Quantum particle on a surface: Catenary surface and paraboloid of revolution

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Abstract
We revisit the Schrödinger equation of a quantum particle that is confined on a curved surface. Inspired by the seminal work of da Costa (1980 Phys. Rev. A 23 1982) we find the field equation in a more convenient notation. The contribution of the principal curvatures in the effective binding potential on the surface is emphasized. Furthermore, using the so-called Monge-Gauge we construct the approximate Schrödinger equation for a flat surface with small fluctuations. Finally, the resulting Schrödinger equation is solved for some specific surfaces. In particular, we give exact solutions for a particle confined on a Catenary surface and a paraboloid of revolution.

1. Introduction
Nowadays, studying the propagation of quantum particles on curved surfaces became of interest in different areas of experimental and theoretical physics. Graphene with only one atom thickness is one of such 2-dimensional surface-like materials [1]. Furthermore, another surface-like material is the so-called carbon nanotube [2] with stronger $sp^2$ bonds which make it one of the best thermal conductors [3]. Another important 2D material that has been the subject of intensive research both experimentally and theoretically is Phosphorene, a monolayer of black phosphorus [4] (see also the references therein). It is a 2D semiconductor material with an anisotropic orthorhombic structure and high optical and UV absorption. To this list of 2-dimensional materials, we would like to add the so-called fluid lipid membranes [5] which have shown a growing interest among physicists, mathematicians, and biologists.

Concerning these highly important lower-dimensional materials, and the quantum phenomenon on these surfaces, one has to create/construct an induced lower-dimensional quantum mechanics which may be applied to such 2-dimensional materials. Such a formalism has been introduced a long time ago in [6, 7]. However, the recent work of Ferrari and Cuoghi [8], proves that the story has not been over yet. To summarize the difference between these three works, one observes the following.

In [6], the action principle has been considered directly in $n$-dimensional curved space such that setting $n = 2$ one gets the quantum on a 2-dimensional curved surface. This means that the Schrödinger equation of a particle moving on a curved surface is constructed directly from the quantization of the classical Hamiltonian. Let us add that, for the $n$-dimensional Euclidean space (flat space) such an approach was introduced in the very early time of the quantum by E Schrödinger in 1926 [9] and Boris Podolsky in 1928 [10]. However, for an $n$-dimensional curved space, including the 2-dimensional curved surface, the noncommutativity of the canonical position and its conjugate momentum operators, causes the so-called factor-ordering ambiguity. We refer to [11] for the full explanation of this ambiguity. Furthermore, one may look at [12, 13] for a different look at the problem using symmetries.

On the other hand, in [7], using the differential geometrical properties of the curved surface, the Schrödinger equation in 3-dimensional Euclidean space has been reduced to the Schrödinger equation on the surface. In [8], the formalism of [7] has been extended by including the electric and magnetic fields. Following these seminal works, there have been significant improvements in the quantum systems in two-dimensional curved surfaces as well as on curves in space. In this line, we may refer to [14] where the effects of the geometry, as well as magnetic...
field on the electronic transport properties of metallic nanotubes, have been numerically investigated. In [15], an electron confined on a torus under the influence of external electric and magnetic fields has been numerically studied. In [16] the authors studied an electron on a catenoid surface. Oliveira et al. in [17] solved the Schrödinger equation on a sphere under non-central potential and Schmidt in [18, 19] introduced exact solutions of the Schrödinger equation for a charged particle confined on a sphere, on a cylinder, and on a torus while is imposed with uniform electric and magnetic fields. Furthermore, electrons confined on a rotating sphere in the presence of a magnetic field have been considered by Lima et al. in [20], and the effects of the rotation were compared with the effects of the magnetic fields. The applications of this formalism have not been limited only to these works, for instance, there is an important application for Da Costa’s paper in recent work by F Impens et al. in [21]. We also refer to [22] for further reading.

We observed that most of the papers published recently are based on the effective Schrödinger equation derived in [7], particularly equation (14). In driving this equation, R C T da Costa used a kind of unfamiliar notation to our new generation of young physicists. For instance, while these days we are very careful on distinguishing between contravariant and covariant vectors especially when the contraction of tensors is in the subject, in [7], it was only a matter of notation. Therefore, many steps in finding the effective Schrödinger equation in [7] are unfamiliar. The aim of this paper is first to construct a full detailed calculation with modern—so to say—notation toward the effective Schrödinger equation. We should add that there are some other works that looked at this issue from another perspective. For instance, a very general, as well as an interesting approach to the constraint motion of a quantum particle in n-dimensional Euclidian space, has been studied by P C Schuster and R L Jaffe in [23]. In addition, the applications of some specific parametrization such as Monge parametrization seem to be missing in the literature. Since for surfaces such as graphene, the small deviation from a flat surface may be of interest the application of Monge gauge becomes important. Hence, we study the quantum particle under Monge parametrization and for small deviation, we present the simplified approximate effective Schrödinger equation. Finally, the exact solutions of the effective Schrödinger equation for quantum particles confined on some important curved surfaces such as one-directional Catenoid and paraboloid of revolution are missing. Therefore, we investigate the possible exact solutions for these two cases.

The organization of this paper is as follows. In section 2 we re-obtain the Schrödinger equation of a quantum particle confined on a curved surface. In section 3, Monge parametrization is considered for a curved surface and the corresponding Schrödinger equation is presented. In sections 4 and 5, we apply the formalism to the surfaces obtained by extrusion of the catenary curve and the surface of Paraboloid of Revolution, respectively. We conclude our paper in section 6.

2. Schrödinger equation on a curved surface

We consider a quantum particle of mass \( m \) confined on a differentiable surface \( S \) in the three-dimensional Euclidean space. We also adopt a local two-dimensional coordinate system \( (u^1, u^2) \) on the surface and mapping \( \mathbf{r} := \mathbf{r}(u^1, u^2) \) which assigns any point on the surface to a point on the three-dimensional space, i.e.,

\[
\mathbf{r} := \mathbf{r}(u^1, u^2) = (x^1(u^1, u^2), x^2(u^1, u^2), x^3(u^1, u^2)).
\]

Herein, the position vector \( \mathbf{r} = (x^1, x^2, x^3) \) represents a point on the surface while a point in the neighborhood of the surface can be described using an additional coordinate i.e., \( u^3 \) in the direction normal to the surface (see figure 1).

Hence, one writes

\[
\mathbf{R}(u^1, u^2, u^3) = \mathbf{r}(u^1, u^2) + u^3 \mathbf{n}.
\]

We note that the tangent vectors

\[
\mathbf{e}_a = \frac{\partial \mathbf{r}}{\partial u^a} = \partial_a \mathbf{r}
\]

with \( a = 1, 2 \), make a local 2-dimensional coordinate system that spans the tangent surface to \( S \) at any point \( P \) on \( S \) where \( \mathbf{e}_a \) are determined. Furthermore, the three-dimensional coordinate system consists of two tangent vectors \( \mathbf{e}_a \) and the unit normal \( \mathbf{n} \) makes a local 3-dimensional coordinate system that describes the space surrounding the surface \( S \) such that

\[
\mathbf{n} = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|}.
\]

We add that, in general, \( \mathbf{e}_a \) are neither unit vectors nor orthogonal, however, \( \mathbf{n} \) is unit vector and normal to \( \mathbf{e}_a \). Furthermore, the 3-dimensional coordinate system as well as the two dimensional one are curvilinear, and based on the tangent vectors \( \mathbf{e}_a \) and the normal vector \( \mathbf{n} \), one constructs the metric tensor of the space and the surface as defined by
and
\[ h_{ab} = \frac{\partial r}{\partial u^a} \cdot \frac{\partial r}{\partial u^b}, \]
respectively, in which \( \mu, \nu = 1, 2, 3 \) and \( a, b = 1, 2 \). Let’s add that \( h_{ab} \) is called the first fundamental form of the surface \( S \) which is an intrinsic geometrical property of the surface. In addition to the first fundamental form, there exists the second fundamental form of the surface which is an extrinsic property of the surface and is defined by
\[ k_{ab} := e_a \cdot \partial_{b} n \]
which due to the fact that \( e_a \cdot n = 0 \), it is also equal to
\[ k_{ab} = -n \cdot \partial_{ab} r. \]

The second fundamental form \( k_{ab} \) is also called the extrinsic curvature tensor because it is defined in terms of the normal vector \( n \) which is an indication of the embedding of the surface in a higher-dimensional space/ambiance.

Coming back to the metric tensor of the space surrounding the surface one writes
\[ g_{ab} = \frac{\partial R}{\partial u^a} \cdot \frac{\partial R}{\partial u^b} = (e_a + u^c \partial_c n) \cdot (e_b + u^c \partial_c n), \]
\[ g_{a3} = g_{3a} = (e_a + u^c \partial_c n) \cdot n \]
and
\[ g_{33} = n \cdot n = 1. \]

To calculate \( g_{ab}, g_{a3} \) and \( g_{3a} \) we apply the so-called equation of Weingarten which states
\[ \partial_c n = k^c_a e_b \]
in which \( k^c_a = h^b_c k_{ab} \) is the mixed form of the extrinsic curvature tensor and \( h^{bc} \) is the inverse of the metric tensor \( h_{bc} \) such that \( h_{ac} h^{ab} = \delta_a^b \). Considering (12) in (9) one finds
\[ g_{ab} = e_a \cdot e_b + u^c \partial_c n \cdot e_b + u^c e_a \cdot \partial_c n + (u^c)^2 \partial_c \partial_a n \cdot \partial_b n \]
or simply
\[ g_{ab} = h_{ab} + u^c k^c_a h_{ab} + u^c k^c_b h_{ac} + (u^c)^2 k^c_d k^d_b h_{cd} \]
and \( g_{a3} = g_{3a} = 0 \). Having the first and the second fundamental forms symmetric, we obtain
\[ g_{ab} = h_{ab} + 2u^c k^c_a h_{ab} + (u^c)^2 k^c_d k^d_b h_{cd}. \]

Using the so-called Laplace-Beltrami operator the Schrödinger equation of a particle in the space spanned with \( g_{\mu \nu} \), is given by
-\frac{\hbar^2}{2m} \frac{1}{\sqrt{\hbar}} \partial_{\omega}(\sqrt{\hbar} g^{\omega\nu} \partial_{\nu}) \psi(u^\alpha, t) + V_0 \tilde{\delta}(u^3) \psi(u^\alpha, t) = i\hbar \frac{\partial}{\partial t} \psi(u^\alpha, t) \tag{16}

in which $V_0 \tilde{\delta}(u^3)$ is a potential which confines the particle on the surface $S$ and $\tilde{\delta}(u^3)$ is an anti-Dirac delta function such that

$$
\tilde{\delta}(u^3) = \begin{cases} 
0 & u^3 = 0 \\
\infty & u^3 \neq 0.
\end{cases}
$$

To simplify (16), one needs to calculate $g = \det g_{\mu\nu}$, i.e.

$$g = \det \begin{bmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \det \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = g_{11}g_{22} - (g_{12})^2 \tag{18}

where

$$g_{11} = h_{11} + 2u^j k_j^i h_{i1} + (u^3)^2 k_j^i k_i^j h_{i1} \\
g_{22} = h_{22} + 2u^j k_j^2 h_{22} + (u^3)^2 k_j^2 k_i^j h_{i2} \tag{19}

and

$$g_{12} = g_{21} = h_{12} + 2u^j k_j^1 h_{22} + (u^3)^2 k_j^1 k_i^j h_{i2} \tag{20}

upon which

$$g = (h_{11} + 2u^j k_j^1 h_{i1} + (u^3)^2 k_j^1 k_i^j h_{i1})(h_{22} + 2u^j k_j^2 h_{22} + (u^3)^2 k_j^2 k_i^j h_{i2}) - (h_{12} + 2u^j k_j^1 h_{22} + (u^3)^2 k_j^1 k_i^j h_{i2})(h_{21} + 2u^j k_j^2 h_{21} + (u^3)^2 k_j^2 k_i^j h_{i2}). \tag{22}

After some manipulation, the latter equation reduces to

$$g = (h_{22} h_{11} - h_{12}^2)(1 + u^j(k_j^1 + k_j^2) + (u^3)^2(k_j^1 k_j^2 - k_j^2 k_j^1))^2. \tag{23}

We remember that the trace and the determinant of the extrinsic curvature tensor are invariant under the coordinate transformation. Therefore, although $k_j^1$ and $k_j^2$ are not diagonal and consequently $k_j^1$ and $k_j^2$ are not the principal curvatures, but $k_j^1 + k_j^2 = T(k^i_a) = \tilde{k}_1^i + \tilde{k}_2^i$ and $k_j^1 k_j^2 - k_j^2 k_j^1 = \det(k^i_a) = \tilde{k}_1^i \tilde{k}_2^i$ in which $\tilde{k}_1^i$ and $\tilde{k}_2^i$ are the principal curvatures of the surface. Considering the above facts one writes

$$g = h(1 + u^j(\tilde{k}_1^i + \tilde{k}_2^i) + (u^3)^2 \tilde{k}_1^i \tilde{k}_2^i)^2 \tag{24}

where

$$h = h_{22} h_{11} - h_{12}^2 = \det(h_{ab}). \tag{25}

We also recall the definition of the Gaussian and total curvatures in terms of the principal curvatures which are defined as

$$K_G = k_1^1 k_2^2 \tag{26}

and

$$K = \tilde{k}_1^1 + \tilde{k}_2^2, \tag{27}

respectively, which simplifies (24) as

$$g = h(1 + u^j K + (u^3)^2 K_G)^2. \tag{28}

Next, we substitute (28) into the Schrödinger equation (16) to get

$$-\frac{\hbar^2}{2m} \frac{1}{\sqrt{\hbar}} \partial_{\omega}(\sqrt{\hbar} g^{\omega\nu} \partial_{\nu}) \psi(u^\alpha, t) - \frac{\hbar^2}{2m} \frac{1}{\omega \sqrt{\hbar}} \partial_{\omega}(\omega \sqrt{\hbar} g^{\omega\nu} \partial_{\nu}) \psi(u^\alpha, t) + V_0 \tilde{\delta}(u^3) \psi(u^\alpha, t) = i\hbar \frac{\partial}{\partial t} \psi(u^\alpha, t)
\quad$$

where $\omega = 1 + u^j K + (u^3)^2 K_G$.

Introducing $\psi(u^\alpha, t) = \frac{1}{\sqrt{\omega}} \tilde{\psi}_1(u^\alpha) e^{-i\omega t/\hbar}$ one obtains

$$-\frac{\hbar^2}{2m} \frac{1}{\sqrt{\hbar}} \partial_{\omega}(\sqrt{\hbar} g^{\omega\nu} \partial_{\nu}) \tilde{\psi}_1(u^\alpha) = \frac{\hbar^2}{2m} \frac{1}{\sqrt{\omega}} \psi_1(u^\alpha) \partial_{\omega}(\omega \partial_{\nu}) \psi_1(u^\alpha) - \frac{\hbar^2}{2m} \frac{1}{\sqrt{\omega}} \psi_1(u^\alpha) \partial_{\omega}(\omega \partial_{\nu}) \psi_1(u^\alpha) + V_0 \tilde{\delta}(u^3) = E.
which after some manipulation becomes

\[-\frac{h^2}{2m} \frac{1}{\sqrt{h}} \frac{\partial^2}{\partial u^2} \left( \sqrt{h} g^{ab} \partial_b \psi(u^a) \right) + \frac{h^2}{2m} \frac{1}{\psi(u^a)} \left( \psi''(u^a) + \left( \frac{\partial \psi}{2 \omega} \right)^2 - \frac{\partial^2 \psi}{2 \omega} \psi(u^a) \right) + V_0 \delta(u^a) = E. \tag{29}\]

Considering the fact that, on the surface where the particle is confined, \(u^3 = 0\) we obtain \(\omega = 1, \partial_3 \omega = K, \) and \(\partial^2_3 \omega = 2K_0.\) Hence, (29) becomes

\[-\frac{h^2}{2m} \frac{1}{\sqrt{h}} \frac{\partial_3}{\partial u^3} \left( \sqrt{h} g^{ab} \partial_b \psi(u^a) \right) + \frac{h^2}{2m} \frac{1}{\psi(u^a)} \left( \psi''(u^a) + \left( \frac{K}{2} \right)^2 - K_0 \right) + V_0 \delta(u^a) = E. \tag{30}\]

After separating the equation to ‘on the surface’ and ‘normal to the surface’ one gets

\[-\frac{h^2}{2m} \frac{1}{\sqrt{h}} \frac{\partial_3}{\partial u^3} \left( \sqrt{h} g^{ab} \partial_b \psi(u^a) \right) - \frac{h^2}{2m} \left( \frac{K}{2} \right)^2 - K_0 \psi(u^a) = E \psi(u^a) \tag{31}\]

and

\[-\frac{h^2}{2m} \psi''(u^a) + V_0 \delta(u^a) \psi(u^a) = E_0 \psi(u^a) \tag{32}\]

respectively. The first equation is the two-dimensional Schrödinger equation of the particle on the surface and the second equation is the one-dimensional Schrödinger equation normal to the surface. The total energy of the particle is given by \(E = E_t + E_n.\)

In this study, we consider

\[\delta(u^3) = \begin{cases} 0 & 0 < u^3 < \epsilon \\ \infty & \text{elsewhere} \end{cases}\]

which yields

\[\psi(u^a) = \frac{\nu}{\sqrt{2}} \sin \left( \frac{\nu \pi u^a}{\epsilon} \right) \tag{34}\]

with energy

\[(E_0)_\nu = \frac{\nu^2 \pi^2 h^2}{2m \epsilon^2} \tag{35}\]

where \(\nu = 1, 2, 3, \ldots\) and \(\epsilon\) is the thickness of the surface.

On the other hand, the tangent Schrödinger equation implies a non-positive effective potential which is purely geometric, i.e.

\[V_S = -\frac{h^2}{2m} \left( \frac{K}{2} \right)^2 - K_0 \tag{36}\]

which in terms of the principal curvatures of the surface becomes

\[V_S = -\frac{h^2}{8m} \left( \frac{k_1}{k_2} - \frac{k_2}{k_1} \right) \leq 0. \tag{37}\]

This effective potential is negative for all surfaces and zero for flat planes and spherical shells. Therefore, we conclude that being the surface curved, in general, causes the particle to be bounded to the surface. The strength of the binding potential depends on the square of the difference between the principal curvatures. In other words, the larger \(|k_1 - k_2|\) the deeper the binding potential.

### 3. Monge parametrization

Monge-Gauge refers to the so-called Monge parametrization which describes a surface by a single function which is called the height function (from a reference plane). Therefore, the Monge parametrization for a curved surface is given by

\[r = (x, y, H(x, y)) \tag{38}\]
in which the surface defined by
\[ z = H(x, y) \]  
(39)
is described by \((x, y)\), and \(H(x, y)\) is called the height function. The first fundamental form is given by
\[
h_{ab} = \begin{pmatrix} r_x & r_x \\ r_x & r_y \\ r_y \end{pmatrix} = \begin{pmatrix} 1 + H_x^2 & H_xH_y \\ H_xH_y & 1 + H_y^2 \end{pmatrix} \]  
(40)
with the corresponding line element on the surface
\[ ds^2 = (1 + H_x^2)dx^2 + (1 + H_y^2)dy^2 + 2H_xH_y dx dy. \]  
(41)
The unit normal vector is obtained to be
\[
n = \frac{r_x \times r_y}{|r_x \times r_y|} = \frac{(-H_{xx}, -H_{xy}, 1)}{\sqrt{1 + H_x^2 + H_y^2}} \]  
(42)
upon which, the second fundamental form is calculated as
\[
k_{ab} = -\mathbf{n} \cdot \partial_{ab} \mathbf{r} = -\frac{1}{\sqrt{1 + H_x^2 + H_y^2}} \begin{pmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{pmatrix}. \]  
(43)
Considering the inverse metric tensor
\[
h^{ab} = \frac{1}{1 + H_x^2 + H_y^2} \begin{pmatrix} 1 + H_y^2 & -H_xH_y \\ -H_xH_y & 1 + H_x^2 \end{pmatrix} \]  
(44)
one finds
\[
k_x^x = \frac{H_{xx}(1 + H_y^2) - H_xH_yH_{yy}}{(1 + H_x^2 + H_y^2)^{3/2}}, \]  
(45)\[
k_y^y = \frac{H_{yy}(1 + H_x^2) - H_xH_yH_{xx}}{(1 + H_x^2 + H_y^2)^{3/2}}, \]  
(46)\[
k_x^y = \frac{H_{xx}H_{yy} - H_{xy}(1 + H_y^2)}{(1 + H_x^2 + H_y^2)^{3/2}} \]  
(47)and
\[
k_y^x = \frac{H_{yy}H_xH_y - H_{xx}(1 + H_x^2)}{(1 + H_x^2 + H_y^2)^{3/2}}. \]  
(48)These imply
\[
K = k^i_i = -\frac{H_{xx}(1 + H_y^2) + H_{yy}(1 + H_x^2) - 2H_xH_yH_{xy}}{(1 + H_x^2 + H_y^2)^{3/2}} \]  
(49)and
\[
K_G = \det k^i_a = \frac{H_{xx}H_{yy} - H_{xy}^2}{(1 + H_x^2 + H_y^2)^2}. \]  
(50)Introducing, \(\nabla H = (H_x, H_y)\) one writes
\[
K = -\mathbf{\nabla} \cdot \left( \frac{\mathbf{\nabla} H}{\sqrt{1 + (\mathbf{\nabla} H)^2}} \right) \]  
(51)and
\[
K_G = \frac{\det \partial^2 H}{(1 + (\mathbf{\nabla} H)^2)^{3/2}} \]  
(52)in which \(\partial^2 H\) is called the Hessian of the function \(H(x, y)\) given by
\[
\partial^2 H = \begin{pmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{pmatrix} \]  
(53)
As a particular case, we may consider $H(x, y) = H(x)$ or $H(x, y) = H(y)$ upon which both result in the same form of

$$K = -\frac{H_i}{(1 + H_i^2)^{3/2}}$$

where $i = x, y$ and

$$K_G = 0.$$  

(54)

As we have mentioned at the beginning of this section, we are going to consider the curved surface to be a small deviation from the flat surface i.e.,

$$|\nabla H| \ll 1$$

(56)

such that

$$K \simeq -\nabla^2 H = -\text{Tr} \partial^2 H$$

(57)

and

$$K_G \simeq \det \partial^2 H.$$  

(58)

We note also that the determinant of the metric tensor i.e.,

$$g = 1 + (\nabla H)^2$$

(59)

is approximately 1 in this gauge i.e.,

$$g \simeq 1.$$  

(60)

Therefore, the resulting time-independent tangential Schrödinger equation of a particle confined on the surface $S$ is obtained to be

$$-\frac{\hbar^2}{2m} \nabla^2 \psi_i(x, y) + V_S \psi_i(x, y) = E_i \psi_i(x, y)$$

(61)

where for the general configuration the effective potential is expressed as

$$V_S = -\frac{\hbar^2}{8m} \left( \nabla \left( \frac{\nabla H}{\sqrt{1 + (\nabla H)^2}} \right) \right)^2 - \frac{4 \det \partial^2 H}{(1 + (\nabla H)^2)^2}.$$  

(62)

For the Monge gauge where $|\nabla H| \ll 1$ one finds

$$V_S \simeq -\frac{\hbar^2}{2m} \left( \frac{\text{Tr} \partial^2 H}{2} \right)^2 - \det \partial^2 H$$

(63)

which after simplification becomes

$$V_S \simeq -\frac{\hbar^2}{8m} ((H_{xx} - H_{yy})^2 + 4H_{xy}^2).$$  

(64)

Moreover, for the case where $H$ is only a function of one coordinate, the latter becomes ($i = x, y$)

$$V_S = -\frac{\hbar^2}{8m} \frac{H_i^2}{(1 + H_i^2)^3}$$

(65)

which in the gauge where $H_i^2 \ll 1$ it reduces to

$$V_S \simeq -\frac{\hbar^2}{8m} H_i^2.$$  

(66)

### 3.1. Monge gauge in polar coordinates with radial symmetry

Considering an axial symmetric surface defined by

$$z(\rho, \theta) = H(\rho)$$

(67)

in the cylindrical coordinates system $\{\rho, \theta, z\}$, admitting radial symmetry, one obtains

$$g_{ab} = \begin{pmatrix} 1 + H_i^2 & 0 \\ 0 & \rho^2 \end{pmatrix}.$$  

(68)
with the line element
\[ ds^2 = (1 + H_\rho^2) d\rho^2 + \rho^2 d\theta^2. \] (69)

The unit normal vector and the second fundamental form are obtained to be
\[ n_a = \frac{1}{\sqrt{1 + H_\rho^2}} (-H_\rho, 0, 1) \] (70)

and
\[ k_{ab} = \begin{pmatrix} -H_\rho & 0 \\ \frac{-H_\rho}{\sqrt{1 + H_\rho^2}} & 0 \\ 0 & -\frac{H_\rho}{\sqrt{1 + H_\rho^2}} \rho \end{pmatrix}. \] (71)

Consequently, we calculate
\[ k^a_b = \begin{pmatrix} -H_\rho & 0 \\ \frac{H_\rho}{(1 + H_\rho^2)^{3/2}} & 0 \\ 0 & -\frac{1}{\sqrt{1 + H_\rho^2}} \rho \end{pmatrix}. \] (72)

Having \( k^a_b \) diagonal, the principal curvatures are given by
\[ \tilde{k}_\rho = \frac{-H_\rho}{(1 + H_\rho^2)^{3/2}} \] (73)

and
\[ \tilde{k}_\theta = \frac{-H_\rho}{\sqrt{1 + H_\rho^2}} \rho. \] (74)

Using (73) and (74), the time-independent tangent Schrödinger equation on the surface becomes
\[ -\frac{\hbar^2}{2m} \frac{1}{\sqrt{h}} \partial_\rho (\sqrt{h} g^{ab} \partial_b) \psi_\rho (\rho, \theta) - \frac{\hbar^2}{8m} \left( \frac{H_\rho}{(1 + H_\rho^2)^{3/2}} - \frac{1}{\sqrt{1 + H_\rho^2}} \rho \right) \psi_\rho (\rho, \theta) = E \psi_\rho (\rho, \theta) \] (75)

in which \( h = \rho^2 (1 + H_\rho^2) \). In the small curvature limit where \(|H_\rho| \ll 1\), we find
\[ V_S = -\frac{\hbar^2}{8m} \left( H_\rho^2 - \frac{H_\rho}{\rho} \right) = -\frac{\hbar^2}{8m} \rho^3 \left( \partial_\rho \left( \frac{H_\rho}{\rho} \right) \right)^2. \] (76)

4. Catenary surface (the surface obtained by extrusion of the catenary)

Consider an infinite flat plane that is bent into a surface obtained by extrusion of the catenary curve. We shall call it Catenary surface, for short, and is defined by
\[ H(x, y) = H(x) = a \cosh \left( \frac{x}{a} \right) \] (77)

where \( a \) is a positive real constant. The corresponding Schrödinger equation of a particle confined on this surface is given by (61). Corresponding to the curved surface, the geometric potential, the metric tensor, and the determinant of the metric tensor are obtained to be
\[ V_S = -\frac{\hbar^2}{8m} \frac{H_\rho^2}{(1 + H_\rho^2)^3} = -\frac{\hbar^2}{8m} \frac{1}{a^2 \cosh^4 \left( \frac{x}{a} \right)}, \] (78)

\[ g_{ab} = \text{diag} \left[ 1 + \sinh^2 \left( \frac{x}{a} \right), 1 \right]. \] (79)
\[ g = 1 + H'(x)^2 = \cosh^2 \left( \frac{x}{a} \right) \]

respectively. In figure 2 we plot the Catenary surface together with the corresponding effective potential \( V_\Sigma \left( \frac{\partial}{\partial y} \right) \).

Considering \( \psi(x, y) = X(x)Y(y) \) one obtains

\[ -\frac{\hbar^2}{2m} \frac{1}{\cosh \left( \frac{z}{a} \right)} \frac{d}{dx} \left( \frac{1}{\cosh \left( \frac{z}{a} \right)} \frac{d}{dx} X(x) \right) - \frac{\hbar^2}{8m} \frac{1}{a^2 \cosh^4 \left( \frac{z}{a} \right)} X(x) = E_x X(x) \]  

\( (80) \)

and

\[ -\frac{\hbar^2}{2m} Y''(y) = E_y Y(y). \]

\( (81) \)

Let’s introduce \( \alpha = \frac{2mE}{\hbar^2} \) and \( \beta = \frac{2mE}{\hbar^2} \) upon which \( (80) \) and \( (81) \) become

\[ -\frac{1}{\cosh \left( \frac{z}{a} \right)} \frac{d}{dx} \left( \frac{1}{\cosh \left( \frac{z}{a} \right)} \frac{d}{dx} X(x) \right) - \frac{1}{4a^2 \cosh^4 \left( \frac{z}{a} \right)} X(x) = \alpha X(x) \]  

\( (82) \)

and

\[ Y''(y) + \beta Y(y) = 0, \]

\( (83) \)

respectively. To solve the \( x \)-component of the field equations, we introduce a change of variable in the following form

\[ q = \sinh \left( \frac{x}{a} \right) \]

\( (84) \)

which after some manipulation \( (82) \) becomes

\[ -X''(q) - \frac{1}{4(1 + q^2)^2} X(q) = E X(q), \]

\( (85) \)

in which \( E = a^2 \alpha \). The latter equation describes a one-dimensional quantum particle in \( q \)-space which undergoes a binding potential of the form.
Further, to solve (85) we introduce another change of variable given as $z = -q^2$ and $X(q) = (1 - z)^4 w(z)$ with \( \lambda = \frac{1}{2} + \frac{\sqrt{5}}{4} \), upon which (85) becomes

$$w'' + \left( \alpha + \frac{\beta + 1}{z} + \frac{\gamma + 1}{z - 1} \right) w' + \left( \frac{\mu}{z} + \frac{\nu}{z - 1} \right) w = 0$$

where

$$\alpha = 0, \beta = -\frac{1}{2}, \gamma = \frac{\sqrt{5}}{2}, \mu = -\frac{4E + 5 + 2\sqrt{5}}{16} \quad \text{and} \quad \nu = \frac{1}{8} \left( \frac{5}{2} + \sqrt{5} \right).$$

Equation (87) is the so-called Confluent Heun Differential equation (CHDE) whose solution is given by

$$w(z) = C_1 \text{HeunC}(\alpha, \beta, \gamma, \delta, \eta, z) + C_2 z^{-\delta} \text{HeunC}(\alpha - \beta, \gamma, \delta, \eta, z).$$

Herein, $C_1$ and $C_2$ are two integration constants and $\delta = \mu + \nu - \frac{\alpha(2 + \gamma + \beta)}{2} = -\frac{E}{4}$ and $\eta = \frac{(\beta + 1)(\alpha - \gamma) - \beta}{2} - \mu = \frac{E}{4} + \frac{9}{16}$. Hence, the general solution of the Schrödinger equation becomes

$$X(q) = C_1 (1 + q^2)^{1/2} + \frac{E}{4} \text{HeunC} \left( 0, -\frac{1}{2}, \frac{\sqrt{5}}{2}, -\frac{E}{4}, \frac{E}{4}, +\frac{9}{16}, -q^2 \right)$$

$$+ C_2 (1 + q^2)^{1/2} + \frac{E}{4} \text{HeunC} \left( 0, -\frac{1}{2}, \frac{\sqrt{5}}{2}, -\frac{E}{4}, \frac{E}{4}, +\frac{9}{16}, -q^2 \right).$$

Our detailed numerical analysis reveals that there exists only one bound state with $E = -0.02892460$, $C_2 = 0$, and $C_1 = \frac{1}{\sqrt{6.7406}}$. Hence, the normalized wave function is found to be

$$X(q) = \frac{1}{\sqrt{6.7406}} (1 + q^2)^{1/2} + \frac{E}{4} \text{HeunC} \left( 0, -\frac{1}{2}, \frac{\sqrt{5}}{2}, -\frac{E}{4}, \frac{E}{4}, +\frac{9}{16}, -q^2 \right).$$

which together with the potential $V(q) = -\frac{1}{4(1 + q^2)^{1/2}}$ are plotted in figure 3. This figure reveals that the particle is localized around the deep of the surface where the total curvature is maximum.

### 5. Paraboloid of revolution

Concerning the results of the polar coordinate with radial symmetry, we consider the surface of the paraboloid of revolution which is defined as
\[ H(\rho) = a \left( \frac{\rho}{a} \right)^2 \]  

where \( a \) is a constant parameter with the dimension of length. In figure 4 we plot the paraboloid of revolution for \( a = 1 \). The line element on the surface of the paraboloid of revolution is obtained to be

\[ ds^2 = \left( 1 + \frac{\rho^2}{a^2} \right)d\rho^2 + \rho^2d\theta^2. \]

The two dimensional Schrödinger equation on the surface is given by

\[ -\frac{h^2}{2m} \frac{1}{\sqrt{h}} \partial_\rho (\sqrt{h} g^{\rho\rho} \partial_\rho) \psi_1(\rho, \theta) + V_\Sigma(\rho) \psi_1(\rho, \theta) = E_1 \psi_1(\rho, \theta) \]

in which

\[ V_\Sigma(\rho) = -\frac{\hbar^2}{8m} \frac{\rho^4}{(a^2 + \rho^2)^3}, \]

\[ h = \det g_{ab} = \rho^2 \left( 1 + \frac{\rho^2}{a^2} \right) \]

and

\[ g^{ab} = \text{diag} \left[ \frac{1}{1 + \frac{\rho^2}{a^2}}, \frac{1}{\rho^2} \right]. \]

Applying the separating method one obtains

\[ \psi_1(\rho, \theta) = R(\rho)e^{i\ell \theta} \]

with \( \ell = 0, \pm 1, \pm 2, \ldots \) and \( R(r) \) satisfying

\[ -R'' - \frac{a^2}{\rho(a^2 + \rho^2)} R' + \left( \frac{\ell^2(a^2 + \rho^2)}{a^2 \rho^2} - \frac{\rho^4}{4a^2(a^2 + \rho^2)^2} \right) R = \frac{a^2 + \rho^2}{a^2} \mathcal{E} R \]

where \( \mathcal{E} = \frac{2mE_1}{\hbar^2} \). Without going into the details, the general solution of the Schrödinger equation is given in terms of the Confluent Heun function, expressed as

\[ \text{Figure 4. The graph of the Paraboloid of Revolution with } a = 1. \]
in which $C_1$ and $C_2$ are the integration constants. From the solution (100) we see that with $\ell \geq 0$ the first solution is regular at the origin while for $\ell \leq 0$ the second solution coincides with the first solution (with $\ell \geq 0$). Knowing also that the sign of $\ell$ doesn’t make any change in the solution, we set $\ell \geq 0$ and eliminate the second solution as it is irregular at the origin. The square integrability of the solution implies
Our numerical calculations revealed that for the ground state where $\ell = 0$ the binding energy is given by $E = -0.0113978$ and $C_1 = 0.27268$. In figure 5 we plot $\rho \psi_\ell (\rho, \theta)$ in terms of $\rho$ and $\theta$. The bright circle is the pick of the wave function indicating the maximum probability radius. In figure 6 we plot the normalized wave function $\rho R(\rho)$ and the potential $V_S(\rho)$ in terms of $\rho$ for $\ell = 0$ and $a = 1$. In figure 7 we plot a three-dimensional picture of the potential which the particle on the surface observes. We comment finally that for $\ell \neq 0$ there is no bound state solution satisfying the boundary conditions.

6. Conclusion

The nonrelativistic quantum particle confined to a curved surface has been revisited in a more familiar notation and more details. We have studied explicitly the case of the Monge parametrization in two different coordinate systems, i.e., Cartesian and Polar coordinates. We have shown that in the Cartesian coordinate system the geometric effective potential for the small perturbation is simply given by $V_\ell \simeq -\frac{\hbar^2}{2m}((H_{xx} - H_{yy})^2 + 4H_{xy}^2)$ which depends on the second derivatives of the height function $H(x, y)$. The effective Schrödinger equation i.e.,

$$-\frac{h^2}{2m} \nabla^2 \psi_\ell (x, y) - \frac{h^2}{8m}((H_{xx} - H_{yy})^2 + 4H_{xy}^2) \psi_\ell (x, y) = E_\ell \psi_\ell (x, y)$$

with $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is significantly simpler than the original one. In many cases where the deviation from a flat surface is small, we believe that this equation is a very good and acceptable approximation. In the last part of the paper, we studied two interesting curved surfaces, particularly a Catenoid and a paraboloid of revolution. We solved the corresponding Schrödinger equation of a particle confined on these surfaces without any external potential. We found only one possible bound state for each surface which localizes the particle around the deep of the point of maximum curvature. The exact normalized wave functions with the potentials and the surfaces have been displayed in a number of figures.

Data availability statement

No new data were created or analyzed in this study.
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