Exact plane gravitational waves and electromagnetic fields

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The behaviour of a “test” electromagnetic field in the background of an exact gravitational plane wave is investigated in the framework of Einstein’s general relativity. We have expressed the general solution to the de Rham equations as a Fourier–like integral. In the general case we have reduced the problem to a set of ordinary differential equations and have explicitly written the solution in the case of linear polarization of the gravitational wave. We have expressed our results by means of Fermi Normal Coordinates (FNC), which define the proper reference frame of the laboratory. Moreover we have provided some gedanken experiments, showing that an external gravitational wave induces measurable effects of non tidal nature via electromagnetic interaction. Consequently it is not possible to eliminate gravitational effects on electromagnetic field, even in an arbitrarily small spatial region around an observer freely falling in the field of a gravitational wave. This is opposite to the case of mechanical interaction involving measurements of geodesic deviation effects. This behaviour is not in contrast with the principle of equivalence, which applies to arbitrarily small region of both space and time.

I. INTRODUCTION

The behaviour of a “test” electromagnetic field (i.e. an electromagnetic field whose stress–energy tensor does not affect the curvature of the underlying space–time) in a gravitational wave background has been widely studied in the framework of Einstein’s general relativity. This topic is very important to conceive possible further experimental verification of general relativity and also to better understand the principles underlying the theory itself. A great deal of efforts have been aimed at solving Maxwell (or de Rham) equations to first order in the gravitational wave amplitude [1–12] or, at most, to second order under geometrical optics limit [13]. However, a general solution within the framework of the full theory of general relativity would highlight the main features of a free electromagnetic field in radiative curved space–time. Indeed any possible ambiguity arising from approximation procedures could be circumvented [11,12]. We point out that such ambiguities exist not only when strong gravitational waves are concerned, but also may appear for weak gravitational fields as well. An example of this is the question related to photon creation [see discussion in Sec. VI after Eq. (6.27)]. Therefore any result obtained in the framework of linearized general relativistic theory has to be handle with care due to the non linear character of the full Einstein theory. The linearized theory is not general relativity but minkowski flat space–time plus a small perturbation.

To solve the problem one can take advantage of the freedom in the choice of a particular system of coordinates in which writing the equations. Obviously it is convenient to choose the reference frame in which the equations are easier to be solved. This is usually accomplished when the metric tensor is expressed in its simplest form. This system attached to the wave will be hereinafter referred to as Wave Reference Frame (WRF). In the limit of linearized gravity, it reduces to the usual Transverse Traceless (TT) gauge [14]. Recently a solution in the WRF has been obtained [15]. However, such a system is not the one where measurements are performed (laboratory frame). Therefore, in order to obtain results that have a direct interpretation from the physics point of view, one should express the electromagnetic field variables in a reference frame attached to an observer. The most natural way to construct such a frame is to consider the observer freely falling in the field of the gravitational wave. Such coordinates are the well–known Fermi Normal Coordinates (FNC) [14,16].

The aim of this paper is to obtain the solution of this problem in FNC and employ this solution to predict a few new physical effects. To this purpose we have re–obtained the solution in WRF in a slightly different way, expressing it as a Fourier–like integral. This form is more suitable for applications to concrete physical situations. Moreover we have obtained the transformation rules between WRF and FNC which hold true in each point of the region of space–time where the plane propagation of the exact gravitational wave is valid.

One of the most important results is that in FNC it is possible to infer the presence of a gravitational wave over an arbitrarily small spatial region in the neighbourhood of the origin. In other words there are both tidal and non–tidal effects in spatial coordinates. Therefore, near the origin, d’Alembertian operator applied to the electromagnetic field is non–vanishing, but rather proportional to the order of magnitude of the electromagnetic field. This is not surprising

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because Maxwell second–order equations involve the Riemann tensor \([14,17]\). Consequently the solution to Maxwell equations cannot be written as a monochromatic plane wave and no photon is created by the interaction. We stress that this behaviour is not in contrast with the principle of equivalence, which applies to arbitrarily small region of space–time \([14]\). This topic is more deeply investigated in \([18]\).

The paper is organized as follows. In Secs. II and III we perform calculation in WRF. In Sec. IV we provide the transformation rules connecting WRF and FNC. In Sec. V we express the four–vector potential in the latter frame. Finally, in Sec. VI the theory is applied to a case that is theoretically interesting and opens up some gedanken experiments.

**II. SOLUTION TO THE FREE DE RHAM EQUATIONS IN WRF**

Let us call \(y^\mu\) the coordinates in WRF. According to \([14,19,20]\) an exact plane gravitational wave propagating along the \(y^3\)–axis is described by the following line element

\[
ds^2 = - (y^0)^2 + (y^3)^2 + \mathcal{P}(y^3 - y^0)(dy^1)^2 + \mathcal{Q}(y^3 - y^0)(dy^2)^2 + 2 \mathcal{S}(y^3 - y^0)dy^1dy^2
\]

We do not make any assumption about the actual form or amplitude of the three quantities \(\mathcal{P}, \mathcal{Q}, \text{ and } \mathcal{S}\) but simply require that the metric tensor is an exact solution to the vacuum Einstein equations. Moreover, were the curvature small, the line element \((2.1)\) would just describe a weak plane waves in a \(TT\) reference frame \([14]\). Since metric coefficients depend on the coordinates only through \(y^3 - y^0\) we are naturally led to perform the following transformation:

\[
Y^0 = y^3 - y^0 \\
Y^1 = y^1 \\
Y^2 = y^2 \\
Y^3 = y^3 + y^0
\]

Since the wave keeps plane in the new set of coordinates, the new reference frame \(Y^\mu\) will still be referred to as \(\text{WRF}\).

The covariant components of the metric tensor become:

\[
g_{\mu\nu} = \begin{pmatrix}
0 & 0 & 0 & \frac{1}{2} \\
0 & \mathcal{P}(Y^0) & \mathcal{S}(Y^0) & 0 \\
0 & \mathcal{S}(Y^0) & \mathcal{Q}(Y^0) & 0 \\
\frac{1}{2} & 0 & 0 & 0
\end{pmatrix}
\]

In order to have a real space–time, the following inequality must hold true:

\[
g = -\frac{1}{4} (\mathcal{P} \mathcal{Q} - \mathcal{S}^2) < 0
\]

First of all we consider the general form of de Rham equations under Lorentz condition, describing the propagation of free electromagnetic field in empty curved space–time (we stress again that the electromagnetic field strength is considered so small as to neglect its effects on the curvature of space–time):

\[
\mathcal{A}^{\alpha;\nu} = 0 \quad \mathcal{A}^{\nu} = 0
\]

By substituting the metric tensor \((2.3)\) in \((2.5)\)–\((2.6)\) we achieve (we use convention and notation as in \([14]\) except for \(p, q, r, \text{ and } s\) which take values 1 and 2):

\[
g^{\mu\nu} \mathcal{A}^{\alpha;\nu} + 2 g^{\alpha\sigma} \mathcal{g}_{qp} \partial_3 \mathcal{A}^p + g^{qp} g^{\sigma\rho} \mathcal{g}_{q} \partial_r \mathcal{A}^0 - 2 \delta^3 g^{qp} \mathcal{g}_{r} \partial_p \mathcal{A}^r
\]

\[
-\frac{1}{2} \delta^3 g^{rs} g^{\rho\sigma} \mathcal{g}_{q} \mathcal{g}_{ps} \mathcal{A}^0 + 2 \frac{d \log \sqrt{(-g)}}{dY^0} \partial_3 \mathcal{A}^0 = 0,
\]

with Lorentz condition:

\[
\partial_\mu \mathcal{A}^\mu + \frac{d \log \sqrt{(-g)}}{dY^0} \mathcal{A}^0 = 0;
\]
dot means derivation with respect to the argument. As is well known in classical electrodynamics \((e.g. [2]1\)) even for potential satisfying Lorentz condition there remains the possibility to perform a restricted gauge transformation such that the resulting new potential still satisfies the Lorentz condition.

Our aim is to solve the system of equations \((2.7)\) taking into account condition \((2.8)\); in order to simplify them, we choose the particular restricted Lorentz gauge in which \(A^0 = 0\). We will show that this choice is always possible, without loss of generality (analogously to the classical charge–free case \([21]\)). In fact, if \(A'^\mu\) satisfies Eq. \((2.7)\) with the Lorentz condition \((2.8)\), then also \(A^\mu + \Lambda^\mu\) does, provided that:

\[
g^{\alpha\nu} \Lambda_{\nu,\alpha} + 2 \frac{d \log \sqrt{(-g)}}{dY_0} \Lambda,3 = 0, \tag{2.9}\]

where \(\Lambda\) is a scalar function. Setting \(A^0 = 0\), then \(\Lambda,3 = \frac{1}{2} A^0\). If this last condition holds, then, by substitution in Eq. \((2.7)\) for \(\alpha = 0\), it follows that \(\Lambda\) indeed does satisfy:

\[
\left( g^{\alpha\nu} \Lambda_{\nu,\alpha} + 2 \frac{d \log \sqrt{(-g)}}{dY_0} \Lambda,3 \right)_3 = 0. \tag{2.10}\]

In conclusion, for the charge–free case, it is always possible to set \(A^0 = 0\).

Because of this choice both Eqs. \((2.7)\) and \((2.8)\) turn into simpler ones. Namely:

\[
g^{\mu\nu} A^\alpha,\mu,\nu + 2 g^{\alpha\nu} \dot{g}_{\nu\rho} \partial_3 A^\rho - 2 \delta^\alpha_3 g^{\nu\rho} \dot{g}_{\nu\rho} \partial_\rho A^\nu + 2 \frac{d \log \sqrt{(-g)}}{dY_0} \partial_\rho A^\alpha = 0 \tag{2.11}\]

and:

\[
\partial_k A^k = 0 \tag{2.12}\]

The advantage of having replaced the old variables \(y^\alpha\) with \(Y^\alpha\) is that in Eqs. \((2.11)\) there is no second–order derivative with respect to either \(Y^1\) or \(Y^3\). This fact, together with the dependence of the coefficients of the metric–tensor components only on \(Y^0\), make it possible to turn the partial derivatives system \((2.11)\) into a simple ordinary one. If we write the solution as a Fourier integral

\[
A^\alpha = \int d^3 \lambda e^{[i(\lambda_1 Y^1)]} a^\alpha (Y^0, \lambda_1, \lambda_2, \lambda_3) \tag{2.13}\]

and substitute it in the first two equations of system \((2.11)\), we get rid of the dependence on \(Y^1, Y^2\) and \(Y^3\) and obtain the following ordinary system for Fourier–coefficients \(a^s\) and \(a^2\):

\[
4i \lambda_3 \frac{d a^s}{dY_0} - \left( \lambda_1^2 + \lambda_2^2 \right) a^s + f^pq \lambda_p \lambda_q a^s + 2i \lambda_3 g^{sq} \dot{g}_{pq} a^p + i \lambda_3 \frac{d \log \sqrt{(-g)}}{dY_0} a^s = 0 \tag{2.14}\]

where we have set \(f_{pq} = \delta_{pq} - g^{pq}\). The above system of equations can still be simplified by setting

\[
a^s = \gamma B^s \tag{2.15}\]

with

\[
\gamma = \frac{1}{(-g)^\frac{1}{4}} \exp \left[ \frac{(\lambda_1^2 + \lambda_2^2) Y^0 - f^{(-1)pq} \lambda_p \lambda_q}{4i \lambda_3} \right] \tag{2.16}\]

where \(f^{(-1)pq}\) is any primitive of \(f^{pq}\). Equations \((2.14)\) yield the following system of equations for the unknown quantities \(B^s\):

\[
\frac{dB^s}{dY_0} + \frac{1}{2} \mathcal{M}^s_p B^p = 0 \tag{2.17}\]

where we have set:

\[
\mathcal{M}^s_p = g^{sq} \dot{g}_{qp} \tag{2.18}\]

before discussing the solutions of system \((2.17)\) we consider last equation of the system \((2.11)\):
\[ 4i\lambda_3 \frac{da_3}{dY^0} - \left( \lambda_1^2 + \lambda_2^2 \right) a_3 + f^{pq} \lambda_p \lambda_q a_3 - 2i\lambda_p g^{pq} \dot{g}_{qs} a_s + i\lambda_3 \frac{d\log (-g)}{dY^0} a_3 = 0 \] (2.19)

The solution of this equation comes directly from Lorentz condition:

\[ a^0 = 0; \quad a^3 = \gamma B^3; \quad B^3 = -\frac{\lambda_r B^r}{\lambda_3} \] (2.20)

Now only Eq. (2.17) needs to be solved: once \( B^1 \) and \( B^2 \) have been obtained, it is trivial to achieve \( A^\alpha \) by means of Eqs. (2.15), (2.20) and (2.13). The solution to Eq. (2.17) can be more or less difficult to be determined depending on the actual form of \( P(Y^0) \), \( Q(Y^0) \), and \( S(Y^0) \). In the general case, an analytical solution to Eqs. (2.17) could not be available. A major advantage of this system is its non–dependence on \( \lambda_k \); therefore a numerical solution can be easily implemented. The general solution to Eqs. (2.17) can be inferred by multiplying the numerical solution by an arbitrary function of \( \lambda_k \).

The results of this Section are in agreement with those obtained in [15]. Yet, we have used a different formalism which is more suitable for application to concrete physical situations. For instance, by choosing properly the Fourier–like coefficients of Eq. (2.13), any possible particular boundary or initial condition can be obtained.

In next Section we will show that it is possible to solve exactly such a system for the case of a linearly polarized gravitational wave.

### III. LINEAR POLARIZATION

An exact plane gravitational wave is linearly polarized when the metric tensor can be put into diagonal form through a fixed rotation in the \( Y^1–Y^2 \) plane.

Consequently, without losing generality, we choose the \( Y^\mu \) reference frame in such a way that the only non–vanishing components of the metric tensor are:

\[ g_{03} = g_{30} = \frac{1}{2}, \quad g_{11} = P(Y^0), \quad g_{22} = Q(Y^0). \] (3.1)

If we set \( P(Y^0) = L^2(Y^0) \exp(2\beta(Y^0)) \) and \( Q(Y^0) = L^2(Y^0) \exp(2\beta(Y^0)) \), \( L(Y^0) \) and \( \beta(Y^0) \) being arbitrary functions, then the condition for the gravitational field equations to be satisfied becomes [14]:

\[ \frac{d^2 L}{d(Y^0)^2} + \left( \frac{d\beta}{dY^0} \right)^2 L = 0 \] (3.2)

One can easily see that, were \( \beta \) small, the metric would reduce to that of a usual weak gravitational wave in linearized theory [14]. According to [14], \( L(Y^0) \) and \( \beta(Y^0) \) are referred to as background and wave factors.

We now proceed to calculate the exact solution to the system of equations (2.17). By using the background and wave factors instead of \( P \) and \( Q \), it reads:

\[ \frac{dB^1}{dY^0} + \frac{1}{L} \left( \frac{dL}{dY^0} + L(Y^0) \frac{d\beta}{dY^0} \right) B^1(Y^0) = 0 \] (3.3)

\[ \frac{dB^2}{dY^0} + \frac{1}{L} \left( \frac{dL}{dY^0} - L(Y^0) \frac{d\beta}{dY^0} \right) B^2(Y^0) = 0. \] (3.4)

If we set:

\[ \mathcal{V}^+ = \beta + \frac{1}{2} \log (L^2), \quad \mathcal{V}^- = \beta - \frac{1}{2} \log (L^2) \] (3.5)

after some manipulation, we obtain:

\[ \frac{dB^1}{B^1} = -d\mathcal{V}^+ \] (3.6)

\[ \frac{dB^2}{B^2} = d\mathcal{V}^-. \] (3.7)
These equations can be easily integrated obtaining:

\[ B^1(Y^0) = b^1 e^{-V^+} \]  \hspace{1cm} (3.8)

\[ B^2(Y^0) = b^2 e^{V^-} \]  \hspace{1cm} (3.9)

where \( b^1 \) and \( b^2 \) are constants. Substitution of Eqs. (3.8) and (3.9) in Eqs. (2.20) yields:

\[ B^3 = -\frac{\lambda_1}{\lambda_3} b^1 e^{-V^+} - \frac{\lambda_2}{\lambda_3} b^2 e^{V^-}. \]  \hspace{1cm} (3.10)

As for Eq. (2.16), we obtain

\[ \gamma = \sqrt{2} e^{-\frac{1}{2}(V^+-V^-)+\frac{i}{4\lambda_3} ((\lambda_1)^2 V^++(\lambda_2)^2 V^-)} \]  \hspace{1cm} (3.11)

where \( V^+ \) and \( V^- \) are functions defined as follows:

\[ \frac{dV^\pm(w)}{dw} = e^{\mp 2V^\pm(w)} \]  \hspace{1cm} (3.12)

IV. FNC FOR AN EXACT LINEARLY POLARIZED WAVE

The results we have obtained in last section are valid in the WRF; though they do not correspond to any measurable quantity. In fact, the proper reference frame where an observer may execute a measurement is the FNC [14,16]. Therefore any tensor expressed in WRF must be written in FNC. To this aim, this section is devoted to linking the WRF to the laboratory reference frame (FNC); this is accomplished by following the procedure outlined in [14,16].

First, we consider an observer moving along a time–like geodesic in WRF; let \( q(\tau) \) be its world line, \( \tau \) its proper time, and \( f^\alpha(0) \) its four–velocity. Thus we can write:

\[ f^\alpha(0) = \frac{dq^\alpha(\tau)}{d\tau}. \]  \hspace{1cm} (4.1)

To achieve the actual form of the geodesic the following equations need to be solved:

\[ \frac{df^\alpha(0)}{d\tau} + \Gamma^\alpha_{\mu\nu} f^\mu(0) f^\nu(0) = 0 \]  \hspace{1cm} (4.2)

\[ g_{\mu\nu} f^\mu(0) f^\nu(0) = -1 \]  \hspace{1cm} (4.3)

The only non–vanishing Christoffel symbols are:

\[ \Gamma^0_{01} = -\frac{\partial V^+}{\partial Y^0} \]  \hspace{1cm} (4.4)

\[ \Gamma^2_{02} = -\frac{\partial V^-}{\partial Y^0} \]

\[ \Gamma^3_{11} = -\frac{\partial}{\partial Y^0} \left( e^{2V^+} \right) \]  \hspace{1cm} (4.4)

\[ \Gamma^3_{22} = -\frac{\partial}{\partial Y^0} \left( e^{-2V^-} \right) \]

After straightforward calculation we obtain:

\[ f^0(\tau) = \varphi^0(\tau) \]  \hspace{1cm} (4.5)

\[ f^1(\tau) = \varphi^1(\tau) e^{-2V^+} \]  \hspace{1cm} (4.6)

\[ f^2(\tau) = \varphi^2(\tau) e^{2V^-} \]  \hspace{1cm} (4.7)

\[ f^3(\tau) = -\frac{\varphi^3(\tau)}{\varphi^0(\tau)} e^{-2V^+} - \frac{\varphi^2(\tau)}{\varphi^0(\tau)} e^{2V^-} \]  \hspace{1cm} (4.8)

where \( \varphi^\alpha(\tau) \) are four constants. The geodesic parameterization may be achieved by performing the following integration:

\[ q^\alpha(\tau) = q^\alpha(0) + \int_0^\tau f^\alpha(\tau') \, d\tau'. \]  \hspace{1cm} (4.9)
Once $f_{(0)}$ has been determined, which is the time–like vector of the orthonormal tetrad carried by the observer, we determine the other three orthonormal space–like vectors $f_{(j)}$, to complete the observer’s orthonormal tetrad $\{f_{(i)}\}$. These vectors have to be parallel transported along $q(\tau)$. Therefore the following set of equations must hold true:

$$\frac{df^\alpha_{(i)}}{d\tau} + \Gamma^\alpha_{\mu\nu} f^\mu_{(i)} f^\nu_{(0)} = 0. \quad (4.10)$$

The solution can be written as:

$$f^0_{(i)}(\tau) = \varphi^0_{(i)} \quad (4.11)$$

$$f^1_{(i)}(\tau) = \frac{\varphi^1_{(i)} \varphi^0_{(0)}}{\varphi^0_{(0)}} e^{-2V^+(\varphi^0_{(0)} \tau)} + \varphi^1_{(i)} e^{-V^+(\varphi^0_{(0)} \tau)} \quad (4.12)$$

$$f^2_{(i)}(\tau) = \frac{\varphi^2_{(i)} \varphi^0_{(0)}}{\varphi^0_{(0)}} e^{2V^-(\varphi^0_{(0)} \tau)} + \varphi^2_{(i)} e^{V^-(\varphi^0_{(0)} \tau)} \quad (4.13)$$

$$f^3_{(i)}(\tau) = -\frac{\varphi^0_{(i)} \varphi^1_{(0)}}{\varphi^0_{(0)}} e^{-2V^+(\varphi^0_{(0)} \tau)} - \frac{\varphi^0_{(i)} \varphi^2_{(0)}}{\varphi^0_{(0)}} e^{2V^-(\varphi^0_{(0)} \tau)}$$

$$+ 2 \frac{\varphi^1_{(i)} \varphi^1_{(0)}}{\varphi^0_{(0)}} e^{-V^+(\varphi^0_{(0)} \tau)} - 2 \frac{\varphi^1_{(i)} \varphi^2_{(0)}}{\varphi^0_{(0)}} e^{V^-(\varphi^0_{(0)} \tau)} + \varphi^3_{(i)} \quad (4.14)$$

where $\varphi_{(j)}$ are twelve constants. Besides, orthonormality conditions imply:

$$g_{\mu\nu} f^\mu_{(i)} f^\nu_{(j)} = 0 \quad g_{\mu\nu} f^\mu_{(i)} f^\nu_{(j)} = \delta_{ij}. \quad (4.15)$$

The conditions given in Eqs. (4.3) and (4.15) hold true provided that:

$$\varphi^3_{(i)} \varphi^0_{(0)} + \varphi^0_{(i)} \varphi^3_{(0)} = 0,$n

$$\varphi^0_{(i)} \varphi^3_{(j)} + \varphi^3_{(i)} \varphi^0_{(j)} + 2 \varphi^1_{(i)} \varphi^1_{(j)} + 2 \varphi^2_{(i)} \varphi^2_{(j)} = 2 \delta_{ij}, \quad (4.16)$$

Next step is construction of a space–like geodesic, originating from an arbitrary point on the observer world line. Let $\sigma$ be the space–like geodesic parameter; therefore $Y(\sigma, \tau)$ is the point parameterized by the $\sigma$–variable on the space–like geodesic whose emanation point is $q(\tau)$. Let $f^\alpha = \frac{dY^\alpha}{d\sigma}$ be the tangent vector in $Y(\sigma, \tau)$ to the space–like geodesic; the geodesic equation reads:

$$\frac{df^\alpha}{d\sigma} + \Gamma^\alpha_{\mu\nu} f^\mu f^\nu = 0. \quad (4.17)$$

We seek out the solution to the above equation satisfying the following conditions at $\sigma = 0$:

$$f^\mu(\sigma = 0) = \alpha^j f^\mu_{(j)}(\tau), \quad (4.18)$$

$$Y^\mu(\sigma = 0) = q^\mu(\tau), \quad (4.19)$$

where $\alpha^j$ are the direction cosines of the geodesic at $q(\tau)$. In order to achieve $f$ we perform a calculation, which is formally very similar to the one we made to solve Eq. (4.2); then we obtain $Y^\alpha(\sigma, \tau)$ by direct integration:

$$Y^\alpha(\sigma, \tau) = q^\alpha(\tau) + \int_0^\sigma f^\alpha d\sigma' \quad (4.20)$$

A natural way to identify an event in the observer’s reference frame is taking the four numbers $\{x^k\}$:

$$x^0 = \tau \quad x^k = \alpha^k \sigma \quad (4.21)$$

Substitution of these identities in Eq. (4.20) yields the desired coordinate transformation:
In the framework of weak–gravitational–field approximation a similar result was found in [16].

\[
Y^1 = \frac{(\varphi_1^0(k)x^k)}{(\varphi_0^0(k)x^k)} e^{V^+(\varphi_0^0x^0)} \left[ V^+(\varphi_0^0x^0 + \varphi_0^0(k)x^k) - V^+(\varphi_0^0x^0) \right] + \frac{\varphi_1^0}{\varphi_0^0} [V^+(\varphi_0^0x^0 + \varphi_0^0(k)x^k) - V^+(0)] + q^1(0)
\]

(4.23)

\[
Y^2 = \frac{(\varphi_2^0(k)x^k)}{(\varphi_0^0(k)x^k)} e^{-V^-(\varphi_0^0x^0)} \left[ V^-(\varphi_0^0x^0 + \varphi_0^0(k)x^k) - V^-(\varphi_0^0x^0) \right] + \frac{\varphi_2^0}{\varphi_0^0} [V^-(\varphi_0^0x^0 + \varphi_0^0(k)x^k) - V^-(0)] + q^2(0)
\]

(4.24)

\[
Y^3 = \frac{(\varphi_3^0(k)x^k)}{(\varphi_0^0(k)x^k)} \left[ V^+(\varphi_0^0x^0 + \varphi_0^0(k)x^k) - V^+(0) \right] - \frac{(\varphi_3^0(k)x^k)^2}{(\varphi_0^0(k)x^k)^2} \left[ V^-(\varphi_0^0x^0 + \varphi_0^0(k)x^k) - V^-(0) \right] - \frac{(\varphi_2^0(k)x^k)}{(\varphi_0^0(k)x^k)} e^{2V^+(\varphi_0^0x^0)} + 2 \frac{\varphi_1^0(k)x^k}{(\varphi_0^0(k)x^k)^2} e^{V^+(\varphi_0^0x^0)} \left[ V^+(\varphi_0^0x^0 + \varphi_0^0(k)x^k) - V^+(\varphi_0^0x^0) \right] - \frac{(\varphi_2^0(k)x^k)}{(\varphi_0^0(k)x^k)} e^{-2V^-(\varphi_0^0x^0)} + 2 \frac{\varphi_1^0(k)x^k}{(\varphi_0^0(k)x^k)^2} e^{-V^-(\varphi_0^0x^0)} \left[ V^-(\varphi_0^0x^0 + \varphi_0^0(k)x^k) - V^-(\varphi_0^0x^0) \right] + \frac{\varphi_3^0(k)x^k}{(\varphi_0^0(k)x^k)} + \frac{(\varphi_1^0(k)x^k)^2}{(\varphi_0^0(k)x^k)^2} + \frac{(\varphi_2^0(k)x^k)^2}{(\varphi_0^0(k)x^k)^2} + \frac{\varphi_3^0}{\varphi_0^0} x^0 + q^3(0)
\]

(4.25)

In the framework of weak–gravitational–field approximation a similar result was found in [16].

The metric tensor components in FNC could be written through the usual relation

\[
\gamma_{\mu\nu} = \frac{\partial Y^\alpha}{\partial x^\mu} \frac{\partial Y^\beta}{\partial x^\nu} g_{\alpha\beta}.
\]

(4.26)

Yet, for our purposes, it is sufficient to know that \( \gamma_{\mu\nu} = \eta_{\mu\nu} + O(\left| x \right|^2) \) and we do not need to calculate them explicitly. It can also be checked that \( \Gamma^\alpha_{\mu\nu} = O(\left| x \right|) \).

Although coordinate transformation (4.22)–(4.23) was determined in the case of a diagonal metric tensor in WRF, the expression we obtained holds true for a generic linearly polarized plane wave. Consequently Eq. (3.1) does not lead to any loss in generality.

V. FOUR–VECTOR POTENTIAL IN FNC

The knowledge of the transformation rules given by Eqs. (4.22)–(4.23) allows one to obtain — via direct differentiation — the expressions of \( \frac{\partial Y^\alpha}{\partial x^\mu} \) needed to calculate covariant components of a generic tensor in the FNC. Therefore the covariant components of the four–vector potential in the WRF are to be determined. From Eqs. (2.3), (2.13), (2.15), (2.20), (3.8)–(3.11) we get:

\[
a_0 = -\frac{1}{\sqrt{2} \lambda_3} \left( \lambda_1 b_1 e^{-Y^+(Y^0)} + \lambda_2 b_2 e^{-Y^-(Y^0)} \right) e^{-\frac{i}{2} (Y^+(Y^0) - Y^-(Y^0))} e^{\frac{i}{4x_3} \left( \lambda_1^2 b_1^2 (Y^+(Y^0) + Y^-(Y^0)) + (\lambda_2 b_2)^2 Y^-(Y^0) \right)}
\]

(5.1)

\[
a_1 = \frac{\sqrt{2} b_1}{\lambda_3} e^\frac{i}{2} (Y^+(Y^0) + Y^-(Y^0)) e^{\frac{i}{4x_3} \left( \lambda_1^2 b_1^2 (Y^+(Y^0) + Y^-(Y^0)) + (\lambda_2 b_2)^2 Y^-(Y^0) \right)}
\]

(5.2)

\[
a_2 = \frac{\sqrt{2} b_2}{\lambda_3} e^\frac{-i}{2} (Y^+(Y^0) + Y^-(Y^0)) e^{\frac{i}{4x_3} \left( \lambda_1^2 b_1^2 (Y^+(Y^0) + Y^-(Y^0)) + (\lambda_2 b_2)^2 Y^-(Y^0) \right)}
\]

(5.3)

\[
a_3 = 0
\]

(5.4)

By setting \( A_\mu \) as the covariant components of the four–vector potential in the FNC, they can be achieved by:

\[
A_\mu = \frac{\partial Y^\alpha}{\partial x^\mu} A_\alpha.
\]

(5.5)

The components are written as:
\[ A_\mu = \int d^3 \lambda \, e^{i \Psi} \bar{a}_\mu, \] (5.6)

where

\[ \bar{a}_0 = \left[ \frac{\varphi_0^{(0)} \lambda_1}{\sqrt{2} \lambda_3} + \sqrt{2} \varphi_{(0)}^{0} \frac{\varphi_0^{(0)} x^k}{\varphi_{(0)}^{0} x^k} e^{V^+(\phi_0)} + \sqrt{2} \varphi_{(0)}^{1} \right] b^1 e^{-\frac{i}{2}(\alpha V^+(\phi) - V^-(\phi))} \] (5.7)

\[ + \left[ \frac{\varphi_0^{(0)} \lambda_2}{\sqrt{2} \lambda_3} + \sqrt{2} \varphi_{(0)}^{0} \frac{\varphi_0^{(0)} x^k}{\varphi_{(0)}^{0} x^k} e^{-V^-(\phi_0)} + \sqrt{2} \varphi_{(0)}^{2} \right] b^2 e^{-\frac{i}{2}(V^+(\phi) - 3 V^-)} \] (5.8)

\[ + \sqrt{2} \varphi_{(0)}^{0} \left[ \dot{V}^+(\phi_0) e^{V^+(\phi_0)} (V^+(\phi) - V^+(\phi_0)) - e^{-V^+(\phi_0)} \right] \frac{\varphi_0^{(0)} x^k}{\varphi_{(0)}^{0} x^k} b^1 e^{\frac{i}{2}(V^+(\phi) + V^-)} \]

\[ - \sqrt{2} \varphi_{(0)}^{0} \left[ \dot{V}^-(\phi_0) e^{-V^-(\phi_0)} (V^- - V^-(\phi_0)) + e^{-V^-(\phi_0)} \right] \frac{\varphi_0^{(0)} x^k}{\varphi_{(0)}^{0} x^k} b^2 e^{\frac{i}{2}(V^+(\phi) + V^-)}, \]

and

\[ \Psi = \varphi_{(k)}^{x^k} e^{V^+(\phi_0)} \left( \lambda_1 - 2 \lambda_3 \varphi_{(0)}^{x^0} \right) D^+ + \varphi_{(k)}^{x^k} e^{-V^-(\phi_0)} \left( \lambda_2 - 2 \lambda_3 \varphi_{(0)}^{x^0} \right) D^- \]

\[ - \frac{\lambda_3}{\varphi_{(0)}^{x^0}} \left( \varphi_{(0)}^{x^0} D^+ + \varphi_{(0)}^{x^0} D^- \right) + \lambda_3 (C^+ + C^-) + \lambda_3 \varphi_{(0)}^{x^0} x^k + \lambda_3 \varphi_{(0)}^{x^0} + \lambda_3 q^3(0) \]

\[ - \frac{(\lambda_1)^2 V^+(\phi) + (\lambda_2)^2 V^-(\phi)}{4 \lambda_3}. \] (5.9)

In the previous equations we set

\[ \phi = \varphi_{(0)}^{x^0} + \varphi_{(k)}^{x^k} \quad \phi_0 = \varphi_{(0)}^{x^0}. \] (5.10)

As for Eq. (5.5), the following functions were introduced:

\[ C^+ = \left( \frac{\varphi_{(k)}^{x^k} x^k}{\varphi_{(0)}^{x^0} x^k} \right)^2 \left[ 1 - e^{2 V^+(\phi)} D^+ \right], \] (5.11)

\[ C^- = \left( \frac{\varphi_{(k)}^{x^k} x^k}{\varphi_{(0)}^{x^0} x^k} \right)^2 \left[ 1 - e^{-2 V^-} D^- \right], \] (5.12)

\[ D^\pm = \frac{\varphi_{(k)}^{x^k} x^k}{\varphi_{(0)}^{x^0} x^k} \phi^\pm(\phi_0), \] (5.13)

\[ D^+ = \frac{\varphi_{(0)}^{x^0}}{\varphi_{(0)}^{x^0}} \left[ V^+(\phi) - V^+(0) \right], \] (5.14)

\[ D^- = \frac{\varphi_{(0)}^{x^0}}{\varphi_{(0)}^{x^0}} \left[ V^-(\phi) - V^-(0) \right]. \] (5.15)
VI. APPLICATIONS AND DISCUSSION

In order to emphasize geometrical effects over cinematic ones we are naturally led to choose the coefficients \( \varphi_{(\mu)}^{\nu} \) as follows:

\[
\varphi_{(0)}^{\mu} = -\delta_{\mu 0} + \delta_{\mu 3} \\
\varphi_{(i)}^{j} = D_{i}^{j} \\
\varphi_{(i)}^{0} = \varphi_{(i)}^{3}
\]

(6.1)

where

\[
D_{i}^{j} D_{j}^{k} = \delta_{i}^{k}.
\]

(6.2)

With this choice the observer is assumed to be at rest in the \( y_{\mu}^{\mu} \) reference frame with given orientation with respect to \( y_{\mu}^{\mu} \) in \( q(\tau) \). Euler angles of rotation matrix \( D_{i}^{j} \) determine the orientation. One can easily see that the conditions (4.16) are met.

In order to obtain the relationship between Fourier coefficients \( \lambda \) and the usual flat space–time wave–vector, we set \( L = 1 \) and \( \beta = 0 \) in formula (5.9):

\[
\Psi = \left( \lambda_{j} D_{j}^{i} k - \frac{(\lambda_{1})^{2} + (\lambda_{2})^{2}}{4 \lambda_{3}} D_{3}^{i} k \right) x^{i} + \left( \lambda_{3} + \frac{(\lambda_{1})^{2} + (\lambda_{2})^{2}}{4 \lambda_{3}} \right) x^{0}.
\]

(6.3)

In a flat space-time, \( \Psi \) is usually defined as

\[
\Psi = \Psi^{\pm} = k_{j} x^{j} \pm k x^{0} \quad k = \sqrt{k_{1}^{2} + k_{2}^{2} + k_{3}^{2}}
\]

(6.4)

From the above equations, by taking into account the orthogonality of matrix \( D_{i}^{j} \) we obtain:

\[
\lambda_{r} = D_{r}^{j} k_{j} \\
\lambda_{3} = \lambda_{3}^{\pm} = \frac{D_{3}^{j} k_{j} \pm k}{2}
\]

(6.5)

The integral of Eq. (5.6) is expressed in terms of \( k \) as a sum of two terms:

\[
A_{\mu} = \int d^{3} k \left[ e^{i\Psi^{+}} \tilde{a}_{\mu}^{+} + e^{i\Psi^{-}} \tilde{a}_{\mu}^{-} \right]
\]

(6.6)

where \( \tilde{a}_{\mu}^{\pm} = \tilde{a}_{\mu} [\lambda^{\pm}(k), x_{\alpha}; b_{\alpha}(k)] \) with \( \tilde{a}_{\mu} \) given by Eqs. (5.7) and (5.8).

In order to better understand the behaviour of the solution, let us suppose:

\[
D_{j}^{i} = \delta_{j}^{i}.
\]

(6.7)

In this case

\[
\lambda_{r} = k_{r} \quad \lambda_{3}^{\pm} = \frac{k_{3} \pm k}{2},
\]

(6.8)

while

\[
\Psi^{\pm} = k_{1} x^{1} e^{i \psi^{+} (-x^{0})} D^{+} + k_{2} x^{2} e^{-i \psi^{-} (-x^{0})} D^{-} + \frac{k_{3} \pm k}{2} \left( x^{3} + x^{0} \right)
\]

\[
- \frac{(k_{1})^{2} V^{+}(x^{3} - x^{0}) + (k_{2})^{2} V^{-}(x^{3} - x^{0})}{2(k_{3} \pm k)} + \frac{k_{3} \pm k}{2} \left( C^{+} + C^{-} \right),
\]

(6.9)

where
\[ D^\pm = \frac{V^\pm (x^3 - x^0) - V^\pm (-x^0)}{x^3} \]

\[ D^\pm = 0 \]

\[ C^+ = \frac{(x^1)^2}{x^3} \left[ 1 - e^{2V^+ (x^3 - x^0)} D^+ \right] \]

\[ C^- = \frac{(x^2)^2}{x^3} \left[ 1 - e^{-2V^- (x^3 - x^0)} D^- \right] \]  

(6.10)

As an example we choose the constants \( b'_\pm \) so that there is a static magnetic field along the \( x^3 \) direction in absence of gravitational wave. This is mathematically accomplished by setting \( b'_\pm = b_\pm \delta_\gamma^1 \), where

\[ b_+ = b_- = -\frac{i E_0}{2 \sqrt{2} k} \delta(k_3 - k') \delta(k_1) \delta(k_2). \]  

(6.11)

By performing the integration and taking the limit \( k' \to 0 \), we achieve:

\[ A_0 = -\frac{x^1 x^2}{x^3} B_0 D^- e^{-V^+ (-x^0)} \left\{ e^{V^+ (-x^0)} e^{-\frac{1}{2} [3V^+ (x^3 - x^0) - V^- (x^3 - x^0) + V^- (x^3 - x^0)]} \right\} \]

\[ A_1 = x^2 B_0 D^+ D^- e^{\frac{1}{2} [V^+ (x^3 - x^0) + V^- (x^3 - x^0)]} e^{3 V^- (x^3 - x^0)} \]

\[ A_2 = 0 \]

\[ A_3 = \frac{x^1 x^2}{x^3} B_0 D^- e^{V^+ (-x^0)} e^{-V^- (-x^0)} \left\{ e^{-\frac{1}{2} [3V^+ (x^3 - x^0) - V^- (x^3 - x^0)]} - D^+ e^{\frac{1}{2} [V^+ (x^3 - x^0) + V^- (x^3 - x^0)]} \right\}. \]  

(6.12)

The above expressions hold true everywhere. In order to show their good behaviour in the neighborhood of the origin we perform a power expansion in \( x^3 \) up to \( |x|^2 \) terms. We obtain:

\[ A_0 = x^1 x^2 B_0 \dot{V}^+ (-x^0) e^{-\frac{1}{2} [V^+ (x^3 - x^0) + V^- (x^3 - x^0)]} \]  

(6.15)

\[ A_1 = x^2 B_0 \left[ 1 - \frac{x^3}{2} \ddot{V}^+ (-x^0) + \frac{3}{2} x^3 \dot{V}^- (-x^0) \right] e^{-[V^+ (-x^0) - 2V^- (-x^0)]} \]  

(6.16)

\[ A_2 = 0 \]  

(6.17)

\[ A_3 = -x^1 x^2 B_0 \dot{V}^+ (-x^0) e^{-\frac{1}{2} [V^+ (x^3 - x^0) - 3V^- (x^3 - x^0)]} \]  

(6.18)

The electromagnetic tensor field components can be obtained from the usual relation:

\[ F_{\mu \nu} = A_{\nu, \mu} - A_{\mu, \nu} \]  

(6.19)

Although it would be possible to calculate \( F_{\mu \nu} \) everywhere, we shall limit ourselves near the origin. Therefore, up to linear terms in \( x^3 \), we achieve:

\[ F_{01} = x^2 B_0 \left[ \dot{V}^+ (-x^0) - 2 \dot{V}^- (-x^0) \right] e^{-[V^+ (-x^0) - 2V^- (-x^0)]} \]

\[ - x^2 B_0 \dot{V}^+ (-x^0) e^{-\frac{1}{2} [V^+ (x^3 - x^0) + V^- (x^3 - x^0)]} \]  

(6.20)

\[ F_{02} = -x^1 B_0 \dot{V}^+ (-x^0) e^{-\frac{1}{2} [V^+ (x^3 - x^0) + V^- (x^3 - x^0)]} \]  

(6.21)

\[ F_{03} = 0 \]  

(6.22)

\[ F_{12} = -B_0 \left[ 1 - \frac{x^3}{2} \ddot{V}^+ (-x^0) + \frac{3}{2} x^3 \dot{V}^- (-x^0) \right] e^{-[V^+ (-x^0) - 2V^- (-x^0)]} \]  

(6.23)

\[ F_{23} = -x^1 B_0 \dot{V}^+ (-x^0) e^{-\frac{1}{2} [V^+ (x^3 - x^0) - 3V^- (x^3 - x^0)]} \]  

(6.24)

\[ F_{31} = x^2 B_0 \left[ -\frac{1}{2} \ddot{V}^+ (-x^0) + \frac{3}{2} \dot{V}^- (-x^0) \right] e^{-[V^+ (-x^0) - 2V^- (-x^0)]} + x^2 B_0 \dot{V}^+ (-x^0) e^{-\frac{1}{2} [V^+ (x^3 - x^0) - 3V^- (x^3 - x^0)]} \]  

(6.25)

We notice that in the laboratory frame the effect of a gravitational wave on an electromagnetic field is not only of tidal nature. In fact when \( x^k \to 0 \) the only non–vanishing component becomes:
\[ F_{12} = -B(x^0) = -B_0 e^{-[\nu^+(-x^0) - 2 \nu^-(x^0)]} \] (6.26)

Two important conclusions may be drawn from this rather simple case. First, the solution can not be interpreted as a photon creation. In fact, the solution described by Eqs. (6.20)–(6.25) has non–vanishing d'Alembertian, proportional to the field itself. Moreover direct computation of the divergence of Pointing vector at the origin reads:

\[ T^{0k}_{,k} = - \frac{d}{dx^0} \left\{ \frac{B_0^2}{8 \pi} e^{-2[\nu^+(-x^0) - 2 \nu^-(x^0)]} \right\} \] (6.27)

It is the time derivative of a function assuming finite values. Its time average vanishes: in fact, if the gravitational wave is a periodic function, the time average of Eq. (6.27) over the period \( T \) is zero; for a generic non–periodic wave, the time average vanishes over an interval of time that is longer than the signal duration (see for instance [13], §34). Therefore there is no net flux of electromagnetic radiation across any close surface. This implies the absence of any radiation field. This conclusion is not surprising since the solution still keeps the tensor form of a static magnetic field.

This is a good example in which the calculation in the framework of the full theory of general relativity settles an open question of the linearized theory. In fact, by using the linear approximation, the time average of the the Pointing vector at the origin vanishes (taking only first order terms). The first non–zero contribution is due to second order terms, which, however, are to be neglected in a linearized theory. One could be led to think that quadratic terms could possibly give rise to a non–vanishing time average of \( T^{0k}_{,k} \). In the full theory, Eq. (6.27) immediately shows that this is not the case.

Another interesting feature is the possibility for an observer in FNC to prove — at least in principle — the presence of a gravitational wave, by performing an experiment involving electromagnetic interaction. The non–tidal nature of the interaction allows one to assess the presence of a gravitational wave irrespective of the apparatus size. This is quite different to mechanical detectors, involving geodesic deviation measurements. In the following we will propose a few gedanken experiments aimed at showing this peculiarity.

**A. Motion of a particle on the \( x^3 = 0 \) plane**

As a first example we consider a charged particle moving on the \( x^3 = 0 \) plane. The exact equations of motion are given by [19]:

\[ mc \left( \frac{du^\mu}{ds} + \Gamma^\mu_{\alpha \beta} u^\alpha u^\beta \right) = \frac{e}{c} F^\mu_{\alpha} u^\alpha. \] (6.28)

As \( \Gamma^\mu_{\alpha \beta} = \mathcal{O}(x^k) \), we can assume that near the origin the leading term is the non–tidal one [see Eq. (6.26)]. Therefore equations of motion become (\( \gamma = (1 - \beta^2)^{-\frac{1}{2}}, \beta = \frac{u^3}{c}, v \) is the speed of the particle):

\[ \frac{du^0}{ds} = 0 \quad \Rightarrow \quad u^0 = \gamma \quad \Rightarrow \quad x^0 = \gamma s, \]
\[ \frac{du^3}{ds} = 0 \quad \Rightarrow \quad u^3 = 0 \quad \Rightarrow \quad x^3 = 0, \]
\[ \frac{du^1}{dx^0} = -\frac{e}{\mathcal{E}} B(x^0) u^2, \]
\[ \frac{du^2}{dx^0} = \frac{e}{\mathcal{E}} B(x^0) u^1, \]

where \( \mathcal{E} = \gamma m c^2 \) is the energy of the particle. We see that the speed of the particle is a constant of motion. Setting \( B dx^0 = dw \) we get:

\[ u^1 = \gamma \beta c_1 \cos \varphi(x^0) + \gamma \beta c_2 \sin \varphi(x^0), \]
\[ u^2 = \gamma \beta c_1 \sin \varphi(x^0) - \gamma \beta c_2 \cos \varphi(x^0), \]

\[ \varphi(x^0) = \frac{e}{\mathcal{E}} \left[ \int_0^{x^0} B(\xi) d\xi + w_0 \right] \quad c_1^2 + c_2^2 = 1 \]

the particle performs a non closed orbit around the origin, with a variable period \( T(x^0) \) given implicitly by:
\[ 2\pi = \int_{x_0}^{x_0 + cT(x^0)} \varphi(\xi) d\xi \quad (6.36) \]

By measurement of the time variation of the period (which is constant in flat space–time) one can infer the presence of a gravitational wave.

### B. Induced e.m.f. in conducting circuits

As a second example we consider a conducting ring with resistance \( R \), lying in the \( x^3 = 0 \) plane with its center at the origin (in this application SI units are used). Assuming the ring sufficiently small, we can neglect all the terms in the electromagnetic tensor except the one given by Eq. (6.26). Therefore Faraday law causes a current to flow. One has:

\[ I(t) = -\frac{S}{R} \frac{\partial B(t)}{\partial t} \quad (6.37) \]

where \( S \) is the ring surface and \( x^0 = ct \).

In general one expects the same kind of effect in RLC circuits. Let us consider a series circuit with a resistance \( R \), an inductance \( L \), and a capacitance \( C \) under the same assumption as before. In this case the equation of motion for the charge on the condenser plates is:

\[ \ddot{Q} + \frac{1}{\tau_0} \dot{Q} + \omega_0^2 Q = -\frac{S}{L} \dot{B} \]  

(6.38)

where \( \tau_0 = \frac{RC}{2}, \omega_0^2 = \frac{1}{LC} \), and in this case the dot means derivation with respect to \( t \). We notice that in flat space–time, where \( \dot{B} = 0 \), there is no current flowing. The effect of the gravitational wave is a current flowing through the circuit. In general the solution to Eq. (6.38) is given by:

\[ Q(t) = -\frac{S}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} f(\omega) d\omega \]

(6.39)

In order to study the main characteristics of the current we assume the gravitational wave to have a typical frequency \( \omega_g \). In this way the Fourier components of the magnetic field are approximately given by:

\[ B(\omega) \approx \sqrt{2\pi} B_0 \delta(\omega) - \frac{i}{2} f_0 \left[ \delta(\omega - \omega_g) - \delta(\omega + \omega_g) \right] \]  

(6.40)

Substitution in Eq. (6.38) gives for the flowing current:

\[ I(t) \approx \frac{S B_0 \omega_g^2 f_0}{2} \left[ \frac{e^{i\omega_g t}}{\omega_0^2 - \omega_g^2 + i \frac{\omega_0}{\tau_0}} \right] \]  

(6.41)

where \( \tau_0 = \frac{2 \sqrt{2\pi}}{\omega_0} \) is the quality factor of the circuit. Therefore it is possible, at least in principle, to detect a gravitational wave by measuring a current. It is noticeable that the effect is not affected by the size of the device, but only by the circuit parameters as it was already known in the framework of linear approximation (e.g. [22,23]).

### C. Paschen Back effect

It is well known that the spectrum of Alkali atoms embedded in a static and homogeneous magnetic field shows frequency shifts. We are here interested in magnetic fields strong enough to let us neglect spin–orbit terms. Besides we assume that the system is so close to the origin that we do not have to take into account tidal terms (i.e. we can use the usual quantum mechanics in flat space–time). Therefore under these assumptions we start from the Hamiltonian describing the interaction with an external electromagnetic field in the non relativistic regime (e.g. [24]). Substituting the value of the four–vector potential \( A_\mu \), given in Eqs. (7.13)–(7.18), and neglecting tidal effects we get:
\[ \hat{H} = \hat{H}_0 + \delta \hat{H} \]  
\[ \hat{H}_0 = \frac{\hat{P}^2}{2m} + V(r) \]  
\[ \delta \hat{H} = -\frac{e}{2mc} B(t) \left( \hat{L}_z + 2 \hat{S}_z \right) \]  
(6.42)  
(6.43)  
(6.44)

where \( e \) is the electron charge, \( m \) the electron mass, and \( V(r) \) the atomic potential energy. One could solve the problem using the standard time-dependent perturbation theory. However for typical frequencies of the gravitational wave much smaller than orbital ones (condition that is expected to hold true in nearly all circumstances), the effect results in a time variation of the frequency shifts of the flat space–time spectrum. Thus we get:

\[ \nu_{nl \rightarrow n'l'}(t) = \nu_{nl}^{(0)} - \frac{\mu_B B(t)}{\hbar} \Delta m_l; \quad \mu_B = \frac{|e|\hbar}{2mc} \]  
(6.45)

Under this assumption gravitational wave causes the distance between two lines of the usual Paschen Back spectrum to change with time.

D. Spin precession

Let us consider a free particle of charge \( q \) and spin \( \frac{1}{2} \). Neglecting all its degrees of freedom except for the spin, the interaction with an external magnetic field can be described introducing the following Hamiltonian:

\[ \hat{H} = -\frac{e}{mc} \mathbf{B}(t) \cdot \hat{\mathbf{S}} = -\frac{e}{mc} B(t) \hat{S}_z \]  
(6.46)

Once again we neglect tidal effects. This is true provided the particle is sufficiently near the origin. Therefore the magnetic field is given in Eq. (6.20). We obtain the evolution operator by solving Schrödinger equation. One has:

\[ i\hbar \frac{\partial}{\partial t} \hat{U} = \hat{H} \hat{U} \quad \Rightarrow \quad \hat{U} = e^{-i \oint \mathbf{B} \cdot d\mathbf{r}} \]  
(6.47)

where

\[ \phi(t) = \frac{eB_0}{2mc} [t + F(t)]; \quad F(t) = \int_0^t dt' \left\{ \frac{B(t)}{B_0} - 1 \right\} \]  
(6.48)

Let us suppose to prepare the particle in the following state (at time \( t = 0 \)):

\[ |\alpha \rangle = c_1 |+ \rangle + c_2 |> \]  
(6.49)

where \(|+ \rangle\) are the eigenkets of the spin operator along \( x^3 \) axis. Thus the state of the particle at time \( t \) is given by:

\[ |\alpha t \rangle = \hat{U}(t) |\alpha \rangle = c_1 e^{-i\phi(t)} |+ \rangle + c_2 e^{i\phi(t)} |> \]  
(6.50)

Let

\[ |x + \rangle = \frac{1}{\sqrt{2}} (|+ \rangle + |>\rangle) \]  
(6.51)

be the eigenket of the spin operator along \( x^1 \) axis with positive eigenvalue \( \frac{3}{2} \). As it is well known, the probability to find the particle in such a state at time \( t \) is given by

\[ p(t) = |\langle x + |\alpha t \rangle|^2 = \frac{1}{2} \left[ 1 + 2 |c_1| |c_2| \cos (\arg c_2 - \arg c_1 + \phi(t)) \right] \]  
(6.52)

If the magnetic field were static (i.e. in flat space–time) this probability would periodically vanish when \( n \) is integer:

\[ t = \tau_n^{(0)} = \frac{(\arg c_1 - \arg c_2) mc}{eB_0} + \frac{mc}{eB_0} \arccos \left( \frac{1}{2 |c_1| |c_2|} \right) + 2 \frac{mc \pi n}{eB_0} \]  
(6.53)

However, because of the presence of a gravitational wave this probability does not vanish in \( T_n^{(0)} \) any longer, but takes values depending on \( n \). Namely:

\[ p(T_n^{(0)}) = \frac{1}{2} \left[ 1 - \cos \left( \frac{eB_0}{mc} F(T_n^{(0)}) \right) - \sqrt{4 |c_1|^2 |c_2|^2 - 1} \sin \left( \frac{eB_0}{mc} F(T_n^{(0)}) \right) \right] \]  
(6.54)
VII. CONCLUSIONS

In this paper we have presented some interesting features of the behaviour of electromagnetic field in exact gravitational wave background by considering de Rham equations with no approximation. We have expressed our results in the laboratory frame (FNC) where any measurable quantity should be referred to.

We have investigated a particular case in order to better understand the main features of the solution. In particular we have shown the appearance of non-tidal effects. Furthermore we have conceived some experimental setups which—in principle—could outline the differences between the response of an electromagnetic device and a mechanical one. In fact the latter is related to the geodesic deviation, while the former includes spatial point-like effects as well. This behaviour is not in contrast with the principle of equivalence, which applies to arbitrarily small region of space-time.

It is also explicitly shown that in FNC the interaction does not create any photons. This is due to the fact that the d’Alembertian of the electromagnetic field does not vanish, being proportional to the field itself.

ACKNOWLEDGMENTS

The authors are pleased to thank V. Guidi for reading of the manuscript. One of the author (E. M.) wish also to thank D. Etro for invaluable help.

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