Luttinger liquid behaviour and superconducting correlations in \(t\)-\(J\) ladders

C.A.Hayward and D.Poilblanc

Lab. de Physique Quantique, Université Paul Sabatier, 31062 Toulouse, France

(September 95)

The low energy behaviour of the isotropic \(t\)-\(J\) ladder system is investigated using exact diagonalization techniques, specifically finding the Drude weight, the charge velocity and the compressibility. By applying the ideas of Luttinger liquid theory, we determine the correlation exponent \(K_p\) which defines the behaviour of the long range correlations in the system. The boundary to phase separation is determined and a phase diagram is presented.

At low electron density, a Tomonaga-Luttinger-like phase is stabilized whilst at higher electron densities a gapped phase with power law pairing correlations is stabilized. A large region of this gapped phase is found to exhibit dominant superconducting correlations.

PACS numbers: 74.72.-h, 71.27.+a, 71.55.-i

1. Introduction

Over the last few years, the behaviour of strongly correlated electrons confined to coupled chains has received widespread attention; the reasons for this are numerous. Firstly, the behaviour of electrons in one-dimension, under \(t\)-\(J\) or Hubbard type interactions, is now relatively well understood and described generally by the term Luttinger Liquid (LL). The coupling of two such Luttinger liquids as in the ladder geometry provides an interesting first step towards the challenge of describing the behaviour in two-dimensional systems. A second reason for interest in these systems lies in the unusual nature of the ground state in the undoped system, namely spin liquid behaviour with a finite gap in the spin excitation spectrum; this behaviour is in contrast to the gapless behaviour of a single chain. The evolution of the spin-gapped state on doping has obvious relevance to gapped superconducting behaviour. In addition, compounds such as \((\mathrm{VO}_2)_2\mathrm{PrO}_4\) and \(\mathrm{SrCu}_2\mathrm{O}_2\mathrm{F}_2\) are believed to be well described by a lattice of coupled chains. Very recently, experiments on \(\mathrm{La}_{1-x}\mathrm{Sr}_x\mathrm{Cu}_2\mathrm{O}_4\) have provided insight into the doping of coupled chain systems. Whilst there is considerable literature on many aspects of the \(t\)-\(J\) ladder behaviour, a complete picture is still far from being realised: Our aim in this paper is to clarify some of the behaviour of the \(t\)-\(J\) ladder system by drawing on some of the ideas used in describing the strictly one-dimensional systems (i.e. LL theory). In conjunction with results from several other techniques, we will then present a speculative phase diagram.

The \(t\)-\(J\) Hamiltonian on the \(2 \times L\) ladder is defined as,

\[
\mathcal{H} = J' \sum_{j} (\mathbf{S}_{j,1} \cdot \mathbf{S}_{j,2} - \frac{1}{4} n_{j,1} n_{j,2}) + J \sum_{j,\beta} (\mathbf{S}_{j,\beta} \cdot \mathbf{S}_{j+1,\beta} - \frac{1}{4} n_{j,\beta} n_{j+1,\beta}) - t \sum_{j,\beta,\gamma} \rho_G(c_{j,\beta,s}^\dagger c_{j+1,\gamma,s} + \text{H.c.}) P_G,
\]

where most notations are standard. \(\beta (=1,2)\) labels the two legs of the ladder (oriented along the \(x\)-axis) while \(j\) is a rung index \((j=1,...,L)\). We shall concentrate on the isotropic case where the intra-ladder (along \(x\)) couplings \(J\) and \(t\) are equal to the inter-ladder (along \(y\)) couplings \(J'\) and \(t'\).

At half filling the hamiltonian reduces to the Heisenberg model and the behaviour is generally relatively well understood. A simple interpretation is given by considering the strong coupling limit \((J=0)\) in which the ground state consists of a singlet on each rung with a spin gap \((\sim J')\) which corresponds to forming a triplet on one of the rungs. With the introduction of intra-chain coupling \(J\), the triplets can propagate and form a coherent band thereby reducing the spin gap. In the isotropic case, the spin gap remains \((\sim 0.5J)\).

The evolution of this spin gapped state on doping is perhaps one of the most interesting aspects of the ladder behaviour. Recent work on this hole-doped phase has indicated a finite spin gap, a single gapless charge mode, hole pairing and possible dominant superconducting correlations. In this paper we shall discuss some new independent results which provide a more complete description of not only this gapped phase but the whole region of parameter space. A possible phase diagram for the isotropic \(t\)-\(J\) ladder as a function of \(J/t\) and doping has been proposed recently, and we use these ideas in our analysis: Away from half filling the spin-gapped phase is stabilized up to \(J/t \sim 2.1\) where the system phase separates. As the system is doped further, a phase with a single gapless spin and a single gapless charge mode is found and, as we shall explain, this behaviour is like that of the one-dimensional Tomonaga-Luttinger-Liquid system. As in the gapped phase, phase separation occurs for large \(J/t\). At very small electron densities, an electron paired phase exists. Note that although in the \(t\)-\(J\) ladder there are four possible zero-momentum gap-
less modes (two spin and two charge), throughout the phase diagram only one gapless charge mode and either zero or one gapless spin modes are observed; this then allows an almost identical treatment to the strictly one-dimensional case.

In section II we discuss briefly the Luttinger Liquid theory used to describe strictly one-dimensional systems and also the possible application of this theory to coupled chains; In section III we present our numerical calculations, and finally in section IV we apply the Luttinger Liquid theory to our results and present a phase diagram for the system.

II. LUTTINGER LIQUID BEHAVIOUR: TOMONAGA-LUTTINGER AND LUTHER-EMERY PHASES

In dealing with strictly one-dimensional interacting fermion systems, one can in general make use of conformal field theory and bosonization which allow a determination of the decay exponents of the various correlation functions to be determined from the low energy behaviour of the model. The general idea is that one-dimensional interacting fermion systems can be mapped onto the Fermi-gas model and the corresponding 'g-ology' weak coupling theory. This Fermi gas model scales to two different regimes, namely the Tomonaga-Luttinger (TL) fixed point and the Luther-Emery (LE) line which are relevant for repulsive ($g_l > 0$) and attractive ($g_l < 0$) backscattering matrix elements respectively; as we shall explain, the important difference between these two universality classes lies in the spin degrees of freedom. The low lying excitations of the Fermi gas model are collective spin or charge density oscillations, which propagate with different velocities, giving rise to spin-charge separation and power law behaviour of the correlation functions. In the TL phase, both a gapless spin and a gapless charge mode are exhibited; In contrast, whilst exhibiting a gapless charge mode, the LE phase has a gap to spin excitations.

Conformal field theory relates the properties of a finite system (such as the compressibility and the Drude weight) with the correlation exponents; One coefficient ($K_\rho$) determines the exponents of all the power law decays, (and similarly the singularity of the momentum distribution function close to $k_f$). Hence if a particular model scales to the TL (or LE) universality class, one can infer the dominant correlations from the low energy behaviour of the system which can be deduced from much smaller system sizes than would be required to calculate the correlation lengths directly.

The relationships between the correlation exponent ($K_\rho$) and the low energy behaviour of the model are given below for a system of size $N$, and length $L$ (we have chosen to give general equations such that $N=L$ for a chain, $N=2L$ for the ladder geometry). Firstly, the ratio of the charge velocity $u_\rho$ to the coefficient $K_\rho$ is proportional to the variation of the ground state energy $E_0$ with particle density $n$, i.e. the inverse compressibility

$$\frac{\pi}{2} u_\rho K_\rho = \frac{1}{n^2 K} = \frac{\partial^2 (E_0/N)}{\partial n^2}$$  (2)

The coefficient $K_\rho$ is also related to the Drude weight $\sigma_0$; this Drude weight is the weight of the zero frequency (dc) peak in the conductivity $\sigma$, and may be obtained by considering the curvature of the ground state energy level as a function of threaded flux, $\Phi$

$$\sigma_0 = 2u_\rho K_\rho = \frac{L^2}{4\pi} \frac{\partial^2 (E_0/N)}{\partial \Phi^2}$$  (3)

We can also determine the charge velocity by considering the dispersion of the energy spectrum

$$u_\rho = \frac{E_{1\rho} - E_0}{(2\pi/L)}$$  (4)

where $E_{1\rho}$ is the lowest lying charge mode to the ground state ($E_0$) with neighbouring $k$ value.

These equations provide us with 3 independent conditions on $K_\rho$ and $u_\rho$ which can be used to check the consistency of the Luttinger liquid relations. Also a calculation of the parameter $K_\rho$ is relatively straightforward and this parameter then determines the exponent coefficients of all the correlation functions.

As explained previously, the essential difference between the TL and LE fixed points lies in the spin degrees of freedom: the LE region gapped whilst the TL gapless. The correlation exponents of the two different cases are summarised in table I (taken from reference [1]). We have omitted the logarithmic corrections and these are detailed in reference [2]. We emphasise that for LL systems, the correlation functions show either power law or exponential decay, with interaction-dependent powers determined by one coefficient $K_\rho$. Also we see that for $K_\rho < 1$ (spin or charge) density waves at $2k_f$ are enhanced and diverge, whereas for $K_\rho > 1$ pairing fluctuations dominate.

Whilst in strictly one-dimensional systems there is a single gapless charge mode, the theory could equally well be applied in a case where more than one gapless mode existed if the excitations were decoupled in the low energy regime. In such a case, each degree of freedom would have an associated operator algebra, i.e. an associated $K_\rho$.

Specifically considering the problem of coupled chains, we note that at present little is known and there is much interest into how to connect the quasi-one-dimensional results to the strictly one-dimensional case. A recent calculation by Schulz[4] has considered the coupling of two Luttinger liquids by a small interchain hopping using a bosonization technique. Interestingly, in the presence of both forward and backward scattering terms, the calculation predicts a gap in all the magnetic excitations and a gapless charge mode (as observed for the hole-doped region of the t–J ladder). This gapped phase however exhibits somewhat different correlations.
to the strictly one-dimensional LE phase described above. Firstly the CDW and SDW correlations decay exponentially along the chains (we use the notation of Schulz where 0 or π indicate the oscillations are in or out of phase between the two chains respectively, i.e. ‘bonding’ or ‘antibonding’). A divergent density-density response, decaying as \( \sim \cos \left( 2k_f r \right) e^{-K_\rho r} \), exists in analogy with the 4\( k_f \) oscillations of a single chain \( k_f^0 \) and \( k_f^\pi \) refer to the Fermi points of the bonding and antibonding quasi particle branches respectively); we note that the coefficient \( K_\rho \) differs by a factor of two to that used by Schulz since we have chosen to define \( \sigma_0 \) and \( \kappa \) as the Drude weight and compressibility per site rather than per rung. The superconducting correlations (cross chain pairing) decay as \( r^{-1/K_\rho} \) and exhibit a ‘d’-like character. Hence, for this gapped phase, we would expect dominant superconducting correlations for \( K_\rho < 1 \). In agreement with these findings, different approaches by Troyer et al., Nagaoa and Balents and Fisher predict similar behaviour in the spin-gapped region: i.e. ‘Luther-Emery-like’ in the sense that there exist two order parameters (analogous to the on-site pairing and 2\( k_f \) CDW in the LE class) whose exponents obey a reciprocal relation: these correspond to the pair field correlations and to a four fermion operator \( \langle n_B(r)n_B(0) \rangle \), where \( n_B \) is the density of ‘bosonic’ hole pairs bound on a rung (analogous to the 4\( k_f \) oscillations of a single chain (i.e.2\( k_f^0 + 2k_f^\pi \))). In table II we summarize the correlation exponents predicted for this spin gapped phase in the ladder geometry.

In the following section we give details of calculations using these ideas to characterize the behaviour of the t–J ladder, finding the parameter \( K_\rho \) and hence the dominant correlation functions. We consider various electron densities and various ratios of \( J/t \) to build up a speculative phase diagram.

III. NUMERICAL CALCULATIONS

Since we require information concerning the low energy properties of the model, the dominant technique we have employed is that of exact diagonalization of finite systems, specifically 2 \( \times \) 5 and 2 \( \times \) 10 double chain rings. Exact diagonalization techniques are particularly well adapted to the investigation of low energy modes since implementation of various quantum numbers is straightforward: the various excitation modes can be obtained by calculating the ground state energy in each symmetry sector. The low energy modes of the system are characterized firstly by their spin: singlet and triplet excitations correspond to charge and spin modes respectively. It is also useful to consider the parity of the states under a reflection in the symmetry axis of the ladder along the direction of the chains: Even \( (R_x = 1) \) or odd \( (R_x = -1) \) excitations corresponding to bonding (B) or anti-bonding (A) modes respectively (0 and \( \pi \) as used by Schulz). Finally the dispersion relation of each mode is determined by the momentum \( k_z = 2\pi n/L \).

In order to ensure that the antiferromagnetic correlations are not frustrated when one goes around each chain, we have chosen the electron number to be always a multiple of four; our results then concern electron densities 0.4 and 0.8 for both system sizes and in addition 0.2 and 0.6 for the larger system size. The absolute ground state is given by the boundary conditions that form a closed shell in the non-interacting Fermi sea (obtained by turning off the interaction, \( J \)) and these are used in the calculations of \( u_\nu \) and \( \kappa \), specifically anti-periodic boundary conditions for \( n < 0.5 \) and periodic boundary conditions for \( n > 0.5 \).

A. Drude Weight and Anomalous Flux Quantization

The first calculation we present concerns the Drude weight, defined by equation 3. The numerical technique involves threading the double chain ring with a flux \( \Phi \) and studying the functional form of the ground state energy with respect to the threaded flux, namely \( E_0(\Phi) \). In general \( E_0(\Phi) \) consists of a series of parabola, corresponding to the curves of the individual many body states \( E_n(\Phi) \): This envelope exhibits a periodicity of one, where we have chosen to measure the flux in units of the flux quantum \( \Phi_0 = \hbar c/e \). Note that the function \( E_0(\Phi) \) also gives a quantitative value of the superfluid density \( D_s \), which is in general different from \( \sigma_0 \). The Drude weight corresponds to curvature of a single ground-state many-body energy level whilst the superfluid density corresponds to the curvature of the envelope of the individual many-body states as a function of flux. However, since the flux \( (\phi_c) \) at which another many body energy level crosses the zero-flux ground state energy level varies as \( \phi_c \sim (\hbar c/e)L^{1-d} \) (where \( d \) is the dimension). in one-dimensional \( \phi_c \) is independent of \( L \); there are only a finite number of energy level crossings in the thermodynamic limit and \( \sigma_0 \) and \( D_s \) are equal (up to a factor of \( 2\pi \)).

In addition to the Drude weight and the superfluid density, the function \( E_0(\Phi) \) also yields information regarding the phenomenon of anomalous flux quantization; this has been explained in a previous publication, so we mention it only briefly here. Whilst in general the ground state envelope \( E_0(\Phi) \) exhibits a periodicity of one, Byers and Yang have shown that in the thermodynamic limit \( E_0(\Phi) \) exhibits local minima at quantized values of flux, the separation of which is \( 1/n \) where \( n \) is the sum of the charges in the basic group. Hence, for a paired superconducting state we would expect minima in \( E_0(\Phi) \) at intervals of \( 1/2 \). These minima are related to the existence of supercurrents which are trapped in metastable states corresponding to the flux minima and are thus unable to decay away. It should be mentioned that this anomalous flux quantization (AFQ) is an indication of pairing and is not in itself sufficient to imply a supercon-
ducting state.

Numerically the application of a flux through the double chain ring is achieved by modifying the kinetic term of the hamiltonian such that

\[ c_{j,\beta;s}^{\dagger} c_{j+1,\beta;s} \rightarrow c_{j,\beta;s}^{\dagger} c_{j+1,\beta;s} e^{\frac{i\Phi}{2\pi}} \]

where \( \Phi \) is the flux through the ring measured in units of \( \Phi_0 \). Hence the application of a flux is numerically equivalent to a change in the boundary conditions of the problem; \( \Phi = 0 \) representing periodic and \( \Phi = 1/2 \) representing anti-periodic boundary conditions. In the thermodynamic limit, \( \sigma \) must be independent of the phase introduced at the ring boundary and therefore we consider the whole of the envelope \( E_0(\Phi) \) as a function of flux (in general consisting of several parabolas).

Choosing the parameters \( n = 0.8 \) and \( J/t = 0.5 \), we show in figure 1a (b) all the possible spin and charge modes of the \( 2 \times 5 \) (\( 2 \times 10 \)) system, for all possible momenta, as a function of applied flux. In the case of the larger system, we show the full spectrum for \( \Phi < 0.25 \) in order to simplify the diagram (this work has been previously published). For both system sizes the minimum energy is formed by charge (spin zero) bonding modes; the excited modes with different quantum numbers move further from the ground state as the system size is increased (a result we have checked by finite size scaling) and hence will not interfere with \( E_0(\Phi) \). The existence of minima at intervals of half a flux quantum (i.e. anomalous flux quantization) clearly indicates the existence of pairing.

The envelope \( L[E_0(\Phi) - E_0(\Phi = 0)] \) has been extracted and is shown in figure 2a (b) along with equivalent plots from other regions of the phase diagram: Figure 3a shows the data for \( J/t = 1.0 \) and electron densities of 0.4 and 0.8, whilst figure 3b shows the data for an electron density of 0.8 for ratios \( J/t = 0.5 \) and 4.0. Apart from the curve with \( n = 0.8, J/t = 4.0 \) (which appears to scale to a flat function, consistent with a phase separated state), all the data appears to show only small finite size effects. Anomalous flux quantization is observed for the larger electron density (for the lower values of \( J/t \)) indicating pairing, and hence consistent with a superfluid state in this region.

In order to determine the Drude weight, we simply calculate the average value of the curvature of \( L[E_0(\Phi) - E_0(\Phi = 0)] \) over all \( \Phi \); a quadratic curve was fitted to each portion. In figure 4a we plot the Drude weight as a function of \( J/t \) for electron densities 0.2, 0.4, 0.6 and 0.8 for the \( 2 \times 10 \) system and electron densities 0.4 and 0.8 for the \( 2 \times 5 \) system. The curves are plotted up to a maximum in \( J/t \) which is determined by the value at which the system phase separates (see compressibility). In figure 4b we plot the Drude weight as a function of electron density for various values of the ratio \( J/t \).

There are several features of the resulting behaviour we should mention: Note firstly that finite size effects are relatively small with the \( 2 \times 5 \) results close to those of the \( 2 \times 10 \) results. The Drude weight increases as the electron density is increased from zero, until it reaches one-quarter filling, then decreases with increasing electron density. Also as we would expect for a spin charge separated state, the Drude weight is effectively independent of \( J/t \).

### B. Charge Velocity

The second quantity we have calculated is the charge velocity, defined by equation 4. Considering the charge bonding modes (the lowest lying charge modes), the energy difference between the ground state energy level and the energy level with neighbouring momentum, \( \Delta k_x = \frac{2\pi}{L} \) was calculated. As an example, we show in figure 4a the charge bonding modes for the case \( n = 0.4, J/t = 2.0 \), indicating with solid lines the specific energy levels whose energy difference gives the charge velocity. We note that the gap (and therefore \( u_s \)) is approximately constant as a function of flux. For consistency with future data however, we have calculated the charge velocity at the particular flux which gives the absolute ground state, i.e. \( \Phi = 0.5 \) for \( n < 0.5 \) and \( \Phi = 0 \) for \( n > 0.5 \). The results are shown in figure 4b as a function of \( J/t \) for various system sizes and various electron densities (the results for \( n = 0.6 \) are not included since the numerics present some difficulty close to one-quarter filling). Note again that finite size effects are relatively small.

### C. Compressibility

The next quantity we calculate is that of the compressibility, defined by equation 5. The finite size equivalent is given by

\[
\frac{1}{n^2}\kappa \approx \frac{1}{2L} \left[ \frac{E_0(n + \Delta n) + E_0(n - \Delta n) - 2E_0(n)}{(\Delta n)^2} \right]
\]

where \( E_0(n) \) is the ground state energy of the finite system of ladder length \( L \) with an electron density \( n \). As for the calculation of the charge velocity, the boundary conditions have been chosen to give the absolute ground state. \( \Delta n \) represents the finite change in electron density, 0.2 and 0.4 for the \( 2 \times 10 \) and \( 2 \times 5 \) systems respectively. In figure 5 we show the results of the calculation of \( 1/n^2\kappa \) for electron densities corresponding to 0.2, 0.4, 0.6 and 0.8 for the \( 2 \times 10 \) ladder and we also plot the result of the \( 2 \times 5 \) system for an electron density of 0.4.

In addition to the determination of the specific values of the inverse compressibility, the boundary to phase separation may also be determined from these results. It is well known that at sufficiently large values of \( J/t \), a system will undergo a separation into two phases; a hole-rich and an electron rich phase. This effect arises principally to minimize the number of broken antiferromagnetic bonds in the system. Phase separation occurs when the compressibility diverges (i.e. a ‘liquid’ to a...
‘solid’ phase) and hence the inverse compressibility vanishes. Whilst there remain some small finite size effects, the general form of the phase separation line is readily observed from figure 6. As electron density is increased, the value of $J/t$ at which phase separation occurs decreases (although electron densities of 0.6 and 0.8 are both indicative phase separation close to $J/t \sim 2.1$). These results agree well with those of Tsunetsugu et al. who used a similar technique but varied both system size and electron density simultaneously. The phase separation curve can be extrapolated to all electron densities and will be shown in the predicted phase diagram, figure 6.

IV. LUTTINGER LIQUID PARAMETERS FOR THE LADDER

In order to explore the validity of the Luttinger liquid relations in our problem, we consider the ratio $\sigma_0/\pi n^2 K u^2_0$ which equals unity for a Luttinger liquid. The results of the numerical calculation of this quantity are shown in figure 7 for various electron densities.

At low electron densities (i.e. for the cases $n = 0.2$ and $n = 0.4$), previous work has suggested that a TL phase is stabilized and the results of this ratio show good agreement with the predicted value of unity; for the case of $n = 0.4$ an increase in system size shows the ratio scaling towards unity. For the hole-doped spin-gapped region ($n = 0.8$), where the behaviour is less well understood, the system phase separates at a much lower value of $J/t$ and hence the curve drops to zero at $J/t \sim 2.1$. Before this phase separation, the data is not inconsistent with the ‘new’ LE-like behaviour which may be described by equations 2-4 (finite size effects are largest for high electron density and low $J/t$). Since the ratio is close to unity, it appears to confirm the earlier justification for using this one-dimensional theory, i.e. only a single gapless charge mode is observed.

With the values of the compressibility and the Drude weight, we can obtain an estimate of the coefficient $K_\rho$ using $K_\rho = 1/2\sqrt{\pi n^2 K \sigma_0}$. This behaviour of $K_\rho$ for different electron densities of the $2 \times 10$ ladder is shown in Fig. 7 and we have also plotted the results for a $2 \times 5$ ladder with electron density 0.4. Note that as $J/t$ is increased, $K_\rho$ increases (for all electron densities) becoming infinite at phase separation as the compressibility diverges. A similar calculation has been performed by Troyer et al. for a specific electron density of 0.857, i.e. two holes on a $2 \times 7$ ladder and the results are consistent in this region.

Before analysing the behaviour further, specifically the importance of $K_\rho$, we present a speculative phase diagram of the isotropic t–J ladder as a function of $J/t$ and electron density. This phase diagram is shown in figure 8 and we discuss briefly the various regions.

At larger values of $J/t$, the system phase separates, and to estimate the value of $J/t$ at which this occurs, we show the data points at which the inverse compressibility vanishes in the $2 \times 10$ system (see figure 6). On doping away from half filling, the spin gap region persists and a phase exhibiting one gapless charge mode is stabilized. On further doping a gapless phase is stabilized: the schematic boundary between these two phases is shown as a dot-dashed line. As for both the one- and two-dimensional t–J case, a gas of electron pairs is formed at low electron densities above a critical value of the ratio $J/t$: other 2p-particle ($p > 1$) boundstates could also become stable at larger $J/t$ in this region. Again the boundary to this paired phase is dot-dashed and is schematic.

From the data in figure 7 we have plotted contours of constant $K_\rho$, in order to allow a determination of the dominant correlation functions. Whilst the parameter $K_\rho$ is continuous for a particular value of $J/t$ as the electron density is varied, at some region between $n = 0.4$ and $n = 0.8$ a gap opens in the spin excitation spectrum and the correlation functions change their form discontinuously, scaling to a different fixed point as explained in section 5. A ‘jump’ in the exponent of the superconducting correlations occurs with the SCd charge exponent changing from $1 + 1/K_\rho$ to $1/K_\rho$; the $2k_f$ CDW correlations jump from power law behaviour (exponent $1 + K_\rho$) to exponential decay. In addition, in the gapped high density state we would expect conjugate ‘four fermion 4$k_f$’ CDW correlations with exponent $K_\rho$.

For both the high density ‘gapped’ phase and the low density TL phase we expect different correlation functions to dominate either side of the contour $K_\rho = 1$; in both cases superconducting correlations dominate for $K_\rho > 1$. However, in the gapped hole-doped phase, this region is much larger and in contrast to other models such as the one-dimensional t–J model, does not just exist as a precursor to phase separation. A physical picture of the behaviour is of the holes pairing up on the rungs, the spins in singlets, and the dominant correlation functions are then associated with the movement of the hole pairs.

We wish to thank H.J. Schulz for many useful discussions and comments; we also gratefully acknowledge many helpful conversations with M. Luchini, F. Mila, W. Hanke and D.J. Scalapino. Laboratoire de Physique Quantique, Toulouse is Unité de Recherche Associée au CNRS No 505. CAH and DP acknowledge support from the EEC Human Capital and Mobility program under Grants ERBCHBICT941392 and CHRX-CT93-0332. We also thank IDRIS (Orsay) for allocation of CPU time on the C94 and C98 CRAY supercomputers.

1 T.Barnes et al, Phys.Rev.B 47, 3196 (1993); S.Gopalan, T.M.Rice and M.Sigrist, Phys.Rev.B 49, 8901 (1994).
FIG. 1. Energy as a function of flux (in units of $\Phi_0 = hc/e$) for a) the $2 \times 5$ and b) the $2 \times 10$ ladders with $J/t = 0.5$ and $n = 0.8$. We show all possible momenta for various quantum numbers: For the charge modes, the solid lines correspond to bonding and the dotted lines to anti-bonding, while for the spin modes the dashed lines correspond to bonding and the dot-dashed lines to anti-bonding. For the larger system size, we give only the charge bonding and the lowest lying spin antibonding mode in full to simplify the diagram.

FIG. 2. $L|E_0(\Phi) - E_0(\Phi = 0)|$ where $L$ is the length of the ladder and $E_0(\Phi)$ is the ground state energy with an applied flux $\Phi$. The dashed lines correspond to $2 \times 5$, the solid lines to $2 \times 10$. a) shows the results for $n = 0.4$ and $n = 0.8$ both with $J/t = 1.0$, while b) shows $J/t = 0.5$ and $J/t = 4.0$ both with $n = 0.8$.

FIG. 3. The Drude weight $\sigma_0$ a) as a function of ratio $J/t$ for various electron densities and system sizes, and b) as a function of electron density for various $J/t$. The smooth lines are a guide to the eye.

FIG. 4. (a) Energy of the charge bonding modes as a function of flux for the $2 \times 10$ system with $n = 0.4, J/t = 2.0$. The solid lines indicate the ground state and the excited states used to calculate the charge velocity. (b) The charge velocity $u_\rho$ as a function of ratio $J/t$ for various electron densities and system sizes. The dotted lines are a guide to the eye.

FIG. 5. $1/n^2 \kappa$ as a function of ratio $J/t$ for various electron densities and system sizes. The dotted lines are a guide to the eye.

FIG. 6. The ratio $\pi n^2 \kappa u_\rho^2 / \sigma_0$ as a function of $J/t$ for various electron densities and system sizes. The dotted lines are a guide to the eye.

FIG. 8. Speculative phase diagram of the $t$–$J$ ladder as a function of $J/t$ and electron density $n$. The circles and crosses represent results from the $2 \times 10$ system and the dotted lines are guides to the eye. The dot-dashed lines separating the different phases are estimated.

TABLE I. Correlation exponents for the Tomonaga Luttinger and Luther Emery cases. SS and TS indicate singlet and triplet superconductivity (pairing) correlations respectively.

| Correlation | TL | LE |
|-------------|----|----|
| $2k_f$ SDW | $1 + K_\rho$ | exponential |
| $2k_f$ CDW | $1 + K_\rho$ | $K_\rho$ |
| SS | $1 + 1/K_\rho$ | $1/K_\rho$ |
| TS | $1 + 1/K_\rho$ | exponential |
| $4k_f$ CDW | $4K_\rho$ | $4K_\rho$ |

TABLE II. Correlation exponents for the spin gapped phase in the ladder geometry. The $\pi$ indicates that the correlations along the two chains are out of phase.

| Correlation | gapped spin |
|-------------|-------------|
| $2k_f$ SDW | exponential |
| $2k_f$ CDW | exponential |
| SCd | $1/K_\rho$ |
| $4k_f$ CDW | $K_\rho$ |
a) 2x5 J/t=0.5 \langle n \rangle = 0.8
b) $2 \times 10^{-1} \ J/t = 0.5 \quad \langle n \rangle = 0.8$
\[ L \{ E(\text{flux}) - E(\text{flux}=0) \} \]

\begin{align*}
\langle n \rangle &= 0.8 \\
J/t &= 1.0 \\
\langle n \rangle &= 0.4 \\
J/t &= 1.0
\end{align*}
The graph shows the Drude weight as a function of $J/t$ for different values of $\langle n \rangle$.

- $\langle n \rangle = 0.2$ (2x10)
- $\langle n \rangle = 0.4$ (2x10)
- $\langle n \rangle = 0.6$ (2x10)
- $\langle n \rangle = 0.8$ (2x10)
- $\langle n \rangle = 0.4$ (2x5)
- $\langle n \rangle = 0.8$ (2x5)
2x10 J/t=2.0 <n>=0.4
phase separation

spin modes gapped

spin & charge modes gapless

gas of electron pairs