BRST-BFV, DIRAC AND PROJECTION OPERATOR QUANTIZATIONS:
CORRESPONDENCE OF STATES

O. Yu. Shvedov

Sub-Dept. of Quantum Statistics and Field Theory,
Department of Physics, Moscow State University
Vorobievy gory, Moscow 119899, Russia

Abstract

The correspondence between BRST-BFV, Dirac and projection operator approaches to quantize
constrained systems is analyzed. It is shown that the component of the BFV wave function with
maximal number of ghosts and antighosts in the Schrödinger representation may be viewed as a wave
function in the projection operator approach. It is shown by using the relationship between different
quantization techniques that the Marnelius inner product for BRST-BFV systems should be in general
modified in order to take into account the topology of the group; the Giulini-Marolf group averaging
prescription for the inner product is obtained from the BRST-BFV method. The relationship between
observables in different approaches is also found.

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1shvedov@qs.phys.msu.su
1. There are different approaches to quantize the constrained systems. One can use the Dirac approach and impose constraints on the states or solve the classical constraints, reduce the phase space and quantize the obtained system \[1\]. An alternative way \[2, 3, 4\] is based on extension of the phase space with the help of ghosts and antighosts and imposing the BRST-BFV condition on the physical states. The most difficult step for both approaches is to introduce an inner product for physical states.

The purpose of this paper is to analyze the relationship between states, observables and inner products in different quantization methods.

2. Consider the constrained system which is specified by the Hamiltonian \( H \) and constraints \( \Lambda_a, a = 1, M \), which obey the Lie-algebra commutation relations:

\[
[\Lambda_a; \Lambda_b] = i f_{ab}^c \Lambda_c. \tag{1}
\]

In the Dirac approach, states are specified by the generalized wave functions \( \Psi \) obeying the usual Schrödinger equation \( i \dot{\Psi} = H \Psi \) and the additional conditions of the form

\[
\Lambda_a \Psi = 0 \tag{2}
\]

(for the case of the unimodular group). It is not easy to introduce an inner product in this approach \[4\]: one can impose additional gauge conditions and integrate \( |\Psi|^2 \) over the gauge-fixed surface.

Instead of imposing the conditions (2) on the wave function, one can take the constraints into account by modifying the inner product of the theory. States are specified by smooth and damping at the infinity wave functions \( \Phi \), but the inner product is modified: for the abelian case and constraints with continuous spectrum, the formula reads

\[
(\Phi, \prod_a 2\pi \delta(\Lambda_a) \Phi). \tag{3}
\]

Such inner products were used in \[3\]; they naturally arise in the BFV approach \[6\]. Similar formulas were introduced in the projection operator approach \[7\]. Since the inner product (3) is degenerate, one should factorize the state space: wave functions \( \Phi_1 \) and \( \Phi_2 \) are equivalent if \( (\Phi, \prod_a 2\pi \delta(\Lambda_a)(\Phi_1 - \Phi_2)) = 0 \) for all \( \Phi \). This means that the following quantum gauge transformation

\[
\Phi \to \Phi + \Lambda_a X^a \tag{4}
\]

is admitted. One can suggest the following correspondence between wave functions \( \Psi \) and \( \Phi \) \[5\]:

\[
\Psi = \prod_a 2\pi \delta(\Lambda_a) \Phi. \tag{5}
\]

Indeed, the wave function \( \Psi \) satisfies the Dirac conditions (2) and does not vary if \( \Phi \) is changed by a zero-norm state. Prescription (3) gives us an inner product for the Dirac approach then.

3. To develop the BRST-BFV approach \[2, 3, 4, 6\], it is necessary to introduce additional degrees of freedom: coordinates and momenta \( \lambda^a, \pi_a, a = 1, M \), ghosts and antighosts \( C^a, \overline{C}_a \), canonically conjugated momenta \( \Pi^a, \overline{\Pi}_a \), \( a = 1, M \). The nontrivial commutation relations are:

\[
[C^a, \overline{C}_b]_+ = [\overline{C}_a, \Pi_b]_+ = \delta^a_b; \quad [\lambda^a, \pi_b]_- = i \delta^a_b.
\]

The following nilpotent BRST-BFV charge \( \Omega \) is introduced:

\[
\Omega = C^a \Lambda_a - \frac{i}{2} f_{bc}^a \overline{\Pi}_a C^b C^c - \frac{i}{2} f_{ba}^c C^b - i \pi_a \Pi^a. \tag{6}
\]

Instead of requirement (2), the BRST-BFV condition is imposed on physical states \( \Upsilon \):

\[
\Omega \Upsilon = 0. \tag{7}
\]
while the gauge freedom is also allowed, the gauge transformation is:

\[ \Upsilon \rightarrow \Upsilon + \Omega X, \]

so that states \( \Upsilon \) and \( e^{i[\Omega, \rho]} \Upsilon \) are also equivalent.

There are several prescriptions to introduce an inner product for physical states (partial cases are considered in [8] in details). The most interesting general formula is the following [6]. One considers the representatives of the equivalence classes which obey the following additional conditions

\[ C^a \Upsilon = 0, \quad \pi_a \Upsilon = 0 \]

which make the state \( \Upsilon \) BRST-BFV-invariant. Unfortunately, the quantity \( (\Upsilon, \Upsilon) \) is ill-defined. However, the expression

\[ (\Upsilon, e^{i[\Omega, \rho]} \Upsilon) \]

which is formally equivalent to \( (\Upsilon, \Upsilon) \) occurs to be well-defined for a certain choice of the gauge fermion \( \rho \),

\[ \rho = -\lambda^a \Pi_a \]

Let us analyze the prescription (10) (cf. [6]). Consider the Schrödinger representation for the BFV wave function \( \Upsilon \), \( \Upsilon = \Upsilon(q, \lambda, \Pi, \Pi) \). The operators are rewritten then as

\[ C^a = \frac{\partial}{\partial \Pi_a}; \quad \pi^a = -i \frac{\partial}{\partial \lambda^a}; \quad p_i = -i \frac{\partial}{\partial q^i}, \]

the left derivatives are considered here. The inner product is indefinite [9]

\[ (\Upsilon_1, \Upsilon_2) = \int dq \prod_{a=1}^{M} d\mu^ad\Pi_a d\Pi^a (\Upsilon_1(q, i\mu, \Pi, \Pi))^* \Upsilon_2(q, -i\mu, \Pi, \Pi). \]

The integration and conjugation rules are \( (\Pi_{a_1} ... \Pi_{a_l} \Pi^{b_1} ... \Pi^{b_s})^* = (-1)^s \Pi^{b_s} ... \Pi^{b_1} \Pi_{a_1} ... \Pi_{a_l}, f d\Pi_a \Pi_a = 1, f d\Pi^a \Pi^a = 1 \). Condition (9) means that \( \Upsilon \) is \( \mu, \Pi, \Pi \)-independent, \( \Upsilon = \Phi(q) \), provided that the ghost number of \( \Upsilon \) is zero (this is a usual assumption of gauge theories [6]). Since

\[ [\Omega, \rho]_+ = -\lambda^a \Lambda_a + \frac{i}{2} \lambda^a f_{ab}^b - i\lambda^a \Pi_b f_{ac}^b \Pi_c - \Pi^a \Pi_a, \]

for the simplest abelian case one has

\[ (e^{i[\Omega, \rho]} \Upsilon)(q, \lambda, \Pi, \Pi) = e^{-t\lambda^a \Lambda_a} \Phi(q) e^{-i\Pi_a \Pi^a}, \]

so that

\[ (\Upsilon, e^{i[\Omega, \rho]} \Upsilon) = \int dq \prod_{a=1}^{M} d\mu^a d\Pi_a d\Pi^a \Phi^*(q) e^{it\mu^a \Lambda_a} e^{-i\Pi_a \Pi^a} \Phi(q). \]

Integration over ghost variables gives us \( t^M \), so that

\[ (\Upsilon, e^{i[\Omega, \rho]} \Upsilon) = \int dq \prod_{a=1}^{M} d\mu^a t^M \Phi^*(q) e^{it\mu^a \Lambda_a} \Phi(q). \]

Integrating over \( \mu \), we obtain expression (13).

One can suggest then that the correspondence between states in BFV and projection operator approaches is the following: one should take the BFV state to the gauge (9) and obtain the state \( \Phi(q) \) in the projection operator approach (3).
4. It is much more convenient to obtain the correspondence in a gauge-independent form. Consider the BFV transformation (8). Let

\[ X = X_{00}(q, \lambda) + X_{01}^a(q, \lambda)\Pi_a + X_{10,a}(q, \lambda)\Pi^a + \ldots \]

For \( \Pi = 0, \Pi = 0 \), one has \( \Omega X_{11,\Pi,\Pi} = (\Lambda_a - \frac{i}{2} f_{ab}^b) X_{01}^a(q, \lambda) \). The wave function \( \Upsilon = \Phi(q) + \Omega X \) has the following form on the surface \( \Pi, \Pi, \lambda = 0 \):

\[ \Upsilon(q, 0, 0, 0) = \Phi(q) + (\Lambda_a - \frac{i}{2} f_{ab}^b) X_{01}^a(q, 0). \]

It coincides with \( \Phi(q) \) up to a gauge transformation (4) for the abelian case. Thus, we find that projection-operator and BFV wave functions are related as follows:

\[ \Phi(q) = \Upsilon(q, 0, 0, 0), \quad (16) \]

while the BFV gauge transformation (8) corresponds to the gauge transformation (4).

If we considered the coordinate ghost representation of the BFV wave function, where \( C^a \) and \( \overline{C}_a \) are multiplicators, the coefficient of \( C^1 \ldots C^M \overline{C}_1 \ldots \overline{C}_M \) at \( \lambda = 0 \) would play a role of the function \( \Phi \).

5. Prescription (14) can be also justified in the [3, 8] approach. For the abelian case, one introduces the creation and annihilation operators \( A_+^a = \frac{1}{\sqrt{\pi}} \int d\pi a \sqrt{M_a} \Lambda_b \) for some Hermitian real positively definite nondegenerate matrix \( M \), shows that it is possible to perform such a gauge transformation (8) that after it

\[ A_+^a \Upsilon = 0. \quad (17) \]

It follows from the BFV condition that \( \frac{\partial}{\partial M_a} + M_b \Pi^b \Upsilon = 0 \) and \( \Upsilon(q, \lambda, \Pi, \Pi) = \exp[-\Pi_a M_b^a \Pi^b] \Upsilon_0(q, \lambda) \).

Condition (17) implies that \( \Upsilon_0(q, \lambda) = \exp[\lambda^a M_a^a \Lambda_b] \Phi(q) \), where \( \Phi(q) \) is of the form (16), so that

\[ \Upsilon(q, \lambda, \Pi, \Pi) = \exp[-\Pi_a M_b^a \Pi^b] \exp[\lambda^a M_a^b \Lambda_b] \Phi(q). \quad (18) \]

Making use of formula (12) for the inner product, we find

\[ (\Upsilon, \Upsilon) = \int dq \prod_{a=1}^{M_a} \prod_{a=1}^{M_b} d\mu^a \Phi^*(q) e^{-2i\mu^a M_a^b \Lambda_b} \Phi(q) \int \prod_{a=1}^{M_a} d\Pi_a d\Pi^a \exp[-2\Pi_a M_b^a \Pi^b]. \]

Integration over Grassmannian variables gives us the factor \( \text{det} 2M \) which is involved to the integration measure after substitution \( 2\mu^a M_a^b = \tilde{\mu}^b \). We obtain formula (8).

6. Starting from formulas (3) and (10), let us try to investigate their range of validity and modify them for the case of discrete spectrum of \( \Lambda_a \).

**Example 1.** Let \( M = 1, q = x, \Lambda = -i\partial/\partial x \). The formulas (3) and (13) take the form \( | \int_0^{+\infty} dx \Phi(x) |^2 \). This expression is well-defined.

**Example 2.** Let \( M = 1, q = \varphi, \Lambda = -i\partial/\partial \varphi, \varphi \in (0, 2\pi) \), the wave functions be periodic with respect to \( \varphi, \Phi(\varphi + 2\pi) = \Phi(\varphi) \). Formula (13) takes then the form:

\[ \int_0^{2\pi} d\mu t \Phi^*(\varphi + \mu t) = \int_0^{2\pi} d\varphi \Phi^*(\varphi) \int_{-\infty}^{+\infty} dy \Phi(y). \]

However, this integral is divergent. To avoid this difficulty, one should perform an integration over the period only, i.e. \( \mu \in (0, 2\pi/t) \). We obtain then the formula \( | \int_0^{2\pi} d\varphi \Phi(\varphi) |^2 \) for the inner product which is a basis of the projection operator quantization [4]. Thus, we see that the topology of the group should certainly be taken into account in defining the inner product (cf. [10]): formula (10) should be corrected in such a way that integration over \( \mu \) in eq. (12) should be performed over the finite interval.
7. Let us modify formula (3) for the nonabelian case. Let \( L_a, a = 1, M \) be generators of the Lie algebra with the following commutation relations \([L_a, L_b] = i f_{ab}^c L_c\). Consider the corresponding Lie group \( G \) and the exponential mapping \( \exp(\mu^a L_a) \). The operators \( \Lambda_a \) perform a representation of the Lie algebra, so that \( \exp(\mu^a L_a) \) will perform a representation of group \( T(\exp(\mu^a L_a)) = \exp(\mu^a \Lambda_a) \).

By \( Ad(L_a) \) we denote the adjoint representation of the Lie algebra, \( (Ad(L_a)\rho)^c = i f_{ab}^c \rho^b \), while \( Ad\{g\} \) is an adjoint representation of the group \( (Ad\{g\})^c = (exp(A))_b^c \rho^b \) with \( A_b^c = -\mu^a f_{ab}^c, g = \exp(i\mu^a L_a) \).

Find a nonabelian analog of formula (14). Let us look for it in the following form:

\[
(e^{[\Omega, \rho]}_+ \Upsilon)(q, \lambda, \Pi, \Xi) = e^{-i\lambda^a \Lambda_a \Phi(q)} e^{\Pi_a B^a_\lambda(B, t)} \Pi^b
\]

with

\[
\tilde{\Lambda}_a = \Lambda_a - \frac{i}{\sqrt{2}} f_{ab}^c.
\]

Making use of the relation \( \frac{d}{dt}(e^{[\Omega, \rho]}_+ \Upsilon) = [\Omega, \rho]_+ e^{[\Omega, \rho]}_+ \Upsilon \), we find the following equation for the matrix \( B, \dot{B}_a^b = -i \lambda^a f_{ac}^b B_c^d - 1 \), so that \( B(\lambda, t) = -\int_0^t d\tau Ad\{exp(-\tau \lambda^a L_a)\} \).

Therefore,

\[
(\Upsilon, e^{[\Omega, \rho]}_+ \Upsilon) = \int dq \prod_a d\mu^a d\Pi_a d\Pi^b \Phi^*(q) e^{i\mu^a(\Lambda_a - \frac{i}{\sqrt{2}} f_{ab})} \Phi(q) e^{-\Pi_a \int_0^t d\tau Ad\{exp(i\tau \mu^c L_c)\}} \Pi^b.
\]

Integration over fermionic variables gives us the group measure

\[
dg = \det \int_0^t d\tau (Ad\{exp(i\tau \mu^c L_c)\})^{\mu_a} = g = \exp(it\mu^c L_c)
\]

It happens that it coincides with the right-invariant Haar measure which has the form (see, for example, (11)) \( dRg = d\mu J(\mu) \) with \( J(\mu) = \det \frac{d\mu}{\delta\mu} \) for

\[
\exp(i(\mu^a + \delta \mu^a) L_a) = \exp(i\delta \mu^a L_a) \exp(i\mu^a L_a)
\]

Without loss of generality, consider the case \( t = 1 \). One finds

\[
\delta \mu^a L_a = \int_0^1 d\alpha e^{i\alpha \mu^a L_a} \delta \mu^a L_a e^{-i\alpha \mu^a L_a} = \int_0^1 d\alpha (Ad\{e^{i\alpha \mu^c L_c}\}) \delta \mu^a L_a,
\]

so that \( dg = dRg \). The multiplicator \( e^{i\frac{1}{2} f_{ab}^c \tau} \) can be presented as \( (det Ad\{g\})^{-1/2} \). The inner product (22) can be rewritten then as an integral over group

\[
\int dq dRg(det Ad\{g\}^{-1/2} \Phi^*(q) T(g) \Phi(q)
\]

with \( T(e^{i\mu^a L_a}) = e^{i\mu^a \Lambda_a} \). Note that \( dRg(det Ad\{g\})^{-1/2} = dLg(det Ad\{g\})^{1/2} \).

Analogously to the abelian case, one can propose that each point of the gauge group should be taken into account once. This means that integration over \( \mu \) in (22) should be performed in general only over some domain.

Contrary to (10), choice (9), (11) of additional conditions and gauge fermion leads to well-defined inner products (23) for the models of (11). Formula (22) coincide with the inner product derived in (12). For the compact groups, analogous formulas were used in (1) and in lattice gauge theories (13).

8. Formulas (4), (3) can be generalized to the nonabelian case (12). Two states \( \Phi_1 \) and \( \Phi_2 \) are gauge-equivalent if their difference satisfies the condition

\[
\int dRg(det Ad\{g\})^{-1/2} T(g)(\Phi_1 - \Phi_2) = 0.
\]
For example, the transformation $\Phi \rightarrow (\text{det} \text{Ad}\{h\})^{-1/2}T(h)\Phi$ is gauge. One can also consider infinitesimal gauge transformations

$$\Phi \rightarrow \Phi + \tilde{\Lambda}_a X^a. \quad (23)$$

The Dirac wave function can be defined as

$$\Psi = \int dR g(\text{det} \text{Ad}\{g\})^{-1/2}T(g)\Phi. \quad (24)$$

It obeys the condition

$$(\text{det} \text{Ad}\{h\})^{1/2}T(h)\Psi = \Psi$$

analogously to eq.(2) \[12\]. It can be also presented in the infinitesimal form

$$\tilde{\Lambda}_a^+ \Psi \equiv (\Lambda_a + \frac{i}{2} f^a_{bc}) \Psi = 0 \quad (25)$$

found in \[13\].

9. It follows from eqs.(20) and (22) that the Dirac wave function can be also presented via the integral over ghost momenta and Lagrange multipliers,

$$\Psi(q) = \prod_a d\mu_a d\Pi_a d\Pi_a (e^{[\Omega_a,\Psi]} + \Phi)(q, -i\mu, \Pi, \Pi). \quad (26)$$

Gauge equivalent states give us identical Dirac wave functions, since the B-charge can be written as a full derivative,

$$\Omega = (\Lambda_a + \frac{i}{2} f^a_{bc}) \frac{\partial}{\partial \Pi_a} - \frac{i}{2} f^a_{bc} \frac{\partial}{\partial \Pi_b} \frac{\partial}{\partial \Pi_c} \Pi_a - \frac{\partial}{\partial \lambda^a} \Pi^a,$$

so that the integral of $\Omega X$ over $\mu, \Pi, \Pi$ vanishes. Furthermore, it follows from the property

$$\int \prod_a d\mu_a d\Pi_a d\Pi_a (\Omega \Pi_a \Psi)(q, -i\mu, \Pi, \Pi)$$

and relation $\Omega \Psi = 0$ that eq.(23) is indeed satisfied for definition \[26\].

Thus, the formal relationship between Dirac and BFV states is obtained analogously to \[10\]. However, integration over $\mu$ should be performed carefully due to topological problems. Note that conjecture \[26\] is indeed valid for the state \[18\].

10. Let us investigate under what conditions the operator $H$ is an observable. In the BFV approach, an observable $H_B$ should commute with BRST-BFV charge \[3\]. Consider the expansion of $H_B$ in ghost momenta \[1\]:

$$H_B = H + \Pi_a H_{10}^a + \Pi^b H_{01,b} + \Pi_a \Pi^b H_{11}^a b + ...$$

We are to order the ghost operators as follows: the ghost momenta should be put to the left, the ghosts are put to the right. One has

$$\Omega H_B = C^c \tilde{\Lambda}_c H + \tilde{\Lambda}_a H_{10}^a + ..., \quad H_B \Omega = HC^c \tilde{\Lambda}_c + ...$$

where ... are terms with ghost momenta, $\tilde{\Lambda}_a$ is of the form \[19\]. Therefore, $H_{10}^a = iR_a^c C^c$, so that the term $H$ should obey the following property:

$$[H; \tilde{\Lambda}_a] = i \tilde{\Lambda}_c R_a^c. \quad (27)$$
The operator $H$ should be identified with the observable in the projection operator approach. Notice that it indeed takes equivalent states to equivalent, i.e. $H\Phi = \tilde{\Lambda}a\tilde{X}^a$, provided that $\Phi = \tilde{\Lambda}aX^a$: it is sufficient to choose $\tilde{X}^a = RX^a + iR_b^aX^b$.

An important feature of the physical observable is that the corresponding evolution operator $e^{-iHt}$ should be unitary with respect to the inner product (22). This means that

$$H^+ \int d_Rg(\det Ad\{g\})^{-1/2}T(g) = \int d_Rg(\det Ad\{g\})^{-1/2}T(g)H.$$  (28)

Formula (28) implies that state $H\Phi$ corresponds to the Dirac wave function

$$\int d_Rg(\det Ad\{g\})^{-1/2}T(g)H\Phi = H^+\Psi.$$  

Therefore, it is the operator $H^+$ that corresponds to the physical observable in the Dirac approach.

Let us illustrate condition (28) for the closed-algebra case, when $R^a_c = const$ and the $B$-extension of the observable $H$ is written explicitly [4]:

$$H_B = H + i\Pi_bR_b^cC^c.$$  (29)

Eq. (28) to be checked can be rewritten as

$$\int d_RgH^+e^{i\mu^a\tilde{\Lambda}_a} = \int d_Rge^{i\mu^a\tilde{\Lambda}_a}H.$$  (30)

One has

$$e^{i\mu^a\tilde{\Lambda}_a}He^{-i\mu^a\tilde{\Lambda}_a} = H + \int_0^1 d\alpha e^{i\alpha\mu^a}\tilde{\Lambda}_a [i\mu^a, H]e^{-i\mu^a}\tilde{\Lambda}_a = H + \int_0^1 d\alpha e^{i\alpha\mu^a}\tilde{\Lambda}_a \mu^a R^b_c\tilde{\Lambda}_b e^{-i\mu^a}\tilde{\Lambda}_a.$$  

Since the commutation relations between generators $\tilde{\Lambda}_a$ coincide with (1), $[\tilde{\Lambda}_a, \tilde{\Lambda}_b] = if^c_{ab}\tilde{\Lambda}_c$, it follows from eq.(21) that

$$e^{i\mu^a\tilde{\Lambda}_a}He^{-i\mu^a\tilde{\Lambda}_a} = H + \frac{1}{i} \frac{d}{d\tau}|_{\tau=0}e^{i(\mu^a + \tau\mu^bR_b^a)\tilde{\Lambda}_a}e^{-i\mu^a}\tilde{\Lambda}_a.$$  

Eq. (31) is taken then to the form $\int d_Rg(H^+ - H)e^{i\mu^a\tilde{\Lambda}_a} = \int d_Rg\frac{1}{i} \frac{d}{d\tau}|_{\tau=0}e^{i(\mu^a + \tau\mu^bR_b^a)\tilde{\Lambda}_a}$, so that

$$H = H^+ - iR_b^a.$$  (31)

Condition (31) is a relationship between observables $H$ and $H^+$ in the projection operator and Dirac approaches. We see that this is in agreement with the condition $H^+_B = H_B$.

11. Thus, the correspondence (16), (26) between states in different approaches to quantize the constrained systems is found. The inner products used in different methods are modified and generalized. The relationship (29), (31) between observables $H_B$, $H$ and $H^+$ in BFV, projection operator and Dirac approaches is also found. Note also that the obtained results can be generalized to the case of open algebras [15].

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