Designing Circle Swimmers: Principles and Strategies

Zhiyu Cao,¹ Huijun Jiang,¹ and Zhonghuai Hou¹,²

Department of Chemical Physics & Hefei National Laboratory for Physical Sciences at Microscales, iChEM, University of Science and Technology of China, Hefei, Anhui 230026, China

(Dated: 22 December 2021)

Various microswimmers move along circles rather than straight lines due to their swimming mechanisms, body shapes or hydrodynamic effects. Here, we adopt the concepts of stochastic thermodynamics to analyze circle swimmers confined in a two-dimensional plane, and study the trade-off relations between various physical quantities such as precision, energy cost and rotational speed. Based on these findings, we predict principles and strategies for designing microswimmers of special optimized functions under limited energy resource conditions, which will bring new experimental inspiration for designing smart motors.

PACS numbers: 05.40.-a, 05.70.Ln, 02.50.Ey

I. INTRODUCTION

Microswimmers are intrinsically far away from equilibrium by continuously transferring internal or external energy into their mechanical motion. From biological motors, bacteria to synthetic active colloids, a large amount of microswimmers have received great attentions due to their nontrivial dynamical behaviors forbidden in equilibrium systems. Due to their self-propulsion motions, microswimmers serve as good candidates for cargo-carriers in the realm of natural or man-made micro-machines, which are of significant interest for applications such as drug delivery, biosensing, or shuttle-transport of living cells and emulsion droplets to list a few.

In particular, circle swimmers moving in curved trajectories have been widely applied in cargo delivery. For instance, the sperm-flagella driven Micro-Bio-Robot (MBR) has been reported to perform precisely point-to-point closed-loop motion under the influence of external magnetic field with applications towards targeted drug delivery and microactuation. Synthetic catalytic bimetallic nanomotors have been engineered to pick up, haul, and release micrometer-scale cargo with constant velocity circular movements to mimic nanoscale biomotors in biological systems. Magnetic swimmers actuated by externally applied field, rotating at a constant speed in a plane, have been envisaged for targeted drug delivery and so on. Since such microswimmers generally function at small scales, their transport dynamics are intrinsically stochastic. Therefore, how to work against such inherent fluctuations with good performance is of great importance for their design.

Here, we address such an issue by studying a general model of Brownian circle swimmer in a 2D plane, subjected to an active force with constant amplitude $F_0$, an internal torque $M$ and an external harmonic potential with strength $k$. By using the framework of stochastic thermodynamics, we have analyzed the heat dissipation of the system as a function of $F_0$, $M$ and $k$, identified as thermodynamic cost of the swimmer. Interestingly, we find that there exists an optimal torque such that the cyclic cost $Q_{cye}$ could reach a minimum value, which serves as a lower bound for the cost to sustain a stable circular motion. In addition, a trade-off relation between the precision of the circular motion and the thermodynamic cost has been found, indicating that an increased energy supply is needed to maintain punctual transport process as expected. By utilizing the thermodynamic uncertainty relations (TURs), we have also obtained an analytical expression for the transport efficiency $\chi_0$, which properly quantifies the performance of the swimmer to work with large speed, high precision, but low cost. With these results, we are able to propose proper strategies for designing swimmers under certain constraints, for instance, when the cyclic energy supply $Q_0$ is limited. Analysis shows that the maximum $\chi_0$ increases with increasing $Q_0$, which can be further enhanced by increasing the external potential strength $k$. Finally, we illustrate our predicted design principles and strategies by numerical simulations.

II. MODEL

As shown in Fig. 1, we consider a microswimmer performing circular motion, carrying cargo along a fluctuating circular orbit in two-dimensional $xy$ plane from $A$ (the start point) to $B$ (the destination) and then going back, which completes a single cyclic transport task. These microswimmers can be natural or synthetic, such as bacteria, rods, or spherical Janus particles, and related scenarios have been reported in environmental remediation, targeted drug delivery or transportation of cells.

For simplicity, we model the circle swimmer by an active spherical particle of radius $R$. The particle is subjected to a self-propelled force with amplitude $F_0$ along the orientation $e = (\cos \varphi, \sin \varphi)$ and an internal torque $\Omega$ (along the $z$ direction) that drives the circular mo-
FIG. 1. Schematic sketch of cargo delivering by various types of Brownian circle swimmers. The circle swimmers deliver the cargo from A (start point) to B (destination) along a fluctuating circular orbit (the black line). One may control the swimmer motion by changing its active torque for circular motion, active force for translational motion or by applying possible external potentials (blue surface). The yellow curved line represents the actual stochastic trajectory, and the black dashed line represents the averaged trajectory.

where we have set Boltzmann constant $k = 1$ throughout the paper, thus the angular velocity of swimmer equals to the value of torque $\Omega$. The fluctuation terms, $\xi = \{\xi_x, \xi_y\}$ and $\zeta$, are both independent Gaussian white noises with zero mean, satisfying $\langle \xi(t)\xi(s) \rangle = 2T\gamma_t T\delta(t-s)$ and $\langle \zeta(t)\zeta(s) \rangle = 2T\delta(t-s)$, where $T$ is the ambient temperature and $I$ the unit tensor. According to (the fluctuation-dissipation relation), the short-time translational and rotational diffusion coefficients of the particle read $D_t = T/\gamma_t$ and $D_r = T$ (where we have set Boltzmann constant $k_B = 1$), with $D_t/D_r = 4R^2/3$ holding for a spherical particle\cite{24}. The corresponding Fokker-Planck equation for the probability density function $P(r, \phi; t)$ is given by

$$\gamma_t \dot{r} = F_k + F_0e + \xi,$$  \hspace{1cm} (1)

$$\dot{\phi} = \Omega + \zeta,$$  \hspace{1cm} (2)

where $\gamma_t$ is the translational friction coefficient. For simplicity, we have set the rotational friction coefficient $\gamma_r = 1$ throughout the paper, thus the angular velocity of swimmer equals to the value of torque $\Omega$. The fluctuation terms, $\xi = \{\xi_x, \xi_y\}$ and $\zeta$, are both independent Gaussian white noises with zero mean, satisfying $\langle \xi(t)\xi(s) \rangle = 2T\gamma_t T\delta(t-s)$ and $\langle \zeta(t)\zeta(s) \rangle = 2T\delta(t-s)$, where $T$ is the ambient temperature and $I$ the unit tensor. According to (the fluctuation-dissipation relation), the short-time translational and rotational diffusion coefficients of the particle read $D_t = T/\gamma_t$ and $D_r = T$ (where we have set Boltzmann constant $k_B = 1$), with $D_t/D_r = 4R^2/3$ holding for a spherical particle. The corresponding Fokker-Planck equation for the probability density function $P(r, \phi; t)$ is given by

$$\partial_t P(r, \phi; t) = -\partial_r \cdot J_r(r, \phi; t) - \partial_{\phi} J_{\phi}(r, \phi; t),$$  \hspace{1cm} (3)

where the probability currents read

$$J_r(r, \phi; t) = \gamma_t^{-1} (F + F_0e - T\partial_r) P(r, \phi; t)$$

and

$$J_{\phi}(r, \phi; t) = (\Omega - T\partial_{\phi}) P(r, \phi; t).$$

respectively. Hereafter, we mainly focus upon the steady state with $\partial_t P_s(r, \phi) = 0$.

To deliver cargo with good performance for the swimmer considered here, it may be required that the swimmer can reach the destination with a high speed and precisely on time, but with relatively low dissipation\cite{15}. Therefore, fast delivery with high accuracy and low cost can be viewed as a design principle of the swimmer.

III. THERMODYNAMIC COST

To establish a quantitative measure of such a principle, firstly, one needs to investigate the cost of the delivery process. Since the swimmer follows stochastic dynamics, here we use the strategy of stochastic thermodynamics\cite{10}, which has been widely studied in the last decade. Note that the stochastic thermodynamics has been successfully generalized to active matter systems\cite{10, 54}. For instance, Pearce and Giomi have studied the linear response to leadership, effective temperature and decision making for a collection of flocking agents.\cite{47} Kyriakopoulos et.al have investigated the nonequilibrium response of polar ordered active fluid to external alignment.\cite{48} Besides, Burkholder and Brady have obtained a generalization of the standard stochastic thermodynamics to self-propelled particles\cite{49}.

The framework of stochastic thermodynamics allows us to calculate the average entropy production and then heat dissipation rate $\Sigma$ along a delivery cycle, which can be identified as the thermodynamic cost for the delivery process. Following Seifert’s spirit\cite{10}, one can define a trajectory-based entropy for the the microswimmer as $\dot{s}(t) = -\ln P(r, \phi; t)$, the change rate of which can be written as

$$\dot{s}(t) = \dot{s}_{tot}(t) - \dot{s}_m(t)$$  \hspace{1cm} (4)

where

$$\dot{s}_m(t) = \langle \dot{s}_{tot}(t) \rangle = \frac{\langle F + F_0e \cdot \dot{r} + \Omega \dot{\phi} \rangle}{T}$$  \hspace{1cm} (5)

is the so called medium entropy production\cite{22} that can be related directly to the heat dissipation rate $\dot{q}$ via $\dot{q} = T\dot{s}_m(t)$. The term $\dot{s}_{tot}(t) = \dot{s}(t) + \dot{s}_m(t) = -\partial_t \ln P(r, \phi; t) + T^{-1} \partial_r \ln P(r, \phi; t) \cdot \dot{r} + T^{-1} \partial_{\phi} \ln P(r, \phi; t) \cdot \dot{\phi}$ is the change rate of the trajectory-dependent total entropy production, where $v_r = J_r/P$ and $v_\phi = J_\phi/P$ are the local mean velocities\cite{10}. Upon averaging over trajectories, one can obtain

$$\dot{S}_{tot} = \langle \dot{s}_{tot} \rangle = T^{-1} \int drd\phi \gamma_t v_r^2 + v_\phi^2 \rangle P(r, \phi; t) \geq 0.$$  \hspace{1cm} (6)

In the steady state, we can use $\Sigma = T \langle \dot{s}_m \rangle = T \dot{S}_{tot}$ to calculate the heat dissipation rate, i.e., the rate of thermodynamic cost for the delivery process. To be specific,
we now set the external potential \( U(r) = kr^2/2 \), where the potential strength \( k \) works as a control parameter. For further purpose, it is convenient to rewrite Eq.\([1]\) in polar coordinates \( (r = re^{i\theta}) \):

\[
i = -\beta D_i kr + \beta D_i F_0 \cos(\theta - \varphi) + \sqrt{2D_i} \xi_r.
\]

\[
\dot{\theta} = -\frac{\beta D_i F_0}{r} \sin(\theta - \varphi) + \sqrt{2D_i \xi_\theta},
\]

where \( \xi_r = \cos \theta \xi_x + \sin \theta \xi_y \), \( \xi_\theta = - (\sin \theta \xi_x - \cos \theta \xi_y) / r \). In the steady state, the angles \( \theta \) and \( \varphi \) would be phase-locked, i.e., \( \theta - \varphi \approx \text{const} \) and \( \langle \dot{\theta} \rangle \approx \Omega \). The circle swimmer will rotate along a stochastic limit cycle, and the mean radius of the circular orbit can be computed by setting \( \beta D_i \dot{r}_m \approx \beta D_i F_0 \cos(\theta - \varphi) \) \( \text{i.e.,} \)

\[
r_m = \frac{F_0}{\sqrt{\gamma_t \Omega^2 + k^2}},
\]

where we have used \( -\beta D_i F_0 \sin(\theta - \varphi) / r_m \approx \Omega \). Using Eq.\([6]\) one can then obtain the cost rate in the steady state as (see Appendix \( A \) for details)

\[
\Sigma \approx \gamma_t \omega^2 r_m^2 + \omega^2 = \frac{F_0^2 \Omega^2}{\gamma_t \Omega^2 + \gamma_t^{-1} k^2} + \Omega^2.
\]

IV. OPTIMAL TORQUE FOR CYCLIC COST

From Eq.\([10]\) one can see that the cost rate depends on all the parameters, the active force \( F_0 \), the torque \( \Omega \), the external force strength \( k \), as well as the friction coefficients \( \gamma_t \). In general, larger activity and torque will lead to larger cost rate, while stronger control (larger \( k \)) will reduce the cost rate. Nevertheless, due to the circular motion of the swimmer, a more relevant quantity in real processes would be the cyclic cost \( Q_{\text{cyc}} \), i.e., the thermodynamic cost per delivery cycle, which is given by

\[
Q_{\text{cyc}} = \left( \frac{2\pi}{\Omega} \right) \Sigma = \frac{2\pi}{\Omega} \left( \frac{F_0^2 \Omega^2}{\gamma_t \Omega^2 + \gamma_t^{-1} k^2} + \Omega^2 \right).
\]

It can be seen that, while smaller torque \( \Omega \) leads to smaller cost rate \( \Sigma \), it also causes longer cyclic process. An interesting result is then there exists a trade-off between speed and thermodynamic cost for the circular motion. In particular, an optimal torque \( \Omega_{\text{opt}} \) exists for the circle swimmer to minimize the cost per cycle, which can be obtained by setting \( dQ_{\text{cyc}}/dM = 0 \), giving

\[
\Omega_{\text{opt}}^2 = \frac{-(\gamma_t^{-1} F_0^2 + 2\gamma_t^{-2} k^2) + \sqrt{\gamma_t^{-2} F_0^4 + 8F_0^2 k^2 \gamma_t^{-3}}}{2}.
\]

Note that the condition for the existence of the optimal torque is that the active and external forces must satisfy \( k \leq k_0 = \sqrt{\gamma_t F_0^2} \), and the minimum \( Q_{\text{cyc}}^{\text{opt}} \) provides a lower bound of the energy to sustain the circular motion. In the simple case when the external potential is absent \( (k = 0) \), the optimal torque reads \( \Omega_{\text{opt}} = F_0/\sqrt{\gamma_t} \), and the corresponding minimum cyclic cost reads \( Q_{\text{cyc}}^{\text{opt}} = 4\pi F_0^2 / \sqrt{\gamma_t} \). With increasing \( k \), it can be observed from Eq.\([11]\) that the minimum cyclic cost can be further reduced. If lowering \( Q_{\text{cyc}} \) is the target for the delivery process, Eqs.\([11]\) and \([12]\) serve as the formula for design principles.

V. PRECISION-COST TRADE-OFF RELATION

Besides thermodynamic cost, another important aspect of the cyclic delivery process is the precision. Similar to biochemical oscillation processes, the precision can be conveniently measured by the phase diffusion constant \( D_\theta = \left( \langle \dot{\theta}^2 \rangle - \langle \theta \rangle^2 \right) / 2t \). Using similar method as in our previous work\([53]\), we can obtain

\[
D_\theta \approx \frac{D_t}{r_m^2} = T \left( \frac{\gamma_t \Omega^2 + \gamma_t^{-1} k^2}{F_0^2} \right).
\]

Combining this result with Eq.\([10]\), we reach an interesting trade-off relation between phase diffusion and dissipation rate, which reads,

\[
D_\theta \approx \frac{T \Sigma_c}{\Sigma - \Sigma_c}.
\]

where \( \Sigma_c = \Omega^2 \) can be identified as the minimum dissipation rate of a circular motion.

A few remarks can be made. Firstly, large dissipation rate is required to reach high precision, which exactly tells the trade-off between cost and gain. Secondly, for a circle swimmer carrying cargo, the dissipation rate needs to exceed a critical value \( \Sigma_c \) to transport along the rotational orbit, which is already indicated in Eq.\([10]\), and extra dissipation could be applied to ensure the swimmer works with a more punctual manner. Thirdly, the inverse law, Eq.\([14]\) between phase diffusion constant \( D_\theta \) and dissipation rate \( \Sigma \) has also been found in biochemical oscillation systems\([54, 55]\), highlighting its ubiquitous importance in living systems.

VI. TRANSPORT EFFICIENCY

To further characterize the performance and design principle of the circle swimmer, here we apply the so-called thermodynamic uncertainty relation (TUR), which was first proposed by Barato and Seifert\([14]\) in Markovian jump processes, and has been extensively studied recently\([56, 57]\). According to Dechant and Sasa\([58]\), a TUR relation holds for general Langevin systems as
transport efficiency for the circle swimmer as with lower dissipation. Combining Eq.10 and 13, noting swimmer to maintain a high-speed, accurate transport potential \( F_0 \) the activity \( (F_0) \) and reduce the torque \( (\Omega) \) or external potential \( (k) \). However, as discussed above, increasing \( F_0 \) will also increase the thermodynamic cost rate \( \Sigma \) or the cyclic cost \( Q_{cyc} \). In real systems, a more relevant question would be how to design the circle swimmer when the energy resource is limited. For instance, when the chemical substances driving the directional movement are exhausted, the microswimmers may fail to self-propelled.\textsuperscript{43}

Here, we focus on a situation when the cyclic supply of energy is limited by a given value \( Q_0 \), and discuss how to maximize the transport efficiency \( \chi_0 \) under such a constraint. Note that to sustain the rotation motion, \( Q_0 \) must be larger than the minimum value \( Q_{cyc}^{\text{opt}} \) given by Eq.\textsuperscript{11}. If not, a feasible way is to increase the external potential strength \( k \) and thus reduce \( Q_{cyc}^{\text{opt}} \) as indicated also in Eq.\textsuperscript{11}. A simple analysis shows that setting the strength \( k \geq \sqrt{\gamma_0 (2\pi F_0^2 \Omega/\gamma_0 - \gamma_0 \Omega^2)} \) can make the swimmer maintain circular motion in the case when \( Q_0 < 4\pi F_0^2 \sqrt{\pi / 4} \).

Given that \( Q_0 > 4\pi F_0^2 \sqrt{\pi / 4} \), circular motion can be sustained in the absence of external potential, i.e., \( k = 0 \). For now, the design principle is to maximize the transport efficiency \( \chi_0 = F_0^2 / (F_0^2 + \gamma_0 \Omega^2) \) under the constraint that \( Q_{cyc}(F_0, \Omega) = 2\pi (F_0^2 / \gamma_0 + \Omega) \leq Q_0 \). The conditional highest transport efficiency, which is achieved when \( Q_{cyc} = Q_0 \), can be obtained as (see Appendix B for details)

\[
\chi_{0,\text{max}} = \left[ 1 + \frac{4}{(C + \sqrt{C^2 - 4})^2} \right]^{-1}
\]

(16)

with \( C = \sqrt{\gamma_0 Q_0 / (2\pi F_0)} \). Then for fixed self-propelled force \( F_0 \), one can get higher transport efficiency with increasing input energy per-cycle \( Q_0 \), and correspondingly the swimmer works with a torque \( \Omega \) determined by the condition \( Q_{cyc}(F_0, \Omega) = Q_0 \). These serve as a reasonable design strategy for the swimmer, although how to transfer the input \( Q_0 \) to torque \( \Omega \) may require specific technic.

As mentioned above, the lower bound for \( Q_{cyc} \) can be decreased with increasing \( k \). Therefore, for given \( Q_0 \), \( Q_0/Q_{cyc}^{\text{opt}} \) will increase with \( k \), i.e., the energy supply becomes more abundant with increasing \( k \). Intuitively, the transport efficiency would become larger if all \( Q_0 \) is transferred to the torque. Unfortunately, it is not possible to get an analytical expression for \( \chi_{0,\text{max}} \) for \( k \neq 0 \). By numerical simulation, however, we indeed find that the maximum transport efficiency can be further improved by increasing \( k \) under the condition \( Q_{cyc} \leq Q_0 \), as illustrated below.

VII. DESIGN STRATEGY FOR LIMITED ENERGY RESOURCE

For better performance of the swimmer regarding cargo delivery, it is suggested that \( \chi_0 \) should be as large as possible. To this goal, it seems that one can simply enhance the activity \( (F_0) \) and reduce the torque \( (\Omega) \) or external potential \( (k) \). However, as discussed above, increasing \( F_0 \) will also increase the thermodynamic cost rate \( \Sigma \) or the cyclic cost \( Q_{cyc} \). In real systems, a more relevant question would be how to design the circle swimmer when the energy resource is limited. For instance, when the chemical substances driving the directional movement are exhausted, the microswimmers may fail to self-propelled.\textsuperscript{43}

Here, we focus on a situation when the cyclic supply of energy is limited by a given value \( Q_0 \), and discuss how to maximize the transport efficiency \( \chi_0 \) under such a constraint. Note that to sustain the rotation motion, \( Q_0 \) must be larger than the minimum value \( Q_{cyc}^{\text{opt}} \) given by Eq.\textsuperscript{11}. If not, a feasible way is to increase the external potential strength \( k \) and thus reduce \( Q_{cyc}^{\text{opt}} \) as indicated also in Eq.\textsuperscript{11}. A simple analysis shows that setting the strength \( k \geq \sqrt{\gamma_0 (2\pi F_0^2 \Omega/\gamma_0 - \gamma_0 \Omega^2)} \) can make the swimmer maintain circular motion in the case when \( Q_0 < 4\pi F_0^2 \sqrt{\pi / 4} \).

Given that \( Q_0 > 4\pi F_0^2 \sqrt{\pi / 4} \), circular motion can be sustained in the absence of external potential, i.e., \( k = 0 \). For now, the design principle is to maximize the transport efficiency \( \chi_0 = F_0^2 / (F_0^2 + \gamma_0 \Omega^2) \) under the constraint that \( Q_{cyc}(F_0, \Omega) = 2\pi (F_0^2 / \gamma_0 + \Omega) \leq Q_0 \). The conditional highest transport efficiency, which is achieved when \( Q_{cyc} = Q_0 \), can be obtained as (see Appendix B for details)

\[
\chi_{0,\text{max}} = \left[ 1 + \frac{4}{(C + \sqrt{C^2 - 4})^2} \right]^{-1}
\]

(16)

with \( C = \sqrt{\gamma_0 Q_0 / (2\pi F_0)} \). Then for fixed self-propelled force \( F_0 \), one can get higher transport efficiency with increasing input energy per-cycle \( Q_0 \), and correspondingly the swimmer works with a torque \( \Omega \) determined by the condition \( Q_{cyc}(F_0, \Omega) = Q_0 \). These serve as a reasonable design strategy for the swimmer, although how to transfer the input \( Q_0 \) to torque \( \Omega \) may require specific technic.

As mentioned above, the lower bound for \( Q_{cyc} \) can be decreased with increasing \( k \). Therefore, for given \( Q_0 \), \( Q_0/Q_{cyc}^{\text{opt}} \) will increase with \( k \), i.e., the energy supply becomes more abundant with increasing \( k \). Intuitively, the transport efficiency would become larger if all \( Q_0 \) is transferred to the torque. Unfortunately, it is not possible to get an analytical expression for \( \chi_{0,\text{max}} \) for \( k \neq 0 \). By numerical simulation, however, we indeed find that the maximum transport efficiency can be further improved by increasing \( k \) under the condition \( Q_{cyc} \leq Q_0 \), as illustrated below.

VIII. NUMERICAL SIMULATIONS

To test our previous theoretical predictions, we numerically solve the Langevin equation\textsuperscript{1} and \textsuperscript{2} with a time step \( 10^{-3} \). The dissipation rate \( \Sigma \) is numerically
FIG. 3. (a) Cyclic cost $Q_{\text{cyc}}$ as a function of the torque $\Omega$ for different values of the potential strength $k$. The optimal torque $\Omega_{\text{opt}}$ to minimize $Q_{\text{cyc}}$ moves to the left, and the cyclic energy cost can be further reduced by increasing the potential strength. Lines: theory. Symbols: simulations. (b) The maximal transport efficiency $\chi_{\theta,\text{max}}$ as a function of the limited energy supply $Q_0$ for different values of the potential strength $k$. The maximal transport efficiency can also be enhanced by increasing the potential strength. The data is obtained from maximizing the analytical results for transport efficiency $\chi_{\theta} = F_0^2/(F_0^2 + \gamma_t \Omega^2 + \gamma_t^{-1}k^2)$ under the constraint $Q_{\text{cyc}} \leq Q_0$. $F_0 = 100$.

The cyclic energy cost $Q_{\text{cyc}}$ is calculated by using $\Sigma = T \langle \dot{s}_m \rangle$, i.e., Eq. (5), and $D_\theta$ is computed according to the definition. The precision-dissipation trade-off relation, Eq. (14), has been shown in Fig. 2. It can be observed that the normalized phase diffusion constant $D = D_\theta/T$ is inversely proportional to the dissipation rate $\Sigma - \Sigma_c$, and the data plotted in log-log scale are fitted very well by a line with slope -1, in good agreement with the theoretical results.

In Fig. 3(a), we show how the cyclic energy cost $Q_{\text{cyc}}$ changes with the internal torque $\Omega$, and the optimal torque $\Omega_{\text{opt}}$ can be clearly identified. As discussed above, the cyclic energy cost can be further reduced by increasing the potential strength $k$. At last, for different values of the potential strength $k$, the maximal transport efficiency $\chi_{\theta,\text{max}}$ under limited energy resource $Q_0$ is shown in Fig. 3(b) for fixed $F_0 = 100$. Clearly, $\chi_{\theta,\text{max}}$ increases with $Q_0$ and approaching the upper bound 1 for very large energy input $Q_0$, and introducing external potential will further improve the maximal transport efficiency, in consistent with our previous predictions.

IX. CONCLUSION

In summary, we have studied the stochastic thermodynamics for the Brownian circle swimmer in a 2D plane. We theoretically obtain an expression of the cost rate $\Sigma$, and an optimal torque for minimal cyclic cost $Q_{\text{cyc}}$ has been found, which provides the lower bound for the thermodynamic cost to sustain a stable circular motion. Meanwhile, it has been revealed that extra energy supply could be applied to enhance the precision of the transport process by a derived trade-off relation. Using the general principles for dissipative processes, the TURs, we also analyze the transport efficiency of cargo transport, which offers quantitative insight into the performance of the swimmer. Furthermore, we predict design strategies under the condition when the cyclic energy resource is limited. Analysis shows that the maximum $\chi_{\theta}$ gradually approaches the upper bound while increasing the cyclic supply, which can be further enhanced by introducing an external potential. By using our approach, it is possible to design the swimmer for other important conditions, which is left for further investigation in future work.

Recently, the collective effects in active systems have attracted much attention. According to our main result (Eq. 15), the transport efficiency reads as $\chi_{\theta} = (\langle \dot{\theta}^4 \rangle)^2/S_{\text{tot}}D_\theta = F_0^2/(F_0^2 + \gamma_t \Omega^2 + \gamma_t^{-1}k^2)$, where $F_0$ is the self-propelled force, $\Omega$ is the torque and $k$ is the potential strength. The steric interactions may induce an effective larger $k$, which will result a decreased transport efficiency. On the other hand, collective interactions could make the swimmers synchronize, which suppress the phase diffusion (smaller phase diffusion constant $D_\theta$) to enhance the transport efficiency. Besides, the macroscopic crowding as a result of the interparticle interactions may also enhance the transport efficiency by suppressing the torque $\chi_{\theta}$.

In our opinion, the generalization to collective active systems is not straightforward, which deserves further study. Since it has been shown that various experimental setups are feasible to realize the swimmer model, we believe that the predicted principles and strategies presented here are testable, and can offer guidelines for experiments.

ACKNOWLEDGMENTS

This work is supported by MOST(2018YFA0208702), NSFC (32090044, 21973085, 21833007, 21790350, 21521001).

DATA AVAILABILITY

The data that support the findings of this study are available within the article and its supplementary material.
SUPPLEMENTARY INFORMATION

Appendix A: Cost rate for the circle swimmer

We now calculate the cost rate in steady state. Following Eq. (6) of the main text, the energy dissipation rate \( \Sigma = \dot{\mathcal{Q}} = T \dot{S}_{\text{tot}} \) reads

\[
\Sigma = \int dr d\varphi \left( \gamma_t v_r^2 + \gamma_r r^2 \right) P_{ss}(r, \varphi) = \Sigma_r + \Sigma_c. 
\]

Here, \( \Sigma_r = \gamma_t \langle v_r^2 \rangle \) and \( \Sigma_c = \gamma_r \langle v^2 \rangle \) denote the cost rate associated with the rotational motion along the circular orbit and the self chiral motion. Due to the time scale separation between the radial motion and the angular/chiral motion, \( \Sigma_r \) can be calculated by approximating \( P_{ss}(r, \varphi) \approx P_{ss}(r) P_{ss}(\varphi) \) and \( \langle r^2 \rangle \approx r_m^2 \).

The explicit expression for \( \Sigma_r \) reads

\[
\Sigma_r \approx \frac{F_0^2 \Omega^2}{\gamma_t \Omega^2 + \gamma_t^{-1} k^2}. 
\]

As \( \Sigma_c \approx \Omega^2 \), the cost rate can be derived as the form of Eq. (10) in the main text.

Appendix B: Design strategies for limited energy resource

By writing \( \rho = \sqrt{\gamma_t \Omega / F_0} \), the design strategies emerges from maximizing the transport efficiency \( \chi_\theta = 1 / (1 + \rho^2) \) under the condition that \( 2 \leq \rho + \rho^{-1} \leq \sqrt{\gamma_t Q_0 / (2\pi F_0)} = C \), where \( C \) is the dissipation normalized by self-propelled force. By solving for the range of the ratio that satisfies the condition, \( \frac{2}{C+\sqrt{C^2-4}} \leq \rho \leq \frac{C+\sqrt{C^2-4}}{2} \), we can obtain the highest transport efficiency

\[
\chi_{\theta, \text{max}} = \frac{1}{1 + \rho_{\text{min}}^2} 
\]

with \( \rho_{\text{min}} = \frac{2}{C+\sqrt{C^2-4}} \). We find that the available supply need to be fully utilized to enhance the transport efficiency.

Further, we start to analyze the effect of the external potential. Similarly, the question emerges from how to maximize the transport efficiency \( \frac{1}{1+F_0^{-1}(\Omega^2+k^2)} \) under the condition \( \frac{F_0^2 \Omega}{(\Omega^2+k^2)} + \Omega \leq \frac{Q_0}{2\pi} \). By simply rewriting, the transport efficiency reads

\[
\chi_\theta = \left( 1 + \frac{\Omega}{\frac{Q_0}{2\pi} - \Omega} \right)^{-1}, 
\]

i.e., the smaller the torque allowed to be selected, the transport efficiency will be higher. For simplicity, we assume the potential strength \( k \) is relatively small. Approximating by Taylor expansion, the condition reads \( \frac{F_0^2 \Omega}{\Omega^2} + \Omega \lesssim \frac{Q_0}{2\pi} + \frac{F_0^2 k^2}{\Omega^2} \), which implies that the allowed torque is smaller by increasing the potential strength, i.e., the maximal transport efficiency can be further enhanced.

Appendix C: Numerical details

The starting point for the numerical simulations are Eqs. (1) and (2), which are discretized according to the Euler’s method as

\[
\gamma_t [r(t + \Delta t) - r(t)] = F_k(r(t))) \Delta t + F_0 e(t) \Delta t + \sqrt{2 \gamma_t T \Delta t} \mathcal{N}_\xi, 
\]
and \( \gamma \) are independent and normally distributed random variables with zero mean and unit variance. The discretized time step \( \Delta t = 10^{-3} \), the translational friction coefficient \( \gamma_t = 1 \), and the statistical data are obtained from averaging over \( 10^4 \) trajectories. Typical trajectory in the xy plane of the numerical results is plotted in Fig. S1 (also can be seen Movie S1).

The most important quantities to be calculated are the thermodynamic cost rate \( \Sigma \) and transport efficiency \( \chi_\theta \). Here, we use \( \Sigma = T \langle \dot{s}_m \rangle = T \dot{S}_{\text{tot}} \) to calculate the thermodynamic cost rate. As discussed in the main text, \( T \dot{d}s_m = (F + F_0 e) \cdot \dot{d}r + \Omega d\phi \). Since \( dr \) and \( d\phi \) can be obtained from the dynamics generating from Eqs. 1 and 2, \( ds_m \) or \( \dot{s}_m \) can be calculated numerically. By averaging over trajectories in steady states, the thermodynamic rate \( \Sigma \) can then be obtained. Moreover, the transport efficiency \( \chi_\theta = (v^2_\theta - \langle v^2 \rangle)/\dot{S}_{\text{tot}} \), are obtained from the numerical data of \( \dot{S}_{\text{tot}} \) and \( v^2_\theta \).

As above, \( \dot{S}_{\text{tot}} \) can be obtained from simulations. As \( r = r e^{i\theta} \), \( r \) and \( \theta \) can be got from the generating dynamics, i.e., \( v^2_\theta = \lim_{t \to \infty}(\dot{\theta}^2 - \langle \dot{\theta}^2 \rangle)/2t \) can also be calculated from the steady state trajectories. Therefore, the transport efficiency can be obtained from the simulated data.

\[
\varphi(t + \Delta t) - \varphi(t) = \Omega \Delta t + \sqrt{2T \Delta t} \mathcal{N}_\zeta
\]  

with \( r = (x, y) \) and \( e = (\cos \phi, \sin \phi) \). \( \mathcal{N}_\zeta \) and \( \mathcal{N}_\zeta \) are independent and normally distributed random variables with zero mean and unit variance. The discretized time step \( \Delta t = 10^{-3} \), the translational friction coefficient \( \gamma_t = 1 \), and the statistical data are obtained from averaging over \( 10^4 \) trajectories. Typical trajectory in the xy plane of the numerical results is plotted in Fig. S1 (also can be seen Movie S1).

The most important quantities to be calculated are the thermodynamic cost rate \( \Sigma \) and transport efficiency \( \chi_\theta \). Here, we use \( \Sigma = T \langle \dot{s}_m \rangle = T \dot{S}_{\text{tot}} \) to calculate the thermodynamic cost rate. As discussed in the main text, \( T \dot{d}s_m = (F + F_0 e) \cdot \dot{d}r + \Omega d\phi \). Since \( dr \) and \( d\phi \) can be obtained from the dynamics generating from Eqs. 1 and 2, \( ds_m \) or \( \dot{s}_m \) can be calculated numerically. By averaging over trajectories in steady states, the thermodynamic rate \( \Sigma \) can then be obtained. Moreover, the transport efficiency \( \chi_\theta = (v^2_\theta - \langle v^2 \rangle)/\dot{S}_{\text{tot}} \), are obtained from the numerical data of \( \dot{S}_{\text{tot}} \) and \( v^2_\theta \). As above, \( \dot{S}_{\text{tot}} \) can be obtained from simulations. As \( r = r e^{i\theta} \), \( r \) and \( \theta \) can be got from the generating dynamics, i.e., \( v^2_\theta = \lim_{t \to \infty}(\dot{\theta}^2 - \langle \dot{\theta}^2 \rangle)/2t \) can also be calculated from the steady state trajectories. Therefore, the transport efficiency can be obtained from the simulated data.

FIG. S1. Typical simulated trajectory in the xy plane (black lines) from Eqs. 1 and 2. Here, \( F_0 = 100 \), \( \Omega = 100 \), \( k = 28 \), \( T = 0.1 \) and \( \gamma = 1 \). The red dashed circle represents the stationary circular orbit with the radius \( r_m = F_0/\sqrt{\gamma t \Omega^2 + k^2} \).
