An enhanced velocity-based algorithm for safe implementations of gain-scheduled controllers

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ABSTRACT
This paper presents an enhanced velocity-based algorithm to implement gain-scheduled controllers for nonlinear and parameter-dependent systems. A new scheme including pre- and post-filtering is proposed with the assumption that the time-derivative of the controller input is not available for feedback control. It is shown that the proposed control structure can preserve the input–output properties of the linearised closed-loop system in the neighbourhood of each equilibrium point, avoiding the emergence of the so-called hidden coupling terms. Moreover, it is guaranteed that this implementation will not introduce unobservable or uncontrollable unstable modes, and hence the internal stability will not be affected. A case study dealing with the design of a pitch-axis missile autopilot is carried out and the numerical simulation results confirm the validity of the proposed approach.

1. Introduction
Gain-scheduling techniques have been successfully used for the design and implementation of a great variety of control systems; the main reason being that gain-scheduling control allows leveraging the well-established linear system control techniques developed for decades (Leith & Leithead, 2000a; Rugh & Shamma, 2000). Basically, a classic point-to-point gain-scheduling design consists in first linearising the nonlinear plant around a finite set of operating points capturing the system behaviour over the operating domain. Then, for each operating point, a linear time invariant (LTI) controller is designed to achieve stability and performance requirements of the linearised closed-loop system. Finally, the LTI controllers are interpolated a posteriori along with the operating point evolving with respect to scheduling signals, yielding a linear parameter varying (LPV) controller (Lawrence & Rugh, 1995; Rugh, 1991; Shamma & Athans, 1990). Nevertheless, ad hoc interpolation methods do not provide any stability and performance guarantee for the closed-loop system except at the operating points used in the synthesis. To solve this problem, more elaborated interpolation strategies have been developed, aimed at guaranteeing that the underlying closed-loop LPV system is stable as long as the rate of variation of the operating point remains below a certain upper bound (Stilwell & Rugh, 2000). Modern LPV approaches, such as Lyapunov-based design, can achieve a LPV controller with guaranteed closed-loop stability and adequate level of performance for a given predefined rate of variation of the scheduling variable (Biannic & Apkarian, 1999; Naus, 2009; Vesel & Ilka, 2013; Wu, Yang, Packard, & Becker, 1995). However, it is worth mentioning that when endogenous signals are used as scheduling parameters, the behaviour of the closed-loop system with the original nonlinear plant still needs to be assessed a posteriori.

The present work addresses an issue related to the implementation of LPV gain-scheduled controllers obtained by either of the aforementioned techniques, which arises when endogenous system variables, such as system outputs or state variables, are used as scheduling signals. Specifically, in this case, the gain-scheduled controller is quasi-LPV (Rugh & Shamma, 2000), exhibiting nonlinear dynamics. Consequently, a naive implementation of the nonlinear gain-scheduled controller on the original nonlinear system may lead to the occurrence of extra terms in the linearised dynamics, called hidden coupling terms (Rugh & Shamma, 2000). The hidden coupling terms generally introduce a discrepancy between the controller dynamics used in the design and the actual dynamics of the implemented gain-scheduled controller. To handle such a mismatch, it is necessary to proceed to an adequate implementation of the nonlinear gain-scheduled controller such that the linearisation of the nonlinear closed-loop dynamics exhibits the same input–output properties as the feedback of the linearised plant and the corresponding linear controller used in the synthesis process (Leith, 1999; Leith & Leithead, 1998).
Moreover, the implementation must be safe in the sense that it will preserve the internal stability property, i.e., it does not introduce any unobservable or uncontrollable unstable modes. However, much attention in the work reported in the literature is only devoted to the controller design. Consequently, an inadequate implementation of the gain-scheduled controller may induce severe performance deterioration or even destabilisation of the closed-loop system once applied to the original nonlinear plant (Leith & Leithead, 2000b; Rugh & Shamma, 2000).

A solution to avoid such a pitfall is to integrate the hidden coupling terms issue in the control design process. Indeed, these terms can be cancelled by an appropriate choice of the controller architecture. The conditions for achieving this objective are given by a set of first-order partial differential equations (Lawrence & Rugh, 1995; Nichols, Reichert, & Rugh, 1993; Rugh & Shamma, 2000). One can also resort to the dynamic gain-scheduled (DSG) technique (Yang, Hammoudi, Herrmann, Lowenberg, & Chen, 2012, 2015; Yang, Herrmann, Lowenberg, & Chen, 2010). In this method, a gain-scheduling design is first applied (e.g., classic point-to-point gain-scheduling with a posteriori interpolation, self-scheduling, LPV design based on linear and bilinear matrix inequalities (LMIs and BMIs)) and, then, the gains of the nonlinear gain-scheduled controller are computed via the resolution of a set of partial differential equations. However, these two aforementioned approaches generally lead to complex controller architectures. In sharp contrast, taking advantage of recently developed self-scheduling methods (Do Valle, Menegaldo, & Simões, 2014; Lhachemi, Saussié, & Zhu, 2015; Magni, Le Gorrec, & Chiappa, 1998; Saussié, Saydy, & Akhrif, 2008; Saussié, Saydy, Akhrif, & Bérard, 2011), another solution developed in Lhachemi, Saussié, and Zhu (2016a, 2016b, 2016c) incorporates explicitly the hidden coupling terms in the synthesis process and avoids to resort to complex controller architectures.

The above solutions for integrating the hidden coupling terms directly in the synthesis phase are at the expense of an increased complexity of the synthesis procedure and are not directly applicable to all the classic or modern gain-scheduling techniques. In this case, assuming that the time-derivative of the controller inputs are available for feedback control, one can resort to the velocity-based implementation (Kaminer, Pascoal, Khargonekar, & Coleman, 1995), which is a generic gain-scheduled controller implementation that allows avoiding the occurrence of the hidden coupling terms. However, as the time-derivative of the controller inputs are not readily available in most of the practical applications, it requires to invoke pseudo-derivations. This approach may fail since pseudo-derivation introduces an extra pole in the controller dynamics, interfering in the closed-loop system dynamics. Such an interference may not preserve the input–output properties, and hence it can induce performance degradation or even the destabilisation of the closed-loop system. In this paper, an enhanced velocity-based implementation is developed to tackle the problems related to pseudo-derivations involved in the standard treatment. The main feature of the proposed approach is that it preserves both input–output properties and internal stability of the linearised closed-loop system in the neighbourhood of each equilibrium point. Consequently, it results in a safe implementation without introducing unobservable or uncontrollable unstable modes.

The rest of the paper is organised as follows. Various notations and problem statement are introduced in Section 2. Then, details of the proposed solution and the corresponding theoretical analysis are presented in Section 3. Finally, the efficiency of the proposed approach is demonstrated via a case study on the design of a benchmark pitch-axis missile autopilot in Section 4, followed by some concluding remarks in Section 5.

2. Problem settings

2.1 System model

In this paper, we deal with nonlinear systems of the following state-space form:

\[
\begin{align*}
S : = & \begin{cases}
\dot{x} = f(x, u, w) \\
y = h(x, u, w)
\end{cases}
\end{align*}
\]

where \(x \in \mathbb{R}^n\) is the state vector, \(u \in \mathbb{R}^m\) the control input vector, \(w \in \mathbb{R}^q\) the exogenous input vector and \(y \in \mathbb{R}^p\) the output vector. The vector field \(f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^n\) represents the plant dynamics and the function \(h : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}^p\) generates the system outputs. It is assumed that \(f\) and \(h\) are both of class \(C^1\).

This paper considers the tracking control problem of slow time-varying reference commands. In this setting, \(r \in \mathbb{R}^{p_1}\) denotes the reference signals to track. System output vector \(y\) can then be split as follows: \(y = [y_1^T \quad y_2^T]^T\), where \(y_1 \in \mathbb{R}^{p_1}\) is the vector of output signals that must track the vector \(r\), \(y_2 \in \mathbb{R}^{p_2}\) gathers all extra signals available for feedback and \(p_1 + p_2 = p\). Accordingly, we have \(y_1 = h_1(x, u, w)\) and \(y_2 = h_2(x, u, w)\). The exogenous input vector is also decomposed as follows: \(w = [d^T \quad w_m^T]^T \in \mathbb{R}^q\), where \(d \in \mathbb{R}^{p_1}\) is the vector of non-measurable disturbances (e.g., sensor noise), \(w_m \in \mathbb{R}^{p_2}\) is the vector of measurable exogenous signals (e.g., altitude, airspeed, dynamic pressure in an aeronautical context) and \(q_1 + q_2 = q\).
As the control objective is to make the output signal $y_1$ track the reference $r$, we are interested in equilibrium points such that their trimmed values, respectively, denoted $y_{1,e}$ and $r_e$, coincide. This requirement can be formulated as a constraint $y_{1,e} = r_e \in \Omega$, where $\Omega$ is an open subset of $\mathbb{R}^p$. Furthermore, we consider in this work small disturbances $d$ around their zero nominal value, i.e. the disturbance trim condition is such that $d_e = 0$. Consequently, the following set of equilibrium points is introduced:

$$\mathcal{E}_q := \{(x_e, u_e, w_e) : f(x_e, u_e, w_e) = 0, \quad h_1(x_e, u_e, w_e) = r_e, \quad d_e = 0, \quad r_e \in \Omega\},$$

where $x_e$, $u_e$ and $w_e$ denote, respectively, the trimmed values of the state vector $x$, the input vector $u$ and the exogenous input vector $w$. We assume that $\mathcal{E}_q$ can be parametrised by a vector, called operating point, $\theta$ such that there exists a set of equilibrium points such that their trimmed values, respectively, denoted $y_e$ and $w_{m,e}$, there should be one and only one pair $(x_e, u_e)$ such that for $(x_e, u_e, w_e) \in \mathcal{E}_q$, $h(x_e, u_e, w_e) = y_e$. Consequently, we assume there exists a $C^1$ bijective function $\nu : \Theta \rightarrow \mathcal{E}_q$ such that:

$$(x_e, u_e, w_e) \in \mathcal{E}_q \iff \exists \theta \in \Theta : (x_e, u_e, w_e) = \nu(\theta).$$

Furthermore, it is assumed that the operating point $\theta$ depends uniquely on the measured system output $y_e$ and on the measurable endogenous vector $w_{m,e}$. This means that this assumption directly implies that for any admissible vectors $y_e$ and $w_{m,e}$ there should be one and only one pair $(x_e, u_e)$ such that for $(x_e, u_e, w_e) \in \mathcal{E}_q$, $h(x_e, u_e, w_e) = y_e$. Consequently, we assume there exists a $C^1$ bijective function $\nu : \mathcal{R} \rightarrow \Theta$, where

$$\mathcal{R} = \{(y_e, w_{m,e}) : h(x_e, u_e, w_e) = y_e, (x_e, u_e, w_e) \in \mathcal{E}_q\},$$

such that

$$(x_e, u_e, w_e) = \nu(\theta) \iff \theta = \nu(y_e, w_{m,e}).$$

For synthesis purposes, the system $S$ is linearised at each operating point $\theta \in \Theta$ (i.e. around the equilibrium point $\nu(\theta)$). Denoting, respectively, by $\delta x$, $\delta u$, $\delta w$ and $\delta y$, the deviations of $x$, $u$, $w$ and $y$ from their equilibrium value $x_e$, $u_e$, $w_e$ and $y_e$, the linearisation of $S$ yields

$$S_1(\theta_e) := \begin{bmatrix} \delta x = A^s_1(\theta_e)\delta x + B^u_1(\theta_e)\delta u + B^w_1(\theta_e)\delta w \\ \delta y = C^s_1(\theta_e)\delta x + D^u_1(\theta_e)\delta u + D^w_1(\theta_e)\delta w \end{bmatrix}$$

(2)

where $A^s_1(\theta_e), B^s_1(\theta_e), D^u_1(\theta_e), D^w_1(\theta_e)$ are matrices of suitable dimensions. Thus, the family of linear models associated to the system $S$ over the operating domain $\Theta$ is defined as follows:

$$S_1 := \{S_1(\theta_e) : \theta_e \in \Theta\}.$$

**2.2 Set of linear controllers**

In the gain-scheduling design, the first objective is to synthesise for each linearised model $S_1(\theta_e) \in S_1$ a controller $C_i(\theta_e)$. In this work, it is supposed that the synthesis of the gain-scheduled controller can be performed based on any gain-scheduling synthesis method, e.g. classical point-to-point synthesis or modern LPV synthesis methods.

Let the signal $\delta r$, $\delta y_1$ and $\delta y_2$ be, respectively, the deviations of $r$, $y_1$ and $y_2$ from $r_e$, $y_{1,e}$ and $y_{2,e}$. We consider a linear controller $C_i(\theta_e)$ of the following form (Figure 1):

$$C_i(\theta_e) := \begin{bmatrix} \delta x_1 = \delta r - \delta y_1 \\ \delta x_2 = A^c(x_e, \theta_e)\delta x_1 + A^c(x_e, \theta_e)\delta x_2 + B^c(x_e, \theta_e)\delta r \\ + B^c(x_e, \theta_e)\delta y_1 + B^c(x_e, \theta_e)\delta y_2 \\ \delta u = C^c(x_e, \theta_e)\delta x_1 + C^c(x_e, \theta_e)\delta x_2 + D^c(x_e, \theta_e)\delta r \\ + D^c(x_e, \theta_e)\delta y_1 + D^c(x_e, \theta_e)\delta y_2 \end{bmatrix}$$

(3)

where the controller state vector is composed of an integral component $\delta x_1 \in \mathbb{R}^n$ and a vector $\delta x_2 \in \mathbb{R}^m$; $A^c(x_e, \theta_e)$, $\ldots$, $D^c(x_e, \theta_e)$ are matrices of suitable dimensions with all the entries being $C^1$ functions of the scheduling parameter $\theta \in \Theta$. Then, the family of linear controllers over the operating domain $\Theta$ is defined as follows:

$$C_i := \{C_i(\theta_e) : \theta_e \in \Theta\}.$$

In this framework, the design objective is to tune the gains of the fixed structure controller such that at each operating point $\theta_e \in \Theta$, the closed-loop linear system (Figure 2) composed of $S_1(\theta_e)$ and $C_i(\theta_e)$, denoted $CL(S_1(\theta_e), C_i(\theta_e))$, is asymptotically stable and presents an appropriate level of performance. In the case of LPV
control design, one may also guarantee both stability and performance of the closed-loop LPV system for a predefined rate of variation of the time-varying parameters.

For the following developments, we introduce a state-space representation (4) of the linear controller $C_l(\theta_e)$ defined in (3) with the controller state vector $X_l^T = [\delta x_1^\top \delta x_2^\top]^\top$, the controller input $U_l^T = [\delta r^\top \delta y_1^\top \delta y_2^\top]^\top$ and the controller output $Y_l = \delta u_l$. Note that for notational simplicity, the dependency of controller matrices over the operating point $\theta_e$ has been omitted. Thus, the linearised dynamics of the controller can be expressed as

$$
\begin{align*}
\dot{X}_l &= A_l X_l + B_l U_l, \\
Y_l &= C_l X_l + D_l U_l
\end{align*}
$$

where

$$
A_l = \begin{bmatrix} 0 & 0 \\ A_l^C & A_l^C \end{bmatrix}, \quad B_l = \begin{bmatrix} I_{P_1} & -I_{P_1} & 0 \\ B_l^C & B_l^C & B_l^C \end{bmatrix}, \quad C_l = \begin{bmatrix} C_l^C \\ C_l^C \end{bmatrix}, \quad D_l = \begin{bmatrix} D_l^C & D_l^C & D_l^C \end{bmatrix}.
$$

### 2.3 Problem statement

In the remainder of the paper, we assume that for a given nonlinear system $S$, a family $C_l$ of LTI controllers with the structure given by (3) has been designed over the operating domain such that, at any operating point $\theta_e \in \Theta$, $\mathcal{CL}(S_l(\theta_e), C_l(\theta_e))$ is stable and presents an adequate level of performance. Again, such a family can be obtained by any method of choice (e.g. classic or modern gain-scheduling synthesis methods). The next step aims at finding a (nonlinear) gain-scheduled controller:

$$
C := \begin{cases}
\dot{x}_k = f_k(x_k, y, r, \theta) \\
u = h_k(x_k, y, r, \theta) \\
\theta = \nu(y, w_m)
\end{cases}
$$

that can preserve the input–output property of the linearised closed-loop systems. In other words, the objective is to find a nonlinear controller $C$ such that, once placed in closed-loop with $S$ and linearised in the vicinity of any operating point $\theta_e \in \Theta$, the resulting dynamic model coincides with the one obtained during the synthesis phase based on the interconnection of $S_l(\theta_e)$ and $C_l(\theta_e)$. This is an essential property guaranteeing that the implementation can fully comply with the controller design. In particular, such an implementation allows avoiding the occurrence of the hidden coupling terms, which is the main source of performance degradation and instability once the controller is applied to the original nonlinear plant. In order to formulate this problem, let $\mathcal{CL}(S_l(\theta_e), C_l(\theta_e)) : (\delta r, \delta w) \rightarrow \delta y$ be the closed-loop system corresponding to the interconnection of the linear system $S_l(\theta_e)$ and the linear controller $C_l(\theta_e)$. The corresponding transfer function is denoted by $T_l(S_l(\theta_e), C_l(\theta_e))$. The nonlinear closed-loop system corresponding to the interconnection of the nonlinear system $S$ and the nonlinear controller $C$ is denoted by $\mathcal{CL}(S, C) : (r, w) \rightarrow y$. Let $CL_l(S, C)(\theta_e)$ be the linearisation of $\mathcal{CL}(S, C)$ at the operating point $\theta_e \in \Theta$ and $T_l(S, C)(\theta_e)$ be the corresponding transfer function. The (nonlinear) gain-scheduled controller implementation problem can be formulated as follows.

**Problem 1.** Safe implementation of the gain-scheduled controller: Find a (nonlinear) gain-scheduled controller $C$ such that for each operating point $\theta_e \in \Theta$, the following properties hold:

1. the closed-loop transfer function $T_l(S, C)(\theta_e)$ coincides with $T_l(S_l(\theta_e), C_l(\theta_e))$;
2. if $\mathcal{CL}(S_l(\theta_e), C_l(\theta_e))$ is internally stable (i.e. do not present any unobservable or uncontrollable unstable mode), then so is $\mathcal{CL}_l(S, C)(\theta_e)$.

In Problem 1, requirement (1) aims at finding a nonlinear gain-scheduled controller $C$ that can preserve, once linearised in the vicinity of any operating point $\theta_e \in \Theta$, the linearised input–output properties of the closed-loop system $\mathcal{CL}(S_l(\theta_e), C_l(\theta_e))$. As the strategy for elaborating such a nonlinear controller $C$ might introduce hidden modes, requirement (2) imposes that such hidden modes must be stable, in order to guarantee the internal stability of the closed-loop system.

If a nonlinear gain-scheduled controller $C$ that solves Problem 1 can be found, the stability and the performance of the closed-loop nonlinear system depend significantly on the design of the LTI controllers $C_l(\theta_e)$. However, assuming that the set of LTI controllers $C_l$ is designed such that for any operating point $\theta_e \in \Theta$, $\mathcal{CL}(S_l(\theta_e), C_l(\theta_e))$ is internally stable, the stability of the resulting closed-loop nonlinear system $\mathcal{CL}(S, C)$ can be guaranteed for slow time variations of the reference input $r$ and the exogenous input $w$ (Lawrence & Rugh, 1990; Rugh & Shamma, 2000).

### 2.4 A motivating example

Before presenting a solution to Problem 1, let us consider the following second-order nonlinear system (Khalil & Grizzle, 1996):

$$
\begin{align*}
\dot{x}_1 &= \tan x_1 + x_2 \\
\dot{x}_2 &= x_1 + u \\
y &= x_2
\end{align*}
$$
where $u$ and $y$ are, respectively, the system input and output. It is assumed that the state variables $x_1$ and $x_2$ are all available for feedback. The objective is to make the system output $y$ track the reference signal $r$.

### 2.4.1 Controller synthesis

The system equilibrium point such that $y_e = \theta_e \in \mathbb{R}$ is characterised by

$$
x_{1,e}(\theta_e) = -\tan^{-1}\theta_e, \quad x_{2,e}(\theta_e) = y_e(\theta_e) = \theta_e,
$$

$$
u_e(\theta_e) = \tan^{-1}\theta_e.
$$

Then, the linearisation of the system around this equilibrium point yields

$$
\begin{align*}
\delta x_1 &= (1 + \theta_e^2)\delta x_1 + \delta x_2 \\
\delta x_2 &= \delta x_1 + \delta u \\
\delta y &= \delta x_2
\end{align*}
$$

(7)

where $\delta x_1$, $\delta x_2$, $\delta y$ and $\delta u$ represent deviations of $x_1$, $x_2$, $y$ and $u$ from $x_{1,e}(\theta_e)$, $x_{2,e}(\theta_e)$, $y_e(\theta_e)$ and $u_e(\theta_e)$. Around a given operating point characterised by $\theta_e$, the following linear controller has been proposed in Khalil and Grizzle (1996):

$$
\begin{align*}
\dot{x}_1 &= r - y \\
u &= k_1(y)x_1 + k_1(y)x_1 + k_2(y) y
\end{align*}
$$

(9)

Linearising the controller dynamics yields

$$
\begin{align*}
\delta \dot{x}_1 &= \delta r - \delta y \\
\delta u &= k^*_1(\theta_e) \delta x_1 + k^*_1(\theta_e) \delta x_1 + \delta x_2(\theta_e) \delta y
\end{align*}
$$

(10)

with $k^*_1 = k_1^* = k_1$ and, denoting $x_{1,e}(\theta_e)$ the controller integral component trim condition,

$$
k^*_1(\theta_e) = k_1(\theta_e) + \frac{dk_1}{dy}|_{\theta_e} x_{1,e}(\theta_e) + \frac{dk_1}{dy}|_{\theta_e} x_{1,e}(\theta_e) + \frac{dk_2}{dy}|_{\theta_e} y_e(\theta_e)
$$

$$
= k_1(\theta_e) - 4(1 + \theta_e^2) + \frac{4}{1 + \theta_e^2}
$$

$$
+ 2 \left[ \theta_1(7 + 3 \theta_e^2) + \frac{2 \theta_e^2}{1 + \theta_e^2} \right] \tan^{-1}(\theta). 
$$

Thus, the considered nonlinear controller (9) admits linearised dynamics (10) different from that given in (8) which is the one used in the design. Although scheduled gains have been selected in order to assign the closed-loop eigenvalues at $-1, -1/2 \pm j\sqrt{3}/2$ for all $\theta_e \in \mathbb{R}$, this is not the case for controller (9). In fact, with the naive approach, the closed-loop system is stable for $|\theta_e| < \theta_{lim}$ with $\theta_{lim} \approx 0.417$ and unstable otherwise. Obviously, such a naive implementation cannot preserve the input–output property of the linear closed-loop systems with linear controllers (8).

### 2.4.3 Velocity-based implementation

In order to preserve the input–output property of the linearised closed-loop systems with linear controllers (8), one may resort to the velocity-based implementation (Kaminer et al., 1995). However, such an implementation requires the temporal derivative of the variables used for feedback, i.e. $\dot{r}$, $\dot{y}_1$ and $\dot{y}_2$, which are not readily available in most of the practical applications. To overcome this issue, it has been proposed in Kaminer et al. (1995) to employ pseudo-derivative of the inputs. Denoting by $\tau > 0$ the pseudo-derivative parameter, signals $\dot{r}$, $\dot{y}_1$ and $\dot{y}_2$ are estimated from $r$, $y_1$ and $y_2$ by resorting to the transfer function $s/(\tau s + 1)$. The resulting implementation scheme for the set of LTI controllers $C_l$ is given by (11), as illustrated in Figure 3. Note that in Figure 3, the transfer function $s/(\tau s + 1)$ has been split for comparison purposes with the subsequent
where $\Delta_1$ introduces an undesired pole in the linearised controller.

Comparing (8) and (13) shows that the pseudo-derivative implementation strategy proposed in Section 3. of the inputs.

Velocity-based implementation with pseudo-derivative of inputs (11) to the motivating example leads to the following implementation of the family of LTI controllers given by (8):

$$
\begin{align*}
\dot{x}_f &= -\tau^{-1}x_{r,f} + \tau^{-1}r \\
\dot{x}_{y,f} &= -\tau^{-1}x_{y,f} + \tau^{-1}y_1 \\
\dot{x}_{y_2,f} &= -\tau^{-1}x_{y_2,f} + \tau^{-1}y_2 \\
\dot{v} &= \tau k_1(y)(r - y) + k_1(y)(x_1 - x_{y,f}) \\
\dot{u} &= \tau^{-1}v \\
\theta &= \nu(y, w_m)
\end{align*}
$$

(11)

The application of the velocity-based implementation with pseudo-derivative of inputs (11) to the motivating example leads to the following implementation of the family of LTI controllers given by (8):

$$
\begin{align*}
\dot{x}_{r,f} &= -\tau^{-1}x_{r,f} + \tau^{-1}r \\
\dot{x}_{y,f} &= -\tau^{-1}x_{y,f} + \tau^{-1}y_1 \\
\dot{x}_{y_2,f} &= -\tau^{-1}x_{y_2,f} + \tau^{-1}y_2 \\
v &= \tau k_1(y)(r - y) + k_1(y)(x_1 - x_{y,f}) \\
\dot{u} &= \tau^{-1}v \\
\theta &= \nu(y, w_m)
\end{align*}
$$

(12)

The linearisation of this controller dynamics at the operating point $\theta_e$ yields in the Laplace domain:

$$
\Delta u(s) = \frac{k_1(\theta_e)}{s} (\Delta r(s) - \Delta y(s)) + \frac{k_1(\theta_e)}{\tau s + 1} \Delta x_1(s) + \frac{k_2(\theta_e)}{\tau s + 1} \Delta y(s)
$$

(13)

where $\Delta u(s)$, $\Delta r(s)$, $\Delta x_1(s)$ and $\Delta y(s)$ denote, respectively, the Laplace transforms of $\delta u$, $\delta r$, $\delta x_1$ and $\delta y$. Comparing (8) and (13) shows that the pseudo-derivative introduces an undesired pole in the linearised controller dynamics. It has been proven in Kaminer et al. (1995) and Khalil and Grizzle (1996) that for any frozen operating point, the closed-loop system with this approximation recovers the nominal performances when $\tau$ tends to zero. However, there is no guarantee that this property holds uniformly over the operating domain. Hence, it may be impossible to find a suitable value of $\tau$ over the operating domain. For instance, based on the Routh Hurwitz criterion, a necessary condition for the closed-loop stability with controller (12) is $\tau < 1/(1 + \theta^2_e)$. Therefore, it is impossible to find a unique $\tau > 0$ such that the closed-loop system is stable for all $\theta_e \in \mathbb{R}$. In practice, it may be sufficient to guarantee the stability over a compact operating domain, e.g. $|\theta_e| \leq \theta_{\text{lim}}$. Even in this case, practical difficulties arise when $\tau$ is too close to zero, because the pseudo-derivative amounts to implementing a first-order filter with a pole located in $-1/\tau$. For instance, at the operating point $\theta_e = 10$, the aforementioned necessary condition implies $\tau < 0.01$ for closed-loop stability.

3. Enhanced velocity-based implementation

In this section, assuming that the time-derivative of the controller inputs are not available for feedback control, we introduce an enhanced velocity-based implementation that preserves both input–output properties and internal stability of the linearised closed-loop system in the neighbourhood of each equilibrium point.

3.1 Proposed strategy

Given the set of linearised controllers $C_i$, designed to stabilise the closed-loop system $\mathcal{CL}(S_i(\theta_e), C_i(\theta_e))$ at each operating point $\theta_e \in \Theta$, the following gain-scheduled controller is proposed for $\tau > 0$, as illustrated in Figure 4:

$$
\begin{align*}
\dot{x}_r &= -\tau^{-1}x_{r,f} + \tau^{-1}r \\
\dot{x}_{y,f} &= -\tau^{-1}x_{y,f} + \tau^{-1}y_1 \\
\dot{x}_{y_2,f} &= -\tau^{-1}x_{y_2,f} + \tau^{-1}y_2 \\
v &= \tau k_1(y)(r - y) + k_1(y)(x_1 - x_{y,f}) \\
\dot{u} &= \tau^{-1}v \\
\theta &= \nu(y, w_m)
\end{align*}
$$

C := $\begin{bmatrix}$
\begin{bmatrix}A \quad B^c \quad B^T_c \quad B^u \quad B^w \end{bmatrix} \\
\begin{bmatrix}C \quad D \quad D^c \quad D^u \quad D^w \end{bmatrix}
\end{bmatrix}$$  \\
\begin{bmatrix}1 \quad 0 \quad 0 \quad 0 \quad 0
\end{bmatrix}$

(14)

Note that for simplicity, the dependency of controller matrices over the operating point $\theta$ has been omitted.
in Figure 4. It can be observed that the proposed strategy presents a similar architecture to the velocity-based implementation with pseudo-derivative of the inputs (see Figure 3). Indeed, the state-space matrices of the set of LTI controllers $C_i$ are involved in an identical manner in these two strategies. Moreover, pre-filtering and an integral component at the controller output are also considered. However, as illustrated in the motivating example via the transfer function given in (13), the pseudo-derivative scheme with pre-filtering components introduces an undesired pole at $-1/\tau$. In order to cancel this harmful pole, the controller output, which was generated by an integral component with gain $\tau^{-1}$ (see Figure 3), is augmented to also include a zero component located at $-1/\tau$. This can be observed in Figure 4 because the transfer function between $v$ and $u$ is given by $(\tau s + 1)/(\tau s)$. Finally, in the implementation shown in Figure 3, the error signal $r - y_1$ is not pre-filtered. Thus, the post-filtering component introduces an uncompensated zero at $-1/\tau$. To avoid such a problem, the error signal is also pre-filtered in the proposed strategy. From the controller architecture $C$, we can draw the following additional observations.

- Every controller input signal $z \in \{r, y_1, y_2\}$ is pre-filtered by a low-pass filter $1/(\tau s + 1)$, resulting in signal $x_{zf}$. Thus, the error signal $z - x_{zf}$, which is used for feedback, is composed of only medium- and high-frequency components of the input signal $z$. Note that for a signal $z$ evolving around a frozen equilibrium value $z_e$, one can expect that $x_{zf} \approx z_e$, thus the error signal is such that $z - x_{zf} \approx z - z_e$. This situation recovers the classic implementation of a linear controller around a given equilibrium point where only the deviations of the signal around the equilibrium point are used for feedback.
- The low-frequency signal $x_{zf} - x_{y1f}$ is also used in the feedback for tracking purposes. As detailed in the remainder of the paper, it is used to guarantee that in closed-loop the equilibrium condition $y_{1,e} = r_e$ is satisfied.
- The controller output is generated via post-filtering. At the equilibrium, one can find that the signal $v$ generated by the controller is null in steady state, i.e. $v_e = 0$. Therefore, the adequate controller output signal $u$ in steady state $u$ is generated by post-filtering via an integral component.
- The filtering parameter $\tau$ is involved in both $x_e$-dynamics and $v$-equation as a multiplying factor of both matrices $A_i(\theta)$ and $C_i(\theta)$. As demonstrated in the next subsection, this multiplying factor is required to preserve the input–output properties of the linearised closed-loop system.

### 3.2 Properties of the enhanced velocity-based implementation

In order to establish the properties of the enhanced velocity-based implementation, the following assumptions are made:

(A1) for any operating point $\theta_e \in \Theta$, the matrix

$$\begin{bmatrix} A_iC_i(\theta_e) & A_iC_i(\theta_e) \\ C_i(\theta_e) & C_i(\theta_e) \end{bmatrix} \in \mathbb{R}^{(n+p) \times (n+p)}$$

is full column rank;

(A2) the dimension of the control input $u$ coincides with the one of the reference signal $r$, i.e. $m = p_1$;

(A3) for any operating point $\theta_e \in \Theta$, the state-space representation of the linear controller $(A_i(\theta_e), B_i(\theta_e), C_i(\theta_e), D_i(\theta_e))$ introduced in (4) is stabilisable and detectable.

In particular, (A2) implies that for $C_i(\theta)$, the number of integrator channels coincides with the dimension of...
the reference signal \( r \). Assumption (A3) implies that all the hidden modes in controller dynamics correspond to stable poles.

### 3.2.1 Input–output properties

The main result of this section is the following theorem.

**Theorem 3.1:** Suppose that (A1) holds and consider the nonlinear gain-scheduled controller \( C \) given in (14). Then, for any operating point \( \theta \in \Theta \), the closed-loop matrix transfer functions \( T_l(\mathcal{S}, C)(\theta, \cdot) \) and \( T(S_l(\theta), C_l(\theta)) \) coincide.

**Proof:** Let \( \theta \in \Theta \) be a given operating point. The equilibrium point of gain-scheduled controller \( C \), placed in closed-loop with \( \mathcal{S} \), is characterised by the following set of algebraic equations:

\[
\begin{align*}
\begin{cases}
x_{r,f,c} = r_c \\
x_{y_1,f,c} = y_{1,c} \\
x_{y_2,f,c} = y_{2,c} \\
0 = A_C^c(\theta_c)x_{c,e} + \tau A_C^c(\theta_c)[r_c - y_{1,c}] \\
0 = C_C^c(\theta_c)x_{c,e} + \tau C_C^c(\theta_c)[r_c - y_{1,c}] \\
0 = v_c \\
u_c = x_{n,f,c}
\end{cases}
\end{align*}
\]  

(16)

In particular, the fourth and fifth rows of the above system can be rewritten as follows:

\[
\begin{bmatrix}
A_C^c(\theta_c) & A_C^c(\theta_c) \\
C_C^c(\theta_c) & C_C^c(\theta_c)
\end{bmatrix}
\begin{bmatrix}
x_{c,e} \\
\tau[r_c - y_{1,c}]
\end{bmatrix} = 0.
\]  

(17)

Based on (A1), it follows that \( r_c = y_{1,c}, x_{c,e} = 0 \).

To prove that the transfer function resulting from the linearisation of the nonlinear gain-scheduled controller \( C \) at the operating point \( \theta \), coincides with the one resulting from \( C_l(\theta) \) given in (3). To this end, the first step is to compute the linearisation of controller \( C \) given by (14). For the given operating point \( \theta_c \), the deviation of the controller signals \( x_{r,f}, x_{y_1,f}, x_{y_2,f}, x_c, v \) and \( x_{n,f} \) from their equilibrium values specified above are denoted, respectively, by \( \delta x_r, \delta x_{y_1}, \delta x_{y_2}, \delta x_c, \delta v \) and \( \delta x_{n,f} \). The linearisation of the pre-filter and the post-filter gives

\[
\begin{align*}
\delta \dot{x}_r &= -\tau^{-1}\delta x_r + \tau^{-1}\delta r \\
\delta \dot{x}_{y_1} &= -\tau^{-1}\delta x_{y_1} + \tau^{-1}\delta y_1 \\
\delta \dot{x}_{y_2} &= -\tau^{-1}\delta x_{y_2} + \tau^{-1}\delta y_2 \\
\delta \dot{x}_c &= \tau^{-1}\delta v \\
\delta u &= \delta x_{n,f} + \delta v
\end{align*}
\]  

(18)

The linearisation of the \( x_c \)-dynamics and \( v \)-equation, with \( \theta = v(y, w_m) \), is given by

\[
\delta \dot{x}_c = \\
A_C^c(\theta_c)x_c + \frac{\partial}{\partial y}A_C^c(\theta_c)x_c \bigg|_e \delta y + \frac{\partial}{\partial w_m}A_C^c(\theta_c)x_c \bigg|_e \delta w_m \\
+ \tau A_C^c(\theta_c)[\delta x_r - \delta x_{y_1}] + \tau \frac{\partial}{\partial y}A_C^c(\theta_c)[x_r - x_{y_1}] \bigg|_e \delta y \\
+ \frac{\partial}{\partial w_m}A_C^c(\theta_c)[x_r - x_{y_1}] \bigg|_e \delta w_m \\
+ B_C^c(\theta_c)[\delta r - \delta x_{y_1}] + \frac{\partial}{\partial y}B_C^c(\theta_c)[r - x_{y_1}] \bigg|_e \delta y \\
+ \frac{\partial}{\partial w_m}B_C^c(\theta_c)[r - x_{y_1}] \bigg|_e \delta w_m \\
+ B_C^c(\theta_c)[\delta y_1] \bigg|_e \delta y_1 \\
+ \frac{\partial}{\partial y}B_C^c(\theta_c)[y_1 - x_{y_1}] \bigg|_e \delta y_1 \\
+ \frac{\partial}{\partial w_m}B_C^c(\theta_c)[y_1 - x_{y_1}] \bigg|_e \delta w_m \\
- B_C^c(\theta_c)[\delta x_{y_1}] + \frac{\partial}{\partial y}B_C^c(\theta_c)[y_2 - x_{y_2}] \bigg|_e \delta y_1 \\
+ \frac{\partial}{\partial w_m}B_C^c(\theta_c)[y_2 - x_{y_2}] \bigg|_e \delta w_m
\]

(19)

Note that the derivatives are evaluated at the equilibrium point characterised by the frozen operating point \( \theta \). In order to simplify (19) and (20), one can note that for \( z \in \{y, w_m\} \):

\[
\frac{\partial}{\partial z}A_C^c(\theta_c)x_c \bigg|_e = \sum_{k=1}^{n_c} x_{c,k}(k) \frac{\partial}{\partial z}A_C^{c,k}(v(y, w_m)) \bigg|_e = 0
\]

(21)
since $x_{c,e} = 0$. In (21), $x_{c, e}(k)$ denotes the $k$-th element of vector $x_{c,e}$ and $A_{C, \Theta}^c$ denotes the $k$-th column of matrix $A_{C}^c$. Similarly, since $x_{r,c,e} - x_{y_1, f,e} = 0$, $x_{c,e} = 0$, $r_{c,e} - x_{r, f,e} = 0$, $y_{r, e} - x_{y_1, f,e} = 0$, and $y_{2,e} - x_{y_2, f,e} = 0$, all the derivatives involved in (19) and (20) are equal to zero except $(\partial/\partial y_1) [B_{1,C}^{c}(\theta)[y_1 - x_{y_1, f}]]$, $(\partial/\partial y_1) [D_{1,C}^{c}(\theta)[y_2 - x_{y_2, f}]]$, and $(\partial/\partial y_2) [D_{2,C}^{c}(\theta)[y_2 - x_{y_2, f}]]$. The first derivative becomes

$$\frac{\partial}{\partial y_1} [B_{1,C}^{c}(\theta)[y_1 - x_{y_1, f}]] = \frac{\partial}{\partial y_1} \left[ \sum_{k=1}^{p_1} [y_1(k) - x_{y_1, f}(k)]B_{1,C}^{c}(v(y, w_m)) \right]$$

$$= \sum_{k=1}^{p_1} \left\{ \frac{\partial}{\partial y_1} \left[ y_1(k) - x_{y_1, f}(k) \right] B_{1,C}^{c}(\theta, \tau) \right\}$$

$$= \sum_{k=1}^{p_1} \left[ \frac{\partial}{\partial y_1} \left[ y_1(k) - x_{y_1, f}(k) \right] B_{1,C}^{c}(\theta, \tau) \right]$$

Similarly, the three other derivatives become

$$(\partial/\partial y_1) [D_{1,C}^{c}(\theta)[y_2 - x_{y_2, f}]] = D_{1,C}^{c}(\theta),$$

$$(\partial/\partial y_2) [B_{2,C}^{c}(\theta)[y_2 - x_{y_2, f}]] = B_{2,C}^{c}(\theta),$$

and

$$(\partial/\partial y_2) [D_{2,C}^{c}(\theta)[y_2 - x_{y_2, f}]] = D_{2,C}^{c}(\theta).$$

Then, (19) and (20) can be simplified as

$$\begin{bmatrix} \dot{\delta x}_c \newline \dot{\delta x}_y \newline \dot{\delta v} \end{bmatrix} = \begin{bmatrix} A_{C}^{c}(\theta) \newline D_{C}^{c}(\theta) \newline D_{C}^{c}(\theta) \end{bmatrix} \begin{bmatrix} \delta x_c \newline \delta x_y \newline \delta v \end{bmatrix} + \begin{bmatrix} B_{C}^{c}(\theta) \newline B_{C}^{c}(\theta) \newline B_{C}^{c}(\theta) \end{bmatrix} \begin{bmatrix} \delta r - \delta x_r \newline \delta y_1 - \delta x_y \newline \delta y_1 - \delta y_2 \end{bmatrix}$$

where $\Delta u(s), \Delta v(s), \Delta \nu(s), \Delta y_1(s), \Delta y_2(s), \Delta x_c(s), \Delta x_y(s)$, and $\Delta y_2(s)$ denote, respectively, the Laplace transforms of $\delta u$, $\delta v$, $\delta \nu$, $\delta y_1$, $\delta y_2$, $\delta x_c$, $\delta x_y$, and $\delta y_2$. Thus, it yields

$$\Delta r(s) - \Delta x_c(s) = \frac{\tau s}{\tau s + 1} \Delta r(s),$$

$$\Delta y_1(s) - \Delta x_y(s) = \frac{\tau s}{\tau s + 1} \Delta y_1(s),$$

$$\Delta y_2(s) - \Delta x_y(s) = \frac{\tau s}{\tau s + 1} \Delta y_2(s),$$

$$\Delta x_c(s) - \Delta x_y(s) = \frac{1}{\tau s + 1} (\Delta r(s) - \Delta y_1(s)).$$

Finally, based on these transfer functions and by taking the Laplace transformation of (22), the linearised dynamics of the gain-scheduled controller $C$ can be expressed by the following transfer function:

$$\Delta u(s) = \frac{1}{s} \left[ C_{C}^{c}(\theta) + C_{C}^{c}(\theta)(s I - A_{C}^{c}(\theta))^{-1} A_{C}^{c}(\theta) \right]$$

$$\times (\Delta r(s) - \Delta y_1(s)) + \left[ D_{C}^{c}(\theta) + C_{C}^{c}(\theta)(s I - A_{C}^{c}(\theta))^{-1} B_{C}^{c}(\theta) \right] \Delta r(s)$$

$$+ \left[ D_{C}^{c}(\theta) + C_{C}^{c}(\theta)(s I - A_{C}^{c}(\theta))^{-1} B_{C}^{c}(\theta) \right] \Delta y_1(s)$$

$$+ \left[ D_{C}^{c}(\theta) + C_{C}^{c}(\theta)(s I - A_{C}^{c}(\theta))^{-1} B_{C}^{c}(\theta) \right] \Delta y_2(s)$$

$$(23)$$

To conclude the proof, it is sufficient to note that, based on the linearised dynamics given by (3), the transfer function of $C_{C}(\theta)$ is identical to the one given by (23). [Q.E.D.]

3.2.2 Internal stability

As shown in Theorem 3.1, the enhanced velocity-based implementation (14) can preserve the input-output properties of the set of linear controllers $C_{\theta}$ since for any $\theta \in \Theta$, $T_{C}(S, C_{\theta}) = T(S(\theta), C_{\theta}(\theta))$. However, it introduces a hidden mode as the pre-filtering pole $-1/\tau$ does not appear in the linearised controller dynamics transfer function. The presence of hidden modes requires the analysis of the internal stability of the closed-loop system. For this purpose, we consider the state-space representation of the linear controller $C_{C}(\theta)$ given in (4) and of the linearised dynamics of controller $C$ which is, based on (18) and (22), given by

$$\begin{bmatrix} \dot{X}_a \newline Y_a \end{bmatrix} = \begin{bmatrix} A_{a} X_a + B_{a} U_a \newline C_{a} X_a + D_{a} U_a \end{bmatrix}$$

where $X_a = \begin{bmatrix} \delta x_c^T \newline \delta x_y^T \newline \delta y_1^T \newline \delta y_2^T \end{bmatrix}$ is the controller state vector, $U_a = \begin{bmatrix} \delta r^T \newline \delta y_1^T \newline \delta y_2^T \end{bmatrix}$ the controller input, $Y_a = \delta u$ the controller output,
From the two last equations of (26), we obtain \((1/\tau + \lambda)x_5 = 0\), which implies \(x_5 = 0\). Thus, it yields

\[
\begin{align*}
A_c^c x_4 &= \lambda x_4 \\
C_c^c x_4 &= 0
\end{align*}
\]  

(27)

As \(X\) is non-zero, it implies that \(x_4\) is non-zero since \(x_1, x_2, x_3\) and \(x_5\) are zero vectors. Consequently, \(\lambda\) is an unobservable mode of \((A_c^c, C_c^c)\). Conversely, if \(\lambda\) is an unobservable mode of \((A_c^c, C_c^c)\), it is sufficient to consider \(x_1 = 0, x_2 = 0, x_3 = 0, x_5 = 0\) and \(x_4 \neq 0\) such that \(A_c^c x_4 = \lambda x_4\) and \(C_c^c x_4 = 0\) to show that \(\lambda\) is an unobservable mode of \((A_c, C_c)\).

It remains now to show that \(\lambda\) is an unobservable mode of \((A_c^c, C_c^c)\) if and only if it is an unobservable mode of \((A_i, C_i)\). We first consider the case \(\lambda \neq 0\). If there exists a non-zero complex vector \(x\) such that \(A_i x = \lambda x\) and \(C_i x = 0\), then \(X = [x^T \ x_i^T]^T\) is a non-zero vector such that \(A_i X = \lambda X\) and \(C_i X = 0\). Conversely, assume that there exists a non-zero complex vector \(X = [x^T \ x_i^T]^T\) such that \(A_i X = \lambda X\) and \(C_i X = 0\). The first equality implies \(x_1 = 0\). As \(\lambda \neq 0\), we have \(x_1 = 0\), which yields \(x_2 \neq 0, A_i^c x_2 = \lambda x_2\) and \(C_i^c x_2 = 0\).

In the case \(\lambda = 0\), \(A_i X = \lambda X = 0\) and \(C_i X = 0\) imply

\[
\begin{bmatrix}
A_i^c \\
C_i^c
\end{bmatrix}
\begin{bmatrix}
x_4 \\
x_i^T
\end{bmatrix} = 0.
\]  

Based on (A1), we have \(X = 0\), and hence \(\lambda = 0\) cannot be an unobservable mode of \((A_i, C_i)\). Similarly, \(\lambda = 0\) cannot be an unobservable mode of \((A_c^c, C_c^c)\) since based on (A1), \(A_i^c x = \lambda x = 0\) and \(C_i^c x = 0\) imply \(x = 0\). ■

**Lemma 3.2:** For any \(\lambda \in \mathbb{C}^*\), \(\lambda\) is an uncontrollable mode of \((A_a, B_a)\) if and only if it is an uncontrollable mode of \((A_i, B_i)\).

**Proof:** Let \(X = [x^T \ x_i^T \ x_3^T \ x_4^T \ x_5^T]^T\) be a non-zero complex vector with suitable dimensions such that \(X^T A_a = \lambda X^T\) and \(X^T B_a = 0\). After some algebra, it is equivalent to the following system:

\[
\begin{align*}
\frac{1}{\lambda} x_4^T A_i^c \cdot x_4^T \\
\frac{1}{\lambda} x_4^T A_i^c \cdot x_i^T
\end{align*}
\]  

\[
\begin{bmatrix}
x_i^T \\
x_2 \ x_3 \ x_5
\end{bmatrix} = 0
\]  

(28)

To conclude that \(\lambda\) is an uncontrollable mode of \((A_i, B_i)\), it is sufficient to note that \(X \neq 0\) if and only
if \( x_4 \neq 0 \), i.e. if and only if \([1/\lambda x_4^TA_i, x_4^T]\) \( \neq 0 \). Conversely, let \( Z = [z_1^T, z_2^T]^T \) be a non-zero complex vector with suitable dimensions such that \( Z^TA_1 = \lambda Z^T \) and \( Z^TB_1 = 0 \). The first equality implies that \( z_1^TA_1^T = \lambda z_1^T \). Consequently, as \( \lambda \neq 0 \) and \( Z \) is a non-zero vector, we have \( z_2 \neq 0 \). Therefore, with \( x_1 = z_2 \), it is sufficient to consider \( x_1, x_2, x_3 \) and \( x_4 \) as defined by (28) to conclude that \( \lambda \) is an uncontrollable mode of \((A_u, B_u)\).

**Lemma 3.3:** Assume that (A1) and (A2) hold. Then, a zero-value \( \lambda \) cannot be an uncontrollable mode of \((A_u, B_u)\), neither \((A_i, B_i)\).

**Proof:** Let \( X = [x_1^T, x_2^T, x_3^T, x_4^T]^T \) be a complex vector with suitable dimensions such that \( X^TA_u = 0 \) and \( X^TB_u = 0 \). After some algebra, it is equivalent to the following system:

\[
\begin{align*}
0 &= [\tau x_4^T, x_3^T] \begin{bmatrix} A_C^C & A_C^C \\ C_C & C_C \end{bmatrix} \\
x_1^T &= -\tau x_4^TB_C^C - x_3^TD_C^C \\
x_2^T &= -\tau x_4^TB_C^C - x_3^TD_C^C \\
x_3^T &= -\tau x_4^TB_C^C - x_3^TD_C^C
\end{align*}
\]

(29)

Based on assumptions (A1) and (A2), matrix (15) is square and full column rank, and hence, it is invertible. Consequently, system (29) is equivalent to \( X = 0 \).

Now, let \( X = [x_1^T, x_2^T]^T \) be a complex vector with suitable dimensions such that \( X^TA_i = 0 \) and \( X^TB_i = 0 \). Then, \( X^TA_i = 0 \) implies

\[
\begin{bmatrix} x_2^T, 0^T \end{bmatrix} \begin{bmatrix} A_C^C \\ C_C \end{bmatrix} = 0
\]

Since matrix (15) is invertible, \( x_2 = 0 \). Furthermore, as \( X^TB_i = 0 \) implies \( x_1^T + x_2^TB_C^C = 0 \), we conclude that \( x_1 = 0 \) and then \( X = 0 \).

We can now introduce the following main result.

**Theorem 3.2:** Assume that (A1), (A2) and (A3) hold. Then, for any \( \tau > 0 \), \((A_u, B_u, C_u, D_u)\) is stabilisable and detectable. Furthermore, assume that \( \mathcal{CL}(S_i(\theta_c), C_i(\theta_c)) \) is internally stable, then so is \( \mathcal{CL}(S, C)(\theta) \).

**Proof:** Based on Lemmas 3.1, 3.2 and 3.3, it is straightforward to conclude that \((A_u, B_u, C_u, D_u)\) is stabilisable and detectable since (A3) assumes that \((A_i, B_i, C_i, D_i)\) is stabilisable and detectable. Furthermore, if we assume that \( \mathcal{CL}(S_i(\theta_c), C_i(\theta_c)) \) is internally stable, we can directly conclude that \( \mathcal{CL}(S, C)(\theta_c) \) is internally stable since the internal stability property does not depend on specific stabilisable and detectable state-space representations of the plant and the controller (Zhou, Doyle, & Glover, 1996).

**3.2.3 Selection of the pre-/post-filtering strategy**

In this subsection, we investigate what kind of pre-/post-filtering can be used in the enhanced velocity-based implementation so that Theorems 3.1 and 3.2 hold.

Based on Figure 5, we are looking for scalar rational transfer functions \( F_i(s) \), \( F_{yi}(s) \), \( F_{yi}(s) \) and \( F_i(s) \) such that the gain-scheduled controller \( \tilde{C} \) given by (30) solves Problem 1. In this setting, the parameter \( \tau \in \mathbb{R}^+ \) is still a scaling factor for both \( A_i^C(\theta) \) and \( C_i^C(\theta) \).

**Figure 5.** Enhanced velocity-based implementation: pre-/post-filtering strategy.

Theorems 3.1 and 3.2 show that the filtering parameter \( \tau > 0 \) has no impact on the controller transfer function and the internal stability of the closed-loop system. Consequently, \( \tau \) can be tuned based on the behaviour of the nonlinear closed-loop system \( \mathcal{CL}(S, C) \). This is a fundamental difference from the classic velocity-based implementation using pseudo-derivative. Furthermore, we have demonstrated that a nonlinear gain-scheduled controller \( \tilde{C} \) solving Problem 1 can always be found. Therefore, assuming that the set of LTI controllers \( C_l \) has been designed such that for any operating point \( \theta_c \in \Theta \), \( \mathcal{CL}(S_c(\theta_c), C_l(\theta_c)) \) is internally stable, the stability of the resulting closed-loop nonlinear system \( \mathcal{CL}(S, C) \) is guaranteed for slow time variations of the reference input \( r \) and the exogenous input \( w \) (D. Lawrence & Rugh, 1990; Rugh & Shamma, 2000).
Analysing the proof of Theorem 3.1, the key point enabling to avoid the emergence of the hidden coupling terms in the linearised dynamics of the gain-scheduled controller lies in the trim conditions (16). Indeed, if these trim conditions are not satisfied, (19) and (20) cannot be simplified, leading to the occurrence of the hidden coupling terms. Therefore, to avoid the emergence of these terms when linearising the gain-scheduled controller \( \tilde{C} \), the trim conditions must satisfy \( x_{r,f,e} = r_e, x_{r,f,e} = \gamma_{1,e}, \) \( x_{y_f,e} = y_{2,e} \) and \( v_e = 0 \). In this case, based on Assumption (A1) and Equation (17), it will imply the two remaining key constraints \( r_e = y_{1,e} \) and \( x_{r,e} = 0 \). The condition \( x_{r,f,e} = r_e \), as \( x_{r,f} = F_r(s)r \), is equivalent to \( F_r(0) = 1 \). Similarly, we have \( F_{y_2}(0) = F_{y_2}(0) = 1 \). Finally, to impose \( v_e = 0 \), as \( u = F_u(s)v, F_u(s) \) must contain an integral component, i.e. \( F_u(s) = F_u(s)/s \) with \( F_u(s) \) a rational transfer function such that \( F_u(s)(0) \neq 0 \). Under these conditions, as they prevent the emergence of the hidden coupling terms, a direct computation, similar to the one achieved in the proof of Theorem 3.1, shows that

\[
\Delta u(s) = \tau F_u(s) \left[ C^2_u(\theta_c) + C^2_c(\theta_c) (sI_n - A^2_c(\theta_c))^{-1} A^2_c(\theta_c) \right] \times (F_u(s) \Delta r(s) - F_u(s) \Delta y_1(s)) \\
+ F_u(s)(1 - F_u(s)) \times \left[ D^2_u(\theta_c) + C^2_c(\theta_c) (sI_n - A^2_c(\theta_c))^{-1} B^2_c(\theta_c) \right] \Delta r(s) \\
+ F_u(s)(1 - F_u(s)) \times \left[ D^2_u(\theta_c) + C^2_c(\theta_c) (sI_n - A^2_c(\theta_c))^{-1} B^2_c(\theta_c) \right] \Delta y_1(s) \\
+ F_u(s)(1 - F_u(s)) \times \left[ D^2_u(\theta_c) + C^2_c(\theta_c) (sI_n - A^2_c(\theta_c))^{-1} B^2_c(\theta_c) \right] \Delta y_2(s)
\]

(31)

The objective is then to select the pre-/post-filtering transfer functions such that (31) coincides with the linear controller dynamics \( C_i(\theta_c) \) given in (23). As we are looking for a generic scheme, this equality must hold for any matrices \( A^2_c(\theta_c), \ldots, D^2_c(\theta_c) \) with suitable dimensions. It allows in the subsequent developments considering specific values of these matrices in order to derive the properties of the desired filters.

Taking \( \Delta r(s) = \Delta y_1(s) = 0 \), since the equality between (31) and (23) must hold for any \( \Delta y_2(s) \), we have

\[
[F_u(s)(1 - F_u(s)) - 1] \times \left[ D^2_u(\theta_c) + C^2_c(\theta_c) (sI_n - A^2_c(\theta_c))^{-1} B^2_c(\theta_c) \right] = 0.
\]

Again, as we aim at finding a generic implementation, this equality must hold for any matrices \( A^2_c(\theta_c), \ldots, D^2_c(\theta_c) \) with suitable dimensions. In particular, it must hold for \( C^2_c(\theta_c) = 0 \) and a non-zero matrix \( D^2_c(\theta_c) \). Thus, we deduce that

\[
F_u(s)(1 - F_u(s)) = 1.
\]

(32)

Then, considering the equality between (23) and (31) while taking \( \Delta y_2(s) = 0, \Delta y_1(s) = -\Delta r(s) \) and \( C^2_c(\theta_c) = 0 \), it yields

\[
[F_u(s)(1 - F_u(s)) - 1] D^2_c(\theta_c) \\
+ [F_u(s)(1 - F_u(s)) - 1] D^2_c(\theta_c) = 0.
\]

In particular, choosing \( D^2_c(\theta_c) = I_{p_1} \) and \( D^2_c(\theta_c) = 0 \), the last matrix equality boils down to

\[
F_u(s)(1 - F_u(s)) = 1.
\]

(33)

Conversely, choosing \( D^2_c(\theta_c) = 0 \) and \( D^2_c(\theta_c) = I_{p_1} \), we have

\[
F_u(s)(1 - F_u(s)) = 1.
\]

(34)

Thus, based on (32)–(34),

\[
F_u(s) = F_{y_2}(s) = F_{y_1}(s) = 1 - \frac{1}{F_u(s)}.
\]

(35)

Finally, considering again the equality between (23) and (31) while taking \( \Delta y_2(s) = 0, \Delta y_1(s) = -\Delta r(s) \) and \( C^2_c(\theta_c) = 0 \), we have, based on (35),

\[
\tau F_u(s) F_u(s) - \frac{1}{s} C^2_c(\theta_c) = 0.
\]

Moreover, as this matrix equality must hold for any matrix \( C^2_c(\theta_c) \) with suitable dimensions, we have

\[
\tau F_u(s) F_u(s) = \frac{1}{s}.
\]

(36)

Combining (35) and (36), simple algebra yields \( F_u(s) = F_{y_1}(s) = F_{y_2}(s) = 1/(\tau s + 1) \) and \( F_u(s) = 1 + 1/(\tau s) \). Obviously, these results are compatible with the trim conditions formerly established, i.e. \( F_r(0) = F_{y_1}(0) = F_{y_2}(0) = 1 \) and \( F_u(s) \) presents an integral component. Consequently, the filters proposed in the enhanced velocity-based implementation (14) are the only possible choice for which Theorem 3.1 holds.

4. Case study

In the following, we illustrate the approach proposed in this paper through the implementation of a pitch-axis missile autopilot.
4.1 Missile nonlinear model and self-scheduled controller

4.1.1 Nonlinear model

The considered pitch-axis model of the missile involving the angle of attack $\alpha$ and the pith rate $q$ is the one given by Reichert (1992):

$$\begin{align*}
\dot{\alpha} &= K_\alpha MC_n(\alpha, \delta, M) \cos(\alpha) + q \\
\dot{q} &= K_q M^2 C_m(\alpha, \delta, M)
\end{align*} \tag{37}$$

where $C_n(\alpha, \delta, M)$ and $C_m(\alpha, \delta, M)$ are, respectively, the lift and the pitching-moment aerodynamic coefficients. The dynamics of the actual tail deflection $\delta$ related to the commanded tail deflection $\delta_c$ are modelled by a second-order system. The system output is the normal acceleration $\eta$ given by

$$\eta = K_\eta M^2 C_n(\alpha, \delta, M). \tag{38}$$

The measured outputs available for feedback are $\eta$ and $q$. The angle of attack $\alpha$ (an endogenous variable) and the Mach number $M$ (an exogenous variable) are used as scheduling parameters. The plant input is the commanded tail deflection $u = \delta_c$. Further description of the physical parameters involved in the model, including their numerical values, are given in Reichert (1992).

4.1.2 Gain-scheduled controller design

The control objective is to design an autopilot allowing to track commanded normal accelerations $\eta_c$ over the flight domain $M \in [2, 4]$ and $\alpha \in [-20^\circ, 20^\circ]$. Among other possibilities, this design can be achieved via a self-scheduling approach based on eigenstructure assignment techniques (Le Gorrec, Magni, Carsten, & Chiappa, 1998). Such a procedure has been applied in Döll, Le Gorrec, Ferreres, and Magni (2001) for the following set of LTI controllers parametrised by the operating point $\theta_c = (\alpha_c, M_c)$:

$$\begin{align*}
\delta \dot{x}_i &= \delta \eta_c - \delta \eta \\
\delta u &= K_\delta(\theta_c) \delta x_i + K_\eta(\theta_c) \delta \eta + K_q(\theta_c) \delta q
\end{align*} \tag{39}$$

where $K_\delta$, $K_\eta$ and $K_q$ are quadratic functions of the flight condition $\theta_c$. The scheduled gains have been tuned based on the six operating points and associated eigenvalue assignment of Table 1. Their numerical values are given in Döll et al. (2001).

4.2 Gain-scheduled controller implementation

This subsection presents the results and the comparison of the performance of the naive, the velocity-based and the enhanced velocity-based implementations.

Table 1. Operating points and associated eigenvalue assignment considered in the synthesis.

| $M_c$ | $\alpha_c$ | $\lambda_e$ | $\lambda_c$ |
|------|------------|------------|------------|
| 4    | $20^\circ$ | $-14$      | $-19 \pm 19j$ |
| 4    | $0^\circ$  | $-13.9$    | $-15 \pm 16j$ |
| 2    | $0^\circ$  | $-12$      | $-12 \pm 12j$ |
| 2    | $20^\circ$ | $-12.5$    | $-12.5 \pm 12.5j$ |
| 4    | $10^\circ$ | $-13.7$    | $-13.7 \pm 13.7j$ |
| 3    | $0^\circ$  | $-12.5$    | No constraint |

4.2.1 Naive implementation

The naive implementation has a structure similar to that of the LTI controllers used for design purposes:

$$C_{\text{naive}} := \left\{ \begin{array}{l}
\dot{x}_i = \eta_c - \eta \\
\dot{u} = K_\alpha(\alpha, M)x_i + K_\eta(\alpha, M)\eta + K_q(\alpha, M)q
\end{array} \right. \tag{40}$$

However, as scheduled gains are varying according to the state signal $\alpha$, the linearisation of $C_{\text{naive}}$ at a given operating point $\theta_c$ brings hidden coupling terms that are not present in (39). For instance, at the operating point $M_c = 4$ and $\alpha_c = 10^\circ$, the assigned poles are located at $-13.7$ and $-13.7 \pm 13.7j$. Nevertheless, with the naive implementation, the actual pole location is very different, i.e. $-47.9$ and $-1.49 \pm 10.6j$. Particularly, the complex pair exhibits a very low damping, leading to large overshoots.

4.2.2 Velocity-based implementation

As the measurement of $\dot{\eta}$ and $\dot{q}$ is not available for feedback, the velocity-based implementation is achieved via the pseudo-derivative of the normal acceleration $\eta$ and the pitch-rate $q$ with a constant $\tau > 0$:

$$C_{\text{vel}} := \left\{ \begin{array}{l}
\dot{\eta}_c = \tau^{-1}\eta_f - \tau^{-1}\eta \\
\dot{q}_f = \tau^{-1}q_f - \tau^{-1}q \\
\dot{u} = \tau^{-1}v
\end{array} \right.$$

As mentioned previously, the pseudo-derivative introduces an extra dynamic component in the controller, which is not considered in the design.

4.2.3 Enhanced velocity-based implementation

The enhanced velocity-based implementation, denoted $C_{\text{vel}+}$, is of the following form for $\tau > 0$:

$$\begin{align*}
\dot{\eta}_c &= \tau^{-1}\eta_{c,f} - \tau^{-1}\eta_c \\
\dot{\eta}_f &= \tau^{-1}\eta_f - \tau^{-1}\eta \\
\dot{q}_f &= \tau^{-1}q_f - \tau^{-1}q \\
\dot{u} &= \tau^{-1}v
\end{align*} \tag{43}$$

$$C_{\text{vel}+} := \left\{ \begin{array}{l}
v = \tau K_v(\alpha, M)(\eta_c - \eta) + K_\eta(\alpha, M)(\eta - \eta_f) \\
+ K_q(\alpha, M)(q - q_f) \\
\dot{x}_c,f = \tau^{-1}v \\
u = x_c,f + v
\end{array} \right.$$
Assuming that $K_i(\alpha, M) \neq 0$ over the operating domain, Theorems 3.1 and 3.2 guarantee that $C_{vel+}$ is a safe implementation of the designed gain-scheduled controller.

### 4.3 Nonlinear simulations

The temporal behaviour of the three closed-loop systems with nonlinear controllers $C_{naive}$, $C_{vel}$ and $C_{vel+}$ is simulated for a realistic profile of the Mach number (Reichert, 1992):

$$
\begin{align*}
M &= \frac{1}{b_s} \left(-|\eta|g \sin(|\alpha|) + A_xM^2 \cos(\alpha)\right) \\
M(0) &= 3.0
\end{align*}
$$

Simulation results are depicted in Figures 6 and 7. Note that the open-loop pitch-axis missile benchmark model, when the normal acceleration $\eta$ is selected as the output, exhibits unstable zeros, resulting in an initial undershoot in their step-responses. As predicted for the naive implementation $C_{naive}$, the hidden coupling terms interfere in the closed-loop dynamics, leading to large overshoots for important command inputs. On the contrary, Figure 6(a) shows that both velocity-based $C_{vel}$ and enhanced velocity-based $C_{vel+}$ implementations work well with a similar behaviour for sufficiently small filtering parameters, e.g. $\tau = 0.002$. The evolution of the Mach number, which is the exogenous scheduling variable, is depicted in Figure 6(b). Nevertheless, small values of $\tau$ may not be suitable in practice particularly due to measurement noises in the closed-loop system. As illustrated, for higher values of the filtering parameters, e.g. $\tau = 0.02$ (see Figure 7), the closed-loop system performance of the velocity-based implementation $C_{vel}$ is significantly degraded due to the interference of the pole introduced by the pseudo-derivative scheme. In sharp contrast, the

![Figure 6. Comparison of the closed-loop response to a series of step commands in acceleration – $\tau = 0.002$.](image)

![Figure 7. Comparison of the closed-loop to a series of step commands in acceleration – $\tau = 0.02$.](image)
enhanced velocity-based implementation $C_{\text{vel}+}$ is mostly insensitive to the variation of the filtering parameter $\tau$.

Finally, the impact of the choice of the filtering parameter on the system in closed loop when white noise is introduced in the feedback loop by the measurement of both system outputs and scheduling variables is illustrated in Figures 8 and 9. As expected, as both implementations are based on a pseudo-derivation scheme employing the transfer function $s/(\tau s + 1)$, a small value of the parameter $\tau$ will induce noise amplification, which may significantly degrade the performance of the closed-loop system for both strategies. Thus, an arbitrary small value of the filtering parameter $\tau$ is not appropriate for practical applications. However, in accordance with the conclusions of the above analysis, a larger value of the pseudo-derivative parameter also degrades the performance of the closed-loop system for the velocity-based implementation $C_{\text{vel}}$ (see Figure 8). Conversely, the enhanced velocity-based implementation $C_{\text{vel}+}$ allows much higher values of the filtering parameter $\tau$. Consequently, it results in an improved closed-loop performance, even in the presence of noise in the feedback loop (see Figure 9).

5. Conclusion

This paper introduced an enhanced velocity-based algorithm for safe implementation of gain-scheduled controllers. Based on a parameter-dependent set of LTI controllers that are designed to ensure the stability and performance of the linear closed-loop system for any frozen operating point, the proposed gain-scheduled controller implementation preserves both internal stability and
input–output properties of the linearised closed-loop system. Furthermore, this implementation is relatively simple, with a gain-scheduled controller presenting an architecture similar to that of the original controller. The efficiency of the proposed approach has been demonstrated on the implementation of a pitch-axis missile autopilot. The simulation results confirmed the performance of the control system using the proposed new approach predicted by theoretical analysis.

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Note

1. Note that this condition is necessary but not sufficient to guarantee the closed-loop stability.

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