Role of mean free path in spatial phase correlation and nodal screening

B.A. van Tiggelen, D. Anache and A. Ghysels(*)

Laboratoire de Physique et Modélisation des Milieux Condensés/CNRS, Maison des Magistères/UFJ, BP 166, 38042 Grenoble, France.

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Abstract. – We study the spatial correlation function of the phase and its derivative, and related, fluctuations of topological charge, in two and three dimensional random media described by Gaussian statistics. We investigate their dependence on the scattering mean free path.

Introduction. – In disordered media the scattering mean free path (notation $\ell$) denotes the characteristic distance that waves need to travel to achieve a random phase [1]. The length $\ell$ sets the scale in all aspects of multiple scattering. “Mesoscopic” samples have sizes larger than $\ell$ and smaller than the decoherence length, beyond which the phase is irreversibly destroyed, by either coupling to environment (quantum) or due to background noise (classical). A third fundamental length, the transport mean free path (notation $\ell^*$), determines the randomization of the direction of propagation. It is closely related to $\ell$ but the physics is subtly different: whereas $\ell$ is determined by superposition, $\ell^*$ is also sensitive to interference.

Despite its conceptual importance, $\ell$ is rather unaccessible. Theoreticians have found it hard to calculate for dense media because infinitely many “Feynman diagrams” contribute [1], and choices have usually been guided by effective medium arguments [2]. In experiments it is mostly $\ell^*$ that appears in observables associated with energy transport [3]. To get $\ell$ directly we must measure the ‘coherent’ field $\langle \Psi \rangle$, either for different sample thicknesses or for different times. This requires a coherent source, phase sensitive detection of an exponentially small signal, and a good ensemble average. Today, these requirements can be met for ultrasound [4] and electromagnetic waves [5]. A final complication is that coherent beam experiments always measure the combination of phase randomization and absorption, quantified by the extinction length $\ell_e$. Here, we investigate the possibility to retrieve the scattering mean free path of a random medium from the spatial phase statistics of the scattered waves, rather than from the coherent field. The phase of a diffuse field can be measured using a “passive” source, whose coherence, position and magnitude are of no relevance, provided it delivers diffuse energy well in excess of the background noise.

(*) present address: Center for Molecular Modelling, Universiteit Gent, Proeftuinstraat 86, 9000, Gent, Belgium

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First we introduce the field correlation function in random media and discuss the role of small absorption. Secondly we discuss the basic results related to phase statistics and give a rapid theoretical support. We finally discuss the relation to topological charge, whose variance has been studied extensively in literature.

**Spatial Field Correlation.** – In a multiply scattering medium far from the localization threshold, it is widely accepted that the complex field obeys circular Gaussian statistics [7]. Since the average field \( \langle \Psi(\mathbf{r}, t) \rangle \) vanishes if the source is far away, the only parameter governing the wave statistics is the two-point correlation function \( \langle \Psi(\mathbf{r} - \frac{1}{2}\mathbf{x}, t - \frac{1}{2}\tau)\Psi(\mathbf{r} + \frac{1}{2}\mathbf{x}, t + \frac{1}{2}\tau) \rangle \) [6]. We consider a heterogeneous medium in either 2D or 3D, in which all waves suffer from the same small absorption rate \( \frac{1}{\tau_a} \) in energy (absorption length \( \ell_a = c\tau_a \)). We assume the medium to be large enough to ignore the boundaries.

We consider a wave field \( \psi_B(\mathbf{r}, t) \) released by a source with spectral density \( S(\omega) \). The subscript \( B \) refers to a band-filtering in some frequency band \( B \) in which diffusion constant \( D \) and absorption time \( \tau_a \) are constant (\( Bt > 1 \) is imposed to resolve the dynamics). Its (ensemble-averaged) spatial correlation at any lapse time \( t \) in the coda takes the following diffuse form [8],

\[
\left\langle \Psi_B(\mathbf{r} - \frac{1}{2}\mathbf{x}, t)\Psi_B(\mathbf{r} + \frac{1}{2}\mathbf{x}, t) \right\rangle \sim \frac{\exp\left(\frac{-r^2}{4Dt^2} - \frac{i}{\tau_a}\right)}{(2\pi)^2} \int \frac{d\omega}{2\pi} S(\omega) \text{Im} \ G(\mathbf{x}, \omega, \ell_e^{-1} \to \ell^{-1})
\]

This formula shows that, once absorption has been acknowledged by the factor \( \exp(-t/\tau_a) \), the field correlation function is proportional to (the imaginary part of) the absorption-free Green function. This is best understood in the frequency domain, where Eq. (11) gives rise to the following product of exponentials

\[
\exp[i(\omega + i/2\tau_a)(t - \tau/2)] \times \exp[i(\omega + i/2\tau_a)(t + \tau/2)]^* = \exp(-i\omega\tau)\exp(-i\tau/\tau_a),
\]

from which infer that the absorption only appears as a function of the lapse time \( t \), and not of the local correlation time \( \tau \). A first observation is thus that in the diffuse time-tail, we can consider the (normalized) spatial correlation of the field to be free of absorption.

In this work we shall ignore the effects of source spectrum and finite bandwidth, which can both straightforwardly be included into Gaussian statistics [18]. If we denote by \( C(x) \) the spatial field correlation function at frequency \( \omega \) [9], in the vicinity of some point \( \mathbf{r} \), deep in the coda at time \( t \), and normalized to \( C(0) = 1 \), we find it to be solely dependent on the local Green’s function, a central issue in recent attempts to image passively [10]. Contrary to chaotic media, the \( C(x) \) of random media always decays exponentially due to dephasing from scattering. In 3D is \( C(x) = \text{sinc}(kx)\exp(-x/2\ell) \), whereas in 2D \( C(x) = J_0(kx)\exp(-x/2\ell) \) [11], with \( k = 2\pi/\lambda \) the wave number. The damped oscillations on the scale of the wavelength \( \lambda \) originate from a superposition of plane waves incident with arbitrary directions but with equal amplitude. It prevents us from seeing the genuine dephasing that occurs on the much longer scale of the scattering mean free path.

**Phase correlation.** – Rather than the field correlation, we propose here to consider the spatial phase correlation \( \Xi_B(\mathbf{x}, t) \equiv \langle \Phi(\mathbf{r} - \frac{1}{2}\mathbf{x}, t)\Phi(\mathbf{r} + \frac{1}{2}\mathbf{x}, t) \rangle \). The phase \( \Phi(\mathbf{r}, t) \) is defined as the complex phase of the wave function \( \Psi_B = A\exp(i\Phi) \). At any point \( \mathbf{x} \) the phase is a flat random number between \( -\pi \) and \( \pi \). Upon moving to another point \( \mathbf{x}' \) it will exhibit discontinuous jumps of \( 2\pi \) between \( \pm\pi \). An occasional jump of a \( \pi \) could occur when a singularity with either positive or negative topological charge [12] is crossed, but such event
dx
an infinitesimal increment
speaking - be equal to the continuous cumulative phase. If we agree to walk straight along
the \( x \)-plateau after a few mean free paths. The plateau value depends logarithmically on the value
rise is visible on the scale of the wavelength. Subsequently, the phase variance reaches the diffuse
\( x \)-against distance
Fig. 3 – Cumulative phase variance
\[ \xi_U(x) \] versus distance \( x \) for a random medium in two dimensions. The distance has been measured with respect to the scattering mean free path \( \ell \), and the correlation function has been normalized to \( k\ell/\pi \) which corresponds to (minus) the asymptotic value for large \( kR \). The curves for \( k\ell = 10, 100, 200 \) overlap and follow an exponential decay (in red).

Fig. 2 – Spatial cumulative phase correlation \( \Xi(x) \) for a random 3D medium, for different values of
\( k\ell \). The asymptotic value varies logarithmically with \( k\ell \). Dashed line is obtained by integrating just
the asymptotic formula (4).

Fig. 3 – Cumulative phase variance \( \langle \Phi^2(x) \rangle \) in 2D random media, divided by the optical path \( kx \),
against distance \( x \), measured in units of the mean free path. Two length scales exist. An initial linear
rise is visible on the scale of the wavelength. Subsequently, the phase variance reaches the diffuse
plateau after a few mean free paths. The plateau value depends logarithmically on the value \( k\ell \).

has zero probability on a 1D line. Let us now “unwrap” the phase, i.e correct by hand for the
\( 2\pi \) discontinuities as was done in the frequency domain [13] and in the time domain [14],
to get the unwrapped phase or cumulative phase \( \Phi_U \) that varies continuously with \( x \), and which
in principle can take all values \([-\infty, \infty]\). This continuation is not topologically invariant,
but depends on the exact path chosen to go from \( r \) to \( r + x \). Gaussian statistics demonstrate
that both the distribution \( P(\Phi/dx) \), and the spatial correlation function \( C(\Phi(\Delta x)) \) of the phase
derivative \( d\Phi/dx \) are smooth, confirming that phase jumps do not affect phase statistics within
an infinitesimal increment \( dx \). So the integral of \( d\Phi/dx \) along the path should - statistically
speaking - be equal to the continuous cumulative phase. If we agree to walk straight along
the \( x \)-axis, this results in,
\[ \Xi_U(x) = \int_0^{-x/2} dx' \int_0^{x/2} dx'' \left( \frac{d\Phi}{dx}(x') \frac{d\Phi}{dx}(x'') \right) = -\int_0^{x/2} dx' \left[ C(\Phi(x')) + C(\Phi(x-x')) \right]. \] (1)

The second equality uses that the phase derivative correlation function \( C(\Phi(x', x'')) \) depends only on \( |x' - x''| \).

The variance of the unwrapped phase has already been discussed in literature in view of its
close connection with fluctuations in the positions of zeros of the wave function [15,16,18,19].
It follows from Eq. (1) that,
\[ \langle \Phi_U^2(x) \rangle = 2 \int_0^x dx' \left( x - x' \right) C(\Phi(x')). \] (2)

The phase derivative correlation function \( C(\Phi(x)) \) can be calculated from the joint probability
function of 4 complex fields at 4 different positions, which involves both \( C(x), C'(x) \) and
The result was earlier reported as an integral [20], but can be put into closed form,

\[ C_\Phi(x) = \frac{2}{\pi} \int_0^\pi d\phi \left[ \sin^2 \phi H'(C \cos \phi) - \left( \frac{C''}{1 - C^2} \cos \phi \right) + \left( \frac{C'}{1 - C^2} \right) \cos \phi \right] \]

with \( H(y) = \arctan(\sqrt{1 + y}/\sqrt{1 - y})/\sqrt{1 - y^2} \). Already for \( x > \lambda \), \( C \) is small enough to approximate \( C_\Phi(x) \) by its asymptotic limit, and we find

\[ C_\Phi(x > \lambda) = \frac{1}{2} (\log C'' \log(1 - C^2)) \] (3)

We see that, contrary to the field correlation \( C \), \( C_\Phi \) exhibits no oscillations on the scale of the wavelength. Equation (4) shows that the mean free path \( \ell \) assures convergence of Eq. (1) in both dimensions, making \( \ell \) the characteristic length scale in spatial phase correlation. In fig. 1 we plot \( \Xi_U(x) \) as a function of \( x/\ell \) for 2D. For all values of \( k\ell \) it follows a universal exponential decay towards the asymptotic value \(-k\ell/\pi\). In 3D the asymptotic value varies only logarithmically with \( k\ell \) in 3D (fig. 2) and two characteristic length scales appear, \( \lambda \) and \( \ell \). A measurement of \( \Xi_U(x) \) would thus give us direct access to \( k\ell \) and \( \ell \) in both cases.

The same study has been performed for the phase variance. We infer from Eq. (2) that the parameter \( \langle \Phi^2_U(x) \rangle/x \) is less sensitive to large values for \( x \) than the correlation \( \Xi_U(x) \). In 2D random media (fig. 3), its plateau value \( D_\Phi \) (the “phase diffusion constant”) depends logarithmically on \( k\ell \), although the diffuse plateau is still reached after typically one mean free path. In 3D (dashed line in fig. 4) we see that after an approximate linear rise, \( \langle \Phi^2_U(x) \rangle/x \) converges already after a few wavelengths to the asymptote \( D_\Phi = 1.922 \times k \), independent of \( k\ell \). We conclude that the cumulative phase correlation is sensitive to the mean free path in both 2D and 3D, unlike the phase variance which depends on \( \ell \) only in 2D.

Statistics of Topological Charge. — It was realized recently by Wilkinson [16] that the spatial phase correlation function is a key element in a old discussion on the statistics of topological charge of Gaussian field inside a closed contour. This discussion goes back to Halperin [17] and Nye and Berry [6], with significant contributions later by Berry and Dennis [18], and Freund and Wilkinson [15]. The zero’s of a complex function in \( d \) dimensions are located on \( d - 2 \) “nodal” hypersurfaces, nodal lines in 3D and nodal points in 2D. The topological charge \( Q \) enclosed by a 2D surface is defined as the number of nodal points weighed by their topological sign. This number is determined by the order of the vortex surrounding the nodal point [12]. For complex Gaussian random fields topological charges different from \( \pm 1 \) are highly improbable [6]. The following relation holds,

\[ \oint_{\Gamma} d\mathbf{r} \cdot \nabla \Phi(\mathbf{r}) = 2\pi Q \] (5)

with \( \Gamma \) the line contour enclosing the surface. This relation is reminiscent of the quantized rotation of superfluids, and is deeply related to the Rouché theorem in complex analysis. Its validity can be understood by applying Stokes’ theorem to the function \( \nabla \log \psi(\mathbf{r}) \), whose imaginary part has been analytically continued along the contour.

Equation (5) facilitates a study of the statistics of the topological charge \( Q \). In the diffuse regime the phase gradient in any direction vanishes on average, so \( < Q > = 0 \). Following Wilkinson [16] we will relate the variance of \( Q \) to the correlation of the phase derivative.
We shall define the screening length $\xi$ are not independent but tend to be perfectly screened [17,18], which affects the quadratic law.

More precisely, if $\theta \times R$ measures the length along a circular contour with radius $R$ we find,

$$\langle Q^2(R) \rangle = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\phi \int_0^{2\pi} d\theta' C_\phi(\Delta \theta) = \frac{1}{2\pi} \int_0^{2\pi} d\Delta \theta \ C_\phi(\Delta \theta)$$

The phase derivative correlation $C_\phi(\Delta \theta) \equiv \langle \partial_\theta \Phi(\theta) \partial_\theta \Phi(\theta') \rangle$ is, mutatis-mutandis, given by Eq. 8 (with $' = \partial_\theta$ and $C = C(2kR \sin[(\Delta \theta)/2])$). One basic feature of $\langle Q^2(R) \rangle$ is already known. If we would assume all nodal points $n$ to have random charges $Q_n = \pm 1$, independent of each other, we would find $\langle (\sum Q_n)^2 \rangle \sim R^2$, i.e. proportional to the surface. Actually, they are not independent but tend to be perfectly screened [17,18], which affects the quadratic law.

We shall define the screening length $\xi$ as the typical length beyond which the asymptotic form of the charge variance is reached. Wilkinson and Freund report a linear, “diffuse” asymptotic form [12],

$$\lim_{R \to \infty} \frac{\langle Q^2(R) \rangle}{R} = \frac{1}{\pi} \int_0^\infty dx \frac{C'(x)^2}{1 - C(x)^2}$$

We have verified that Eq. 10 is consistent with this diffusion law, with the same phase diffusion constant obtained earlier for the phase accumulated along a straight line (Fig. 3 and 4). Berry and Dennis [18] calculate $\langle Q^2(R) \rangle$ from the topological charge correlation function, which is, due to screening, sensitive to “edge effects”. When the charge distribution is smoothed with
the Gaussian $\exp(-R^2/R_0^2)$ (with smoothed surface $\pi R_0^2$), the quadratic law valid for small $R_0$ turns over into the asymptote,

$$\lim_{R_0 \to \infty} \langle Q_{sm}^2(R_0) \rangle = \int_0^\infty dx \frac{C'(x)^2}{1 - C(x)^2}$$  \hspace{1cm} (8)

provided the integral converges. In figs. 4 and 5 we have calculated $\langle Q^2 \rangle$ for a disk in 2D and 3D random media. For small $kR$ we see that $\langle Q^2 \rangle \sim k^2 R^2$ for both dimensions. In 3D $\langle Q^2 \rangle$ at large $R$ depends very weakly on the mean free path $\ell$ since the integral in Eq. (8) converges even for $\ell = \infty$. For the field correlation $C(x) = 2J_1(kx)/kx$ [12] the curve is very similar though with a somewhat smaller phase diffusion constant. In both cases this yields a screening length proportional to the wavelength. For 2D random media we can see that $\langle Q^2 \rangle$ depends logarithmically on $k\ell$. The topological screening length now depends on the mean free path. The variance of smoothed charge $\langle Q_{sm}^2 \rangle$ however (fig. 6), converges to $k\ell/\pi$ in 2D and to $0.54 \log k\ell$ in 3D, with a screening length typically equal to the $\ell$. Note that in 2D chaotic media, for which $C(x) = J_0(kx)$, the integrals (7) and (8) would diverge and the topological charge variance would actually increase as $\langle Q^2 \rangle \propto kR \log(kR)$, i.e. is slightly superdiffuse, whereas its smoothed version $\langle Q_{sm}^2 \rangle$ no longer reaches a constant but becomes diffuse: $\langle Q_{sm}^2 \rangle \to kR_0/(2\sqrt{\pi})$.

**Conclusions.** – We have shown that in the late coda of waves propagating in 2D and 3D random media, the scattering mean free path governs the spatial fluctuations of both the phase derivative $d\Phi/dx$, cumulative phase and the topological charge, measured inside a surface of size $A$. Screening makes topological charge fluctuations grow slower than $A$. The screening length is defined as the typical length beyond which they reach their final behavior. In 2D it is proportional to the mean free path, in 3D to the wavelength. If the charge is smoothed with a Gaussian the screening length is typically equal to the mean free path, in both dimensions. This highlights the subtle role of smoothing screened charge.

The spatial correlation of $d\Phi/dx$, whose measurement requires only four phase-sensitive detectors, decays exponentially with the scattering mean free path $\ell$ as the sole length scale. In seismic measurements, often dominated by 2D Rayleigh waves, this may be a novel opportunity to measure the scattering mean free path of surface waves. Such measurement would not be sensitive to absorption, neither to the precise source location, quite opposed to a measurement of the coherent beam.

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