THE COHOMOLOGICAL CREPANT RESOLUTION CONJECTURE FOR
THE HILBERT-CHOW MORPHISMS

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ABSTRACT. In this paper, we prove that Ruan’s Cohomological Crepant Resolution Conjecture holds for the Hilbert-Chow morphisms. There are two main ideas in the proof. The first one is to use the representation theoretic approach proposed in [QW] which involves vertex operator techniques. The second is to prove certain universality structures about the 3-pointed genus-0 extremal Gromov-Witten invariants of the Hilbert schemes by using the indexing techniques from [LiJ], the product formula from [Beh2] and the co-section localization from [KL1, KL2, LL]. We then reduce Ruan’s Conjecture from the case of an arbitrary surface to the case of smooth projective toric surfaces which has already been proved in [Che].

1. Introduction

In [ChR], Chen and Ruan defined the orbifold cohomology ring $H^*_{CR}(Z)$ for an orbifold $Z$. Motivated by orbifold string theory from physics, Ruan [Ruan] proposed the Cohomological Crepant Resolution Conjecture. It eventually evolved into the Crepant Resolution Conjecture after the work of Bryan-Graber, Coates-Corti-Iritani-Tseng and Coates-Ruan [BG, CCIT, CoR]. Roughly speaking, assuming that an orbifold $Z$ has a crepant resolution $W$, then the Crepant Resolution Conjecture predicts that the orbifold Gromov-Witten theory of $Z$ is ring isomorphic (in the sense of analytic continuations, symplectic transformations and change of variables of type $q = e^{i\theta}$) to the ordinary cohomology ring of $W$ plus those quantum corrections on $W$ which are related to curves contracted by the crepant resolution. We refer to [BG, Che, Coa] and the references there for other excellent examples confirming the Crepant Resolution Conjecture.

In this paper, we prove that Ruan’s Cohomological Crepant Resolution Conjecture holds for the Hilbert-Chow morphisms. Let $X$ be a smooth projective complex surface, and $X^{[n]}$ be the Hilbert scheme of points in $X$. Sending an element in $X^{[n]}$ to its support in the symmetric product $X^{(n)}$, we obtain the Hilbert-Chow morphism $\rho_n : X^{[n]} \to X^{(n)}$, which is a crepant resolution of singularities. Let $H^*_{\rho_n}(X^{[n]})$ be the quantum corrected cohomology ring (see Sect. 4 for details).

Theorem 1.1. Let $X$ be a simply connected smooth projective surface. Then, Ruan’s Cohomological Crepant Resolution Conjecture holds for the Hilbert-Chow morphism $\rho_n$, i.e., the rings $H^*_{\rho_n}(X^{[n]})$ and $H^*_{CR}(X^{(n)})$ are isomorphic.

This theorem has been proved earlier when $n = 2, 3$ [ELQ, LQ], when $K_X$ is trivial [FG, LS], and when $X$ is a smooth toric surface [Che]. We also refer to [LQWA, MO, OP, QW, Zho] for discussions when $X$ is quasi-projective.

There are two main ingredients in our proof of Theorem 1.1. The first one is the axiomatization approach originated from [Leh, LQW1] and formulated in [QW]. This approach involves Heisenberg algebra actions and vertex operator techniques pioneered in [Gro, Nak]. We recall that a graded Frobenius algebra over a field $k$ is a finite dimensional graded vector space $A$ with a graded associative multiplication $A \otimes A \to A$ and unit element $1_A$ together with a linear form

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$T: A \to k$ such that the induced bilinear form $(a, b) := T(ab)$ is nondegenerate. For $k \geq 1$, the $k$-th co-product $\tau_k: A \to A \otimes k$ is defined by requiring $\langle \tau_k(a), b_1 \otimes \cdots \otimes b_k \rangle = T(ab_1 \cdots b_k)$. Now the axiomatization in $[QW]$ states that the algebra structure on each $A^n$ in a sequence of graded Frobenius algebras $A^{[n]}$ ($n \geq 0$) is determined if

(A1) the direct sum $\bigoplus_n A^n$ affords the structure of the Fock space of a Heisenberg algebra modeled on $A := A^{[1]}$.

(A2) There exists a sequence of elements $G_k(\alpha, n) \in A^n$ depending on $\alpha \in A$ (linearly) and a non-negative integer $k$. Define the operators $\tilde{G}_k(\alpha)$ on $\bigoplus_n A^n$ which act on the component $A^n$ via multiplication by $G_k(\alpha, n) \in A^n$. The operators $\tilde{G}_k(\alpha)$ and the Heisenberg generators satisfy:

$$\tilde{G}_1(\alpha) = -\frac{1}{6} : a^3 :_0 (\tau_3 1_A),$$  \hspace{1cm} \text{(1.1)}

$$\tilde{G}_k(\alpha), a_{-1}(\beta) = \frac{1}{k!} \alpha_{(k)} (\alpha \beta)$$  \hspace{1cm} \text{(1.2)}

where $: a^3 :_0$ is the zero mode in the normally ordered product $: a^3 :$, and $a_{(k)} = \frac{1}{k!} a_{(k)}$ denotes the $k$-th derivative with $\alpha_{(k)} = \alpha_{-1}(\alpha)$ and $a_{(k)} (\alpha) = \tilde{G}_1(\alpha), a_{(k-1)}(\alpha)$ for $k \geq 1$.

When (A1) and (A2) are satisfied, the algebra $A^n$ is generated by the elements $G_k(\alpha, n) \in A^n,$ \hspace{0.5cm} $\alpha \in A, k \geq 0$.

In addition, the product is determined by (1.1) and (1.2). On one hand, with $A^n = H^*_{CR}(X(n))$ (viewed as an algebra over $\mathbb{C}$), the results in $[QW]$ (see Theorem 4.1 below) indicate that (A1) and (A2) hold for the rings $H^*_{CR}(X(n))$. On the other hand, by $[\text{Grod} \text{ Nak}]$ and $[\text{LL}]$, the rings $A^n = H^*_{\rho_n}(X(n)) = H^*(X(n))$ as vector spaces) also satisfy (A1) and (1.1). Moreover, using $[\text{Che}]$, we prove that (1.2) holds when $X$ is a smooth projective toric surface.

To prove that the rings $A^n = H^*_{\rho_n}(X(n))$ satisfy (1.2), for an arbitrary surface $X$, our main ingredient comes into play. It involves finer analysis of the virtual fundamental cycle using the method in $[\text{LL}]$ and the co-section localization technique in $[\text{KL1} \text{ KL2} \text{ LL}]$. Let $X[n,d]$ be the moduli space of 3-pointed genus-0 degree-$d$ stable maps to $X^n$. By $[\text{LL}]$, every stable map $(\varphi, C) \in X[n,d]$ has a standard decomposition $\varphi = (\varphi_1, \ldots, \varphi_l) \in X[n,d]$ where the stable reduction $\varphi^\circ$ is contained in $X[n,d]$ for some $n_i$ and $d_i$, $\rho_n(\text{Im}(\varphi_i)) = n_i x_i$, the points $x_1, \ldots, x_l$ are distinct, and $\varphi(p) = \sum_i \varphi_i(p)$ for all $p \in C$. We use the ideas from $[\text{LL}]$ to index the support of $\rho_n(\text{Im}(\varphi)) = \sum_i n_i x_i \in X^n$. This is done by introducing the notion of 3-pointed genus-0 degree-$d$ $\alpha$-stable maps to $X^n$, where $\alpha = (\alpha_1, \ldots, \alpha_l)$ denotes a partition of the set $[n] = \{1, \ldots, n\}$ and $d = (d_1, \ldots, d_l)$ with $d_i$'s being nonnegative integers. The set of such pairs $(\alpha, d)$ with $\sum_i d_i = d$ is denoted by $\mathcal{P}[n,d]$. The techniques in $[\text{LL}]$ and the product formula in $[\text{Beh2}]$ for Gromov-Witten invariants enable us to express the virtual fundamental cycle $[X[n,d]]^{vir}$ in terms of certain discrepancy cycles $[\Theta^{[\alpha,d]}], (\alpha, d) \in \mathcal{P}[n,d]$. In fact, one of the key points in the paper is to study such decomposition of $[X[n,d]]^{vir}$ as a sum of cycles indexed by the partition type of $\rho_n(\text{Im}(\varphi)) \in X^n$. However, this cannot be done on the moduli space $X[n,d]$. The technique to overcome this impasse is to introduce the Hilbert scheme of $\alpha$-points $X[n]$ and an non-separated space $X^{\leq n}$ following $[\text{LL}]$. Then the cycle $ev_*([X[n,d]]^{vir})$ is a sum of various $[\Theta^{[\alpha,d]}]$ in $(X^{\leq n})^{\delta}$. Even though the space $X^{\leq n}$ is not Hausdorff in analytic topology, all the operations involving $X^{\leq n}$ in this paper are all algebraic topological, such as pullbacks of cohomology classes and cap products, which are defined on any topological spaces. Combining with the co-section localization theory in $[\text{KL1} \text{ KL2} \text{ LL}]$, pairs with $[\Theta^{[\alpha,d]}]$ can be studied via $C^\infty$-maps from $X$ to the Grassmannians. For $d \geq 1$, we assemble those $[\Theta^{[\alpha,d]}], (\alpha, d) \in \mathcal{P}[n,d]$ with $d_i > 0$ for every $i$ into a homology class $3_{n,d} \in H_*(X[n])^3$. Note that we are back to the original Hilbert scheme $X[n]$. Now the structure of the 3-pointed genus-0 extremal Gromov-Witten invariants of $X[n]$ is given by the following two theorems.
Theorem 1.2. Let $A_1, A_2, A_3 \in H^*(X^{[n]})$ be Heisenberg monomial classes, and $\pi_{m,i}$ be the $i$-th projection on $(X^{[n]})^3$. Then, $(A_1, A_2, A_3)_{0,d_n}$ is equal to

$$\sum_{m \leq n} \sum_{A_{1,1}=A_{1,2}=A_1} \langle A_{1,1}, A_{2,1}, A_{3,1} \rangle \cdot \left\langle 3_{m,d}, \prod_{i=1}^{3} \pi_{m,i}^* A_{i,2} \right\rangle.$$  \hspace{1cm} (1.3)

Theorem 1.3. Let $A_1, A_2, A_3 \in H^*(X^{[n]})$ be Heisenberg monomial classes.

(i) If $A_i$ contains a factor $a_{-\lambda}(x)$ for some $i$, then $\langle 3_{n,d}, \prod_{i=1}^{3} \pi_{m,i}^* A_i \rangle = 0$.

(ii) For $1 \leq i \leq 3$, let $A_i = a_{-\lambda}(x)a_{-\mu}(x)a_{-\lambda}(x)\cdots a_{-\mu}(x)$ where $u_i \geq 0$ and $|\alpha_{i,1}| = \ldots = |\alpha_{i,u_i}| = 2$. Then,

$$\left\langle 3_{n,d}, \prod_{i=1}^{3} \pi_{m,i}^* A_i \right\rangle = \prod_{i=1}^{3} \prod_{j=1}^{u_i} \langle K_X, \alpha_{i,j} \rangle \cdot p$$  \hspace{1cm} (1.4)

where $p$ is a polynomial in $\langle K_X, K_X \rangle$ whose degree is at most $(n - \sum_{i,j} n_{i,j})/2$, and whose coefficients depend only on $n, \lambda^{(i)}, n_{i,j}$ (and hence are independent of the surface $X$ and the classes $\alpha_{i,j}$).

We refer to Definition 2.9 for the operation $\circ$ appearing in (1.2), and to Definition 2.2 for the notation $a_{-\lambda}(x)$ appearing in Theorem 1.3 (ii). Geometrically, we may think of the pairing $(3_{n,d}, \prod_{i=1}^{3} \pi_{m,i}^* A_i)$ in (1.3) as the contributions of the non-constant components $\varphi_i$ in the standard decomposition of $\varphi = (\varphi_1, \ldots, \varphi_i, \ldots) \in X^{[n,d]}$, while those constant components $\varphi_i$ contribute to the factor $(A_{1,1}, A_{2,1}, A_{3,1})$ in (1.3).

Using Theorem 1.2 and Theorem 1.3 we are able to reduce the proof of (1.2) for $A^{[n]} = H^*_{\text{CR}}(X^{[n]})$ from an arbitrary surface $X$ to the case when $X$ is a smooth projective toric surface. This proves (1.2) for $A^{[n]} = H^*_{\text{CR}}(X^{[n]})$ and hence completes the proof of Theorem 1.1.

Finally, this paper is organized as follows. In Sect. 2, we review the Hilbert schemes of points on surfaces and Heisenberg algebras. In Sect. 3, we recall from [QW] the results regarding $H^*_{\text{CR}}(X^{[n]})$. In Sect. 4, we review Ruan’s Cohomological Crepant Resolution Conjecture. In Sect. 5, we prove Theorem 1.2 and Theorem 1.3. In Sect. 6, we verify (1.2) and Theorem 1.3.

Conventions: All the homology and cohomology groups are in $\mathbb{C}$-coefficients unless otherwise specified. For a subvariety $Z$ of a smooth projective variety $Y$, we will use $Z$ or $[Z]$ to denote the corresponding cycle/cohomology class, and use $1_Y$ to denote the fundamental cohomology class of $Y$. The symbol $A \cdot B$ denotes the cup product for $A, B \in H^*(Y)$. For $A_1, \ldots, A_k \in H^*(Y)$, let $\langle A_1, \ldots, A_k \rangle = \int_Y A_1 \cdots A_k$. By abuse of notation, for $A \in H_*(W)$ and $B \in H^*(W)$ of an arbitrary topological space $W$, $\langle A, B \rangle$ also stands for the natural paring between the homology group and the cohomology group. For subsets $A$ and $B$ of $W$, $A \cap B \subset W$ stands for the intersection of the two subsets; for $A \in H_*(W)$ and $B \in H^*(W)$, $A \cap B \in H_*(W)$ stands for the cap product.

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2. Hilbert schemes of points on surfaces

Let $X$ be a smooth projective complex surface with the canonical class $K_X$ and the Euler class $e_X$, and $X^{[n]}$ be the Hilbert scheme of points in $X$. An element in $X^{[n]}$ is represented by a length-$n$ 0-dimensional closed subscheme $\xi$ of $X$. It is well known that $X^{[n]}$ is smooth. For a subset $Y \subset X$, define

$$M_n(Y) = \{ \xi \in X^{[n]} | \text{Supp}(\xi) = \{ x \} \text{ for some } x \in Y \}.$$
Let $Z_n = \{(\xi, x) \subset X^{[n]} \times X \mid x \in \text{Supp}(\xi)\}$ be the universal codimension-2 subscheme of $X^{[n]} \times X$. Let $p_1$ and $p_2$ be the two projections of $X^{[n]} \times X$. Let

$$
\mathbb{H}_X = \bigoplus_{n=0}^{+\infty} H^*(X^{[n]})
$$

be the direct sum of total cohomology groups of the Hilbert schemes $X^{[n]}$.

For $m \geq 0$ and $n > 0$, let $Q^{[m,m]} = \emptyset$ and define $Q^{[m+n,m]}$ to be the closed subset:

$$
\{ (\xi, x, \eta) \in X^{[m+n]} \times X \times X^{[m]} \mid \xi \supset \eta \text{ and Supp}(I_\eta/I_\xi) = \{ x \}\}.
$$

We recall Nakajima’s definition of the Heisenberg operators [Nak]. Let $n > 0$. The linear operator $a_{-n}(\alpha) \in \text{End}(\mathbb{H}_X)$ with $\alpha \in H^*(X)$ is defined by

$$
a_{-n}(\alpha)(a) = \tilde{p}_1 \ast ([Q^{[m+n,m]}] \cdot \tilde{\rho}^* \alpha \cdot \tilde{p}_2^* a)
$$

for $a \in H^*(X^{[m]})$, where $\tilde{p}_1, \tilde{\rho}, \tilde{p}_2$ are the projections of $X^{[m+n]} \times X \times X^{[m]}$ to $X^{[m+n]}, X, X^{[m]}$ respectively. Define the linear operator $a_n(\alpha) \in \text{End}(\mathbb{H}_X)$ to be $(-1)^n$ times the operator obtained from the definition of $a_{-n}(\alpha)$ by switching the roles of $\tilde{p}_1$ and $\tilde{p}_2$. We also set $a_0(\alpha) = 0$.

For $n > 0$ and a homogeneous class $\alpha \in H^*(X)$, let $|\alpha| = s$ if $\alpha \in H^s(X)$, and let $G_i(\alpha, n)$ be the homogeneous component in $H^{[\alpha]+2i}(X^{[n]})$ of

$$
G(\alpha, n) = p_{1*}(\text{ch}(\mathcal{O}_{Z_n}) \cdot p_2^* \text{td}(X) \cdot p_2^* \alpha) \in H^*(X^{[n]})
$$

where $\text{ch}(\mathcal{O}_{Z_n})$ denotes the Chern character of $\mathcal{O}_{Z_n}$ and $\text{td}(X)$ denotes the Todd class. Set $G_i(\alpha, 0) = 0$. We extend the notion $G_i(\alpha, n)$ linearly to an arbitrary class $\alpha \in H^*(X)$. The Chern character operator $\Theta_1(\alpha) \in \text{End}(\mathbb{H}_X)$ is defined to be the operator acting on the component $H^*(X^{[n]})$ by the cup product with $G_i(\alpha, n)$. It was proved in [LQW1] that the cohomology ring of $X^{[n]}$ is generated by the classes $G_i(\alpha, n)$ where $0 \leq i < n$ and $\alpha$ runs over a linear basis of $H^*(X)$. Let $\delta = \Theta_1(1_X)$ where $1_X$ is the fundamental cohomology class of $X$. The operator $\delta$ was first introduced in [Leh]. For a linear operator $f \in \text{End}(\mathbb{H}_X)$, define its derivative $f'$ by $f' = [\delta, f]$. The $k$-th derivative $f^{(k)}$ is defined inductively by $f^{(k)} = [\delta, f^{(k-1)}]$.

Let $a_{m_1}a_{m_2} : = a_{m_1}a_{m_2}$ when $m_1 \leq m_2$ and $a_{m_1}a_{m_2}$ when $m_1 > m_2$. For $k \geq 1$, $\tau_{k*} : H^*(X) \to H^*(X^k)$ is the linear map induced by the diagonal embedding $\tau_k : X \to X^k$, and $a_{m_1} \cdots a_{m_k}(\tau_{k*}(\alpha))$ denotes $\sum_j a_{m_1}(\alpha_{j,1}) \cdots a_{m_k}(\alpha_{j,k})$ when $\tau_{k*} \alpha = \sum_j \alpha_{j,1} \otimes \cdots \otimes \alpha_{j,k}$ via the Künneth decomposition of $H^*(X^k)$.

The following is a combination of various theorems from [Nak, Gro, Leh, LQW1]. Our notations and convention of signs are consistent with [LQW2].

**Theorem 2.1.** Let $k \geq 0, n, m \in \mathbb{Z}$ and $\alpha, \beta \in H^*(X)$. Then,

(i) the operators $a_n(\alpha)$ satisfy a Heisenberg algebra commutation relation:

$$
[a_m(\alpha), a_n(\beta)] = -m \delta_{m,-n} \cdot \langle \alpha, \beta \rangle \cdot \text{Id}_{\mathbb{H}_X}.
$$

The space $\mathbb{H}_X$ is an irreducible module over the Heisenberg algebra generated by the operators $a_{-1}(\alpha)$ with a highest weight vector $|0\rangle = 1 \in H^0(X^{[0]}) \cong \mathbb{C}$.

(ii) $\Theta_1(\alpha) = -\frac{1}{6} : a^2 : 0 \cdot (\tau_{2*} \alpha) - \sum_{n>0} \frac{n-1}{2} : a_{-n} : (\tau_{2*}(K_X \alpha))$.

(iii) $[\Theta_k(\alpha), a_{-1}(\beta)] = \frac{1}{k!} a_{-(k)}(\alpha) \beta$.

The Lie brackets in Theorem 2.1 are understood in the super sense according to the parity of the degrees of the cohomology classes involved. Also, Theorem 2.1 (i) implies that $\mathbb{H}_X$ is linearly spanned by the cohomology classes $a_{-n_1}(\alpha_1) \cdots a_{-n_k}(\alpha_k)|0\rangle$ where $k \geq 0$ and $n_1, \ldots, n_k > 0$. These classes are called Heisenberg monomial classes.
Theorem 2.3. Let $k \geq 0$, $n \in \mathbb{Z}$, and $\alpha \in H^*(X)$. Then, $a_n^{(k)}(\alpha)$ equals

$$(-n)^k k! \left( \sum_{\ell(\lambda)=k+1, |\lambda|=n} \frac{1}{\lambda!} a_\lambda(\tau_\alpha) - \sum_{\ell(\lambda)=k-1, |\lambda|=n} \frac{s(\lambda)-1}{24\lambda!} a_\lambda(\tau_\epsilon(\epsilon_X \alpha)) \right)$$

$$+ \sum_{e \in \{K_X,K_X^2\}} f_{|e|}(\lambda) \frac{1}{\lambda!} a_\lambda(\tau_\epsilon(e\alpha))$$

where all the numbers $f_{|e|}(\lambda)$ are independent of $X$ and $\alpha$.

Theorem 2.4. Let $k \geq 0$ and $\alpha \in H^*(X)$. Then, $\Theta_k(\alpha)$ is equal to

$$- \sum_{\ell(\lambda)=k+2, |\lambda|=0} \frac{1}{\lambda!} a_\lambda(\tau_\alpha) + \sum_{\ell(\lambda)=k, |\lambda|=0} \frac{s(\lambda)-2}{24\lambda!} a_\lambda(\tau_\epsilon(\epsilon_X \alpha))$$

$$+ \sum_{e \in \{K_X, K_X^2\}} g_{|e|}(\lambda) \frac{1}{\lambda!} a_\lambda(\tau_\epsilon(e\alpha))$$

where all the numbers $g_{|e|}(\lambda)$ are independent of $X$ and $\alpha$.

Theorem 2.5. Let $n \geq 1$, $k \geq 0$, and $\alpha \in H^*(X)$. Then, $G_k(\alpha, n)$ is equal to

$$\sum_{0 \leq j \leq k} \sum_{\ell(\lambda)=k+j-1} \frac{(-1)^{|\lambda|-1}}{\lambda! \cdot |\lambda|!} \cdot 1_{-(n-j-1)} a_{-\lambda}(\tau_\alpha)[0]$$

$$+ \sum_{0 \leq j \leq k} \sum_{\ell(\lambda)=k+j-1} \frac{(-1)^{|\lambda|}}{\lambda! \cdot |\lambda|!} \cdot \frac{|\lambda|+s(\lambda)-2}{24} \cdot 1_{-(n-j-1)} a_{-\lambda}(\tau_\epsilon(\epsilon_X \alpha))$$

$$+ \sum_{e \in \{K_X, K_X^2\}} \sum_{0 \leq j \leq k} \frac{(-1)^{|\lambda|} g_{|e|}(\lambda+(1+j^2+1))}{\lambda! \cdot |\lambda|!} \cdot 1_{-(n-j-1)} a_{-\lambda}(\tau_\epsilon(e\alpha))$$

where $1_{-(n-j-1)}$ denotes $a_{-1}(1_X)^{n-j-1}/(n-j-1)!$ when $(n-j-1) \geq 0$ and is 0 when $(n-j-1) < 0$, the universal function $g_{|e|}$ is from Theorem 2.4, and $\lambda+(1+j^2+1)$ is the partition obtained from $\lambda$ by adding $(j+1)$ to the multiplicity of 1.

Lemma 2.6. $[a_{n_1} \cdots a_{n_s}(\tau_k \alpha), a_{m_1} \cdots a_{m_t}(\tau_s \beta)]$ is equal to

$$- \sum_{i=1}^{k} \sum_{j=1}^{s} n_i \delta_{n_i,-m_j} \left( \prod_{l=1}^{j-1} a_{m_l} \prod_{1 \leq u \leq s, u \neq t} a_{n_u} \prod_{l=j+1}^{m_t} a_{m_l} \right) (\tau_{(k+s-2)s}(\alpha \beta))$$

The above lemma was proved in [LQW2], and will be used implicitly in many proofs throughout the paper. The following geometric result was proved in [LQW3].
Lemma 2.8. Let the classes $\alpha_1, \ldots, \alpha_k \in \bigoplus_{i=1}^k H^i(X)$ be respectively represented by the cycles $X_1, \ldots, X_k \subset X$ in general position. Then, the Heisenberg monomial class

$$\left( \prod_{i=1}^t \frac{a_{-i}(1_X)^{s_i}}{s_i!} \right) \left( \prod_{j=1}^k a_{-n_j}(\alpha_j) \right) [0]$$

is represented by the closure of the subset consisting of elements of the form

$$\sum_{i=1}^t (\xi_{i,1} + \ldots + \xi_{i,s_i}) + \sum_{j=1}^k \xi_j$$

where $\xi_{i,m} \in M_i(x_{i,m})$ for some $x_{i,m} \in X$, $\xi_j \in M_{n_j}(x_j)$ for some $x_j \in X_j$, and all the points $x_{i,m}, 1 \leq i \leq t$, $1 \leq m \leq s_i$ and $x_j, 1 \leq j \leq k$ are distinct.

Theorem 2.9 in [LQW4] expresses a Heisenberg monomial class in terms of a polynomial of the classes $G_k(\gamma, n)$. The following lemma is a special case.

Lemma 2.8. Let $\lambda \vdash n_0$, $\alpha \in H^* (X)$ with $|\alpha| = 2$, and $m \geq 1$.

(i) Then, the class $1_{-\lambda}(\alpha, \lambda(x)[0] \in H^*(X[n])$ can be written as a polynomial of the classes $G_k(x, n), k \geq 0$. Moreover, the coefficients and the integers $k$ depend only on $\lambda$ (hence, are independent of $n$ and $X$);

(ii) If the odd Betti numbers of the surface $X$ are equal to zero, then

$$1_{-\lambda}(\alpha, \lambda(x)[0] = (K_{X, \alpha}) \cdot F_1(n) + \sum_{i} G_{k_i}(\alpha, n_i) \cdot F_{2,i}(n)$$

where $F_1(n)$ and $F_{2,i}(n)$ are polynomials of the classes $G_k(x, n), k \geq 0$. Moreover, the coefficients of $F_1(n)$, $F_{2,i}(n)$ and the integers $k_i$ depend only on $\lambda$ and $m$ (hence, are independent of $n, \alpha$ and $X$).

Proof. These follow from the same proof of Theorem 2.9 in [LQW4] by setting $I = C \cdot x + H^*(X)$ and $J = C \cdot x + C \cdot \alpha \in H^*(X)$ respectively.

3. The ring $H^*_\text{CR}(X^{(n)})$

For an orbifold $Z$, the ring $H^*_\text{CR}(Z)$ was defined by Chen and Ruan [ChR]. For a global orbifold $M/G$ where $M$ is a complex manifold with a finite group $G$ action, the ring structure of $H^*_\text{CR}(M/G)$ was further clarified in [FG] [CR].

Next, let $X$ be a closed complex manifold, and let $X^{(n)} = X^n/S_n$ be the $nth$ symmetric product of $X$. An explicit description of the ring structure of $H^*_\text{CR}(X^{(n)})$ has been obtained in [FG]. An alternative approach to the ring structure of $H^*_\text{CR}(X^{(n)})$ is given in [QW] via Heisenberg algebra actions. Put

$$F_X = \bigoplus_{n=0}^{+\infty} H^*_\text{orb}(X^{(n)}).$$

In [QW], for $\alpha \in H^*(X)$ and $n \in \mathbb{Z}$, the Heisenberg operators $p_n(\alpha) \in \text{End}(F_X)$ were defined via the restriction and induction maps. Moreover for $k \geq 0$, the elements $O^k(\alpha, n) \in H^*_\text{CR}(X^{(n)})$
were introduced via the Jucys-Murphy elements in the symmetric groups. Let $O_k(\alpha, n) = 1/k! \cdot O^k(\alpha, n)$. Let the operator $\Omega_k(\alpha) \in \text{End}(F_X)$ be the orbifold ring product with $O_k(\alpha, n)$ in $H^*_\text{CR}(X^{(n)})$ for every $n \geq 0$. The operator $\Omega_1(1_X)$ plays the role of the boundary operator $\partial = \partial_1(1_X)$ for the Hilbert schemes. Define $p_n^{(k)}(\alpha)$ inductively by putting $p_n^{(0)}(\alpha) = p_m(\alpha)$ and $p_n^{(k)}(\alpha) = [\Omega_1(1_X), p_{n-k-1}^{(k)}(\alpha)]$ for $k \geq 1$. The following result was proved in [QW].

**Theorem 3.1.** Let $X$ be a closed complex manifold. Then,

(i) the operators $p_n(\alpha) \in \text{End}(F_X)$ for $n \in \mathbb{Z}$, $\alpha \in H^*(X)$ generate a Heisenberg (super)algebra with commutation relations given by

$$[p_m(\alpha), p_n(\beta)] = m \delta_{m,-n} \cdot \langle \alpha, \beta \rangle \cdot \text{Id}_{F_X}$$

where $m, n \in \mathbb{Z}$, $\alpha, \beta \in H^*(X)$, and $F_X$ is an irreducible representation of the Heisenberg algebra with the vacuum vector $|0\rangle = 1 \in H^*(pt) \cong \mathbb{C}$.

(ii) $\Omega_1(1_X) = -\frac{1}{6} : p^3 : (\tau_1 X)$. In general, $\Omega_k(\alpha)$ is equal to

$$(-1)^k \cdot \left( \sum_{\ell(\lambda) = k+1, |\lambda| = 0} \frac{1}{\lambda!} p_\lambda(\tau_1 X) + \sum_{\ell(\lambda) = k, |\lambda| = 0} \frac{s(\lambda) - 24}{24 \lambda!} p_\lambda(e_X \alpha) \right).$$

(iii) $[\Omega_k(\alpha), p_{-1}(\beta)] = \frac{1}{k!} p_{-1}^{(k)}(\alpha, \beta)$, and both sides are equal to

$$(-1)^k \cdot \left( \sum_{\ell(\lambda) = k+1, |\lambda| = -1} \frac{1}{\lambda} p_\lambda(\tau_1 X) + \sum_{\ell(\lambda) = k-1, |\lambda| = -1} \frac{s(\lambda) - 1}{24 \lambda!} p_\lambda(e_X \alpha) \right).$$

Note that there is a fundamental sign difference in the two commutators of Theorems 2.1 (i) and Theorems 3.1 (i). Since $O_k(\alpha, n) = \Omega_k(\alpha) p_{-1}(1_X)^n/|n|!$, we see from formula [3.1] that the class $O_k(\alpha, n)$ is equal to

$$(-1)^k \cdot \left( \sum_{0 \leq j \leq k} \sum_{\ell(\lambda) = k-j+1} \frac{1}{\lambda!} \cdot \frac{1}{|\lambda|!} \cdot 1_{-(n-j-1)} p_{-1}(\tau_1 X) |0\rangle \right)$$

$$+ \sum_{0 \leq j \leq k} \sum_{\ell(\lambda) = k-j-1} \frac{1}{\lambda!} \cdot \frac{1}{|\lambda|!} \cdot \frac{|\lambda| + s(\lambda) - 2}{24} \cdot \frac{1}{24 \lambda!} \cdot 1_{-(n-j-1)} p_{-1}(\tau_1 X |0\rangle).$$

Moreover, as noted in [QW], the ring $H^*_\text{CR}(X^{(n)})$ is completely determined by Theorem 3.1 (i), the formula of $\Omega_1(1_X)$ in Theorem 3.1 (ii), and Theorem 3.1 (iii). In particular, the ring $H^*_\text{CR}(X^{(n)})$ is generated by the classes $O_k(\alpha, n)$ where $k \geq 0$ and $\alpha$ runs over a fixed linear basis of $H^*(X)$.

4. **Ruan’s Cohomological Crepant Resolution Conjecture**

In this section, we briefly review the definition of Gromov-Witten invariants, and recall Ruan’s Cohomological Crepant Resolution Conjecture for the Hilbert-Chow morphisms.

Let $Y$ be a smooth projective variety. For a fixed homology class $\beta \in H_2(Y, \mathbb{Z})$, let $\overline{M}_{g,k}(Y, \beta)$ be the coarse moduli space parameterizing all the stable maps $[\mu : (D; p_1, \ldots, p_k) \rightarrow Y]$ such that $\mu_*([D]) = \beta$ and the arithmetic genus of $D$ is $g$. The $i$-th evaluation map $ev_i : \overline{M}_{g,k}(Y, \beta) \rightarrow Y$ is defined by $ev_i([\mu : (D; p_1, \ldots, p_k) \rightarrow Y]) = \mu(p_i) \in Y$. It is known [FP, LT1, LT2, Beh1, BF] that $\overline{M}_{g,k}(Y, \beta)$ is projective and has a virtual fundamental cycle $[\overline{M}_{g,k}(Y, \beta)]^\text{vir} \in A_0(\overline{M}_{g,k}(Y, \beta))$ where $d_0 = -(K_Y \cdot \beta) + (\text{dim}(Y) - 3)(1-g) + k$. Let $\alpha_1, \ldots, \alpha_k \in H^*(Y)$, and $ev = ev_1 \times \cdots \times ev_k : \overline{M}_{g,k}(Y, \beta) \rightarrow Y^k$. Then, the $k$-pointed Gromov-Witten invariant is defined by

$$\langle \alpha_1, \ldots, \alpha_k \rangle_{g, \beta} = \int_{[\overline{M}_{g,k}(Y, \beta)]^\text{vir}} ev^*(\alpha_1 \otimes \cdots \otimes \alpha_k).$$

(4.1)
Next, let $X$ be a smooth complex projective surface. Define the homology class
\[ \beta_n = M_2(x_1) + x_2 + \ldots + x_{n-1} \in H_2(X[n]; \mathbb{Z}) \]  \hspace{1cm} (4.2)
where $x_1, \ldots, x_{n-1}$ are fixed distinct points in $X$. An irreducible curve $C \subset X[n]$ is contracted to a point by $\rho_n$ if and only if $C \sim d\beta_n$ for some integer $d > 0$. Let $q$ be a formal variable. For $w_1, w_2, w_3 \in H^*(X[n])$, define a function of $q$:
\[ \langle w_1, w_2, w_3 \rangle_{\rho_n}(q) = \sum_{d \geq 0} \langle w_1, w_2, w_3 \rangle_{0, d\beta_n} q^d \]

**Definition 4.1.** The quantum corrected cohomology ring $H^*_{\rho_n}(X[n])$ is the group $H^*(X[n])$ together with the quantum corrected product $w_1 \cdot_{\rho_n} w_2$ defined by
\[ \langle w_1 \cdot_{\rho_n} w_2, w_3 \rangle = \langle w_1, w_2, w_3 \rangle_{\rho_n}(-1) \]  \hspace{1cm} (4.3)

**Conjecture 4.2.** (Ruan’s Cohomological Crepant Resolution Conjecture) The quantum corrected cohomology ring $H^*_{\rho_n}(X[n])$ is ring isomorphic to $H^*_{\text{CR}}(X(n))$.

Our idea to deal with Conjecture 1.2 is to use the axiomatization approach mentioned in the Introduction. On one hand, letting $A^{[n]} = H^*_{\text{CR}}(X(n))$ and $G_k(\alpha, n) = O_k(\alpha, n)$, we see from Theorem 2.5 that both (A1) and (A2) in the Introduction hold for the rings $H^*_{\text{CR}}(X(n))$. On the other hand, by [CGR] Nak, the rings $A^{[n]} = H^*_{\rho_n}(X[n])$ also satisfy (A1) with $A = A^{[1]} = H^*(X)$. To deal with Axiom (A2) for $H^*_{\rho_n}(X[n])$, we now define the elements $\tilde{G}_k(\alpha, n) \in H^*_{\rho_n}(X[n])$.

**Definition 4.3.** Let $k \geq 0$ and $\alpha \in H^*(X)$. Define $\tilde{G}_k(\alpha, n) \in H^*_{\rho_n}(X[n])$ to be
\[ \sum_{0 \leq j \leq k} \sum_{\ell(\lambda) = k-j+1} (-1)^{|\lambda|-1} \chi! \cdot \lambda! \cdot 1_{-(n-j-1)a-\lambda}(\tau_*\alpha)|0) \]
\[ + \sum_{0 \leq j \leq k} \sum_{\ell(\lambda) = k-j+1} (-1)^{|\lambda|} \chi! \cdot |\lambda| + s(\lambda) - 2 \cdot 1_{-(n-j-1)a-\lambda}(\tau_*e_X\alpha)|0) \]  \hspace{1cm} (4.4)

**Remark 4.4.** By definition, $\tilde{G}_0(\alpha, n) = 1_{-(n-1)a-1}(\alpha)|0) = G_0(\alpha, n)$. Also,
\[ \tilde{G}_1(\alpha, n) = -\frac{1}{2} 1_{-(n-2)a-2}(\alpha)|0) = G_1(\alpha, n). \]

In general, we see from Theorem 2.5 that the class $\tilde{G}_k(\alpha, n)$ consists of those terms in $G_k(\alpha, n)$ which do not contain the canonical divisor $K_X$.

Note from the definition of the operator $\tilde{\mathfrak{S}}_k(\alpha)$ on $\oplus_n H^*_{\rho_n}(X[n])$ that
\[ \langle \tilde{\mathfrak{S}}_k(\alpha)w_1, w_2 \rangle = \langle \tilde{G}_k(\alpha, n)\cdot_{\rho_n} w_1, w_2 \rangle = \langle \tilde{G}_k(\alpha, n), w_1, w_2 \rangle_{\rho_n}(-1) \]
for $w_1, w_2 \in H^*_{\rho_n}(X[n])$. For convenience, we introduce the operator $\tilde{\mathfrak{S}}_k(\alpha; q)$ by
\[ \langle \tilde{\mathfrak{S}}_k(\alpha; q)w_1, w_2 \rangle = \sum_{d \geq 0} \langle \tilde{G}_k(\alpha, n), w_1, w_2 \rangle_{0, d\beta_n} q^d \]  \hspace{1cm} (4.5)

In the rest of this section, let the smooth projective surface $X$ be simply connected. By Remark 4.4 $G_1(1_X, n) = G_1(1_X, n)$. Thus by [LL],
\[ \tilde{\mathfrak{S}}_1(1_X) = -\frac{1}{6} : a^3 : 0 (r_3, 1_X). \]  \hspace{1cm} (4.6)

So (1.1) holds for the rings $H^*_{\rho_n}(X[n])$ as well. To verify Ruan’s conjecture for $\rho_n$, it remains to show that (1.2) holds for $H^*_{\rho_n}(X[n])$. For the right-hand-side of (1.2), we have the following which follows from (1.0) and the same proof of Theorem 2.5 (i.e., Theorem 4.4 in [LQW3]).
Lemma 4.5. Let \( k \geq 0, m \in \mathbb{Z} \), and \( \alpha \in H^*(X) \). Then, \( a_{m}^{(k)}(\alpha) \) is equal to

\[
(-m)^k k! \left( \sum_{\lambda: \ell(\lambda)=k+1, |\lambda|=m} \frac{1}{\lambda} \alpha_\lambda \tau_\alpha(\tau_e(\alpha)) - \sum_{\lambda: \ell(\lambda)=k-1, |\lambda|=m} \frac{s(\lambda) - 1}{24\lambda} \alpha_\lambda (\tau_e(\tau_e(\alpha))) \right). \]

\[
\square
\]

Comparing with Theorem 2.3, we see that the operator \( a_{m}^{(k)}(\alpha) \) consists of those terms in \( a_{m}^{(k)}(\alpha) \) which do not contain the canonical divisor \( K_X \).

Lemma 4.6. Let \( X \) be a smooth toric surface. Then (1.2) holds for \( X^{[n]} \).

Proof. Recall that \( \mathbb{P}^2 \) and the Hirzebruch surfaces \( \mathbb{F}_a \) are smooth toric surfaces, and admit \( \mathbb{T} = (\mathbb{C}^*)^2 \)-actions. By the Proposition in Subsection 2.5 of [Ful], \( X \) is obtained from \( \mathbb{P}^2 \) or \( \mathbb{F}_a \) by a succession of blow-ups at \( \mathbb{T} \)-fixed points.

Now let \( \eta_{a}^T(\alpha, H^{*}_{\mathcal{C}}(X^{[n]}) \) and \( \eta_{a}^T(\alpha, H^{*}_{\mathcal{C}}(X^{(n)}) \) be the equivariant versions of \( a_{m}^{(k)}(\alpha, H^{*}_{\mathcal{C}}(X^{[n]}) \) and \( a_{m}^{(k)}(\alpha, H^{*}_{\mathcal{C}}(X^{(n)}) \) respectively. By [Che], the equivariant version of Conjecture 4.2 holds for \( X \), i.e., there exists a ring isomorphism

\[
\Psi_{T} : H^{*}_{\mathcal{C}}(X^{[n]}) \to H^{*}_{\mathcal{C}}(X^{[n]})
\]

sending \( \sqrt{-1}^{n_1+\ldots+n_{s}} \eta_{a}^T(\alpha_1) \cdots \eta_{a}^T(\alpha_s) \) to \( \eta_{a}^T(\alpha_1) \cdots \eta_{a}^T(\alpha_s) \). Note that up to a scalar factor which depends only on the partition \( \lambda = (n_1, \ldots, n_s) \) and the tuple \( \alpha = (\alpha_1, \ldots, \alpha_s) \), our notation \( \eta_{a}^T(\alpha_1) \cdots \eta_{a}^T(\alpha_s) \) coincides with the notation \( \lambda(\alpha) \) used in [Che]. Also, our notation \( \eta_{a}^T(\alpha) \) coincides with the notation \( p_{a}(\alpha) \) used in [Che]. The integer \( n_1 + \ldots + n_s - s \) is the age. Passing the map \( \Psi_{T} \) to the ordinary cohomology, we obtain a ring isomorphism

\[
\Psi_{n} : H^{*}_{\mathcal{C}}(X^{[n]}) \to H^{*}_{\mathcal{C}}(X^{[n]})
\]

which sends \( \sqrt{-1}^{n_1+\ldots+n_{s}} \eta_{a}^T(\alpha_1) \cdots \eta_{a}^T(\alpha_s) \) to \( a_{n_1}(\alpha_1) \cdots a_{n_1}(\alpha_s) \). Using 3.2 and 4.4, we see that \( \Psi_{n}(\sqrt{-1}^{k} O_k(a, n)) = G_k(a, n) \).

Next, let \( A = a_{n_1}(\alpha_1) \cdots a_{n_1}(\alpha_s) \) and \( a = n_1 + \ldots + n_s - s \). Let \( \bullet \) denote the orbifold ring product. Then, \( \Psi_{n}(\sqrt{-1}^{k} O_k(a, n-1) \bullet \sqrt{-1}^{a} P) \) equals

\[
a_{-1}(\beta) \Psi_{n}(\sqrt{-1}^{k} O_k(a, n-1) \bullet \sqrt{-1}^{a} P) = a_{-1}(\beta)(G_k(a, n-1) \bullet A),
\]

and \( \Psi_{n}(\sqrt{-1}^{k} O_k(a, n) \bullet \sqrt{-1}^{a} P) = a_{-1}(\beta). \) So \( [\mathcal{G}_k(a), a_{-1}(\beta)] \) is equal to

\[
\Psi_{n}(\sqrt{-1}^{k} O_k(a, n) \bullet \sqrt{-1}^{a} P) - \Psi_{n}(\sqrt{-1}^{k} O_k(a, n-1) \bullet \sqrt{-1}^{a} P).
\]

By Theorem 3.1 (iii), we conclude that \( [\mathcal{G}_k(a), a_{-1}(\beta)] \) is equal to

\[
\sum_{\ell(\lambda)=k+1, |\lambda|=a_1} \frac{1}{\lambda} \mathcal{P}_{\lambda}(\tau_\alpha(\alpha)) + \sum_{\ell(\lambda)=k-1, |\lambda|=a_0} \frac{s(\lambda) - 1}{24\lambda} \mathcal{P}_{\lambda}(\tau_\alpha(e\alpha))P.
\]

Finally, by the definition of \( \Psi_{n} \) and Lemma 4.3, \( [\mathcal{G}_k(a), a_{-1}(\beta)] \) is equal to

\[
\sum_{\ell(\lambda)=k+1, |\lambda|=a_1} \frac{1}{\lambda} \mathcal{P}_{\lambda}(\tau_\alpha(\alpha))A - \sum_{\ell(\lambda)=k-1, |\lambda|=a_0} \frac{s(\lambda) - 1}{24\lambda} \mathcal{P}_{\lambda}(\tau_\alpha(e\alpha))A = \frac{1}{k!} a_{-1}^{(k)}(\alpha \beta)A.
\]
Therefore, $[\tilde{\mathcal{G}}_k(\alpha), a_{-1}(\beta)] = 1/k! \cdot a^{(k)}_{-1}(\alpha \beta)$. Hence (1.2) holds.

## 5. Extremal Gromov-Witten invariants of Hilbert schemes

In this section, we study the structure of extremal Gromov-Witten invariants of $X^{[n]}$ for a smooth projective surface $X$. We will use the ideas and approaches in \cite{LiJ}, and adopt many presentations, notations and results directly from \cite{LiJ}. In addition, the product formula in \cite{Beh2} and the co-section localization in \cite{KL1, KL2, LL} for Gromov-Witten theory will play important roles. For convenience, we assume that $X$ is simply connected.

### 5.1. Hilbert schemes of $\alpha$-points and partial equivalence

In this subsection, we introduce some new spaces related to Hilbert schemes to provide a platform where, in the subsequent subsections, we can construct cycles $\mathfrak{I}_{n,d} \in H_*(X^{[n]}_n)$ derived from various virtual cycles of moduli spaces of stable maps to these new spaces.

Let $Y \to T$ be a smooth family of projective surfaces over a smooth, projective base $T$. The relative Hilbert scheme of length-$n$ 0-dimensional closed subschemes is denoted by $Y_T^{[n]}$. It is over $T$ and for any $t \in T$, $Y_T^{[n]} ×_T \{t\} = (Y_t)^{[n]}$. Define its relative fiber product $Y_T^n = Y ×_T \cdots ×_T Y$ ($n$ times), and its relative symmetric product $Y_{T^n} = Y_T^n / S_n$.

Let $\Lambda$ be a finite set with $|\Lambda| = n$. We define $Y_T^{[\Lambda]} = Y_T^{[n]}$, $Y_T^{(\Lambda)} = Y_T^{(n)}$, and for accounting purpose, denote

$$Y_T^{\Lambda} = \{(x_\alpha)_{\alpha \in \Lambda} \mid x_\alpha \in Y_t \text{ for some } t \in T\}.$$ 

Using the Hilbert-Chow morphism $\rho_\Lambda := \rho_n : Y_T^{[\Lambda]} \to Y_T^{(\Lambda)}$, we define the Hilbert scheme of $\Lambda$-points to be

$$Y_T^{[\Lambda]} = Y_T^{[\Lambda]} ×_{Y_T^{(\Lambda)}} Y_T^{\Lambda}. \tag{5.1}$$

These spaces $Y_T^{[\Lambda]}$ can be thought of as Hilbert schemes of ordered points.

Let $\mathcal{P}_\Lambda$ be the set of partitions or equivalence relations on $\Lambda$. When $\alpha \in \mathcal{P}_\Lambda$ consists of $l$ equivalence classes $\alpha_1, \ldots, \alpha_l$, we write $\alpha = (\alpha_1, \ldots, \alpha_l)$. For such $\alpha$, we form the relative Hilbert scheme of $\alpha$-points as follows:

$$Y_T^{[\alpha]} = \prod_{i=1}^l Y_T^{[\alpha_i]}, \quad Y_T^{[\alpha]} = \prod_{i=1}^l Y_T^{[\alpha_i]} / \prod_{i=1}^l Y_T^{[\alpha_i]}, \quad Y_T^{[\alpha]} = \prod_{i=1}^l Y_T^{[\alpha_i]} \tag{5.2}$$

where the products are taken relative to $T$. Note that $Y_T^{[\alpha]} = Y_T^{[\alpha]} ×_{Y_T^{[\alpha]}} Y_T^{\Lambda}$. The “indexing” morphism is defined to be the second projection

$$\text{in} : Y_T^{[\alpha]} \longrightarrow Y_T^{\Lambda}. \tag{5.3}$$

The spaces $Y_T^{[\alpha]}$ and $Y_T^{[\beta]}$ are birational. To make this precise, we first fix our convention on a partial ordering on $\mathcal{P}_\Lambda$. We agree

"$\alpha \geq \beta" \iff "a \sim_\alpha b \Rightarrow a \sim_\beta b".$

Namely, $\alpha \geq \beta$ if $\beta$ is finer than $\alpha$. When $\beta = (\beta_1, \ldots, \beta_r)$, we put

$$\alpha \land \beta = (\alpha_1 \cap \beta_1, \ldots, \alpha_l \cap \beta_r)$$

which is the largest element among all that are less than or equal to both $\alpha$ and $\beta$. Note that $\mathcal{P}_\Lambda$ contains a maximal and a minimal element. The maximal element is $\Lambda$ consisting of a single equivalence class $\Lambda$; the minimal element is $1^\Lambda$ whose equivalence classes are single element sets.

For $\alpha > \beta \in \mathcal{P}_\Lambda$, define

$$\Xi_\beta^\alpha = \{x \in Y_T^{\Lambda} \mid \exists a, b \in \Lambda \text{ so that } x_a = x_b, \ a \sim_\alpha b, \ a \not\sim_\beta b\}.$$
For $\alpha \neq \beta \in \mathcal{P}_\Lambda$, define $\Xi^\alpha = \Xi^\alpha \wedge \Xi^\beta$. The discrepancy between $Y_T^{[\alpha]}$ and $Y_T^{[\beta]}$ (in $Y_T^{[\alpha]}$) and its complement are defined to be

$$\Xi^\beta = Y_T^{[\alpha]} \times_{Y_T^{[\alpha]}} \Xi^\alpha, \quad \text{and} \quad Y_T^{[\alpha]} = Y_T^{[\alpha]} - \Xi^\beta.$$  \hfill (5.4)

More precisely, by Lemma 1.2 in \cite{LL}, there exists a functorial open embedding $\xi_\beta : Y_T^{[\alpha]} \to Y_T^{[\beta]}$ induced by the universal property of the respective moduli spaces such that $\mathsf{Im}(\xi_{\alpha}^\beta) = Y_T^{[\beta]}$. Thus we obtain an isomorphism (equivalence) $\xi_\beta : Y_T^{[\beta]} \xrightarrow{\sim} Y_T^{[\beta]}$. We define

$$Y_T^{[\alpha \leq \beta]} = \left( \prod_{\beta \leq \alpha} Y_T^{[\beta]} \right) / \sim,$$  \hfill (5.5)

where the equivalence is by identifying $Y_T^{[\beta]} \subset Y_T^{[\alpha]}$ and $Y_T^{[\gamma]} \subset Y_T^{[\beta]}$ via $\xi_{\alpha}^\beta$ for all $\beta, \gamma \leq \alpha$. Note that $Y_T^{[\leq \beta]}$ is non-separated (except when $\alpha = 1^\Lambda$), and contains the spaces $Y_T^{[\beta]}$, $\beta \leq \alpha$, as open subschemes.

Even though the non-separated space $Y_T^{[\leq \beta]}$ comes into the picture, in later subsections, we only perform standard algebraic topological operations on these non-Hausdorff spaces such as pull-backs of cohomology classes and cap products. These operations are allowed on any topological spaces (see \cite{GH} \cite{Iy} \cite{Sp}).

5.2. Stable maps to Hilbert schemes of ordered points. We incorporate stable maps into the above constructions. This is motivated by the standard decompositions of stable morphisms introduced in \cite{LL}. For $d \geq 0$, we let

$$Y_T^{[n,d]} := \mathfrak{M}_{0,3}(Y_T^{[n]}, d\beta_n)$$

be the relative moduli space of 3-pointed genus-0 stable maps to $Y_T^{[n]}$ of class $d\beta_n$.

We study the standard decomposition of $[u,C] \in Y_T^{[n,d]}$. Given $[u,C] \in Y_T^{[n,d]}$, composed with the Hilbert-Chow morphism $\rho_n$, we obtain $\rho_n \circ u : C \to Y_T^{[n]}$. Since the fundamental class of $u(C)$ is a multiple of the null class $\beta_n$, and $C$ is connected, $\rho_n \circ u$ is a constant map. We express $\rho_n \circ u(C) = \sum_{i=1}^l n_i x_i$, where $n_i \in \mathbb{N}$, such that $\sum n_i = n$, and $x_i$ are distinct. With such data, for $p \in C$, we can decompose $u(p) = z_1(p) \cup \cdots \cup z_l(p)$ such that $z_i(p) \in Y_T^{[n_i]}$, and $\rho_n(z_i(p)) = n_i x_i$. Because $x_i$ are distinct, such decomposition is unique. We define

$$u_i : C \to Y_T^{[n_i]}, \quad u_i(p) = z_i(p).$$  \hfill (5.6)

Because of the uniqueness of the decomposition, one checks that $u_i$ are morphisms; since $u_i[C] = d\beta_n$, we have $u_i[C] = d_i \beta_n$, for some $d_i \geq 0$ such that $\sum d_i = d$. Using such data, we can define the Hilbert-Chow map from $Y_T^{[n,d]}$ to the weighted symmetric product of $Y$.

For the pair $(n,d)$, we define the weighted symmetric product of $Y$ to be

$$Y_T^{(n,d)} = \{ \sum_{i=1}^l d_i [n_i x_i] | 1 \leq l \leq n, \ x_1, \cdots, x_l \in Y_t \text{ distinct, for a } t \in T \}.$$  

Here the formal summation $\sum d_i [n_i x_i]$ is subject to the constraints $d_i \in \mathbb{N}$, $\sum d_i = d$, $n_i \in \mathbb{N}_+$ and $\sum n_i = n$. Also, $[n_i x_i]$ represents the multiplicity-$n_i$ 0-cycle supported at $x_i$, and $d_i$ is its weight. Thus $d_i [n_i x_i] \neq d_i n_i x_i$ and $0[x_i]$ is non-trivial. Endow $Y_T^{(n,d)}$ with the obvious topology so that it is a stratified space such that the forgetful map $Y_T^{(n,d)} \to Y_T^{(n)}$ is continuous, proper and having finite fibers.

We define the Hilbert-Chow map:

$$\mathfrak{hc} : Y_T^{[n,d]} \longrightarrow Y_T^{(n,d)}, \quad [u] \mapsto \sum_{i=1}^l d_i [n_i x_i],$$  \hfill (5.7)
where \((d_i, n_i, x_i)\) are data associated to \((u_i)\) from \(\ref{X}.\) Define \(\mathfrak{h}c_1 : Y_T^{[n,d]} \rightarrow Y_T^{(n)}\) to be the composite of \(\mathfrak{h}c\) with the forgetful map \(Y_T^{(n,d)} \rightarrow Y_T^{(n)}\). For a finite set \(\Lambda\) (of order \(n\)), define
\[
Y_T^{[\Lambda,d]} = Y_T^{[n,d]} \times_{Y_T^{(n)}} Y_T^{\Lambda}.
\]
(5.8)

**Definition 5.1.** We call \((\alpha, \delta)\) a weighted partition of \(\Lambda\) if \(\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathcal{P}_\Lambda\) and \(\delta = (\delta_1, \ldots, \delta_l), \delta_i \geq 0\) for every \(i\). We define \(\sum \delta_i\) to be the total weight of \((\alpha, \delta)\). For \((\Lambda, d)\), we denote by \(\mathcal{P}_{\Lambda,d}\) the set of all weighted partitions of \(\Lambda\) with total weight \(d\). We say that \((\alpha, \delta) \geq (\beta, \eta)\) if \(\alpha \geq \beta\) and \(\sum \delta_i < \eta_i = \delta_j\) for every \(j\).

For \((\alpha, \delta) \in \mathcal{P}_{\Lambda,d}\), we form the relative moduli space of 3-pointed genus-0 degree-\(\delta\) \(\alpha\)-stable morphisms to the Hilbert scheme of points:
\[
Y_T^{[\alpha, \delta]} = Y_T^{[\alpha, \delta_1]} \times_T \cdots \times_T Y_T^{[\alpha, \delta_l]}.
\]
To simplify notations, the composition of \(Y_T^{[\Lambda,d]} \rightarrow Y_T^{[\Lambda,d]}\) and \(\mathfrak{h}c_1 : Y_T^{[\Lambda,d]} \rightarrow Y_T^{(n)}\) will again be denoted by \(\mathfrak{h}c_1\).

### 5.3. Birationality

The key result Lemma \(\text{5.3}\) provides the comparison between \(Y_T^{[\alpha, \delta]}\) and \(Y_T^{[\beta, \eta]}\), which will be used in later subsections for the comparison of normal cones for \(Y_T^{[\alpha, \delta]}\) and \(Y_T^{[\beta, \eta]}\).

For \((\alpha, \delta) > (\beta, \eta)\), the pair \(Y_T^{[\alpha, \delta]}\) and \(Y_T^{[\beta, \eta]}\) are “birational”. To make this more precise, we introduce some notations. Given an element
\[
\xi = ([u,C], (y_a)) \in Y_T^{[\alpha, \delta]} = Y_T^{[n,d]} \times_{Y_T^{(n)}} Y_T^{\Lambda},
\]
where \(\mathfrak{h}c([u]) = \sum_{i=1}^{l} d_i [n_i, x_i]\) and such that \(\sum n_i x_i = \sum \eta_a\) (as 0-cycles in \(Y_T^{(n)}\)), we define a pair \((a(\xi), \varnothing(\xi))\) \(\in \mathcal{P}_{\Lambda,d}\) by
\[
a(\xi) = (a_1, \ldots, a_l), \quad a_i = \{a \in \Lambda \mid y_a = x_i\}; \quad \varnothing(\xi) = (d_1, \ldots, d_l).
\]

**Definition 5.2.** For \((\beta, \eta) \in \mathcal{P}_{\Lambda,d}\), we define
\[
Y_T^{[\beta, \eta]} = \{\xi \in Y_T^{[\alpha, \delta]} \mid (a(\xi), \varnothing(\xi)) \leq (\beta, \eta)\},
\]
and
\[
Y_T^{[\beta, \eta]} = \{((\xi_1, \ldots, \xi_r)) \in Y_T^{[\beta, \eta]} \mid \mathfrak{h}c_1(\xi_1), \ldots, \mathfrak{h}c_1(\xi_r)\) mutually disjoint\}.
\]

For \((\beta, \eta) \leq (\alpha, \delta)\), we define (as fiber products over \(T\))
\[
Y_T^{[\alpha, \delta]} = \prod_{i=1}^{l} Y_T^{[\alpha_i, \delta_i]} \text{ and } Y_T^{[\beta, \eta]} = \prod_{i=1}^{l} Y_T^{[\beta_i, \eta_i]}.
\]

**Lemma 5.3.** For \((\alpha, \delta) > (\beta, \eta)\), we have a natural, proper surjective morphism
\[
\zeta^{\beta, \eta}_{\alpha, \delta} : Y_T^{[\alpha, \delta]} \rightarrow Y_T^{[\beta, \eta]}.
\]
(5.10)

**Proof.** By definition, we only need to prove the case \((\alpha, \delta) = (\Lambda, d)\). Let \(\xi = ([u,C, p_1], (y_a)) \in Y_T^{[\Lambda,d]}\), with \(\mathfrak{h}c([u]) = \sum_{i=1}^{l} d_i [n_i, x_i]\). Let \(u_\xi : C \rightarrow Y_T^{[n]}\) be as in \(\ref{X}\). Denote \(a(\xi) = (a_1, \ldots, a_l)\) and \(\varnothing(\xi) = (d_1, \ldots, d_l)\). Since \(\xi \in Y_T^{[\lambda, \delta]}\), we have \((a(\xi), \varnothing(\xi)) \leq (\beta, \eta)\). Thus we can form
\[
u_\beta : C \rightarrow Y_T^{[n]}; \quad u_\beta(p) = \bigcup_{\delta_i \leq \beta_i} u_j(p) \in Y_T^{[n]}.
\]

Because the degree of \(u_j\) is \(d_j\), and \((a(\xi), \varnothing(\xi)) \leq (\beta, \eta)\), the degree of \(u_\beta\) is \(\eta_i\). For \(1 \leq l \leq r\), replace \(u_\beta : C = \bigcup_{i=1}^{l} \bigcup_{\delta_i \leq \beta_i} u_j : C \rightarrow Y_T^{[n]}\) by the stabilization of \([u_\beta, C, p_1]\). Then \((u_\beta^{st} : \cdots, u_\beta^{st}) \in Y_T^{[\beta, \eta]}\). It is routine to check that
\[
\zeta^{\beta, \eta}_{\alpha, \delta} : Y_T^{[\alpha, \delta]} \rightarrow Y_T^{[\beta, \eta]}; \quad ([u,C], (y_a)) \mapsto (u_\beta^{st}, \ldots, u_\beta^{st}),
\]
\footnote{Without further mentioning \(\alpha = (\alpha_1, \ldots, \alpha_l)\) and \(\beta = (\beta_1, \ldots, \beta_r)\).}
defines a morphism. By the definition of \( Y^{[\beta, \eta]} \), we have \( \text{Im}(\zeta) \subset Y^{[\beta, \eta]} \).

We now show that \( \text{Im}(\zeta) = Y^{[\beta, \eta]} \). Note that a closed point in \( Y^{[\beta, \eta]} \) is an \( r \)-tuple \((\xi_1, \cdots, \xi_r)\) with \( \xi_i \in Y^{[\beta, \eta]}_T \) such that \( h_{\zeta_i}(\xi_1), \cdots, h_{\zeta_1}(\xi_r) \) are mutually disjoint. Let \( \xi_i = [u_i, C_i, p_i] \). Since \([C_i, p_i]\) are 3-pointed genus-0 nodal curves, we can find a 3-pointed genus-0 \([C, p]\) and contraction morphisms \( \phi_i : C \to C_i \) so that \( \phi_i(p_j) = p_{ij} \), \( j = 1, 2, 3 \). Since \( h_{\zeta_i}(\xi_1), \cdots, h_{\zeta_1}(\xi_r) \) are mutually disjoint, the assignment \( p \mapsto u(p) = u_1 \circ \phi_1(p) \cup \cdots \cup u_r \circ \phi_r(p) \in Y^{[\eta]}_T \) defines a morphism \( u : C \to Y^{[\eta]}_T \). We let \( \xi = [u, C, p_j] \) be its stabilization. Then \( \xi \in Y^{[\beta, \eta]}_T \), and \( \zeta = (\xi_1, \cdots, \xi_r) \). Hence \( \text{Im}(\zeta) = Y^{[\beta, \eta]}_T \).

We check that \( \zeta \) is proper. Let \( s_0 \in S \) be a pointed smooth curve over \( T \); let \( S^* = S - s_0 \). Suppose \( \xi^* \) is an \( S^* \)-family in \( Y^{[\beta, \eta]}_T \) so that \( \zeta(\xi^*) = (\xi_1^*, \cdots, \xi_r^*) \) extends to an \( S \)-family \((\xi_1, \cdots, \xi_r)\), we need to show that, possibly after a base change, \( \xi^* \) extends to \( \xi \) so that \( \zeta(\xi^*) = (\xi_1, \cdots, \xi_r) \).

Since \( Y^{[\beta, \eta]}_T \) is \( T \)-proper, possibly after a base change, we can extend \( \xi^* \) to an \( S \)-family \( \xi \) in \( Y^{[\beta, \eta]}_T \). Let \( \xi \) be given by \(([u, C, p_j], (y_a))\), where each term explicitly is an \( S \)-family. Let \( y_{\beta_i} = \sum_{a \in \beta_i} y_a : S \to Y^{[\beta_i]}_T \). By definition, \( \xi(s_0) = \xi \times S \{ s_0 \} \in Y^{[\beta, \eta]}_T \) if \( y_{\beta_i}(s_0), \cdots, y_{\beta_i}(s_0) \) are mutually disjoint. Since \( \zeta(\xi^*) = (\xi_1^*, \cdots, \xi_r^*) \), we have \( y_{\beta}(s^*) = h_{\zeta_1}(\xi^*(s_0)) \). Since \( Y^{[\eta]}_T \) is separated, we have \( y_{\beta}(s_0) = h_{\zeta_1}(\xi_1(s_0)) \). Further, since \( \xi_1(s_0), \cdots, \xi_r(s_0) \in Y^{[\beta, \eta]}_T \), \( h_{\zeta_i}(\xi_i(s_0)) \), (see the diagram (3) in [Beh2] for details.)

We denote by \( M_{0,3}(d) \) the Artin stack of total weights \( d \) \( \mathbb{N} \)-weighted 3-pointed genus-0 nodal curves. We denote by \( \mathcal{D}(d_1, d_2) \) the Artin stack of the data

\[
\{ (C, p_j, \tau) \to (C_1, p_{1j}, \tau_1), (C, p_j, \tau) \to (C_2, p_{2j}, \tau_2) \}
\]

so that \((C, p_j, \tau)\) is a stable total weight \((d_1, d_2)\) \( \mathbb{N} \)-weighted 3-pointed genus-0 nodal curve, \((C, p_{ij}, \tau_i) \in M_{0,3}(d_i)\), and the two arrows induce isomorphisms \((C, p_j, \tau) \cong (C_1, p_{1j}, \tau_1)\), where \( p_{ij} : \mathbb{N}^2 \to N \) is the \( i \)-th projection. (See the diagram (3) in [Beh2] for details.)

**Lemma 5.4.** Let \( \beta = (\beta_1, \beta_2) \) be a partition of length two, and let \( \eta = (d_1, d_2) \) with \( d = d_1 + d_2 \). We have a Cartesian diagram

\[
\begin{array}{ccc}
Y^{[\beta, \eta]}_T & \longrightarrow & Y^{[\beta, \eta]}_T \\
\downarrow & & \downarrow \\
\mathcal{D}(d_1, d_2) & \xrightarrow{(\epsilon_1, \epsilon_2)} & M_{0,3}(d_1) \times M_{0,3}(d_2)
\end{array}
\]

Further, \((\epsilon_1, \epsilon_2)\) is proper and birational.

**Proof.** The proof is a direct application of Proposition 5 in [Beh2] plus the definition of \( Y^{[\beta, \eta]}_T \). Note that the second vertical arrow is induced by \( Y^{[\beta, \eta]}_T \subset Y^{[\beta_1, d_1]}_T \times_T Y^{[\beta_2, d_2]}_T \) and the forgetful morphism \( Y^{[\beta, \eta]}_T \to M_{0,3}(d_i) \). □
5.4. Virtual classes and comparison of normal cones. As \( Y^n_T \to Y^{(n)}_T \) is a finite quotient map by a finite group, it is flat. Let \( [Y^n_T]^{\text{vir}}_\alpha \) be the virtual class of \( Y^n_T \) \( \alpha \). We define \( [Y^n_T]^{\text{vir}}_\alpha \) to be the flat pullback of \( [Y^n_T]^{\text{vir}}_\alpha \). Our goal is to inductively construct cycle representatives of the virtual classes of \( Y^n_T \) \( \alpha \) that are compatible via the comparison \( \psi^{\alpha,\beta} \).

We recall the construction of virtual cycles in \([BF, LT]\). Let \( (\mathbb{E}_\alpha^{[a,\delta]} )^T \to L_{Y^n_T} / T \times \mathcal{M}_0 \), be the standard perfect relative obstruction theory\( \mathbb{E}_\alpha \) of \( Y^n_T \to T \times \mathcal{M}_0 \); let \( C_{\alpha,\delta} \subset F_{\alpha,\delta} := h^1 / h^0(\mathbb{E}_\alpha) \) be its intrinsic normal cone. To use analytic Gysin map, we put it in a vector bundle. Following \([BF, LT]\), we can find a vector bundle (locally free sheaf) \( E_{\alpha,\delta} \) on \( Y^n_T \) and a surjection of bundle-stack \( E_{\alpha,\delta} \to h^1 / h^0(\mathbb{E}_\alpha) \). Let \( C_{\alpha,\delta} \subset C_{\alpha,\delta} \) be the flat pullback of \( C_{\alpha,\delta} \). Then \( [Y^n_T]^{\text{vir}}_\alpha \) is the intrinsic normal cone of \( E_{\alpha,\delta} \). Let \( \rho_{\alpha,\delta} : Y^n_T \to Y^n_T \) be the intrinsic normal cone. Note that \( \rho_{\alpha,\delta} \) is the Gysin homomorphism of the zero section of \( E_{\alpha,\delta} \), and \([Y^n_T]^{\text{vir}}_\alpha \) is the coarse moduli space of \( Y^n_T \). Also, put \( E_{\alpha,\delta} \), and let \( F_{\alpha,\delta} = h^1 / h^0(\mathbb{E}_\alpha) = \rho_{\alpha,\delta}^*F_{\alpha,\delta} \) be the flat pullback. Let \( C_{\alpha,\delta} \subset C_{\alpha,\delta} \) be the flat pullback of \( C_{\alpha,\delta} \) via \( F_{\alpha,\delta} \to F_{\alpha,\delta} \).

We now compare the cycles \( C_{\alpha,\delta} \) using \( C_{\alpha,\delta} \) and \( C_{\alpha,\delta} \) are not comparable. Thus we will state the comparison using cycles in \( F_{\alpha,\delta} \), and later will use the obstruction sheaf for accounting purpose.

**Lemma 5.5.** For pairs \((\alpha, \beta) \succ (\beta, \eta)\), we have canonical isomorphisms

\[
\varphi^{\alpha,\beta} : (\zeta^{\alpha,\beta})^* (\varphi^{\beta,\eta})^* (\varphi^{\gamma,\varepsilon})^* \mathbb{F}_{\beta,\eta,\varepsilon}^\alpha \xrightarrow{\cong} \mathbb{F}_{\alpha,\delta}\]

that satisfy the cocycle condition: we have \( \varphi^{\alpha,\beta} \circ (\zeta^{\alpha,\beta})^* \varphi^{\gamma,\varepsilon} = \varphi^{\gamma,\varepsilon} \) for any triple \((\alpha, \beta) \succ (\beta, \eta) \succ (\gamma, \varepsilon)\). Further, let \( \varphi^{\alpha,\beta} : \mathbb{F}_{\alpha,\delta} \to \mathbb{F}_{\beta,\eta,\varepsilon} \) be the projection induced by \( \varphi^{\alpha,\beta} \), which is proper by Lemma 5.3. Then

\[
(\varphi^{\alpha,\beta})^* |_{Y^n_T} \circ (\zeta^{\alpha,\beta})^* |_{Y^n_T} = |_{Y^n_T} \circ (\zeta^{\alpha,\beta})^* |_{Y^n_T} \]

**Proof.** By induction, we only need to prove the case where \( \ell(\beta) = \ell(\alpha) + 1 \); by definition this follows from the case \((\alpha, \delta) = (\Lambda, \delta)\) and \( \beta = (\beta_1, \beta_2) \) with \( \eta = (d_1, d_2) \), which we suppose in the remainder of this proof.

Let \( y = (y_\alpha) \in Y^n_T \) be a closed point so that \( y_\beta_1 = \rho_\beta_1 ((y_\alpha)_{\alpha \beta_1}) \in Y^{(\beta_1)} \) and \( y_\beta_2 \in Y^{(\beta_2)} \) (defined similarly) are disjoint. We then form

\[
V_1 = Y^{(\beta_1)}_{Y^n_T} \times Y^{(\beta_2)} \{ y_\beta_2 \} \quad \text{and} \quad V = Y^{(\delta)}_{Y^n_T} \times Y^{(\alpha)} \{ \rho_\Lambda(y) \}.
\]

Note that \( y_\beta_1 \cap y_\beta_2 = \emptyset \) implies that \( V_1 \times V_2 \subset Y^{(\beta)}_T \). Also, there exists a canonical isomorphism \( \zeta^{\beta}_1 : V_1 \times V_2 \to V \). Let \( \tilde{V}_1 \) (respectively, \( \tilde{V} \)) be the formal completion of \( Y^{(\beta_1)}_T \) (respectively, \( Y^{(\delta)}_T \)) along \( V_1 \) (respectively, \( V \)). The isomorphism \( \zeta^{\beta}_1 \) induces

\[
\tilde{V}_1 \times \tilde{V} = \tilde{V}_1 \times \tilde{V}_2 \to \tilde{V},
\]

which is injective and smooth.

For notational simplicity, we denote \( \mathfrak{M}(V_1) = \mathfrak{M}_{0,3}(V_1, d_i) \) with \( d_i \in \{ 0, 1 \} \) being the tautological morphism induced by \( \tilde{V}_1 \to Y^{(\beta_1)} \); we let

\[
\mathfrak{M}(V_1 \times V_2) = \mathfrak{M}_{0,3}(V_1 \times V_2, (d_1, d_2))
\]

\[2\]Here \( E_{\alpha,\delta} \) is a derived object locally presented as a two-term complex of locally free sheaves placed at \([0, 1] \).
Let \( W \) smooth locally closed DM stacks \( C \).

Multi-sections and pseudo-cycle representatives. restricted to each stratum \( W \) stratified sections; we denote the space of such sections by \( \mathcal{W}(\mathcal{T}) \) with \( \phi \) canonical isomorphism and identity that \( F \) is injective and smooth, we have (5.12)

\[
\begin{array}{c}
Y_{(\beta, \eta)}^{[A,d]} \xrightarrow{\phi} Y_{(\Lambda, d)}^{[A]} \\
\downarrow \psi_1 \quad \downarrow \psi_2 \\
\mathfrak{M}(\hat{V}_1 \times \hat{V}_2) \xrightarrow{\phi'} \mathfrak{M}(\hat{V}_1) \times_{\mathfrak{M}(\hat{V}_2)} \mathfrak{M}(\hat{V}_2) \\
\downarrow \psi_1 \quad \downarrow \psi_2 \\
\mathfrak{M}(V_1 \times V_2) \xrightarrow{\phi} \mathfrak{M}(V_1) \times_{\mathfrak{M}(V_2)} \mathfrak{M}(V_2) \\
\end{array}
\]

We recall the notion of multi-sections, following \([\text{FO, LT}2]\). We first consider the case where \( C \) be a vector bundle over a DM stack \( W \). In case \( W \) is singular, we stratify \( C \) and use continuous sections that are smooth when restricted to each stratum \( W \). Without further commenting, all sections used in this section are \( \in (5.12) \) being the tautological morphism induced by \( \hat{V}_1 \times \hat{V}_2 \to Y_{(A,d)}^{[A]} \).

Similarly, letting \( C_2 \subset F_2 \) be the intrinsic normal cone in the bundle stack of the obstruction complex of the prefect relative obstruction theory of \( \mathfrak{M}(V_1) \times_{\mathfrak{M}(V_2)} \mathfrak{M}(V_2) \to T \times M_{0,3} \), we have \( \phi_1 \) and \( \phi_2 \) are injective and smooth, since \( \phi' \) is proper, since the top square is commutative, and since the image of \( \phi_1 \) (respectively, of \( \phi_2 \)) covers \( Y_{(\beta, \eta)}^{[A,d]} \) (respectively, \( Y_{(\Lambda, d)}^{[A]} \)) for \( y \) varying through \( Y_{(\beta, \eta)}^{[A]} \) satisfying \( y_3 \cap y_2 = \emptyset \), to prove that \( F_{[A,d]} = (\zeta_{A, d})^* F_{[A,d]} \) and \( (\zeta_{A, d})^* C_{[A,d]} = [C_{[A,d]}] \), it suffices to show that we have the canonical isomorphism and identity

\[
F_1 \cong \phi^* F_2 \quad \text{and} \quad \phi_* [C_1] = [C_2], \tag{5.13}
\]

where \( \phi : F_1 \to F_2 \) is the induced projection. But this follows from the Cartesian square

\[
\begin{array}{c}
\mathfrak{M}(V_1 \times V_2) \xrightarrow{\phi} \mathfrak{M}(V_1) \times_{\mathfrak{M}(V_2)} \mathfrak{M}(V_2) \\
\downarrow \quad \downarrow \\
T \times \mathcal{O}(d_1, d_2) \longrightarrow T \times M_{0,3}(d_1) \times M_{0,3}(d_2),
\end{array}
\]

similar to the one stated in Lemma \([5, 4]\) (originally constructed in Proposition 5 of \([\text{Bch}2]\)). Since the lower horizontal line is birational, and \( T \) is smooth and projective, by Theorem 5.0.1 in \([\text{Cos}]\), we have the isomorphism and identities in \([5, 13]\). This proves the lemma.

5.5. Multi-sections and pseudo-cycle representatives. In this subsection, we use multi-sections to intersect the cycles \( C_{[A,d]} \) to obtain pseudo-cycle representatives of \( Y_{(A,d)}^{[A,d]} \).

In the remainder of this section, we will work with analytic topology and smooth \( (C^\infty) \) sections. Let \( V \) be a vector bundle over a DM stack \( W \). In case \( W \) is singular, we stratify \( W \) into a union of smooth locally closed DM stacks \( W = \bigsqcup W_a \), and use continuous sections that are smooth when restricted to each stratum \( W_a \). Without further commenting, all sections used in this section are stratified sections; we denote the space of such sections by \( C(W,V) \). Also, we will use \( |W| \) and \( |V| \) to denote the coarse moduli of \( W \) and \( V \).

We recall the notion of multi-sections, following \([\text{FO, LT}2]\). We first consider the case where \( W = U/G \) is a quotient stack and \( V \) is a \( G \)-vector bundle on \( U \). Let \( S^n(V) \to U \) be the
A lifetable multi-section \( s \) of \( V \) (of multiplicity \( n \)) is a \( G \)-equivariant section \( s \in \mathcal{C}(U, S^n(V))^G \) such that there are \( n \) sections \( s_1, \ldots, s_n \in \mathcal{C}(U, V) \) so that \( s \) is the image of \((s_1, \ldots, s_n)\). For a multi-section \( s \in \mathcal{C}(U, S^n(V))^G \) that is the image of \((s_1, \ldots, s_n)\), we define its integer multiple \( ms \in \mathcal{C}(U, S^{mn}(V))^G \) be the image of \((s_1, \ldots, s_1, \ldots, s_n, \ldots, s_n)\), where each \( s_i \) is repeated \( m \) times. Given two multi-sections \( s \) and \( s' \) of multiplicities \( n \) and \( n' \), we say that \( s \approx s' \), if \( n's = ns' \) as multi-sections.

In general, since \( W \) is a DM-stack, it can be covered by (analytic) open quotient stacks \( U_\alpha/G_\alpha \subset W \), and the restriction \( V|_{U_\alpha/G_\alpha} = V_\alpha/G_\alpha \) for \( G_\alpha \)-vector bundles \( V_\alpha \) on \( U_\alpha \). A multi-section \( s \) of \( V \) consists of an analytic open covering \( U_\alpha/G_\alpha \) of \( W \) and a collection of liftable multi-sections \( s_\alpha \) of \( V|_{U_\alpha/G_\alpha} \) so that for any pair \((\alpha, \beta)\), the pullbacks of \( s_\alpha \) and \( s_\beta \) to \( U_\alpha \times_W U_\beta \) are equivalent. We denote the space of multi-sections of \( V \) by \( \mathcal{C}_{mul}(W, V) \). (Thus multi-sections in this paper are always locally liftable.)

The space of multi-sections of \( V \) has the same extension property as the space of sections of a vector bundle on a manifold. The usual extension property of vector bundles on manifolds is proved by using the partition of unity and the addition structure of the vector bundles. For multi-sections, over a chart \( U \), multi-sections, \( V \), multi-sections, over a chart is proved by using the partition of unity and the addition structure of the vector bundles.

We pick stratifications \( C = \bigcup_{\alpha,i} [C_{\alpha,i}] \) with \( \dim C_{\alpha,i} \leq k \) and \( \dim C = k \). If \( s \in \mathcal{C}_{mul}(W, V) \) intersects \([C]\) transversally, then \( p^*s \in \mathcal{C}_{mul}(W', p^*V) \) intersects \([C']\) transversally.

Lemma 5.6. Let \( p : W' \to W \) be a proper morphism of DM-stacks; let \( V \) be a vector bundle on \( W \) and \( \tilde{p} : p^*V \to V \) be the induced projection. Suppose \([C']\in Z_*(p^*V)\) is an algebraic cycle and \([C] = \tilde{p}_*(C')\). If \( s \in \mathcal{C}_{mul}(W, V) \) intersects \([C]\) transversally, then \( p^*s \in \mathcal{C}_{mul}(W', p^*V) \) intersects \([C']\) transversally.

Proof. We pick stratifications \( W = \coprod W_{\alpha} \) and \( W' = \coprod W'_{\alpha} \) so that \( p(W'_{\alpha}) = W_{\alpha} \) and \( p_{\alpha} = p|_{W'_{\alpha}} : W'_{\alpha} \to W_{\alpha} \) are smooth. We then pick a stratification \( C' = \coprod C'_{\beta} \) so that each \( C'_{\beta} \) lies over a stratum of \( W' \), and that \( \tilde{p}|_{C'_{\beta}} : C'_{\beta} \to \tilde{p}(C'_{\beta}) \) is smooth. Therefore, by the definition of transversal to \( C \), we are reduced to check when \( p : W' \to W \) and \( C' \to \tilde{p}(C') \) are smooth. In this case, the statement of the lemma holds by direct local coordinate checking. This proves the lemma. \( \square \)

We now construct pseudo-cycle representatives of the topological Gysin map

\[ 0_V^*: Z_*V \to H_*(|W|, \mathbb{Q}), \]

via intersecting with multi-sections [FO] LT2 LT3 [MD] Zin.

We assume \( W \) is proper. Let \( \pi : V \to W \) and \( \tilde{\pi} : |V| \to |W| \) be the projections. Given a closed integral algebraic substack \( C \subset V \), we find a multi-section \( s \) of \( V \) so that it intersects \( C \) transversally. Let \( k = 2(\text{rank } V - \dim C) \). By slightly perturbing \( s \) if necessary, we can assume that there is a closed (stratifiable) subset \( R \subset |V| \) of \( \dim R \leq k - 2 \) and an (analytic) open covering of \( W \) by quotient stacks \( U_\alpha/G_\alpha \) so that, letting \( q_\alpha : V_\alpha \to |V| \) be the projections,

1. \( s|_{U_\alpha/G_\alpha} \) are images of \( s_{a,1}, \ldots, s_{a,m_a} \) in \( \mathcal{C}(U_\alpha, V_\alpha) \);
2. there are topological spaces \( S_{a,i} \) and proper embeddings \( f_{a,i} : S_{a,i} \to V_\alpha \) such that
   a. there are dense open subsets \( S^\circ_{a,i} \subset S_{a,i} \) so that \( S^\circ_{a,i} \) are smooth manifolds and \( f_{a,i}|_{S^\circ_{a,i}} : S^\circ_{a,i} \to V_\alpha \) are smooth embeddings;
   b. \( s_{a,i} \cap (C \times V V_\alpha - q^{-1}_\alpha(R)) = f_{a,i}(S^\circ_{a,i}) \);
   c. \( f_{a,i}(S_{a,i} - S^\circ_{a,i}) \subset q^{-1}_\alpha(R) \).

via intersecting with multi-sections [FO] LT2 LT3 [MD] Zin.
Since \( s \in C_{\text{mou}}(W, V) \), by definition, \( \sum_{i=1}^{m_\alpha} f_{\alpha,i}(S_{\alpha,i}) \) is \( G_\alpha \)-equivariant. Define
\[
(C \cap s)|_{V_{\alpha}} = \frac{1}{m_\alpha} \left( \sum_{i=1}^{m_\alpha} f_{\alpha,i}(S_{\alpha,i}) \right) / G_\alpha,
\] (5.15)
viewed as a sum of piecewise smooth \( k \)-dimensional \( \mathbb{Q} \)-currents away from a \((k-2)\)-dimensional subset. Since \( (s_{\alpha,i}) \) are local lifts of a global multi-section \( s \), the \( \mathbb{Q} \)-currents \((5.14)\) patch to form a piecewise smooth \( \mathbb{Q} \)-currents with vanishing boundary in \( |V| - R \). We denote this current by \( C \cap s \).

Since \( |W| \) is compact, the current \( C \cap s \) defines a homology class in \( H_k(|V|, R; \mathbb{Q}) = H_k(|V|; \mathbb{Q}) \).

Applying the projection \( \pi : |V| \to |W| \), we obtain the image \( \mathbb{Q} \)-current \( \pi(C \cap s) \) and its associated homology class \( [\pi(C \cap s)] \in H_k(|W|; \mathbb{Q}) \).

Following the topological construction of Gysin map of intersecting with the zero-section of \( V \),
\[
0_V^* [C] = [\pi(C \cap s)] \in H_* (|W|; \mathbb{Q})
\]
is the image of \([C]\) under the topological Gysin map \( 0_V^* \). By the linearity of Gysin map, this defines the topological \( 0_V^* \) in \((5.14)\).

The current \( \pi(C \cap s) \) is called a pseudo-cycle representative of the Gysin map.

We now assume in addition that \( \mathcal{F} \) is a quotient sheaf \( \mathcal{O}_W(V) \to \mathcal{F} \), and the cycle \([C] = \sum n_i [C_i] \in \mathbb{Z}_W \) has the property
\[
\text{(P) for each } C_i, \text{ and any closed } z \in W \text{ and } a \in \mathcal{F}|_z, \text{ letting } \phi_z : V_z \to \mathcal{F}|_z \text{ be } \phi \text{ restricting to } z, \text{ we have either } \phi_z^{-1}(a) \cap C_i = \emptyset \text{ or } \phi_z^{-1}(a) \cap C_i = \phi_z^{-1}(a). \]

**Definition 5.7.** Two multi-sections \( s \) and \( s' \) of \( V \) are \( \mathcal{F} \)-equivalent, denoted by \( s \sim_{\mathcal{F}} s' \), if for any \( x \in W \), as \( \mathbb{Q} \)-zero-cycles, we have \( (\phi_x)_*(s(x)) = (\phi_x)_*(s'(x)) \). A multi-section of \( \mathcal{F} \) is an \( \sim_{\mathcal{F}} \) equivalence class of multi-sections of \( V \). We say a multi-section \( s \) of \( \mathcal{F} \) intersects \( C \subset V \) transversally if a representative \( s \) of \( s \) intersects \( C \) transversally.

We comment that when \( C \) satisfies property (P), the notion that a multi-section of \( \mathcal{F} \) intersects \( C \) transversally is well-defined, after we pick the stratification of \( W \) so that \( \mathcal{F} \) restricts to each stratum is locally free, which we always assume in the remaining discussion.

We apply this discussion to \( C_{[\alpha, \delta]} \subset E_{[\alpha, \delta]} \). Let \( \mathcal{F}_{[\alpha, \delta]} = H^1(\mathcal{E}_{[\alpha, \delta]}), \) a coherent sheaf on \( Y_T^{[\alpha, \delta]} \), and let \( \mathcal{F}_{[\alpha, \delta]} = \rho_{\alpha, \delta}^* \mathcal{F}_{[\alpha, \delta]} \), the pullback sheaf on \( Y_T^{[\alpha, \delta]} \). (Note that \( \mathcal{F}_{[\alpha, \delta]} \) is the obstruction sheaf of the relative obstruction theory of \( Y_T^{[\alpha, \delta]} \).) Then \( \mathcal{F}_{[\alpha, \delta]} \) is the quotient sheaf of \( E_{[\alpha, \delta]} \) via
\[
\phi_{[\alpha, \delta]} : E_{[\alpha, \delta]} \to F_{[\alpha, \delta]} \xrightarrow{\text{h}} H^0(\rho_{\alpha, \delta}^* \mathcal{E}_{[\alpha, \delta]})) \to H^1(\rho_{\alpha, \delta}^* \mathcal{E}_{[\alpha, \delta]}) = \mathcal{F}_{[\alpha, \delta]}.
\]
Since \( C_{[\alpha, \delta]} \) is the pullback of the cycle \( C_{[\alpha, \delta]} \) in \( F_{[\alpha, \delta]} \), the cycle \( C_{[\alpha, \delta]} \) satisfies property (P) for the pair \( E_{[\alpha, \delta]} \to \mathcal{F}_{[\alpha, \delta]} \). Thus we can speak of multi-sections \( s \) of \( \mathcal{F}_{[\alpha, \delta]} \) intersecting \( C_{[\alpha, \delta]} \subset E_{[\alpha, \delta]} \) transversally.

In the future, we will call a multi-section of \( \mathcal{F}_{[\alpha, \delta]} \) intersecting \( C_{[\alpha, \delta]} \) transversally a good multi-section. Let \( k_{[\alpha, \delta]} \) be the virtual dimension of \( Y_T^{[\alpha, \delta]} \). For a good multi-section \( s_{[\alpha, \delta]} \) of \( \mathcal{F}_{[\alpha, \delta]} \), we denote
\[
D(s_{[\alpha, \delta]}) = \pi(C_{[\alpha, \delta]} \cap s_{[\alpha, \delta]}),
\]
where \( s_{[\alpha, \delta]} \) is a representative of \( s_{[\alpha, \delta]} \), and \( D(s_{[\alpha, \delta]}) \) is a piecewise smooth \( k_{[\alpha, \delta]} \)-dimensional \( \mathbb{Q} \)-current away from a subset of dimension at most \( k_{[\alpha, \delta]} - 2 \). (Note that \( D(s_{[\alpha, \delta]}) \) is independent of the choice of \( s_{[\alpha, \delta]} \).) We denote
\[
[D(s_{[\alpha, \delta]})] \in H_{k_{[\alpha, \delta]}} (Y_T^{[\alpha, \delta]} ; \mathbb{Q})
\]
the homology class it represents.

Applying the pseudo-cycle representative of Gysin maps, we obtain

---

3As argued in [CL], this means that \( C \) is a pull back of a “substack” of \( \mathcal{F} \).
Proposition 5.8. Given a good multi-section $s_{[α, δ]}$ of $F_{[α, δ]}$, we have

$$[D(s_{[α, δ]}]] = [Y^{[α, δ]}_T]_{vir} \in H_*([Y^{[α, δ]}_T]; \mathbb{Q}).$$

5.6. Comparison of virtual cycles. Our goal in this subsection is to compare the virtual cycles in terms of pseudo-cycle representatives. We will prove the analogue of Lemma 5.6 in [LJ].

To begin with, we recall $α$-diagonals, their tubular neighborhoods, and the associated partitions from [LJ]. For $α ∈ P_Λ$, we form the strict $α$-diagonal:

$$Δ_α = Δ^Y_α = \{ x \in Y^A_α | a ∼_α b ⇒ x_a = x_b \};$$  \hspace{1cm} (5.16)

it is closed in $Y^A_α$ and isomorphic to $Y^p_α$ when $α = (α_1, \ldots, α_l)$. Fix a sufficiently small number $c > 0$ and a large real $N$, and pick a function $ε : P_Λ → (0, c)$ whose values on any ordered pair $α > β$ satisfy $ε(α) > N \cdot ε(β)$. After fixing a Riemannian metric on $Y$, we define the $ε$-neighborhood of $Δ_α$ in $Y^A_μ$ to be

$$Δ_{α, ε} = Δ^Y_α \cap \{ x \in Y^A_μ | \text{dist}(x, Δ_α) < ε(α) \}.$$

(5.17)

For a pair $α ≥ β$, we define $Δ^F_α = \cup_{α ≥ γ ≥ β} Δ^F_{γ, ε}$ and $Q^F_α = Δ^F_α \setminus \cup_{α ≥ γ > β} Δ^F_{γ, ε} = Δ^F_β \setminus \cup_{α ≥ γ > β} Δ^F_{γ, ε}$. Then, $Q^F_α$ is a closed subset of $Δ^F_β$. By Lemma 5.5 of [LJ], if $Δ_{β, ε} \cap Q^F_β ≠ \emptyset$ for some $β_1, β_2 ≤ α$, then

$$β_1 ≤ β_2.$$

(5.18)

It follows that $Δ^F_β = \bigcup_{α ≥ γ ≥ β} Q^F_α$. In particular, for any $α$, by taking $β = 1^A$, we get $Y^A_β = \bigcup_{γ ≤ α} Q^F_γ$. Further, letting $Q^{[α, δ]}_β = \bigcup_{(β, δ) ≤ (α, δ)} Y^{[α, δ]}_β$, we obtain $Y^{[α, δ]}_T = \bigcup_{β ≤ α} Q^{[α, δ]}_β$. Note that for fixed $β$ with $β ≤ α$, we have $Q^{[α, δ]}_β = \bigcup_{(β, δ) ≤ (α, δ)} Y^{[α, δ]}_β$. Define $Q^{[α, δ]}_{(β, δ), ε} = Q^{[α, δ]}_β \cap Y^{[α, δ]}_{(β, δ)}$ for $(β, δ) ≤ (α, δ)$. Then, we obtain a partition:

$$Y^{[α, δ]}_T = \bigcup_{(β, δ) ≤ (α, δ)} Q^{[α, δ]}_{(β, δ), ε}.$$  \hspace{1cm} (5.19)

Lemma 5.9. For sufficiently small $ε$, we can find a collection of good multi-sections $s_{[α, δ]}$ of $F_{[α, δ]}$ that satisfy the properties

(i) each $s_{[α, δ]}$ intersects transversally with the cycle $C_{[α, δ]} \subset E_{[α, δ]}$;

(ii) for $(β, δ) < (α, δ)$, the pseudo-cycles (as $Q$-currents)

$$(C^{[α, δ]}_{β, δ})_* (D(s_{[α, δ]})) \cap Q^{[α, δ]}_{(β, δ), ε} = D(s_{[β, δ]}) \cap C^{[α, δ]}_{(β, δ), ε}(Q^{[α, δ]}_{(β, δ), ε}).$$

Lemma 5.5.1. We follow the proof of [LJ] Lemma 5.6 line by line, with $Q^{[α, δ]}_{(β, δ), ε}$ (respectively, $s_{[α, δ]}$) in [LJ] p. 2156 replaced by $Q^{[α, δ]}_{(β, δ), ε}$ (respectively, $s_{[α, δ]}$).

To carry the argument in [LJ] p. 2156 through in the current situation, two modifications are necessary. The first is using multi-sections of $F_{[α, δ]}$, etc. The two properties of sections we used in the proof of [LJ] Lemma 5.6 are the existence of extensions and general position results. For multiple-sections, similar results hold as we have mentioned before.

The other is to choose multi-section $s_{[α, δ]}|_{[β, δ]}$ of $F_{[α, δ]}|_{[β, δ], ε}$ to be the pullback

$$s_{[α, δ]}|_{[β, δ]} = (C^{[α, δ]}_{β, δ})^*(s_{[β, δ]}|_{C^{[α, δ]}_{β, δ}}).$$

(Compare the construction of $s_{[α, δ]}|_{[β, δ]} = s_{[β]}|_{Q^{[α, δ]}_{[β, δ], ε}}$ in [LJ] p. 2156.) Since $F_{[α, δ]}|_{[β, δ]} = (C^{[α, δ]}_{β, δ})^*F_{[β, δ]}$, such pullback is well-defined.

What we need to make sure is that the section $s_{[α, δ]}|_{[β, δ]}$ of $F_{[α, δ]}|_{[β, δ], ε}$ intersects transversally with the cycle $C_{[α, δ]}|_{[β, δ]}$; this is true, following Lemma 5.5 and Lemma 5.6. This completes the proof of the lemma.\[\square\]
5.7. **Approximating virtual cycles.** In this subsection, we define the pseudo-cycle $\Theta^{[\alpha,\delta]}$ and study its properties. The formula (5.22) below can be roughly thought of as a decomposition of the virtual cycle $\text{ev}_* [Y_T^{[\alpha,\delta]}]_{\text{vir}}$ as a sum of cycles $\Theta^{[\beta,\eta]}$ supported near $\alpha$-diagonals. The ideal situation is that we have a similar decomposition for $\text{ev}_* [Y_T^{[\alpha,\delta]}]_{\text{vir}}$ in $(Y_T^{[\alpha]})^3$. Unfortunately, such a decomposition doesn’t exist. However, the decomposition (5.22) works equally well as if we had a decomposition for $\text{ev}_* [Y_T^{[\alpha,\delta]}]_{\text{vir}}$. This is carried out in subsections 5.11 and 5.12.

In fact, the main reason for introducing Hilbert schemes of $\alpha$-points $Y_T^{[\alpha]}$, non-separated spaces $Y_T^{[\leq \alpha]}$, and moduli spaces $Y_T^{[\alpha,\delta]}$ of $\alpha$-stable maps to Hilbert schemes is to provide appropriate spaces where we can define $\Theta^{[\alpha,\delta]}$.

Let $(\beta, \eta) \leq (\alpha, \delta) \in \mathcal{P}_{\Lambda,d}$. Define

$$
\phi_{\beta,\alpha} : Y_T^{[\beta]} \rightarrow Y_T^{[\leq \alpha]}, \quad \bar{\phi}_{\beta,\alpha} : Y_T^{[\leq \beta]} \rightarrow Y_T^{[\leq \alpha]}
$$

to be the open immersions induced from the construction (5.5). The evaluation map $\text{ev}_i : Y_T^{[\Lambda,d]} \rightarrow Y_T^{[\Lambda]}$ induces an evaluation map $Y_T^{[\alpha,\delta]} \rightarrow Y_T^{[\alpha]}$ which will be denoted again by $\text{ev}_i$. Let $ev = \text{ev}_1 \times \text{ev}_2 \times \text{ev}_3 : Y_T^{[\alpha,\delta]} \rightarrow (Y_T^{[\alpha]})^3$. Since $ev = Y_T^{[\alpha,\delta]} = Y_T^{[\alpha]} \times (\gamma,\rho) Y_T^{[\delta]} = Y_T^{[\alpha] \times (\gamma,\rho)} Y_T^{[\delta]}$, we use induction on the order of $\Delta = (\gamma,\rho) \subset (\alpha,\delta)$. By Proposition 5.8, we obtain

$$
\iota_{\Lambda} : Y_T^{[\Lambda]} \rightarrow (Y_T^{[\Lambda]})^3
$$
is the diagonal embedding. Define the indexing morphism to be

$$
in : \bigcup_{(\beta, \eta) \leq (\alpha, \delta)} (\phi_{\beta,\alpha})_{\text{ev}}(Y_T^{[\beta,\eta]}) \rightarrow \iota_{\Lambda}(Y_T^{[\Lambda]}) \cong Y_T^{[\Lambda]}.
$$

**Definition 5.10.** Define the pseudo-cycle $\Theta^{[\alpha,\delta]} \subset (Y_T^{[\leq \alpha]})^3$ inductively by

$$
\Theta^{[\alpha,\delta]} = (\phi_{\alpha,\alpha})_{\text{ev}} D(s_{[\alpha,\delta]}) - \sum_{(\beta, \eta) < (\alpha, \delta)} (\bar{\phi}_{\beta,\alpha})_* [\Theta^{[\beta,\eta]}].
$$

(5.21)

By Proposition 5.8, we obtain

$$
(\phi_{\alpha,\alpha})_{\text{ev}} [Y_T^{[\alpha,\delta]}]_{\text{vir}} = \sum_{(\beta, \eta) \leq (\alpha, \delta)} (\bar{\phi}_{\beta,\alpha})_* [\Theta^{[\beta,\eta]}].
$$

(5.22)

Further properties of the pseudo-cycles $\Theta^{[\alpha,\delta]}$ are contained in the next two lemmas which are the analogues of Lemmas 5.7 and 5.9 in [61].

**Lemma 5.11.** $\Theta^{[\alpha,\delta]} \subset (Y_T^{[\leq \alpha]})^3 \times Y_T^{[\Lambda]^{\beta}} \iota_{\Lambda}(\Delta_{\alpha,\epsilon})$ for sufficiently small $\epsilon > 0$.

**Proof.** We use induction on the order of $(\alpha, \delta) \in \mathcal{P}_{\Lambda,d}$. Assume that $(\alpha, \delta)$ is minimal. Then for each $i$, we have either $\delta_i = 1$, or $\alpha_i = 2$ and $\delta_i > 0$. So $Y_T^{[\alpha,\delta]} = Y_T^{[\alpha]} \times Y_T^{[\delta]} \Delta_{\alpha} = Y_T^{[\alpha]} \times Y_T^{[\alpha]} \Delta_{\alpha,\epsilon}$. Thus, $\Theta^{[\alpha,\delta]} = (\phi_{\alpha,\alpha})_{\text{ev}} D(s_{[\alpha,\delta]}) \subset (Y_T^{[\leq \alpha]})^3 \times Y_T^{[\Lambda]^{\beta}} \iota_{\Lambda}(\Delta_{\alpha,\epsilon})$.

Next, we assume that our lemma is true for every $(\gamma, \rho)$ with $(\gamma, \rho) < (\alpha, \delta)$. Recall that $Y_T^{[\alpha]} = \coprod_{\beta \leq \alpha} Q_{\beta,\epsilon}$ and $Q_{\beta,\epsilon} = \Delta_{\alpha,\epsilon}$. So to prove the lemma, it suffices to verify $\Theta^{[\gamma,\rho]} \cap (((Y_T^{[\leq \alpha]})^3 \times (Y_T^{[\Lambda]^{\beta}}) \iota_{\Lambda}(Q_{\beta,\epsilon}^{[\alpha]})) = \emptyset$ for every $\beta < \alpha$. By (5.21), this is equivalent to proving that the intersection

$$
(\phi_{\alpha,\alpha})_{\text{ev}} D(s_{[\alpha,\delta]}) \cap \left( (Y_T^{[\leq \alpha]})^3 \times (Y_T^{[\Lambda]^{\beta}}) \iota_{\Lambda}(Q_{\beta,\epsilon}^{[\alpha]}) \right)
$$

$$
= \sum_{(\gamma, \rho) < (\alpha, \delta)} (\bar{\phi}_{\beta,\alpha})_* [\Theta^{[\gamma,\rho]}] \cap \left( (Y_T^{[\leq \alpha]})^3 \times (Y_T^{[\Lambda]^{\beta}}) \iota_{\Lambda}(Q_{\beta,\epsilon}^{[\alpha]}) \right).
$$

(5.23)

On one hand, if $(\gamma, \rho) < (\alpha, \delta)$, then $\Theta^{[\gamma,\rho]} \subset (Y_T^{[\leq \gamma]})^3 \times (Y_T^{[\Lambda]^{\beta}}) \iota_{\Lambda}(\Delta_{\gamma,\epsilon})$ by induction. Thus, a nonempty $(\phi_{\alpha,\alpha})_{\text{ev}} D(s_{[\alpha,\delta]}) \cap \left( (Y_T^{[\leq \alpha]})^3 \times (Y_T^{[\Lambda]^{\beta}}) \iota_{\Lambda}(Q_{\beta,\epsilon}^{[\alpha]}) \right)$ forces $\Delta_{\gamma,\epsilon} \cap Q_{\beta,\epsilon}^{[\alpha]} \neq \emptyset$ which in turn implies...
\( \gamma \leq \beta \) by (5.18). Therefore, the right-hand-side of (5.23) equals
\[
\sum_{(\gamma, \rho) \leq (\alpha, \delta), \gamma \leq \beta} (\hat{\phi}_{3}^{3})_{*} (\Theta^{|\gamma, \rho|} \cap ((Y_{T}^{[\leq \alpha])})^{3} \times (Y_{T}^{[\leq \alpha])})^{3}) = \sum_{(\beta, \eta) \leq (\alpha, \delta)} (\hat{\phi}_{3}^{3})_{*} (\phi_{\beta, \alpha}^{3})_{*} (\Theta^{|\gamma, \rho|} \cap ((Y_{T}^{[\leq \alpha])})^{3} \times (Y_{T}^{[\leq \alpha])})^{3}) \tag{5.24}
\]
Since \( (\hat{\phi}_{3}^{3})_{*} (\Theta^{|\gamma, \rho|} \cap ((Y_{T}^{[\leq \alpha])})^{3} \times (Y_{T}^{[\leq \alpha])})^{3}) \) is contained in \( (\phi_{3}^{3})_{*} (\Theta^{|\gamma, \rho|} \cap ((Y_{T}^{[\leq \alpha])})^{3} \times (Y_{T}^{[\leq \alpha])})^{3}) \), so (5.25) (hence the right-hand-side of (5.23)) equals
\[
\sum_{(\beta, \eta) \leq (\alpha, \delta)} (\phi_{3}^{3})_{*} (\Theta^{|\gamma, \rho|} \cap ((Y_{T}^{[\leq \alpha])})^{3} \times (Y_{T}^{[\leq \alpha])})^{3}) \tag{5.25}
\]
Since \( Q_{\beta, \xi}^{\alpha} = \Delta_{\beta, \xi} - \cup_{\alpha \geq \beta} \Delta_{\alpha}^{\alpha}, \) we see that \( (\phi_{3}^{3})_{*} (\Theta^{|\gamma, \rho|} \cap ((Y_{T}^{[\leq \alpha])})^{3} \times (Y_{T}^{[\leq \alpha])})^{3}) \) is contained in \( (\phi_{3}^{3})_{*} (\Theta^{|\gamma, \rho|} \cap ((Y_{T}^{[\leq \alpha])})^{3} \times (Y_{T}^{[\leq \alpha])})^{3}) \), and the subsets \( \iota_{\Lambda}(\gamma, \xi, \alpha) \leq \alpha \) are disjoint. So the left-hand-side of (5.23) is equal to
\[
\sum_{(\beta, \eta) \leq (\alpha, \delta)} (\phi_{3}^{3})_{*} (\Theta^{|\gamma, \rho|} \cap ((Y_{T}^{[\leq \alpha])})^{3} \times (Y_{T}^{[\leq \alpha])})^{3}) \tag{5.26}
\]
where we have used Lemma 5.19 (ii) in the last step. Combining with (5.26), we get (5.23). \( \Box \)

**Lemma 5.12.** Let \( (\alpha, \delta) \in \mathcal{P}_{\Lambda, d} \) with \( \alpha = (\alpha_{1}, \ldots, \alpha_{l}) \). Then, \( \Theta^{[\alpha, \delta]} = \prod_{i=1}^{l} \Theta^{[\alpha_{i}, \delta_{i}]} \) via the natural identification \( (Y_{T}^{[\leq \alpha])})^{3} = \prod_{i=1}^{l} (Y_{T}^{[\leq \alpha_{i})})^{3} \).

**Proof.** First of all, since \( Y_{T}^{[\alpha, \delta]} = \prod_{i=1}^{l} Y_{T}^{[\alpha_{i}, \delta_{i}]} \), we have
\[
D(s^{[\alpha, \delta]}_{\Lambda, \eta}) = \prod_{i=1}^{l} D(s^{[\alpha_{i}, \delta_{i}]}_{\Lambda, \eta}). \tag{5.27}
\]

Next, to prove the lemma, we use induction on the size \( |\Lambda| \) and on the order of \( (\alpha, \delta) \in \mathcal{P}_{\Lambda, d} \). Assume that \( (\alpha, \delta) \) is minimal in \( \mathcal{P}_{\Lambda, d} \). Then \( (\alpha_{1}, \delta_{1}) \) is minimal in \( \mathcal{P}_{\alpha_{1}, \delta_{1}} \). By (5.21) and (5.27),
\[
\Theta^{[\alpha, \delta]} = (\phi_{3}^{3})_{*} D(s^{[\alpha, \delta]}_{\Lambda, \eta}) = \prod_{i=1}^{l} (\phi_{3}^{3})_{*} D(s^{[\alpha_{i}, \delta_{i}]}_{\Lambda, \eta}) = \prod_{i=1}^{l} \Theta^{[\alpha_{i}, \delta_{i}]}.
\]
In particular, the lemma holds for \(|\Lambda| = 1\) (necessarily, \((\alpha, \delta) = (1, 0)\)). Next, assume that 
\[ \Theta_{\beta, \eta} = \prod_{i} \Theta_{[\beta, \eta]} \] 
for every \((\beta, \eta) < (\alpha, \delta)\). By \((5.27)\) and \((5.21)\),
\[
(\tilde{\phi}_{\beta, \alpha}^3)_{ev} D(s_{[\alpha, \delta]}) = \prod_{i=1}^{l} (\phi_{\beta, \alpha}^3)_{ev} D(s_{[\alpha, \delta]}) \\
= \prod_{i=1}^{l} \sum_{(\beta(i), \eta(i)) \leq (\alpha, \delta)} (\phi_{\beta(i), \alpha}^3)_{ev} \Theta_{[\beta(i), \eta(i)]} \\
= \prod_{i=1}^{l} \Theta_{[\alpha, \delta]} + \sum_{(\beta, \eta) < (\alpha, \delta)} (\tilde{\phi}_{\beta, \alpha}^3)_{ev} \Theta_{[\beta, \eta]}
\]
noting that induction has been used in the last step to handle those \(\beta(i)\) which have length greater than 1. Applying \((5.21)\) again, we obtain the lemma.

5.8. Co-section localizations. We now apply the co-section localization techniques from [KL2] to the constructions in the previous subsections. Let \(\theta\) be a meromorphic section of \(\mathcal{O}_{X}(K_X)\), and let \(D_0\) and \(D_\infty\) be the vanishing and pole divisors of \(\theta\) respectively. For simplicity, we assume that \(D_0\) and \(D_\infty\) are smooth irreducible curves intersecting transversally. Let \(X_\theta^{[n,d]}\) be the subset of \(X^{[n,d]}\) consisting of those \(\varphi\) whose standard decomposition \((\varphi_1, \ldots, \varphi_l)\) have the property that for each \(i\), either \(\varphi_i\) is constant or the support of \(\varphi_i\) lies in \(D_0 \cup D_\infty\). The meromorphic section \(\theta\) induces a meromorphic section \(\theta_{[n]}\) of \(\Omega^2_{X^{[n]}}\). By [KL2], we obtain the localized virtual fundamental cycle \([X^{[n,d]}]_{loc}^{vir} \in A_*(X_{\theta}^{[n,d]})\) of \(X^{[n,d]}\) such that \(\iota_* [X^{[n,d]}]_{loc}^{vir} = [X^{[n,d]}]_{loc}^{vir}\) where \(\iota_*\) is the map induced by the inclusion map \(\iota : X_\theta^{[n,d]} \hookrightarrow X^{[n,d]}\). For simplicity of notations, we write \([X^{[n,d]}]_{loc}^{vir} = [X^{[n,d]}]^{vir}\).

The constructions in [KL2] and Subsections 5.1-5.7 are canonical. Applying the constructions in [KL2] to Subsections 5.1-5.7, we obtain localized cycles \([X^{[\alpha, \delta]}]_{loc}^{vir} \in H_*(X_{\theta}^{[\alpha, \delta]}; \mathbb{Q})\), \([D(s_{[\alpha, \delta]})]_{loc}^{vir}, \text{ and } \Theta_{[\alpha, \delta]}^{loc} \subset U(\beta, \eta) \leq (\alpha, \delta) \tilde{\phi}_{\beta, \alpha}^{3, ev}(X_{\theta}^{[\beta, n]})\) with \([X^{[\alpha, \delta]}]_{loc}^{vir} = [X^{[\alpha, \delta]}]^{vir}\) and \([\Theta_{[\alpha, \delta]}^{loc}] = [\Theta_{[\alpha, \delta]}^{3}]\) in \(H_*(X_{\theta}^{[\alpha, \delta]}; \mathbb{Q})\) and \(H_*(X^{[\alpha, \delta]}; \mathbb{Q})\) respectively. Here the subset \(X_\theta^{[\alpha, \delta]} \subset X^{[\alpha, \delta]}\) is defined similarly as \(X^{[n,d]} \subset X^{[n,d]}\).

5.9. Extensions of Heisenberg monomial classes. Let \((\beta, \eta) \in \mathcal{P}_{[n,d]}\). To study the pairings with \((\tilde{\phi}_{\beta, \alpha}^3)_{ev}(\Theta_{[\beta, \eta]}^{[\alpha, \delta]})\), we need to extend the classes \((f^{[n]})^\ast w \in H^*(X^{[n]})\) from \(X^{[n]}\) to \(X^{[\leq n]}\), where \(f^{[n]} : X^{[n]} \rightarrow X^{[n]}\) is the tautological map. Let \(f^\ast = \prod_{i} f^{[\beta_i]}\).

Lemma 5.13. Let \(\alpha_t \in H^*(X)\) be homogeneous with \(|\alpha_t| > 0\), and \(\alpha_{t,i,j} = 1\). Let
\[
w = \left( \prod_{i=1}^{l} \prod_{j=1}^{s_i} a_{-i} \right) \left( \prod_{i=1}^{k} a_{-n_i} \alpha_t \right) |0\rangle \in H^*(X^{[n]}). \tag{5.28}
\]
Then there exists a class \(w^{[\leq n]} \in H^*(X^{[\leq n]})\) such that \((\phi_{[n]}^{[\eta]} w^{[\leq n]} = (f^{[n]})^\ast w, \text{ and that if }\)
\(\beta = (\beta_1, \ldots, \beta_t) \leq [n], \text{ then via the identification } X^{[\leq \beta]} = \prod_{i=1}^{l} X^{[\leq \beta_i]}, \)
\[
(\tilde{\phi}_{\beta, [n]}^3)_{ev} w^{[\leq n]} = \sum_{w_1 \circ \cdots \circ w_t = w} \xi_{l} w^{[\leq \beta_t]} \tag{5.29}
\]
where each \(w_i \in H^*(X^{[\leq \beta_i]})\) is a Heisenberg monomial class.

Proof. We use induction on \(n\). The lemma is trivially true when \(n = 1\). In the following, assume that the lemma holds for all \(X^{[m]}\) with \(m < n\).
Let $S$ be the set consisting of all $\beta < [n]$ such that there does not exist $\gamma < [n]$ with $\beta < \gamma$. Then $X^{[\leq n]}$ is covered by the open subsets $\phi_{[n],[n]}(X^{[n]})$ and $\bar{\phi}_{[n]}(X^{[\leq n]}), \beta \in S$. Define

$$w^{[\leq \beta]} = \sum_{w_1 \circ \cdots \circ w_{i-1} = w} \otimes_{i=1}^{\beta} w_i^{[\leq \beta]} \in H^*(X^{[\leq \beta]}) = H^*(\bar{\phi}_{[n]}(X^{[\leq \beta]}))$$

for each $\beta \in S$. Applying the Mayer-Vietoris sequence successively, we see that to prove the existence of $w^{[\leq n]} \in H^*(X^{[\leq n]})$, it suffices to show that $(f^{[n]})^*w$ and $w^{[\leq \beta]}$, $\beta \in S$ are equal on the overlaps of the open subsets $\phi_{[n],[n]}(X^{[n]})$ and $\bar{\phi}_{[n]}(X^{[\leq n]}), \beta \in S$.

First of all, let $\beta, \gamma = (\gamma_1, \ldots, \gamma_r) \in S$ and $\beta \neq \gamma$. Then,

$$\bar{\phi}_{[n]}(X^{[\leq \beta]}) \cap \bar{\phi}_{[n]}(X^{[\leq \gamma]}) = \bar{\phi}_{\beta \wedge \gamma,[n]}(X^{[\leq \beta \wedge \gamma]}) = \bar{\phi}_{\beta \wedge \gamma,[n]}(X^{[\leq \beta \wedge \gamma]}).$$

Let $\beta_i \wedge \gamma = (\beta_i \cap \gamma_1, \ldots, \beta_i \cap \gamma_r) \in P_{\beta_i}$. Then $(\bar{\phi}_{\beta \wedge \gamma,[n]}(X^{[\leq \beta]}))^*w^{[\leq \beta]}$ is equal to

$$(\bar{\phi}_{\beta \wedge \gamma,[n]}(X^{[\leq \beta]}))^* w^{[\leq \beta]} = \sum_{w_1 \circ \cdots \circ w_{i-1} = w} \otimes_{i=1}^{\beta} w_i^{[\leq \beta]} = \sum_{w_1 \circ \cdots \circ w_{i-1} = w} \otimes_{i=1}^{\beta} (\bar{\phi}_{\beta \wedge \gamma,[n]}(X^{[\leq \beta]}))^* w_i^{[\leq \beta]}.$$ 

Applying induction to the classes $w_i^{[\leq \beta_i]}$, we see that

$$(\bar{\phi}_{\beta \wedge \gamma,[n]}(X^{[\leq \beta]}))^* w^{[\leq \beta]} = \sum_{w_1 \circ \cdots \circ w_{i-1} = w} \otimes_{i=1}^{\beta} \left( \sum_{w_1 \circ \cdots \circ w_{i-1} = w} \otimes_{j=1}^{\gamma} w_{i,j}^{[\leq \beta \wedge \gamma_i]} \right) = \sum_{w_1 \circ \cdots \circ w_{i-1} = w} \otimes_{i=1}^{\beta} \otimes_{j=1}^{\gamma} w_{i,j}^{[\leq \beta \wedge \gamma_i]}.$$ 

(5.30)

It follows immediately that $(\bar{\phi}_{\beta \wedge \gamma,[n]}(X^{[\leq \beta]}))^* w^{[\leq \beta]} = (\bar{\phi}_{\beta \wedge \gamma,[n]}(X^{[\leq \beta]}))^* w^{[\leq \gamma]}.$

Next, we claim that the restrictions of $(f^{[n]})^*w$ and $w^{[\leq \beta]}$, $\beta \in S$ are equal on $\phi_{[n],[n]}(X^{[n]}) \cap X^{[\leq \beta]}$ are equal. Note that $X^{[\leq \beta]}$ is covered by the open subsets $\phi_{[n],[n]}(X^{[n]})$, $\beta \leq \gamma$, and $\phi_{[n],[n]}(X^{[n]}) \cap \phi_{[1],[n]}(X^{[\beta]})$ is identified with the images of $X^{[n]}_\beta \cong X^{[n]}_{\beta \cap \gamma}$. So it suffices to prove that

$$(f^{[n]})^*w|_{X^{[n]}_\beta} = (c_{[n]}^{\beta})^* \left( (\phi_{[n],[n]}(X^{[n]}) \cap X^{[\leq \beta]}) \right) = (f^{[n]})^*w|_{X^{[n]}_{\beta \wedge \gamma}}.$$ 

(5.31)

To see this, represent each $\alpha_i \in H^*(X)$ by a cycle $X_i$ such that $X_1, \ldots, X_k$ are in general position. By Proposition 2.7, the class $w_{[n]} := w/\prod_{i=1}^{p} s_i$ is represented by the closure $W$ of the subset consisting of elements of the form $[22]$. Then, $(f^{[n]})^*w_{[n]}$ is represented by $(f^{[n]})^{-1}(W)$. By Proposition 2.7 again, the closure of $f^{[n]}((f^{[n]})^{-1}(W) \cap X^{[n]}_{\beta})$ in $X^{[\beta]}$ represents the class

$$w_{\beta} := \sum_{w_1 \circ \cdots \circ w_l = w} \prod_{i=1}^{l} \frac{1}{n_{i,j}} \cdot w_1 \otimes \cdots \otimes w_l \in H^*(X^{[\beta]}) \cong \bigotimes_{i=1}^{l} H^*(X^{[\beta_i]})$$

where each $w_j \in H^*(X^{[\beta_j]})$ contains exactly $s_{i,j}$ copies of $a_{1,i}$. Note that $\sum_{j=1}^{l} s_{i,j} = s_i$.

Also, the class $(f^{[n]})^*w_{\beta} \in H^*(X^{[\beta]})$ is represented by the closure of $c_{[n]}^{\beta}((f^{[n]})^{-1}(W) \cap X^{[n]}_{\beta})$. So $(f^{[n]})^*w_{[n]}|_{X^{[n]}_{\beta}} = (c_{[n]}^{\beta})^*\left( (f^{[n]})^*w_{\beta} \right)$.

Note that for fixed integers $s_{i,j}$, the number of choices of $w_1, \ldots, w_l$ satisfying $w_1 \circ \cdots \circ w_l = w$ is precisely equal to $\prod_{i=1}^{n} s_i / \prod_{i=1}^{n} \prod_{k=1}^{l} s_{i,j}$.

Therefore, $(f^{[n]})^*w_{[n]}|_{X^{[n]}_{\beta}}$ is equal to

$$(c_{[n]}^{\beta})^*\left( (f^{[n]})^*w_{\beta} \right)|_{X^{[n]}_{\beta}} = \left( c_{[n]}^{\beta} \right)^* \left( f^{[\beta]} \right)^* \sum_{w_1 \circ \cdots \circ w_l = w} w_1 \otimes \cdots \otimes w_l |_{X^{[n]}_{\beta}}.$$ 

(5.32)
On the other hand, since \( \phi_{\beta, \gamma} = \bar{\phi}_{\beta, \gamma} \circ \phi_{\beta, \beta} \), we obtain from (5.30) that
\[
(\phi_{\beta, \gamma})^* w_{[\leq |\gamma|]} = (\phi_{\beta, \beta})^* \sum_{w_{1} \cdots w_{|\gamma|} = w} \otimes_{i=1}^{|\gamma|} w_{i}^{[\leq |\beta|]} = \sum_{w_{1} \cdots w_{|\gamma|} = w} \otimes_{i=1}^{|\gamma|} (\phi_{\beta, \beta})^* w_{i}^{[\leq |\beta|]} = \sum_{w_{1} \cdots w_{|\gamma|} = w} \otimes_{i=1}^{|\gamma|} (f[|\beta|])^* w_{i} = \sum_{w_{1} \cdots w_{|\gamma|} = w} \otimes_{i=1}^{|\gamma|} w_{i}
\]
where we have used induction in the third equality. Combining with (5.32) verifies (5.31).

Our next lemma says that even though the extension \( w_{[\leq n]} \in H^*(X_{[\leq n]}) \) may not be unique, it does not affect the pairings with \( \langle \Theta^{[n, d]} \rangle \). Recall that the tautological map \( \rho_{\alpha, \delta} : X^{[\alpha, \delta]} \to X^{[\alpha]} \) is a finite map of degree \( n! \).

**Lemma 5.14.** Let \( A_1, A_2, A_3 \in H^*(X^{[n]}) \) be Heisenberg monomial classes. Then, the pairing
\[
\langle \langle \Theta^{[n, d]} \rangle, A_1^{[\leq n]} \otimes A_2^{[\leq n]} \otimes A_3^{[\leq n]} \rangle
\]
is independent of the choices of \( A_1^{[\leq n]}, A_2^{[\leq n]}, A_3^{[\leq n]} \).

**Proof.** Since \( [X^{[n, d]}]_{\text{vir}} = \rho_{*}[X^{[n, d]}]_{\text{vir}}, \langle [X^{[n, d]}]_{\text{vir}}, ev^*(A_1 \otimes A_2 \otimes A_3) \rangle \) is equal to
\[
\frac{1}{n!} \langle [X^{[n, d]}]_{\text{vir}}, \rho_{*}[X^{[n, d]}]_{\text{vir}}, ev^*(A_1 \otimes A_2 \otimes A_3) \rangle = \frac{1}{n!} \langle [X^{[n, d]}]_{\text{vir}}, ev^* \bigotimes_{i=1}^3 (f[n])^* A_i \rangle.
\]
By Lemma 5.13 and 5.22, \( \langle [X^{[n, d]}]_{\text{vir}}, ev^*(A_1 \otimes A_2 \otimes A_3) \rangle \) is equal to
\[
\frac{1}{n!} \left( \langle [X^{[n, d]}]_{\text{vir}}, ev^* \bigotimes_{i=1}^3 \phi_{3_{[n], [n]}}, A_i^{[\leq n]} \rangle \right)
= \frac{1}{n!} \left( \langle \phi_{3_{[n], [n]}}, ev^*[X^{[n, d]}]_{\text{vir}}, \bigotimes_{i=1}^3 A_i^{[\leq n]} \rangle \right)
= \frac{1}{n!} \left( \sum_{(\alpha, \delta) \leq ([n], d)} \langle \Theta^{[\alpha, \delta]}, \bar{\phi}_{\alpha, [n]}^* A_1^{[\leq n]} \otimes \bar{\phi}_{\alpha, [n]}^* A_2^{[\leq n]} \otimes \bar{\phi}_{\alpha, [n]}^* A_3^{[\leq n]} \rangle \right).
\]
Next, to prove the lemma, we use induction on \( n \). When \( n = 1 \), the lemma is trivially true since \( A_1^{[\leq n]} = A_1 \). Assume that the lemma holds for all \( X^{[m]} \) with \( m < n \). Let \( (\alpha, \delta) \leq ([n], d) \).

By Lemma 5.12 and 5.29, \( \langle \phi_{3_{[n], [n]}}, \Theta^{[\alpha, \delta]}, A_1^{[\leq n]} \otimes A_2^{[\leq n]} \otimes A_3^{[\leq n]} \rangle \) is equal to
\[
\sum_{A_1, A_2, \ldots, A_3} \prod_{i=1}^l \langle \Theta^{[\alpha, \delta], [\alpha], [\alpha]}, A_1^{[\leq |\alpha|]} \otimes A_2^{[\leq |\alpha|]} \otimes A_3^{[\leq |\alpha|]} \rangle.
\]
Now our lemma follows from (5.33) and induction.

**Remark 5.15.** Note that for \( A \in H_k(W) \) and \( B \in H^k(W) \) on a topological space \( W \), the pairing \( \langle A, B \rangle \) is the degree of the 0-cycle \( A \cap B \in H_0(W) \). As 0-cycles, \( \langle \phi_{3_{[n], [n]}}, ev_*[D(s[n, d])] \cap ((f[n])^* A_1 \otimes (f[n])^* A_2 \otimes (f[n])^* A_3) \rangle \) is equal to
\[
\sum_{(\alpha, \delta) \leq ([n], d)} \langle \phi_{3_{[n], [n]}}, \sum_{A_1, A_2, \ldots, A_3} \prod_{i=1}^l \langle \Theta^{[\alpha, \delta], [\alpha], [\alpha]}, A_1^{[\leq |\alpha|]} \otimes A_2^{[\leq |\alpha|]} \otimes A_3^{[\leq |\alpha|]} \rangle \rangle.
\]
Next, we extend the notation of Heisenberg monomial classes to a smooth family \( Y \to T \) of quasi-projective surfaces.

**Definition 5.16.** Fix positive integers \( s_1, \ldots, s_t \) with \( \sum_i i s_i = n \). Define

\[
w^Y = \prod_{i=1}^t a^Y_{X_i}(1_X)^{s_i} |0 \in H^*(Y^n_T) \tag{5.35}\]

\( Y \) to be the cohomology class represented by the cycle \( \prod_{i=1}^t s_i! \cdot [W] \in A_*(Y^n_T) \) where \( W \subset Y^n_T \) is the closure of the subset consisting of elements of the form

\[
\sum_{i=1}^t (\xi_{i,1} + \ldots + \xi_{i,s_i}) \in (Y_u)^{[n]} \text{, } u \in T
\]

where \( \xi_{i,m} \in M_i(x_{i,m}) \) for some \( x_{i,m} \in Y_u \), and all the points \( x_{i,m} \) are distinct.

The following lemma is similar to Lemma \([5.13] \) and its proof is omitted.

**Lemma 5.17.** Let \( w = w^Y \) be as in \((5.35) \). Then there exists \( w^{[\leq n]} \in H^*(Y^n_T) \) such that \((w_{[n],[n]})_{*}w^{[\leq n]} = (f[n])_{*}w \), and that if \( \beta = (\beta_1, \ldots, \beta_t) \leq [n] \), then \((w_{[\beta,t]},[n])_{*}w^{[\leq n]} = \sum_{w_{[\beta,t]},[n]} = w_{[\beta,t]}^{[\leq \beta]} \) via the identification \( Y^n_T = \prod_{i=1}^t Y^n_{T_i} \).

### 5.10. Normal slices and universal families

This subsection mainly provides a necessary set-up for the proof of the universality result Lemma \([5.21] \) in the next subsection.

By Lemma \([5.14] \), we have \( \Theta^{[\alpha, \beta]} \subset (Y^n_T)^3 \times (Y^n_T)^3 \chi_{A_{\alpha, \beta}} \). In this subsection, with \( Y = X \) and \( \alpha = [n] \), we will describe an analytic space, independent of \( \epsilon \), which contains \((X^{[\leq n]})^3 \times (X^n)^3 \chi_{A_{[n], \epsilon}} \) whenever \( \epsilon \) is sufficiently small.

To begin with, let \( Y \to T \) be the total space of a rank-2 vector bundle, viewed as a smooth family of affine schemes. Define the fiber-wise averaging morphism

\[
aw : Y_T^{(n)} \to Y; \sum m_i x_i \in Y_T^{(n)} \to \frac{1}{n} \sum m_i x_i \in Y_t, \text{ } t \in T.
\]

Here \( \sum m_i x_i \) is the sum using the fiber-wise linear structure of \( Y/T \). Using \( Y^n_T \to Y_T^{(n)} \) and \( Y_T^{[n]} \to Y_T^{(n)} \), we obtain the averaging maps \( aw : Y^n_T \to Y_T^{[n]} \) and \( Y_T^{[n]} \to Y_T^{(n)} \). We define the relative Hilbert scheme of centered \( \alpha \)-points to be

\[
Y_{T,0}^n = Y_T^n \times_{aw,Y} 0_Y,\tag{5.36}
\]

where \( 0_Y \subset Y \) is the zero-section of \( Y \to T \).

Next, like in \([5.15] \), we need to express an open neighborhood of the diagonal \( [2] \subset X \times X \) a vector bundle structure, using the first projection. As this is impossible in general, we will content to have a \( C^\infty \)-vector bundle structure. For this reason, we will again work with the analytic category. We will use differentiable map to mean a \( C^\infty \)-map; and an open subset will be open in analytic topology; we will use regular function and Zariski open set to stand for their original meanings in algebraic geometry.

Consider the total space of the tangent bundle \( TX \), and its zero-section \( 0_X \subset TX \). For an open \( U \subset X \times X \), we view it as a space over \( X \) via (that induced by the first projection) \( pr_1|_U : U \to X \). By Lemma 2.4 in \([5.16] \), there exists a diffeomorphism

\[
\varphi : U \to V \tag{5.37}
\]

of a tubular neighborhood \( U \) of \( X_{[2]} \subset X \times X \) and a tubular neighborhood \( V \) of \( 0_X \subset TX \), both considered as fiber bundles over \( X \), such that

(A-i) restricting to each fiber \( U_x = (pr_1|_U)^{-1}(x) \), the map \( \varphi_x = \varphi|_{U_x} : U_x \to V_x \) is a bihomorphism,

(A-ii) \( \varphi_x(x) = 0 \in T_{X,x} \), and \( d\varphi_x : T_{U_x,x} \to T_{V_x,0} \) is the identity map.
Since $\mathcal{V} \subset T_X$ (over $X$), we define

$$\mathcal{V}_X^{[\alpha]} = \{ (\xi_1, \cdots, \xi_l) \in (T_X)^{[\alpha]}_X | \text{Supp}(\xi_i) \in \mathcal{V} \}.$$ 

For $U$ over $X$, we define $U_X^{[\alpha]} = \bigcup_{x \in X} (U_x)^{[\alpha]}$ endowed with the obvious smooth structure. By Lemma 2.5 in [LiJ], $\varphi$ induces a differentiable isomorphism

$$\varphi^{[\alpha]} : U_X^{[\alpha]} \rightarrow \mathcal{V}_X^{[\alpha]}$$

(5.38)
as stratified spaces. Both $\mathcal{V}_X^{[\alpha]}$ and $U_X^{[\alpha]}$ are bundles over $X$:

$$\mathcal{V}_X^{[\alpha]} \rightarrow X \quad \text{and} \quad U_X^{[\alpha]} \rightarrow X.$$ (5.39)

The first is induced by the bundle $\mathcal{V} \subset T_X \rightarrow X$, and the second is via $(U_x)^{[\alpha]} \rightarrow \{x\}$. As $T_X \rightarrow X$ is a vector bundle, we obtain $(T_X)^{[\alpha]}_X \subset (T_X)^{[\alpha]}$ as in (5.38). Let $\mathcal{V}_X^{[\alpha]} = \mathcal{V}_X^{[\alpha]} \cap (T_X)^{[\alpha]}_X$, and let $U^{[\alpha]}_X \subset X^{[\alpha]}$ be the image of $\mathcal{V}_X^{[\alpha]}$ under the composition

$$\varphi : \mathcal{V}_X^{[\alpha]} \subset \mathcal{V}_X^{[\alpha]} \cong U_X^{[\alpha]} \rightarrow X^{[\alpha]} \times X \xrightarrow{pr_2} X^{[\alpha]},$$

where the first factor of $U_X^{[\alpha]} \rightarrow X^{[\alpha]} \times X$ is induced by the inclusion $(U_x)^{[\alpha]} \subset X^{[\alpha]}$, and the second is (5.39). By the Lemma 2.6 and Lemma 2.7 in [LiJ], after shrinking $\mathcal{V}$ if necessary, $U^{[\alpha]}$ is an open neighborhood of $X^{[\alpha]} \times X^n \cap \Delta_{\alpha[n]} \subset X^{[\alpha]}$, and

$$\varphi : \mathcal{V}_X^{[\alpha]} \rightarrow U^{[\alpha]}_X$$

(5.40)
is a smooth isomorphism of stratified spaces fibered over $\Delta_{\alpha[n]}$, via the map

$$U^{[\alpha]}_X \subset U_X^{[\alpha]} \rightarrow X^{[\alpha]} \times X \xrightarrow{pr_2} X,$$

and preserves the partial equivalences of $\mathcal{V}_X^{[\alpha]}$ and $U^{[\alpha]}_X$. Note that

$$X^{[\leq n]} \times X^n \cap \Delta_{\alpha[n]} = \bigcup_{\alpha \leq \alpha[n]} \phi_{\alpha[n], \alpha} (X^{[\alpha]} \times X^n \Delta_{\alpha[n]}).$$

(5.41)

So $U^{[\leq n]} := \bigcup_{\alpha \leq \alpha[n]} \phi_{\alpha[n], \alpha} (U^{[\alpha]}_X)$ is an open neighborhood of $X^{[\leq n]} \times X^n \cap \Delta_{\alpha[n]}$ in $X^{[\leq n]}$. Since $\epsilon$ is sufficiently small, $X^{[\leq n]} \times X^n \cap \Delta_{\alpha[n], \epsilon} \subset U^{[\leq n]}$. Thus,

$$(X^{[\leq n]})^3 \times (X^n)^3 \cap \Delta_{\alpha[n], \epsilon} \subset (U^{[\leq n]})^3$$

(5.42)

noting that by our convention, $(U^{[\leq n]})^3$ is a fibered product over $\Delta_{\alpha[n]}$. Since $\mathcal{V}_X^{[\alpha]} \subset (T_X)^{[\alpha]}_X$, we put $\mathcal{V}_{X,0}^{[\leq n]} = \bigcup_{\alpha \leq \alpha[n]} \phi_{\alpha[n], \alpha} (\mathcal{V}_{X,0}^{[\alpha]} \subset (T_X)^{[\alpha]}_X).$ Then, the smooth isomorphisms $\varphi$ from (5.40) induces a smooth isomorphism

$$\varphi_{[\leq n]} : \mathcal{V}_{X,0}^{[\leq n]} \rightarrow U^{[\leq n]}_X$$

(5.43)
of stratified spaces fibered over $X \cong \Delta_{\alpha[n]}$. Combining with (5.42), we have

$$(T_X)^{[\leq n]}_X \supset \mathcal{V}_{X,0}^{[\leq n]} \supset (U^{[\leq n]}_X)^3 \supset (X^{[\leq n]})^3 \times (X^n)^3 \cap \Delta_{\alpha[n], \epsilon}.$$ (5.44)

To prove universality results later on, we pick a differentiable map

$$g : X \rightarrow Gr = Gr(2, \mathbb{C}^N)$$

(5.45)

with $N \gg 0$ so that $T_X \cong \text{Gr}^*F$ as smooth vector bundles, where $F \rightarrow Gr$ is the total space of the universal quotient rank-$2$ bundle over $Gr$. Let $F_{Gr,0}^{[\alpha]} \rightarrow Gr$ be the associated relative Hilbert scheme of centered $\alpha$-points. By Lemma 2.8 in [LiJ], $g$ induces isomorphisms (as stratified spaces) of fiber bundles over $X$:

$$g^{\alpha} : (T_X)^{\alpha}_{X,0} \rightarrow g^*F_{Gr,0}^{[\alpha]} \quad \text{and} \quad g^{[\leq n]} : (T_X)^{[\leq n]}_X \rightarrow g^*F_{Gr,0}^{[\leq n]}.$$
5.11. Pairings with \([\Theta^{[n,d]}]\) when \(d > 0\). Here we study more properties of \([\Theta^{[n,d]}]\) which are the crucial ingredients for the universality results of extremal Gromov-Witten invariants of \(X^n\).

**Convention 5.18.** Fix \(d > 0\) and Heisenberg monomial classes

\[
A_i = a_{-\lambda^I}(x)a_{-\mu_{i,j}}(\alpha_{i,j}) \cdots a_{-\mu_{i,j}}(\alpha_{i,j}) |0\rangle \in H^*(X^n)
\]

where \(1 \leq i \leq 3\), \(u_i \geq 0\), and \(|\alpha_{i,j}| > 0\). When \(|\alpha_{i,j}| = 4\), we let \(\alpha_{i,j} = x\) (the cohomology class of a point). Moreover, if \(|\alpha_{i,j}| = 2\), then \(\alpha_{i,j}\) can be represented by a Riemann surface intersecting transversally with \(D_0 \cup D_\infty\). For simplicity, put \(A^{[\leq n]} = A^{[\leq n]}_1 \otimes A^{[\leq n]}_2 \otimes A^{[\leq n]}_3\).

Our goal is to understand the pairing \(\langle [\Theta^{[n,d]}], A^{[\leq n]} \rangle\) when \(d > 0\).

**Lemma 5.19.** Fix \(d > 0\). Then, \(\langle [\Theta^{[n,d]}], A^{[\leq n]} \rangle = 0\) if one of the following holds:

(i) \(|\alpha_{i,j}| = 4\) for some \((i,j)\);

(ii) \(|\alpha_{i,j}| = 2\) for two different pairs \((i,j)\).

**Proof.** (i) We begin with \(d \geq 0\). Consider the 0-cycle \([\Theta^{[n,d]}_{loc}] \cap A^{[\leq n]}\) in \((X^{[\leq n]})^3\). Choose the point presentation \(x \in X\) of \(\alpha_{i,j}\) such that \(x \notin D_0 \cup D_\infty\). By Proposition 2.7, \(A_i\) can be represented by a cycle \(W_i \subset X^n\) such that \(x \in \text{Supp}(\xi_i)\) for every \(\xi_i \in W_i\). Thus for every \(\xi_2\) contained in the 0-cycle \((\phi_{(n,d)}^3,\iota_\Lambda)\), \(ev(D[s_{n,d}]_{loc}) \cap (f^{[n]}_1)^*A_1 \otimes (f^{[n]}_2)^*A_2 \otimes (f^{[n]}_3)^*A_3)\), the point \(x\) is a component of \(\text{in}(\xi_2)\) where \(x\) is from \(\Theta_0^{[n,d]}\). By the localized version of Remark 5.16 and induction, we conclude that \(x\) is a component of \(\text{in}(\xi)\) if \(\xi\) is contained in \([\Theta^{[n,d]}_{loc}] \cap A^{[\leq n]}\).

Now let \(d > 0\). By the localized version of Lemma 5.11, we have

\[
\Theta^{[n,d]}_{loc} \subset \left( (X^{[\leq n]})^3 \times (X^n) \times (\Lambda^{[n,d]}_{loc}) \right) \setminus \bigcup_{(\beta,\gamma) \leq (\leq n,d)} \phi_{(n,d)}^3(\text{ev}(X^{[\beta,\gamma]}_{0})).
\]

Thus, since \(d > 0\), if \(\xi \in \Theta^{[n,d]}_{loc}\), then \(\text{in}(\xi) \in \Lambda_{loc}^{[n,d]}\) and \(y \in D_0 \cup D_\infty\) for some component \(y\) of \(\text{in}(\xi)\). Since \(\epsilon\) is sufficiently small, we see from the previous paragraph that \([\Theta^{[n,d]}_{loc}] \cap A^{[\leq n]}\) is empty. Hence as pairings, \(\langle [\Theta^{[n,d]}], A^{[\leq n]} \rangle = 0\).

(ii) Let \(|\alpha_{i_1,j_1}| = |\alpha_{i_2,j_2}| = 2\) where \((i_1,j_1) \neq (i_2,j_2)\). Represent \(\alpha_{i_1,j_1}\) and \(\alpha_{i_2,j_2}\) by Riemann surfaces \(C_{i_1,j_1}\) and \(C_{i_2,j_2}\) respectively so that \(\alpha_{i_1,j_1}, C_{i_2,j_2}\) and \(D_0 \cup D_\infty\) are in general position. As the second \((i,j)\), we see that if \(\xi \in \Theta^{[n,d]}_{loc} \cap A^{[\leq n]}\), then \(\text{in}(\xi) \in \Lambda_{loc}^{[n,d]}\) and the components of \(\text{in}(\xi)\) contain three points \(x_1 \in C_{i_1,j_1}, x_2 \in C_{i_2,j_2}\) and \(x_3 \in D_0 \cup D_\infty\). This is impossible since \(\epsilon\) is sufficiently small and \(C_{i_1,j_1}, C_{i_2,j_2}, D_0 \cup D_\infty\) are in general position. So the 0-cycle \([\Theta^{[n,d]}_{loc}] \cap A^{[\leq n]}\) is empty.

**Lemma 5.20.** Let \(u_1 = 1, u_2 = u_3 = 0\), and \(|\alpha_{1,1}| = 2\) in \([\mathbb{P}_X,\alpha_{1,1}]\) where \(p\) is a constant depending only on \(n_{1,1}\) and the partitions \(\lambda^{(i)}\).

**Proof.** Represent \(\alpha_{1,1}\) by a Riemann surface \(C_{1,1}\) intersecting transversally with \(D_0 \cup D_\infty\). Let

\[
C_{1,1} \cap D_0 = \{x_1, \ldots, x_{s_++s_-+t_++t_-}, x_{s_++s_++t_++t_+}\},
\]

\[
C_{1,1} \cap D_\infty = \{x_{s_++s_++t_++t_-}, \ldots, x_{s_++s_-+t_++t_+}\}
\]

so that the points \(x_1, \ldots, x_{s_++s_-+t_++t_-}\) are distinct, the intersection of \(C_{1,1}\) and \(D_0\) at \(x_i\) for \(1 \leq i \leq s_+\) (respectively, for \(s_+ + 1 \leq i \leq s_++s_-\)) is equal to 1 (respectively, –1), and the intersection of \(C_{1,1}\) and \(D_\infty\) at \(x_i\) for \(s_++s_-+t_++1 \leq i \leq s_++s_-+t_+\) (respectively, for \(s_++s_-+t_++1 \leq i \leq s_++s_-+t_++t_-\)) is equal to 1 (respectively, –1). So \((s_+-s_-) = (D_0,\alpha_{1,1})\) and \(t_++t_- = (D_\infty,\alpha_{1,1})\). Let \(x_i \in X_i\) be a small analytic open neighborhood of \(x_i\) such that \(X_1, \ldots, X_{s_++s_-+t_++t_-}\) are mutually disjoint. As in the proof of Lemma 5.19 (i), we see that the 0-cycle \([\Theta^{[n,d]}_{loc}] \cap A^{[\leq n]}\) is a disjoint union of \(W_1, \ldots, W_{s_++s_-+t_++t_-}\) such that \(\text{in}(W_i) \subset (X_i)^n\) for every \(i\). Let \(e_i\) be the contribution of each \(W_i\) to the pairing \(\langle [\Theta^{[n,d]}_{loc}], A^{[\leq n]} \rangle\). Then,

\[
\langle [\Theta^{[n,d]}], A^{[\leq n]} \rangle = \langle [\Theta^{[n,d]}_{loc}], A^{[\leq n]} \rangle = e_1 + \ldots + e_{s_++s_-+t_++t_-}.
\]
Lemma 5.21. Let \( d > 0 \) and \( A_i = a_{\lambda_i}(1_X)|0 \) for \( i \in \{1, 2, 3\} \). Then, \( \langle [\Theta^{[n,d]}], A^{[\leq n]} \rangle = p \cdot \langle K_X, K_X \rangle \) where \( p \) is a constant depending only on the partitions \( \lambda^i \).

Proof. By Lemma 5.11, \( \Theta^{[n,d]} \subset (X^{[\leq n]} \times (X^n)^3 \subset t_n(\Delta_{[n],e}) \). Using (5.43) and the smooth isomorphism (5.43), we transport the 0-cycle \( [\Theta^{[n,d]}] \cap A^{[\leq n]} \) in \((X^{[\leq n]} \times (X^n)^3 \subset t_n(\Delta_{[n],e}) \) to the following 0-cycle in \((V_X^{[\leq n]} \times (X^n)^3 \subset (T_X)^{[\leq n]} \times (X^n)^3 \subset t_n(\Delta_{[n],e}) \):

\[
\langle (q_3^{[\leq n]})^* [\Theta^{[n,d]}], A^{[\leq n]} \rangle = \left( (q_3^{[\leq n]})^* [\Theta^{[n,d]}], (q_3^{[\leq n]})^* (A^{[\leq n]}|(X^{[\leq n]} \times (X^n)^3 \subset t_n(\Delta_{[n],e})|) \right).
\] (5.48)

Note that these two 0-cycles have the same degree. So as pairings,

\[
\langle [\Theta^{[n,d]}], A^{[\leq n]} \rangle = \left( (q_3^{[\leq n]})^* [\Theta^{[n,d]}], (q_3^{[\leq n]})^* (A^{[\leq n]}|(X^{[\leq n]} \times (X^n)^3 \subset t_n(\Delta_{[n],e})|) \right).
\] (5.49)

The intersection \( \Theta^{[n,d]} \cap (q_3^{[\leq n]})^* [\Theta^{[n,d]}] \) is transversal, and

\[
(q_3^{[\leq n]})^{-1} (\Theta^{[n,d]}) = \Theta^{[n,d]} \cap (q_3^{[\leq n]})^* [\Theta^{[n,d]}] \times G_X \]

via \((T_X^{[\leq n]} \times (TX^n)^3 \subset t_n(\Delta_{[n],e}) \), which is pullback of \((A^{[n]}_{TX,0})^{[\leq n]} \subset H^*(X^{[\leq n]}_X) \) via the inclusion \((TX)^{[\leq n]}_X \subset (TX)^{[\leq n]}_X \). Let \( S \) denote the intersection

\[
\left( (T_X^{[\leq n]} \times (TX^n)^3 \subset t_n(\Delta_{[n],e}) \right) \cap (q_3^{[\leq n]})^{-1} (X^{[\leq n]} \times (X^n)^3 \subset t_n(\Delta_{[n],e})|).
\] (5.51)
Then, \( S = (\varphi_{l \leq n})^{-1}(X^{l \leq n})^3 \times (X^{n})^3 \iota_n(\Delta_{[n]})) \). We claim that
\[
\left(\varphi_{l \leq n}\right)^* \left(A_{l \leq n}^3 \times (X^{n})^3 \iota_n(\Delta_{[n]}))\right)|_S = (A_{T_X,0}^3)^l \leq n|_S, \tag{5.52}
\]
i.e., \( (\varphi_{l \leq n})^* \left(A_{l \leq n}^3 \times (X^{n})^3 \iota_n(\Delta_{[n]}))\right) = (A_{T_X,0}^3)^l \leq n|_S \). It suffices to prove that
\[
\varphi_{l \leq n}^* \left(A_{l \leq n}^3 \times (X^{n})^3 \iota_n(\Delta_{[n]}))\right) = (A_{T_X,0}^3)^l \leq n|_S, \tag{5.53}
\]
Indeed, for every \( \alpha \leq [n] \), we conclude from Lemma \( 5.13 \) and Lemma \( 5.14 \) that the same subvariety in \( \varphi_{l \leq n}^* \left(\phi_{[n]}(X^{l \leq n}) \times (X^{n})^3 \iota_n(\Delta_{[n]}))\right) \) represents the cohomology classes
\[
\varphi_{l \leq n}^* \left(A_{l \leq n}^3 \times (X^{n})^3 \iota_n(\Delta_{[n]}))\right) \quad \text{and} \quad (A_{T_X,0}^3)^l \leq n|_S^l \leq 1 \left(\phi_{[n]}(X^{l \leq n}) \times (X^{n})^3 \iota_n(\Delta_{[n]}))\right).
\]
Since \( X^{l \leq n} \times (X^{n})^3 \iota_n(\Delta_{[n]}) = \coprod_{\alpha \leq [n]} \phi_{[n]}(X^{l \leq n}) \times (X^{n})^3 \iota_n(\Delta_{[n]}) \), we obtain \( 5.53 \).

By \( 5.45 \), \( 5.52 \) and \( 5.51 \), as pairings, we have
\[
\langle \Theta^\alpha, \mathcal{A}_{l \leq n} \rangle = \left(\varphi_{l \leq n}\right)^* \left(\Theta^\alpha \right) \left(A_{T_X,0}^3 \iota_n(\Delta_{[n]}))|_S = \left(\left(\varphi_{l \leq n}\right)^* \left(\Theta^\alpha \right) \right)^l \leq n|_S \right. \frac{A_{l \leq n}^3 \times (X^{n})^3 \iota_n(\Delta_{[n]}))}{A_{T_X,0}^3} \tag{5.54}
\]
Let \( (g_{[l \leq n]}^3 : ((T_X)^{l \leq n} \times (X^{n})^3 \iota_n(\Delta_{[n]})) \rightarrow \left(F_{s,0}^{l \leq n}\right)^{l \leq n} \times (X^{n})^3 \iota_n(\Delta_{[n]})) \) be the isomorphism induced by \( g \). By Lemma \( 5.13 \), \( (A_{l \leq n}^3 \iota_n(\Delta_{[n]})) \) can be taken to be \( (g_{[l \leq n]}^3)^{l \leq n} \frac{A_{l \leq n}^3 \times (X^{n})^3 \iota_n(\Delta_{[n]}))}{A_{T_X,0}^3 \times (X^{n})^3 \iota_n(\Delta_{[n]}))} \).

Combining with \( 5.50 \) and putting \( W_\epsilon = \left(F_{s,0}^{l \leq n}\right)^{l \leq n} \times (X^{n})^3 \iota_n(\Delta_{[n]})) \rightarrow W \), we get
\[
\left\langle \Theta^\alpha, \mathcal{A}_{l \leq n} \right\rangle = \left(\left(\varphi_{l \leq n}\right)^* \left(\Theta^\alpha \right) \right)^l \leq n|_S, \frac{A_{l \leq n}^3 \times (X^{n})^3 \iota_n(\Delta_{[n]}))}{A_{T_X,0}^3} \right. \tag{5.54}
\]
where \( \left(g_{[l \leq n]}^3 : ((T_X)^{l \leq n} \times (X^{n})^3 \iota_n(\Delta_{[n]})) \rightarrow W \right. \) is the morphism induced by \( g \). By \( 5.49 \), \( \Theta^\alpha \) is supported on \( \pi \circ PD^{-1} \left[ g(X) \right] \), \( (A_{l \leq n}^3 \iota_n(\Delta_{[n]})) \rightarrow W \). Therefore, by \( 5.50 \), we obtain
\[
\left\langle \Theta^\alpha, \mathcal{A}_{l \leq n} \right\rangle = \left(\left(\varphi_{l \leq n}\right)^* \left(\Theta^\alpha \right) \right)^l \leq n|_S, \frac{A_{l \leq n}^3 \times (X^{n})^3 \iota_n(\Delta_{[n]}))}{A_{T_X,0}^3} \right. \tag{5.54}
\]
where \( \pi : W \rightarrow Gr \) is the tautological projection. Here a bit of topological argument is needed for the first equality. Observe that \( W \) is in fact a disjoint union of Hausdorff spaces. Then we use the Zariski local triviality of the bundle \( F \) over \( Gr \), the Künneth decomposition for Borel-Moore homology \( H_\mathbf{E} \), and the properties of cap products \( \mathbf{G}_{H} \mathbf{I}_{V} \mathbf{S}_{P} \).

The Poincaré dual of \( \pi_* \left(\left(\Theta^\alpha \right)^l \iota_n(\Delta_{[n]})) \right) \) is a polynomial \( P \) in the Chern classes \( c_i(F) \).

Hence,
\[
\left\langle \Theta^\alpha, \mathcal{A}_{l \leq n} \right\rangle = p \cdot \langle K_X, K_X \rangle + q \cdot \deg(e_X) \tag{5.54}
\]
where \( p \) and \( q \) are constants depending only on the partitions \( \lambda^{(i)} \).

Finally, it remains to prove that \( q = 0 \) in \( 5.54 \). To see this, choose the surface \( X \) such that \( |K_X| \) contains a smooth divisor \( D \). Let \( \theta \) be a holomorphic section of \( O_X(K_X) \) such that the vanishing divisor of \( \theta \) is \( D = 0 \). By \( 5.51 \), \( \left(\left(\varphi_{l \leq n}\right)^* \left(\Theta^\alpha \right) \right)^l \leq n|_S, \frac{A_{l \leq n}^3 \times (X^{n})^3 \iota_n(\Delta_{[n]}))}{A_{T_X,0}^3} \right. \tag{5.54}
\]
where \( p' \) depends only on the partitions \( \lambda^{(i)} \). Since there exist two surfaces \( X \) with smooth \( D \in |K_X| \) such that the pairs \( (\langle K_X, K_X \rangle, \deg(e_X)) \) are linearly independent, \( p = p' \) and \( q = 0 \). \( \square \)
5.12 Proofs of Theorem 1.2 and Theorem 1.3 In this subsection, we introduce a new class \(3_{n,d} \in H_2((X^n)^3)\) in terms of cycles \(\Theta^{[\alpha,\delta]}\) studied intensively in previous subsections. Note that now \(3_{n,d}\) is on the Hilbert scheme \((X^n)^3\), not on the non-separated spaces \((X^{[\leq n]}))\).

Let \(B = \{\beta_1, \ldots, \beta_\ell\}\) be a basis of \(H^2(X)\). Then, \(\{1_X, x, \beta_1, \ldots, \beta_\ell\}\) is a basis of \(H^*(X)\), and \(H^*(X^n)\) has a basis \(B^{[n]}\) consisting of the elements \(a_{-\lambda}(1_X a_{-\mu}(x) a_{-\nu(\delta)}(\beta_1) \cdots a_{-\nu(\delta)}(\beta_\ell))(0)\) where \(|\lambda| + |\mu| + \sum |\nu(\delta)| = n\). Via the Künneth decomposition, a basis of \(H^*((X^n)^3)\) consists of the elements \(A_1 \otimes A_2 \otimes A_3 = \prod_{i=1}^3 \pi_{n,i}^* A_i\), where \(A_1, A_2, A_3 \in B^{[n]}\) and \(\pi_{n,i}\) denotes the \(i\)-th projection \((X^n)^3 \to X^n\).

**Definition 5.22.**

(i) Let \(d \geq 1\), and let \(P^{+}_{[n],d}\) be the subset of \(P_{[n],d}\) consisting of all the weighted partitions \((\alpha, \delta)\) such that \(\delta_i > 0\) for every \(i\).

(ii) For \(d \geq 1\), define the class \(3_{n,d} = 3_{n,d}^{[n]} \in H_2((X^n)^3)\) by putting

\[
\left\langle 3_{n,d}, \prod_{i=1}^3 \pi_{n,i}^* A_i \right\rangle = \frac{1}{n!} \sum_{(\alpha, \delta) \in P^{+}_{[n],d}} \left(\tilde{\phi}_{\alpha,\delta}^3\right)_{*} [\Theta^{[\alpha,\delta]}], \ A_1^{[\leq n]} \otimes A_2^{[\leq n]} \otimes A_3^{[\leq n]} \right) \tag{5.55}
\]

for the basis elements \(A_1, A_2, A_3 \in B^{[n]}\).

Next, we prove Theorem 1.2 and Theorem 1.3 which determine the structure of the 3-pointed genus-0 extremal Gromov-Witten invariants of \(X^n\). Note from Theorem 1.3 that the class \(3_{n,d} = 3_{n,d}^{[n]} \in H_2((X^n)^3)\) is independent of the choice of the basis \(B = B^2(X)\). So from now on, the basis \(B\) of \(H^2(X)\) will be implicit in our presentation.

**Proof of Theorem 1.2.** By (5.44), the invariant \(\left\langle (A_1, A_2, A_3)_{0,d,\beta_n}\right\rangle\) is equal to

\[
\frac{1}{n!} \sum_{(\alpha, \delta) \in P^{+}_{[n],d}} \left(\tilde{\phi}_{\alpha,\delta}^3\right)_{*} [\Theta^{[\alpha,\delta]}], \ A_1^{[\leq n]} \otimes A_2^{[\leq n]} \otimes A_3^{[\leq n]} \right) \tag{5.56}
\]

Define \(\alpha = ((\alpha_i)_i, |\delta_i| = 0)\), and let \((\alpha', \delta')\) be the weighted partition such that all the weights are equal to 0. Let \(\beta = (\alpha', \delta')\) be the weighted partition obtained from \((\alpha, \delta)\) by deleting all the \(\alpha_i\) and \(\delta_i\) with \(\delta_i = 0\). Let \(|\alpha'| = m\), \(\Lambda_{\alpha'} = \prod_i (\alpha')_i\), and \(\Lambda_{\alpha'} = \prod_i (\alpha_i')\). Then, \(\alpha = (\alpha', 0) \prod_i (\alpha', \delta')\), and \(|\alpha'| = n - m\), and \(|\alpha| = m\). By (5.44), \(\left(\tilde{\phi}_{\alpha,\delta}^3\right)_{*} [\Theta^{[\alpha,\delta]}], \ A_1^{[\leq n]} \otimes A_2^{[\leq n]} \otimes A_3^{[\leq n]} \right)\) equals

\[
\frac{1}{n!} \sum_{(\alpha, \delta) \in P^{+}_{[n],d}} \prod_{i=1}^l \left(\Theta^{[\alpha_i,\delta_i]}, \ A_1^{[\leq n]} \otimes A_2^{[\leq n]} \otimes A_3^{[\leq n]} \right) \tag{5.57}
\]

Put \(\Lambda = \Lambda_{\alpha'}\). By (5.56), \(\left(\Lambda_{\alpha}, A_1, A_2, A_3\right)_{0,d,\beta_n}\) is equal to

\[
\frac{1}{n!} \sum_{m \leq n} \sum_{A_1, A_2, A_3} \sum_{(\alpha', \delta') \in P^{+}_{\Lambda, d}} \left(\tilde{\phi}_{\alpha',\delta'}^3\right)_{*} [\Theta^{[\alpha',\delta']}], \ A_1^{[\leq m]} \otimes A_2^{[\leq m]} \otimes A_3^{[\leq m]} \right) \tag{5.57}
\]
In particular, setting \( d = 0 \) in (5.57), we see that \( \langle A_1, A_2, A_3 \rangle \) is equal to
\[
\frac{1}{n!} \sum_{\alpha \in P_{[n]}} \left( \langle \theta^3_{\alpha, [n]} \rangle_{\alpha, \emptyset}, A_1^{\leq [n]} \otimes A_2^{\leq [n]} \otimes A_3^{\leq [n]} \right).
\]
Therefore, by (5.57), \( \langle A_1, A_2, A_3 \rangle_{0, d, \delta_0} \) is equal to
\[
\frac{1}{n!} \sum_{m \leq n} \sum_{A_1, I_1, A_2, I_2, A_3, I_3} \sum_{(\alpha, \delta') \in P_{[m], d}} \sum_{(\alpha', \delta) \in P_{[m], d}} (n-m)! \cdot \langle A_1, A_2, A_3, I_1, I_2, I_3 \rangle
\]
\[
\cdot \left( \langle \theta^3_{\alpha', \delta}, A_1^{\leq [m]} \otimes A_2^{\leq [m]} \otimes A_3^{\leq [m]} \rangle \right)
\]
\[
= \frac{1}{n!} \sum_{m \leq n} \sum_{A_1, I_1, A_2, I_2, A_3, I_3} \sum_{(\alpha, \delta') \in P_{[m], d}} \sum_{(\alpha', \delta) \in P_{[m], d}} \left( n-m \right) (n-m)! \cdot \langle A_1, A_2, A_3, I_1, I_2, I_3 \rangle
\]
\[
\cdot \left( \langle \theta^3_{\alpha', \delta}, A_1^{\leq [m]} \otimes A_2^{\leq [m]} \otimes A_3^{\leq [m]} \rangle \right).
\]
Using the definition of \( 3_m, d \), we complete the proof of our theorem. \( \square \)

**Proof of Theorem 5.23.** Let \( A_1 = a_{-\lambda(1)}(x), a_{-\mu(1)}(x), \ldots, a_{-n_{i,j}}(x) \) with \( |\alpha_{i,j}| = 2 \). By linearity, we may assume \( \alpha_{i,j} \in \mathcal{B} \) for every \( i \) and \( j \). By (5.55) and (5.54),
\[
\left\langle 3_{n, d}, \prod_{i=1}^l \pi^a_{n,i} A_i \right\rangle = \frac{1}{n!} \cdot \sum_{(\alpha, \delta) \in P_{[n], d}} \sum_{A_1, I_1, A_2, I_2, A_3, I_3} \prod_{i=1}^l \left( \langle \theta^3_{\alpha, \delta}, A_1^{\leq [\alpha,i]} \otimes A_2^{\leq [\alpha,i]} \otimes A_3^{\leq [\alpha,i]} \rangle \right).
\]
So our theorem, except the degree of \( p \) in (ii), follows from Lemma 5.19, Lemma 5.20 and Lemma 5.21. To see the degree of \( p \) in (ii), consider a nonzero term in (5.58):
\[
\prod_{i=1}^l \left( \langle \theta^3_{\alpha, \delta}, A_1^{\leq [\alpha,i]} \otimes A_2^{\leq [\alpha,i]} \otimes A_3^{\leq [\alpha,i]} \rangle \right).
\]
By Lemma 5.19, for each \( i \) in (5.59), the classes \( A_1, A_2, A_3 \) together contain at most one Heisenberg factor of the form \( a_{-n_{i,j}}(\alpha_{j,k}) \). By Lemma 5.20 and Lemma 5.21, the degree of (5.59) as a monomial of \( (K_X, X^d) \) is equal to \( |I| \) where \( I \) is the set consisting of the index \( i \in \{1, \ldots, l\} \) such that the classes \( A_1, A_2, A_3 \) together do not contain any Heisenberg factor of the form \( a_{-n_{i,j}}(\alpha_{j,k}) \). Now for each \( i \in I \), \( |\alpha_i| \geq 2 \) since \( \delta_i \geq 1 \). So we conclude that
\[
|I| \leq \frac{1}{2} \sum_{i \in I} |\alpha_i| = \frac{1}{2} \left( n - \sum_{i \not\in I} |\alpha_i| \right) \leq \frac{1}{2} (n - \sum_{j,k} n_{j,k}).
\]
Hence the degree of \( p \) as a polynomial of \( (K_X, X^d) \) is at most \( (n - \sum_{i,j} n_{i,j})/2 \). \( \square \)

**Corollary 5.23.** Let \( d \geq 1 \), and let \( A_1, A_2, A_3 \in H^*(X^{[n]}) \) be Heisenberg monomial classes.

(i) If \( A_1 = a_{-1}(1)^{n-1} a_{-\alpha}(0) [0] \), then \( \left\langle 3_{n, d}, \prod_{i=1}^l \pi^a_{n,i} A_i \right\rangle = 0 \).

(ii) If \( A_1 = a_{-1}(1)^{n-1-|\lambda|} a_{-\alpha}(x)[0] \) for some \( \lambda \), then \( \langle A_1, A_2, A_3 \rangle_{0, d, \delta_0} = 0 \).

**Proof.** (i) First of all, if \( \alpha = x \), then \( \left\langle 3_{n, d}, \prod_{i=1}^l \pi^a_{n,i} A_i \right\rangle = 0 \) by Theorem 5.23 (i).

Next, let \( \alpha = 1_X \). Use induction on \( n \). Since \( d \geq 1 \), the conclusion is trivially true when \( n = 1 \). Let \( n > 1 \). Recall that \( 1/n! \cdot A_1 \) is the fundamental class \( 1_{X^{[n]}} \) of \( X^{[n]} \). By Theorem 5.22 and the Fundamental Class Axiom of Gromov-Witten theory,
\[
\sum_{m=2}^n \sum_{A_1, I_1, A_2, I_2, A_3, I_3} \langle A_1, A_2, A_3 \rangle \cdot \left\langle 3_{m, d}, \prod_{i=1}^3 \pi^a_{m+i} A_i \right\rangle = 0.
\]
Since $A_1 = a_{-1}(1_X)^n|0\rangle$, we have $A_{1,2} = a_{-1}(1_X)^m|0\rangle$. By induction, $\langle 3_{n,d}, \prod_{i=1}^{3} \pi^*_m, A_{1,2} \rangle = 0$ if $2 \leq m \leq n - 1$. It follows that $\langle 3_{n,d}, \prod_{i=1}^{3} \pi^*_m, A_i \rangle = 0$.

Now let $|a| = 2$. By the Divisor Axiom of Gromov-Witten theory and $\langle A_1, \beta_n \rangle = 0$, we have $\langle A_1, A_2, A_3 \rangle_{0,d\beta_n} = 0$. Using an argument similar to the one in the previous paragraph, we conclude that $\langle 3_{n,d}, \prod_{i=1}^{3} \pi^*_m, A_i \rangle = 0$.

(ii) We compute $\langle A_1, A_2, A_3 \rangle_{0,d\beta_n}$ by using (1.3). Note that the class $A_{1,2}$ in (1.3) is equal to $a_{-1}(1_X)^m|0\rangle$, or equal to $a_{-1}(1_X)^{m-1}a_{-1}(1)|0\rangle$, or contains a factor $a_{-i}(x)$ for some $i > 0$. By (i) and Theorem 1.1 (1), we get $\langle A_1, A_2, A_3 \rangle_{0,d\beta_n} = 0$.

6. Proofs of (1.2) and Theorem 1.1

Let $X$ be a simply connected smooth projective surface. Our goal in this section is to prove (1.2) and Theorem 1.1 for $A^{[n]} = H^*_{CR}(X^{[n]})$. The proof of (1.2) divided into three cases depending on the cohomology degree of the class $\alpha$ in (1.2) and leading to Proposition 6.1 and Proposition 6.12. Assuming these three propositions, we now prove Theorem 1.1.

Proof of Theorem 1.1. Note that the shift number (or the age) of the class $p_{-n_1}(\alpha) \cdots p_{-n_s}(\alpha_s)|0\rangle$ is equal to $n_1 + \ldots + n_s - s$. Define a linear isomorphism

$$\Psi : \mathcal{F}_X \rightarrow \mathbb{H}_X$$

(6.1)

by sending $\sqrt{-1}^{n_1 + \ldots + n_s - s} p_{-n_1}(\alpha_1) \cdots p_{-n_s}(\alpha_s)|0\rangle$ to $a_{-n_1}(\alpha_1) \cdots a_{-n_s}(\alpha_s)|0\rangle$. This induces a linear isomorphism $\Psi : H^*_{CR}(X^{(n)}) \rightarrow H^*(X^{[n]})$ for each $n$. Moreover, $\Psi_1$ is simply the identity map on the cohomology group of the surface $X$.

By (1.3), Proposition 6.3, Proposition 6.9 and Proposition 6.12, the two formulas (1.1) and (1.2) hold for $A^{[n]} = H^*_{CR}(X^{[n]})$. By the proof of Theorem 2.4 (i.e., Theorem 4.7 in [LQW3]),

$$\bar{G}_k(\alpha) = -\sum_{\ell(\lambda) = k+2, |\lambda| = 0} \frac{1}{\lambda} \alpha(\tau_\alpha) + \sum_{\ell(\lambda) = k+1, |\lambda| = 0} \frac{s(\lambda) - 2}{24\lambda} \alpha(\tau_\alpha(\varepsilon_\alpha)).$$

(6.2)

Combining this with formula (5.1), we check directly that

$$\Psi_n(\sqrt{-1}^k O_k(\alpha, n) \cdot \sqrt{-1}^{n_1 + \ldots + n_s - s} p_{-n_1}(\alpha_1) \cdots p_{-n_s}(\alpha_s)|0\rangle)$$

$$= \Psi_n(\sqrt{-1}^{n_1 + \ldots + n_s - s} D_k(\alpha)p_{-n_1}(\alpha_1) \cdots p_{-n_s}(\alpha_s)|0\rangle)$$

$$= \bar{G}_k(\alpha, n) \cdot a_{-n_1}(\alpha_1) \cdots a_{-n_s}(\alpha_s)|0\rangle$$

where $n_1 + \ldots + n_s = n$. In particular, letting $s = n, n_1 = \ldots = n_s = 1$ and $\alpha_1 = \ldots = \alpha_s = 1_X$, we obtain $\Psi_n(\sqrt{-1}^k O_k(\alpha, n)) = \bar{G}_k(\alpha, n)$. Thus,

$$\Psi_n(\sqrt{-1}^k O_k(\alpha, n) \cdot \sqrt{-1}^{n_1 + \ldots + n_s - s} p_{-n_1}(\alpha_1) \cdots p_{-n_s}(\alpha_s)|0\rangle)$$

$$= \Psi_n(\sqrt{-1}^{n_1 + \ldots + n_s - s} p_{-n_1}(\alpha_1) \cdots p_{-n_s}(\alpha_s)|0\rangle).$$

Since the classes $O_k(\alpha, n)$ with $k \geq 0, \alpha \in H^*(X)$ generate the ring $H^*_{CR}(X^{(n)})$, we conclude that $\Psi_n : H^*_{CR}(X^{(n)}) \rightarrow H^*(X^{[n]})$ is a ring isomorphism.

Remark 6.1. Using Heisenberg monomial classes, one checks that the ring isomorphism $\Psi_n$ preserves the pairings on $H^*_{CR}(X^{(n)})$ and $H^*(X^{[n]})$.

In the next three subsections, we will verify (1.2) by proving Propositions 6.3, 6.9 and 6.12 used in the proof of Theorem 1.1. For simplicity, put $\langle w_1, w_2, w_3 \rangle_d = \langle w_1, w_2, w_3 \rangle_{0,d\beta_n}$. In addition, $w_1, w_2$ and $w_3$ will stand for Heisenberg monomial classes.
6.1. The case $\alpha = x$. We begin with a setup for the proof of (1.2) for arbitrary $\alpha, \beta \in H^*(X)$. To prove (1.2), it is equivalent to verify that

$$\langle [\tilde{G}_k(\alpha), a_{-1}(\beta)]w_1, w_2 \rangle = \frac{1}{k!} \langle a^{(k)}_{-1}(\alpha \beta)w_1, w_2 \rangle. \quad (6.3)$$

for $w_1 \in H^*_p(X^{[n-1]}) = H^*(X^{[n-1]})$ and $w_2 \in H^*_p(X^{[n]}) = H^*(X^{[n]})$. Put

$$D^a_{\beta}(w_1, w_2; q) := \langle [\tilde{G}_k(\alpha; q), a_{-1}(\beta)]w_1, w_2 \rangle - \frac{1}{k!} \langle a^{(k)}_{-1}(\alpha \beta)w_1, w_2 \rangle \quad (6.4)$$

where we have omitted $k$ in $D^a_{\beta}(w_1, w_2; q)$ since it will be clear from the context.

**Lemma 6.2.** The difference $D^a_{\beta}(w_1, w_2; q)$ is equal to

$$\sum_{0 \leq j \leq k} \sum_{\ell(\lambda) = k-j+1} \frac{(-1)^{|\lambda|-1}}{\lambda! \cdot |\lambda|!} \sum_{d \geq 1} \left( \langle 1_{-(n-j-1)} a_{-\lambda}(\tau_\alpha) | 0 \rangle, a_{-1}(\beta)^{\dagger} w_2 \right)_d q^d$$

$$+ \sum_{\epsilon \in \lbrace K, K^\perp \rbrace} \sum_{\ell(\lambda) = k+1+1+|\epsilon|-1/2} \tilde{f}_\epsilon(\lambda) \cdot \langle a_{\lambda}(\tau_\epsilon(\alpha \beta))w_1, w_2 \rangle$$

$$- \sum_{\epsilon \in \lbrace K, K^\perp \rbrace} \sum_{0 \leq j \leq k} \sum_{\ell(\lambda) = k-j+1+|\epsilon|-1/2} \tilde{g}_\epsilon(\lambda) \cdot \left( \langle 1_{-(n-j-1)} a_{-\lambda}(\tau_\epsilon(\alpha)) | 0 \rangle, a_{-1}(\beta)^{\dagger} w_2 \right)_d$$

$$- \langle 1_{-(n-j-2)} a_{-\lambda}(\tau_\epsilon(\alpha)) | 0 \rangle, w_1, a_{-1}(\beta)^{\dagger} w_2 \rangle \quad (6.5)$$

where $a_{-1}(\beta)^{\dagger} = -a_1(\beta)$ is the adjoint operator of $a_{-1}(\beta)$, and the functions $\tilde{f}_\epsilon(\lambda)$ and $\tilde{g}_\epsilon(\lambda)$ depend only on $k, |\epsilon|$ and $\lambda$.

**Proof.** By (1.3), $\langle [\tilde{G}_k(\alpha; q), a_{-1}(\beta)]w_1, w_2 \rangle$ is equal to

$$\langle \tilde{G}_k(\alpha; q)(a_{-1}(\beta)w_1), w_2 \rangle - \langle a_{-1}(\beta)\tilde{G}_k(\alpha; q)(w_1), w_2 \rangle$$

$$= \langle \tilde{G}_k(\alpha; q)(a_{-1}(\beta)w_1), w_2 \rangle - \langle \tilde{G}_k(\alpha; q)(w_1), a_{-1}(\beta)^{\dagger} w_2 \rangle$$

$$= \sum_{d \geq 0} \left( \langle \tilde{G}_k(\alpha, n), a_{-1}(\beta)w_1, w_2 \rangle_d - \langle \tilde{G}_k(\alpha, n-1), w_1, a_{-1}(\beta)^{\dagger} w_2 \rangle_d \right) q^d. \quad (6.6)$$

If $d \geq 1$, then we see from (4.4) and Corollary 5.23 (ii) that

$$\langle \tilde{G}_k(\alpha, n), a_{-1}(\beta)w_1, w_2 \rangle_d$$

$$= \sum_{0 \leq j \leq k} \sum_{\ell(\lambda) = k-j+1+1} \frac{(-1)^{|\lambda|-1}}{\lambda! \cdot |\lambda|!} \langle 1_{-(n-j-1)} a_{-\lambda}(\tau_\alpha) | 0 \rangle, a_{-1}(\beta)w_1, w_2 \rangle_d. \quad (6.7)$$

Similarly, if $d \geq 1$, then $\langle \tilde{G}_k(\alpha, n-1), w_1, a_{-1}(\beta)^{\dagger} w_2 \rangle_d$ is equal to

$$\sum_{0 \leq j \leq k} \sum_{\ell(\lambda) = k-j+1+1} \frac{(-1)^{|\lambda|-1}}{\lambda! \cdot |\lambda|!} \langle 1_{-(n-j-2)} a_{-\lambda}(\tau_\epsilon(\alpha)) | 0 \rangle, w_1, a_{-1}(\beta)^{\dagger} w_2 \rangle_d. \quad (6.8)$$

Next, we study the two terms with $d = 0$ in (6.6). By (4.4) and Theorem 2.5,

$$\tilde{G}_k(\alpha, n) = G_k(\alpha, n) - \sum_{\epsilon \in \lbrace K, K^\perp \rbrace} \sum_{0 \leq j \leq k} \sum_{\ell(\lambda) = k-j+1+1+1/2} \tilde{g}_\epsilon(\lambda) \cdot \langle 1_{-(n-j-1)} a_{-\lambda}(\tau_\epsilon(\alpha)) | 0 \rangle$$
Thus, \( g_{\alpha}(\lambda) \) depends only on \( k, |\epsilon| \) and \( \lambda \). By Theorem 2.3 (iii), Theorem 2.3 and Lemma 4.5,
\( \langle G_k(\alpha, n), a_{-1}(\beta) w_1, w_2 \rangle - \langle G_k(\alpha, n - 1), w_1, a_{-1}(\beta)^t w_2 \rangle \)
equals
\( \langle G_k(\alpha, n) \cdot a_{-1}(\beta) w_1, w_2 \rangle - \langle G_k(\alpha, n - 1) \cdot w_1, a_{-1}(\beta)^t w_2 \rangle \)

\( = \langle \Phi_k(\alpha) a_{-1}(\beta) w_1, w_2 \rangle - \langle a_{-1}(\beta) \Phi_k(\alpha) w_1, w_2 \rangle \)

\( = \langle G_k(\alpha) a_{-1}(\beta) w_1, w_2 \rangle = \frac{1}{k!} \langle a_{-1}(\alpha^* \beta) w_1, w_2 \rangle \)

\( \frac{1}{k!} \sum_{\epsilon \in \{K_X, K^*_X\}} \sum_{(\lambda) = \frac{k}{k+1-|\epsilon|/2}} \tilde{g}_{\epsilon|\lambda}(\lambda) \cdot \langle a_{\lambda}(\tau_{\epsilon}(\alpha^\beta)) w_1, w_2 \rangle. \) (6.9)

Thus, \( \langle G_k(\alpha, n), a_{-1}(\beta) w_1, w_2 \rangle - \langle G_k(\alpha, n - 1), w_1, a_{-1}(\beta)^t w_2 \rangle \) is equal to
\( \frac{1}{k!} \langle a_{-1}(\alpha^* \beta) w_1, w_2 \rangle + \sum_{\epsilon \in \{K_X, K^*_X\}} \sum_{(\lambda) = \frac{k}{k+1-|\epsilon|/2}} \tilde{g}_{\epsilon|\lambda}(\lambda) \cdot \langle a_{\lambda}(\tau_{\epsilon}(\alpha^\beta)) w_1, w_2 \rangle\)

\( - \sum_{\epsilon \in \{K_X, K^*_X\}} \sum_{(\lambda) = \frac{k}{k+1-|\epsilon|/2}} \tilde{g}_{\epsilon|\lambda}(\lambda) \cdot \left( \langle 1_{\lambda-\mu} a_{-1}(\tau_{\epsilon}(\alpha)) w_1, w_2 \rangle \right) \).

Finally, our lemma follows from (6.6), (6.7), (6.8) and (6.9).

Now we deal with the simplest case when \( \alpha = x \) and \( \beta \) is arbitrary.

**Proposition 6.3.** If \( \alpha = x \) is the cohomology class of a point, then (1.2) is true.

**Proof.** By Corollary 2.20 (ii), every term in (6.6) is equal to zero. So \( D^*_\beta(w_1, w_2; q) = 0 \). Setting \( q = -1 \), we conclude immediately that (1.2) is true.

6.2. The case \( |\alpha| = 2 \). We begin with two lemmas about the structures of the intersections in \( H^*(X^n) \).

**Lemma 6.4.** Let \( \lambda \) be a partition with \( |\lambda| \leq n \). For \( i = 1 \) and \( 2 \), let
\( w_1 = a_{-\lambda(1)}(x) a_{-\mu(i)}(1_X) \cdots a_{-\lambda(i)}(1_X) \).

where \( |\alpha_{i,j}| = 2 \) for all \( i \) and \( j \). Then, \( \langle a_{-1}(1_X)^{n-|\lambda|} a_{-\lambda}(x) w_1, w_2 \rangle \) is equal to
\( \delta_{u_1, u_2} \cdot \sum_{\sigma \in \text{Perm}(1, \ldots, u_1)} \prod_{i=1}^{u_1} (\alpha_{1,i}, \alpha_{2,\sigma(i)}) \cdot p(\sigma) \) (6.11)

where \( p(\sigma) \) depends only on \( \sigma, n, \lambda \) and all the \( \lambda(i), \mu(i), n_{i,j} \).

**Proof.** By Lemma 2.3 (i), \( a_{-1}(1_X)^{n-|\lambda|} a_{-\lambda}(x) w_1 \) is a polynomial of the classes \( G_k(x, n), k \geq 0 \) whose coefficients are independent of \( X \). In addition, the integers \( k \) involved depend only on \( \lambda \).

Note that
\( \langle G_{k_1}(x, n) \cdots G_{k_1}(x, n), w_1, w_2 \rangle \)

\( = \langle a_{-\lambda(1)}(x) a_{-\lambda(1)}(1_X) \cdots a_{-\lambda(1)}(1_X) \rangle \).

So by Theorem 2.4 and Theorem 2.1 (i), \( \langle G_{k_1}(x, n) \cdots G_{k_1}(x, n), w_1, w_2 \rangle \) equals
\( \delta_{u_1, u_2} \cdot \sum_{\sigma \in \text{Perm}(1, \ldots, u_1)} \prod_{i=1}^{u_1} (\alpha_{1,i}, \alpha_{2,\sigma(i)}) \cdot \tilde{p}(\sigma) \) (6.12)

where \( \tilde{p}(\sigma) \) depends only on \( \sigma, n, k_1, \ldots, k_t \) and all the \( \lambda(i), \mu(i), n_{i,j} \).
Lemma 6.5. Let $n_0 \geq 1$, $|\alpha| = 2$, and $\lambda$ be a partition. Let $w_1$ and $w_2$ be given by $[\alpha,0]$. Then, \( \langle 1_{-\{\lambda\}}a_{-\lambda}(x)a_{-n_0}(\alpha)|0\rangle, w_1, w_2 \rangle \) is equal to

\[
\langle K_X, \alpha \rangle \cdot \delta_{u_1,u_2} \cdot \sum_{\sigma_1 \in \text{Perm}\{1,\ldots,u_1\}} \prod_{i=1}^{u_1} (\alpha_{1,i}, \alpha_{2,\sigma_1(i)}) \cdot p_1(\sigma_1)
\]

\[
+ \sum_{j=1}^{u_1} (\alpha, \alpha_{1,j}) \cdot \delta_{u_1-1,u_2} \cdot \prod_{\sigma_2 \neq j} (\alpha_{1,i}, \alpha_{2,\sigma_2(i)}) \cdot p_2(\sigma_2)
\]

\[
+ \sum_{j=1}^{u_2} (\alpha, \alpha_{2,j}) \cdot \delta_{u_1,u_2-1} \cdot \prod_{\sigma_3} (\alpha_{1,i}, \alpha_{2,\sigma_3(i)}) \cdot p_3(\sigma_3)
\]

(6.13)

where $\sigma_2$ runs over all bijections $\{1, \ldots, u_1\} \rightarrow \{1, \ldots, u_2\}$, $\sigma_3$ runs over all bijections $\{1, \ldots, u_1\} \rightarrow \{1, \ldots, u_2\} \setminus \{j\}$, and $p_1(\sigma_1)$ (respectively, $p_2(\sigma_2), p_3(\sigma_3)$) depend only on $\sigma_1$ (respectively, $\sigma_2, \sigma_3$), $n, n_0, \lambda$ and all the $\lambda^{(i)}, \mu^{(i)}, n_{i,j}$.

Proof. By Lemma 2.4 (ii), $\langle 1_{-\{\lambda\}}a_{-\lambda}(x)a_{-n_0}(\alpha)|0\rangle$ can be written as

\[
\langle K_X, \alpha \rangle \cdot F_1(n) + \sum_i G_{k_i}(\alpha, n) \cdot F_{2,i}(n)
\]

where $F_1(n)$ and $F_{2,i}(n)$ are polynomials of $G_{k_i}(x,n)$, $k \geq 0$ whose coefficients are independent of $n$ and $\alpha$. Moreover, the integers $k$ and $k_i$ depend only on $\lambda$ and $n_0$. Thus,

\[
\langle 1_{-\{\lambda\}}a_{-\lambda}(x)a_{-n_0}(\alpha)|0\rangle, w_1, w_2 \rangle = \langle K_X, \alpha \rangle \cdot \langle F_1(n), w_1, w_2 \rangle + \sum_i \langle G_{k_i}(\alpha, n) \cdot F_{2,i}(n), w_1, w_2 \rangle.
\]

As in the proof of Lemma 6.4 $\langle F_1(n), w_1, w_2 \rangle$ is of the form

\[
\delta_{u_1,u_2} \cdot \sum_{\sigma_1 \in \text{Perm}\{1,\ldots,u_1\}} \prod_{i=1}^{u_1} (\alpha_{1,i}, \alpha_{2,\sigma_1(i)}) \cdot \hat{p}_{1,1}(\sigma_1)
\]

(6.15)

where $\hat{p}_{1,1}(\sigma_1)$ depends only on $\sigma_1, n, n_0, \lambda$ and all the $\lambda^{(i)}, \mu^{(i)}, n_{i,j}$. Also,

\[
\langle G_{k_i}(\alpha, n)G_{s_1}(x,n) \cdots G_{s_i}(x,n), w_1, w_2 \rangle = \langle \varnothing_{s_1}(x) \cdots \varnothing_{s_i}(x)\varnothing_{k_i}(\alpha)w_1, w_2 \rangle.
\]

By Theorem 2.4 and Lemma 2.6 $\langle G_{k_i}(\alpha, n)G_{s_1}(x,n) \cdots G_{s_i}(x,n), w_1, w_2 \rangle$ equals

\[
\langle K_X, \alpha \rangle \cdot \delta_{u_1,u_2} \cdot \sum_{\sigma_1 \in \text{Perm}\{1,\ldots,u_1\}} \prod_{i=1}^{u_1} (\alpha_{1,i}, \alpha_{2,\sigma_1(i)}) \cdot \hat{p}_{1,1}(\sigma_1)
\]

\[
+ \sum_{j=1}^{u_1} (\alpha, \alpha_{1,j}) \cdot \delta_{u_1-1,u_2} \cdot \prod_{\sigma_2 \neq j} (\alpha_{1,i}, \alpha_{2,\sigma_2(i)}) \cdot \hat{p}_{2}(\sigma_2)
\]

\[
+ \sum_{j=1}^{u_2} (\alpha, \alpha_{2,j}) \cdot \delta_{u_1,u_2-1} \cdot \prod_{\sigma_3} (\alpha_{1,i}, \alpha_{2,\sigma_3(i)}) \cdot \hat{p}_3(\sigma_3)
\]

(6.16)

where $\sigma_2$ runs over all the bijections $\{1, \ldots, u_1\} \setminus \{j\} \rightarrow \{1, \ldots, u_2\}$, and $\sigma_3$ runs over all the bijections $\{1, \ldots, u_1\} \rightarrow \{1, \ldots, u_2\} \setminus \{j\}$. Hence \( \sum_i \langle G_{k_i}(\alpha, n) \cdot F_{2,i}(n), w_1, w_2 \rangle \) is of the form (6.16) as well. Combining with (6.14) and (6.15), we obtain (6.13). \( \square \)

Next, we introduce the notion of universal polynomials $P(K_X, S_1, S_2)$ in $\langle K_X, K_X \rangle$ of degree at most $m$ and of type $(u_1, u_2)$, and prove a vanishing lemma.
Definition 6.6. Fix three integers \(m, u_1, u_2 \geq 0\). Then a universal polynomial \(P(K_X, S_1, S_2)\) in \(\langle K_X, K_X \rangle\) of degree at most \(m\) and of type \((u_1, u_2)\) is of the form
\[
\sum_{1 \leq i_1 < \cdots < i_s \leq u} \prod_{j \in \{j_1, \ldots, j_s\}} \langle K_X, \alpha_{1,j} \rangle \cdot \prod_{i \in \{i_1, \ldots, i_s\}} \langle K_X, \alpha_{2,i} \rangle \\
\cdot \sum_{\sigma \in \text{Perm}(i_1, \ldots, i_s)} \prod_{i=1}^{s} \langle \alpha_{1,j_i}, \alpha_{2,\sigma(i_i)} \rangle \cdot p(j_1, \ldots, j_s; l_1, \ldots, l_s; \sigma) \tag{6.17}
\]
where \(S_i = \{\alpha_{i,1}, \ldots, \alpha_{i,u_i}\} \subset H^2(X)\), and \(p(j_1, \ldots, j_s; l_1, \ldots, l_s; \sigma)\) is a polynomial in \(\langle K_X, K_X \rangle\) whose degree is at most \(m\) and whose coefficients are independent of \(X\) and the classes \(\alpha_{i,j}\).

Lemma 6.7. Fix \(m, u_1, u_2 \geq 0\). Let \(P(K_X, S_1, S_2)\) be a universal polynomial in \(\langle K_X, K_X \rangle\) of degree at most \(m\) and of type \((u_1, u_2)\). Assume that \(P(K_X, S_1, S_2) = 0\) whenever \(X\) is a smooth projective toric surface. Then \(P(K_X, S_1, S_2) = 0\) for every smooth projective surface \(X\).

Proof. Let \(r \gg m + u_1 + u_2\), and let \(X_r\) be a smooth toric surface obtained from \(\mathbb{P}^2\) as an \(r\)-fold blow-up. Let \(L_0\) be a line in \(\mathbb{P}^2\), and let \(e_1, \ldots, e_r\) be the exceptional divisors. Then, \(K_{X_r} = -3L_0 + e_1 + \cdots + e_r\). For fixed \(j_1, \ldots, j_s, l_1, \ldots, l_s\) and \(\sigma\), let
\[
\{\alpha_{1,i} | i \in [1, \ldots, u_1] - \{j_1, \ldots, j_s\}\} = \{-e_1, \ldots, -e_{u_1-s}\},
\{\alpha_{2,i} | i \in [1, \ldots, u_2] - \{l_1, \ldots, l_s\}\} = \{-e_{u_1-s+1}, \ldots, -e_{u_1-s+u_2-s}\},
\]
and \(\alpha_{1,j_i} = \alpha_{2,\sigma(i_i)} = e_{u_1-s+u_2-s+2i-1} - e_{u_1-s+u_2-s+2i-1}\) for \(i = 1, \ldots, s\). Then,
\[
0 = P(K_{X_r}, S_1, S_2) = (-2)^s \cdot p(j_1, \ldots, j_s; l_1, \ldots, l_s; \sigma) \tag{6.18}
\]
by (6.17). It follows that \(p(j_1, \ldots, j_s; l_1, \ldots, l_s; \sigma)\) is 0 for all the surfaces \(X_r\) with \(r \gg m + u_1 + u_2\). Since \(p(j_1, \ldots, j_s; l_1, \ldots, l_s; \sigma)\) is a polynomial in \(\langle K_X, K_X \rangle\) whose degree is at most \(m\), we conclude that as polynomials, \(p(j_1, \ldots, j_s; l_1, \ldots, l_s; \sigma) = 0\). Therefore, \(P(K_X, S_1, S_2) = 0\) for every smooth projective surface \(X\). \(\square\)

Our next lemma is about the structure of certain 3-pointed extremal Gromov-Witten invariants, and provides the motivation for Definition 6.6.

Lemma 6.8. Let \(d, n_0 \geq 1\) and \(|\alpha| = 2\). Let \(w_1\) and \(w_2\) be given by (1.11). Then,
\[
\langle 1 - (n - |\alpha| - n_0) a_{-\lambda}(x) a_{-n_0}(\alpha)[0], w_1, w_2 \rangle_d = \langle K_X, \alpha \rangle \cdot P(K_X, S_1, S_2) \tag{6.19}
\]
where \(S_1 = \{\alpha_{1,1}, \ldots, \alpha_{1,u_1}\}, S_2 = \{\alpha_{2,1}, \ldots, \alpha_{2,u_2}\}\), and \(P(K_X, S_1, S_2)\) is a universal polynomial in \(\langle K_X, K_X \rangle\) of degree at most \((n - n_0)/2\) and of type \((u_1, u_2)\).

Proof. For simplicity, let \(w_0 = 1 - (n - |\lambda| - n_0) a_{-\lambda}(x) a_{-n_0}(\alpha)[0]\). Also, for \(i = 1, 2\), let \(\tilde{w}_i = a_{-\mu(i)}(1_x) a_{-n_1}(\alpha_{i,1}) \cdots a_{-n_{\lambda}(i)}(\alpha_{i,\lambda(i)})[0]\). We compute \(\langle w_0, w_1, w_2 \rangle_d\) by using (1.3). Consider the following term from (1.3):
\[
\langle B_0, B_1, B_2 \rangle \cdot \left( \frac{3_{m,d}}{B_0} \cdot \pi_{m,1} \left( \frac{w_0}{B_0} \right) \cdot \pi_{m,2} \left( \frac{w_1}{B_1} \right) \cdot \pi_{m,3} \left( \frac{w_2}{B_2} \right) \right) \tag{6.20}
\]
where \(m \leq n\), \(B_0, B_1, B_2 \in H^*(X^{[n-m]}), B_0 \subset w_0, B_1 \subset w_1,\) and \(B_2 \subset w_2\). By Theorem 1.3 (i) and Corollary 5.23(i), such a term is nonzero only if \(B_0 = a_{-1}(1_x)^j a_{-\lambda}(x)[0]\) with \(j \leq (n - |\lambda| - n_0)\), \(B_1 = a_{-\lambda(i)}(x) B_1\) with \(B_1 \subset \tilde{w}_1\), and \(B_2 = a_{-\lambda(\alpha)}(x) B_2\) with \(B_2 \subset \tilde{w}_2\). In this situation, (6.20) can be rewritten as
\[
\langle a_{-1}(1_x)^j a_{-\lambda}(x)[0], a_{-\lambda(i)}(x) B_1, a_{-\lambda(\alpha)}(x) B_2 \rangle \tag{6.21}
\]
\[
\cdot \left( \frac{3_{m,d}}{B_0} \cdot \pi_{m,1} \left( \frac{1 - (n - |\lambda| - n_0) a_{-n_0}(\alpha)[0]}{a_{-1}(1_x)^j[0]} \right) \cdot \pi_{m,2} \left( \frac{\tilde{w}_1}{B_1} \right) \cdot \pi_{m,3} \left( \frac{\tilde{w}_2}{B_2} \right) \right) \tag{6.22}
\]
Note that \(B_1 = a_{-\mu(i)}(1_x) a_{-n_{\lambda(i)}}(\alpha_{1,j_1}) \cdots a_{-n_{\lambda(j)}(\alpha_{1,j_\lambda})}[0]\) for some \(1 \leq j_1 < \cdots < j_\lambda \leq u_1\) and some sub-partition \(\mu(i)\) of \(\mu(1)\) (i.e., every part of \(\mu(i)\) is a part of \(\mu(1)\)). Similarly, \(B_2 = \)
whose coefficients are independent of the surface $X$ that the factor $(6.22)$ is equal to

$$p = \prod_{i,j} (\alpha_{i,j} \cdot p_1(j_1, \ldots, j_s; l_1, \ldots, l_s; \sigma))$$

where $p_1$ is a number independent of the surface $X$ and the classes $\alpha_{i,j}$. By Theorem 1.3 we see that the factor $(6.22)$ is equal to

$$\langle K_X, \alpha \rangle \cdot \prod_{i \in \{1, \ldots, J_s\}} (\langle K_X, \alpha_{i,1} \rangle \cdot \prod_{i \in \{1, \ldots, J_s\}} \langle K_X, \alpha_{i,2} \rangle \cdot p_2(j_1, \ldots, j_s; l_1, \ldots, l_s; \sigma))$$

where $p_2$ is a polynomial in $\langle K_X, K_X \rangle$ whose degree is at most $(m - n_0)/2 \leq (n - n_0)/2$, and whose coefficients are independent of the surface $X$ and the classes $\alpha_{i,j}$. Combining this with $(6.20)$, $(6.21)$, $(6.22)$ and $(6.23)$, we obtain $(6.19)$. \hfill \Box

**Proposition 6.9.** If $|\alpha| = 2$, then $(6.2)$ is true.

**Proof.** Recall that $(1.2)$ is equivalent to $(6.3)$, and the difference $D_\beta^\alpha(w_1, w_2; q)$ from $(6.4)$ is computed by Lemma 6.2. Let $w_1$ and $w_2$ be given by $(6.10)$. Let $u'_1 = \delta_{2,\beta} + u_1$ and $S_2 = \{\alpha_{2,1}, \ldots, \alpha_{2,w_2}\}$. Let $S_1 = \{\alpha_{1,1}, \ldots, \alpha_{1,w_1}\}$ if $|\beta| \neq 2$, and $S_1 = \{\beta, \alpha_{1,1}, \ldots, \alpha_{1,w_1}\}$ if $|\beta| = 2$.

By Lemma 6.6 and Lemma 6.7 it suffices to prove that

$$D_\beta^\alpha(w_1, w_2; -1) = \langle K_X, \alpha \rangle \cdot P(K_X, S_1, S_2)$$

where $P(K_X, S_1, S_2)$ is a universal polynomial in $\langle K_X, K_X \rangle$ of degree at most $(n - 1)/2$ and of type $(u'_1, w_2)$. This follows if we can prove that

$$D_\beta^\alpha(w_1, w_2; q) = \langle K_X, \alpha \rangle \cdot \sum_{d \geq 0} P(K_X, S_1, S_2; d) q^d$$

where every $P(K_X, S_1, S_2; d)$ is a universal polynomial in $\langle K_X, K_X \rangle$ of degree at most $(n - 1)/2$ and of type $(u'_1, w_2)$. We remark that $d$ has been inserted into the notation $P(K_X, S_1, S_2; d)$ to emphasise its dependence on $d$.

In the following, we will show that the contribution of every term in $(6.5)$ is of the form $P(K_X, S_1, S_2; d)$ for a suitable $d \geq 0$. Note that in $H^*(X^i)$,

$$\tau_\alpha(x) = \alpha \otimes x \otimes \cdots \otimes x + x \otimes \alpha \otimes x \otimes \cdots \otimes x + \cdots + x \otimes \cdots \otimes x \otimes \alpha.$$  

Thus, by Lemma 6.8

$$\langle 1_{-(n-j-1)} a_{-\lambda}(\tau_\alpha)|0\rangle, a_{-1}(\beta)w_1, w_2 \rangle_d$$

is equal to

$$\langle K_X, \alpha \rangle \cdot P_1(K_X, S_1, S_2; d)$$

where $P_1(K_X, S_1, S_2; d)$ is a universal polynomial in $\langle K_X, K_X \rangle$ of degree at most $(n - 1)/2$ and of type $(u'_1, w_2)$. Similarly, since $a_{-1}(\beta)^\dagger w_2 = -a_1(\beta)w_2$, we see from Theorem 2.1 (i) and Lemma 6.8 that

$$\langle 1_{-(n-j-2)} a_{-\lambda}(\tau_\alpha)|0\rangle, w_1, a_{-1}(\beta)^\dagger w_2 \rangle_d = \langle K_X, \alpha \rangle \cdot P_2(K_X, S_1, S_2; d).$$

Next, we move to the term $\langle a_\lambda(\tau_\epsilon(x)\beta)|0\rangle, w_1, w_2 \rangle$ in $(6.5)$, where $\epsilon \in \{K_X, K_X^2\}$. Such a term is zero unless $\epsilon = K_X$ and $|\beta| = 0$. In this case, we may assume that $\beta = 1_X$. So let $\epsilon = K_X$ and $\beta = 1_X$. Then,

$$\langle a_\lambda(\tau_\epsilon(x)\beta)|0\rangle, w_1, w_2 \rangle = \langle K_X, \alpha \rangle \cdot \langle a_\lambda(x)w_1, w_2 \rangle = \langle K_X, \alpha \rangle \cdot P_3(K_X, S_1, S_2; 0).$$

by Theorem 2.1 (i), where $P_3(K_X, S_1, S_2; 0)$ is a universal polynomial in $\langle K_X, K_X \rangle$ of degree 0 (i.e., $(K_X, K_X)$ does not appear) and of type $(u'_1, w_2)$.

Finally, $\tau_\epsilon(x)$ is zero unless $\epsilon = K_X$. Let $\epsilon = K_X$. By Lemma 6.4

$$\langle 1_{-(n-j-1)} a_{-\lambda}(\epsilon \alpha)|0\rangle, a_{-1}(\beta)w_1, w_2 \rangle$$

$$= \langle K_X, \alpha \rangle \cdot \langle 1_{-(n-j-1)} a_{-\lambda}(x)|0\rangle, a_{-1}(\beta)w_1, w_2 \rangle$$
where \( P_{\lambda}(K_X, S_1, S_2; 0) \) is a universal polynomial in \( \langle K_X, K_X \rangle \) of degree 0 and of type \((u_1', u_2)\). Similarly, since \( a_{-1}(\beta)w_2 = -a_1(\beta)w_2 \), we obtain
\[
\langle 1 - (n - j - 2) a_{-\lambda}(\tau_*(\alpha)) \rangle |0\rangle, \quad w_1, a_{-1}(\beta)w_2 = \langle K_X, \alpha \rangle \cdot P_{\beta}(K_X, S_1, S_2; 0).
\]

### 6.3. The case \( \alpha = 1_X \)

**Lemma 6.10.** Let \( d, n_0 \geq 1 \). Let \( w_1 \) and \( w_2 \) be given by \((6.10)\). Then,
\[
\langle 1 - (n - |\lambda| - n_0) a_{-\lambda}(x) a_{n_0}(1_X) |0\rangle, \quad w_1, w_2 \rangle = P(K_X, S_1, S_2)
\]
where \( S_1 = \{ \alpha_1, \ldots, \alpha_{n_1}, \alpha_1, \ldots, \alpha_{n_2} \} \), and \( P(K_X, S_1, S_2) \) is a universal polynomial in \( \langle K_X, K_X \rangle \) of degree at most \( n/2 \) and of type \((u_1, u_2)\).

**Proof.** This follows from the proof of Lemma 6.8 by replacing \( \alpha \) by \( 1_X \) (and then by noticing that the factor \( \langle K_X, \alpha \rangle \) there will not appear here).

### 6.11. Let \( d \geq 1 \) and \( |\lambda| \leq n \). Let \( w_1 \) and \( w_2 \) be given by \((6.10)\). Then,
\[
\langle 1 - (n - |\lambda|) a_{-\lambda}(\tau_1 1_X) |0\rangle, \quad w_1, w_2 \rangle = P(K_X, S_1, S_2)
\]
where \( S_1 = \{ \alpha_1, \ldots, \alpha_1, \alpha_1, \ldots, \alpha_{n_2} \} \), and \( P(K_X, S_1, S_2) \) is a universal polynomial in \( \langle K_X, K_X \rangle \) of degree at most \( n/2 \) and of type \((u_1, u_2)\).

**Proof.** For \( i = 1 \) and \( 2 \), let \( \tilde{w}_i = a_{\mu(i)}(1_X) a_{-n_1} a_{-n_2} \alpha_{\ldots, \alpha_{n_1}, \alpha_1, \ldots, \alpha_{n_2}} |0\rangle \). Note that if the Künneth decomposition of \( \tau_{2,1} X \in H^*(X^2) \) is given by
\[
\tau_{2,1} X = x \otimes 1_X + 1_X \otimes x + \sum_j \gamma_{j,1} \otimes \gamma_{j,2}
\]
where \( |\gamma_{j,1}| = |\gamma_{j,2}| = 2 \), then up to permutations of factors, a typical term in the Künneth decomposition of \( \tau_{i} x_1 \in H^*(X^i) \) with \( i \geq 3 \) is either \( x \otimes \cdots \otimes x \otimes 1_X \) or \( x \otimes \cdots \otimes x \otimes \gamma_{j,1} \otimes \gamma_{j,2} \).

In view of Lemma 6.10 it suffices to verify that
\[
\sum_j \langle 1 - n a_{-\lambda}(x) a_{-n_1} a_{-n_2} \alpha_{\ldots, \alpha_{n_1}, \alpha_1, \ldots, \alpha_{n_2}} |0\rangle, \quad w_1, w_2 \rangle = P_1(K_X, S_1, S_2)
\]
where \( n = |\lambda| - n_1 - n_2 \), and \( P_1(K_X, S_1, S_2) \) is a universal polynomial in \( \langle K_X, K_X \rangle \) of degree at most \( n/2 \) and of type \((u_1, u_2)\). For simplicity, let
\[
w_0 = 1 - n a_{-\lambda}(x) a_{-n_1} a_{-n_2} \alpha_{\ldots, \alpha_{n_1}, \alpha_1, \ldots, \alpha_{n_2}} |0\rangle.
\]
We see from \((6.28)\) that to prove \((6.29)\), it suffices to show that
\[
\sum_j \langle B_0, B_1, B_2 \rangle ; \left\langle \mathcal{M}, \pi_{m_1}^* \left( \frac{w_0}{B_0} \right) \cdot \pi_{m_2}^* \left( \frac{w_1}{B_1} \right) \cdot \pi_{m_3}^* \left( \frac{w_2}{B_2} \right) \right\rangle
\]
is equal to \( P_2(K_X, S_1, S_2) \), where \( m \leq n, B_0, B_1, B_2 \in H^* (X^{[m-n]}), B_0 \subset w_0, B_1 \subset w_1, \) and \( B_2 \subset w_2 \). By Theorem 1.3 (i) and Corollary 5.24 (i), such a term is nonzero only if \( B_1 = a_{-\lambda(\gamma)}(x) B_1 \) with \( B_1 \subset \tilde{w}_1 \), \( B_2 = a_{-\lambda(\gamma)}(x) B_2 \) with \( B_2 \subset \tilde{w}_2 \), and \( B_0 = a_{-1}(1_X)^* a_{-\lambda(\gamma)}(x) \) or \( a_{-1}(1_X)^* a_{-\lambda(\gamma)}(x) \alpha_{\ldots, \alpha_{n_1}, \alpha_1, \ldots, \alpha_{n_2}} |0\rangle \) where \( s \leq n \). In the following, we assume that \((6.29)\) is nonzero. By symmetry, we need only to consider two cases for \( B_0 \):
\[
B_0 = a_{-1}(1_X)^* a_{-\lambda(\gamma)}(x) \quad \text{or} \quad B_0 = a_{-1}(1_X)^* a_{-\lambda(\gamma)}(x) \alpha_{\ldots, \alpha_{n_1}, \alpha_1, \ldots, \alpha_{n_2}} |0\rangle.
\]
We begin with the case \( B_0 = a_{-1}(1_X)^* a_{-\lambda(\gamma)}(x) \). Then \((6.29)\) becomes
\[
\sum_j \langle a_{-1}(1_X)^* a_{-\lambda(\gamma)}(x) |0\rangle, \quad a_{-\lambda(\gamma)}(x) B_1, a_{-\lambda(\gamma)}(x) B_2 \rangle \times \left\langle \mathcal{M}, \pi_{m_1}^* \left( \frac{1 - n a_{-n_1} a_{-n_2} \alpha_{\ldots, \alpha_{n_1}, \alpha_1, \ldots, \alpha_{n_2}} |0\rangle}{a_{-1}(1_X)^* |0\rangle} \right) \cdot \pi_{m_2}^* \left( \frac{\tilde{w}_1}{B_1} \right) \cdot \pi_{m_3}^* \left( \frac{\tilde{w}_2}{B_2} \right) \right\rangle.
\]
Applying the same arguments as in the computations of (6.21) and (6.22), we conclude that the term (6.29) is equal to $$\sum_j \langle K_X, \gamma_j, 1 \rangle \cdot \langle K_X, \gamma_j, 2 \rangle \cdot P_3(K_X, S_1, S_2)$$ where $$P_3(K_X, S_1, S_2)$$ is a universal polynomial in $$\langle K_X, K_X \rangle$$ of degree at most $$(m - n_1 - n_2)/2$$ and of type $$(u_1, u_2)$$. Note that for $$\beta_1, \beta_2 \in H^2(X)$$, we have $$\sum_j \langle \beta_1, \gamma_j, 1 \rangle \cdot \langle \beta_2, \gamma_j, 2 \rangle = \langle \beta_1, \beta_2 \rangle.$$ (6.30)

Therefore, (6.29) is equal to $$\langle K_X, K_X \rangle \cdot P_3(K_X, S_1, S_2)$$ which is a universal polynomial in $$\langle K_X, K_X \rangle$$ of degree at most $$m/2 \leq n/2$$ and of type $$(u_1, u_2)$$.

Next, let $$B_0 = a_{-1}(1_X)^s a_{-\lambda}(x) a_{-\lambda}(\gamma_j, 1)|0\rangle$$. This time, (6.29) becomes $$\sum_j \langle a_{-1}(1_X)^s a_{-\lambda}(x) a_{-\lambda}(\gamma_j, 1)|0\rangle, a_{-\lambda}(x) B_1, a_{-\lambda}(\gamma_j, 1) B_2 \rangle \cdot \left(3_{m, d}, \pi_m^1 \left(1 - \frac{a_{-\lambda}(\gamma_j, 1)|0\rangle}{a_{-1}(1_X)^s|0\rangle}\right), \pi_m^2 \left(\frac{w_1}{B_1}\right), \pi_m^3 \left(\frac{w_2}{B_2}\right) \right).$$

Using Lemma 6.3, Theorem 1.3 and (6.30), we conclude that (6.29) is equal to $$P_4(K_X, S_1, S_2)$$ which is a universal polynomial in $$\langle K_X, K_X \rangle$$ of degree at most $$(m - n_2)/2 + 1 \leq ((n - n_1) - n_2)/2 + 1 \leq n/2$$ and of type $$(u_1, u_2)$$. This completes the proof of (6.27).

Proposition 6.12. If $$\alpha = 1_X$$, then (1.3) is true.

Proof. We adopt the same notations and approaches as in the proof of Proposition 6.1. By Lemma 4.6 and Lemma 6.7, it suffices to prove that $$D_{\beta}^1(\alpha_{1, w_1}; -1) = P(K_X, S_1, S_2)$$ (6.31)

where $$P(K_X, S_1, S_2)$$ is a universal polynomial in $$\langle K_X, K_X \rangle$$ of degree at most $$(n + 1)/2$$ and of type $$(u_1, u_2)$$. This follows if we can prove that $$D_{\beta}^1(\alpha_{1, w_1}; q) = \sum_{d \geq 0} P(K_X, S_1, S_2; d) q^d$$ (6.32)

where $$P(K_X, S_1, S_2; d)$$ is a universal polynomial in $$\langle K_X, K_X \rangle$$ of degree at most $$(n + 1)/2$$ and of type $$(u_1', u_2)$$.

In the following, we will show that the contribution of every term in (6.31) is of the form $$P(K_X, S_1, S_2; d)$$ for a suitable $$d \geq 0$$.

First of all, when $$d \geq 1$$, we conclude from Lemma 6.11 that $$\langle 1_{- (n - j - 1)} a_{-\lambda}(\tau, 1_X)|0\rangle, a_{-1}(\beta) w_1, w_2 \rangle d - \langle 1_{- (n - j - 2)} a_{-\lambda}(\tau, 1_X)|0\rangle, w_1, a_{-1}(\beta) w_2 \rangle d$$ is equal to $$P_4(K_X, S_1, S_2; d)$$ which is a universal polynomial in $$\langle K_X, K_X \rangle$$ of degree at most $$n/2$$ and of type $$(u_1', u_2)$$.

Next, consider $$\langle a_{\lambda}(\tau, (\epsilon \beta)|w_1, w_2\rangle = \langle a_{\lambda}(\tau, (\epsilon \beta)|w_1, w_2\rangle$$ from (6.5), where $$\epsilon \in \{K_X, K_X^2\}$$. It is zero unless $$\epsilon = K_X^2$$ and $$\beta = 1_X$$ (when $$|\beta| = 0$$, we let $$\beta = 1_X$$), or $$\epsilon = K_X$$ and $$|\beta| = 2$$, or $$\epsilon = K_X^2$$ and $$|\beta| = 2$$. If $$\epsilon = K_X^2$$ and $$|\beta| = 2$$, then $$\langle a_{\lambda}(\tau, (\epsilon \beta)|w_1, w_2\rangle = \langle K_X, K_X \rangle \cdot \langle a_{\lambda}(x) w_1, w_2\rangle = \langle K_X, K_X \rangle \cdot P_2(K_X, S_1, S_2; 0)$$ by Theorem 2.2 (i), where $$P_2(K_X, S_1, S_2; 0)$$ is a universal polynomial in $$\langle K_X, K_X \rangle$$ of degree 0 and of type $$(u_1', u_2)$$. If $$\epsilon = K_X$$ and $$|\beta| = 2$$, then $$\langle a_{\lambda}(\tau, (\epsilon \beta)|w_1, w_2\rangle = \langle K_X, K_X \rangle \cdot \langle a_{\lambda}(x) w_1, w_2\rangle = \langle K_X, K_X \rangle \cdot P_3(K_X, S_1, S_2; 0)$$
which is a universal polynomial in $\langle K_X, K_X \rangle$ of degree 0 and of type $(u'_1, u_2)$. If $\epsilon = K_X$ and $\beta = 1_X$, then we obtain $\langle a_\lambda(\tau_*(\epsilon \beta))w_1, w_2 \rangle = \langle a_\lambda(\tau_*K_X)w_1, w_2 \rangle$ which again is a universal polynomial in $\langle K_X, K_X \rangle$ of degree 0 and of type $(u'_1, u_2)$.

Finally, let $\epsilon \in \{ K_X, K_X^2 \}$. We have $\tau_*(\epsilon \beta) = \tau_*(\epsilon) \beta$. Let $I_\epsilon$ be the difference

\[ I_\epsilon = \langle K_X, K_X \rangle \cdot \langle 1_{-(n-j-1)}a - \lambda(\epsilon)0, a_1 - \lambda(\epsilon)w_1, w_2 \rangle - \langle K_X, K_X \rangle \cdot \langle 1_{-(n-j-2)}a - \lambda(\epsilon)0, w_1, a_2 - \lambda(\epsilon)w_2 \rangle \]

from (6.5). When $\epsilon = K_X^2$, we see from Lemma 6.2 that

\[ I_\epsilon = \langle K_X, K_X \rangle \cdot P_4(K_X, S_1, S_2; 0) \]

is a universal polynomial in $\langle K_X, K_X \rangle$ of degree 0 and of type $(u'_1, u_2)$. When $\epsilon = K_X^2$, we see from Lemma 6.3 that

\[ I_\epsilon = \langle 1_{-(n-j-1)}a - \lambda(K_X)0, a_1 - \lambda(K_X)w_1, w_2 \rangle - \langle 1_{-(n-j-2)}a - \lambda(K_X)0, w_1, a_2 - \lambda(K_X)w_2 \rangle \]

is a universal polynomial in $\langle K_X, K_X \rangle$ of degree at most 1 and of type $(u'_1, u_2)$. $\square$

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