Resolution of The Linear-Bounded Automata Question

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Abstract

This paper resolves a famous and longstanding open question in automata theory, i.e., the linear-bounded automata question (or shortly, LBA question), which can also be phrased succinctly in the language of computational complexity theory as

\[ \text{NSPACE}[n] \not\subseteq \text{DSPACE}[n]. \]

In fact, we prove a more general result that

\[ \text{DSPACE}[S(n)] \subsetneq \text{NSPACE}[S(n)] \]

where \( S(n) \geq n \) is a space-constructible function. Our proof technique is based on diagonalization against deterministic \( S(n) \) space-bounded Turing machines with a universal nondeterministic Turing machine and on other novel and interesting new techniques. Our proof also implies the following consequences, which resolve some famous open questions in complexity theory:

1. \( \text{DSPACE}[n] \subsetneq \text{NSPACE}[n] \);
2. \( L \subseteq NL \);
3. \( L \subsetneq P \);
4. There exists no deterministic Turing machine working in \( O(\log n) \) space deciding the \( st \)-connectivity question (STCON).

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1 Introduction

1.1 Background

The field of automata theory (see e.g. [A1]), which was more or less influenced by Turing’s abstraction [Tur37] when viewing Turing machines as the most general kind of automata, has witnessed two fundamental branches in the last eight decades, the first of which is the study of the notion of computability; see e.g. Rogers [Rog67], dealing with the main topics of what is computable by Turing machines. This kind of study, including other important questions related to Turing machines, has been extended to different types of automata such as finite automata, pushdown automata, linear-bounded automata, and so on; see e.g. Ginsburg [Gin66], Hopcroft et al. [HU79], and Kuroda [Kur64]. It is worth noting that, in the context of formal language theory, a Turing machine is an automaton capable of enumerating some arbitrary subset of valid strings of an alphabet (see e.g. [A3]).

As already observed by Stearns et al. [SHL65], an important goal of automata theory is a basic understanding of computational processes for various classes of problems. Thus, the theory of automata expanded beyond the notion of computability to include some measure of the difficulty of the computation and how that difficulty is related to the organization of the machine that performs the computation, which formed the second branch of automata theory known as complexity theory; see e.g. [SHL65].

Perhaps the most basic and fundamental measures of difficulty in complexity theory that appear particularly important are time and space (memory). Clearly, the study of space complexity classes DSPACE[S(n)], NSPACE[S(n)] and coNSPACE[S(n)] was made possible by the basic measure of space, which also enabled the theory of PSPACE – completeness; see e.g. [GJ79, AB09, Pap94, Sip13]. Today, these notions are important building blocks in complexity theory.

As is well-known, nondeterminism is another important notion in complexity theory. The question of whether nondeterminism adds power to a space-bounded Turing machine was apparently first posed by Kuroda [Kur64], who asked whether deterministic and nondeterministic
linear bounded automata are equivalent. Indeed, in the 20th century 60’s, in his seminal paper [Kur64], Kuroda, who showed that the class of languages accepted by nondeterministic linear-bounded automata (i.e., the complexity class NSPACE[n]) is right the context-sensitive languages, stated two research challenges, which subsequently became famously known as the LBA questions (see e.g. [HH73, HH74] for a discussion of the history and importance of these questions), the first of which is whether the class of languages accepted by nondeterministic linear-bounded automata is equal to the class of languages accepted by deterministic linear-bounded automata, i.e., the following

**Question 1:** Are the complexity classes NSPACE[n] and DSPACE[n] the same?

If NSPACE[n] and DSPACE[n] are the same, then the family of context-sensitive languages is closed under complement. On the other hand, it still could happen that

$$\text{DSPACE}[n] \neq \text{NSPACE}[n]$$

but the family of context-sensitive languages is closed under complement. Thus we have the second LBA question, i.e., the following

**Question 2:** Are the complexity classes NSPACE[n] and coNSPACE[n] the same?

The above two questions stated respectively can be phrased succinctly in the language of computational complexity theory as (see e.g. [A2]):

$$\text{NSPACE}[n] \supseteq \text{DSPACE}[n];$$
$$\text{NSPACE}[n] \supseteq \text{coNSPACE}[n].$$

Both of these questions are basically problems about the minimal amount of memory needed to perform a computation. As said in [HH73], such questions, in general, are quite difficult. Although the second LBA question (i.e., the Question 2) has an affirmative answer — implied by the famous Immerman–Sénizergues theorem (see [Imm88, Sze88] — proved 20 years after the question was raised, the first LBA question (i.e., the Question 1) still remains open. It is worth noting that Kuroda [Kur64] showed that the class of languages accepted by nondeterministic linear-bounded automata (i.e., the complexity class NSPACE[n]) is right the context-sensitive languages, so the first LBA question is equivalent to asking whether the whole of the context-sensitive languages can be accepted by deterministic linear-bounded automata; see e.g. [A2]. In fact, Question 1 has become one of the oldest open questions in automata theory and complexity theory; see e.g. [Sav70, RCH91]. At the same time, our inability to answer this question indicates that we have not yet understood the nature of nondeterministic computation (see [HH73]). In what follows, as the title indicated, we will call the first LBA question, which is still open, the LBA question.
Interestingly, Savitch’s algorithm [Sav70] gives us an initial insight into the LBA question: nondeterministic space Turing machines can be simulated efficiently by deterministic space Turing machines, with only a quadratic loss in space usage. That is,

\[ \text{NSPACE}[S(n)] \subseteq \text{DSpace}[S(n)^2] \]

for \( S(n) \geq \log n \) where \( S(n) \) is a space-constructible function. As can be seen, the technique used in the proof of Savitch’s algorithm [Sav70] is an interesting application of divide-and-conquer (see e.g. p. 369 in [AHU74]).

In addition, although most of the interesting questions related to the power of nondeterminism remain open and we still do not know whether nondeterministic space is equal to deterministic space, we believe that nondeterministic space is more powerful than deterministic space. Indeed, Cook et al. [CRS80] have already given some evidence that NSPACE[log \( n \)] is more powerful than DSPACE[log \( n \)] by showing that the limited model of log-space Turing machine (JAG machines) cannot recognize the Threadable Mazes set, where the Threadable Mazes set introduced by Savitch [Sav70] is log-space complete for NSPACE[log \( n \)]. Nonetheless, we still lack proof of the exact relation between the complexity class DSPACE[log \( n \)] (i.e., \( L \)) and the complexity class NSPACE[log \( n \)] (i.e., \( NL \)), so we have the following

**Question 3:** Are the complexity classes \( L \) and \( NL \) the same?

The directed st-connectivity question, denoted STCON, is one of the most widely studied questions in theoretical computer science and simple to state: Given a directed graph \( G \) together with vertices \( s \) and \( t \), the st-connectivity question is to determine if there is a directed path from \( s \) to \( t \). Interestingly, STCON plays an important role in complexity theory as it is complete for the complexity class nondeterministic logspace \( NL \) under log-space reductions, see e.g. [AB09, Mic92, Pap94, Sip13]. Currently, the best known space upper bound is \( O(\log^2 n) \) using Savitch’s algorithm [Sav70]. For the undirected st-connectivity question, in a breakthrough result, Reingold [Rei08] showed that the undirected st-connectivity question can be solved in \( O(\log n) \) space, which renews the enthusiasm to improve Savitch’s bound for STCON, since we are aware that one of the obvious directions is to extend Reingold’s algorithm to the directed case. However, it is considered that proving any nontrivial \( \Omega(\log n) \) space lower bound on a general Turing machine is beyond the reach of current techniques. So, we have the following open question:

**Question 4:** Whether there is a deterministic \( O(\log n) \) space-bounded Turing machine deciding the STCON?

\(^2\)Throughout this paper, \( \log n \) stands for \( \log_2 n \).
1.2 Main Results

In this paper, we will work with the technique of diagonalization against deterministic space-bounded Turing machines with a universal nondeterministic Turing machine to resolve the LBA question (for more information about the diagonalization, see e.g. [For00, FS07] or Turing’s original article [Tur37]). We remark that the technique of diagonalization against deterministic Turing machines by a universal nondeterministic Turing machine will be further used in the author’s other works, such as [Lin21a, Lin21b], since both of them handle the topic of determinism versus nondeterminism. Before getting into the main contributions, let us make some digressions. The inspiration of diagonalization against deterministic Turing machines by a universal nondeterministic Turing machine originated from two concurrent matters: At that moment, the author was reading the proof of the Space Hierarchy Theorem for Deterministic Turing Machines in [AHU74] (see Theorem 11.1 in standard textbook [AHU74], p. 408–410); and simultaneously, the author was having the LBA question in his mind. Then, a very natural idea appears: If using a universal nondeterministic Turing machine to diagonalize against deterministic $S(n)$ space-bounded Turing machine, is it possible to obtain a language in $\text{NSPACE}[S(n)]$? If so, setting $S(n) = n$ will yield a negative answer to the LBA question. In fact, all conclusions in this paper are the direct or indirect products of this inspiration and crack.

Indeed, the following is our first main result:

**Theorem 1.1** Let $S(n) \geq n$ be a space-constructible function. Then there is a language

$$L_d \in \text{NSPACE}[S(n)]$$

but

$$L_d \notin \text{DSPACE}[S(n)].$$

That is,

$$\text{DSPACE}[S(n)] \subsetneq \text{NSPACE}[S(n)].$$

Focus our attention on the LBA question; only the limited case, i.e., the Turing machines are limited to the one-way case but not the two-way, was shown in [HU67] that the nondeterministic models are more powerful than deterministic models. However, for two-way Turing machines of all other complexity classes, it is an open question as to whether or not the deterministic and nondeterministic models are equivalent. In fact, set $S(n) = n$, then the above Theorem 1.1 immediately yields a proof of a negative answer to the LBA question:

**Theorem 1.2** There exists a language $L_d$ such that

$$L_d \in \text{NSPACE}[n]$$

but

$$L_d \notin \text{DSPACE}[n].$$

That is,

$$\text{DSPACE}[n] \subsetneq \text{NSPACE}[n].$$

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3Around an afternoon of early October 2021.
For the complexity classes $L$, $NL$, and the deterministic polynomial-time class $P$, and so on, there is a well-known tower of inclusions, see e.g. [AB09, Pap94, Sip13]:

$$L \subseteq NL \subseteq P \subseteq NP \subseteq \text{PSPACE}. $$

It is unknown whether $L \subsetneq NL$ and $L \subsetneq P$; see e.g. [A1]. With the above Theorem 1.2 at hand, we are able to resolve the famous open question $L \nsubseteq NL$ (see e.g. [AB09, Pap94, Sip13]) by showing the following:

**Theorem 1.3** $L \subseteq NL$. That is, $L \neq NL$.

From which it immediately follows that:

**Corollary 1.4** $L \neq P$. That is, $L \subsetneq P$.

However, currently it is unknown whether $NL = P$, and to the best of our knowledge, it is also an open question in theoretical computer science.

Furthermore, the question of $L \nsubseteq NL$ and STCON are closely correlated, since STCON is log-space complete for $NL$ (see e.g. [AB09, Pap94, Sip13]). Thus, if STCON can be decided in space $O(\log n)$ deterministically, then $L = NL$ will follow, and vice versa. It is clear that the above Theorem 1.3 together with some basic knowledge, can show the following:

**Theorem 1.5** There exists no deterministic $O(\log n)$ space-bounded Turing machine deciding the STCON.

### 1.3 Overview

The remainder of this paper is organized as follows: For the convenience of the reader, in the next section, we will review some notation and notions that are closely associated with our introductions appearing in Section 1. In Section 3 and Section 4 we will prove our more general results, which lead to our main results. The proofs of Theorem 1.1 and Theorem 1.2 are put in Section 5. In Section 6 we resolve the famous open question of $L$ vs. $NL$ by giving the proof of Theorem 1.3. Section 7 is dedicated to solving STCON by showing Theorem 1.5. Finally, a brief conclusion is drawn, and some open questions are introduced in the last section.
2 Preliminaries

In this section, we describe some notions and notation needed in the following context.

Let $\mathbb{N}$ denote the natural numbers

\[ \{0, 1, 2, 3, \cdots \} \]

where $+\infty \not\in \mathbb{N}$. Further, $\mathbb{N}_1$ denotes the set of

\[ \mathbb{N} - \{0\}, \]

i.e., the set of positive integers.

For convenience, we also denote the set of positive integers which are greater than or equal to 3 by $\mathbb{N}_3$, i.e.,

\[ \mathbb{N}_3 \overset{\text{def}}{=} \mathbb{N} - \{0, 1, 2\}. \]

Throughout the paper, the computational modes used are Turing machines and linear-bounded automata. We follow the definition of a Turing machine given in standard textbook [AHU74] and redefine the linear-bounded automata as per the notion of space-bounded Turing machine.

**Definition 2.1 (k-tape deterministic Turing machine, [AHU74])** A k-tape deterministic Turing machine (shortly, DTM) $M$ is a seven-tuple $(Q, T, I, \delta, b, q_0, q_f)$ where:

1. $Q$ is the set of states.
2. $T$ is the set of tape symbols.
3. $I$ is the set of input symbols; $I \subseteq T$.
4. $b \in T - I$, is the blank.
5. $q_0$ is the initial state.
6. $q_f$ is the final (or accepting) state.
7. $\delta$ is the next-move function, maps a subset of $Q \times T^k$ to

\[ Q \times (T \times \{L, R, S\})^k. \]

That is, for some $(k + 1)$-tuples consisting of a state and k tape symbols, it gives a new state and k pairs, each pair consisting of a new tape symbol and a direction for the tape head. Suppose

\[ \delta(q, a_1, a_2, \cdots, a_k) = (q', (a'_1, d_1), (a'_2, d_2), \cdots, (a'_k, d_k)), \]

and the deterministic Turing machine is in state $q$ with the $i$th tape head scanning tape symbol $a_i$ for $1 \leq i \leq k$. Then in one move the deterministic Turing machine enters state $q'$, changes symbol $a_i$ to $a'_i$, and moves the $i$th tape head in the direction $d_i$ for $1 \leq i \leq k$. 7
The notion of a nondeterministic Turing machine is similar to that of a deterministic Turing machine, except that the next-move function \( \delta \) is a mapping from \( Q \times T^k \) to subsets of \( Q \times (T \times \{L, R, S\})^k \), stated as follows:

**Definition 2.2** (\( k \)-tape nondeterministic Turing machine, [AHU74]) A \( k \)-tape nondeterministic Turing machine (shortly, NTM) \( M \) is a seven-tuple \( (Q, T, I, \delta, b, q_0, q_f) \) where all components have the same meaning as for the ordinary deterministic Turing machine, except that here the next-move function \( \delta \) is a mapping from \( Q \times T^k \) to subsets of \( Q \times (T \times \{L, R, S\})^k \).

The language accepted by the Turing machine \( M \), denoted as \( L(M) \), is the set of words \( w \) in \( I^* \) such that \( M \) enters an accepting state. If for every input word of length \( n \), deterministic (resp. nondeterministic) Turing machine \( M \) scans at most \( S(n) \) tape cells on any storage tape (work-tape), then \( M \) is said to be a deterministic (resp. nondeterministic) \( S(n) \) space-bounded Turing machine, or of space complexity \( S(n) \). The language recognized by \( M \) (i.e., \( L(M) \)) is also said to be of space complexity \( S(n) \).

We should note that, if a Turing machine can rewrite on its input tape, then it is an on-line Turing machine (as in Definition 2.1 and Definition 2.2). Otherwise, it is an off-line Turing machine. An on-line Turing machine \( M \) is of space complexity \( S(n) \) if for each input of length \( n \), \( M \) uses at most \( S(n) \) tape cells of its storage tape (including input tape). For an off-line Turing machine \( M \), it has a read-only input tape and \( k \) semi-infinite storage tapes (work tapes). If for every input word \( w \) of length \( n \), \( M \) scans at most \( S(n) \) cells on any storage tape, then \( M \) is said to be an \( S(n) \) space-bounded off-line Turing machine, or of space complexity \( S(n) \). Then, a language is of space complexity \( S(n) \) for some machine model if it is defined by a Turing machine of that model, which is of space complexity \( S(n) \).

Also note that off-line Turing machines enable us to consider space bounds of less than linear growth. For example, in Section 6, we will deal with off-line Turing machines, since their space bounds are less than linear growth, i.e., to be \( \log n \). It is worth noting that, if a Turing machine could rewrite on its input tape, then the length of the input would have to be included in calculating the space bound. Thus no space bound could be less than linear (see e.g. [HU79]), i.e., its space bounds \( \geq n \).

The notation \( \text{DSPACE}[S(n)] \) and \( \text{NSPACE}[S(n)] \) denote the class of languages accepted by deterministic \( S(n) \) space-bounded Turing machines and nondeterministic \( S(n) \) space-bounded Turing machines, respectively. In particular, \( L \) is the class \( \text{DSPACE}[\log n] \) and \( NL \) is the class \( \text{NSPACE}[\log n] \), i.e., the class of languages accepted by deterministic and nondeterministic \( \log n \) space-bounded Turing machines, respectively.

The original notion of a deterministic (nondeterministic) linear-bounded automaton is a deterministic (nondeterministic) single-tape (i.e., on-line) Turing machine whose read/write head never leaves those cells on which the input was placed, see [Myh60, Lan63, Kur64]. The equivalent definition of linear-bounded automaton is as follows:

**Definition 2.3** Formally, a deterministic (nondeterministic) linear-bounded automaton is an on-line deterministic (nondeterministic) \( S(n) = n \) space-bounded Turing machine.

The following two lemmas are convenient tools needed in proof of the main results, whose proof can be found in page 372 of [AHU74]:
Lemma 2.1 (Corollary 3, [AHU74], p. 372) If $L$ is accepted by a $k$-tape deterministic Turing machine of space complexity $S(n)$, then $L$ is accepted by a single-tape deterministic Turing machine of space complexity $S(n)$.  

Lemma 2.2 (Corollary 2, [AHU74], p. 372) If $L$ is accepted by a $k$-tape nondeterministic Turing machine of space complexity $S(n)$, then $L$ is accepted by a single-tape nondeterministic Turing machine of space complexity $S(n)$.  

By Lemma 2.2 we know that if a language $L$ is accepted by an on-line $k$-tape deterministic Turing machine of space complexity $S(n) \geq n$, then $L$ is accepted by an on-line single-tape deterministic $S(n)$ space-bounded Turing machine. Thus we can restrict ourselves to (on-line) single-tape deterministic Turing machines when studying the LBA question.

For a complexity class $C$, its complement is denoted by $coC$ (see e.g. [Pap94]), i.e.,

$$coC = \{ \overline{L} : L \in C \},$$

where $L$ is a decision problem, and $\overline{L}$ is the complement of $L$. Note that, the complement of a decision problem $L$ is defined as the decision problem whose answer is “yes” whenever the input is a “no” input of $L$ and vice versa.

The following famous result, which resolved the second LBA question, says that the working space for both on-line and off-line nondeterministic Turing machines is closed under complement:

Lemma 2.3 (Immerman-Sénizergues Theorem, [Imm88, Sze88]) Let $S(n) \geq \log n$ be a space-constructible function. Then the nondeterministic space of $S(n)$ is closed under complement. That is

$$NSPACE[S(n)] = coNSPACE[S(n)].$$

Other background information and notions will be given along the way in proving our main results stated in Section 1.

3 Diagonalization against Deterministic $S(n)$ Space-Bounded Turing Machines

To obtain the result of Theorem 1.1, we need to enumerate the deterministic Turing machines, that is, assign an ordering to DTMs so that for each nonnegative integer $i$, there is a unique DTM associated with $i$.

By Lemma 2.1, we can restrict ourselves to single-tape deterministic Turing machines. So, in the following context, by DTMs we mean single-tape DTMs. Thus, these DTMs are on-line Turing machines.

We will use the method presented in [AHU74], p. 407, to encode a single-tape deterministic Turing machine into an integer. Moreover, without loss of generality, we can make the following

4In this paragraph, $L$ is not the complexity class $\text{DSPACE}[\log n]$, but a language accepted by some Turing machine.
assumptions about the representation of a single-tape deterministic Turing machine with input alphabet \(\{0, 1\}\) because that will be all we need:

1. The states are named \(q_1, q_2, \cdots, q_s\) for some \(s\), with \(q_1\) the initial state and \(q_s\) the accepting state.

2. The input alphabet is \(\{0, 1\}\).

3. The tape alphabet is \(\{X_1, X_2, \cdots, X_t\}\) for some \(t\), where \(X_1 = \#\), \(X_2 = 0\), and \(X_3 = 1\).\(^5\)

4. The next-move function \(\delta\) is a list of quintuples of the form

   \[(q_i, X_j, q_k, X_l, D_m),\]

   meaning that

   \[\delta(q_i, X_j) = (q_k, X_l, D_m),\]

   and \(D_m\) is the direction, \(L, R,\) or \(S\), if \(m = 1, 2,\) or \(3\), respectively. We assume this quintuple is encoded by the string

   \[10^i10^j10^k10^m1.\]

5. The deterministic Turing machine itself is encoded by concatenating in any order the codes for each of the quintuples in its next-move function. Additional 1’s may be prefixed to the string if desired. The result will be some string of 0’s and 1’s, beginning with 1, which we can interpret as an integer.

By this method, any integer that cannot be decoded is assumed to represent the trivial Turing machine with an empty next-move function by this encoding. It is obvious that such a representation of a deterministic Turing machine defines a function that is surjective

\[e : \mathbb{N}_1 \to T\]

where \(T\) is the set of all single-tape deterministic Turing machines. Hence, \(e\) is an enumeration of the set of all single-tape deterministic Turing machines. In addition, we denote the \(i^{th}\) Turing machine in enumeration \(e\) as \(e(i)\). It should be noted that every DTM will appear infinitely often in the enumeration, since given a DTM, we may prefix 1’s at will to find larger and larger integers representing the same set of quintuples.

We can now design a four-tape NTM \(M\) that treats its input string \(x_i\) both as an encoding of a DTM \(M_i\) and also as the input to \(M_i\). One of the capabilities possessed by \(M\) is the ability to simulate a Turing machine, given its specification. We shall have \(M\) determine whether the

\(^5\)By this condition, it is clear that \(t\), the number of tape symbols of a single-tape Turing machine, is a fixed positive integer in \(\mathbb{N}_3\). And, different single-tape Turing machine may be with different \(t \in \mathbb{N}_3\).
Turing machine $M_i$ accepts the input $x_i$ without using more than $S(|x_i|)$ tape cells for some space-constructible function $S$. If $M_i$ accepts $x_i$ in space $S(|x_i|)$, then $M$ does not. Otherwise, $M$ accepts $x_i$. Thus, for all $i$, either $M$ disagrees with the behavior of the $i$th DTM on that input $x_i$ which is the binary representation of $i$, or the $i$th DTM uses more than $S(|x_i|)$ tape cells on input $x_i$. We first show the following important result:

**Theorem 3.1** Let $S(n) \geq n$ be a space-constructible function. Then, there exists a language $L_d$ accepted by a nondeterministic Turing machine $M$ running within space

$$(1 + \lceil \log t \rceil)S(n) \quad \text{for all } t \in \mathbb{N},$$

but by no deterministic $S(n)$ space-bounded Turing machines.

**Proof.** Let $M$ be a four-tape NTM which operates as follows on an input string $x$ of length of $n$.

1. $M$ decodes the deterministic Turing machine encoded by $x$ and determines $t$, the number of tape symbols used by this deterministic Turing machine, and $s$, its number of states. The third tape of $M$ can be used as “scratch” memory to calculate $t$. If $x$ is not the encoding of some single-tape deterministic Turing machine, $M$ halts without accepting.

2. $M$ marks off $(1 + \lceil \log t \rceil)S(n)$ cells on each tape. After doing so, if any tape head of $M$ attempts to move off the marked cells, $M$ halts without accepting.

3. Otherwise let $M_i$ be the deterministic Turing machine encoded by $x$. Then $M$ lays off on its second tape $S(n)$ blocks of $\lceil \log t \rceil$ cells each, the blocks being separated by single cells holding a marker #, i.e., there are $(1 + \lceil \log t \rceil)S(n)$ cells in all. Each tape symbol occurring in a cell of $M_i$’s tape will be encoded as a binary number in the corresponding block of the second tape of $M$. Initially, $M$ places its input, in binary coded form, in the blocks of tape 2, filling the unused blocks with the code for the blank.

4. On tape 3, $M$ sets up a block of $\lceil \log s \rceil + \lceil \log S(n) \rceil + \lceil \log t \rceil S(n)$ cells, initialized to all $0$’s, provided again that this number of cells does not exceed $(1 + \lceil \log t \rceil)S(n)$, which is possible, because

$$\frac{\lceil \log s \rceil + \lceil \log S(n) \rceil + \lceil \log t \rceil S(n)}{(1 + \lceil \log t \rceil)S(n)} \rightarrow \frac{\lceil \log t \rceil}{1 + \lceil \log t \rceil} < 1$$

as

$$n \rightarrow +\infty$$

and since there are arbitrary long binary string representing the same Turing machine $M_i$. Tape 3 is used as a counter to count up to $sS(n)t^{S(n)}$.\footnote{If a single-tape deterministic/nondeterministic Turing machine $N$ (with $t$ tape symbols) is of constructible space complexity $S(n)$, then there is a number of distinct configurations that $N$ needs to enter when started with an input of length $n$, and this number is at most $|Q| \times S(n) \times t^{S(n)}$,}

where $Q$ is the set of states of $N$. 11
5. $M$ simulates $M_i$, using tape 1, its input tape, to determine the moves of $M_i$ and using tape 2 to simulate the tape of $M_i$. The moves of $M_i$ are counted in binary in the block of tape 3, and tape 4 is used to hold the state of $M_i$. If $M_i$ accepts, then $M$ halts without accepting. $M$ accepts if $M_i$ halts without accepting, if the simulation of $M$ attempts to use more than the allotted cells on tape 2, or if the counter on tape 3 overflows, i.e., the number of moves made by $M_i$ exceeds $sS(n)tS(n)$.

The nondeterministic Turing machine $M$ described above is of space complexity 
\[(1 + \lceil \log t \rceil)S(n) \text{ for all } t \in \mathbb{N}_3,\]

since $M_i$ has $t$ tape symbols with $t \in \mathbb{N}_3$ (see footnote 5). It is worth noting that $M$ running within space 
\[(1 + \lceil \log t \rceil)S(n) \text{ for all } t \in \mathbb{N}_3\]

includes the most extreme case, i.e., the case that for each $t \in \mathbb{N}_3$, there exists a single-tape deterministic $S(n)$ space-bounded Turing machine with $t$ tape symbols.

Then, by Lemma 2.2, it is equivalent to a single-tape NTM of space complexity
\[O((1 + \lceil \log t \rceil)S(n)) \text{ for all } t \in \mathbb{N}_3,\]

and it of course accepts some language $L_d$.

Suppose now $L_d$ were accepted by some deterministic $S(n)$ space-bounded Turing machine $M_j$. By Lemma 2.1 we may assume that $M_j$ is a single-tape Turing machine. Let $M_j$ have $s$ states and $t$ tape symbols, we can find a string $w$ of length $n$ representing $M_j$ such that
\[\frac{\lceil \log s \rceil + \lceil \log S(n) \rceil + \lceil \log t \rceil S(n)}{(1 + \lceil \log t \rceil)S(n)} < 1.\]

This is also possible, because
\[\lim_{n \to +\infty} \frac{\lceil \log s \rceil + \lceil \log S(n) \rceil + \lceil \log t \rceil S(n)}{(1 + \lceil \log t \rceil)S(n)} = \frac{\lceil \log t \rceil}{1 + \lceil \log t \rceil} < 1.\]

So, $M$ has enough spaces to simulate $M_j$ and accepts $w$ if and only if $M_j$ does not accept it. But we assumed that $M_j$ accepted $L_d$, i.e., $M_j$ agreed with $M$ on all inputs. We thus conclude that $M_j$ does not exist, i.e., $L_d$ is not accepted by any deterministic $S(n)$ space-bounded Turing machine.

Now, for convenience, if we denote by
\[\text{NSPACE}[(1 + \lceil \log t \rceil)S(n) \text{ for all } t \in \mathbb{N}_3]\]

the set of languages accepted by nondeterministic Turing machines which run within space
\[(1 + \lceil \log t \rceil)S(n) \text{ for all } t \in \mathbb{N}_3,\]

we thus can conclude, by the above arguments, that:
\[\text{DSPACE}[S(n)] \subset \neq \text{NSPACE}[(1 + \lceil \log t \rceil)S(n) \text{ for all } t \in \mathbb{N}_3].\]

This completes the proof. $\square$
Remark 3.1 In fact, we can design our universal nondeterministic Turing machine $M$ more complicated. For example, since we can also encode any nondeterministic Turing machine into a binary string (see e.g. [Lin21b]), the input to $M$ can be classified into three types: If the input is a deterministic Turing machine, then $M$ does the work specified in the proof of Theorem 3.1. If the input is a nondeterministic Turing machine $N$, then $M$ determines $t$, the number of tape symbols used by $N$, and $s$, its number of states, then $M$ marks off
\[(1 + \lceil \log t \rceil)S(n)\]
cells on each tape and sets the counter of tape 3 to count up to (see footnote 6)
\[sS(n)t^{S(n)}\]
then $M$ simulates $N$ nondeterministically, using tape 1, its input tape, to determine the moves of $N$ and using tape 2 to simulate the tape of $N$. The moves of $M$ are counted in binary in the block of tape 3, and tape 4 is used to hold the state of $N$. $M$ accepts the input if and only if $N$ accepts. However, if the input can not be decoded to some deterministic Turing machine or nondeterministic Turing machine, $M$ rejects the input. Note that such a design does not change $M$’s space complexity.

4 Proof of $L_d \in \text{NSPACE}[S(n)]$

Since the nondeterministic Turing machine $M$ constructed above, which accepts the language $L_d$, runs within space $(1 + \lceil \log t \rceil)S(n)$ for all $t \in \mathbb{N}_3$, for the goal of this section, we define the following family of languages $\{L_d^i\}_{i \in \mathbb{N}_3}$:

$L_d^i \overset{\text{def}}{=} \text{language accepted by } M \text{ running within } (1 + \lceil \log i \rceil)S(n) \text{ space for fixed } i \in \mathbb{N}_3$.

That is, $M$ first marks off $(1 + \lceil \log i \rceil)S(n)$ cells on each tape after acting item (1) and before acting item (2) in the proof of Theorem 1.3. Then, $M$ turns itself off mandatorily if any tape head of $M$ attempts to move off the marked $(1 + \lceil \log i \rceil)S(n)$ cells without accepting,

which means that when $M$ is acting on item (2), $M$ will reject the input if $t > i$. This is not hard to prove, since it is clear that give $M$ only $(1 + \lceil \log i \rceil)S(n)$ space; if $t > i$, then it is impossible to simulate $M_{xt}$, which requires $(1 + \lceil \log t \rceil)S(n)$ space, i.e., item (3) in the proof of Theorem 1.3 will not fit.

Then, we next show the following theorem:

\[\text{In other words, when limited to use only } (1 + \lceil \log i \rceil)S(n) \text{ tape cells, } M \text{ only can simulate those single-tape deterministic } S(n) \text{ space-bounded Turing machines that the number } t \text{ of tape symbols less than or equal to } i.\]
Theorem 4.1

\[ L_d = \bigcup_{i \in \mathbb{N}_3} L_d^i. \]

Furthermore, the following relations hold:

\[ L_d^3 \subseteq L_d^4 \subseteq \cdots \subseteq L_d^i \subseteq L_d^{i+1} \subseteq \cdots. \]

Proof. Note first that to accept the language \( L_d \), \( M \) works within space \( (1 + \lceil \log t \rceil)S(n) \) for all \( t \in \mathbb{N}_3 \), this yields

\[ L_d = \bigcup_{i \in \mathbb{N}_3} L_d^i. \]

It is clear that for any fixed \( i \in \mathbb{N}_3 \),

\[ L_d^i \subseteq L_d^{i+1}. \]

Since suppose \( w \in L_d^i \), i.e., \( M \) accepts \( w \) within space \( (1 + \lceil \log i \rceil)S(n) \), \( M \) of course can work within space \( (1 + \lceil \log (i+1) \rceil)S(n) \) to accept \( w \), because the \( (1 + \lceil \log (i+1) \rceil)S(n) \) space is larger than the \( (1 + \lceil \log i \rceil)S(n) \) space. Hence, the proof is completed.

Next, for any language \( L_d^i \) with \( i \in \mathbb{N}_3 \), we can show the following specified Tape Compression Theorem:

**Theorem 4.2** Let \( i \in \mathbb{N}_3 \) be a fixed positive integer, then

\[ L_d^i \in \text{NSPACE}[S(n)]. \]

Proof. Note that for any fixed \( i \in \mathbb{N}_3 \),

\[ L_d^i \in \text{NSPACE}[(1 + \lceil \log i \rceil)S(n)], \]

i.e., on input \( w \) of length \( n \), \( M \) scans at most \( T(n) \) cells, where

\[ T(n) = (1 + \lceil \log i \rceil)S(n). \]

It is convenient to make, without loss of generality, the assumption about \( M \) that the tape cells of \( M \) are enumerated by \( 1, 2, 3, \) etc., we construct a new nondeterministic Turing machine \( N \) that, on input \( w \) of length \( n \), simulates the computation of \( M(w) \) but works in space \( cS(n) \) with constant \( c > 0 \).

Suppose \( N \) packs \( r \) old symbols into a new symbol. Then \( N \) will have a much larger tape alphabet, i.e., the tape alphabet of \( N \) is \( \Phi \cup \Phi^r \) where \( \Phi \) is the tape alphabet of \( M \); it consists of the original symbols and the packed symbols. Each symbol in \( N \) is now considered to occupy one tape cell of \( N \). The new machine \( N \) will also have one more work tape, i.e., 5 tapes, and will have a much larger control states than \( M \).
We can view M’s tape as being subdivided into blocks of \( r \) adjacent cells each, i.e., for any input \( w = w_1w_2 \cdots w_n \) of \( M \), the blocks are the pairs of cells with numbers

\[
(w_{jr+1}, w_{jr+2}, \cdots, w_{jr+r})
\]

for \( j = 0, 1, 2, \ldots \). So,

\[
(w_{lr+1}, w_{lr+2}, \cdots, w_{lr+r})
\]

and

\[
(w_{(l+1)r+1}, w_{(l+1)r+2}, \cdots, w_{(l+1)r+r})
\]

are considered to be two adjacent blocks.\(^8\) For example, see following Fig. 1 if the input to the input tape of \( M \) is on the left, then after the encoding the tape of \( N \) is on the right for \( r = 3 \).

\[\begin{align*}
\text{[ _ a b b a b b a _ …]} &\quad \text{[ (_a,b) (b,a,b) (b,a,) …]} \\
\end{align*}\]

Figure 1: Adjacent blocks for \( r = 3 \)

Now, we let \( N \) do the following initializations:

- \( N \) copies the input of \( M \) onto the additional work-tape in compressed form: \( r \) successive symbols of \( M \) will be represented by one symbol in \( N \).

- \( N \) uses the additional work-tape as “input tape”. The head positions of \( N \), within the \( r \) symbols represented by current cells, is stored in finite control.

Then, \( N(w) \) simulates the computation of \( M(w) \), except that \( N \) moves its head to the left or to the right only when \( M \)’s head crosses a boundary between two blocks to the left or to the right. All steps of \( M \) within any one block can be simulated by \( N \)’s finite control. This is why \( N \) needs a much larger control states than \( M \).

Clearly, \( N \) use no more than

\[
[T(n)/r] = \left\lfloor \frac{(1 + \lceil \log i \rceil)S(n)}{r} \right\rfloor \leq \lceil c \times S(n) \rceil.
\]

By setting \( c = 1 \),

\[
\frac{1 + \lceil \log i \rceil}{r} \leq c = 1
\]

yields

\[
r \geq 1 + \lceil \log i \rceil
\]

Thus, packing \( r = 1 + \lceil \log i \rceil \) old symbols of \( M \) to a new symbol of \( N \) yields that \( N \) scans at most \( S(n) \) tape cells for each work-tape of \( N \) when simulating \( M \) on input \( w \) of length \( n \). This completes the proof.

\( \square \)

Now, we are ready to show the following important theorem:

\(^8\)The index \( lr + k \) stands for \( l \times r + k \) for \( l = 0, 1, 2, \ldots \) and \( 1 \leq k \leq r \).
**Theorem 4.3** The language $L_d$ accepted by $M$ is in $\text{NSPACE}[S(n)]$.

**Proof.** It is clear that the proof of Theorem 4.2 holds true for arbitrary positive integer $t \in \mathbb{N}_3$. That is to say, for all $t \in \mathbb{N}_3$,

$$L_d^t \in \text{NSPACE}[S(n)].$$

Thus, Theorem 4.1 together with Theorem 4.2 and the above observations, yields

$$L_d \in \text{NSPACE}[S(n)].$$

This finishes the proof.

**Remark 4.1** The above proof, which confirms our conjecture (i.e., the parameter $t$ is irrelevant to the input length $n$), shows that when considering only the growth rates of space functions on the input length of $n$, the parameter $t$, i.e., the number of tape symbols of a deterministic $S(n)$ space-bounded Turing machine, is irrelevant to the language $L_d$’s space complexity.

## 5 Proofs of Theorem 1.1 and Theorem 1.2

Our goal in this section is to establish Theorem 1.1, which states that for $S(n) \geq n$ a space-constructible function,

$$\text{DSPACE}[S(n)] \subsetneq \text{NSPACE}[S(n)],$$

and to establish Theorem 1.2, which gives a negative answer to the LBA question. Indeed, if the above Theorem 3.1 in Section 3 and Theorem 4.3 in Section 4 were not proved, we do not know how to show Theorem 1.1. But at this point with the above two theorems at hand, the proof of Theorem 1.1 can be made naturally as follows.

**Proof of Theorem 1.1** It immediately follows from Theorem 3.1 and Theorem 4.3. This finishes the proof.

Indeed, Theorem 1.2 is a special case of Theorem 1.1. To see this, we can set $S(n) = n$. Hence, setting $S(n) = n$ in the proof of Theorem 1.1, we can obtain the proof of Theorem 1.2.

**Proof of Theorem 1.2** It clearly follows from Theorem 1.1 by setting $S(n) = n$. Thus, the proof is completed.

## 6 Proof of $L \subsetneq NL$

In this section, the main objective is to prove the following theorem with respect to the question of $L$ vs. $NL$, which is a famous and important open question (see e.g. [AB09, Pap94, Sip13]), based on the padding argument like that in Lemma 12.2 in [HU79] (see p. 302, [HU79]). Note
that our main computing model in this section is the off-line deterministic/nondeterministic Turing machine, because it is with space bounds less than linear, i.e., to be $\log n$. Note again that

$$L = \text{DSPACE}[\log n],$$

and

$$NL = \text{NSPACE}[\log n].$$

**Theorem 1.3** (restatement) It holds that:

$$\text{DSPACE}[\log n] \subsetneq \text{NSPACE}[\log n].$$

That is, $L \subsetneq NL$.

**Proof.** We show Theorem 1.3 by contradiction. That is, we assume that

$$\text{DSPACE}[\log n] = \text{NSPACE}[\log n]$$

and obtain a contradiction. So, assuming the result

$$\text{DSPACE}[\log n] = \text{NSPACE}[\log n],$$

then we will show by padding argument that

$$\text{DSPACE}[n] = \text{NSPACE}[n],$$

which contradicts the Theorem 1.2.

Now, let $L_1 \in \text{NSPACE}[n]$, i.e., $L_1$ is accepted by a nondeterministic $n$ space-bounded Turing machine $M_1$. Pad the strings in $L_1$ as follows:

$$L_2 = \left\{ xS^{(2|x| - |x|)} \mid x \in L_1 \right\}$$

where $S$ is a new symbol not in the alphabet of $L_1$. Then the padded version $L_2$ is accepted by a nondeterministic Turing machine $M_2$ as follows: first, $M_2$ checks if the input $w$ is of the form $xS^{(2|x| - |x|)}$.

If not then $M_2$ rejects, which requires

$$\log\left(|x| + (2^{2|x|} - |x|)\right) = \log 2^{2|x|}
= |x|$$

space; otherwise, $M_2$ marks off $|x|$ cells, then $M_2$ simulates $M_1$ on $x$, accepting if and only if $M_1$ accepts without using more than $|x|$ cells. Note that, the length of input $w$ is

$$|w| = |x| + (2^{2|x|} - |x|)
= 2^{2|x|}.$$
Hence, $M_2$ works in space $|x| = \log |w|$, i.e.,

$$L_2 \in \text{NSPACE}[\log n].$$

By the hypothesis that

$$\text{NSPACE}[\log n] = \text{DSPACE}[\log n],$$

there is a deterministic $\log n$ space-bounded Turing machine $M_3$ accepting $L_2$. Finally, we construct a deterministic Turing machine $M_4$ accepting the original set $L_1$ within space $n$, which means that

$$\text{NSPACE}[n] = \text{DSPACE}[n].$$

For an input $x$ to $M_4$, $M_4$ marks off $|x|$ cells, which it may do since $n$ is space constructible. $M_4$ has not used more than $|x|$ cells.

Next $M_4$ on input $x$ simulates $M_3$ on $xS(2^{|x|} - |x|)$. To do this, $M$ must keep track of the head location of $M_3$ on $xS(2^{|x|} - |x|)$. If the head of $M_3$ is within $x$, $M_4$’s head is at the corresponding point on its input. Whenever the head of $M_3$ moves into the $S$’s, $M_4$ records the location in a counter. The length of the counter is at most

$$\log(2^{|x|} - |x|) \leq \log 2^{|x|}$$

which is less than $|x|$.

If during the simulation, $M_3$ accepts, then $M_4$ accepts. If $M_3$ does not accept, then $M_4$ increases the counter until the counter no longer fits on $|x|$ tape cells. Then $M_4$ halts. Now, if $x$ is in $L_1$, then $xS(2^{|x|} - |x|)$ is in $L_2$. Thus the counter requires $\log(2^{|x|} - |x|)$ space. It follows that the counter will fit. Thus $x$ is in $L(M_4)$ if and only if $xS(2^{|x|} - |x|)$ is in $L(M_3)$. Therefore $L(M_4) = L_1$, and $L_1$ is in $\text{DSPACE}[n]$, which yields

$$\text{NSPACE}[n] = \text{DSPACE}[n],$$

a contradiction to Theorem 1.2. This completes the proof.

7 Proof of Theorem 1.5

Our objective in this section is to establish Theorem 1.5 which states that there exists no deterministic $O(\log n)$ space-bounded Turing machine deciding the STCON.

Before doing the aforementioned work, let us first give some definitions on “reduction” and “log-space reduction,” which have not appeared in Section 2. As noted in [Mon81], there has been great interest in finding complete problems for various complexity classes defined by Turing machines. This work is important in two aspects. First, to find a complete language for such a class is to show that a single problem represents the complexity of the whole class. Hence, the complexity of the class is better understood. Secondly, to identify a “natural” problem as being complete for a class is to classify the complexity of this problem; see e.g. [Mon81].
Definition 7.1 (cf. [Mon81]) Let \( \Sigma \) and \( \Delta \) be two alphabets and let \( f : \Sigma^* \to \Delta^* \) be a function. \( f \) is log-space computable if there is a deterministic Turing machine with a read-only input tape, an output tape and some work-tapes, which when started with \( w \in \Sigma^* \) on its input tape will halt having written \( f(w) \in \Delta^* \) on its output tape and having scanned at most \( \log |w| \) tape cells on each of its work-tapes.

Definition 7.2 (cf. [Mon81]) Let \( F \) be a class of functions and let \( A \subseteq \Sigma^* \), \( B \subseteq \Delta^* \) be arbitrary sets. \( A \) is \( F \)-reducible to \( B \), denoted by \( A \leq_F B \), if there is a function \( f \in F \) with \( f : \Sigma^* \to \Delta^* \) such that
\[
\forall w \in \Sigma^*, \; w \in A \iff f(w) \in B.
\]
We use the terms “log-space reducible” (\( \leq_{log} \)) if \( F \) is the class of log-space computable functions.

Similarly, we have that a question \( Q \) is \( \leq_{log} \)-complete for a complexity class \( C \) if the question \( Q \) is in \( C \) and any other questions in \( C \) can be log-space reducible to \( Q \).

Lemma 7.1 (cf. Lemma 4.17, [AB09]) Let \( Q_i \), \( i = 1, 2, 3 \), be questions.

1. If \( Q_1 \leq_{log} Q_2 \) and \( Q_2 \leq_{log} Q_3 \), then \( Q_1 \leq_{log} Q_3 \).

2. If \( Q_1 \leq_{log} Q_2 \) and \( Q_2 \in L \), then \( Q_1 \in L \).

Proof of Theorem 7.3. Since STCON is \( \leq_{log} \)-complete for the complexity class \( NL \) (see e.g. [AB09, Pap94, Sip13]), from which we know that
\[
L = NL
\]
if and only if
\[
STCON \in L, \quad \text{(by Lemma 7.1)}
\]
i.e., if and only if there is a deterministic algorithm deciding STCON in space \( O(\log n) \).

Now, if STCON has deterministic algorithms with space complexity \( O(\log n) \), we have
\[
L = NL,
\]
a contradiction to Theorem 1.3. This completes the proof.

8 Conclusions

To summarize, we have shown that the class of languages accepted by nondeterministic linear-bounded automata (i.e., the context-sensitive languages) is not equal to the class of languages accepted by deterministic linear-bounded automata. Thus, we resolve the LBA question, which is a famous and longstanding open question in automata theory.
In fact, we have shown a more general result, i.e., for any space-constructible function $S(n) \geq n$,

$$\text{DSPACE}[S(n)] \subseteq \text{NSPACE}[S(n)].$$

We achieved this by enumerating all deterministic $S(n)$ space-bounded Turing machines, then diagonalizing against them by a universal nondeterministic Turing machine $M$. After that, we use novel and interesting methods to show that the language $L_d$ accepted by the simulating machine $M$ is in fact in $\text{NSPACE}[S(n)]$.

Next, we study the relationship between $\text{DSPACE}[\log n]$ and $\text{NSPACE}[\log n]$ and show by padding argument that

$$\text{DSPACE}[\log n] \nsubseteq \text{NSPACE}[\log n],$$

i.e.,

$$L \nsubseteq NL,$$

which also resolves a famous open question in complexity theory. It is clear that the result of

$$L \nsubseteq P$$

follows obviously from

$$L \nsubseteq NL.$$

As a special by-product of our result, the STCON has been resolved as well. That is, no deterministic $\log n$ space-bounded Turing machine can determine the $st$-connectivity of a directed graph, where $n$ is the number of vertices in that directed graph. Because the STCON is $NL$-complete under the log-space reduction.

Finally, we should say that there are many important questions we did not touch on in this paper; see e.g. [A1]. Moreover, we have only shown

$$\text{DSPACE}[S(n)] \nsubseteq \text{NSPACE}[S(n)]$$

for $S(n) \geq n$ and $S(n) = \log n$, and leaving open the case for

$$\log n < S(n) < n.$$

In addition, it was shown in [RCH91] that we could separate nondeterministic space from deterministic space by showing a nondeterministically fully space-constructible function below $\log n$. In particular, the work [Gef91] pointed out that if $\log \log n$ were fully space constructible by a nondeterministic Turing machine, then we would have

$$\text{DSPACE}[S(n)] \neq \text{NSPACE}[S(n)]$$

for $\log \log n \leq S(n) < \log n$.

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