Approximations of a complex Brownian motion
with processes constructed from a process with
independent increments

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Abstract

In this paper, we show an approximation in law of the complex Brownian motion by processes constructed from a stochastic process with independent increments. We give sufficient conditions for the characteristic function of the process with independent increments that ensure the existence of the approximation. We apply these results to Lévy processes. Finally we extend this results to the $m$-dimensional complex Brownian motion.

1 Introduction and main result.

The purpose of this paper is to investigate a weak approximation of a complex Brownian motion. The most typical processes taken as approximations to Gaussian processes are usually based on Donsker approximations (the functional central limit theorem) or on Kac-Stroock type approximations. In this paper, we will deal with this family of approximations.

In 1974, Kac [7] described the solution of the telegrapher's equation in terms of a Poisson process. Eight years later, Stroock [9] showed the weak convergence of this solution to a Brownian motion. More precisely, given $\{N_t, t \geq 0\}$ a

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MSC2010: 60F17, 60G15.
standard Poisson process, the laws of the processes \( x_\varepsilon \)

\[ \{ x_\varepsilon(t) = \varepsilon \int_0^t (-1)^{N_s} ds, \quad t \in [0, T] \} \]

converge weakly towards the law of a standard Brownian motion in the space of continuous functions on \([0, T]\).

This result have been extended in order to obtain approximations of other processes as, among others: \( m \)-dimensional Brownian process \([5]\), SPDE driven by Gaussian white noise \([2]\), fractional SDE \([4]\), multiple Wiener integrals \([3]\) or complex Brownian motion \([1]\).

Although all this cases are built beginning from a Poisson process, a detailed study of the proofs shows that the authors use only some properties of the Poisson process that can be found in a bigger class of processes as Lévy processes. In this paper we will show how to do this extension.

More precisely, we will deal with approximations of the complex Brownian motion built from an unique stochastic process with independent increments. Let us recall that \( \{ B_t, t \in [0, T] \} \) is a complex Brownian motion if it takes values on \( \mathbb{C} \) and its real part and its imaginary part are two independent standard Brownian motions.

We consider the processes

\[ \{ x_\varepsilon^{\theta}(t) = \varepsilon c(\theta) \int_0^t e^{i\theta X_s} ds, \quad t \in [0, T] \}, \tag{1} \]

where \( \{ X_t, t \geq 0 \} \) is a stochastic processes with independent increments and \( c(\theta) \) is a constant, depending on \( \theta \), that we will determine latter. Let us recall that our approximations can be written as

\[ x_\varepsilon^{\theta}(t) = \varepsilon c(\theta) \int_0^t \cos(\theta X_s) ds + i\varepsilon c(\theta) \int_0^t \sin(\theta X_s) ds. \]

Notice that when \( X \) is a Poisson process it has been proved in \([1]\) that for \( \theta \neq 0 \) and \( \theta \neq \pi \) the limit is a complex Brownian motion. When \( \theta = \pi \) we obtain an alternative version of Stroock’s results since in this case \( e^{i\theta X_s} = (-1)^{X_s} \).

The aim of this paper is to study the weak limits of the processes \( (1) \) when \( \varepsilon \) tends to zero depending on the value of \( \theta \), showing that Lévy processes can be used to approximate a complex Brownian motion.

In section 2, we recall some basic facts on Lévy processes and we present the classical methodology to get weak approximations of Gaussian processes.

Section 3 is devoted to give the main results of the paper. First we give some conditions on the characteristic functions of the process \( X \) that ensures the weak convergence of \( (1) \) to a complex Brownian motion. Then, we show when the characteristic functions of Lévy processes satisfy such conditions.
In section 4 we study the \( m \)-dimensional case, showing how we can obtain a \( m \)-dimensional complex Brownian motion from an unique \( \text{Lévy} \) process.

Along the paper \( K \) denote positive constants, not depending on \( \varepsilon \), which may change from one expression to another one. The real part and the imaginary part of a complex number will be denoted by \( \text{Re}[\cdot] \) and \( \text{Im}[\cdot] \).

### 2 Preliminaries

#### 2.1 Lévy processes

Set \( \{X_s, s \geq 0\} \) a \( \text{Lévy} \) process, that is, \( X \) has stationary and independent increments, is continuous in probability, is càdlàg and \( X_0 = 0 \), and it is defined on a complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). There are many important examples of \( \text{Lévy} \) processes: Brownian motion, Poisson Process, jump-diffusion processes, stable processes, subordinators, etc.

Consider \( \phi_{X_t}(u) \) its characteristic function. Remember that it can be written as

\[
\phi_{X_t}(u) = E(e^{iuX_t}) = e^{-t\psi_X(u)},
\]

where \( \psi_X(u) \) is called the \( \text{Lévy} \) exponent of \( X \).

It is well known that the \( \text{Lévy} \) exponent can be expressed, by the \( \text{Lévy-Khinchine} \) formula, as

\[
\psi_X(u) = -aiu + \frac{1}{2}\sigma^2 u^2 - \int_{\mathbb{R}\setminus\{0\}} (e^{iux} - 1 - iuxI_{|x|<1})\eta(dx),
\]

where \( a \in \mathbb{R}, \sigma \geq 0 \) and \( \eta \) is a Lévy measure, that is, \( \int_{\mathbb{R}\setminus\{0\}} \min\{x^2, 1\}\eta(dx) < \infty \).

For notation and simplicity along the paper we will write

\[
a(u) := \text{Re}[\psi_X(u)] = \frac{1}{2}\sigma^2 u^2 - \int_{\mathbb{R}\setminus\{0\}} (\cos(ux) - 1)\eta(dx),
\]

and

\[
b(u) = \text{Im}[\psi_X(u)] = -au - \int_{\mathbb{R}\setminus\{0\}} (\sin(ux) - uxI_{|x|<1})\eta(dx).
\]

Notice that \( a(-u) = a(u) \) and \( b(-u) = -b(u) \).

We refer the reader to \[8\] for more information about \( \text{Lévy} \) processes.

#### 2.2 Weak approximations of the complex Brownian motion

For any \( \varepsilon > 0 \), set \( \{x_\varepsilon(t), t \in [0, T]\} \) a complex stochastic process with \( x_\varepsilon(0) = 0 \). Consider \( P_\varepsilon \) the image law of \( x_\varepsilon \) in the Banach space \( C([0, T], \mathbb{C}) \) of continuous functions on \( [0, T] \).

In order to prove that \( P_\varepsilon \) converges weakly as \( \varepsilon \) tends to zero towards the law on \( C([0, T], \mathbb{C}) \) of a complex Brownian motion we have to check that the
family $P_ε$ is tight and that the law of all possible weak limits of $P_ε$ is the law of two independent standard Brownian motions.

In order to prove that the family $P_ε$ is tight, we need to prove that the laws corresponding to the real part and the imaginary part of the processes $x_ε$ are tight. Using the Billingsley criterium (see Theorem 12.3 of [6]) and that our processes are null on the origin, it suffices to prove that there exists a constant $K$ such that for any $s < t$

$$\sup_ε \{E((\text{Re}[x_ε(t)] - x_ε(t))^4) + E((\text{Im}[x_ε(t)] - x_ε(t))^4)\} \leq K(t-s)^2. \quad (5)$$

The second part of the proof consists in the identification of the limit law. Let $\{P_ε\}_n$ be a subsequence of $\{P_ε\}$ (that we will also denote by $\{P_ε\}$) weakly convergent to some probability $P$. We want to see that the canonical process $Y = \{Y_t(x) = y(t)\}$ is a complex Brownian motion under the probability $P$, that is, the real part and the imaginary part of this process are two independent Brownian motions. Using Paul Lévy’s theorem it suffices to prove that under $P$, the real part and the imaginary part of the canonical process are both martingales with respect to the natural filtration, $\{F_t\}$, with quadratic variations $<\text{Re}[Y], \text{Re}[Y]> = t$, $<\text{Im}[Y], \text{Im}[Y]> = t$ and covariance $<\text{Re}[Y], \text{Im}[Y]> = 0$.

To see that under $P$ the real part and the imaginary part of the canonical process $X$ are martingales with respect to its natural filtration, $\{F_t\}$, we have to prove that for any $s_1 \leq s_2 \leq \cdots \leq s_n \leq s$ and for any bounded continuous function $φ : \mathbb{C}^n \rightarrow \mathbb{R}$,

$$E_P [φ(X_{s_1}, \ldots, X_{s_n})(\text{Re}[Y_t] - \text{Re}[Y_s])] = 0,$$

$$E_P [φ(X_{s_1}, \ldots, X_{s_n})(\text{Im}[Y_t] - \text{Im}[Y_s])] = 0.$$

Since $P_ε \xrightarrow{w} P$, we have that

$$\lim_{ε \to 0} E_{P_ε} [φ(y(s_1), \ldots, y(s_n))(\text{Re}[y(t)] - \text{Re}[y(s)])] = E_P [φ(y(s_1), \ldots, y(s_n))(\text{Re}[y(t)] - \text{Re}[y(s)])],$$

and we get the same with the imaginary part. So, it suffices to see that

$$\lim_{ε \to 0} E(φ(x_ε(s_1), \ldots, x_ε(s_n))(\text{Re}[x_ε(t)] - \text{Re}[x_ε(s)])) = 0, \quad (6)$$

$$\lim_{ε \to 0} E(φ(x_ε(s_1), \ldots, x_ε(s_n))(\text{Im}[x_ε(t)] - \text{Im}[x_ε(s)])) = 0. \quad (7)$$

To check the quadratic variation, it is enough to prove that for any $s_1 \leq s_2 \leq \cdots \leq s_n \leq s$ and for any bounded continuous function $φ : \mathbb{C}^n \rightarrow \mathbb{R}$,

$$\lim_{ε \to 0} E[φ(x_ε(s_1), \ldots, x_ε(s_n))((\text{Re}[x_ε(t)] - \text{Re}[x_ε(s)])^2 - (t-s))] = 0, \quad (8)$$

$$\lim_{ε \to 0} E[φ(x_ε(s_1), \ldots, x_ε(s_n))((\text{Im}[x_ε(t)] - \text{Im}[x_ε(s)])^2 - (t-s))] = 0. \quad (9)$$
Finally to prove that \( < \text{Re}[Y], \text{Im}[Y] >_{\varepsilon} = 0 \), it suffices to check that for any \( s_1 \leq s_2 \leq \cdots \leq s_n \leq s \) and for any bounded continuous function \( \varphi : \mathbb{C}^n \rightarrow \mathbb{R} \),

\[
\lim_{\varepsilon \to 0} E[\varphi(x_\varepsilon(s_1), \ldots, x_\varepsilon(s_n))(\text{Re}[x_\varepsilon(t)] - \text{Re}[x_\varepsilon(s)])(\text{Im}[x_\varepsilon(t)] - \text{Im}[x_\varepsilon(s)])] = 0.
\]

(10)

3 Approximations to a complex Brownian motion

As we have explained, we built our approximations from a stochastic process \( X \) with independent increments. We will deal with \( X \) using the study of its characteristic function \( \phi_X \). Let us introduce a set of useful hypothesis \( (H^\theta) \) for the characteristic function \( \phi_X \) of a process \( X \):

- \((H^\theta_1)\) there exists a constant \( K(\theta) \) such that
  \[
  \varepsilon^2 \int_0^{2\varepsilon t} \int_0^y \|\phi_{X_y - X_x}(\theta)\| \, dx \, dy \leq K(\theta)(t - s),
  \]

- \((H^\theta_2)\) there exists a constant \( c(\theta) \) such that
  \[
  \lim_{\varepsilon \to 0} \varepsilon^2 c(\theta)^2 \int_0^{2\varepsilon t} \int_0^x \left[ \phi_{X_y - X_x}(\theta) + \phi_{X_y - X_x}(-\theta) \right] \, dy \, dx = 2(t - s),
  \]

- \((H^\theta_3)\)
  \[
  \lim_{\varepsilon \to 0} \varepsilon^2 \int_0^{2\varepsilon t} \int_0^y \|\phi_{X_y - X_x}(\theta)\| \|\phi_{X_y - X_x}(2\theta)\| \, dx \, dy = 0.
  \]

In the next Theorem we give some sufficient conditions on the characteristic function of the process \( \{X_s, s \geq 0\} \) to get the convergence to a complex Brownian motion.

**Theorem 3.1** Let \( \{X_s, s \geq 0\} \) be a stochastic process with independent increments and characteristic function \( \phi_X \). Set \( C_X = \{\theta, \text{ such that } \phi_X \text{ satisfies } (H^\theta)\} \).

Define for any \( \varepsilon > 0 \) and \( \theta \in C_X \)

\[
\{x_\varepsilon^\theta(t) = \varepsilon c(\theta) \int_0^{2\varepsilon t} e^{i\theta X_s} ds, \quad t \in [0, T]\}
\]

where \( c(\theta) \) is the constant given by hypothesis \((H^\theta_2)\).

Consider \( P_\varepsilon^\theta \) the image law of \( x_\varepsilon^\theta \) in the Banach space \( \mathcal{C}([0, T], \mathbb{C}) \) of continuous functions on \([0, T]\). Then, \( P_\varepsilon^\theta \) converges weakly as \( \varepsilon \) tends to zero, towards the law on \( \mathcal{C}([0, T], \mathbb{C}) \) of a complex Brownian motion.
Proof: We will follow the method explained in Subsection 2.2.

Step 1: Tightness. We have to check (5), that is, that there exists a constant $K(\theta)$ such that for any $s < t$

$$\sup_{\varepsilon} \left( E(\varepsilon c(\theta) \int_{\frac{s}{\varepsilon}}^{\frac{t}{\varepsilon}} \cos(\theta N_x)dx)^4 + E(\varepsilon c(\theta) \int_{\frac{s}{\varepsilon}}^{\frac{t}{\varepsilon}} \sin(\theta N_x)dx)^4 \right) \leq K(\theta)(t-s)^2.$$  

From the properties of the complex numbers we have that

$$E(\varepsilon c(\theta) \int_{\frac{s}{\varepsilon}}^{\frac{t}{\varepsilon}} \cos(\theta X_x)dx)^4 + E(\varepsilon c(\theta) \int_{\frac{s}{\varepsilon}}^{\frac{t}{\varepsilon}} \sin(\theta X_x)dx)^4$$

$$\leq 2E\|x_\varepsilon^\theta(t) - x_\varepsilon^\theta(s)\|^4$$

$$= 2c(\theta)^4\varepsilon^4 \int_{\frac{s}{\varepsilon}}^{\frac{t}{\varepsilon}} e^{i\theta X_u} du \int_{\frac{s}{\varepsilon}}^{\frac{t}{\varepsilon}} e^{-i\theta X_u} du \right)^2$$

$$= 2c(\theta)^4\varepsilon^4 \int_{\frac{s}{\varepsilon}}^{\frac{t}{\varepsilon}} E \left( e^{i\theta(X_{x_1} - X_{x_2}) + (X_{x_2} - X_{x_1})} \right) dv_1 dv_2 du_1 du_2. \quad (11)$$

Using that for $x_1 < x_2 < x_3 < x_4$ and $\rho_i \in \{0, 1\}$ for $i = 1, 2, 3, 4$ with $\sum_{i=1}^4 \rho_i = 2$ we can write

$$(-1)^{\rho_1} X_{x_1} + (-1)^{\rho_2} X_{x_3} + (-1)^{\rho_3} X_{x_2} + (-1)^{\rho_4} X_{x_4}$$

$$= (-1)^{\rho_1} (X_{x_1} - X_{x_3}) + ((-1)^{\rho_2} + (-1)^{\rho_3}) (X_{x_3} - X_{x_2})$$

$$+ ((-1)^{\rho_3} + (-1)^{\rho_4}) (X_{x_2} - X_{x_1}),$$

and the last expression (11) can be written as the sum of 24 integrals of the type

$$2c(\theta)^4\varepsilon^4 \int_{\frac{s}{\varepsilon}}^{\frac{t}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{t}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{t}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{t}{\varepsilon}} E \left( e^{i\theta[c_1(X_{x_4} - X_{x_3}) + c_2(X_{x_3} - X_{x_2}) + c_3(X_{x_2} - X_{x_1})]} \right)$$

$$\times dx_1 dx_2 dx_3 dx_4. \quad (12)$$

where $c_1 \in \{1, -1\}$, $c_2 \in \{-2, 0, 2\}$ and $c_3 \in \{1, -1\}$. Notice that since the process $X$ has independent increments, we have that

$$E \left( e^{i\theta c_1(X_{x_4} - X_{x_3}) + c_2(X_{x_3} - X_{x_2}) + c_3(X_{x_2} - X_{x_1})} \right)$$

$$= E \left( e^{i\theta c_1(X_{x_4} - X_{x_3})} \right) E \left( e^{i\theta c_2(X_{x_3} - X_{x_2})} \right) E \left( e^{i\theta c_3(X_{x_2} - X_{x_1})} \right),$$

and we obtain,

$$\|E \left( e^{i\theta c_1(X_{x_4} - X_{x_3}) + c_2(X_{x_3} - X_{x_2}) + c_3(X_{x_2} - X_{x_1})} \right)\|$$

$$\leq \|\phi_{X_{x_4} - X_{x_3}}(c_1\theta)\| \|\phi_{X_{x_2} - X_{x_3}}(c_2\theta)\|$$

$$\leq \|\phi_{X_{x_4} - X_{x_3}}(\theta)\| \|\phi_{X_{x_2} - X_{x_3}}(\theta)\|.$$
where we have used that for any random variable $Z$, $\|\phi_Z(-u)\| = \|\phi_Z(u)\|$. So, each one of the 24 integrals of the type (12) is bounded by
\[
c(\theta)\varepsilon^2 \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \|\phi_{X_{x_3}}(\theta)\| dx_3 dx_4 \varepsilon^2 \int_0^{2\pi} \int_0^{2\pi} \|\phi_{X_{x_2}}(\theta)\| dx_1 dx_2.
\]
Clearly, hypothesis $(H^\theta 1)$ completes the proof of this step.

**Step 2: Martingale property.** It is enough to check (6) and (7). So, it suffices to see that
\[
E(\phi(x_\theta(s_1), ..., x_\theta(s_n))\varepsilon c(\theta) \int_0^{2\pi} \cos(\theta X_s) dx
\]
and,
\[
E(\phi(x_\theta(s_1), ..., x_\theta(s_n))\varepsilon c(\theta) \int_0^{2\pi} \sin(\theta X_s) dx
\]
converge to zero when $\varepsilon$ tends to zero.

Thus, it is enough to prove that
\[
\|E(\phi(x_\theta(s_1), ..., x_\theta(s_n))\varepsilon c(\theta) \int_0^{2\pi} e^{i\theta X_s} dx)\|
\]
converges to zero when $\varepsilon$ tends to zero.

But applying the Schwartz inequality and using that the function $\phi$ is bounded it is enough to prove the convergence to zero of
\[
\|E(\varepsilon c(\theta) \int_0^{2\pi} e^{i\theta X_s} dx)^2\|
\]
\[
= \|E(\varepsilon^2 c(\theta)^2 \int_0^{2\pi} \int_0^{2\pi} e^{i\theta(X_s+X_y)} dxdy)\|
\]
\[
= \|E(\varepsilon^2 c(\theta)^2 \int_0^{2\pi} \int_0^{2\pi} e^{i\theta(X_y-X_s)} e^{i2\theta(X_s-X_y)} e^{i2\theta X_s} \frac{1}{\pi^2} dxdy)\|
\]
\[
= \|E(\varepsilon^2 \frac{1}{\pi^2} e^{i2\theta X_s} \cdot \phi_{X_{x_3}}(\theta) \cdot \phi_{X_{x_3}}(\theta) dxdy)\|
\]
Notice that this last expression can be bounded by
\[
\varepsilon^2 c(\theta)^2 \int_0^{2\pi} \int_0^{2\pi} \|\phi_{X_{x_3}}(\theta)\| \|\phi_{X_{x_3}}(\theta)\| dxdy
\]
that from $(H^\theta 3)$ converges to zero when $\varepsilon$ goes to zero.
Step 3: Quadratic variations. It is enough to check (8) and (9), that is to prove that for any $s_1 \leq s_2 \leq \cdots \leq s_n \leq s$ and for any bounded continuous function $\varphi : \mathbb{C}^n \rightarrow \mathbb{R}$,

$$a_\varepsilon := E[\varphi(x^0_\varepsilon(s_1), \ldots, x^0_\varepsilon(s_n)) \{(Re[x^0_\varepsilon(t)] - Re[x^0_\varepsilon(s)])^2 - (t - s)\}]$$

and

$$b_\varepsilon := E[\varphi(x^0_\varepsilon(s_1), \ldots, x^0_\varepsilon(s_n)) \{(Im[x^0_\varepsilon(t)] - Im[x^0_\varepsilon(s)])^2 - (t - s)\}]$$

converge to zero when $\varepsilon$ tends to zero.

In order to prove that $a_\varepsilon$ and $b_\varepsilon$ converge to zero, when $\varepsilon$ goes to zero, it is enough to show that $a_\varepsilon + b_\varepsilon$ and $a_\varepsilon - b_\varepsilon$ converge to zero. But,

$$a_\varepsilon + b_\varepsilon$$

$$= E[\varphi(x^0_\varepsilon(s_1), \ldots, x^0_\varepsilon(s_n)) \{||x^0_\varepsilon(t) - x^0_\varepsilon(s)||^2 - 2(t - s)\}]$$

$$= E[\varphi(x^0_\varepsilon(s_1), \ldots, x^0_\varepsilon(s_n)) \{\varepsilon^2 c(\theta)^2 \int_\varepsilon^{2\varepsilon} \int_\varepsilon^{2\varepsilon} e^{i\theta(X_s - X_u)} dy du - 2(t - s)\}]$$

$$= E[\varphi(x^0_\varepsilon(s_1), \ldots, x^0_\varepsilon(s_n)) E[\varepsilon^2 c(\theta)^2 \int_\varepsilon^{2\varepsilon} \int_\varepsilon^{2\varepsilon} e^{i\theta(X_s - X_u)} dy du - 2(t - s)\}]$$

$$= E[\varphi(x^0_\varepsilon(s_1), \ldots, x^0_\varepsilon(s_n)) E[\varepsilon^2 c(\theta)^2 \int_\varepsilon^{2\varepsilon} \int_\varepsilon^{2\varepsilon} e^{-i\theta(X_s - X_u)} dy du - 2(t - s)\}]$$

$$+ E[\varepsilon^2 c(\theta)^2 \int_\varepsilon^{2\varepsilon} \int_\varepsilon^{2\varepsilon} e^{-i\theta(X_s - X_u)} dy du - 2(t - s)\}]$$

$$= E[\varphi(x^0_\varepsilon(s_1), \ldots, x^0_\varepsilon(s_n)) \times \varepsilon^2 c(\theta)^2 \int_\varepsilon^{2\varepsilon} \int_\varepsilon^{2\varepsilon} [\phi_{X_s - X_u}(\theta) + \phi_{X_s - X_u}(-\theta)] dy du - 2(t - s)\}]$$

Clearly, $(H^0)$ yields that $\lim_{\varepsilon \rightarrow 0} (a_\varepsilon + b_\varepsilon) = 0$.

It remains to see that $a_\varepsilon - b_\varepsilon$ converges to zero. But

$$a_\varepsilon - b_\varepsilon = E[\varphi(x^0_\varepsilon(s_1), \ldots, x^0_\varepsilon(s_n)) \{(\varepsilon c(\theta) \int_\varepsilon^{2\varepsilon} \cos(\theta X_s) dx)^2$$

$$- (\varepsilon c(\theta) \int_\varepsilon^{2\varepsilon} \sin(\theta X_s) dx)^2\}]$$

$$= \frac{1}{2} E[\varphi(x^0_\varepsilon(s_1), \ldots, x^0_\varepsilon(s_n)) \{(\varepsilon c(\theta) \int_\varepsilon^{2\varepsilon} e^{i\theta X_s} dx)^2$$

$$+(\varepsilon c(\theta) \int_\varepsilon^{2\varepsilon} e^{-i\theta X_s} dx)^2\}]$$.
where in the last step we have used that $2(\alpha^2 - \beta^2) = (\alpha + \beta i)^2 + (\alpha - \beta i)^2$. We will show that this two last terms go to zero. For the first one we have that,

$$\frac{1}{2} E[\varphi(x^0_\varepsilon(s_1), ..., x^0_\varepsilon(s_n))(\varepsilon c(\theta) \int_0^{2\pi} e^{i\theta X_x} dx)^2]$$

$$= E[\varphi(x^0_\varepsilon(s_1), ..., x^0_\varepsilon(s_n))\varepsilon^2 c(\theta)^2 \int_0^{2\pi} \int_0^{2\pi} e^{2i\theta(x_x - x_y + 2i\theta(x_x - x_{2\pi}) + 2\theta x_{2\pi})} dx dy]$$

$$= E[\varphi(x^0_\varepsilon(s_1), ..., x^0_\varepsilon(s_n))e^{2i\theta X_{2\pi}}]$$

$$\times \varepsilon^2 c(\theta)^2 \int_0^{2\pi} \int_0^{2\pi} \phi_{X_{2\pi}}(\theta) \phi_{X_{2\pi}}(2\theta) dx dy.$$  

Notice that this last expression can be bounded by

$$K \varepsilon^2 c(\theta)^2 \int_0^{2\pi} \int_0^{2\pi} \|\phi_{X_{2\pi}}(\theta)\| \|\phi_{X_{2\pi}}(2\theta)\| dx dy$$

that from (H\theta3) converges to zero when $\varepsilon$ goes to zero. Following the same computations, and using that, in general, for any random variable $Z$, $\|\phi_Z(-u)\| = \|\phi_Z(u)\|$ we obtain the same bound and the convergence to zero, for the second term of expression (13).

**Step 4: Quadratic covariation.** It is enough to check (10). Using that

$$\alpha \beta = \frac{1}{4} i[(\alpha - \beta i)^2 - (\alpha + \beta i)^2],$$

the term in the right side of (10) is equal to

$$E(\varphi(x^0_\varepsilon(s_1), ..., x^0_\varepsilon(s_n))(\varepsilon c(\theta) \int_0^{2\pi} \cos(\theta X_x) dx)(\varepsilon c(\theta) \int_0^{2\pi} \sin(\theta X_x) dx))$$

$$= \frac{1}{4} i E[\varphi(x^0_\varepsilon(s_1), ..., x^0_\varepsilon(s_n))[(\varepsilon c(\theta) \int_0^{2\pi} e^{-i\theta X_x} dx)^2$$

$$- (\varepsilon c(\theta) \int_0^{2\pi} e^{i\theta X_x} dx)^2].$$

We have already shown in the study of (13) that this term goes to zero.

Let us state now the main result of the paper. We prove that the approximations built from a Lévy process converge to a complex Brownian motion.
Theorem 3.2 Define for any \( \varepsilon > 0 \)
\[
\{ x^\theta_\varepsilon(t) = \varepsilon c(\theta) \int_0^{t \varepsilon} e^{i t X_s}ds, \quad t \in [0,T] \}
\]
where \( \{ X_s, s \geq 0 \} \) is a Lévy process with Lévy exponent \( \psi_X \) and
\[
c(u) = \sqrt{\frac{\| \psi_X(u) \|^2}{2 \text{Re}[\psi_X(u)]}}.
\]
Consider \( P^\theta_\varepsilon \) the image law of \( x^\theta_\varepsilon \) in the Banach space \( C([0,T], \mathbb{C}) \) of continuous functions on \( [0,T] \). Then, for \( \theta \) such that \( \text{Re}[\psi_X(\theta)] \neq 0 \), \( P^\theta_\varepsilon \) converges weakly as \( \varepsilon \) tends to zero, towards the law on \( C([0,T], \mathbb{C}) \) of a complex Brownian motion.

Proof: The results follows as a particular case of Theorem 3.1. It suffices to check that the characteristic function \( \phi_X \) of the Lévy process \( X \) satisfies \((H^\theta)\) for any \( \theta \) such that \( a(\theta) a(2\theta) \neq 0 \) (recall definitions (3) and (4)).

Proof of \((H^\theta 1)\): We can write
\[
\varepsilon^2 \int_{\frac{2t}{\varepsilon}}^{\frac{2t}{\varepsilon}} \int_{\frac{2t}{\varepsilon}}^{\frac{2t}{\varepsilon}} \| \phi_{X_s - X_y} (\theta) \| dx dy = \varepsilon^2 \int_{\frac{2t}{\varepsilon}}^{\frac{2t}{\varepsilon}} \int_{\frac{2t}{\varepsilon}}^{\frac{2t}{\varepsilon}} e^{-(y-x)a(\theta)} dx dy \leq \frac{2}{a(\theta)} (t-s).
\]
Using that \( a(\theta) > 0 \) we complete the proof of \((H^\theta 1)\).

Proof of \((H^\theta 2)\): Note first that
\[
\varepsilon^2 c(\theta)^2 \int_{\frac{2t}{\varepsilon}}^{\frac{2t}{\varepsilon}} \int_{\frac{2t}{\varepsilon}}^{\frac{2t}{\varepsilon}} \phi_{X_s - X_y} (\theta) dy dx
\]
\[
= \varepsilon^2 c(\theta)^2 \int_{\frac{2t}{\varepsilon}}^{\frac{2t}{\varepsilon}} \int_{\frac{2t}{\varepsilon}}^{\frac{2t}{\varepsilon}} e^{-(x-y)(a(\theta) + b(\theta)i)} dy dx
\]
\[
= \varepsilon^2 \frac{c(\theta)^2}{a(\theta) + b(\theta)i} \int_{\frac{2t}{\varepsilon}}^{\frac{2t}{\varepsilon}} (1 - e^{-(x - \frac{2t}{\varepsilon})(a(\theta) + b(\theta)i)}) dx
\]
\[
= o(\varepsilon) + 2(t-s) \frac{c(\theta)^2}{a(\theta) + b(\theta)i}.
\]
Following the same computations and taking into account that \( a(-\theta) = a(\theta) \), and that \( b(-\theta) = -b(\theta) \) we obtain that
\[
\varepsilon^2 c(\theta)^2 \int_{\frac{2t}{\varepsilon}}^{\frac{2t}{\varepsilon}} \int_{\frac{2t}{\varepsilon}}^{\frac{2t}{\varepsilon}} \phi_{X_s - X_y} (-\theta) dy dx
\]
\[
= o(\varepsilon) + 2(t-s) \frac{c(\theta)^2}{a(\theta) - b(\theta)i}.
\]
So
\[ \varepsilon^2 c(\theta)^2 \int_{\mathbb{R}} \int_{\mathbb{R}} [\phi_{X_t - X_s}(\theta) + \phi_{X_t - X_s}(-\theta)]dydx \]
\[ = o(\varepsilon) + 2(t - s) \left( \frac{c(\theta)^2}{a(\theta) + b(\theta)i} + \frac{c(\theta)^2}{a(\theta) - b(\theta)i} \right) \]
\[ = o(\varepsilon) + 2(t - s), \]
and \((H^\theta 2)\) is clearly true.

Proof of \((H^\theta 3)\): Notice that
\[ K\varepsilon^2 \int_{\mathbb{R}} \int_{\mathbb{R}} \|\phi_{X_t - X_s}(\theta)\| \|\phi_{X_t - X_s}(-\theta)\| dxdy \]
\[ = K\varepsilon^2 \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(y-x)a(\theta)} e^{-(x-\frac{y}{2}\theta)a(2\theta)} dxdy \]
\[ \leq K\varepsilon^2 \frac{1}{a(\theta)} \int_{\mathbb{R}} e^{-(x - \frac{y}{2}\theta)a(2\theta)} dx \]
\[ \leq K\varepsilon^2 \frac{1}{a(\theta)a(2\theta)}, \]
that converges to zero when \(\varepsilon\) goes to zero.

\[ \square \]

Remark 3.3 Given a Lévy process with characteristic function given by the Lévy-Khinchine formula \([3]\), the condition \(\text{Re}[\psi_X(\theta)] = 0\) is equivalent to
\[ \frac{1}{2} \sigma^2 \theta^2 - \int_{\mathbb{R}\backslash\{0\}} (\cos(\theta x) - 1)\eta(dx) = 0, \]
that is, \(\sigma = 0\) and
\[ \int_{\mathbb{R}\backslash\{0\}} (\cos(\theta x) - 1)\eta(dx) = 0. \]
So, the condition \(\text{Re}[\psi_X(\theta)]\text{Re}[\psi_X(2\theta)] \neq 0\) can be written as \(\sigma \neq 0\) or
\[ \int_{\mathbb{R}\backslash\{0\}} (\cos(\theta x) - 1)\eta(dx)(\int_{\mathbb{R}\backslash\{0\}} (\cos(2\theta x) - 1)\eta(dx)) \neq 0. \]

Remark 3.4 When we consider \(\{X_t, t \geq 0\}\) a standard Poisson process it is well-known that it is a Lévy process with Lévy exponent
\[ \psi_X(u) = -(\cos(u) - 1) - i\sin(u) \]
that corresponds to the Lévy-Khinchine formula \([3]\) with \(a = 0, \sigma = 0\) and \(\eta = \delta_{\{1\}}\). Then the condition \(\text{Re}[\psi_X(\theta)]\text{Re}[\psi_X(2\theta)] \neq 0\) yields that \(\theta \neq k\pi\) for any \(k \geq 1\).
When \( \theta = (2k+1)\pi \), we have that
\[
x_\varepsilon^\theta(t) = c((2k+1)\pi)\varepsilon \int_0^{2\varepsilon} \cos((2k+1)\pi X_s) ds = \varepsilon \int_0^{2\varepsilon} (-1)^X_s ds,
\]
that is a real process that can not converge to a complex Brownian motion. Nevertheless part of the same proof done in Theorem 3.1 (steps 1 and 2 and study of \( a_\varepsilon \), note that \( b_\varepsilon = 0 \)) works to prove that the processes defined by (14) converge weakly to a standard Brownian motion.

On the other hand, when \( \theta = 2k\pi \), we have that
\[
x_\varepsilon^\theta(t) = c(2k\pi)\varepsilon \int_0^{2\varepsilon} \cos(2k\pi X_s) ds = 0.
\]

4 The \( m \)-dimensional case

The aim of this section is to extend this result to a \( m \)-dimensional case for any \( m \geq 1 \). We will give the extensions of Theorem 3.1 and 3.2.

We define for any \( \varepsilon > 0 \) and for any \( 1 \leq j \leq m \)
\[
\left\{ x_\varepsilon^\theta_j(t) = \varepsilon \int_0^{2\varepsilon} e^{i\theta_j X_s} ds, \; t \in [0,T] \right\},
\]
where \( \{ X_s, s \geq 0 \} \) is a stochastic process with independent increments and we consider
\[
\{ x_\varepsilon^\theta(t) = (x_\varepsilon^\theta_1, \ldots, x_\varepsilon^\theta_m)(t), \; t \in [0,T] \}.
\]

In order to simplify calculus and notation we will denote by \( \theta \) the \( m \) values \( \theta_1, \theta_2, \ldots, \theta_m \). Since we have to control more quadratic covariations we will need to introduce new hypothesis on \( \theta \), \( (\tilde{H}_{\theta_j, \theta_h}) \) for a characteristic function \( \phi_X \):
\[
(\tilde{H}_{\theta_j, \theta_h}) \text{ For any } c_1 \in \{-1, 1\}
\]
\[
\lim_{\varepsilon \to 0} \varepsilon^2 \int_0^{2\varepsilon} \int_0^{2\varepsilon} \Phi_{X_s} - X_\varepsilon(\theta_j) \| \Phi_{X_s} - X_\varepsilon(\theta_j + c_1\theta_h) \| dx dy = 0.
\]

Then, the extension of Theorem 3.1 reads as follows:

**Theorem 4.1** Let \( \{ X_s, s \geq 0 \} \) be a stochastic process with independent increments and characteristic function \( \phi_X \). Set \( C^m_X = \{ \theta \in \mathbb{R}^m, \text{ such that } \phi_X \text{ satisfies } (H_{\theta_j}) \text{ for any } j = 1, \ldots, m \text{ and satisfies } (\tilde{H}_{\theta_j, \theta_h}) \text{ for any } h \neq j \} \).

Define for any \( \varepsilon > 0 \) and for any \( 1 \leq j \leq m \)
\[
\left\{ x_\varepsilon^\theta_j(t) = \varepsilon c(\theta_j) \int_0^{2\varepsilon} e^{i\theta_j X_s} ds, \; t \in [0,T] \right\},
\]
where \( c(\theta_j) \) is the constant given by hypothesis \( (H_{\theta_j}) \).
Consider $P^\theta_\varepsilon$ the image law of $x^\theta_\varepsilon = (x^\theta_1, \ldots, x^\theta_m)$ in the Banach space $\mathcal{C}([0,T], \mathbb{C}^m)$ of continuous functions on $[0,T]$. Then, if $\theta \in C^m_X$, $P^\theta_\varepsilon$ converges weakly as $\varepsilon$ tends to zero towards the law on $\mathcal{C}([0,T], \mathbb{C}^m)$ of an $m$-dimensional complex Brownian motion.

Proof: The proof follows applying the computations done for the one-dimensional case combined to the method used in [5]. We will only give some hints of the proof.

Notice that the proof of the tightness, the martingale property of each component and the quadratic variations can be done following exactly the proof of the one-dimensional case. So, it remains only to study all the covariations. As it can be seen in Section 3.1 in [5], it suffices to prove that for $j \neq h$ and for any $s_1 \leq s_2 \leq \cdots \leq s_k \leq s < t$ and for any bounded continuous function $\varphi : \mathbb{C}^m \to \mathbb{R}$,

$$E \left( \varphi \left( x^\theta_\varepsilon(s_1), \ldots, x^\theta_\varepsilon(s_k) \right) \left( \varepsilon \int_{\frac{s_1}{\varepsilon}}^{\frac{s_2}{\varepsilon}} c(\theta_j) \cos(\theta_j X_x) dx \right) \right. $$

$$\times \left. \left( \varepsilon \int_{\frac{s_1}{\varepsilon}}^{\frac{s_2}{\varepsilon}} c(\theta_h) \cos(\theta_h X_y) dy \right) \right) ,$$

$$E \left( \varphi \left( x^\theta_\varepsilon(s_1), \ldots, x^\theta_\varepsilon(s_k) \right) \left( \varepsilon \int_{\frac{s_1}{\varepsilon}}^{\frac{s_2}{\varepsilon}} c(\theta_j) \sin(\theta_j X_x) dx \right) \right. $$

$$\times \left. \left( \varepsilon \int_{\frac{s_1}{\varepsilon}}^{\frac{s_2}{\varepsilon}} c(\theta_h) \sin(\theta_h X_y) dy \right) \right)$$

and

$$E \left( \varphi \left( x^\theta_\varepsilon(s_1), \ldots, x^\theta_\varepsilon(s_k) \right) \left( \varepsilon \int_{\frac{s_1}{\varepsilon}}^{\frac{s_2}{\varepsilon}} c(\theta_j) \cos(\theta_j X_x) dx \right) \right. $$

$$\times \left. \left( \varepsilon \int_{\frac{s_1}{\varepsilon}}^{\frac{s_2}{\varepsilon}} c(\theta_h) \sin(\theta_h X_y) dy \right) \right)$$

converge to zero when $\varepsilon$ tends to zero. But, using that $\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$, and the symmetry between $x$ and $y$ (interchanging the roles of $j$ and $h$), it is enough to show that

$$\lim_{\varepsilon \to 0} \varepsilon^2 \|E \left( \varphi \left( x^\theta_\varepsilon(s_1), \ldots, x^\theta_\varepsilon(s_k) \right) \int_{\frac{s_1}{\varepsilon}}^{\frac{s_2}{\varepsilon}} \int_{\frac{s_1}{\varepsilon}}^{\frac{s_2}{\varepsilon}} e^{i(c_1 \theta_j X_x + c_2 \theta_h X_y)} dx dy \right) \| = 0,$$

for any $c_1, c_2 \in \{-1, 1\}$. But,

$$\|E \left( \varphi \left( x^\theta_\varepsilon(s_1), \ldots, x^\theta_\varepsilon(s_k) \right) \int_{\frac{s_1}{\varepsilon}}^{\frac{s_2}{\varepsilon}} \int_{\frac{s_1}{\varepsilon}}^{\frac{s_2}{\varepsilon}} e^{i(c_1 \theta_j X_x + c_2 \theta_h X_y)} dx dy \right) \|$$

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Theorem 4.2 Assume now that \( \psi \) is a characteristic function for \( K \) continuous functions on \( [0, T] \) and set \( m \), \( \theta \)
\[ c(u) = \sqrt{\frac{\|\psi_X(u)\|^2}{2Re[\psi_X(u)]}}.\]

Consider \( P_\varepsilon^\theta \) the image law of \( x_\varepsilon^\theta \) in the Banach space \( C([0, T], \mathbb{C}^m) \) of continuous functions on \( [0, T] \). Then, for \( \theta \) such that \( Re[\psi_X(\theta_j)]Re[\psi_X(2\theta_j)] \neq 0 \) for all \( j \in \{1, \ldots, m\} \) and \( Re[\psi_X(\theta_j + c_1\theta_h)] \neq 0 \) for all \( j, h \in \{1, \ldots, m\} \) and \( c_1 \in \{-1, 1\} \), \( P_\varepsilon^\theta \) converges weakly as \( \varepsilon \) tends to zero, towards the law on \( C([0, T], \mathbb{C}^m) \) of a \( m \)-dimensional complex Brownian motion.

Proof: As in the one-dimensional case it suffices to check that the characteristic function \( \phi_X \) of the Lévy process satisfies \( (H^\theta_j) \) for any \( j = 1, \ldots, m \) and satisfies \( (H^\theta_j, \theta_h) \) for any \( j \neq h \). It remains only to see the second part and can be easily checked that, for \( c_1 \in \{-1, 1\} \)
\[ \int_{\frac{T}{2}}^{\frac{T}{2}} \int_{\frac{T}{2}}^{\frac{T}{2}} \|\phi(x_s-x_r)(\theta_j)\| \|\phi(x_s-x_r)(\theta_j + c_1\theta_h)\| dxdy \]
\[ \leq K\varepsilon^2 \frac{1}{a(\theta_j)a(\theta_j + c_1\theta_h)}. \]

\[ \Box \]

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