REFINEMENTS OF MİTRINOVİĆ–CUSA INEQUALITY

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Abstract. The Mitrinović-Cusa inequality states that for \(x \in (0, \pi/2)\)

\((\cos x)^{1/3} < \frac{\sin x}{x} < \frac{2 + \cos x}{3}\)

hold. In this paper, we prove that

\((\cos x)^{1/3} < (\cos px)^{1/(3p^2)} < \frac{\sin x}{x} < (\cos qx)^{1/(3q^2)} < \frac{2 + \cos x}{3}\)

hold for \(x \in (0, \pi/2)\) if and only if \(p \in (p_1, 1)\) and \(q \in (0, 1/\sqrt{5}]\), where
\(p_1 = 0.45346830977067\ldots\). And the function \(p \mapsto (\cos px)^{1/(3p^2)}\) is decreasing
on \((0, 1]\). Our results greatly refine the Mitrinović-Cusa inequality.

1. Introduction

In the recent past, the following double inequality

\((1.1) (\cos x)^{1/3} < \frac{\sin x}{x} < \frac{2 + \cos x}{3} \quad (0 < x < \pi/2)\)

has attracted the attention of many scholars.

The left hand side inequality \([1.1]\) was first proved by Mitrinović in \([10]\) (see also
\([11]\), pages 238-240]), and so we call it as Mitrinović’s inequality. While the right
hand side inequality \([1.1]\) was found by the German philosopher and theologian
Nicolaus de Cusa (1401-1464) and proved explicitly by Huygens (1629–1695) when
he approximated \(\pi\), and it is now known as Cusa’s inequality \([18]\), \([23]\), \([12]\), \([13]\), \([5]\). Hence \([1.1]\) can be called as Mitrinović-Cusa inequality.

A nice refinement of the Mitrinović-Cusa inequality \([1.1]\) appeared in \([11, 3.4.6]\).
For convenience, we record it as follows.

**Theorem M.** For \(x \in (0, \pi/2)\),

\((1.2) \cos px \leq \frac{\sin x}{x} \leq \cos qx\)

with the best possible constants

\(p = \frac{1}{\sqrt{3}}\) and \(q = \frac{2}{\pi} \arccos \frac{2}{\pi}\).

Also, the following inequalities hold:

\((1.3) \cos x \leq \frac{\cos x}{1 - x^2/3} \leq (\cos x)^{1/3} \leq \cos \frac{x}{\sqrt{3}} \leq \frac{\sin x}{x} \leq \cos qx \leq \cos \frac{x}{2} \leq 1.\)
Recently, Klén et al. [8] Theorem 2.4 showed that the function \( p \mapsto (\cos px)^{1/p} \) is decreasing on \((0, 1)\) and improved Cusa’s inequality (the right hand side inequality in (1.4)), which is stated as follows.

**Theorem K.** For \( x \in \left(-\sqrt{\frac{27}{5}}, \sqrt{\frac{27}{5}}\right) \)

\[
\cos^2 \frac{x}{2} \leq \frac{\sin x}{x} \leq \cos^3 \frac{x}{3} \leq \frac{2 + \cos x}{3}.
\]

The following sharp bounds for \((\sin x)/x\) due to Lv et al. [9] give another refinement of the Mitrinović’s inequality.

**Theorem L.** For \( x \in (0, \pi/2) \) inequalities

\[
(\cos \frac{x}{2})^{4/3} < \frac{\sin x}{x} < (\cos \frac{x}{2})^\theta
\]

hold, where \( \theta = 2 (\ln \pi - \ln 2) / \ln 2 = 1.3030... \) and \( 4/3 \) are the best possible constants.

Other results involving Mitrinović’s and Cusa’s inequality can be found in [7], [13], [23], [22], [16], [13], [5], [12], [14] and related references therein.

This paper is motivated by these studies and is aimed at giving sharp bounds \((\cos px)^{1/(3p^2)}\) for \((\sin x)/x\) to establish interpolated inequalities of (1.4), that is, for \( x \in (0, \pi/2)\), determine the best \( p, q \in (0, 1) \) such that

\[
(\cos x)^{1/3} < (\cos px)^{1/(3p^2)} < \frac{\sin x}{x} < (\cos qx)^{1/(3q^2)} < \frac{2 + \cos x}{3}
\]

hold.

The organization of this paper is as follows. Some useful lemmas are given in section 2. In section 3, the sharp bounds \((\cos px)^{1/(3p^2)}\) for \((\sin x)/x\) and its relative error estimates are established. In the last section, some precise estimates for certain integrals are presented.

2. **Lemmas**

**Lemma 1 ([19], [1]).** Let \( f, g : [a, b] \to \mathbb{R} \) be two continuous functions which are differentiable on \((a, b)\). Further, let \( g' \neq 0 \) on \((a, b)\). If \( f'/g' \) is increasing (or decreasing) on \((a, b)\), then so are the functions

\[
x \mapsto \frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad x \mapsto \frac{f(x) - f(b)}{g(x) - g(b)}.
\]

**Lemma 2 ([2]).** Let \( a_n \) and \( b_n \) \((n = 0, 1, 2, \ldots)\) be real numbers and let the power series \( A(t) = \sum_{n=1}^\infty a_n t^n \) and \( B(t) = \sum_{n=1}^\infty b_n t^n \) be convergent for \(|t| < R\). If \( b_n > 0 \) for \( n = 0, 1, 2, \ldots \), and \( a_n/b_n \) is strictly increasing (or decreasing) for \( n = 0, 1, 2, \ldots \), then the function \( A(t)/B(t) \) is strictly increasing (or decreasing) on \((0, R)\).

**Lemma 3 ([6] pp.227-229]).** We have

\[
\cot x = \frac{1}{x} - \sum_{n=1}^\infty \frac{2^{2n}}{(2n)!} B_{2n} x^{2n-1}, \quad |x| < \pi,
\]

\[
\tan x = \sum_{n=1}^\infty \frac{2^{2n-1} - 1}{(2n)!} 2^{2n} B_{2n} x^{2n-1}, \quad |x| < \pi/2,
\]

\[
\frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{n=1}^\infty \frac{(2n - 1) 2^{2n}}{(2n)!} B_{2n} x^{2n-2}, \quad |x| < \pi,
\]
where $B_n$ is the Bernoulli numbers.

**Lemma 4.** Let $F_p$ be the function defined $(0, \pi/2)$ by

$$(2.4) \quad F_p(x) = \frac{\ln \sin x}{\ln \cos px}.$$  

Then $F_p$ is strictly increasing on $(0, \pi/2)$ if $p \in (0, \sqrt{5}/5]$ and decreasing on $(0, \pi/2)$ if $p \in [1/2, 1]$. Moreover, we have

$$(2.5) \quad \frac{\ln 2 - \ln x}{\ln(\cos \frac{\pi}{4}p)} \ln \cos px < \ln \frac{\sin x}{x} < \frac{1}{3p} \ln \cos px$$

if $p \in (0, \sqrt{5}/5]$. The inequalities (2.5) are reversed if $p \in [1/2, 1]$.

**Proof.** For $x \in (0, \pi/2)$, we define $f(x) = \ln \frac{\sin x}{x}$ and $g(x) = \ln \cos px$, where $p \in (0, 1]$. Note that $f(0^+) = g(0^+) = 0$, then $F_p(x)$ can be written as

$$F_p(x) = \frac{f(x) - f(0^+)}{g(x) - g(0^+)}.$$

Differentiation and using (2.1) and (2.2) yield

$$f'(x) = \frac{p(\frac{1}{x} - \cot x)}{\tan px} = \frac{\sum_{n=1}^{\infty} \frac{2^n}{(2n)!} B_{2n} |x^{2n-1}|}{\sum_{n=1}^{\infty} \frac{2^{n-1}}{(2n)!} p^{2n-2} 2^n |B_{2n}| x^{2n-1}} := \frac{\sum_{n=1}^{\infty} a_n x^{2n-1}}{\sum_{n=1}^{\infty} b_n x^{2n-1}},$$

where

$$a_n = \frac{2^{2n}}{(2n)!} |B_{2n}|, \quad b_n = \frac{2^{2n} - 1}{(2n)!} p^{2n-2} 2^n |B_{2n}|.$$

Clearly, if the monotonicity of $a_n/b_n$ is proved, then by Lemma 2 it is deduced the monotonicity of $f'/g'$, and then the monotonicity of the function $F_p$ easily follows from Lemma 1. Now we prove the monotonicity of $a_n/b_n$. Indeed, elementary computation yields

$$\frac{b_{n+1}}{a_{n+1}} \frac{b_n}{a_n} = \frac{(2^{2n+2} - 1) p^{2n} - (2^{2n} - 1) p^{2n-2}}{(4^{n+1} - 1) p^{2n-2} \left(p^2 - \frac{1}{4} + \frac{3}{4 (4^{n+1} - 1)}\right)},$$

from which it is easy to obtain that for $n \in \mathbb{N}$

$$\frac{b_{n+1}}{a_{n+1}} \frac{b_n}{a_n} \begin{cases} \leq 0 & \text{if } p^2 < \frac{1}{4}, \\ > & \text{if } p^2 \geq \frac{1}{4}. \end{cases}$$

It is seen that $b_n/a_n$ is decreasing if $0 < p \leq \sqrt{5}/5$ and increasing if $1/2 \leq p \leq 1$, which together with $a_n, b_n > 0$ for $n \in \mathbb{N}$ leads to $a_n/b_n$ is strictly increasing if $0 < p \leq \sqrt{5}/5$ and decreasing if $1/2 \leq p \leq 1$.

By the monotonicity of the function $F_p$ and notice that

$$F_p(0^+) = \frac{1}{3p} \quad \text{and} \quad F_p(\frac{\pi}{2}) = \frac{\ln 2 - \ln \pi}{\ln(\cos \frac{\pi}{4}p)},$$

the inequalities (2.5) follow immediately. \endproof

**Remark 1.** Lemma 4 contains many useful and interesting inequalities for trigonometric functions. For example, put $p = 1/\sqrt{3}$,  $\frac{\pi}{2} \arccos \frac{2}{3} \in [1/2, 1]$ in (2.5) yield the second and first inequality of (1.2), respectively: put $p = 1/2 \in [1/2, 1]$ leads to (1.3). Similarly, by virtue of Lemma 4 we will easily prove our most main results in the sequel.
Lemma 5. For \( x \in (0, \pi/2) \), let the function \( U : (0, 1] \mapsto (-\infty, 0) \) be defined by

\[
U(p) = \frac{1}{3p^2} \ln \cos px.
\]

Then \( U \) is decreasing on \((0, 1]\) with the limit \( U(0^+) = -x^2/6 \).

Proof. Differentiation yields

\[
3p^3U'(p) = -2 \ln (\cos px) - \frac{p x \sin px}{\cos px} := V(p),
\]

\[
V'(p) = \frac{x}{2 \cos^2 px} (\sin 2px - 2px) < 0.
\]

It follows that \( V(p) < V(0) = 0 \), and therefore \( U'(p) > 0 \), that is, \( U \) is decreasing on \((0, 1]\).

Simple computation leads to \( U(0^+) = -x^2/6 \).

Thus the proof ends. \( \square \)

Lemma 6. For \( p \in (0, 1] \), let the function \( f_p \) be defined on \((0, \pi/2)\) by

\[
f_p(x) := \ln \frac{\sin x}{x} - \frac{1}{3p^2} \ln \cos px.
\]

(i) If \( f_p(x) < 0 \) holds for all \( x \in (0, \pi/2) \) then \( p \in (0, \sqrt{5}/5] \).

(ii) If \( f_p(x) > 0 \) for all \( x \in (0, \pi/2) \), then \( p \in [p_1, 1] \), where \( p_1 = 0.45346830977067... \) is the unique root of equation

\[
f_p\left(\frac{\pi}{2}\right) = \ln \frac{2}{\pi} - \frac{1}{3p^2} \ln \cos \frac{p\pi}{2} = 0
\]

on \((0, 1]\).

Proof. At first, We assert that there is a unique \( p_1 \in (0, 1) \) to satisfy equation \( 2.8 \) such that \( f_p\left(\frac{\pi}{2}\right) < 0 \) for \( p \in (0, p_1) \) and \( f_p\left(\frac{\pi}{2}\right) > 0 \) for \( p \in (p_1, 1] \).

In fact, Lemma 5 indicates that \( U \) is decreasing on \((0, 1)\), and so \( p \mapsto f_p\left(\frac{\pi}{2}\right) \) is increasing on \((0, 1)\). Since

\[
f_{1/3}\left(\frac{\pi}{2}\right) = \ln \frac{2}{\pi} - 3 \ln \frac{\sqrt{5}}{2} < 0,
\]

\[
f_{1/2}\left(\frac{\pi}{2}\right) = \ln \frac{2}{\pi} - \frac{4}{3} \ln \frac{\sqrt{5}}{2} > 0,
\]

so the equation \( 2.8 \) has a unique solution \( p_1 \) on \((0, 1)\) and \( p_1 \in (1/3, 1/2) \) such that \( f_p\left(\frac{\pi}{2}\right) < 0 \) for \( p \in (0, p_1) \) and \( f_p\left(\frac{\pi}{2}\right) > 0 \) for \( p \in (p_1, 1] \). Numerical calculation yields \( p_1 = 0.45346830977067... \).

Now, if inequality \( f_p(x) < 0 \) holds for \( x \in (0, \pi/2) \), then we have

\[
\left\{ \begin{array}{l}
\lim_{x \to 0^+} \frac{f_p(x)}{x^2} = \lim_{x \to 0^+} \frac{\ln \sin px - \frac{1}{3p^2} \ln \cos px}{x^2} = \frac{1}{3p^2} \ln \cos \frac{p\pi}{2} \leq 0, \\
\frac{f_p\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \ln \frac{2}{\pi} - \frac{1}{3p^2} \ln \cos \frac{p\pi}{2} \leq 0.
\end{array} \right.
\]

Solving the inequalities for \( p \) yields

\( p \in (0, \sqrt{5}/5] \cap (0, p_1] = (0, \sqrt{5}/5] \).

In the same way, if inequality \( f_p(x) > 0 \) holds for all \( x \in (0, \pi/2) \), then

\( p \in \left[\frac{\sqrt{5}}{5}, 1\right] \cap [p_1, 1] = [p_1, 1], \)

which completes the proof. \( \square \)
3. Main Results

Now we state and prove the sharp upper bound \((\cos px)^{1/(3p^2)}\) for \((\sin x)/x\).

**Theorem 1.** For \(p \in (0, 1]\), the inequality

\[
(3.1) \quad \frac{\sin x}{x} < (\cos px)^{1/(3p^2)}
\]

holds for all \(x \in (0, \pi/2)\) if and only if \(p \in (0, \sqrt{5}/5]\). Moreover, we have

\[
(3.2) \quad \left(\cos \frac{x}{\sqrt{5}}\right)^\alpha < \frac{\sin x}{x} < \left(\cos \frac{x}{\sqrt{5}}\right)^{5/3},
\]

where \(\alpha = (\ln 2) / \ln \left(\cos \frac{\sqrt{5}x}{10}\right) = 1.6714... and 5/3 = 1.6667... are the best possible constants.

**Proof.** From Lemma \(\square\) the necessity follows. The second inequality of (2.5) implies that the condition \(p \in (0, \sqrt{5}/5]\) is sufficient.

Put \(p = \sqrt{5}/5\) in (2.5) yields (3.1).

Thus the proof is completed. \(\square\)

From the corollary, in order to prove the last inequality in (1.6), it suffices to compare \(e^{-x^2/6}\) with \((2 + \cos x)/3\). We have

**Theorem 2.** The inequality

\[
(3.3) \quad e^{-x^2/6} < \frac{2 + \cos x}{3}
\]

holds for \(x \in (0, \infty)\). Moreover, for \(x \in (0, a)\) (\(a > 0\)) we have

\[
(3.4) \quad \frac{2 + \cos x}{(2 + \cos a)e^{a^2/6}} < e^{-x^2/6} < \frac{2 + \cos x}{3}.
\]

**Proof.** Considering the function \(g\) defined by

\[
g(x) = \ln \frac{2 + \cos x}{3} + \frac{x^2}{6},
\]

and differentiation yields

\[
(3.5) \quad g'(x) = \frac{x}{3} - \frac{\sin x}{\cos x + 2},
\]

\[
g''(x) = \frac{1}{3} \frac{(\cos x - 1)^2}{(\cos x + 2)^2} \geq 0,
\]

which implies that for \(x \in (0, \infty)\), \(g'(x) > g'(0^+) = 0\), then, \(g'(x) > 0\), that is, \(g\) is increasing on \((0, \infty)\). Hence, we have \(g(x) > g(0^+) = 0\) for \(x \in (0, \infty)\), that is, \((3.3)\) is true.

For \(x \in (0, a)\) we have

\[
0 = g(0^+) < g(x) < g(a) = \ln \left(\frac{2 + \cos a}{3}e^{a^2/6}\right),
\]

which proves (3.4). \(\square\)

Next we establish the sharp lower bound for \((\sin x)/x\).
Theorem 3. Let \( p \in (0, 1] \). Then the inequality
\[
\frac{\sin x}{x} > (\cos px)^{1/(3p^2)}
\]
holds for all \( x \in (0, \pi/2) \) if and only if \( p \in [p_1, 1] \), where \( p_1 = 0.45346830977067... \)
is the unique root of equation (2.3) in \( p \in (0, 1) \). Moreover, we have
\[
(\cos p_1 x)^{1/(3p_1^2)} < \frac{\sin x}{x} < \beta (\cos p_1 x)^{1/(3p_1^2)},
\]
where 1 and \( \beta \approx 1.0002 \) are the best possible constants.

Proof. Necessity. Lemma 6 implies necessity.

Sufficiency. Due to Lemma 5, it suffices to show that \( f_{p_1}(x) > 0 \) for all \( x \in (0, \pi/2) \), where \( f_p \) is defined by (2.7). To this end, we introduce an auxiliary function \( h \) defined on \( (0, \pi/2) \) by
\[
h(x) = \frac{f'_{p_1}(x)}{x^4} = \frac{\left(\cot x - \frac{1}{x}\right) + \frac{1}{3p_1^2} \tan p_1 x}{x^4}.
\]
We will show that \( h \) is decreasing on \( (0, \pi/2) \).

Differentiation and simplifying yield
\[
x^4 h'(x) = \frac{4}{3} \sum_{n=1}^{\infty} \frac{(2n-1)x^{2n}}{2n-1} |B_{2n}| (2p_1)^{2n-2} x^{2n-1} - \frac{4}{3} \sum_{n=1}^{\infty} \frac{(2n-1)x^{2n}}{2n-1} |B_{2n}| p_1^{2n-2} x^{2n-1}
\]
\[= \sum_{n=1}^{\infty} \frac{2n|B_{2n}| x^{2n-1}}{3(2n)!} u_n x^{2n-1},
\]
where
\[u_n = (2^{2n-1} (2n-10) p_1^{2n-2} - 3 (2n-1)).\]
Clearly, \( u_n < 0 \) for \( n = 1, 2, 3, 4, 5 \). We now show that \( u_n < 0 \) for \( n \geq 6 \). For this purpose, it needs to prove that for \( n \geq 6 \)
\[p_1 < \left( \frac{3 (2n-1)}{(2^{2n-1} (2n-10))} \right)^{1/(n-6)} := h_1(n).
\]
Since \( (2n-1) > (2n-10) \), we have
\[h_1(n) > \left( \frac{3}{2^{2n-1}} \right)^{1/(2n-2)} := k(n).
\]
Considering the function \( k : (1, \infty) \to (0, \infty) \) defined by
\[
k(x) = \left( \frac{3}{2^{x-1}} \right)^{1/(2x-2)},
\]
and differentiation leads to
\[
\frac{2(x-1)^2}{k(x)} k'(x) = \ln(2^{2x} - 1) - \ln 3 - 2 \ln 2 \frac{(x-1)^{2x}}{2^{2x} - 1} := k_1(x),
\]
\[
k'_1(x) = \frac{2^{2x+2} \ln^2 2}{(2^{2x} - 1)^2} (x-1),
\]
which reveals that \( k_1 \) is increasing on \((1, \infty)\), and so \( k_1(x) > k_1(1^+) = 0 \), then \( k'(x) > 0 \), that is, \( k \) is increasing on \((1, \infty)\). Therefore for \( n \geq 6 \)
\[0.485 \, 83 \approx 1365^{-1/10} = k(6) \leq k(n) < k(\infty) = \frac{1}{2}.
\]It follows that for \( n \geq 6 \)
\[h_1(n) > k(n) > 0.485 \, 83 > p_1,
\]
which indicates that \( u_n \) is a unique \( x \) for \((0, \pi/2)\), which is increasing on \((0, \pi/2)\).

On the other hand, it is clear that
\[h(0^+) = \lim_{x \to 0^+} \left( \frac{\cot x - \frac{1}{x} + \frac{1}{3p_1} \tan p_1 x}{x^3} \right) = \frac{1}{9} \left( \frac{p_1^2 - 1}{\sqrt{3}} \right) > 0.
\]
And we claim that \( h\left(\frac{\pi}{2}^-\right) < 0 \). If \( h\left(\frac{\pi}{2}^+\right) \geq 0 \), then there must be \( h(x) > 0 \) for all \( x \in (0, \pi/2) \), which, by \((3.8)\), implies that \( f_{p_1}'(x) > 0 \), then \( f_{p_1} \) is increasing on \((0, \pi/2)\). It yields
\[f_{p_1}(x) > f_{p_1}\left(0^+\right) = 0 \text{ and } f_{p_1}(x) < f_{p_1}\left(\frac{\pi}{2}^-\right) = \ln \frac{\pi}{2} - \frac{1}{3p_1} \ln (\cos \frac{\pi}{2}p_1 \pi) = 0,
\]
which is a contradiction. Consequently, \( h(0^+) > 0 \) and \( h\left(\frac{\pi}{2}^-\right) < 0 \).

Make use of the monotonicity of the auxiliary function \( h \) it is showed that there is a unique \( x_0 \in (0, \pi/2) \) to satisfy \( h(x_0) = 0 \) such that \( h(x) > 0 \) for \( x \in (0, x_0) \) and \( h(x) < 0 \) for \( x \in (x_0, \pi/2) \). Then, by \((3.8)\), it is seen that \( f_{p_1} \) is increasing on \((0, x_0)\) and decreasing on \((x_0, \pi/2)\). It is concluded that
\[0 = f_{p_1}\left(0^+\right) < f_{p_1}(x) < f_{p_1}(x_0) \text{ for } x \in (0, x_0),
\]
\[0 = f_{p_1}\left(\frac{\pi}{2}^-\right) < f_{p_1}(x) < f_{p_1}(x_0) \text{ for } x \in (x_0, \pi/2),
\]
that is, \( 0 < f_{p_1}(x) < f_{p_1}(x_0) \) for \( x \in (0, \pi/2) \).

Solving the equation \( h(x) = 0 \) which is equivalent with
\[f_{p_1}'(x) = \left( \cot x - \frac{1}{x} \right) + \frac{1}{3p_1} \tan p_1 x = 0
\]
by using mathematical computer software, we find that
\[x_0 \in (1.31187873615727632, 1.31187873615727633),
\]
and \( \beta = \exp(f_{p_1}(x_0))) \approx 1.0002 \), which proves the sufficiency and \((3.7)\).

Letting \( p = \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{\sqrt{3}}{2}, 1 \) in Theorem \((3)\) and \( p = \frac{1}{\sqrt{6}}, \frac{1}{3}, \frac{1}{2\sqrt{3}}, \frac{1}{4}, \ldots, \to 0 \) in Theorem \((1)\) together with Theorem \((2)\) we have
Corollary 1. For \( x \in (0, \pi/2) \), we have

\[
\cos^{1/3} x < \cdots < \left( \cos \frac{\sqrt{5}x}{3} \right)^{1/2} < \left( \cos \frac{x}{\sqrt{3}} \right)^{2/3} < \cos \frac{x}{\sqrt{3}} < \left( \cos \frac{\pi}{3} \right)^{1/3}
\]

\[
< \left( \cos p_1 x \right)^{1/(3p_1^2)} < \frac{\sin x}{x} < \left( \cos \frac{x}{\sqrt{5}} \right)^{5/3} < \left( \cos \frac{x}{\sqrt{5}} \right)^2 < \left( \cos \frac{\pi}{3} \right)^3
\]

where \( p_1 = 0.45346830977067 \).

Thus it can be seen that our results greatly refine Milinović-Cusa inequality \((1.1)\).

The following give a relative error estimating \( (\sin x) / x \) by \( (\cos px)^{1/(3p^2)} \).

**Theorem 4.** For \( p \in (0, 1) \), let \( f_p \) be defined on \( (0, \pi/2) \) by \((2.7)\). Then \( f_p \) is decreasing if \( p \in (0, \sqrt{5}/5) \) and increasing if \( p \in [1/2, 1] \).

Moreover, if \( p \in (0, \sqrt{5}/5) \) then for \( x \in (0, c) \) with \( c \in (0, \pi/2) \)

\[
\gamma_p (c) \cos px^{1/(3p^2)} < \frac{\sin x}{x} < (\cos px)^{1/(3p^2)}
\]

with the best possible constants \( \gamma_p (c) = c^{-1} (\sin c) (\cos pc)^{1/(3p^2)} \) and 1. The inequalities \((3.10)\) are reversed if \( p \in [1/2, 1] \).

**Proof.** Differentiation and using \((2.1)\) and \((2.2)\) yield

\[
f'_p (x) = \left( \cot x - \frac{1}{x} \right) + \frac{1}{3p} \tan px
\]

\[
= -\sum_{n=1}^{\infty} \frac{22^n}{(2n)!} |B_{2n}| x^{2n-1} + \frac{1}{3} \sum_{n=1}^{\infty} \frac{22^n - 1}{(2n)!} p^{2n-2} 22^n |B_{2n}| x^{2n-1}
\]

\[
= \sum_{n=1}^{\infty} \frac{(22^n - 1) 22^n}{3 (2n)!} |B_{2n}| \left( p^{2n-2} - \frac{3}{2^{2n-1}} \right) x^{2n-1} = \sum_{n=2}^{\infty} s_n t_n x^{2n-1},
\]

where

\[
s_n = \frac{(22^n - 1) 22^n |B_{2n}|}{3 (2n)!} p^{2n-2} \frac{3}{2^{2n-1}} t_n > 0,
\]

\[
t_n = p - k (n)
\]

for \( n \geq 2 \) and \( p \in (0, 1) \), where the function \( k \) is defined by \((3.9)\). As showed in the proof of Theorem 3 \( k \) is increasing on \((1, \infty)\), and so for \( n \geq 2 \)

\[
1/\sqrt{5} \leq k (2) \leq k (n) \leq k (\infty) = \lim_{n \rightarrow \infty} \left( \frac{3}{2^{2n-1}} \right)^{1/(2n-2)} = \frac{1}{2}
\]

and then, \( t_n = p - k (n) \leq 0 \) if \( p \in (0, \sqrt{5}/5) \) and \( t_n = p - k (n) \geq 0 \) if \( p \in [1/2, 1] \). Thus, if \( p \in (0, \sqrt{5}/5) \) then \( f'_p (x) < 0 \), that is, \( f_p \) is decreasing, and it is derived that for \( x \in (0, c) \) with \( c \in (0, \pi/2) \)

\[
\ln \gamma_p (c) = f_p (a) < f_p (x) < \lim_{x \rightarrow 0^+} f_p (x) = 0,
\]

which yields \((3.10)\).

Likewise, if \( p \in [1/2, 1] \) then \( f'_p (x) > 0 \), then, \( f_p \) is increasing, and \((3.10)\) is reversed, which completes the proof. \( \square \)
Letting \( c \to \frac{\pi}{2} \) and putting \( p = \sqrt{5}/5, \sqrt{6}/6, 1/3, 0^+ \) in Theorem 4, we get

**Corollary 2.** The following inequalities

\[
\begin{align*}
\gamma_{1/\sqrt{5}} \left( x^3 \right) \left( \cos \frac{2x}{\sqrt{5}} \right) &< \frac{\sin x}{x} < \left( \cos \frac{2x}{\sqrt{5}} \right)^{5/3}, \\
\gamma_{1/\sqrt{6}} \left( x^2 \right) \left( \cos \frac{2x}{\sqrt{6}} \right)^2 &< \frac{\sin x}{x} < \left( \cos \frac{2x}{\sqrt{6}} \right)^2, \\
\gamma_{1/3} \left( x^3 \right) \left( \cos \frac{2x}{3} \right) &< \frac{\sin x}{x} < \left( \cos \frac{2x}{3} \right)^3, \\
\gamma_{0+} \left( x^2 \right) e^{-x^2/6} &< \frac{\sin x}{x} < e^{-x^2/6}
\end{align*}
\]

hold true for \( x \in (0, \pi/2) \), where \( \gamma_{1/\sqrt{5}} (\pi/2) = 0.99872..., \gamma_{1/\sqrt{6}} (\pi/2) = 0.99141..., \gamma_{1/3} (\pi/2) = 16\sqrt{3}/(9\pi), \gamma_{0+} (\pi/2) = 2e^{\pi^2/24}/\pi \) are the best possible constants.

Letting \( c \to \pi/2 \) and putting \( p = 1/2 \) in Theorem 4, we obtain

**Corollary 3.** For \( x \in (0, \pi/2) \), the double inequality

\[
\left( \cos \frac{x}{2} \right)^{4/3} < \frac{\sin x}{x} < \gamma_{1/2} \left( \frac{x}{2} \right) \left( \cos \frac{x}{2} \right)^{4/3}
\]

holds, where 1 and \( \gamma_{1/2} (\pi/2) = 2^{5/3}/\pi = 1.0106... \) are the best constants.

**Remark 2.** Note that the first inequality of (3.15) also holds for \( x \in (0, \pi) \), Indeed, differentiation yields

\[
\frac{x \sin x}{\cos x + 2} f'_{1/2} (x) = \frac{x}{3} - \frac{\sin x}{\cos x + 2} = g' (x).
\]

From the proof of Theorem 4, we see that for \( x \in (0, \infty), g' (x) > 0 \), which yields for \( x \in (0, \pi) \), \( f'_{1/2} (x) > 0 \), and then \( f_{1/2} (x) > f_{1/2} (0^+) = 0 \), that is, the first inequality of (3.15) holds for \( x \in (0, \pi) \).

4. Applications

As simple applications of main results, we will present some precise estimates for certain integrals in this section. The following is a direct corollary of Theorem 4

**Application 1.** We have

\[
\frac{1}{p} \left( \frac{2}{p} \right)^{3p^2} \tan \frac{p\pi}{2} \leq \int_0^{\pi/2} \left( \frac{\sin x}{x} \right)^{3p^2} dx < \int_0^{\pi/2} (\cos px) = \frac{1}{p} \sin \frac{p\pi}{2}
\]

if \( p \in (0, \sqrt{5}/5) \). Inequalities (4.1) is reversed if \( p \in [1/2, 1] \).

By integrating both sides of (3.14) over \([0, a]\) and simple computation, we have

**Application 2.** For \( a > 0 \) the following inequalities

\[
f(a) = \frac{2a + \sin a}{(2 + \cos a) e^{x^2/6}} < \int_0^a e^{-x^2/6} dx < \frac{2a + \sin a}{3}
\]
are valid. Particularly, we have

\[
\frac{\pi + 1}{2e^{\pi^2/24}} < \int_0^{\pi/2} e^{-x^2/6} dx < \frac{\pi + 1}{3},
\]

\[
\frac{(4 - \sqrt{2}) (\pi + \sqrt{2})}{14e^{\pi^2/90}} < \int_0^{\pi/4} e^{-x^2/6} dx < \frac{\pi + \sqrt{2}}{6}.
\]

For the estimate for the sine integral defined by

\[\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt,\]

there has some results, for example, Qi \[15\] showed that

\[1.3333... = \frac{4}{3} < \text{Si} \left( \frac{\pi}{2} \right) < \frac{\pi + 1}{3} = 1.3805...;\]

the following two estimations are due to Wu \[20], \[21]:

\[1.3569... = \frac{\pi + 5}{6} < \text{Si} \left( \frac{\pi}{2} \right) < \frac{\pi + 1}{3} = 1.3805...;\]

\[1.3688... = \frac{92 - \pi^2}{60} < \text{Si} \left( \frac{\pi}{2} \right) < \frac{8 + 4\pi}{15} = 1.3711...;\]

Now we give a more better one.

**Application 3.** We have

\[
\frac{\sqrt{3}}{4} \pi < \int_0^{\pi/2} \frac{\sin x}{x} dx < \frac{7}{16} \pi
\]

**Proof.** By Corollary 1 we see that the inequalities

\[
\cos \frac{x}{\sqrt{3}} < \frac{\sin x}{x} < \cos \frac{x}{\sqrt{6}}
\]

hold for \(x \in [0, \pi/2]\). Integrating both sides over \([0, \pi/2]\) and simple calculation yield

\[
\sqrt{3} \sin \frac{\pi}{2\sqrt{3}} < \int_0^{\pi/2} \frac{\sin x}{x} dx < \frac{\pi}{4} + \frac{\sqrt{6}}{4} \sin \frac{\pi}{\sqrt{6}}.
\]

Using (4.4) again gives

\[
\sin \frac{\pi}{2\sqrt{3}} > \frac{\pi}{2\sqrt{3}} \cos \frac{\pi}{2\sqrt{3}} = \frac{\pi}{4},
\]

\[
\sin \frac{\pi}{\sqrt{6}} < \frac{\pi}{\sqrt{6}} \cos \frac{\pi}{\sqrt{6}} = \frac{3\pi}{4\sqrt{6}}
\]

which implies that the left hand side of (4.5) is grater than \(\sqrt{3}\pi/4\) and the right hand side is less than

\[
\frac{\pi}{4} + \frac{\sqrt{6}}{4} \frac{3\pi}{4\sqrt{6}} = \frac{7}{16} \pi.
\]

Thus (4.2) follows. \(\square\)

It is known that

\[
\int_0^{\pi/2} \ln (\sin x) dx = -\frac{\pi}{2} \ln 2.
\]

We now evaluate the integral \(\int_c^\pi \ln (\sin x) dx (c \in (0, \pi/2)).\)
Application 4. For \( c \in (0, \pi/2) \), we have

\[
(4.6) \quad c \ln (\sin c) - c + \frac{1}{9}c^3 < \int_0^c \ln (\sin x) \, dx < c \ln c - \frac{1}{18}c^3.
\]

Particularly, we get

\[
(4.7) \quad -\frac{\pi}{72} (36 - \pi^2) < \int_0^{\pi/2} \ln (\sin x) \, dx < -\frac{\pi}{72} (\ln 2 + \frac{\pi^2}{72} + 1),
\]

\[
(4.8) \quad -\frac{\pi}{8} (2 + \ln 2 - \frac{\pi^2}{72}) < \int_0^{\pi/4} \ln (\sin x) \, dx < -\frac{\pi}{8} (2 \ln 2 + 1 + \frac{\pi^2}{288} - \ln \pi).
\]

Proof. Letting \( p \to 0^+ \) in \((3.10)\) gives

\[
\gamma_0^+ (c) e^{-x^2/6} < \frac{\sin x}{x} < e^{-x^2/6},
\]

where \( \gamma_0^+ (c) = e^{-1} (\sin c) e^{c^2/6} \). Multiplying both sides by \( x \) and taking the logarithm and next integrating \([0, c]\) yield

\[
\int_0^c \ln \left( \gamma_0^+ (c) xe^{-x^2/6} \right) \, dx < \int_0^c \ln (\sin x) \, dx < \int_0^c \ln \left( xe^{-x^2/6} \right) \, dx.
\]

Simple integral computation leads to desired result. □

The Catalan constant \([4]\)

\[
G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = 0.9159655941772190...
\]

is a famous mysterious constant appearing in many places in mathematics and physics. Its integral representations \([3]\) contain the following

\[
G = \int_0^1 \frac{\arctan x}{x} \, dx = \frac{1}{2} \int_0^{\pi/2} \frac{x}{\sin x} \, dx
\]

\[
= -2 \int_0^{\pi/4} \ln (2 \sin x) \, dx = \frac{\pi^2}{16} - \frac{\pi}{4} \ln 2 + \int_0^{\pi/4} \frac{x^2}{\sin^2 x} \, dx.
\]

We next prove three accurate estimations for \( G \).

Application 5. We have

\[
(4.9) \quad \frac{\sqrt{6\pi}}{2\sqrt{16\sqrt{3} - \pi^2}} < G < \frac{3}{32} \pi^2,
\]

\[
(4.10) \quad \frac{\pi}{2} \left( 2 \ln 2 - \ln \pi + \frac{\pi^2}{288} + 1 \right) < G < \frac{\pi}{4} \left( 2 \ln 2 - \frac{\pi^2}{72} \right),
\]

\[
(4.11) \quad \frac{\pi^2}{16} - \frac{\pi}{4} \ln 2 + \frac{8}{5} \left( 172 - 99\sqrt{3} \right) < G < \frac{\pi^2}{320} \left( 37 + 6\sqrt{3} \right) - \frac{\pi}{4} \ln 2.
\]

Proof. (i) \((3.12)\) implies that for \((0, \pi/2)\)

\[
\frac{1}{\cos^2 \frac{x}{\sqrt{6}}} < \frac{x}{\sin x} < \frac{1}{\gamma_1/\sqrt{6} \left( \frac{\pi}{2} \right) \cos^2 \frac{x}{\sqrt{6}}},
\]
where $\gamma_{1/\sqrt{3}}(\pi/2) = 2\pi^{-1} \cos^2\left(\sqrt{6}\pi/12\right)$. Integrating both sides over $[0, \pi/2]$ yields
\[
\sqrt{6} \tan \frac{\pi}{2\sqrt{6}} < \int_{0}^{\pi/2} \frac{x}{\sin x} \, dx < \frac{\sqrt{6}}{4} \pi \sin \frac{\pi}{\sqrt{6}}.
\]
By Corollary 1 it is seen that for $x \in (0, \pi/2)$
\[
\sin x > x \left(\cos \frac{\sqrt{6}x}{3}\right)^{1/2}
\]
holds, and so
\[
\tan^{2} \frac{\pi}{2\sqrt{6}} = 1 - \sin^{2} \frac{\sqrt{6}x}{3} < x^2 < \cos^{2} \frac{\pi}{\sqrt{6}} = \frac{\sqrt{6}}{8} \pi.
\]
Hence,
\[
\frac{\sqrt{6}x}{\sqrt{16\sqrt{3}-\pi^2}} < \sqrt{6} \tan \frac{\pi}{2\sqrt{6}} < \int_{0}^{\pi/2} \frac{x}{\sin x} \, dx < \frac{\sqrt{6}}{4} \pi \sin \frac{\pi}{\sqrt{6}} < \frac{3}{16} \pi^2,
\]
which, from the third integral representation for $G$, implies (4.9).

(ii) By (7) it is derived that
\[
\frac{\pi}{8} \left(\ln 2 + \frac{1}{72} \pi^2 - 2\right) < \int_{0}^{\pi/4} \ln(2 \sin x) \, dx < \frac{\pi}{4} \left(\ln \pi - \ln 2 - 1 - \frac{1}{288} \pi^2\right),
\]
it follows from the third integral representation for $G$ that (4.9) holds.

(iii) Lastly, we use the fourth integral representation for $G$ to prove (4.10). Employing Theorem 4, we have
\[
\gamma_{1/3} \left(\frac{\pi}{4}\right) \cos^{3} \frac{x}{3} < \sin \frac{x}{x} < \cos^{3} \frac{x}{3},
\]
where $\gamma_{1/3} \left(\pi/4\right) = 16 \left(3\sqrt{3} - 5\right)/\pi$. It is obtained that
\[
\cos^{-6} \frac{x}{3} < \frac{x^2}{\sin^2 x} < \frac{\pi^2}{512} \left(15\sqrt{3} + 26\right) \cos^{-6} \frac{x}{3},
\]
and integrating both sides over $[0, \pi/4]$ leads to
\[
\int_{0}^{\pi/4} \cos^{-6} \frac{x}{3} \, dx < \int_{0}^{\pi/4} \frac{x^2}{\sin^2 x} \, dx < \frac{\pi^2}{512} \left(15\sqrt{3} + 26\right) \int_{0}^{\pi/4} \cos^{-6} \frac{x}{3} \, dx.
\]
Integral computation reveals that
\[
\int_{0}^{\pi/4} \left(\cos \frac{x}{3}\right)^{-6} \, dx = \frac{8}{5} \left(172 - 99\sqrt{3}\right),
\]
and therefore
\[
\frac{8}{5} \left(172 - 99\sqrt{3}\right) < \int_{0}^{\pi/4} \frac{x^2}{\sin^2 x} \, dx < \frac{\pi^2}{320} \left(6\sqrt{3} + 17\right).
\]
Application of the fourth integral representation for $G$ the desired inequality (4.10) follows. □
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