Digit Polynomials and their Application to Integer Factorization

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Abstract
This paper presents the concept of digit polynomials. We will develop a deterministic and unconditional integer factorization algorithm with the runtime complexity $O(N^{(4+\frac{4C}{\log N - \epsilon})^{-1} + \epsilon})$, $C \approx 0.962$, and with potential for further improvement. We will also prove criteria for primality.

1 Introduction

We consider the problem of computing the prime factorization of a given natural number $N$. Currently, the best publicly known deterministic and unconditional factorization algorithms all have a runtime complexity of the form $O(N^{1/4+\epsilon})$ (See p.240 in [W]). A simple method which achieves this complexity is the approach of Strassen, based on the idea to compute the factorial of $[N^{1/2}]$, which was presented in [S]. A recent improvement of the logarithmic factor in the complexity can be found in [CH]. For a general overview, the reader can consult [P].

In this paper we present a new approach, which yields a deterministic and unconditional factorization algorithm of the complexity

$O(N^{(4+\frac{4C}{\log N - \epsilon})^{-1} + \epsilon})$

with $C = 2 \cdot (\log(1 + \sqrt{5}) - \log 2) \approx 0.962$. The main idea is to construct a polynomial $g \in \mathbb{Z}[X]$ such that as many $x \in \mathbb{N}$, $0 \leq x \leq N - 1$, as possible
satisfy $1 < \gcd(g(x), N) < N$. Several $b$-adic representations of $N$ are used in Theorem 2.3 which yields a method to construct such a polynomial of degree $d$ with the runtime complexity $O(d^{1+\epsilon})$. In the factorization algorithm we will not only make use of the cardinality, but also of the position of those $x$ with the property above.

It appears possible that one can significantly improve our algorithm, for example by solving the problem stated in Section 4. The current version is not appropriate for factorizing large numbers. In practice there are used probabilistic algorithms with much lower complexity. We refer the reader to [R] and [CP].

## 2 The Idea

Throughout this paper, $\mathbb{P}$ denotes the set of primes. We call a natural number semiprime if and only if it is the product of two primes. Let $n \in \mathbb{N}$. We denote the complete residue system $\{0, ..., n-1\}$ modulo $n$ by $\mathbb{Z}_n$. We denote the residue class ring $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}_n$. For $f \in \mathbb{Z}[X]$ we denote the leading coefficient of $f$ by $\text{lc} f$. Until further notice, let $N \in \mathbb{N}$ be fixed.

**Definition 2.1.** Let $b \in \mathbb{Z}$. We denote the set of polynomials $f \in \mathbb{Z}[X]$ with the property $f(b) = N$ by $\mathcal{D}_{N,b}$. The elements of $\mathcal{D}_{N,b}$ are called *digit polynomials of $N$ to base $b$*.

**Definition 2.2.** Let $b \in \mathbb{N}$, $b \geq 2$. Let $N = \sum_{i \geq 0} n_i b^i$ be the unique $b$-adic representation of $N$, $n_i \in \{0, ..., b-1\}$. Define

$$P_b := \sum_{i \geq 0} n_i X^i \in \mathbb{Z}[X].$$

We call $P_b$ the *$b$-adic digit polynomial of $N$*. Clearly, we have $P_b \in \mathcal{D}_{N,b}$.

**Lemma 2.3.** Let $b \in \mathbb{Z}$ and $f \in \mathcal{D}_{N,b}$. Then, for every $x \in \mathbb{Z}$, we have $N \equiv f(x) \mod x - b$.

**Proof.** We know that $b$ is a zero of the polynomial $f - N$, hence $X - b$ divides $f - N$ in $\mathbb{Z}[X]$ and the congruence holds for every evaluation. \(\square\)

**Corollary 2.4.** Let $b \in \mathbb{Z}$ and $f \in \mathcal{D}_{N,b}$. We conclude for every $x \in \mathbb{Z}$ that $x - b \mid N$ iff $x - b \mid f(x)$ and that $\gcd(N, x - b) = \gcd(f(x), x - b)$.

**Lemma 2.5.** Let $b \in \mathbb{Z}$ and let $u, v$ be nontrivial and coprime divisors of $N$. Let $f \in \mathcal{D}_{N,b}$ with the properties that $d := \deg f$ is smaller than the largest prime factor of $v$, $\gcd(\text{lc} f, N) = 1$. Then there exists $x \in \mathbb{Z}$ with $u \mid f(x)$ and $v \nmid f(x)$.
Proof. Let \( y \in \mathbb{Z} \) be arbitrary. Let \( x \in \mathbb{Z} \) with \( uy = x - b \). From Lemma 2.3 we derive \( u \mid f(x) \), hence \( u \mid f(uy + b) \) for any \( y \in \mathbb{Z} \). We have to show that there exists \( y \in \mathbb{Z} \) with \( v \nmid f(uy + b) \).

Assume to the contrary that \( f(uy + b) \equiv 0 \mod v \) for all \( y \in \mathbb{Z} \). Write \( f(uy + b) \) as \( f(b) + u \cdot g(y) \) for \( g \in \mathbb{Z}[X] \). It is easy to verify that \( \deg g = d \) and \( \text{lc} g = u^{d-1} \text{lc} f \). Let \( p \) be the largest prime factor of \( v \). Then, for every \( y \in \mathbb{Z} \), it follows that

\[
f(uy + b) = u \cdot g(y) + f(b) \equiv u \cdot g(y) \equiv 0 \mod p.
\]

The fact \( p \nmid u \) implies \( g(y) \equiv 0 \mod p \) for every \( y \in \mathbb{Z} \). But, since \( \gcd(\text{lc} f, N) = 1 \), we get \( p \nmid \text{lc}(g) \). Therefore, \( g \) is of degree \( d \) in \( \mathbb{Z}_p[X] \) and, for this reason, has at most \( d \) zeros in \( \mathbb{Z}_p[X] \). From \( d < p \) the contradiction follows.

In the proof of the preceding lemma we have seen that, if \( N \) is a composite number and if \( f \in D_{N,b} \) is chosen with appropriate degree, we get various integers \( x \in \mathbb{Z}_N \) such that \( 1 < \gcd(f(x), N) < N \).

Definition 2.6. Let \( g \in \mathbb{Z}[X] \). An element \( x \in \mathbb{Z}_N \) is called suitable in \( g \), if and only if \( 1 < \gcd(g(x), N) < N \). We also define

\[
\nu(g) := | \{ x \in \mathbb{Z}_N : x \text{ is suitable in } g \} |.
\]

If we multiply two polynomials \( f, g \in \mathbb{Z}[X] \), it can happen that \( x \in \mathbb{Z}_N \) suitable in \( f \) and in \( g \), but not in \( f \cdot g \).

Definition 2.7. Let \( d \in \mathbb{N} \) and \( f_i \in \mathbb{Z}[X] \), \( 1 \leq i \leq d \). An element \( x \in \mathbb{Z}_N \) vanishes in \( g := \prod_{i=1}^{d} f_i \), if and only if \( \gcd(g(x), N) = N \) and there is at least one \( i \) such that \( x \) is suitable in \( f_i \).

Theorem 2.8. Let \( N \in \mathbb{N} \) be a semiprime number with distinct prime factors \( p, q \) and assume \( p < q \). Let \( f \in \mathbb{Z}[X] \) and \( d := \deg f \). Let \( n \) be the number of distinct zeros of \( f \) modulo \( p \) and \( m \) be the number of distinct zeros of \( f \) modulo \( q \). Then:

1. \( \nu(f) = mp + nq - 2nm \).

2. Let \( f \neq 0 \) in \( \mathbb{Z}_p[X] \) and in \( \mathbb{Z}_q[X] \). If \( d < p/2 \), then \( \nu(f) \leq dp + dq - 2d^2 \).
Proof. Let \( x \in \mathbb{Z}_N \) be suitable in \( f \). Then \( x \) is a zero of \( f \) either modulo \( p \) or modulo \( q \). Let \( \alpha_1, \ldots, \alpha_n \) be the distinct zeros of \( f \) modulo \( p \) and \( \beta_1, \ldots, \beta_m \) be the distinct zeros of \( f \) modulo \( q \). For \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \) we consider
\[
py + \alpha_i, \text{ for } y = 0, \ldots, q - 1,
qy + \beta_j, \text{ for } y = 0, \ldots, p - 1.
\]
Every \( x \) which is suitable in \( f \) is of that form, and these are a priori \( mp + nq \) values in \( \mathbb{Z}_N \). But some of them might be equal. First, we show that the values of the form \( py + \alpha_i \) are distinct modulo \( N \). We assume that there are \( y_1, y_2 \in \mathbb{Z}_q \) with \( py_1 + \alpha_i \equiv py_2 + \alpha_k \mod N \) for some \( i, k \in \{1, \ldots, n\} \). For \( i \neq k \) this is not possible, because we get \( \alpha_i \equiv \alpha_j \mod p \), which contradicts the assumption that the zeros are distinct modulo \( p \). For \( i = k \), it follows that \( y_1 \equiv y_2 \mod q \). Hence, the congruence only holds if we compare the value \( py_1 + \alpha_i \) with itself. For this reason, all these values are distinct. By similar arguments, one can show that this also holds for the values of the form \( qy + \beta_j \).

Next, we consider the case that some value of the form \( py + \alpha_i \) is congruent to some value of the form \( qy + \beta_j \). Then this value is a zero of \( f \) modulo \( N \). By the Chinese Remainder Theorem, one can easily verify that \( f \) must have exactly \( nm \) distinct zeros modulo \( N \). Since any zero \( z \) of \( f \) modulo \( N \) is also a zero of \( f \) modulo \( p \) and modulo \( q \), we can write \( z = py_1 + \alpha_i = qy_2 + \beta_j \) for some \( y_1, y_2 \) and \( i, j \). Hence, at every zero of \( f \) modulo \( N \) exactly two equal values of our list above coincide. The other values all satisfy \( 1 < \gcd(f(x), N) < N \). Therefore, we get \( \nu(f) = mp + nq - 2nm \).

Consider \( h = -2XY + Xq + Yp \in \mathbb{Z}[X,Y] \). Since \( f \) has at most \( d \) distinct zeros modulo \( p \) and modulo \( q \), we want to maximize this function for \((x, y) \in [0, d]^2 \). We get
\[
h_X(x, y) = -2y + q \text{ and } h_Y(x, y) = -2x + p
\]
as partial derivatives. Hence, the only critical point is \((x, y) = (p/2, q/2)\). But this point is not in \([0, d]^2 \), so we consider \( h \) on the boundary and get
\[
g_1(x) = xq, \ g_2(x) = xp, \ g_3(x) = x(q - 2d) + dp \text{ and } g_4(x) = x(p - 2d) + dq,
\]
for \( x \in [0, c] \). Since \( q > 2d \) and \( p > 2d \), the maximum is \( g_3(d) = g_4(d) = dp + dq - 2d^2 \). \( \square \)

Now we know that, for any polynomial \( f \) with appropriate degree \( d \), there are at most \( dp + dq - 2d^2 \) integers which are suitable in \( f \). We are interested
in efficient methods to construct polynomials which are best possible in this sense. The following theorem will yield a method with runtime complexity of the form $O(d^{1+\epsilon})$. We will use this idea in the factorization algorithm in Section 3. Therefore, the further details will be explained in Remark 3.3.

**Theorem 2.9.** Let $N \in \mathbb{N}$ be semiprime with distinct prime factors $p, q$. Let $d \in \mathbb{N}$ and $b_i \in \mathbb{Z}$, $1 \leq i \leq d$. Let $f_i \in \mathcal{D}_{N,b_i}$ such that $\deg f_i = 1$ and write $f_i = l_iX + c_i$ for every $i$. If $\gcd(c_i, N) = 1$ for every $i$ and also $\gcd(b_j - b_k, N) = 1$ for every choice of $j, k \in \{1, ..., d\}$, $j \neq k$, then

$$\nu\left(\prod_{i=1}^{d} f_i\right) = dp + dq - 2d^2.$$ 

**Proof.** For $1 \leq i \leq d$, consider $f_i$. Since $\gcd(c_i, N) = 1$, $f_i \neq 0$ as polynomial in $\mathbb{Z}_p[X]$ and in $\mathbb{Z}_q[X]$. Therefore, $b_i$ is the only zero of $f_i$ modulo $p$ and modulo $q$.

Now consider $g := \prod_{i=1}^{d} f_i$. Obviously, every $b_i$, $1 \leq i \leq d$, is a zero of $g$ modulo $p$ as well as modulo $q$. Since $\gcd(b_j - b_k, N) = 1$ for every choice of $j, k \in \{1, ..., d\}$, $j \neq k$, these zeros are distinct. For this reason, $g$ has $d$ distinct zeros modulo $p$ and $d$ distinct zeros modulo $q$. Now we apply Theorem 2.8.

**Remark 2.10.** For every polynomial $f_i$ in the theorem above, there are $p + q - 2$ integers which are suitable in $f_i$. But, if we multiply all these polynomials, we do not get $d(p + q - 2)$ suitable integers in the product $g$. It is easy to see that there are $4 \cdot \binom{d}{2}$ integers vanishing in $g$. We get

$$\nu\left(\prod_{i=1}^{d} f_i\right) = dp + dq - 2d^2 = d(p + q - 2) - 4 \cdot \binom{d}{2}.$$ 

### 3 The Algorithm

Let $N \in \mathbb{N}$ be a composite number and $p \leq \lfloor N^{1/2} \rfloor$ be a prime factor of $N$.

**Lemma 3.1.** Let $d \in \mathbb{N}$ and $b_i \in \mathbb{Z}$, $1 \leq i \leq d$. Let $f_i \in \mathcal{D}_{N,b_i}$ be injective modulo $N$ for every $i$ and set $g := \prod_{i=1}^{d} f_i$. If $x \in \mathbb{Z}_N$ with $x \not\equiv b_i \mod N$ for every $i$, then:

$$\gcd(g(x), N) = N \iff x \text{ vanishes in } g.$$
Proof. Assume \( \gcd(g(x), N) = N \). This implies \( g(x) \equiv 0 \mod N \). For every \( i \), we have \( f_i(b_i) \equiv 0 \mod N \). Since \( f_i \) is injective modulo \( N \), we derive \( f_i(x) \not\equiv 0 \mod N \) for every \( i \). But if \( N \) divides \( g(x) \), and if there is no \( i \) such that \( N \) divides \( f_i(x) \), then there has to exist an \( i \) such that \( x \) is suitable in \( f_i \). This means that \( x \) vanishes in \( g \).

Now assume that \( x \) vanishes in \( g \). Then \( \gcd(g(x), N) = N \) follows from the definition.

For \( d := \lceil N^{1/4} \rceil \) and \( i \in \{1, \ldots, d\} \), we choose \( b_i \in \{\lceil N^{1/2} \rceil, \ldots, N-d\} \) such that \( b_{i+1} = b_i + 1 \). We compute the \( b_i \)-adic digit polynomials \( P_{b_i} \) for every \( i \). Since \( \deg P_{b_i} = 1 \) for this choice of bases, we write \( P_{b_i} = l_i X + c_i \). Assume that \( \gcd(c_i, N) = 1 \) for every \( i \). Then these polynomials are injective modulo \( N \), since \( \gcd(c_i, N) = 1 \) immediately implies \( \gcd(l_i, N) = 1 \) and, therefore, \( P_{b_i}(x) \equiv P_{b_i}(y) \mod N \) yields \( x \equiv y \mod N \).

Next, set \( g := \prod_{i=1}^{d} P_{b_i} \) and choose any \( y \in \{1, \ldots, N/p-1\} \). For every \( i \in \{1, \ldots, d\} \) we get \( f_i(py+b_i) \equiv 0 \mod p \) from Lemma 2.3, but we also have \( f_i(py+b_i) \not\equiv 0 \mod N \), since \( f_i(b_i) \equiv 0 \mod N \) and since \( f_i \) is injective modulo \( N \). Hence, the reduction modulo \( N \) of \( py+b_i = py+b_1+i-1 \) is suitable in \( f_i \) for every \( i \). Therefore, the reduction modulo \( N \) of the integers

\[
py+b_1, py+b_1+1, \ldots, py+b_1+d-1
\]

is either suitable in \( g \) or vanishes in \( g \). Now define the sequence

\[
x_j = x_0 + d \cdot j \quad \text{for } j \in \{0, \ldots, d-1\}
\]

and let \( x_0 \in \mathbb{Z}_N \) be chosen such that there is no \( x_j \) and no \( i \in \{0, \ldots, d-1\} \) with \( x_j \equiv b_1+i \mod N \). (For example, we can set \( x_0 := \lceil N^{1/2} \rceil + b_1 + d-1 \).) Then there is \( j \in \{0, \ldots, d-1\} \) such that for some \( y \in \{1, \ldots, N/p-1\} \), the integer \( x_j \) is equal to one of the integers in the string \( (1) \). This is true because of two facts:

1. There is no possibility that we "jump" over such a string \( (1) \), since these strings consist of \( d \) consecutive integers and the increment of our arithmetic sequence \( (2) \) is exactly \( d \).

2. The gaps between the lists are of length \( p-d \). Since \( N^{1/4} \leq \lceil N^{1/4} \rceil \), we get \( N^{1/2} \leq (\lceil N^{1/4} \rceil)^2 \) and, therefore, \( \lceil N^{1/2} \rceil \leq (\lceil N^{1/4} \rceil)^2 \). Consequently, the distance between \( x_0 \) and \( x_{d-1} \) is \( d^2-d \geq \lceil N^{1/2} \rceil - d > p-d \).

Now we use the fact that we can identify integers which vanish in \( g \) by Lemma 3.1. Let \( j \in \{0, \ldots, d-1\} \) such that \( x_j \) is in one of the strings \( (1) \).
If \( x_j \) is suitable in \( g \), then \( 1 < \gcd(g(x_j), N) < N \), and we have found a nontrivial divisor. If \( x_j \) vanishes in \( g \), which is the case if and only if \( \gcd(g(x_j), N) = N \), we compute \( \gcd(P_{b_i}(x_j), N) \) for every \( i \in \{1, \ldots, d\} \). For some \( i \) we get \( 1 < \gcd(P_{b_i}(x_j), N) < N \), and we have also found a nontrivial divisor. These arguments show that the following algorithm is correct.

**Algorithm 3.2.** Let \( N \in \mathbb{N} \). Set \( a_1 := 0, a_2 := 0 \) and take the following steps to factorize \( N \):

1. Choose any \( b \in \lceil N^{1/2} \rceil, \ldots, N - d \) and compute \( P_b, \ldots, P_{b+d-1} \). If there is any \( i \in \{0, \ldots, d-1\} \) such that \( \gcd(c_i, N) > 1 \) for the constant coefficient \( c_i \) of \( P_{b+i} \), we have found a nontrivial factor of \( N \) and the algorithm terminates. Otherwise, set \( x_0 := \lceil N^{1/2} \rceil + b + d - 1 \) and compute
   \[
   g := \prod_{i=0}^{d-1} P_{b+i} \mod N.
   \]
2. For \( k = 0, \ldots, d-1 \), compute \( y_k := g(x_0 + d \cdot k) \mod N \).
3. Set \( j := a_1 \).
4. If \( j \geq d \), \( N \) is a prime number and the algorithm terminates. Otherwise, compute \( G := \gcd(y_j, N) \). If \( G = 1 \), set \( a_1 := j + 1 \) and go to Step 3. If \( 1 < G < N \), we have found a nontrivial factor of \( N \) and the algorithm terminates. If \( G = N \), go to Step 5.
5. Set \( i := a_2 \).
6. If \( i \geq d \), \( N \) is a prime number and the algorithm terminates. Otherwise, compute \( H := \gcd(P_{b+i}(x_0 + d \cdot j), N) \). If \( H = 1 \), set \( a_2 := i + 1 \) and go to Step 5. If \( 1 < H < N \), we have found a nontrivial factor of \( N \) and the algorithm terminates.

**Remark 3.3.** Let us discuss the runtime complexity of the algorithm above. Note that the multiplication time \( M(d) \) for multiplying two integers of length \( n \) can be bounded by \( \mathcal{O}(d \cdot \log d \cdot \log(\log d)) \).

Step 1: We have to perform \( d \) divisions and, if \( \gcd(c_i, N) = 1 \) for every \( i \), we have to multiply the \( d \) polynomials \( P_{b+i} \) of degree 1. There are well known methods to do this by \( \mathcal{O}(M(d) \cdot \log d) \) arithmetic operations.

Step 2: Here we have to evaluate the polynomial \( g \) of degree \( d \) in \( d \) points. This can be done by \( \mathcal{O}(M(d) \cdot \log d) \) arithmetic operations, using the well known methods for multipoint evaluation of polynomials.
For detailed information concerning Step 1 and Step 2, we refer the reader to [GG, Ch.10]. In particular, see algorithms 10.3 and 10.5.

Step 4 and Step 6: We have to compute at most $d$ greatest common divisors in each of these steps. For this task, we employ the Euclidean algorithm.

To summarize, our algorithm has a runtime complexity of the form $O(N^{1/4+\epsilon})$. If we know that there is a prime factor smaller than $\lceil N^{1/m} \rceil$, which for instance has to be the case if $N$ has at least $m$ nontrivial factors, then it is easy to see that our algorithm will also work if we put $d := \lceil N^{1/m} \rceil$. Hence, we get a runtime complexity of the form $O(N^{1/m+\epsilon})$ in these cases.

4 Practical and theoretical Improvements

We have already seen that a significant improvement of the runtime complexity of the algorithm can only be achieved if we are able to prove that the algorithm, or some variant of it, also works for a smaller choice of $d$. We exhibit such an improvement in the following theorem, which yields the runtime complexity stated in the abstract. We will see that our earlier choice of $x_0$ was not really random. Also note that square numbers are easy to factorize, which allows us to make the following assumption.

**Theorem 4.1.** Let $N \in \mathbb{N}$, $N \geq 18$ be not a square and set $C := 2 \cdot (\log(1+\sqrt{5}) - \log 2) \approx 0.962$. Then Algorithm 3.2 also works for the choice of

$$d := \left\lceil N^{(4+4C)/(\log N - C)} \right\rceil.$$ 

**Proof.** We consider the worst case, which is to say, $N$ is semiprime with distinct prime factors $p, q$. We assume $p \leq \lceil N^{1/2} \rceil < q$ and write $d := \lceil N^{1/x} \rceil$ for $x \in \mathbb{R}^+$, to be determined later. First we show that if there is any $m \in \mathbb{N}$ in the interval $[N^{1/2} + b, N^{1/2} + N^{2/x} + b]$ which can be written as $m = py + b$ or $m = qy + b$ for some $y \in \mathbb{N}$, we are able to find some element of the related string (1) with the sequence (2).

Consider the first element $x_0 = \lceil N^{1/2} \rceil + b + d - 1$ and the last element $x_{d-1} = \lceil N^{1/2} \rceil + b - 1 + d^2 \geq \lceil N^{1/2} \rceil + \lceil N^{2/x} \rceil + b - 1$ of the sequence (2). If there is any $m \in \{\lceil N^{1/2} \rceil + b + d - 1, \ldots, \lceil N^{1/2} \rceil + \lceil N^{2/x} \rceil + b - 1\}$ of the form $m = py + b$ or $m = qy + b$ for some $y \in \mathbb{N}$, we already know from previous arguments that the sequence will find an element of the related string. Furthermore, if there is any $m \in \{\lceil N^{1/2} \rceil + b, \ldots, \lceil N^{1/2} \rceil + b + d - 2\}$
of the form \( m = py + b \) or \( m = qy + b \) for some \( y \in \mathbb{N} \), \( x_0 \) itself is an element of the related string.

Next, we show that we have considered the entire subset of natural numbers in the interval above. Assume that \( \lceil \frac{N_1}{2} \rceil + \lceil \frac{N_2}{x} \rceil + b - 1 < N_1/2 + N_2/x + b \) and assume that there is a natural number in the interval \( (\lceil \frac{N_1}{2} \rceil + \lceil \frac{N_2}{x} \rceil + b - 1, N_1/2 + N_2/x + b] \). Since \( \lceil \frac{N_1}{2} \rceil + \lceil \frac{N_2}{x} \rceil + b - 1 \) is a natural number itself, this implies

\[
N^{1/2} + N^{2/x} + b - (\lceil \frac{N_1}{2} \rceil + \lceil \frac{N_2}{x} \rceil + b - 1) \geq 1.
\]

Since \( N \) is not a square number, we can write \( \lceil \frac{N_{1/2}}{2} \rceil = \frac{N_{1/2}}{2} + \delta_1 \) for some \( 0 < \delta_1 < 1 \). Also write \( \lceil \frac{N_2}{x} \rceil = \frac{N_2}{x} + \delta_2 \) for some \( 0 \leq \delta_2 < 1 \). Hence, the inequality above is equivalent to the contradiction \( \delta_1 + \delta_2 \leq 0 \). Furthermore, since \( N \) is not a square number, there is also no natural number in the interval \( \lceil \frac{N_1}{2} + b, \lceil \frac{N_1}{2} \rceil + b \rceil \).

For this reason, we know that we only have to consider the following question: For which \( x \in \mathbb{R}^+ \) can we be sure that there is a multiple of \( p \) or a multiple of \( q \) in the interval \( [N_1^{1/2}, N_1^{1/2} + N_2^{2/x}] \)? We assume that there is no multiple of the factors in the interval, which obviously implies \( p > N_2^{2/x} \) and \( q > N_1^{1/2} + N_2^{2/x} \). We conclude \( N > N_2^{2/x}(N_1^{1/2} + N_2^{2/x}) \). Therefore, we obtain a contradiction to our assumption for all \( x \in \mathbb{R}^+ \) for which \( N \leq N_2^{2/x}(N_1^{1/2} + N_2^{2/x}) \) holds. For example, \( x = 4 \) satisfies this condition with \( N \leq 2N \). This shows that there must exist \( x > 4 \), depending on \( N \), which always is a solution to the inequality. Indeed, one can verify that \( x = 4 \cdot \frac{\log N}{\log N - c} = 4 + \frac{4c}{\log N - c} \) is the optimization of this problem, since it satisfies the condition with equality.

This proves the statement of the theorem for the worst case. We know that \( N \geq 18 \), hence \( x < 6 \) and the algorithm also works with the same \( d \), if \( N \) has more than two nontrivial factors.

Theorem 4.1 was a first successful approach to improve the runtime complexity of the algorithm, based on the fact that this version of the algorithm easily detects too small divisors and divisors directly above the square root of \( N \). Some further minor improvements may be possible by an appropriate choice of the first base \( b \). Another approach that might lead to better results is to choose another kind of sequence in (2). With this idea in mind, we consider the following general and open problem, also formulated for the worst case that \( N \) is semiprime.
**Problem 4.2.** Let \( N \in \mathbb{N} \) be semiprime with the unknown, distinct prime factors \( p,q \). Let \( d \in \mathbb{N}, d < \lceil \frac{N}{4} \rceil \). Let \( b \in \mathbb{N} \), like in the algorithm. We search for \( f \in \mathbb{Z}[X] \) such that, for at least one pair \((x, y) \in \{0, ..., d - 1\}^2\), precisely one of the following properties holds:

1. \( f(x) \equiv b + y \mod p \),
2. \( f(x) \equiv b + y \mod q \).

*Note that this implies* \( f(x) \neq b + y \mod N \).

If we could find any \( f \) that satisfies these conditions, we would construct the sequence \( x_j = f(j) \) for \( j = 0, ..., d - 1 \). We know that then there is some \( j \) with either \( x_j = pk + b + y \) or \( x_j = qk + b + y \) for some \( k \in \mathbb{N} \) and some \( y \in \{0, ..., d - 1\} \). Therefore, \( x_j \) would be an element of some string (1) and we could use the sequence in our algorithm.

It is the next goal to find such \( f \) for the choice of \( d := \lceil \frac{N}{5} \rceil \). For now, we will give the following reformulation of the problem in the terms of this paper.

**Lemma 4.3.** Let \( d \in \mathbb{N}, d < \frac{N}{4} \). Let \( b \in \mathbb{N} \) like in the algorithm. Let \( f \in \mathbb{Z}[X] \) and define \( g := f(X) - b - Y \in \mathbb{Z}[X,Y] \). Then the following are equivalent:

1. \( f \) is a solution to the problem.
2. There exists \((x, y) \in \{0, ..., d - 1\}^2\) such that \( x \) is suitable in \( g(\cdot, y) \in \mathbb{Z}[X] \).

*Proof.* Clearly, \( f \) is a solution to the problem if and only if there exists \((x, y) \in \{0, ..., d - 1\}^2\) with either \( g(x, y) \equiv 0 \mod p \) or \( g(x, y) \equiv 0 \mod q \).

This is equivalent to \( 1 < \gcd(g(x, y), N) < N \), which already means that \( x \) is suitable in \( g(\cdot, y) \).

The preceding lemma is a link between the algorithm and the theory developed in Section 2. For the rest of this section we will focus on some further theoretical results. We will prove a result to ensure the maximal amount of suitable integers in the product of digit polynomials of degree 2, which may be compared to Theorem 2.9. We will see that the \( b \)-adic digit polynomials will be especially useful in this case, not only because it is easy to compute them, but also because of their uniqueness and their special construction.
**Theorem 4.4.** Let \( N \in \mathbb{N} \) be semiprime with distinct prime factors \( p, q \).

Let \( d \in \mathbb{N} \) and \( b_i \in \mathbb{Z} \), \( 1 \leq i \leq d \). Let \( f_i \in D_{N,b_i} \) such that \( \deg f_i = 2 \) and write \( f_i = n_{2,i}X^2 + n_{1,i}X + n_{0,i} \) for every \( i \).

If \( \gcd(n_{2,i}, b_i, N) = 1 \) for every \( i \) and if for \( b_{d+i} := n_{0,i} \cdot n_{2,i}^{-1} \cdot b_i^{-1} \mod N \), \( 1 \leq i \leq d \), we have \( \gcd(b_j - b_k, N) = 1 \) for every choice of \( j, k \in \{1, ..., 2d\}, j \neq k \), then

\[
\nu(\prod_{i=1}^{d} f_i) = 2dp + 2dq - 8d^2.
\]

**Proof.** For \( 1 \leq i \leq d \), consider \( f_i \). Since \( \gcd(n_{2,i}, N) = 1 \), \( f_i \) is a polynomial of degree 2 modulo \( p \). Therefore \( f_i \) has at most two zeros modulo \( p \). One of them is \( b_i \). But since \( \mathbb{Z}_p \) is a field, there has to be another zero modulo \( p \).

We know from Vieta’s Theorem that this zero has to be the solution of

\[
n_{2,i}b_i \cdot x \equiv n_{0,i} \mod p.
\]

Since \( b_{d+i} \equiv n_{0,i} \cdot n_{2,i}^{-1} \cdot b_i^{-1} \mod p \), \( b_{d+i} \) is this zero of \( f_i \) modulo \( p \). With similar arguments, one can also show that \( b_i \) and \( b_{d+i} \) are the zeros of \( f_i \) modulo \( q \).

Now consider \( g := \prod_{i=1}^{d} f_i \). Obviously, every \( b_i, 1 \leq i \leq 2d \) is a zero of \( g \) modulo \( p \) as well as modulo \( q \). Since \( \gcd(b_j - b_k, N) = 1 \) for every choice of \( j, k \in \{1, ..., 2d\}, j \neq k \), these zeros are distinct. For this reason, \( g \) has \( 2d \) distinct zeros modulo \( p \) and \( 2d \) distinct zeros modulo \( q \). Now we apply Theorem 2.8.

If we set \( d = 1 \) in the Theorem above, the following statement is an immediate consequence.

**Corollary 4.5.** Let \( N \in \mathbb{N} \) be semiprime with distinct prime factors \( p, q \).

Let \( b \in \mathbb{Z} \) and \( f \in D_{N,b} \) with \( \deg f = 2 \) and \( f = n_2X^2 + n_1X + n_0 \). If \( \gcd(n_2 \cdot b, N) = 1 \) and \( \gcd(N, n_2b^2 - n_0) = 1 \), then \( \nu(f) = 2p + 2q - 8 \).

We want to make Theorem 4.4 applicable. Hence, we have to find digit polynomials for which the condition of distinct zeros modulo the factors of \( N \) can be verified in \( O(d) \) steps. For the linear polynomials in Theorem 2.9 this was feasible, since we were able to choose appropriate bases, for example consecutive integers. Here, every base \( b_i \) we choose comes with a second integer \( b_{d+i} \), which we have to control.
Lemma 4.6. Let $d \in \mathbb{N}$ and let $b_i \in \{\lceil N^{1/2}/\sqrt{2} \rceil, \ldots, \lceil N^{1/2} \rceil\}$, $1 \leq i \leq d$, be coprime to $N$ such that $b_{i+1} = b_i + 1$. Set $D := b_1 + \lfloor N/b_d \rfloor \mod N$.

If $\gcd(D + z, N) = 1$ for every $z \in \{0, \ldots, 2d - 2\}$ and if the $b$-adic digit polynomials $P_b$ satisfy $n_{1,i} \leq n_{0,i} + 1$ for every $i$, then they also satisfy the conditions in Theorem 4.4.

Proof. Let $i \in \{1, \ldots, d\}$ be arbitrary. It is easy to see that $n_{2,i} = 1$ for this choice of bases. Since $\gcd(b_i, N) = 1$, the first condition of Theorem 4.4 is satisfied. Now set $b_{d+i} := n_{0,i}b_i^{-1} \mod N$. Consider the division with remainder of $N$ with respect to $b_i$ and write $m_i b_i + n_{0,i} = N$. We get

$$-m_i b_i \equiv n_{0,i} \equiv b_{d+i} b_i \mod N,$$

hence $-m_i \equiv b_{d+i} \mod N$. Next, we consider $N = (m_i - 1)(b_i + 1) + r = m_i b_i + m_i - b_i - 1 + r$ for some $r \in \mathbb{Z}$. Assume that $r \geq b_i + 1$. Then it follows that $N \geq m_i b_i + m_i$. But since $b_i \leq \lceil N^{1/2} \rceil$, it is easy to see that $m_i \geq b_i$. By $n_{0,i} < b_i$ we conclude

$$N \geq m_i b_i + m_i \geq m_i b_i + b_i > m_i b_i + n_{0,i} = N,$$

hence a contradiction. Now we assume that $r < 0$. Then it follows that $N < m_i b_i + m_i - b_i - 1$. But this yields that $n_{0,i} + b_i + 1 < m_i$, and by $N > b_i(n_{0,i} + b_i + 1) + n_{0,i} = b_i^2 + (n_{0,i} + 1)b_i + n_{0,i}$ we conclude $n_{1,i} > n_{0,i} + 1$, which contradicts our assumption. As a consequence, we get $0 \leq r < b_i + 1$. Because of the uniqueness of the division with remainder, there has to be $r = n_{0,i+1}$ and $m_{i+1} = m_i - 1$. Altogether we derive

$$b_{d+i+1} \equiv -m_{i+1} \equiv -m_i + 1 \equiv b_{d+i} + 1 \mod N.$$

Now assume that there exist $j, k \in \{1, \ldots, d\}$ such that $b_{d+k} \equiv b_j \mod p$. We write $b_j = b_1 + m$ for some $m \in \{0, \ldots, d-1\}$ and, as we just have shown, we can write

$$b_{d+k} \equiv b_{2d} - l \equiv -m_d - l \equiv -\lfloor N/b_d \rfloor - l \mod p,$$

for some $l \in \{0, \ldots, d-1\}$. It follows that $-\lfloor N/b_d \rfloor - l \equiv b_1 + m \mod p$.

Therefore, we get

$$0 \equiv b_1 + \lfloor N/b_d \rfloor + m + l \equiv D + z \mod p,$$

for some $z \in \{0, \ldots, 2d - 2\}$. But this contradicts our assumption. Hence, for every choice of $j, k \in \{1, \ldots, d\}$, the integers $b_{d+j}$ are different from the
integers $b_k$ modulo $p$. It is also impossible that there exist $j, k \in \{1, \ldots, d\}$, $j \neq k$ with $b_{d+j} \equiv b_{d+k} \mod p$ or with $b_k \equiv b_j \mod p$, because this would imply $p \leq d$, which as well contradicts the assumption $\gcd(D + z, N) = 1$ for $z \in \{0, \ldots, 2d - 2\}$. By similar arguments, one can show that the zeros are all distinct modulo $q$.

This lemma allows to work with digit polynomials of degree 2 in practice.

## 5 Some Results for Primes

Finally, we present some characterizations of primality by digit polynomials. The major work for the following proofs is already done. Let $N \in \mathbb{N}$ be a fixed odd number. Note that it is easy to detect powers of prime numbers, which allows us to assume that $N$ is either prime or composite with at least two different prime factors.

**Theorem 5.1.** Let $b \in \mathbb{Z}$ and $f \in D_{N,b}$ with $d := \deg f$. Let $d$ be smaller than $q := \max\{q' \in \mathbb{P} : q' \mid N\}$ and $\gcd(lc f, N) = 1$. Then the following holds:

$$N \in \mathbb{P} \iff \forall x \in \mathbb{Z}_N : f^{N-1}(x) \mod N \in \{0, 1\}.$$ 

**Proof.** Assume that $N$ is prime. Then the statement immediately follows from Fermat’s little Theorem.

Assume that $N$ is a composite number. Let $p$ be a prime factor of $N$ such that $p \neq q$. According to Lemma 2.5 there exists $x \in \mathbb{Z}$ with $p \mid f(x)$ and $q \nmid f(x)$. Write $pj = f(x)$ for some $j \in \mathbb{Z}$. Then we get

$$f^{N-1}(x) \equiv (pj)^{N-1} \not\equiv 1 \mod N,$$

because otherwise there would exist $k \in \mathbb{Z}$ with $(pj)^{N-1} - 1 = pk$; hence $p \mid 1$. Since $f^{N-1}(x) \not\equiv 0 \mod q$, we also derive $f^{N-1}(x) \not\equiv 0 \mod N$. Therefore, we have found $x \in \mathbb{Z}$ with $f^{N-1}(x) \not\equiv 1 \mod N$ and $f^{N-1}(x) \not\equiv 0 \mod N$, which yields a contradiction.

**Corollary 5.2.** Let $b \in \mathbb{Z}$ and $f \in D_{N,b}$ with $d := \deg f$. Let $d$ be smaller than $q := \max\{q' \in \mathbb{P} : q' \mid N\}$ and $\gcd(lc f, N) = 1$. Then the following holds:

$$N \in \mathbb{P} \iff \forall x \in \mathbb{Z}_N : f^{N-1}(x) / 2 \mod N \in \{-1, 0, 1\}.$$ 

Proof. Assume that $N$ is prime. Then the statement immediately follows from Euler’s Criterion.

Assume that $N$ is a composite number. According to Theorem 5.1 there is $x \in \mathbb{Z}_N$ such that $f^{N-1}(x) \mod N$ is neither 0 nor 1. Then $f^{N-1}(x) \mod N$ is different from $-1, 0$ and 1. Hence, this implies a contradiction.

Example 5.3. Let $b = N$ and let $f = X \in D_{N,b}$. Then all the conditions of Theorem 5.1 and Corollary 5.2 are satisfied. We derive the well known results

$$N \in \mathbb{P} \iff \forall x \in \mathbb{Z}_N : x^{N-1} \mod N \in \{0, 1\}$$

$$\iff \forall x \in \mathbb{Z}_N : x^{\frac{N-1}{2}} \mod N \in \{-1, 0, 1\}.$$ 

It would be interesting to deduce a characterization of primality with a closed formula for digit polynomials. As possible condition we could consider the congruence $\forall x \in \mathbb{Z}_N : f^{N-1}(x) \equiv f^{2N-2}(x) \mod N$, since 0 and 1 are idempotent elements. Unfortunately, this statement is not only true for prime numbers, which we will prove in the following theorem.

Theorem 5.4. Let $f \in \mathbb{Z}[X]$. Then the following holds:

If $N$ is Carmichael, then $f^{N-1}(x) \equiv f^{2N-2}(x) \mod N$ for every $x \in \mathbb{Z}_N$.

Proof. Let $x \in \mathbb{Z}_N$ be arbitrary. Assume $\gcd(f(x), N) = 1$. Since $N$ is Carmichael, we get $f^{N-1}(x) \equiv 1 \mod N$, hence the congruence holds. Assume $\gcd(f(x), N) =: d > 1$. For some $j \in \mathbb{Z}$ with $\gcd(j, N) = 1$ we can write $dj = f(x)$. Therefore, we get

$$f^{N-1}(x) \equiv (dj)^{N-1} \equiv d^{N-1} \mod N,$$

and

$$f^{2N-2}(x) \equiv (dj)^{2N-2} \equiv d^{2N-2}(j^{N-1})^2 \equiv d^{2N-2} \mod N.$$

We have to show that $N$ is a divisor of $d^{2N-2} - d^{N-1} = d^{N-1}(d^{N-1} - 1)$. Because $N$ is Carmichael, $N$ is squarefree and we can write $N = p_1 \cdots p_k$ with $p_i \neq p_j$ for $i \neq j$. Without loss of generality, we write $d = p_1 \cdots p_m$ for some $m \leq k$. Now every prime factor $p_n$ of $N$ with $m < n \leq k$ is coprime to $d$. It follows that

$$d^{p_n - 1} \equiv 1 \mod p_n.$$

$N$ is Carmichael. Therefore, we know that $p_n - 1 \mid N - 1$. Hence, $p_n$ is a divisor of $d^{N-1} - 1$. This yields the result.
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