NON-COMMUTATIVE HOLOMORPHIC SEMICOYCLES

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Abstract. This paper studies holomorphic semicocycles over semigroups in the unit disk, which take values in an arbitrary unital Banach algebra. We prove that every such semicocycle is a solution to a corresponding evolution problem. We then investigate the linearization problem: which semicocycles are cohomologous to constant semicocycles? In contrast with the case of commutative semicocycles, in the non-commutative case non-linearizable semicocycles are shown to exist. Simple conditions for linearizability are derived and are shown to be sharp.

Key words: holomorphic semicocycle, nonlinear semigroup, evolution problem, linearization.

1. Introduction

The notion of (semi-)cocycle over a group or semigroup of transformations is important in many areas of the theory of dynamical systems (see, e.g., [10]), in particular in connection with the study of non-autonomous differential equations (see, e.g., [4]). These objects, in addition to their intrinsic interest, have applications to the study of invariant subspaces of Banach spaces of holomorphic mappings and semigroups of weighted composition operators on these spaces; see, e.g., [3, 4, 11, 7].

Several works have studies holomorphic semicocycles over semigroups of holomorphic self-mappings of the open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$, with values in $\mathbb{C}$, which are commutative semicocycles. In particular, it was shown in [8] (see also [11]) that each $\mathbb{C}$-valued semicocycle is smooth; the question whether a semicocycle is a coboundary was considered in [11, 9].

Our aim in this work is to study non-commutative semicocycles over semigroups of holomorphic self-mappings of $\mathbb{D}$, a primary example of such semicocycles being those with values in $M_n(\mathbb{C})$, the space of $n \times n$ matrices. We wish, on the one hand, to extend some results known for ($\mathbb{C}$-valued) commutative semicocycles to the non-commutative case — as we will see, while some such extensions are straightforward, others require some new ideas. On the other hand, we wish to stress some essential differences between the commutative and the non-commutative cases. We will see that interesting
new phenomena arise in the non-commutative case, even when we consider semicocycles over semigroups of the simplest kind, that consist of linear functions.

To introduce our basic notions, we start with some standard notations. Let $D$ be a domain in the complex plane $\mathbb{C}$ and let $\Omega$ be a domain in a complex Banach space. By $\text{Hol}(D, \Omega)$ we denote the set of all holomorphic mappings on $D$ with values in $\Omega$ and $\text{Hol}(D) := \text{Hol}(D, D)$ is the set of all holomorphic self-mappings of $D$. The open unit disk in $\mathbb{C}$ is denoted by $D$ and $D_r = rD$, the open disk of radius $r$. We also denote by $A$ a unital Banach algebra over $\mathbb{C}$ equipped with the norm $\| \cdot \|$ such that the unit element $1_A$ satisfies $\|1_A\| = 1$. The set of all invertible elements of $A$ is denoted by $A^*$. An important example is the algebra of bounded linear operators on a Banach space; in particular, the case of a finite-dimensional space corresponds to $A = M_n(\mathbb{C})$.

**Definition 1.1.** A family $F = \{F_t\}_{t \geq 0} \subset \text{Hol}(D)$ is called a one-parameter continuous semigroup (semigroup, for short) on $D$ if it is continuous in $(t, z) \in [0, \infty) \times D$ and the following properties hold

(i) $F_{t+s} = F_t \circ F_s$ for all $t, s \geq 0$;
(ii) $F_0(z) = z$ for all $z \in D$.

It was proven in [3] that every one-parameter continuous semigroup $F = \{F_t\}_{t \geq 0}$ on $D$ is differentiable with respect to its parameter $t \geq 0$. In this case the limit

\[
\lim_{t \to 0^+} \frac{F_t(z) - z}{t}, \quad z \in D,
\]

exists. The mapping $f \in \text{Hol}(D, \mathbb{C})$ is called the (infinitesimal) generator of the semigroup $F$. In turn, the generated semigroup $F$ can be reproduced as the solution of the Cauchy problem

\[
\begin{cases}
\frac{\partial u(t, z)}{\partial t} = f(u(t, z)) \\
u(0, z) = z,
\end{cases}
\]

where we set $u(t, z) = F_t(z)$.

Another important way to represent and to study semigroups is by means of a linearization model of the form $F_t(z) = h^{-1}(G_t \circ h(z))$, where $h \in \text{Hol}(D, \mathbb{C})$ is a biholomorphic mapping, and $\{G_t\}_{t \geq 0}$ is an affine semigroup (see [6] for a survey of this problem). In particular, it is known that if for some point $z_0 \in \overline{D}$, the semigroup generator $f \in \text{Hol}(D, \mathbb{C})$ satisfies $f(z_0) = 0$ and $\text{Re} \, f'(z_0) < 0$, then there is a spirallike function $h \in \text{Hol}(D, \mathbb{C})$, called
the Koenigs function, such that
\begin{equation}
F_t(z) = h^{-1}\left(e^{tf'(z_0)}h(z)\right).
\end{equation}

We now introduce the central object to be studied in this paper.

**Definition 1.2.** Let \( F = \{F_t\}_{t \geq 0} \subset \text{Hol}(\mathbb{D}) \) be a semigroup on the open unit disk. Let \( \Gamma_t : \mathbb{R}^+ \rightarrow \text{Hol}(\mathbb{D}, \mathcal{A}) \) be such that the mapping \( (t, z) \mapsto \Gamma_t(z) \in \mathcal{A} \) is continuous. The family \( \{\Gamma_t\}_{t \geq 0} \) is called a semicocycle over \( F \) if it satisfies the following properties:

(a) the chain rule: \( \Gamma_t(F_s(z))\Gamma_s(z) = \Gamma_{t+s}(z) \) for all \( t, s \geq 0 \) and \( z \in \mathbb{D} \);

(b) \( \Gamma_0(z) = 1_{\mathcal{A}} \) for every \( z \in \mathbb{D} \).

Since both semigroup elements and semicocycle elements depend on the real parameter \( t \geq 0 \) and the point \( z \in \mathbb{D} \), we usually denote the derivative with respect to the parameter \( t \) by \( \frac{\partial}{\partial t} \) while the derivative with respect to \( z \) is denoted by \( F_t'(z) \), \( \Gamma_t'(z) \) and so on.

For \( \mathcal{A} = \mathbb{C} \), the most natural example of a semicocycle over \( F \) is \( \Gamma_t(z) = F_t'(z) \). Another simple example is a constant (not dependent on \( z \)) semicocycles of the form \( \Gamma_t(z) = e^{tB_0} \), where \( B_0 \in \mathcal{A} \).

In this work we study inter alia two general methods to construct semicocycles:

(i) If \( B : \mathbb{D} \rightarrow \mathcal{A} \) is a holomorphic mapping, a semicocycle \( \Gamma_t(z) \) can be generated by solving the evolution problem
\begin{equation}
\frac{d}{dt}\Gamma_t(z) = B(F_t(z))\Gamma_t(z), \quad \Gamma_0(t) = 1_{\mathcal{A}}
\end{equation}
(see Theorem 2.2(a) below), which can be considered a semicocycle analog of the Cauchy problem (1.2) for semigroups. Thus the mapping \( B \) serves as a 'generator' of a semicocycle.

(ii) If \( M : \mathbb{D} \rightarrow \mathcal{A}^* \) is holomorphic and \( B_0 \in \mathcal{A} \), then
\begin{equation}
\Gamma_t(z) = M(F_t(z))^{-1}e^{tB_0}M(z)
\end{equation}
is a semicocycle (see Lemma 3.1 below). In view of the analogy between (1.5) and (1.2), representing a given semicocycle \( \{\Gamma_t\}_{t \geq 0} \) by (1.5) will be called 'linearization' of \( \{\Gamma_t\}_{t \geq 0} \). Note that the choice \( B_0 = 0 \) leads to semicocycles of the form \( \Gamma_t(z) = M(F_t(z))M(z)^{-1} \), which are sometimes referred to as coboundaries; see, for example [9].

These two constructions are of central importance: they immediately suggest questions concerning their degree of generality:

(Q1) Is every semicocycle differentiable? In the case of affirmative answer, is it generated, that is, can it be reproduced solving the Cauchy problem (1.4) for some \( B \in \text{Hol}(\mathbb{D}, \mathcal{A}) \)?
(Q2) Is every semicocycle linearizable, that is, can be represented in the form (1.5)?

The last question is intimately connected to the problem of classifying semicocycles up to the equivalence relation of cohomology: a semicocycle \( \Gamma_t \) is said to be cohomologous to a semicocycle \( \tilde{\Gamma}_t \) if there exists a holomorphic mapping \( M : \mathbb{D} \to \mathcal{A}_* \) (called ‘transfer mapping’) such that
\[
\Gamma_t(z) = M(F_t(z))^{-1} \tilde{\Gamma}_t(z) M(z) \tag{1.10}
\]
Hence (Q2) can be reformulated: is every semicocycle cohomologous to a constant semicocycle?

These problems are the central focus of this work and are studied in Sections 2 and 3.

For question (Q1), we will give a positive answer in Section 2, proving that every semicocycle is generated by an appropriate holomorphic mapping \( B \) (see Theorem 2.2 below). The main step here is to prove that a semicocycle is automatically differentiable with respect to its parameter \( t \). For the case \( \mathcal{A} = \mathbb{C} \), this was proved in [8], but the proof given there does not seem easy to generalize to the non-commutative case. We therefore develop an alternative proof, using an approach which appears to us simpler, even in the case \( \mathcal{A} = \mathbb{C} \). In this section, we also apply Theorem 2.2 to derive growth estimates for semicocycles.

When investigating question (Q2), we discover real differences between the commutative and non-commutative cases. For \( \mathcal{A} \) commutative, we show that the answer is positive, and an explicit expression for the mapping \( M \) is constructed in terms of the ‘generator’ of the semicocycle. The proof of this result, which slightly generalizes the results of [8] for the case \( \mathcal{A} = \mathbb{C} \), relies very strongly on commutativity. In fact, as soon as we move to the non-commutative case, a simple example with \( \mathcal{A} = M_2(\mathbb{C}) \) shows that there are semicocycles which are not linearizable (see Example 3.2).

To understand the obstructions to linearization, we develop an approach based on power series expansions (with coefficients in \( \mathcal{A} \)), around the interior fixed point \( z_0 \) of the semigroup \( \mathcal{F} \). This allows us to show that the key factor is the spectral behavior of the element \( B_0 = B(z_0) \), where \( B \) is the ‘generator’ of \( \Gamma_t \). We provide a simple and sharp condition on the spectrum of \( B_0 \) under which the semicocycle is linearizable. This condition, in the case \( \mathcal{A} = M_n(\mathbb{C}) \), holds for all matrices \( B_0 \) except for a ‘thin’ set. Thus, while the answer to (Q2) is negative as stated, it is positive in the generic sense. Our results thus provide a quite precise answers to the questions posed.
2. Generation of semicocycles

In this section we prove that every holomorphic semicocycle is differentiable with respect to \( t \), and can be generated as the unique solution of an evolution problem the form (1.4).

We will need some auxiliary results on general evolution problems in a Banach algebra \( A \). Given \( a : \mathbb{R}^+ \to A \), consider the evolution problem

\[
\begin{cases}
\frac{dv(t)}{dt} = a(t)v(t) \\
v(0) = 1_A,
\end{cases}
\]

where \( v \in \mathbb{R}^+ \to A \). The general theory of such problems is well-known (see, for example, [5, 12] and references therein), and the following theorem collects some basic facts.

**Theorem 2.1.** Let \( a(t) \in C(\mathbb{R}^+, A) \). The following assertions hold.

(i) The evolution problem (2.1) is equivalent to the integral equation

\[
v(t) = 1_A + \int_0^t a(s)v(s)ds.
\]

(ii) The evolution problem (2.1) (hence, the integral equation (2.2)) has a unique solution \( u \in C((0, \infty), A) \). Moreover, defining for every element \( A \in A \) the quantity

\[
\mu(A) = \lim_{t \to 0^+} \frac{1}{t} \left[ \|1_A + tA\| - 1 \right],
\]

we have

\[
\|u(t)\|_U \leq \exp \left( \int_0^t \mu(a(s))ds \right).
\]

(iii) The element \( u(t) \) is invertible for every \( t \geq 0 \).

In particular, assertion (i) as well the uniqueness part of assertion (ii) can be found in different monographs (see, for example, [12, 5]), while estimate (2.4) was proven in [1].

Our specific interest is to the special case of (2.1) in which the function \( a \) has a ‘generator’ structure. More precisely, we assume that \( a(t) = B(F_t(z)) \), where \( F = \{F_t\}_{t \geq 0} \subset \text{Hol}(\mathbb{D}) \) is a one-parameter continuous semigroup and \( B \in \text{Hol}(\mathbb{D}, A) \). In this case it is natural to replace an \( A \)-valued function \( v(t) \) above by a mapping \( v(t, z) : \mathbb{R}^+ \times \mathbb{D} \to A \). By Theorem 2.1, the corresponding evolution problem has the unique solution \( u(t) := u(t, z) \) for every fixed \( z \in \mathbb{D} \). It turns out that the solutions of these problems are precisely all the semicocycles over the semigroup \( F \).
Theorem 2.2. Let $\mathcal{F} = \{F_t\}_{t \geq 0} \subset \text{Hol}(\mathbb{D})$ be a semigroup on the open unit disk $\mathbb{D}$.

(a) Assume that $B \in \text{Hol}(\mathbb{D}, \mathcal{A})$. Denote by $u(t, z)$, $t \geq 0$, $z \in \mathbb{D}$, the unique solution to the evolution problem

\begin{equation}
\begin{cases}
\frac{du(t, z)}{dt} = B(F_t(z))u(t, z) \\
u(0, z) = 1_{\mathcal{A}}.
\end{cases}
\end{equation}

Then the family $\{\Gamma_t(\cdot) := u(t, \cdot)\}_{t \geq 0}$ is a semicocycle over $\mathcal{F}$.

(b) Let $\{\Gamma_t\}_{t \geq 0}$ be a semicocycle over $\mathcal{F}$. Then $\Gamma_t(z)$ is differentiable with respect to $t$ for every $z \in \mathbb{D}$. Defining

\begin{equation}
B(z) = \frac{d}{dt} \Gamma_t(z) \bigg|_{t=0},
\end{equation}

we have $B \in \text{Hol}(\mathbb{D}, \mathcal{A})$ and $\Gamma_t(z)$ is the solution to the evolution problem (2.5).

Proof. (a) Assume $u(t, z)$ satisfies (2.5). We first note that $u(t, \cdot) \in \text{Hol}(\mathbb{D}, \mathcal{A})$ by the theorem on the differentiability of solutions of differential equations with respect to parameters.

Fix $z \in \mathbb{D}$, $s \geq 0$ and define

$$h(t) = u(t + s, z) - u(t, F_s(z))u(s, z).$$

We have $h(0) = u(s, z) - u(0, F_s(z))u(s, z) = 0$. Using (2.5) and the semigroup property of $\mathcal{F}$, we compute

$$h'(t) = \frac{\partial}{\partial t} u(t + s, z) - \frac{\partial}{\partial t} u(t, F_s(z))u(s, z)$$

$$= B(F_{t+s}(z))u(t + s, z) - B(F_t(F_s(z)))u(t, F_s(z))u(s, z)$$

$$= B(F_{t+s}(z))u(t + s, z) - B(F_{t+s}(z))u(t, F_s(z))u(s, z)$$

$$= B(F_{t+s}(z))(u(t + s, z) - u(t, F_s(z))u(s, z))$$

$$= B(F_{t+s}(z))h(t).$$

Thus, the function $h(t)$ satisfies the linear differential equation

$$h'(t) = B(F_{t+s}(z))h(t).$$

Since $h(0) = 0$, the uniqueness theorem for linear differential equations implies that

$$h(t) = 0 \quad \text{for all} \quad t \geq 0.$$ 

This proves that $\Gamma_t(z) = u(t, z)$ satisfies the chain rule. In addition, $\Gamma_0(z) = u(0, z) = 1_{\mathcal{A}}$, hence the family $\{\Gamma_t\}_{t \geq 0}$ is a semicocycle.

(b) Assume that $\{\Gamma_t\}_{t \geq 0}$ is a semicocycle. We fix $0 < r < 1$ and first prove that $t \mapsto \Gamma_t(z)$ is differentiable for any $z \in \mathbb{D}_r$. 

We define

\[ V(t, z) = \int_0^t \Gamma_s(z) ds. \]  

Note that:

(i) By the continuity of the function \( s \mapsto \Gamma_s(z) \), the mapping \( V \) is differentiable with respect to \( t \), namely, \( \frac{d}{dt} V(t, z) = \Gamma_t(z) \).

(ii) \( \Gamma'_t(z) \), the derivative of \( \Gamma_t(z) \) with respect to \( z \), is jointly continuous with respect to \((t, z)\), so that differentiation of the expression (2.7) with respect to \( z \) under the integral sign is valid, hence \( V(t, z) \) is holomorphic with respect to \( z \).

(iii) We have that \( \frac{1}{t} \int_0^t \Gamma_s(z) ds \rightarrow 1_A \) as \( t \rightarrow 0 \), uniformly with respect to \( z \) in compact sets of \( \mathbb{D} \). Therefore we can choose \( t_0 > 0 \) so that \( \frac{1}{t_0} \int_0^t \Gamma_s(z) ds \), hence also \( V(t_0, z) \), is invertible for all \( z \in \mathbb{D}_r \).

Using the semicocycle property \( \Gamma_{t+s}(z) = \Gamma_s(F_t(z))\Gamma_t(z) \), we get the identity

\[ V(t_0, F_t(z))\Gamma_t(z) = \int_0^{t_0} \Gamma_s(F_t(z))\Gamma_t(z) ds \]

\[ = \int_0^{t+t_0} \Gamma_s(z) ds = \int_t^{t+t_0} \Gamma_s(z) ds = V(t+t_0, z) - V(t, z). \]  

(2.8)

Now, fixing \( z \in \mathbb{D}_r \), we can choose \( t_1 > 0 \) so that \( F_t(z) \in \mathbb{D}_r \) whenever \( 0 \leq t \leq t_1 \). Therefore by (iii) above, \( V(t_0, F_t(z)) \) is invertible for \( 0 \leq t \leq t_1 \). Thus we can rewrite (2.8) as

\[ \Gamma_t(z) = \left[ V(t_0, F_t(z)) \right]^{-1} \left[ V(t+t_0, z) - V(t, z) \right]. \]

By (i) and (ii) above, both terms in the product on the right-hand side are differentiable with respect to \( t \). This implies that \( t \mapsto \Gamma_t(z) \) is differentiable on \([0, t_1] \) and, in particular, at \( t = 0 \). In fact, we can obtain an explicit equation for the derivative \( B(z) = \left. \frac{d}{dt} \Gamma_t(z) \right|_{t=0} \). Differentiating both sides of (2.8) and denoting by \( f \) the infinitesimal generator of \( \mathcal{F} \), we obtain

\[ V_z(t_0, F_t(z))f(F_t(z))\Gamma_t(z) + V(t_0, F_t(z))\frac{d}{dt} \Gamma_t(z) = \Gamma_{t+t_0}(z) - \Gamma_t(z), \]

and then setting \( t = 0 \) and rearranging we have

\[ B(z) = V(t_0, z)^{-1} [\Gamma_{t_0}(z) - 1_A - f(z)V_z(t_0, z)]. \]

In particular, this expression shows that the function \( B \) is holomorphic in \( \mathbb{D}_r \). Since the choice of \( 0 < r < 1 \) was arbitrary, we have shown that \( B \) is a
well-defined holomorphic function on $\mathbb{D}$. Moreover, we have, for any $t$,
\[
\frac{\partial}{\partial t} \Gamma_t(z) = \lim_{s \to 0^+} \frac{1}{s} \left( (\Gamma_{t+s}(z) - \Gamma_t(z)) \right) = \lim_{s \to 0^+} \frac{1}{s} \left( (\Gamma_s(F_t(z)) - 1_A) \Gamma_t(z) \right) = B(F_t(z)) \Gamma_t(z).
\]
(2.9)
Therefore, for every $z \in \mathbb{D}$, the $A$-valued mapping $v(t) = \Gamma_t(z)$ is differentiable with respect to $t$, and is the unique solution of (2.5). Note, finally, that by setting $t = 0$ in (2.9) we obtain the representation (2.6) of the generator $B$.

In the remainder of this section, we present a few results about holomorphic semicocycles, which can be derived using the above theorem.

Theorems 2.1(iii) and 2.2 immediately imply the following fact.

**Corollary 2.1.** Let $\{\Gamma_t\}_{t \geq 0} \subset \text{Hol}(\mathbb{D}, A)$ be a semicocycle over some semigroup $F \subset \text{Hol}(\mathbb{D})$. Then $\Gamma_t(z)$ is invertible for any $t \geq 0$ and $z \in \mathbb{D}$.

The consequence of Theorem 2.2, which seems to be new even in the one-dimensional case $A = \mathbb{C}$, shows that the function $z \to \Gamma_t(z)$, for $t$ fixed, satisfies a differential equation. One can compare the formula below with a known one for semigroup elements:

\[
F_t'(z) = f(z) \left[ B(F_t(z)) \Gamma_t(z) - \Gamma_t(z) B(z) \right].
\]

**Corollary 2.2.** Let $F = \{F_t\}_{t \geq 0} \subset \text{Hol}(\mathbb{D})$ be a semigroup generated by $f \in \text{Hol}(\mathbb{D}, \mathbb{C})$ and $\{\Gamma_t, t \geq 0\}$ be a semicocycle over $F$. Denote $B(z) = \frac{\partial}{\partial t} \Gamma_t(z) \big|_{t=0}$. Then
\[
\Gamma_t'(z) = \frac{1}{f(z)} \left[ B(F_t(z)) \Gamma_t(z) - \Gamma_t(z) B(z) \right].
\]

**Proof.** It follows by Theorem 2.2 that $\Gamma_t(z)$ is differentiable at every point, hence the limit
\[
\lim_{s \to 0} \frac{1}{s} \left[ \Gamma_{t+s}(z) - \Gamma_t(z) \right] = \lim_{s \to 0} \frac{1}{s} \left[ \Gamma_t(F_s(z)) \Gamma_s(z) - \Gamma_t(z) \right]
\]
exists. Since $\Gamma_t \in \text{Hol}(\mathbb{D}, A)$, we conclude that
\[
\Gamma_t(F_s(z)) = \Gamma_t(z) + \Gamma_t'(z)(F_s(z) - z) + o(F_s(z) - z).
\]
Therefore,
\[
\lim_{s \to 0} \frac{1}{s} \left[ \Gamma_{t+s}(z) - \Gamma_t(z) \right] = \Gamma_t(z) B(z) + \Gamma_t'(z) f(z).
\]
On the other hand, the same limit equals $B(F_t(z)) \Gamma_t(z)$ by (2.9). So, the conclusion follows. \qed
We now use Theorem 2.2 to estimate the growth of semicocycles. For this we need the following notation. Given a semicocycle $\{\Gamma_t\}_{t \geq 0}$ and a subset $D \subseteq \mathbb{D}$, denote
\[ \|\Gamma_t\|_D := \sup \{\|\Gamma_t(z)\| , z \in D\} \in [0, \infty]. \]

**Theorem 2.3.** Let $\{\Gamma_t\}_{t \geq 0} \subset \text{Hol}(\mathbb{D}, A)$ be a semicocycle over a semigroup $\mathcal{F} = \{F_t\}_{t \geq 0}$ and $D \subseteq \mathbb{D}$ be an $\mathcal{F}$-invariant domain. For any $K \in \mathbb{R}$, the following assertions are equivalent

(i) $\|\Gamma_t(z)\| \leq e^{Kt}$ for all $z \in D$;

(ii) $\limsup_{t \to 0^+} \frac{1}{t} [\|\Gamma_t\|_D - 1] \leq K$;

(iii) $\mu(B(z)) \leq K$ for all $z \in D$, where $B(z) = \frac{d}{dt} \Gamma_t(z) \big|_{t=0}$, and $\mu$ is defined by (2.3).

**Proof.** Suppose (i) holds. Then
\[ \limsup_{t \to 0^+} \frac{1}{t} \left[ \|\Gamma_t\|_D - 1 \right] \leq \limsup_{t \to 0^+} \frac{1}{t} \left[ e^{Kt} - 1 \right] = K, \]
that is, (ii) is satisfied.

Assume now that (ii) holds. Fix $z \in D$. By definition of $B$, we have
\[ \Gamma_t(z) = 1_A + tB(z) + o(t) \quad \text{as } t \to 0. \]
Therefore
\[ \limsup_{t \to 0^+} \frac{1}{t} \left[ \|1_A + tB(z)\| - 1 \right] = \limsup_{t \to 0^+} \frac{1}{t} \left[ \|\Gamma_t(z)\| - 1 \right] \leq \limsup_{t \to 0^+} \frac{1}{t} \left[ \|\Gamma_t\|_D - 1 \right] \leq K. \]
Thus (iii) follows.

Finally, if (iii) holds, then by Theorem 2.1(ii) we have (i). \[\square\]

**Example 2.1.** Denote
\[ \Gamma_t(z) = \frac{1 + \sqrt{1 - e^{-t}z}}{1 + \sqrt{1 - z}} e^t, \]
where the root branches are chosen such that $\sqrt{1} = 1$. A direct calculation shows that the family $\{\Gamma_t\}_{t \geq 0}$ is a semicocycle over the linear semigroup $\mathcal{F} = \{e^{-t}\}_{t \geq 0}$. One can easily see that
\[ \|\Gamma_t\|_D := e^t \sup_{z \in D} \left| \frac{1 + \sqrt{1 - e^{-t}z}}{1 + \sqrt{1 - z}} \right| \begin{cases} \leq \left(1 + \sqrt{1 + e^{-t}}\right) e^t \leq 3e^t, \\
\geq |\Gamma_t(1)| = \left(1 + \sqrt{1 - e^{-t}}\right) e^t. \end{cases} \]
The upper inequality means that each $\Gamma_t$ is bounded on $\mathbb{D}$. At the same time, the lower inequality implies that

$$\frac{1}{t} [\|\Gamma_t\|_{\mathbb{D}} - 1] \geq \left(1 + \sqrt{1 - e^{-t}}\right) \frac{e^t - 1}{t} = \frac{e^t - 1}{t} + \sqrt{\frac{e^t - 1}{t}} \cdot \frac{e^t}{\sqrt{t}}.$$ 

Thus

$$\limsup_{t \to 0^+} \frac{1}{t} [\|\Gamma_t\|_{\mathbb{D}} - 1] = \infty.$$ 

Therefore by Theorem 2.3, $\{\Gamma_t\}_{t \geq 0}$ does not admit estimate of the form $e^{Kt}$.

It is interesting to compare Theorem 2.3 with the following assertion which was proven in [11, Lemma 2.1] for the case where $\mathcal{D} = \mathbb{D}$, the open unit disk, and $\mathcal{A} = \mathbb{C}$. The proof in the general case is just a repetition of that proof.

**Proposition 2.1.** Let $\{\Gamma_t\}_{t \geq 0}$ be a semicocycle over some semigroup and let $\mathcal{D}$ be an $\mathcal{F}$-invariant domain. The following assertions are equivalent:

(i) $\|\Gamma_t\|_{\mathcal{D}} < \infty$ for every $t \geq 0$.

(ii) $\limsup_{t \to 0^+} \|\Gamma_t\|_{\mathcal{D}} < \infty$;

(iii) $\|\Gamma_t\|_{\mathcal{D}} \leq Me^{Kt}$ for some real $M$ and $K$.

This criterion of boundedness is analogous to a known estimate for strongly semigroups of linear operators while inequality (i) in Theorem 2.3 is similar to estimates of uniformly continuous semigroups; see, for example, [12, 15].

**Example 2.2.** Let $\mathcal{A} = \mathbb{C}$ and $B(z) = \frac{1}{1 - z}$. Clearly, the growth of semicocycles depends on the asymptotic behavior of semigroups.

We start with the linear semigroup $\mathcal{F} = \{e^{-t}\}_{t \geq 0}$ converging to the interior point $z_0 = 0$. In this case the invariant sets are just disks $\mathbb{D}_r$, $r \leq 1$.

Solving the evolution problem (2.5), one finds that

$$\Gamma_t(z) = \frac{e^t - z}{1 - z}.$$ 

Fix some $r < 1$. Then

$$\|\Gamma_t\|_{\mathbb{D}_r} := \sup_{z \in \mathbb{D}_r} \left|\frac{e^t - z}{1 - z}\right| = \frac{e^t - r}{1 - r}.$$ 

and

$$\limsup_{t \to 0^+} \frac{1}{t} [\|\Gamma_t\|_{\mathbb{D}_r} - 1] = \frac{1}{1 - r}.$$ 

Therefore $\|\Gamma_t\|_{\mathbb{D}_r} \leq \exp(K_r t)$ with $K_r = \frac{1}{1 - r}$. At the same time, for every $t > 0$, $\Gamma_t$ is unbounded on the whole unit disk $\mathbb{D}$. 

Now we consider the affine semigroup on \( \mathbb{D} \) defined by
\[
F_t(z) = 1 - (1 - z)e^{-t}.
\]
It converges to the boundary point \( z_0 = 1 \) and the invariant sets are disks tangent to the unit circle at this point. Solving again (2.5), we obtain
\[
\Gamma_t(z) := \exp \left[ \frac{e^t - 1}{1 - z} \right].
\]
In this case, for every \( z_0 \in \mathbb{D} \),
\[
\Gamma_t(F_s(z_0)) = \exp \left[ \frac{e^s(e^t - 1)}{1 - z_0} \right].
\]
So, any \( \Gamma_t \) is unbounded on every semigroup trajectory, and hence cannot satisfy estimate of the form \( M e^{Kt} \) considered in Proposition 2.7.

3. Linearization of semicocycles

3.1. Statement of the problem. As above, let \( \mathcal{F} = \{F_t\}_{t \geq 0} \) be a semigroup on the open unit disk \( \mathbb{D} \). In Section 2, we have seen that each semicocycle can be generated by solving a non-autonomous linear differential equation. A different way to obtain semicocycles is based on the following simple assertion.

**Lemma 3.1.** Let \( M \in \text{Hol}(\mathbb{D}, \mathcal{A}_s) \) and \( B_0 \in \mathcal{A} \). Then the family \( \{\Gamma_t\}_{t \geq 0} \subset \text{Hol}(\mathbb{D}, \mathcal{A}_s) \) defined by
\[
(3.1) \quad \Gamma_t(z) = M(F_t(z))^{-1}e^{tB_0}M(z)
\]
is a semicocycle over \( \mathcal{F} \).

**Proof.** Obviously, \( \Gamma_t \in \text{Hol}(\mathbb{D}, \mathcal{A}) \), is jointly continuous with respect to \((t, z)\) and \( \Gamma_0(z) = 1_\mathcal{A} \). In addition, we have
\[
\Gamma_t(F_s(z))\Gamma_s(z) = M(F_t(F_s(z)))^{-1}e^{tB_0}M(F_s(z))M(F_s(z))^{-1}e^{sB_0}M(z)
= M(F_t(F_s(z)))^{-1}e^{(t+s)B_0}M(z) = \Gamma_{t+s}(z).
\]
So, \( \{\Gamma_t\}_{t \geq 0} \) is a semicocycle. \( \square \)

A semicocycle obtained in this way is cohomologous to the constant semicocycle \( \tilde{\Gamma}_t(z) = e^{tB_0} \).

Our central question in this section is:

- Which semicocycles admit the representation (3.1), for an appropriate choice of \( B_0 \in \mathcal{A} \) and \( M \in \text{Hol}(\mathbb{D}, \mathcal{A}_s) \)?
We will call a representation of the form (3.1) a ‘linearization’ of the semicocycle $\Gamma_t$, and if such a representation exists we will say that the semicocycle is ‘linearizable’. As explained in Introduction, this terminology is intended to convey an analogy with the linearization of semigroups.

It was shown in [11, 9] that if $A = C$ and the semigroup $F$ has no interior fixed point then each semicocycle can be represented in the form

$$\Gamma_t(z) = M(F_t(z))^{-1} M(z),$$

where $M \in \text{Hol}(\mathbb{D}, \mathbb{C} \setminus \{0\})$. For the case where $F$ has an interior fixed point $z_0 \in \mathbb{D}$ the same representation was deduced with $M$ analytic in $\mathbb{D} \setminus \{z_0\}$ — but not in $\mathbb{D}$. The following example shows that the generalized representation (3.1) is more relevant since it allows to describe a larger family of semicocycles using functions analytic in the whole disk.

**Example 3.1.** Let a semigroup $F = \{F_t\}_{t \geq 0} \subset \text{Hol}(\mathbb{D})$ preserve zero and be generated by a function $f \in \text{Hol}(\mathbb{D}, \mathbb{C})$ with $\text{Re} f'(0) < 0$. Define the family $\{\Gamma_t\}_{t \geq 0}$ as follows:

$$\Gamma_t(z) = \left( \frac{F_t(z)}{z} \right)^\beta (m(F_t(z)))^{-1} m(z),$$

where $\beta$ is any real number and a function $m \in \text{Hol}(\mathbb{D}, A^*)$ is normalized by $m(0) = 1_A$. It can be verified directly that this family is a semicocycle over $F$. In order to represent it by formula (3.1), recall (see (1.3)) that for such semigroup there is the unique solution $h$ of the Schröder functional equation

$$h(F_t(z)) = e^{tf'(0)} h(z)$$

normalized by $h'(0) = 1$. Hence, taking $M(z) = \left( \frac{h(z)}{z} \right)^\beta m(z)$, we get

$$\Gamma_t(z) = e^{\beta f'(0)} (M(F_t(z)))^{-1} M(z),$$

which coincides with (3.1).

We note that the representation (3.1), if it exists, is not unique: if we choose any $A \in A^*$ and define

$$\tilde{M}(z) = AM(z), \quad \tilde{B}_0 = AB_0 A^{-1},$$

then

$$\tilde{\Gamma}_t(z) = \tilde{M}(F_t(z))^{-1} e^{t\tilde{B}_0} \tilde{M}(z) = M(F_t(z))^{-1} A^{-1} e^{tA B_0 A^{-1}} A M(z) = M(F_t(z))^{-1} e^{tB_0} M(z) = \Gamma_t(z).$$

This fact allows us to choose the representation in a convenient form depending on the location of the Denjoy–Wolff point of $F$. 
If the semigroup $\mathcal{F}$ has a boundary Denjoy–Wolff point $z_0 \in \partial \mathbb{D}$, then there exists a univalent function $h$ that satisfies Abel’s functional equation

$$h \circ F_t = h + t \quad \text{on } \mathbb{D}$$

(see, e.g., [6]). Hence, if a semicocycle $\{\Gamma_t\}_{t \geq 0}$ over $\mathcal{F}$ admits representation (3.1), then

$$\Gamma_t(z) = M(F_t(z))^{-1} e^{tB_0} M(z) = M(F_t(z))^{-1} e^{(h(F_t(z)) - h(z))B_0} M(z) = M_1(F_t(z))^{-1} M_1(z), \quad \text{where } M_1(z) = e^{-h(z)B_0} M(z).$$

Recall that semicocycles which can be represented in the form (3.1) with $B_0 = 0$, that is $\Gamma_t(z) = M(F_t(z))^{-1} M(z)$, are known as coboundaries. Thus

**Proposition 3.1.** If $\mathcal{F}$ has a boundary Denjoy–Wolff point and a semicocycle over $\mathcal{F}$ admits representation of the form (3.1), then this semicocycle is a coboundary.

This explains why in the above-mentioned result from [11, 9] there was no need to introduce a term $e^{B_0 t}$ in the representation in the case of a boundary Denjoy–Wolff point.

On the other hand, if $z_0 \in \mathbb{D}$ is an interior fixed point of $\mathcal{F}$, then we may choose $A = M(z_0)^{-1}$ in (3.2), so that we obtain $M(z_0) = 1_A$. Therefore we may assume, without loss of generality, that the mapping $M$ in (3.1) satisfies the normalization

$$M(z_0) = 1_A,$$

which we will henceforth do.

As above, denote $B(z) = \frac{d}{dt} \Gamma_t(z) \Big|_{t=0}$. From (3.1)–(3.3) we have

$$\Gamma_t(z_0) = M(z_0)^{-1} e^{tB_0} M(z_0) = e^{tB_0},$$

hence

$$B(z_0) = \frac{d}{dt} \Gamma_t(z_0) \Big|_{t=0} = \frac{d}{dt} e^{tB_0} \Big|_{t=0} = B_0.$$

Thus $B_0$ is uniquely determined by

$$B_0 = B(z_0).$$

From this and (3.2) we see that a necessary condition for a semicocycle $\Gamma_t$ to be a coboundary is that its generator $B$ satisfy $B(z_0) = 0$.

We now show that it is not true, in general, that semicocycles are linearizable.
Example 3.2. Let $\mathcal{A}$ be the Banach algebra $M_2(\mathbb{C})$ of $2 \times 2$ matrices, equipped with the operator norm. Consider the linear semigroup $\mathcal{F} = \{e^{-t}\}_{t \geq 0}$ acting on $\mathbb{D}$ and define the family $\{\Gamma_t\}_{t \geq 0}$ by

$$\Gamma_t(z) = \left( \begin{array}{cc} e^t & zte^t \\ 0 & e^{2t} \end{array} \right).$$

A straightforward computation shows that $\{\Gamma_t\}_{t \geq 0}$ is a semicocycle over $\mathcal{F}$. Differentiating this semicocycle at $t = 0$, one sees that its generator is $B(z) = \left( \begin{array}{cc} 1 & z \\ 0 & 2 \end{array} \right)$, so that $B_0 = B(0) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 2 \end{array} \right)$.

Assume that $\Gamma_t(z)$ can be represented by \ref{3.1} with some

$$M(z) = \left( \begin{array}{cc} \eta_{11}(z) & \eta_{12}(z) \\ \eta_{21}(z) & \eta_{22}(z) \end{array} \right) \in \text{Hol}(\mathbb{D}, \mathcal{A}^*),$$

that is, $M(F_t(z)) \Gamma_t(z) = e^{tB_0} M(z)$. Multiplying these matrices and equating elements of the first row we get

$$\eta_{11}(e^{-t}z) \cdot e^t = e^t \cdot \eta_{11}(z)$$

$$\eta_{11}(e^{-t}z) \cdot zte^t + \eta_{12}(e^{-t}z) \cdot e^{2t} = e^{2t} \cdot \eta_{12}(z).$$

By taking $t \to \infty$ in the first equality and using the normalization $\eta_{11}(0) = 1$, we conclude that $\eta_{11}(z)$ is a constant function, $\eta_{11} = 1$, so that the second equality can be written as

$$z \cdot t \cdot e^{-t} + \eta_{12}(e^{-t}z) = \eta_{12}(z),$$

which leads to a contradiction when taking $t \to \infty$, since $\eta_{12}(0) = 0$. Therefore, $\Gamma_t(z)$ cannot be represented by \ref{3.1}.

In the following we will analyze the problem of linearization, and gain a good understanding of the obstructions to linearizability, which in particular will give us a deeper explanation for the non-linearizability in the above example.

We will first show that in the case of semicocycles whose range $\mathcal{A}$ is commutative, every semicocycle is linearizable, and the linearizing mapping $M$ can be represented in a simple explicit way. Moving to the non-commutative case, we will obtain a sufficient condition for linearizability, depending only on the value $B_0 = B(z_0)$ of the semicocycle generator at the interior fixed point of the semigroup. This condition will also be shown to be sharp. In the case where $\mathcal{A}$ is finite-dimensional, it will be seen that the condition for linearizability is equivalent to a simple condition on the eigenvalues of $B_0$, from which it will follow that it holds for ‘generic’ $B_0 \in \mathcal{A}$. 
3.2. **The commutative case.** In this subsection we assume that the algebra \( A \) is commutative. We show that every semicocycle is linearizable. Our results somewhat generalize results in [11, 9], which considered \( A = \mathbb{C} \).

We begin with the case where \( \mathcal{F} \) has no interior fixed point.

**Theorem 3.1.** Let \( \mathcal{F} = \{F_t\}_{t \geq 0} \subset \text{Hol}(\mathbb{D}) \) be a semigroup generated by \( f \in \text{Hol}(\mathbb{D}, \mathbb{C}) \) with no interior fixed point. Let \( \{\Gamma_t\}_{t \geq 0} \subset \text{Hol}(\mathbb{D}, A) \) be a semicocycle over \( \mathcal{F} \), generated by \( B \). Then \( \Gamma_t(z) \) is linearizable, namely, \( \Gamma_t(z) = M(F_t(z))^{-1}M(z) \), where

\[
(3.5) \quad M(z) = \exp \left( -\int_0^z \frac{1}{f(w)}B(w)dw \right).
\]

**Proof.** We first note that \( M(z) \) in (3.5) is well-defined and \( M \in \text{Hol}(\mathbb{D}, A^*) \). Denote

\[
\tilde{\Gamma}_t(z) := M(F_t(z))^{-1}M(z) = \exp \left( \int_z^{F_t(z)} \frac{1}{f(w)}B(w)dw \right).
\]

Because \( A \) is commutative, differentiation of this equality gives

\[
\frac{\partial \tilde{\Gamma}_t(z)}{\partial t} = B(F_t(z))\tilde{\Gamma}_t(z).
\]

Since \( \Gamma_t \) satisfies the same differential equation and \( \tilde{\Gamma}_0(z) = \Gamma_0(z) = 1_A \), the uniqueness theorem completes the proof. \( \square \)

We proceed with the case where \( \mathcal{F} \) has an interior fixed point. Here, in contrast with [11, 9], we prove that every semicocycle is linearizable by a function \( M \) holomorphic in the whole unit disk \( \mathbb{D} \).

**Theorem 3.2.** Let \( \mathcal{F} = \{F_t\}_{t \geq 0} \subset \text{Hol}(\mathbb{D}) \) be a semigroup with an attractive fixed point \( z_0 \in \mathbb{D} \). Let \( A \) be commutative and \( \{\Gamma_t\}_{t \geq 0} \subset \text{Hol}(\mathbb{D}, A) \) be a semicocycle over \( \mathcal{F} \). Then \( \Gamma_t(z) \) is linearizable, namely,

\[
\Gamma_t(z) = M(F_t(z))^{-1}e^{tB_0}M(z),
\]

where \( B(z) = \frac{d}{dt} \Gamma_t(z)|_{t=0} \), \( B_0 = B(z_0) \), and

\[
M(z) = \exp \left[ \int_0^\infty (B(F_t(z)) - B_0) \, dt \right].
\]

**Proof.** Without loss of generality we can assume that \( z_0 = 0 \). Denote by \( f \in \text{Hol}(\mathbb{D}, \mathbb{C}) \) the generator of \( \mathcal{F} \), so that \( f(0) = 0 \) and \( \lambda := -f'(0) \) satisfies \( \text{Re} \lambda > 0 \). Therefore, we have

\[
|F_t(z)| \leq |z| \exp \left( -\lambda t \frac{1 - |z|}{1 + |z|} \right), \quad z \in \mathbb{D}, \quad t \geq 0.
\]
We already know by Theorem 2.2 that \( \{ \Gamma_t \} \geq 0 \) is differentiable, hence \( B(z) := \frac{d}{dt} \Gamma_t(z) \mid_{t=0} \) exists. First we show that the integral

\[
N(z) := \int_0^\infty (B(F_t(z)) - B_0) \, dt
\]

converges for any \( z \in \mathbb{D} \) and defines a holomorphic function \( N \in \text{Hol}(\mathbb{D}, \mathcal{A}_*) \).

Indeed, fix \( R \in (0, 1) \). There exists \( L > 0 \) such that \( \| B(z) - B_0 \| \leq L \) whenever \( |z| < R \). Then by the Schwarz Lemma

\[
\| B(z) - B_0 \| \leq \frac{L |z|}{R}, \quad z \in \mathbb{D}_R.
\]

For any \( z \in \mathbb{B} \) there is \( t_1 > 0 \) such that \( |F_{t_1}(z)| < R \). Then for all \( t > 0 \),

\[
\| B(F_{t_1+t}(z)) - B_0 \| \leq \frac{L |F_{t_1+t}(z)|}{R} \leq \frac{L |F_{t_1}(z)|}{R} \exp \left( -\lambda t \frac{1 - |F_{t_1}(z)|}{1 + |F_{t_1}(z)|} \right) = M \exp \left( -\lambda t \frac{1 - R}{1 + R} \right).
\]

Therefore, the integral (3.6) is absolutely convergent at any point \( z \in \mathbb{D} \) and uniformly convergent on \( \mathbb{D}_R \).

We now consider the integral

\[
(3.7) \quad \int_0^\infty (B(F_t(z)) - B_0)' \, dt = \int_0^\infty B'(F_t(z))F_t'(z) \, dt.
\]

To estimate it, we note that by the Cauchy inequality \( \| B'(z) \| \leq \frac{L}{R-R_1} \) whenever \( |z| < R_1, \ R_1 \in (0, R) \). We choose \( R_1 \) such that such that \( \text{Re } f'(z) \leq \frac{\lambda}{2} \) for all \( z \in \mathbb{D}_{R_1} \).

In addition, differentiating (1.2) we get that the function \( v \) defined by \( v(t) := F_{t}'(z) \), is the solution of the evolution problem

\[
\left\{ \begin{array}{l}
\frac{\partial v(t)}{\partial t} = f'(F_t(z))v(t) \\
v(0) = 1
\end{array} \right.
\]

Therefore \( |F_{t}'(z)| \leq e^{-\frac{\lambda t}{2}} \) for instance, it follows from assertion (ii) of Theorem 2.1, hence the integral (3.7) converges uniformly on \( \mathbb{D}_{R_1} \). Therefore
one can differentiate (3.6) as follows:

\[ N'(z) = \int_0^\infty B'(F_t(z))F_t'(z)\,dt \]

\[ = \frac{1}{f(z)} \int_0^\infty B'(F_t(z))f(F_t(z))\,dt \]

\[ = \frac{1}{f(z)} \int_0^\infty \frac{\partial}{\partial t} B(F_t(z)) \, dt \]

\[ = \frac{B_0 - B(z)}{f(z)}. \]

Thus

(3.8) \[ B(F_s(z)) = B_0 - N'(F_s(z))f(F_s(z)). \]

Note that since \( A \) is commutative and using the uniqueness of solutions of differential equations,

\[ \Gamma_t(z) = \exp \left( \int_0^t B(F_s(z))\,ds \right). \]

(because the right-hand side in the last formula satisfies (2.5)). Now using (3.8), we rewrite this as follows:

\[ \Gamma_t(z) = \exp \left( \int_0^t (B_0 - N'(F_s(z))f(F_s(z)))\,ds \right) \]

\[ = \exp \left[ tB_0 - N(F_t(z)) + N(z) \right]. \]

This completes the proof. \( \square \)

3.3. The general case. Henceforth in this section we will assume that the semigroup has an interior fixed point \( z_0 \in \mathbb{D} \), and thus we can assume the normalization (3.3), hence (3.4) holds. As we have seen in Example 3.2 above, general non-commutative semicocycles are not always linearizable. We will now analyze the problem of finding a linearizing mapping \( M \), and develop a procedure to construct it using a power series expansion, which will also reveal the nature of the obstructions to linearizability.

We write (3.1) in the form

(3.9) \[ M(F_t(z))\Gamma_t(z) = e^{tB_0}M(z), \]

and consider it as a linear functional equation, which is to be solved for \( M \in \text{Hol}(\mathbb{D}, A_s) \).

As a first step, it will be useful to simplify (3.9) as follows. We denote by \( h \in \text{Hol}(\mathbb{D}, \mathbb{C}) \) the Koenigs function associated to the semigroup, so that

\[ h(F_t(z)) = e^{-\lambda t}h_z(z), \]
where $\lambda = -f'(0)$, with $\text{Re}(\lambda) > 0$, and $h(z_0) = 0, h'(z_0) = 1$. Recall that
\[ \Omega = h(D) \] is a spirallike domain. Set $z = h^{-1}(w), w \in \Omega$, in (3.9) to obtain
\[ M(h^{-1}(e^{-\lambda t}w))\Gamma_t(h^{-1}(w)) = e^{tB_0}M(h^{-1}(w)), \]
or, setting
\[ \gamma_t(w) = \Gamma_t(h^{-1}(w)), \quad m(w) = M(h^{-1}(w)), \]
(3.10)
(3.11)
\[ m(e^{-\lambda t}w)\gamma_t(w) = e^{tB_0}m(w). \]
Thus, finding a representation of $\Gamma_t$ in the form (3.1) is equivalent to finding $m \in \text{Hol}(\Omega, A^*)$ such that (3.11) holds. We shall therefore treat the problem of solving (3.11) for $m \in \text{Hol}(\Omega, A^*)$.

Our strategy is to solve first (3.11) for $m$ locally in a neighborhood of $w = 0$, and then note that any such solution automatically extends to all of $\Omega$.

**Lemma 3.2.** Let $r_0$ be such that $D_{r_0} \subset \Omega = h(D)$. A holomorphic mapping $m : D_{r_0} \rightarrow A$, which satisfies (3.11) for all $w \in D_{r_0}$ and $t \geq 0$, can be extended to a holomorphic mapping satisfying (3.11) for all $w \in \Omega$.

**Proof.** Fix $r > r_0$. Since the domain $\Omega$ is spirallike, the set $\Omega \cap D_r$ is connected. To extend $m$ to $\Omega \cap D_r$, choose $t_0 > 0$ so that $|e^{-\lambda t_0}r < r_0$, and define
\[ \tilde{m}(w) = e^{-t_0B_0}m(e^{-\lambda t_0}w)\gamma_{t_0}(w), \quad w \in \Omega \cap D_r. \]
In view of (3.11), $\tilde{m}(w) = m(w)$ for $w \in D_{r_0}$, so that $\tilde{m}$ extends $m$ holomorphically to $\Omega \cap D_r$. By holomorphicity, the equation (3.11) will continue to hold in the larger domain. Since $r > 0$ is arbitrary, we can extend $m$ to the whole of $\Omega$. \qed

We now begin the process of solving (3.11) locally around $w = 0$. Setting
\[ b(w) = B(h^{-1}(w)), \]
we note that
\[ \frac{d}{dt}\gamma_t(w) = \frac{d}{dt}\Gamma_t(h^{-1}(w)) = B\left(F_t(h^{-1}(w))\right)\Gamma_t(h^{-1}(w)) \]
(3.12)
\[ = B\left(h^{-1}(e^{-\lambda t}w)\right)\gamma_t(w) = b(e^{-\lambda t}w)\gamma_t(w). \]
Differentiating (3.11) with respect to $t$ and then putting $t = 0$, we have
\[ \lambda w m'(w) = m(w)b(w) - B_0m(w). \]
(3.13)
We claim that in fact equation (3.13) is equivalent to (3.10):

**Lemma 3.3.** A mapping $m \in \text{Hol}(D, A^*)$ is a solution of (3.10) if and only if it is a solution to (3.13).
Proof. We have already shown that (3.10) implies (3.13). Now assume \( m \in \text{Hol}(\Omega, \mathcal{A}_*) \) satisfies (3.13). Define \( \tilde{\gamma}_t \) by
\[
(3.14) \quad \tilde{\gamma}_t(w) = m(e^{-\lambda t} w)^{-1}e^{tB_0} m(w).
\]
Multiplying both sides by \( m(e^{-\lambda t} w) \), and differentiating with respect to \( t \), we have
\[
(3.15) \quad -\lambda e^{-\lambda t}w m'(e^{-\lambda t} w) \tilde{\gamma}_t(w) + m(e^{-\lambda t} w) \frac{d}{dt} \tilde{\gamma}_t(w) = e^{tB_0} B_0 m(w).
\]
Substituting \( e^{-\lambda t} w \) in place of \( w \) in (3.13) we get
\[
(3.16) \quad \lambda e^{-\lambda t} w m'(e^{-\lambda t} w) = m(e^{-\lambda t} w)b(e^{-\lambda t} w) - B_0 m(e^{-\lambda t} w).
\]
From (3.14)–(3.16), we have
\[
-m(e^{-\lambda t} w)b(e^{-\lambda t} w) \tilde{\gamma}_t(w) + m(e^{-\lambda t} w) \frac{d}{dt} \tilde{\gamma}_t(w) = 0,
\]
that is,
\[
\frac{d}{dt} \tilde{\gamma}_t(w) = b(e^{-\lambda t} w) \tilde{\gamma}_t(w).
\]
We therefore see that the mapping \( t \mapsto \tilde{\gamma}_t(w) \) satisfies the same differential equations as does \( t \mapsto \gamma_t(w) \) (see (3.12)), and since \( \tilde{\gamma}_0(w) = \gamma_0(w) = 1_\mathcal{A} \), the uniqueness of solutions for initial-value problems implies that \( \tilde{\gamma}_t = \gamma_t \), hence by (3.14) we have the representation (3.10). \( \square \)

In the following we thus investigate equation (3.13), obtaining necessary conditions and sufficient conditions for its solvability. Note that, in view of Lemma 3.2, it suffices to solve (3.13) in a neighborhood of \( w = 0 \). Since (3.13) is a first-order differential equation for \( m(w) \) with a singularity at \( w = 0 \), we cannot appeal to the existence theorem for differential equations to obtain local solvability (and indeed the equation is not always solvable).

To proceed, we expand \( b(w) \) and \( m(w) \) in power series:
\[
b(w) = \sum_{k=0}^{\infty} b_k w^k,
\]
and
\[
m(w) = \sum_{k=0}^{\infty} m_k w^k,
\]
where all the coefficients are elements of \( \mathcal{A} \). Equation (3.13) then becomes
\[
\lambda \sum_{k=0}^{\infty} k m_k w^k = \sum_{k=0}^{\infty} w^k \sum_{l=0}^{k} m_l b_{k-l} - \sum_{k=0}^{\infty} B_0 m_k w^k.
\]
Equating corresponding coefficients, we get
\[ k\lambda m_k = \sum_{l=0}^{k} m_l b_{k-l} - B_0 m_k \]
or, noting that \( b_0 = B_0 \),
\[
(3.18) \quad k\lambda m_k - (m_k B_0 - B_0 m_k) = \sum_{l=0}^{k-1} m_l b_{k-l}, \quad k \geq 0.
\]
We therefore have a sequence of recursive equations, with the equation for \( k = 0 \) holding trivially (since \( m_0 = 1_A \)), and the equation for \( k \geq 1 \), if it is uniquely solvable, defining \( m_k \). Thus the solvability of the sequence of equations \( (3.18) \) is a necessary condition for the existence of a mapping \( m \) satisfying \( (3.13) \). Conversely, if the sequence of equations \( (3.18) \) is solvable, and if the series \( (3.17) \) can be shown to converge in a neighborhood of \( w = 0 \), then it defines a solution of \( (3.13) \) in a neighborhood of 0, and, by Lemma \( 3.2 \) on the whole of \( \Omega \). We have therefore shown:

**Proposition 3.2.** The semicocycle generated by \( B \in \text{Hol}(\mathbb{D}, A) \) is linearizable if and only if the sequence of equations \( (3.18) \) with
\[
B_0 + \sum_{k=1}^{\infty} b_k w^k = B(h^{-1}(w)),
\]
is solvable.

To study the solvability of \( (3.18) \), we define the bounded linear operator \( \text{ad}_{B_0} : A \to A \) by
\[
\text{ad}_{B_0}(a) = aB_0 - B_0 a.
\]
Then \( (3.18) \) can be written as
\[
(3.19) \quad k\lambda m_k - \text{ad}_{B_0}(m_k) = \sum_{l=0}^{k-1} m_l b_{k-l}.
\]
We therefore conclude that, for given \( k \), a sufficient condition for \( (3.19) \) to be solvable is that \( k\lambda \) is not in the spectrum of \( \text{ad}_{B_0} \). If we assume the condition
\[
(3.20) \quad \sigma(\text{ad}_{B_0}) \cap (\lambda \mathbb{N}) = \emptyset,
\]
then all the equations \( (3.19) \) will be solvable:
\[
(3.21) \quad m_k = (k\lambda \cdot 1_A - \text{ad}_{B_0})^{-1}\sum_{l=0}^{k-1} m_l b_{k-l}, \quad k \in \mathbb{N}.
\]
It is known that the spectrum of \( \text{ad}_{B_0} \) can be related to the spectrum of \( B_0 \).
Lemma 3.4. We have

$$\sigma(\text{ad}_{B_0}) \subseteq \sigma(B_0) - \sigma(B_0) := \{\lambda_1 - \lambda_2 : \lambda_1, \lambda_2 \in \sigma(B_0)\}.$$ 

If $\mathcal{A}$ is finite-dimensional, then the inclusion sign can be replaced by the equality sign.

For proof see [13, Theorem 11.23], and for the finite-dimensional case [2, Exercise 5.15.1].

In view of this lemma, we can see that (3.20) follows from the condition

$$\sigma(B_0) - \sigma(B_0) \cap (\lambda \mathbb{N}) = \emptyset,$$

and in the case in which $\mathcal{A}$ is finite-dimensional the two conditions (3.20) and (3.22) are equivalent.

We now show that if (3.20) holds then not only are the equations (3.19) solvable, but the series (3.17) corresponding to the solutions $\{m_k\}_{k=0}^{\infty}$ converges in a neighborhood of $w = 0$, and thus by Lemma 3.2 defines a solution of (3.13).

We note that the condition $\lambda k \not\in \sigma(\text{ad}_{B_0})$ obviously holds for all $k > |\lambda|^{-1}\|\text{ad}_{B_0}\|_{L(A)}$. This observation implies that, under (3.22), there exists a uniform bound

$$\|(k\lambda \cdot 1_A - \text{ad}_{B_0})^{-1}\| \leq C_1, \quad k \in \mathbb{N}.$$ 

Therefore

$$\|m_k\| \leq C_1 \sum_{l=0}^{k-1} \|m_l\| \cdot \|b_{k-l}\|, \quad k \in \mathbb{N}.$$ 

Since $b$ is holomorphic around zero, we can choose $r, C_2 > 0$ so that we have a bound $\|b_k\| \leq C_2 \cdot \frac{1}{r^k}$ for all $k \in \mathbb{N}$. Setting $C_3 = C_1 C_2$, we have

$$\|m_k\| \leq \frac{C_3}{r^k} \sum_{l=0}^{k-1} r^l \|m_l\|.$$ 

This can be used to prove, by induction, that

$$\|m_k\| \leq \left(\frac{C_3 + 1}{r} \right)^k.$$
Indeed, since \( m_0 = 1, A \), (3.24) holds for \( k = 0 \), and, assuming (3.24) for \( k = 1, 2, \ldots, n \), we have

\[
\| m_{n+1}\| \leq \frac{C_3}{r^{n+1}} \sum_{l=0}^{n} r^l \| m_l \| \leq \frac{C_3}{r^{n+1}} \sum_{l=0}^{n} (C_3 + 1)^l \leq \frac{C_3}{r} (C_3 + 1)^{n+1} \cdot (C_3 + 1) - 1 \leq \left( \frac{C_3 + 1}{r} \right)^{n+1}.
\]

Thus the series (3.17) converges at least when \( |w| < \frac{r}{C_3 + 1} \). In view of Lemma 3.2, we have proven the main result of this section.

**Theorem 3.3.** Let a semigroup \( F \) be generated by \( f \in \text{Hol}(\mathbb{D}, \mathbb{C}) \) such that \( f(z_0) = 0 \) for some \( z_0 \in \mathbb{D} \) and \( \lambda := -f'(z_0) \) satisfy \( \text{Re} \lambda > 0 \). Let \( \{ \Gamma_t \}_{t\geq 0} \) be a semicocycle over \( F \). If \( B_0 := \frac{d}{dt}\Gamma_t(z)\big|_{t=0} \) satisfies condition (3.20), that is,

\[
\sigma(\text{ad}_B) \cap (\lambda \mathbb{N}) = \emptyset,
\]

then the semicocycle \( \{ \Gamma_t \}_{t\geq 0} \) generated by \( B \), is linearizable, that is, it can be represented in the form

\[
\Gamma_t(z) = M(F_t(z))^{-1} e^{tB_0} M(z)
\]

with some \( M \in \text{Hol}(\mathbb{D}, A_\ast) \).

Note that the equality \( B_0 = B(0) = 0 \) implies that \( \text{ad}_B = 0 \), so that (3.20) trivially holds, hence \( \Gamma_t(z) = M(F_t(z))^{-1} M(z) \). Thus we have the following.

**Corollary 3.1.** A semicocycle \( \{ \Gamma_t \}_{t\geq 0} \) is a coboundary if and only if \( B_0 = \frac{d}{dt}\Gamma_t(z_0)\big|_{t=0} = 0 \).

We conclude with some remarks concerning Theorem 3.3:

1. It is noteworthy that the sufficient condition given in the theorem is only in terms of the value \( B_0 \) of \( B \) at the interior fixed point of the semigroup.

2. We also recall that the condition (3.20) follows from the more ‘explicit’ condition (3.22) on the spectrum of \( B_0 \), and that in the case that \( A \) is finite-dimensional, it is equivalent to this condition.

3. In the case that \( A \) is finite-dimensional, the condition (3.22) holds ‘generically’, in the sense that, given \( B_0 \), the set of \( \lambda \)’s for which it does not hold is a countable set.

4. In case \( A \) is commutative, we have \( \text{ad}_B = 0 \), so that (3.20) trivially holds, and linearizability is assured. We thus recapture the conclusion of Theorem 3.2 above, but the value of that theorem is that, for commutative
algebras, it gives us a simple explicit representation for $M$, which is not available in the non-commutative case.

(5) The condition (3.20) is a sharp one, in the following sense: assuming that $B_0 \in \mathcal{A}$ is such that (3.22) does not hold, we can always find function $B : \mathbb{D} \to \mathcal{A}$ such that $B(0) = B_0$, and the semicocycle generated by $B$ is not linearizable. Indeed, let $k$ be an integer such that $k\lambda \in \sigma(\text{ad}_{B_0})$. Then by Lemma 3.4 we have $k\lambda \in \sigma(\text{ad}_{B_0})$, hence (using the finite-dimensionality of $\mathcal{A}$) there exists $A \in \mathcal{A}$ which is not in the range of $k\lambda \cdot 1_{\mathcal{A}} - \text{ad}_{B_0}$. Take $b(w) = B_0 + Aw^k$ (that is, $B(z) = B_0 + h(z)^k A$). We thus have $b_j = 0$ for $j \neq 0, k$. Then equation (3.19) (for the chosen value of $k$) becomes

$$(k\lambda \cdot 1_{\mathcal{A}} - \text{ad}_{B_0})(m_k) = A,$$

which is unsolvable, since $A$ is not in the range of $k\lambda \cdot 1_{\mathcal{A}} - \text{ad}_{B_0}$. Therefore the sequence of equations (3.21) is not solvable, so that the semicocycle generated by $B$ is not linearizable.

(6) Using the above observation, we can easily construct simple examples of nonlinearizable semicocycles. For instance, take the semigroup $F_t(z) = e^{-t}z$, so that $\lambda = -f'(0) = 1$ and the Koenigs function is the identity $h(z) = z$ with $\Omega = \mathbb{D}$. Take $\mathcal{A} = M_2(\mathbb{C})$. Let the semicocycle generator be

$$B(z) = \begin{pmatrix} \alpha_1 + \beta_{11} z & \beta_{12} z \\ \beta_{21} z & \alpha_2 + \beta_{22} z \end{pmatrix}.$$ 

Then

$$b(z) = B(z) = b_0 + b_1 z,$$

where

$$b_0 = B_0 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \quad b_1 = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}.$$ 

By Theorem 3.3 if $|\alpha_1 - \alpha_2|$ is not a natural number, then the semicocycle generated by $B$ is linearizable. Therefore, to obtain non-linearizable semicocycles, we can take, e.g., $\alpha_2 = \alpha_1 + 1$. Write

$$m_1 = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix}.$$ 

Equation (3.22), for $k = 1$, is then

$$\begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} - \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} + \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}.$$
which, after simplifying and using the assumption $\alpha_2 = \alpha_1 + 1$, becomes
\[
\begin{pmatrix}
\eta_{11} & 0 \\
2\eta_{21} & \eta_{22}
\end{pmatrix} = -\begin{pmatrix}
\beta_{11} & \beta_{12} \\
\beta_{21} & \beta_{22}
\end{pmatrix}.
\]

We thus see that the equation is not solvable whenever $\beta_{12} \neq 0$. This explains Example 3.2 above.

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