Emergence of Quantum Correlations from Non-Localy Swapping

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By studying generalized non-signalling theories, the hope is to find out what makes quantum mechanics so special. In the present paper, we revisit the paradigmatic model of non-signalling boxes and introduce the concept of a genuine box. This will allow us to present the first generalized non-signalling model featuring quantum-like dynamics. In particular, we present the coupler, a device enabling non-locality swapping, the analogue of quantum entanglement swapping, as well as teleportation. Remarkably, part of the boundary between quantum and post-quantum correlations emerges in our study.

Quantum correlations cannot be ascribed to a local theory [1], as confirmed by all experiments performed to date [2]. However, Quantum Mechanics (QM) predicts an upper bound on the non-locality of allowed correlations, as shown by Tsirelson [3]. In trying to understand this bound Popescu and Rohrlich [4] asked whether it was a direct consequence of relativity — whether correlations more non-local would lead to signalling — and surprisingly found this not to be the case.

This discovery prompted the study of general models, containing more non-locality than QM, but still respecting the no-signaling principle [5]. The ultimate goal of this line of research is to find out what is special about QM; what distinguishes it from other non-signaling theories. Among the fundamental questions is the following: What physical principle limits quantum non-locality? This is still unknown today, but there is no doubt that answering this question will bring deeper understanding of the foundations of QM, as well as further developments in quantum information science.

Studying the information theoretic properties of generalizes non-signaling models has already provided insight to these questions [2, 7, 8, 9]. On the one hand, many astonishing features of QM, such as no-cloning, no broadcasting and monogamy of correlations, have been shown to be general properties of any non-signaling model [3, 10, 11]. Hence these properties do not indicate any separation between QM and post-quantum theories. On the other hand, van Dam [12] and Brassard et al. [13] showed that particular classes of post-quantum models allow for a dramatic increase of communication power compared to QM. Moreover, Linden et al. [14] showed that the same post-quantum theories allow for non-local computation while QM does not, here providing a tight separation between QM and post-quantum models.

More importantly however, there is one crucial aspect of QM that generalized models have failed to reproduce until now, namely its dynamics; in particular, the ability to perform joint measurements on two systems, which is the key ingredient for fascinating quantum processes such as teleportation [15] and entanglement swapping [16]. In fact, Short et al. [17] and Barrett [9] showed that there are no joint measurements in theories constrained only by no-signaling, thus suggesting the existence of another fundamental principle inherent to QM, that generalized models fail to capture.

Here we take a new conceptual perspective on generalized non-signalling models, which allows us to implement joint measurements. We revisit the paradigmatic model of Popescu-Rohrlich (PR) boxes [4] and introduce the concept of a genuine box. This allows us to present a model featuring rich dynamics, such as non-locality swapping, the analogue of quantum entanglement swapping, and teleportation. Joint measurements are implemented using an imaginary device called a coupler. Finally, and probably most surprisingly, we show that the set of quantum correlations partially emerges in our model.

Genuine boxes. As we shall work with generalized non-signaling theories, the quantum formalism is no longer relevant; here bipartite states are not given by vectors in a Hilbert space but by bipartite joint probability distributions; i.e. probabilities of a pair of results (outputs) given a pair of measurements (inputs). In other words, quantum correlations will be replaced by more general “boxes” (i.e. input-output devices).

Here we shall focus on the simplest possible scenario, namely the case of two possible measurements for each party (inputs \(x, y \in \{0, 1\}\)); each measurement providing a binary

![Diagram](https://via.placeholder.com/150)

**FIG. 1:** The set of allowed states is restricted to (i) the local polytope \(\mathcal{L}\) and (ii) the PR box. The other PR boxes are discarded since they are not genuine, and should therefore not be considered for non-locality swapping. The left axis is the CH value. Local states satisfy \(0 \leq \text{CH} \cdot P_{L} \leq 1\), the CH Bell inequality. The coupler (right axis) is a re-scaling of the CH value (see text). Note that the polytope is 8-dimensional; the figure is a 2-dimensional illustration.
The black box consists of a quantum system and measuring devices (see Fig. 2a). For the case of two polarized photons, we would require one polarizer on each side of the box, with two possible orientations, and detectors recording the measurement outcome and outputting the corresponding bit. Here, the quantum state is the genuine part of the box, i.e. the non-local resource; the measurement is then a processing. Indeed, by changing the orientation of the polarizers one can produce many different black-boxes starting from the same initial quantum state, but they are clearly not genuinely different. Moreover, it is also possible to produce the same black-box by using two different quantum states, subjected to appropriate measurements.

In the case of the PR box there are clearly no quantum states and polarizers in the box, and so it is more delicate to separate what is genuine in the box from what is not. Note that as long as we do not need to look inside the box, as is the case in most of the scenarios considered so far, we need make no distinction between genuine and non-genuine.

However, when dynamics are introduced in the model, things change. Let us first think of how a joint measurement would be implemented in the quantum case; importantly it is performed on quantum particles, and not on measurement circuitry (see Fig. 2b). In order to perform a joint measurement, one should first open the box, remove the circuitry, and connect the coupler directly to the genuine PR box.

Let us first re-examine the standard “black box” approach to quantum correlations, where they are stripped back to their purest form; measurement choices and outcomes are both reduced to single bits of information. It is instructive to think about how such a setup would in reality be produced.

FIG. 2: Genuine boxes. (a) A ‘quantum’ black-box contains a quantum state and measurement devices (polarizers, detectors). The orientation of the polarizers depends on the input values $x, y$. The genuine part of the box is the quantum state; before performing a joint measurement, one must remove the measurement devices. (b) A non-genuine PR box contains the genuine PR box and classical circuitry. Importantly, upon applying the coupler, one should first open the box, remove the circuitry, and connect the coupler directly to the genuine PR box.
a PR box to Alice and Charlie. With probability

the (reduced)

PR

box of Alice and Charlie must be independent of whether Bob

not signal by applying the coupler. Therefore the (reduced)

failure box

\[ P \]

is applied the coupler or not, i.e.

\[ P(ac|xz) = \sum_{b_1,b_2} P(ab_1|x_1)P(b_2c|y_2z) = \sum_{b'} P(ab'c|xz). \quad (4) \]

In case Bob shares a PR box with both Alice and Charlie, one has that \( P(ac|x) = \mathbb{1}(ac|x) = \frac{1}{2} \forall a,c,x,z, \) the fully mixed state. The requirement that \( P^f \) is an allowed box imposes a limit on the probability of success; here we make the optimal choice \( q = \frac{1}{3} \). Thus we have \( P^f(ac|x) = \frac{3}{2} (\mathbb{1}(ac|x) - \frac{1}{2}P^{PR}(ac|x)) \) and \( \text{CH} \cdot P^f = 0 \).

Next it must be checked that the coupler acts consistently when applied directly to any allowed box; not only when it is applied between two boxes. For example, the output probabilities must be positive when the coupler is connected to both ends of a single PR box. Here we just sketch the argument; the full proof can be be found in Appendix A. The proof is based on the following observation: if Bob, after applying the coupler, learns from Alice and Charlie their respective inputs and outputs, he should get the same result as if he learned Alice’s and Charlie’s inputs and outputs first and then applied the coupler. We find that the coupler outputs \( b' = 0 \) with a probability proportional to the CH value of the box it is applied to, i.e.

\[ P(b' = 0|P(ab|xy)) = \frac{2}{3} \text{CH} \cdot \bar{P}(ab|xy). \quad (5) \]

The constant of proportionality is here crucial, since it ensures that the coupler outputs with a valid probability when applied to any allowed box (see Fig. 1). Notably, upon applying the coupler directly to a PR box, one always obtains the outcome \( b' = 0 \). This is exactly what happens in the quantum case: when Bob holds a singlet and performs a joint measurement, he always projects onto \( |\psi^\mp \rangle \).

Note that the inconsistency of the potential coupler presented in Ref. [17] becomes now clear, since it output with a probability equal to the CH value; thus, when applied onto the PR box, it output with a non-valid probability of \( \frac{2}{3} \). Along the same line, it is also clear why our coupler runs into inconsistencies if we try to reintroduce disallowed (non-genuine) PR boxes; for instance the anti-PR box (given by \( a \oplus b \oplus 1 = xy \)) would lead to negative probabilities.

Finally to be consistent, the coupler must take any two genuine boxes to a genuine box. It is straightforward to check that this is the case, by applying the coupler to all pairs of vertices.

**Emergence of quantum correlations.** The coupler enables **perfect** swapping of two PR boxes; this means that the final state of Alice and Charlie, which is a PR box, is as non-local as the initial states shared by Alice-Bob and Bob-Charlie. Now a natural question to ask is whether Bob, by applying the coupler, can also swap non-locality starting from imperfect boxes. Here we consider a natural section of the polytope, which includes PR, PR$_2$ (another PR box given by \( a \oplus b = xy \oplus x \)), and the identity \( \mathbb{1} \). Thus we have noisy boxes of the form

![Image](image-url)
with \( \xi + \gamma \leq 1 \). Note that these boxes are genuine as long as \( \xi \leq \frac{1}{2} \) \cite{22}. Of particular interest are isotropic boxes \( P_{\xi,0} \), that lie on the line joining the PR box and the identity. One finds that \( \text{CH} \cdot P_{\xi,\gamma} = \xi + \frac{1}{2} \).

Using the linearity of the coupler one can check that when Bob succeeds in swapping non-locality (i.e. he gets \( b = 0 \)) starting from two \( P_{\xi,\gamma}^{PR} \) boxes, the final state of Alice and Charlie has CH value \( \text{CH} \cdot \tilde{P}(ac|xz) = \xi^2 + \gamma^2 + \frac{1}{2} \). Thus, the coupler enables perfect swapping only for noiseless PR boxes; two noisy boxes can only be swapped to an even noisier box.

Remarkably, non-locality can be swapped using two boxes \( P_{\xi,\gamma}^{PR} \) if and only if \( P_{\xi,\gamma}^{PR} \) is post-quantum; that is if \( \text{CH} \cdot P_{\xi,\gamma}^{PR} \) violates the Tsirelson-Landau-Masanes (TLM) inequality \cite{23,24}, a necessary and (here) sufficient condition for a box to be quantum. Thus, when the two initial boxes \( P_{\xi,\gamma}^{PR} \) are noisy enough to have been produced quantum mechanically, the resulting box shared by Alice and Charlie is so noisy as to become local. For isotropic boxes, this condition reduces to \( \text{CH} \cdot P_{\xi,\gamma}^{PR} > B_Q = \frac{1}{2} + \frac{1}{\sqrt{2}} \), where \( B_Q \) is the Tsirelson bound of the CH inequality.

**Proof.** Boxes \( P_{\xi,\gamma}^{PR} \) useless for non-locality swapping, i.e. leading to \( \text{CH} \cdot P_{\xi,\gamma}(ac|xz) \leq 1 \), are characterized by the relation

\[
\xi^2 + \gamma^2 \leq \frac{1}{2}.
\]

The TLM criteria is written here in the form of Landau \cite{24}

\[
|E_{00}E_{01} - E_{10}E_{11}| \leq \sqrt{(1 - E_{00}^2)(1 - E_{01}^2)} + \sqrt{(1 - E_{10}^2)(1 - E_{11}^2)}
\]

where \( E_{xy} = P(a = b|x,y) - P(a \neq b|x,y) \) is the correlator associated to the pair of measurements \( x,y \). For noisy states \( P_{\xi,\gamma}^{PR} \), the four correlators are given by \( E_{00} = E_{01} = \xi + \gamma \) and \( E_{10} = -E_{11} = \xi - \gamma \). Inserting these last expressions in \((8)\), we get exactly the relation \((7)\), which completes the proof.

Let us point out however that not the entire quantum versus post-quantum boundary emerges in this way: on other sections of the polytope the coupler ceases to swap non-locality before reaching the quantum bound.

**Conclusion and Perspectives.** In summary, we revisited the post-quantum model of PR boxes, introducing the concept of genuine boxes. This allowed us to consider a restricted space of non-signalling boxes; this space features much richer dynamics than the full non-signalling space. We presented the coupler, a device enabling non-locality swapping. The coupler also implements teleportation (see Appendix B). Even more surprisingly, quantum correlations partially emerged from the coupler. Though we do not understand its full significance at this stage, we believe this intimate connection is tantalizing, since it links a dynamical process in a natural non-signalling model directly to QM. In the future we plan to investigate further on this link, and look for a fundamental principle potentially underlying it. Studying other information theoretic tasks from the new perspective of genuine boxes may help us understand what is so special about QM.

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Appendix A: Deriving the action of the coupler on allowed states

Here we derive the action of the coupler on any allowed box (see Fig. 5). Since all consistent couplers are linear functions of the inputs and outputs of a box \[17\], it is sufficient here to consider only extremal boxes.

Let us start with the deterministic ones. Note first that the output of the coupler must be consistent regardless of the timings of Alice’s and Charlie’s inputs and of Bob’s application of the coupler. That is, if Bob, after applying the coupler, learns from Alice and Charlie their respective inputs and outputs, he should get the same result as if he learnt Alice’s and Charlie’s inputs and outputs of Alice’s and Charlie’s inputs and of Bob’s application of the coupler to the extremal boxes in the following ways:

\[ P(b' = 0|P_{\alpha\beta\gamma\delta}) = \begin{cases} \frac{2}{3} & \text{if } \alpha\gamma \oplus \beta \oplus \delta = 0 \\ 0 & \text{otherwise} \end{cases} \]  

This again can be understood geometrically – if the local box is on the facet \(CH = 1\) the coupler outputs \(b' = 0\) with probability \(\frac{2}{3}\), whilst if the box is on the facet \(CH = 0\) then it deterministically outputs \(b' = 1\).

Next, let us find the action of the coupler on the PR box. In order to do this, we decompose a given probability distribution in two different ways. We consider the point \(P(c)\) in the centre of the CH = 1 facet, half way between the PR box and the identity, which can be written as a convex combination of the extremal boxes in the following ways:

\[ P(c) = \frac{1}{8} \sum_{\alpha\beta\gamma\delta} P_{\alpha\beta\gamma\delta}(c_{\alpha\beta\gamma\delta}) (b_1 b_2 | y_1 y_2) \]

\[ = \frac{1}{2} \left( P^{\text{PR}}(b_1 b_2 | y_1 y_2) + \mathbb{I}(b_1 b_2 | y_1 y_2) \right) \]

Thus upon applying the coupler directly to a PR box, Bob always obtains the outcome \(b' = 0\); exactly as in the quantum case.

From inspection of equations \(9\) and \(12\) it is clear that the coupler outputs \(b' = 0\) with a probability that is proportional to the CH value of the box it is applied to, i.e.

\[ P(b' = 0) = \frac{2}{3} \text{CH} \cdot \tilde{P}(ab|xy) . \]  

This constant of proportionality ensures that the coupler outputs a valid probability when applied to any allowed box. Note that the inconsistency of the potential coupler presented in Ref. \[17\] becomes now clear, since it outputted with a probability equal to the CH value; therefore, when applied onto the PR box, it gave a non-valid probability of \(\frac{3}{4}\). Along the same line, it is also clear why the coupler runs into inconsistencies if we try to reintroduce the seven disallowed non-genuine PR boxes; for instance the anti-PR, defined by the relation \(a \oplus b \oplus 1 = xy\) would lead to negative probabilities.

Appendix B: Teleportation.

When Alice-Bob share a PR box, and Bob holds a deterministic box \(P_{\alpha\beta}\), the coupler implements the transformation

\[ P^{\text{PR}}(ab|xy) P_{\alpha\beta}(b_2|y_2) \xrightarrow{b' = 0} P_{\alpha\beta}(a|x) . \]

Therefore the final box held by Alice (given that the joint measurement succeeded) is \(P_{\alpha\beta}(a|x)\) (see Fig. 6). Thus, Bob can teleport to Alice any single-party box \(B(b|y) = \sum_{\alpha\beta} p_{\alpha\beta}P_{\alpha\beta}(b|y)\), with \(\sum_{\alpha\beta} p_{\alpha\beta} = 1\), which can be seen by using the linearity of the coupler. Here the PR provides the teleportation channel, as does the maximally entangled state in the quantum protocol.