A note on the Krein–Rutman theorem for sectorial operators

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Abstract
In this paper, we present some generalized versions of the Krein–Rutman theorem for sectorial operators. They are formulated in a fashion that can be easily applied to elliptic operators. Another feature of these generalized versions is that they contain some information on the generalized eigenspaces associated with nonprincipal eigenvalues, which are helpful in the study of the dynamics of evolution equations in ordered Banach spaces.

KEYWORDS
elliptic eigenvalue problem, Krein–Rutman theorem, sectorial operator

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1 | INTRODUCTION

It is well known that the Krein–Rutman–type theorems (see, e.g., [3, 16, 21]) for bounded operators (particularly for compact operators) play a crucial role in the discussion of the principal eigenvalue problem of elliptic operators via their resolvents. But generally only part of the information concerning the principal eigenvalue and eigenvectors can be obtained. One reason is that there may be no one-one correspondence between the boundary spectrum of an elliptic operator $A$ and the peripheral spectrum of its resolvent $R_\lambda(A)$. As a remedy, one has to do some tricky partial differential equation (PDE) argument when refined information on the principal eigenvalue and eigenvectors is needed; see, for example, Du [5, Theorem 1.4], Evans [6, section 6.5, Theorem 3], and Ni [19]. It is therefore of particular interest to extend the Krein–Rutman–type theorems from bounded operators to unbounded ones.

In Greiner et al. [9], it was proved that the spectral bound $\text{spb}(A)$ of the generator $A$ of a positive $C_0$-semigroup on a Banach space with a normal reproducing cone that is contained in $\sigma(A)$. The monograph [2] contains a far-reaching theory concerning the above question in the framework of generators of $C_0$-semigroups on the functional space $C_0(X)$ consisting of continuous functions vanishing at infinity, where $X$ is a locally compact space; see [2, Chapter B-III]. In a recent paper [12], the authors developed some generalizations of the Krein–Rutman theorem for generators of $C_0$-semigroups from the point of view of tangentially positive operators. Relevant results can also be found in the references cited in the above-mentioned works. These extensions make the Krein–Rutman theorem more efficient in studying the spectral properties of unbounded operators.

When dealing with the elliptic differential operator $L$ given by

$$Lu = -\sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x)u \quad (1.1)$$

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via the $L^p$-theory of PDEs, the frequently used spaces in which $L$ may generate a $C_0$-semigroup are the Sobolev ones. However, in many cases, the cone of nonnegative functions in a Sobolev space may fail to have an interior or to be normal (these two properties were required in most abstract results in the literature). Of course, $L$ can be considered as an operator in the space $C^{k,α}(Ω)$, in which the cone $C$ of nonnegative functions has a nonempty interior. But as it was pointed in Kielhöfer [14], $L$ can fail to generate a $C_0$-semigroup in this space. To cover this situation, Nussbaum [20] made an effort to extend a major part of the Krein–Rutman theorem to operators that may not generate a $C_0$-semigroup.

In a recent paper [18], the first author and his colleague Jia reformulated the classical Perron–Frobenius theory and Krein–Rutman theorem by using an elementary dynamical approach. Inspired by this work, in this paper, we present another generalization of the Krein–Rutman theorem for sectorial operators in a formalism that seems to be more natural and suitable for elliptic operators. It is different in some ways from those mentioned above and can be easily applied to elliptic operators, which enables us to reduce significantly the technical PDE argument involved in the investigation of elliptic eigenvalue problems. Another feature of these generalized versions is that they contain some information on the generalized eigenspaces associated with nonprincipal eigenvalues, which are helpful in the study of the dynamics of evolution equations in ordered Banach spaces (see, e.g., [11]). As an illustrating example, the principal eigenvalue problem of the general elliptic operator $L$ as given in (1.1) is discussed under degenerate mixed boundary conditions.

We mention that [18] was mainly devoted to the principal eigenvalue problem of bounded operators. Although this work focuses on unbounded operators, the fundamental results in [18] form a basis of the present one. Our basic idea is to transform the principal eigenvalue problem of an unbounded operator into that of a bounded one via appropriate spectral decompositions.

The remaining part of this paper is organized as follows. In Section 2, we do some preliminary work, and in Section 3, we prove generalized Krein–Rutman–type theorems (Theorems 3.1, 3.3, and 3.5) for sectorial operators. Section 4 consists of an example illustrating the theoretical results mentioned above.

## 2 | PRELIMINARIES

Let $X$ be a real Banach space with norm $\| \cdot \|$. Given $M \subset X$, the interior and closure of $M$ are denoted, respectively, by $\overset{\circ}{M}$ and $\overline{M}$. When we need to emphasize in which space the interior and closure are taken, we also use the notations $\overset{\circ}{M}_X$ and $\overline{M}_X$ in place of $\overset{\circ}{M}$ and $\overline{M}$, respectively. For $x \in X$, set

$$d(x, M) = \inf_{y \in M} \| x - y \|.$$

### 2.1 | Some basic knowledge in the spectral theory of operators

Let $A$ be a closed densely defined operator in $X$. Denote by $\sigma(A)$ and $\varrho(A)$ the spectrum and resolvent set of $A$, respectively. For $\lambda \in \varrho(A)$, let $R_\lambda(A) := (\lambda - A)^{-1}$ be the resolvent of $A$. Let $X = X + iX$ be the complexification of $X$ (see, e.g., [9, section 3] for details), and define the complexification $\mathbb{A}$ of $A$ as

$$\mathbb{A}u = Ax + iAy, \quad \forall u = x + iy \in X.$$

Given $\mu \in \sigma(A) = \sigma(\mathbb{A})$, let

$$\text{GE}_\mu(\mathbb{A}) = \{ \xi \in X : (\mathbb{A} - \mu) \xi = 0 \text{ for some } j \geq 1 \}.$$

Then, $\text{GE}_\mu(\mathbb{A})$ is an invariant subspace of $\mathbb{A}$. For each $\xi \in \text{GE}_\mu(\mathbb{A}) \setminus \{0\}$, it is clear that there is an integer $k \geq 1$ such that

$$(\mathbb{A} - \mu)^j \xi \neq 0 \quad (0 \leq j \leq k - 1), \quad (\mathbb{A} - \mu)^k \xi = 0. \quad (2.1)$$

For convenience in statement, we call the number $k$ in (2.1) the rank of $\xi$, denoted by $\text{rank}(\xi)$. 

For \( \mu \in \sigma(A) \), it follows by the invariance of \( GE_\mu(A) \) that

\[
GE_\mu(A) := \{ \text{Re } \xi : \xi \in GE_\mu(A) \} = \{ \text{Im } \xi : \xi \in GE_\mu(A) \}
\] (2.2)

is an invariant subspace of \( A \) in \( X \), which will be referred to as the \textit{generalized eigenspace} of \( A \) pertaining to \( \mu \). (The second equality in (2.2) is due to the fact that if \( \xi \in GE_\mu(A) \), then \( \pm i \xi \in GE_\mu(A) \).)

\textbf{Lemma 2.1.} Let \( \mu \in \sigma(A) \) and \( \lambda \in \rho(A) \). Then,

\[
R_\lambda(A)GE_\mu(A) = GE_\mu(A) = GE_{\lambda-\mu}^{-1}(R_\lambda(A)).
\] (2.3)

\textit{Proof.} We believe this is a basic knowledge in the spectral theory of linear operators. Here, we give a proof for the readers’ convenience.

To prove (2.3), it suffices to check that

\[
R_\lambda(A)GE_\mu(A) = GE_\mu(A) = GE_{\lambda-\mu}(R_\lambda(A)),
\]

where \( \lambda_\mu := (\lambda - \mu)^{-1} \).

Let \( \xi \in GE_\mu(A) \). Then, \((A - \mu)^k \xi = 0\) for some \( k \geq 1 \). Hence,

\[
(A - \mu)^k (R_\lambda(A) \xi) = R_\lambda(A) (A - \mu)^k \xi = R_\lambda(A) 0 = 0.
\]

It follows that \( R_\lambda(A) \xi \in GE_\mu(A) \). Simple calculations also yield

\[
(\lambda - A)^k (R_\lambda(A) - \lambda_\mu)^k \xi = \lambda_\mu^k (A - \mu)^k \xi = 0.
\] (2.4)

Since \( \lambda \in \rho(A) \), (2.4) implies that \( (R_\lambda(A) - \lambda_\mu)^k \xi = 0 \). Therefore, \( \xi \in GE_{\lambda_\mu}(R_\lambda(A)) \). In conclusion, we have

\[
R_\lambda(A)GE_\mu(A) \subset GE_\mu(A) \subset GE_{\lambda_\mu}(R_\lambda(A)).
\]

The verification of the inverse inclusions is similar. We omit it. \( \square \)

Denote by \( \sigma_e(A) \) the \textit{essential spectrum} of \( A \) in the terminology of Browder [4, pp. 107–108, Definition 11]. Then, each \( \mu \in \sigma(A) \setminus \sigma_e(A) \) is isolated in \( \sigma(A) \) with \( GE_\mu(A) \) being a finite-dimensional subspace of \( X \); see [4, p. 108].

The \textit{spectral bound} \( \text{spb}(A) \) and \textit{essential spectral bound} \( \text{spb}_e(A) \) of \( A \) are defined as

\[
\text{spb}(A) = \sup \{ \text{Re } \mu : \mu \in \sigma(A) \}, \quad \text{spb}_e(A) = \sup \{ \text{Re } \mu : \mu \in \sigma_e(A) \}.
\]

(We assign \( \text{spb}_e(A) = -\infty \) if \( \sigma_e(A) = \emptyset \).) Set

\[
\sigma_b(A) = \sigma(A) \cap \{ z \in \mathbb{C} : \text{Re } z = \text{spb}(A) \}.
\]

\( \sigma_b(A) \) is called the \textit{boundary spectrum} of \( A \).

Let \( \mathcal{L}(X) \) be the space of bounded linear operators on \( X \). If \( A \in \mathcal{L}(X) \), we define the \textit{spectral radius} \( r(A) \) and \textit{essential spectral radius} \( r_e(A) \) as

\[
r(A) = \sup \{ |\mu| : \mu \in \sigma(A) \}, \quad r_e(A) = \sup \{ |\mu| : \mu \in \sigma_e(A) \}.
\]

It is basic knowledge that \( r(A) = \lim_{k \to \infty} \| A^k \|^{1/k} \).
2.2 Cones and positive operators

Let $X$ be a Banach space. A wedge in $X$ is a closed set $K \subset X$ with $K \neq \{0\}$ such that $tK \subset K$ for all $t \geq 0$.

A convex wedge $K$ in $X$ with $K \cap (-K) = \{0\}$ is called a cone. Let $K$ be a cone in $X$. We say that $K$ is total if $\overline{K - K} = X$.

It is said to be solid if $K \neq \emptyset$.

From now on, we assume that there has been given a cone $K$ in $X$.

An operator $A \in \mathcal{L}(X)$ is called positive (resp. strongly positive) if $AK \subset K$ (resp. $A(K \setminus \{0\}) \subset K$).

Definition 2.2 [18]. In case $K$ is a solid cone, a positive operator $A$ is called weakly irreducible, if the boundary $\partial K$ of $K$ contains no eigenvectors of $A$ pertaining to nonnegative real eigenvalues.

Remark 2.3. It is almost obvious that strongly positive operators and primitive operators are weakly irreducible. (Recall that $A$ is said to be primitive, if there is an integer $m \geq 1$ such that $A^m(K \setminus \{0\}) \subset K$; see [17, p. 285].) We also infer from the argument following Definition 7.5 in [18] that irreducibility (in the terminology of [17]) implies weak irreducibility defined as above.

We refer the interested readers to [2, 3, 15, 16, 20, 21] for example for fundamental results on the principal eigenvalue problem of positive operators. Here, we recall some generalized versions of the classical Krein–Rutman theorem given in [18] to conclude this section.

Theorem 2.4 [18, Theorem 7.3]. Let $A \in \mathcal{L}(X)$ be a positive operator with

$$r_e := r_e(A) < r(A) := r.$$  \hfill (2.5)

If $K$ is total, then the following assertions hold true:

1. $r$ is an eigenvalue of $A$ with a principal eigenvector $u \in K$.
2. If $K$ contains a principal eigenvector $v$ of $A$, then the algebraic and geometric multiplicities of $r$ coincide.
3. If $\mu$ is a complex eigenvalue with $|\mu| > r_e$, then $\mathrm{GE}_\mu(A) \cap K = \{0\}$.
4. All eigenvectors of $A$ pertaining to eigenvalues $\mu \neq r$ with $|\mu| > r_e$ are contained in $X \setminus \overline{K}$.

Theorem 2.5 [18, Theorems 7.7 and 7.9]. Let $A \in \mathcal{L}(X)$ be a positive operator satisfying (2.5). Suppose $K$ is solid, and that $A$ is weakly irreducible. Then,

1. $r$ is a simple eigenvalue of $A$ with a principal eigenvector $w \in \overline{K}$;
2. $\mathrm{GE}_\mu(A) \cap K = \{0\}$ for any $\mu \in \sigma(A) \setminus \{r\}$ with $|\mu| > r_e$;
3. If $A$ is strongly positive, then

$$|\mu| < r, \quad \forall \mu \in \sigma(A) \setminus \{r\}.$$

3 KREIN–RUTMAN–TYPE THEOREMS FOR SECTORIAL OPERATORS

Let $X, Y$ be two real Banach spaces with $Y \hookrightarrow X$; moreover, $Y$ is dense in $X$. Denote by $\| \cdot \|$ and $\| \cdot \|_1$ the norms of $X$ and $Y$, respectively.

Let $K$ be a cone in $Y$, and $A$ a closed densely defined operator in $X$ with $-A$ being sectorial (see [10, Chapter 1] for definition). We always assume that the following standing assumptions are fulfilled:

1. $R_\lambda(A)K \subset K$ for $\lambda > 0$ sufficiently large.
2. $s_e := \mathrm{spb}_e(A) < \mathrm{spb}(A) := s$.
3. $\mathrm{GE}_\mu(A) \subset Y$ for every $\mu \in \sigma(A)$ with $\text{Re} \mu > s_e$. 

(A1) 

(A2) 

(A3)
As usual, if \( s \in \sigma(A) \), then we call \( s \) the \textit{principal eigenvalue} of \( A \). Consequently eigenvectors pertaining to \( s \) are referred to as \textit{principal eigenvectors}.

One of our main results is the following general Krein–Rutman–type theorem.

**Theorem 3.1.** Assume \( K \) is total in \( Y \). Then, the following assertions hold:

1. \( s \) is an eigenvalue of \( A \) admitting a principal eigenvector \( u \in K \).
2. If \( \operatorname{int}_Y K \neq \emptyset \) and contains a principal eigenvector of \( A \), then \( s \) shares the same algebraic and geometric multiplicities.
3. All eigenvectors of \( A \) corresponding to other eigenvalues \( \mu \neq s \) with \( \Re \mu > s_e \) are contained in \( Y \setminus \operatorname{int}_Y K \).
4. If \( \mu \in \sigma(A) \), \( \Re \mu > s_e \) and \( \Im \mu \neq 0 \), then

\[
\operatorname{GE}_\mu(\mathcal{A}) \cap K = \{0\}.
\]

**Proof.** For \( t \in \mathbb{R} \), \( t > s_e \), set

\[
\Sigma_1(t) := \{ \mu \in \sigma(A) : \Re \mu \geq t \}.
\]

Since \( -A \) is sectorial, \( \Sigma_1(t) \) is a compact subset of \( \mathbb{C} \) for every \( t > s_e \) (by the definition of sectorial operators). As every \( \mu \in \sigma(A) \setminus \sigma_e(A) \) is isolated in \( \sigma(A) \), one concludes that \( \Sigma_1(t) \) consists of a finite number of elements.

Let \( \eta \in (s_e, s] \) and \( \Sigma_0(\eta) = \sigma(A) \setminus \Sigma_1(\eta) \). By the finiteness of \( \Sigma_1(t) \) \( (t > s_e) \), one trivially checks that for some \( \delta = \delta(\eta) > 0 \),

\[
\Re \mu \leq \eta - \delta, \quad \forall \mu \in \Sigma_0(\eta).
\] (3.1)

Hence, \( \Sigma_0(\eta) \) and \( \Sigma_1(\eta) \) form a spectral decomposition of \( \sigma(A) \). Denote by

\[
X = X_0(\eta) \oplus X_1(\eta)
\] (3.2)

the corresponding decomposition of \( X \). Then, \( X_1(\eta) = \bigoplus \mu \in \Sigma_1(\eta) \operatorname{GE}_\mu(\mathcal{A}) \) is a finite-dimensional subspace of \( X \). By (A3) we have \( X_1(\eta) \subset Y \).

For notational simplicity, we rewrite \( X_i(\eta) := X_i \) \( (i = 0, 1) \). Let us split the argument below into several steps.

**Step 1.** We show that

\[
X_1 \cap K \neq \{0\}.
\] (3.3)

Let \( P = \operatorname{Cl}_X K \), the closure of \( K \) in \( X \). Obviously \( P \) is a cone in \( X \). Recalling that \( X_1 \subset Y \), to prove (3.3), it suffices to check that

\[
X_1 \cap P \neq \{0\}.
\]

For this purpose, put \( \tilde{A} = A - \eta + \delta \), where \( \delta \) is the positive number in (3.1). \( \sigma(\tilde{A}) \) has a corresponding spectral decomposition \( \sigma(\tilde{A}) = \tilde{\Sigma}_0(\eta) \cup \tilde{\Sigma}_1(\eta) \) with

\[
\tilde{\Sigma}_i(\eta) = \Sigma_i(\eta) - \eta + \delta, \quad i = 0, 1.
\]

We observe that

\[
\sup\{\Re \mu : \mu \in \tilde{\Sigma}_0(\eta)\} \leq -\delta, \quad \inf\{\Re \mu : \mu \in \tilde{\Sigma}_1(\eta)\} \geq \delta.
\] (3.4)

The direct sum decomposition of \( X \) corresponding to the above spectral decomposition of \( \sigma(\tilde{A}) \) remains the same as in (3.2).

We claim that \( P \not\subset X_0 \). Indeed, suppose on the contrary that \( P \subset X_0 \). Then, since \( K \) is total in \( Y \), one would have

\[
Y = \operatorname{Cl}_Y (K - K) \subset \operatorname{Cl}_X (X - K) \subset \operatorname{Cl}_X (P - K) \subset X_0.
\]
We emphasize that the closures $\text{Cl}_Y$ and $\text{Cl}_X$ are taken with respect to the topologies of $Y$ and $X$, respectively.) Because $Y$ is dense in $X$, we therefore have $X = \text{Cl}_Y Y \subset \text{Cl}_X X_0 = X_0$, a contradiction.

Take a $u_0 \in P \setminus X_0$. Write $u_0 = x_0 + x_1$, where $x_i \in X_i$. Clearly $x_1 \neq 0$. Let $u(t) = e^{t\hat{A}}u_0 \ (t \geq 0)$, where $e^{t\hat{A}}$ is the $C_0$-semigroup generated by $\hat{A}$. Then,

$$u(t) = e^{t\hat{A}}x_0 + e^{t\hat{A}}x_1 := x_0(t) + x_1(t).$$

We infer from (3.4) that

$$\lim_{t \to \infty} \|x_0(t)\| = 0, \quad \lim_{t \to \infty} \|x_1(t)\| = \infty.$$  \hspace{1cm} (3.5)

By (A1) we have $R_\lambda(A)K \subset K \subset P$ for $\lambda > 0$ sufficiently large. Therefore,

$$R_\lambda(A)P = R_\lambda(A)\overline{K} = \overline{R_\lambda(A)K} \subset \overline{K} \subset P,$$ \hspace{1cm} (3.6)

where the closures are taken in $X$. (The second equality in (3.6) is due to the fact that $R_\lambda(A) \in \mathcal{L}(X)$.) This guarantees that $A$ is semigroup positive, that is, $e^{tA}P \subset P$ for $t \geq 0$ (see, e.g., Kato [13, Lemma 5.1]). Hence,

$$e^{t\hat{A}}P = e^{t(\delta-\eta)}e^{t\hat{A}}P \subset P, \quad t \geq 0.$$

In particular, we have $u(t) = e^{t\hat{A}}u_0 \in P$ for all $t \geq 0$.

Now we show that $X_1 \cap P \neq \{0\}$ and complete the proof of (3.3). First, by the first equality in (3.5), we see that $\lim_{t \to \infty} d(u(t),X_1) = 0$. Now suppose on the contrary that $X_1 \cap P = \{0\}$. Then, by [18, Lemma 2.4], one deduces that $\lim_{t \to \infty} \|u(t)\| = 0$. This contradicts (3.5).

**Step 2.** The verification of assertions (1) and (2).

Take $\eta = s$. Then, $\Sigma_1(\eta) = \sigma_\partial(A)$. Let $K_1 = X_1 \cap K$, where $X_i = X_i(\eta) = X_i(s) \ (i = 0, 1)$ are given as in (3.2). Since $X_1$ is a finite-dimensional subspace of $Y$, (3.3) implies that $K_1$ is a cone in $X_1$. As $K$ is total in $Y$, we have

$$\text{Cl}_{X_1}(K_1 - K_1) = \text{Cl}_Y(K_1 - K_1) = \text{Cl}_Y(X_1 \cap K - X_1 \cap K) \subset \text{Cl}_Y(X_1 \cap (K - K)) = X_1 \cap \text{Cl}_Y(K - K) = X_1 \cap Y = X_1.$$

That is, $K_1$ is total in $X_1$. Let $A_1 = A|_{X_1}$. For $\lambda > 0$ sufficiently large, we infer from (2.3) that $R_\lambda(A_1)X_1 = X_1$. Thus, by (A1) one easily verifies that

$$R_\lambda(A_1)K_1 \subset K_1.$$ \hspace{1cm} (3.7)

Note that $\sigma(A_1) = \Sigma_1(\eta) = \sigma_\partial(A)$. Let

$$\sigma(A_1) = \{\mu_i = s + i\beta_i : 0 \leq i \leq n\}.$$

We may assume that $|\beta_0| = \min_{0 \leq i \leq n} |\beta_i|$. Fix a number $\lambda > s$ such that (3.7) holds. Then, $|\lambda - \mu_0| = \min_{0 \leq i \leq n} |\lambda - \mu_i|$, and hence

$$r(R_\lambda(A_1)) = \sup\{1/|\lambda - \mu_i| : 0 \leq i \leq n\} = 1/|\lambda - \mu_0| := r.$$\hspace{1cm} By Theorem 2.4, one concludes that $r$ is an eigenvalue of $R_\lambda(A_1)$ with an eigenvector $w \in K_1$. On the other hand, since $1/|\lambda - \mu_i| < r$ for $\mu_i \in \sigma(A_1)$ with $\mu_i \neq \mu_0, \overline{\mu_0}$, we see that the circle $S_r = \{z \in \mathbb{C} : |z| = r\}$ in the complex plane $\mathbb{C}$ contains at most two eigenvalues of $R_\lambda(A_1)$, that is, $1/(\lambda - \mu_0)$ and $1/(\lambda - \overline{\mu_0})$. Thus, one necessarily has $1/(\lambda - \mu_0) = 1/(\lambda - \overline{\mu_0}) = r$, which implies $\beta_0 = 0$. It follows that $\mu_0 = s$ is an eigenvalue of $A$; furthermore, $w$ is an eigenvector of $A$ corresponding to $s$. This completes the proof of (1).

If $\text{int}_{X_1} K_1 \neq \emptyset$ and contains a principal eigenvector $v$, one easily verifies that $\text{int}_{X_1} K_1$ is nonvoid and $v \in \text{int}_{X_1} K_1$. Thus, by Theorem 2.4, we deduce that the eigenvalue $1/(\lambda - s)$ of $R_\lambda(A_1)$ has the same algebraic and geometric multiplicities. Consequently by (2.3) the algebraic and geometric multiplicities of the principal eigenvalue $s$ of $A$ coincide. Hence, assertion (2) holds true.
Step 3. The verification of assertions (3) and (4).

Let \( \mu \in \sigma(A) \setminus \{s\} \), \( \Re \mu > s_e \). Take a real number \( \eta \) with \( s_e < \eta < s \) such that \( \mu \in \Sigma_1(\eta) \). Let \( X_1 = X_1(\eta) \), \( K_1 = X_1 \cap \mathcal{K} \), and \( A_1 = A|_{X_1} \). Then, as in Step 2, it can be shown that \( K_1 \) is a total cone in \( X_1 \). Furthermore, (3.7) remains valid for \( \lambda > 0 \) sufficiently large. Take a \( \lambda > s \) such that (3.7) holds and consider the resolvent operator \( R_\lambda(A_1) \) of \( A_1 \) on \( X_1 \).

Then, by Theorem 2.4 (4), we deduce that \( \text{int}_X K_1 \) does not contain eigenvectors of \( R_\lambda(A_1) \) pertaining to the eigenvalue \( \lambda_{\mu} := 1/(\lambda - \mu) \). Now if \( A \) has an eigenvector \( v \in \text{int}_Y K \) corresponding to \( \mu \), one easily verifies that \( v \in \text{int}_X K_1 \) and is an eigenvector of \( R_\lambda(A_1) \) corresponding to \( \lambda_{\mu} \). This leads to a contradiction and proves assertion (3).

If \( \Im \mu \neq 0 \), Theorem 2.4 (3) asserts that \( \text{GE}_\mu(A) \cap \mathcal{K} = \{0\} \). We also infer from (2.3) that \( \text{GE}_\mu(A) = \text{GE}_\mu(R_\lambda(A_1)) \). Therefore,

\[
\text{GE}_\mu(A) \cap \mathcal{K} = \text{GE}_\mu(A_1) \cap K_1 = \{0\}.
\]

This completes the proof of assertion (4).

\( \square \)

Remark 3.2. Note that we do not require that \( e^{tA}Y \subset Y \) for \( t \geq 0 \) in the proof of the above theorem. This allows us to avoid deriving higher regularity results on the corresponding parabolic equations when applying the theory to elliptic differential operators.

Theorem 3.3. In addition to (A1)–(A3), we also assume that

(A4) \( K \) is a solid cone in \( Y \), and \( R_\lambda(A)(K \setminus \{0\}) \subset \text{int}_Y K \).

Then, \( s \) is a simple eigenvalue with a unique normalized eigenvector \( w \in \text{int}_Y K \). Moreover, for any \( \mu \in \sigma(A) \setminus \{s\} \) with \( \Re \mu > s_e \),\n
\[
\text{GE}_\mu(A) \cap \mathcal{K} = \{0\}.
\]

Proof. The proof follows a fully analogous manner as the one for Theorem 2.5, and is thus omitted. The interested readers may consult [18, Theorems 7.3 and 7.7] for details.

\( \square \)

Remark 3.4. Under the hypotheses of Theorem 3.3, one may expect that the boundary spectrum \( \sigma_b(A) \) consists of exactly one eigenvalue of \( A \). Unfortunately the easy counterexample below indicates that this may fail to be true.

Example 3.1. Let \( X = Y = \mathbb{R}^3 \). For computational convenience, here we make a convention that \( \mathbb{R}^3 \) consists of column vectors. Denote \( v' \) the transpose of a row vector \( v = (x, y, z) \). Define a cone \( K \) in \( X \) as

\[
K = \left\{ (x, y, z)' \in X : z \geq \sqrt{x^2 + y^2} \right\}.
\]

Then, \( \text{int}_Y K = \left\{ (x, y, z)' \in X : z > \sqrt{x^2 + y^2} \right\} \). Let

\[
A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

For \( \lambda > 0 \), simple computations yield

\[
R_\lambda(A) := (\lambda - A)^{-1} = \begin{pmatrix} B & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \text{where} \quad B = \frac{1}{1 + \lambda^2} \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}.
\]
Therefore,
\[ R_\lambda(A)u = \frac{1}{1 + \lambda^2} \left( \lambda x - y, x + \lambda y, \frac{1 + \lambda^2}{\lambda} z \right)' = \frac{1}{1 + \lambda^2} (\tilde{x}, \tilde{y}, \tilde{z})'. \]

Observe that
\[ \tilde{x}^2 + \tilde{y}^2 = (x^2 + y^2) + \lambda^2 (x^2 + y^2) = (1 + \lambda^2)(x^2 + y^2). \] (3.8)

Now let \( u = (x, y, z)' \in K \). Then, \( x^2 + y^2 \leq z^2 \). Since \( (1 + \lambda^2)/\lambda^2 > 1 \), by (3.8) we deduce that
\[ \tilde{x}^2 + \tilde{y}^2 \leq (1 + \lambda^2) z^2 \leq \left( \frac{1 + \lambda^2}{\lambda} \right)^2 z^2 = \tilde{z}^2. \]

This implies \( (\tilde{x}, \tilde{y}, \tilde{z})' \in K \). Thus, we see that \( R_\lambda(A)K \subset K \).

If \( u \in \partial K, u \neq 0 \), then \( x^2 + y^2 = z^2 \neq 0 \). By (3.8) we find that
\[ \tilde{x}^2 + \tilde{y}^2 = (1 + \lambda^2) z^2 < \left( \frac{1 + \lambda^2}{\lambda} \right)^2 z^2 = \tilde{z}^2. \]

Hence, \( (\tilde{x}, \tilde{y}, \tilde{z})' \in \text{int}_Y K \). Therefore, \( R_\lambda(A)u \in \text{int}_Y K \). This indicates that the operator given by \( A \) satisfies all the requirements in Theorem 3.3. However, all the eigenvalues of \( A \) have the same real part \( s = 0 \).

To guarantee the uniqueness of elements in \( \sigma_b(A) \), Nussbaum [20] used the notion of “\( u_0 \)-positivity” due to Krasnosel’skii [15]; see [20, Theorem 1.3]. Here, we remark that if the semigroup \( e^{tA} \) has some strong positivity property, then one can still ensure the uniqueness of elements in \( \sigma_b(A) \).

**Theorem 3.5.** In addition to (A1)–(A4), assume that

(A5) for any \( t > 0 \) and \( \mu \in \sigma(A) \) with Re \( \mu > s \),
\[ e^{tA}(K_\mu \setminus \{0\}) \in \text{int}_Y K, \quad \text{where } K_\mu = \text{GE}_\mu(A) \cap K. \]

Then, \( \sigma_b(A) = \{s\} \).

**Proof.** Let \( Y' = \Phi_{\mu \in \sigma_b(A)} \text{GE}_\mu(A), A' = A|_{Y'} \). Denote \( K' = Y' \cap K \). Then, by Theorem 3.3, we see that \( K' \neq \{0\} \). Hence, \( K' \) is a cone in \( Y' \). Let \( v \in K', v \neq 0 \). By (A5), we have \( e^{tA}v \in \text{int}_Y K \) for \( t > 0 \). Since \( e^{tA}v \in Y' \), one trivially verifies that \( e^{tA}v \in \text{int}_Y K' \). Therefore, \( K' \) is a solid cone in \( Y' \) and
\[ e^{tA}(K' \setminus \{0\}) \subset \text{int}_Y K', \quad t > 0. \] (3.9)

Now let \( \mu := s + i\beta \in \sigma_b(A) \). Then, \( \lambda := e^{it} = e^{it} e^{i\beta t} \) is an eigenvalue of \( e^{tA'} \) with \( |\lambda| = e^{it} := r(t) \). But (3.9) implies that \( r(t) \) is the unique eigenvalue of \( e^{tA'} \) on the circle \( \mathbb{S}_{r(t)} : = \{z \in \mathbb{C} : |z| = r(t)\} \) for \( t > 0 \). Hence, we necessarily have \( e^{i\beta t} = 1 \), and therefore \( \beta t \in \{2k\pi : k \in \mathbb{Z}\} \) for all \( t > 0 \). But this is impossible unless \( \beta = 0 \). This proves what we desired. \( \square \)

### 4 PRINCIPAL EIGENVALUE PROBLEM OF ELLIPTIC OPERATORS ASSOCIATED WITH DEGENERATE MIXED BOUNDARY CONDITIONS

As an illustrating example, we consider the principal eigenvalue problem of the elliptic operator \( L \) on a smooth bounded domain \( \Omega \subset \mathbb{R}^n (n \geq 1) \):
\[ Lu = -\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u, \]
which is associated with the mixed boundary condition:

\[ Bu := \alpha(x)u + \beta(x) \frac{\partial u}{\partial \nu} = 0 \]  

(4.1)

on the boundary \( \Gamma := \partial \Omega \) of \( \Omega \), where \( \nu \) stands for the unit outward normal vector field on \( \Gamma \). The coefficients of \( L \) and \( B \) are assumed to be \( C^\infty \) functions satisfying the hypotheses below:

(H1) \( a_{ij} = a_{ji} \) (\( 1 \leq i, j \leq n \)), and there is \( \theta > 0 \) such that

\[ \sum_{i,j=1}^{n} a_{ij}(x)\zeta_i \zeta_j \geq \theta |\zeta|^2, \quad \forall \zeta \in \mathbb{R}^n, \ x \in \Omega; \]

(H2) \( c, \alpha, \beta \) are nonnegative functions satisfying that

\[ \alpha(x) + \beta(x) > 0, \quad \forall x \in \Gamma. \]  

(4.2)

In the case of the Dirichlet boundary condition or the Robin boundary condition (regular case), this problem has already been well understood; see, for example, [1, Theorem 12.1], [5, Theorem 1.4], and also [8, 20]. Here, we are interested in a degenerate case where \( \beta \) may vanish on a proper subset of \( \Gamma \). In such a situation, if \( L \) has a divergence form (hence \( L \) enjoys some symmetric properties), one can find some nice results concerning the principal eigenvalue problem of \( L \) in Taira [23, Theorem 1.2]. As an application of the abstract results given in Section 3, we deal with the general case and present a less involved argument on the problem.

- Some fundamental results

First, making use of the classical Hopf’s lemma, one can easily verify the comparison result below:

**Lemma 4.1.** Let \( u \in C^1(\overline{\Omega}) \cap C^2(\Omega), u \neq 0 \). Assume that

\[ Lu + \lambda u \geq 0 \text{ in } \Omega, \quad Bu \geq 0 \text{ on } \Gamma, \]

where \( \lambda \geq 0 \). Then, \( u(x) > 0 \) for \( x \in \Omega \).

Denote by \( W^{s,p}(\Omega) \) (\( s \in \mathbb{R}_+, \ 1 \leq p < \infty \)) the Sobolev spaces equipped with the standard norms. We infer from Taira [23, p. 5, Theorem 1] that the following existence and uniqueness result holds true.

**Lemma 4.2.** Let \( 1 < p < \infty, s > 1 + 1/p, \) and let \( \lambda \geq 0 \). Then, for any \( f \in W^{s-2,p}(\Omega) \), the homogeneous boundary value problem

\[ Lu + \lambda u = f \text{ in } \Omega, \quad Bu = 0 \text{ on } \Gamma, \]  

(4.3)

has a unique solution \( u \in W^{s,p}(\Omega) \). Here, the boundary condition is understood in the sense that \( B \) can be viewed as a linear operator from \( W^{s,p}(\Omega) \) to Besov space \( B^{s-1-1/p,p}_{\infty}(\Gamma) \) (see [23], p. 3, for details).

Note that Lemma 4.2 implies that if \( f \in C^1(\overline{\Omega}) \), then the solution \( u \) of (4.3) belongs to \( C^2(\overline{\Omega}) \), and hence it solves (4.3) in the classical sense. Indeed, if \( f \in C^1(\overline{\Omega}) \), then \( f \in W^{1,p}(\Omega) \) for any \( 1 < p < \infty \). Lemma 4.2 then asserts that \( u \in W^{3,p}(\Omega) \). Taking a number \( p > 1 \) sufficiently large so that \( W^{3,p}(\Omega) \hookrightarrow C^2(\overline{\Omega}) \), one immediately concludes that \( u \in C^2(\overline{\Omega}) \).

By virtue of [23, p. 4, Theorem 1], we also deduce that

\[ \|u\|_{C^2(\Omega)} \leq C\|u\|_{W^{3,p}(\Omega)} \leq C\|f\|_{W^{3,p}(\Omega)} \leq C\|f\|_{C^1(\Omega)} \]  

(4.4)

for all \( f \in C^1(\overline{\Omega}) \), where \( C \) denotes a general constant independent of \( f \).
• Resolvent strong positivity of the operator $A = -L$

Let $X = L^2(\Omega)$, and set

$$ Y = \left\{ u \in C^1(\overline{\Omega}) : u \text{ satisfies (4.1)} \right\}. $$

$Y$ is equipped with the usual norm of $C^1(\overline{\Omega})$. Clearly $Y \hookrightarrow X$.

Let $K$ be the positive cone in $Y$ consisting of nonnegative functions.

Denote by $A$ the operator $-L$ with domain

$$ D(A) = \{ u \in H^2(\Omega) : Bu = 0 \}, $$

where the boundary condition $Bu = 0$ is understood in the same sense as in Lemma 4.2. Invoking [23, p. 5, Theorem 2], we deduce that $-A$ is a sectorial operator in $X$ with compact resolvent. Hence, by [12, Example 2.4, (i)] it is easy to see that $\text{spb}(A) = -\infty < \text{spb}(A) < \infty$. Thus, $A$ fulfills (A2) in Section 3.

We infer from Lemma 4.2 and (4.4) that $R_\lambda(A)Y \subset Y$ for $\lambda \geq 0$; furthermore, $R_\lambda(A)|Y$ is compact as an operator on the space $Y$.

The following result indicates that $A$ fulfills hypotheses (A1) and (A4) in Section 3. The proof of such a result is somewhat standard. We include the details in the Appendix for the readers’ convenience.

**Lemma 4.3.** $R_\lambda(A)(K \setminus \{0\}) \subset \text{int}_Y K$ for each $\lambda \geq 0$.

• Regularity of the generalized eigenfunctions

Let $\mathcal{A}$ be the complexification of $A$ with $D(\mathcal{A}) = D(A) + iD(A)$. We start with the eigenfunctions of $\mathcal{A}$. Let $\mu = a + ib \in \sigma(\mathcal{A})$, and let $w = u + iv$ be a corresponding eigenfunction of $\mathcal{A}$, where $u, v \in D(\mathcal{A})$. Then, $\mathcal{A}w = \mu w$ amounts to say that

$$ Au = au - bv, \quad Av = av + bu. \quad (4.5) $$

Since $u, v \in H^2(\Omega)$, by Lemma 4.2 and (4.5), one finds that $u, v \in H^4(\Omega)$. Further by a simple bootstrap argument, we finally conclude that $u, v \in H^s(\Omega)$ for all $s \geq 0$. It follows by the Sobolev embeddings that $u, v \in C^\infty(\overline{\Omega})$.

Now let $g \in GE_{\mu}(\mathcal{A})$ and $\text{rank}(g) \geq 2$. Set $k = \text{rank}(g) - 1$. Then, $(\mathcal{A} - \mu)^k g := w$ is an eigenfunction of $\mathcal{A}$. Hence, $w$ is a $C^\infty$ function on $\overline{\Omega}$.

Note that it is readily implied in $(\mathcal{A} - \mu)^k g = w$ that $(\mathcal{A} - \mu)^j g \in D(\mathcal{A})$ for all $j \leq k$. In particular,

$$ (\mathcal{A} - \mu)^{k-1} g := f_1 \in D(\mathcal{A}) \subset H^2(\Omega), $$

here $H^s(\Omega) := H^s(\Omega) + iH^s(\Omega)$. Therefore, by $(\mathcal{A} - \mu)f_1 = w$, we find that

$$ \mathcal{A}f_1 = w + \mu f_1 := f_1 \in H^2(\Omega). \quad (4.6) $$

It follows by Lemma 4.2 that $f_1 \in H^4(\Omega)$. This in turn implies that $f_1 \in H^4(\Omega)$. Equation (4.6) and Lemma 4.2 then assert that $f_1 \in H^s(\Omega)$ for all $s \geq 0$.

Repeating the above argument with $w$ and $f_1$ therein replaced by $f_1$ and $(\mathcal{A} - \mu)^{k-2} g := f_2$, respectively, one deduces that $f_2 \in H^s(\Omega)$ for $s \geq 0$. Continuing this procedure, we finally obtain that $g = f_k \in H^s(\Omega)$ for all $s \geq 0$. The Sobolev embeddings then immediately imply that $g$ is a $C^\infty$ function.

It follows from the above results that $\text{GE}_{\mu}(A) \subset Y$ for any $\mu \in \sigma(\mathcal{A})$. Hence, $A$ fulfills hypothesis (A3).

By far we have seen that the operator $A$ satisfies hypotheses (A1)–(A4).

• The verification of hypothesis (A5)

Let $\mu \in \sigma(A)$. Denote by $\mathcal{A}_\mu$ the restriction of $\mathcal{A}$ on $\text{GE}_{\mu}(\mathcal{A})$. Given $g \in \text{GE}_{\mu}(\mathcal{A})$, let $u = u(t)$ be the solution of equation $\dot{u} = \mathcal{A}u$ with $u(0) = g$. Then, $u(t) = e^{t\mathcal{A}} g = e^{t\mathcal{A}_\mu} g$. Since $\mathcal{A}_\mu$ is a bounded operator on $\text{GE}_{\mu}(\mathcal{A})$, we have
\[ u(t) = e^{t\lambda g} = e^{\mu t} e^{(\lambda - \mu) t} g = e^{\mu t} \sum_{j=0}^{\infty} \frac{t^j}{j!}(\lambda - \mu)^j g \]

\[ = e^{\mu t} \left( I + \frac{t}{1!}(\lambda - \mu) + \cdots + \frac{t^{k-1}}{(k-1)!}(\lambda - \mu)^{(k-1)} \right) g, \]

where \( k = \text{rank}(g) \).

Noticing that \((\lambda - \mu)^j g \in \text{GE}_\mu(\mathcal{A})\) for any integer \( j \geq 0 \), by what we have proved above, it is clear that \((\lambda - \mu)^j g\) is a \( C^\infty \) function. Consequently foreach \( t \geq 0 \) fixed, \( u(t) \) is a \( C^\infty \) function in the space variable on \( \bar{\Omega} \). We write \( u(t, x) = u(t)(x) \) for \((t, x) \in \mathbb{R}_+ \times \bar{\Omega}\). Then it can be easily seen that \( u = u(t, x) \) is a complex \( C^\infty \) function on \( \mathbb{R}_+ \times \bar{\Omega} \).

Now we come back to the real situation. The above result implies that for each \( v_0 \in \text{GE}_\mu(\mathcal{A}) \), the function \( v(t, x) := v(t)(x) \), where \( v(t) = e^{t\lambda} v_0 \), is a \( C^\infty \) function on \( \mathbb{R}_+ \times \bar{\Omega} \). Therefore, \( v \) is a classical solution of the parabolic equation:

\[ \frac{\partial v}{\partial t} + L v = 0, \quad x \in \Omega, \quad t > 0 \]

associated with the boundary condition \( Bv = 0 \) on \( \Gamma \). Thanks to the Hopf’s lemma for parabolic equations (see, e.g., Friedman [7, Chapter 2, Theorem 14] or Smith [22, Chapter 7, Theorem 2.2]), using almost the same argument as in the proof of Lemma 4.3 (see the Appendix), it can be shown that \( v(t, \cdot) \in \text{int}_f K \) if \( v_0 \in K, v_0 \neq 0 \). This is precisely what we desired.

Now that \( A \) satisfies hypotheses (A1)–(A5), as a straightforward application of Theorems 3.3 and 3.5, one immediately obtains the following result.

**Theorem 4.4.** The following assertions hold true:

(1) The spectral bound \( s \) of \( A \) is an algebraically simple eigenvalue with a corresponding eigenvector \( w \in \text{int}_f K \).

(2) \( \text{GE}_\mu(A) \cap K = \{0\} \) for any \( \mu \in \sigma(A) \setminus \{s\} \).

(3) \( \sigma_b(A) = \{s\} \).

**Remark 4.5.** We mention that in the case of the Dirichlet (or Robin) boundary condition, almost all the facts concerning the operator \( A \) needed in proving Theorem 3.5 are well known and need not be checked. Therefore, the theorem becomes nearly an immediate consequence of Theorems 3.3 and 3.5.

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APPENDIX: PROOF OF LEMMA 4.3

Proof. Let \( f \in K \setminus \{0\} \) and \( u = R_\lambda(A)f \). Then, \( u \in C^2(\overline{\Omega}) \). By Lemma 4.1, we deduce that \( u(x) > 0 \) for \( x \in \Omega \). Set \( \Gamma_0 = \{ x \in \Gamma : u(x) = 0 \} \). Then, the classical Hopf's lemma asserts that \( \frac{\partial u}{\partial \nu}(x) < 0 \) for \( x \in \Gamma_0 \). Hence, by compactness of \( \Gamma_0 \), there is \( \varepsilon_0 > 0 \) such that \( \frac{\partial u}{\partial \nu}(x) \leq -3\varepsilon_0 \) for \( x \in \Gamma_0 \).

Denote \( \| \cdot \|_1 = \| \cdot \|_{C^1(\overline{\Omega})} \). Take a neighborhood \( \mathcal{W} \) of \( \Gamma_0 \) in \( \Gamma \) such that \( \frac{\partial u}{\partial \nu}(x) \leq -2\varepsilon_0 \) for \( x \in \mathcal{W} \).

Then, there exists \( \delta > 0 \) such that for all \( h \in Y \) with \( \| h \|_1 < \delta \),

\[
\frac{\partial (u + h)}{\partial \nu}(x) \leq -\varepsilon_0 < 0 \quad \text{for} \quad x \in \mathcal{W}.
\] (A2)

We claim that \( \alpha(x) > 0 \) for \( x \in \Gamma_0 \). Indeed, if \( \alpha(x) = 0 \), then by (4.2) we have \( \beta(x) > 0 \). Thus, by (A1), one deduces that \( Bu(x) = \beta(x)\frac{\partial u}{\partial \nu}(x) \neq 0 \), a contradiction. Hence, the claim holds true. By compactness of \( \Gamma_0 \), we deduce that \( \alpha(x) \geq 2\varepsilon_1 > 0 \) for all \( x \in \Gamma_0 \). Therefore, by continuity of \( \alpha \), it can be assumed that the neighborhood \( \mathcal{W} \) of \( \Gamma_0 \) is chosen sufficiently small so that

\[
\alpha(x) \geq \varepsilon_1 > 0 \quad \text{for} \quad x \in \mathcal{W}.
\] (A3)

Now for any \( h \in Y \) with \( \| h \|_1 < \delta \), we have at any point \( x \in \mathcal{W} \) that

\[
\alpha(x)(u + h)(x) = -\beta(x)\frac{\partial (u + h)}{\partial \nu}(x) \geq 0,
\]
here the last inequality follows from (A2). Hence, by (A3), we see that

\[(u + h)(x) \geq 0 \quad \text{for } x \in W.\]  \hspace{1cm} (A4)

Using (A2) and (A4), it is not difficult to deduce that there is a neighborhood \(U\) of \(\Gamma_0\) in \(\overline{\Omega}\) such that for any \(h \in Y\) with \(\|h\|_1 < \delta\),

\[(u + h)(x) \geq 0 \quad \text{for } x \in U.\]  \hspace{1cm} (A5)

We may assume that \(U\) is open relative to \(\overline{\Omega}\). Hence, \(\Gamma_1 := \Gamma \setminus U\) is compact. Because \(u\) is positive on \(\Gamma_1\), there is \(\varepsilon_2 > 0\) such that \(u(x) \geq 2 \varepsilon_2\) for \(x \in \Gamma_1\). This allows us to pick a neighborhood \(V\) of \(\Gamma_1\) in \(\overline{\Omega}\) such that \(u(x) \geq \varepsilon_2\) for \(x \in V\). Further, one can restrict \(\delta\) sufficiently small so that

\[(u + h)(x) \geq 0, \quad \forall \ x \in V\]  \hspace{1cm} (A6)

for all \(h \in Y, \|h\|_1 < \delta\). Note that \(G := U \cup V\) is a neighborhood of \(\Gamma\) in \(\overline{\Omega}\).

It can be assumed that both \(U, V\) are open relative to \(\overline{\Omega}\). Hence, \(G\) is open in \(\overline{\Omega}\). Consequently, \(F := \overline{\Omega} \setminus G\) is a compact subset of \(\overline{\Omega}\). Since \(u\) is positive on \(F\), there exists \(\varepsilon_3 > 0\) such that \(u(x) \geq 2 \varepsilon_3\) for \(x \in F\). Therefore, if \(\delta\) is sufficiently small, then \(u + h\) is positive on \(F\) for all \(h \in Y\) with \(\|h\|_1 < \delta\). Combining this with (A5) and (A6), it follows that \(u + h \geq 0\) in \(\overline{\Omega}\) for all \(h \in Y\) with \(\|h\|_1 < \delta\), that is, \(u + h \in K\). Hence, \(u \in \text{int}_Y K\). \(\square\)