A NOTE ON QUANTUM ONE-WAY PERMUTATIONS

ELHAM KASHEFI
Centre for Quantum Computation, Clarendon Laboratory, University of Oxford, Parks Road
Oxford OX1 3PU, England
Optics Section, The Blackett Laboratory, Imperial College
London SW7 2BZ, England

HARUMICHI NISHIMURA
Centre for Quantum Computation, Clarendon Laboratory, University of Oxford, Parks Road
Oxford OX1 3PU, England

VLATKO VEDRAL
Optics Section, The Blackett Laboratory, Imperial College
London SW7 2BZ, England

Abstract

We discuss the question of the existence of quantum one-way permutations. First, we prove the equivalence between inverting a permutation and that of constructing a polynomial size network for reflecting about a given quantum state. Next, we consider the question: if a state is difficult to prepare, is the operator reflecting about that state difficult to construct? By revisiting Grover’s algorithm, we present the relationship between this question and the existence of one-way permutations. Moreover, we compare our method to Grover’s algorithm and discuss possible applications of our results.

1 Introduction

Quantum computation is a rapidly growing field which explores the relationship between quantum physics and computation [1]. We have two strong indications that quantum systems are potentially more efficient than their classical counter-parts at performing computational tasks. One is Shor’s algorithm [2], which solves the factoring problem and the discrete logarithm problem in quantum polynomial time. The other is Grover’s algorithm [3], which works quadratically faster than any classical algorithm for the search problem in the oracle setting. On the other hand, Bennett, Bernstein, Brassard, and Vazirani [4] have shown that with probability 1 there exists a quantum one-way permutation relative to a random permutation oracle.

The existence of one-way functions is one of the most important open problems in classical computation. For example, it is well-known that one-way functions have applications in cryptography [5]. Loosely speaking, a one-way function is one that is easy to compute but hard to invert. To make this notion precise, we define a function $f$ to be (quantum) one-way, if $f$ is one-one, $f$ is honest, $f$ can be computed in (quantum) polynomial time, and $f^{-1}$ is not computable in (quantum) polynomial time. By $f$ being honest we mean that there exists a polynomial $p$ such that $|x| \leq p(|f(x)|)$, where $|.|$ denotes the length of binary strings. Note that in this paper we are discussing one-way functions in the setting of the worst case complexity [6]. It is thought that the proof of existence of one-way function is a difficult problem, since it is equivalent to the separation between the complexity classes $P$ and $UP$ [6].

We address the question of the existence of quantum one-way permutations which is a restricted type of one-way functions. First, we consider a necessary and sufficient condition for inverting efficiently a polynomial time computable permutation. In the classical case, Hemaspaandra and Rothe [7] presented a necessary and sufficient condition for the existence of one-way permutations. We show that in the quantum setting, the problem of inverting a permutation in polynomial time is equivalent
to the problem of constructing polynomial size networks for the reflections about some quantum states \(|\psi\rangle\), i.e., \(2|\psi\rangle\langle\psi| - I\). In the proof of this equivalence, we present a quantum algorithm for inverting a permutation efficiently under the condition that the reflections about their quantum states are efficiently implementable. Similar to Grover’s algorithm, our algorithm also consists of the iteration of the tagging and reflection operators [3]. We show that the exponential speed-up over Grover’s algorithm is possible if and only if the efficient reflections about some quantum states that we will define in the paper, are possible.

Next, we consider the relationship between the complexity of preparing a state and the reflection about that state. We define a unitary operator on \(n\) qubits to be easy if there exists a polynomial size network implementing the operator up to a global phase. The \(n\)-qubit state \(|\phi\rangle\) is defined to be easy if there exists a polynomial size network which produces the state \(|\phi\rangle\) up to a global phase. It is straightforward to see that if a state is easy, the reflection about that state is also easy. We consider the other direction, which seems to hold at first glance. If the reflection about a state is easy, the state \(|x\rangle\) enable us to mark all the states \(|y\rangle\) such that \(\langle x|f(x) + y\rangle\), where \(|x\rangle\) and \(|y\rangle\) each consist of \(n\) qubits. We consider the following problem called hereafter INVERT: for any given \(x \in \{0,1\}^n\), find \(f^{-1}(x)\). In the setting where \(f\) is given as an oracle, Grover’s algorithm can solve INVERT with quadratic speed-up over any classical algorithm [3]. In its algorithm, Grover uses the tagging operator \(O\) defined as

\[
O|x\rangle|y\rangle = (-1)^{\delta_{x,f(y)}}|x\rangle|y\rangle
\]

and the reflection about the uniform superposition defined as

\[
|\psi\rangle = \sum_{y \in \{0,1\}^n}|y\rangle,
\]

i.e. \(2|\psi\rangle\langle\psi| - I\), which is also called the inversion about the average amplitude. The operator \(O\) can be simulated by two applications of \(U_f\) and \(n\) controlled-not gates. Moreover, if \(f\) is polynomial time computable, then it is also possible to efficiently construct the unitary operator \(O[k]\) defined by

\[
O[k]|x\rangle|y\rangle = (-1)^{\delta_{x,f_k(y)}\delta_{x_{k+1},f_{k+1}(y)}}|x\rangle|y\rangle,
\]

where \(x_i\) and \(f_i(y)\) for \(i = 1, \ldots, n\) represent the \(i\)-th bits of \(x\) and \(f(y)\). This operator \(O[k]\) will enable us to mark all the states \(|y\rangle\) such that \(2\) qubits of \(|f(y)\rangle\) are equal to the corresponding qubits of \(|x\rangle\). Geometrically, \(O[k]\) can be considered to be the reflection about the hyper-plane spanned by the vectors \(|y\rangle\) for \(f(y)(k,k+1) \neq x(k,k+1)\). We show that if we can efficiently implement \(O[k]\)’s and the unitary operator

\[
Q_j = \sum_{x \in \{0,1\}^n} |x\rangle\langle x| \otimes (2|\psi_{j,x}\rangle\langle\psi_{j,x}| - I),
\]

2 Main Result

For any permutation \(f\) on \(n\)-bit strings, let \(U_f\) denote the unitary operator mapping the basis state \(|x\rangle|y\rangle\) to \(|x\rangle|f(x) \oplus y\rangle\), where \(|x\rangle\) and \(|y\rangle\) each consist of \(n\) qubits. We consider the following problem called hereafter INVERT: for any given \(x \in \{0,1\}^n\), find \(f^{-1}(x)\). In the setting where \(f\) is given as an oracle, Grover’s algorithm can solve INVERT with quadratic speed-up over any classical algorithm [3]. In his algorithm, Grover uses the tagging operator \(O\) defined as

\[
O|x\rangle|y\rangle = (-1)^{\delta_{x,f(y)}}|x\rangle|y\rangle
\]

and the reflection about the uniform superposition defined as

\[
|\psi\rangle = \sum_{y \in \{0,1\}^n}|y\rangle,
\]

i.e. \(2|\psi\rangle\langle\psi| - I\), which is also called the inversion about the average amplitude. The operator \(O\) can be simulated by two applications of \(U_f\) and \(n\) controlled-not gates. Moreover, if \(f\) is polynomial time computable, then it is also possible to efficiently construct the unitary operator \(O[k]\) defined by

\[
O[k]|x\rangle|y\rangle = (-1)^{\delta_{x,f_k(y)}\delta_{x_{k+1},f_{k+1}(y)}}|x\rangle|y\rangle,
\]

where \(x_i\) and \(f_i(y)\) for \(i = 1, \ldots, n\) represent the \(i\)-th bits of \(x\) and \(f(y)\). This operator \(O[k]\) will enable us to mark all the states \(|y\rangle\) such that \(2\) qubits of \(|f(y)\rangle\) are equal to the corresponding qubits of \(|x\rangle\). Geometrically, \(O[k]\) can be considered to be the reflection about the hyper-plane spanned by the vectors \(|y\rangle\) for \(f(y)(k,k+1) \neq x(k,k+1)\). We show that if we can efficiently implement \(O[k]\)’s and the unitary operator

\[
Q_j = \sum_{x \in \{0,1\}^n} |x\rangle\langle x| \otimes (2|\psi_{j,x}\rangle\langle\psi_{j,x}| - I),
\]
where
\[ |\psi_{j,x}\rangle = \frac{1}{\sqrt{2^{n-2j}}} \sum_{y : f(y)_{(1,2)} = x_{(1,2)}} |y\rangle, \]
then we can efficiently invert \( f \) by polynomial size network. Conversely, we also can show that if \( f \) is difficult to invert, then \( Q_j \)'s are also difficult to construct.

Now we state and prove this result formally. We say that a set \( F \) of unitary operators is easy if every \( U \in F \) is easy.

**Theorem 1:** A function \( f : \{0,1\}^n \rightarrow \{0,1\}^n \) is a quantum one-way permutation if and only if the set \( F_n = \{Q_j\}_{j=0,1,...,\frac{n}{2}-1} \) of unitary operators is not easy.

**Proof:** Without loss of generality, we can assume that \( n \) is even.

(\(\Rightarrow\)) Suppose that \( F_n \) is easy. Then we show that \( f^{-1} \) is computable by a polynomial size quantum network. A quantum algorithm (Algorithm A below) computing \( f^{-1} \) is as follows. Assume that \( x \) is given as the input in the first register of the quantum network to be constructed.

**ALGORITHM A**

Step 1 (Preparation).
Prepare the second register in the uniform superposition
\[ |\psi_0\rangle = \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} |y\rangle. \]

Step 2 (Iteration).
For \( j = 0 \) to \( \frac{n}{2} - 1 \), implement the following steps 2.j.1–2.j.2.

Step 2.j.1. Carry out \( O[2^{j+1}] \) on the first and the second registers.
Step 2.j.2. Carry out \( Q_j \) on the first and the second registers.

Step 2.j.1 can be implemented through the following 3 steps: (1) Carry out \( U_f : |y\rangle|z\rangle \rightarrow |y\rangle|f(y) \oplus z\rangle \) on the second and third registers. (2) Compare the 2\( j+1 \)-th and the 2\( j+2 \)-th qubits of the first register with the corresponding qubits of the third register, and apply a phase shift of \( -1 \) if they are same; otherwise do nothing. (3) Carry out \( U_f \) on the second and third registers.

Now we show that Algorithm A computes \( f^{-1} \). After Step 1, the state of the system is
\[ \frac{1}{\sqrt{2^n}} |x\rangle \sum_{y \in \{0,1\}^n} |y\rangle. \]

We show that after Step 2.j.2 the state of the system is
\[ \frac{2^{j+1}}{\sqrt{2^n}} |x\rangle \sum_{y : f(y)_{(1,2)} = x_{(1,2)+2}} |y\rangle, \]
which means that Algorithm A computes \( f^{-1} \) after \( \frac{n}{2} \) iterations. In the case \( j = 0 \), the state evolves as follows (note that for any \( x \) we have \( |\psi_{0,x}\rangle = |\psi_0\rangle \))
\[ \frac{1}{\sqrt{2^n}} |x\rangle \sum_{y \in \{0,1\}^n} |y\rangle \]
\[ \rightarrow \frac{2^{0.1}}{\sqrt{2^n}} |x\rangle \left( \sum_{y : f(y)_{(1,2)} \neq x_{(1,2)}} |y\rangle - \sum_{y : f(y)_{(1,2)} = x_{(1,2)}} |y\rangle \right) \]
A can be implemented by a polynomial size quantum network. Thus, the case \( j \) can be implemented by a polynomial size quantum network. According to the assumption, quantum polynomial time computable. The following operator evolves as follows:

\[
\begin{align*}
\frac{2k}{\sqrt{2^n}} |x\rangle \left( \sqrt{2^n} \psi_0 \right) &= \sqrt{2^n} |\psi_0\rangle - 2 \sum_{y : f(y) = x} |y\rangle \\
\frac{2k}{\sqrt{2^n}} |x\rangle (2 |\psi_0\rangle \langle \psi_0| - I) &\left( \sqrt{2^n} \langle \psi_0 | - 2 \sum_{y : f(y) = x} |y\rangle \right) \\
&= \frac{2k}{\sqrt{2^n}} |x\rangle \left( 2 \sqrt{2^n} |\psi_0\rangle - \sqrt{2^n} |\psi_0\rangle - 4 |\psi_0\rangle \sum_{y : f(y) = x} \langle \psi_0 | y \rangle \right) \\
&\quad + 2 \sum_{y : f(y) = x} |y\rangle \\
&= \frac{2k}{\sqrt{2^n}} |x\rangle \sum_{y : f(y) = x} |y\rangle.
\end{align*}
\]

On the other hand, suppose that the case \( j = k \) holds. Then, following Steps 2.k.1–2.k.2, the state evolves as follows:

\[
\begin{align*}
\frac{2k}{\sqrt{2^n}} |x\rangle &\sum_{y : f(y) = x} |y\rangle \\
\frac{2k}{\sqrt{2^n}} |x\rangle \left( \sum_{y : f(y) = x} |y\rangle - 2 \sum_{y : f(y) = x} |y\rangle \right) \\
&= \frac{2k}{\sqrt{2^n}} |x\rangle \left( \sqrt{2^n - 2^k} |\psi_{k,x}\rangle - 2 \sum_{y : f(y) = x} |y\rangle \right) \\
\frac{2k}{\sqrt{2^n}} |x\rangle (2 |\psi_{k,x}\rangle \langle \psi_{k,x}| - I) &\left( \sqrt{2^n - 2^k} |\psi_{k,x}\rangle - 2 \sum_{y : f(y) = x} |y\rangle \right) \\
&= \frac{2k}{\sqrt{2^n}} |x\rangle \left( 2 \sqrt{2^n - 2^k} |\psi_{k,x}\rangle - \sqrt{2^n - 2^k} |\psi_{k,x}\rangle - 4 |\psi_{k,x}\rangle \sum_{y : f(y) = x} \langle \psi_{k,x} | y \rangle \right) \\
&\quad + \frac{2k}{\sqrt{2^n}} |x\rangle \sum_{y : f(y) = x} |y\rangle \\
&= \frac{2k+1}{\sqrt{2^n}} |x\rangle \sum_{y : f(y) = x} |y\rangle.
\end{align*}
\]

Thus, the case \( j = k \) holds. From the assumption that \( \{Q_j\} \) is easy, it is simple to see that Algorithm A can be implemented by a polynomial size quantum network.

(\( \Leftarrow \)) Suppose that \( f \) is not a one-way permutation. Then we show that \( \{Q_j\}_{j=0,1,...,2^n-1} \) can be implemented by a polynomial size quantum network. According to the assumption, \( f \) and \( f^{-1} \) are quantum polynomial time computable. The following operator

\[
M_f : |x\rangle \mapsto |f(x)\rangle
\]

can be implemented by a polynomial size quantum network \([8, 9]\). To see why note that:

\[
M_f \otimes I = (U_{f^{-1}})^{-1} \otimes S \otimes U_f,
\]
where the swap gate $S$ is defined by $S : |a\rangle \otimes |b\rangle \rightarrow |b\rangle \otimes |a\rangle$.

We now show that the unitary operator $Q'_j = (I \otimes M_f)Q_j(I \otimes M_f)^\dagger$ can be implemented by a polynomial size quantum network, which means that $Q_j$ can also be implemented by a polynomial size quantum network. The operator $Q'_j$ can be rewritten as follows

$$Q'_j = (I \otimes M_f) \left\{ \sum_{x \in \{0,1\}^n} |x\rangle \langle x| \otimes \left( 2 \left( \frac{1}{2^{n-2}} \sum_{y,y'} |y\rangle \langle y'| - I \right) \right) \right\} (I \otimes M_f)^\dagger$$

$$= \sum_{x \in \{0,1\}^n} |x\rangle \langle x| \otimes \left( 2 \sum_{y,y'}^* |f(y)\rangle \langle f(y')| - I \right)$$

$$= \sum_{x \in \{0,1\}^n} |x\rangle \langle x| \otimes \left( 2|x(1,2j)\rangle \langle x(1,2j)| \frac{1}{2^{n-2}} \sum_{y,y'} |f(y)(2j+1,n)\rangle \langle f(y')(2j+1,n)| - I \right)$$

$$= \sum_{x \in \{0,1\}^n} |x\rangle \langle x| \otimes \left( 2|x(1,2j)\rangle \langle x(1,2j)| \otimes |\psi_j\rangle \langle \psi_j| - I \right)$$

$$= \sum_{x \in \{0,1\}^n} |x\rangle \langle x| \otimes \left( |x(1,2j)\rangle \langle x(1,2j)| \otimes (2|\psi_j\rangle \langle \psi_j| - I) + \sum_{y : y \neq x(1,2j)} |y\rangle \langle y| \otimes I \right).$$

Here, $\sum_{y,y'}^*$ denotes $\sum_{y,y':f(y)(1,2j)=f(y')(1,2j)=x(1,2j)}$ and $|\psi_j\rangle$ denotes

$$|\psi_j\rangle = \frac{1}{\sqrt{2^{n-2}j}} \sum_{i \in \{0,1\}^{n-2}} |i\rangle.$$

Thus, we can implement $Q'_j$ by comparing the first $2j$ qubits of the first register with the corresponding qubits of the second register and applying $2|\psi_j\rangle \langle \psi_j| - I$ if they are the same and applying the identity otherwise (i.e. conditional-$2|\psi_j\rangle \langle \psi_j| - I$). The operator $2|\psi_j\rangle \langle \psi_j| - I$ is easy, since $2|\psi_j\rangle \langle \psi_j| - I = H^{\otimes n-2j}(2|0\rangle \langle 0| - I)H^{\otimes n-2j}$, where $H$ is the Hadamard gate and the superscript $n-2j$ indicates that Hadamard gate is applied to the last $n-2j$ qubits. Therefore, $Q'_j$ is easy and this completes the proof. \square

Note that all unitary operators $U_k$ are easy if and only if the operation

$$\sum_k |k\rangle \langle k| \otimes U_k,$$

which implements $U_k$ conditionally, is easy. The operator $Q_j$ implements the reflection about the state $|\psi_j,x\rangle$ conditionally, therefore Theorem 1 gives a necessary and sufficient condition for quantum one-way permutations in terms of the reflection about a quantum state.

3 General View

It is well-known that if a state is easy, then the reflection about the state is easy [1]. Does the inverse hold? We call its inverse, i.e., the statement “if the reflection about a state is easy, the state itself is easy”, Assumption A. In this section, we revisit both Grover’s and our algorithm from the viewpoint of complexity of a state and the reflection about the state, and discuss the relationship between the existence of one-way permutations and Assumption A.

First, let us revisit Grover’s search problem and algorithm. In [3], Grover considered the following problem called hereafter SEARCH. Let $f : \{0,1\}^n \rightarrow \{0,1\}$ be a function such that $|f^{-1}(\{1\})| = 1.$
Then, the goal is to find $f^{-1}(1)$. Grover’s algorithm for this problem (Algorithm B below) consists of the following steps.

**ALGORITHM B**

Step 1 (Preparation).
Prepare the uniform superposition

$$|\psi\rangle = \sum_{x \in \{0,1\}^n} |x\rangle.$$

Step 2 (Iteration).
Iterate Step 2.1 and Step 2.2.

Step 2.1. Carry out the tagging operation given by

$$\sum_{x \in \{0,1\}^n} (-1)^{f^{-1}(1)}|x\rangle \langle x| = I - 2|f^{-1}(1)\rangle \langle f^{-1}(1)|.$$

Step 2.2. Carry out the reflection about the state $|\psi\rangle$ (i.e., the inversion about the average amplitude).

Here, Step 1 and Step 2.2 are easy, and Step 2.1 can be implemented by querying the oracle

$$U_f : |x\rangle |b\rangle \rightarrow |x\rangle |f(x) \oplus b\rangle$$

twice, since for the reflection about the target state $|f^{-1}(1)\rangle$, we have

$$\{(2|f^{-1}(1)\rangle \langle f^{-1}(1)| - I) \otimes I |0\rangle = \{U_f(I \otimes (2|1\rangle \langle 1| - I))U_f\}|x\rangle |0\rangle.$$

By $O(\sqrt{2^n})$ iterations of Step 2, i.e., by $O(\sqrt{2^n})$ queries, we can get $f^{-1}(1)$ with high probability. Thus, when $f$ is given as an oracle, this algorithm works quadratically faster than any possible classical algorithm. However, this algorithm is shown to be optimal \[ 4, 10, 11 \], so that we cannot solve this problem by using at most queries polynomial in $n$. This implies that even if the reflection about the state $|f^{-1}(1)\rangle$ is assumed to be easy, the state $|f^{-1}(1)\rangle$ itself is not easy.

Next, we shall relate Assumption A to the existence of quantum one-way permutations by revisiting Grover’s algorithm for \textsc{Invert} considered in the previous section. Grover’s algorithm for \textsc{Invert} (Algorithm C below) is as follows.

**ALGORITHM C**

Step 1 (Preparation).
Prepare the uniform superposition

$$|\psi\rangle = \sum_{y \in \{0,1\}^n} |y\rangle.$$

Step 2 (Iteration).
Iterate Step 2.1 and Step 2.2.

Step 2.1. Carry out the tagging operator

$$O = I - 2|f^{-1}(x)\rangle \langle f^{-1}(x)|.$$

Step 2.2. Carry out the reflection about the state $|\psi\rangle$. 

6
Similar to Algorithm A, Step 1 and Step 2.2 are easy. The operator $O$ in Step 2.1 is a tagging operator and can be implemented by using the operator $U_f$:

$$|y⟩|z⟩ → |y⟩|f(y) ⊕ z⟩.$$ 

In fact, for any $y \in \{0,1\}^n$ we have

$$\{(I - 2|f^{-1}(x)⟩⟨f^{-1}(x)|) \otimes I\}|y⟩|0⟩ = U_f(I \otimes (I - 2|x⟩⟨x|))U_f|y⟩|0⟩.$$ 

Thus, given $U_f$ as an oracle, we can compute $f^{-1}(x)$ with high probability by $O(\sqrt{2n})$ queries. This algorithm is also shown to be optimal [12]. Note that the operator $2|f^{-1}(x)⟩⟨f^{-1}(x)| - I$ is performing the reflection about the state $|f^{-1}(x)⟩$. Thus, Algorithm C shows that even if the reflection about the state $|f^{-1}(x)⟩$ is assumed to be easy, the state itself is not necessarily easy.

Now, let us consider the case when $f$ is a quantum one-way permutation. Then, the operator $U_f$ is easy, so that $2|f^{-1}(x)⟩⟨f^{-1}(x)| - I$ is also easy. Therefore, from Algorithm C, we can infer the following interesting fact: If there exists a quantum one-way permutation, there exists a counter-example to Assumption A.

Compared to Algorithm C, the Algorithm A for inverting $f$ is exponentially faster, providing that $Q_j$’s are easy. The operator $Q_j$ is a unitary operator which reflects any given state about the state $|\psi_{j,x}⟩$, where $x$ is the value of the first register. Does the state $|\psi_{j,x}⟩$ also provide a counter-example to Assumption A? The answer is no, since it is shown in the proof of Theorem 1 that if $F_n$ is easy, we can generate the state $|\psi_{j,x}⟩$ by a polynomial size network. Therefore we have the following corollary from Theorem 1.

**Corollary 2:** The following conditions are equivalent.

- $f$ is not a quantum one-way permutation.
- All states $|\psi_{j,x}⟩$, where $j \in \{0, \frac{n}{2} - 1\}$ and $x \in \{0,1\}^n$, are easy.
- The reflections about all states $|\psi_{j,x}⟩$ are easy.

So far we have considered only the exact setting. However, using the diamond metric and its properties [13], the similar results also hold in the bounded error setting. In the latter setting we define the notions of easy operator, easy state, and quantum one-way function as follows. A trace-preserving completely positive superoperator (CPSO) $U$ is defined to be approximately easy if there exists a family of polynomial size quantum networks $\{N_ε\}$ such that

$$||U - U_ε||_∞ \leq \epsilon,$$

where $U_ε$ is the operator implemented by $N_ε$ exactly. A mixed state $ρ$ is defined to be approximately easy if there exists an approximately easy CPSO $U$ such that

$$U(|0⟩⟨0|) = ρ.$$ 

Now we can give the definition of quantum one-way function in the bounded-error setting. A function $f$ is quantum one-way if $f$ is one-one, $f$ is honest, $f$ is approximately easy, and $f^{-1}$ is not approximately easy, where $f$ is quantum approximately easy if the unitary operator $U_f : |x⟩|z⟩ → |x⟩|z ⊕ f(x)⟩$ is approximately easy. It is straightforward to check that if there exists a quantum one-way permutation, then there exists a counter-example for the Assumption A in the bounded error setting.
4 Discussions

We have reduced the problem of the existence of a quantum one-way permutation to the problem of constructing a polynomial size network for performing the specific task of the reflection about a given state. Ambainis [12] proved that inverting a permutation on the $n$-bit strings in the standard query model requires $\Omega(\sqrt{2^n})$ queries. In the standard query model [14], a quantum computation with $T$ queries is a sequence of unitary operators

$$U_0 \rightarrow O \rightarrow U_1 \rightarrow O \cdots \rightarrow U_{T-1} \rightarrow O \rightarrow U_T,$$

where $U_j$’s are arbitrary unitary operators independent of the input qubits, and $O$ is the standard query operator. However, our algorithm is consistent with Ambainis’ result, since we consider the case that $U_j$’s depend on the input qubits and this does not fit his model.

Another related issue is the work of Chen and Diao [15] where they attempted to present an efficient quantum algorithm for the problem SEARCH, which is similar to our algorithm for the problem INVERT. They mentioned that the tagging operation and the reflection about a given state which varies dynamically can be constructed by polynomial size networks, but they did not show the construction for their operations. (This construction is, of course, impossible given Grover’s black box, since it would violate the optimality proof of Grover’s algorithm [16].) For our problem INVERT we have given the polynomial size network of the tagging operation and we have shown that the difficulty of the construction of the reflection operation is equivalent to the existence of the quantum one-way permutation. Furthermore it is an interesting open problem whether there exists a reduction from other types of one-way functions to constructing a polynomial size network for performing the reflection about a given state.

On the other hand, we have seen that Grover’s algorithm gives us an example of states that are difficult to prepare but the reflections about these states are easy, i.e., it provides a counter-example to Assumption A assuming the existence of one-way permutations. This investigation of Assumption A seems to be useful for cryptographic applications since recently, quantum bit commitment protocols based on quantum one-way permutations have been proposed [16, 17]. Moreover, it is interesting to find such a concrete counter-example without the existence of quantum one-way permutations. Presenting such examples of states may provide us with more ideas for constructing novel quantum algorithms.

Acknowledgements This work was supported by EPSRC, the European grant EQUIP and the QUIPROCONE grant.

References

[1] M.A. Nielsen and I.L. Chuang (2000), Quantum Computation and Quantum Information, Cambridge University Press.

[2] P.W. Shor (1994), Algorithms for quantum computation: Discrete logarithms and factoring, in Proceedings of 35th IEEE Symposium on Foundations of Computer Science, pp. 124-134.

[3] L.K. Grover (1996), A fast quantum mechanical algorithm for database search, in Proceedings of 28th ACM Symposium on the Theory of Computing, pp. 212-219.

[4] C.H. Bennett, E. Bernstein, G. Brassard, and U.Vazirani (1997), Strengths and weaknesses of quantum computing, SIAM J. Comput., 26, pp.1510-1523.

[5] C.H. Papadimitriow (1994), Computational Complexity, Addison-Wesley.
[6] J. Grollmann and A.L. Selman (1988), Complexity measures for public-key cryptosystems, SIAM J. Comput., 17, pp. 309-335.

[7] L. Hemaspaandra and J. Rothe (2000), Characterizing the existence of one-way permutation, Theoret. Comput. Sci., 244, pp. 257-261.

[8] C.H. Bennett (1973), Logical reversibility of computations, IBM J. Res. Develop., 17, pp. 525-532.

[9] E. Kashefi, A. Kent, V. Vedral, and K. Banaszek (2001), On the power of quantum oracles, quant-ph/0109104.

[10] M. Boyer, G. Brassard, P. Høyer, and A. Tapp (1998), Tight bounds on quantum searching, Fortsch. Phys., 46, pp. 493-505.

[11] C. Zalka (1999), Grover’s quantum searching algorithm is optimal, Phys. Rev. A, 60, pp. 2746-2751.

[12] A. Ambainis. (2000), Quantum lower bounds by quantum arguments, in Proceedings of 32th ACM Symposium on the Theory of Computing, pp. 636-643.

[13] D. Aharonov, A. Kitaev and N. Nisan (1998), Quantum circuits with mixed states, in Proceedings of 30th ACM Symposium on the Theory of Computing, pp. 20-30.

[14] R. Beal, H. Buhrman, R. Cleve, M. Mosca and R. de Wolf. (1998), Quantum lower bounds by polynomials, in Proceedings of 39th IEEE Symposium on Foundations of Computer Science, pp. 352-361.

[15] G. Chen and Z. Diao (2000), An exponentially fast quantum search algorithm, quant-ph/0011109.

[16] P. Dumais, D. Mayers and L. Salvail (2000), Perfectly concealing quantum bit commitment from any one-way permutation, Advances in Cryptology – EUROCRYPT 2000, B. Preneel (Ed.), Lecture Note in Computer Science 1807, Springer-Verlag, pp. 300-315.

[17] M. Adcock and R. Cleve. (2001), A quantum Goldreich-Levin theorem with cryptographic applications, quant-ph/0108095.