A critical assessment on Kassapoglou’s statistical model for composites fatigue

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Abstract

Kassapoglou recently proposed a model for fatigue of composite materials which seems to suggest that fatigue SN curve can be entirely predicted on the basis of the statistical distribution of static strengths. The original abstract writes “Expressions for the cycles to failure as a function of R ratio are derived. These expressions do not require any curve fitting and do not involve any experimentally determined parameters. The fatigue predictions do not require any fatigue tests for calibration”. These surprisingly ambitious claims and attractive results deserve careful scrutiny. We contend that the results seem to be due to a number of approximations and incorrect derivations, and one particular misleading assumption, which make the model not conform to a fatigue testing in a given specimen with resulting SN curve distribution. The quantitative agreement of some predictions (the scatter of distribution of fatigue lives being close to the mode value found in typical composites of aeronautical interest in the large Navy database) should not motivate any enthusiasm. It is believed that a proper statistical treatment of the fatigue process should not
make wear-out constants disappear, and hence the SN curves would depend on them, and not just on scatter of static data. These serious concerns explain the large discrepancies found by 3 independent studies which tried to apply Kassapoglou’s model to composite fatigue data, and to other well known results.

Keywords
Composite materials, fatigue, wearout models, Kassapoglou model, strength-life equal rank, statistics, Weibull distribution

1 Introduction

The “strength-life equal rank assumption” wear-out models for fatigue of composite materials were first presented by Hahn and Kim [1], and later as a fitting approach to fatigue data by Sendeckyj [2]. A significantly different model has been proposed more recently by Kassapoglou [3,4,5] (in the following, K is Kassapoglou), which in fact claims an extremely strong result, which defies the efforts of more than a century of research in fatigue: that of predicting SN curve of a material from just the static data. The effort started already in the end of the 1800’s for metals, and even today only crude approximations can be made on fatigue limit over static strength (the so-called, fatigue ratio), which are generally based on hardness tests. The use of simplified equations for SN curve also is well known in any fatigue textbook [6], but it is always clearly shown that any such empirical equation is limited, and in general makes only a very crude estimate. This justifies the industry of fatigue machine testing, which is by no means less flourishing in composite materials, although composites are known to suffer more crucially to impact than to fatigue. Hence, no aircraft flying today is there without having passed a very serious fatigue testing certification procedure, and for a good reason. Fatigue testing is required by any certification agency to get airworthiness certificates, and the cost of testing is huge. The idea to obtain even approximate results for fatigue from just static data is therefore still obviously attractive, since experience on static strength is so much easier and cheaper to obtain. Therefore, it is surprising to read in K’s theory [3] that “Expressions for the cycles to failure as a function of R ratio are derived. These expressions do not require any curve fitting and do not involve any experimentally determined parameters. The fatigue predictions do not require
any fatigue tests for calibration”. Further, that “comparison to several test cases found in the Literature show this first simple model to be very promising”, where for “several test cases”, K intends a few references (references [34-41] in his paper [3], which are [7-14] here), where the error is said to be small, but which in fact is not necessarily so. Take Fig.6 of K’s paper [3], where the agreement is said to be “very good”: K’s curve, which should be the median value, is seen to pass close to the lowest data, and hence the error in terms of life can be easily of 2 orders of magnitude. It is surprising that, after so much emphasis to statistical treatment, the assessment of the quality of data fit is so poor, and so much in favour of his own proposal. Not much better can be said regarding Fig.9 (where the author admits the “agreement to be not so good”), obviously the author prefers to measure the error based on stress, and claims a 17% error is found – it is not clear what kind of crude estimate he is making given the so few data, but clearly the error in terms of live can be various orders of magnitude. Similar problems were found in Fig.10, and Fig.11, where the data are so few that do not really deserve much attention.

Therefore the sentence in the conclusion “The approach allows analytical determination of the ratio of mean to B- or A-basis life which can be used in designing certification of qualification programs” seems premature. The data show that the link with static scatter is not as strong as to make any remote estimates of the SN curve slopes.

More recent assessment of K’s model confirm this much more cautious view, and indeed find naïf that anyone could expect much from static data statistical analysis, or when using K’s method as “predictive” as such, warn of the surprisingly poor results [15-17]. We shall draw attention in this paper to the fact that we do not expect theoretically any reason for a good predictive capability to be realistic in general.

Certainly, there is some connection between static data and fatigue limit (and therefore the SN curve slope tend to be similar for similar materials), but the fact that the SN curve slope should depend purely on the scatter of static data is an interesting and very potent result, if only it were true. This happens to be what predicted by K’s model, particularly simple in the form assuming a Weibull distribution of static strengths in which case the distribution of fatigue life is always an exponential one, independent on either strength or scatter of strength of the base material. The tendency to exponential distribution of fatigue lives, which was observed before [8, 18] also in much larger set of data, such a Navy database, is not a result of K’s
2 A view on firm evidence from large databases

Fleck Kang and Ashby [18], in an authoritative review which contains also data on composite materials, produce a large set of maps covering a huge number of references, and in particular show in Figure 1 the well-known fact that the endurance limit $\sigma_e$ scales in a roughly linear way with the yield strength, $\sigma_y$. The fatigue ratio, defined as $\sigma_e/\sigma_y$ (but more classically for metals, $\sigma_e/\sigma_{fs}$) at load ratio $R = -1$, appears as a set of diagonal contours. “The value of fatigue ratio, for engineering materials, usually lies between 0.3 and 1. Generally speaking, it is near 1 for monolithic ceramics, about 0.5 for metals and elastomers, and about 0.3 for polymers, foams and wood; the values for composites vary more widely—from 0.1 to 0.5”. Naturally, for fatigue limit in composites (as well as light alloys), it is often intended the value at a given fixed number of cycles. This wide variation already makes one wonder that for composites the fatigue properties depend less on static properties than for other materials. A first alarm bells rings, with respect to Kassapoglou’s claim. Moreover, Fleck Kang and Ashby [17] remark “The wide range of fatigue ratios shown by composites relates, in part, to the wide spectrum of materials used to make them, and to the necessarily broad definition of failure: in particulate composites, failure means fracture; in fibrous composites it means major loss of stiffness”.

Clearly, the exact mechanisms for fatigue limit, if there is one, are microscopic and however may interact with geometry, loading conditions, etc. For composite materials and structures in general, the failure mechanism can vary. For materials for which the endurance limit depends on formation of slip bands, it is obvious to find a correlation with yield strength, but a full microscopic model for the shape of the SN curve is more difficult. Fleck et al.[] summarize about SN curve: “It is the failure envelope associated with a sequence of interdependent phenomena: cyclic hardening, crack nucleation and cyclic growth, and final fast fracture”. For composites, the actual nature of each phenomenon is very different, but their interdependence is also clear.

K’s model is based on some statistical reasoning over the distribution of static strength, and the successive application of cycles. Making a certain number of (reasonable) assumption, he seems to derive apparently simple and clear results, which then he combines with a calculation of probability of failure. We shall discuss in the present note the basic results of K model
Figure 1: From Fleck Kang and Ashby [18] (with permission). Classical plot of static vs fatigue strength for many classes of materials.
in details.

3 Main assumptions in K’s model

K’s model in the original form [3] makes a certain number of assumptions, including that the probability of failure stays constant, cycle-by-cycle. Since the probability of failure during the first cycle is determined by a probability distribution function for the static strength, the author concludes that it remains the same for all subsequent cycles. In the later paper [4], this was obtained “rigorously”. He then has to compute the probability of failure, and essentially, he consider each cycle as an independent event without any discussion of this assumption. In fact, even if the probability of failure did remained constant cycle-per-cycle, this is incorrect calculation for a fatigue experiment.

As an illustrative example of this incorrect calculations let us consider the following discrete analogue:

1. There is a bucket full of balls that are numbered with 1 (which is analogous to failure of the specimen under a certain applied stress, “success”) and 0 (unsuccessful event, no failure of the specimen). Assume that the probability of failure is $p$, so that the probability of no-success is $q = 1 - p$.

2. One can perform a series of experiments counting the number $k$ of trials until the picked ball has a number 1 (success) on it. This event is called “first success”. After each attempt, in order to implement an analogue of the main assumption of the K’s model, one must put the picked ball back into the bucket: then, and only then the probability of success is exactly the same at each single trial. The resultant distribution of number of trials until “first success” is described by the geometric distribution density function: the probability $p_1$ that the “first success” occurs at the $k$th trial (cycle) is

$$p_1 (k) = p (1 - p)^{k-1}.$$  (1)

Eq. (1) is simply the probability of success (failure) at the last attempt (cycle) $k$ times the probability of no-failure at previous $k - 1$ attempts (cycles). The average expected number $\langle n \rangle$ of trials until the first
success is then given by

\[ \langle n \rangle = \sum_{k=1}^{+\infty} kp (1 - p)^{k-1} = \frac{1}{p}, \]  

(2)

as indeed should be expected.

3. We can also calculate the probability that the failure of the sample occurs during the first \( n \) experiments (cycles), i.e. that the event “first success” happens at any cycle \( k \) in between 1 and \( n \). This probability is the cumulative distribution of (1)

\[ P(n) = \sum_{k=1}^{n} p_1(k) = \sum_{k=1}^{n} p (1 - p)^{k-1} = 1 - (1 - p)^n \]  

(3)

\[ = 1 - \exp\left(-n \log \frac{1}{1-p}\right). \]

If now the number of trials \( n \) is treated as a continuous variable, this function represents the so-called cumulative exponential distribution with the mean value

\[ \langle n \rangle = \frac{1}{\log \frac{1}{1-p}} = -\frac{1}{\log(1 - p)}, \]  

(4)

and if \( p \) is small, the two averages (2) and (4) become close.

This trivial example implements the main assumption of K’s model. However, a randomly picked specimen is tested each time, each test is an independent static test, being the probability of failure \( p \) equal to the probability that the static strength of the specimen is less than the applied static stress \( \sigma \). Obviously, this type of test is irrelevant to the fatigue phenomenon. The resulting relation between the mean number of cycles and the applied load, mistakenly claimed by the author to be the SN curve, is actually the mean number of tested specimens until failure! If the same specimen undergoes subsequent loadings, it may fail only if its static strength decreases with cycles. However, this static strength degradation should be a material specific function and is not uniquely determined by the statistics of the static strength.

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While dispersion in the static strength of a material reflects the possible level of initial damage observed in the material, the fatigue failure phenomenon is the result of the initial damage growth and accumulation with cyclic loading. This growth would eventually appear in any specimen independently of the strength in other specimens and the statistics that describes the static strength scatter.

4  K’s model with strength degradation

In his 2012 PhD thesis [5] and in his 2011 paper [4] the author extends the model and incorporates the residual strength degradation. Suppose that a constant amplitude load with maximum stress $\sigma (R = \sigma_{\text{min}}/\sigma = 0)$ is applied to a composite structure. If the static failure strength $\sigma_{fs}$ of this structure is less or equal to $\sigma$ (i.e. $\sigma_{fs} \leq \sigma$) the structure will fail at $n = 0$, i.e. before the first cycle is completed, whilst if $\sigma_{fs} > \sigma$ the structure will fail at the cycle $n = N$. Within the K’s wear-out model the quantity $N$ is treated as constant and no failure can occur for $0 < n < N$. These assumptions are, as we show in the sequel, the most critical flaws of K’s model.

If the test is stopped at any cycle level $n < N$ the structure would not have failed and would still be able to carry load. However, a strength test on the structure would show a failure strength $\sigma_{fs} > \sigma_r > \sigma$, where $\sigma_r$ is the residual strength. Hence, during cycling, $\sigma_r$ decreases from the static failure strength $\sigma_{fs} > \sigma$ at $n = 0$, to $\sigma$ after $N$ cycles. K’s model starts with the assumption that the change in residual strength is proportional to the current residual strength, which in the simple case of zero fatigue limit can be written in the form

$$\frac{d\sigma_r}{dn} = -A\sigma_r,$$

where $A > 0$ is independent of $n$ and $\sigma_r$. This wear-out model assumes that under a fixed amplitude the strength of a specimen with higher residual strength stress will degrade faster than one with lower residual strength, which is physically unreasonable. That is why a typical wear-out model would define the strength degradation rate as a reciprocal to the current residual strength. The above expression in K’s model results in a residual strength

$$\sigma_r = \sigma_{fs} \left( \frac{\sigma}{\sigma_{fs}} \right)^{n/(N-1)}.$$

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Treating $\sigma$ as a constant would result, following K’s arguments, in a Weibull cumulative distribution $P_W(\sigma_r; \beta_r, \alpha_r)$ for residual strength with shape and scale parameters

$$
\alpha_r = \alpha \frac{N - 1}{N - n - 1}; \quad \beta_r = \sigma^n (N - 1) \frac{N - n - 1}{N - 1},
$$

which vary with $n$ in a simple manner, implying a clear reduction in experimental scatter with lower stress level of testing (or longer lives). In particular, for $n \to N - 1$, $\alpha_r \to \infty$ which means that the distribution converges to the Dirac Delta function, the residual strength becomes a deterministic function, and $\beta_r = \sigma$ consistently to the fact that the residual strength tends exactly to the SN curve.

There are many critical inconsistencies that one can spot immediately: Firstly, $N$ from the SN curve should itself be a variate, instead of being a deterministic and constant quantity. Secondly, with a constant amplitude $\sigma$, during cycling the residual strength distribution cannot approach the SN curve from below, since a specimen that has strength below $\sigma$ should have failed at an earlier stage and for any $n$ the residual strength distribution should be truncated from below by the applied stress $\sigma$. One can even and easily show that, in contrast with what has been claimed by K, K’s model cannot lead to a cycle-by-cycle constant probability of failure $p(n)$, which indeed, even within the hypothesis of K’s wear out model, would be

$$
p(n) = \delta_{0n} P_W(\sigma; \beta, \alpha) + \delta_{nN} [1 - P_W(\sigma; \beta, \alpha)],
$$

where $\delta_{kh}$ is the Kronecker delta (see Appendix II). The discussed inconsistencies in the model development refute the claim of rigorous proof of constant cycle-by-cycle probability of failure.

5 Comparisons with experimental data

The original K model did not find an exact Weibull distribution for the life, but over a very wide range of $p$ values ($p < 0.1$ hence, unless the applied load is very high, and close to the static strength) the two ratios of mean and modal lives to B-Basis life (respectively 17.86 and 8.93) are essentially constant. This suggested the author of the original K model that the average of the two ratios, 13.4 is very close to the value of 13.6 determined in the NAVY reports [6] after statistical analysis of thousands of data points.
Using a slightly different re-derivation (see Appendix I), we show that K’s approach essentially obtains an exponential distribution of fatigue lives, for a Weibull starting point in scatter of static data. Hence, it is correct to say that K’s approach, in a slightly re-elaborated form (see Appendix I), we easily obtain the estimate on fatigue ratio (FR, defined as the ratio between fatigue limit at $10^6$ cycle, and the “static value” at $N = 1$), as

$$FR(\alpha) = \frac{\sigma_{\text{max,lim},10^{-6}}}{\sigma_{\text{max,1}}} = 10^{(-6/\alpha)},$$  \hspace{1cm} (9)

where \(\alpha\) is a Weibull shape parameter in a 2 parameters Weibull distribution, of the static strength distributions. Notice that in this form, K’s model includes also the R-ratio effect, for \(0 < R < 1\) in the cases of pure tensile loading. For \(\alpha\) we can make use not of few sparse references like K, but the thousands of test done by the Navy. The distribution of fatigue lives is not as unique as to be a exponential \(\alpha_L = 1\). A full distribution was found, which we may define the distribution of “scatter of fatigue lives”, \(\alpha_L\). A generally accepted approximation is to take \(\alpha_L = 1.25\), but it is clear that its distribution is relatively wide, and depends also on the method used for analysis.

Also, the distribution of static strength scatter \(\alpha\) has in turn a distribution, which we can denominate \(\alpha_a\), see Figure 2.

We can use the mean value of \(\alpha\) which results from the Figure 2 to be 26. This corresponds to a value

$$FR(\alpha_{\text{mean}}) = 10^{(-6/26)} = 0.5878,$$  \hspace{1cm} (10)

which is outside the known values of Ashby fatigue ratios. This suggests, as confirmed by most experimental data we shall describe, that K’s method would tend to give unconservative estimates, as too high FR. This however depends very much in the type of materials under examinations. If the materials are of “poor” quality, full of defects, tending to having low \(\alpha\), then the very steep SN curve predicted by K may neglect the possible phase in SN curve where the degradation is not so evident, resulting extremely conservative. On the other hand, for materials having very high \(\alpha\) like close to a metal, K’s theory fails to capture the wearout at all, and results in too optimistic SN curve. Unidirectional laminates will tend to have very horizontal fatigue lines, yet their static scatter may be significant.

Hence, even if the agreement with Navy experiments has some very loose qualitative agreement in terms of scatter of fatigue lives, this is an oversim-
Figure 2: (a) distribution of fatigue lives scatter $\alpha_L$ and (b) of scatter of static strength $\alpha$
plification (a single mode value instead of the full distribution) and the huge risks of using this approximation even as a crude estimate, is evident.

The recent investigation on K’s method by the FAA (Tomblin and Seneviratne, 2011 [5], Appendix A) finds also the SN curve predicted by the original K model (which we found here is the mean life curve) to have rather erratic comparison with experimental data. In particular, in 14 sets of data, K’s model was found

- accurate only for 2 sets
- conservative only for 2 sets (both by 1-2 orders of magnitude)
- unconservative for the vast majority of data (10 sets), of which 5 perhaps by 1-2 orders of magnitude, 3 by 2-3 orders of magnitude, and 2 by 4-5 orders of magnitude.

Clearly, although this statistics don’t say much, it shows a tendency of K’s method to overpredict fatigue life by large factors. Examples given in chapt.6 of [3] and in [19] show that more sophisticated methods with variable $p$ function may improve the situation, although the examples given tend to predict longer lives than the original K method — we are not able to judge if the errors and approximations from the original K theory with ”constant $p$ value” continue to manifest their negative effect here. S/N curves based on the Sendeckyj analysis [2] were found generally accurate and conservative, but this is to be expected since that method is a fitting method of SN data.

In a recent book, Vassilopoulos & Keller [16] compare 4 methods to make a statistical analysis of fatigue data, which is a problem of enormous industrial interest since aeronautical structures are designed and certified using SN curves that correspond to high reliability levels in the range above 90% and conform with design codes, but without an impossibly expensive program of fatigue testing on a population of full scale structures. The method based on the normal lifetime distribution (“NLD”) was found non-conservative, giving a median SN curve which is closer to the median SN curve of ASTM than the 95% reliability one. Whitney’s pooling scheme and Sendeckyj’s wear-out model are found to produce similar SN curves, with Whitney’s easier to implement, as not requiring any optimization process, and Sendeckyj being also less conservative. However, this is mainly due to the need of multiple fatigue results at each stress level, and no capability to consider static data. Some significant problems were found in the fitting of Sendeckyj’s constant
process, with strange slopes of the SN curve predicted, particularly when
disregarding static strength data. A discussion follows on the appropriateness
of including the static data in the fitting.

K’s method is not even compared to the previous four, mainly because
the static data were not enough to fit Weibull distributions. It is discussed
however in its extension to describe mean stress effect, in a later chapter
on Constant Life Diagrams. However, its assumptions are negatively judged
“*This assumption oversimplifies the reality and masks the effect of the dif-
ferent damage mechanisms that develop under static loading and at different
stages of fatigue loading.*”, and “*the restricted use of static data disregards
the different damage mechanisms that develops during fatigue loading and in
many cases leads to erroneous results*”. In the evaluation of K’s model for
one database the model “proved to be inaccurate for the examined material’s
fatigue data”.

6 Conclusion

In the original K’s model there is a confusion between what we call fatigue
and statistics of the static strength of a number of specimens. Fatigue life
(number of cycles) is mistakenly replaced with the number of tested speci-
mens to find a specimen with strength less than applied load. This number
of specimen indeed solely depends on the initial statistical distribution of the
static strength, while fatigue is related to damage accumulation in a speci-
men and its strength degradation with cycles, which contradicts the main
assumption. One can also mistakenly deduce from the proposed model that
if there is no dispersion in the static strength, for instance, all the specimen
have exactly the same static strength, there is no such thing as an SN curve.
K model is an interesting attempt of using wear-out models with degrada-
tion deterministic equations to predicting SN curves from static data only
for composites (something which is not easy even with metals). However,
its results don’t look realistic at all, and indeed we have explained here why.
Not surprisingly, SN curves found in many independent assessments were
found to be (generally) *unconservative* for the vast majority of data (10 sets)
considered in FAA 2011 report [15], at least by 1-2 orders of magnitude.
We have given reasons for this effect, both theoretically and with additional
estimates from large set of results from databases of composite materials.

Only "fitting" models can be considered reliable, as discussed by Vasil-
ilopoulos & Keller [16], and it should be remarked in this respect that an
additional interesting wearout model is [21-23].

Appendix I - SN curve of K’s model in slightly different form

We have shown that K’s model is incorrect. However, a simpler form can be adapted for comparative form in a much simpler form. In particular, using this equation for a SN curve at any quantile $Q$

$$N = \left( \frac{\beta}{\sigma} \right)^{\alpha} \left[ -\log (1 - Q) \right], \quad (11)$$

which obviously has mean value $N_m = (\beta/\sigma)^{\alpha}$ and modal value $N_{mod} = (\beta/\sigma)^{\alpha} \log 2$, but mode value zero (because the distribution of lives is an exponential distribution $\alpha_L = 1$), we obtain a closed form version of the "incorrect" K model, which can be used more easily than the original K model which is not in closed form, and which obtains only the mode life $N_c$

$$N_c = \left( \frac{\beta}{\sigma} \right)^{\alpha}, \quad (12)$$

which in the present result, coincides with the present mean value. The distribution in terms of stress for given number of cycles, $P_W(\sigma; \frac{\beta}{N^{1/\alpha}}, \alpha)$

$$\sigma = \left( \frac{\beta}{N^{1/\alpha}} \right) \left[ -\log (1 - Q) \right]^{1/\alpha}; \quad (13)$$

this has obviously mean value $\sigma_m = (\beta/N^{1/\alpha}) \Gamma \left( 1 + \frac{1}{\alpha} \right)$, whereas median value $\sigma_{med} = (\beta/N^{1/\alpha}) \left( \log 2 \right)^{1/\alpha}$. In other words, in this new form, the SN derives from Weibull distribution both in terms of stress and life at all levels of stress including the original static distribution $P_W(\sigma; \beta, \alpha)$ which is obtained consistently for $N^{1/\alpha} = 1$.

Appendix II - Inconsistencies of Kassapoglou’s model

Suppose we want to calculate the cumulative probability distribution of failure $P(n)$ during the first $n$ cycles assuming K’s wear-out model

$$\frac{\sigma_r}{\sigma_{fs}} = \left( \frac{\sigma}{\sigma_{fs}} \right)^{n/(N-1)}, \quad (14)$$
where the $\sigma_r$ is the residual strength, $\sigma_{fs}$ is the static strength, $\sigma$ is the applied fatigue load, $n$ is the actual cycle number, and $N$ is the number of cycle at which the samples fails (assuming its static strength is larger than $\sigma$). Equation (14) simply states that: (i) all sample which have a static strength $\sigma_{fs}$ larger than the fatigue stress $\sigma$ will fail at the same given number of cycles $n = N$, and (ii) samples with static strength $\sigma_{fs}$ less than $\sigma$ will all fail at cycle $n = 0$. Therefore samples may fail either at $n = 0$ when $\sigma_{sf} \leq \sigma$ or at $n = N$ when $\sigma_{sf} > \sigma$, no failure may occur in between i.e. for $0 < n < N$. Within the K’s wear-out model we have: (a) the probability that failure occurs at $n = 0$ is $p_0 = P_W(\sigma; \beta, \alpha)$, (b) the probability that failure occurs at $n = N$ is $p_N = P(\sigma_S > \sigma) = 1 - P_W(\sigma; \beta, \alpha)$, (c) the probability that failure occurs at $n$ satisfying the condition $0 < n < N$ must be equal to the probability that failure occurs at cycle 0, i.e.

$$P_n = p_0 = P_W(\sigma; \beta, \alpha); \quad 0 \leq n < N,$$

whereas considering that for $n \geq N$ failures has necessarily occurred one as

$$P_n = 1; \quad n \geq N.$$

Therefore the cumulative distribution presents two steps one of amplitude $P_W(\sigma)$ at $n = 0$ ant the other of amplitude $1 - P_W(\sigma; \beta, \alpha)$ at $n = N$. In between the cumulative probability distribution is constant. We stress that, as already shown, the probability of failure cycle per cycle is not constant indeed it is:

$$p_n = P_W(\sigma; \beta, \alpha); \quad n = 0,$$

$$p_n = 0; \quad 0 < n < N,$$

$$p_n = 1 - P_W(\sigma; \beta, \alpha); \quad n = N,$$

$$p_n = 0; \quad n > N,$$

in compact notation

$$p(n) = \delta_{0n}P_W(\sigma; \beta, \alpha) + \delta_{nN}[1 - P_W(\sigma; \beta, \alpha)],$$

(15)
where $\delta_{jk}$ is Kronecker’s delta. Eq. (15) shows that the probability of failure cycle per cycle is zero for $0 < n < N$, thus revealing one of the serious mistakes of the Kassapoglou model where the cycle per cycle probability of failure was assumed different from zero and equal to $P_W(\sigma; \beta, \alpha)$. Indeed, Eq. (15) shows that the sample life $n$ is a discrete statistical quantity which only takes two different values $n = 0$ and $n = N$, and failure at $n = 0$ occurs with probability $P_W(\sigma; \beta, \alpha)$ whereas failure at $n = N$ occurs with probability $1 - P_W(\sigma; \beta, \alpha)$. This allows to calculate within the wear-out model (14) the expected life of the samples as

$$\langle n \rangle = N \left[ 1 - P_W(\sigma; \beta, \alpha) \right].$$

which as expected differs from the value obtained by K.

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