Measures related to \((\epsilon, n)\)-complexity functions

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Abstract

The \((\epsilon, n)\)-complexity functions describe total instability of trajectories in dynamical systems. They reflect an ability of trajectories going through a Borel set to diverge on the distance \(\epsilon\) during the time interval \(n\). Behavior of the \((\epsilon, n)\)-complexity functions as \(n \to \infty\) is reflected in the properties of special measures. These measures are constructed as limits of atomic measures supported at points of \((\epsilon, n)\)-separated sets. We study such measures. In particular, we prove that they are invariant if the \((\epsilon, n)\)-complexity function grows subexponentially.

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1 Introduction

The instability of orbits in dynamical systems is quantitatively reflected by complexity functions. Topological complexity reflects pure topological features of dynamics \([4]\), symbolic complexity (see, for instance, \([9]\)) deals with symbolic systems, and the \((\epsilon, n)\)-complexity (see definition below) depends on a distance in the phase space. If a dynamical system is generated by a map \(f : X \to X\) where \(X\) is a metric space with a distance \(d\), one can introduce the sequence of distances \((7)\)

\[d_n(x, y) = \max_{0 \leq i \leq n-1} d(f^i x, f^i y), n \in \mathbb{N},\]

and study the \(\epsilon\)-complexity with respect to the distance \(d_n\) as a function of “time” \(n\). This function describes the evolution of instability of orbits in time (see, for instance, \([3], [10], [1], [2]\)). It depends not only on \(n\) but on \(\epsilon\) as well.

In fact, the \((\epsilon, n)\)-complexity \(C_{\epsilon,n}\) is the maximal number of \(\epsilon\)-distinguishable pieces of trajectories of temporal length \(n\). It is clear that this number is growing as \(\epsilon\) is decreasing. If a system possesses an amount of instability then this number is also growing as \(n\) is increasing: the opportunity for trajectories to diverge on the distance \(\epsilon\) during the temporal interval \(n + 1\) is greater than to

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do it during \( n \) temporal units. It is known (see, for instance, \cite{14}) that
\[
b := \lim_{n \to \infty} \limsup_{\epsilon \to 0} \frac{\ln C_{\epsilon, n}}{-\ln \epsilon}
\]
is the fractal (upper box) dimension of \( X \). Moreover (see, for instance, \cite{8})
\[
h := \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{\ln C_{\epsilon, n}}{n}
\]
is the topological entropy of the dynamical system \((X, f)\). Following Takens \cite{14}, people say that the system \((X, f)\) is deterministic and possesses dynamical chaos if \( 0 < b < \infty \), \( 0 < h < \infty \).

Thus, the \((\epsilon, n)\)-complexity is a global characteristic of the evolution of instability allowing one to single out systems with dynamical chaos.

Generally, this process of evolution occurs very non-uniformly: there are regions in the phase space (red spots) where the divergence of trajectories is developing very fast, for other pieces of initial conditions (green spots) trajectories manifest the distal behavior for long intervals of time and only after that have a possibility to diverge on the distance \( \epsilon \). In other words, for every fixed \( n \) there is a “distribution” of initial points according to their ability to diverge to the distance \( \epsilon \) during the time interval \( n \).

In our article we prove that these distributions converge to the limiting ones as \( n \to \infty \). We call them the measures related to the \((\epsilon, n)\)-complexity. We study main properties of the measures and consider some examples that allow us to hypothesize that generally for systems with positive topological entropy these measures are non-invariant. We also prove that for systems with zero topological entropy these measures are invariant.

The article is a continuation of the study we started in our previous work \cite{1} where we proved the existence of special measures reflecting the asymptotic behavior of the \((\epsilon, n)\)-complexity as \( \epsilon \to 0 \).

## 2 Set-up and definitions

Let \((X, d)\) be a compact metric space with a distance \( d \), \( S \subset X \) and \( f : X \setminus S \to X \) a continuous map. If \( S = \emptyset \), the map is continuous on \( X \); if not, it can be discontinuous. We assume that \( X \setminus S \) is open dense in \( X \) set. Further properties of \( S \) will be specified below.

Let \( D = \bigcup_{i=0}^{\infty} f^{-i} S \), the set of all preimages of \( S \). The dynamical system \((f^i, X \setminus D)\) and the distances \( d_n \) are well-defined.

The notion of the \( \epsilon \)-separability was first introduced by Kolmogorov and Tikhomirov \cite{12} and was applied to study dynamical systems by Bowen \cite{7}.
Definition 1. Two points \( x \) and \( y \) in \( X \setminus \mathcal{D} \) are said to be \((\epsilon, n)\)-separated if 
\[ d_n(x, y) \geq \epsilon. \]

It means that the pieces of the orbits of temporal length \( n \) going through \( x \) and \( y \) are \( \epsilon \)-distinguishable at the instant \( i \) of time, \( 0 \leq i \leq n - 1 \).

Definition 2. (i) A set \( Y \subset X \setminus \mathcal{D} \) is said to be \((\epsilon, n)\)-separated if any pair \( x, y \in Y, x \neq y \), is \((\epsilon, n)\)-separated.

(ii) Given \( A \subset X \setminus \mathcal{D} \), the quantity 
\[ C_{\epsilon,n}(A) = \max\{|Y|: Y \subset A \text{ is } (\epsilon, n) - \text{separated}\} \]
where \( |Y| \) is the cardinality of \( Y \), is called the \((\epsilon, n)\)-complexity of the set \( A \). As the function of \( n \) it is called the \( \epsilon \)-complexity function of \( A \).

The following proposition is proved exactly en the same way as in [1].

Proposition 1. Given \( B_1, B_2 \subset X \setminus \mathcal{D} \) and \( \epsilon > 0 \), the following inequality holds 
\[ C_{\epsilon,n}(B_1 \cup B_2) \leq C_{\epsilon,n}(B_1) + C_{\epsilon,n}(B_2). \]

Definition 3. Given \( Z \subset X \setminus \mathcal{D} \), an \((\epsilon, n)\)-separated set \( Y \subset Z \) is called \((\epsilon, n)\)-optimal in \( Z \) if \( |Y| = C_{\epsilon,n}(Z) \).

3 Measures

Here we define some measures reflecting the asymptotic behavior of the \((\epsilon, n)\)-complexity as \( n \to \infty \). For that, we use a technique of ultrafilters (see Appendix 1) and also the Marriage Lemma (Appendix 2).

Given \( \epsilon > 0, n \in \mathbb{Z} \), consider an optimal (in \( X \setminus \mathcal{D} \) \( (\epsilon, n) \)-separated set \( A_{\epsilon,n} \). Allowing \( n \to \infty \) we fix a sequence of \((\epsilon, n)\)-separated sets. Introduce the following functional 
\[ I_{\epsilon,n}(\phi) = \frac{1}{C_{\epsilon,n}} \sum_{x \in A_{\epsilon,n}} \phi(x). \]

where \( \phi: X \to \mathbb{R} \) is a continuous function. It is clear that \( I_{\epsilon,n} \) is a positive bounded linear functional on \( C(X) \). Moreover, for any fixed \( \epsilon > 0 \) the sequence \( I_{\epsilon,n}(\phi) \) is bounded. Fix an arbitrary non-proper ultrafilter \( \mathcal{F} \), see Appendix I. Consider 
\[ I_{\epsilon}(\phi) = \lim_{\mathcal{F}} I_{\epsilon,n}(\phi). \]

\( I_{\epsilon} \) is a positive bounded linear functional on \( C(X) \) that may depend on the choice of \( \epsilon \), the ultrafilter \( \mathcal{F} \) and optimal sets \( A_{\epsilon,n} \). We denote by \( \mu = \mu_{\epsilon,\mathcal{F},A_{\epsilon,n}} \) the corresponding regular Borel measure on \( X \).

Remark. As one can see, the functional \( I_{\epsilon}(\cdot) \) is defined for any bounded function, in particular, for the characteristic function \( \chi_{Y} \) of a set \( Y \). Generally, \( I_{\epsilon}(\chi_{Y}) \neq \)
µ(Y). But if C is a compact set and W is an open set then \( I_\varepsilon(\chi_C) \leq \mu(C) \) and \( I_\varepsilon(\chi_W) \geq \mu(W) \), see [10, 11].

**Definition 4.** The measures \( \{\mu\} \) will be called the measures related to the \((\varepsilon, n)\)-complexity.

In the further consideration we will use the following property of a measure \( \mu \).

**Proposition 2.** If \( \mu(S) = 0 \) then for any sequence of positive numbers \( \delta_n \), \( \delta_n \to 0 \) as \( n \to \infty \), one has

\[
\lim_{n \to \infty} \frac{1}{C_{\varepsilon,n}} |A_{\varepsilon,n} \cap O_{\delta_n}(S)| = 1, \tag{1}
\]

where \( A_{\varepsilon,n} \) are the \((\varepsilon, n)\)-optimal sets used in the definition of \( \mu \) and \( O_{\delta}(S) \) is the \( \delta \)-neighborhood (in the metric \( d \)) of the set \( S \).

**Proof.** In fact, the validity of (1) follows directly from the definition of \( \mu \). Indeed, for any small \( \delta > \alpha > 0 \)

\[
\lim_{n \to \infty} \frac{1}{C_{\varepsilon,n}} |A_{\varepsilon,n} \cap O_{\alpha}(S)| = I_\varepsilon(\chi_{O_{\alpha}(S)}) \leq L_\varepsilon(\chi_{O_{\alpha}(S)}) \leq \mu(O_{\alpha}(S)) \leq \mu(O_{\delta}(S)),
\]

and \( \mu(O_{\delta}(S)) \to 0 \) as \( \delta \to 0 \) (\( \mu \) is a regular measure). Moreover, \( \delta_n < \delta \) if \( n \) is large enough. Therefore,

\[
\lim_{n \to \infty} \frac{1}{C_{\varepsilon,n}} |A_{\varepsilon,n} \cap O_{\delta_n}(S)| \leq \mu(O_{\delta}(S)).
\]

It implies the desired result. \( \square \)

The following proposition is proved in the same way as Proposition 6 in [11], (one should just replace the distance \( d \) by the distance \( d_n \) and apply the Marriage Lemma (see Appendix II). For completeness we present the proof here.

**Proposition 3.** Let \( A \) and \( B \) be the \((\varepsilon, n)\)-separated sets and \( A \) is optimal. Then there exists an injection map \( \alpha_n : B \to A \) such that \( d_n(x, \alpha_n(x)) < \varepsilon \) for any \( x \in B \). If \( |B| = |A| \) then \( \alpha_n \) is bijection.

**Proof.** Recall that \( O_\varepsilon(x) = \{y : d_n(x, y) < \varepsilon\} \), the ball of radius \( \varepsilon \) centered at \( x \). Given \( Y \subseteq X \) let \( O_\varepsilon(Y) = \bigcup_{x \in Y} O_\varepsilon(x) \).

For any \( x \in B \) let \( A_x = O_\varepsilon(x) \cap A \). If we show that for any \( S \subseteq B \) the following inequality holds

\[
| \bigcup_{x \in S} A_x | \geq |S|, \tag{2}
\]

then the proposition follows from the Marriage lemma. To prove the inequalities (2), suppose that \( | \bigcup_{x \in S} A_x | = |O_\varepsilon(S) \cap A| < |S| \) for some \( S \subseteq B \). Then

\[
|S \cup (A \setminus (O_\varepsilon(S) \cap A))| = |S| + (|A| - |O_\varepsilon(S) \cap A|) > |A| = C_{\varepsilon,n}.
\]
On the other hand, the set \( S \cup (A \setminus (O_r(S) \cap A) \) is \((\epsilon, n)\)-separated. We have a contradiction with optimality of \( A \).

For an arbitrary map \( f \) the validity of the inequalities \( d_n(x, \alpha_n(x)) < \epsilon, n \in \mathbb{N} \), does not imply that \( d(x, \alpha_n(x)) \to 0 \). For example, for distal dynamical system it is not true. As a corollary, we have an unpleasant fact that the functional \( I_\epsilon \) and the corresponding measure \( \mu \) may depend on the choice of optimal sets. In the next section we introduce a class of maps for which it is not so.

## 4 Measures for \( \epsilon \)-expansive maps

We begin with the following definition.

**Definition 5.** (i) We say that the map \( f \) is \( \epsilon \)-expansive if for any \( \delta > 0 \) and any pair \( x, y \in X \setminus \mathcal{D}, x \neq y \), there exists \( N = N(x, y, \delta) \) such that the inequality \( d_n(x, y) \leq \epsilon, n \geq N \), implies that \( d(x, y) \leq \delta \).

(ii) The map \( f \) is uniformly \( \epsilon \)-expansive if there exists a sequence of non-negative numbers \( \delta_n \to 0 \) as \( n \to \infty \) such that for any pair \( x, y \in X \setminus \mathcal{D} \) with \( d_n(x, y) \leq \epsilon \) one has \( d(x, y) \leq \delta_n, n = 1, 2, \ldots \).

**Lemma 1.** A continuous \( \epsilon \)-expansive map \((S = \emptyset)\) is uniformly \( \epsilon \)-expansive.

**Proof.** Assume that it is not true, i.e. there exists a sequence \( n_k \to \infty \) as \( k \to \infty \) and a sequence of pairs \( x_k \neq y_k \) such that \( d_{n_k}(x_k, y_k) \leq \epsilon \), \( d(x_k, y_k) > \beta > 0 \). Since \( X \) is compact, then without loss of generality one may assume that there exist \( \lim_{k \to \infty} x_k = x_0, \lim_{k \to \infty} y_k = y_0 \) (in the metric \( d \)) and \( d(x_0, y_0) \geq \beta \). Since \( d_{n_k}(x_k, y_k) \leq \epsilon \), then \( d(x_k, y_k) \leq \epsilon \) and \( d(x_0, y_0) \leq \epsilon \). Also, \( d(f(x_k), f(y_k)) \leq \epsilon, \) so, \( d(f(x_0), f(y_0)) \leq \epsilon, \) because of the continuity of \( f \). In the same way, one may show that \( d(f^{m-1}(x_0), f^{m-1}(y_0)) \leq \epsilon \) (if one chooses \( n_k > m \)), thus \( d_m(x_0, y_0) \leq \epsilon \) for any \( m \in \mathbb{N} \). Since \( f \) is \( \epsilon \)-expansive \( x_0 = y_0 \), a contradiction.

For uniformly \( \epsilon \)-expansive maps the following fact takes place.

**Proposition 4.** If \( f \) is uniformly \( \epsilon \)-expansive, \( y, z \in f^{-1}x, x \in X \setminus \mathcal{D}, \) and \( d(z, y) \leq \epsilon \) then \( z = y \).

**Proof.** Since \( fz = fy \) then \( d_k(z, y) \leq \epsilon \) for every \( k \in \mathbb{N} \). Hence, \( d(z, y) \leq \delta_k, \) i.e. \( z = y \).

**Theorem 1.** If \( f \) is uniformly \( \epsilon \)-expansive then the functional \( I_\epsilon \) (and the corresponding measure) is independent of the choice of optimal sets \( A_{\epsilon, n} \).
Proof. Let $A_{\epsilon,n}, B_{\epsilon,n}$ be optimal $(\epsilon,n)$-separated sets, $n \in \mathbb{N}$. Because of Proposition 3 there exists a bijection $\alpha_n : A_{\epsilon,n} \rightarrow B_{\epsilon,n}$ such that $d_n(x, \alpha_n(x)) < \epsilon$. It implies the existence of $\delta_n \geq 0$ such that $d(x, \alpha_n(x)) \leq \delta_n$. Thus,

$$\frac{1}{C_{\epsilon,n}} \left| \sum_{x \in A_{\epsilon,n}} \phi(x) - \sum_{x \in B_{\epsilon,n}} \phi(x) \right| \leq \frac{1}{C_{\epsilon,n}} \sum_{x \in A_{\epsilon,n}} \left| \phi(x) - \phi(\alpha_n(x)) \right| \leq \omega_{\delta_n}(\phi),$$

where

$$\omega_{\delta_n}(\varphi) = \sup_{d(x,y) \leq \delta_n} | \varphi(x) - \varphi(y) |,$$

the modulus of continuity of $\phi$.

Since $X$ is compact and $\phi$ is continuous, then $\omega_{\delta_n}(\phi) \rightarrow 0$ as $\delta_n \rightarrow 0$.

\[\square\]

5 Non-invariance of the measures

As it was mentioned in Introduction, we have studied in II behavior of $C_{\epsilon,n}$ as $\epsilon \rightarrow 0$. In particular, we proved that for any sequence $\epsilon_k \rightarrow 0$, $k \rightarrow \infty$, there exists a regular Borel measure corresponding to the functional $I(\cdot) = \lim_{\mathcal{F}} I_{\delta_n,1}$, where $\mathcal{F}$ is a nonproper ultrafilter. We call here such measures the $\epsilon$-measures. The measures constructed in Section 4 will be called the $n$-measures. Neither $\epsilon$-measures nor $n$-measures are not obliged to be $f$-invariant. The following example shows that it is really so. In the example $n$-measure will coincide with $\epsilon$-measure.

The item (ii) of Definition 5 could be rewritten as "$d(x,y) > \delta_n$ implies $d_n(x,y) > \epsilon$. It means that any $(\delta_n, 1)$-separated set is an $(\epsilon, n)$-separated set. So, suppose that $f$ is continuous and satisfies the following stronger condition:

(ii*): There exists a sequence of non-negative numbers $\delta_n \rightarrow 0$ as $n \rightarrow \infty$ such that for any pair $x, y \in X$ one has $d_n(x,y) \leq \epsilon$ if and only if $d(x,y) \leq \delta_n$, $n = 1, 2, \ldots$.

In this case set $A$ is $(\epsilon, n)$-separated if and only if it is $(\delta_n, 1)$-separated and, $A$ is an optimal $(\epsilon, n)$-separated set if and only if it is an optimal $(\delta_n, 1)$-separated set. So, the $\epsilon$-measure, corresponding to $\lim_{\mathcal{F}} I_{\delta_n,1}$ equals the $n$-measure, corresponding to $\lim_{\mathcal{F}} I_{\epsilon,n}$.

It is easy to check that any symbolic dynamical system with finite alphabet satisfies (ii*), but the corresponding $\epsilon$-measure, constructed in II is not shift-invariant. Let us describe an example.

Example. Let $X = \Omega_M$ be a topological Markov chain, defined by a finite matrix $M : \{0,1, \ldots, p-1\}^2 \rightarrow \{0,1\}$, i.e. $\Omega_M = \{(x_0, x_1, \ldots) \mid x_i \in \{1, 2, \ldots, p-1\} \}$ and $M(x_i, x_{i+1}) = 1$. The set $\Omega_M$ is endowed with the metric $d(x,y) = 2^{-n}$, where $n = \min\{i \mid x_i \neq y_i\}$, and map $f$ is the shift: $f(x_0, x_1, x_2, \ldots) = (x_1, x_2, \ldots)$. The map $f$ satisfies (ii*) with $\delta_n = 2^{-n+1}\epsilon$. So, here $n$-measure and $\epsilon$-measures coincide and are given by the following proposition (see II).
Proposition 5. Let $M$ be a primitive matrix and $C \subset \Omega_M$ is an admissible cylinder of length $n$, ending by $i$. Then $\mu(C) = \lambda^{-n} \epsilon_i$, where $(\epsilon_0, \epsilon_1, ..., \epsilon_{p-1})$ is the positive eigenvector of $M$, with $\epsilon_0 + \epsilon_1 + ... \epsilon_{p-1} = 1$, and $\lambda > 1$ is the corresponding eigenvalue.

If $\lambda$ is not an integer such a measure cannot be invariant.

Other properties of the measures we want to discuss here are related to the group $GI(X)$ of isometries of $X$.

Definition 6. We will say that $f$ commute with a group $G$ of transformations of $X$ iff for any $g \in G$ there exists $h \in G$ such that $f \circ g = h \circ f$.

Example. The map $f : x \rightarrow 2x \mod 1$, of the circle commutes with the group of rotations $x \rightarrow x + \omega, \mod 1$.

One can check that if $f$ commutes with the group $GI(X)$ of isometries of $X$, then elements of $GI(X)$ are isometries for $d_n(\cdot, \cdot)$, and the following proposition is true:

Proposition 6. If $f$ commutes with $GI(X)$ then the corresponding $n$-measure is $GI(X)$ invariant.

It follows that in the example above the $n$-measure is just the Lebesgue measure.

Let us present an easy example where the $\epsilon$-measure is different from $n$-measure. The example in some sense is artificial but it shows that if the expansivity of a map is different at different points, then the $n$-measure may be different from the $\epsilon$-measure. Let $X = \{0,1\} \times [0,1)$ (disjoint union of to unit intervals, considered as a circles, $d((i,x),(i,y)) = \min\{|x-y|,|x-y|-1/2\}$, $d((0,x),(1,y)) = 1$. Let $f(0,x) = (0, 2x \mod 1)$, $f(1,x) = (1, 3x \mod 1)$. Then the $\epsilon$-measure is a Lebesgue measure, such that $\mu(\{0\} \times [0,1)) = \mu(\{1\} \times [0,1)) = 1/2$, for the $\epsilon$-measure of the circle is a Lebesgue measure and $C_\epsilon(\{0\} \times [0,1)) = C_\epsilon(\{1\} \times [0,1))$ But the $n$-measure $\mu(\{0\} \times [0,1)) = 0$, since $C_{\epsilon,n}(\{0\} \times [0,1)) = C_{\epsilon,n}(\{1\} \times [0,1)) \rightarrow 0$ as $n \rightarrow \infty$.

We think that generally for dynamical systems with positive topological entropy, $n$-measures are not invariant. But for systems with zero entropy, they may be invariant in a general enough situation.

6 Invariance of the measures

For many subexponential functions $C_{\epsilon,n}$, the following equality holds

$$\lim_{n \rightarrow \infty} \frac{C_{\epsilon,n} - C_{\epsilon,n-1}}{C_{\epsilon,n}} = 0.$$ (3)

Remark 1. It follows from the definition of the topological entropy that the equality (3) is not satisfied if the topological entropy $h_{top}(f|\mathcal{X} \setminus \mathcal{D}) > 0$ and $\epsilon$ is small enough.
In fact, (3) could not be satisfied even if $h_{top} = 0$. Suppose that $C_{\epsilon,n} = 2^{[\sqrt{n}]}$, where $[\cdot]$ means the integer part of the number. For this sequence

$$\frac{C_{\epsilon,n} - C_{\epsilon,n-1}}{C_{\epsilon,n}} = \begin{cases} \frac{1}{2}, & \text{if } n = m^2 \\ 0, & \text{if } n \text{ is not a full square} \end{cases}$$

So, for subexponential functions $C_{\epsilon,n}$ limit (3) could not exist. But, for any subexponential $C_{\epsilon,n}$ there exists the lower limit:

$$\liminf_{n \to \infty} \frac{C_{\epsilon,n} - C_{\epsilon,n-1}}{C_{\epsilon,n}} = 0,$$

(it equals 0 since if the lower limit $> 0$ then $C_{\epsilon,n}$ grows exponentially, the contradiction). It implies that there exists an ultrafilter such that the corresponding limit with respect to this ultrafilter is 0. So, we replace (3) by the following more weak assumption:

$$\lim_{\mathcal{F}} \frac{1}{C_{\epsilon,n}} (C_{\epsilon,n} - C_{\epsilon,n-1}) = 0,$$

where $\mathcal{F}$ is a non-proper ultrafilter.

The assumption (4) imply

**Proposition 7.** Let

$$\{ n \mid C_{\epsilon,n-1} \leq b_n \leq C_{\epsilon,n}, \} \in \mathcal{F}.$$  

Then

$$b_n = C_{\epsilon,n}(1 - q_n) \quad (5)$$

where $\lim_{\mathcal{F}} q_n = 0$.

**Proof.** Defining $q_n$ as $q_n := 1 - \frac{b_n}{C_{\epsilon,n}}$ we need to show only that $\lim_{\mathcal{F}} q_n = 0$. Assume not, i.e. $\lim_{\mathcal{F}} q_n = \rho > 0$. Then

$$C_{\epsilon,n} - b_n = q_n C_{\epsilon,n} = C_{\epsilon,n}(\rho + \xi_n)$$

with $\lim_{\mathcal{F}} \xi_n = 0$. Thus,

$$b_n = C_{\epsilon,n} - q_n C_{\epsilon,n} = C_{\epsilon,n}(1 - \rho - \xi_n) \geq C_{\epsilon,n-1}.$$

Hence,

$$\frac{C_{\epsilon,n} - C_{\epsilon,n-1}}{C_{\epsilon,n}} \geq \rho + \xi_n, \quad (6)$$

a contradiction with (4). \qed
Corollary 7.1. In particular if
\[ \{ n \mid C_{e,n-1} = C_{e,n}(1-q_n) \} \in \mathcal{F}. \]
then \( \lim_{\mathcal{F}} q_n = 0. \)

From now on we assume that \( f \) is uniformly \( \epsilon \)-expansive.

Proposition 8. If \( A_{e,n-1} \) is \((\epsilon, n-1)\)-separated then \( f^{-1}A_{e,n-1} \) is \((\epsilon, n)\)-separated, \( n = 2, 3, \ldots \)

Proof. If \( y, z \in f^{-1}x, x \in A_{e,n-1}, y \neq z \), then, because of Proposition 4, \( d(y, z) > \epsilon \), and \( y \) and \( z \) are \((\epsilon, n)\)-separated.

If \( y \) and \( z \) belong to \( f^{-1}A_{e,n-1} \) and \( fy \neq fz \) then they are \((\epsilon, n)\)-separated, since \( fy \) and \( fz \) are \((\epsilon, n-1)\)-separated.

\( \square \)

Corollary 8.1. Proposition implies that \( |f^{-1}A_{e,n-1}| \leq C_{e,n} \).

Let us repeat that since \( f \) is uniformly \( \epsilon \)-expansive, the sequence \( \delta_n \) is defined.

We restrict our attention now to a class of maps that could be discontinuous but possess a large amount of continuity.

Definition 7. We say that \( f : X \setminus S \to X \) is almost uniformly continuous if there exist \( \delta_0 > 0 \) such that for every \( 0 < \delta < \delta_0 \) and \( 0 < \sigma < \delta \) the modulus of continuity
\[ \omega_{\sigma}(f \mid X \setminus O_{\delta}(S)) \leq \eta(\sigma) \]
where the function \( \eta(\sigma) \) is independent of \( \delta \) and goes to 0 as \( \sigma \to 0 \).

In other words
\[ d(fx, fy) \leq \eta(\sigma) \]
if \( d(x, S) \geq \delta, d(y, S) \geq \delta \) and \( d(x, y) \leq \sigma \).

As an example, one may consider a smooth map \( f \) on a subset \( X \subset \mathbb{R}^n \) for which \( \sup_{x \in X \setminus D} ||Df(x)|| < \infty \).

The main result of this section is the following theorem.

Theorem 2. Let \( f : X \setminus S \to X \) be an almost uniformly continuous, uniformly \( \epsilon \)-expansive map and \( \mu \) be the measure related to the \((\epsilon, n)\)-complexity corresponding to the ultrafilter \( \mathcal{F} \) satisfying the equation (4). Suppose, that \( \mu(S) = 0 \) then \( \mu \) is \( f \)-invariant.

Proof. If is enough to show that \( I_\epsilon(\varphi) = I_\epsilon(\varphi \circ f) \) for every \( \varphi \in C(X) \) where
\[ I_\epsilon(\varphi \circ f) = \lim_{\mathcal{F}} \frac{1}{C_{e,n}} \sum_{x \in A_{e,n}} \varphi(fx), \quad A_{e,n} \subset X \setminus D. \]
Given an \((\epsilon, n)\)-optimal \(A_{\epsilon,n}\), let \(A_{\epsilon,n-1}\) be an arbitrary \((\epsilon, n - 1)\)-optimal set. Then

\[
\frac{1}{C_{\epsilon,n}} \left| \sum_{x \in A_{\epsilon,n}} \varphi(x) - \sum_{x \in A_{\epsilon,n}} \varphi(f(x)) \right| \leq \\
\frac{1}{C_{\epsilon,n}} \left\{ \left| \sum_{x \in A_{\epsilon,n}} \varphi(x) - \sum_{x \in A_{\epsilon,n-1}} \varphi(x) \right| + \left| \sum_{x \in A_{\epsilon,n-1}} \varphi(x) - \sum_{x \in A_{\epsilon,n}} \varphi(f(x)) \right| \right\}
\]

The first sum. The set \(A_{\epsilon,n-1}\) is \((\epsilon, n - 1)\)-separated, therefore it is \((\epsilon, n)\)-separated. Proposition 3 implies that there exists an injection \(\alpha_n : A_{\epsilon,n-1} \to A_{\epsilon,n}\) such that \(d_n(x, \alpha_n(x)) \leq \epsilon\), and because of the uniform \(\epsilon\)-expansiveness of \(f\), \(d(x, \alpha_n(x)) \leq \delta_n\) for any \(x \in A_{\epsilon,n-1}\). Thus,

\[
\Delta^{(1)} := \frac{1}{C_{\epsilon,n}} \left| \sum_{x \in A_{\epsilon,n}} \varphi(x) - \sum_{x \in A_{\epsilon,n}} \varphi(x) \right| \leq \\
\frac{1}{C_{\epsilon,n}} \left\{ \sum_{x \in A_{\epsilon,n-1}} \left| \varphi(x) - \varphi(\alpha_n(x)) \right| + \sum_{x \in A_{\epsilon,n} \setminus \alpha_n(A_{\epsilon,n-1})} \left| \varphi(x) \right| \right\} \leq (8)
\]

where \(\omega_{\delta_n}(\varphi)\) is the modulus of continuity of \(\varphi\). Since \(\varphi\) is continuous, \(\omega_{\delta_n}(\varphi) \to 0\) as \(\delta_n \to 0\).

The second sum. We use the identity \(A_{\epsilon,n-1} = f(f^{-1}A_{\epsilon,n-1})\) and the following representation

\[
f^{-1}A_{\epsilon,n-1} = A^{(1)}_{\epsilon,n-1} \cup A^{(2)}_{\epsilon,n-1}
\]

where \(f(A^{(1)}_{\epsilon,n-1}) = A_{\epsilon,n-1}, \ |A^{(1)}_{\epsilon,n-1}| = \ |A_{\epsilon,n-1}| = C_{\epsilon,n-1}\) and \(A^{(2)}_{\epsilon,n-1} = f^{-1}A_{\epsilon,n-1} \setminus A^{(1)}_{\epsilon,n-1}\), so \(A^{(2)}_{\epsilon,n-1} = f^{-1}A_{\epsilon,n-1} \setminus C_{\epsilon,n-1}\). Because of Proposition \(f^{-1}A_{\epsilon,n-1}\) is \((\epsilon, n)\)-separated, therefore, \(|f^{-1}A_{\epsilon,n-1}| \leq C_{\epsilon,n}\) and \(|A^{(2)}_{\epsilon,n-1}| \leq C_{\epsilon,n} - C_{\epsilon,n-1}\). Now,

\[
\sum_{x \in A_{\epsilon,n-1}} \varphi(x) = \sum_{x \in A^{(1)}_{\epsilon,n-1}} \varphi(x) = \sum_{x \in f^{-1}A_{\epsilon,n-1}} \varphi(x) - \sum_{x \in A^{(2)}_{\epsilon,n-1}} \varphi(f(x)),
\]

hence,

\[
\Delta^{(2)} := \frac{1}{C_{\epsilon,n}} \left| \sum_{x \in A_{\epsilon,n-1}} \varphi(x) - \sum_{x \in A_{\epsilon,n}} \varphi(f(x)) \right| \leq \\
\frac{1}{C_{\epsilon,n}} \left\{ \left| \sum_{x \in f^{-1}A_{\epsilon,n-1}} \varphi(f(x)) - \sum_{x \in A_{\epsilon,n}} \varphi(f(x)) \right| + \left| \sum_{x \in A^{(2)}_{\epsilon,n-1}} \varphi(f(x)) \right| \right\}.
\]
Since $f^{-1}A_{ε,n-1}$ is $(ε, n)$-separated, there exists an injection $β_n : f^{-1}A_{ε,n-1} → A_{ε,n}$ such that $d_n(β_n(x), x) ≤ ε$, i.e. $d(x, β_n(x)) ≤ δ_n$ for any $x ∈ f^{-1}A_{ε,n-1}$. Therefore,

$$\Delta^{(2)} ≤ \frac{1}{C_{ε,n}} \left\{ \sum_{x ∈ f^{-1}A_{ε,n-1}} |φ(x) - φ(β_n(x))| + \left| \sum_{x ∈ A_{ε,n} \setminus β_n(f^{-1}A_{ε,n-1})} φ(x) \right| + \left| \sum_{x ∈ A_{ε,n} \setminus β_n(f^{-1}A_{ε,n-1})} φ(x) \right| \right\}. $$

Since $|A_{ε,n} \setminus β_n(f^{-1}A_{ε,n-1})| ≤ C_{ε,n} - C_{ε,n-1}$, we obtain

$$\Delta^{(2)} ≤ \frac{1}{C_{ε,n}} \sum_{x ∈ f^{-1}A_{ε,n-1}} |φ(x) - φ(β_n(x))| + 2 \frac{C_{ε,n} - C_{ε,n-1}}{C_{ε,n}} \cdot ||φ||. $$

Because of the almost uniform continuity of $f$, we know that if $x, y ∈ X \setminus O_{2δ_n}(S)$ and $d(x, y) ≤ δ_n$, then $d(f(x), f(y)) ≤ η(δ_n)$. Therefore

$$\sum_{x ∈ f^{-1}A_{ε,n-1} \setminus O_{2δ_n}(S)} |φ(x) - φ(β_n(x))| ≤ \sum_{x ∈ f^{-1}A_{ε,n-1} \setminus O_{2δ_n}(S)} |φ(x) - φ(β_n(x))| + 2 |f^{-1}A_{ε,n-1} \cap O_{2δ_n}(S)| \cdot ||φ|| ≤ η(δ_n) |f^{-1}A_{ε,n-1} \setminus O_{2δ_n}(S)| + 2 |A_{ε,n} \cap O_{2δ_n}(S)| \cdot ||φ|| ≤ C_{ε,n} \cdot η(δ_n) + 2 |A_{ε,n} \cap O_{2δ_n}(S)| \cdot ||φ||$$

and finally, we obtain

$$\Delta^{(2)} ≤ η(δ_n) + 2 ||φ|| \cdot \frac{|A_{ε,n} \cap O_{2δ_n}(S)|}{C_{ε,n}} + 2 ||φ|| \cdot \frac{C_{ε,n} - C_{ε,n-1}}{C_{ε,n}}. $$

Thus,

$$\Delta^{(1)} + \Delta^{(2)} ≤ ω_{δ_n}(φ) + η(δ_n) + 2 ||φ|| \frac{|A_{ε,n} \cap O_{2δ_n}(S)|}{C_{ε,n}} + 3 ||φ|| \frac{C_{ε,n} - C_{ε,n-1}}{C_{ε,n}}$$

and

$$\lim_{x} \left( \Delta^{(1)} + \Delta^{(2)} \right) = 0.$$

(the third term goes to 0 because of Proposition 2)
7 Interval exchange transformation

Dynamical systems generated by interval exchange transformations are basic ones among systems with zero topological entropy. They possess some amount of instability (generally, they are weak mixing [15]) and it is not difficult to calculate their $(\epsilon, n)$-complexity functions (see below).

An interval exchange transformation on the interval $I = [0, 1]$ can be written as follows

$$x = f(x) \equiv x + c_i \quad \text{for} \quad x \in [a_i, a_{i+1}),$$

$i = 0, 1, \ldots, m - 1$, $a_0 = 0$, $a_m = 1$, where $c_i \neq c_{i+1}$.

The set of discontinuity here $S = \{ x = a_i, i = 1, \ldots, m - 1 \}$. We assume that: (i) the map $f$ is one-to-one; (ii) the set $D = \bigcup_{k=0}^{\infty} f^{-k}S$ is dense in $I$; (iii) the set $D$ does not contain $f$-periodic points. These assumptions imply ([5], [15]) that the only invariant measure is the Lebesgue measure.

**Proposition 9.** Under the assumptions (i)-(iii), there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$ the map $f$ is uniformly $\epsilon$-expansive.

**Proof.** First of all one can find $\epsilon_0 > 0$ satisfying the following condition:

C1. If $d(x, y) \leq \epsilon_0$ and interval $[x, y]$ contains a point of discontinuity of $f$, then $d(f(x), f(y)) > \epsilon_0$.

Now take $0 < \epsilon \leq \epsilon_0$. It is clear that this $\epsilon$ also satisfies the condition C1.

Let $D_n = \bigcup_{k=0}^{n-2} f^{-k}S$, for $n \geq 2$. One can order the set $D_n = \{ r_1, r_2, \ldots, r_{k_n} \}$, $r_1 < r_2 < \ldots < r_{k_n}$.

Take $\delta_1 = \epsilon$ and $\delta_n = 2 \min\{ r_{i+1} - r_i \}$. It is clear that $\delta_n \to 0$ when $n \to \infty$. We have to show only that if $d(x, y) > \delta_n$ then $d_n(x, y) > \epsilon$, or $d(f^i(x), f^i(y)) > \epsilon$ for some $0 \leq i < n$. But $[x, y] \cap D_n \neq \emptyset$.

So, there exists $i$, $0 \leq i < n - 1$ such that $[f^i(x), f^i(y)]$ contains a point of discontinuity of $f$. But then, if $d(f^i(x), f^i(y)) \leq \epsilon$ then $d(f^{i+1}(x), f^{i+1}(y)) > \epsilon$ by property C1.

Thus, measure $\mu$ related to the $(\epsilon, n)$-complexity is independent of the choice of the $(\epsilon, n)$-optimal sets. Moreover, if $\mu(S) = 0$ then it is invariant and, hence, the Lebesgue measure.

In fact, we can calculate the $(\epsilon, n)$-complexity function.

**Proposition 10.** There exists $\epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$ there is $n_0 = n_0(\epsilon)$ such that for $n > n_0$.

$$C_{\epsilon, n} = (m - 1)(n - 1) + 1.$$
Proof. Let $\epsilon_0 = \min\{\min_i (a_{i+1} - a_i), \min_j (c_{j+1} - c_j)\}$. This number is positive. Given $0 < \epsilon < \epsilon_0$, consider the set $A_n = \bigcup_{k=0}^{n-2} f^{-k}S$. Under our assumptions (i) and (ii), $A_n$ does not contain the end points $\{0\}$ and $\{1\}$ and $N_n := |A_n| = (m-1)(n-1)$. We denote by $b_j$, $j = 1, \ldots, N_n$, the points in $A_n$ ordered in such a way that $b_0 := 0 < b_1 < b_2 < \cdots < b_{N_n} < 1 := b_{N_n+1}$. The intervals $I_j = \{b_j \leq x \leq b_{j+1}\}$, $j = 0, \ldots, N_n$ form a partition of $I$. Under the assumption (ii), there exists $n_0 = n_0(\epsilon)$ such that for any $j$, $b_{j+1} - b_j < \epsilon$ for all $n \geq n_0$.

Given $n \geq n_0$, consider an $(\epsilon, n)$-optimal set $A_{\epsilon,n} = \{p_0 < p_1 < \cdots < p_r\}$, $r = C_{\epsilon,n} - 1$. If the pair $(p_i, p_{i+1})$ belong to the same interval $I_j$, then $d(f^k p_i, f^k p_{i+1}) = d(p_i, p_{i+1})$ for $k = 0, 1, \ldots, n-1$, and we have a contradiction. On the other side, if they belong to different intervals then there is $s$, $0 \leq s \leq n - 1$, such that $d(f^s p_i, f^s p_{i+1}) \geq \epsilon_0 > \epsilon$. Thus, $C_{\epsilon,n}$ is equal to the number of different intervals $I_j$, i.e.

$$C_{\epsilon,n} = (m-1)(n-1) + 1.$$ 

\[\square\]

7.1 Appendix I

Now we give some known results and definitions that can be found, for instance, in [6].

Definition 8. A set $F \subset 2^\mathbb{N}$ is said to be a filter over $\mathbb{N}$ iff it satisfies the following conditions:

- If $A \in F$ and $B \in F$, then $A \cap B \in F$,
- If $A \in F$ and $A \subset B$ then $B \in F$,
- $\emptyset \notin F$.

Let $a_n$ be a sequences of real numbers, $a$ is called to be a limit of $a_n$ with respect to a filter $F$, $a = \lim_{F} a_n$, if for any $\epsilon > 0$ one has $\{n \mid |a_n - a| < \epsilon\} \in F$.

From the definition of a filter it follows that $\lim_{F} a_n$ is unique, if exists.

Example Let $F_F = \{A \subseteq \mathbb{N} \mid \mathbb{N} \setminus A \text{ is finite}\}$. $F_F$ is said to be a Frechét filter. One can check that it is, indeed, a filter. A limit with respect to $F_F$ coincides with ordinary limit.

Definition 9. A filter $F$ is called to be ultrafilter iff for any set $A \subseteq \mathbb{N}$ one has $A \in F$ or $\mathbb{N} \setminus A \in F$.

Theorem 3. A bounded sequences has a limit with respect to an ultrafilter. This limit is unique.
Example For \( i \in \mathbb{N} \) let \( \mathcal{F}_i = \{ A \subseteq \mathbb{N} \mid i \in A \} \). It is an ultrafilter. Such an ultrafilter is called proper for \( i \). One can check that \( \lim_{n \to \infty} a_n = a_i \). So, limits with respect to a proper ultrafilter are not interesting.

Proposition 11. An ultrafilter \( \mathcal{F} \) is proper (for some \( i \in \mathbb{N} \)) if and only if it contains a finite set.

This proposition implies that an ultrafilter is non-proper if and only if it is an extension of the Frechét filter \( \mathcal{F}_F \). On the other hand, it follows from the Zorn lemma that any filter can be extended to an ultrafilter.

Proposition 12. There is an ultrafilter \( \mathcal{F} \supseteq \mathcal{F}_F \). Any such an ultrafilter is non-proper.

7.2 Appendix II

The Marriage Lemma of P. Hall (see, for instance, [13]) is formulated as follows.

Lemma 2. For an indexed collections of finite sets \( F_1,F_2,\ldots,F_k \) the following conditions are equivalent:

1. there exists an injective function \( \alpha : \{1,2,\ldots,k\} \to \bigcup_{i=1}^{k} F_i \) such that \( \alpha(i) \in F_i \);
2. For all \( S \subseteq \{1,2,\ldots,k\} \) one has \( |\bigcup_{i \in S} F_i| \geq |S| \).

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