Entanglement polygon inequalities for pure states in qudit systems

Xian Shi

College of Information Science and Technology, Beijing University of Chemical Technology, Beijing 100029, China

Abstract Entanglement is one of the important resources in many quantum tasks. And the issue of high-dimensional entangled systems is intriguing. Here we consider the entanglement distribution of higher-dimensional multipartite systems. Specifically, we show that the $n$-qudit pure states satisfy the entanglement polygon inequality (EPI) in terms of geometrical entanglement measure, then we offer an entanglement indicator for three-qubit pure states based on the geometrical entanglement measure. At last, we show that the EPI is not generally valid for pure states in higher-dimensional systems in terms of negativity. Nevertheless, the above inequality is valid for higher-dimensional systems in terms of concurrence.

1 Introduction

Quantum entanglement is an essential feature of quantum mechanics. It plays a vital role in quantum information and quantum computation theory [1], such as superdense coding [2], teleportation [3], and the speedup of quantum algorithms [4].

One of the essential properties of multipartite entanglement is that entanglement cannot be freely shared. For a tripartite entangled state $\rho_{ABC}$, there are six different bipartite entanglements $E_{A|B}, E_{A|C}, E_{B|C}, E_{A|BC}, E_{B|AC}$ and $E_{C|AB}$. Here $E$ is an entanglement measure for bipartite systems, $E_{A|BC}$ can be seen as a $one-to-group$ entanglement, and $E_{A|B}$ can be seen as a $one-to-one$ entanglement. In 2000, Coffman, Kundu and Wootters showed a famous inequality for a three-qubit system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ in terms of concurrence [5].

\[ E_{A|BC} \geq E_{A|B} + E_{A|C}. \]  

(1)

The inequality means that entanglement between a singled-out qubit to a group of qubits is bounded by the sum of the entanglement between the singled-out qubit to each qubit in the group. And this relation is generalized to $n$-qubit($n > 3$) systems in terms of concurrence [6], the squashed entanglement measure [7], the squared entanglement of formation [8], the squared Tsallis-$q$ entanglement measure [9], and the squared Renyi entanglement measure [10]. However, (1) is invalid for multipartite higher-dimensional systems in terms of generic bipartite entanglement measures [11]. Nevertheless, higher-dimensional entanglement plays fascinating roles in quantum information science [12], quantum communications [13–15], and technological advances [16–18]. Hence the problem on characterizing the entanglement distribution in multipartite higher-dimensional systems is meaningful. Recently other types of monogamy of entanglement have been presented [19–21]. Besides, strict constraints between the reduced states in terms of some quantum correlations of the multipartite higher systems were shown in [22, 23].

Recently, entanglement polygon inequality (EPI) was proposed for a multipartite entangled pure state $|\phi_n\rangle$ in $\otimes_{i=1}^n \mathcal{H}_i$ in terms of an entanglement measure $E$ [27, 28],

\[ E_{j|\overline{j}}(|\phi_n\rangle) \leq \sum_{k \neq j} E_{k|\overline{j}}(|\phi_n\rangle), \]  

(2)

here $j$ stands for any party of $\otimes_{i=1}^n \mathcal{H}_i$, and $\overline{j}$ is on the remaining systems. This inequality was proved for pure states in multi-qubit systems in terms of bipartite entanglement measures [27] and in arbitrary dimensional systems in terms of the $q$-concurrence when $q \geq 2$ and the unified-$(q, s)$ entangled measure when $q \geq 1$, $s \geq 0$ [28]. However, whether the EPI is valid for pure states in higher-dimensional systems in terms of concurrence or negativity is unknown [28]. Here we will show the answer to the problem and provide other advances on the EPI for pure states in multipartite systems. Specifically, first, we present that the EPI is valid for pure states in terms of geometrical entanglement measure (GEM). We also show that the EPI is valid in terms of the $\alpha$-th power of GEM when $\alpha \in (0, 1]$. And due to the proof method, we show that the EPI is valid for pure states in higher-dimensional systems in terms of concurrence is valid. Next based on the EPI in terms of GEM, we propose an indicator of whether a three-qubit pure state is...
generally entangled. At last, we offer a class of tripartite pure states which does not satisfy the EPI in terms of negativity. We also show an important class of states, the generalized W class states, meet the EPI in terms of the $\alpha$-th power of negativity when $\alpha \in (0, 1]$.

This paper is organized as follows. In Sec. ii, we present the preliminary knowledge needed here; in Sec. iii, we present our main results; first we present a generalized EPI is valid in terms of $\alpha$-th power of GEM for pure states in arbitrary dimensional systems; we also propose entanglement indicators for three-qubit pure states. Next, we offer a class of pure states that does not satisfy the EPI in terms of negativity, which answers the problem proposed in [28]. Moreover, we give a class of pure states in higher-dimensional systems meeting the EPI in terms of negativity. At last, we show that the EPI is valid in terms of concurrence for pure states in arbitrary dimensional systems, in Sec. iv, we end with a summary.

2 Preliminaries

An $n$-partite pure state $|\psi\rangle_{A_1A_2...A_n} \in \bigotimes_{i=1}^{n} \mathcal{H}_{A_i}$ is fully product if it can be written as

$$|\psi\rangle_{A_1A_2...A_n} = |\phi_1\rangle_{A_1}|\phi_2\rangle_{A_2}...|\phi_n\rangle_{A_n},$$

(3)

where $|\phi_i\rangle_{A_i}$ is a pure state in $\mathcal{H}_{A_i}$. $i = 1, 2, \ldots, n$, respectively. Moreover, a multipartite pure state is called genuinely entangled if

$$|\psi\rangle_{A_1A_2...A_n} \neq |\phi\rangle_{S}|\zeta\rangle_{\overline{S}}$$

(4)

for any partite $S/\overline{S}$ up to local operations, here $S$ is a subset of $A = \{A_1, A_2, \ldots, A_n\}$, and $\overline{S} = A - S$.

Next we recall some entanglement measures for a bipartite state $\rho_{AB}$. A bipartite pure state $|\phi\rangle_{AB} \in \mathcal{H}_{AB}$ can be always written as

$$|\phi\rangle_{AB} = \sum_{i=0}^{d-1} \sqrt{\lambda_i}|i\rangle_A|\bar{i}\rangle_B,$$

where $\lambda_i \geq \lambda_{i+1} \geq 0$, $i = 0, 1, \ldots, d-2$, and $\sum_i \lambda_i = 1$. The GEM for a bipartite pure state $|\phi\rangle_{AB}$ is defined as

$$G(|\phi\rangle) := 1 - \max_{|\psi\rangle = |\psi\rangle_{AB}} \langle\psi|\phi\rangle^2,$$

(5)

where the maximum takes over all the product states $|\psi\rangle = |a\rangle|b\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. The GEM for a mixed state $\rho_{AB}$ is generally defined as [29],

$$G(\rho_{AB}) = \min_{\{p_i, |\phi_i\rangle_{AB}\}} \sum_i p_i G(|\phi_i\rangle_{AB}),$$

(6)

where the minimum takes over all the decompositions of $\rho_{AB}$ with $\sum_i p_i |\phi_i\rangle = \rho_{AB}$.

Negativity is an important entanglement measure for bipartite systems. It is defined as follows for a pure state $|\phi\rangle_{AB}$ [30],

$$N(|\phi\rangle_{AB}) = \frac{\|\phi\|_{T_A} - 1}{2},$$

(7)

here $\langle ij|(|\phi\rangle T_A |kl\rangle = \langle kj|(|\phi\rangle |il\rangle$, $(\cdot) T_A$ denotes the partial transposition. $\|X\| = tr\sqrt{X^\dagger X}$.

Concurrence is the other entanglement measure for bipartite mixed states $\rho_{AB}$. The concurrence of a pure state $|\psi\rangle_{AB}$ is defined as

$$C(|\psi\rangle_{AB}) = \sqrt{2(1 - Tr \rho^2_A)},$$

(8)

Here $\rho_A = Tr_B |\psi\rangle_{AB}\langle\psi|$. For a mixed state $\rho_{AB}$, it is defined as

$$C(\rho_{AB}) = \min_{\{p_i, |\phi_i\rangle_{AB}\}} \sum_i p_i C(|\phi_i\rangle_{AB}),$$

(9)

where the minimum takes over all the decompositions of $\rho_{AB} = \sum_i p_i |\phi_i\rangle_{AB} \langle\phi_i|$ with $p_i \geq 0$ and $\sum_i p_i = 1$.

For a two-qubit mixed state $\rho_{AB}$, Wootters derived an analytical formula of its concurrence [31]:

$$C(\rho_{AB}) = \max\{\sqrt{\mu_1} - \sqrt{\mu_2} - \sqrt{\mu_3} - \sqrt{\mu_4}, 0\},$$

(10)

where $\mu_1, \mu_2, \mu_3$, and $\mu_4$ are the eigenvalues of the matrix $\rho_{AB}(\sigma_x \otimes \sigma_x)\rho^*_{AB}(\sigma_y \otimes \sigma_y)$ with nonincreasing order.

A parametrized entanglement measure $C_q(\cdot)$ for a pure state $|\psi\rangle_{AB}$ [32] is defined as

$$C_q(|\psi\rangle_{AB}) = 1 - Tr \rho^q_A,$$

(11)
here $\rho_A = tr_B |\psi\rangle_{AB} \langle \psi|$. And the $q$-concurrence satisfies the EPI for pure states in multipartite higher-dimensional systems when $q \geq 2$ [28].

Lemma 1 [28] For any $n$-qudit pure entangled state $|\psi\rangle$ in $\bigotimes_i \mathcal{H}_i$. When $q \geq 2$, the following inequality holds,

$$C^j_q \leq \sum_{k \neq j \neq k} C^k_q (|\psi\rangle), \quad (12)$$

here $k, j$ are on behalf of some partite of the multipartite system.

3 Main results

In this section, we first present that the tripartite pure states satisfies the EPI in terms of the $\alpha$-th power of GEM when $\alpha \in (0, 1]$, then we propose an entanglement indicator for three-qubit pure states. At last, we show that EPI in terms of negativity is not always valid for three qudit pure states, whereas the inequality is valid in terms of concurrence for multi-qudit pure states.

3.1 EPI in terms of geometric entanglement measure

Assume $|\psi\rangle_{AB} = \sum_{i=0}^{d-1} \sqrt{\lambda_i} |ii\rangle$ is a pure state with $\lambda_i \geq \lambda_{i+1} \geq 0$, $i = 0, 1, \ldots, d-2$, and $\sum_i \lambda_i = 1$. By the formula (5), we have

$$G(|\psi\rangle) = 1 - \lambda_0, \quad (13)$$

Next we recall the Schatten $p$-norm of a bounded operator $M$ on the Hilbert space $\mathcal{H}$ with finite dimensions,

$$\|M\|_p := [\text{Tr}(|M|^p)]^{\frac{1}{p}} \leq \left( \sum_{i=0}^{d-1} s_i^p (M) \right)^{\frac{1}{p}}, \quad (14)$$

here $s_i (M)$ are the singular values of $M$ with decreasing order. And when $p \to \infty$, $\|M\|_p \to s_0 (M)$.

Lemma 2 [33] For any bipartite state $\rho_{AB}$ on a Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, the inequality

$$1 + \|\rho_{AB}\|_q \geq \|\text{Tr}_{A} \rho_{AB}\|_q + \|\text{Tr}_{B} \rho_{AB}\|_q, \quad (15)$$

holds for $q > 1$.

Clearly, the inequality (15) can be written as

$$(1 - \|\text{Tr}_{A} \rho_{AB}\|_q) + (1 - \|\text{Tr}_{B} \rho_{AB}\|_q) \geq 1 - \|\rho_{AB}\|_q, \quad (16)$$

when $q > 1$.

Theorem 3 Assume $|\psi\rangle_{A_1 A_2 \ldots A_n}$ is a pure state in the following system $\mathcal{H}_{A_1 A_2 \ldots A_n}$, then we have

$$G(|\psi\rangle_{P_i|\mathcal{T}}) \leq \sum_{j \neq i} G(|\psi\rangle_{P_j|\mathcal{T}}), \quad (17)$$

here $\{P_1, P_2, \ldots, P_k\}$ is a partition of the set $\{A_1, A_2, \ldots, A_n\}$ with $k \leq n$.

Proof As

$$G(|\psi\rangle_{AB}) = 1 - s_0 (\rho_A) = \lim_{p \to \infty} [1 - \|\rho_A\|_p], \quad (18)$$

then we have

$$G(|\psi\rangle_{P_i|\mathcal{T}}) = \lim_{p \to \infty} [1 - \|\rho_{P_i}\|_p]$$

$$\leq \lim_{p \to \infty} [1 - \|\rho_{A_i \neq P_i}\|_p]$$

$$\leq \sum_{j \neq i} [1 - \|\rho_{P_j}\|_q]$$

$$= \sum_{j \neq i} G(|\psi\rangle_{P_j|\mathcal{T}}). \quad (19)$$
Fig. 1 EPI for Example 6. When 
\( \alpha \in (0, 1] \), the inequality (22) is 
bigger than 0

Here the first equality is due to (18), the first inequality is due to the inequality (16), and the last equality is due to the equality (18).

Next we propose a generalized EPI for \( n \)-qudit pure states in terms of the GEM.

**Lemma 4** When \( a, \ b, \ c \in (0, 1] \), and \( a + b \geq c \), then
\[
a^\alpha + b^\alpha \geq c^\alpha, \]
when \( \alpha \in (0, 1] \).

**Proof** Here we can always assume \( a \leq b \), and then when \( 0 < \alpha \leq 1 \), \( \left( \frac{a}{b} + 1 \right)^\alpha - \left( \frac{a}{b} \right)^\alpha \leq 1 \), then we have
\[
a^\alpha + b^\alpha \geq (a + b)^\alpha \geq c^\alpha. \]

**Corollary 5** Assume \( |\psi\rangle_{A_1A_2...A_n} \) is a pure state, then we have
\[
G(\langle \psi|_{P_i}\langle P_i|) \leq \sum_{j \neq i} G(\langle \psi|_{P_j}\langle P_j|)^\alpha, \tag{20}
\]
here \( \{P_1, \ P_2, \ldots, \ P_k\} \) is a partition of the set \( \{A_1, \ A_2, \ldots, A_n\} \) with \( k \leq n \), and \( \alpha \in (0, 1] \).

The inequality (20) is due to the Theorem 3 and Lemma 4.

**Example 6** Assume \( |\psi\rangle_{ABC} \) is a tripartite pure state,
\[
|\psi\rangle = \frac{3}{5}|102\rangle + \frac{2\sqrt{3}}{5}|200\rangle + \frac{2}{5}|010\rangle + \frac{\sqrt{3}}{5}|020\rangle + \frac{\sqrt{3}}{5}|001\rangle,
\]
through computation, we have
\[
G(|\psi\rangle_{A|BC}) = 16\frac{25}{25}, \ G(|\psi\rangle_{B|AC}) = 6\frac{25}{25}, \ G(|\psi\rangle_{C|AB}) = 11\frac{25}{25}. \tag{21}
\]

Next let \( g = G(|\psi\rangle_{C|BA})^\alpha + G(|\psi\rangle_{B|AC})^\alpha - G(|\psi\rangle_{A|BC})^\alpha \), then
\[
g = \left( \frac{11}{25} \right)^\alpha + \left( \frac{6}{25} \right)^\alpha - \left( \frac{16}{25} \right)^\alpha, \tag{22}
\]
we plot (22) in Fig. 1. From the figure, we see that the inequality (22) is bigger than 0 when \( \alpha \in (0, 1] \).
Consider a tripartite star quantum network consisting of two EPR states. It can be seen as a pure state in a $4 \otimes 2 \otimes 2$ system represented as

$$|\psi\rangle_{ABC} = \frac{1}{2}(|000\rangle + |101\rangle + |210\rangle + |311\rangle),$$

through computation, we have

$$G(|\psi\rangle_{B|AC}) = G(|\psi\rangle_{C|AB}) = \frac{1}{2},$$

$$G(|\psi\rangle_{A|BC}) = \frac{3}{4},$$

then let

$$g_1 = G(|\psi\rangle_{A|BC})^\alpha + G(|\psi\rangle_{B|AC})^\alpha - G(|\psi\rangle_{C|AB})^\alpha = \left(\frac{3}{4}\right)^\alpha,$$

$$g_2 = G(|\psi\rangle_{A|BC})^\alpha + G(|\psi\rangle_{C|AB})^\alpha - G(|\psi\rangle_{B|AC})^\alpha = \left(\frac{3}{4}\right)^\alpha,$$

$$g_3 = G(|\psi\rangle_{C|AB})^\alpha + G(|\psi\rangle_{B|AC})^\alpha - G(|\psi\rangle_{A|BC})^\alpha = 2 \times \left(\frac{1}{2}\right)^\alpha - \left(\frac{3}{4}\right)^\alpha.$$

From Fig. 3, $g_1$, $g_2$ and $g_3$ are not equal 0, and $|\psi\rangle_{ABC}$ is entangled.

3.2 Applications of the EPI in terms of GEM

Assume $|\psi\rangle_{A_1A_2...A_n}$ is a pure state in $\otimes_i \mathcal{H}_{A_i}$, then we consider an entanglement indicator defined as [28, 34],

$$\delta^G_{A_\alpha}(|\psi\rangle) = \min_i \left[ \sum_{j \neq i} G_A^\alpha(|\psi\rangle_{A_j|A_i}) - G_A^\alpha(|\psi\rangle_{A_i|A_j}) \right], \quad (23)$$

here $i$ in (23) stands for some partite $A_i$ of $|\psi\rangle_{A_1A_2...A_n}$, and the minimum takes over all the partite $A_i$.

By Corollary 5, we have that $\delta^G_{A_\alpha} \geq 0$, $\alpha \in (0, 1]$. When limiting $|\psi\rangle$ to be a three-qubit pure state, we could get a sufficient and necessary condition on its biseparability.

**Theorem 8** Assume $|\psi\rangle_{A_1A_2A_3}$ is a three-qubit pure state, then $\delta^G_{A_\alpha} = 0$, $\forall \alpha \in (0, 1]$ if and only if $|\psi\rangle_{A_1A_2A_3}$ is a biseparable state.

**Proof** $\Rightarrow$: When $\alpha \in (0, 0.5)$, as $2\alpha \in (0, 1]$, we have...
Here we consider the superposition of a generalized W class state and the vacuum, \(|\Delta_1\rangle\_{\mathcal{A}_1}\) and the vacuum, \(|\Delta_1\rangle\_{\mathcal{A}_1}\) and the vacuum, \(|\Delta_1\rangle\_{\mathcal{A}_1}\) and the vacuum, \(|\Delta_1\rangle\_{\mathcal{A}_1}\) and the vacuum, \(|\Delta_1\rangle\_{\mathcal{A}_1}\)

\[
G^{2a}(|\psi\rangle_{A_j|\mathcal{A}_1}) - G^{2a}(|\psi\rangle_{A_j|\mathcal{A}_2}) - G^{2a}(|\psi\rangle_{A_k|\mathcal{A}_2})
= \left(\frac{G^a(|\psi\rangle_{A_j|\mathcal{A}_1} + G^a(|\psi\rangle_{A_j|\mathcal{A}_2})}{2}\right)
- G^{2a}(|\psi\rangle_{A_j|\mathcal{A}_1}) - G^{2a}(|\psi\rangle_{A_k|\mathcal{A}_2})
= 2G^a(|\psi\rangle_{A_j|\mathcal{A}_1})G^a(|\psi\rangle_{A_k|\mathcal{A}_2}) = 0.
\]

Here \(j \neq k \neq i\). From the above, we have \(G(|\psi\rangle_{A_j|\mathcal{A}_1}) = 0\) or \(G(|\psi\rangle_{A_k|\mathcal{A}_2}) = 0\).

A three-qubit pure state \(|\psi\rangle_{A_1A_2A_3}\) can be written as follows due to the generalized Schmidt decomposition [35]:

\[
|\psi\rangle = l_0|000\rangle + l_1e^{i\theta}|100\rangle + l_2|101\rangle + l_3|110\rangle + l_4|111\rangle,
\]

where \(\theta \in [0, \pi]\), \(l_i \geq 0\) (\(i = 0, 1, 2, 3, 4\), and \(\sum_{i=0}^{4} l_i^2 = 1\). From simple computation, we have the Schmidt coefficients of \(|\psi\rangle\) in terms of \(A_1BC\) is

\[
\text{Sch}(|\psi\rangle_{A_1|A_2A_3}) = (l_0^2, 1 - l_0^2).
\]

the Schmidt coefficients of \(|\psi\rangle\) in terms of \(B_1AC\) is

\[
\text{Sch}(|\psi\rangle_{A_2|A_1A_3}) = \left(\frac{1 + \sqrt{1 - 4\Delta_0}}{2}, \frac{1 - \sqrt{1 - 4\Delta_0}}{2}\right).
\]

here \(\Delta_0 = l_0^2l_2^2 + l_0^2l_3^2 + l_1^2l_4^2 + l_2^2l_3^2 - 2l_1l_2l_3l_4 \cos \theta\), the Schmidt coefficients of \(|\psi\rangle\) in terms of \(C_1AB\) is

\[
\text{Sch}(|\psi\rangle_{A_3|A_1A_2}) = \left(\frac{1 + \sqrt{1 - 4\Delta_1}}{2}, \frac{1 - \sqrt{1 - 4\Delta_1}}{2}\right).
\]

here \(\Delta_1 = l_0^2l_2^2 + l_1^2l_4^2 + l_2^2l_3^2 - 2l_1l_2l_3l_4 \cos \theta\).

When \(G(|\psi\rangle_{A_1A_2A_3}) = 0\), \(l_0 = 0\) or \(l_0 = 1\), then \(|\psi\rangle_{A_1A_2A_3}\) is biseparable. When \(G(|\psi\rangle_{A_2A_1A_3}) = 0\), that is, \(\Delta_0 = 0\), as \(\Delta_0 = l_0^2(l_2^2 + l_3^2) + l_1l_4 - l_2l_3)^2 + 4l_1l_2l_3l_4 \sin^2 \theta = 0\), then \(l_3 = l_4 = 0\), or \(l_0 = 0\), then \(|\psi\rangle_{A_1A_2A_3}\) is biseparable. The case \(G(|\psi\rangle_{A_1A_2A_3}) = 0\) is similar to \(G(|\psi\rangle_{A_2A_1A_3}) = 0\).

\(\Leftrightarrow\): Here we can always assume \(|\psi\rangle = |\phi\rangle_{A_1}|\psi\rangle_{A_2A_3}\), then \(G(|\psi\rangle_{A_1A_2A_3}) = 0\), \(G(|\psi\rangle_{A_2A_1A_3}) = G(|\psi\rangle_{A_1A_2A_3})\), then \(\delta^G = 0\), \(\forall \alpha \in (0, 1]\).

**Example 9** Here we consider the superposition of a generalized W class state and the vacuum,

\[
|\phi\rangle = \frac{1}{90}(|100\rangle + 4|010\rangle + 8|001\rangle) + \sqrt{0.99}|000\rangle.
\]

Then
Fig. 4 $\delta^G_\alpha$ as a function of $\alpha (\alpha \in (0, 1])$.

From Fig. 4, one gets that $\delta^G_\alpha (|\phi\rangle) > 0$ and $|\phi\rangle$ is genuinely entangled.

3.3 EPI in term of negativity and concurrence

In this subsection, we present a class of counterexamples that does not satisfy the EPI in terms of negativity, which answers the problem proposed in [28]. We also show that the generalized W class states meet the EPI in terms of negativity. At last, we show that pure states in multipartite arbitrary dimensional systems satisfy the EPI in terms of concurrence.

Before presenting a class of states which does not satisfy the EPI in terms of negativity, we consider a concrete example that does not satisfy the EPI.

Example 10

$|\psi\rangle_{ABC} = \frac{1}{3} (|000\rangle + |101\rangle + |202\rangle + |310\rangle + |411\rangle + |512\rangle + |620\rangle + |721\rangle + |822\rangle).$ (25)

Through computation, we have that $N(|\psi\rangle_A|_{BC}) = 4$, $N(|\psi\rangle_B|_{AC}) = 1$, $N(|\psi\rangle_C|_{AB}) = 1$. Clearly, $N(|\psi\rangle_A|_{BC}) \geq N(|\psi\rangle_B|_{AC}) + N(|\psi\rangle_C|_{AB})$, that is, the EPI is invalid for $|\psi\rangle_{ABC}$ in terms of negativity.

Remark 11 In the above example, $|\psi\rangle_{ABC}$ is a purification of a product state $\rho_{BC} = \frac{1}{3} I_3 \otimes I_3$. Through using this property, we propose a class of states that does not satisfy the EPI in terms of negativity.

Theorem 12 Assume $\rho = \sum_i c_i |i\rangle \langle i|$ and $\sigma = \sum_j c_j |j\rangle \langle j|$ are two states on the system $\mathcal{H}_d$ with rank($\rho$), rank($\sigma$) > 1, here $\{|i\rangle\}$ and $\{|j\rangle\}$ are the orthonormal bases of $\mathcal{H}_d$. Let

$|\psi\rangle_{ABC} = \sum_{ij} \sqrt{c_i c_j} |i\rangle_A |j\rangle_B |ij\rangle_C$

be a purification of $\rho_A \otimes \sigma_B$, then we have

$N(|\psi\rangle_C|_{AB}) \geq N(|\psi\rangle_A|_{BC}) + N(|\psi\rangle_B|_{AC}).$ (26)
Then we have

\[ N(\phi_{AB}) = \frac{1}{2} \left( \sum_i \sqrt{\lambda_i} \right)^2 - 1. \]  

Due to (27), we have

\[ N(|\psi\rangle_{C|AB}) = N(|\psi\rangle_{A|BC}) - N(|\psi\rangle_{B|AC}) \]

\[ = \frac{1}{2} \left[ \left( \sum_{ij} \sqrt{a_i b_j} \right)^2 - 1 - \left( \sum_i \sqrt{a_i} \right)^2 + 1 \right] - \frac{1}{2} \left[ \left( 1 - \left( \sum_i \sqrt{a_i} \right)^2 \right) \left( 1 - \left( \sum_j \sqrt{b_j} \right)^2 \right) \right]. \]  

(28)

As \( \sum_i a_i = \sum_j b_j = 1 \), and \( \text{rank } \rho, \text{rank } \sigma > 1 \), then (28) > 0. Hence we finish the proof.

Next we present a class of pure states, the generalized W class (GW) states, which satisfy the EPI in terms of negativity. These states were first studied on the problem of monogamy relations in terms of concurrence for higher-dimensional systems [36]. Recently, the general monogamy relations of the states attracts much attention of the relevant researchers [37, 38].

Now let us recall the definition of the GW states \( |W^d_n\rangle \),

\[ |W^d_n\rangle_{A_1\cdots A_n} = \sum_{i=1}^d (a_{i1}|00\cdots 0\rangle + \cdots + a_{in}|00\cdots i\rangle), \]  

(29)

where we assume \( \sum_{i=1}^d \sum_{j=1}^n |a_{ji}|^2 = 1 \). Next we present a lemma and show some meaningful properties of the GW states.

**Lemma 13** [36] Assume \( |\psi\rangle_{AB_1B_2\cdots B_{n-1}} \) is a GW state, then for an arbitrary partition \( \{P_1, P_2, \ldots, P_m\} \) of the set \( S = \{A, B_1, B_2, \ldots, B_{n-1}\} \), the state \( |\psi\rangle_{P_1P_2\cdots P_m} \) is also a GW state, here we assume \( P_i \cap P_j = \emptyset \) (\( i \neq j \)) and \( \cup P_j = S \), \( m \leq n \).

**Theorem 14** Assume \( |\psi\rangle_{A_1\cdots A_n} \) is a GW state, and here we denote \( \{P_1, P_2, P_3\} \) is a partition of the set \( \{A_1, A_2, \ldots, A_n\} \), \( n \geq 3 \). Then we have

\[ N(|\psi\rangle_{P_1P_2P_3}) \leq N(|\psi\rangle_{P_1P_2P_3}) + N(|\psi\rangle_{P_1P_3}). \]  

(30)

**Proof** Due to the above lemma, we have that \( |\psi\rangle_{P_1P_2P_3} \) can also be seen a GW state, that is, \( |\psi\rangle_{P_1P_2P_3} \) can be written as

\[ |\psi\rangle_{P_1P_2P_3} = \sum_{i=1}^d (a_{i1}|00\rangle + a_{i2}|01\rangle + a_{i3}|0i\rangle), \]  

(31)

through computation, we have that

\[ N(|\psi\rangle_{P_1P_2P_3}) = \sqrt{\sum_i |a_{i1}|^2} \times \sqrt{\sum_i |a_{i2}|^2 + |a_{i3}|^2}, \]

\[ N(|\psi\rangle_{P_2P_1P_3}) = \sqrt{\sum_i |a_{i2}|^2} \times \sqrt{\sum_i |a_{i1}|^2 + |a_{i3}|^2}, \]

\[ N(|\psi\rangle_{P_3P_1P_2}) = \sqrt{\sum_i |a_{i3}|^2} \times \sqrt{\sum_i |a_{i2}|^2 + |a_{i1}|^2}, \]  

(32)

let \( a = \sum_i |a_{i1}|^2, \ b = \sum_i |a_{i2}|^2, \ c = \sum_i |a_{i3}|^2 \), as

\[ \sqrt{a(b + c)} \leq \sqrt{ab + ac + 2bc} \leq \sqrt{ab(a + c)(a + b)}, \]

\[ = \sqrt{(\sqrt{ab(a + c)} + \sqrt{bc(a + b)})^2}. \]

Then we finish the proof.

By Lemma 4, we can generalize the Theorem 14 to the following.
Fig. 5 EPI for Example 16. Here we can see the inequality (34) is bigger than 0 when $\alpha \in (0, 1]$

**Corollary 15** Assume $|\psi\rangle_{A_1\cdots A_n}$ is a GW state, and we denote $\{P_1, P_2, \ldots, P_k\}$ is a partition of the set $\{A_1, A_2, \ldots, A_n\}$, $n \geq 3$. When $\alpha \in (0, 1]$, 

$$N(|\psi\rangle_{P_j}|P_j)^\alpha \leq \sum_{j \neq i} N(|\psi\rangle_{P_i}|P_i)^\alpha.$$  \hspace{1cm} (33)

**Example 16** Here we consider a tripartite pure state, 

$$|\psi\rangle_{ABCD} = 0.3|0001\rangle + 0.4|0020\rangle + 0.5|0100\rangle + \sqrt{0.5}|1000\rangle,$$

when we take $P_1 = \{A\}$, $P_2 = \{BC\}$, $P_3 = \{D\}$, and let 

$$h = N(|\psi\rangle_{P_2|P_1P_3})^\alpha + N(|\psi\rangle_{P_3|P_1P_2})^\alpha - N(|\psi\rangle_{P_1|P_2P_3})^\alpha,$$

then 

$$h = 0.2419^\alpha + 0.0819^\alpha - 0.5^\alpha.$$ \hspace{1cm} (34)

From Fig. 5, we see that (34) is bigger than 0 when $\alpha \in (0, 1]$.

At last, we present that the EPI is valid in terms of concurrence for pure states in arbitrary dimensional systems. Comparing (11) with (8), when $|\psi\rangle_{AB}$ is a bipartite pure state, $C(|\psi\rangle_{AB}) = \sqrt{2C_2(|\psi\rangle_{AB})}$. Then combing Lemma 1 and Lemma 4, we have the following result,

**Theorem 17** For an $n$-qudit pure entangled state $|\psi\rangle$ in $\otimes_i \mathcal{H}_i$, we have 

$$C_{j,k}(|\psi\rangle) \leq \sum_{k \neq j \neq k} C_k(|\psi\rangle),$$ \hspace{1cm} (35)

4 Conclusion

In this article, we mainly investigated the EPI for pure states in multipartite systems. First, we presented that the EPI is valid in terms of the $\alpha$-th power of GEM for pure states in arbitrary dimensional systems when $\alpha \in (0, 1]$. Then we presented an entanglement indicator in terms of GEM for pure states in three-qubit systems. At last, we showed that the EPI may be invalid for $n$-qudit pure states in terms of negativity. Nevertheless, we also presented the GW states meets the EPI in terms of negativity. Furthermore, we pointed out that the EPI is valid in terms of concurrence for pure states in arbitrary dimensional systems. Due to the importance of the study on the higher-dimensional multipartite entanglement systems, our results can provide a reference for future work on the study of multiparty quantum entanglement.

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