GOTZMANN SQUAREFREE IDEALS
ANDREW H. HOEFEL AND JEFF MERMIN

Abstract: We classify the squarefree ideals which are Gotzmann in a polynomial ring.

1. Introduction

Let $S = k[x_1, \ldots, x_n]$ and $R$ be either $S/(x_1^2, \ldots, x_n^2)$ or the exterior algebra on the same variables.

The question of what numerical functions can arise as the Hilbert functions of homogeneous ideals these rings was answered by Macaulay [Ma27] (for $S$) and Kruskal [Kr63] and Katona [Ka68] (for $R$). Macaulay’s theorem and the Kruskal-Katona theorem have seen widespread application in both commutative algebra and combinatorics. These theorems provide lower bounds on the Hilbert function growth of homogeneous ideals, and describe special ideals, called lexicographic or lex ideals, which attain these bounds. Lex ideals are defined combinatorially, and are very well understood; for example, their minimal free resolutions are known explicitly.

Lex ideals are important tools in many contexts. In geometry, Hartshorne’s proof that the Hilbert scheme is connected [Ha66] uses lex ideals in an essential way. More combinatorially, lex ideals and Macaulay’s and Kruskal and Katona’s Theorems have arisen in Sperner theory, network reliability, and other graph problems; see [En97] or [BL05].

In many of these settings, the only relevant property of the lex ideals is the slow growth of their Hilbert functions, so it is worthwhile to consider other ideals whose Hilbert functions achieve the bounds of Macaulay’s (or Kruskal and Katona’s) Theorem. Such ideals are called Gotzmann. Gotzmann ideals share many nice properties with lex ideals; for example, they have componentwise linear resolutions and maximal graded Betti numbers among ideals with the same Hilbert function [HH99].

However, only a few classes of Gotzmann ideals are known. Murai and Hibi show in [MH08] that all Gotzmann ideals of $S$ with at most $n$ generators have a very specific form. The problem of understanding monomial Gotzmann ideals has proven slightly more tractable. Bonanzinga classifies the principal Borel ideals generated in degree at most four which are Gotzmann [Bo03]. Mermin enumerates in [Me06] the lexlike ideals, ideals which are generated by initial segments of “lexlike” sequences and which share many properties with lex ideals, including minimal Hilbert function growth. Murai studies Hilbert functions for which the only Gotzmann monomial

2000 Mathematics Subject Classification: 13F20.
Keywords and Phrases: Edge ideals, Gotzmann ideals
ideals are lex, and classifies the Gotzmann monomial ideals of $k[a, b, c]$, in [Mu07], and Olteanu, Olteanu, and Sorrenti [OOS08] classify the Gotzmann ideals which are generated by (not necessarily initial) segments in the lex order. Finally, in [Ho09], Hoefel shows that a graph has a Gotzmann edge ideal if and only if it is star-shaped. In this paper, we generalize Hoefel’s result by classifying all the Gotzmann ideals of $S$ which are generated by squarefree monomials. An immediate consequence of our classification is that all such ideals have at most $n$ generators, so they have the form prescribed by Murai and Hibi.

In section 2, we define notation which will be used throughout the paper, and recall background information about Gotzmann ideals, squarefree ideals, and the relationship between $R$ and $S$. In section 3, we classify the squarefree Gotzmann ideals of $S$. Finally, in section 4, we begin to study the monomial Gotzmann ideals of $R$. Our main result is that a Gotzmann ideal has Gotzmann Alexander dual if and only if all of its degreewise components are lex in some order.

Acknowledgements: We thank Chris Francisco, Huy Tài Hà and Gwyn Whieldon for helpful discussions. The first author also thanks NSERC and the Killam Trusts for their financial support.

2. Background and Notation

Let $S = k[x_1, \ldots, x_n]$ be the polynomial ring and $R = S/(x_1^2, \ldots, x_n^2)$ be the associated “squarefree ring”. Let $\mathfrak{m} = (x_1, \ldots, x_n)$ be the homogeneous maximal ideal.

In this paper all ideals are homogeneous. A monomial ideal is an ideal with a generating set consisting of monomials. Every monomial $I$ has a canonical minimal set of monomial generators which we denote $\text{gens}(I)$. When we refer to the generators of a monomial ideal — for example, to count them — we mean those in this canonical generating set. If all the generators of $I$ are squarefree monomials, then we say that $I$ is squarefree.

For an ideal $I$, we write $I_d$ for the vector space of degree $d$ forms in $I$. When $I$ is a monomial ideal, a basis for $I_d$ is given by the monomials of degree $d$ that are in $I$.

An important vector space is $S_d$ (or $R_d$), the space of all forms of degree $d$. We call subspaces of $S_d$ or $R_d$ monomial vector spaces when they have a monomial basis. The (unique) monomial basis of a monomial vector space $V_d$ will be denoted $\text{gens}(V_d)$. Usually, we indicate the degree of a monomial vector space with a subscript in this way.

The ideal $(V_d)$ generated by a monomial vector space $V_d \subseteq S_d$ is a monomial ideal with with $\text{gens}((V_d)) = \text{gens}(V_d)$. We will often need to consider the monomial vector space

$$\mathfrak{m}_1 V_d = \text{span}_k \{ x_i m \mid m \in \text{gens}(V_d) \}.$$ 

While we treat this as a product of monomial vector spaces, it has a natural interpretation in terms of ideals. If $I = (V_d)$ is the ideal generated by $V_d$, then $\mathfrak{m}_1 V_d = I_{d+1}$.

We will write $|V_d|$ or sometimes $|V|_d$ to denote the vector space dimension of $V_d$. 
Definition 2.1. The Hilbert function of an ideal $I$ is the function $HF_I : \mathbb{N} \to \mathbb{N}$ which gives the dimension of each component of $I$:

$$HF_I(d) = |I_d|.$$ 

When $I$ is a monomial ideal $|I_d| = |\text{gens}(I_d)|$ is simply the number of degree $d$ monomials in $I$. If $I$ is squarefree, we write $I^sf = IR$ for the corresponding ideal of $R$. Thus $|I^sf_d|$ is the number of squarefree monomials in $I$ of degree $d$.

The Hilbert series and squarefree Hilbert series of $I$ are $HS_I(t) = \sum_{d=0}^{\infty} |I_d| t^d$ and $HS^sf_I(t) = \sum_{d=0}^{\infty} |I^sf_d| t^d$. If $I$ is squarefree, these are related by the formula $HS_I(t) = HS^sf_I(t^{\frac{1}{1-t}})$.

Definition 2.2. We say that an ideal $I$ of $S$ (or, with the obvious changes, of $R$) is Gotzmann if for all degrees $d$ and all ideals $J$ satisfying $|I_d| = |J_d|$ we have $|m_1 I_d| \leq |m_1 J_d|$.

The Gotzmann property may be viewed degreewise as a property of vector spaces. We say that a vector space $V_d \subseteq S_d$ (or $R_d$) is Gotzmann if for all subspaces $W_d \subseteq S_d$ (resp. $R_d$) with $|V_d| = |W_d|$ we have $|m_1 V_d| \leq |m_1 W_d|$. Thus an ideal $I$ is Gotzmann if and only if each component $I_d$ is Gotzmann.

Definition 2.3. A vector space $L_d \subseteq S_d$ (or $R_d$) is a lex segment if it is spanned by an initial segment of the degree $d$ monomials in lexicographic order. An ideal is lex if each of its components are lex segments. We say that a squarefree ideal $L$ is squarefree lex if $L^sf$ is lex in $R$.

Lex ideals are important because they are very well understood combinatorially, and because of the following theorem of Macaulay [Ma27] (in $S$) and Kruskal and Katona [Kr63, Ka68] (in $R$):

Theorem 2.4 (Macaulay, Kruskal, Katona). Lex ideals are Gotzmann.

Corollary 2.5. For every homogeneous ideal $I$ there is a unique lex ideal $L$ with $HF(L) = HF(I)$.

Macaulay’s theorem allows the following alternative characterization of Gotzmann ideals:

Proposition 2.6. A degree $d$ monomial vector space $V_d$ is Gotzmann if and only if $|m_1 V_d| = |m_1 L_d|$ where $L_d$ is the degree $d$ lex segment of dimension $|V_d|$.

Corollary 2.7. Let $L$ be the lex ideal with the same Hilbert function as $I$. Then $I$ is Gotzmann if and only if $L$ and $I$ have the same number of generators in each degree.

Proof. The number of degree $d$ generators of $I$ and $L$ is given by $|I_d| - |m_1 I_{d-1}|$ and $|L_d| - |m_1 L_{d-1}|$, respectively. Since $I$ and $L$ have the same Hilbert function, they have the same number of generators in degree $d$ if and only if $I_{d-1}$ is Gotzmann. □
Gotzmann’s persistence theorem [Go78] states that if $V_d$ is a Gotzmann vector space then $m_1 V_d$ is also Gotzmann. Thus, to check if an ideal is Gotzmann, one only needs to check that its components are Gotzmann in the degrees of its minimal generators.

**Theorem 2.8** (Gotzmann’s persistence theorem, vector space version). Suppose that $I_d$ is a Gotzmann vector space. Then $m_1 I_d$ is Gotzmann.

**Theorem 2.9** (Gotzmann’s persistence theorem, ideal version). Suppose that every generator of $I$ has degree at most $d$, and let $L$ be the lex ideal with the same Hilbert function as $I$. If $L$ has no generators of degree $d + 1$, then all generators of $L$ have degree at most $d$. In particular, if $I$ and $L$ have the same number of generators in every degree less than or equal to $d + 1$, then $I$ is Gotzmann.

The persistence theorem holds in both $S$ and $R$ [AHH97, FG86].

3. Gotzmann Squarefree Ideals of the Polynomial Ring

We will classify the squarefree ideals of $S$ which are Gotzmann. To do this, we compare squarefree ideals with their squarefree lexifications and exploit the interaction between $S$ and $R$.

In [Ho09], Hoefel proved that a squarefree quadric ideal is Gotzmann if and only if it is the edge ideal of a star-shaped graph. We generalize this result as follows:

**Definition 3.1.** Let $H$ be a pure $d$-dimensional hypergraph. We say that $H$ is star-shaped if there exists a $(d - 1)$-simplex which is contained in every edge of $H$. More generally, we say that a $d$-dimensional simplicial complex $\Delta$ is a supernova if there exists a chain of faces $\emptyset \subset F_0 \subset F_1 \subset \cdots \subset F_{d-1}$ such that every $i$-dimensional facet of $\Delta$ contains the $(i - 1)$-dimensional face $F_{i-1}$.

We show in Theorem 3.9 that a squarefree ideal is Gotzmann if and only if it is the edge ideal of a supernova. In particular, a purely $d$-generated squarefree ideal is Gotzmann if and only if it is the edge ideal of a $(d - 1)$-dimensional supernova.

A consequence of Theorem 3.9 is that all Gotzmann squarefree ideals of $S = \mathbb{k}[x_1, \ldots, x_n]$ have at most $n$ generators. The Gotzmann ideals of $S$ with at most $n$ generators are classified by Murai and Hibi [MH08]; it is clear from their classification that any Gotzmann squarefree ideal with at most $n$ generators must have the form prescribed by Theorem 3.9. Thus, if this bound on the number of generators could be easily proved, Theorem 3.9 would be a corollary of [MH08, Theorem 1.1]. We have been unable to find a simple proof of this bound. Regardless, the smaller scope of our investigation allows a simpler proof than that given in [MH08].

**Definition 3.2.** The squarefree lexification of a squarefree ideal $I \subseteq S$ is the squarefree lex ideal $L$ in $S$ with the same Hilbert function as $I$.

The existence of squarefree lexifications follows from the following construction: Let $J^sf \subseteq R$ be the lex ideal having the same Hilbert function as $I^sf$. Then let $L$
be the ideal of $S$ with $I^sf = J^sf$ (that is, $L$ is generated by the monomials of $J^sf$). Then $L$ is squarefree lex and has the same Hilbert function as $I$ because

$$HS_I(t) = HS_{I^sf}(\frac{t}{1-t}) = HS_{J^sf}(\frac{t}{1-t}) = HS_L(t).$$

The following proposition is a consequence of Kruskal and Katona’s theorem:

**Proposition 3.3** (Aramova, Avramov, Herzog [AAH00]). If $I \subseteq S$ is a Gotzmann squarefree ideal then $I^sf$ is Gotzmann in $R$.

**Lemma 3.4.** If $I \subseteq S$ is a Gotzmann squarefree ideal then its squarefree lexification $L$ is Gotzmann.

**Proof.** By Proposition 3.3, $I^sf$ is Gotzmann in $R$. Thus, applying Corollary 2.7, $I^sf$ and $L^sf$ have the same number of minimal generators in every degree. Now $I$ and $I^sf$ have the same generating set, as do $L$ and $L^sf$, so $I$ and $L$ have the same number of generators in every degree. Applying Corollary 2.7 again, $L$ must be Gotzmann in $S$.

The next general lemma is the squarefree version of a common inductive technique for studying Hilbert functions in terms of lex ideals.

**Lemma 3.5.** Let $I \subseteq S$ be a squarefree ideal and let $L$ be its squarefree lexification. Then $L \subseteq (x_1)$ if and only if $I \subseteq (x_1)$ for some variable $x_1$.

**Proof.** If $I \subseteq (x_1)$ then $I^sf \subseteq (x_1)^sf$ and hence

$$|L^sf_d| = |I^sf_d| \leq |(x_1)^sf_d| = |(x_1)^sf_d|.$$

As $(x_1)^sf_d$ is a lex segment in $R$, we have $L^sf_d \subseteq (x_1)^sf_d$ and hence every generator of $L$ is divisible by $x_1$.

Conversely, assume that $L \subseteq (x_1)$. We have $|L^sf_{n-1}| \leq n - 1$, so there is at least one squarefree monomial $m$ of degree $n - 1$ which is not in $I$. Write $m = \frac{x_1 \cdots x_n}{x_i}$. We claim that $I \subseteq (x_1)$. Indeed, every squarefree monomial outside of $(x_1)$ divides $m$, so no such monomial can appear in $I^sf$. Thus every generator of $I$ is in $(x_1)$ and, in particular, $I \subseteq (x_1)$.

**Lemma 3.6.** If $I \subseteq S$ is a squarefree Gotzmann ideal then either $I \subseteq (x_i)$ for some variable $x_i$ or $(x_i) \subseteq I$ for some variable $x_i$.

**Proof.** Suppose to the contrary that $I$ is Gotzmann but, for all $i$, $I \not\subseteq (x_i)$ and $(x_i) \not\subseteq I$. We will show that $L$, the squarefree lexification of $I$, is not Gotzmann, contradicting Lemma 3.4.

It follows from Lemma 3.5 that $L \not\subseteq (x_1)$. Therefore we may choose a generator $m$ of $L$ which is not divisible by $x_1$. Let $d$ be the degree of $m$.

Since $I$ contains no variable, $L$ cannot contain $x_1$. Choose a squarefree monomial $m' \in (x_1) \setminus L$ of maximal degree $d'$. As $L$ is squarefree lex and $m$ is not divisible by $x_1$, $L$ contains all squarefree monomials that are divisible by $x_1$ and have degree $d$ or larger. Thus, $d' < d$.

Let $T$ be the ideal generated by $\text{gens}(L) \cup \{x_1^{d-d'}m'\} \setminus \{m\}$. Note that $|T_d| = |L_d|$.
Let $A = \text{gens}(m_1 L_d)$ and $B = \text{gens}(m_1 T_d)$ be the sets of degree $d + 1$ monomials lying above $L_d$ and $T_d$ respectively. If $L$ were Gotzmann, it would follow that $|A| \leq |B|$. We will show that instead $|B| < |A|$.

We claim that $B \setminus A = \{x_1^{d-d'}m'x_i : x_i \text{ divides } m'\}$. Indeed, let $\mu \in B \setminus A$ be a monomial. Then $\mu$ is divisible by $x_1^{d-d'}m'$, so it has the form $x_1^{d-d'}m'x_i$ for some $i$. If $x_i$ divides $m'$ then the support of $x_1^{d-d'}m'x_i$ is $m'$ and hence $\mu$ is not in $A$. On the other hand, if $x_i$ does not divide $m'$, then $m'x_i$ is a squarefree monomial of degree $d' + 1$ which is divisible by $x_1$. By the choice of $m'$, we have $m'x_i \in L$ and hence $x_1^{d-d'}m'x_i \in A$, proving the claim. In particular, $|B \setminus A| = d'$.

Similarly, monomials in $A \setminus B$ must have the form $x_i m$ for some $i$. If $x_i$ divides $m$ then $x_i m$ has support $m$ and hence is not in $B$. Thus

$$A \setminus B \supseteq \{x_i m : x_i \text{ divides } m\}$$

which has cardinality at least $d$.

As $|B \setminus A| = d' < d \leq |A \setminus B|$, it follows that $|m_1 T_d| = |B| < |A| = |m_1 L_d|$, and so $L$ is not Gotzmann.

\[ \square \]

**Lemma 3.7.** Let $I \subseteq S$ be a Gotzmann squarefree monomial ideal with $I \subseteq (x_i)$. Then $\frac{1}{x_i} I$ is Gotzmann in $S$.

**Proof.** Let $L$ be the (non-squarefree) lexification of $I$. It is clear that $L \subseteq (x_i)$: $(x_1)$ is the lexification of $(x_i)$, which contains $I$.

Now multiplication by $x_i$ is a degree one module isomorphism from $\frac{1}{x_i} I$ to $I$, and similarly for $L$. Applying Corollary 2.7 twice, we have that $\frac{1}{x_i} I$ is Gotzmann with the same Hilbert function as $\frac{1}{x_i} L$.

\[ \square \]

**Lemma 3.8.** Let $I \subseteq S$ be a Gotzmann squarefree monomial ideal with $(x_i) \subseteq I$. The image of $I$ in the quotient ring $S/(x_i)$ is a Gotzmann squarefree monomial ideal.

**Proof.** By renaming the variables if necessary, we may assume that $(x_1) \subseteq I$. Let $\tilde{I}$ be the image of $I$ in $S/(x_1)$ (or, equivalently, the squarefree monomial ideal of $k[x_2, \ldots, x_n]$ generated by every generator of $I$ other than $x_1$).

Let $L$ be the (non-squarefree) lexification of $I$ in $S$. We have $(x_1) \subseteq L$. Let $\check{L}$ be the image of $L$ in $S/(x_1)$. Then $\check{L}$ is the lexification of $\tilde{I}$. Observe that $\text{gens}(\check{I}) = \text{gens}(I) \setminus \{x_1\}$ and similarly for $L$. Thus, applying Corollary 2.7 twice, $\tilde{I}$ is Gotzmann.

\[ \square \]

Lemma 3.6 allows us to characterize the squarefree ideals which are Gotzmann.

**Theorem 3.9.** Suppose $I \subseteq S$ is a squarefree ideal. Then $I$ is Gotzmann if and only if

$$I = m_1(x_{i_1,1}, \ldots, x_{i_1,r_1}) + m_1 m_2(x_{i_2,1}, \ldots, x_{i_2,r_2}) + \cdots + m_1 \cdots m_s(x_{i_s,1}, \ldots, x_{i_s,r_s})$$

for some squarefree monomials $m_1, \ldots, m_s$ and variables $x_{i,j}$ all having pairwise disjoint support.
Proof. Suppose that $I$ is Gotzmann. By Lemma 3.6, either $(x_j) \subseteq I$ or $I \subseteq (x_j)$ for some $j$.

If $I \subseteq (x_j)$ then $\frac{1}{x_j}I$ is Gotzmann in $S$ and its generators are supported on \{\!\!\{x_1, \ldots, \hat{x_j}, \ldots, x_n\}\!\!\}. Inducting on the number of variables, $\frac{1}{x_j}I$ may be written as

$$m_1(x_{i_1,1}, \ldots, x_{i_1,r_1}) + m_1m_2(x_{i_2,1}, \ldots, x_{i_2,r_2}) + \cdots + m_1 \cdots m_s(x_{i_s,1}, \ldots, x_{i_s,r_s})$$

where $x_j$ does not appear in this expression. Thus, $I$ can be expressed in the desired form by replacing $m_1$ with $x_jm_1$.

Alternatively, suppose that $(x_j) \subseteq I$, so, without loss of generality, $I = (x_j) + J$, where $J$ is Gotzmann in the ring $k[x_1, \ldots, \hat{x_j}, \ldots, x_n]$. By induction on the number of variables, $J$ may be written in the desired form and so $I = (x_j) + J$ has the desired form as well (with $m_1 = 1$).

Using Theorem 3.9, it is possible, if difficult, to count the Gotzmann squarefree ideals of $S$. We begin by counting these ideals up to symmetry. (This is the same as counting the “universally squarefree lex” ideals: the squarefree lex ideals which are still squarefree lex in $S[y]$.)

Proposition 3.10. If $n \geq 2$, the following are all equal to $2^{n-2}$:

(i) The number of ordered partitions of $n$ into an even number of summands.

(ii) The number of ordered partitions of $n$ into an odd number of summands.

(iii) The number of Gotzmann squarefree ideals which contain no linear forms and are not contained in any monomial subalgebra of $S$, up to a reordering of the variables.

(iv) The number of nonunit Gotzmann squarefree ideals which contain linear forms and are not contained in any monomial subalgebra of $S$, up to a reordering of the variables.

(v) The number of Gotzmann squarefree ideals which contain no linear forms and are contained in some monomial subalgebra of $S$, up to a reordering of the variables.

(vi) The number of Gotzmann squarefree ideals which contain linear forms and are contained in some monomial subalgebra of $S$, up to a reordering of the variables.

In particular, there are $2^n$ nonunit Gotzmann squarefree ideals up to symmetry.

Proof. For (iii) through (vi), we describe a bijection to the ordered partitions. Given a partition, we partition the variables, in order, into sets of the given sizes. Using the notation of Theorem 3.9, we will alternate these sets between the supports of the monomials $m_i$ and the sets $\{x_{i,j,1}, \ldots, x_{i,j,r_j}\}$. We begin with the monomial if we are counting without linear forms, and with the set if we are counting with linear forms (because $m_1 = 1$ in these cases). If we are counting ideals contained in a subalgebra, we do not use the last summand. Note that the parity of the partition is fixed in each case.

When $n = 1$, the two nonunit Gotzmann squarefree ideals are (0) and $(x_1)$. □
We use the same idea of partitioning the variables and alternating between monomials $m_i$ and sets $\{x_{i,j,1}, \ldots, x_{i,j,r_j}\}$ to count the Gotzmann squarefree ideals of $S$ without symmetry. The difficulty is that it is easy to overcount those ideals with $r_s = 1$.

Let $G$ be the set of all squarefree Gotzmann ideals in polynomial rings of the form $\mathbb{k}[x_1, \ldots, x_n]$ for some $n$. We define a weight function $\omega : G \to \mathbb{N}$ by $\omega(I) = n$ if $I \subset \mathbb{k}[x_1, \ldots, x_n]$.

We will show that the exponential generating function (e.g.f.) of $G$ is

$$g(t) = \sum_{I \in G} \frac{t^{\omega(I)}}{\omega(I)!} = e^t \left( \frac{2(1-t)}{2-e^t} + t \right).$$

The coefficients this e.g.f. count the number of squarefree Gotzmann ideals in polynomial rings in $n$ variables for each value of $n$.

We begin with notation for ordered set partitions. In Proposition 3.13, we relate them to set $H \subset G$ of all squarefree Gotzmann ideals with full support (i.e. $I \in H$ uses all $n$ variables for $n = \omega(I)$).

**Notation 3.11** (Ordered Set Partitions). An ordered set partition of $[n] = \{1, \ldots, n\}$ is an ordered sequence $\sigma = (\sigma_1, \ldots, \sigma_k)$ of sets $\sigma_i$ which partition $[n]$. Each $\sigma_i$ is called a block of $\sigma$.

Let $P_n$ be the set of ordered set partitions of $[n]$ and $P$ be the union of all $P_n$. On the set $P$ we define a weight function $\nu : P \to \mathbb{N}$ where $\nu(\sigma)$ is the number of elements that $\sigma$ partitions.

We will frequently use the e.g.f. of $P$ which counts the number of ordered set partitions of $[n]$: 

$$f(t) = \sum_{\sigma \in P} \frac{t^{\nu(\sigma)}}{\nu(\sigma)!} = \frac{1}{2-e^t}.$$ 

This e.g.f. is entry A670 in the On-Line Encyclopedia of Integer Sequences [Sl03].

**Lemma 3.12.** Let $P'$ be the set of ordered set partitions that have last blocks of size greater than one. The e.g.f. of $P'$ with weight $\nu$ is $(1-t)/(2-e^t)$.

**Proof.** Let $\sigma \in P \setminus P'$ be an ordered set partition with $\omega(\sigma) = n + 1$. Then the last block of $\sigma$ is the singleton $\{i\}$ for some $i = 1, \ldots, n + 1$. Removing the last block from this partition gives a bijection between ordered set partitions ending in $\{i\}$ and ordered set partitions of a set of size $n$. Thus, the exponential generating function of $P \setminus P'$ is $tf(t) = \frac{t}{2-e^t}$ and hence the e.g.f. of $P'$ is $f(t) - tf(t) = \frac{1}{2-e^t}$. \hfill $\square$

The next proposition describes the relationship between ordered set partitions and the set $H$ of Gotzmann squarefree ideals with full support.

**Proposition 3.13.** The e.g.f. of $H$ with weight $\omega$ is 

$$h(t) = \sum_{I \in H} \frac{t^{\omega(I)}}{\omega(I)!} = \frac{2(1-t)}{2-e^t} + t.$$
Proof. Every ideal in $H$ is of the form
\[
m_1(x_{i_1}, \ldots, x_{i_{r_1}}) + m_2(x_{i_2}, \ldots, x_{i_{r_2}}) + \cdots + m_s(x_{i_s_1}, \ldots, x_{i_{s_r}})\]
for some $m_j$ and $x_{i_j}$ all distinct. Let $\beta_{0,I}(I)$ be the number of generators of $I$ of degree $d$. We partition $H$ into five subsets $H = \cup_{i=0}^{4} H_i$ where
\[
H_0 = \{(x_1)\},
\]
\[
H_1 = \{I \in H \mid I \text{ contains a linear form and } \beta_{0,\text{reg}}(I) = 1 \text{ and } \text{reg} I \neq 1\},
\]
\[
H_2 = \{I \in H \mid I \text{ contains a linear form and } \beta_{0,\text{reg}}(I) > 1\},
\]
\[
H_3 = \{I \in H \mid I \text{ does not contain a linear form and } \beta_{0,\text{reg}}(I) = 1\}, \text{ and}
\]
\[
H_4 = \{I \in H \mid I \text{ does not contain a linear form and } \beta_{0,\text{reg}}(I) > 1\}.
\]
Recall we use $P'$ to denote the set of ordered set partitions whose last block is not a singleton. There is a weight preserving bijection between $P'$ and $H_1 \cup H_2$ given by
\[
(\sigma_1, \ldots, \sigma_k) \mapsto \begin{cases} 
(\sigma_1 + (\prod_{i=1}^{k-1} \sigma_2)(\sigma_2) + \cdots + (\prod_{i=1}^{k-1/2} \prod_{s=1}^{k/2} \sigma_{2s}))(\sigma_k) & k \text{ odd}, \\
(\sigma_1 + (\prod_{i=1}^{k-1} \sigma_2)(\sigma_2) + \cdots + (\prod_{i=1}^{k/2} \sigma_{2i}))(\sigma_k) & k \text{ even}.
\end{cases}
\]
Similarly, there is a weight preserving bijection between $P'$ and $H_3 \cup H_4$;
\[
(\sigma_1, \ldots, \sigma_k) \mapsto \begin{cases} 
(\prod_{i=1}^{k} \sigma_1)(\sigma_2) + (\prod_{i=1}^{k} \sigma_1)(\prod_{s=1}^{k} \sigma_{3s})(\sigma_4) + \cdots + (\prod_{i=1}^{k/2} \prod_{s=1}^{k/2} \sigma_{2s-1})(\sigma_k) & k \text{ even}, \\
(\prod_{i=1}^{k} \sigma_1)(\sigma_2) + (\prod_{i=1}^{k} \sigma_1)(\prod_{s=1}^{k} \sigma_{3s})(\sigma_4) + \cdots + (\prod_{i=1}^{k+1/2} \prod_{s=1}^{k+1/2} \sigma_{2s-1})(\sigma_k) & k \text{ odd}.
\end{cases}
\]
The desired formula follows from Lemma \[3.12\] \]

Corollary 3.14. The exponential generating function for $G$, the set of all Gotzmann squarefree monomial ideals, is
\[
g(t) = \sum_{I \in G} \frac{t^{\omega(I)}}{\omega(I)!} = e^t \left( \frac{2(1-t)}{2-e^t+t} \right)
\]

Proof. For each Gotzmann squarefree monomial ideal with full support in a polynomial ring over $k$ variables there are $\binom{n}{k}$ Gotzmann squarefree monomial ideals in a polynomial ring over $n$ variables with support of size $k$. Thus, we apply the inverse binomial transform to the previous proposition (i.e. multiply the e.g.f. by $e^{t}$). $\square$

From this generating function, one can extract the number of squarefree Gotzmann ideals in a polynomial ring in $n$ generators. For the first few values of $n$, these numbers are $2, 3, 6, 19, 96, 669$ (i.e. for $n = 0, 1, \ldots, 5$).

4. Gotzmann Ideals of the Squarefree Ring

The problem of classifying all Gotzmann monomial ideals of the squarefree ring $R$ turns out to be much more difficult. We might hope to prove some squarefree analog of Lemma 3.6, but, arguing as in the previous section, we would be able to prove that Gotzmann ideals of $R$ are lex segments (if generated in one degree) or initial segments in a lexlike tower (see [Mc06]) in general. Unfortunately such an approach is doomed to fail, as the following examples show.
Example 4.1. The ideal $I = (ab, ac, bd, cd)$ is Gotzmann in $R$ but is not lex.

The ideal $I$ above is (up to symmetry) the only monomial Gotzmann ideal of $k[a, b, c, d]/(a^2, b^2, c^2, d^2)$ which is not lex in some order. Thus we might hope that it is the only such ideal, or at least is the first instance of a one-parameter family of exceptions. This hope is dashed as well as soon as we add a fifth variable.

Example 4.2. The ideal $I = (abc, abd, abe, acd, ace, bcd, bce)$ is Gotzmann in $R$ but is not lex.

Since the Alexander duals of lex ideals are lex, we might hope that the Alexander duals of Gotzmann ideals are Gotzmann. However, the duals of the two examples above are not Gotzmann. We will see in Theorem 4.16 that a Gotzmann ideal has Gotzmann dual if and only if it is in some sense morally lex.

Throughout the section, all ideals will be monomial ideals of $R$. Since we no longer work with the polynomial ring, we can dispense with the notation $I_{sf}$ to indicate that an ideal lives in $R$, and will simply write $I$, $J$, etc. Many of our arguments are technical, so for ease of notation we work mostly with monomial vector spaces rather than ideals. Recall that a vector space $V \subseteq \mathbb{R}^d$ is Gotzmann if $|m|_V$ is minimal given $|V|$ and $d$, and that an ideal $I$ is Gotzmann if and only if $|I_d|$ is Gotzmann for all $d$.

4.1. Decomposing Gotzmann Ideals of $R$. In this section we show every Gotzmann monomial vector space $V \subseteq \mathbb{R}^d$ can be decomposed as the direct sum of two monomial vector spaces which are Gotzmann in a squarefree ring with one fewer generator. This decomposition relates to the operation of compression (see [MP06] or [Me08]). We begin by recalling the necessary notation.

Given a (fixed) variable $x_i$, let $n = (x_1, \ldots, \hat{x_i}, \ldots, x_n)$ be the maximal ideal in $Q = R/(x_i)$ which is a squarefree ring on $n - 1$ variables.

Definition 4.3 ($x_i$-decomposition). Let $V \subseteq \mathbb{R}^d$ be a monomial vector space and fix a variable $x_i$. The monomial basis of $V$ can be partitioned as $A \cup B$ where $A$ contains the monomials divisible by $x_i$ and $B$ contains those not divisible by $x_i$.

Let $V_0$ be the monomial vector space spanned by $B$ and let $V_1$ be the monomial vector space spanned by $\{m \mid x_im \in A\}$. We write $V$ as the direct sum

$$V = V_0 \oplus x_iV_1$$

which we call the $x_i$-decomposition of $V$.

We view the monomial vector spaces $V_0$ and $V_1$ as subspaces of $Q_d$ and $Q_{d-1}$ respectively.

Definition 4.4 ($x_i$-compression). Let $V = V_0 \oplus x_iV_1$ be the $x_i$-decomposition of the monomial vector space $V$. Let $L_0$ and $L_1$ be the squarefree lex-segments in $Q$ with the same degrees and dimensions as $V_0$ and $V_1$. The $x_i$-compression of $V$ is the monomial vector space

$$T = L_0 \oplus x_iL_1.$$
We recall the following important fact about compressions from [MP06]:

**Proposition 4.5** ([MP06]). If $T$ is the $x_i$-compression of the monomial vector space $V \subseteq R_d$, then

$$|m_i T| \leq |m_1 V|.$$ 

**Lemma 4.6.** If $V = V_0 \oplus V_1$ then the $x_i$-decomposition of $m_1 V$ is

$$m_1 V = n_1 V_0 \oplus x_i (V_0 + n_1 V_1).$$

**Proof.** Since $x_i^2 = 0$, we have $m_1 (x_i V_1) = n_1 (x_i V_1)$. Thus,

$$m_1 V = m_1 (V_0 + x_i V_1) = n_1 V_0 + x_i V_0 + x_i n_1 V_1 = n_1 V_0 \oplus x_i (V_0 + n_1 V_1).$$

This sum is direct since the second summand is contained in $(x_i)$ while the first summand is not. □

**Proposition 4.7.** Let $V \subseteq R_d$ be a Gotzmann monomial vector space and let $V = V_0 \oplus x_i V_1$ be its $x_i$-decomposition. Then $V_0$ is Gotzmann in $Q$.

**Proof.** Let $L$ be the $x_i$-compression of $V$. As $V$ is Gotzmann $|m_1 V| \leq |m_1 L|$ and so $|m_1 V| = |m_1 L|$ by Proposition 4.5.

Thus we have

$$(*) \quad |n_1 V_0| + |V_0 + n_1 V_1| = |n_1 L_0| + |L_0 + n_1 L_1|$$

from the previous lemma.

Since $L_1$ and $n_1 L_0$ are lex segments of the same degree, it follows that one contains in the other. If $n_1 L_1 \subseteq L_0$ then

$$|L_0 + n_1 L_1| = |L_0| = |V_0| \leq |V_0 + n_1 V_1|.$$ 

Similarly, if $L_0 \subseteq n_1 L_1$ then

$$|L_0 + n_1 L_1| = |n_1 L_1| \leq |n_1 V_1| \leq |V_0 + n_1 V_1|.$$ 

In both cases $|L_0 + n_1 L_1| \leq |V_0 + n_1 V_1|$. From the equality above we see that $|n_1 V_0| \leq |n_1 L_0|$ and hence $V_0$ is Gotzmann by Proposition 2.6. □

**Lemma 4.8.** Let $V$ be Gotzmann in $R$ with $x_i$-decomposition $V = V_0 \oplus x_i V_1$ and let $L = L_0 \oplus x_i L_1$ be its $x_i$-compression. Then either $V_1$ is Gotzmann in $Q$ or $n_1 L_1 \subset L_0$.

**Proof.** We know from the previous proposition that $V_0$ is Gotzmann in $Q$ and hence $|n_1 V_0| = |n_1 L_0|$. Thus, the equality $(*)$ gives

$$|V_0 + n_1 V_1| = |L_0 + n_1 L_1|.$$ 

If $n_1 L_1 \not\subset L_0$ then $L_0 \subseteq n_1 L_1$ as they are both lex segments. Thus

$$|n_1 V_1| \leq |V_0 + n_1 V_1| = |L_0 + n_1 L_1| = |n_1 L_1|$$

which proves that $V_1$ is Gotzmann. □
If \( \mathbf{n}_1 L_1 \subseteq L_0 \), then \( V_1 \) need not be Gotzmann. For example,

\[
V = \text{span}_k \{abc, abd, acd, bcd, bce, bde, cde\}
\]

is Gotzmann in \( R = \mathbb{k}[a, b, c, d, e]/(a^2, \ldots, e^2) \), but \( V_1 = \text{span}_k \{bc, bd, cd\} \) from the \( a \)-decomposition of \( V \) is not Gotzmann in \( Q = R/(a) \).

However, we will see that it is always possible to choose \( x_i \) such that \( V_1 \) is Gotzmann.

**Lemma 4.9.** Let \( V \) be Gotzmann with \( x_i \)-decomposition \( V = V_0 \oplus x_i V_1 \) and compression \( L = L_0 \oplus x_i L_1 \). If \( \mathbf{n}_1 L_1 \subseteq L_0 \), then \( V \) satisfies the property:

Let \( m \in V \) be a monomial such that \( x_i \) divides \( m \), and let \( x_j \) be any variable not dividing \( m \). Then \( \frac{x_j}{x_i} m \in V \).

**Proof.** Applying (\( \star \)), we have \(|\mathbf{n}_1 V_1 + V_0| = |\mathbf{n}_1 L_1 + L_0| = |L_0| = |V_0|\), i.e., \( \mathbf{n}_1 V_1 \subseteq V_0 \). The desired property follows. \( \square \)

**Theorem 4.10.** Suppose \( V \subset R_d \) is a Gotzmann monomial vector space. Then \( x_i \) may be chosen so that both summands \( V_1 \) and \( V_0 \) of the \( x_i \)-decomposition of \( V \) are Gotzmann in \( Q \) and \( V_0 \subseteq \mathbf{n}_1 V_1 \).

**Proof.** Suppose that \( x_i \) cannot be chosen so that the summands \( L_1 \) and \( L_0 \) of the \( x_i \)-compression satisfy \( L_0 \subseteq \mathbf{n}_1 L_1 \). Then Lemma 4.9 applies for all \( x_i \), so \( V \) satisfies the property:

Let \( m \in V \) be a monomial, and suppose that \( x_i \) divides \( m \) and \( x_j \) does not. Then \( \frac{x_j}{x_i} m \in V \) as well.

The only subspaces of \( R_d \) satisfying this property are \((0)\) and \( R_d \). In either case, we have \( L_0 \subseteq \mathbf{n}_1 L_1 \) for any \( x_i \).

Thus, \( x_i \) may be chosen such that \( L_0 \subseteq \mathbf{n}_1 L_1 \). Then by Lemma 4.8 \( V_1 \) and \( V_0 \) are Gotzmann in \( Q \). Applying (\( \star \)), we have \(|V_0 + \mathbf{n}_1 V_1| = |L_0 + \mathbf{n}_1 L_1| = |\mathbf{n}_1 L_1| = |\mathbf{n}_1 V_1|\), i.e., \( V_0 \subseteq \mathbf{n}_1 V_1 \). \( \square \)

In fact, the obvious choice of variable works:

**Lemma 4.11.** Suppose \( V \subset R_d \) is a Gotzmann monomial vector space, and let \( x_i \) be such that \(|V \cap \langle x_i \rangle|\) is maximal. Let \( V = V_0 \oplus x_i V_1 \) be the \( x_i \)-decomposition of \( V \). Then \( V_0 \) and \( V_1 \) are both Gotzmann in \( Q \) and \( V_0 \subseteq \mathbf{n}_1 V_1 \).

**Proof.** Let \( L_0 \) and \( L_1 \) be the lexifications in \( Q \) of \( V_0 \) and \( V_1 \), respectively.

By Theorem 4.10, there exists a variable \( x_j \) such that we may decompose \( V = W_0 \oplus x_j W_1 \) with both \( W_0 \) and \( W_1 \) Gotzmann in \( Q \) and \( W_0 \subseteq \mathbf{n}_1 W_1 \).

We have

\[
|L_0| \leq |W_0| \leq |\mathbf{n}_1 W_1| \leq |\mathbf{n}_1 L_1|,
\]

the first inequality by construction, the second by Theorem 4.10 and the third because \( |W_1| \leq |L_1| \) and both are Gotzmann. By Lemma 4.8 \( V_1 \) is Gotzmann. Applying (\( \star \)) again, we obtain \( V_0 \subseteq \mathbf{n}_1 V_1 \). \( \square \)
Unfortunately the converse to Theorem 4.10 does not hold in general. For example, let \( V = \text{span}_k \{ ab, ac, bc \} \) in \( R = k[a, b, c, d]/(a^2, \ldots, d^2) \). Then \( V \) is not Gotzmann in \( R \) but, decomposing with respect to \( a \), \( V_0 = \text{span}_k \{ bc \} \) and \( V_1 = \text{span}_k \{ b, c \} \) are both Gotzmann in \( Q = k[b, c, d]/(b^2, c^2, d^2) \).

However, we can prove the following partial converse.

**Theorem 4.12.** Let \( V_0 \) and \( V_1 \) be Gotzmann monomial vector spaces in \( Q \) with \( V_0 = n_1V_1 \). Then \( V = V_0 \oplus n_1V_1 \) is Gotzmann in \( R \).

*Proof.* Choose any lex order in which \( x_i \) comes last, and let \( L = L_0 + x_iL_1 \) be the \( x_i \)-compression of \( V \). We have \( |m_1V| = |n_1V_0| + |V_0 + n_1V_1| = |n_1V_0| + |V_0| = |n_1L_0| + |L_0| = |n_1L_0| + |L_0 + n_1L_1| = |m_1L| \). Thus, it suffices to show that \( L \) is lex.

Indeed, suppose that \( u \in L \) and \( v \) is a monomial of the same degree which precedes \( u \) in the lex order. If both or neither of \( u, v \) are divisible by \( x_i \), then clearly \( v \in L \). Now suppose that \( u \) is divisible by \( x_i \) but \( v \) is not. Then we may write \( u = u'x_i \). By construction, \( v \) precedes \( u' \) in the (ungraded) lex order. Let \( v' = \frac{v}{x_j} \), where \( x_j \) is the lex-last variable dividing \( v \). Then \( v' \) precedes \( u' \) in the lex order as well, so \( u' \in L_1 \) implies \( v' \in L_1 \) and in particular \( v \in n_1L_1 = L_0 \). A similar argument shows that \( v \in L \) if \( v \) is divisible by \( x_i \) but \( u \) is not. \[\square\]

**Example 4.13.** Consider the Gotzmann vector space

\[ V_1 = \text{span}_k \{ ab, bc, cd, ad \} \]

in \( Q = k[a, b, c, d]/(a^2, \ldots, d^2) \). Let \( V_0 = n_1V_1: \)

\[ V_0 = \text{span}_k \{ abc, abd, acd, bcd \}. \]

In \( R = k[a, b, c, d, e]/(a^2, \ldots, e^2) \), the monomial vector space \( V = V_0 + eV_1 \) is Gotzmann but is not lex with respect to any order of the variables.

4.2. **Alexander Duality.** Recall that for a monomial vector space \( V \subseteq R_d \), the Alexander dual of \( V \) is the subspace \( V^\vee \subseteq R_{n-d} \) spanned by the monomials \( \{ \frac{x^m}{m} : m \notin V \} \) where \( x \) is the product of all the variables. For a monomial ideal \( I \subseteq R \), the Alexander dual is \( I^\vee = \oplus(I_d)^\vee \). This duality corresponds to topological Alexander duality under the Stanley-Reisner correspondence, and turns out to have many nice algebraic properties. For example, duality turns generators into associated primes, and the duals of lex or Borel ideals are always lex or Borel, respectively. Thus, we would like to understand ideals whose duals are Gotzmann.

**Definition 4.14.** We say that a monomial vector space \( V \) is Nnamztog if \( V^\vee \) is Gotzmann.

**Theorem 4.15.** Let \( V \) be Nnamztog in \( R \). Then \( x_i \) may be chosen so that both summands \( V_0 \) and \( V_1 \) of the \( x_i \)-decomposition are Nnamztog in \( Q \), and \( (V_0 : n_1) \subseteq V_1 \).

*Proof.* Let \( W = V^\vee \). Then Theorem 4.10 applies to \( W \), so we may choose \( x_i \) such that \( W_0 \) and \( W_1 \) are Gotzmann in \( Q \) and \( W_0 \subseteq n_1W_1 \).

We compute \( V_0 = (W_1)^\vee \) and \( V_1 = (W_0)^\vee \). In particular, \( V_0 \) and \( V_1 \) are Nnamztog. Finally, suppose that \( m \in (V_0 : n_1) \). We will show that \( m \in V_1 \). By construction,
Thus, any recursive enumeration of Nnamztog ideals should look similar to any recursive enumeration of Gotzmann ideals. However, they will not be identical. In fact, ideals which are simultaneously Gotzmann and Nnamztog are quite rare, as the next theorem shows.

**Theorem 4.16.** Suppose that \( V \subset R_d \) is both Gotzmann and Nnamztog. Then \( V \) is lex in some order.

**Proof.** Suppose not. Then there exists a counterexample \( V \subset R_d \) where \( R = k[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2) \) with \( n \) minimal. Let \( x_i \) be such that \( |V \cap (x_i)| \) is maximal. Then \( |V^\vee \cap (x_i)| \) is maximal as well, and Lemma 4.11 applies to both \( V \) and \( V^\vee \). Thus \( V_0 \) and \( V_1 \) are both Gotzmann and Nnamztog, so, by the minimality of \( n \), both are lex in \( Q \). Since \( V \) is not lex, we have \( V_0 \neq 0 \) and \( V_1 \neq Q_{d-1} \). Since \( V_0 \neq 0 \), we have \( m_{n-d-1}V = R_{n-1} \). Thus the lexification of \( V \) (in any order where \( x_i \) comes first) must contain at least one monomial not divisible by \( x_i \). Similarly, the lexification of \( V^\vee \) must contain at least one monomial not divisible by \( x_i \). Thus, if \( L \) and \( L^\vee \) are the lexifications of \( V \) and \( V^\vee \), respectively, we have

\[
|L| + |L^\vee| \geq |Q_{d-1}| + 1 + |Q_{n-d-1}| + 1 \\
\geq |Q_{d-1}| + |Q_d| \\
= |R_d|.
\]

On the other hand, \( |L| + |L^\vee| = |V| + |V^\vee| = |R_d| \). Thus, such a minimal counterexample cannot exist. \( \square \)

Note that Theorem 4.16 is not a theorem about ideals. If an ideal \( I \) is both Gotzmann and Nnamztog, then Theorem 4.16 guarantees that every degreewise component \( I_d \) is lex in some order, but does not guarantee a consistent order. For example, the ideal \( I = (bc, abd, abe, acd, ace, ade) \subset k[a, b, c, d, e]/(a^2, b^2, c^2, d^2, e^2) \) is Gotzmann and Nnamztog, but is not lex in any order. The component \( I_p \) is lex with respect to the order \( a > b > c > d > e \) for \( p \neq 2 \), and with respect to the order \( b > c > a > d > e \) for \( p < 3 \), but no lex order works in both degrees two and three.

**References**

[AAH00] Annetta Aramova, Luchezar L. Avramov, and Jürgen Herzog, *Resolutions of monomial ideals and cohomology over exterior algebras*, Trans. Amer. Math. Soc. 352 (2000), no. 2, 579–594. MR 1603874 (2000c:13021)

[AHH97] Annetta Aramova, Jürgen Herzog, and Takayuki Hibi, *Gotzmann theorems for exterior algebras and combinatorics*, J. Algebra 191 (1997), no. 1, 174–211. MR 1444495 (98c:13025)

[BL05] Sergei L. Bezrukov and Uwe Leck, *Macaulay posets*, The Electronic Journal of Combinatorics DS (2005), no. 12, 1–40.

[Bo03] Vittoria Bonanzinga, *Principal Borel ideals and Gotzmann ideals*, Arch. Math. (Basel) 81 (2003), no. 4, 385–396. MR 2057059 (2005b:13041)
[En97] Konrad Engel, *Sperner theory*, Encyclopedia of Mathematics and its Applications, vol. 65, Cambridge University Press, Cambridge, 1997. MR 1429390 (98m:05187)

[FG86] Zoltan Füredi and Jerrold R. Griggs, *Families of finite sets with minimum shadows*, Combinatorica 6 (1986), no. 4, 355–363. MR 879339 (88d:05002)

[Go78] Gerd Gotzmann, *Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes*, Math. Z. 158 (1978), no. 1, 61–70. MR 0480478 (58 #641)

[Ha66] Robin Hartshorne, *Connectedness of the Hilbert scheme*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 29, 5–48. MR 0213368 (35 #4232)

[HH99] Jürgen Herzog and Takayuki Hibi, *Componentwise linear ideals*, Nagoya Math. J. 153 (1999), 141–153. MR 1684555 (2000i:13019)

[Ho09] Andrew H. Hoefel, *Gotzmann edge ideals*, preprint. arXiv:0908.1946v1

[Ka68] Gyula O. H. Katona, *A theorem of finite sets*, Theory of graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, pp. 187–207. MR 0290982 (45 #76)

[Kr63] Joseph B. Kruskal, *The number of simplices in a complex*, Mathematical optimization techniques, Univ. of California Press, Berkeley, Calif., 1963, pp. 251–278. MR 0154827 (27 #4771)

[Ma27] F. S. Macaulay, *Some properties of enumeration in the theory of modular systems*, Proc. London Math. Soc. 26 (1927), 531–555.

[Me06] Jeffrey Mermin, *Lexlike sequences*, J. Algebra 303 (2006), no. 1, 295–308. MR 2253664 (2007d:13023)

[Me08] Jeff Mermin, *Compressed ideals*, Bull. Lond. Math. Soc. 40 (2008), no. 1, 77–87. MR 2409180 (2009d:13027)

[MH08] Satoshi Murai and Takayuki Hibi, *Gotzmann ideals of the polynomial ring*, Math. Z. 260 (2008), no. 3, 629–646. MR 2434473 (2010f:13025)

[MP06] Jeffrey Mermin and Irena Peeva, *Lexifying ideals*, Math. Res. Lett. 13 (2006), no. 2-3, 409–422. MR 2231127 (2007b:13023)

[Mu07] Satoshi Murai, *Gotzmann monomial ideals*, Illinois J. Math. 51 (2007), no. 3, 843–852. MR 2379725 (2009b:13039)

[OOS08] Anda Olteanu, Oana Olteanu, and Loredana Sorrenti, *Gotzmann lexsegment ideals*, Mathematische (Catania) 63 (2008), no. 2, 229–241 (2009). MR 2531664 (2010h:13029)

[Slo3] Neil J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, published electronically at http://oeis.org, 2010. MR 1992789 (2004f:11151)

ANDREW H. HOEFEL, DEPARTMENT OF MATHEMATICS AND STATISTICS, DALHOUSSIE UNIVERSITY, HALIFAX, NOVA SCOTIA B3H 3J5

E-mail address: andrew.hoefel@mathstat.dal.ca

JEFF MERMIN, DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OK 74078

E-mail address: mermin@math.okstate.edu