On the Approximability and Hardness of the Minimum Connected Dominating Set with Routing Cost Constraint

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Abstract

In the problem of minimum connected dominating set with routing cost constraint, we are given a graph $G = (V, E)$, and the goal is to find the smallest connected dominating set $D$ of $G$ such that, for any two non-adjacent vertices $u$ and $v$ in $G$, the cost of routing between $u$ and $v$ through $D$ (the number of internal nodes on the shortest path between $u$ and $v$ in $G[D \cup \{u, v\}]$) is at most $\alpha$ times that through $V$. For general graphs, the only known previous approximability result is an $O(\log n)$-approximation algorithm ($n = |V|$) for the case $\alpha = 1$ by Ding et al. For $1 < \alpha < 5$, no non-trivial approximation algorithm was previously known even on special graphs like unit disk graphs. When $\alpha > 1$, we give an $O(n^{1-\frac{1}{\alpha}}(\log n)^{\frac{1}{\alpha}})$-approximation algorithm. When $\alpha \geq 5$, we give an $O(\sqrt{n \log n})$-approximation algorithm. Finally, we prove that, when $\alpha = 2$, unless $NP \subseteq DTIME(n^{poly \log n})$, the problem admits no polynomial-time $2^{2^{2^{1-\epsilon} \cdot n}}$-approximation algorithm, for any constant $\epsilon > 0$, improving upon the $\Omega(\log n)$ bound by Du et al. (albeit under a stronger hardness assumption).

Keywords — Connected dominating set, spanner, set cover with pairs, MIN-REP problem

1 Introduction

1.1 Motivation

In wireless networks, a source can send packets to a destination through relays in the virtual backbone. The virtual backbone consists of nodes in the wireless network, and every node in the wireless network is adjacent to some node in the virtual backbone, i.e., the virtual backbone is a dominating set. Once a node in the virtual backbone receives a packet, a routing path from the node to a node adjacent to the destination is established. Moreover, all the nodes on the routing path belong to the virtual backbone. Hence, the virtual backbone forms a connected dominating set. One of the concerns in constructing the virtual backbone is the routing cost. Specifically, the routing cost of sending a packet from the source $src$ to the destination $dst$ is the number of internal nodes (relays) in the routing path from $src$ to $dst$. When packets are only allowed to be routed through the virtual backbone, the routing cost should not be expanded too much (compared to that through the original graph). Next, we give the formal definition of the problem.

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1.2 Problem Definition

Let $G[S]$ be the subgraph of $G$ induced by $S$. Let $m_G(u,v)$ be the number of internal vertices on the shortest path between $u$ and $v$ in $G$. For example, if $u$ and $v$ are adjacent, then $m_G(u,v) = 0$. If $u$ and $v$ are not adjacent and have a common neighbor, then $m_G(u,v) = 1$. Furthermore, given a vertex subset $D$ of $G$, $m^D_G(u,v)$ is defined as $m_G[D\cup\{u,v\}](u,v)$, i.e., is the number of internal vertices on the shortest path between $u$ and $v$ through $D$. We use $n(G)$ to denote the number of vertices in graph $G$. When the graph we are referring to is clear from the context, we simply write $n$, $m(u,v)$, and $m^D(u,v)$ instead of $n(G)$, $m_G(u,v)$, and $m^D_G(u,v)$, respectively.

Definition 1. Given a connected graph $G$ and a positive integer $\alpha$, the **Connected Dominating set problem with Routing cost constraint** (CDR-$\alpha$) asks for the smallest connected dominating set $D$ of $G$, such that, for every two vertices $u$ and $v$, if $u$ and $v$ are not adjacent in $G$, then $m^D(u,v) \leq \alpha \cdot m(u,v)$.

Note that, if two vertices $u$ and $v$ are adjacent, then they can communicate with each other without the help of nodes in $D$. Therefore, in the above definition, we only consider pairs of non-adjacent vertices.

In the CDR-$\alpha$ problem, we need to consider all the pairs of non-adjacent nodes. Ding et al. discovered that to solve the CDR-$\alpha$ problem, it suffices to consider vertex pairs $(u, v)$ such $m(u,v) = 1$, i.e., $u$ and $v$ are not adjacent but have a common neighbor [5]. We call the corresponding problem the 1-DR-$\alpha$ problem.

Definition 2. Given a connected graph $G = (V, E)$ and a positive integer $\alpha$, the **1-DR-$\alpha$ problem** asks for the smallest dominating set $D$ of $G$, such that, for every two vertices $u$ and $v$, if $m(u,v) = 1$, then $m^D(u,v) \leq \alpha$.

We say that $u$ and $v$ form a **target couple**, denoted by $[u, v]$, if $m(u,v) = 1$. We say that a set $S$ **covers** a target couple $[u, v]$ if $m^S(u,v) \leq \alpha$. Hence, the 1-DR-$\alpha$ asks for the smallest dominating set that covers all the target couples.

The equivalence between the CDR-$\alpha$ problem and the 1-DR-$\alpha$ problem is stated in the following theorem.

Theorem 1 (Ding et al. [5]). $D$ is a feasible solution of the CDR-$\alpha$ problem with input graph $G$ if and only if $D$ is a feasible solution of the 1-DR-$\alpha$ problem with input graph $G$.

Corollary 1. Any $r$-approximation algorithm of the 1-DR-$\alpha$ problem is an $r$-approximation algorithm of the CDR-$\alpha$ problem.

1.3 Previous Result on the CDR-$\alpha$ Problem

**General graphs:** When $\alpha = 1$, the 1-DR-$\alpha$ problem can be transformed to the set cover problem, i.e., cover all the vertices (dominating set) and cover all the target couples. Observe that each target couple can be covered by a single vertex. The resulting approximation ratio is $O(\log n)$ [5]. When $\alpha$ is sufficiently large, e.g., $\alpha \geq n$, any connected dominating set is feasible for the CDR-$\alpha$ problem. Note that, for any $\alpha$, the size of the minimum connected dominating set is a lower bound of the CDR-$\alpha$ problem. Since the connected dominating set can be approximated within a factor of $O(\log n)$ [12][21], the CDR-$\alpha$ problem can be approximated within a factor of $O(\log n)$. If $\alpha$ falls between these two extremes, e.g., $\alpha = 2$, the only known previous result is the trivial $O(n)$-approximation algorithm. On the hardness side, it has been proved that it is impossible to approximate the CDR-$\alpha$ problem within a factor of $\rho \log \delta$ ($0 < \delta < 1$) for $\alpha = 1$ [5] and $\alpha \geq 2$ [8][10], unless $NP \subseteq DTIME(n^{\log \log n})$, where $\delta$ is the maximum degree of $G$. 

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Open Question 1 (Du and Wan [8]). Is there a polynomial-time $O(\log n)$-approximation for the CDR-$\alpha$ problem for $\alpha \geq 2$?

Unit disk graphs (UDG): Most of the studies on the CDR-$\alpha$ problem focus on UDGs [5,8,10,11,19]. UDG exhibits many nice properties that enables constant factor approximation algorithms (or PTAS) in many problems where only $O(\log n)$-approximation algorithms (or worse) are known in general graphs, e.g., the minimum (connected) dominating set problem and the maximum independent set problem [3,4,15,20,24]. All the previous researches on the CDR-$\alpha$ problem on UDGs leverage constant bounds of minimum independent sets or maximum dominating sets. However, all the previous research can only solve the case where $\alpha \geq 5$. The best result so far is a PTAS for the CDR-$\alpha$ problem on UDGs [11]. For the case where $1 < \alpha < 5$, the only known previous result is the trivial $O(n)$-approximation algorithm.

Next, we give the basic idea of solving the 1-DR-$\alpha$ problem ($\alpha \geq 5$) used in previous researches. One of our algorithms still uses this idea. First, find a dominating set $D$. Thus, for any target couple $[u,v]$, there exists $u'$ and $v'$ in $D$, such that $u'$ and $v'$ dominate $u$ and $v$, respectively. Let $D' = D$. Second, we add more vertices to $D'$. Specifically, for any two vertices $u'$ and $v'$ in $D$, if $m(u',v') \leq 3$, we connect $u'$ and $v'$ by a shortest path in $G$ (and add the internal vertices to $D'$). Observe that $m(u',v') \leq 3$. Hence, after connecting $u'$ and $v'$ be a shortest path in $G$, we have $m^{D'}(u,v) \leq 5$. We state the result formally in the following lemma.

Lemma 1. Let $D$ be a dominating set of $G$. Let $D' \supseteq D$ be a subset of $G$ such that, for any two vertices $u'$ and $v'$ in $D$, if $m(u',v') \leq 3$, then $m^{D'}(u,v') \leq 3$. Then, $D'$ is a feasible solution of the 1-DR-$\alpha$ problem with input $G$ and $\alpha \geq 5$.

1.4 Our Result and Basic Ideas

In this paper, we first give an approximation algorithm of the 1-DR-$\alpha$ problem on general graphs for the case $\alpha > 1$. A critical observation is that the 1-DR-2 problem is a special case of the Set Cover with Pairs (SCP) problem [13]. Hassin and Segev proposed an $O(\sqrt{t \log t})$-approximation algorithm for the SCP problem, where $t$ is the number of targets to be covered. However, since $t = O(n^2)$ in the 1-DR-2 problem, directly applying the $O(\sqrt{t \log t})$-approximation bound yields a trivial upper bound for the 1-DR-2 problem. We re-examine the analysis in [13] and find that, when applying the algorithm to the 1-DR-2 problem, the approximation ratio can also be expressed as $O(\sqrt{n \log n})$. Nevertheless, in this paper, we give a slightly simplified algorithm with an easier analysis for the SCP problem. We also consider the generalized SCP problem and obtain the following result, which is the first non-trivial result for $\alpha > 1$ in general graphs and for $1 < \alpha < 5$ in UDGs.

Theorem 2. When $\alpha > 1$, there exists an $O(n^{1-\frac{1}{2\alpha}}(\log n)^{\frac{1}{\alpha}})$-approximation algorithm for the 1-DR-$\alpha$ problem.

Apparently, the above performance guarantee deteriorates quickly as $\alpha$ increases. In our second algorithm, we apply the aforementioned idea of finding a feasible solution when $\alpha \geq 5$, i.e., Lemma 1. We have the following result.

Theorem 3. When $\alpha \geq 5$, there exists an $O(\sqrt{n \log n})$-approximation algorithm for the 1-DR-$\alpha$ problem.

Finally, we answer Open Question 1 negatively. We improve upon the $\Omega(\log n)$ hardness result for the 1-DR-2 problem (albeit under a stronger hardness assumption) [8,10]. In this paper, we give a reduction from the MIN-REP problem [16]. The hardness result is stated as follows.
1.5 Other Related Work

When we ignore the constraint that any feasible solution must be a connected dominating set, the CDR-α problem is similar to the basic k-spanner problem. For completeness, we give the formal definition of the basic k-spanner problem. Given a graph $G = (V, E)$, a k-spanner is a subgraph $H$ of $G$ such that $d_H(u,v) \leq kd_G(u,v)$ for all $u$ and $v$ in $V$, where $d_G(u,v)$ is the number of edges in the shortest path between $u$ and $v$ in $G$. The basic k-spanner problem asks for the k-spanner that has the fewest number of edges. The CDR-α problem differs with the basic k-spanner problem in the following three aspects: First, in the CDR-α problem, we find a set of vertices $D$, and all the edges in the subgraph induced by $D$ can be used for routing; while in the basic k-spanner problem, only edges in $H$ can be used. Second, in the CDR-α problem, the objective is to minimize the number of chosen vertices; while in the basic k-spanner problem, the objective is to minimize the number of chosen edges. Finally, in the basic k-spanner problem, the distance is measured by the number of edges; while in the CDR-α problem, the distance is measured by the number of internal nodes. Despite the above differences, these two problems share similar approximability and hardness results. Althöfer et al. proved that every graph has a k-spanner of at most $n^{1 + \frac{1}{\lceil (k+1)/2 \rceil}}$ edges, and such a k-spanner can be constructed in polynomial time \[1\]. Since the number of edges in any k-spanner is at least $n - 1$, this yields an $O(n^{\frac{1}{\lceil (k+1)/2 \rceil}})$-approximation algorithm for the basic k-spanner problem. For $k = 2$, there is an $O(\log n)$-approximation algorithm due to Kortsarz and Peleg \[17\], and this is the best possible \[16\]. For $k = 3$, Berman et al. proposed an $O(n^{1/3})$-approximation algorithm \[2\]. For $k = 4$, Dinitz and Zhang proposed an $O(n^{1/3})$-approximation algorithm \[7\]. On the hardness side, it has been proven that for any constant $\epsilon > 0$ and for $3 \leq k \leq \log^{1-\epsilon} n$, unless $NP \subseteq BPTIME(2^{poly\log n})$, there is no polynomial-time algorithm that approximates the basic k-spanner problem to a factor better than $2^{(\log^{1-\epsilon} n)/k}$ \[6\].

2 Two Algorithms for the 1-DR-α Problem

2.1 The First Algorithm

In this subsection, we first give the formal definition of the Set Cover with Pairs (SCP) problem. We give an approximation algorithm for the SCP problem in Section 2.1.1. A transformation from the 1-DR-2 problem to the SCP problem is given in Section 2.1.2. To solve the 1-DR-2 problem, we generalize the SCP problem and propose the Set Cover with α-Tuples (SCT) problem in Section 2.1.3. An approximation algorithm for the SCT problem and a transformation from the 1-DR-α problem to the SCT problem are given in Section 2.1.4.

**Definition 3.** Let $T$ be a set of $t$ targets. Let $V$ be a set of $n$ elements. For every two elements $v_i$ and $v_j$ in $V$ ($v_i$ and $v_j$ may be the same element), $C(v_i, v_j)$ denotes the set of targets covered by the pair $\{v_i, v_j\}$. The Set Cover with Pairs (SCP) problem asks for the smallest subset $S$ of $V$ such that $\bigcup_{v_i, v_j \subseteq S} C(v_i, v_j) = T$.

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1 If the graph is triangle-free, then any two vertices with a common neighbor form a target couple.

2 One might want to drop the constraint that the solution must be a dominating set, and focus on minimizing the number of vertices to cover all the target couples. This theorem also applies to such a problem.
2.1.1 Approximating the SCP Problem

Our algorithm is a simple greedy algorithm: in each round, we choose two elements $u$ and $v$ that maximize the number of covered targets. Specifically, $S$ is an empty set initially. In each round, we select two elements $u$ and $v$ in $V \setminus S$ ($u$ and $v$ may be the same element, e.g., when there is only one element in $V \setminus S$), such that the pair $\{u, v\}$ increases the number of covered targets the most, i.e., $\{u, v\} = \arg\max_{u', v' \in V \setminus S, u' \neq v'} g(\{u', v'\})$, where

$$g(\{u', v'\}) = | \bigcup_{\{v_j \} \subseteq S \cup \{u', v'\}} C(v_j, v) | - | \bigcup_{\{v_j \} \subseteq S} C(v_j, v) |.$$ 

We then add $u$ and $v$ to $S$ and repeat the above process until all the targets are covered.\footnote{In the algorithm proposed by Hassin and Segev, $g(\{u', v'\}) \leftarrow \frac{g'(u', v')}{|\{u', v'\}|}$, where $|\{u', v'\}| = 1$ if $u' = v'$.}

**Theorem 5.** The above algorithm is an $O(\sqrt{n \log t})$-approximation algorithm for the SCP problem.

**Proof.** Let $R_i$ be the number of uncovered elements after round $i$. In the first round, some elements $u$ and $v$ in the optimal solution can cover at least $t/\binom{OPT}{2}$ targets, where $OPT$ is the number of elements in the optimum. Since we choose two vertices greedily in each round, $R_1 \leq t(1 - 1/(\binom{OPT}{2}))$. In the second round, there exist two elements in the optimal solution that can cover at least $R_1/(\binom{OPT}{2})$ targets among the $R_1$ uncovered targets. Again, we choose the two vertices that can increase the number of covered targets the most. Hence, $R_2 \leq R_1 - R_1/(\binom{OPT}{2}) \leq t(1 - 1/(\binom{OPT}{2}))^2$. In general, $R_i \leq t(1 - 1/(\binom{OPT}{2}))^i$. After $r = \binom{OPT}{2} \ln t$ rounds, the number of uncovered elements is at most $t(1 - 1/(\binom{OPT}{2}))^r \leq t(e^{-1/(\binom{OPT}{2})})^r \leq te^{-\ln t} = 1$. Hence, after $OPT^2 \ln t$ rounds, all targets are covered. Let $ALG$ be the number of elements chosen by the algorithm. Since we choose at most two elements in each round, $ALG \leq 2OPT^2 \ln t$. Finally, since $ALG \leq n$, $ALG \leq \sqrt{n \cdot 2OPT^2 \ln t} = \sqrt{2n \ln t/OPT}$. \hfill \Box

Note that, in Theorem 5 we can replace $n$ with any upper bound of solutions obtained by any polynomial time algorithm $A$ for the SCP problem. This is achieved by executing both $A$ and our algorithm. Choosing the best between the two outputs yields the desired approximation ratio. An example is replacing $n$ with $2t$ \footnote{In the algorithm proposed by Hassin and Segev, $g(\{u', v'\}) \leftarrow \frac{g'(u', v')}{|\{u', v'\}|}$, where $|\{u', v'\}| = 1$ if $u' = v'$.}.

2.1.2 Approximating the 1-DR-2 Problem

To transform the 1-DR-2 problem to the SCP problem, we treat each target couple and each vertex (to form a dominating set) as a target. The set of elements $V$ in the SCP problem is the vertex set of $G$. A target couple $[u, v]$ can be covered by a pair of vertices $W = \{w_1, w_2\}$ if $m^W(u, v) \leq 2$. Finally, a target vertex can be covered by itself or any of its neighbors in $G$. It is easy to see that the SCP problem instance is equivalent to the 1-DR-2 problem instance. Note that in this SCP instance, $n = n(G)$ and $t = O(n(G)^2)$. Hence, we obtain the following result.

**Theorem 6.** There exists an $O(\sqrt{n \log n})$-approximation algorithm for the 1-DR-2 SCP problem.

2.1.3 The Set Cover with $\alpha$-Tuples (SCT) Problem

In the 1-DR-2 problem, a target couple is covered by at most two vertices. In the 1-DR-$\alpha$ problem, a target couple is covered by at most $\alpha$ vertices. Hence, we consider the following generalization of the SCP problem.
Definition 4. Let $T$ be a set of $t$ targets. Let $V$ be a set of $n$ elements. Let $\alpha$ be a positive integer greater than one. For every $\alpha$ elements $v_1, v_2, \ldots, v_\alpha$ in $V$ (not necessarily distinct), $C(v_1, v_2, \ldots, v_\alpha)$ denotes the set of targets covered by the $\alpha$-tuple $\{v_1, v_2, \ldots, v_\alpha\}$. The Set Cover with $\alpha$-Tuples (SCT) problem asks for the smallest subset $S$ of $V$ such that $\bigcup_{\{v_1, v_2, \ldots, v_\alpha\} \subseteq S} C(v_1, v_2, \ldots, v_\alpha) = T$.

2.1.4 Approximating the SCT Problem and the 1-DR-$\alpha$ Problem

The algorithm for the SCT problem is a straightforward generalization of the algorithm for the SCP problem. The only difference is that, in each round, we choose the $\alpha$ elements that can increase the number of covered targets the most. The transformation from the 1-DR-$\alpha$ Problem to the SCT problem is also similar to the previous transformation. The only difference is that a target couple $[u, v]$ can be covered by an $\alpha$-Tuple $W = \{w_1, w_2, \ldots, w_\alpha\}$ if $m_W(u, v) \leq \alpha$. Hence, Theorem 2 is a direct result of the following theorem.

Theorem 7. There exists an $O(n^{1-\frac{1}{\alpha}} \cdot (\ln t)^{\frac{1}{\alpha}})$-approximation algorithm for the SCT problem.

We have the following claim.

Claim 1. When $c = \frac{1}{\alpha} - \frac{\ln \ln(t^\alpha)}{\alpha \ln n}$, $n^{1-c} = \sqrt{n \cdot \alpha} (n^c)^{\alpha-2} \ln t = n^{1-\frac{1}{\alpha}} \cdot (\alpha \ln t)^{\frac{1}{\alpha}}$.

Proof of Theorem 7. Let $R_i$ be the number of uncovered elements after round $i$. By a similar argument in the proof of Theorem 2, we get that $R_i \leq t (1 - 1/(OPT^\alpha))^i$. After $r = (OPT^\alpha) \ln t$ rounds, the number of uncovered elements is at most one. Hence, after $OPT^\alpha \ln t$ rounds, all targets are covered. Let $ALG$ be the number of elements chosen by the algorithm. Since we choose at most $\alpha$ elements in each round, $ALG \leq \alpha OPT^\alpha \ln t$. Since $ALG \leq n$, $ALG \leq \sqrt{n \cdot \alpha OPT^\alpha \ln t}$.

Let $c = \frac{1}{\alpha} - \frac{\ln \ln(t^\alpha)}{\alpha \ln n}$. When $OPT \geq n^c$, the approximation ratio is $n^{1-c}$. When $OPT \leq n^c$, $ALG \leq \sqrt{n \cdot \alpha OPT^\alpha - 2} \ln t OPT \leq \sqrt{n \cdot \alpha} (n^c)^{\alpha-2} \ln tOPT$. The proof then follows from Claim 1 and $\alpha^{\frac{1}{\alpha}} = O(1)$.

Proof of Claim 1:

$$n^{1-c} = \sqrt{n \cdot \alpha} (n^c)^{\alpha-2} \ln t$$
$$\Leftrightarrow n^{2-2c} = n \cdot \alpha (n^c)^{\alpha-2} \ln t \text{ (both sides are non-negative)}$$
$$\Leftrightarrow n^{2-2c} = (1 + c(\alpha-2)) = \alpha \ln t$$
$$\Leftrightarrow n^{1-c} = \alpha \ln t.$$

When $c = \frac{1}{\alpha} - \frac{\ln \ln(t^\alpha)}{\alpha \ln n}$,

$$n^{1-c} = n^{1 - \left(\frac{\ln \ln(t^\alpha)}{\ln n}\right)} = n^{\frac{\ln \ln(t^\alpha)}{\ln n}} = (n^{\ln \ln(t^\alpha)})^{\frac{1}{\ln n}} = \left((\ln(t^\alpha))^{\ln n}\right)^{\frac{1}{\ln n}} = \left((\alpha \ln t)^{\ln n}\right)^{\frac{1}{\ln n}} = \alpha \ln t.$$

Hence, when $c = \frac{1}{\alpha} - \frac{\ln \ln(t^\alpha)}{\alpha \ln n}$, $n^{1-c} = \sqrt{n \cdot \alpha} (n^c)^{\alpha-2} \ln t$. 

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Finally, when \( c = \frac{1}{\alpha} - \frac{\ln \ln(t^\alpha)}{\alpha \ln n} \),
\[
    n^{1-c} = n^{1-\frac{1}{\alpha} + \frac{\ln \ln(t^\alpha)}{\alpha \ln n}}
    = n^{1-\frac{1}{\alpha} \cdot n^{\frac{\ln \ln(t^\alpha)}{\alpha \ln n}}}
    = n^{1-\frac{1}{\alpha} \cdot (n^{\frac{\ln \ln(t^\alpha)}{\alpha \ln n}})^{\frac{1}{\alpha}}}
    = n^{1-\frac{1}{\alpha} \cdot (\alpha \ln t)^{\frac{1}{t}}}. 
\]

In the last equality, we reuse Eq. (1)–Eq. (4). \( \square \)

2.2 The Second Algorithm

Our second algorithm is designed for the 1-DR-\( \alpha \) problem when \( \alpha \geq 5 \). It has a better approximation ratio than that of the previous algorithm when \( \alpha \geq 5 \). The algorithm is suggested in Lemma 1.

We first find a dominating set \( D \) by any \( O(\log n) \)-approximation algorithm. Let \( D' = D \). For any two vertices \( u \) and \( v \) in \( D \), if \( m(u, v) \leq 3 \), we then add at most three vertices to \( D' \) so that \( m_{D'}(u, v) \leq 3 \).

**Proof of Theorem 3** Let \( OPT_{DS} \) be the size of the minimum dominating set in \( G \). Let \( OPT \) be the size of the optimum of the 1-DR-\( \alpha \) problem. Since any feasible solution of the 1-DR-\( \alpha \) problem must be a dominating set, \( OPT_{DS} \leq OPT \). \( |D'| \leq 3^{(\frac{|D|}{3})} = O((\log(n)OPT_{DS})^2) = O((\log(n)OPT)^2) \). Since \( |D'| \leq n \), we have, \( |D'| = O(\sqrt{n} \cdot (\log(n)OPT)^2) = O(\sqrt{n} \log n)OPT \). \( \square \)

3 Inapproximability Result

3.1 The MIN-REP Problem

We prove Theorem 3 by a reduction from the MIN-REP problem [16]. The input of the MIN-REP problem consists of a balanced bipartite graph \( G = (X, Y, E) \) and partitions of \( X \) and \( Y \), \( \mathcal{P}_X = \{X_1, X_2, \ldots, X_k\} \) and \( \mathcal{P}_Y = \{Y_1, Y_2, \ldots, Y_k\} \), such that \( X = \bigcup_{i=1}^{k} X_i \), \( Y = \bigcup_{i=1}^{k} Y_i \), and \( |X_i| = |Y_j| = \frac{|X|}{k} = \frac{|Y|}{k} \). We can view \( X_1, X_2, \ldots, X_k \) and \( Y_1, Y_2, \ldots, Y_k \) as super nodes, and we say that two super nodes \( X_i \) and \( Y_j \) are adjacent if some vertex in \( X_i \) is adjacent to some vertex in \( Y_j \). In this case, \( X_i \) and \( Y_j \) form a super edge. In the MIN-REP problem, our task is to choose representatives for super nodes so that if \( X_i \) and \( Y_j \) form a super edge, then some representative for \( X_i \) is adjacent to some representative for \( Y_j \). Note that a super node may have multiple representatives. Specifically, the goal of the MIN-REP problem is to find a smallest subset \( R \subseteq X \cup Y \) such that if \( X_i \) and \( Y_j \) form a super edge, then \( R \) must contain two vertices \( x \) and \( y \) such that \( x \in X_i \), \( y \in Y_j \) and \( (x, y) \in E \). In this case, we say that \( \{x, y\} \) covers the super edge \((X_i, Y_j)\).

We say that a MIN-REP problem instance is an YES instance if there is a solution of size \( 2k \), and a MIN-REP instance is a NO instance if every solution has size at least \( 2k \cdot 2^{\log^{1-\epsilon} n} \), where \( n \) is the number of vertices in the input graph of the MIN-REP problem, i.e., \( |X| + |Y| \). The inapproximability result of the MIN-REP problem is stated as the following theorem.

**Theorem 8** (Kortsarz [16]). Unless \( NP \subseteq DTIME(n^{\text{poly} \log n}) \), for any constant \( \epsilon > 0 \), there is no polynomial time algorithm that can distinguish between YES and NO instances of the MIN-REP problem. Hence, unless \( NP \subseteq DTIME(n^{\text{poly} \log n}) \), the MIN-REP problem admits no polynomial-time \( 2^{\log^{1-\epsilon} n} \)-approximation algorithm, for any constant \( \epsilon > 0 \).
3.2 The Reduction

Given inputs $G = (X, Y, E)$, $P_X$ and $P_Y$ of the MIN-REP problem, we construct a corresponding graph $G'(G, P_X, P_Y)$ of the 1-DR-2 problem. When $G, P_X$, and $P_Y$ are clear from the context, we simply write $G'$ instead of $G'(G, P_X, P_Y)$. Initially, $G' = G$. Hence, $G'$ contains $X$ and $Y$. For each super node $X_i$ (respectively, $Y_i$), we create two corresponding vertices $px_i^1$ and $px_i^2$ (respectively, $py_i^1$ and $py_i^2$) in $G'$. If $x$ is in super node $X_i$ (respectively, $y$ is in super node $Y_i$), then we add two edges $(x, px_i^1)$ and $(x, px_i^2)$ (respectively, $(y, py_i^1)$ and $(y, py_i^2)$) in $G'$. If $X_i$ and $Y_j$ form a super edge, then we add two vertices $i_{i,j}^1$ and $i_{i,j}^2$ to $G'$, and we add four edges $(px_i^1, i_{i,j}^1), (i_{i,j}^1, py_j^1), (px_i^2, i_{i,j}^2), (i_{i,j}^2, py_j^2)$ to $G'$. $i_{i,j}^1$ (respectively, $i_{i,j}^2$) is called the relay of $px_i^1$ and $py_j^1$ (respectively, $px_i^2$ and $py_j^2$).

Before we complete the construction of $G'$, we briefly explain the idea behind the construction so far. If two super nodes $X_i$ and $Y_j$ form a super edge, then $px_i^1$ and $py_j^1$ ($I \in \{1, 2\}$) have a common neighbor, i.e., the relay $i_{i,j}^1$ in $G'$. Hence, $px_i^1$ and $py_j^1$ form a target couple. To transform a solution $D$ of the 1-DR-2 problem to a solution of the MIN-REP problem, we need to transform $D$ to another feasible solution $D'$ for the 1-DR-2 problem so that none of the relays is chosen, and only vertices in $X \cup Y$ are used to connect $px_i^1$ and $py_j^1$. This is the reason that we have two corresponding vertices for each super nodes (and thus two relays for each super edge). Under this setting, to connect $px_i^1$ to $py_j^1$ and $px_i^2$ to $py_j^2$, choosing two vertices in $X \cup Y$ is no worse than choosing the relays.

Let $PX = \{px_1^1, px_2^1, \ldots, px_k^1\} \cup \{px_1^2, px_2^2, \ldots, px_k^2\}$ be the set of vertices in $G'$ corresponding to super nodes in $X$. Similarly, let $PY = \{py_1^1, py_2^1, \ldots, py_k^1\} \cup \{py_1^2, py_2^2, \ldots, py_k^2\}$. Let $R$ be the sets of all relays. To complete the construction, we add four vertices (hubs) $h_{X.R}, h_{Y.R}, h_{P.X},$ and $h_{P.Y}$ to $G'$. In $G'$, all the vertices in $X, Y, PX,$ and $PY$ are adjacent to $h_{X.R}, h_{Y.R}, h_{P.X},$ and $h_{P.Y}$, respectively. Moreover, every relay is adjacent to $h_{X.R}$ and $h_{Y.R}$. These four hubs induce a 4-cycle $(h_{P.X}, h_{Y.R}, h_{P.Y}, h_{X.R}, h_{P.X})$ in $G'$. Finally, for each hub $h$, we create two dummy nodes $d_1$ and $d_2$, and add two edges $(h, d_1)$ and $(h, d_2)$ to $G'$. This completes the construction of $G'$. Fig.1 shows an example of the reduction. Let $H$ and $M$ be the set of hubs and dummy nodes, respectively. Hence, the vertex set of $G'$ is $X \cup Y \cup PX \cup PY \cup R \cup H \cup M$. We have the following facts about $G'$.

**Lemma 2.** $n(G') = O(n(G)^2)$.

**Proof.** Since $|PX| + |PY| = O(n(G))$, $|R| = O(n(G)^2)$, and $|H| + |M| = O(1)$, we have $n(G') = O(n(G)^2)$.

Let $N(u)$ be the set of neighbors of $u$ in $G'$. We then have

- $N(px) \subseteq X \cup R \cup \{h_{P.X}\}$ if $px \in PX$.
- $N(py) \subseteq Y \cup R \cup \{h_{P.Y}\}$ if $py \in PY$.
- $N(x) \subseteq PX \cup Y \cup \{h_{X.R}\}$ if $x \in X$.
- $N(y) \subseteq PY \cup X \cup \{h_{Y.R}\}$ if $y \in Y$.
- $N(r) \subseteq PX \cup PY \cup \{h_{X.R}, h_{Y.R}\}$ if $r \in R$.
- $N(h_{X.R}) \setminus M = X \cup R \cup \{h_{P.X}, h_{P.Y}\}$.
- $N(h_{Y.R}) \setminus M = Y \cup R \cup \{h_{P.X}, h_{P.Y}\}$.
- $N(h_{P.X}) \setminus M = PX \cup \{h_{X.R}, h_{Y.R}\}$.
- $N(h_{P.Y}) \setminus M = PY \cup \{h_{X.R}, h_{Y.R}\}$.
- $N(m) \subseteq H$ if $m \in M$. 

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Lemma 3. $G'$ is triangle-free.

Proof. It is easy to check that, for any two adjacent vertices $u$ and $v$ in $G'$, $u$ and $v$ have no common neighbor. \qed

We say that $[a, b]$ is in $[A, B]$ if $a \in A$ and $b \in B$.

Lemma 4. $H$ must be chosen to cover all the target couples in $[M, M]$.

Lemma 5. $H$ covers all the target couples except those in $PX, PY$.

Proof. If $[u, v]$ is in $[PX, PX \cup \{h_{X,R}, h_{Y,R}\}]$, $[PY, PY \cup \{h_{X,R}, h_{Y,R}\}]$, $[X, X \cup R \cup \{h_{X,R}, h_{Y,R}\}]$, $[Y, Y \cup R \cup \{h_{X,R}, h_{Y,R}\}]$, $[R, R \cup \{h_{X,R}, h_{Y,R}\}]$, or $[X, Y \cup \{h_{X,R}, h_{Y,R}\}]$, then $[u, v]$ can be covered by one vertex in $H$. If $[u, v]$ is in $[PX, X \cup R \cup \{h_{X,R}, h_{Y,R}\}]$, $[PY, Y \cup R \cup \{h_{X,R}, h_{Y,R}\}]$, or $[X, Y \cup \{h_{X,R}, h_{Y,R}\}]$, then $[u, v]$ can be covered by an edge in $H$. If $[u, v]$ is in $[PX, X \cup R \cup \{h_{X,R}, h_{Y,R}\}]$, $[PY, Y \cup R \cup \{h_{X,R}, h_{Y,R}\}]$, or $[X, Y \cup \{h_{X,R}, h_{Y,R}\}]$, then $[u, v]$ cannot be a target couple (since $u$ and $v$ do not have a common neighbor). If $[u, v]$ is in $[X, \{h_{X,R}\}]$, $[Y, \{h_{Y,R}\}]$, or $[R, \{h_{X,R}, h_{Y,R}\}]$, then $[u, v]$ cannot be a target couple (since $u$ and $v$ are adjacent). Moreover, it is easy to see that $H$ covers all the target couples in $[V(G'), M]$, where $V(G')$ is the vertex set of $G'$. Finally, note that if $[u, v]$ is in $[PX, PY]$, then $H$ cannot cover $[u, v]$. \qed

Let $px$ and $py$ be vertices in $PX$ and $PY$, respectively. Observe that, if $(px, x, y, py)$ is a path in $G'$, then $x \in X$ and $y \in Y$. We then have the following lemma.

Lemma 6. $D$ covers target couples $[px_i^1, py_i^1]$ and $[px_i^2, py_i^2]$ if and only if at least one of the following conditions is satisfied.

1. There exist $x \in X$ and $y \in Y$ such that $(px_i^1, x, y, py_i^1)$ is a path in $G'$ ($i \in \{1, 2\}$) and \( \{x, y\} \subseteq D \).

2. $\{r_{i,j}^1, r_{i,j}^2\} \subseteq D$.

\(^4\)In addition, by Lemma 3, $u$ and $v$ do not have a common neighbor.
3.3 The Analysis

Let $I_{MR}$ be an instance of the MIN-REP problem with inputs $G$, $P_X$, and $P_Y$. Let $I_D$ be the instance of the 1-DR-2 problem with input $G'(G, P_X, P_Y)$. To prove the inapproximability result, we use the following two lemmas.

**Lemma 7.** If $I_{MR}$ has a solution of size $s$, then $I_D$ has a solution of size $s + 4$.

**Lemma 8.** If every solution of $I_{MR}$ has size at least $s \cdot 2^{\log^{1-\epsilon} n(G)}$, then every solution of $I_D$ has size at least $s \cdot 2^{\log^{1-\epsilon} n(G)} + 4$.

**Proof of Theorem 4:** By the above two lemmas and Theorem 8, we get that, unless $NP \subseteq DTIME(n^{polylog n})$, the 1-DR-2 admits no $\frac{2k}{2k + 4} \cdot 2^{\log^{1-\epsilon} n(G)}$-approximation algorithm for any constant $\epsilon > 0$. Lemma 2 implies that there is no $2^{\log^{1-\epsilon} n(G) + o(1)}$-approximation algorithm for the 1-DR-2 problem, where $c_1 > 0$ and $c_2 > 0$ are constants. By considering sufficiently large instances and a smaller value of $\epsilon$, we have the hardness result claimed in Theorem 4. On the other hand, by Lemma 4 even if we drop the constraint that any feasible solution must be a dominating set and only consider the problem of covering target couples, the inapproximability result still holds. Finally, the proof follows from Lemma 3.

**Lemma 7** is a direct result of the following claim.

**Claim 2.** Let $S$ be any feasible solution of $I_{MR}$. $S \cup H$ is a feasible solution of $I_D$.

**Proof.** Since $H$ is a dominating set, by Lemma 5, it suffices to prove that every target couple $[u, v] = [px^1, py^2]$ in $[PX, PY]$ is covered by $S$. Note that $[px^1, py^2]$ cannot be a target couple if $I_1 \neq I_2$. This is because $px^1$ and $py^2$ do not have a common neighbor if $I_1 \neq I_2$. If $I_1 = I_2$, then the common neighbor must be $r_{ij}$. By the construction of $G'$, this implies that $X_i$ and $Y_j$ form a super edge. Since $S$ is a feasible solution of $I_{MR}$, there exists $x \in X_i$ and $y \in Y_j$ such that $(x, y) \in G$ and $\{x, y\} \subseteq S$. Again by the construction of $G'$, $(u, x, y, v)$ is a path in $G'$. Hence, $S \supseteq \{x, y\}$ covers $[u, v]$.

To prove Lemma 8 we use the following claim.

**Claim 3.** There is an optimal solution $D^*$ of $I_D$, such that $D^* \setminus H$ is a feasible solution of $I_{MR}$.

**Proof of Lemma 8.** Let $S^*$ be the optimal solution of $I_{MR}$. By the assumption, we have $|S^*| \geq s \cdot 2^{\log^{1-\epsilon} n}$. It suffices to prove that $S^* \cup H$ is an optimal solution for $I_D$, which implies that every feasible solution of $I_D$ has size at least $\log^{1-\epsilon} n + 4 \geq s \cdot 2^{\log^{1-\epsilon} n} + 4$. The feasibility of $S^* \cup H$ follows from Claim 2. For the sake of contradiction, assume that the optimal solution of $I_D$ has size smaller than $|S^* \cup H| = |S^*| + 4$. Claim 3 and Lemma 4 then lead to a contradiction.

**Proof of Claim 3.** Let $D_{OPT}$ be any optimal solution of $I_D$. By Lemmas 4 and 8, $D_{OPT}$ only needs to cover every target couple $[u, v]$ in $[PX, PY]$. Since $[u, v]$ is a target couple, we can assume that $u = px^1$ and $v = py^2$. Hence, the only common neighbor of $u$ and $v$ is $r_{ij}$. By the construction of $G'$, this implies that there exist $x \in X_i$ and $y \in Y_j$ such that $(x, y, v)$ is a path in $G'$. By Lemma 6, $D_{OPT} \subseteq H \cup X \cup Y \cup R$; otherwise, we can remove vertices that are not in $H \cup X \cup Y \cup R$ from $D_{OPT}$. The resulting solution is still feasible and is smaller than $D_{OPT}$ (feasibility follows from the Lemmas 5 and 6).

If $D_{OPT} \cap R = \emptyset$, then (by Lemma 6) each target couple $[px^1, py^2]$ is covered by some $x \in X$ and some $y \in Y$. By the construction of $G'$, such $x$ and $y$ also cover the super edge $(X_i, Y_j)$ in $I_{MR}$. Hence, $D_{OPT} \setminus H$ is a feasible solution of $I_{MR}$.
If $D_{OPT} \cap R \neq \emptyset$, then some $r_{i,j}^1 \in D_{OPT}$. We can further assume that both $r_{i,j}^1$ and $r_{i,j}^2$ are in $D_{OPT}$; otherwise, by Lemma 6, we can remove $r_{i,j}^1$ from $D_{OPT}$, the resulting solution is smaller and is still feasible. Since both $r_{i,j}^1$ and $r_{i,j}^1$ are in $D_{OPT}$, we can replace $r_{i,j}^1$ and $r_{i,j}^2$ by some $x \in X$ and some $y \in Y$ satisfying the first condition in Lemma 6. The resulting solution still has size $|D_{OPT}|$. Moreover, the resulting solution is feasible. This is because, by Lemma 6, the target couples that are not covered by $D_{OPT} \setminus \{r_{i,j}^1, r_{i,j}^2\}$ can only be $[px_i^1, py_j^1]$ and $[px_i^2, py_j^2]$. Obviously, $[px_i^1, py_j^1]$ and $[px_i^2, py_j^2]$ can be covered by the resulting solution. Repeat the above replacing process until the resulting solution does not contain any relay. The proof then follows from the argument of the case where $D_{OPT} \cap R = \emptyset$. □

4 Reduction of the CDR-\(\alpha\) Problem to Other Related Problems

Submodular Cost Set Cover Problem: The CDR-\(\alpha\) problem can also be considered as a special case of the submodular cost set cover problem \[9,11,23\]. In the set cover problem, we are given a set of targets \(T\) and a set of objects \(S\). Each object in \(S\) can cover a subset of \(T\) (specified in the input). The goal is to choose the smallest subset of \(S\) that covers \(T\). In the submodular cost set cover problem, there is a non-negative submodular function \(c\) that maps each subset of \(S\) to a cost, and the goal is to find a set cover with the minimum cost. To transform the CDR-\(\alpha\) problem with input \(G = (V, E)\) to the submodular cost set cover problem, let \(T\) be the union of \(V\) and the set of all target couples, and let \(S\) be the set of all subsets of \(V\) with size at most \(\alpha\). An object \(S \in S\) can cover a vertex \(v\) if \(v\) is adjacent to some vertex in \(S\) or \(v \in S\). An object \(S \in S\) can cover a target couple \([u, v]\) if \(m^S(u, v) \leq \alpha\). The cost of a subset \(S'\) of \(S\) is simply the size of the union of objects in \(S'\), i.e., the number of distinct vertices in \(S'\).

Iwata and Nagano proposed a \(|T|\)-approximation algorithm and an \(f\)-approximation algorithm, where \(f\) is the maximum frequency, \(\arg \max_{T \in T} |\{S \in S|S\text{ covers }T\}|\) \[13\]. Koufogiannakis and Young also proposed an \(f\)-approximation algorithm when the cost function \(c\) is non-decreasing \[18\]. It is easy to see that these algorithms give trivial bounds for the CDR-\(\alpha\) problem. When \(c\) is integer-valued, non-decreasing, and satisfies \(c(\emptyset) = 0\), Wan et al. propose a \(\rho H(\gamma)\)-approximation algorithm, where \(\rho = \min_{S \in S^*} c(S^*)\) is an optimal solution \(\sum_{S \in S^*} c(S^*) / c(S)\), \(\gamma\) is the largest number of targets that can be covered by an object in \(S\), and \(H(k)\) is the \(k\)-th Harmonic number \[23\]. Du et al. proved that the value of \(\rho\) for the CDR-\(\alpha\) problem on UDGs is \(O(1)\) \[9\]. To the best of our knowledge, the only known upper bound of \(\rho\) for the CDR-\(\alpha\) problem on general graphs is the trivial bound \(O(n^{\alpha-1})\).

Minimum Rainbow Subgraph Problem on Multigraphs Finally, the 1-DR-2 problem can be transformed to the minimum rainbow subgraph problem on multigraphs: Given a set of \(p\) colors and a multigraph \(G = (V, E)\), where each edge is colored with one of the \(p\) colors, the minimum rainbow subgraph problem asks for the smallest subset \(D\) of \(V\), such that each of the \(p\) colors appears in some edge induced by \(D\). We can transform the 1-DR-2 problem with input \(G = (V, E)\) to the minimum rainbow subgraph problem with input \(G' = (V, E')\) as follows: Let \(T\) be the union of \(V\) and the set of all target couples. The set of colors is \(\{c_i| i \in T\}\). For each \(v \in V\), if \((u, v) \in E\), then we add a loop \((u, u)\) with color \(c_u\) to \(E'\). For each \(v \in V\), we also add a loop \((v, v)\) with color \(c_v\) to \(E'\). For each target couple \([u, v]\), if \(w\) is a common neighbor of \(u\) and \(v\), we add a loop \((w, w)\) with color \(c_{[u,v]}\) to \(E'\). Finally, for each target couple \([u, v]\), if \((u, w_1, w_2, v)\) is a path in \(G\), then we add an edge \((u, v)\) with color \(c_{[u,v]}\) to \(E'\). The only known approximation algorithm for this problem is by transforming it to the SCP problem. When the graph is simple, Tirodkar and Vishwanathan proposed an \(O(n^{1/3} \log n)\)-approximation algorithm \[22\].
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