Quantum-inspired classical sublinear-time algorithm for solving low-rank semidefinite programming via sampling approaches

Nai-Hui Chia∗ Tongyang Li† Han-Hsuan Lin∗ Chunhao Wang∗

Abstract

Semidefinite programming (SDP) is a central topic in mathematical optimization with extensive studies on its efficient solvers. Recently, quantum algorithms with superpolynomial speedups for solving SDPs have been proposed assuming access to its constraint matrices in quantum superposition. Mutually inspired by both classical and quantum SDP solvers, in this paper we present a sublinear classical algorithm for solving low-rank SDPs which is asymptotically as good as existing quantum algorithms. Specifically, given an SDP with $m$ constraint matrices, each of dimension $n$ and rank $\text{poly}(\log n)$, our algorithm gives a succinct description and any entry of the solution matrix in time $O(m \cdot \text{poly}(\log n, 1/\epsilon))$ given access to a sample-based low-overhead data structure of the constraint matrices, where $\epsilon$ is the precision of the solution. In addition, we apply our algorithm to a quantum state learning task as an application.

Technically, our approach aligns with both the SDP solvers based on the matrix multiplicative weight (MMW) framework and the recent studies of quantum-inspired machine learning algorithms. The cost of solving SDPs by MMW mainly comes from the exponentiation of Hermitian matrices, and we propose two new technical ingredients (compared to previous sample-based algorithms) for this task that may be of independent interest:

• Weighted sampling: assuming sampling access to each individual constraint matrix $A_1, \ldots, A_\tau$, we propose a procedure that gives a good approximation of $A = A_1 + \cdots + A_\tau$.

• Symmetric approximation: we propose a sampling procedure that gives low-rank spectral decomposition of a Hermitian matrix $A$. This improves upon previous sampling procedures that only give low-rank singular value decompositions, losing the signs of eigenvalues.

∗Department of Computer Science, University of Texas at Austin. Email: {nai,linhh,chunhao}@cs.utexas.edu
†Department of Computer Science, Institute for Advanced Computer Studies, and Joint Center for Quantum Information and Computer Science, University of Maryland. Research supported by IBM PhD Fellowship, QISE-NET Triplet Award (NSF DMR-1747426), and the U.S. Department of Energy, Office of Science, Office of Advanced Scientific Computing Research, Quantum Algorithms Teams program. Email: tongyang@cs.umd.edu
1 Introduction

1.1 Motivations

Semidefinite programming (SDP) is a central topic in the studies of mathematical optimization and theoretical computer science, with a wide range of applications including algorithm design, machine learning, operations research, etc. The importance of SDP mutually comes from its generality that contains the better-known linear programming (LP) and the fact that it admits polynomial-time solvers. Mathematically, an SDP is defined as follows:

\[
\begin{align*}
\text{max} & \quad \text{Tr}[CX] \\
\text{s.t.} & \quad \text{Tr}[A_iX] \leq b_i \quad \forall i \in [m]; \\
& \quad X \succeq 0,
\end{align*}
\]

where \( m \) is the number of constraints in the SDP, \( A_1, \ldots, A_m, C \) are \( n \times n \) Hermitian matrices, and \( b_1, \ldots, b_m \in \mathbb{R} \); Eq. (3) restricts the variable matrix \( X \) to be positive semidefinite (PSD), i.e., \( X \) is an \( n \times n \) Hermitian matrix with non-negative eigenvalues. An \( \varepsilon \)-approximate solution of this SDP is an \( X^\ast \) that satisfies Eqs. (2) and (3) while the maximum in Eq. (1) is smaller than \( \text{Tr}[CX^\ast] + \varepsilon \).

There is rich literature on solving SDPs. Ellipsoid methods gave the first polynomial-time SDP solvers [Kha80, GLS81]. Then, the complexities of the SDP solvers had been subsequently improved by the interior-point method [NN92] and the cutting-plane method [Ans00, Mit03]; see also the survey paper [VB96]. Currently, the state-of-the-art SDP solver [LSW15] improved the previous cutting-plane methods with running time \( \tilde{O}(m(m^2 + n^\omega + mn^2)\text{poly}(\log 1/\varepsilon)) \), where \( \omega \approx 2.373 \) is the current exponent of matrix multiplication.\(^1\) However, if we tolerate polynomial dependence in \( 1/\varepsilon \), Ref. [AK07] gave an SDP solver with better complexities in \( m \) and \( n \):

\[
\tilde{O}(mn^2(R_p R_d/\varepsilon)^4 + n^2(R_p R_d/\varepsilon)^7),
\]

where \( R_p, R_d \) are upper bounds on the norm of the optimal primal and dual solutions, respectively.

However, these SDP solvers all use the standard, entry-wise access to matrices \( A_1, \ldots, A_m, C \). In contrast, a common methodology in algorithm design is to assume a certain natural \textit{preprocessed data structure} such that the problem can be solved in sublinear time, perhaps even poly-logarithmic time given queries to the preprocessed data structure (e.g., see the examples discussed in Section 1.4). Such methodology is extensively exploited in quantum algorithms, where we are given entry-wise access to matrices in \textit{superposition}, a fundamental feature in quantum mechanics and the essence of quantum speedups. In particular, quantum algorithms for solving SDPs have been studied in [BS17, AGGW17, BKL+17, AG18], and the state-of-the-art quantum SDP solver runs in time

\[
\tilde{O}\left((\sqrt{m} + \sqrt{n} R_p R_d/\varepsilon)s(R_p R_d/\varepsilon)^4\right),
\]

where \( s \) is sublinear in \( m \) and \( n \) and polynomially faster than the classical counterparts.

Mutually inspired by both quantum and classical SDP solvers, in this paper we study the impact of alternative classical models on solving SDPs. Specifically, we ask:

\[Can we solve SDP with sublinear time and queries to a reasonable classical data structure?\]

1.2 Contributions

We give an affirmative answer to the above question under a low-overhead data structure based on sampling.

\(^1\)Throughout the paper, \( \tilde{O}(f(\cdot)) \) hides factors that are polynomial in \( \log f(\cdot) \).
**Definition 1** (Sampling access). Let $M \in \mathbb{C}^{n \times n}$ be a matrix. We say that we have the sampling access to $M$ if we can

1. sample a row index $i \in [n]$ of $M$ where the probability of row $i$ being chosen is

   $$\frac{\|M(i, \cdot)\|_2^2}{\|M\|_F^2},$$

   and

2. for all $i \in [n]$, sample an index $j \in [n]$ where the probability of $j$ being chosen is

   $$\frac{|M(i, j)|^2}{\|M(i, \cdot)\|_2^2},$$

with time and query complexity $O(\text{poly}(\log n))$ for each sampling.

A low-overhead data structure that allows for this sampling access is shown in Section 2.1. Our main result is as follows.

**Theorem 2** (informal; see Theorems 5 and 9). Assume $\text{rank}(C), \max_{i \in [n]} \text{rank}(A_i) \leq r$. Given the sampling access of $A_1, \ldots, A_m, C$ in Definition 1, there is an algorithm that gives a succinct description and any entry of an $\varepsilon$-approximate solution of the SDP in Eqs. (1) to (3) with probability at least $2/3$; the algorithm runs in time $O(m \cdot \text{poly}(\log n, r, R_p R_d/\varepsilon))$.

Our result aligns with the studies of sample-based algorithms for solving linear algebraic problems. In particular, [FKV04] gave low-rank approximations of a matrix in poly-logarithmic time with query access to the matrix as in Definition 1. Recently, Tang extended the idea of [FKV04] to give a poly-logarithmic time algorithm for solving recommendation systems [Tan18a]. Subsequently, still under the same sampling assumption, Ref. [Tan18b] gave poly-logarithmic algorithms for principal component analysis and clustering assignments, and Refs. [GLT18, CLW18] gave poly-logarithmic algorithms for solving low-rank linear systems. However, all these sample-based sublinear algorithms directly exploit the sampling approach in [FKV04] (see Section 1.3 for details); to solve SDPs, we derive an augmented sampling toolbox which includes two novel techniques: weighted sampling and symmetric approximation.

As a corollary, our SDP solver can be applied to learning quantum states efficiently. A particular task of learning quantum states is shadow tomography [Aar18], where we are asked to find a description of an unknown quantum state $\rho$ such that we can approximate $\text{Tr}[\rho E_i]$ up to error $\varepsilon$ for a specific collection of Hermitian matrices $E_1, \ldots, E_m$ where $0 \preceq E_i \preceq I$ and $E_i \in \mathbb{C}^{n \times n}$ for all $i \in [m]$ (such $E_i$ is also known as a POVM measurement in quantum computing). Mathematically, shadow tomography can be formulated as the following SDP feasibility problem:

\[
\text{Find } \sigma \text{ such that } |\text{Tr}[\sigma E_i] - \text{Tr}[\rho E_i]| \leq \varepsilon \quad \forall i \in [m]; \\
\sigma \succeq 0, \quad \text{Tr}[\sigma] = 1.
\]

Under a quantum model proposed by [BKL+17] where $\rho, E_1, \ldots, E_m$ are stored as quantum states, the state-of-the-art quantum algorithm [AG18] solves shadow tomography with time $O(\sqrt{m} + r^{3.5}/\varepsilon^{7.5})$ where $r = \max_{i \in [m]} \text{rank}(E_i)$; in other words, quantum algorithms achieve poly-logarithmic complexity in $n$ for low-rank shadow tomography. We observe the same phenomenon under our classical sample-based model:

\[2^\text{A quantum state } \rho \text{ is a PSD matrix with trace one.}\]
Corollary 1 (informal; see Corollary 12). Given the sampling access of $E_1, \ldots, E_m$ as in Definition 1 and real numbers $\text{Tr}[\rho E_1], \ldots, \text{Tr}[\rho E_m]$, there is a classical algorithm that gives a succinct description and any entry of an $\epsilon$-approximate solution of the shadow tomography problem in Eqs. (4) and (5) with probability at least $2/3$; the algorithm runs in time $O(m \cdot \text{poly}(\log n, r, 1/\epsilon))$.

1.3 Techniques

Matrix multiplicative weight method (MMW). We study a normalized SDP feasibility testing problem defined as follows:

Definition 3 (Feasibility of SDP). Given an $\epsilon > 0$, $m$ real numbers $a_1, \ldots, a_m \in \mathbb{R}$, and Hermitian $n \times n$ matrices $A_1, \ldots, A_m$ where $-I \preceq A_i \preceq I, \forall j \in [m]$, define $S_\epsilon$ as the set of all $X$ such that

$$\text{Tr}[A_i X] \leq a_i + \epsilon \quad \forall i \in [m];$$  
(6)

$$X \succeq 0;$$  
(7)

$$\text{Tr}[X] = 1.$$  
(8)

For $\epsilon$-approximate feasibility testing of the SDP, we require that:

- If $S_\epsilon = \emptyset$, output “infeasible”;
- If $S_0 \neq \emptyset$, output an $X \in S_\epsilon$.

It is a well-known fact that one can use binary search to reduce $\epsilon$-approximation of the SDP in Eqs. (1) to (3) to $O(\log(R_p R_d / \epsilon))$ calls of the feasibility problem in Definition 3 with $\epsilon = \epsilon/(R_p R_d)$.

Therefore, throughout the paper we focus on solving feasibility testing of SDPs.

To solve the feasibility testing problem in Definition 3, we follow the matrix multiplicative weight (MMW) framework [AHK12]. To be more specific, we follow the approach of regarding MMW as a zero-sum game with two players (see, e.g., [Haz06, Wu10, GW12, LRS15, BKL+17]), where the first player wants to provide a feasible $X \in S_\epsilon$, and the second player wants to find any violation $j \in [m]$ of any proposed $X$, i.e., $\text{Tr}[A_j X] > a_j + \epsilon$. At the $t^{th}$ round of the game, if the second player points out a violation $j_t$ for the current proposed solution $X_t$, the first player proposes a new solution

$$X_{t+1} \leftarrow \exp[-(A_{j_1} + \cdots + A_{j_t})]$$  
(9)

(up to normalization); such solution by matrix exponentiation is formally named as a Gibbs state. A feasible solution is actually an equilibrium point of the zero-sum game, which can also be approximated by the MMW method [AHK12]; formal discussions are given in Section 2.2.

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3 If $S_\epsilon \neq \emptyset$ and $S_0 = \emptyset$, either output is acceptable.

4 For the normalized case $R_p R_d = 1$, we first guess a candidate value $c_1 = 0$ for the objective function, and add that as a constraint $\text{Tr}[CX] \geq c_1$ to the optimization problem. If this problem is feasible, the optimum is larger than $c_1$ and we accordingly take $c_2 = c_1 + \frac{1}{2}$; if this problem is infeasible, the optimum is smaller than $c_1$ and we accordingly take $c_2 = c_1 - \frac{1}{2}$; we proceed similarly for all $c_i$. As a result, we could solve the optimization problem with precision $\epsilon$ using $\lceil \log_2 \frac{1}{\epsilon} \rceil$ calls to the feasibility problem in Definition 3. For renormalization, it suffices to take $\epsilon = \epsilon/(R_p R_d)$.
Improved sampling tools. Before we describe our improved sampling tools, let us give a brief review of the techniques introduced by [FKV04]. The basic idea of [FKV04] comes from the fact that a low-rank matrix $A$ can be well-approximated by sampling a small number of its rows. More precisely, suppose that $A$ is an $n \times n$ matrix with rank $r$, where $n \gg r$. Because $n$ is large, it is preferable to obtain a “description” of $A$ without using $\text{poly}(n)$ resources. If we have the sampling access to $A$ in the form of Definition 1, we can sample rows from $A$ according to statement 1 of Definition 1. Suppose $S$ is the $p \times n$ submatrix of $A$ formed by sampling $p = \text{poly}(r)$ rows from $A$; it is shown in [FKV04] that $S^tS \approx A^tA$ with high probability. As a result, we can record these $p$ indices sampled from $A$ as a succinct description of $S$. Note that the size of this description, $p$, is independent of $n$. This description of $S$ is made useful by utilizing statement 2 of Definition 1 in sample-based calculations. For example, it is possible to efficiently calculate the $p \times p$ matrix $SS^\dagger$.

Building on the fact that $S^tS \approx A^tA$, it is possible to efficiently calculate a description of the two matrices $D$ and $V$ such that $VDV^\dagger \approx A^tA$, where $D$ is an $r \times r$ real positive diagonal matrix, and $V$ is an $n \times r$ matrix described by the linear combinations of the rows of $A$. $D$ corresponds to the singular values of $A$.

To implement the MMW framework, we need an approximate description of the matrix exponentiation

$$A := \exp \left[ -\sum_{i=1}^r A_{ji} \right]$$

in Eq. (9). We achieve this in two steps. First, we get an approximate description of the spectral decomposition of $A$: $A \approx \tilde{V} \tilde{D} \tilde{V}^\dagger$, where $\tilde{V}$ is an $n \times r$ matrix and $\tilde{D}$ is an $r \times r$ real diagonal matrix. Then, we approximate the matrix exponentiation $e^{-A}$ by $V e^{-D} V^\dagger$.

There are two main technical difficulties that we overcome with new tools while following the above strategy. First, since $A$ changes dynamically throughout the MMW method, we cannot assume the sampling access to $A$; a more reasonable assumption is to have sampling access to each individual constraint matrix $A_{ji}$, but it is hard to directly sample from $A$ by sampling from each individual $A_{ji}$. In Section 3, we sidestep this difficulty by devising the weighted sampling procedure which gives a succinct description of a low-rank approximation of $A = \sum_i A_{ji}$ by sampling each individual $A_{ji}$. In other words, We cannot sample according to $A$, but we can still find a succinct description of a low-rank approximation of $A$.

Second, the original sampling procedure of [FKV04] gives $VDDV^\dagger \approx A^tA$ instead of a spectral decomposition $\tilde{V} \tilde{D} \tilde{V}^\dagger \approx A$, even if $A$ is Hermitian. For our purpose of matrix exponentiation, singular value decomposition is problematic because the singular values ignore the signs of the eigenvalues; specifically, we get a large error if we approximate $e^{-A}$ by naively exponentiate the singular value decomposition: $e^{-A} \neq V e^{-D} V^\dagger$. Note that this issue of missing negative signs is intrinsic to the tools in [FKV04] because they are built upon the approximation $S^tS \approx A^tA$; Suppose that $A$ has the decomposition $A = UDV^\dagger$, where $D$ is a diagonal matrix, and $U$ and $V$ are isometries. Then $A^tA = VD^tDV^\dagger$, cancelling out any phase on $D$. We resolve this issue by a

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4. If we have the description of the three matrices $U$, $D$, and $V$ such that $UDV^\dagger \approx A$, we have an approximate singular value decomposition. However, the method in [FKV04] gives the description of either $U$ or $V$, not both.

5. For example, assume we have $A = A_1 + A_2$ such that $A_2 = -A_1 + \epsilon$, where $\epsilon$ is a matrix with small entries. In this case, $A_1$ and $A_2$ mostly cancel out each other and leave $A = \epsilon$. Since $\epsilon$ can be arbitrarily small compared to $A_1$ and $A_2$, it is hard to sample from $\epsilon$ by sampling from $A_1$ and $A_2$.

6. Gilyén [Gil19] and Tang [Tan19] pointed out to us that one might be able to sample from $A$ by lower-bounding the cancellation and doing a rejection sampling. We did not explore this approach in detail, but this is a possible alternative to weighted sampling.
novel approximation procedure, symmetric approximation. Symmetric approximation is based on the result $A \approx AVV^\dagger$ shown by [FKV04]. It then holds that $A \approx V(V^\dagger AV)V^\dagger$, since $(V^\dagger AV)$ is a small matrix of size $r \times r$, we can calculate it explicitly and diagonalize it, getting approximate eigenvalues of $A$. Together with the description of $V$, we get the desired description of the spectral decomposition of $A$. See Section 4 for more details.\footnote{It might be illustrative to describe some of our failed attempts before achieving symmetric approximation. We tried to separate the exponential function into even and odd parts; unfortunately that decomposes $e^{-x}$ into $e^{-x} = \cosh x - \sinh x$, resulting in large cancellation and unbounded error. We also tried to obtain the eigenvectors of $A$ from $V$; this approach faces multiple difficulties, the most serious one being the “fake degeneracy” as shown by the following example. Suppose $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. $A$ has two distinct eigenvectors. However, $A^\dagger A = VDV^\dagger$ can be satisfied by taking $D = I$ together with any unitary $V$. In this case, $V$ does not give any information about the eigenvectors.}

1.4 Related work

Our work follows the general methodology of leveraging preprocessed data structures; more specifically, we use sample-based data structures to fulfill the MMW framework in our SDP solver. In this subsections, we delve into related works about preprocessed data structures and MMW-based SDP solvers.

Preprocessed data structures. Preprocessed data structures are ubiquitous in algorithm design, which enable further computation tasks to be completed within sublinear or even poly-logarithmic time. For the task of nearest neighbor search, we are given a set $P$ of $n$ points in $\mathbb{R}^d$ and the goal is to preprocess a data structure such that given any point $q$, it returns a point in $P$ that is closest to $q$. In the case $d = 2$, there exists a data structure using $O(n)$ space with $O(\log n)$ time for each query [LT77]; more general cases are discussed in the survey paper [AIR18]. A related problem is orthogonal range search, where the goal is to preprocess a data structure such that one can efficiently report the points contained in an axis-aligned query box. When $d = 2$, Ref. [CLPT1] preprocessed a data structure with $O(n \log \log n)$ space and only $O(\log \log n)$ query time; for larger $d$, the query time $O(\log \log n)$ can be kept with a slight overhead on its space complexity. If preprocessed data structures are not exploited for these problems, we have to check all $n$ points in brute-force, inefficient for applications in data analytics, machine learning, computer vision, etc.

This methodology is also widespread in graph problems. It concerns fully dynamic graphs, where we start from an empty graph on $n$ fixed vertices and maintain a data structure such that edge insertions and deletions only take sublinear update time for specific graph properties. For instance, the data structure in [AOS18] maintains the maximal independent set of the graph deterministically in $O(m^{3/4})$ amortized update time ($m$ being the dynamic number of edges). There also exist data structures with sublinear update time for minimum vertex cover size [ORRR12] and all-pairs shortest paths [Tho05, ACK17]; furthermore, data structures with poly-logarithmic update time can be constructed for connectivity, minimum spanning tree, and bipartiteness [HKK99, HLT01]; maximum matching [Sol16, BHN17], graph coloring [BCHN18], etc.

Solving SDPs by the MMW framework. As introduced in Section 1.1, many SDP solvers use cutting-plane methods or interior-point methods with complexity poly(log(1/\epsilon)) but larger complexities in $m$ and $n$. In contrast, our SDP solver follows the MMW framework, and we briefly summarize such SDP solvers in existing literature. They mainly fall into two categories as follows.
First, MMW is adopted in solvers for positive SDPs, i.e., $A_1, \ldots, A_m, C \succeq 0$. In this case, the power of MMW lies in its efficiency of having only $\tilde{O}(\text{poly}(1/\epsilon))$ iterations and the fact that it admits width-independent solvers whose complexities are independent of $R_p$ and $R_d$. Ref. [LN93] first gave a width-independent positive LP solver that runs in $O(\log^2(mn)/\epsilon^4)$ iterations; [JY11] subsequently generalized this result to give the first width-independent positive SDP solver, but the number of iterations can be as large as $O(\log^{14}(mn)/\epsilon^{13})$. The state-of-the-art positive SDP solver was proposed by [AZLO16] with only $O(\log^2(mn)/\epsilon^3)$ iterations.

Second, as far as we know, the vast majority of quantum SDP solvers use the MMW framework. The first quantum SDP solver was proposed by [BS17] with worst-case running time $\tilde{O}(\sqrt{mn}s^2(R_pR_d/\epsilon)^{32})$, where $s$ is the sparsity of input matrices, i.e., every row or column of $A_1, \ldots, A_m, C$ has at most $s$ nonzero elements. Subsequently, the quantum complexity of solving SDPs was improved by [AGGW17, BKL17], and the state-of-the-art quantum SDP solver runs in time $\tilde{O}((\sqrt{m} + \sqrt{n}R_pR_d/\epsilon)s(R_pR_d/\epsilon)^4)$ [AG18]. This is optimal in the dependence of $m$ and $n$ because [BS17] proved a quantum lower bound of $\Omega(\sqrt{m} + \sqrt{n})$ for constant $R_p, R_d, s,$ and $\epsilon$.

### 1.5 Open questions

Our paper raises a few natural open questions for future work. For example:

- Can we give faster sample-based algorithms for solving LPs? Note that a recent breakthrough by [CLS18] solves LPs with complexity $\tilde{O}(n^\omega)$, significantly faster than the state-of-the-art SDP solver [LSW15] with complexity $\tilde{O}(m(m^2 + n^{\omega} + mn^2))$.

- Can we prove lower bounds on sample-based methods? In particular, a lower bound in the rank $r$ can help us understand the limit of our current approach. It is also of interest to prove a lower bound in $1/\epsilon$; if one can prove a $\text{poly}(1/\epsilon)$ lower bound for sample-based SDP solvers, then the answer to the first open question becomes negative and there must be a trade-off between $1/\epsilon$ and $n, r$.

- How is the empirical performance of our sample-based method? Admittedly, the exponents of our poly-logarithmic factors are large; nevertheless, it is common that numerical experiments perform better than theoretical guarantees, and we wonder if this phenomenon can be observed when applying our method.

**Notations.** Throughout the paper, we denote by $m$ and $n$ the number of constraints and the dimension of constraint matrices in SDPs, respectively. We denote by $R_p, R_d$ the upper bounds on the norm of the optimal primal and dual solutions of the SDP in Eqs. (1) to (3) respectively, and denote by $\epsilon$ the precision of its solution. We use $\epsilon$ to denote the precision of the solution of the SDP feasibility problem in Definition 3; $\epsilon = \epsilon/(R_pR_d)$ (see Footnote 4).

We let $[n]$ denote the set $\{1, \ldots, n\}$. For a vector $v \in \mathbb{C}^n$, we use $\mathcal{D}_v$ to denote the probability distribution on $[n]$ where the probability of $i$ being chosen is $\mathcal{D}_v(i) = |v(i)|^2 / \|v\|^2$ for all $i \in [n]$. When it is clear from the context, a sample from $\mathcal{D}_v$ is often referred to as a sample from $v$. For a matrix $A \in \mathbb{C}^{n \times n}$, we use $\|A\|$ and $\|A\|_F$ to denote its spectral norm and Frobenius norm, respectively; we use $A(i, \cdot)$ and $A(\cdot, j)$ to denote the $i^{\text{th}}$ row and $j^{\text{th}}$ column of $A$, respectively.

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Footnote 4: Without loss of generality, we can assume $m \leq n$ for LPs by deleting overcomplete constraints. The result $\tilde{O}(n^\omega)$ only holds for the current matrix multiplication exponent $\omega \approx 2.373$; when $\omega = 2$, the complexity becomes $\tilde{O}(n^{13/6})$. 

6
We use \text{rows}(A) to denote the \(n\)-dimensional vector formed by the norms of its row vectors, i.e., 
\(\text{rows}(A))(i) = \|A(i, \cdot)\|\), for all \(i \in [n]\).

\textbf{Organization.} The rest of the paper is organized as follows. We formulate the sample-based data structure and the SDP feasibility problem in Section 2. Our two techniques, weighted sampling and symmetric approximation, are presented in Section 3 and Section 4, respectively. Subsequently, we apply these techniques to estimate traces with respect to Gibbs states in Section 5. Our main results on solving SDP with the application to shadow tomography are proven in Section 6.

\section{Preliminaries}
\subsection{Sample-based data structure}

As we develop sublinear-time algorithms for solving SDP in this paper, the whole constraint matrices cannot be loaded into memory since storing them requires at least linear space and time. Instead, we assume the \textit{sampling access} of each constraint matrix as defined in Definition 1. This sampling access relies on a natural probability distribution that arises in many machine learning applications [CLW18, GLT18, KP17, KP18, Tan18a, Tan18b].

Technically, Ref. [FKV04] used this sampling access to develop sublinear algorithms for low-rank matrix approximation. It is well-known (as pointed out by [KP17] and also used in [CLW18, GLT18, KP18, Tan18a, Tan18b]) that there exist low-overhead preprocessed data structures that allow for the sampling access. More precisely, the existence of the data structures for the sampling access defined in Definition 1 is stated as follows.

\textbf{Theorem 4} ([KP17]). \textit{Given a matrix \(M \in \mathbb{C}^{n \times n}\) with \(s\) non-zero entries, there exists a data structure storing \(M\) in space \(O(s \log^2 n)\), which supports the following operators:}

1. \textit{Reading and writing \(M(i, j)\) in \(O(\log^2 n)\) time.}
2. \textit{Evaluating \(\|M(i, \cdot)\|\) in \(O(\log^2 n)\) time.}
3. \textit{Evaluating \(\|M\|_F^2\) in \(O(1)\) time.}
4. \textit{Sampling a row index of \(M\) according to statement 1 of Definition 1 in \(O(\log^2 n)\) time.}
5. \textit{For each row, sampling an index according to statement 2 of Definition 1 in \(O(\log^2 n)\) time.}

Readers may refer to [KP17, Theorem A.1] for the proof of Theorem 4. In the following, we give the intuition of the data structure, which is demonstrated in Fig. 1. We show how to sample from a row vector: we use a binary tree to store the date of each row. The square of the absolute value of each entry, along with its original value are store in the leaf nodes. Each internode contains the sum of the values of its two immediate children. It is easy to see that the root node contains the square of the norm of this row vector. To sample an index and to query an entry from this row, logarithmic steps suffice. To fulfill statement 1 of Definition 1, we treat the norms of rows as a vector \((\|M(1, \cdot)\|, \ldots, \|M(n, \cdot)\|)\) and organize the data of this vector in a binary tree.
Figure 1: Illustration of a data structure that allows for sampling access to a row of $M \in \mathbb{C}^{4 \times 4}$.

2.2 Feasibility testing of SDPs

We adopt the MMW framework to solve SDPs under the zero-sum approach [Haz06, Wu10, GW12, LRS15, BKL+17]. This is formulated as the following theorem:

**Theorem 5** (Master algorithm). Feasibility of the SDP in Eqs. (6) to (8) can be tested by Algorithm 1.

**Algorithm 1:** MMW for testing feasibility of SDPs.

1. Set the initial Gibbs state $\rho_1 = \frac{I_n}{n}$, and number of iterations $T = \frac{16\ln n}{\epsilon^2}$;
2. for $t = 1, \ldots, T$ do
3.   Find a $j_t \in [m]$ such that $\text{Tr}[A_{j_t}\rho_t] > a_{j_t} + \epsilon$. If we cannot find such $j_t$, claim that $\rho_t \in \mathcal{S}_\epsilon$ and terminate the algorithm;
4.   Define the new weight matrix $W_{t+1} := \exp\left[\frac{\epsilon}{2} \sum_{i=1}^{t} A_{j_i}\right]$ and Gibbs state $\rho_{t+1} := \frac{W_{t+1}}{\text{Tr}[W_{t+1}]}$;
5. end
6. Claim that the SDP is infeasible and terminate the algorithm;

This theorem is proved in [BKL+17, Theorem 2.3]; note that the weight matrix therein is $W_{t+1} = \exp\left[-\frac{\epsilon}{2} \sum_{\tau=1}^{t} M_{\tau}\right]$ where $M_{\tau} = \frac{1}{2}(I_n - A_{j_\tau})$, but this gives the same Gibbs state as in Line 4 since for any Hermitian matrix $W \in \mathbb{C}^{n \times n}$ and real number $c \in \mathbb{R}$,

$$\frac{e^{W+cl}}{\text{Tr}[e^{W+cl}]} = \frac{e^WE^cI}{\text{Tr}[e^WE^cI]} = \frac{e^W}{\text{Tr}[e^W]}.$$  \hfill (10)

It should also be understood that this master algorithm is **not** the final algorithm; the step of trace estimation with respect to the Gibbs state (Line 3) will be fulfilled by our sample-based approach.

3 Weighted sampling

The objective of this section is to provide a method for sampling a small submatrix of $A$ of the form $A = A_1 + \cdots + A_\ell$ where the sampling access of each $A_\ell$ is given. Note that the standard FKV sampling method [FKV04] is not capable of this task, as the sampling access of each $A_\ell$ does not trivially imply the sampling access of $A$. In the following, we propose the **weighted sampling** method. The intuition is assigning each $A_\ell$ a different weight when computing the probability distribution, and then sampling a row/column index of $A$ according to this probability distribution.
Let $N_t$ for all $t \in [\tau]$. Then for all $t \in [\tau]$. Define the matrix $S$ as

$$S(S)^\top \approx A(S)^\top A.$$

After applying Procedure 2, we obtain the row indices $i_1, \ldots, i_p$. Let $S_1, \ldots, S_\tau$ be matrices such that $S_\ell(t, \cdot) = A_\ell(i_t, \cdot)/\sqrt{pP_j}$ for all $t \in [p]$ and $\ell \in [\tau]$. Define the matrix $S$ as

$$S = S_1 + \cdots + S_\tau.$$  \hfill (11)

Next, we give the method for sampling column indices of $S$ as in Procedure 3: we need to sample a submatrix $W$ from $S$ such that $WW^\top \approx SS^\top$.

**Procedure 3:** Weighted sampling of columns.

input : $A = \sum_{\ell=1}^{\tau} A_\ell$ where each $A_\ell$ has the sampling access as in Definition 1; integer $p$.

1 Do the following $p$ times independently to obtain samples $j_1, \ldots, j_p$. begin

2 Sample a row index $t \in [p]$ uniformly at random;

3 Sample a column index $j \in [n]$ from the probability distribution $\{Q_{1j_t}, \ldots, Q_{n|j_t}\}$ where

$$Q_{j|t} = \sum_{k=1}^{\tau} D_{Ak(t, \cdot)}(j) \|A_k(i_t, \cdot)\|^2 / \left(\sum_{\ell=1}^{\tau} \|A_\ell(i_t, \cdot)\|^2\right);$$

4 end

Now, we obtained column indices $j_1, \ldots, j_p$. Let $W_1, \ldots, W_\tau$ be matrices such that $W_\ell(\cdot, t) = S_\ell(\cdot, j_t)/\sqrt{pP_{j_t}}$ for all $t \in [p]$ and $\ell \in [\tau]$, where $P'_j = \frac{1}{p} \sum_{\ell=1}^{\tau} Q_{j|t}$ for $j \in [n]$. Define the matrix $W$ as

$$W = W_1 + \cdots + W_\tau.$$ \hfill (12)

Before showing $S(S)^\top \approx A(S)^\top A$ and $SS^\top \approx WW^\top$, we first prove the following general result.

**Lemma 2.** Let $M_1, \ldots, M_\tau \in \mathbb{C}^{n \times n}$ be a matrices. Independently sample $p$ rows indices $i_1, \ldots, i_p$ from $M = M_1 + \cdots + M_\tau$ according to the probability distribution $\{P_1, \ldots, P_n\}$ where

$$P_\ell \geq \frac{\sum_{j=1}^{\tau} D_{rows(M_j)}(i) \|M_j\|^2_F}{(\tau + 1) \sum_{\ell=1}^{\tau} \|M_\ell\|^2_F}.$$ \hfill (13)

Let $N_1, \ldots, N_\tau \in \mathbb{C}^{n \times n}$ be matrices with

$$N_\ell(i_t, \cdot) = \frac{M_\ell(i_t, \cdot)}{\sqrt{P_{j_t}}}.$$ \hfill (14)

for $t \in [p]$ and $\ell \in [\tau]$. Define $N = N_1 + \cdots + N_\tau$. Then for all $\theta > 0$, it holds that

$$\Pr \left(\|M^\dagger M - N^\dagger N\|_F \geq \theta \sum_{\ell=1}^{\tau} \|M_\ell\|^2_F\right) \leq \frac{(\tau + 1)^2}{\theta^2 p}. $$ \hfill (15)
Proof. We first show that the expected value of each entry of $N^\dagger N$ is the corresponding entry of $M^\dagger M$ as follows.

$$
\mathbb{E}\left(N^\dagger(i, \cdot)N(\cdot, j)\right) = \sum_{t=1}^{p} \mathbb{E}\left(N^*(t, i)N(t, j)\right)
= \sum_{t=1}^{p} \sum_{k=1}^{n} P_k \frac{M^*(k, i)M(k, j)}{pP_k}
= M^\dagger(i, \cdot)M(\cdot, j).
$$

(16)

Furthermore, we have

$$
\mathbb{E}\left(|N^\dagger(i, \cdot)N(\cdot, j) - M^\dagger(i, \cdot)M(\cdot, j)|^2\right) \leq \sum_{t=1}^{p} \mathbb{E}\left((N^*(t, i)N(t, j))^2\right)
= \sum_{t=1}^{p} \sum_{k=1}^{n} P_k \frac{(M^*(k, i))^2(M(k, j))^2}{p^2P_k^2}
\leq \frac{(\tau + 1) \sum_{t=1}^{\tau} \|M_t\|_F^2}{p} \sum_{k=1}^{n} \frac{(M^*(k, i))^2(M(k, j))^2}{\sum_{t'=1}^{\tau} \|M_{t'}(\cdot, \cdot)\|_F^2}
= \frac{(\tau + 1) \sum_{t=1}^{\tau} \|M_t\|_F^2}{p} \sum_{k=1}^{n} \frac{(M^*(k, i))^2(M(k, j))^2}{\sum_{t'=1}^{\tau} \|M_{t'}(\cdot, \cdot)\|_F^2}
$$

(19)

$$
\mathbb{E}\left(\|M^\dagger M - N^\dagger N\|_F^2\right) = \sum_{i,j=1}^{n} \mathbb{E}\left(|N^\dagger(i, \cdot)N(\cdot, j) - M^\dagger(i, \cdot)M(\cdot, j)|^2\right)
\leq \frac{(\tau + 1) \sum_{t=1}^{\tau} \|M_t\|_F^2}{p} \sum_{k=1}^{n} \frac{\sum_{i,j=1}^{n} (M^*(k, i))^2(M(k, j))^2}{\sum_{t'=1}^{\tau} \|M_{t'}(\cdot, \cdot)\|_F^2}
= \frac{(\tau + 1) \sum_{t=1}^{\tau} \|M_t\|_F^2}{p} \sum_{k=1}^{n} \frac{\|M(k, \cdot)\|^4}{\sum_{t'=1}^{\tau} \|M_{t'}(\cdot, \cdot)\|^2}
= \frac{(\tau + 1) \sum_{t=1}^{\tau} \|M_t\|_F^2}{p} \sum_{k=1}^{n} \frac{\|M(k, \cdot)\|^2 \left(\tau \sum_{t'=1}^{\tau} \|M_{t'}(\cdot, \cdot)\|^2\right)}{\sum_{t''=1}^{\tau} \|M_{t''}(\cdot, \cdot)\|^2}
= \frac{\tau(\tau + 1) \sum_{t=1}^{\tau} \|M_t\|_F^2}{p} \|M\|_F^2
\leq \frac{\tau(\tau + 1) \left(\sum_{t=1}^{\tau} \|M_t\|_F^2\right)^2}{p}
\leq \frac{(\tau + 1)^2 \left(\sum_{t=1}^{\tau} \|M_t\|_F^2\right)^2}{p}.
$$

(23)

(24)

(25)

(26)

(27)

(28)

(29)

Now, we bound the expected distance between $N^\dagger N$ and $M^\dagger M$:

Consequently, the result of this lemma follows from Markov’s inequality.
The following technical claim will be used multiple times in this paper. It relates the three quantities: $\sum_{\ell=1}^\tau \| A_\ell \|_F^2$, $\sum_{\ell=1}^\tau \| S_\ell \|_F^2$, and $\sum_{\ell=1}^\tau \| W_\ell \|_F^2$.

**Claim 3.** Let $A = A_1 + \cdots + A_m$ be a matrix with the sampling access for each $A_\ell$ as in Definition 1. Let $S$ and $W$ be defined by Eqs. (11) and (12). Then, with probability at least $1 - 2\tau^2/p$ it holds that

$$\frac{1}{\tau + 1} \sum_{\ell=1}^\tau \| A_\ell \|_F^2 \leq \sum_{\ell=1}^\tau \| S_\ell \|_F^2 \leq \frac{2\tau + 1}{\tau + 1} \sum_{\ell=1}^\tau \| A_\ell \|_F^2, \quad (30)$$

and

$$\frac{1}{\tau + 1} \sum_{\ell=1}^\tau \| S_\ell \|_F^2 \leq \sum_{\ell=1}^\tau \| W_\ell \|_F^2 \leq \frac{2\tau + 1}{\tau + 1} \sum_{\ell=1}^\tau \| S_\ell \|_F^2, \quad (31)$$

**Proof.** We first evaluate $\mathbb{E}(\| S_\ell \|_F^2)$ as follows. For all $\ell \in [\tau]$,

$$\mathbb{E} \left( \| S_\ell \|_F^2 \right) = \sum_{i=1}^p \mathbb{E} \left( \| S_\ell(i, \cdot) \|_2^2 \right) = \sum_{i=1}^p \sum_{j=1}^n \mathbb{E} \left( \frac{|A_\ell(i, j)|^2}{pP_j} \right) = \sum_{j=1}^n \| A_\ell(j, \cdot) \|_2^2 = \| A_\ell \|_F^2. \quad (32)$$

Then we have

$$\| S_\ell(i, \cdot) \|_2^2 = \sum_{j=1}^n \frac{|A_\ell(i, j)|^2}{pP_j} \leq \sum_{j=1}^n \frac{2|A_\ell(i, j)|^2}{p} \sum_{i=1}^\tau \| A_\ell \|_F^2 \left( \sum_{j=1}^\tau \| A_\ell \|_F^2 \right) \sum_{i=1}^n \| A_\ell(j, \cdot) \|_2^2 \leq \frac{2 \sum_{\ell=1}^\tau \| A_\ell \|_F^2}{p}. \quad (33)$$

Note that the quantity $\| S_\ell \|_F^2$ can be viewed as a sum of $p$ independent random variables $\| S_\ell(1, \cdot) \|_2^2, \ldots, \| S_\ell(p, \cdot) \|_2^2$. As a result,

$$\text{Var}(\| S_\ell \|_F^2) = p\text{Var}(\| S_\ell(i, \cdot) \|_2^2) \leq p\mathbb{E}(\| S_\ell(i, \cdot) \|_2^4) \quad (34)$$

$$\leq p \sum_{i=1}^n P_i \left( \frac{2 \sum_{j=1}^\tau \| A_\ell \|_F^2}{p} \right)^2 = \frac{2 \left( \sum_{\ell=1}^\tau \| A_\ell \|_F^2 \right)^2}{p}. \quad (36)$$

According to Chebyshev’s inequality, we have

$$\Pr \left( \frac{\| S_\ell \|_F^2 - \| A_\ell \|_F^2}{\sum_{\ell=1}^\tau \| A_\ell \|_F^2} \right) \leq \frac{2 \left( \sum_{\ell=1}^\tau \| A_\ell \|_F^2 \right)^2}{p} = \frac{2\tau^2}{p}. \quad (37)$$

Therefore, with probability at least $1 - 2\tau^2/p$, it holds that

$$-\frac{1}{\tau + 1} \sum_{j \neq \ell} \| A_j \|_F^2 + \frac{\tau}{\tau + 1} \| A_\ell \|_F^2 \leq \| S_\ell \|_F^2 \leq \frac{1}{\tau + 1} \sum_{j \neq \ell} \| A_j \|_F^2 + \frac{\tau + 2}{\tau + 1} \| A_\ell \|_F^2, \quad (38)$$

which implies that

$$\frac{1}{\tau + 1} \sum_{\ell=1}^\tau \| A_\ell \|_F^2 \leq \sum_{\ell=1}^n \| S_\ell \|_F^2 \leq \frac{2\tau + 1}{\tau + 1} \sum_{\ell=1}^\tau \| A_\ell \|_F^2. \quad (39)$$

Eq. (31) can be proven in a similar way. □
Now, the main result of the weighted sampling method, namely, \( A^\dagger A \approx S^\dagger S \) and \( WW^\dagger \approx SS^\dagger \) is a consequence of Lemma 2:

**Corollary 4.** Let \( A = A_1 + \cdots + A_m \) be a matrix with the sampling access for each \( A_\ell \) as in Definition 1. Let \( S \) and \( W \) be defined by Eqs. (11) and (12). Letting \( \theta = (\tau + 1)\sqrt{\frac{100}{\mathbf{p}}} \), then, with probability at least \( 9/10 \), the following holds:

\[
\|A^\dagger A - S^\dagger S\|_F \leq \theta \sum_{\ell=1}^\tau \|A_\ell\|_F^2, \text{ and} \tag{40}
\]

\[
\|SS^\dagger - WW^\dagger\|_F \leq \theta \sum_{\ell=1}^\tau \|S_\ell\|_F^2 \leq 2\theta \sum_{\ell=1}^\tau \|A_\ell\|_F^2. \tag{41}
\]

**Proof.** First note that Eq. (40) follows from Lemma 2. For the second statement, we need the probability \( P_j' \) to satisfy Eq. (13) in Lemma 2. In fact,

\[
P_j' = \sum_{t=1}^p \frac{Q(j|t)}{p} = \frac{1}{p} \sum_{t=1}^p \frac{\sum_{k=1}^\tau D_k(i_\ell, j) \|A_k(i_\ell, \cdot)\|^2}{\sum_{\ell=1}^\tau \|A_\ell(i_\ell, \cdot)\|^2} \tag{42}
\]

\[
= \frac{1}{p} \sum_{t=1}^p \frac{\sum_{k=1}^\tau |A_k(i_\ell, j)|^2}{\sum_{\ell=1}^\tau \|A_\ell(i_\ell, \cdot)\|^2} \tag{43}
\]

\[
= \frac{1}{p} \sum_{t=1}^p \frac{pP_{it} \sum_{k=1}^\tau |S_k(i_\ell, j)|^2}{\sum_{\ell=1}^\tau \|A_\ell(i_\ell, \cdot)\|^2} \tag{44}
\]

\[
= \sum_{t=1}^p \frac{\sum_{k=1}^\tau |A_j(i_\ell, \cdot)|^2}{\sum_{\ell=1}^\tau \|A_\ell\|_F^2} \sum_{k=1}^\tau \frac{|S_k(i_\ell, j)|^2}{\sum_{\ell=1}^\tau \|A_\ell(i_\ell, \cdot)\|^2} \tag{45}
\]

\[
= \sum_{t=1}^p \frac{\sum_{k=1}^\tau |S_k(i_\ell, j)|^2}{\sum_{\ell=1}^\tau \|A_\ell\|_F^2} \tag{46}
\]

\[
\geq \frac{\sum_{k=1}^\tau |S_k(i_\ell, \cdot)|^2}{(\tau + 1)\sum_{\ell=1}^\tau \|S_\ell\|_F^2}. \tag{47}
\]

where the last inequality follows from Claim 3. Note that the probability satisfies Eq. (13); as a result of Lemma 2, Eq. (41) holds. \( \square \)

With the weighted sampling method, we obtained a small submatrix \( W \) from \( A \). Now, we use the singular values and singular vectors of \( W \) to approximate the ones of \( A \). This is shown in Algorithm 4.

An important result of Algorithm 4 is that the vectors \( u_1, \ldots, u_\tilde{r} \) are approximately orthonormal, as stated in the following lemma:

**Lemma 5.** Let \( A = A_1 + \cdots + A_r \) be a matrix with the sampling access to each \( A_\ell \) as in Definition 1. Assume \( \|A_\ell\| \leq 1 \) and \( \text{rank}(A_\ell) \leq r \) for all \( \ell \in [\tilde{r}] \). Take \( A \) and error parameter \( \epsilon \) as the input of Algorithm 4 and obtain the \( \sigma_1, \ldots, \sigma_\tilde{r} \) and \( u_1, \ldots, u_\tilde{r} \). Let \( V \in \mathbb{C}^{n \times \tilde{r}} \) be the matrix such that \( V(:, j) = \frac{S^\dagger}{\sigma_j} u_j \) for \( j \in \{1, \ldots, \tilde{r}\} \). Then, with probability at least \( 9/10 \), the following statements hold:
Algorithm 4: Approximation of singular vectors.

\textbf{input} : $A = A_1 + \cdots + A_\tau$ with the sampling access as in Definition 1 for each $A_\ell$ and $\text{rank}(A_\ell) \leq r$; error parameter $\epsilon$.
1. Set $p = 2 \cdot 10^{20}\frac{12 \sqrt{F}}{\epsilon^2}$, $\gamma = \frac{1}{3\times10^{10}\epsilon^{2+\tau}}$ ;
2. Use Procedure 2 to obtain row indices $i_1, \ldots, i_p$;
3. Let $S_1, \ldots, S_\tau$ be matrices such that $S_\ell(t, \cdot) = A_\ell(i_t, \cdot)/\sqrt{pP_i}$ for all $t \in [p]$ and $\ell \in [\tau]$, where $P_i$ is defined in Line 1 in Procedure 2. Let $S = S_1 + \cdots + S_\tau$;
4. Use Procedure 3 to obtain column indices $j_1, \ldots, j_p$;
5. Let $W_1, \ldots, W_\tau$ be matrices such that $W_\ell(\cdot, t) = S_\ell(\cdot, j_t)/\sqrt{pP_j}$ for all $t \in [p]$ and $\ell \in [\tau]$,
   where $P_j = \frac{1}{p} \sum_{t=1}^p Q_{jij}$, $Q_{jij}$ is defined in Line 3 in Procedure 3. Let $W = W_1 + \cdots + W_\tau$;
6. Compute the top $\hat{r}$ singular values $\sigma_1, \ldots, \sigma_{\hat{r}}$ and their corresponding left singular vectors $u_1, \ldots, u_{\hat{r}}$;
7. Discard the singular values and their corresponding singular vectors satisfying $\sigma^2 < \gamma \sum_{\ell=1}^m \|W_\ell\|_F^2$. Let the remaining number of singular values be $\tilde{r}$;
8. Output $\sigma_1, \ldots, \sigma_{\tilde{r}}$ and $u_1, \ldots, u_{\tilde{r}}$;

1. There exists an isometry $U \in \mathbb{C}^{n \times \tilde{r}}$ whose column vectors span the column space of $V$ satisfying $\|U - V\|_F \leq \frac{\epsilon}{300n^2(\tau + 1)}$.
2. $\|V\| - 1 \leq \frac{\epsilon}{300n^2(\tau + 1)}$.
3. Let $\Pi_V$ be the projector on the column space of $V$, then it holds that $\|VV^\dagger - \Pi_V\|_F \leq \frac{\epsilon}{300n^2(\tau + 1)}$.
4. $\|V^\dagger V - I\|_F \leq \frac{\epsilon}{300n^2(\tau + 1)}$.

We postpone the proof of this lemma in Appendix C. Note that following the proof, one can get a tight bound which is $\frac{\sqrt{\epsilon}}{\gamma \tau} + O(\epsilon^2)$. However, for the convenience of the analysis in the rest of the paper, we choose a looser bound $\frac{\epsilon}{300n^2(\tau + 1)}$ as in Lemma 5.

Algorithm 4 is similar to the main algorithm in [FKV04] except for the different sampling method used here. In terms of the low-rank approximation, a similar result holds as follows.

Lemma 6. Let $A = A_1 + \cdots + A_\tau \in \mathbb{C}^{n \times n}$ be a Hermitian matrix where $A_\ell \in \mathbb{C}^{n \times n}$ is Hermitian, $\|A_\ell\| \leq 1$, and $\text{rank}(A_\ell) \leq r$ for all $\ell \in [\tau]$. The sampling access each $A_\ell$ is given as in Definition 1. Take $A$ and error parameter $\epsilon$ as the input of Algorithm 4 to obtain the $\sigma_1, \ldots, \sigma_{\tilde{r}}$ and $u_1, \ldots, u_{\tilde{r}}$. Let $V \in \mathbb{C}^{n \times \tilde{r}}$ be the matrix such that $V(\cdot, j) = \frac{\hat{S}_1}{\sqrt{\tau}} u_j$ for $j \in \{1, \ldots, \tilde{r}\}$. Then, with probability at least $9/10$, it holds that $\|AVV^\dagger - A\|_F \leq \frac{\epsilon}{300n^2}$.

The proof of this lemma mostly follows the proof of the FKV algorithm but with the weighted sampling method. We prove this lemma in Appendix B.

To our purpose, the main consequence of Algorithm 4 is summarized in the following theorem.

Theorem 6. Let $A = A_1 + \cdots + A_\tau \in \mathbb{C}^{n \times n}$ be a Hermitian matrix where $A_\ell \in \mathbb{C}^{n \times n}$ is Hermitian, $\|A_\ell\| \leq 1$, and $\text{rank}(A_\ell) \leq r$ for all $\ell \in [\tau]$. The sampling access each $A_\ell$ is given as in Definition 1.
Take $A$ and error parameter $\epsilon$ as the input of Algorithm 4 to obtain the $\sigma_1, \ldots, \sigma_r$ and $u_1, \ldots, u_r$. Let $V \in \mathbb{C}^{n \times \tilde{r}}$ be the matrix such that $V(\cdot,j) = \frac{\sigma_j}{\sigma_j} u_j$ for $j \in \{1, \ldots, \tilde{r}\}$. Then with probability at least $9/10$, it holds that $\|VV^\dagger AVV^\dagger - A\|_F \leq \frac{\epsilon}{300r^2}(1 + \frac{\epsilon}{300r^2(\tau+1)}) + \frac{\epsilon}{300r^2}$.

Proof. By Lemma 6, we have

$$\|AVV^\dagger - A\|_F \leq \frac{\epsilon}{300r^2}. \quad (49)$$

By taking adjoint, we have

$$\|VV^\dagger A - A\|_F \leq \frac{\epsilon}{300r^2}. \quad (50)$$

Then,

$$\|VV^\dagger AVV^\dagger - A\|_F \leq \|VV^\dagger AVV^\dagger - AVV^\dagger\|_F + \|AVV^\dagger - A\|_F \leq \frac{\epsilon}{300r^2}(1 + \frac{\epsilon}{300r^2(\tau+1)}) + \frac{\epsilon}{300r^2}, \quad (51)$$

where the last inequality follows from Lemma 5. Then the result follows.

4 Symmetric approximation of low-rank Hermitian matrices

In this section, we show that the spectral decomposition of the sum of low-rank Hermitian matrices can be approximated in time logarithmic in the dimension with the given data structure. We call this technique symmetric approximation.

Briefly speaking, suppose we are given the approximated left singular vectors $V$ of $A$ from Algorithm 4 such that $\|VV^\dagger AVV^\dagger - A\|$ is bounded as in Theorem 6, then we can approximately do spectral decomposition of $A$ as follows. First, we approximate the matrix $V^\dagger AV$ by sampling. Then, since $V^\dagger AV$ is a matrix with low dimension, we can do spectral decomposition of the matrix efficiently as $UDU^\dagger$. Finally, we show that $(VU)$ is close to an isometry. Therefore, $(VU)D(VU)^\dagger$ is an approximation to the spectral decomposition of $A$.

The algorithm for approximating the spectral decomposition of $A$ is as follows:

**Algorithm 5:** Approximation of the spectral decomposition of $A$.

**input:** $A = A_1 + \cdots + A_\tau$ with the query and sampling access as in Definition 1 for each $A_\ell$ and query access to $V$; error parameter $\epsilon$.

1. Compute the matrix $\tilde{B}$ according to Lemma 8.;
2. Compute the spectral decomposition $UDU^\dagger$ of matrix $\tilde{B}$;
3. Output an isometry $U$ and a diagonal matrix $D$ such that $UDU^\dagger$ is the spectral decomposition of $\tilde{B}$. $U$ and $\tilde{B}$ satisfy Lemma 9.

We first introduce a useful Claim from [GLT18, Lemma 11].

**Claim 7** (Trace inner product estimation). Let $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times n}$ be two Hermitian matrices. Given sampling and query access to $A$ and query access to $B$. Then one can estimate $\text{Tr}[AB]$ with the additive error $\epsilon_s$ with probability at least $1 - \delta$ by using

$$O\left(\frac{\|A\|_F \|B\|_F}{\epsilon^2_s} (Q(A) + Q(B) + S(A) + N(A)) \log \frac{1}{\delta}\right)$$
time and queries, where \( Q(B) \) is the cost of query access to \( B \), and \( Q(A), S(A), N(A) \) are the cost of query access, sampling access and norm access to \( A \).

By using Claim 7, we approximate \( V^\dagger AV \) as follows.

**Lemma 8.** Let \( V \in \mathbb{C}^{n \times r} \) and \( A = \sum_\ell A_\ell \in \mathbb{C}^{n \times n} \) be a Hermitian matrix. Given query access and sampling access to \( A_\ell \) for \( \ell \in [\tau] \), and query access to \( V \). Then, one can output a Hermitian matrix \( \tilde{B} \in \mathbb{C}^{r \times r} \) such that \( \|V^\dagger AV - \tilde{B}\|_F \leq \epsilon_s \) with probability \( 1 - \delta \) by using \( O((p + \log n)^{\frac{2\log \frac{3r}{\epsilon}}{2}} \log \frac{1}{\delta}) \) samples and time.

**Proof.** Let \( B_t = V^\dagger A_t V \) for \( t \in [\tau] \) and \( B = \sum_{t=1}^\tau B_t \). \( B_t(i, j) = V^\dagger(i, \cdot)A_t(V(\cdot, j)) \). Then, by Claim 7, one can estimate \( V^\dagger(i, \cdot)A_t V(\cdot, j) \) with error at most \( \epsilon_s/r\sqrt{\tau} \) with probability \( 1 - \frac{2\delta}{\tau(r^2 + r)} \) by using \( O(||A_t||_F||V(i, \cdot)||V(\cdot, j)||r^2\tau \log \frac{(r^2 + r)\tau}{2\delta}) \) queries. We denote the estimation to \( B_t(i, j) \) as \( \tilde{B}(i, j) \).

Since \( A_t \) is a Hermitian matrix, we only need to compute \( (r^2 + r)/2 \) elements. Hence,

\[
\text{Pr}[|B_t(i, j) - \tilde{B}_t(i, j)| \leq \epsilon_s/r\sqrt{\tau} \text{ for all } i, j \in [\tau]] \geq 1 - \delta/\tau.
\] (53)

Then, let us consider \( B_1, \ldots, B_t \),

\[
\text{Pr}[|B_t(i, j) - \tilde{B}_t(i, j)| \leq \epsilon_s/r\sqrt{\tau} \text{ for all } i, j \in [\tau], t \in [\tau]] \geq 1 - \delta.
\] (54)

Now, we are guaranteed that for all \( t \in [\tau] \), \(-\epsilon I \leq B_t - \tilde{B}_t \leq \epsilon I \) with probability at least \( 1 - \delta \). Let \( \tilde{B} = \sum_t \tilde{B}_t \). With probability \( 1 - \delta \),

\[
\|B - \tilde{B}\|_F \leq \sqrt{r^2\tau(\epsilon_s^2/r^2\tau)} = \epsilon_s.
\] (55)

Then, we prove that the matrix multiplication of an isometry and a matrix satisfying Lemma 5 is still close to an isometry.

**Lemma 9.** Let \( U \in \mathbb{C}^{r \times r} \) be a unitary matrix and \( V \in \mathbb{C}^{n \times r} \) be a matrix which satisfies Lemma 5 with error parameter \( \frac{\epsilon}{300r^2(\tau + 1)} \). Then the following properties hold for the matrix \( UV \).

1. There exists an isometry \( W \in \mathbb{C}^{n \times r} \) such that \( W \) spans the column space of \( UV \) and \( ||VU - W||_F \leq \frac{\epsilon}{300r^2(\tau + 1)} \).

2. \( ||VU - I_r||_F \leq \frac{\epsilon}{300r^2(\tau + 1)} \).

3. \( ||(VU)^\dagger(VU) - I_r||_F \leq \frac{\epsilon}{300r^2(\tau + 1)} \).

4. Let \( \Pi_{VU} \) be the projector of the column space of \( UV \). Then \( ||(VU)(VU)^\dagger - \Pi_{VU}||_F \leq \frac{3\epsilon}{300r^2(\tau + 1)} \).

**Proof.** By Lemma 5, there exists an isometry \( W' \in \mathbb{C}^{n \times r} \) such that \( W' \) spans the column space of \( V \) and \( ||V - W'||_F \leq \frac{\epsilon}{300r^2(\tau + 1)} \). Let \( W = W'U \),

\[
||VU - W||_F = ||VU - W'U||_F \leq ||V - W'||_F ||U|| \leq \frac{\epsilon}{300r^2(\tau + 1)}.
\] (56)

Note that \( W \) is also an isometry.
For the second property, by Eq. (56), we can get the following inequality

$$\| VU \| - 1 | = \| VU \| - \| W \| \leq \| VU - W \| \leq \frac{\epsilon}{300r^2(\tau + 1)}. \quad (57)$$

For the third inequality,

$$\|(VU)^\dagger(VU) - I\|_F = \| U^\dagger VU - U^\dagger U \|_F \leq \| U^\dagger \| \| V - I \|_F \| U^\dagger \| \leq \frac{\epsilon}{300r^2(\tau + 1)}. \quad (58)$$

The last inequality holds because of Lemma 5.

Lastly,

$$\| VUU^\dagger - \Pi_{VU} \|_F = \| VUU^\dagger V^\dagger - WW^\dagger \|_F \quad (59)$$

$$= \| VUU^\dagger V^\dagger - VUU^\dagger W^\dagger + VUW^\dagger - W^\dagger W^\dagger \|_F \quad (60)$$

$$\leq \| VU \| \| U^\dagger \| \| V^\dagger - W^\dagger \|_F + \| VU \| \| W^\dagger \| \quad (61)$$

$$\leq \left( 1 + \frac{\epsilon}{300r^2(\tau + 1)} \right) \frac{\epsilon}{300r^2(\tau + 1)} + \frac{\epsilon}{300r^2(\tau + 1)} \quad (62)$$

$$\leq \frac{3\epsilon}{300r^2(\tau + 1)}. \quad (63)$$

We conclude by the following main theorem of this section:

**Theorem 7.** Let $A_1, \ldots, A_r \in \mathbb{C}^{n \times n}$ be Hermitian matrices with rank at most $r$, and $A = \sum_{\ell=1}^{r} A_\ell$. Suppose given $V$ which satisfies $\| AV \|_F \leq \frac{\epsilon}{300r^2}$ and statements 1 to 4 in Lemma 5. Then, there exists an algorithm which outputs a Hermitian matrix $\tilde{B} \in \mathbb{C}^{r \times r}$ with probability at least $1 - \delta$ with time and query complexity $O((p + \log n)^{\frac{216r^3\tau^3}{\epsilon^2}} \log \frac{1}{\delta})$ such that the following properties holds.

1. $\| V\tilde{B}V^\dagger - A \| \leq (1 + \frac{\epsilon}{300r^2(\tau + 1)})^2 \frac{\epsilon}{400r^2} + (2 + \frac{\epsilon}{300r^2(\tau + 1)}) \frac{\epsilon}{300r^2}$.

2. Let $UDU^\dagger$ be the spectral decomposition of $\tilde{B}$ and, then statements 1 to 4 in Lemma 9 hold for $UV$.

**Proof.** By Lemma 8, we can compute $\tilde{B}$ in time $O((p + \log n)^{\frac{216r^3\tau^3}{\epsilon^2}} \log \frac{1}{\delta})$.

For the first statement, we have

$$\| V\tilde{B}V^\dagger - VV^\dagger AVV^\dagger + VV^\dagger AVV^\dagger - A \| \leq \| V\tilde{B}V^\dagger - VV^\dagger AVV^\dagger \| + \| VV^\dagger AVV^\dagger - A \| \quad (64)$$

$$\leq \left( 1 + \frac{\epsilon}{300r^2(\tau + 1)} \right)^2 \frac{\epsilon}{400r^2}$$

$$+ \left( 2 + \frac{\epsilon}{300r^2(\tau + 1)} \right) \frac{\epsilon}{300r^2} \quad (65)$$

The first term of the last inequality comes from Lemma 8 with $\epsilon_s = \frac{\epsilon}{300r^2}$. The second statement directly follows from Lemma 9. \qed
Algorithm 6: Approximation of the trace.

input : Given query and sampling access to a constraint $A_\ell$, query access to $U$, and the matrix $D$ where $UDU^\dagger$ is an spectral decomposition of $\tilde{B}$ such that $(VU)D(VU)^\dagger$ is an approximated spectral decomposition of $A = \sum_i A_i$ as in Theorem 7.

1. Compute $\text{Tr}[e^{-\frac{\epsilon}{2}D}]$;
2. Approximate $\text{Tr}[A_\ell(VU)(e^{-\frac{\epsilon}{2}D}/\text{Tr}[e^{-\frac{\epsilon}{2}D]})(VU)^\dagger]$ by $\zeta$ according to Claim 7;
3. Output $\zeta$.

5 Gibbs states

In this section, we combine our techniques from Section 3 and Section 4 to give a sample-based estimator of the traces of a Gibbs state times a constraint $A_\ell$. This is formulated as Algorithm 6.

We show that the output of Algorithm 6 $\epsilon$-approximates $\text{Tr}[A_\ell \rho]$ for $\rho = e^{-\frac{\epsilon}{2}A}/\text{Tr}[e^{-\frac{\epsilon}{2}A}]$ in the following two subsections.

5.1 Estimating matrices inner product

Lemma 10. Let $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times n}$, and $B' \in \mathbb{C}^{n \times n}$ be Hermitian matrices. Suppose $\|B - B'\| \leq \frac{\epsilon}{300r^2(\tau + 1)}$. Then

$$|\text{Tr}[AB] - \text{Tr}[AB']| \leq \frac{\epsilon}{300r^2(\tau + 1)} \text{Tr}|A|.$$ (66)

Proof. Let $A = \sum_i \sigma_i v_i v_i^\dagger$. We have

$$|\text{Tr}[AB] - \text{Tr}[AB']| = \sum_i \sigma_i (B - B') v_i \leq \sum_i |\sigma_i| \|B - B'\| \leq \frac{\epsilon}{300r^2(\tau + 1)} \text{Tr}|A|.$$ (67)

Lemma 11. Let $A$ and $B$ be Hermitian matrices, and $\|A - B\| \leq \epsilon$. Let $\rho_A(\frac{\epsilon}{2}) = e^{-\frac{\epsilon}{2}A}/\text{Tr}[e^{-\frac{\epsilon}{2}A}]$ and $\rho_B(\frac{\epsilon}{2}) = e^{-\frac{\epsilon}{2}B}/\text{Tr}[e^{-\frac{\epsilon}{2}B}]$. Then $F(\rho_A(\frac{\epsilon}{2}), \rho_B(\frac{\epsilon}{2})) \geq e^{-\frac{\epsilon}{2}}$, where $F(\rho_A, \rho_B) := \text{Tr}[\sqrt{\rho_A \rho_B \sqrt{\rho_A}}]$ is the fidelity between $\rho_A$ and $\rho_B$.

Proof. This lemma has been proven in [PW09, Appendix C]. We leave the proof to Appendix A for completeness.

Lemma 11 implies that the trace distance between $\rho_A$ and $\rho_B$ is

$$\frac{1}{2} \text{Tr}|\rho_A - \rho_B| \leq \sqrt{1 - e^{-2\frac{\epsilon}{2}}},$$ (68)

and the spectral distance is

$$\|\rho_A(\epsilon/2) - \rho_B(\epsilon/2)\| \leq 2\sqrt{1 - e^{-2\frac{\epsilon}{2}}}. $$ (69)
5.2 Approximating the Gibbs state

Let $\tilde{A} = VV^\dagger AVV^\dagger$ and $U$, $D$, and $\tilde{B}$ be outputs of Algorithm 5. In this section, we suppose $\|\tilde{A} - A\| \leq (1 + \frac{\epsilon}{300r^2(\tau+1)})^2 \frac{\epsilon}{300r^2} + (2 + \frac{\epsilon}{300r^2(\tau+1)}) \frac{\epsilon}{300r^2}$ as in Theorem 6.

**Theorem 8.** Let $\rho = \frac{e^{-\frac{1}{2}A}}{Tr e^{-\frac{1}{2}A}}$ and $\tilde{\rho} = \frac{\tilde{V}e^{-\frac{1}{2}D}V^\dagger}{Tr e^{-\frac{1}{2}D}}$. Suppose $\|A - \tilde{A}\|_F \leq (1 + \frac{\epsilon}{300r^2(\tau+1)})^2 \frac{\epsilon}{300r^2} + (2 + \frac{\epsilon}{300r^2(\tau+1)}) \frac{\epsilon}{300r^2}$. Let $A_\ell$ be a Hermitian matrix with the promise that $\|A_\ell\| \leq 1$ and rank$(A_\ell) \leq r$. Then Algorithm 6 outputs $\zeta$ such that

$$|\text{Tr}[A_\ell \rho] - \zeta| \leq \epsilon \quad (70)$$

with probability $1 - \delta$ in time $O(\frac{A}{\epsilon^2} (\log^2 n + \tau pr) \log \frac{1}{\delta})$.

**Proof.** As we have proven in Lemma 9, there exists an isometry $\tilde{U}$ such that $\|\tilde{U} - \tilde{V}\| \leq \frac{\epsilon}{300r^2(\tau+1)}$ and $\tilde{U}$ spans the column space of $\tilde{V}$. We define two additional Gibbs states $\rho' = \frac{\tilde{U}e^{-\frac{1}{2}D} \tilde{U}^\dagger}{Tr e^{-\frac{1}{2}D}}$ and $\tilde{\rho} = \frac{e^{-\frac{1}{2}A}}{Tr e^{-\frac{1}{2}A}}$.

$$|\text{Tr}[A_\ell \rho] - \text{Tr}[A_\ell \tilde{\rho}]| = |\text{Tr}[A_\ell \rho] - \text{Tr}[A_\ell \tilde{\rho}] + \text{Tr}[A_\ell \tilde{\rho}] - \text{Tr}[A_\ell \rho' + \text{Tr}[A_\ell \rho'] - \text{Tr}[A_\ell \tilde{\rho}] + \text{Tr}[A_\ell \tilde{\rho}] - \zeta| \leq |\text{Tr}[A_\ell \rho] - \text{Tr}[A_\ell \tilde{\rho}]| + |\text{Tr}[A_\ell \tilde{\rho}] - \text{Tr}[A_\ell \rho']| + |\text{Tr}[A_\ell \rho'] - \text{Tr}[A_\ell \tilde{\rho}]| + |\text{Tr}[A_\ell \tilde{\rho}] - \zeta|. \quad (71)$$

We give bounds on each term as follows. First,

$$|\text{Tr}[A_\ell \rho] - \text{Tr}[A_\ell \tilde{\rho}]| \leq \text{Tr}[A|\rho - \tilde{\rho}|] \leq 2\text{Tr}[A]\sqrt{1 - e^{-2\frac{1}{2}(1 + \frac{\epsilon}{300r^2(\tau+1)})^2 \frac{\epsilon}{400r^2} + (2 + \frac{\epsilon}{300r^2(\tau+1)}) \frac{\epsilon}{300r^2}}}. \quad (73)$$

For $|\text{Tr}[A_\ell \tilde{\rho}] - \text{Tr}[A_\ell \rho']|$, we first compute an upper bound on $\|\tilde{V}D\tilde{V}^\dagger - \tilde{U}D\tilde{U}^\dagger\|$.

$$\|\tilde{V}D\tilde{V}^\dagger - \tilde{U}D\tilde{U}^\dagger\| \leq \|\tilde{U} - \tilde{V}\||D||\|\tilde{V}\| + \|\tilde{U}\| \leq 3\frac{\epsilon}{300r^2(\tau+1)} \|D\|. \quad (75)$$

Then, by applying Lemmas 10 and 11 again, we get

$$|\text{Tr}[A_\ell \tilde{\rho}] - \text{Tr}[A_\ell \rho']| \leq \text{Tr}[A]|\tilde{\rho} - \rho'| \leq \text{Tr}[A]|2\sqrt{1 - e^{-6\frac{1}{2}(1 + \frac{\epsilon}{300r^2(\tau+1)}) \|D\|}}. \quad (76)$$

For the second last term $|\text{Tr}[A_\ell \rho'] - \text{Tr}[A_\ell \tilde{\rho}]|$, it is not hard to show that

$$\|\tilde{U}e^{-\frac{1}{2}D} \tilde{U}^\dagger - \tilde{V}e^{-\frac{1}{2}D} \tilde{V}^\dagger\| \leq 2\|\tilde{U} - \tilde{V}\||Tr[e^{-\frac{1}{2}D}]. \quad (77)$$

Then,

$$|\text{Tr}[A_\ell \rho'] - \text{Tr}[A_\ell \tilde{\rho}]| \leq \text{Tr}[A_\ell]|\rho' - \tilde{\rho}| \leq (2\|\tilde{U} - \tilde{V}\||\text{Tr}[A_\ell] \leq 2\frac{\epsilon}{300r^2(\tau+1)} \|D\|. \quad (78)$$

The last term follows from Claim 7 by setting the precision to be $\epsilon/5$. Hence

$$|\text{Tr}[A_\ell \rho] - \zeta| \leq \epsilon/5. \quad (79)$$
By adding Eqs. (74), (76) and (78) together,

\[ |\text{Tr}[A_\ell \rho] - \zeta| \leq \epsilon. \]  

(80)

\(\text{Tr}[A_\ell \rho]\) can be approximated with precision \(\epsilon/5\) with probability \(1 - \delta\) in time

\[ O\left(4 (Q(A_\ell) + Q(VU) + S(A_\ell) + N(A_\ell)) \log \frac{1}{\delta}\right) = O\left(4 (\log^2 n + \tau pr \log \frac{1}{\delta})\right), \]

(81)

where \(p\) is the number of rows sampled in Algorithm 4 and the maximum rank of the Gibbs state is \(\tau r\). The last equality is true since one can compute \((VU)(i,j)\) by computing \((S^\dagger(i,\cdot)u_j/\sigma_j\) and then compute the inner product \(V(i,\cdot)U(\cdot,j)\), which takes \(O(p \tau r)\) time.

6 Main results: sample-based SDP and shadow tomography solvers

We finally prove our main results on solving SDPs via sampling.

**Theorem 9.** Given Hermitian matrices \(\{A_1, \ldots, A_m\}\) with the promise that each of \(A_1, \ldots, A_m\) has rank at most \(r\), spectral norm at most \(1\), and the sampling access of each \(A_i\) is given as in Definition 1. Also given \(a_1, \ldots, a_m \in \mathbb{R}\). Then for any \(\epsilon > 0\), Algorithm 7 gives a succinct description and any entry (see Remark 1) of the solution of the SDP feasibility problem

\[ \text{Tr}[A_i X] \leq a_i + \epsilon \quad \forall i \in [m]; \]  

(82)

\[ X \succeq 0; \]  

(83)

\[ \text{Tr}[X] = 1 \]  

(84)

with probability at least \(2/3\) in \(O(\frac{mr57 \ln 37 n}{\epsilon^2})\) time.

**Algorithm 7:** Feasibility testing of SDPs by our sample-based approach.

1. Set the initial Gibbs state \(\rho_1 = \frac{I_n}{n}\), and number of iterations \(T = \frac{16\ln n}{\epsilon^2}\);
2. for \(t = 1, \ldots, T\) do
3. Find a \(j_t \in [m]\) such that \(\text{Tr}[A_j \rho_t] > a_j + \epsilon\) using Algorithms 4 to 6. If we cannot find such \(j_t\), claim that \(\rho_t \in S_\epsilon\) and terminate the algorithm;
4. Define the new weight matrix \(W_{t+1} := \exp[\frac{\epsilon}{2} \sum_{i=1}^t \lambda_i]\) and Gibbs state \(\rho_{t+1} := \frac{W_{t+1}}{\text{Tr}[W_{t+1}]}\);
5. end
6. Claim that the SDP is infeasible and terminate the algorithm;

The algorithm follows the master algorithm in Theorem 5. The main challenge is to estimate \(\text{Tr}[A_j \rho_t]\) where \(\rho_t\) is the Gibbs state at iteration \(t\); this is achieved by Theorem 8 in Section 5.

**Proof.** We prove Theorem 9 by showing the correctness and the time complexity of Algorithm 7.

**Correctness:** The correctness of Algorithm 7 directly follows from Theorem 8. Specifically, we have shown that one can estimate the quantity \(\text{Tr}[A_j \rho_t]\) with precision \(\epsilon\) with high probability by applying Algorithms 4 to 6.
The succinct representation is given by Algorithm 7, where \( t \) (this suffices because the SDP feasibility problem of deciding an element of the matrix \( \mathcal{S} \) can be solved with probability 1) has rank at most \( P \). SDP optimization problem in Eqs. (1) to (3) in time \( O(\tau \log n) \). Specifically, we generate a random number \( p \in [0, 1] \), and then do the binary search in the data structure to find the index \( i \) such that \( p \in [\sum_{j=1}^{i-1} \mathcal{P}_j, \sum_{j=1}^{i} \mathcal{P}_j] \). Similarly, we can implement Procedure 3 in time \( O(\tau \log n) \). Hence, the time complexity to construct the matrix \( \mathcal{W} \) and compute its SVD is \( O(\tau \log n + p^3) \). Algorithm 4 succeeds with probability 9/10.

Then, Algorithms 5 and 6 take \( O((p + \log n)^{216}/\varepsilon^4 \log 1/\delta) \) and \( O(4\varepsilon^2 (\log^2 n + \tau p \log 1/\delta)) \) and succeed with probability at least 1 – \( 2\delta \). By setting \( \delta \) as a small enough constant (say \( \delta = 1/6 \)), Algorithm 7 succeeds with probability at least 2/3 in time \( O(\tau n^3) = O(\tau n^3) \). \( \square \)

Remark 1. Theorem 9 solves the SDP feasibility problem, i.e., to decide \( \mathcal{S}_0 = \emptyset \) or \( \mathcal{S}_\varepsilon \neq \emptyset \). For the SDP optimization problem in Eqs. (1) to (3), the optimal value can be approximated by binary search (see Footnote 4); however, writing down the approximate solution would take \( n^2 \) space, ruining the poly-logarithmic complexity in \( n \). Nevertheless,

- we have its succinct representation

 \[
 \exp\left[\frac{t}{2} \sum_{i=1}^{t} A_{j_i}\right] / \Tr\left[\exp\left[\frac{t}{2} \sum_{i=1}^{t} A_{j_i}\right]\right], \quad \text{and} \]

- we can query any entry of the solution matrix.

The succinct representation is given by Algorithm 7, where \( t \leq T \) and \( j_t \in [m] \) for all \( t \in [T] \). Storing all \( j_t \) takes \( t \log_2 m = O(\log m \log n/\varepsilon^2) \) bits. A query to the solution is accessed by computing the element of \( (VU)(e^{-\frac{1}{2}D}/\Tr[e^{-\frac{1}{2}D}])VU^\dagger \), which is an \( \epsilon \)-approximation to the solution by Theorem 8 (this suffices because the SDP feasibility problem of deciding \( \mathcal{S}_0 = \emptyset \) or \( \mathcal{S}_\varepsilon \neq \emptyset \) is \( \epsilon \)-approximate).

Shadow tomography. As a corollary of Theorem 9, we have:

Corollary 12. Given Hermitian matrices \( \{E_1, \ldots, E_m\} \) with the promise that each of \( E_1, \ldots, E_m \) has rank at most \( r \), \( 0 \leq E_i \leq I \) and the sampling access to \( E_i \) is given as in Definition 1 for all \( i \in [m] \). Also given \( p_1, \ldots, p_m \in \mathbb{R} \). Then for any \( \epsilon > 0 \), the shadow tomography problem

\[
\text{Find } \sigma \text{ such that } \quad |\Tr[\sigma E_i] - p_i| \leq \epsilon \quad \forall i \in [m];
\]

\[
\sigma \succeq 0, \quad \Tr[\sigma] = 1
\]

can be solved with probability \( 1 - \delta \) with cost \( O(m \cdot \text{poly}(\log n, 1/\epsilon, \log(1/\delta), r)) \).
Here, $p_i = \text{Tr}[\rho E_i]$ in Eq. (4) for all $i \in [m]$. Notice that the assumption of knowing $p_1, \ldots, p_m$ makes our problem slightly different from the shadow tomography problem in [Aar18, BKL+17, AG18] where we are only given copies of the quantum state $\rho$ without the knowledge of $\text{Tr}[\rho E_1], \ldots, \text{Tr}[\rho E_m]$. However, quantum state is a concept without a counterpart in classical computing, hence we follow the conventional assumption in SDPs that these real numbers are given.

Proof. We denote $A_i = E_i$ for all $i \in [m]$ and $A_i = -E_i - m$ for all $i \in \{m + 1, \ldots, 2m\}$; also denote $a_i = p_i$ for all $i \in [m]$ and $a_i = -p_i - m$ for all $i \in \{m + 1, \ldots, 2m\}$. As a result, $\text{Tr}[\sigma E_i] - p_i \leq \epsilon$ is equivalent to $\text{Tr}[\sigma A_i] \leq a_i + \epsilon$ for all $i \in [2m]$; therefore, the shadow tomography problem in Eqs. (86) and (87) is equivalent to the following SDP feasibility problem:

Find $\sigma$ such that

\[
\begin{align*}
\text{Tr}[A_i \sigma] &\leq a_i + \epsilon \quad \forall i \in [2m]; \\
\sigma &\succeq 0, \quad \text{Tr}[\sigma] = 1.
\end{align*}
\]

Consequently, Corollary 12 reduces to the SDP in Eqs. (82) and (84) with $2m$ constraints; the result hence follows from Theorem 9.

Remark 2. Similar to Remark 1, $\sigma$ can be stored as a succinct representation. This is because

\[
\sigma = \frac{\exp\left[\frac{\epsilon}{2} \sum_{\tau=1}^{t} (-1)^{i_{\tau}} A_{j_{\tau}} \right]}{\text{Tr}\left[\exp\left[\frac{\epsilon}{2} \sum_{\tau=1}^{t} (-1)^{i_{\tau}} A_{j_{\tau}} \right]\right]},
\]

by the proof of Corollary 12, where $t \leq T$ and $i_{\tau} \in \{0, 1\}$, $j_{\tau} \in [m]$ for all $\tau \in [t]$. Storing all $i_{\tau}, j_{\tau}$ takes $t(\log_2 m + 1) = O(\log m \log n/\epsilon^2)$ bits.

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A Bounding the distance between Gibbs states

We prove the following restated lemma for bounding the distance between Gibbs states:

Lemma 11. Let $A$ and $B$ be Hermitian matrices, and $\|A - B\| \leq \epsilon$. Let $\rho_A(\frac{\epsilon}{2}) = \frac{e^{-\frac{\epsilon}{2} A}}{\text{Tr}[e^{-A}]}$ and $\rho_B(\frac{\epsilon}{2}) = \frac{e^{-\frac{\epsilon}{2} B}}{\text{Tr}[e^{-B}]}$. Then $F(\rho_A(\frac{\epsilon}{2}), \rho_B(\frac{\epsilon}{2})) \geq e^{-\frac{\epsilon}{2} \epsilon}$, where $F(\rho_A, \rho_B) := \text{Tr}[\sqrt{\sqrt{\rho_A \rho_B} \sqrt{\rho_A}}]$ is the fidelity between $\rho_A$ and $\rho_B$.

Proof. Let $\sigma_1(A) > \cdots > \sigma_n(A)$ and $\sigma_1(B) > \cdots > \sigma_n(B)$ be eigenvalues of $A$ and $B$. Then, by the definition of spectral norm,

$$\max_{j \in [n]} |\sigma_j(A) - \sigma_j(B)| \leq \|A - B\|. \quad (91)$$

This fact implies that

$$e^{-\|A - B\|} \leq e^{\sigma_j(A) - \sigma_j(B)} \leq e^{\|A - B\|}, \quad (92)$$

for all $j \in [n]$. Therefore,

$$e^{-\|A - B\|} \text{Tr}[e^A] \leq e^B \leq e^{\|A - B\|} \text{Tr}[e^A]. \quad (93)$$

Now, we define a function

$$f(p) = \text{Tr}(e^{pD/2}e^{pC}e^{pD/2})^{1/p}. \quad (94)$$

It has been shown in [PW09, FS94] that the function $f$ has two properties: $f$ is an increasing function for Hermitian matrices $C$ and $D$ in $p \in (0, \infty)$, and when $p \to 0$, $f(p) = \text{Tr}[e^{C+D}]$. Hence,

$$\text{Tr}[e^{C+D}] \leq \text{Tr}[(e^{pD/2}e^{pC}e^{pD/2})^{1/p}] \quad (95)$$

holds for $p > 0$.

Let $p = 2$, $C = -\beta A/2$, and $D = -\beta B/2$, we bound the fidelity as follows:

$$F(\rho_A(\beta), \rho_B(\beta)) = \frac{\text{Tr}(e^{-\beta B/2}e^{-\beta A/2}e^{-\beta B/2})^{1/2}}{\sqrt{\text{Tr}\rho_A(\beta)\text{Tr}\rho_B(\beta)}} \geq \frac{\text{Tr}e^{-\beta(A+B)/2}}{\sqrt{\text{Tr}\rho_A(\beta)\text{Tr}\rho_B(\beta)}}. \quad (96)$$

Note that $\|B - \frac{A+B}{2}\| \leq \epsilon/2$. By applying Eq. (93),

$$\frac{\text{Tr}e^{-\beta(A+B)/2}}{\sqrt{\text{Tr}\rho_A(\beta)\text{Tr}\rho_B(\beta)}} \geq \frac{e^{-\beta \epsilon/2}\text{Tr}e^{-\beta B}}{\sqrt{e^{\beta \epsilon}\text{Tr}e^{-\beta B}}} \geq e^{-\beta \epsilon}. \quad (97)$$

B The FKV approximation with the new weighted sampling method

In this section, we show that the FKV approximation still holds for our weighted sampling method. The main lemma we are going to show is the following:
Lemma 6. Let $A = A_1 + \cdots + A_\tau \in \mathbb{C}^{n \times n}$ be a Hermitian matrix where $A_\ell \in \mathbb{C}^{n \times n}$ is Hermitian, $\|A_\ell\| \leq 1$, and $\text{rank}(A_\ell) \leq \tau$ for all $\ell \in [\tau]$. The sampling access each $A_\ell$ is given as in Definition 1. Take $A$ and error parameter $\epsilon$ as the input of Algorithm 4 to obtain the $\sigma_1, \ldots, \sigma_{\tilde{r}}$ and $u_1, \ldots, u_{\tilde{r}}$. Let $V \in \mathbb{C}^{n \times \tilde{r}}$ be the matrix such that $V(:,j) = S_\ell^\dagger u_j$ for $j \in \{1, \ldots, \tilde{r}\}$. Then, with probability at least 9/10, it holds that $\|AVV^\dagger - A\|_F \leq \frac{\epsilon}{300\tilde{r}^2}$.

To begin with, we first prove the following lemma, which is adapted from [FKV04].

Lemma 13. Let $A = A_1 + \cdots + A_\tau \in \mathbb{C}^{n \times n}$ be a Hermitian matrix where $A_\ell \in \mathbb{C}^{n \times n}$ is Hermitian and $\text{rank}(A_\ell) \leq \tau$ for $\ell \in [\tau]$. The sampling access each $A_\ell$ is given as in Definition 1. Apply Algorithm 2 on $A$ (to obtain row indices $i_1, \ldots, i_p$) and let $S$ be defined as Eq. (11). Then with probability at least 9/10, there exists an orthonormal set of vectors $\{y_1, \ldots, y_{\tilde{r}}\}$ in the row space of $S$ such that

$$
\|A - A \tilde{r}' \sum_{j=1}^{\tilde{r}'} y_j y_j^\dagger \|_F \leq \frac{10\tilde{r}r}{p} \sum_{\ell=1}^{\tau} \|A_\ell\|_F^2,
$$

where $\tilde{r}'$ and $\tilde{r}$ satisfy $\tilde{r}' \leq \tilde{r} \leq \tau r$.

Proof. Denote the rank of $A$ by $\hat{r}$. First note that $\hat{r} \leq \tau r$. We write the singular value decomposition of $A$ as

$$
A = \sum_{t=1}^{\hat{r}} \sigma_t u_t v_t^\dagger.
$$

For the convenience of analysis, we define $S$ as the set of row indices $\{i_1, \ldots, i_p\}$ that was sampled from Line 1 of Procedure 2. For $t \in [\hat{r}]$, we define the vector-valued random variable

$$
w_t = \frac{1}{p} \sum_{i \in S} \frac{u_t(i)}{P_t} A(i, \cdot).
$$

Note that $w_t$ can be viewed as the average of $p$ i.i.d. random variables $X_1, \ldots, X_p$ where each $X_j$ is defined as

$$
X_j = \frac{u_t(i)}{P_t} A(i, \cdot), \quad \text{with probability } P_t, \quad \text{for } i \in [n].
$$

The expected value of $X_j$ can be computed as

$$
E(X_j) = \sum_{i=1}^{n} \frac{u_t(i)}{P_t} A(i, \cdot) P_t = u_t^\dagger A = \sigma_t v_t^\dagger.
$$

Hence, we have

$$
E(w_t) = \sigma_t v_t^\dagger.
$$

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As in Procedure 2, \( P_i = \sum_{\ell=1}^\tau \| A_\ell(i, \cdot) \|^2 / \sum_{\ell'=1}^\tau \| A_{\ell'} \|^2_F \). It follows that

\[
E\left( \| w_t - \sigma_t v_t^\dagger \|^2 \right) = \frac{1}{p} \sum_{i=1}^\tau \frac{|u_t(i)|^2 \| A(i, \cdot) \|^2}{P_i} - \frac{\sigma_t^2}{p} \quad (104)
\]

\[
\leq \frac{1}{p} \sum_{i=1}^n |u_t(i)|^2 \left( \sum_{\ell=1}^\tau \| A_\ell(i, \cdot) \|^2 \right) \left( \sum_{\ell'=1}^\tau \| A_{\ell'} \|^2_F \right) \sum_{\ell=1}^\tau \| A_\ell(i, \cdot) \|^2_F \quad (105)
\]

\[
\leq \frac{1}{\tau} \sum_{t=1}^\tau \| A_\ell \|^2_F \quad (106)
\]

\[
= \frac{\tau}{p} \sum_{t=1}^\tau \| A_\ell \|^2_F, \quad (107)
\]

where the second inequality follows from the Cauchy–Schwarz inequality, and the last equality follows from the fact that \( u_t \) is a unit vector.

If all the random variables \( w_t \) happen to be \( \sigma_t v_t^\dagger \), then \( \| A - A \sum_{j=1}^{\hat{r}} w_t w_t^\dagger \|^2_F = 0 \). In the following, we bound the distance of \( w_t \) from its expected value. For \( t \in [\hat{r}] \), define \( \hat{y}_t = \frac{1}{\sigma_t} w_t^\dagger \). Let \( \{ y_1, \ldots, y_n \} \) be an orthonormal basis of \( \mathbb{C}^n \) with \( \text{span}(\hat{y}_1, \ldots, \hat{y}_{\hat{r}}) = \text{span}(y_1, \ldots, y_{\hat{r}'}) \), where \( \hat{r}' \) is the dimension of \( \text{span}(\hat{y}_1, \ldots, \hat{y}_{\hat{r}}) \). Define the matrix \( F \) as

\[
F = \sum_{t=1}^{\hat{r}'} A y_t y_t^\dagger, \quad (108)
\]

which will be used to approximate \( A \). We also define an matrix \( \hat{F} \) as follows that will be used in the intermediate steps to bound the distance:

\[
\hat{F} = \sum_{t=1}^{\hat{r}} A v_t \hat{y}_t^\dagger. \quad (109)
\]

We have

\[
\| A - F \|^2_F = \sum_{t=1}^n \| (A - F) y_t \|^2 = \sum_{t=\hat{r}'+1}^n \| A y_t \|^2 = \sum_{t=\hat{r}'+1}^n \| (A - \hat{F}) y_t \|^2 \leq \| A - \hat{F} \|^2_F, \quad (110)
\]

where the third equality follows from the fact that \( \hat{y}_t^\dagger y_j = 0 \) for all \( i < \hat{r} \) and \( j > \hat{r}' \). In addition, we have

\[
\| A - \hat{F} \|^2_F = \sum_{t=1}^n \| u_t^\dagger (A - \hat{F}) \|^2 = \sum_{t=1}^{\hat{r}} \| \sigma_t v_t^\dagger - w_t \|^2. \quad (111)
\]

Taking the expected value \( \| A - \hat{F} \|^2_F \) and using Eq. (107), we have

\[
E(\| A - \hat{F} \|^2_F) \leq \frac{\hat{r} \tau}{p} \sum_{t=1}^{\tau} \| A_\ell \|^2_F. \quad (112)
\]
Therefore,

\[
\Pr \left( \| A - F \|_F \leq \frac{10^{\hat{r}r}}{p} \sum_{\ell=1}^{\tau} \| A_{\ell} \|_F^2 \right) \leq \Pr \left( \| A - \tilde{F} \|_F \leq \frac{10^{\hat{r}r}}{p} \sum_{\ell=1}^{\tau} \| A_{\ell} \|_F^2 \right) \leq \frac{1}{10}.
\]  

(113)

Similarly, from S to W using Procedure 3, we have the following corollary.

**Corollary 14.** Let \( A = A_1 + \cdots + A_{\tau} \in \mathbb{C}^{n \times n} \) be a Hermitian matrix where \( A_{\ell} \in \mathbb{C}^{n \times n} \) is Hermitian and \( \text{rank}(A_{\ell}) \leq r \) for \( \ell \in [\tau] \). The sampling access each \( A_{\ell} \) is given as in Definition 1. Apply Procedure 2 and Procedure 3 on \( A \) (to obtain row indices \( i_1, \ldots, i_p \) and column indices \( j_1, \ldots, j_p \)) and let \( S \) be defined as Eq. (11) and \( W \) be defined as in Eq. (12). Then with probability at least \( \frac{9}{10} \), there exists an orthonormal set of vectors \( \{y_1, \ldots, y_{\tilde{r}} \} \) in the row space of \( S \) such that

\[
\| S - S \sum_{j=1}^{\tilde{r}} y_i y_i^\dagger \|_F \leq \frac{10^{\hat{r}r}(\tau + 1)}{p} \sum_{\ell=1}^{\tau} \| S_{\ell} \|_F^2,
\]

(114)

where \( \hat{r} \) and \( \tilde{r} \) satisfy \( \tilde{r} \leq \hat{r} \leq \tau r \).

The proof of this corollary is similar to that of Lemma 13, except that the sampling probability satisfies

\[
P_j \geq \frac{\sum_{\ell'=1}^{\hat{r}} \| S_{\ell'}(\cdot, j) \|_F^2}{(\tau + 1) \sum_{\ell=1}^{\tau} \| S_{\ell} \|_F^2}.
\]

(115)

Now, we introduce a new notation which is defined in [FKV04] for the proofs in this section. For a matrix \( M \) and a set of vectors \( x_i, i \in I \),

\[
\Delta(M; x_i, i \in I) := \| M \|_F^2 - \| M - M \sum_{i \in I} x_i x_i^\dagger \|_F^2.
\]

(116)

Note that when \( x_i \) forms a set of orthogonal unit vectors,

\[
\Delta(M; x_i, i \in I) = \sum_{i \in I} x_i^\dagger M^\dagger M x_i.
\]

(117)

The following lemma adapted from [FKV04, Lemma 3] will be useful for our analysis.

**Lemma 15.** Let \( A = \sum_{\ell=1}^{\tau} A_{\ell} \) where \( A_{\ell} \in \mathbb{C}^{n \times n} \) for \( \ell \in [\tau] \). Let \( A \) and \( S \in \mathbb{C}^{k \times n} \) be matrices with same number of columns, and \( \| A^\dagger A - S^\dagger S \|_F \leq \theta \sum_{\ell=1}^{\tau} \| A_{\ell} \|_F^2 \). Then,

1. For any unit vectors \( z \) and \( z' \) in the row space of \( A \),

\[
|z^\dagger A^\dagger A z' - z^\dagger S^\dagger S z'| \leq \theta \sum_{\ell} \| A_{\ell} \|_F^2.
\]

(118)

2. For any set of unit vectors \( z_1, \ldots, z_h \) in the row space of \( A \) and \( h \leq k \),

\[
|\Delta(A; z_i, i \in [h]) - \Delta(S; z_i, i \in [h])| \leq k^2 \theta \left( \sum_{\ell=1}^{\tau} \| A_{\ell} \|_F^2 \right).
\]

(119)
Proof. The first part is true by following the submultiplicity of matrix norms.
\[
|z A^\dagger A z' - z S^\dagger S z'| = |z (A^\dagger A - S^\dagger S) z'|
\]
\[
\leq \|z\| \|A^\dagger A - S^\dagger S\|
\]
\[
\leq \|A^\dagger A - S^\dagger S\| \leq \theta \sum_{\ell=1}^{\tau} \|A_{\ell}\|^2_F.
\]

For the second part of the lemma, we see that
\[
\Delta(A; z_i, i \in [h]) = \|A\|^2_F - \|A - A \sum_i z_i z_i^\dagger\|^2_F
\]
\[
= \text{Tr}(AA^\dagger) - \text{Tr} \left( A - A \sum_i z_i z_i^\dagger \right) (A^\dagger - \sum_i z_i z_i^\dagger A^\dagger)\]
\[
= 2 \text{Tr} \left( A \sum_i z_i z_i^\dagger A^\dagger \right) - \text{Tr} \left( A (\sum_i z_i z_i^\dagger) (\sum_i z_i z_i^\dagger) A^\dagger \right)
\]
\[
= 2 \sum_i z_i^\dagger A^\dagger A z_i - \sum_i z_i^\dagger A^\dagger A z_i - \sum_i (z_i^\dagger z_{i'}^\dagger) z_{i'}^\dagger A^\dagger A z_i
\]
\[
= \sum_i z_i^\dagger A^\dagger A z_i - \sum_{i \neq i'} (z_i^\dagger z_{i'}^\dagger) z_{i'}^\dagger A^\dagger A z_i.
\]

Similarly, we have
\[
\Delta(S; z_i, i \in [h]) = \sum_i z_i^\dagger S^\dagger S z_i - \sum_{i \neq i'} (z_i^\dagger z_{i'}^\dagger) z_{i'}^\dagger S^\dagger S z_i.
\]

Then, by applying the first part of the lemma,
\[
|\Delta(A; z_i, i \in [h]) - \Delta(S; z_i, i \in [h])| = \left| (\sum_i z_i^\dagger A^\dagger A z_i - \sum_{i \neq i'} (z_i^\dagger z_{i'}^\dagger) z_{i'}^\dagger A^\dagger A z_i) 
\right.
\]
\[
- \left(\sum_i z_i^\dagger S^\dagger S z_i - \sum_{i \neq i'} (z_i^\dagger z_{i'}^\dagger) z_{i'}^\dagger S^\dagger S z_i)\right|
\]
\[
\leq \left| \sum_i z_i^\dagger A^\dagger A z_i - \sum_i z_i^\dagger S^\dagger S z_i\right|
\]
\[
- \left| \sum_{i \neq i'} (z_i^\dagger z_{i'}^\dagger) z_{i'}^\dagger A^\dagger A z_i - \sum_{i \neq i'} (z_i^\dagger z_{i'}^\dagger) z_{i'}^\dagger S^\dagger S z_i\right|
\]
\[
\leq k\theta \sum_{\ell} \|A_{\ell}\|^2_F + (k^2 - k)\theta \sum_{\ell} \|A_{\ell}\|^2_F
\]
\[
= k^2 \theta \sum_{\ell} \|A_{\ell}\|^2_F.
\]

Now, we are ready to prove Lemma 6.
Proof of Lemma 6. In this proof, we assume that Eqs. (40) and (41) hold. (Note that Corollary 4 states that these hold with high probability.)

We first show two facts that will be used later. The first fact states that the vectors $v_t$ are almost unit vectors:

**Fact 1.**

$$
\|v_t\|^2 \leq 1 + \frac{\theta(\tau + 1)}{\gamma}.
$$

(133)

To see this, observe first that the first part of Lemma 15 implies

$$
\|S^\dagger u_t\|^2 - \|W^\dagger u_t\|^2 \leq \theta \sum_\ell \|S_\ell\|^2_F.
$$

(134)

Therefore,

$$
\left| \frac{\|S^\dagger u_t\|}{\|W^\dagger u_t\|^2} - 1 \right| \leq \frac{\theta \sum_\ell \|S_\ell\|^2_F}{\gamma \sum_\ell \|W_\ell\|^2_F} \leq \frac{\theta(\tau + 1)}{\gamma}.
$$

(135)

by the bound in Algorithm 4 and Claim 3. Eq. (133) follows immediately.

**Fact 2.**

$$
\Delta(S; v_t, t \in T) \geq \Delta(S; u_t, t \in T) - \left( \frac{\theta(\tau + 1)}{\gamma} + \frac{6\tau(\tau + 1)^2\theta^2}{\gamma^2} \right) \sum_\ell \|A_\ell\|^2_F.
$$

(136)

To show this, observe that by Corollary 4 we have

$$
\|SS^\dagger - WW^\dagger WW^\dagger\|_F \leq \|SS^\dagger (SS^\dagger - WW^\dagger)\|_F + \|(SS^\dagger - WW^\dagger)WW^\dagger\|_F
$$

(137)

$$
\leq \theta \sum_\ell \|S_\ell\|^2_F(\|S_\ell\|^2_F + \|W_\ell\|^2_F).
$$

(138)

and that for $T \neq T' \in T$,

$$
u_t^\dagger WW^\dagger u_{t'} = u_t^\dagger WW^\dagger u_{t'} = 0.
$$

(139)

Now consider $T \neq T' \in T$. Recall that $v_t = \frac{SS^\dagger u_t}{\|W^\dagger u_t\|}$, we have

$$
(v_t^\dagger v_{t'}) (v_t^\dagger SS^\dagger v_{t'}) = \frac{(u_t^\dagger SS^\dagger u_t')(u_t^\dagger SS^\dagger SS^\dagger u_{t'})}{\|W^\dagger u_t\|^2 \|W^\dagger u_{t'}\|^2}.
$$

(140)

This is the cross term to be used in the calculation of Eq. (136). We now bounds its norm. First note that

$$
\|u_t^\dagger SS^\dagger u_{t'}\| = \|u_t^\dagger (SS^\dagger - WW^\dagger) u_{t'}\| \leq \theta \sum_\ell \|S_\ell\|^2_F
$$

(141)
where we used Eq. (139) in the equality. Similarly, using Eq. (137),

\[
\|u_t^\dagger SS^\dagger u_t\| = \|u_t^\dagger (SS^\dagger SS^\dagger - WW^\dagger WW^\dagger)u_t\| \leq \theta \sum_\ell \|S_\ell\|_F^2 (\|S\|_F^2 + \|W\|_F^2).
\]

Using Eq. (141), Eq. (142), and the bound in Algorithm 4 in Eq. (140),

\[
\left\| (v_t^\dagger v_t)(v_t^\dagger S v_t) \right\| \leq \frac{\theta^2 \left( \sum_\ell \|S_\ell\|_F^2 \right)^2 \left( \|S\|_F^2 + \|W\|_F^2 \right)}{\gamma^2 \sum_\ell (\|W_\ell\|_F^2)^2}
\leq \frac{\theta^2 \left( \sum_\ell \|S_\ell\|_F^2 \right)^2 \left( \tau \sum_\ell \|S_\ell\|_F^2 + \tau \sum_\ell \|W_\ell\|_F^2 \right)}{\gamma^2 \sum_\ell (\|W_\ell\|_F^2)^2}
\leq \frac{\tau \theta^2 \left( \sum_\ell \|S_\ell\|_F^2 \right)^2 \left( \sum_\ell \|S_\ell\|_F^2 + 2 \sum_\ell \|S_\ell\|_F^2 \right)}{\gamma^2 \sum_\ell \left( \frac{1}{\tau+1}\right) \|S_\ell\|_F^2}
\leq \frac{6\tau(\tau + 1)^2\theta^2}{\gamma^2} \sum_\ell \|A_\ell\|_F^2,
\]

where we used Cauchy-Schwarz inequality in Eq. (144) and Claim 3 in Eqs. (145) and (146), respectively. Next, we bound the diagonal terms. By Cauchy-Schwarz inequality

\[
\|u\| \|SS^\dagger u\| \geq u^\dagger SS^\dagger u = \|S^\dagger u\|^2.
\]

So for \( t \in [\bar{\gamma}] \),

\[
v_t^\dagger S v_t = \frac{u_t^\dagger SS^\dagger S v_t u_t}{\|W^\dagger u_t\|^2} \geq \frac{\|S^\dagger u_t\|^4}{\|W^\dagger u_t\|^2}.
\]

We then have

\[
\sum_{t \in [\bar{\gamma}]} v_t^\dagger S v_t \geq \sum_{t \in [\bar{\gamma}]} \frac{\|S^\dagger u_t\|^4}{\|W^\dagger u_t\|^2}
= \sum_{t \in [\bar{\gamma}]} \left( u_t^\dagger S S u_t \right) \left( \frac{\|S^\dagger u_t\|^2}{\|W^\dagger u_t\|^2} \right)
\geq \left( 1 - \frac{\theta(\tau + 1)}{\gamma} \right) \sum_{t \in [\bar{\gamma}]} u_t^\dagger SS^\dagger u_t
= \left( 1 - \frac{\theta(\tau + 1)}{\gamma} \right) \Delta(S^\dagger; u_t, t \in [\bar{\gamma}]),
\]
According to the statement 2 of Lemma 15, we have

\[ \Delta(S; v_t, t \in [\hat{r}]) = \sum_{t \in [\hat{r}]} v_t^* S v_t - \sum_{t \neq t'} (v_t^* c_{t'}) v_{t'}^* S v_t \]  

where we used Eq. (135) in Eq. (151). Putting Eqs. (146) and (152) into Eq. (128) of Lemma 15,

\[ \geq \left( 1 - \frac{\theta(\tau + 1)}{\gamma} \right) \Delta(S^t; u_t, t \in [\hat{r}]) - \frac{6\tau(\tau + 1)^2\theta 2\gamma^2}{\gamma^2} \sum_{t} \| A_t \|^2_F \]  

Putting Eqs. (146) and (152) into Eq. (128) of Lemma 15,

\[ \geq \Delta(S^t; u_t, t \in [\hat{r}]) - \left( \frac{2\theta(\tau + 1)}{\gamma} + \frac{6\tau(\tau + 1)^2\theta 2\gamma^2}{\gamma^2} \right) \sum_{t} \| A_t \|^2_F, \]  

where we used \( \Delta(S^t; u_t, t \in [\hat{r}]) \leq \| S \|^2_F \leq 2\tau \sum_{t} \| A_t \|^2_F \) in the last line. This completes the proof of Eq. (136).

By Lemma 6, with probability at least 9/10, there exist orthonormal vectors \( x_t \) for \( t \in [\hat{r}'] \) (for \( \hat{r}' \leq \hat{r} \)) in the row space of \( S \) such that

\[ \Delta(A; x_t, t \in [\hat{r}']) \geq \| A \|^2_F - \| A - A \sum_{t=1}^{\hat{r}'} x_t x_t^* \|_F^2 \geq \| A \|^2_F - \frac{10\hat{r}\tau}{p} \sum_{t=1}^{\tau} \| A_t \|^2_F. \]  

According to the statement 2 of Lemma 15, we have

\[ \Delta(S; x_t, t \in [\hat{r}']) \geq \Delta(A; x_t, t \in [\hat{r}']) - \hat{r}'^2 \theta \sum_{t=1}^{\tau} \| A_t \|^2_F \geq \| A \|^2_F - \left( \frac{10\hat{r}\tau}{p} + \hat{r}'^2 \theta \right) \sum_{t=1}^{\tau} \| A_t \|^2_F. \]  

Since \( S \) and \( S^t \) have the same singular values, there exist orthonormal vectors \( y_t \) for \( t \in [\hat{r}'] \) in the row space of \( S^t \) satisfying

\[ \Delta(S^t; y_t, t \in [\hat{r}']) \geq \| A \|^2_F - \left( \frac{10\hat{r}\tau}{p} + \hat{r}'^2 \theta \right) \sum_{t=1}^{\tau} \| A_t \|^2_F. \]  

Now, applying Corollary 14, it holds that with probability at least 9/10, there exist orthonormal vectors \( z_t \) for \( t \in [\hat{r}'] \) in the row space of \( W \) such that

\[ \Delta(S^t; z_t, t \in [\hat{r}']) = \| S \|^2_F - \| S - S \sum_{j=1}^{\hat{r}'} z_j z_j^* \|_F^2 \]  

where we used Eq. (135) in Eq. (151). Putting Eqs. (146) and (152) into Eq. (128) of Lemma 15,

\[ \geq \| S \|^2_F - \frac{10\hat{r}\tau(\tau + 1)}{p} \sum_{t=1}^{\tau} \| S_t \|^2_F \]  

Putting Eqs. (146) and (152) into Eq. (128) of Lemma 15,

\[ \geq \| S \|^2_F - \| S - S \sum_{j=1}^{\hat{r}'} y_j y_j^* \|_F^2 - \frac{10\hat{r}\tau(\tau + 1)}{p} \sum_{t=1}^{\tau} \| S_t \|^2_F \]  

where we used \( \Delta(S^t; y_t, t \in [\hat{r}']) \leq \| S \|^2_F \leq 2\tau \sum_{t} \| A_t \|^2_F \) in the last line. This completes the proof of Eq. (136).
Again, by the statement 2 of Lemma 15, we have

\[
\Delta(W^\dagger; z_t, t \in [\hat{r}]) \geq \Delta(S^\dagger; z_t, t \in [\hat{r}]) - \hat{r}^2 \gamma \sum_{\ell=1}^\tau \|S_\ell\|^2_F
\]  

(164)

\[
\geq \|A\|^2_F - \left(\frac{20\hat{r}^2\tau(\tau + 1)}{p} + 3\hat{r}^2\theta + 4\hat{r}\gamma\right) \sum_{\ell=1}^\tau \|A_\ell\|^2_F.
\]  

(165)

\[
\geq \|A\|^2_F - \left(\frac{20\hat{r}^2\tau(\tau + 1)}{p} + 3\hat{r}^2\theta + 4\hat{r}\gamma\right) \sum_{\ell=1}^\tau \|A_\ell\|^2_F.
\]  

(166)

As \(u_t\) for \(t \in [\hat{r}]\) (computed in Algorithm 4) are the left singular vectors of \(W\), we have

\[
\Delta(W^\dagger; u_t, t \in [\hat{r}]) \geq \Delta(W^\dagger; z_t, t \in [\hat{r}]) \geq \|A\|^2_F - \left(\frac{20\hat{r}^2\tau(\tau + 1)}{p} + 3\hat{r}^2\theta\right) \sum_{\ell=1}^\tau \|A_\ell\|^2_F.
\]  

(167)

Recall that \(u_t\) for \(t \in [\hat{r}]\) are the singular vectors after the filter in Algorithm 4 and \(u_t\) for \(t \in [\hat{r}]\) are orthonormal vectors. It holds that

\[
\Delta(W^\dagger; u_t, t \in [\hat{r}]) \geq \Delta(W^\dagger; u_t, t \in [\hat{r}]) - \hat{r}^2 \gamma \sum_{\ell=1}^\tau \|W_\ell\|^2_F
\]  

(168)

\[
\geq \|A\|^2_F - \left(\frac{20\hat{r}^2\tau(\tau + 1)}{p} + 3\hat{r}^2\theta + 4\hat{r}\gamma\right) \sum_{\ell=1}^\tau \|A_\ell\|^2_F.
\]  

(169)

Now, we apply the statement 2 of Lemma 15 one more time. It follows that

\[
\Delta(S^\dagger; u_t, t \in [\hat{r}]) \geq \Delta(W^\dagger; u_t, t \in [\hat{r}]) - \hat{r}^2 \theta \sum_{\ell=1}^\tau \|S_\ell\|^2_F
\]  

(170)

\[
\geq \|A\|^2_F - \left(\frac{20\hat{r}^2\tau(\tau + 1)}{p} + 5\hat{r}^2\theta + 4\hat{r}\gamma\right) \sum_{\ell=1}^\tau \|A_\ell\|^2_F.
\]  

(171)

Lemma 15 implies that

\[
\Delta(A; v_t, t \in [\hat{r}]) \geq \Delta(S; v_t, t \in [\hat{r}]) - \left(1 + \frac{\theta(\tau + 1)}{\gamma}\right) \hat{r}^2 \theta \sum_{\ell=1}^\tau \|A\|^2_F
\]  

(172)

\[
\geq \|A\|^2_F - \left(\frac{20\hat{r}^2\tau(\tau + 1)}{p} + 5\hat{r}^2\theta + 4\hat{r}\gamma + \left(1 + \frac{\theta(\tau + 1)}{\gamma}\right) \hat{r}^2 \theta + \frac{2\tau(\tau + 1)\theta}{\gamma^2} + \frac{6\tau(\tau + 1)^2\theta^2\hat{r}^2}{\gamma^2}\right) \sum_{\ell=1}^\tau \|A_\ell\|^2_F.
\]  

(173)

(174)

By the definition of the \(\Delta\) function, we conclude that

\[
\|A - A \sum_{t \in [\hat{r}]} v_t v_t^\dagger\|_F^2 \leq \left(\frac{20\hat{r}^2\tau(\tau + 1)}{p} + 5\hat{r}^2\theta + 4\hat{r}\gamma + \left(1 + \frac{\theta(\tau + 1)}{\gamma}\right) \hat{r}^2 \theta + \frac{2\tau(\tau + 1)\theta}{\gamma^2} + \frac{6\tau(\tau + 1)^2\theta^2\hat{r}^2}{\gamma^2}\right) \sum_{\ell=1}^\tau \|A_\ell\|^2_F.
\]  

(175)

(176)

The claim bound follows by the choice of \(p\) and \(\gamma\) in Line 1 of Algorithm 4, \(\theta\) in Corollary 4, and the fact that \(\hat{r} \leq \tau r\).
C Approximate orthogonality of the approximate singular vectors

In this section, we prove the following restated lemma.

Lemma 5. Let $A = A_1 + \cdots + A_r$ be a matrix with the sampling access to each $A_\ell$ as in Definition 1. Assume $\|A_\ell\| \leq 1$ and $\text{rank}(A_\ell) \leq r$ for all $\ell \in [r]$. Take $A$ and error parameter $\epsilon$ as the input of Algorithm 4 and obtain the $\sigma$ and $\tilde{u}_j$ for $j \in \{1, \ldots, \tilde{r}\}$. Let $V \in \mathbb{C}^{n \times \tilde{r}}$ be the matrix such that $V(:,j) = \frac{\tilde{u}_j}{\tilde{\sigma}_j}$ for $j \in \{1, \ldots, \tilde{r}\}$. Then, with probability at least $9/10$, the following statements hold:

1. There exists an isometry $U \in \mathbb{C}^{n \times \tilde{r}}$ whose column vectors span the column space of $V$ satisfying $\|U - V\|_F \leq \frac{600\epsilon}{(\tau+1)}$.
2. $|\|V\| - 1| \leq \frac{600\epsilon}{(\tau+1)}$.
3. Let $\Pi_V$ be the projector on the column space of $V$, then it holds that $\|VV^\dagger - \Pi_V\|_F \leq \frac{600\epsilon}{(\tau+1)}$.
4. $\|V^\dagger V - I\|_F \leq \frac{600\epsilon}{(\tau+1)}$.

Proof. Most of the arguments in this proof are similar to the proofs in [Tan18a, Lemma 6.6, Corollary 6.7, Proposition 6.11]. Let $v_j \in \mathbb{C}^n$ denote the column vector $V(:,j)$, i.e., $v_j = \frac{\tilde{u}^\dagger}{\tilde{\sigma}}\tilde{u}_j$. Choose $\theta = (\tau+1)\sqrt{\gamma/\rho}$. When $i \neq j$, with probability at least $9/10$, it holds that

$$|v_i^\dagger v_j| = \frac{|\tilde{u}_i^\dagger SS^\dagger \tilde{u}_j|}{\tilde{\sigma}_i \tilde{\sigma}_j} \leq \frac{|\tilde{u}_i^\dagger (SS^\dagger - WW^\dagger) \tilde{u}_j|}{\tilde{\sigma}_i \tilde{\sigma}_j} \leq \frac{\theta \sum_{\ell=1}^{\tilde{r}} \|S\|_F^2}{\gamma \sum_{\ell=1}^{\tilde{r}} \|W\|_F^2} \leq \frac{(\tau+1)\theta}{\gamma},$$

(177)

where the second inequality follows from Corollary 4, and the last inequality uses Claim 3. Similarly, when $i = j$, the following holds with probability at least $9/10$.

$$|\|v_i\| - 1| = \frac{|\tilde{u}_i^\dagger SS^\dagger \tilde{u}_i - \tilde{\sigma}_i^2|}{\tilde{\sigma}_i^2} \leq \frac{|\tilde{u}_i^\dagger (SS^\dagger - WW^\dagger) \tilde{u}_i|}{\tilde{\sigma}_i^2} \leq \frac{(\tau+1)\theta}{\gamma}.$$

(178)

Since $|(V^\dagger V)(i,j)| = |v_i^\dagger v_j|$, each diagonal entry of $V^\dagger V$ is at most $(\tau+1)\theta/\gamma$ away from 1 and each off-diagonal entry is at most $(\tau+1)\theta/\gamma$ away from 0. More precisely, let $M \in \mathbb{C}^{n \times n}$ be the matrix with all ones, i.e., $M(i,j) = 1$ for all $i, j \in \{1, \ldots, n\}$, then for all $i, j \in \{1, \ldots, n\}$, we have

$$\left(I - \frac{(\tau+1)\theta}{\gamma} M\right)(i,j) \leq (V^\dagger V)(i,j) \leq \left(I + \frac{(\tau+1)\theta}{\gamma} M\right)(i,j).$$

(179)

To prove statement 1, we consider the QR decomposition of $V$. Let $Q \in \mathbb{C}^{n \times n}$ be a unitary and $R \in \mathbb{C}^{n \times k}$ be upper-triangular with positive diagonal entries satisfying $V = QR$. Since $V^\dagger V = R^\dagger R$, we have

$$\left(I - \frac{(\tau+1)\theta}{\gamma} M\right)(i,j) \leq (R^\dagger R)(i,j) \leq \left(I + \frac{(\tau+1)\theta}{\gamma} M\right)(i,j).$$

(180)

Let $\hat{R}$ be the upper $k \times k$ part of $R$. Since $R$ is upper-triangular, $\hat{R}^\dagger \hat{R} = R^\dagger R$. Hence, $\hat{R}$ can be viewed as an approximate Cholesky factorization of $I$ with error $\frac{(\tau+1)\theta}{\gamma}$. As a consequence of [CPS96, Theorem 1], we have $\|R - I\|_F \leq \frac{r(\tau+1)\theta}{\sqrt{2}\gamma} + O((\tilde{r}(\tau+1)\theta/\gamma)^2)$. Now, we define $R' \in \mathbb{C}^{n \times k}$
as the matrix with $I$ on the upper $k \times k$ part and zeros everywhere else. Let $U = QR'$. Clearly, $U$ is isometry as it contains the first $k$ columns of $Q$. To see the column vectors of $V$ span the column space of $V$, note that $UU^\dagger = QR'R'^\dagger Q^\dagger$ where $R'R'^\dagger$ only contains $I$ on its upper-left $(k \times k)$-block, and $VV^\dagger = QRR'^\dagger$ where $R'R$ only contains a diagonal matrix on its upper-left $(k \times k)$-block. To bound the distance between $U$ and $V$, we have $\|U - V\|_F = \|QR'(R' - R)\|_F = \|R' - R\|_F \leq \frac{\tilde{r}(\tau + 1)\theta}{\sqrt{2} \gamma} + O((\tilde{r}(\tau + 1)\theta / \gamma)^2)$. Then statement follows from the choice of $p$ and $\gamma$ in Line 1 of Algorithm 4 and the fact that $\tilde{r} \leq \tau r$.

Statement 2 follows from the triangle inequality:

\begin{align*}
\|V\| - 1 &= \|V - U\| \leq \|V - U\|_F, \quad \text{and} \\
1 - \|V\| &= \|U\| - \|V\| \leq \|U - V\| \leq \|U - V\|_F. \quad (181)
\end{align*}

For statement 3, we have

\begin{align*}
\|VV^\dagger - \Pi V\|_F &= \|VV^\dagger - UU^\dagger\|_F \\
&\leq \|V(V^\dagger - U^\dagger)\|_F + \|(V - U)U^\dagger\|_F \\
&\leq \|V\| \|V^\dagger - U^\dagger\|_F + \|V - U\|_F \|U^\dagger\| \\
&\leq \frac{\sqrt{2} \tilde{r}(\tau + 1)\theta}{\gamma} + O\left(\frac{\sqrt{2} \tilde{r}(\tau + 1)\theta}{\gamma}\right)^2, \quad (186)
\end{align*}

and statement 3 follows from the choice of $p$ and $\gamma$ and the fact that $\tilde{r} \leq \tau r$ in Line 1 of Algorithm 4.

Similarly, statement 4 follows by bounding the distance $\|V^\dagger V - U^\dagger U\|_F$. \qed