Constructions of Subsystem Codes 
over Finite Fields
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Abstract—Subsystem codes protect quantum information by
encoding it in a tensor factor of a subspace of the physical
state space. Subsystem codes generalize all major quantum error
protection schemes, and therefore are especially versatile. This
paper introduces numerous constructions of subsystem codes. It
is shown how one can derive subsystem codes from classical
cyclic codes. Methods to trade the dimensions of subsystem and
cosubsystem are introduced that maintain or improve the
minimum distance. As a consequence, many optimal subsystem
codes are obtained. Furthermore, it is shown how given subsystem
codes can be extended, shortened, or combined to yield new
subsystem codes. These subsystem code constructions are used
to derive tables of upper and lower bounds on the subsystem
code parameters.

I. INTRODUCTION
Quantum information processing as a growing exciting field
has attracted researchers from different disciplines. It utilizes
the laws of quantum mechanical operations to perform exponential
speedy computations. In an open system, one might
wonder how to perform such computations in the presence of
decoherence and noise that disturb quantum states storing
quantum information. Ultimately, the goals of quantum error-
correcting codes are to protect quantum states and to allow
recovery of quantum information processed in computational
operations of a quantum computer. Henceforth, one seeks to
design good quantum codes that can be efficiently utilized for
these goals.

A well-known approach to derive quantum error-correcting
codes from self-orthogonal (or dual-containing) classical codes
is called stabilizer codes, which were introduced a decade ago.
The stabilizer codes inherit some properties of clifford group
theory, i.e., they are stabilized by abelian finite groups. In the
seminal paper by Calderbank et. al [7], [20], [22], various
methods of stabilizer code constructions are given, along with
their propagation rules and tables of upper bounds on their
parameters. In a similar tactic, we also present subsystem
code structures by establishing several methods to derive them
easily from classical codes. Subsystem codes inherit their
name from the fact that the quantum codes are decomposed
into two systems as explained in Section II. The classes of
subsystem codes that we will derive are superior because
they can be encoded and decoded using linear shift-register
operations. In addition, some of these classes turned out to be
optimal and MDS codes.

Subsystem codes as we prefer to call them were mentioned
in the unpublished work by Knill [14], [15], in which he
attempted to generalize the theory of quantum error-correcting
codes into subsystem codes. Such codes with their stabilizer
formalism were reintroduced recently [6], [12], [16], [17],
[19]. An \((n, K, R, d))_q\) subsystem code is a \(KR\)-dimensional
subspace \(Q\) of \(\mathbb{C}^n\) that is decomposed into a tensor product
\(Q = A \otimes B\) of a \(K\)-dimensional vector space \(A\) and an \(R\)-
dimensional vector space \(B\) such that all errors of weight less
than \(d\) can be detected by \(A\). The vector spaces \(A\) and \(B\) are
respectively called the subsystem \(A\) and the co-subsystem \(B\).
For some background on subsystem codes see the next section.

This paper is structured as follows. In section II, we present
a brief background on subsystem code structures and present
the Euclidean and Hermitian constructions. In section III,
we derive cyclic subsystem codes and provide two generic
methods of their constructions from classical cyclic codes.
Consequently in section IV, we construct families of subsystem
BCH and RS codes from classical BCH and RS over \(F_q\)
and \(F_{q^2}\) defined using their defining sets. In Sections V,VI, VII,
we establish various methods of subsystem code constructions
by extending and shortening the code lengths and combining
pairs of known codes, in addition, tables of upper bounds on
subsystem code parameters are given. Finally, the paper is
concluded with a discussion and future research directions in
section VIII.

Notation. If \(S\) is a set, then \(|S|\) denotes the cardinality of
the set \(S\). Let \(q\) be a power of a prime integer \(p\). We denote
by \(F_q\) the finite field with \(q\) elements. We use the notation
\((x|y) = (x_1, \ldots, x_n|y_1, \ldots, y_n)\) to denote the concatenation
of two vectors \(x\) and \(y\) in \(F_q^n\). The symplectic weight of
\((x|y) \in F_{q^n}^2\) is defined as
\[ \text{swt}(x|y) = \{(x_i, y_i) \neq (0, 0) \mid 1 \leq i \leq n\}. \]
We define \(\text{swt}(X) = \min\{\text{swt}(x) \mid x \in X, x \neq 0\}\) for any
nonempty subset \(X \neq \{0\}\) of \(F_q^n\).
The trace-symplectic product of two vectors \(u = (a|b)\) and
\(v = (a'|b')\) in \(F_{q^n}^2\) is defined as
\[ \langle u|v \rangle_s = \text{tr}_{q/p}(a' \cdot b - a \cdot b'), \]
where \(x \cdot y\) denotes the dot product and \(\text{tr}_{q/p}\) denotes the
trace from \(F_q\) to the subfield \(F_p\). The trace-symplectic dual
of a code $C \subseteq F_q^{2n}$ is defined as
\[ C^\perp = \{ v \in F_q^{2n} \mid \langle v | w \rangle_s = 0 \text{ for all } w \in C \}. \]

We define the Euclidean inner product $\langle x | y \rangle = \sum_{i=1}^{n} x_i y_i$ and the Euclidean dual of $C \subseteq F_q^n$ as
\[ C^\perp = \{ x \in F_q^n \mid \langle x | y \rangle = 0 \text{ for all } y \in C \}. \]

We also define the Hermitian inner product for vectors $x, y$ in $F_q^n$ as $\langle x | y \rangle_h = \sum_{i=1}^{n} x_i^* y_i$ and the Hermitian dual of $C \subseteq F_q^n$ as
\[ C^\perp_h = \{ x \in F_q^n \mid \langle x | y \rangle_h = 0 \text{ for all } y \in C \}. \]

II. BACKGROUND ON SUBSYSTEM CODES

In this section we give a quick overview of subsystem codes. We assume that the reader is familiar with the theory of stabilizer codes over finite fields, see [7], [11], [20] and the references therein.

A. Errors

Let $F_q$ denote a finite field with $q$ elements of characteristic $p$. Let $\{ \{ x \} \mid x \in F_q \}$ be a fixed orthonormal basis of $C_q^q$ with respect to the standard hermitian inner product, called the computational basis. For $a, b \in F_q$, we define the unitary operators $X(a)$ and $Z(b)$ on $C_q^q$ by
\[ X(a)|x\rangle = |x + a\rangle, \quad Z(b)|x\rangle = \omega^{tr(bx)}|x\rangle, \]
where $\omega = \exp(2\pi i/p)$ is a primitive $p$th root of unity and $tr$ is the trace operation from $F_q$ to $F_p$. The set $E = \{ X(a)Z(b) \mid a, b \in F_q \}$ forms an orthogonal basis of the operators acting on $C_q^q$ with respect to the trace inner product, called the error basis.

The state space of $n$ quantum digits (or qudits) is given by $C_q^n = C_q^q \otimes C_q^q \otimes \cdots \otimes C_q^q$. An error basis $E$ on $C_q^n$ is obtained by tensoring $n$ operators in $E$; more explicitly, $E = \{ X(a)Z(b) \mid a, b \in F_q^n \}$, where
\[ X(a) = X(a_1) \otimes \cdots \otimes X(a_n), \]
\[ Z(b) = Z(b_1) \otimes \cdots \otimes Z(b_n) \]
for $a = (a_1, \ldots, a_n) \in F_q^n$ and $b = (b_1, \ldots, b_n) \in F_q^n$. The set $E$ is not closed under multiplication, whence it is not a group. The group $G$ generated by $E$ is given by
\[ G = \{ \omega^c E \mid \omega^c X(a)Z(b) \mid a, b \in F_q^n, c \in F_p \}, \]
and $G$ is called the error group of $C_q^n$. The error group is an extraspecial $p$-group. The weight of an error in $G$ is given by the number of nonidentity tensor components; hence, the weight of $\omega^c X(a)Z(b)$ is given by the symplectic weight $\text{swt}(a|b)$.

B. Subsystem Codes

An $(n, (K, R, d))_q$ subsystem code is a subspace $Q = A \otimes B$ of $C_q^n$ that is decomposed into a tensor product of two vector spaces $A$ and $B$ of dimension $\dim A = K$ and $\dim B = R$ such that all errors in $G$ of weight less than $d$ can be detected by $A$. We call $A$ the subsystem and $B$ the co-subsystem. The information is exclusively encoded in the subsystem $A$. This yields the attractive feature that errors affecting co-subsystem $B$ alone can be ignored.

A particularly fruitful way to construct subsystem codes proceeds by choosing a normal subgroup $N$ of the error group $G$, and this choice determines the dimensions of subsystem and co-subsystem as well as the error detection and correction capabilities of the subsystem code, see [12]. One can relate the normal subgroup $N$ to a classical code, namely $N$ modulo the intersection of $N$ with the center $Z(G)$ of $G$ yields the classical code $X = N/(N \cap Z(G))$. This generalizes the familiar case of stabilizer codes, where $N$ is an abelian normal subgroup. It is remarkable that in the case of subsystem codes any classical additive code $X$ can occur. It is most convenient that one can also start with any classical additive code and obtain a subsystem code, as is detailed in the following theorem from [12]:

**Theorem 1.** Let $C$ be a classical additive subcode of $F_q^{2n}$ such that $C \neq \{0\}$ and let $D$ denote its subcode $D = C \cap C^\perp$. If $x = |C|$ and $y = |D|$, then there exists a subsystem code $Q = A \otimes B$ such that
\[ i) \dim A = q^n/(xy)^{1/2}, \]
\[ ii) \dim B = (x/y)^{1/2}. \]
The minimum distance of subsystem $A$ is given by
\[ (a) d = \text{swt}((C + C^\perp) - C) = \text{swt}(D^\perp - C) \text{ if } D^\perp \neq C; \]
\[ (b) d = \text{swt}(D^\perp) \text{ if } D^\perp = C. \]

Thus, the subsystem $A$ can detect all errors in $E$ of weight less than $d$, and can correct all errors in $E$ of weight $\leq [(d-1)/2]$. □

**Proof:** See [12, Theorem 5].

A subsystem code that is derived with the help of the previous theorem is called a Clifford subsystem code. We will assume throughout this paper that all subsystem codes are Clifford subsystem codes. In particular, this means that the existence of an $(n, (K, R, d))_q$ subsystem code implies the existence of an additive code $C \leq F_q^{2n}$ with subcode $D = C \cap C^\perp$ such that $|C| = q^n R/K$, $|D| = q^n/(KR)$, and $d = \text{swt}(D^\perp - C)$. A subsystem code derived from an additive classical code $C$ is called pure to $d'$ if there is no element of symplectic weight less than $d'$ in $C$. A subsystem code is called pure if it is pure to the minimum distance $d$. We require that an $(n, (1, R, d))_q$ subsystem code must be pure.

We also use the bracket notation $[[n, k, r, d]]_q$ to write the parameters of an $(n, q^k, q^d)_q$ subsystem code in simpler form. Some authors say that an $[[n, k, r, d]]_q$ subsystem code has $r$ gauge qudits, but this terminology is slightly confusing, as the co-subsystem typically does not correspond to a state space of $r$ qudits except perhaps in trivial cases. We will avoid this misleading terminology. An $(n, (K, 1, d))_q$ subsystem code is also an $(n, (K, d))_q$ stabilizer code and vice versa.
Subsystem codes can be constructed from the classical codes over \( F_q \) and \( F_q^2 \). We recall the Euclidean and Hermitian constructions from [3], which are easy consequences of the previous theorem.

**Lemma 2** (Euclidean Construction). If \( C \) is a \( k' \)-dimensional \( F_q \)-linear code of length \( n \) that has a \( k'' \)-dimensional subcode \( D = C \cap C^+ \) and \( k' + k'' < n \), then there exists an
\[
[[n, n - (k' + k''), k' - k'', \text{wt}(D^+) \setminus C)]_q
\]
subsystem code.

**Lemma 3** (Hermitian Construction). If \( C \) is a \( k' \)-dimensional \( F_q^2 \)-linear code of length \( n \) that has a \( k'' \)-dimensional subcode \( D = C \cap C^{+\perp} \) and \( k' + k'' < n \), then there exists an
\[
[[n, n - (k' + k''), k' - k'', \text{wt}(D^{+\perp} \setminus C)]_q
\]
subsystem code.

### III. CYCLIC SUBSYSTEM CODES

In this section we shall derive subsystem codes from classical cyclic codes. We first recall some definitions before embarking on the construction of subsystem codes. For further details concerning cyclic codes see for instance [10] and [18].

Let \( n \) be a positive integer and \( F_q \) a finite field with \( q \) elements such that \( \gcd(n, q) = 1 \). Recall that a linear code \( C \subseteq F_q^n \) is called cyclic if and only if \( (c_0, \ldots, c_{n-1}) \) in \( C \) implies that \( (c_{n-1}, c_0, \ldots, c_{n-2}) \) in \( C \).

For \( g(x) \) in \( F_q[x] \), we write \( (g(x)) \) to denote the principal ideal generated by \( g(x) \) in \( F_q[x] \). Let \( \pi \) denote the vector space isomorphism \( \pi : F_q^n \rightarrow F_q[x]/(x^n - 1) \) given by
\[
\pi((c_0, \ldots, c_{n-1})) = c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} + (x^n - 1).
\]

A cyclic code \( C \subseteq F_q^n \) is mapped to a principal ideal \( \pi(C) \) of the ring \( R_n \). For a cyclic code\( C \), the unique monic polynomial \( g(x) \) in \( F_q[x] \) of the least degree such that \( (g(x)) = \pi(C) \) is called the **generator polynomial of** \( C \). If \( C \subseteq F_q^n \) is a cyclic code with generator polynomial \( g(x) \), then
\[
\dim_{F_q} C = n - \deg(g(x)).
\]

Since \( \gcd(n, q) = 1 \), there exists a primitive \( n^{\text{th}} \) root of unity \( \alpha \) over \( F_q^2 \); that is, \( F_q[\alpha] \) is the splitting field of the polynomial \( x^n - 1 \) over \( F_q \). Let us henceforth fix this primitive \( n^{\text{th}} \) primitive root of unity \( \alpha \). Since the generator polynomial \( g(x) \) of a cyclic code \( C \subseteq F_q^n \) is of minimal degree, it follows that \( g(x) \) divides the polynomial \( x^n - 1 \) in \( F_q[x] \). Therefore, the generator polynomial \( g(x) \) of a cyclic code \( C \subseteq F_q^n \) can be uniquely specified in terms of a subset \( T \) of \( \{0, \ldots, n-1\} \) such that
\[
g(x) = \prod_{t \in T} (x - \alpha^t).
\]

The set \( T \) is called the **defining set** of the cyclic code \( C \) (with respect to the primitive \( n^{\text{th}} \) root of unity \( \alpha \)). Since \( g(x) \) is a polynomial in \( F_q[x] \), a defining set is the union of cyclotomic cosets \( C_\kappa \), where
\[
C_\kappa = \{ x \kappa^i \mod n \mid i \in \mathbb{Z}, i \geq 0 \}, \quad 0 \leq x < n
\]
The following lemma recalls some well-known and easily proved facts about defining sets (see e.g. [10]).

**Lemma 4.** Let \( C_1 \) be a cyclic code of length \( n \) over \( F_q \) with defining set a \( T_i \) for \( i = 1, 2 \). Let \( N = \{0, 1, \ldots, n-1\} \) and \( T_i^a = \{ a \mod n \mid t \in T_i \} \) for some integer \( a \). Then
\begin{itemize}
  \item[i)] \( C_1 \cap C_2 \) has defining set \( T_1 \cup T_2 \).
  \item[ii)] \( C_1 \cup C_2 \) has defining set \( T_1 \cap T_2 \).
  \item[iii)] \( C_1 \subseteq C_2 \) if and only if \( T_2 \subseteq T_1 \).
  \item[iv)] \( C_1^{\perp} \) has defining set \( N \setminus T_1^{\perp} \).
  \item[v)] \( C_1^{\perp,2} \) has defining set \( N \setminus T_1^{\perp,2} \) provided that \( q = r^2 \) for some positive integer \( r \).
\end{itemize}

**Notation.** Throughout this section, we denote by \( N \) the set \( N = \{0, \ldots, n-1\} \). The cyclotomic coset of \( x \) will be denoted by \( C_x \). If \( T \) is a defining set of a cyclic code of length \( n \) and \( a \) is an integer, then we denote henceforth by \( T^a \) the set
\[
T^a = \{ a \mod n \mid t \in T \},
\]
as in the previous lemma. We use a superscript, since this notation will be frequently used in set differences, and arguably \( N \setminus T^q \) is more readable than \( N \setminus -qT \).

Now, we shall give a general construction for subsystem cyclic codes. We say that a code \( C \) is self-orthogonal if and only if \( C \subseteq C^{\perp} \). We show that if a classical cyclic code is self-orthogonal, then one can easily construct cyclic subsystem codes.

**Proposition 5.** Let \( D \) be a \( k \)-dimensional self-orthogonal cyclic code of length \( n \) over \( F_q \). Let \( T_D \) and \( T_{D^+} \) respectively denote the defining sets of \( D \) and \( D^+ \). If \( T \) is a subset of \( T_D \cup T_{D^+} \) that is the union of cyclotomic cosets, then one can define a cyclic code \( C \) of length \( n \) over \( F_q \) by the defining set \( T_C = T_D \setminus (T \cup T^{-1}) \). If \( r = |T \cup T^{-1}| \) is in the range \( 0 \leq r < n - 2k \), and \( d = \min \text{wt}(D^+ \setminus C) \), then there exists a subsystem code with parameters \( [[n, n - 2k - r, r, d]]_q \).

**Proof:** Since \( D \) is a self-orthogonal cyclic code, we have \( D \subseteq D^+ \), whence \( T_{D^+} \subseteq T_D \) by Lemma 4 iii). Observe that if \( s \) is an element of the set \( S = T_D \setminus T_{D^+} = T_D \setminus (N \setminus T_D) \), then \(-s\) is an element of \( S \) as well. In particular, \( T^{-1} \) is a subset of \( T_D \setminus T_{D^+} \).

By definition, the cyclic code \( C \) has the defining set \( T_C = T_D \setminus (T \cup T^{-1}) \); thus, the dual code \( C^{\perp} \) has the defining set
\[
T_{C^{\perp}} = N \setminus T_{C^{\perp}} = T_D \cup (T \cup T^{-1}).
\]
Furthermore, we have
\[
T_C \cup T_{C^{\perp}} = (T_D \setminus (T \cup T^{-1})) \cup (T_D \cup T \cup T^{-1}) = T_D;
\]
therefore, \( C \cap C^{\perp} = D \) by Lemma 4 i).

Since \( n - k = |T_D| \) and \( r = |T \cup T^{-1}| \), we have \( \dim_{F_q} D = n - |T_D| = k \) and \( \dim_{F_q} C = n - |T_C| = k + r \). Thus, by Lemma 2 there exists an \( F_q \)-linear subsystem code with parameters \( [[n, k, \rho, d]]_q \), where
\begin{itemize}
  \item[i)] \( k = \dim D^+ - \dim C = n - k - (k + r) = n - 2k - r \),
  \item[ii)] \( \rho = \dim C - \dim D = k + r - k = r \),
  \item[iii)] \( d = \min \text{wt}(D^+ \setminus C) \),
\end{itemize}
as claimed.  ■
We can also derive subsystem codes from cyclic codes over $\mathbb{F}_{q^2}$ by using cyclic codes that are self-orthogonal with respect to the Hermitian inner product.

**Proposition 6.** Let $D$ be a cyclic code of length $n$ over $\mathbb{F}_{q^2}$ such that $D \subseteq D^\perp$. Let $T_D$ and $T_{D^\perp}$ respectively be the defining set of $D$ and $D^\perp$. If $T$ is a subset of $T_D \setminus T_{D^\perp}$ that is a union of cyclotomic cosets, then one can define a cyclic code $C$ of length $n$ over $\mathbb{F}_{q^2}$ with defining set $T_C = T_D \setminus (T \cup T^\perp)$. If $n - k = |T_D|$ and $r = |T \cup T^\perp|$ with $0 \leq r < n - 2k$, and $d = \text{wt}(D^\perp \setminus C)$, then there exists an $[[n, n - 2k - r, d]]_q$ subsystem code.

**Proof:** Since $D \subseteq D^\perp$, their defining sets satisfy $T_{D^\perp} \subseteq T_D$ by Lemma 4 iii). If $s$ is an element of $T_D \setminus T_{D^\perp}$, then one easily verifies that $-qs$ (mod $n$) is an element of $T_D \setminus T_{D^\perp}$.

Let $N = \{0, 1, \ldots, n - 1\}$. Since the cyclic code $C$ has the defining set $T_C = T_D \setminus (T \cup T^\perp)$, its dual code $C^\perp$ has the defining set $T_{C^\perp} = N \setminus T^\perp = T_{D^\perp} \cup (T \cup T^\perp)$. We notice that

$$T_C \cap T_{C^\perp} = (T_D \setminus (T \cup T^\perp)) \cup (T_{D^\perp} \cup T \cup T^\perp) = T_D;$$

thus, $C \cap C^\perp = D$ by Lemma 4 i). Since $n - k = |T_D|$ and $r = |T \cup T^\perp|$, we have $\dim D = n - |T_D| = k$ and $\dim C = n - |T_C| = k + r$. Thus, by Lemma 3 there exists an $[[n, k, \rho, d]]_q$ subsystem code with

1. $\kappa = \dim D^\perp - \dim C = (n - k) - (k + r) = n - 2k - r$,
2. $\rho = \dim C - \dim D = k + r - k = r$,
3. $d = \min \text{wt}(D^\perp \setminus C)$,

as claimed.

We include an example to illustrate the construction given in the previous proposition.

**Example 7.** Consider the narrow-sense BCH code $D^{1/2}$ of length $n = 31$ over $\mathbb{F}_4$ with designed distance $5$. The defining set $T_{D^{1/2}}$ of $D^{1/2}$ is given by $T_{D^{1/2}} = C_1 \cup C_2 \cup C_3 \cup C_4 = C_1 \cup C_3$, where the cyclotomic cosets of 1 and 3 are given by

$$C_1 = \{1, 4, 16, 2, 8\} \quad \text{and} \quad C_3 = \{3, 12, 17, 6, 24\}.$$

If $N = \{0, 1, \ldots, 30\}$, then the defining set of the dual code $D$ is given by $T_D = N \setminus (C_{15} \cup C_7) = C_0 \cup C_1 \cup C_3 \cup C_5 \cup C_{11}$. Therefore, $D \subseteq D^\perp$, $\dim D^\perp = 21$ and $\dim D = 10$. If we choose $T = C_0$, then $T^\perp = C_{11}$, whence the defining set $T_C$ of the code $C$ is given by $T_C = T_D \setminus (C_5 \cup C_{11}) = C_0 \cup C_1 \cup C_3$. It follows that $\dim C = 20$ and $\dim C^\perp = 11$. Therefore, the construction of the previous proposition yields a BCH subsystem code with parameters $[[31, 11, 10, 5]]_2$.

The general principle behind the previous example yields the following simple recipe for the construction of subsystem codes: Choose a cyclic code (such as a BCH or Reed-Solomon code) with known lower bound $\delta$ on the minimum distance that contains its (hermitian) dual code, and use Proposition 5 (or Proposition 6) to derive subsystem codes. This approach allows one to control the minimum distance $d$ of the subsystem code, since $d \geq \delta$ is guaranteed. Another advantage is that one can exploit the cyclic structure in encoding and decoding algorithms.

For example, if we start with primitive, narrow-sense BCH codes, then Proposition 5 yields the following family of subsystem codes:

**Corollary 8.** Consider a primitive, narrow-sense BCH code of length $n = q^m - 1$ with $m \geq 2$ over $\mathbb{F}_q$ with designed distance $\delta$ in the range

$$2 \leq \delta \leq q^{\lceil m/2 \rceil} - 1 - (q - 2)[q \text{ is odd}]. \quad (1)$$

If $T$ is a subset of $N \setminus (\bigcup_{a=1}^{q^m-1} (C_a \cup C_{-a}))$ that is a union of cyclotomic cosets and $r = |T \cup T^\perp|$ with $0 \leq r < n - 2k$, where $k = m \lfloor (\delta - 1)(1 - 1/q) \rfloor$, then there exists an $[[q^m - 1, q^m - 1 - 2m \lfloor (\delta - 1)(1 - 1/q) \rfloor, r, \geq \delta]]_q$ subsystem code.

**Proof:** By [5, Theorem 2], a primitive, narrow-sense BCH code $D^\perp$ with designed distance $\delta$ in the range (1) satisfies $D \subseteq D^\perp$. By [5, Theorem 7], the dimension of $D^\perp$ is given by $\dim D^\perp = q^m - 1 - \lfloor m(\delta - 1)(1 - 1/q) \rfloor = n - k$, whence $k = \dim D$. Let $T_D$ and $T_{D^\perp}$ respectively denote the defining sets of $D$ and $D^\perp$. It follows from the definitions that $T_{D^\perp} = \bigcup_{a=1}^{q^m-1} C_a$ and that $T$ is a subset of

$$N \setminus (T_{D^\perp} \cup T_{D^\perp}^{-1}) = (N \setminus T_D^{-1}) \setminus T_{D^\perp} = T_D \setminus T_{D^\perp}.$$
Corollary 9. Consider a primitive, narrow-sense BCH code of length \( n = q^{2m} - 1 \) with \( m \neq 2 \) over \( \mathbb{F}_q \) with designed distance \( \delta \) in the range
\[
2 \leq \delta \leq q^m - 1
\]
If \( T \) is a subset of the set \( N \setminus \left( \bigcup_{a=1}^{m-1} (C_a \cup C_{qa}) \right) \) that is a union of cyclotomic cosets and \( r = \left| T \cup T^{-q} \right| \) with \( 0 \leq r < n - 2k \), where \( k = m \left( \delta - 1 \right) \left( 1 - 1/q^q \right) \), then there exists a \( \left(q^{2m} - 1, q^{2m} - 1 - 2m \left( \delta - 1 \right) \left( 1 - 1/q^q \right) \right) \) subsystem code.

Proof: The proof is similar to the proof of the previous corollary, and is a consequence of [5, Theorems 4 and 7] and Proposition 6.

It is straightforward to generalize the previous two corollaries to the case of non-primitive BCH codes using the results given in [4, 9].

One of the disadvantages of the cyclic constructions is that the parameter \( r \) is restricted to values dictated by the possible cardinalities of the sets \( T \cup T^{-1} \) or \( T \cup T^{-q} \), where \( T \) is confined to be a union of cyclotomic cosets. In the next section, we will see how one can overcome this limitation.

We conclude this section by giving some examples of the parameters of subsystem BCH codes in Tables I and II.

### IV. Trading Dimensions of Subsystem and Co-subsystem Codes

In this section we show how one can trade the dimensions of subsystem and co-subsystem to obtain new codes from a given subsystem or stabilizer code. The results are obtained by exploiting the symplectic geometry of the space. A remarkable consequence is that nearly any stabilizer code yields a series of subsystem codes.

Our first result shows that one can increase the dimension of the co-subsystem while keeping or increasing the minimum distance of the subsystem code.

**Theorem 10.** Let \( q \) be a power of a prime \( p \). If there exists an \( \left( n, K, R, d \right)_q \) subsystem code with \( K > p \) that is pure to \( d' \), then there exists an \( \left( n, K/p, \ell R, \geq d \right)_q \) subsystem code that is pure to \( \min \{ d, d' \} \). If a pure \( \left( n, p, R, d \right)_q \) subsystem code exists, then there exists a \( \left( n, 1, pR, d \right)_q \) subsystem code.

**Proof:** By definition, an \( \left( n, K, R, d \right)_q \) Clifford subsystem code is associated with a classical additive code \( C \subseteq \mathbb{F}^n_q \) and its subcode \( D = C \cap C^\perp \) such that \( x = |C|, \ y = |D|, \ K = q^{m}/(xy)^{1/2}, \ R = (x/y)^{1/2}, \ d = \text{swt}(D^\perp - C) \) if \( C \neq D^\perp, \) otherwise \( d = \text{swt}(D^\perp) \) if \( D^\perp = C. \)

We have \( q = p^m \) for some positive integer \( m \). Since \( K \) and \( R \) are positive integers, we have \( x = p^{m+2r} \) and \( y = p^r \) for some integers \( r \geq 1, \) and \( s \geq 0. \) There exists an \( \mathbb{F}_p \)-basis of \( C \) of the form
\[
C = \text{span}_{\mathbb{F}_p} \{ z_1, \ldots, z_s, x_{s+1}, \ldots, x_{s+r}, x_{s+r+1}, \ldots, z_{s+r}\}
\]
that can be extended to a symplectic basis \( \{ x_1, \ldots, x_{nm}, z_{nm} \} \) of \( \mathbb{F}^n_q \), that is, \( \langle x_k \ | \ x_r \rangle_s = 0 \), \( \langle x_k \ | \ z_r \rangle_s = 0 \), \( \langle x_{k+i} \ | \ z_{r+i} \rangle_s = \delta_{k,l} \) for all \( 1 \leq k, l \leq nm \), see [8, Theorem 8.10.1].

Define an additive code
\[
C_m = \text{span}_{\mathbb{F}_p} \{ z_1, \ldots, z_s, x_{s+1}, \ldots, x_{s+r}, x_{s+r+1}, \ldots, z_{s+r+1} \}.
\]

It follows that
\[
C_m = \text{span}_{\mathbb{F}_p} \{ z_1, \ldots, z_s, x_{s+r+2}, \ldots, x_{nm}, z_{nm} \}
\]
and
\[
D = C_m \cap C^\perp = \text{span}_{\mathbb{F}_p} \{ z_1, \ldots, z_s \}.
\]

By definition, the code \( C \) is a subset of \( C_m \).

The subsystem code defined by \( C_m \) has the parameters \( (n, K_m, R_m, d_m) \), where \( K_m = q^{m}/(p^{m+2r+2}p^r)^{1/2} = K/p \) and \( R_m = (p^{m+2r+2}/p^r)^{1/2} = pR \). For the claims concerning minimum distance and purity, we distinguish two cases:

(a) If \( C_m \neq D^\perp, \) then \( K > p \) and \( d_m = \text{swt}(D^\perp - C_m) \geq \text{swt}(D^\perp - C) = d. \) Since by hypothesis \( \text{swt}(D^\perp - C) = d \) and \( \text{swt}(C) \geq d', \) \( D \subseteq C \subset C_m \subseteq D^\perp \) by construction, we have \( \text{swt}(C_m) \geq \min \{ d, d' \} \); thus, the subsystem code is pure to \( \min \{ d, d' \} \).

(b) If \( C_m = D^\perp, \) then \( K_m = 1 = K/p, \) that is, \( K = p; \) it follows from the assumed purity that \( d = \text{swt}(D^\perp - C) = \text{swt}(D^\perp) = d_m \).

This proves the claim.

For \( \mathbb{F}_q \)-linear subsystem codes there exists a variation of the previous theorem which asserts that one can construct the resulting subsystem code such that it is again \( \mathbb{F}_q \)-linear.

**Theorem 11.** Let \( q \) be a power of a prime \( p. \) If there exists an \( \left[ n, k, r, d \right]_q \) subsystem code with \( k > 1 \) that is pure to \( d' \), then there exists an \( \left[ n, k - 1, r + 1, \geq d \right]_q \) subsystem code that is pure to \( \min \{ d, d' \} \). If a pure \( \left[ n, 1, r, d \right]_q \) subsystem code exists, then there exists an \( \left[ n, 1, r, d \right]_q \) subsystem code.

**Proof:** The proof is analogous to the proof of the previous theorem, except that \( \mathbb{F}_q \)-bases are used instead of \( \mathbb{F}_p \)-bases.

### Table II

**Subsystem Codes that are Derived with the Help of the Hermitian Construction**

| Subsystem Code | Parent BCH Code C | Designed distance |
|----------------|-------------------|-------------------|
| [14, 1, 3, 4]_2 | [14, 8, 5]_2      | 6                 |
| [15, 1, 2, 5]_2 | [15, 8, 6]_2      | 6                 |
| [15, 5, 2, 3]_2 | [15, 6, 7]_2      | 7                 |
| [16, 5, 2, 3]_2 | [16, 6, 7]_2      | 7^+               |
| [17, 8, 1, 4]_2 | [17, 5, 9]_2      | 4                 |
| [21, 6, 3, 3]_2 | [21, 9, 7]_2      | 6                 |
| [21, 9, 7]_2    | [21, 8, 9]_2      | 8                 |
| [24, 10, 1, 9]_2| [24, 11, 11]_2    | 8                 |
| [31, 10, 1, 5]_2| [31, 12, 11]_2    | 8                 |
| [32, 20, 1, 3]_2| [32, 6, 15]_2     | 12                |
| [25, 12, 3, 3]_3| [25, 8, 12]_3     | 9                 |
| [26, 6, 2, 5]_3| [26, 8, 13]_3     | 8                 |
| [26, 13, 2, 4]_3| [26, 14, 14]_3    | 9                 |
| [26, 15, 2, 4]_3| [26, 7, 14]_3     | 14                |
| [80, 1, 17, 20]_3| [80, 48, 21]_3    | 21                |
| [80, 1, 17, 20]_3| [80, 46, 22]_3    | 22                |

* punctured code
+ extended code
There exists a partial converse of Theorem 10, namely if the subsystem code is pure, then it is possible to increase the dimension of the subsystem and decrease the dimension of the co-subsystem while maintaining the same minimum distance.

**Theorem 12.** Let \( q \) be a power of a prime \( p \). If there exists a pure \(((n, K, R, d))_q\) subsystem code with \( R > 1 \), then there exists a pure \(((n, pK, R/p, d'))_q\) subsystem code.

**Proof:** Suppose that the \(((n, K, R, d))_q\) Clifford subsystem code is associated with a classical additive code

\[
C_m = \text{span}_{F_p} \{ z_1, \ldots, z_s, x_{s+1}, \ldots, x_{s+r+1}, z_{s+r+1} \}
\]

Let \( D = C_m \cap C_m^\perp \). We have \( x = |C_m| = p^{s+2r+2} \), \( y = |D| = p^s \), hence \( K = q^n/p^r + s \) and \( R = p^{r+1} \). Furthermore,

\[
d = \text{swt}(D^{1+}) \text{.}
\]

The code

\[
C = \text{span}_{F_p} \{ z_1, \ldots, z_s, x_{s+1}, \ldots, x_{s+r}, z_{s+r} \}
\]

has the subcode \( D = C \cap C^{1+} \). Since \( |C| = |C_m|/p^2 \), the parameters of the Clifford subsystem code associated with \( C \) are \(((n, pK, R/p, d'))_q\). Since \( C \subset C_m \), the minimum distance \( d' \) satisfies

\[
d' = \text{swt}(D^{1+} - C) \leq \text{swt}(D^{1+} - C_m) = \text{swt}(D^{1+}) = d.
\]

On the other hand, \( d' = \text{swt}(D^{1+} - C) \geq \text{swt}(D^{1+}) = d \), whence \( d = d' \). Furthermore, the resulting code is pure since \( d = \text{swt}(D^{1+}) = \text{swt}(D^{1+} - C) \).

Replacing \( F_p \)-bases by \( F_q \)-bases in the proof of the previous theorem yields the following variation of the previous theorem for \( F_q \)-linear subsystem codes.

**Theorem 13.** Let \( q \) be a power of a prime \( p \). If there exists a pure \( F_q \)-linear \(((n, k, r, d))_q\) subsystem code with \( r > 0 \), then there exists a pure \( F_q \)-linear \(((n, k + 1, r - 1, d))_q\) subsystem code.

The purity hypothesis in Theorems 12 and 13 is essential, as the next remark shows.

**Remark 14.** The Bacon-Shor code is an impure \(((9, 1, 4, 3))_2\) subsystem code. However, there does not exist any \(((9, 5, 3))_2\) stabilizer code. Thus, in general one cannot omit the purity assumption from Theorems 12 and 13.

An \(((n, k, d))_q\) stabilizer code can also be regarded as an \(((n, k, 0, d))_q\) subsystem code. We record this important special case of the previous theorems in the next corollary.

**Corollary 15.** If there exists an \( (F_q \)-linear) \(((n, k, d))_q\) stabilizer code that is pure to \( d' \), then there exists for all \( r \) in the range \( 0 \leq r < k \) an \( (F_q \)-linear) \(((n, k - r, r, d))_q\) subsystem code that is pure to \( \min\{d, d'\} \). If a pure \( (F_q \)-linear) \(((n, k, r, d))_q\) subsystem code exists, then a pure \( (F_q \)-linear) \(((n, k + r, d))_q\) stabilizer code exists.

We have shown in [4], [5] that a (primitive or non-primitive) narrow sense BCH code of length \( n \) over \( F_q \) contains its dual code if the designed distance \( d \) is in the range

\[
2 \leq d \leq \delta_{\max} = \frac{n}{q^m - 1} (q^{m/2} - 1 - (q - 2)|m \text{ odd}|).
\]

For simplicity, we will proceed our work for primitive narrow sense BCH codes, however, the generalization for non-primitive BCH codes is a straightforward.

**Corollary 16.** If \( q \) is a power of a prime, \( m \) is a positive integer, and \( 2 \leq \delta \leq q^{m/2} - 1 - (q - 2)|m \text{ odd}| \), then there exists a subsystem BCH code with parameters \([q^m - 1, n - 2m[(\delta - 1)(1/q)] - r, \delta]_q\) where \( 0 \leq r < n - 2m[(\delta - 1)(1/q)] \).

**Proof:** We know that if \( 2 \leq \delta \leq q^{m/2} - 1 - (q - 2)|m \text{ odd}| \), then there exists a stabilizer code with parameters \([q^m - 1, n - 2m[(\delta - 1)(1/q)] + \delta]_q\). Let \( r \) be an integer in the range \( 0 \leq r < n - 2m[(\delta - 1)(1/q)] \). From [1, Theorem 2], then there must exist a subsystem BCH code with parameters \([q^m - 1, n - 2m[(\delta - 1)(1/q)] - r, r, \delta]_q\).

We can also construct subsystem BCH codes from stabilizer codes using the Hermitian constructions.

**Corollary 17.** If \( q \) is a power of a prime, \( m \) is a positive integer, and \( \delta \) is an integer in the range \( 2 \leq \delta \leq \delta_{\max} = q^{m/2} - 1 - (q - 2)|m \text{ even}| \), then there exists a subsystem code \( Q \) with parameters \([q^{2m} - 1, q^{2m} - 1 - 2m[(\delta - 1)(1/q^2)] - r, d_Q \geq \delta]_q\) that is pure up to \( \delta \), where \( 0 \leq r < q^{2m} - 1 - 2m[(\delta - 1)(1/q^2)] \).

**Proof:** If \( 2 \leq \delta \leq \delta_{\max} = q^{m/2} - 1 - (q - 2)|m \text{ even}| \), then exists a classical BCH code with parameters \([q^m - 1, q^m - 1 - m[(\delta - 1)(1/q)] + \delta]_q\) which contains its dual code. From [1, Theorem 2], then there must exist a subsystem code with the given parameters.

V. MDS Subsystem Codes

Recall that an \(((n, k, r, d))_q\) subsystem code derived from an \( F_q \)-linear classical code \( C \subseteq F_q^n \) satisfies the Singleton bound \( k + r \leq n - 2d + 2 \), see [13, Theorem 3.6]. A subsystem code attaining the Singleton bound with equality is called an MDS subsystem code.

An important consequence of the previous theorems is the following simple observation which yields an easy construction of subsystem codes that are optimal among the \( F_q \)-linear Clifford subsystem codes.

**Theorem 18.** If there exists an \( F_q \)-linear \(((n, k, d))_q\) MDS stabilizer code, then there exists a pure \( F_q \)-linear \(((n, k - r, r, d))_q\) MDS subsystem code for all \( r \leq 0 \leq k \).

**Proof:** An MDS stabilizer code must be pure, see [20, Theorem 2] or [11, Corollary 60]. By Corollary 15, a pure \( F_q \)-linear \(((n, k, r, d))_q\) stabilizer code implies the existence of an \( F_q \)-linear \(((n, k - r, r, d))_q\) subsystem code that is pure up to \( d \) for any \( r \) in the range \( 0 \leq r \leq k \). Since the stabilizer code is MDS, we have \( k = n - 2d + 2 \). By the Singleton bound, the parameters of the resulting \( F_q \)-linear \(((n, n - 2d + 2 - r, r, d))_q\) subsystem codes must satisfy \( n - 2d + 2 - r \leq n - 2d + 2 \), which shows that the minimum distance \( d_r = d \), as claimed.
Remark 19. We conjecture that \( F_q \)-linear MDS subsystem codes are actually optimal among all subsystem codes, but a proof that the Singleton bound holds for general subsystem codes remains elusive.

In the next corollary, we give a few examples of MDS subsystem codes that can be obtained from Theorem 18. These are the first families of MDS subsystem codes (though sporadic examples of MDS subsystem codes have been established before, see e.g. [3], [6]).

Corollary 20. i) An \( F_q \)-linear pure \([n, n-2d+2-r, r, d]_q \) MDS subsystem code exists for all \( n, d, r \) and \( \delta \) such that \( 3 \leq n \leq q, 1 \leq d \leq n/2+1, \) and \( 0 \leq r \leq n-2d+1 \).

ii) An \( F_q \)-linear pure \([\nu+1]q, (\nu+1)q-2\nu-2r, r, \nu+2]_q \) MDS subsystem code exists for all \( n, \nu, r \) such that \( 0 \leq \nu \leq q-2 \) and \( 0 \leq r \leq (\nu+1)q-2\nu-3 \).

iii) An \( F_q \)-linear pure \([q-1, q-1-2\delta-r, r, \delta+1]_q \) MDS subsystem code exists for all \( r, \delta \) such that \( 0 \leq \delta < (q-1)/2 \) and \( 0 \leq r \leq q-2\delta-1 \).

iv) An \( F_q \)-linear pure \([q, q-2\delta-2-r', r', \delta+2]_q \) MDS subsystem code exists for all \( \delta \leq (q-1)/2 \) and \( 0 \leq r' \leq q-2\delta-2 \).

v) An \( F_q \)-linear pure \([q^2-1, q^2-2\delta-1-r, r, \delta+1]_q \) MDS subsystem code exists for all \( \delta \leq (q-1)/2 \) and \( 0 \leq r \leq q-2\delta-1 \).

vi) An \( F_q \)-linear pure \([q^2, q^2-2\delta-2-r', r', \delta+2]_q \) MDS subsystem code exists for all \( \delta \leq (q-1)/2 \) and \( 0 \leq r' \leq q^2-2\delta-2 \).

Proof: i) By [9, Theorem 14], there exist \( F_q \)-linear \([n, n-2d+2,d]_q \) stabilizer codes for all \( n, d \) such that \( 3 \leq n \leq q \) and \( 1 \leq d \leq n/2+1 \). The claim follows from Theorem 18.

ii) By [21, Theorem 5], there exist a \([\nu+1]q, (\nu+1)q-2\nu-2, r, d+1]_q \) stabilizer code. In this case, the code is derived from an \( F_q \)-linear code \( X \) of length \( n \) over \( F_q \) such that \( X \subseteq X^{1,b} \). The claim follows from Lemma 29 and Theorem 18.

iii), iv) There exist \( F_q \)-linear stabilizer codes with parameters \([q-1, q-2\delta-1, \delta+1]_q \) and \([q, q-2\delta-2, \delta+1]_q \) for \( 0 \leq \delta < (q-1)/2 \). The claim follows from Theorem 18.

v) vi) There exist \( F_q \)-linear stabilizer codes with parameters \([q^2-1, q^2-2\delta-1, \delta+1]_q \) and \([q^2, q^2-2\delta-2, \delta+2]_q \) for \( 0 \leq \delta < q-1 \). The claim follows from Theorem 18.

The existence of the codes in i) are merely established by a non-constructive Gilbert-Varshamov type counting argument. However, the result is interesting, as it asserts that there exist for example \([6, 1, 1, 3]_q \) subsystem codes for all prime powers \( q \geq 7 \). \([7, 1, 2, 3]_q \) subsystem codes for all prime powers \( q \geq 7 \), and other short subsystem codes that one should compare with a \([5, 1, 3]_q \) stabilizer code. If the syndrome calculation is simpler, then such subsystem codes could be of practical value.

The subsystem codes given in ii)-vi) of the previous corollary are constructively established. The subsystem codes in ii) are derived from Reed-Muller codes, and in iii)-vi) from Reed-Solomon codes. There exists an overlap between the parameters given in ii) and in iv), but we list here both, since each code construction has its own merits.

Remark 21. By Theorem 13, pure MDS subsystem codes can always be derived from MDS stabilizer codes, see Table III. Therefore, one can derive in fact all possible parameter sets of pure MDS subsystem codes with the help of Theorem 18.

Remark 22. In the case of stabilizer codes, all MDS codes must be pure. For subsystem codes this is not true, as the \([9, 1, 4, 3]_2 \) subsystem code shows. Finding such impure \( F_q \)-linear \([n, k, r, d]_q \) MDS subsystem codes with \( k + r = \frac{n}{2} + 2 \) is a particularly interesting challenge.

Recall that a pure subsystem code is called perfect if and only if it attains the Hamming bound with equality. We conclude this section with the following consequence of Theorem 18:

Corollary 23. If there exists an \( F_q \)-linear pure \([n, k, d]_q \) stabilizer code that is perfect, then there exists a pure \( F_q \)-linear \([n, k-r, r, d]_q \) perfect subsystem code for all \( r \) in the range \( 0 \leq r \leq k \).

VI. EXTENDING AND SHORTENING SUBSYSTEM CODES

In Section IV, we showed how one can derive new subsystem codes from known ones by modifying the dimension of the subsystem and co-subsystem. In this section, we derive new subsystem codes from known ones by extending and shortening the length of the code.

Theorem 24. If there exists an \((n, K, R, d)_q \) Clifford subsystem code with \( K > 1 \), then there exists an \((n+1, K, R, \geq d)_q \) subsystem code that is pure to 1.

Proof: We first note that for any additive subcode \( X \subseteq F_{2^n} \), we can define an additive code \( X' \subseteq F_{2^n+2} \) by

\[ X' = \{(a \alpha | b) | (a | b) \in X, \alpha \in F_q \} \]
We have $|X'| = q|X|$. Furthermore, if $(c,d) \in X^{±1}$, then $(c\alpha|d\alpha))$ is contained in $(X')^{±1}$ for all $\alpha \in \mathbb{F}_q$, whence $(X')^{±1} \subseteq (X)^{±1}$. By comparing cardinalities we find that equality must hold; in other words, we have

$$(X^{±1})' = (X')^{±1}.$$  

By Theorem 1, there are two additive codes $C$ and $D$ associated with an $((n, K, R, d))_q$ Clifford subsystem code such that

$$|C| = q^n R/K$$

and

$$|D| = |C \cap C^{±1}| = q^n/(KR).$$

We can derive from the code $C$ two new additive codes of length $2n + 2$ over $\mathbb{F}_q$, namely $C'$ and $D' = C' \cap (C^{±1})'$. The codes $C'$ and $D'$ determine a $((n + 1, K', R', d'))_q$ Clifford subsystem code. Since

$$D' = C' \cap (C^{±1})' = (C \cap C^{±1})',$$

we have $|D'| = q|D|$. Furthermore, we have $|C'| = q|C|$. It follows from Theorem 1 that

1. $K' = q^{n+1}/\sqrt{|C'||D'|} = q^n/\sqrt{|C||D|} = K,$
2. $R' = (|C'|/|D'|)^{1/2} = (|C|/|D|)^{1/2} = R,$
3. $d' = \text{swt}((D'\cap C')^1) \geq \text{swt}((D^{±1} \cap C)^1) = d.$

Since $C'$ contains a vector $(00|00)$ of weight 1, the resulting subsystem code is pure to 1.

**Corollary 25.** If there exists an $[[n, k, r, d]]_q$ subsystem code with $k > 0$ and $0 \leq r < k$, then there exists an $[[n + 1, k, r, \geq d]]_q$ subsystem code that is pure to 1.

We can also shorten the length of a subsystem code in a simple way as shown in the following Theorem.

**Theorem 26.** If a pure $((n, K, R, d))_q$ subsystem code exists, then there exists a pure $((n-1, qK, R, d-1))_q$ subsystem code.

**Proof:** By [3, Lemma 10], the existence of a pure Clifford subsystem code with parameters $((n, K, R, d))_q$ implies the existence of a pure $((n, K'R, d'))_q$ stabilizer code. It follows from [11, Lemma 70] that there exists a pure $((n-1, qK'R, d'-1))_q$ stabilizer code, which can be regarded as a pure $((n-1, qK'R, d-1))_q$ subsystem code. Thus, there exists a pure $((n-1, qK, R, d-1))_q$ subsystem code by Theorem 12, which proves the claim.

In bracket notation, the previous theorem states that the existence of a pure $[[n, k, r, d]]_q$ subsystem code implies the existence of a pure $[[n-1, k+1, r, d-1]]_q$ subsystem code.

**VII. COMBINING SUBSYSTEM CODES**

In this section we show how one can obtain a new subsystem code by combining two given subsystem codes in various ways.

**Theorem 27.** If there exists a pure $[[n_1, k_1, r_1, d_1]]_q$ subsystem code and a pure $[[n_2, k_2, r_2, d_2]]_q$ subsystem code such that $k_2 + r_2 \leq n_1$, then there exists subsystem codes with parameters

$$[[n_1 + n_2 - k_2 - r_2, k_1 + r_1 - r, d_2]]_q$$

for all $r$ in the range $0 \leq r < k_1 + r_1$, where the minimum distance $d \geq \min\{d_1, d_2 - k_2 - r_2\}$.

**Proof:** Since there exist pure $[[n_1, k_1, r_1, d_1]]_q$ and $[[n_2, k_2, r_2, d_2]]_q$ subsystem codes with $k_2 + r_2 \leq n_1$, it follows from Theorem 12 that there exist stabilizer codes with the parameters $[[n_1, k_1 + r_1, d_1]]_q$ and $[[n_2, k_2 + r_2, d_2]]_q$ such that $k_2 + r_2 \leq n_1$. Therefore, there exists an $[[n_1 + n_2 - k_2 - r_2, k_1 + r_1, d_2]]_q$ stabilizer code with minimum distance

$$d \geq \min\{d_1, d_2 - k_2 - r_2\}$$

by [7, Theorem 8]. It follows from Theorem 10 that there exists $[[n_1 + n_2 - k_2 - r_2, k_1 + r_1 - r, r, \geq d]]_q$ subsystem codes for all $r$ in the range $0 \leq r < k_1 + r_1$.

**Theorem 28.** Let $Q_1$ and $Q_2$ be two pure subsystem codes with parameters $[[n_1, k_1, r_1, d_1]]_q$ and $[[n_2, k_2, r_2, d_2]]_q$, respectively. If $Q_2 \subseteq Q_1$, then there exists pure subsystem codes with parameters

$$[[2n, k_1 + k_2 + r_1 - r - 2, r, d]]_q$$

for all $r$ in the range $0 \leq r \leq k_1 + k_2 + r_1 + r_2$, where the minimum distance $d \geq \min\{d_1, 2d_2\}$.

**Proof:** By assumption, there exists a pure $[[n_1, k_1, r_1, d_1]]_q$ subsystem code, which implies the existence of a pure $[[n_1 + r_1, d_1]]_q$ stabilizer code by Theorem 12, where $i \in \{1, 2\}$. By [11, Lemma 74], there exists a pure stabilizer code with parameters $[[2n, k_1 + k_2 + r_2 + 1, d_2]]_q$ such that $d' \geq \min\{2d_2, d_1\}$. By Theorem 10, there exist a pure subsystem code with parameters $[[2n, k_1 + k_2 + r_1 + r_2 - r, r, d_1]]_q$ for all $r$ in the range $0 \leq r \leq k_1 + k_2 + r_1 + r_2$, which proves the claim.

Further analysis of propagation rules of subsystem code constructions, tables of upper and lower bounds, and short subsystem codes are presented in [2].

**VIII. CONCLUSION AND DISCUSSION**

Subsystem codes are among the most versatile tools in quantum error-correction, since they allow one to combine the passive error-correction found in decoherence free subspaces and noiseless subsystems with the active error-control methods of quantum error-correcting codes. In this paper we demonstrate several methods of subsystem code constructions over binary and nonbinary fields. The subclass of Clifford subsystem codes that was studied in this paper is of particular interest because of the close connection to classical error-correcting codes. As Theorem 1 shows, one can derive from each additive code over $\mathbb{F}_q$ an Clifford subsystem code. This offers more flexibility than the slightly rigid framework of stabilizer codes.

We showed that any $\mathbb{F}_q$-linear MDS stabilizer code yields a series of pure $\mathbb{F}_q$-linear MDS subsystem codes. These
codes are known to be optimal among the \( F_q \)-linear Clifford subsystem codes. We conjecture that the Singleton bound holds in general for subsystem codes. There is quite some evidence for this fact, as pure Clifford subsystem codes and \( F_q \)-linear Clifford subsystem codes are known to obey this bound. We have established a number of subsystem code constructions. In particular, we have shown how one can derive subsystem codes from stabilizer codes. In combination with the propagation rules that we have derived, one can easily create tables with the best known subsystem codes. Further propagation rules and examples of such tables are given in [2], and will appear in an expanded version of this paper.

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APPENDIX

We recall that the Hermitian construction of stabilizer codes yields \( F_q \)-linear stabilizer codes, as can be seen from the following reformulation of [5, Corollary 2].

**Lemma 29** ([5]), If there exists an \( F_q^2 \)-linear code \( X \subseteq F_q^2 \) such that \( X \subseteq X^⊥ \), then there exists an \( F_q \)-linear code \( C \subseteq F_q^n \) such that \( C \subseteq C^⊥ \). If \( |X| = |C| \), \( \text{swt}(C^⊥ \setminus C) = \text{wt}(X^⊥ \setminus X) \) and \( \text{swt}(C) = \text{wt}(X) \).

**Proof:** Let \( \{1, \beta\} \) be a basis of \( F_q^2/F_q \). Then \( \text{tr}_{q^2/q}(\beta) = \beta + \beta^q \) is an element \( \beta_0 \) of \( F_q \), hence, \( \beta^q = -\beta + \beta_0 \). Let

\[
C = \{(u|v) | u, v \in F_q^n, u + \beta v \in X\}.
\]

It follows from this definition that \( |X| = |C| \) and that \( \text{wt}(X) = \text{swt}(C) \). Furthermore, if \( u + \beta v \) and \( u' + \beta v' \) are elements of \( X \) with \( u, v, u', v' \in F_q^n \), then

\[
0 = (u + \beta v)^q \cdot (u' + \beta v') = u \cdot u' + \beta v \cdot u' + \beta(u \cdot v' - v \cdot u').
\]

On the right hand side, all terms but the last are in \( F_q \), hence we must have \( (u \cdot v' - v \cdot u') = 0 \), which shows that \( (u|v) \bot (u'|v') \), whence \( C \subseteq C^⊥ \). Expanding \( X^⊥ \) in the basis \( \{1, \beta\} \) yields a code \( C' \subseteq C^⊥ \), and we must have equality by a dimension argument. Since the basis expansion is isometric, it follows that

\[
\text{swt}(C^⊥ \setminus C) = \text{wt}(X^⊥ \setminus X).
\]

The \( F_q \)-linearity of \( C \) is a direct consequence of the definition of \( C \).

\[ \Box \]