FAULT-TOLERANT MOSAIC ENCODING IN KNOT-BASED CRYPTOGRAPHY

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Abstract: The cryptographic protocol based on topological knot theory, recently proposed by the authors, is improved for what concerns the efficiency of the encoding of knot diagrams and its error robustness. The standard Dowker–Thistlethwaite code, based on the ordered assignment of two numbers to each crossing of a knot diagram and not unique for some classes of knots, is replaced by a system of eight prototiles (knot mosaics) which, once assembled according to a set of combinatorial rules, reproduces unambiguously any unoriented knot diagram. A Reed–Muller scheme is used to encode with redundancy the eight prototiles into blocks and, once the blank tile is added and suitably encoded, the knot diagram is turned into an $N \times N$ mosaic, uniquely associated with a string of length $4N^2$ bits. The complexity of the knot, measured topologically by the number of crossings, is in turn polynomially related to the number of tiles of the associated mosaic, and for knot diagrams of higher complexity the mosaic encoding provides a design of the knot–based protocol which is fault–tolerant under random 1-bit flips. It is also argued that the knot mosaic alphabet might be used in other applications which require high–capacity data transmission.

1 KNOT-BASED PROTOCOL: REVIEW OF THE DT CODE

The theoretically secure protocol proposed in (Marzuoli \& Palumbo, 2011) is based on purely topological knot theory. The scheme relies on the ‘easy’ problem of associating with prime knots listed in Knot Tables their Dowker–Thistlethwaite codes, numerical sequences which are different for inequivalent knots. Then the ‘difficulty’ of factorizing complex knots generated by composing prime knots is exploited to securely encode the given message. The scheme combines an asymmetric public key protocol with symmetric private ones and is briefly reviewed in the Appendix. In the following two sections the protocol will be improved for what concerns the efficiency of the encoding of knot diagrams and its error robustness.

In order to explain the DT coding used in the original proposal for the protocol, a few basic notions of topological knot theory have to be recalled (Rolfsen, 1976). A knot $K$ is a continuous embedding of the circle $S^1$ into the Euclidean 3–space $\mathbb{R}^3$. Knots can be oriented or unoriented, and collections of a finite number of interlaced knots are called links (in the cryptographic protocol only knots will be used). Referring for simplicity to the unoriented case, two knots $K_1$ and $K_2$ are said to be equivalent, $K_1 \sim K_2$, if and only if they are (ambient) isotopic. An isotopy is a continuous deformation of the shape of, say, $K_2 \subset \mathbb{R}^3$ which makes $K_2$ identical to $K_1$ without cutting and gluing back the ‘closed string’ $K_2$.

The diagram of a knot $K$ is its projection on a plane $\mathbb{R}^2 \subset \mathbb{R}^3$, in such a way that no point belongs to the projection of three segments, namely the singular points in the diagram are only transverse double points. Such a projection, together with ‘over’ and ‘under’ information at the crossing points –depicted in figures by breaks in the under–passing segments– is denoted by the same symbol $K$. In Knots Tables (see Hoste et al., 1998 and the Knot Atlas on Wikipedia) standard (i.e. associated with minimal projections) diagrams of unoriented ‘prime’ knots are listed by increasing crossing numbers as $\chi_n$, where $\chi$ is the number of crossings and $n = 1,2,\ldots$ enumerates in a conventional way the knots with the same $\chi$. The
\textit{Knot Tables}) with its mirror image the 6-crossing 'granny' knot. Connected sum of the trefoil knot sum, denoted by \( K_1 \# K_2 \). Below it is shown the connected sum of the trefoil knot \( K_1 \) (configuration 3\textsubscript{1} in Knot Tables) with its mirror image \( K_2 \), giving rise to the 6-crossing 'granny' knot.

\[ K_1 \# K_2 \]

The Dowker–Thistlethwaite (DT) notation (or code) is defined for oriented knots and assigns to each planar diagram its (minimal) DT sequence. Given for instance an oriented alternating knot with \( \chi \) crossings (namely a diagram with an alternating sequence of over and under–crossings) the associated DT sequence is built iteratively: i) start labeling an arbitrarily chosen crossing with 1; ii) then, following the given orientation, go down the strand to the next crossing and denote it by 2; iii) continue around the knot until each crossing has been numbered twice. Thus each crossing is decorated with a pair of even/odd positive numbers, running from 1 to \( 2\chi \), as shown below for the knot listed as 5\textsubscript{1}.

![Diagram of knot labeled by DT sequence](image)

For generic, non–alternating prime knots (which actually appear in tables for crossing numbers greater than 7), the DT coding is slightly modified by making the sign of the even numbers positive if the crossing is on the top strand, and negative if it is on the bottom strand. Since any sequence is dependent on both a minimal projection and the choice of a starting point, the mapping between knots and their DT sequences is in general one–to–many. In the following section a new type of encoding which overcome these ambiguities is proposed.

\section{Knot Mosaics and Their Block Encoding}

Kauffman and Lomonaco introduced in (Lomonaco & Kauffman, 2008) a \textit{mosaic system} –made of eleven elementary building blocks–with the aim of addressing what they call ‘quantum knots’, namely quantum observables arising in the framework of 3-dimensional topological quantum field theories. The prototiles we are going to employ here are a subset of Lomonaco–Kauffman mosaics which suffices to reconstruct the cores of diagrams of (prime or composite) knots on the basis of purely combinatorial rules to be addressed in the following section.

Let \( \mathcal{M}_{\{8\}} \) denotes the ordered set of eight knot mosaics (prototiles) \( m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8 \) depicted in Fig. 1. Similarly to what happens in tiling (a portion of) a plane with a given set of prototiles, the single mosaics will be assembled as they stand (neither rotation nor reflection allowed). Note however that, unlike most commonly used tiling prescriptions, the set that is being used here is closed under rotations of the single mosaics (the most economical set would include just \( m_1, m_2, m_6 \), but then rotations should be allowed).

![Diagram of eight knot prototiles](image)

These objects can be associated with digital sequences through encoding maps \( \mathcal{M}_{\{8\}} \rightarrow \{0,1\}^5 \) that can be chosen in many different ways (Pless, 1982). For instance the code space \( \Sigma_{(3)} := \{0,1\}^3 \) of 3-bit sequences would provide a very simple (but not fault–tolerant) one–to–one encoding of the eight mosaics.
Variable–length codes do not seem particularly suitable in the present context, where each mosaic is in principle on the same footing as each other, and the encoding of long sequences of tiles would become overwhelming. To achieve a sufficient degree of redundancy (the basic requirement of any fault–tolerant coding protocol) still keeping a block design, consider the injective map

\[ E : \mathcal{M}(8) \to \Sigma(4) := \{0, 1\}^4 \]

defined by the correspondences given in Table 1.

Table 1: Encoding of the eight knot mosaics into 4-bits strings

| \(m_1\) | \(m_2\) | \(m_3\) | \(m_4\) |
|---|---|---|---|
| 0000 | 1010 | 1111 |
| \(m_5\) | \(m_6\) | \(m_7\) | \(m_8\) |
| 0011 | 0110 | 1001 | 1100 |

The valid words in the code space \(\Sigma(4)\) are a sub-set characterized by the property of being identical to the eight sequences of the binary Reed–Muller code \(R(1, 2)\). Mutual Hamming distances between these codewords are easily evaluated:

\[ d(m_1, m_4) = d(m_2, m_3) = d(m_5, m_6) = d(m_7, m_8) = 4, \]

all others are equal to 2. The minimal distance associated with the encoding (1) is

\[ d_{\min} [E(\mathcal{M}(8)) \subset \Sigma(4)] = 2, \]  

thus providing a random error detection ability equal to 1 (a single bit–flip cannot turn one codeword into another).

In order to carry out the assembling of knot mosaics, the empty prototile (blank tile) has to be added and suitably represented in the codespace \(\Sigma(4)\). Actually, as will be described in the following section, a single blank tile does not suffice to achieve redundancy, but rather four empty mosaics endowed with double arrows are needed. Denoting by \(\mathcal{B}(4)\) the ordered set of the four mosaics \(b_1, b_2, b_3, b_4\), the domain of the encoding map is extended to

\[ E : \mathcal{B}(4) \cup \mathcal{M}(8) \to \Sigma(4), \]

where the \(b\)'s are associated with previously unassigned codewords according to the list in Table 2.

Table 2: Encoding of the four blank mosaics

| \(b_1\) | \(b_2\) | \(b_3\) | \(b_4\) |
|---|---|---|---|
| 1000 | 0010 | 0100 | 0001 |

3 ENCODING PROCESS OF DIAGRAMS INTO \(N \times N\) MOSAICS

As a first instance of what is meant by an assembled blank diagram made out of \(m\) and \(b\) tiles, in Fig. 2 the composite 6-crossing granny knot already shown in section 1 is depicted.

![Figure 2: The mosaic of the granny knot.](image)

This rectangular \(6 \times 4\) mosaic can be completed with two more rows of blank tiles to get a \(6 \times 6\) square and, as can be easily checked, also all 6-crossing prime knots can be arranged into such a square by suitably combining the \(m\)-type and blank tiles, see Fig. 3, 4, 5 (recall that there are exactly three prime knots with 6 crossings, listed as \(6_1, 6_2, 6_3\) in Knot Tables).

![Figure 3: The mosaic of the 6_1 knot.](image)
While the above examples can be handled quite easily, in case of a generic diagram of some prime knot $K$ with $\chi(K)$ crossings, global combinatorial rules are needed to reconstruct (and coding) the associated mosaic. Note first that the extension of the square (rectangle), not known \textit{a priori}, can be evaluated since any tile containing a crossing ($m_1, m_4$ of Fig. 1) is topologically interconnected to eight tiles surrounding it. Then an upper bound on the size of an $N \times M$ mosaic is given by

$$N \times M \leq 8\chi,$$

so that the operation of converting any standard knot diagram into a mosaic can be efficiently performed.

Focusing on square mosaics, since their assembling must proceed with no reference to the size of the resulting table, it is worth starting from an internal tile and following a spiral path, moving e.g. in the clockwise direction, as shown in Fig. 4 for a $6 \times 6$ mosaic.

As for blank tiles, placed at the corners of $6 \times 6$ mosaics in Fig. 3, their decorations with double arrows introduced in Table 2 can now be explained. Thus $b_1$ must be used to fill the empty squares around the upper right corner, $b_2$ around the lower right, $b_3$ around the lower left, and $b_4$ around the upper left. Referring, e.g., to $b_1$ in connection with Fig. 3, both arrows of tile $b_1$ are activated in the empty square located at the upper right corner; the rightward arrow agrees with the spiral pathway one mosaic left to the corner and the downward arrow agrees with the spiral pathway one mosaic below the corner.

Before going through an analysis of topological prescriptions which will further enforce the coding procedure with respect to fault tolerance, it is necessary to extend the encoding map (3) to deal with $N \times N$ mosaics. Denoting by $\ast$ the set of all words based on the two alphabets (12 knot tiles and 16 4-bit sequences) define

$$E^{(N \times N)} : (B_{(4)} \cup M_{(8)})^{\ast} \rightarrow \Sigma_{(4)},$$

where accepted codewords (of length $4N^2$ bits) are those associated with knot mosaics that can be arranged into the given square.

The topological (combinatorial) prescriptions that have to be taken into account traveling along the spiral encoding path of a given $N \times N$ knot mosaic are summarized as follows.

i) The knot is a continuous closed path (over and under–crossing points are artifacts due to the fact that we deal with a planar projection) and thus each mosaics added to the previous one must match correctly the knot strand. In other words, any arrangement of mosaics giving rise at some point to disconnected arcs or lines is forbidden.

ii) Most critical situations may occur at the crossings points (knot mosaics $m_1$ and $m_4$) because an accidental swap would change the topology of the knot. The encoding given in section 2, Table 1 is such that $d(m_1, m_4) = 4$ and thus this type of error is actually highly suppressed.
iii) Away from crossing points, but still in the core region of the mosaic, another kind of mismatching can occur whenever a double–arc tile \((m_6, m_7)\) is turned accidentally into a single-arc one \((m_2, m_3, m_5, m_8)\): then either a discontinuity (see prescription i)) or an improper closure of the knot string is created. In the latter case the resulting configuration would correspond to a link, namely a multi-component knot, contrary to the basic assumption which must hold true in the knot–based cryptographic protocol.

iv) Finally, going through the most external layer of the mosaic, accidental swaps could occur between single–arc tiles, between the latter and blank tiles and between blank tiles. In all these situations disconnected patterns must be ruled out again on the basis of i) (recall also that the minimum distances among the \(m\)’s and among the \(b\)’s is 2). The encoding prescription for the blank tiles (Table 2) together with the observation that \(m\)-tiles are arranged in pairs in the boundary layer \((m_3-m_2, m_2-m_8, m_8-m_5, m_5-m_3)\) ensure that distances between anyone of these \(m\) and the two admissible contiguous blank tiles are equal to 2.

4 CONCLUSIONS AND OUTLOOK

It has been shown that the mosaic encoding of the knot–based cryptographic protocol proposed in (Marzuoli & Palumbo, 2011) is efficient (with respect to increasing complexity of knot diagrams) and robust against random 1-bit flips of the encoded string of \(4N^2\) bits. Such procedure can be applied to both prime knots in standard Knot Tables and composite knots used in the protocol. The rules described in the previous section, in particular the spiral pathway in the \(N\times N\) mosaic and the connection property i), enforce fault–tolerance since are related to topological, global features: most probable errors would turn connected knot diagrams into collections of tiles still joined together but not corresponding to a connected knotted curve.

The system of knot mosaics, introduced in (Lomonaco & Kauffman, 2008) and used here for specific encoding purposes, might be employed in at least two more contexts. The first one is related to the fact that they are ‘prototiles’, so that it would be interesting to ask whether they constitute a Wang set, namely if they are able to generate aperiodic tiling of the plane (Grunbaum & Shepard, 1987). It does not seem so, but work is in progress to improve (or dis-prove) this conjecture, which could have interesting consequences also for open problems in topological knot theory. Recall that Wang tiles have not only a foundational interest in logic and in information theory, but are used also in applied computer science, see e.g. (Cohen et al., 2003) in connection with image and texture generation.

A second remark resorts to the observation that the mosaic system is actually an alphabet, as the mapping \(\mathcal{E}\) suggests. Large knot mosaics, with varying \(N\) and embedded knotted curves, might be used, in turn, to modeling and encoding large sets of data. Once shown that the procedure described in the previous section provides a sort of ‘topologically–protected’ encoding, it can be argued that such further extensions could define a new efficient method for data transmission.

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APPENDIX

As is well known most RSA–type protocols are based on the computational complexity of factorization of prime numbers, because the generators are two large prime numbers (p and q) and the public key is the product of them (N = pq). Once given N, decrypting the message needs the knowledge of its prime factors, and this is of course a computationally hard problem. Note however that public key algorithms are very costly in terms of computational resources. The time it takes the message to be encoded and decoded is relatively high and this is actually the main drawback of (any) asymmetric decoding. This problem can be overcome or even solved by using a symmetric key together with the asymmetric one, as done in (Marzuoli & Palumbo, 2011) for the knot–based cryptosystem. The following brief review refers to the original formulation based on the DT coding. The translation in terms of the (most reliable and fault–tolerant) mosaic encoding proposed in this paper would be straightforward.

A sender A must prepare a secret message for the receiver B and they share the same finite list of prime knots K’s. The message M will be built by resorting to a finite sequence of (not necessarily prime) knots L1,….LN according to the following steps.

I) Through a standard RSA protocol, B sends to A an ordered sublist of N prime knots taken from current available Knot Tables, K1,....KN, together with mutation instructions to be applied to each Ki. (The operation called ‘mutation’ amounts to remove a portion of the knot diagram with four external legs, replace it with the configuration obtained by rotating the original one, and then gluing back the tips of the strands.)

A second list K’1,…,K’N is generated by picking up definite mutations of the original sequence.

II) A takes K’1,…,K’N and performs a series of ordered connected sums

\[ L_1 \# K'_1, L_2 \# K'_2, \ldots, L_N \# K'_N \]

with the knots L1,....,LN associated with the message to be sent.

These composite knots are now translated (efficiently) into Dowker–Thistlethwaite sequences and sent to B. Obviously at this stage everyone has access to these strings of relative integers.

III) B receives the (string of) composite knots. Since he knows the DT sub–codes for the prime knots of the shared list, he can decompose the composite knots, thus obtaining the DT code for every Li. Then the planar diagrams of L1,....,LN can be uniquely recovered.

Basically we are using in the protocol both a public key (step I) and a private key (step II). In fact the message is encrypted (by A) and decrypted (by B) using the same key, the sequence of prime knots that they share (secretly) thanks to step I).