Global Existence of Classical Solutions to Full Compressible Navier-Stokes System with Large Oscillations and Vacuum in 3D Bounded Domains

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Abstract

The full compressible Navier-Stokes system describing the motion of a viscous, compressible, heat-conductive, and Newtonian polytropic fluid is studied in a three-dimensional simply connected bounded domain with smooth boundary having a finite number of two-dimensional connected components. For the initial-boundary-value problem with slip boundary conditions on the velocity and Neumann boundary one on the temperature, the global existence of classical and weak solutions which are of small energy but possibly large oscillations is established. In particular, both the density and temperature are allowed to vanish initially. Finally, the exponential stability of the density, velocity, and temperature is also obtained. Moreover, it is shown that for the classical solutions, the oscillation of the density will grow unboundedly in the long run with an exponential rate provided vacuum appears (even at a point) initially. This is the first result concerning the global existence of classical solutions to the full compressible Navier-Stokes equations with vacuum in general three-dimensional bounded smooth domains.

Keywords: full compressible Navier-Stokes system; global existence; slip boundary condition; vacuum; large oscillations.

1 Introduction

The motion of a compressible viscous, heat-conductive, and Newtonian polytropic fluid occupying a spatial domain $\Omega \subset \mathbb{R}^3$ is governed by the following full compressible
Navier-Stokes system:

\[
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
(u\rho)_t + \text{div}(\rho u \otimes u) + \nabla P &= \text{div}\mathcal{S}, \\
(\rho E)_t + \text{div}(\rho E u + Pu) &= \text{div}(\kappa \nabla \theta) + \text{div}(S u),
\end{aligned}
\]

(1.1)

where \(\mathcal{S}\) and \(E\) are respectively the viscous stress tensor and the total energy given by

\[
\mathcal{S} = 2\mu \mathcal{D}(u) + \lambda \text{div}u I_3, \quad E = e + \frac{1}{2} |u|^2,
\]

with \(\mathcal{D}(u) = (\nabla u + (\nabla u)^{\text{tr}})/2\) and \(I_3\) denoting the deformation tensor and the \(3 \times 3\) identity matrix respectively. Here, \(t \geq 0\) is time, \(x \in \Omega\) is the spatial coordinate, and \(\rho, u = (u_1, u_2, u_3)^{\text{tr}}, e, P,\) and \(\theta\) represent respectively the fluid density, velocity, specific internal energy, pressure, and absolute temperature. The viscosity coefficients \(\mu\) and \(\lambda\) are constants satisfying the physical restrictions:

\[
\mu > 0, \quad 2\mu + 3\lambda \geq 0.
\]

(1.2)

The heat-conductivity coefficient \(\kappa\) is a positive constant. We consider the ideal polytropic fluids so that \(P\) and \(e\) are given by the state equations:

\[
P(\rho, e) = (\gamma - 1)\rho e = R \rho \theta, \quad e = \frac{R \theta}{\gamma - 1},
\]

(1.3)

where \(\gamma > 1\) is the adiabatic constant and \(R\) is a positive constant.

Let \(\Omega \subset \mathbb{R}^3\) be a simply connected bounded domain. Note that for the classical solutions, the system (1.1) can be rewritten as

\[
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
\rho(u_t + u \cdot \nabla u) &= \mu \Delta u + (\mu + \lambda) \text{div}(\nabla u) - \nabla P, \\
\frac{R}{\gamma - 1} \rho(\theta_t + u \cdot \nabla \theta) &= \kappa \Delta \theta - P \text{div}u + \lambda (\text{div}u)^2 + 2\mu |\mathcal{D}(u)|^2.
\end{aligned}
\]

(1.4)

We consider the system (1.4) subjected to the given initial data

\[
(\rho, \rho u, \rho \theta)(x, t = 0) = (\rho_0, \rho_0 u_0, \rho_0 \theta_0)(x), \quad x \in \Omega,
\]

(1.5)

and boundary conditions

\[
u \cdot n = 0, \quad \text{curl}u \times n = 0, \quad \nabla \theta \cdot n = 0 \quad \text{on } \partial \Omega \times (0, T),
\]

(1.6)

where \(n = (n_1, n_2, n_3)^{\text{tr}}\) is the unit outward normal vector on \(\partial \Omega\).

There is a lot of literature on the global existence and large time behavior of solutions to (1.1). The one-dimensional problem with strictly positive initial density and temperature has been studied extensively by many people (see [2, 18, 19] and the references therein). For the multi-dimensional case, the local existence and uniqueness of classical solutions are known in [24, 29] in the absence of vacuum. The global classical solutions were first obtained by Matsumura-Nishida [23] for initial data close to a non-vacuum equilibrium in some Sobolev space \(H^s\). Later, Hoff [12] studied the global weak solutions with strictly positive initial density and temperature for discontinuous initial
data. On the other hand, in the presence of vacuum, this issue becomes much more complicated. Concerning viscous compressible fluids in a barotropic regime, where the state of these fluids at each instant \( t > 0 \) is completely determined by the density \( \rho = \rho(x,t) \) and the velocity \( u = u(x,t) \), the pressure \( P \) being an explicit function of the density, the major breakthrough is due to Lions \[22\] (see also Feireisl \[9,10\]), where he obtained global existence of weak solutions, defined as solutions with finite energy, when the pressure \( P \) satisfies \( P(\rho) = a\rho^\gamma \) \((a > 0, \gamma > 1)\) with suitably large \( \gamma \). The main restriction on initial data is that the initial energy is finite, so that the density vanishes at far fields, or even has compact support. Recently, Huang-Li-Xin \[16\] and Li-Xin \[21\] established the global well-posedness of classical solutions to the Cauchy problem for the 3D and 2D barotropic compressible Navier-Stokes equations in whole space with smooth initial data that are of small energy but possibly large oscillations, in particular, the initial density is allowed to vanish. More recently, for slip boundary condition in bounded domains, Cai-Li \[6\] obtained the global classical solutions with initial vacuum, provided that the initial energy is suitably small.

Compared with the barotropic flows, it seems much more difficult and complicated to study the global well-posedness of solutions to full compressible Navier-Stokes system \((1.1)\) with vacuum, where some additional difficulties arise, such as the degeneracy of both momentum and energy equations, the strong coupling between the velocity and temperature, et al. For specific pressure laws excluding the perfect gas equation of state, the question of existence of so-called “variational” solutions in dimension \( d \geq 2 \) has been recently addressed in \[8,9\], where the temperature equation is satisfied only as an inequality which justifies the notion of variational solutions. Moreover, for a very particular form of the viscosity coefficients depending on the density, Bresch-Desjardins \[5\] obtained global stability of weak solutions. For the global well-posedness of classical solutions to the full compressible Navier-Stokes system \((1.1)\), it is shown in Xin \[35\] that there is no solution in \( C^1 \left( \mathbb{R}^d \right) \) for large \( s \) to the Cauchy problem for the full compressible Navier-Stokes system without heat conduction provided that the initial density has compact support. See also the recent generalizations to the case for non-compact but rapidly decreasing at far field initial densities \((27)\). Recently, Huang-Li \[14\] established the global existence and uniqueness for the classical solutions to the 3D Cauchy problem with interior vacuum provided the initial energy is small enough. Later, Wen-Zhu \[32\] obtained the global existence and uniqueness of the classical solutions for vanishing far-field density under the assumption that the initial mass is sufficiently small or both viscosity and heat-conductivity coefficients are large enough. It should be mentioned here that the results of \[14,32\] hold only for the Cauchy problem. However, the global existence of classical solutions or even weak ones with vacuum to multi-dimensional full compressible Navier-Stokes system \((1.1)\) in general bounded domains remains completely open except for spherically or cylindrically symmetric initial data \((33,34)\). In fact, one of the aims of this paper is to study the global well-posedness of classical solutions to full compressible Navier-Stokes system \((1.1)\) in general bounded domains.

Before stating the main results, we explain the notations and conventions used throughout this paper. We denote

\[
\int f dx \triangleq \int_\Omega f dx,
\]
Theorem 1.1. Let 

\[ \bar{T} = \frac{1}{|\Omega|} \int_{\Omega} f \, dx, \]

which is the average of a function \( f \) over \( \Omega \). For \( 1 \leq p \leq \infty \) and integer \( k \geq 0 \), we adopt the simplified notations for Sobolev spaces as follows:

\[ \begin{align*}
L^p = L^p(\Omega), \quad W^{k,p} = W^{k,p}(\Omega), \quad H^k = W^{k,2}(\Omega), \\
H^i_\omega = \{ f \in H^i \mid f \cdot n = 0, \text{curl} f \times n = 0 \text{ on } \partial \Omega \} \quad (i = 1, 2). 
\end{align*} \]

Without loss of generality, we assume that

\[ \bar{\rho}_0 = \frac{1}{|\Omega|} \int_{\Omega} \rho_0 \, dx = 1. \tag{1.7} \]

We then define the initial energy \( C_0 \) as follows:

\[ C_0 = \frac{1}{2} \int \rho_0 |u_0|^2 \, dx + R \int (1 + \rho_0 \log \rho_0 - \rho_0) \, dx \]
\[ + \frac{R}{\gamma - 1} \int \rho_0 (\theta_0 - \log \theta_0 - 1) \, dx. \tag{1.8} \]

The first main result in this paper can be stated as follows:

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^3 \) be a simply connected bounded smooth domain, whose boundary \( \partial \Omega \) has a finite number of 2-dimensional connected components. For given numbers \( M > 0 \) (not necessarily small), \( q \in (3, 6) \), \( \hat{\rho} > 2 \), and \( \hat{\theta} > 1 \), suppose that the initial data \( (\rho_0, u_0, \theta_0) \) satisfies

\[ \rho_0 \in W^{2,q}, \quad u_0 \in H^2, \quad \theta_0 \in \{ f \in H^1 \mid \nabla f \cdot n = 0 \text{ on } \partial \Omega \}, \tag{1.9} \]
\[ 0 < \inf \rho_0 \leq \sup \rho_0 < \hat{\rho}, \quad 0 \leq \inf \theta_0 \leq \sup \theta_0 \leq \hat{\theta}, \quad \| \nabla u_0 \|_{L^2} \leq M, \tag{1.10} \]

and the compatibility condition

\[ - \mu \Delta u_0 - (\mu + \lambda) \nabla \text{div} u_0 + R \nabla (\rho_0 \theta_0) = \sqrt{\rho_0} g, \tag{1.11} \]

with \( g \in L^2 \). Then there exists a positive constant \( \varepsilon \) depending only on \( \mu, \lambda, \kappa, R, \gamma, \hat{\rho}, \hat{\theta}, \Omega, \) and \( M \) such that if

\[ C_0 \leq \varepsilon, \tag{1.12} \]

the problem (1.4)–(1.6) admits a unique global classical solution \((\rho, u, \theta)\) in \( \Omega \times (0, \infty) \) satisfying

\[ 0 \leq \rho(x,t) \leq 2 \hat{\rho}, \quad \theta(x,t) \geq 0, \quad x \in \Omega, \quad t \geq 0, \tag{1.13} \]

and

\[ \begin{align*}
\rho &\in C([0,T];W^{2,q}), \\
u &\in C([0,T];W^{1,p}) \cap \mathcal{L}^{\infty}(0,T;H^2) \cap L^{\infty}(\tau,T;W^{3,q}), \\
\theta &\in \mathcal{L}^{\infty}(\tau,T;H^1) \cap C([\tau,T];W^{3,p}), \\
u_t &\in L^{2}(0,T;H^1) \cap L^{\infty}(\tau,T;H^2) \cap \mathcal{L}^{1}(\tau,T;H^{1}), \\
\theta_t &\in \mathcal{L}^{\infty}(\tau,T;H^{1}) \cap \mathcal{L}^{1}(\tau,T;H^{1}),
\end{align*} \tag{1.14} \]
for any $0 < \tau < T < \infty$ and $\tilde{p} \in [1,6)$. Moreover, for any $p \in [1,\infty)$ and $r \in [1,6]$, there exist positive constants $C, \alpha_0, \infty$, and $\theta_\infty$ depending only on $\mu, \lambda, R, \gamma, \hat{\rho}, \hat{\theta}, \Omega, p, r,$ and $M$ such that for any $t \geq 1$,

$$\|\rho - 1\|_{L^p} + \|u\|^2_{W^{1,r}} + \|\theta - \theta_\infty\|^2_{H^2} \leq Ce^{-\alpha_0 t}. \quad (1.15)$$

The next result of this paper concerns weak solutions whose definition is as follows.

**Definition 1.1.** We say that $(\rho, u, E = \frac{1}{2}|u|^2 + \frac{R}{\gamma - 1}\theta)$ is a weak solution to Cauchy problem $(1.1)$ $(1.5)$ $(1.6)$ provided that

$$\rho \in L^\infty_{\text{loc}}([0,\infty); L^\infty(\Omega)), \quad u, \theta \in L^2_{\text{loc}}([0,\infty); H^1(\Omega)),$$

and that for all test functions $\psi \in D(\Omega \times (-\infty, \infty))$,

$$\int_\Omega \rho_0 \psi(\cdot,0)dx + \int_0^\infty \int_\Omega (\rho \psi_t + \rho u \cdot \nabla \psi) dx dt = 0, \quad (1.16)$$

$$\int_\Omega \rho_0 u_0 \psi(\cdot,0)dx + \int_0^\infty \int_\Omega \left(\rho u_0 \psi_t + \rho u \cdot \nabla \psi + P(\rho, \theta)\psi_x\right) dx dt$$

$$- \int_0^\infty \int_\Omega \left(\mu \nabla u_j \cdot \nabla \psi + (\mu + \lambda)(\text{div} u)\psi_x\right) dx dt = 0, \quad j = 1, 2, 3, \quad (1.17)$$

$$\int_\Omega \left(\frac{1}{2}\rho_0 |u_0|^2 + \frac{R}{\gamma - 1}\rho \theta_0\right) \psi(\cdot,0)dx$$

$$+ \int_0^\infty \int_\Omega \left(\rho E \psi_t + (\rho E + P)u \cdot \nabla \psi\right) dx dt$$

$$- \int_0^\infty \int_\Omega \left(\kappa \nabla \theta + \frac{1}{2\mu} \nabla (|u|^2) + \mu u \cdot \nabla u + \lambda \text{div} u\right) \cdot \nabla \psi dx dt = 0. \quad (1.18)$$

Then we state our second main result as follows:

**Theorem 1.2.** Under the conditions of Theorem 1.1 except $(1.11)$, where the condition $(1.2)$ is replaced by

$$u_0 \in H^1_\omega, \quad (1.19)$$

assume further that $C_0$ as in $(1.8)$ satisfies $(1.13)$ with $\varepsilon$ as in Theorem 1.1 Then there exists a global weak solution $(\rho, u, E = \frac{1}{2}|u|^2 + \frac{R}{\gamma - 1}\theta)$ to the problem $(1.1)$ $(1.5)$ $(1.6)$ satisfying

$$\rho \in C([0,\infty); L^p), \quad (\rho u, \rho |u|^2, \rho \theta) \in C([0,\infty); H^{-1}), \quad (1.20)$$

$$u \in L^\infty(0,\infty; H^1) \cap C((0,\infty); L^2), \quad \theta \in C((0,\infty); W^{1,\tilde{p}}), \quad (1.21)$$

$$\text{curl} u(\cdot, t), \quad ((\text{div} u - (P - \tilde{P}))(\cdot, t), \quad \nabla \theta(\cdot, t) \in H^1, \quad t > 0, \quad (1.22)$$

$$\rho \in [0,2\hat{\rho}] \quad \text{a.e.}, \quad \theta \geq 0 \quad \text{a.e.}, \quad (1.23)$$

and the exponential decay property $(1.11)$ with $p \in [1,\infty)$, $\tilde{p} \in [1,6)$, and $r \in [1,6]$. In addition, there exists some positive constant $C$ depending on $\mu, \lambda, r, \gamma, \hat{\rho}, \hat{\theta}, \Omega, \tilde{p}$, and $M$ such that, for $\sigma(t) \triangleq \min\{1,t\}$, the following estimates hold

$$\sup_{t \in (0,\infty)} \|u\|^2_{H^1} + \int_0^\infty \int \left|(\rho u)_t + \text{div}(\rho u \otimes u)\right|^2 dx dt \leq C, \quad (1.24)$$
\[
\sup_{t \in (0, \infty)} \int \left( (\rho - 1)^2 + \rho |u|^2 + \rho (R \theta - \overline{\theta})^2 \right) \, dx \\
+ \int_0^\infty \left( \|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) \, dt \leq CC_0^{1/4},
\]

\[
\sup_{t \in (0, \infty)} \left( \sigma \|\nabla u\|_{L^6}^2 + \sigma^2 \|\theta\|_{H^2}^2 \right) \\
+ \int_0^\infty \left( \sigma \|u_t\|_{L^2}^2 + \sigma^2 \|\nabla u\|_{L^2}^2 + \sigma^2 \|\theta_t\|_{H^1}^2 \right) \, dt \leq C.
\] (1.26)

Moreover, \((\rho, u, \theta)\) satisfies (1.4) in the weak form, that is, for any test function \(\psi \in \mathcal{D}(\Omega \times (-\infty, \infty))\),

\[
\frac{R}{\gamma - 1} \int \rho_0 \theta_0 \psi(\cdot, 0) \, dx + \frac{R}{\gamma - 1} \int_0^\infty \int \rho \theta (\psi_t + u \cdot \nabla \psi) \, dx \, dt \\
= \kappa \int_0^\infty \int \nabla \theta \cdot \nabla \psi \, dx \, dt + R \int_0^\infty \int \rho \theta \, u \cdot \nabla \psi \, dx \, dt \\
- \int_0^\infty \int (\lambda (\text{div} u)^2 + 2\mu |\nabla \theta|^2) \psi \, dx \, dt.
\] (1.27)

Next, as a direct application of (1.15), the following Corollary 1.3, whose proof is similar to that of [6, Theorem 1.2], shows that the oscillation of the density will grow unboundedly in the long run with an exponential rate provided vacuum appears (even at a point) initially.

**Corollary 1.3.** In addition to the conditions of Theorem 1.1, assume further that there exists some point \(x_0 \in \Omega\) such that \(\rho_0(x_0) = 0\). Then for any \(\hat{p} > 3\), there exists some positive constant \(C\) depending on \(\mu, \lambda, \kappa, R, \gamma, \hat{p}, \hat{R}, \hat{\theta}, \Omega, \hat{\rho}, \hat{\theta}\) and \(M\) such that the unique global classical solution \((\rho, u, \theta)\) to the problem (1.4)–(1.6) obtained in Theorem 1.1 satisfies that for any \(t \geq 1\),

\[
\|\nabla \rho(\cdot, t)\|_{L^\hat{p}} \geq C e^{Ct}.
\]

A few remarks are in order:

**Remark 1.1.** It is easy to deduce from (1.14) and the Sobolev imbedding theorem that for any \(0 < \tau < T < \infty\),

\[
(\rho, \nabla \rho, u) \in C(\Omega \times [0, T]), \quad (\theta, \nabla \theta, \nabla^2 \theta) \in C(\Omega \times (0, T)),
\]

and

\[
(\nabla u, \nabla^2 u) \in C(\tau, T; L^2) \cap L^\infty(\tau, T; W^{1,q}) \hookrightarrow C(\Omega \times [\tau, T]),
\]

which together with (1.4) and (1.28) shows

\[
\rho_t \in C(\Omega \times [\tau, T]).
\]

(1.30)

Analogously, we have

\[
(u_t, \theta_t) \in C(\Omega \times [\tau, T]),
\]

which along with (1.28)–(1.30) arrives at the solution \((\rho, u, \theta)\) obtained in Theorem 1.1 is a classical one to the problem (1.4)–(1.6) in \(\Omega \times (0, \infty)\).
Remark 1.2. It seems that Theorem 1.1, which extends the global existence result of the barotropic flows studied in [6] to the full compressible Navier-Stokes system, is the first result concerning the global existence of classical solutions with initial vacuum to (1.1) in general bounded domains. Although its energy is small, the oscillations could be arbitrarily large.

Remark 1.3. To obtain the global existence and uniqueness of classical solutions with vacuum, we only need the compatibility condition on the velocity (1.11) as in [20], which is much weaker than those in [7, 14, 32] where not only (1.11) but also the following compatibility condition on the temperature

\[ \kappa \Delta \theta_0 + \frac{\mu}{2} \nabla u_0 + (\nabla u_0)^{tr} + \lambda (\text{div} u_0)^2 = \sqrt{\rho_0} g_1, \quad g_1 \in L^2 \]  

is needed. This reveals that the compatibility condition on the temperature (1.31) is not necessary for establishing the classical solutions with vacuum to the full Navier-Stokes equations, which is just the same as the barotropic case [16, 21].

Remark 1.4. It should be mentioned here that the boundary condition for velocity \( u \):

\[ u \cdot n = 0, \quad \text{curl} u \times n = 0 \text{ on } \partial \Omega, \]  

is a special case of the following general Navier-type slip condition (see Navier [25])

\[ u \cdot n = 0, \quad (2 \mathcal{D}(u) n + \partial u)_{\text{tan}} = 0 \text{ on } \partial \Omega, \]  

which as indicated by [6, Remark 1.1], is in fact a particular case of the following slip boundary one:

\[ u \cdot n = 0, \quad \text{curl} u \times n = -Au \text{ on } \partial \Omega, \]  

where \( \vartheta \) is a scalar friction function, the symbol \( v_{\text{tan}} \) represents the projection of tangent plane of the vector \( v \) on \( \partial \Omega \), and \( A = A(x) \) is a given \( 3 \times 3 \) symmetric matrix defined on \( \partial \Omega \). Indeed, our result still holds for more general slip boundary condition (1.33) with \( A \) being semi-positive and regular enough. The proof is similar to [6] and omitted here.

We now comment on the analysis of this paper. We mainly take the strategy that we first extend the standard local classical solutions with strictly positive initial density (see Lemma 2.1) globally in time just under the condition that the initial energy is suitably small (see Proposition 5.1), then let the lower bound of the initial density go to zero. To do so, one needs to establish global a priori estimates, which are independent of the lower bound of the density, on smooth solutions to (1.4)–(1.6) in suitable higher norms. As indicated in [13, 14], the key issue is to obtain the time-independent upper bound of the density.

The first main difficulty arises in deriving the basic energy estimate, which indeed is obtained directly for the Cauchy problem [14]. However, in our case, the basic energy equality reads:

\[ E'(t) + \int \left( \frac{\lambda (\text{div} u)^2 + 2\mu |\mathcal{D}(u)|^2}{\vartheta} + \kappa |\nabla \theta|^2 \right) dx = -\mu \int \left( |\text{curl} u|^2 + 2(\text{div} u)^2 - 2|\mathcal{D}(u)|^2 \right) dx, \]  

(1.34)
where the basic energy $E(t)$ is defined by

$$E(t) \triangleq \int \left( \frac{1}{2} \rho |u|^2 + R(1 + \rho \log \rho - \rho) + \frac{R}{\gamma - 1} \rho(\theta - \log \theta - 1) \right) dx.$$  

(1.35)

Note that the right-hand term in (1.34) is sign-undetermined due to the slip boundary condition (1.32), thus it seems difficult to obtain directly the usual standard energy estimate

$$E(t) \leq CC_0,$$

(1.36)

where the smallness (with the same order of initial energy) of basic energy plays a key role in the whole analysis of the global existence of classical solutions with vacuum not only for Cauchy problem/IBVP of barotropic flows [6, 16, 21] but also for Cauchy problem of full compressible Navier-Stokes equations [14]. To overcome this difficulty, we first assume that $A_2(T)$ (see (3.3)) a priori satisfies $A_2(T) \leq 2C_1^{1/4}$ (see (3.9)) and obtain the following “weaker” basic energy estimate (see also (3.15)):

$$E(t) \leq CC_0^{1/4},$$

(1.37)

which compared with (1.36), however, is not enough and indeed will bring us some essential difficulties to obtain all the a priori estimates (see Proposition 3.1). Then, the first observation is that the average of the pressure $\bar{P}$ is uniformly bounded with positive lower and upper bounds (see (3.11)) by both “weaker” basic energy estimate (1.37) and Jensen’s inequality (see Lemma 2.2). Combining this with the fact that the quantity $\bar{P}$ plays a similar role as $\bar{\theta}$ (see (3.22)) implies that we can replace $\bar{\theta}$ by $\bar{P}$ whose positive lower and upper bounds play an important role in further analysis.

Next, the second difficulty lies in the estimation on the energy-like term $A_2(T)$ (see (3.2)) which includes the key bounds on the $L^2(\Omega \times (0, T))$-norm of the spatial derivatives of both the velocity and the temperature. To proceed, first, we adopt the ideas due to [6, 12] to estimate $\dot{u}$ and $\dot{\theta}$ (see Lemma 3.3), where $\dot{f} \triangleq f_t + u \cdot \nabla f$ denotes the material derivative of $f$. Indeed, in this process, one needs to deal with the boundary integrals in (3.43), for example

$$\int_{\partial \Omega} G(u \cdot \nabla)u \cdot \nabla n \cdot udS,$$

which by the classical trace theorem seems to be bounded by some good terms and the $L^p$-norm of $\nabla^2 u$, which is unavailable in this step. To overcome this difficulty, we adopt some idea due to [6], that is, $u = u^\perp \times n$ with $u^\perp \triangleq -u \times n$, which combined with the following fact:

$$\text{div}(\nabla u^i \times u^\perp) = -\nabla u^i \cdot \nabla \times u^\perp,$$  

(1.38)

yields that the above boundary integral can be indeed bounded by some suitable norms on both $\nabla u$ and $\nabla G$ (see (3.45) for details). Next, after observing that the evolution of $\bar{P}$ can be derived from the temperature equation (see (3.26)), combining a careful analysis on the system (1.4) with the $L^1(0, \min\{1, T\}; L^\infty)$-norm of the temperature gives the desired basic energy estimate for small time (see Lemma 3.5). Moreover, we observe that $R\theta - \bar{P}$ can be bounded by the combination of the initial energy with
the spatial $L^2$-norm of the spatial derivatives of the temperature (see (3.74)), which together with a suitable combination of kinetic energy and thermal energy (see (3.91)) yields $A_2(T) \leq C C_0^{7/24}$ (see (3.99)) which implies $A_2(T) \leq C_0^{1/4}$ (see (3.88)), provided the initial energy is suitably small.

Next, the third difficulty is to obtain the key time-independent upper bound of the density. It should be noted that the methods used in Cauchy problem [14], which heavily relies on the nontrivial far field states, can not be applied to the IBVP directly. Here, by inserting the key quantity $\mathcal{P}$, we rewrite the continuity equation in the following way

$$(2\mu + \lambda)D_t \rho = -\mathcal{P} \rho (\rho - 1) - \rho^2 (R\theta - \mathcal{P}) - \rho G,$$

where $D_t$ is the material derivative and $G$ is the effective viscous flux, defined by

$$D_t f \triangleq \dot{f} \triangleq f_t + u \cdot \nabla f, \quad G \triangleq (2\mu + \lambda) \text{div} u - (P - \mathcal{P}),$$

(1.39)

respectively. With the aid of the uniform bound of $\mathcal{P}$ (see (3.11)), the upper bound of the density follows directly by applying the Grönwall-type inequality (see Lemma 2.8) and using the estimates on $R\theta - \mathcal{P}$ and $G$.

Finally, with the lower-order estimates including the time-independent upper bound of the density at hand, we can obtain the higher-order estimates just under the compatibility condition on the velocity (1.11). Note that all the a priori estimates are independent of the lower bound of the density, thus after a standard approximate procedure, we can obtain the global existence of classical solutions with vacuum. Moreover, we can as well establish the global weak solutions almost the same way as we established the classical one with a new modified approximate initial data.

The rest of the paper is organized as follows: In Section 2, we collect some basic facts and inequalities which will be used later. Section 3 is devoted to deriving the lower-order a priori estimates on classical solutions which are needed to extend the local solutions to all time. The higher-order estimates are established in Section 4. Finally, with all a priori estimates at hand, the main results, Theorems 1.1 and 1.2, are proved in Section 5.

## 2 Preliminaries

First, the following local existence theory with strictly positive initial density can be shown by the standard contraction mapping arguments as in [7, 23, 30].

**Lemma 2.1.** Let $\Omega$ be as in Theorem 1.1. Assume that $(\rho_0, u_0, \theta_0)$ satisfies

$$\begin{cases}
(\rho_0, u_0, \theta_0) \in H^3, & \inf_{x \in \Omega} \rho_0(x) > 0, \quad \inf_{x \in \Omega} \theta_0(x) > 0, \\
u_0 \cdot \mathbf{n} = 0, & \text{curl} u_0 \times \mathbf{n} = 0, \quad \nabla \theta_0 \cdot \mathbf{n} = 0 \quad \text{on } \partial \Omega.
\end{cases}$$

(2.1)

Then there exist a small time $0 < T_0 < 1$ and a unique classical solution $(\rho, u, \theta)$ to the problem (1.4)–(1.6) on $\Omega \times (0, T_0]$ satisfying

$$\inf_{(x, t) \in \Omega \times (0, T_0]} \rho(x, t) \geq \frac{1}{2} \inf_{x \in \Omega} \rho_0(x),$$

(2.2)
and

\[
\begin{aligned}
(p, u, \theta) &\in C([0, T_0]; H^3), \quad \rho_t \in C([0, T_0]; H^2), \\
(u_t, \theta_t) &\in C([0, T_0]; H^1), \quad (u, \theta) \in L^2(0, T_0; H^4).
\end{aligned}
\]  

(2.3)

**Remark 2.1.** Applying the same arguments as in [14, Lemma 2.1], one can deduce that the classical solution \((p, u, \theta)\) obtained in Lemma 2.2 satisfies

\[
\begin{aligned}
(tu_t, t\theta_t) &\in L^2(0, T_0; H^3), \quad (tu_{tt}, t^2\theta_{tt}) \in L^2(0, T_0; H^1), \\
(t^2u_{tt}, t^2\theta_{tt}) &\in L^2(0, T_0; H^2), \quad (t^2 u_{ttt}, t^2 \theta_{ttt}) \in L^2(0, T_0; L^2).
\end{aligned}
\]  

(2.4)

Moreover, for any \((x, t) \in \Omega \times [0, T_0]\), the following estimate holds:

\[
\theta(x, t) \geq \inf_{x \in \Omega} \theta_0(x) \exp\left\{- (\gamma - 1) \int_0^{T_0} \|\text{div} u\|_{L^\infty} dt \right\}.
\]  

(2.5)

Next, we state the classical Jensen’s inequality (see [28, Theorem 3.3]), which guarantees the key uniform upper and lower bounds of \(\overline{r}_\mu\).

**Lemma 2.2.** Let \(\mu\) be a positive measure on a \(\sigma\)-algebra \(\mathcal{M}\) in a set \(\Omega\), so that \(\mu(\Omega) = 1\). If \(f\) is a real function in \(L^1(\mu)\), \(a < f(x) < b\) for all \(x \in \Omega\), and \(\Phi\) is convex on \((a, b)\), then

\[
\Phi\left(\int_{\Omega} f d\mu\right) \leq \int_{\Omega} (\Phi \circ f) d\mu.
\]  

(2.6)

Next, the following well-known Gagliardo-Nirenberg-Sobolev-type inequality (see [26]) will be used later frequently.

**Lemma 2.3.** Assume that \(\Omega \subset \mathbb{R}^3\) is a bounded Lipschitz domain. For \(r \in [2, 6]\), \(p \in (1, \infty)\), and \(q \in (3, \infty)\), there exist positive constants \(C, C'\), and \(C''\) which may depend on \(r, p, q,\) and \(\Omega\) such that for \(f \in H^1\), \(g \in L^p \cap W^{1,q}\), and \(\varphi, \psi \in H^2\),

\[
\|f\|_{L^r} \leq C \|f\|_{L^2}^{(6-r)/2r} \|\nabla f\|_{L^2}^{3r-6}/2r + C' \|f\|_{L^2},
\]  

(2.7)

\[
\|g\|_{C(\overline{\Omega})} \leq C \|g\|_{L^p}^{p(q-3)/(3q+p(q-3))} \|\nabla g\|_{L^q}^{3q/(3q+p(q-3))} + C'' \|g\|_{L^2},
\]  

(2.8)

\[
\|\varphi \psi\|_{H^2} \leq C \|\varphi\|_{H^2} \|\psi\|_{H^2}.
\]  

(2.9)

Moreover, if \(f \cdot n|_{\partial \Omega} = 0\) or \(\overline{f} = 0\), one has \(C' = 0\). Similarly, if \(g \cdot n|_{\partial \Omega} = 0\) or \(\overline{g} = 0\), it holds \(C'' = 0\).

Then, the following div-curl type inequality (see [31, Theorem 3.2]) is used to get the estimates on the spatial derivatives of velocity.

**Lemma 2.4.** Assume that \(\Omega \subset \mathbb{R}^3\) is a simply connected bounded domain with \(C^{1,1}\) boundary \(\partial \Omega\). Then, for \(v \in W^{1,q}\) with \(q \in (1, \infty)\) and \(v \cdot n|_{\partial \Omega} = 0\), there exists a positive constant \(C = C(q, \Omega)\) such that

\[
\|v\|_{W^{1,q}} \leq C (\|\text{div} v\|_{L^q} + \|\text{curl} v\|_{L^q}).
\]  

(2.10)
Now, we deduce from (1.4)–(1.6) that $G$ defined in (1.39) satisfies the following elliptic equation:

$$
\begin{aligned}
\Delta G &= \text{div}(\rho \dot{u}), & x \in \Omega, \\
\nabla G \cdot n &= \rho \ddot{u} \cdot n, & x \in \partial \Omega.
\end{aligned}
$$

(2.11)

The standard $L^p$-estimate for (2.11) together with the div-curl type inequalities (see [3][31]) yields the following essential estimates (see also [3] Lemma 2.9)).

**Lemma 2.5.** Let $\Omega \subset \mathbb{R}^3$ be the same as in Theorem 1.1 and $(\rho, u, \theta)$ a smooth solution of (1.4) – (1.6). Then there exists a generic positive constant $C$ depending only on $p, \mu, \lambda, \text{ and } \Omega$ such that, for any $p \in [2, 6]$,

$$
\|\nabla u\|_{L^p} \leq C\left(\|\text{div} u\|_{L^p} + \|\text{curl} u\|_{L^p}\right),
$$

(2.12)

$$
\|\nabla G\|_{L^p} \leq C\|\rho \dot{u}\|_{L^p},
$$

(2.13)

$$
\|\nabla \text{curl} u\|_{L^p} \leq C\left(\|\rho \ddot{u}\|_{L^p} + \|\nabla u\|_{L^2}\right),
$$

(2.14)

$$
\|G\|_{L^p} \leq C\|\rho \dot{u}\|_{L^2}^{(3p-6)/(2p)} \left(\|\nabla u\|_{L^2} + \|P - \overline{P}\|_{L^2}\right)^{(6-p)/(2p)},
$$

(2.15)

$$
\|\text{curl} u\|_{L^p} \leq C\|\rho \dot{u}\|_{L^2}^{(3p-6)/(2p)} \|\nabla u\|_{L^2}^{(6-p)/(2p)} + C\|\nabla u\|_{L^2}.
$$

(2.16)

Moreover, it holds that

$$
\|G\|_{L^p} + \|\text{curl} u\|_{L^p} \leq C\left(\|\rho \ddot{u}\|_{L^2} + \|\nabla u\|_{L^2}\right),
$$

(2.17)

$$
\|\nabla u\|_{L^p} \leq C\left(\|\rho \dot{u}\|_{L^2}^{(3p-6)/(2p)} \left(\|\nabla u\|_{L^2} + \|P - \overline{P}\|_{L^2}\right)^{(6-p)/(2p)} + C\|\nabla u\|_{L^2} + \|P - \overline{P}\|_{L^p}\right).
$$

(2.18)

Next, we state the following estimates on $\dot{u}$ with $u \cdot n|_{\partial \Omega} = 0$, whose proof can be found in [3] Lemma 2.10).

**Lemma 2.6.** Let $\Omega \subset \mathbb{R}^3$ with $C^{1,1}$ boundary. Assume that $u$ is smooth enough and $u \cdot n|_{\partial \Omega} = 0$, then there exists a generic positive constant $C$ depending only on $\Omega$ such that

$$
\|\dot{u}\|_{L^6} \leq C\left(\|\nabla \dot{u}\|_{L^2} + \|\nabla u\|_{L^2}^2\right),
$$

(2.19)

$$
\|\nabla \dot{u}\|_{L^2} \leq C\left(\|\text{div} \dot{u}\|_{L^2} + \|\text{curl} \dot{u}\|_{L^2} + \|\nabla u\|_{L^4}^2\right).
$$

(2.20)

Furthermore, by the classical elliptic theory owing to Agmon-Douglis-Nirenberg [11], one has the following estimates for smooth solution to the Lamé’s system:

$$
\begin{aligned}
-\mu \Delta u - (\lambda + \mu) \nabla \text{div} u &= -\rho \ddot{u} - \nabla P, & x \in \Omega, \\
\n u \cdot n &= 0, \text{curl} \dot{u} \times n = 0, & x \in \partial \Omega,
\end{aligned}
$$

(2.21)

where $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain, and $\mu, \lambda$ satisfy the condition [11].

**Lemma 2.7.** Let $u$ be a smooth solution of the Lamé’s system (2.21). Then for $p \in [2, 6]$ and $k \geq 2$, there exists a positive constant $C$ depending only on $\lambda, \mu, p, k, \text{ and } \Omega$ such that

$$
\|u\|_{W^{k,p}} \leq C\left(\|\rho \dot{u}\|_{W^{k-2,p}} + \|\nabla P\|_{W^{k-2,p}} + \|\nabla u\|_{L^2}\right).
$$

(2.22)
Next, the following Grönwall-type inequality will be used to get the uniform (in time) upper bound of the density $\rho$, whose proof can be found in [14, Lemma 2.5].

**Lemma 2.8.** Let the function $y \in W^{1,1}(0, T)$ satisfy

$$y'(t) + \alpha(t)y(t) \leq g(t) \text{ on } [0, T], \quad y(0) = y_0,$$

where $0 < \alpha_0 \leq \alpha(t)$ for any $t \in [0, T]$ and $g \in L^p(0, T_1) \cap L^q(T_1, T)$ for some $p, q \geq 1$, $T_1 \in [0, T]$. Then it has

$$\sup_{0 \leq t \leq T} y(t) \leq |y_0| + (1 + \alpha_0^{-1}) \left( \|g\|_{L^p(0, T_1)} + \|g\|_{L^q(T_1, T)} \right). \quad (2.23)$$

Next, to derive the exponential decay property of the solutions, we consider the following auxiliary problem

$$\begin{cases}
\text{div } v = f, & x \in \Omega, \\
v = 0, & x \in \partial \Omega.
\end{cases} \quad (2.24)$$

**Lemma 2.9.** [11, Theorem III.3.1] There exists a linear operator $B = [B_1, B_2, B_3]$ enjoying the properties:

1) The operator $B : \{ f \in L^p \mid f = 0 \} \mapsto W^{1, p}_0$ is a bounded linear one, that is,

$$\|B[f]\|_{W^{1, p}_0} \leq C(p)\|f\|_{L^p}, \text{ for any } p \in (1, \infty).$$

2) The function $v = B[f]$ solves the problem (2.24).

3) If, moreover, for $f = \text{div } g$ with a certain $g \in L^r$, $g \cdot n|_{\partial \Omega} = 0$, then for any $r \in (1, \infty)$,

$$\|B[f]\|_{L^r} \leq C(r)\|g\|_{L^r}.$$

Finally, in order to estimate $\|\nabla u\|_{L^\infty}$ for the further higher order estimates, we need the following Beale-Kato-Majda-type inequality, which was first proved in [4,17] when $\text{div } u \equiv 0$, whose detailed proof in the case of slip boundary condition can be found in [6, Lemma 2.7] (see also [13,15]).

**Lemma 2.10.** Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain. For $3 < q < \infty$, assume that $u \in \{ f \in W^{2,q} \mid f \cdot n = 0, \text{curl} f \times n = 0 \text{ on } \partial \Omega \}$, then there is a positive constant $C = C(q, \Omega)$ such that

$$\|\nabla u\|_{L^\infty} \leq C(\|\text{div } u\|_{L^\infty} + \|\text{curl } u\|_{L^\infty}) \ln(\epsilon + \|\nabla^2 u\|_{L^q}) + C\|\nabla u\|_{L^2} + C.$$

### 3 A priori estimates (I): lower-order estimates

In this section, we will establish a priori bounds for the local-in-time smooth solution to problem (1.4)–(1.6) obtained in Lemma 2.1.
Let \((\rho, u, \theta)\) be a smooth solution to the problem \((1.4)-(1.6)\) on \(\Omega \times (0, T]\) for some fixed time \(T > 0\), with initial data \((\rho_0, u_0, \theta_0)\) satisfying \((2.1)\). For \(\sigma(t) \triangleq \min\{1, t\}\), we define \(A_i(T) (i = 1, 2, 3)\) as follows:

\[
A_1(T) \triangleq \sup_{t \in [0, T]} \|\nabla u\|_{L^2}^2 + \int_0^T \int \rho |\dot{u}|^2 \, dx \, dt, \tag{3.1}
\]

\[
A_2(T) \triangleq \frac{1}{2(\gamma - 1)} \sup_{t \in [0, T]} \int \rho (R\theta - \overline{P})^2 \, dx + \int_0^T (\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) \, dt, \tag{3.2}
\]

\[
A_3(T) \triangleq \sup_{t \in (0, T]} \left( \sigma \|\nabla u\|_{L^2}^2 + \sigma^2 \int \rho |\dot{u}|^2 \, dx + \sigma^2 \|\nabla \theta\|_{L^2}^2 \right) + \int_0^T \int \left( \sigma |\dot{u}|^2 + \sigma^2 |\nabla \dot{u}|^2 + \sigma^2 \rho |\dot{\theta}|^2 \right) \, dx \, dt. \tag{3.3}
\]

We have the following key a priori estimates on \((\rho, u, \theta)\).

**Proposition 3.1.** For given numbers \(M > 0, \hat{\rho} > 2,\) and \(\hat{\theta} > 1\), assume further that \((\rho_0, u_0, \theta_0)\) satisfies

\[
0 < \inf \rho_0 \leq \sup \rho_0 < \hat{\rho}, \quad 0 < \inf \theta_0 \leq \sup \theta_0 \leq \hat{\theta}, \quad \|\nabla u_0\|_{L^2} \leq M. \tag{3.4}
\]

Then there exist positive constants \(K, C^*, \alpha, \theta_\infty,\) and \(\varepsilon_0\) all depending on \(\mu, \lambda, \kappa, R, \gamma, \hat{\rho}, \hat{\theta}, \Omega,\) and \(M\) such that if \((\rho, u, \theta)\) is a smooth solution to the problem \((1.4)-(1.6)\) on \(\Omega \times (0, T]\) satisfying

\[
0 < \rho \leq 2\hat{\rho}, \quad A_1(T) \leq 3K, \quad A_2(T) \leq 2C^*_0^{1/4}, \quad A_3(T) \leq 2C^*_0^{1/6}, \tag{3.5}
\]

the following estimates hold:

\[
0 < \rho \leq 3\hat{\rho}/2, \quad A_1(T) \leq 2K, \quad A_2(T) \leq C^*_0^{1/4}, \quad A_3(T) \leq C^*_0^{1/6}, \tag{3.6}
\]

and for any \(t \geq 1,\)

\[
\|\rho - 1\|_{L^2} + \|u\|_{W^{1, 6}}^2 + \|\theta - \theta_\infty\|_{H^2}^2 \leq C^* e^{-\alpha t}. \tag{3.7}
\]

provided

\[
C_0 \leq \varepsilon_0. \tag{3.8}
\]

**Proof.** Proposition 3.1 is a straight consequence of the following Lemmas 3.2, 3.4, 3.6, 3.7 and 3.9 with \(\varepsilon_0\) as in \((3.124).\)

In this section, we always assume that \(C_0 \leq 1\) and let \(C\) denote some generic positive constant depending only on \(\mu, \lambda, \kappa, R, \gamma, \hat{\rho}, \hat{\theta}, \Omega,\) and \(M,\) and we write \(C(\alpha)\) to emphasize that \(C\) may depend on \(\alpha.\)

To begin with, we have the following uniform estimate on \(\overline{P}\), which plays an important role in the whole analysis.
Lemma 3.1. Under the conditions of Proposition 3.1, there exists a positive constant $C$ depending only on $\mu$, $R$, and $\hat{\rho}$ such that if $(\rho, u, \theta)$ is a smooth solution to the problem (1.4)–(1.6) on $\Omega \times (0, T]$ satisfying

$$0 < \rho \leq 2\hat{\rho}, \quad A_2(T) \leq 2C_0^{1/4}, \quad (3.9)$$

the following estimates hold:

$$\sup_{0 \leq t \leq T} \int (\rho |u|^2 + (\rho - 1)^2) \, dx \leq CC_0^{1/4}, \quad (3.10)$$

and

$$0 < \pi_1 \leq \mathcal{P}(t) \leq \pi_2, \quad \text{for any } t \in [0, T], \quad (3.11)$$

where $\pi_1$ and $\pi_2$ are positive constants depending only on $\mu$, $\gamma$, $R$, and $\Omega$.

Proof. First, it follows from (3.4) and (2.5) that, for all $(x,t) \in \Omega \times (0, T)$,

$$\theta(x,t) > 0. \quad (3.12)$$

Note that

$$\Delta u = \nabla \text{div} u - \nabla \times \text{curl} u,$$

one can rewrite (1.4) as

$$\rho(u_t + u \cdot \nabla u) = (2\mu + \lambda)\nabla \text{div} u - \mu \nabla \times \text{curl} u - \nabla P. \quad (3.13)$$

Adding (3.13) multiplied by $u$ to (1.4) multiplied by $1 - \theta^{-1}$ and integrating the resulting equality over $\Omega$ by parts, we obtain after using (1.4), (1.2), (3.12), and the boundary conditions (1.6) that

$$E'(t) = - \int \left( \frac{\lambda (\text{div} u)^2}{\theta} + 2\mu |\mathfrak{D}(u)|^2 \right) + \kappa \frac{|\nabla \theta|^2}{\theta^2} \, dx$$

$$- \mu \int (|\text{curl} u|^2 + 2(\text{div} u)^2 - 2|\mathfrak{D}(u)|^2) \, dx \quad (3.14)$$

$$\leq 2\mu \int |\nabla u|^2 \, dx,$$

where $E(t)$ is the basic energy defined by (1.35).

Then, integrating (3.14) with respect to $t$ over $(0, T)$ and using (3.9), one has

$$\sup_{0 \leq t \leq T} E(t) \leq C_0 + 2\mu \int_0^T \int |\nabla u|^2 \, dx \, dt \leq CC_0^{1/4}, \quad (3.15)$$

which together with

$$(\rho - 1)^2 \geq 1 + \rho \log \rho - \rho = (\rho - 1)^2 \int_0^1 \frac{1 - \alpha}{\alpha(\rho - 1) + 1} \, d\alpha \geq \frac{(\rho - 1)^2}{2(2\rho + 1)} \quad (3.16)$$

gives (3.10).
Next, it is easy to deduce from (1.4) and (1.7) that for any \( t \in [0,T] \),
\[
\rho(t) = \rho_0 = 1. \tag{3.17}
\]
Denote \( d\mu \triangleq |\Omega|^{-1}\rho dx \). Then \( d\mu \) is a positive measure satisfying \( \mu(\Omega) = 1 \) due to (3.9) and (3.17). Moreover, observe that \( y - \log y - 1 \) is a convex function in \((0, \infty)\), it thus follows directly from Jensen’s inequality (2.6) that for any \( t \in [0,T] \),
\[
\rho \theta(t) - \log \rho \theta(t) - 1 \leq \int (\theta - \log \theta - 1) \frac{\rho dx}{|\Omega|} \leq C
\]
due to (3.15). This in particular gives (3.11) and finishes the proof of Lemma 3.1.

\[\Box\]

**Remark 3.1.** It should be pointed out that the following term in (3.14)
\[
- \mu \int (|\nabla u|^2 + 2(\text{div } u)^2 - 2|\mathcal{D}(u)|^2) \, dx \tag{3.18}
\]
is a sign-undetermined term due to the slip boundary condition (1.32), which is in sharp contrast to the Cauchy problem [14] where the term (3.18) vanishes after integration by parts. Thus, in this case, we can not bound the basic energy only by the initial energy. However, this term obviously can be bounded by \( C \int |\nabla u|^2 \, dx \), which implies a “weaker” basic energy estimate (3.15).

The next lemma provides an estimate on the term \( A_1(T) \).

**Lemma 3.2.** Under the conditions of Proposition 3.1 there exist positive constants \( K \) and \( \varepsilon_1 \) both depending only on \( \mu, \kappa, \gamma, \hat{\rho}, \hat{\theta}, \Omega \), and \( M \) such that if \( (\rho, u, \theta) \) is a smooth solution to the problem (1.4)–(1.6) on \( \Omega \times (0,T] \) satisfying
\[
0 < \rho \leq 2\hat{\rho}, \quad A_2(T) \leq 2C_0^{1/4}, \quad A_1(T) \leq 3K, \tag{3.19}
\]
the following estimate holds:
\[
A_1(T) \leq 2K, \tag{3.20}
\]
provided \( C_0 \leq \varepsilon_1 \).

**Proof.** First, integrating (3.13) multiplied by \( 2u_t \) over \( \Omega \) by parts gives
\[
\frac{d}{dt} \int (\mu|\nabla u|^2 + (2\mu + \lambda)(\text{div } u)^2) \, dx + \int \rho |\dot{u}|^2 \, dx \\
\leq 2 \int P \text{div } u \, dx + \int \rho |\nabla u|^2 \, dx \\
= 2 \frac{d}{dt} \int (P - \mathcal{P}) \text{div } u \, dx - 2 \int (P - \mathcal{P}) \text{div } u \, dx + \int \rho |\nabla u|^2 \, dx \\
= 2 \frac{d}{dt} \int (P - \mathcal{P}) \text{div } u \, dx - \frac{1}{2\mu + \lambda} \frac{d}{dt} \int (P - \mathcal{P})^2 \, dx \\
- \frac{2}{2\mu + \lambda} \int (P - \mathcal{P}) \text{div } u \, dx + \int \rho |\nabla u|^2 \, dx,
\]
\[\Box\]
where in the last equality we have used \((1.39)\).

Next, straight calculations show that for any \(p \in [2, 6]\),
\[
\|R\theta - \overline{\theta}\|_{L^p} \leq R\|\theta - \overline{\theta}\|_{L^p} + C|R\overline{\theta} - \overline{\theta}| \leq C(\hat{\rho})\|\nabla \theta\|_{L^2},
\]
where one has used \((2.7)\) and the following fact:
\[
|R\overline{\theta} - \overline{\theta}| = \frac{R}{|\Omega|} \left| \int (1 - \rho)\theta dx \right| = \frac{R}{|\Omega|} \int (1 - \rho)(\theta - \overline{\theta}) dx \leq C\|\rho - 1\|_{L^2}\|\theta - \overline{\theta}\|_{L^2} \leq C(\hat{\rho})C_0^{1/8}\|\nabla \theta\|_{L^2}
\]
due to \((3.17)\) and \((3.10)\). Thus, it follows from \((3.11)\), \((3.19)\), \((3.10)\), and \((3.22)\) that for any \(p \in [2, 6]\),
\[
\|P - \overline{\theta}\|_{L^p} = \|\rho(R\theta - \overline{\theta}) + (\rho - 1)\overline{\theta}\|_{L^p} \\
\leq \|\rho(R\theta - \overline{\theta})\|_{L^p}^{(6-p)/(2p)}\|\rho(R\theta - \overline{\theta})\|_{L^p}^{3(p-2)/(2p)} + C\|\rho - 1\|_{L^p} \leq C(\hat{\rho})C_0^{1/(4p)},
\]
which together with \((2.18)\) and \((3.19)\) yields
\[
\|\nabla u\|_{L^6} \leq C(\hat{\rho}) \left(\|u^{1/2}\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla \theta\|_{L^2} + C_0^{1/24}\right).
\]

Note that \((1.4)\) implies
\[
P_t = -\text{div}(P u) - (\gamma - 1)P\text{div}u + (\gamma - 1)\kappa \Delta \theta \]
\[
+ (\gamma - 1) \left(\lambda(\text{div}u)^2 + 2\mu|\mathcal{D}(u)|^2\right),
\]
which along with \((1.6)\) gives
\[
\overline{P}_t = - (\gamma - 1)\overline{P}\text{div}u + (\gamma - 1) \left(\lambda(\text{div}u)^2 + 2\mu|\mathcal{D}(u)|^2\right),
\]
We thus obtain after using integration by parts, \((3.11)\), \((2.7)\), \((2.13)\), and \((3.23)-(3.25)\) that
\[
\left|\int P_t G dx\right| \leq C\int P(|G|\nabla u + |u|\nabla G) dx + C\int (|\nabla \theta|\nabla G + |\nabla u|^2|G|) dx \\
\leq C\int |P - \overline{\theta}|(|G|\nabla u + |u|\nabla G) dx + C\overline{\theta}\int (|G|\nabla u + |u|\nabla G) dx \\
+ C\|\nabla G\|_{L^2}\|\nabla \theta\|_{L^2} + C\|G\|_{L^6}\|\nabla u\|_{L^2}^{3/2}\|\nabla u\|_{L^6}^{1/2} \leq C(\hat{\rho})\|\nabla \theta\|_{L^2}^{1/2} + C\|G\|_{L^2}\|\nabla u\|_{L^2} + C\|\nabla G\|_{L^2}\|\nabla \theta\|_{L^2} \\
+ C(\hat{\rho})\|\nabla G\|_{L^2}\|\nabla u\|_{L^2}^{3/2} \left(\|u^{1/2}\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla \theta\|_{L^2} + 1\right)^{1/2} \leq C(\hat{\rho})\|\nabla \theta\|_{L^2}^{1/2} + C(\hat{\rho})\|\nabla u\|_{L^2}^2 + C(\hat{\rho})\|\nabla \theta\|_{L^2}^2 + C(\hat{\rho})\|\nabla u\|_{L^2}^2 \leq C(\hat{\rho})\|\nabla \theta\|_{L^2}^{1/2} + C(\hat{\rho})\|\nabla \theta\|_{L^2}^2 + C(\hat{\rho})\|\nabla u\|_{L^2}^2.
and
\[ \left| \int P_t G \, dx \right| \leq C(\hat{\rho})(\|\nabla u\|_{L^2} + \|\nabla u\|^2_{L^2}) \|G\|_{L^2} \]
\[ \leq \delta \|\nabla G\|_{L^2}^2 + C(\delta, \hat{\rho})(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4) \]
\[ \leq C(\hat{\rho})\delta \|\rho^1\hat{u}\|_{L^2}^2 + C(\delta, \hat{\rho})(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4), \tag{3.28} \]
where one has used
\[ |P_t| \leq C\|P - \overline{P}\|_{L^2} \|\nabla u\|_{L^2} + C\|\nabla u\|_{L^2}^2 \leq C(\hat{\rho})(C_0^{1/8}\|\nabla u\|_{L^2} + \|\nabla u\|_{L^2}^2) \tag{3.29} \]
owing to (3.26) and (3.23).

Then, it follows from (2.7), (3.19), and (3.24) that
\[ \int \rho |\nabla u|^2 \, dx \leq C(\hat{\rho})\|\nabla u\|_{L^2}^2 \|\nabla u\|_{L^6} \]
\[ \leq \delta \|\rho^1\hat{u}\|_{L^2}^2 + C(\delta, \hat{\rho}) \left( \|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^6 \right). \tag{3.30} \]

Finally, substituting (3.27), (3.28), and (3.30) into (3.21), one obtains after choosing \( \delta \) suitably small that
\[ \frac{d}{dt} \int (\mu |\nabla u|^2 + (2\mu + \lambda) |\text{div} u|^2) \, dx + \frac{1}{2\mu + \lambda} \frac{d}{dt} \|P - \overline{P}\|_{L^2}^2 + \frac{1}{2} \int \rho |\hat{u}|^2 \, dx \]
\[ \leq 2 \frac{d}{dt} \int (P - \overline{P}) \text{div} u \, dx + C(\hat{\rho}) \left( \|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^6 \right). \tag{3.31} \]

Note that it holds
\[ \|P - \overline{P}\|_{L^2}^2(0) = R^2 \int (\rho_0 \theta_0 - \overline{\rho_0 \theta_0})^2 \, dx \]
\[ \leq C(\hat{\rho}) \int \rho_0(\theta_0 - \overline{\rho_0 \theta_0})^2 \, dx + C \int (\rho_0 - 1)^2 \, dx \]
\[ \leq C(\hat{\rho}, \hat{\theta}) C_0, \tag{3.32} \]
where we have used (3.11), (3.16), and the following fact:
\[ \int \rho_0(\theta_0 - \overline{\rho_0 \theta_0})^2 \, dx \leq C \int \rho_0(\theta_0 - 1)^2 \, dx + C|1 - \overline{\rho_0 \theta_0}|^2 \]
\[ \leq C \int \rho_0(\theta_0 - 1)^2 \, dx + C \left| \int \rho_0(1 - \theta_0) \, dx \right|^2 \]
\[ \leq C(\hat{\rho}, \hat{\theta}) \int \rho_0(\theta_0 - \log \theta_0 - 1) \, dx \]
\[ \leq C(\hat{\rho}, \hat{\theta}) C_0 \tag{3.33} \]
due to (1.7) and
\[ \theta - \log \theta - 1 = (\theta - 1)^2 \int_0^1 \frac{\alpha}{\alpha(\theta - 1) + 1} \, d\alpha \geq \frac{1}{2(||\theta(\cdot, t)||_{L^\infty} + 1)} (\theta - 1)^2. \tag{3.34} \]
Then, integrating \((3.31)\) over \((0, T)\), one deduces from \((2.12), (3.32), (3.19),\) and \((3.23)\) that
\[
\sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 + \int_0^T \int \rho|\dot{u}|^2 \, dx \, dt \\
\leq CM^2 + C(\hat{\rho}, \hat{\theta})C_0^{1/4} + C(\hat{\rho})C_0^{1/4} \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^4 + C(\hat{\rho})C_0^{1/8} \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^2 \\
\leq CM^2 + C(\hat{\rho}, \hat{\theta})C_0^{1/12} + C(\hat{\rho})C_0^{1/4} \sup_{0 \leq t \leq T} \|\nabla u\|_{L^2}^4 \\
\leq K + 9C(\hat{\rho})C_0^{1/4}K^2 \\
\leq 2K,
\]
with \(K \triangleq CM^2 + C(\hat{\rho}, \hat{\theta}) + 1\), provided
\[
C_0 \leq \varepsilon_1 \triangleq \min \left\{ 1, (9C(\hat{\rho})K)^{-1} \right\}.
\]
The proof of Lemma \ref{lem3.2} is completed.

Next, to estimate \(A_3(T)\), we adopt the approach due to Hoff \cite{12} (see also Huang-Li \cite{14}) to establish the following elementary estimates on \(\dot{u}\) and \(\dot{\theta}\), where the boundary terms are handled by the ideas due to \cite{6}. The estimate of \(A_3(T)\) will be postponed to Lemma \ref{lem3.4}.

**Lemma 3.3.** Under the conditions of Proposition \ref{prop3.1}, let \((\rho, u, \theta)\) be a smooth solution to the problem \ref{eq1.2}–\ref{eq1.4} on \(\Omega \times (0, T)\) satisfying \ref{eq3.2} with \(K\) as in Lemma \ref{lem3.2}. Then there exist positive constants \(C, C_1,\) and \(C_2\) depending only on \(\mu, \lambda, k, R, \gamma, \hat{\rho}, \hat{\theta}, \Omega,\) and \(M\) such that, for any \(\eta \in (0, 1)\) and \(m \geq 0\), the following estimates hold:
\[
(\sigma B_1)'(t) + \frac{1}{2} \sigma \int \rho|\dot{u}|^2 \, dx \leq CC_0^{1/4} \sigma' + C \left( \|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right), \tag{3.35}
\]
\[
\left( \sigma^m \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right)_t + C_1 \sigma^m \|\nabla \dot{u}\|_{L^2}^2 \\
\leq -2 \left( \int_{\partial\Omega} \sigma^m (u \cdot \nabla n \cdot u) GdS \right)_t + C(\sigma^{m-1}\sigma' + \sigma^m) \|\rho^{1/2} \dot{u}\|_{L^2}^2 \\
+ C_2 \sigma^m \|\rho^{1/2} \dot{\theta}\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 + C \sigma^m \|\nabla u\|_{L^2}^4 + C \sigma^m \|\theta \nabla u\|_{L^2}^2,
\tag{3.36}
\]
and
\[
(\sigma^m B_2)'(t) + \sigma^m \int \rho|\dot{\theta}|^2 \, dx \leq C \eta \sigma^m \|\nabla \dot{u}\|_{L^2}^2 + C \|\nabla \theta\|_{L^2}^2 + C \sigma^m \|\nabla u\|_{L^2}^4 + C(\eta) \sigma^m \|\theta \nabla u\|_{L^2}^2,
\tag{3.37}
\]
where
\[
B_1(t) \triangleq \mu \|\text{curl}u\|_{L^2}^2 + (2\mu + \lambda) \|\text{div}u\|_{L^2}^2 \\
+ \frac{1}{2\mu + \lambda} \|P - \mathcal{P}\|_{L^2}^2 - 2 \int \text{div}(P - \mathcal{P}) \, dx,
\tag{3.38}
\]
and
\[
B_2(t) \triangleq \frac{\gamma - 1}{R} \left( \kappa \|\nabla \theta\|_{L^2}^2 - 2 \int (\lambda (\text{div}u)^2 + 2\mu |\nabla u|^2) \, dx \right),
\tag{3.39}
\]
Proof. First, multiplying (3.31) by \( \sigma \) and using (3.23) and (3.5) give (3.35) directly.

Now, we will prove (3.36).

First, one can rewrite (1.39) as

\[
\rho \ddot{u} = \nabla G - \mu \nabla \times \text{curl} u, \quad (3.40)
\]

with \( G \) defined in (1.39). For \( m \geq 0 \), operating \( \sigma^m \dot{u}[\partial / \partial t + \text{div}(u \cdot )] \) to (3.40) and integrating the resulting equality over \( \Omega \) by parts lead to

\[
\left( \sigma^m / 2 \int \rho |\dot{u}|^2 \, dx \right)_{t} - m \sigma^m \sigma' \int \rho |\dot{u}|^2 \, dx = \int_{\partial \Omega} \sigma^m nG \cdot dS - \int \sigma^m [\text{div} \dot{u} G_t + u \cdot \nabla \dot{u} \cdot \nabla G] \, dx
\]

\[
- \mu \int \sigma^m \dot{u}^j \left[ (\nabla \times \text{curl} u)_t^j + \text{div}(u \times \nabla \text{curl} u)^j \right] \, dx \triangleq \sum_{i=1}^{3} N_i. \tag{3.41}
\]

Noticing that

\[
u \cdot \nabla u \cdot n = - u \cdot \nabla n \cdot u \quad \text{on} \ \partial \Omega \tag{3.42}
\]
due to \( u \cdot n |_{\partial \Omega} = 0 \), one can deduce from (1.6) and (3.42) that

\[
N_1 = - \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) G \, dS
\]

\[
= - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) G \, dS \right)_{t} + m \sigma^m \sigma' \int_{\partial \Omega} (u \cdot \nabla n \cdot u) G \, dS
\]

\[
+ \int_{\partial \Omega} \sigma^m (\dot{u} \cdot \nabla n \cdot u) G \, dS + \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot \dot{u}) G \, dS
\]

\[
- \int_{\partial \Omega} \sigma^m G (u \cdot \nabla) u \cdot \nabla n \cdot u \, dS - \int_{\partial \Omega} \sigma^m G u \cdot \nabla n \cdot (u \cdot \nabla) u \, dS \tag{3.43}
\]

\[
\leq - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) G \, dS \right)_{t} + C \sigma^m \sigma' \| \nabla u \|_{L^2} \| \nabla G \|_{L^2}
\]

\[
+ \delta \sigma^m \| \dot{u} \|_{H^1}^2 + C(\delta) \sigma^m \| \nabla u \|_{L^2} \| \nabla G \|_{L^2}^2
\]

\[
- \int_{\partial \Omega} \sigma^m G (u \cdot \nabla) u \cdot \nabla n \cdot u \, dS - \int_{\partial \Omega} \sigma^m G u \cdot \nabla n \cdot (u \cdot \nabla) u \, dS,
\]

where one has used

\[
\left| \int_{\partial \Omega} (\dot{u} \cdot \nabla n \cdot u + u \cdot \nabla n \cdot \dot{u}) G \, dS \right| \leq C \| \dot{u} \|_{H^1} \| u \|_{H^1} \| G \|_{H^1}
\]

\[
\leq C \| \dot{u} \|_{H^1} \| \nabla u \|_{L^2} \| \nabla G \|_{L^2},
\]

and

\[
\left| \int_{\partial \Omega} (u \cdot \nabla n \cdot u) G \, dS \right| \leq C \| \nabla u \|_{L^2}^2 \| \nabla G \|_{L^2} \tag{3.44}
\]

Now, we will adopt the idea in [6] to deal with the last two boundary terms in (3.43).
In fact, denote \( u^\perp \triangleq -u \times n \), it follows from \( u \cdot n |_{\partial \Omega} = 0 \) that

\[
u = u^\perp \times n \quad \text{on} \ \partial \Omega,
\]

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which along with \((2.7)\), \((1.38)\), and integration by parts yields

\[
- \int_{\partial \Omega} G(u \cdot \nabla)u \cdot \nabla n \cdot u dS
\]

\[
= - \int_{\partial \Omega} Gu^\perp \times n \cdot \nabla u^\perp \nabla n \cdot u dS
\]

\[
= - \int_{\partial \Omega} Gn \cdot (\nabla u^\perp \times u^\perp) \nabla n \cdot u dS
\]

\[
= - \int \text{div}(G(\nabla u^\perp \times u^\perp) \nabla n \cdot u)dx
\]

\[
= - \int \nabla (\nabla n \cdot uG) \cdot (\nabla u^\perp \times u^\perp) dx - \int \text{div}(\nabla u^\perp \times u^\perp) \nabla n \cdot u G dx
\]

\[
= - \int \nabla (\nabla n \cdot uG) \cdot (\nabla u^\perp \times u^\perp) dx + \int G \nabla u^\perp \cdot \nabla \times u^\perp \nabla n \cdot u dS
\]

\[
\leq C \int |\nabla G| |\nabla u||u^2 dx + C \int |G|(|\nabla u|^2 |u| + |\nabla u||u^2) dx
\]

\[
\leq C ||\nabla G||^2_{L^6} ||\nabla u||_{L^2} ||u||_{L^4}^2 + C ||G||^2_{L^3} ||\nabla u||_{L^4} ||u||_{L^6} + C ||G||^2_{L^6} ||\nabla u||_{L^2} ||u||_{L^2}^2
\]

\[
\leq \delta ||\nabla G||^2_{L^6} + C(\delta) ||\nabla u||_{L^2}^4 + C ||\nabla u||_{L^4}^4 + C ||\nabla G||^2_{L^2} (||\nabla u||_{L^2}^2 + 1).
\]

Similarly, it holds that

\[
- \int_{\partial \Omega} Gu \cdot \nabla n \cdot (u \cdot \nabla) u dS
\]

\[
\leq \delta ||\nabla G||^2_{L^6} + C(\delta) ||\nabla u||_{L^2}^4 + C ||\nabla G||^2_{L^2} (||\nabla u||_{L^2}^2 + 1).
\]

Next, it follows from \((1.39)\) that

\[
G_t = (2\mu + \lambda) \text{div} u_t - (P_t - \overline{P_t})
\]

\[
= (2\mu + \lambda) \text{div} \dot{u} - (2\mu + \lambda) \text{div}(u \cdot \nabla u) - R \rho \dot{\theta} + \text{div}(Pu) + \overline{R \rho \dot{\theta}}
\]

\[
= (2\mu + \lambda) \text{div} \dot{u} - (2\mu + \lambda) \nabla u : (\nabla u)^{tr} - u \cdot \nabla G + P \text{div} u - R \rho \dot{\theta} + \overline{R \rho \dot{\theta}},
\]

where one has used

\[
P_t = (R \rho \dot{\theta})_t = R \rho \dot{\theta} - \text{div}(Pu), \quad \overline{P_t} = \overline{R \rho \dot{\theta}}.
\]

Then, integration by parts combined with \((3.47)\) gives

\[
N_2 = - \int \sigma^m [\text{div} \dot{u} G_t + u \cdot \nabla \dot{u} \cdot \nabla G] dx
\]

\[
= - (2\mu + \lambda) \int \sigma^m (\text{div} \dot{u})^2 dx + (2\mu + \lambda) \int \sigma^m \text{div} \dot{u} \nabla : (\nabla u)^{tr} dx
\]

\[
+ \int \sigma^m \text{div} \dot{u} u \cdot \nabla G dx - \int \sigma^m \text{div} \dot{u} Pu dx
\]

\[
+ R \int \sigma^m \text{div} \dot{u} \rho \dot{\theta} dx - R \rho \dot{\theta} \int \sigma^m \text{div} \dot{u} dx - \int \sigma^m u \cdot \nabla \dot{u} \cdot \nabla G dx
\]

\[
\leq - (2\mu + \lambda) \int \sigma^m (\text{div} \dot{u})^2 dx
\]

\[
+ C \sigma^m ||\nabla \dot{u}||_{L^2} ||\nabla u||_{L^4} + C \sigma^m ||\nabla \dot{u}||_{L^2} ||\nabla G||_{L^6}^{1/2} ||\nabla G||_{L^6}^{1/2} ||u||_{L^6}
\]

\[
+ C(\hat{\rho}) \sigma^m ||\nabla \dot{u}||_{L^2} ||\theta \nabla u||_{L^2} + C(\hat{\rho}) \sigma^m ||\nabla \dot{u}||_{L^2} ||\rho^{1/2} \dot{\theta}||_{L^2}.
\]
Note that
\[ \text{curl} u_t = \text{curl} \dot{u} - u \cdot \text{curl} u - \nabla u^i \times \nabla_i u, \]
which together with some straightforward calculations yields
\[ N_3 = -\mu \int \sigma^m |\text{curl} \dot{u}|^2 dx + \mu \int \sigma^m \text{curl} \dot{u} \cdot (\nabla u^i \times \nabla_i u) dx \]
\[ + \mu \int \sigma^m u \cdot \text{curl} u \cdot \text{curl} \dot{u} dx + \mu \int \sigma^m u \cdot \nabla \dot{u} \cdot (\nabla \times \text{curl} u) dx \]
\[ \leq -\mu \int \sigma^m |\text{curl} \dot{u}|^2 dx + \delta \sigma^m (|\nabla \dot{u}|^2 + \|\text{curl} u\|^2_{L^2}) \]
\[ + C(\delta) \sigma^m \|\nabla u\|^4_{L^4} + C(\delta) \sigma^m \|\nabla u\|^4_{L^2} \|\text{curl} u\|^2_{L^2} . \]

Finally, it is easy to deduce from Lemmas 2.5 and 2.6 that
\[ \|G\|_{H^1} + \|\text{curl} u\|_{H^1} \leq C(\|\rho \dot{u}\|_{L^2} + \|\nabla u\|_{L^2}) , \]
and that
\[ \|\nabla G\|_{L^6} + \|\nabla \text{curl} u\|_{L^5} + \|\dot{u}\|_{H^1} \leq C(\|\rho \dot{u}\|_{L^6} + \|\nabla u\|_{L^2}) + \|\dot{u}\|_{H^1} \]
\[ \leq C(\hat{\rho})(\|\nabla \dot{u}\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla u\|^2_{L^2}) . \]

Hence, submitting (3.44), (3.49), and (3.50) into (3.44), one obtains after using (3.45), (3.46), (3.5), (3.51), and (3.52) that
\[ \left( \frac{\sigma^m}{2} \|\rho^{1/2} \dot{u}\|^2_{L^2} \right)_t + (2\mu + \lambda) \sigma^m \|\text{div} \dot{u}\|^2_{L^2} + \mu \sigma^m \|\text{curl} \dot{u}\|^2_{L^2} \]
\[ \leq - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) G dS \right)_t + C(\hat{\rho}) \delta \sigma^m \|\nabla \dot{u}\|^2_{L^2} \]
\[ + C(\delta, \hat{\rho}, M) \|\rho^{1/2} \dot{\theta}\|^2_{L^2} + C(\delta, \hat{\rho}, M) (\sigma^{m-1} \sigma' + \sigma^m) \|\rho^{1/2} \dot{u}\|^2_{L^2} \]
\[ + C(\delta, \hat{\rho}, M) \|\nabla u\|^2_{L^2} + C(\delta) \sigma^m \|\nabla u\|^4_{L^4} + C(\delta, \hat{\rho}) \sigma^m \|\theta \nabla u\|^2_{L^2} . \]

Applying (2.20) to (3.53) and choosing \( \delta \) small enough infer (3.36) directly.

Finally, we will prove (3.37).

For \( m \geq 0 \), multiplying (1.3) by \( \sigma^m \dot{\theta} \) and integrating the resulting equality over \( \Omega \) yield that
\[ \frac{\kappa \sigma^m}{2} \left( \|\nabla \theta\|^2_{L^2} \right)_t + \frac{R \sigma^m}{\gamma - 1} \int \rho \dot{\theta}^2 dx \]
\[ = -\kappa \sigma^m \int \nabla \theta \cdot (u \cdot \nabla \theta) dx + \lambda \sigma^m \int (\text{div} u)^2 \dot{\theta} dx \]
\[ + 2\mu \sigma^m \int |\nabla (u)|^2 \dot{\theta} dx - R \sigma^m \int \rho \text{div} u \theta dx = \sum_{i=1}^4 I_i . \]

First, combining (2.7) and (3.5) gives
\[ I_1 \leq C \sigma^m \|\nabla u\|_{L^2} \|\nabla \theta\|_{L^2}^{1/2} \|\nabla^2 \theta\|_{L^2}^{3/2} \]
\[ \leq \delta \sigma^m \|\rho^{1/2} \dot{\theta}\|^2_{L^2} + \sigma^m \left( \|\nabla u\|^4_{L^4} + \|\theta \nabla u\|^2_{L^2} \right) + C(\delta, \hat{\rho}, M) \sigma^m \|\nabla \theta\|^2_{L^2} , \]
where in the last inequality we have used the following estimate:

$$\|\nabla^2 \theta\|_{L^2} \leq C(\hat{\rho}) \left( \|\rho^{1/2} \dot{\theta}\|_{L^2} + \|\nabla u\|_{L^4}^2 + \|\theta \nabla u\|_{L^2}^2 \right),$$

(3.56)

which is derived from the standard $L^2$-estimate to the following elliptic problem:

$$\begin{cases}
\kappa \Delta \theta = \frac{R}{\gamma - 1} \rho \dot{\theta} + R \rho \theta \text{div} u - \lambda (\text{div} u)^2 - 2\mu |\mathcal{D}(u)|^2, \\
\nabla \theta \cdot n|_{\partial \Omega \times (0,T)} = 0.
\end{cases}$$

(3.57)

Next, it holds that for any $\eta \in (0, 1]$,

$$I_2 = \lambda \sigma^m \int (\text{div} u)^2 \theta_t dx + \lambda \sigma^m \int (\text{div} u)^2 u \cdot \nabla \theta dx$$

$$= \lambda \sigma^m \left( \int (\text{div} u)^2 \theta dx \right)_t - 2\lambda \sigma^m \int \theta \text{div}(\dot{u} - u \cdot \nabla u) dx$$

$$+ \lambda \sigma^m \int (\text{div} u)^2 u \cdot \nabla \theta dx$$

$$= \lambda \sigma^m \left( \int (\text{div} u)^2 \theta dx \right)_t - 2\lambda \sigma^m \int \theta \text{div}(\dot{u}v - u \cdot \nabla u) dx$$

$$+ 2\lambda \sigma^m \int \theta \text{div}u \dot{u} + \lambda \sigma^m \int u \cdot \nabla (\theta (\text{div} u)^2) dx$$

$$\leq \lambda \left( \sigma^m \int (\text{div} u)^2 \theta dx \right)_t - \lambda m \sigma^{m-1} \sigma' \int (\text{div} u)^2 \theta dx$$

$$+ \eta \sigma^m \|\nabla \dot{u}\|_{L^2}^2 + C(\eta) \sigma^m \|\theta \nabla u\|_{L^2}^2 + C \sigma^m \|\nabla u\|_{L^4}^4,$$

(3.58)

and

$$I_3 \leq 2\mu \left( \sigma^m \int |\mathcal{D}(u)|^2 \theta dx \right)_t - 2\mu m \sigma^{m-1} \sigma' \int |\mathcal{D}(u)|^2 \theta dx$$

$$+ \eta \sigma^m \|\nabla \dot{u}\|_{L^2}^2 + C(\eta) \sigma^m \|\theta \nabla u\|_{L^2}^2 + C \sigma^m \|\nabla u\|_{L^4}^4.$$

(3.59)

Finally, Cauchy’s inequality gives

$$|I_4| \leq \delta \sigma^m \int \rho|\theta|^2 dx + C(\delta, \hat{\rho}) \sigma^m \|\theta \nabla u\|_{L^2}^2.$$

(3.60)

Substituting (3.55) and (3.58)–(3.60) into (3.54), we obtain (3.37) after using (1.2) and choosing $\delta$ suitably small. The proof of Lemma 3.3 is completed.

With the estimates (3.35)–(3.37) (see Lemma 3.3) at hand, we are now in a position to prove the following estimate on $A_3(T)$.

**Lemma 3.4.** Under the conditions of Proposition 3.1, there exists a positive constant $\varepsilon_2$ depending only on $\mu, \lambda, \kappa, R, \gamma, \hat{\rho}, \hat{\theta}, \Omega$, and $M$ such that if $(\rho, u, \theta)$ is a smooth solution to the problem (1.4)–(1.6) on $\Omega \times (0,T]$ satisfying (3.5) with $K$ as in Lemma 3.2, the following estimate holds:

$$A_3(T) \leq C_0^{1/6},$$

(3.61)

provided $C_0 \leq \varepsilon_2$. 

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Proof. First, it follows from (3.38), (2.12), and (3.23) that

$$B_1(t) \geq C \| \nabla u \|_{L^2}^2 - C\|P - \overline{P}\|_{L^2}^2 \geq C\|\nabla u\|_{L^2}^2 - C(\hat{\rho})C_0^{1/4},$$

which together with (3.35) and (3.5) implies that

$$\sup_{t \in (0, T)} (\sigma \|\nabla u\|_{L^2}^2) + \int_0^T \sigma \rho |\dot{u}|^2 dx dt \leq C(\hat{\rho}, M)C_0^{1/4}. \quad (3.62)$$

For $C_2$ as in (3.36), adding (3.37) multiplied by $C_2 + 1$ to (3.36) and choosing $\eta$ suitably small gives

$$(\sigma^m \varphi)'(t) + \sigma^m \int \left( \frac{C_1}{2} |\nabla \dot{u}|^2 + \rho |\dot{\theta}|^2 \right) dx \leq -2 \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) G dS \right) + C(\hat{\rho}, M)(\sigma^{m-1} \sigma' + \sigma^m)\|\rho^{1/2}\dot{u}\|_{L^2}^2$$

$$+ C(\hat{\rho}, M)(\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) + C\sigma^m \|\nabla u\|_{L^4}^4 + C(\hat{\rho})\sigma^m \|\theta \nabla u\|_{L^2}^2,$$

where $\varphi(t)$ is defined by

$$\varphi(t) \equiv \|\rho^{1/2} \dot{u}\|_{L^2}^2 + (C_2 + 1)B_2(t). \quad (3.64)$$

Then it follows from (3.39) that

$$\varphi(t) \geq \frac{1}{2} \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \frac{\kappa(\gamma - 1)}{2R} \|\nabla \theta\|_{L^2}^2 - C(\hat{\rho}, M)\|\nabla u\|_{L^2}^2, \quad (3.65)$$

where one has used that for any $\delta \in (0, 1]$,

$$\int \theta |\nabla u|^2 dx \leq C \int |R\theta - \overline{P}| |\nabla u|^2 dx + C\overline{P} \int |\nabla u|^2 dx \leq C\|R\theta - \overline{P}\|_{L^6} \|\nabla u\|_{L^6}^{3/2} \|\nabla u\|_{L^2}^{1/2} + C\|\nabla u\|_{L^2}^2$$

$$\leq C(\hat{\rho}) \|\nabla \theta\|_{L^2} \|\nabla u\|_{L^2}^{3/2} \left( \|\rho^{1/2} \dot{u}\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla \theta\|_{L^2} + 1 \right)^{1/2}$$

$$+ C\|\nabla u\|_{L^2}^2 \leq \delta \left( \|\nabla \theta\|_{L^2}^2 + \|\rho^{1/2} \dot{u}\|_{L^2}^2 \right) + C(\delta, \hat{\rho}, M)\|\nabla u\|_{L^2}^2,$$

due to (3.11), (3.22), (3.24), and (3.35).

Next, it follows from (3.11), (3.22), (3.5), and (3.24) that

$$\|\theta \nabla u\|_{L^2}^2 \leq C\|R\theta - \overline{P}\|_{L^6}^2 \|\nabla u\|_{L^2} \|\nabla u\|_{L^6} + C\overline{P}^2 \|\nabla u\|_{L^2}^2$$

$$\leq C(\hat{\rho}, M) \left( \|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) \left( \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + 1 \right).$$

And, by virtue of (2.18), (3.22), and (3.5), one gets

$$\|\nabla u\|_{L^4}^4 \leq C\|\rho \dot{u}\|_{L^2}^3 \left( \|\nabla u\|_{L^2} + \|P - \overline{P}\|_{L^2} \right) + C(\|\nabla u\|_{L^2}^4 + \|P - \overline{P}\|_{L^4}^4)$$

$$\leq C(\hat{\rho}) \|\rho^{1/2} \dot{u}\|_{L^2}^3 \left( \|\nabla u\|_{L^2} + 1 \right) + C(\hat{\rho})\|\nabla \theta\|_{L^2}^3 + C\|\rho - 1\|_{L^4}^4 + C\|\nabla u\|_{L^2}^4$$

$$\leq C(\hat{\rho}, M) \left( \|\rho^{1/2} \dot{u}\|_{L^2}^3 + \|\nabla \theta\|_{L^2}^3 \right) + C(\hat{\rho})\|\rho - 1\|_{L^2}^2 + C(\hat{\rho}, M)\|\nabla u\|_{L^2}^2,$$
which together with (3.5) yields
\[ \sigma \| \nabla u \|^4_{L^4} \leq C(\hat{\rho}, M) \left( \| \rho^{1/2} u \|^2_{L^2} + \| \nabla \theta \|^2_{L^2} + \| \nabla u \|^2_{L^2} \right) + C(\hat{\rho}) \sigma \| \rho - 1 \|^2_{L^2}. \]  
(3.69)

Thus, taking \( m = 2 \) in (3.63), one obtains after using (3.5), (3.67), and (3.69) that
\[ \left( \sigma^2 \varphi \right)'(t) + \sigma^2 \int \left( \frac{C_1}{2} |\nabla u|^2 + \rho |\dot{\theta}|^2 \right) dx \leq -2 \left( \int_{\partial \Omega} \sigma^2 (u \cdot \nabla n \cdot u)GdS \right)_t + C(\hat{\rho}, M) \sigma \| \rho^{1/2} u \|^2_{L^2} \]
\[ + C(\hat{\rho}, M) \left( \| \nabla u \|^2_{L^2} + \| \nabla \theta \|^2_{L^2} \right) + C(\hat{\rho}) \sigma \| \rho - 1 \|^2_{L^2}. \]  
(3.70)

Now, we deduce from (2.13), (3.5), and (3.44) that
\[ \sup_{0 \leq t \leq T} \left| \int_{\partial \Omega} \sigma^2 u \cdot \nabla n \cdot u GdS \right| \leq C(\hat{\rho}) \sup_{0 \leq t \leq T} \sigma \| \nabla u \|^2_{L^2} \sup_{0 \leq t \leq T} \sigma \| \rho^{1/2} u \|^2_{L^2} \]
\[ \leq C(\hat{\rho}) C_0^{1/4}. \]  
(3.71)

Furthermore, note that (1.39) is equivalent to
\[ \overline{P}(\rho - 1) = -G + (2\mu + \lambda) \text{div}u - \rho (R \theta - \overline{P}), \]
this together with (3.11), (3.5), (2.17), (3.22), and (3.62) implies
\[ \int_0^T \sigma \| \rho - 1 \|^2_{L^2} dt \leq C \int_0^T \sigma (\| G \|^2_{L^2} + \| \nabla u \|^2_{L^2}) dt + C(\hat{\rho}) \int_0^T \| R \theta - \overline{P} \|^2_{L^2} dt \]
\[ \leq C(\hat{\rho}) \int_0^T \left( \sigma \| \rho^{1/2} u \|^2_{L^2} + \| \nabla u \|^2_{L^2} + \| \nabla \theta \|^2_{L^2} \right) dt \]
\[ \leq C(\hat{\rho}, M) C_0^{1/4}. \]  
(3.72)

Thus, integrating (3.70) over (0, T), one obtains after using (3.62), (3.65), (3.71), (3.72), and (3.5) that
\[ A_3(T) \leq C(\hat{\rho}, M) C_0^{1/4} \leq C_0^{1/6}, \]
provided
\[ C_0 \leq \varepsilon_2 \triangleq \min \{ 1, (C(\hat{\rho}, M))^{-12} \}. \]

The proof of Lemma 3.4 is completed. \( \Box \)

Next, in order to control \( A_2(T) \), we first re-establish the basic energy estimate for short time \([0, \sigma(T)]\), and then show that the spatial \( L^2 \)-norm of \( R \theta - \overline{P} \) could be bounded by the combination of the initial energy and the spatial \( L^2 \)-norm of \( \nabla \theta \), which is indeed the key ingredient to estimate \( A_2(T) \).
Lemma 3.5. Under the conditions of Proposition 3.1, there exist positive constants $C$ and $\varepsilon_{3,1}$ depending only on $\mu, \lambda, \kappa, R, \gamma, \hat{\rho}, \hat{\theta}, \Omega,$ and $M$ such that if $(\rho, u, \theta)$ is a smooth solution to the problem (1.4)-(1.6) on $\Omega \times (0, T]$ satisfying (3.5) with $K$ as in Lemma 3.2, the following estimates hold:

\[ \sup_{0 \leq t \leq \sigma(T)} \int (\rho |u|^2 + (\rho - 1)^2 + \rho(\theta - \log \theta - 1)) \, dx \leq CC_0, \quad (3.73) \]

and

\[ \| (R\theta - \bar{\theta})(\cdot, t) \|_{L^2} \leq C \left( C_{1/2}^{1/2} + C_0^{1/3} \| \nabla \theta(\cdot, t) \|_{L^2} \right), \quad (3.74) \]

for all $t \in (0, \sigma(T)]$, provided $C_0 \leq \varepsilon_{3,1}$.

Proof. The proof is divided into the following two steps.

Step 1: The proof of (3.73).

First, multiplying (3.13) by $u$, one deduces from integration by parts, (1.4)$_1$, and (3.34) that

\[ \frac{d}{dt} \int \left( \frac{1}{2} \rho |u|^2 + R(1 + \rho \log \rho - \rho) \right) \, dx + \int (\mu |\nabla u|^2 + (2\mu + \lambda)(\div u)^2) \, dx \]

\[ = R \int \rho(\theta - 1) \div u \, dx \]

\[ \leq \delta \| \nabla u \|_{L^2}^2 + C(\delta, \hat{\rho}) \int \rho(\theta - 1)^2 \, dx \]

\[ \leq \delta \| \nabla u \|_{L^2}^2 + C(\delta, \hat{\rho})(\| \theta(\cdot, t) \|_{L^\infty} + 1) \int \rho(\theta - \log \theta - 1) \, dx. \quad (3.75) \]

Using (2.12) and choosing $\delta$ small enough in (3.75), it holds that

\[ \frac{d}{dt} \int \left( \frac{1}{2} \rho |u|^2 + R(1 + \rho \log \rho - \rho) \right) \, dx + C_3 \int |\nabla u|^2 \, dx \]

\[ \leq C(\hat{\rho})(\| \theta(\cdot, t) \|_{L^\infty} + 1) \int \rho(\theta - \log \theta - 1) \, dx. \quad (3.76) \]

Then, adding (3.76) multiplied by $(2\mu + 1)C_3^{-1}$ to (3.14), one has

\[ ((2\mu + 1)C_3^{-1} + 1) \frac{d}{dt} \int \left( \frac{1}{2} \rho |u|^2 + R(1 + \rho \log \rho - \rho) \right) \, dx \]

\[ + \frac{R}{\gamma - 1} \frac{d}{dt} \int \rho(\theta - \log \theta - 1) \, dx + \int |\nabla u|^2 \, dx \]

\[ \leq C(\hat{\rho})(\| \theta(\cdot, t) \|_{L^\infty} + 1) \int \rho(\theta - \log \theta - 1) \, dx. \quad (3.77) \]

Next, we claim that

\[ \int_0^{\sigma(T)} \| \theta \|_{L^\infty} \, dt \leq C(\hat{\rho}, M). \quad (3.78) \]

Combining this with (3.77), (3.16), and Grönwall inequality implies (3.73) directly.
Finally, it remains to prove (3.78). Taking $m = 1$ in (3.63) and integrating the resulting inequality, one deduces from (3.65), (3.67), (3.69), (3.5), (3.72), (2.13), and (3.41) that

$$\sigma \varphi + \int_0^t \sigma \int \left( \frac{C_1}{2} |\nabla \dot{u}|^2 + \rho |\dot{\vartheta}|^2 \right) \, dx \, d\tau$$

$$\leq 2\sigma \left| \int_{\partial \Omega} (u \cdot \nabla v \cdot u) G ds \right| (t) + C(\hat{\rho}, M) \int_0^t (|\rho^{1/2} \dot{u}|_{L^2}^2 + |\nabla \vartheta|_{L^2}^2 + |\nabla \theta|_{L^2}^2) \, d\tau$$

$$+ C(\hat{\rho}) \int_0^t \sigma \|\rho - 1\|_{L^2}^2 \, d\tau + \sigma \int_0^t \left( |\nabla u|^2_{L^2} + |\nabla \theta|^2_{L^2} \right) \, d\tau$$

$$\leq C(\hat{\rho}) (|\nabla u|^2_{L^2} |\rho^{1/2} \dot{u}|_{L^2}) (t) + C(\hat{\rho}, M)$$

$$+ C(\hat{\rho}, M) \int_0^t \left( |\nabla u|^2_{L^2} + |\nabla \theta|^2_{L^2} \right) \sigma \varphi \, d\tau$$

$$\leq C(\hat{\rho}, M) + C(\hat{\rho}, M) \int_0^t \left( |\nabla u|^2_{L^2} + |\nabla \theta|^2_{L^2} \right) \sigma \varphi \, d\tau.$$

Then Grönwall inequality together with (3.56) and (3.65) yields

$$\sup_{0 \leq t \leq T} \sigma \left( \int |\dot{u}|^2 \, dx + |\nabla \theta|^2_{L^2} \right) + \int_0^T \sigma \int \left( |\nabla \dot{u}|^2 + \rho |\dot{\vartheta}|^2 \right) \, dx \, d\tau \leq C(\hat{\rho}, M). \quad (3.79)$$

Next, it follows from (3.56), (3.69), (3.67), (3.70), (3.5), and (3.72) that

$$\int_0^T \sigma |\nabla^2 \theta|^2_{L^2} \, dt \leq C(\hat{\rho}, M) \int_0^T \left( |\rho^{1/2} \dot{\vartheta}|_{L^2}^2 + |\rho^{1/2} \dot{u}|_{L^2}^2 \right) \, dt$$

$$+ C(\hat{\rho}, M) \int_0^T \left( |\nabla u|^2_{L^2} + |\nabla \theta|^2_{L^2} + \sigma |\rho - 1|^2_{L^2} \right) \, dt \quad (3.80)$$

Furthermore, one deduces from (2.8), (2.7), and (3.22) that

$$|R \theta - \overline{P}|_{L^\infty} \leq C |R \theta - \overline{P}|_{L^6}^{1/2} |\nabla \theta|_{L^6}^{1/2} + |R \theta - \overline{P}|_{L^2}$$

$$\leq C(\hat{\rho}) |\nabla \theta|_{L^2}^{1/2} |\nabla^2 \theta|_{L^2}^{1/2} + C(\hat{\rho}) |\nabla \theta|_{L^2}$$

which together with (3.5) and (3.80) gives that

$$\int_0^{\sigma(T)} |R \theta - \overline{P}|_{L^\infty} \, dt$$

$$\leq C(\hat{\rho}) \int_0^{\sigma(T)} |\nabla \theta|_{L^2}^{1/2} \left( \sigma |\nabla^2 \theta|_{L^2}^2 \right)^{1/4} \sigma^{-1/4} \, dt + C(\hat{\rho}) \left( \int_0^{\sigma(T)} |\nabla \theta|_{L^2}^2 \, dt \right)^{1/2} \quad (3.82)$$

$$\leq C(\hat{\rho}) \left( \int_0^{\sigma(T)} |\nabla \theta|_{L^2}^2 \, dt \right)^{1/4} + C(\hat{\rho}) C_0^{1/8}$$

$$\leq C(\hat{\rho}, M) C_0^{1/16}.$$

Combining this with (3.11) yields (3.78) directly.
Step 2: The proof of (3.74).

Direct calculations together with (3.34) lead to

\[ \theta - \log \theta - 1 \geq \frac{1}{8}(\theta - 1)1(\theta(\cdot,t) > 2) + \frac{1}{12}(\theta - 1)^21(\theta(\cdot,t) < 3), \]

with \((\theta(\cdot,t) > 2) \triangleq \{ x \in \Omega | \theta(x,t) > 2 \} \) and \((\theta(\cdot,t) < 3) \triangleq \{ x \in \Omega | \theta(x,t) < 3 \} \). Combining this with (3.73) gives

\[
\sup_{0 \leq t \leq \sigma(T)} \int \left( \rho(\theta - 1)1(\theta(\cdot,t) > 2) + \rho(\theta - 1)^21(\theta(\cdot,t) < 3) \right) dx \leq C(\hat{\rho}, M)C_0. \quad (3.83)
\]

Next, it follows from (3.83), (3.73), and the Sobolev inequality that for \( t \in (0, \sigma(T)) \),

\[
\| \theta - 1 \|_{L^2(\theta(\cdot,t) < 3)}^2 \\
\leq \int \rho(\theta - 1)^21(\theta(\cdot,t) < 3)dx + \int (\rho - 1)(\theta - 1)^2dx \\
\leq C(\hat{\rho}, M)C_0 + C\| \rho - 1 \|_{L^2}\| \theta - 1 \|_{L^6}^2\| \theta - 1 \|_{L^3}^3 \\
\leq C(\hat{\rho}, M)C_0 + C(\hat{\rho}, M)C_0^{1/2}\| \theta - 1 \|_{L^2}^{1/2}\| \theta - 1 \|_{L^2} + \| \theta \|_{L^2}^{3/2} \\
\leq C(\hat{\rho}, M) \left( C_0 + C(\delta)C_0^{2/3}\| \nabla \theta \|_{L^2}^2 + (\delta + C_0^{1/2})\| \theta - 1 \|_{L^2}^2 \right),
\]

and

\[
\| \theta - 1 \|_{L^2(\theta(\cdot,t) > 2)}^2 \\
\leq \| \theta - 1 \|_{L^1(\theta(\cdot,t) > 2)}^{4/5}\| \theta - 1 \|_{L^6}^{6/5} \\
\leq C(\hat{\rho}, M) \left( C_0 + C_0^{1/2}\| \theta - 1 \|_{L^2} \right)^{4/5}\left( \| \theta - 1 \|_{L^2} + \| \nabla \theta \|_{L^2} \right)^{6/5} \\
\leq C(\hat{\rho}, M) \left( C_0 + C(\delta)C_0^{2/3}\| \nabla \theta \|_{L^2}^2 + (\delta + C_0^{2/5})\| \theta - 1 \|_{L^2}^2 \right),
\]

where in the second inequality one has used

\[
\| \theta - 1 \|_{L^1(\theta(\cdot,t) > 2)} \leq \int \rho(\theta - 1)1(\theta(\cdot,t) > 2)dx + \int (\rho - 1)(\theta - 1)dx \\
\leq C(\hat{\rho}, M)(C_0 + C_0^{1/2}\| \theta - 1 \|_{L^2}).
\]

Hence, adding (3.84) with (3.85) together and choosing \( \delta \) small enough in the resulting inequality, one has for any \( t \in (0, \sigma(T)) \),

\[
\| \theta - 1 \|_{L^2} \leq C(\hat{\rho}, M) \left( C_0 + C_0^{2/3}\| \nabla \theta \|_{L^2}^2 + C_0^{2/5}\| \theta - 1 \|_{L^2}^2 \right),
\]

which implies that

\[
\| \theta - 1 \|_{L^2}^2 \leq C(\hat{\rho}, M) \left( C_0 + C_0^{2/3}\| \nabla \theta \|_{L^2}^2 \right), \quad (3.86)
\]

provided

\[
C_0 \leq \varepsilon_{3,1} \triangleq \min \left\{ 1, (2C(\hat{\rho}, M))^{-5/2} \right\}. \quad (3.87)
\]
Finally, note that
\[
\| R\theta - \overline{P} \|_{L^2} \leq R\| \theta - 1 \|_{L^2} + C|1 - \overline{\rho}\|
\]
\[
\leq R\| \theta - 1 \|_{L^2} + C \left| \int \rho(1 - \theta) dx \right|
\]
\[
\leq C(\overline{\rho})\| \theta - 1 \|_{L^2},
\]
this together with (3.86) yields (3.74).

The proof of Lemma 3.5 is completed. \(\square\)

Next, with the help of (3.74), the estimate on 
\(A_2(T)\) will be handled smoothly.

**Lemma 3.6.** Under the conditions of Proposition 3.1, there exists a positive constant \(\varepsilon_3\) depending only on \(\mu, \lambda, \kappa, R, \gamma, \overline{\rho}, \overline{\theta}, \Omega, \) and \(M\) such that if \((\rho, u, \theta)\) is a smooth solution to the problem (1.4)–(1.6) on \(\Omega \times (0, T)\) satisfying (3.5) with \(K\) as in Lemma 3.2, the following estimate holds:
\[
A_2(T) \leq C_0^{1/4},
\] (3.88)
provided \(C_0 \leq \varepsilon_3\).

**Proof.** To begin with, multiplying (3.13) by \(u\) and integrating by parts give that
\[
\frac{d}{dt} \left( \frac{1}{2} \rho |u|^2 + \overline{P}(1 + \rho \log \rho - \rho) \right) dx
\]
\[
+ \int (\mu |\nabla u|^2 + (2\mu + \lambda)(\text{div} u)^2) dx
\]
\[
= \overline{P_t} \int (1 + \rho \log \rho - \rho) dx + \int \rho(R\theta - \overline{P}) \text{div} u dx.
\] (3.89)

Next, multiplying (1.4)3 by \(\overline{P}^{-1} (R\theta - \overline{P})\), one obtains after integrating the resulting equality over \(\Omega\) by parts that
\[
\frac{1}{2(\gamma - 1)} \frac{d}{dt} \int \overline{P}^{-1} \rho(R\theta - \overline{P})^2 dx + \kappa R \overline{P}^{-1} \| \nabla \theta \|_{L^2}^2
\]
\[
= -\frac{1}{\gamma - 1} \overline{P_t} \int \rho(R\theta - \overline{P}) dx - \frac{1}{2(\gamma - 1)} \overline{P_t} \int \rho(R\theta - \overline{P})^2 dx
\]
\[
- \overline{P}^{-1} \int \rho(R\theta - \overline{P})^2 \text{div} u dx - \int \rho(R\theta - \overline{P}) \text{div} u dx
\]
\[
+ \overline{P}^{-1} \int (R\theta - \overline{P})(\lambda(\text{div} u)^2 + 2\mu |\mathbf{D}(u)|^2) dx.
\] (3.90)
Adding (3.89) and (3.90) together yields that
\[
\frac{d}{dt} \int \left( \frac{1}{2} \rho |u|^2 + \overline{P} (1 + \rho \log \rho - \rho) + \frac{1}{2(\gamma - 1)} \rho \overline{P}^{-1} (R\theta - \overline{P})^2 \right) \, dx \\
+ \mu \|\text{curl}u\|_{L^2}^2 + (2\mu + \lambda) \|\text{div}u\|_{L^2}^2 + \kappa R \overline{P}^{-1} \|\nabla \theta\|_{L^2}^2 \\
= -\frac{1}{\gamma - 1} \overline{P}^{-1} \overline{P} t \int \rho (R\theta - \overline{P}) \, dx - \frac{1}{2(\gamma - 1)} \overline{P}^{-2} \overline{P} t \int \rho (R\theta - \overline{P})^2 \, dx \\
+ \overline{P} t \int (1 + \rho \log \rho - \rho) \, dx - \overline{P}^{-1} \int \rho (R\theta - \overline{P})^2 \text{div}u \, dx \\
+ \overline{P}^{-1} \int (R\theta - \overline{P}) (\lambda (\text{div}u)^2 + 2\mu |\mathcal{D}(u)|^2) \, dx \triangleq \sum_{i=1}^{5} J_i.
\]

The terms $J_i$ ($i = 1, \cdots, 5$) can be estimated as follows.

It follows from (3.11), (3.29), (3.5), and (3.22) that
\[
J_1 + J_2 \leq C |\overline{P} t| \left( \|\rho (R\theta - \overline{P})\|_{L^2} + \|\rho^{1/2} (R\theta - \overline{P})\|_{L^2}^2 \right) \\
\leq C (\hat{\rho}) C_0^{1/8} \|\nabla u\|_{L^2} + \|\nabla u\|_{L^2}^2 \|\rho^{1/2} (R\theta - \overline{P})\|_{L^2} \\
\leq C (\hat{\rho}) C_0^{1/8} (\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2),
\]
and
\[
J_4 \leq C \|\rho^{1/2} (R\theta - \overline{P})\|_{L^2}^{1/2} \|\rho^{1/2} (R\theta - \overline{P})\|_{L^2}^{3/2} \|\nabla u\|_{L^2} \\
\leq C (\hat{\rho}) A_2^{1/4} (T) \|\nabla \theta\|_{L^2}^{3/2} \|\nabla u\|_{L^2} \\
\leq C (\hat{\rho}) M C_0^{1/10} (\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2).
\]

Furthermore, by virtue of (3.16), (3.29), and (3.10), we have
\[
J_3 \leq |\overline{P} t| \left( \int (1 + \rho \log \rho - \rho) \, dx \right) \\
\leq C (\hat{\rho}) C_0^{1/8} \|\nabla u\|_{L^2} + \|\nabla u\|_{L^2}^2 \|\rho - 1\|_{L^2}^2 \\
\leq C (\hat{\rho}) C_0^{1/4} \|\nabla u\|_{L^2}^2 + C (\hat{\rho}) C_0^{1/4} \|\rho - 1\|_{L^2}^2.
\]

Now, we will estimate the term $J_5$ for the short time $t \in [0, \sigma(T))$ and the large time $t \in [\sigma(T), T]$, respectively.

For $t \in [0, \sigma(T))$, it follows from (3.11), (3.24), (3.22), (5.24), and (3.5) that
\[
J_5 \leq C \int |R\theta - \overline{P}| \|\nabla u\|_{L^2}^2 \, dx \\
\leq C \|R\theta - \overline{P}\|_{L^2}^{1/2} \|R\theta - \overline{P}\|_{L^2}^{1/2} \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla \theta\|_{L^2} \|\rho^{1/2} \hat{u}\|_{L^2} \\
\leq C (\hat{\rho}) \|R\theta - \overline{P}\|_{L^2}^{1/2} \|\nabla \theta\|_{L^2}^{1/2} \|\nabla u\|_{L^2} \|\rho^{1/2} \hat{u}\|_{L^2} \\
\leq C (\hat{\rho}) \|R\theta - \overline{P}\|_{L^2} \|\nabla \theta\|_{L^2} \|\nabla u\|_{L^2} \|\rho^{1/2} \hat{u}\|_{L^2} \\
+ C (\hat{\rho}) M C_0^{1/24} (\|\nabla \theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) \\
\leq C (\hat{\rho}) M C_0^{7/24} \|\rho^{1/2} \hat{u}\|_{L^2}^2 + C (\hat{\rho}) M C_0^{1/24} (\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2).
\]
where we have used following calculations:
\[
\| \mathbf{R} \mathbf{\theta} - \mathbf{F} \|^1/2 \| \nabla \mathbf{\theta} \|^1/2 \| \nabla u \|_{L^2} \| \rho \|^1/2 \| \hat{u} \|_{L^2} \\
\leq C(\hat{\rho}, M)(C_0^{1/4} \| \nabla \mathbf{\theta} \|^1/2 + C_0^{1/6} \| \nabla \mathbf{\theta} \|_{L^2}) \| \nabla u \|_{L^2} \| \rho \|^1/2 \| \hat{u} \|_{L^2} \\
\leq C(\hat{\rho}, M)C_0^{7/24} \| \rho \|^1/2 \| \hat{u} \|_{L^2}^2 + C(\hat{\rho}, M)C_0^{1/12} \| \nabla u \|_{L^2}^2 + C(\hat{\rho}, M)C_0^{1/24} \| \nabla \mathbf{\theta} \|^2_{L^2}
\]
owing to (3.74).

For \( t \in [\sigma(T), T] \), it holds that
\[
J_5 \leq C \| \mathbf{R} \mathbf{\theta} - \mathbf{F} \|_{L^4} \| \nabla u \|_{L^6} \| \nabla \mathbf{\theta} \|_{L^6} \leq C(\hat{\rho})C_0^{1/24} (\| \nabla u \|_{L^2}^2 + \| \nabla \mathbf{\theta} \|^2_{L^2}),
\]
where one has used (3.22) and the following fact:
\[
\sup_{0 \leq t \leq T} \| \nabla u \|_{L^6} \leq C(\hat{\rho})C_0^{1/24}
\]
due to (3.5) and (3.24).

Finally, substituting (3.92)–(3.96) into (3.91), one obtains after using (3.5), (3.11), and (3.33) that
\[
\sup_{0 \leq t \leq T} \int_0^T \left( \frac{1}{2} \rho |u|^2 + \mathbf{F}(1 + \rho \log \rho - \rho) + \frac{1}{2(\gamma - 1)} \rho \mathbf{F}^{-1} (R \mathbf{\theta} - \mathbf{F})^2 \right) dt \\
+ \int_0^T (\mu \| \nabla u \|_{L^2}^2 + (2\mu + \lambda) \| \nabla u \|_{L^2}) dt + R \kappa \mathbf{F}^{-1} \| \nabla \mathbf{\theta} \|_{L^2}^2 dt \\
\leq C(\hat{\rho}, M)C_0^{1/24} \int_0^T (\| \nabla u \|_{L^2}^2 + \| \nabla \mathbf{\theta} \|_{L^2}^2) dt + C(\hat{\rho})C_0^{1/4} \int_0^T \| \rho - 1 \|_{L^2}^2 dt \\
+ C(\hat{\rho}, M)C_0^{7/24} \int_0^{\sigma(T)} \| \rho \|_{L^2}^2 + C(\hat{\rho}, \hat{\theta})C_0 \\
\leq C(\hat{\rho}, \hat{\theta}, M)C_0^{7/24},
\]
where one has used
\[
\int_0^T \| \rho - 1 \|_{L^2}^2 dt \leq \sup_{0 \leq t \leq \sigma(T)} \| \rho - 1 \|_{L^2}^2 + \int_0^{\sigma(T)} \| \rho - 1 \|_{L^2}^2 dt \leq C(\hat{\rho}, M)C_0^{1/4}
\]
due to (3.73) and (3.72). Thus, one deduces from (3.97), (2.12), and (3.11) that
\[
A_2(T) \leq C(\hat{\rho}, \hat{\theta}, M)C_0^{7/24}
\]
which implies (3.88) provided
\[
C_0 \leq \varepsilon_3 \triangleq \min \left\{ \varepsilon_{3,1}, (C(\hat{\rho}, \hat{\theta}, M))^{-24} \right\},
\]
with \( \varepsilon_{3,1} \) as in (3.87). The proof of Lemma 3.6 is completed. \( \square \)

**Remark 3.2.** It’s worth noticing that the energy-like estimate \( A_2(T) \) is a little subtle, since \( A_2(T) \) is not a conserved quantity for the full Navier-Stokes system owing to the nonlinear coupling of \( \theta \) and \( u \). Thus, further consideration is needed to handle this issue. More precisely,
on the one hand, while deriving the kinetic energy (see (3.89)), we need to deal with the following term
\[ \int \rho (R \theta - \overline{P}) \text{div} u \, dx. \]
Unfortunately, this term is troublesome for large time \( t \in [\sigma(T), T] \). In fact, this term could be bounded by \( \| \nabla \theta \|_{L^2} \| \nabla u \|_{L^2} \), which will only be of the same order as \( C_0^{1/4} \) with the help of all a priori estimates (3.5). Therefore, we cannot handle this term directly. Here, based on careful analysis on system (1.4), we find that this term can be cancelled by a suitable combination of kinetic energy and thermal energy, see (3.89)–(3.91);

on the other hand, while deriving the thermal energy (see (3.90)), we need to handle the following term
\[ \int (R \theta - \overline{P}) (\text{div} u)^2 \, dx. \]
Note that for short time \( t \in [0, \sigma(T)) \), the “weaker” basic energy estimate (3.10) is not enough, hence it’s necessary to re-establish the basic energy estimate (3.73), which is obtained by the a priori \( L^1(0, \sigma(T); L^\infty) \)-norm of \( \theta \) (see (3.78)). Consequently, we can obtain (3.74) as a consequence of (3.73) and then handle this term for short time \( t \in [0, \sigma(T)) \) (see (3.95)).

Moreover, it should be mentioned that the uniform positive lower and upper bounds of \( \overline{P} \) also play a critical role in estimating \( A_2(T) \).

We now proceed to derive a uniform (in time) upper bound for the density, which turns out to be the key to obtaining all the higher order estimates and thus extending the classical solution globally.

**Lemma 3.7.** Under the conditions of Proposition 3.1, there exists a positive constant \( \varepsilon_4 \) depending only on \( \mu, \lambda, \kappa, R, \gamma, \hat{\rho}, \hat{\theta}, \Omega, \) and \( M \) such that if \((\rho, u, \theta)\) is a smooth solution to the problem (1.4)–(1.6) on \( \Omega \times (0, T] \) satisfying (3.5) with \( K \) as in Lemma 3.2, the following estimate holds:
\[
\sup_{0 \leq t \leq T} \| \rho(\cdot, t) \|_{L^\infty} \leq \frac{3\hat{\rho}}{2},
\]
provided \( C_0 \leq \varepsilon_4 \).

**Proof.** First, it follows from (3.80), (3.81), and (3.5) that
\[
\int_{\sigma(T)}^T \| R \theta - \overline{P} \|_{L^\infty} \, dt \leq C(\hat{\rho}) \left( \int_{\sigma(T)}^T \| \nabla \theta \|_{L^2}^2 \, dt \right)^{1/2} \left( \int_{\sigma(T)}^T \| \nabla^2 \theta \|_{L^2}^2 \, dt \right)^{1/2} + C(\hat{\rho}) \int_{\sigma(T)}^T \| \nabla \theta \|_{L^2}^2 \, dt
\]
\[
\leq C(\hat{\rho}, M)C_0^{1/8}.
\]
Next, it follows from (2.13), (2.19), (3.79), and (3.5) that
\[
\int_0^{\sigma(T)} \|G\|_{L^\infty} dt \\
\leq C \int_0^{\sigma(T)} \|\nabla G\|_{L^2}^{1/2} \|\nabla G\|_{L^\infty}^{1/2} dt \\
\leq C(\hat{\rho}) \int_0^{\sigma(T)} \|\hat{\rho} u\|_{L^2}^{1/2} (\|\nabla \hat{u}\|_{L^2} + \|\nabla u\|_{L^2}^2)^{1/2} dt \\
\leq C(\hat{\rho}) \int_0^{\sigma(T)} (\sigma\|\hat{\rho} u\|_{L^2})^{1/4} (\sigma\|\hat{\rho} u\|_{L^2}^2)^{1/8} (\sigma\|\nabla \hat{u}\|_{L^2})^{1/4} \sigma^{-5/8} dt \\
+ C(\hat{\rho}) \int_0^{\sigma(T)} (\sigma\|\hat{\rho} u\|_{L^2})^{1/2} \|\nabla u\|_{L^2} \sigma^{-1/2} dt \\
\leq C(\hat{\rho}, M) C_0^{1/48} \left( \int_0^{\sigma(T)} \sigma\|\nabla \hat{u}\|_{L^2}^2 dt \right)^{1/4} \left( \int_0^{\sigma(T)} \sigma^{-5/8} dt \right)^{3/4} \\
+ C(\hat{\rho}, M) C_0^{1/24} \int_0^{\sigma(T)} \sigma^{-1/2} dt,
\]
and
\[
\int_{\sigma(T)}^T \|G\|_{L^\infty}^2 dt \\
\leq C \int_{\sigma(T)}^T \|\nabla G\|_{L^2} \|\nabla G\|_{L^\infty} dt \\
\leq C(\hat{\rho}, M) \int_{\sigma(T)}^T \left( \|\rho^{1/2} \hat{u}\|_{L^2}^2 + \|\nabla \hat{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right) dt \\
\leq C(\hat{\rho}, M) C_0^{1/6}.
\]

Denoting \( D_t \rho = \rho_t + u \cdot \nabla \rho \) and using (1.39), one can rewrite (1.41) as follows
\[
(2\mu + \lambda) D_t \rho = -\tau_\rho (\rho - 1) - \rho^2 (R\theta - \tau_\rho) - \rho G \\
\leq -\tau_\rho (\rho - 1) + C(\hat{\rho}) \|R\theta - \tau_\rho\|_{L^\infty} + C(\hat{\rho}) \|G\|_{L^\infty},
\]
which gives
\[
D_t (\rho - 1) + \frac{\tau_\rho}{2\mu + \lambda} (\rho - 1) \leq C(\hat{\rho}) \|R\theta - \tau_\rho\|_{L^\infty} + C(\hat{\rho}) \|G\|_{L^\infty}.
\]

Finally, applying Lemma [2.8] with
\[
y = \rho - 1, \quad \alpha = \frac{\tau_\rho}{2\mu + \lambda}, \quad g = C(\hat{\rho}) \|R\theta - \tau_\rho\|_{L^\infty} + C(\hat{\rho}) \|G\|_{L^\infty}, \quad T_1 = \sigma(T),
\]
we thus deduce from (3.101), (3.82), (3.101)–(3.103), (2.23), and (3.11) that
\[
\rho \leq \hat{\rho} + 1 + C \left( \|g\|_{L^1(0,\sigma(T))} + \|g\|_{L^2(\sigma(T),T)} \right) \leq \hat{\rho} + 1 + C(\hat{\rho}, M) C_0^{1/48},
\]
which gives (3.100) provided
\[
C_0 \leq \varepsilon_4 \triangleq \min \left\{ 1, \left( \frac{\hat{\rho} - 2}{2C(\hat{\rho}, M)} \right)^{48} \right\}.
\]
The proof of Lemma [3.7] is completed. \( \square \)
Next, we summarize some uniform estimates on \((\rho, u, \theta)\) which will be useful for higher-order ones in the next section.

**Lemma 3.8.** Under the conditions of Proposition 3.7, there exists a positive constant \(C\) depending only on \(\mu, \lambda, \kappa, R, \gamma, \hat{\rho}, \hat{\theta}, \Omega,\) and \(M\) such that if \((\rho, u, \theta)\) is a smooth solution to the problem (1.2)–(1.6) on \(\Omega \times (0, T)\) satisfying (3.3) with \(K\) as in Lemma 3.2 the following estimate holds:

\[
\sup_{0 < t \leq T} \sigma^2 \int \rho |\dot{\theta}|^2 dx + \int_0^T \sigma^2 \|\nabla \dot{\theta}\|_{L^2}^2 dt \leq C. \tag{3.105}
\]

Moreover, it holds that

\[
\sup_{0 < t \leq T} \left( \sigma \|\nabla u\|_{L^6}^2 + \sigma^2 \|\theta\|_{H^2}^2 \right)
+ \int_0^T \left( \sigma \|\nabla u\|_{L^4}^4 + \sigma \|\nabla \theta\|_{H^1}^2 + \sigma \|u_t\|_{L^2}^2 + \sigma^2 \|\theta_t\|_{H^1}^2 + \|\rho - 1\|_{L^2}^2 \right) dt \leq C. \tag{3.106}
\]

**Proof.** First, applying the operator \(\partial_t + \text{div}(u \cdot )\) to (1.4) and using (1.4), one gets

\[
\frac{R}{2(\gamma - 1)} \left( \int \rho |\dot{\theta}|^2 dx \right)_t + \kappa \|\nabla \dot{\theta}\|_{L^2}^2 \leq C \int |\nabla \dot{\theta}| (|\nabla^2 \theta| |u| + |\nabla \theta| |\nabla u|) dx + C \int \rho |\dot{\theta} - P||\nabla \dot{u}||\dot{\theta}| dx
+ C(\hat{\rho}) \left( \int \nabla u^2 |\dot{\theta}| \left( |\nabla u| + |\dot{\theta} - P| \right) dx + C \int \nabla \dot{u} |\dot{\theta}| dx
+ C(\hat{\rho}) \left( \int (|\nabla u|^2 |\dot{\theta}| + \rho |\dot{\theta}|^2 |\nabla u| + |\nabla u| |\nabla \dot{u}||\dot{\theta}| dx
\right) \leq C \|\nabla u\|_{L^6}^{1/2} \|\nabla u\|_{L^6}^{1/2} \|\nabla^2 \theta\|_{L^2} \|\nabla \dot{u}\|_{L^2} + C(\hat{\rho}) \|\nabla \theta\|_{L^6} \|\nabla \dot{u}\|_{L^2} \|\dot{\theta}\|_{L^6} \tag{3.109}
+ C(\hat{\rho}) \|\nabla u\|_{L^6} \|\nabla u\|_{L^6} \|\nabla \theta\|_{L^6} \|\dot{\theta}\|_{L^6} + C(\hat{\rho}) \|\nabla \dot{u}\|_{L^2} \|\dot{\theta}\|_{L^6} \|\rho \|_{L^6} \|\rho\|_{L^6} \|\nabla \dot{u}\|_{L^2} \right) 
\]

\[
\leq \frac{\kappa}{2} \|\nabla \dot{\theta}\|_{L^2}^2 + C(\hat{\rho}) \|\nabla u\|_{L^6} \|\nabla \theta\|_{L^2} \left( \|\nabla u\|_{L^6} + \|\nabla \theta\|_{L^2} \right) + C(\hat{\rho}, M) \|\nabla u\|_{L^6} \|\nabla u\|_{L^2}^2 \tag{3.109}
+ C(\hat{\rho}, M) \left( 1 + \|\nabla u\|_{L^6} + \|\nabla \theta\|_{L^2} \right) \left( \|\nabla^2 \theta\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2}^2 + \|\rho \|_{L^2}^2 \right),
\]
where we have used (3.108), (2.7), (2.8), (3.5), (3.105), and the following Poincaré-type inequality (9, Lemma 3.2):

\[ \|f\|_{L^p} \leq C(\hat{\rho})(\|\rho^{1/2} f\|_{L^2} + \|\nabla f\|_{L^2}), \quad p \in [2, 6], \]  

(3.110)

for any \( f \in \{ h \in H^1 \mid \rho^{1/2} h \in L^2 \} \).

Multiplying (3.109) by \( \sigma^2 \) and integrating the resulting inequality over \((0, T)\), we obtain after integrating by parts that

\[
\sup_{0 \leq t \leq T} \sigma^2 \int_0^T \rho |\dot{\theta}|^2 dt + \int_0^T \sigma^2 \|\nabla \dot{\theta}\|_{L^2}^2 dt \\
\leq C(\hat{\rho}) \sup_{0 \leq t \leq T} \left( \sigma^2 \left( \|\nabla u\|_{L^6}^4 + \|\nabla \theta\|_{L^2}^4 \right) \right) \int_0^T \|\nabla u\|_{L^2}^2 dt \\
+ C(\hat{\rho}, M) \sup_{0 \leq t \leq T} \left( \sigma \left( 1 + \|\nabla u\|_{L^6} + \|\nabla \theta\|_{L^2} \right) \right) \\
\cdot \int_0^T \sigma \left( \|\nabla^2 \theta\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2}^2 + \|\rho^{1/2} \dot{\theta}\|_{L^2}^2 \right) dt \\
+ C(\hat{\rho}, M) \sup_{0 \leq t \leq T} \left( \sigma \|\nabla u\|_{L^6}^2 \right) \int_0^T \|\nabla u\|_{L^2}^2 dt + C \int_0^T \sigma \|\rho^{1/2} \dot{\theta}\|_{L^2}^2 dt \\
\leq C(\hat{\rho}, M),
\]

where we have used (3.5), (3.79), (3.80), and the following fact:

\[
\sup_{0 \leq t \leq T} (\sigma \|\nabla u\|_{L^6}^2) \leq C(\hat{\rho}, M)
\]

(3.111)

due to (3.24), (3.79), and (3.5).

Next, it follows from (3.5), (3.79), (3.80), (3.105), (3.98), (3.86), and (3.80) that

\[
\sup_{0 \leq t \leq T} (\sigma^2 \|\theta\|_{H^2}^2) + \int_0^T \left( \sigma \|\nabla u\|_{L^4}^4 + \sigma \|\nabla \theta\|_{H^1}^2 + \|\rho - 1\|_{L^2}^2 \right) dt \leq C(\hat{\rho}, M),
\]

(3.112)

which along with (3.5), (3.79), (3.80), (3.110), (3.111), and (3.105) gives

\[
\int_0^T \|u\|_{L^2}^2 dt \leq C \int_0^T \sigma (\|\dot{u}\|_{L^2}^2 + |u \cdot \nabla u|_{L^2}^2) dt \\
\leq C(\hat{\rho}) \int_0^T \sigma (\|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|\nabla \dot{u}\|_{L^2}^2 + \|u\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2) dt \\
\leq C(\hat{\rho}, M),
\]

(3.113)

\[
\int_0^T \sigma^2 \|\theta_t\|_{L^2}^2 dt \leq C \int_0^T \sigma^2 (\|\dot{\theta}\|_{L^2}^2 + |u \cdot \nabla \theta|_{L^2}^2) dt \\
\leq C(\hat{\rho}) \int_0^T \sigma^2 (\|\rho^{1/2} \dot{\theta}\|_{L^2}^2 + \|\nabla \dot{\theta}\|_{L^2}^2 + \|u\|_{L^6}^2 \|\nabla \theta\|_{L^3}^2) dt \\
\leq C(\hat{\rho}, M),
\]

(3.114)
and
\[\int_0^T \sigma^2 \|
abla \theta_t\|^2_{L^2} dt \leq C \int_0^T \sigma^2 \|
abla \theta\|^2_{L^2} dt + C \int_0^T \sigma^2 \|\nabla (u \cdot \nabla \theta)\|^2_{L^2} dt\]
\[\leq C(\hat{\rho}, M) + C \int_0^T \sigma^2 (\|\nabla u\|^2_{L^2} + \|u\|_{L^\infty}^2) \|\nabla^2 \theta\|^2_{L^2} dt\]
\[\leq C(\hat{\rho}, M).\] (3.115)

Hence, (3.106) is derived from (3.111)–(3.115) immediately. The proof of Lemma 3.8 is finished. \qed

Finally, we end this section by establishing the exponential decay-in-time for the classical solutions.

**Lemma 3.9.** Under the conditions of Proposition 3.7, there exist positive constants \(\varepsilon_0, C^*, \alpha, \) and \(\theta_\infty\) depending only on \(\mu, \lambda, \kappa, R, \gamma, \hat{\rho}, \hat{\theta}, \Omega, \) and \(M\) such that if \((\rho, u, \theta)\) is a smooth solution to the problem (1.4)–(1.6) on \(\Omega \times (0, T)\) satisfying (3.5) with \(K\) as in Lemma 3.2 (7.4) holds for any \(t \geq 1\), provided \(C_0 \leq \varepsilon_0\).

**Proof.** First, it follows from (3.91), (3.5), (3.11), (3.16), (3.29), (3.22), (3.8), and (3.106) that for any \(t \geq 1\),
\[\frac{1}{2} W'(t) + \mu \|\nabla \theta\|^2_{L^2} + (2\mu + \lambda) \|\nabla u\|^2_{L^2} + \kappa R \bar{\rho}^{-1} \|\nabla \theta\|^2_{L^2} \leq C \bar{\theta}_1(\|R \theta - \bar{\rho}\|^2_{L^2} + \|R \theta - \bar{\rho}\|^2_{L^2}) + |\bar{\theta}_1| \|\rho - 1\|^2_{L^2} \]
\[\leq C \bigl( C_0^{1/8} \|\nabla u\|^2_{L^2} + \|\nabla \theta\|^2_{L^2} (\|\nabla \theta\|^2_{L^2} + \|\nabla \theta\|^2_{L^2} + \|\rho - 1\|^2_{L^2})\bigr) + C \|\nabla \theta\|^2_{L^2} \|\nabla^2 \theta\|^2_{L^2} (\|\nabla \theta\|^2_{L^2} + \|\nabla u\|^2_{L^2}) \]
\[\leq CC_0^{1/24} (\|\nabla u\|^2_{L^2} + \|\nabla \theta\|^2_{L^2} + \|\rho - 1\|^2_{L^2}),\] (3.116)

where
\[W(t) \triangleq \int (\rho |u|^2 + 2R \rho \log \rho - \rho) + \frac{1}{\gamma - 1} \rho \bar{\rho}^{-1} (R \theta - \bar{\rho})^2 dx\] (3.117)

owing to (3.3), (3.11), (3.16), and (3.22). Combining (3.116) with (2.12) and (3.11) yields that
\[W'(t) + \hat{\theta}_3 (\|\nabla u\|^2_{L^2} + \|\nabla \theta\|^2_{L^2}) \leq \hat{\theta}_2 C_0^{1/24} (\|\nabla u\|^2_{L^2} + \|\nabla \theta\|^2_{L^2} + \|\rho - 1\|^2_{L^2}).\] (3.118)

Next, rewriting (1.4) as
\[(\rho u)_t + \text{div}(\rho u \otimes u) = \mu \Delta u + (\mu + \lambda) \nabla (\text{div} u) - \nabla (\rho (R \theta - \bar{\rho})) - \bar{\rho} \nabla (\rho - 1),\]

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By virtue of (3.11), (3.16), and Lemma 2.9, it holds provided

\[
\begin{align*}
\|\rho - 1\|_{L^2}^2 & \leq \frac{2}{\pi_1} \left( \int \rho u \cdot \mathcal{B}[\rho - 1] \, dx \right)_t + \hat{C}_4 (\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2). \quad (3.119)
\end{align*}
\]

By virtue of (3.11), (3.16), and Lemma 2.9 it holds

\[
\left| \int \rho u \cdot \mathcal{B}[\rho - 1] \, dx \right| \leq C (\|\rho u\|_{L^2}^2 + \|\rho - 1\|_{L^2}^2)
\]

\[
\leq \hat{C}_5 \left( \|\rho^{1/2} u\|_{L^2}^2 + 2\mathcal{P}(1 + \rho \log \rho - \rho) \right). \quad (3.120)
\]

Adding (3.118) to (3.119) multiplied by \(\hat{\mathcal{C}}_6\) with \(\hat{\mathcal{C}}_6 = \min\{\frac{\pi_1}{4\hat{C}_5}, \frac{\hat{C}_1}{4\hat{C}_4}\}\) yields

\[
W_1'(t) + \frac{3\hat{\mathcal{C}}_1}{4} (\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) + \hat{\mathcal{C}}_6 \|\rho - 1\|_{L^2}^2
\]

\[
\leq \hat{C}_2 C_0^{1/24} (\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|\rho - 1\|_{L^2}^2), \quad (3.121)
\]

where

\[
W_1(t) \triangleq W(t) - \frac{2\hat{\mathcal{C}}_6}{\pi_1} \int \rho u \cdot \mathcal{B}[\rho - 1] \, dx,
\]

satisfies

\[
\frac{1}{2} W(t) \leq W_1(t) \leq 2W(t) \quad (3.122)
\]

due to (3.120). Thus we infer from (3.121) that

\[
W_1'(t) + \frac{\hat{C}_1}{2} (\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) + \frac{\hat{\mathcal{C}}_6}{2} \|\rho - 1\|_{L^2}^2 \leq 0, \quad (3.123)
\]

provided

\[
C_0 \leq \varepsilon_0 \triangleq \min \left\{ \varepsilon_1, \cdots, \varepsilon_4, \left( \frac{\hat{\mathcal{C}}_6}{2C_2} \right)^{24}, \left( \frac{\hat{C}_1}{4C_2} \right)^{24} \right\}. \quad (3.124)
\]
Then by (3.117), one derives that for \( \alpha = \frac{1}{2} \min \{ \frac{C_1}{2C_3}, \frac{\dot{C}_2}{2C_3} \} \),

\[
W_1'(t) + 3\alpha W_1(t) \leq 0,
\]

which along with (3.122), (3.16), (3.11), (3.117), and (3.5) shows that for any \( t \geq 1 \),

\[
\|\rho^{1/2}u\|_{L^2}^2 + \|\rho - 1\|_{L^2}^2 + \|\rho^{1/2}(R\theta - \mathcal{P})\|_{L^2}^2 \leq CW_1(t) \leq Ce^{-3\alpha t}.
\]  \hspace{1cm} (3.125)

Moreover, we deduce from (3.123) and (3.125) that for any \( t \geq 1 \),

\[
\frac{1}{2} e^{\alpha t} \int \rho|\dot{u}|^2 \, dx \leq Ce^{\alpha t} \left( \|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|P - \mathcal{P}\|_{L^2}^2 \right).
\]  \hspace{1cm} (3.127)

Note that by (3.5), (3.11), and (3.125),

\[
\|P - \mathcal{P}\|_{L^2} \leq \|\rho(R\theta - \mathcal{P})\|_{L^2} + \|\rho - 1\|_{L^2} \leq Ce^{-\alpha t},
\]  \hspace{1cm} (3.128)

which together with (3.126), (3.127), and (2.12) gives for any \( 1 \leq t \leq T < \infty \),

\[
\sup_{1 \leq t \leq T} \left( e^{\alpha t}\|\nabla u\|_{L^2}^2 \right) + \int_1^T e^{\alpha t}\|\rho^{1/2}\dot{u}\|_{L^2}^2 \, dt \leq C.
\]  \hspace{1cm} (3.129)

Furthermore, choosing \( m = 0 \) in (3.69), it follows from (3.67), (3.68), and (3.5) that for any \( t \geq 1 \),

\[
\varphi'(t) + 2 \left( \int_{\partial\Omega} (u \cdot \nabla n \cdot u) \, dG \right) + \int \left( \frac{C_1}{2} |\nabla \dot{u}|^2 + \rho|\ddot{\theta}|^2 \right) \, dx \leq C \left( \|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|\rho - 1\|_{L^2}^2 \right),
\]

where \( \varphi(t) \) is defined in (3.64). Multiplying this by \( e^{\alpha t} \) along with (3.39), (3.64), (3.65), (3.41), (3.125), (3.126), (3.129), (2.13), and (3.5) yields that for any \( 1 \leq t \leq T < \infty \),

\[
\sup_{1 \leq t \leq T} \left( e^{\alpha t}\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) + \int_1^T e^{\alpha t}\|\nabla \dot{u}\|_{L^2}^2 + \|\rho^{1/2}\ddot{\theta}\|_{L^2}^2 \, dt \leq C.
\]  \hspace{1cm} (3.130)

Adopting the analogous method and applying (3.109), (3.106), (3.125), (3.126), (3.129), (3.130), (3.56), (3.67), (3.5), and (3.68), we obtain that

\[
\sup_{1 \leq t \leq T} \left( e^{\alpha t}\|\rho^{1/2}\ddot{\theta}\|_{L^2}^2 + \|\nabla^2 \theta\|_{L^2}^2 \right) + \int_1^T e^{\alpha t}\|\nabla \ddot{\theta}\|_{L^2}^2 \, dt \leq C.
\]  \hspace{1cm} (3.131)

Finally, it remains to determine the limit of \( \theta \) as \( t \) tends to infinity. Combining (3.26), (3.128), and (3.129) shows that for any \( t \geq 1 \),

\[
|\mathcal{P}_t| \leq C\left( \|\nabla u\|_{L^2}^2 + \|P - \mathcal{P}\|_{L^2}^2 \right) \leq Ce^{-\alpha t},
\]
which implies there exists a constant $P_\infty$ such that $\lim_{t \to \infty} P = P_\infty$ and
\[
|P - P_\infty| \leq Ce^{-\alpha t}.
\] (3.132)

Denoting $\theta_\infty \triangleq P_\infty / R$, we have
\[
\|\theta - \theta_\infty\|^2_{L^2} \leq C\|R \theta - P\|^2_{L^2} + C|P - R \theta_{\infty}|^2 \leq Ce^{-\alpha t},
\] (3.133)
where we have used (3.22), (3.130), and (3.132). Therefore, the combination of (3.125), (3.129)–(3.131), (3.133), (2.18), and (3.23) concludes (3.124) and finishes the proof of Lemma 3.9.

4 A priori estimates (II): higher-order estimates

In this section, we will derive the higher-order estimates of smooth solution $(\rho, u, \theta)$ to problem (1.4)–(1.6) on $\Omega \times (0, T]$ with initial data $(\rho_0, u_0, \theta_0)$ satisfying (1.9) and (3.4).

We shall assume that (3.5) and (3.8) both hold as well. To proceed, we define $\tilde{g}$ as
\[
\tilde{g} \triangleq \rho_0^{-1/2} (-\mu \Delta u_0 - (\mu + \lambda)\nabla \text{div} u_0 + R\nabla (\rho_0 \theta_0)).
\] (4.1)

Then it follows from (1.9) and (3.4) that
\[
\tilde{g} \in L^2.
\] (4.2)

From now on, the generic constant $C$ will depend only on $T, \|\tilde{g}\|_{L^2}, \|\rho_0\|_{W^{2,q}}, \|\nabla u_0\|_{H^1}, \|\nabla \theta_0\|_{L^2},$
besides $\mu, \lambda, \kappa, R, \gamma, \hat{\rho}, \hat{\theta}, \Omega,$ and $M.$

We begin with the following estimates on the spatial gradient of the smooth solution $(\rho, u, \theta)$.

Lemma 4.1. The following estimates hold:
\[
\sup_{0 \leq t \leq T} \left( \|\rho^{1/2} \tilde{u}\|^2_{L^2} + \|\rho^{1/2} \tilde{\theta}\|^2_{L^2} + \|\tilde{\theta}\|^2_{H^1} + \sigma\|\nabla^2 \theta\|^2_{L^2} \right)
+ \int_0^T \left( \|\nabla \tilde{u}\|^2_{L^2} + \|\rho^{1/2} \tilde{\theta}\|^2_{L^2} + \|\nabla^2 \theta\|^2_{L^2} + \sigma\|\nabla \tilde{\theta}\|^2_{L^2} \right) dt \leq C,
\] (4.3)

and
\[
\sup_{0 \leq t \leq T} (\|u\|^2_{H^2} + \|\rho\|^2_{H^2}) + \int_0^T \left( \|\nabla u\|^2_{L^\infty} + \sigma\|\nabla^3 \theta\|^2_{L^2} + \|\tilde{u}\|^2_{H^3} \right) dt \leq C.
\] (4.4)

Proof. The proof is divided into the following two steps.
Step 1: The proof of (4.3). First, for \( \varphi(t) \) as in (3.64), taking \( m = 0 \) in (3.63), one gets

\[
\varphi'(t) + \int \left( \frac{C_1}{2} |\nabla \hat{u}|^2 + \rho |\hat{\theta}|^2 \right) dx \\
\leq -2 \left( \int_{\partial \Omega} G(u \cdot \nabla n \cdot u) dS \right)_{t} + C \left( \|\rho^{1/2} \hat{u}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) \\
+ C \left( \|\rho^{1/2} \hat{u}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^3 \right) + \|\nabla u\|_{L^2}^2 + \|\rho - 1\|_{L^2}^2 \right) \\
+ C \left( \|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) \left( \|\rho^{1/2} \hat{u}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + 1 \right)
\]

(4.5)
due to (3.5), (3.65), (3.68), and (3.67). Taking into account the compatibility condition (1.1), we can define

\[
\sqrt{\rho} \hat{u}(x, t = 0) \triangleq \tilde{g},
\]

which along with (3.39), (3.66), and (4.2) yields that

\[
|\varphi(0)| \leq C \|\hat{g}\|_{L^2}^2 + C \leq C.
\]

(4.6)

Then, integrating (4.5) over \((0, t)\), one obtains after using (3.5), (3.44), (2.13), (3.65), (3.68), and (4.6) that

\[
\varphi(t) + \int_{0}^{t} \int \left( \frac{C_1}{2} |\nabla \hat{u}|^2 + \rho |\hat{\theta}|^2 \right) dx ds \\
\leq 2 \left| \int_{\partial \Omega} G(u \cdot \nabla n \cdot u) dS \right| (t) + C \int_{0}^{t} \left( \|\rho^{1/2} \hat{u}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) \varphi ds + C \\
\leq C(\|\nabla u\|_{L^2}^2 \|\rho^{1/2} \hat{u}\|_{L^2}^2)(t) + C \int_{0}^{t} \left( \|\rho^{1/2} \hat{u}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) \varphi ds + C \\
\leq \frac{1}{2} \varphi(t) + C \int_{0}^{t} \left( \|\rho^{1/2} \hat{u}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) \varphi ds + C.
\]

(4.7)

Applying Grönwall’s inequality to (4.7) and using (3.5) and (3.65), it holds

\[
\sup_{0 \leq t \leq T} \left( \|\rho^{1/2} \hat{u}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 \right) + \int_{0}^{T} \int \left( |\nabla \hat{u}|^2 + \rho |\hat{\theta}|^2 \right) dx dt \leq C,
\]

(4.8)

which together with (3.11) and (4.22) implies

\[
\|\theta\|_{L^2} \leq C(\|R \theta - \overline{\theta}\|_{L^2} + \overline{\theta}) \leq C(\|\nabla \theta\|_{L^2} + 1) \leq C.
\]

(4.9)

Next, multiplying (3.109) by \( \sigma \) and integrating over \((0, T)\) lead to

\[
\sup_{0 \leq t \leq T} \sigma \int \rho |\hat{\theta}|^2 dx + \int_{0}^{T} \sigma \|\nabla \hat{\theta}\|_{L^2}^2 dt \\
\leq C \int_{0}^{T} \left( \|\nabla \theta\|_{L^2}^2 + \|\nabla \hat{u}\|_{L^2}^2 + \|\rho^{1/2} \hat{\theta}\|_{L^2}^2 \right) dt + C
\]

(4.10)

\leq C;
where we have used (1.8), (3.5), (3.67), (3.68), (3.56), and the following fact:
\[ \| \nabla u \|_{L^6} \leq C \] (4.11)
due to (3.24), (3.5), and (4.8). Then, it follows from (3.56), (4.8), (4.10), (4.11), (3.67), and (3.5) that
\[ \sup_{0 \leq t \leq T} \sigma \| \nabla^2 \theta \|_{L^2}^2 + \int_0^T \| \nabla^2 \theta \|_{L^2}^2 \, dt \leq C, \]
which along with (4.8)–(4.10) gives (4.13).

**Step 2: The proof of (4.7).** First, standard calculations show that for \( 2 \leq p \leq 6, \)
\[
\partial_t \| \nabla \rho \|_{L^p} \leq C \| \nabla u \|_{L^\infty} \| \nabla \rho \|_{L^p} + C \| \nabla^2 u \|_{L^p}
\leq C \left( 1 + \| \nabla u \|_{L^\infty} + \| \nabla^2 \theta \|_{L^2} \right) \| \nabla \rho \|_{L^p}
+ C \left( 1 + \| \nabla \dot{u} \|_{L^2} + \| \nabla^2 \theta \|_{L^2} \right),
\]
where we have used
\[
\| \nabla^2 u \|_{L^p} \leq C(\| \rho \dot{u} \|_{L^p} + \| \nabla P \|_{L^p} + \| \nabla u \|_{L^2})
\leq C \left( 1 + \| \nabla \dot{u} \|_{L^2} + \| \nabla \theta \|_{L^p} + \| \theta \|_{L^\infty} \| \nabla \rho \|_{L^p} \right)
\leq C \left( 1 + \| \nabla \dot{u} \|_{L^2} + \| \nabla^2 \theta \|_{L^2} + \left( \| \nabla^2 \theta \|_{L^2} + 1 \right) \| \nabla \rho \|_{L^p} \right)
\] (4.13)
due to (2.21), (2.19), (2.22), (3.5), and (4.3). It follows from Lemma 2.10, (3.5), and (4.13) that
\[
\| \nabla u \|_{L^\infty} \leq C \left( \| \text{div} u \|_{L^\infty} + \| \nabla u \|_{L^6} \right) \log(e + \| \nabla^2 u \|_{L^6} ) + C \| \nabla u \|_{L^2} + C \\
\leq C \left( \| \text{div} u \|_{L^\infty} + \| \text{curl} u \|_{L^\infty} \right) \log(e + \| \nabla \dot{u} \|_{L^2} + \| \nabla^2 \theta \|_{L^2})
\leq C \left( \| \text{div} u \|_{L^\infty} + \| \text{curl} u \|_{L^\infty} \right) \log(e + \| \nabla \rho \|_{L^6} ) + C.
\]

Denote
\[
\begin{align*}
f(t) & \triangleq e + \| \nabla \rho \|_{L^6}, \\
h(t) & \triangleq 1 + \| \text{div} u \|_{L^\infty}^2 + \| \text{curl} u \|_{L^\infty}^2 + \| \nabla \dot{u} \|_{L^2}^2 + \| \nabla^2 \theta \|_{L^2}^2.
\end{align*}
\]
One obtains after submitting (4.14) into (4.12) with \( p = 6 \) that
\[
f'(t) \leq Ch(t) f(t) \ln f(t),
\]
which implies
\[
(\ln(\ln f(t)))' \leq Ch(t).
\]
(4.15)

Note that by virtue of (1.39), (3.11), (4.3), (3.5), (3.51), and (3.52), one gets
\[
\begin{align*}
\int_0^T (\| \text{div} u \|_{L^\infty}^2 + \| \text{curl} u \|_{L^\infty}^2 ) \, dt \\
\leq C \int_0^T (\| G \|_{L^\infty}^2 + \| \text{curl} u \|_{L^\infty}^2 + \| P - \mathcal{P} \|_{L^\infty}^2 ) \, dt \\
\leq C \int_0^T (\| G \|_{W^{1,6}}^2 + \| \text{curl} u \|_{W^{1,6}}^2 + \| \theta \|_{L^\infty}^2 ) \, dt + C
\leq C \int_0^T (\| \nabla G \|_{L^6}^2 + \| \nabla \text{curl} u \|_{L^6}^2 + \| \nabla^2 \theta \|_{L^2}^2 ) \, dt + C
\leq C \int_0^T (\| \nabla \dot{u} \|_{L^2}^2 + \| \nabla^2 \theta \|_{L^2}^2 ) \, dt + C
\leq C,
\end{align*}
\] (4.16)
which as well as (4.15) and (4.3) yields that
\[
\sup_{0 \leq t \leq T} \| \nabla \rho \|_{L^6} \leq C. \tag{4.17}
\]
Combining this with (4.14), (4.16), and (4.3) leads to
\[
\int_{0}^{T} \| \nabla u \|_{L^6}^{3/2} \, dt \leq C. \tag{4.18}
\]
Moreover, it follows from (2.22), (3.5), (4.17), and (4.3) that
\[
\sup_{0 \leq t \leq T} \| u \|_{H^2} \leq C \sup_{0 \leq t \leq T} \left( \| \rho \dot{u} \|_{L^2} + \| \nabla P \|_{L^2} + \| \nabla u \|_{L^2} \right) \leq C. \tag{4.19}
\]
Next, applying operator \( \partial_{ij} \) (1 \( \leq i, j \leq 3 \)) to (1.4) gives
\[
(\partial_{ij} \rho) + \text{div}(\partial_{ij} \rho u) + \text{div}(\rho \partial_{ij} u) + \text{div}(\partial_{i} \rho \partial_{j} u + \partial_{j} \rho \partial_{i} u) = 0. \tag{4.20}
\]
Multiplying (4.20) by \( 2 \partial_{ij} \rho \) and integrating the resulting equality over \( \Omega \), it holds
\[
\frac{d}{dt} \| \nabla \rho \|_{L^2}^{2} \leq C(1 + \| \nabla u \|_{L^6}) \| \nabla \rho \|_{L^2}^{2} + C\| \nabla u \|_{H^2}^{2}
\leq C(1 + \| \nabla u \|_{L^6} + \| \nabla \theta \|_{L^2}^{2})(1 + \| \nabla \rho \|_{L^2}^{2}) + C\| \nabla \dot{u} \|_{L^2}^{2}, \tag{4.21}
\]
where one has used (3.5), (4.17), and the following estimate:
\[
\| u \|_{H^3} \leq C(\| \nabla (\rho \dot{u}) \|_{L^2} + \| \rho \dot{u} \|_{L^2} + \| \nabla P \|_{H^1} + \| \nabla u \|_{L^2})
\leq C\| \nabla \rho \|_{L^3} \| \dot{u} \|_{L^6} + C\| \nabla \dot{u} \|_{L^2} + C\| \nabla \theta \|_{H^1}
\leq C\| \nabla \dot{u} \|_{L^2} + C(1 + \| \nabla \theta \|_{L^2}^{2})(1 + \| \nabla \rho \|_{L^2}^{2}) + C \tag{4.22}
\]
due to (2.22), (2.19), (4.17), and (3.5). Then applying Grönwall’s inequality to (4.21) and using (4.13), (4.18) yield
\[
\sup_{0 \leq t \leq T} \| \nabla \rho \|_{L^2} \leq C, \tag{4.23}
\]
which together with (4.22) and (4.3) gives
\[
\int_{0}^{T} \| u \|_{H^1}^{2} \, dt \leq C. \tag{4.24}
\]
Finally, applying the standard \( H^1 \)-estimate to elliptic problem (3.57), one derives from (3.5), (4.3), (4.17), (3.110), and (4.19) that
\[
\| \nabla^2 \theta \|_{H^1} \leq C \left( \| \rho \dot{\theta} \|_{H^1} + \| \rho \theta \|_{H^1} + \| \nabla u \|_{H^1}^{2} \right)
\leq C \left( 1 + \| \nabla \theta \|_{L^2} + \| \rho \theta \|_{L^2} + \| \nabla \rho \theta \|_{L^2} + \| \nabla u \|_{H^1}^{2} \right) \tag{4.25}
\leq C \left( 1 + \| \nabla \theta \|_{L^2} + \| \rho \theta \|_{L^2} + \| \nabla \rho \|_{L^2} \right),
\]
which along with (3.5), (4.23), (4.24), (4.17), (4.19), and (4.3) yields (4.4).

The proof of Lemma 4.1 is finished.
Lemma 4.2. The following estimates hold:

\[
\sup_{0 \leq t \leq T} \| \rho_t \|_{H^1} + \int_0^T (\| u_t \|_{H^1}^2 + \| \theta_t \|_{H^1}^2 + \| \mu u_t \|_{H^1}^2 + \| \eta \|_{H^1}^2) \, dt \leq C, \tag{4.26}
\]

and

\[
\int_0^T \sigma (\| \rho u_t \|_{H^{-1}}^2 + \| \rho_t \|_{H^{-1}}^2) \, dt \leq C. \tag{4.27}
\]

Proof. First, it follows from (4.3) and (4.4) that

\[
\sup_{0 \leq t \leq T} \int (\rho |u_t|^2 + \sigma \rho \dot{\theta}_t^2) \, dx + \int_0^T (\| \nabla u_t \|_{L^2}^2 + \sigma \| \nabla \theta_t \|_{L^2}^2) \, dt
\]

\[
\leq \sup_{0 \leq t \leq T} \int \left( (\rho \dot{u}_t^2 + \sigma \rho |\dot{\theta}_t|^2) \, dx + \int_0^T (\| \nabla \dot{u}_t \|_{L^2}^2 + \sigma \| \nabla \dot{\theta}_t \|_{L^2}^2) \, dt 
\right. \\
\left. + \sup_{0 \leq t \leq T} \int \rho (|u \cdot \nabla u|^2 + \sigma |u \cdot \nabla \theta|^2) \, dx 
\right)
\]

\[
\leq C,
\]

which together with (4.3) and (4.4) gives

\[
\int_0^T (\| \nabla (\rho u_t) \|_{L^2}^2 + \sigma \| \nabla (\rho \theta_t) \|_{L^2}^2) \, dt
\]

\[
\leq C \int_0^T \left( \| \nabla u_t \|_{L^2}^2 + \| \nabla \rho \|_{L^2} \| u_t \|_{L^6} + \sigma \| \nabla \theta_t \|_{L^2}^2 + \sigma \| \nabla \rho \|_{L^2} \| \theta_t \|_{L^6} \right) \, dt
\]

\[
\leq C,
\]

where we have used

\[
\| \theta_t \|_{L^6} \leq C \| \rho^{1/2} \theta_t \|_{L^2} + C \| \nabla \theta_t \|_{L^2},
\]

due to (3.110).

Next, one deduces from (1.4), (1.4), and (2.9) that

\[
\| \rho_t \|_{H^1} \leq \| \text{div}(\rho u) \|_{H^1} \leq C \| u \|_{H^2} \| \rho \|_{H^2} \leq C,
\]

which as well as (4.28)–(4.30) shows (4.26).

Finally, differentiating (1.4) with respect to \( t \) yields that

\[
(\rho u_t)_t = - (\rho u \cdot \nabla u)_t + (\rho D u + (\mu + \lambda) \nabla \rho)_t - \nabla P_t.
\]

It follows from (4.29), (4.4), (4.3), (3.56), and (4.30) that

\[
\|[\rho u \cdot \nabla u]_t\|_{L^2} = \| \rho u \cdot \nabla u + \rho u_t \cdot \nabla u + \rho u \cdot \nabla u_t \|_{L^2}
\]

\[
\leq C \| \rho_t \|_{L^6} \| \nabla u \|_{L^3} + C \| u_t \|_{L^6} \| \nabla u \|_{L^3} + C \| u \|_{L^\infty} \| \nabla u_t \|_{L^2}
\]

\[
\leq C + C \| \nabla u_t \|_{L^2},
\]
and
\[ \| \nabla P_t \|_{L^2} = R \| \rho_t \nabla \theta + \rho \nabla \theta_t + \nabla \rho \theta + \nabla \rho \theta_t \|_{L^2} \]
\[ \leq C (\| \rho_t \|_{L^6} \| \nabla \theta \|_{L^3} + \| \nabla \theta_t \|_{L^2} + \| \theta \|_{L^\infty} \| \nabla \rho_t \|_{L^2} + \| \nabla \rho \|_{L^6} \| \theta_t \|_{L^3}) \]
\[ \leq C + C \| \nabla \theta_t \|_{L^2} + C \| \rho^{1/2} \theta_t \|_{L^2}. \]
Combining (4.31)–(4.33) with (4.26) shows
\[ \int_0^T \sigma \left( \| \rho \theta_t \|_{L^2} \right)^2 dt \leq C. \]
Similarly, we have
\[ \int_0^T \sigma \left( \| \rho \theta_t \|_{L^2} \right)^2 dt \leq C, \]
which combined with (4.31) implies (4.27). The proof of Lemma 4.2 is completed. □

**Lemma 4.3.** The following estimate holds:
\[ \sup_{0 \leq t \leq T} \sigma \left( \| \nabla u_t \|_{L^2}^2 + \| \rho u_t \|_{L^2}^2 + \| u_t \|_{H^1}^2 \right) + \int_0^T \sigma \left( \| \rho^{1/2} u_t \|_{L^2}^2 + \| \nabla u_t \|_{H^1}^2 \right) dt \leq C. \]

**Proof.** Differentiating (3.13) with respect to \( t \) leads to
\[ \begin{cases} (2\mu + \lambda) \nabla \mathbf{div} u_t - \mu \nabla \times \text{curl} u_t = \rho u_{tt} + \rho_t u_t + \rho u \cdot \nabla u + \rho u_t \cdot \nabla u + \rho \nabla \theta_t + \nabla \rho_t, & \text{in } \Omega \times [0, T], \\ u_t \cdot n = 0, \ \text{curl} u_t \times n = 0, & \text{on } \partial \Omega \times [0, T]. \end{cases} \]
Multiplying (4.36) by \( u_{tt} \) and integrating the resulting equality by parts, one gets
\[ \frac{1}{2} \frac{d}{dt} \int (\mu |\nabla u_t|^2 + (2\mu + \lambda)(\mathbf{div} u_t)^2) \, dx + \int \rho |u_{tt}|^2 \, dx \]
\[ = \frac{d}{dt} \left( -\frac{1}{2} \int \rho_t |u_t|^2 \, dx - \int \rho_t u \cdot \nabla u \cdot u_t \, dx + \int P_t \mathbf{div} u_t \, dx \right) \]
\[ + \frac{1}{2} \int \rho_t |u_t|^2 \, dx + \int \rho u_t \cdot \nabla u \cdot u_t \, dx - \int \rho u_t \cdot \nabla u \cdot u_{tt} \, dx \]
\[ - \int \rho u \cdot \nabla u_t \cdot u_{tt} \, dx - \int (P_{tt} - \kappa(\gamma - 1)\Delta \theta_t) \mathbf{div} u_t \, dx \]
\[ + \kappa(\gamma - 1) \int \nabla \theta_t \cdot \nabla \mathbf{div} u_t \, dx \triangleq \frac{d}{dt} \tilde{I}_0 + \sum_{i=1}^6 \tilde{I}_i. \]

Each term \( \tilde{I}_i (i = 0, \ldots, 6) \) can be estimated as follows:
First, it follows from simple calculations, (1.4)\textsuperscript{1}, (1.26), (4.2), (4.3), and (4.28) that
\[ |\tilde{I}_0| = \left| -\frac{1}{2} \int \rho_t |u_t|^2 \, dx - \int \rho_t u \cdot \nabla u \cdot u_t \, dx + \int P_t \mathbf{div} u_t \, dx \right| \]
\[ \leq C \int \rho |u_t|^2 \, dx + C \| \rho_t \|_{L^3} \| \nabla u_t \|_{L^2} \| u_t \|_{L^6} + C \| (\rho \theta_t)_{tt} \|_{L^2} \| \nabla u_t \|_{L^2} \]
\[ \leq C \| \rho^{1/2} u_t \|_{L^2} \| \nabla u_t \|_{L^2} + C(1 + \| \rho^{1/2} \theta_t \|_{L^2} + \| \rho_t \|_{L^3} \| \theta \|_{L^6}) \| \nabla u_t \|_{L^2} \]
\[ \leq C(1 + \| \rho^{1/2} \theta_t \|_{L^2}) \| \nabla u_t \|_{L^2}, \]
\[ 2|\tilde{I}_1| = \left| \int \rho u_t |u_t|^2 \, dx \right| \leq C \|\rho u_t\|_{L^2} \|u_t\|_{L^4}^2 \leq C \|\rho u_t\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^4, \quad (4.39) \]

\[ |\tilde{I}_2| = \left| \int (\rho u \cdot \nabla u) u_t \, dx \right| = \left| \int (\rho u_t \cdot \nabla u - \rho u_t \cdot \nabla u \cdot u_t + \rho u_t \cdot \nabla u \cdot u_t) \, dx \right| \leq C \|\rho u_t\|_{L^2} \|\nabla u\|_{L^6} \|u\|_{L^6} \|u_t\|_{L^6} + C \|\rho u_t\|_{L^2} \|u_t\|_{L^2} \|\nabla u\|_{L^6} \|u_t\|_{L^6} \leq C \|\rho u_t\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^2, \quad (4.40) \]

and

\[ |\tilde{I}_3| + |\tilde{I}_4| = \left| \int \rho u_t \cdot \nabla u \cdot u_t \, dx \right| + \left| \int \rho u \cdot \nabla u_t \cdot u_t \, dx \right| \leq C \|\rho^{1/2} u_t\|_{L^2} (\|u_t\|_{L^6} \|\nabla u\|_{L^3} + \|u\|_{L^\infty} \|\nabla u_t\|_{L^2}) \quad (4.41) \]

Then, by virtue of (3.25), (4.26), (4.33), and Lemma 4.1, it holds

\[ \|P_t - \kappa(\gamma - 1) \Delta \theta_t\|_{L^2} \leq C \|(u \cdot \nabla P)\|_{L^2} + C \|(P \text{div} u)\|_{L^2} + C \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} + C \|\rho u_t\|_{L^2} + C \|\rho u\|_{L^2} \|\nabla u\|_{L^3} + C \|P\|_{L^\infty} \|\nabla u_t\|_{L^2} + C \|\nabla u\|_{L^\infty} \|\nabla u_t\|_{L^2} \leq C \left(1 + \|\nabla u\|_{L^\infty} + \|\nabla^2 \theta\|_{L^2}\right) \|\nabla u_t\|_{L^2} + C \left(1 + \|\nabla \theta_t\|_{L^2} + \|\rho^{1/2} \theta_t\|_{L^2}\right), \]

which yields

\[ |\tilde{I}_5| = \left| \int (P_t - \kappa(\gamma - 1) \Delta \theta_t) \text{div} u_t \, dx \right| \leq \|P_t - \kappa(\gamma - 1) \Delta \theta_t\|_{L^2} \|\nabla u_t\|_{L^2} \leq C \left(1 + \|\nabla u\|_{L^\infty} + \|\nabla^2 \theta\|_{L^2}\right) \|\nabla u_t\|_{L^2}^2 + C \left(1 + \|\nabla \theta_t\|_{L^2}^2 + \|\rho^{1/2} \theta_t\|_{L^2}^2\right). \quad (4.42) \]

Next, combining Lamé’s system (4.36) with Lemma 2.7, (4.4), (4.26), and (4.33) gives

\[ \|\nabla^2 u_t\|_{L^2} \leq C \|\tilde{f}\|_{L^2} + C \|\nabla u_t\|_{L^2} \leq C \|\rho u_{tt}\|_{L^2} + C \|\rho\|_{L^3} \|u_t\|_{L^6} + C \|\rho\|_{L^3} \|\nabla u\|_{L^6} \|u\|_{L^\infty} + C \|\rho u_t\|_{L^2} \|u_t\|_{L^2} + C \|\rho\|_{L^3} \|\nabla u_t\|_{L^2} \|u\|_{L^\infty} + C \|P_t\|_{L^2} \leq C \left(\|\rho u_{tt}\|_{L^2} + \|\nabla u_t\|_{L^2} + \|\rho^{1/2} \theta_t\|_{L^2} + \|\nabla \theta_t\|_{L^2} + 1\right), \quad (4.43) \]

which immediately leads to

\[ |\tilde{I}_6| = \kappa(\gamma - 1) \left| \int \nabla \theta_t \cdot \text{div} u_t \, dx \right| \leq C \|\nabla^2 u_t\|_{L^2} \|\nabla \theta_t\|_{L^2} \leq C \|\rho^{1/2} u_{tt}\|_{L^2}^2 + C \left(1 + \|\nabla u_t\|_{L^2}^2 + \|\rho^{1/2} \theta_t\|_{L^2}^2 + \|\nabla \theta_t\|_{L^2}^2\right). \quad (4.44) \]
Putting (4.39)–(4.42) and (4.44) into (4.37) yields
\[
\frac{d}{dt} \int (\mu |\text{curl} u_t|^2 + (2\mu + \lambda)(\text{div} u_t)^2 - 2I_0) \, dx + \int \rho |u_t|^2 \, dx \\
\leq C (1 + \|
abla u_t\|_{L^\infty} + \|
abla u_t\|_{L^2}^2 + \|
abla^2 \theta\|_{L^2}^2) \|
abla u_t\|_{L^2}^2 \\
+ C \left(1 + \|ho_t\|_{L^2}^2 + \|ho^{1/2} \theta_t\|_{L^2}^2 + \|
abla \theta_t\|_{L^2}^2\right).
\]
(4.45)

Furthermore, we infer from (1.3), (4.4), and (4.25) that
\[
\|ho_t\|_{L^2} = \|\text{div}(\rho u_t)\|_{L^2} \\
\leq C \left(\|ho_x\|_{L^6} \|
abla u_t\|_{L^3} + \|
abla u_t\|_{L^2} + \|u_t\|_{L^6} \|
abla \rho\|_{L^3} + \|
abla \rho_t\|_{L^2}\right) \\
\leq C + C \|
abla u_t\|_{L^2}.
\]
(4.46)

Multiplying (4.45) by \(\sigma\) and integrating the resulting inequality over \((0, T)\), one thus deduces from (2.16), (4.3), (4.4), (4.28), (4.33), (4.46), and Grönewall’s inequality that
\[
\sup_{0 \leq t \leq T} \sigma \|
abla u_t\|_{L^2}^2 + \int_0^T \sigma \int \rho |u_t|^2 \, dx \, dt \leq C.
\]
(4.47)

Finally, by Lemma 4.1, (4.46), (4.22), (4.43), (4.28), and (4.47), we have
\[
\sup_{0 \leq t \leq T} \sigma \left(\|ho_t\|_{L^2}^2 + \|u_t\|_{H^1}^2\right) + \int_0^T \sigma \|
abla u_t\|_{H^1}^2 \, dt \leq C,
\]
which along with (4.47) gives (4.35). We complete the proof of Lemma 4.3.

Lemma 4.4. For \(q \in (3, 6)\) as in Theorem 1.1, it holds that
\[
\sup_{0 \leq t \leq T} \|ho\|_{W^{2, q}} + \int_0^T \|
abla^2 u\|_{W^{1, q}}^p \, dt \leq C,
\]
where
\[
1 < p_0 < \frac{4q}{5q - 6} \in (1, 4/3).
\]
(4.49)

Proof. First, it follows from (2.22), (2.19), and Lemma 4.1 that
\[
\|
abla^2 u\|_{W^{1, q}} \leq C \left(\|ho \dot{u}\|_{W^{1, q}} + \|
abla P\|_{W^{1, q}} + \|
abla u\|_{L^2}\right) \\
\leq C \left(\|
abla \dot{u}\|_{L^2} + \|
abla (\rho \dot{u})\|_{L^q} + \|
abla^2 \theta\|_{L^2} + \|	heta \nabla^2 \rho\|_{L^q} \\
+ \|
abla \rho\| \|
abla \theta\|_{L^q} + \|
abla^2 \theta\|_{L^q} + 1\right) \\
\leq C \left(\|
abla \dot{u}\|_{L^2} + \|
abla (\rho \dot{u})\|_{L^q} + \|
abla^2 \theta\|_{L^q} + 1\right) \\
+ C \left(1 + \|
abla^2 \theta\|_{L^2}\right) \|
abla^2 \rho\|_{L^q}.
\]
(4.50)

Next, multiplying (4.20) by \(q|\partial_{ij} \rho|^{q-2} \partial_{ij} \rho\) and integrating the resulting equality over \(\Omega\), we obtain after using (4.4) and (4.50) that
\[
\frac{d}{dt} \|
abla^2 \rho\|_{L^q}^q \\
\leq C \|
abla u\|_{L^\infty} \|
abla^2 \rho\|_{L^q}^q + C \|
abla^2 u\|_{W^{1, q}} \|
abla^2 \rho\|_{L^q}^{q-1} \|
abla \rho\|_{L^q} + 1 \\
\leq C \|
abla \dot{u}\|_{L^2} \|
abla^2 \rho\|_{L^q}^q + C \|
abla^2 u\|_{W^{1, q}} \|
abla^2 \rho\|_{L^q}^{q-1} \\
\leq C \left(\|
abla \dot{u}\|_{L^2} + \|
abla (\rho \dot{u})\|_{L^q} + \|
abla^2 \theta\|_{L^q} + 1\right) (\|
abla^2 \rho\|_{L^q}^q + 1).
\]
(4.51)
Note that Lemma 4.1, (2.7), (2.19), and (4.35) imply
\[
\|\nabla (\rho \dot{u})\|_{L^6} \leq C \|\nabla \rho\|_{L^6} \|\dot{u}\|_{L^{6q/(6-q)}} + C \|\nabla \dot{u}\|_{L^6} \\
\leq C \|\dot{u}\|_{W^{1,6q/(6-q)}} + C \|\nabla \dot{u}\|_{L^6} \\
\leq C \|\nabla u_t\|_{L^6} + C \|\nabla (u \cdot \nabla u)\|_{L^6} + C \\
\leq C \|\nabla u_t\|_{L^6}^{(6-q)/2q} \|\nabla u\|_{L^6}^{3(q-2)/2q} \\
+ C \|\nabla u\|_{L^6} \|\nabla u\|_{L^\infty} + C \|u\|_{L^\infty} \|\nabla^2 u\|_{L^6} + C \\
\leq C \sigma^{-1/2} (\|\nabla u_t\|_{H^1}^{2})^{3(q-2)/4q} + C \|u\|_{H^3} + C,
\]
and
\[
\|\nabla^2 \theta\|_{L^6} \leq C \|\nabla^2 \theta\|_{L^6}^{(6-q)/2q} \|\nabla^3 \theta\|_{L^2}^{3(q-2)/2q} + C \|\nabla^2 \theta\|_{L^2} \\
\leq C \sigma^{-1/2} (\|\nabla^3 \theta\|_{L^2}^{2})^{3(q-2)/4q} + C \|\nabla^2 \theta\|_{L^2},
\]
which combined with Lemma 4.1 and (4.35) shows that, for \(p_0\) as in (4.49),
\[
\int_0^T \left( \|\nabla (\rho \dot{u})\|_{L^6}^{p_0} + \|\nabla^2 \theta\|_{L^6}^{p_0} \right) dt \leq C.
\]
Finally, applying Grönwall’s inequality to (4.51), we obtain after using Lemma 4.1 and (4.53) that
\[
\sup_{0 \leq t \leq T} \|\nabla^2 \rho\|_{L^6} \leq C,
\]
which together with Lemma 4.1, (4.53), and (4.50) gives (4.48). We finish the proof of Lemma 4.1. \(\square\)

**Lemma 4.5.** For \(q \in (3, 6)\) as in Theorem 1.1, the following estimate holds:
\[
\sup_{0 \leq t \leq T} \sigma (\|\theta_t\|_{H^1} + \|\nabla^2 \theta\|_{H^1} + \|u_t\|_{H^2} + \|u\|_{W^{3,q}}) + \int_0^T \sigma^2 \|\nabla u_t\|_{L^2}^2 dt \leq C.
\]

**Proof.** First, differentiating (4.36) with respect to \(t\) gives
\[
\begin{align*}
\rho_t u_t + \rho u \cdot \nabla u_t - (2\mu + \lambda) \nabla \div u_t + \mu \nabla \times \curl u_t \\
= 2 \div (\rho u) u_t + \div (\rho u)_t u_t - 2(\rho u)_t \cdot \nabla u_t \\
- (\rho u_t + 2\rho u_t) \cdot \nabla u - \rho u_t \cdot \nabla u - \nabla P_t, \quad & \text{in } \Omega \times [0, T], \\
\end{align*}
\]
\[
\begin{align*}
u_t \cdot n = 0, \quad & \curl u_t \times n = 0, \quad & \text{on } \partial \Omega \times [0, T].
\end{align*}
\]
Multiplying (4.55) \(1\) by \(u_t\) and integrating the resulting equality over \(\Omega\) by parts imply that
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int \rho |u_t|^2 dx + \int \left((2\mu + \lambda) (\div u_t)^2 + \mu |\curl u_t|^2\right) dx \\
= -4 \int \rho u_t \cdot \nabla u_t^i dx - \int (\rho u_t) \cdot (\nabla (u_t \cdot u_t)) + 2 \nabla u_t \cdot u_t dx \\
- \int (\rho u_t + 2\rho u_t) \cdot \nabla u - \nabla u \cdot u_t dx - \int \rho u_t \cdot \nabla u \cdot u_t dx \\
+ \int P_t \div u_t dx \triangleq \sum_{i=1}^5 \tilde{J}_i.
\end{align*}
\]

It follows from Lemmas 4.1–4.3, (4.28), (4.30), and (4.46) that, for $\eta \in (0, 1]$,

\[
|\tilde{J}_1| \leq C\|\rho^{1/2}u_t\|_{L^2}\|\nabla u_t\|_{L^2}\|u\|_{L^\infty} \leq \eta\|\nabla u_t\|_{L^2}^2 + C(\eta)\|\rho^{1/2}u_t\|_{L^2}^2,
\]

(4.57)

\[
|\tilde{J}_2| \leq C \left( \|\rho u_t\|_{L^3} + \|\rho_t u\|_{L^3} \right) \left( \|\nabla u_t\|_{L^2}\|u_t\|_{L^6} + \|u_t\|_{L^6}\|\nabla u_t\|_{L^2} \right)
\leq C \left( \|\rho^{1/2}u_t\|_{L^2}^2 + \|\rho_t\|_{L^6}\|u_t\|_{L^6} \right) \|\nabla u_t\|_{L^2}\|\nabla u_t\|_{L^2}
\leq \eta\|\nabla u_t\|_{L^2}^2 + C(\eta)\|\nabla u_t\|_{L^2}^3 + C(\eta)
\leq \eta\|\nabla u_t\|_{L^2}^2 + C(\eta)\sigma^{-3/2},
\]

(4.58)

\[
|\tilde{J}_3| \leq C \left( \|\rho u_t\|_{L^2}\|u\|_{L^6} + \|\rho_t\|_{L^2}\|u_t\|_{L^6} \right) \|\nabla u\|_{L^6}\|u_t\|_{L^6}
\leq \eta\|\nabla u_t\|_{L^2}^2 + C(\eta)\sigma^{-1}
\]

(4.59)

and

\[
|\tilde{J}_4| + |\tilde{J}_5|
\leq C\|\rho u_{tt}\|_{L^2}\|\nabla u\|_{L^2}\|u_{tt}\|_{L^6} + C\|\rho_t\|_{L^2}\|\rho_t\|_{L^2}\|\nabla u_t\|_{L^2}
\leq \eta\|\nabla u_t\|_{L^2}^2 + C(\eta) \left( \|\rho^{1/2}u_{tt}\|_{L^2}^2 + \|\rho_t\|_{L^2}^2 + \|\rho_t\|_{L^2}^2 + \|\rho^{1/2}\theta_t\|_{L^2}^2 \right)
\leq \eta\|\nabla u_t\|_{L^2}^2 + C(\eta) \left( \|\rho^{1/2}u_{tt}\|_{L^2}^2 + \|\nabla \theta_t\|_{L^2}^2 + \|\rho^{1/2}\theta_t\|_{L^2}^2 + \sigma^{-2} \right).
\]

(4.60)

Substituting (4.57)–(4.60) into (4.55), we obtain after using (2.10) and choosing $\eta$ suitably small that

\[
\frac{d}{dt} \int \rho|u_{tt}|^2 dx + C_4 \int |\nabla u_{tt}|^2 dx
\leq C\sigma^{-2} + C\|\rho^{1/2}u_{tt}\|_{L^2}^2 + C\|\nabla \theta_t\|_{L^2}^2 + C_5\|\rho^{1/2}\theta_t\|_{L^2}^2.
\]

(4.61)

Next, differentiating (3.57) with respect to $t$ infers

\[
\left\{ \begin{array}{l}
-\frac{\kappa(\gamma - 1)}{R}\Delta \theta_t + \rho\theta_{tt} = -\rho\rho_t + \rho \left( u \cdot \nabla \theta + (\gamma - 1)\theta \text{div} u \right) - \rho \left( u \cdot \nabla \theta + (\gamma - 1)\theta \text{div} u \right)_t + 
\frac{2\gamma - 1}{R} \left( \lambda(\text{div} u)^2 + 2\mu|\mathcal{O}(u)|^2 \right)_t, \\
\nabla \theta_t \cdot n|_{\partial \Omega \times (0, T)} = 0.
\end{array} \right.
\]

(4.62)

Multiplying (4.62) by $\theta_{tt}$ and integrating the resulting equality over $\Omega$ lead to

\[
\left( \frac{\kappa(\gamma - 1)}{2R} \|\nabla \theta_t\|_{L^2}^2 + H_0 \right)_t + \int \rho\theta_{tt}^2 dx
= \frac{1}{2} \int \rho t \left( \theta_t^2 + 2 \left( u \cdot \nabla \theta + (\gamma - 1)\theta \text{div} u \right) \theta_t \right) dx
+ \int \rho_t \left( u \cdot \nabla \theta + (\gamma - 1)\theta \text{div} u \right)_t \theta_t dx
- \int \rho \left( u \cdot \nabla \theta + (\gamma - 1)\theta \text{div} u \right)_t \theta_{tt} dx
- \frac{\gamma - 1}{R} \int \left( \lambda(\text{div} u)^2 + 2\mu|\mathcal{O}(u)|^2 \right)_t \theta_t dx
\equiv \sum_{i=1}^4 H_i.
\]

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where
\[
H_0 = \frac{1}{2} \int \rho_t \theta_t^2 dx + \int \rho_t (u \cdot \nabla \theta + (\gamma - 1) \theta \text{div} u) \theta_t dx
- \frac{\gamma - 1}{R} \int (\lambda (\text{div} u)^2 + 2\mu |\nabla (u)|^2) \theta_t dx.
\]

It follows from (4.4), (4.28), (4.30), and (4.35) that
\[
|H_0| \leq C \int \rho |u| |\theta_t| |\nabla \theta_t| dx + C \|\rho_t\|_{L^\infty} \|\theta_t\|_{L^6} (\|\nabla \theta_t\|_{L^2} u_{\infty} + \|\theta\|_{L^6} \|\nabla u\|_{L^3})
+ C \|\nabla u\|_{L^3} \|\nabla u_t\|_{L^2} \|\theta_t\|_{L^6}
\leq C (\|\rho^{1/2} \theta_t\|_{L^2} + \|\nabla \theta_t\|_{L^2}) (\|\rho^{1/2} \theta_t\|_{L^2} + \|\nabla u_t\|_{L^2} + 1)
\leq \frac{\kappa(\gamma - 1)}{4R} \|\nabla \theta_t\|^2_{L^2} + C \sigma^{-1},
\]  
and
\[
|H_1| \leq C \|\rho u_t\|_{L^2} (\|\theta_t\|^2_{L^4} + \|\theta_t\|_{L^6} (\|u \cdot \nabla \theta\|_{L^3} + \|\theta \text{div} u\|_{L^3}))
\leq C \|\rho u_t\|_{L^2} (\|\rho^{1/2} \theta_t\|^2_{L^2} + \|\nabla \theta_t\|^2_{L^2} + \sigma^{-1})
\leq C (1 + \|\nabla u_t\|_{L^2}) \|\nabla \theta_t\|^2_{L^2} + C \sigma^{-3/2}.
\]
Combining Lemma 4.4 with 4.30 gives
\[
\| (u \cdot \nabla \theta + (\gamma - 1) \theta \text{div} u)_t \|_{L^2}
\leq C (\|u\|_{L^6} \|\nabla \theta\|_{L^4} + \|\nabla u\|_{L^2} + \|\theta\|_{L^6} \|\nabla u_t\|_{L^2} + \|\theta\|_{L^{\infty}} \|\nabla u_t\|_{L^2})
\leq C \|\nabla u_t\|_{L^2} (\|\nabla^2 \theta\|_{L^2} + 1) + C \|\nabla \theta_t\|_{L^2} + C \|\rho^{1/2} \theta_t\|_{L^2},
\]
which together with (4.3), (4.26), (4.28), and (4.30) shows
\[
|H_2| + |H_3| \leq C \left( \sigma^{-1/2} (\|\nabla u_t\|_{L^2} + 1) + \|\nabla \theta_t\|_{L^2} \right) (\|\rho_t\|_{L^3} \|\theta_t\|_{L^6} + \|\rho \theta_t\|_{L^2})
\leq \frac{1}{2} \int \rho \theta_t^2 dx + C \|\nabla \theta_t\|^2_{L^2} + C \sigma^{-1} \|\nabla u_t\|^2_{L^2} + C \sigma^{-1}.
\]
One deduces from (4.4), (4.28), (4.30), and (4.35) that
\[
|H_4| \leq C \int (\|\nabla u_t\|^2_{L^2} + \|\nabla u_t \| \|\nabla u_t\|_{L^2}) |\theta_t| dx
\leq C \left( \|\nabla u_t\|^3_{L^2} \|\nabla u_t\|_{L^6} \|\nabla u_t\|_{L^2} \right) \|\theta_t\|_{L^6}
\leq \delta \|\nabla u_t\|^2_{L^2} + C (\|\nabla u_t\|^2_{L^2} + C(\delta) (\|\nabla \theta_t\|^2_{L^2} + \sigma^{-1}) + C \sigma^{-2} \|\nabla u_t\|^2_{L^2}.
\]
Substituting (4.65), (4.67), and (4.68) into (4.63) gives
\[
\left( \frac{\kappa(\gamma - 1)}{2R} \|\nabla \theta_t\|^2_{L^2} + H_0 \right)_t + \frac{1}{2} \int \rho \theta_t^2 dx
\leq \delta \|\nabla u_t\|^2_{L^2} + C(\delta)((1 + \|\nabla u_t\|_{L^2}) \|\nabla \theta_t\|^2_{L^2} + \sigma^{-3/2})
+ C (\|\nabla^2 u_t\|^2_{L^2} + \sigma^{-2} \|\nabla u_t\|^2_{L^2}).
\]

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Finally, for $C_5$ as in (4.61), adding (4.69) multiplied by $2(C_5 + 1)$ to (4.61) and choosing $\delta$ suitably small yields that

\[
\left[2(C_5 + 1) \left( \frac{\kappa(\gamma - 1)}{2R} \|\nabla \theta_t\|_{L^2}^2 + H_0 \right) + \int \rho|u_{tt}|^2 dx \right]_t \\
\int \rho \theta_{tt}^2 dx + \frac{C_4}{2} \int |\nabla u_{tt}|^2 dx \\
\leq C(1 + \|\nabla u_t\|_{L^2}^2)(\sigma^{-2} + \|\nabla \theta_t\|_{L^2}^2) + C\|\rho^{1/2} u_{tt}\|_{L^2}^2 + C\|\nabla^2 u_t\|_{L^2}^2.
\]

Multiplying this by $\sigma^2$ and integrating the resulting inequality over $(0, T)$, we obtain after using (4.61), (4.35), (4.26), and Grönwall’s inequality that

\[
\sup_{0 \leq t \leq T} \sigma^2 \int (|\nabla \theta_t|^2 + \rho|u_{tt}|^2) dx + \int_0^T \sigma^2 \int (\rho \theta_{tt}^2 + |\nabla u_{tt}|^2) dxdt \leq C,
\]

which together with Lemmas 4.4.1, 4.3.4, 4.25, (4.28), (4.50), and (4.52) gives

\[
\sup_{0 \leq t \leq T} \sigma \left( \|\nabla u_t\|_{H^1} + \|\nabla^2 \theta\|_{H^1} + \|\nabla^2 u\|_{W^{1,4}} \right) \leq C.
\]

We thus derive (4.55) from (4.70), (4.71), (4.28), (4.30), and (4.4). The proof of Lemma 4.5 is completed.

**Lemma 4.6.** The following estimate holds:

\[
\sup_{0 \leq t \leq T} \sigma^2 \left( \|\nabla^2 \theta\|_{H^2} + \|\theta_t\|_{H^2} + \|\rho^{1/2} \theta_{tt}\|_{L^2} \right) + \int_0^T \sigma^4 \|\nabla \theta_t\|_{L^2}^2 dt \leq C.
\]

**Proof.** First, differentiating (4.62) with respect to $t$ yields

\[
\begin{align*}
\rho \theta_{tt} - \frac{\kappa(\gamma - 1)}{R} \Delta \theta_t &= -\rho u \cdot \nabla \theta_t + 2 \text{div}(\rho u) \theta_t - \rho (\theta_t + u \cdot \nabla \theta + (\gamma - 1) \text{div} u) \\
-2 \rho_t (u \cdot \nabla \theta + (\gamma - 1) \text{div} u)_t &+ \theta_t (u \cdot \nabla \theta + (\gamma - 1) \text{div} u) \\
-\rho (u_{tt} \cdot \nabla \theta + 2 u_t \cdot \nabla \theta_t + (\gamma - 1) (\theta \text{div} u)_{tt}) \\
+ \frac{\gamma - 1}{R} (\lambda (\text{div} u)^2 + 2 \mu |\nabla u|^2)_{tt} + \nabla \theta_t \cdot n |_{\partial \Omega \times (0, T)} &= 0.
\end{align*}
\]

Multiplying (4.73) by $\theta_{tt}$ and integrating the resulting equality over $\Omega$ yield that

\[
\frac{1}{2} \frac{d}{dt} \int \rho \theta_{tt}^2 dx + \frac{\kappa(\gamma - 1)}{R} \int |\nabla \theta_t|^2 dx \\
= -4 \int \theta_{tt} \rho u \cdot \nabla \theta_t dx - \int \rho (\theta_t + u \cdot \nabla \theta + (\gamma - 1) \text{div} u) \theta_{tt} dx \\
-2 \int \rho_t (u \cdot \nabla \theta + (\gamma - 1) \text{div} u)_t \theta_{tt} dx \\
- \int \rho (u_{tt} \cdot \nabla \theta + 2 u_t \cdot \nabla \theta_t + (\gamma - 1) (\theta \text{div} u)_{tt}) \theta_{tt} dx \\
+ \frac{\gamma - 1}{R} \int (\lambda (\text{div} u)^2 + 2 \mu |\nabla u|^2)_{tt} \theta_{tt} dx = \sum_{i=1}^{5} K_i.
\]
It follows from Lemmas 4.1–4.3, 4.5, (4.70), and (4.28) that

\[ \sigma^4|K_1| \leq C \sigma^4 \| \rho^{1/2} \theta_t \|_{L^2} \| \nabla \theta_t \|_{L^2} \| u \|_{L^\infty} \]
\[ \leq \delta \sigma^4 \| \nabla \theta_t \|_{L^2}^2 + C(\delta) \sigma^4 \| \rho^{1/2} \theta_t \|_{L^2}^2, \]  
(4.75)

\[ \sigma^4|K_2| \leq C \sigma^4 \| \rho \theta_t \|_{L^2} \| \theta_t \|_{L^6} \left( \| \theta_t \|_{H^1} + \| \nabla \theta \|_{L^3} + \| \nabla u \|_{L^6} \| \theta \|_{L^6} \right) \]
\[ \leq C \sigma^2 \left( \| \nabla \theta_t \|_{L^2} + \| \rho^{1/2} \theta_t \|_{L^2} \right) \]
\[ \leq \delta \sigma^4 \| \nabla \theta_t \|_{L^2}^2 + C(\delta) \sigma^4 \| \rho^{1/2} \theta_t \|_{L^2}^2 + 1, \]  
(4.76)

\[ \sigma^4|K_4| \leq C \sigma^4 \| \theta_t \|_{L^6} \left( \| \nabla \theta \|_{L^3} \| \rho \theta_t \|_{L^2} + \| \nabla \theta_t \|_{L^2} \| u \|_{L^3} \right) \]
\[ + C \sigma^4 \| \theta_t \|_{L^6} \left( \| \nabla u \|_{L^3} \| \rho \theta_t \|_{L^2} + \| \nabla u \|_{L^2} \| \theta_t \|_{L^3} \right) \]
\[ + C \sigma^4 \| \theta \|_{L^\infty} \| \rho \theta_t \|_{L^2} \| \nabla u_t \|_{L^2} \]
\[ \leq \delta \sigma^4 \| \nabla \theta_t \|_{L^2}^2 + C(\delta) \left( \sigma^4 \| \rho^{1/2} \theta_t \|_{L^2}^2 + \sigma^3 \| \nabla u_t \|_{L^2}^2 \right) + C(\delta), \]  
(4.77)

\[ \sigma^4|K_5| \leq C \sigma^4 \| \theta_t \|_{L^6} \left( \| \nabla u_t \|_{L^3}^{3/2} \| \nabla u_t \|_{L^2}^{1/2} + \| \nabla u \|_{L^3} \| \nabla u_t \|_{L^2} \right) \]
\[ \leq \delta \sigma^4 \| \nabla \theta_t \|_{L^2}^2 + C(\delta) \sigma^4 \left( \| \rho^{1/2} \theta_t \|_{L^2}^2 + \| \nabla u_t \|_{L^2}^2 \right) + C(\delta), \]  
(4.78)

and

\[ \sigma^4|K_3| \leq C \sigma^4 \| \rho \|_{L^3} \| \theta_t \|_{L^6} \left( \sigma^{-1/2} \| \nabla u_t \|_{L^2} + \| \rho^{1/2} \theta_t \|_{L^2} + \| \nabla \theta_t \|_{L^2} \right) \]
\[ \leq \delta \sigma^4 \| \nabla \theta_t \|_{L^2}^2 + C \sigma^4 \| \rho^{1/2} \theta_t \|_{L^2}^2 + C(\delta), \]  
(4.79)

where in the last inequality we have used (4.66).

Then, multiplying (4.70) by \( \sigma^4 \), substituting (4.75)–(4.79) into the resulting equality and choosing \( \delta \) suitably small, one obtains

\[
\frac{d}{dt} \int \sigma^4 \rho |\theta_t|^2 dx + \frac{\kappa(g - 1)}{R} \int \sigma^4 |\nabla \theta_t|^2 dx \\
\leq C \sigma^2 \left( \| \rho^{1/2} \theta_t \|_{L^2}^2 + \| \nabla u_t \|_{L^2}^2 \right) + C,
\]

which together with (4.70) gives

\[
\sup_{0 \leq t \leq T} \sigma^4 \int \rho |\theta_t|^2 dx + \int_0^T \sigma^4 \int |\nabla \theta_t|^2 dx dt \leq C. \]  
(4.80)

Finally, applying the standard \( L^2 \)-estimate to (4.62), one obtains after using Lemmas 4.1–4.3, 4.5, (4.28), and (4.80) that

\[
\sup_{0 \leq t \leq T} \sigma^2 \| \nabla^2 \theta_t \|_{L^2} \\
\leq C \sup_{0 \leq t \leq T} \sigma^2 \left( \| \rho \theta_t \|_{L^2} + \| \rho \theta_t \|_{L^3} \| \theta_t \|_{L^3} + \| \rho \theta_t \|_{L^6} \left( \| \nabla \theta \|_{L^3} + \| \theta \|_{L^6} \| \nabla u \|_{L^6} \right) \right) \\
+ C \sup_{0 \leq t \leq T} \sigma^2 \left( \| \rho^{1/2} \theta_t \|_{L^2} + \| \nabla \theta_t \|_{L^2} + (1 + \| \nabla^2 \theta_t \|_{L^2}) \| \nabla u_t \|_{L^2} + \| \nabla u_t \|_{L^6} \right) \\
\leq C.
\]  
(4.81)
Moreover, it follows from the standard $H^2$-estimate to (3.57), (2.9), (4.35), and Lemma 4.1 that
\[
\|\nabla^2 \theta\|_{H^2} \leq C (\|\rho\|_{H^2} + \|\rho \cdot \nabla \theta\|_{H^2} + \|\rho \text{div} u\|_{H^2} + \|u\|_{H^2}^2 )
\]
\[
\leq C (\|\rho\|_{H^2} \|\theta\|_{H^2} + \|\rho\|_{H^2} \|u\|_{H^2} \|\nabla \theta\|_{H^2} )
+ C \|\rho\|_{H^2} \|\theta\|_{H^2} \|\text{div} u\|_{H^2} + C \|\nabla u\|_{H^2}^2 + C
\]
\[
\leq C \sigma^{-1} + C \|\nabla^2 \theta\|_{L^2} + C \|\theta\|_{H^2}.
\]
Combining this with (4.54), (4.81), and (4.80) shows (4.72). The proof of Lemma 4.6 is completed.

5 Proof of Theorems 1.1 and 1.2

With all the a priori estimates in Sections 3 and 4 at hand, we are ready to prove the main results of this paper in this section.

**Proposition 5.1.** For given numbers $M > 0$ (not necessarily small), $\hat{\rho} > 2$, and $\hat{\theta} > 1$, assume that $(\rho_0, u_0, \theta_0)$ satisfies (2.1), (3.4), and (3.8). Then there exists a unique classical solution $(\rho, u, \theta)$ of problem (1.4)–(1.6) in $\Omega \times (0, \infty)$ satisfying (2.3)–(2.5) with $T_0$ replaced by any $T \in (0, \infty)$. Moreover, (3.6), (3.10), (3.105), and (3.106) hold for any $T \in (0, \infty)$ and (3.7) holds for any $t \geq 1$.

**Proof.** First, by the standard local existence result (Lemma 2.1), there exists a $T_0 > 0$ which may depend on $\inf_{x \in \Omega} \rho_0(x)$, such that the problem (1.4)–(1.6) with initial data $(\rho_0, u_0, \theta_0)$ has a unique classical solution $(\rho, u, \theta)$ on $\Omega \times (0, T_0]$ satisfying (2.2)–(2.5) with $T_0$ replaced by any $T \in (0, \infty)$. Moreover, (3.6), (3.10), (3.105), and (3.106) hold for any $T \in (0, \infty)$ and (3.7) holds for any $t \geq 1$. It follows from (3.1)–(3.4) and (3.8) that
\[
A_1(0) \leq M^2, \quad A_2(0) \leq C_0^{1/4}, \quad A_3(0) = 0, \quad \rho_0 < \hat{\rho}, \quad \theta_0 \leq \hat{\theta},
\]
which implies there exists a $T_1 \in (0, T_0]$ such that (3.5) holds for $T = T_1$. We set
\[
T^* = \sup \left\{ T \left| \sup_{t \in [0, T]} \|(\rho, u, \theta)\|_{H^3} < \infty \right. \right\},
\]
and
\[
T_* = \sup \{ T \leq T^* \mid (3.5) \text{ holds} \}. \quad (5.1)
\]
Then $T^* \geq T_* \geq T_1 > 0$. Next, we claim that
\[
T_* = \infty. \quad (5.2)
\]
Otherwise, $T_* < \infty$. Proposition 3.1 shows (3.6) holds for all $0 < T < T_*$, which together with (3.8) yields Lemmas 4.1–4.6 still hold for all $0 < T < T_*$. Note here that all constants $C$ in Lemmas 4.1–4.6 depend on $T_*$ and $\inf_{x \in \Omega} \rho_0(x)$, and are in fact independent of $T$. Then, we claim that there exists a positive constant $\bar{C}$ which may depend on $T_*$ and $\inf_{x \in \Omega} \rho_0(x)$ such that, for all $0 < T < T_*$,
\[
\sup_{0 \leq t \leq T} \|\rho\|_{H^3} \leq \bar{C}, \quad (5.3)
\]
which together with Lemmas 4.1, 4.3, 4.5, (2.5), and (3.4) gives
\[ \| (\rho(x, T_*), u(x, T_*), \theta(x, T_*)) \|_{H^3} \leq \tilde{C}, \quad \inf_{x \in \Omega} \rho(x, T_*) > 0, \quad \inf_{x \in \Omega} \theta(x, T_*) > 0. \]

Thus, Lemma 2.1 implies that there exists some \( T^{**} > T_* \), such that (3.5) holds for \( T = T^{**} \), which contradicts (5.1). Hence, (5.2) holds. This along with Lemmas 2.1, 3.1, 3.8, and Proposition 3.1, thus finishes the proof of Proposition 5.1.

Finally, it remains to prove (5.3). Using (1.43) and (2.2), we can define
\[ \theta_t(\cdot, 0) \triangleq -u_0 \cdot \nabla \theta_0 + \frac{\gamma - 1}{R} \rho_0^{-1} \left( \kappa \Delta \theta_0 - R \rho_0 \theta_0 \text{div} u_0 + \lambda (\text{div} u_0)^2 + 2\mu |\Sigma(\text{u})|^2 \right), \]

which along with (2.1) gives
\[ \| \theta_t(\cdot, 0) \|_{L^2} \leq \tilde{C}. \] (5.4)

Thus, one deduces from (3.109), (2.1), (5.4), and Lemma 4.1 that
\[ \sup_{0 \leq t \leq T} \int_0^T \rho |\theta|^2 dx + \int_0^T \| \nabla \theta \|^2_{L^2} dt \leq \tilde{C}, \] (5.5)

which together with (3.56) and Lemma 4.1 yields
\[ \sup_{0 \leq t \leq T} \| \nabla^2 \theta \|_{L^2} \leq \tilde{C}. \] (5.6)

Using (1.4) and (2.2), we can define
\[ u_t(\cdot, 0) \triangleq -u_0 \cdot \nabla u_0 + \rho_0^{-1} \left( \mu \Delta u_0 + (\mu + \lambda) \text{div} u_0 - R \nabla (\rho_0 \theta) \right), \]

which along with (2.1) gives
\[ \| \nabla u_t(\cdot, 0) \|_{L^2} \leq \tilde{C}. \] (5.7)

Thus, it follows from Lemmas 4.1, 4.2, (4.43), (4.46), (5.5) – (5.7), and Grönwall’s inequality that
\[ \sup_{0 \leq t \leq T} \| \nabla u_t \|_{L^2} + \int_0^T \int_0^T |u_{tt}|^2 dx dt \leq \tilde{C}, \] (5.8)

which as well as (4.22), (5.6), and (4.4) yields
\[ \sup_{0 \leq t \leq T} \| u \|_{H^3} \leq \tilde{C}. \] (5.9)

Combining this with Lemma 4.1, 4.2, 4.3, (4.51), (5.5) – (5.7), and (5.9) gives
\[ \int_0^T \left( \| \nabla^3 \theta \|^2_{L^2} + \| \nabla u_t \|^2_{H^3} \right) dt \leq \tilde{C}. \] (5.10)

Then, applying (2.22), (2.9), (2.5), (5.5), and Lemma 4.1 one has
\[ \| \nabla^2 u \|_{H^2} \leq \tilde{C} \left( \| \rho u \|_{H^2} + \| \nabla P \|_{H^2} + \| \nabla u \|_{L^2} \right) \]
\[ \leq \tilde{C} \left( \| \rho \|_{H^2} \| u \|_{H^2} + \| \rho \|_{H^2} \| u \|_{H^2} \| \nabla u \|_{H^2} \right) \]
\[ + \tilde{C} \left( \| \nabla \rho \|_{H^2} \| \theta \|_{H^2} + \| \rho \|_{H^2} \| \nabla \theta \|_{H^2} + 1 \right) \]
\[ \leq \tilde{C} (1 + \| \nabla^2 u_t \|_{L^2} + \| \nabla^3 \rho \|_{L^2} + \| \nabla^3 \theta \|_{L^2}), \]

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which along with some standard calculations leads to

\[
\left(\|\nabla^3 \rho\|_{L^2}\right)_t \leq \tilde{C} \left(\|\nabla^3 u\|_{L^2} + \|\nabla^2 u\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla^3 \theta\|_{L^2} + \|\nabla^4 u\|_{L^2}\right)
\]

\[
\leq \tilde{C} \left(\|\nabla^3 u\|_{L^2} \|\nabla \rho\|_{H^2} + \|\nabla^2 u\|_{L^6} \|\nabla \nabla^2 \rho\|_{L^2} + \|\nabla u\|_{L^\infty} \|\nabla^3 \rho\|_{L^2} + \|\nabla^4 u\|_{L^2}\right)
\]

\[
\leq \tilde{C}(1 + \|\nabla^3 \rho\|_{L^2} + \|\nabla^2 u\|_{L^2} + \|\nabla^3 \theta\|_{L^2}),
\]

where we have used (5.9) and Lemma 4.1. Combining this with (5.10) and Grönwall’s inequality yields

\[
\sup_{0 \leq t \leq T} \|\nabla^3 \rho\|_{L^2} \leq \tilde{C},
\]

which together with (4.4) gives (5.3). The proof of Proposition 5.1 is completed. \(\square\)

With Proposition 5.1 at hand, we are now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \((\rho_0, u_0, \theta_0)\) satisfying (1.9)–(1.11) be the initial data in Theorem 1.1. Assume that \(C_0\) satisfies (1.12) with

\[
\varepsilon \triangleq \varepsilon_0 / 2,
\]

where \(\varepsilon_0\) is given in Proposition 3.1.

First, we construct the approximate initial data \((\rho_0^m, \eta_0^m, \theta_0^m)\) as follows. For constants

\[
m \in \mathbb{Z}^+, \quad \eta \in (0, \eta_0), \quad \eta_0 \triangleq \min \left\{1, \frac{1}{2} \left(\hat{\rho} - \sup_{x \in \Omega} \rho_0(x)\right)\right\},
\]

we define

\[
\rho_0^m = \rho_0 + \eta, \quad u_0^m = \frac{u_0}{1 + \eta}, \quad \theta_0^m = \frac{\theta_0}{1 + 2\eta},
\]

where \(\rho_0^m\) satisfies

\[
0 \leq \rho_0^m \in C^\infty, \quad \lim_{m \to \infty} \|\rho_0^m - \rho_0\|_{W^{2,q}} = 0,
\]

\(u_0^m\) is the unique smooth solution to the following elliptic equation:

\[
\begin{cases}
\Delta u_0^m = \Delta \bar{u}_0^m, & \text{in } \Omega, \\
u_0^m \cdot n = 0, \quad \text{curl}(u_0^m \times n) = 0, & \text{on } \partial \Omega,
\end{cases}
\]

with \(\bar{u}_0^m \in C^\infty\) satisfying \(\lim_{m \to \infty} \|\bar{u}_0^m - u_0\|_{H^2} = 0\), and \(\theta_0^m\) satisfying \(\int_{\Omega} \theta_0^m dx = \int_{\Omega} \theta_0 dx\) is the unique smooth solution to the following Poisson equation:

\[
\begin{cases}
\Delta \theta_0^m = \Delta \bar{\theta}_0^m - \Delta \bar{\theta}_0^m, & \text{in } \Omega, \\
\nabla \theta_0^m \cdot n = 0, & \text{on } \partial \Omega,
\end{cases}
\]

with \(0 \leq \bar{\theta}_0^m \in C^\infty\) satisfying \(\lim_{m \to \infty} \|\bar{\theta}_0^m - \theta_0\|_{H^2} = 0\).
Then for any \( \eta \in (0, \eta_0) \), there exists \( m_1(\eta) \geq 1 \) such that for \( m \geq m_1(\eta) \), the approximate initial data \((\rho_0^{m,\eta}, u_0^{m,\eta}, \theta_0^{m,\eta})\) satisfies

\[
\begin{aligned}
\left\{ \begin{array}{l}
(\rho_0^{m,\eta}, u_0^{m,\eta}, \theta_0^{m,\eta}) \in C^\infty,
\eta \leq \rho_0^{m,\eta} < \hat{\rho}, \quad \frac{\eta}{4} \leq \theta_0^{m,\eta} \leq \hat{\theta},
\|\nabla u_0^{m,\eta}\|_{L^2} \leq M,
\end{array} \right.
\end{aligned}
\tag{5.13}
\]

and

\[
\lim_{\eta \to 0} \lim_{m \to \infty} (\|\rho_0^{m,\eta} - \rho_0\|_{W^{2,q}} + \|u_0^{m,\eta} - u_0\|_{H^2} + \|\theta_0^{m,\eta} - \theta_0\|_{H^2}) = 0.
\tag{5.14}
\]

Moreover, the initial norm \( C_0^{m,\eta} \) for \((\rho_0^{m,\eta}, u_0^{m,\eta}, \theta_0^{m,\eta})\), which is defined by the right-hand side of (1.5) with \((\rho_0, u_0, \theta_0)\) replaced by \((\rho_0^{m,\eta}, u_0^{m,\eta}, \theta_0^{m,\eta})\), satisfies

\[
\lim_{\eta \to 0} \lim_{m \to \infty} C_0^{m,\eta} = C_0.
\]

Therefore, there exists an \( \eta_1 \in (0, \eta_0) \) such that, for any \( \eta \in (0, \eta_1) \), we can find some \( m_2(\eta) \geq m_1(\eta) \) such that

\[
C_0^{m,\eta} \leq C_0 + \varepsilon_0/2 \leq \varepsilon_0,
\tag{5.15}
\]

provided that

\[
0 < \eta < \eta_1, \quad m \geq m_2(\eta).
\tag{5.16}
\]

We assume that \( m, \eta \) satisfy (5.10), Proposition 5.1 together with (5.15) and (5.18) thus yields that there exists a smooth solution \((\rho^{m,\eta}, u^{m,\eta}, \theta^{m,\eta})\) of problem (1.4)–(1.6) with initial data \((\rho_0^{m,\eta}, u_0^{m,\eta}, \theta_0^{m,\eta})\) on \( \Omega \times (0, T) \) for all \( T > 0 \). Moreover, one has (1.13), (3.6), (3.7), (3.10), (3.105), and (3.106) with \((\rho, u, \theta)\) being replaced by \((\rho^{m,\eta}, u^{m,\eta}, \theta^{m,\eta})\).

Next, for the initial data \((\rho_0^{m,\eta}, u_0^{m,\eta}, \theta_0^{m,\eta})\), the function \( \tilde{g} \) in (4.1) is

\[
\tilde{g} \triangleq (\rho_0^{m,\eta})^{-1/2} (\|\mu\|_{L^\infty} u_0^{m,\eta} - \mu + \lambda) \nabla \text{div} u_0^{m,\eta} + R\nabla (\rho_0^{m,\eta} \theta_0^{m,\eta})
\]

\[
= (\rho_0^{m,\eta})^{-1/2} \sqrt{\rho_0} g + \mu (\rho_0^{m,\eta})^{-1/2} \Delta (u_0^{m,\eta} - u_0^{m,\eta}) + (\mu + \lambda) (\rho_0^{m,\eta})^{-1/2} \nabla \text{div} (u_0^{m,\eta} - u_0^{m,\eta}) + R (\rho_0^{m,\eta})^{-1/2} \nabla (\rho_0^{m,\eta} \theta_0^{m,\eta} - \rho_0 \theta_0),
\tag{5.17}
\]

where in the second equality we have used (1.11). Since \( g \in L^2 \), one deduces from (5.11), (5.13), (5.14), and (1.9) that for any \( \eta \in (0, \eta_1) \), there exist some \( m_3(\eta) \geq m_2(\eta) \) and a positive constant \( C \) independent of \( m \) and \( \eta \) such that

\[
\|\tilde{g}\|_{L^2} \leq \|g\|_{L^2} + C \eta^{-1/2} \delta(m) + C \eta^{1/2},
\tag{5.18}
\]

with \( 0 \leq \delta(m) \to 0 \) as \( m \to \infty \). Hence, for any \( \eta \in (0, \eta_1) \), there exists some \( m_4(\eta) \geq m_3(\eta) \) such that for any \( m \geq m_4(\eta) \),

\[
\delta(m) < \eta.
\tag{5.19}
\]

We thus obtain from (5.18) and (5.19) that there exists some positive constant \( C \) independent of \( m \) and \( \eta \) such that

\[
\|\tilde{g}\|_{L^2} \leq \|g\|_{L^2} + C,
\tag{5.20}
\]

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provided that

\[ 0 < \eta < \eta_1, \ m \geq m_4(\eta). \]  

(5.21)

Now, we assume that \( m, \eta \) satisfy \( (5.21) \). It thus follows from \((5.13)-(5.15), (5.20), (5.21)\), Proposition 3.1 and Lemmas 3.8, 4.3, 4.4 that for any \( T > 0 \), there exists some positive constant \( C \) independent of \( m \) and \( \eta \) such that \((1.13), (3.6), (3.10), (3.105), (3.106), (4.3), (4.4), (4.41), (4.27), (4.38), (4.51), (4.72)\), and \((4.72)\) hold for \((\hat{\rho}^{m,\eta}, \hat{u}^{m,\eta}, \hat{\theta}^{m,\eta})\). Then passing to the limit first \( m \to \infty \), then \( \eta \to 0 \), together with standard arguments yields that there exists a solution \((\rho, u, \theta)\) of the problem \((1.1)-(1.6)\) on \( \Omega \times (0, T) \) for all \( T > 0 \), such that the solution \((\rho, u, \theta)\) satisfies \((1.13), (3.10), (3.105), (3.106), (4.3), (4.4), (4.27), (4.38), (4.51), (4.72)\), and the estimates of \( A_i(T) \) \((i = 1, 2, 3)\) in \((5.6)\). Hence, \((\rho, u, \theta)\) satisfying \((1.13)\) and \((1.14)\) refers to \([14]\) for the detailed proof. Moreover, one deduces from Proposition 3.1 that the desired exponential decay property \((1.15)\).

Finally, the proof of the uniqueness of \((\rho, u, \theta)\) is similar to that of \([7, \text{Theorem 1}]\) and will be omitted here for simplicity. The proof of Theorem 1.1 is completed. \( \square \)

**Proof of Theorem 1.2** We will prove Theorem 1.2 in two steps.

**Step 1. Construction of approximate solutions.** Assume \((\rho_0, u_0, \theta_0)\) satisfying \((1.10)\) and \((1.19)\) is the initial data in Theorem 1.2. and \( C_0 \) satisfies \((1.12)\) with \( \varepsilon \) as in \((5.11)\). For \( j_{m-1}(x) \) being the standard mollifying kernel of width \( m^{-1} \), we construct

\[ \hat{\rho}_0^{m,\eta} = (\rho_01_\Omega) \ast j_{m-1}1_\Omega + \eta, \quad \hat{u}_0^{m,\eta} = \frac{u_0^m}{1+\eta}, \quad \hat{\theta}_0^{m,\eta} = \frac{(\rho_0\theta_01_{\Omega_m}) \ast j_{m-1} + \eta}{(\rho_01_{\Omega_m}) \ast j_{m-1} + \eta}, \]

where \( \Omega_m = \{x \in \Omega | \text{dist}(x, \partial \Omega) > 2/m \} \) and \( u_0^m \) satisfies

\[ u_0^m \in C^\infty \cap H^1_\omega \quad \text{and} \quad \lim_{m \to \infty} ||u_0^m - u_0||_{H^1} = 0. \]

Then for any \( \eta \in (0, \eta_0) \) with \( \eta_0 \) as in \((5.12)\), there exists \( m(\eta) \geq 1 \) such that for \( m \geq m(\eta) \), the approximate initial data \((\hat{\rho}_0^{m,\eta}, \hat{u}_0^{m,\eta}, \hat{\theta}_0^{m,\eta})\) satisfies

\[ \left\{ \begin{array}{l} (\hat{\rho}_0^{m,\eta}, \hat{u}_0^{m,\eta}, \hat{\theta}_0^{m,\eta}) \in C^\infty, \\ \eta \leq \hat{\rho}_0^{m,\eta} < \hat{\rho}, \quad \frac{\eta}{\hat{\rho} + \eta} \leq \hat{\theta}_0^{m,\eta} \leq \hat{\theta}, \quad ||\nabla \hat{u}_0^{m,\eta}||_{L^2} \leq M, \\
\hat{u}_0^{m,\eta} \cdot n = 0, \quad \text{curl} \hat{u}_0^{m,\eta} \times n = 0, \quad \nabla \hat{\theta}_0^{m,\eta} \cdot n = 0, \quad \text{on} \partial \Omega, \end{array} \right. \]  

(5.22)

and for any \( p \geq 1 \),

\[ \lim_{\eta \to 0} \lim_{m \to \infty} \left( ||\hat{\rho}_0^{m,\eta} - \rho_0||_{L^p} + ||\hat{u}_0^{m,\eta} - u_0||_{H^1} + ||\hat{\rho}_0^{m,\eta} \hat{\theta}_0^{m,\eta} - \rho_0\theta_0||_{L^2} \right) = 0 \]  

(5.23)

owing to \((1.10)\) and \((1.12)\).

Now, we claim that the initial norm \( \hat{C}_0^{m,\eta} \) for \((\hat{\rho}_0^{m,\eta}, \hat{u}_0^{m,\eta}, \hat{\theta}_0^{m,\eta})\), i.e., the right hand side of \((1.3)\) with \((\rho_0, u_0, \theta_0)\) replaced by \((\hat{\rho}_0^{m,\eta}, \hat{u}_0^{m,\eta}, \hat{\theta}_0^{m,\eta})\), satisfies

\[ \lim_{\eta \to 0} \lim_{m \to \infty} \hat{C}_0^{m,\eta} \leq C_0, \]  

(5.24)

which leads to that there exists an \( \tilde{\eta} \in (0, \eta_0) \) such that, for any \( \eta \in (0, \tilde{\eta}) \), there exists some \( \tilde{m}(\eta) \geq m(\eta) \) such that

\[ \hat{C}_0^{m,\eta} \leq C_0 + \varepsilon_0/2 \leq \varepsilon_0, \]  

(5.25)
provided
\[ 0 < \eta < \hat{\eta}, \quad m \geq \hat{m}(\eta). \] (5.26)

Then if we assume (5.26) holds, it directly follows from Proposition 5.1, (5.22) and (5.25) that there exists a classical solution \((\hat{\rho}^{m,\eta}, \hat{u}^{m,\eta}, \hat{\theta}^{m,\eta})\) of problem (1.4)–(1.6) with initial data \((\rho_0^{m,\eta}, u_0^{m,\eta}, \theta_0^{m,\eta})\) on \(\Omega \times (0,T)\) for all \(T > 0\). Furthermore, \((\hat{\rho}^{m,\eta}, \hat{u}^{m,\eta}, \hat{\theta}^{m,\eta})\) satisfies (1.13), (3.6), (3.10), (3.11), (3.105), (3.106), and (3.7) respectively for any \(T > 0\) and \(t \geq 1\) with \((\rho, u, \theta)\) replaced by \((\hat{\rho}^{m,\eta}, \hat{u}^{m,\eta}, \hat{\theta}^{m,\eta})\).

It remains to prove (5.24). Indeed, we just need to infer
\[
\lim_{\eta \to 0} \lim_{m \to \infty} \int \rho_0^{m,\eta} (\hat{\theta}_0^{m,\eta} - \log \hat{\theta}_0^{m,\eta} - 1) \, dx \leq \int \rho_0 (\theta_0 - \log \theta_0 - 1) \, dx,
\] (5.27)
since the other terms in (5.24) can be proved in a similar and even simpler way. Note that
\[
\rho_0^{m,\eta} (\hat{\theta}_0^{m,\eta} - \log \hat{\theta}_0^{m,\eta} - 1) = \rho_0^{m,\eta} (\hat{\theta}_0^{m,\eta} - 1)^2 \int_0^1 \frac{\alpha}{\alpha (\hat{\theta}_0^{m,\eta} - 1) + 1} \, d\alpha
\]
\[
= \rho_0^{m,\eta} \left( j_{m-1} \ast (\rho_0 (\theta_0 - 1) 1_{\Omega_m})^2 \right)
\]
\[
\cdot \int_0^{1} \frac{\alpha j_{m-1} \ast (\rho_0 (\theta_0 - 1) 1_{\Omega_m}) + j_{m-1} \ast (\rho_0 1_{\Omega_m})}{\alpha j_{m-1} \ast (\rho_0 (\theta_0 - 1) 1_{\Omega_m}) + \rho_0 1_{\Omega_m} + \eta} \, d\alpha
\]
\[
\in \left[ 0, \rho_0^{-2} (j_{m-1} \ast (\rho_0 (\theta_0 - 1) 1_{\Omega_m}))^2 \right],
\]
which combined with Lebesgue’s dominated convergence theorem yields that
\[
\lim_{m \to \infty} \int \rho_0^{m,\eta} (\hat{\theta}_0^{m,\eta} - \log \hat{\theta}_0^{m,\eta} - 1) \, dx
\]
\[
= \int (\rho_0 + \eta) \left( \frac{\rho_0 \theta_0 + \eta}{\rho_0 + \eta} - \log \frac{\rho_0 \theta_0 + \eta}{\rho_0 + \eta} - 1 \right) \, dx
\]
\[
= \int (\rho_0 \theta_0 - \rho_0 + (\rho_0 + \eta) \log (\rho_0 + \eta)) \, dx
\]
\[
- \int (\rho_0 \log (\rho_0 \theta_0 + \eta) + \eta \log (\rho_0 \theta_0 + \eta)) \, dx
\]
\[
\leq \int (\rho_0 \theta_0 - \rho_0 + (\rho_0 + \eta) \log (\rho_0 + \eta)) \, dx - \int (\rho_0 \log (\rho_0 \theta_0 + \eta) + \eta \log \eta) \, dx
\]
\[
\to \int \rho_0 (\theta_0 - \log \theta_0 - 1) \, dx, \quad \text{as } \eta \to 0.
\]

It thus gives (5.27).

**Step 2. Compactness results.** With the approximate solutions \((\hat{\rho}^{m,\eta}, \hat{u}^{m,\eta}, \hat{\theta}^{m,\eta})\) obtained in the previous step at hand, we can derive the global existence of weak solutions by passing to the limit first \(m \to \infty\), then \(\eta \to 0\). Since the two steps are similar, we will only sketch the arguments for \(m \to \infty\). For any fixed \(\eta \in (0, \hat{\eta})\), we simply denote \((\hat{\rho}^{m,\eta}, \hat{u}^{m,\eta}, \hat{\theta}^{m,\eta})\) by \((\hat{\rho}^{m}, \hat{u}^{m}, \hat{\theta}^{m})\). Then the combination of Aubin-Lions Lemma with (3.6), (3.10), (3.11), (3.105), and Lemma 2.5 yields that there exists some appropriate
subsequence $m_j \to \infty$ of $m \to \infty$ such that, for any $0 < \tau < T < \infty$, $p \in [1, \infty)$, and $\tilde{p} \in [1, 6)$,

\begin{equation}
    u^{m_j} \rightharpoonup u \text{ weakly star in } L^\infty(0, T; H^1),
\end{equation}

\begin{equation}
    \theta^{m_j} \rightharpoonup \theta \text{ weakly in } L^2(0, T; H^1),
\end{equation}

\begin{equation}
    \rho^{m_j} \to \rho \text{ in } C([0, T]; L^p\text{-weak}) \cap C([0, T]; H^{-1}),
\end{equation}

\begin{equation}
    \rho^{m_j} u^{m_j} \rightharpoonup \rho u, \rho^{m_j} \theta^{m_j} \rightharpoonup \rho \theta \text{ in } C([0, T]; L^2\text{-weak}) \cap C([0, T]; H^{-1}),
\end{equation}

\begin{equation}
    G^{m_j} \to G, \text{ curl } u^{m_j} \rightharpoonup \text{ curl } u \text{ in } C([\tau, T]; H^1\text{-weak}) \cap C([\tau, T]; L^{\tilde{p}}),
\end{equation}

\begin{equation}
    u^{m_j} \to u \text{ in } C([\tau, T]; W^{1,6}\text{-weak}) \cap C(\Omega \times [\tau, T]),
\end{equation}

and

\begin{equation}
    \theta^{m_j} \to \theta \text{ in } C([\tau, T]; H^2\text{-weak}) \cap C([\tau, T]; W^{1,\tilde{p}}),
\end{equation}

referring to [14] for the detailed proof. Now we consider the approximate solutions $(\rho^{m_j}, u^{m_j}, \theta^{m_j})$ in the weak forms, i.e. (1.16)–(1.18), then take appropriate limits. Standard arguments as well as (5.23) and (5.28)–(5.35) thus conclude that the limit $(\rho, u, \theta)$ is a weak solution of (1.1)–(1.5) in the sense of Definition 1.1 and satisfies (1.20)–(1.23) and the exponential decay property (1.15). Moreover, we obtain the estimates (1.24)–(1.26) with the aid of (3.6), (3.10), (3.106), and (5.28)–(5.35). Finally, (1.27) shall be obtained by adopting the same way as in [14]. The proof of Theorem 1.2 is finished.

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**References**

[1] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II. *Commun. Pure Appl. Math.* 17(1)(1964), 35–92.

[2] S. N. Antontsev, A. V. Kazhikhov, V. N. Monakhov, *Boundary value problems in mechanics of nonhomogeneous fluids.* North-Holland Publishing Co., Amsterdam, 1990.

[3] J. Aramaki, *$L^p$ theory for the div-curl system.* *Int. J. Math. Anal.* 8(6)(2014), 259–271.

[4] J.T. Beale, T. Kato, A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Commun. Math. Phys.* 94(1984), 61–66.
[5] D. Bresch, B. Desjardins, On the existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids. *J. Math. Pures Appl.* **87**(9)(2007), 57–90.

[6] G.C. Cai, J. Li, Existence and exponential growth of global classical solutions to the compressible Navier-Stokes equations with slip boundary conditions in 3D bounded domains. *Indiana Univ. Math. J.* in press.

[7] Y. Cho, H. Kim, Existence results for viscous polytropic fluids with vacuum. *J. Differ. Eqs.* **228**(2006), 377–411.

[8] E. Feireisl, On the motion of a viscous, compressible, and heat conducting fluid. *Indiana Univ. Math. J.* **53**(2004), 1707–1740.

[9] E. Feireisl, *Dynamics of Viscous Compressible Fluids*. Oxford Science Publication, Oxford, 2004.

[10] E. Feireisl, A. Novotny, H. Petzeltová, On the existence of globally defined weak solutions to the Navier-Stokes equations. *J. Math. Fluid Mech.* **3**(2001), 358–392.

[11] G.P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Steady-State Problems. Second Edition*. Springer, New York, 2011.

[12] D. Hoff, Discontinuous solutions of the Navier-Stokes equations for multidimensional flows of heat-conducting fluids. *Arch. Rational Mech. Anal.* **139**(1997), 303–354.

[13] X.D. Huang, J. Li, Serrin-type blowup criterion for viscous, compressible, and heat conducting Navier-Stokes and magnetohydrodynamic flows. *Commun. Math. Phys.* **324**(2013), 147–171.

[14] X.D. Huang, J. Li, Global classical and weak solutions to the three-dimensional full compressible Navier-Stokes system with vacuum and large oscillations. *Arch. Rational Mech. Anal.* **227**(2018), 995–1059.

[15] X.D. Huang, J. Li, Z.P. Xin, Serrin type criterion for the three-dimensional compressible flows. *SIAM J. Math. Anal.* **43**(4)(2011), 1872–1886.

[16] X.D. Huang, J. Li, Z.P. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations. *Comm. Pure Appl. Math.* **65**(4)(2012), 549–585.

[17] T. Kato, Remarks on the Euler and Navier-Stokes equations in $\mathbb{R}^2$. *Proc. Symp. Pure Math. Amer. Math. Soc. Providence*. **45**(1986), 1–7.

[18] A. V. Kazhikhov, Cauchy problem for viscous gas equations. *Siberian Math. J.* **23**(1982), 44–49.

[19] A. V. Kazhikhov, V. V. Shelukhin, Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas. *J. Appl. Math. Mech.* **41**(1977), 273–282.
[20] S.H. Lai, H. Xu, J.W. Zhang, Well-posedness and exponential decay for the Navier-Stokes equations of viscous compressible heat-conductive fluids with vacuum. arxiv: 2103.16332.

[21] J. Li, Z.P. Xin, Global well-posedness and large time asymptotic behavior of classical solutions to the compressible Navier-Stokes equations with vacuum. *Annals of PDE*. 5(2019), 7.

[22] P. L. Lions, *Mathematical topics in fluid mechanics. Compressible models. Vol. 2*. Oxford University Press, New York, 1998.

[23] A. Matsumura, T. Nishida, The initial value problem for the equations of motion of viscous and heat-conductive gases. *J. Math. Kyoto Univ.* 20(1980), 67–104.

[24] J. Nash, Le problème de Cauchy pour les équations différentielles d’un fluide général. *Bull. Soc. Math. France*. 90 (1962), 487–497.

[25] C. L. M. H. Navier, Sur les lois de léquilibre et du mouvement des corps élastiques. *Mem. Acad. R. Sci. Inst. France*. 6 (1827), 369.

[26] L. Nirenberg, On elliptic partial differential equations. *Ann. Scuola Norm. Sup. Pisa*. 13(3)(1959), 115–162.

[27] O. Rozanova, Blow up of smooth solutions to the compressible Navier-Stokes equations with the data highly decreasing at infinity. *J. Differ. Eqs*. 245 (2008), 1762–1774.

[28] W. Rudin, *Real and complex analysis. Third Edition*. McGraw-Hill Book Company, New York, 1987.

[29] J. Serrin, On the uniqueness of compressible fluid motion. *Arch. Rational Mech. Anal.* 3 (1959), 271–288.

[30] A. Tani, On the first initial-boundary value problem of compressible viscous fluid motion. *Publ. Res. Inst. Math. Sci. Kyoto Univ.* 13(1977), 193–253.

[31] W. von Wahl, Estimating $\nabla u$ by $\text{div} u$ and $\text{curl} u$. *Math. Meth. Appl. Sci.* 15(1992), 123–143.

[32] H.Y. Wen, C.J. Zhu, Global solutions to the three-dimensional full compressible Navier-Stokes equations with vacuum at infinity in some classes of large data. *SIAM J. Math. Anal*. 49(2017), 162–221.

[33] H.Y. Wen, C.J. Zhu, Global classical large solutions to Navier-Stokes equations for viscous compressible and heat-conducting fluids with vacuum. *SIAM J. Math. Anal*. 45(2013), 431–468.

[34] H.Y. Wen, C.J. Zhu, Global symmetric classical solutions of the full compressible Navier-Stokes equations with vacuum and large initial data. *J. Math. Pures Appl*. 102(2014), 498–545.

[35] Z. P. Xin, Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density. *Comm. Pure Appl. Math*. 51(1998), 229–240.