Exponential Decay and Fermi’s Golden Rule from an Uncontrolled Quantum Zeno Effect

P. W. Bryant
IBM Research-Brazil, Av. Pasteur 138/146, Botafogo, Rio de Janeiro, CEP 22290-240, Brazil
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Abstract

The prediction of the Quantum Zeno Effect is based on a product of autocorrelation functions and on the evolution of these functions at short times. We show why in quantum mechanics the autocorrelation between a state and itself some time later cannot be a surrogate for the standard probability. We then reformulate the Quantum Zeno Effect based on the physically meaningful probability, and we find as a condition to observe an inhibition of decay that, following each measurement in a sequence, the system must be prepared in a common eigenstate of the same set of commuting observables. This leads to a simple model of spontaneous decay, in which an excited atom continuously interacts with and is monitored by an incoherent sum of radiation fields. Regardless of the system’s evolution otherwise, while monitored the excited atom has exponentially decreasing survival probability at all times. We derive the decay rate and find Fermi’s Golden Rule. Continuous monitoring may thus be a mechanism by which to explain the exponential decay law.

1 Introduction

The Quantum Zeno Effect is the apparent slowing, changing, or even stopping of a quantum system’s normal time evolution away from the eigenstate of an observable, resulting from frequent or continuous measurements. It is well known that the act of measurement can change the state of a quantum mechanical system, and the effect of repeated measurements has been seen in a variety of experiments [15, 5, 7, 12]. As formulated in [9], however, the effect was deemed a paradox because it predicted that a continuously monitored unstable state will not decay.

In Section 2 we review the quantity calculated in [9] and interpreted as the probability for a continuously monitored state not to decay. We show how it is equivalent to a product of autocorrelation functions [1], which also forms the basis of the typical intuitive explanation [11, 4] of the Quantum Zeno Effect. The autocorrelation is the expectation value
between a state and itself some time later. It is different from the standard probability, which contains for the observable an operator fixed in time in the Schrödinger picture, because a probability amplitude has a time-independent, complex phase absent from the amplitude of the autocorrelation. In Section 3 we study the time evolution of the autocorrelation function, and we find that the autocorrelation cannot be a surrogate for a probability meant to be compared with observations. Simply stated, the observable time evolution of a quantum system is not described by the time evolution of its state vector alone, and the initially quadratic time evolution of the autocorrelation function is observable only in special cases.

Starting from the intuitive picture, in Section 4 we reformulate the Quantum Zeno Effect based upon the physically significant probabilities. We find that a condition to observe an inhibition of decay is that the system is repeatedly measured, or prepared, to be in a common eigenstate of the same set of commuting observables. The predicted stopping of decay, or the Quantum Zeno’s Paradox, thus arises in the special case treated in [9], when all measurements find the system to be in the same subspace. In Section 5 we consider a more general case for when measurements are not perfectly controlled. We present a model for an excited atom continuously monitored by an incoherent sum of radiation fields, which represents the interaction between the atom’s nucleus and electrons. The atom is shown to decay with exponential survival probability, regardless of the time evolution of the system in the absence of monitoring or in the presence of coherent radiation. Furthermore, we derive the decay rate and find Fermi’s Golden Rule.

To the extent that a system’s uncontrolled interactions can be modeled or approximated by such continuous monitoring, it may be a candidate to explain the exponential decay law. For state vectors in a Hilbert space and for systems bounded from below in energy, it was understood long ago [8] that for uninterrupted time evolution the survival probability cannot be exactly exponential. We will discuss in Section 6 how this argument does not apply to our model of decay, because a monitored system is not represented by the same state vector at all times. In the absence of measurement, energy wave functions are not generally Lorentzian. With continuous measurement, the Lorentzian line shape and exponential decay emerge. This may explain why the predicted deviation from the exponential has never been observed for spontaneous decay.

2 Modeling Continuous Measurement

The article [9], from which the Quantum Zeno’s Paradox (now often called Quantum Zeno Effect) was given its name, demonstrates that the continuous monitoring of a spontaneously decaying system stops the system’s time evolution for general, kinematical reasons related to the quantum state’s time evolution at small times. This result, as well as the more transparent, intuitive explanations [11, 4], is based on a product of autocorrelation functions between a state and itself a short time later. To illustrate how, we will need to use the time evolution of a density operator, \( \rho \), in the absence of measurement:

\[
\rho(t) = U(t - t_0)\rho(t_0)U^\dagger(t - t_0),
\]  

(1)
where
\[ U^\dagger(t) = e^{iHt/\hbar} = e^{-iH(-t)/\hbar} = U(-t), \tag{2} \]
and \( H \) is the self-adjoint Hamiltonian operator. \( U(t) \) is a unitary group of operators, such that
\[ 1 = U^\dagger(t)U(t), \tag{3} \]
and
\[ U(t)U(t') = U(t + t'). \tag{4} \]

To treat the ideal measurement of the unstable, decaying part of a quantum system, the authors of [9] use a projection, \( \mathcal{E} = \mathcal{E}^2 \), onto the subspace of undecayed states. For initial conditions at the time \( t_0 \), the system is prepared in an unstable state, such that it is represented by the density operator satisfying
\[ \rho(t_0) = \mathcal{E}\rho(t_0)\mathcal{E}, \tag{5} \]
and
\[ \text{Tr}(\mathcal{E}\rho(t_0)) = 1. \tag{6} \]

To model repeated measurements, the authors introduce the quantity
\[ T_n(t - t_0) = [\mathcal{E}U((t - t_0)/n)\mathcal{E}]^n = [\mathcal{E}U(\Delta t)\mathcal{E}]^n; \tag{7} \]
with which they construct the unnormalized density operator,
\[ \rho(n, t) = T_n(t - t_0)\rho(t_0)T_n^\dagger(t - t_0), \tag{8} \]
to represent the state of the undecayed part of the system after the \( n \)th measurement. As used in [8], \( T_n(t - t_0) \) and \( T_n^\dagger(t - t_0) \) operationally represent a sequence of \( n \) projections, each representing an ideal measurement, onto the undecayed subspace, \( \mathcal{E} \). Each projection is then followed by normal time evolution of the resulting state for a duration \( (t - t_0)/n \equiv \Delta t \). The limit of continuous measurement is reached by letting \( n \to \infty \) but maintaining a finite duration of monitoring, \( t - t_0 \). In [9] the authors prove the existence of an operator
\[ \lim_{n \to \infty} T_n(t - t_0) = T(t - t_0), \tag{9} \]
with which one can construct the density operator of the undecayed part of the system following continuous measurement from \( t_0 \) until \( t \).

The density operator in [8] is normalized as
\[ \left( \text{Tr}(T_n(t - t_0)\rho(t_0)T_n^\dagger(t - t_0)) \right)^{-1}\rho(n, t), \tag{10} \]
and it is claimed in [9] that this normalization factor,
\[ Q(n, [t_0, t]; \rho) \equiv \text{Tr}(T_n(t - t_0)\rho(t_0)T_n^\dagger(t - t_0)), \tag{11} \]
is the probability that no decay occurs throughout the interval \([t_0, t]\), during which the sequence of ideal measurements occurs. The interpretation of the Quantum Zeno Effect for continuous monitoring then follows from this quantity in the limit \(n \rightarrow \infty\). It is shown that \(Q\) \[Q([t_0, t]; \rho) = \text{Tr}(T(t - t_0)\rho(t_0)T^\dagger(t - t_0)) = 1,\] which indicates that no decay is ever found during the interval \([t_0, t]\).

Note from (11) and (12) that \(Q(n, [t_0, t]; \rho)\) and \(Q([t_0, t]; \rho)\) are normalization factors, which statically may be interpreted to represent a probability, but which are fixed in time and thus do not represent probabilities as systems evolve in time. As noted by the authors of [9], however, these quantities must be distinguished from typical probabilities, because \(Q(n, [t_0, t]; \rho)\) and \(Q([t_0, t]; \rho)\) represent the cumulative effect of the \(n\) measurements, with each measurement finding the system to be undecayed.

Given the nonstandard meaning of the quantities \(Q(n, [t_0, t]; \rho)\) and \(Q([t_0, t]; \rho)\), we wish to study more closely the extent to which they can be compared with observable results. To simplify things, note that we are calculating the probability of occurrence of a sequence of measurements in which the system is found each time to be in an undecayed state, given that it was also found (and thus prepared) in an undecayed state at the previous measurement. We are concerned with spontaneous decay, for example the decay of an atom from an excited state, and not with Quantum Zeno Dynamics in a multi-dimensional subspace [3]. For each measurement in the sequence we will therefore ask the question: what is the probability for an atom to be found undecayed, given that the very same atom was also undecayed at the previous measurement? Because a single atom cannot be found or prepared in a mixed state, for now we will associate \(E\) with a pure state and choose it to project into a one-dimensional subspace of the space of undecayed states. Here we make this choice for simplicity, but in Section 4 we will discuss its generality for the phenomenon of decay.

When \(E\) projects into a one-dimensional subspace, from (5) and (6),
\[
\rho(t_0) = \text{Tr}(E\rho(t_0))E = E. \tag{13}
\]
It is now easier to understand \(Q(n, [t_0, t]; \rho)\) if we recast the model of \(n\) measurements by repeatedly using
\[
E U(\Delta t)\rho(t_0)U^\dagger(\Delta t)E = \text{Tr}(E U(\Delta t)\rho(t_0)U^\dagger(\Delta t))E = \text{Tr}(E \rho(t_0 + \Delta t))\rho(t_0). \tag{14}
\]
For example, with (14), one can write \(\rho(n, t)\) from (8) as
\[
\rho(n, t) = \text{Tr}(E \rho(t_0 + \Delta t))^n\rho(t_0). \tag{15}
\]
Recall that \(t - t_0 = n\Delta t\).

\(^1\) Now we will also consider \(E\) to project at every measurement into the same one-dimensional subspace. Later we will need to generalize to the case when there are multiple \(E_i\), and the subscript \(i\) labels different measurements from the sequence.
Let \( \mathcal{H} \) be the Hilbert space for the decaying system. It is apparent from (15) that the repeated measurement process for \( n \) ideal measurements, as developed in [9], can be modeled by one projection of a single state in the product Hilbert space

\[
\tilde{\mathcal{H}} = \prod_{j=1}^{n} \otimes \mathcal{H}.
\]  

The density operator in \( \tilde{\mathcal{H}} \) is

\[
\tilde{\rho}(t) = \prod_{j=1}^{n} \otimes \rho(t).
\]

The projection onto the one-dimensional subspaces of undecayed states is

\[
\tilde{\mathcal{E}} = \prod_{j=1}^{n} \otimes \mathcal{E},
\]

and \( \tilde{\rho}(t_0) = \tilde{\mathcal{E}} \), which means that each of \( n \) independent systems is identically prepared in the same excited state. Finally,

\[
Q(n, [t_0, t]; \rho) = \text{Tr}(\tilde{\mathcal{E}}\tilde{\rho}(\Delta t)).
\]  

From (19) we see that the desired quantity, \( Q(n, [t_0, t]; \rho) \), meant to be the probability that the system is repeatedly found undecayed over a duration \( n\Delta t \), is the product

\[
Q(n, [t_0, t]; \rho) = q(\Delta t)^n,
\]  

where

\[
q(\Delta t) = \text{Tr}(\mathcal{E}\rho(t_0 + \Delta t)) = \text{Tr}(\rho(t_0)\rho(t_0 + \Delta t)).
\]  

Thus \( q(\Delta t) \) is an autocorrelation function, or the expectation value between a state at \( t_0 \) and itself a duration \( \Delta t \) later. That the sequence of measurements for a one-dimensional subspace is equivalent to the product in (20) is already understood [1], and in Section 4 we will revisit it as the basis of the common intuitive explanation of the Quantum Zeno Effect.

### 3 Time Evolution of the Autocorrelation

As is well known, for ideal, isolated systems the fundamental connection between quantum theory and the numbers measured in experiments is the probability

\[
\mathcal{P}_\Lambda(\rho(t)) = \text{Tr}(\Lambda\rho(t)),
\]

where \( \Lambda \) is an operator representing an observable. \( \mathcal{P}_\Lambda(\rho(t)) \) is the probability to observe at time \( t \) the quantity represented by \( \Lambda \) in the state of a system represented by \( \rho(t) \). Continuing in the Schrödinger picture, \( \Lambda \) does not evolve in time.

Let us momentarily ignore the repeated measurements described in Section 2 and consider the uninterrupted time evolution of quantities such as \( q(\Delta t) \) in (21). We use the
duration, \( \delta \equiv t - t_0 \), which need not be small, and define the autocorrelation function, which requires two time arguments,

\[
C(t_0, t_0 + \delta) \equiv \text{Tr}(\rho(t_0)\rho(t_0 + \delta)). \tag{23}
\]

\( C(t_0, t_0 + \delta) \) is equivalent to \( C(t_0, t) \), and in this section we will attempt to write the one that makes the most sense contextually\(^2\).

If the prediction of the Quantum Zeno Effect is to be observable, one must assume that the autocorrelation function in (23) can be a surrogate for a probability \( (22) \). To identify \( C(t_0, t_0 + \delta) \) with \( \mathcal{P}_\Lambda(\rho(t_0 + \delta)) \), one apparently uses

\[
\text{Tr}(\Lambda \rho(t_0)) = 1 \tag{24}
\]

either to choose the state, \( \rho(t_0) \), or to define the observable, \( \Lambda \).\(^3\) Something similar was done above in (3) to model repeated measurement.

Now we will study the time evolution of the autocorrelation function to demonstrate why it is not a probability. First, use (11) and insert (3) twice into (23) to calculate

\[
C(t_0, t_0 + \delta) = \text{Tr}(\rho(t_0)\rho(t_0 + \delta))
\]

\[
= \text{Tr}(U(t'-t_0)U(t'-t_0)\rho(t_0)U(t'-t_0)\rho(t_0 + \delta))
\]

\[
= \text{Tr}(\rho(t')\rho(t' + \delta))
\]

\[
= C(t', t' + \delta). \tag{25}
\]

This is valid for all \( t_0 \) and \( t' \), so the correlation between a quantum state and itself after a duration \( \delta \) is the same for all times. There is no analogue of (25) for the probability:

\[
\mathcal{P}_\Lambda(\rho(t_0 + \delta)) \neq \mathcal{P}_\Lambda(\rho(t' + \delta)) \tag{26}
\]

for general \( t' \), unless \( [\rho(t), H] = 0 \) and dynamics is trivial.

An autocorrelation does not evolve in time as does a probability. The result in (25) leads also to an ambiguity in the physical meaning of the derivative of \( C(t_0, t_0 + \delta) \). Use (11) and (2) to rearrange (23):

\[
C(t_0, t_0 + \delta) = \text{Tr}(\rho(t_0)\rho(t_0 + \delta))
\]

\[
= \text{Tr}(\rho(t_0)U(\delta)\rho(t_0)U(\delta))
\]

\[
= \text{Tr}(\rho(t_0)U(\delta)\rho(t_0))
\]

\[
= \text{Tr}(\rho(t_0)\rho(t_0 - \delta))
\]

\[
= C(t_0, t_0 - \delta). \tag{27}
\]

\(^2\) In many treatments, the amplitude, \( a(t_0, t_0 + \delta) \), of the autocorrelation, \( C(t_0, t_0 + \delta) = |a(t_0, t_0 + \delta)|^2 \), is identified as a probability amplitude. Note that in Chapter 2 of [11], however, \( a(t_0, t_0 + \delta) \) is correctly called the correlation amplitude instead of the probability amplitude.

\(^3\) Equation (24) explains why there is one time argument in \( \mathcal{P}_\Lambda(\rho(t)) \) and two time arguments in \( C(t_0, t) \). The observable, \( \Lambda \), is associated with the state at \( t_0 \), \( \rho(t_0) \).
If \( C(t_0, t_0 + \delta) \) is a continuous function, it is an even function of the duration, \( \delta \). Its derivative with respect to the duration is therefore odd, and it follows that
\[
\left. \frac{dC(t_0, t_0 + \delta)}{d\delta} \right|_{\delta=0} = \left. \frac{dC(t_0, t)}{dt} \right|_{t=t_0} = 0. \quad (28)
\]
The initial rate of change of the autocorrelation is zero. Note also that, from (25) and (28), we have
\[
\left. \frac{dC(t_0, t)}{dt} \right|_{t=t_0} = \left. \frac{dC(t', t)}{dt} \right|_{t=t'} = 0, \quad \text{for all } t_0, t'. \quad (29)
\]
The derivative of the autocorrelation is initially zero always and not just at preparation.

The ambiguity in the meaning of (29) arises because, with a simple change of variables, there are multiple ways to calculate the same derivative. For example, the probability rate at a duration \( \delta^* \) from a time of zero can be
\[
\left. \frac{dP_\Lambda(\rho(\delta))}{d\delta} \right|_{\delta=\delta^*} \quad (30)
\]
or
\[
\left. \frac{dP_\Lambda(\rho(\delta^* + h))}{dh} \right|_{h=0}, \quad (31)
\]
both of which result in the same value. In the case that \( P_\Lambda(\rho) \) represents a decay probability, the physically meaningful probability for a transition during the interval \([\delta^*, \delta^* + \Delta t]\),
\[
\left( \frac{dP_\Lambda(\rho(\delta))}{d\delta} \right|_{\delta=\delta^*} \right) \Delta t \quad \text{is independent of one’s choice, (30) or (31), for calculating a derivative, as required. If } C(t_0, t_0 + \delta) \text{ is interpreted as a probability, however, both}
\]
\[
\left( \frac{dC(0, 0 + \delta)}{d\delta} \right|_{\delta=\delta^*} \right) \Delta t \neq 0 \quad \text{in general} \quad (32)
\]
and
\[
\left( \frac{dC(\delta^*, \delta^* + h)}{dh} \right|_{h=0} \right) \Delta t = 0 \quad (33)
\]
would have the same physical interpretation but result in different values.

The autocorrelation is only physically meaningful and consistent if one fixes the first time argument in the autocorrelation, \( C(t_0^{\text{fixed}}, t) \), and prohibits the otherwise allowed manipulations in (25) and (27). To do so, one must replace the density operator \( \rho(t_0) \) with an observable, which does not evolve in time in the Schrödinger picture. This effectively changes the autocorrelation into a probability. It is straightforward to check that this is not merely a cosmetic change, because it endows the amplitude with a phase that is not

\footnote{This result is found differently in [11, 1]. If one assumes that \( C(t_0, t_0 + \delta) \) represents a probability, then the result in (28) implies for spontaneous decay that the survival probability will not be exponential at short times after preparation at \( t_0 \). See also Section 4.}
proportional to time. For example, if \( a(t_0, t_0 + \delta) \) is the amplitude of the autocorrelation, such that \( C(t_0, t_0 + \delta) = |a(t_0, t_0 + \delta)|^2 \), then \( a^*(t_0, t_0 + \delta) = a(t_0, t_0 - \delta) \). A probability amplitude does not have this property.

As will become clear in the following sections, the important but subtle conclusion is that the probabilities representing the observable time evolution of a quantum system require an operator representing an observable, and are not given solely by the time evolution of the state vector or density operator of the system. The quadratic evolution at early times of a system’s autocorrelation function is only observable in special cases.

4 The Intuitive Picture

The intuitive explanation [11, 4] often given for the Quantum Zeno Effect derives from the product \( q(\Delta t)^n \) in (20), which is equivalent to the quantity calculated in [9] before moving to the continuous limit by taking \( n \to \infty \). As shown below, the result in (28) for the initial derivative of the autocorrelation is necessary for the prediction of the inhibition of decay. Using the autocorrelation, however, does not necessarily lead to an observable result. In this section we will follow the standard intuitive development, but we will identify \( q(\Delta t) \), or \( q(\delta) \), instead with the probability, \( P_u(\rho(t_0 + \delta)) \), to observe the system in its undecayed state at a duration \( \delta \) after preparation.

The subscript \( u \) on \( P_u(\rho(t_0 + \delta)) \) denotes the use of an operator representing the observable, or observation, that the state is undecayed. We will discuss the operator representing this observable later. For now, assume the excited state is prepared at \( t_0 \), such that \( P_u(\rho(t_0)) = 1 \), and expand the probability for small durations, \( \delta \):

\[
P_u(\rho(t_0 + \delta)) = 1 + \left. \frac{d}{d\delta} P_u(\rho(t_0 + \delta)) \right|_{\delta=0} \delta + \mathcal{O}(\delta^2).
\] (34)

To move to the limit of continuous measurement over a finite duration \( t-t_0 \), one calculates the product \( q(\delta)^n \) after letting \( t-t_0 = n\delta \) and taking the limit \( n \to \infty \). Use \( \delta = (t-t_0)/n \) and continue the expansion begun in (34):

\[
P_u(\rho(t_0 + \delta)) = 1 + \frac{x}{n},
\] (35)

where

\[
x = \frac{-(t-t_0)}{\tau} + \sum_{\ell=2}^{\infty} \frac{D^{\ell}(0)}{\ell!n^{\ell-1}} (t-t_0)^{\ell}.
\] (36)

Here \( \tau \) is the negative inverse of the initial rate of change of the probability,

\[
\tau^{-1} = \left. \frac{d}{d\delta} P_u(\rho(t_0 + \delta)) \right|_{\delta=0},
\] (37)

and \( D^{\ell}(0) \) is the \( \ell \)th time derivative of \( P_u(\rho(t_0 + \delta)) \), evaluated at \( \delta = 0 \).
Note that we have assumed $\mathcal{P}_u(\rho(t_0 + \delta))$ and thus $\tau$ do not change from measurement to measurement in the sequence. We have not given $\tau$ a label corresponding to any particular measurement. We also assume that $\mathcal{P}_u(\rho(t_0 + \delta))$ is analytic in the neighborhood of $t_0$. After a finite time $t - t_0 = n\delta$, and in the limit of an infinite number, $n$, of ideal measurements, the probability for the system to remain undecayed is

$$
\lim_{n \to \infty} \mathcal{P}_u(\rho(t_0 + \delta))^n = \lim_{n \to \infty} \left[ 1 + \frac{x}{n} \right]^n
= e^{-\frac{(t-t_0)}{\tau}} \exp \left( \sum_{\ell=2}^{\infty} \left( \frac{D^\ell(0)}{\ell!n^{\ell-1}} \right)(t-t_0)^\ell \right) \bigg|_{n \to \infty}
= e^{-\frac{(t-t_0)}{\tau}}. \tag{38}
$$

The probability that there occurs a continuous sequence of measurements in which the system is always found undecayed, is exponentially decreasing in the time duration of the sequence. Continuous measurement as a limit of a sequence of ideal measurements, as imagined in [9], therefore leads to exactly exponential decay with a decay rate, $\tau^{-1}$, given by (37), regardless of the time evolution of the unmonitored state.

This intuitive picture shows why, when using the autocorrelation, $C(t_0, t_0 + \delta)$, as a surrogate for a probability, one predicts an inhibition of decay. In this case, the decay rate in (37) is given not by the initial probability rate, but instead by the initial time derivative of the autocorrelation, $\left. \frac{dC(t_0, t_0 + \delta)}{d\delta} \right|_{\delta=0}$, which by (28) is known to be zero. It follows that the survival probability calculated in (38) equals one for the duration of continuous monitoring.

As discussed in Section 3, however, the autocorrelation cannot be a surrogate for the physically meaningful probabilities, $\mathcal{P}_\Lambda(\rho(t))$. To verify an observable Quantum Zeno Effect, we insert the probability (22) for a general state and a general observable into (37), which results in

$$
\tau^{-1} = \frac{i}{\hbar} \left( \text{Tr}(\rho(t_0)\Lambda H) - \text{Tr}(\Lambda \rho(t_0) H) \right). \tag{39}
$$

There are two possibilities for the decay rate at $t_0$ to be zero. The first is for trivial dynamics, $[\rho(t), H] = 0$, as expected. The second is when the state of the system at preparation commutes with the observable:

$$
[\rho(t_0), \Lambda] = 0. \tag{40}
$$

Equation (40) is the condition for a vanishing initial probability rate in a standard experiment without continuous monitoring. Here we wish to study its application to repeated measurement and to spontaneous decay, which is an incoherent process. It will be useful to distinguish operators representing a coarse “decayed” or “undecayed” observation for the system, from infinitesimal projection operators into undecayed subspaces. For example, we will use the operator $\Lambda_u$, which corresponds to the observation that the system is undecayed. It is defined by the characteristics of a real or hypothetical detector built to detect undecayed states. If $\hat{E}_\ell$ are one-dimensional projection operators into undecayed subspaces labeled by
\( \ell \), then in the discrete case:

\[
\Lambda_u = \sum_{\ell} w_{\ell} \hat{E}_{\ell}.
\]  

(41)

Here the \( \hat{E}_{\ell} \) included in the sum represent the pure states to which the real or hypothetical registration device is sensitive, and sensitivities to different pure states can be described by the weights, \( w_{\ell} \). For a perfectly efficient detection device, and when the \( \hat{E}_{\ell} \) span the entire space of undecayed states, an undecayed system is described as always by a density operator satisfying

\[
\text{Tr}(\Lambda_u \rho(t)) = 1.
\]  

(42)

Of course, if a registration device is isolated enough not to perturb the system, its actual presence is physically irrelevant, and an excited system will decay whether or not its decay products are recorded in an experiment.

Continuing to think in terms of consecutive measurements performed on a single atom at a time, as was done in Section 2, the \( \hat{E}_{\ell} \) are useful because a single atom cannot be prepared in a mixed state. As a result of any given measurement in which it was found to be undecayed, the state of the atom will be represented initially by a density operator satisfying \( \text{Tr}(\hat{E}_{\ell} \rho(t_0)) = 1 \) for some value of the index \( \ell \). We will label the density operator for such a pure state \( \rho_{\ell}(t_0) \). If at the next measurement the atom is found to be in the undecayed subspace \( \hat{E}_k \), then the relevant condition for there to have been a vanishing initial probability rate is

\[
[\rho_{\ell}(t_0), \hat{E}_k] = 0.
\]  

(43)

The commutation of \( \rho_{\ell}(t_0) \) with all the other \( \hat{E}_{j \neq k} \) is irrelevant.

Fortunately, when experimentally detecting if a quantum system has decayed or not, one can continue to treat one atom at a time. In an experiment with many atoms, for example, each atom may individually undergo a different measurement sequence, and thereby be prepared in a different sequence of pure states, \( \hat{E}_1, \hat{E}_2, \ldots \hat{E}_n \). With a hypothetical, perfectly efficient detector, however, all undecayed atoms will be represented by a mixed density operator satisfying (42), as required. The only consequence is that the monitoring interaction causes any initially pure state to become mixed. When using the model we create in the next section, we must think of the decay of an excited state as an incoherent process, which agrees with observations and the phenomenological understanding.

The condition in (43) can be satisfied only if the system is prepared at \( t_0 \) to be in the eigenstate of \( \hat{E}_k \). In the case of repeated measurements, (43) will be satisfied only as long as the atom is repeatedly prepared to be in a common eigenstate of the same set of commuting observables. Rewriting (43) for the sequence \( \hat{E}_1, \hat{E}_2, \ldots \hat{E}_n \), where \( n \to \infty \) represents continuous monitoring, we have as the condition for stopping decay

\[
[\rho_{\ell}(t_0), \hat{E}_{i+1}] = 0.
\]  

(44)

5 In equation (46) is an example with a continuous spectrum.

6 We will use the hat on \( \hat{E}_{\ell} \) to remind ourselves that the \( \hat{E}_{\ell} \) cannot be a density operator or be used to create an autocorrelation function, as was the \( \mathcal{E} \) introduced in Section 2.
for all $i$, $0 \leq i \leq n - 1$. For a controlled experiment on a perfectly isolated physical system, one can in principle perform a sequence of measurements such that equation (44) is always satisfied. For the article [9] discussed in Section 2, the sequence envisioned is $\rho_i(t_0) = \mathcal{E}$ and $\hat{\mathcal{E}}_i = \mathcal{E}$ for all $i$. In this case, equation (44) is clearly satisfied, and an inhibition of decay is physically observable, even though the autocorrelation was used instead of the probability.

For more general sequences and certainly in uncontrolled situations, one expects (44) not to be satisfied, and that there will be no inhibition of decay. Consider the following two possibilities:

1. $[\rho_i(t_0), \hat{\mathcal{E}}_{i+1}] \neq 0$, and the probability $P_u(\rho(t_0 + \delta))$ changes from measurement to measurement: $P_u(\rho_i(t_0 + \delta)) \neq P_u(\rho_{i+1}(t_0 + \delta))$. Then the value of the decay rate will change throughout the sequence, and the calculation of the survival probability in (38) is not correct. To proceed in such a case, one must specify the sequence of measurements described by $\hat{\mathcal{E}}_1, \hat{\mathcal{E}}_2, \ldots \hat{\mathcal{E}}_n$.

2. Though $[\rho_i(t_0), \hat{\mathcal{E}}_{i+1}] \neq 0$, the probability $P_u(\rho(t_0 + \delta))$ is indeed constant throughout the sequence, and the calculation in (38) is valid.

As exemplified in the next section, it is not difficult to imagine systems being monitored such that $[\rho_i(t_0), \hat{\mathcal{E}}_{i+1}] \neq 0$ and such that the probability remains constant. In these cases the result for continuous monitoring, as the limit of a sequence of ideal measurements, is exactly exponential decay (38). Interestingly, continuous monitoring causes exponential decay even for systems that otherwise do not exhibit exponential time evolution.

5 A Model for Spontaneous Decay

Upon the incoherent, spontaneous decay of atoms from an excited state, each atom emits a photon in a random direction, and each atom is left in a state allowed by the conservation of angular momentum. This suggests a simple model in which the excited atom interacts continuously with an incoherent radiation field, or an incoherent sum of many radiation fields, which provides the measurement interaction and physically represents the interaction between the atom’s nucleus and electrons. With each measurement interaction, the system of the excited atom and a field mode is left in a state that does not commute with the state in which it will be left following the next measurement in the sequence. For example, interactions with radiation modes polarized in different directions will leave the atom in eigenstates of angular momentum aligned differently, such that total angular momentum is conserved. In this scenario, to each combination of radiation mode and undecayed atom corresponds one of the $\hat{\mathcal{E}}_i$. Following each measurement of an undecayed system, one has

$$\text{Tr}(\hat{\mathcal{E}}_i \rho_i(t_0)) = 1. \quad \text{(45)}$$

We will assume that the atom interacts with one of the incoherent field modes at a time, and that there exist enough modes that the ideal measurements are all separated in time by arbitrarily small durations. Based on the randomness of the directions followed by decay
products, we will also assume that the radiation field modes are randomly aligned such that there is a vanishingly small probability for the observables of consecutive projective measurements to commute: \([\rho_i(t_0), \hat{E}_{i+1}] \neq 0\) for almost all \(i\).

With the conditions for exponential decay met, let us calculate the decay rate. As with actual experiments we will assume the presence of a real or hypothetical detector built to observe the outgoing photon associated with decay, and not the system directly in its undecayed state. The probability to observe the photon associated with decay is \(P_d(\rho(t)) = \text{Tr}(\Lambda_d \rho(t)) = 1 - P_u(\rho(t))\). The operator representing the observable, \(\Lambda_d\), is therefore

\[
\Lambda_d = \sum_{\eta} \int d^2 \Omega \, dE \, \lambda(E, \Omega, \eta) \Lambda_{E, \Omega, \eta},
\]

where

\[
\Lambda_{E, \Omega, \eta} = |E, \Omega, \eta\rangle(E, \Omega, \eta) \equiv |\psi_{E, \Omega, \eta}\rangle(\psi_{E, \Omega, \eta}|
\]

are projection operators onto the one-dimensional or infinitesimal subspaces\(^7\) representing the state of the emitted photon and the available decayed states of the atom, and \(\lambda(E, \Omega, \eta)\) is the density of observable final states describing the sensitivity of a hypothetical detector, which we will assume to be perfectly efficient. Here \(E\) is the energy, and \(\Omega\) describes the direction of momentum of the decay products. Finally, \(\eta\) represents the excitation level of the atom, as well as any other discrete or internal quantum numbers required to specify the state of the system, such as angular momentum and polarization. The density of states, \(\lambda(E, \Omega, \eta)\), is a number per unit energy and unit solid angle. By choosing \(\Lambda_d\) in (46) so that \(\lambda(E, \Omega, \eta)\) controls the integration and summation, we must construct the \(\Lambda_{E, \Omega, \eta}\) from dimensionless, discrete basis vectors (see (49) below.) Our reason for choosing this form will soon become clear.

To calculate the decay rate in (37) or (39), we will use perturbation theory for the continuous spectrum. Let the Hamiltonian,

\[
H = H_0 + V,
\]

be divided into a free part, \(H_0\), and an interaction potential, \(V\). The \(|E, \Omega, \eta\rangle\) in (47) are the discrete basis vectors of the free Hamiltonian:

\[
H_0 |E, \Omega, \eta\rangle = E |E, \Omega, \eta\rangle.
\]

As usual, eigenkets, or continuous basis vectors, of \(H_0\) are \(|E, \Omega, \eta\rangle\), and eigenkets of \(H\) are \(|E, \Omega, \eta^+\rangle\). The \(|E, \Omega, \eta^+\rangle\) are the Lippmann-Schwinger kets,

\[
|E, \Omega, \eta^+\rangle = |E, \Omega, \eta\rangle + \lim_{\epsilon \to 0} \frac{1}{(E - H_0) + i\epsilon} V |E, \Omega, \eta^+\rangle.
\]

At every moment the single atom is measured, or prepared, it is prepared to be in a pure state with one mode of the radiation field and is under the influence of the interaction potential. Therefore, we have for the state of the system when the atom and field interact

\[
\rho(t) = |\phi^+(t)\rangle \langle \phi^+(t)|,
\]

\(^7\) We will write \(|E . . .\rangle\) for discrete basis vectors and \(|E . . .\rangle\) for the eigenkets, or continuous basis vectors.
where \(|\phi^+(t_0)\rangle\) is expanded in terms of the eigenkets of the total Hamiltonian:

\[
|\phi^+(t_0)\rangle = \sum_\eta \int d^2\Omega dE |E, \Omega, \eta\rangle \langle +E, \Omega, \eta|\phi^+(t_0)\rangle. \tag{52}
\]

This fixes the dimensionality of the eigenkets \(|E, \Omega, \eta\rangle\). As usual, we relate \(|\phi^+\rangle\) to the system without interaction, \(|\phi\rangle\), by

\[
\langle +E, \Omega, \eta|\phi^+\rangle = \langle E, \Omega, \eta|\phi\rangle, \tag{53}
\]

where

\[
|\phi\rangle = \sum_\eta \int d^2\Omega dE |E, \Omega, \eta\rangle \langle E, \Omega, \eta|\phi\rangle. \tag{54}
\]

Using (51) and (46), and the fact that \(\frac{d}{dt} \mathcal{P}_u(\rho(t)) = -\frac{d}{dt} \mathcal{P}_d(\rho(t))\), we can rewrite the decay rate in (37) as

\[
\tau^{-1} = \frac{2}{\hbar} \text{Im} \sum_{\eta_f} \int d^2\Omega_f dE_f \langle \psi_{E_f,\Omega_f,\eta_f} |V| \phi^+(t_0)\rangle \langle \phi^+(t_0)|\psi_{E_f,\Omega_f,\eta_f}\rangle \lambda(E_f, \Omega_f, \eta_f), \tag{55}
\]

where we have used \([\Lambda_d, H_0] = 0\). Expanding the states with energy eigenkets, and using (50) once, we have for the decay rate,

\[
\tau^{-1} = \lim_{\epsilon \to 0} \frac{2}{\hbar} \text{Im} \sum_{\eta_f} \int d^2\Omega dE d^2\Omega_f dE_f \lambda(E_f, \Omega_f, \eta_f) \\
\langle \psi_{E_f,\Omega_f,\eta_f} |V| \phi(t_0)\rangle \langle \phi(t_0)|E, \Omega, \eta\rangle \langle E, \Omega, \eta|\psi_{E_f,\Omega_f,\eta_f}\rangle \\
+ \langle \psi_{E_f,\Omega_f,\eta_f} |V| \phi(t_0)\rangle \langle \phi(t_0)|E, \Omega, \eta\rangle \langle +E, \Omega, \eta|V\frac{1}{E - H_0 - i\epsilon} |\psi_{E_f,\Omega_f,\eta_f}\rangle. \tag{56}
\]

Considering (53), the first term on the right hand side of (56) contains the product

\[
\langle \phi(t_0)|E, \Omega, \eta\rangle \langle E, \Omega, \eta|\psi_{E_f,\Omega_f,\eta_f}\rangle. \tag{57}
\]

Because \(\eta\) represents the excitation level of the atom, this term will vanish. Using (49), we are left with

\[
\tau^{-1} = \frac{2\pi}{\hbar} \sum_{\eta_f} \int d^2\Omega dE d^2\Omega_f dE_f \langle E_f, \Omega_f, \eta_f|V|\phi(t_0)\rangle \langle \phi(t_0)|E, \Omega, \eta\rangle \\
\langle +E, \Omega, \eta|V|E_f, \Omega_f, \eta_f\rangle \lambda(E_f, \Omega_f, \eta_f) \delta(E - E_f). \tag{58}
\]

The delta function in (58) enforces energy conservation. Recall that the final energy, \(E_f\), includes the energy of the photon. One can write, for instance, \(E_f = E_{\eta_f} + \hbar \omega_f\). Equation (58) is our most general result for the decay rate, \(\tau^{-1}\), of our model system.
When the excited state of the system at each preparation, $|\phi(t_0^+)\rangle$, is characterized by sharp values, $E_i$, $\Omega_i$, and $\eta_i$, such that

$$\sum_\eta \int dE d\Omega dE_f |E, \Omega, \eta^+\rangle \langle E, \Omega, \eta|\phi^+\rangle \delta(E - E_f) \approx \int dE_f |\phi^+_{E_i, \Omega_i, \eta_i}\rangle \delta(E_i - E_f).$$

we can write (58) in infinitesimal form. Keeping in mind that we have already used the Lippmann-Schwinger kets once in (56),

$$\tau^{-1} = \frac{2\pi}{\hbar} \sum_{\eta_f} \int dE_f d^2\Omega_f |(\psi_{E_f, \Omega_f, \eta_f}|V|\phi^+_{E_i, \Omega_i, \eta_i}(t_0))|^2 \lambda(E_f, \Omega_f, \eta_f) \delta(E_i - E_f).$$

Because of the random orientation of radiation modes causing the measurement, or preparation, we will ignore $\Omega_i$ and let $|\phi^+_{E_i, \Omega_i, \eta_i}(t_0)\rangle \rightarrow |\phi^+_{E_i, \eta_i}(t_0)\rangle$ represent the average prepared state. Assuming an isotropic distribution of radiation fields and that the hypothetical registration device surrounds the system, we choose $\lambda(E, \Omega, \eta)$ such that the density of final states is isotropic:

$$|\psi_{E, \eta}\rangle(\psi_{E, \eta}| \lambda(E, \eta) = 4\pi \lambda(E, \Omega, \eta)|\psi_{E, \Omega, \eta}\rangle(|\psi_{E, \Omega, \eta}|.$$  

(61)

We then have for the partial decay rate, $\tau^{-1}_{\eta_i \rightarrow \eta_f}$, for a channel labeled by the initial and final internal quantum numbers $\eta_i$ and $\eta_f$,

$$\tau^{-1}_{\eta_i \rightarrow \eta_f} = \frac{2\pi}{\hbar} |(\psi_{E, \eta_f}|V|\phi^+_{E, \eta_i}(t_0))|^2 \lambda(E, \eta_f),$$

(62)

where the notation reflects energy conservation. Thus we have derived Fermi’s Golden Rule for the decay rate. Recall that $|\phi^+_{E, \eta_i}(t_0)\rangle$ represents the excited atom and radiation at preparation, when the interaction potential is active. In this infinitesimal form, it is a Lippmann-Schwinger ket, which can be iteratively expanded to obtain the Born series.

Finally, note that the effects of uncontrolled monitoring have been considered in the past (see \cite{1, 10, 6} and references therein), though to the best of our knowledge, conclusions have differed, and have often been drawn from the product of autocorrelation functions. For our example of an excited atom, we are imagining as a measurement mechanism the presence of incoherent electromagnetic field modes corresponding to the force between an atom’s nucleus and its electrons. The presence of an analogous system for nuclear decay has been mentioned in \cite{10}. In that case, however, it was an argument against the link between measurement and the predicted inhibition of decay (based on the autocorrelation), because it implied that “a neutron bound in a nucleus does not decay because it is under constant observation by its fellow nucleons.” Here we understand that constant observation by an incoherent system can in fact cause exponential decay.

6 Discussion

There are reasons related to the stability of matter to expect that, if a spontaneously decaying state has a Lorentzian energy wave function and is represented by a state vector from
a Hilbert space, then its survival probability cannot be perfectly exponential \[8\]. Any deviation from exponential decay, however, has never been observed. Our results indicate that exponential decay may be thought of as emerging from the process of continuous, uncontrolled measurement. The arguments in \[8\] for a deviation from exponential decay therefore do not apply, because the system is not represented continuously by a single state vector during the decay process. Indeed, between discrete measurements, or when continuous monitoring switches off, \(\phi^+(E)\) is not generally a Lorentzian. As with the exponential decay in time, in our model the Lorentzian line shape only emerges as a consequence of continuous measurement.

Bypassing the arguments against exponential decay comes at the price of invoking the instantaneous collapse mechanism that is normally imagined for the Quantum Zeno Effect. Though the measurement process is not dynamically described in theory, it is treated by a postulate of quantum mechanics \[2, 13\]. At this time, it is simply not understood physically what occurs during the measurement of a quantum system. The continuous limit of ideal measurements, as taken in \[38\], strongly constrains any measurement scheme and any collapse mechanism. Our model suggests that spontaneous decay may be another phenomenon useful in the study of the extent to which measurement must be treated dynamically.

We note that our model also satisfies one’s intuition about the qualitative differences between exponential decay and Rabi oscillations. Both phenomena affect the same states, but decay results from interaction with incoherent radiation modes, and Rabi oscillations follow from interaction with coherent radiation. Unfortunately the model does not help to clarify the nature of the metastable state itself, as discrete eigenstates of the Hamiltonian remain stable (see Equation \[39\].)

To conclude, we have shown why an autocorrelation function cannot be a surrogate for a physically meaningful probability in quantum mechanics. The time evolution of a quantum system depends on the observable as well as on the state of the system. An important consequence is that the autocorrelation’s initially quadratic time evolution does not generally correspond to the observable time evolution of a quantum system. Replacing the autocorrelation with a probability, we reformulated the model of continuous measurement described in \[9\] and found that a condition to observe the Quantum Zeno Paradox is that, following each measurement in the sequence, the system must be prepared in a common eigenstate of the same set of commuting observables. The prediction of the inhibition of decay therefore applies only to special cases, including the case in \[9\].

We also created a model of an excited atom in the presence of an incoherent sum of radiation fields that monitor the atom’s state. We found that, regardless of the time evolution of the atom otherwise, continuous measurement as the limit of ideal, discrete measurements in fact causes exactly exponential decay. Furthermore, the derived decay rate is Fermi’s Golden Rule. If continuous measurement over a finite \(t = n\delta\) can be realized in nature, especially passively by an incoherent radiation field, then it may be a candidate to explain quantum mechanically the normally empirically or approximately \[14\] derived exponential decay law.

An understanding of the effects of repeated or continuous measurement is of great fun-
damental importance, especially for open quantum systems. Our results may also provide insight into the treatment of a quantum system when it suffers interference simultaneously from multiple environmental systems that are in non-commuting states.

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