RISK-SENSITIVE CONTROL AND AN ABSTRACT COLLATZ–WEILANDT FORMULA

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Abstract. The 'value' of infinite horizon risk-sensitive control is the principal eigenvalue of a certain positive operator. This facilitates the use of Chang's extension of the Collatz–Weilandt formula to derive a variational characterization thereof. For the uncontrolled case, this reduces to the Donsker–Varadhan functional.

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1. INTRODUCTION

We consider the infinite horizon risk-sensitive control problem for a controlled reflected diffusion in a bounded domain. This seeks to minimize the asymptotic growth rate of the expected 'exponential of integral' cost, which in turn coincides with the principal eigenvalue of a quasi-linear elliptic operator defined as the pointwise envelope of a family of linear elliptic operators parametrized by the 'control' parameter. The Kreǐn-Rutman theorem has been widely applied to study the time-asymptotic behavior of linear parabolic equations [15, Chapter 7]. A recent extension of the Kreǐn-Rutman theorem to positively 1-homogeneous compact (nonlinear) operators and the ensuing variational formulation for the principal eigenvalue extends the classical Collatz–Weilandt formula for the Perron-Frobenius eigenvalue of irreducible non-negative matrices. Using this, we are able to obtain a variational formulation for the principal eigenvalue that reduces to the celebrated Donsker–Varadhan characterization thereof in the linear case. This establishes interesting connections between theory of risk-sensitive control, nonlinear Kreǐn-Rutman theorem, and Donsker–Varadhan theory.

2. RISK-SENSITIVE CONTROL

Let $Q \subset \mathbb{R}^d$ be an open bounded domain with a $C^3$ boundary $\partial Q$ and $\bar{Q}$ denote its closure. Consider a reflected controlled diffusion $X(\cdot)$ taking values in the bounded domain

...
\( Q \) satisfying
\[
dX(t) = b(X(t), v(t)) \, dt + \sigma(X(t)) \, dW(t) - \gamma(X(t)) \, d\xi(t),
\]
(2.1)
\[
d\xi(t) = I\{X(t) \in \partial Q\} \, d\xi(t)
\]
for \( t \geq 0 \), with \( X(0) = x \) and \( \xi(0) = 0 \). Here:
(a) \( b : Q \times V \to \mathbb{R}^d \) for a prescribed compact metric control space \( V \) is continuous and Lipschitz in its first argument uniformly with respect to the second,
(b) \( \sigma : Q \to \mathbb{R}^{d \times d} \) is continuously differentiable, its derivatives are Hölder continuous with exponent \( \beta_0 > 0 \), and is uniformly non-degenerate in the sense that the minimum eigenvalue of
\[
a(x) = [\lfloor a_{ij}(x)\rfloor] := \sigma(x)\sigma^T(x)
\]
is bounded away from zero.
(c) \( \gamma : \mathbb{R}^d \to \mathbb{R}^d \) is co-normal, i.e., \( \gamma(x) = [\gamma_1(x), \ldots, \gamma_d(x)]^T \), where
\[
\gamma_i(x) = \sum_{i=1}^d a_{ij}(x) \cos (\langle n(x), e_j \rangle),
\]
\( n(x) \) is the unit outward normal, and \( e_j \) is the \( j \)th unit coordinate vector,
(d) \( W(\cdot) \) is a \( d \)-dimensional standard Wiener process,
(e) \( v(\cdot) \) is a \( V \)-valued measurable process satisfying the non-anticipativity condition:
for \( t > s \geq 0 \), \( W(t) - W(s) \) is independent of \( \{v(y), W(y) : y \leq s\} \). A process \( v \) satisfying this property is called an ‘admissible control’.

Let \( r : Q \times V \to \mathbb{R} \) be a continuous ‘running cost’ function which is Lipschitz in its first argument uniformly with respect to the second. We define
\[
r_{\max} := \max_{(x,v) \in Q \times V} |r(x,v)|.
\]
The infinite horizon risk-sensitive problem aims to minimize the cost
\[
\limsup_{T \uparrow \infty} \frac{1}{T} \log E \left[ e^{\int_0^T r(X(s), v(s)) \, ds} \right],
\]
(2.2)
i.e., the asymptotic growth rate of the exponential of the total cost. See [16] for background and motivation.

The notation used in the paper is summarized below.

**Notation 2.1.** The standard Euclidean norm in \( \mathbb{R}^d \) is denoted by \(| \cdot |\). The set of nonnegative real numbers is denoted by \( \mathbb{R}_+ \) and \( \mathbb{N} \) stands for the set of natural numbers. The closure, the boundary and the complement of a set \( A \subset \mathbb{R}^d \) are denoted by \( \overline{A} \), \( \partial A \) and \( A^c \), respectively.

We adopt the notation \( \partial_t := \frac{\partial}{\partial t} \), and for \( i, j \in \mathbb{N} \), \( \partial_i := \frac{\partial}{\partial x_i} \) and \( \partial_{ij} := \frac{\partial^2}{\partial x_i \partial x_j} \). For a nonnegative multi-index \( \alpha = (\alpha_1, \ldots, \alpha_d) \) we let \( D^\alpha := \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} \) and \( |\alpha| := \alpha_1 + \cdots + \alpha_d \). For a domain \( Q \) in \( \mathbb{R}^d \) and \( k = 0, 1, 2, \ldots \), we denote by \( C^k(Q) \) the set of functions \( f : Q \to \mathbb{R} \) whose derivatives \( D^\alpha f \) for \( |\alpha| \leq k \) are continuous and bounded. For \( k = 0, 1, 2, \ldots \), we define
\[
[f]_{k;Q} := \max_{|\alpha| = k} \sup_Q |D^\alpha f| \quad \text{and} \quad \|f\|_{k;Q} := \sum_{j=0}^k [f]_{k;Q}.
\]
Also for \( \delta \in (0, 1) \) we define

\[
[g]_{\delta; Q} := \sup_{x, y \in Q, x \neq y} \frac{|g(x) - g(y)|}{|x - y|^{\delta}} \quad \text{and} \quad ||f||_{k+\delta; Q} := ||f||_{k; Q} + \max_{|\alpha|=k} |D^\alpha f|_{\delta; Q}.
\]

For \( k = 0, 1, 2, \ldots \), and \( \delta \in (0, 1) \) we denote by \( C^{k+\delta}(Q) \) the space of all real-valued functions \( f \) defined on \( Q \) such that \( ||f||_{k+\delta; Q} < \infty \). Unless indicated otherwise, we always view \( C^{k+\delta}(Q) \) and \( C^k(Q) \) as topological spaces under the norms \( \| \cdot \|_{k+\delta; Q} \) and \( \| \cdot \|_{k; Q} \) respectively. We also write \( C^{k+\delta}(Q) \) and \( C^k(Q) \) if the derivatives up to order \( k \) are continuous on \( Q \). Thus \( C^\delta(Q) \) stands for the Banach space of real-valued functions defined on \( Q \) that are Hölder continuous with exponent \( \delta \in (0, 1) \).

Let \( G \) be a domain in \( \mathbb{R}_+ \times \mathbb{R}^d \). Recall that \( C^{1,2}(G) \) stands for the set of bounded continuous real-valued functions \( \varphi(t, x) \) defined on \( G \) such that the derivatives \( D^\alpha \varphi \), \( |\alpha| \leq 2 \) and \( \partial_t \varphi \) are bounded and continuous in \( G \). Let \( \delta \in (0, 1) \). We define

\[
[\varphi]_{s/2, \delta; G} := \sup_{(t, x) \neq (s, y)} \frac{|\varphi(t, x) - \varphi(s, y)|}{|t - s|^{s/2}}.
\]

\[
||\varphi||_{s/2, \delta; G} := ||\varphi||_{0; G} + [\varphi]_{s/2, \delta; G}.
\]

By \( C^{s/2, \delta}(G) \) we denote the space of functions \( \varphi \) such that \( ||\varphi||_{s/2, \delta; G} < \infty \). The parabolic Hölder space \( C^{1+s/2, 2+\delta}(G) \) is the set of all real-valued functions defined on \( G \) for which

\[
||\varphi||_{1+s/2, 2+\delta; G} := \max_{|\alpha| \leq 2} ||D^\alpha \varphi||_{s/2, \delta; G} + ||\partial_t \varphi||_{s/2, \delta; G}
\]

is finite. It is well known that \( C^{1+s/2, 2+\delta}(G) \) equipped with the norm \( ||\varphi||_{1+s/2, 2+\delta; G} \) is a Banach space.

For a Banach space \( \mathcal{Y} \) of continuous functions on \( Q \) we denote by \( \mathcal{Y}_+ \) its positive cone and by \( \mathcal{Y}_\gamma \) the subspace of \( \mathcal{Y} \) consisting of the functions \( f \) satisfying \( \nabla f \cdot \gamma = 0 \) on \( \partial Q \). For example

\[
C^2_{\gamma, +}(Q) := \{ f \in C^2(Q) : f \geq 0, \nabla f \cdot \gamma = 0 \text{ on } \partial Q \}.
\]

Also let \( \mathcal{Y}^* \) denote the dual of \( \mathcal{Y} \) and \( \mathcal{Y}^*_\gamma \) the dual cone of \( \mathcal{Y}_\gamma \). For example \((C^2_\gamma(Q))^*_\gamma\) is defined by

\[
(C^2_\gamma(Q))^*_\gamma := \left\{ \Lambda \in (C^2_\gamma(Q))^* : \Lambda(f) \geq 0 \quad \forall f \in C^2_\gamma(Q) \right\}.
\]

We define the operator \( L_v \) on \( C^2(Q) \) by

\[
L_v f(\cdot) := \frac{1}{2} \text{tr} \left( a(\cdot) \nabla^2 f(\cdot) \right) + \langle b(\cdot, v), \nabla f(\cdot) \rangle, \quad v \in \mathcal{V},
\]

where \( \nabla^2 \) denotes the Hessian.

3. The Nisio semigroup

Associated with the above control problem, define for each \( t \geq 0 \) the operator \( S_t : C(Q) \to C(Q) \) by

\[
S_t f(x) := \inf_{v(\cdot)} E_x \left[ e^{\int_0^t r(X(s), v(s)) ds} f(X(t)) \right],\]

where \( \mathcal{V} \) is the set of control functions.
Theorem 3.1. Let \( \psi \) be a solution of the parabolic p.d.e. \( \mathcal{G} \), where \( \mathcal{G} f(x) := \frac{1}{2} \text{tr} (a(x) \nabla^2 f(x)) + \mathcal{H}(x, f(x), \nabla f(x)) \), with \( \mathcal{H}(x, f, p) := \min_{v \in V} \left[ b(x,v),p \right] + r(x,v)f \).

We can say more by invoking p.d.e. theory. We start with the following theorem that characterizes \( S_t \) as the solution of a parabolic p.d.e.

Theorem 3.2. For each \( f \in C^{2+\delta}_\gamma(\bar{Q}), \delta \in (0,\beta_0), \) and \( T > 0 \), the quasi-linear parabolic p.d.e.

\[
\frac{\partial}{\partial t} \psi(t,x) = \inf_{v \in V} \left( \mathcal{L}_v \psi(t,x) + r(x,v)\psi(t,x) \right) \quad \text{in } (0,T] \times \bar{Q},
\]

with \( \psi(0,x) = f(x) \) for all \( x \in \bar{Q} \) and \( (\nabla \psi(t,x), \gamma(x)) = 0 \) for all \( (t,x) \in (0,T] \times \partial Q \), has a unique solution in \( C^{1+\delta/2,2+\delta}(\bar{Q}_T) \). The solution \( \psi \) has the stochastic representation

\[
\psi(t,x) = \inf_{v(t)} E_x \left[ e^{\int_0^T r(X(s),v(s)) \, ds} f(X(T)) \right] \quad \forall (t,x) \in [0,T] \times \bar{Q}.
\]

Moreover,

\[
\|\psi\|_{1,2,\infty;[0,T] \times \bar{Q}} \leq K_1,
\]

\[
\|\nabla^2 \psi(s,\cdot)\|_{\delta;\bar{Q}} \leq K_2 \quad \text{for all } s \in [0,T],
\]

where the constants \( K_1, K_2 > 0 \) depend only on \( T,\|a\|_{1+\beta_0,Q}, \) the Lipschitz constants of \( b, r, \) the lower bound on the eigenvalues of \( a, \) the boundary \( \partial Q \) and \( \|f\|_{2+\delta,Q} \).

Proof. This follows by \cite{11} Theorem 7.4, p. 491] and \cite{11} Theorem 7.2, pp. 486–487. \( \square \)

Lemma 3.1. Let \( \delta \in (0,\beta_0) \). For each \( t > 0 \), the map \( S_t : C^{2+\delta}_{\gamma}(\bar{Q}) \to C^{2+\delta}_{\gamma}(\bar{Q}) \) is compact.
Proof. Suppose \( f \in C^{2+\delta}_\gamma(\bar{Q}) \) for some \( \delta \in (0, \beta_0) \). Fix any \( T > 0 \). Let \( g : [0, \infty) \rightarrow [0, \infty) \) be a smooth function such that \( g(0) = 0 \) and \( g(s) = 1 \) for \( s \in [T/2, \infty) \). Define \( \bar{\psi}(t, x) = g(t)\psi(t, x) \), with \( \psi \) as in Theorem 3.2. Then \( \psi \) satisfies

\[
(3.6) \quad \frac{\partial}{\partial t} \psi(t, x) - \frac{1}{2} \text{tr} \left( a(x)\nabla^2 \psi(t, x) \right) = \frac{\partial g}{\partial t}(t)\psi(t, x) + g(t)\mathcal{H}(x, \psi(t, x), \nabla \psi(t, x))
\]

in \((0, \infty) \times Q, \bar{\psi}(0, x) = 0 \) on \( Q \) and \( \langle \nabla \bar{\psi}(t, x), \gamma(x) \rangle = 0 \) for all \((t, x) \in (0, \infty) \times \partial Q \). It is well known that \( \partial/\partial s \) is a bounded operator from \( C^{1+\delta/2, 2+\delta}(\bar{Q}_T) \) to \( C^{(1+\delta)/2, 1+\delta}(\bar{Q}_T) \) \([10]\) p. 126]. In particular

\[
\sup_{x \in Q} \sup_{s \neq t} \left| \frac{\partial_s \psi(s, x) - \partial_x \psi(t, x)}{|s-t|^{(1+\delta)/2}} \right| < \infty.
\]

Since \( \mathcal{H} \) is Lipschitz in its arguments and \( g \) is smooth it follows that the r.h.s. of \((3.6)\) is in \( C^{\beta/2, \beta}(\bar{Q}_T) \) for any \( \beta \in (0, 1) \). Then it follows by the interior estimates in \([11]\) Theorem 10.1, pp. 351-352] that \( \bar{\psi} \in C^{1+\beta/2, 2+\beta}((T, T+1) \times Q) \) for all \( \beta \in (0, \beta_0) \). Since \( \psi = \bar{\psi} \) on \([T, T+1] \) it follows that \( \delta_T f \in C^{2+\beta}(\bar{Q}) \) for all \( \beta \in (0, \beta_0) \). Since the inclusion \( C^{2+\beta}(\bar{Q}) \hookrightarrow C^{2+\delta}(\bar{Q}) \) is compact for \( \beta > \delta \), the result follows. \( \square \)

4. AN ABSTRACT COLLATZ–WEILANDT FORMULA

The classical Collatz–Weilandt formula (see \([5, 17]\)) characterizes the principal (i.e., the Perron-Frobenius) eigenvalue \( \kappa \) of an irreducible non-negative matrix \( Q \) as (see \([13]\), Chapter 8])

\[
\kappa = \max_{\{x=(x_1, \ldots, x_d) : x_i \geq 0\}} \min_{\{i : x_i > 0\}} \left( \frac{(Qx)_i}{x_i} \right) = \min_{\{x=(x_1, \ldots, x_d) : x_i > 0\}} \max_{\{i : x_i > 0\}} \left( \frac{(Qx)_i}{x_i} \right).
\]

An infinite dimensional version of this was recently given by Chang \([4]\) as follows. Let \( X \) be a real Banach space with order cone \( P \), i.e., a nontrivial closed subset of \( X \) satisfying

(a) \( tP \subset P \) for all \( t \geq 0 \), where \( tP = \{tx : x \in P\} \);
(b) \( P + P \subset P \);
(c) \( P \cap (-P) = \{\theta\} \), where \( \theta \) denotes the zero vector of \( X \) and \( -P = \{-x : x \in P\} \).

Let \( \hat{P} := P\setminus\{\theta\} \). Write \( x \preceq y \) if \( y - x \in P \). Define the dual cone

\[
P^* := \{x \in X^* : \langle x^*, x \rangle \geq 0 \quad \forall x \in P\}.
\]

A map \( T : X \rightarrow X \) is said to be increasing if \( x \preceq y \Rightarrow T(x) \preceq T(y) \), and strictly increasing if \( x < y \Rightarrow T(x) < T(y) \). If \( \text{int}(P) \neq \emptyset \), and \( T : \hat{P} \rightarrow \text{int}(P) \), then \( T \) is called strongly positive, and if \( x < y \Rightarrow T(y) - T(x) \in \text{int}(P) \) it is called strongly increasing. It is called positively 1-homogeneous if \( T(tx) = tT(x) \) for all \( t > 0 \) and \( x \in X \). Also, a map \( T : X \rightarrow X \) is called completely continuous if it is continuous and compact. The following generalization of the Krein-Rutman theorem is proved in \([12]\):

**Theorem 4.1.** Let \( T : X \rightarrow X \) be an increasing, positively 1-homogeneous, completely continuous map such that for some \( u \in P \) and \( M > 0 \), \( MT(u) \succeq u \). Then there exist \( \lambda > 0 \)
and \( \hat{x} \in \hat{P} \) such that \( T(\hat{x}) = \lambda \hat{x} \). Moreover, if \( T \) is strongly positive then \( \lambda \) is the unique eigenvalue with an eigenvector in \( P \).

Remark 4.1. The conclusion that the eigenvalue described in Theorem 4.1 is geometrically simple and is the principal eigenvalue as claimed in [12, Theorem 2] and [4, Theorems 1.4, 4.8, and 4.13] is not correct. For a detailed discussion on this see the forthcoming paper [1].

The following is proved in [4]:

**Theorem 4.2.** Let \( T \) and \( \lambda \) be as in the preceding theorem. Define:

\[
P^*(x) := \{ x^* \in P^* : \langle x^*, x \rangle > 0 \},
\]

\[
r_*(T) := \sup_{x \in \hat{P}} \inf_{x^* \in P^*(x)} \frac{\langle x^*, T(x) \rangle}{\langle x^*, x \rangle},
\]

\[
r^*(T) := \inf_{x \in \hat{P}} \sup_{x^* \in P^*(x)} \frac{\langle x^*, T(x) \rangle}{\langle x^*, x \rangle}.
\]

If \( T \) is strongly positive and strictly increasing, then \( \lambda = r^*(T) = r_*(T) \).

Uniqueness of the positive eigenvector can be obtained under additional assumptions. In this paper we are concerned with superadditive operators \( T \), in other words operators \( T \) which satisfy

\[
T(x + y) \succeq T(x) + T(y) \quad \forall x, y \in \mathcal{X}.
\]

We have the following simple assertion:

**Corollary 4.1.** Let \( T : \mathcal{X} \to \mathcal{X} \) be a superadditive, positively 1-homogeneous, strongly positive, completely continuous map. Then there exist a unique \( \lambda > 0 \) and a unique \( \hat{x} \in P \) with \( \| \hat{x} \| = 1 \), where \( \| \cdot \| \) denotes the norm in \( \mathcal{X} \), such that \( T(\hat{x}) = \lambda \hat{x} \).

Proof. It is clear that strong positivity implies that for any \( x \in \mathcal{X} \) there exists \( M > 0 \) such that \( MT(x) \succeq x \). By superadditivity \( T(x - y) \succeq T(x) - T(y) \). Hence if \( x > y \), by strong positivity we obtain \( T(x) - T(y) \in \text{int}(P) \). Therefore every superadditive, strongly positive map is strongly increasing. Existence of a unique eigenvalue with an eigenvector in \( P \) then follows by Theorem 4.1. Suppose \( \hat{x} \) and \( \hat{y} \) are two distinct unit eigenvectors in \( P \). Since, by strong positivity \( \hat{x} \) and \( \hat{y} \) are in \( \text{int}(P) \) there exists \( \alpha > 0 \) such that \( \hat{x} - \alpha \hat{y} \in \hat{P} \setminus \text{int}(P) \). Since \( T \) is strongly increasing we obtain

\[
\lambda(\hat{x} - \alpha \hat{y}) = T(\hat{x}) - T(\alpha \hat{y}) \in \text{int}(P),
\]

a contradiction. Uniqueness of a unit eigenvector in \( P \) follows. \( \Box \)

An application of Theorem 4.1 and Corollary 4.1 provides us with the following result for strongly continuous semigroups of operators.

**Corollary 4.2.** Let \( \mathcal{X} \) be a Banach space with order cone \( P \) having non-empty interior. Let \( \{ S_t, t \geq 0 \} \) be a strongly continuous semigroup of superadditive, strongly positive, positively 1-homogeneous, completely continuous operators on \( \mathcal{X} \). Then there exists a unique \( \rho \in \mathbb{R} \) and a unique \( \hat{x} \in \text{int}(P) \), with \( \| \hat{x} \| = 1 \), such that \( S_t \hat{x} = e^{\rho t} \hat{x} \) for all \( t \geq 0 \).
**Proof.** By Theorem 4.1 and Corollary 4.1 there exists a unique \( \lambda(t) > 0 \) and a unique \( x_t \in P \) such that \( \|x_t\| = 1 \), such that \( S_t x_t = \lambda(t) x_t \). By the uniqueness of a unit eigenvector in \( P \) and the semigroup property it follows that there exists \( \hat{x} \in X \) such that \( x_t = \hat{x} \) for all dyadic rational numbers \( t > 0 \). On the other hand, from the strong continuity it follows that if a sequence of dyadic rationals \( t_n \geq 0, n \geq 1 \), converges to some \( t > 0 \), then \( \lambda(t_n) \) is a Cauchy sequence and its limit point \( \lambda \) is an eigenvalue of \( S_t \) corresponding to the eigenvector \( \hat{x} \) and therefore \( \lambda(t) = \lambda \) and \( x_t = \hat{x} \) by the uniqueness thereof. Strong continuity then implies that \( \lambda(\cdot) \) is continuous and by the semigroup property and positive 1-homogeneity we have \( \lambda(t+s) = \lambda(t) \lambda(s) \) for all \( t, s > 0 \). It follows that \( \lambda(t) = e^{\rho t} \) for some \( \rho \in \mathbb{R} \). 

Concerning the time-asymptotic behavior of \( S_t x \) we have the following.

**Theorem 4.3.** Let \( X, \{S_t\}, \rho \) and \( \hat{x} \) be as in Corollary 4.2. Then

(i) The set
\[
0_1 := \{e^{-\rho t} S_t x : x \in P, \|x\| \leq 1, \ t \geq 1\}
\]
is relatively compact in \( X \).

(ii) There exists \( \alpha^*(x) \in \mathbb{R}_+ \) such that
\[
\lim_{t \to \infty} \|e^{-\rho t} S_t x - \alpha^*(x) \hat{x}\| \xrightarrow{t \to \infty} 0 \quad \forall x \in \hat{P}.
\]

(iii) Suppose that additionally the following properties hold:

(P1) For every \( M > 0 \) there exist \( \tau (0,1) \) and a positive constant \( \zeta_0 = \zeta_0(M) \) such that
\[
\|S_t (\hat{x} - z)\| + \|S_t z\| \geq \zeta_0
\]
for all \( z \in P \) such that \( z \leq \hat{x} \) and \( \|z\| \leq M \).

(P2) For every compact set \( K \subset P \) there exists a constant \( \zeta_1 = \zeta_1(K) \) such that
\[
x \in K \text{ and } x \leq \alpha \hat{x} \implies \|x\| \leq \alpha \zeta_1.
\]

Then the convergence is exponential: there exists \( M_0 > 0 \) and \( \theta_0 > 0 \) such that
\[
\|e^{-\rho t} S_t x - \alpha^*(x) \hat{x}\| \leq M_0 e^{-\theta_0 t} \|x\| \quad \text{for all } t \geq 0 \quad \text{and all } x \in \hat{P}.
\]

**Proof.** Without loss of generality we can assume \( \rho = 0 \). For \( t \geq 0 \) and \( x \in P \) we define
\[
\underline{\alpha}(x) := \sup \{a \in \mathbb{R} : x - a \hat{x} \in P\},
\]
\[
\overline{\alpha}(x) := \inf \{a \in \mathbb{R} : a \hat{x} - x \in P\}.
\]

Since \( \hat{x} \in \text{int}(P) \) it follows that \( \underline{\alpha}(x) \) and \( \overline{\alpha}(x) \) are finite and \( \overline{\alpha}(x) \geq \underline{\alpha}(x) \geq 0 \). Note also that for \( x \in \hat{P} \) we have \( \overline{\alpha}(x) > 0 \) and since \( S_t x \in \text{int}(P) \) we have \( \underline{\alpha}(S_t x) > 0 \) for all \( t > 0 \). It is also evident from the definition that
\[
\lambda(\overline{\alpha}(x)) = \lambda \underline{\alpha}(x) \quad \text{and} \quad \overline{\alpha}(\lambda x) = \lambda \overline{\alpha}(x) \quad \text{for all } x \in \hat{P}, \lambda \in \mathbb{R}_+.
\]

By the increasing property and the positive 1-homogeneity of \( S_t \) we obtain \( S_{t+s} x - \underline{\alpha}(S_s x) \hat{x} \in P \) for all \( x \in P \) and \( t \geq 0 \) and this implies that \( \underline{\alpha}(S_{t+s} x) \geq \underline{\alpha}(S_s x) \) for all \( t \geq 0 \) and \( x \in P \). It follows that for any \( x \in P \) the map \( t \mapsto \underline{\alpha}(S_t x) \) is non-decreasing. Similarly, the map \( t \mapsto \overline{\alpha}(S_t x) \) is non-increasing.

We next show that the orbit \( 0 \) of the unit ball in \( P \) defined by
\[
0 := \{S_t x : x \in P, \|x\| \leq 1, \ t \geq 0\}
\]

is bounded. Suppose not. Then we can select a sequence \( \{x_n\} \subset \hat{P} \) with \( \|x_n\| = 1 \), and an increasing sequence \( \{t_n, \ n \in \mathbb{N}\} \) such that \( \|S_{t_n}x_n\| \to \infty \) as \( n \to \infty \) and such that \( \|S_{t_n}x_n\| \geq \|S_t x_n\| \) for all \( t \leq t_n \). By the properties of the sequence \( \{S_{t_n}\} \) the sequence \( \{\frac{S_{t_n - 1}x_n}{\|S_{t_n}x_n\|}\} \) is bounded and this implies that \( \{\frac{S_{t_n - 1}x_n}{\|S_{t_n}x_n\|}\} \) is relatively compact. Let \( y \in \mathcal{X} \) be any limit point of \( \frac{S_{t_n - 1}x_n}{\|S_{t_n}x_n\|} \) as \( n \to \infty \). By continuity of \( S_1 \) it follows that \( \|S_{t_n}x_n\| \leq k_1\|S_{t_n - 1}x_n\| \) for some \( k_1 > 0 \). This implies that \( \|y\| \geq k_1^{-1} \). Therefore \( y \in \hat{P} \) which in turn implies that \( \mathcal{A}(S_1y) > 0 \). It is straightforward to show that the map \( x \mapsto \mathcal{A}(x) \) is continuous. Therefore, we have

\[
\mathcal{A}\left(\frac{S_{t_n}x_n}{\|S_{t_n}x_n\|}\right) = \mathcal{A}\left(S_1\left(\frac{S_{t_n - 1}x_n}{\|S_{t_n}x_n\|}\right)\right) \xrightarrow{n \to \infty} \mathcal{A}(S_1y).
\]

On the other hand, it holds that

\[
\mathcal{A}(S_{t_n}x_n) = \|S_{t_n}x_n\| \mathcal{A}\left(\frac{S_{t_n}x_n}{\|S_{t_n}x_n\|}\right).
\]

Since \( \hat{x} \in \text{int}(P) \) the constant \( \kappa_1 \) defined by

\[
\kappa_1 := \sup_{x \in P, \|x\| = 1} \overline{\mathcal{A}}(x)
\]

is finite. Since \( \mathcal{A}(S_1y) > 0 \) and \( \|S_{t_n}x_n\| \) diverges, (4.1)–(4.2) imply that \( \mathcal{A}(S_{t_n}x_n) \) diverges which is impossible since

\[
\mathcal{A}(S_{t_n}x_n) \leq \overline{\mathcal{A}}(S_{t_n}x_n) \leq \overline{\mathcal{A}}(x_n) \leq \kappa_1.
\]

Since \( \mathcal{O} \) is bounded in \( \mathcal{X} \), there exists a constant \( k_0 \) such that

\[
\|S_tx\| \leq k_0\|x\| \quad \forall t \in [0, 1], \quad \forall x \in P.
\]

That the set \( \mathcal{O}_1 \) is relatively compact for each \( x \in \mathcal{X} \) now easily follows. Indeed, since \( \mathcal{O}(x) \) is bounded, by the semigroup property we obtain

\[
\mathcal{O}_1 = \{S_1(S_{t-1}x) : x \in P, \|x\| = 1, \ t \geq 1\} \subset S_1(\mathcal{O}),
\]

and the claim follows since by hypothesis \( S_1 \) is a compact map.

For all \( t \geq s \geq 0 \) we have

\[
S_t(S_s x - \mathcal{A}(S_s x) \hat{x}) \leq S_{t+s} x - \mathcal{A}(S_s x) \hat{x},
\]

\[
S_t(\overline{\mathcal{A}}(S_s x) \hat{x} - S_s x) \leq \overline{\mathcal{A}}(S_s x) \hat{x} - S_{t+s} x.
\]

Let \( s = t_n \) in (4.5) and take limits along some converging sequence \( S_{t_n}x \to \bar{x} \) as \( n \to \infty \) to obtain

\[
\mathcal{A}^*(x) \hat{x} + S_t(\bar{x} - \mathcal{A}^*(x) \hat{x}) \leq S_t \bar{x},
\]

where \( \mathcal{A}^*(x) := \lim_{t \to \infty} \mathcal{A}(S_t x) \). Since \( \bar{x} \) is an \( \omega \)-limit point of \( S_t x \) it follows that \( \mathcal{A}(S_t \bar{x}) = \mathcal{A}^*(x) \) for all \( t \geq 0 \). Therefore \( S_t \bar{x} - \mathcal{A}^*(x) \hat{x} \notin \text{int}(P) \) for all \( t \geq 0 \), which implies by (4.7) and the strong positivity of \( S_t \) that \( \bar{x} - \mathcal{A}^*(x) \hat{x} = 0 \). A similar argument shows that \( \bar{x} = \overline{\mathcal{A}}^*(x) \hat{x} \), where \( \overline{\mathcal{A}}^*(x) := \lim_{t \to \infty} \overline{\mathcal{A}}(S_t x) \). We let \( \mathcal{A}^* = \overline{\mathcal{A}}^* = \mathcal{A}^* \).
It remains to prove that convergence is exponential. Since the orbit \( O \) is bounded and \( \hat{x} \in \text{int}(P) \) it follows that the set \( \{ \overline{\alpha}(S_k x) : t \geq 0, \ x \in P, \ |x| \leq 1 \} \) is bounded. Therefore since the orbit \( O_1 \) is also relatively compact, it follows that the set
\[
\mathcal{K}_1 := \{ S_k x - \alpha(S_k x) \hat{x}, \overline{\alpha}(S_k x) \hat{x} - S_k x : k \geq 1, \ x \in P, \ |x| \leq 1 \}
\]
is a relatively compact subset of \( P \). Define
\[
\eta(S_k x) := \overline{\alpha}(S_k x) - \alpha(S_k x), \quad k = 1, 2, \ldots
\]
By property (P2), since
\[
S_k x - \alpha(S_k x) \hat{x} \leq \eta(S_k x) \hat{x},
\]
\[
\overline{\alpha}(S_k x) \hat{x} - S_k x \leq \eta(S_k x) \hat{x},
\]
it follows that for some \( \zeta_1 = \zeta_1(\mathcal{K}_1) \) we have
\[
(4.8) \quad \max \{ \| S_k x - \alpha(S_k x) \hat{x} \|, \| \overline{\alpha}(S_k x) \hat{x} - S_k x \| \} \leq \zeta_1 \eta(S_k x)
\]
for all \( k \geq 1 \) and \( x \in P \) with \( |x| \leq 1 \). Define
\[
\overline{Z}_k(x) := \frac{S_k x - \alpha(S_k x) \hat{x}}{\eta(S_k x)}, \quad \underline{Z}_k(x) := \frac{\overline{\alpha}(S_k x) \hat{x} - S_k x}{\eta(S_k x)},
\]
provided \( \eta(S_k x) \neq 0 \), which is equivalent to \( S_k x \neq \hat{x} \). By \((4.8)\) the set
\[
\mathcal{K}_1 := \{ \overline{Z}_k(x), \underline{Z}_k(x) : k \geq 1, \ x \in \hat{P} \setminus \{ \hat{x} \}, \ |x| \leq 1 \}
\]
lies in the ball of radius \( \zeta_1 \) centered at the origin of \( \mathcal{X} \). Therefore, since \( \overline{Z}_k(x) = \hat{x} - \underline{Z}_k(x) \), by property (P1) there exists \( \zeta_0 = \zeta_0(\mathcal{K}_1) > 0 \) and \( \tau \in (0, 1) \) such that
\[
(4.9) \quad \| S_{\tau} \overline{Z}_k(x) \| + \| S_{\tau} \underline{Z}_k(x) \| \geq \zeta_0 \quad \forall k = 1, 2, \ldots, \ \forall x \in \hat{P} \setminus \{ \hat{x} \}, \ |x| \leq 1
\]
Let
\[
A_k(x) := \sup \{ \alpha \in \mathbb{R} : \{ S_1 \overline{Z}_k(x) - \alpha \hat{x} \} \cup \{ S_1 \underline{Z}_k(x) - \alpha \hat{x} \} \subset P \}.
\]
We claim that
\[
(4.10) \quad \zeta_2 := \inf \{ A_k(x) : k \geq 1, \ x \in \hat{P} \setminus \{ \hat{x} \}, \ |x| \leq 1 \} > 0.
\]
Indeed if the claim is not true then by \((4.9)\) and the definition of \( A_k \) there exists a sequence \( z_k \) taking values in
\[
\{ \overline{Z}_k(x), \underline{Z}_k(x) : x \in \hat{P} \setminus \{ \hat{x} \}, \ |x| \leq 1 \}
\]
for each \( k = 1, 2, \ldots \), such that \( \| S_{\tau} z_k \| \geq \zeta_0/2 \) and such that \( \overline{\alpha}(S_1 z_k) \to 0 \) as \( k \to \infty \). However, since \( \mathcal{K}_1 \) is bounded, it follows that \( S_{\tau}(\mathcal{K}_1) \) is a relatively compact subset of \( \text{int}(P) \). Therefore the limit set of \( S_{\tau} z_k \) is nonempty and any limit point \( y \in P \) of \( S_{\tau} z_k \) satisfies \( \| y \| \geq \zeta_0/2 \). Since \( \overline{\alpha}(S_1 z_k) = \overline{\alpha}(S_{1-\tau} S_{\tau} z_k) \) and \( z \mapsto \overline{\alpha}(S_{1-\tau} z) \) is continuous on \( P \) any such limit point \( y \) satisfies \( \overline{\alpha}(S_{1-\tau} y) = 0 \) which contradicts the strong positivity hypothesis.

Equation \((4.10)\) implies that
\[
(4.11) \quad \overline{\alpha}(S_1 (\overline{\alpha}(S_k x) \hat{x} - S_k x)) + \alpha(S_1 (S_k x - \alpha(S_k x) \hat{x})) \geq \zeta_2 (\overline{\alpha}(S_k x) - \alpha(S_k x))
\]
for all \( x \in \hat{P} \setminus \{ \hat{x} \} \) with \( |x| \leq 1 \), and by 1-homogeneity, for all \( x \in \hat{P} \setminus \{ \hat{x} \} \).
By (4.5)–(4.6) we have
\[
S_1(S_k x - \alpha(S_k x) \hat{x}) \preceq S_{k+1} x - \alpha(S_k x) \hat{x},
\]
\[
S_1(\bar{\alpha}(S_k x) \hat{x} - S_k x) \preceq \bar{\alpha}(S_k x) \hat{x} - S_{k+1} x.
\]
(4.12)

In turn (4.12) implies that
\[
\alpha(S_{k+1} x) \geq \alpha(S_k x) + \alpha(S_1(S_k x - \alpha(S_k x) \hat{x})),
\]
\[
\bar{\alpha}(S_{k+1} x) \leq \bar{\alpha}(S_k x) - \alpha(S_1(\bar{\alpha}(S_k x) \hat{x} - S_k x)).
\]
(4.13)

By (4.11) and (4.13) we obtain that
\[
\eta(S_k x) - \eta(S_{k+1} x) \geq \zeta_2 \eta(S_k x),
\]
which we write as
(4.14)

We add the inequalities
\[
\|S_k x - \alpha^*(x) \hat{x}\| \leq \|S_k x - \alpha(S_k x) \hat{x}\| + \alpha^*(x) - \alpha(S_k x),
\]
\[
\|\alpha^*(x) \hat{x} - S_k x\| \leq \|\bar{\alpha}(S_k x) \hat{x} - S_k x\| + \bar{\alpha}(S_k x) - \alpha^*(x)
\]
and use (4.8) and (4.14) to obtain
(4.15)

We have
(4.16)

where \(k_0\) is the continuity constant in (4.4) and \(\kappa_1\) is defined in (4.3). Let \([t]\) denote the integral part of a number \(t \in \mathbb{R}_+\). We define
\[
M_0 := \frac{\kappa_1 k_0^2 (2\zeta_1 + 1)}{2} \quad \text{and} \quad \theta_0 := -\log(1 - \zeta_2),
\]
and combine (4.15)–(4.16) to obtain
\[
\|S_t x - \alpha^*(x) \hat{x}\| \leq \frac{M_0}{k_0} (1 - \zeta_2)^{[t]-1} \|S_{t-[t]} x\|
\]
\[
\leq M_0 e^{-\theta_0 t} \|x\|.
\]
The proof is complete. \(\square\)
Remark 4.2. Recall that the cone $P$ is called normal if there exists a constant $K$ such that $\|x\| \leq K\|y\|$ whenever $0 \leq x \leq y$. Hence property (P2) is weaker than normality of the cone. Also strong positivity of $S_t$, $t > 0$, implies that (P1) is automatically satisfied over compact subsets of $\{x \in P : x \leq \hat{x}\}$.

We now return to the Nise semigroup in (3.1).

**Lemma 4.1.** There exists a unique pair $(\rho, \varphi) \in \mathbb{R} \times C^2_{\gamma,+}(\bar{Q})$ satisfying $\|\varphi\|_{0,\bar{Q}} = 1$ such that

$$S_t\varphi = e^{\rho t}\varphi, \quad t \geq 0.$$  

The pair $(\rho, \varphi)$ is a solution to the p.d.e.

$$\rho \varphi(x) = \mathcal{G}\varphi(x) = \inf_{v \in V} \left( \mathcal{L}_v \varphi(x) + r(x,v)\varphi(x) \right) \quad \text{in } Q, \quad \langle \nabla \varphi, \gamma \rangle = 0 \quad \text{on } \partial Q,$$

where (4.17) specifies $\rho$ uniquely in $\mathbb{R}$ and $\varphi$, with $\|\varphi\|_{0,\bar{Q}} = 1$, uniquely in $C^2_{\gamma,+}(\bar{Q})$.

**Proof.** It is clear that $S_t$ is superadditive. If $f \in C^2_{\gamma,+}(\bar{Q})$ then (3.5) implies that the solution $\psi$ of (3.4) is non-negative. Moreover by the strong maximum principle [9, Theorem 3, p. 38] and the Hopf boundary lemma [9, Theorem 14, p. 49] it follows that $\psi(t, \cdot) > 0$ for all $t > 0$. Hence the strong positivity hypothesis in Corollary 4.2 is satisfied. Since also the compactness hypothesis holds by Lemma 3.1, the first statement follows by Corollary 4.2. That (4.17) holds follows from (6) of Theorem 3.1 (see also [3, pp. 73–75]). Uniqueness follows from the following argument. Suppose $\hat{\rho} \in \mathbb{R}$ and $\hat{\varphi} \in C^2_{\gamma,+}(\bar{Q})$ solve

$$\hat{\rho} \hat{\varphi}(x) = \inf_{v \in V} \left( \mathcal{L}_v \hat{\varphi}(x) + r(x,v)\hat{\varphi}(x) \right).$$

Then by direct substitution we have

$$\frac{\partial}{\partial t} (e^{\hat{\rho} t} \hat{\varphi}(x)) = \hat{\rho} e^{\hat{\rho} t} \hat{\varphi}(x) = \inf_{v \in V} \left[ \mathcal{L}_v (e^{\hat{\rho} t} \hat{\varphi}(x)) + r(x,v)\{e^{\hat{\rho} t} \hat{\varphi}(x)\} \right].$$

Therefore, $S_t \hat{\varphi} = e^{\hat{\rho} t} \hat{\varphi}$, and by the uniqueness assertion in Corollary 4.2 we have $\hat{\rho} = \rho$ and $\hat{\varphi} = C\varphi$ for some positive constant $C$. \hfill $\square$

**Remark 4.3.** Consider the operator $R_t : C^2_{\gamma}(\bar{Q}) \to C^2_{\gamma}(\bar{Q})$ defined by $R_t f = -S_t (-f)$. Then by same arguments as in the proof of Lemma 4.1 using Corollary 4.2 there exists a unique $\beta \in \mathbb{R}$ and $\psi > 0$ in $C^2_{\gamma}(\bar{Q})$ such that

$$R_t \psi = e^{\beta t} \psi.$$

Hence the pair $(e^{\beta t}, -\psi)$ is an eigenvalue-function pair of $S_t$. Now the same arguments as in the proof of Lemma 4.1 lead to the conclusion that $(\beta, \psi)$ is the unique positive solution pair of

$$\beta \psi(x) = \sup_{v \in V} \left( \mathcal{L}_v \psi(x) + r(x,v)\psi(x) \right) \quad \text{in } Q, \quad \langle \nabla \psi, \gamma \rangle = 0 \quad \text{on } \partial Q,$$

Hence $(\beta, -\psi)$ is the unique solution pair of (4.17) satisfying $-\psi < 0$. Moreover it is easy to see that $\rho \leq \beta$ and that $\beta$ is the principal eigenvalue of both operators $R_t$, $S_t$. This leads to the conclusion that the risk-sensitive control problem where the controller tries to maximize the risk-sensitive cost (2.2) leads to the value $\beta$ which is the principal eigenvalue.
Lemma 4.2. Let $\mathcal{M}(\bar{Q})$ denote the space of finite Borel measures on $\bar{Q}$. Then

$$(C^2_\gamma(\bar{Q}))^* = \mathcal{M}(\bar{Q}).$$

Proof. Let $\Lambda \in (C^2_\gamma(\bar{Q}))^*_+$. Then for $f \in C^2_\gamma(\bar{Q})$ by positivity of $\Lambda$ we have

$$|\Lambda(f)| = |\Lambda(f + \|f\|_{0,Q} \cdot 1) - \Lambda(\|f\|_{0,Q} \cdot 1)|$$

$$\leq \max \{\Lambda(f + \|f\|_{0,Q} \cdot 1), \Lambda(\|f\|_{0,Q} \cdot 1)\}$$

$$\leq \Lambda(2\|f\|_{0,Q} \cdot 1)$$

$$= 2\|f\|_{0,Q}\Lambda(1).$$

It follows that $\Lambda$ is a bounded linear functional on the linear subspace $C^2_\gamma(\bar{Q})$ of $C(\bar{Q})$. By the Hahn-Banach theorem $\Lambda$ can be extended to some $\psi \in C^*(\bar{Q})$. Clearly $\psi$ is a positive linear functional. By the Riesz representation theorem there exists $\mu \in \mathcal{M}(\bar{Q})$ such that $\psi(f) = \int_Q f \, d\mu$ for all $f \in C(\bar{Q})$. Therefore $\Lambda(f) = \int_Q f \, d\mu$ for all $f \in C^2_\gamma(\bar{Q})$. This shows that $(C^2_\gamma(\bar{Q}))^*_+ \subset \mathcal{M}(\bar{Q})$. It is clear that $\mathcal{M}(\bar{Q}) \subset (C^2_\gamma(\bar{Q}))^*_+$, so equality follows. \hfill $\square$

Lemma 4.3. Let $\delta \in (0, \beta_0)$. Then for any $f \in C^{2+\delta}_\gamma(\bar{Q})$ we have

$$\limsup_{t \downarrow 0} \inf_{\mu \in \mathcal{M}(\bar{Q})} \int_Q \frac{S_t f(x) - f(x)}{t} \mu(dx) = \inf_{\mu \in \mathcal{M}(\bar{Q})} \int_Q G f(x) \mu(dx)$$

and

$$\liminf_{t \downarrow 0} \sup_{\mu \in \mathcal{M}(\bar{Q})} \int_Q \frac{S_t f(x) - f(x)}{t} \mu(dx) = \sup_{\mu \in \mathcal{M}(\bar{Q})} \int_Q G f(x) \mu(dx).$$

Proof. Note that

$$\lim_{t \downarrow 0} \frac{S_t f(x) - f(x)}{t} = G f(x), \quad x \in \bar{Q}.$$ 

Hence using the dominated convergence theorem\footnote{Note that

$$\left| \frac{S_t f(x) - f(x)}{t} \right| \leq \inf_{v_1, \ldots, v_n} \frac{1}{t} E_x \left[ \int_0^t e^{\int_0^s r(X_v, v_x) \, ds} \left| \mathcal{L}_{v_x} f(X_s) + r(X_s, v_s) f(X_s) \right| \, ds \right]$$

$$\leq Ke^{r_{\max}}, \quad 0 \leq t \leq 1,$$

for some constant $K > 0.$} we get for all $\mu \in \mathcal{M}(\bar{Q})$ satisfying $\int f \, d\mu = 1$,

$$\lim_{t \downarrow 0} \int_Q \frac{S_t f(x) - f(x)}{t} \mu(dx) = \int_Q G f(x) \mu(dx).$$
Therefore
\[
\limsup_{t \downarrow 0} \inf_{\mu \in M(Q)} \int_Q \frac{S_t f(x) - f(x)}{t} \mu(dx) \leq \lim_{t \downarrow 0} \int_Q \frac{S_t f(x) - f(x)}{t} \mu(dx) = \int_Q \mathcal{G} f(x) \mu(dx)
\]
for all \( \mu \in M(Q) \) satisfying \( \int f \, d\mu = 1 \). Hence
\[
\limsup_{t \downarrow 0} \inf_{\mu \in M(Q)} \int_Q \frac{S_t f(x) - f(x)}{t} \mu(dx) \leq \inf_{\mu \in M(Q)} \int_Q \mathcal{G} f(x) \mu(dx).
\] (4.18)

Since for each \( t > 0 \) the map \( \mu \mapsto \int_Q \frac{S_t f(x) - f(x)}{t} \mu(dx) \) from \( M(Q) \to \mathbb{R} \) is continuous, there exists a \( \mu_t \in M(Q) \) satisfying \( f \, d\mu = 1 \) such that
\[
\inf_{\mu \in M(Q)} \int_Q \frac{S_t f(x) - f(x)}{t} \mu(dx) = \int_Q \frac{S_t f(x) - f(x)}{t} \mu_t(dx).
\]
Clearly \( \{ \mu_t \} \) is tight. Let \( \hat{\mu} \) be a limit point of \( \mu_t \) as \( t \to 0 \). Suppose \( \mu_{t_n} \to \hat{\mu} \) in \( M(Q) \) as \( t_n \downarrow 0 \). Then \( \int f \, d\hat{\mu} = 1 \). Note that for \( f \in C^{2+\delta}_{r+}(Q) \),
\[
\frac{S_t f(x) - f(x)}{t} = \frac{1}{t} \int_0^t \partial_s u^f(s, x) \, ds,
\] (4.19)
with \( u^f(t, \cdot) := S_t f(\cdot) \). By the Hölder continuity of \( \partial_s u^f \) on \( [0, 1] \times \bar{Q} \), there exists \( k_1 > 0 \) such that
\[
|\partial_s u^f(s, x) - \partial_s u^f(s, y)| < k_1 |x - y|^\delta \quad \forall x, y \in \bar{Q}, s \in [0, 1].
\] (4.20)
Therefore by (4.19) and (4.20) \( x \mapsto \frac{S_t f(x) - f(x)}{t} \) is Hölder equicontinuous over \( t \in (0, 1] \), and the convergence
\[
\lim_{t \downarrow 0} \frac{S_t f(x) - f(x)}{t} = \mathcal{G} f(x)
\]
is uniform in \( \bar{Q} \). Hence from
\[
\int_Q \frac{S_{t_n} f(x) - f(x)}{t_n} \mu_{t_n}(dx) = \int_Q \left( \frac{S_{t_n} f(x) - f(x)}{t_n} - \mathcal{G} f(x) \right) \mu_{t_n}(dx)
\]
\[ + \int_Q \mathcal{G} f(x) \mu_{t_n}(dx) - \int_Q \mathcal{G} f(x) \hat{\mu}(dx) \]
it follows that
\[
\lim_{n \to \infty} \int_Q \left( \frac{S_{t_n} f(x) - f(x)}{t_n} \right) \mu_{t_n}(dx) = \int_Q \mathcal{G} f(x) \hat{\mu}(dx)
\]
greater or equal than
\[
\inf_{\mu \in M(Q)} \int_Q \mathcal{G} f(x) \mu(dx).
\]
Hence

\[
\limsup_{t \downarrow 0} \int_{\overline{Q}} \left( \frac{S_t f(x) - f(x)}{t} \right) \mu(dx) \geq \inf_{\mu \in \mathcal{M}(Q)} \int_{f} \mathcal{G} f(x) \mu(dx).
\]

From (4.18) and (4.21), the result follows. The proof of the second limit follows by a symmetric argument.

\[\Box\]

**Theorem 4.4.** The scalar \( \rho \) in Lemma 4.1 satisfies

\[
\rho = \inf_{f \in C^2_{\gamma, +}(\tilde{Q})} \sup_{\mu \in \mathcal{M}(\tilde{Q})} \int \mathcal{G} f \, d\mu = 1 \int \mathcal{G} f \, d\mu,
\]

or equivalently

\[
\rho = \sup_{f \in C^2_{\gamma, +}(\tilde{Q}), f > 0} \inf_{\nu \in \mathcal{P}(\tilde{Q})} \int \mathcal{G} f \, d\nu,
\]

where \( \mathcal{P}(\tilde{Q}) \) denotes the space of probability measures on \( \tilde{Q} \) with the Prohorov topology.

**Proof.** Let \( \delta \in (0, \beta_0) \). Since \( \rho \varphi = \mathcal{G} \varphi \) by Lemma 4.1, we obtain

\[
\rho = \inf_{\mu \in \mathcal{M}(\tilde{Q})} \int \mathcal{G} \varphi \, d\mu \geq \sup_{f \in C^2_{\gamma, +}(\tilde{Q})} \inf_{\mu \in \mathcal{M}(\tilde{Q})} \int \mathcal{G} f \, d\mu.
\]

To show the reverse inequality we use Theorem 4.2 and Lemma 4.2. We have

\[
e^{\rho t} = \sup_{g \in C^2_{\gamma, +}(\tilde{Q})} \inf_{\mu \in \mathcal{M}(\tilde{Q})} \int S_t g \, d\mu.
\]
Therefore, using Lemma 4.3 we obtain
\[
\rho = \lim_{t \downarrow 0} \sup_{g \in C^{2,\delta}_{\gamma,+}(Q)} \inf_{\mu \in \mathcal{M}(Q)} \int \frac{S_t g - g}{t} \, d\mu
\]
\[
\geq \limsup_{t \downarrow 0} \inf_{\mu \in \mathcal{M}(Q)} \int \frac{S_t f - f}{t} \, d\mu
\]
\[
= \inf_{\mu \in \mathcal{M}(Q)} \int G f \, d\mu
\]
for all \( f \in C^{2,\delta}_{\gamma,+}(Q) \). Therefore,
\[
\rho \geq \sup_{f \in C^{2,\delta}_{\gamma,+}(Q)} \inf_{\mu \in \mathcal{M}(Q)} \int G f \, d\mu.
\]
Using a symmetric argument to establish the first equality in (4.24) below, we obtain
\[
\rho = \inf_{f \in C^{2,\delta}_{\gamma,+}(Q)} \sup_{\mu \in \mathcal{M}(Q)} \int G f \, d\mu
\]
\[
= \sup_{f \in C^{2,\delta}_{\gamma,+}(Q)} \inf_{\mu \in \mathcal{M}(Q)} \int G f \, d\mu
\]
for all \( \delta \in (0, \beta_0) \). Note that the outer ‘inf’ and ‘sup’ in (4.24) are realized at the function \( \varphi \) in Lemma 4.1. Therefore, since \( \varphi > 0 \), equation (4.24) remains valid if we restrict the outer ‘inf’ and ‘sup’ on \( f > 0 \). Hence using the probability measure \( d\nu = f \, d\mu \) we can write (4.24) as
\[
\rho = \inf_{f \in C^{2,\delta}_{\gamma,+}(Q), f > 0} \sup_{\nu \in \mathcal{P}(Q)} \int \frac{G f}{f} \, d\nu
\]
\[
= \sup_{f \in C^{2,\delta}_{\gamma,+}(Q), f > 0} \inf_{\nu \in \mathcal{P}(Q)} \int \frac{G f}{f} \, d\nu.
\]
Therefore
\[
\inf_{f \in C^2_{\gamma,+}(Q), f > 0} \sup_{\nu \in \mathcal{P}(Q)} \int \frac{G f}{f} \, d\nu \leq \rho \leq \sup_{f \in C^2_{\gamma,+}(Q), f > 0} \inf_{\nu \in \mathcal{P}(Q)} \int \frac{G f}{f} \, d\nu.
\]
Suppose that the inequality on the r.h.s. of (4.25) is strict. Then for some \( f \in C^2_{\gamma,+}(Q) \) we have
\[
\inf_{\nu \in \mathcal{P}(Q)} \int \frac{G f}{f} \, d\nu > \rho.
\]
Since \( G : C^2_{\gamma,+}(Q) \to C^0(\bar{Q}) \) is continuous and since \( C^{2,\delta}_{\gamma,+}(Q) \) is dense in \( C^2_{\gamma,+}(Q) \) in the \( \| \cdot \|_{2,Q} \) norm, there exists \( g \in C^{2,\delta}_{\gamma,+}(Q) \), \( g > 0 \), such that \( \min_{\bar{Q}} \frac{g f}{g} > \rho \). However this contradicts Theorem 4.4 which means that the first equality in (4.23) must hold. The proof...
of the second equality in (4.23) is similar. The last assertion of the theorem follows via the change of measure $f \, \text{d}\mu = d\nu$. \hfill \Box

**Remark 4.5.** As pointed out in the proof of Theorem 4.4 the outer ‘inf’, resp. ‘sup’ in (4.22)–(4.23) above are in fact ‘min’, ‘max’ attained by $\varphi$. Concerning the stability of the semigroup we have the following lemma.

**Lemma 4.4.** There exist $M > 0$ and $\theta > 0$ such that for any $f \in C_{\gamma,+}^{2}(\bar{Q})$ we have

$$
\|e^{-\theta t}S_{t}f - \alpha^{*}(f)\|_{0,\bar{Q}} \leq Me^{-\theta t}\|f\|_{0,\bar{Q}} \quad \forall t \geq 1,
$$

for some $\alpha^{*}(f) \in \mathbb{R}_{+}$.

**Proof.** Without loss of generality we assume $\varrho = 0$. We first verify that property (P1) of Theorem 4.3 holds. Let $\tau = 1/2$. We claim that there exists a constant $c_{0} > 0$ such that

$$
(E_{x}^{0}[f(X_{\tau})])^{2} \leq c_{0} E_{x}^{0}[f(X_{\tau})] \quad \forall f \in C(\bar{Q}), \ 0 \leq f \leq \varphi,
$$

and for all Markov controls $v$, $v'$ and $x \in \bar{Q}$. The proof of (4.26) is as follows. To distinguish between processes, let $X, Y$ denote the processes corresponding to the controls $v, v'$ respectively. Then using Girsanov’s theorem, it follows that if we define

$$
F(t) := \int_{0}^{t} \sigma^{-1}(Y_{t})[b(Y_{t},v_{t}) - b(Y_{t},v'_{t})]dW_{t} - \frac{1}{2} \int_{0}^{t} \|\sigma^{-1}(Y_{t})[b(Y_{t},v_{t}) - b(Y_{t},v'_{t})]\|^{2}dt,
$$

then

$$
E_{x}[f(Y_{\tau})] = E_{x}[e^{F(\tau)}f(X_{\tau})]
\leq (E_{x}[f^{2}(X_{\tau})])^{1/2}(E_{x}[e^{2F(\tau)}])^{1/2}
\leq (E_{x}[f^{2}(X_{\tau})])^{1/2}(E_{x}[e^{\int_{0}^{\tau}\|\sigma^{-1}(Y_{t})[b(Y_{t},v_{t}) - b(Y_{t},v'_{t})]\|^{2}dt}]^{1/2}
\leq c_{1}(E_{x}[f^{2}(X_{\tau})])^{1/2}
\leq c_{1}\|\varphi\|_{0,\bar{Q}} E_{x}[f(X_{\tau})]^{1/2}
$$

where $c_{1} > 0$ is a constant which only depends on the bounds of $\sigma^{-1}$ and $b$. This proves (4.26). For $f \in C(\bar{Q})$ satisfying $0 \leq f \leq \varphi$ and for any fixed $v$ we have

$$
S_{\tau}(\varphi - f)(x) \geq e^{\tau\min}E_{x}^{v_{1}}[\varphi(X_{\tau}) - f(X_{\tau})] \geq e^{\tau\min}c_{0}^{-1}(E_{x}^{v}[\varphi(X_{\tau}) - f(X_{\tau})])^{2}
$$

and

$$
S_{\tau}(f)(x) \geq e^{\tau\min}E_{x}^{v_{2}}[f(X_{\tau})] \geq e^{\tau\min}c_{0}^{-1}(E_{x}^{v}[f(X_{\tau})])^{2},
$$

where $v_{1}, v_{2}$ are the corresponding minimizers. Note that\footnote{\text{The first part of the inequality below follows from the fact that $(a - x)^{2} + x^{2}, 0 \leq x \leq a$ attains it minimum at $x = \frac{a}{2}$}}

$$
(E_{x}^{v}[\varphi(X_{1}) - f(X_{1})])^{2} + (E_{x}^{v}[f(X_{1})])^{2} \geq \frac{1}{2}(E_{x}^{v}[\varphi(X_{1})])^{2} \geq \frac{1}{2}(\min \varphi)^{2}.
$$
Adding (4.27) and (4.28) and using (4.29), it follows that
\[ \|S_r(\varphi - f)\| + \|S_rf\| \geq \frac{e^{\tau_{\min}}}{2c_0} (\min \varphi)^2, \]
which establishes property (P1). On the other hands, property (P2) of Theorem 4.4 is
trivially satisfied under the \( \| \cdot \|_{\nu;Q} \) norm. Hence the result follows by Theorem 4.4 (iii).

Let \( U = \{ u \} \), i.e., a singleton, and \( v(\cdot) \equiv v := \delta_u \), thus reducing the problem to an
uncontrolled one. Thus \( G = L_v + r(x, v) \) is a linear operator. By [6] Lemma 2, pp. 781–782,
the first of the above expressions then also equals the Donsker–Varadhan functional
\[ \sup_{\nu \in \mathcal{P}(Q)} \left( \int_Q r(x, v) \nu(dx) - I(\nu) \right), \]
where
\[ I(\nu) = - \inf_{f \in C^2_{\gamma,\nu}(Q), f > 0} \int \frac{L_v f}{f} d\nu. \]
More generally if \( r(x, v) \) does not depend on \( v \), say \( r(x, v) = r(x) \) and \( A \) is defined by
\[ Af(x) := \frac{1}{2} \text{tr} (a(x) \nabla^2 f(x)) + \min_{w \in \mathcal{V}} \left( b(x, v), \nabla f(x) \right), \]
then
\[ \rho = \sup_{\nu \in \mathcal{P}(Q)} \left( \int_Q r(x, v) \nu(dx) - I(\nu) \right), \]
\[ I(\nu) = - \inf_{f \in C^2_{\gamma,\nu}(Q), f > 0} \int \frac{Af}{f} d\nu. \]
Our results thus provide a counterpart of the Donsker–Varadhan functional for the nonlinear
case arising from control.

It is also interesting to consider the substitution \( f = e^\psi \). Then we obtain
\[ \rho = \inf_{\psi \in C^2(Q)} \sup_{\nu \in \mathcal{P}(Q)} \int \inf_{w \in \mathbb{R}^d} \sup_{v \in \mathcal{V}} \left( r(\cdot, v) - \frac{1}{2} ||w||^2 + L_v \psi + \langle \nabla \psi, \sigma w \rangle \right) d\nu \]
\[ = \sup_{\psi \in C^2(Q)} \inf_{\nu \in \mathcal{P}(Q)} \int \inf_{w \in \mathbb{R}^d} \sup_{v \in \mathcal{V}} \left( r(\cdot, v) - \frac{1}{2} ||w||^2 + L_v \psi + \langle \nabla \psi, \sigma w \rangle \right) d\nu \]
\[ = \inf_{\psi \in C^2(Q)} \sup_{\nu \in \mathcal{P}(Q)} \int \sup_{w \in \mathbb{R}^d} \inf_{v \in \mathcal{V}} \left( r(\cdot, v) - \frac{1}{2} ||w||^2 + L_v \psi + \langle \nabla \psi, \sigma w \rangle \right) d\nu \]
\[ = \sup_{\psi \in C^2(Q)} \inf_{\nu \in \mathcal{P}(Q)} \int \sup_{w \in \mathbb{R}^d} \inf_{v \in \mathcal{V}} \left( r(\cdot, v) - \frac{1}{2} ||w||^2 + L_v \psi + \langle \nabla \psi, \sigma w \rangle \right) d\nu, \]
where the last two expressions follow from the standard Ky Fan min-max theorem [7].
This is the standard logarithmic transformation to convert the Hamilton-Jacobi-Bellman
equation for risk-sensitive control to the Hamilton-Jacobi-Isaacs equation for an associated
zero sum ergodic stochastic differential game [8], given by
\[ \inf_{v \in \mathcal{V}} \sup_{w \in \mathbb{R}^d} \left( r(\cdot, v) - \frac{1}{2} ||w||^2 + L_v \psi + \langle \nabla \psi, \sigma w \rangle \right) = \rho \]
in $Q$, with $\langle \nabla \psi, \gamma \rangle = 0$ on $\partial Q$. The expressions above bear the same relationship with (4.30) as what Lemma 4.1 and Remark 4.4 spell out for (4.17).

5. Risk-sensitive control with periodic coefficients

In this section we consider risk-sensitive control with periodic coefficients. Consider a controlled diffusion $X(\cdot)$ taking values in $\mathbb{R}^d$ satisfying

(5.1) \[ dX(t) = b(X(t), v(t)) \, dt + \sigma(X(t)) \, dW(t) \]

for $t \geq 0$, with $X(0) = x$.

We assume that

1. The functions $b(x, v)$, $\sigma(x)$ and the running cost $r(x, v)$ are periodic in $x_i$, $i = 1, 2, \ldots, d$. Without loss of generality we assume that the period equals 1.
2. $b : \mathbb{R}^d \times \mathcal{V} \to \mathbb{R}^d$ is continuous and Lipschitz in its first argument uniformly with respect to the second,
3. $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is continuously differentiable, its derivatives are Hölder continuous with exponent $\beta_0 > 0$, and is non-degenerate,
4. $r : \mathbb{R}^d \times \mathcal{V} \to \mathbb{R}$ is continuous and Lipschitz in its first argument uniformly with respect to the second. We let $r_{\text{max}} := \max_{(x,v) \in Q \times \mathcal{V}} |r(x,v)|$.

Admissible controls are defined as in (e).

We consider here as well the infinite horizon risk-sensitive problem which aims to minimize the cost in (5.2) under the controlled process governed by (5.1). Recall the notation defined in Section 2 and note that $C^0(\mathbb{R}^d)$ is the space of all continuous and bounded real-valued functions on $\mathbb{R}^d$. We define the semigroups of operators $\{S_t, t \geq 0\}$ and $\{T_t^u, t \geq 0\}$ acting on $C^0(\mathbb{R}^d)$ as in (3.1)–(3.2) relative to the controlled process governed by (5.1). Also the operators $\mathcal{L}_u : C^2(\mathbb{R}^d) \to C^0(\mathbb{R}^d)$ are as defined in (2.3).

Let $C_p(\mathbb{R}^d)$ denote the set of all $C(\mathbb{R}^d)$ functions with period 1 and in general if $\mathcal{X}$ is a subset of $C(\mathbb{R}^d)$ we let $\mathcal{X}_p(\mathbb{R}^d) := \mathcal{X} \cap C_p(\mathbb{R}^d)$.

We start with the following theorem which is analogous to Theorem 3.1.

**Theorem 5.1.** $\{S_t, t \geq 0\}$ acting on $C^0(\mathbb{R}^d)$ satisfies the following properties:

1. Boundedness: $\|S_t f\|_{0; \mathbb{R}^d} \leq e^{r_{\text{max}} t} \|f\|_{0; \mathbb{R}^d}$. Furthermore, $e^{r_{\text{max}} t} S_t \mathbf{1} \geq 1$, where $\mathbf{1}$ is the constant function $\equiv 1$.
2. Semigroup property: $S_0 = I$, $S_t \circ S_s = S_{t+s}$ for $s, t \geq 0$.
3. Monotonicity: $f \geq (\text{resp., } >) g \implies S_t f \geq (\text{resp., } >) S_t g$.
4. Lipschitz property: $\|S_t f - S_t g\|_{0; \mathbb{R}^d} \leq e^{r_{\text{max}} t} \|f-g\|_{0; \mathbb{R}^d}$.
5. Strong continuity: $\|S_t f - S_s f\|_{0; \mathbb{R}^d} \to 0$ as $t \to s$.
6. Envelope property: $T_t^u f \geq S_t f$ for all $u \in U$ and $S_t f \geq S_t^u f$ for any other $\{S_t^u\}$ satisfying this along with the foregoing properties.
7. Generator: the infinitesimal generator of $\{S_t\}$ is given by (3.3).
8. For $f \in C_p(\mathbb{R}^d)$, $S_t f \in C_p(\mathbb{R}^d), t \geq 0$.

**Proof.** Properties (1)–(4) and (6) follow by standard arguments from (3.1) and the bound on $r$. That $S_t : C^0(\mathbb{R}^d) \to C^0(\mathbb{R}^d)$ is well known. See Remark 5.1 below. Property (8) follows from (3.1) and the periodicity of the data. □
Theorem 5.2. For $f \in C^{2+\delta}_p(\mathbb{R}^d)$, $\delta \in (0, \beta_0)$, the p.d.e. 
\begin{equation}
\frac{\partial}{\partial t} u(t, x) = \inf_{v \in V} \left( \mathcal{L}_v u(t, x) + r(x, v)u(t, x) \right) \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^d, 
\end{equation}
with $u(0, x) = f(x)$ for all $x \in \mathbb{R}^d$, has a unique solution in $C^{1+\delta,2+\delta}_p([0, T] \times \mathbb{R}^d)$, $T > 0$. The solution $\psi$ has the stochastic representation
\begin{equation}
\begin{aligned}
u(t, x) &= \inf_{v \in V} E_x \left[ e^{\int_0^\tau r(X(s), v(s)) \, ds} f(X(t)) \right] \quad \forall (t, x) \in [0, \infty) \times \mathbb{R}^d. 
\end{aligned}
\end{equation}
Moreover, for some $K_T > 0$ depending on $T$, $\delta$, $\|f\|_{2+\delta,\mathbb{R}^d}$ and the bounds on the data, we have
\[ \|u\|_{1+\delta,2+\delta;[0,T] \times B_R} \leq K_T. \]

**Proof.** Consider the p.d.e.
\begin{equation}
\frac{\partial}{\partial t} u^R(t, x) = \inf_{v \in V} \left( \mathcal{L}_v u^R(t, x) + r(x, v)u^R(t, x) \right) \quad \text{in } \mathbb{R}_+ \times B_R, 
\end{equation}
with $u^R = 0$ on $\mathbb{R}_+ \times \partial B_R$ and with $u^R(0, x) = f(x)g(R^{-1}x)$ for all $x \in B_R$, where $g$ is a smooth non-negative function equals 1 on $B_{\frac{R}{2}}$ and 0 on $B_{\frac{R}{4}}$. From [11, Theorem 6.1, pp. 452–453], the p.d.e. (5.2) has a unique solution $u^R$ in $C^{1+\delta,2+\delta}_p([0, T] \times B_R)$, $T > 0$. This solution has the stochastic representation
\begin{equation}
\begin{aligned}
u^R(t, x) &= \inf_{v \in V} E_x \left[ e^{\int_0^\tau r(X(s), v(s)) \, ds} f(X(t \wedge \tau_R))g(R^{-1}X(t \wedge \tau_R)) \right] 
\end{aligned}
\end{equation}
for all $(t, x) \in [0, \infty) \times \mathbb{R}^d$, where $\tau_R$ denotes the first exit time from the ball $B_R$. By [11, Theorem 5.2, p. 320] for each $T > 0$ there exists a constant $K_T$
\[ \|u^R\|_{1+\delta,2+\delta;[0,T] \times B_R} \leq K_T. \]

Therefore $u^R$ converges to a function $u \in C^{1+\delta,2+\delta}_p([0, T] \times \mathbb{R}^d)$, as $R \to \infty$, which satisfies (5.2)–(5.3). The periodicity of $u(t, x)$ in $x$ follows by (5.3) and the periodicity of the coefficients. \hfill \square

**Remark 5.1.** The regularity of the initial condition $f$ is only needed to obtain continuous second derivatives at $t = 0$. It is well known that for each $f \in C^0(\mathbb{R}^d)$ (5.2) has a solution in $C([0, T] \times \mathbb{R}^d) \cap C^{1+\delta,2+\delta}_{\text{loc}}(\mathbb{R}^d)$, for $T > 0$.

**Theorem 5.3.** There exists a unique $\rho \in \mathbb{R}$ and a $\varphi > 0$ in $C^2_p(\mathbb{R}^d)$ unique up to a scalar multiple such that
\[ S_t \varphi = e^{\rho t} \varphi, \quad t \geq 0. \]

**Proof.** Using Theorem 5.2, one can show as in the proof of Lemma 4.1 that $S_t : C^2_p(\mathbb{R}^d) \to C^2_p(\mathbb{R}^d)$ is compact for each $t \geq 0$. Now with $X = C^2_p(\mathbb{R}^d)$ and $P = \{ f \in C^2_p(\mathbb{R}^d) : f \geq 0 \}$ and $T = S_t$ for some $t \geq 0$, the conditions of Theorems 4.1 and 4.2 are easily verified using Theorem 5.1. Repeating the same argument as in the proof of Corollary 4.2 completes the proof. \hfill \square
Lemma 5.1. The pair \((\rho, \varphi)\) given in Theorem 5.3 is a solution to the p.d.e.
\[
\rho \varphi(x) = \inf_v \left( L_v \varphi(x) + r(x, v)\varphi(x) \right),
\]
where (5.4) specifies \(\rho\) uniquely in \(\mathbb{R}\) and \(\varphi\) uniquely in \(C_p^2(\mathbb{R}^d)\) up to a scalar multiple. Moreover, \(\inf_{\mathbb{R}^d} \varphi > 0\).

Proof. The proof is directly analogous to that of Lemma 4.1. \(\square\)

Lemma 5.2. \((C_p^2(\mathbb{R}^d))^* \simeq \mathcal{M}(Q),\) with \(Q = [0,1)^d\).

Proof. Let \(\pi\) denote the projection of \(\mathbb{R}^d\) to \([0,1)^d\). Set
\[
\mathcal{D} = \{ f \circ \pi \in C(Q) : f \in C_p(\mathbb{R}^d) \}.
\]
Then \(\mathcal{D}\) is a linear subspace of \((C^0(Q))^*\).

For \(\Lambda \in (C_p(\mathbb{R}^d))^*\), define the linear map \(\tilde{\Lambda} : \mathcal{D} \to \mathbb{R}\) by
\[
\tilde{\Lambda}(f \circ \pi) = \Lambda(f).
\]
Then
\[
|\tilde{\Lambda}(f \circ \pi)| \leq \|\Lambda\| \|f\|_{0;\mathbb{R}^d} \leq \|\Lambda\| \|f \circ \pi\|_{0, Q}.
\]
i.e. \(\tilde{\Lambda} \in \mathcal{D}^*\). Using the Hahn-Banach theorem, there exists a continuous linear extension \(\Lambda' : C^0(Q) \to \mathbb{R}\) of \(\tilde{\Lambda}\) such that \(\|\Lambda'\| = \|\tilde{\Lambda}\|\).

Since \((C^0(Q))^* = \mathcal{M}(Q)\), the set of all finite signed Radon measures, we have \((C_p(\mathbb{R}^d))^* \subseteq \mathcal{M}(Q)\). The reverse inequality follows easily. Hence \((C_p(\mathbb{R}^d))^* = \mathcal{M}(Q)\). Now the analogous argument in Lemma 4.2 can be used to complete the proof. \(\square\)

Now by closely mimicking the proofs of Lemma 4.3 and Theorem 4.4, we have

Theorem 5.4. \(\rho\) satisfies
\[
\rho = \inf_{f \in C^2_p(Q) \cap \mathcal{D}} \sup_{\mu \in \mathcal{M}(Q) : \|f\|_{\mathcal{M}(Q)} = 1} \int \mathcal{G} f \, d\mu
\]
\[
= \sup_{f \in C^2_p(Q) \cap \mathcal{D}} \inf_{\mu \in \mathcal{M}(Q) : \|f\|_{\mathcal{M}(Q)} = 1} \int \mathcal{G} f \, d\mu,
\]
where \(\mathcal{G}\) given in Theorem 5.1.

The stability of the semigroup also follows as in Lemma 4.4. It is well known that (5.1) has a transition probability density \(p(t, x, y)\) which is bounded away from zero, uniformly over all Markov controls \(v\), for \(t = 1\) and \(x, y\) in a compact set. It is straightforward to show that this implies property (P1). Therefore exponential convergence follows by Theorem 4.3 (iii).

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