Linear Algebra on investment portfolio optimization model

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Abstract. In this paper we discuss the issue of linear algebra on the investment portfolio optimization models. It was assumed that stock returns are analyzed have a certain distribution, so that the mean and variance and covariance between the separation can be determined. Return of some stock used to form a vector averaging, and the number of shares used as the basis to form a unit vector. While the variance of each stock as well as the covariance between stocks, is used to form a covariance matrix. The investment portfolio was formed consisting of several stocks, in order to maximize the expected return and minimize risk. The portfolio optimization was performed using linear algebra approach. The result is a formula used to determine the optimum composition of the portfolio weights. The resulting formula is very useful for the analysis of the investment portfolio optimization.

1. Introduction

Model of portfolio formation, the first appeared in 1952 by Markowitz, famous for Mean-Variance models of Markowitz [1]. In the establishment of a portfolio, investors always want to maximize the expected returns with a certain level of risk that is willing to bear, or are looking for a portfolio which offers the lowest risk with a certain rate of return expectations [2]. Characteristics of such a portfolio is called efficient portfolio. To establish an efficient portfolio, had to hold on assumptions about how the behavior of investors in making investment decisions to be taken. One of the most important assumptions is that all investors have the risk tolerance level [3,4].

The issue that is fundamental in determining the optimal portfolio is how to determine the weight of the composition (proportion) allocations of funds in each asset in the formation of the investment portfolio. To determine the optimum portfolio weights, the mathematics plays an important role. Many techniques to determine the optimum portfolio weights can be used. For examples by using ordinary algebra techniques, and application of linear algebraic equations [2]. In ordinary algebra techniques, portfolio objective function is expressed as a function of some of the weight of the portfolio is a weighted mean and variance of each stock return, wherein the amount of weight equal to one. While in linear algebra, portfolio objective function is expressed as equations involving weight vector, the mean vector, unit vector, and the covariance matrix. Wherein the weight vectors transpose times the unit vector must be equal to one [5]. Both in engineering and the use of ordinary algebra linear algebra, optimization of the portfolio can be done by using Lagrange multiplier and Kuhn-Tucker method [6,7].

The research in this paper is intended conduct investment portfolio optimization model formulation with risk tolerance approach using linear algebra. This model formulation aimed at creating a mathematical model that can be used to determine the weight of the composition (proportion) allocations
of funds in each asset, in the formation of the optimal portfolio. As a numerical illustration, the model has been formulated subsequently used to analyze five stocks that are traded on the capital market in Indonesia. The goal is to get the weight of the allocation of funds for each stock in a portfolio optimum formation [8].

2. Mathematical model
In this part of the mathematical model is discussed about the equation for determining stock returns, and formulating weight optimization of portfolio investment.

2.1 Stocks return.
Suppose \( P_{it} \) stock price \( i \) at time \( t \), and \( r_{it} \) stock return \( i \) at time \( t \). The value of \( r_{it} \) can be calculated using the following equation.

\[
    r_{it} = \ln \left( \frac{P_{it}}{P_{it-1}} \right),
\]

where \( i = 1, ..., N \) with \( N \) number of stocks that were analyzed, and \( t = 1, ..., T \) with \( T \) the number of stock price data observed [9].

2.2 Weight optimization of portfolio investment.
Suppose \( r_i \) stock return \( i \) where \( i = 1, ..., N \) with \( N \) the number of stocks that were analyzed. Suppose also \( w' = (w_1, ..., w_N) \) weight vector, \( r' = (\eta, \mu_2, ..., \mu_N) \) vector stock returns, and \( e' = (1, 1, ..., 1) \) unit vector.

Portfolio return can be expressed as \( r_p = (w_1, w_2, ..., w_N) \begin{bmatrix} \eta \\ \mu_2 \\ \mu_N \end{bmatrix} \) with \( (w_1, w_2, ..., w_N) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \)

Suppose \( \mu' = (\mu_1, \mu_2, ..., \mu_N) \), expectations of portfolio \( \mu_p \) can be expressed as [2,10]:

\[
    \mu_p = E[r_p] = (w_1, w_2, ..., w_N) \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_N \end{bmatrix}.
\]

Suppose given covariance matrix \( \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & ... & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & ... & \sigma_{2N} \\ M & M & M & M \\ \sigma_{N1} & \sigma_{N2} & ... & \sigma_{NN} \end{bmatrix} \), where \( \sigma_{ij} = \text{Cov}(r_i, r_j) \). Variance of the portfolio return can be expressed as follows:

\[
    \sigma_p^2 = (w_1, w_2, ..., w_N) \begin{bmatrix} \sigma_{11} & \sigma_{12} & ... & \sigma_{1N} \\ \sigma_{21} & \sigma_{22} & ... & \sigma_{2N} \\ M & M & M & M \\ \sigma_{N1} & \sigma_{N2} & ... & \sigma_{NN} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ M \\ w_N \end{bmatrix}.
\]
Means, for investors with risk tolerance $\tau$ ($\tau \geq 0$) need to resolve the problem of portfolio

$$\text{Maximize} \left( \frac{\mu_1}{\mu_2} \right) \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \ldots & \sigma_{1N} \\ \sigma_{21} & \sigma_2^2 & \ldots & \sigma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{N1} & \sigma_{N2} & \ldots & \sigma_N^2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix}$$

$$2\tau(w_1, w_2, \ldots, w_N) - (w_1, w_2, \ldots, w_N) \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_N \end{pmatrix}$$

(4)

the condition $w_1, w_2, \ldots, w_N = 1$,

Equation (3) is the optimization problem of quadratic convex [4]. Lagrange multiplier function of the portfolio optimization problem is given by

$$L(w, \lambda) = 2\tau(w_1, w_2, \ldots, w_N) - (w_1, w_2, \ldots, w_N) \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_N \end{pmatrix}$$

$$+ \lambda(w_1, w_2, \ldots, w_N) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

(5)

Based on the Kuhn-Tucker theorem, the optimality condition of equation (5) is $\frac{\partial L}{\partial w} = 0$ and $\frac{\partial L}{\partial \lambda} = 0$. Completed two conditions of optimality mentioned equation, the equation would be the optimal portfolio weights as follows:

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_N \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1N} \\ a_{21} & a_{22} & \ldots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \ldots & a_{NN} \end{pmatrix}$$

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_N \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

(6)
Where

\[
\begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1N} \\
a_{21} & a_{22} & \ldots & a_{2N} \\
M & M & M & M \\
a_{N1} & a_{N2} & \ldots & a_{NN}
\end{pmatrix}
\begin{pmatrix}
\sigma_1^2 & \sigma_{12} & \ldots & \sigma_{1N} \\
\sigma_{21} & \sigma_2^2 & \ldots & \sigma_{2N} \\
M & M & M & M \\
\sigma_{N1} & \sigma_{N2} & \ldots & \sigma_N^2
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
M & M & M & M \\
0 & 0 & \ldots & 1
\end{pmatrix}
\]

If denoted as

\[
\Sigma^{-1} = \begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1N} \\
a_{21} & a_{22} & \ldots & a_{2N} \\
M & M & M & M \\
a_{N1} & a_{N2} & \ldots & a_{NN}
\end{pmatrix}
\]

It was clear that \( \Sigma^{-1} \) is the inverse of a matrix \( \Sigma \).

Furthermore, with substituting \( \mathbf{w}^T = (w_1, \ldots, w_N) \) from equation (6), into equation (2) and (3), respectively obtained the values of the expectation and variance of the portfolio

3. Result and analysis of numerical illustration

The data used in the analysis of numerical illustration here consists of five stocks, i.e. stocks which is symbolized as \( S_1, S_2, S_3, S_4, \) and \( S_5 \). These stocks were subsequently determined the value of return using equation (1). Mean return of stocks is given in the form of vector the mean as:

\[
\mathbf{\mu}^T = (0.001407, 0.004919, 0.014656, 0.004940, 0.004406).
\]

Since the number of shares analyzed consisted of five stocks, then compiled as a unit vector

\[
\mathbf{e}^T = (1, 1, 1, 1).
\]

Furthermore, by using the values of variance and together with the values of the covariance between stocks, used to form the covariance matrix \( \Sigma \) and the inverse matrix \( \Sigma^{-1} \) as follows:

\[
\Sigma = \begin{pmatrix}
0.001200 & 0.000136 & 0.000251 & 0.000113 & 0.000401 \\
0.000136 & 0.001840 & 0.000092 & 0.000315 & 0.000225 \\
0.000251 & 0.000092 & 0.001078 & 0.000512 & 0.001333 \\
0.000113 & 0.000315 & 0.000512 & 0.000956 & 0.000757 \\
0.000401 & 0.000225 & 0.000133 & 0.000956 & 0.001399
\end{pmatrix}
\quad \text{and} \quad
\Sigma^{-1} = 10^3 \times \begin{pmatrix}
0.9613 & -0.0345 & -0.2020 & 0.0257 & -0.2522 \\
-0.0345 & 0.5904 & 0.0738 & -0.2237 & -0.0801 \\
-0.2020 & 0.0738 & 1.3037 & -0.6955 & -0.0406 \\
0.0257 & -0.2237 & -0.6955 & 1.4880 & 0.0150 \\
-0.2522 & -0.0801 & -0.0406 & 0.0150 & 0.8030
\end{pmatrix}
\]

Optimization done in order to determine the composition of the portfolio weights, and thus the portfolio weight vector is determined by using equation (5). The weight vector calculation process, the values of risk tolerance \( \tau \) determined by the simulation begins value \( \tau = 0.000 \) with an increase of 0.001. If it is assumed that short sales are not allowed, then the simulation is stopped when the value of \( \tau = 0.036 \), because it has resulted in a portfolio weight at least there is a negative value. The portfolio weights calculation results are given:
The optimization process results, can be calculated ratio value $\hat{\mu}_p$ towards $\hat{\sigma}_p^2$ for each level of risk tolerance. The ratio calculation results can be shown as in column $\hat{\mu}_p/\hat{\sigma}_p^2$. This ratio shows the relationship between the optimum portfolio return expected with variance as a measure of risk. Based on the results of the calculation of portfolio optimization, the optimum value is achieved when the value of the portfolio's risk tolerance $\tau = 0.027$. The portfolio produces mean value of $\hat{\mu}_p = 0.0099$ with the
value of risk as the variance $\tilde{\sigma}_p^2 = 0.00053808$.

Composition weight of the maximum portfolio respectively is 0.0608, 0.1555, 0.5398, 0.0619, and 0.1821. This provides reference to investors that invest in stocks of $S_1$, $S_2$, $S_3$, $S_4$, and $S_5$, in order to achieve the maximum value of the portfolio, the composition of the portfolio weights are as mentioned above.

4. Conclusion
In this paper has discussed the issue of linear algebra on the investment portfolio optimization models. Discussion has formulated the optimum weight vector expressed as a function of linear algebra, as given in equation (5). Furthermore, the resulting formula is used to analyze five stocks in the formation of an investment portfolio. Based on the results of the calculation of portfolio optimization, produced that the optimum is Achieved when the composition of the investment portfolio weights in Islamic stocks of $S_1$, $S_2$, $S_3$, $S_4$, and $S_5$, respectively are: 0.0608, 0.1555, 0.5398, 0.0619, and 0.1821. The composition of the portfolio weights will thereby produce a portfolio with a mean value of 0.0099 and the value of risk, measured as the variance of 0.00053808.

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