A QUASI-LOCAL MASS FOR 2 SPHERES WITH NEGATIVE GAUSS CURVATURE

XIAO ZHANG

ABSTRACT. We extend our previous definition of quasi-local mass to 2-spheres whose Gauss curvature is negative and prove its positivity.

1. INTRODUCTION

In [7], Liu and Yau propose a definition of quasi-local mass for any smooth spacelike, topological 2-sphere with positive Gauss curvature. In particular, Liu and Yau [7, 8] are able to use Shi-Tam’s result [10] to prove its positivity. When the Gauss curvature of a 2-sphere is allowed to be negative, Wang and Yau [14] use Pogorelov’s result [9] to embed the 2-sphere into the hyperbolic space to generalize Liu-Yau’s definition, and prove its positivity by using a spinor argument of the positive mass theorem for asymptotically hyperbolic manifolds [15, 4, 16]. Wang-Yau’s result is improved in certain sense by Shi and Tam [11].

In attempting to resolve the decreasing monotonicity of Brown-York’s quasi-local mass [1, 2], the author [18] propose a new quasi-local mass and prove its positivity essentially for 2-spheres with positive Gauss curvature. It is still open when the 2-spheres have nonnegative Gauss curvature because the isometric embedding into \(\mathbb{R}^3\) in this case is only proved to be \(C^{1,\alpha}\) by Guan-Li and Hong-Zuily [5, 6]. However, we expect the \(C^{1,\alpha}\) regularity is sufficient for our propose, and we address it elsewhere.

In this note, we use the idea of Wang and Yau to extend the quasi-local mass in [18] to the case of 2-spheres with negative Gauss curvature. We embed such 2-spheres into the (spacelike) hyperbola in the Minkowski spacetime which has the nontrivial second fundamental form. By using the constant spinors in the Minkowski spacetime, we can solve a boundary problem for the Dirac-Witten equation. Then,

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the method in [18] gives rise to the quasi-local mass as well as its positivity. We would like to point out that our quasi-local mass is only one quantity, while the one defined by Wang and Yau is a 4-vectors. This difference is due to the hyperbola in our approach goes to null infinity in the Minkowski spacetime, and the one in Wang-Yau’s approach goes to spatial infinity in the Anti-de Sitter spacetime, which has trivial second fundamental form. The positive mass theorem near null infinity in asymptotically Minkowski spacetimes was established in [16, 17].

2. Dirac-Witten equations

In this section, we will review the existences of the Dirac-Witten equations proved in [18]. Let \((N, \tilde{g})\) be a 4-dimensional spacetime which satisfies the Einstein fields equations. Let \((M, g, p)\) be a smooth initial data set. Fix a point \(p \in M\) and an orthonormal basis \(\{e_\alpha\}\) of \(T_p N\) with \(e_0\) future-time-directed normal to \(M\) and \(e_i\) tangent to \(M\) \((1 \leq i \leq 3)\).

Denote by \(\mathbb{S}\) the (local) spinor bundle of \(N\). It exists globally over \(M\) and is called the hypersurface spinor bundle of \(M\). Let \(\tilde{\nabla}\) and \(\nabla\) be the Levi-Civita connections of \(\tilde{g}\) and \(g\) respectively, the same symbols are used to denote their lifts to the hypersurface spinor bundle. There exists a Hermitian inner product \((\cdot, \cdot)\) on \(\mathbb{S}\) along \(M\) which is compatible with the spin connection \(\tilde{\nabla}\). The Clifford multiplication of any vector \(\tilde{X}\) of \(N\) is symmetric with respect to this inner product. However, this inner product is not positive definite and there exists a positive definite Hermitian inner product defined by \(\langle \cdot, \cdot \rangle = (e_0, \cdot)\) on \(\mathbb{S}\) along \(M\).

Define the second fundamental form of the initial data set \(p_{ij} = \tilde{g}(\tilde{\nabla}_i e_0, e_j)\). Suppose that \(M\) has boundary \(\Sigma\) which has finitely many connected components \(\Sigma^1, \cdots, \Sigma^l\), each of which is a topological 2-sphere, endowed with its induced Riemannian and spin structures. Fix a point \(p \in \Sigma\) and an orthonormal basis \(\{e_i\}\) of \(T_p \Sigma\) with \(e_r = e_1\) outward normal to \(\Sigma\) and \(e_a\) tangent to \(\Sigma\) for \(2 \leq a \leq 3\). Let \(h_{ab} = \langle \nabla_a e_r, e_b \rangle\) be the second fundamental form of \(\Sigma\). Let \(H = tr(h)\) be its mean curvature. \(\Sigma\) is a future/past apparent horizon if

\[
H = tr(p|_\Sigma) \geq 0 \quad (2.1)
\]

holds on \(\Sigma\). When \(\Sigma\) has multi-components, we require that (2.1) holds (with the same sign) on each \(\Sigma_i\). The spin connection has the following relation

\[
\tilde{\nabla}_a = \nabla_a + \frac{1}{2} h_{ab} e_r \cdot e_b \cdot -\frac{1}{2} p_{aj} e_0 \cdot e_j . \quad (2.2)
\]

The Dirac-Witten operator along \(M\) is defined by \(\tilde{D} = e_i \cdot \tilde{\nabla}_i\). The Dirac operator of \(M\) but acting on \(\mathbb{S}\) is defined by \(D = e_i \cdot \nabla_i\). Denote
by $\nabla$ the lift of the Levi-Civita connection of $\Sigma$ to the spinor bundle $\mathbb{S}|\Sigma$. Let $D = e_a \cdot \nabla_a$ be the Dirac operator of $\Sigma$ but acting on $\mathbb{S}|\Sigma$. The Weitzenböck type formula gives rise to

$$\int_M |\tilde{\nabla} \phi|^2 + \langle \phi, T \phi \rangle - |\tilde{D} \phi|^2 = \int_\Sigma \langle \phi, (e_r \cdot D - \frac{H}{2} + \frac{tr(p|\Sigma)}{2}) e_0 \cdot e_r - \frac{P_{ar}}{2} e_0 \cdot e_a \cdot \phi \rangle. \quad (2.3)$$

where $T = \frac{1}{2}(T_{00} + T_{0i} e_0 \cdot e_i)$. If the spacetime satisfies the dominant energy condition, then $T$ is a nonnegative operator. Let

$$P_\pm = \frac{1}{2}(Id \pm e_0 \cdot e_r)$$

be the projective operators on $\mathbb{S}|\Sigma$. In [18], we prove the following existences:

(i) If $tr_g(p) \geq 0$ and $\Sigma$ is a past apparent horizon, then the following Dirac-Witten equation has a unique smooth solution $\phi \in \Gamma(\mathbb{S})$

$$\begin{cases} 
\tilde{D} \phi &= 0 \quad \text{in} \quad M \\
P_+ \phi &= P_+ \phi_0 \quad \text{on} \quad \Sigma_{i_0} \\
P_+ \phi &= 0 \quad \text{on} \quad \Sigma_i \ (i \neq i_0) 
\end{cases} \quad (2.4)$$

for any given $\phi_0 \in \Gamma(\mathbb{S}|\Sigma)$ and for fixed $i_0$;

(ii) If $tr_g(p) \leq 0$ and $\Sigma$ is a future apparent horizon, then the following Dirac-Witten equation has a unique smooth solution $\phi \in \Gamma(\mathbb{S})$

$$\begin{cases} 
\tilde{D} \phi &= 0 \quad \text{in} \quad M \\
P_- \phi &= P_- \phi_0 \quad \text{on} \quad \Sigma_{i_0} \\
P_- \phi &= 0 \quad \text{on} \quad \Sigma_i \ (i \neq i_0) 
\end{cases} \quad (2.5)$$

for any given $\phi_0 \in \Gamma(\mathbb{S}|\Sigma)$ and for fixed $i_0$.

3. Embedding 2-spheres

Let $(M, g, p)$ be a smooth initial data set where $M$ has boundary $\Sigma$ which has finitely many connected components $\Sigma_1, \cdots, \Sigma_l$, each of which is a topological 2-sphere. Suppose that some $\Sigma_{i_0}$ can be smoothly isometrically embedded into a smooth spacelike hypersurface $\tilde{M}^3$ in the Minkowski spacetime $\mathbb{R}^{3,1}$ and denote by $\tilde{\mathcal{R}}$ the isometric embedding. Let $\tilde{\Sigma}_{i_0}$ be the image of $\Sigma_{i_0}$ under the map $\tilde{\mathcal{R}}$. Let $\tilde{e}_r$ be the unit vector outward normal to $\tilde{\Sigma}_{i_0}$ and $\tilde{h}_{ij}$, $\tilde{H}$ are the second fundamental form, the
mean curvature of $\Sigma_{i_0}$ respectively. Denote by $p_0 = \tilde{p} \circ \mathcal{R}$, $H_0 = \tilde{H} \circ \mathcal{R}$ the pullbacks to $\Sigma$.

The isometric embedding $\mathcal{R}$ also induces an isometry between the (intrinsic) spinor bundles of $\Sigma_{i_0}$ and $\tilde{\Sigma}_{i_0}$ together with their Dirac operators which are isomorphic to $e_r \cdot D$ and $\tilde{e}_r \cdot \tilde{D}$ respectively. This isometry can be extended to an isometry over the complex 2-dimensional sub-bundles of their hypersurface spinor bundles. Denote by $\tilde{S}^{\Sigma_{i_0}}$ this sub-bundle of $\tilde{S}|_{\tilde{\Sigma}_{i_0}}$. Let $\tilde{\phi}$ be a constant section of $\tilde{S}^{\Sigma_{i_0}}$ and denote $\phi_0 = \tilde{\phi} \circ \mathcal{R}$. Denote by $\tilde{\Xi}$ the set of all these constant spinors $\tilde{\phi}$ with the unit norm. This set is isometric to $S^3$.

Let $\tilde{D}$ be the (induced) Dirac operator on $\tilde{\Sigma}_{i_0}$ which acts on the hypersurface spinor bundle $\tilde{S}$ of $\tilde{M}$. Let $\tilde{\phi}$ be the covariant constant spinor of the trivial spinor bundle on $\mathbb{R}^{3,1}$ with unit norm taking by the positive Hermitian metric on $\tilde{S}$. Then (2.2) implies

$$\nabla_a \tilde{\phi} + \frac{1}{2} \tilde{h}_{ab} \tilde{e}_r \cdot \tilde{e}_b \cdot \tilde{\phi} - \frac{1}{2} \tilde{p}_{a_0} \tilde{e}_0 \cdot \tilde{e}_j \cdot \tilde{\phi} = 0$$

over $\tilde{\Sigma}_{i_0}$. Pullback to $\Sigma_{i_0}$, we obtain

$$e_r \cdot D \phi_0 = \frac{H_0}{2} \phi_0 - \frac{1}{2} p_{0aa} e_0 \cdot e_r \cdot \phi_0 + \frac{1}{2} p_{0ar} e_0 \cdot e_a \cdot \phi_0 \quad (3.1)$$

over $\Sigma_{i_0}$. Denote $\phi_0^\pm = P_\pm \phi_0$. Since $e_r \cdot D \circ P_\pm = P_\mp \circ e_r \cdot D$, (3.1) gives rise to

$$e_r \cdot D \phi_0^+ = \frac{H_0}{2} \phi_0^- + \frac{1}{2} p_{0aa} \phi_0^- + \frac{1}{2} p_{0ar} e_0 \cdot e_a \cdot \phi_0^+,$$

$$e_r \cdot D \phi_0^- = \frac{H_0}{2} \phi_0^+ - \frac{1}{2} p_{0aa} \phi_0^+ + \frac{1}{2} p_{0ar} e_0 \cdot e_a \cdot \phi_0^-.$$

Therefore, using

$$\int_{\Sigma_{i_0}} \langle \phi_0^-, e_r \cdot D \phi_0^+ \rangle = \int_{\Sigma_{i_0}} \langle e_r \cdot D \phi_0^-, \phi_0^+ \rangle,$$

we obtain

$$\int_{\Sigma_{i_0}} (H_0 - p_{0aa}) || \phi_0^- ||^2 = \int_{\Sigma_{i_0}} (H_0 + p_{0aa}) || \phi_0^+ ||^2. \quad (3.2)$$

In this paper, we introduce the following conditions on $M$:

(i) $tr_g(p) \geq 0$, $H|_{\Sigma_i} + tr(p|_{\Sigma_i}) \geq 0$ for all $i$;

(ii) $tr_g(p) \leq 0$, $H|_{\Sigma_i} - tr(p|_{\Sigma_i}) \geq 0$ for all $i$. 

Lemma 1. Let \((N^{3,1}, \bar{g})\) be a spacetime which satisfies the dominant energy condition. Let \((M, g, p)\) be a smooth spacelike (orientable) hypersurface which has boundary \(\Sigma\) with finitely many multi-components \(\Sigma_i\), each of which is a topological sphere. Suppose that \(\Sigma_{i_0}\) can be smoothly isometrically embedded into some spacelike hypersurface \((\tilde{M}, \tilde{g}, \tilde{p})\) in the Minkowski spacetime \(\mathbb{R}^{3,1}\). Let \(\mathcal{H}\) be the isometric embedding and let \(\tilde{\Sigma}_{i_0}\) be the image of \(\Sigma_{i_0}\). Suppose either condition (i) holds and \(\tilde{\Sigma}_{i_0}\) are past apparent horizons, i.e.,
\[
\bar{H} + \text{tr}(\tilde{p}|_{\tilde{\Sigma}_{i_0}}) \geq 0,
\]
or condition (ii) holds and \(\tilde{\Sigma}_{i_0}\) are future apparent horizons, i.e.,
\[
\bar{H} - \text{tr}(\tilde{p}|_{\tilde{\Sigma}_{i_0}}) \geq 0.
\]
Let \(\phi\) be the unique solution of (2.4) or (2.5) for some \(\tilde{\phi} \in \mathcal{H}\). Then
\[
\int_{\Sigma_{i_0}} \langle \phi, e_r \cdot D\phi \rangle \leq \frac{1}{2} \int_{\Sigma_{i_0}} \langle \phi, (H_0 - p_{0aa}e_0 \cdot e_r + p_{0ar}e_0 \cdot e_a)\phi \rangle.
\]

**Proof**: Assume condition (i) holds and \(\tilde{\Sigma}_{i_0}\) are past apparent horizons. Let \(\phi\) be the smooth solution of (2.4) with the prescribed \(\phi_0\) on \(\Sigma_{i_0}\). Denote \(\phi^\pm = P_{\pm} \phi\). Denote \(\phi^\pm_0 = P_{\pm} \phi_0\). By the boundary condition, we have \(\phi^+ = \phi^+_0\). Thus
\[
\int_{\Sigma_{i_0}} \langle \phi, e_r \cdot D\phi \rangle = 2\Re \int_{\Sigma_{i_0}} \langle \phi^-, e_r \cdot D\phi^+_0 \rangle
\]
\[
= \Re \int_{\Sigma_{i_0}} \langle \phi^-, H_0\phi^- + p_{0aa}\phi^-_0 + p_{0ar}e_0 \cdot e_r \cdot \phi^+_0 \rangle
\]
\[
\leq \frac{1}{2} \int_{\Sigma_{i_0}} (H_0 + p_{0aa}) (|\phi^-|^2 + |\phi^-_0|^2)
\]
\[
+ \Re \int_{\Sigma_{i_0}} \langle \phi^-, p_{0ar}e_0 \cdot e_a \cdot \phi^+_0 \rangle
\]
\[
= \frac{1}{2} \int_{\Sigma_{i_0}} (H_0 + p_{0aa}) |\phi^-|^2 + (H_0 - p_{0aa}) |\phi^+_0|^2
\]
\[
+ \Re \int_{\Sigma_{i_0}} \langle \phi^-, p_{0ar}e_0 \cdot e_a \cdot \phi^+_0 \rangle
\]
\[
= \frac{1}{2} \int_{\Sigma_{i_0}} H_0 |\phi^-|^2 + p_{0aa} (|\phi^-|^2 - |\phi^+_0|^2)
\]
\[
+ \Re \int_{\Sigma_{i_0}} \langle \phi^-, p_{0ar}e_0 \cdot e_a \cdot \phi^+_0 \rangle.
\]
Note that
\[ \langle \phi, p_{0ad}e_0 \cdot e_a \cdot \phi \rangle = p_{0aa}(|\phi^+|^2 - |\phi^-|^2). \]
Moreover, that \( e_0 \cdot e_a \cdot P_{\pm} = P_{\pm} \cdot e_0 \cdot e_a \cdot \) gives rise to
\[ \langle \phi, p_{0ar}e_0 \cdot e_a \cdot \phi \rangle = 2\Re(\langle \phi^-, p_{0ar}e_0 \cdot e_a \cdot \phi^+ \rangle). \]
Same argument is applied under condition \((ii)\). We finally prove the lemma. Q.E.D.

4. QUASI-LOCAL MASS

Now we use the idea of Wang and Yau [14] (see also [11]) to extend the definition of quasi-local mass in [18] to the case of 2-spheres with negative Gauss curvature.

We first review the definition for 2-spheres with nonnegative Gauss curvature in [18]: Suppose some \( \Sigma_{i_0} \) can be smoothly isometrically embedded into \( \mathbb{R}^3 \) in the Minkowski spacetime \( \mathbb{R}^{3,1} \) and denote \( \tilde{\Sigma}_{i_0} \) its image. (It exists if \( \Sigma_{i_0} \) has positive Gauss curvature.) In this case, \( \hat{p} = 0 \).

Let \( \phi \) be the unique solution of (2.4) or (2.5) for some \( \tilde{\phi} \in \tilde{\Xi} \). Denote
\[
m(\Sigma_{i_0}, \tilde{\phi}) = \frac{1}{8\pi} \int_{\Sigma_{i_0}} \left[ (H_0 - H)|\phi|^2 
+ tr(p|_{\Sigma_{i_0}}) \langle \phi, e_0 \cdot e_a \cdot \phi \rangle 
- p_{ar} \langle \phi, e_0 \cdot e_a \cdot \phi \rangle \right].
\]
(4.1)
The quasi local mass of \( \Sigma_{i_0} \) is defined as
\[
m(\Sigma_{i_0}) = \min_{\tilde{\Xi}} m(\Sigma_{i_0}, \tilde{\phi}).
\]
(4.2)
If all \( \Sigma_i \) can be isometrically embedded into \( \mathbb{R}^3 \) in the Minkowski spacetime \( \mathbb{R}^{3,1} \), we define the quasi local mass of \( \Sigma \) as
\[
m(\Sigma) = \sum_i m(\Sigma_i).
\]
(4.3)

If the mean curvature of \( \tilde{\Sigma}_{i_0} \) is further nonnegative (it is true if \( \Sigma_{i_0} \) has positive Gauss curvature), we can prove the positivity of the quasi-local mass \((4.2)\) (Theorem 1 in [18]).

Now suppose some \( \Sigma_{i_0} \) has negative Gauss curvature and let
\[
K_{\Sigma_{i_0}} \geq -\kappa^2
\]
(\kappa > 0) where \(-\kappa^2\) is the minimum of the Gauss curvature. (Here we must choose the minimum of the Gauss curvature instead of arbitrary lower bound, otherwise the quasi-local mass defined in the following way might depend on this arbitrary lower bound.) By \([9, 3]\), \(\Sigma_{i0}\) can be smoothly isometrically embedded into the hyperbolic space \(\mathbb{H}^3_{-\kappa^2}\) with constant curvature \(-\kappa^2\) as a convex surface which bounds a convex domain in \(\mathbb{H}^3_{-\kappa^2}\). Let \((t, x_1, x_2, x_3)\) be the spacetime coordinates of \(\mathbb{R}^3\).

Then \(\mathbb{H}^3_{-\kappa^2}\) is one-fold of the spacelike hypersurfaces

\[
\{(t, x_1, x_2, x_3) \mid t^2 - x_1^2 - x_2^2 - x_3^2 = \frac{1}{\kappa^2}\}.
\]

The induced metric of \(\mathbb{H}^3_{-\kappa^2}\) is

\[
\tilde{g}_{\mathbb{H}^3_{-\kappa^2}} = \frac{1}{1 + \kappa^2 r^2} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\psi^2)
\]

It has the second fundamental form \(\tilde{p}^+_{\mathbb{H}^3_{-\kappa^2}} = \kappa \tilde{g}_{\mathbb{H}^3_{-\kappa^2}}\) for the upper-fold \(\{t > 0\}\) and \(\tilde{p}^-_{\mathbb{H}^3_{-\kappa^2}} = -\kappa \tilde{g}_{\mathbb{H}^3_{-\kappa^2}}\) for the lower-fold \(\{t < 0\}\) with respect to the future-time-directed normal. Denote also \(\tilde{\Sigma}_{i0}\) its image.

Let \(\phi\) be the unique solution of \((2.4)\) or \((2.5)\) for some \(\tilde{\phi} \in \tilde{\Xi}\). Denote

\[
\hat{m}_\pm(\Sigma_{i0}, \tilde{\phi}) = \frac{1}{8\pi} \mathcal{R} \int_{\Sigma_{i0}} \left[ (H_0 - H)|\phi|^2 - (tr(p_0|_{\Sigma_{i0}}) - tr(p|_{\Sigma_{i0}})) \langle \phi, e_0 \cdot e_r \cdot \phi \rangle 
- (p_{0ar} - p_{ar}) \langle \phi, e_0 \cdot e_a \cdot \phi \rangle \right] \quad (4.4)
\]

where

\[
p_0 = \begin{cases} 
\text{pullback of } \tilde{p}^+_{\mathbb{H}^3_{-\kappa^2}} & \text{if } \Sigma_{i0} \text{ is isometrically embedded into the upper-fold } \{t > 0\}, \\
\text{pullback of } \tilde{p}^-_{\mathbb{H}^3_{-\kappa^2}} & \text{if } \Sigma_{i0} \text{ is isometrically embedded into the lower-fold } \{t < 0\}.
\end{cases}
\]

It is easy to see that \(tr(p_0|_{\Sigma_{i0}}) = \pm 2\), thus

\[
\hat{m}_\pm(\Sigma_{i0}, \tilde{\phi}) = \frac{1}{8\pi} \mathcal{R} \int_{\Sigma_{i0}} \left[ (H_0 - H)|\phi|^2 
+ tr(p|_{\Sigma_{i0}}) \langle \phi, e_0 \cdot e_r \cdot \phi \rangle 
- p_{0ar} \langle \phi, e_0 \cdot e_a \cdot \phi \rangle \right] 
+ \frac{\kappa}{4\pi} \int_{\Sigma_{i0}} \langle \phi, e_0 \cdot e_r \cdot \phi \rangle.
\]
Now we define the quasi local mass of $\Sigma_{i_0}$ under conditions $(i)$, $(ii)$ which are introduced in the previous section.

If condition $(i)$ holds, we embed $\Sigma_{i_0}$ into upper-fold $\{t > 0\}$. Since $\tilde{\Sigma}_{i_0}$ is convex, we have

$$\tilde{H} + \text{tr}(\tilde{p}|_{\Sigma_{i_0}}) > 0.$$  

If condition $(ii)$ holds, we embed $\Sigma_{i_0}$ into lower-fold $\{t < 0\}$. We have

$$\tilde{H} - \text{tr}(\tilde{p}|_{\Sigma_{i_0}}) > 0$$

in this case.

The quasi local mass of $\Sigma_{i_0}$ is defined as

$$\hat{m}(\Sigma_{i_0}) = \begin{cases} 
\min_{\tilde{\phi}} \hat{m}_+(\Sigma_{i_0}, \tilde{\phi}) & \text{if condition (i) holds,} \\
\min_{\tilde{\phi}} \hat{m}_-(\Sigma_{i_0}, \tilde{\phi}) & \text{if condition (ii) holds.}
\end{cases}$$  

(4.5)

Note that it might have two different values via embedding to the upper-fold and to the lower-fold respectively when $\text{tr}(\tilde{p}) = 0$. However, since $\tilde{D}\phi = 0$, $\tilde{D}(e_0 \cdot \phi) = -\text{tr}_g(p)\phi = 0$, we have

$$\int_{\Sigma} \langle e_r \cdot \phi, e_0 \cdot \phi \rangle = \int_{M} \langle \tilde{D}\phi, e_0 \cdot \phi \rangle - \langle \phi, \tilde{D}(e_0 \cdot \phi) \rangle = 0.$$  

This implies $\hat{m}_+(\Sigma_{i_0}, \tilde{\phi}) = \hat{m}_-(\Sigma_{i_0}, \tilde{\phi})$. Hence $\hat{m}(\Sigma_{i_0})$ is unique in this case. Furthermore, (4.5) approaches (4.2) when $\kappa \to 0$.

If $\Sigma_1, \ldots, \Sigma_{i_0}$ can be isometrically embedded into $\mathbb{R}^3$ in the Minkowski spacetime $\mathbb{R}^{3,1}$, and $\Sigma_{i_0+1}, \ldots, \Sigma_l$ can be isometrically embedded into $\mathbb{H}^{3,-\kappa_{i_0+1}^2}, \ldots, \mathbb{H}^{3,-\kappa_l^2}$ in the Minkowski spacetime $\mathbb{R}^{3,1}$ respectively, we define the quasi local mass of $\Sigma$ as

$$\hat{m}(\Sigma) = \sum_{1 \leq i \leq i_0} m(\Sigma_i) + \sum_{l_0+1 \leq i \leq l} \hat{m}(\Sigma_i).$$  

(4.6)

**Theorem 1.** Let $(N, \tilde{g})$ be a spacetime which satisfies the dominant energy condition. Let $(M, g, p)$ be a smooth initial data set with the boundary $\Sigma$ which has finitely many multi-components $\Sigma_i$, each of which is topological 2-sphere. Suppose that some $\Sigma_{i_0}$ has negative Gauss curvature and let $K_{\Sigma_{i_0}} \geq -\kappa^2$ ($\kappa > 0$) where $-\kappa^2$ is the minimum of the Gauss curvature. If either condition (i) or condition (ii) holds, then

1. $\hat{m}(\Sigma_{i_0}) \geq 0$;
2. that $\hat{m}(\Sigma_{i_0}) = 0$ implies the energy-momentum of spacetime satisfies

$$T_{00} = |f||\phi|^2, \quad T_{0i} = f\langle \phi, e_0 \cdot e_i \cdot \phi \rangle$$
along $M$, where $f$ is a real function, $\phi$ is the unique solution of (2.4) or (2.5) for some $\bar{\phi} \in \bar{\Xi}$.

(3) Furthermore, if $p_{ij} = 0$, then $\hat{m}(\Sigma_{i0}) = 0$ implies that $M$ is flat with connected boundary; if $p_{ij} = \pm \kappa g_{ij}$, then $\hat{m}(\Sigma_{i0}) = 0$ implies that $M$ has constant curvature $-\kappa^2$.

Proof: By Lemma 1, statements (1), (2) and the first part of statement (3) can be proved by the same argument as the proof of Theorem 1 in [18]. For the proof of the second part of the statement (3), the vanishing quasi local mass implies

$$\nabla_i \phi \pm \frac{\kappa}{2} e_0 \cdot e_i \cdot \phi = 0.$$  

Since $\nabla_i (e_0 \cdot \phi) = e_0 \cdot \nabla_i \phi$, we find the $M$ has constant Ricci curvature with the scalar curvature $-6\kappa^2$. Therefore $M$ has constant curvature $-\kappa^2$ because the dimension is 3. Q.E.D.

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References

[1] J.D. Brown and J.W. York, Quasilocal energy in general relativity, Mathematical aspects of classical field theory (Seattle, WA, 1991), 129-142, Contemp. Math., 132, Amer. Math. Soc., Providence, RI, 1992.
[2] J.D. Brown and J.W. York, Quasilocal energy and conserved charges derived from the gravitational action, Phys. Rev. D(3) 47, 1407-1419 (1993).
[3] M.P. do Carmo and F.W. Warner, Rigidity and convexity of hypersurfaces in spheres, J. Diff. Geom. 4, 133-144 (1970).
[4] P. Chruściel, M. Herzlich, The mass of asymptotically hyperbolic Riemannian manifolds, Pacific J. Math., 212, 231-264 (2003).
[5] F. Guan and Y.Y. Li, The Weyl problem with nonnegative Gauss curvature, J. Diff. Geom., 39, 331-342 (1994).
[6] J.X. Hong and C. Zuily, Isometric embedding of the 2-sphere with nonnegative curvature in $\mathbb{R}^3$, Math. Z., 219, 323-334 (1995).
[7] C-C.M. Liu and S.T. Yau, Positivity of quasilocal mass, Phys. Rev. Lett. 90, 231102 (2003).
[8] C-C.M. Liu and S.T. Yau, Positivity of quasilocal mass II, J. Amer. Math. Soc. 19, 181 (2006).
[9] A.V. Pogorelov, Some results on surface theory in the large, Adv. Math. 1, 191-264 (1964).
[10] Y. Shi and L-F. Tam, Positive mass theorem and the boundary behavior of compact manifolds with nonnegative scalar curvature, J.Diff.Geom. 62, 79 (2002).
[11] Y. Shi and L-F. Tam, Boundary behaviors and scalar curvature of compact manifolds, math/0611253.
[12] R. Schoen, S.T. Yau, On the proof of the positive mass conjecture in general relativity, Commun. Math. Phys. 65, 45-76 (1979).
[13] R. Schoen, S.T. Yau, *Proof of the positive mass theorem. II*, Commun. Math. Phys. 79, 231-260 (1981).

[14] M.T. Wang and S.T. Yau, *A generalization of Liu-Yau’s quasi-local mass*, math.DG/0602321.

[15] X. Wang, *The mass of asymptotically hyperbolic manifolds*, J. Diff. Geom., 57, 273-299 (2001).

[16] X. Zhang, *A definition of total energy-momenta and the positive mass theorem on asymptotically hyperbolic 3-manifolds I*, Commun. Math. Phys., 249, 529-548 (2004).

[17] X. Zhang, *The positive mass theorem near null infinity*, Proceedings of ICCM 2004, December 17-22, Hong Kong (eds. S.T. Yau, etc.), AMS/International Press, Boston, to appear, math/0604154.

[18] X. Zhang, *A new quasi-local mass and positivity*, Acta Mathematica Sinica, English Series, to appear.

Institute of Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China

E-mail address: xzhang@amss.ac.cn