The RAM equivalent of P vs. RP

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Abstract

One of the fundamental open questions in computational complexity is whether the class of problems solvable by use of stochasticity under the Random Polynomial time (RP) model is larger than the class of those solvable in deterministic polynomial time (P). However, this question is only open for Turing Machines, not for Random Access Machines (RAMs).

Simon (1981) was able to show that for a sufficiently equipped Random Access Machine, the ability to switch states nondeterministically does not entail any computational advantage. However, in the same paper, Simon describes a different (and arguably more natural) scenario for stochasticity under the RAM model. According to Simon’s proposal, instead of receiving a new random bit at each execution step, the RAM program is able to execute the pseudofunction $\text{RAND}(y)$, which returns a uniformly distributed random integer in the range $[0, y)$. Whether the ability to allot a random integer in this fashion is more powerful than the ability to allot a random bit remained an open question for the last 30 years.

In this paper, we close Simon’s open problem, by fully characterising the class of languages recognisable in polynomial time by each of the RAMs regarding which the question was posed. We show that for some of these, stochasticity entails no advantage, but, more interestingly, we show that for others it does.

1 Introduction

The Turing machine (TM), first introduced in [19], is undoubtedly the most familiar computational model. However, for algorithm analysis it often fails to adequately represent real-life complexities, for which reason the random access machine (RAM), closely resembling the intuitive notion of an idealised computer, has become the common choice in algorithm design. Ben-Amram and Galil [2] write “The RAM is intended to model what we are used to in conventional programming, idealized in order to be better accessible for theoretical study.”

Here, “what we are used to in conventional programming” refers, among other things, to the ability to manipulate high-level objects by basic commands. However, this ability comes with some unexpected side effects. For example, it was shown regarding many RAMs working with fairly limited instruction sets that they are able to recognise any PSPACE problem in
deterministic polynomial time \[13, 7, 8, 11\]. A unit-cost RAM equipped only with arithmetic operations, Boolean operations and bit shifts can, in fact, recognise in constant time any language that is recognised by a TM in time and/or space constrained by any elementary function of the input size \[5\]. In polynomial time, such a RAM can recognise a class of languages which we denote \(\text{PEL}\) and define below.

However, just like \(\text{PSPACE} = \text{NPSPACE}\), and for basically the same reasons, it was shown that nondeterminism does not make any of these RAM models more powerful \[11, 15, 16\].

The class of problems that can be solved using stochastic computation in the RP model is an intermediate class between deterministic computation and nondeterministic computation. As such, the results that nondeterminism confers no advantage apply also to stochasticity.

The basic idea is that any machine that spans all of \(\text{PSPACE}\) can enumerate over an exponential number of possibilities (denoted by a polynomial number of bits), and can therefore simulate any assignment of a polynomial number of stochastic or nondeterministic bits.

Simon \[16\], however, suggested a different approach to the definition of stochasticity. Because RAMs work natively with nonnegative integers, and because they derive their power from this ability to manipulate in constant time large operands, it seems unnatural that stochasticity in them will be limited to bits alone. Instead, Simon suggested to equip the RAM with a pseudofunction, \(\text{RAND}(y)\), whose output is an integer random variable, \(x\), uniformly distributed in the range \(0 \leq x < y\).

The question of whether polynomial time computation using this pseudofunction, and using the acceptance criteria of RP (no input is falsely accepted, and any input that is in the language has at most probability \(1/2\) of being falsely rejected), a model that we refer to as RP-RAM, is a more powerful model than the deterministic \(\text{PTIME-RAM}\), remained open for the past 30 years, and is the RAM equivalent of the Turing machine question “\(P \neq \text{RP}\)”.

This paper analyses specific examples of RAMs whose native operation set is a subset of the arithmetic, bitwise Boolean and bit-shift operations, and shows that for some of these stochasticity is advantageous and for others not. The RAMs considered are those analysed by Simon \[16\] as well as by Simon and Szegedy \[17\]. To the best of the author’s knowledge, the examples given here where stochasticity confers an advantage are the first known examples for natural, general-purpose computational models in which stochastic computation under RP acceptance criteria is provably stronger than deterministic polynomial time computation.

(The results most comparable to it, in this respect, are those of Heller \[8\], in the context of comparing relativised polynomial hierarchies with relativised RP hierarchies.)

In order to present our results in greater detail, we first redefine, briefly, the RAM model. (See \[1\] for a more formal introduction.)

A Random Access Machine, denoted \(\text{RAM}[\text{op}]\), is a computation model that affords all that we expect from a modern computer in terms of flow control (loops, conditional jump instructions, etc.) and access to variables (direct and indirect addressing). The operations it can perform are those belonging to the set \(\text{op}\), and these are assumed to execute in a single unit of time each. A comparator for equality is also assumed to be available, and this also executes in a single unit of time. The variables (or registers) of a RAM contain nonnegative integers and are also indexable by addresses that are nonnegative integers.

Because the result of RAM operations must be storable in a register, and therefore be a nonnegative integer, operations such as subtraction (which may lead to a negative result) cannot be supported directly. Instead, we use natural subtraction denoted “\(-\)” and defined

\[1\] Natural subtraction will be taken here to share the same properties as normal subtraction in terms of
as

\[ a \downarrow b \overset{\text{def}}{=} \max(a - b, 0). \]

By the same token, regular bitwise negation is not allowed, and \( \neg a \) is tweaked to mean that the bits of \( a \) are negated only up to and including its most significant \( "1" \) bit. We use \( X \text{ clr} Y \) to denote what would have been \( X \land \neg Y \) if the standard \( \neg \) operation had been available.

Furthermore, we assume, following e.g. [10], that all explicit constants used as operands in RAM programs belong to the set \( \{0, 1\} \). This assumption does not make a material difference to the results, but it simplifies the presentation.

In order for our RAMs to be directly comparable to TMs, we consider only RAMs that take a single integer as input. This is the value of some pre-designated register (e.g. \( R[0] \)) at program start-up. All other registers are initialised to zero. A RAM is considered to “accept” the input if the RAM program ultimately terminates, and the final value of a pre-designated output register (e.g. \( R[0] \)) is nonzero.

The flavour of Turing machine most directly comparable to this design is one that works on a one-sided-infinite tape over a binary alphabet, where “0” doubles as the blank. All TMs used in this paper are of this type.

To introduce our results, we first define the function class PEL and the eponymous complexity class.

**Definition 1 (PEL).** PEL (standing for “Polynomial Expansion Limit”) is the class of functions that can be described by

\[ f(n) = p(n)^2, \]

where \( p \) can be any polynomial and the left superscript denotes tetration.

The complexity class PEL is the class of languages recognisable by a TM in time \( f(n) \), where \( f(n) \) is in PEL and \( n \) is the bit-length of the input. Equivalently, PEL is the class of languages recognisable by a TM working on a tape of size \( f(n) \), where \( f(n) \) is in PEL.

PEL can therefore also be described as either PEL-TIME or PEL-SPACE.

The equivalence of the time-constrained and the tape-constrained definitions is given by the well-known relation \( f\text{-TIME} \subseteq f\text{-SPACE} \subseteq \exp(f)\text{-TIME} \).

Our results are as follows.

**Theorem 1.**

\[ \text{RP-RAM}[+, [\downarrow], [\times], \leftarrow, \rightarrow, \text{Bool}] = \text{PEL}. \]

In the above theorem, operations that appear in brackets inside the operation list are optional, in the sense that the theorem continues to be true both with and without them. The operation "\( \downarrow \)" is integer division, "\( \leftarrow \)" is left shifting \( (a \leftarrow b \overset{\text{def}}{=} a \times 2^b) \), "\( \rightarrow \)" is right shifting \( (a \rightarrow b \overset{\text{def}}{=} \lfloor a/2^b \rfloor) \) and “Bool” is shorthand for the set of all bitwise Boolean operations.\(^2\)

Two direct corollaries from Theorem 1 are

**Corollary 1.1.**

\[ \text{RP-RAM}[+, [\downarrow], [\times], \leftarrow, \rightarrow, \text{Bool}] \neq \text{P-RAM}[+, [\downarrow], [\times], \leftarrow, \rightarrow, \text{Bool}]. \]

\(^2\)We define \( \neg 0 \) to be zero.

\(^3\)For assignment, we use “\( \leftarrow \)”, so as to disambiguate it from left shifting.
Corollary 1.2.

\[ \text{RP-RAM}[^+, [\times], /, [\div], \leftarrow, [\rightarrow], \text{Bool}] = \text{P-RAM}[^+, [\times], /, [\div], \leftarrow, [\rightarrow], \text{Bool}]. \]

Here, “/”, known as exact division, is a weaker form of division. The result of \(a/b\) is the same as that of integer division (“\(a \div b\)”), but \(a/b\) is only defined when \(a\) is a multiple of \(b\).

These corollaries can be derived from Theorem 1 by making use of the following known facts.

Theorem 2 ([5]).

\[ \text{P-RAM}[^+, [\times], /, [\div], \leftarrow, [\rightarrow], \text{Bool}] = \text{PEL} \]

and

Theorem 3 ([5, 17]).

\[ \text{P-RAM}[^+, [\times], \leftarrow, [\rightarrow], \text{Bool}] = \text{PSPACE}, \]

given that \(\text{PEL} \neq \text{PSPACE}\) is known from [18, 6, 14].

Together, Corollaries 1.1 and 1.2 show the surprising fact that for unit-cost RAMs the answer to the P vs. RP question can go either way, depending on the choice of a basic operation set: by simply adding division as a basic operation, the answer is reversed.

The rest of this paper is dedicated to a proof of Theorem 1 and some corollaries. It is arranged as follows.

We begin, in Section 2, by constructing the basic tools used in the proof.

The main body of the proof of Theorem 1 consists of two parts, which relate to a new computational model introduced here, which we call the \(\text{BRP-RAM}\). This model is identical to the \(\text{RP-RAM}\) model, except that the instruction

\[ x \leftarrow \text{RAND}(y) \]

is now replaced by

\[ x \leftarrow \text{RAND}(2^k), \]

a pseudofunction assigning to \(x\) an integer random variable uniformly distributed in the range \(0 \leq x < 2^k\).

The first part of the proof, handled in Section 3, proves

Lemma 1.

\[ \text{BRP-RAM}[^+, [\times], [\div], \leftarrow, [\rightarrow], \text{Bool}] = \text{PEL}. \]

The second part, handled in Section 4, then completes the proof by establishing

Lemma 2.

\[ \text{RP-RAM}[^+, [\times], [\div], \leftarrow, [\rightarrow], \text{Bool}] = \text{BRP-RAM}[^+, [\times], [\div], \leftarrow, [\rightarrow], \text{Bool}]. \]

After completing this main proof, we turn in Section 5 to reuse the machinery developed in order to sharpen these results, characterising both the power of RAMs working in any specific time complexity and the power of RAMs working under BPP acceptance criteria.

A short conclusions section follows.
2 Preliminaries

We begin by describing the basic tools used in the construction.

2.1 Some redundant operations

In describing the algorithms in this paper we use “≤” and “→”, as well as all comparators, freely, even when the only non-optional operations available are “+”, “←” and “Bool” and the only comparator available is “=” testing for equality. In this section we justify this presentation, by showing that “≤”, “→” and all comparators can be simulated given the available operations.

To see this, note first that had these been RAMs working on general integers (not necessarily non-negative), and had numbers been stored in registers in standard two’s complement notation (see [2]), then Bool would have included the standard bitwise negation operator ¬, and standard arithmetic subtraction (“−”) would have been implementable as a−b = a + ¬b + 1. Because we deal with RAMs working over nonnegative integers, we have a negation operator that only works up to (and including) the most-significant “1” bit of its operand. This means, for example, that for any a, a + ¬a is a number of the form 2^m − 1, with the minimal m such that 2^m − 1 ≥ a. We therefore define the function

\[ \text{SET}(a) \overset{\text{def}}{=} a + ¬a. \]

To determine whether a ≤ b, for some a and b, without utilisation of either “≤” or any comparison operator other than equality, consider two cases. First, it may be that a and b differ in their bit-length. In this case, it is enough to check that SET(a) < SET(b), which can be implemented by SET(a) ∨ SET(b) ≠ SET(a). Alternatively, it may be that SET(a) = SET(b). In this case, it is known that a and ¬b are both in the range [0, SET(b)], so the question whether a ≤ b can be formulated equivalently as whether a + ¬b (which by definition of SET also equals SET(b) + (a − b)) is in the range [0, SET(b)] or in the range [SET(b) + 1, 2SET(b)]. These two ranges can be differentiated by their value in a single bit position, namely that indicated by SET(b) + 1, so, altogether, we can define

\[ a ≤ b \overset{\text{def}}{=} \text{SET}(a) ∨ \text{SET}(b) ≠ \text{SET}(a) \]

\[ \text{or } (\text{SET}(a) = \text{SET}(b) \text{ and } (a + ¬b) ∧ (\text{SET}(b) + 1) = 0). \]

All other comparison operators can now be computed from “≤”, making their inclusion in the RAM’s instruction set superfluous.

Consider now the case b < a. Let c = b + SET(a). Once again, by definition a + ¬c = SET(c) + (a − c) = (SET(c) − SET(a)) + (a − b). If we wish to calculate a ∙ b, which, in this case, equals a − b, we can now utilise the fact that the result must be in the range [0, SET(a)]. Because c ≥ a, we know that (SET(c) − SET(a)) ∧ SET(a) = 0, so a ∙ b can be calculated as (a + ¬c) ∧ SET(a).

Putting it all together, we can define a function to calculate a ∙ b as follows.

\[ a ∙ b \overset{\text{def}}{=} \begin{cases} 
0 & \text{if } a ≤ b, \\
(a + ¬(b + \text{SET}(a))) ∧ \text{SET}(a) & \text{otherwise}.
\end{cases} \]

Right shifting is not as universally replaceable. However, it can be simulated under appropriate circumstances, as outlined by Lemma [3].
Lemma 3. For op = \{\leftarrow, \rightarrow, [+] , [\times] , \text{Bool}\}, if a RAM[op] does not use indirect addressing and is restricted to shifts by bounded amounts, it can be simulated by a RAM[op\{\rightarrow}\] without loss in time complexity. This result remains true also if the RAM can apply “a \rightarrow b” when b is the (unbounded) contents of a register, provided that the calculation of b does not involve use of the “\rightarrow” operator.

Proof. We begin by considering the case of bounded shifts.

A RAM that does not use indirect addressing is inherently able to access only a finite set of registers. Without loss of generality, let us assume that these are \(R[0], \ldots, R[k]\). The simulating RAM will have \(R'[0], \ldots, R'[k+1]\) satisfying the invariant

\[
\forall i : 0 \leq i \leq k, R[i] = R'[i] / R'[k+1].
\]

To do this, we initialise \(R'[k+1]\) to be 1, and proceed with the simulation by translating any action by the simulated RAM on \(R[i]\), for any \(i\), to the same action on \(R'[i]\). We do this for all actions except \(R[i] \rightarrow X\), which is an operation that is unavailable to the simulating RAM.

To simulate \(R[j] \leftarrow R[i] \rightarrow X\), we perform the following.

1. \(R'[j] \leftarrow R'[i]\).
2. \(\forall x : x \neq j, R'[x] \leftarrow R'[x] \leftarrow X\).
3. \(R'[j] \leftarrow R'[j] \text{ clr} \neg R'[k+1]\).

We note regarding the second step that this operation is performed also for \(x = k+1\). The fact that \(k\) is bounded ensures that this step is performed in \(O(1)\) time.

Essentially, if “\(R[j] \leftarrow R[i] \rightarrow X\)” is thought of as “\(R[j] \leftarrow [R[i] / 2^X]\)” , Step 1 performs the assignment, Step 2 the division, and Step 3 the truncation.

In order to support “\(R[j] \leftarrow R[i] \rightarrow X\)” also when \(X\) is the product of a calculation, the simulating RAM also performs, in parallel to all of the above, a direct simulation that keeps track of the register’s native values. In this alternate simulation, right shifts are merely ignored. Any calculation performed by the simulated RAM that does not involve right shifts will, however, be calculated correctly, so the value of \(X\) will always be correct.

Because the conditions of Lemma 3 hold for all constructions described here, we assume the availability of “\(\rightarrow\)” throughout.

2.2 Vectors and tableaux

Consider the following definitions, following [4]:

Definition 2 (Vectors). A triplet \((m, V, n)\) of integers will be called an encoded vector. We refer to \(m\) as the width of the vector, \(V\) as the contents of the vector and \(n\) as the length of the vector. If \(V = \sum_{i=0}^{n-1} 2^{mi}k_i\) with \(\forall i : 0 \leq i < n \Rightarrow 0 \leq k_i < 2^m\), then \([k_0, \ldots, k_{n-1}]\) will be called the vector (or, the decoded vector), and the \(k_i\) will be termed the vector elements. Notably, vector elements belong to a finite set of size \(2^m\) and are not general integers. It is

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4An action involving an explicit “1” (except for shifting by 1) will have the “1” replaced by \(R'[k+1]\) in the simulation.
well-defined to consider the most-significant bits (MSBs) of vector elements. Nevertheless, any \( n \) integers can be encoded as a vector, by choosing a large enough \( m \).

Actions described as operating on the vector are mathematical operations on the encoded vector (typically, on the vector contents, \( V \)). However, many times we will be more interested in analysing these mathematical operations in terms of the effects they have on the vector elements. Where this is not ambiguous, we will name vectors by their contents. For example, we can talk about the “decoded \( V \)” to denote the decoded vector corresponding to some encoded vector whose contents are \( V \).

**Definition 3 (Instantaneous Description).** Let \( T \) be a TM working on a bounded tape of size \( s \) and having a state space that can be described by \( c \) bits.

An instantaneous description of \( T \) at any point in its execution is the tuple

\[
(tape, 2^{headpos}, state \times 2^{headpos}, 1, 2^s)
\]

at that point in its execution, where \( tape \) is the instantaneous content of the TM’s tape, translated into an integer, \( headpos \) is the distance from the position of the head to the end of the tape, and \( state \) is an integer indicating the current state of the finite control, where a mapping from states to integers can be arbitrary subject to the requirement that all integers must be at most \( c \) bits in length. By convention, the initial state of the TM is mapped to zero.

(The constants 1 and \( 2^s \) are needed because they denote the endpoints of the portion of the tape that can be occupied by the TM’s head.)

This particular format to describe the instantaneous state of a TM was chosen because advancing the TM to the next instantaneous state can be done using it in constant time with only bit shifts and bitwise Boolean operations (but without the tweaked “¬”). Furthermore, only bit shifts by a constant amount are required. That is, one can avoid all bit shift operations in which the right-hand operand depends on \( tape, headpos, state \) or \( s \). No constants are required in the calculation.

Let \( X_i \) (a tuple of 5 integers) denote the instantaneous description of \( T \) after \( i \) execution steps. We denote by \( \hat{T} \) the function that calculates in constant time \( X_{i+1} = \hat{T}(X_i) \) using only bitwise Boolean operations and shifts by a constant amount.

**Definition 4 (Tableau).** Let \( T, s \) and \( c \) be as in Definition 3.

A tableau is a tuple \((m, T, H, S, n)\), where \( m = s + c - 1 \) and the vectors \((m, T, n), (m, H, n)\) and \((m, S, n)\) are such that \((T[i], H[i], S[i], 1, 2^s)\) is the instantaneous description of \( T \) after \( i \) execution steps.

A technique used extensively in [16, 5] to verify the validity of an entire tableau in constant time is to define vector \((m, I, n)\), such that \( I[i] = 1 \) for every \( i \), as well as the scalar \( I_s = I \leftarrow s \), and then to calculate

\[
(T', H', S', I', I_s') = \hat{T}(T, H, S, I, I_s).
\]

With a little care, the fact that \( \hat{T} \) works only with bitwise Boolean functions and constant shifts can be leveraged to make it work independently for each instantaneous description inside the vector contents that comprise the tableau candidate. Specifically, if \((m, T, H, S, n)\) is a valid tableau for \( T \) working on input \( inp \), then for every \( i \) except \( i = n - 1 \) we should have \( T'[i] = T[i+1], H'[i] = H[i+1], S'[i] = S[i+1] \).
These equalities, in turn, can be verified by confirming 
\((T' \leftarrow m) \oplus T) \land (1 \leftarrow (nm) \oplus 1) = \text{inp}, ((H' \leftarrow m) \oplus H) \land (1 \leftarrow (nm) \oplus 1) = 1 \text{ and } ((S' \leftarrow m) \oplus S) \land (1 \leftarrow (nm) \oplus 1) = 0,\) where 
\(\oplus\) signifies the exclusive OR (XOR) bitwise operation. To see this, consider, for example, 
\((T' \leftarrow m) \oplus T.\) In a correct tableau, this will equal \(T[0] + (T[n - 1] \leftarrow (nm)).\) The later “\(\land\) takes care of the part dependent on \(T[n - 1].\) The tableau is valid if it starts with the correct element, \(T[0] = \text{inp},\) and every subsequent element satisfies the relation \(T'[i] = T[i + 1].\) The same verification can now be repeated for \(H\) and \(S.\)

It is also possible to verify the correctness of \(I\) and \(I_s\) in a similar way: \((I \leftarrow m) \oplus I = 1 \leftarrow (nm) + 1, I_s = I \leftarrow s.\)

In practice we will always pick \(n\) to be a known power of two, \(n = 2^k,\) so that \(nm\) can be calculated as \(m \leftarrow k,\) with no need for a multiplication operation.

Once a tableau is verified, examining \((T[n - 1], H[n - 1], S[n - 1])\) can answer the question of how the execution described by the tableau terminates. Options are

1. Termination and acceptance of the input.
2. Termination and rejection of the input.
3. Nontermination.
4. The TM has exceeded its allotted tape requirements.

In constant time, we are able to tell these options apart.

Let \(\mathcal{R}\) be the constant time RAM program that verifies an input tableau and returns its validity and termination status, as described above. The function \(\mathcal{R}\) takes as inputs, in addition to the original tableau and \(s,\) also the helper vector \(I.\) The vector \(I_s\) does not need to be given as an extra input, because it can be calculated as \(I \leftarrow s.\)

In [5], the point is made that the total number of possible instantaneous descriptions is bounded by \(B \overset{\text{def}}{=} 2^{3(s+c-1)}.\) Thus, there is no need to test the validity of any tableau longer than \(B.\) If a TM working on a tape of size \(s\) has not terminated on its first \(B\) steps, it is guaranteed to be in an infinite loop.

The paper then goes on to make the point that this limit makes the grand total of possible tableaux one needs to test also finite. It is the set of all possible tableaux of length \(B.\)

2.3 Maps

**Definition 5** (Map). A map (or, an encoded map) is a triplet, \((\hat{L}, \hat{I}, \hat{w}),\) of integers, where \(\hat{L}\) is the contents of the map, \(\hat{I}\) is the domain of the map and \(\hat{w}\) is the width of the map. The domain, \(\hat{I},\) must satisfy the criterion that any two “1”s in its binary representation are at least \(\hat{w}\) bit positions apart. The contents, \(\hat{L},\) must satisfy that all bit-positions of \(\hat{L}\) are zero, except the \(\hat{w}\) positions immediately following a position that is a “1” in the domain.\(^5\) For simplicity we assume \(\hat{w} > 1.\)

Conceptually, a map represents a mapping from a list of indices \(\text{ind}_1, \ldots\) to a list of map elements \(k_1, \ldots,\) in the following way: \(\hat{I} = \sum(i \leftarrow \text{ind}_i)\) and \(\hat{L} = \sum_i(k_i \leftarrow \text{ind}_i).\) The \(k_i\) are required to be in the range \(0 \leq k_i < 2^w.\) The \(\text{ind}_i\) are required to satisfy that for every \(i, \hat{w} + \text{ind}_i \leq \text{ind}_{i+1}.\) Under these restrictions, there is a one-to-one correspondence between mappings and encoded maps. Such mappings will be termed decoded maps. A map will often

\(^5\)The latter criterion can be worded in a formula as \(\hat{L} = \hat{L} \wedge ((\hat{I} \leftarrow \hat{w}) - \hat{I}).\)
be referred to by the name of the encoded map contents. This is done, for example, in the
following notation, which we introduce to signify the relation between an encoded map and
its underlying mapping: \( k_i = \hat{L}[\text{ind}_i] \).

Note that the map elements belong to a finite set, the elements of which are representable
by \( \hat{w} \)-bit strings. As such, it is well-defined to consider their most significant bits (MSBs).

Operations on maps are RAM operations performed on the encoded maps. We will,
however, mostly be interested in the effects these have on the map elements. Such operations
may include multiple maps that share the same \( \hat{I} \) and \( \hat{w} \), in which case, when we say that
we apply a function \( f \) on two maps, \( V \) and \( U \), we typically mean that we wish to produce a
result, \( W \), such that for all indices \( W[\text{ind}_i] = f(V[\text{ind}_i], U[\text{ind}_i]) \).

In Section 2.2, we have shown that a suitably-designed function, \( \hat{T} \) that was designed
to work on integers can be run with vector contents as input, with the effect that it will
function independently on each vector element. The function \( \hat{R} \), also introduced in the same
section, has similar properties and can be given as input a tuple of map contents, and will
act separately on each element of the maps. The effect is that if each map element houses a
tableau candidate, function \( \hat{R} \) can determine, in constant time, which of the candidates is a
valid tableau and which not, with valid tableaux being signified by zeroes in the outputs and
non-valid tableaux being signified by nonzeros.

We make use of several variants of this parallelised \( \hat{R} \). The one described above we de-
note \( \hat{R}_{\text{valid}} \), because a zero in the output indicates that the tableau is valid. Other variants
used here are \( \hat{R}_{\text{valid-and-accepting}} \), \( \hat{R}_{\text{valid-and-rejecting}} \) and \( \hat{R}_{\text{valid-and-exceeded-tape}} \), which are RAM
programs in which an output of zero indicates that the input tableau is “valid and accept-
ing”, “valid and rejecting” or “valid and indicating that the simulated TM exceeded its tape
allocation”, respectively. Each of these can be programmed by applying the same techniques
as shown here.

The most salient difference between the parallelised version of \( \hat{R} \) and its standard version
is that the standard version used freely constants such as 1 and \( \text{inp} \) in calculating and verifying
\( T[0], H[0] \) and \( S[0] \). For these to be usable by the parallelised version of \( \hat{R} \), they have to
be multiplied by the map domain, \( \hat{I} \). The constant 1 becomes \( 1 \times \hat{I} = \hat{I} \), which is available
already, but calculating \( \text{inp} \times \hat{I} \) may not be as straightforward. Whereas the rest of the
computations described here can be performed in constant time, calculating \( \text{IN} = \text{inp} \times \hat{I} \)
requires \( \Theta(n) \) time, where \( n \) is the bit-length of \( \text{inp} \). The calculation algorithm itself is the
classic long-multiplication algorithm. It is described in Algorithm 1.

The tool needed in order to work conveniently with such an output is an element-by-
element comparator. We would like to check for each element equality to zero and signify
this by a “1”, with non-equality being signified by zero. A sequence of function definitions
working on maps, culminating in a general element-by-element equality operation that works
in constant time, parallelising over all map elements, is described below.

Let \( (V, I, w) \) and \( (U, I, w) \) be two maps.

The total set of bits that can be used by the map contents is

\[
\text{MASK}(I, w) \overset{\text{def}}{=} (I \leftarrow w) \downarrow I.
\]

For convenience, we divide these into two subsets. We take the lowest \( w - 1 \) bits of each
map element to be its “data” bits, and the MSB to be the “flag” bit. The following functions
extract these positions.

\[
\text{FLAGS}(I, w) \overset{\text{def}}{=} I \leftarrow (w \downarrow 1),
\]
Algorithm 1 Calculating \( IN = \hat{I} \times inp \)

1: \( IN \leftarrow 0 \)
2: \( acc \leftarrow \hat{I} \)
3: \( r \leftarrow inp \)
4: while \( r \neq 0 \) do
5:   if \( r \land 1 = 1 \) then
6:     \( IN \leftarrow IN + acc \)
7:   end if
8:   \( r \rightarrow 1 \)
9:   \( acc \leftarrow 1 \)
10: end while
11: return \( IN \)

\[
DATA(I, w) \overset{\text{def}}{=} FLGS(I, w) \land I.
\]

To actually extract the data from the positions we use

\[
FLGS(V, I, w) \overset{\text{def}}{=} V \land FLGS(I, w),
\]

\[
DATA(V, I, w) \overset{\text{def}}{=} V \land DATA(I, w).
\]

Summing the data of two maps can be performed by

\[
ADD(V, U, I, w) \overset{\text{def}}{=} (DATA(V, I, w) + DATA(U, I, w)) \oplus FLGS(V, I, w) \oplus FLGS(U, I, w).
\]

The ADD function, as implemented here, avoids overflows by calculating the sum modulo \( 2^w \). In order to find out if an overflow occurred, we can calculate the carry bit.

\[
CARRY(V, U, I, w) \overset{\text{def}}{=} (V + U \oplus ADD(V, U, I, w)) \rightarrow w.
\]

The following function implements bitwise negation:

\[
NEG(V, I, w) \overset{\text{def}}{=} MASK(I, w) \land \neg V,
\]

and, as a last helper function, the following function calculates a map, \( RC \), such that for every \( i \), \( RC[ind_i] \) is 1 if \( V[ind_i] > U[ind_i] \), and is 0, otherwise, this being an element-by-element “greater than” comparison:

\[
GT(V, U, I, w) \overset{\text{def}}{=} CARRY(V, NEG(U, I, w), I, w).
\]

With this build-up, we can finally implement element-by-element equality as follows:

\[
EQ(V, U, I, w) \overset{\text{def}}{=} I \oplus GT(V, U, I, w) \oplus GT(U, V, I, w).
\]

In order to simultaneously check and verify many candidate tableaux, what we need are integers \( \hat{L} \) and \( \hat{I} \), as well as the length, \( s \), of the tape on which TM \( \mathcal{T} \) runs, and the TM’s input, \( inp \). The inputs \( \hat{L} \) and \( \hat{I} \) can be interpreted as follows.

Let \( w = s + c \land 1 \) be the width of the tableau vectors being checked, \( B = 1 \leftarrow (w + w + w) \) will be the length of the tableau vectors being checked. The total bit-length of these vectors
is \( w_m = B \times w = w \leftarrow (w + w + w) \). Therefore, they can be stored as elements inside maps of such width. Specifically, let \( \hat{w} = 4w_m \). The map \((\hat{L}, \hat{I}, \hat{w})\) holds the information of all tableaux to be verified, by keeping in each element, \( i \), of the map the following composite:

\[
L[i] = (T_i \leftarrow 3w_m) + (H_i \leftarrow 2w_m) + (S_i \leftarrow w_m) + I_i,
\]

where \((w, T_i, H_i, S_i, B)\) is the \( i \)’th tableau to be verified, and \((w, I_i, B)\) is the additional vector (the vector whose elements are all 1) required to execute \( \hat{R} \).

In order to perform the actual verification, we begin by calculating \( w_m \), and then

\[
T \leftarrow (\hat{L} \rightarrow 3w_m) \land \text{MASK}(w_m)
\]
\[
H \leftarrow (\hat{L} \rightarrow 2w_m) \land \text{MASK}(w_m)
\]
\[
S \leftarrow (\hat{L} \rightarrow w_m) \land \text{MASK}(w_m)
\]
\[
I \leftarrow \hat{L} \land \text{MASK}(w_m),
\]

this unpacking all \( T_i, H_i, S_i \) and \( I_i \) simultaneously as the \( i \)’th elements of the maps \((T, \hat{I}, w_m)\), \((H, \hat{I}, w_m)\), \((S, \hat{I}, w_m)\) and \((I, \hat{I}, w_m)\), respectively.

As a last preparation step, we calculate \( IN = \hat{I} \times \text{inp} \) by means of Algorithm 1.

We are now ready to run \( \hat{R}_{\text{valid}} \) on the inputs \((T, H, S, I, s)\). This is essentially the same program as \( \hat{R} \), with the only differences being that “1”s in \( \hat{R} \)’s code are replaced by “\( \hat{I} \)”s in \( \hat{R}_{\text{valid}} \) (except when shifting by 1) and “inp”s are replaced by “IN”s. The program’s output is a map, \((R, \hat{I}, w_m)\), such that \( R[i] \) is zero if and only if the \( i \)’th element of the input is a valid tableau candidate, and nonzero otherwise.

By running \( EQ \) over this vector, to compare it with the zero vector, we reverse the zero/nonzero distinction. Now, nonzero (one) elements represent valid tableau candidates and zero elements represent invalid candidates. Consider the contents of this map. If it is nonzero, this should be interpreted as “some of the tableau candidates in the input are valid”.

We can, similarly, create such vectors for valid-and-accepting tableaux and valid-and-rejecting tableaux, etc.. Here, the meaning is “some of the tableau candidates are valid and accepting” (or valid and rejecting), from which we can conclude that \( T \) accepts (rejects).

The full verification algorithm is given as Algorithm 2.

### 3 BRP-RAM = PEL

Arguably, the more difficult direction in proving Lemma 1 is

\[
\text{PEL} \subseteq \text{BRP-RAM}[+, \leftarrow, \text{Bool}].
\]

We prove this by means of an explicit algorithm that simulates any TM running in space PEL, on a BRP-RAM. The top level of this algorithm is depicted in Algorithm 3. This algorithm simulates a TM, \( T \), the state of whose finite control can be described in \( c \) bits, running on input \( \text{inp} \).

We require Algorithm 3 to be a BRP-RAM algorithm. This can be broken down into three conditions.

1. If the run of \( T \) has tape requirements bounded by PEL, the algorithm must run, deterministically, in polytime.
Algorithm 2  \textit{Verify}(\hat{L}, \hat{I}, s, inp): Simultaneous verification of tableau candidates
1: \( w \leftarrow s + c \downarrow 1 \)
2: \( B \leftarrow 1 \leftarrow (w + w + w) \)
3: \( w_m \leftarrow w \leftarrow (w + w + w) \)
4: \( T \leftarrow (L \rightarrow (w_m + w_m + w_m)) \land \text{MASK}(w_m) \)
5: \( H \leftarrow (L \rightarrow (w_m + w_m)) \land \text{MASK}(w_m) \)
6: \( S \leftarrow (L \rightarrow w_m) \land \text{MASK}(w_m) \)
7: \( I \leftarrow L \land \text{MASK}(w_m) \)
8: \( \text{IN} \leftarrow \hat{I} \times \text{inp} \) \quad \triangleright Multiplication performed using Algorithm 1
9: \( R_{\text{valid}} \leftarrow \hat{R}_{\text{valid}}(T, H, S, I, \text{IN}, s) \)
10: \( R_{\text{valid-and-accepting}} \leftarrow \hat{R}_{\text{valid-and-accepting}}(T, H, S, I, \hat{I}, \text{IN}, s) \)
11: \( R_{\text{valid-and-rejecting}} \leftarrow \hat{R}_{\text{valid-and-rejecting}}(T, H, S, I, \hat{I}, \text{IN}, s) \)
12: \( R_{\text{valid-and-exceeded-tape}} \leftarrow \hat{R}_{\text{valid-and-exceeded-tape}}(T, H, S, I, \hat{I}, \text{IN}, s) \)
13: if \( \text{EQ}(R_{\text{valid-and-accepting}}, 0, I, w_m) \neq 0 \) then
14: \quad \text{Output: } T \text{ accepts on input } inp.
15: else if \( \text{EQ}(R_{\text{valid-and-rejecting}}, 0, I, w_m) \neq 0 \) then
16: \quad \text{Output: } T \text{ rejects on input } inp.
17: else if \( \text{EQ}(R_{\text{valid-and-exceeded-tape}}, 0, \hat{I}, w_m) \neq 0 \) then
18: \quad \text{Output: } T \text{ requires more than } s \text{ tape elements on input } inp.
19: else if \( \text{EQ}(R_{\text{valid}}, 0, \hat{I}, w_m) \neq 0 \) then
20: \quad \text{Output: } T \text{ rejects input } inp \text{ by entering an infinite loop.}
21: else
22: \quad \text{Output: } Simulation \text{ failed. No valid tableau found. It was not determined whether } T \text{ accepts input } inp.
23: end if

Algorithm 3  Top level of a BRP-RAM simulating a PEL TM, \( T \), working on input \( \text{inp} \).
1: \( \text{maxstep} \leftarrow 1 \)
2: \textbf{loop}
3: \quad \( s \leftarrow \text{maxstep}^2 \)
4: \quad Generate \( \hat{L} \) and \( \hat{I} \) for simulation on tape of length \( s \). \quad \triangleright See Section 6.1
5: \quad \text{Verify}(\hat{L}, \hat{I}, s, \text{inp}) \quad \triangleright Probabilistic verification. See Algorithm 2
6: \quad Check if the simulation succeeded. Halt and reject if not.
7: \quad Check if the simulated \( T \) has exceeded its tape allocation. Halt and report the final state if not. (If it has not reached a final state, reject.)
8: \quad \text{maxstep} \leftarrow \text{maxstep} + \text{maxstep}
9: \textbf{end loop}
2. If the input is not in the language accepted by $T$, it must be rejected.

3. If the input is in the language, it must be accepted with probability at least $1/2$.

Let us begin by considering the first condition. Algorithm 3 contains a loop on Step 2. The complexity of the algorithm is determined by the number of times the algorithm goes through the loop before halting at either Step 6 or Step 7 and by the complexities of the individual steps.

In Algorithm 3, Step 3 can be implemented by the straightforward algorithm, which acts in $O(\maxstep)$ time. Step 5 acts in $O(n)$ time, where $n$ is the bit-length of the output. This time requirement stems from the application of Algorithm 1. All other parts of this step run in constant time. The greatest challenge in constructing the algorithm, to which Sections 3.1 and 3.2 are dedicated, is to implement Step 4 efficiently. Specifically, we implement it by use of an $O(\maxstep)$ time algorithm. The rest of the body of the loop of Step 2 runs in constant time, for a total of $O(\maxstep + n)$ time execution.

Ultimately, the combined complexity of the algorithm is $\Theta(m + n \log m)$, where $m$ is the ultimate value of $\maxstep$ in running the algorithm, which is also $\Theta(m' + n \log m')$, where $m'$ is the penultimate value of $\maxstep$. Because we know that the algorithm did not halt during its penultimate cycle through the loop, we know that the simulation of $T$ over a tape of length $s' = m'2$ was successful (Step 6) and showed that $T$ requires more than $s'$ tape elements (Step 7). Because the space requirement for $T$ is known to be more than $s'$, and because by assumption it is in PEL, by definition $m'$ is a polynomial in $n$. Hence, the first condition, that of the RAM running in polynomial time, is satisfied.

The second condition is that all inputs not in the language must be rejected. The only way to accept an input in Algorithm 3 is in Step 7, when the simulation shows that $T$ has halted and accepted the input, after the simulation was verified as correct in Step 6. Thus, the second condition is also satisfied.

The third condition is that if $\text{inp}$ is in the language, its probability of being rejected should be at most $1/2$. The only place where Algorithm 3 can falsely reject is at Step 4. If the simulation is correct, Step 7 is able to determine $T$’s true final state. It neither falsely rejects nor falsely accepts.

Suppose now that we are able to devise our simulation of $T$ so that in the first iteration through the loop it has probability $p_1$ of failing, in the second iteration it has probability $p_2$, etc., such that the sum of the entire $p_i$ sequence is no more than $1/2$. (For example, we can set $p_i = 2^{-1-i}$. ) The probability that at least one failure occurred is certainly no more than the sum of all $p_i$, indicating that the probability of false rejection is properly bounded.

Sections 3.1 and 3.2 describe how to generate $\hat{L}$ and $\hat{I}$ in Step 4 of the algorithm, so as to meet the requirement that a failure of the algorithm occurs in probability bounded by $2^{-1-i}$, thus completing the description of the simulating algorithm.

### 3.1 Calculating $\hat{L}$ and $\hat{I}$

Summarising the conditions required of the map $(\hat{L}, \hat{I}, \hat{w})$ to be generated at iteration $i$ of the loop on Step 2 of Algorithm 3

1. In the binary representation of the output $\hat{I}$, any two “1”s should be at least $\hat{w}$ bit positions apart. (We refer to this condition as $\hat{w}$-sparseness.)
2. The correct tableau, which is a bit-string of length \( \hat{w} \) bits, should appear with probability of at least \( 1 - 2^{-i-1} \) as a substring of the bits of \( \hat{L} \) beginning at some “1” position of \( \hat{I} \).

3. All bits of \( \hat{L} \) which are more than \( \hat{w} \) bit positions from the preceding “1” position of \( \hat{I} \) should be zeroes.

4. The generation of \( I \) and \( L \), with \( \hat{w} \) in \( \Theta (\text{maxstep}) \), 2, should be performed in \( O(\text{maxstep}) \) time.

The general framework we use to meet these requirements is as follows. First, we choose a number, \( k \), as a function of \( \text{maxstep} \). Then, we generate \( \hat{I} \) to be a valid map domain in the range \([0, 2^k - \hat{w}) \). Last, we generate \( \hat{L} \) to meet with the remaining requirements.

Generating \( \hat{L} \) is ultimately done by

\[
\hat{L} \leftarrow \text{RAND}(2^k) \land \text{MASK}(\hat{I}, \hat{w}).
\]

This constant time procedure ensures that if \( \hat{I} \) is a valid map domain of Hamming weight \( h \), the resulting map will be valid, and will include \( h \) independent, uniformly chosen elements. The correct tableau is one of the possible elements for the map. Thus, the greater \( h \) the higher the probability that one of the randomly-chosen elements is the correct one. We will design \( \hat{I} \) and \( k \) so that \( h \) will be, probabilistically, high enough to ensure that this probability is at least \( 1 - p_i > 1 - 2^{-i-1} \).

In this section we discuss the generation of \( \hat{I} \) given a choice of \( k \), and in Section 3.2 the generation of \( k \).

Consider the sparseness condition for \( \hat{I} \). We note that unlike \( \hat{I} \)'s Hamming weight, which may be probabilistically chosen to meet with the probabilistic success criteria, the condition of \( \hat{w} \)-sparseness is a deterministic condition. We cannot generate \( I \) by a random process that is simply biased towards “0” bits. Even if such a process has high probability of meeting the criterion, it still admits the possibility that the condition will not be satisfied and, as a result, that the calculation will be incorrect. Such an approach can create a positive false-accept probability, which is not acceptable in the BRP model.

We construct \( \hat{I} \) by means of a procedure that iteratively dilutes the “1”s in \( \hat{I} \). After \( i \) steps, \( \hat{I} \) is guaranteed to be \( \hat{w}_i \)-sparse, with \( \hat{w}_i > i \cdot 2 \). In this way, generating a \( \hat{w} \)-sparse \( \hat{I} \) can be done in \( O(\text{maxstep}) \) steps. Each step will be accomplished by a constant time procedure, so the total time complexity of the algorithm is as required.

Let \( R_i \), for \( i = 0, \ldots \), be independent random values generated by calls to \( \text{RAND}(2^k) \) for our chosen \( k \). In the definition of maps we assumed that the width of any map is at least 2, so we bootstrap the dilution process by Algorithm 4.

\begin{algorithm}
\begin{algorithmic}
1: \( \hat{I} \leftarrow R_0 \)
2: \( \hat{I} \leftarrow \hat{I} \land (\hat{I} \leftarrow 1) \)
3: \( \hat{I} \leftarrow \hat{I} \rightarrow 1 \)
4: \text{return } \hat{I}
\end{algorithmic}
\end{algorithm}

This procedure generates a \( \hat{w}_0 \)-sparse candidate for \( \hat{I} \). We label it \( I_0 \). From this point on we iteratively begin, in step \( i + 1 \), with a pair \( (I_i, w_i) \), where \( I_i \) is a candidate for \( \hat{I} \) satisfying \( w_i \)-sparseness, and generate \( (I_{i+1}, w_{i+1}) \) for the next iteration. Algorithm 5 describes how to
do this with \( w_{i+1} = (w_i \leftarrow w_i) + 1 \), which is a growth rate that meets with the requirements of the algorithm.

**Algorithm 5** Creating \((I_{i+1}, w_{i+1})\) from \((I_i, w_i)\)

1. \( R_i \leftarrow \text{RAND}(2^k) \)
2. \( I_{\text{begin}} \leftarrow I_i \lor (I_i \leftarrow w_i) \)
3. \( I_{\text{end}} \leftarrow (I_i \leftarrow w_i) \lor I_i \)
4. \( I_{\text{middle}} \leftarrow I_i = I_{\text{begin}} \)
5. \( I_{\text{goodbegin}} \leftarrow \text{EQ}(R_i \land \text{MASK}(I_{\text{begin}}, w_i), 0, I_{\text{begin}}, w_i) \)
6. \( I_{\text{goodend}} \leftarrow \text{EQ}((R_i \leftarrow w_i) \land \text{MASK}(I_{\text{end}}, w_i), 0, I_{\text{end}}, w_i) \)
7. \( I_{\text{goodmiddle}} \leftarrow \text{EQ}(\text{ADD}((R_i \leftarrow w_i) \land \text{MASK}(I_{\text{middle}}, w_i)), I_{\text{middle}}, I_{\text{middle}}, w_i), R_i \land \text{MASK}(I_{\text{middle}}, w_i)) \)
8. \( w_{i+1} \leftarrow (w_i \leftarrow w_i) + 1 \)
9. \( I_{i+1} = ((\text{MASK}(I_{\text{goodbegin}} + I_{\text{goodmiddle}}, w_i) + I_{\text{goodbegin}}) \land I_{\text{goodend}}) \rightarrow (w_{i+1} + 1) \)
10. return \((I_{i+1}, w_{i+1})\)

Algorithm 5 is the core algorithm, at the heart of this paper’s entire construction. We therefore examine and explain it line by line.

The idea behind Algorithm 5 is to use the same tools we developed in order to verify a tableau, only this time to use them in order to verify the separation between “1” bits. Specifically, we begin by assuming that every “1” bit in \( I_i \) is followed by \( w_i - 1 \) zero bits. This entire integer can therefore be thought of as occurrences of the repeating pattern \( “0^{w_i-1}1” \), separated by zeroes. We want to measure the lengths of these repeating patterns. It is straightforward to find where such a pattern begins, where it ends, and which “1” bits are its middle bits. These are designated \( I_{\text{begin}}, I_{\text{end}} \) and \( I_{\text{middle}} \), respectively. The question is only how to count the number of consecutive “middle” bits.

To do this, we treat \( R_i \) as an Oracle string. Specifically: a counter. We verify (in \( I_{\text{goodbegin}}, I_{\text{goodend}} \) and \( I_{\text{goodmiddle}} \), respectively) that the counter begins with a zero, ends with \( 2^w - 1 \) and increments each time by one. In the last step, the addition by \( I_{\text{goodbegin}} \) sends a carry bit through the entire verified part of the vector. If it reaches \( I_{\text{goodend}} \), this is an indication that the counting was correct. We can therefore now take one bit \( (I_{\text{goodend}}) \) from the sequence, knowing that there must be at least \( w_i \times 2^w \) zero bits preceding it.

We remark that because all we are interested in is to verify that the “1” bits are spaced far enough apart, it is not important to check, for example, that the counter did not go through several entire revolutions, instead of just one, between the beginning and the ending of each repetition. The omitted checks are all for conditions which, if invalidated, merely extend the number of zero bits that separate the “1” bits.

### 3.2 Calculating \( k \)

In Section 3.1 \( \hat{L} \) and \( \hat{I} \) are calculated based on a chosen \( k \). We now complete their construction by determining which value of \( k \) to use in the \( i \)’th iteration over the loop of Step 2 of Algorithm 3. A good value would be one for which the correct tableau, describing the run of \( T \) on a tape of size \( s \) for its first \( B \) steps, is an element of \((\hat{L}, \hat{I}, \hat{w})\) with probability of \( 1 - p_i \), where \( p_i < 2^{-i-1} \), and therefore \( \sum_i p_i \leq 1/2 \).

Let \( m \) be the number of iterations required by Algorithm 5 before reaching \( w_m \geq \hat{w} \). (The value of \( m \) is \( \text{maxstep} + O(1) \).)
Consider a specific bit-position of $\hat{I}$, $b$, neither close to the LSB nor to the $k$'th bit position. For $\hat{I}$'s value in such a bit position to be 1, each of $R_0$ through $R_m$ should have exactly a prescribed set of values in each of $O(w_m)$ bit positions surrounding $b$. (If there is more than one such possibility, pick one arbitrarily.) Given that the value of $\hat{I}$ is 1 in this bit position, we require that the value of $\hat{L}$ in a further $\hat{w}$ bit positions be exactly the correct tableau. Altogether, we need $Z = O(w_m \times m)$ bits to be randomly chosen to exactly the appropriate values.

Let us divide the set of bit positions of $\hat{I}$ into segments of size $W = O(w_m)$, chosen such that there will not be any overlap between the bit positions in $R_0, \ldots, R_m$ required for the least bit inside each segment to be 1 in $\hat{I}$. The probability that this bit position of $\hat{I}$ is 1, and the associated element in $\hat{L}$ is a correct tableau is $2^{-Z}$. Suppose we pick $k$ to be $W \times (i + 1) \times 2^Z$. This ensures that there are $(i + 1) \times 2^Z$ concurrent attempts, each of which has a $2^{-Z}$ independent probability of providing a correct tableau. Altogether, the probability that no such tableau exists (and the simulation will therefore fail) is on the order of $p_i \approx e^{-i-1} < 2^{-k-1}$, as desired.

In practice, we cannot use this value for $k$ because we cannot compute it for lack of a multiplication operation. However, we can compute values that are guaranteed to be no smaller than it, for example by switching every $a \times b$ in the calculation to $a \leftarrow b$. A larger value for $k$ results in a smaller probability of false acceptance.

Thus, the program meets the RP-RAM acceptance criteria.

### 3.3 Completing the proof

**Proof of Lemma 1.** The techniques developed thus far provide most of the proof of the lemma: Algorithm 3 provides the top level of a BRP simulator for PEL, the details of which are provided in the sections following it.

To complete the proof, we now discuss the reverse direction:

$$\text{BRP-RAM} \subseteq \text{PEL}.$$  \hfill (1)

To show this, we first note that, just like PEL can equally be thought of as PEL-TIME or as PEL-SPACE, so can it be thought of as PEL-NSPACE. This is a direct result of Savitch’s Theorem [12]. This indicates that for PEL we can replace the deterministic TM with a nondeterministic one, without this affecting its computational power. The power of the class of randomised TMs is clearly sandwiched between the deterministic and the nondeterministic classes, so it must equal both. For this reason, in demonstrating Equation (1), it is enough that we show that a PEL-SPACE randomised TM can simulate a BRP-RAM.

In order to accomplish this simulation, we retain on the TM’s tape the status of the RAM’s registers encoded as address-value pairs for the registers whose value is nonzero. The number of these is linear and the size of each cannot exceed $O(\text{step}) \max(2, \text{inp})$ on the $\text{step}'$th execution step. The total space required, including a scratchpad area to perform the actual calculations, is therefore no more than PEL-SPACE, as necessary.

The reason the simulation was performed using a randomised TM is in order to simulate

$$X \leftarrow \text{RAND}(2^k)$$

instructions, which are handled by writing $k$ of the TM’s random bits into $X$. \hfill \square
4 RP vs. BRP

The discussion above pertained to a RAM model that incorporates an $X \leftarrow \text{RAND}(2^k)$ pseudofunction. This still leaves the question of whether the more general $X \leftarrow \text{RAND}(Y)$ is perhaps more powerful. We complete the proof of Theorem 1 by showing that this is not the case.

Proof of Lemma 2. We construct a BRP-RAM simulation of an RP-RAM.

First, due to Lemma 1 we know that we can assume the existence of “÷”, “×” and “±”, from which we can further assume the existence of a modulo operation.

Second, we can reuse a technique that was showcased in Algorithm 3, wherein simulation to an unknown number of steps is performed by a sequence of simulations to a bounded and exponentially increasing number of steps (denoted by “maxstep”) without this affecting runtime complexities. This method allows to convert a finite-time simulation of an RP-RAM on a BRP-RAM to a general simulation. We can, therefore, limit ourselves to constructing a simulation bounded by maxstep steps.

During maxstep steps, a randomised RAM can invoke $\text{RAND}(Y_i)$ at most maxstep times, and the $Y_i$ parameter used in each invocation is at most $M = O(\text{maxstep}) \max(2, \text{inp})$.

The first step in the proof is to collate all calls to $\text{RAND}(Y_i)$ in the maxstep-step simulation into a single call. If we were able to know in advance the $Y_i$ parameter used in each call, it would have been possible to make all calls into a single call whose parameter is the product of all $Y$ parameters used. The individual random results can then be separated by applications of “÷” and “mod”.

In fact, we do not know the parameters in advance, but can bound their product, $Y = \prod Y_i$, by $M^{\text{maxstep}}$. Clearly, it is possible in polynomial time to generate a value, $2^{\tilde{k}}$, that exceeds this limit.

Let us now continue with the result from the call to $\text{RAND}(2^{\tilde{k}})$ as though it was a call to $\text{RAND}(Y)$. The extraction process of the individual $X_i$ from $X$ depends only on $X \mod Y$. As such, the termination probabilities that it affords are the same as those of the RP-RAM if we condition over $X < 2^k - (2^k \mod Y)$. For higher $X$ values, the BRP-RAM will not accept the input if it is not in the language, and will accept the input with some probability if it is in the language.

It is not difficult to see that the worst-case for the total probability of acceptance for an input that is within the language is $p/(2-p)$ for the BRP-RAM, if this probability is $p$ for the RP-RAM. This worst-case is attained when $2^k$ is approximately $(2-p)Y$, and the accepting computations are those that work with $X \geq Y(1-p)$.

To meet with the acceptance criteria of the BRP-RAM model, we simply run the simulation three times for each maxstep value. If in each of three independent runs the probability of acceptance is at least $p/(2-p)$, the probability of acceptance in at least one of the three is at least

$$1 - \left(1 - \frac{p}{2-p}\right)^3.$$ 

A little calculus shows that this is never less than $p$ for the entire range $0 \leq p \leq 1$, so the success rate of the simulating BRP-RAM is always at least as good as that of the simulated

*The total RP-RAM algorithm must succeed with probability 0.5 or more, but this does not mean that the same is true for each one of the bounded-step simulations individually. This is the reason why a general parameter, $p$, is required.
RP-RAM.

On the reverse direction, an RP-RAM can simulate a BRP-RAM trivially, because it has “←” as a basic operation.

5 Additional results

We introduce two corollaries to Theorem 1. First, Theorem 1 relates only to polynomial-time execution. We sharpen this result by proving

Corollary 1.3. For any function \( f(n) \), a RAM \([+, [\cdot], [\div], ←, →, \text{Bool}]\) with access to a RAND(\(n\)) instruction, working in \( \Theta(f(n)) \) time, where \( n \) is the bit-length of the input, can accept a language \( S \) in the RP sense (all inputs not in the language are rejected, all inputs in the language are accepted with probability at least \( 1/2 \)) if and only if the language can be accepted by a Turing machine working in \( \Theta(f(n)) \max(2, n) \) time.

Second, we consider the more relaxed acceptance criteria (and therefore, the ostensibly stronger computational model) of BPP. Here, we ask only that all inputs not in the language be rejected with probability at least \( 2/3 \) and that all inputs in the language will be accepted with probability at least \( 2/3 \). We claim:

Corollary 1.4. For any function \( f(n) \), a RAM \([+, [\cdot], [\div], ←, →, \text{Bool}]\) with access to a RAND(\(n\)) instruction, working in \( \Theta(f(n)) \) time, where \( n \) is the bit-length of the input, can accept a language \( S \) in the BPP sense if and only if the language can be accepted by a Turing machine working in \( \Theta(f(n)) \max(2, n) \) time.

In particular, \( \text{BPP-RAM\([+, [\cdot], [\div], ←, →, \text{Bool}\]} = \text{PEL} \), where BPP-RAM denotes a RAM working in polynomial time, using BPP acceptance criteria.

Proof of Corollary 1.3. Most parts of the construction used in the proof of Theorem 1 can be used as-is in the present proof. By replacing arbitrary polynomial bounds by concrete \( f(n) \) bounds, we conclude that a RAM machine working in \( \Theta(f(n)) \) time can be simulated by a Turing machine working in \( O(f(n)) \max(2, n) \) time, as desired\(^7\). In the opposite direction, a Turing machine working in \( \Theta(f(n)) \max(2, n) \) time can be simulated by a RAM in \( O(f(n) + n \log f(n)) \) time. For this, the only alteration required in Algorithm 3 is for Step 3 to be changed to

\[
s \Leftarrow \text{maxstep} \max(2, n).
\]

For \( f(n) \) functions even as slow-growing as \( n^2 \), the \( f(n) \) factor is the dominant part of the complexity, meeting the conditions of the lemma. However, when \( f(n) \) is small (for example, if it were a constant), the \( n \log f(n) \) becomes the dominant factor. We show that this factor can be eliminated.

The extraneous \( n \log f(n) \) factor comes from Algorithm 1 which requires \( O(n) \) time. We remind that the purpose of Algorithm 1 was to perform the multiplication \( IN = \hat{I} \times \text{inp} \). In verifying the correctness of an accepting tableau, one needs to ascertain that it contains the correct initialisation, a correct progression from state to state, and an accepting final state.

\(^7\)In the construction of Section 3.3, we use \( O(f(n)) \max(2, \text{inp}) \), rather than \( O(f(n)) \max(2, n) \), but it is not difficult to ascertain that the two are the same. Let us take, for simplicity, \( \text{inp} \) to be \( 2^n \), with \( n \geq 2 \). We have \( \hat{N} \text{inp} = \hat{N}(2^n) \leq \hat{N}(n^n) \leq 2^{\hat{N}n} \), which is \( \#(X)_{\hat{N}} \), as required.
Without IN, we are able to verify the correct progression and the properties of the final state, but we cannot, given the means described so far, ascertain that its initialisation matches the given input.

In a correct initialisation we have

\[
T[0] = \text{inp} \\
H[0] = 1 \\
S[0] = 0 \\
I[0] = 1.
\]

The construction in the proof of Lemma 1 already provides means to verify the correctness of \(H[0], S[0]\) and \(I[0]\) in constant time. The difficulty is in verifying \(T[0]\), for which the method of Lemma 1 requires \(IN = I \times \text{inp}\).

Where the present construction differs from the original construction is that instead of verifying \(T[0]\) for a randomly generated tableau, we actively set \(T[0]\) to its desired value. We will do this simultaneously in all tableau candidates in the entire \((\hat{L}, \hat{I}, \hat{w})\) map.

We begin by generalising somewhat the notion of a tableau. Specifically, we will use tableaux that are element-wise reversed. The last element in the tableau will signify the initial state of the TM, the fore-last element will be the instantaneous description of the TM after a single step, and so on. Reviewing the proof of Theorem 1, it is not difficult to ascertain that the same results derived for the original tableaux are equally relevant for order-reversed tableaux.

Under this new definition of tableau, what we are trying to create is a map, \((\hat{L}, \hat{I}, \hat{w})\), where the topmost \(w = s + c - 1\) bits of each element are set to the value \(\text{inp}\), and which meets, in all other respects, the criteria we previously required of such a map: a number exponential in \(\hat{w}\) of independent, uniformly distributed map elements.

Consider now that the tools which we have already developed suffice in order to simulate a TM working without input. In this case, \(\text{inp} = 0, \text{inp} \times \hat{I} = 0\).

Let us therefore, for now, ignore the original TM that we intend to simulate and ignore its input. Instead, let us assume that we have at our disposal a pair \((I_x, w_x)\), where \(w_x\) is not equal to \(w_m\), the last \(w_i\) value computed by Algorithm 5, but rather larger than \((2^{w_m}) \leftarrow (2^{w_m})\). (Such a pair can be found simply by running Algorithm 5 two additional iterations.) Consider, now, Algorithm 6 where \(w\) is the width of the tableau vectors to be verified (the bit length of each of their elements), and \(\hat{w}\) is the width of the map that can store it as an element.

Let us analyse this algorithm line by line.

Steps 4 through 7 generate a map \((L_{\text{const}}, I_{\text{good}}, \text{width})\) that has the bit-string

\[
(0^{\text{elementwidth}-11})^{2^{\text{elementwidth}}}
\]

as each one of its elements. The program performs this by considering all elements in the map \((L_{\text{const}}, I_x, \text{width})\) and filtering out first those indices whose elements do not begin with the substring

\[
0^{(\text{elementwidth}-1)}1
\]

(Step 5), and then those elements which are not composed entirely of repetitions of a constant string of length \text{elementwidth}. The latter is tested in Step 6 by verifying equality between
Algorithm 6 Verifying the input

1: function INPUTVERIFY($I_x, w_x, \hat{w}, \text{inp}, w$)
2:     elementwidth $\leftarrow 1 \leftarrow \hat{w}$
3:     width $\leftarrow$ elementwidth $\leftarrow$ elementwidth
4:     $L_{\text{const}} \leftarrow \text{RAND}(2^k) \land \text{MASK}(I_x, \text{width})$
5:     $I_{\text{goodbegin}} \leftarrow \text{EQ}(L_{\text{const}} \land \text{MASK}(I_x, \text{elementwidth}), I_x, I_x, \text{elementwidth})$
6:     $I_{\text{goodtransition}} \leftarrow (\text{MASK}(I_{\text{goodbegin}}, \text{width}) \text{clr} (L_{\text{const}} \oplus (L_{\text{const}} \leftarrow \text{elementwidth}))) \lor \text{MASK}(I_{\text{goodbegin}}, \text{elementwidth})$
7:     $I_{\text{good}} \leftarrow I_{\text{goodbegin}} \land ((I_{\text{goodtransition}} + I_{\text{goodbegin}}) \rightarrow \text{width})$
8:     $L_{\text{counter}} \leftarrow \text{RAND}(2^k) \land \text{MASK}(I_{\text{good}}, \text{width})$
9:     $I_{\text{goodbegin}} \leftarrow I_{\text{good}} \land \text{EQ}(L_{\text{counter}} \land \text{MASK}(I_x, \text{elementwidth}), 0, I_x, \text{elementwidth})$
10:    temp $\leftarrow \text{ADD}(L_{\text{counter}}, L_{\text{const}}, L_{\text{const}}, \text{elementwidth}) \leftarrow \text{elementwidth}$
11:    gbmask $\leftarrow \text{MASK}(I_{\text{goodbegin}}, \text{elementwidth})$
12:    $I_{\text{goodtransition}} \leftarrow (\text{MASK}(I_{\text{goodbegin}}, \text{width}) \text{clr} (L_{\text{counter}} \oplus \text{temp})) \lor \text{gbmask}$
13:    $I_{\text{good}} \leftarrow I_{\text{goodbegin}} \land ((I_{\text{goodtransition}} + I_{\text{goodbegin}}) \rightarrow \text{width})$
14:    $M \leftarrow \text{MASK}(I_{\text{good}} \leftarrow (\text{inp} \leftarrow (\hat{w} + \hat{w} \cdot w)), \text{elementwidth} \leftarrow (\hat{w} \cdot w))$
15:    $L_{\text{output}} \leftarrow L_{\text{counter}} \land M$
16:    $I_{\text{output}} \leftarrow L_{\text{const}} \land M$
17:    return $(L_{\text{output}}, I_{\text{output}}, \hat{w})$
18: end function

Each substring of length $\text{elementwidth}$ of $L_{\text{const}}$ and the substring of the same length following it immediately. As was done in Algorithm 5, an addition operation, carried out on Step 7 propagates a carry bit through every element. The good elements, remaining in the final index set, $I_{\text{good}}$, are those for which the carry propagated through the entire element, thereby verifying that all the element’s bits are correct.

A similar technique, used in Steps 8 through 13, filters $I_{\text{good}}$ even further, until it is known additionally that every element of the map $(L_{\text{counter}}, I_{\text{good}}, \text{width})$ is a counter of width $\text{elementwidth}$. That is to say, its first $\text{elementwidth}$ bits are all zero, its next $\text{elementwidth}$ bits are the binary representation of the number 1, and so on, in arithmetic progression, until the last element, being $2^{\text{elementwidth}} - 1$. This second phase of filtering on $I_{\text{good}}$ is, once again, performed by verifying first the lowest $\text{elementwidth}$ bits (which, in this case, must equal 0) and then the relation between each element and the next (which is here incrementation). Simultaneous incrementation of all substrings of length $\text{elementwidth}$ is done by adding $L_{\text{const}}$ to $L_{\text{counter}}$.

In addition to the two maps generated, $(L_{\text{const}}, I_{\text{good}}, \text{width})$ and $(L_{\text{counter}}, I_{\text{good}}, \text{width})$, consider, now, the following new map: $(L_{\text{counter}}, L_{\text{const}}, \text{elementwidth})$. If this map has any elements at all, then it has every possible element of bit-length $\text{elementwidth}$. In particular, it would have our desired tableau.

However, while this procedure has so far presented an alternate method for producing tableau candidates, it still has not addressed the main problem of verifying that the candidate tableaux begin with the correct bit-string, $\text{inp}$.

The method by which this entire construction can now overcome the problem of verifying $\text{inp}$ is by noting that in the new structure the value of each tableau candidate is determined completely by its bit-position relative to the “1” bit of $I_{\text{good}}$ immediately preceding it. Specifically, the mask $M$, built in Step 14, is able to mask out all candidates whose most significant
bits do not match the desired value.
Thus, the algorithm generates exactly that subset of the possible tableaux that have the correct $T_{[0]}$.

Two remarks regarding this algorithm.

1. In Step 2 of the algorithm we define the width of the map the algorithm constructs.
Ostensibly, only a map of width $\hat{w}$ is required, and there is no need to define $elementwidth$ to be any higher. While this is true for most of Algorithm 6, we do need $elementwidth$ to be a power of 2 for step 14 to work properly. The expression
   $$inp \leftarrow (\hat{w} + \hat{w} \times w)$$
which appears in it is really a rewriting of
   $$(inp \times elementwidth) \leftarrow (\hat{w} \times w),$$
which is necessary because “$\times$” is not assumed to be available. This rewrite requires using a known power of 2 for $elementwidth$.

2. The algorithm actually builds the map $(L_{output}, I_{output}, elementwidth)$ with desired values. However, it returns only $(L_{output}, I_{output}, \hat{w})$, truncating the size of its elements to only $\hat{w}$ bits, as the final output should be. The reason this can be done is that in all elements, in all bits higher than $\hat{w}$, the bit values are set to zero by the algorithm. This is the “desired value” for these bit positions.

The new algorithm differs from the original one in its error (false rejection) probabilities. We complete the proof, therefore, by verifying that the new error probabilities for iteration $i$ of the loop in Step 2 of Algorithm 3, which we denote $p_i$, can still be made to satisfy
   $$\sum_i p_i \leq 1/2.$$  
Let us bound $p_i$ from above. A false reject occurs in the new algorithm only in one situation: when $I_{good} = 0$ in Step 13 of Algorithm 6. In all other cases, every possible tableau with the correct initialisation is generated and tested.

The remainder of the argument is the same as in the original proof: for a bit of $I_{good}$ to be 1, a total of $Z = O(w_x \times maxstep)$ bits in a total of $W = O(w_x)$ consecutive bit positions in $O(maxstep)$ randomly chosen integers are required to attain specific bit values. That being the case, a choice of $k$ as $W \times (i + 1) \times 2^Z$ will ensure $p_i \approx e^{-(i+1)}$, leading to $\sum_i p_i < 1/2$, as desired. Choosing a larger $k$, so as to avoid the need for multiplication, only lowers the error probability further.

Proof of Corollary 1.4. To extend Corollary 1.3 from RP acceptance criteria to BPP acceptance criteria we first note that any RP problem is by definition also a BPP problem. (Running an RP algorithm twice results in a 0 false acceptance rate and 1/4 false rejection rate, both of which are better than what is required by BPP.) We therefore only need to prove that a TM can simulate a RAM under the appropriate time constraints. Doing so is essentially done as in the proof of Theorem 1. The part of the proof corresponding to Lemma 1 remains unchanged: we use the same simulation of a randomised RAM by a randomised TM. The only difference is that we use, for both the RAM and the TM, BPP acceptance criteria, rather than RP ones.

For the equivalent of Lemma 2 showing that a generic random function is not more powerful than $RAND(2^k)$, we use a slightly different construction.
In the original construction, we picked as $2^k$ a value in excess of $M^{\text{maxstep}}$. This time we will pick a $\tilde{k}$ larger by 2. Whereas the original choice of $\tilde{k}$ ensured that instead of a false reject probability of $p_i$ the restricted-RAND algorithm will have a false reject of $p_i/(2 - p_i)$, the new choice of $\tilde{k}$ now ensures $4p_i/(5 - p_i)$.

A little arithmetic now shows that for error probabilities lower than $1/3$ simply running the algorithm 3 times and taking a majority vote attains acceptance and rejection error probabilities that are better than the original, and therefore certainly, over the entire algorithm, within the parameters of BPP.

6 Conclusions

This work introduced the new complexity class, PEL, and showed that PEL arises naturally in both deterministic and randomised PTIME RAM computations. The power of both the RP-RAM and the BPP-RAM with several interesting operation sets was characterised as PEL, this characterisation of the RP-RAM closing a 30 year old open question.

However, perhaps the most important point of this paper is in pointing out that P-RAM = RP-RAM for some RAMs (specifically, those whose basic operation sets include division), whereas for others this is not the case.

Although these conclusions seem in no way applicable to the central question of P vs. RP in TMs, it still sheds interesting light on this problem, in pointing out that the answer of whether stochasticity adds computational power under RP criteria does not have a single universal answer. Rather, it is dependent on the details of the computational model examined.

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