Maximal (120, 8)-arcs in projective planes of order 16 and related designs

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Abstract
The resolutions and maximal sets of compatible resolutions of all 2-(120, 8, 1) designs arising from maximal (120, 8)-arcs in the known projective planes of order 16 are computed. It is shown that each of these designs is embeddable in a unique way in a projective plane of order 16.

Keywords: resolvable design, maximal arc, projective plane.
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1 Introduction

A 2-(v, k, λ) design (or shortly, a 2-design) is a pair $D = \{X, B\}$ of a set $X$ of $v$ points and a collection of subsets of $X$ of size $k$ called blocks, such that every two points appear together in exactly $\lambda$ blocks [2], [7]. Every point of a 2-(v, k, λ) design is contained in $r = \lambda(v - 1)/(k - 1)$ blocks, and the total number of blocks is $b = v(v - 1)\lambda/k(k - 1)$.

The incidence matrix of a design $D$ is a $(0, 1)$-matrix $A = (a_{ij})$ with rows labeled by the blocks, columns labeled by the points, where $a_{i,j} = 1$ if the $i$th block contains the $j$th point, and $a_{i,j} = 0$ otherwise. If $p$ is a prime, the $p$-rank of a design $D$ is the rank of its incidence matrix of the finite field of order $p$.

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Two designs are *isomorphic* if there is a bijection between their point sets that maps every block of the first design to a block of the second design. An *automorphism* of a design is any isomorphism of the design to itself. The set of all automorphisms of $D$ form the automorphism group $Aut(D)$ of $D$.

The *dual* design $D^\perp$ of a design $D$ has as points the blocks of $D$, and as blocks the points of $D$. A 2-$(v, k, \lambda)$ design is *symmetric* if $b = v$, or equivalently, $r = k$. The dual design $D^\perp$ of a symmetric 2-$(v, k, \lambda)$ design $D$ is a symmetric design with the same parameters as $D$. A symmetric design $D$ is *self-dual* if $D$ and $D^\perp$ are isomorphic.

A design with $\lambda = 1$ is called a *Steiner* design. An affine plane of order $n (n \geq 2)$, is a Steiner 2-$(n^2, n, 1)$ design. A projective plane of order $n$ is a symmetric Steiner 2-$(n^2 + n + 1, n + 1, 1)$ design with $n \geq 2$. The classical (or Desarguesian) plane $PG(2, p^t)$ of order $n = p^t$, where $p$ is prime and $t \geq 1$, has as points the 1-dimensional subspaces of the 3-dimensional vector space $V_3$ over the finite field of order $p^t$, and as blocks (or lines), the 2-dimensional subspaces of $V_3$.

Let $D = \{X, B\}$ be a Steiner 2-$(v, k, 1)$ design with point set $X$, collection of blocks $B$, and let $v$ be a multiple of $k$, $v = nk$. Since every point of $X$ is contained in $r = (v - 1)/(k - 1) = (nk - 1)/(k - 1)$ blocks, it follows that $k - 1$ divides $n - 1$. Thus, $n - 1 = s(k - 1)$ for some integer $s \geq 1$, and

$$v = nk = (sk - s + 1)k.$$  

A *parallel class* $P$ is a set of $v/k = n$ pairwise disjoint blocks, and a *resolution* of $D$ is a partition of the collection of blocks $B$ into $r = (v - 1)/(k - 1) = sk + 1$ parallel classes. A design is *resolvable* if it admits a resolution.

Any 2-(((sk - s + 1)k, k, 1) design with $s = 1$ is an affine plane of order $k$, and admits exactly one resolution. If $s > 1$, a resolvable 2-(((sk - s + 1)k, k, 1) design may admit more than one resolution.

Following [17], we call two resolutions $R_1, R_2$, $R_1 = P_1^{(1)} \cup P_2^{(1)} \cup \cdots P_r^{(1)}, R_2 = P_1^{(2)} \cup P_2^{(2)} \cup \cdots P_r^{(2)}$ compatible if they share one parallel class, $P_i^{(1)} = P_j^{(2)}$, and

$$|P_i^{(1)} \cap P_j^{(1)}| \leq 1$$

for $i' \neq i$ and $j' \neq j$.

More generally, a set of $m$ resolutions $R_1, \ldots, R_m$ is compatible if every two of these resolutions are compatible.

Suppose that $\mathcal{P}$ is a projective plane of order $q = sk$. A *maximal* $\{(sk - s + 1)k, k\}$-*arc* [11], is a set $\mathcal{A}$ of $(sk - s + 1)k$ points of $\mathcal{P}$ such that every line
of \( \mathcal{P} \) is either disjoint from \( \mathcal{A} \) or meets \( \mathcal{A} \) in exactly \( k \) points. The collection of lines of \( \mathcal{P} \) which have no points in common with \( \mathcal{A} \) determines a maximal \( \{(sk - k + 1)s, s\} \)-arc \( \mathcal{A}^{\perp} \) in the dual plane \( \mathcal{P}^{\perp} \).

Maximal arcs with \( 1 < k < q \) do not exist in any Desarguesian plane of odd order \( q \) [11], and are known to exist in every Desarguesian plane of order \( q = 2^i \) [8], for any \( k = 2^i, 1 \leq i < t \), as well as in some non-Desarguesian planes of even order [10], [16].

If \( k > 1 \), the nonempty intersections of a maximal \( \{(sk - s + 1)k, k\} \)-arc \( \mathcal{A} \) with the lines of a projective plane \( \mathcal{P} \) of order \( q = sk \) are the blocks of a resolvable \( 2-((sk - s + 1)k, k, 1) \) design \( D \). Similarly, if \( s > 1 \), the corresponding \( \{(sk - k + 1)s, s\} \)-arc \( \mathcal{A}^{\perp} \) in the dual plane is the point set of a resolvable \( 2-((sk - k + 1)s, s, 1) \) design \( D^{\perp} \). We will refer to \( D \) (resp. \( D^{\perp} \)) as a design embeddable in \( \mathcal{P} \) (resp. \( \mathcal{P}^{\perp} \)) as a maximal arc. The points of \( D^{\perp} \) determine a set of \( (sk - k + 1)s \) mutually compatible resolutions of \( D \). Respectively, the points of \( D \) determine a set of \( (sk - s + 1)k \) mutually compatible resolutions of \( D^{\perp} \).

Two maximal arcs \( \mathcal{A}', \mathcal{A}'' \) in a projective plane \( \mathcal{P} \) are equivalent if there is an automorphism of \( \mathcal{P} \) that maps \( \mathcal{A}' \) to \( \mathcal{A}'' \). We note that the designs associated with equivalent arcs are necessarily isomorphic, while the converse is not true in general.

In [17], one of the authors of this paper proved the following upper bound on the number of pairwise compatible resolutions of a \( 2-((sk - s + 1)k, k, 1) \) design.

**Theorem 1.1** Let \( S = \{R_1, \ldots, R_m\} \) be a set of \( m \) mutually compatible resolutions of a \( 2-((sk - s + 1)k, k, 1) \) design \( D = \{X, \mathcal{B}\} \). Then

\[
m \leq (sk - k + 1)s.
\]

The equality

\[
m = (sk - k + 1)s
\]

holds if and only if there exists a projective plane \( \mathcal{P} \) of order \( sk \) such that \( D \) is embeddable in \( \mathcal{P} \) as a maximal \( \{(sk - s + 1)k, k\} \)-arc.

This paper summarizes the computation of all parallel classes, resolutions, and compatible sets of resolutions of maximum size of the \( 2-(120, 8, 1) \) associated with maximal \((120, 8) \)-arcs in projective planes of order 16. The main result can be formulated as follows.

**Theorem 1.2** Every \( 2-(120, 8, 1) \) design associated with a maximal \((120, 8) \)-arc in a known projective plane \( \mathcal{P} \) of order 16 admits exactly one compatible set of resolutions of maximal size, meeting the bound of Theorem [7, and consequently, is uniquely embeddable in \( \mathcal{P} \).
2 Maximal (120,8)-arcs and related 2-(120, 8, 1) designs

There are 22 nonisomorphic projective planes of order 16 that are known currently. Four planes, $PG(2, 16)$, SEMI2, SEMI4, and BBH1 (in the notation of [16]) are self-dual, and there are nine planes which are not self-dual: HALL, LMRH, JOWK, DSFP, DEMP, BBH2, JOHN, BBS4, and MATH [16]. Lists with the collections of lines of these planes are available at Eric Moorhouse’s web page [14].

In [16], Penttila, Royle, and Simpson enumerated and classified up to equivalence all hyperovals in the known planes of order 16. A hyperoval $A$ in a plane $P$ of order 16 is a maximal $(18,2)$-arc, and its dual arc $A^\perp$ is a maximal $(120,8)$-arc in the dual plane $P^\perp$. Since two maximal arcs $A'$, $A''$ in a plane $P$ are equivalent if and only if their dual arcs $(A')^\perp$, $(A'')^\perp$ are equivalent, the results from [16] imply the classification of all maximal $(120,8)$-arcs in the known projective planes of order 16, up to equivalence.

We used the data about the inequivalent hyperovals graciously provided to the authors by Gordon F. Royle, to compute the corresponding dual $(120,8)$-arcs and the related 2-(120, 8, 1) designs. The 93 inequivalent hyperovals give rise to 93 inequivalent $(120,8)$-arcs. For each 2-(120, 8, 1) design $D$ associated with an arc in the dual plane of the plane containing the corresponding hyperoval, we computed all parallel classes of $D$, all resolutions of $D$, and all compatible sets of maximal size 18. The parallel classes were found as 13-cliques in a graph $\Gamma$ having as vertices the blocks of $D$, where two blocks are adjacent in $\Gamma$ if they are disjoint. The resolutions were computed as 17-cliques in a graph $\Delta$ having as vertices the parallel classes of $D$, where two parallel classes are adjacent in $\Delta$ if they do not share any block. Finally, we computed the compatible sets of maximal size 18 as 18-cliques in a graph $\mathcal{E}$ having as vertices the resolutions of $D$, where adjacency is defined according to the definition of compatible resolutions given in the preceding section. For these computations, we wrote algorithms using Magma [3] and Cliquer [15].

The results of these computations are summarized in Table 2.4.

Note 2.1 The number of parallel classes ranges from 153 to 221, while the number of resolutions is 18 in all but one notable exception, namely, the 2-(120, 8, 1) design corresponding to the dual $(120,8)$-arc of the regular hyperoval in $PG(2, 16)$, with group of order 16,320. The number of resolutions of this particular design is 137 (cf. Table 2.4). However, among those 137 resolutions, there is only one set of 18 pairwise compatible resolutions, thus, this design, as well as all remaining designs, are embeddable in a unique way in a projective plane of order 16.
Note 2.2 The 2-ranks of the 2-(120, 8, 1) designs associated with maximal (120,8)-arcs, as well as their groups, were previously computed by Laurel Carpenter [6]. The 2-ranks of the 93 designs range from 65 to 94, and the minimum 65 is achieved only by the two designs in the Desarguesian plane, $PG(2,16)$, corresponding to the regular hyperoval with a group of order 16,320, and the Lunelli-Sce hyperoval [12] (also known as the Hall hyperoval [9]), with a group of order 144 (cf. Table 2.5). This supports a conjecture from [6], stating that the 2-rank of any design associated with a hyperoval in $PG(2,2^t)$ is $3^t - 2^t$, as well as the following stronger conjecture formulated in [17], which generalizes a conjecture by A. E. Brouwer [4].

Conjecture 2.3 If $D$ is a 2-$(2^{2t-1} - 2^{t-1}, 2^{t-1}, 1)$ design ($t \geq 2$), with an incidence matrix $A$, then

$$\text{rank}_2(A) \geq 3^t - 2^t,$$

and the equality

$$\text{rank}_2(A) = 3^t - 2^t$$

holds if and only if $D$ is embeddable as a maximal $(2^{2t-1} - 2^{t-1}, 2^{t-1})$-arc in $PG(2,2^t)$.

Conjecture 2.3 is trivially true for $t = 2$, and its validity for $t = 3$ follows from the results of [13].

Table 2.4

| Hyperoval | Plane     | $\text{Aut}(D)$ | 2-rank | # Par. Cl. | # Resolutions |
|-----------|-----------|-----------------|--------|------------|---------------|
| 1         | $PG(2,16)$| 144             | 65     | 153        | 18            |
| 2         | $PG(2,16)$| 16320           | 65     | 221        | 137           |
| 1         | SEMI$_2$  | 3               | 81     | 153        | 18            |
| 2         | SEMI$_2$  | 8               | 81     | 153        | 18            |
| 3         | SEMI$_2$  | 8               | 81     | 153        | 18            |
| 4         | SEMI$_2$  | 16              | 81     | 153        | 18            |
| 5         | SEMI$_2$  | 16              | 81     | 153        | 18            |
| 6         | SEMI$_2$  | 16              | 80     | 153        | 18            |
| 7         | SEMI$_2$  | 16              | 81     | 153        | 18            |
| 8         | SEMI$_2$  | 16              | 80     | 153        | 18            |
| 9         | SEMI$_2$  | 16              | 81     | 153        | 18            |
| 10        | SEMI$_2$  | 16              | 80     | 153        | 18            |
| 11        | SEMI$_2$  | 16              | 81     | 153        | 18            |
| Hyperoval | Plane | \(|\text{Aut}(D)|\) | 2-rank | # Par. Cl. | # Resolutions |
|-----------|-------|----------------|--------|-----------|---------------|
| 12        | SEMI2 | 16             | 81     | 153       | 18            |
| 13        | SEMI2 | 16             | 80     | 153       | 18            |
| 14        | SEMI2 | 16             | 81     | 153       | 18            |
| 15        | SEMI2 | 16             | 81     | 153       | 18            |
| 16        | SEMI2 | 32             | 81     | 157       | 18            |
| 17        | SEMI2 | 32             | 80     | 157       | 18            |
| 1         | SEMI4 | 16             | 81     | 153       | 18            |
| 2         | SEMI4 | 16             | 81     | 153       | 18            |
| 3         | SEMI4 | 16             | 81     | 153       | 18            |
| 1         | HALL  | 16             | 80     | 153       | 18            |
| 2         | HALL  | 64             | 81     | 157       | 18            |
| 3         | HALL  | 64             | 80     | 157       | 18            |
| 4         | HALL  | 320            | 80     | 173       | 18            |
| 1         | HALL.d| 2              | 81     | 153       | 18            |
| 2         | HALL.d| 2              | 81     | 153       | 18            |
| 3         | HALL.d| 6              | 81     | 153       | 18            |
| 1         | LMRH  | 16             | 83     | 153       | 18            |
| 2         | LMRH  | 16             | 86     | 153       | 18            |
| 3         | LMRH  | 16             | 86     | 153       | 18            |
| 4         | LMRH  | 16             | 86     | 153       | 18            |
| 5         | LMRH  | 64             | 86     | 157       | 18            |
| 6         | LMRH  | 112            | 82     | 153       | 18            |
| 1         | LMRH.d| 14             | 89     | 153       | 18            |
| 1         | JOWK  | 16             | 82     | 153       | 18            |
| 2         | JOWK  | 16             | 82     | 153       | 18            |
| 3         | JOWK  | 16             | 83     | 153       | 18            |
| 4         | JOWK  | 16             | 82     | 153       | 18            |
| 5         | JOWK  | 64             | 82     | 157       | 18            |
| 6         | JOWK  | 112            | 82     | 153       | 18            |
| 1         | JOWK.d| 14             | 83     | 153       | 18            |
| 1         | DSFP  | 16             | 86     | 153       | 18            |
| 2         | DSFP  | 16             | 86     | 153       | 18            |
| 3         | DSFP  | 16             | 86     | 153       | 18            |
| 4         | DSFP  | 16             | 86     | 153       | 18            |
| 5         | DSFP  | 16             | 86     | 153       | 18            |
| 6         | DSFP  | 16             | 86     | 153       | 18            |
| Hyperoval | Plane | |Aut(D)| 2-rank | # Par. Cl. | # Resolutions |
|-----------|-------|-----|------|--------|-------------|--------------|
| 7         | DSFP  |     | 16   | 86     | 153         | 18           |
| 8         | DSFP  |     | 16   | 86     | 153         | 18           |
| 9         | DSFP  |     | 16   | 86     | 153         | 18           |
| 10        | DSFP  |     | 16   | 86     | 153         | 18           |
| 11        | DSFP  |     | 16   | 86     | 153         | 18           |
| 12        | DSFP  |     | 16   | 86     | 153         | 18           |
| 13        | DSFP  |     | 16   | 86     | 153         | 18           |
| 14        | DSFP  |     | 16   | 86     | 153         | 18           |
| 15        | DSFP  |     | 16   | 86     | 153         | 18           |
| 16        | DSFP  |     | 16   | 85     | 153         | 18           |
| 17        | DSFP  |     | 16   | 86     | 153         | 18           |
| 18        | DSFP  |     | 16   | 86     | 153         | 18           |
| 19        | DSFP  |     | 16   | 86     | 153         | 18           |
| 20        | DSFP  |     | 16   | 86     | 153         | 18           |
| 21        | DSFP  |     | 16   | 86     | 153         | 18           |
| 22        | DSFP  |     | 64   | 86     | 157         | 18           |
| 1         | DEMP  |     | 16   | 85     | 153         | 18           |
| 2         | DEMP  |     | 16   | 84     | 153         | 18           |
| 3         | DEMP  |     | 16   | 85     | 153         | 18           |
| 4         | DEMP  |     | 16   | 85     | 153         | 18           |
| 5         | DEMP  |     | 16   | 83     | 153         | 18           |
| 6         | DEMP  |     | 16   | 85     | 153         | 18           |
| 7         | DEMP  |     | 16   | 84     | 153         | 18           |
| 8         | DEMP  |     | 16   | 85     | 153         | 18           |
| 9         | DEMP  |     | 16   | 84     | 153         | 18           |
| 10        | DEMP  |     | 16   | 85     | 153         | 18           |
| 11        | DEMP  |     | 16   | 84     | 153         | 18           |
| 12        | DEMP  |     | 64   | 83     | 157         | 18           |
| 13        | DEMP  |     | 64   | 83     | 157         | 18           |
| 1         | DEMP.d|     | 2    | 85     | 153         | 18           |
| 2         | DEMP.d|     | 6    | 85     | 153         | 18           |
| 1         | JOHN  |     | 16   | 94     | 157         | 18           |
| 1         | BBS4  |     | 16   | 94     | 157         | 18           |
| 1         | BBH2  |     | 4    | 94     | 153         | 18           |
| 2         | BBH2  |     | 4    | 94     | 153         | 18           |
| 1         | MATH  |     | 8    | 90     | 153         | 18           |
| Hyperoval | Plane   | $|\text{Aut}(D)|$ | 2-rank | # Par. Cl. | # Resolutions |
|-----------|---------|----------------|--------|-----------|---------------|
| 1         | MATH.d  | 4              | 91     | 153       | 18            |
| 2         | MATH.d  | 4              | 91     | 153       | 18            |
| 3         | MATH.d  | 8              | 92     | 153       | 18            |
| 1         | BBH1    | 8              | 90     | 153       | 18            |
| 2         | BBH1    | 16             | 92     | 153       | 18            |
| 3         | BBH1    | 32             | 90     | 157       | 18            |

Table 2.5 2-ranks of 2-(120, 8, 1) designs

| 2-rank | 65 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 89 | 90 | 91 | 92 | 94 |
|--------|----|----|----|----|----|----|----|----|----|----|----|----|----|
| Frequency | 2  | 8  | 19 | 6  | 7  | 4  | 10 | 25 | 1  | 3  | 2  | 2  | 4  |

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