Unifying the Anderson Transitions in Hermitian and Non-Hermitian Systems

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Non-Hermiticity enriches the 10-fold Altland-Zirnbauer symmetry class into the 38-fold symmetry class, where critical behavior of the Anderson transitions (ATs) has been extensively studied recently. Here, we propose a correspondence of the universality classes of the ATs between Hermitian and non-Hermitian systems. We illustrate that the critical exponents of the length scale in non-Hermitian systems coincide with the critical exponents in the corresponding Hermitian systems with additional chiral symmetry. A remarkable consequence of the correspondence is superuniversality, i.e., the ATs in some different symmetry classes of non-Hermitian systems are characterized by the same critical exponent. In addition to the comparisons between the known critical exponents for non-Hermitian systems and their Hermitian counterparts, we obtain the critical exponents in symmetry classes AI, AII, AII†, CII, and DIII in two and three dimensions. Estimated critical exponents are consistent with the proposed correspondence. According to the correspondence, some of the exponents also give useful information of the unknown critical exponents in Hermitian systems, paving a way to study the ATs of Hermitian systems by the corresponding non-Hermitian systems.

Introduction.— Scattering, transmission, and interference of waves in dissipative media lead to a rich variety of physical phenomena. A prime example is localization, where a propagating wave and its counter-propagating wave caused by scattering form a standing wave. After Anderson’s seminal work [1], which predicted delocalization-localization transitions of electron wavefunctions in disordered solids, a general scaling theory of localization was introduced [2 3]. Subsequent development of field theory descriptions, as well as renormalization-group analyses, clarified the universality classes of the Anderson transitions (ATs) in three fundamental symmetry classes of time-reversal symmetry: Wigner-Dyson classes [1 2]. Furthermore, chiral symmetry [6 7] and particle-hole symmetry enrich the universality classes into the ten-fold symmetry classification [8].

Like other continuous phase transitions, a universality class of the ATs is characterized by scaling properties of an effective theory. Based on the single-parameter-scaling hypothesis, the critical exponents of the ATs in the ten symmetry classes have been numerically studied [19 25]. It is commonly believed that the universality classes are determined solely by spatial dimension and symmetry, being independent from details of Hamiltonians. In some cases, two distinct symmetry classes share the same scaling property, which is called superuniversality [26 32]. Superuniversality can be numerically observed by precisely determining critical exponents and other universal scaling properties. In this paper, we show that superuniversality emerges also in non-Hermitian disordered systems.

Recently, the ATs in non-Hermitian disordered systems attract considerable research interest [33 40]. Non-Hermitian disordered systems describe random media with amplification or dissipation, which include open classical systems [47–51], as well as quantum systems of quasi-particles with finite lifetime [52–55]. In contrast to Hermitian systems, non-Hermitian systems are classified into 38 symmetry classes [56–58]. However, universality classes of the ATs in these 38 symmetry classes have yet to be understood clearly.

In this paper, we propose a correspondence between the ATs in Hermitian systems and those in non-Hermitian systems and develop a unified understanding about the ATs. We argue that the critical behavior of the length scale in non-Hermitian systems is identical to the critical behavior in the corresponding Hermitian system with additional chiral symmetry. To examine the proposed correspondence, we carry out extensive numerical studies of critical exponents in non-Hermitian disordered systems. In particular, we study in this paper the universal critical behavior of the ATs for non-Hermitian models in classes AI, AII, AII†, CII, and DIII in two dimensions (2D) and three dimensions (3D). We calculate the localization lengths of these models by the transfer matrix method, analyze them by the finite size scaling [46], and determine values of the critical exponents of the ATs, as summarized in Table I. Combining with the critical exponents for classes A and AI† previously obtained in Refs. [15 40], we show that the critical exponents in these non-Hermitian symmetry classes are consistent with the known critical exponents in the corresponding Hermitian symmetry classes, supporting the correspondence of the ATs between Hermitian and non-Hermitian systems. Notably, estimated critical exponents in some non-
Hermitian systems also provide useful information of critical behaviour in Hermitian symmetry classes with chiral or particle-hole symmetry, where the critical exponents were previously difficult to estimate.

Unified universality classes.— Our correspondence of the ATs between Hermitian and non-Hermitian systems is based on Hermitization [34, 35, 39, 57, 60]. A non-Hermitian Hamiltonian $\mathcal{H}$ with complex energy $E \in \mathbb{C}$ is mapped to the Hermitian Hamiltonian $\tilde{\mathcal{H}}$ by

$$\tilde{\mathcal{H}} = \begin{pmatrix} 0 & \mathcal{H} - E \end{pmatrix} \begin{pmatrix} \mathcal{H}^\dagger - E^* & 0 \end{pmatrix} .$$

By construction, the Hermitian Hamiltonian $\tilde{\mathcal{H}}$ respects additional chiral symmetry $\tau_3 \tilde{\mathcal{H}} \tau_3 = -\tilde{\mathcal{H}}$. Let $|\varphi_r\rangle$ and $|\varphi_l\rangle$ be a right eigenmode and a left eigenmode of the non-Hermitian Hamiltonian $\mathcal{H}$ with eigenenergy $E$, respectively: $\mathcal{H} |\varphi_r\rangle = E |\varphi_r\rangle$ and $\mathcal{H}^\dagger |\varphi_l\rangle = E^* |\varphi_l\rangle$. Then, $(0 |\varphi_l\rangle)^T$ and $(|\varphi_r\rangle 0)^T$ comprise doubly degenerate zero modes of the Hermitized Hamiltonian $\tilde{\mathcal{H}}$ (i.e., $\tilde{\mathcal{H}} (0 |\varphi_l\rangle)^T = \tilde{\mathcal{H}} (|\varphi_r\rangle 0)^T = 0$). This is the Hermitization, which associates the non-Hermitian Hamiltonian $\mathcal{H}$ with the Hermitian Hamiltonian $\tilde{\mathcal{H}}$ with chiral symmetry. Hermitization is relevant to non-Hermitian random matrices [34] and topological phases [39, 57], as well as topological characterization [60, 61] of the anomalous boundary physics due to non-Hermiticity (i.e., non-Hermitian skin effect [62, 63]). However, the significance of Hermitization has been unclear for the ATs.

We argue that Hermitization unifies the ATs in Hermitian and non-Hermitian systems. The ATs are continuous phase transitions that are characterized by the universal scaling properties of the localization lengths. As shown above, eigenmodes of $\mathcal{H}$ and the corresponding zero modes of $\tilde{\mathcal{H}}$ share the same spatial profiles, including the localization lengths. Therefore, the universal scaling properties of the localization lengths in non-Hermitian systems, as well as the absence or presence of the ATs, are generally the same as those in the Hermitian counterparts. Notably, the right eigenmode $|\varphi_r\rangle$ and the corresponding left eigenmode $|\varphi_l\rangle$ exhibit similar localization properties with the same localization length, since they correspond to zero modes in the Hermitized Hamiltonian $\tilde{\mathcal{H}}$ with opposite chiralities. Notably, although the Hermitization procedure always maps non-Hermitian Hamiltonians to Hermitian Hamiltonians with chiral symmetry, nonchiral symmetry classes can appear in the Hermitized Hamiltonians. Even if the Hermitized Hamiltonians respect chiral symmetry, they can respect additional unitary symmetry and then be block diagonalized. In such a case, the relevant symmetry classes (or equivalently, classifying spaces) are not necessarily chiral classes [65].

For several non-Hermitian symmetry classes, we summarize the correspondence in Table I (see Ref. [57] and the Supplemental Material [65] for the correspondence of all the 38 symmetry classes). For these classes in 2D and 3D in Table I we illustrate the correspondence by...
numerical evaluations of the critical exponents, as shown below.

Model and symmetry class.— To study the AT in class AI, we introduce the following O(1) tight-binding model on 3D cubic lattice:

$$\mathcal{H} = \sum_i \varepsilon_i c_i^\dagger c_i + \sum_{\langle i,j \rangle} V_{i,j} c_i^\dagger c_j,$$

where $\varepsilon_i$ is the random potential characterized by the uniform distribution in $[-W/2, W/2]$ with the disorder strength $W$. Here, $\langle i,j \rangle$ denotes nearest-neighbor lattice sites. $V_{i,j}$ is set to either $-1$ or $+1$ randomly with the equal probability, and $V_{i,j}$ and $V_{j,i}$ are treated as independent random numbers. Hermiticity is broken because of $V_{i,j}^\dagger \neq V_{j,i}$, and reciprocity is absent in each disorder realization ($\mathcal{H}^T \neq \mathcal{H}$). Still, $\mathcal{H}$ is statistically reciprocal in a sense that $\mathcal{H}$ and $\mathcal{H}^T$ appear with the equal probability in the ensemble. Eigenstates of $\mathcal{H}$ at real and complex energy $E$ belong to non-Hermitian symmetry classes AI and A respectively. For the real and complex $E$, the Hermitian Hamiltonian $\tilde{\mathcal{H}}$ belongs to symmetry classes BDI and AIII, respectively.

To study the ATs in classes AII, AII†, CII†, and DIII, we introduce the following non-Hermitian extension of the SU(2) model on 2D square and 3D cubic lattices,

$$\mathcal{H} = \sum_{i,\sigma} \varepsilon_{i,\sigma} c_{i,\sigma}^\dagger c_{i,\sigma} + \sum_{\langle i,j \rangle, \sigma, \sigma'} R(i,j)_{\sigma,\sigma'} c_{i,\sigma}^\dagger c_{j,\sigma'},$$

with $\sigma = \uparrow, \downarrow$. The spin-dependent nearest-neighbor hoppings are parametrized by the SU(2) matrix

$$R(i,j) = \begin{pmatrix} e^{i\alpha_{i,j}} \cos(\beta_{i,j}) & e^{i\gamma_{i,j}} \sin(\beta_{i,j}) \\ -e^{-i\gamma_{i,j}} \sin(\beta_{i,j}) & e^{-i\alpha_{i,j}} \cos(\beta_{i,j}) \end{pmatrix},$$

where $i$ is the imaginary unit, $\alpha_{i,j}$ and $\gamma_{i,j}$ are uniformly distributed in $[0, 2\pi]$, and $\beta_{i,j}$ is distributed in $[0, \pi/2]$ according to the probability density $P(\beta)d\beta = \sin(2\beta)d\beta$. The hopping terms satisfy $R^\dagger(i,j) = R(j,i)$ for classes AII, AII†, and CII† ($\alpha_{i,j} = -\alpha_{j,i}$, $\gamma_{i,j} = \gamma_{j,i} + \pi$), while they satisfy $\sigma_z R(i,j)\sigma_z = -R(j,i)$ for class DIII ($\alpha_{i,j} = -\alpha_{j,i} + \pi$, $\gamma_{i,j} = \gamma_{j,i} + \pi$). The on-site potentials $\varepsilon_{i,\sigma} = \omega_{i,\sigma}^r + i\omega_{i,\sigma}^i$ are complex-valued, letting $\mathcal{H}$ be non-Hermitian. The complex-valued potentials are realized in classical optical systems with random amplification and dissipation. $\omega_{i,j}^r$ and $\omega_{i,j}^i$ are independent for each site $j$, and are uniformly distributed in $[-W_r/2, W_r/2]$ and $[-W_i/2, W_i/2]$, respectively. A relation between $\varepsilon_{i,\uparrow}$ and $\varepsilon_{i,\downarrow}$, as well as $W_r$ and $W_i$, is chosen approximately so that $\mathcal{H}$ will belong to the different symmetry classes among classes AII, AII†, CII†, and DIII. The SU(2) models are reciprocal in classes AII†, CII†, and DIII; the SU(2) model in class AII is reciprocal only statistically, similarly to the O(1) model.

Transfer matrix study and polynomial fitting.— Localization length and conductance of non-Hermitian systems were previously calculated by the transfer matrix method. Thereby, the critical exponents of the ATs in classes A and AII† were determined precisely by the finite-size scaling analysis. In this paper, the localization lengths for the five symmetry classes are calculated for different complex-valued energies in a quasi-one-dimensional geometry ($L \times L_z$ in 2D and $L \times L \times L_z$ in 3D with $L_z \gg L$). The quasi-one-dimensional localization length $\xi(L)$ along the $z$ direction is normalized by the system size $L$ along the transverse direction. Being dimensionless, the normalized length $\Lambda = \xi(W, L)/L$ shows scale-invariant behavior at the AT as a function of $L$.

The single-parameter scaling has been demonstrated to be successful in analyses of the quantum criticality of the ATs in Hermitian systems and in non-Hermitian systems. Apart from fine-tuned critical points such as multicritical points, critical properties of a generic continuous phase transition must be controlled by a saddle-point fixed point with only one relevant scaling variable. The scaling argument dictates that the dimensionless normalized localization length $\Lambda$ follows a scaling function that depends on the relevant scaling variable and possibly many other irrelevant scaling variables. The universal critical exponent $\nu$ associated with the relevant scaling variable can be estimated based on a polynomial expansion of the scaling function in terms of the scaling variables.

Numerical results.— The normalized quasi-one-dimensional localization lengths $\Lambda$ for classes AI, AII,
AII\textsuperscript{†}, CII\textsuperscript{†}, and DIII in 2D or 3D are calculated at different complex energies \cite{65}. As an illustration, Fig. \ref{fig:1} shows \(\Lambda\) around the critical point at \(E = 0\) for 3D class AII with different system sizes \(L\) and disorder strength \(W\). As \(L\) increases, \(\Lambda\) increases below the critical point (delocalized phase) and decreases above the critical point (localized phase). In terms of numerical fitting based on the polynomial expansion \cite{65}, universal critical parameters of the ATs in the five non-Hermitian symmetry classes are obtained. The critical exponents \(\nu\) as well as normalized localization lengths \(\Lambda_c\) at the critical point are summarized in Table \ref{tab:1}. Fitted critical parameters are confirmed to be stable against changing the system sizes and/or expansion orders \cite{65}.

Universal critical exponents of the ATs in the non-Hermitian symmetry classes are mostly consistent with the known exponents in the corresponding Hermitian symmetry classes (Table \ref{tab:1}). On the other hand, we also found discrepancies in the exponents between the 3D class AII with different system sizes \(L\). As \(L\) increases, \(\Lambda\) increases below the critical point (delocalized phase) and decreases above the critical point (localized phase). In class AII, by contrast, time-reversal symmetry creates a soft gap in class AII (inset of Fig. \ref{fig:1}). This behavior is consistent with the random-matrix behavior in classes AI and AII \cite{35, 65, 76, 77}, and originates from the difference of time-reversal symmetry. In class AI, time-reversal symmetry imposes a constraint on each real eigenenergy. Because of this constraint, real eigenenergies remain real unless they are mixed with other real eigenenergies. Consequently, some of them are stable even against non-Hermitian perturbations, forming the sharp peak of the density of states. In class AII, by contrast, time-reversal symmetry creates Kramers pairs with real eigenenergies. In the presence of non-Hermitian perturbations, they are fragile and form complex-conjugate pairs \cite{78}. Hence, eigenenergies tend to be away from the real axis, which leads to the soft gap of the density of states. We give other heuristic discussions in the Supplemental Material \cite{65}.

**Nonreciprocity.**— In the numerical studies, we focused on statistically reciprocal models as illustrative examples. Nonreciprocity can give rise to unique non-Hermitian topology \cite{39, 57} and further change the universal critical properties. Nevertheless, our correspondence of the ATs in Hermitian and non-Hermitian systems should remain valid even in the presence of nonreciprocity since it is based solely on Hermitization. We conjecture that even if nonreciprocity changes critical behavior of the ATs in non-Hermitian systems because of an additional mechanism such as topology, the critical behavior in the corresponding Hermitian systems should also change by the same mechanism and thus coincides with the non-Hermitian counterpart. As an example of this, the ATs in one-dimensional nonreciprocal systems are characterized by \(\nu = 1\) \cite{33, 44}, which coincides with the critical behavior in the corresponding Hermitized systems \cite{79, 80}.
The conjecture can be argued for the case of symmetry-conserving energy [65]. It is worthwhile to further confirm our correspondence for higher-dimensional nonreciprocal systems.

**Summary and concluding remarks.**— In this paper, we propose a correspondence of the ATs between Hermitian and non-Hermitian systems. The 38-fold non-Hermitian symmetry class is mapped to the 10-fold Hermitian symmetry class in terms of the universal scaling properties of the length scale. Consequently, superuniversality emerges in non-Hermitian systems: the ATs in several distinct symmetry classes share the same universal scaling properties around their critical points. To test this correspondence, we study the ATs in classes AII, AIII, and DIII in 2D and 3D, and estimate the critical exponents by the transfer matrix method. The estimated critical exponents are consistent with the correspondence and superuniversality in non-Hermitian disordered systems. From the correspondence, we also provide useful information of the unknown critical exponents for 2D class AIII and 3D class CII in Hermitian systems. Investigating non-Hermitian systems is a new efficient way to study critical behavior of the ATs in Hermitian systems since non-Hermitian matrices are often half the size of the corresponding Hermitian matrices. We note that conformal invariance [81–83] should emerge at the ATs in 2D non-Hermitian systems from the correspondence. The multifractality at the ATs [84] in 2D and 3D non-Hermitian systems should also be unified with the Hermitian counterparts.

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SUPPLEMENTARY MATERIAL FOR “UNIFYING THE ANDERSON TRANSITIONS IN HERMITIAN AND NON-HERMITIAN SYSTEMS”

SYMMETRY

Non-Hermiticity ramifies and unifies symmetries in Hermitian physics \[57, 58\]. In non-Hermitian systems, symmetries are defined by

\[
\begin{align*}
\text{time-reversal symmetry (TRS)} & : \quad U_{\tau_x} \hat{H}^\dagger U_{\tau_x}^\dagger = \hat{H}, \quad U_{\tau_x} U_{\tau_x}^\dagger = \pm 1, \\
\text{particle-hole symmetry (PHS)} & : \quad U_{P_+} \hat{H} U_{P_+}^\dagger = - \hat{H}, \quad U_{P_+} U_{P_+}^\dagger = \pm 1, \\
\text{time-reversal symmetry}^\dagger \ (\text{TRS}^\dagger) & : \quad U_{\tau_x} \hat{H}^\dagger U_{\tau_x} = \hat{H}, \quad U_{\tau_x} U_{\tau_x}^\dagger = \pm 1, \\
\text{particle-hole symmetry}^\dagger \ (\text{PHS}^\dagger) & : \quad U_{P_+} \hat{H}^\dagger U_{P_+} = - \hat{H}, \quad U_{P_+} U_{P_+}^\dagger = \pm 1, \\
\text{chiral symmetry (CS)} & : \quad U_C \hat{H} U_C^\dagger = - \hat{H}, \quad U_C^2 = 1, \\
\text{sublattice symmetry (SLS)} & : \quad U_S \hat{H} U_S^\dagger = - \hat{H}, \quad U_S^2 = 1,
\end{align*}
\]

where \(U_{\tau_x}, U_{\tau_x}, U_C,\) and \(U_S\) are unitary matrices. TRS and PHS\(^\dagger\) are unified: if \(\hat{H}\) respects TRS, \(i\hat{H}\) respects PHS\(^\dagger\), and vice versa \[58\]. It is useful to group the non-Hermitian symmetry classes according to the number \(N\) of independent anti-unitary symmetries (TRS, PHS, TRS\(^\dagger\), and PHS\(^\dagger\)) \[57\]. In the simultaneous presence of multiple symmetries, the commutation or anticommutation relations between them are relevant.

Hermitization is a powerful tool to analyze non-Hermitian systems. It is relevant to understand topological phases of non-Hermitian systems \[57, 58\]. In this work, we provide a universal understanding about the Anderson transitions (ATs) by Hermitization. Here, we will explain in detail some of the mappings between the 38 non-Hermitian symmetry classes and 10 Hermitian symmetry classes, and relate the localization properties of non-Hermitian systems with those in Hermitian systems. The comprehensive correspondence between the 38-fold non-Hermitian symmetry class and 10-fold symmetry class is summarized in Table II and Table III. Note also that the correspondence table was previously derived also in the context of the point-gap classification of non-Hermitian topological phases \[57\].

Hermitization is a local operation which can be regarded as a local introduction of a sublattice structure. When a non-Hermitian Hamiltonian only has short-range hoppings in \(d\)-dimensional space, so does the Hermitian Hamiltonian obtained from Hermitization. Eigenmodes of the non-Hermitian Hamiltonian are the same as zero modes of the Hermitized Hamiltonian; both eigenmodes share the same localization properties. Thus, universal critical properties of disorder-driven quantum phase transitions are shared by the non-Hermitian Hamiltonians and Hermitian Hamiltonians that are related to each other through Hermitization.

Hermitization maps a non-Hermitian Hamiltonian to a Hermitian Hamiltonian with CS (SLS). When the original non-Hermitian Hamiltonian respects CS or SLS, the Hermitized Hamiltonian commutes with a unitary matrix and can be further block diagonalized. Even in the presence of disorder, the Hermitized Hamiltonian can be block diagonalized, given that disorder does not break any symmetries. The relevant symmetry class (classifying space) of the Hermitized Hamiltonian is determined by the symmetry class of its irreducible block, not by the symmetries of the Hermitized Hamiltonian itself.

In the following, we summarize relevant symmetry classes of the Hermitized Hamiltonians for several non-Hermitian symmetry classes in an elementary manner. For clarity of the following description, we always use \(\hat{x}\) for a Hermitian matrix and \(x\) for a non-Hermitian matrix (\(x\) can be different symbols).

Class AIII

Non-Hermitian Hamiltonians \(\hat{H}\) in class AIII respect CS: \(CH^\dagger C = -\hat{H}\) with a unitary matrix \(C\) satisfying \(C^2 = 1\) and \(C^\dagger = C\). A Hermitian Hamiltonian is introduced through Hermitization,

\[
\hat{H} := \begin{pmatrix} 0 & \hat{H} \\ \hat{H}^\dagger & 0 \end{pmatrix},
\]

where the reference energy is set to \(E = 0\) without loss of generality. CS of the non-Hermitian Hamiltonian \(\hat{H}\) leads to CS of the Hermitized Hamiltonian \(\hat{H}\):

\[
(\tau_x \otimes C) \hat{H} (\tau_x \otimes C)^{-1} = -\hat{H},
\]
TABLE II. 38-fold non-Hermitian symmetry class (NHSC). Any universality class of disorder-driven quantum phase transitions in a NHSC can be realized in the corresponding Hermitian symmetry class (HSC). The blank entries mean the absence of the symmetries. For the anti-unitary symmetries (i.e., TRS, PHS, TRS†, and PHS†), the entries ±1 denote the signs of the symmetry operations. N is the number of the independent anti-unitary symmetries. A symmetry class labelled as ...+S± or ...
+ S±± has sublattice symmetry (SLS) in addition to the symmetries in the symmetry class specified by “...”. For N = 0, the subscript of S± specifies the commutation (+1) or anti-commutation (−1) relation between SLS and chiral symmetry (CS). For N = 2, the subscript of S± specifies the commutation (+1) or anti-commutation (−1) relation between SLS and TRS or PHS. For N = 3, only three anti-unitary symmetries are independent of one another, where the first subscript of S±± specifies the commutation or anti-commutation relation between SLS and TRS, and the second one specifies the commutation or anti-commutation relation between SLS and PHS. In the presence of both CS and SLS, the commutation or anti-commutation relation between CS and SLS is respectively specified by + or − in the column of [C, S]± = 0.

| NHSC | TRS (T+) | PHS (P-) | TRS†(P+) | PHS†(T-) | CS (C) | SLS (S) | [C, S]±=0 | HSC |
|------|----------|----------|----------|----------|--------|---------|----------|-----|
| N=0  |          |          |          |          |        |         |          |     |
| A    |          |          |          |          |        |         |          | AHH |
| AHH  |          |          |          |          |        |         |          | AIII|
| AIII†(A+S) |          |          |          |          |        |         |          | AIII|
| AIII + S+ |          |          |          |          |        |         |          | AIII|
| AIII + S− |          |          |          |          |        |         |          | A  |
| N=1  |          |          |          |          |        |         |          |     |
| Al   | 1        |          |          |          |        |         |          | BDI |
| Al†  |          |          |          |          |        |         |          | CII |
| D    | -1       |          |          |          |        |         |          | DIII|
| C    |          |          |          |          |        |         |          | CI  |
| Al†  |          |          |          |          |        |         |          | CI  |
| Al†  | -1       |          |          |          |        |         |          | DIII|
| N=2  |          |          |          |          |        |         |          |     |
| BDI  | 1        | 1        |          |          |        |         |          | D   |
| CI   | 1        |          |          |          |        |         |          | AI  |
| DIII | -1       | 1        |          |          |        |         |          | CII |
| CH   |          | -1       |          |          |        |         |          | C   |
| BDI† |          |          |          |          |        |         |          | AI  |
| CI†  | -1       |          |          |          |        |         |          | D   |
| DIII†| -1       | -1       |          |          |        |         |          | DIII|
| CH†  |          |          |          |          |        |         |          | CII |
| D + S+ | 1      | 1        |          |          |        |         |          | AI  |
| C + S− | -1     | 1        |          |          |        |         |          | CI  |
| D + S− | 1      | -1       |          |          |        |         |          | DIII|
| C + S+ | -1     | -1       |          |          |        |         |          | AI  |
| Al + S+ | 1  | 1        |          |          |        |         |          | BDI |
| Al + S− | 1  | -1       |          |          |        |         |          | AIII|
| Al† + S− | -1 |          |          |          |        |         |          | CI  |
| N=3  |          |          |          |          |        |         |          |     |
| BDI + S++ | 1  | 1        | 1        | 1        |        |         |          | BDI |
| BDI + S−− | 1  | 1        | -1       | -1       |        |         |          | DIII|
| DIII + S++ | -1 | 1        | 1        | -1       |        |         |          | CII |
| Cl + S++ | 1      | -1       | -1       | -1       |        |         |          | BDI |
| Cl + S−− | 1      | -1       | 1        | 1        |        |         |          | CI  |
| CH + S++ | -1     | -1       | -1       | -1       |        |         |          | CII |
| BDI + S−− | 1  | 1        | -1       | 1        |        |         |          | D   |
| BDI + S++ | 1  | 1        | -1       | 1        |        |         |          | AI  |
| DIII + S−− | -1 | 1        | -1       | 1        |        |         |          | A   |
| Cl + S++ | 1      | -1       | 1        | 1        |        |         |          | A   |
| Cl + S−− | 1      | -1       | -1       | -1       |        |         |          | C   |
| CH + S+ | -1     | -1       | 1        | -1       |        |         |          | C   |
where $\tau_z$ is the Pauli matrix that exchanges $\mathcal{H}$ and $\mathcal{H}^\dagger$ in Eq. (S.7). By the construction of Hermitization, $\tilde{\mathcal{H}}$ also respects another CS:

$$\tau_z \tilde{\mathcal{H}} \tau_z^{-1} = -\tilde{\mathcal{H}}.$$  \hfill (S.9)

The simultaneous presence of the two CSs allows $\tilde{\mathcal{H}}$ to commute with a unitary symmetry:

$$(\tau_y \otimes C) \tilde{\mathcal{H}} (\tau_y \otimes C)^{-1} = \mathcal{H}.$$  \hfill (S.10)

The unitary symmetry enables block diagonalization of $\tilde{\mathcal{H}}$.

The CS operator can be written as $\mathcal{C} = \mathcal{C}_+ - \mathcal{C}_-$ with the projection operators $\mathcal{C}_+$ and $\mathcal{C}_-$ on subspaces in which eigenenergies of $\mathcal{C}$ are $+1$ and $-1$, respectively. Here are some useful properties of the projection operators:

$$C_+ + C_- = 1, \quad C_+ C_- = 0, \quad C_-^2 = C_+, \quad C_+^2 = C_-, \quad C_+ C = CC_+ = C_+, \quad C_- C = CC_- = -C_-.$$  \hfill (S.11)

Then, we have

$$C_+ \mathcal{H}^\dagger C_+ = -C_+ (\mathcal{H} C) C_+ = -C_+ \mathcal{H} C_+, \quad (C_- \mathcal{H} C_+)^\dagger = C_+ \mathcal{H}^\dagger C_- = -C_+ (\mathcal{H} C) C_- = C_+ \mathcal{H} C_-.$$  \hfill (S.12, 13)

In the basis that diagonalizes $\mathcal{C}$, the Hamiltonian $\mathcal{H}$ can be written as

$$\mathcal{H} = \begin{pmatrix} \tilde{h}_1 & h_{12} \\ h_{12}^\dagger & \tilde{h}_2 \end{pmatrix},$$  \hfill (S.14)

where the matrices $\tilde{h}_1$, $\tilde{h}_2$, and $h_{12}$ satisfy

$$\tilde{h}_1^\dagger = \tilde{h}_1, \quad \tilde{h}_2^\dagger = \tilde{h}_2, \quad C_+ \mathcal{H} C_+ = \tilde{h}_1, \quad C_- \mathcal{H} C_- = \tilde{h}_2, \quad C_+ \mathcal{H} C_- = h_{12}, \quad C_- \mathcal{H} C_+ = h_{21} = h_{12}^\dagger.$$  \hfill (S.15)

The following argument holds true for arbitrary dimensions of $\mathcal{C}_+$ and $\mathcal{C}_-$. For simplicity, let us assume that $\mathcal{C}_+$ and $\mathcal{C}_-$ have the same dimension and take $\mathcal{C} = \sigma_z$. In this basis, we can block diagonalize the Hermitized Hamiltonian $\tilde{\mathcal{H}}$ by

$$\tilde{\mathcal{H}} = \begin{pmatrix} 0 & 0 & \tilde{h}_1 & h_{12} \\ 0 & 0 & h_{12}^\dagger & \tilde{h}_2 \\ -\tilde{h}_1 & h_{12} & 0 & 0 \\ h_{12}^\dagger & -\tilde{h}_2 & 0 & 0 \end{pmatrix} \rightarrow \mathcal{U} \tilde{\mathcal{H}} \mathcal{U}^\dagger = \begin{pmatrix} -\tilde{h}_1 & -i h_{12} & 0 & 0 \\ i h_{12} & \tilde{h}_2 & 0 & 0 \\ 0 & 0 & \tilde{h}_1 & i h_{12} \\ 0 & 0 & -i h_{12}^\dagger & -\tilde{h}_2 \end{pmatrix},$$  \hfill (S.16)

where $\mathcal{U}$ is the following unitary matrix:

$$\mathcal{U} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -i & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & i & 0 \\ 0 & 1 & 0 & -i \end{pmatrix}. \hfill (S.17)$$

The two irreducible blocks have no symmetry except for Hermiticity,

$$\tilde{\mathcal{H}}_{\text{block}} := \begin{pmatrix} -\tilde{h}_1 & -i h_{12} \\ i h_{12}^\dagger & \tilde{h}_2 \end{pmatrix}. \hfill (S.18)$$

Thus, the relevant symmetry class of the Hermitized Hamiltonian $\tilde{\mathcal{H}}$ obtained from the non-Hermitian Hamiltonian $\mathcal{H}$ in class AIII is class A.

**Class CII$^\dagger$**

Non-Hermitian Hamiltonians $\mathcal{H}$ in class CII$^\dagger$ have TRS$^\dagger$ $\mathcal{U}_{P_z} \mathcal{H}^\dagger \mathcal{U}_{P_z}^\dagger = \mathcal{H}$ with $\mathcal{U}_{P_z} \mathcal{U}_{P_z}^\dagger = -1$ and PHS$^\dagger$ $\mathcal{U}_{\mathcal{T}} \mathcal{H}^* \mathcal{U}_{\mathcal{T}}^\dagger = -\mathcal{H}$ with $\mathcal{U}_{\mathcal{T}} \mathcal{U}_{\mathcal{T}}^\dagger = -1$. The combination of TRS$^\dagger$ and PHS$^\dagger$ gives CS. From the above argument for class AIII, we can write $\mathcal{H}$ into Eq. (S.14) in the basis that diagonalizes $\mathcal{C} = \sigma_z$. Here, we choose the symmetry operators to be $\mathcal{U}_{P_z} = \sigma_0 \otimes \mathcal{V}$ and $\mathcal{U}_{\mathcal{T}} = \sigma_z \otimes \mathcal{V}$ with $\mathcal{V} \mathcal{V}^* = -1$. A unitary matrix $\mathcal{V}$ respects

$$\mathcal{V} \tilde{h}_1^\dagger \mathcal{V}^\dagger = \tilde{h}_1, \quad \mathcal{V} \tilde{h}_2^\dagger \mathcal{V}^\dagger = \tilde{h}_2, \quad \mathcal{V} h_{12} \mathcal{V}^\dagger = h_{12},$$  \hfill (S.19)

where $\tilde{h}_1$, $\tilde{h}_2$, and $h_{12}$ are defined by Eq. (S.14). Thus, the Hermitian block given by Eq. (S.18) respects TRS, ($\sigma_z \otimes \mathcal{V}$) $\tilde{\mathcal{H}}_{\text{block}}$ $(\sigma_z \otimes \mathcal{V})^{-1} = \tilde{\mathcal{H}}_{\text{block}}$, and belongs to class AII.
Class DIII

Non-Hermitian Hamiltonians $\mathcal{H}$ in class DIII have TRS $U_{\tau_x}^* \mathcal{H} U_{\tau_x}^\dagger = \mathcal{H}$ with $U_{\tau_x} U_{\tau_x}^\dagger = -1$ and PHS $U_{\mathcal{P}}^\dagger \mathcal{H} U_{\mathcal{P}} = -\mathcal{H}$ with $U_{\mathcal{P}}^\dagger U_{\mathcal{P}} = 1$. The combination of TRS and PHS gives CS. From the above argument for class AIII, we can write $\mathcal{H}$ into Eq. (S.14) in the diagonal basis of $C = \sigma_z$. Here, we choose the symmetry operators to be $U_{\tau_x} = \sigma_y \otimes \mathcal{V}$ and $U_{\mathcal{P}} = \sigma_x \otimes \mathcal{V}$ with $\mathcal{V}^\dagger \mathcal{V} = 1$. A unitary matrix $\mathcal{V}$ respects $\mathcal{V} \tilde{h}_1 \mathcal{V}^\dagger = -\tilde{h}_2$, $\mathcal{V} \tilde{h}_2 \mathcal{V}^\dagger = -\tilde{h}_1$, and $\mathcal{V} h_{12} \mathcal{V}^\dagger = -h_{12}$, \hfill (S.20)

where $\tilde{h}_1$, $\tilde{h}_2$, and $h_{12}$ are defined by Eq. (S.14). Thus, the Hermitian block given in Eq. (S.18) respects TRS, $(\sigma_y \otimes \mathcal{V}) \tilde{H}_{\text{block}} (\sigma_y \otimes \mathcal{V})^{-1} = \tilde{H}_{\text{block}}$. The irreducible block of the Hermitized Hamiltonian belongs to class AII.

Non-Hermitian Hamiltonians with anti-commutative CS and SLS: class AIII + $S_-$

Let us consider a generic non-Hermitian Hamiltonian with CS and SLS which anti-commute with each other. We can choose the symmetry operators to be $U_C = \sigma_y$ and $U_S = \sigma_z$ in a certain basis. In this basis, the non-Hermitian Hamiltonian takes a form of

$$
\mathcal{H} = \begin{pmatrix} 0 & h_1 \\ h_2 & 0 \end{pmatrix}
$$

with $h_1 = h_1^\dagger$ and $h_2 = h_2^\dagger$. Then, the Hermitized Hamiltonian $\tilde{\mathcal{H}}$ can be block diagonalized

$$
\tilde{\mathcal{H}} = \begin{pmatrix} 0 & 0 & 0 & h_1 \\ 0 & 0 & h_2 & 0 \\ 0 & h_2 & 0 & 0 \\ h_1 & 0 & 0 & 0 \end{pmatrix} \rightarrow \mathcal{U} \tilde{\mathcal{H}} \mathcal{U}^\dagger = \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & -h_1 & 0 & 0 \\ 0 & 0 & h_2 & 0 \\ 0 & 0 & 0 & -h_2 \end{pmatrix},
$$

(S.22)

by the unitary matrix

$$
\mathcal{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}.
$$

(S.23)

Note that $h_1$ and $h_2$ have no symmetry restrictions other than Hermiticity. Thus, the relevant symmetry class of the Hermitized Hamiltonian is class A.

Properties of Hermitized Hamiltonians

Though non-Hermitian Hamiltonian $\mathcal{H}$ in Eq. (1) in the main text takes a general form of a non-Hermitian symmetry class, it is not obvious whether irreducible blocks of the Hermitized Hamiltonian take a general Hamiltonian form in its corresponding Hermitian symmetry class. To complement this point partially (see a subsection of 'symmetry-conserving energy and symmetry-breaking energy' below for why we use “partially” here), in the following, we start from a general Hermitian Hamiltonian $\tilde{\mathcal{H}}$ of a Hermitian symmetry class, an energy $\tilde{E}$ that respects all the symmetries of $\tilde{\mathcal{H}}$, and eigenmodes $|\psi\rangle$ at $\tilde{E}$,

$$
\tilde{\mathcal{H}}|\psi\rangle = \tilde{E}|\psi\rangle.
$$

(S.24)

Then, for each of its corresponding non-Hermitian symmetry classes, we introduce a non-Hermitian Hamiltonian $\mathcal{H}$ of the symmetry class, an energy $E$ that respects all the symmetries of the symmetry class of $\mathcal{H}$ ('symmetry-conserving energy'), and right eigenmodes $|\phi_r\rangle$ at the symmetry-conserving energy $E$,

$$
\mathcal{H}|\phi_r\rangle = E|\phi_r\rangle.
$$

(S.25)

We show that $|\phi_r\rangle$ is given by $|\psi\rangle$; $|\phi_r\rangle$ and $|\psi\rangle$ share the same localization properties. Importantly, the non-Hermitian Hamiltonian $\mathcal{H}$ thus introduced takes a generic Hamiltonian form under the symmetries of the symmetry class. Here,
TABLE III. 10-fold Hermitian symmetry class (HSC) and 38-fold non-Hermitian symmetry class (NHSC). Any universality class of disorder-driven quantum phase transitions in a Hermitian symmetry class can be realized in the corresponding non-Hermitian symmetry classes. TRS, PHS, and CS in the Hermitian symmetry classes are time-reversal, particle-hole, and chiral symmetries, respectively. $T_+$, $P_-$, $P_+$, $T_-$, $C$, and $S$ in the NHSCes are time-reversal, particle-hole, time-reversal†, particle-hole†, chiral, and sublattice symmetries defined in Eqs. (S.1)-(S.6). The blank entries, ±1 entries, ± entries, and the names of the 38-fold NHSC are defined in the caption of Table II. “*” in the remark means that the dimension of random Hermitian matrices must be even, which is always the case with the symplectic class, three chiral classes, and four BdG classes. “**” in the remark means that the dimension of the Hermitian matrices must be a multiple of 4.

| HSC | TRS | PHS | CS | NHSC | $T_+$ | $P_-$ | $P_+$ | $T_-$ | $C$ | $S$ | $C,S|_{\pm}=0$ | remark |
|-----|-----|-----|----|------|------|------|------|------|----|----|----------------|-------|
| unitary | $A$ | $A$ | $A$ | $A$ | 1 | 1 | 1 | -1 | $\checkmark$ | $\checkmark$ | $\checkmark$ | - | |
| orthogonal | $A$ | 1 | $B$ | $C$ | 1 | -1 | 1 | $\checkmark$ | - | |
| symplectic | $A$ | -1 | $B$ | $C$ | -1 | 1 | -1 | $\checkmark$ | - | |
| chiral unitary | $A$ | 1 | $B$ | $C$ | 1 | 1 | 1 | $\checkmark$ | $\checkmark$ | + | $*$ | |
| chiral orthogonal | $B$ | 1 | $B$ | $C$ | 1 | -1 | 1 | $\checkmark$ | $\checkmark$ | + | $*$ | |
| chiral symplectic | $C$ | -1 | $C$ | $C$ | -1 | 1 | 1 | $\checkmark$ | $\checkmark$ | + | $*$ | |
| BdG | $D$ | 1 | $D$ | $D$ | 1 | 1 | 1 | $\checkmark$ | - | |
| | $D$ | -1 | $D$ | $D$ | -1 | 1 | 1 | $\checkmark$ | + | $*$ | |
| | $C$ | 1 | $C$ | $C$ | 1 | -1 | 1 | $\checkmark$ | - | |

by ‘takes a generic Hamiltonian form’, we mean that when $\tilde{\mathcal{H}}$ takes all the possible Hamiltonian defined by the symmetries of the Hermitian symmetry class, $\mathcal{H}$ takes all the possible Hamiltonian defined by the symmetries of the non-Hermitian symmetry class, e.g. compare Eq. (S.27) with Eq. (S.26).

The 38-fold non-Hermitian symmetry class contains more symmetry classes than the 10-fold Hermitian symmetry class. Consequently, a set of different non-Hermitian symmetry classes can be mapped to the same Hermitian symmetry class by the Hermitization. Table III summarizes all the corresponding non-Hermitian symmetry classes for the 10 Hermitian symmetry classes. A similar correspondence between classifying spaces was obtained on the basis of the $K$-theory [57].
Symmetry-conserving energy and symmetry-breaking energy

Note that the following argument focuses only on those cases where the energy $E$ respects all the symmetries of $\mathcal{H}$ (symmetry-conserving energy case). The argument does not cover the other cases where $E$ breaks (some) symmetries of $\mathcal{H}$ (symmetry-breaking energy case). In the latter cases, $\mathcal{H}$ together with $E$ belongs to a given non-Hermitian symmetry class, while $\mathcal{H}$ itself has higher symmetries than the symmetry class and $E$ breaks some of the symmetries of $\mathcal{H}$ such that $\mathcal{H}$ with $E$ belongs to the symmetry class. For example, eigenmodes of non-Hermitian class-AI (or AII) Hamiltonian at complex energy $E \neq E^*$ belong to non-Hermitian symmetry class A. According to the Hermitization, the class-AI (or AII) Hamiltonian with the complex energy maps to a Hermitian Hamiltonian whose relevant symmetry class is Hermitian class AIII. The mapping is related to numerical results in the 2D class AII model with $E \neq E^*$.

In spite of the presence of such mapping, the argument in this section does not discuss how to construct a generic non-Hermitian class-AI (or AII) Hamiltonian whose eigenmodes at the complex energy are given by zero modes of a general Hermitian class-AIII Hamiltonian. Due to the lack of arguments for symmetry-breaking energy cases, we regard the proposed correspondence as a conjecture instead of as a mathematically-proven fact.

In the next subsection, we first begin with Hermitian class A case, whose corresponding non-Hermitian symmetry classes are class AIII, AIII + $S_-$, BDI + $S_+$, and CI + $S_-$. A general Hermitian class-A Hamiltonian $\tilde{\mathcal{H}}_A$ is given in Eq. (S.26), while Hamiltonians in these non-Hermitian symmetry classes $\mathcal{H}$ are given by Eqs. (S.27,S.30,S.32,S.34) respectively. The symmetry-conserving energies $E$ in these four non-Hermitian classes are $E = iE_1$ with $E_1 \in \mathbb{R}$, $E = 0$, $E = 0$, and $E = 0$ respectively.

**Hermitian class A**

A general Hermitian Hamiltonian in class A can always be written as

$$\tilde{\mathcal{H}}_A = \begin{pmatrix} \tilde{h}_1 & \tilde{h}_{12} \\ \tilde{h}_{12} & \tilde{h}_2 \end{pmatrix}, \quad (S.26)$$

with $\tilde{h}_1, \tilde{h}_2 \in \mathbb{R}$, and $\tilde{h}_1, \tilde{h}_2, \tilde{h}_{12} \neq 0$. Let $n_1$ and $n_2$ be the dimensions of the square matrices $\tilde{h}_1$ and $\tilde{h}_2$, respectively. In general, $n_1$ and $n_2$ can be different. Let $\tilde{E} \in \mathbb{R}$ be an eigenenergy of $\tilde{\mathcal{H}}_A$ and $|\psi\rangle$ be the corresponding eigenstate.

**From Hermitian class A to non-Hermitian class AIII:** For the given Hermitian Hamiltonian $\tilde{\mathcal{H}}_A$ in Eq. (S.26), we construct a non-Hermitian Hamiltonian $\mathcal{H}$ in class AIII, which takes a generic form of class AIII Hamiltonians and whose eigenmode at symmetry-conserving energy shares the same localization properties with $|\psi\rangle$ of $\tilde{\mathcal{H}}_A$. Here the symmetry-conserving energy of class AIII is pure imaginary, $E = iE_1$ and $E_1 \in \mathbb{R}$. Then, consider a non-Hermitian Hamiltonian $\mathcal{H}$ and its chiral symmetry by

$$\text{AIII} : \mathcal{H} \equiv \begin{pmatrix} -i(\tilde{h}_1 - \tilde{E}) + iE_1 & i\tilde{h}_{12} \\ -i\tilde{h}_{12} & i(\tilde{h}_2 - \tilde{E}) + iE_1 \end{pmatrix}, \quad \mathcal{U}_C \equiv \begin{pmatrix} 1_{n_1 \times n_1} & 0 \\ 0 & -1_{n_2 \times n_2} \end{pmatrix}, \quad (S.27)$$

The non-Hermitian Hamiltonian $\mathcal{H}$ respects chiral symmetry $\mathcal{H} = -\mathcal{U}_C \mathcal{H} \mathcal{U}_C^\dagger$, and hence belongs to class AIII. Moreover, $\mathcal{H}$ has a right eigenstate at $E = iE_1$, that share the same localization properties as $|\psi\rangle$ of $\tilde{\mathcal{H}}_A$. In fact, $\mathcal{U}_C |\psi\rangle$ is such an eigenstate,

$$\mathcal{H} \mathcal{U}_C |\psi\rangle = iE_1 \mathcal{U}_C |\psi\rangle. \quad (S.28)$$

Since $\mathcal{U}_C$ is a local transformation, $\mathcal{U}_C |\psi\rangle$ has the same localization properties as $|\psi\rangle$. Under the chiral symmetry defined in Eq. (S.27), non-Hermitian Hamiltonians in class AIII take following general form,

$$\mathcal{H}_{AIII} = \begin{pmatrix} i\tilde{g}_1 & i\tilde{g}_{12} \\ -i\tilde{g}_{12} & i\tilde{g}_2 \end{pmatrix}, \quad (S.29)$$

with $\tilde{g}_1, \tilde{g}_2 \in \mathbb{R}$. Since $\tilde{E}, E_1 \in \mathbb{R}$, $\mathcal{H}$ in Eq. (S.27) indeed takes this general form.
From Hermitian class $A$ to non-Hermitian class $AIII + S_-$: Let us introduce two random Hermitian Hamiltonians in class $A$ defined by Eq. (S.26), $\hat{H}_A$ and $\hat{H}'_A$, which have the same dimension and are independent of each other. For example, one can pick up two different disorder realizations in the same Hermitian lattice model with the same disorder strength in class $A$. Let $\hat{E}$ and $\hat{E}'$ be real eigenenergies of $\hat{H}_A$ and $\hat{H}'_A$, and $|\psi\rangle$ and $|\psi'\rangle$ be the corresponding eigenstates, respectively. Then, we introduce a non-Hermitian Hamiltonian $\hat{H}$ of class $AIII + S_-$ together with its CS and SLS by

$$AIII + S_− : \hat{H} = \begin{pmatrix} 0 & \hat{H}'_A - \hat{E}' \\ \hat{H}_A - \hat{E} & 0 \end{pmatrix}, \quad U_C = \tau_y, \quad U_S = \tau_z.$$

(3.30)

Here, $\tau_y$ and $\tau_z$ are Pauli matrices. Thus, $\hat{H}$ respects $U_C\hat{H}U_C^\dagger = \hat{H}$ and $U_S\hat{H}U_S^\dagger = \hat{H}$ with $[U_C, U_S]_− = 0$, and hence belongs to class $AIII + S_−$. The symmetry-conserving energy of class $AIII + S_−$ is zero, $E = 0$. $\hat{H}$ in Eq. (3.30) has right eigenmodes at the zero energy, $(0 \ | \psi\rangle)^T$ and $(0 \ | \psi'\rangle)^T$, which have the same localization properties as $|\psi\rangle$ and $|\psi'\rangle$ of $\hat{H}_A$ and $\hat{H}'_A$. Under the chiral and sublattice symmetries in Eq. (3.30), non-Hermitian Hamiltonians in class $AIII + S_−$ take following general form,

$$\hat{H}_{AIII+S_−} = \begin{pmatrix} 0 & \hat{G}_A \\ \hat{G}_A^* & 0 \end{pmatrix} ,$$

(3.31)

with two independent Hermitian Hamiltonians $\hat{G}_A$ and $\hat{G}_A^*$. Since $\hat{H}_A$ and $\hat{H}'_A$ in Eq. (3.30) are two independent Hermitian class-A Hamiltonians, Eq. (3.30) takes the general form of Eq. (3.31).

From Hermitian class $A$ to non-Hermitian class $BDI + S_−$: For a generic Hermitian Hamiltonian $\hat{H}_A$, we introduce a non-Hermitian Hamiltonian $\hat{H}$ in class $BDI + S_−$ together with its SLS, TRS, and PHS by

$$BDI + S_− : \hat{H} = \begin{pmatrix} 0 & i(\hat{H}_A - \hat{E}) \\ -i(\hat{H}_A - \hat{E})^T & 0 \end{pmatrix}, \quad U_S = \tau_z, \quad U_{T_\pi} = \tau_x, \quad U_{T_π} = \tau_0.$$

(3.32)

Here, $\tau_x$ and $\tau_z$ are Pauli matrices. The non-Hermitian Hamiltonian $\hat{H}$ respects $U_S\hat{H}U_S^\dagger = -\hat{H}$, $U_{T_\pi}\hat{H}U_{T_\pi}^\dagger = \hat{H}$, and $U_{T_π}\hat{H}U_{T_π}^T = -\hat{H}$ with $U_{T_\pi}U_{T_\pi}^T = +1$, $U_{T_π}U_{T_π}^* = -1$, and $[U_S, U_{T_\pi}]_- = [U_S, U_{T_π}]_+ = 0$. Thus, $\hat{H}$ belongs to class $BDI + S_−$. Symmetry-conserving energy of class $BDI + S_−$ is zero, $E = 0$. $\hat{H}$ in Eq. (3.32) has right eigenmodes at the zero energy, $(|\psi\rangle^* \ 0)^T$ and $(0 \ |\psi\rangle)^T$, which have the same localization properties as the original state $|\psi\rangle$ of $\hat{H}_A$. Under the symmetries in Eq. (3.32), non-Hermitian Hamiltonians in class $BDI + S_−$ take following general form,

$$\hat{H}_{BDI+S_−} = \begin{pmatrix} 0 & i\hat{G}_A \\ -i\hat{G}_A^* & 0 \end{pmatrix} ,$$

(3.33)

with general Hermitian Hamiltonian $\hat{G}_A$. With $\hat{H}_A$ in Eq. (3.26), Eq. (3.32) takes this general form.

From Hermitian class $A$ to non-Hermitian class $CI + S_−$: For a general Hermitian Hamiltonian $\hat{H}_A$ in class $A$, we introduce a non-Hermitian Hamiltonian $\hat{H}$ in class $CI + S_−$ together with its SLS, TRS, and PHS by

$$CI + S_− : \hat{H} = \begin{pmatrix} 0 & \hat{H}_A - \hat{E} \\ -\sigma_y(\hat{H}_A - \hat{E})^* \sigma_y & 0 \end{pmatrix}, \quad U_S = \tau_z, \quad U_{T_π} = \tau_y \otimes \sigma_y, \quad U_{T_π} = \sigma_y.$$

(3.34)

Then, $\hat{H}$ respects $U_S\hat{H}U_S^\dagger = -\hat{H}$, $U_{T_π}\hat{H}U_{T_π}^\dagger = \hat{H}$, and $U_{T_π}\hat{H}U_{T_π}^T = -\hat{H}$ with $U_{T_π}U_{T_π}^T = +1$, $U_{T_π}U_{T_π}^* = -1$, and $[U_S, U_{T_π}]_- = [U_S, U_{T_π}]_+ = 0$. Thus, $\hat{H}$ belongs to class $CI + S_−$. Here we assume that the dimension of $\hat{H}_A$ is even so that $\hat{H}_A$ can be regarded as a Hamiltonian with local pseudospin-$\frac{1}{2}$ degree of freedom, $\sigma_y$ in Eq. (3.34) flips the spin locally. Symmetry-conserving energy of class $CI + S_−$ is zero, $\tilde{E} = 0$. $\hat{H}$ in Eq. (3.34) has right eigenmode at the zero energy, $(\sigma_y |\psi\rangle^* \ 0)^T$ and $(0 \ |\psi\rangle)^T$, which have the same localization properties as the original state $|\psi\rangle$ of $\hat{H}_A$. Under the symmetries in Eq. (3.34), non-Hermitian Hamiltonians in class $CI + S_−$ take following general form,

$$\hat{H}_{CI+S_−} = \begin{pmatrix} 0 & \hat{G}_A \\ -\sigma_y\hat{G}_A^* \sigma_y & 0 \end{pmatrix} ,$$

(3.35)

with general Hermitian Hamiltonian $\hat{G}_A$. Eq. (3.34) takes this general form.
In summary, for a generic Hermitian class-A Hamiltonian $\hat{H}_A$ and its eigenstate $|\psi\rangle$, we construct a non-Hermitian Hamiltonian $\hat{\mathcal{H}}_A$ in its corresponding symmetry classes whose right eigenmode $|\phi_r\rangle$ at symmetry-conserving energy $E$ has the same localization properties as $|\psi\rangle$ of $\hat{H}_A$. The non-Hermitian Hamiltonian thus introduced takes a generic Hamiltonian form of respective non-Hermitian symmetry classes.

In the next subsection, we consider Hermitian class AI case, whose corresponding non-Hermitian symmetry classes are class CI, BDI, and CI + $S_+$. A general Hermitian class-AI Hamiltonian $\hat{H}_{\text{AI}}$ is given in Eq. (S.36), while Hamiltonians in these non-Hermitian symmetry classes $\mathcal{H}$ are given by Eqs. (S.38, S.40, S.43) respectively. The symmetry-conserving energies $E$ in these non-Hermitian classes are $E = 0$, $E = iE_1$ with $E_1 \in \mathbb{R}$, and $E = 0$ respectively.

**Hermitian class AI**

A generic Hermitian Hamiltonian $\hat{H}_{\text{AI}}$ in class AI respects TRS $\mathcal{V}_T \hat{H}_{\text{AI}} \mathcal{V}_T^\dagger = \hat{H}_{\text{AI}}^\ast$ with a unitary matrix $\mathcal{V}_T$ satisfying $\mathcal{V}_T \mathcal{V}_T^\ast = 1$. Without loss of generality, $\hat{H}_{\text{AI}}$ and $\mathcal{V}_T$ can be put into the following form,

$$\hat{\mathcal{H}}_{\text{AI}} \equiv \begin{pmatrix} \hat{h}_1 & \hat{h}_{12} \\ \hat{h}_{12}^\dagger & \hat{h}_2 \end{pmatrix}, \quad \mathcal{V}_T = \begin{pmatrix} 1_{n_1 \times n_1} & 0 \\ 0 & -1_{n_2 \times n_2} \end{pmatrix},$$

(S.36)

with $\hat{h}_1 = \hat{h}_1^\ast$, $\hat{h}_2 = \hat{h}_2^\ast$, and $\hat{h}_{12} = -\hat{h}_{12}^\ast$. Let $n_1$ and $n_2$ be the dimensions of the square matrices $\hat{h}_1$ and $\hat{h}_2$, respectively.

**From Hermitian class AI to non-Hermitian class CI**: For a generic Hermitian Hamiltonian $\hat{H}_{\text{AI}}$ in class AI with even rows and columns, we construct a non-Hermitian Hamiltonian $\hat{\mathcal{H}}_A$ in class CI whose $|\phi_r\rangle$ shares the same localization properties with $|\psi\rangle$ of $\hat{H}_{\text{AI}}$. Since the rows and columns are assumed to be even, we can take $n_1 = n_2 = n$ without loss of generality. Then, under an appropriate unitary transformation, we can choose $\hat{\mathcal{H}}_{\text{AI}}$ and $\mathcal{V}_T$ to be

$$\hat{\mathcal{H}}_{\text{AI}} \equiv \begin{pmatrix} \hat{h}_1' & \hat{h}_{12}' \\ \hat{h}_{12}'^\dagger & \hat{h}_2' \end{pmatrix}, \quad \mathcal{V}_T = \begin{pmatrix} 0 & 1_{n \times n} \\ 1_{n \times n} & 0 \end{pmatrix} \equiv \sigma_z,$$

(S.37)

with $\hat{h}_1' = \hat{h}_1^\ast$, $\hat{h}_2' = \hat{h}_2^\ast$, and $\hat{h}_{12}' = \hat{h}_{12}^\ast$. Let $\tilde{E} \in \mathbb{R}$ be an eigenenergy of $\hat{\mathcal{H}}_{\text{AI}}$ in Eq. (S.37) and $|\psi\rangle$ be the corresponding eigenstate. We introduce a non-Hermitian Hamiltonian $\hat{\mathcal{H}}_A$ in class CI with its symmetry operations,

$$\text{CI} : \hat{\mathcal{H}} = \begin{pmatrix} -i(\hat{h}_1' - \tilde{E}) & i\hat{h}_{12}' \\ -i\hat{h}_{12}'^\dagger & i(\hat{h}_2' - \tilde{E}) \end{pmatrix}, \quad \mathcal{U}_{\text{T}_+} = \sigma_x, \quad \mathcal{U}_{\text{T}_-} = \sigma_y,$$

(S.38)

with $\mathcal{U}_{\text{T}_+} \hat{\mathcal{H}}^\dag \mathcal{U}_{\text{T}_+} = \hat{\mathcal{H}}$, $\mathcal{U}_{\text{T}_-} \hat{\mathcal{H}}^\dag \mathcal{U}_{\text{T}_-} = -\hat{\mathcal{H}}$, $\mathcal{U}_{\text{T}_+} \mathcal{U}_{\text{T}_-} = 1$, and $\mathcal{U}_{\text{T}_-} \mathcal{U}_{\text{T}_-} = -1$. Symmetry-conserving energy of class CI is zero, $E = 0$. $\hat{\mathcal{H}}$ in Eq. (S.38) has a zero-energy right eigenmode $\sigma_x |\psi\rangle$, which has the same localization properties as the original eigenstate $|\psi\rangle$ of $\hat{\mathcal{H}}_{\text{AI}}$. Under the symmetries in Eq. (S.38), a generic non-Hermitian Hamiltonian in class CI takes a following Hamiltonian form,

$$\hat{\mathcal{H}}_{\text{CI}} = \begin{pmatrix} \hat{g}^\dagger & f \\ f^\dagger & -\hat{g} \end{pmatrix}.$$ 

(S.39)

Here $g$ and $f$ are $n$ by $n$ square matrices satisfying $\hat{g}^\dagger = \hat{g}$ and $f^\dagger = f$. With Eq. (S.37), one can see that Eq. (S.38) indeed takes this general form.

**From Hermitian class AI to non-Hermitian class BDI**: For the given Hermitian Hamiltonian $\hat{H}_{\text{AI}}$ in Eq. (S.36), we construct a non-Hermitian Hamiltonian $\hat{\mathcal{H}}$ in class BDI whose $|\phi_r\rangle$ shares the same localization properties with $|\psi\rangle$ of $\hat{H}_{\text{AI}}$. The symmetry-conserving energy of class BDI takes pure imaginary values, $E = iE_i$. Let $\tilde{E} \in \mathbb{R}$ be an eigenenergy of $\hat{\mathcal{H}}_{\text{AI}}$ in Eq. (S.36) and $|\psi\rangle$ be the corresponding eigenstate. We introduce a non-Hermitian Hamiltonian $\hat{\mathcal{H}}$ in class BDI with its symmetry operations by

$$\text{BDI} : \hat{\mathcal{H}} = \begin{pmatrix} -i(\hat{h}_1 - \tilde{E}) + iE_i & i\hat{h}_{12} \\ -i\hat{h}_{12}^\dagger & i(\hat{h}_2 - \tilde{E}) + iE_i \end{pmatrix}, \quad \mathcal{U}_{\text{T}_+} = 1, \quad \mathcal{U}_{\text{T}_-} = \begin{pmatrix} 1_{n_1 \times n_1} & 0 \\ 0 & -1_{n_2 \times n_2} \end{pmatrix},$$

(S.40)

with $\mathcal{U}_{\text{T}_+} \hat{\mathcal{H}}^\dag \mathcal{U}_{\text{T}_+} = \hat{\mathcal{H}}$, $\mathcal{U}_{\text{T}_-} \hat{\mathcal{H}}^\dag \mathcal{U}_{\text{T}_-} = -\hat{\mathcal{H}}$, and $\mathcal{U}_{\text{T}_+} \mathcal{U}_{\text{T}_-} = \mathcal{U}_{\text{T}_-} \mathcal{U}_{\text{T}_+} = 1$. $\hat{\mathcal{H}}$ has a right eigenmode at $E = iE_i$, which has the same localization properties as the original eigenstate $|\psi\rangle$ of $\hat{\mathcal{H}}_{\text{AI}}$. In fact, $\mathcal{U}_{\text{T}_-} |\psi\rangle$ is such an eigenmode,

$$\mathcal{U}_{\text{T}_-} |\psi\rangle = iE_i \mathcal{U}_{\text{T}_-} |\psi\rangle.$$ 

(S.41)
Since $\mathcal{U}_T$ is a local transformation, $\mathcal{U}_T \ket{\psi}$ and $\ket{\psi}$ share the same localization properties. Under the symmetries in Eq. (S.40), non-Hermitian Hamiltonians in class BDI$^\dagger$ take the following general form,

$$\mathcal{H}_{\text{BDI}} = \begin{pmatrix} ig_1 & g_{12} \\ g_{12}^* & ig_2 \end{pmatrix}. \quad (S.42)$$

Here $g_1$ and $g_2$ are $n_1 \times n_1$ and $n_2 \times n_2$ general real symmetric Hamiltonians respectively, and $g_{12}$ is $n_1 \times n_2$ general real Hamiltonian $g_{12} = g_{12}^*$. Given $h_1$, $h_2$, and $h_{12}$ in Eq. (S.36) and real $\tilde{E}$ and $E_1$, one can see that Eq. (S.40) takes this general form.

From Hermitian class AI to non-Hermitian class CI + $S_{+-}$: For generic Hermitian Hamiltonians $\hat{\mathcal{H}}_{\text{AI}}$ in Eqs. (S.36,S.37), we construct a non-Hermitian Hamiltonian in class CI + $S_{+-}$ whose $\ket{\phi_i}$ shares the same localization properties with $\ket{\psi}$ of $\hat{\mathcal{H}}_{\text{AI}}$. Let us have two independent random Hermitian Hamiltonians in class AI, $\hat{\mathcal{H}}_{\text{AI}}$ and $\hat{\mathcal{H}}'_{\text{AI}}$. They have the same dimensions and respect the same symmetry as in Eq. (S.36) or Eq. (S.37). Let $E \in \mathbb{R}$ and $E' \in \mathbb{R}$ be eigenenergies of $\hat{\mathcal{H}}_{\text{AI}}$ and $\hat{\mathcal{H}}'_{\text{AI}}$, and $\ket{\psi}$ and $\ket{\psi'}$ be the corresponding eigenstates, respectively. Out of the two, we introduce a non-Hermitian Hamiltonian in class CI + $S_{+-}$ together with SLS, TRS, and PHS operations by

$$\mathcal{H} = \begin{pmatrix} \hat{\mathcal{H}}_{\text{AI}} - E' \\ 0 \end{pmatrix}, \quad \mathcal{U}_S = \tau_z \otimes 1_{N \times N}, \quad \mathcal{U}_T = \tau_0 \otimes \mathcal{V}_T, \quad \mathcal{U}_P = \tau_y \otimes \mathcal{V}_T, \quad (S.43)$$

with $\mathcal{U}_S \mathcal{H} \mathcal{U}_S^\dagger = -\mathcal{H}$, $\mathcal{U}_T \mathcal{H} \mathcal{U}_T^\dagger = \mathcal{H}$, $\mathcal{U}_P \mathcal{H} \mathcal{T} \mathcal{U}_P^\dagger = -\mathcal{H}$, $[\mathcal{U}_S, \mathcal{U}_T]_+ = [\mathcal{U}_S, \mathcal{U}_T]_- = 0$ and $N \equiv n_1 + n_2$. $\mathcal{V}_T$ is defined in Eqs. (S.36,S.37). Here $\tau_0$, $\tau_z$, and $\tau_y$ act in the enlarged space, and $\tau_y$ exchanges $\hat{\mathcal{H}}_{\text{AI}}$ and $\hat{\mathcal{H}}'_{\text{AI}}$ in $\mathcal{H}$. Symmetry-conserving energy of class CI + $S_{+-}$ is zero, $E = 0$. Then, $\mathcal{H}$ in Eq. (S.43) has right eigenmodes at the zero energy, $(\ket{\psi})^T$ and $(\ket{\psi'})^T$, which have the same localization properties as the original eigenstates $\ket{\psi}$ and $\ket{\psi'}$ of $\hat{\mathcal{H}}_{\text{AI}}$ and $\hat{\mathcal{H}}'_{\text{AI}}$. Under the symmetries in Eq. (S.43), non-Hermitian Hamiltonians in class CI + $S_{+-}$ take the following general form,

$$\mathcal{H}_{\text{CI}+S_{+-}} = \begin{pmatrix} 0 & \hat{\mathcal{G}}_{\text{AI}} \\ \hat{\mathcal{G}}_{\text{AI}}^* & 0 \end{pmatrix}. \quad (S.44)$$

Here two independent Hermitian Hamiltonians $\hat{\mathcal{G}}_{\text{AI}}$ and $\hat{\mathcal{G}}_{\text{AI}}'$ respect the class AI symmetry; $\mathcal{V}_T \hat{\mathcal{G}}_{\text{AI}} \mathcal{V}_T^\dagger = \hat{\mathcal{G}}_{\text{AI}}$ and $\mathcal{V}_T \hat{\mathcal{G}}_{\text{AI}}' \mathcal{V}_T^\dagger = \hat{\mathcal{G}}_{\text{AI}}'$. Since $\hat{\mathcal{H}}_{\text{AI}}$ and $\hat{\mathcal{H}}'_{\text{AI}}$ are two independent class-AI Hamiltonians, Eq. (S.43) takes this general form.

In summary, for a generic Hermitian class-AI Hamiltonian $\hat{\mathcal{H}}_{\text{AI}}$ and its eigenstate $\ket{\psi}$, we constructed a non-Hermitian Hamiltonian $\mathcal{H}$ in its corresponding symmetry classes whose right eigenmode $\ket{\phi_i}$ at the symmetry-conserving energy $E$ has the same localization properties as $\ket{\psi}$ of $\hat{\mathcal{H}}_{\text{AI}}$. The non-Hermitian Hamiltonian thus constructed also takes a generic Hermitian form of respective symmetry classes.

In the next subsection, we consider Hermitian chiral unitary case, whose corresponding non-Hermitian symmetry classes are class A, AIII$^\dagger$, AIII + $S_+$, D + $S_+$, C + $S_+$, and AI + $S_-$. A generic Hamiltonian of the chiral unitary class is given in Eq. (S.45). Hamiltonians in the last five non-Hermitian classes are given by Eqs. (S.47,S.48,S.49,S.50,S.51). Symmetry-conserving energy of the chiral unitary class is zero, $E = 0$. Symmetry-conserving energy of non-Hermitian class A takes a complex value, while symmetry-conserving energies of the other non-Hermitian classes are all zero, $E = 0$.

**Hermitian class AIII**

A generic Hermitian Hamiltonian $\hat{\mathcal{H}}_{\text{AIII}}$ in class AIII respects $\mathcal{C} \mathcal{S} \mathcal{V}_S \hat{\mathcal{H}}_{\text{AIII}} \mathcal{V}_S^\dagger = -\hat{\mathcal{H}}_{\text{AIII}}$. Without loss of generality, it takes the following form with $\mathcal{V}_S = \sigma_z$,

$$\hat{\mathcal{H}}_{\text{AIII}} = \begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix}. \quad (S.45)$$

$h$ is a general non-Hermitian Hamiltonian (i.e., $h^\dagger \neq h$) and has no symmetries. The symmetry-conserving energy of class AIII is zero, $E = 0$. We assume the presence of zero modes in $\hat{\mathcal{H}}_{\text{AIII}}$. Because of $\mathcal{C} \mathcal{S}$, a zero mode with positive (negative) chirality can be written as $(\ket{\psi_+})^T [0 \ket{\psi_-}]^T$, satisfying $h^\dagger \ket{\psi_+} = 0 (h \ket{\psi_-} = 0)$.
There clearly exists a non-Hermitian Hamiltonian in class $A$ whose right eigenstate at complex energy $E$ shares the same localization properties with the zero mode of $\hat{\mathcal{H}}_{\text{AIII}}$: $h' \equiv h + E$, $h'|\psi_-\rangle = E|\psi_-\rangle$. Since $h$ is general non-Hermitian, $h'$ also takes the general Hamiltonian form of the non-Hermitian symmetry class $A$.

**From Hermitian class $\text{AIII}$ to non-Hermitian class $\text{AIII}^\dagger$:** For generic Hermitian Hamiltonians in class $\text{AIII}$, we construct a non-Hermitian Hamiltonian $\hat{\mathcal{H}}$ in class $\text{AIII}^\dagger$ that shares the same localization properties of zero modes. To this end, we consider two independent Hermitian random Hamiltonians in class $\text{AIII}$,

$$\hat{\mathcal{H}}_{\text{AIII}} = \begin{pmatrix} 0 & h \\ h^\dagger & 0 \end{pmatrix}, \quad \hat{\mathcal{H}}'_{\text{AIII}} = \begin{pmatrix} 0 & h' \\ h'^\dagger & 0 \end{pmatrix},$$

and assume the presence of zero modes in $\tilde{\mathcal{H}}_{\text{AIII}}$ and/or $\hat{\mathcal{H}}'_{\text{AIII}}$. Let $(0|\psi_-\rangle)^T$ and/or $(0|\psi'_-\rangle)^T$ be a zero mode of $\hat{\mathcal{H}}_{\text{AIII}}$ ($\hat{\mathcal{H}}'_{\text{AIII}}$) with negative chirality. Out of these two, we introduce a non-Hermitian Hamiltonian $\mathcal{H}$ that belongs to class $\text{AIII}^\dagger$,

$$\text{AIII}^\dagger : \mathcal{H} = \begin{pmatrix} 0 & h \\ h'^\dagger & 0 \end{pmatrix}. \quad (S.47)$$

By construction, $\mathcal{H}$ has PHS $\mathcal{U}_S\mathcal{H}\mathcal{U}_S^\dagger = -\mathcal{H}$ with $\mathcal{U}_S = \tau_z$ and belongs to class $\text{AIII}^\dagger$. Symmetry-conserving energy of class $\text{AIII}^\dagger$ is zero, $E = 0$. $\mathcal{H}$ in Eq. (S.47) has right eigenmodes at the zero energy, $(0|\psi_-\rangle)^T$ and/or $(0|\psi'_-\rangle)^T$, which have the same localization properties as the zero modes of $\hat{\mathcal{H}}_{\text{AIII}}$ and $\hat{\mathcal{H}}'_{\text{AIII}}$. Since $h$ and $h'$ are two independent non-Hermitian Hamiltonians, Eq. (S.47) takes a general Hamiltonian form of class $\text{AIII}^\dagger$ under the SLS with $\mathcal{U}_S = \tau_z$.

**From Hermitian class $\text{AIII}$ to non-Hermitian class $\text{AIII} + \mathcal{S}_+$:** For the given Hermitian Hamiltonian $\hat{\mathcal{H}}_{\text{AIII}}$ in Eq. (S.45), we consider the following non-Hermitian Hamiltonian $\mathcal{H}$,

$$\text{AIII} + \mathcal{S}_+ : \mathcal{H} = \begin{pmatrix} 0 & h \\ -h'^\dagger & 0 \end{pmatrix}, \quad \mathcal{U}_C = \tau_0, \mathcal{U}_S = \tau_z. \quad (S.48)$$

$\mathcal{H}$ satisfies CS $\mathcal{U}_C\mathcal{H}\mathcal{U}_C^\dagger = -\mathcal{H}$ and SLS $\mathcal{U}_S\mathcal{H}\mathcal{U}_S^\dagger = -\mathcal{H}$ and hence belongs to class $\text{AIII} + \mathcal{S}_+$. Symmetry-conserving energy of class $\text{AIII} + \mathcal{S}_+$ is zero, $E = 0$. $\mathcal{H}$ in Eq. (S.48) has right eigenmodes at the zero energy, $(|\psi_+\rangle)^T$ and $(0|\psi_-\rangle)^T$, which are identical to the original zero modes of $\hat{\mathcal{H}}_{\text{AIII}}$. Since $\mathcal{H}$ is a general non-Hermitian Hamiltonian from Eq. (S.45), $\mathcal{H}$ in Eq. (S.48) takes a general form of class $\text{AIII} + \mathcal{S}_+$ under the CS and SLS symmetries.

**From Hermitian class $\text{AIII}$ to non-Hermitian class $\text{D} + \mathcal{S}_+$:** For the given Hermitian Hamiltonian $\hat{\mathcal{H}}_{\text{AIII}}$ in Eq. (S.45), we introduce the following non-Hermitian Hamiltonian $\mathcal{H}$ in class $\text{D} + \mathcal{S}_+$ and its symmetry operations,

$$\text{D} + \mathcal{S}_+ : \mathcal{H} = \begin{pmatrix} 0 & h \\ h^T & 0 \end{pmatrix}, \quad \mathcal{U}_{P_\perp} = \mathcal{U}_S = \tau_z. \quad (S.49)$$

$\mathcal{H}$ has PHS $\mathcal{U}_{P_\perp}\mathcal{H}\mathcal{U}_{P_\perp}^\dagger = -\mathcal{H}$, TRS$^\dagger \mathcal{H}^T = \mathcal{H}$, and SLS $\mathcal{U}_S\mathcal{H}\mathcal{U}_S^\dagger = -\mathcal{H}$, and hence belongs to class $\text{D} + \mathcal{S}_+$. Symmetry-conserving energy of class $\text{D} + \mathcal{S}_+$ is zero, $E = 0$. $\mathcal{H}$ in Eq. (S.49) has right eigenmodes at the zero energy, $(|\psi_+\rangle)^T$ and $(0|\psi_-\rangle)^T$, which have the same localization properties as the original zero modes $(|\psi_+\rangle)^T$ and $(0|\psi_-\rangle)^T$ of $\hat{\mathcal{H}}_{\text{AIII}}$. Since $\mathcal{H}$ is a general Hamiltonian from Eq. (S.45), $\mathcal{H}$ in Eq. (S.49) takes a general form of class $\text{D} + \mathcal{S}_+$ under the PHS, TRS$^\dagger$ and SLS symmetries.

**From Hermitian class $\text{AIII}$ to non-Hermitian class $\text{C} + \mathcal{S}_+$:** For the given Hermitian Hamiltonian $\hat{\mathcal{H}}_{\text{AIII}}$ in Eq. (S.45), we define the following non-Hermitian Hamiltonian $\mathcal{H}$ in class $\text{C} + \mathcal{S}_+$ and its symmetry operations,

$$\text{C} + \mathcal{S}_+ : \mathcal{H} = \begin{pmatrix} 0 & h \\ \sigma_y h^T \sigma_y & 0 \end{pmatrix}, \quad \mathcal{U}_{P_\perp} = \tau_z \otimes \sigma_y, \quad \mathcal{U}_{P_+} = \tau_0 \otimes \sigma_y, \quad \mathcal{U}_S = \tau_z \otimes 1. \quad (S.50)$$

By construction, the non-Hermitian Hamiltonian $\mathcal{H}$ has PHS $\mathcal{U}_{P_\perp}\mathcal{H}\mathcal{U}_{P_\perp}^\dagger = -\mathcal{H}$, TRS$^\dagger \mathcal{U}_{P_\perp}\mathcal{H}^T\mathcal{U}_{P_\perp}^\dagger = \mathcal{H}$, and SLS $\mathcal{U}_S\mathcal{H}\mathcal{U}_S^\dagger = -\mathcal{H}$, and hence belongs to class $\text{C} + \mathcal{S}_+$. Here we assume that the dimension of $h$ is even so that $h$ can be regarded as a Hamiltonian with local pseudospin-$\frac{1}{2}$ degree of freedom. $\sigma_y$ in Eq. (S.50) flips the spin locally. Symmetry-conserving energy of class $\text{C} + \mathcal{S}_+$ is zero, $E = 0$. Then $\mathcal{H}$ in Eq. (S.50) has right eigenmodes at the zero energy, $(\sigma_y |\psi_+\rangle)^T$ and $(0|\psi_-\rangle)^T$, which have the same localization properties as the original zero modes $(|\psi_+\rangle)^T$ and $(0|\psi_-\rangle)^T$ of the Hermitian Hamiltonian $\hat{\mathcal{H}}_{\text{AIII}}$. Since $\mathcal{H}$ is a general Hamiltonian from Eq. (S.45), $\mathcal{H}$ in Eq. (S.50) takes a general Hamiltonian form of class $\text{C} + \mathcal{S}_+$ under the PHS, TRS$^\dagger$ and SLS symmetries.
From Hermitian class AIII to non-Hermitian class AI + $S_-$: For the given Hermitian Hamiltonian $\tilde{H}_{\text{AIII}}$ in Eq. (S.45), we introduce the following non-Hermitian Hamiltonian in class AI + $S_-$ and its symmetry operations,

$$ AI + S_- : \mathcal{H} = \begin{pmatrix} 0 & h^* \\ h & 0 \end{pmatrix}, \quad U_{\tau_x} = \tau_x, \quad U_{\tau_y} = \tau_y, \quad U_S = \tau_z. \quad (S.51) $$

$\mathcal{H}$ belongs to class AI + $S_-$ because of TRS $U_{\tau_x} \mathcal{H}^* U_{\tau_x}^\dagger = \mathcal{H}$, PHS $U_{\tau_y} \mathcal{H}^* U_{\tau_y}^\dagger = -\mathcal{H}$, SLS $(U_S \mathcal{H} U_S^\dagger = -\mathcal{H})$, and the anti-commutation relation between $U_{\tau_x}$ and $U_S$. Symmetry-conserving energy of class AI + $S_-$ is zero, $E = 0$. $\mathcal{H}$ in Eq. (S.51) has right eigenmodes at the zero energy, $(|\psi_-\rangle)^T = (0 \ 0^T)$ and $(|\psi_-\rangle)^T$, which have the same localization properties as the original zero mode $(0 \ |\psi_-\rangle)^T$ of the Hermitian Hamiltonian $\tilde{H}_{\text{AIII}}$. $\mathcal{H}$ in Eq. (S.51) takes a general Hamiltonian form of class AI + $S_-$ under the TRS, PHS$^T$ and SLS symmetries.

In the next subsection, we consider Hermitian chiral orthogonal case, whose corresponding non-Hermitian symmetry classes are class AI, AI + $S_+$, BDI + $S_+$, and CI + $S_++$. A generic Hamiltonian of chiral orthogonal class is given in Eq. (S.52). Hamiltonians in the last three non-Hermitian classes are given by Eqs. (S.54,S.55,S.56). Symmetry-conserving energy of the chiral orthogonal class is zero, $E = 0$. Symmetry-conserving energy of non-Hermitian class AI takes a real value ($E = E^*$), while symmetry-conserving energies of the other non-Hermitian classes are all zero, $E = 0$.

Hermitian class BDI

A Hermitian Hamiltonian $\tilde{H}_{\text{BDI}}$ in class BDI respects TRS, PHS, and CS, where the signs of TRS and PHS are +1. Under an appropriate unitary transformation, $\mathcal{H}$ can be put into the following form together with the three symmetry operations,

$$ \mathcal{H}_{\text{BDI}} = \begin{pmatrix} 0 & h \dagger \\ h & 0 \end{pmatrix}, \quad \nu_T = 1, \quad \nu_P = \nu_S = \sigma_z. \quad (S.52) $$

$h$ is real non-Hermitian satisfying $h = h^*$. $\mathcal{H}_{\text{BDI}}$ respects $\tilde{H}_{\text{BDI}}^* = \mathcal{H}_{\text{BDI}}$ and $\nu_P \tilde{H}_{\text{BDI}}^\dagger \nu_P = \nu_S \tilde{H}_{\text{BDI}} \nu_S = -\mathcal{H}_{\text{BDI}}$. Symmetry-conserving energy of BDI class is zero, $E = 0$. Let us assume the presence of zero modes in $\tilde{H}_{\text{BDI}}$. A zero mode with positive (negative) chirality can be written as $(|\psi_+\rangle)^T (0 \ 0^T)$, satisfying $h^\dagger |\psi_+\rangle = (h |\psi_-\rangle) = 0$.

Then, there exists a non-Hermitian Hamiltonian in class AI, whose right eigenmode at real energy $E$ shares the same localization properties with the zero mode of $\mathcal{H}_{\text{BDI}}$: $h' \equiv h + E$, $h' |\psi_-\rangle = E |\psi_-\rangle$. Since $h$ is general real non-Hermitian Hamiltonian and the symmetry-conserving energy $E$ is real in class AI, $h' \equiv h + E$ also takes the general form of non-Hermitian Hamiltonian in the symmetry class AI.

From Hermitian class BDI to non-Hermitian class AI + $S_+$: Symmetry-conserving energy of class AI + $S_+$ is zero, $E = 0$. From generic Hermitian Hamiltonians in class BDI, we construct a non-Hermitian Hamiltonian $\mathcal{H}$ in class AI + $S_+$ whose right eigenmode at the zero energy shares the same localization properties with the zero modes of the BDI Hamiltonians. To this end, we consider two independent Hermitian Hamiltonians in class BDI,

$$ \mathcal{H}_{\text{BDI}} = \begin{pmatrix} 0 & h \dagger \\ h & 0 \end{pmatrix}, \quad \tilde{\mathcal{H}}_{\text{BDI}} = \begin{pmatrix} 0 & h' \dagger \\ h' & 0 \end{pmatrix}. \quad (S.53) $$

with $h = h^*$, $h' = h''$, $\neq h^\dagger$, and and assume the presence of their zero modes. Let $(0 \ |\psi_-\rangle)^T$ and $(0 \ |\psi_-\rangle)^T$ be zero modes of $\mathcal{H}_{\text{BDI}}$ and $\tilde{\mathcal{H}}_{\text{BDI}}$ with negative chirality, respectively. From Eq. (S.53), we construct the following non-Hermitian Hamiltonian:

$$ AI + S_+ : \mathcal{H} = \begin{pmatrix} 0 & h \\ h' & 0 \end{pmatrix}. \quad (S.54) $$

$\mathcal{H}$ belongs to class AI + $S_+$ because of TRS $\mathcal{H}^* = \mathcal{H}$ and SLS $\tau_z \mathcal{H} \tau_z = -\mathcal{H}$. $\mathcal{H}$ in Eq. (S.54) has right eigenmodes at the zero energy, $(0 \ |\psi_-\rangle)^T$ and $(|\psi_-\rangle \ 0)^T$, which share the same localization properties as the original zero modes of $\tilde{\mathcal{H}}_{\text{BDI}}$ and $\tilde{\mathcal{H}}_{\text{BDI}}$. Since real $h$ and $h'$ are independent of each other, $\mathcal{H}$ in Eq. (S.54) takes a general Hamiltonian form of class AI + $S_+$ under the TRS and SLS symmetries.
From Hermitian class BDI to non-Hermitian class BDI + $S_{++}$: From the given Hermitian Hamiltonian $\tilde{H}_{BDI}$ in Eq. (S.52), we introduce the following non-Hermitian Hamiltonian,

$$BDI + S_{++} : \mathcal{H} = \begin{pmatrix} 0 & h \\ -h^\dagger & 0 \end{pmatrix}.$$  

$\mathcal{H}$ has TRS with $U_T = 1$ ($\mathcal{H}^* = \mathcal{H}$), PHS with $U_P = \tau_0 (U_P^T U_P^\dagger = -\mathcal{H})$, and SLS with $U_S = \tau_z$ ($U_S \mathcal{H} U_S^\dagger = -\mathcal{H}$), and hence belongs to class BDI + $S_{++}$. Symmetry-conserving energy of class BDI + $S_{++}$ is zero, $E = 0$. $\mathcal{H}$ in Eq. (S.55) has right eigenmodes at the zero energy, $(|\psi_+\rangle \ 0^T)$ and $(0 \ |\psi_-\rangle)^T$, which are identical to the original zero modes of $\tilde{H}_{BDI}$. Since $h$ is real non-Hermitian without any other symmetries, $\mathcal{H}$ in Eq. (S.55) takes a generic Hamiltonian form of class BDI + $S_{++}$ under the TRS, PHS and SLS symmetries.

From Hermitian class BDI to non-Hermitian class CI + $S_{++}$: From the given Hermitian Hamiltonian $\tilde{H}_{BDI}$ in Eq. (S.52), we consider the following non-Hermitian Hamiltonian,

$$CI + S_{++} : \mathcal{H} = \begin{pmatrix} 0 & h \\ -\sigma_y h^\dagger \sigma_y & 0 \end{pmatrix}.$$  

$\mathcal{H}$ respects TRS with $U_T = 1$ ($\mathcal{H}^* = \mathcal{H}$), PHS with $U_P = \sigma_y (U_P^T U_P^\dagger = -\mathcal{H})$, and SLS with $U_S = \tau_z$ ($U_S \mathcal{H} U_S^\dagger = -\mathcal{H}$), and hence belongs to class CI + $S_{++}$. Here we assume that the dimension of $h$ is even so that $\mathcal{H}_A$ can be regarded as a Hamiltonian with local pseudospin-$1/2$ degree of freedom. $\sigma_y$ in Eq. (S.56) flips the spin locally. Symmetry-conserving energy of class CI + $S_{++}$ is zero, $E = 0$. $\mathcal{H}$ in Eq. (S.56) has right eigenmodes at the zero energy, $(\sigma_y |\psi_+\rangle \ 0^T)$ and $(0 \ |\psi_-\rangle)^T$, which have the same localization properties as the original zero modes $(|\psi_+\rangle \ 0^T)$ and $(0 \ |\psi_-\rangle)^T$ of $\tilde{H}_{BDI}$. With real non-Hermitian $h$, $\mathcal{H}$ in Eq. (S.56) takes a generic Hamiltonian form of class CI + $S_{++}$ under the TRS, PHS and SLS symmetries.

Other Hermitian symmetry classes

For general Hermitian Hamiltonians in classes AII, C, and D, where only one anti-unitary symmetry is relevant, we can construct the corresponding non-Hermitian Hamiltonians in a similar manner to class AI. For general Hermitian Hamiltonians in classes CI, CII, and DIII, where two anti-unitary symmetries are relevant, we can construct the corresponding non-Hermitian Hamiltonians in a similar manner to class BDI.

MODEL, LOCALIZATION LENGTH, AND POLYNOMIAL FITTING

To study the AT in class AI, we introduce the following O(1) tight-binding model on 2D square and 3D cubic lattices:

$$H = \sum_i \varepsilon_i c_i^\dagger c_i + \sum_{\langle i,j \rangle} V_{i,j} c_i^\dagger c_j,$$  

where $\varepsilon_i$ is the random potential with the uniform distribution in $[-W/2, W/2]$ with the disorder strength $W$. Here, $\langle i,j \rangle$ stands for nearest neighbor lattice sites. $V_{i,j}$ is set to $-1$ or $+1$ randomly with the equal probability. $V_{i,j}$ and $V_{j,i}$ are treated as independent random variables, so that Hermiticity will be broken by $V_{i,j}^* \neq V_{j,i}$. According to the symmetry classification, $H$ belongs to class AI with $H = H^*$. To study the ATs in classes AII, AII†, CII†, and DIII, we introduce the following non-Hermitian extension of the SU(2) model [13] [14] [66] on 2D square and 3D cubic lattices,

$$H = \sum_{i,\sigma} \varepsilon_{i,\sigma} c_{i,\sigma}^\dagger c_{i,\sigma} + \sum_{\langle i,j \rangle, \sigma, \sigma'} R(i,j)_{\sigma,\sigma'} c_{i,\sigma}^\dagger c_{j,\sigma'}.$$  

We parametrize the matrix $R(i,j)$ as

$$R(i,j) = \begin{pmatrix} e^{i \alpha_{i,j}} \cos(\beta_{i,j}) & e^{i \gamma_{i,j}} \sin(\beta_{i,j}) \\ -e^{-i \gamma_{i,j}} \sin(\beta_{i,j}) & e^{-i \alpha_{i,j}} \cos(\beta_{i,j}) \end{pmatrix}.$$  

(S.59)
TABLE IV. Polynomial fitting results for the normalized localization lengths around the Anderson transition points for three-dimensional classes AI, AII, and AIII at different complex energies $E$. The goodness of fit (GOF), critical disorder $W_c$, critical exponents $\nu$, scaling dimensions $-\gamma$ of the least irrelevant scaling variable, and critical normalized localization length $\Lambda_c$ are shown for the different system sizes and for the different orders of the expansion $(m_1, n_1, m_2, n_2)$. The square bracket is the 95% confidence interval.

| Class | $E$   | $L$  | $m_1$ | $n_1$ | $m_2$ | $n_2$ | GOF   | $W_c$           | $\nu$   | $\gamma$ | $\Lambda_c$ |
|-------|-------|------|-------|-------|-------|-------|-------|-----------------|---------|----------|-------------|
| AI    | 0.5i  | 8-20 | 3     | 3     | 0     | 1     | 0.12  | 12.84[12.834, 12.852] | 0.988[0.965, 1.008] | 0.94[0.78, 1.10] | 0.584[0.571, 0.593] |
|       |       | 10-20| 3     | 3     | 0     | 1     | 0.124| 12.841[12.835, 12.847] | 0.980[0.959, 0.999] | 1.31[1.15, 1.48]  | 0.593[0.588, 0.598] |
|       | 0     | 10-20| 3     | 3     | 0     | 1     | 0.21 | 21.540[21.471, 21.564] | 0.933[0.799, 1.041] | 0.512[0.468, 0.668] | 0.269[0.259, 0.293] |
|       |       | 10-20| 3     | 3     | 0     | 1     | 0.22 | 21.576[21.503, 21.616] | 0.943[0.816, 1.068] | 0.439[0.372, 0.588] | 0.253[0.234, 0.282] |
| AII   | 4-18  | 1     | 3     | 0     | 1     | 0.69 | 8.068[8.063, 8.072]   | 1.021[0.997, 1.042] | 0.48[0.42, 0.54]  | 0.528[0.505, 0.548] |
|       | 4-18  | 2     | 3     | 0     | 1     | 0.70 | 8.067[8.062, 8.072]   | 1.021[0.997, 1.041] | 0.49[0.43, 0.55]  | 0.532[0.510, 0.551] |
|       | 8-18  | 1     | 3     | 0     | 1     | 0.30 | 8.066[8.050, 8.082]   | 0.996[0.879, 1.058] | 0.50[0.22, 0.97]  | 0.537[0.397, 0.602] |
|       | 8-18  | 2     | 3     | 0     | 1     | 0.25 | 8.055[8.042, 8.074]   | 1.005[0.910, 1.044] | 0.79[0.32, 1.36]  | 0.585[0.473, 0.623] |
| AII†  | 0     | 10-18| 2     | 3     | 0     | 1     | 0.11 | 7.706[7.703, 7.708]   | 0.903[0.896, 0.908] | 2.65[2.25, 3.15]  | 0.581[0.576, 0.586] |
|       |       | 10-18| 2     | 3     | 0     | 1     | 0.13 | 7.712[7.706, 7.720]   | 0.908[0.908, 0.922] | 1.75[1.16, 2.44]  | 0.565[0.541, 0.579] |
|       |       | 12-18 | 2     | 3     | 0     | 1     | 0.19 | 7.712[7.708, 7.718]   | 0.899[0.876, 0.914] | 1.90[1.09, 2.82]  | 0.566[0.542, 0.577] |
|       |       | 12-18 | 3     | 0     | 1     | 0.18 | 7.711[7.702, 7.719]   | 0.899[0.880, 0.914] | 2.06[1.06, 4.02]  | 0.569[0.537, 0.580] |

with the imaginary unit $i$. Here, we distribute $\alpha_{i,j}$ and $\gamma_{i,j}$ with the uniform probability in the range $[0, 2\pi)$, and $\beta_{i,j}$ according to the probability density $P(\beta) d\beta = \sin(2\beta) d\beta$ in the range $[0, \pi/2]$. The parameters in the hopping terms satisfy $\alpha_{i,j} = -\alpha_{j,i}$, $\beta_{i,j} = \beta_{j,i}$, and $\gamma_{i,j} = \gamma_{j,i} + \pi$ for classes AI, AII, and AIII, leading to $R^i \downarrow R(j, i)$. For class DIII, on the other hand, we have $\alpha_{i,j} = -\alpha_{j,i} + \pi$ and $\gamma_{i,j} = \gamma_{j,i} + \pi$, leading to $\sigma \bar{R}^i \downarrow \sigma R(j, i)$. We set the on-site potential as $\varepsilon_{i,\sigma} = \omega_{i,\sigma}^r + i \omega_{i,\sigma}^i$, where $\omega_{i,\sigma}^r$ and $\omega_{i,\sigma}^i$ are independently and uniformly distributed random numbers in $[-W_c/2, W_c/2]$ and $[-W_c/2, W_c/2]$, respectively.

With $\varepsilon_{i,\uparrow} = \varepsilon_{i,\downarrow}$, $H$ satisfies $\sigma_y H^* \sigma_y = H$. If we set $W_c \neq 0$ and $W_c \neq 0$, $H$ belongs to class AII†. If we set $W_c = 0$ and $W_c \neq 0$, $H$ on the bipartite lattice also has CS, $\mu_\xi \sigma_H^* \sigma_y \mu_\xi = -H$, where $\mu_\xi$ is diagonal in the sublattice degrees of freedom, taking different signs on the different sublattices. Thus $H$ belongs to class CII†. With $\varepsilon_{i,\uparrow} = \varepsilon_{i,\downarrow}$, $H$ satisfies $\sigma_y H^* \sigma_y = H$. If we set $W_c \neq 0$ and $W_c \neq 0$, $H$ belongs to class AI. If we set $W_c = 0$, $W_c \neq 0$, $\varepsilon_{i,\uparrow} = -\varepsilon_{i,\downarrow}$, and require $\sigma \bar{R}^i \downarrow \sigma R(j, i)$, $H$ belongs to class DIII, $\sigma \bar{H}^i \sigma_y = H$ and $\sigma \bar{H}^i \sigma_z = -H$.

In order to extract the critical exponents, localization lengths $\xi(L, W)$ are calculated by the transfer matrix method. We note that the transfer matrix along the transmission direction $z$ can be put into the unit matrix by proper gauge transformations. In classes AI and DIII, however, the onsite disorder is different for spin up and spin down, and the spin-dependent hopping along the transmission direction cannot be gauged away.

For class AI, the localization lengths at $E = 0$ for different disorder strength are calculated with the transmission length $L_z = 10^8$ for $L = 6, 8, 10, 12, 16, 18, 20$ [Fig. S1(a)]. Similarly, the localization lengths at $E = 0.5i$ are calculated.
with the transmission length $L_z = 10^7$ for $L = 6, 8, 10, 12, 16, 18, 20$ [Fig. S.1(b)].

We set $W_r = W_l = W$ for the following calculations of the SU(2) models in classes AII and AII. For 2D class AII at $E = 0$, the localization lengths are calculated with $L_z = 10^8$ for $L = 60, 100$, $L_z = 4 \times 10^7$ for $L = 150$, $L_z = 2 \times 10^7$ for $L = 200$, and $L_z = 10^7$ for $L = 250$ [Fig. S.2(a)]. For 3D class AII at $E = 0$, the localization lengths are calculated with $L_z = 10^7$ for $L = 4, 6, 8, 10, 12, 14, 16$, and $L_z = 6 \times 10^6$ for $L = 18$ [Fig. S.2(b)]. For 2D class AII at $E = 0.01i$, the localization lengths are calculated with $L_z = 5 \times 10^7$ for $L = 20, 60, 80, 100, 120, 150$ [Fig. S.3(a)]. For 3D class AII at $E = 0$, the localization lengths are calculated with $L_z = 10^6$ for $L = 6, 8, 10, 12, 16, 18$ [Fig. S.3(b)]. For 3D class AII at $E = i$, the localization length for $L = 4, 6, 8, 10, 12, 14, 16, 18$ has been calculated with $L_z = 5 \times 10^6$ [Fig. S.3(c)].

We set $W_r = 0$ and $W_l = W$ for the calculations of the SU(2) models in classes CII and DIII. For 2D class CII at $E = 0$, the localization lengths are calculated with $L_z = 10^8$ for $L = 16, 24, 32, 48, 64, 96, 144$ [Fig. S.4(a)]. For 2D class DIII at $E = 0$, the localization lengths are calculated with $L_z = 10^7$ for $L = 6, 8, 16, 24, 32, 48, 64, 96$ [Fig. S.4(b)].

According to the finite-size scaling, the dimensionless normalized localization length \( \Lambda = \xi(W, L)/L \) follows a scaling function that depends on the relevant scaling variable and possibly many other irrelevant scaling variables. Empirically, we do not need to consider most of the irrelevant scaling variables, and numerical data of \( \Lambda \) for the larger system size $L$ depend only on the relevant scaling variable $\phi_1$ and the least irrelevant scaling variable $\phi_2$:

$$\Lambda(W, L) = f(\phi_1, \phi_2).$$  \hfill (S.60)

The scaling argument leads to

$$\phi_1 = u_1(w) L^{1/\nu}, \quad \phi_2 = u_2(w) L^{-y}$$  \hfill (S.61)

with $w \equiv (W - W_c)/W_c$ and some functions $u_1$, $u_2$. Here, $\nu$ is the critical exponent for the correlation length, and $-y$ is the scaling dimension of the least irrelevant scaling variable. Taking account of the non-linearity of $u_i(w)$ in $w$, these functions can be expanded in small $w$:

$$u_i(w) \equiv \sum_{j=0}^{m_i} b_{i,j} w^j$$  \hfill (S.62)

with $b_{1,0} = 0$. For sufficiently small $w$ and large $L$, the scaling function can be also expanded by

$$f(\phi_1, \phi_2) = \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} a_{j_1,j_2} \phi_1^{j_1} \phi_2^{j_2}.$$  \hfill (S.63)

To fix the ambiguity, we set $a_{1,0} = a_{0,1} = 1$. Then, the numerical data of $\Lambda$ for different $L$ and $W$ are fitted by the polynomial with the fitting parameters $W_c$, $\nu$, $y$, $a_{i,j}$, and $b_{i,j}$. We minimize

$$\chi^2 = \sum_{k=1}^{N_D} \frac{(\Lambda_k - f_k)^2}{\sigma_{\Lambda_k}^2}$$  \hfill (S.64)

in terms of the fitting parameters, where $N_D$ is the number of total data points, $\Lambda_k$ the $k$-th data point calculated with precision $\sigma_{\Lambda_k}$, and $f_k$ the $k$-th fitted data point by the polynomial. Goodness of fit \cite{9} is calculated to quantify the quality of the fits. The critical parameters, such as $W_c$, $\nu$, $y$, and $\Lambda_c$, are estimated with 95% confidence interval \cite{9}. The results are shown in Table IV for 3D and Table V for 2D.

**PHASE DIAGRAMS FOR CLASSES AI AND AII**

The ATs at real energy and non-real energy for classes AI and AII belong to the different symmetry and universality classes. In this section, we show the phase diagrams for the O(1) model in class AI and the SU(2) model in class AII in terms of the disorder strength $W$ and the imaginary part $\text{Im} \ E$ of energy. We set $\text{Re} \ E = 0$ and take several values of $\text{Im} \ E$. For each value of $\text{Im} \ E$, we calculate the localization lengths with different system sizes and disorder strength, to determine the critical disorder for the ATs. This gives the phase diagrams shown in Fig. S.5.

As shown in Fig. S.5(b) and (c), when the disorder strength $W$ increases, the 2D and 3D SU(2) models in class AII exhibit reentrant behavior of the localization-delocalization-localization transition for small $\text{Im} \ E$. This is because...
the models at \( W_c = W_i = W = 0 \) reduce to Hermitian models that have no eigenstates at nonzero \( \text{Im } E \). When weak complex-valued disorder \( W \) is introduced in these Hermitian models, eigenstates acquire nonzero but small \( \text{Im } E \) and tend to be localized. When increasing the disorder strength, the density of states (DoS) at \( \text{Im } E \neq 0 \) increases, and states in this energy region may undergo a localization-delocalization transition. Further increase of the disorder leads to localization. Consequently, the reentrant localization-delocalization-localization transition is observed in the SU(2) models in class AII. This situation is similar to reentrant behavior of the localization-delocalization-localization transition near band edges of Hermitian band insulators \[85, 86\].

On the other hand, the 3D O(1) model in class AI without the on-site potential disorder \( W \) is a non-Hermitian system with the random hopping \( V_{i,j} \neq V_{j,i} \). This model shows no reentrant behavior but a delocalization-localization transition at \( \text{Im } E \neq 0 \) and \( W = 0 \) \[Fig. S.5 (a)\]. A delocalized phase with the larger DoS appears for small \( \text{Im } E \), and a localized phase with the smaller DoS appears for large \( \text{Im } E \) in the outer region. The mobility edge decreases as a function of \( \text{Im } E \) on introducing the on-site potential disorder \( W \).

**DISTRIBUTION OF EIGENENERGIES, AND INVERSE PARTICIPATION RATIO**

The distribution of eigenenergies provides complementary information about criticality of disorder-driven quantum phase transitions in Hermitian systems. A prime example is non-Anderson transitions in disordered semimetals \[87\], where the DoS plays a role of the order parameter of a semimetal-metal quantum phase transition \[88\], and the scaling property of the DoS is characterized by a dynamical critical exponent \[89\]. The DoS in the complex energy plane may also provide important information about quantum criticality of the ATs in non-Hermitian system. Motivated by this anticipation, we study the distribution of eigenenergies in the complex energy plane for the O(1) model in class AI and SU(2) model in classes AII and AII\(^*\). We numerically diagonalize the 2D and 3D Hamiltonians with smaller system size under periodic boundary conditions. We take an average of the distribution over many different disorder realizations. To characterize eigenstates \( \psi(\mathbf{r}) \), we also calculate the inverse participation ratio (IPR):

\[
I = \frac{\sum_{\mathbf{r}} |\psi(\mathbf{r})|^4}{\left( \sum_{\mathbf{r}} |\psi(\mathbf{r})|^2 \right)^2}.
\]

For extended states, \( 1/I \) scales with the system’s volume \( L^d \), where \( L \) is the system’s length and \( d \) is the spatial dimension. For localized states, it remains to be around \( \xi^d \) with a localization length \( \xi \). In the following, we summarize the results for 3D class AI, 2D and 3D class AII, and 2D and 3D class AII\(^*\).

**3D class AI**

In Fig. S.6(a), (c), and (e), we show the distributions of eigenenergies for the 3D O(1) model in class AI for different values of the disorder strength \( W \). For \( W = 0 \), the distribution is statistically symmetric with respect to the exchange between \( \text{Re } E \) and \( \text{Im } E \). The symmetry comes from a gauge transformation that assigns \( +i \) \((+1)\) to one \( (\text{the other})\) of the sublattice sites in the cubic lattice. On increasing the real-valued on-site potential disorder \( W \), eigenenergies collapse into the real axis. In Fig. S.6(b), (d), and (f), we show the DoS as a function of the imaginary part of eigenenergies. The DoS shows a singular peak on the real axis. The peak becomes sharper for larger \( W \). The IPR shows that eigenmodes near \( E = 0 \) are more delocalized than the other eigenmodes.

**2D and 3D class AII**

Figure S.7 shows that in the presence of nonzero complex-valued potential disorder \( W \), the DoS shows a soft gap around the real axis and gets largest away from the real axis. The soft gap in the DoS is observed both in 2D and 3D. Note that the 2D model does not show any ATs at \( E = 0 \), but undergoes the AT for \( \text{Im } E \neq 0 \) \[Fig. S.5 (c)\]; the disorder-driven localization-delocalization and delocalization-localization transition points are located at \( W_{c,1} \approx 0.46 \) and \( W_{c,2} \approx 2.62 \) for \( E = 0.01i \), respectively \[Fig. S.5(d)\]. On the other hand, the 3D model shows the AT at \( E = 0 \) as well as \( \text{Im } E \neq 0 \) \[Fig. S.5(b)\]. We emphasize that the soft gap of the DoS around the real axis appears both in 2D and 3D although the phase diagrams are qualitatively different.
2D and 3D class AII

Figure S.8 shows that eigenenergies in the 2D and 3D SU(2) models in class AII have no singular structure around the real axis and are quite equally distributed in the complex plane. The IPR shows that both in 2D and 3D the AT occurs at $E = 0$ as well as $\text{Im} E \neq 0$. We also confirm that the critical behavior on the real axis is similar to the critical behavior far away from the real axis.

DENSITY OF STATES AROUND THE REAL AXIS

In this section, we heuristically discuss the DoS around the real axis (i.e., $\text{Im} E = 0$), assuming that the non-Hermitian disorder is weak and treating it as a perturbation.

Class AII

For the non-Hermitian SU(2) model in class AII, we observe the soft gap of DoS around the real axis. To explain it heuristically, we begin with a disordered Hermitian Hamiltonian $H$ in class AII, where we have a Kramers pair on the real axis, $\psi$ and $\psi' = T\psi$, for eigenenergy $\varepsilon$. Here, $T$ is the time-reversal anti-unitary operator. In the real space basis, they are expressed explicitly as $\psi = [a_1, b_1, a_2, b_2, \cdots, a_N, b_N]^T$ and $\psi' = [b_1^*, -a_1^*, b_2^*, -a_2^*, \cdots, b_N^*, -a_N^*]^T$, where $a$ and $b$ refer to the amplitudes on $\uparrow$-spin and $\downarrow$-down, and $N = L^d$ with the spatial dimension $d$. We then introduce a non-Hermitian on-site potential that respects the time-reversal symmetry,

$$V = i \text{diag}[w_1, w_2, \cdots, w_N] \otimes \sigma_z,$$

(S.66)

with real $w_i$. To study how the Kramers pair on the real axis is split by the non-Hermitian on-site potential $V$, let us consider a $2 \times 2$ effective Hamiltonian $h$,

$$h = [\psi, \psi']^\dagger (H + V) [\psi, \psi'] = \begin{pmatrix} \varepsilon + \langle \psi | V | \psi \rangle & \langle \psi | V | \psi' \rangle \\ \langle \psi' | V | \psi \rangle & \varepsilon + \langle \psi' | V | \psi' \rangle \end{pmatrix} \equiv \begin{pmatrix} \varepsilon + i h_{1,1} & i h_{1,2} \\ i h_{2,1} & \varepsilon + i h_{2,2} \end{pmatrix}. $$

(S.67)

The matrix elements are calculated as

$$h_{1,1} = \sum_i w_i (|a_i|^2 - |b_i|^2) =: \Delta_z,$$

(S.68)

$$h_{1,2} = 2 \sum_i w_i a_i b_i^* =: \Delta_x - i \Delta_y,$$

$$h_{2,1} = 2 \sum_i w_i a_i b_i = \Delta_x + i \Delta_y,$$

$$h_{2,2} = \sum_i w_i (-|a_i|^2 + |b_i|^2) = -\Delta_x,$$

which are expressed as

$$h = \varepsilon + i (\sigma_x \Delta_x + \sigma_y \Delta_y + \sigma_z \Delta_z).$$

(S.69)

Here, $\Delta_i (i = x, y, z)$ are all real and random numbers. Consistently, $h$ respects time-reversal symmetry $\sigma_y h^\dagger \sigma_y = h$.

The eigenenergies of $h$ are $\varepsilon \pm i \sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}$, and the energy splitting is along the imaginary axis and equals $2\sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2}$. Assuming that each of the three $\Delta_i$ obeys the independent Gaussian distributions due to the central limit theorem, the probability distribution of the energy splitting from the real axis is estimated as

$$P(s) = \frac{1}{N} \int_{-\infty}^{\infty} d\Delta_x \int_{-\infty}^{\infty} d\Delta_y \int_{-\infty}^{\infty} d\Delta_z \delta \left( s - \sqrt{\Delta_x^2 + \Delta_y^2 + \Delta_z^2} \right) \exp \left( -A (\Delta_x^2 + \Delta_y^2 + \Delta_z^2) \right),$$

$$N \equiv \int_{-\infty}^{\infty} d\Delta_x \int_{-\infty}^{\infty} d\Delta_y \int_{-\infty}^{\infty} d\Delta_z \exp \left( -A (\Delta_x^2 + \Delta_y^2 + \Delta_z^2) \right) = \left( \frac{\pi}{A} \right)^3,$$
where $A > 0$ is a normalization constant. This leads to the Wigner distribution for the unitary ensemble:

$$P(s) = \frac{4A^3}{\sqrt{\pi}} s^2 \exp \left(-As^2\right).$$  \hspace{1cm} (S.70)$$

Notably, $P(s)$ vanishes for small $s$, i.e., $P(s) \propto s^2 \to 0$ for $s \to 0$. This means that the probability of the energy levels having a small imaginary part is significantly suppressed, hence the small density of states around the real axis.

**Class AI**

For the non-Hermitian O(1) model in class AI, we observe the peak of DoS around the real axis, $\text{Im} E = 0$. To explain it, we begin with a disordered Hermitian Hamiltonian $H$ in class AI, introduce a non-Hermitian on-site potential $V$ that respects time-reversal symmetry, and study two nearest-neighbor eigenmodes. The non-Hermitian on-site potential reads

$$V = i \text{diag}[w_1, w_2, \cdots, w_N] \otimes \sigma_z$$  \hspace{1cm} (S.71)$$

with real random number $w_i$ ($i = 1, 2, \cdots, N$). The system respects time-reversal symmetry:

$$\sigma_x (H + V)^* \sigma_x = H + V,$$  \hspace{1cm} (S.72)$$

which imposes a constraint on each of the two eigenmodes of the disordered Hermitian Hamiltonian $H$ as $\psi_1 = [a_1, a_1^*, a_2, a_2^*, \cdots, a_N, a_N^*]^T$ and $\psi_2 = [b_1, b_1^*, b_2, b_2^*, \cdots, b_N, b_N^*]^T$. Let $\varepsilon_1$ and $\varepsilon_2$ be the corresponding eigenenergies of $\psi_1$ and $\psi_2$, respectively. The two by two effective Hamiltonian $h$ reads

$$h = [\psi_1, \psi_2]^T (H + V) [\psi_1, \psi_2] = \begin{pmatrix} \varepsilon_1 + \langle \psi_1 | V | \psi_1 \rangle & \langle \psi_1 | V | \psi_2 \rangle \\ \langle \psi_2 | V | \psi_1 \rangle & \varepsilon_2 + \langle \psi_2 | V | \psi_2 \rangle \end{pmatrix} = \begin{pmatrix} \varepsilon_1 + i h_{1,1} & i h_{1,2} \\ i h_{2,1} & \varepsilon_2 + i h_{2,2} \end{pmatrix}.$$  \hspace{1cm} (S.73)$$

The matrix elements are calculated as

$$h_{1,1} = \sum_i w_i (a_i^* a_i - a_i a_i^*) = 0,$$  \hspace{1cm} (S.74)$$

$$h_{1,2} = 2 \sum_i w_i (a_i^* b_i - a_i b_i^*) = i \Delta_0,$$ $$h_{2,1} = 2 \sum_i w_i (b_i^* a_i - b_i a_i^*) = -i \Delta_0,$$ $$h_{2,2} = \sum_i w_i (b_i^* b_i - b_i b_i^*) = 0,$$

which are expressed as

$$h = \begin{pmatrix} \varepsilon_1 & -\Delta_0 \\ \Delta_0 & \varepsilon_2 \end{pmatrix}.$$  \hspace{1cm} (S.75)$$

Here, $\Delta_0$ is a real random number. Then, $h$ is a real matrix and indeed respects time-reversal symmetry $h^* = h$.

The two eigenenergies of $h$ are

$$\varepsilon_1 + \varepsilon_2 = \frac{\varepsilon_1 - \varepsilon_2}{2} \pm \sqrt{\left(\frac{\varepsilon_1 - \varepsilon_2}{2}\right)^2 - \Delta_0^2},$$  \hspace{1cm} (S.76)$$

which are real for $|\varepsilon_1 - \varepsilon_2|/2 \geq |\Delta_0|$. In class AII, by contrast, the eigenenergies cannot be real unless the stronger constraint $\Delta_x = \Delta_y = \Delta_z = 0$ is satisfied. This means that real eigenenergies are more stable against non-Hermitian perturbations in class AI than in class AII. Consequently, the eigenenergies remain real more easily in class AI, which corresponds to the sharp peak of DoS on the real axis.
DENSITY OF STATES FOR THE GINIBRE ORTHOGONAL AND SYMPLECTIC ENSEMBLES

The Ginibre ensembles are ensembles of non-Hermitian random matrices [76]. They are useful for understanding the energy level statistics and the ATs in non-Hermitian disordered systems. We have three kinds of the Ginibre ensembles: Ginibre unitary ensemble (GinUE) (no restriction on $H$; class A), Ginibre orthogonal ensemble (GinOE) ($H^* = H$; class AI), and Ginibre symplectic ensemble (GinSE) ($\sigma_y H^* \sigma_y = H$; class AII). Real and imaginary parts of each element of non-Hermitian random matrices are independent and produced by the same Gaussian distribution. The GinOE is the ensemble of non-Hermitian but real random matrices. Each element of real random matrices is independent and produced by the same Gaussian distribution. For the GinSE, non-Hermitian random matrices are defined to satisfy $\sigma_y H^* \sigma_y = H$ with

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$  \hspace{1cm} (S.77)

so that $H$ has the following symplectic structure:

$$H = \begin{pmatrix} X & Y \\ -Y^* & X^* \end{pmatrix},$$  \hspace{1cm} (S.78)

where $X, Y$ are generic non-Hermitian random matrices with no constraint.

As shown in Figs. S.6 and S.7, the DoS of non-Hermitian disordered systems in class AI has a sharp peak on the real axis, while the DoS in class AII has a soft gap on the real axis. In order to understand these behavior, we study the DoS of the GinOE and GinSE. The eigenenergy distribution and the DoS along the imaginary axis are shown in Fig. S.9. The DoS around the real axis for non-Hermitian disordered Hamiltonians in classes AI and AII is qualitatively consistent with the DoS of the GinOE and GinSE, respectively.
FIG. S.1. Normalized localization lengths $\Lambda$ as a function of the disorder strength $W$ for the 3D O(1) model in class AI at (a) $E = 0$ and (b) $E = 0.5i$. The points with the error bars are the numerical data with the different system sizes $L$. The colored curves are the fitted curves with the expansion order $(m_1, n_1, m_2, n_2)$.

FIG. S.2. Normalized localization lengths $\Lambda$ as a function of the disorder strength $W = W_r = W_i$ for the (a) 2D and (b) 3D SU(2) model in class AII$^*$ with $E = 0$. The points with the error bars are the numerical data with the different system sizes $L$. The colored curves are the fitted curves with the expansion order $(m_1, n_1, m_2, n_2)$. 
FIG. S.3. Normalized localization lengths $\Lambda$ as a function of the disorder strength $W = W_r = W_i$ for (a) the 2D SU(2) model in class AII with $E = 0.01i$, (b) the 3D SU(2) model in class AII with $E = 0$, and (c) the 3D SU(2) model in class AII with $E = i$. The points with the error bars are the numerical data with the different system sizes $L$. The colored curves are the fitted curves with the expansion order $(m_1, n_1, m_2, n_2)$. 
FIG. S.4. Normalized localization lengths $\Lambda$ as a function of the disorder strength $W \equiv W_i$ for the 2D SU(2) model in (a) class CII$^\dagger$ and (b) class DIII. The points with the error bars are the numerical data with the different system sizes $L$. The colored curves are the fitted curves with the expansion order $(m_1, n_1, m_2, n_2)$. 

(a) $E = 0$, 2D, class CII$^\dagger$, $(1,3,0,0)$
(b) $E = 0$, 2D, class DIII, $(1,2,0,0)$
FIG. S.5. Phase diagrams of the O(1) and SU(2) models for (a) 3D class AI, (b) 3D class AII, and (c) 2D class AII in terms of the disorder strength $W$ and the imaginary part of eigenenergies $E$. The real part of $E$ is set to 0. The blue squares and red circles are the phase boundaries for the Anderson transitions. The phase boundaries are determined by the localization lengths. (d) Normalized localization length as a function of $W = W_r = W_i$ for the 2D class AII model at $E = 0.01i$. 
FIG. S.6. (a), (c), (e) Eigenenergy distributions in the complex plane. Each point corresponds to each eigenenergy in one sample. The color of the points describes $1/I$ with the inverse participation ratios $I$ for the corresponding eigenmodes. (b), (d), (f) Density of states (DoS) for the imaginary part of eigenenergies. The numerical calculations are performed for the 3D O(1) model in class AI with the cubic system size $L = 20$, under the periodic boundary conditions, and with the disorder strength $W = 0, 10, 20$. 
FIG. S.7. (a), (c) Heat maps of eigenenergy density in the complex plane. (b), (d) Density of states (DoS) for the imaginary part of eigenenergies. Eigenenergies are calculated for the 2D and 3D SU(2) model in class AII under the periodic boundary conditions, with $W_r = W_i = 1$, and over the 640 samples with different disorder realizations. The system sizes are $70 \times 70$ for 2D and $16 \times 16 \times 16$ for 3D.
FIG. S.8. Eigenenergy distributions in the complex plane. Each point corresponds to each eigenenergy in one sample. The color of the points describes $1/I$ with the inverse participation ratios $I$ for the corresponding eigenmodes. The eigenenergies are calculated for the SU(2) models in class AII$^*$ with the system sizes $60 \times 60$ for 2D and $16 \times 16 \times 16$ for 3D. The periodic boundary conditions are imposed in all the directions for both 2D and 3D. The disorder strength is set to $W_r = W_i = 4$ for 2D and $W_r = W_i = 7$ for 3D.
FIG. S.9. Heat maps of eigenenergy density in the complex plane for (a) the Ginibre orthogonal ensemble and (c) the Ginibre symplectic ensemble. The color describes the density of states (DoS) in the complex plane. The DoS for the imaginary part of eigenenergies of (b) the Ginibre orthogonal ensemble and (d) the Ginibre symplectic ensemble. The numerical data come from the 640 realizations of random matrices with the size $10^4 \times 10^4$. 