THE PRODUCT FORMULA FOR REGULARIZED FREDHOLM DETERMINANTS: TWO NEW PROOFS

NIKOLAOS KOUTSONIKOS-KOULOUMPIS AND MATTHIAS LESCH

Abstract. For an $m$-summable operator $A$ in a separable Hilbert space the higher regularized Fredholm determinant $\text{det}_m(I + A)$ generalizes the classical Fredholm determinant. Recently, Britz et al. presented a proof of a product formula

$$\text{det}_m((I + A) \cdot (I + B)) = \text{det}_m(I + A) \cdot \text{det}_m(I + B) \cdot \exp \text{Tr}(X_m(A, B)),$$

where $X_m(A, B)$ is an explicit polynomial in $A, B$ with values in the trace class operators. If $m = 1$ then $X_1(A, B) = 0$, hence the formula generalizes the classical determinant product formula.

One of the purposes of this note is to present two very simple alternative proofs of the formula. The first proof is a priori analytic and makes use of the fact that $z \mapsto \text{det}_m(I + zA)$ is holomorphic, while the second proof is completely algebraic. The algebraic proof has, in our opinion, some interesting aspects in its own about the trace and commutators.

Secondly, we extend the above mentioned formula to several factors

$$\text{det}_m \left( \prod_{l=1}^r (I + A_l) \right) = \left( \prod_{l=1}^r \text{det}_m(I + A_l) \right) \cdot \exp \text{Tr}(X_{m,r}(A_1, \ldots, A_r)).$$

The latter is more than just a straightforward generalization as we will gain more insights into the combinatorics behind it. Also we will present an algebraized version of the analytic proof in the language of formal power series. The upshot is that the two identities are just combinatorial in nature.

Contents

1. Introduction 2

Part 1. The case of 2 factors 7

2. An analytic proof of the product formula 7

3. Algebraic approach to the product formula 8

Part 2. Generalization to the multi-factor case 12

4. Algebraic and combinatorial preparations 12

5. Reduction to a combinatorial problem in the trace class case 18

6. Two proofs of Theorem 11 (2) 19

References 19

2020 Mathematics Subject Classification. Primary: 47B10; Secondary: 47B02.

Key words and phrases. Trace ideals, regularized Fredholm determinant, determinant product formula.

This paper is based on the first named author’s Master’s thesis [KK21] written under the supervision of the second named author. Both authors gratefully acknowledge the support of the Bonn International Graduate School (BIGS).
1. Introduction

1.1. Notations and conventions. We denote by $\mathbb{N} = \{0, 1, 2, \ldots\}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ the natural numbers, integers, real and complex numbers resp. The cardinality of a set $\xi$ is denoted by $|\xi|$.

In Part 2 we will also make use of multiindex notation. For a multiindex $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$ we write $|\alpha| = \alpha_1 + \ldots + \alpha_r$.

When dealing with noncommuting variables, product notation means that the product is taken in the given order, i.e. $\prod_{i=1}^{N} x_i = x_1 \cdot \ldots \cdot x_N$ in this order.

Concerning Hilbert spaces and trace ideals our standard references are [Sim05] and [Ped89]. Hilbert spaces will be denoted by calligraphic letters, e.g. $\mathcal{H}$. $\mathcal{B}(\mathcal{H}), \mathcal{K}(\mathcal{H})$ denote the algebra of bounded resp. compact linear operators on $\mathcal{H}$. $\text{Tr} = \text{Tr}_{\mathcal{H}}$ denotes the trace on nonnegative operators in $\mathcal{B}(\mathcal{H})$ resp. on all operators in the trace ideal $\mathcal{B}^1(\mathcal{H})$. For a compact operator $K \in \mathcal{K}(\mathcal{H})$ we adopt the convention of Simon’s book [Sim05] and denote by $\lambda_n(K), 1 \leq n \leq N \in \mathbb{N} \cup \{\infty\}$, the $n$-th nonzero eigenvalue (in some enumeration with $|\lambda_n|$ in decreasing order, counted with multiplicity) of $K$. $N$ is a natural number or $\infty$.

By $\mathcal{B}^p(\mathcal{H})$ we denote the von Neumann–Schatten ideal of $p$–summable operators. Recall that $T \in \mathcal{B}(\mathcal{H})$ is $p$–summable if

$$\text{Tr}(|T|^p) = \text{Tr}((T^*T)^{p/2}) = \sum_{n=1}^{\infty} \lambda_n(T^*T)^{p/2} < \infty.$$  

1.2. Main result. This paper is inspired by the recent work [BCG+21] of Britz et al. on the product formula for regularized Fredholm determinants. Namely, the classical theory of trace ideals (see [Sim05, Sim77]) but see also the more historical quotes after (1.3) in [BCG+21]) allows to generalize the notion of Fredholm determinant to operators which differ from the identity by a $p$–summable operator. More concretely, let first $A$ be a trace class operator in a separable Hilbert space $\mathcal{H}$. Then

$$\text{det}_\mathcal{F}(I + A) = \sum_{k=0}^{\infty} \text{Tr}(A^k) = \prod_{n=1}^{N} (1 + \lambda_n(A))$$  

is called the Fredholm determinant of $I + A$, sometimes the latter being called of determinant class. The Fredholm determinant retains two important properties of the ordinary Linear Algebra determinant. Firstly, there is a product formula

$$\text{det}_\mathcal{F}((I + A) \cdot (I + B)) = \text{det}_\mathcal{F}(I + A) \cdot \text{det}_\mathcal{F}(I + B),$$

(1.2)

to which we come back in due course and secondly the function $z \mapsto \text{det}_\mathcal{F}(I + A)$ is an entire function of genus 0 [Ahl78, Chap. 5, Sec. 2.3] with zeros exactly in $-1/\lambda_n(A)$, where $\lambda_n(A), 1 \leq n \leq N$, are the nonzero eigenvalues of $A$ counted with multiplicity as explained above in Section 1.1. The second property supports the intuition that $z \mapsto \text{det}(I + z \cdot A)$ plays the role of the “characteristic polynomial” of $A$.

If $A \in \mathcal{B}^m(\mathcal{H})$ is only $m$–summable, $m \in \mathbb{N}, m \geq 2$, then the product (1.1) will not converge. One therefore employs the classical Weierstrass method of convergence generating factors. Namely, it turns out that the operator

$$(I + A) \cdot \exp\left(\sum_{j=1}^{m-1} \frac{(-A)^j}{j}\right)$$

(1.3)

is an entire function of genus 0 [Ahl78, Chap. 5, Sec. 2.3] with zeros exactly in $-1/\lambda_n(A)$, where $\lambda_n(A), 1 \leq n \leq N$, are the nonzero eigenvalues of $A$ counted with multiplicity as explained above in Section 1.1. The second property supports the intuition that $z \mapsto \text{det}(I + z \cdot A)$ plays the role of the “characteristic polynomial” of $A$.

---

*Of course, if $N < \infty$ it is even a polynomial.*
is of determinant class, i.e. differs from the identity by a trace class operator and hence one obtains a regularized version of the Fredholm determinant by putting

$$
\det_m(I + A) := \det \left( (I + A) \cdot \exp \left( \sum_{j=1}^{m-1} \frac{(-A)^j}{j} \right) \right)
$$

$$
= \prod_{n=1}^{N} \left( (1 + \lambda_n) \cdot \exp \left( \sum_{j=1}^{m-1} \frac{(-\lambda_n)^j}{j} \right) \right), \quad \lambda_n = \lambda_n(A).
$$

(1.4)

Note that the function of \( z \)

$$
\det_m(I + z \cdot A) = \prod_{n=1}^{N} \left( (1 + z \cdot \lambda_n) \cdot \exp \left( \sum_{j=1}^{m-1} \frac{(-\lambda_n z)^j}{j} \right) \right)
$$

(1.5)

is a canonical Weierstraß product of genus \( m - 1 \) [AHL78, Chap. 5, Sec. 2.3] with zeros exactly in \(-1/\lambda_n\), hence it is an entire function of order at most \( m \), cf. also [HALE22, Sec. 4]. Of this we will only use that the function is holomorphic in a neighborhood of \( z = 0 \).

The product formula, however, is more involved.

**Theorem 1** (Product Formula [BCG+21, Theorem 1.1]). There exist explicit (see below) polynomials \( X_m(a, b) \) over \( \mathbb{Q} \) in the free polynomial algebra \( \mathbb{Q}(a, b) \) in two noncommuting indeterminates \( a, b \) such that the following holds:

Let \( \mathcal{H} \) be a separable Hilbert space and let \( A, B \in \mathcal{B}(\mathcal{H}), m \in \mathbb{N}, m \geq 1 \), be \( m \)-summable operators. Then \( X_m(A, B) \) is of trace class and one has

$$
\det_m \left( (I + A) \cdot (I + B) \right) = \det_m(I + A) \cdot \det_m(I + B) \cdot \exp \text{Tr}(X_m(A, B)).
$$

(1.6)

**Remark. 1.** Theorem 1 is essentially a reformulation of [BCG+21, Theorem 1.1]. But there are predecessors. As pointed out to us by Rupert Frank, Hansmann [HAN10, Lemma 1.5.10] considered the case where \( A \) (or \( B \)) is of finite rank.

2. Of course for an element \( f(a, b) \in \mathbb{Q}(a, b) \) we denote by \( f(A, B) \) the operator obtained by inserting \( A \) for \( a \) and \( B \) for \( b \).

To describe the polynomials \( X_m(a, b) \) we need to introduce one more notation from [BCG+21]. For a subset \( \xi \subset \{1, \ldots, j\} \) one puts

$$
y_{\xi,j}^k(a, b) := \begin{cases} ab, & k \in \xi, \\ a + b, & k \notin \xi, \end{cases} \quad y_{\xi,j}(a, b) := \prod_{k=1}^{j} y_{\xi,j}^k(a, b).
$$

(1.7)

Then

$$
(a + b + ab)^j = \sum_{\xi \subset \{1, \ldots, j\}} y_{\xi,j}(a, b).
$$

(1.8)

**Proposition 2.** The polynomial \( X_m(a, b) \) in Theorem 1 is explicitly given by

$$
X_m(a, b) = \sum_{j=1}^{m-1} \frac{(-1)^j}{j} \sum_{\xi \subset \{1, \ldots, j\} \atop |\xi|+j \geq m} y_{\xi,j}(a, b).
$$

(1.9)

**Remark.** Note that \( y_{\xi,j}(a, b) \) is a product of \(|\xi| + j \) factors. Hence \( y_{\xi,j}(A, B) \in \mathcal{B}^{m}(\mathcal{H}) \) if \( A, B \in \mathcal{B}^{m}(\mathcal{H}) \) and \(|\xi| + j \geq m\). We will from now on reserve the notation \( X_m(a, b) \) for the right hand side of (1.9). It is then to be proved that (1.6) holds.

Furthermore, \( X_m(a, b) \) has degree \( 2m - 2 \) and furthermore each monomial in \( X_m(a, b) \) has degree at least \( m \). Introducing another indeterminate \( t \) over \( \mathbb{Q} \) which commutes with
a and b this could equivalently be expressed in the algebra $\mathbb{Q}(a, b)[t]$ as $X_m(ta, tb) = O(t^m)$. This observation will play a prominent role below.

Finally, we note that when comparing with [BCG+21] one needs to take into account that we have a different sign convention due to the fact that we consider $\det_m(I + A)$ instead of $\det_m(I - A)$.

1.3. Reduction to a combinatorial problem in the trace class case. The following reduction is the same as in the first part of the proof of Theorem 3.1 in [BCG+21]. To be self-contained we briefly explain this here. Thereafter we will be able to formulate the result for which we will be giving two independent proofs further below.

1.3.1. First reduction to the trace class case. The first reduction is rather trivial. Namely, suppose the product formula in Theorem 1 is proved for $A, B$ of trace class. Since the monomials occurring in $X_m(A, B)$ are at least of degree $m$ it follows that $\text{Tr}(X_m(A, B))$ is a continuous function on $\mathcal{B}^m(\mathcal{H}) \times \mathcal{B}^m(\mathcal{H})$. Consequently, both sides of the product formula depend continuously on $A, B \in \mathcal{B}^m(\mathcal{H})$. Since $\mathcal{B}^1(\mathcal{H})$ is dense in $\mathcal{B}^m(\mathcal{H})$ it therefore suffices to prove the product formula for $A, B \in \mathcal{B}^1(\mathcal{H})$ of trace class.

1.3.2. Second reduction. Thus suppose that $A, B \in \mathcal{B}^1(\mathcal{H})$. Then by (1.3) and [Sti77, Theorem 6.2]

$$\det_m(I + A) = \det_{\mathcal{F}}(I + A) \cdot \exp \text{Tr}\left(\sum_{j=1}^{m-1} \frac{(-A)^j}{j}\right),$$

which implies that

$$\det_m((I + A) \cdot (I + B)) = \det_m(I + A + B + AB) = \det_m(I + A) \cdot \det_m(I + B) \cdot \exp \text{Tr}(\tilde{X}_m(A, B)), \tag{1.10}$$

with, see (1.8),

$$\tilde{X}_m(A, B) = \sum_{j=1}^{m-1} \frac{(-1)^j}{j} ((A + B + AB)^j - A^j - B^j) = \sum_{j=1}^{m-1} \frac{(-1)^j}{j} \left( \sum_{\xi \in \{1, \ldots, j\}} y_{\xi,j}(A, B) - A^j - B^j \right). \tag{1.11}$$

Hence for $A, B$ of trace class it is rather simple to derive a product formula (i. e. (1.10)) and hence an explicit candidate for the correction factor on the right hand side of (1.6). The problem, however, is that if $A, B \in \mathcal{B}^m(\mathcal{H})$ then $\tilde{X}_m(A, B)$ is not even well-defined as it contains monomials of degree $< m$. Those monomials are not of trace class if $A, B$ are only in $\mathcal{B}^m(\mathcal{H})$.

The heart of the proof of the product formula therefore is the following Theorem.

**Theorem 3.1.** For $A, B \in \mathcal{B}^1(\mathcal{H})$ we have

$$\text{Tr}(X_m(A, B)) = \text{Tr}(\tilde{X}_m(A, B)). \tag{1.12}$$

2. In the free polynomial $\mathbb{Q}$-algebra $\mathbb{Q}(a, b)$ generated by two noncommuting indeterminates $a, b$ the difference $\tilde{X}_m(a, b) - X_m(a, b)$ is a sum of commutators.

(1.10), (1.12) and the reduction to the trace class case Section 1.3.1 imply Theorem 1 (and Proposition 2). Clearly, the second part of Theorem 3 implies the first. Theorem 3 is implicit in [BCG+21, Lemma 2.4].
This is the point where our exposition deviates from the source. As announced before we will present two independent proofs of Theorem 3 and hence of the Product Formula Theorem 1.

For future reference we put $Z_m(ta, tb) = X_m(ta, tb) - X_m(ta, tb)$, explicitly

$$Z_m(ta, tb) = \tilde{X}_m(ta, tb) - X_m(ta, tb)$$

$$= \sum_{j=1}^{m-1} \left[ \frac{(-t)^j}{j!} \left( (a + b + t ab)^j - a^j - b^j - \sum_{\xi \subseteq \{1, \ldots, j\}} \sum_{|\xi| \geq m} y_{\xi, j}(a, b) t^{|\xi|} \right) \right]$$

$$= \sum_{j=1}^{m-1} \frac{(-t)^j}{j!} \left( \sum_{\xi \subseteq \{1, \ldots, j\}} \sum_{|\xi| \geq m} y_{\xi, j}(a, b) t^{|\xi| - a^j - b^j} \right),$$

where again (1.8) was used. The $t$ is just a convenient marker to count orders.

1.4. The multi-factor case. The generalization to several factors of Theorem 1 now reads as follows

**Theorem 4.** There exist explicit polynomials $X_{m,r}(a_1, \ldots, a_r)$ in the free polynomial algebra $\mathbb{Q}(a_1, \ldots, a_r)$ over $\mathbb{Q}$ in $r$ noncommuting indeterminates $a_1, \ldots, a_r$ such that the following holds:

1. $X_{m,r}$ is of degree at most $r \cdot (m - 1)$.
2. All monomials occurring in $X_{m,r}$ have degree at least $m$.
3. Let $\mathcal{H}$ be a separable Hilbert space and let $A_1, \ldots, A_r \in \mathcal{B}^m(\mathcal{H})$, $m \in \mathbb{N}, m \geq 1$, be $m$-summable operators. Then $X_{m,r}(A_1, \ldots, A_r)$ is of trace class and one has

$$\det_m \left( \prod_{l=1}^{r} (I + A_l) \right) = \left( \prod_{l=1}^{r} \det_m (I + A_l) \right) \cdot \exp(\text{Tr}(X_{m,r}(A_1, \ldots, A_r))).$$

**Remark.**

1. We learned from Rupert Frank that in [Fra18, Lemma C.1] he proves a formula for three factors if one of the operators is of finite rank. He kindly suggested to us to consider the multi-factor case which is gratefully acknowledged.

2. We emphasize that the second condition is crucial for $X_{m,r}(A_1, \ldots, A_r)$ being of trace class.

3. The explicit formula for $X_{m,r}$ will be given after some preparations in (4.10) in Section 4 in the text below.

4. The proof in the multi-factor case requires a bit more algebraic and combinatorial preparations which make the proof appear unduly lengthy. We emphasize, however, that both the algebraic as well as the combinatorial facts outlined in some detail are very elementary and that the basic idea of proof is very simple as the short final section shows.

1.5. The paper is organized as follows. In the first part we present the two-factor case. This is basically the first version of this paper which was put on arXiv in Feb. 2022. In this part we give two independent proofs of the product formula, one using a tiny bit of complex analysis and the other being completely algebraic.

In the second part we present the multi-factor case. This requires a bit more algebraic preparations which are given in Section 4. Section 5 then reduces the problem to a combinatorial problem for trace class operators. Finally, in Section 6 we present again

*Recall from the Remark above that we work in $\mathbb{Q}(a, b)\{t\}$, where $t$ is an auxiliary indeterminate commuting with $a$ and $b$. 
two proofs of the multi-factor case. The two proofs are formally analogues of the two proofs given in the two-factor case. However, both proofs are formulated in the language of formal power series without reference to complex analysis.

In principle the two parts are independent and could have been either presented as paper I and paper II or they could have been more streamlined into one slightly shorter paper dealing with the most general case only. However, we wanted to keep the original flavor and therefore chose the current presentation.

1.6. Acknowledgment. The first part of this paper is based on the first named author’s Master’s thesis [KK21] written under the supervision of the second named author within the Master’s program of the Mathematical Institute of the University of Bonn. A first version on the two-factor case was put on arxiv as arXiv:2202.12923v1. To consider the multi-factor case was suggested to us by Rupert Frank whose insight and interest is appreciated.
Part 1. The case of 2 factors

**Summary.** In this first part we present our two new proofs of Theorem 1, cf. [BCG+21, Theorem 1.1].

## 2. An analytic proof of the product formula

### 2.1. Preliminaries.

The following Lemma can be found between the lines in [Sim77] and it can even be traced back to Poincare [Sim77, Second paragraph in Sec. 6].

**Lemma 5.** Let $A \in \mathcal{B}(H)$ with $\|A\| < 1$. Then the series $\sum_{j=m}^{\infty} \frac{(-A)^j}{j}$ converges in the trace norm and

$$
\det_m(I + A) = \exp\left(-\sum_{j=m}^{\infty} \frac{\text{Tr}(-A)^j}{j}\right).
$$

(2.1)

Consequently, $A \mapsto \sum_{j=m}^{\infty} \frac{(-A)^j}{j}$ is an analytic mapping.

$$
\{A \in \mathcal{B}(H) \mid \|A\| < 1\} \rightarrow \mathcal{B}(H).
$$

Furthermore, for any $A \in \mathcal{B}(H)$ the entire function $z \mapsto \det_m(I + z \cdot A)$ satisfies

$$
\det_m(I + z \cdot A) = 1 + O(z^m), \quad \text{as } z \to 0.
$$

(2.2)

**Proof.** This follows immediately from the inequality

$$
|\text{Tr}(-A)^j| \leq \|A\|^j_1 \leq \|A\|^m \|A\|^{j-m}, \quad \text{for } j \geq m,
$$

and, cf. (1.4),

$$(I + A) \cdot \exp\left(\sum_{j=1}^{m-1} \frac{(-A)^j}{j}\right) = \exp\left(\sum_{j=1}^{\infty} \frac{(-A)^j}{j}\right) \cdot \exp\left(\sum_{j=m}^{\infty} \frac{(-A)^j}{j}\right)
$$

$$
= \exp\left(\sum_{j=m}^{\infty} \frac{(-A)^j}{j}\right).
$$

Clearly, the formula now gives (again using [Sim77, Theorem 6.2]) for arbitrary $A \in \mathcal{B}(H)$ and $z$ in a sufficiently small neighborhood of 0 (i.e. such that $|z||A\| < 1$)

$$
\det_m(I + z \cdot A) = \exp\left(-\sum_{j=m}^{\infty} \frac{\text{Tr}(-A)^j}{j}z^j\right) = 1 + O(z^m), \quad \text{as } z \to 0.
$$

(2.3)

As remarked in the Introduction after (1.5), $z \mapsto \det_m(I + z \cdot A)$ is a canonical Weierstrass product of genus $m$, hence entire. □

### 2.2. Proof of Theorem 3.1.

It now follows from Lemma 5 that, replacing $A$ by $z \cdot A$, and $B$ by $z \cdot B$, the left hand side of (1.10) satisfies

$$
\det_m((I + z \cdot A) \cdot (I + z \cdot B)) = 1 + O(z^m), \quad \text{as } z \to 0,
$$

and the first two-factors on the right hand side of (1.10) satisfy

$$
\det_m(I + z \cdot A) = 1 + O(z^m), \quad \det_m(I + z \cdot B) = 1 + O(z^m)
$$

as well. Thus

$$
\exp\left(\text{Tr}\left(\tilde{X}_m(zA,zB)\right)\right) = 1 + O(z^m), \quad \text{as } z \to 0.
$$
Taking the principal branch of log on both sides we find
\[
\text{Tr}(\tilde{X}_m(zA, zB)) = O(z^m), \text{ as } z \to 0,
\]
and further, since each monomial in \(X_m(zA, zB)\) has degree \(\geq m\) we have
\[
\text{Tr}(X_m(zA, zB)) = O(z^m)
\]
on the nose. Taking differences we therefore find (see (1.13))
\[
\text{Tr}(Z_m(zA, zB)) = O(z^m), \text{ as } z \to 0.
\]
But \(\text{Tr}(Z_m(zA, zB))\) is a polynomial of degree \(\leq m - 1\) in \(z\). Being \(O(z^m)\) means it must vanish. This proves Theorem 3.1 and hence the first proof of Theorem 1 is finished. □

**Remark.** We emphasize that the decisive (easy) fact which is needed for proving the product formula is that \(\det_m(I + z \cdot A) = 1 + O(z^m)\) or equivalently that
\[
\log \det_m(I + z \cdot A) = O(z^m), \text{ as } z \to 0. \tag{2.4}
\]

## 3. Algebraic approach to the product formula

### 3.1. Elementary commutative algebra considerations

We elaborate here a bit more than would be absolutely necessary to prove the second part of Theorem 3. Rather we indulge a little into elementary commutative algebra considerations.

Let \(K\) be a field of characteristic 0 and let \(A\) be a unital \(K\)-algebra. By \(CA := [A, A] \subset A\) we denote the commutator **subspace**, i.e. the linear span of commutators \([a, b] := ab - ba, a, b \in A\). \([A, A]\) is a module over the **center** \(ZA\) of \(A\).

By \(\tau_A\) we denote the quotient map (the “trace”)
\[
\tau_A : A \to A/\mathbb{C}A :=: \mathcal{Q}A.
\]
A priori \(\mathcal{Q}A\) is only a \(K\)-vector space. E.g. in the case of \(A = \mathcal{B}(H)\) we have the commutative diagram
\[
\begin{array}{c}
A \xrightarrow{\tau_A} \mathcal{Q}A \\
\downarrow \text{Tr} \quad \downarrow \text{Tr} \quad \downarrow \text{Tr} \\
\mathbb{C}.
\end{array}
\]

**Lemma 6.1.** \([A, A]\) and \(\mathcal{Q}A\) are natural \(ZA\)-modules and \(\tau_A\) is a \(ZA\)-module map satisfying
\[
\tau_A(a \cdot b) = \tau_A(b \cdot a) \text{ for all } a, b \in A.
\]

1. If \(D\) is a derivation of \(A\) then \(D\) descends to a \(ZA\)-derivation \(^3\) \(\bar{D}\) of \(\mathcal{Q}A\) such that the following diagram commutes
\[
\begin{array}{c}
A \xrightarrow{\tau_A} \mathcal{Q}A \\
\downarrow \tau_A \quad \downarrow D \quad \downarrow \bar{D} \\
\mathcal{Q}A \xrightarrow{\tau_A} \mathcal{Q}A.
\end{array}
\]

**Proof.** 1. follows immediately from \(z[a, b] = [za, zb] = [a, zb]\) for \(a, b \in A\) and \(z \in ZA\). Clearly, \(\tau_A(a \cdot b) = \tau_A(b \cdot a)\) holds by construction.

2. \(D[a, b] = [Da, b] + [a, Db]\), hence \(D\) leaves \([A, A]\) invariant. Furthermore, for \(z \in ZA\) and \(a \in A\)
\[
(Dz)a = D(za) - zDa = D(az) - [Da]z = aDz,
\]
hence \(D\) leaves \(ZA\) invariant as well. Consequently, \(\bar{D}\) is a well-defined \(ZA\)-derivation on the quotient \(\mathcal{Q}A\). □

\(^3\)More precisely, \(\bar{D}\) is a linear endomorphism of the \(ZA\)-module \(\mathcal{Q}A\) satisfying \(\bar{D}(za) = D(z) \cdot a + z \cdot \bar{D}a\) for \(z \in ZA\) and \(a \in \mathcal{Q}A\).
For a polynomial \( f(t) \in \mathbb{K}[t] \) and \( a \in \mathcal{A} \) we denote by \( f(a) \in \mathcal{A} \) the element obtained by replacing \( t \) by \( a \) (the “insertion homomorphism”). Similarly, if \( f(t), g(t) \in \mathbb{K}[t] \) and \( g(a) \) is invertible in \( \mathcal{A} \) then
\[
(f/g)(a) := f(a)g(a)^{-1} = g(a)^{-1}f(a).
\]
If \( \tilde{f}(t), \tilde{g}(t) \in \mathbb{K}[t] \) with \( \tilde{g}(a) \) invertible and \( (f/g)(t) = (\tilde{f}/\tilde{g})(t) \) then indeed \( (f/g)(a) = (\tilde{f}/\tilde{g})(a) \). By \( \partial_t \) we denote the usual derivation on polynomial algebras with indeterminate \( t \) obtained by formal differentiation by \( t \). For \( f(t) \in \mathbb{K}[t] \) we also write \( f'(t) \) for \( \partial_t f(t) \).

**Lemma 7.** Let \( a, x \in \mathcal{A} \) with \([a, x] = 0\) and \( f(t), g(t) \in \mathbb{K}[t] \) with \( g(a) \) invertible. Then
\[
\tau_\mathcal{A}(x \cdot D(\frac{f}{g}(a))) = \tau_\mathcal{A}(x \cdot (\frac{f}{g})'(a) \cdot Da),
\]
in particular
\[
\tilde{D}\tau_\mathcal{A}(\frac{f}{g}(a)) = \tau_\mathcal{A}((\frac{f}{g})'(a) \cdot Da).
\]

**Proof.** For a monomial \( a^n \) we have
\[
x \cdot D(a^n) = \sum_{j=0}^{n-1} x \cdot a^j \cdot (Da) \cdot a^{n-1-j},
\]
thus since \( x \) commutes with \( a \)
\[
\tau_\mathcal{A}(x \cdot D(a^n)) = \tau_\mathcal{A}(x \cdot na^{n-1} \cdot Da),
\]
thus the first claim for monomials. By linearity it follows for polynomials \( f(t) \in \mathbb{K}[t] \).

For \( g(t) \in \mathbb{K}[t] \) with \( g(a) \) invertible we infer \(^4\) from applying \( D \) to \( I_\mathcal{A} = g(a) \cdot g(a)^{-1} \) that
\[
D(g(a)^{-1}) = -g(a)^{-1} \cdot (Dg(a)) \cdot g(a)^{-1}
\]
hence
\[
\tau_\mathcal{A}(x \cdot D(\frac{f}{g}(a))) = \tau_\mathcal{A}(x \cdot (\frac{f}{g})'(a) \cdot Da).
\]
Since \( xg(a)^{-1} \) and \( xf(a)g(a)^{-2} \) commute with \( a \) we may apply the proven part to the polynomials \( f(t), g(t) \) and obtain further
\[
\tau_\mathcal{A}(x \cdot (\frac{f}{g})'(a) \cdot Da).
\]
The second claim now follows with \( x = 1_\mathcal{A} \) together with the commutative diagram in Lemma 6. □

### 3.2. Application to the product formula.
We now apply the previous considerations to the free unital polynomial \( \mathbb{K} \)-algebra \( \mathcal{A} = \mathbb{K}(a, b) \) on two noncommuting variables \( a, b \).

Let further \( t \) be an indeterminate commuting with \( a, b \) and consider the polynomial algebra \( \mathcal{A}[t] = \mathbb{K}(a, b)[t] \). On \( \mathcal{A}[t] \) we have the derivation \( D := \frac{\partial}{\partial t} \) sending \( t^k \) to \( kt^{k-1} \). Furthermore, we fix \( m \in \mathbb{N}, m \geq 1 \). Note that for \( x \in \mathcal{A} \) the polynomial \( 1 + xt \) is invertible in the quotient \( \mathcal{A}[t]/(t^m) \) with inverse \(^5\)

\(^4\)The Leibniz-rule implies \( D(1_\mathcal{A}) = D(1_\mathcal{A}) + D(1_\mathcal{A}) = 0 \).

\(^5\)Alternatively, one could work in the formal power series in \( t \) with coefficients in \( \mathcal{A} \) where \( 1 + xt \) is invertible on the nose, cf. Part 2.
\[ (1 + xt + \mathcal{O}(t^m))^{-1} = \sum_{j=0}^{m-1} (-1)^j x^j t^j + \mathcal{O}(t^m), \]

where \( \xi + \mathcal{O}(t^m) \) stands for the class of \( \xi \mod t^m \).

As before we denote by \( \tau_{A[t]} : A[t] \to \Omega(A[t]) \) the quotient map. It is elementary to check that the spaces \( \Omega(A[t]) \) and \( \Omega(A)[t] \) are isomorphic (as \( \mathbb{K}\)-vector spaces resp. \( \mathbb{Z}\)-modules) via the correspondence
\[
\tau_{A[t]}(a_0 + a_1 t + ... + a_n t^n) \leftrightarrow \tau_{A}(a_0) + \tau_{A}(a_1)t + ... + \tau_{A}(a_n)t^n.
\]

Recall from Proposition 2 and Section 1.3 the definition of \( X_m(ta, tb), \tilde{X}_m(ta, tb) \), and \( Z_m(ta, tb) \). From the isomorphism (3.2) and the fact that \( X_m(ta, tb) = \mathcal{O}(t^m) \) the claim 2. in Theorem 3 is equivalent to
\[
\tau_{A[t]}(\tilde{X}_m(ta, tb)) = \mathcal{O}(t^m),
\]
meaning that the left hand side is a polynomial in \( t \) which is divisible by \( t^m \). Since \( \tilde{X}(ta, tb) \) has no constant term (3.3) is equivalent to
\[
\partial_1 \tau_{A[t]}(\tilde{X}_m(ta, tb)) = \mathcal{O}(t^{m-1}).
\]

To see this we introduce the polynomial
\[
f(x) := \sum_{j=1}^{m-1} \frac{(-1)^j}{j} x^j \in \mathbb{K}[x].
\]

Clearly \( \mod x^{m-1} \) we find
\[
f'(x) + \mathcal{O}(x^{m-1}) = -\sum_{j=0}^{m-2} \frac{(-x)^j}{j} \mathcal{O}(x^{m-1}) = -(1 + x + \mathcal{O}(x^{m-1}))^{-1}.
\]

By slight abuse of notation, to save space, we will write \((1 + x)^{-1} \mod x^{m-1}\) for \((1 + x + \mathcal{O}(x^{m-1}))^{-1}\). With \( f \) we have
\[
\tilde{X}_m(ta, tb) = f(ta + tb + t^2ab) - f(ta) - f(tb),
\]
and applying Lemma 7 we find
\[
\partial_1 \tau_{A[t]}(\tilde{X}_m(ta, tb)) \mod t^{m-1} = \tau_{A[t]}(f'(ta + tb + t^2ab)(a + b + 2tab) - f'(ta)a - f'(tb)b) = \tau_{A[t]}(f'(ta + tb + t^2ab)(a + b + 2tab) - f'(ta)a - f'(tb)b).
\]

We note that
\[
f'(ta + tb + t^2ab) \mod t^{m-1} = -(1 + ta)^{-1}(1 + ta)^{-1} \mod t^{m-1}
\]
\[
a + b + 2tab = (1 + ta)b + a(1 + tb),
\]
\[
f'(ta)a \mod t^{m-1} = -(1 + ta)^{-1}a \mod t^{m-1},
\]
\[
f'(tb)b \mod t^{m-1} = -(1 + tb)^{-1}b \mod t^{m-1}.
\]

Plugging this into the previous expression (3.5) and exploiting that \( \tau_{A[t]} \) vanishes on commutators implies \( \partial_1 \tau_{A[t]}(\tilde{X}_m(ta, tb)) \mod t^{m-1} = 0 \) and hence the claim.
3.3. **Comparison to [BCG+21]**. We briefly discuss how the decisive combinatorial Lemma [BCG+21, Lemma 2.3], which a priori seems more general than Theorem 2., can be obtained quite easily from our method as well.

Denote by $\Pi_j$ the set of partitions $\pi = (\pi_1, \pi_2, \pi_3)$ of the set $\{1, \ldots, j\}$ into three subsets and by $\Pi_{j,k_1,k_2}$ the set of those $\pi = (\pi_1, \pi_2, \pi_3) \in \Pi_j$ with $|\pi_1| + |\pi_3| = k_1$ and $|\pi_2| + |\pi_3| = k_2$. Furthermore put

$$z_{\pi,k}(a, b) := \begin{cases} a, & k \in \pi_1, \\ b, & k \in \pi_2, \\ ab, & k \in \pi_3, \end{cases} \quad z_\pi(a, b) := \prod_{k=1}^j z_{\pi,k}(a, b). \quad (3.6)$$

$\pi_3$ plays the role of $\xi$ in (1.7). With this notation one has

$$y_{\xi,j}(a, b) = \sum_{\pi \in \Pi_j \atop \pi_3 = \xi} z_\pi(a, b)$$

and further, see (1.13)

$$Z_m(a, b) = \sum_{j=1}^{m-1} \frac{(-1)^j}{j} \left( \sum_{\pi \in \Pi_j} \sum_{j+|\pi_3| < m} z_\pi(a, b) - a^j - b^j \right),$$

$$= \sum_{k_1,k_2 > 0 \atop k_1 + k_2 < m} \sum_{j=1}^{k_1 + k_2} \frac{(-1)^j}{j} \sum_{\pi \in \Pi_{j,k_1,k_2}} z_\pi(a, b). \quad (3.7)$$

Note that

$$z_{0,0}(a, b) = 0, \quad z_{0,k_2}(a, b) = \frac{(-b)^{k_2}}{k_2}, \quad k_2 > 0, \quad z_{k_1,0}(a, b) = \frac{(-a)^{k_1}}{k_1}, \quad k_1 > 0.$$

This explains the conditions $k_1, k_2 > 0$ in the last formula.

3.4. **Lemma [BCG+21, Lemma 2.3]** now states that $z_{k_1,k_2}$ is a sum of commutators for $k_1, k_2 \geq 1$. This is seemingly stronger than the corresponding statement for $Z_m(a, b)$. However, introducing the commuting indeterminates $s, t$ commuting with $a, b$ we have in the algebra $\mathbb{K}\langle a, b \rangle [s, t]$

$$Z_m(sa, tb) = \sum_{k_1,k_2 > 0 \atop k_1 + k_2 < m} z_{k_1,k_2}(a, b) \cdot s^{k_1} t^{k_2}$$

and hence by the proven Theorem 3 (inserting $sa$ for $a$ and $tb$ for $b$) and by (3.2)

$$0 = \text{tr}_{\mathbb{K}\langle a, b \rangle [s, t]} \left( Z_m(sa, tb) \right) = \sum_{k_1,k_2 > 0 \atop k_1 + k_2 < m} \text{tr}_{\mathbb{K}\langle a, b \rangle} \left( z_{k_1,k_2}(a, b) \right) \cdot s^{k_1} t^{k_2}$$

and hence all $\text{tr}_{\mathbb{K}\langle a, b \rangle}(z_{k_1,k_2}(a, b))$ must vanish for $k_1, k_2 > 0, k_1 + k_2 < m$. 

13.03.2024 13:58
Part 2. Generalization to the multi-factor case

Summary. Here we present the multi-factor case. At the same time we streamline further the algebraic as well as the combinatorial aspects of the problem.

4. Algebraic and combinatorial preparations

4.1. The free polynomial algebra in $r$ noncommuting variables. We discuss here results analogous to Section 3.1 in the context of formal power series. For general facts on formal power series, see [LAN02, IV.9].

Let $K$ be a field of characteristic 0 and let $A := K(t_1, \ldots, t_r)$ be the free polynomial algebra in the $r$ noncommuting indeterminates $a_1, \ldots, a_r$. The monomials are words in the alphabet $\{a_1, \ldots, a_r\}$, i.e. a priori there are no relations between the $a_1, \ldots, a_r$.

As in Section 3.1 $\Omega A$ denotes the quotient of $A$ by the commutator subspace $\mathcal{C}A = [A, A]$.

In the sequel we will need to adjoin central indeterminates. A single one will usually be denoted by $x$. The letter $t$ is reserved for an $r$-tuple $(t_1, \ldots, t_r)$ of indeterminates. In order to avoid unnecessary repetitions, we will mostly work with $t$ with the understanding that the results also hold for a single indeterminate $x$. For $t$ we will also make use of multiindex notation, $t^\alpha$ then stands for $t_1^{\alpha_1} \cdots t_r^{\alpha_r}$; in this context multiindices will be denoted by greek letters.

Let then $t := (t_1, \ldots, t_r)$ be central indeterminates. I.e. the $t_1, \ldots, t_r$ commute and the $t_1, \ldots, t_r$ commute with the $a_1, \ldots, a_r$. $A[t_1, \ldots, t_r] := A[t]$ then denotes the usual polynomial algebra over $A$ obtained by adjoining $t = (t_1, \ldots, t_r)$.

Similarly, $A[t_1, \ldots, t_r] := A[t]$ denote the formal power series in $t_1, \ldots, t_r$ over $A$, i.e. formal sums of the form $\sum_{\alpha \in \mathbb{N}^r} f_\alpha \cdot t^\alpha$ with $f = g$ iff all coefficients $f_\alpha = g_\alpha$ are equal. $A[t]$ is an algebra with Cauchy-product

$$f(t) \cdot g(t) = \sum_{\alpha} \left( \sum_{\beta \leq \alpha} f_\beta \cdot g_{\alpha - \beta} \right) \cdot t^\gamma,$$

(4.1)

where $\beta \leq \alpha$ means $\alpha - \beta \in \mathbb{N}^r$. The center $Z\left(A[[t]]\right)$ equals $ZA[t]$. The $ZA[[t]]$-modules $\Omega A[t]$ are defined accordingly. Indeed, in view of Lemma 6 the Cauchy-product (4.1) of formal power series turns $\Omega A[t]$ into a $ZA[[t]]$-module.

For any vector space $B$ ($B$ stands for $A, ZA, QA$ etc.) with corresponding formal power series vector space $B[t]$ the formal partial derivative $\frac{\partial}{\partial t_j}$ acts as a linear map reducing the degree by one. $\frac{\partial}{\partial t_j}$ satisfies the Leibniz-rule $\frac{\partial}{\partial t_j}(f \cdot g) = (\frac{\partial}{\partial t_j} f) \cdot g + f \cdot \frac{\partial}{\partial t_j} g$, whenever the product $f \cdot g$ makes sense (e.g. when $B$ is an algebra, or for $f \in ZA[[t]]$ and $g \in QA[[t]]$).

Furthermore, we denote by $B_N[[t]]$ the set of those $f = \sum_{\alpha} f_\alpha \cdot t^\alpha \in B[[t]]$ with $f_\alpha = 0$ for $|\alpha| < N$, i.e. $f$ takes the form $f = \sum_{|\alpha| \geq N} f_\alpha \cdot t^\alpha$.

Clearly, $A_N[[t]]$ is an ideal in $A[[t]]$ and more generally

$$A_N[[t]] \cdot A_M[[t]] \subset A_{N+M}[[t]], \quad ZA_N[[t]] \cdot QA_M[[t]] \subset ZA_{N+M}[[t]],$$
in particular $\Omega A_N[[t]]$ is a $ZA[[t]]$-module.

Recall from Section 3.1 the facts about the trace map $\tau_A$. In contrast to (3.2), we may not expect $\Omega(A[t])$ to be isomorphic to $QA[t]$. We circumvent this by working solely
with $\Omega A[[t]]$, the formal power series over the quotient $\Omega A$, and define the map

$$\sigma_{A[[t]]} : A[[t]] \longrightarrow \Omega A[[t]], \sum_{\alpha} a_{\alpha} \cdot t^{\alpha} \mapsto \sum_{\alpha} \tau_{A}(a_{\alpha}) \cdot t^{\alpha}. \quad (4.2)$$

**Lemma 8.** 1. $\sigma_{A[[t]]}$ is a well-defined, surjective $\mathbb{Z}A[[t]]$-module map.

2. $\sigma_{A[[t]]}$ satisfies $\sigma_{A[[t]]}(f \cdot g) = \sigma_{A[[t]]}(g \cdot f)$ for $f, g \in A[[t]]$.

3. The kernel of $\sigma_{A[[t]]}$ equals $\mathcal{C}A[[t]]$, that is if $f = \sum_{\alpha} f_{\alpha} t^{\alpha} \in A[[t]]$ with $\sigma_{A[[t]]}(f) = 0$ then each $f_{\alpha} \in \mathcal{C}A$ is a sum of commutators.

4. The formal partial derivative $\frac{\partial}{\partial t_j}$ commutes with $\sigma_{A[[t]]}$, i.e. the following diagram commutes

$$\begin{array}{ccc}
A[[t]] & \xrightarrow{\frac{\partial}{\partial t_j}} & A \\
\downarrow{\sigma_{A[[t]]}} & & \downarrow{\sigma_{A[[t]]}} \\
\Omega A[[t]] & \xrightarrow{\frac{\partial}{\partial t_j}} & \Omega A[[t]].
\end{array}$$

**Proof.** 1. is straightforward and 2. follows by applying $\tau_{A}(a \cdot b) = \tau_{A}(b \cdot a)$ term by term.

3. is equally obvious as $\sigma_{A[[t]]}(f) = 0$ implies that $\tau_{A}(f_{\alpha}) = 0$ for all $\alpha$ and hence the claim is just a reformulation of the latter.

Finally, 4. follows from a term by term check. $\square$

We now have the following formal power series analogue of Lemma 7:

**Lemma 9.** Let $f \in \mathbb{K}[x]$ and let $g \in A[[t]]$ be without constant term. Denote by $f(g(t)) \in A[[t]]$ the element obtained by inserting $g$ into $f$, which is well-defined thanks to $g(0) = 0$. Then we have for the formal partial derivative $\frac{\partial}{\partial t_j}$ and the “trace” $\sigma_{A[[t]]}$

$$\frac{\partial}{\partial t_j} \sigma_{A[[t]]}(f(g(t))) = \sigma_{A[[t]]}(f'(g(t)) \cdot \frac{\partial}{\partial t_j} g(t)).$$

**Proof.** 1. Note first, that the claim is true if $f(x) = x^n$ is a monomial. This follows already from Lemma 7 applied with the $A$ there being $A[[t]]$, the $D$ there being $\frac{\partial}{\partial t_j}$, and the $a$ there being $g(t)$; alternatively

$$\frac{\partial}{\partial t_j} g(t)^n = \sum_{j=0}^{n-1} g(t)^j \cdot \left( \frac{\partial}{\partial t_j} g(t) \right) \cdot g(t)^{n-1-j},$$

hence after taking $\sigma_{A[[t]]}$, and using Lemma 8,

$$\frac{\partial}{\partial t_j} \sigma_{A[[t]]}(g(t)^n) = \sigma_{A[[t]]}(n \cdot g(t)^{n-1} \cdot \frac{\partial}{\partial t_j} g(t)) = \sigma_{A[[t]]}(f'(g(t)) \cdot \frac{\partial}{\partial t_j} g(t))$$

in this case.

2. Given a general $f \in \mathbb{K}[x]$ write for arbitrary $N$

$$f(x) = \sum_{n=0}^{N} f_n \cdot x^n + x^{N+1} \cdot R_N(x)$$

with $R_N \in \mathbb{K}[x]$. Then

$$f(g(t)) = \sum_{n=0}^{N} f_n \cdot g(t)^n + g(t)^{N+1} \cdot R_N(g(t)).$$
The remainder $g(t)^{N+1} \cdot R^N(g(t))$ lies in the ideal $A_{N+1}[t]$ and hence we have by the first part of this proof

$$\frac{\partial}{\partial t_j} \sigma_{A[t]}(f(g(t))) = \sum_{n=0}^{N} n \cdot f_n \cdot \sigma_{A[t]}(g(t)^{n-1} \cdot \frac{\partial}{\partial t_j} g(t)) \mod \Omega A_{N+1}$$

$$= \sigma_{A[t]}(f'(g(t)) \cdot \frac{\partial}{\partial t_j} g(t)) \mod \Omega A_{N+1}.$$

Since $N$ is arbitrary and since $\bigcap_{N} \Omega A_{N+1} = \{0\}$ the claim follows. \qed

4.2. Combinatorial preparations.

1. Recall from 1.1 that for noncommuting variables, product notation means that the product is taken in the given order, i.e. $\prod_{j=1}^{r} a_j = a_1 \cdot \ldots \cdot a_N$ in this order.

For a subset $\xi \subseteq \{1, \ldots, r\}$ with $\xi = \{l_1, \ldots, l_p\}$, let

$$z_{\xi}(a_1, \ldots, a_r) := \prod_{k=1}^{p} a_{l_k}, \quad z_{\emptyset}(a_1, \ldots, a_r) := 1. \quad (4.3)$$

With this notation we have

$$\prod_{l=1}^{r}(1 + a_l) = \sum_{\xi \subseteq \{1, \ldots, r\}} z_{\xi}.$$

2. Denote by $\mathcal{M}(j, r)$ the set of maps $f : \{1, \ldots, j\} \rightarrow \mathcal{P}(1, \ldots, r) \setminus \{\emptyset\}$, where $\mathcal{P}(\{1, \ldots, r\})$ denotes the power set of $\{1, \ldots, r\}$. For $f \in \mathcal{M}(j, r)$ let

$$z_f(a_1, \ldots, a_r) := \prod_{k=1}^{j} z_{f(k)}.$$

Let us compare this to the $z_{\pi}$ of Section 3.3 above. There $r = 2, a_1 = a, a_2 = b$ and to a partition $\pi = (\pi_1, \pi_2, \pi_3)$ of the set $\{1, \ldots, j\}$ there corresponds a map $f : \{1, \ldots, j\} \rightarrow (\{1\}, \{2\}, \{1, 2\})$, where

$$f(k) := \begin{cases} \{1\}, & k \in \pi_1, \\ \{2\}, & k \in \pi_2, \\ \{1, 2\}, & k \in \pi_3, \end{cases} \quad z_{f(k)} = \begin{cases} a, & f(k) = \{1\}, \\ b, & f(k) = \{2\}, \\ ab, & f(k) = \{1, 2\}. \end{cases}$$

Consequently $z_{\pi}(a, b) = z_f(a_1, a_2)$. With this notation we now have

$$\left(\prod_{l=1}^{r}(1 + a_l) - 1\right)^j = \left(\sum_{\emptyset \neq \xi \subseteq \{1, \ldots, r\}} z_{\xi}(a_1, \ldots, a_r)\right)^j = \sum_{f \in \mathcal{M}(j, r)} z_f(a_1, \ldots, a_r). \quad (4.4)$$

3. Orders. For a monomial $w$ (word) in the algebra $A$ we denote by $L(w)$ its length or order, explicitly $w = x_1 \cdot \ldots \cdot x_{L(w)}$ where $x_i \in \{a_1, \ldots, a_r\}$. Given $f \in \mathcal{M}(j, r)$ we have $L(f(k)) = |f(k)| = \text{cardinality}(f(k))$ and

$$L(z_f(a_1, \ldots, a_r)) = \sum_{k=1}^{j} L(z_{f(k)}) = \sum_{k=1}^{j} |f(k)|.$$

Note that for $f$ one clearly has $j = \sum_{\xi \subseteq \{1, \ldots, r\}} |f^{-1}(\{\xi\})|$. 

13.03.2024 13:38
4. Now we define the analogues of the \( z_{k_1,k_2}(a,b) \) of Section 3.3 in the multi-factor case. For a multiindex \( \alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r \) and \( f \in \mathcal{M}(j,r) \) let \( \alpha(f,1) \) be the number of \( \alpha_i \) occuring in the word \( z_{\alpha}(a_1, \ldots, a_r) \), i.e.

\[
\alpha(f,1) := \sum_{l \in \iota \subset \{1, \ldots, r\}} |f^{-1}(\iota)| = \sum_{k=1}^{j} 1 = |\{k \mid l \in f(k)\}| \leq j, \tag{4.5}
\]

and further \( \alpha(f) := (\alpha(f,1), \ldots, \alpha(f,r)) \).

For \( f \in \mathcal{M}(j,r) \) we have from 4. and (4.5)

\[
j \cdot r \geq \alpha(f,1) + \ldots + \alpha(f,r) = \sum_{l=1}^{r} \sum_{\iota \subset \{1, \ldots, r\}} |f^{-1}(\iota)| \geq \sum_{\iota \subset \{1, \ldots, r\}} |f^{-1}(\iota)| = j. \tag{4.6}
\]

4.3. The correction term in the multi-factor product formula. Now we are ready to define the multivariate analogues of the polynomials \( X_m(a,b) \), \( \tilde{X}(a,b) \), and \( Z_m(a,b) \), cf. (1.9), (1.11), and (1.13). Namely, put for \( a = (a_1, \ldots, a_r) \)

\[
\tilde{X}_{m,r}(a) = \sum_{j=1}^{m-1} \frac{(-1)^j}{j} \left[ \left( \prod_{l=1}^{r} (1 + a_l) - 1 \right)^j - \sum_{l=1}^{r} a_l^j \right]
\]

\[
= \sum_{j=1}^{m-1} \frac{(-1)^j}{j} \left[ \sum_{f \in \mathcal{M}(j,r)} z_{\iota}(\alpha) - \sum_{l=1}^{r} a_l^j \right] \tag{4.7}
\]

\[
= \sum_{j=1}^{m-1} \frac{(-1)^j}{j} \sum_{\alpha \in \mathbb{N}^r} \sum_{f \in \mathcal{M}(j,r) \atop \alpha(f) = \alpha} z_{\iota}(\alpha),
\]

Here,

\[
\mathbb{N}^{r^*} := \mathbb{N}^r \setminus \{ \alpha \in \mathbb{N}^r \mid \exists \iota \alpha_l = |\alpha| \}. \tag{4.8}
\]

In words, tuples of the form \((0, \ldots, 0, |\alpha|, 0, \ldots, 0)\) are excluded from \( \mathbb{N}^{r^*} \). The reason for this are the easy to verify formulas

\[
z_{\alpha=(0, \ldots, 0, j, \ldots, 0)} = \frac{(-1)^{j-1}}{j} a_l^j, \tag{4.9}
\]

\[
z_{(0, \ldots, 0)}(a) = 0,
\]

where \( \alpha = (0, \ldots, 0, j, 0, \ldots, 0) \in \mathbb{N}^r \setminus \mathbb{N}^{r^*}, j > 0 \) (\( j \) in the l-th slot).

Furthermore, from the second to the third line in (4.7) we have used (4.4) and the disjoint decomposition

\[
\mathcal{M}(j,r) = \bigcup_{\alpha \in \mathbb{N}^r} \{ f \in \mathcal{M}(j,r) \mid \alpha(f) = \alpha \}.
\]

\( X_{m,r}(a) \) is now obtained from \( \tilde{X}_{m,r}(a) \) by removing all summands of order \( < m \):

\[
X_{m,r}(a) := \sum_{j=1}^{m-1} \frac{(-1)^j}{j} \sum_{\alpha \in \mathbb{N}^{r^*}} \sum_{f \in \mathcal{M}(j,r) \atop \alpha(f) = \alpha} z_{\iota}(\alpha) \tag{4.10}
\]

\(^6\text{cf. Section 1.1, recall } 0 \in \mathbb{N} \text{ and } |\alpha| = \alpha_1 + \ldots + \alpha_r\)
and finally, cf. (1.13)
\[
Z_{m,r}(a) = \tilde{X}_{m,r}(a) - X_{m,r}(a) = \sum_{j=1}^{m-1} \frac{(-1)^j}{j} \prod_{\substack{\alpha \in N^+ \cap \mathbb{N}(j,r) \ni \alpha(f) = \alpha \leq m}} z_\ell(a)
\]
\[
= \sum_{\substack{\alpha \in N^+ \cap \mathbb{N}(j,r) \ni \alpha(f) = \alpha \leq m}} \left( -1 \right)^{|\alpha|} \prod_{f \in \mathbb{N}(j,r)} z_\ell(a) = \sum_{\alpha \in N^+ \cap \mathbb{N}(j,r) \ni \alpha(f) = \alpha} z_\alpha(a),
\]
where
\[
\begin{align*}
z_\alpha(a) &= \sum_{\substack{j=1 \leq m}} \frac{(-1)^j}{j} \prod_{f \in \mathbb{N}(j,r) \ni \alpha(f) = \alpha} z_\ell(a) =: \sum_{\alpha \in N^+ \cap \mathbb{N}(j,r) \ni \alpha(f) = \alpha} z_\alpha(a), \\
\end{align*}
\]
\[
\text{The upper limit } |\alpha| \text{ in the inner sum follows from (4.6).}
\]

4.4. **The log of the regularized determinant as formal power series.** We return to formal power series: for \( b \in \mathcal{A} \) and a formal power series indeterminate \( z \), cf. Section 4.1, let
\[
\log(1 + z \cdot b) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} b^j \cdot z^j
\]
resp. in analogy to (1.4), (2.3) let
\[
\log_m(1 + z \cdot b) := \log(1 + z \cdot b) - \sum_{j=1}^{m-1} \frac{(-1)^{j-1}}{j} b^j \cdot z^j
\]
\[
= \sum_{j=m}^{\infty} \frac{(-1)^{j-1}}{j} b^j \cdot z^j,
\]
as elements of the formal power series algebra \( \mathcal{A}[z] \). Finally, taking the “trace” map \( \sigma_{\mathcal{A}[z]} \)
\[
\logdet^A_m(1 + z \cdot b) := \sigma_{\mathcal{A}[z]} (\log_m(1 + z \cdot b))
\]
\[
\logdet^A(1 + z \cdot b) := \sigma_{\mathcal{A}[z]} (\log(1 + z \cdot b)) = \logdet^A_m(1 + z \cdot b),
\]
as an element of \( \mathcal{Q}\mathcal{A}[z] \). \( \logdet^A_m \) is the formal power series analogue of the function \( \log \det_m(1 + z \cdot \Lambda) \) in the operator case and with obvious identifications, for \( \Lambda \in \mathcal{B}^m(\mathcal{H}) \) the function \( \log \det_m(1 + z \cdot \Lambda) \) is obtained by inserting \( \Lambda \) into \( \logdet^A_m(1 + z \cdot a) \) for \( a \) and taking the Hilbert space trace, cf. the commutative diagram (3.1). In sum, the formal power series (4.13), (4.14) allow for a completely algebraic treatment of the product formula for regularized Fredholm determinants. The next result is the formal power series analogue of the product formula for regularized determinants.
Proposition 10. In the formal power series algebra $A[t]$ resp. in $QA[t]$ we have the following identities:

(1)
\[
\log \left( \prod_{l=1}^{r}(1 + t_1 \cdot a_l) \right) = \sum_{\alpha \in \mathbb{N}^r} z_\alpha(a) \cdot t^\alpha,
\]
\[
\text{cf. (4.12)}.
\]

(2)
\[
\log \det^A \left( \prod_{l=1}^{r}(1 + t_1 \cdot a_l) \right) = \sum_{l=1}^{r} \log \det^A (1 + t_1 \cdot a_l)
\]
\[
in the formal power series $QA[t]$.
\]

(3)
\[
\log \det^A_m \left( \prod_{l=1}^{r}(1 + t_1 \cdot a_l) \right) - \sum_{l=1}^{r} \log \det^A_m (1 + t_1 \cdot a_l)
\]
\[
= \sigma_{A[t]}(X_{m,r}(t_1 \cdot a_1, \ldots, t_r \cdot a_r)) + \sigma_{A[t]}(Z_{m,r}(t_1 \cdot a_1, \ldots, t_r \cdot a_r)).
\]

Proof. 1. Using (4.4) and (4.7)-(4.12) we find
\[
\log \left( \prod_{l=1}^{r}(1 + t_1 \cdot a_l) \right) = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \left( \prod_{l=1}^{r}(1 + t_1 \cdot a_l) - 1 \right)^j
\]
\[
= \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \sum_{\{t \in \mathcal{D}(j,r)\}} z_t(t_1 a_1, \ldots, t_r a_r)
\]
\[
= \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \sum_{a \in \mathbb{N}^r} \sum_{t \in \mathcal{D}(j,r)} z_t(a) t^\alpha = \sum_{\alpha \in \mathbb{N}^r} z_\alpha(a) \cdot t^\alpha,
\]
\[
\text{(4.19)}.
\]

2. The left hand side of (4.17) equals
\[
\sigma_{A[t]} \left( \log \left( 1 + \left( \prod_{l=1}^{r}(1 + t_1 \cdot a_l) - 1 \right) \right) \right).
\]

To take the formal partial derivative w.r.t. $t_j$ we may therefore apply Lemma 9 with $f(z) = \log(1 + z)$ and $g(t) = \prod_{l=1}^{r}(1 + t_1 \cdot a_l) - 1$. We have
\[
\frac{\partial}{\partial t_j} \left( \prod_{l=1}^{r}(1 + t_1 \cdot a_l) - 1 \right) = \prod_{1 \leq i < j} (1 + t_1 \cdot a_i) \cdot a_j \cdot \prod_{j < l \leq r} (1 + t_1 \cdot a_l).
\]
and
\[
f'(g(t)) = \prod_{l=1}^{r}(1 + t_1 \cdot a_l)^{-1},
\]

hence using the tracial property of $\sigma_{A[t]}$, Lemma 8,
\[
\sigma_{A[t]} \left( f'(g(t)) \cdot \frac{\partial}{\partial t_j} g(t) \right) = \sigma_{A[t]} \left( (1 + t_1 \cdot a_1)^{-1} \cdot a_j \right).
\]

On the other hand we find for the formal partial derivative w.r.t. $t_j$ of the right hand side of (4.17)
\[
\frac{\partial}{\partial t_j} \sigma_{A[t]} \left( \log(1 + t_1 \cdot a_j) \right) = \sigma_{A[t]} \left( (1 + t_1 \cdot a_1)^{-1} \cdot a_j \right),
\]
\[13.03.2024 13:58]
again by invoking Lemma 9. Hence the formal partial derivatives w.r.t. to \( t_j, 1 \leq j \leq r \), of the left and right hand sides of (4.17) coincide. Since both sides have no constant term they must hence coincide as well.

3. From the definition (4.14), the proven part (2) and (4.7) we see that the left hand side equals \( \sigma_{\alpha[t]}(\tilde{X}_{m,r}(t_1 \cdot a_1, \ldots, t_r \cdot a_r)) \) and hence, by (4.11) the claim. \( \square \)

5. Reduction to a combinatorial problem in the trace class case

Having defined the polynomials \( \tilde{X}_{m,r}, X_{m,r}, Z_{m,r} \) we now proceed in parallel to the two-factor case and reduce the proof of Theorem 4 to a purely combinatorial problem in the trace class case.

5.1. First reduction to the trace class case. The argument here is the same as in Section 1.3.1. Namely, since the monomials in \( X_{m,r}(a_1, \ldots, a_r) \) are at least of degree \( m \) it follows that \( \text{Tr}(X_{m,r}(A_1, \ldots, A_r)) \) is a continuous function on the \( r \)-fold cartesian product \( (B^m(H))^r \). Consequently, both sides of the product formula (1.14) depend continuously on \( A_1, \ldots, A_r \). Since \( B^1(H) \) is dense in \( B^m(H) \) it therefore suffices to prove (1.14) for \( A_1, \ldots, A_r \in B^1(H) \) of trace class.

5.2. Second reduction. In view of the algebraic and combinatorial preparations of Section 4 the second reduction in parallel to Section 1.3.2 is now straightforward.

Let \( A_1, \ldots, A_r \in B^1(H) \) be trace class operators. Then by the very definition of the higher Fredholm determinant (1.4) we compute analogously to (4.10)

\[
\begin{align*}
\det_m \left( \prod_{l=1}^r (I + A_{l}) \right) \cdot \prod_{l=1}^r \det_m (I + A_{l})^{-1} \\
= \exp \text{Tr}(\tilde{X}_{m,r}(A_1, \ldots, A_r)), \\
= \exp \text{Tr}(X_{m,r}(A_1, \ldots, A_r)) \cdot \exp \text{Tr}(Z_{m,r}(A_1, \ldots, A_r)).
\end{align*}
\] (5.1)

with \( \tilde{X}_{m,r}, X_{m,r}, Z_{m,r} \) defined in (4.7), (4.10), (4.11) resp. Cf. also (4.18).

After these preparations the natural analogue of Theorem 3 is

**Theorem 11.**

1. For \( A_1, \ldots, A_r \in B^1(H) \) we have

\[
\text{Tr}(X_{m,r}(A_1, \ldots, A_r)) = \text{Tr}(\tilde{X}_{m,r}(A_1, \ldots, A_r)).
\] (5.2)

2. In the free polynomial algebra \( \mathbb{Q}\langle a_1, \ldots, a_r \rangle \) generated by \( r \) noncommuting indeterminates \( a_1, \ldots, a_r \) the polynomial

\[
Z_{m,r}(a_1, \ldots, a_r) = \sum_{\alpha \in \mathbb{N}^r, |\alpha| \leq m} z_\alpha(a_1, \ldots, a_r)
\] (5.3)

is a sum of commutators.

3. For \( \alpha \in \mathbb{N}^r \) the polynomial \( z_\alpha(a_1, \ldots, a_r) \) (4.12) is a homogeneous polynomial in \( a_1, \ldots, a_r \) of degree \( |\alpha| \).

For \( \alpha \in \mathbb{N}^{r*} \) the polynomial \( z_\alpha(a_1, \ldots, a_r) \) is a sum of commutators.

**Remark. 1.** Clearly we have the implications (3) \( \Rightarrow \) (2) \( \Rightarrow \) (1) of the Theorem.

2. (5.1), (5.2) imply Theorem 4. Note that (1), (2) of Theorem 4 follow immediately from (4.7) and (4.10).
3. The argument in Section 3.4 shows that (2) and (3) of the Theorem are in fact equivalent. Namely, replacing $a_l$ by $t_l \cdot a_l$ and assuming (2) we have

$$Z_{m,r}(t_1a_1, \ldots, t_r a_r) = \sum_{\alpha \in \mathbb{N}^r, |\alpha| < m} z_\alpha(a) \cdot t^\alpha$$

and thus

$$0 = \sigma_{A[t]} \left( Z_{m,r}(t_1a_1, \ldots, t_r a_r) \right) = \sum_{\alpha \in \mathbb{N}^r, |\alpha| < m} \tau_A(z_\alpha(a)) \cdot t^\alpha,$$

and hence $\tau_A(z_\alpha(a))$ vanishes for $\alpha \in \mathbb{N}^r$. The formula also immediately implies that $z_\alpha$ is homogeneous of degree $|\alpha|$.

4. Summing up to prove the main Theorem 4 it remains to prove (2) of Theorem 11.

6. Two proofs of Theorem 11 (2)

6.1. Formal power series (analytic) proof. Here we argue along the pattern of Section 2, but using formal power series instead of analytic functions. Namely, looking at (4.18) and inserting $x$ for each $t_l$ the left hand side of (4.18) is divisible by $x^m$ (i.e. $O(x^m)$). As $X_{m,r}$ contains only monomials of order $\geq m$ also $X_{m,r}(x \cdot a_1, \ldots, x \cdot a_r)$ is divisible by $x^m$. Thus the last summand $\sigma_{A[x]} \left( Z_{m,r}(x \cdot a_1, \ldots, x \cdot a_r) \right)$ on the right of (4.18) must be divisible by $x^m$ as well. However, since $Z_{m,r}$ is of order $< m$, this summand is a polynomial in $x$ of degree $< m$ and hence must vanish.

6.2. Algebraic proof. Combining (1) and (2) of Proposition 10 gives, using the definition (4.2),

$$\sum_{\alpha \in \mathbb{N}^r} \tau_A(z_\alpha(a)) \cdot t^\alpha = \sum_{l=1}^r \sigma_{A[t]} \left( \log(1 + t_l a_l) \right)$$

$$= \sum_{j=1}^\infty \frac{(-1)^{j-1}}{j} \sum_{l=1}^r \tau_A(a_l^j) \cdot t_l^j,$$

from which the claim follows by comparing the coefficients of $t^\alpha$.

REFERENCES

[AHL78] L. V. Ahlfors, Complex analysis, third ed., McGraw-Hill Book Co., New York, 1978, An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics. MR 510197 (80c:30001)

[BCG+21] T. Britz, A. Carey, F. Gesztesy, R. Nichols, F. Sukochev, and D. Zanin, The product formula for regularized Fredholm determinants, Proc. Amer. Math. Soc. Ser. B 8 (2021), 42–51. MR 4213516

[Fra18] R. L. Frank, Eigenvalue bounds for Schrödinger operators with complex potentials. III, Trans. Amer. Math. Soc. 370 (2018), no. 1, 219–240. MR 3717979

[HAE22] L. Hartmann and M. Lesch, Zeta and Fredholm determinants of self-adjoint operators, J. Funct. Anal. 283 (2022), no. 1, Paper No. 109491, 27. MR 4404074

[HAN10] M. Hansmann, On the discrete spectrum of linear operators in Hilbert spaces, Dissertation, TU Clausthal 2010, https://d-nb.info/1001898664/34, 2010.

[KK21] N. Koutsonikonos-Kouloumpis, The product formula for regularized Fredholm determinants, Master’s thesis, Universität Bonn, 2021.

[LAN02] S. Lang, Algebra, third ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002. MR 1878556

[Ped86] G. K. Pedersen, Analysis now, Graduate Texts in Mathematics, vol. 118, Springer-Verlag, New York, 1989. MR 971256

[Sim77] B. Simon, Notes on infinite determinants of Hilbert space operators, Advances in Math. 24 (1977), no. 3, 244–273. MR 0482328
[Sim05] ____, Trace ideals and their applications, second ed., Mathematical Surveys and Monographs, vol. 120, American Mathematical Society, Providence, RI, 2005. MR 2154153

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PATRAS, 26504 PATRAS, GREECE
Email address: up1019669@ac.upatras.gr

MATHEMatisches INSTITUT, UNIVERSITät BONN, ENDELICHER ALLE 60, 53115 BONN, GERMANY
Email address: lesch@math.uni-bonn.de
URL: www.math.uni-bonn.de/people/lesch