Analytic operator-valued generalized Feynman integral on function space

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Abstract
In this paper an analytic operator-valued generalized Feynman integral was studied on a very general Wiener space \( C_{a,b}[0,T] \). The general Wiener space \( C_{a,b}[0,T] \) is a function space which is induced by the generalized Brownian motion process associated with continuous functions \( a \) and \( b \). The structure of the analytic operator-valued generalized Feynman integral is suggested and the existence of the analytic operator-valued generalized Feynman integral is investigated as an operator from \( L^1(\mathbb{R}, \nu_{\delta,a}) \) to \( L^\infty(\mathbb{R}) \) where \( \nu_{\delta,a} \) is a \( \sigma \)-finite measure on \( \mathbb{R} \) given by

\[
d\nu_{\delta,a} = \exp\{\delta \text{Var}(a)u^2\}du,
\]

where \( \delta > 0 \) and \( \text{Var}(a) \) denotes the total variation of the mean function \( a \) of the generalized Brownian motion process. It turns out in this paper that the analytic operator-valued generalized Feynman integrals of functionals defined by the stochastic Fourier–Stieltjes transform of complex measures on the infinite dimensional Hilbert space \( C'_{a,b}[0,T] \) are elements of the linear space

\[
\bigcap_{\delta > 0} \mathcal{L}(L^1(\mathbb{R}, \nu_{\delta,a}), L^\infty(\mathbb{R})).
\]

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1. Introduction
Before giving a basic survey and a motivation for our topic we fix some notation. Let \( \mathbb{C} \), \( \mathbb{C}_+ \) and \( \mathbb{C}_+ \) denote the set of complex numbers, complex
numbers with positive real part and nonzero complex numbers with nonnegative real part, respectively. For all \( \lambda \in \mathbb{C}_+ \), \( \lambda^{1/2} \equiv \sqrt{\lambda} \) (or \( \lambda^{-1/2} \)) is always chosen to have positive real part. Furthermore, let \( C[0,T] \) denote the space of real-valued continuous functions \( x \) on \([0,T]\) and let \( C_0[0,T] \) denote those \( x \) in \( C[0,T] \) such that \( x(0) = 0 \). The function space \( C_0[0,T] \) is referred to as one-parameter Wiener space, and we let \( m_w \) denote Wiener measure. Given two Banach spaces \( X \) and \( Y \), let \( \mathcal{L}(X,Y) \) denote the space of continuous linear operators from \( X \) to \( Y \).

Let \( F \) be a \( \mathbb{C} \)-valued measurable functional on \( C[0,T] \). For \( \lambda > 0 \), \( \psi \in L^2(\mathbb{R}) \), and \( \xi \in \mathbb{R} \), consider the Wiener integral

\[
(I_\lambda(F)\psi)(\xi) \equiv \int_{C_0[0,T]} F(\lambda^{-1/2} x + \xi) \psi(\lambda^{-1/2} x(T) + \xi) dm_w(x).
\]  

(1.1)

In the application of the Feynman integral to quantum theory, the function \( \psi \) in (1.1) corresponds to the initial condition associated with Schrödinger equation.

In [1], Cameron and Storvick considered the following natural and interesting questions. Under what conditions on \( F \) will the linear operator \( I_\lambda(F) \) given by (1.1) be an element of \( \mathcal{L}(L^2(\mathbb{R}),L^2(\mathbb{R})) \)? If so, does the operator valued function \( \lambda \to I_\lambda(F) \) have an analytic extension, write \( I_\lambda^{an}(F) \) (it is called the analytic operator-valued Wiener integral of \( F \) with parameter \( \lambda \)), to \( \mathbb{C}_+ \)? If so, for each nonzero real number \( q \), does the limit

\[
J_q^{an}(F) \equiv \lim_{\lambda \to iq} I_\lambda^{an}(F)
\]

exist in some topological (normed) structure? The functional \( J_q^{an}(F) \) (if it exists) is called the analytic operator-valued Feynman integral of \( F \) with parameter \( q \).

Cameron and Storvick in [1] introduced the analytic operator-valued function space “Feynman integral” \( J_q^{an}(F) \), which mapped an \( L^2(\mathbb{R}) \) function \( \psi \) into an \( L^2(\mathbb{R}) \) function \( J_q^{an}(F)\psi \). In [1] and several subsequent papers [2, 3, 14, 15, 16, 17, 18, 19, 21, 22], the existence of this integral as an element of \( \mathcal{L}(L^2(\mathbb{R}),L^2(\mathbb{R})) \) was established for various functionals. Next, the existence of the integral as an element of \( \mathcal{L}(L^1(\mathbb{R}),L^\infty(\mathbb{R})) \) was established in [1, 3, 13, 22]. Finally, the \( L_p \to L_{p'} \) theory (\( 1 < p \leq 2 \) and \( 1/p + 1/p' = 1 \)) was developed as an element of \( \mathcal{L}(L^p(\mathbb{R}),L^{p'}(\mathbb{R})) \) in [22].

The Wiener space \( C_0[0,T] \) can be considered as the space of sample paths of standard Brownian motion process (SBMP). Thus, in various Feynman integration theories, the integrand \( F \) of the Feynman integral (1.1) is a functional of the SBMP, see [1, 2, 3, 4, 5, 6, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23].

Let \( D = [0,T] \) and let \( (\Omega, \mathcal{F}, P) \) be a probability space. By the definition, a generalized Brownian motion process (GBMP) on \( D \times \Omega \) is a Gaussian process \( Y \equiv \{ Y_t \}_{t \in D} \) such that \( Y_0 = 0 \) almost surely and for any \( 0 \leq s < t \leq T \),

\[
Y_t - Y_s \sim N(\alpha(t) - \alpha(s), \beta(t) - \beta(s)),
\]

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where \( N(m, \sigma^2) \) denotes the normal distribution with mean \( m \) and variance \( \sigma^2 \), \( a(t) \) is a continuous real-valued function on \([0, T]\) and \( b(t) \) is an increasing continuous real-valued function on \([0, T]\). Thus a GBMP is determined by the continuous functions \( a(t) \) and \( b(t) \). The function space \( C_{a,b}[0,T] \), induced by GBMP, was introduced by Yeh [24, 25] and was used extensively in [6,7,8,9,10,11,12]. The function space \( C_{a,b}[0,T] \) used in [6,7,8,9,10,11,12] can be considered as the space of sample paths of the GBMP. The generalized Feynman integral studied in [6,7,9,10] are scalar-valued. In this paper, the analytic operator-valued generalized Feynman integral \((AOVG'Feynman'I)\) of functionals \( F \) on the general Wiener space \( C_{a,b}[0,T] \) is investigated as an element of \( L(L^1(\mathbb{R},\nu_{\delta,a}),L^\infty(\mathbb{R})) \), where \( \nu_{\delta,a} \) is a measure on \( \mathbb{R} \) given by

\[
d\nu_{\delta,a} = \exp(\delta \text{Var}(a)u^2)du,
\]

and where \( \delta > 0 \) and \( \text{Var}(a) \) denotes the total variation of the mean function \( a \) of the GBMP. It turns out in this paper that the AOVG’Feynman’Is of functionals \( F \) defined by the stochastic Fourier–Stieltjes transform of complex measures on the infinite dimensional Hilbert space \( C_{a,b}'[0,T] \), the space of absolutely continuous functions in \( C_{a,b}[0,T] \), are elements of the linear space

\[
\bigcap_{\delta > 0} L(L^1(\mathbb{R},\nu_{\delta,a}),L^\infty(\mathbb{R})).
\]

Note that when \( a(t) \equiv 0 \) and \( b(t) = t \), the GBMP is an SBMP, and so the function space \( C_{a,b}[0,T] \) reduces to the classical Wiener space \( C_{0}[0,T] \). But we are obliged to point out that an SBMP used in [1,2,3,4,5,13,14,15,16,17,18,19,20,21,22,23] is stationary in time and is free of drift. While, the GBMP used in this paper, as well as in [6,7,8,9,10,11,12], is nonstationary in time and is subject to a drift \( a(t) \). It turns out, as noted in Remark 4.2 below, that including a drift term \( a(t) \) makes establishing the existence of the analytic operator-valued generalized function space integral \((AOVGFSI)\) and AOVG’Feynman’I of functionals on \( C_{a,b}[0,T] \) very difficult.

The results in this paper are quite a lot more complicated because the GBMP is neither stationary nor centered.

2. Preliminaries

In this section, we briefly list some of the preliminaries from [6,7,9,10] that we need to establish our results in next sections; for more details, see [6,7,9,10].

Let \((C_{a,b}[0,T],B(C_{a,b}[0,T]),\mu)\) denote the function space induced by the GBMP \( Y \) determined by continuous functions \( a(t) \) and \( b(t) \), where \( B(C_{a,b}[0,T]) \) is the Borel \( \sigma \)-field induced by the sup-norm, see [24, 25]. We assume in this paper that \( a(t) \) is an absolutely continuous real-valued function on \([0, T]\) with \( a(0) = 0 \), \( a'(t) \in L^2[0,T] \), and \( b(t) \) is an increasing, continuously differentiable real-valued function with \( b(0) = 0 \) and \( b'(t) > 0 \) for each \( t \in [0,T] \). We complete this function space to obtain the complete probability measure space.
Stieltjes measures on \([0, T]\), \(W(C_{a,b}[0, T])\), \(\mu\) where \(W(C_{a,b}[0, T])\) is the set of all \(\mu\)-Carathéodory measurable subsets of \(C_{a,b}[0, T]\).

We can consider the coordinate process \(X: [0, T] \times C_{a,b}[0, T] \to \mathbb{R}\) given by \(X(t, x) = x(t)\) which is a continuous process. The separable process \(X\) induced by \(Y\) also has the following properties:

(i) \(X(0, x) = x(0) = 0\) for every \(x \in C_{a,b}[0, T]\).

(ii) For any \(s, t \in [0, T]\) with \(s \leq t\) and \(x \in C_{a,b}[0, T]\),

\[
x(t) - x(s) \sim N(a(t) - a(s), b(t) - b(s)).
\]

Thus it follows that for \(s, t \in [0, T]\), \(\text{Cov}(X(s, x), X(t, x)) = \min\{b(s), b(t)\}\).

A subset \(B\) of \(C_{a,b}[0, T]\) is said to be scale-invariant measurable provided \(\rho B\) is \(W(C_{a,b}[0, T])\)-measurable for all \(\rho > 0\), and a scale-invariant measurable set \(N\) is said to be scale-invariant null provided \(\mu(\rho N) = 0\) for all \(\rho > 0\). A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s.a.e.). A functional \(F\) is said to be scale-invariant measurable provided \(F\) is defined on a scale-invariant measurable set and \(F(\rho \cdot)\) is \(W(C_{a,b}[0, T])\)-measurable for every \(\rho > 0\).

Let \(L^2_{a,b}[0, T]\) be the separable Hilbert space of functions on \([0, T]\) which are Lebesgue measurable and square integrable with respect to the Lebesgue–Stieltjes measures on \([0, T]\) induced by \(b(t)\) and \(a(t)\): i.e.,

\[
L^2_{a,b}[0, T] = \left\{ v : \int_0^T v^2(s)db(s) < +\infty \text{ and } \int_0^T v^2(s)d|a|(s) < +\infty \right\}
\]

where \(|a|(t)\) denotes the total variation function of \(a(t)\) on \([0, T]\). The inner product on \(L^2_{a,b}[0, T]\) is defined by \((u, v)_{a,b} = \int_0^T u(t)v(t)db(t) + |a|(t)\). Note that \(\|u\|_{a,b} = \sqrt{(u, u)_{a,b}} = 0\) if and only if \(u(t) = 0\) a.e. on \([0, T]\) and that all functions of bounded variation on \([0, T]\) are elements of \(L^2_{a,b}[0, T]\). Also note that if \(a(t) \equiv 0\) and \(b(t) = t\), then \(L^2_{a,b}[0, T] = L^2[0, T]\). In fact,

\[
(L^2_{a,b}[0, T], \| \cdot \|_{a,b}) \subset (L^2_{0,b}[0, T], \| \cdot \|_{0,b}) = (L^2[0, T], \| \cdot \|_2)
\]

since the two norms \(\| \cdot \|_{0,b}\) and \(\| \cdot \|_2\) are equivalent.

Throughout the rest of this paper, we consider the linear space

\[
C'_{a,b}[0, T] = \left\{ w \in C_{a,b}[0, T] : w(t) = \int_0^t z(s)db(s) \text{ for some } z \in L^2_{a,b}[0, T] \right\}.
\]

For \(w \in C'_{a,b}[0, T]\), with \(w(t) = \int_0^t z(s)db(s)\) for \(t \in [0, T]\), let \(D : C'_{a,b}[0, T] \to L^2_{a,b}[0, T]\) be defined by the formula

\[
Dw(t) = z(t) = \frac{w'(t)}{b'(t)}. \tag{2.1}
\]
Then $C_{a,b} = C_{a,b}[0,T]$ with inner product

$$(w_1, w_2)_{C_{a,b}} = \int_0^T Dw_1(t) Dw_2(t) db(t) = \int_0^T z_1(t) z_2(t) db(t)$$

is also a separable Hilbert space.

Note that the two separable Hilbert spaces $L^2_{a,b}[0,T]$ and $C_{a,b}[0,T]$ are topologically homeomorphic under the linear operator given by equation (2.1). The inverse operator of $D$ is given by

$$(D^{-1} z)(t) = \int_0^t z(s) db(s)$$

for $t \in [0,T]$.

In this paper, in addition to the conditions put on $a(t)$ above, we now add the condition

$$\int_0^T |a'(t)|^2 d|a|(t) < +\infty. \quad (2.2)$$

Then, the function $a : [0,T] \to \mathbb{R}$ satisfies the condition (2.2) if and only if $a(\cdot)$ is an element of $C'^{r}_{a,b}[0,T]$, see [11, 12]. Under the condition (2.2), we observe that for each $w \in C'_{a,b}[0,T]$ with $Dw = z$,

$$(w, a)_{C_{a,b}} = \int_0^T Dw(t) Da(t) db(t) = \int_0^T z(t) a'(t) b(t) db(t) = \int_0^T z(t) da(t).$$

Next we will define a Paley–Wiener–Zygmund (PWZ) stochastic integral. Let $\{g_j\}_{j=1}^{\infty}$ be a complete orthonormal set in $C'_{a,b}[0,T]$ such that for each $j = 1, 2, \ldots$, $D g_j = \alpha_j$ is of bounded variation on $[0,T]$. For each $w = D^{-1} z \in C'_{a,b}[0,T]$, the PWZ stochastic integral $(w, x)^\sim$ is defined by the formula

$$(w, x)^\sim = \lim_{n \to \infty} \int_0^T \sum_{j=1}^n (w, g_j)_{C_{a,b}} D g_j(t) dx(t)$$

$$= \lim_{n \to \infty} \int_0^T \sum_{j=1}^n \int_0^T z(s) \alpha_j(s) db(s) \alpha_j(t) dx(t)$$

for all $x \in C_{a,b}[0,T]$ for which the limit exists.

It is known that for each $w \in C'_{a,b}[0,T]$, the PWZ stochastic integral $(w, x)^\sim$ exists for $\mu$-a.e. $x \in C_{a,b}[0,T]$. If $Dw = z \in L^2_{a,b}[0,T]$ is of bounded variation on $[0,T]$, then the PWZ stochastic integral $(w, x)^\sim$ equals the Riemann–Stieltjes integral $\int_0^T z(t) dx(t)$. It also follows that for $w, x \in C'_{a,b}[0,T]$, $(w, x)^\sim = (w, x)_{C_{a,b}}$. For each $w \in C'_{a,b}[0,T]$, the PWZ stochastic integral $(w, x)^\sim$ is a Gaussian random variable on $C_{a,b}[0,T]$ with mean $(w, a)_{C_{a,b}}$ and variance $\|w\|_{C_{a,b}}^2$. Note that for all $w_1, w_2 \in C_{a,b}[0,T]$,

$$\int_{C_{a,b}[0,T]} (w_1, x)^\sim (w_2, x)^\sim d\mu(x) = (w_1, w_2)_{C_{a,b}} + (w_1, x)_{C_{a,b}} (w_2, x)_{C_{a,b}}.$$
Hence we see that for \( w_1, w_2 \in C'_{a,b}[0,T], (w_1, w_2)_{C'_{a,b}} = 0 \) if and only if \((w_1, x)\) and \((w_2, x)\) are independent random variables. We thus have the following function space integration formula: let \( \{e_1, \ldots, e_n\} \) be an orthonormal set in \((C'_{a,b}[0,T], \| \cdot \|_{C'_{a,b}})\), and given a Lebesgue measurable function \( r : \mathbb{R}^n \to \mathbb{C} \), let \( R : C_{a,b}[0,T] \to \mathbb{C} \) be given by equation

\[
R(x) = r((e_1, x)^\sim, \ldots, (e_n, x)^\sim).
\]

Then

\[
\int_{C_{a,b}[0,T]} R(x)d\mu(x) \equiv \int_{C_{a,b}[0,T]} r((e_1, x)^\sim, \ldots, (e_n, x)^\sim)d\mu(x)
\]

\[
= (2\pi)^{-n/2} \int_{\mathbb{R}^n} r(u_1, \ldots, u_n) \exp\left\{-\sum_{j=1}^n \frac{(u_j - (e_j, a)_{C'_{a,b}})^2}{2}\right\}du_1 \cdots du_n
\]

in the sense that if either side of equation \( 2.3 \) exists, both sides exist and equality holds.

The following integration formula is also used in this paper:

\[
\int_{\mathbb{R}} \exp\{-a u^2 + \beta u\} du = \sqrt{\frac{\pi}{a}} \exp\left\{\frac{\beta^2}{4a}\right\}
\]

for complex numbers \( \alpha \) and \( \beta \) with \( \text{Re}(\alpha) > 0 \).

3. Analytic operator-valued generalized function space integral

In this section, we introduce the definition of the AOVGFSI as an element of \( \mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R})) \). The definition below is based on the previous definitions in [3, 4, 5, 13, 21, 22, 23].

**Definition 3.1.** Let \( F : C[0,T] \to \mathbb{C} \) be a measurable functional and let \( h \) be an element of \( C'_{a,b}[0,T]\setminus\{0\} \). Given \( \lambda > 0, \psi \in L^1(\mathbb{R}) \) and \( \xi \in \mathbb{R} \), let

\[
(I_{\lambda}(F;h)\psi)(\xi) \equiv \int_{C_{a,b}[0,T]} F(\lambda^{-1/2}x + \xi, \psi(\lambda^{-1/2}(h, x)^\sim + \xi)d\mu(x). \tag{3.1}
\]

If \( I_{\lambda}(F;h)\psi \) is in \( L^\infty(\mathbb{R}) \) as a function of \( \xi \) and if the correspondence \( \psi \to I_{\lambda}(F;h)\psi \) gives an element of \( \mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R})) \), we say that the operator-valued generalized function space integral (OVGFSI) \( I_{\lambda}(F;h) \) exists.

Let \( \Gamma \) be a region in \( \mathbb{C}_+ \) such that \( \text{Int}(\Gamma) \) is a simply connected domain in \( \mathbb{C}_+ \) and \( \text{Int}(\Gamma) \cap (0, +\infty) \) is a nonempty open interval of positive real numbers. Suppose that there exists an \( \mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R})) \)-valued function which is analytic in \( \lambda \) on \( \text{Int}(\Gamma) \) and agrees with \( I_{\lambda}(F;h) \) on \( \text{Int}(\Gamma) \cap (0, +\infty) \), then this \( \mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R})) \)-valued function is denoted by \( I_{\lambda}^{\text{an}}(F;h) \) and is called the AOVGFSI of \( F \) associated with \( \lambda \).

The notation \( \| \cdot \|_o \) will be used for the norm of operators in \( \mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R})) \).
Remark 3.2. (i) In equation (3.1) above, choosing \( h(t) = \int_0^t db(s) = b(t) \in C_{a,b}^r[0,T] \), we obtain
\[
(h,x)^\sim = (b,x)^\sim = \int_0^T Db(t)dx(t) = \int_0^T dx(t) = x(T).
\]
In this case, equation (3.1) is given by
\[
(I_\lambda(\mathcal{F};b)(\psi))(\xi) = \int_{C_{a,b}[0,T]} F(\lambda^{-1/2} x + \xi) \psi(\lambda^{-1/2} x(T) + \xi) d\mu(x).
\]
Moreover, if \( a(t) \equiv 0 \) and \( b(t) = t \) on \([0,T]\), then the function space \( C_{a,b}[0,T] \) reduces to the classical Wiener space \( C_0[0,T] \) and the definition of the OVGFSI \( I_\lambda(F;b) \) in equation (3.2) agrees with the definitions of the analytic function space integrals \( I_\lambda(F) \) with \( \lambda > 0 \) defined in [1, 2, 3, 4, 5, 7, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23].

(ii) In the case that \( a(t) \equiv 0 \) and \( h(t) = b(t) = t \) on \([0,T]\), choosing \( \Gamma = C_+ \cap \{ \lambda \in \mathbb{C} : |\lambda| < \lambda_0 \} \) for some \( \lambda_0 \in (0, +\infty) \), then the definition of the AOVGFSI \( I_{\lambda}^{\text{an}}(\mathcal{F};b) \) (if it exists) agrees with the definitions of the analytic operator-valued function space integral \( I_{\lambda}^{\text{an}}(\mathcal{F}) \) associated with \( \lambda > 0 \) defined in [24, 25].

4. The \( \mathcal{F}(C_{a,b}[0,T]) \) theory

In [6, 8], Chang, Choi and Lee introduced a Banach algebra \( \mathcal{F}(C_{a,b}[0,T]) \) of functionals on function space \( C_{a,b}[0,T] \), each of which is a stochastic Fourier transform of \( \mathbb{C} \)-valued Borel measure on \( C_{a,b}[0,T] \), and showed that it contains many functionals of interest in Feynman integration theory. In [6], Chang and Choi showed that the analytic (but scalar-valued) generalized Feynman integral exists for functionals in \( \mathcal{F}(C_{a,b}[0,T]) \). In this section, we show that the AOVGFSI \( I_{\lambda}^{\text{an}}(F;\mu) \) is in \( L^1(\mathbb{R}), L^\infty(\mathbb{R}) \) for functionals \( F \) in \( \mathcal{F}(C_{a,b}[0,T]) \).

Let \( \mathcal{M}(C_{a,b}^r[0,T]) \) denote the space of \( \mathbb{C} \)-valued, countably additive (and hence finite) Borel measures on \( C_{a,b}^r[0,T] \). We define the Fresnel type class \( \mathcal{F}(C_{a,b}[0,T]) \) of functionals on \( C_{a,b}[0,T] \) as the space of all stochastic Fourier–Stieltjes transforms of elements of \( \mathcal{M}(C_{a,b}^r[0,T]) \); that is, \( F \in \mathcal{F}(C_{a,b}[0,T]) \) if and only if there exists a measure \( f \) in \( \mathcal{M}(C_{a,b}^r[0,T]) \) such that
\[
F(x) = \int_{C_{a,b}^r[0,T]} \exp\{i(w,x)^\sim\} df(w)
\]
for s-a.e. \( x \in C_{a,b}[0,T] \).

More precisely, since we shall identify functionals which coincide s-a.e. on \( C_{a,b}[0,T], \mathcal{F}(C_{a,b}[0,T]) \) can be regarded as the space of all s-equivalence classes of functionals having the form (4.1).

We note that \( \mathcal{M}(C_{a,b}^r[0,T]) \) is a Banach algebra under the total variation norm and with convolution as multiplication. The Fresnel type class \( \mathcal{F}(C_{a,b}[0,T]) \) is a Banach algebra with norm
\[
\|F\| = \|f\| = \int_{C_{a,b}^r[0,T]} |df(w)|.
\]
Throughout this paper, we will use the following notations for convenience:

where \( s-a.e. \) defined on \( C \) and by (4.1). For a more detailed study of functionals in \( F \) for (4.7) above, \( \lambda, \xi, v, h, w, \psi \) be a (fixed) function in \( F_{a,b}[0, T] \), we obtain an orthonormal set \( \{ e_1, e_2(w) \} \) in \( C_{a,b}'[0, T] \), by the Gram–Schmidt process, such that \( h = \| h \|_{C_{a,b}'} e_1 \) and

\[
\begin{align*}
  w = (w, e_1)_{C_{a,b}'} e_1 + \beta_w e_2(w)
\end{align*}
\]

where

\[
\begin{align*}
  \beta_w = \| w - (w, e_1)_{C_{a,b}'} e_1 \|_{C_{a,b}'} = \left[ \| w \|_{C_{a,b}'}^2 - (w, e_1)_{C_{a,b}'}^2 \right]^{1/2}.
\end{align*}
\]

Remark 4.1. If \( F \) is in \( F(C_{a,b}[0, T]) \), then \( F \) is scale-invariant measurable and \( s-a.e. \) defined on \( C_{a,b}[0, T] \). If \( x \) in \( C_{a,b}[0, T] \) is such that \( F(x) \) is defined, then by (4.1) and the definition of the PWZ stochastic integral, it follows that \( F(x + \xi) = F(x) \) for all \( \xi \in \mathbb{R} \).

Let \( h \) be a (fixed) function in \( C_{a,b}'[0, T] \{0\} \). Then for any function \( w \) in \( C_{a,b}'[0, T] \), we obtain an orthonormal set \( \{ e_1, e_2(w) \} \) in \( C_{a,b}'[0, T] \), by the Gram–Schmidt process, such that \( h = \| h \|_{C_{a,b}'} e_1 \) and

\[
\begin{align*}
  w = (w, e_1)_{C_{a,b}'} e_1 + \beta_w e_2(w)
\end{align*}
\]

where

\[
\begin{align*}
  \beta_w = \| w - (w, e_1)_{C_{a,b}'} e_1 \|_{C_{a,b}'} = \left[ \| w \|_{C_{a,b}'}^2 - (w, e_1)_{C_{a,b}'}^2 \right]^{1/2}.
\end{align*}
\]

Throughout this paper, we will use the following notations for convenience:

\[
\begin{align*}
  M(\lambda; h) &= \left( \frac{\lambda}{2\pi \| h \|_{C_{a,b}'}^2} \right)^{1/2}, \quad (4.3) \\
  V(\lambda; \xi, v; h, w) &= \exp \left\{ \frac{1}{2\lambda \| h \|_{C_{a,b}'}^2} \left[ (i\lambda(v - \xi) + (h, w)_{C_{a,b}'})^2 - \| h \|_{C_{a,b}'}^2 \| w \|_{C_{a,b}'}^2 \right] \right\}, \quad (4.4) \\
  L(\lambda; \xi, v; h) &= \exp \left\{ \frac{\lambda}{2 \| h \|_{C_{a,b}'}^2} \right\}, \quad (4.5) \\
  H(\lambda; \xi, v; h) &= \exp \left\{ \frac{\lambda}{2 \| h \|_{C_{a,b}'}^2} \left( \sqrt{\lambda}(v - \xi) - (h, a)_{C_{a,b}'} \right)^2 \right\}, \quad (4.6) \\
  A(\lambda; w) &= \exp \left\{ \frac{i}{\sqrt{\lambda}} \beta_w(e_2(w), a)_{C_{a,b}'} \right\} \\
  &= \exp \left\{ \frac{i}{\sqrt{\lambda}} \left[ \| w \|_{C_{a,b}'}^2 - (w, e_1)_{C_{a,b}'}^2 \right]^{1/2} (e_2(w), a)_{C_{a,b}'} \right\} \quad (4.7) \\
\end{align*}
\]

and

\[
\begin{align*}
  (K_\lambda(F; h)\psi)(\xi)
  &= M(\lambda; h) \int_{C_{a,b}'[0, T]} \int_\mathbb{R} \psi(v) V(\lambda; \xi, v; h, w) \\
  &\quad \times L(\lambda; \xi, v; h) H(\lambda; \xi, v; h) A(\lambda; w) dv df(w)
\end{align*}
\]

for \( (\lambda, \xi, v, h, w, \psi) \in \tilde{C}_+ \times \mathbb{R} \times (C_{a,b}'[0, T] \{0\}) \times C_{a,b}'[0, T] \times L^1(\mathbb{R}) \). In equation (4.8) above, \( w, e_1 \) and \( e_2 \) are related by equation (4.2).
Remark 4.2. Clearly, for $\lambda > 0$, $|H(\lambda; \xi, v; h)| \leq 1$ for all $(\xi, v, h) \in \mathbb{R}^2 \times (C_{a,b}^1[0,T] \setminus \{0\})$. But for $\lambda \in \mathbb{C}_+$, $|H(\lambda; \xi, v; h)|$ is not necessarily bounded by 1. Note that for each $\lambda \in \mathbb{C}_+$, $\text{Re}(\lambda) \geq 0$ and $\text{Re}(\sqrt{\lambda}) \geq |\text{Im}(\sqrt{\lambda})| \geq 0$. Hence for each $\lambda \in \mathbb{C}_+$,

$$H(\lambda; \xi, v; h) = \exp \left\{ - \frac{[\text{Re}(\lambda) + i\text{Im}(\lambda)](v - \xi)^2}{2\|h\|_{C_{a,b}^1}^2} + \frac{[\text{Re}(\sqrt{\lambda}) + i\text{Im}(\sqrt{\lambda})](v - \xi)(h, a)_{C_{a,b}^1}}{\|h\|_{C_{a,b}^1}^2} - \frac{(h, a)_{C_{a,b}^1}}{2\|h\|_{C_{a,b}^1}^2} \right\},$$

and so

$$|H(\lambda; \xi, v; h)| = \exp \left\{ - \frac{\text{Re}(\lambda)(v - \xi)^2}{2\|h\|_{C_{a,b}^1}^2} + \frac{\text{Re}(\sqrt{\lambda})(v - \xi)(h, a)_{C_{a,b}^1}}{\|h\|_{C_{a,b}^1}^2} - \frac{(h, a)_{C_{a,b}^1}}{2\|h\|_{C_{a,b}^1}^2} \right\}.$$  

Note that for $\lambda \in \mathbb{C}_+$, the case we consider throughout Section 4, $\text{Re}(\sqrt{\lambda}) > |\text{Im}(\sqrt{\lambda})| \geq 0$, which implies that $\text{Re}(\lambda) = [\text{Re}(\sqrt{\lambda})]^2 - [\text{Im}(\sqrt{\lambda})]^2 > 0$. Hence for each $\lambda \in \mathbb{C}_+$, $0 < |\text{Arg}(\lambda)| < \pi/2$ and so

$$|\text{Re}(\sqrt{\lambda})|^2 = 2 \left( \left| \frac{\lambda}{\text{Re}(\lambda)} \right| + 1 \right) = \frac{1}{2}(\text{sec } \text{Arg}(\lambda) + 1).$$

For $(\lambda, h) \in \mathbb{C}_+ \times (C_{a,b}^1[0,T] \setminus \{0\})$, let

$$S(\lambda; h) = \exp \left\{ (\text{sec } \text{Arg}(\lambda) + 1) \frac{(h, a)_{C_{a,b}^1}}{4\|h\|_{C_{a,b}^1}^2} \right\}.$$  

Using (4.10), (4.11), and (4.12), we obtain that for all $\lambda \in \mathbb{C}_+$,

$$|H(\lambda; \xi, v; h)| = \exp \left\{ - \frac{\text{Re}(\lambda)(v - \xi)^2}{2\|h\|_{C_{a,b}^1}^2} + \frac{\text{Re}(\sqrt{\lambda})(v - \xi)(h, a)_{C_{a,b}^1}}{\|h\|_{C_{a,b}^1}^2} - \frac{(h, a)_{C_{a,b}^1}}{2\|h\|_{C_{a,b}^1}^2} \right\}$$

$$\leq S(\lambda; h).$$

These observations are critical to the development of the existence of the AOVGFSI $I_\lambda^\alpha(F; h)$. 

9
One can see that for all \((\lambda, \xi, v, h, w) \in \mathbb{C}_+ \times \mathbb{R}^2 \times (\mathcal{C}_{a,b}^\prime[0, T] \setminus \{(0)\}) \times \mathcal{C}_{a,b}[0, T],\)

\[|V(\lambda; \xi, v; h, w) L(\lambda; \xi, v; h)|\]

\[= \exp \left\{ \left[ \frac{(i\lambda(v - \xi) + (h, w)C_{a,b}^\prime)^2}{2\lambda||h||_C_{a,b}^\prime^2} - \frac{\lambda}{2} \left( \frac{|v - \xi|^2}{||h||_C_{a,b}^\prime^2} \right) \right] \right\}

= \exp \left\{ - \frac{\text{Re}(\lambda)}{2|\lambda|^2||h||_C_{a,b}^\prime^2} \left[ ||h||^2_{C_{a,b}^\prime} ||w||^2_{C_{a,b}^\prime} - (h, w)^2_{C_{a,b}^\prime} \right] \right\}

\leq 1,

\text{(4.14)}

because \((h, w)^2_{C_{a,b}^\prime} \leq ||h||^2_{C_{a,b}^\prime} ||w||^2_{C_{a,b}^\prime} .\) However, the expression \((4.4)\) is an unbounded function of \(w\) for \(w \in \mathcal{C}_{a,b}^\prime[0, T],\) because \(\beta_w(e_2(w), a)C_{a,b}^\prime\) with

\[e_2(w) = \frac{1}{\beta_w} \left[ w - (w, e_1)C_{a,b}^\prime e_1 \right] = \frac{1}{\beta_w} \left[ w - \frac{1}{||h||_{C_{a,b}^\prime}} (h, w)C_{a,b}^\prime h \right]

\text{(4.15)}

is an unbounded function of \(w\) for \(w \in \mathcal{C}_{a,b}^\prime[0, T].\) Throughout this section, we thus will need to put additional restrictions on the complex measure \(f\) corresponding to \(F\) in order to obtain the existence of our AOVGFSI \(I^w_\lambda(F; h)\) of \(F\) in \(\mathcal{F}(\mathcal{C}_{a,b}[0, T]).\)

In order to obtain the existence of the AOVGFSI, we need to impose additional restrictions on the functionals in \(\mathcal{F}(\mathcal{C}_{a,b}[0, T]).\)

For a positive real number \(q_0,\) let

\[k(q_0; w) = \exp \left\{ (2q_0)^{-1/2}||w||_{C_{a,b}^\prime} ||a||_{C_{a,b}^\prime} \right\}

\text{(4.16)}

and let

\[\Gamma_{q_0} = \left\{ \lambda \in \mathbb{C}_+ : |\text{Im}(\lambda^{-1/2})| = \sqrt{|\lambda - \text{Re}(\lambda)|^2 - 2|q_0|} < (2q_0)^{-1/2} \right\}.

\text{(4.17)}

Define a subclass \(\mathcal{F}_{q_0}\) of \(\mathcal{F}(\mathcal{C}_{a,b}[0, T])\) by \(F \in \mathcal{F}_{q_0}\) if and only if

\[\int_{\mathcal{C}_{a,b}^\prime[0, T]} k(q_0; w) d|f|(w) < +\infty.

\text{(4.18)}

Then for all \(\lambda \in \Gamma_{q_0},\)

\[|A(\lambda; w)| < k(q_0; w).

\text{(4.19)}

**Remark 4.3.** The region \(\Gamma_{q_0}\) given by \text{(4.17)} satisfies the conditions stated in Definition \text{(4.4)} i.e., \(\text{Int}(\Gamma_{q_0})\) is a simple connected domain in \(\mathbb{C}_+\) and \(\text{Int}(\Gamma_{q_0}) \cap (0, +\infty)\) is an open interval. We note that for all real \(q\) with \(|q| > q_0,\)

\[(-iq)^{-1/2} = \frac{1}{\sqrt{2|q|}} + \frac{i \text{sign}(q)}{\sqrt{2|q|}}.

\text{Also, by a close examination of (4.17), it follows that \(-iq\) is an element of the region \(\Gamma_{q_0}.\) In fact, \(\Gamma_{q_0}\) is a simple connected neighborhood of \(-iq\) in \(\mathbb{C}_+.\)
Lemma 4.4. Let $q_0$ be a positive real number and let $F$ be an element of $\mathcal{F}^{q_0}$. Let $h$ be an element of $C_{a, b}[0, T] \setminus \{0\}$ and let $\Gamma_{q_0}$ be given by (4.17). Let $(K_{\lambda}(F; h)\psi)(\xi)$ be given by equation (4.13) for $(\lambda, \xi, \psi) \in \Gamma_{q_0} \times \mathbb{R} \times L^1(\mathbb{R})$. Then $K_{\lambda}(F; h)$ is an element of $\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R}))$ for each $\lambda \in \text{Int}(\Gamma_{q_0})$.

Proof. Let $\Gamma_{q_0}$ be given by (4.17). Using (4.13), (4.14), (4.15), (4.16), (4.17), (4.18), the Fubini theorem, (4.15), and (4.19), we observe that for all $(\lambda, \xi, \psi) \in \text{Int}(\Gamma_{q_0}) \times \mathbb{R} \times L^1(\mathbb{R})$,

$$\begin{align*}
| (K_{\lambda}(F; h)\psi)(\xi) | & \leq M(|\lambda|; h) \int_{C_{a, b}} |\psi(v)| \left| V(\lambda; \xi, v; h, w)L(\lambda; \xi, v; h) \right| \\
& \quad \times \left| H(\lambda; \xi, v; h) \right| |A(\lambda; w)| dvdf(w) \\
& \leq M(|\lambda|; h) \int_{C_{a, b}} |\psi(v)| \left| H(\lambda; \xi, v; h) \right| |A(\lambda; w)| dvdf(w) \\
& \leq \|\psi\|_1 S(\lambda; h) M(|\lambda|; h) \int_{C_{a, b}} k(q_0; w) df(w) \\
& < +\infty,
\end{align*}$$

where $S(\lambda; h)$ is given by equation (4.12). Clearly $K_{\lambda}(F; h) : L_1(\mathbb{R}) \to L_\infty(\mathbb{R})$ is linear. Thus, for all $\lambda \in \text{Int}(\Gamma_{q_0})$,

$$\|K_{\lambda}(F; h)\|_\sigma \leq S(\lambda; h) M(|\lambda|; h) \int_{C_{a, b}} k(q_0; w) df(w)$$

and the lemma is proved.

Lemma 4.5. Let $q_0$, $F$, $h$, $\Gamma_{q_0}$ and $(K_{\lambda}(F; h)\psi)(\xi)$ be as in Lemma 4.4. Then $(K_{\lambda}(F; h)\psi)(\xi)$ is an analytic function of $\lambda$ on $\text{Int}(\Gamma_{q_0})$.

Proof. Let $\lambda \in \text{Int}(\Gamma_{q_0})$ be given and let $\{\lambda_l\}_{l=1}^{\infty}$ be a sequence in $\mathbb{C}_+$ such that $\lambda_l \to \lambda$. Clearly, $0 \leq |\text{Arg}(\lambda_l)| < \pi/2$. Thus there exist $\theta_0 \in (\text{Arg}(\lambda), \pi/2)$ and $n_0 \in \mathbb{N}$ such that $\lambda_l \in \text{Int}(\Gamma_{q_0})$ and $0 < |\text{Arg}(\lambda_l)| < \theta_0$ for all $l > n_0$. We first note that for each $l > n_0$,

$$\frac{|\text{Re}(\sqrt{\lambda_l})|^2}{\text{Re}(\lambda_l)} = \frac{1}{2} \left( \frac{1}{\text{Re}(\lambda_l)} + 1 \right) = \frac{1}{2} (\text{sec} \theta_l + 1) < \frac{1}{2} (\text{sec} \theta_0 + 1).$$

Using this and the Cauchy–Schwarz inequality, it follows that for all $l > n_0$ and $\psi \in L^1(\mathbb{R})$,

$$\begin{align*}
|\psi(v)| & \left| V(\lambda_l; \xi, v; h, w)L(\lambda_l; \xi, v; h)H(\lambda_l; \xi, v; h)A(\lambda_l; w) \right| \\
& = |\psi(v)| \exp \left\{ - \frac{\text{Re}(\lambda_l)(v - \xi)^2}{2|h|_{C_{a, b}}^2} - \frac{\text{Re}(\lambda_l)}{2|\lambda_l|^2 |h|_{C_{a, b}}^2} \|h\|_{C_{a, b}}^2 \|w\|_{C_{a, b}}^2 - (h, w)_{C_{a, b}}^2 \right\}
\end{align*}$$

(4.21)
ψ ∈ \( (4.18) \), the last expression of \( (4.21) \) is integrable on the product space \( \mathbb{R}^2 \), where the Fubini theorem, and the Morera theorem, it follows that for every rectifiable right-hand side of equation \( (4.8) \) is a continuous function of \( \lambda \) for all \( (a,b) \) and \( L \times \mathbb{R} \). Hence by the dominated convergence theorem, we see that the \( \{ C_{a,b} \} \) and \( L \times \mathbb{R} \). Thus using \( (4.8) \), the corresponding measure of \( F \) by \( (1.16) \), satisfies condition \( (1.18) \), the last expression of \( (4.21) \) is integrable on the product space \( \mathbb{R}^2 \), where \( m_L \) denotes the Lebesgue measure on \( \mathbb{R} \). Hence by the dominated convergence theorem, we see that the right-hand side of equation \( (1.15) \) is a continuous function of \( \lambda \) on \( \text{Int}(\Gamma_{q_0}) \). Next we note that for all \( (\xi, v, h, w) \in \mathbb{R}^2 \times (C'_{a,b}[0,T]\setminus\{0\}) \times C'_{a,b}[0,T] \),

\[
V(\lambda; \xi, v, h, w) L(\lambda; \xi, v, h) H(\lambda; \xi, v, h) A(\lambda; w)
\]

is an analytic function of \( \lambda \) throughout the domain \( \text{Int}(\Gamma_{q_0}) \). Thus using \( (1.8) \), the Fubini theorem, and the Morera theorem, it follows that for every rectifiable
Since using the Fubini theorem, we can change the order of integration in (4.22),

\[ \int_{\Delta} K_\lambda(F; h)\psi(\xi)d\lambda \]

\[ = M(\lambda; h) \int_{C_{a,b}[0,T]} \int_{\mathbb{R}} \psi(v) \]

\[ \times \left( \int_{\Delta} V(\lambda; \xi, v; h, w)L(\lambda; \xi, v; h)H(\lambda; v, h)A(\lambda; w)d\lambda \right)dvdf(w) \]

\[ = 0. \]

Therefore for all \((\xi, h, \psi) \in \mathbb{R} \times (C_{a,b}[0,T] \setminus \{0\}) \times L^1(\mathbb{R})\), \((K_\lambda(F; h))\psi(\xi)\) is an analytic function of \(\lambda\) throughout the domain \(\text{Int}(\Gamma_{q_0})\).

**Theorem 4.6.** Let \(q_0, F, h\) and \(\Gamma_{q_0}\) be as in Lemma [4.2]. Then for each \(\lambda \in \text{Int}(\Gamma_{q_0})\), the AOVGFSI \(I^\lambda_{q_0}(F; h)\) exists and is given by the right-hand side of equation (4.8). Thus, \(K_\lambda(F; h)\) is an element of \(\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R}))\) for each \(\lambda \in \text{Int}(\Gamma_{q_0})\).

**Proof.** Let \((\lambda, \xi, \psi) \in (0, +\infty) \times \mathbb{R} \times L^1(\mathbb{R})\). We begin by evaluating the function space integral

\[ (I_\lambda(F; h))\psi(\xi) \]

\[ = \int_{C_{a,b}[0,T]} F(\lambda^{-1/2}x + \xi)\psi(\lambda^{-1/2}(h, x)^- + \xi)d\mu(x) \]

\[ = \int_{C_{a,b}[0,T]} \int_{C_{a,b}[0,T]} \exp\{i\lambda^{-1/2}(w, x)^-\} \psi(\lambda^{-1/2}(h, x)^- + \xi)dvdf(w)d\mu(x). \]

Using the Fubini theorem, we can change the order of integration in (4.22). Since \(\psi \in L^1(\mathbb{R})\), \(f \in M(C_{a,b}[0,T])\), and \((h, x)^-\) is a Gaussian random variable with mean \((h, a)_{C_{a,b}}\) and variance \(\|h\|_{C_{a,b}}^2\), it follows that for \(\lambda > 0\),

\[ |(I_\lambda(F; h))\psi(\xi)| \leq \int_{C_{a,b}[0,T]} \int_{C_{a,b}[0,T]} |\psi(\lambda^{-1/2}(h, x)^- + \xi)|d\mu(x)d|f|(w) \]

\[ \leq M(|\lambda|; h) \int_{C_{a,b}[0,T]} \int_{\mathbb{R}} |\psi(v)||H(\lambda; v, h)|dvdf(w) \]

\[ \leq M(|\lambda|; h) \int_{C_{a,b}[0,T]} \int_{\mathbb{R}} |\psi(v)||dvd|f|(w) \]

\[ = M(|\lambda|; h)||\psi||_1||f|| \]

\[ < +\infty. \]

Next, using (4.22), the Fubini theorem, (4.2), (4.3), (2.3), (2.4), (4.5), (4.6), (4.7), (4.8).
Hence we see that the OVGFSI \((I_{\lambda}(F; h)\psi)(\xi)\)

\[
= \int_{C_{a,b}[0, T]} \int_{C_{a,b}[0, T]} \psi(\lambda^{-1/2}\|h\|C'_{a,b}(e_1, x)\sim + \xi) \\
\times \exp \left\{ i\lambda^{-1/2}(w, e_1)C'_{a,b}(e_1, x)\sim + i\lambda^{-1/2}\beta_w(e_2(w), x)\sim \right\} d\mu(x) df(w) \\
= \left( \frac{\lambda}{2\pi} \right)^{1/2} \int_{C_{a,b}[0, T]} \int_{\mathbb{R}^2} \psi(\|h\|C'_{a,b}u_1 + \xi) \\
\times \exp \left\{ i(w, e_1)\lambda^{-1/2}(e_1, a)C'_{a,b} - \frac{(\sqrt{\lambda}u_1 - (e_1, a)C'_{a,b})^2}{2} \right\} du_1 \\
\times \exp \left\{ - \frac{1}{2\lambda}\beta_w^2 + \frac{i}{\sqrt{\lambda}}\beta_w(e_2(w), a)C'_{a,b} \right\} df(w) \\
= M(\lambda; h) \int_{C'_{a,b}[0, T]} \int_{\mathbb{R}} \psi(v) \exp \left\{ i\frac{(w, e_1)\lambda^{-1/2}(e_1, a)C'_{a,b}}{\|h\|C'_{a,b}}(v - \xi) \\
- \frac{(\sqrt{\lambda}(v - \xi) - \|h\|C'_{a,b}(e_1, a)C'_{a,b})^2}{2\|h\|C'_{a,b}} \right\} dv \\
\times \exp \left\{ - \frac{1}{2\lambda}\beta_w^2 + \frac{i}{\sqrt{\lambda}}\beta_w(e_2(w), a)C'_{a,b} \right\} df(w) \\
= M(\lambda; h) \int_{C'_{a,b}[0, T]} \int_{\mathbb{R}} \psi(v)V(\lambda; \xi, v; h, w)L(\lambda; \xi, v; h) \\
\times H(\lambda; \xi, v; h) A(\lambda; w) dv df(w) \\
= (K_{\lambda}(F; h)\psi)(\xi).
\]

Hence we see that the OVGFSI \(I_{\lambda}(F; h)\) exists for all \((\lambda, h) \in (0, +\infty) \times (C'_{a,b}[0, T] \setminus \{0\})\).

Let \(I_{\lambda}^{\text{avg}}(F; h)\psi = K_{\lambda}(F; h)\psi\) for all \(\lambda \in \text{Int}(\Gamma_{ab})\). Then by Lemma 4.4 and Lemma 4.5, we obtain the desired result.

**5. The analytic operator-valued generalized Feynman integral**

In this section we study the AOVG-Feynman Integral \(J_{\lambda}^{\text{avg}}(F; h)\) for functionals \(F\) in \(\mathcal{F}(C_{a,b}[0, T])\). First of all, we note that for any \(q \in \mathbb{R} \setminus \{0\}\) and any \((\xi, v, h, w) \in \mathbb{R}^2 \times (C'_{a,b}[0, T] \setminus \{0\}) \times C'_{a,b}[0, T],\)

\[|V(-iq; \xi, v; h, w)L(-iq; \xi, v; h)| = 1.\]
Let \( \lambda = -iq \in \mathbb{C}_+ \). Then
\[
\sqrt{\lambda} = \sqrt{-iq} = \sqrt{|q|/2} - i\text{sign}(q)\sqrt{|q|/2}.
\]
Hence for \( \lambda = -iq \) with \( q \in \mathbb{R} \setminus \{0\} \), \([\text{Re}(\sqrt{-iq})]^2 - [\text{Im}(\sqrt{-iq})]^2 = 0\), and so
\[
|H(-iq; \xi, v; h)| = \exp \left\{ \frac{\sqrt{2}q(h, a)\sigma_{\alpha, q}(v - \xi) - (h, a)^2_{\alpha, q}}{2|h|_\sigma^2} \right\}
\]
which is not necessarily in \( L^p(\mathbb{R}) \), as a function of \( v \), for any \( p \in [1, +\infty] \). Hence \( K_{-iq}(F; h) \) might not exist as an element of \( \mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R})) \).

Let \( q = -1 \) and let \( h \) be an element of \( C_{a, b}^0[0, T] \) with \( ||h||_{C_{a, b}^0} = 1 \) and with \( (h, a)_{a, b} > 0 \) (we can choose \( h \) to be \( a/||a||_{C_{a, b}^0} \) in \( C_{a, b}^0[0, T] \)). Let \( \psi : \mathbb{R} \to \mathbb{C} \) be defined by the formula
\[
\psi(v) = v\chi_{[0, +\infty)}(v)\exp \left\{ \frac{ih^2}{2} - \frac{i\sqrt{2}(h, a)\sigma_{\alpha, q}v}{2} + \frac{(h, a)^2_{\alpha, q}}{2} - \frac{\sqrt{2}(h, a)\sigma_{\alpha, q}v}{4} \right\}.
\]
We note that
\[
|\psi(v)| = v\chi_{[0, +\infty)}(v)\exp \left\{ \frac{(h, a)^2_{\alpha, q}}{2} - \frac{\sqrt{2}(h, a)\sigma_{\alpha, q}v}{4} \right\},
\]
and hence \( \psi \in L^p(\mathbb{R}) \) for all \( p \in [1, +\infty] \). In fact, \( \psi \) is also an element of \( C_0(\mathbb{R}) \), the space of bounded continuous functions on \( \mathbb{R} \) that vanish at infinity.

Let \( F(x) \equiv 1 \). Then \( F \) is an element of \( \mathcal{F}^{q_0} \) for all \( q_0 \in (0, +\infty) \), and \( (K_{-iq}(F; h)\psi)(\xi) \) with \( q = -1 \) is given by
\[
(K_i(1; h)\psi)(\xi) = \left( \frac{i}{2\pi} \right)^{1/2} \int_{\mathbb{R}} \psi(v)H(i; \xi, v; h)dv.
\]
Next, using equation (5.2) with \( \lambda = i \) and \( \sqrt{\lambda} = \sqrt{i} = (1+i)/\sqrt{2} \), we observe that
\[
H(i; \xi, v; h) = \exp \left\{ -\frac{i(v - \xi)^2}{2} + \frac{2(h, a)\sigma_{\alpha, q}(v - \xi)}{\sqrt{2}} + \frac{i(h, a)\sigma_{\alpha, q}v}{\sqrt{2}} - \frac{(h, a)^2_{\alpha, q}}{2} \right\},
\]
and hence,
\[
\psi(v)H(i; \xi, v; h) = v\chi_{[0, +\infty)}(v)\exp \left\{ \frac{\sqrt{2}(h, a)\sigma_{\alpha, q}v}{4} + i\xi v - \frac{i\xi^2}{2} - \frac{1}{\sqrt{2}}(h, a)\sigma_{\alpha, q} \xi \right\},
\]
which is not an element of \( L^p(\mathbb{R}) \), as a function of \( v \), for any \( p \in [1, +\infty] \).
Then, using equations (5.1) and (5.2), we see that
\[
(K_i(1;h)\psi)(\xi) = \left(\frac{i}{2\pi}\right)^{1/2} \exp \left\{ -\frac{i\xi^2}{2} - \frac{(1+i)}{\sqrt{2}} (h, a)C_{u,b}^\prime \xi \right\} \\
\times \int_{\mathbb{R}} v \chi_{[0,\infty)}(v) \exp \left\{ \frac{\sqrt{2}(h, a)C_{u,b}^\prime v}{4} + i\xi v \right\} dv.
\]
Hence, choosing \(\xi = 0\), and using the fact that \((h, a)C_{u,b}^\prime\) is positive, we see that
\[
|(K_i(1;h)\psi)(0)| = (2\pi)^{-1/2} \int_0^{\infty} v \exp \left\{ \frac{\sqrt{2}(h, a)C_{u,b}^\prime v}{4} \right\} dv = +\infty.
\]
In fact, for each fixed \(\xi \in \mathbb{R}\), we observe that
\[
|(K_i(1;h)\psi)(\xi)| = (2\pi)^{-1/2} \exp \left\{ -\frac{1}{\sqrt{2}} (h, a)C_{u,b}^\prime \xi \right\} \\
\times \left| \int_{\mathbb{R}} v \chi_{[0,\infty)}(v) \exp \left\{ \frac{\sqrt{2}(h, a)C_{u,b}^\prime v}{4} + i\xi v \right\} dv \right| = +\infty,
\]
and so \((K_i(1;h)\psi)\) is not an element of \(L^\infty(\mathbb{R})\) even though \(\psi\) was an element of \(L^1(\mathbb{R})\). Hence \(K_{-iq}(F;h)\psi \equiv K_i(1;h)\psi\) is not in \(\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R}))\).

In this section, we thus clearly need to impose additional restrictions on \(\psi\) for the existence of our AOVG-Feynman I.

For any positive real number \(\delta\), let \(\nu_{\delta,a}\) be a measure on \(\mathbb{R}\) with
\[
d\nu_{\delta,a} = \exp\{\delta \text{Var}(a)u^2\} du
\]
where \(\text{Var}(a) = |a|(T)\) denotes the total variation of \(a\), the mean function of the GBMP, on \([0,T]\) and let \(L^1(\mathbb{R}, \nu_{\delta,a})\) be the space of \(\mathbb{C}\)-valued Lebesgue measurable functions \(\psi\) on \(\mathbb{R}\) such that \(\psi\) is integrable with respect to the measure \(\nu_{\delta,a}\) on \(\mathbb{R}\). Let \(\| \cdot \|_{1,\delta}\) denote the \(L^1(\mathbb{R}, \nu_{\delta,a})\)-norm. Then for all \(\delta > 0\), we have the following inclusion
\[
L^1(\mathbb{R}, \nu_{\delta,a}) \subseteq L^1(\mathbb{R}) \tag{5.3}
\]
as sets, because \(\|\psi\|_{1,\delta} \leq \|\psi\|_{1}\) for all \(\psi \in L^1(\mathbb{R})\).

Let \(\mathcal{L}(L^1(\mathbb{R}, \nu_{\delta,a}), L^\infty(\mathbb{R}))\) be the space of continuous linear operators form \(L^1(\mathbb{R}, \nu_{\delta,a})\) to \(L^\infty(\mathbb{R})\). In Theorem 4.6 we proved that for all \(\psi \in L^1(\mathbb{R})\), \(I_{\lambda}^{an}(F;h)\psi\) is in \(L^\infty(\mathbb{R})\). From the inclusion (5.3), we see that for all \(\psi \in L^1(\mathbb{R}, \nu_{\delta,a})\), \(I_{\lambda}^{an}(F;h)\psi\) is in \(L^\infty(\mathbb{R})\). Furthermore, for all \(\delta > 0\),
\[
\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R})) \subseteq \mathcal{L}(L^1(\mathbb{R}, \nu_{\delta,a}), L^\infty(\mathbb{R})), \tag{5.4}
\]
as sets.
Now, the notation $\| \cdot \|_{o, \delta}$ will be used for the norm on $\mathcal{L}(L^1(\mathbb{R}, \nu_{\delta, a}), L^\infty(\mathbb{R}))$. We already shown in (4.20) that for all $(\lambda, \xi, \psi) \in \text{Int}(\Gamma_{q_0}) \times \mathbb{R} \times L^1(\mathbb{R}),$

$$|(K_{\lambda}(F; h)\psi)(\xi)| \leq M(|\lambda|; h) \int_{\mathbb{R}} \psi(v) |H(\lambda; \xi, v; h)| dv \int_{C_{a, \delta}([0, T])} |A(\lambda; w)| df(w).$$

But, by the same method, (4.13), and (4.19), it also follows that for any $\delta > 0$ and all $(\lambda, \xi, \psi) \in \text{Int}(\Gamma_{q_0}) \times \mathbb{R} \times L^1(\mathbb{R}, \nu_{\delta, a}),$

$$|(K_{\lambda}(F; h)\psi)(\xi)| \leq M(|\lambda|; h) \int_{\mathbb{R}} |\psi(v)| |H(\lambda; \xi, v; h)| dv \int_{C_{a, \delta}([0, T])} |A(\lambda; w)| df(w)$$

$$\leq M(|\lambda|; h) \int_{\mathbb{R}} |\psi(v)| \exp\{\delta \text{Var}(a)v^2\} |H(\lambda; \xi, v; h)| dv \int_{C_{a, \delta}([0, T])} |A(\lambda; w)| df(w)$$

$$\leq M(|\lambda|; h) S(\lambda; h) \int_{\mathbb{R}} |\psi(v)| \exp\{\delta \text{Var}(a)v^2\} dv \int_{C_{a, \delta}([0, T])} k(q_0, w)|f|(w)$$

and so

$$\|K_{\lambda}(F; h)\|_{o, \delta} \leq S(\lambda; h) M(|\lambda|; h) \int_{C_{a, \delta}([0, T])} k(q_0, w)|f|(w).$$

Thus we have the following definition.

**Definition 5.1.** Let $\Gamma$ be as in Definition 5.7 and let $q$ be a nonzero real number with $-iq \in \Gamma$. Suppose that there exists an operator $J_q^{an}(F; h)$ in $\mathcal{L}(L^1(\mathbb{R}, \nu_{\delta, a}), L^\infty(\mathbb{R}))$ for some $\delta > 0$ such that for every $\psi$ in $L^1(\mathbb{R}, \nu_{\delta, a}),$

$$\|J_q^{an}(F; h)\psi - I_{\lambda}^{an}(F; h)\psi\|_{\infty} \to 0$$

as $\lambda \to -iq$ through $\text{Int}(\Gamma)$, then $J_q^{an}(F; h)$ is called the AOVG’Feynman’I of $F$ with parameter $q$.

**Theorem 5.2.** Let $q_0, F, h$ and $\Gamma_{q_0}$ be as in Lemma 4.4. Then for all real $q$ with $|q| > q_0$, the AOVG’Feynman’I of $F$, $J_q^{an}(F; h)$, exists as an element of $\mathcal{L}(L^1(\mathbb{R}, \nu_{\delta, a}), L^\infty(\mathbb{R}))$ for any $\delta > 0$, and is given by the right-hand side of equation (4.18) with $\lambda = -iq$. 

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Proof. First, we will show that $K_{-iq}(F; h)$ is an element of $\mathcal{L}(L^1(\mathbb{R}, \nu_\delta,a), L^\infty(\mathbb{R}))$. Note that for every $\delta > 0$, $|H(-iq; \xi, v; h)| \exp\{-\delta \text{Var}(a)u^2\}$ is bounded by 1. Hence for any $\delta \in (0, +\infty)$ and every $\psi \in L^1(\mathbb{R}, \nu_\delta)$,

$$\int_{\mathbb{R}} |\psi(v)| |H(-iq; \xi, v; h)| dv = \int_{\mathbb{R}} |\psi(v)| \exp\{\delta \text{Var}(a)u^2\} |H(-iq; \xi, v; h)| \exp\{-\delta \text{Var}(a)u^2\} dv \leq \|\psi\|_{1,\delta}.$$ 

Also, by a simple calculation, it follows that

$$|V(-iq; \xi, v; h, w)||L(-iq; \xi, v; h)| = 1.$$ 

Thus, using these and (4.19), it also follows that for all real $q$ with $|q| > q_0$,

$$|(K_{-iq}(F; h)\psi)(\xi)| \leq M(|q|; h) \int_{C_{a,b}(0,T)} |\psi(v)||V(-iq; \xi, v; h, w)||L(-iq; \xi, v; h)|$$

$$\times |H(-iq; \xi, v; h)||A(-iq; w)| dv d|f|(w) \leq M(|q|; h) \int_{C_{a,b}(0,T)} |A(-iq; w)| d|f|(w)\quad(5.6)$$

Therefore we have that

$$\|K_{-iq}(F; h)\psi\|_\infty \leq \|\psi\|_{1,\delta} \left(M(|q|; h) \int_{C_{a,b}(0,T)} k(q_0; w) d|f|(w)\right)$$

and

$$\|K_{-iq}(F; h)\|_{2,\delta} \leq M(|q|; h) \int_{C_{a,b}(0,T)} k(q_0; w) d|f|(w),$$

and implies that $K_{-iq}(F; h) \in \mathcal{L}(L^1(\mathbb{R}, \nu_\delta,a), L^\infty(\mathbb{R}))$.

We now want to show that the AOVG-Feynman $I_{-iq}^\text{an}(F; h)$ of $F$ exists and is given by the right-hand side of (4.8) with $\lambda = -iq$. To do this, it suffices to show that for every $\psi$ in $L^1(\mathbb{R}, \nu_\delta,a)$,

$$\|K_{-iq}(F; h)\psi - I_{-iq}^\text{an}(F; h)\psi\|_\infty \to 0$$

as $\lambda \to -iq$ through $\text{Int}(\Gamma_{q_0})$, where $\Gamma_{q_0}$ is given by equation (4.17). But, in view of Lemmas 4.4, 4.5, Theorem 4.6 and equation (5.4), we already proved that $I_{-iq}^\text{an}(F; h) = K_{\lambda}(F; h)$ for all $\lambda \in \text{Int}(\Gamma_{q_0})$ and that $K_{\lambda}(F; h)$ is an element of
\[ \mathcal{L}(L^1(\mathbb{R}, \nu_{\delta,a}), L^\infty(\mathbb{R})). \] Next, by \cite{5.5} and \cite{5.6}, we obtain that for all \((\lambda, \xi, \psi) \in \Gamma_{\eta_0} \times \mathbb{R} \times L^1(\mathbb{R}, \nu_{\delta}),\)

\[
|((K_\lambda(F; h)\psi)(\xi)| \\
\leq \left\{ \begin{array}{ll}
||\psi||_{1,\delta} \{ S(\lambda; \xi)M(|\lambda|) \int_{C_{\nu_{\delta}}[0,T]} k(q_0; w)d|f|(w) \}, & \lambda \in \text{Int}(\Gamma_{\eta_0}) \\
||\psi||_{1,\delta} \{ M(|\xi|; \eta) \int_{C_{\nu_{\delta}}[0,T]} k(q_0; w)d|f|(w) \}, & \lambda = iq, q \in \mathbb{R}\setminus\{0\}
\end{array} \right.
< +\infty.
\]

Moreover, using the techniques similar to those used in the proof of Lemma \cite{4.3}, one can easily verify that there exists a sufficiently small \(\varepsilon_0 > 0\) satisfying the inequality:

\[
|((K_\lambda(F; h)\psi)(\xi)| \\
\leq ||\psi||_{1,\delta} \left( \exp \left\{ \frac{(h, a)^2}{\varepsilon_0 + 1} \right\} M(1 + |q|; h) \int_{C_{\nu_{\delta}}[0,T]} k(q_0; w)d|f|(w) \right) \\
< +\infty
\]

for all \(\lambda \in \Gamma_{\eta_0} \cap \{ \lambda \in \mathbb{C} : |\lambda - (-iq)| < \varepsilon_0 \}. \) Hence by the dominated convergence theorem, we have

\[
\lim_{\lambda \to -iq} (I_\lambda^\alpha(F; h)\psi)(\xi) = \lim_{\lambda \to -iq} (K_\lambda(F; h)\psi)(\xi) = (K_{-iq}(F; h)\psi)(\xi)
\]

for each \(\xi \in \mathbb{R}. \) Thus \(J_q^\alpha(F; h)\) exists as an element of \(\mathcal{L}(L^1(\mathbb{R}, \nu_{\delta,a}), L^\infty(\mathbb{R})).\)

It is clear that given two positive real number \(\delta_1\) and \(\delta_2\) with \(\delta_1 < \delta_2,\)

\[
L^1(\mathbb{R}, \nu_{\delta_2,a}) \subseteq L^1(\mathbb{R}, \nu_{\delta_1,a}) \subseteq L^1(\mathbb{R}).
\]

Thus it follows that

\[
\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R})) \subseteq \mathcal{L}(L^1(\mathbb{R}, \nu_{\delta_1,a}), L^\infty(\mathbb{R})) \subseteq \mathcal{L}(L^1(\mathbb{R}, \nu_{\delta_2,a}), L^\infty(\mathbb{R})).
\]

Let

\[
L^{1,a}(\mathbb{R}) = \bigcup_{\delta > 0} L^1(\mathbb{R}, \nu_{\delta,a})
\]

and let

\[
\mathfrak{B}(L^{1,a}(\mathbb{R}), L^\infty(\mathbb{R})) = \bigcap_{\delta > 0} \mathcal{L}(L^1(\mathbb{R}, \nu_{\delta,a}), L^\infty(\mathbb{R})).
\]

We note that \(L^{1,a}(\mathbb{R})\) and \(\mathfrak{B}(L^{1,a}(\mathbb{R}), L^\infty(\mathbb{R})).\) are not normed spaces. However we can suggest set theoretic structures between themselves as follows: since \(L^1(\mathbb{R}, \nu_{\delta,a}) \subset L^{1,a}(\mathbb{R}) \subset L^1(\mathbb{R})\) for any \(\delta > 0\), it follows that

\[
\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R})) \subset \mathfrak{B}(L^{1,a}(\mathbb{R}), L^\infty(\mathbb{R})) \subset \mathcal{L}(L^1(\mathbb{R}, \nu_{\delta,a}), L^\infty(\mathbb{R})).
\]

From this observation and Theorem \cite{5.2} we can obtain the following assertion.
Theorem 5.3. Let $q_0$, $F$, $h$ and $\Gamma_{q_0}$ be as in Lemma 4.4. Then for all real $q$ with $|q| > q_0$, the AOVG Feynman integral $J_q^{an}(F; h)$ exists as an element of $\mathfrak{B}(L^{1,0}(\mathbb{R}), L^\infty(\mathbb{R}))$.

Remark 5.4. If $b(t) = t$ and $a(t) \equiv 0$ on $[0, T]$, the function space $C_{a,b}[0,T]$ reduces to the classical Wiener space $C_0[0,T]$. In this case, the three linear spaces $L^1(\mathbb{R})$, $L^1(\mathbb{R}, \nu_{\delta, 0})$ and $L^{1,0}(\mathbb{R})$ coincide each other. Furthermore, the three classes $\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R}))$, $\mathfrak{B}(L^{1,0}(\mathbb{R}), L^\infty(\mathbb{R}))$, and $\mathcal{L}(L^1(\mathbb{R}, \nu_{\delta, 0}), L^\infty(\mathbb{R}))$ also coincide.

6. Examples

In this section, we present interesting examples to which our results in previous sections can be applied.

Let $\mathcal{M}(\mathbb{R})$ be the class of complex-valued, countably additive measures on $\mathcal{B}(\mathbb{R})$. For $\eta \in \mathcal{M}(\mathbb{R})$, the Fourier transform $\hat{\eta}$ of $\eta$ is a $\mathbb{C}$-valued function defined on $\mathbb{R}$, given by the formula

$$\hat{\eta}(u) = \int_{\mathbb{R}} \exp\{iuv\} d\eta(v).$$

(1) Let $w_0 \in C'_{a,b}[0,T]$ and let $\eta \in \mathcal{M}(\mathbb{R})$. Define $F_1 : C_{a,b}[0,T] \to \mathbb{C}$ by

$$F_1(x) = \hat{\eta}((w_0, x)^\sim).$$

Define a function $\phi : \mathbb{R} \to C'_{a,b}[0,T]$ by $\phi(v) = vw_0$. Let $f = \eta \circ \phi^{-1}$. It is quite clear that $f$ is in $\mathcal{M}(C'_{a,b}[0,T])$ and is supported by $\{w_0\}$, the subspace of $C'_{a,b}[0,T]$ spanned by $\{w_0\}$. Now for $s$-a.e. $x \in C_{a,b}[0,T]$,

$$\int_{C'_{a,b}[0,T]} \exp\{i(w, x)^\sim\} df(w) = \int_{C'_{a,b}[0,T]} \exp\{i(w, x)^\sim\} d(\eta \circ \phi^{-1})(w)$$

$$= \int_{\mathbb{R}} \exp\{i(\phi(v), x)^\sim\} d\eta(v)$$

$$= \int_{\mathbb{R}} \exp\{i(w_0, x)^\sim v\} d\eta(v)$$

$$= \hat{\eta}((w_0, x)^\sim).$$

Thus $F_1$ is an element of $\mathcal{F}(C_{a,b}[0,T])$.

Suppose that for a fixed positive real number $q_0 > 0$,

$$\int_{\mathbb{R}} \exp \{ (2q_0)^{-1/2} \|w_0\|_{C'_{a,b}} \|a\|_{C'_{a,b}} \|v\| \} d|\eta|(v) < +\infty. \quad (6.1)$$

It is easy to show that condition (6.1) is equivalent to condition (4.18) with $f = \eta \circ \phi^{-1}$. Thus, under condition (6.1), $F_1$ is an element of $\mathcal{F}^m$ and so, by Theorem 5.2, $J_q^{an}(F_1; h)$ exists as an element of $\mathcal{L}(L^1(\mathbb{R}, \nu_{\delta, a}), L^\infty(\mathbb{R}))$ for all real $q$ with $|q| > q_0$, all $h \in C_{a,b}[0,T]\{0\}$, and any $\delta > 0$. Moreover $J_q^{an}(F_1; h)$ is an element of the space $\mathfrak{B}(L^{1,0}(\mathbb{R}), L^\infty(\mathbb{R}))$ by Theorem 5.3.
Next, we present more explicit examples of functionals in $\mathcal{F}(\mathcal{C}_{a,b}[0,T])$ whose associated measures satisfy condition (6.1).

(2) Let $S : \mathcal{C}'_{a,b}[0,T] \to \mathcal{C}''_{a,b}[0,T]$ be the linear operator defined by $Sw(t) = \int_0^t w(s)db(s)$. Then the adjoint operator $S^*$ of $S$ is given by

$$S^*w(t) = \int_0^t (w(T) - w(s))db(s)$$

and for $x \in C_{a,b}[0,T]$, $(S^*b, x)^\sim = \int_0^T x(t)db(t)$ by an integration by parts formula.

Given $m$ and $\sigma^2$ in $\mathbb{R}$ with $\sigma^2 > 0$, let $\eta_{m,\sigma^2}$ be the Gaussian measure given by

$$\eta_{m,\sigma^2}(B) = (2\pi\sigma^2)^{-1/2} \int_B \exp \left\{ -\frac{(v-m)^2}{2\sigma^2} \right\}dv, \quad B \in \mathcal{B}(\mathbb{R}). \quad (6.2)$$

Then $\eta_{m,\sigma^2} \in \mathcal{M}(\mathbb{R})$ and

$$\eta_{m,\sigma^2}(u) = \int_{\mathbb{R}} \exp(izu) d\eta_{m,\sigma^2}(v) = \exp \left\{ -\frac{1}{2} \sigma^2 u^2 + imu \right\}.$$

The complex measure $\eta_{m,\sigma^2}$ given by equation (6.2) satisfies condition (6.1) for all $q_0 > 0$. Thus we can apply the results in argument (1) to the functional $F_2 : C_{a,b}[0,T] \to \mathbb{C}$ given by

$$F_2(x) = \eta_{m,\sigma^2}((w_0, x)^\sim)$$

$$= \exp \left\{ -\frac{1}{2} \sigma^2 \left| (w_0, x)^\sim \right|^2 + im(w_0, x)^\sim \right\}. \quad (6.3)$$

For example, if we choose $w_0 = S^*b$, $m = 0$ and $\sigma^2 = 2$ in (6.3), we have

$$F_3(x) = \exp \left\{ -\left| (S^*b, x)^\sim \right|^2 \right\} = \exp \left\{ -\left( \int_0^T x(t)db(t) \right)^2 \right\}$$

for $x \in C_{a,b}[0,T]$.

We note that the functional $F_3$ is in $\mathcal{F}_{q_0>0}$, and so that for every nonzero real number $q$, the AOVG Feynman’I $J_q^{an}(F_3; h)$ exists as an element of $\mathfrak{B}(L^{1,a}(\mathbb{R}), L^\infty(\mathbb{R}))$.

(3) Let $F_4 : C_{a,b}[0,T] \to \mathbb{C}$ be given by

$$F_4(x) = \exp \left\{ i \int_0^T x(t)db(t) \right\}.$$

Then $F_4$ is a functional in $\mathcal{F}(\mathcal{C}_{a,b}[0,T])$, because

$$F_4(x) = \exp\{i(S^*b, x)^\sim\} = \int_{\mathcal{C}'_{a,b}[0,T]} \exp\{i(w, x)^\sim\}d\zeta(w)$$

for s.a.e. $x \in C_{a,b}[0,T]$, where $\zeta$ is the Dirac measure concentrated at $S^*b$ in $C'_{a,b}[0,T]$. The Dirac measure $\zeta$ also satisfies condition (6.1) with $f$ replaced with $\zeta$ for all $q_0 > 0$; that is, $F_4 \in \mathcal{F}_{q_0>0}$, and so that for every nonzero real number $q$, the AOVG Feynman’I $J_q^{an}(F_4; h)$ exists as an element of $\mathfrak{B}(L^{1,a}(\mathbb{R}), L^\infty(\mathbb{R}))$. 

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References

[1] R.H. Cameron, D.A. Storvick, An operator valued function space integral and a related integral equation, J. Math. Mech. 18 (1968), 517–552.

[2] R.H. Cameron, D.A. Storvick, An integral equation related to the Schrödinger equation with an application to integration in function space, in Problems in Analysis (Papers dedicated to Salomon Bochner, 1969), Princeton Univ. Press, Princeton, New Jersey (1970), 175–193.

[3] R.H. Cameron, D.A. Storvick, An operator valued function space integral applied to integrals of functions of class $L_2$, J. Math. Anal. Appl. 42 (1973), 330–372.

[4] R.H. Cameron, D.A. Storvick, An operator-valued function space integral applied to integrals of functions of class $L_1$, Proc. London Math. Soc. 27 (1973), 345–360.

[5] R.H. Cameron, D.A. Storvick, An operator valued function space integral applied to multiple integrals of functions of class $L_1$, Nagoya Math. J. 51 (1973), 91–122.

[6] S.J. Chang, J.G. Choi, Effect of drift of the generalized Brownian motion process: an example for the analytic Feynman integral, Arch. Math. 106 (2016), 591–600.

[7] S.J. Chang, J.G. Choi, A.Y. Ko, Multiple generalized analytic Fourier–Feynman transform via rotation of Gaussian paths on function space, Banach J. Math. Anal. 9 (2015), 58–80.

[8] S.J. Chang, J.G. Choi, S.D. Lee, A Fresnel type class on function space, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. 16 (2009), 107–119.
[9] S.J. Chang, J.G. Choi, D. Skoug, Integration by parts formulas involving generalized Fourier–Feynman transforms on function space, Trans. Amer. Math. Soc. 355 (2003), 2925–2948.

[10] S.J. Chang, D. Skoug, Generalized Fourier–Feynman transforms and a first variation on function space, Integral Transforms Spec. Funct. 14 (2003), 375–393.

[11] J.G. Choi, H.S. Chung, S.J. Chang, Sequential generalized transforms on function space, Abstr. Appl. Anal. 2013 (2013), Article ID: 565832.

[12] J.G. Choi, D. Skoug, Further results involving the Hilbert space $L^2_{a,b}[0,T]$, J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. 27 (2020), 1–11.

[13] D.M. Chung, C. Park, D. Skoug, Operator-valued Feynman integrals via conditional Feynman integrals, Pacific J. Math. 146 (1990), 21–42.

[14] G.W. Johnson, D.L. Skoug, Operator-valued Feynman integrals of certain finite-dimensional functionals, Proc. Amer. Math. Soc. 24 (1970), 774–780.

[15] G.W. Johnson, D.L. Skoug, Operator-valued Feynman integrals of finite-dimensional functionals, Pacific J. Math. 34 (1970), 415–425.

[16] G.W. Johnson, D.L. Skoug, An operator valued function space integral: A sequel to Cameron and Storvick’s paper, Proc. Amer. Math. Soc. 27 (1971), 514–518.

[17] G.W. Johnson, D.L. Skoug, A Banach algebra of Feynman integrable functionals with application to an integral equation formally equivalent to Schroedinger’s equation, J. Funct. Anal. 12 (1973), 129–152.

[18] G.W. Johnson, D.L. Skoug, Feynman integrals of non-factorable finite-dimensional functionals, Pacific J. Math. 45 (1973), 257–267.

[19] G.W. Johnson, D.L. Skoug, Cameron and Storvick’s function space integral for certain Banach spaces of functionals, J. London Math. Soc. 9 (1974), 103–117.

[20] G.W. Johnson, D.L. Skoug, A function space integral for a Banach space of functionals on Wiener space, Proc. Amer. Math. Soc. 43 (1974), 141–148.

[21] G.W. Johnson, D.L. Skoug, Cameron and Storvick’s function space integral for a Banach space of functionals generated by finite-dimensional functionals, Annali di Matematica Pura ed Applicata 104 (1975), 67–83.

[22] G.W. Johnson, D.L. Skoug, The Cameron-Storvick function space integral: The $L_1$ theory, J. Math. Anal. Appl. 50 (1975), 647–667.

[23] G.W. Johnson, D.L. Skoug, The Cameron-Storvick function space integral: An $L(L_p, L_p')$ theory, Nagoya Math. J. 60 (1976), 93–137.
[24] J. Yeh, Singularity of Gaussian measures on function spaces induced by Brownian motion processes with non-stationary increments, Illinois J. Math. 15 (1971), 37–46.

[25] J. Yeh, Stochastic Processes and the Wiener Integral, Marcel Dekker, Inc., New York, 1973.