COBOUNDARIES OF NONCONVENTIONAL ERGODIC AVERAGES

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Abstract. Let \((X, A, \mu)\) be a probability measure space and let \(T_i, 1 \leq i \leq H\), be invertible bi measurable measure preserving transformations on this measure space. We give a sufficient condition for the product of \(H\) bounded functions \(f_1, f_2, ..., f_H\) to be a coboundary. This condition turns out to be also necessary when one seeks bounded coboundaries.

1. Introduction

The purpose of this short article is to answer a question brought to our attention by S. Donoso \(^1\) during the 2017 ETDS workshop held at Chapel Hill.

To this end we refine the setting in \([2]\).

Definition 1.1. A probability measure preserving system \((X, F, \mu, T_1, T_2, \ldots, T_H)\) is a combination of a probability measure space \((X, F, \mu)\) and \(T_i, 1 \leq i \leq H\ bi-measurable invertible measure preserving maps acting on this probability space.

Given a probability measure preserving system \((X, F, \mu, T_1, T_2, \ldots, T_H)\), \(\mu_\Delta\) is the diagonal measure on \(X^H\), \(\Phi = T_1 \times T_2 \times \cdots \times T_H\), and \(\nu\) is the diagonal-orbit measure of \(\Phi\), i.e.

\[\nu(A) = \frac{1}{3} \sum_{n \in \mathbb{Z}} \frac{1}{2^{2n}} \mu_\Delta(\Phi^{-n}A).\]

We note that \(\nu\) is nonsingular, since \(\frac{1}{3} \nu(A) \leq \nu(\Phi^{-1}A) \leq 2 \nu(A)\).

Definition 1.2. The diagonal orbit system of the probability measure preserving system \((X, F, \mu, T_1, T_2, \ldots, T_H)\) is the system \((X^H, F^H, \nu, \Phi)\).

Remarks

(1) The maps \(T_i\) do not necessarily commute.

(2) The nonsingularity of \(\Phi\) with respect to \(\nu\) implies the following simple but key lemma (This lemma does not seem to hold when one replaces \(\nu\) with the diagonal measure \(\mu_\Delta\) on \((X^H, F^H)\), defined by the equation \(\int F(x_1, x_2, \ldots, x_H) d\mu_\Delta = \int F(x, x, \ldots, x) d\mu\).)

Lemma 1.3. Let \(F_n\) be a sequence of measurable functions defined on \(X^H\). If \(F_n\) converges \(\nu\) a.e. then the sequence \(G_n = F_n \circ \Phi\) converges \(\nu\) a.e. as well.

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\(^1\)He indicated that this question was mentioned to him by J.P. Conze and Y. Kifer
Lemma 1.5. Let \( A = \{ z \in X^H; F_n(z) \text{ converges} \} \) and \( B = \{ z \in X^H; G_n(z) \text{ converges} \} \). We have \( B = \Phi^{-1}(A) \). Therefore if \( \nu(A^c) = 0 \) we have \( \nu(\Phi^{-1}(A^c)) = 0 \) by the non singularity of \( \nu \).

We wish to prove the following proposition.

**Proposition 1.4.** Let \( (X, F, \mu, T_1, \ldots, T_H) \) be a measure preserving system, and \( f_1, f_2, \ldots, f_H \in L^\infty(\mu) \) and \( 1 \leq p < \infty \).

1. Let us assume that the supremum of the nonconventional ergodic sums is \( L^p \)-bounded, i.e.

\[
\sup_N \left\| \sum_{n=1}^N \prod_{i=1}^H f_i \circ T^n_i \right\|_{L^p(\nu)} < \infty.
\]

2. Then the product of the functions is a coboundary in \( L^p(X^H, \nu) \), i.e. if \( \Phi = T_1 \times T_2 \times \cdots \times T_H \), there exists \( V \in L^p(X^H, \nu) \) such that

\[
\bigotimes_{i=1}^H f_i = V - V \circ \Phi, \text{\( \nu \)-a.e.}
\]

Therefore, for \( \mu_\Delta \)-a.e. \( (x_1, x_2, \ldots, x_H) \in X^H \), we have

\[
f_1(x_1)f_2(x_2) \cdots f_H(x_H) = V(x_1, x_2, \ldots, x_H) - V(T_1x_1, T_2x_2, \ldots, T_Hx_H).
\]

Proof. Let \( A = \{ z \in X^H; f_n(z) \text{ converges}\} \) and \( \Phi^{-1}(A) \). Therefore if \( \nu(A^c) = 0 \) we have \( \nu(\Phi^{-1}(A^c)) = 0 \) by the non singularity of \( \nu \).

We give only the proof for the case \( p = 1 \). We use the following a.e.-convergence result obtained by Komlós in 1967. When \( 1 < p < \infty \) the reflexivity of \( L^p(\nu) \) allows to bypass this lemma. For \( p = \infty \) the assumptions (1) and (2) in the statement of Theorem 1.4 are equivalent. We state it separately as a corollary.

**Lemma 1.5** (\[3\]). Let \( (X, F, \mu) \) be a probability measure space, and \( (g_n) \) be a sequence in \( L^1(\mu) \). Assume that

\[
\liminf_n \| g_n \|_{L^1(\mu)} < \infty.
\]

Then there exists a subsequence \( (g_{n_k})_k \) and a function \( g \in L^1(\mu) \) such that for \( \mu \)-a.e. \( x \in X \),

\[
\lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^K g_{n_k}(x) = g(x).
\]

Proof of Proposition 1.4. We show that the techniques used for an invariant measure can be applied to our current nonsingular setting. The assumption made tells us that, if \( F = f_1 \otimes f_2 \otimes \cdots \otimes f_H \), we have

\[
\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^N F \circ \Phi^n \right\|_{L^1(\nu)} = 0.
\]

Therefore, there exists a subsequence \( N_k \) of natural numbers such that

\[
\lim_{N_k \to \infty} \frac{1}{N_k} \sum_{n=1}^{N_k} F \circ \Phi^n(z) = 0
\]
for \(\nu\text{-a.e. } z \in X^H\). Because \(\Phi\) is nonsingular with respect to \(\nu\), we also know that

\[
\lim_{N_k \to \infty} \frac{1}{N_k} \sum_{n=1}^{N_k} F \circ \Phi^{n+1}(z) = 0
\]

for \(\nu\text{-a.e. } z \in X^H\) for the same subsequence \((N_k)\). Set

\[
D_{N_k} = \frac{1}{N_k} \sum_{n=1}^{N_k} F \circ \Phi^n.
\]

Since \(\sup_n \left\| \sum_{j=0}^{n-1} F \circ \phi^j \right\|_{L^1(\nu)} < \infty\), \(\lim \inf_k \left\| D_{N_k} \right\|_{L^1(\nu)} < \infty\). Thus, we may apply Lemma 1.3 to show that there exists a subsequence of \((D_{N_k})\) (which remain denoted as \((D_{N_k})\)) such that the averages \(\frac{1}{K} \sum_{k=1}^{K} D_{N_k}\) converge \(\nu\text{-a.e.}\) to a function \(V \in L^1(\nu)\). Similarly, by Lemma 1.3 we have

\[
V \circ \Phi(z) = \lim_{K} \frac{1}{K} \sum_{k=1}^{K} D_{N_k} \circ \Phi(z)
\]

for \(\nu\text{-a.e. } z \in X^H\). Therefore,

\[
V - V \circ \Phi = \lim_{K} \left( \frac{1}{K} \sum_{k=1}^{K} F - \frac{1}{K} \sum_{k=1}^{K} \left( \frac{1}{N_k} \sum_{n=1}^{N_k} F \circ \Phi^n \right) \right) = F.
\]

Since \(V - V \circ \Phi = F\) for \(\nu\text{-a.e.}\), the equality certainly holds for \(\mu_\Delta\text{-a.e.}\) (the construction of \(\nu\) guarantees that \(V \in L^1(\mu_\Delta)\)).

\[\square\]

**Corollary 1.6.** Let \((X, \mathcal{F}, \mu, T_1, \ldots, T_H)\) be a measure preserving system, and \(f_1, f_2, \ldots, f_H \in L^\infty(\mu)\) The following statements are equivalent.

1. The supremum of the nonconventional ergodic sums is \(L^\infty\)-bounded, i.e.

\[
\sup_N \left\| \sum_{n=1}^{N} \prod_{i=1}^{H} f_i \circ T_i^n \right\|_{L^\infty(\nu)} < \infty.
\]

2. The product of the functions is a coboundary in \(L^\infty(X^H, \nu)\), i.e. if \(\Phi = T_1 \times T_2 \times \cdots \times T_H\), there exists \(V \in L^\infty(X^H, \nu)\) such that

\[
\bigotimes_{i=1}^{H} f_i = V - V \circ \Phi, \nu\text{-a.e.}
\]

Therefore, for \(\mu\text{-a.e. } x \in X\), we have

\[
f_1(x)f_2(x) \cdots f_H(x) = V(x, x, \ldots, x) - V(T_1x, T_2x, \ldots, T_Hx).
\]

**Proof.** The implication 1) implies 2) can be obtained by following the same path as in the proof of the previous proposition. The only thing to check is that \(V \in L^\infty(\nu)\). But this follows from the fact that if

\[
\sup_N \left\| \sum_{n=1}^{N} \prod_{i=1}^{H} f_i \circ T_i^n \right\|_{L^\infty(\nu)} < C < \infty
\]
then $\|D_{N_k}\|_{L^\infty} \leq C$. From this observation one can conclude that the limit function $V$ is also in $L^\infty(\nu)$.

For the reverse implication, if $F = f_1 \times f_2 \times \cdots \times f_H$ is a coboundary in $L^\infty(\nu)$ (i.e. $F = V - V \circ \Phi$ where $V \in L^\infty(\nu)$) then

$$\sum_{n=1}^{N} F \circ \Phi^n = V - V \circ \Phi^{N+1}$$

and

$$\|\sum_{n=1}^{N} F \circ \Phi^n\|_{L^\infty(\nu)} = \|V - V \circ \Phi^{N+1}\|_{L^\infty(\nu)} \leq 2\|V\|_{L^\infty(\nu)}$$

\[\square\]

**Remark**

The assumption

$$\sup_{N} \left\| \sum_{n=1}^{H} \prod_{i=1}^{H} f_i \circ T_i^n \right\|_{L^p(\nu)} < \infty.$$ 

is satisfied when

$$\sup_{N, m \in \mathbb{Z}} \left\| \sum_{n=1}^{H} \prod_{i=1}^{H} f_i \circ T_i^{n+m} \right\|_{L^p(\mu)} < \infty.$$ 

This last condition is equivalent to

$$\sup_{N \in \mathbb{Z}} \left\| \sum_{n=0}^{H} \prod_{i=1}^{H} f_i \circ T_i^n \right\|_{L^p(\mu)} < \infty.$$ 

which may be easier to check in the applications.

**References**

[1] **I. Assani:** “A note on the equation $y = (I - T)x$ in $L^1$” Illinois J. of Math. vol 43, 3, (1999) p.540-541.

[2] **I. Assani:** “Pointwise recurrence for commuting measure preserving transformations” Preprint, arXiv:1312.5270v2, 2015

[3] **J. Komlos:** “A generalization of a problem of Steinhaus” Acta Math. Acad. Sci. Hungar. 18:217-229, 1967.