KAWAGUCHI-SILVERMAN CONJECTURE ON BIRATIONAL AUTOMORPHISMS OF PROJECTIVE THREEFOLDS

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Abstract. We first prove that there is no any Zariski dense $f$-orbit for a birational automorphism $f$ of a normal projective variety $X$ with non-negative Kodaira dimension admitting a $f$-equivariant morphism over $\mathbb{P}^1$. Under the framework of dynamics on projective varieties by Kawamata, Nakayama and Zhang [16, 30, 32, 41], Hu and the author [11], we may reduce Kawaguchi-Silverman conjecture for automorphisms $f$ on projective threefolds $X$ with either trivial canonical divisor or negative Kodaira dimension to the following three cases: (i) weak Calabi-Yau threefolds and $f$ is primitive (ii) rationally connected threefolds and (iii) uniruled threefolds admitting a special MRC fibration over an elliptic curve. And we prove Kawaguchi-Silverman conjecture is true for birational automorphisms of normal projective varieties $X$ with the irregularity $q(X) \geq \dim X - 1$. We also discuss Kawaguchi-Silverman conjecture on projective varieties with Picard number two.

1. Introduction

1.1. Kawaguchi-Silverman conjecture. Let $X$ be a projective variety of dimension $n \geq 1$ over a global field $K$ of characteristic 0. Let $f : X \rightarrow X$ be a dominant rational map. Then the first dynamical degree of $f$ is the quantity

$$d_1(f) := \lim_{n \rightarrow \infty} \left( (f^n)^* H \cdot H^{\dim X-1} \right)^{1/n},$$

where $H$ is a choice of ample divisor on $X$. A result of Dinh and Sibony [7] says that this limit exists and is independent of the choice of ample divisor $H$.

In [18, 19], Kawaguchi and Silverman studied an analogous arithmetic degree, which we now describe. Assume that $X$ and $f$ are defined over $\overline{\mathbb{Q}}$, and write $X(\overline{\mathbb{Q}})_f$ for the set of points $P$ whose forward $f$-orbit

$$\mathcal{O}_f(P) = \{ P, f(P), f^2(P), \cdots \}$$

is well defined. Further, let

$$h_X : X(\overline{\mathbb{Q}}) \rightarrow [0, \infty)$$

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be a Weil height on $X$ associated with an ample divisor, and let $h^+_X = \max\{1, h_X\}$. The arithmetic degree of $f$ at $P \in X(\overline{\mathbb{Q}})_f$ is the quantity

$$\alpha_f(P) = \lim_{n \to \infty} h^+_X(f^n(P))^{1/n},$$

if the limit exists.

The following Kawaguchi-Silverman conjecture (KSC for short) asserts that for a dominant self-map $f : X \to X$ of a projective variety $X$ over $\mathbb{Q}$, the arithmetic degree $\alpha_f(x)$ of any point $x$ with Zariski dense $f$-orbit is equal to the first dynamical degree $\delta_f$ of $f$.

**Conjecture 1.1.** [19, Conjecture 6] Let $f : X \to X$ be a dominant rational map of a projective variety $X$ over $\overline{\mathbb{Q}}$, and let $P \in X(\overline{\mathbb{Q}})_f$. If $\mathcal{O}_f(P)$ is Zariski dense in $X$, then $\alpha_f(P) = d_1(f)$.

**Remark 1.2.** It is known that any dominant rational map of a projective variety with positive Kodaira dimension does not have any Zariski dense $f$-orbits (cf. [14, Remark 1.3]). So we will consider the case of varieties $X$ with the Kodaira dimension $\kappa(X) \leq 0$ admitting an equivariant fibration over varieties $Y$ with $\kappa(Y) \leq 0$.

1.2. **Historical note.** Let $f$ be a surjective endomorphism of a projective variety $X$. By taking the normalization which is $f$-equivariant, we may assume that $X$ is normal.

When $f$ is an automorphism and $\dim X = 2$, one can further take an $f$-equivariant resolution (cf. [3, Theorem 13.2]). Kawaguchi and Silverman showed in [17, Theorem 2(c)] that KSC holds for such $f$. When $f$ is non-isomorphic and $X$ is smooth, Matsuzawa, Sano and Shibata [24, Theorem 1.3] proved that KSC holds for $f$, by reducing the problem to three precise cases: $\mathbb{P}^1$-bundles, hyperelliptic surfaces, and surfaces of Kodaira dimension one. For a singular projective surface $X$, running an $f$-equivariant minimal model program (MMP) after iterating $f$, Meng and Zhang [28, Theorem 1.3] proved that KSC holds for any surjective endomorphism of a projective surface.

Assume that $f$ is a surjective endomorphism of a projective variety $X$. Kawaguchi and Silverman [19, Theorem 5] proved that KSC when $f$ is polarized, i.e. there is an ample Cartier divisor $D$ and integer $q > 1$ such that $f^*D \sim qD$. Let such $D$ is a big $\mathbb{R}$-Cartier divisor. By a result of Meng and Zhang [27, Proposition 1.1], $f$ is actually polarized and KSC also holds for such $f$. Note that $f^*|_{\text{NS}_X}$ is diagonalizable with all eigenvalues of modulus $q$ by [27, Proposition 2.9]. Morover, when only assume that $\kappa(X, D) > 0$, then KSC also holds by Matsuzawa and the author [14, Proposition 5.2(1)].

Silverman [37, Theorem 1.2] proved that KSC holds for any dominant self-map on abelian varieties. And then KSC also holds for any self-morphism on semi-abelian varieties by Matsuzawa and Sano [25, Theorem 1.1].
Matsuzawa [20] proved that KSC is true when $X$ is a projective toric variety, linear algebraic group or variety of Fano type, and moreover established in [20, Theorem 4.1] that KSC is true for any surjective endomorphism $f$ on $X$ when assume that $\text{NS}_Q(X) \cong \text{Pic}_Q(X)$ and the nef cone is generated by finitely many semi-ample integral divisors.

In [15, Theorem 1.2 and Proposition 1.7], Lesieutre and Satriano proved that KSC is true when $X$ is a hyper-Kähler variety, a smooth projective threefold with $\kappa(X) = 0$ and $\deg f > 1$ or a smooth minimal threefold of Kodaira dimension zero with sufficiently large Picard number (contingent on certain conjectures in the minimal model program). Moreover, KSC is true when $f$ is an automorphism of a smooth projective variety $X$ with Picard number $\rho(X) = 2$ by Shibata [36, Theorem 4.2] or [15, Theorem 2.30].

When $X$ is a smooth rationally connected variety admitting an int-amplified endomorphism, KSC holds for every surjective endomorphism of such $X$ by Meng and Zhang [28, Theorem 1.11] and Matsuzawa and Yoshikawa [26, Theorem 1.1]. The task of their proof is to exclude the Case TIR as in [22, Section 6]. More precisely, let $X$ be a $\mathbb{Q}$-factorial klt projective variety with the algebraic fundamental group $\pi_1^{alg}(X_{\text{reg}})$ of the smooth locus $X_{\text{reg}}$ of $X$ being finite. Suppose $X$ admits an int-amplified endomorphism. Then Case TIR will not occur during any MMP starting from $X$ by Meng, Matsuzawa, Shibata and Zhang [22, Theorem 6.3]. Therefore, KSC holds for any surjective endomorphism of such $X$ by [28, Theorem 1.7].

Let $\pi : X \to Y$ be a surjective endomorphism of normal projective variety. Suppose $f$ (resp. $g$) is a surjective endomorphisms $f$ (resp. $Y$) of $X$ (resp. $Y$) such that $\pi \circ f = g \circ \pi$. It is known that KSC holds for $f$ when $d_1(f) = d_1(g)$ and KSC holds for $g$ by [15, Theorem 3.4]. Now let $X$ be a projective bundle over a smooth projective variety $Y$ with the Picard number one. Lesieutre and Satriano [15] proved that KSC is true for any surjective endomorphism of $X$ when $Y = \mathbb{P}^1$. And then Matsuzawa and the author [14] proved that KSC is true for any surjective endomorphism of $X$ when $Y$ is Fano. Recently, Nasserden showed in [31] that Conjecture 1.1 is true for any surjective endomorphism of $X$ when $Y$ is an elliptic curve.

Chen, Lin and Oguiso [4] showed that KSC is true for $f$ which is a birational automorphism of a smooth projective variety $X$ with $\kappa = 0$ and the irregularity $q(X) \geq \dim X - 1$ or $X$ is an irregular smooth threefold (modulo the case that $X$ is covered by rational surfaces).

### 1.3. Main results

Let $f$ be a surjective endomorphism of a projective variety $X$. By taking the normalization which is $f$-equivariant, we may assume that $X$ is normal. We say that a rational map $\pi : X \dasharrow Y$ is $f$-equivariant, if there exists a surjective endomorphism $f_Y$ such that $\pi \circ f = g \circ \pi$. When $f$ is a birational automorphism, there is a
resolution of singularities $\pi : X' \to X$ and a birational automorphism $f'$ on $X'$ such that $\pi \circ f' = f \circ \pi$ (cf. [3, Theorem 13.2]). Therefore, KSC for $f$ reduces to KSC for $f'$, which is a birational automorphism on a smooth projective variety. This makes the problem easier sometime but it is sometime better to work on singular variety because it might have better birational geometric properties. However, there is no $f$-equivariant minimal model program for (birational) automorphism groups of projective varieties in general (cf. [11, Remark 1.3(1)]). This makes dynamics on projective varieties still challenging. Therefore, it is reasonable to assume that $X$ is minimal, e.g. $K_X \sim 0$ when $\dim X = 3$ and $\kappa(X) = 0$.

Due to work of Höring and Peternell [12, Theorem 1.5], we have have the Beauville-Bogomolov decomposition for minimal models with trivial canonical class as follows. Let $X$ be a normal projective variety at most klt singularities such that $K_X \equiv 0$. Then there exists a finite cover, étale in codimension one $\pi : \tilde{X} \to X$ such that
$$
\tilde{X} \cong A \times \prod Y_j \times \prod Z_k
$$
where $A$ is an abelian variety, the $Y_j$ are singular Calabi-Yau varieties and the $Z_k$ are singular irreducible holomorphic symplectic varieties (see [9, Definition 1.3]). However, it is still unclear whether we can always lift the automorphisms of $X$ to some splitting cover $\tilde{X}$ (cf. [11, Remark 3.5]). Instead of utilizing their strong decomposition theorem, we use a weak version (cf. [11, Lemma 2.7]) due to Kawamata [16], and developed by Nakayama-Zhang [32]. For more details about automorphisms of projective varieties with trivial canonical divisor, we refer to [41, Theorem 1.1 and 2.4] and [11, Theorem 1.2].

For automorphisms on projective varieties with negative Kodaira dimension, we can use special MRC fibration due to Nakayama [30] which have the descent property. We refer to [11, Lemma 2.11] for more details about it.

We first give the following result which will be used frequently in this paper.

**Theorem 1.3.** (cf. Theorem 3.5) Let $f$ be a birational automorphism on a normal projective variety $X$ with non-negative Kodaira dimension. If $X$ admits a $f$-equivariant morphism $\pi : X \to \mathbb{P}^1$, then $X$ does no have any dense $f$-orbit.

The notion of a primitive birational automorphism was introduced by Zhang [39] as follows.

**Definition 1.4.** A birational automorphism $f : X \to X$ is imprimitive if there exists a variety $B$ with $1 \leq \dim B < \dim X$, a birational map $g : B \dashrightarrow B$, and a dominant rational map $\pi : X \dashrightarrow B$ such that $\pi \circ f = g \circ \pi$. The map $f$ is called primitive if it is not imprimitive.
Now we give our main result of KSC for projective threefolds as follows.

**Theorem 1.5.** Let $X$ be a normal projective threefold $X$ and an abelian subgroup $G$ of $\text{Aut}(X)$. Then the following statements hold.

1. Suppose $K_X \sim 0$ and $f \in G$. Then to prove KSC, we may assume that $X$ is weak Calabi-Yau and $f$ is primitive.

2. Suppose $\kappa(X) = -\infty$. Then to prove KSC, we may assume that $X$ either is a rationally connected threefold or a uniruled threefold admitting a special MRC fibration over an elliptic curve.

**Remark 1.6.** Now let $X$ be a smooth uniruled threefold admitting a special MRC fibration over an elliptic curve. Then the irregularity $q(X) > 0$. So KSC is true for any automorphism of $X$ by [4, Theorem 1.6(2)]. Moreover, Chen, Lin and Oguiso proved in [4, Theorem 1.5(2)] that KSC is true for any automorphism of a smooth irregular threefold.

**Remark 1.7.** Let $f$ be a birational automorphism of a weak Calabi-Yau threefold. To prove KSC for $(X, f)$, then we may assume that $f$ is primitive (cf. Proposition 3.7).

Notice that a birational automorphism $f$ on a minimal Calabi-Yau threefold $X$ of Picard number $\rho(X) \geq 2$ is primitive if the action $f^*|_{\text{NS}_Q(X)_Q}$ is irreducible over $\mathbb{Q}$ (cf. [33, Corollary 1.3]). This motivates the following question.

**Question 1.8.** Let $f$ be a birational automorphism of a weak Calabi-Yau variety $X$ with $\rho(X) \geq 2$. Suppose that $f^*|_{\text{NS}_Q(X)}$ is irreducible over $\mathbb{Q}$. Then is KSC true for $(X, f)$?

Finally, we may extend [4, Theorems 1.4] to singular cases as follows.

**Theorem 1.9.** (cf. Theorem 3.4) Let $f$ be a birational automorphism of a normal projective variety $X$. Then KSC is true for $(X, f)$ if $q(X) \geq \dim X - 1$.

The paper is organized as follows. In Section 2, we collect some basic facts on KSC. In Section 3, we study invariant fibrations on projective varieties and prove Theorems 1.3 and 1.9. In Section 4, we prove Theorems 1.5. We study KSC on projective varieties with Picard number two in Section 5.

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2. Preliminaries

Lemma 2.1. [28, Lemma 2.5] Let \( \pi : X \to Y \) be a dominant rational map of projective varieties. Let \( f : X \to X \) and \( g : Y \to Y \) be surjective endomorphisms such that \( g \circ \pi = \pi \circ f \). Then the following hold.

1. Suppose \( \pi \) is generically finite. Then KSC holds for \( f \) if and only if KSC holds for \( g \).
2. Suppose \( d_1(f) = d_1(g) \) and KSC holds for \( g \). Then KSC holds for \( f \).

Proposition 2.2. Let \( X \) be a projective variety and a subgroup \( G \) of \( \text{Aut}(X) \). Let \( G_1 \) be a finite index subgroup of \( G \). Then KSC holds for any automorphism \( g \in G \) if and only if KSC holds for any automorphism \( g_1 \in G_1 \).

Proof. It follows from [14, Lemma 2.3]. \( \square \)

Lemma 2.3. [35, Lemma 3.2] Let \( f : X \to X \) and \( g : Y \to Y \) be two surjective endomorphisms of projective varieties. Suppose KSC is true for both \( f \) and \( g \). Then KSC is true for \( f \times g \).

A normal projective variety \( X \) is called \( Q \)-abelian if there is an abelian variety \( A \) and a finite surjective morphism \( A \to X \) which is quasi-étale, i.e. étale in codimension one.

Theorem 2.4. [28, Theorem 2.8] Let \( X \) be a \( Q \)-abelian variety. Then KSC holds for any surjective endomorphism of \( X \).

Proposition 2.5. [14, Proposition 5.2] Let \( X \) be a \( Q \)-factorial normal projective variety and \( f : X \to X \) a surjective morphism with \( \delta_f > 1 \).

1. If there is an \( \mathbb{R} \)-divisor \( D \) which is \( \mathbb{R} \)-linearly equivalent to at least two effective \( \mathbb{R} \)-divisors such that \( f^*D \sim_{\mathbb{R}} \delta_f D \), then KSC holds for \( f \).
2. Suppose \( f \) is an automorphism. Suppose further that there are \( \mathbb{R} \)-divisors \( D_+ \) and \( D_- \) such that \( f^*D_+ \sim_{\mathbb{R}} \delta_f D_+ \), \( (f^{-1})^*D_- \sim_{\mathbb{R}} \delta_{f^{-1}} D_- \) and \( D_+ + D_- \) is \( \mathbb{R} \)-linearly equivalent to at least two effective \( \mathbb{R} \)-divisors. Then KSC holds for \( f \).

Definition 2.6. Let \( f \) be a surjective endomorphism (or an automorphism) on a projective variety \( X \) with \( \dim X \geq 3 \) and some \( \mathbb{R} \)-divisor \( D \) (or \( D_+ \) and \( D_- \)) on \( X \).

1. We say \( (X, D) \) is positive on \( (X, f) \) if \( D \) is \( \mathbb{R} \)-linearly equivalent to at least two effective \( \mathbb{R} \)-divisors such that \( f^*D \sim_{\mathbb{R}} \delta_f D \).
2. We say \( (X, (D_+, D_-)) \) is positive on \( (X, f) \) if \( f \) is an automorphism, \( f^*D_+ \sim_{\mathbb{R}} \delta_f D_+ \), \( (f^{-1})^*D_- \sim_{\mathbb{R}} \delta_{f^{-1}} D_- \) and \( D_+ + D_- \) is \( \mathbb{R} \)-linearly equivalent to at least two effective \( \mathbb{R} \)-divisors.
Remark 2.7. Lesieutre and Satriano established in [15] that KSC is true on hyper-Kähler varieties using the pair \((X, D_+, D_-)\) which is positive on \((X, f)\) when \(\kappa(X, D_+ + D_-) = \dim X\). So an interesting question is asked as follows.

Question 2.8. Could we find new examples of projective varieties \(X\) with the pair \((X, D)\) (or \((X, D_+, D_-)\)) which is positive on \((X, f)\) such that \(\kappa(X, D) < \dim X\) (or \(\kappa(X, D_+ + D_-) < \dim X\))?

The following theorem is due to Matsuzawa, Meng, Shibata, Zhang and Zhong.

Theorem 2.9. [23, Theorem 1.9(1)] Let \(f : X \to X\) be a surjective endomorphism of a normal projective variety over a field \(k\) of arbitrary characteristic. Assume \(d_1(f) > 1\). Then \(f^* D \sim_R d_1(f) D\) for some nef \(\mathbb{R}\)-Cartier divisor \(D\).

Definition 2.10. We say such \(D\) in Theorem 2.9 is a leading real eigendivisor of \((X, f)\).

Remark 2.11. Note that KSC is true for \((X, f)\) when the leading real eigendivisor \(D\) on \((X, f)\) has \(\kappa(X, D) > 0\) by Proposition 2.5.

Now we quote the definition of a weak Calabi-Yau variety in [11, Definition 2.4].

Definition 2.12. A normal projective variety \(X\) is called a weak Calabi-Yau variety, if

- \(X\) has only canonical singularities;
- the canonical divisor \(K_X \sim 0\); and
- the augmented irregularity \(\tilde{q}(X) = 0\).

Here, the augmented irregularity \(\tilde{q}(X)\) of \(X\) is defined as the supremum of \(q(Y)\) of all normal projective varieties \(Y\) with finite surjective morphism \(Y \to X\), étale in codimension one. Namely,

\[
\tilde{q}(X) = \sup \{ q(Y) : Y \to X \text{ is a finite surjective and étale in codimension one} \}.
\]

Proposition 2.13. Let \(f\) be a surjective endomorphism of a weak Calabi-Yau threefold \(X\). To prove KSC is true for \((X, f)\), we only need to show that the leading real eigendivisor \(D\) of \((X, f)\) has \(\kappa(X, D) \geq 0\).

Proof. Assume that \(\kappa(X, D) \geq 0\). Since \(K_X \sim 0\), we know that every \(\mathbb{Q}\)-divisor is semi-ample by the log abundance for threefolds. And by [8, Proposition 4.7.6], \(D\) is semi-ample. So the proof follows from Proposition 2.5.

Theorem 2.14. (cf. [15, Theorem 1.8]) Let \(f\) be an automorphism of a normal non-uniruled projective threefold \(X\). If the second Chern class \(c_2(X)\) is strictly positive on \(\text{Nef}(X)\) and \(q(X) = 0\), then \(f\) has finite order. In particular, KSC is true for \((X, f)\).
Proof. By [2, Lemma 7.1] we know that \( \{ D \in \text{Nef}(X) \mid c_2(X) \cdot D \leq m \} \) is compact for all \( m \geq 0 \). So the function \( D \mapsto c_2(X) \cdot D \) achieves a minimum positive value on \( \text{Nef}(X) \cap \text{Amp}(X) \) and this value is achieved by only finitely many \( D_i \). Taking the sum of these finitely many \( D_i \), we obtain an ample class \( A \) that is fixed by \( f^* \). Then by a Fujiki-Liberman type theorem (cf. [13, Theorem 1.4]), some iterate \( f^n \) lies in the connected component of the identity \( \text{Aut}^0(X) \subseteq \text{Aut}(X) \). For a smooth model \( X' \) of \( X \), the birational automorphism group \( \text{Bir}(X') \) contains \( \text{Aut}^0(X) \) as a subgroup. By [10, Theorem 2.1], \( \text{Bir}(X') \) is a disjoint union of abelian varieties of dimension equal to \( q(X') = q(X) = 0 \). And we conclude that \( f \) has finite order. \( \square \)

3. Invariant fibrations on projective varieties

Invariant fibrations play an important role in the study of rational maps in higher dimension, and the product formula of Dinh-Nguyễn-Truong [6] is useful in dealing with their dynamical degrees. Let \( \pi : X \to Y \) be a dominant rational map of projective varieties and \( f : X \to X \) a dominant rational map. Fix an ample divisor \( H \) on \( X \) and an ample divisor \( H' \) on \( Y \). Suppose that \( f \) preserves this fibration, i.e., \( f \) sends generic fiber of \( \pi \) to fibers of \( \pi \). This property is equivalent to the existence of a dominant rational map \( g : Y \to Y \) such that \( \pi \circ f = g \circ \pi \). Now we may give the following definition.

Definition 3.1. The first dynamical degree of \( f \) relative to \( \pi \) is the limit

\[
d_1(f|_{\pi}) = \lim_{n \to \infty} \left( (f^n)^* H \cdot \pi^*((H')^{\dim Y}) \cdot H^{\dim X - \dim Y - 1} \right)^{1/n}.
\]

The following is due to the product formula (cf. [5, Theorem 1.1]).

Proposition 3.2. [15, Theorem 3.4] Let \( \pi : X \to Y \) be a surjective morphism of a normal projective variety. Suppose \( f \) (resp. \( g \)) is a surjective endomorphism of \( X \) (or \( Y \)) such that \( g \circ \pi = \pi \circ f \). If \( d_1(f|_{\pi}) \leq d_1(g) \) and KSC holds for \( g \), then KSC also holds for \( f \). The condition \( d_1(f|_{\pi}) \leq d_1(g) \) holds in particular if \( f \) is birational and \( \dim Y = \dim X - 1 \).

Proposition 3.3. [14, Proposition 3.7] Let \( f \) be a surjective endomorphism of a normal projective variety \( X \). To prove KSC is true for \( (X, f) \), then we may assume the Albanese morphism \( \pi : X \to A \) is surjective.

Theorem 3.4. Let \( f \) be a birational automorphism of a normal projective variety \( X \). Then KSC is true for \( (X, f) \) if \( q(X) \geq \dim X - 1 \).

Proof. By Proposition 3.3, the Albanese morphism \( \pi : X \to A \) is surjective and \( \dim A = q(X) \). Notice \( f \) descents to \( f_A \) of \( A \) by the universal of the Albanese morphism. If \( \dim A = \dim X \), then KSC are true for \( (X, f) \) by Proposition 2.1 and Theorem 2.4. If \( \dim A = \dim X - 1 \), then KSC are true for \( (X, f) \) by Proposition 3.2 and Theorem 2.4. \( \square \)
The following is motivated by [15, Proof of Proposition 1.7].

**Theorem 3.5.** Let $f$ be a birational automorphism on a normal projective variety $X$ with non-negative Kodaira dimension. If $X$ admits a $f$-equivariant morphism $\pi : X \to \mathbb{P}^1$, then $X$ does no have any dense $f$-orbit. In particular, KSC holds for $f$.

**Proof.** By [3, Theorem 13.2], there exists a resolution $\varphi : \tilde{X} \to X$ and a birational automorphism $\tilde{f}$ of $\tilde{X}$ such that $\tilde{f} \circ \varphi = \varphi \circ f$. Then we may assume that $X$ is smooth. Now assume that $f$ descents to an automorphism $g$ of $\mathbb{P}^1$ as $\dim \mathbb{P}^1 = 1$. Let $Z \subset \mathbb{P}^1$ be the locus of points $t$, where the fiber $X_t$ is singular. Then $g(Z) = Z$. Since $Z$ is a finite set, after replacing $f$ by a further iterate, we can assume $g$ fixes $Z$ point-wise. By [38, Theorem 0.2], we know that $Z$ contains at least three points. It follows that $g$ is the identity since it fixes at least three points of $\mathbb{P}^1$. In other words, there exists a rational function on $X$ that is invariant under some iterate of $f$, which contradicts the fact that $X$ has a point with a dense orbit.

**Proposition 3.6.** Let $\pi : X \to Y$ be a surjective endomorphism of normal projective varieties with $3 = \dim X > \dim Y \geq 1$. Let $f : X \to X$ and $g : Y \to Y$ be surjective endomorphisms such that $g \circ \pi = \pi \circ f$. Suppose $f$ is a birational automorphism. Then to show KSC holds for $(X, f)$, we only need to assume that $Y$ is $\mathbb{P}^1$ or an elliptic curve. In particular, if $\kappa(X) \geq 0$, then $Y$ is an elliptic curve.

**Proof.** When $\dim Y = 2$, then Conjecture 1.1 holds for $(Y, g)$ by [28, Theorem 1.3]. Then Conjecture 1.1 holds for $(X, f)$ by Proposition 3.2. Then we may assume that $\dim Y = 1$. If $g(Y) \geq 2$ (i.e., $\kappa(Y) > 0$), then $Y$ does no have any dense $f$-orbit by Remark 1.2. So Conjecture 1.1 is true for $(X, f)$. Therefore, we may assume that $Y$ is $\mathbb{P}^1$ or an elliptic curve. If $\kappa(X) \geq 0$, then we end the proof of Proposition 3.6 by Theorem 3.5.

**Proposition 3.7.** Let $f$ be a birational automorphism of a weak Calabi-Yau threefold. To prove KSC for $(X, f)$, then we may assume that $f$ is primitive.

**Proof.** Assume that $X$ has a $f$-equivariant rational map $\pi : X \to Y$. Take $W$ as the normalization of the graph $\Gamma_\pi$ of $\pi$ which admits a natural faithful $G|_W$-action and then $q : W \to X$ is $G|_W$-equivariant. So we assume that $\pi$ is morphism by Proposition 2.1. By Proposition 3.6 we assume that $Y$ is an elliptic curve. This completes the proof of Proposition 3.7 since $X$ has trivial Albanese.

4. Automorphisms on projective threefolds

We first give a result of a projective 3-fold with the Kodaira dimension vanishes.
Proposition 4.1. Let $X$ be a normal projective threefold with the Kodaira dimension $\kappa(X) = 0$ and an abelian subgroup $G$ of $\text{Aut}(X)$. To prove Conjecture 1.1, we may assume that $G \cong \mathbb{Z}$ and is of positive entropy, i.e. every element $g \in G$ has $d_1(g) > 1$.

Proof. Let $f$ be an automorphism of $X$. It is well-known that $\alpha_f(P) \leq d_1(f)$ for $P \in X(\mathbb{Q})$ by [24, Remark 2.2]. So we may assume that $f$ is of positive entropy, i.e. $d_1(f) > 1$. Hence, $G$ is of positive entropy. After replacing $G$ by a finite index subgroup, it is known that $G = \mathbb{Z}^{\oplus r}$ with $r \leq 2$ (cf. [40]). If $G \cong \mathbb{Z} \times \mathbb{Z}$, then by [11, Theorem 1.1], after replacing $G$ by a finite-index subgroup, $X$ is $G$-birational to a $Q$-abelian variety $Y$. Then KSC holds for $f \in G$ by Lemma 2.1, Proposition 2.2 and Theorem 2.4. So we may assume that $G \cong \mathbb{Z}$ and $G$ is of positive entropy. This ends the proof of Proposition 4.1. □

Given a $G$-action on an algebraic variety $W$, i.e., there is a group homomorphism $G \to \text{Aut}(W)$, we denote by $G|_W$ the image of $G$ in $\text{Aut}(W)$. The action of $G$ is faithful, if $G \to \text{Aut}(W)$ is injective.

Now we divide the proof of Theorem 1.5 into two parts as follows.

Proof of Theorem 1.5(1). We may assume that $G \cong \mathbb{Z}$ by Proposition 4.1. Therefore, we may assume that $X$ is a weak Calabi-Yau variety by [14, Theorem 3.6(2)]. Now we assume that $X$ has a $f$-equivariant rational map $\pi : X \to Y$. We take $W$ as the normalization of the graph $\Gamma_\pi$ of $\pi$ which admits a natural faithful $G|_W$-action and then $q : W \to X$ is $G|_W$-equivariant. So we may assume $\pi$ is morphism by Proposition 2.1. By Proposition 3.2, we may assume that $Y$ is an elliptic curve. This completes the proof of 1.5(1) as $X$ has trivial Albanese. □

The following special MRC fibration is due to Nakayama [30].

Definition 4.2. [11, Definition 2.10] Given a projective variety $X$, a dominant rational map $\pi : X \dashrightarrow Z$ is called the special MRC fibration of $X$, if it satisfies the following conditions:

(1) The graph $\Gamma_\pi \subseteq X \times Z$ of $\pi$ is equidimensional over $Z$.
(2) The general fibers of $\Gamma_\pi \to Z$ are rationally connected.
(3) $Z$ is a non-uniruled normal projective variety.
(4) If $\pi' : X \dashrightarrow Z'$ is a dominant rational map satisfying (1)-(3), then there is a birational morphism $v : Z' \to Z$ such that $\pi = v \circ \pi'$.

The existence and the uniqueness (up to isomorphism) of the special MRC fibration is proved in [30, Theorem 4.18]. One of the crucial advantages of the special MRC is the following descent property.
Theorem 4.3. Let $\pi : X \dasharrow Z$ be the special MRC fibration, and $f \in \text{Aut}(X)$. Then there exists a birational morphism $p : W \to X$ and an automorphism $f_W \in \text{Aut}(W)$ and an equidimensional surjective morphism $q : W \to Z$ satisfying the following conditions:

1. $W$ is a normal projective variety.
2. A general fiber of $q$ is rationally connected.
3. $W$ admits $f_W$-equivariant fibration over $X$ and $Z$.

Proof. It follows from [11, Lemma 2.11].

Now we resume the proof of Theorem 1.5.

Proof of Theorem 1.5(2). Consider the special MRC fibration $\pi : X \dasharrow Z$. By Theorem 4.3 and Proposition 2.1, we may assume that $\pi : X \to Z$ is $f$-equivariant morphism and the general fiber of $q$ is rationally connected. If $\dim Y = 0$, then $X$ is rationally connected. Now assume that $\dim Y > 0$. Then the proof follows from Proposition 3.6 as $Y$ is non-uniruled.

5. On projective varieties with Picard number two

It is well-known that KSC holds for any surjective endomorphism of a projective variety $X$ with the Picard number $\rho(X) = 1$. So we may assume that $\rho(X) \geq 2$. An interesting example of projective varieties with Picard number two is a projective bundle $X$ over a projective variety $Y$ with $\rho(Y) = 1$. Then we ask the following question.

Question 5.1. Is KSC true for any surjective endomorphism of a projective bundle $X$ over a normal projective variety $Y$ with $\rho(Y) = 1$?

Remark 5.2. Question 5.1 is affirmative when either $\dim Y = 1$ (cf. [15, 31]) or $Y$ is smooth Fano (cf. [14]).

We say that a surjective endomorphism $f$ of a projective variety $X$ is int-amplified if $f^*L - L$ is ample for some ample Cartier divisor $L$ (cf. [21]).

To study KSC on projective varieties with Picard number two, we first give the following result, which is motivated by Sano’s theorem [34] on arithmetic and dynamical degrees.

Proposition 5.3. Let $f$ be a surjective endomorphism of a normal projective variety $X$ with the Picard number $\rho(X) = 2$. Let $\lambda_1$ and $\lambda_2$ be two eigenvalues of $f^*|_{\text{NS}_{\mathbb{R}}(X)}$ such that $|\lambda_1| \geq |\lambda_2|$. To prove KSC is true for $(X, f)$, then we may assume that $f$ is int-amplified (i.e. $d_1(f) = \lambda_1 > \lambda_2 > 1$) or $\lambda_2 = 1$. Moreover, we may assume that $f^*|_{\text{NS}_{\mathbb{R}}(X)}$ is diagonalizable with positive eigenvalues $p > q \geq 1$. 

Proof. It is known that \( 1 \leq \alpha_f(x) \leq d_1(f) \) by [24, Remark 2.2]. Now we may assume that \( d_1(f) > 1 \). Note that \( |\lambda_1| = d_1(f) \). If \( |\lambda_1| = |\lambda_2| > 1 \) or \( |\lambda_1| > 1 > |\lambda_2| \), then the arithmetic degree \( \alpha_f(x) = d_1(f) \) by [34, Theorem 1.1]. Then we may assume that \( |\lambda_1| > |\lambda_2| \geq 1 \). Note that the characteristic polynomial \( p(x) \) of \( f^*|_{\NS_{\mathbb{R}}(X)} \) over \( \mathbb{C} \) may written as \( p(x) = (x - \lambda_1)(x - \lambda_2) \). If \( p(x) \) is irreducible over \( \mathbb{R} \), then \( \lambda_1 \) and \( \lambda_2 \) are conjugate. However, \( |\lambda_1| \neq |\lambda_2| \). It leads a contradiction. Therefore, \( \lambda_1, \lambda_2 \) are real numbers. As a result, replace \( f \) by \( f^2 \), we know that \( d_1(f) = \lambda_1 > \lambda_2 \geq 1 \). Therefore, \( f^*|_{\NS_{\mathbb{R}}(X)} \) is diagonalizable with positive eigenvalues \( d_1(f) > \lambda_2 \geq 1 \) as \( d_1(f) \neq \lambda_2 \) and \( \rho(X) = 2 \). If \( \lambda_2 > 1 \), then \( f \) is int-amplified by [21, Theorem 1.1]. This completes the proof of Proposition 5.3.

Corollary 5.4. Let \( f \) be a surjective endomorphism of a weak Calabi-Yau variety \( X \) with \( \rho(X) = 2 \). Let \( \lambda_1 \) and \( \lambda_2 \) be two eigenvalues of \( f^*|_{\NS_{\mathbb{R}}(X)} \) such that \( |\lambda_1| \geq |\lambda_2| \). To prove KSC is true for \( (X, f) \), then we may assume that \( \lambda_2 = 1 \). Moreover, we may assume that \( f^*|_{\NS_{\mathbb{R}}(X)} \) is diagonalizable with positive eigenvalues \( p > q \geq 1 \).

Proof. Since \( K_X \) is pseudoeffective, the proof follows from Proposition 5.3, [21, Theorem 1.9] and Theorem 2.4.

The following is due to [28, Lemma 10.4].

Proposition 5.5. Let \( f : X \to X \) be a surjective endomorphism of a projective variety \( X \). Suppose \( f^*|_{\NS^1(X)} \) is diagonalizable with positive eigenvalues \( p \geq q \geq 1 \), an no other eigenvalues. Let \( H \) be an ample Cartier divisor. Then \( H = A + B \) for some nef \( \mathbb{R} \)-Cartier divisors \( A \) and \( B \) such that \( f^*A \equiv pA \) and \( f^*B \equiv qB \).

Proof. If \( p = q \), then \( f^*|_{\NS^1(X)} = p \) id and we may take \( A = H \) and \( B = 0 \). Assume \( p > q \). Let \( \varphi := f^*|_{\NS^1(X)} \). Let \( A = \lim_{i \to +\infty} \varphi^i(H)/p^i \) and \( B = \lim_{i \to +\infty} q^i\varphi^{-i}(H) \). Since \( \varphi \) is diagonalizable with only eigenvalues \( p \) and \( q \), the above limits are \( \mathbb{R} \)-Cartier and \( H = A + B \). It is clear that \( \varphi(A) = pA \) and \( \varphi(B) = qB \). Note that \( A \) and \( B \) are limits of ample divisors. So \( A \) and \( B \) are nef.

Remark 5.6. To prove KSC is true for projective varieties with Picard number two, we wish construct a canonical height function \( \hat{h}_D \) associated with some divisors \( D \) (e.g. the lead real eigendivisor of \( (X, f) \))

\[
\hat{h}_D(x) = \lim_{n \to \infty} \frac{h_D(f^n(x))}{d_1(f)^{\dim X}}
\]

which is positive at every point \( p \in X(\mathbb{C}) \) with a Zariski dense \( f \)-orbit by using \( A \) and \( B \) in Proposition 5.5.

As a start to addressing Question 5.1 when \( \dim Y > 1 \), we show the following result.
Theorem 5.7. Let $f$ be a surjective endomorphism of a projective bundle $X$ over a normal projective variety $Y$ with $\rho(Y) = 1$, $\pi : X \to Y$ the projection and $f$ be a surjective endomorphism. Replacing $f$ with its iterate, $f$ descents to $g$ on $Y$. To prove KSC is true for $(X, f)$, we may assume that one of the following case holds:

1. $d_1(f) = d_1(f|_\pi) > d_1(g) > 1$, and so $f$ is int-amplified.
2. $d_1(f) = d_1(f|_\pi) > d_1(g) = 1$, and the morphisms between fibers of $\pi$ induced by $f$ are not isomorphism.

Here, $d_1(f|_\pi)$ is the first dynamical degree of $f$ relative to $\pi$ (cf. Definition 3.1). Notice that the degree of $f$ is greater than one.

Proof. Replacing $f$ with its iterate, by [14, Lemma 2.3] we may assume that $f$ induces an endomorphism $g : Y \to Y$ such that $g \circ \pi = \pi \circ f$ (cf. discussion before [1, Theorem 2]). Since $\rho(Y) = 1$, $g$ is polarized and KSC is true for $g$ by [19, Theorem 5]. Therefore, we may assume $d_1(f) = d_1(f|_\pi) > d_1(g)$ by Lemma 2.1 and the product formula on dynamical degrees. Note that $\text{NS}_R(X) = \mathbb{R}O_X(1) \oplus \pi^*\text{NS}_R(Y)$. If $d_1(g) > 1$, then the eigenvalues of $f^* : \text{NS}_R(X) \to \text{NS}_R(X)$ are $d_1(f)$ and $d_1(g)$ and, then they have modulus larger than one. Thus $f$ is int-amplified by [21, Theorem 1.1]. As a result, $\deg f > 1$ by [21, Lemma 3.7]. Now Suppose $d_1(f) > d_1(g) = 1$. Then the morphisms between fibers of $\pi$ induced by $f$ are not isomorphism. Indeed, let $\pi^{-1}(y)$ be a closed fiber. Since $\text{NS}_R(X) = \mathbb{R}O_X(1) \oplus \pi^*\text{NS}_R(Y) \cong \mathbb{R}^2$ and $f^*$ fixes $\pi^*\text{NS}_R(Y)$, we see that $f^*O_X(1) = d_1(f)O_X(1) + \pi^*D$ for some divisor $D$ on $Y$.

\[
(f_*(\pi^{-1}(y) \cdot O_X(1)^{\dim X - \dim Y}) = ([\pi^{-1}(y)] \cdot f^*O_X(1)^{\dim X - \dim Y}) = ([\pi^{-1}(y)] \cdot (d_1(f)O_X(1))^{\dim X - \dim Y}) = d_1(f)^{\dim X - \dim Y}.
\]

This shows that the degree of the morphism $f : \pi^{-1}(y) \to \pi^{-1}(g(y))$ is $d_1(f)^{\dim X - \dim Y}$ which is greater than one. So we also have $\deg f > 1$. $\square$

Remark 5.8. The two reduced cases in Theorem 5.7 are very difficult.

1. In the first case we may reduce KSC to the TIR case in Theorem 5.11 as below.
2. In the second case we need to study the relations of arithmetic degrees and relatively dynamical degrees on projective bundles, but the morphisms on fibers induced by $f$ may not be an endomorphism. In general, let $\pi : X \to Y$ be a surjective morphism and a surjective endomorphism $f$ of $X$ descending to a surjective endomorphism $g$ on $Y$. Then a question is asked as follows.
Question 5.9. Let $\pi : X \to Y$ be a surjective morphism between projective varieties $X$ and $Y$ with $\dim X > \dim Y > 0$ and a surjective endomorphism $f$ of $X$ descents to $g$ on $Y$. Is KSC true for $(X, f)$ when $d_1(f) > d_1(g)$?

Remark 5.10. If $\dim Y = 1$, Question 5.9 is affirmative when $Y = \mathbb{P}^1$ and $f$ is a birational automorphism (cf. Theorem 3.5).

Finally, we quote KSC for the TIR case on normal projective varieties admitting an int-amplified endomorphism in [22] as follows.

Theorem 5.11. Let $f$ be a surjective endomorphism of a normal projective variety $X$ admitting an int-amplified endomorphism. To show KSC holds for $f$, it suffices to show KSC is true for that $X$ is the TIR case as follows:

1. $\kappa(X, -K_X) = 0$;
2. $f^*D_1 = d_1(f)D_1$ for some reduced effective Weil divisor $D_1 \sim_{\mathbb{Q}} -K_X$;
3. There is an $f$-equivariant Fano contraction $\pi : X \to Y$ with $d_1(f) > d_1(f_Y) (\geq 1)$, where $f_Y$ is induced by $f$ on $Y$.

Proof. It follows from [28, Theorem 1.7] and [22, Lemma 6.4].

Remark 5.12. When $X$ admits an int-amplified endomorphism, it is very difficult to show that KSC is true for any surjective endomorphism on $X$ even in dimension three.

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