BOUNDARY STABILIZATION AND OBSERVATION OF A MULTI-DIMENSIONAL UNSTABLE HEAT EQUATION *

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Abstract. In this paper, we consider the boundary stabilization and observation of the multi-dimensional unstable heat equation. Since we consider the heat equation in a general domain, the usual partial differential equation backstepping method is hard to apply to the considered problems. The unstable dynamics of the heat equation are treated by combining the finite-dimensional spectral truncation method and the dynamics compensation method. By introducing additional finite-dimensional actuator/sensor dynamics, the unbounded stabilization/observation turns to be a bounded one. As a result, the controller/observer design becomes more easier. Both the full state feedback stabilizer and the state observer are designed. The exponential stability of the closed-loop and the well-posedness of the observer are obtained.

Key words. Dynamic compensation, unstable heat equation, observer, stabilization.

AMS subject classifications. 93B07, 37N35, 34C28, 35L10.

1. Introduction. When there is at least one point spectrum in the right-half complex plane, the system is referred to as an unstable system. Owing to the pole assignment theorem, stabilization of the finite-dimensional unstable system is almost trivial. However, the problem may become pretty difficult for the infinite-dimensional unstable system. When there are only finite point spectrums in the right-half complex plane, the system can be stabilized by the finite spectral truncation technique [16], [17]. Since heat equation with the boundary or interior source term is usually an unstable system which has finite unstable point spectrums, the finite spectral truncation technique can be used to stabilize the unstable heat system. See, for instance, [3], [12] and [13], to name just a few.

The partial differential equation (PDE) backstepping method is another way to stabilize the infinite-dimensional unstable system. It has been used to stabilize the unstable heat equations in [14], [18] and [20]. The PDE backstepping method can also cope with other infinite-dimensional systems such as the first order hyperbolic equation system [10], the unstable wave equation system [9] and even for ODE-PDE cascade system [21]. Although the PDE backstepping is powerful, it is still hard to apply to the general multi-dimensional infinite-dimensional system. Very recently, [7] has considered the stabilization and observation of the unstable heat equation in a general multi-dimensional region. By using the dynamic compensation method [4], [5] and the finite dimensional spectral truncation method [3], both the full state feedback and the state observer are proposed for the unstable multi-dimensional heat equation under the assumption that the unstable point spectrums are algebraic simple. In this paper, we shall extend the results in [7] to the more general case.

Let \( \Omega \subset \mathbb{R}^n (n \geq 2) \) be a bounded domain with \( C^2 \)-boundary \( \Gamma \). Suppose that \( \Gamma \) consists of two parts: \( \Gamma_0 \) and \( \Gamma_1 \), \( \Gamma_0 \cup \Gamma_1 = \Gamma \), with \( \Gamma_0 \) is a non-empty connected...
open set in $\Gamma$. Let $\nu$ be the unit outward normal vector of $\Gamma_1$ and let $\Delta$ be the usual Laplacian. We consider the following system

\begin{align}
\begin{cases}
w_t(x,t) &= \Delta w(x,t) + \mu w(x,t), \quad (x,t) \in \Omega \times (0, +\infty), \\
w(x,t) &= 0, \quad (x,t) \in \Gamma_0 \times (0, +\infty), \\
\frac{\partial w(x,t)}{\partial \nu} &= u(x,t), \quad (x,t) \in \Gamma_1 \times (0, +\infty), \\
y(x,t) &= w(x,t), \quad (x,t) \in \Gamma_1 \times (0, +\infty),
\end{cases}
\end{align}

where $\mu > 0$ is a positive constant, $u$ is the control and $y$ is the output. Owing to the source term $\mu w(x,t)$, system (1.1) is unstable provided $\mu$ is large enough. The main goal of this paper is to design an output feedback to stabilize system (1.1) exponentially. By the separation principle of the linear systems, the output feedback will be available once we address the following two problems: (i) stabilize system (1.1) by a full state feedback; (ii) design a state observer to estimate the state online. We will consider these two problems separately.

We consider system (1.1) in state space $L^2(\Omega)$. Define

\begin{equation}
\begin{cases}
Af = \Delta f, \quad \forall f \in D(A) = \left\{ f \in H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega) \mid \frac{\partial f}{\partial \nu}(x) = 0, x \in \Gamma_1 \right\}, \\
H^1_{\Gamma_0}(\Omega) = \left\{ f \in H^1(\Omega) \mid f(x) = 0, x \in \Gamma_0 \right\}.
\end{cases}
\end{equation}

Then $A$ generates an exponentially stable analytic semigroup on $L^2(\Omega)$. It is well known (e.g. [11, p.668]) that $D((-A)^{1/2}) = H^1_{\Gamma_0}(\Omega)$ and $(-A)^{1/2}$ is a canonical isomorphism from $H^1_{\Gamma_0}(\Omega)$ onto $L^2(\Omega)$. Let $[D((-A)^{1/2})]' = H^{-1}_{\Gamma_0}(\Omega)$ be the dual space of $H^1_{\Gamma_0}(\Omega)$ with the pivot space $L^2(\Omega)$. We obtain the following Gelfand triple inclusions:

\begin{equation}
D((-A)^{1/2}) \hookrightarrow L^2(\Omega) = [L^2(\Omega)]' \hookrightarrow [D((-A)^{1/2})]'.
\end{equation}

An extension $\tilde{A} \in \mathcal{L}(H^1_{\Gamma_0}(\Omega), H^{-1}_{\Gamma_0}(\Omega))$ of $A$ is defined by

\begin{equation}
(\tilde{A}x, z)_{H^{-1}_{\Gamma_0}(\Omega), H^1_{\Gamma_0}(\Omega)} = -(\langle -A \rangle^{1/2}x, (-A)^{1/2}z)_{L^2(\Omega)}, \quad \forall x, z \in H^1_{\Gamma_0}(\Omega).
\end{equation}

By a simple computation, the eigenpairs $\{(\phi_j(\cdot), \lambda_j)\}_{j=1}^{\infty}$ of $A$ satisfy

\begin{equation}
\begin{cases}
\Delta \phi_j = \lambda_j \phi_j, \quad x \in \Omega, \\
\phi_j(x) = 0, x \in \Gamma_0; \quad \frac{\partial \phi_j(x)}{\partial \nu} = 0, x \in \Gamma_1, \\
\end{cases}
\end{equation}

Since the operator $A$ defined by (1.2) is self-adjoint and negative with compact resolvents, it follows from [19, p.76, Proposition 3.2.12] that the eigenvalues $\{\lambda_j\}_{j=1}^{\infty}$ are real and we can repeat each eigenvalue according to its finite multiplicity to get

\begin{equation}
0 > \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N > \lambda_{N+1} \ldots \rightarrow -\infty.
\end{equation}

Without loss of the generality, we assume that

**Assumption 1.1.** Let the operator $A$ be given by (1.2). Suppose that the eigenpairs $\{(\phi_j(\cdot), \lambda_j)\}_{j=1}^{\infty}$ of $A$ satisfy (1.6) and $\|\phi_j\|_{L^2(\Omega)} = 1$. Suppose that there exists a constant $N > 0$ such that

\begin{equation}
\lambda_N + \mu \geq 0 \quad \text{and} \quad \lambda_{N+1} + \mu < 0.
\end{equation}
Define the Neumann map $\Upsilon \in L(L^2(\Gamma_1), H^{3/2}(\Omega))$ \cite[p.668]{11} by $\Upsilon u = \psi$ if and only if

\begin{equation}
\begin{cases}
\Delta \psi = 0 & \text{in } \Omega, \\
\psi|_{\Gamma_0} = 0; & \frac{\partial \psi}{\partial \nu}|_{\Gamma_1} = u.
\end{cases}
\end{equation}

Using the Neumann map, one can write (1.1) abstractly in $H^{-1}_{\Gamma_0}(\Omega)$ as

\begin{equation}
\begin{cases}
w_t(\cdot, t) = (\tilde{A} + \mu)w(\cdot, t) + Bu(\cdot, t), & t > 0, \\
y(\cdot, t) = B^*w(\cdot, t), & t \geq 0,
\end{cases}
\end{equation}

where $\tilde{A}$ is the extension of $A$ given by (1.4), $B \in L(L^2(\Gamma_1), H^{-1}_{\Gamma_0}(\Omega))$ is defined by

\begin{equation}
Bu = -\tilde{A}\Upsilon u, \quad \forall \ u \in L^2(\Gamma_1),
\end{equation}

and $B^*$ is the adjoint of $B$, given by

\begin{equation}
B^*f = f|_{\Gamma_1}, \quad \forall \ f \in H^1_{\Gamma_0}(\Omega).
\end{equation}

The rest of the paper is organized as follows: In Section 2, we give some preliminaries for the full state feedback design. Section 3 is devoted to the full state feedback design. The exponential stability of the closed-loop system is also demonstrated in Section 3. Section 4 gives some preliminaries for the observer design which will be considered in Section 5. Section 6 concludes our paper and presents an outlook for future work. Some results that are less relevant to stabilizer and observer design are arranged in the Appendix.

Throughout of this paper, the space $\mathcal{L}(X_1, X_2)$ represents all the bounded linear operators from the space $X_1$ to $X_2$. The space of bounded linear operators from $X$ to itself is denoted by $\mathcal{L}(X)$. The spectrum, resolvent set and the domain of the operator $A$ are denoted by $\sigma(A)$, $\rho(A)$ and $D(A)$, respectively. The point spectrum of $A$ is represented by $\sigma_p(A)$. The set of positive integers is denoted by $\mathbb{Z}_+$. We define inner product in $\mathbb{R}^N$ by

\begin{equation}
(a, b)_{\mathbb{R}^N} = \sum_{i=1}^{N} a_i b_i, \quad \forall \ a = (a_1, a_2, \cdots, a_N)^T, \ b = (b_1, b_2, \cdots, b_N)^T \in \mathbb{R}^N.
\end{equation}

### 2. Preliminaries for full state feedback design.
In this section, we will give some preliminaries that are very important to the state feedback design for system (1.1). For any positive integer $i$, we can define, in terms of the function $\phi_i(x)$ of (1.5), the operator $P_{\phi_i} : \mathbb{R}\setminus\sigma(A) \to L^2(\Omega)$ by

\begin{equation}
P_{\phi_i}\theta = \zeta_{\phi_i}, \quad \forall \ \theta \in \mathbb{R}\setminus\sigma(A), \quad i = 1, 2, \cdots, N,
\end{equation}

where $\zeta_{\phi_i}$ is the solution of following elliptic equation:

\begin{equation}
\begin{cases}
\Delta \zeta_{\phi_i}(x) = \theta \zeta_{\phi_i}(x), & x \in \Omega, \\
\zeta_{\phi_i}(x) = 0, & x \in \Gamma_0; \quad \frac{\partial \zeta_{\phi_i}(x)}{\partial \nu} = \phi_i(x), \ x \in \Gamma_1.
\end{cases}
\end{equation}
Lemma 2.1. Suppose that $\theta \in \mathbb{R}$ satisfies
\begin{equation}
\theta \neq \lambda_j, \quad j \in \mathbb{Z}_+.
\end{equation}
Then the function $P_{\phi_i}$ defined by (2.1) satisfies
\begin{equation}
\langle P_{\phi_i}, \phi_j \rangle_{L^2(\Omega)} = \frac{1}{\theta - \lambda_j} \langle \phi_i, \phi_j \rangle_{L^2(\Gamma_1)}, \quad i, j \in \mathbb{Z}_+.
\end{equation}

Proof. A straightforward computation shows that
\begin{equation}
\theta \int_{\Omega} \zeta \phi_i(x) dx = \int_{\Omega} \Delta \zeta \phi_i(x) dx = \int_{\Gamma_1} \phi_i(x) \phi_j(x) dx + \lambda_j \int_{\Omega} \zeta \phi_j(x) dx,
\end{equation}
which, together with (2.1), yields (2.4) easily.

Inspired by [7], the controller design is closely related to the following system:
\begin{equation}
\begin{cases}
z_t(x,t) = \Delta z(x,t) + \mu z(x,t) + \sum_{i=1}^{N} (P_{\phi_i \theta})(x)u_i(t), & (x,t) \in \Omega \times (0, +\infty) \\
z(x,t) = 0, & (x,t) \in \Gamma_0 \times (0, +\infty), \\
\partial z(x,t) / \partial \nu = 0, & (x,t) \in \Gamma_1 \times (0, +\infty),
\end{cases}
\end{equation}
where $P_{\phi_i \theta}$ is defined by (2.1), $\mu > 0$, $N$ is a positive integer satisfies Assumption 1.1 and $u_1, u_2, \cdots, u_N$ are new controls. Since the sequence $\{\phi_j(\cdot)\}_{j=1}^{\infty}$ under the Assumption 1.1 forms an orthonormal basis for $L^2(\Omega)$, $P_{\phi_i \theta}$ and $z(\cdot, t)$ can be represented by
\begin{equation}
P_{\phi_i \theta} = \sum_{k=1}^{\infty} f_{ki} \phi_k, \quad f_{ki} = \int_{\Omega} (P_{\phi_i \theta})(x) \phi_k(x) dx
\end{equation}
and
\begin{equation}
z(\cdot, t) = \sum_{k=1}^{\infty} z_k(t) \phi_k(\cdot).
\end{equation}
In view of (2.6), the function $z_k(t)$ in (2.8) satisfies
\begin{equation}
\dot{z}_k(t) = \int_{\Omega} z_t(x,t) \phi_k(x) dx \\
= \int_{\Omega} \left[ \Delta z(x,t) + \mu z(x,t) + \sum_{i=1}^{N} (P_{\phi_i \theta})(x)u_i(t) \right] \phi_k(x) dx \\
= (\lambda_k + \mu) z_k(t) + \sum_{i=1}^{N} f_{ki} u_i(t), \quad k = 1, 2, \cdots.
\end{equation}
Since $\lambda_k + \mu < 0$ provided $k > N$, $z_k(t)$ is stable for all $k > N$. Consequently, it is sufficient to consider $z_k(t)$ for $k \leq N$, which satisfy the following finite-dimensional
system:

\[
(2.10) \quad \dot{Z}_N(t) = \Lambda_N Z_N(t) + F_N u(t), \quad Z_N(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_N(t) \end{pmatrix}, \quad u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_N(t) \end{pmatrix},
\]

where \( \Lambda_N \) and \( F_N \) are defined by

\[
(2.11) \quad \left\{ \begin{array}{l}
\Lambda_N = \text{diag}(\lambda_1 + \mu, \ldots, \lambda_N + \mu), \\
F_N = \begin{pmatrix}
\begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1N} \\
f_{21} & f_{22} & \cdots & f_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
f_{N1} & f_{N2} & \cdots & f_{NN}
\end{pmatrix}_{N \times N} \\
\end{array}, \\
f_{kl} = \int_{\Omega} (P_{\phi_l}(x)\phi_k(x))dx, \quad i, k = 1, 2, \ldots, N.
\right.
\]

**Lemma 2.2.** In addition to Assumption 1.1, assume that \( \theta \in \mathbb{R} \) satisfies (2.3). Then, there exists a matrix \( L_N = (l_{ij})_{N \times N} \) such that \( \Lambda_N + F_N L_N \) is Hurwitz, where \( \Lambda_N \) and \( F_N \) are defined by (2.11). Moreover, the operator \( A + \mu + \sum_{i=1}^{N} (P_{\phi_i}\theta)K_i \) generates an exponentially stable \( C_0 \)-semigroup on \( L^2(\Omega) \), where \( P_{\phi_i}\theta \) is given by (2.1) and \( K_i \) is given by

\[
(2.12) \quad K_i : g \rightarrow \int_{\Omega} g(x) \sum_{k=1}^{N} l_{ik} \phi_k(x)dx, \quad \forall g \in L^2(\Omega), \quad \forall i = 1, 2, \ldots, N.
\]

**Proof.** Since the sequence \( \{\phi_j(\cdot)\}_{j=1}^{\infty} \) under the Assumption 1.1 are linearly independent on \( L^2(\Omega) \), it follows from Lemmas 7.1 and 7.2 in Appendix and (2.4) that the pair \( (\Lambda_N, F_N) \) is controllable. Hence, there exists a matrix \( L_N \) such that \( \Lambda_N + F_N L_N \) is Hurwitz. Since \( A + \mu \) generates an analytic semigroup \( e^{(A+\mu)t} \) on \( L^2(\Omega) \) and \( \sum_{i=1}^{N} P_{\phi_i}\theta K_i \in \mathcal{L}(L^2(\Omega)) \), it follows from [15, Corollary 2.2, p.81] that \( A + \mu + \sum_{i=1}^{N} P_{\phi_i}\theta K_i \) generates an analytic semigroup on \( L^2(\Omega) \) as well. By a straightforward computation, \( A + \mu + \sum_{i=1}^{N} P_{\phi_i}\theta K_i \) is the inverse of a compact operator. Thanks to [15, Theorem 4.3, p.118], the proof will be accomplished if we can show that the point spectrum of \( A + \mu + \sum_{i=1}^{N} P_{\phi_i}\theta K_i \) satisfies

\[
(2.13) \quad \sigma_p(A + \mu + \sum_{i=1}^{N} P_{\phi_i}\theta K_i) \subset \{ s \mid \text{Re } s < 0 \}.
\]

Actually, for all \( \lambda \in \sigma_p(A + \mu + \sum_{i=1}^{N} P_{\phi_i}\theta K_i) \), we consider following characteristic equation \( (A + \mu + \sum_{i=1}^{N} P_{\phi_i}\theta K_i)g = \lambda g \) with \( g \neq 0 \). Since the sequence \( \{\phi_j(\cdot)\}_{j=1}^{\infty} \) forms an orthonormal basis for \( L^2(\Omega) \), we can suppose that

\[
(2.14) \quad 0 \neq g = \sum_{k=1}^{\infty} g_k \phi_k, \quad g_k = \langle g, \phi_k \rangle_{L^2(\Omega)}, \quad k = 1, 2, \ldots.
\]
As a result, the characteristic equation becomes

\[(2.15) \quad \sum_{k=1}^{\infty} \lambda g_k \phi_k = \sum_{k=1}^{\infty} g_k (A + \mu) \phi_k + \sum_{i=1}^N \sum_{k=1}^{\infty} (P_{\psi_i} \theta) g_k K_i \phi_k \]

\[
= \sum_{k=1}^{\infty} g_k (\lambda_k + \mu) \phi_k + \sum_{i,k=1}^N (P_{\phi_i} \theta) l_{ik} g_k.
\]

When \((g_1, g_2, \cdots, g_N) \neq 0\), we take the inner product with \(\phi_j, j = 1, 2, \cdots, N\) on equation (2.15) to obtain

\[(2.16) \quad \lambda g_j = g_j (\lambda_j + \mu) + \sum_{i,k=1}^N f_{ji} l_{ik} g_k,
\]

which, together with (2.11), leads to

\[(2.17) \quad (\lambda - \Lambda N - F_N L_N)(g_1, g_2, \cdots, g_N)^T = 0.
\]

Since \((g_1, g_2, \cdots, g_N)^T \neq 0\), (2.17) implies that

\[(2.18) \quad \text{Det}(\lambda - \Lambda N - F_N L_N) = 0.
\]

Hence, \(\lambda \in \sigma(\Lambda N + F_N L_N) \subset \{ s \mid \text{Re} s < 0 \}\) due to that \(\Lambda N + F_N L_N\) is Hurwitz.

When \((g_1, g_2, \cdots, g_N) = 0\), there exists a \(j_0 > N\) such that \(\int_{\Omega} g(x) \phi_{j_0}(x) dx \neq 0\) due to \(g \neq 0\), and at the same time, (2.15) is reduced to

\[(2.19) \quad \sum_{k=1}^{\infty} \lambda g_k \phi_k = \sum_{k=1}^{\infty} g_k (\lambda_k + \mu) \phi_k.
\]

Take the inner product with \(\phi_{j_0}\) on equation (2.19) to get

\[(2.20) \quad (\lambda_{j_0} + \mu) g_{j_0} = \lambda g_{j_0},
\]

which, together with (1.7), implies that \(\lambda = \lambda_{j_0} + \mu < 0\). So \(\text{Re} \lambda < 0\). Therefore, we can get (2.13).

3. State feedback. This section is devoted to the full state feedback design for system (1.1). To this end, we first define, in terms of the eigenfunctions \(\phi_1, \phi_2, \cdots, \phi_N\) that are given by (1.5), the operator \(B_v \in L(R^N, L^2(\Gamma_1))\) by following equation

\[(3.1) \quad B_v c = \sum_{j=1}^N c_j \phi_j(x), \quad x \in \Gamma_1, \quad \forall c = (c_1, c_2, \cdots, c_N)^T \in R^N.
\]

Inspired by [4], we consider the following dynamics feedback

\[(3.2) \quad \begin{cases} u(x, t) = v(x, t), & x \in \Gamma_1, \\
v_1(\cdot, t) = -\alpha I v(\cdot, t) + B_v u_v(t) & \text{in } L^2(\Gamma_1), \end{cases}
\]

where \(\alpha > 0\) is a tuning parameter, \(I\) is the identity operator in \(L^2(\Gamma_1)\) and

\[(3.3) \quad u_v(t) = (u_1(t), u_2(t), \cdots, u_N(t))^T \in R^N,
\]
is a new control to be designed later. Under the controller (3.2), the control plant (1.1) becomes

\[
\begin{align*}
    w_i(\cdot, t) &= (\hat{A} + \mu)w(\cdot, t) + Bv(\cdot, t) \quad \text{in} \quad H^{-1}_0(\Omega), \\
    v_i(\cdot, t) &= -\alpha v(\cdot, t) + B_v u_v(t) \quad \text{in} \quad L^2(\Gamma_1),
\end{align*}
\]

(3.4)

where \( \hat{A} \) is the extension of \( A \) given by (1.4) and \( B \in L(L^2(\Gamma_1), H^{-1}(\Omega)) \) is given by (1.10).

Since (3.4) is a cascade system, the “v-system” can be regarded as the actuator dynamics of the control plant “w-system”. Therefore, we can use the actuator dynamics compensation method proposed in [4] to stabilize the system (3.4). Actually, the controller (3.3) can be designed as

\[
u_i(t) = -[K_i w(\cdot, t) + K_i S v(\cdot, t)], \quad i = 1, 2, \cdots, N,
\]

(3.5)

where \( K_i \) are given by (2.12), \( S \in L(L^2(\Gamma_1), L^2(\Omega)) \) solves the Sylvester equation

\[
(\hat{A} + \mu)S + \alpha S = B.
\]

(3.6)

Combining (2.12) and (3.11), the controller (3.5) turns to be

\[
u_i(t) = -\int_\Omega [w(x, t) - \varphi_v(x, t)] \left[ \sum_{k=1}^N l_{ik} \phi_k(x) \right] dx,
\]

(3.7)

where \( \varphi_v \) satisfies following equation

\[
\begin{align*}
    \Delta \varphi_v(\cdot, t) &= (-\alpha - \mu)\varphi_v(\cdot, t) \quad \text{in} \quad \Omega, \\
    \varphi_v(x, t) &= 0, \quad x \in \Gamma_0, \\
    \frac{\partial \varphi_v(x, t)}{\partial \nu} &= v(x, t), \quad x \in \Gamma_1.
\end{align*}
\]

(3.8)

Combining (1.1), (3.1), (3.2), (3.3) and (3.7), we obtain the closed-loop system

\[
\begin{align*}
    w_i(x, t) &= \Delta w(x, t) + \mu w(x, t), \quad (x, t) \in \Omega \times (0, +\infty), \\
    w(x, t) &= 0, \quad (x, t) \in \Gamma_0 \times (0, +\infty), \\
    \frac{\partial w(x, t)}{\partial \nu} &= v(x, t), \quad (x, t) \in \Gamma_1 \times (0, +\infty), \\
    v_i(x, t) &= -\alpha v(x, t) - \sum_{j=1}^N \phi_j(x) \int_\Omega [w(x, t) - \varphi_v(x, t)] \left[ \sum_{i=1}^N l_{ij} \phi_i(x) \right] dx, \quad (x, t) \in \Gamma_1 \times (0, +\infty), \\
    \Delta \varphi_v(\cdot, t) &= (-\alpha - \mu)\varphi_v(\cdot, t) \quad \text{in} \quad \Omega, \\
    \varphi_v(x, t) &= 0, \quad x \in \Gamma_0, \\
    \frac{\partial \varphi_v(x, t)}{\partial \nu} &= v(x, t), \quad x \in \Gamma_1.
\end{align*}
\]

(3.9)

**Lemma 3.1.** Let the operators \( \hat{A} \) and \( B \) be given by (1.4) and (1.10), respectively. Suppose that \( \alpha \) satisfies

\[
(\alpha + \mu) \in \rho(-A).
\]

(3.10)
Then the solution of Sylvester equation (3.6) satisfies
\begin{equation}
Sg = -\varphi_g \in L^2(\Omega), \quad \forall g \in L^2(\Gamma_1),
\end{equation}
where \( \varphi_g \) satisfies following equation
\begin{equation}
\begin{cases}
\Delta \varphi_g(x) = (-\alpha - \mu) \varphi_g(x), & x \in \Omega, \\
\varphi_g(x) = 0, & x \in \Gamma_0; \\
\frac{\partial \varphi_g(x)}{\partial \nu} = g(x), & x \in \Gamma_1.
\end{cases}
\end{equation}
Moreover, for any \( c = (c_1, c_2, \ldots, c_N)^\top \in \mathbb{R}^N \)
\begin{equation}
SB_vc = -\sum_{i=1}^{N} c_i P_{\varphi_i} \theta, \quad \theta = -\alpha - \mu,
\end{equation}
where \( B_v \) is given by (3.1) and \( P_{\varphi_i} \theta \) are given by (2.1).

\textbf{Proof.} By (3.6) and (3.10), we get
\begin{equation}
S = (\alpha + \mu + \hat{A})^{-1} B.
\end{equation}
It follows from (1.8), (1.10) and (3.12) that
\begin{equation}
(\alpha + \mu + \hat{A}) \varphi_g = (\alpha + \mu + \hat{A}) \varphi_g - \hat{A}Yg + \hat{A}Yg
\end{equation}
\begin{equation}
= (\alpha + \mu) \varphi_g + \hat{A}(\varphi_g - Yg) + \hat{A}Yg = \hat{A}Yg = -Bg,
\end{equation}
which together with (3.14), leads to \( Sg = -\varphi_g \). Consequently, we combine (2.1) and (3.1) to get (3.13).

**Theorem 3.2.** In addition to Assumption 1.1, suppose that \( \alpha > 0 \) satisfies
\begin{equation}
\alpha + \mu + \lambda_j \neq 0, \quad j \in \mathbb{Z}_+.
\end{equation}
Then, there exists a matrix \( L_N = (l_{ij})_{N \times N} \) such that \( \Lambda_N + F_N L_N \) is Hurwitz, where \( \Lambda_N \) and \( F_N \) are defined by (2.11). Moreover, for any initial value \( (w(\cdot, 0), v(\cdot, 0))^\top \in L^2(\Omega) \times L^2(\Gamma_1) \), system (3.9) admits a unique solution \( (w, v)^\top \in C([0, \infty); L^2(\Omega) \times L^2(\Gamma_1)) \) that decays to zero exponentially in \( L^2(\Omega) \times L^2(\Gamma_1) \) as \( t \to +\infty \).

\textbf{Proof.} We combine the (3.3) and (3.5) to get
\begin{equation}
u(t) = -Kw(\cdot, t) - KSv(\cdot, t),
\end{equation}
where \( K \in L^2(\Omega, \mathbb{R}^N) \) is defined by
\begin{equation}
Kg = (K_1g, K_2g, \ldots, K_N g)^\top, \quad \forall g \in L^2(\Omega).
\end{equation}
In view of the operators given by (1.2), (1.10), (3.1), (3.11) and (3.18), the closed-loop system (3.9) can be written as the abstract form:
\begin{equation}
\frac{d}{dt}(w(\cdot, t), v(\cdot, t))^\top = \mathcal{A}(w(\cdot, t), v(\cdot, t))^\top,
\end{equation}
where the operator \( \mathcal{A} : D(\mathcal{A}) \subset L^2(\Omega) \times L^2(\Gamma_1) \to L^2(\Omega) \times L^2(\Gamma_1) \) is defined by
(3.20) \[ A = \begin{pmatrix} A + \mu & B \\ -B_v K & -B_v KS - \alpha I \end{pmatrix}, \quad D(A) = D(A) \times L^2(\Gamma_1). \]

It is sufficient to prove that the operator \( A \) generates an exponentially stable \( C_0 \)-semigroup on \( L^2(\Omega) \times L^2(\Gamma_1) \).

Inspired by [4], we introduce following transformation

(3.21) \[ S(f, g) = (f + Sg, g)^\top, \quad (f, g) \in L^2(\Omega) \times L^2(\Gamma_1), \]

where \( S \in \mathcal{L}(L^2(\Gamma_1), L^2(\Omega)) \) solves the Sylvester equation (3.6). By a simple computation, \( S \in L^2(\Omega) \times L^2(\Gamma_1) \) is invertible and its inverse is

(3.22) \[ S^{-1}(f, g) = (f - Sg, g)^\top, \quad (f, g) \in L^2(\Omega) \times L^2(\Gamma_1). \]

Moreover,

(3.23) \[ \mathcal{S}A\mathcal{S}^{-1} = \mathcal{A}_S, \quad D(\mathcal{A}_S) = \mathcal{S}D(A), \]

where \( \mathcal{A}_S \) satisfies

(3.24) \[ \mathcal{A}_S = \begin{pmatrix} A + \mu - SB_v K & 0 \\ B_v K & -\alpha \end{pmatrix}. \]

Here \( SB_v \) is given by (3.13) and \( K \) is given by (3.18). According to the Lemma 2.2, the operator \( A + \mu - SB_v K = A + \mu + \sum_{i=1}^N P_i \phi_i \theta K_i \) generates an exponentially stable \( C_0 \)-semigroup on \( L^2(\Omega) \) with \( \theta = -\alpha - \mu \). Owing to the block-triangle structure and [22, Lemma 5.1], the operator \( \mathcal{A}_S \) generates an exponentially stable \( C_0 \)-semigroup \( e^{\mathcal{A}_S t} \) on \( L^2(\Omega) \times L^2(\Gamma_1) \). Therefore, the operator \( A \) also generates an exponentially stable \( C_0 \)-semigroup on \( L^2(\Omega) \times L^2(\Gamma_1) \) due to the similarity (3.23).

**Remark 3.1.** We point out that Theorems 3.2 is better than the results in [1] and [2], where the additional assumption that the eigenfunctions \( \phi_j, \ j \leq N \) are independent on \( L^2(\Gamma_1) \) must be required.

### 4. Preliminaries for observer design.

This section is devoted to the preliminaries on the observer design that is closely related to the adjoint of the operator \( A \) given by (3.20). We first compute the adjoint operators of the \( A, B, B_v, K \) and \( S \).

Since the adjoint of \( B \) has been given by (1.11) and \( A^* = A \), we only need to compute the adjoint operators of \( B_v, K \) and \( S \).

By (3.1), the adjoint operator \( B_v^* \in \mathcal{L}(L^2(\Gamma_1), \mathbb{R}^N) \) of \( B_v \) satisfies

\[
\langle c, B_v^* g \rangle_{\mathbb{R}^N} = \langle B_vc, g \rangle_{L^2(\Gamma_1)} = \int_{\Gamma_1} \sum_{j=1}^N c_j \phi_j(x)g(x)dx
\]

(4.1)

\[
= \sum_{j=1}^N c_j \int_{\Gamma_1} \phi_j(x)g(x)dx
\]

for all \( c = (c_1, c_2, \cdots, c_N)^\top \in \mathbb{R}^N \) and \( g \in L^2(\Gamma_1) \). As a result,

\[
B_v^* g = \begin{pmatrix} \int_{\Gamma_1} \phi_1(x)g(x)dx \\ \vdots \\ \int_{\Gamma_1} \phi_N(x)g(x)dx \end{pmatrix}^\top, \quad \forall \ g \in L^2(\Gamma_1).
\]

(4.2)
Similarly, it follows from (3.18) that \( K^* \in \mathcal{L}(\mathbb{R}^N, L^2(\Omega)) \) satisfies

\[
\langle f, K^* c \rangle_{L^2(\Omega)} = \langle f, c \rangle_{\mathbb{R}^N} = \sum_{i=1}^{N} c_i \int_{\Omega} f(x) \sum_{j=1}^{N} l_{ij} \phi_j(x)dx
\]

(4.3)

\[
= \int_{\Omega} f(x) \sum_{i,j=1}^{N} c_i l_{ij} \phi_j(x)dx
\]

for all \( c = (c_1, c_2, \cdots, c_N)^\top \in \mathbb{R}^N \) and \( f \in L^2(\Omega) \). (4.3) implies that

\[
K^* c = \sum_{i,j=1}^{N} c_i l_{ij} \phi_j(x), \quad \forall \, c = (c_1, c_2, \cdots, c_N)^\top \in \mathbb{R}^N.
\]

(4.4)

To compute \( S^* \), we suppose that \( \xi_f(x) \) is the solution of the following elliptic equation

\[
\begin{aligned}
\Delta \xi_f(x) &= -(\alpha + \mu) \xi_f(x) + f(x) \quad x \in \Omega, \\
\xi_f(x) &= 0, \quad x \in \Gamma_0; \quad \frac{\partial \xi_f(x)}{\partial \nu} = 0, \quad x \in \Gamma_1,
\end{aligned}
\]

(4.5)

where \( \alpha \) and \( \mu \) satisfy (3.10). Owing to Fredholm alternative theorem, equation (4.5) admits a unique solution \( \xi_f \) for each inhomogeneous term \( f \in L^2(\Omega) \). So the function \( \xi_f(x) \) makes sense. In view of (3.11), for any \( g \in L^2(\Gamma_1) \) and \( f \in L^2(\Omega) \), the adjoint operator \( S^* \in \mathcal{L}(L^2(\Omega), L^2(\Gamma_1)) \) satisfies

\[
\langle g, S^* f \rangle_{L^2(\Gamma_1)} = \langle Sg, f \rangle_{L^2(\Omega)} = \int_{\Omega} -\varphi_g(x)f(x)dx
\]

\[
= \int_{\Omega} -\varphi_g(x)(\Delta \xi_f(x) + (\alpha + \mu) \xi_f(x))dx
\]

\[
= \int_{\Omega} (\Delta \varphi_g(x) \xi_f(x) - \varphi_g(x) \Delta \xi_f(x))dx
\]

\[
= \int_{\Gamma} \left( \frac{\partial \varphi_g(x)}{\partial \nu} \xi_f(x) - \frac{\partial \xi_f(x)}{\partial \nu} \varphi_g(x) \right)dx
\]

\[
= \int_{\Gamma_1} g(x)\xi_f(x)dx,
\]

which yields

\[
S^* f = \xi_f, \quad \forall \, f \in L^2(\Omega),
\]

(4.7)

where \( \xi_f \) satisfies (4.5).

With the operators \( B^*, A^*, \beta_B^*, K^* \) and \( S^* \) at hand, a simple computation shows that the operator \( A^* : D(A^*) \subset L^2(\Omega) \times L^2(\Gamma_1) \rightarrow L^2(\Omega) \times L^2(\Gamma_1) \), the adjoint of the operator \( A \) given by (3.20), is

\[
A^* = \begin{pmatrix}
A + \mu & -K^* B^*_c \\
B^* & -\alpha I - S^* K^* B^*_c
\end{pmatrix}, \quad D(A^*) = D(A) \times L^2(\Gamma_1).
\]

(4.8)

**Lemma 4.1.** In addition to Assumption 1.1, suppose that \( \alpha > 0 \) satisfies

\[
-\alpha - \mu \neq \lambda_j, \quad j \in \mathbb{Z}_+.
\]

(4.9)
Then, the operator $A^*$ given by (4.8) generates an exponentially stable $C_0$-semigroup on $L^2(\Omega) \times L^2(\Gamma_1)$, where $B^*$, $B^*_v$, $K^*$ and $S^*$ are given by (1.11), (4.2), (4.4) and (4.7), respectively.

Proof. Similarly to the proof in Theorem 3.2, we introduce the following transformation

\[ T(f, g)^\top = (f, g - S^* f)^\top, \quad \forall (f, g)^\top \in L^2(\Omega) \times L^2(\Gamma_1), \]

where $S^* \in \mathcal{L}(L^2(\Gamma_1), L^2(\Omega))$ is given by (4.7). By a simple computation, we can conclude that $T \in L^2(\Omega) \times L^2(\Gamma_1)$ is invertible and its inverse is

\[ T^{-1}(f, g)^\top = (f, g + S^* f)^\top, \quad \forall (f, g)^\top \in L^2(\Omega) \times L^2(\Gamma_1). \]

Furthermore, we obtain

\[ TA^*T^{-1} = A^*_v, \quad D(A^*_v) = TD(A^*), \]

where $A^*_v$ satisfies

\[ A^*_v = \begin{pmatrix} A + \mu - K^*B^*_vS^* & -K^*B^*_v \\ -S^*(A + \mu) - \alpha S^* + B^* & -\alpha \end{pmatrix}. \]

Owing to (3.6), we have $-S^*(A + \mu) - \alpha S^* + B^* = 0$ and hence

\[ A^*_v = \begin{pmatrix} A + \mu - K^*B^*_vS^* & -K^*B^*_v \\ 0 & -\alpha \end{pmatrix}. \]

Since $A + \mu$ generates an analytic semigroup $e^{(A+\mu)t}$ on $L^2(\Omega)$ and $K^*B^*_vS^*$ is bounded, it follows from [15, Corollary 2.2, p.81] that $A + \mu - K^*B^*_vS^*$ also generates an analytic semigroup on $L^2(\Omega)$. The point spectrum satisfies $\sigma_p(A + \mu - K^*B^*_vS^*) = \sigma_p(A + \mu - SB_vK) \subset \{ s \mid \text{Re } s < 0 \}$. Noting that $A + \mu - K^*B^*_vS^*$ is the inverse of a compact operator, it follows from [15, Theorem 4.3, p.118] that the operator $A + \mu - K^*B^*_vS^*$ generates an exponentially stable $C_0$-semigroup on $L^2(\Omega)$. Owing to the block-triangle structure and [22, Lemma 5.1], the operator $A^*_v$ generates an exponentially stable $C_0$-semigroup $e^{A^*_v t}$ on $L^2(\Omega) \times L^2(\Gamma_1)$. Therefore, the operator $A^*$ also generates an exponentially stable $C_0$-semigroup on $L^2(\Omega) \times L^2(\Gamma_1)$ due to the similarity (4.12).

5. Observer design. Inspired by the dynamic compensation method in [7], we add, in terms of $\phi_1, \phi_2, \cdots, \phi_N$ given by (1.5), the sensor dynamic to system (1.1):

\[
\begin{align*}
&w_t(x, t) - \Delta w(x, t) - \mu w(x, t) = 0, \quad (x, t) \in \Omega \times (0, +\infty), \\
&w(x, t) = 0, \quad (x, t) \in \Gamma_0 \times (0, +\infty), \\
&\frac{\partial w(x, t)}{\partial \nu} = u(x, t), \quad (x, t) \in \Gamma_1 \times (0, +\infty), \\
p_e(\cdot, t) = -\alpha p(\cdot, t) + B^*w(\cdot, t), \quad \text{in } \Gamma_1, \\
y_p(t) = (y_1(t), y_2(t), \cdots, y_N(t))^\top, \quad t \in (0, +\infty),
\end{align*}
\]

where $\alpha > 0$ is a tuning parameter, $p(\cdot, t)$ is an extended state, $B^*$ is given by (1.11) and

\[ y_i(t) = \int_{\Gamma_1} p(x, t)\phi_i(x)dx, \quad i = 1, 2, \cdots, N. \]
By (1.2), (1.10), (1.11) and (4.2), system (5.1) can be written abstractly as

$$
\begin{cases}
 w_t(\cdot, t) = (\hat{A} + \mu)w(\cdot, t) + Bu(\cdot, t), \\
p_t(\cdot, t) = -\alpha p(\cdot, t) + B^*w(\cdot, t), \\
y_p(t) = B^*_p p(\cdot, t).
\end{cases}
$$

(5.3)

Inspired by the method in [7], the observer of system (5.3) can be designed as

$$
\begin{cases}
 \hat{w}_t(x, t) = \Delta \hat{w}(x, t) + \mu \hat{w}(x, t) + K^*B^*_p[p(\cdot, t) - \hat{p}(\cdot, t)], \\
\hat{w}(x, t) = 0, \quad x \in \Gamma_0, \\
\frac{\partial \hat{w}(x, t)}{\partial \nu} = u(x, t), \quad x \in \Gamma_1, \\
\hat{p}_t(x, t) = -\alpha \hat{p}(x, t) + B^*\hat{w}(x, t) + S^*K^*B^*_p[p(\cdot, t) - \hat{p}(\cdot, t)] \quad \text{in } \Gamma_1,
\end{cases}
$$

(5.4)

where $B^*_p$, $K^*$ and $S^*$ are given by (4.2), (4.4) and (4.7), respectively. Combining (1.11), (4.2), (4.4), (4.5), (4.7) and (5.1), the abstract observer (5.4) can be written concretely

$$
\begin{cases}
 \hat{w}_t(x, t) = \Delta \hat{w}(x, t) + \mu \hat{w}(x, t) \\
+ \sum_{i,j=1}^N l_{ij} \phi_j(x) \int_{\Gamma_1} \phi_i(x)(p(x, t) - \hat{p}(x, t))dx, \quad x \in \Omega, \\
\hat{w}(x, t) = 0, \quad x \in \Gamma_0, \\
\frac{\partial \hat{w}(x, t)}{\partial \nu} = u(x, t), \quad x \in \Gamma_1, \\
\hat{p}_t(x, t) = -\alpha \hat{p}(x, t) + \hat{w}(x, t) \\
+ \sum_{i,j=1}^N l_{ij} \xi_{\phi_j}(x) \int_{\Gamma_1} \phi_i(x)(p(x, t) - \hat{p}(x, t))dx, \quad x \in \Gamma_1,
\end{cases}
$$

(5.5)

where $\xi_{\phi_j}(x)$, satisfies

$$
\begin{cases}
 \Delta \xi_{\phi_j}(x) = (-\alpha - \mu)\xi_{\phi_j}(x) + \phi_j(x), \quad x \in \Omega, \\
\xi_{\phi_j}(x) = 0, \quad x \in \Gamma_0; \quad \frac{\partial \xi_{\phi_j}(x)}{\partial \nu} = 0, \quad x \in \Gamma_1, \quad j = 1, 2, \cdots, N.
\end{cases}
$$

(5.6)

**Theorem 5.1.** In addition to Assumption 1.1, let $\alpha > 0$ satisfy

$$
-\alpha - \mu \neq \lambda_j, \quad j \in \mathbb{Z}_+.
$$

(5.7)

Then, for any initial value $(w(\cdot, 0), p(\cdot, 0), \hat{w}(\cdot, 0), \hat{p}(\cdot, 0))^T \in [L^2(\Omega) \times L^2(\Gamma_1)]^2$ and $u \in L^2_{loc}([0, \infty), L^2(\Gamma_1))$, the observer (5.5) of system (5.1) admits a unique solution $(\hat{w}, \hat{p})^T \in C([0, \infty), L^2(\Omega) \times L^2(\Gamma_1))$ such that

$$
\lim_{t \to \infty} e^{\omega t} \|w(\cdot, t) - \hat{w}(\cdot, t), p(\cdot, t) - \hat{p}(\cdot, t)\|_{L^2(\Omega) \times L^2(\Gamma_1)} = 0,
$$

(5.8)

where $\omega$ is a positive constant that is independent of $t$. 


it is well known that the control plant (5.1) admits a unique solution \((w, p)^T \in C([0, \infty), L^2(\Omega) \times L^2(\Gamma_1))\) such that \(y_j \in L^{2}_{\text{loc}}[0, \infty)\) for any \(j = 1, 2, \ldots, N\). Let

\[
\begin{align*}
\dot{w}(x, t) &= w(x, t) - \hat{w}(x, t), \quad (x, t) \in \Omega \times [0, \infty), \\
\dot{p}(s, t) &= p(s, t) - \hat{p}(s, t), \quad (s, t) \in \Gamma_1 \times [0, \infty).
\end{align*}
\]

(5.9)

Then the errors are governed by

\[
\begin{align*}
\dot{w}(x, t) &= \Delta \tilde{w}(x, t) + \mu \tilde{w}(x, t) - K^* B^*_v \tilde{p}(\cdot, t), \quad x \in \Omega, \\
\dot{w}(x, t) &= 0, \quad x \in \Gamma_0, \\
\frac{\partial \tilde{w}(x, t)}{\partial \nu} &= 0, \quad x \in \Gamma_1, \\
\tilde{p}(\cdot, t) &= -\alpha \tilde{p}(\cdot, t) + B^* \tilde{w}(\cdot, t) - S^* K^* B^*_v \tilde{p}(\cdot, t) \quad \text{in} \quad \Gamma_1.
\end{align*}
\]

(5.10)

In terms of the operator \(A^*\) given by (4.8), system (5.10) can be written as

\[
\frac{d}{dt}(\tilde{w}(\cdot, t), \tilde{p}(\cdot, t))^T = A^*(\tilde{w}(\cdot, t), \tilde{p}(\cdot, t))^T.
\]

(5.11)

By Lemma 4.1, the operator \(A^*\) generates an exponentially stable analytic semigroup \(e^{A^*t}\) on \(L^2(\Omega) \times L^2(\Gamma_1)\). Hence, the error system (5.10) with initial state \((\tilde{w}(0), \tilde{p}(0))^T \in L^2(\Omega) \times L^2(\Gamma_1)\) admits a unique solution \((\tilde{w}(\cdot, t), \tilde{p}(\cdot, t))^T \in C([0, \infty); L^2(\Omega) \times L^2(\Gamma_1))\) such that

\[
\lim_{t \to \infty} e^{\omega t} ||\tilde{w}(\cdot, t), \tilde{p}(\cdot, t)||_{L^2(\Omega) \times L^2(\Gamma_1)} = 0,
\]

(5.12)

where \(\omega\) is a positive constant that is independent of \(t\). Let \((\hat{w}(\cdot, t), \hat{p}(\cdot, t)) = (w(\cdot, t) - \tilde{w}(\cdot, t), p(\cdot, t) - \tilde{p}(\cdot, t))\), it shows that such a defined \((\hat{w}, \hat{p})^T \in C([0, \infty); L^2(\Omega) \times L^2(\Gamma_1))\) is a solution of system (5.5) or equivalently system (5.4). Moreover, (5.8) holds due to (5.9) and (5.12). Owing to the linearity of system (5.5), the solution is unique. \(\blacksquare\)

6. Conclusions. In this paper, we extend the results in [7] to the general multi-domain. By introducing the ODE actuator/sensor dynamics, the difficulties caused by instability can be solved by the newly developed dynamics compensation approach [4, [5]. Since both the full state feedback law and the state observer are designed, the observer based output feedback is actually proposed to stabilize exponentially the unstable multi-dimensional heat system. The dynamics compensation approach may also used to the other multi-dimensional PDEs. Our future work is the stabilization and observation of the multi-dimensional unstable wave equation.

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7. Appendix. Lemma 7.1. Let the operator $A$ be given by (1.2). For any $\lambda \in \sigma(A)$, suppose that $\psi_1, \psi_2, \cdots, \psi_{m_\lambda}$ are the eigenfunctions corresponding to eigenvalue $\lambda$ of $A$, where $m_\lambda$ is the geometric multiplicity of $\lambda$. If $\psi_1, \psi_2, \cdots, \psi_{m_\lambda}$ are linearly independent on $L^2(\Omega)$, then the following matrix is invertible

\[
\begin{pmatrix}
\langle \psi_1, \psi_1 \rangle_{L^2(\Gamma_1)} & \langle \psi_1, \psi_2 \rangle_{L^2(\Gamma_1)} & \cdots & \langle \psi_1, \psi_{m_\lambda} \rangle_{L^2(\Gamma_1)} \\
\langle \psi_2, \psi_1 \rangle_{L^2(\Gamma_1)} & \langle \psi_2, \psi_2 \rangle_{L^2(\Gamma_1)} & \cdots & \langle \psi_2, \psi_{m_\lambda} \rangle_{L^2(\Gamma_1)} \\
\vdots & \vdots & \ddots & \vdots \\
\langle \psi_{m_\lambda}, \psi_1 \rangle_{L^2(\Gamma_1)} & \langle \psi_{m_\lambda}, \psi_2 \rangle_{L^2(\Gamma_1)} & \cdots & \langle \psi_{m_\lambda}, \psi_{m_\lambda} \rangle_{L^2(\Gamma_1)}
\end{pmatrix}
\]

(7.1)

Proof. Since the operator $A$ is self-adjoint and negative with compact resolvents, $\sigma(A)$ consists of isolated eigenvalues of finite geometric multiplicity only. Hence, $\psi_k$ satisfies

\[
\begin{align*}
\Delta \psi_k(x) &= \lambda \psi_k(x), & x \in \Omega, \\
\psi_k(x) &= 0, & x \in \Gamma_0, \\
\frac{\partial \psi_k(x)}{\partial \nu} &= 0, & x \in \Gamma_1,
\end{align*}
\]

(7.2)
Noting that the matrix in (7.1) happens to be a Gram matrix of the sequence
\(\psi_1, \psi_2, \cdots, \psi_m\), the proof will be accomplished if we can prove that \(\psi_1, \psi_2, \cdots, \psi_m\)
are linearly independent on \(L^2(\Gamma_1)\) ([8, p.441, Theorem 7.2.10]).

Actually, suppose there exist \(\ell_1, \ell_2, \cdots, \ell_m \in \mathbb{R}\) such that
\[
(7.11) \quad \ell_1 \psi_1(x) + \ell_2 \psi_2(x) + \cdots + \ell_m \psi_m(x) = 0, \quad x \in \Gamma_1.
\]

If we let
\[
(7.4) \quad \beta_\ell(x) = \ell_1 \psi_1(x) + \ell_2 \psi_2(x) + \cdots + \ell_m \psi_m(x), \quad x \in \Omega,
\]
than it follows from (7.2) that
\[
(7.5) \quad \begin{cases}
\Delta \beta_\ell(x) = \lambda \beta_\ell(x), & x \in \Omega, \\
\beta_\ell(x) = 0, & x \in \Gamma, \\
\frac{\partial \beta_\ell(x)}{\partial \nu} = 0, & x \in \Gamma_1,
\end{cases}
\]
which admits a unique solution \(\beta_\ell(x) \equiv 0, x \in \Omega\) ([6, Theorem 2.4]). Since \(\psi_1, \psi_2, \cdots, \psi_m\)
are linearly independent on \(L^2(\Omega)\), we can conclude that \(\ell_1 = \ell_2 = \cdots = \ell_m = 0\). As
a result, \(\psi_1, \psi_2, \cdots, \psi_m\) are linearly independent on \(L^2(\Gamma_1)\).

**Lemma 7.2.** Let \(N\) be a positive integer, \(F_N \in \mathbb{R}^{N \times N}\) and
\[
\Lambda_N = \text{diag}(J_1, \cdots, J_p), \quad J_j = \text{diag}(\lambda_j, \cdots, \lambda_j) \in \mathbb{R}^{n_j \times n_j}, \quad j = 1, 2, \cdots, p,
\]
where \(n_1, n_2, \cdots, n_p\) are \(p\) positive integers such that \(n_1 + n_2 + \cdots + n_p = N\). Suppose
that
\[
(7.7) \quad \lambda_k \neq \lambda_j \quad \text{if and only if} \quad k \neq j, \quad k, j = 1, 2, \cdots, p
\]
and the matrix \(F_N\) can be written as
\[
(7.8) \quad F_N = \begin{pmatrix}
P_1 & \ast & \cdots & \ast \\
\ast & P_2 & \cdots & \ast \\
\vdots & \vdots & \ddots & \vdots \\
\ast & \cdots & \ast & P_p
\end{pmatrix}, \quad P_j \in \mathbb{R}^{n_j \times n_j}, \quad j = 1, 2, \cdots, p.
\]

Then, the pair \((\Lambda_N, F_N)\) is controllable provided the matrix determinant of \(P_j\) satisfies
\[
(7.9) \quad |P_j| \neq 0, \quad j = 1, 2, \cdots, p.
\]

**Proof.** Otherwise, we can conclude that \((\Lambda_N^T, F_N^T)\) is not observable. By the
Hautus test [19, p.15, Remark 1.5.2], there exist \(0 \neq V \in \mathbb{R}^N\) and \(k \in \{1, 2, \cdots, p\}\)
such that
\[
(7.10) \quad \Lambda_N V = \lambda_k V \quad \text{and} \quad F_N^T V = 0.
\]

Without loss of generality, we suppose that
\[
(7.11) \quad V = \begin{pmatrix}
V_1 \\
V_2 \\
\vdots \\
V_p
\end{pmatrix}, \quad V_j = (v_1^j, \cdots, v_{n_j}^j)^T, \quad j = 1, 2, \cdots, n_j.
\]
The first equation of (7.10) becomes

\[
\Lambda N V - \lambda k V = \begin{pmatrix}
(\lambda_1 - \lambda_k)V_1 \\
\vdots \\
(\lambda_k - \lambda_k)V_2 \\
\vdots \\
(\lambda_p - \lambda_k)V_p
\end{pmatrix} = 0.
\]

(7.12)

By the assumption (7.7), \(V_i = 0\) when \(i \neq k\). As a result, \(F_N^T V = 0\) implies that \(P_k^T V_k = 0\). Owing to the assumption (7.9), we can conclude that \(V_k = 0\) and hence \(V = 0\). This is contradict to the fact \(V \neq 0\). Therefore, the pair \((\Lambda_N, F_N)\) is controllable. \(\Box\)