Encoded Universality Of Quantum Computations
On The Multi-Atomic Ensembles In The QED Cavity

Farid Ablayev  Sergey Andrianov  Sergey Moiseev  Alexander Vasiliev

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Abstract

We propose an effective set of elementary quantum gates which provide an encoded universality and demonstrate the physical feasibility of these gates for the solid-state quantum computer based on the multi-atomic systems in the QED cavity. We use the two-qubit encoding and swapping-based operations to simplify a physical realization of universal quantum computing and add the immunity to a number of errors. This approach allows to implement any encoded single-qubit operation by three elementary gates and the encoded controlled-NOT operation can be performed in a single step. The considerable advantages are also shown for implementing some commonly used controlled gates.

Keywords: quantum computer; encoded universality; swapping gates; multi-atomic ensembles.

1 Introduction

During the last two decades different types of quantum computer and its physical implementations have been considered [1, 2, 3, 4], where single natural or artificial atoms, ions, molecules etc., are used for encoding of the qubits. For these physical models a lot of universal sets of elementary unitary transformations had been proposed [5, 6, 7, 8]. And though there are infinitely many of them [7], only few sets of the quantum gates are usually realized in experimental implementation. The most commonly used universal set is that consisting of CNOT and single-qubit gates (e.g., see [1]). Moreover, it is well-known that even smaller discrete set can be used to approximate any unitary operator to arbitrary accuracy. Namely, this is the set of three one qubit gates $H$, $S$, $T$ and two qubit CNOT gate, which is usually referred to as the standard set [1].

Physical implementation of the quantum computing on many qubits remains a main huge challenge leading to intensive search of the novel experimental approaches. The promising approach is the one using Heisenberg-like interactions between spin qubits. The Heisenberg interaction yields fast two-qubit quantum gates but single-qubit gates are still the problem since they rely on weak local interactions and hence are slow. It was shown [9, 10, 11] that the set of quantum gates universal for some Hilbert subspace can be built of Heisenberg-only interactions by encoding each logical qubit by several physical qubits. This approach was termed as encoded universality [12]. This type of interaction is universal for the case of encoding a logical qubit by at least three physical qubits. Additionally, the proof of this fact is non-constructive and the exact sequence of elementary gates is obtained from extensive numerical optimization [13]. The implementation of the encoded controlled-NOT operation (up to single qubit operations) in such sequences is rather complicated since it uses seven parallel exchange interactions or 19 serial gates.

In [14] it was also shown that controlled $\pi$-phase shift can be achieved with Heisenberg interactions in two steps only using the encoding of logical qubits by pairs of physical qubits.

Recently a new physical realization of a quantum computer has been proposed which uses multi-atomic ensembles for encoding of single qubits [15, 16]. Multi-atomic coherent ensembles provide a huge enhancement of the effective dipole moment that leads to a considerable acceleration of the quantum processing rate. However, here excess excited quantum states in the multi-atomic ensemble should be blocked in order to realize an effective two-level system providing perfect encoding of the qubits based on the multi-atomic ensemble states. A dipole-dipole interaction is intensively discussed for the blockade of the excess quantum states [17].

However, the dipole blockade mechanism suffers from the decoherence problem arising due to a strong dependence of the dipole-dipole interaction on a spatial distance between the interacting atoms. Recently another blockade mechanism has been proposed [18] based on using a light-shift imbalance in a Raman transition. We have also proposed a novel decoherence free blockade mechanism [19, 20] based on the collective interactions in QED cavity.
Rapid development of physics and technology of the microcavities [21, 22, 23] makes this blockade mechanism a quite promising though not very simple for experimental realization.

In this paper we propose an effective set of unitary transformations and demonstrate that logical single and two-qubit gates can be realized naturally in the quantum computer based on the multi-atomic ensembles in the QED cavity [20]. We explicitly show that this set possesses an encoded universality when using the 2-qubit encoding mentioned in [24]: 

\[ |0_L\rangle = |01\rangle, \quad |1_L\rangle = |10\rangle. \]

In [25] it was shown that this encoding allows to solve two major problems of solid-state quantum computing. First of all, it eliminates the problems with implementation of single-qubit operations. Additionally, this encoding forms a decoherence-free subspace (DFS) [10], which allows to prevent a number of computational errors [25]. In the physical model we consider here [20], there is a specific type of error, which comes out when the swapping operations are applied to the basis state \(|11\rangle\). This error can be suppressed by using collective blockade, which slows the computation. On the other hand, it is obvious that the qubit encoding we use here has the immunity to this type of error.

The main advantage of our approach is the ability to perform the controlled-NOT operation in a single step. This is achieved by additional nonlinear frequency shift naturally arising in the QED cavity with Heisenberg-type interaction. We also show that the proposed set of quantum gates is efficient for implementing complex controlled operations, which are at the heart of many efficient quantum algorithms (e.g., creating fingerprints for the technique of quantum fingerprinting [26]).

2 Quantum Computer Based on Multi-Atomic Ensembles in Resonator

We discuss a novel architecture of quantum computer based on the multi-atomic ensembles of two level atoms in QED cavity [19, 20]. Here, we can introduce the following collective states in m-th processing node:

\[ |0\rangle_m = |g\rangle_1 |g\rangle_2 \cdots |g\rangle_{N_m}, \]

corresponds to the ground state of the node,

\[ |1\rangle_m = \sqrt{1/N_m} \sum_{j} |g\rangle_1 |g\rangle_2 \cdots |e\rangle_j \cdots |g\rangle_{N_m}, \]

and

\[ |2\rangle_m = \sqrt{2/N_m(N_m-1)} \sum_{i \neq j} |g\rangle_1 |g\rangle_2 \cdots |e\rangle_j \cdots |e\rangle_j \cdots |g\rangle_{N_m}, \]

are the collective states of m-th node with single and two atomic excitations (where \(|g\rangle_j\) and \(|e\rangle_j\) are the ground and excited states of j-th atom and \(N_m\) is a number of atoms in the m-th node).

We denote the collective state of a pair of the processing nodes 1 and 2 by \(|00\rangle = |0\rangle_1 |0\rangle_2\). Similarly, we introduce the states \(|01\rangle, |10\rangle, |11\rangle, |02\rangle, |20\rangle\). The proposed architecture provides the means of performing the following quantum gates:

\[
iSWAP = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & i & 0 \\
0 & i & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
C(SWAP) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

It follows also from [19, 20] that, using appropriate times, it is possible to perform a gate which generalizes the iSWAP gate. Specifically, this gate is
Figure 1: Pairwise qubit encoding. Small circles denote the processing nodes in the indicated quantum states.

\[
i\text{SWAP} (\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} & 0 \\
0 & i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} & 0 \\
0 & 0 & 0 & 1 \end{pmatrix} \tag{6}
\]

for arbitrary (experimentally possible) \(\theta\). When \(\theta = \pi\), this gate is exactly \(i\text{SWAP}\).

The gate we denote here as \(i\text{SWAP} (\theta)\) describes an anisotropic exchange interaction and is known to be universal in an encoded universality setting [12]. However, we include it in a larger universal set for the reasons mentioned earlier.

The proposed architecture also allows us to perform the following gate:

\[
\text{PHASE} (\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & e^{-i\theta/2 + i\varphi/2} & 0 & 0 \\
0 & 0 & e^{i\theta/2 + i\varphi/2} & 0 \\
0 & 0 & 0 & e^{i\varphi} \end{pmatrix}, \tag{7}
\]

where the phases \(\theta\) and \(\varphi\) can be controlled by the different values of the external magnetic (or electric) fields on the spatially distinct nodes that provides, respectively, different Zeeman (or Stark) frequency shifts for the two-level atoms localized in the nodes [19].

The implementation of universal quantum computations based on the set of \(i\text{SWAP}\) and single qubit gates in the proposed model meets some obstacles:

- while the two-qubit gates can be implemented quite quickly, the single-qubit gates require blockade, which leads to a significant loss of performance;
- the application of the “swapping” gates \(i\text{SWAP} (\theta)\) and \(C(\text{SWAP})\) to the basis state \(|11\rangle\) turns it into \((|02\rangle + |20\rangle)/\sqrt{2}\) and backwards. This side effect can be avoided by using collective blockade, which tunes down the performance as well.

### 2.1 Implementing universal quantum computations with \(C(\text{SWAP}), i\text{SWAP} (\theta), \text{PHASE} (\theta)\)

In this section we present an approach for implementing universal quantum computations using the set of elementary quantum gates available in the considered physical model of a quantum computer. At the heart of our approach lies the idea of using logical qubits encoded by two physical qubits. This idea was proposed in [24].

Such encoding implies that we consider pairs of processing nodes in states \(|01\rangle, |10\rangle\) as single logical qubits in states \(|0_L\rangle, |1_L\rangle\) respectively. This encoding is somewhat similar to dual rail logic in classical computations for it adds robustness to the system forming a decoherence-free subspace [10, 25].

In this case any single qubit state \(\alpha |0_L\rangle + \beta |1_L\rangle\) is actually stored as an entangled two qubit state \(\alpha |01\rangle + \beta |10\rangle\), that is the basis state of such a composite qubit is determined by the basis state of the first qubit of a pair.

Note also, that this encoding excludes the usage of the single qubit operations, plus it prohibits the habits of processing nodes to be in state \(|11\rangle\). Hence it solves both of physical problems stated earlier.

The following property explicitly shows that in such encoding of qubits the two and three qubit operations available in our physical model are sufficient to implement the standard set, thus proving their encoded universality.

**Property 2.1.** The set of quantum gates \(\{C(\text{SWAP}), i\text{SWAP} (\theta), \text{PHASE} (\theta)\}\) is universal for the Hilbert subspace spanned by encoded states \(|0_L\rangle = |01\rangle, |1_L\rangle = |10\rangle\).

**Proof.** First of all we show the effect of our elementary quantum gates when acting on pairs of nodes in basis states \(|01\rangle, |10\rangle\) and their linear combinations.
For example, the SWAP operation turns $|01⟩$ into $|10⟩$ and backwards, thus acting on a pair like the gate $X$ (the NOT gate):

In the similar manner C(SWAP) actually implements the CNOT gate:

More formally:

$$CNOT_L (\alpha_1 |0⟩_L |0⟩_L + \alpha_2 |0⟩_L |1⟩_L + \alpha_3 |1⟩_L |0⟩_L + \alpha_4 |1⟩_L |1⟩_L) =$$

$$C(SWAP) (\alpha_1 |01⟩ + \alpha_2 |01⟩ |10⟩ + \alpha_3 |10⟩ |01⟩ + \alpha_4 |10⟩ |10⟩) =$$

$$\alpha_1 |01⟩ + \alpha_2 |01⟩ |10⟩ + \alpha_3 |10⟩ |01⟩ + \alpha_4 |10⟩ |10⟩.$$

The physical implementation of these gates based on the controlled interaction between the ensembles of the two-level atoms and field cavity modes is described in [20].

If we look at the matrix for the generalized iSWAP ($\theta$) gate, it’s middle part (responsible for transforming $|01⟩$ and $|10⟩$ basis states) is actually a rotation by the angle $-\theta$ about the $\hat{x}$ axis of the Bloch sphere, i.e. iSWAP ($\theta$) corresponds to the following operator:

$$\text{iSWAP} (\theta) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} & 0 \\
0 & -i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix} \quad \rightarrow \quad R_x (\theta) = \begin{pmatrix}
\cos \frac{\theta}{2} & -i \sin \frac{\theta}{2} \\
-i \sin \frac{\theta}{2} & \cos \frac{\theta}{2} \\
\end{pmatrix}. \quad (9)$$

Similarly, PHASE ($\theta$) turns our composite qubit around the axis $\hat{z}$ (up to the phase factor of $e^{i\theta/2}$):

$$\text{PHASE} (\theta) = e^{i\frac{\theta}{2}} \begin{pmatrix}
e^{-i\frac{\theta}{2}} & 0 & 0 & 0 \\
0 & e^{-i\frac{\theta}{2}} & 0 & 0 \\
0 & 0 & e^{i\frac{\theta}{2}} & 0 \\
0 & 0 & 0 & e^{i\frac{\theta}{2}} \\
\end{pmatrix} \quad \rightarrow \quad R_z (\theta) = \begin{pmatrix}
e^{-i\frac{\theta}{2}} & 0 \\
0 & e^{i\frac{\theta}{2}} \\
\end{pmatrix}. \quad (10)$$

Since an arbitrary rotation of a single qubit (and thus any single qubit gate) can be decomposed into the product of three rotations about orthogonal axes (say, $R_x$ and $R_z$), our basis allows to avoid using operations of single processing nodes and thus blockage. All of the single qubit gates are performed by the means two node operations.
For instance, the Hadamard transform can be implemented as follows:

\[ H = e^{i\pi/2} R_z \left( \frac{\pi}{2} \right) R_y \left( \frac{\pi}{2} \right). \]  

(11)

The other two single qubit gates \( S \) and \( T \) from the standard set up to the phase factor are rotations about \( \hat{z} \) axis:

\[
S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = e^{i\pi/4} \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = e^{i\pi/4} R_z \left( \frac{\pi}{2} \right)
\]

(12)

\[
T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = e^{i\pi/8} \begin{pmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{pmatrix} = e^{i\pi/8} R_z \left( \frac{\pi}{4} \right)
\]

(13)

Therefore, the set of gates \( \{ C(SWAP), iSWAP(\theta), PHASE(\theta) \} \) allows to implement the standard set of quantum gates, which proves it’s encoded universality.

\( \Box \)

Note, that even though the SWAP is not in our basis set it can be simulated by the \( iSWAP = iSWAP(\pi) \) gate up to unimportant global phase \( e^{i\pi/2} \) when acting on basis states \( |01\rangle \) and \( |10\rangle \).

Note also, that the universality of the proposed set of elementary gates rely on the presence of “continuous” operations \( iSWAP(\theta) \) and \( PHASE(\theta) \). This fact requires a higher precision of the hardware, but excludes the approximation algorithms for implementing arbitrary single qubit operations using the standard set of CNOT, \( H \), \( S \), and \( T \). Conversely, we may restrict ourselves with using only \( iSWAP \left( \frac{\pi}{2} \right) \), \( PHASE \left( \frac{\pi}{2} \right) \), and \( PHASE \left( \frac{\pi}{4} \right) \) (which is enough for implementing gates \( H \), \( S \), and \( T \)) and exploit standard approximation schemes for arbitrary single qubit gates.

It is also well-known [1] that the C(SWAP) (Fredkin) gate is universal for classical computations, since it can be used to perform logical NOT, AND, and FANOUT operations:

2.2 Universal set based on controlled phase gate

We have already mentioned that there is another popular approach to constructing universal quantum computations based on the Controlled Phase gate:

\[
e^{-i\phi} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{-i\phi} \end{pmatrix}.
\]

(14)

When \( \phi = \pi \) this gate becomes the Controlled-\( Z \) gate

\[
C(Z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},
\]

(15)

which can be used to construct the CNOT gate:

\[
\begin{array}{c}
\ \ \ \ \ H \\
\ \ \ \ Z \\
\ \ \ \ \ H
\end{array}
\]

(16)

It can be easily verified that we can implement the Controlled Phase gate in our pairwise qubit encoding by simply applying it to the first processing nodes of each pair:

Hence, using the results of the previous subsection we can conclude that the set \( \{ C(S), iSWAP(\theta), PHASE(\theta) \} \) is also universal for quantum computation with respect to our qubit encoding.
2.3 Improved constructions for useful quantum gates

One of the most commonly used quantum gate is the Toffoli (also known as CCNOT) gate. Using the universal set of CNOT and single qubit gates it can be implemented up to relative phase factor using the following circuit from [1]:

We can construct a more efficient “low-level” circuit (in a sense that we are transforming the physical qubits rather than logical ones) using C(SWAP) gates and an extra processing node in the state $|0\rangle$. The following circuit implements a controlled-controlled-SWAP operation, which is equivalent to the encoded CCNOT gate:

This approach can be generalized to improve constructions of the general $C^t(U)$ gate, defined by the following equation in [1]:

$$C^t(U) |c_1c_2\ldots c_t⟩|ψ⟩= |c_1c_2\ldots c_t⟩U_{c_1c_2\ldots c_t}|ψ⟩.$$  \hfill (17)

The usual construction exploiting ancillary qubits in state $|0\rangle$ is the following (demonstrated for $t = 4$):

This circuit requires $2(t−1)$ Toffoli gates and one controlled-$U$ operation plus $t−1$ pairs of qubits (initially in the state $|0\rangle$) for temporary storage. As we already know each encoded Toffoli gate can be implemented using three “physical” C(SWAP) gates and additional node in the state $|0\rangle$. Of course, for non-parallel Toffoli gates we can use the same ancillary node, for it remains in the state $|0\rangle$.

On the other hand, we can use the following scheme for implementing the encoded $C^t(U)$ gate:

Here we have made use of $2(t−1)$ C(SWAP)s (which are elementary gates in our model) and $t−1$ ancillary processing nodes instead of pairs of qubits.

2.4 Physical realization of swapping gates

Below we describe the realization of swapping gates controlled by a photon qubit. Let’s consider two atomic ensembles situating in two separate nodes in a common resonator (Fig. 10). We can introduce signal (denoted by $E_{\text{in}}$ and $E_{\text{out}}$) and control ($E_c$) photons through a beam splitter into the system. The photons are stored for a processing time in quantum memory situating also in common resonator [27, 28]. After storage of the

$$|c_{1,L}\rangle \quad |c_{1,L}\rangle \quad |c_{1,L}\rangle \quad |c_{1,L}\rangle \quad |c_{1,L}\rangle \quad |c_{1,L}\rangle \quad |c_{1,L}\rangle \quad |c_{1,L}\rangle \quad |c_{1,L}\rangle$$

$$|c_{2,L}\rangle \quad |c_{2,L}\rangle \quad |c_{2,L}\rangle \quad |c_{2,L}\rangle \quad |c_{2,L}\rangle \quad |c_{2,L}\rangle \quad |c_{2,L}\rangle \quad |c_{2,L}\rangle \quad |c_{2,L}\rangle$$

$$|ψ_L\rangle \quad |ψ_L\rangle \quad |ψ_L\rangle \quad |ψ_L\rangle \quad |ψ_L\rangle \quad |ψ_L\rangle \quad |ψ_L\rangle \quad |ψ_L\rangle \quad |ψ_L\rangle$$

$$|0\rangle \quad |0\rangle \quad |0\rangle \quad |0\rangle \quad |0\rangle \quad |0\rangle \quad |0\rangle \quad |0\rangle \quad |0\rangle$$

Figure 7: Efficient implementation of the encoded Toffoli gate using the physical C(SWAP) operation.
Figure 8: Implementation of the gate $U$ controlled by four qubits.

Figure 9: Improved implementation of the gate $U$ controlled by four qubits.
photons, we raise reflectivity of input-output mirror in order to make resonator perfect. First, signal photon is transferred from quantum memory to two processing nodes that leads to the following quantum state of the nodes $|\psi_L\rangle = \alpha |0_L\rangle + \beta |1_L\rangle$. Then the frequency of atomic transitions in processing nodes is tuned out of resonance with the cavity mode. We release a control photon from the quantum memory and detune the memory from resonance with the released photon. In that case, the control photon can not be absorbed by the memory and processing nodes or released from cavity during its lifetime in the cavity.

Such a system of two nodes and control photon in the cavity is described by the Hamiltonian $H = H_0 + H_1$, where $H_0 = H_a + H_r$ is a main part and $H_1 = H_{r-a}$ is a perturbation part $H_1 = H_{r-a} = H^{(1)}_{r-a} + H^{(2)}_{r-a}$, $H_a = H_{a_1} + H_{a_2}$ is Hamiltonian of atoms in the nodes 1 and 2. With that, $H_{a_1} = \hbar \omega_0 \sum_{j_1} S_{j_1}^z$, and $H_{a_2} = \hbar \omega_0 \sum_{j_2} S_{j_2}^z$, where $\omega_0$ is the frequency of working transitions in atoms, $S_{j_1}^z$ and $S_{j_2}^z$ operators of effective spin $1/2$ $z$-projection in two-level model for atoms in sites $j_1$ and $j_2$ of nodes 1 and 2; $H_r = \hbar \omega_0 a_{k_\alpha}^+ a_{k_\alpha}^-$, where $\omega_0$ is frequency of photons with wave vector $k_0$, $a_{k_\alpha}^+$ and $a_{k_\alpha}^-$ are creation and annihilation operators for photons with wave vector $k_0$ and polarizations $\sigma$.

We have the following expressions $H_{r-a}^{(\sigma)} = H_{r-a}^{(1)} + H_{r-a}^{(2)}$ for interaction of atoms with photons of polarizations $\sigma$ in nodes 1 and 2, where
\begin{align}
H_{r-a}^{(1)} &= \sum_{j_1} \left( g_{k_{1\alpha}}^{(1)} e^{i k_0 \vec{r}_{j_1}} S_{j_1}^+ a_{k_\alpha} + g_{k_{1\alpha}}^{(1)*} e^{-i k_0 \vec{r}_{j_1}} S_{j_1}^- a_{k_\alpha}^+ \right), \\
H_{r-a}^{(2)} &= \sum_{j_2} \left( g_{k_{2\alpha}}^{(2)} e^{i k_0 \vec{r}_{j_2}} S_{j_2}^+ a_{k_\alpha} + g_{k_{2\alpha}}^{(2)*} e^{-i k_0 \vec{r}_{j_2}} S_{j_2}^- a_{k_\alpha}^+ \right),
\end{align}
where $g_{k_{\alpha}}^{(\sigma)}$ are interaction constants, $S_{j_\alpha}^+$ are raising and lowering operators for spin $1/2$ in two level model, $\vec{r}_{j_\alpha}$ are radius vectors for atoms in sites $j_\alpha$ of nodes $\alpha = 1, 2$.

We perform unitary transformation [29] of Hamiltonian $H_s = e^{-s} H e^s$ that yields the following result in the second order on small perturbation:
\begin{equation}
H_s = H_0 + \frac{1}{2} [H_1, s],
\end{equation}
when relation $H_1 + [H_0, s] = 0$ is valid. Using relation [20] we find $s = s_1 + s_2$ where
\begin{equation}
s_1 = \sum_{j_1} \left( \alpha_1 g_{k_{1\alpha}}^{(1)} e^{i k_0 \vec{r}_{j_1}} S_{j_1}^+ a_{k_\alpha} + \beta_1 g_{k_{1\alpha}}^{(1)*} e^{-i k_0 \vec{r}_{j_1}} S_{j_1}^- a_{k_\alpha}^+ \right),
\end{equation}
where $\alpha_{1,2} = -\beta_{1,2} = -h^{-1}/(\omega_0 - \omega_{k_0}) = -h^{-1}/\Delta$.

Substituting (21), (22) into (20), we get

$$H_s = \hbar \omega_{k_0} a_{k_0}^+ a_{k_0} + \sum_{m,j_m} \hbar \omega_m S_{j_m}^z + 2 \sum_{m,j_m} g_{1m}^{|}\frac{|g_{m}|^2}{\Delta_{m}} a_{k_0}^+ a_{k_0} S_{j_m}^z +$$

$$\sum_{m,j_m} \sum_{i,m,j_m} \frac{|g_{m}|^2}{\Delta_{m}} S_{j_m}^+ S_{j_m}^- + \frac{1}{2\hbar} \left( \frac{\Delta_1}{\Delta_1 + \Delta_2} \right) \sum_{j_1,j_2} \left[ g_{k_0}^{|}\frac{|g_{1}^{|}}{\Delta_1} a_{k_0}^+ S_{j_1,j_2}^+ S_{j_2,j_1}^- + h.c. \right].$$

The first two terms are unchanged energy of photons, the third term is unchanged energy of atoms in nodes 1 and 2, the forth and the fifths terms are atomic energy shifts due to photons with polarization $\sigma$, the sixes and sevens terms are atomic intra-node swap energies, the eights term is atomic inter-node swap energy and the nines term is atomic mediated polarization swapping energy of photons.

Using Hamiltonian (23) and basic lowest atomic states $\psi_1 = |0\rangle_1 |0\rangle_2$, $\psi_2 = |1\rangle_1 |0\rangle_2$, $\psi_3 = |0\rangle_1 |1\rangle_2$, $\psi_4 = |1\rangle_1 |1\rangle_2$, $\psi_5 = |2\rangle_1 |0\rangle_2$ and $\psi_6 = |0\rangle_1 |2\rangle_2$, we get the following wave function of the atoms and control field excited in the Fock state with $n$ photons $|n\rangle$

$$|\psi(n,t)\rangle = \left\{ \sum_{j=1}^{6} c_j^{(n)}(t) \psi_j \right\} |n\rangle,$$

that leads to the following Schrödinger equation

$$\frac{d(|n\rangle\langle n|)}{dt} = -i \hbar \langle H_s |\psi(n,t)\rangle = \frac{1}{2} \left\{ N_1 \omega_1 + N_2 \omega_2 + 2n \left( N_1 \Omega_1 + N_2 \Omega_2 \right) \right\} c_1^{(n)} \psi_1 +$$

$$+ i \left\{ \left( \frac{N_1}{2} - 1 \right) \omega_1 + 2n \Omega_1 \right\} c_2^{(n)} \psi_2 - i \sqrt{N_1 N_2} c_3^{(n)} c_4^{(n)} \psi_2 +$$

$$+ i \left\{ \left( \frac{N_1}{2} - 1 \right) \omega_1 + 2n \Omega_1 \right\} c_3^{(n)} \psi_3 - i \sqrt{N_1 N_2} c_4^{(n)} c_3^{(n)} \psi_3 +$$

$$+ i \left\{ \left( \frac{N_1}{2} - 1 \right) \omega_1 + 2n \Omega_1 \right\} c_4^{(n)} \psi_4 - i \Omega_s \sqrt{2N_1 (N_1 - 1) c_5^{(n)}} \psi_4 +$$

$$+ i \left\{ \left( \frac{N_1}{2} - 2 \right) \omega_1 + 2n \Omega_1 \right\} c_5^{(n)} - \Omega_s \sqrt{2N_1 (N_1 - 1) c_4^{(n)}} \psi_5 +$$

$$+ i \left\{ \left( \frac{N_1}{2} - 2 \right) \omega_1 + 2n \Omega_1 \right\} c_6^{(n)} - \Omega_s \sqrt{2N_1 (N_1 - 1) c_4^{(n)}} \psi_6,$$

where $\Omega_1 = \frac{|g_{1,0}|^2}{\hbar \Delta_1}$, $\Omega_2 = \frac{|g_{2,0}|^2}{\hbar \Delta_2}$ and $\Omega_s = \frac{|g_{1,0}|^2 |g_{2,0}|^2}{2 \hbar \Delta_1 \Delta_2} \left( \frac{1}{\Delta_1} + \frac{1}{\Delta_2} \right)$.

Below we are interested only in the dynamics of the amplitudes $c_2^{(n)}$ and $c_3^{(n)}$ controlled by the presence or absence of the cavity photon state $|n\rangle$.

$$\frac{dc_2^{(n)}}{dt} = \frac{i}{\hbar} E_2(n) c_2^{(n)} - i \sqrt{N_1 N_2} \Omega_s c_3^{(n)},$$

$$\frac{dc_3^{(n)}}{dt} = \frac{i}{\hbar} E_3(n) c_3^{(n)} - i \sqrt{N_1 N_2} \Omega_s c_2^{(n)},$$

where $E_{2,3}(n) = \bar{E}(n) + \delta E_{2,3}(n)$, $\bar{E}(n) = \frac{1}{2} \left\{ N_1 (\omega_1 + 2n \Omega_1) + N_2 (\omega_2 + 2n \Omega_2) \right\}$, $\delta E_2(n) = -\hbar \left\{ (\omega_1 + 2n \Omega_1) + N_1 \Omega_1 \right\}$, $\delta E_3(n) = -\hbar \left\{ (\omega_2 + 2n \Omega_2) + N_2 \Omega_2 \right\}$ with a solution

$$c_2^{(n)} = e^{i \Delta(n)t} \left\{ A_1(n) e^{i S(n) t} + A_2(n) e^{-i S(n) t} \right\},$$

$$c_3^{(n)} = e^{i \Delta(n)t} \left\{ B_1(n) e^{i S(n) t} + B_2(n) e^{-i S(n) t} \right\},$$

where $\Delta(n) = \frac{1}{2\hbar} [E_2(n) + E_3(n)]$, $S(n) = \sqrt{\left\{ \delta E_2(n) - \delta E_3(n) \right\}^2 + 4N_1 N_2 \hbar^2 \Omega_s^2}$.

By taking into account the initial conditions $c_2^{(n)} = 1$ and $c_3^{(n)} = 0$, we get $B_1(n) = -B_2(n) = -\frac{\sqrt{N_1 N_2 \Omega_s^2}}{2S(n)}$, \hspace{1cm} (24)

$$c_3^{(n)}(t) = 2ie^{i \Delta(n)t} B_1(n) \sin[S(n)t],$$

and

$$A_1(n) = \frac{1}{2} \left( 1 + \frac{\left\{ \delta E_2(n) - \delta E_3(n) \right\}}{S(n)} \right),$$

(30)
\[
A_2(n) = \frac{1}{2} \left( 1 - \frac{(\delta E_2(n) - \delta E_3(n))}{S(n)} \right). \tag{31}
\]

For equal resonant frequencies of the two nodes in vacuum cavity mode state \( \omega_1 + N_1 \Omega_1 = \omega_2 + N_2 \Omega_2, \delta E_2(n) - \delta E_3(n) = 2n \hbar (\Omega_1 - \Omega_2), S(n) = n^2 (\Omega_1 - \Omega_2)^2 + N_1 N_2 \Omega_s^2. \) Expression (24) simplifies to the following
\[
B_1(n) = -B_2(n) = -\frac{S(0)}{2S(n)}. \tag{32}
\]

and expressions (29), (30) simplify to
\[
A_1(n) = \frac{1}{2} \left( 1 + \frac{n (\Omega_2 - \Omega_1)}{S(n)} \right), \tag{33}
\]
\[
A_2(n) = \frac{1}{2} \left( 1 - \frac{n (\Omega_2 - \Omega_1)}{S(n)} \right). \tag{34}
\]

Below we are interested in the quantum dynamics of the processing nodes controlled by a single photon field where the obtained solutions are

1. **CSWAP gate based on dynamical elimination of** \( c_3^{(1)} \)

\(|n = 0\rangle \text{- photon state:}\)
\[
c_2^{(0)}(t) = e^{i \Delta(0)T} \cos \{S(0)t\}, \tag{35}
\]
\[
c_3^{(0)}(t) = -ie^{i \Delta(0)T} \sin \{S(0)t\}. \tag{36}
\]

\(|n = 1\rangle \text{- photon state:}\)
\[
c_2^{(1)}(t) = e^{i \Delta(1)T} \{\cos[S(1)t] + i \frac{S(1)}{S(1)} \sin[S(1)t]\}, \tag{37}
\]
\[
c_3^{(1)}(t) = -ie^{i \Delta(1)T} \frac{S(0)}{S(1)} \sin[S(1)t], \tag{38}
\]

where \( S(0) = \sqrt{N_1 N_2 \Omega_s}, S(1) = \sqrt{\Omega_2 - \Omega_1}^2 + N_1 N_2 \Omega_s^2. \)

We can realize the C(SWAP) gate by using the quantum evolution of the processing gates during fixed temporal interval \( t = t_{C(SWAP)} = T \) where \( S(0)T = \pi/2 + n \pi \) and \( S(1)T = \pi + m \pi \) which leads to the following relation
\[
S(1) = \frac{1 + m}{(1/4 + n)} S(0) \quad (m = 0, 1, 2, \ldots; n = 1, 2, \ldots). \]

By assuming \( m = 0 \) and \( n = 1 \), we get \( |\Omega_1 - \Omega_2| = \frac{1}{2} \sqrt{5} N_1 N_2 \Omega_s. \)

At equal values of atom numbers in the nodes \( N_1 = N_2 = N \) and frequency offsets \( \Delta_1 = \Delta_2 = \Delta \), we have \( \Omega_s^2 = \Omega_1 \Omega_2 \) and \( \Omega_1 = 1.25 N^2 \Omega_s \).

2. **CSWAP gate based on strong blockade of** \( c_3^{(1)} \)

Here, we see that if \( n = 1 \) and \( |\Omega_2 - \Omega_1| >> \Omega_s \sqrt{N_1 N_2} \) we have \( B_1 = -B_2 = c_3^{(0)} = 0, A_1 = 1 \) and \( A_2 = 0. \)

If \( n = 0 \) we have \( B_1 = -B_2 = -\frac{1}{2} \) and \( A_1 = A_2 = \frac{1}{2}. \) In the first case no swap occurs and in the second case we have swapping solution
\[
c_3^{(1)} \approx -ie^{\pi \delta E_3/N_2 \Omega_s t} \sin \left( \sqrt{N_1 N_2 \Omega_s t} \right), \tag{39}
\]
and
\[
c_2^{(1)} \approx e^{\pi \delta E_2/N_2 \Omega_s t} \cos \left( \sqrt{N_1 N_2 \Omega_s t} \right). \tag{40}
\]

State \( \psi_2 \) is converted into state \( \psi_3 \) on time interval \( t_{C(SWAP)} = \pi/(2 \Omega_s \sqrt{N_1 N_2}) \). Thus, we have swap gate controlled by photon state (C(SWAP) gate). By substituting the blockade condition we find that C(SWAP) gate operates with single atomic rate \( t_{C(SWAP)} = \pi/(2 \Omega_s N) = t_{C(SWAP)} = \pi/(2 \Omega_s N) \approx \pi/(2 \Omega_1) \). While the SWAP gate does operate \( N \) times faster in multi-atomic case in accordance with Eqs. (39), (40). Therefore, it is reasonable to use single atoms (molecules) or quantum dots with a large dipole moment for C(SWAP) operation, while using multi-atomic ensembles for simple SWAP operation. They can yield CNOT and single qubit gates with the qubit encoding from subsection 2.1.

According to subsection 2.1, we need the have a swap operation in the pair of atomic ensembles (two green nodes in Fig. 10) controlled by another pair of atomic ensembles (two blue nodes in Fig. 10). This can be achieved by releases photon from one node of blue node pair by equalizing the node frequency with the frequency QED cavity mode and switching of the node frequency from the resonance in a proper moment of time. After the C(SWAP) gate realization, we can return the photon to the chosen blue node by equalizing the node frequency with the cavity.
This C(SWAP) gate is equivalent to CNOT gate with qubit encoding introduced in section 2.1. Such gates and other necessary gates can be incorporated in common cavity in a quantity that is needed for implementation of one or another quantum algorithm. All of them can be initiated through a single quantum memory that is essentially multi-qubit in photon echo approach [27].

3 Conclusion

Summarizing, we have proposed an approach for constructing encoded universal quantum computations based on swapping operations. The main idea of the proposed set of quantum gates is encoding logical qubits by two physical qubits. It allows to explicitly implement any encoded single-qubit gate by 3 elementary operations and to perform encoded controlled-NOT gate by a single C(SWAP) operation on pairs of atomic ensembles. We have shown that our basis of quantum gates is efficient (in terms of the number of elementary gates) for implementing complex controlled operations, which are at the core of many efficient quantum algorithms.

This approach considerably simplifies physical implementation of quantum computer on multi-atomic ensembles in the QED resonator at the price of doubling the number of qubits for computation. Besides, it permits to avoid the necessity of implementing blockade of excess states in the multi-atomic ensembles. The physical implementation of the basic gates is sufficiently robust and provides fast single qubit operations based on multi-atomic ensembles.

Note also, that using of two atomic ensembles for encoding of a single qubit state will be convenient for the quantum computer interface with the external quantum information carried by using photon polarization qubits since the two polarization components of the photon qubit can be coupled directly with the relevant pair of the atomic ensemble state.

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References

[1] Michael A. Nielsen and Isaac L. Chuang. Quantum Computation and Quantum Information. Cambridge University Press, 1 edition, October 2000.
[2] Mikio Nakahara and Tetsuo Ohmi. Quantum Computing: From Linear Algebra to Physical Realizations. CRC Press, Taylor & Francis, 2008.
[3] Pieter Kok, W. J. Munro, Kae Nemoto, T. C. Ralph, Jonathan P. Dowling, and G. J. Milburn. Linear optical quantum computing with photonic qubits. Rev. Mod. Phys., 79(1):135–174, Jan 2007.
[4] T. D. Ladd, F. Jelezko, R. Laflamme, Y. Nakamura, C. Monroe, and J. L. O’Brien. Quantum computers. Nature, 464(7285):45–53, March 2010.
[5] David Deutsch. Quantum computational networks. Royal Society of London Proceedings Series A, 425:73–90, sep 1989.
[6] David P. DiVincenzo. Two-bit gates are universal for quantum computation. Phys. Rev. A, 51(2):1015–1022, Feb 1995.
[7] David Deutsch, Adriano Barenco, and Artur Ekert. Universality in Quantum Computation. Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences, 449(1937):669–677, 1995.
[8] P. Oscar Boykin, Tal Mor, Matthew Pulver, Vwani Roychowdhury, and Farrokh Vatan. A new universal and fault-tolerant quantum basis. Information Processing Letters, 75(3):101–107, 2000.
[9] David P. DiVincenzo, Dave Bacon, Julia Kempe, Guido Burkard, and K. Birgitta Whaley. Universal quantum computation with the exchange interaction. Nature, 408:339–342, 2000.
[10] Dave Bacon, Julia Kempe, Daniel A. Lidar, and K. Birgitta Whaley. Universal fault-tolerant quantum computation on decoherence-free subspaces. Phys. Rev. Lett., 85(8):1758–1761, Aug 2000.
[11] Julia Kempe, Dave Bacon, Daniel A. Lidar, and K. Birgitta Whaley. Theory of decoherence-free fault-tolerant universal quantum computation. Phys. Rev. A, 63(4):042307, Mar 2001.

[12] Julia Kempe, Dave Bacon, David P. DiVincenzo, and K. Birgitta Whaley. Encoded universality from a single physical interaction. In R. Clark, G. Milburn, R. Hughes, A. Imamoglu, and P. Delsing, editors, Quantum Computation and Information, volume 1, pages 33–55. Rinton Press, New Jersey, 2001.

[13] Julia Kempe and K. Birgitta Whaley. Exact gate sequences for universal quantum computation using the xy interaction alone. Phys. Rev. A, 65(5):052330, May 2002.

[14] Jeremy Levy. Universal quantum computation with spin-1/2 pairs and heisenberg exchange. Phys. Rev. Lett., 89(14):147902, Sep 2002.

[15] E. Brion, K. Mølmer, and M. Saffman. Quantum computing with collective ensembles of multilevel systems. Phys. Rev. Lett., 99(26):260501, Dec 2007.

[16] M. Saffman and K. Mølmer. Scaling the neutral-atom rydberg gate quantum computer by collective encoding in holmium atoms. Phys. Rev. A, 78(1):012336, Jul 2008.

[17] M. Saffman, T. G. Walker, and K. Mølmer. Quantum information with rydberg atoms. Rev. Mod. Phys., 82(3):2313–2363, Aug 2010.

[18] M. S. Shahriar, G. S. Pati, and K. Salit. Quantum communication and computing with atomic ensembles using a light-shift-imbalance-induced blockade. Phys. Rev. A, 75(2):022323, Feb 2007.

[19] Sergey A. Moiseev, Sergey N. Andrianov, and Firdus F. Gubaidullin. Solid state multi-ensemble quantum computer in waveguide circuit model. Technical Report arXiv:1009.5771, Cornell University Library, Sep 2010.

[20] Sergey N. Andrianov and Sergey A. Moiseev. Fast and robust two- and three-qubit swapping gates on multi-atomic ensembles in quantum electrodynamical cavity. Electronic Proceedings in Theoretical Computer Science, 52:13–21, 2011.

[21] L.-M. Duan and H. J. Kimble. Scalable photonic quantum computation through cavity-assisted interactions. Phys. Rev. Lett., 92(12):127902, Mar 2004.

[22] Takao Aoki, Barak Dayan, E. Wilcut, W. P. Bowen, A. S. Parkins, T. J. Kippenberg, K. J. Vahala, and H. J. Kimble. Observation of strong coupling between one atom and a monolithic microresonator. Nature, 443(7112):671–674, 2006.

[23] J. Majer, J. M. Chow, J. M. Gambetta, Jens Koch, B. R. Johnson, J. A. Schreier, L. Frunzio, D. I. Schuster, A. A. Houck, A. Wallraff, A. Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf. Coupling superconducting qubits via a cavity bus. Nature, 449(7161):443–447, September 2007.

[24] G. Massimo Palma, Kalle-Antti Suominen, and Artur K. Ekert. Quantum computers and dissipation. Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences, 452(1946):567–584, 1996.

[25] Mark S. Byrd and Daniel A. Lidar. Comprehensive encoding and decoupling solution to problems of decoherence and design in solid-state quantum computing. Phys. Rev. Lett., 89(4):047901, Jul 2002.

[26] Farid Ablayev and Alexander Vasiliev. Algorithms for quantum branching programs based on fingerprinting. Electronic Proceedings in Theoretical Computer Science, 9:1–11, 2009.

[27] Sergey A. Moiseev, Sergey N. Andrianov, and Firdus F. Gubaidullin. Efficient multimode quantum memory based on photon echo in an optimal qed cavity. Phys. Rev. A, 82(2):022311, Aug 2010.

[28] Sergey A. Moiseev and Sergey N. Andrianov. Multi-qubit quantum memory integrated in quantum computer. In Proceedings of the 5-th International Scientific school “Science and innovation-2010”, ISS “SI-2010”: 19 – 24 July 2010, Ioshkar-Ola, pages 156–164, 2010.

[29] A. Imamoglu, D. D. Awschalom, G. Burkard, D. P. DiVincenzo, D. Loss, M. Sherwin, and A. Small. Quantum information processing using quantum dot spins and cavity-qed. Physical Review Letters, 83(20):4204–4207, 1999.