A residual acceleration effect due to an inhomogeneity

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Abstract. The perturbed Friedmann-Lemaître-Robertson-Walker models allow many different possibilities for the 3-manifold of the comoving spatial section of the Universe. It used to be thought that global properties of the spatial section, including the topology of the space, have no feedback effect on dynamics. However, an elementary, weak-limit calculation shows that in the presence of a density perturbation, a gravitational feedback effect that is algebraically similar to dark energy does exist. Moreover, the effect differs between different 3-spaces. Among the well-proportioned spaces, the effect disappears down to third order in several cases, and down to fifth order for the Poincaré dodecahedral space \( S^3/I^* \). The Poincaré space, that which also is preferred in many observational analyses, is better-balanced than the other spaces.

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INTRODUCTION

What is the nature of the dark energy (for simplicity, let us use the term “dark energy” to generically signify either a form of dark energy or a cosmological constant, i.e. the parameter \( \Omega_\Lambda \)) that has been detected empirically to very high significance in observations of the Universe with several different telescopes and different types of surveys? This is one of the two key questions of this conference. However, the “detection” of dark energy would be meaningless without the underlying general-relativistic models of the Universe. These models are the Friedmann-Lemaître-Robertson-Walker models.

What did de Sitter [1], Friedmann [2, 3], Lemaître [4] and Robertson [5] state were the two properties of the constant curvature 3-manifold of spatial sections of the Universe that would need to be determined by observations? They referred to both curvature and topology. It is through the link between curvature and the various density parameters (e.g. the total density \( \Omega_{tot} \)) that we infer from observations that dark energy must (according to the exact-FLRW model) exist. What is the role of topology? We now know that theoretically, there are many different possibilities for the 3-manifold of the comoving spatial section of the Universe for any of the three curvatures [6, 7, 8, 9, 10] (for a shorter introduction, see [11]). For many years, it was thought that topology has no effect on the Friedmann equation of an FLRW model, since the Einstein equations are, by definition, local, while topology (used in this context to mean the \( \pi_1 \) homotopy group of a 3-manifold) is a global property of a 3-manifold. Indeed, for a perfectly homogeneous FLRW model, this argument appears to be correct.

However, this parallel session is about the fact that the Universe is not perfectly homogeneous. The planet Earth, the Solar System, the Galaxy, clusters of galaxies and the cosmic web of large-scale structure exist and constitute violations of perfect homogeneity. Many of the contributions in this session constitute work that may lead to (or already has led to) claims that the present “dark energy” is an approximation error rather than a physical phenomenon. That is, the Einstein equations should in principle be solved by an inhomogeneous solution rather than by an homogeneous solution with perturbations added afterwards.

Is there a topological feedback effect on the Friedmann equation in an almost-FLRW Universe, i.e. in one that is close to homogeneous but contains perturbations? An elementary, weak-limit calculation for a flat universe with one simply-connected direction shows that in the presence of a density perturbation, a gravitational feedback effect does exist [12]. Moreover, it is algebraically similar to dark energy. In more general 3-manifolds, several further interesting characteristics are found [12, 13].

\( T^1 \): A FLAT UNIVERSE WITH ONE SIMPLY-CONNECTED DIRECTION

First consider a flat spatial section with one simply-connected direction, i.e. with spatial section \( S^1 \times \mathbb{R}^2 \), informally known as the three-dimensional 1-torus \( T^1 \) [Sect. 3.1, 12]. Figure[1] shows this schematically. Place a particle of mass...
\[ \ddot{x} = \frac{G M}{L} \left( \frac{1}{(L-x)^2} - \frac{1}{(L+x)^2} \right) \]

\[ \approx \frac{4GM}{L^2} \frac{x}{L} \]  

(1)

to first order in \( x/L \ll 1 \), where \( G \) is the gravitational constant.

Thus, in the presence of a density perturbation, a gravitational feedback effect due to global topology can exist. Spherically symmetric solutions to the Einstein equations cannot represent this effect, because the distribution of matter in the covering space is not perfectly spherically symmetric. The multiple topological images of a single, physical, massive object prevent any possibility of spherical symmetry.

Algebraically, the residual acceleration effect derived in Eq. (1) is similar to that of dark energy:

\[ \ddot{x} \approx c x. \]  

(2)

The test particle sees an effect that opposes its “normal” attraction to the nearby instance of the massive particle. The further the test particle is displaced from the (nearby instance of the) massive particle, the stronger the effect. A fully relativistic, and necessarily non-spherically-symmetric, derivation of this effect remains to be carried out, but as derived here in the weak limit, it is algebraically similar to the effect of dark energy: it is an acceleration effect.

The effect is unlikely to explain the presently observed dark energy. As derived heuristically in Sect. 4.1 of ref [12], the dark energy equation-of-state parameter \( w := p/(\rho c^2) \), for pressure \( p \), density \( \rho \) and space-time unit conversion constant \( c \), would be \( w \sim -\left(\chi/L\right)^3 \), where \( \chi \) is the comoving separation such that \( x = a\chi \) and \( a \) is the scale factor. For a typical distance of an object inside a “unit” of large-scale structure, i.e. a typical distance from a massive cluster of \( x \sim 20\, h^{-1} \) Mpc, and a typical value of \( L \sim 14h^{-1} \) Gpc for the 3-manifold favoured from Wilkinson Microwave
Anisotropy Probe (WMAP) data, either for the Poincaré dodecahedral space \([14, 15, 16, 17, 18, 19, 20]\) or the 3-torus \(T^3\) \([21, 22, 23, 24, 25]\), we have \(w \sim -10^{-9}\). Thus, the effect is small at the present.

On the other hand, its role may have been important during early epochs. For example, in inflationary scenarios, the pre-inflationary Universe might have been highly inhomogeneous. The existence of an effect of topology on the dynamics of an almost-FLRW model shows that top-down effects, from global to local, can in principle play a role in physical models of the Universe.

**WELL-PROPORTIONED MODELS**

The simplest way to explain the near-absence of structure in the WMAP sky maps on scales above 10\(h^{-1}\) Gpc (comoving), i.e. the nearly zero auto-correlation function on these scales (see in particular refs \([26, 27]\)), is that the compactness scale of the Universe is close to this scale \([14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25]\). The missing fluctuations interpretation is a more conservative interpretation of the WMAP cosmological signal than what is frequently called the “low quadrupole” problem or the “axis of evil” problem \([26, 27]\). The type of 3-manifold that statistically can be expected to best match the data is one that is well-proportioned, i.e. that has approximately equal fundamental lengths of its fundamental domain in different directions \([28]\). Indeed, it is the regular 3-torus and the Poincaré dodecahedral space, both of which are well-proportioned, that are at present the favoured models, although some authors find that an infinite, flat model is preferred \([29, 30]\). Among the spherical spaces, there also exist two other well-proportioned spaces, the octahedral space \(S^3/T^*\) and the truncated-cube space \(S^3/O^*\) \([31]\). What is the nature of the residual gravitational acceleration effect, as derived above, in these spaces?

**\(T^3\): a flat universe with three simply-connected directions**

In a regular \(T^3\) perturbed-FLRW model, i.e. containing a massive particle as before, consider Fig. 1. In addition to the adjacent topological images to the “left” and “right”, consider images of the massive particle that are “below, above, in front of, and behind”, relative to the 2-plane of the page. All four of these images are in the 2-plane that exactly bisects the “left-hand” and “right-hand” images. Hence, the additional gravitational effect of these four images, in the weak limit, is to pull the test particle back towards the left, i.e. it opposes the effect given in Eq. (1). By how much is the effect weakened? Also, more generally, we need to consider a test particle offset from the massive particle in an arbitrary direction, rather than along one of the fundamental axes. In Sect. 3.2 of ref \([12]\) and Sect. 3.1 of ref \([13]\), this more general calculation is given to first order for slightly irregular \(T^3\) models and to third order for regular \(T^3\) models, respectively, using Taylor expansions in the displacement from the massive particle.

For the regular \(T^3\) model, the opposition between the four “extra” images and the initial images considered in the \(T^1\) case leads to the cancellation of the first-order term in the residual acceleration. The lowest order terms are the third order terms

\[
\ddot{x} = \frac{7GM}{L^2} \left[ 2\epsilon_x^3 - 3\epsilon_x (\epsilon_x^2 + \epsilon_y^2), 2\epsilon_y^3 - 3\epsilon_y (\epsilon_y^2 + \epsilon_z^2), 2\epsilon_z^3 - 3\epsilon_z (\epsilon_z^2 + \epsilon_x^2) \right],
\]

where the massive particle is at \((0,0,0)\) in the covering space \(\mathbb{R}^3\) and the test particle is at \((x = \epsilon_x L, y = \epsilon_y L, z = \epsilon_z L)\) (Eq. (12), \([13]\)).

In this sense, we can say that a regular \(T^3\) model is not only well-proportioned, but it is also well-balanced: the first-order terms balance each other perfectly, i.e. they cancel, leaving the third-order terms to dominate the expression. This relies on perfect regularity.

As shown in Sect. 3.2 of \([12]\), if the three side-lengths are unequal, then the linear terms fail to cancel, so that the effect is again a linear effect, similar to the \(T^1\) case. The effect is strongest in the shortest direction. It appears that this effect is a stabilising effect. Given the heuristic argument in \([12]\), an irregular \(T^3\) model will have slightly anisotropic scale factors in the different directions in the sense that opposes the irregularity. Shorter side-lengths will expand a little faster and longer side-lengths a little slower. Before this result was found, perfectly regular \(T^3\) models were the preferred \(T^3\) model discussed in the literature (e.g. \([12, 23]\)), but without any physical motivation. The residual gravity effect suggests a physical motivation for a regular \(T^3\) model: it appears to be an equilibrium point (a stable attractor) in the phase space \((L_a, L_c, L_u, L_a, L_c, L_u)\) at which the residual gravity effect drops from a linear expression (in terms of the dimensionless displacement) to a third-order expression. Here, \(L_a, L_c, L_u\) are the three comoving side-lengths and \(L_a, L_c, L_u\) are their respective derivatives with respect to proper time (cosmological time).
Well-proportioned spherical spaces: the octahedral space $S^3/T^*$, the truncated-cube space $S^3/O^*$ and the Poincaré dodecahedral space $S^3/I^*$

What happens in the spherical well-proportioned spaces? Calculating the residual acceleration effect for the spherical well-proportioned spaces, i.e. the octahedral space $S^3/T^*$, the truncated-cube space $S^3/O^*$ and the Poincaré dodecahedral space $S^3/I^*$, is most easily done by embedding the covering space $S^3$ in Euclidean 4-space $\mathbb{R}^4$. As explained in Sect. 2.1 of ref [13], weak-limit gravity in a spherical covering space is proportional to $[R_C \sin(r/R_C)]^{-2}$, where $R_C$ is the curvature radius, rather than to $r^{-2}$, in order to satisfy Stokes’ theorem.

In Sect. 3.3 of ref [13] it is shown numerically that the octahedral space and the truncated-cube space are also well-balanced. The residual gravity effect due to the adjacent images of the massive object balance well enough that again, the linear term disappears and the effect is dominated by third order terms.

The effect in the Poincaré dodecahedral space is even more finely balanced than for the other well-proportioned spaces. In ref [13], this is shown numerically, and also analytically. Both the linear and the third-order terms cancel.

Writing the displacement as a four-vector $\mathbf{r}$ in $\mathbb{R}^4$, necessarily lying in the covering space, i.e. $\mathbf{r} \in S^3 \subset \mathbb{R}^4$, the dominant term is the fifth-order term, which can be written as a vector in $\mathbb{R}^4$

$$\mathbf{r} = \frac{12\sqrt{7}}{125\sqrt{5} - \sqrt{5}} \left( \frac{GM}{R_C^2} \right) \left( \frac{r}{R_C} \right)^5 \left\{ 70y^4 + (42\sqrt{5} + 70)x^2y^2 - (14\sqrt{5} + 70)y^2x^2 - 28\sqrt{5}x^4 + 28\sqrt{5}y^4 + 7\sqrt{5} + 5 \right\} x,$$

$$70z^4 + (42\sqrt{5} + 70)y^2z^2 - (14\sqrt{5} + 70)z^2y^2 - 28\sqrt{5}y^4 + 7\sqrt{5} + 5 \right\} y,$$

$$70x^4 + (42\sqrt{5} + 70)x^2z^2 - (14\sqrt{5} + 70)x^2y^2 - 28\sqrt{5}x^4 + 28\sqrt{5}z^4 + 7\sqrt{5} + 5 \right\} z,$$

$$0 \} , \quad (4)$$

where the curvature radius is $R_C$, the massive particle is at $(0,0,0,R_C)$, the nearby test particle is at $\mathbf{r} := R_C[\sin(r/R_C)x, \sin(r/R_C)y, \sin(r/R_C)z, \cos(r/R_C)]$, and $x^2 + y^2 + z^2 = 1$ {cf. Eq. (21), [13]}. Although it appears in Eq. (3) that the residual acceleration is exactly tangent to the observer at $(0,0,0,R_C)$, this is only an artefact of showing only the dominant (fifth-order) term in the residual acceleration. The dominant term in the fourth $(w)$ direction in the embedding space is a sixth-order term, and the residual acceleration does indeed lie in the tangent 3-plane to the 3-sphere at $\mathbf{r}$, as it must.

Hence, the Poincaré dodecahedral space $S^3/I^*$, the spatial 3-section that is preferred in many observational analyses, is better balanced than the other well-proportioned spaces. General 3-manifolds are dominated by the linear term, most of the (regular) well-proportioned spaces are dominated by the third-order term, and the Poincaré space is dominated by the fifth-order term.

**CONCLUSION**

Throughout the history of cosmology since ancient times, preferences of the model of space have mostly shifted between finite spaces with a boundary and infinite unbounded spaces. Riemannian geometry made it possible to have flat (or hyperbolic) finite 3-spaces of constant curvature without boundaries. Pseudo-Riemannian geometry extends these to the FLRW space-time models of the Universe. Infinity is not a real number, and for the Universe to be a physical object, it would be most reasonable for the comoving volume, and hence total mass-energy, of the Universe to be finite.

The preferred models of comoving space, given the most recent cosmological observational catalogues, especially including the WMAP data, are summarised in Table [1] The other models discussed above are included as well. The displacement is written as a scalar $x$ in all three cases for simplicity. Full expressions of the dominant terms for $T^3$ and $S^3/I^*$ are given in Eq. (3) and Eq. (4) respectively. While $\mathbb{R}^3$ is frequently considered to be the implicit spatial section of the Concordance model, this is rarely stated explicitly, since it is not usually meant to be interpreted literally as a global model. It has the physically undesirable characteristic of giving the Universe an infinite amount of mass-energy, and has difficulty in explaining the lack of fluctuations above $10h^{-1}\text{Gpc}$ (in projection, above 60 degrees on the sky) in the WMAP cosmic microwave background maps.
Among the finite spaces, the well-proportioned spaces are, in general, good candidates for explaining the lack of > 10 h⁻¹ Gpc fluctuations, but constraints on the curvature radius $R_c$ make the fundamental domains of the octahedral and truncated-cube spaces a little too large compared to the surface of last scattering. For this reason, “no” is written for these spaces in Table 1. The best candidates would appear to be the regular 3-torus $T^3$ and the Poincaré dodecahedral space $S^3/I^*$. The latter appears to constitute an extremum among the class of possible 3-manifolds, in that the residual gravity effect is exceptionally well balanced, down to fifth order. We could say that “some spaces are more equal than others” and that the Poincaré space is the space that is “the most equal” [34].

This characteristic is a gravitational, geometrical, topological property of the Poincaré space, provided that an inhomogeneity exists in the space, and it was clearly unknown to the group that first proposed the Poincaré space as the best fit to the WMAP data [14].

Is this just a coincidence? Or is it possible that gravity, geometry, topology and inhomogeneity together determine the most likely space to have been realised by early universe physical processes, in the way derived above, and that this most likely space is the one first proposed in 2003 in order to fit the WMAP data [14]?

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