U |\vec{H}_{\mathbb{R}^N}|^2 d\mu$$  

(3.5)

here $\vec{H}_{\mathbb{R}^N}$ is the mean curvature vector of $F : \mathbb{D}(r) \to M \hookrightarrow \mathbb{R}^N$.

**Proposition 3.1** For any $p \in (1, 2)$, $R > r > 0$ and $N \in \mathbb{N}$, there are constants $\varepsilon_0 > 0$ and $C > 0$ depending on $p, R, r, N$, such that if $F : \mathbb{D}(R) \to \mathbb{R}^N$ is a branched conformal immersion into $\mathbb{R}^N$ with

$$W_{\mathbb{R}^N}(F, \mathbb{D}(R)) < \varepsilon_0^2.$$  

(3.6)

then

$$\|\nabla F\|_{W^{1,p}(\mathbb{D}(r))} \leq C \|\nabla F\|_{L^2(\mathbb{D}(r))}.$$  

(3.7)

Since $(M, \bar{g})$ is isometrically embedded into $\mathbb{R}^N$, Proposition 3.1 is applicable when the image of $F$ is contained in a bounded region $K$ in $M$ by observing

$$\vec{H}_{\mathbb{R}^N} = \vec{H} + A^M(e_1, e_1) + A^M(e_2, e_2),$$

where $\{e_1, e_2\}$ is some orthonormal basis of $F_* T_r \mathbb{D}(r)$.

Next we show that the small energy condition (3.6) is sufficient to control all higher derivatives of a branched HSL immersion:

**Theorem 3.1** Let $F : \mathbb{D}(1) \to M$ be a smooth branched conformal HSL immersion into a complete Kähler manifold $M$, which is isometrically embedded in $\mathbb{R}^N$. Assume that the image of $F$ lies in a compact set $K$ and (3.6) is satisfied for $F$ and $p = 12/7 < 2$. Then for any $k \in \mathbb{N}$, there is $C_k$ depending only on $k, (M, \bar{g})$ and $K$ such that

$$\|F\|_{C^k, \beta(\mathbb{D}(1/2))} \leq C_k \left( \int_{\mathbb{D}(1)} |\nabla F|^2 d\mu + \int_{\mathbb{D}(1)} |\vec{H}|^2 d\mu + 1 \right)^{\frac{1}{2}}.$$  

(3.8)

for all $k \in \mathbb{N}$. Here $\beta \in (0, 1)$ is fixed.
Proof We will assume that $\alpha$ is smoothly defined across the branched points (This will be proved later, see Theorem 3.2). We can write (3.4) and (2.9) as

$$\Delta F = \alpha \ast J(F) \ast \nabla F + A^M(F) \ast \nabla F \ast \nabla F.$$  \hspace{1cm} (3.9)

\[
\begin{cases}
\, d\alpha = \text{Rc}(F) \ast \nabla F \ast \nabla F \\
\, d^*\alpha = 0.
\end{cases} \hspace{1cm} (3.10)
\]

We have used $A \ast B$ to abbreviate a linear combination of components of quantities $A, B$ respectively. Note that we have $\nabla F \ast \nabla F$ on the RHS of (3.10) since $\text{Rc}$ is a 2-form on $M$. Thus the pair $(F, \alpha)$ satisfies an elliptic system. We will show that (3.9) and (3.10) are sufficient for a bootstrapping process.

In the sequel, we will use $\| \cdot \|_{k,p}$ and $\| \cdot \|_p$ to denote the Sobolev norms and $L^p$-norms respectively. It is also understood that in the following inequalities, the smaller terms denote norms evaluated at a smaller open sets (which still strictly contain $\mathbb{D}(1/2)$), since we are applying Sobolev or the interior Schauder estimates. Note also that we can apply Proposition 3.1 for any $1 < p \leq 12/7$ by Hölder’s inequality.

First we recall the Sobolev inequality [16, (7.26)]: if $p < 2$, then

$$\| u \|_{2p/(2-p)} \leq C(p) \| u \|_{1,p}.$$  \hspace{1cm} (3.11)

Note that (3.11) together with Hölder’s inequality implies that for any $q > 1$,

$$\| u \|_{q} \leq C(q) \| u \|_{1,2}.$$  \hspace{1cm} (3.12)

By Proposition 3.1, there is $C$ so that

$$\| \nabla F \|_{1,4/3} \leq C \| \nabla F \|_2.$$  

By (3.11) with $p = 4/3$,

$$\| \nabla F \|_{4} \leq C \| \nabla F \|_2$$

therefore the RHS of (3.10) satisfies

$$\| \text{Rc}(F) \ast \nabla F \ast \nabla F \|_2 \leq C \| \nabla F \|_2.$$  

By the a priori estimates [15, Theorem 6.28] applied to (3.10),

$$\| \alpha \|_{1,2} \leq C(\| \nabla F \|_2 + \| \alpha \|_2).$$

Using (3.12) with $q = 12$,

$$\| \alpha \|_{12} \leq C(\| \nabla F \|_2 + \| \alpha \|_2).$$  \hspace{1cm} (3.13)

On the other hand, by Proposition 3.1 with $p = 12/7$,

$$\| \nabla F \|_{1,12/7} \leq C \| \nabla F \|_2.$$  

and (3.11) with $p = 12/7$,

$$\| \nabla F \|_{12} \leq C \| \nabla F \|_2.$$  \hspace{1cm} (3.14)

Hence (3.13), (3.14) together with (3.9) imply that

$$\| \Delta F \|_6 \leq C(\| \nabla F \|_{12} \| \alpha \|_{12} + \| \nabla F \|_{12}^2)
\leq C(\| \nabla F \|_2^2 + \| \alpha \|_2^2).$$  \hspace{1cm} (3.15)
The $L^p$-estimates [16, Theorem 9.11] yield
\[
\|F\|_{2,6} \leq C \left( \|F\|_6 + \|\nabla F\|_2^2 + \|\alpha\|_2^2 \right) \leq C \left( 1 + \|\nabla F\|_2^2 + \|\alpha\|_2^2 \right).
\]
(3.16)

Then
\[
\|Rc \ast \nabla F \ast \nabla F\|_{1,2} \leq C \left( 1 + \|\nabla F\|_2^2 + \|\alpha\|_2^2 \right)^2
\]
and it follows
\[
\|\alpha\|_{2,2} \leq C \left( 1 + \|\nabla F\|_2^2 + \|\alpha\|_2^2 \right)^2
\]
(3.17)
by the a priori estimates [15, Theorem 6.28] applied to (3.10).

Using (3.16), (3.17) and the Sobolev embedding theorem, we can bound the $C^{0,\beta}$-norm of both $\nabla F$ and $\alpha$ for some $\beta \in (0, 1)$. Applying the interior Schauder estimates [16, Corollary 6.3] to (3.9), (3.10), the corollary is proved. \qed

Next, we discuss removability of a point singularity of an HSL immersion from a punctured disk. The result is similar to [8, Proposition 3.1].

**Theorem 3.2** Let $F : \mathbb{D} \setminus \{0\} \to M$ be a smooth branched conformal HSL immersion into a complete Kähler manifold $(M, \omega, \bar{g}, J)$ with finite area, finite Willmore energy and finitely many branch points. Then $F$ and $\alpha$ can be smoothly extended to $\mathbb{D}$.

**Proof** By assumption, $F$ has only finitely many branch points. Shrinking and translating $\mathbb{D}$ if necessary, we can assume that $F$ has no branch points in $\mathbb{D} \setminus \{0\}$, i.e., 0 is the only possible branched point. Let $(x, y)$ be the local coordinates of $\mathbb{D}$. Note that $\alpha$ is smooth on the punctured disk $\mathbb{D} \setminus \{0\}$ and satisfies
\[
\begin{cases}
  d\alpha = F^*Rc \\
  \text{div} \alpha = 0.
\end{cases}
\]
(3.18)
on $\mathbb{D} \setminus \{0\}$ as $d^* = \frac{1}{2} \text{div}$. As the Willmore energy of $F$ is finite by assumption, using a cutoff function argument as in [34, p.41] we see (3.18) is satisfied in the sense of distribution on the whole disk $\mathbb{D}$.

Next we use the bootstrapping argument as in the proof of Theorem 3.1 to show that $F, \alpha$ are smoothly defined at 0. First of all, since $F$ has finite area and Willmore energy, by Proposition 2.4 in [6], $F$ can be extended to $\mathbb{D}$ so that $F \in W^{2,p}(\mathbb{D})$ for $1 < p < 4/3$. By (3.11), we have
\[
\nabla F \in L^s_{loc}(\mathbb{D}), \quad \text{for all } 1 < s < 4.
\]
(3.19)
We then proceed in two steps:

**Step 1** $\alpha, F \in W^{1,q}_{loc}(\mathbb{D})$ for all $q > 1$: By (3.19), $F^*Rc \in L^s_{loc}(\mathbb{D})$ for all $1 < s < 2$. Then by [20, Theorem 7.9.7] applied to (3.18), $\alpha \in W^{1,s}_{loc}(\mathbb{D})$ for all $1 < s < 2$. Together with (3.11) we have $\alpha \in L^s_{loc}(\mathbb{D})$ for all $q > 1$. Using (3.19), the RHS of (3.9) is in $L^s_{loc}(\mathbb{D})$ for all $1 < s < 2$. Hence $F \in W^{2,s}_{loc}(\mathbb{D})$ for all $1 < s < 2$ by [16, Lemma 9.16]. With (3.11) this implies $F \in W^{1,q}_{loc}(\mathbb{D})$ for all $q > 1$, so the RHS of (3.18) is in $L^q_{loc}(\mathbb{D})$ for all $q > 1$. By [20, Theorem 7.9.7],
\[
\alpha \in W^{1,q}_{loc}(\mathbb{D}), \quad \text{for all } q > 1.
\]
(3.20)
Step 2 $\alpha, F$ are smooth at 0: By Step 1, the RHS of (3.9) is in $L^q_{loc}(\mathbb{D})$ for all $q > 1$. Again the $L^p$-theory [16, Lemma 9.16] implies that $F \in W^{2,q}_{loc}(\mathbb{D})$ for all $q > 1$. Together with (3.20), the RHS of (3.9) is in $W^{1,q}_{loc}(\mathbb{D})$ for all $q > 1$. Thus $F \in W^{3,q}_{loc}(\mathbb{D})$ for all $q > 1$ by [16, Theorem 9.19]. Using (3.18), this implies $\alpha \in W^{3,q}_{loc}(\mathbb{D})$ for all $q > 1$. Now one can argue similarly to see that

$$F, \alpha \in W^{k, q}_{loc}(\mathbb{D}), \quad \text{for all } k \in \mathbb{N}, q > 1. \tag{3.21}$$

Thus $F, \alpha$ can be both smoothly extended across $0 \in \mathbb{D}$. \qed

An immediate consequence is the following rigidity:

Corollary 3.1 Let $F : S^2 \to M$ be a smooth branched conformal HSL immersion to a Kähler-Einstein surface with finite Willmore energy. Then $F$ is a branched minimal immersion. When $M = \mathbb{C}^2$, there does not exists any branched conformal HSL sphere.

Proof Let $F : S^2 \to M$ be such an immersion. By Theorem 3.2, the mean curvature 1-form $\alpha$ extends smoothly to a smooth 1-form on $S^2$, and it is harmonic since $M$ is Kähler-Einstein. By the Hodge theorem, since $S^2$ is simply connected $\alpha$ is zero and thus $F : S^2 \to M$ is a branched minimal immersion. The last statement is true since in $\mathbb{C}^2$ there is no closed branched conformal minimal immersion. \qed

4 Bubble tree convergence: Proof of Theorem 1.1

In this section we prove Theorem 1.1.

Proof of Theorem 1.1 Let $\{F_n\}$ be a sequence as described in Theorem 1.1. We isometrically embed $(M, \omega, J, g)$ into $\mathbb{R}^N$. Thus when treated as immersions to $\mathbb{R}^N$ the areas of $F_n$ and the Willmore energies (in $\mathbb{R}^N$) are uniformly bounded as $F_n(\Sigma)$ all lie in a fixed compact set $K$ in $M$. Thus Theorem 1 in [6] is applicable. In particular, there is a stratified surface $\Sigma_{\infty}$ and a branched conformal immersion $F_{\infty} : \Sigma_{\infty} \to \mathbb{R}^N$ such that a subsequence of $\{F_n(\Sigma)\}$ converges in Hausdorff measure to $F_{\infty}(\Sigma_{\infty})$; consequently, the image of $F_{\infty}$ is in $M$.

Now we show that $\{F_n\}$ converges to $F_{\infty}$ in the sense of bubble tree as in Definition 2.4 and $F_{\infty}$ is a branched conformal HSL immersion on each component. Following [6] and supplementing further detailed construction of various domains which will be used in showing convergence of HSL immersions, we now divide the construction of the bubble tree and convergence into six steps.

Step 1 Principal component $\overline{\Sigma}_0$. First we discuss the convergence on the principal components. We consider only the case of high genus ($g_{\Sigma} \geq 2$) with the possible degeneration of conformal structures. The case for $g_{\Sigma} = 0, 1$ are easier and details can be found in [6]. Let $h_n$ be the Riemannian metric on $\Sigma$ conformal to $F_n^* \hat{g}$ and with constant Gauss curvature $-1$. We closely follow the Hyperbolic case in [6, Section 2.5].

By Proposition 5.1 in [21], there exists a nodal surface $\Sigma_0$ with nodal points $N = \{a_1, \ldots, a_m\}$ and a maximal collection $\Gamma_n = \{\gamma_n^1, \ldots, \gamma_n^m\}$ of pairwise disjoint, simple closed geodesics in $(\Sigma, h_n)$. The geodesics $\gamma_n^j$ satisfy $\ell_n^j := \text{Length}(\gamma_n^j) \to 0$ as $n \to \infty$. Moreover, by passing to a subsequence the followings hold:

1. There are continuous maps $\varphi_n^0 : \Sigma \to \Sigma_0$ for $n \in \mathbb{N}$ such that $\varphi_n^0 : \Sigma \setminus \Gamma_n \to \Sigma_0 \setminus N$ are diffeomorphic and $\varphi_n^0(\gamma_n^j) = a_j$ for $j = 1, \ldots, m$. Springer
(2) For the inverse diffeomorphisms \( \psi_n^p : \Sigma_0 \setminus \mathcal{N} \rightarrow \Sigma \setminus \Gamma_n \) of \( \varphi^p \), we have \( (\psi_n^p)^* h_n \rightarrow h_0 \) locally smoothly in \( \Sigma_0 \setminus \mathcal{N} \).

Here \( h_0 \) is a hyperbolic structure on \( \Sigma_0 \); that is, a smooth complete metric on \( \Sigma \setminus \mathcal{N} \) with finite volume and Gauss curvature \(-1\).

Consider the sequence of mappings
\[
\widetilde{F}_n := F_n \circ \psi_n^p : \Sigma_0 \setminus \mathcal{N} \rightarrow M. \tag{4.1}
\]

Let \( z \in \Sigma_0 \setminus \mathcal{N} \) be fixed. By Lemma 1.2 in [14], since \( \{ (\psi_n^p)^* h_n \} \) converges locally smoothly to \( h_0 \) in \( \Sigma_0 \setminus \mathcal{N} \), there exist neighborhoods \( D^n_z \), \( D_0^0 \) in \( \Sigma_0 \setminus \mathcal{N} \) of \( z \) and conformal diffeomorphisms
\[
\theta_n : \mathbb{D} \rightarrow D^n_z \quad \text{for } (\psi_n^p)^* h_n \text{ on } D^n_z \text{ such that } \theta_n(0) = z \text{ and } \{ \theta_n \} \text{ converges smoothly to a conformal diffeomorphism }
\]
\[
\theta_\infty : \mathbb{D} \rightarrow D_0^0.
\]

We may further assume that the geodesic disk \( D_0^0(r_0) \) in \( \Sigma_0 \setminus \mathcal{N} \) in the metric \( h_0 \) for some \( r_0 > 0 \) in contained in all \( D^n_z \) for large \( n \). Define
\[
\widetilde{F}_n := \widetilde{F}_n \circ \theta_n : \mathbb{D} \rightarrow M. \tag{4.2}
\]

To summarize,
\[
\begin{array}{c}
\xymatrix{ \mathbb{D} \ar[r]^{\theta_n} & \Sigma_0 \setminus \mathcal{N} \ar[r]^{\psi_n^p} & \Sigma \setminus \Gamma_n \ar[r]^{F_n} & M. }
\end{array}
\]

Let \( C((\widetilde{F}_n)) \) be the blowup set of the sequence \( \{ \widetilde{F}_n \} \) in \( \Sigma_0 \setminus \mathcal{N} \) defined as
\[
C((\widetilde{F}_n)) := \left\{ y \in \Sigma_0 \setminus \mathcal{N} : \lim_{r \to 0} \liminf_{n \to \infty} W_{\mathbb{R}^N} (\widetilde{F}_n, D^n_y (r)) > \varepsilon_2^2 \right\}, \tag{4.3}
\]
where \( D^n_y (r) \) is the disk centered at \( y \) of radius \( r \) in the metric \( h_0 \) and \( \varepsilon_2 < \varepsilon_0 \) is given as in the Decay estimate [6, Proposition 2.3] which is used for the bubble tree construction in [6] (Fig. 1).\(^1\)

The principal component of the bubble tree is constructed away from the blowup set as follows. Assume \( z \notin C((\widetilde{F}_n)) \). Then there is \( r > 0, \ell \in \mathbb{N} \) so that
\[
D^n_z (r) \subset \Sigma_0 \setminus \mathcal{N}, \quad W_{\mathbb{R}^N} (\widetilde{F}_n, D^n_z (r)) < \varepsilon_0^2, \quad \forall n \geq \ell.
\]

Thus \( \{ \widetilde{F}_n \} \) is a sequence of branched conformal HSL immersions \( \mathbb{D} \rightarrow M \) with
\[
W_{\mathbb{R}^N} (\widetilde{F}_n, \mathbb{D}) < \varepsilon_0^2, \quad \forall n \geq \ell
\]
due to the conformal invariance of the Willmore energy. By Theorem 3.1, for each \( k \in \mathbb{N} \), there is \( C_k > 0 \) so that
\[
\| \widetilde{F}_n \|_{C^k,\mathbb{R}^2(\mathbb{D})(1/2))} \leq C_k. \tag{4.4}
\]

\(^1\) In the above illustration, \( \Sigma \) is a genus two surface, the nodal surface \( \Sigma_0 \) has two nodal points \( \mathcal{N} = \{ a_1, a_2 \} \), the mapping \( \varphi^p : \Sigma \rightarrow \Sigma_0 \) maps the two geodesics \( y_{a_1}^n, y_{a_2}^n \) on \((\Sigma, h_n)\) to \( a_1, a_2 \) respectively, and \( C((\widetilde{F}_n)) = \{ z \} \).
Fig. 1 The principal component

Hence a subsequence of \( \{ \hat{F}_n \} \) converges smoothly in \( \mathbb{D}(1/2) \) to some \( \hat{F}_\infty : \mathbb{D}(1/2) \to M \) which satisfies

\[
\hat{F}_\infty^* \hat{g} = \frac{1}{2} |\nabla \hat{F}_\infty|^2 \delta \quad \text{and} \quad \hat{F}_\infty^* \omega = 0. \tag{4.5}
\]

Hence \( \hat{F}_\infty \) is a branched conformal Lagrangian immersion if it is non-constant.

Let \( \hat{\alpha}_n \) be the mean curvature 1-form of \( \hat{F}_n \). Using (4.4) and (3.10), we have

\[
\| \hat{\alpha}_n \|_{W^{k,2}(\mathbb{D}(1/2))} \leq C_k, \quad \text{for all} \ n \geq \ell, \ k \in \mathbb{N}.
\]

Thus a subsequence of \( \{ \hat{\alpha}_n \} \) converges smoothly to a 1-form \( \hat{\alpha}_\infty \) on \( \mathbb{D}(1/2) \). Note that \( (\hat{F}_n, \hat{\alpha}_n) \) satisfies (3.4) and (2.9) for all \( n \). Taking \( n \to \infty \), we have

\[
\Delta \hat{F}_\infty = (\alpha_\infty)_x J(\hat{F}_\infty) \frac{\partial \hat{F}_\infty}{\partial x} + (\alpha_\infty)_y J(\hat{F}_\infty) \frac{\partial \hat{F}_\infty}{\partial y} + A^M \left( \frac{\partial \hat{F}_\infty}{\partial x}, \frac{\partial \hat{F}_\infty}{\partial y} \right) + A^M \left( \frac{\partial \hat{F}_\infty}{\partial y}, \frac{\partial \hat{F}_\infty}{\partial x} \right) \tag{4.6}
\]

and

\[
\begin{cases}
\ d\alpha_\infty = \hat{F}_\infty^* Rc \\
\ d^*\alpha_\infty = 0.
\end{cases} \tag{4.7}
\]

From (4.6) and (4.5), if \( \hat{F}_\infty \) is non-constant, we see that \( \alpha_\infty = \hat{F}_\infty^* \mu_\infty \omega \), where \( \vec{H}_\infty \) is the mean curvature vector of \( \hat{F}_\infty \). Then (4.7) implies that \( \hat{F}_\infty \) is a branched conformal HSL immersion.

Define \( F_\infty : \theta_\infty(\mathbb{D}(1/2)) \to M \) by \( F_\infty = \hat{F}_\infty \circ \theta_\infty^{-1} \). Then the convergence \( \theta_n \to \theta_\infty \) implies that \( \{ F_n \} \) converges smoothly to \( F_\infty \) in \( \theta_\infty(\mathbb{D}(1/2)) \). Since \( z \in \Sigma_0 \setminus (\mathcal{N} \cup C(\{ F_n \})) \) is arbitrary, by choosing a countable cover by open balls and picking a diagonal subsequence if necessary, there is a smooth mapping \( F_\infty : \Sigma_0 \setminus (\mathcal{N} \cup C(\{ F_n \})) \to M \) so that the sequence \( \{ F_n \} \) converges locally smoothly to \( F_\infty \).

From the construction of the nodal surface \( \Sigma_0 \), for each \( a \in \mathcal{N} \) and \( \delta \) small, \( B_\delta(\delta) \setminus \{ a \} \subset \Sigma_0 \setminus \mathcal{N} \) is a union of two punctured disks. For each punctured disk \( \mathbb{D}_+^*, \mathbb{D}_-^* \), we add the points...
Define

\[ \Sigma_0 = (\Sigma_0 \setminus \mathcal{N}) \cup \{a^+, a^- : a \in \mathcal{N}\}. \]  

(4.8)

As the set \( \Sigma_0 \setminus \mathcal{N} \) decomposes into finitely many connected components \( \{\Sigma_0^i\}_{i \in I} \) for some finite index set \( I \),

\[ \Sigma_0 = \bigcup_{i \in I} \Sigma_0^i \]  

(4.9)

where each \( \Sigma_0^i \) is a connected closed Riemann surface, and \( \Sigma_0^{i_1} \cap \Sigma_0^{i_2} \) is finite whenever \( i_1 \neq i_2 \).

For each \( i \in I \), \( F_\infty \) is defined in \( \Sigma_0^i \setminus (\{a^+, a^- : a \in \mathcal{N}\} \cup \mathcal{C}(\{F_n\})) \).

If \( F_\infty \) is constant in this set, then clearly \( F_\infty \) extends to a constant map on \( \Sigma_0^i \). If not, then \( F_\infty \) restricts to a branched conformal HSL immersion. By Theorem 3.2, \( F_\infty \) can be smoothly extended to a branched conformal HSL immersion \( \Sigma_0^i \). Since \( i \in I \) is arbitrary, \( F_\infty \) can be extended continuously to entire \( \Sigma_0 \) and it is smooth on each \( \Sigma_0^i \); for simplicity, we still denote \( F_\infty : \Sigma_0 \to M \) for the extended mapping.

**Step 2** The first level of bubbles at \( C(\{\tilde{F}_n\}) \). Let \( z \in C(\{\tilde{F}_n\}) \subset \Sigma_0 \setminus \mathcal{N} \). We now construct the first level of the bubble tree at \( z \). Let \( \theta_n, \tilde{F}_n \) and \( \hat{F}_n \) be defined as in Step 1. For each \( n \), let \( z_n \in \mathbb{D}, r_n > 0 \) with \( z_n \to 0, r_n \to 0 \) chosen as in [6, Section 2.3, Step 1]:

\[ W(\hat{F}_n, \mathbb{D}, z_n(r_n)) = \frac{\epsilon_2^2}{2}. \]  

(4.10)

Define \( \phi_{z,n} : \mathbb{S}^1 \times [0, T_n] \to M \) with \( T_n = -\ln r_n \) by

\[ \phi_{z,n}(\theta, t) = z_n + (e^{-t}, \theta). \]  

(4.11)

Recall [6, Lemma 2.7], there are numbers \( l = l(z) \) and \( d_n^0, \ldots, d_n^l \) so that

\[ 0 = d_n^0 < d_n^1 < \cdots < d_n^l = T_n, \]

\[ \lim_{n \to +\infty} (d_n^j - d_n^{j-1}) = +\infty, \]  

(4.12)

\[ W_{\mathbb{R}^N}(\hat{F}_n \circ \phi_{z,n}, \mathbb{S}^1 \times [d_n^j, d_n^j + 1]) \geq \epsilon_2^2, \quad j \neq 0, l \]

and

\[ \lim \liminf_{T \to +\infty} \lim_{k \to +\infty} \sup_{t \in [d_k^j + T, d_k^j - T]} W_{\mathbb{R}^N}(\hat{F}_n \circ \phi_{z,n}, \mathbb{S}^1 \times [t, t + 1]) \leq \epsilon_2^2, \quad j = 1, \ldots, l. \]  

(4.13)

Choose \( c_n^0, \ldots, c_n^l, e_n^0, \ldots, e_n^{l-1} \) so that

\[ d_n^i < e_n^i < c_n^{i+1} < d_n^{i+1} \]

\[ \lim_{n \to +\infty} (d_n^i - c_n^i) = \lim_{n \to +\infty} (e_n^i - d_n^i) = +\infty. \]  

(4.14)
Next we show that there is no loss of area in the region \((\widehat{F}_n \circ \phi_{z,n})(S^1 \times [e^i_n, e^{i+1}_n])\) when \(i = 0, \ldots, l - 1\):

\[
\lim_{n \to +\infty} \int_0^{2\pi} \int_{e^i_n}^{e^{i+1}_n} |\nabla(\widehat{F}_n \circ \phi_{z,n})|^2 \, dt \, d\theta = 0. \tag{4.15}
\]

By (4.13), we can apply [6, Proposition 2.6 (2)], that is

\[
\lim_{T \to +\infty} \lim_{n \to +\infty} \int_0^{2\pi} \int_{d^i_n + T}^{d^{i+1}_n - T} |\nabla(\widehat{F}_n \circ \phi_{z,n})|^2 \, dt \, d\theta = 0. \tag{4.16}
\]

Then for any \(\epsilon > 0\), there is \(T\) so that

\[
\lim_{n \to +\infty} \int_0^{2\pi} \int_{d^i_n + T}^{d^{i+1}_n - T} |\nabla(\widehat{F}_n \circ \phi_{z,n})|^2 \, dt \, d\theta < \epsilon/2 \tag{4.17}
\]

and hence

\[
\int_0^{2\pi} \int_{d^i_n + T}^{d^{i+1}_n - T} |\nabla(\widehat{F}_n \circ \phi_{z,n})|^2 \, dt \, d\theta < \epsilon \tag{4.18}
\]

for \(n\) large enough. By (4.14), we have \(d^i_n + T < e^i_n < e^{i+1}_n < d^{i+1}_n - T\) for \(n\) large, hence

\[
\int_0^{2\pi} \int_{e^i_n}^{e^{i+1}_n} |\nabla(\widehat{F}_n \circ \phi_{z,n})|^2 \, dt \, d\theta < \epsilon \tag{4.19}
\]

for \(n\) large enough and this proves (4.15).

Fix a conformal diffeomorphism \(\Phi : S^2 \setminus \{\pm 1\} \to S^1 \times \mathbb{R}\) with \(\lim_{z \to \pm 1} \Phi(z) = \pm \infty\), and let \(T_{r_0} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}\) be the translation \(T_{r_0}(\theta, r) = (\theta, r + r_0)\). For each \(i = 1, \ldots, l - 1\) define

\[
F^i_{z,n} : \Phi^{-1}(S^1 \times (c^i_n - d^i_n, e^i_n - d^i_n)) \to M, \tag{4.20}
\]

and

\[
F^i_{z,n} = \widehat{F}_n \circ \phi_{z,n} \circ T_{d^i_n} \circ \Phi. \tag{4.21}
\]

By (4.14), the domain of \(F^i_{z,n}\) exhausts \(S^2 \setminus \{\pm 1\}\) as \(n \to +\infty\). Let \(\Phi_0 : S^2 \setminus \{1\} \to \mathbb{R}^2\) be the stereographic projection from \(-1\). Define

\[
F^i_{z,n} : \Phi_0^{-1}(\mathbb{D})(n) \to M, \tag{4.22}
\]

\[
F^i_{z,n}(y) = \widehat{F}_n \left( z_n + \frac{1}{n e^{i_n}^2} \Phi_0(y) \right). \tag{4.23}
\]

So for each fixed \(i = 1, \ldots, l\), \(\{F^i_{z,n}\}\) is a sequence of branched conformal HSL immersions from a sequence of exhausting domains in a fixed Riemann sphere \(S^2_{z,i}\). Let \(C(\{F^i_{z,n}\})\) be the blowup set of the sequence \(\{F^i_{z,n}\}\):

\[
C(\{F^i_{z,n}\}) := \left\{ y \in S^2_{z,i} \setminus \{\pm 1\} : \lim_{r \to 0} \liminf_{n \to +\infty} W_{R^N}(F^i_{z,n}, D_y(r)) > \epsilon^2_2 \right\}. \tag{4.24}
\]

Using Theorems 3.1 and 3.2, a subsequence of \(\{F^i_{z,n}\}\) (without changing notation) converges locally smoothly in \(S^2_{z,i} \setminus (\{\pm 1\} \cup C(\{F^i_{z,n}\}))\) to a smooth mapping \(F^i_{z,\infty} : S^2 \to M\). Arguing
as in Step 1, $F_{z,\infty}^i$ is either constant or a branched conformal HSL immersion. Moreover, by [6, (2.14)], we have

$$F_{z,\infty}^i(1) = F_{z,\infty}^{i+1}(-1), \quad 1 \leq i \leq l - 1,$$

$$F_{z,\infty}^1(-1) = F_{\infty}(z). \quad (4.23)$$

This will be used in Step 4.

**Step 3** The first level of bubbles at $\mathcal{N}$. Let $a \in \mathcal{N}$. For each $n \in \mathbb{N}$, there is a simple closed geodesic $\gamma_n^a$ in $(\Sigma, h_n)$ so that its length $L(\gamma_n^a) \to 0$ and $\gamma_n^a \to a$ as $n \to \infty$. By the Collar Lemma ([38], see also [6, Lemma 2.9]), there is a collar neighborhood $\mathcal{C}^a_n \subset \Sigma$ containing $\gamma_n^a$ and a conformal diffeomorphism

$$\phi_{a,n} : \mathbb{S}^1 \times (-l_n, l_n) \to (\mathcal{C}^a_n, h_n) \quad (4.24)$$

with $l_n \to \infty$ as $n \to \infty$. By [6, Lemma 2.7], there is $\bar{l} = \bar{l}(a) \in \mathbb{N}$ so that

$$-l_n = \bar{d}_0^0 < \bar{d}_1^0 < \cdots < \bar{d}_n^j = l_n$$

$$\lim_{n \to +\infty} \left( \bar{d}_n^j - \bar{d}_n^{j-1} \right) = \infty. \quad (4.25)$$

Also (2.17), (2.18) in [6] are satisfied. Then we can choose $\tilde{c}_n^j, \tilde{e}_n^j$ which satisfy similar conditions satisfied by $c_n^j, \epsilon_n^j$ in Step 2. Note that for any $j = 1, \ldots, \bar{l} - 1$, as in Step 2, define

$$F_{a,n}^j : \Phi^{-1}(\mathbb{S}^1 \times (\tilde{c}_n^j - \bar{d}_n^j, \tilde{e}_n^j - \bar{d}_n^j)) \to M,$$

$$F_{a,n}^j = F_n \circ \phi_{a,n} \circ T_{\bar{d}_n^j} \circ \Phi. \quad (4.26)$$

For each fixed $j = 1, \ldots, \bar{l} - 1$, $\{F_{a,n}^j\}$ is a sequence of branched conformal HSL immersions from a fixed Riemann sphere $\mathbb{S}^2_{a,j}$. Also, the sequence $\{F_{a,n}\}$ subconverges locally smoothly in $\mathbb{S}^2_{a,j} \setminus (\{0,1\} \cup C(F_{a,n}))$ to a smooth mapping $F_{a,\infty}^j : \mathbb{S}^2_{a,j} \to M$, which is either constant or a branched conformal HSL immersion and we have

$$F_{a,\infty}^j(1) = F_{a,\infty}^{j+1}(-1), \quad 1 \leq j \leq \bar{l} - 2,$$

$$F_{a,\infty}^1(-1) = F_{\infty}(a^-), \quad (4.27)$$

$$F_{a,\infty}^1(-1) = F_{\infty}(a^+).$$

**Step 4** Attaching the first level of bubbles to $\Sigma_0$. Let $\Sigma_{L_1}$ be the topological space given by

$$\Sigma_{L_1} := \left( \Sigma_0 \cup \bigcup_{z \in C(\{\tilde{F}_n\})} \bigcup_{i=1}^{l(z)} \mathbb{S}^2_{z,i} \cup \bigcup_{a \in \mathcal{N}} \bigcup_{j=1}^{\bar{l}(a)-1} \mathbb{S}^2_{a,j} \right) / \sim, \quad (4.28)$$

where $\sim$ identifies

1. for each $z \in C(\{\tilde{F}_n\})$: $z$ with $-1 \in \mathbb{S}^2_{z,1}$, and $+1 \in \mathbb{S}^2_{z,i}$ with $-1 \in \mathbb{S}^2_{z,i+1}$ for $i = 1, \ldots, l(z) - 1$;
2. for each $a \in \mathcal{N}$: $a^-$ with $-1$ in $\mathbb{S}^2_{a,1}$, $a^+$ with $+1$ in $\mathbb{S}^2_{a,\bar{l}(a)-1}$, and $+1 \in \mathbb{S}^2_{a,j}$ with $-1 \in \mathbb{S}^2_{a,j+1}$ for $j = 1, \ldots, \bar{l}(a) - 2$ (Fig. 2).
Then $F_\infty$ can be extended to a continuous mapping on $\Sigma_{L_1}$, by setting

$F_\infty|_{\Sigma_{L_1}} = F_i\n \quad \forall z \in C((\overline{F_i}))$, \quad i = 1, \ldots, l(z),$

$F_\infty|_{\Sigma_{a,j}} = F_j\a, \quad \forall \a \in N$, \quad j = 1, \ldots, \tilde{l}(\a) - 1$. \hfill (4.29)

Moreover, for each $z, i$ (resp. $a, j$) and $n \in \mathbb{N}$, let $V^i_{z,n} \subset S^2_{z,i}$ (resp. $V^j_{a,n} \subset S^2_{a,j}$) be the domain of $F^i_{z,n}$ (resp $F^j_{a,n}$). Then $\{V^i_{z,n}\}_{i=1}^{l(z)}$, $\{V^j_{a,n}\}_{j=1}^{\tilde{l}(\a)}$ are pairwise disjoint open sets in $\Sigma_{L_1}$ and

$\bigcup_{n} V^i_{z,n} = S^2_{z,i}\setminus \{\pm 1\}$, \quad 1 \leq i \leq l(z) - 1,

$\bigcup_{n} V^l_{z,n} = S^2_{z,l}\setminus \{-1\}$, \quad l = l(z),

$\bigcup_{n} V^j_{a,n} = S^2_{a,j}\setminus \{\pm 1\}$, \quad 1 \leq j \leq \tilde{l}(\a) - 1$. \hfill (4.30)

For each $z \in C((\overline{F_i}))$, $i \in \{1, \ldots, l(z)\}$ and $n \in \mathbb{N}$, there is an open set $U^i_{z,n} \subset \Sigma$ and a diffeomorphism $\varphi^i_{z,n} : U^i_{z,n} \to V^i_{z,n}$ so that $F^i_{z,n} = F_n \circ (\varphi^i_{z,n})^{-1}$. In fact, when $i \neq l(z)$, we have

$\varphi^i_{z,n} = (\psi^p_n \circ \theta_n \circ \phi_{z,n} \circ T_{d^i_n} \circ \Phi)^{-1}$

by (4.1), (4.2) and (4.20). Similarly, for each $a \in N$, $j = 1, \ldots, \tilde{l}(a) - 1$, there is an open set $U^j_{a,n}$ in $\Sigma$ and a diffeomorphism $\varphi^j_{a,n} : U^j_{a,n} \to V^j_{a,n}$ so that $F^j_{a,n} = F_n \circ (\varphi^j_{a,n})^{-1}$ by (4.26). Lastly, define

$U^p_n = \Sigma \setminus \bigcup_{z \in C((\overline{F_i}))} \theta_n(\mathbb{D}_{z,n}(e^{-e_n^p})) \cup \bigcup_{a \in N} \overline{\phi_{a,n}(S^1 \times [c^0_a, c^{-e_n^p}_a])}$, \quad $V^p_n = \varphi^p_n(U^p_n)$. \hfill (4.31)
Let $U^0_n \subset \Sigma$, $V^1_n \subset \Sigma_L$ and $\varphi^{0,1}_n : U^0_n \to V^1_n$ be given by (Fig. 3)

\[
U^0_n = U^0_n \cup \bigcup_{i=1}^l U^i_{z,n} \cup \bigcup_{j=1}^{i-1} U^j_{a,n},
\]
\[
\varphi^{0,1}_n = \varphi^0_n \cup \bigcup_{z,i} \varphi^i_{z,n} \cup \bigcup_{a,j} \varphi^j_{a,n},
\]
\[
V^1_n = \varphi^{0,1}_n(U^0_n).
\]

**Step 5** Higher levels of bubbles. Note that $\Sigma_L$ decomposes into the principal component $\Sigma_0$ and the bubbling components $\{S^2_{z,i}\}, \{S^2_{a,j}\}$. On each component there is a fixed conformal structure, given by $h_0$ on $\Sigma_0$ and the round metric on each bubbling component. We call this a conformal structure on $\Sigma_L$ and is denoted $h^1$.

From Step 4, let the sequence $\{F^1_n : V^1_n \to M\}$ be given by $F^1_n := F_n \circ (\varphi^{0,1}_n)^{-1}$. Let $C_1 := C(F^1_n)$ be the blowup set of this new sequence. From Step 2 and Step 3,

\[
C_1 = \bigcup_{z,i} C(F^i_{z,n}) \cup \bigcup_{a,j} C(F^j_{a,n}).
\]

Moreover, by construction in Steps 1–3, $\{F^1_n\}$ converges locally smoothly to $F_\infty$ in $\Sigma_L \setminus (C_1 \cup P_1)$, where $P_1$ is the set of non-smooth points in $\Sigma_L$.

Now we repeat Steps 1–4 for the sequence $\{F^1_n\}$. Then the followings hold:

1. There is a stratified surface $\Sigma_{L_2}$ formed by attaching finitely many $S^2$’s to $\Sigma_{L_1}$ at $C_1$.
2. For each $n \in \mathbb{N}$, there are $U^1_n \subset V^1_n, V^2_n \subset \Sigma_{L_2}$ and a diffeomorphism $\varphi^{1,2}_n : U^1_n \to V^2_n$.
3. $\bigcup_n U^1_n = \Sigma_{L_1} \setminus (P_1 \cup C_1)$ and $\bigcup_n V^2_n = \Sigma_{L_2} \setminus P_2$.

---

**Fig. 3** Construction of mappings at the first level
(4) Identifying $\Sigma_{L_1} \subset \Sigma_{L_2}$, $F_\infty$ extends to a continuous mapping on $\Sigma_{L_2}$. When restricted to each component, $F_\infty$ is either constant or a branched conformal HSL immersion, and

(5) The sequence $\{F^2_n : V^2_n \to M\}$ defined by $F^2_n = F^1_n \circ (\varphi_n^{1,2})^{-1}$ for each $n \in \mathbb{N}$ converges locally smoothly to $F_\infty$ on $\Sigma_{L_2} \setminus (P_2 \cup C_2)$, where $C_2 = \mathcal{C}((F^2_n))$.

Indeed, the constructions in Steps 1–4 imply that there is $\delta_n \to 0$ so that

$$V^1_n \setminus \bigcup_{z_1 \in C_1} B_{\delta_n}(z_1) \subset U^1_n \tag{4.34}$$

and $\varphi_n^{1,2}$ can be chosen to be the identity map in this open set (after identifying $\Sigma_{L_1} \subset \Sigma_{L_2}$).

Now assume that $\Sigma_{L_k}, V^k_n, F^k_n : V^k_n \to M$ have been defined for some $k$ and all $n$. If the blowup set $C_k := \mathcal{C}((F^k_n))$ is nonempty, we repeat Steps 1–4 to construct $\Sigma_{L_{k+1}}$, the open sets $U^k_n \subset V^k_n, V^{k+1}_n \subset \Sigma_{L_{k+1}}$ and diffeomorphisms $\varphi^{k,k+1}_n : U^k_n \to V^{k+1}_n$; and define $F^{k+1}_n = F^k_n \circ (\varphi^{k,k+1}_n)^{-1}$.

Note that the above procedure must stop in finite steps: that is, there is $k_0$ so that the blowup set $C_{k_0} := \mathcal{C}((F^{k_0}_n))$ is empty. This is true from the construction in Step 2. If $z$ is in the blowup set $C_k$ for some $k$, we choose $z_n, r_n$ as in [6, (2.15)]. In particular, by (4.10), the outermost bubbles has no blowup point. In particular, if $C$ is the bound of the Willmore energy of $\{F^k_n\}$, then the number of elements in $C_{k+1}$ is less than $2C/e^2_2 - 1$ (Fig. 4).

---

**Fig. 4** A bubble tree of three levels
**Step 6** Collapsing ghost components. Let $k_0 \in \mathbb{N}$ such that $C_{k_0}$ is empty. Note $\Sigma_{L_{k_0}}$ is a union of $\Sigma_0$ and finitely many bubbling components indexed by some finite set $J = J(k_0)$. Then

$$\Sigma_{L_{k_0}} = \bigcup_{i \in I} \Sigma^i_0 \cup \bigcup_{j \in J} S^2_j,$$

where $\Sigma^i_0$ and $I$ are defined in (4.9). Moreover, $V^k_{i,n} \subset \Sigma_{L_{k_0}}$ decomposes into connected components $\{V^p_{i,n}\}_{i \in I}, \{V^j_{n}\}_{j \in J}$ so that $V^p_i \subset \Sigma^i_0$ for each $i \in I$ and $V^j_n \subset S^2_j$ for each $j \in J$. Define

$$\varphi^{k_0}_n = \varphi^{k_0-1}_n \circ \ldots \circ \varphi^1_n \circ \varphi^0_n,$$

$$U^p_{i,n} = (\varphi^{k_0}_n)^{-1} V^p_{i,n}, \quad i \in I,$$

$$U^j_n = (\varphi^{k_0}_n)^{-1} V^j_n, \quad j \in J.$$

When restricted to some $\Sigma^i_0$ and $S^2_j$, $F_\infty$ might be constant, in this case we call those components the ghost components (when the component is a bubble component, it is called a ghost bubble in [33]). In constructing the stratified surface, we delete the ghost components. To this end, define

$$I_0 = \{i \in I : F_\infty|_{\Sigma^i_0} \text{ is non-constant}\},$$

$$J_0 = \{j \in J : F_\infty|_{S^2_j} \text{ is non-constant}\}.$$

Let $\Sigma_\infty$ be defined by collapsing $\Sigma^i_0, i \notin I_0$ and $S^2_j, j \notin J_0$ in $\Sigma_{L_{k_0}}$. Let

$$\Pi : \Sigma_{L_{k_0}} \to \Sigma_\infty$$

be the projection. Lastly, define $U_n \subset \Sigma, V_n \subset \Sigma_\infty$ and $\varphi_n$ by

$$U_n = \bigcup_{i \in I_0} U^p_{i,n} \cup \bigcup_{j \in J_0} U^j_n,$$

$$V_n = \Pi \left( \bigcup_{i \in I} V^p_{i,n} \cup \bigcup_{j \in J} V^j_n \right),$$

$$\varphi_n = \Pi \circ \varphi^{k_0}_n.$$

Let $P$ be the set of non-smooth point of $\Sigma_\infty$. Then $P = \Pi(P_{k_0})$ and $\Sigma_\infty$ is a stratified surface and $U_n, V_n, \varphi_n$ satisfy (1)–(3) in Definition 2.4.

From Step 1 to Step 6, $\{F_n\}$ converges to $F_\infty$ in the sense of bubble tree. The area identity (1.1) follows from [6, Proposition 2.6 (2)]. By Step 6, $F_\infty$ is non-constant on each component. Thus on each component, $F_\infty$ is a branched conformal HSL immersion.

It remains to show (1.2). Recall that the sequence $\{F_n\}$ has uniformly bounded Willmore energies. By [6, Remark 3.3], the $L^2$-norms of the second fundamental form is also uniformly bounded. Then the bubble tree convergence implies

$$\int_{\Sigma_\infty} |A_\infty|^2 d\mu_\infty \leq \lim_{n \to \infty} \int_{\Sigma} |A_n|^2 d\mu_n < +\infty,$$

where $d\mu_\infty$ is the area element in the metric $g_\infty = F_\infty^* \bar{g}$. 

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Let $S$ be any component of $\Sigma_\infty$ and let $x \in F_\infty(S)$. Let $F_\infty^{-1}|_S(x) = \{y_1, \ldots, y_k\}$. By [24, Theorem 3.1] (see also [17]), we have
\[
\lim_{r \to 0} \frac{\mu_\infty(F_\infty^{-1}|_S(B^0_r(x)))}{\pi r^2} = \sum_{i=1}^k m_i, \tag{4.41}
\]
where $m_i$ is the branching order of $F_\infty$ at $y_i$. In particular, [7, Lemma 2.2] is applicable and thus $F_\infty(S)$ is a rectifiable integral 2-varifold with generalized mean curvature in $L^2$. From the proof of [7, Lemma 2.2] the generalized mean curvature equals the usual mean curvature vector $\vec{\mathbf{H}}_\infty$ away from the branch points.

Let $f \in C_\infty^\infty(M)$. Then from (2.7),
\[
\int_S \bar{g}(\vec{\mathbf{H}}_\infty, J\nabla f) d\mu_\infty = -\int_S \langle \alpha_\infty, d(f|_S) \rangle g_\infty d\mu_\infty = 0 \tag{4.42}
\]
since $d^*\alpha_\infty = 0$ in the sense of distribution [34, p.41]. Thus we establish (1.2) and conclude Theorem 1.1.

5 Hamiltonian stationary Lagrangian Tori in $\mathbb{C}^2$ and $\mathbb{C}P^2$

Every Riemannian torus $(T^2, h)$ is conformal to $\mathbb{C}/\Lambda$ with the Euclidean metric for some lattice
\[
\Lambda = \text{span}_\mathbb{Z} \left\{ 1, \tau_1 + \tau_2 \sqrt{-1} \right\},
\]
where $\tau = \tau_1 + \tau_2 \sqrt{-1}$ satisfies (2.11). When $(M, \omega)$ is Kähler-Einstein and $F : \mathbb{C}/\Lambda \to M$ is a branched conformal HSL immersion, the mean curvature 1-form $\alpha$ is harmonic on $\mathbb{C}/\Lambda$ equipped with the flat metric descended from the Euclidean metric on $\mathbb{C}$. It is easy to check that every harmonic 1-form on the torus is constant, i.e.
\[
\alpha = \alpha_x dx + \alpha_y dy
\]
for some constants $\alpha_x, \alpha_y$, where $dx$ and $dy$ are globally defined 1-forms on $\mathbb{C}/\Lambda$. Let $\vec{\alpha} = (\alpha_x, \alpha_y)$. With this identification,
\[
\mathcal{W}(F) = \frac{1}{4} \int_{\mathbb{C}/\Lambda} |\alpha|^2 dxdy = \frac{1}{4} |\vec{\alpha}|^2 A(\mathbb{C}/\Lambda), \tag{5.1}
\]
where $A(\mathbb{C}/\Lambda) = \tau_2$ is the Euclidean area of $\mathbb{C}/\Lambda$ (cf. [28]).

**Proof of Theorem 1.2** Assume that the sequence $\{F_n\}$ does not converge to a point. Using that $0 \in F_n(T^2)$ and Simon’s diameter estimates [35], there is $R > 0$ so that $F_n(T^2) \subset B_0(R)$ for all $n \in \mathbb{N}$. Thus we can apply Theorem 1.1 to extract a subsequence of $\{F_n\}$ converging in the sense of bubble tree to a stratified surface $F_\infty : \Sigma_\infty \to \mathbb{C}^2$. Note that by construction $F_\infty$ is a branched conformal HSL immersion when restricted on each component. In particular, there can not be any bubbling component by Corollary 3.1.

Next we show that the sequence of conformal structures $\{h_n\}$ does not degenerate. Since we assume that $\{F_n\}$ does not converge to a point, $F_\infty$ is non-constant. Arguing by contradiction, if the sequence of conformal structures $\{h_n\}$ degenerates, the principal component $\Sigma_0$ is a union of 2-spheres. Again by Corollary 3.1, this is impossible.

From the above discussion, we have shown that $F_\infty$ is a branched conformal HSL immersed torus and $h_n \to h_\infty$ as $n \to \infty$. It remains to show the smooth convergence.
(Note that, unlike the case for harmonic maps [33], the smooth convergence does not follow from the absent of non-trivial bubbles).

Let \( \alpha_n = \alpha H^k = \alpha^x dx + \alpha^y dy \) be the corresponding mean curvature 1-forms and \( \tilde{\alpha}_n = (\alpha^x_n, \alpha^y_n) \). The Willmore energies are uniformly bounded above, by (5.1) we have

\[
|\tilde{\alpha}_n|^2 \leq \frac{C}{\tau_2^2}, \quad \text{for all } n \in \mathbb{N}.
\]

From (2.11) we have \( \tau_2 \geq \sqrt{\frac{3}{2}} \). Thus

\[
|\tilde{\alpha}_n| \leq C, \quad \text{for all } n \in \mathbb{N}.
\]

In particular, for any set \( U \subset \mathbb{T}^2 \) we have

\[
\int_U |\tilde{H}_n|^2 d\mu = |\tilde{\alpha}_n|^2 \int_U dxdy \leq C A_n(U),
\]

where \( A_n(U) \) is the area of \( U \) in \( (\mathbb{T}^2, h_n) \). Since \( \{h_n\} \) converges smoothly to \( h_\infty \), from (5.3) there is no Willmore energy concentration. Thus we can cover \( \mathbb{T} \) by finitely open balls such that on each ball, we have a uniform bound on \( \|F_n\|_{C^k, \delta(C/A_n)} \) for each \( k \in \mathbb{N} \) by Theorem 3.1. We can therefore extract a smooth convergent subsequence. \( \square \)

Next, we consider \( M = \mathbb{CP}^2 \) endowed with the Fubini-Study metric.

**Proof of Theorem 1.3** We assume that \( \{F_n\} \) does not converge to a point. By the assumption of Theorem 1.3, \( \{F_n\} \) has uniformly bounded areas and Willmore energies. By Theorem 1.1, a subsequence of \( \{F_n\} \) converges to \( F_\infty \) in the sense of bubble tree. By the area identity (1.1), we have

\[
\text{Area}(F_\infty) < 2\text{Area}(\mathbb{RP}^2).
\]

Next we argue that there is no nontrivial bubble at the limit. By Theorem 1.1, if \( f: \mathbb{S}^2 \to \mathbb{CP}^2 \) is one of the nontrivial bubbles, then it is a branched conformal HSL 2-sphere. By Corollary 3.1, \( f: \mathbb{S}^2 \to \mathbb{CP}^2 \) is a branched conformal minimal Lagrangian immersion. A theorem of Yau [37] asserts that \( f \) is totally geodesics. Note that the aforementioned theorem in [37] is proved for immersions. In general, one can use the argument in [12]: For any branched conformal minimal immersion into a Kähler manifold of constant holomorphic sectional curvature, the cubic differential

\[
C = C(z) \, dz^3, \quad C(z) = 4\omega(\nabla_{\partial_z} \partial_{\bar{z}}, \partial_{\bar{z}})
\]

is holomorphic (In [12] they only consider immersed surface, but the cubic form \( C \) is clearly smooth even when there are branch points). However, there is no nontrivial holomorphic cubic differential on \( \mathbb{S}^2 \) so \( C \) is identically zero. This implies that the second fundamental form \( A \) is identically zero, and hence \( f \) is totally geodesic. The fact that \( f \) is Lagrangian implies that \( f: \mathbb{S}^2 \to \mathbb{CP}^2 \) is a branched cover of the totally geodesic \( \mathbb{RP}^2 \) in \( \mathbb{CP}^2 \).

In particular, the degree \( d \) of \( f \) is at least two and

\[
\text{Area}(f) = d \, \text{Area}(\mathbb{RP}^2) \geq 2 \, \text{Area}(\mathbb{RP}^2).
\]

But this contradicts (5.4). Thus, there cannot be any nontrivial bubbles.

Finally, arguments similar to that in the proof of Theorem 1.2 assert non-degeneracy of conformal structure \( \{h_n\} \) and smooth convergence of a subsequence of \( \{F_n\} \) to \( F_\infty \). \( \square \)
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