Relativistic Diffusions  
and Schwarzschild Geometry  

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Abstract  

The purpose of this article is to introduce and study a relativistic motion whose acceleration, in proper time, is given by a white noise. We deal with general relativity, and consider more closely the problem of the asymptotic behaviour of paths in the Schwarzschild geometry example.  

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1 Introduction

The classical theory of Brownian motion is not compatible with relativity, as it appears clearly from the fact that the heat flow propagates instantaneously to infinity. A Lorentz invariant generalized Laplacian was defined by Dudley (cf [D1]) on the tangent bundle of the Minkowski space, and it was shown that there is no other adequate definition than this one ¹ as long as Lorentz invariance is assumed. An intuitive description of the associated diffusion (i.e. continuous Markov process) is that boosts are continuously applied in random directions of space. We show that this process is induced by a left invariant Brownian motion on the Poincaré group. The asymptotic behaviour of the paths of this process was studied (cf [D3]).

Considering the importance of heat kernels in Riemannian geometry and the extensive use that is made of their probabilistic representation via sample paths, it is somewhat surprising that Dudley’s first studies were not pursued and extended to the general context, namely to Lorentz manifolds. It is indeed easy to check that the “relativistic diffusion” can be defined on any Lorentz manifold using a development, as done below. The infinitesimal generator is the generator of the geodesic flow perturbed by the vertical Laplacian. But such an extension would have little appeal, if some natural questions such as the asymptotic behaviour and the nature of harmonic functions could not be solved in some examples of interest.

Here we provide a rather complete study of this question in the case of Schwarzschild and Kruskal-Szekeres manifolds, which are used in physics to represent “black holes”. The specific interest of these manifolds comes from the vanishing of Ricci curvature, their symmetry, and the integrability of the geodesic flow.

The picture that comes out in the Kruskal-Szekeres case appears quite remarkable, with paths confined in a neighborhood of the singularity, while their velocity increases, and an infinity of $SO_3$-invariant harmonic functions.

One difficulty of the study (and it might explain why Dudley had few followers) is that no explicit solution was found. The reason is that, even after reduction using the symmetries, the operator cannot involve less than three coordinates (even in Minkowski space), instead of one for the Laplacian on Riemann spaces of constant curvature. Estimations and comparison techniques of stochastic analysis are the main tools we use to prove our results. They do not include yet a full determination of the Poisson boundary, but they suggest

¹Note however that some physical models of diffusion in a relativistic fluid are not Lorentz invariant since the frame of “the fluid at rest” plays a specific role: cf [D] and its references.
that for general Lorentz manifolds bounded harmonic functions could be characterised by classes of light rays, i.e. null geodesics.

Let us now explain more precisely the content of this article.

We consider diffusions, namely continuous strong Markov processes. We start, in Section 2.1, with the flat case of Minkowski space $\mathbb{R}^{1,d}$, and therefore with the Brownian motion of its unit pseudo-sphere, integrated then to yield the only true relativistic diffusion, according to [D1]. We get then its asymptotic behaviour, somewhat simplifying the point of view of [D3].

In Section 2.2 below, we present an extension of the preceding construction to the framework of general relativity, that is to say of a generic Lorentz manifold. The process is first defined at the level of pseudo-orthonormal frames, with Brownian noise only in the vertical directions, and projects into a diffusion on the pseudo-unit tangent bundle. The infinitesimal generator we get in Theorem 1 decomposes into the sum of the vertical Laplacian and of the horizontal vector field generating the geodesic flow.

In Sections 2.3 and 2.4, we deal in detail with the Schwarzschild space, which is the most classical example of curved Lorentz manifold, used in physics to model the space outside a black hole or a spherical body.

Using the symmetry, and introducing the energy and the angular momentum, which are constants of the geodesic motion, we reduce the problem to the study of a degenerate three-dimensional diffusion. We then establish in Theorem 2 that almost surely the diffusion either hits the hole or wanders out to infinity, both events occurring with positive probability. We prove also in Theorem 2 that almost surely, conditionally on the non-hitting of the hole, the relativistic diffusion goes away to infinity in some random asymptotic direction, asymptotically with the velocity of light. Then we prove in Theorem 3 that almost surely, conditionally on the hitting of the hole, the relativistic diffusion reaches the essential singularity at the center of the hole within a finite proper time, and we describe the limit.

We show then in Theorem 4 that the story can be continued further: namely, the Schwarzschild relativistic diffusion, a priori defined till its hitting of the center of the hole (the essential singularity of the so-called Kruskal-Szekeres space, we also call full Schwarzschild space), can be extended to a diffusion which crosses this singularity. This extended diffusion reaches soon the restricted Schwarzschild space again, where it evolves as before, but maybe running backward in time. Thus such hole crossing can happen then again and again, but without accumulation, according to Theorem 4 below, so that the extended Schwarzschild relativistic diffusion is well defined for all positive proper times.

We finally study the asymptotic behaviour of this extended Schwarzschild relativistic diffusion, and show mainly in Theorem 5 below that there is a unique alternative possibility to the escape to infinity: there is indeed a positive probability that the relativistic diffusion becomes endlessly confined in a spherical neighborhood of the hole, with an increasing velocity, and a trajectory becoming asymptotically planar, with an asymptotic shape. This implies the existence of an infinity of $SO_3$-invariant harmonic functions.
2 Statement of the results

2.1 A relativistic diffusion in Minkowski space

Let us consider an integer \( d \geq 2 \) and the Minkowski space \( \mathbb{R}^{1,d} := \{ \xi = (\xi^o, \vec{\xi}) \in \mathbb{R} \times \mathbb{R}^d \} \), endowed with the Minkowski pseudo-metric \( \langle \xi, \xi \rangle := |\xi^o|^2 - |\vec{\xi}|^2 \).

Let \( G \) denote the connected component of the identity in \( O(1,d) \), and denote by \( \mathbb{H}^d := \{ \xi \in \mathbb{R}^{1,d} | \xi^o > 0 \text{ and } \langle \xi, \xi \rangle = 1 \} \) the positive half of the unit pseudo-sphere.

The opposite of the Minkowski pseudo-metric induces a Riemannian metric on \( \mathbb{H}^d \), namely the hyperbolic one, so that \( \mathbb{H}^d \) is a model for the \( d \)-dimensional hyperbolic space.

A convenient parametrization of \( \mathbb{H}^d \) is \( (\varrho, \theta) \in \mathbb{R}_+ \times S^{d-1} \), given by \( \varrho := \arcsinh(\xi^o) \) and \( \theta := \vec{\xi} / \sqrt{|\xi^o|^2 - 1} \). In these coordinates the hyperbolic metric writes \( d\varrho^2 + \sinh^2 \varrho \mathsf{d}\theta^2 \), and the hyperbolic Laplacian is \( \Delta_{\mathbb{H}} := \frac{\partial^2}{\partial \varrho^2} + (d - 1) \coth \varrho \frac{\partial}{\partial \varrho} + \sinh^{-2} \varrho \Delta_{\theta} \), \( \Delta_{\theta} \) denoting the Laplacian of \( S^{d-1} \). The associated volume measure is \( |\sinh \varrho|^{d-1} d\varrho d\theta \).

Note that \( G \) acts isometrically on \( \mathbb{R}^{1,d} \) and on \( \mathbb{H}^d \), and that the Casimir operator on \( G \) induces on \( \mathbb{H}^d \) the hyperbolic Laplacian.

Fix \( \sigma > 0 \), and denote by \( \mathcal{L}\sigma \) the \( \sigma \)-relativistic Laplacian, defined on \( \mathbb{R}^{1,d} \times \mathbb{H}^d \) by

\[
\mathcal{L}\sigma f(\xi, p) := \sigma^2 \frac{\partial f}{\partial \xi^o}(\xi, p) + \sum_{j=1}^d p^j \frac{\partial f}{\partial \xi^j}(\xi, p) + \frac{\sigma^2}{2} \Delta_{\mathbb{H}} f(\xi, p),
\]

that is to say \( \mathcal{L}\sigma f := \langle p, \text{grad}(f) \rangle + \frac{\sigma^2}{2} \Delta_{\mathbb{H}} f \). This is a hypoelliptic operator.

Given any \( (\xi_0, p_0) \in \mathbb{R}^{1,d} \times \mathbb{H}^d \), there exists a unique (in law) diffusion process \( (\xi_s, p_s), s \in \mathbb{R}_+ \), solving the \( \mathcal{L}\sigma \)-martingale problem, that is to say such that for any compactly supported \( f \in C^2(\mathbb{R}^{1,d} \times \mathbb{H}^d) \),

\[
f(\xi_s, p_s) - (\xi_0, p_0) - \int_0^s \mathcal{L}\sigma f(\xi_t, p_t) \, dt \text{ is a martingale.}
\]

Note that \( p_s \) is a hyperbolic Brownian motion, and that \( \xi_s = \xi_0 + \int_0^s p_t \, dt \).

Note also that \( \xi_s \) is parametrized by its arc length. Mechanically, \( \xi_s \) describes the trajectory of a relativistic particle of small mass indexed by its proper time, submitted to a white noise acceleration (in proper time). Its law is invariant under any Lorentz transformation.

Note that if we denote by \( (e_0, e_1, ..., e_d) \) the canonical base of \( \mathbb{R}^{1,d} \), and by \( (e^*_j) \) the dual base (with respect to \( \langle \cdot, \cdot \rangle \)), the matrices \( E_j := e_0 \otimes e^*_j + e_j \otimes e^*_0 \) belong to the Lie algebra of \( G \), and generate the boost transformations. Given \( d \) independent real Wiener processes \( w^j_s, p_s = (p^o_s, \vec{p}_s) \) can be defined by \( p_s := \Lambda_s e_0 \), where the matrix \( \Lambda_s \in G \) is defined by the following stochastic differential equation:

\[
\Lambda_s = \Lambda_0 + \sigma \sum_{j=1}^d \int_0^s \Lambda_t E_j \circ dw^j_t.
\]
This means in fact that the relativistic diffusion process \((\xi_s, p_s)\) is the projection of some diffusion process having independent increments, namely a Brownian motion with drift, living on the Poincaré group. This group is the analogue in the present Lorentz-Minkowski setup of the classical group of rigid motions, and can be seen as the group of \((d + 2, d + 2)\) real matrices having the form \(\begin{pmatrix} \Lambda & \xi \\ 0 & 1 \end{pmatrix}\), with \(\Lambda \in G\), \(\xi \in \mathbb{R}^{1,d}\) (written as a column), and \(0 \in \mathbb{R}^{1+d}\) (written as a row). Its Lie algebra is the set of matrices \(\begin{pmatrix} \beta & x \\ 0 & 0 \end{pmatrix}\), with \(\beta \in so(1, d)\) and \(x \in \mathbb{R}^{1,d}\). The Brownian motion with drift we consider on the Poincaré group solves the stochastic differential equation
d
\[
d\begin{pmatrix} \Lambda_s & \xi_s \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \Lambda_s & \xi_s \\ 0 & 1 \end{pmatrix} \circ d\begin{pmatrix} \beta_s & e_0 s \\ 0 & 0 \end{pmatrix},
\]
where \((\beta_s = \sigma \sum_{j=1}^d E_j w_s^j)\) is a Brownian motion on \(so(1, d)\). This equation is equivalent to \(d\Lambda_s = \Lambda_s \circ d\beta_s\) and \(d\xi_s = \Lambda_s e_0 ds\), so that \((\Lambda_s)\) is a Brownian motion on \(G\). On functions of \(p = \Lambda e_0\), its infinitesimal generator \(\sum_{j=1}^d (L E_j)^2\) coincides with a Casimir operator, and induces the hyperbolic Laplacian, so that \((p_s = \Lambda_s e_0)\) is a Brownian motion on \(\mathbb{H}^d\), as required.

Then it is well known that \(\theta_s := \tilde{\rho}_s/\sqrt{\lvert p_0^s\rvert^2 - 1}\) converges almost surely in \(S^{d-1}\) to some random limit \(\theta_\infty\), and that \(p_0^s\) increases to infinity. Set also \(\varrho_s := \argch(p_0^s)\).

The Euclidian trajectory \(Z(t)\) is defined by \(\xi_{s(t)}\), where \(s(t)\) is determined by \(\xi^o_{s(t)}(t) = t\). Let us note that the Euclidian velocity \(\frac{dZ(t)}{dt} = \theta_s(t) \varrho_{s(t)}\) has norm < 1 \((1\) is here the velocity of light). Moreover we have the following.

**Remark 1** The mean Euclidian velocity \(\frac{Z(t)}{t}\) converges almost surely to \(\theta_\infty \in S^{d-1}\).

**Proof** We have \(\lim_{t \to \infty} s(t) = +\infty\), so that \(\varrho_{s(t)} = \sqrt{1 - (p_0^s(t))^2}\) goes to 1. Thus we get almost surely \(\lim_{t \to \infty} \frac{dZ(t)}{dt} = \theta_\infty\), and the result follows at once. ♦

**Remark 2** The scattering amplitude, id est the law of \(\theta_\infty\) given \(p_0\), is given by the hyperbolic harmonic measure in the unit ball of \(\mathbb{R}^d\) (taken as model for \(\mathbb{H}^d\)), which has density proportional to \(P(p_0, \cdot)^{d-1}\) with respect to the uniform measure of \(S^{d-1}\), \(P\) denoting the classical Poisson kernel of the unit ball of \(\mathbb{R}^d\). See for example ([E-F-LJ], case \(\delta = 0\)).

### 2.2 Extension to general Lorentz manifolds

Let us now see how the preceding construction can be naturally extended to the framework of manifolds.

Let \(\mathcal{M}\) be a \((d+1)\)-dimensional manifold, equipped with a pseudo-Riemannian metric of signature \((+, -, \ldots, -)\), together with an orientation and a time direction, and its Levi-Civita connection.
For notational convenience, $T^1\mathcal{M}$ will always denote the positively oriented half of the unit tangent bundle of $\mathcal{M}$. As in the construction of Brownian motion on Riemannian manifolds, we have to use the frame bundle (see [M]).

So let $G(\mathcal{M})$ be the bundle of direct pseudo-orthonormal frames, with first element in the positive half of the unit pseudo sphere (in the tangent space), which has its fibers modelled on the special Lorentz group $G$. Let $V_j$ be the canonical vertical vector field associated with the preceding matrix $E_j$, and $H_0$ be the first canonical horizontal vector field.

Set $\mathcal{L} := H_0 + \frac{\alpha^2}{2} \sum_{j=1}^{d} V_j^2$.

Let $\pi_1$ denote the canonical projection from $G(\mathcal{M})$ onto the tangent bundle $T^1\mathcal{M}$, which to each frame associates its first element. The canonical vertical vector fields $V_{kl}$ associated with the matrices $E_{kl} := e_k \otimes e_l^* - e_l \otimes e_k^* \in so(1,d)$, for $1 \leq k < l \leq d$, generate an action of $SO_d$ on $G(\mathcal{M})$, which leaves $T^1\mathcal{M}$ invariant and then allows the identification $T^1\mathcal{M} \equiv G(\mathcal{M})/SO_d$. The Casimir operator is $\mathcal{C} = \sum_{j=1}^{d} V_j^2 - \sum_{1 \leq k < l \leq d} V_{kl}^2$.

Note that the matrices $\{E_j, E_{kl}; \, 1 \leq j \leq d, 1 \leq k < l \leq d\}$ constitute a pseudo-orthonormal base of $so(1,d)$ (endowed with its Killing form).

**Lemma 1** The operators $H_0$, $\sum_{j=1}^{d} V_j^2$, $\mathcal{C}$, $\mathcal{L}$ do act on $C^2$ functions on the pseudo-unit tangent bundle $T^1\mathcal{M}$, inducing respectively: the vector field $\mathcal{L}_0$ generating the geodesic flow on $T^1\mathcal{M}$, the so-called vertical Laplacian $\Delta_v$, $\Delta_v$ again, and the generator $\mathcal{G} := \mathcal{L}_0 + \frac{\alpha^2}{2} \Delta_v$.

More precisely, for any test-function $F$ on $T^1\mathcal{M}$, we have on $G(\mathcal{M})$:

$$(\mathcal{L}_0 F) \circ \pi_1 = H_0 (F \circ \pi_1), \quad (\Delta_v F) \circ \pi_1 = \mathcal{C} (F \circ \pi_1).$$

Besides, in local coordinates $(x^i, e_j^k)$, with $e_j = e_j^k \partial / \partial x^k : V_j = e_j^k \partial / \partial x^k + e_0^k \partial / \partial x^0$, and $((g^{kl})$ denoting in these coordinates the inverse matrix of the pseudo-Riemannian metric of $\mathcal{M}$):

$$(\Delta_v F) \circ \pi_1 = \sum_{j=1}^{d} V_j^2 (F \circ \pi_1) = \left( (e_0^k e_l^i - g^{kl}) \frac{\partial^2}{\partial e_0^i \partial e_l^k} + d e_l^k \frac{\partial}{\partial e_l^k} \right) F \circ \pi_1.$$

**Proof** Let us observe that for any $u = (x, e_0, ..., e_d) \in SO_{1,d}\mathcal{M}$, if $(u_s)$ denotes the horizontal curve such that $u_0 = u$ and $\pi^* u_0 = \dot{x}_0 = e_0$, then $(\pi_1(u_s))$ is the geodesic generated by $\pi_1(u) = (x, e_0)$. Hence for any differentiable function $F$ on $T^1\mathcal{M}$, we have

$$H_0 (F \circ \pi_1) (u) = \frac{d}{ds} F \circ \pi_1 (u_s) = \frac{d}{ds} F (\pi_1(u_s)) = \mathcal{L}_0 F (\pi_1(u)).$$

Another way of expressing this is to recall that $H_0$ commutes with the rotation vertical vectors $V_{kl}$. It is also classical that the Casimir operator $\mathcal{C}$ commutes with all vertical vectors $V_{kl}, V_j$. Moreover, since the rotation vectors $V_{kl}$ act trivially on $T^1\mathcal{M}$, the operators $\mathcal{C}$ and $\sum_{j=1}^{d} V_j^2$ induce the same operator $\Delta_v$ on $T^1\mathcal{M}$.
Then as \( e^k_0[e^{tV_j}(x,e)] = e^k_0(x,e) \tanh t + e^k_0(x,e) \sinh t \) and \( e^k_0[e^{tV_j}(x,e)] = e^k_0(x,e) \tanh t + e^k_0(x,e) \sinh t \), we have indeed \( V_jf(x,e) = \frac{d}{dt} f[e^{tV_j}(x,e)] = e^k_j \frac{\partial}{\partial e_0^k} f(x,e) + e^k_0 \frac{\partial}{\partial e_j} f(x,e) \), id est \( V_j = e^k_j \frac{\partial}{\partial e_0^k} + e^k_0 \frac{\partial}{\partial e_j} \). Using that \( e^k_0 e^l_j - \sum_{1}^{d} e^k_j e^l_j = g^{kl} \), we deduce immediately:
\[
\sum_{j=1}^{d} V_j^2 = (e^k_0 e^l_j - g^{kl}) \frac{\partial^2}{\partial e_0^k \partial e_0^l} + \sum_{j=1}^{d} e^k_j \frac{\partial}{\partial e_j} + d e^k_0 \frac{\partial}{\partial e_0^k} + 2 \sum_{j=1}^{d} e^k_0 e^l_j \frac{\partial^2}{\partial e_0^k \partial e_j} + \sum_{j=1}^{d} e^k_0 e^l_j \frac{\partial^2}{\partial e_j \partial e_j} ,
\]
which reduces to the formula of the statement in the particular case of a function depending only on \((x,e_0)\). In accordance with the commutation relations arguments above, \( \sum_{j=1}^{d} V_j^2 (F \circ \pi_1) \) is a function depending only on \((x,e_0)\), id est a function on \( T^1 \mathcal{M} \).

Now, according to Section 2.1, the relativistic motion we will consider lives on \( T^1 \mathcal{M} \) and admits as infinitesimal generator the operator \( \mathcal{G} = \mathcal{L}_0 + \frac{a^2}{2} \Delta_v \) of Lemma 1 above.

If \( \mathcal{M} \) is the Minkowski flat space of special relativity, it coincides with the diffusion defined in Section 2.1 above.

To construct this general relativistic diffusion, we use a kind of stochastic development to produce a stochastic flow on the bundle \( G(\mathcal{M}) \), as classically done to construct the Brownian motion on a Riemannian manifold. But we have now to project on \( T^1 \mathcal{M} \) and no longer on the base manifold \( \mathcal{M} \), and to put the white noises on the acceleration, id est on the vertical vectors, and no longer on the velocity, id est on the horizontal vectors.

To proceed, let us simply fix \( \Psi_0 \in G(\mathcal{M}) \) and a \( \mathbb{R}^d \)-valued Brownian motion \( w = (w^j_t) \), and let us consider the \( G(\mathcal{M}) \)-valued diffusion \( \Psi = (\Psi_s) \in G(\mathcal{M}) \) solving the following Stratonovitch stochastic differential equation:
\[
(*) \quad \Psi_s = \Psi_0 + \int_0^s H_0(\Psi_t) \, dt + \sigma \int_0^s \sum_{j=1}^{d} V_j(\Psi_t) \circ dw^j_t .
\]

By Lemma 1, the stochastic flow defined by \( (*) \) commutes with the action of \( \text{SO}_d \) on \( G(\mathcal{M}) \), and therefore the projection \( (\xi_s, \xi_v^s) := (\xi_s, e_0(s)) = \pi_1(\Psi_s) \) defines a diffusion on \( T^1 \mathcal{M} \); namely this is the relativistic diffusion we intended to define and construct.

The following theorem defines the relativistic diffusion \( (\xi_s, \xi_v^s) \), possibly till some explosion time. The vector field \( \mathcal{L}_0 \) denotes the generator of the geodesic flow, which operates on the position \( \xi \)-component, and \( \Delta_v \) denotes the vertical Laplacian (restriction to \( T^1 \mathcal{M} \) of the Casimir operator on \( G(\mathcal{M}) \)), which operates on the velocity \( \xi \)-component.

**Theorem 1** 1) The \( G(\mathcal{M}) \)-valued Stratonovitch stochastic differential equation
\[
(*) \quad d\Psi_s = H_0(\Psi_s) \, ds + \sigma \sum_{j=1}^{d} V_j(\Psi_s) \circ dw^j_s
\]
defines a diffusion \( (\xi_s, \xi_v^s) := \pi_1(\Psi_s) \) on \( T^1 \mathcal{M} \), whose infinitesimal generator is \( \mathcal{L}_0 + \frac{a^2}{2} \Delta_v \).
2) If $\overline{\xi}(s) : T_{\xi_0}M \rightarrow T_{\xi_0}M$ denotes the inverse parallel transport along the $C^1$ curve $(\xi_s', 0 \leq s' \leq s)$, then $\xi_s := \overline{\xi}(s)$ is an hyperbolic Brownian motion on $T_{\xi_0}M$.

Therefore the path $(\xi_s)$ is the development of a relativistic diffusion path in the Minkowski space $T_{\xi_0}M$.

Remark 3 In local coordinates $(x^i, e^k_s)$, with $e_j = e_j^k \frac{\partial}{\partial x^k}$, $\Psi_s = (\xi, \xi_0(s), ..., \xi_d(s))$, $\Gamma^k_{il}$ denoting as usual the Christoffel coefficients, the equation $(\ast)$ writes:

$$\begin{align*}
\frac{dx^i_s}{ds} &= e^i_0(s) ds; \\
\frac{dx^k_s}{ds} &= -\Gamma^k_{il}(\xi_s) e^l_j(s) dx^i_s + 1_{(j \neq 0)} e^i_0(s) \circ dw^j_s + 1_{(j = 0)} \sigma \sum_{i=1}^d e^i_k(s) \circ dw^i_s,
\end{align*}$$

or equivalently in the Itô form:

$$\begin{align*}
\frac{dx^k_s}{ds} &= e^k_0(s) ds; \\
\frac{dx^k_s}{ds} &= -\Gamma^k_{il}(\xi_s) e^l_j(s) dx^i_s + \sigma \sum_{i=1}^d e^i_k(s) dw^i_s + \frac{\sigma^2}{2} e^k_0(s) ds,
\end{align*}$$

and

$$\begin{align*}
\frac{dx^j_s}{ds} &= -\Gamma^j_{il}(\xi_s) e^i_k(s) dx^l_s + \sigma e^k_0(s) dw^j_s + \frac{\sigma^2}{2} e^j_0(s) ds \quad \text{for } j \geq 1, 0 \leq k \leq d.
\end{align*}$$

Note that the martingales in the above equations for $e_0(s)$, that is to say the differentials $dM^k_s := \sigma \sum_{j=1}^d e^j_k(s) dw^j_s$, $0 \leq k \leq d$, have the following quadratic covariation matrix:

$$K^{kl}_s := \langle (dM^k_s, dM^l_s) \rangle_{ds} = \sigma^2 \sum_{j=1}^d e^j_k(s) e^j_l(s) = \sigma^2 (e^k_0(s) e^l_0(s) - g^{kl}(\xi_s)),$$

id est $K_s = \sigma^2 (e_0(s) e_0(s) - g^{-1}(\xi_s))$, in accordance with Lemma 1. (Here $\xi_0$ denotes the transpose of the column-vector $e_0$, and $g^{-1}$ denotes the inverse matrix of the pseudo-metric.)

Note that $K_s$ does not depend on the other frame vectors $e_j (j \geq 1)$, proving again that the projection $(\xi_s, \xi_s)$ is a diffusion on the tangent bundle, as Theorem (1,1) asserts.

Proof of Theorem 1 1) does not need any further proof.

2) The process $(\xi_s = \overline{\xi}(s) \xi_s)_{s \geq 0}$ is continuous and lives on the fixed unit pseudosphere $T^1_{\xi_0}M$. Let $\overline{\xi}(s) : T_{\xi_0}M \rightarrow T_{\xi_0}M$ denote the parallel transport along the $C^1$ curve $(\xi_s', 0 \leq s' \leq s)$, and recall that

$$\begin{align*}
\frac{d}{ds} \overline{\xi}(s)^j_k &= -\overline{\xi}(s)^l_k \Gamma^l_{km}(\xi_s) \overline{\xi}_s^m, \quad \text{which implies that } \frac{d}{ds} \overline{\xi}(s)^i_k = \overline{\xi}(s)^j_k \Gamma^j_{km}(\xi_s) \overline{\xi}_s^m;
\end{align*}$$

indeed

$$\begin{align*}
\left(\frac{d}{ds} \overline{\xi}(s)^i_k\right) \overline{\xi}(s)^j_k &= \overline{\xi}(s)^l_k \overline{\xi}(s)^j_l \Gamma^l_{km}(\xi_s) \overline{\xi}_s^m = \overline{\xi}(s)^k \overline{\xi}(s)^i_k \Gamma^k_{lm}(\xi_s) \overline{\xi}_s^m,
\end{align*}$$

whence

$$\begin{align*}
\frac{d}{ds} \overline{\xi}(s)^i_k &= \overline{\xi}(s)^j_k \overline{\xi}(s)^j_l \Gamma^j_{km}(\xi_s) \overline{\xi}_s^m = \overline{\xi}(s)^i_k \Gamma^i_{km}(\xi_s) \overline{\xi}_s^m.
\end{align*}$$
Recall then from Remark 3 that (for $0 \leq D \leq d$)
\[
d\xi_s^\ell = \sigma \sum_{k=1}^d e_s^k(s) \circ dw_s^k - \Gamma_{j k}^\ell (\xi) \dot{\xi}_s^j \dot{\xi}_s^k ds.
\]

Therefore we get
\[
d\zeta_s^\ell = (\circ d \xi)_{s}^\ell + \xi_s^\ell(s) \circ d \xi_s^\ell
\]
\[
= \xi_s^i(s) \Gamma_{i m}^q (\xi) \dot{\xi_s}^m ds + \sigma \xi_s^i(s) \sum_{k=1}^d e_s^k(s) \circ dw_s^k - \xi_s^i(s) \Gamma_{j k}^\ell (\xi) \dot{\xi}_s^j \dot{\xi}_s^k ds
\]
\[
= \sigma \sum_{k=1}^d \xi_s^i(s) e_k(s) \circ dw_s^k = \sigma \sum_{k=1}^d e_k(s) \circ dw_s^k,
\]
where $e_k(s) := \xi_s^i(s) e_k(s)$, for $1 \leq k \leq d$ and $s \geq 0$.

Similarly, we have $d\tilde{e}_k(s) = \sigma \xi_s^i(s) \tilde{e}_k(s) \circ dw_s^k$, i.e., $d\tilde{e}_k(s) = \sigma \zeta(s) \circ dw_s^k$.

Observe that, for any $s \geq 0$, $\left(\xi_s, \tilde{e}_1(s), ..., \tilde{e}_d(s)\right)$ constitutes a pseudo-orthonormal basis of the fixed tangent space $T_{\xi_0}M$. Hence, owing to Section 2.1 and Remark 3, we find that the velocity process $\zeta$ defines a hyperbolic Brownian motion on the hyperbolic space $T_{\xi_0}M$, isometric to $\mathbb{H}^d$.

In the reverse direction, we have of course $\dot{\xi}_s = \xi_s(s) \zeta_s$, meaning indeed that we recover the $C^1$ curve $\xi$, as the deterministic development of the flat relativistic diffusion $\int_{0}^{t} \zeta ds$.

Remark 4 The equation $(\ast)$ can be expressed intrinsically, in Stratonovitch or in Itô form, by using the covariant differential $D$, which is defined in local coordinates $(x^i, e^\ell)$ by $(De_j)^k := de_k^j + \Gamma_{i k}^j e^\ell_i dx^\ell$, for $0 \leq j, k \leq d$. The equation $(\ast)$ is indeed equivalent to:
\[
\dot{\xi}_s = e_0(s) \quad ; \quad D e_0(s) = \sigma \sum_{j=1}^d e_j(s) \circ dw_s^j \quad ; \quad D e_j(s) = \sigma e_0(s) \circ dw_s^j \quad \text{for } 1 \leq j \leq d.
\]

See for example ([B], page 30) or ([Em], page 427).

2.3 The restricted Schwarzschild space $S_0$

This space is commonly used in physics to model the complement of a spherical body, star or black hole; see for example [DF-C], [F-N], [L-L], [M-T-W], [S].

We take $\mathcal{M} = S_0 := \{\xi = (t, r, \theta) \in \mathbb{R} \times [R, +\infty] \times \mathbb{S}^2\}$, where $R \in \mathbb{R}_+$ is a parameter of the central body, endowed with the radial pseudo-metric:
\[
(1 - \frac{R}{r}) dt^2 - (1 - \frac{R}{r})^{-1} dr^2 - r^2 |d\theta|^2.
\]

The coordinate $t$ represents the absolute time, and $r$ the distance from the origin.
In spherical coordinates $\theta = (\varphi, \psi) \in [0, \pi] \times (\mathbb{R}/2\pi \mathbb{Z})$, we have $|d\theta|^2 = d\varphi^2 + \sin^2 \varphi d\psi^2$. The geodesics are associated with the Lagrangian $L(\xi, \xi)$, where

$$2L(\xi, \xi) = (1 - \frac{R}{r}) \dot{t}^2 - (1 - \frac{R}{r})^{-1} \dot{r}^2 - \dot{\varphi}^2 - r^2 \sin^2 \varphi \dot{\psi}^2,$$

and the non-vanishing Christoffel symbols are:

$$\Gamma^t_{rt} = \frac{R}{2r(r - R)}; \quad \Gamma^r_{tt} = \frac{R(r - R)}{2r^3}; \quad \Gamma^r_{\varphi\varphi} = R - r; \quad \Gamma^r_{\psi\psi} = (R - r) \sin^2 \varphi;$$

$$\Gamma^\varphi_{r\varphi} = \Gamma^\psi_{r\psi} = r^{-1}; \quad \Gamma^\varphi_{\varphi\psi} = -\sin \varphi \cos \varphi; \quad \Gamma^\psi_{\varphi\psi} = \cot \varphi.$$

The Ricci tensor vanishes, the space $S_0$ being empty. A theorem of Birkhoff (see [M-T-W]) asserts that there is no other radial pseudo-metric in $S_0$ which satisfies this constraint. The limiting case $R = 0$ is the flat case of special relativity, considered in section 2.1.

### 2.3.1 The stochastic differential system in spherical coordinates

Let us take as local coordinates the global spherical coordinates:

$$\xi = (\xi^0, \xi^1, \xi^2, \xi^3) := (t, r, \varphi, \psi).$$

According to Remark 3, the system of Itô stochastic differential equations governing the relativistic diffusion $(\xi_s, \dot{\xi}_s)$ writes here as follows:

$$dt_s = e_0^0(s) ds, \quad dr_s = e_0^1(s) ds, \quad d\varphi_s = e_0^2(s) ds, \quad d\psi_s = e_0^3(s) ds,$$

$$d\sigma^0_0(s) = 2\sigma^2_2 e_0^1(s) ds - \frac{R}{r_s(r_s - R)} e_0^0(s) e_0^1(s) ds + dM^0_s,$$

$$d\sigma^1_0(s) = \frac{3\sigma^2_2}{2} e_0^1(s) ds + \frac{R}{2r_s(r_s - R)} e_0^1(s) e_0^2(s) ds - \frac{R(r_s - R)}{2r_s^3} e_0^0(s) e_0^1(s) ds + (r_s - R) e_0^2(s) ds$$

$$+ (r_s - R) \sin^2 \varphi_s e_0^3(s)^2 ds + dM^1_s,$$

$$d\sigma^2_0(s) = \frac{3\sigma^2_2}{2} e_0^1(s) ds - \frac{2}{r_s} e_0^1(s) e_0^2(s) ds + \sin \varphi_s \cos \varphi_s e_0^3(s) ds + dM^2_s,$$

$$d\sigma^3_0(s) = \frac{3\sigma^2_2}{2} e_0^1(s) ds - \frac{2}{r_s} e_0^1(s) e_0^2(s) ds - 2 \cot \varphi_s e_0^2(s) e_0^3(s) ds + dM^3_s,$$

where the martingale $M_s := (M^{0}_s, M^{1}_s, M^{2}_s, M^{3}_s)$ has the following rank 3 quadratic covariation matrix: $K_s = \sigma^2 (e_0(s) e_0(s) - g^{-1}(\xi_s)).$

### 2.3.2 Energy and angular momentum

We shall use widely the angular momentum $\vec{b} := r^2 \theta \wedge \tilde{\theta}$, the energy $a := (1 - \frac{R}{r}) \dot{t} = \frac{M}{r}$, and the norm of $\vec{b}$: $b := |\vec{b}| = r^2 U$, with $U := |\dot{\theta}|$.

Set also $T := \dot{r}$, and accordingly

$$T_s := \dot{r}_s = e_0^1(s), \quad U_s := |\tilde{\theta}_s| = \sqrt{e_0^0(s)^2 + \sin^2 \varphi_s e_0^3(s)^2}, \quad \text{and} \quad D := \min\{s > 0 | r_s = R\}.$$

Standard computations yield the following:
Lemma 2  
There exist a standard real Brownian motion \( w_s \) and a real process \( \eta_s \), almost surely converging in \( \mathbb{R} \) as \( s \nearrow D \), such that \( a_s = \exp(\sigma^2 s + \sigma w_s + \eta_s) \) for all \( s \in [0,D] \). In particular \( a_s \) almost surely cannot vanish, which means that time \( t_s \) is always strictly increasing.

Proof  
Proposition 1 above shows that \( (a_s^2 - 1) \sigma^2 ds \leq \langle dM^a_s \rangle \leq a_s^2 \sigma^2 ds \), for \( 0 \leq s < D \).

So that we have almost surely (as \( s \to \infty \), when \( D = \infty \)):

\[
\log a_s - \log a_0 = 3\sigma^2 s/2 - \frac{1}{2} \int_0^s a_t^{-2} \langle dM^a_t \rangle + \int_0^s a_t^{-1} dM^a_t \geq \sigma^2 s + \int_0^s a_t^{-1} dM^a_t
\]

\[
= \sigma^2 s + o\left(\int_0^s a_t^{-2} \langle dM^a_t \rangle\right) = \sigma^2 s + o(s).
\]

Since \( (1 - R/r_s) \leq a_s^2 \), this implies \( \int_0^D (1 - R/r_s) a_s^{-2} ds < \infty \) almost surely.

Proposition 1  
1) The unit pseudo-norm relation (which expresses that the parameter \( s \) is precisely the arc length, id est the so-called proper time) writes

\[ T_s^2 = a_s^2 - (1 - R/r_s)(1 + b_s^2/r_s^2). \]

2) The process \( (r_s, a_s, b_s, T_s) \) is a degenerate diffusion, with lifetime \( D \), which solves the following system of stochastic differential equations:

\[
dr_s = T_s ds, \quad dT_s = dM^T_s + \frac{3\sigma^2}{2} T_s ds + (r_s - \frac{3}{2} R) \frac{b_s^2}{r_s^4} ds - \frac{R}{2r_s^2} ds,
\]

\[
da_s = dM^a_s + \frac{3\sigma^2}{2} a_s ds, \quad db_s = dM^b_s + \frac{3\sigma^2}{2} b_s ds + \frac{\sigma^2 r_s^2}{2 b_s} ds,
\]

with quadratic covariation matrix of the local martingale \( (M^a, M^b, M^T) \) given by

\[ K'_s := \sigma^2 \begin{pmatrix}
a_s^2 - 1 + \frac{R}{r_s} & a_s b_s & a_s T_s \\
a_s b_s & b_s^2 + r_s^2 & b_s T_s \\
a_s T_s & b_s T_s & T_s^2 + 1 - \frac{R}{r_s}
\end{pmatrix}. \]

We get in particular the following statement, in which the dimension is reduced.

Corollary 1  
The process \( (r_s, b_s, T_s) \) is a diffusion, with lifetime \( D \) and infinitesimal generator

\[
\mathcal{G}' := T \frac{\partial}{\partial r} + \frac{\sigma^2}{2} (b^2 + r^2) \frac{\partial^2}{\partial b^2} + \frac{\sigma^2}{2b} (3b^2 + r^2) \frac{\partial}{\partial b} + \sigma^2 b T \frac{\partial^2}{\partial b \partial T} + \frac{\sigma^2}{2} \left(T^2 + 1 - \frac{R}{r}\right) \frac{\partial^2}{\partial T^2} + \left(\frac{3\sigma^2}{2} T + (r - \frac{3}{2} R) \frac{b^2}{r^4} - \frac{R}{2r^2}\right) \frac{\partial}{\partial T}.
\]

We have the following result on the behaviour of coordinate \( a_s \).

Lemma 2  
There exist a standard real Brownian motion \( w_s \), and a real process \( \eta_s \), almost surely converging in \( \mathbb{R} \) as \( s \nearrow D \), such that \( a_s = \exp(\sigma^2 s + \sigma w_s + \eta_s) \) for all \( s \in [0,D] \). In particular \( a_s \) almost surely cannot vanish, which means that time \( t_s \) is always strictly increasing.
Consider then the standard real Brownian motion \( w \) defined by
\[
dM_s = \sigma \sqrt{a_s^2 - (1 - R/r_s)} \, dw_s,
\]
and the process \( \eta \) defined by the formula in the statement. We have:
\[
d\eta_s = d(\log a_s) - \sigma^2 \, ds - \sigma \, dw_s = \frac{1}{2} (1 - R/r_s) a_s^{-2} \sigma^2 \, ds + \left( \sqrt{1 - (1 - R/r_s) a_s^{-2}} - 1 \right) \, \sigma \, dw_s,
\]
and then for any \( s < D \):
\[
\eta_s = \eta_0 + \frac{\sigma^2}{2} \int_0^s (1 - R/r_t) a_t^{-2} \, dt - \sigma \int_0^s \frac{(1 - R/r_t) a_t^{-2}}{1 + \sqrt{1 - (1 - R/r_t) a_t^{-2}}} \, dw_t,
\]
which converges almost surely to a finite limit as \( s \nearrow D \), since almost surely for all \( s \in ]0, D[ \):
\[
\left\langle \int_0^s \frac{(1 - R/r_t) a_t^{-2}}{1 + \sqrt{1 - (1 - R/r_t) a_t^{-2}}} \, dw_t \right\rangle \leq \int_0^s \left( (1 - \frac{R}{r_t}) a_t^{-2} \right)^2 \, dt < \int_0^D (1 - \frac{R}{r_t}) a_t^{-2} \, ds < \infty.
\]

2.3.3 Asymptotic behaviour of the relativistic diffusion \((\xi_s, \dot{\xi}_s)\)

We see in the appendix (Section 4.1) that in the geodesic case \( \sigma = 0 \) five types of behaviour can occur, owing to the trajectory of \((r_s)\); it can be:
- running from \( R \) to \( +\infty \), or in the opposite direction;
- running from \( R \) to \( R \) in finite proper time;
- running from \( R \) to some \( R_1 \) or from \( R_1 \) to \( +\infty \), or idem in the opposite direction;
- running for endlessly in a bounded region away from \( R \).

More detailed results can be found in [L-L] and mainly [M-T-W]. A full treatment is given, for future reference, in the appendix below (Section 4).

The stochastic case \( \sigma \neq 0 \) can be seen as a perturbation of the geodesic case \( \sigma = 0 \); however the asymptotic behaviour classification is quite different.

**Theorem 2**  
1) For any initial condition, the radial process \((r_s)\) almost surely reaches \( R \) within a finite proper time \( D \) or goes to \( +\infty \) as \( s \to +\infty \) (equivalently: as \( t(s) \to +\infty \) if \( a_0 > 0 \), and as \( t(s) \to -\infty \) if \( a_0 < 0 \)).

2) Both events in 1) above occur with positive probability, from any initial condition.

3) Conditionally on the event \( \{D = \infty\} \) of non-reaching the central body, the Schwarzschild relativistic diffusion \((\xi_s, \dot{\xi}_s)\) goes almost surely to infinity in some random asymptotic direction of \( \mathbb{R}^3 \), asymptotically with the velocity of light.

Note in particular that the relativistic diffusion almost surely cannot explode before the finite proper time \( D \).

The proof we give for this theorem is rather long. It is postponed till Section 3 below.
2.4 The full Schwarzschild space $S$

The full Schwarzschild space $S$, also known as the Kruskal-Szekeres space (see [DF-C], [F-N], [S], and especially [M-T-W] and its historical account page 822), can be defined by extending the previous restricted Schwarzschild space $S_0$ as follows. On $S_0$, set

$$u := \sqrt{\frac{r}{R} - 1} \times e^{r/2R} \times \text{ch} \left( \frac{1}{2R} \right) \quad \text{and} \quad v := \sqrt{\frac{r}{R} - 1} \times e^{r/2R} \times \text{sh} \left( \frac{1}{2R} \right).$$

Note that $(\frac{r}{R} - 1) \times e^{r/R} = u^2 - v^2$, and that the Schwarzschild pseudo-metric expresses in the Kruskal-Szekeres coordinates $(u, v, \theta)$ as:

$$\left( * * \right) \quad \frac{4R^3}{r} e^{r/R} (dv^2 - du^2) - r^2 |d\theta|^2,$$

where $r = r(u^2 - v^2)$, $r$ denoting the inverse function of $[r \mapsto (\frac{r}{R} - 1) e^{r/R}]$ (which is an increasing diffeomorphism from $\mathbb{R}_+$ onto $[-1, +\infty]$).

In those Kruskal-Szekeres coordinates, we have $S_0 = \{ (u, v, \theta) \in \mathbb{R}^2 \times S^2 \mid u > |v| \}$.

The full Schwarzschild space $S$ is now defined as

$$S := \{ (u, v, \theta) \in \mathbb{R}^2 \times S^2 \mid u^2 - v^2 > -1 \},$$

and is equipped with the pseudo-metric defined by $(**)$ above and by $r = r(u^2 - v^2)$.

$S$ contains $S_0$, $-S_0$ (isometric to $S_0$), two isometric copies of the hole:

$$\mathcal{H} := \{ (u, v, \theta) \in S \mid v > |u| \} \quad \text{and} \quad -\mathcal{H},$$

and the boundary between $\pm S_0$ and $\pm \mathcal{H}$, which is $\{ r = R \} = \{ u = \pm v \}$.

It is a Lorentz manifold, to which our general construction of section 2.2 applies.

The energy and angular momentum are extended to $T^1S$ by setting

$$a := \frac{2R^2}{r} e^{-r/R} (u\dot{v} - v\dot{u}) \quad \text{and} \quad \vec{b} := r^2 \theta \wedge \dot{\theta}. $$

As before we set $T := \dot{r} = \frac{2R^2}{r} e^{-r/R} (u\dot{v} - v\dot{u}) \quad \text{and} \quad b := |\vec{b}| = r^2 U$. Recall that $a$ and $\vec{b}$ are constant along geodesics. The unit pseudo-norm relation writes as before:

$$a^2 - T^2 + \left( \frac{R}{r} - 1 \right) \left( \frac{b^2}{r^2} + 1 \right) = 0,$$

or equivalently $\dot{u}^2 - \dot{v}^2 + \frac{r e^{-r/R}}{4R^3} \left(\frac{b^2}{r^2} + 1 \right) = 0$, which implies $|\dot{v}| > |\dot{u}|$, whence $\dot{v} > |\dot{u}|$ along a timelike path. The following correspondences between a line-element $(\xi, \dot{\xi}) \in T^1S$ and its projection $\xi \in S$ are easily deduced: $(\xi, \dot{\xi}) \in \{ r > R; a > 0 \} \Leftrightarrow \xi \in S_0,$

$(\xi, \dot{\xi}) \in \{ r > R; a < 0 \} \Leftrightarrow \xi \in -S_0$, $(\xi, \dot{\xi}) \in \{ r < R; T < 0 \} \Leftrightarrow \xi \in -\mathcal{H},$
\((\xi, \dot{\xi}) \in \{r < R; T > 0\} \Leftrightarrow \xi \in \mathcal{H}, (\xi, \dot{\xi}) \in \{r = R; a = T = 0\} \Leftrightarrow \xi \in \{u = v = 0\}\).}

Now two other coordinates, namely the so-called inward and outward Eddington-Finkelstein coordinates \(u^-\) and \(u^+\), prove to be very convenient for performing calculations on \(\mathcal{S}\). They are defined, not on the whole \(\mathcal{S}\), but \(u^-\) on \(\mathcal{S} \cap \{u + v \neq 0\}\) and \(u^+\) on \(\mathcal{S} \cap \{u - v \neq 0\}\), by:

\[
u^- := 2R \log |u + v| \quad \text{and} \quad u^+ := -2R \log |u - v|.
\]

In those Eddington-Finkelstein coordinates, the metric expresses as:

\[
(1 - \frac{R}{r}) (du^\pm)^2 \pm 2 du^\pm dr - r^2 d\theta^2,
\]

and the energy expresses as

\[
a = (1 - \frac{R}{r}) \dot{u}^\pm \pm i = \frac{\partial L}{\partial \dot{u}^\pm}.
\]

We shall need (from Section 2.4.2) to complete the space \(\mathcal{S}\), in \(\overline{\mathcal{S}} := \mathcal{S} \cup \partial \mathcal{S}\), where

\[
\partial \mathcal{S} := \left\{ \pm (u, v, \theta) \mid (u, v, \theta) \in \mathbb{R}^2 \times S^2 ; u^2 - v^2 = -1 \right\},
\]

meaning that we identify the opposite points above the singularity \(\{r = 0\}\). Note that \(r, u^+, u^-\) are naturally continued to \(\overline{\mathcal{S}}\), that \(r = 0 \iff v^2 = u^2 + 1 \iff u^+ = u^-\) in \(\overline{\mathcal{S}}\), and that

\[
u^+ + r + R \log |\frac{r}{R} - 1| = u^- - r - R \log |\frac{r}{R} - 1| \quad \text{on} \quad \overline{\mathcal{S}} \cap \{|u| \neq |v|\},
\]

this quantity being equal to \(t\) on \(\pm \mathcal{S}_0\). The \(\mathbb{R}\)-valued absolute time \(t\) is continuously extended to \(\mathcal{S} \cap \{u = v = 0\}\), by setting \(t = +\infty\) on \(\{u = v \neq 0\}\), \(t = -\infty\) on \(\{u = -v \neq 0\}\), \(t = 2R \text{argth}(u/v)\) on \(\pm \mathcal{H}\), and \(t = 2R \text{argth}(v/u)\) on \(-\mathcal{S}_0\) (as on \(\mathcal{S}_0\)).

Note that the region \(\mathcal{A} := \{u = v = 0\}\) appears right away as exceptional, as the only part of \(\overline{\mathcal{S}}\) where \(t\) cannot be continued, and the only part of \(\overline{\mathcal{S}}\) where both \(u^\pm\) explode.

In the Kruskal-Szekeres coordinates, a path \((u_s, v_s, \theta_s)\) is timelike if and only if \(\dot{v}_s > |\dot{u}_s|\). This implies that any timelike path started in \(\mathcal{H}\) has to hit \(\{v = \sqrt{u^2 + 1}\} \subset \{r = 0\}\), and that any timelike path started in \(-\mathcal{H}\) has to hit \(\{r = R\}\). In particular, a timelike path started in \(-\mathcal{H}\) and avoiding \(\mathcal{A}\) has to enter the region \(\{r > R\}\) through \(\{t = -\infty\}\), appearing either as a particle born and then evolving in \(\mathcal{S}_0\) (for ever, or entering then \(\mathcal{H}\) through \(\{t = +\infty\}\)), or the analogue through \(-\mathcal{S}_0\), which could be viewed as the case of an antiparticle through \(\mathcal{S}_0\).

These dynamics of the full Schwarzschild space \(\mathcal{S}\) show that the inward Eddington-Finkelstein coordinate \(u^-\) is appropriate to the study of timelike paths started in \(\mathcal{S}_0 \cup \mathcal{H}\), till they hit \(\{r = 0\}\) (and even to extend them further, see section 2.4.2 below), and that the outward Eddington-Finkelstein coordinate \(u^+\) is appropriate to the study of timelike paths started in \(-\mathcal{H}\) and entering \(\pm \mathcal{S}_0\), till they hit \(\mathcal{H}\).
2.4.1 Diffusion in the full space $S$: hitting the singularity

We follow the same route as in the restricted Schwarzschild space, to express the relativistic diffusion on $T^1 S$. Let us proceed, using the Eddington-Finkelstein coordinates $u^\pm$. The Lagrangian $L(\xi, \eta)$ writes

$$2 L(\xi, \eta) = (1 - \frac{R^3}{r^3}) (u^\pm)^2 \pm 2 u^\pm \dot{r} - r^2 \dot{\varphi}^2 - r^2 \sin^2 \varphi \dot{\psi}^2.$$

We apply then Remark 3, to get the Itô stochastic differential equations of the relativistic diffusion in the full Schwarzschild space:

$$\begin{align*}
    d\dot{u}_s^\pm &= \sigma dM_s^\pm + \frac{3a^2}{2} \dot{u}_s^\pm ds \pm \frac{R}{2r_s^2} (\dot{u}_s^\pm)^2 ds \mp r_s (\dot{\varphi}_s^2 + \sin^2 \varphi_s \dot{\psi}_s^2) ds; \\
    d\dot{r}_s &= \sigma dM_s^r + \frac{3a^2}{2} \dot{r}_s ds + (\frac{R}{r_s} - 1) \frac{R}{2r_s^2} (\dot{u}_s^\pm)^2 ds \mp \frac{R}{r_s^2} \dot{u}_s^\pm \dot{r}_s ds + (r_s - R) (\dot{\varphi}_s^2 + \sin^2 \varphi_s \dot{\psi}_s^2) ds; \\
    d\dot{\varphi}_s &= \sigma dM_s^{\varphi} + \frac{3a^2}{2} \dot{\varphi}_s ds - \frac{2}{r_s} \dot{r}_s \dot{\varphi}_s ds + \sin \varphi_s \cos \varphi_s \dot{\psi}_s^2 ds; \\
    d\dot{\psi}_s &= \sigma dM_s^{\psi} + \frac{3a^2}{2} \dot{\psi}_s ds - \frac{2}{r_s} \dot{r}_s \dot{\psi}_s ds - 2 \cot \varphi_s \dot{\varphi}_s \dot{\psi}_s ds,
\end{align*}$$

for some continuous local martingale $(M_s^\pm, M_s^r, M_s^{\varphi}, M_s^{\psi})$, having quadratic covariance matrix (according to Lemma 1 and Section 2.3.1):

$$\begin{pmatrix}
    (\dot{u}_s^\pm)^2 & \dot{u}_s^\pm \dot{r}_s + 1 & \dot{u}_s^\pm \dot{\varphi}_s & \dot{u}_s^\pm \dot{\psi}_s \\
    \dot{u}_s^\pm \dot{r}_s + 1 & \dot{r}_s^2 + 1 - \frac{R}{r_s} & \dot{r}_s \dot{\varphi}_s & \dot{r}_s \dot{\psi}_s \\
    \dot{u}_s^\pm \dot{\varphi}_s & \dot{r}_s \dot{\varphi}_s & \dot{\varphi}_s^2 + r_s^{-2} & \dot{\varphi}_s \dot{\psi}_s \\
    \dot{u}_s^\pm \dot{\psi}_s & \dot{r}_s \dot{\psi}_s & \dot{\varphi}_s \dot{\psi}_s & \dot{\psi}_s^2 + (r_s \sin \varphi_s)^{-2}
\end{pmatrix}.$$

We have again a reduced diffusion $(r_s, b_s, T_s)$ (with minimal dimension), solving the same system of Itô stochastic differential equations as before.

This system of stochastic differential equations has been derived using Eddington-Finkelstein coordinates, so that it is valid a priori outside $\{u = v = 0\}$. But the smooth functions $(r, a, b, T)$ of the relativistic diffusion have an Itô decomposition with continuous coefficients, so that the formulas involving them hold without restriction.

From the pseudo-norm relation, we see that $T_s$ cannot vanish in the region $\{r < R\}$. As $r_s$ enters this region necessarily almost surely with derivative $T_D < 0$ (indeed $|T_D| = |a_D| > 0$ by Lemma 2), $r_s$ is then necessarily strictly decreasing. Precisely, we have the following.

**Theorem 3** The relativistic diffusion in $T^1 S$ either escapes to infinity, or enters above $\mathcal{H}$ at time $D$ and converges to the singularity within some finite proper time $D'$. Moreover, in the second case we have almost surely:

1) for $s \geq D$, $r_s$ decreases and hits 0 at proper time $D'$, with $D < D' \leq D + \frac{4}{7} R$; moreover $\lim_{s \nearrow D'} T_s = -\infty$;

2) $\theta_s$, $a'_s$, $b_s$ and $a_s$ converge to finite limits as $s \nearrow D'$, and $b_{D'}$ cannot vanish;
3) as \( s \searrow D' \), we have the following equivalents:

\[
\begin{align*}
  r_s &\sim \left[ \frac{5}{2} b_{D'} \sqrt{R} (D' - s) \right]^{3/5} \\
  T_s &\sim -b_{D'} \sqrt{R} \times \left[ \frac{5}{2} b_{D'} \sqrt{R} (D' - s) \right]^{-3/5}.
\end{align*}
\]

**Remark 5** The equivalents in Theorem (3,3) above can be specified further. Indeed, we have almost surely, as \( s \searrow 0 \):

\[
\begin{align*}
  r_{D' - s} &\sim \left[ \frac{5}{2} b_{D'} \sqrt{R} s \right]^{2/5} \times \left( 1 + \left( \frac{5b_{D'}}{2R^2} s \right)^{2/5} + O\left( \sqrt{s \log | \log s |} \right) \right),
  \\
  T_{D' - s} &\sim -b_{D'} \sqrt{R} \times \left[ \frac{5}{2} b_{D'} \sqrt{R} s \right]^{-3/5} \times \left( 1 - 2\left( \frac{5b_{D'}}{2R^2} s \right)^{2/5} + O\left( \sqrt{s \log | \log s |} \right) \right).
\end{align*}
\]

Indeed, using the stochastic differential equation of \( T_s \) and the iterated logarithm law, together with the equivalents in Theorem (3,3), we deduce easily these more precise asymptotic expansions near \( D' \).

**Remark 6** We know from Theorem 1 that the relativistic diffusion can start from any initial condition in the full space \( T^1 S \). When it starts above \( -\mathcal{H} \), the pseudo-norm relation forbids any vanishing of \( T_s \) (which has then to remain \( > 0 \)), till the level \( \{ r = R \} \) is hit, which takes a proper time less than \( \pi R/2 \), for the very same reason as in Theorem (3,1). When the diffusion starts above \( \{ r = R \} \), it enters \( \{ r \neq R \} \) at once, as any timelike path. Note that above \( A \equiv \{ u = v = 0 \} \subset \{ r = R \} \), we have necessarily \( T = a = 0 \). Moreover it can be proved that \( T^1 A \) is polar for the relativistic diffusion. So, when starting above \( -\mathcal{H} \), the relativistic diffusion enters then above \( \pm S_0 \), before possibly entering later above \( \mathcal{H} \).

We postpone the proof of Theorem 3 till Section 3 below.

A part of this proof is based on the following proposition, which allows to recover the whole relativistic diffusion from the reduced relativistic diffusion \( (r_s, b_s, T_s) \).

**Proposition 2** The spherical coordinate \( \theta_s \) satisfies the following stochastic differential equation (conditionally on the reduced relativistic diffusion \( (r_s, b_s, T_s) \)):

\[
d\left( \frac{\dot{\theta}_s}{U_s} \right) = \left( \frac{r_s}{b_s} \sigma \, d\beta_s \right) \dot{\theta}_s \wedge \frac{\dot{\theta}_s}{U_s} - \left( \frac{b_s}{r^2_s} \, ds \right) \theta_s - \left( \frac{\sigma^2 r^2_s}{2 b^2_s} \, ds \right) \frac{\dot{\theta}_s}{U_s},
\]

for some standard real Brownian motion \( \beta_s \), which is independent of \( (r_s, b_s, T_s) \).

Moreover, \( \dot{\theta}_s/U_s \) converges in \( S^2 \) as \( s \searrow D' \), almost surely.

We postpone the proof of this proposition till Section 3 below.

**Corollary 2** The curve in the full space \( S \) defined by the image of the trajectory \( \{(r_s, u_s, \theta_s) | s \leq D'\} \) admits almost surely a semi-tangent at the center of the hole.
Proof Using the strict monotonicity of $r_s$ near the singularity, we see that it is sufficient to verify the left-differentiability of the curve $(r_s \mapsto (u_s^-, r_s \theta_s))$ at $s = D'$. Now using Theorem (2.3), as $s \nearrow D'$, on one hand we have
\[
\frac{\partial u_s^-}{\partial r_s} = \frac{\dot{u}_s^-}{T_s} = (a_s/T_s + 1) r_s/(r_s - R) \sim -r_s/R \rightarrow 0,
\]
and on the other hand
\[
\frac{\partial (r_s \theta_s)}{\partial r_s} = \theta_s + (\dot{\theta}_s r_s/T_s) \rightarrow \theta_{D'} \in S^2, \text{ since}
\]
\[
|\dot{\theta}_s| r_s/T_s = b_s/(r_s T_s) \sim -R^{-1/2} \left[ \frac{\sqrt{2}}{2} b_{D'} \sqrt{R} (D' - s) \right]^{1/5} \rightarrow 0.
\]

Remark 7 Since we have $|\dot{\theta}_s| = U_s = b_s/r_s^2$, we observe the explosion of the spherical speed $\dot{\theta}_s$, as well as of the radial speed $T_s$, and as of the speed $\dot{u}_s^-$, at proper time $D'$, i.e., at the hitting of the singularity $\{r = 0\}$. However we also just saw that $(r_s, \vec{b}_s, a_s, \theta_s, u_s^-)$ is left-continuous at $D'$, and that moreover the curve in the space $S$ defined by the image of the trajectory $\{(r_s, u_s^-, \theta_s) \mid s \leq D'\}$ admits almost surely a tangent at any point. Indeed it happens that the explosion of the derivatives does not forbid to define a continuation of the relativistic diffusion after the finite hitting proper time $D'$. This will be indeed the purpose of Section 2.4.2 below.

2.4.2 Regeneration through the singularity: entrance law after $D'$

We see from Theorem 3 and Remark 7 that the set of endpoints of the relativistic paths (at proper hitting time $D'$ of $\{r = 0\}$) identifies with the boundary $\partial S$, which we defined in Section 2.4, identifying pointwise the outward $\partial H$ and inward $\partial(-H)$ boundaries. Thus we have indeed $\partial S \equiv \partial T^1 S$.

In this identification, a differentiable inward path ending at $\{r = 0\}$ can be continued by a differentiable outward path, so that the $\mathbb{R}^3$-valued curve $r \theta$ is differentiable at any point. In particular, geodesics are thus well defined for any proper time, and there are geodesics which cross endlessly the singularity $\{r = 0\}$, namely those which are described in case 1.2 (and are met also in cases 2.2.1, 2.2.2, 2.5.2, and 2.6), completed by Remark 13, of Section 4. For generic values of parameters $(a, \vec{b}, k)$, such geodesics are dense in some disk of $\mathbb{R}^3$ (centred at 0).

Since the diffusion can hit the singularity $\{r = 0\}$ in finite time, it is natural to look for an entrance law, allowing to continue it after time $D'$. Clearly it as to enter above $-H$. Thus we have to define for the diffusion on $T^1 S$ a family of entrance laws above the singularity $\{r = 0\}$, and more precisely on the boundary $\partial S$.

Let $\mathcal{G}$ denote the generator of the relativistic diffusion, acting on $C^2(T^1 S)$. Theorem 3 allows us to extend the relativistic diffusion to a continuous strong Markov process on $T^1 S \cup \partial T^1 S$, provided we establish the following proposition.

Proposition 3 The martingale problem associated with $\mathcal{G}$ has a unique continuous solution, starting from any point of $\partial S$.

We postpone the proof of this proposition till Section 3 below.
2.4.3 The relativistic diffusion for all positive proper times

The existence and uniqueness of the entrance law (in Proposition 3) allows to prove the first assertion of the following.

**Theorem 4** There exists a unique continuous strong Markov process on \( \mathcal{S} \) with positive lifetime inducing the relativistic diffusion on \( \mathcal{S} \).

The lifetime of this extended relativistic diffusion is almost surely infinite.

For such a process, we define an increasing sequence of hitting proper times \( D_k \) as follows. Let \( D_0 := D \in [0, \infty] \) denote the hitting time of \( \{ r \leq R \} \), \( D_1 := D' \) denote the hitting time of \( \{ r = 0 \} \), and set by induction, for any \( n \in \mathbb{N}^* \):

- \( D_{3n} := \inf \{ s > D_{3n-1} | r_s = R \} \), \( D_{3n+1} := \inf \{ s > D_{3n} | r_s = 0 \} \), and
- \( D_{3n+2} := \inf \{ s > D_{3n+1} | r_s = R \} \).

Finally consider \( D_\infty := \sup_n D_n \).

This is obviously an increasing sequence of stopping times, strictly increasing as long as it is finite. The preceding section 2.4.2 extends in a unique way the law of the relativistic diffusion to the proper time interval \([0, D_\infty] \). It is clear that the process cannot be extended continuously beyond \( D_\infty \).

This proves the first assertion of Theorem 4, within the lifetime \( D_\infty \). We postpone till Section 3 the proof of the second assertion of this theorem: \( D_\infty \) is almost surely infinite.

**Remark 8** We saw in Theorem (3.1) that \( D_1 \leq D_0 + \frac{\pi}{2} R \). Now exactly the same reason shows that \( D_{3n+1} - D_{3n} \) and \( D_{3n+2} - D_{3n+1} \) are \( \leq \frac{\pi}{2} R / 2 \), as long as these times are finite. The time intervals \([D_{3n-1}, D_{3n}]\) correspond to the excursions outside the hole, and the time intervals \([D_{3n}, D_{3n+2}]\) correspond to the excursions inside the hole. Moreover we see that \( D_k \) becomes infinite if and only if \( k = 3n \) and the process escapes to infinity during its \( n \)-th excursion outside the hole.

Recall that (according to Lemma 2) during every excursion outside the hole \( \{ r \leq R \} \), the diffusion can have its absolute time coordinate \( t = t_s \) strictly increasing from \(-\infty\) to \(+\infty\) or strictly decreasing from \(+\infty\) to \(-\infty\) (this case can be seen as the antiparticle case), depending on the sign of \( a \) at the exit of the hole.

Moreover, the \( \mathbb{R}^3 \)-valued curve \((s \mapsto r_s \theta_s)\) is differentiable at any proper time \( s \), whereas the \( \mathbb{R} \)-valued curves \((s \mapsto u_s)\) present a cusp (with half a tangent: this appears in the proof of Theorem 3, where we saw that \( \dot{u} \sim -r T / R \to \pm \infty \) near \( D' \)) at proper times \( D_{3n+1} \) (and are differentiable at any time \( s \neq D_{3n+1} \)).

**Remark 9** The Liouville measure is invariant for the extended relativistic diffusion on \( \mathcal{S} \). It induces the invariant measure \( dr \, da \, dT \) of the autonomous diffusion \((r_s, a_s, T_s)\).
2.4.4 Capture of the diffusion by a neighbourhood of the hole

The preceding section leads naturally to the following question: can the extended relativistic diffusion cross infinitely many times the hole, as some geodesics do?

Note first that it was clear from the preceding sections 2.3.3, 2.4.2, and 2.4.3, that there is, for any \( n \in \mathbb{N} \) and any initial condition, a positive probability that the extended relativistic diffusion crosses exactly \( n \) times the hole and thereafter goes away to infinity. So the following theorem asserts essentially that the limiting case of \( n = \infty \) crossings of the hole can also happen, thereby completing the picture of all possible asymptotic behaviours of the extended relativistic diffusion.

Moreover it says that this last case corresponds to an asymptotic confinement of the relativistic diffusion in the vicinity of the hole.

**Theorem 5** Almost surely, from any initial condition, the extended relativistic diffusion can have only two types of asymptotic behaviour, each occurring with positive probability:

Either

1) \( r \) and \( |a| \) go away to infinity, and \( \theta \) converges;

or

2) \( \rho := \limsup_{s \to \infty} r_s \in [R, 3R/2], \liminf_{s \to \infty} r_s = 0 \), \( b \) goes away to infinity, \( \bar{b}/b \) converges, and \( a/b \) converges to \( \ell = \pm \frac{1}{\rho} \sqrt{1 - \frac{R}{\rho}} \).

Moreover

(i) For any \( \varepsilon > 0 \), if \( r_0 > 3R/2 \) and if \( T_0 \) is large enough, then the probability that the relativistic diffusion goes away to infinity is at least \( 1 - \varepsilon \).

(ii) For any \( \varepsilon > 0 \), if \( r_0 \in ]R, 3R/2[, T_0 = 0 \), and \( b_0 \) is large enough, then
\[
\mathbb{P}(|\rho - r_0| < \varepsilon) > 1 - \varepsilon.
\]

The proof of this theorem is rather delicate. It will be presented in Section 3 below.

**Remark 10** For any open interval \( ]x, y[ \subset ]R, 3R/2[, \mathbb{P}(x < \rho < y) \) is a non-trivial \( SO_3 \)-invariant \( (\mathcal{L}_0 + \frac{\sigma_2}{2} \Delta_v) \)-harmonic function on \( T^1 \mathcal{M} \).

The following result describes more precisely what happens when the diffusion is captured by a neighbourhood of the hole: while (according to Theorem 5) they are asymptotically planar, they exhibit progressively another type of regularity.

**Corollary 3** In the second case of Theorem 5, and more precisely conditionally on the event \( \rho < 3R/2 \), the times \( D'_n \) of the first maxima of the radius \( r \) at each excursion out of the hole are such that \( \lim_{n \to \infty} r_{D'_n} = \rho \) and that \( \int_{D'_{3n+1}}^{D'_n} d\theta_s \) converges as \( n \to \infty \), towards
\[
\pm \int_0^\rho \frac{dr}{\sqrt{[R - r + \ell^2 r^3] r}} \quad \text{(which is well defined if and only if} \ \rho < 3R/2)\).
The proof of this corollary will close Section 3 below.

Remark 11 The result of Corollary 3 concerns the time intervals \([D_{3n+1}, D'_{n+1}],\) that is to say the upcrossings from the singularity to the successive tops of the limiting trajectories. It is very likely that the same result is valid as well for the downcrossings, that is to say the time intervals \([D'_{n-1}, D_{3n+1}],\) yielding the same angular random limit (the sign of \(T_s\) compensating for the interchange of the bounds \(D'_{n-1}\) and \(D_{3n+1}\) in the integral).

So another statement in the spirit of Corollary 3, but which demands some more work, should be: almost surely \(\lim_{n \to \infty} \int_{D'_{n-1}}^{D'_{n+1}} d\theta_s = \pm 2 \Psi,\) where

\[
\Psi := \int_0^e \frac{dr}{\sqrt{r[R-r+t^2r^3]}} = \int_0^1 \frac{dr}{\sqrt{r[(1-r^3)(R/g)-(1-r^2)r]}}
\]

is a strictly increasing continuous function of \(g/R,\) from \([1, \frac{3}{2}]\) onto \([\frac{\pi}{2}, \infty[.\)

It is thus likely that the shape of the excursions should approach more and more the null geodesics, id est the light rays. See the appendix, Section 4.2.

3 Proofs

3.1 Proof of Theorem 2

In the proof of this theorem, we shall use the following very simple lemma.

Lemma 3 Let \(M\) be a continuous local martingale, and \(A\) a process such that \(\liminf \frac{A_s}{\langle M \rangle_s} > 0\) almost surely on \(\{\langle M \rangle = \infty\}.\) Then \(\lim_{s \to \infty} (M_s + A_s) = +\infty\) almost surely on \(\{\langle M \rangle = \infty\}.\)

Proof Writing \(M_s = W(\langle M \rangle_s)\), for some real Brownian motion \(W,\) we find almost surely some \(\varepsilon > 0\) and some \(s_0 \geq 0\) such that \(A_s \geq 2\varepsilon \langle M \rangle_s\) and \(|M_s| \leq \varepsilon \langle M \rangle_s\) for \(s \geq s_0.\) Whence \(M_s + A_s \geq \varepsilon \langle M \rangle_s\) for \(s \geq s_0.\)

We prove now successively the 3 assertions of Theorem 2.

1) Almost sure convergence on \(\{D = \infty\}\) of \(r_s\) to \(\infty.\)

This proof will be split into six parts.

Let us denote by \(A\) the set of paths with infinite lifetime \(D\) such that the radius \(r_s\) does not go to infinity. We have to show that it is negligible for any initial condition \(x = (r, b, T) = (r_0, b_0, T_0)\) belonging to the state space \([R, \infty] \times \mathbb{R}_+ \times \mathbb{R}.\)

The cylinder \(\{r = 3R/2\}\) plays a remarkable rôle in Schwarzschild geometry. In particular, it contains light lines. We see in the following first part of proof that we have to deal with this cylinder.
(i) \( r_s \) must converge to \( 3R/2 \), almost surely on \( A \).

Observe from the unit pseudo-norm relation (Proposition 1, 1) that \( |T_s|/a_s \) is bounded by 1. Let us apply Itô’s formula to \( Y_s := (1 - \frac{3R}{2r_s}) T_s/a_s \):

\[
Y_s = M_s + \frac{3R}{2} \int_0^s \frac{T^2_t}{a_t r^2_t} \, dt + \frac{b^2}{a_s} \int_0^s \frac{b^2}{a_t r^2_t} \, dt - \sigma^2 \int_0^s \frac{Y_t}{a^2_t} \, dt - \frac{b^2}{2a_t r^2_t} \, dt ,
\]

with some local martingale \( M \) having quadratic variation:

\[
\langle dM \rangle = (1 - \frac{3R}{2r_s})^2 (1 - \frac{R}{r_s}) \sigma^2 \frac{a^2_s \, ds}{a^3_s} \leq \sigma^2 a^2_s \, ds .
\]

Now \( |Y_s| \) is also bounded by 1. Hence Lemma 2 implies that the last two terms in the expression of \( Y_s \) above have almost surely finite limits as \( s \to \infty \). Idem for \( \langle M_s \rangle \), and then for \( M_s \). Moreover the two remaining bounded variation terms in the expression of \( Y_s \) above increase. As a consequence, we get that \( Y_s, \int_0^s \frac{T^2_{t}}{a_t r^2_t} \, dt, \text{ and } \int_0^s (1 - \frac{3R}{2r_s})^2 \frac{b^2}{a_t r^2_t} \, dt \)

converge almost surely in \( \mathbb{R} \) as \( s \to \infty \). So does also \( \int_0^s \frac{dt}{a_t r^2_t} \).

Now using that \( \frac{a}{r^2} \leq \left( a + \frac{R}{r^2} \left( \frac{b^2}{a r^2} + \frac{1}{a} \right) \right) r^{-2} = \frac{T^2}{a r^2} + \frac{b^2}{a r^2} + \frac{1}{a} r^2 \) by the unit pseudo-norm relation, we deduce that almost surely

\[
\int_0^\infty (1 - \frac{3R}{2r_s})^2 \frac{d}{dt} \frac{1}{r_s} \right) \, dt = \int_0^\infty (1 - \frac{3R}{2r_s})^2 \frac{|T_t|}{r^2_t} \, dt \leq \int_0^\infty (1 - \frac{3R}{2r_s})^2 \, dt < \infty .
\]

This implies the almost sure convergence of \( \left( 1 - \frac{3R}{2r_s} + \frac{3R^2}{4r^2_s} \right) /r_s \), and therefore of \( (1/r_s) \).

Since \( \lim_{s \to \infty} (1/r_s) \) cannot be 0 on \( A \), we have necessarily \( \lim_{s \to \infty} r_s = 3R/2 \) almost surely on \( A \), from the convergence of \( \int_0^\infty (1 - \frac{3R}{2r_s})^2 \, dt \).

(ii) \( b_s/a_s \) converges to \( 3R\sqrt{3}/2 \), and \( T_s/b_s \) goes to 0, almost surely on \( A \).

Indeed, Itô’s formula gives (for some real Brownian motion \( w \))

\[
\frac{b^2}{a^2} = \frac{b^2_0}{a^2_0} + 2\sigma \int_0^s \frac{b^2}{a^2} \frac{r^2_t}{b^2_s} - \frac{1}{r^2_t} \, dw_s + 2\sigma^2 \int_0^s \frac{r^2_t}{a^2} \, ds - 3\sigma^2 \int_0^s (1 - \frac{R}{r_s}) \frac{b^2}{a^2} \, ds .
\]

Since by the unit pseudo-norm relation we have \( \frac{b^2}{a^2} < r^2/(1 - \frac{R}{r_s}) \), whence \( b^2/a^2 \) bounded on \( A \), the above formula and Lemma 2 imply the almost sure convergence of \( b^2/a^2 \) on \( A \). Indeed the bounded variation terms converge, and as \( b^2/a^2 \) is positive, the martingale part has to converge also. Using the unit pseudo-norm relation again, we deduce that

\[
\frac{T^2_s}{a^2} = 1 - (1 - \frac{R}{r_s}) \left( \frac{b^2_s}{a^2 r^2_s} + \frac{1}{a^2} \right) \text{ has also to converge, necessarily to 0, since otherwise we}
\]
would have an infinite limit for $T_s$, which is clearly impossible on $A$. The value of the limit of $b_s/a_s$ follows now directly from this and from (i).

(iii) We have almost surely on $A$: $\int_0^\infty (r_t - \frac{3R}{2})^2 b_t^2 \, dt < \infty$, and $\int_0^\infty T_t^2 \, dt < \infty$.

Let us write Itô’s formula for $Z_s := (r_s - \frac{3R}{2}) T_s = \frac{1}{2} \frac{d}{ds} (r_s - \frac{3R}{2})^2$:

$$Z_s = Z_0 + M_s + \frac{3a^2}{4} (r_s - r_0)(r_s + r_0 - 3R) + \int_0^s T_t^2 \, dt + \int_0^s (r_t - \frac{3R}{2})^2 b_t^2 \frac{dt}{r_t^3} - \frac{R}{2} \int_0^s (r_t - \frac{3R}{2}) \frac{dt}{r_t^2},$$

where $M_s$ is a local martingale having quadratic variation given by:

$$\langle M \rangle_s = \sigma^2 \int_0^s (r_t - \frac{3R}{2})^2 (1 - \frac{R}{r_t} + T_t^2) \, dt.$$

Note that if $\langle M \rangle_\infty = \infty$, then by (ii) above $\lim_{s \to \infty} \int_0^s (r_t - \frac{3R}{2})^2 b_t^2 \frac{dt}{r_t^3} / \langle M \rangle_s = \infty$.

Note moreover that in this case $\int_0^s |r_t - \frac{3R}{2}| \frac{dt}{r_t^3} \leq \sqrt{\int_0^s (r_t - \frac{3R}{2})^2 b_t^2 \frac{dt}{r_t^3}} \times \sqrt{\int_0^s \frac{dt}{b_t^2}}$ is also negligible with respect to $\int_0^s (r_t - \frac{3R}{2})^2 b_t^2 \frac{dt}{r_t^3}$.

On the other hand, we must have $\liminf_{s \to \infty} |Z_s| = 0$ on $A$.

Therefore we deduce from Lemma 3 that necessarily $\langle M \rangle_\infty < \infty$, and then that $M_s$ has to converge, almost surely on $A$.

Using again that $\liminf_{s \to \infty} |Z_s| = 0$, we deduce the almost sure boundedness and convergence on $A$ of $\int_0^\infty T_t^2 \, dt$ and of $\int_0^\infty (r_t - \frac{3R}{2})^2 b_t^2 \frac{dt}{r_t^3}$.

(iv) $(r_s - \frac{3R}{2})^2 b_s$ and $T_s^2/b_s$ go to 0 as $s \to \infty$, almost surely on $A$.

Indeed, on one hand we deduce from (iii) that (for some real Brownian motion $W$)

$$(r_s - \frac{3R}{2})^2 b_s = \sigma W \left[ \int_0^s (r_t - \frac{3R}{2})^4 (b_t^4 + r_t^4) \, dt \right] + 2 \int_0^s (r_t - \frac{3R}{2}) T_t b_t \, dt + \frac{a^2}{2} \int_0^s (r_t - \frac{3R}{2})^2 (b_t^4 + r_t^4) \, dt$$

has to converge almost surely on $A$ as $s \to \infty$, necessarily to 0 since it is integrable with respect to $s$.

On the other hand we have for some real Brownian motion $W'$, by Itô formula:

$$\frac{T_s^2}{b_s} = \frac{T_0^2}{b_0} + \sigma W' \left[ \int_0^s \left( \frac{T_t^4}{b_t^4} + r_t^4 \frac{T_t^4}{b_t^4} + 4 \left( 1 - \frac{R}{r_t} \right) \frac{T_t^2}{b_t^4} \right) \, dt \right] + \frac{a^2}{2} \int_0^s \frac{T_t^2}{b_t^4} \, dt + 2 \int_0^s \frac{T_t^2}{b_t} \, dt + \int_0^s (r_t - \frac{3R}{2}) T_t b_t \frac{dt}{r_t^3}$$

$$- \int_0^s \frac{R T_t}{r_t^2 b_t} \, dt + \sigma^2 \int_0^s \left( 1 - \frac{R}{r_t} \right) \frac{dt}{b_t} + \frac{a^2}{2} \int_0^s \frac{r_t^2 T_t^2}{b_t^4} \, dt.$$

Recall from (i) that $\frac{T_t}{b_t} \to 0$ and that $b_t \sim \frac{3R}{2} a_t$. Thus using (iii) we see easily that all
integrals in the above formula converge. Hence we deduce the almost sure convergence of 
$s \mapsto T^2_s/b_s$ on $A$, necessarily to 0, since it is integrable.

(v) It is sufficient to show that $\int_0^\infty |r_t - 3R| |T_t| b_t^2 dt < \infty$, and that $\int_0^\infty T_t^4 dt < \infty$, almost surely on $A$.

Indeed, assuming that these 2 integrals are finite, Itô’s formula shows that we have for some real Brownian motion $W''$:

$$T^2_s = T^2_0 + 2\sigma W'' \left[ \int_0^s (T^2_t + 1 - \frac{R}{r_t}) T_t^2 dt \right] + 4\sigma^2 \int_0^s T_t^2 dt + 2 \int_0^s (r_t - \frac{3R}{r_t}) T_t b_t^2 \frac{dt}{r_t}$$

$$+ \sigma^2 \int_0^s (1 - \frac{R}{r_t}) dt - R \int_0^s \frac{T_t}{r_t} dt$$

$$= \gamma_s + \sigma^2 \int_0^s (1 - \frac{R}{r_t}) dt - R \int_0^s \frac{T_t}{r_t} dt = \gamma_s + \int_0^s \left[ \frac{1}{2} + \frac{2}{3r_t} (r_t - \frac{3R}{r_t}) \right] dt + \frac{R}{r_s} - \frac{R}{r_0} = \gamma_s + s/3,$$

where $\gamma, \gamma'$ (since $|r_t - \frac{3R}{r_t}| = o(b_t^{-1/2}) = o(a_t^{-1/2})$ by (iv) and (ii)) are bounded converging processes on $A$. Whence $\lim_{s \to \infty} T^2_s = \infty$ almost surely on $A$, which with (iii) above implies that $A$ must be negligible.

(vi) End of the proof of the convergence of $r_s$ to 0 on $\{ D = \infty \}$.

By Schwarz inequality, the first bound in (v) above will follow from $\int_0^s T_t^2 b_t dt < \infty$ and from $\int_0^s (r_t - \frac{3R}{r_t})^2 b_t^3 dt < \infty$. Now these two terms appear in the Itô expression for $Z^1_s := (r_s - \frac{3R}{r_s}) T_s b_s$:

$$Z^1_s = Z^1_0 + M^1_s + \sigma^2 \int_0^s \left[ 8 + \frac{r_t^2}{b_t^2} \right] Z_t^1 dt + \int_0^s T_t^2 b_t dt + \int_0^s (r_t - \frac{3R}{r_t})^2 b_t^3 \frac{dt}{r_t} - \frac{R}{r_s} \int_0^s (r_t - \frac{3R}{r_t}) b_t^3 \frac{dt}{r_t} ,$$

with a local martingale $M^1$ having quadratic variation:

$$\langle M^1 \rangle_s = \sigma^2 \int_0^s (r_t - \frac{3R}{r_t})^2 b_t^2 \times \left( 1 - \frac{R}{r_t} + [4 + r_t^2 b_t^{-2}] T_t^2 \right) dt .$$

Note that by Schwarz inequality, (iii) above implies that $\int_0^\infty |Z_t^1| dt < \infty$, and then that $\int_0^s \left[ 8 + \frac{r_t^2}{b_t^2} \right] Z_t^1 dt$ is bounded and converges, almost surely on $A$, as $s \to \infty$.

Using the first assertion of (iv), observe that $\lim_{s \to \infty} \int_0^s (r_t - \frac{3R}{r_t})^2 b_t^3 \frac{dt}{r_t} + \int_0^s T_t^2 b_t dt = \langle M^1 \rangle_s = \infty$ if $\langle M^1 \rangle_\infty = \infty$. Note moreover that in this case

$$\int_0^s (r_t - \frac{3R}{r_t})^2 b_t \frac{dt}{r_t} \leq \sqrt{\int_0^s (r_t - \frac{3R}{r_t})^2 b_t^3 \frac{dt}{r_t^4}} \times \sqrt{\int_0^s \frac{dt}{b_t}}$$

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is also negligible with respect to \( \int_0^s (r_t - \frac{3R}{2})^2 b_i^2 \frac{dt}{r_t^4} + \int_0^s T_t^2 b_t dt \).

Therefore we deduce from Lemma 3 and from the integrability of \( t \mapsto |Z_t| \), that necessarily \( \langle M^1 \rangle_\infty < \infty \), and then that \( M^1_t \) has to converge, almost surely on \( A \).

Hence \( Z^1_t \) must have a limit almost surely on \( A \), which must be 0, owing to the integrability of \( Z^1_t \). This forces clearly \( \int_0^\infty (r_t - \frac{3R}{2})^2 b_t^2 \frac{dt}{r_t^4} + \int_0^\infty T_t^2 b_t dt \) to be finite, almost surely on \( A \), showing the first bound in \( (v) \) above.

Finally, the integrability of \( T_t^2 b_t \) and the second convergence of \( (iv) \) imply the second bound in \( (v) \) above: \( \int_0^\infty T_t^4 dt < \infty \) almost surely on \( A \).

This concludes the proof of the first assertion in Theorem 2.

2) \( r_s \to R \) and \( r_s \to \infty \) occur both with positive probability, from any initial condition.

Let us use the support theorem of Stroock and Varadhan (see for example ([I-W], Theorem VI.8.1)) to show that the diffusion \( (r, b, T) \) of Corollary 1 is irreducible. Since we can decompose further the equations given in Proposition 1 for \( (r, b, T) \), using a standard Brownian motion \( (w, \beta, \gamma) \in \mathbb{R}^3 \), as follows:

\[
\begin{align*}
    dr_s &= T_s ds, \quad db_s = \sigma b_s dw_s + \sigma r_s d\beta_s + \frac{3\sigma^2}{2} b_s ds + \frac{\sigma^2 r_s^2}{2} ds, \\
    dT_s &= \sigma T_s dw_s + \sigma \sqrt{1 - \frac{R}{r_s}} d\gamma_s + \frac{3\sigma^2}{2} T_s ds + (r_s - \frac{R}{2}) \frac{b_s^2}{r_s^4} ds - \frac{R}{2r_s^2} ds,
\end{align*}
\]

we see that trajectories moving the coordinate \( b \) without changing the others, and trajectories moving the coordinate \( T \) without changing the others, belong to the support of \( (r, b, T) \). Moreover we see from Section 4 that there are timelike geodesics, and then trajectories in the support, which link \( r \) to \( r' \), and then considering the velocities also, which link say \( (r, b'', T'') \) to \( (r', b'', T'') \). So, for given \( (r, b, T) \) and \( (r', b', T') \) in the state space, we can, within the support of \( (r, b, T) \), move \( (r, b, T) \) to \( (r, b'', T'') \), then \( (r, b'', T'') \) to \( (r', b'', T'') \), and finally move \( (r', b'', T'') \) to \( (r', b', T') \), thereby showing the irreducibility of \( (r, b, T) \).

This implies that it is enough to show that for large enough \( r_0, T_0 \), the convergence to \( \infty \) occurs with probability \( \geq 1/2 \), and that for \( r_0 \) close enough from \( R \) and \( T_0 \) negative enough, the convergence to \( R \) occurs with probability \( \geq 1/2 \) as well. Now this can be done by a classical supermartingale argument using the process \( 1/|T_s| \), stopped at some hitting time. Indeed we see from Proposition 1 that

\[
\frac{1}{|T_s|} + \int_0^s \left( \frac{\sigma^2}{2} T_t^2 - \sigma^2 (1 - \frac{R}{r_s}) - \frac{RT_t}{2r_s^2} + (2r_t - 3R) \frac{b_t^2 T_t}{r_t^4} \right) \frac{dt}{|T_t|^2}
\]

is a local martingale.
Take first \( r_0 \geq 3R/2 \), \( T_0 \geq 4 + \frac{4}{R_\infty} \), and \( \tau := \inf \{ s > 0 \mid T_s = 2 + \frac{2}{R_\infty} \} \): \( r_s \) increases on \( \{ 0 \leq s < \tau \} \) and then we see that \( 1/|T_s \wedge \tau| \) is a supermartingale, which implies that 
\[
(2 + \frac{2}{R_\infty})^{-1} \mathbb{P}(\tau < \infty) \leq \liminf_{s \to \infty} \mathbb{E}(\frac{1}{|T_s \wedge \tau|} 1_{\{\tau < \infty\}}) \leq \liminf_{s \to \infty} \mathbb{E}(\frac{1}{|T_s \wedge \tau|}) \leq \mathbb{E}(\frac{1}{T_0}) \leq (4 + \frac{4}{R_\infty})^{-1},
\]
and then that \( \mathbb{P}(\lim_{s \to \infty} r_s = +\infty) \geq \mathbb{P}(\tau = \infty) \geq 1/2. \)

Conversely take \( r_0 \leq 3R/2 \), \( T_0 \leq -2 \), and \( \tau' := \inf \{ s > 0 \mid T_s = -\sqrt{2} \} \): \( r_s \) decreases on \( \{ 0 \leq s < \tau' \} \) and then we see that \( 1/|T_s \wedge \tau'| \) is a supermartingale, which implies that 
\[
2^{-1/2} \mathbb{P}(\tau' < \infty) \leq \liminf_{s \to \infty} \mathbb{E}(\frac{1}{|T_s \wedge \tau'|} 1_{\{\tau' < \infty\}}) \leq \liminf_{s \to \infty} \mathbb{E}(\frac{1}{|T_s \wedge \tau'|}) \leq \mathbb{E}(\frac{1}{|T_0|}) \leq 1/2,
\]
and then that \( \mathbb{P}(D < \infty) \geq \mathbb{P}(\tau' = \infty) \geq 1/\sqrt{2}. \)

This concludes the proof of the second assertion in Theorem 2.

3) Existence of an asymptotic direction for the relativistic diffusion, on \( \{ D = \infty \} \).

We want to generalize the observation made in Section 2.1 for \( R = 0 \), see Remark 1. Recall from Lemma 2 that it does not matter for this asymptotic behaviour whether we consider the trajectories as function of \( s \) or of \( t(s) \) (id est as viewed from a fixed point).

We shall use Remark 1 and proceed by comparison between the flat Minkowski case \( R = 0 \) and the Schwarzschild case \( R > 0 \). Let us split this proof into four parts.

(i) We have \( \int_0^\infty \frac{a_i}{r_i^2} dt < \infty \) and \( \int_0^\infty \frac{U_i}{r_i} dt < \infty \), almost surely on \( \{ D = \infty \} \).

We know from 1) above that \( r_s \to \infty \) almost surely on \( \{ D = \infty \} \).

The very beginning of this proof remains valid: Using (1,i) again, we have almost surely
\[
\int_0^\infty \frac{T_i^2}{a_i r_i^2} dt \quad \text{and} \quad \int_0^\infty (1 - \frac{3R^2}{2r^2}) \frac{b_i^2}{a_i r_i^2} dt \quad \text{finite, whence} \quad \int_0^\infty \frac{b_i^2}{a_i r_i^2} dt \quad \text{finite, and then, since}
\]
\[
a_i \leq \frac{T_i^2}{a_i r_i^2} + \frac{b_i^2}{a_i r_i^2} + \frac{1}{a_i r_i^2}, \quad \text{also} \quad \int_0^\infty a_i \frac{r_i^2}{r_i^2} dt \quad \text{finite, almost surely on} \quad \{ D = \infty \}.
\]

Now by the unit pseudo-norm relation, we have \( \frac{U_i}{r_i} = \frac{b_i}{r_i^3} \leq \frac{a_i}{r_i^3 \sqrt{1 - \frac{R}{r}}} \), whence
\[
\int_0^\infty \frac{U_i}{r_i} dt \quad \text{finite, almost surely on} \quad \{ D = \infty \}.
\]

(ii) The perturbation of the Christoffel symbols due to \( R \) is \( O(r^{-2}) \).

Recall from the beginning of Section 2.3 the values of the Christoffel symbols \( \Gamma^i_{jk} \). Denote by \( \tilde{\Gamma}^i_{jk} \) the difference between these symbols and their analogues for \( R = 0 \), which is a tensor, has only five non-vanishing components in spherical coordinates, and then is easily computed in Euclidian coordinates \( (x_1 = r \sin \varphi \cos \psi; \ x_2 = r \sin \varphi \sin \psi; \ x_3 = r \cos \varphi) \):

we find
\[
\tilde{\Gamma}^i_{jk} \left|_{x_j, x_k} \right. = \frac{\partial x_i}{\partial r} \times \left( \frac{\partial r}{\partial x_j} \frac{\partial r}{\partial x_k} \Gamma^r_{rr} + \frac{\partial \varphi}{\partial x_j} \frac{\partial \varphi}{\partial x_k} \Gamma^\varphi_{\varphi \varphi} + \frac{\partial \psi}{\partial x_j} \frac{\partial \psi}{\partial x_k} \Gamma^\psi_{\psi \psi} \right).
\]
towards (1
since \(a_\infty\) could explode within \(s \to \infty\) would this is clearly forbidden by the simple stochastic differential equation governing \(a_\infty\). This proves the non-explosion of the Schwarzschild relativistic diffusion, quoted directly after the statement of Theorem 2.

\section*{3.2 Proof of Theorem 3}

\(i\) The first sentence is clear from Theorem 2 when the diffusion starts above \(S_0\), except the finiteness of \(D'\), proved in (ii) below. This is the same when the diffusion
starts above \(-S_0\), Theorem 2 being valid as well in this very similar case (it is sufficient to change the signs of \(a \) and \(t\)). The other cases are reviewed in Remark 6. So we just have to establish the assertions (1), (2), (3) of the statement, assuming \(D\) finite.

(ii) Let us prove (1) first. Since \(T_D < 0\) and \(T_s\) cannot vanish in the region \(\{r < R\}\), \(r_s\) must decrease and then converge to some \(g \in [0, R]\), with \(\limsup T_s = \frac{\sqrt{R}}{r_s} - 1 < 0\), which in turn forces \(g = 0\), hit within a finite proper time \(D'\), and \(\limsup_{s \nearrow D'} T_s = -\infty\).

Only the upper bound for \(D'\) remains to be proved. It is a consequence of the pseudo-norm equation, which implies \(T^2 \geq \frac{R}{r} - 1\) and then \(T_s \leq -\frac{\sqrt{R}}{r_s} - 1\) on \(\{D \leq s \leq D'\}\). Indeed, consider

\[g(r) := \frac{R}{2} \arcsin \left[ \frac{2}{R} \sqrt{r(R-r)} \right] - \sqrt{r(R-r)} \quad \text{if} \quad 0 \leq r \leq \frac{R}{2},\]

and

\[g(r) := \frac{R}{2} \left( \pi - \arcsin \left[ \frac{2}{R} \sqrt{r(R-r)} \right] \right) - \sqrt{r(R-r)} \quad \text{if} \quad \frac{R}{2} \leq r \leq R.\]

We have \(g'(r) = (\frac{R}{r} - 1)^{-1/2}\), \(g''(r) = \frac{R}{2r^2} \times (\frac{R}{r} - 1)^{-3/2}\), \(g(0) = g'(0) = 0\), \(g(R) = \frac{\pi}{2} R\), \(g'(R) = +\infty\). Hence \(g'(r_s) \times T_s \leq -1\) implies by integration on \([D, s]\), for \(D \leq s \leq D'\):

\[g(r_s) \leq \frac{\pi}{2} R + D - s;\] taking \(s = D'\), this proves (1).

(iii) Let us then prove the non-explosion of \(\log b\) at \(D'\). For that, let us apply the comparison theorem (see ([I-W], VI, th 4.1)) to \(b_{|D, D'|}\): we get so a real diffusion process \(\beta\) solving

\[\beta_s = b_D + \sigma \int_D^s \sqrt{\beta_s^2 + R^2} \, dw_s + \frac{3s^2}{2} \int_D^s (\beta_s^2 + R^2) \, ds,\]

and such that almost surely \(\sup_{D \leq s < D'} b_s \leq \sup_{D \leq s < D'} \beta_s\). Indeed the ratio of the drift coefficient and of the squared diffusion coefficient of \(b\) is maximal for \(r_s = 0\). Now there is a real Brownian motion \(B\). such that

\[\beta_{D+s'} = B^{-1/2} \left[ \inf \left\{ s \mid \int_0^s \frac{dt}{(1 + R^2 B_t^2)} > 4s' \right\} \right],\]

so that \(\beta\) cannot diverge at a finite time; indeed it is immediately seen that

\[\int_0^s \frac{dt}{B_t^2} = 2 \log(B_0/B_s) + 2 \int_0^s \frac{dB_t}{B_t}\]

must almost surely diverge as \(s\) approaches the hitting time of 0 by \(B\). This proves that \(b\) almost surely cannot explode at \(D'\), and thus must be continuous on \([0, D']\).

Moreover, since the differential equation governing \(b\) can be written

\[\log(b_s/b_0) = \sigma W \left[ \int_0^s \left( 1 + \frac{r_i^2}{b_i^2} \right) dt \right] + \sigma^2 s,\]

for some standard Brownian motion \(W\), we see that \(b_{D'} = 0\) would imply \(\int_0^{D'} \frac{r_i^2}{b_i^2} dt = \infty\), and then \(\limsup_{s \nearrow D'} b_s = +\infty\), a contradiction.

(iv) Let us now prove the statement (3). For that, let us write again the unit pseudo-norm relation: near \(D'\) it writes
We have
\[ \frac{3}{2} T_s r_s^{3/2} = -\sqrt{(R - r_s)b_s^2 + R r_s^2 + (a_s^2 - 1)r_s^3} \rightarrow -b_D \sqrt{R}. \]
So that for any \( \varepsilon > 0 \) we have:
\[ -b_D \sqrt{R} (1 + \varepsilon) \leq T_s r_s^{3/2} \leq -b_D \sqrt{R} (1 - \varepsilon), \]
for \( s \) sufficiently close to \( D' \). Integrating this, we get immediately, for \( s \) sufficiently close to \( D' \):
\[ -b_D \sqrt{R} (1 + \varepsilon)(D' - s) \leq -\frac{2}{3} r_s^{5/2} \leq -b_D \sqrt{R} (1 - \varepsilon)(D' - s), \]
which means the first equivalent in the statement (3). The second equivalent follows at once by using again the unit pseudo-norm relation. This proves (3).

As a consequence, we deduce at once that \( |\dot{\theta}_s| = U_s = b_s/r_s^2 \) is integrable near \( D' \), which implies the convergence of \( \theta_s \) in \( \mathbb{S}^2 \) as \( s \nearrow D' \).

(v) End of the proof of the statement (2).

To establish the non-explosion of \( a_s \), let us rewrite its stochastic differential equation with two independent real standard Brownian motions \( w, \beta \):
\[ da_s = \sigma a_s \, dw_s + \sigma \left( \frac{R}{r_s} - 1 \right) \, d\beta_s + \frac{3\sigma^2}{2} a_s \, ds, \]
and consider \( X_s := a_s \times e^{-\sigma w_s-D-\sigma^2(s-D)}. \)
We have
\[ dX_s = e^{-\sigma w_s-D-\sigma^2(s-D)} \sigma \left( \frac{R}{r_s} - 1 \right) \, d\beta_s, \]
whence for \( D \leq s \leq D' \):
\[ a_s = e^{\sigma w_s-D+\sigma^2(s-D)} \left( a_D + \sigma \int_D^s e^{-\sigma w_t-D-\sigma^2(t-D)} \left( \frac{R}{r_t} - 1 \right) \, d\beta_t \right), \]
for some real standard Brownian motion \( W \). The equivalent seen above for \( r_s \) near \( D' \) shows the almost sure convergence of the integral \( \int_D^{D'} dt \), and then by the above formula, of \( a_s \) as \( s \nearrow D' \).

The same equivalent again shows the almost sure integrability of \( \dot{u}_s^- \) near \( D' \): indeed
\[ \dot{u}_s^- = (a_s + T_s)/(1 - R/r_s) \sim [2 b_D^4/(5 R^3 (D' - s))]^{-1/5}, \]
which finally proves the convergence of \( u_s^- \) as \( s \nearrow D' \). Likewise for \( u_s^+ \).

Finally it remains to show the convergence of \( \tilde{b}_s \), or equivalently of \( \tilde{b}_s/b_s = \theta_s \wedge \dot{\theta}_s/U_s \), since we already saw above the convergence of \( b_s \) in \( \mathbb{R}^+ \). Since we also saw the convergence of \( \theta_s \) in \( \mathbb{S}^2 \), it remains to get the convergence of \( \dot{\theta}_s/U_s \). Now this is the last assertion of Proposition 2. \( \diamond \)

### 3.3 Proof of Proposition 2

We have
\[ d\left( \frac{\dot{\theta}}{U_s} \right) = 2 r_s b_s^{-1} T_s \dot{\theta} + r_s^2 b_s^{-1} \, d\dot{\theta} - (r_s^2 b_s^{-2} \, db_s) \dot{\theta} + r_s^2 b_s^{-3} \, (db_s, d\theta_s) \dot{\theta} - r_s^2 b_s^{-2} \, (db_s, d\theta_s). \]

To perform the computations, let us use the basis \( (u, v, k) \) of \( \mathbb{R}^3 \) defined by:
\[ u = (\cos \psi, \sin \psi, 0); \quad v = (-\sin \psi, \cos \psi, 0); \quad k = (0, 0, 1), \]
so that \( \theta = u \sin \varphi + k \cos \varphi \), and
\[
\dot{\theta} = (u \cos \varphi - k \sin \varphi) \dot{\varphi} + v \dot{\psi} \sin \varphi ; \quad \theta \wedge \dot{\theta} = v \dot{\varphi} - (u \cos \varphi - k \sin \varphi) \dot{\psi} \sin \varphi .
\]

Then \( d\dot{\theta} = (u \cos \varphi - k \sin \varphi) \, d\dot{\varphi} + v \sin \varphi \, d\dot{\psi} - \dot{\varphi}^2 \theta + 2 \, v \dot{\varphi} \dot{\psi} \cos \varphi - u \dot{\psi}^2 \sin \varphi \),
whence
\[
\langle d\dot{\theta}, db \rangle = (u \cos \varphi - k \sin \varphi) \langle d\dot{\varphi}, db \rangle + v \sin \varphi \langle d\dot{\psi}, db \rangle
= \left( (u \cos \varphi - k \sin \varphi) \dot{\varphi} + v \sin \varphi \dot{\psi} \right) \sigma^2 b^{-1} (b^2 + r^2) = b^{-1} \langle db, \dot{\theta} \rangle,
\]
so that the last two terms in the expression of \( d\left( \frac{\dot{\theta}}{U_s} \right) \) above cancel. Hence
\[
d\left( \frac{\dot{\theta}}{U_s} \right) = \frac{2 r_s}{b_s} T_s \dot{\theta}_s - r_s^2 b_s^{-2} \left( \sigma \, dM_s^b + \frac{3 k^2}{4} b_s \, ds + \frac{\sigma^2 r_s^2}{2 b_s} \, ds \right) \dot{\theta}_s
+ U_s^{-1} \left( u_s \cos \varphi_s - k \sin \varphi_s \right) \left( \sigma \, dM_s^\varphi + \frac{3 k^2}{4} \varphi_s \, ds - \frac{2}{r_s} T_s \, \dot{\varphi}_s \, ds + \sin \varphi_s \cos \varphi_s \dot{\psi}_s^2 \, ds \right)
+ U_s^{-1} v_s \sin \varphi_s \left( \sigma \, dM_s^\psi + \frac{3 k^2}{4} \psi_s \, ds - \frac{2}{r_s} T_s \, \dot{\psi}_s \, ds - 2 \cotg \varphi_s \dot{\varphi}_s \dot{\psi}_s \, ds \right)
+ U_s^{-1} \left( 2 v_s \, \dot{\varphi}_s \dot{\psi}_s \cos \varphi_s - \dot{\varphi}_s^2 \theta_s - u_s \dot{\psi}_s^2 \sin \varphi_s \right) ds
= \frac{\sigma r_s^2}{b_s^3} \theta_s \, ds - U_s^{-1} \dot{\varphi}_s^2 \theta_s \, ds + U_s^{-1} \sin \varphi_s \left( (u_s \cos \varphi_s - k \sin \varphi_s) \cos \varphi_s - u_s \right) \dot{\psi}_s^2 \, ds .
\]

Observe now that the definition of \( b = r^2 U \) implies
\[
dM_s^b = r_s^2 U_s^{-1} \left( \dot{\varphi}_s \, dM_s^\varphi + \sin^2 \varphi_s \dot{\psi}_s \, dM_s^\psi \right) .
\]

Therefore, expressing all in the basis \((\theta_s, \dot{\theta}_s/U_s, \theta_s \wedge \dot{\theta}_s/U_s)\), we get
\[
d\left( \frac{\dot{\theta}_s}{U_s} \right) = \sigma U_s^{-2} \sin \varphi_s \left( \dot{\varphi}_s \, dM_s^\varphi - \psi_s \, dM_s^\psi \right) \theta_s \wedge \frac{\dot{\theta}_s}{U_s} - \frac{\sigma^2 r_s^4}{2 b_s^3} \dot{\theta}_s \, ds - U_s \theta_s \, ds ,
\]
that is to say the formula of the statement, with \( d\beta_s := r_s U_s^{-1} \sin \varphi_s \left( \dot{\varphi}_s \, dM_s^\varphi - \psi_s \, dM_s^\psi \right) \).

Now it is straightforward to verify (from the covariation matrix given about the beginning of Section 2.4.1, before Theorem 3) that
\[
\langle d\beta_s \rangle = ds , \quad \langle d\beta_s, dM_s^b \rangle = 0 , \quad \langle d\beta_s, dM_s^\varphi \rangle = 0 , \quad \langle d\beta_s, dM_s^\psi \rangle = 0 ,
\]
which shows that indeed \( \beta_\cdot \) is a standard Brownian motion and is independent from \((r_\cdot, b_\cdot, T)\).

To establish the last assertion of the statement, we deduce from this expression for \( d\left( \frac{\dot{\theta}_s}{U_s} \right) \) that the bounded variation part of \( \dot{\theta}_s/U_s \) is not larger than \( b_s r_s^{-2} + \sigma^2 b_s^{-2} r_s^2 \), which (as \( r_s^{-2} \), recall Theorem (2.3)) is almost surely integrable on \([0, D]\), while its martingale part has quadratic variation not larger than \( \sigma^2 b_s^{-2} r_s^2 \), and thus is almost surely integrable on \([0, D']\) as well by Theorem (2.2). \( \diamond \)
3.4 Proof of Proposition 3

1) To establish this, we need a system of stochastic differential equations relative to the whole relativistic diffusion \((\xi, \dot{\xi})\), and not only to its projection \((r_s, b_s, T_s)\) on the coordinates \((r, b, T)\). Recall that the whole relativistic diffusion \((\xi, \dot{\xi})\) lives in a 7-dimensional space. For our purpose the following system of coordinates is convenient: 

\[(r, \theta, u^-, a, T, \dot{\theta}) \in \mathbb{R}_+ \times S^2 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times T \mathbb{S}^2\].

Recall that

\[b = r^2 |\dot{\theta}| = r^2 U = r^2 \sqrt{\dot{\varphi}^2 + \dot{\psi}^2 \sin^2 \varphi}.

Now we have the following simplification: we can recover the coordinate \(u^-\) from the other ones by integration, since its value is fixed at the origin. Hence, to recover the whole relativistic diffusion \((\xi, \dot{\xi})\), it is sufficient to get the reduced diffusion \((r_s, a_s, b_s, T_s)\), and to recover \((\theta_s, \dot{\theta}_s)\) conditionally from \((r_s, a_s, b_s, T_s)\).

2) Let us deal with the reduced diffusion \((r_s, a_s, b_s)\): we have to show first that for any \((a_0, b_0) \in \mathbb{R} \times \mathbb{R}_+^*\) there exists a unique (in law) diffusion process \((r_s, a_s, b_s) \in [0, R] \times \mathbb{R} \times \mathbb{R}_+^*\) defined up to proper time \(D'' := \inf\{s | r_s = R\}\), starting with initial condition \((0, a_0, b_0)\), and having infinitesimal generator (see Proposition 1)

\[G'' := T \frac{\partial}{\partial r} + \frac{\sigma^2}{2} \left((a^2 - 1 + \frac{R}{r}) \frac{\partial^2}{\partial a^2} + (b^2 + r^2) \frac{\partial^2}{\partial b^2} + 2ab \frac{\partial^2}{\partial a \partial b} + 3a \frac{\partial}{\partial a} + (3b + \frac{r^2}{b}) \frac{\partial}{\partial b}\right),\]

where \(T = T(a, b) := \sqrt{a^2 + \left(\frac{R}{r} - 1\right)(\frac{b^2}{r^2} + 1)}\) is chosen positive.

Clearly \(r_s\) must increase strictly as \(s \not\searrow D''\), so that \(s\) can be expressed as random continuous strictly increasing function of \(r \in [0, R]: s = s(r)\). Let us set \(\bar{a}_r := a_s(r)\) and \(\bar{b}_r := b_s(r)\). In other words, we consider here the radial coordinate \(r\) as an alternative time coordinate. Then \((\bar{a}_r, \bar{b}_r)\) has to be a time inhomogeneous diffusion on \(\mathbb{R} \times \mathbb{R}_+^*\) started from \((a_0, b_0)\) and with infinitesimal generator

\[\mathcal{G} := \frac{\sigma^2}{2T(\bar{a}, \bar{b})} \left((\bar{a}^2 - 1 + \frac{R}{\bar{a}}) \frac{\partial^2}{\partial \bar{a}^2} + (\bar{b}^2 + r^2) \frac{\partial^2}{\partial \bar{b}^2} + 2\bar{a}\bar{b} \frac{\partial^2}{\partial \bar{a} \partial \bar{b}} + 3\bar{a} \frac{\partial}{\partial \bar{a}} + (3\bar{b} + \frac{r^2}{\bar{b}}) \frac{\partial}{\partial \bar{b}}\right).

Conversely, given such a diffusion, the inverse time change yields a diffusion with generator \(G''\). Therefore it is enough to prove existence and uniqueness for the \(\mathcal{G}\)-diffusion stopped at \(r = R\). Now it is well known that a sufficient condition is locally boundedness and continuity of the coefficients of the associated stochastic differential equation, together with a local Lipschitz condition on these coefficients with respect to \((\bar{a}, \bar{b})\).

Now the term \(r^2/\bar{b}\) causes no trouble since \(\bar{b}_0 \neq 0\) and since the proof of Theorem \((2,2)\) insures that \(\bar{b}_r\) stays in \(\mathbb{R}_+^*\), and then we observe that \((\bar{a}^2 - 1 + \frac{R}{\bar{a}})/T\) goes to 0 as \(r \not\searrow 0\), and that \(T^{-1/2}, T^{-1}\), and their derivatives with respect to \((\bar{a}, \bar{b})\), stay bounded as well; indeed (for example) \(T^{-3/2} |\frac{\partial T}{\partial \bar{b}}| = 2T^{-5/2} (\frac{R}{r} - 1) \frac{\dot{\bar{b}}}{r^3} \leq 2T^{-1/2}/\bar{b}\) is bounded.

3) It remains to prove that we can recover \((\theta_s, \dot{\theta}_s)\) conditionally from \((r_s, b_s, T_s)\), once \((\theta_0, \dot{\theta}_0)\) is fixed, in a unique way. Now Proposition 2 displays the equation we have to
solve. To solve it, let us complete the equation of Proposition 2 into a linear system in the variables \( V_s := (\theta_s, (\hat{\theta}_s/U_s), \theta_s \wedge (\hat{\theta}_s/U_s)) \), by adjoining the equation \( d\theta_s = U_s (\hat{\theta}_s/U_s) \, ds \), and the following one (immediately deduced from Proposition 2):

\[
d(\theta_s \wedge \hat{\theta}_s/U_s) = -\left( \frac{r_s}{b_s} \sigma \, d\beta_s \right) \hat{\theta}_s - \left( \frac{\sigma^2 r_s^2}{2b_s^2} \right) \theta_s \wedge \hat{\theta}_s/U_s.
\]

So we get a linear differential system:

\[
dV_s = V_s \, dA_s,
\]

where the matrix-valued differential \( dA_s \) is given by:

\[
dA_s := \begin{pmatrix} 0 & -(b_s/r_s^2) \, ds & 0 \\ (b_s/r_s^2) \, ds & -\left( \sigma^2 r_s^2 / 2b_s^2 \right) \, ds & -(\sigma r_s/b_s) \, d\beta_s \\ 0 & (\sigma r_s/b_s) \, d\beta_s & -\left( \sigma^2 r_s^2 / 2b_s^2 \right) \, ds \end{pmatrix}.
\]

Note that this differential system can be equivalently written in the Stratonovitch form

\[
dV_s = V_s \circ d\tilde{A}_s,
\]

where the matrix-valued differential \( d\tilde{A}_s \) is given by:

\[
d\tilde{A}_s := \begin{pmatrix} 0 & -(b_s/r_s^2) \, ds & 0 \\ (b_s/r_s^2) \, ds & 0 & -(\sigma r_s/b_s) \, d\beta_s \\ 0 & (\sigma r_s/b_s) \, d\beta_s & 0 \end{pmatrix},
\]

so that any solution takes its values in the rotation group.

Let us solve this linear equation by means of the following series:

\[
V_s = V_{D'} \left( 1 + \sum_{k \in \mathbb{N}^*} J_k(s) \right),
\]

where for each \( k \quad J_k(s) := \int_{\{D'<s_1<...<s_k<s\}} dA_{s_1} \times \ldots \times dA_{s_k} \).

To justify that, let us choose on the space of \((3,3)\)-matrices the Euclidian operator norm, and fix \( C = C(\omega) \geq 1 \), measurable with respect to \((r,b,T)\), such that for \( D' \leq s \leq \min\{D'', D'+1\} \) we have:

\[
b_s/r_s^2 \leq C (s-D')^{-4/5}, \quad \sigma r_s/b_s \leq C (s-D')^{2/5}, \quad \sigma^2 r_s^2/b_s^2 \leq C;
\]

this is possible by Theorem 2. Let us suppose that \( \mathbb{E}[\|J_k(s)\|^2|\mathcal{F}_0]\leq \frac{(5C)^{2k}}{k!} \times (s-D')^{2k/5} \), where \( \mathcal{F}_0 \) denotes the \( \sigma \)-field generated by the reduced diffusion \((r,b,T)\). Then we have

\[
\mathbb{E}[\|J_{k+1}(s)\|^2|\mathcal{F}_0] = \mathbb{E}[\|\int_{D'} J_k(s') \, dA_{s'}\|^2|\mathcal{F}_0]
\]

\[
\leq 2 \int_{D'} \mathbb{E}[\|J_k(s')\|^2|\mathcal{F}_0] \frac{\sigma r_s}{b_s} ds' + 2 \int_{D'}^s \frac{b_{s'}}{r_{s'}} ds' \times \int_{D'}^s \mathbb{E}[\|J_k(s')\|^2|\mathcal{F}_0] \frac{b_{s'}}{r_{s'}} ds'
\]

\[
+ 2 (s-D') \times \int_{D'}^s \mathbb{E}[\|J_k(s')\|^2|\mathcal{F}_0] \frac{\sigma^2 r_s^2}{2b_s^2} ds'
\]

\[
\leq 2 C^2 \frac{(5C)^{2k}}{k!} \left[ \int_{D'}^s (s' - D')^{(2k+4)/5} ds' + 5(s-D')^{1/5} \int_{D'}^{s'} (s' - D')^{(2k-4)/5} ds' \right.
\]

\[
\quad \left. + (s-D') \int_{D'}^{s'} (s' - D')^{2k/5} ds' \right]
\]

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3.5 Proof of the second assertion of Theorem 4: \( D_\infty = \infty \) a. s.

For any \( \varepsilon \in ]0,1[ \), set \( A_\varepsilon := \left\{ \sup_{s \leq D_\infty} b \leq \varepsilon^{-1} \right\} \). By continuity of \( b \), we have
\[
\mathbb{P}(\{ D_\infty < \infty \} \cap A_\varepsilon^c) < \varepsilon^t,
\]
for any fixed \( \varepsilon > 0 \) and small enough fixed \( \varepsilon \).

For any \( n \in \mathbb{N} \), set \( \tau_n^* := \inf\{ s > D_{3n+2} \mid T_s = 0 \} \) and \( \tau_n := \inf\{ s > D_{3n+3} \mid r_s = R/2 \} \). Note that \( D_{3n+2} < \tau_n^* < D_{3n+3} < \tau_n < D_{3n+4} \) on \( \{ D_\infty < \infty \} \), and \( \sum_n (\tau_n - \tau_n^*) < D_\infty \).

Moreover \( \tau_n - \tau_n^* = \tau \circ \Theta^{\tau_n^*} \), where \( \tau \) denotes the hitting time of \( \{ r = R/2 \} \).

Setting \( A_\varepsilon^\prime := \{ \tau < c\varepsilon \} \cap \left\{ \sup_{s \leq D_\infty} b \leq \varepsilon^{-1} \right\} \) (for some constant \( c = c(\sigma, R) > 0 \)), we have
\[
\mathbb{P}(D_\infty < \infty) - \varepsilon' < \mathbb{P}(\{ D_\infty < \infty \} \cap A_\varepsilon) \leq \mathbb{P}\left( \sum_n (1_{A_\varepsilon} \times \tau) \circ \Theta^{\tau_n^*} < \infty \right)
\]
\[
\leq \mathbb{P}\left( \liminf_n (\Theta_{\tau_n^*})^{-1}(A_\varepsilon^\prime) \right) \leq \sum_n \lim_{p \to \infty} \mathbb{P}\left( \bigcap_{m=n}^p (\Theta_{\tau_n^*})^{-1}(A_\varepsilon^\prime) \right)
\]
\[
= \sum_n \lim_{p \to \infty} \mathbb{E}\left( \mathbb{P}_{\tau_n^*}(A_\varepsilon^\prime) \times \prod_{m=n}^{p-1} (1_{A_\varepsilon} \circ \Theta_{\tau_n^*}) \right)
\]
by the strong Markov property. Hence we see that this proof is achieved if we show that
$$\lim_{p \to \infty} \mathbb{E}\left( \mathbb{P}_{t^p} (A'_c) \times \prod_{m=n}^{p-1} (1_{A'_c} \circ \Theta^m_n) \right) = 0,$$
for any \(n\). Now this follows immediately, by induction, from the following lemma. \(\Box\)

**Lemma 4**  For small enough \(\varepsilon > 0\), and for any initial condition \(r_0 > R\) and \(b_0 > 0\), provided \(T_0 = 0, A'_c\) being as in the proof of Theorem 4 above, we have \(\mathbb{P}(A'_c) < 1/2\).

**Proof** Let us write again the stochastic equation of \(T_s\), under the following form, for some real standard Brownian motion \(W\):

$$T_s = \sigma W\left[ \int_0^s (T_t^2 + 1 - \frac{R}{r_t}) \, dt \right] + \frac{3\sigma^2}{2} \int_0^s T_t \, dt + \int_0^s (r_t - \frac{3R}{2} \frac{b^2_t}{r_t^2}) \, dt - \int_0^s \frac{R}{2r_t^2} \, dt.$$

Consider \(\tau := \inf\{s \mid r_s = R/2\}\) and \(\sigma'_c := \inf\{s \mid |T_s| = \varepsilon^{-1}\}\), and some constant \(q\).

On the event \(A''_c := \left\{ \max |W|((0, (\varepsilon^{-2} + 1)q)) < (2\sigma\varepsilon)^{-1}\right\} \cap \left\{ \sup_{s \leq b'} b < \varepsilon^{-1}\right\}\), we have for \(0 \leq s \leq \min\{s, \sigma'_c, \tau\}\):

\[-\frac{1}{2\varepsilon} - \frac{3\sigma^2}{2\varepsilon} q - 3(2/R)^3\varepsilon^{-2} q - (2/R) q < T_s < \frac{1}{2\varepsilon} + \frac{3\sigma^2}{2\varepsilon} q + (2/R)^3\varepsilon^{-2} q,\]

whence \(|T_s| < \frac{1}{2\varepsilon} + \frac{1}{2\varepsilon} (3\sigma^2 + 4 R^{-1}\varepsilon + 48 R^{-3}\varepsilon^{-1}) q\).

Hence \(q < \min\{\sigma'_c, \tau\}\) on \(A''_c\) if \(q \leq \left(3\sigma^2 + 4 R^{-1}\varepsilon + 48 R^{-3}\varepsilon^{-1}\right)^{-1}\) and \(q \leq R\varepsilon/2\).

Thus there exists a constant \(c = c(\sigma, R) > 0\) such that \(\min\{\sigma'_c, \tau\} > q = c\varepsilon\) on \(A''_c\).

Therefore \(\\{ \max |W|((0, (\varepsilon^{-2} + 1)q)) < (2\sigma\varepsilon)^{-1}\} \cap A'_c = \emptyset\), and then

$$\mathbb{P}(A'_c) \leq \mathbb{P}\left[ \max |W|((0, (\varepsilon^{-2} + 1)q)) \geq (2\sigma\varepsilon)^{-1}\right] = \mathbb{P}\left[ \max |W|((0, q)) \geq (2\sigma \sqrt{\varepsilon^2 + 1})^{-1}\right]$$

$$\leq 2 \mathbb{P}\left[ \max |W|(0, q) \geq (2\sigma \sqrt{\varepsilon^2 + 1})^{-1}\right] = 2 \mathbb{P}(|W| \geq (2\sigma \sqrt{\varepsilon^2 + 1})^{-1}) / \sqrt{2\pi}$$

$$= 4 \int_0^{\infty} e^{-x^2/2} \, dx / \sqrt{2\pi} \leq \frac{12\sigma}{\sqrt{2\pi}} \sqrt{c\varepsilon} e^{-1/(12\sigma^2 c\varepsilon)}.$$

\(\Box\)

### 3.6 Proof of Theorem 5

The proof of this theorem will be split into 8 parts and a series of lemmas.

1) Let us begin by the dichotomy of the first assertion: there is no other possibility than the obvious one: \(r\) goes to infinity, and the confinement exhibited here: \(r\) remains endlessly bounded.

**Lemma 5**  Almost surely, if \(r\) does not go to infinity, then it is bounded.
Proof Consider the double sequence of hitting times \( \{\Lambda_n, \Lambda'_n \mid n \in \mathbb{N}\} \) defined by \( \Lambda_0 = \Lambda'_0 = 0 \), \( \Lambda'_n := \inf \{s > \Lambda_n \mid r_s < R\} \), \( \Lambda_n := \inf \{s > \Lambda'_{n-1} \mid r_s > e^n \} \) and \( T_s = 0 \), and set \( E_n := \{\Lambda_n < \infty\} \), for \( n \in \mathbb{N}\).

By Theorem (2.1) the event \( \bigcap_n E_n \) contains all trajectories such that \( r_s \) does not go to infinity, but is unbounded. Thus we want precisely to prove that \( \mathbb{P}(\bigcap_n E_n) = 0 \).

Let us apply the comparison theorem (see for example ([I-W], Theorem 4.1)): there exist comparison processes \( X^+_s \) and \( X^-_s \) on the same probability space, such that almost surely on \( E_n \):

\[
\max_{\Lambda_n \leq s \leq \Lambda_n+1} T_s \geq \max_{\Lambda_n \leq s \leq \Lambda_n+1} X^+_s, \quad X^+_{\Lambda_n} = 0 \quad \text{and} \quad \min_{\Lambda_n \leq s \leq \Lambda_n+1} T_s \geq \min_{\Lambda_n \leq s \leq \Lambda_n+1} X^-_s, \quad X^-_{\Lambda_n} = 0,
\]

where \( X^+_s \) and \( X^-_s \) are real diffusions given by their stochastic differential equations, which are deduced from the equation governing \( T_s \) by bringing down the drift term and the diffusion coefficient, and by bringing the diffusion coefficient down (in the \( X^+ \) case) or up (in the \( X^- \) case). Recall that

\[
dT_s = \sigma \sqrt{T_s^2 + 1} R dW_s + \frac{3\sigma^2}{2} T_s ds + (r_s - \frac{3R}{2}) \frac{b_s}{r_s^4} ds - \frac{R}{2r_s^2} ds.
\]

Thus, as long as \( r_s \geq 3R/2 \), we can take \( \frac{1}{3} \leq T^2 + \frac{1}{3} \leq T^2 + 1 - \frac{R}{r_s} \leq T^2 + 1 \) and then:

\[
dx^+_s = \frac{2}{\sqrt{3}} dw_s + \frac{3\sigma^2}{2} (X^+_s \wedge 0) ds - \frac{b_s}{9R} ds,
\]

and

\[
dx^-_s = \sigma \sqrt{(X^-_s)^2 + 1} dw_s + \left( \frac{3\sigma^2}{2} (X^-_s \wedge 0) - \frac{2}{3R} \right) ds.
\]

Now, for \( A := 4 + \frac{4}{R^2} \), fix \( \varepsilon := \mathbb{P}(\max_{\Lambda_n \leq s \leq \Lambda_n+1} X^+_s > A) \), which is \( > 0 \), \( N > 0 \) such that \( \mathbb{P}(\min_{\Lambda_n \leq s \leq \Lambda_n+1} X^-_s < -N) < \varepsilon/2 \), and \( n \) such that \( r_{\Lambda_n} > N + 3R \). Note that the event \( E'_n := \{\max_{\Lambda_n \leq s \leq \Lambda_n+1} X^+_s > A\} \cap \{\min_{\Lambda_n \leq s \leq \Lambda_n+1} X^-_s < -N\} \) has probability \( > \varepsilon/2 \).

Setting \( \Lambda''_n := \inf \{s > \Lambda_n \mid r_s \leq 3R/2\} \) and applying the comparison theorem, we get:

\[
\min_{\Lambda_n \leq s \leq (\Lambda_n+1) \land \Lambda''_n} T_s \geq \min_{\Lambda_n \leq s \leq \Lambda_n+1} X^-_s \geq -N \quad \text{on } E_n \cap E'_n,
\]

and then

\[
\min_{\Lambda_n \leq s \leq (\Lambda_n+1) \land \Lambda''_n} r_s \geq r_{\Lambda_n} - N \geq 3R \quad \text{on } E_n \cap E'_n,
\]

showing that \( \Lambda''_n > 1 \) on \( E_n \cap E'_n \). Hence, applying the comparison theorem again, we get also:

\[
\max_{\Lambda_n \leq s \leq \Lambda_n+1} T_s \geq \max_{\Lambda_n \leq s \leq \Lambda_n+1} X^+_s \geq A \quad \text{on } E_n \cap E'_n.
\]

Thus, using the strong Markov property, we find that for large enough \( n \):

\[
\mathbb{P}(\exists s > \Lambda_n \mid T_s > 4 + \frac{4}{R^2} \text{ and } r_s > 3R/2 \mid E_n) > \varepsilon/2.
\]
Let us use now the proof of Theorem (2,2), where we proved that for \( r_0 \geq 3R/2 \) and \( T_0 \geq 4 + \frac{4}{R \sigma t} \), then \( \mathbb{P}( \lim_{s \to \infty} r_s = \infty ) \geq 1/2 \), together with the Markov property: we obtain

\[
\mathbb{P}( E_{n+1} \mid E_n ) \leq \mathbb{P}( \Lambda'_{n+1} < \infty \mid E_n ) < 1 - \epsilon/4, \quad \text{for } n \text{ larger than some } n_0 .
\]

Therefore we get \( \mathbb{P}( E_{n0+k+1} ) < (1 - \epsilon/4)^k \) and then \( \mathbb{P}( \cap_n E_n ) = 0 \), as wanted. \( \diamond \)

2) The first case in the theorem was already handled in Lemma 2 and Theorem 2. The irreducibility of the relativistic diffusion is clear from Theorem (2,2).

Let us now focus on the second case in the theorem, supposing therefore that \( r \) is bounded. Set \( \tau_M := \inf \{ s \mid r_s = M \} \), for \( M \) of the form \( M = k \times 3R/2 \), with \( k \in \mathbb{N}^* \).

The proof is divided in several distinct lemmas. We shall always let the relativistic diffusion start at proper time \( D_{-1} = 0 \) from level \( \{ r = R \} \) with \( T_0 > 0 \). We let the hitting times \( D_j \) be as in Section 2.4.3. Fix also some \( \epsilon \) in \( [0, \frac{4}{3}] \).

3) Estimates related to \( b \).

Let us begin by proving that (when \( r \) is bounded) \( b \) has to go to infinity, which by means of the Markov property will allow then to consider only large enough \( b_0 \).

**Lemma 6** Almost surely, if \( r_s \) is bounded, then \( \lim_{s \to \infty} b_s e^{-(1-\epsilon)\sigma^2 s} = +\infty \).

**Proof** Note that \( \{ \sup_{s \geq 0} r_s < \infty \} = \bigcup_M \{ \tau_M = \infty \} \).

Let us recall the logarithmic form of the stochastic differential equation of \( b \):

\[
\log(b_s e^{-(1-\epsilon)\sigma^2 s}) = \log b_0 + \epsilon \sigma^2 s + \sigma W \left[ s + \int_0^s \frac{r_t^2}{b_t^2} dt \right] \quad \text{(for some Brownian motion } W).\]

A straightforward consequence is that \( \limsup_{s \to \infty} (b_s e^{-(1-\epsilon)\sigma^2 s}) = +\infty \) almost surely, so that the stopping time \( t_n := \inf \{ s > 0 \mid b_s e^{-(1-\epsilon)\sigma^2 s} > n \} \) is finite for any \( n \in \mathbb{N} \).

Then let us write the equation governing \( b \) in the following linearized form (for some other Brownian motion \( \tilde{W} \)):

\[
b_s e^{-(1-\epsilon)\sigma^2 s} = e^{\sigma w_s + \epsilon \sigma^2 s} \left( b_0 + \sigma \tilde{W} \left[ \int_0^s e^{2\sigma w_t - 2\sigma^2 t} r_t^2 dt \right] + \int_0^s e^{-\sigma w_t - \sigma^2 t} \frac{\sigma^2 r_t^2}{2b_t} dt \right).\]

It is clear from this expression that for any fixed \( M \) the probability of the asymptotic event \( A_M := \{ \tau_M = \infty \} \cap \{ b_s e^{-(1-\epsilon)\sigma^2 s} \text{ does not go to } \infty \} \) can be made arbitrarily close to 0 by choosing \( b_0 \) large enough. Using that \( A_M = \Theta_{t_n}^{-1}(A_M) \), we apply the strong Markov property at the sequence of stopping times \( \{ t_n \} \) to conclude that \( \mathbb{P}(A_M) = 0 \), which yields the result. \( \diamond \)

We estimate then the increase of \( b \), when started from some large value \( b_0 \).
Lemma 7. Fix $\varepsilon > 0$, and recall that $\tau_M := \inf \{ s \mid r_s = M \}$. Then there exists a lower bound $b(M, \varepsilon)$ such that for $b_0 \geq b(M, \varepsilon)$ we have:

$$\mathbb{P}\left( b_0^{(1-\varepsilon)/2} e^{(1-\varepsilon)\sigma^2 s} \leq b_s \leq b_0^{(1+\varepsilon)/2} e^{(1+\varepsilon)\sigma^2 s} \quad \text{for all } s \leq \tau_M \right) > 1 - 2b_0^{-\varepsilon^2/2} > 1 - \varepsilon.$$  

Proof. For any $q \in]0, 1[$, set $\nu_q := \inf \{ s \mid b_s e^{-(1-\varepsilon)\sigma^2 s} \leq q b_0 \} \in]0, \infty[$.

Recall that the equation governing $b$ can be expressed in the following form:

$$\log(b_s/b_0) = \sigma W\left[ \int_0^s \left(1 + \frac{r_t^2}{b_t^2}\right) dt \right] + \sigma^2 s,$$

for some standard Brownian motion $W$.

Hence for $0 \leq s \leq \nu_q \wedge \tau_M$ we have:

$$\log(b_s e^{-(1-\varepsilon)\sigma^2 s}/b_0) \geq \varepsilon \sigma^2 s + \sigma \min W[s, (1 + (M/qb_0)^2) s],$$

whence

$$\mathbb{P}(\nu_q < \tau_M) \leq \mathbb{P}\left( \exists s' \quad \varepsilon s' + \min W[s, (1 + (M/qb_0)^2) s] \leq \log q \right)$$

$$= \mathbb{P}\left( \exists s' \quad \frac{\varepsilon s'}{1 + (M/qb_0)^2} + \min W\left[ \frac{s'}{1 + (M/qb_0)^2}, s' \right] \leq \log q \right)$$

$$\leq \mathbb{P}\left( \exists s \quad \frac{\varepsilon s}{1 + (M/qb_0)^2} + W_s \leq \log q \right) = \exp\left( \frac{2\varepsilon \log q}{1 + (M/qb_0)^2} \right).$$

Taking $q = b_0^{-\varepsilon^2/2}$, this yields:\n\[ \mathbb{P}(\nu_q < \tau_M) \leq b_0^{-(\varepsilon^2/[1 + M^2 b_0^{\varepsilon^2 - 2}])} < b_0^{-\varepsilon^2/2} \text{ for } b_0 \geq b(R, \varepsilon). \]

This proves the lower control of the statement. The upper control is obtained exactly in the same way. ⋄

4) We prove now that $\varrho := \limsup_{s \to \infty} r_s$ cannot be strictly between $3R/2$ and $+\infty$.

Lemma 8. Almost surely, if $r_s$ is bounded, then $\varrho := \limsup_{s \to \infty} r_s \leq 3R/2$.

Proof. Let us proceed somewhat as for Lemma 5. Fix $\varepsilon \in]0, 1[$, $M > 3\varepsilon + 3R$, and consider the double sequence of hitting times \{ $\Lambda_n, \Lambda_n' \mid n \in \mathbb{N}$ \} defined by $\Lambda_0 = \Lambda_0' = 0$,

$$\Lambda_n := \inf \{ s > \Lambda_{n-1} \mid r_s > 2\varepsilon + 3R/2 \text{ and } T_s = 0 \text{ and } b_s > e^{2n} \},$$

$$\Lambda_n' := \inf \{ s > \Lambda_n \mid r_s < \varepsilon + 3R/2 \text{ or } b_s < e^n \}.$$  

Recall that $\tau_M := \inf \{ s > 0 \mid r_s > M \}$, and set $E_n := \{ \Lambda_n < \tau_M \}$, for $n \in \mathbb{N}^*$.

By Theorem (2,1) and by Lemma 6, the event $\cap_n E_n$ contains all trajectories such that $r_s$ is bounded by $M$ and $\limsup_{s \to \infty} r_s > 2\varepsilon + 3R/2$. Thus we want precisely to prove that $\mathbb{P}(\cap_n E_n) = 0$.

Let us apply the comparison theorem (see for example ([I-W], Theorem 4.1)): there exist comparison processes $X^+_s, X^-_s$ and $Y^-_s$ on the same probability space, such that
almost surely on $E_n$:

$$\min_{\Lambda_n \leq s \leq \Lambda_n + 1} b_s \geq \min_{\Lambda_n \leq s \leq \Lambda_n + 1} Y_s^-, \quad Y_{\Lambda_n}^- = b_{\Lambda_n} > e^{2n}, \quad \text{and}$$

$$\max_{\Lambda_n \leq s \leq \Lambda_n + 1} T_s \geq \max_{\Lambda_n \leq s \leq \Lambda_n + 1} X_s^+, \quad X_{\Lambda_n}^+ = 0 \quad \text{and} \quad \min_{\Lambda_n \leq s \leq \Lambda_n + 1} T_s \geq \min_{0 \leq s \leq 1} X_s^-, \quad X_{\Lambda_n}^- = 0,$$

where $X_s^+$ and $X_s^-$ are real diffusions given by their stochastic differential equations, which are deduced from the equation governing $T_s$ by bringing down the ratio between the drift term and the diffusion coefficient, and by bringing the diffusion coefficient down (in the $X^+$ case) or up (in the $X^-$ case), and similarly for $b_s, Y_s^-$. Recall that

$$dB_s = \sigma \sqrt{b_s^2 + r_s^2} \, dw_s + \frac{3\sigma^2}{2} b_s \, ds + \frac{\sigma^2 r_s^2}{2 b_s} \, ds,$$

and

$$dT_s = \sigma \sqrt{T_s^2 + 1} \, dw_s + \frac{3\sigma^2}{2} T_s \, ds + \left( r_s - \frac{3\sigma^2}{2 r_s^2} \right) \rho_s^2 \, ds - \frac{R}{2r_s^2} \, ds.$$ 

Thus, on $[\Lambda_n, \Lambda'_n]$ we can use $b^2 + r^2 \leq b^2 + M^2$ and $T^2 + \frac{1}{R} \leq T^2 + 1 - \frac{R}{r} \leq T^2 + 1$, and then, for $n$ such that $\varepsilon M^{-4} e^{2n} - \frac{2}{3R} > 3 e^n$:

$$dY_s^- = \sigma \sqrt{(Y_s^-)^2 + M^2} \, dw_s + \frac{3\sigma^2}{2} Y_s^- \, ds,$$

and

$$dX_s^+ = \sigma \sqrt{(X_s^+)^2 + \frac{1}{3}} \, dw_s + \frac{3\sigma^2}{2} (X_s^+ \wedge 0) \, ds + e^n \, ds,$$

and

$$dX_s^- = \sigma \sqrt{(X_s^-)^2 + 1} \, dw_s + \frac{9\sigma^2}{2} (X_s^- \wedge 0) \, ds + 3 e^n \, ds.$$

Now $P\left( \min_{\Lambda_n \leq s \leq \Lambda_n + 1} Y_s^- > e^n \right)$, $P\left( \min_{\Lambda_n \leq s \leq \Lambda_n + 1} X_s^- > -\varepsilon \right)$, and (for $A := 4 + \frac{4}{R_0 \sigma^2}$)

$$P\left( \max_{\Lambda_n \leq s \leq \Lambda_n + 1} X_s^+ > A \right)$$

are (for large enough $n$) arbitrary near from 1, so that the event

$$F_n := \left\{ \min_{\Lambda_n \leq s \leq \Lambda_n + 1} Y_s^- > e^n \right\} \cap \left\{ \min_{\Lambda_n \leq s \leq \Lambda_n + 1} X_s^- > -\varepsilon \right\} \cap \left\{ \max_{\Lambda_n \leq s \leq \Lambda_n + 1} X_s^+ > A \right\}$$

has probability arbitrary near from 1.

Hence, applying the comparison theorem, we get:

$$\min_{\Lambda_n \leq s \leq (\Lambda_n + 1) \wedge \Lambda_n' \wedge \tau_M} b_s \geq \min_{\Lambda_n \leq s \leq \Lambda_n + 1} Y_s^- > e^n \quad \text{on} \quad F_n \cap E_n,$$

$$\min_{\Lambda_n \leq s \leq (\Lambda_n + 1) \wedge \Lambda_n' \wedge \tau_M} T_s \geq \min_{\Lambda_n \leq s \leq \Lambda_n + 1} X_s^- > -\varepsilon \quad \text{on} \quad F_n \cap E_n,$$

and then

$$\min_{\Lambda_n \leq s \leq (\Lambda_n + 1) \wedge \Lambda_n' \wedge \tau_M} \tau_{\Lambda_n + s} > r_{\Lambda_n} - \varepsilon > \varepsilon + 3R/2 \quad \text{on} \quad F_n \cap E_n,$$

showing that $\Lambda'_n > \Lambda_n + 1$ on $F_n \cap \{\tau_M \geq \Lambda'_n\} \cap E_n$. Hence, applying the comparison theorem again, we get also:

$$\max_{\Lambda_n \leq s \leq \Lambda_n + 1} T_s \geq \max_{\Lambda_n \leq s \leq \Lambda_n + 1} X_s^+ > A \quad \text{on} \quad F_n \cap \{\tau_M \geq \Lambda'_n\} \cap E_n.$$
Thus, using the strong Markov property, we find that for large enough $n$:
\[
\mathbb{P}\left(\tau_M < \Lambda'_n \mid \exists s \in [\Lambda_n, \Lambda'_n[ \right) \mathbb{P}(T_s > 4 + \frac{4}{R^2}) \mathbb{P}(r_s > 3R/2) / E_n) > 1 - \varepsilon.
\]
Let us use now the proof of Theorem (2.2), where we proved that for $r_0 \geq 3R/2$ and $T_0 \geq 4 + \frac{4}{R^2}$, then $\mathbb{P}(r_s)$ increases towards $\infty$) $\geq 1/2$, together with the Markov property: we obtain
\[
\mathbb{P}\left(\tau_M < \Lambda'_n / E_n) > 1/2 - \varepsilon, \text{ for } n \text{ larger than some } n_0.
\]
Therefore, noting that $E_{n+1} \subset \{\tau_M > \Lambda'_n\}$, we get $\mathbb{P}(E_{n_0+k+1}) < [1/2 + \varepsilon]^k$ and then $\mathbb{P}(\cap_n E_n) = 0$, as wanted. $\diamond$

5) Lower estimates related to $D_{3n+2}$

The aim of the following lemma is to get a lower bound (in an optimal way, owing to Remark 12 below) on the duration of an excursion in the hole, outside a given event of small probability. The idea is that when $T$, vanishes, with an acceleration of order $b^2$ it takes a lapse of time of order $1/b$ to make a non-infinitesimal move.

**Lemma 9** Suppose $r_0 = R$, $T_0 > 0$, and $b_0$ large enough.

Then we have $\mathbb{P}(D_1 > \frac{c}{b_0} \wedge \tau_M) > 1 - \varepsilon^{b_0/(64\sigma^2c)}$, for $c := R^3 / (512M)$.  

**Proof** Consider $D'_0 := \inf\{s > 0 \mid T_s = 0\} \leq D_0$, $D'_1 := \inf\{s > 0 \mid r_s = \frac{R}{b}\} \in [D'_0, D'_1]$, 
\[
\eta := \inf\{s > 0 \mid \log\left(\frac{b_0}{b_0}\right) = \pm \log 2\}, \text{ and } \lambda := \inf\{s > D'_1 \mid T_s = \pm \frac{b_0}{R}\} \leq D_0.
\]
Set $E := \{D'_0 < \frac{c}{b_0} \wedge \tau_M\}$, $E' := \{\sigma W^*[2c/b_0] < \frac{1}{2}\}$, and $E'' := \{\sigma \tilde{W}^*[2cb_0/R^2] < \frac{b_0}{4R}\}$, 
where $W$ and $\tilde{W}$ are standard real Brownian motions used to write the stochastic equations governing $b$ and $T$, respectively, and $W^*$, $\tilde{W}^*$ are their maximum processes: $W^*(s) := \max |W|([0, s])$ for any $s > 0$ and similarly for $\tilde{W}^*$.

Recall from the proofs of Lemmas 6 and 7 that $\log(b_s/b_0) = \sigma W\left[\int_0^s \left(1 + \frac{r_t^2}{b_t^2}\right) dt \right] + \sigma^2 s$, and (as already used for example in the proof of Lemma 4) that for $s \geq D'_0$:
\[
T_s = \sigma \tilde{W}\left[\int_{D'_0}^s (T_t^2 + 1 - \frac{R}{r_t}) dt \right] + \frac{3\sigma^2}{2} \int_{D'_0}^s T_t dt + \int_{D'_0}^s (r_t - \frac{3R}{2}) \frac{R^2}{r_t} dt - \int_{D'_0}^s \frac{R^2}{2r_t} dt.
\]
Therefore on $E \cap E'$, for $0 \leq s \leq \frac{c}{b_0} \wedge \tau_M \wedge \eta$ we have:
\[
\left|\log(b_s/b_0)\right| \leq \sigma W^*\left[\left(1 + \frac{4M^2}{b_0}\right) \frac{c}{b_0}\right] + \frac{\sigma^2 c}{b_0} < \frac{1}{2} + \frac{\sigma^2 c}{b_0} < \log 2, \text{ whence } \eta \geq \frac{c}{b_0} \wedge \tau_M,
\]
and on the other hand, on $E \cap E' \cap E''$, for $D'_0 \leq s \leq \frac{c}{b_0} \wedge \tau_M \wedge \lambda \wedge D'_1$ we have:
\[
|T_s| < \sigma W^*\left[\left(\frac{b_0^2}{R^2} + 1\right) \frac{c}{b_0}\right] + \frac{3\sigma^2}{2} \frac{b_0}{R} \frac{c}{b_0} + 2M \frac{64b_0^2}{R^2} c \frac{c}{b_0} + 2R \frac{c}{b_0} \frac{c}{b_0} < \frac{b_0}{4R} + \frac{3\sigma^2 c}{2R} + \frac{128Mcb_0}{R^2} + \frac{2c}{Rb_0} < \frac{b_0}{R},
\]

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since \( c = R^3/(512M) \), for \( b_0 \) large enough. Since \( c \leq R^2/2 \), this implies that
\[
r_s = R + \int_0^s T_t \, dt > R - \frac{b_0}{R} c \geq \frac{R}{2}, \quad \text{still for } D'_0 \leq s \leq \frac{c}{b_0} \wedge \tau_M \wedge \lambda \wedge D'_1.
\]
Hence we find that \( E \cap E' \cap E'' \subset \{ D'_1 > \frac{c}{b_0} \wedge \tau_M \} \), and then also \( E' \cap E'' \subset \{ D'_1 > \frac{c}{b_0} \wedge \tau_M \} \).
\[
\mathbb{P}(\tilde{\tau} > \frac{c}{b_0}) = \mathbb{P}(\tilde{\tau} \wedge \frac{c}{b_0} < \tilde{\tau}) = \mathbb{P}\left( \sup_{s < \tilde{\tau} \wedge \frac{c}{b_0}} |\log(b_0)| < \log 3, \sup_{s < \tilde{\tau} \wedge \frac{c}{b_0}} |T_s| < \frac{b_0}{R}, \sup_{s < \tilde{\tau} \wedge \frac{c}{b_0}} |r_s - R| < \frac{R}{2} \right).
\]

Now, as already used in the proof of Lemma 4, we have for \( b_0 \) large enough:
\[
\mathbb{P}[E'] > 1 - 8\sigma \sqrt{c/(\pi b_0)} e^{-b_0/(16\sigma^2 c)} > 1 - \frac{1}{2} e^{-b_0/(16\sigma^2 c)}
\]
and
\[
\mathbb{P}[E''] > 1 - 16\sigma \sqrt{c/(\pi b_0)} e^{-b_0/(64\sigma^2 c)} > 1 - \frac{1}{2} e^{-b_0/(64\sigma^2 c)},
\]
whence finally:
\[
\mathbb{P}\left( D_1 > \frac{c}{b_0} \wedge \tau_M \right) \geq \mathbb{P}\left( D'_1 > \frac{c}{b_0} \wedge \tau_M \right) \geq \mathbb{P}(E' \cap E'') > 1 - e^{-b_0/(64\sigma^2 c)}.
\]

Let us apply now the preceding lemma, to get jointly a lower bound on all proper hitting times at which an excursion in the hole begins.

\textbf{Lemma 10} \quad Fix \( \varepsilon > 0 \), and recall that \( \tau_M := \inf\{ s \mid r_s = M \} \). Then there exists a lower bound \( b(\sigma, R, M, \varepsilon) \) such that for \( b_0 \geq b(\sigma, R, M, \varepsilon) \) we have:
\[
\mathbb{P}\left( D_{3n-1} \geq \tau_M \wedge \left( \frac{1}{(1+\varepsilon)\sigma^2} \log(1 + \frac{(1+\varepsilon)c}{b_0^2} \varepsilon^2 n) \right) \right) \text{ for all } n \in \mathbb{N} > 1 - \varepsilon.
\]

\textbf{Proof} \quad Set \( \delta_k := e^{(1+\varepsilon)c^2} D_{3k+2} \), for \( k \in \mathbb{N} \), and consider the following events, indexed by the integer \( n \geq 0 \):
\[
B_n := \left\{ b_0^{1-(\varepsilon/2)} e^{(1-\varepsilon)\sigma^2 s} \leq b_s \leq b_0^{1+(\varepsilon/2)} e^{(1+\varepsilon)\sigma^2 s} \quad \text{for all } s \leq D_{3n-1} \right\},
\]
\[
A_n := \left\{ D_{3k-1} \geq \tau_M \wedge \left( \frac{1}{(1+\varepsilon)\sigma^2} \log(1 + \frac{(1+\varepsilon)c}{b_0^2} \varepsilon^2 ) k \right) \right\} \quad \text{for all integers } k \leq n,
\]
and
\[
A'_n := B_n \cap \left\{ \tau_M \geq D_{3n-1} \right\}, \quad B'_n := B_n \cap \left\{ \tau_M \geq D_{3n-1} \right\}.
\]

Finally let us consider also
\[
B := \left\{ b_0^{1-(\varepsilon/2)} e^{(1-\varepsilon)\sigma^2 s} \leq b_s \leq b_0^{1+(\varepsilon/2)} e^{(1+\varepsilon)\sigma^2 s} \quad \text{for all } s \leq \tau_M \right\}.
\]
We have

\[ A'_{n+1} = \left\{ D_{3n+2} \geq \frac{1}{(1+\varepsilon)c^2} \log(1 + \frac{(1+\varepsilon)c\sigma^2}{b_0 + (\varepsilon/2)}) (n+1) \right\} = \left\{ \delta_n \geq 1 + \frac{(1+\varepsilon)c\sigma^2}{b_0 + (\varepsilon/2)} (n+1) \right\} \]

\[ \sup \left\{ \delta_n \geq \delta_{n-1} + \frac{(1+\varepsilon)c\sigma^2}{b_0 + (\varepsilon/2)} \right\} \cap \left\{ \delta_{n-1} \geq 1 + \frac{(1+\varepsilon)c\sigma^2}{b_0 + (\varepsilon/2)} n \right\} \]

\[ \sup \left\{ (D_{3n+2} - D_{3n-1}) \delta_{n-1} \geq \frac{c}{b_0 + (\varepsilon/2)} \right\} \cap \left\{ \delta_{n-1} \geq 1 + \frac{(1+\varepsilon)c\sigma^2}{b_0 + (\varepsilon/2)} (n-1) \right\}, \]

and

\[ \left\{ (D_{3n+2} - D_{3n-1}) \delta_{n-1} \geq \frac{c}{b_0 + (\varepsilon/2)} \right\} \sup \left\{ D_{3n+2} - D_{3n-1} \geq \frac{c}{b_{D_{3n-1}}} \right\} \cap B_n, \]

whence, using that \( A_{n+1} = (A'_{n+1} \cap A_n) \cup \{ D_{3n+2} > \tau_M \} \cap A_n \):

\[ A_{n+1} \cap B'_n \supset \left\{ D_{3n+2} - D_{3n-1} \geq \frac{c}{b_{D_{3n-1}}} \text{ or } D_{3n+2} > \tau_M \right\} \cap A_n \cap B'_n. \]

Now by Lemma 9 and the strong Markov property we have (denoting by \( \mathcal{F} \) the natural filtration of the diffusion, and setting \( \alpha := (64\sigma^2c)^{-1} \)):

\[ \mathbb{P} \left( D_{3n+2} - D_{3n-1} \geq \frac{c}{b_{D_{3n-1}}} \text{ or } D_{3n+2} > \tau_M \mid \mathcal{F}_{D_{3n-1}} \right) \geq \mathbb{E} \left( 1 - e^{-\alpha b_{D_{3n-1}}} \mid \mathcal{F}_{D_{3n-1}} \right). \]

Hence

\[ \mathbb{P} \left( A_{n+1} \cap B'_n \right) \geq \mathbb{P} \left( \left\{ D_{3n+2} - D_{3n-1} \geq \frac{c}{b_{D_{3n-1}}} \text{ or } D_{3n+2} > \tau_M \right\} \cap A_n \cap B'_n \right) \]

\[ \geq \mathbb{E} \left[ \left( 1 - e^{-\alpha b_{D_{3n-1}}} \right) \times 1_{A_n \cap B'_n} \right] \geq \mathbb{E} \left[ \left( 1 - e^{-\alpha b_0 + (\varepsilon/2)} \left( \frac{\varepsilon}{b_0 + (\varepsilon/2)} \right) \right) \times 1_{A_n \cap B'_n} \right] \]

\[ \geq \left( 1 - \exp \left[ -\alpha b_0 + (\varepsilon/2) \left( \left( 1 + \varepsilon \right) c \sigma^2 / b_0 + (\varepsilon/2) \right) \right] \right) \times \mathbb{P} \left( A_n \cap B'_n \right) \]

\[ = \left( 1 - \exp \left[ -\frac{1}{64} \left( \sigma^2 c \right) \frac{1+\varepsilon}{1+\varepsilon} \left( 1 + \varepsilon \right) \frac{c b_0 + (\varepsilon/2)}{b_0 + (\varepsilon/2)} \frac{\varepsilon}{b_0 + (\varepsilon/2)} \right] \right) \times \mathbb{P} \left( A_n \cap B'_n \right). \]

Set \( \varepsilon_n := \exp \left[ -\frac{1}{64} \left( \sigma^2 c \right) \frac{1+\varepsilon}{1+\varepsilon} \left( 1 + \varepsilon \right) \frac{c b_0 + (\varepsilon/2)}{b_0 + (\varepsilon/2)} \frac{\varepsilon}{b_0 + (\varepsilon/2)} \right] \), for \( n \geq 1 \), and \( \varepsilon_0 := e^{-\alpha b_0} \), so that we have clearly, for \( b_0 \geq b(\sigma, R, \varepsilon) \), on one hand \( \sum_{n \geq 0} \varepsilon_n < b_0 - b(\varepsilon(2-\varepsilon) < \varepsilon \), and on the other hand:

\[ \mathbb{P} \left( A_{n+1} \cap B'_n \right) \geq (1 - \varepsilon_n) \times \mathbb{P} \left( A_n \cap B'_n \right) \geq \mathbb{P} \left( A_n \cap B'_n \right) - \varepsilon_n, \]

or equivalently

\[ \mathbb{P} \left( A_n \cap B'_n \cap (A_{n+1})^c \right) \leq \varepsilon_n. \]

Note that \( B'_n \supset \{ \tau_M \geq D_{3n-1} \} \cap B \). Therefore

\[ \mathbb{P} \left( A_{n+1} \cap B \right) = \mathbb{P} \left( A_n \cap \{ D_{3n-1} > \tau_M \} \cap B \right) + \mathbb{P} \left( A_{n+1} \cap \{ \tau_M \geq D_{3n-1} \} \cap B \right) \]
\[
\begin{align*}
= \mathbb{P}(A_n \cap \{D_{3n-1} > \tau_M\} \cap B) + \mathbb{P}(A_n \cap \{\tau_M \geq D_{3n-1}\} \cap B) - \mathbb{P}(A_n \cap \{\tau_M \geq D_{3n-1}\} \cap B \cap (A_{n+1})^c) \\
\geq \mathbb{P}(A_n \cap \{D_{3n-1} > \tau_M\} \cap B) + \mathbb{P}(A_n \cap \{\tau_M \geq D_{3n-1}\} \cap B) - \mathbb{P}(A_n \cap B' \cap (A_{n+1})^c) \\
\geq \mathbb{P}(A_n \cap \{D_{3n-1} > \tau_M\} \cap B) + \mathbb{P}(A_n \cap \{\tau_M \geq D_{3n-1}\} \cap B) - \varepsilon_n = \mathbb{P}(A_n \cap B) - \varepsilon_n ,
\end{align*}
\]
whence finally by Lemma 7, for \( b_0 \geq b(\sigma, R, \varepsilon) : \\
\mathbb{P}\left( \bigcap_{n \geq b} A_n \right) \geq \mathbb{P}(B) - \sum_{n \in \mathbb{N}} \varepsilon_n > 1 - 2b_0^{-\varepsilon^2/2} - b_0^{-\varepsilon(2-\varepsilon)} > 1 - \varepsilon . \diamond
\]

6) Estimates related to \(|a.|/b.

We need next to control the integral \( \int \frac{dt}{r_t b_t^2} \), which occurs in the Itô expression of the crucial quantity \( \frac{a_s}{b_s} \). The following lemma estimates its contribution due to an excursion in the hole.

**Lemma 11** During any excursion in the hole, id est during any proper time interval \([D_{3n}, D_{3n+2}]\), we have the following control: 
\[
\int_{D_{3n}}^{D_{3n+2}} \frac{dt}{r_t} \leq \frac{\pi R}{\min_{[D_{3n}, D_{3n+2}]} b}.
\]

**Proof** Firstly, it is sufficient to consider the proper time interval \([D_{3n}, D_{3n+1}]\), since the estimates are exactly the same on the other half \([D_{3n+1}, D_{3n+2}]\). Since \( T_t r_t^{3/2} = -\sqrt{(R - r_t)b_t^2 + Rr_t^2 + (a_t^2 - 1)r_t^3} \leq -\sqrt{R - r} \cdot b_t \) for \( D_{3n} \leq t \leq D_{3n+1} \), we get
\[
\int_{D_{3n}}^{D_{3n+1}} \frac{dt}{r_t} \leq - \int_{D_{3n}}^{D_{3n+1}} \sqrt{\frac{r_t}{R - r}} \frac{dr_t}{b_t} \leq \frac{1}{\min b[D_{3n}, D_{3n+2}]} \times \int_0^R \sqrt{\frac{r}{R - r}} dr \\
= \frac{\pi R}{2 \min b[D_{3n}, D_{3n+2}]} . \diamond
\]

**Remark 12** The same argument yields also the following estimate on the duration of an excursion in the hole (showing that the estimate from below in Lemma 9 is essentially optimal):
\[
D_{3n+2} - D_{3n} = \int_{D_{3n}}^{D_{3n+2}} dt \leq \frac{2}{\min b[D_{3n}, D_{3n+2}]} \times \int_0^R \sqrt{\frac{r}{R - r}} \cdot r dr = \frac{3\pi R^2}{4 \min b[D_{3n}, D_{3n+2}]} .
\]

We can now deduce the control on the integral \( \int \frac{dt}{r_t b_t^2} \), which we shall need below.

**Lemma 12** For any \( \varepsilon \in \left[0, \frac{1}{2}\right] \), there exists a lower bound \( b(\sigma, R, M, \varepsilon) \) such that for \( b_0 \geq b(\sigma, R, M, \varepsilon) \) we have:
\[
\int_0^{\tau_M} \frac{dt}{b_t^2} + \int_0^{\tau_M} \frac{R dt}{r_t b_t^2} \leq b_0^{2\varepsilon - 2} , \text{ with probability larger than } 1 - \varepsilon .
\]
Proof. Recall from Lemma 7 that with large probability we have $b_s \geq b_0^{1-(\varepsilon/2)} e^{(1-\varepsilon)\sigma^2 s}$ for $s \leq \tau_M$, whence $\int_0^{\tau_M} b_t^{-2} dt \leq \frac{b_0^{-2}}{2(1-\varepsilon)^{\varepsilon^2}}$. Moreover, using Lemma 11 we have:

$$
\int_0^{\tau_M} \frac{R dt}{r_t b_t^2} - \int_0^{\tau_M} \frac{dt}{b_t^2} < R \sum_{D_{3n} < \tau_M} \int_{D_{3n}} \frac{dt}{r_t b_t^2} < \pi R^2 \sum_{D_{3n} < \tau_M} (\min b[D_{3n}, D_{3n+2}])^{-3} < \pi R^2 b_0^{3-3} \sum_{n \in \mathbb{N}} (1 + n)^{-3(1-\varepsilon)\sigma^2} < b_0^{2\varepsilon-3} \text{ (for } b_0 \text{ large enough).} \phantom{a}
$$

We establish then the crucial control on $a_s/b_s$.

**Lemma 13.** Almost surely, if $r_s$ is bounded, then $a_s/b_s$ converges as $s \to \infty$, to some random limit $\ell \in \mathbb{R}$. Moreover for any $\varepsilon > 0$ and any $M \geq 3R/2$ there exists a lower bound $b(\sigma, R, M, \varepsilon)$ such that for $b_0 \geq b(\sigma, R, M, \varepsilon)$ we have:

$$
P \left( \sup_{0 \leq s < \tau_M} \frac{|a_s|}{b_s} < \left( \frac{|a_0|}{b_0} + 1 \right) \varepsilon^2 \right) > 1 - \varepsilon.
$$

Proof. a) Let us prove the second assertion first.

Set $\beta := \inf \{ s > 0 \mid b_s < b_0^{-\varepsilon}(1-\varepsilon)\sigma^2 s \}$, and $\eta := \inf \{ s > 0 \mid |a_s| - |a_0| > \varepsilon^2 b_0 \}$.

Fix $0 < \varepsilon < 1/3$ and $M \geq 3R/2$. Recall that we set $\tau_M := \inf \{s \mid r_s = M \}$, and that, for $b_0 \geq b(M, \varepsilon)$, the event $E_1 := \{ \beta \leq \tau_M \}$ has probability at least $1 - \varepsilon$ by Lemma 7, and that the event $E_2 := \{ \int_0^{\tau_M} \frac{dt}{b_t^2} + \int_0^{\tau_M} \frac{R dt}{r_t b_t^2} \leq b_0^{2\varepsilon-2} \}$ has probability at least $1 - \varepsilon$ by Lemma 12. Now, the equation governing $a_s/b_s$ writes for some Brownian motion $W$ and for any $s \geq 0$:

$$
\frac{a_s}{b_s} = \frac{a_0}{b_0} + W \left[ \sigma^2 \int_0^s \left( \frac{a_t}{b_t} \right)^2 \left( \frac{R}{b_t} \right)^2 dt + \sigma^2 \int_0^s \left( \frac{R}{l_t b_t^2} - \frac{1}{b_t^2} \right) dt \right] + \frac{\sigma^2}{2} \int_0^s \left( \frac{a_t}{b_t^2} \right)^2 dt,
$$

whence

$$
\left| \frac{a_s}{b_s} - \frac{a_0}{b_0} \right| \leq W^* \left[ \sigma^2 \int_0^s \left( \frac{a_t}{b_t} \right)^2 \left( \frac{R}{b_t} \right)^2 dt + \sigma^2 \int_0^s \frac{R}{l_t b_t^2} dt \right] + \frac{\sigma^2}{2} \int_0^s \left( \frac{a_t}{b_t^2} \right)^2 dt
$$

(recall that $W^*$ denotes the two-sided maximum process of $W$:

$W^*(s) := \max |W|([0, s])$) so that we have on $E_1 \cap E_2$, for $b_0 \geq b(\sigma, R, M, \varepsilon)$ and $0 \leq s < \tau_M \wedge \eta$:

$$
\left| \frac{a_s}{b_s} - \frac{a_0}{b_0} \right| \leq W^* \left[ \sigma^2 M^2 \left( \frac{|a_0|}{b_0} + \varepsilon^2 \right)^2 \int_0^{\tau_M} \frac{dt}{b_t^2} + \sigma^2 \int_0^{\tau_M} \frac{R}{r_t b_t^2} dt \right] + \frac{\sigma^2 M^2}{2} \left( \frac{|a_0|}{b_0} + \varepsilon^2 \right) \int_0^{\tau_M} \frac{dt}{b_t^2}
$$

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\[ W^* \left[ \frac{|a|}{b_0} + 1 \right]^2 b_0^{3 \varepsilon - 2} + \left( \frac{|a|}{b_0} + 1 \right) b_0^{3 \varepsilon - 2} < 2 \left( \frac{|a|}{b_0} + 1 \right) b_0^{3 \varepsilon - 1} < \varepsilon^2 \left( \frac{|a|}{b_0} + 1 \right), \]

on an event \( E_3 \) of probability larger than \( 1 - \varepsilon \), for \( b_0 \) large enough.

Hence we find that for \( b_0 \geq b(\sigma, R, M, \varepsilon) \) we have \( \eta \geq \tau_M \) on \( E_1 \cap E_2 \cap E_3 \), whence:

\[ \mathbb{P} \left( \sup_{0 < s < \tau_M} \left| \frac{a_s}{b_s} - \frac{a_0}{b_0} \right| < \left( \frac{|a|}{b_0} + 1 \right) \varepsilon^2 \right) > 1 - 3 \varepsilon. \]

b) To prove the first assertion of the statement, let us consider then, for any \( k \in \mathbb{N}^* \), the hitting time \( \sigma_k := \inf \{ s > 0 \mid b_s = b(\sigma, R, k, 2^{-k}) \} \), which is almost surely finite by Lemma 6. Moreover, since \( r \) is bounded, there exists almost surely some random integer \( \kappa \) such that \( \tau_\kappa = \infty \). Applying the strong Markov property to the above, we get

\[ \sum_{k \in \mathbb{N}^*} \mathbb{P} \left( \sup_{\sigma_k \wedge \kappa \leq s < \tau_k} \left| \frac{a_s}{b_s} - \frac{a_{\sigma_k}}{b_{\sigma_k}} \right| \geq 4^{-k} \left( \frac{|a_{\sigma_k}|}{b_{\sigma_k}} + 1 \right) \right) \leq 3 \sum_{k \in \mathbb{N}^*} 2^{-k} < \infty, \]

showing by the Borel-Cantelli lemma that almost surely, there exists some random integer \( \kappa' \) such that \( \sup_{s \geq \sigma_k \wedge \kappa \leq \tau_k} \left| \frac{a_s}{b_s} - \frac{a_{\sigma_k}}{b_{\sigma_k}} \right| < 4^{-k} \left( \frac{|a_{\sigma_k}|}{b_{\sigma_k}} + 1 \right) \) for \( k \geq \kappa' \). Hence we get almost surely \( \sup_{s \geq \sigma_k} \left| \frac{a_s}{b_s} - \frac{a_{\sigma_k}}{b_{\sigma_k}} \right| < 4^{-k} \left( \frac{|a_{\sigma_k}|}{b_{\sigma_k}} + 1 \right) \) for \( k \geq k_0 := \kappa \vee \kappa' \). This implies at once that

\[ \sup_{s \geq \sigma_{k_0}} \left| \frac{a_s}{b_s} \right| < \left( 2 \frac{|a_{\sigma_{k_0}}|}{b_{\sigma_{k_0}}} + 1 \right), \]

and then that \( \sup_{s \geq \sigma_k} \left| \frac{a_s}{b_s} - \frac{a_{\sigma_k}}{b_{\sigma_k}} \right| < 4^{-k} \left( 2 \frac{|a_{\sigma_k}|}{b_{\sigma_k}} + 1 \right) \) for \( k \geq k_0 \), thereby showing that \( a_s/b_s \) satisfies the Cauchy criterion, and then converges as \( s \to \infty \).

\[ \diamond \]

7) End of description of the second case in Theorem 5.

**Lemma 14** Let \( \varrho := \limsup_{s \to \infty} r_s \). Then almost surely, if \( r \) is bounded, \( a/\varrho \) converges to \( \pm \frac{1}{\sqrt{1 - \frac{\varrho}{\rho}}} \).

**Proof** Lemma 13 insures that \( a_s/b_s \) converges to some \( \ell \in \mathbb{R} \) as \( s \to \infty \), and we can find a sequence \( s_n \) of proper times increasing to \( \infty \) such that \( \varrho = \lim_{n \to \infty} r_{s_n} \) and \( T_{s_n} = 0 \) for every \( n \), which by the pseudo-norm relation implies:

\[ \frac{a^2_{s_n}}{b^2_{s_n}} = (1 - \frac{\varrho}{r_{s_n}}) (r_{s_n}^{-2} + b_{s_n}^{-2}), \]

so that using Lemma 6 we find as wanted:

\[ \ell^2 = (1 - \frac{\varrho}{\rho})(\ell_{s_n}^{-2} + \varrho_{s_n}^{-2}). \]

\[ \diamond \]

We prove next that when the relativistic diffusion becomes eventually captured by a neighbourhood of the hole, it tends to stay in some asymptotic random plane of \( \mathbb{R}^3 \).

**Lemma 15** Almost surely, if \( r \) is bounded, the direction \( \vec{b}_s/b_s \) of the angular momentum \( \vec{b}_s \) converges in \( \mathbb{S}^2 \) as \( s \to \infty \): the trajectories are asymptotically planar.
Proof. As we already saw in the third part of the proof of Theorem 3, it is easily deduced from Proposition 2 that
\[ d\left( \frac{\tilde{b}_s}{b_s} \right) = d\left( \theta_s \wedge \frac{\hat{\theta}_s}{U_s} \right) = - \left( \frac{r_s}{b_s} \sigma d\beta_s \right) \frac{\hat{\theta}_s}{U_s} - \left( \frac{\sigma^2 r_s^2}{2 b_s^2} ds \right) \theta_s \wedge \frac{\hat{\theta}_s}{U_s} \cdot \]

Then for any \( k \in \mathbb{N} \) and \( s \geq k \) we have (for some standard Brownian motion \( B_k \)):
\[
\left| \frac{\tilde{b}_s}{b_s} - \frac{\tilde{b}_k}{b_k} \right| = \sup_{u \in \mathbb{S}^2} \int_k^s \left[ \frac{\tilde{b}_u}{b_u} \right] \sigma d\beta_u + \frac{\sigma^2 r_u^2}{2 b_u^2} \left( \frac{\tilde{b}_u}{b_u} \cdot u \right) dt \leq \sigma B_k^* \left[ \int_k^s \frac{r_u^2}{b_u^2} dt \right] + \int_k^s \frac{\sigma^2 r_u^2}{2 b_u^2} dt.
\]

Now Lemma 6 insures that \( \int_k^\infty \frac{r_u^2}{b_u^2} dt \leq e^{-\sigma^2 k/2} \) for \( k \) larger than some finite random \( \kappa \), and the Borel-Cantelli lemma insures that \( B_k^* (e^{-\sigma^2 k/2}) \leq e^{-\sigma^2 k/6} \) for \( k \) larger than some finite random \( \kappa' \). Hence for \( k \geq \kappa \lor \kappa' \) we get \( \sup_{s \geq k} \left| \frac{\tilde{b}_s}{b_s} - \frac{\tilde{b}_k}{b_k} \right| \leq \sigma e^{-\sigma^2 k/6} + \sigma^2 e^{-\sigma^2 k/2} \), showing that \( \tilde{b}_s/b_s \) satisfies almost surely the Cauchy criterion and thus converges in \( \mathbb{S}^2 \).

8) End of the proof of Theorem 5.

The following lemma proves the statement (ii) of Theorem 5.

Lemma 16. Fix \( \varepsilon \in ]0,1[ \), and suppose \( r_0 \in ]R,3R/2[ \) and \( T_0 = 0 \). Fix \( \alpha, \beta \) such that \( R \leq \alpha < r_0 < \beta < 3R/2 \). Then for \( b_0 \) large enough, with probability larger than \( 1 - \varepsilon \), \( \varrho = \limsup_{s \to \infty} r_s \) belongs to \( ]\alpha, \beta[ \).

Proof. Set \( f(r) := r^{-1/2} \sqrt{1 - \frac{R}{r}} \). The function \( f \) is continuous and strictly increasing from \( [R, \frac{3R}{2}] \) onto \( \left[ 0, \frac{2}{3\sqrt{3}R} \right] \). For \( b_0 \) large enough, by the pseudo-norm relation we have
\[
|a_0| = f(r_0) \times \sqrt{1 + \frac{r_0^2}{b_0^2}} \in ]f(\alpha), f(\beta)[, \quad \text{and then by Lemma 13 the event}
\]
\[
E := \{ f(\alpha) < \inf_{0 \leq s < \tau_{3R/2}} |a_s| < \sup_{0 \leq s < \tau_{3R/2}} |a_s| < f(\beta) \} \quad \text{has probability larger than } 1 - \varepsilon.
\]

Now the pseudo-norm relation implies that \( f(r_s) 1_{\{r_s \geq R \}} \leq \frac{|a_s|}{b_s} \) for any \( s \geq 0 \). So that \( (\text{by continuity of } f^{-1}) \), \( r_s \) does never hit \( \beta \) on \( E \), and then \( E \subset \{ \tau_{3R/2} = \infty \} \). Finally by Lemma 14 and by continuity of \( f^{-1} \), we have \( f(\alpha) < f(\varrho) < f(\beta) \) and then \( \alpha < \varrho < \beta \) on \( E \).

The statement (i) in Theorem 5 is seen as in the proof of Theorem (2,2). Indeed, we observed there that if \( r_0 \geq 3R/2 \), \( T_0 \geq 4 + \frac{1}{R \sigma \alpha} \), and \( \lambda := \inf \left\{ s > 0 \left| T_s = 2 + \frac{2}{R \sigma \alpha} \right. \right\} \), then \( 1/|T_{s \wedge \lambda}| \) is a supermartingale. Now this implies, if \( T_0 \) is large enough:
\[
(2 + \frac{2}{R \sigma \alpha})^{-1} \mathbb{P}(\lambda < \infty) \leq \liminf_{s \to \infty} \mathbb{E}(|T_{s \wedge \lambda}|^{-1}_{\{\lambda < \infty\}}) \leq \liminf_{s \to \infty} \mathbb{E}(|T_{s \wedge \lambda}|^{-1}) \leq T_0^{-1},
\]
and then that \( \mathbb{P}(\lim_{s \to \infty} r_s = +\infty) \geq \mathbb{P}(\lambda = \infty) \geq 1 - (2 + \frac{9}{6\sigma^2})/T_0 > 1 - \varepsilon \).

Then the irreducibility of the relativistic diffusion follows at once from the proof of Theorem (2,2). With Statements (i) and (ii), this proves that both cases in Theorem 5 indeed occur with strictly positive probability, from any initial condition. Of course we already proved the dichotomy in Lemma 5.

Finally the description of the first case in Theorem 5 follows directly from Lemma 2 and from Theorem (2,3). \( \diamondsuit \)

### 3.7 Proof of Corollary 3

Let us begin by a lemma which specifies at which times the relativistic diffusion approaches the top of its excursions outside the hole.

**Lemma 17** Let us consider the following stopping times, for any \( n \in \mathbb{N} \) and any \( \varepsilon > 0 \):

\[
D_n' := \min\{s > D_{3n+1} | T_s = 0\} \quad \text{and} \quad D_n^\varepsilon := \min\{s > D_{3n+1} | T_s \leq \varepsilon b_s\} < D_n'.
\]

Then almost surely, when \( r. \) is bounded, we have: \( \lim_{n \to \infty} r_{D_n'} = \varrho \), and \( \lim_{n \to \infty} r_{D_n^\varepsilon} = \varrho_\varepsilon < \varrho \), where \( \varrho_\varepsilon \) is the unique solution between 0 and \( \frac{3R}{2} \) of the equation \( (1 - R/\varrho_\varepsilon) \varrho_\varepsilon^{-2} = \ell^2 - \varepsilon^2 \).

**Proof** This is only an additional precision to the second case in Theorem 5: applying the pseudo-norm relation at time \( D_n' \), we get

\[
(1 - R/r_{D_n'}) r_{D_n'}^{-2} \sim (1 - R/r_{D_n'}) (r_{D_n'}^{-2} + b_{D_n'}^{-2}) = a_{D_n'}^2/b_{D_n'}^2 \to \ell^2 \quad \text{as} \quad n \to \infty,
\]

so that, by Lemma 8, owing to the function \( f \) used in the proof of Lemma 16, we must have the unique possibility: \( \lim_{n \to \infty} r_{D_n'} = \varrho \). Similarly, at time \( D_n^\varepsilon \) we get

\[
(1 - R/r_{D_n^\varepsilon}) r_{D_n^\varepsilon}^{-2} \sim (1 - R/r_{D_n^\varepsilon}) (r_{D_n^\varepsilon}^{-2} + b_{D_n^\varepsilon}^{-2}) = \frac{a_{D_n^\varepsilon}^2}{b_{D_n^\varepsilon}^2} - \frac{T_{D_n^\varepsilon}^2}{b_{D_n^\varepsilon}^2} \to \ell^2 - \varepsilon^2 \quad \text{as} \quad n \to \infty,
\]

so that we must have the unique possibility: \( \lim_{n \to \infty} r_{D_n^\varepsilon} = \varrho_\varepsilon \). \( \diamondsuit \)

We shall need to control the angular contribution around the top of the excursions outside the hole. This is the aim of the following lemma. Notations are as above.

**Lemma 18** We have \( \limsup_{n \to \infty} \int_{D_n^\varepsilon} b_s \, ds \leq \frac{12}{3R - 2\varepsilon} \left( \frac{3R}{\sqrt{2}} \right)^4 \varepsilon \) on the intersection of the event \( \{ \varrho < \frac{3R}{2} \} \) and of an event of probability larger than \( 1 - \varepsilon \).

**Proof** Set \( E_0 := \{ \varrho < \frac{3R}{2} \} \). Fix \( M = M(\varepsilon) \geq \frac{3R}{2} \) such that the event \( E_0 \setminus \{ \sup r. < M \} \) has probability smaller than \( \varepsilon/8 \). Set \( E_1 = E_1(\varepsilon) := \{ \sup r. < M \} \). By Lemma 6 we can find some \( \gamma = \gamma(\varepsilon) \) such that, setting \( E_2 = E_2(\varepsilon) := \{ b_\gamma > \gamma e^{(1-\varepsilon)\sigma^2 s} \text{ for all } s \geq 0 \} \), the event \( E_0 \setminus E_2 \) has probability smaller than \( \varepsilon/8 \). Consider then the stopping time \( \nu = \nu(\varepsilon) \) at which \( b \) hits the lower bound \( b(\sigma, R, M, \varepsilon/4) \) of Lemma 10, which is finite on \( E_2 \). Applying now Lemma 10 and the strong Markov property (applied at time \( \nu \)), we find...
some integer \( k = k(\varepsilon) \) such that, setting \( E_3 = E_3(\varepsilon) := \cap_{n > k} \{ D_{3n+1} \geq \frac{1}{(1+\varepsilon)^{n}} \log(n/k) \} \) and \( E_4 = E_1 \cap E_2 \cap E_3 \), the event \( E_0 \setminus E_4 \) has probability smaller than \( \varepsilon/2 \).

Set \( \beta_n := b_{D_{3n+1}} \). We have obviously

\[
E_4 \subset B = B(\varepsilon) := \bigcap_{n \geq k} \{ \sqrt{\beta_n} > \gamma \exp[\frac{1-\varepsilon}{2(1+\varepsilon)} \log(n/k)] \}.
\]

Consider also for all \( n > k \) : \( B_n = B_n(\varepsilon) := \bigcap_{m=K+1}^n \{ \sqrt{\beta_n} > \gamma \exp[\frac{1-\varepsilon}{2(1+\varepsilon)} \log(n/k)] \} \), and let us proceed now somewhat as in Lemma 9, to control the variations of \( b \) and \( T \) between \( D_{3n+1} \) and \( D_n' \).

Set \( \eta_n := \inf \{ s > D_{3n+1} \mid |\log(b_s/\beta_n)| = \log 2 \} \), and \( \tau := \inf \{ s > 0 \mid r_s = M \} \) (which is infinite on \( E_4 \)). For \( D_{3n+1} \leq s \leq (D_{3n+1} + \beta_n^{-1/2}) \wedge \eta_n \wedge \tau \) and \( n \) large enough, we have:

\[
|\log(b_s/\beta_n)| \leq \sigma W_n^\ast \left[ (1 + \frac{4M^2}{\beta_n}) \frac{1}{\beta_n^{1/2}} \right] + \frac{\sigma^2}{\beta_n} < \frac{1}{2} + \frac{\sigma^2}{\beta_n^{1/2}} < \log 2 \quad \text{on} \quad B_n \cap F_n,
\]

where \( F_n := \{ \sigma W_n^\ast[2\beta_n^{-1/2}] < \frac{1}{2} \} \). Whence \( B_n \cap F_n \subset \{ \eta_n > (D_{3n+1} + \beta_n^{-1/2}) \wedge \tau \} \).

Recalling Remark 12, we observe then that for large enough \( n \) we have

\[
D_{3n+2} \wedge (D_{3n+1} + \beta_n^{-1/2}) \wedge \tau \leq D_{3n+1} + \frac{3\pi R^2}{4\beta_n} < D_{3n+1} + \beta_n^{-1/2} \quad \text{on} \quad B_n \cap F_n,
\]

whence for any constant \( \alpha \) : \( B_n \cap F_n \subset \{ \eta_n > (D_{3n+1} + \alpha/\beta_n) \wedge \tau \} \) for large enough \( n \).

Then for any deterministic \( q, j \in \mathbb{N}^\ast \), set \( A_{q,j} := \{(j+1)2^{-q} < \frac{3\beta_n}{2} - \eta \} \subset E_0 \), and fix a deterministic integer \( m \) such that each \( A_{q,j}^m := A_{q,j} \cap \{ \sup_{s > D_{3n+1}} r_s < \eta + j2^{-q} \} \) satisfies

\[
\mathbb{P}(A_{q,j} \setminus A_{q,j}^m) < \frac{2^{2+q}}{3R}. \quad \text{Set} \quad \tau_{q,j}^m := \inf \{ s > D_{3n+1} \mid r_s = \frac{3\beta_n}{2} - \frac{j}{2} \} \quad \text{which is infinite on} \quad A_{q,j}^m \).
\]

Consider \( \Lambda_n^\varepsilon := \inf \{ s > D_n^\varepsilon \mid T_s \geq \beta_n \} \), and fix some \( \alpha > 0 \) (to be specified below). The equation governing \( T \) implies that for \( D_n^\varepsilon \leq s \leq \eta_n \wedge D_n' \wedge \Lambda_n^\varepsilon \wedge (D_n^\varepsilon + \alpha/\beta_n) \wedge \tau_{q,j}^m \)
and for \( n \) large enough we have:

\[
T_s = \varepsilon b_{D_n} + \sigma \bar{W}_n \left[ \int_{D_n}^s \left( T_t^2 + 1 - \frac{R}{r_t} \right) dt \right] + \frac{3\sigma^2}{2} \int_{D_n}^s T_t \left( \frac{R}{r_t} \right) dt + \frac{\beta_n^3}{4} \frac{1}{r_t} dt - \int_{D_n}^s \frac{R}{2r_t} dt
\]

\[
\leq 2\varepsilon \beta_n + \sigma \bar{W}_n^\ast \left[ (\beta_n^2 + 1) \frac{1}{\beta_n} \right] + \frac{3\sigma^2}{2} \alpha < 2\varepsilon \beta_n + \beta_n^{3/4} + \frac{3\sigma^2}{2} \alpha < \beta_n
\]
on \( F'_n := \{ \sigma \bar{W}_n^\ast[2\alpha/\beta_n] < \beta_n^{3/4} \} \), so that \( F'_n \subset \{ \Lambda_n^\varepsilon > \eta_n \wedge D_n' \wedge (D_n + \alpha/\beta_n) \wedge \tau_{q,j}^m \} \).

The equation of \( T \) implies also (in the same way) that, for large enough \( n \) and for \( D_n^\varepsilon \leq s \leq \eta_n \wedge D_n' \wedge \Lambda_n^\varepsilon \wedge (D_n^\varepsilon + \alpha/\beta_n) \wedge \tau_{q,j}^m \):

\[
T_s \leq 2\varepsilon \beta_n + \sigma \bar{W}_n^\ast \left[ 2\alpha \beta_n \right] + \frac{3\sigma^2}{2} \alpha - j \frac{(2/3R)^4}{4} \beta_n^2 (s - D_n^\varepsilon);
\]
while on $F'_n$, for $s = D_n^\epsilon + \alpha/\beta_n$, taking $\alpha := \frac{3\epsilon 2^q}{j} \times \left(\frac{3R}{\sqrt{2}}\right)^4$, the right hand side above is $< 0$ for $n$ large enough. Hence we find that

\[ B_n \cap F_n \cap F'_n \subset \{D'_n \wedge \tau^m_{q,j} < D^\epsilon_n + \alpha/\beta_n\}, \text{ for large enough } n. \]

Therefore so far we get on $B_n \cap F_n \cap F'_n$, almost surely for large enough $n$:

\[ \int_{D'_n \wedge \tau^m_{q,j}} b_s \, ds \leq 2\alpha = \frac{6 \epsilon 2^q}{j} \left(\frac{3R}{\sqrt{2}}\right)^4. \]

Moreover, since the standard Brownian motions $W_n$ and $\bar{W}_n$ are independent from the $\sigma$-field $\mathcal{F}_{D_{n+1}}$, we have:

\[ \mathbb{P}(F^c_n \mid \mathcal{F}_{D_{n+1}}) < e^{-\sqrt{\beta_n/(16\sigma^2)}} \quad \text{and} \quad \mathbb{P}((F'_n)^c \mid \mathcal{F}_{D_{n+1}}) < e^{-\sqrt{\beta_n/(4\sigma^2)}}. \]

Therefore

\[ \sum_{n>k} \mathbb{P}(F^c_n \cap B) \leq \sum_{n>k} \mathbb{P}(F^c_n \cap B_n) < \sum_{n>k} \exp \left( -\frac{n}{16\sigma^2} \exp\left(\frac{1-e^{-1}}{2(1+e)} \log(n/k)\right) \right) < \infty, \]

and the Borel-Cantelli lemma implies that $B$ is almost surely included in $\liminf_n F_n$. This is obviously the same for $F'_n$, proving that almost surely for any $q, j$ we have on $B \cap A^m_{q,j}$, for large enough $n$:

\[ \int_{D'_n} b_s \, ds \leq \frac{6 \epsilon 2^q}{j} \left(\frac{3R}{\sqrt{2}}\right)^4. \]

Finally, since the event $E_0 \setminus \left\{ \limsup_{n \to \infty} \int_{D'_n} b_s \, ds \leq \frac{12}{3R-2\epsilon} \left(\frac{3R}{\sqrt{2}}\right)^4 \epsilon \right\}$ is the increasing union (for $q \in \mathbb{N}^*$) of the events $\bigcup_{j=1}^{3R 2^{q-1}} A_{q,j} \setminus \left\{ \limsup_{n \to \infty} \int_{D'_n} b_s \, ds \leq \frac{6 \epsilon 2^q}{j} \left(\frac{3R}{\sqrt{2}}\right)^4 \right\}$, we get

\[ \mathbb{P}\left[E_0 \setminus \left\{ \limsup_{n \to \infty} \int_{D'_n} b_s \, ds \leq \frac{12}{3R-2\epsilon} \left(\frac{3R}{\sqrt{2}}\right)^4 \epsilon \right\}\right] \leq \mathbb{P}[E_0 \setminus B] + \sup_{q \in \mathbb{N}^*} \sum_{j=1}^{3R 2^{q-1}} \mathbb{P}[A_{q,j} \setminus A^m_{q,j}] < \epsilon, \]

which concludes the proof. \(\diamond\)

We can now establish the second assertion of Corollary 3. (The first assertion is just Lemma 17.)

Set $g(r) := R - r + \ell^2 r^3$, so that $g(\ell) = 0$ and $g'(r) = (\sqrt{3} |\ell| r + 1)(\sqrt{3} |\ell| r - 1)$. We have $\varrho \leq \frac{3R}{2} \Leftrightarrow \varrho \leq \frac{1}{|\ell|\sqrt{3}}$, and $\varrho = \frac{3R}{2} \Leftrightarrow \varrho = \frac{1}{|\ell|\sqrt{3}}$. So that the root $r = \varrho$ of $g$ is double if and only if $\varrho = \frac{3R}{2}$. So that the integral $\int_0^\varrho \frac{dr}{\sqrt{rg'(r)}}$ converges if and only if $\varrho < \frac{3R}{2}$, thereby justifying our assumption $\varrho < \frac{3R}{2}$. 

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Then the pseudo-norm relation and Lemmas 6 and 14 imply the convergence of 
\[ \frac{T_n^2}{\omega_n^2} + (1 - \frac{R}{r_s})r_s^{-2} \] towards \( \ell^2 = (1 - \frac{R}{\varrho})\varrho^{-2} \). By the definition of \( b \), this can be written:

\[ U_s \, ds = \text{sgn}(T_s) \left( r_s \times \left[ R - r_s + (\ell^2 + \varepsilon_s)r_s^3 \right] \right)^{-1/2} \, dr_s, \]

with \( \lim_{s \to \infty} \varepsilon_s = 0 \). Lemma 15 means that we can restrict to the limit plane orthogonal to \( \overrightarrow{\beta} := \lim_{s \to \infty} \overrightarrow{b_s}/ds \in S^2 \). Indeed, we have \( \dot{\theta}_s/U_s = (\overrightarrow{b_s}/ds) \wedge \theta_s \), whence \( \dot{\theta}_s = U_s (\overrightarrow{\beta} \wedge \theta_s + \varepsilon_s) \), with \( \varepsilon_s \perp \theta_s \) and \( \lim_{s \to \infty} \varepsilon_s = 0 \); so that writing \( \theta_s = \lambda_s \overrightarrow{\beta} + \overrightarrow{V}_s \) with \( \overrightarrow{V}_s \perp \overrightarrow{\beta} \), we have as \( s \to \infty \):

\[ |\overrightarrow{V}_s| = |\overrightarrow{\beta} \wedge \overrightarrow{V}_s| = |\overrightarrow{\beta} \wedge \theta_s| = |\theta_s/U_s - \varepsilon_s| \longrightarrow 1, \]

whence \( \lambda_s \longrightarrow 0 \).

Moreover \( \overrightarrow{\beta} \wedge \overrightarrow{V}_s = \overrightarrow{\beta} \wedge \theta_s \) and then, setting \( \overrightarrow{v}_s := \overrightarrow{V}_s/|\overrightarrow{V}_s| \), we have on one hand:

\[ |\theta_s - v_s| \leq |\lambda_s| + |\overrightarrow{V}_s| - 1 \longrightarrow 0 \]

almost surely as \( s \to \infty \), and on the other hand:

\[
\begin{align*}
d\overrightarrow{v}_s &= d \frac{\overrightarrow{V}_s}{|\overrightarrow{V}_s|} = \frac{d\theta_s - d\lambda_s \overrightarrow{\beta}}{|\overrightarrow{V}_s|^2} \overrightarrow{V}_s - \frac{\overrightarrow{V}_s}{|\overrightarrow{V}_s|^2} d|\overrightarrow{V}_s| = \overrightarrow{\beta} \wedge \overrightarrow{V}_s \, U_s \, ds - \overrightarrow{\beta} \wedge \overrightarrow{V}_s \, d\lambda_s + \frac{U_s}{|\overrightarrow{V}_s|} \varepsilon_s \, ds - \frac{\overrightarrow{V}_s}{|\overrightarrow{V}_s|^2} \, d|\overrightarrow{V}_s| \\
&= (\overrightarrow{\beta} \wedge \overrightarrow{v}_s) \, U_s \, ds + (\varepsilon_s \cdot [\overrightarrow{\beta} \wedge \overrightarrow{v}_s]) \left( \overrightarrow{\beta} \wedge \overrightarrow{v}_s \right) \frac{U_s}{|\overrightarrow{V}_s|} \, ds \\
&= (\overrightarrow{\beta} \wedge \overrightarrow{v}_s) \left( 1 + \det \left[ \frac{\overrightarrow{V}_s}{|\overrightarrow{V}_s|}, \overrightarrow{\beta}, \overrightarrow{v}_s \right] \right) \, U_s \, ds.
\end{align*}
\]

Let us now denote by \( \phi_s \) the angular coordinate of \( \overrightarrow{v}_s \) in the constant plane \( \overrightarrow{\beta} \); then the preceding equation writes equivalently (using the expression of \( U_s \) seen at the beginning of this proof, and choosing the orientation in the plane \( \overrightarrow{\beta} \) in order to have a positive sign for the remainder of this proof):

\[ d\phi_s = \left( 1 + \det \left[ \frac{\overrightarrow{V}_s}{|\overrightarrow{V}_s|}, \overrightarrow{\beta}, \overrightarrow{v}_s \right] \right) U_s \, ds = \left( 1 + \det \left[ \frac{\overrightarrow{V}_s}{|\overrightarrow{V}_s|}, \overrightarrow{\beta}, \overrightarrow{v}_s \right] \right) \frac{\text{sgn}(T_s) \, dr_s}{\sqrt{r_s \left[ R - r_s + (\ell^2 + \varepsilon_s)r_s^3 \right]}}, \]

whence (setting \( \delta_s := \det \left[ \frac{\overrightarrow{V}_s}{|\overrightarrow{V}_s|}, \overrightarrow{\beta}, \overrightarrow{v}_s \right] \) for any \( n \in \mathbb{N} \) and \( \varepsilon > 0 \):

\[ \int_{D_n^\varepsilon}^D d\phi_s = \int_{D_{n+1}^\varepsilon}^{D_n^\varepsilon} d\phi_s = \int_{D_{n+1}^\varepsilon}^{D_n^\varepsilon} \frac{(1 + \delta_s) \, dr_s}{\sqrt{r_s \left[ R - r_s + (\ell^2 + \varepsilon_s)r_s^3 \right]}}, \]

and idem with \( D_n' \) instead of \( D_n^\varepsilon \). Since we have \( \lim_{n \to \infty} \int_{D_{n+1}^\varepsilon}^{D_n^\varepsilon} d\theta_s - \int_{D_{n+1}^\varepsilon}^{D_n^\varepsilon} d\overrightarrow{v}_s = 0 \) by the beginning of this proof, we have henceforth to deal with

\[ \int_{D_{n+1}^\varepsilon}^{D_n^\varepsilon} \frac{(1 + \delta_s) \, dr_s}{\sqrt{r_s \left[ R - r_s + (\ell^2 + \varepsilon_s)r_s^3 \right]}}, \]

(and analogously with \( D_n' \) instead of \( D_n^\varepsilon \)). Recall that \( \lim_{n \to \infty} \frac{1}{r_{D_n^\varepsilon}} = \varrho \varepsilon < \varrho \) by Lemma 17.
Now, pushing somewhat further the observation we already used for proving Theorem 3, we can use the strictly increasing radius \( r_s \) as another "time" variable on the proper time interval \([D_{3n+1}, D'_n]\). So we set \( s = s_n(r) , \varepsilon_n := \varepsilon_{s_n(r)} , \tilde{\delta}_n := \delta_{s_n(r)} \) on \([D_{3n+1}, D'_n]\), and we get almost surely:

\[
\int_{D_{3n+1}}^{D'_n} d\phi_s = \int_0^{r_{D'_n}} \frac{(1 + \tilde{\delta}_n) \, dr}{\sqrt{r [R - r + (\ell^2 + \varepsilon_n)^3]}} .
\]

Note that we have \(|\varepsilon_n| = |\varepsilon_{s(r)}| \to 0\) and \(|\tilde{\delta}_n| = O(|\varepsilon_{s(r)}|) \to 0\), uniformly on \([D_{3n+1}, D'_n]\) as \( n \to \infty \), and that we have for large enough \( n \) : \( r_{D'_n} < (\varrho_{\varepsilon} + \varrho)/2 < \varrho \).

Hence, using the function \( g \) introduced at the beginning of this proof, we see that \( 0 \leq r \leq r_{D'_n} \Rightarrow g(r) \geq g((\varrho_{\varepsilon} + \varrho)/2) > 0 \), and then that dominated convergence holds, for any \( \varepsilon > 0 \), showing that we have almost surely:

\[
\lim_{n \to \infty} \int_{D_{3n+1}}^{D'_n} d\phi_s = \int_0^{\varepsilon} \frac{dr}{\sqrt{r [R - r + \ell^2 r^3]}} .
\]

Since it is clear that \( \int_0^{\varepsilon} \frac{dr}{\sqrt{r [R - r + \ell^2 r^3]}} \) converges to \( \int_0^{\varepsilon} \frac{dr}{\sqrt{r [R - r + \ell^2 r^3]}} \) as \( \varepsilon \searrow 0 \), we are left with the remainder \( \int_{D'_n}^{D'_n} d\phi \), which we must control uniformly. Note that such control is not obvious at all, since we have a root of the denominator precisely at time \( D'_n \) (at which we are at the top of the excursion), according to the fact that around this same time the angular move is much more rapid than the radial one.

Now we have, for large enough \( n \) (using that \( \varepsilon < \frac{2}{R} \Rightarrow (1 - R/\varrho_{\varepsilon}) \varrho_{\varepsilon} - 2 > 0 \Rightarrow \varrho_{\varepsilon} > \frac{R}{2} \Rightarrow r_{D'_n} \geq \frac{R}{3} \)):

\[
\left| \int_{D'_n}^{D'_n} d\phi_s \right| = \int_{D'_n}^{D'_n} \frac{(1 + \tilde{\delta}_n) \, dr}{\sqrt{r [R - r + (\ell^2 + \varepsilon_n)^3]}} \leq 2 \int_{D'_n}^{D'_n} \frac{dr}{\sqrt{r [R - r + (\ell^2 + \varepsilon_n)^3]}} \leq 2 \int_{D'_n}^{D'_n} b_{s_n(r)} d(s_n(r)) = \frac{18}{R} \int_{D'_n}^{D'_n} b_{s_n(r)} d(s_n(r)) = \frac{18}{R} \int_{D'_n}^{D'_n} b_{s_n(r)} ds .
\]

Therefore, applying Lemma 18, we conclude that we have with probability larger than \( 1 - \varepsilon \):

\[
\limsup_{n \to \infty} \left| \int_{D_{3n+1}}^{D'_n} d\phi_s \right| \leq \frac{12}{3R - 2\varrho} \left( \frac{3R}{\sqrt{2}} \right)^4 \varepsilon , \text{ and then}
\]

\[
\limsup_{n \to \infty} \left| \int_{D_{3n+1}}^{D'_n} d\phi_s - \int_0^{\varepsilon} \frac{dr}{\sqrt{r [R - r + \ell^2 r^3]}} \right| \leq \frac{12}{3R - 2\varrho} \left( \frac{3R}{\sqrt{2}} \right)^4 \varepsilon + \int_0^{\varepsilon} \frac{dr}{\sqrt{r [R - r + \ell^2 r^3]}} ,
\]

which goes to 0 as \( \varepsilon \searrow 0 \), while the left hand side does not depend upon \( \varepsilon \). This proves that indeed we have almost surely:

\[
\lim_{n \to \infty} \int_{D_{3n+1}}^{D'_n} d\theta_s = \lim_{n \to \infty} \int_{D_{3n+1}}^{D'_n} d\phi_s = \int_0^{\varepsilon} \frac{dr}{\sqrt{r [R - r + \ell^2 r^3]}} ,
\]

\( \diamond \)
4 Appendix: Study of timelike and null geodesics

4.1 Timelike geodesics

The case \( \sigma = 0 \) in Section 2.3 corresponds to geodesics; precisely it is the case of timelike geodesics having speed 1. The equations we obtained in Section 2.3 are here simplified as follows:

\[
a, b \text{ constant } ; \quad \ddot{r}_s = (r_s - \frac{3}{2}R) \frac{b^2}{r_s^4} - \frac{R}{2r_s^2}.
\]

Integrating this last equation (after multiplication by \( \dot{r}_s \)) leads simply, up to some constant, to the unit pseudo-norm relation \( t'c_0 \geq g e_0 = 1 \):

\[
\dot{r}_s^2 = a^2 - (1 - \frac{R}{r_s})(1 + \frac{b^2}{r_s^2}),
\]

or equivalently to:

\[
|s| = \int_{r(0)}^{r(s)} \frac{r \ dr}{\sqrt{a^2r^2 - (1 - \frac{R}{r})(r^2 + b^2)}}.
\]

The equations relative to \( \varphi_s \) and to \( \psi_s \) yield easily a real constant \( k \) such that:

\[
\varphi_s^2 = \frac{b^2}{r_s^4} - \frac{k^2}{r_s^2 \sin^2 \varphi_s} , \quad \psi_s = -\frac{k}{r_s^2 \sin^2 \varphi_s}.
\]

They are equivalent to the equation \( \frac{d}{ds}(r_s^2 \dot{\varphi}_s) = -b^2 r_s^{-2} \theta_s \). (This is also a consequence of Proposition 2 in Section 2.4.1 below, taking there \( \sigma = 0 \).) They are also equivalent to the constancy of the angular momentum \( \vec{\hat{b}} \). This means in particular that \( \dot{\varphi} = -U^2 \theta - 2(\dot{r}/r) \hat{\theta} \) along geodesics, and therefore that every geodesic is included in some plane containing the origin \( \{ r = 0 \} \).

Set \( u := 1/r \) and \( P(u) := (1 - Ru)(1 + b^2u^2) \). The variation of \( P \) as \( r \) increases from \( R \) to \( \infty \) is as follows:

- if \( b \leq R\sqrt{3} \): \( a^2 - P(u) \) decreases (as \( r \) increases from \( R \) to \( \infty \)) from \( a^2 \) to \( a^2 - 1 \);
- if \( b > R\sqrt{3} \): (as \( r \) increases from \( R \) to \( \infty \)) \( a^2 - P(u) \) decreases firstly from \( a^2 \) to \( a^2 - P(u_1) \), then increases from \( a^2 - P(u_1) \) to \( a^2 - P(u_2) \), then decreases from \( a^2 - P(u_2) \) to \( a^2 - 1 \), where \( \frac{2}{3R} > u_1 := \frac{1+\sqrt{1-3R^2b^2}}{3R} > u_2 := \frac{1-\sqrt{1-3R^2b^2}}{3R} > 0 \).

Note that for \( b > R\sqrt{3} \) \( P(u_1) = \frac{8}{9} + \frac{2}{27R^2}(b^2 - 3R^2)(1 + \sqrt{1 - 3R^2b^2}) \in ]\frac{8}{9}, +\infty [ \), and \( P(u_2) = \frac{8}{9} + \frac{2}{27R^2}(b^2 - 3R^2)(1 - \sqrt{1 - 3R^2b^2}) \in ]\frac{8}{9}, \min\{1, P(u_1)\}[ \).

Besides, \( \int_{a^2/r_0^2 - [1 - \frac{2}{3}(r^2 + b^2)]}^r \frac{r \ dr}{\sqrt{a^2r^2 - [1 - \frac{2}{3}(r^2 + b^2)]}} \) is integrable near a simple root of \( a^2 - P(1/r) \), but not near a double root.

Hence, using the relation \( \dot{r}^2 = a^2 - P(u) \) (which does not allow any value for \( r_0 \)), we get the following classification of timelike geodesics. (See [L-L], § 100, problems 1,2, for a partial resolution, and [M-T-W], section 25.5, for a more explicit one).
Note that we restrict first to the case of $S_0$. The extension to the full Schwarzschild space $S$ is then easy: see Remark 13 below.

1. $P$ monotone

**Case 1.1:** $b \leq R\sqrt{3}$ and $|a| \geq 1$: $a^2 - P$ has no root, so that $(s \mapsto r_s)$ runs, in one direction or in the other one, an increasing trajectory from $R$ to $+\infty$, slowing down (in the increasing case) till the limit speed $\sqrt{a^2 - 1}$. In the particular case $|a| = 1$, we have $r_s \sim (9R^2s^2/4)^{1/3}$ as $s \to \infty$.

**Case 1.2:** $b \leq R\sqrt{3}$ and $a^2 < 1$ (but $b \neq R\sqrt{3}$ or $a^2 \neq 8/9$): $a^2 - P$ has a simple root, so that $(s \mapsto r_s)$ runs a bounded trajectory, which has 2 ends at $R$, and increases firstly then decreases, with a unique maximum at $R_0 \in ]R, \infty[$. $r_0 > R_0$ is impossible.

**Case 1.3:** $b = R\sqrt{3} = P(u_1) = 8/9$: $a^2 - P$ has a triple root at $\frac{1}{u_2} = 3R$, so that we have here either a geodesic which runs monotonically (during an infinite time) the interval $[R, 3R]$, in one direction or in the other one, or a geodesic included in a circle (centred at 0) of radius $3R$. Indeed, it is easily verified (looking at $\dot{r}$) that such circular geodesics correspond precisely to multiple roots of $a^2 - P$. $r_0 > 3R$ is impossible.

2. $P$ non-monotone

**Case 2.1:** $b > R\sqrt{3}$ and $a^2 > P(u_1)$: $a^2 - P$ has a unique (simple) root, $\dot{r}$ vanishes at $r = R_0 \in ]R, \frac{1}{u_2}[$, and we are brought back to the case 1.2.

**Case 2.2:** $b > R\sqrt{3}$ and $P(u_1) < a^2 < 1$: $a^2 - P$ has a unique (simple) root, $\dot{r}$ vanishes at $r = R_2 \in ]\frac{1}{u_2}, \infty[$, and we are brought back to the case 1.2, alternatively with acceleration and slackening phases, the unique maximum being here at $R_2$.

**Case 2.3:** $b > R\sqrt{3}$ and $a^2 = P(u_1) \geq 1$: $a^2 - P$ has a double root at $\{ r = \frac{1}{u_1} \}$ (and no other root), so that the trajectory $(s \mapsto r_s)$ needs an infinite time to reach this level. Such geodesics run monotonically, in one direction or in the other one, either the interval $[R, \frac{1}{u_2}[$, or the interval $]\frac{1}{u_2}, \infty[$. There are again here also geodesics included in a circle (centred at 0) of radius $\frac{1}{u_1}$.

**Case 2.4:** $b > R\sqrt{3}$ and $1 \leq a^2 < P(u_1)$: $a^2 - P$ has a two (simple) roots, $\dot{r}$ vanishes at $r = R_0 \in ]R, \frac{1}{u_1}[$, and at $r = R_1 \in ]\frac{1}{u_2}, \frac{1}{u_4}[$; we are brought back to the case 1.2 if $r_0 \in [R, R_0]$: $r_0 \in ]R_0, R_1[$ is impossible; and if $r_0 \in [R_1, \infty[$, then $(s \mapsto r_s)$ runs an unbounded trajectory the 2 ends of which are at $\infty$, with a unique minimum at $R_1$. In this last subcase we have $r_s \sim |s|\sqrt{a^2 - 1}$ (and even $r_s = |s|\sqrt{a^2 - 1} + \log(1 + |s|) + r_0 + o(1)$) as $s \to \infty$ if $a^2 > 1$, and $r_s \sim (9R^2s^2/4)^{1/3}$ if $a^2 = 1$. Such (projection of) geodesic runs approximately a parabola or a branch of hyperbola.

**Case 2.5:** $b > R\sqrt{3}$ and $a^2 = P(u_1) < 1$: $a^2 - P$ has a double root at $\{ r = \frac{1}{u_1} \}$ and a simple root at $r = R_0 > \frac{1}{u_2}$. Hence the trajectory $(s \mapsto r_s)$ needs an infinite time to reach the level $\{ r = \frac{1}{u_1} \}$, so that there are on one hand (projection of) geodesics which run
monotonically the interval $[R, \frac{1}{u_1}], \text{ in one direction or in the other one, and on the other hand (projection of) geodesics which run (during an infinite time) from $\frac{1}{u_1}$ to $\frac{1}{u_2}$ via a unique maximum at $R_2$. Again, there are also here geodesics included in a circle (centred at $0$) of radius $\frac{1}{u_2}$. $r_0 > R_2$ is impossible.

Case 2.5.2: $b > R\sqrt{3}$ and $a^2 = P(u_2)$: $a^2 - P$ has a double root at $\{r = \frac{1}{u_2}\}$ and a simple root at $r = R_0 < \frac{1}{u_1}$. Hence we are brought back to the case 1.2 if $r_0 \in [R, R_0]$, and we have also centred circular geodesics of radius $\frac{1}{u_2}$. $r_0 \in [R_0, \frac{1}{u_2} \cup \frac{1}{u_2}, \infty]$ is impossible.

Case 2.6: $b > R\sqrt{3}$ and $P(u_2) < a^2 < \min\{P(u_1), 1\}$: $a^2 - P$ has 3 distinct simple roots, so that $\dot{r}$ vanishes at $r = R_0 \in [R, \frac{1}{u_1}], \text{ at } r = R_1 \in [\frac{1}{u_1}, \frac{1}{u_2}]$ and at $r = R_2 \in [\frac{1}{u_2}, \infty]$; we are brought back to the case 1.2 if $r_0 \in [R, R_0]$: if $r_0 \in [R_1, R_2]$, then $(s \mapsto r_s)$ oscillates periodically, increasing from $R_1$ to $R_2$ then decreasing from $R_2$ to $R_1$, running approximately an ellipse. $r_0 \in [R_0, R_1] \cup [R_2, \infty]$ is impossible.

Note that the set of limiting radii $\frac{1}{u_1}$ such that there exists a geodesic which winds asymptotically (either from inside or from outside) around a circle (centred at $0$) of radius $\frac{1}{u_1}$ equals the interval $[3R/2, 3R]$ . Note also that the set of radii $\frac{1}{u_1}$ or $\frac{1}{u_2}$ of circles (centred at $0$) which contain geodesics equals the interval $[3R/2, \infty]$.

This gives a geometrical intrinsic meaning to the particular radii $3R$ and $3R/2$.

Remark 13. The extension to the full space $\mathcal{S}$ of the preceding classification and description of timelike geodesics in the restricted space $\mathcal{S}_0$ is more or less straightforward. Indeed the results of Theorem 3 are clearly valid for geodesics as well, with the major simplification that $a_s$ and $b_s$ are constant when $\sigma = 0$.

Otherwise, there exist timelike geodesics included in the cylinder $\{r = R\}$: looking at the above and taking $a = 0 < |k| < b$, we find easily such solutions, which satisfy $\varphi_s = \text{Arccos}(\sqrt{1 - k^2/b^2} \sin(\pm b(s - s_0)/b^2))$. This completes for the space $\mathcal{S}$ the picture of timelike geodesics we have just drawn above for the strict Schwarzschild space $\mathcal{S}_0$.

Note finally the following intrinsic characterisation of the radius $R$ in the space $\mathcal{S}$: $R$ is the minimal radius which can be reached by a timelike geodesic which does not hit the singularity.

### 4.2 Null geodesics

For these null geodesics, or light rays, the unit pseudo-norm relation is replaced by

$$\alpha^2 - \bar{r}^2 = (1 - R/r) r^{-2}.$$ 

Note that the proper time does not make sense any more, so that the new “time” parameter or abscissa $\lambda$ makes sense only up to an affine transform $(\lambda \mapsto q\lambda + q')$, and the constant parameters $a$ and $b$ do not make sense both anymore, but only their quotient $\alpha := a/b$. This unique “impact parameter” $\alpha$ of the null geodesic is of course a constant of the
geodesic. As for the timelike geodesics, the equations relative to \( \theta = (\varphi, \psi) \) show that they are planar, so that by means of a trivial change of axis we may consider that \( \varphi \equiv \pi/2 \).

Thus every null geodesic is determined by the equations (in its own plane, the derivatives being relative to the abcissa \( \lambda \)) (see also [M-T-W], page 674):

\[
\alpha \text{ constant;} \quad r^2 + (1 - R/r) r^{-2} = \alpha^2; \quad \dot{t} = \alpha/(1 - R/r); \quad \dot{\psi} = r^{-2}.
\]

Eliminating the abcissa \( \lambda \), we get:

\[
d\psi = \pm r [\alpha^2 - (1 - R/r)r^{-2}]^{-1/2} dr; \quad dt = -\alpha r (1 - R/r)^{-1} [\alpha^2 - (1 - R/r)r^{-2}]^{-1/2} dr.
\]

As \( r \) increases from 0 to \( 3R/2 \), \( [\alpha^2 - (1 - R/r)r^{-2}] \) decreases from infinity to \( \alpha^2 - \frac{4}{27R^2} \), and then increases from \( \alpha^2 - \frac{4}{27R^2} \) to \( \alpha^2 \) as \( r \) increases from \( 3R/2 \) to infinity. Therefore, owing to their projection on the coordinate \( r \), we find three cases for the null geodesics:

- **Case 0:** \( |\alpha| = \frac{2}{3\sqrt{3}R} \): The null geodesic can be either included in \( \{ r = 3R/2 \} \), or it can be asymptotic to \( \{ r = 3R/2 \} \), either growing strictly from \( r = 0 \) to \( r = 3R/2 \), or growing strictly from \( r = 3R/2 \) to infinity (or in the reverse direction).

- **Case 1:** \( |\alpha| > \frac{2}{3\sqrt{3}R} \): The null geodesic runs from 0 to infinity (with an asymptotic velocity: \( r_\lambda \sim \alpha \lambda \)), or in the reverse direction.

- **Case 2:** \( |\alpha| < \frac{2}{3\sqrt{3}R} \): The null geodesic can either be reminiscent from a parabola, with a minimal radius \( \rho' > 3R/2 \), or indefinitely oscillate between \( r = 0 \) and a maximal radius \( \rho \in [R, 3R/2] \).

This last sort of null geodesics, which are recurrent at the singularity \( r = 0 \), is the most interesting for us here, since the confined diffusion trajectories of Section 2.4.4 seem to have their shape asymptotic to the shape of one such geodesic. Indeed, considering the impact parameter \( \alpha \) such that \( |\alpha| < \frac{2}{3\sqrt{3}R} \), the maximal radius \( \rho < 3R/2 \) solves \( \alpha^2 \rho^2 + R/\rho = 1 \), so that \( \alpha \) stands for \( \ell = \lim_{r \to \rho} a_s / b_s \) in Theorem (5, 2) of Section 2.4.4, and the angular deviation \( \Psi \) during each increase from \( r = 0 \) to \( r = \rho \) (or decrease from \( r = \rho \) to \( r = 0 \)) has exactly the expression found for confined diffusion paths in Corollary 3 (and Remark 11) of Section 2.4.4.

Moreover, \( t = -\alpha \int_{\rho}^{r} \frac{r \, dr}{(1 - R/r)\sqrt{\alpha^2 r^2 - 1 + R/r}} \) shows that each confined null geodesic has its graph invariant under the symmetry \( (r, t) \mapsto (r, -t) \). Indeed this is clear in \( S_0 \), and remains true everywhere by analytic continuation. In the Kruskal-Szekeres coordinates, this means invariance under the symmetry \( (u, v) \mapsto (u, -v) \). The gluing we defined at the singularity \( r = 0 \) implies at once that null geodesics must also be symmetric with respect to \( \{ u = v = 0 \} \).

As a conclusion, we see that the confined null geodesics run indefinitely a closed analytic curve, which in the coordinate plane \( (r, \psi) \) appears as a figure eight (symmetrical and centred at the origin), and in the Kruskal-Szekeres coordinate plane \( (u, v) \) looks like a pair of round brackets (symmetrical with respect to the coordinate axes and joining the two branches \( \{ r = 0 \} \)).
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