Adaptivity is exponentially powerful for testing monotonicity of halfspaces

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Abstract

We give a poly(log n, 1/ε)-query adaptive algorithm for testing whether an unknown Boolean function \( f : \{-1,1\}^n \rightarrow \{-1,1\} \), which is promised to be a halfspace, is monotone versus \( \epsilon \)-far from monotone. Since non-adaptive algorithms are known to require almost \( \Omega(n^{1/2}) \) queries to test whether an unknown halfspace is monotone versus far from monotone, this shows that adaptivity enables an exponential improvement in the query complexity of monotonicity testing for halfspaces.

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1 Introduction

Monotonicity testing has been a touchstone problem in property testing for more than fifteen years [DGL+99, GGL+00, EKK+00, FLN+02, Fis04, BKR04, ACCL07, HK08, RS09, BMM12, BCGSM12, RRS+12, CS13a, CS13b, CS14, BRY13, CST14, KMS15, CDST15, BB16], with many exciting recent developments leading to a greatly improved understanding of the problem in just the past few years. The seminal work of [GGL+00] introduced the problem and gave an $O(n/\varepsilon)$-query algorithm that tests whether an unknown and arbitrary function $f : \{-1,1\}^n \to \{-1,1\}$ is monotone versus $\varepsilon$-far from every monotone function. While steady progress followed for non-Boolean functions and for functions over other domains, the first improved algorithm for Boolean-valued functions over $\{-1,1\}^n$ was only achieved in [CS13a], who gave a $\tilde{O}(n^{7/8}) \cdot \text{poly}(1/\varepsilon)$-query non-adaptive testing algorithm. A slightly improved $\tilde{O}(n^{5/6}) \cdot \text{poly}(1/\varepsilon)$-query non-adaptive algorithm was given by [CST14], and subsequently [KMS15] gave a $\tilde{O}(n^{1/2}) \cdot \text{poly}(1/\varepsilon)$-query non-adaptive algorithm.

On the lower bounds side, the fundamental class of halfspaces has played a major role in non-adaptive lower bounds for monotonicity testing to date. We discuss lower bounds for two-sided error monotonicity testing of Boolean-valued functions over $\{-1,1\}^n$, and refer the reader to the above references for lower bounds on other variants of the monotonicity testing problem. The first (two-sided) lower bound was established by Fischer et al [FLN+02], who used a slight variant of the majority function to give an $\Omega(\log n)$ lower bound for non-adaptive monotonicity testing. More recently, the lower bound of [CDST15], strengthening [CST14], shows that for any constant $\delta > 0$, there is a constant $\varepsilon = \varepsilon(\delta) > 0$ such that $\Omega(n^{1/2-\delta})$-non-adaptive queries are required to distinguish whether a Boolean function $f$ — which is promised to be a halfspace — is monotone or $\varepsilon$-far from every monotone function. Together with the $\tilde{O}(n^{1/2}) \cdot \text{poly}(1/\varepsilon)$-query non-adaptive monotonicity testing algorithm of [KMS15], this shows that halfspaces are “as hard as the hardest functions” to non-adaptively test for monotonicity. Halfspaces are also commonly referred to as “linear threshold functions” or LTFs; for brevity we shall subsequently refer to them as LTFs.

The role of adaptivity. While the above results largely settle the query complexity of non-adaptive monotonicity testing, the situation is less clear when adaptive algorithms are allowed. More generally, the power of adaptivity in property testing is not yet well understood, despite being a natural and important question. A recent breakthrough result of Belovs and Blais [BB16] gives a $\tilde{\Omega}(n^{1/4})$ lower bound on the query complexity of adaptive algorithms that test whether $f : \{-1,1\}^n \to \{-1,1\}$ is monotone versus $\varepsilon$-far from monotone, for some absolute constant $\varepsilon > 0$. This result was then improved by [CWX17] to $\Omega(n^{1/3})$. [BB16] also shows that when $f$ is promised to be an “extremely regular” LTF, with regularity parameter at most $O(1)/\sqrt{n}$, then $\log n + O(\varepsilon)$ adaptive queries suffice. (We define the “regularity” of an LTF in part (a) of Definition 1.2 below. Here we note only that every $n$-variable LTF has regularity between $1/\sqrt{n}$ and 1, so $O(1)/\sqrt{n}$-regular LTFs are “extremely regular” LTFs.)

A very compelling question is whether adaptivity helps for monotonicity testing of Boolean functions: can adaptive algorithms go below the $\Omega(n^{1/2-\delta})$-query lower bound for non-

\footnote{For monotonicity testing of functions $f : [n]^2 \to \{0,1\}$, Berman et al. [BRY14] showed that adaptive algorithms are strictly more powerful than non-adaptive ones (by a factor of $\log 1/\varepsilon$). For unateness testing of real-valued functions $f : \{0,1\}^n \to \mathbb{R}$, a natural generalization of monotonicity, [BCP+17] showed that adaptivity helps by a logarithmic factor. We remark that for another touchstone class in property testing, the class of Boolean juntas, it was only very recently shown [STW15, CST+17] that adaptive algorithms are strictly more powerful than non-adaptive algorithms.}
adaptive algorithms? While we do not know the answer to this question for general Boolean functions, in this work we give a strong positive answer in the case of LTFs, generalizing the upper bound of [BB16] from “extremely regular” LTFs to arbitrary unrestricted LTFs. The main result of this work is an adaptive algorithm with one-sided error that can test any LTF for monotonicity using $\text{poly}(\log n, 1/\varepsilon)$ queries:

**Theorem 1.1** (Main). There is a $\text{poly}(\log n, 1/\varepsilon)$-query adaptive algorithm with the following property: given $\varepsilon > 0$ and black-box access to an unknown LTF $f : \{-1,1\}^n \to \{-1,1\}$,

- If $f$ is monotone then the algorithm outputs “monotone” with probability 1;
- If $f$ is $\varepsilon$-far from every monotone function then the algorithm outputs “non-monotone” with probability at least $2/3$.

Recalling that the $\Omega(n^{1/2-\delta})$ non-adaptive lower bound from [CDST15] is proved using LTFs as both the yes- and no- functions, Theorem 1.1 shows that adaptive algorithms are exponentially more powerful than non-adaptive algorithms for testing monotonicity of LTFs. Together with the $\Omega(n^{1/3})$ adaptive lower bound from [CWX17], it also shows that LTFs are exponentially easier to test for monotonicity than general Boolean functions using adaptive algorithms.

1.1 A very high-level overview of the algorithm

The adaptive algorithm of [BB16] for testing monotonicity of “extremely regular” LTFs is essentially based on a simple binary search over the hypercube $\{-1,1\}^n$ to find an anti-monotone edge. [BB16] succeeds in analyzing such an algorithm, taking advantage of some of the nice structural properties of regular LTFs, but it is not clear how to carry out such an analysis for general LTFs.

To deal with general LTFs, our algorithm is more involved and employs an iterative stage-wise approach, running for up to $O(\log n)$ stages. Entering the $(t+1)$-th stage, the algorithm maintains a restriction $\rho(t)$ that fixes some of the input variables to $f$, and in the $(t+1)$-th stage the algorithm queries $f_{\rho(t)}$, where we write $f_{\rho(t)}$ to denote the function $f$ after the restriction $\rho(t)$. At a very high level, in the $(t+1)$-th stage the algorithm either

(i) Obtains definitive evidence (in the form of an anti-monotone edge) that $f_{\rho(t)}$, and hence $f$, is not monotone. In this case the algorithm halts and outputs “non-monotone.” Or, it

(ii) Extends the restriction $\rho(t)$ to obtain $\rho(t+1)$. This is done by fixing a random subset of the variables of expected density $1/2$ that are not fixed under $\rho(t)$, and possibly some additional variables, in such a way as to maintain an invariant described later. Or, it

(iii) Fails to achieve (i) or (ii), which we show is very unlikely to happen. In this case the algorithm simply halts and outputs “monotone.”

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2 For very special functions such as truncated anti-dictators, it is known [FLN+02] that adaptive algorithms are known to be much more efficient than nonadaptive algorithms ($O(\log n)$ versus $\Omega(\sqrt{n})$ queries) in finding a violation to monotonicity.

3 See Theorem 5.14 of Section 5 for a detailed description of the algorithm’s query complexity; we have made no effort to optimize the particular polynomial dependence on $\log n$ and $1/\varepsilon$ that the algorithm achieves.

4 A bi-chromatic edge of $f : \{-1,1\}^n \to \{-1,1\}$ is a pair $(x, y)$ of points such that $x, y \in \{-1,1\}^n$ differ at exactly one coordinate and satisfy $f(x) \neq f(y)$. An anti-monotone edge of $f$ is a bi-chromatic edge $(x, y)$ that also satisfies $x_i = -1, y_i = 1$ for some $i \in [n]$ and $f(x) = 1, f(y) = -1$. 

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We describe the invariant of $\rho^{(t)}$ maintained in Case (ii) in Section 1.2. One of its implications in particular is that $f_{\rho^{(t)}}$ is $\varepsilon'$-far from monotone, where $\varepsilon'$ has a polynomial dependence on $\varepsilon$. As a result, when the number of surviving variables under $\rho^{(t)}$ at the beginning of a stage $t^*$ is at most $\text{poly}(\log n)$, the algorithm can run the simple “edge tester” of [GGL*00] on $f_{\rho^{(t^*)}}$ to find an anti-monotone edge with high probability. Although the “edge tester” has query complexity linear in the number of variables, this is affordable since $f_{\rho^{(t^*)}}$ only has $\text{poly}(\log n)$ many variables left. Case (ii) ensures that there are at most $O(\log n)$ stages overall. We will also see that each stage makes at most $\text{poly}(\log n, 1/\varepsilon)$ queries; hence the overall query complexity is $\text{poly}(\log n, 1/\varepsilon)$.

1.2 A more detailed overview of the algorithm and why it works

In this section we give a more detailed overview of the algorithm and a high-level sketch of its analysis. The algorithm only outputs “non-monotone” if it identifies an anti-monotone edge, so it will correctly output “monotone” on every monotone $f$ with probability 1. Hence, establishing correctness of the algorithm amounts to showing that if $f$ is an LTF that is $\varepsilon$-far from monotone, then with high probability the algorithm will output “non-monotone” when it runs on $f$. Thus, for the remainder of this section, $f(x) = \text{sign}(w_1x_1 + \cdots + w_nx_n - \theta)$ should be viewed as being an LTF that is $\varepsilon$-far from monotone.

A crucial notion for understanding the algorithm is that of a $(\tau, \gamma, \lambda)$-non-monotone LTF.

**Definition 1.2.** Given an LTF $f : \{-1, 1\}^S \to \{-1, 1\}$ of the form $f(x) = \text{sign}(w \cdot x - \theta)$ over a set of variables $S$, we say it is a $(\tau, \gamma, \lambda)$-non-monotone LTF with respect to the weights $w$ if it satisfies the following three properties:

(a) $f$ is $\tau$-weight-regular\(^5\) with respect to $w$, i.e.,

$$\max_{i \in S} |w_i| \leq \tau \cdot \sqrt{\sum_{j \in S} w_j^2};$$

(b) $f$ is $\gamma$-balanced, i.e., $|E_{x \in \{-1, 1\}^n}[f(x)]| \leq 1 - \gamma$; and

(c) $f$ has $\lambda$-significant squared negative weights in $w$, i.e.,

$$\frac{\sum_{i \in S, w_i < 0} (w_i)^2}{\sum_{i \in S} (w_i)^2} \geq \lambda.$$

Looking ahead, an insight that underlies this definition (as well as our algorithm) is that, when $f = \text{sign}(w \cdot x - \theta)$ is a weight-regular LTF that is far from monotone, $f$ must satisfy (c) above for some large value of $\lambda$ (see Lemma 3.1 for a precise formulation). The converse also holds, i.e., an LTF that satisfies all three conditions above must be $\varepsilon$-far from monotone for some large value of $\varepsilon$ (see Lemma 3.3). This is indeed the reason why we call such functions $(\tau, \gamma, \lambda)$-non-monotone LTFs. An additional motivation for the regularity condition (a) is that, when $f$ satisfies (c) for some value $\lambda \gg \tau$ (the parameter in (a)), a random restriction $\rho$ (that randomly fixes half of the variables to uniform values from $\{-1, 1\}$) would have $f_{\rho}$ still satisfy (c) with essentially the same $\lambda$. The balance condition (b), on the other hand, may be viewed as a technical condition that makes it possible for

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\(^5\)Our terminology “weight-regular” means the same thing as [BB16]’s “regular.” We use the terminology “weight-regular” to distinguish it from the different notion of “Fourier-regularity” which we also require, see Section 2.2.
our various subroutines to work efficiently and correctly; we note that if \( f \) is not \( \gamma \)-balanced, then \( f \) is trivially \((\gamma/2)\)-close to either the monotone function \( 1 \) or the monotone function \( -1 \).

With Definition 1.2 in hand, we proceed to a more detailed overview of the algorithm (still at a rather conceptual level). The algorithm takes as input black-box access to \( f: \{-1,1\}^n \rightarrow \{-1,1\} \) and a parameter \( \varepsilon > 0 \). We remind the reader that in the subsequent discussion \( f \) should be viewed as an \( \varepsilon \)-far-from-monotone LTF. For the analysis of the algorithm, we also assume that \( f \) takes the form of \( f(x) = \text{sign}(w_1x_1 + \cdots + w_nx_n - \theta) \), for some unknown (but fixed\(^6\)) weight vector \( w \) and threshold \( \theta \). They are unknown to the algorithm and will be used in the analysis only.

Our algorithm has two main phases: first an initialization phase, and then the phase consisting of the main procedure.

**Initialization.** The algorithm runs an initialization procedure called Regularize-and-Balance. Roughly speaking, it with high probability either identifies \( f \) as a non-monotone LTF by finding an anti-monotone edge and halts, or constructs a restriction \( \rho^{(0)} \) such that \( f_{\rho^{(0)}} \) becomes a \((\tau, \gamma, \lambda_0)\)-non-monotone LTF for suitable parameters \( \tau, \gamma, \lambda_0 \), with \( \tau = \text{poly}(1/\log n, \varepsilon) \), \( \gamma = \varepsilon \), \( \lambda_0 = \text{poly}(\varepsilon) \) and \( \tau \ll \lambda_0 \). In the latter case the algorithm continues with \( f_{\rho^{(0)}} \).

**Main Procedure.** As sketched earlier in Section 1.1 the main procedure operates in a sequence of \( O(\log n) \) stages. In its \((t+1)\)th stage, it operates on the restricted function \( f_{\rho^{(t)}} \) which is assumed to be a \((\tau, \gamma, \lambda_t)\)-non-monotone LTF, and with high probability either identifies \( f \) as non-monotone and halts, or constructs an extension \( \rho^{(t+1)} \) of the restriction \( \rho^{(t)} \) such that \( f_{\rho^{(t+1)}} \) remains \((\tau, \gamma, \lambda_{t+1})\)-non-monotone (for some parameter \( \lambda_{t+1} \) that is only slightly smaller than \( \lambda_t \)) while the number of free variables in \( \rho^{(t+1)} \) drops by a constant factor.

To describe each stage in more detail, we need the following notation for restrictions. Given a restriction \( \rho \in \{-1,1,\ast\}^n \), we use \( \text{STARS}(\rho) \) to denote the set of indices that are not fixed in \( \rho \), i.e., the set of \( i \) such that \( \rho(i) = \ast \). Given \( f: \{-1,1\}^n \rightarrow \{-1,1\} \) of the form \( f(x) = \text{sign}(\sum w_ix_i - \theta) \), we let \( f_{\rho}: \{-1,1\}^{\text{STARS}(\rho)} \rightarrow \{-1,1\} \) denote the function \( f \) after the restriction \( \rho \):

\[
f_{\rho}(x) = \text{sign} \left( \sum_{i \in \text{STARS}(\rho)} w_i \cdot x_i + \sum_{j \notin \text{STARS}(\rho)} w_j \cdot \rho(j) - \theta \right).
\]

We stress that the weights of \( f_{\rho} \) remain \( w_i \) while the threshold is \( \theta - \sum_{j \notin \text{STARS}(\rho)} w_j \cdot \rho(j) \).

Now for the \((t+1)\)th stage, where \( t = 0, 1, 2, \ldots \), the main procedure carries out the following sequence of steps (we defer discussion of how these steps are implemented to Section 5). Below for convenience we let \( g \) denote \( f_{\rho^{(t)}} \), the function that the algorithm operates on in the \((t+1)\)th stage.

1. Draw a random subset \( A_t \subseteq \text{STARS}(\rho^{(t)}) \), which consists of roughly half of its variables.
   
   Assuming that \( \tau \ll \lambda_t \), we have that, with high probability, \( A_t \) partitions the positive and negative weights roughly evenly and the collection of weights of variables in \( \text{STARS}(\rho^{(t)}) \setminus A_t \) has \( \lambda_{t+1} \)-significant squared negative weights for some \( \lambda_{t+1} \) that is only slightly smaller than \( \lambda_t \). (This also justifies the assumption of \( \tau \ll \lambda_t \) at the beginning.)

2. Find a restriction \( \rho' \in \{-1,1,\ast\}^{\text{STARS}(\rho^{(t)})} \) that fixes the variables in \( A_t \) in such a way that \( g_{\rho'} \) is 0.96-balanced. The exact constant 0.96 here is not important as long as it is close enough to 1. Note that \( g_{\rho'} \) is more balanced than \( g \) is promised to be (i.e., \((\gamma = \varepsilon)\)-balanced and we may assume that \( \varepsilon \leq 0.5 \)). This helps in the last step of the stage. Our analysis shows that if \( g \) is \((\tau, \gamma, \lambda_t)\)-non-monotone, then this step succeeds with high probability.

\(^6\)Note that \((w, \theta)\) is not unique for a given \( f \). Here we pick any such pair and stick to it throughout the analysis.
3. Find a set $H_t \subset \text{STARS}(\rho(t)) \setminus A_t$ that contains those variables $x_i$ that have “high influence” in $g_{\rho}$. Intuitively, $H_t$ contains variables of $g_{\rho}$ that violate the $\tau$-weight-regularity condition; after its removal, the collection of weights of variables in $\text{STARS}(\rho(t)) \setminus (A_t \cup H_t)$ becomes $\tau$-weight-regular again.

4. For each $i \in H_t$, find a bi-chromatic edge of $g_{\rho}$ on the $i$th coordinate (this can be done efficiently because the variables in $H_t$ all have high influence in $g_{\rho}$), which reveals the sign of $w_i$. If an anti-monotone edge is found, halt and output “non-monotone;” otherwise, we know that the weight of every variable in $H_t$ is positive.

5. Finally, find a restriction $\rho'' \in \{-1, 1, *\}^{\text{STARS}(\rho(t))}$, which extends $\rho'$ and fixes the variables in $A_t \cup H_t$, such that $g_{\rho''}$ is $\gamma$-balanced. Our analysis shows that if $g$ is $(\tau, \gamma, \lambda_t)$-non-monotone and $g_{\rho}$ is 0.96-balanced, then this step succeeds with high probability. By Step 3, $g_{\rho''}$ is $\tau$-weight-regular. In addition, $g_{\rho''}$ has $\lambda_{t+1}$-significant squared negative weights because of Step 1 and Step 4 (which makes sure that all variables in $H_t$ have positive weights). At the end, we set $\rho^{(t+1)}$ to be the composition of $\rho^{(t)}$ and $\rho''$ and move on to the next stage.

To summarize, our analysis shows that if $f_{\rho(t)}$ is $(\tau, \gamma, \lambda_t)$-non-monotone (entering the $(t+1)$th stage) then with high probability the algorithm in the $(t+1)$th stage either finds an anti-monotone edge and halts, or finds an extension $\rho^{(t+1)}$ of $\rho^{(t)}$ such that

i) The new function $f_{\rho^{(t+1)}}$ is $(\tau, \gamma, \lambda_{t+1})$-non-monotone (entering the $(t+2)$th stage), where the parameter $\lambda_{t+1}$ is only slightly smaller than $\lambda_t$ (more on this below); and

ii) The number of surviving variables in $\rho^{(t+1)}$ is only about half of that of $\rho^{(t)}$.

This implies that, with high probability, the main procedure within $O(\log n)$ stages either finds an anti-monotone edge and returns the correct answer “non-monotone” or constructs a restriction $\rho^{(t)}$ such that $f_{\rho(t)}$ is $(\tau, \gamma, \lambda_t)$-non-monotone and the number of surviving variables under $\rho^{(t)}$ is at most $m = \text{poly}(\log n, 1/\varepsilon)$. For the latter case, our analysis (Lemma 3.3) together with the fact that $\lambda_t$ drops only slightly in each stage show that $f_{\rho(t)}$ remains $\varepsilon'$-far from monotone. Thus, the algorithm concludes by running the “edge tester” from [GGL+00] to $\varepsilon'$-test the $m$-variable function $f_{\rho(t)}$, which uses $O(m/\varepsilon') = \text{poly}(\log n, 1/\varepsilon)$ queries to $f_{\rho(t)}$ and finds an anti-monotone edge with high probability. To summarize, when $f$ is an LTF that is $\varepsilon$-far from monotone, our algorithm finds an anti-monotone edge and outputs “non-monotone” with high probability. As discussed earlier at the beginning of Section 1.2 about its one-sideness, the correctness of the algorithm follows.

1.3 Relation to previous work

We have already discussed how our main result, Theorem 1.1, relates to the recent upper and lower bounds of [KMS15, CDST15, BB16] for monotonicity testing. At the level of techniques, several aspects of our algorithm are reminiscent of some earlier work in property testing of Boolean functions and probability distributions as we describe below.

At a high level, the poly$(1/\varepsilon)$-query algorithm of [MORS10] for testing whether a function is an LTF identifies high-influence variables and “deals with them separately” from other variables, as does our algorithm. The more recent algorithm of [RS15], for testing whether a function is a signed majority function, like our algorithm proceeds in a series of stages which successively builds up a restriction by fixing more and more variables. Like our algorithm the [RS15] algorithm makes only
poly(log n, 1/ε) adaptive queries, but there are many differences both between the two algorithms and between their analyses. To briefly note a few of these differences, the [RS15] algorithm has two-sided error while our algorithm has one-sided error; the former also heavily leverages both the very “rigid” structure of the degree-1 Fourier coefficients of any signed majority function and the near-perfect balancedness of any signed majority function between the two outputs 1 and −1, neither of which hold in our setting. Finally, we note that the general approach of iteratively selecting and retaining a random subset of the remaining “live” elements, then doing some additional pruning to identify, check, and discard a small number of “heavy” elements, then proceeding to the next stage is reminiscent of the APPROX-EVAL-SIMULATOR procedure of [CRS15], which deals with testing probability distributions in the “conditional sampling” model.

1.4 Organization

In Section 2 we recall the necessary background concerning monotonicity, LTFs, and restrictions, and state a few useful algorithmic and structural results from prior work. In Section 3 we establish several new structural results about “regular” LTFs: we first show that its distance to monotonicity corresponds (approximately) to its total amount of squared negative coefficient weights; we also prove that its distance to monotonicity is preserved under a random restriction to a set of its non-decreasing variables. In Section 4 we present and analyze some simple algorithmic subroutines that will be used to identify high influence variables and check that they are non-decreasing. Finally in Section 5, we give a detailed description of our overall algorithm for testing monotonicity of LTFs, and prove its correctness, establishing our main result (Theorem 1.1).

2 Background

We write [n] for {1, . . . , n}, and use boldface letters (e.g., x and X) to denote random variables.

We briefly recall some basic notions. A function f: {−1, 1}n → {−1, 1} is monotone (short for “monotone non-decreasing”) if x ⪯ y implies f(x) ≤ f(y), where “x ⪯ y” means that x_i ≤ y_i for all i ∈ [n]. A function f is unate if there is a bit vector a ∈ {−1, 1}^n such that f(a_1x_1, . . . , a_nx_n) is monotone. It is well known that every LTF (defined below) is unate.

We measure distance between functions f, g: {−1, 1}^n → {−1, 1} with respect to the uniform distribution, so we say that f and g are ε-close if

\[ \text{dist}(f, g) := \Pr_{x \in \{-1,1\}^n} [f(x) \neq g(x)] \leq \varepsilon, \]

and that f and g are ε-far otherwise. A function f is ε-far from monotone if it is ε-far from every monotone function g. We write dist(f, MONO) to denote the minimum value of dist(f, g) over all monotone functions g. Throughout the paper all probabilities and expectations are with respect to the uniform distribution over {−1, 1}^n unless otherwise indicated. As indicated in Definition 1.2, we say that a {−1, 1}-valued function f is γ-balanced if

\[ \Pr_{x \in \{-1,1\}^n} [f(x)] \leq 1 - \gamma. \]

A function g: {−1, 1}^n → {−1, 1} is a junta over S ⊆ [n] if g depends only on the coordinates in S. We say f is ε-close to a junta over S if f is ε-close to g for some g that is a junta over S.
2.1 LTFs and weight-regularity

A function \( f: \{-1, 1\}^n \to \{-1, 1\} \) is an LTF (also commonly referred to as a halfspace) if there exist real weights \( w_1, \ldots, w_n \in \mathbb{R} \) and a real threshold \( \theta \in \mathbb{R} \) such that

\[
f(x) = \begin{cases} 
1 & \text{if } w_1 x_1 + \cdots + w_n x_n \geq \theta, \\
-1 & \text{if } w_1 x_1 + \cdots + w_n x_n < \theta.
\end{cases}
\]

We say that \( w = (w_1, \ldots, w_n) \) are the weights and \( \theta \) the threshold of the LTF, and we say that \((w, \theta)\) represents the LTF \( f \), or simply \( f(x) \) is the LTF given by \( \text{sign}(w \cdot x - \theta) \). Note that for any LTF \( f \) there are in fact infinitely many pairs \((w, \theta)\) that represent \( f \); we fix a particular pair \((w, \theta)\) for each \( n \)-variable LTF \( f \) and work with it in what follows.

An important notion in our arguments is that of weight-regularity. As indicated in Definition 1.2, given a weight vector \( w \in \mathbb{R}^n \), we say that \( w \) is \( \tau \)-weight-regular if no more than a \( \tau \)-fraction of the 2-norm of \( w = (w_1, \ldots, w_n) \) comes from any single coefficient \( w_i \), i.e.,

\[
\max_{i \in [n]} |w_i| \leq \tau \cdot \sqrt{w_1^2 + \cdots + w_n^2}. \tag{1}
\]

If we have fixed a representation \((w, \theta)\) for \( f \) such that \( w \) is \( \tau \)-weight-regular, we frequently abuse the terminology and say that \( f \) is \( \tau \)-weight-regular.

2.2 Fourier analysis of Boolean functions and Fourier-regularity

Given a function \( f: \{-1, 1\}^n \to \mathbb{R} \), we define its Fourier coefficients by \( \hat{f}(S) = \mathbb{E}[f \cdot x_S] \) for each \( S \subseteq [n] \), where \( x_S \) denotes \( \prod_{i \in S} x_i \), and we have that \( f(x) = \sum_S \hat{f}(S) \cdot x_S \). We will be particularly interested in \( f \)'s degree-1 coefficients, i.e., \( \hat{f}(S) \) for \( |S| = 1 \); we will write these as \( \hat{f}(i) \) rather than \( \hat{f}(|i|) \). We recall Plancherel’s identity \( \langle f, g \rangle = \sum_S \hat{f}(S) \hat{g}(S) \), which has as a special case Parseval's identity, \( \mathbb{E}_x[f(x)^2] = \sum_S \hat{f}(S)^2 \). It follows that every \( f: \{-1, 1\}^n \to \{-1, 1\} \) has \( \sum_S \hat{f}(S)^2 = 1 \).

We further recall that, for any unate function \( f: \{-1, 1\}^n \to \{-1, 1\} \) (and hence any LTF), we have \( |\hat{f}(i)| = \text{Inf}_i(f) \), where the influence of variable \( i \) on \( f \) is

\[
\text{Inf}_i(f) = \Pr_{x \in \{-1, 1\}^n} [f(x) \neq f(x^\oplus i)],
\]

where \( x^\oplus i \) is the vector obtained from \( x \) by flipping coordinate \( i \).

We say that \( f: \{-1, 1\}^n \to \{-1, 1\} \) is \( \tau \)-Fourier-regular if \( \max_{i \in [n]} |\hat{f}(i)| \leq \tau \). Section 2.5 below summarizes some useful relationships between weight-regularity and Fourier-regularity of LTFs.

2.3 Restrictions

A restriction \( \rho \) is an element of \( \{-1, 1, \ast\}^{[n]} \); we view \( \rho \) as a partial assignment to the \( n \) variables \( x_1, \ldots, x_n \), where \( \rho(i) = \ast \) indicates that variable \( x_i \) is unassigned. We write \( \text{supp}(\rho) \) to denote the set of indices \( i \) such that \( \rho(i) \in \{-1, 1\} \) and \( \text{STARS}(\rho) \) to denote the set of \( i \) such that \( \rho(i) = \ast \) (and thus, \( \text{STARS}(\rho) \) is the complement of \( \text{supp}(\rho) \)).

Given restrictions \( \rho, \rho' \in \{-1, 1, \ast\}^{[n]} \) we say that \( \rho' \) is an extension of \( \rho \) if \( \text{supp}(\rho) \subseteq \text{supp}(\rho') \) and \( \rho'(i) = \rho(i) \) for all \( i \in \text{supp}(\rho) \). If \( \rho \) and \( \rho' \) are restrictions with disjoint support we write \( \rho \rho' \) to denote the composition of these two restrictions (that has support \( \text{supp}(\rho) \cup \text{supp}(\rho') \)).
2.4 Useful algorithmic tools from prior work

We recall some algorithmic tools for working with black-box functions \( f: \{-1,1\}^n \to \{-1,1\} \).

**Estimating sums of squares of degree-1 Fourier coefficients.** We first recall Corollary 16 of [MORS10] (slightly specialized to our context):

**Lemma 2.1** (Corollary 16 [MORS10]). There is a procedure \( \text{Estimate-Sum-of-Squares}(f, T, \eta, \delta) \) with the following properties. Given as input black-box access to \( f: \{-1,1\}^n \to \{-1,1\} \), a subset \( T \subseteq [n] \), and parameters \( \eta, \delta > 0 \), it runs in time \( O(n \cdot \log(1/\delta)/\eta^4) \), makes \( O(\log(1/\delta)/\eta^4) \) queries, and with probability at least \( 1 - \delta \) outputs an estimate of \( \sum_{i \in T} \hat{f}(i)^2 \) that is accurate to within an additive \( \pm \eta \).

**Checking Fourier regularity.** We recall Lemma 18 of [MORS10], which is an easy consequence of Lemma 2.1:

**Lemma 2.2** (Lemma 18 [MORS10]). There is a procedure \( \text{Check-Fourier-Regular}(f, T, \tau, \delta) \) with the following properties. Given as input black-box access to \( f: \{-1,1\}^n \to \{-1,1\} \), \( T \subseteq [n] \), and parameters \( \tau, \delta > 0 \), it runs in time \( O(n \cdot \log(1/\delta)/\tau^{16}) \), makes \( O(\log(1/\delta)/\tau^{16}) \) queries, and

- If \( |\hat{f}(i)| \geq \tau \) for some \( i \in T \) then it outputs “not regular” with probability \( 1 - \delta \);
- If every \( i \in T \) has \( |\hat{f}(i)| \leq \tau^2/4 \) then it outputs “regular” with probability \( 1 - \delta \).

**Estimating the mean.** For completeness we recall the following simple fact (which follows from a standard Chernoff bound):

**Fact 2.3.** There is a procedure \( \text{Estimate-Mean}(f, \varepsilon, \delta) \) with the following properties. Given as input black-box access to \( f: \{-1,1\}^n \to \{-1,1\} \) and parameters \( \varepsilon, \delta > 0 \), it makes \( O(\log(1/\delta)/\varepsilon^2) \) queries and with probability at least \( 1 - \delta \) it outputs a value \( \tilde{\mu} \) such that \( |\tilde{\mu} - \mu| \leq \varepsilon \), where \( \mu = E_{x \in \{-1,1\}^n}[f(x)] \).

**The edge tester of [GGL+00].** We recall the performance guarantee of the “edge tester” (which works by querying both endpoints of uniform random edges and outputting “non-monotone” if and only if it encounters an anti-monotone edge):

**Theorem 2.4** ([GGL+00]). There is a procedure \( \text{Edge-Tester}(f, \varepsilon, \delta) \) with the following properties: Given black-box access to \( f: \{-1,1\}^n \to \{-1,1\} \) and parameters \( \varepsilon, \delta > 0 \), it makes \( O(n \log(1/\delta)/\varepsilon) \) queries and outputs either “monotone” or “non-monotone” such that

- If \( f \) is monotone then it outputs “monotone” with probability 1;
- If \( f \) is \( \varepsilon \)-far from monotone then it outputs “non-monotone” with probability at least \( 1 - \delta \).

2.5 Useful structural results from prior work

**Gaussian distributions and the Berry–Esséen theorem.** Recall that the p.d.f. of the standard Gaussian distribution \( \mathcal{N}(0,1) \) with mean 0 and variance 1 is given by

\[
\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.
\]
Let \( w \) for all \( \tau > 0 \) that \( f - \) Fourier-regular when their highest-influence variables are fixed to constants:

\[
\left( \frac{1}{t - 1} \right) \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-t^2/2} \leq \Pr_{z \sim N(0,1)}[|z| \geq t] \leq e^{-t^2/2}. \tag{2}
\]

We also need the following Gaussian anti-concentration bound.

**Fact 2.5 (Gaussian anti-concentration).** Let \( z \) be a random variable drawn from a Gaussian distribution with variance \( \sigma^2 \). Then for all \( \varepsilon > 0 \), we have \( \sup_{\theta \in \mathbb{R}} \{ \Pr[|z - \theta| \leq \varepsilon \sigma] \} \leq \varepsilon \).

The Berry–Esséen theorem (see e.g., [Fel68]) is a version of the central limit theorem for sums of independent random variables (stating that such a sum converges to a normal distribution) that provides a quantitative error bound. It is useful for analyzing weight-regular LTFs and we recall it below (as well as the standard Hoeffding inequality).

**Theorem 2.6 (Berry–Esséen).** Let \( \ell(x) = c_1 x_1 + \cdots + c_n x_n \) be a linear form of \( n \) unbiased, independent random \( \{\pm 1\} \)-valued variables \( x_i \). Let \( \tau \) be such that \( |c_i| \leq \tau \) for all \( i \), and let \( \sigma = (\sum c_i^2)^{1/2} \). Write \( F \) for the c.d.f. of \( \ell(x)/\sigma \), i.e., \( F(t) = \Pr[|\ell(x)|/\sigma \leq t] \). Then for all \( t \in \mathbb{R} \), we have that \( |F(t) - \Phi(t)| \leq \tau/\sigma \), where \( \Phi \) denotes the c.d.f. of a standard \( N(0,1) \) Gaussian random variable.

**Theorem 2.7 (Hoeffding’s Inequality).** Let \( x \) be a random variable drawn uniformly from \( \{-1, 1\}^n \). Let \( w \in \mathbb{R}^d \) and \( t > 0 \). Then we have

\[
\Pr_x[|x \cdot w| \geq t] \leq 2 \exp \left( -\frac{t^2}{2\|w\|^2} \right) \quad \text{and} \quad \Pr_x[|x \cdot w| \geq t] \leq \exp \left( -\frac{t^2}{2\|w\|^2} \right).
\]

Weight-regularity versus Fourier-regularity for LTFs. An easy argument, using the Berry–Esséen inequality above, shows that weight-regularity always implies Fourier-regularity for LTFs:

**Theorem 2.8 (Theorem 38 of [MORS10]).** Let \( f : \{-1, 1\}^n \rightarrow \{-1, 1\} \) be a \( \tau \)-weight-regular LTF. Then \( f \) is \( O(\tau) \)-Fourier-regular.

The converse is not always true; for example, the constant 1 function, which is \( \tau \)-Fourier-regular for all \( \tau > 0 \), may be written as \( f(x) = \text{sign}(x_1 + 2) \). However, if we additionally impose the condition that \( f \) is not too biased towards +1 or −1, then a converse holds. Sharpening an earlier result (Theorem 39 of [MORS10]), Dzindzalieta has proved the following:

**Theorem 2.9 (Theorem 20 of [Dzi14]).** Let \( f(x) = \text{sign}(w \cdot x - \theta) \) be an LTF such that \( \|E_x[f(x)]\| \leq 1 - \gamma \). If \( f \) is \( \tau \)-Fourier-regular, then it is also \( O(\tau/\gamma) \)-weight-regular.

Making LTFs Fourier-regular by fixing high-influence variables. Finally, we will need the following simple result (Proposition 62 from [MORS10]), which shows that LTFs typically become Fourier-regular when their highest-influence variables are fixed to constants:

**Proposition 2.10.** Let \( f : \{-1, 1\}^n \rightarrow \{-1, 1\} \) be an LTF and let \( J \supseteq \{i : |\hat{f}(i)| \geq \beta\} \). Then \( f_{\rho} \) is not \((\beta/\eta)\)-Fourier-regular for at most an \( \eta \)-fraction of all \( 2^{|J|} \) restrictions \( \rho \) that fix variables in \( J \).
3 New structural results about LTFs

Our analysis requires a few new structural results about LTFs. We collect and prove these results in this section. Readers who are eager to proceed to the algorithm and its analysis of correctness can skip this section and refer back to it as needed.

3.1 Far-from-monotone weight-regular LTFs have significant squared negative weights, and vice versa

The main idea of this subsection is that for weight-regular LTFs, the distance to monotonicity corresponds (approximately) to its total amount of squared weights of negative coefficients (under any representation \((w, \theta)\)). Lemma 3.1 shows that if \(f\) is far from monotone then this quantity is large, and Lemma 3.3 establishes a converse (both for weight-regular LTFs). We note that Lemma 3.1 is essentially equivalent to a lemma proved in [BB16].

We introduce some notation. Given an LTF \(f: \{-1,1\}^n \rightarrow \{-1,1\}\) with \(f(x) = \text{sign}(w \cdot x - \theta)\), we use \(P = P(f)\) and \(N = N(f)\) to denote the set of non-negative and negative indices, respectively: \(P = \{i \in [n]: w_i \geq 0\}\) and \(N = \{j \in [n]: w_j < 0\}\). We also let \(\text{pos}(f)\) and \(\text{neg}(f)\) denote the sum of squared weights of positive and negative coefficients, respectively:

\[
\text{pos}(f) = \sum_{i \in P} w_i^2 \quad \text{and} \quad \text{neg}(f) = \sum_{j \in N} w_j^2.
\]

Recall that we say \(f\) has \(\lambda\)-significant squared negative weights if \(\text{neg}(f)/(\text{pos}(f) + \text{neg}(f)) \geq \lambda\).

We start with the proof of Lemma 3.1.

**Lemma 3.1.** Let \(f: \{-1,1\}^n \rightarrow \{-1,1\}\) be an LTF given by \(f(x) = \text{sign}(w \cdot x - \theta)\). If \(f\) is both \(\varepsilon\)-far from monotone and \(\tau\)-weight-regular for some \(\tau \leq \varepsilon/16\), then \(f\) must have \(\lambda\)-significant squared negative weights, where \(\lambda = \varepsilon^2/(16 \ln(8/\varepsilon))\).

**Proof.** For convenience, we normalize all the weights so that \(\text{pos}(f) + \text{neg}(f) = 1\). Then it suffices to show that \(\text{neg}(f) \geq \varepsilon^2/(16 \ln(8/\varepsilon))\). Since \(f\) is \(\tau\)-weight-regular, we have that \(\max_i |w_i| \leq \tau\). We also assume that \(\text{pos}(f) > 1/2\), since otherwise \(\text{neg}(f) \geq 1/2\) and we are already done.

Let \(g: \{-1,1\}^n \rightarrow \{-1,1\}\) be the LTF obtained by removing the negative weights from \(f\). By independence of \((x_i)_{i \in P}\) and \((x_j)_{j \in N}\) for uniform \(x \in \{-1,1\}^n\),

\[
\Pr_x[f(x) = g(x)] \geq \Pr_x \left[ \sum_{i \in P} w_i \cdot x_i - \theta \right] \geq \frac{\varepsilon \sqrt{\text{pos}(f)}}{2} \cdot \Pr_x \left[ \sum_{j \in N} w_j \cdot x_j \right] \leq \frac{\varepsilon \sqrt{\text{pos}(f)}}{2}. \quad (3)
\]

We can lower bound the first probability by

\[
\Pr_x \left[ \sum_{i \in P} w_i \cdot x_i - \theta \right] \geq \frac{\varepsilon \sqrt{\text{pos}(f)}}{2} \geq \Pr_{z \sim \mathcal{N}(0, \text{pos}(f))} \left[ |z - \theta| \geq \frac{\varepsilon \sqrt{\text{pos}(f)}}{2} \right] - \frac{2\tau}{\sqrt{\text{pos}(f)}} \quad (4)
\]

\[
\geq 1 - \frac{\varepsilon}{2} - \frac{2\tau}{\sqrt{\text{pos}(f)}} \geq 1 - \frac{3\varepsilon}{4}, \quad (5)
\]

by using the Berry–Esséen theorem for (4). Note that even though the error term from the Berry–Esséen theorem is \(\tau/\sqrt{\text{pos}(f)}\), the set we are interested in is indeed the union of two intervals, giving
us $2\tau/\sqrt{\text{pos}(f)}$. In (5) we used Gaussian anti-concentration, $\text{pos}(f) \geq 1/2$, and $\tau \leq \varepsilon/16$. On the other hand, we can lower bound the second probability of (3) by

$$
\Pr_x \left[ \left| \sum_{j \in N} w_j \cdot x_j \right| \leq \frac{\varepsilon \sqrt{\text{pos}(f)}}{2} \right] \geq 1 - 2 \cdot \exp \left( -\frac{\varepsilon^2 \text{pos}(f)}{8 \text{neg}(f)} \right),
$$

using Hoeffding’s Inequality. Moreover, since $g$ is a monotone function and $f$ is $\varepsilon$-far from monotone we have $\Pr_x [f(x) = g(x)] \leq 1 - \varepsilon$. Combining all these inequalities, we get

$$
1 - \varepsilon \geq \left( 1 - \frac{3\varepsilon}{4} \right) \left( 1 - 2 \cdot \exp \left( -\frac{\varepsilon^2 \text{pos}(f)}{8 \text{neg}(f)} \right) \right),
$$

which, together with $\text{pos}(f) \geq 1/2$, implies that $\text{neg}(f) \geq \varepsilon^2/(16 \ln(8/\varepsilon))$ as claimed.

The following lemma, which will be used to prove a converse to Lemma 3.1, says that if $f$ is an LTF that is close to a monotone function, then $f$ must be close to the LTF obtained by erasing all its negative weights. Recall $\text{dist}(f, \text{MONO})$ is the distance from $f$ to the closest monotone function.

**Lemma 3.2.** Let $f : \{-1,1\}^n \to \{-1,1\}$ be an LTF given by $f(x) = \text{sign}(\sum_{i \in [n]} w_i \cdot x_i - \theta)$ and let $g$ denote the (monotone) LTF given by $g(x) = \text{sign}(\sum_{i \in P} w_i \cdot x_i - \theta)$. Then

$$
\text{dist}(f, g) = \text{dist}(f, \text{MONO}).
$$

**Proof.** We view an assignment $x \in \{-1,1\}^n$ as the concatenation of $x' \in \{-1,1\}^P$ and $y \in \{-1,1\}^N$, and we write $f(x', y)$ for $f(x)$. We denote

$$
p_{x'} = \mathbb{E}_{y} \left[ \frac{1}{2} + \frac{f(x', y)}{2} \right]
$$

as the fraction of 1 inputs. It is clear that $g$ depends only on $x'$ (so we may write $g(x', y)$ simply as $g(x')$) and that $g(x')$ is monotone. Additionally, (by symmetry) we have that

$$
g(x') = 1 \iff p_{x'} \geq \frac{1}{2}.
$$

Thus, $\text{dist}(f, g) = \mathbb{E}_{x'} \left[ \min\{p_{x'}, 1 - p_{x'}\} \right]$. Implicitly in Lemma 3.11 in [KMS15], it is shown that when $f$ is unate (which is the case for LTFs), $\text{dist}(f, \text{MONO}) = \mathbb{E}_{x'} \left[ \min\{p_{x'}, 1 - p_{x'}\} \right]$ which finishes the proof.

Here is the converse to Lemma 3.1:

**Lemma 3.3.** Let $f(x) = \text{sign}(\sum_{i \in [n]} w_i \cdot x_i - \theta)$ be $(\tau, \gamma, \lambda)$-non-monotone with $\tau \leq \sqrt{\lambda}/16$. Then

$$
\text{dist}(f, \text{MONO}) \geq \min \left\{ \Omega(\sqrt{\lambda} \gamma^2) - O(\tau), \Omega \left( \frac{\gamma^3}{\ln(8/\gamma)} \right) - O(\tau \gamma) \right\}.
$$

**Proof.** We may assume without loss of generality that $\text{pos}(f) + \text{neg}(f) = 1$ (by definition, we have $\text{pos}(f) \leq 1 - \lambda$ and $\text{neg}(f) \geq \lambda$) and $\theta \geq 0$. Using Lemma 3.2, it suffices to lower bound $\text{dist}(f, g)$ where $g(x) = \text{sign}(\sum_{i \in P} w_i x_i - \theta)$. Similar to the proof of Lemma 3.2, we view $x \in \{-1,1\}^n$ as the concatenation of $x' \in \{-1,1\}^P$ and $y \in \{-1,1\}^N$. 


The proof has two cases depending on whether or not \( \text{pos}(f) \geq 2/3 \):

**Case 1**: \( \text{pos}(f) \geq 2/3 \). We begin by observing that

\[
\Pr_y \left[ \sum_{j \in N} w_j \cdot y_j < -\sqrt{\text{neg}(f)} \right] \geq \frac{1}{8} - \frac{\tau}{\sqrt{\text{neg}(f)}} \geq \frac{1}{8} - \frac{\tau}{\sqrt{\lambda}} \geq \frac{1}{16},
\]

using the Berry–Esséen theorem as well as the fact that a Gaussian distribution has at least 1/4 of its mass at least one standard deviation away from its mean. Next we establish the following:

**Claim 3.4.** If \( \text{pos}(f) \geq 2/3 \) then \( \Pr_{x'} \left[ 0 \leq \sum_{i \in P} w_i \cdot x'_i - \theta < \sqrt{\text{neg}(f)} \right] \geq \Omega(\sqrt{\lambda \gamma^2} - 3\tau). \)

**Proof.** This holds because (using Berry–Esséen) we have

\[
\Pr_{x'} \left[ 0 \leq \sum_{i \in P} w_i \cdot x'_i - \theta < \sqrt{\text{neg}(f)} \right] \geq \Pr_{z \sim N(\theta, \text{pos}(f))} \left[ z \in \left[ \theta, \theta + \sqrt{\text{neg}(f)} \right] \right] - \frac{2\tau}{\sqrt{\text{pos}(f)}} \\
\geq \sqrt{\text{neg}(f)} \left( \frac{1}{\sqrt{2\pi \cdot \text{pos}(f)}} \cdot \exp \left( -\frac{(\theta + \sqrt{\text{neg}(f)})^2}{2\text{pos}(f)} \right) \right) - 3\tau \\
\geq \sqrt{\frac{\lambda}{2\pi}} \cdot \exp \left( -\frac{(\theta + \sqrt{1/3})^2}{4/3} \right) - 3\tau = \Omega(\sqrt{\lambda \gamma^2} - 3\tau).
\]

where we used \( \text{pos}(f) \in [2/3, 1] \), \( \text{neg}(f) \leq 1/3 \), \( \theta \leq \sqrt{2\ln(2/\gamma)} \) (which we prove immediately) and

\[
\exp \left( -\frac{(\theta + \sqrt{1/3})^2}{4/3} \right) = \Omega(\gamma^2).
\]

This can be shown by using \( \theta \leq \sqrt{2\ln(2/\gamma)} \) and considering the two cases of \( \theta = O(1) \) and \( \theta = \omega(1) \). The case when \( \theta = O(1) \) is trivial since \( \gamma \leq 1 \), and when \( \theta = \omega(1) \), we have \( 3(\theta + \sqrt{1/3})^2/4 < \theta^2 \).

Finally, the upper bound of \( \sqrt{2\ln(2/\gamma)} \) on \( \theta \) follows directly from

\[
\frac{\gamma}{2} \leq \Pr_x \left[ \sum_{i \in [n]} w_j \cdot x_j > \theta \right] \leq e^{-\theta^2/2},
\]

where we have used Hoeffding’s inequality and the fact that \( \text{pos}(f) + \text{neg}(f) = 1 \).

Combining Claim 3.4 and (7), we get that if \( \text{pos}(f) \geq 2/3 \) then

\[
\Pr_x[f(x) \neq g(x)] \geq \Pr_y \left[ \sum_{j \in N} w_j \cdot y_j < -\sqrt{\text{neg}(f)} \right] \cdot \Pr_{x'} \left[ 0 \leq \sum_{i \in P} w_i \cdot x'_i - \theta < \sqrt{\text{neg}(f)} \right],
\]

which is at least \( \Omega(\sqrt{\lambda \gamma^2}) - O(\tau) \) as desired.

**Case 2**: \( \text{pos}(f) < 2/3 \). We can assume \( g \) satisfies \( \|E[g]\| \leq 1 - \gamma/2 \) since otherwise \( \text{dist}(f, g) \geq \gamma/4 \) by virtue of the difference in their expectations. Because \( \|E[g]\| \leq 1 - \gamma/2 \), at least a \( \gamma/4 \) fraction of \( x' \in \{-1, 1\}^P \) satisfy \( g(x') = 1 \), i.e., \( \theta \leq \sum_{i \in P} w_i \cdot x'_i \). By Hoeffding’s inequality, we have

\[
\Pr_{x'} \left[ \sum_{i \in P} w_i \cdot x'_i \geq \sqrt{2\ln(8/\gamma) \cdot \sqrt{\text{pos}(f)}} \right] \leq \gamma/8.
\]
This means that at least a $\gamma/8$ fraction of $x' \in \{-1, 1\}^P$ satisfy (since $\theta \geq 0$)

$$\theta \leq \sum_{i \in P} w_i \cdot x'_i \leq \theta + \sqrt{2 \ln(8/\gamma)} \cdot \sqrt{\text{pos}(f)}.$$ 

Next, recall that the variance of $\sum_{j \in N} w_j \cdot y_j$ is $\text{neg}(f)$. So the Gaussian tail lower bound (2) from Section 2.5 together with Berry–Esséen as well as $\text{neg}(f) \geq 1/3 > \text{pos}(f)/2$ gives

$$\Pr_y \left[ \sum_{j \in N} w_j \cdot y_j < -\sqrt{2 \ln(8/\gamma)} \cdot \text{pos}(f) \right] \geq \Pr_y \left[ \sum_{j \in N} w_j \cdot y_j < -2\sqrt{\ln(8/\gamma)} \cdot \text{neg}(f) \right]$$

$$\geq \Omega \left( \frac{\gamma^2}{\ln(8/\gamma)} \right) - \frac{\tau}{\sqrt{\text{neg}(f)}}$$

$$\geq \Omega \left( \frac{\gamma^2}{\ln(8/\gamma)} \right) - 2\tau.$$  \hfill (9)

As a result, $\Pr_x[f(x) \neq g(x)]$ is at least the product of (8) and (9), concluding the proof. \hfill \Box

### 3.2 Restrictions of monotonically non-decreasing variables

Our goal in this subsection is to show that for any LTF $f: \{-1, 1\}^n \to \{-1, 1\}$, a random restriction that fixes variables of $f$ that are monotonically non-decreasing has, in expectation, the same distance to monotonicity as the original function $f$. We will use this in the proof of Lemma 5.3.

**Lemma 3.5.** Let $f: \{-1, 1\}^n \to \{-1, 1\}$ be an LTF and let $S \subseteq [n]$ be a set of variables of $f$ that are monotonically non-decreasing. Then a random restriction $\rho$ that fixes each variable in $S$ independently and uniformly to a random element of $\{-1, 1\}$ satisfies

$$\mathbb{E}_\rho \left[ \text{dist}(f_\rho, \text{Mono}) \right] = \text{dist}(f, \text{Mono}).$$

**Proof.** We let $g: \{-1, 1\}^n \to \{-1, 1\}$ be the LTF from Lemma 3.2.

$$\mathbb{E}_\rho \left[ \text{dist}(f_\rho, \text{Mono}) \right] = \mathbb{E}_\rho \left[ \text{dist}(f_\rho, g_\rho) \right] = \text{dist}(f, g) = \text{dist}(f, \text{Mono}).$$

\hfill \Box

### 4 Algorithmic tools for LTFs

Our algorithm uses a few simple subroutines that may be viewed as relatively low-level algorithmic tools for working with LTFs. We present and analyze those tools in this section.

#### 4.1 Finding high-influence variables

We start with a subroutine that finds high-influence variables of a function.
Subroutine \textbf{Find-Hi-Influence-Vars}(f, \rho, \tau, \delta)

\textbf{Input:} Black-box oracle access to \(f: \{-1, 1\}^n \to \{-1, 1\}\), a restriction \(\rho \in \{-1, 1, \ast\}^n\), and parameters \(\tau, \delta > 0\).

\textbf{Output:} Set \(H \subseteq \text{STARS}(\rho)\).

1. If \(|\text{STARS}(\rho)|\) is not a power of 2, augment \(\text{STARS}(\rho)\) with “dummy” variables (that are irrelevant in \(f_\rho\)) to bring its size to the next power of 2. Let \(X'\) be this set of variables.

2. Initialize the collection \(\mathcal{S}\) of sets of variables to be \(\mathcal{S} = \{X'\}\).

3. While \(\mathcal{S}\) contains an element (i.e., a set of variables) which is of size > 1:
   
   a. Remove an arbitrary element \(X\) from \(\mathcal{S}\) that is of maximum size (any one will do).
   
   b. Partition \(X\) into equal-size subsets \(A\) and \(B\) (any partition will do).
   
   c. Let \(\delta' = \tau^2\delta/(8 \log n)\). Call \textbf{Estimate-Sum-of-Squares}\((f_\rho, A, \tau^2/10, \delta')\) and let \(\hat{a}\) be the value it returns, and call \textbf{Estimate-Sum-of-Squares}\((f_\rho, B, \tau^2/10, \delta')\) and let \(\hat{b}\) be the value it returns. (If the total number of calls made to \textbf{Estimate-Sum-of-Squares} ever exceeds \(8\log n/\tau^2\), halt and output “fail.”)
   
   d. If \(\hat{a} > 3\tau^2/4\) then add \(A\) to \(\mathcal{S}\). Similarly if \(\hat{b} > 3\tau^2/4\) then add \(B\) to \(\mathcal{S}\).

4. Return the set \(H\) that consists of all elements \(i\) such that the set \(\{i\}\) is an element of \(\mathcal{S}\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Subroutine \textbf{Find-Hi-Influence-Vars}.}
\end{figure}

\textbf{Lemma 4.1.} Suppose that the subroutine \textbf{Find-Hi-Influence-Vars}(\(f, \rho, \tau, \delta\)) is called on a function \(f: \{-1, 1\}^n \to \{-1, 1\}\), a restriction \(\rho \in \{-1, 1, \ast\}^n\), and parameters \(\tau, \delta > 0\). Then it runs in \(O(\log n \cdot \log(1/\delta)/\tau^{10}) \cdot n\) time, makes at most \(O(\log n \cdot \log(1/\delta)/\tau^{10})\) queries, and with probability at least \(1 - \delta\) it outputs a set \(H \subseteq \text{STARS}(\rho)\) such that

- If \(|\hat{f}_\rho(i)| \geq \tau\) then \(i \in H\);
- If \(|\hat{f}_\rho(i)| < \tau/2\) then \(i \notin H\).

\textbf{Proof.} The query complexity and running time of the subroutine are immediate given the “escape condition” in Step 3(c) and the bounds of Lemma 2.1. Below we establish the claimed performance guarantee. First notice that given the “escape condition”, with probability at least \(1 - \delta\), every call to \textbf{Estimate-Sum-of-Squares} returns an estimate that is additively \(\pm\tau^2/10\)-accurate. We assume this is the case and show below that the subroutine returns a set \(H\) with the claimed property.

Next, since the size of the initial set \(X'\) is a power of two (say \(2^r\)), it is clear that every set \(X\) that ever belongs to \(\mathcal{S}\) will have size \(2^\ell\) for some integer \(\ell \leq r\). It is also clear that the elements of \(\mathcal{S}\) are always disjoint sets.

We may divide the execution of the Step 3 loop into \(r\) stages \(0, \ldots, r - 1\) where in the \(t\)-th stage each execution of Step 3(a) selects a set \(X\) of size \(2^r - t\). Right before the start of the \(t\)-th stage every element of \(\mathcal{S}\) is a set of size \(2^r - t\), and after the \(t\)-th stage every element is of size \(2^{r - t - 1}\).

Consider the state of the collection \(\mathcal{S}\) right before the beginning of stage \(t\). Because all calls to \textbf{Estimate-Sum-of-Squares} in stage \(t - 1\) returned estimates that are additively \(\pm\tau^2/10\)-accurate,
every $X \in S$ right before the beginning of stage $t$ has $\sum_{i \in X} \hat{f}(i)^2 \geq 0.65\tau^2$. By Parseval’s identity we have that $\sum_{i \in \text{STARS}(\rho)} \hat{f}(i)^2 \leq 1$ so since the elements of $S$ are disjoint sets, it follows that before the beginning of Stage $t$ the number of elements of $S$ is at most $1/(0.65\tau^2) \leq 2/\tau^2$, and hence there are at most $4/\tau^2$ calls to Estimate-Sum-of-Squares made in stage $t$. Since the number of stages $r$ is at most $1 + \log n < 2 \log n$, there are at most $8 \log n/\tau^2$ calls to Estimate-Sum-of-Squares made in total (so the “escape condition” in Step 3(c) is not triggered).

Finally, given that all calls to Estimate-Sum-of-Squares return estimates that are additively $\pm\tau^2/10$-accurate, it is clear that the set $H$ will have the claimed property. 

4.2 Checking that high-influence variables have positive weight

Given an LTF, the next subroutine checks whether the weight of a variable is positive.

Subroutine Check-Weight-Positive($f, \rho, i, \tau, \delta$)

Input: Black-box oracle access to an LTF $f(x) = \text{sign} (\sum_{i=1}^{n} w_i \cdot x_i - \theta)$, a restriction $\rho$, an element $i \in \text{STARS}(\rho)$, and parameters $\tau, \delta > 0$.

Output: Either “negative,” “positive,” or “fail.”

1. Draw $O(\log(1/\delta)/\tau)$ uniform random edges from the $2^{\text{STARS}(\rho)} - 1$ edges in direction $i$ that are consistent with $\rho$.

2. If $f$ (equivalently, $f_{\rho}$) is bi-chromatic and monotone on any of these edges, return “positive;” if $f$ is bi-chromatic and anti-monotone on any of these edges, return “negative;” if $f$ is not bi-chromatic on any of these edges, return “fail.”

(Note that since $f$ is an LTF, it is a unate function and thus it is impossible for the set of edges drawn in Step 1 to contain both a monotone edge and an anti-monotone edge.)

Lemma 4.2. Suppose that Check-Weight-Positive is called on an LTF $f(x) = \text{sign} (\sum_{i=1}^{n} w_i \cdot x_i - \theta)$, a restriction $\rho \in \{-1, 1, *\}^n$, $i \in \text{STARS}(\rho)$, and $\tau, \delta > 0$ such that $|\hat{f}_{\rho}(i)| \geq \tau$ (note that the latter implies that $w_i \neq 0$). Then it runs in $O(\log(1/\delta)/\tau) \cdot n$ time, makes $O(\log(1/\delta)/\tau)$ queries, and

- If it does not output “fail”, which happens with probability at most $\delta$,
- It outputs “positive” if $w_i > 0$, and it outputs “negative” if $w_i < 0$.

Proof. This follows from $\text{Inf}_i(f) = |\hat{f}(i)|$ for any LTF as well as the fact that if $\hat{f}(i) \neq 0$ in an LTF $f = \text{sign} (\sum_{i=1}^{n} w_i x_i - \theta)$, then $\hat{f}(i) > 0$ iff $w_i > 0$ and $\hat{f}(i) < 0$ iff $w_i < 0$. 

5 Detailed description of the algorithm

We present our algorithm and its analysis in this section.

5.1 The algorithm

Our main testing algorithm, Mono-Test-LTF, is presented in Figure 2. Its main components are two procedures called Regularize-and-Balance and Main-Procedure, described and analyzed in Sections 5.2 and 5.3. As will become clear later, Mono-Test-LTF is one-sided, i.e., it always outputs
Algorithm Mono-Test-LTF($f, \varepsilon$)

**Input:** Black-box oracle access to an LTF $f: \{-1,1\}^n \to \{-1,1\}$ and a parameter $\varepsilon > 0$.

**Output:** Returns “monotone” or “non-monotone.”

1. Call Regularize-and-Balance($f, \varepsilon$). If it returns a restriction $\rho \in \{-1,1,\ast\}^n$ then continue to Step 2; if it returns “non-monotone,” halt and output “non-monotone;” if it returns “monotone,” halt and output “monotone.”

2. Call Main-Procedure($f, \rho, \varepsilon$). If it returns “non-monotone,” halt and output “non-monotone;” if it returns “monotone,” halt and output “monotone.”

Figure 2: Main algorithm Mono-Test-LTF. If $f$ is monotone it outputs “monotone” with probability 1; if $f$ is $\varepsilon$-far from monotone, it outputs “non-monotone” with probability $\geq 2/3$.

“monotone” when the input function $f$ is monotone (because it only outputs “non-monotone” when an anti-monotone edge is found, via Check-Weight-Positive or Edge-Tester). Thus, our analysis of correctness below focuses on the case when $f$ is an LTF that is $\varepsilon$-far from monotone, and shows that in this case Mono-Test-LTF outputs “non-monotone” with probability at least 2/3.

### 5.2 Key properties of procedure Regularize-and-Balance

Let $f: \{-1,1\}^n \to \{-1,1\}$ be an LTF, given by $f(x) = \text{sign}(w \cdot x - \theta)$. Assume that $f$ is $\varepsilon$-far from monotone. The goal of the procedure Regularize-and-Balance($f, \varepsilon$) is to return a restriction $\rho \in \{-1,1,\ast\}^n$ such that $f_\rho$ is a $(\tau, \varepsilon, \lambda)$-non-monotone LTF (with respect to $(w, \theta)$), where

$$\lambda = \frac{\varepsilon^2}{36 \ln(12/\varepsilon)} \quad \text{and} \quad \tau = \frac{\lambda \varepsilon}{\log^2 n}. \quad (10)$$

Here is some intuition that may be helpful in understanding Regularize-and-Balance. If the procedure halts and outputs “monotone” in Step 2, this signals that the (low-probability) failure event of Find-Hi-Influence-Variables has taken place (since it has spuriously identified more variables as having high influence than is possible given Parseval’s identity; see Lemma 4.1). The procedure halts and outputs “non-monotone” in Step 3 only if Check-Weight-Positive has unambiguously found an anti-monotone edge. If the procedure outputs “monotone” in Step 3, this signals the (low-probability) event that Check-Weight-Positive failed to identify some index $i \in H$ (which was supposed to have high influence) as either having $w_i > 0$ or $w_i < 0$. Finally if it outputs “monotone” in Step 4, this signals that $f$ appears to be close to monotone.\(^7\)

It is clear that Regularize-and-Balance is one-sided.

**Fact 5.1.** Regularize-and-Balance($f, \varepsilon$) never returns “non-monotone” if $f$ is monotone.

We also have the following upper bound for the number of queries it uses (which can be straightforwardly verified by tracing through procedure calls and parameter settings):

\(^7\)This will become clear later in the proof of Lemma 5.3 where we show that Step 4 fails with low probability when $f$ is far from monotone.
Procedure \textbf{Regularize-and-Balance}(f, \varepsilon)

**Input:** Parameter $\varepsilon > 0$ and black-box oracle access to an LTF $f: \{-1, 1\}^n \to \{-1, 1\}$ of the form $f(x) = \text{sign}(w \cdot x - \theta)$, with unknown weights $w$ and threshold $\theta$.

**Output:** Either “non-monotone,” “monotone,” or a restriction $\rho \in \{-1, 1, *\}^n$.

1. Let $C_{RB} > 0$ be a large enough constant, and let $\tau'$ and $\delta$ be the following parameters:
   $$\tau' = \frac{\tau^2 \varepsilon^3}{C_{RB}} \quad \text{and} \quad \delta = \frac{\tau'^2}{C_{RB}}.$$

2. Call \textbf{Find-Hi-Influence-Vars}(f, (\ast)^n, \tau', \delta) and let $H$ be the set it returns.
   If $|H| > \frac{4}{\tau'^2}$, halt and output “monotone.”

3. For each $i \in H$, call \textbf{Check-Weight-Positive}(f, (\ast)^n, i, \tau'/2, \delta). If any call returns “negative,” halt and output “non-monotone;” if any call returns “fail,” halt and output “monotone;” otherwise (the case that every call returns “positive”) continue to Step 4.

4. Repeat $C_{RB}/\varepsilon$ times:
   - Draw a restriction $\rho$, which has support $H$ and is obtained by selecting a random assignment from $\{-1, 1\}^H$. Call \textbf{Check-Fourier-Regular}(f$_\rho$, [n] \setminus H, $\sqrt{12/\tau'/\varepsilon}$, $\delta/2$) and \textbf{Estimate-Mean}(f$_\rho$, $\varepsilon/6$, $\delta/2$).
   - Halt and output the first $\rho$ where \textbf{Check-Fourier-Regular} outputs “regular” and \textbf{Estimate-Mean} returns a number of absolute value $\leq 1 - 7\varepsilon/6$. If the procedure fails to find such a restriction $\rho$, halt and output “monotone.”

Figure 3: Procedure \textbf{Regularize-and-Balance}. Our analysis (Lemma 5.3) focuses on the case when $f$ is $\varepsilon$-far from monotone.

**Fact 5.2.** The number of queries used by \textbf{Regularize-and-Balance}(f, \varepsilon) is $\tilde{O}(\log^{41} n/\varepsilon^{90})$.

We now prove the main property of the procedure \textbf{Regularize-and-Balance}.

**Lemma 5.3.** If $f(x) = \text{sign}(w \cdot x - \theta)$ is $\varepsilon$-far from monotone, then with probability at least 9/10, \textbf{Regularize-and-Balance}(f, \varepsilon) returns either “non-monotone,” or a restriction $\rho$ such that $f_\rho$ is a $(\tau, \varepsilon, \lambda)$-non-monotone LTF with respect to $(w, \theta)$.

**Proof.** Using Lemma 4.1, with probability $1 - \delta$, \textbf{Find-Hi-Influence-Vars} in Step 2 returns a set $H \subseteq [n]$ of indices that satisfies the following property:

$$|\hat{f}(i)| \geq \tau' \quad \text{then} \quad i \in H; \quad |\hat{f}(i)| < \tau'/2 \quad \text{then} \quad i \notin H. \quad (11)$$

When this happens, we have by Parseval $|H| \leq 4/\tau'^2$, and the procedure continues to Step 3.

We consider two subevents: $E'_0$: $H$ satisfies (11) but contains an elements $i$ with $w_i < 0$; and $E_0$: $H$ satisfies (11) and every $i \in H$ has $w_i > 0$. We have $\Pr[E'_0] + \Pr[E_0] \geq 1 - \delta$ as discussed above. Below we show that the procedure returns “non-monotone” with high probability, conditioning on $E'_0$, and it returns a restriction with the desired property with high probability, conditioning on $E_0$.  

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By the end we combine the two cases to conclude that
\[
\Pr[E_0'] \cdot \Pr\left[\text{the procedure returns "non-monotone" } \mid E_0'\right] + \Pr[E_0] \cdot \Pr\left[\text{it returns } \rho \text{ such that } f_\rho \text{ is } (\tau, \varepsilon, \lambda)\text{-non-monotone} \mid E_0\right] \geq 9/10.
\]

We first address the (easier) case of \(E_0'\). Assume \(i \in H\) satisfies \(w_i < 0\). From (11), \(|\hat{f}(i)| \geq \tau' / 2\) and thus, \text{Check-Weight-Positive}(f, (\tau')^n, i, \tau' / 2, \delta) in Step 3 returns "negative" with probability \(1 - \delta\), and the procedure returns "non-monotone" with probability \(1 - \delta\), conditioning on \(E_0'\).

Next we address the (harder) case of \(E_0\). First we use \(E_1\) to denote the event that every call to \text{Check-Weight-Positive} in Step 3 returns the correct answer, i.e., it returns "positive" for every \(i \in H\). By a union bound we have \(\Pr[E_1 \mid E_0] \geq 1 - 4\delta / \tau^2\).

Assuming that \(E_1\) happens, the procedure then proceeds to Step 4 and we use \(E_2\) to denote the event that \text{Check-Fourier-Regular} and \text{Estimate-Mean} in Step 4 return the correct answer, i.e.:

1. \text{Check-Fourier-Regular} outputs "not regular" if \(|\hat{f}_\rho(i)| \geq \sqrt{12\tau' / \varepsilon}\) for some \(i \in [n] \setminus H\), and outputs "regular" if \(|\hat{f}_\rho(i)| \leq 3\tau' / \varepsilon\) for all \(i \in [n] \setminus H\), for every \(\rho\) in Step 4, and
2. \text{Estimate-Mean} returns a number \(a\) with \(|a - E[f_\rho]| \leq \varepsilon / 6\), for every \(\rho\) in Step 4.

We also write \(E_3\) to denote the event that one of the restrictions \(\rho\) drawn in Step 4 satisfies that \(f_\rho\) is both \((2\varepsilon / 3)\)-far from monotone and \((3\tau' / \varepsilon)\)-Fourier-regular. By a union bound we have
\[
\Pr[E_2 \mid E_0 \land E_1] \geq 1 - C_{RB}\delta / \varepsilon.
\]

In the rest of the proof we show that 1) \(\Pr[|E_3| \mid E_0 \land E_1] \geq 99 / 100\) and 2) given \(E_0, E_1, E_2\) and \(E_3\) the procedure always returns a restriction \(\rho\) such that \(f_\rho\) is \((\tau, \varepsilon, \lambda)\)-non-monotone. Together we have that the procedure returns such a \(\rho\) with probability at least (conditioning on \(E_0\))
\[
(1 - 4\delta / \tau^2) \cdot (1 - C_{RB}\delta / \varepsilon - 1 / 100).
\]

Summarizing the two cases of \(E_0'\) and \(E_0\) we have that \text{Regularize-and-Balance} returns either "non-monotone" or a restriction \(\rho\) such that \(f_\rho\) is \((\tau, \varepsilon, \lambda)\)-non-monotone with probability at least
\[
\Pr[E_0'] \cdot (1 - \delta) + \Pr[E_0] \cdot (1 - 4\delta / \tau^2) \cdot (1 - C_{RB}\delta / \varepsilon - 1 / 100) > 9 / 10,
\]
using \(\Pr[E_0'] + \Pr[E_0] \geq 1 - \delta\) and our choice of \(\delta\) (by letting \(C_{RB}\) be large enough).

We use the following claim to show that \(\Pr[|E_3| \mid E_0 \land E_1] \geq 99 / 100\).

\textbf{Claim 5.4.} A random restriction \(\rho\) over \(H\) satisfies that \(f_\rho\) is both \((2\varepsilon / 3)\)-far from monotone and \((3\tau' / \varepsilon)\)-Fourier-regular with probability at least \(\varepsilon / 3\).

\textbf{Proof.} For each of the two properties, we have

1. Proposition 2.10: with probability at least \(1 - (\varepsilon / 3)\), \(f_\rho\) is \((3\tau' / \varepsilon)\)-Fourier-regular.

2. Lemma 3.5: with probability at least \(2\varepsilon / 3\), \(f_\rho\) is \((2\varepsilon / 3)\)-far from monotone. To see this, let \(c\) denote the probability of \(f_\rho\) being \((2\varepsilon / 3)\)-far from monotone. Then \(c \geq 2\varepsilon / 3\) follows from
\[
(1 - c) \cdot (2\varepsilon / 3) + c \cdot (1 / 2) \geq \varepsilon,
\]
where we used the fact that distance to monotonicity is always at most \(1 / 2\).
The claim then follows from a union bound.

By choosing \( C_{RB} \) to be a large enough constant, we have \( \Pr[E_3 | E_0 \land E_1] \geq 99/100. \)

Finally we show that conditioning on all four events \( E_0, E_1, E_2, E_3 \) the procedure always returns a restriction \( \rho \) such that \( f_\rho \) is a \((\tau, \varepsilon, \lambda)\)-non-monotone LTF. We do this in two steps:

1. First, given \( E_3 \), one of the restrictions \( \rho \) drawn in Step 4 is both \((2\varepsilon/3)\)-far from monotone and \((3\tau'/\varepsilon)\)-Fourier-regular. Given \( E_2 \), \( \rho \) must pass both tests, i.e., \textbf{Check-Fourier-Regular} outputs “regular” and \textbf{Estimate-Mean} returns a number of absolute value at most \( 1 - 7\varepsilon/6 \) in Step 4. The former is trivial; to see the latter, note that being \((2\varepsilon/3)\)-far from monotone implies that \( |E[f_\rho]| \leq 1 - 4\varepsilon/3 \) and therefore, the number returned by \textbf{Estimate-Mean} is at most \( 1 - 7\varepsilon/6 \), given \( E_2 \).

2. Second, we show that if a restriction \( \rho \) passes both tests in Step 4 of the procedure, then \( f_\rho \) must be \((\tau, \varepsilon, \lambda)\)-non-monotone. One can think of this as a soundness property, saying that if the procedure halts and returns some restriction \( \rho \), that it returns a correct one. To see this, note that by \( E_2 \), \( f_\rho \) is both \( \sqrt{12\tau'/\varepsilon} \)-Fourier regular and \( \varepsilon \)-balanced. By Theorem 2.9, \( f_\rho \) is \( \Omega(\sqrt{\tau'/\varepsilon^3}) \)-weight-regular, and \( \tau \)-weight-regular by letting \( C_{RB} \) be large enough. It also follows from Lemma 3.1 that \( f_\rho \) has \( \lambda \)-significant squared negative weights.

This finishes the proof of the lemma.

5.3 Key properties of \textbf{Main-Procedure}

\textbf{Main-Procedure} is presented in Figure 4. Given Lemma 5.3 we may assume that the input \((f, \rho, \varepsilon)\) satisfies that \( f_\rho \) is a \((\tau, \varepsilon, \lambda)\)-non-monotone LTF (see the choices of \( \tau \) and \( \lambda \) in (10)).

We prove the following main lemma in this section.

\textbf{Lemma 5.5.} \textbf{Main-Procedure} \((f, \rho, \varepsilon)\) never returns “non-monotone” when \( f \) is monotone. When \( f_\rho \) is a \((\tau, \varepsilon, \lambda)\)-non-monotone LTF, it returns “non-monotone” with probability at least \( 81/100 \).

The procedure only returns “non-monotone” when it finds an anti-monotone edge in the subroutine \textbf{Check-Weight-Positive}. Hence we may focus on the case when \( f_\rho \) is \((\tau, \varepsilon, \lambda)\)-non-monotone. For this purpose, we analyze the three steps 2(a), 2(b), 2(c) of each while loop of \textbf{Main-Procedure}, and prove the following lemma.

\textbf{Lemma 5.6.} Let \( t \leq 4 \log n \). Suppose that at the beginning of the \((t+1)\)th loop of \textbf{Main-Procedure}, \( f_{\rho(t)} \) is \((\tau, \varepsilon, \lambda(1 - t/(8 \log n)))\)-non-monotone. Then with probability at least \( 1 - 1/(40 \log n) \), it either returns “non-monotone” within this loop or obtains a set \( A_t \subseteq [n] \setminus \text{supp}(\rho(t)) \) and a restriction \( \rho^{(t+1)} \) extending \( \rho(t) \) at the end of this loop such that

1. \(|A_t| \geq |\text{STARS}(\rho(t))|/4; \)
2. \( \text{supp}(\rho(t)) \cup A_t \subseteq \text{supp}(\rho(t+1)); \) and
3. \( f_{\rho(t+1)} \) is a \((\tau, \varepsilon, \lambda(1 - (t + 1)/(8 \log n)))\)-non-monotone LTF.

We use Lemma 5.6 to prove Lemma 5.5.
Procedure \textbf{Main-Procedure}(f, \rho, \varepsilon)

\textbf{Input:} Parameter \(\varepsilon > 0\), black-box oracle access to an LTF \(f: \{-1, 1\}^n \rightarrow \{-1, 1\}\) of the form \(f(x) = \text{sign}(w \cdot x - \theta)\) with unknown weights \(w\) and threshold \(\theta\), and a restriction \(\rho\).

\textbf{Output:} Either “non-monotone” or “monotone.”

1. Set \(t = 0\) and \(\rho^{(0)} = \rho\).

2. While \(|\text{stars}(\rho^{(t)})| \geq 1/\tau^2\), repeat the following steps:

   (a) Construct a subset \(A_t \subseteq \text{stars}(\rho^{(t)})\) by independently putting each index \(i \in \text{stars}(\rho^{(t)})\) into \(A_t\) with probability \(1/2\).

   (b) Call \textbf{Find-Balanced-Restriction}(f, \rho^{(t)}, A_t, \varepsilon). If it returns “monotone” then halt and return “monotone;” otherwise, it returns a restriction \(\rho'\) with \(\text{supp}(\rho') = \text{supp}(\rho^{(t)}) \cup A_t\).

   (c) Call \textbf{Maintain-Regular-and-Balance}(f, \rho', \varepsilon). If it returns “non-monotone” then halt and output “non-monotone;” if it returns “monotone” then halt and output “monotone;” otherwise, it returns a restriction \(\eta\) and we set \(\rho^{(t+1)} = \rho'\eta\).

   (d) Increment \(t\) by 1. If \(t > 4 \log n\), halt and output “monotone;” otherwise proceed to the next iteration of step (a) of the loop.

3. Let \(\varepsilon' = \varepsilon^3/(C\log(1/\varepsilon))\) for some large enough constant \(C\); run \textbf{Edge-Tester}(f_{\rho^{(t)}}, \varepsilon', 1/10) and output what it outputs (either “monotone” or “non-monotone”).

Figure 4: Procedure \textbf{Main-Procedure}. Our analysis in Section 5.3 focuses on the case when \(f_{\rho}\) is a \((\tau, \varepsilon, \lambda)\)-non-monotone LTF.

\textbf{Proof of the Second Part of Lemma 5.5 using Lemma 5.6}. We consider the event \(E\) where the conclusion of Lemma 5.6 holds for every iteration of the while loop of \textbf{Main-Procedure}. As the condition of Lemma 5.6 holds for the first loop (\(t = 0\)) and there are at most \(4 \log n\) many loops, this happens with probability at least 9/10. Since \(E\) implies \(|A_t| \geq |\text{stars}(\rho^{(t)})|)/4\), we can also assume that the procedure never halts and outputs “monotone” due to line 2(d).

Given \(E\), \textbf{Main-Procedure} either returns “non-monotone” as desired or reaches line 3. Furthermore, if it reaches line 3, \(f_{\rho^{(t)}}\) must be \((\tau, \varepsilon, \lambda/2)\)-non-monotone by Lemma 5.6 and have at most \(1/\tau^2\) variables. It follows from Lemma 3.3 that \(f_{\rho^{(t)}}\) is \(\varepsilon'\)-far from monotone, where \(\varepsilon' = \varepsilon^3/(C\log(1/\varepsilon))\) for some large enough constant \(C\). Finally, by Theorem 2.4, \textbf{Edge-Tester} outputs “non-monotone” (by finding an anti-monotone edge) with probability at least 9/10 and the proof is complete.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{Procedure \textbf{Main-Procedure}. Our analysis in Section 5.3 focuses on the case when \(f_{\rho}\) is a \((\tau, \varepsilon, \lambda)\)-non-monotone LTF.}
\end{figure}

\textbf{5.3.1 Proof of Lemma 5.6}

The proof of Lemma 5.6 consists of three lemmas, one for each steps 2(a), 2(b) and 2(c). Below we assume that the condition of Lemma 5.6 holds at the beginning of the \((t + 1)\)th loop, for some \(t \leq 4 \log n\). We introduce the following notation for convenience. We let \(I = \text{stars}(\rho^{(t)})\), with \(m = |I|\). Given the random subset \(A_t\) of \(I\) found in Step 2(a), we let \(B_t = I \setminus A_t\). Also note that \(m \geq 1/\tau^2\).
Subroutine \texttt{Find-Balanced-Restriction}(f, \rho^{(t)}, A_t, \varepsilon)

**Input:** oracle access to \( f \colon \{-1,1\}^n \to \{-1,1\} \), restriction \( \rho^{(t)} \), \( A_t \subseteq \text{STARS}(\rho^{(t)}) \), and \( \varepsilon > 0 \).

**Output:** “monotone” or a restriction \( \rho' \) with \( \text{supp}(\rho') = \text{supp}(\rho^{(t)}) \cup A_t \) that extends \( \rho^{(t)} \).

Repeat \( C_{BR} \cdot \log n / \varepsilon^3 \) times for some large enough constant \( C_{BR} \):

1. Draw a restriction \( \rho^* \), which has support \( A_t \) and is obtained by selecting a random assignment from \( \{-1,1\}^{A_t} \), and let \( \rho' = \rho^{(t)} \rho^* \). Call \texttt{Estimate-Mean}(\( f_{\rho'}, 0.01, \delta \)), where \( \delta = \varepsilon^3 / (200 C_{BR} \log^2 n) \). If it returns a number of absolute value at most 0.03, halt and output \( \rho' \).

Otherwise, output “monotone.”

Figure 5: Subroutine \texttt{Find-Balanced-Restriction}. We are interested in the case when \( f_{\rho^{(t)}} \) is a \((\tau, \varepsilon, \lambda(1-t/(8 \log n)))\)-non-monotone LTF, and \( A_t \) satisfies the conditions of Lemma 5.8.

We start with the lemma for Step 2(a), which states that with high probability, \( A_t \) is large and splits the weights (both positive and negative) in \( I \) evenly.

**Lemma 5.7.** Assume that \( f_{\rho^{(t)}} \) is a \((\tau, \varepsilon, \lambda(1-t/(8 \log n)))\)-non-monotone LTF. With probability at least \( 1 - \exp(-\Omega(\log^2 n)) \), \( A_t \) and \( B_t \) satisfy \( |A_t| \geq m/4 \),

\[
\frac{1}{2} - \frac{1}{32 \log n} \leq \sum_{i \in A_t} w_i^2 \sum_{i \in I} w_i^2 \leq \frac{1}{2} + \frac{1}{32 \log n} \quad \text{and} \quad \frac{\sum_{i \in B_t : w_i < 0} w_i^2}{\sum_{i \in B_t} w_i^2} \geq \lambda \left(1 - \frac{t + 1}{8 \log n}\right). \tag{12}
\]

**Proof.** We consider the three events separately and then apply a union bound.

First by Chernoff bound, \( |A_t| \geq m/4 \) holds with probability at least \( 1 - e^{-\Omega(m)} \).

Next for the first inequality in (12), assume without loss of generality that \( \sum_{i \in I} w_i^2 = 1 \) (as \( f_{\rho^{(t)}} \) cannot be all-1 or all-\((-1)) \). By Hoeffding bound the probability that it does not hold is at most

\[
2 \exp \left(-\Omega \left( \frac{1}{\log n} \sum_{i \in I} w_i^4 \right) \right).
\]

Since \( f_{\rho^{(t)}} \) is \( \tau \)-weight-regular (over \( I \)), we have that \( |w_i| \leq \tau \) for all \( i \in I \) and thus,

\[
\sum_{i \in I} w_i^4 \leq \tau^2 \cdot \sum_{i \in I} w_i^2 = \tau^2.
\]

As a result, the second inequality holds with probability at least \( 1 - \exp(-\Omega(1/((\tau^2 \log^2 n)))) \).

For the last inequality, note that \( \sum_{i \in I : w_i < 0} w_i^2 \geq \lambda(1-t/(8 \log n)) \). Similarly by Hoeffding,

\[
\Pr \left[ \sum_{i \in B_t : w_i < 0} w_i^2 < \left( \frac{\lambda}{2} \right) \left( 1 - \frac{t + 0.5}{8 \log n} \right) \right] \leq \exp \left(-\Omega \left( \frac{\lambda^2}{\log n} \sum_{i \in I : w_i < 0} w_i^4 \right) \right) \leq \exp \left(-\Omega \left( \log^2 n \right) \right).
\]

Combining the above with the analysis of the first inequality in (12), the last inequality holds with probability at least \( 1 - \exp(-\Omega((\log^2 n))) \). The lemma follows from a union bound.
We give Find-Balanced-Restriction in Figure 5 and show the following lemma for Step 2(b). (The Find-Balanced-Restriction subroutine is similar to Algorithm 1 of [RS15], and Lemma 5.8 and its proof are reminiscent of Lemma 7 of [RS15]; however, because of some technical differences we cannot directly apply those results, so we give a self-contained presentation here.)

**Lemma 5.8.** Assume that \( f_{\rho(t)} \) is a \((\tau, \varepsilon, \lambda(1-t/(8 \log n)))\)-non-monotone LTF, and sets \( A_t \) and \( B_t \) satisfy \(|A_t| \geq m/4 \) and (12). With probability at least \( 1/(100 \log n) \), Find-Balanced-Restriction outputs a \( \rho' \) with \( \supp(\rho') = \supp(\rho(t)) \cup A_t \) such that \( \rho' \) extends \( \rho(t) \) and \( f_{\rho'} \) is 0.96-balanced.

**Proof.** For convenience we use \( f' \) to denote \( f_{\rho(t)} \), \( w' \) to denote the weight vector \( w \) but restricted on \( I \), and \( \theta' \) to denote the new threshold, i.e.,

\[
\theta' = \theta - \sum_{i \in \supp(\rho(t))} \rho(t)(i) \cdot w_i.
\]

Without loss of generality we assume that \( \sum_{i \in I} w_i^2 = 1 \). We may additionally assume that \( \theta' \geq 0 \). This assumption is without loss of generality, because 1) if \( \rho' \) is a 0.96-balanced restriction when \(-\theta' \geq 0 \), then \(-\rho' \) is a 0.96-balanced restriction for \( \theta' \leq 0 \), and 2) Find-Balanced-Restriction will test the only take into account the absolute value of the output of Estimate-Mean. Let

\[
\alpha = \sum_{i \in A_t} w_i^2 \quad \text{and} \quad \beta = \sum_{i \in B_t} w_i^2.
\]

We also use \( a = b \pm c \) to denote the inequalities \( b - c \leq a \leq b + c \). Then from (12) we have that \( \alpha, \beta = 1/2 \pm O(1/\log n) \). By the assumption of the lemma, \( f' \) is \( \tau \)-weight-regular and \( \varepsilon \)-balanced.

For the analysis we define two events \( E_1 \) and \( E_2 \). Here \( E_1 \) denotes the event that every call to Estimate-Mean returns a number \( a \) such that \( |a - \mathbb{E}[f'_{\rho'}]| \leq 0.01 \). By a union bound, this happens with probability \( 1 - 1/(200 \log n) \). Let \( E_2 \) be the event that one of the restrictions \( \rho^* \) drawn has \( f'_{\rho^*} \) being 0.98-balanced. When \( E_1 \) and \( E_2 \) both occur, the subroutine outputs a restriction \( \rho' \) such that \( f_{\rho'} \) is 0.96-balanced. In the rest of the proof we show that \( E_2 \) happens with high probability.

To analyze the probability of \( f'_{\rho^*} \) being 0.98-balanced, we use \( x_i \) to denote an independent and unbiased random \( \{-1, 1\} \)-variable for each \( i \in I \), and let

\[
x_A = \sum_{i \in A_t} x_i \cdot w_i', \quad x_B = \sum_{i \in B_t} x_i \cdot w_i' \quad \text{and} \quad x = x_A + x_B.
\]

By Hoeffding bound and the assumption that \( f' \) is \( \varepsilon \)-balanced, we have

\[
2\varepsilon = \Pr[x \geq \theta'] \leq \exp(-\theta'^2/2).
\]

Using Berry–Esséen \( x_A + x_B \) is \( O(\tau) \)-close to a standard \( \mathcal{N}(0, 1) \) Gaussian random variable, denoted by \( G \), \( x_A \) is \( O(\tau) \)-close to \( \sqrt{\alpha} G \), and \( x_B \) is \( O(\tau) \)-close to \( \sqrt{\beta} G \).

Let \( \theta^* > 0 \) be the threshold such that \( \Pr[|\sqrt{\beta} G| \leq \theta^*] = 0.01 \). Then

\[
\Pr[f'_{\rho^*} \text{ is 0.98-balanced}] \geq \Pr[x_A \in [\theta' - \theta^*, \theta' + \theta^*]].
\]

This is because, for any number \( x_A \in [\theta' - \theta^*, \theta' + \theta^*] \), we have

\[
0.495 - O(\tau) \leq \Pr[x_B \geq \theta' - x_A] = \Pr\left[\sqrt{\beta} G \geq \theta' - x_A\right] \pm O(\tau) \leq 0.505 + O(\tau),
\]

The rest of the proof follows similar ideas as the proof of Theorem 5.1 in [RS15].
Subroutine \textsc{Maintain-Regular-and-Balanced}(f, \rho', \varepsilon)

\textbf{Input:} oracle access to function \(f: \{-1, 1\}^n \rightarrow \{-1, 1\}\), restriction \(\rho'\), parameter \(\varepsilon > 0\).

\textbf{Output:} “non-monotone,” “monotone,” or a restriction \(\eta\) with \(\text{supp}(\eta) \subseteq B_t\) extending \(\rho'\).

1. Let \(C_M > 0\) be a large enough constant; let \(\tau', \delta\) and \(\tau^*\) be the following parameters:

   \[\tau' = (\varepsilon/\theta C_M) \cdot \sqrt{\lambda}, \quad \delta = \tau^2/(C_M \log n) \quad \text{and} \quad \tau^* = \tau'/\sqrt{\lambda}.\]

2. Call \textsc{Find-Hi-Influence-Vars}(\(f, \rho', \tau', \delta\)) and let \(H\) be the set that it returns.

   If \(|H| > 4/\tau^2\), halt and return “monotone.”

3. For each \(i \in H\), call \textsc{Check-Weight-Positive}(\(f, \rho', i, \tau'/2, \delta\)). If any call returns “negative” then halt and output “non-monotone”; if any call returns “fail” then halt and output “monotone;” otherwise (every call returns “positive”) continue to Step 4.

4. Repeat \(C_M \log n/\sqrt{\lambda}\) times:

   Draw a restriction \(\eta\) with support \(H\), by selecting a random assignment from \(\{-1, 1\}^H\). Call \textsc{Check-Fourier-Regular}(\(f, \rho', \eta, \tau'/2, \delta/2\)) and \textsc{Estimate-Mean}(\(f, \rho', \varepsilon/6, \delta/2\)).

   Halt and output the first restriction \(\eta\) where \textsc{Check-Fourier-Regular} outputs “regular” and \textsc{Estimate-Mean} returns a number of absolute value \(\leq 1 - 7\varepsilon/6\). If the procedure fails to find such a restriction \(\eta\), halt and output “monotone.”

Figure 6: Subroutine \textsc{Maintain-Regular-and-Balanced}. Lemma 5.9 assumes that \(f_{\rho(t)}\) is an \((\tau, \varepsilon, \lambda(1 - t/(8 \log n)))\)-non-monotone LTF, \(|A_t| \geq m/4\) and (12), and \(f_{\rho'}\) is 0.96-balanced.

in which case the function \(f'_{\rho'}\) is \(0.99 - O(\tau) = 0.98\)-balanced. To bound \(\Pr[\{x_A \in [\theta' - \theta^*, \theta' + \theta^*]\}]\), we note that \(\theta' \geq 0\) (by assumption) and \(\theta^* = \Omega(1)\) (by our choice of \(\theta^*\) and \(\beta > 1/3\)). As a result,

\[\Pr[\{x_A \in [\theta' - \theta^*, \theta']\}] \geq \Pr[\{\sqrt{\alpha} \mathcal{G} \in [\theta' - \theta^*, \theta']\}] - O(\tau) = \Omega(1) \cdot \Omega(\varepsilon^3) - O(\tau) = \Omega(\varepsilon^3),\]

where we used \(\alpha > 1/3\) by (12), \(\tau = \alpha(\varepsilon^3)\), and \(\exp(-\theta^2/2) = \Omega(\varepsilon)\) from (13) to obtain

\[
\min \left( \exp\left(-\theta^*^2/(2\alpha)\right), \exp\left(-\theta^2/(2\alpha)\right) \right) = \Omega(\varepsilon^3).\]

As a result, a random restriction \(\rho^*\) is 0.98-balanced with probability at least \(\Omega(\varepsilon^3)\). Thus with probability \(1 - 1/n\) (by choosing a large enough constant \(C_{BR}\)), \textsc{Find-Balanced-Restriction} gets such a restriction that would pass the \textsc{Estimate-Mean} test. By a union bound on the two events \(E_1\) and \(E_2\), \textsc{Find-Balanced-Restriction} returns a 0.96-balanced \(\rho'\) with probability at least

\[
1 - 1/(200 \log n) - 1/n > 1 - 1/(100 \log n).
\]

This finishes the proof of the lemma. \hfill \Box

For Step 2(c) of \textsc{Main-Procedure}, the subroutine \textsc{Maintain-Regular-and-Balanced} is given in Figure 6. It is very similar to \textsc{Regularize-and-Balance} except the number of rounds in Step 4 and the choice of parameters \(\tau'\) and \(\delta\). We show the following lemma.
Lemma 5.9. Assume that $f_{\rho(t)}$ is a $(\tau, \varepsilon, \lambda(1-t/(8 \log n)))$-non-monotone LTF, $A_t$ and $B_t$ satisfy both $|A_t| \geq m/4$ and (12), and $f_{\rho}$ is 0.96-balanced. Then with probability at least $1 - 1/(100 \log n)$, Maintain-Regular-and-Balance returns either “non-monotone,” or a restriction $\eta$ with $\supp(\eta) \subseteq B_t$ such that $f_{\rho(t+1)}$, where $\rho(t+1) = \rho^t \eta$, is $(\tau, \varepsilon, \lambda(1 - (t + 1)/(8 \log n)))$-non-monotone.

Proof. Most of the proof is similar to that of Lemma 5.3. Let $\lambda' = \lambda(1 - (t + 1)/(8 \log n))$.

First it follows from Lemma 4.1 that with probability at least $1 - \delta$, $H$ satisfies

$$\text{If } |\hat{f}_{\rho'}(i)| \geq \tau', \text{ then } i \in H; \text{ if } |\hat{f}_{\rho'}(i)| < \tau'/2, \text{ then } i \notin H. \tag{14}$$

When this happens, we have by Parseval $|H| \leq 4/\tau'^2$, and the subroutine proceeds to Step 3.

Similar to the proof of Lemma 5.3, we consider two subevents: $E_0': H$ satisfies (14) but has an element $i$ with $w_i < 0$; and $E_0$: $H$ satisfies (14) and all $i \in H$ have $w_i > 0$. Then $\Pr[E_0'] + \Pr[E_0] \geq 1 - \delta$. It is easy to show that, given $E_0'$, the subroutine returns “non-monotone” with probability at least $1 - \delta$. So in the rest of the proof, we focus on the event of $E_0$ and lowerbound the probability that the subroutine returns an $\eta$ such that $f_{\rho(t+1)}$ is $(\tau, \varepsilon, \lambda')$-non-monotone, conditioning on $E_0$.

First we let $E_1$ be the event that Check-Weight-Positive returns the correct answer in Step 3, i.e., it returns “positive” for all $i \in H$. By a union bound, we have $\Pr[E_1 | E_0] \geq 1 - O(\delta/\tau'^2)$.

Conditioning on $E_0$ and $E_1$ we proceed to Step 4 and introduce two events $E_2$ and $E_3$. Here $E_2$ is the event that every call to Check-Fourier-Regular and Estimate-Mean returns the correct answer (see the proof of Lemma 5.3). By a union bound, $\Pr[E_2 | E_0 \land E_1] \geq 1 - O(\tau^2/\sqrt{\lambda})$. To define $E_3$, we state the following claim to introduce the constant $a > 0$, but delay its proof to the end.

Claim 5.10. Assume $E_0$ holds. Then there is an absolute constant $a > 0$ such that when $\eta$ is drawn uniformly at random from $\{-1, 1\}^H$, $f_{\rho, \eta}$ is $(4\varepsilon/3)$-balanced with probability at least $a\sqrt{\lambda}$.

Next, we let $E_3$ be the event that one of the restrictions $\eta$ drawn in Step 4 has $f_{\rho, \eta}$ being both $(4\varepsilon/3)$-balanced and $2\tau'/a = 2\tau'/(a\sqrt{\lambda})$-Fourier-regular. Using Proposition 2.10, $f_{\rho, \eta}$ is $2\tau'/(a\sqrt{\lambda})$-Fourier-regular with probability at least $1 - a\sqrt{\lambda}/2$. It follows from Claim 5.10 and a union bound that, for an $\eta$ drawn from $\{-1, 1\}^H$, $f_{\rho, \eta}$ is both $(4\varepsilon/3)$-balanced and $(2\tau'/a)$-Fourier-regular with probability at least $a\sqrt{\lambda}/2$ and thus, $\Pr[E_3 | E_0 \land E_1] \geq 1 - 1/n$, by choosing a large enough $C_M$.

In the rest of the proof, we first show that all four events $E_0, E_1, E_2, E_3$ together are sufficient to imply that the subroutine returns a restriction $\eta$ such that $f_{\rho, \eta}$ is $(\tau, \varepsilon, \lambda')$-non-monotone, and then prove Claim 5.10. Combining everything together, we have that the subroutine returns either “non-monotone” or a restriction $\eta$ with the desired property with probability at least

$$\Pr[E_0'] \cdot (1 - \delta) + \Pr[E_0] \cdot (1 - O(\delta/\tau'^2)) \cdot (1 - O(\tau^2/\sqrt{\lambda}) - (1/n)) > 1 - 1/(100 \log n),$$

using $\Pr[E_0'] + \Pr[E_0] \geq 1 - \delta$ and by choosing a large enough constant $C_M$.

Now we prove the sufficiency of $E_0, E_1, E_2, E_3$. Similar to the proof of Lemma 5.3, we split the proof into two steps. We start with the following claim.

Claim 5.11. Suppose $f_{\rho, \eta}$ is both $(4\varepsilon/3)$-balanced and $(2\tau'/a)$-Fourier-regular. Then assuming $E_2$ and the choice of a large enough $C_M$, $\rho \eta$ passes both tests in Step 4 and $f_{\rho, \eta}$ is $\tau$-weight-regular.

Proof. Since $f_{\rho, \eta}$ is $(2\tau'/a)$-Fourier-regular, by choosing a large enough $C_M$, Check-Fourier-Regular will output “not regular”, given $E_2$. Since $f_{\rho, \eta}$ is $(4\varepsilon/3)$-balanced, Estimate-Mean will return a value which is at least $1 - 4\varepsilon/3 - \varepsilon/6 \geq 1 - 7\varepsilon/6$, given $E_2$. Therefore, $\rho \eta$ passes both tests. On the other hand, by Theorem 2.9, $f_{\rho, \eta}$ is $O(\tau/(ae))$-weight-regular, and thus $\tau$-weight-regular. \qed
Given \( E_3 \), at least one of the \( \eta \) sampled in Step 4 satisfies the assumption of Claim 5.11. It then follows from Claim 5.11 that \( \rho' \eta \) of such an \( \eta \) will pass both tests in Step 4. Moreover, we claim that \( f_{\rho' \eta} \) is a \((\tau, \varepsilon, \lambda')\)-non-monotone LTF. Here the only missing part is that \( f_{\rho' \eta} \) has \( \lambda' \)-significant squared negative weights, which follows from (12) and \( w_i > 0 \) for all \( i \in H \), given \( E_0 \). (Indeed they imply that any restriction \( \eta \) over \( H \) has this property, which we also use in the next claim.)

We finish the proof of the sufficiency of \( E_0, E_1, E_2, E_3 \) with the following claim:

**Claim 5.12.** Assuming \( E_2 \) and the choice of a large enough \( C_M \), any restriction \( \eta \) on \( H \) that passes both the Check-Fourier-Regular test and the Estimate-Mean test must be \((\tau, \varepsilon, \lambda')\)-non-monotone.

**Proof.** Given \( E_2 \), we can conclude that a restriction \( \rho' \eta \) that passes both tests satisfies that \( f_{\rho' \eta} \) is \( \sqrt{C_M} \tau^\varepsilon \)-Fourier-regular and \( \varepsilon \)-balanced. As discussed earlier, \( f_{\rho' \eta} \) always has \( \lambda' \)-significant squared negative weights. By Theorem 2.9 and the choice of a large enough \( C_M \), \( f_{\rho' \eta} \) is \( \tau \)-weight-regular. \( \Box \)

Therefore we have shown that the four events \( E_0, E_1, E_2, E_3 \) together imply that the subroutine returns a restriction \( \eta \) with \( f_{\rho' \eta} \) being \((\tau, \varepsilon, \lambda')\)-non-monotone. Finally we prove Claim 5.10.

**Proof of Claim 5.10.** Assume without loss of generality \( \sum_{i \in B_t} w_i^2 = 1 \) and let \( \theta' \geq 0 \) be the threshold of \( f_{\rho'} \). Let

\[
\alpha = \sum_{i \in H} w_i^2 \quad \text{and} \quad \beta = \sum_{i \in B_t \setminus H} w_i^2,
\]

with \( \alpha + \beta = 1 \). By (12) and \( E_0 \), we have \( \beta \geq \lambda' = \lambda(1 - (t + 1)/(8 \log n)) = \Omega(\lambda) \) as \( t \leq 4 \log n \).

For each \( i \in B_t \), let \( x_i \) denote an independent and unbiased \( \{-1,1\} \)-variable and let

\[
x = \sum_{i \in H} w_i \cdot x_i \quad \text{and} \quad x' = \sum_{i \in B_t \setminus H} w_i \cdot x_i.
\]

Using the fact that \( f_{\rho(t)} \) was \( \tau \)-weight regular and (12), \( f_{\rho'} \) is \( O(\tau) \)-weight regular. Thus, by Berry-Esséen, \( x + x' \) is \( O(\tau) \)-close to an \( \mathcal{N}(0,1) \) Gaussian random variable, denoted by \( G \). \( x \) is \( O(\tau/\sqrt{\alpha}) \)-close to \( \sqrt{\alpha} G \), and \( x' \) is \( O(\tau/\sqrt{\beta}) \)-close to \( \sqrt{\beta} G \). Since \( f_{\rho'} \) is 0.96-balanced,

\[
\Pr [G \leq \theta'] \leq \Pr [x + x' \leq \theta'] + O(\tau) \leq 0.52 + O(\tau)
\]

and thus, \( \theta' < 0.06 \). Let \( \theta^* > 0 \) be the threshold such that \( \Pr [\sqrt{\beta} G \leq \theta^*] = 1 - 3\varepsilon/2 \).

We will use the following inequality:

\[
\Pr [f_{\rho' \eta} \text{ is } (4\varepsilon/3)\text{-balanced}] \geq \Pr [x \in [\theta' - \theta^*, \theta']] .
\] (15)

The idea is that if \( \sum_{i \in H} w_i \cdot x_i \in [\theta' - \theta^*, \theta'] \), and we write

\[
f_{\rho' \eta}(x') = \text{sign} \left( \sum_{i \in H} w_i \cdot x_i' - \left( -\sum_{i \in H} w_i \cdot x_i + \theta' \right) \right),
\]

then the new threshold is nonnegative and at most \( \theta^* \). Thus \( \mathbb{E}[f_{\rho' \eta}(x')] = 1 - 2 \Pr [f_{\rho' \eta}(x') = 1] \).

Using \( \beta = \Omega(\lambda) \) and \( \tau/\sqrt{\lambda} = o(\varepsilon) \), we also have

\[
\Pr [f_{\rho' \eta}(x') = 1] \geq \Pr [x' \geq \theta^*] \geq \Pr [\sqrt{\beta} G \geq \theta^*] - O(\tau/\sqrt{\beta}) = (3\varepsilon/4) - o(\varepsilon).
\]

This implies that \( f_{\rho' \eta} \) is \((3\varepsilon/2 - o(\varepsilon))\)-balanced, and thus, \((4\varepsilon/3)\)-balanced.

Finally we bound the probability of \( x \in [\theta' - \theta^*, \theta'] \) by considering the following two cases.
Case 1: $\alpha \geq 0.02$. We have

$$\Pr [\sqrt{\alpha} G \in [\theta' - \theta^*, \theta']] = \Pr [G \in [(\theta' - \theta^*)/\sqrt{\alpha}, \theta'/\sqrt{\alpha}]] \geq \min(1/\sqrt{\alpha}, \theta^*/\sqrt{\alpha}) \cdot \Omega(1) = \Omega(\sqrt{\lambda}),$$

as $\theta^* = \Omega(\sqrt{\lambda})$ (using our choice of $\theta^*$, $\varepsilon < 1/2$, and $\beta = \Omega(\lambda)$). It follows that

$$\Pr [x \in [\theta' - \theta^*, \theta']] \geq \Pr [\sqrt{\alpha} G \in [\theta' - \theta^*, \theta']] - O(\tau/\sqrt{\alpha}) = \Omega(\sqrt{\lambda}).$$

Case 2: $\alpha < 0.02$ and thus, $\beta > 0.98$. Combining $\theta' < 0.06$ and $\theta^* > \sqrt{\beta} \cdot 0.31 > 0.3$:

$$\Pr [x \in [\theta' - \theta^*, \theta']] \geq \Pr [x \in [0, \theta^* - \theta']] \geq \frac{1}{2} - \exp(-25(\theta^* - \theta')^2),$$

by Hoeffding inequality. Plugging in $\theta^* - \theta' > 0.24$, the probability above is $\Omega(1)$.

Summarizing the two cases, $f_{\rho'; \eta}$ is $(4\varepsilon/3)$-balanced with probability at least $\Omega(\sqrt{\lambda})$.

This finishes the two cases, $f_{\rho'; \eta}$ is $(4\varepsilon/3)$-balanced with probability at least $\Omega(\sqrt{\lambda})$.

Lemma 5.6 follows directly from Lemmas 5.7, 5.8, and 5.9.

5.4 Final analysis of the algorithm

**Theorem 5.13.** The algorithm Mono-Test-LTF($f, \varepsilon$) correctly tests whether a given LTF is monotone or $\varepsilon$-far from monotone.

**Proof.** The algorithm is one-sided because it outputs “non-monotone” only when an anti-monotone edge is found. The only interesting case is when the input LTF $f$ is $\varepsilon$-far from monotone. Combining Lemmas 5.3 and 5.5, the algorithm Mono-Test-LTF($f, \varepsilon$) outputs “non-monotone” with probability at least $(9/10)(81/100) > 2/3$. This completes the proof.

**Theorem 5.14.** The algorithm Mono-Test-LTF($f, \varepsilon$) makes $\tilde{O}(\log^{42} n/\varepsilon^{90})$ queries.

**Proof.** From Fact 5.2, the number of queries used by Regularize-and-Balance is $\tilde{O}(\log^{41} n/\varepsilon^{90})$, since the main bottleneck is the call to Find-Hi-Influence-Vars. In Main-Procedure, the bottleneck is the $O(\log n)$ calls to Find-Hi-Influence-Vars in Maintain-Regular-and-Balance, each of query complexity $\tilde{O}(\log^{41} n/\varepsilon^{90})$, despite the slightly different parameters. Note that we run the edge tester when there are fewer than $1/\tau^2$ many stars, so it makes $\tilde{O}(\log^{4} n/\varepsilon^{9})$ many queries.

Theorem 1.1 follows as an immediate consequence of Theorems 5.13 and 5.14.

References

[ACCL07] N. Ailon, B. Chazelle, S. Comandur, and D. Liu. Estimating the distance to a monotone function. Random Structures and Algorithms, 31(3):371–383, 2007.

[BB16] A. Belovs and E. Blais. A polynomial lower bound for testing monotonicity. In Proceedings of the 48th ACM Symposium on Theory of Computing, 2016.

[BBM12] E. Blais, J. Brody, and K. Matulef. Property testing lower bounds via communication complexity. Computational Complexity, 21(2):311–358, 2012.
[BCGSM12] J. Břet, S. Chakraborty, D. García-Soriano, and A. Matsliah. Monotonicity testing and shortest-path routing on the cube. Combinatorica, 32(1):35–53, 2012.

[BCP+17] Roksana Baleshzar, Deeparnab Chakrabarty, Ramesh Krishnan S. Pallavoor, Sofya Raskhodnikova, and C. Seshadhri. Optimal unateness testers for real-values functions: Adaptivity helps. In Proceedings of the 44th International Colloquium on Automata, Languages and Programming (ICALP ’2017), 2017.

[BKR04] T. Batu, R. Kumar, and R. Rubinfeld. Sublinear algorithms for testing monotone and unimodal distributions. In Proceedings of the 36th ACM Symposium on Theory of Computing, pages 381–390, 2004.

[BRY13] E. Blais, S. Raskhodnikova, and G. Yaroslavtsev. Lower bounds for testing properties of functions on hypergrid domains. Electronic Colloquium on Computational Complexity (ECCC), 20:36, 2013.

[BRY14] Piotr Berman, Sofya Raskhodnikova, and Grigory Yaroslavtsev. L_p-testing. In Symposium on Theory of Computing, STOC 2014, New York, NY, USA, May 31 - June 03, 2014, pages 164–173, 2014.

[CDST15] X. Chen, A. De, R.A. Servedio, and L.-Y. Tan. Boolean function monotonicity testing requires (almost) $n^{1/2}$ non-adaptive queries. In Proceedings of the 47th ACM Symposium on Theory of Computing, pages 519–528, 2015.

[CRS15] C. Canonne, D. Ron, and R. Servedio. Testing probability distributions using conditional samples. SIAM Journal on Comput., 44(3):540–616, 2015.

[CS13a] D. Chakrabarty and C. Seshadhri. A $o(n)$ monotonicity tester for boolean functions over the hypercube. In Proceedings of the 45th ACM Symposium on Theory of Computing, pages 411–418, 2013.

[CS13b] D. Chakrabarty and C. Seshadhri. Optimal bounds for monotonicity and Lipschitz testing over hypercubes and hypergrids. In Proceedings of the 45th ACM Symposium on Theory of Computing, pages 419–428, 2013.

[CS14] D. Chakrabarty and C. Seshadhri. An optimal lower bound for monotonicity testing over hypergrids. Theory of Computing, 10(17):453–464, 2014.

[CST14] X. Chen, R.A. Servedio, and L.-Y. Tan. New algorithms and lower bounds for monotonicity testing. In Proceedings of the IEEE 55th Annual Symposium on Foundations of Computer Science, pages 286–295, 2014.

[CST+17] Xi Chen, Rocco A. Servedio, Li-Yang Tan, Erik Waingarten, and Jinyu Xie. Settling the query complexity of non-adaptive junta testing. In Proceedings of the 32nd Conference on Computational Complexity (CCC ’2017), 2017.

[CWX17] Xi Chen, Erik Waingarten, and Jinyu Xie. Beyond talagrand functions: new lower bounds for testing monotonicity and unateness. In Proceedings of the 49th ACM Symposium on the Theory of Computing (STOC ’2017), 2017.
[DGL+99] Y. Dodis, O. Goldreich, E. Lehman, S. Raskhodnikova, D. Ron, and A. Samorodnitsky. Improved testing algorithms for monotonicity. In Proceedings of the 3rd International Workshop on Randomization and Approximation Techniques in Computer Science, pages 97–108, 1999.

[Dzi14] D. Dzindzalieta. Tight Bernoulli tail probability bounds. Technical Report Doctoral Dissertation, Physical Sciences, Mathematics (01 P), Vilnius University, 2014.

[EKK+00] F. Ergün, S. Kannan, S.R. Kumar, R. Rubinfeld, and M. Vishwanthan. Spot-checkers. Journal of Computer and System Sciences, 60:717–751, 2000.

[Fel68] W. Feller. An introduction to probability theory and its applications. John Wiley & Sons, 1968.

[Fis04] E. Fischer. On the strength of comparisons in property testing. Information and Computation, 189(1):107–116, 2004.

[FLN+02] E. Fischer, E. Lehman, I. Newman, S. Raskhodnikova, R. Rubinfeld, and A. Samorodnitsky. Monotonicity testing over general poset domains. In Proceedings of the 34th Annual ACM Symposium on the Theory of Computing, pages 474–483, 2002.

[GGL+00] O. Goldreich, S. Goldwasser, E. Lehman, D. Ron, and A. Samordinsky. Testing monotonicity. Combinatorica, 20(3):301–337, 2000.

[HK08] S. Halevy and E. Kushilevitz. Testing monotonicity over graph products. Random Structures and Algorithms, 33(1):44–67, 2008.

[KMS15] S. Khot, D. Minzer, and M. Safra. On monotonicity testing and boolean isoperimetric type theorems. In Proceedings of the 56th Annual Symposium on Foundations of Computer Science, pages 52–58, 2015.

[MORS10] K. Matulef, R. O’Donnell, R. Rubinfeld, and R. Servedio. Testing halfspaces. SIAM Journal on Comput., 39(5):2004–2047, 2010.

[RRS+12] D. Ron, R. Rubinfeld, M. Safra, A. Samorodnitsky, and O. Weinstein. Approximating the influence of monotone Boolean functions in $O(\sqrt{n})$ query complexity. ACM Transactions on Computation Theory, 4(4):1–12, 2012.

[RS09] R. Rubinfeld and R.A. Servedio. Testing monotone high-dimensional distributions. Random Structures and Algorithms, 34(1):24–44, 2009.

[RS15] D. Ron and R.A. Servedio. Exponentially improved algorithms and lower bounds for testing signed majorities. Algorithmica, 72(2):400–429, 2015.

[STW15] R.A. Servedio, L.-Y. Tan, and J. Wright. Adaptivity helps for testing juntas. In Proceedings of the 50th IEEE Conference on Computational Complexity, pages 264–279, 2015.

[Wai] M. Wainwright. Basic tail and concentration bounds. available online at www.stat.berkeley.edu/~mjawain/stat210b/Chap2_TailBounds_Jan22_2015.pdf.