From Linear Codes to Hyperplane Arrangements via THOMAS Decomposition

W. Plesken, T. Bächler*

Abstract

We establish a connection between linear codes and hyperplane arrangements using the THOMAS decomposition of polynomial systems and the resulting counting polynomial. This yields both a generalization and a refinement of the weight enumerator of a linear code. In particular, one can deal with infinitely many finite fields simultaneously by defining a weight enumerator for codes over infinite fields.

Keywords: linear codes, hyperplane arrangements, weight enumerator, THOMAS decomposition

1 Introduction

Up to now, hyperplane arrangements, cf. [OrT 92], and linear codes, cf. [MaS 77], [NRS 06], have been studied independently. We establish a connection between the two theories, which, in particular, relates the weight enumerator of a linear code with the characteristic polynomial of hyperplane arrangements. In doing so, we define a generalization and a refinement of the weight enumerator of a linear code. At the same time, we show that the characteristic polynomial of a hyperplane arrangement counts the number of points in its complement.

The key concept behind this is the THOMAS decomposition of polynomial systems, cf. [Tho 37], [BGLR 11] and the resulting counting polynomial, cf. [Ple 09a]. This decomposition splits up a system of algebraic equations and inequations into simple systems, which are special triangular systems with disjoint sets of solutions. Over algebraically closed fields in characteristic zero, this decomposition easily allows an enumeration of the solutions which can be captured in the counting polynomial.

In this note, we restrict ourselves to linear equations and inequations. In this case, one can deal with arbitrary fields. In particular, for finite fields \( F \), the counting polynomial yields the number of solutions over any finite extension \( F' \) of \( F \), when one substitutes \( |F'| \) for the indeterminate, cf. [Ple 09b].

As for hyperplane arrangements, their characteristic polynomials, cf. [OrT 92, pg. 43], turn out to be a special case of counting polynomials. As such, they enumerate the points of the complement of the arrangement over arbitrary fields, and count those points for finite fields. Historically, the characteristic polynomial was interpreted topologically, cf. [OrT 92, pg. 195], as Betti numbers of the complement in the case of the field of complex numbers and by means of \( l \)-adic cohomology in the case of finite fields, cf. [Leh 92]. The latter also exhibited the number of points in the complement over a finite field, cf. [Leh 92, Rem. 2.9] using the GROTHENDIECK-LEFSCHETZ fixed points formula.

As for linear codes, one gets a generalization of the weight enumerator, called comprehensive weight enumerator, using the characteristic polynomials of various (central) hyperplane arrangements associated with the code. The comprehensive weight enumerator can be defined over arbitrary fields. If defined for a linear code \( C \) over a finite field \( \mathbb{F}_q \), it yields the classical weight enumerator not only for \( C \) but also for all the scalar extensions \( \mathbb{F}_q^m \otimes_{\mathbb{F}_q} C \). For a linear code over the rationals \( \mathbb{Q} \) it yields the weight enumerator for a related linear code over \( \mathbb{F}_p \) for all but finitely many prime numbers \( p \).

In Section 2 the precise connection between THOMAS decomposition, counting polynomials, hyperplane arrangements, and linear codes over arbitrary fields is explained. Section 3 associates a certain

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lattice with a linear code inspired by the lattice of intersections of hyperplanes in an arrangement. Section 4 gives the definition of two new weight enumerators, the comprehensive and the refined weight enumerator, and introduces their main properties. Finally, Section 5 gives examples demonstrating the new concepts, in new situations such as codes over \( \mathbb{Q} \) and in classical situations such as the Golay- and Hamming-codes.

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2 Linear equations and inequations, hyperplane arrangements, and codes

The usual definition of a hyperplane arrangement \((A, V)\) is that of a finite set \(A\) of (affine) hyperplanes in a finite dimensional vector space \(V\) over some field \(F\), cf. [OrT 92, Def. 1.1]. Equivalently one could simply view it as a finite subset \(H\) of the projective space \(P(V^* \oplus F)\) not containing \([(0, 1)]\), where \(V^*\) denotes the dual space of \(V\). For centered hyperplane arrangements the hyperplanes contain a common point which can without loss of generality be chosen to be \(0 \in V\). In the latter case the arrangements are called central and a central hyperplane arrangement will simply be viewed as a finite subset of the projective space \(P(V^*)\) of \(V^*\).

In contrast, a linear code, more precisely a linear \([n, k]\) code, is defined to be a \(k\)-dimensional subspace \(C\) of \(F^n\) for some \(n, k \in \mathbb{N}\). Thinking of the projections of \(C\) into the various coordinate positions, an equivalent definition is that of a \(k\)-dimensional vector space \(C\) over a field \(F\) together with an \(n\)-tuple \(\varphi \in (C^*)^n\) such that \(\cap_{i=1}^n \ker(\varphi_i) = \{0\}\). In particular, a generator matrix for the code \(C\) is given by \((\varphi_j(b_i))_{i=1,...,k,j=1,...,n} \in F^{k \times n}\), where \((b_1, \ldots, b_k) \in C^k\) is an \(F\)-basis of \(C\). To avoid trivialities we usually assume \(\varphi_i \neq 0\) for all \(i = 1, \ldots, n\).

**Definition 2.1.** Let \((C, \varphi)\) as above be a linear code over the field \(F\), then \((H(\varphi), C)\) with

\[
H(\varphi) := \{F\varphi_i | i = 1, \ldots, n\} = \langle \varphi_i \rangle_F | i = 1, \ldots, n\} \subseteq P(C^*) .
\]

is called the associated (central) hyperplane arrangement of \((C, \varphi)\).

So, in passing to the hyperplane arrangement one looses some less essential information, e. g. one considers linear functionals only up to non-zero scalar multiples, one deletes each linear functional which has already occurred, and one disregards the order in which they come. The view taken for the hyperplane arrangement is to study the complement of the union of the hyperplanes, i. e. to turn all \(\varphi_i\) into inequations. The view taken for the code is to allow both, turning the \(\varphi_i\) into equations or inequations in all possible ways.

**Definition 2.2.** An index set \(S \subseteq \{1, \ldots, n\}\) defines the following system of linear equations and inequations for \(v \in C\):

\[
EQ(S) : \quad \varphi_i(v) = 0 \text{ for } i \in S \text{ and } \varphi_i(v) \neq 0 \text{ for } i \in \{1, \ldots, n\} - S .
\]

Unfortunately it is not sufficient to look for solutions in \(C\) only, in case the field \(F\) is finite. Hence one might consider the solutions in the scalar extension \(C_{\overline{F}} := \overline{F} \otimes_F C\), where \(\overline{F}\) denotes the algebraic closure of \(F\). When dealing with these scalar extensions \(C_{\overline{F}}\) we also write \(\varphi_i\) instead of \(\overline{F} \otimes_F \varphi_i\).

Here, the Thomas-decomposition (cf. [BGLR 11]) yields a way to enumerate the solutions and come up with a counting polynomial: Fix an \(F\)-basis \(b\) of \(C\) as above and rewrite \(EQ(S)\) in terms of \(v = \sum_{i=1}^k x_i b_i\). Then \(EQ(S)\) turns into a system \(EQ_b(S)\) of linear (homogeneous) equations and inequations in \(x_1, \ldots, x_k\). In concrete terms

\[
EQ_b(S) = \{y_i = 0 | i \in S\} \cup \{y_i \neq 0 | i \in \{1, \ldots, n\} - S\}
\]

where

\[
(y_1, \ldots, y_n) = (x_1, \ldots, x_k) \cdot (\varphi_j(b_i))_{i=1,...,k,j=1,...,n} .
\]
The indices in $S$ mark the zero sets of the code words. In coding theory however, it is common to emphasize the support of code words instead. However, in our context, this would complicate matters in section 3.

From [Ple 09a], [Ple 09b] one obtains immediately:

**Lemma 2.3.** The counting polynomial (cf. [Ple 09a, Def. 3.1]) of $EQ_b(S)$ is independent of the choice of the basis $b$ and the order of the $x_i$.

**Definition 2.4.** 1.) Denote the counting polynomial in Lemma 2.3 by $\zeta_S = \zeta_S(q) \in \mathbb{Z}[q]$. (In view of the application in coding theory the variable is called $q$ instead of the symbol $\infty$ as used in [Ple 09a].) 2.) Call $S \subseteq \{1, \ldots, n\}$ saturated if $\zeta_S \neq 0$.

**Remark 2.5.** Let $F$ be a finite field. By [Ple 09b], for any finite extension $L$ of $F$, the number of solutions of $EQ(S)$ in $C_L$ is $\zeta_S(|L|) \in \mathbb{Z}_{\geq 0}$.

Whereas the definition of $\zeta_S(q)$ depends on the rather general Thomas-decomposition, the next section will show that the saturated subsets $S$, which can be easily computed from linear algebra, form a lattice whose M"obius-function also determines the $\zeta_S$ even without referring to any basis of $C$. The definitions above were mainly made in the context of codes, but they clearly can also be made in the context of (not necessarily central) hyperplane arrangements.

## 3 The lattice of saturated subsets

Here is an obvious characterization of the saturated subsets.

**Remark 3.1.** Let $(C, \varphi)$ be a code over $F$ as above with generator matrix

$$ G := (\varphi_j(b_i))_{i=1, \ldots, k, j=1, \ldots, n} \in F^{k \times n}, $$

where $(b_1, \ldots, b_k) \in C^k$ is an $F$-basis of $C$. For a subset $S \subseteq \{1, \ldots, n\}$, denote by $\text{col}(G, S)$ the set of the columns $G_{-i}$ of $G$ with $i \in S$. The following conditions for $S \subseteq \{1, \ldots, n\}$ are equivalent:

1.) $S$ is saturated, i.e. $\zeta_S(q) \neq 0$.

2.) There exists an element $v \in C_F$ with $S = \text{zero}(v) := \{i \in \{1, \ldots, n\} | \varphi_i(v) = 0\}$,

where $\overline{F}$ is the algebraic closure in case $F$ is finite, but $\overline{F} = F$ otherwise.

3.) For any $\varphi_i$ linearly dependent on $\{\varphi_j | j \in S\}$ one has $i \in S$.

3.’) For any column $G_{-i}$ linearly dependent on $\text{col}(G, S)$ one has $i \in S$.

Clearly, the last condition is the best for computational purposes, since it only involves linear algebra over $F$. The next result is also obvious.

**Proposition 3.2.** Let $(C, \varphi)$ be a code over $F$ as above.

1.) The intersection of two saturated sets is also saturated.

2.) For any subset $S \subseteq \{1, \ldots, n\}$ there is a unique minimal saturated subset $\overline{S} \subseteq \{1, \ldots, n\}$ containing $S$.

3.) The set $L(C) := \{S \subseteq \{1, \ldots, n\} | S = \overline{S}\}$ of all saturated subsets of $\{1, \ldots, n\}$ forms a lattice with respect to intersection $\cap$ and saturated union $\hat{\cup}$ defined by

$$ S_1 \hat{\cup} S_2 := \overline{S_1 \cup S_2} $$

with reversed inclusion as order relation, i.e. $S \leq_{L(C)} S' \text{ iff } S' \subseteq S$.  

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3
Define the dimension of a saturated set as \( \dim(S) := \dim_F \bigcap_{i \in S} \ker(\varphi_i) \). Now the road is open to identify the counting polynomial \( \zeta_S(q) \) of \( EQ(S) \) as characteristic polynomial of a restriction of the associated arrangement.

**Proposition 3.3.** Let \( S \subseteq \{1, \ldots, n\} \) be saturated.

1. \( \deg(\zeta_S(q)) = \dim(S) \).
2. \[ \sum_{T \supseteq S, T \in L(C)} \zeta_T(q) = q^{\dim(S)}. \]
3. \[ \zeta_S(q) = \sum_{T \supseteq S, T \in L(C)} \mu(S,T)q^{\dim(T)}. \]

where \( \mu \) denotes the MÖBIUS-function of the lattice \( L(C) \), cf. [Aig 79]. In particular \( \zeta_S(q) \) is monic and depends only on the lattice \( L(C) \).

4. If \( \dim(S) > 0 \) then \( q - 1 | \zeta_S(q) \).

**Proof.** Let \( \Lambda(S) \) denote the set of solutions of \( EQ(S) \) in \( C_T \). Since

\[ \bigcup_{T \supseteq S, T \in L(C)} \Lambda(T) = \bigcap_{i \in S} \ker(\varphi_i) \]

and since the set of solutions of \( \varphi_i(v) = 0 \) with \( i \in S \) is a vector space and thus clearly has counting polynomial \( q^{\dim(S)} \), part 2.) follows. Part 3.) follows from 2.) by MÖBIUS-inversion, cf. [Aig 79], pg. 152. Now 1.) also follows. That \( \zeta_S(q) \) depends only on the lattice \( L(C) \) now follows from the formula and the fact that the \( \dim(S) \) can be read off from the lattice \( L(C) \). This is because \( \dim(S') = \dim(S) - 1 \) for any two \( S, S' \in L(C) \) with \( S \subsetneq S' \) maximal.

Since the set of \( T \)-solutions of \( EQ(S) \) is invariant under multiplication by elements of \( T' \), part 4.) is a general property of counting polynomials, cf. [Ple 09a].

For hyperplane arrangements, a definition of a characteristic polynomial is given in [OrT 92], pg. 43. Now it follows that our counting polynomial \( \zeta_S(q) \) coincides with this characteristic polynomial.

**Corollary 3.4.** Let \( S \subseteq \{1, \ldots, n\} \) be saturated and let \( C(S) := \cap_{i \in S} \ker(\varphi_i) \). The counting polynomial \( \zeta_S(q) \) is equal to the characteristic polynomial of the arrangement

\[ \{\{\varphi_i|_{C(S)}| \mid i \notin S\} \subseteq P(C(S)^*), C(S)\}, \]

which in the case of \( S = \emptyset \) is the associated arrangement of the code \( (C, \varphi) \).

**Proof.** Clearly the intersection poset of the arrangement \( \{\{\varphi_i|_{C(S)}| \mid i \notin S\}, C(S)\} \) as defined in [OrT 92, Ch. 2.1], is isomorphic to the interval \([S, \{1, \ldots, n\}] \) in the lattice \( L(C) \).

This relation between hyperplane arrangements and the counting polynomial shows that the value of the characteristic polynomial of a hyperplane arrangement at \( q = |F| \) can be interpreted as the number of points in the complement of the arrangement over a finite field \( F \).

### 4 The comprehensive weight enumerator

In this section a definition of the comprehensive weight enumerator is given and some of its properties are discussed. The notation of the previous sections is kept.

**Definition 4.1.** Let \( (C, \varphi) \) be a linear \([n, k]\) code over the field \( F \). The comprehensive weight enumerator \( \omega_C = \omega_C(q, x, y) \in \mathbb{Z}[q, x, y] \) is defined as

\[ \omega_C(q, x, y) := \sum_{S \subseteq \{1, \ldots, n\}} \zeta_S(q)x^{|S|}y^{n-|S|}. \]
Of course it suffices to sum over the saturated subsets $S$ of $\{1, \ldots, n\}$ only. Each element $v$ of $C$ or $C_\rho$ belongs to a unique saturated subset $S$ of $\{1, \ldots, n\}$, namely its zero set $\text{zero}(v)$, i.e. the complement of its support. In taking the summation the information might get lost. Here is an attempt to keep at least the dimension $\dim(S)$ of the subspace of $C_\rho$ whose elements have zero set containing $S$, which might be smaller than $n - |S|$.

**Definition 4.2.** The refined weight enumerator $\rho_C = \rho_C(q, z, x, y) \in \mathbb{Z}[q, z, x, y]$ is defined as

$$\rho_C(q, z, x, y) := \sum_S z^{\dim(S)} \zeta_S(q)x^{|S|}y^{n-|S|}. $$

where the sum is taken over all saturated subsets $S \subseteq \{1, \ldots, n\}$ only.

Clearly $\rho_C(q, 1, x, y) = \omega_C(q, x, y)$. The comprehensive weight enumerator has better theoretical properties than the refined weight enumerator. But often in computing the comprehensive weight enumerator one already has all the information needed for the refined weight enumerator.

**Theorem 4.3.** Let $F$ be a finite field and the code $C$ as above. For any finite extension $L$ of $F$ the classical weight enumerator of $C_L$ is given by

$$W_{C_L}(x, y) = \omega_C(|L|, x, y) = \rho_C(|L|, 1, x, y). $$

**Proof.** Immediate from Remark 2.5 and the definition. \qed

For finite ground fields, the MacWilliams identity carries over to the comprehensive weight enumerators. We have not found a way to generalize this to refined weight enumerators.

**Corollary 4.4.** Let $F$ be a finite field and the code $C$ of dimension $k$ as above. Denote the dual code, cf. [MaS 77], by $C^\perp$. Then

$$\omega_{C^\perp}(q, x, y) = q^{-k}\omega_C(q, x + (q - 1)y, x - y). $$

**Proof.** Immediate from Theorem 4.3 by quoting the classical MacWilliams identity, cf. [NRS 06] for finitely many finite extensions of $F$. \qed

One of the most fundamental concepts of codes is the minimum Hamming distance. It can be characterized via the maximal saturated sets $\neq \{1, \ldots, n\}$ and – as G. Nebe pointed out – is independent of the field:

**Theorem 4.5.** Let $F$ be an arbitrary field and $C$ a code as above. Then the minimum Hamming distance of $C$ is equal to

$$n - \max\{|S| \mid S \subsetneq \{1, \ldots, n\} \text{ saturated}\}. $$

In particular the minimum distance of $C$ and any of its scalar extensions $C_L$ are equal.

**Proof.** This follows from the definition of the refined weight enumerator: We are looking for the terms with $n - |S| > 0$ minimal. Clearly, $\dim(S) = 1$ for these cases, because the $\dim(S)$ can only jump by one and $\dim(\{1, \ldots, n\}) = 0$. Also, $\zeta_S(q) = q - 1$ in this case because all non-zero elements in $\bigcap_{i \in S} \ker \rho_i$ are multiples of one vector or by 3.3 1) and 4).

**Proposition 4.6.** Both $\omega_C(q, x, y)$ and $\rho_C(q, z, x, y)$ only depend on the lattice $L(C)$ and the cardinalities of the elements of $L(C)$. In other words, if $D$ is another code, possibly over a different field, such that there is a lattice isomorphism $\alpha : L(C) \to L(D)$ with $|\alpha(S)| = |S|$ for all $S \in L(C)$, then $\omega_C(q, x, y) = \omega_D(q, x, y)$ and $\rho_C(q, z, x, y) = \rho_D(q, z, x, y)$.

**Proof.** Immediate from Proposition 3.3 and the definitions of the enumerators. \qed

A code over $\mathbb{Q}$ gives rise to a family of codes over all residue class fields of $\mathbb{Z}$, most of which have the same weight enumerator. The following theorem shows this in a more general setting:
**Theorem 4.7.** Let $F$ be the field of fractions of a principal ideal domain $R$ and $C \leq F^n$ a code over $F$ as given above. For any residue class field $f$ with natural epimorphism $\nu : R \rightarrow f$, let $C_\nu$ be the code over $f$ defined by

$$C_\nu := \nu(C \cap R^n) \leq f^n$$

where $\nu$ is applied componentwise. Then for all but finitely many $\nu$ one has

$$\rho_{C_\nu}(q, z, x, y) = \rho_C(q, z, x, y).$$

**Proof.** Note first that, by the structure theorem for finitely generated modules over principal ideal domains, $C \cap R^n$ has an $R$-basis which is an $F$-basis of $C$. Therefore $C_\nu$ has the same dimension over $f$ as $C$ over $F$ and we have a possibly new generator matrix $G(R) \in R^{k \times n}$, which has elementary divisors 0’s and 1’s only and componentwise application of $\nu$ yields a generator matrix for $C_\nu$.

Secondly a saturated subset $S$ of $\{1, \ldots, n\}$ with respect to $C$ over $F$ is also a saturated subset of $C_\nu$ over $f$ with the same dimension $\dim(S)$ for all but finitely many $\nu$. To see this first look at the elementary divisors of the submatrix $G(R)_{[\ldots,k]} \times S$. Next look at the elementary divisors of $G(R)_{[\ldots,k]} \times (S \cup(s))$ with $s \in \{1, \ldots, n\} - S$. All these have only finitely many prime divisors. Those $\nu$ having one of these prime divisors in their kernels have to be removed.

Thirdly a non saturated subset $S$ of $\{1, \ldots, n\}$ with respect to $C$ over $F$ is also a non saturated subset of $C_\nu$ over $f$ for all but finitely many $\nu$. To see this first look at the elementary divisors of the submatrix $G(R)_{[\ldots,k]} \times S$. Next look at the elementary divisors of $G(R)_{[\ldots,k]} \times (S \cup(s))$ with $s \in \text{sat}_F(S) - S$, where sat$_F(S)$ denotes the smallest $F$-saturated subset containing $S$. There are only finitely many primes such that these two matrices with the corresponding $\nu$ applied to each entry differ in rank over $f$, as a comparison of the two sets of elementary divisors shows.

As a consequence, the lattice of saturated subsets of $\{1, \ldots, n\}$ does not change when passing from $F$ to $f$ in all but finitely many cases. The result follows from Proposition 4.6. \hfill \Box

**Remark 4.8.** In the last theorem, it suffices to assume that $R$ is a Dedekind-domain. The changes in the proof are marginal, because one has the invariant factor theorem, cf. [CuR 62] pg. 150, available in this case as well.

### 5 Examples

The first two examples the reader might check by hand.

**Example 5.1.** Code $C_6$ over $\mathbb{F}_2$ of length 6 and dimension 3 with minimum weight 3:

$$\omega_{C_6}(q, x, y) = x^6 + 4(q - 1)x^3y^3 + 3(q - 1)x^2y^4 + 6(q - 1)(q - 2)xy^5 + (q - 3)(q - 2)(q - 1)y^6$$

The selfdual code $C_8$ over $\mathbb{F}_2$ of length 8 and dimension 4 with minimum weight 4:

$$\omega_{C_8}(q, x, y) = x^8 + 14(q - 1)x^4y^4 + 28(q - 1)(q - 2)x^2y^6 + 8(q - 1)(q - 2)(q - 4)xy^7 + (q - 1)(q^3 - 7q^2 + 21q - 21)y^8$$

**Example 5.2 (Hexacode).** Selfdual code $L$ of length 6 over $\mathbb{F}_4$ with $\alpha^2 + \alpha + 1 = 0$. Generator matrix:

$$
\begin{pmatrix}
1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & \alpha \\
0 & 0 & 1 & 1 & -1 - \alpha \\
\end{pmatrix}
$$

$$\omega_L(q, x, y) = x^6 + 15(q - 1)x^2y^4 + 6(q - 1)(q - 4)xy^5 + (q - 1)(q^2 - 5q + 10)y^6$$
The comprehensive weight enumerator cannot come from a code over $\mathbb{F}_2$, because inserting $q := 2$ yields a negative coefficient. In this context, it is interesting to note that the generator matrix can also be treated without a relation on $\alpha$, i.e. in the field $\mathbb{F}_2(\alpha)$ of rational functions. The computation yields the same comprehensive weight enumerator, as it does for any specialization of $\alpha$ except for $\alpha = 0$ or $\alpha = 1$. Note this is an example for Theorem 4.7 with ground field $\mathbb{F}_2(\alpha) = \text{Quot}(\mathbb{F}_2[\alpha])$.

Example 5.3 (Hamming code). Let $\binom{m}{i}_p$ denote the Gaussian binomial coefficient. Let $C$ be the Hamming code of length $n = \binom{m}{1}_p$ over a field $F$ of $p$ elements where $p$ is a prime or prime power. The refined weight enumerator of the dual code $C^\perp$ is given by

$$\rho_{C^\perp}(q, z, x, y) = \sum_{i=0}^{m} \binom{m}{i}_p x^{(i)}_p \cdot y^{(m-i)}_p \cdot z^{m-i} \prod_{j=0}^{i} (q - p^j).$$

For the proof note that the associated arrangement of the dual Hamming-code is the arrangement of all hyperplanes through the origin in $F^m$, whose characteristic polynomial is computed in [OrT 92, Pf. of Prop. 2.53]. To obtain the comprehensive weight enumerator for the Hamming code $C$, we use

$$\omega_C(q, x, y) = q^{-m} \rho_{C^\perp}(q, 1, x + (q - 1)y, x - y).$$

For $m = 3$ and $p = 2$, $\rho_{C^\perp}(q, z, x, y)$ is given by

$$(q - 1)(q - 2)(q - 4)y^7 z^3 + 7(q - 1)(q - 2)xy^6 z^2 + 7(q - 1)x^3 y^4 z + x^7$$

and $\omega_C(q, x, y)$ is

$$(q - 1)(q^3 - 6q^2 + 15q - 13)y^7 + 7(q - 1)(q - 2)(q - 3)xy^6 + 21(q - 1)(q - 2)x^2 y^5 + 7(q - 1)x^3 y^4 + 7(q - 1)x^4 y^3 + x^7$$

The refined weight enumerator $\rho_C(q, z, x, y)$ is

$$(q - 1)(q^3 - 6q^2 + 15q - 13)y^7 z^4 + 7(q - 1)(q - 2)(q - 3)xy^6 z^3 + 21(q - 1)(q - 2)x^2 y^5 z^2 + (7(q - 1)x^3 y^4 + 7(q - 1)x^4 y^3)z + x^7$$

but it cannot be obtained with this method. The coefficient of $x^iy^{n-i}z^r$ is a polynomial of degree $r$ in $q$. Its leading coefficient is the number of saturated sets $S$ with $|S| = i$ and $\dim(S) = r$. In general, this number cannot be read off from $\omega_C(q, x, y)$.

Example 5.4. The extended GOLAY-code $G_3$ over $\mathbb{F}_3$ of length 12 and dimension 6 with generator matrix

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 2 & 1 & 2 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 1 & 0 & 2 & 1 & 2 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 & 2 & 1 & \end{pmatrix}$$

yields

$$\rho_{G_3}(q, z, x, y) = (q - 1)(q^5 - 11q^4 + 55q^3 - 165q^2 + 330q - 330)y^{12}z^6 + 12(q - 1)(q - 3)(q - 4)(q^2 - 3q + 12)y^{11}xz^5 + 66(q - 1)(q - 3)(q^2 - 6q + 18)y^{10}x^2z^4 + 220(q - 1)(q - 4)^2y^9x^3z^3 + 495(q - 1)(q - 3)y^8x^4z^2 + 132(q - 1)y^6x^6z + x^{12}.$$

So, for instance there are no codewords with Hamming-weight 10 or 11 over $\mathbb{F}_3$ but over any proper extension.
Example 5.5 (The binary Golay-code). The generator matrix can be obtained by inserting the companion matrix of $x^{23} - 1 \in \mathbb{F}_2[x]$ in one of the two irreducible factors of $x^{23} - 1$ of degree 11, adding a column of ones as last column, and taking the first 12 rows. The automorphism group is the Mathieu group $M_{24}$. The table below shows representatives $S$ of the orbits of $M_{24}$ on the set of saturated subsets of $\{1, \ldots, 24\}$, the lengths of the orbits and the polynomials $\zeta_S(q)$.

| representative $S$ | $|S|$ | $\dim(S)$ | $|M_{24}S|$ | $\zeta_S(q)$ |
|-------------------|------|----------|-------------|-------------|
| $\{\}$           | 0    | 0        | 1           | $\alpha_1 p_{11.1}$ |
| $\{1\}$          | 1    | 12       | 1           | $x^{24} z^{12}$ |
| $\{1, 2\}$       | 2    | 10       | 276         | $\alpha_2 p_{8.2}$ |
| $\{1, 2, 3\}$    | 3    | 9        | 2024        | $\alpha_3 p_{6.2}$ |
| $\{1, 2, 3, 4\}$ | 4    | 8        | 10626       | $\alpha_2 p_{6.1}$ |
| $\{1, 2, 3, 4, 5\}$ | 5    | 7        | 42504       | $\alpha_3 p_{4.1}$ |
| $\{1, 2, 3, 4, 5, 6\}$ | 6    | 6        | 113344      | $\alpha_4 p_{2.4}$ |
| $\{1, 2, 3, 4, 5, 6, 7\}$ | 7    | 5        | 340032      | $\alpha_3 p_{4.2}$ |
| $\{1, \ldots, 7\}$ | 8    | 4        | 759         | $\alpha_1 p_{4.4}$ |
| $\{1, \ldots, 8\}$ | 9    | 3        | 637560      | $\alpha_2 p_{4.2}$ |
| $\{1, \ldots, 9\}$ | 10   | 2        | 12144       | $\alpha_3 p_{3.4}$ |
| $\{1, \ldots, 10\}$ | 10   | 3        | 170016      | $\alpha_2 p_{3.2}$ |
| $\{1, \ldots, 11\}$ | 12   | 2        | 35420       | $\alpha_1 p_{2.4}$ |
| $\{1, \ldots, 14\}$ | 12   | 1        | 2576        | $\alpha_2 p_{1.2}$ |
| $\{1, \ldots, 24\} - O$ | 16   | 1        | 759         | $\alpha_1 p_{3.3}$ |
| $\{1, \ldots, 24\}$ | 24   | 0        | 1           | $x^{24}$ |

where $\alpha_i := \alpha_i(q) := \prod_{j=0}^{i-1}(q - 2^j)$ and

\[
p_{11.1} := q^{11} - 23q^{10} + 253q^9 - 1771q^8 + 8855q^7 - 33649q^6 + 100947q^5 - 244398q^4 + 478170q^3 - 726110q^2 + 754446q - 384307
\]
\[
p_{8.1} := q^8 - 16q^7 + 127q^6 - 650q^5 + 2399q^4 - 6740q^3 + 14728q^2 - 23702q + 21784
\]
\[
p_{8.2} := q^8 - 19q^7 + 172q^6 - 986q^5 + 4013q^4 - 12246q^3 + 28188q^2 - 45054q + 37152
\]
\[
p_{6.2} := q^6 - 14q^5 + 98q^4 - 440q^3 - 1400q^2 - 3080q + 3680
\]
\[
p_{6.1} := q^6 - 17q^5 + 137q^4 - 690q^3 + 2352q^2 - 5124q + 5334
\]
\[
p_{4.1} := q^4 - 12q^3 + 72q^2 - 252q + 432
\]
\[
p_{4.2} := q^4 - 15q^3 + 100q^2 - 345q + 490
\]
\[
p_{4.3} := q^4 - 14q^3 + 92q^2 - 316q + 448
\]
\[
p_{2.1} := q^2 - 9q + 28
\]
\[
p_{4.4} := q^4 - 15q^3 + 105q^2 - 315q + 315
\]
\[
p_{2.2} := q^2 - 9q + 21
\]

Multiplying the corresponding elements in the last three columns of the table and adding them up yields the refined weight enumerator $p_C(q, z, x, y)$.

Clearly, the group action is essential for the computation of $p_C(q, z, x, y)$. To obtain the table, first, the $M_{24}$-orbits of saturated subsets are computed using GAP (cf. [GAP]). Then, for a representative $S$ of each orbit, the $\zeta_S(q)$ is computed using the Thomas-decomposition with the program from [BL08], provided that $|S| \geq 4$. To get the $\zeta_S(q)$ for $|S| \leq 3$ one uses the MacWilliams-identity for selfdual codes and the 5-fold transitivity of the group action.

Example 5.6 (MDS Codes). Maximum distance separable codes of dimension $k$ as defined in [MaS 77, Ch. 11] are codes where any $k$ of the $\varphi_i$ are linearly independent, i.e. if every $k \times k$ submatrix of the generator matrix $(\varphi_j(h_i))_{i=1, \ldots, k, j=1, \ldots, n}$ has rank $k$. The refined weight enumerator of any MDS code is

\[
p(q, z, x, y) = x^n + \sum_{i=0}^{k-1} W(n-i)x^iy^{n-i}z^{k-i}.
\]
with
\[
W(n-i) := \binom{n}{i} \sum_{j=1}^{k-1} (-1)^{k-i-j} \binom{n-i}{k-i-j} (q^j - 1)
\]

The proof is an easy computation that follows from the fact that the lattice of saturated subsets of an MDS code is the lattice of all subsets \(S\) of \(\{1, \ldots, n\}\) with cardinality \(|S| < k\) and \(\{1, \ldots, n\}\) itself. The expression given in [MaS 77, Ch. 11, Thm. 6] can easily be obtained from \(W(i)\).

The dual Hamming code of dimension 2 over \(\mathbb{F}_p\) is an example for a MDS code of dimension 2 and length \(p+1\), where \(p\) is any prime power. Other examples of MDS codes are some Reed-Solomon codes, for details see the discussion in [MaS 77, Ch. 11, Cor. 5].

The final example demonstrates Theorem 4.7 for the ground field \(\mathbb{Q} = \text{Quot}(\mathbb{Z})\).

**Example 5.7.** The permutation module \(P\) over \(\mathbb{Q}_5\) with respect to the natural action on the set of 2-element subsets of \(\{1, 2, \ldots, 5\}\) splits into a direct sum of a 5-dimensional, 4-dimensional, and a 1-dimensional submodule \(P_5, P_4, P_1\). In this example first the 5-dimensional submodule \(P_5\) is taken as a code over \(\mathbb{Q}\) and then the 4-dimensional one \(P_4\). Here is a possible generator matrix for \(P_5\):

\[
\begin{pmatrix}
1 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0
\end{pmatrix} \in \mathbb{Z}^{5 \times 10}
\]

It turns out that for \(\mathbb{Q}\) and any finite prime field \(\mathbb{F}_p\), even for any field, the refined weight enumerator is

\[
\rho_{P_5}(q,z,x,y) = (q-1)(q^4 - 9q^3 + 36q^2 - 69q + 51)y^{10}z^5 + 10(q-1)(q-2)(q^2 - 6q + 10)xy^9z^4 + 45(q-1)(q-2)(q-3)x^2y^8z^3 + 60(q-1)(q-2)x^3y^7z^2 + 15(q-1)(q-2)x^4y^6z + 15(q-1)x^4y^6z + 15(q-1)x^6y^4z + x^{10}
\]

Here is a possible generator matrix for \(P_4\):

\[
\begin{pmatrix}
1 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & -1 & -1 \\
0 & 1 & 0 & 2 & -1 & -1 & -2 & 0 & -1 & 1 \\
0 & 0 & 1 & 2 & -2 & -1 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 3 & -2 & -2 & 1 & -2 & 1 & 1
\end{pmatrix} \in \mathbb{Z}^{4 \times 10}
\]

For any field of characteristic \(\neq 2, 3\) the refined weight enumerator is

\[
\rho_{P_4,\text{gen}}(q,z,x,y) = (q-1)(q^3 - 9q^2 + 36q - 59)y^{10}z^4 + 10(q-1)(q^2 - 8q + 18)xy^9z^3 + 30(q-1)(q-4) + 15(q-1)(q-3)x^2y^8z^2 + 20(q-1)x^3y^7z + 25(q-1)x^4y^6z + x^{10}
\]

In characteristic 2 it is given by

\[
\rho_{P_4,2}(q,z,x,y) = (q-1)(q-2)(q-3)(q-4)y^{10}z^4 + 10(q-1)(q-2)(q-3)xy^9z^3 + 15(q-1)(q-2)x^2y^8z^2 + 10(q-1)(q-2)x^3y^7z^2 + 10(q-1)x^4y^6z + 5(q-1)x^5y^6z + x^{10}
\]

and in characteristic 3 by

\[
\rho_{P_4,3}(q,z,x,y) = (q-1)(q-3)(q^2 - 6q + 18)y^{10}z^4 + 10(q-1)(q-4)xy^9z^3 + 45(q-1)(q-3)x^2y^8z^2 + 30(q-1)x^4y^6z + x^{10}
\]
We conclude this section of examples with some remarks. Quite often the $\zeta_S(q)$ factorize in linear polynomials. In [OrT 92], Theorem 2.63 gives a sufficient criterion for this to happen, namely that the associated arrangement is supersolvable. We have checked supersolvability in all examples where a splitting into linear factors was observed, except for the arrangements connected to the binary GOLAY code. So in all these cases the factorization is explained by this result.

In most cases we did not have to make use of the explicit THOMAS decomposition, which of course gives much more information than we record in the various weight enumerators. However, in the case of the binary GOLAY code, we had to use it. The alternative would have been a more thorough group theoretical analysis of the situation to get the structure of the complete lattice of saturated subsets.

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Authors’ addresses:

W. Plesken, T. Bächler
Lehrstuhl B für Mathematik
RWTH Aachen
Templergraben 64
52062 Aachen, Germany
E-mail: plesken@momo.math.rwth-aachen.de, thomas@momo.math.rwth-aachen.de