HARMLESS OVERPARAMETRIZATION IN TWO-LAYER NEURAL NETWORKS

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Overparametrized neural networks, where the number of active parameters is larger than the sample size, prove remarkably effective in modern deep learning practice. From the classical perspective, however, much fewer parameters are sufficient for optimal estimation and prediction, whereas overparametrization can be harmful even in the presence of explicit regularization. To reconcile this conflict, we present a generalization theory for overparametrized ReLU networks by incorporating an explicit regularizer based on the scaled variation norm. Interestingly, this regularizer is equivalent to the ridge from the angle of gradient-based optimization, but is similar to the group lasso in terms of controlling model complexity. By exploiting this ridge-lasso duality, we show that overparametrization is generally harmless to two-layer ReLU networks. In particular, the overparametrized estimators are minimax optimal up to a logarithmic factor. By contrast, we show that overparametrized random feature models suffer from the curse of dimensionality and thus are suboptimal.

1. Introduction. Deep learning has shown great superiority to classical machine learning methods in terms of representation of raw data and prediction in many tasks, including image recognition in computer vision (He et al., 2016; Voulodimos et al., 2018), machine translation and text generation in natural language processing (Sutskever, Vinyals and Le, 2014; Wu et al., 2016; Kannan et al., 2016), general game playing (Mnih et al., 2015; Silver et al., 2017), and disease diagnosis in clinical research (De Fauw et al., 2018; Tang et al., 2019; Esteva et al., 2019). In such a wide range of applications, neural networks are preferred to be overparametrized in the sense that the number of active (nonzero) network parameters is much larger than the sample size. One example is the Alex-Net (Krizhevsky, Sutskever and Hinton, 2012). There is convincing evidence showing that overparametrization can help optimization (Arora, Cohen and Hazan, 2018; Safran, Yehudai and Shamir, 2020), however, overparametrized deep neural networks can easily fit random labels even in the presence of explicit regularization (Zhang et al., 2016). This indicates that overparametrized models have the capability to overfit, but do not necessarily lead to bad testing performance. Thus two intriguing questions come out: why can overparametrized neural networks exhibit good testing performance and how good can the testing performance be?

Overparametrization, where the number of active parameters is larger than the sample size, is not new in statistics. For example, high dimensional linear models can be viewed to be overparametrized. In high dimensional linear regression, prediction is closely related to the estimation of the true parameter and overparametrization can be harmful to estimation even in the presence of optimal regularization. For example, the prediction risk of the optimal regularized ridge estimator can have a strictly positive limit in the overparametrized settings (Dobriban et al., 2018; Hastie et al., 2019). But if we assume the true parameter is sparsely...
supported or the design matrix is approximately of low rank, the Lasso method (Tibshirani, 1996) or the ridge(less) regression (Shao et al., 2012; Bartlett et al., 2020), respectively, can produce consistent estimators. Essentially, the Lasso method is a model selection procedure whose solution is indeed not overparametrized, and the low rank assumption controls the number of effective parameters, then both of the two assumptions can alleviate the difficulty of estimation caused by overparametrization. It seems that overparametrization is harmful to the prediction. However, this may be not true for pure prediction algorithms (Efron, 2020).

Deep learning is a pure prediction algorithm since it is a nonparametric method and parameters of neural networks are not identifiable. From this viewpoint, different from linear models, overparametrization can be harmless in neural networks without the burden of estimation.

In general nonparametric regression, there exists a trade-off between the approximation and the complexity of candidate prediction rules (Györfi et al., 2006; Wasserman, 2006; Tsybakov, 2008). When the complexity is measured in terms of the number of active parameters, the trade-off leads to the preference of parsimonious models, in which the number of active parameters is much smaller than the sample size. In contrast to overparameterization, we refer to these models as underparametrized models. Underparametrized neural networks can be minimax optimal and break the curse of dimensionality when estimating functions with special structures. For example, Schmidt-Hieber et al. (2020) considered a space whose elements are a composition of several multivariate Hölder functions and showed that highly sparse deep ReLU networks can utilize the composite structure to attain the minimax optimality up to a logarithmic factor while wavelet series estimators are suboptimal. For the function class consisting of sparse linear combinations of some orthogonal wavelets, similar results hold as well (Hayakawa and Suzuki, 2019). In Schmidt-Hieber et al. (2020), Schmidt-Hieber (2019), and Hayakawa and Suzuki (2019), neural networks with optimal performance are solutions to the empirical risk minimization under some network architecture constraints that enforce the underparametrization. It is difficult to get such structured sparse estimators without knowing prior population information. Nevertheless, the optimal performance of underparametrized neural networks can be treated as a baseline for overparametrized ones.

The aforementioned prior work suggests seeking a different approach to control the complexity of overparametrized neural networks. Neural tangent kernel method (Arora et al., 2019; Ghorbani et al., 2019a; Jacot, Gabriel and Hongler, 2018) controls the complexity by showing that the parameters of networks are close to the initialization during the training dynamics. It can be viewed as a first-order Taylor approximation of neural networks around the initialization (Chizat, Oyallon and Bach, 2019; Ghorbani et al., 2019a) and requires $O(n^4)$ hidden nodes to learn $n$ pairs of samples. In this case, overparameterization serves to simplify the interested model. The simplification has limitations (Ghorbani et al., 2019b; Wei et al., 2019) and can only partially illustrate the reason why overparameterized models generalize well because only global minima around the initialization are considered. A generalization guarantee for all global minima of the empirical risk possibly with proper regularization is needed.

In this work, we consider a nonparametric regression model with random predictors, where $n$ i.i.d. vectors $X_i \sim \mu$ whose support set lies in $\mathbb{R}^d$, the unit ball of $\mathbb{R}^d$, and $n$ responses $Y_i \in \mathbb{R}$ are collected from the model

$$Y_i = f^*(X_i) + \varepsilon_i,$$

for $i \in \{1, \ldots, n\}$. Denote by $\sigma(\cdot) = \max\{0, \cdot\}$ the ReLU activation function. A two-layer ReLU network with $m$ hidden units $g(\theta; \cdot): \mathbb{R}^d \rightarrow \mathbb{R}$ can be represented by $g(\theta; x) = \sum_{k=1}^m a_k \sigma(\langle v_k, x \rangle + b_k) + c$, with parameters $\theta = (m, v_1, \ldots, v_m, b_1, \ldots, b_m, a_1, \ldots, a_m, c)$. Its scaled variation norm is defined as $\nu(\theta) = \sum_{k=1}^m |a_k| \|w_k\|_2$, where $w_k = (v_k^T, b_k^T)^T, 1 \leq k \leq m$, which appears but is not primarily studied in Bach (2017a) and Pilanci and Ergen...
We consider the target function class $G_\sigma$ to be the closure of any two-layer ReLU networks with a finite scaled variation norm, where the closure is taken under the $L_2(\mu)$ norm. The detailed assumptions of $\mu$, $f^*(\cdot)$, and $\varepsilon; s$ are listed in Section 2.3. To recover $f^*$, we adopt the following empirical risk minimization penalized by the scaled variation norm

$$J_n(\theta_m; \lambda_n) = \frac{1}{2n} \sum_{i=1}^{n} \left( y_i - \sum_{k=1}^{m} a_k \sigma(\langle (x_i, 1)\rangle^T, w_k) \right)^2 + \lambda_n \sum_{k=1}^{m} |a_k| \|w_k\|_2.$$

In the presence of the explicit regularization, we can answer the two questions in the first paragraph. We show that, for properly chosen $\lambda_n$, the global minima enjoy the generalization rate $O(\sqrt{\log n}/n)$ if $m \geq n + 1$, which achieves the minimax optimality up to a logarithmic factor. In contrast, random feature models suffer from the curse of dimensionality even when estimating subsets of $G_\sigma$. Our result is independent of the algorithms and holds for all global minima. To control the complexity of overparametrized neural networks, we borrow some technical tools from the norm-based regularization theory (Neyshabur, Tomioka and Srebro, 2015), where the path norm defined as $\|\theta\|_P = \sum_{k=1}^{m} |a_k| \|w_k\|_1$ (Ma et al., 2019a) is mostly relevant to our work. The path norm, due to the presence of $\ell_1$-norm, prefers sparse solutions, which enforces certain parameters to be zero and reduces the effect of overparametrization. By contrast, the scaled variation norm enjoys the intriguing ridge-lasso duality. After a simple reparametrization that relies on the homogeneity of ReLU, the scaled variation regularization is equivalent to the ridge regularization, which would induce no sparsity. On other hand, by carefully analyzing the geometry of global minima, we can show that the scaled variation penalty is similar to the group-lasso penalty, and the model complexity can be sharply controlled. We also characterize the positive colinearity of parameters residing in the same convex cone for global minima to further illustrate the rationale of ridge-lasso duality. Combining good properties of both the ridge and the group-lasso, we provide a unified analysis of the learning problem of two-layer ReLU networks and show that overparametrization can be harmless in pure prediction algorithms. Our results can be easily extended to variants of ReLU, such as leaky ReLU.

1.1. Related work. For the generalization ability of two-layer neural networks, Barron (1993, 1994) considered using certain covering entropy based complexity criterion to derive underparametrized network estimators with nearly minimax optimality (Klusowski and Barron, 2017). The number of hidden units is required to be much smaller than the sample size. Bach (2017a) approached this problem via a constrained empirical minimization and utilized incremental conditional gradient algorithms (Frank et al., 1956; Harchaoui, Juditsky and Nemirovski, 2015) to transform the constrained optimization into a convex program over an infinite-dimensional space of functions. The complexity of overparametrized neural networks can be controlled by the constraint level which is independent on the number of parameters. The number of hidden nodes is equivalent to the number of iterations that the incremental conditional gradient algorithm would run, hence underparametrized estimators can be inconsistent. The constraint empirical minimization is difficult to be analyzed under the framework of gradient based algorithms. Klusowski and Barron (2016) considered the path norm regularization and introduced a countable neural network set to determine the level of regularization to derive the resolvability bound. The optimal width therein is much smaller than the sample size. Ma et al. (2019a) also considered the path norm regularization with an unbounded width; however, their estimators are likely to be underparametrized in effect.

For the optimization of finitely wide neural networks, Pilanci and Ergen (2020) showed that $\ell_2$-regularized empirical risk minimization of two-layer ReLU networks is equivalent to a finite dimensional convex group-lasso problem, to which our work is related. However, we focus on the generalization ability of two-layer networks, which they did not.
de Dios and Bruna (2020) and Lacotte and Pilanci (2020) showed that there exists a continuous path from any initial point to the global minima on which the empirical risk with weight decay is non-increasing. Lacotte and Pilanci (2020) also showed the positive collinearity of parameters of Clarke stationary points of $\ell_2$-regularized empirical risk minimization of two-layer ReLU networks, and they referred to this property as nearly minimality. Other work related to the details of this work would be discussed thereafter.

1.2. Organization of the paper. In Section 2, we introduce the mathematical definitions of two-layer ReLU networks and the target function class. We also list our assumptions and some discussions of the assumptions. In Section 3–5, we introduce our main results, including the approximation results, the ridge-lasso duality of the scaled variation norm, and the generalization guarantees of overparametrized estimators. In Section 6, we provide some discussions of our work and future work. Theoretical proofs are deferred to the Appendix and Supplementary Materials.

2. Preliminaries. In this section, we introduce some related notations, the mathematical definitions of two-layer ReLU networks and the target function class, the underlying model, and associated assumptions.

2.1. Notations. In the following, we denote by $\sigma(\cdot)$ the ReLU activation function. Define $\mathbb{B}^d(r) = \{ x \in \mathbb{R}^d : \| x \|_2 \leq r \}$ and $\mathbb{B}^d$ is the abbreviation for $\mathbb{B}^d(1)$. $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^{d+1}$ or any other well-defined Hilbert space and $\| \cdot \|_2$ denotes its associated norm. For any grouped vector $\beta = (\beta_1^T, \ldots, \beta_p^T)^T$, denote $\| \beta \|_{2,1} = \sum_{i=1}^p \| \beta_i \|_2$. $\| \cdot \|_{L^2(\mu)}$ denotes the $L^2$ norm with respect to measure $\mu$. Denote by $M(A)$ the space of signed measures $\alpha$ defined on $(v, b) \in A$ with finite total variation norm $\int_A d|\alpha| < \infty$, and $\delta_x$ denotes the Dirac mass at a point $x$. Denote by $\mu$ the distribution of the covariate $X$ and let $\text{subG}(\sigma^2_x)$ be some sub-gaussian distribution with variance being $\sigma^2_x$.

2.2. Two-layer ReLU networks and the target function class. Consider a two-layer neural network activated by ReLU, $g(\theta; \cdot): \mathbb{R}^d \to \mathbb{R}$ defined by

\begin{equation}
(2.1) \quad g(\theta; x) = \sum_{k=1}^m a_k \sigma(\langle v_k, x \rangle + b_k) + c,
\end{equation}

with parameters $\theta = (m, \mathbf{V} = [v_1, \ldots, v_m], b = [b_1, \ldots, b_m]^T, \mathbf{a} = [a_1, \ldots, a_m], c)$. where the width $m \in \mathbb{N}$ can be unbounded. In some cases, we will use $\theta_m$ to emphasize that the width is $m$. Let $\Theta$ be the collection of all such parameters vectors $\theta$. For simplicity, we will denote by $\mathbf{w}$ the concatenated vector $(v^T, b)^T \in \mathbb{R}^{d+1}$ and still use $x$ to represent $(x^T, 1)^T$ if there is no ambiguity in the context. Define the scaled variation norm of finitely wide neural networks $g(\theta_m; \cdot)$ as

\begin{equation}
(2.2) \quad \nu(\theta_m) = \sum_{k=1}^m |a_k| \| w_k \|_2.
\end{equation}

The definition of the scaled variation norm is intuitive. Notice that $\sum_{k=1}^m (|a_k|^2 + \| w_k \|_2^2)$, the summation of squared $L^2$ norm of parameters, is lower bounded by $2\nu_{\theta_m}$ by the inequality of arithmetic and geometric means. The condition $\nu_{\theta_m} < \infty$ is the minimal requirement for the network estimators being well-behaved. From another viewpoint, if assume $\| w_k \|_2 = 1, 1 \leq k \leq m$, the scaled variation norm is reduced to be the variation norm in Klusowski and Barron (2016), Ongie et al. (2019), and Bach (2017a).
Two-layer neural networks have an integral representation (Bach, 2017a; Ongie et al., 2019; Dou and Liang, 2020). In particular, if define \( \alpha_m = \sum_{k=1}^{m} a_k \delta_{w_k} \) as a discrete signed measure, we have

\[
g(\theta_m; x) = \sum_{k=1}^{m} a_k \sigma(\langle v_k, x \rangle + b_k) + c
\]

(2.3)

\[
g(\theta_m; x) = \int_{\mathbb{R}^{d+1}} (\sigma(\langle v, x \rangle + b) - \sigma(b)) \, d\alpha_m(w) + g(\theta_m; 0).
\]

Motivated by this, we can define an infinitely wide neural network as

\[
g_\alpha(x) = \int_{\mathbb{R}^{d+1}} (\sigma(\langle v, x \rangle + b) - \sigma(b)) \, d\alpha(w) + g_\alpha(0),
\]

(2.4)

which is determined by the signed measure \( \alpha \in \mathcal{M}(\mathbb{R}^{d+1}) \). For \( g_\alpha(x) \) to be well-defined, a sufficient condition is

\[
\int_{\mathbb{R}^{d+1}} \|v\|_2 \, d|\alpha|(w) < \infty,
\]

(2.5)

since by the Lipschitz continuity of the ReLU, \( |\sigma(\langle v, x \rangle + b) - \sigma(b)| \leq |\langle v, x \rangle| \leq \|v\|_2 \|x\|_2 \).

From this perspective, define the interested function space as

\[
\mathcal{G}_\sigma = \left\{ f(x) = \int_{\mathbb{R}^{d+1}} (\sigma(\langle v, x \rangle + b) - \sigma(b)) \, d\alpha(w) + f(0) : \int_{\mathbb{R}^{d+1}} \|v\|_2 \, d|\alpha|(w) < \infty, x \in \mathbb{R}^d \right\}.
\]

(2.6)

We can show that there is a one-to-one correspondence between \( f \) and its associated signed measure \( \alpha \). \( \int_{\mathbb{R}^{d+1}} \|v\|_2 \, d|\alpha|(w) \) can be viewed as the scaled variation norm of infinitely wide neural networks. Following the notations in Ongie et al. (2019), we provide the following definition.

**Definition 2.1.** The \( \mathcal{R} \)-norm for \( f \in \mathcal{G}_\sigma \) is defined as \( \|f\|_\mathcal{R} = \int_{\mathbb{R}^{d+1}} \|v\|_2 \, d|\alpha_f|(w) \), where the signed measure \( \alpha_f \) is determined uniquely by

\[
f(x) = \int_{\mathbb{R}^{d+1}} (\sigma(\langle v, x \rangle + b) - \sigma(b)) \, d\alpha_f(v, b) + f(0).
\]

(2.7)

There is another interpretation of \( \mathcal{G}_\sigma \). By some modifications of Lemma 10 in Ongie et al. (2019), we can show that a Lipschitz function \( f \) can be uniformly approximated with error up to \( \varepsilon \) by a ReLU network with finite scaled variation norm over \( \|x\|_2 \leq \varepsilon^{-1} \) if and only if \( f \in \mathcal{G}_\sigma \). See Corollary D.1 for details.

Barron (1993, 1994) and Ma et al. (2019a) considered the function class with finite Barron norm, i.e., \( \int_{\mathbb{R}^{d+1}} |w|_1 |\mathcal{F}f| \, dw < \infty \), where \( \mathcal{F}f \) denotes the Fourier transform of \( f \). By comparison, the inverse Radon transform (Helgason and Helgason, 1980) of \( \alpha_f \) is closely related to \( f \) (Ongie et al., 2019) and it can be shown that our function space is larger.

**2.3. Model and assumptions.** Consider the data-generating model

\[
Y_i = f^*(X_i) + \varepsilon_i, \quad i = 1, 2, \ldots, n.
\]

(2.8)

We impose the following conditions:

(C.1) \( f^* \in \mathcal{G}_\sigma \);
(C.2) \( X_i \sim \mu \) independently, the support set of \( \mu \) is contained in \( \mathbb{R}^d \), and \( \mu \) is absolutely continuous with respect to the Lebesgue measure;

(C.3) \( \epsilon_i \sim \mathcal{N}(0, \sigma_i^2) \) independently, and \( \{\epsilon_i\}_{i=1}^n \) are independent of \( \{X_i\}_{i=1}^n \).

The third condition is standard in nonparametric models. The first condition requires that the target function \( f^* \) has a finite \( R \)-norm. This assumption can be relaxed and it suffices to assume that \( \|f^*\|_R \) grows with the sample size, in which case \( f^* \) may not be the target function but the best approximator of the target function over a ball in \( G_\sigma \). For simplicity, we only consider the case where \( f^* \) is the target function. The second condition is commonly used in the machine learning literature since \( \{X_i\}_{i=1}^n \) are usually normalized in practice. There are also some technical considerations. The second condition can be viewed as a restriction of \( f^*(x) \) from \( \mathbb{R}^d \) to \( \mathbb{B}^d \). We can show that restricted function class \( \{f \in G_\sigma : x \in \mathbb{B}^d\} \) is equivalent to

\[
\mathcal{F}_\sigma(\mathbb{B}^d) = \left\{ f(x) = \int_{\mathbb{S}^{d-1} \times [-1,1]} \sigma(\langle v, x \rangle + b) d\alpha(w) + c : \right. \\
\left. \int_{\mathbb{S}^{d-1} \times [-1,1]} d|\alpha|(w) < \infty, x \in \mathbb{B}^d \right\}.
\]

(2.9)

Informally, for \( f \in \mathcal{F}_\sigma(\mathbb{B}^d) \),

\[
\|f\|_R = \int_{\mathbb{S}^{d-1} \times [-1,1]} \|v\|_2 d|\alpha|(w) \leq \int_{\mathbb{S}^{d-1} \times [-1,1]} \|w\|_2 d|\alpha|(w) \leq \sqrt{2}\|f\|_R
\]

holds since \( w = (v^T, b)^T \) and \( |b| \leq 1 = \|v\|_2 \). Notice that the signed measure in \( \mathcal{F}_\sigma(\mathbb{B}^d) \) is no longer the original one that defines \( f \), the above argument is not rigorous. However, the conclusion still holds with a different constant. See rigorous proofs in Theorem D.1 of the Supplementary. Therefore, the scaled variation norm of finitely wide ReLU networks is consistent with its infinitely wide counterpart \( R \)-norm in format.

Notice that finitely wide two-layer ReLU networks belong to \( G_\sigma \). Under Assumption (A.2), the scaled variation norm of finitely wide two-layer ReLU networks coincides with their \( R \)-norm up to a multiplicative constant. Let \( \mathcal{G}_\sigma^R = \{f \in G_\sigma : \|f\|_R \leq R\} \). It can be shown that functions in \( \mathcal{G}_\sigma^R \) are \( R \)-Lipschitz continuous and \( \bigcup_{R \geq 0} \mathcal{G}_\sigma^R \) is dense in \( \mathcal{C}(\mathbb{R}^d) \), the space consisting of all continuous functions defined on \( \mathbb{R}^d \). Functions with larger \( R \)-norm are more complex. In Section 3, we show that wide two-layer ReLU networks can approximate \( f^* \) efficiently. Then it is natural to control the scaled variation norm of overparametrized estimators to mitigate overfitting. From this perspective, we adopt the following empirical risk minimization penalized by the scaled variation norm

\[
\min J_n(\theta_m, \lambda_n) = \frac{1}{2n} \sum_{i=1}^n \left( y_i - \sum_{k=1}^m a_k \sigma(w_k, x_i) \right)^2 + \lambda_n \sum_{k=1}^m |a_k| \|w_k\|_2.
\]

(2.10)

Considering the minimization (2.10) enjoys another statistical advantage. In the infinite dimensional case, the minimization of \( J_n(\theta; \lambda_n) \) turns out to find an optimal measure \( \alpha^* \) such that

\[
\alpha^* \in \arg\min_{\alpha} J_n(\alpha; \lambda_n) = \frac{1}{2n} \left\| Y - \int_{\mathbb{R}^d} \sigma(Xw) d\alpha(w) \right\|_2^2 + \lambda_n \|\alpha\|_{TV},
\]

where \( \|\alpha\|_{TV} \) denotes the total variation norm of \( \alpha \). Rosset et al. (2007) showed that the above infinite dimensional minimization always has a discrete solution defined on \( m^* \) points with \( m^* \leq n + 1 \). In this vein, for any \( m \geq n + 1 \geq m^* \), the minimal value of the infinite
dimensional minimization \( J_p(\alpha; \lambda_n) \) can be achieved by minimizing \( J_p(\theta_m; \lambda_n) \), meanwhile we can study the effect of overparametrization by increasing \( m \) arbitrarily greater than \( n \) since the most parsimonious solution only have \( m^* \leq n + 1 \) neurons.

3. Sharp approximation rates. Two-layer ReLU networks enjoy the universal approximation property (Pinkus, 1999). In the subsection, we focus on the convergence rate for two-layer ReLU networks with \( m \) hidden units approximating functions in \( \mathcal{G}_\sigma^R = \{ f \in \mathcal{G}_\sigma: \| f \|_\mathcal{R} \leq R \} \), where \( R \) can be arbitrarily large.

Any function in \( \mathcal{G}_\sigma \) is determined by its associated signed measure \( \alpha \). Heuristically, if we view \( d\alpha = (d\alpha/d|\alpha|)|d|\alpha| \) in (2.6) as the distribution over weights, then two-layer ReLU networks with \( m \) hidden units can achieve the approximation rate \( O(1/\sqrt{m}) \) under the \( L^2(\mu) \) metric by the Monte Carlo argument (Barron, 1993; Ma et al., 2019a). The approximation rate is not optimal. By utilizing the localized Monte Carlo argument developed in Makovoz (1996) and Klusowski and Barron (2018), we provide a sharper approximation rate which scales as \( m^{-(d+2)/(2d)} \) and show that it is nearly optimal. Here we relax the assumption (A.2) and it suffices to assume the distribution \( \mu \) is supported in \( \mathbb{B}^d \), i.e., \( \mu(\mathbb{B}^d) = 1 \).

**Theorem 3.1.** Denote by \( \mathcal{G}_\sigma^R \) the set \( \{ f \in \mathcal{G}_\sigma: \| f \|_\mathcal{R} \leq R \} \) and by \( \Theta_m^R \) the set \( \{ \theta_m: \sum_{i=1}^m |a_i| \| w_i \|_2 \leq R \} \). Assume \( R \geq 1 \) and \( \mu \) is a probability distribution with \( \mu(\mathbb{B}^d) = 1 \). Then for any arbitrary small \( \eta > 0 \), there exists a \( c = c(d, \eta) > 0 \) and a constant \( C \) independent of \( m, R, \eta, \) and \( d \) such that

\[
cR m^{-1/2 - 1/d - \eta} \leq \sup_{f^* \in \mathcal{G}_\sigma^R} \inf_{\theta_m \in \Theta_m^R} \| f^*(x) - g(\theta_m; x) \|_{L^2(\mu)} \leq C R m^{-1/2 - 1/d}
\]

for any \( d \geq 1 \).

The lower bound is identical to Barron (1992); Makovoz (1996). In fact, we can show that the function class considered in Barron (1992) is a subspace of \( \mathcal{G}_\sigma \) and we prove the lower bound using the same technical lemma with Makovoz (1996). However, owing to the homogeneity and the Lipschitz continuity of ReLU, we can improve the upper bound from \( O \left( m^{-1/2 - 1/(2d)} \right) \) to \( O \left( m^{-1/2 - 1/d} \right) \). Similar results can be found in Klusowski and Barron (2018), where \( l_1 \) norm of the weights at the first layer is considered. For some fixed \( f^* \in \mathcal{G}_\sigma^R \), replacing \( R \) by \( \| f^* \|_\mathcal{R} \) gives the following corollary.

**Corollary 3.1.** Under the assumptions of Theorem 3.1, for any function \( f^* \in \mathcal{G}_\sigma \) with \( \| f^* \|_\mathcal{R} < \infty \), we have

\[
\inf_{\theta_m} \| f^*(x) - g(\theta_m; x) \|_{L^2(\mu)} \leq C \| f^* \|_\mathcal{R} m^{-1/2 - 1/d}
\]

for all \( d \geq 1 \), where \( g(\theta_m; \cdot) \) is of the form Equation (2.1), with \( \sum_{i=1}^m |a_i| \| w_i \|_2 = 1, 1 \leq i \leq m \) and \( C \) is a universal absolute constant independent of \( m, \| f^* \|_\mathcal{R}, \) and \( d \).

Denote by \( g(\theta_m^*; \cdot) \) the two-layer ReLU network with the optimal approximation performance and by \( \theta_m^* \) its parameters. From Corollary 3.1, the weights of \( g(\theta_m^*; \cdot) \) at the first layer can have a unit \( \ell_2 \) norm, in which case the scaled variation norm is reduced to the \( \ell_1 \) norm of parameters at the second layer. This property plays a fundamental role in the analyses of properties of global minima.

4. Ridge-Lasso duality of the scaled variation regularization. In this section, we shall illustrate the Ridge-Lasso duality of the scaled variation regularization, which is due to several transformations of parameters of neural networks.
4.1. Equivalence to the ridge. In this subsection, we show that, the solution set of $J_n(\theta_m; \lambda_n)$ is equivalent to that of empirical risk penalized by squared $L_2$-norm penalty up to a reparametrization which does not change the value and the architecture of networks.

Consider a reparametrization, which is defined as a mapping $\mathcal{T}: \Theta_m \to \Theta_m$. For any $(w, a) \in \Theta_m$, define $\mathcal{T}(w, a) = (m, \bar{w}_1, \ldots, \bar{w}_m, \bar{a}_1, \ldots, \bar{a}_m, c)$, where

$$\bar{w}_k = w_k \sqrt{\frac{|a_k|}{\|w_k\|_2}}, \quad \bar{a}_k = a_k \sqrt{\frac{\|w_k\|_2}{|a_k|}}, \quad k = 1, \ldots, m,$$

for $w_k \neq 0_d$ and $a_k \neq 0$, and if either $w_k$ or $a_k$ equals to zero, let $\bar{w}_k = 0$ and $\bar{a}_k = 0$. After the reparametrization, we have $\|\bar{w}_k\|_2 = |\bar{a}_k|$, $k = 1, \ldots, m$, and the scaled variation norm turns out to be $\sum_{k=1}^m |\bar{a}_k|\|\bar{w}_k\|_2 = \sum_{k=1}^m |\bar{a}_k|\|\bar{w}_k\|_2 = \sum_{k=1}^m \|w_k\|_2^2$. By the homogeneity of the ReLU, we also have $\sum_{k=1}^m a_k \sigma(\langle w_k, x \rangle) = \sum_{k=1}^m \bar{a}_k \sigma(\langle \bar{w}_k, x_i \rangle)$. The mapping $\mathcal{T}$ does not change the values of the networks and the scaled variation norm, but selects a parameter representation with the minimal summed $L_2$ norm. From this perspective, we have

$$J_n(\theta_m; \lambda_n) = \frac{1}{2\lambda_n} \sum_{i=1}^n \left( y_i - \sum_{k=1}^m \bar{a}_k \sigma(\langle \bar{w}_k, x_i \rangle) \right)^2 + \frac{\lambda_n}{2} \sum_{k=1}^m (|\bar{a}_k|^2 + \|\bar{w}_k\|_2^2).$$

Then the scaled variation regularization is reduced to the ridge. Let $J_n^{(2)}(\theta_m; \lambda_n)$ denotes the empirical risk penalized by the ridge without requiring $|a_k| = \|w_k\|_2, k = 1, \ldots, m$. If $\hat{\theta}_{m,n} = \arg\min J_n^{(2)}(\theta_m; \lambda_n)$, then $\hat{\theta}_m,n \in \arg\min J_n(\theta_m; \lambda_n)$ holds since

$$J_n(\hat{\theta}_m,n; \lambda_n) \leq J_n(\mathcal{T}(\hat{\theta}_m,n); \lambda_n) = J_n^{(2)}(\hat{\theta}_m,n; \lambda_n) \leq J_n(\mathcal{T}(\hat{\theta}_m,n); \lambda_n),$$

where $\hat{\theta}_m,n \in \arg\min_{\theta_m} J_n(\theta_m; \lambda_n)$. Since the ridge regularization would not shrink the parameters to zero, there exists at least one global minimum of $J_n(\theta_m; \lambda_n)$ is overparametrized. In the following proposition, we show that for gradient flow, the scaled variation regularization is equivalent to the ridge regularization if the initialization is reparametrized by $\mathcal{T}$.

**Proposition 4.1.** Consider the following gradient flow dynamics for optimization procedure (1.2)

$$\frac{d}{dt} a_j(t) = -\frac{\partial J_n(\theta_m; \lambda_n)}{\partial a_j}, \quad \frac{d}{dt} w_j(t) = -\frac{\partial J_n(\theta_m; \lambda_n)}{\partial w_j},$$

for $j = 1, \ldots, m$. For the initialization $\theta(0) = \mathcal{T}(\theta_0)$, where $\theta_0 \in \Theta_m$ can be arbitrary, we have

$$|a_j(t)|^2 = \|w_j(t)\|_2^2, \quad j = 1, 2, \ldots, m$$

hold for all $t$. Moreover, the trajectory of the above gradient flow dynamics (4.2) coincides with the trajectory of the gradient flow dynamics for optimization $\min J_n^{(2)}(\theta_m; \lambda_n)$ with the same initialization $\theta(0)$, i.e.,

$$\frac{d}{dt} a_j(t) = -\frac{\partial J_n^{(2)}(\theta_m; \lambda_n)}{\partial a_j}, \quad \frac{d}{dt} w_j(t) = -\frac{\partial J_n^{(2)}(\theta_m; \lambda_n)}{\partial w_j}.$$

The reparametrization of the initialization is referred as the balanced condition in Dou and Liang (2020); Maenlle, Bousquet and Gelly (2018). We show that, if the gradient flow with initialization reparametrized by $\mathcal{T}$ for minimizing $J_n(\theta_m; \lambda_n)$ can find the global minima, the global minima must be overparametrized. The extra assumption on the gradient flow can find the global minima is reasonable. Chizat and Bach (2018); Wei et al. (2019).
showed that for infinitely wide ReLU networks, Wasserstein gradient flow and its variants can reach the global minima of the empirical risk with weight decay regularization. For finitely wide neural networks, Pilanci and Ergen (2020) showed that squared $\ell_2$-regularized empirical risk minimization of two-layer ReLU networks is equivalent to block $\ell_1$-penalized convex optimization which can be solved with a total number of iterations polynomial in $n$ and $m$, but exponential in $d$ unless $P = NP$.

4.2. Equivalence to the group lasso. The results in this subsection are similar to Lacotte and Pilanci (2020) and Pilanci and Ergen (2020). However, we focus on the scaled variation penalty while they studied on the $\ell_2$-regularization. As we can see, owing to the homogeneity of the scaled variation norm, deriving our results are much more straightforward.

Given the predictors $\{X_k\}_{k=1}^n$, define the design matrix as $X = (X_1, \ldots, X_n)^T \in \mathbb{R}^{n \times d}$. To utilize the rectified linearity of ReLU, we follow the notations in Lacotte and Pilanci (2020), where structures of two-layer ReLU networks are investigated. Consider the partitions $\{I_+(u), I_-(u)\}$ of $\{1, 2, \ldots, n\}$ such that for the vector $u \in \mathbb{R}^d$, $I_+ = \{i: (X_i) > 0\}$ and $I_- = \{i: (X_i) \leq 0\}$. The cardinality of all possible $(I_+(u), I_-(u))$, $p = \text{Card}\{(I_+(u), I_-(u)): u \in \mathbb{R}^d\}$, is trivially bounded by $2^n$, and can be improved to $2d (\frac{n^p}{d})$ (Cover, 1965). For the $i$-th partition $(I_+(i), I_-(i))$, define the $n \times n$ diagonal matrix $D_i$ with $j$-th diagonal element $(D_i)_{jj} = 1$ if $j \in I_+(i)$ and $(D_i)_{jj} = 0$ if $j \in I_- (i)$ for $1 \leq i \leq p$ and $D_i = -D_{i-p}$ for $p + 1 \leq i \leq 2p$. We further define $P_i = \{u: I_+(u) = I_+(i), I_-(u) = I_- (i)\}$ for $1 \leq i \leq p$ and $P_j = P_{j-p}$ for $j = p + 1, \ldots, 2p$. Then we can define a partition $\{A_1, \cdots, A_{2p}\}$ of $\mathbb{R}^d \times \mathbb{R}$ as

$$A_i = P_i \times \mathbb{R}_+, \quad \text{for } i = 1, \ldots, p$$

$$A_i = P_i \times \mathbb{R}_-, \quad \text{for } i = p + 1, \ldots, 2p.$$  

The definition of $\{A_i\}_{i=1}^{2p}$ helps us deal with the nonlinearity of ReLU. In fact, the ReLU activation function behaves linearly when it is restricted to some cone $A_k, 1 \leq s \leq 2p$.

**Lemma 4.1.** For any $1 \leq k \leq 2p$ and $1 \leq i \leq n$, if $\{(w_j, a_j)\}_{j=1}^J \subset A_k$ for some $J = 1, 2, \ldots$, then

$$\sum_{j=1}^J a_j \sigma (w_j^T X_i) = \text{sign}(a_1) \sigma \left( \sum_{j=1}^J |a_j| w_j^T X_i \right).$$

Since every parameter at each neuron, say $(w_k, a_k)$, can find a unique cone $A_s$ for some $1 \leq s \leq 2p$ that it belongs to, we can define the mapping $\beta(\cdot): \Theta_m \rightarrow \mathbb{R}^{2pd}$ that $\beta(\theta_m) = (\beta_1^T, \ldots, \beta_{2p}^T)^T$ with

$$\beta_i = \sum_{j=1}^m |a_j| w_j \mathbb{I}\{(w_j, a_j) \in A_i\}.$$  

Owing to the mapping $\beta(\cdot)$ and Lemma 4.1, the neural network has another representation,

$$\sum_{k=1}^m a_k \sigma (X w_k) = \sum_{k=1}^m \sum_{i=1}^{2p} \mathbb{I}\{(w_j, a_j) \in A_i\} a_k \sigma (X w_k) = \sum_{i=1}^{2p} D_i X \beta_i.$$  

We can also define the inverse mapping $\theta(\cdot): \mathbb{R}^{2pd} \rightarrow \Theta_m$ as $\theta(\beta) = (w = (w_1, \ldots, w_m), a = (a_1, \ldots, a_m), m)$ with $m = \sum_{j=1}^{2p} \mathbb{I}\{\exists \beta_i, \text{ s.t. } \beta_i \in P_j\}$ and

$$w_j = \sum_{i=1}^{2p} \mathbb{I}\{\exists \beta_i \text{ s.t. } \beta_i \in P_j\}, \quad a_j = 2\mathbb{I}\{k_j \leq p\} - 1,$$
where \( k_1 < k_2 < \cdots < k_m \) denotes the indexes of cones that \( \beta_i \)'s belong to. By Lemma 4.1, we can prove that both \( \theta(\cdot) \) and \( \beta(\cdot) \) do not change the values of the neural network on each predictors, arriving at the following proposition.

**Proposition 4.2.** Define

\[
J_n^2(\beta; \lambda_n) = \frac{1}{2n} \left\| Y - \sum_{i=1}^{2p} D_i X \beta_i \right\|_2^2 + \lambda_n \sum_{i=1}^{2p} \| \beta_i \|_2.
\]

If \( m \geq \sum_{j=1}^{2p} \mathbb{I}\{\exists \beta_i, \text{s.t. } \beta_i \in P_j\} \), we have, for any \( \theta_m \in \Theta_m \) and \( \beta \in \mathbb{R}^{2dp} \),

\[
J_n(\theta(\beta); \lambda_n) \leq J_n^2(\beta; \lambda_n), \quad \text{and } J_n^2(\beta(\theta); \lambda_n) \leq J_n(\theta; \lambda_n).
\]

In particular, if \( \hat{\theta} \in \arg\min_{\theta_m} J_n(\theta_m; \lambda_n) \) and \( \hat{\beta} \in \arg\min_{\beta} J_n^2(\beta; \lambda_n) \), then for any \( m \geq n + 1 \), we have

\[
J_n(\theta(\hat{\beta}); \lambda_n) = J_n^2(\hat{\beta}; \lambda_n) = J_n^2(\beta(\hat{\theta}); \lambda_n) = J_n(\hat{\theta}; \lambda_n),
\]

and

\[
\nu(\hat{\theta}) = \| \hat{\beta} \|_{2,1}.
\]

It seems contradictory since the solution to \( J_n^2(\beta; \lambda_n) \) is sparse in view of the group lasso penalty; however, the solution to \( J_n(\theta_m; \lambda_n) \) is dense no matter how large \( m \) can be. This is, in fact, a direct consequence of the homogeneity of ReLU. Let \( \hat{\theta}_n = (\hat{w} = (\hat{w}_1, \ldots, \hat{w}_m), \hat{a} = (\hat{a}_1, \ldots, \hat{a}_m)) \) be any global minimum of \( J_n(\theta_m; \lambda_n) \). If \( (\tilde{w}_i, \tilde{a}_i) \in A_s \) and \( (\tilde{w}_j, \tilde{a}_j) \in A_s \) simultaneously, then \( \tilde{w}_i \) and \( \tilde{w}_j \) are positively colinear, i.e., \( \tilde{w}_i = c \tilde{w}_j \) for some \( c > 0 \). See Lemma B.1 for details. In this vein, although the total number of hidden neurons can be arbitrarily large, by merging the parameters in the same cone, the effective number of hidden nodes is still as small as \( m^* \), the minimum cardinality of the solutions to \( \min_{\alpha} J_n(\alpha; \lambda_n) \). The rigorous formulations and results are deferred to the Appendix.

**4.3. Norm control via the group lasso.** By Proposition 4.2, we have \( \nu(\hat{\theta}_n) = \| \hat{\beta} \|_{2,1} \). We can bound the \( R \)-norm of the global minima of \( J_n(\theta_m; \lambda_n) \) by bounding \( \| \hat{\beta} \|_{2,1} \).

**Proposition 4.3.** Let \( \hat{\theta}_n \in \arg\min_{\theta_m} J_n(\theta_m; \lambda_n) \) and denote by \( g(\hat{\theta}_n; \cdot) \) the corresponding ReLU network. Under the Assumption (A.1)-(A.3), if \( m \geq n + 1 \), \( d = o(n) \), and \( \lambda_n = 4\sigma \sqrt{\frac{\tau \log(n/d)}{n}} \) for some \( \tau \geq 1 \), then with probability greater than \( 1 - O\left(\frac{d}{n}\right)^\tau \),

\[
\| g(\hat{\theta}_n; \cdot) \|_R \leq 3\| f^* \|_R.
\]

Proposition 4.3 can be proven via applying the classical group lasso theory to bounding \( \| \hat{\beta} \|_{2,1} \) (Bühlmann and Van De Geer, 2011). It is noteworthy that the \( R \)-norm of \( g(\hat{\theta}_n; \cdot) \) saturates as the width \( m \) increases. In other words, large \( m \) may not introduce too much model complexity, which is crucial to the following generalization theory.

**5. Optimal generalization rates.** The out-of-sample prediction risk of an estimator \( g(\hat{\theta}; \cdot) \) can be defined as

\[
E \left[ \left( g(\hat{\theta}; X_0) - Y_0 \right)^2 \mid \mathcal{D}^n \right],
\]

(5.1)
where $\mathcal{D}^n = \{(X_i, Y_i)\}_{i=1}^n$ denotes the training samples, $(X_0, Y_0)$ is a new sample that comes from the same data-generating model (2.8) with $\mathcal{D}^n$. Since $Y_0 = f^*(X_0) + \varepsilon$, it suffices to consider the excess risk

$$
\mathbb{E}_{X_0} \left[ (g(\hat{\theta}; X_0) - f^*(X_0))^2 \right].
$$

We will bound the excess risk of the approximately global minima in expectation as follows.

**Theorem 5.1.** Under the model (2.8) and conditions imposed in Section 2.3. Let $\hat{\theta}_{m,n} \in \arg\min_{\theta_n} J_n(\theta_n; \lambda_n)$ and denote by $g(\hat{\theta}_{m,n}; \cdot)$ the corresponding ReLU network. Choose $\lambda_n = 4\sigma_n \sqrt{\frac{\tau \log(n/d)}{n}}$ for some $\tau \geq 1$. If $m \geq n + 1$, then $g(\hat{\theta}_{m,n}; \cdot)$ satisfies

$$
\mathbb{E} \left[ \left\| f^*(x) - g(\hat{\theta}_{m,n}; x) \right\|_{L^2(\mu)}^2 \right] \leq c \|f^*\|_\mathcal{R} \left( \|f^*\|_\mathcal{R} m^{-1-2/d} + \sqrt{\frac{\log n}{n}} \right),
$$

for some constant $c > 0$, where $\mathbb{E}[\cdot]$ denotes the expectation with respect to $n$ pairs of i.i.d. samples $\{(X_i, Y_i)\}_{i=1}^n$.

The condition $m \geq n + 1$ requires overparametrization. The leading term of our convergence rate for overparametrized networks is $\|f^*\|_\mathcal{R} \sqrt{(\log n)/n}$, which does not depend on the number of hidden nodes. This implies overparametrization is harmless to the generalization ability.

We can compare our results with Arora et al. (2019) and Ma et al. (2019b), in which non-asymptotic generalization properties of the neural tangent kernel method are considered. In the neural tangent kernel regime, two-layer neural networks travel around their initialization during the gradient descent dynamics, and can be efficiently approximated by functions in a reproducing kernel Hilbert space. When the initial values of the last layer are sufficiently small, the induced kernel of the first layer is dominated by that of the second layer (Ma et al., 2019b). It suffices to consider a reproducing kernel Hilbert space $\mathcal{H}$ which can be uniquely determined by its associated positive definite kernel induced by the inner product of the derivatives of the second layer

$$
H^\infty_{\mathcal{N}}(x, y) = \mathbb{E}_{w \sim \mathcal{N}(0, d^{-1}I_d)} \left[ \sigma(\langle w, x \rangle) \sigma(\langle w, y \rangle) \right].
$$

$f \in \mathcal{H}$ is equivalent to $f(x) = \int v(w) \sigma(\langle w, x \rangle) d\mu_{\mathcal{N}}(w)$ for some function $v(\cdot) \in L^2(\mu_{\mathcal{N}})$, where $\mu_{\mathcal{N}}(\cdot)$ denotes the multivariate normal distribution with zero mean and covariance matrix being $d^{-1}I_d$, which is also studied in Ji and Telgarsky (2019); Sun, Gilbert and Tewari (2018). Since $\mu_{\mathcal{N}}$ is an element of $\mathcal{M}_2(\mathbb{R}^d) = \{\alpha : \int \|w\|_2 d\alpha(w) < \infty\}$, we have $\mathcal{H} \subset \mathcal{G}_\sigma$. By the proofs in Arora et al. (2019); Ma et al. (2019b), the upper bound of the generalization error scales as $\sqrt{\mathbb{E}(H^\infty_n)^{-1} y)/n \sim \|f\|_\mathcal{H}/\sqrt{n}}$. Note that by Cauchy’s inequality, we have

$$
\|f\|^2_\mathcal{R} = \left( \int \|v(w)\|_2 d\mu_{\mathcal{N}} \right)^2 \leq \int \|v(w)\|^2_2 d\mu_{\mathcal{N}} = \|f\|^2_\mathcal{H}.
$$

Our result agrees with Arora et al. (2019) and Ma et al. (2019b) up to a logarithmic factor. By comparison, we consider the estimation problem from the whole signed measure space $\mathcal{M}_2(\mathbb{R}^d)$ under the condition $m \geq n + 1$ while they only considered the measure at the initialization with the requirement $m \geq O(n^4)$. Ma et al. (2019b) showed that neural tangent kernel methods are equivalent to the random feature model under mild conditions. In fact, random feature methods suffer from a curse of dimensionality for estimating functions in $\mathcal{G}_\sigma$. See Section 5.1 for details.
We can also compare our results with Barron (1993); Ma et al. (2019a), and Klusowski and Barron (2016), where either a sparsity-induced regularization or a direct complexity control of the number of hidden units is considered. Barron (1993) penalized the covering entropy of two-layer sigmoidal networks which is an increasing function of the width. By transforming the results of Barron (1993) into our settings\footnote{In the original paper, the convergence rate scales as $O\left(\sqrt{(d \log n)/n}\right)$ for $x \in [-1,1]^d$. Since in our work, we consider the case $x \in \mathbb{B}^d$, we can improve the rate to $O\left(\sqrt{(\log n)/n}\right)$.}, we can show that the convergence rate scales as $\|f^*\|_R \sqrt{(\log n)/n}$ with the optimal width being of the order $\sqrt{n}$. Our results agree with the optimal performance of the underparametrized estimators. In fact, we can show our overparametrized estimator is minimax optimal up to a logarithmic factor.

**Theorem 5.2.** Given $n$ pairs of i.i.d. samples $\{(X_i, Y_i)\}_{i=1}^n$ with $Y_i = f^*(X_i) + \epsilon_i$ for some $f^* \in \mathcal{G}_\sigma$ and $\epsilon_i \sim \mathcal{N}(0,1)$ independently. Assume $X_i \sim \text{Uniform}(\mathbb{B}^d)$ independently. Then there exists a constant $C'$, for any estimator $\hat{f}_n$, such that

$$\inf_{\hat{f}_n} \sup_{f^* \in \mathcal{G}_\sigma} \mathbb{E} \left[ |\hat{f}_n(X) - f^*(X)|^2 \right] \geq \frac{1}{8} C'n^{-\frac{1}{2}}(\log n)^{-\frac{1}{2}}. \tag{5.4}$$

Therefore, our two-layer-network estimators attain the minimax optimality and overparametrization is almost harmless except for a logarithmic factor.

We argue that this lower bound is effective. For a set of signed measures which are all absolute continuous with respect to a given probability distribution $\mu$, i.e., $a(v, b) = \frac{dv}{d\mu}$, consider the set $\mathcal{H}_\alpha$ consisting of all possible integral representations

$$f(x) = \int_{\mathcal{G}^{d-1} \times \mathbb{R}} a(v, b)\sigma(\langle v, x \rangle + b) d\mu(v, b) + c$$

with $\int_{\mathcal{G}^{d-1} \times \mathbb{R}} |a(v, b)|^2 d\mu(v, b) < \infty$. $\mathcal{H}_\mu$ is a reproducing kernel Hilbert space with the induced kernel being $H^\infty_\mu(x, y) = \mathbb{E}_{\mu \sim \mu}[\sigma(\langle v, x \rangle + b)\sigma(\langle v, y \rangle + b)]$. Note that $\cup_{\mu} \mathcal{H}_\mu \subseteq \mathcal{G}_\sigma$. If the target function $f^* \in \mathcal{H}_\mu$, for some known $\mu^*$, the problem of recovering $f^*$ is reduced to a kernel ridge regression. By the work of Capponnetto and De Vito (2007), both upper bound and lower bound of learning functions in a RKHS are dependent on the vanishing order $l^{-q}$ of $l$-th eigenvalues of $H^\infty_\mu$, and the upper bound scales as $(\log n/n)^{(q+1)/q}$ while the lower bound scales as $n^{-B/(B+1)}$ for any $B > q$. Note that $q > 1$ is needed to guarantee the kernel matrix is well-defined. In $\cup_{\mu} \mathcal{H}_\mu$, the underlying distribution $\mu^*$ associated with $f^*$ is unknown, then it is not a surprise that regularized two-layer neural networks yield a convergence upper bound which coincides with the worst case $q \to 1$ up to a logarithmic factor. Since kernel ridge regression can not identify the underlying distribution of $f^*$, it could be suboptimal for kernel estimators to learn $f^*$. A reproducing kernel can be reformulated as random features (Rahimi and Recht, 2008, 2009; Bach, 2017b). We can show the suboptimality via Proposition 5.1.

### 5.1. Suboptimality of random feature models

Consider the random feature model

$$g_{m, \pi_0}(x, a) = \frac{1}{\sqrt{m}} \sum_{k=1}^m a_k \sigma(\langle v_k, x \rangle + b_k) + c$$

with $\{(v_k^T, b_k)\}_{k=1}^m \sim \pi_0$ independently for some fixed $\alpha_0 \in \mathcal{M}_2(\mathbb{G}^{d-1} \times [-1,1])$ with $\pi_0(w) = |\alpha_0|(w)/(\int d|\alpha_0|(w))$ and $a = (a_1, \cdots, a_m)^T$ denotes the parameter that needs to
be learned. Given \( n \) pairs of samples \( \{(X_i, Y_i)\}_{i=1}^n \) from model (2.8), \( f^* \) is recovered from the following empirical minimization with weight decay regularization:

\[
\frac{1}{2n} \sum_{i=1}^n (Y_i - g_{m,\pi_0}(X_i, \mathbf{a}))^2 + \frac{\lambda_n}{2} \| \mathbf{a} \|^2.
\]

(5.5)

For simplicity, assume \( f^*(0) = c = 0 \). The minimization (5.5) is convex with respect to \( \mathbf{a} \), and the unique solution is given by

\[
\hat{\mathbf{a}} = (K_m^{\pi_0}(X, X) + n \lambda_n I_n)^{-1} Y,
\]

and the associated function is

\[
g_{m,\pi_0}(x; \hat{\mathbf{a}}) = K_m^{\pi_0}(x, X)(K_m^{\pi_0}(X, X) + n \lambda_n I_n)^{-1} Y,
\]

where \( K_m^{\pi_0}(x, y) = m^{-1} \sum_{k=1}^m \sigma((x, 1)^T w_k) \sigma((y, 1)^T w_k) \), \( K_m^{\pi_0}(x, X) = [K_m^{\pi_0}(x, X_i)]_i \), and \( K_m^{\pi_0}(X, X) = [K_m^{\pi_0}(X_i, X_j)]_{i,j} \in \mathbb{R}^{n \times n} \). Let \( m \to \infty \), \( K_m^{\pi_0}(x, y) \to H^{\infty}_{\pi_0} \) almost surely, which is exactly the kernel for infinitely wide networks. From this perspective, random feature estimators can be viewed as a kind of kernel estimators.

**Proposition 5.1.** Denote by \( \hat{\mathbf{a}} \) the unique solution to the minimization (5.5). Under the condition (C.1)–(C.3), there exists an absolute constant \( \kappa > 0 \), such that

\[
\sup_{f^* \in \mathcal{G}_x} \mathbb{E}_\mu \left[ \left| f^*(X) - g_{m,\pi_0}(X; \hat{\mathbf{a}}) \right|^2 \right] \geq \frac{\kappa}{d(\min(m, n) + 1)^{1/d}}.
\]

(6.6)

The proof mainly use the Lemma 6 in Barron (1993) where the approximation lower bound is considered. We focus on the estimation lower bound and can be viewed as a slightly extension. When \( d \geq 3 \) and \( m = O(n^\nu) \) for some \( \nu \geq 0 \), we can see the random feature method suffers from the curse of dimensionality, since it cannot identify the underlying signed measure of the parameters at the first layer.

6. **Discussion.** The Ridge-Lasso duality of the scaled variation regularization exhibits a clean and natural mechanism of how overparametrized two-layer ReLU networks avoid overfitting. The specification of the equivalence between the scaled variation penalty and the group-lasso penalty verifies a longstanding conjecture that overparametrized models indeed admit low-dimensional representations, which further formalizes the minimax optimality of two-layer ReLU networks developed in this article. The main message in this article that overparametrization can be harmless to pure prediction algorithms conforms to the deep learning practice. It opens up the possibilities of providing minimax optimal bounds for deep neural networks and convolutional neural networks, following the general idea of our work that the penalized nonconvex empirical risk minimization can be transformed to a convex optimization problem to analyze the intrinsic symmetry structures in overparametrization.

Overparametrized networks are easier to optimize (Arora, Cohen and Hazan, 2018; Safran, Yehudai and Shamir, 2020). Our work simplify the learning problem of regularized two-layer ReLU networks, by separating its optimization part and generalization part, since we show that all global minima enjoy the same rate owing to the explicit regularization. However, for unregularized empirical risk minimization, generalization rate is closely related to the underlying optimization algorithm. For example, small learning rate and large scaling constant can enforce the overparametrized networks converge to the global minimum around the initialization, yielding a model similar to the neural tangent kernel methods that is equivalent to learning with a specific positive definite kernel (Chizat, Oyallon and Bach, 2019). How can we design an algorithm that can efficiently escape the global minima of suboptimal testing performance? A regularization free analysis can be left for future work.
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APPENDIX A: APPROXIMATION RESULTS

This section consists of proofs of Section 3. Define the space formed by all ReLU networks with $m$ hidden units and finite $\mathcal{R}$-norm as

\[(A.1)\]

\[\mathcal{P}_\sigma = \left\{ f(x) = \sum_{i=1}^{m} a_i (\langle v_i, x \rangle + b_i) - \sigma(b_i) + c : \sum_{i=1}^{m} |a_i||v_i|_2 < \infty, m \in \mathbb{N} \cup \{\infty\} \right\}.\]

We will prove the approximation results for $x \in D \subset \mathbb{R}^d$, where $D$ is an arbitrary compact set, then treat $\mathbb{B}^d$ as a special case.

**Lemma A.1.** For any compact set $D \subset \mathbb{R}^d$, $\mathcal{G}_\sigma$ is the closure of $\mathcal{P}_\sigma$ under $L_2(\mu)$ norm with $\mu(D) = 1$.

**Proof.** By the definition of $\mathcal{G}_\sigma$, given $\theta_m = ([v_1, \ldots, v_m], b_1, \ldots, b_m)$, consider a discrete measure

\[\alpha_m = \sum_{k=1}^{m} a_i |v_i|_2 \delta(\overline{v}_i, \overline{b}_i) \mathbb{I}\{||v_i||_2 \neq 0\},\]

where $\overline{v}_i = v_i/||v_i||_2$ and $\overline{b}_i = b_i/||v_i||_2$. We have for any $g(\theta_m; x) \in \mathcal{P}_\sigma$,

\[g(\theta_m; x) = \sum_{k=1}^{m} a_i \sigma(\langle v_k, x \rangle + b_k) - \sigma(b_k) + g(\theta_m; 0)\]

\[= \sum_{k=1}^{m} a_i |v_i|_2 \left[ \sigma(\langle \overline{v}_k, x \rangle + \overline{b}_k) - \sigma(\overline{b}_k) \right] + g(\theta_m; 0)\]

\[= \int_{\mathbb{S}^{d-1} \times \mathbb{R}} (\sigma(\langle v, x \rangle + b) - \sigma(b)) d\alpha_m(v, b) + g(\theta_m; 0) \in \mathcal{G}_\sigma,\]

since the sum $\sum_{k=1}^{m} a_i |v_k||v_k||_2$ is finite. We conclude that $\mathcal{P}_\sigma \subset \mathcal{G}_\sigma$.

It suffices to show that $\mathcal{P}_\sigma$ is dense under $L_2(\mu)$ norm. Let $w = (v^T, b)^T$. Given any $\alpha_{a, c} \in \mathcal{G}_\sigma$, define $\overline{\alpha} = |\alpha|/\int_{\mathbb{S}^{d-1} \times \mathbb{R}} |\alpha|(w)$, a probability distribution on $\mathbb{S}^{d-1} \times \mathbb{R}$, and denote by $a(w) = \int_{\mathbb{S}^{d-1} \times \mathbb{R}} d\alpha(A)\overline{\alpha}(w)$ the Radon-Nikodym derivative for the signed measure such that $\alpha(A) = \int_A a(w) d\alpha(w)$ for any measurable set $A \subset \mathbb{S}^{d-1} \times \mathbb{R}$.

If we sample $w_{n,k} = (v_{n,k}^T, b_{n,k})^T \sim \overline{\alpha}$ independently for $n \geq 1, 1 \leq k \leq n$. Let

\[g_n(x) = \frac{1}{n} \sum_{k=1}^{n} a(v_{n,k}, b_{n,k}) (\sigma(\langle v_{n,k}, x \rangle + b_{n,k}) - \sigma(b_{n,k})) + c.\]

Note that $g_n \in \mathcal{P}_\sigma$, and

\[\mathbb{E}_{w_{n,k}}[a(v_{n,k}, b_{n,k}) (\sigma(\langle v_{n,k}, x \rangle + b_{n,k}) - \sigma(b_{n,k}))] \leq \sup_{x \in D} \|x\|_2 \int_{\mathbb{S}^{d-1} \times \mathbb{R}} d\alpha(A)(v, b) < \infty.\]

Then by strong law of large numbers

\[\lim_{n \to \infty} g_n(x) = \mathbb{E}_{\overline{\alpha}} \{a(v, b) |\sigma(\langle v, x \rangle + b) - \sigma(b)|\} + c = g_{a,c}(x),\]
almost surely for any fixed \( x \in D \). Then there exist at least one sequence of functions \( g_n(x), n = 1, 2, \ldots \) converging pointwise to \( g_{a,c} \). Since \( D \) is compact, and \( \{g_n(x)\}_{n=1}^{\infty} \) as well as \( g_{a,c} \) is continuous hence uniformly bounded, the convergence holds under \( L_2(\mu) \) norm.

Theorem 3.1 can be viewed as a corollary of Theorem 1 of Makovoz (1996). For completeness, we state and prove a stronger theorem following the arguments of Makovoz (1996).

**Theorem A.2.** Assume the distribution \( \mu \) of input \( X \) is supported on a closed and bounded set \( D \). Consider the set \( K_\sigma = \{\sigma_w(x): \sigma_w(x) = \sigma(v, x) - b) - \sigma(-b), \|v\|_2 = 1, b \in \mathbb{R}\} \). For any function \( f \) of the form

\[
f(x) = \sum_{l=1}^{\infty} c_l \phi_l(x), \phi_l \in K_\sigma, \sum_{l=1}^{\infty} |c_l| < \infty,
\]

and for every natural number \( m \), there is a \( g_m(x) = \sum_{k=1}^{m} a_k \varphi_k(x) \) with \( \varphi_k \in K_\sigma \) and \( \sum_k |a_k| \leq \sum_{l=\infty}^{\infty} |c_l| \), for which

\[
\|f - g_m\|_{L_2(\mu)} \leq c \sup_{x \in D} \|x\|_2 \max \left\{ \sum_{l=1}^{\infty} |c_l|, 1 \right\} m^{-1/2-1/d},
\]

for some universal constant \( c \) independent of \( m, d, \) and \( f \).

**Proof of Theorem (A.2).** Without loss of generality, assume \( c_l > 0 \) for all \( l \geq 1 \), since \( f \) either enjoys the property or is a difference of two functions that have it. For a given \( m \) and fixed \( \varepsilon > \mathcal{E}_m(K_\sigma) \), where

\[
\mathcal{E}_m(K_\sigma) = \inf \{\varepsilon > 0: K_\sigma \text{ can be covered by at most } m \text{ sets of diameter } \leq \varepsilon \text{ under } L_\infty \text{ norm}\},
\]

we can partition the set \( K_\sigma \) into \( m \) non-empty subsets \( \Phi_v, v = 1, 2, \ldots, m \) such that for any \( 1 \leq v \leq m \),

\[
\sup_{\varphi_1, \varphi_2 \in \Phi_v} \|\varphi_1 - \varphi_2\|_{L_\infty} \leq \varepsilon.
\]

We denote by \( i \in I_v \) the event \( \varphi_i \in \Phi_v \).

We first approximate each \( f_v = \sum_{i \in I_v} c_i \phi_i \) by a linear combination \( \sum_{i \in I_v} a_i \phi_i \) with a small number \( n_v \) of nonzero \( a_i \).

Set \( S_v = \sum_{i \in I_v} c_i, n_v = \lceil m S_v \rceil \) and define the random elements

\[
\hat{f}_v = \frac{S_v}{n_v} \left( \hat{\varphi}_1^{(v)} + \cdots + \hat{\varphi}_{n_v}^{(v)} \right),
\]

\[
\hat{f} = \hat{f}_1 + \cdots + \hat{f}_m,
\]

where \( \hat{\varphi}_k^{(v)}, k = 1, \ldots, n_v \) are i.i.d. from a discrete distribution over the set \( \Phi_v = \{\phi_i: \phi_i \in I_v\} \) such that \( \hat{\varphi}_k^{(v)} \) equals to one of \( \phi_i \in \Phi_v \) with the probability \( p_i^{(v)} = c_i / S_v \). Then

\[
\mathbb{E}[\hat{f}_v] = \frac{S_v}{n_v} n_v \sum_{i: \phi_i \in I_v} \frac{c_i}{S_v} \phi_i = f_v,
\]

and \( \mathbb{E}(f - \hat{f}) = 0 \). Note that

\[
\text{Var}(f - \hat{f}) = \sum_{v=1}^{m} \text{Var}(\hat{f}_v) \leq \sum_{v=1}^{m} \frac{\varepsilon^2 S_v^2}{n_v} \leq \frac{\varepsilon^2}{m} \sum_{i=1}^{\infty} c_i.
\]
The first inequality comes from the fact that the diameter of $I_v$ is less than $\epsilon$ and the first equality comes from the independence of $\hat{\varphi}_k^{(v)}$’s. We conclude that there exists a realization of $(\hat{\varphi}_k^{(v)})_{v,k} \subset K_\sigma$ with at most $k(m) = (\sum_{i=1}^{\infty} |c_i| + 1) m$ many nonzero $a_i$’s such that $f^* = \sum_{i=1}^{k(m)} a_i \hat{\varphi}_k^{(v)}$ and

$$|f^*(x) - f(x)|^2 \leq \frac{2}{m} \sum_{i=1}^{\infty} |c_i|, \forall x \in D.$$  

Then for any probability measure $\mu$ defined on $x \in D$, we have (after proper scaling of $m$)

$$\|f^* - f\|_{L_2(\mu)} \leq 2\epsilon \frac{\max\{\sum_{k=1}^{\infty} |c_k|, 1\}}{\sqrt{m}}.$$  

To finish the proof, we need to estimate the value of $\mathcal{E}_m(K_\sigma)$. Note that if $\|v_1 - v_2\| \leq \epsilon/2$, $|b_1 - b_2| < \epsilon/2$ for $|b_1| \leq 1 + \epsilon/2$ or $|b_2| \leq 1 + \epsilon/2$, by the Lipschitz continuity of ReLU, we have

$$\sup_{x \in D} |\sigma(\langle v_1, x \rangle - b_1) - \sigma(-b_1) - (\sigma(\langle v_2, x \rangle - b_2) - \sigma(-b_2))| \leq \frac{3\sup_{x \in D} \|x\|_2 \epsilon}{2}.$$  

For $|b_1| > 1$ and $|b_2| > 1$, we have

$$\sup_{x \in D} |\sigma(\langle v_1, x \rangle - b_1) - \sigma(-b_1) - (\sigma(\langle v_2, x \rangle - b_2) - \sigma(-b_2))| \leq \sup_{x \in D} \|x\|_2 \frac{\epsilon}{2},$$  

if we let $\|v_1 - v_2\|_2 \leq \epsilon/2$ as well.

From the above observation, we conclude that in order to construct an $\epsilon$-net for $K_\sigma$, it suffices to construct an $\epsilon/\sup_{x \in D} \|x\|_2$-net for $\mathbb{S}^{d-1} \times [-1 - \epsilon, 1 + \epsilon]$, which requires $O((\sup_{x \in D} \|x\|_2/\epsilon)^d)$ sets to cover. Then $\mathcal{E}_m(K_\sigma) \asymp \sup_{x \in D} \|x\|_2 m^{-\frac{d}{2}}$ and since $\epsilon$ can be arbitrarily close to $\mathcal{E}_m(K_\sigma)$, we have

$$\|f^* - f\|_{L_2(\mu)} \leq c \sup_{x \in D} \|x\|_2 \max\left\{\sum_{i=1}^{\infty} |c_i|, 1\right\} m^{-1/2-1/d}.$$  

\[ \square \]

**Proof of Theorem 3.1.** In this proof, we only prove the upper bound, and the proof of the lower bound is deferred to Section G of the Supplementary. For any function $f \in P_\sigma^R = \{f \in P_\sigma : \|f\|_R \leq R\}$, there exists a sequence of elements $\{\phi_i\}_{i=1}^{\infty}$ in $K_\sigma$ such that

$$f = \sum_{k=1}^{\infty} a_k \phi_k, \text{ with } \sum_{k=1}^{\infty} |a_k| \leq R,$$

where $a_k$ can be zero for some $k$. To see this, note that any function that can be formulated as a linear combination of functions in $K_\sigma$ is in $P_\sigma$. Conversely, if $f(x) = \sum_{i=1}^{m} a_i \sigma(\langle v_i, x \rangle - b_i) + c_i \sum_{i=1}^{m} |a_i| \|v_i\|_2 < \infty$, we can rewrite it as $g_m(x) = \sum_{i=1}^{m} a_i (\sigma(\langle v_i, x \rangle + b_i) - \sigma(b_i)) + f(0) = f(x)$ and for $\|v_i\|_2 \geq 1$,

$$g_m(x) = \sum_{i=1}^{m} a_i (\sigma(\langle v_i, x \rangle + b_i) - \sigma(b_i)) + f(0) = \sum_{i : \|v_i\| \neq 0} a_i (\sigma(x, v_i + b_i) - \sigma(b_i)) + f(0),$$
where \( a'_i = a_i \| v_i \|_2, \) \( \overline{b}_i = b_i / \| v_i \|_2, \) and \( \overline{\pi}_i = v_i / \| v_i \|_2. \) Note that
\[
\sum_{i=1}^{m} |a_i| \| v_i \|_2 = \sum_{i=1}^{m} |a'| \| \overline{\pi}_i \|_2 = \sum_{i=1}^{m} |a'|.
\]

Then \( g_m \in \text{Span}(K_\sigma). \)
By the application of Theorem (A.2) and denseness of \( \mathcal{P}_\sigma \) for \( \mathcal{G}_\sigma, \) we accomplish the proof. \( \square \)

**APPENDIX B: PROPERTIES OF GLOBAL MINIMA**

In this section, we give results on the properties of global minima, including bounds of their scaled variation norm and nonsparsity. The bounds of their scaled variation norm relies on the group lasso property of the scaled variation regularization, while the nonsparsity relies on the ridge property.

We first state and prove the positive colinearity of parameters in the same cone. Let \( B(w, a) \) denotes the cone which \((w, a)\) belongs to, i.e., if \((w, a) \in A_s\) for some \(1 \leq s \leq 2p,\) then \( B(w, a) = A_s.\)

**LEMMA B.1.** Let \( \hat{\theta} \in \argmin_{\theta} J_n(\theta; \lambda_n), \) and \( \hat{\theta} = (\hat{w} = (\hat{w}_1, \ldots, \hat{w}_m), \hat{a} = (\hat{a}_1, \ldots, \hat{a}_m)). \) If \( B(\hat{w}_j, \hat{a}_j) = B(\hat{w}_k, \hat{a}_k), \) we have
\[
\hat{w}_j = c_0 \hat{w}_k
\]
for some positive constant \( c_0 > 0.\)

**PROOF OF LEMMA B.1.** Without loss of generality, assume \( B(w_i, a_i) = B(w_j, a_j) = A_s \) for some \(1 \leq s \leq 2p.\) For any \( X_k, 1 \leq k \leq n,\)
\[
a_i \sigma(w_i^T X_k) + a_j \sigma(w_j^T X_k) = a_i w_i^T X_k \{w_i^T X_k > 0\} + a_j w_j^T X_k \{w_j^T X_k > 0\}
\]
\[
= |a_i| w_i^T X_k (D_s)_{kk} + |a_j| w_j^T X_k (D_s)_{kk}
\]
\[
= |a_i| w_i + |a_j| w_j^T X_k (D_s)_{kk}
\]
\[
= \text{sign}(a_i) \sigma(|a_i| w_i + |a_j| w_j^T X_k)
\]
\[
= c \text{sign}(a_i) \sigma(|a_i| w_i + |a_j| w_j^T X_k)
\]
\[
+ (1 - c) \text{sign}(a_i) \sigma(|a_i| w_i + |a_j| w_j^T X_k)
\]
for any \( c \in (0, 1), \) since \( \text{sign}(a_i) = \text{sign}(a_j) = (D_s)_{kk} \) by assumption and \( \{w_i^T X_k > 0\} = \{a_i |w_i|^T X_k > 0\}. \) However,
\[
|c \text{sign}(a_i)| \|a_i w_i + a_j w_j\|_2 + |(1 - c) \text{sign}(a_i)| \|a_i w_i + a_j w_j\|_2 = \|a_i w_i + a_j w_j\|_2
\]
\[
\leq |a_i| \|w_i\|_2 + |a_j| \|w_j\|_2,
\]
with the equality holds if and only if \( |a_i| w_i = c_0 |a_j| w_j \) for some \( c_0 \geq 0.\) In our case, if \( (\hat{w}_j, \hat{a}_j) \) and \( (\hat{w}_k, \hat{a}_k) \) are nonzero parameters of some global minimum sharing the same cone \( A_s, \) then \( \hat{w}_j = c_0 \hat{w}_k \) for some \( c_0 > 0 \) must hold. Otherwise, holding the rest of parameters intact, let \( \hat{w}_j' = |\hat{a}_j| \hat{w}_j + |\hat{a}_k| \hat{w}_k, \hat{a}_j' = c \text{sign}(a_i), \hat{w}_j' = |\hat{a}_j| \hat{w}_j + |\hat{a}_k| \hat{w}_k, \) and \( \hat{a}_k' = (1 - c) \text{sign}(a_i), \) for any \( c \in (0, 1), \) and denote by \( \hat{\theta}' \) the new parameter. Then we have \( J_n(\hat{\theta}'; \lambda_n) < J_n(\hat{\theta}; \lambda_n) \) since
\[
\hat{a}_j \sigma(\hat{w}_j^T x_k) + \hat{a}_k \sigma(\hat{w}_k^T x_k) = \hat{a}_j' \sigma(X_k^T \hat{w}_j') + \hat{a}_k' \sigma(X_k^T \hat{w}_k'),
\]
and
but
\[ |\hat{a}_j| \|\hat{w}_j\|_2 + |\hat{a}_k| \|\hat{w}_k\|_2 = \|\hat{a}_j|\hat{w}_j + |\hat{a}_k|\hat{w}_k\|_2 \]
\[ < |\hat{a}_j| \|\hat{w}_j\|_2 + |\hat{a}_k| \|\hat{w}_k\|_2, \]
if \( \hat{w}_j \neq c_0\hat{w}_k \) for any \( c_0 > 0 \).

**Proof of Lemma 4.1.**

\[
\sum_{j=1}^J a_j \sigma(w_j^T X_i) = \sum_{j=1}^J a_j w_j^T X_i \mathbb{1}\{w_j^T X_i > 0\} \\
= \sum_{j=1}^J |a_j| w_j^T X_i (D_k)_{ii} \\
= \text{sign}(a_1) \left( \sum_{j=1}^J |a_j| w_j^T X_i \right) \mathbb{1}\{w_j^T X_i > 0\} \\
= \text{sign}(a_1) \sigma \left( \sum_{j=1}^J |a_j| w_j^T X_i \right),
\]
where the second line comes from the definition of \((D_k)_{ii}\) and note that \( \mathbb{1}\{w_j^T X_i > 0\} = (D_k)_{ii} \) which does not depend on \( j \).

**Proof of Proposition 4.1.** Consider the gradient flow (B.1) of the empirical risk penalized by the scaled variation norm.

(B.1)
\[
\frac{d}{dt} a_j(t) = \hat{E}_n \left[ \left( \sum_{k=1}^m a_k(t) \sigma(\langle w_k(t), x \rangle) - y \right) \sigma(\langle w_j(t), x \rangle) \right] - \lambda_n \partial |a_j(t)| \|w_j(t)\|_2, \\
\frac{d}{dt} w_j(t) = \hat{E}_n \left[ \left( \sum_{k=1}^m a_k(t) \sigma(\langle w_k(t), x \rangle) - y \right) a_j(t) \mathbb{1}\{\langle w_j(t), x \rangle \geq 0\} \right] - \lambda_n |a_j(t)| \partial \|w_j(t)\|_2,
\]
where \( \hat{E}_n \) denotes the empirical expectation and \( \partial \) denotes the sub-gradient operator.

Left multiplying \( a_j(t) \) and \( w_j(t) \) respectively gives
\[
\frac{1}{2} \frac{d |a_j(t)|^2}{dt} = \frac{1}{2} \frac{d \|w_j(t)\|_2^2}{dt},
\]
since \( a \partial |a| = |a| \) and \( \langle w, \partial \|w\|_2 \rangle = \|w\|_2^2 \). If the initialization is reparametrized, then \( |a_j(0)|^2 = \|w_j(0)\|_2^2, j = 1, \ldots, m \). We then have
\[
|a_j(t)|^2 - \|w_j(t)\|_2^2 = 0, \forall j, t.
\]
Now consider the gradient flow (B.2) of the empirical risk penalized by the weight decay.

(B.2)
\[
\frac{d}{dt} a_j^{(2)}(t) = \hat{E}_n \left[ \left( \sum_{k=1}^m a_k^{(2)}(t) \sigma(\langle w_k^{(2)}(t), x \rangle) - y \right) \sigma(\langle w_j^{(2)}(t), x \rangle) \right] - \lambda_n a_j^{(2)}(t), \\
\frac{d}{dt} w_j^{(2)}(t) = \hat{E}_n \left[ \left( \sum_{k=1}^m a_k^{(2)}(t) \sigma(\langle w_k^{(2)}(t), x \rangle) - y \right) a_j^{(2)}(t) \mathbb{1}\{\langle w_j^{(2)}(t), x \rangle \geq 0\} \right] - \lambda_n w_j^{(2)}(t),
\]
Note that if \(|a_j(t)|^2 - \|w_j(t)\|^2 = 0, \forall j, t\), then \(\partial|a_j(t)| \|w_j(t)\|_2 = \partial|a_j(t)| |a_j(t)| = a_j(t)\), and \(a_j(t) \partial\|w_j(t)\|_2 = \|w_j(t)\|_2 \partial\|w_j(t)\|_2 = w_j(t)\). If the reparametrized initialization of both gradient flow coincide, so do the whole trajectories. Then the conclusion follows. \(\square\)

**Proof of Proposition 4.2.** By Lemma 4.1 and by the definition of \(\beta(\cdot)\), we have

\[
\sum_{k=1}^m a_k \sigma(Xw_k) = \sum_{k=1}^{2p} \sum_{i=1}^2 \mathbb{I}\{w_k, a_k\} \in A_i\} a_k \sigma(Xw_k) = \sum_{i=1}^{2p} D_i X \beta_i.
\]

Note that in this case,

\[
\sum_{i=1}^{2p} \|\beta_i\|_2 = \sum_{i=1}^{2p} \sum_{j=1}^m a_j |w_j| \mathbb{I}\{w_j, a_j\} \in A_i\} \|w_k\|_2 \leq \sum_{k=1}^m |a_k| \|w_k\|_2.
\]

Then

\[
J_n(\theta_m; \lambda_n) = \frac{1}{2n} \left\| Y - \sum_{k=1}^m a_k \sigma(Xw_k) \right\|_2^2 + \lambda_n \sum_{k=1}^m |a_k| \|w_k\|_2 \geq \frac{1}{2n} \left\| Y - \sum_{i=1}^{2p} D_i X \beta_i \right\|_2^2 + \lambda_n \sum_{i=1}^{2p} \|\beta_i\|_2 = J_n^\theta(\beta; \lambda_n).
\]

Similarly, by Lemma 4.1 and the definition of \(\theta(\cdot)\), we have

\[
\sum_{i=1}^{2p} D_i X \beta_i = \sum_{j=1}^m \sum_{i: \beta_i \in P_{kj}} D_k X \beta_i = \sum_{j=1}^m a_j \sigma(Xw_j),
\]

and

\[
\sum_{j=1}^m |a_j| \|w_j\|_2 = \sum_{i=1}^{2p} \left\| \sum_{j=1}^m a_j \mathbb{I}\{\beta_i \in P_{kj}\} \right\|_2 \leq \sum_{i=1}^{2p} \|\beta_i\|_2,
\]

where \(m \geq \sum_{j=1}^{2p} \mathbb{I}\{\exists \beta_i, \text{ s.t. } \beta_i \in P_j\}\).

We have that

\[
J_n(\theta(\beta); \lambda_n) \leq J_n^\theta(\beta; \lambda_n).
\]

Let \(\hat{\theta} \in \text{argmin}_{\beta_n} J_n(\theta; \lambda_n)\), \(\hat{\beta} \in \text{argmin}_{\beta_n} J_n^\theta(\beta; \lambda_n)\). In view of Eq. B.4 and Eq. B.3, we immediately have

\[
J_n(\theta(\hat{\beta}); \lambda_n) \leq J_n^\theta(\hat{\beta}; \lambda) \leq J_n^\theta(\beta(\hat{\theta}); \lambda_n) \leq J_n(\hat{\theta}; \lambda_n),
\]

and

\[
\nu(\hat{\theta}) = \|\hat{\beta}\|_{2,1} = \sum_{i=1}^{2p} \|\hat{\beta}_i\|_2.
\]

Note that in this case, we require that \(m \geq \inf_{\beta \in \text{argmin} J_n(\beta; \lambda_n)} \sum_{j=1}^{2p} \mathbb{I}\{\exists \hat{\beta}_i, \text{ s.t. } \hat{\beta}_i \in P_j\}\). Lemma 1 of Liu and Zhang (2009) showed that the number of nonzero groups of the solutions to \(\min J_n(\beta; \lambda_n)\) is upper bounded by \(n\) for any \(\lambda_n > 0\). Then \(m \geq n + 1\) suffices to guarantee the above statement to hold. \(\square\)
Proof of Proposition 4.3. Denote by $g_{\tilde{\theta}_m}$ the two-layer ReLU network with $m$ hidden units that best approximates $f^*$ with respect to the empirical measure $\mu_n = n^{-1} \sum_{k=1}^{n} \delta_{X_k}$ in Corollary 3.1 and by $\tilde{\theta}_m = (m, \tilde{a}^* = [\tilde{a}^*_1, \ldots, \tilde{a}^*_m], \tilde{w}^* = (\tilde{w}_1^*, \ldots, \tilde{w}_m^*), f^*(0))$ its parameters. Let $\tilde{\beta} = \tilde{\beta}_m$. Notice that $\tilde{\beta}_i \in A_i$. For simplicity, denote $X = (D_1X_1, \ldots, D_{2p}X)$ \in $\mathbb{R}^{n \times 2dp}$, and $\|\beta\|_{2,1} = \sum_{i=1}^{2p} \|\beta_i\|_2$. By the basic inequality, we have

$$\frac{1}{2n} \|X(\tilde{\beta} - \bar{\beta})\|_2^2 \leq \lambda_n \left( \|\beta\|_{2,1} - \|\bar{\beta}\|_{2,1} \right) + \frac{1}{n} (Y^* - X\tilde{\beta})^T X(\tilde{\beta} - \bar{\beta}) + \frac{1}{n} \varepsilon^T X(\tilde{\beta} - \bar{\beta}),$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^T$, and $Y^* = (f^*(X_1), \ldots, f^*(X_n))^T$.

We will first bound the term $I_1$. Note that by Cauchy’s inequality,

$$|I_1| = \left| \frac{1}{2n} (Y^* - X\tilde{\beta})^T X(\tilde{\beta} - \bar{\beta}) \right| \leq \frac{1}{2n} \|Y^* - X\tilde{\beta}\|_2 \|X(\tilde{\beta} - \bar{\beta})\|_2 \leq C \|f^*\|_R \|m^{-1/2-1/d}.

for some universal positive constant $C$. Note that the function $f(x) = x^2/2 - ax$ for some $a > 0$ is minimized at $x = a$ with the minimal value being $f(a) = -a^2/2$. We conclude that

$$\frac{1}{2n} \|X(\tilde{\beta} - \bar{\beta})\|_2^2 - |I_1| \geq - \frac{C^2}{2} \|f^*\|_R^2 m^{-1-2/d}.$$

We then bound the term $I_2$. Let $V_j^T = \varepsilon^T D_j X/\sqrt{n}$, and $A_j = D_j X^T D_j/n$. We have

$$|I_2| = \frac{1}{n} \left| \varepsilon^T \sum_{i=1}^{2p} D_i X(\tilde{\beta} - \bar{\beta}) \right| \leq \max_{1 \leq j \leq 2p} \|V_j\|_2 \sqrt{n^{-1/2}} \sum_{i=1}^{2p} \|\beta_j - \bar{\beta}_j\|_2.$$

By choosing $\lambda_n \geq 2 \max_{1 \leq j \leq 2p} \|V_j\|_2 / \sqrt{n}$, we have

$$|I_2| \leq \frac{\lambda_n}{2} \|\tilde{\beta}\|_{2,1} + \frac{\lambda_n}{2} \|\bar{\beta}\|_{2,1}.$$

Combining the two inequalities, we have

$$\frac{\lambda_n}{2} \|\bar{\beta}\|_{2,1} \leq \frac{3\lambda_n}{2} \|\beta\|_{2,1} + \frac{C^2}{2} \|f^*\|_R^2 m^{-1-2/d}.$$

Note that $V_j^T V_j = \varepsilon^T A_j \varepsilon$ is of a quadratic form. Then by the tail bound of quadratic forms of subgaussian vectors (Hsu et al., 2012), we have

$$\mathbb{P} \left( \|V_j\|_2^2 \geq \sigma_j^2 + 2\sigma_j^2 \sqrt{\frac{t}{d} + 2\sigma_j t / d} \right) \leq \exp \left( - \frac{t}{d} \right).$$

By a union bound, we have

$$\mathbb{P} \left( \max_{1 \leq j \leq 2p} \|V_j\|_2^2 \geq \sigma_j^2 + 2\sigma_j^2 \sqrt{\frac{t + \log(2p)}{d} + 2\sigma_j^2 \frac{t + \log(2p)}{d}} \right) \leq \exp \left( - \frac{t}{d} \right).$$

Recall that $p \leq d \left( \frac{n}{d} \right)^d$ and choose $t = \tau \log(2d) + \tau d \log(n/d) \geq \tau \log(2p)$. With a probability greater than $1 - O \left( \frac{d}{n} \right)^T$, we have

$$4\sigma_j^2 \log(n/d) \geq \max_{1 \leq j \leq 2p} \|V_j\|_2^2.$$
Choose \( \lambda_n = 4\sigma \sqrt{\frac{r \log(n/d)}{n}} \). Then we conclude that with probability greater than \( 1 - O\left(\frac{1}{n}\right) \),

\[
\nu(\hat{\theta}) = \|\hat{\beta}\|_{2,1} \leq 3\|\hat{\beta}\|_{2,1} \leq 3\nu(\hat{\theta}^*) \leq 3\|f^*\|_\mathcal{R}.
\]

\( \square \)

**APPENDIX C: GENERALIZATION RESULTS**

**PROOF OF THEOREM 5.1.** Let \( f_n(x) = g(\hat{\theta}_n; x) \) for simplicity, where \( \hat{\theta}_n \) denote some global minimum. Define \( T_{\beta_n}(x) = \max\{\min\{x, \beta_n\}, -\beta_n\} \) and choose \( \beta_n = c_\beta \sqrt{\log n} \) for some sufficiently large constant \( c_\beta \). Denote by \( \mathbb{D}^n = \{(X_k, Y_k)\}_{k=1}^n \) our collected training samples. Define \( f_{\beta_n}(x) = \mathbb{E}[T_{\beta_n}(Y)|X=x]\).

Note that \( \mathbb{E}_{X \sim \mu} \left( |f^*(X) - f_n(X)|^2 \right) \mathbb{D}^n \) have the following decomposition.

\[
\mathbb{E}_{X \sim \mu} \left( |f^*(X) - f_n(X)|^2 \right) \mathbb{D}^n = \left\{ \mathbb{E} \left[ |f_n(X) - Y|^2 \right] \mathbb{D}^n \right\} - \mathbb{E} \left[ |f_n(X) - T_{\beta_n}(Y)|^2 \right] \mathbb{D}^n
\]

Since \( \mathbb{E} \left[ |f^*(X) - Y|^2 \right] \mathbb{D}^n \) hold for some \( c_1 \) and \( c_2 \).

Bound \( T_{1,n} \): Note that

\[
T_{1,n} = \mathbb{E} \left[ |f_n(X) - Y|^2 - |f_n(X) - T_{\beta_n}(Y)|^2 \right] \mathbb{D}^n
\]

Since

\[
\mathbb{I}_{\{Y > \beta_n\}} \leq e^{-\left(c_e / 2\right) \beta_n^2 e^{(c_e / 2)|Y|^2}},
\]
We have that there exists a constant $c$ for some constant $C$.

Combining both of them, we get

$$E \leq \sqrt{E[|Y|^2 e^{(c_1/2)|Y|^2}]} \cdot e^{-(c_2^2/4) c_3 b_n} \quad \text{(Markov inequality).}$$

Note that

$$E \left[ |Y|^2 e^{(c_1/2)|Y|^2} \right] \leq E \left[ \frac{2}{c_1} e^{c_1 Y^2} \right] \leq c_4.$$ 

We have that there exists a constant $c_5$ such that

$$|T_{4,n}| \leq \sqrt{c_4 c_3 b_n e^{-c_5 b_n^2}} \leq c_6 \frac{\log n}{n}.$$

Similarly, by substituting $E[\cdot]$ to its empirical counterpart $E_n[\cdot]$, we have

$$E[|T_{5,n}|] \leq \sqrt{E_n[|Y|^2 \mathbb{I}_{|Y| \geq \beta_n}]} \sqrt{E_n[18 \beta_n^2 + 2|Y|^2 \mathbb{I}_{|Y| \geq \beta_n}]} \leq E \left[ \sqrt{E_n[|Y|^2 \mathbb{I}_{|Y| \geq \beta_n}]} \right] 6 \beta_n + 2 E_n \left[ |Y|^2 \mathbb{I}_{|Y| \geq \beta_n} \right]$$

by $\sqrt{a^2 + b^2} \leq \sqrt{2(a + b)}$. Note that by Jensen’s inequality, we have

$$E \left[ \left( \frac{1}{n} \sum_{k=1}^{n} |Y_k|^2 \mathbb{I}_{|Y_k| > \beta_n} \right)^{1/2} \right] \leq \left( E \left[ \left( \frac{1}{n} \sum_{k=1}^{n} |Y_k|^2 \mathbb{I}_{|Y_k| > \beta_n} \right) \right]^{1/2} \right) \leq \sqrt{E \left[ |Y|^2 \mathbb{I}_{|Y| > \beta_n} \right]} \leq c_7 \frac{1}{n},$$

for some constant $c_7$. Then by a similar argument with that of bounding $T_{4,n}$, we have

$$E[|T_{5,n}|] \leq c_8 \frac{\log n}{n}.$$

Combining both of them, we get

$$T_{1,n} \leq c_9 \frac{\log n}{n}.$$ 

Bound $T_{2,n}$: Let

$$T_{6,n} = \frac{1}{n} \sum_{i=1}^{n} \left( E \left[ |f_n(X) - T_{\beta_n}(Y)|^2 \mathbb{I} \right] - |f_n(X_i) - T_{\beta_n}(Y)|^2 \right)$$

$$T_{7,n} = \frac{1}{n} \sum_{i=1}^{n} \left( E \left[ |f^*(X) - Y|^2 \right] - |f^*(X_i) - Y_i|^2 \right).$$

Note that by Proposition 4.3, the global minima satisfy

$$\|g_{\hat{m},n}\|_R \leq C \|f^*\|_R$$

for some $C$ independent of $m$ and $n$. If we define

$$\mathcal{F}^M = \left\{ f_m = \frac{m}{k=1} \sum_{k=1}^{m} a_k \sigma\left( \langle v_k, x \rangle + b_k \right) : \sum_{k=1}^{m} \|a_k\|_2 \leq M, m \in \mathbb{N} \cup \{ \infty \} \right\},$$
for some \( M \), we have \( g_{\theta_{m,n}} \in \mathcal{F}_d \). Using the Rademacher complexity, by Lemma E.1 and Lemma E.2, we have

\[
\mathbb{E}[|T_{0,n}|] \leq 2C||f^*||_\mathcal{R} \sqrt{\log \frac{n}{n}},
\]

since \( l(x,y) = \frac{1}{2}(x-y)^2 \) is \( \beta_n \)-Lipschitz continuous on \( \{(x,y) : |x| \leq \beta_n, |y| \leq \beta_n\} \).

As for \( T_{7,n} \), note that \( Y - f^*(X) \) is sub-gaussian. Then \( (Y - f^*(X))^2 \) is sub-exponential and by Bernstein inequality, we have for some constant \( c_{10} \),

\[
\mathbb{P} \left( \left| \frac{1}{n} \sum_{i=1}^{n} (\mathbb{E}[f^*(X) - Y_i^2] - |f^*(X_i) - Y_i|^2) \right| \geq t \right) \leq 2e^{-\frac{nt^2}{c_{10}}},
\]

Then by the formula \( \mathbb{E}[T] = \int_{0}^{\infty} \mathbb{P}(T > t) dt \), we have

\[
\mathbb{E}[T_{7,n}] \leq \frac{c_{11}}{\sqrt{n}}.
\]

Combining them together, we have

\[
\mathbb{E}[T_{2,n}] \leq c_{12}||f^*||_\mathcal{R} \sqrt{\log \frac{n}{n}}.
\]

Bound \( T_{3,n} \): Let \( \tilde{f}_{m,n}^*(x) \) be the best approximator with respect to \( \mu \).

\[
\mathbb{E}_{D_n}[T_{3,n}] = \mathbb{E} \left[ \frac{1}{n} \sum_{k=1}^{n} \left| f_{n}(X_i) - Y_i \right|^2 - \frac{1}{n} \sum_{k=1}^{n} \left| \tilde{f}_{m,n}^*(X_i) - Y_i \right|^2 - \lambda_n ||f^*||_\mathcal{R} \right]
+ \frac{1}{n} \sum_{k=1}^{n} \left| \tilde{f}_{m,n}^*(X_i) - Y_i \right|^2 - \frac{1}{n} \sum_{k=1}^{n} \left| f^*(X_i) - Y_i \right|^2 + \lambda_n ||f^*||_\mathcal{R} \right]
\leq \mathbb{E} \left[ \frac{1}{n} \sum_{k=1}^{n} \left| \tilde{f}_{m,n}^*(X_i) - Y_i \right|^2 - \frac{1}{n} \sum_{k=1}^{n} \left| f^*(X_i) - Y_i \right|^2 + \lambda_n ||f^*||_\mathcal{R} \right]
\leq \mathbb{E} \left\{ \mathbb{E} \left[ \frac{1}{n} \sum_{k=1}^{n} \left| \tilde{f}_{m,n}^*(X_i) - Y_i \right|^2 - \frac{1}{n} \sum_{k=1}^{n} \left| f^*(X_i) - Y_i \right|^2 \bigg| X^n \right] \right\}
+ \lambda_n ||f^*||_\mathcal{R}
\leq c_{17} ||f^*||^2_\mathcal{R} m^{-1 - 2/d} + \lambda_n ||f^*||_\mathcal{R},
\]

We then conclude that, choosing \( \lambda_n = c_{13} \sqrt{\frac{\log(n/d)}{n}} \) with \( c_{13} \) sufficiently large,

\[
\mathbb{E}_{D_n} \left[ \int_{\mathbb{R}^d(1)} |g(\hat{\theta}_n; x) - f^*(x)|^2 d\mu(x) \right]
= \mathbb{E} \left\{ \mathbb{E}_{\mu} \left[ |f^*(X) - g(\hat{\theta}_n; X)|^2 \bigg| X^n \right] \right\}
\leq \mathbb{E} \left( T_{1,n} + T_{2,n} + T_{3,n} \right)
\leq c_{19} ||f^*||_\mathcal{R} \left( ||f^*||_\mathcal{R} m^{-1 - \frac{2}{d}} + \sqrt{\frac{\log(n)}{n}} \right).
\]

\( \square \)
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SUPPLEMENTARY MATERIAL

Supplementary material for “Harmless Overparametrization in Two-layer Networks” This supplementary material contains detailed proofs and mathematical background materials.

D. Detailed Backgrounds and Related Proofs.

D.1. Signed measures. In the following, the term measure refers to a finite signed measure on $\mathbb{R}^d$ for any $d \geq 1$, endowed with its Borel $\sigma$-algebra. We denote by $\mathcal{M}(D)$ the set of such measures concentrated on a measurable set $D \subset \mathbb{R}^d$. Any finite signed measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ has a decomposition
\[
\mu = \mu_+ - \mu_-,
\]
where $\mu_+, \mu_- \in \mathcal{M}_+(\mathbb{R}^d)$ owing to the Jordan decomposition theorem. If $\mu_+$ and $\mu_-$ are chosen with minimal total mass, then variation of $\mu$ is the nonnegative measure $|\mu| = \mu_+ + \mu_-$, and the total variation norm can be defined as $|\mu|(\mathbb{R}^d)$.

D.2. Radon transformation. The Radon transform $R$ represents a function $f$ in terms of its integrals over all possible hyperplanes in $\mathbb{R}^d$:
\[
R\{f\}(v,b) = \int_{\langle v, x \rangle = b} f(x) ds(x) \text{ for all } (v,b) \in S^{d-1} \times \mathbb{R},
\]
where $ds(x)$ stands for the $(d-1)$-dimensional surface measure on the hyperplane $\langle v, x \rangle = b$. Consider the dual Radon transform $R^*$ mapping a function $\alpha : S^{d-1} \times \mathbb{R} \to \mathbb{R}$ over $x \in \mathbb{R}^d$ by integrating over the subset of coordinates $(v, b)$ corresponding to all hyperplanes passing through $x$:
\[
R^*\{\alpha\}(x) = \int_{S^{d-1}} \alpha(v, \langle v, x \rangle) dv \text{ for all } x \in \mathbb{R}^d.
\]
See more details in Helgason and Helgason (1980).

D.3. Motivations of the target function space. In the following, we introduce the motivation of choosing $\mathcal{G}_\sigma$ as the target function, which mainly follows Ongie et al. (2019).

We first consider a two-layer neural network activated by ReLU that is defined over the whole Euclidean space, $g(\theta; \cdot) : \mathbb{R}^d \to \mathbb{R}$ defined by
\[
g(\theta; x) = \sum_{k=1}^{m} a_k [\sigma(\langle v_k, x \rangle + b_k) - \sigma(b_k)] + c, \text{ for all } x \in \mathbb{R}^d
\]
with parameters $\theta = (m, v_1, \ldots, v_m, b_1, \ldots, b_m, a_1, \ldots, a_m, c)$, where the width $m \in \mathbb{N}$ is unbounded. Let $\Theta$ be the collection of all such parameters vectors $\theta$ and denote by $C(\theta)$ be the sum of the squared Euclidean norm of the weights excluding the bias term defined by
\[
C(\theta) = \frac{1}{2} \sum_{k=1}^{m} (\|v_k\|^2_2 + |a_k|^2).
\]
Owing to the universal approximation ability of neural networks, given any accuracy $\varepsilon$ in advance, any continuous function $f : \mathbb{R}^d \to \mathbb{R}$ can be approximated uniformly by a two-layer neural network $g(\theta; \cdot)$ on any compact subset $K$ of $\mathbb{R}^d$ such that
\[
\sup_{x \in K} |f(x) - g(\theta; x)| \leq \varepsilon.
\]
We say a two-layer neural network $g(\theta; \cdot)$ (possibly with infinite width) is regular if $C(\theta) < \infty$. Define the representational cost of any continuous function $f$ as

$$\mathcal{R}(f) = \lim_{\varepsilon \to 0} \left( \inf_{\theta \in \Theta} C(\theta) \text{ s.t. } |g(\theta; x) - f(x)| \leq \varepsilon, \forall ||x|| \leq \frac{1}{\varepsilon} \text{ and } g_\theta(0) = f(0) \right).$$

From the definition we can see that $\mathcal{R}(f)$ is the minimal limiting $L_2$ norm of weights among all sequences of networks converging to $f$ uniformly.

For the infinitely wide two-layer ReLU network $f(x)$ defined by

$$f(x) = \int_{\mathbb{S}^{d-1} \times \mathbb{R}} (\sigma(v, x) - b) d\alpha_f(v, b) + f(0),$$

Ongie et al. (2019) proved that $\mathcal{R}(f) = \|\alpha_f\|_1$ if $\alpha_f$ is even in a sense $\alpha_f(b, v) = \alpha_f(-v, -b)$ and $\|\nabla f(\infty)\| = \lim_{r \to \infty} \frac{1}{\|v\|_2 = r} \int_{\|x\|_2 = r} \|\nabla f(x)\| ds(x) = 0$. The relation between two-layer ReLU networks and the Radon transformation comes from the following observations.

Denote by $\mathcal{M}(\mathbb{S}^{d-1} \times \mathbb{R})$ the space of signed measures $\alpha$ defined on $(v, b) \in \mathbb{S}^{d-1} \times \mathbb{R}$ with finite total variation measure $\int_{\mathbb{S}^{d-1} \times \mathbb{R}} |\alpha| < \infty$. Define the infinite width two-layer ReLU network $h_{\alpha, c}$ by

$$h_{\alpha, c}(x) = \int_{\mathbb{S}^{d-1} \times \mathbb{R}} (\sigma(v, x) - b) d\alpha(v, b) + c,$$

where coefficients of $x$ at the first layer are normalized without loss of generality and $\alpha \in \mathcal{M}(\mathbb{S}^{d-1} \times \mathbb{R})$. If the signed measure $\alpha$ has a smooth density $\varphi(v, b)$ over $\mathbb{S}^{d-1} \times \mathbb{R}$, $d\alpha = \varphi(v, b)dvdb$.

Then taking Laplacian with respect to $x$ gives

$$\Delta h_{\alpha, c}(x) = \int_{\mathbb{S}^{d-1}} \varphi(v, \langle v, x \rangle) dv = \mathcal{R}^*\{\varphi\}. $$

By the inverse formula of Radon transformation (Solmon, 1987), for any function $\phi \in \mathcal{S}(\mathbb{S}^{d-1} \times \mathbb{R})$, the space of Schwartz functions on $\mathbb{S}^{d-1} \times \mathbb{R}$, with $\phi(v, b) = \phi(-v, -b)$,

$$\gamma_d \mathcal{R}\{(-\Delta)^{(d-1)/2}\mathcal{R}^*\{\phi\}\} = \phi$$

holds. Then

$$\varphi = -\gamma_d \mathcal{R}\{(-\Delta)^{(d+1)/2}\mathcal{R}^*\{h_{\alpha, c}\}\},$$

where $\gamma_d = \frac{1}{2(2\pi)^{d-1}}$. The representational cost $\mathcal{R}(f)$ of infinitely wide neural networks is

$$\mathcal{R}(f) = \|\alpha_f\|_1 = \| -\gamma_d \mathcal{R}\{(-\Delta)^{(d+1)/2}\mathcal{R}^*\{h_{\alpha, c}\}\} \|_1.$$ 

For functions that are not smooth enough, Ongie et al. (2019) define the $\mathcal{R}_\alpha$-norm as

$$\|f\|_{\mathcal{R}_\alpha} = \sup \left\{-\gamma_d \int_{\mathbb{R}^d} f(x)(-\Delta)^{(d+1)/2}\mathcal{R}^*\{\psi\}(x)dx : \psi \in \mathcal{S}(\mathbb{S}^{d-1} \times \mathbb{R}), \psi \text{ even}, \|\psi\|_{\infty} \leq 1 \right\},$$

which can be viewed as a generalized $L_1$ norm of $\gamma_d \mathcal{R}\{(-\Delta)^{(d+1)/2}\mathcal{R}^*\{\psi\}\}$. The notation $\mathcal{R}_\alpha$ is used to distinguish from our definition. Motivated by this observation, the following lemma can be derived.
**Lemma D.1** (Lemma 10 of Ongie et al. (2019)). Let \( f \in \text{Lip}(\mathbb{R}^d) \). Then \( \|f\|_{\mathcal{R}_o} < \infty \) if and only if there exists a unique even signed measure \( \alpha_f \in M(\mathbb{S}^{d-1} \times \mathbb{R}) \), a unique \( z \in \mathbb{R}^d \), and a unique constant \( c \) such that

\[
f(x) = \int_{\mathbb{S}^{d-1} \times \mathbb{R}} (\sigma(\langle v, x \rangle - b) - \sigma(-b)) \, d\alpha_f(v, b) + \langle z, x \rangle + c,
\]

in which case \( \|f\|_{\mathcal{R}_o} = \int_{\mathbb{S}^{d-1} \times \mathbb{R}} d|\alpha_f|(v, b) \) and \( \|f\|_{\mathcal{R}_o} \leq \overline{\mathcal{R}}(f) \leq \|f\|_{\mathcal{R}_o} + 2\|\nabla f(\infty)\| \).

The definition of \( \mathcal{R}_o \)-norm in Ongie et al. (2019) annihilates affine functions \( f_z(x) = \langle z, x \rangle + c \). But \( f_z \) belongs to \( \mathcal{G}_o \) since

\[
\langle z, x \rangle = \sigma(\langle z, x \rangle) - \sigma(-\langle z, x \rangle).
\]

For affine functions \( f_z \), the representation cost \( \overline{\mathcal{R}}(f_z) = 2\|z\|_2 \). Since we focus on the learning problem of two-layer neural networks, we shall show in the next subsection that our definition of \( \mathcal{R} \)-norm accommodates both \( \mathcal{R}_o \)-norm and affine functions.

**D.4. Omitted proofs of Section 2.3.** In practice, we cannot expect the parameters at the first layer to have a unit \( L_2 \) norm, and we cannot exclude the linear functions as well. It is necessary to extend results in Lemma D.1 to more general cases. We consider the space of signed measures as

\[
\mathcal{M}(\mathbb{R}^{d+1}) = \left\{ \alpha(v, b) : \int_{\mathbb{R}^{d+1}} d|\alpha|(v, b) < \infty \right\},
\]

and show the intrinsic connection to \( \mathcal{M}(\mathbb{S}^{d-1} \times \mathbb{R}) \) that is considered in Lemma D.1. In fact, we have the following corollary which can be viewed as a generalization of Lemma D.1.

**Corollary D.1.** Let \( f \in \text{Lip}(\mathbb{R}^d) \). Then \( \|f\|_{\mathcal{R}_o} < \infty \) if and only if there exists a unique signed measure \( \alpha_f \in \mathcal{M}_2(\mathbb{R}^{d+1}) \), where

\[
\mathcal{M}_2(\mathbb{R}^{d+1}) = \left\{ \alpha \in \mathcal{M}(\mathbb{R}^{d+1}) : \int_{\mathbb{R}^{d+1}} \|v\|_2 d|\alpha|(v, b) < \infty \right\}
\]

and a unique constant \( c \) such that

\[
f(x) = \int_{\mathbb{R}^{d+1}} (\sigma(\langle v, x \rangle - b) - \sigma(-b)) \, d\alpha_f(v, b) + c,
\]

in which case \( \|f\|_{\mathcal{R}_o} \leq \overline{\mathcal{R}}(f) \leq \int_{\mathbb{R}^{d+1}} \|v\|_2 d|\alpha_f|(v, b) \).

**Proof.** In light of Lemma D.1, for the necessity part, it suffices to show that \( \mathcal{M}(\mathbb{S}^{d-1} \times \mathbb{R}) \) is a subspace of \( \mathcal{M}_2(\mathbb{R}^{d+1}) \) under some appropriate norm. Consider the norm \( \int_{\mathbb{R}^{d+1}} \|v\|_2 d|\alpha|(v, b) \) of \( \alpha \). Then

\[
\mathcal{M}(\mathbb{S}^{d-1} \times \mathbb{R}) \subset \mathcal{M}_2(\mathbb{R}^{d+1}).
\]

Then there exists a unique even signed measure \( \alpha_f \in M(\mathbb{S}^{d-1} \times \mathbb{R}) \), a unique \( z \in \mathbb{R}^d \), and a unique constant \( c \) such that

\[
f(x) = \int_{\mathbb{S}^{d-1} \times \mathbb{R}} (\sigma(\langle v, x \rangle - b) - \sigma(-b)) \, d\alpha_f(v, b) + \langle z, x \rangle + c.
\]

Recall that in D.9, there exists a unique signed measure

\[
\alpha_z(v, b) = I\{z \neq 0\} \left( \|z\|_2 \delta(z/\|z\|_2, 0) - \|z\|_2 \delta(-z/\|z\|_2, 0) \right) \in \mathcal{M}(\mathbb{S}^{d-1} \times \mathbb{R})
\]
with $\int_{S^{d-1} \times R} d|z| = 2\|z\|_2$ such that $\langle z, x \rangle = \int_{S^{d-1} \times R} (\sigma(\langle v, x \rangle - b) - \sigma(-b)) d\alpha_z(v, b)$. Then

$$f(x) = \int_{S^{d-1} \times R} (\sigma(\langle v, x \rangle - b) - \sigma(-b)) d(\alpha_f + \alpha_z)(v, b) + c \in M_2(\mathbb{R}^{d+1}).$$

which completes the necessity part.

As for the sufficient part, consider the lifting operator $L$ to map signed measure $\alpha \in M_2(\mathbb{R}^{d+1})$ to $M(S^{d-1} \times \mathbb{R})$. For any signed measure $\alpha \in M(\mathbb{R}^{d+1})$, by the positive homogeneity of ReLU activation function $\sigma(\cdot)$,

$$(D.11) \quad \int_{\mathbb{R}^{d+1}} (\sigma(\langle v, x \rangle + b) - \sigma(b)) d\alpha(v, b)$$

$$= \int_{\mathbb{R}^{d+1} \setminus \{0\}} (\sigma(\langle v/\|v\|_2, x \rangle + b/\|v\|_2) - \sigma(b/\|v\|_2)) \|v\|_2 d\alpha(v, b)$$

$$= \int_{S^{d-1} \times \mathbb{R}} (\sigma(\langle v', x \rangle + b') - \sigma(b')) dL\{\alpha\}(v', b'),$$

provided $\int_{\mathbb{R}^{d+1}} \|v\|_2 d|\alpha|(v, b) < \infty$, where $L\{\cdot\} : M_2(\mathbb{R}^{d+1}) \rightarrow M(S^{d-1} \times \mathbb{R})$ is defined in a distributional version, i.e., for any set $A \subset S^{d-1} \times \mathbb{R}$,

$$L\{\alpha\}(A) = \int_{\|v\|_2^2(v, b) \in A} \|v\|_2 d\alpha(v, b).$$

Consider the decomposition of $L\{\alpha\} = L^+\{\alpha\} + L^-\{\alpha\}$, where $L^+\{\alpha\}$ is even and $L^-\{\alpha\}$ is odd. Due to $\sigma(x) - \sigma(-x) = x$, we have

$$\frac{1}{2} \int_{S^{d-1} \times \mathbb{R}} (\sigma(\langle v', x \rangle + b') - \sigma(b')) dL^-\{\alpha\}(v', b')$$

$$= \frac{1}{2} \int_{S^{d-1} \times \mathbb{R}} (\sigma(\langle v', x \rangle + b') - \sigma(b')) dL^-\{\alpha\}(v', b')$$

$$- \frac{1}{2} \int_{S^{d-1} \times \mathbb{R}} (\sigma(\langle -v', x \rangle - b') - \sigma(-b')) dL^-\{\alpha\}(v', b')$$

$$= \frac{1}{2} \int_{S^{d-1} \times \mathbb{R}} v' dL^-\{\alpha\}(v', b').$$

Define $z = \frac{1}{2} \int_{S^{d-1} \times \mathbb{R}} v' dL^-\{\alpha\}(v', b')$, which is well-defined since $\int_{\mathbb{R}^{d+1}} \|v\|_2 d|\alpha|(v, b) < \infty$. Then if there exists a $\alpha_f \in M_2(\mathbb{R}^{d+1})$ such that $f = h_{\alpha_f, c}$, then $f(x) = h_{L^+\{\alpha_f\}, c}(x) + \langle z, x \rangle$ with

$$\int_{S^{d-1} \times \mathbb{R}} d|L^+\{\alpha_f\}| \leq \int_{\mathbb{R}^{d+1}} \|v\|_2 d|\alpha_f|(v, b) < \infty.$$

Then $\|f\|_{\mathcal{R}_s} \leq \int_{\mathbb{R}^{d+1}} \|v\|_2 d|\alpha_f|(v, b) < \infty.$

To prove $\overline{\mathcal{R}}(f) \leq \int_{\mathbb{R}^{d+1}} \|v\|_2 d|\alpha_f|(v, b)$, note that

$$\nabla f(\infty) = \frac{1}{2} \int_{S^{d-1} \times \mathbb{R}} v' dL^-\{\alpha_f\}(v', b').$$

See Lemma 11 of Ongie et al. (2019) for details. Then by the definition of $\overline{\mathcal{R}}(f)$, following the proofs of Theorem 2 of Ongie et al. (2019), we have

$$\overline{\mathcal{R}}(f) = \min_{\tilde{\alpha}^{\text{odd}}} \|L^+\{\alpha_f\} + \tilde{\alpha}^-\|_1$$

s.t. $\int_{S^{d-1} \times \mathbb{R}} v' d\tilde{\alpha}^-(v', b') = \int_{S^{d-1} \times \mathbb{R}} v' dL^-\{\alpha_f\}(v', b').$
Note that \( \mathcal{L}^- \{ \alpha_f \} \) is a feasible solution, then we have
\[
\mathcal{R}(f) \leq \int_{\mathbb{R}^{d+1}} \|v\|_2 d|\alpha_f|(v, b).
\]
\( \Box \)

Corollary D.1 motivates us to define \( \int_{\mathbb{R}^{d+1}} \|v\|_2 d|\alpha|(v, b) \) as a new norm, as in Section 2.2, is denoted by \( \mathcal{R} \)-norm following Ongie et al. (2019). It enjoys several advantages. Firstly, for affine functions \( f_z(x) = \langle z, x \rangle + c \), \( \|f_z\|_\mathcal{R} = 2\|z\|_2 \) instead of being zero in Ongie et al. (2019). Secondly, \( \mathcal{R} \)-norm defined on \( \mathcal{M}_2(\mathbb{R}^{d+1}) \) keeps the essential property that for any Lipschitz function \( f \) with \( \mathcal{R}(f) < \infty \), there exists a unique signed measure \( \alpha_f \in \mathcal{M}_2(\mathbb{R}^{d+1}) \) with
\[
f(x) = \int_{\mathbb{R}^{d+1}} (\sigma(\langle v, x \rangle - b) - \sigma(-b)) d\alpha_f(v, b) + c
\]
such that \( \mathcal{R}(f) \leq \|f\|_\mathcal{R} < \infty \). Finally, \( \mathcal{R} \)-norm is a norm defined on the space \( \mathcal{G}_\sigma \), where
\[
\mathcal{G}_\sigma = \left\{ f(x) = \int_{\mathbb{R}^{d+1}} (\sigma(\langle v, x \rangle + b) - \sigma(b)) d\alpha(v, b) + f(0) : \int_{\mathbb{R}^{d+1}} \|v\|_2 d|\alpha|(v, b) < \infty, x \in \mathbb{R}^d \right\}.
\]
(D.12)

In contrast, \( \mathcal{R}_\sigma \)-norm in Ongie et al. (2019) is a semi-norm since it annihilates affine functions.

Note that this definition holds for any \( x \in \mathbb{R}^d \). In practice, collected samples \( \{X_i\}_{i=1}^n \) are normalized with a uniformly bounded norm. Define a functional \( \mathcal{T}_D : \mathcal{G}_\sigma \rightarrow \mathcal{F}_\sigma(D) \) as
\[
\mathcal{T}_D[f](x) = f(x), \quad x \in D,
\]
where \( D \) can be any compact set in \( \mathbb{R}^d \). \( \mathcal{F}_\sigma(D) \) is formed by functions that can be viewed as a restriction of \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) with finite \( \|f\|_\mathcal{R} \) from \( \mathbb{R}^d \) to \( D \). In the following proposition, we show that \( \mathcal{R} \)-norm is consistent in format with its finite counterpart, the scaled variation norm, for \( D = \mathbb{B}^d \).

**PROPOSITION D.1.** Consider \( \mathcal{F}_\sigma(\mathbb{B}^d) \). Functions in \( \mathcal{F}_\sigma(\mathbb{B}^d) \) admit the following integral representation
\[
f(x) = \int_{\mathbb{B}^{d-1} \times [-1, 1]} \sigma(\langle v, x \rangle + b) d\alpha_f(w) + c
\]
with \( \int_{\mathbb{B}^{d-1} \times [-1, 1]} \|w\|_2 d|\alpha_f|(w) < \infty \), where \( w = (v^T, b)^T \). In particular,
\[
\int_{\mathbb{R}^{d+1}} \|w\|_2 d|\alpha_f|(w) \leq 4\|f\|_\mathcal{R}.
\]

**PROOF.** If we confine \( x \) to the unit ball \( \mathbb{B}^d \), note that
\[
\sigma(\langle v, x \rangle + b) - \sigma(b) = [\sigma(\langle v, x \rangle + b) - \sigma(b)] \mathbb{I}\{b \leq \|v\|_2\} + \langle v, x \rangle \mathbb{I}\{b > \|v\|_2\},
\]
since \( \sigma(\langle v, x \rangle + b) = 0 \) if \( b < -\|v\|_2 \) and \( \langle v, x \rangle \leq \|v\|_2 \). Easy to see that
\[
\int \langle v, x \rangle \mathbb{I}\{b > \|v\|_2\} d|\alpha|(v, b) \leq \int \|v\|_2 d|\alpha|(v, b) < \infty.
\]
We can rewrite the representation \( D.10 \) as

\[
    f(x) = \int_{\mathbb{R}^{d+1}} (\sigma(v, x) + b) \, d\alpha_1(v, b) + c,
\]

\[
    = \int_{\mathbb{S}^{d-1} \times \mathbb{R}} (\sigma(v, x) + b) \, d\alpha(v, b) + c
\]

\[
    = \int_{\mathbb{S}^{d-1} \times \mathbb{R}} (\sigma(v, x) + b) \, d\mathcal{L}(\alpha)(v, b)
\]

\[
    - x^T \int_{\mathbb{S}^{d-1} \times \mathbb{R}} v \, d\mathcal{L}(\alpha)(v, b) + c
\]

\[
    = \int_{\mathbb{S}^{d-1} \times [-1, 1]} \sigma(v, x) + b \, d\alpha(v, b) + x^T z_f + c_f
\]

where \( d\alpha_1(v, b) = \mathbb{E}\{\|b\| \leq \|v\|_2\} \, d\mathcal{L}(\alpha)(v, b) \), \( z_f = \int_{\mathbb{S}^{d-1} \times \mathbb{R}} v \, d\mathcal{L}(\alpha)(v, b) \) and \( c_f = c - \int_{\mathbb{S}^{d-1} \times \mathbb{R}} \sigma(b) \, d\mathcal{L}(\alpha)(v, b) \). Note that \( \int_{\mathbb{S}^{d-1} \times [-1, 1]} \|w\|_2 d\alpha_1(v, b) \leq \sqrt{2} \int_{\mathbb{S}^{d-1} \times [-1, 1]} d\alpha_1(v, b) \leq \sqrt{2}\|f\|_\mathcal{R} \).

Recall that there exists a unique signed measure

\[
    \alpha_{z_f}(v, b) = \mathbb{E}\{z_f \neq 0\} \left(\|z_f\|_2^2 \delta_{z_f/\|z_f\|_2, 0} - \|z_f\|_2^2 \delta_{-z_f/\|z_f\|_2, 0}\right) \in \mathcal{M}(\mathbb{S}^{d-1} \times \mathbb{R})
\]

with \( \int_{\mathbb{S}^{d-1} \times \mathbb{R}} d\alpha_{z_f} = 2\|z_f\|_2 \leq 2\|f\|_\mathcal{R} \) such that \( \langle z_f, x \rangle = \int_{\mathbb{S}^{d-1} \times \mathbb{R}} (\sigma(v, x) + b) \, d\alpha_{z_f}(v, b) \).

Then

\[
    f(x) = \int_{\mathbb{S}^{d-1} \times [-1, 1]} (\sigma(v, x) + b) \, d\alpha_1(v, b) + x^T z_f + c_f
\]

\[
    \int_{\mathbb{S}^{d-1} \times [-1, 1]} \|w\|_2 d\alpha_1 + \alpha_{z_f} \leq (\sqrt{2} \cdot \sqrt{2} + 2) \int_{\mathbb{R}^{d+1}} \|v\|_2 d\alpha(v, b) = 4\|f\|_\mathcal{R}.
\]

Then the proof is complete. \( \square \)

In this vein, the additional term \( -\sigma(b) \) in the integrand \( D.10 \) can be absorbed into the bias term when \( x \in \mathbb{B}^d \). This arises naturally, since the hyperplane \( \langle v, x \rangle = b \) induced by Radon transform indicates that \( |b| = |\langle v, x \rangle| \leq \|v\|_2 \). We can view the condition \( |b| \leq \|v\|_2 \) as a restriction from \( b \in \mathbb{R} \), which is parallel to the restriction of \( x \).

\section*{E. Rademacher complexity.}

We first introduce some basic results with respect to Rademacher complexity. See more details in \textcite{Shalev-Shwartz and Ben-David 2014}.

Given \( n \) pairs of samples \( S = \{(x_i, y_i)\}_{i=1}^n \) from distribution \( \mathcal{D} \), and a set of functions \( \mathcal{F} \), let \( \mathcal{F} \circ S \) be the set of all possible evaluations a function \( f \in \mathcal{F} \) can achieve on a sample \( S \),

\[
    \mathcal{F} \circ S = \{(f(x_1), \cdots, f(x_n))\}.
\]

Then, the Rademacher complexity of \( \mathcal{F} \) with respect to \( S \) is defined as

\[
    R(\mathcal{F} \circ S) = \frac{1}{n} \mathbb{E}_{\xi_i \sim \{+1, -1\}} \left[ \sup_{f \in \mathcal{F}} \sum_{k=1}^n \xi_k f(x_k) \right],
\]

with \( \xi_i \sim \text{Uniform}\{+1, -1\} \) independently for all \( 1 \leq i \leq n \).
LEMMA E.1 (Lemma (26.2) of Shalev-Shwartz and Ben-David (2014)). Let $\mu$ be the marginal distribution of $X$.

(E.2) $$\mathbb{E}_{S \in \mu^n} \left[ \sup_{f \in \mathcal{F}} \mathbb{E}_{X \sim \mu} [f(X)] - \frac{1}{m} \sum_{k=1}^{m} f(X_k) \right] \leq 2 \sup_{S \in \mu^n} [R(\mathcal{F} \circ S)].$$

Note that in our optimization program 1.2, our candidate functions form a function space

$$\mathcal{F}_\sigma^M = \left\{ f_m = \sum_{k=1}^{m} a_k \sigma(\langle v_k, x \rangle + b_k) : \sum_{k=1}^{m} \| a_k \|_2 \leq M, m \in \mathbb{N} \cup \{\infty\} \right\},$$

for some $M$. In the next, we estimate the Rademacher complexity of $\mathcal{F}_\sigma^M$.

LEMMA E.2.

$$R(\mathcal{F}_\sigma^M \circ S) \leq \frac{2M}{\sqrt{n}}.$$

PROOF.

$$n R(\mathcal{F}_\sigma^M \circ S) = \mathbb{E}_\xi \left[ \sup_{\sum_{k=1}^{m} \| a_k \|_2 \leq M} \sum_{i=1}^{n} \xi_i f_m(x_i) \right]$$

$$= \mathbb{E}_\xi \left[ \sup_{\sum_{k=1}^{m} \| a_k \|_2 \leq M} \sum_{i=1}^{n} \xi_i \sum_{k=1}^{m} a_k \sigma(\langle w_k, i \rangle) \right]$$

$$\leq \mathbb{E}_\xi \left[ \sup_{\sum_{k=1}^{m} \| a_k \|_2 \leq M : \| u \|_2 = 1} \sum_{i=1}^{n} \xi_i \sum_{k=1}^{m} a_k \| w_k \|_2 \sigma(\langle u, i \rangle) \right]$$

$$\leq \mathbb{E}_\xi \left[ \sum_{k=1}^{m} \| a_k \|_2 \sup_{\| u \|_2 = 1} \sum_{i=1}^{n} \xi_i \sigma(\langle u, i \rangle) \right]$$

$$\leq 2M \mathbb{E}_\xi \left[ \sup_{\| u \|_2 = 1} \sum_{i=1}^{n} \xi_i \sigma(\langle u, i \rangle) \right]$$

$$\leq 2M \mathbb{E}_\xi \left[ \sup_{\| u \|_2 = 1} \sum_{i=1}^{n} \xi_i \langle u, i \rangle \right] \leq 2M \sqrt{n},$$

where the fifth line comes form symmetry of $\xi$ and the last line results form the contraction lemma and Lemma (26.10) of Shalev-Shwartz and Ben-David (2014). \hfill \Box

F. Orthogonal polynomials on d-dimensional balls. Let $\mathbb{B}^d$ be the unit ball of $\mathbb{R}^d$ and $\mathbb{S}^d$ be the unit sphere on $\mathbb{R}^{d+1}$, that is,

$$\mathbb{B}^d = \{ x \in \mathbb{R}^d : \| x \|_2 \leq 1 \} \quad \text{and} \quad \mathbb{S}^d = \{ y \in \mathbb{R}^{d+1} : \| y \|_2 = 1 \}.$$

Let $\mathbb{N}_0$ be the set of nonnegative integers. For $p = (p_1, \ldots, p_d) \in \mathbb{N}_0^d$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we write $x^p = x_1^{p_1} \cdots x_d^{p_d}$. The number $|p|_1 = p_1 + \cdots + p_d$ is called the total degree of $x^p$. Denote by $P_n^d$ the space of homogeneous polynomials of total degree at most $n$ and let

$$r_n^d = \dim P_n^d,$$

then $r_n^d = \binom{n+d-1}{n}$. 

Let \( W \) be a non-negative weight function on \( \mathbb{B}^d \) and assume \( \int_{\mathbb{B}^d} W(x) dx < \infty \). It is known that for each \( n \in \mathbb{N}_0 \), the set of polynomials of degree \( n \) that are orthogonal to all polynomials of lower degree forms a vector space \( \mathcal{V}_n \) whose dimension is \( r_n^d \) (Dunkl and Xu, 2014). Let
\[
\{ \phi_k^n \}, 1 \leq k \leq r_n^d, n \in \mathbb{N}_0
\]
be one family of orthogonal polynomials with respect to \( W \) on \( \mathbb{B}^d \) satisfying
\[
\int_{\mathbb{B}^d} \phi_k^n(x) \phi_j^m(x) W(x) dx = \delta_{j,k} \delta_{m,n}.
\]
\( \{ \phi_k^n \}_{k,n} \) can be derived by a transformation of orthonormal polynomials on \( \mathbb{S}^d \). See Theorem 3.8 in (Xu, 1998) for details. For simplicity, let \( D \) denote \( \mathbb{B}^d \).

Consider a smooth function
\[
h_\varepsilon(x) = \varepsilon^{-d} \exp \left\{ -\frac{\varepsilon^2}{\varepsilon^2 - \|x\|^2} \right\} \mathbb{I}_{\{\|x\| \leq \varepsilon\}},
\]
known as a mollifier and the cut-off function \( \mathbb{I}_D * h_\varepsilon \) for set \( D \). Then for sufficiently small \( \varepsilon \)
\[
g_\varepsilon,k,n = \phi_k^n(x) \mathbb{I}_D * h_\varepsilon(x) \in C^\infty_c(\mathbb{B}^d(2)),
\]
hence belongs to \( \mathcal{G}_\sigma \) (Ongie et al., 2019).

Since function in \( \mathcal{G}_\sigma \) are uniquely determined by its signed measure, there exists a signed measure \( \alpha_{\varepsilon,k,n} \) with \( \int_{\mathbb{S}^{d-1} \times \mathbb{R}} \|v\|^2 d\alpha_{\varepsilon,k,n}(v, b) < \infty \) such that
\[
g_\varepsilon,k,n(x) = \int_{\mathbb{S}^{d-1} \times \mathbb{R}} (\sigma(\langle v, x \rangle - b) - \sigma(-b)) d\alpha_{\varepsilon,k,n}(v, b) + c, \quad x \in \mathbb{R}^d.
\]
We have for \( x \in D \)
\[
\phi_k^n(x) = \int_{\mathbb{S}^{d-1} \times \mathbb{R}} (\sigma(\langle v, x \rangle - b) - \sigma(-b)) d\alpha_{\varepsilon,k,n}(v, b) + c, \quad \text{for any } 1 \leq k \leq r_n^d, n \in \mathbb{N}_0.
\]
We conclude that
\[
\{ \phi_k^n \}_{k,n} \subset \mathcal{F}_\sigma(\mathbb{B}^d).
\]
Note that \( W(x) \) can be specified arbitrarily. The functions in \( \mathcal{F}_\sigma(\mathbb{B}^d) \) can represent functions in a strongly adaptive manner.

G. Proofs of the lower bound.

G.1. Lower bound of Theorem 3.1. Following Makovoz (1996), we prove the lower bound part of Theorem 3.1 via a contradiction of the upper bound and lower bound of the covering number of \( \mathcal{P}_\sigma^R \).

Proof of Theorem 3.1, lower bound part. In the following, we assume that the dimension of input samples is fixed. Let \( D = \mathbb{B}^d \subset \mathbb{R}^d \). Suppose that
\[
\sup_{f \in \mathcal{G}_\sigma} \inf_{\theta_n \in \Theta_n} \| f - g(\theta_n; \cdot) \|_{L_2(\mu)} \leq C m^{-\alpha},
\]
for some \( C, \alpha > 0 \). Let \( \delta = C m^{-\alpha} \). Denote by \( \mathcal{K}^\delta_\sigma \) a \( \delta \)-net of \( \mathcal{K}_\sigma \) under \( L_2(\mu) \)-norm and by \( \Lambda^\delta \) a \( \delta \)-net for \( \sum_{k=1}^m |a_k| \leq R \) in the space \( l_1^m \) under the \( l_1 \)-norm. We have the cardinality
\[
\# \mathcal{K}^\delta_\sigma = O\left( \frac{1}{\delta^d} \right) \quad \text{and} \quad \# \Lambda^\delta = O\left( \left( \frac{R}{\delta} \right)^m \right).
\]
Then an $O(\delta)$-net for the set of all linear combinations $P^R_\sigma$ can be given as

$$P^R_\sigma(\delta) = \left\{ g = \sum_{i=1}^{m} c_i \phi_i : \phi_i \in K^\delta, (c_1, \ldots, c_m)^T \in \Delta^\delta \right\}.$$ 

The cardinality of $P^R_\sigma(\delta)$ is at most $cR^m \delta^{-\infty - m \delta}$ with $c$ unrelated with $m$ and $d$. Then the log-covering number of $P^R_\sigma$,

$$\log \mathcal{N}(\delta, P^R_\sigma, \| \cdot \|_{L_2(\mu)}) \leq c_d \log m .$$

We want to estimate the lower bound of the log-covering number of $P^R_\sigma$.

Let $q_w(x) = \frac{\sin(2\delta x \langle w,x \rangle)}{4\pi \|w\|^2}$ for $\|w\|_2 \neq 0$. Note that if $w$ takes values in integers entrywise, $\{q_w(x)\}_{w \in \mathbb{Z}}$ are pairwise orthogonal for $L_2[\text{Leb}([-1/(2\sqrt{d}), 1/(2\sqrt{d})]^d)]$. Let $S = [-1/(2\sqrt{d}), 1/(2\sqrt{d})]^d$. Consider a smooth function

$$h_\varepsilon(x) = \varepsilon^{-d} \exp \left\{ -\frac{\varepsilon^2}{2(1 - \|x\|^2)} \right\} \mathbb{I}(\|x\| \leq \varepsilon),$$

known as a mollifier and the cut-off function $\mathbb{I}_D * h_\varepsilon$ for set $D$. Then for sufficiently small $\varepsilon$, we have

$$g_{\varepsilon,w}(x) = q_w(x) \mathbb{I}_D * h_\varepsilon(x) \in C^\infty_c(\mathbb{R}^d(2)), \quad \text{and} \quad g_{\varepsilon,w}(x) \mathbb{I}_S = q_w(x) \mathbb{I}_S$$

hence belongs to $\mathcal{G}_\sigma$ (Ongie et al., 2019). Since function in $\mathcal{G}_\sigma$ are uniquely determined by its signed measure, there exists a signed measure $\alpha_{\varepsilon,w}$ with $\int_{\mathbb{R}^{d-1} \times \mathbb{R}} d \langle \alpha_{\varepsilon,w} \rangle (v, b) < \infty$ such that

$$g_{\varepsilon,w}(x) = \int_{\mathbb{R}^{d-1} \times \mathbb{R}} (\sigma(\langle v, x \rangle + b) - \sigma(b)) d\alpha_{\varepsilon,w}(v, b), \quad x \in \mathbb{R}^d .$$

We have for $x \in [-1/\sqrt{d}, 1/\sqrt{d}]^d$

$$q_w(x) = \int_{\mathbb{R}^{d-1} \times \mathbb{R}} (\sigma(\langle v, x \rangle + b) - \sigma(b)) d\alpha_{\varepsilon,w}(v, b) + c , \quad \text{for any } w \in \mathbb{R}^d .$$

Let $w_k \in W = \{ w : \|w\|_2 \leq \delta^{-2/(2+d)}, w_ki \in \mathbb{N}, 1 \leq i \leq d \}$. Let the distribution of $X$, $\mu$, to be the uniform distribution on $S$. In this case, $q_{w_k}$ are pair-wise orthogonal, i.e.,

$$\langle q_{w_1}, q_{w_k} \rangle = \int_S q_{w_1}(x) q_{w_k}(x) dx = c \delta_{(w_1 = w_k)} .$$

We have

$$\sum_{k=1,k\neq l}^n |\langle q_{w_k}, q_{w_l} \rangle| \leq \frac{1}{2} |\langle q_{w_l}, q_{w_l} \rangle| , 1 \leq l \leq n ,$$

where $n$ is the number of all possible $w_k \in W$, and $n \asymp C_d \delta^{-2d/(2+d)}$. Note that $\min \sqrt{\langle q_{w_1}, q_{w_1} \rangle} = \delta^{2/(2+d)}$. Apply Lemma (3) of Makovoz (1996) with $n^{-1/2} \min \sqrt{\langle q_{w_1}, q_{w_1} \rangle} \asymp \delta$, $m \asymp m^{1/2+1/\gamma}$, we have

$$C_d \log m \geq \log \mathcal{N}(\delta, P^R_\sigma, \| \cdot \|_{L_2(\mu)}) \geq c_d m^{1/2+1/\gamma} .$$

If we now let $\alpha = 1/2 + 1/d + \eta$, we come to a contradiction. \qed
G.2. **Lower bound of Theorem 5.2.** We prove the minimax lower bound of Theorem 5.2 by applying the following theorem in 
Yang and Barron (1999). Technical details are partially borrowed from Donoho (1996) and Hayakawa and Suzuki (2019).

**Theorem G.1 (Theorem 1 of Yang and Barron (1999)).** In the Gaussian regression model

\[ Y_i = f(X_i) + \xi_i, \quad i = 1, 2, \cdots, n \]

with \( \xi_i \sim N(0, \sigma^2) \) independently and \( f \in \mathcal{F} \subset L^2([-1, 1]^d) \), suppose there exist \( \delta, \varepsilon > 0 \) such that

\[ V(\varepsilon) \leq \frac{n\varepsilon^2}{2\sigma^2}, \quad M(\delta) \geq \frac{2n\varepsilon^2}{\sigma^2} + 2\log 2, \]

where \( V(\varepsilon) = V(\mathcal{F}, \| \cdot \|_{L^2}) (\varepsilon) \) denotes the covering \( \varepsilon \)-entropy for the class of true functions \( \mathcal{F} \) under \( L^2 \) metric and \( M(\varepsilon) = M(\mathcal{F}, \| \cdot \|_{L^2}) (\varepsilon) \) denotes the packing \( \varepsilon \)-entropy for \( \mathcal{F} \) under \( L^2 \) metric. Then we have

\[ \inf_{f} \sup_{f \in \mathcal{F}} \mathbb{P}_f \left( \| \hat{f} - f \|_{L^2} \geq \frac{\delta}{2} \right) \geq \frac{1}{2}, \quad \inf_{f} \sup_{f \in \mathcal{F}} \mathbb{E}_f \left[ \| \hat{f} - f \|_{L^2}^2 \right] \geq \frac{\delta^2}{8}, \]

where \( \mathbb{P}_f \) is the probability law with true function being \( f \), and \( \mathbb{E}_f \) is the expectation determined by \( \mathbb{P}_f \).

We also introduce some facts about lower bounds of metric entropy.

**Lemma G.2 (Lemma 4 of Donoho (1993)).** Let \( C_k \subset l^2 \) be a \( k \)-dimensional hypercube of side \( 2\delta > 0 \) defined as

\[ C_k = \{ a \in l^2 : \max \{ |a_1|, \cdots, |a_k| \} \leq \delta, |a_{k+1}| = |a_{k+2}| = \cdots = 0 \}. \]

Then there exists a constant \( A > 0 \) such that

\[ V(C_k, \| \cdot \|_{l^2}) \left( \frac{\delta \sqrt{k}}{2} \right) \geq Ak, \quad k \in \mathbb{N}. \]

**Proof of Theorem 5.2.** Let \( D = \mathbb{B}^d \) and \( W(x) = d\mu/dx(x) \), where \( \mu \) denotes the distribution of \( X \). We have shown in Section (F) that for the weight function \( W(x) \) on \( D \), one family of orthogonal polynomials \( \{ \phi_k^{(n)} \}_{k,n}, 1 \leq k \leq r_n^d, n \in \mathbb{N}_0 \) is contained in \( \mathcal{F}_\sigma (\mathbb{B}^d) \). After proper enumeration (for instance, in an increasing order of \( R \)-norm), denote by \( \{ \phi_j \}_{j=1}^\infty \) the set \( \{ \phi_k^{(n)} \}_{k,n} \). We have

\[ \int_\mathbb{B}^{d(1)} \phi_i(x)\phi_j(x)d\mu(x) = \delta_{i,j}. \]

Consider the sparse \( l^1 \)-approximated set \( \mathcal{I}_\phi \) that defined as

\[ \mathcal{I}_\phi(C_1, C_2) = \left\{ \sum_{k=1}^{\infty} a_k \phi_k \sum_{k=1}^{\infty} |a_k| \leq C_1, \sum_{k=m+1}^{\infty} a_k^2 \leq C_2 m^{-1}, m \in \mathbb{N} \right\}. \]

The constraint \( \sum_{k=m+1}^{\infty} a_k^2 \leq C_2 m^{-1} \) is called \( 1 \)-minimally tail compactness of \( \{ a_k \}_{k=1}^\infty \), which is required to make the set compact in the \( L^2 \) metric. Recall that \( \{ \phi_k \}_{k=1}^\infty \subset \mathcal{F}_\sigma (\mathbb{B}^d) \),
then we have $\mathcal{I}_\phi(C_1, C_2) \subset \mathcal{F}_\sigma(\mathbb{B}^d)$. We can focus on the minimax lower bound for functions in $\mathcal{I}_\phi(C_1, C_2)$ since

$$\inf_{\hat{f}_n} \sup_{f^* \in \mathcal{F}_\sigma(\mathbb{B}^d)} \mathbb{E}_X \left[ \left| \hat{f}_n(X) - f^*(X) \right|^2 \right] \geq \inf_{\hat{f}_n} \sup_{f \in \mathcal{I}_\phi(C_1, C_2)} \mathbb{E}_X \left[ \left| \hat{f}_n(X) - f^*(X) \right|^2 \right].$$

To apply Theorem (G.1), we need to consider the covering $\varepsilon$-entropy and the packing $\varepsilon$-entropy of $\mathcal{I}_\phi(C_1, C_2)$. Note that $\phi_i$s are orthonormal, it suffices to evaluate $V(\varepsilon) = V(A, \|\cdot\|_2)(\varepsilon)$ under $l^2$ metric, where

$$A = \left\{ a \in l^2 : \sum_{k=1}^{\infty} |a_k| \leq C_1, \sum_{i=m+1}^{\infty} a_i^2 \leq C_2m^{-1}, m \in \mathbb{N} \right\}.$$ 

Following the proof of Lemma (4.7) of Hayakawa and Suzuki (2019), we can provide an estimate of $V(\varepsilon)$ that

$$C_l\varepsilon^{-2} \leq V(\varepsilon) \leq C_6u\varepsilon^{-2}(1 + \log(1/\varepsilon))$$

for some constants $C_l, C_u > 0$. In the following, we provide technical details.

For each $q = 1, 2, \ldots$, let $a^{(q)} \in l^2$ defined as

$$a_k^{(q)} = \begin{cases} q^{-1} & 1 \leq k \leq q; \\ 0 & k \geq q + 1. \end{cases}$$

Then $\sum_{k=1}^{\infty} |a_k^{(q)}| = 1$. As for the second condition, for $1 \leq k \leq q$,

$$k \sum_{i=k+1}^{\infty} \left( a_i^{(q)} \right)^2 = k(q - k)q^{-2} \leq \frac{1}{4},$$

for any $k$ and $q$. Define

$$C = \min \left\{ C_1, 2\sqrt{C_2} \right\},$$

then each $Ca^{(q)}$ is an element of $A$.

For each $q$, define the hyper-rectangle $A_q$ as

$$A_q = \{ a \in l^2 : |a_i| \leq Ca_i^{(q)}, i \in \mathbb{N} \}.$$ 

Note that each $A_q \subset A$. Then $V(\varepsilon) \geq V(A_q, \|\cdot\|_2)(\varepsilon)$. For each pair of distinct vertices of $A_q$, the $l^2$ distance between the two is at least $C q^{-1}$, and by letting $\delta = C q^{-1}$ in Lemma (G.2), we have

$$V\left( \frac{C}{2} q^{-1/2} \right) \geq V(A_2, \|\cdot\|_2) \left( \frac{C}{2} q^{1/2 - 1} \right) \geq Aq$$

for each $q > 0$. Then for $\varepsilon \in [C^{-2}q^{-(j+1)/2 - 1}, C^{-2}q^{-j/2 - 1}]$,

$$V(\varepsilon) \geq V\left( \frac{C}{2} 2^{-j/2} \right) \geq 2^j A \geq \frac{A}{2} \left( \frac{2\varepsilon}{C} \right)^{-2} = C_l \varepsilon^{-2}.$$ 

As for the upper bound, it suffices to estimate the metric entropy of $V(\varepsilon) = V(A, \|\cdot\|_2)(\varepsilon)$. For an arbitrary element $a \in A$, by the 1-minimal tail compactness condition, we have

$$\sum_{i=q+1}^{\infty} a_i^2 \leq C_2q^{-1}.$$
Let \( b = (a_1, \cdots, a_q, 0, 0, \cdots) \), then we have \( \|a - b\|_2^2 \leq C_2 q^{-1} \). Consider a \( l_2 \)-grid approximation of \( b \),

\[
\tilde{b} = \left( \frac{\text{sign}(b_i) |q b_i|}{q} \right)_{i=1}^{\infty},
\]

we have \( \|b - \tilde{b}\|_2 \leq q^{-1/2} \). Note that the number of such \( \tilde{b} \) is at most \( (2C_1 q + 1)^q \) and by letting \( q \sim \varepsilon^{-2} \) we have an upper bound of covering \( \varepsilon \)-entropy as

\[
V(\varepsilon) \leq C_0 \log ((2C_1 q + 1)^q) \leq q \log (2C_1 + 1) + 2q \log q \leq C^n \varepsilon^{-2} (-\log(\varepsilon) + 1).
\]

To this end, applying Theorem (G.1) yields desired results. On one hand, let \( \varepsilon_n = c((\log n)/n)^{1/4} \) for some constant \( c > 0 \). Then

\[
V(\varepsilon_n) \leq C n^{-2} \left( \frac{\log n}{n} \right)^{-1/2} \left( 1 + \frac{1}{4} \log \left( \frac{n}{\log n} \right) \right) \lesssim C_{n}^{-2} n^{1/2}(\log n)^{1/2}.
\]

Then we have

\[
\frac{V(\varepsilon)}{n \varepsilon_n^2} \leq C n^{-4} \leq \frac{1}{2 \sigma^2}
\]

for \( c \geq (2C_{n} \sigma^2)^{1/4} \). On the other hand, let \( \delta_n = C' n^{-1/4} (\log n)^{-1/4} \) for some constant \( C' \). Then

\[
M(\delta_n) \geq V(\delta_n) \geq C_l C^{-2} n^{-1/2}(\log n)^{1/2} \geq C^l c_l \left( \frac{2n \varepsilon_n^2}{\sigma^2} + 2 \log 2 \right)
\]

for some constant \( c_l > 0 \) independent of \( C' \).

Then, by Theorem (G.1) and inequality (G.2), we conclude that

\[
\inf_{\tilde{f}_n} \sup_{f^* \in \mathcal{F}_\sigma(\mathbb{R}^d)} \mathbb{E} \left[ \left| \tilde{f}_n(X) - f^*(X) \right|^2 \right] \geq \frac{1}{8} C' n^{-\frac{1}{2}} (\log n)^{-\frac{1}{2}}.
\]

\[\square\]

G.3. Lower bound of random feature model. Theorem 5.1 is motivated by the following lemma from Barron (1993).

**Lemma G.3 (Theorem 6 of Barron (1993)).** Let

\[
\Gamma_C(\mathbb{B}^d) = \left\{ f(x) = \int e^{i(w,x) + i\theta(w)} F(dw) : \int \|w\|_2 F(dw) \leq C, x \in \mathbb{B}^d \right\},
\]

where \( F(dw) \) denotes the magnitude distribution and \( \theta(w) \) denotes the phase at the frequency \( w \). For every choice of fixed basis functions \( h_1, \ldots, h_l \),

\[
\sup_{f \in \Gamma_C} d(f, \text{span}(h_1, \ldots, h_l)) \geq \frac{C}{d} l^{-1/d},
\]

where \( d(f, \text{span}(h_1, \ldots, h_l)) = \inf_{h \in \text{span}(h_1, \ldots, h_l)} \mathbb{E}_\mu \left[ |f(X) - h(X)|^2 \right] \).

The connection between Lemma G.3 and our work is that \( \Gamma_C(\mathbb{B}^d) \subset \mathcal{F}_\sigma(\mathbb{B}^d) \). This is straightforward since \( \Gamma_C(\mathbb{B}^d) \subset \text{conv}(G_{\cos}) \subset \mathcal{F}_\sigma(\mathbb{B}^d) \), where

\[
G_{\cos} = \left\{ \frac{\gamma}{\|w\|_2} (\cos(\langle w, x \rangle + b) - \cos(b)) : w \neq 0, |\gamma| < C, b \in \mathbb{R} \right\},
\]

and \( \text{conv}(\cdot) \) denotes the convex hull of a set. The first set relation comes from Barron (1993), and the second set relation comes from the proof of the lower bound part of Theorem 3.1.
PROOF OF THEOREM 5.1. Recall that
\[ g_{m,\pi}(x; \hat{a}) = K^\pi_m(x, X) (K^\pi_m(X, X) + n\lambda_n I_n)^{-1} Y, \]
where \( K^\pi_m(x, y) = \frac{1}{m} \sum_{k=1}^m \sigma((x, y)^T, w_k)) \sigma((y, 1)^T, w_k)), K^\pi_m(x, X) = [K^\pi_m(x, X_i)]_{i,i} \in \mathbb{R}^{1 \times n}, \) and \( K^\pi_m(X, X) = [K^\pi_m(x_i, X_j)]_{i,j} \in \mathbb{R}^{n \times n}. \)

Consider the spectral decomposition of \( K^\pi_m(x, y) \). Define the linear functional \( T: L^2_{\mu}(x) \to L^2_{\mu}(x) \)
\[
(Tf)(x) = \int_{\mathbb{R}^d} f(y) K^\pi_m(x, y) d\mu(y),
\]
and denote by \( \{e_1, \ldots, e_p\} \) (\( p \) can be infinity) its eigenfunction set, i.e., \((Te_i)(x) = \lambda_i e_i(x), \) and \( \int_{\mathbb{R}^d} e_i(x)e_j(x)d\mu(x) = \delta_{ij}. \) Our first claim is that \( p \leq m. \)

Notice that \( e_j(x) = m^{-\frac{1}{2}} \sum_{k=1}^m \sigma((x, 1)^T w_k) \int_{\mathbb{R}^d} \sigma((y, 1)^T w_k)e_j(y)d\mu(y). \) Let
\[
a_{jk} = m^{-\frac{1}{2}} \int_{\mathbb{R}^d} \sigma((y, 1)^T w_k)e_j(y)d\mu(y)
\]
and \( A = [a_{jk}]_{1 \leq j \leq m, 1 \leq k \leq m}, \) \( \Sigma(x) = (\sigma((x, 1)^T w_1), \ldots, \sigma((x, 1)^T w_m))^T. \) We have
\[
(e_1(x), \ldots, e_p(x))^T = A \Sigma(x).
\]
Then \( I_p = A \int_{\mathbb{R}^d} \Sigma(x) \Sigma^T(x) d\mu(x) A^T. \) We conclude that \( p \leq m. \)

From the spectral decomposition, we have
\[
K^\pi_m(x, y) = \sum_{i=1}^p \lambda_i e_i(x)e_i(y).
\]
Let \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_p), \) and \( e(X) = [e_i(X_j)]_{1 \leq i \leq p, 1 \leq j \leq n}. \) Then
\[
K^\pi_m(X, X) = e(X)^T \Lambda e(X).
\]
\( g_{m,\pi}(x; \hat{a}) \) can be rewritten as
\[
g_{m,\pi}(x; \hat{a}) = \sum_{i=1}^p c_i e_i(x), \quad (c_1, \ldots, c_p)^T \Lambda e(X) (e(X)^T \Lambda e(X) + n\lambda_n I_n)^{-1} Y.\]

Note that when \( m > n, \) \( r = \text{rank}(e(X)) \leq n. \) For \( p > n, \) there exists a matrix \( B \in \mathbb{R}^{p \times n} \) such that \( (c_1, \ldots, c_p)^T = Bd, \) where \( d \in \mathbb{R}^n. \) For example, one can choose \( B = \Lambda Q e \in \mathbb{R}^{p \times n}, B = D e R^T, (e(X)^T \Lambda e(X) + n\lambda_n I_n)^{-1} Y \in \mathbb{R}^n, \) where \( e(X) = Q e D e R^T \) with \( D e \in \mathbb{R}^{n \times n} \) being the diagonal matrix whose top \( r \) diagonal entries are nonzero. Let \( u_i = (u_{i1}, \ldots, u_{ip})^T \in \mathbb{R}^p \) be the \( i \)-th orthonormal basis of \( \text{Col}(B), 1 \leq i \leq n, \) and \( \hat{e}_i(x) = u_{i1} e_1(x) + \ldots, u_{ip} e_p(x). \)

Note that \( \int_{\mathbb{R}^d} \hat{e}_i(x)\hat{e}_j(x)d\mu(x) = \delta_{ij}. \) Then
\[
g_{m,\pi}(x; \hat{a}) = \sum_{i=1}^n c'_i \hat{e}_i(x),
\]
for some \( c' = (c'_1, \ldots, c'_n)^T. \) Then without loss of generality, assume \( p \leq n. \) By Lemma G.3, we have
\[
\sup_{f \in C_b} \mathbb{E}_\mu \left[ |f^*(X) - g_{m,\pi}(X; \hat{a})|^2 \right] \geq \sup_{f \in C_b} d(f, \text{span}(e_1, \ldots, e_p)) \geq \frac{R}{d} (\min(m, n) \kappa)^{-1/d},
\]
for some constant \( \kappa. \)