Affine Quantum Gravity*

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Abstract

A sketch of the affine quantum gravity program illustrates a different perspective on several difficult issues of principle: metric positivity; quantum anomalies; and nonrenormalizability.

Quantum gravity is under study from several viewpoints. Brane world scenarios and loop quantum gravity represent the two most popular approaches [1], yet alternative viewpoints may also deserve consideration. One such alternative — called affine quantum gravity — rests on just a few basic principles, outlined below. First, we recall some hurdles that any approach to quantum gravity must face.

Our list of problems that must be faced is short, but each problem is major. These problems are: (1) maintaining the physical nature of the metric field; (2) dealing with a set of open first-class classical constraints that becomes partially second class on quantization; and (3) overcoming the perturbative nonrenormalizability of conventional quantum gravity (i.e., Einstein’s theory, to which we confine our attention). Affine quantum gravity looks at these major problems from a fresh perspective [2]. What follows is a brief overview of this program.

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Metric positivity

Most operator descriptions of quantum theory adopt (or are equivalent to) a Hamiltonian point of view. Specifically, there are several kinematical operators obeying a basic algebra, and for quantum gravity we choose so-called affine commutation relations, which are not equivalent to canonical commutation relations. The determining factor in this choice is ensuring commutation relations among self-adjoint field operators that preserve the spectral properties implicit in metric positivity (strictly positive spatial distances). A one-parameter analog provides a useful clarification. Let $\hat{p}$ and $\hat{q}$ denote Hermitian quantum operators obeying the Heisenberg commutation relation $[\hat{q}, \hat{p}] = i\hbar I$. Insist that $\hat{q}$ be self adjoint with a positive spectrum, $\hat{q} > 0$. Then $\hat{p}$ cannot simultaneously be self adjoint for otherwise it would serve to generate unitary translations of $\hat{q}$ thus violating spectral positivity. The remedy is simple: multiply the Heisenberg relation by $\hat{q}$ to obtain $[\hat{q}, \hat{d}] = i\hbar \hat{q}$, where $\hat{d} = (\hat{q}\hat{p} + \hat{p}\hat{q})/2$. Choose the new, affine commutation relation as basic; then $\hat{q}$ and $\hat{d}$ can both be self adjoint since $\hat{d}$ serves to dilate $\hat{q}$, thereby preserving spectral positivity.

Affine commutation relations for the spatial metric tensor operator $\hat{g}_{ab}(x)$ $[= \hat{g}_{ba}(x)]$ and its dilation partner $\hat{\pi}_{cd}(x)$ entail self-adjoint field operators (after smearing) that have the property of respecting metric positivity. Here $\hat{\pi}_{ab}(x)$ is the quantum field associated with the classical “momentric” field $\pi_{ab}(x) = \pi^{cb}(x) g_{cd}(x)$, where $\pi^{cb}(x) [= \pi^{bc}(x)]$ is the ADM classical momentum conjugate to the classical spatial metric tensor $g_{ab}(x) [= \hat{g}_{ba}(x)]$. The affine commutation relations for quantum gravity are then nothing more than commutation expressions (modulo the usual $i\hbar$) of the associated Poisson brackets for the classical fields $g_{ab}(x)$ and $\pi_{cd}(y)$ that follow directly from the usual Poisson brackets for the classical fields $g_{ab}(x)$ and $\pi_{cd}(y)$. Classically, canonical kinematical variables ($g_{ab}$ and $\pi_{cd}$) are entirely equivalent to affine kinematical variables ($\hat{g}_{ab}$ and $\hat{\pi}_{cd}$); quantum mechanically, only the affine kinematical variables ($\hat{g}_{ab}$ and $\hat{\pi}_{cd}$) preserve metric positivity, and their adoption precludes the canonical momentum from even existing as an operator!
Constraints

Let us turn attention to a brief account of a unified procedure to deal with all kinds of quantum constraints. Classically, constraints are a set of real functions, \( \{ \phi_\alpha \}_{\alpha=1}^A \), of the classical phase space variables, and their vanishing restricts the system to a subset of the original phase space, called the constraint hypersurface, determined by \( \phi_\alpha = 0 \) for all \( \alpha \). Quantum mechanically, constraints are self-adjoint operators, \( \{ \hat{\phi}_\alpha \}_{\alpha=1}^A \), that are related to \( \{ \phi_\alpha \}_{\alpha=1}^A \) by quantization. While \( \sum_\alpha \phi_\alpha^2 \) vanishes on the constraint hypersurface, it does not follow that \( X \equiv \sum_\alpha \hat{\phi}_\alpha^2 \) vanishes on any subspace, but generally \( X \rightarrow 0 \) if \( \hbar \rightarrow 0 \). We focus on a subspace of the original Hilbert space, \( \mathcal{E} \subset \mathcal{H} \), where \( \mathcal{E} = \mathcal{E}(\sum_\alpha \hat{\phi}_\alpha^2 \leq \delta(h)^2) \) is a projection operator that satisfies \( 0 \leq \mathcal{E} \mathcal{X} \mathcal{E} \leq \delta(h)^2 I \). By choosing \( \delta(h)^2 \) suitably, various types of constraints may be covered, e.g.: (a) if \( \hat{\phi}_\alpha = J_\alpha \), \( \alpha = 1, 2, 3 \), where \( J_\alpha \) are generators of \( \text{SO}(3) \), then \( \delta(h)^2 = h^2/2 \) forces \( J_\alpha = 0 \) for all \( \alpha \); (b) if \( \hat{\phi}_1 = \hat{p} \) and \( \hat{\phi}_2 = \hat{q} \), then \( \delta(h)^2 = \hbar \) leads to \( \mathcal{E} \) as a projection operator onto states \( |\psi\rangle \) that satisfy \( (\hat{q} + i\hat{p})|\psi\rangle = 0 \); (c) if \( \hat{\phi}_1 = \hat{q} \) alone, then we may choose (say) \( \delta(h)^2 = 10^{-1000} \). For case: (a) \( X \) has a discrete spectrum including zero, and corresponds to first-class constraints; (b) \( X \) has a discrete spectrum not including zero, and corresponds to second-class constraints; and (c) \( X \) has a continuous spectrum including zero, and \( \delta \) is chosen ridiculously small so that highly unphysical energies would be needed to induce excitations above the lowest level for that (sub)system (it is also possible to let \( \delta \rightarrow 0 \) as a suitable limit to enforce \( \hat{q} = 0 \) exactly). These few examples illustrate how general constraint operators can be treated within the projection operator formalism [3] in a remarkably similar fashion.

For gravity, all physics enters through four constraint fields expressing invariance under coordinate transformations. Mutual consistency of the constraints as expressed by their Poisson brackets leads to an open set of first-class constraints (not characterized by a Lie algebra). When quantized, the consistency of these constraints exhibits an anomaly, i.e., they are partially second class. While others change the theory to avoid this conclusion, we accept it. The projection operator approach to quantum constraints outlined above incorporates second-class constraints as readily as it does first class ones. Therefore, affine quantum gravity formally extends to incorporate the four constraint operator fields of gravity. To incorporate such constraints, some sort of regularization is required.
Nonrenormalizability

To deal with nonrenormalizability it helps to understand what it means for an interaction to be perturbatively nonrenormalizable. Consider the schematic functional integral

\[ S_\lambda(h) = \int e^{i\int h f} e^{-Q(f) - \lambda N(f)} \mathcal{D}f, \]

where \( f \) denotes generic fields (\( h \) denotes source fields), and \( Q \) and \( N \) denote quadratic and nonquadratic components of the action, respectively. If, for all relevant \( f \), \( Q(f) < \infty \implies N(f) < \infty \), then \( \lim_{\lambda \to 0^+} S_\lambda(h) = S_0(h) \), and this situation corresponds to (super)renormalizable interactions. However, if \( Q(f) < \infty \not\implies N(f) < \infty \), then \( \lim_{\lambda \to 0^+} S_\lambda(h) = S'_0(h) \neq S_0(h) \), and this situation corresponds to nonrenormalizable interactions. This result holds because \( N(f) \) acts partially as a hard core in field space, and the expression \( S'_0(h) \) differs from \( S_0(h) \) because it retains the hard-core effects even after \( \lambda \to 0^+ \). Note that the interacting theory is not even continuously connected to the noninteracting theory! Consequently, it is not surprising that a power series expansion of a (regularized) hard-core, nonrenormalizable interaction leads to an ever increasing variety of divergent contributions.

The hard-core picture for nonrenormalizable interactions is quite general [4], and formally it applies to various nonrenormalizable models, including gravity. However, incorporating a hard core within a functional integral is challenging, and only recently [5] has a fully specific proposal been advanced regarding how one might achieve nontrivial results for a quartic self-interacting scalar theory in five (or more) spacetime dimensions. Demonstration of the hard-core philosophy in such models would provide a big boost to their use in other problems such as quantum gravity.

Comment

One of the currently important questions in quantum gravity deals with establishing the discreteness of space, and thus the quantization of areas and volumes. Affine quantum gravity is not sufficiently advanced to address this question, but it is useful to note that there seem to be the roots of just such a topic. Although the metric tensor \( \hat{g}_{ab} \) commutes with itself,

\[ [\hat{g}_{ab}(x), \hat{g}_{cd}(y)] = 0, \]
it must be remembered that this metric tensor is not a physical observable. Instead, the physical metric is given by \( \hat{g}_{ab}^E(x) \equiv E \hat{g}_{ab}(x) E \), and it follows almost surely that

\[
[\hat{g}_{ab}^E(x), \hat{g}_{cd}^E(y)] \neq 0.
\]

While the result of such a commutator is not yet in hand, it is quite possible that it could lead to discrete units of space.

**Summary**

In this brief note we have tried to suggest how one may approach various problems in quantum gravity in new ways. First, we have discussed the affine commutation relations for the basic kinematical field operators that (i) are self-adjoint (after smearing), and (ii) retain the positivity of the metric tensor so that spatial distances are strictly positive. Second, we illustrated the projection operator approach to quantum constraints as a way to accommodate quantum anomalies, i.e., constraints that are partially second class, without having to abandon the original Einstein theory. Third, we have presented the hard-core theory of nonrenormalizability as a way to understand why nonrenormalizable interactions behave as they do, as well as to offer, in principle, a way to develop a nonperturbative procedure to overcome such problems.

The program of affine quantum gravity is not easy, but nevertheless it offers novel ways to attack some otherwise seemingly intractable problems.

**Acknowledgments**

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**References**

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