Given a binary treatment and a binary mediator, mediation analysis decomposes the total effect of the treatment on an outcome variable into direct and indirect effects. However, the existing decompositions are “path-dependent”, and consequently, there appeared different versions of direct and indirect effects. Differently from these, this paper proposes a “path-free” decomposition of the total effect into three sub-effects: direct, indirect, and treatment-mediator interaction effects. Whereas the interaction effect has been part of the indirect effect in the existing two-effect decompositions, it is separately identified in our three-effect decomposition. All effects are found using conditional means, but not conditional densities, and are estimated with ordinary least squares estimators. Simulation and empirical analyses are provided as well.

Key Words: mediation, total effect, direct effect, indirect effect, interaction effect.

Compliance with ethical standard & no conflict of interest: no human or animal subject is involved in this research, and there is no conflict of interest to disclose.
1 Introduction

Given a binary treatment $D$, a binary mediator $M$ and an outcome/response variable $Y$, the causal chain of interest in mediation analysis is

$$
D \rightarrow M \rightarrow Y
$$

where the total effect of $D$ on $Y$ consists of the direct effect of $D$ on $Y$ and the indirect effect of $D$ on $Y$ through $M$. This is an important issue in various disciplines of science, as reviewed in MacKinnon et al. (2007), Pearl (2009), Imai et al. (2010), TenHave and Joffe (2012), Preacher (2015), VanderWeele (2015, 2016) and Nguyen et al. (2021).

Finding the total effect of $D$ on $Y$ can be done in various ways such as matching, regression adjustment, weighting, etc. Traditionally, decomposing the total effect has been done straightforwardly, using linear structural-form (SF) models for $Y$ as a function of $(D, M)$ and $M$ as a function of $D$ (unless the interaction term $DM$ appears in the $Y$ SF). However, this type of traditional approaches are model-dependent, and not exactly causal from the viewpoint of modern causal analysis. Once we leave linear SF’s for nonparametric approaches to avoid misspecifications while introducing potential variables for $(M, Y)$, decomposing the total effect is no more straightforward.

Consider two potential versions $M^d, d = 0, 1$, of $M$ corresponding to $D = 0, 1$, and the four potential responses $Y^{dm}$ for $D = d$ and $M = m$ with $d, m = 0, 1$. Also define the potential responses $Y_d, d = 0, 1$, corresponding to $D = 0, 1$ "when $M$ is allowed to take its natural course given $D = d$":

$$
Y_d \equiv Y^{d,M^d}.
$$

Then we have the ‘total effect’ $\tau$:

$$
total \text{ effect} : \tau \equiv Y_1 - Y_0 = Y^{1,M^1} - Y^{0,M^0}.
$$

The ‘natural direct effect’ of Pearl (2001) is ($\delta$ in $\delta(d)$ is from $d$ for ‘direct’):

$$
natural \text{ direct effect with } M^d : \delta(d) \equiv Y^{1,M^d} - Y^{0,M^d} \quad (1.1)
$$

$$
\implies \delta(0) \equiv Y^{1,M^0} - Y^{0,M^0}, \quad \delta(1) \equiv Y^{1,M^1} - Y^{0,M^1};
$$
Robins (2003) called $\delta(d)$ the ‘pure or total direct effect’. The ‘natural indirect effect’ of Pearl (2001) is ($\mu$ in $\mu(d)$ is from m in ‘mediator’):

$$
\text{natural indirect effect with } d : \mu(d) \equiv Y^{d,M^1} - Y^{d,M^0}
$$

$$
\implies \mu(0) \equiv Y^{0,M^1} - Y^{0,M^0}, \quad \mu(1) \equiv Y^{1,M^1} - Y^{1,M^0};
$$

Robins (2003) called $\mu(d)$ the ‘pure or total indirect’ effect.

These effect identification and estimation have been addressed by Pearl (2001), Robins (2003), Peterson (2006) and Tchetgen Tchetgen and Shpitser (2012, 2014), among others. Differently from the natural effects, however, central to our paper are

$$
\text{controlled direct effect with } m : Y^{1,m} - Y^{0,m},
$$

$$
\text{controlled mediator effect with } d : Y^{d,1} - Y^{d,0};
$$

the name ‘controlled mediator effect’ is adopted from TenHave and Joffe (2012).

The two well-known ways to decompose the total effect $\tau$ is

$$
\tau = Y^{1,M^1} - Y^{1,M^0} + Y^{1,M^0} - Y^{0,M^0} = \mu(1) + \delta(0),
$$

$$
\tau = Y^{1,M^1} - Y^{0,M^1} + Y^{0,M^1} - Y^{0,M^0} = \delta(1) + \mu(0).
$$

These two decompositions can be written succinctly as

$$
\tau = \mu(d) + \delta(1 - d) \quad \text{for } d = 0, 1,
$$

which is, however, ‘$d$- or path-dependent’. This is one problem, and another problem is that some effects are relative to $d = 0$ while some others are relative to $d = 1$; e.g., $\delta(0)$ in (1.4) is for the change in $D$ relative to the untreated mediator $M^0$, but $\mu(1)$ in (1.4) is for the change in $M$ relative to $d = 1$, not to $d = 0$. Hence, Pearl (2009, p. 132) even stated “the total effect $TE$ of a transition is equal to the difference between the direct effect of that transition and the indirect effect of the reverse transition”.

There are many generalizations of (1.6), and we could not cover possibly all of them here. Just to mention a few, effects of ‘stochastic interventions’ on $D$ or $M$ (i.e., ‘interventional effects’) appeared in VanderWeele et al. (2014), Lok (2016), Vansteelandt and Daniel (2017), VanderWeele and Tchetgen Tchetgen (2017), Diaz and Hejazi (2020),
Diaz et al. (2021) and Nguyen et al. (2021), among others. Instruments for $D$ were considered in mediation analysis, e.g., by Joffe et al. (2008), Fröhlich and Huber (2017) and Rudolph et al. (2021). Forastiere et al. (2018) adopted principal stratifications for mediation analysis, building on Rubin (2004), Jo and Stuart (2009) and Ding and Lu (2017), which are related to our approach because $M^0 \leq M^1$ will be invoked sometimes.

In (1.4) and (1.5), decomposing $\tau$ (or $E(\tau)$) into direct and indirect effects was done by subtracting and adding a “cross-world” potential outcome such as $Y^{1,M^0}$ or $Y^{0,M^1}$, and depending on which was used, different decompositions were obtained, which is a path-dependence. This is an important issue, and in a nutshell, the goal of this paper is to propose a path-independent or ‘path-free’ decomposition of $\tau$ or $E(\tau)$.

Our approach has two advantages compared with the existing approaches: the first is the aforementioned path independence, and the second is that we obtain a more informative three-effect decomposition, not two as in the existing approaches. The extra effect is the effect of the interaction term $DM$ on $Y$, which has been buried in the indirect effect in the existing decompositions. Our approach also has a limitation: only a single binary $D$ and a single binary $M$ are allowed, neither multi-valued nor multiple treatments or mediators. Nevertheless, binary $D$ and $M$ are building blocks for more general $D$ and $M$, to which our approach may get extended in the future.

In the remainder of this paper, Section 2 introduces our path-free three-effect decompositions. Section 3 explains how the effects in our decomposition can be identified. Section 4 examines simple linear SF’s for $M$ and $Y$ to exemplify what the effects actually look like. Section 5 introduces two estimators for our three-effect decompositions. Section 6 conducts a simulation study to show that the estimators work well, and then provides an empirical analysis. Finally, Section 7 concludes this paper.

2 Path-Free Decomposition of Total Effect

For our path-free decomposition, the first step is rewriting $Y_0$ and $Y_1$:

$$Y_0 \equiv Y^{0,M^0} = Y^{00} + (Y^{01} - Y^{00})M^0, \quad Y_1 \equiv Y^{1,M^1} = Y^{10} + (Y^{11} - Y^{10})M^1;$$  (2.1)

the equalities can be seen by substituting $M^0 = 0, 1$ and $M^1 = 0, 1$. 

4
2.1 Basic Three-Effect Decomposition

With (2.1), the total effect \( E(\tau) \equiv E(Y_1 - Y_0) \) can be decomposed “path-freely”:

\[
E(Y_1 - Y_0) = E[ Y^{10} + (Y^{11} - Y^{10})M^1 - \{Y^{00} + (Y^{01} - Y^{00})M^0 \} ] \\
= E(Y^{10} - Y^{00}) + E\{(Y^{11} - Y^{10})M^1\} - E\{(Y^{01} - Y^{00})M^0\}.
\]

(2.2)

We call \((M^0 = 0, M^1 = 0)\) ‘never taker’, \((M^0 = 0, M^1 = 1)\) ‘complier’, \((M^0 = 1, M^1 = 0)\) ‘defier’, and \((M^0 = 1, M^1 = 1)\) ‘always taker’. These terms were used in Imbens and Angrist (1994) when \(M\) is an endogenous treatment and \(D\) is an instrument, but we use those terms for mediator \(M\) and treatment \(D\) as in Lee (2012, 2017) where \(M\) is participation in an activity and \(Y\) is a performance in the activity.

Now, subtract and add \(E\{(Y^{11} - Y^{10})M^0\}\) to (2.2) to rewrite (2.2) as

\[
E(Y^{10} - Y^{00}) + E\{(Y^{11} - Y^{10})(M^1 - M^0)\} + E\{(Y^{11} - Y^{10} - Y^{01} + Y^{00})M^0\}. \tag{2.3}
\]

The first term is the controlled direct effect with \(m = 0\) in (1.3). The second term can be called the ‘controlled indirect effect’, because \(M^1 - M^0\) is the effect of \(D\) on \(M\) and \(Y^{11} - Y^{10}\) is the controlled mediator effect with \(d = 1\) in (1.3). The third term is the ‘controlled interaction effect’ of \(D\) and \(M\), as is explained below.

To relate the indirect effect in (2.3) to the traditional ‘product approach’, consider

\[
M = \alpha_1 + \alpha_dD + \varepsilon \quad \text{and} \quad Y = \beta_1 + \beta_dD + \beta_mM + U
\]

where the \(\alpha\)’s and \(\beta\)’s are parameters, \(\varepsilon\) and \(U\) are error terms. Here, the effect of \(D\) on \(M\) is \(\alpha_d\), the effect of \(M\) on \(Y\) is \(\beta_m\), and consequently, the indirect effect of \(D\) on \(Y\) through \(M\) is \(\alpha_d\beta_m\). The second term of (2.3) is a nonparametric version of \(\beta_m\alpha_d\).

To understand the third term of (2.3) intuitively, consider \(Y = \beta_1 + \beta_dD + \beta_mM + \beta_{dm}DM + U\) with \(\beta_{dm}DM\) extra, compared with the preceding SF for \(Y\). Observe

\[
Y^{11} - Y^{10} - Y^{01} + Y^{00} = Y^{11} - Y^{00} - (Y^{10} - Y^{00}) - (Y^{01} - Y^{00})
\]

which is the ‘gross effect \((\beta_d + \beta_m + \beta_{dm})\) \(Y^{11} - Y^{00}\) of \((D, M)\)’ minus the ‘separate effect \((\beta_d)\) \(Y^{10} - Y^{00}\) of \(D\)’ minus the ‘separate effect \((\beta_m)\) \(Y^{01} - Y^{00}\) of \(M\)’. That is, the double difference in the third term of (2.3) removes the separate effects \((\beta_d\) and \(\beta_m\)\) of \(D\) and \(M\) from the gross effect to isolate only their interaction effect \(\beta_{dm}\).
2.2 Three-Effect Decomposition under Monotonicity

To better interpret (2.3), we can rule out defier under the assumption

\[ \text{Monotonicity} : M^0 \leq M^1 \implies M^1 - M^0 = 0, 1. \]

This makes the indirect and interaction effects of (2.3) equal to, respectively,

\[
E(Y^{11} - Y^{10} | M^1 = M^0 = 1) P(M^1 - M^0 = 1) = E(Y^{11} - Y^{10} | CP) P(CP),
\]
\[
E(Y^{11} - Y^{10} - Y^{01} + Y^{00} | AT) P(AT),
\]

where ‘CP’ and ‘AT’ are shorthands for complier and always taker. Then (2.3) becomes

\[
E(Y^{10} - Y^{00}) + E(Y^{11} - Y^{10} | CP) P(CP) + E(Y^{11} - Y^{10} - Y^{01} + Y^{00} | AT) P(AT)
\]
\[
= \text{direct effect} + \text{mediator effect for CP} + \text{interaction effect for AT}; \quad (2.4)
\]

being CP represents the effect of \(D\) on \(M\), as will become clear later.

We summarize our findings for the three-effect decomposition:

**THEOREM 1.** A path-free three-effect decomposition of the total effect \(E(Y_1 - Y_0)\) is (2.3), consisting of (i) the controlled direct effect \(E(Y^{10} - Y^{00})\), (ii) the controlled indirect effect \(E\{Y^{11} - Y^{10} | (M^1 - M^0)\}\), and (iii) the controlled interaction effect \(E\{Y^{11} - Y^{10} - Y^{01} + Y^{00} | M^0\}\). If \(M^0 \leq M^1\) holds extra, then (2.3) becomes (2.4).

In (2.4), if \(E(Y^{11} - Y^{10} | CP) = 0\) or \(P(CP) = 0\), then the second term is zero. If \(E(Y^{11} - Y^{10} - Y^{01} + Y^{00} | AT) = 0\) or \(P(AT) = 0\), then the third term is zero. The fact that only the compliers appear in the controlled mediator effect under the monotonicity is natural, because the defiers \((M^1 = 0\) and \(M^0 = 1\)) are ruled out and \(M\) changes neither (i.e., \(M^0 = M^1\)) for always takers nor for never takers.

One may define the ‘controlled total effect’ \(E(Y^{11} - Y^{00})\), and decompose it as in

\[
E(Y^{11} - Y^{00}) = E(Y^{11} - Y^{10}) + E(Y^{10} - Y^{00}), \quad (2.5)
\]
\[
E(Y^{11} - Y^{00}) = E(Y^{11} - Y^{01}) + E(Y^{01} - Y^{00}). \quad (2.6)
\]

Although these are similar to (2.3) in that they are based on controlled effects, there are two critical differences. The first is that \(E(Y^{11} - Y^{00})\) differs from the total effect.
$E(\tau) \equiv E(Y_1 - Y_0)$ for (2.3), and $E(Y^{11} - Y^{00})$ is not germane to mediation analysis, because both $D$ and $M$ are controlled as two causal factors of equal standing with none preceding the other. The second is that (2.3) is path-free, while (2.5) and (2.6) are not.

3 Identification of All Effects

3.1 Identification Conditions

Let $X$ be covariates not affected by $D$. Our conditions are (‘$\Pi$’ for independence):

$C(a) : D \Pi (M^0, M^1, Y^{00}, Y^{01}, Y^{10}, Y^{11})|X$;

$C(b) : (M^0, M^1) \Pi (Y^{00}, Y^{01}, Y^{10}, Y^{11})|(D, X)$;

$C(c) : 0 < P(D = d, M = m|X)$ for all $d, m = 0, 1$ and $X$;

$C(d) : M^0 \leq M^1|X$.

$C(a)$ to $C(c)$ are essential for (2.3). $C(d)$ is for (2.4), which is not essential.

$C(a)$ and $C(b)$ are the ignorability of confounders in the treatment-mediator, treatment-outcome, and mediator-outcome relationships. $C(a)$ and $C(b)$ operate in two stages: as $D$ precedes $M$ which in turn precedes $Y$, the first stage is $D$ being independent of all potential future variables given $X$, and the second stage is $(M^0, M^1)$ being independent of all potential future versions of $Y$ given $(D, X)$. $C(c)$ is a support-overlap condition for $D|X, D|(M, X)$ and $M|(D, X)$; e.g., $C(c)$ implies

$P(D = d|X) = P(D = d, M = 0|X) + P(D = d, M = 1|X) > 0$ for $d = 0, 1$;

$P(M = m|D = d, X) = P(D = d, M = m|X)/P(D = d|X) > 0$ for $d, m = 0, 1$.

Slightly weaker conditions than $C(a)$ and $C(b)$ appeared in Imai et al. (2010):

$D \Pi (M^d, Y^{d\cdot m})|X \quad \text{and} \quad M^d \Pi Y^{d\cdot m}|(D, X)$ for all $d, d', m = 0, 1$

where the joint distributions of $(M^0, M^1)$ and $(Y^{00}, Y^{01}, Y^{10}, Y^{11})$ do not appear, differently from $C(a)$ and $C(b)$. Analogously, Petersen et al. (2006) assumed

$D \Pi M^d|X, \quad D \Pi Y^{d\cdot m}|X, \quad M \Pi Y^{d\cdot m}|(D, X)$ for all $d, m = 0, 1$. 
Using marginal independence instead of joint independence, we can relax $C(a)$ and $C(b)$, but we continue to assume $C(a)$ and $C(b)$ for simplicity.

In $C(b)$, $(M^0, M^1)$ is allowed to be related to $(Y^{00}, Y^{01}, Y^{10}, Y^{11})$ through $(D, X)$, but $D \Pi (M^0, M^1, Y^{00}, Y^{01}, Y^{10}, Y^{11})|X$ in $C(a)$. Hence, $C(a)$ and $C(b)$ imply $C(e)$ next.

$C(e) \land (D, M^0, M^1) \Pi (Y^{00}, Y^{01}, Y^{10}, Y^{11})|X \iff (D, M) \Pi (Y^{00}, Y^{01}, Y^{10}, Y^{11})|X$; the implication arrow holds as $M = M^0 + (M^1 - M^0)D$ is determined by $(M^0, M^1, D)$.

### 3.2 Effect Identification

First, the total effect $E(\tau) \equiv E(Y_1 - Y_0|X)$ is easily identified:

\[
E(Y|D = 1, X) - E(Y|D = 0, X) = E(Y_1|D = 1, X) - E(Y_0|D = 0, X) \quad (3.1)
\]

\[
= E\{Y^{10} + (Y^{11} - Y^{10})M^1|D = 1, X\} - E\{Y^{00} + (Y^{01} - Y^{00})M^0|D = 0, X\}
\]

\[
= E\{Y^{10} + (Y^{11} - Y^{10})M^1|X\} - E\{Y^{00} + (Y^{01} - Y^{00})M^0|X\} = E(Y_1 - Y_0|X);
\]

the third equality is due to $C(a)$. This is sensible, as only $D$ changes from 0 to 1 in (3.1).

Second, the direct effect $E(Y^{10} - Y^{00}|X)$ of $D$ with $m = 0$ is also easily identified:

\[
E(Y|D = 1, M = 0, X) - E(Y|D = 0, M = 0, X) \quad (3.2)
\]

\[
= E(Y^{10}|D = 1, M^1 = 0, X) - E(Y^{00}|D = 0, M^0 = 0, X) = E(Y^{10} - Y^{00}|X)
\]

due to $C(e)$. This is sensible, as only $D$ changes from 0 to 1 with $M = 0$.

Third, due to $C(e)$, the indirect effect given $X$ can be written as

\[
E\{(Y^{11} - Y^{10})(M^1 - M^0)|X\} = E(Y^{11} - Y^{10}|X) \cdot E(M^1 - M^0|X)
\]

\[
= \{E(Y|D = 1, M = 1, X) - E(Y|D = 1, M = 0, X)\}
\]

\[
\cdot \{E(M|D = 1, X) - E(M|D = 0, X)\} \quad (3.3)
\]

because, respectively due to $C(e)$ and $C(a)$,

\[
E(Y|D = 1, M = 1, X) - E(Y|D = 1, M = 0, X) = E(Y^{11} - Y^{10}|X); \quad (3.4)
\]

\[
E(M|D = 1, X) - E(M|D = 0, X) = E(M^1|X) - E(M^0|X) = E(M^1 - M^0|X).
\]
Fourth, the interaction effect is identified as the remainder $(3.1) - (3.2) - (3.3)$.

Now, under C(d), apply C(e) to (3.3) to get the effect of $M$ with $d = 1$ on compliers:

$$E(Y|D = 1, M = 1, X) - E(Y|D = 1, M = 0, X) = E(Y^{11} - Y^{10}|X)$$

$$= E(Y^{11} - Y^{10}|M^1 - M^0 = 1, X) = E(Y^{11} - Y^{10}|CP, X).$$

(3.5)

Also in (3.3), due to C(d) and C(a),

'effect of $D$ on $M' = E(M|D = 1, X) - E(M|D = 0, X)$$

$$= E(M^1|X) - E(M^0|X) = P(M^1 = 1, M^0 = 1|X) + P(M^1 = 1, M^0 = 0|X)$$

$$- P(M^1 = 1, M^0 = 1|X) = P(M^1 = 1, M^0 = 0|X) = P(CP|X).$$

The next theorem summarizes the main findings for effect identification:

**THEOREM 2.** Under C(a) and C(c), the identification findings are:

- **total effect**: $E(Y|D = 1, X) - E(Y|D = 0, X)$;
- **controlled direct**: $E(Y|D = 1, M = 0, X) - E(Y|D = 0, M = 0, X)$;
- **controlled indirect**: $\{E(Y|D = 1, M = 1, X) - E(Y|D = 1, M = 0, X)\}$
  $$\cdot \{E(M|D = 1, X) - E(M|D = 0, X)\}$$ (for CP under C(d));
- **controlled interaction**: total − controlled direct − controlled indirect.

### 4 Effect Comparison in Linear Model

#### 4.1 Structural Form and Reduced Form

To understand various effect decompositions and identifications better, here we illustrate the effects using a randomized $D$ with $P(D = 1) = 0.5$ and linear SF’s:

$$M^d = 1[1 < \alpha_1 + \alpha_d d + X'\alpha_x + \epsilon],$$

$$Y^{dm} = \beta_1 + \beta_d d + \beta_m m + \beta_{dm} dm + X'\beta_x + U,$$

where $1[A] \equiv 1$ if $A$ holds and 0 otherwise, the error terms $(e, U)$ are independent of each other and $X$, and $e \sim Uni[0, 1]$ with $Uni[0, 1]$ standing for the uniform distribution
on $[0,1]$. The reason for $\text{Uni}[0,1]$ and the threshold 1 for $M^d$ is to have a linear model for $E(M^d|X)$, as is shown next.

Assuming $0 < \alpha_1 + \alpha_d d + X'\alpha_x < 1$ for all $X$, it holds that

$$E(M^d|X) = P(e > 1 - \alpha_1 - \alpha_d d - X'\alpha_x|X) = P(e < \alpha_1 + \alpha_d d + X'\alpha_x|X)$$

$$= \alpha_1 + \alpha_d d + X'\alpha_x \implies M^d = \alpha_1 + \alpha_d d + X'\alpha_x + \varepsilon_d$$

(4.2)

where $\varepsilon_d \equiv M^d - \alpha_1 - \alpha_d d - X'\alpha_x \implies E(\varepsilon_d|X) = 0$.

$M^d = \alpha_1 + \alpha_d d + X'\alpha_x + \varepsilon_d$ is a reduced form (RF) in contrast to the SF for $M^d$ in (4.1), and $E(M^d|X) = \alpha_1 + \alpha_d d + X'\alpha_x$ will be used often. ‘$e \sim \text{Uni}[0,1]$’ is restrictive, but not much more restrictive than the probit assumption as the following shows.

Consider a $M^d$ SF with an error $\nu$ having an invertible distribution function $F$:

$$M^d = [\nu > F^{-1}(1 - \alpha_1 - \alpha_d d - X'\alpha_x)] = 1[F(\nu) > 1 - \alpha_1 - \alpha_d d - X'\alpha_x].$$

(4.3)

Since $F(\nu) \sim \text{Uni}[0,1]$ as $e$ is, (4.3) is restrictive only in that the regression function part is assumed to be $F^{-1}(1 - \alpha_1 - \alpha_d d - X'\alpha_x)$ when $\nu$ is the error term. Compare this to the probit assumption: with the $N(0,1)$ distribution function $\Phi(\cdot)$,

$$M^d = 1[N(0,1) > 1 - \alpha_1 - \alpha_d d - X'\alpha_x] = 1[\text{Uni}[0,1] > \Phi(1 - \alpha_1 - \alpha_d d - X'\alpha_x)],$$

(4.4)

which is almost as strong an assumption as (4.3) is.

### 4.2 Various Effects for Linear Model

The linear SF for $Y^{dm}$ in (4.1) and the linear RF for $M^d$ in (4.2) render

$$\delta(d) = \beta_d + \beta_{dm} M^d, \quad \mu(d) = \beta_m (M^1 - M^0) + \beta_{dm} (M^1 - M^0)$$

(4.5)

$$\implies E(\tau) = \beta_d + \beta_m \alpha_d + \beta_{dm} \{\alpha_1 + \alpha_d + E(X')\alpha_x\};$$

(4.6)

the proof is in the appendix. In contrast, the three-effect decomposition (2.3) is

$$E\{Y^{10} - Y^{00} + (Y^{11} - Y^{10})(M^1 - M^0) + (Y^{11} - Y^{10} - Y^{11} - Y^{01} + Y^{00})M^0\}$$

$$= E\{\beta_d + (\beta_m + \beta_{dm}) (\alpha_d + \varepsilon^1 - \varepsilon^0) + \beta_{dm} (\alpha_1 + X'\alpha_x + \varepsilon^0)\}$$

$$= \beta_d + (\beta_m + \beta_{dm}) \alpha_d + \beta_{dm} \{\alpha_1 + E(X')\alpha_x\}.$$

(4.7)
The two-effect decomposition in (4.6) takes \( \beta_m\alpha_d + \beta_{dm}\{\alpha_1 + \alpha_d + E(X')\alpha_x}\) as the indirect effect. In contrast, our three-effect decomposition (4.7) takes only \( \beta_m\alpha_d + \beta_{dm}\alpha_d\) as the indirect effect while classifying \( \beta_{dm}\{\alpha_1 + E(X')\alpha_x\}\) as the interaction effect, where \( \beta_{dm} = E(Y^{11} - Y^{10} - Y^{01} + Y^{00})\) and \( \alpha_1 + E(X')\alpha_x = P(M^0 = 1)\) are for

\[
E\{(Y^{11} - Y^{10} - Y^{01} + Y^{00})M^0\} = E(Y^{11} - Y^{10} - Y^{01} + Y^{00}) \cdot P(M^0 = 1).
\]

Under \( \beta_{dm} = 0 \) (no interaction effect), both (4.6) and (4.7) become \( \beta_d + \beta_m\alpha_d\) which is the “traditional decomposition” of the total effect into the direct effect \( \beta_d\) and the indirect effect \( \beta_m\alpha_d\). Although \( \alpha_1 + E(X')\alpha_x\) is irrelevant in the traditional decomposition, they do matter in (4.6) and (4.7). Intuitively explaining interaction effect only with \( \beta_{dm}\) as was done earlier is not exactly correct, as the interaction effect in (4.7) reveals.

We also mentioned other decompositions in (2.5) and (2.6), which are, respectively,

\[
\{\beta_1 + \beta_d + \beta_m + \beta_{dm} - (\beta_1 + \beta_d)\} + (\beta_1 + \beta_d - \beta_1) = \beta_m + \beta_{dm} + \beta_d,
\]

\[
\{\beta_1 + \beta_d + \beta_m + \beta_{dm} - (\beta_1 + \beta_m)\} + (\beta_1 + \beta_m - \beta_1) = \beta_d + \beta_{dm} + \beta_m. \quad (4.8)
\]

Whereas all the preceding decompositions involve the \( \alpha \) parameters to reveal how \( M \) is affected by \( D \), no \( \alpha \) parameter appears in (4.8). Since \( M \) is affected by \( D \) in reality, the decompositions in (4.8) without any \( \alpha \) parameter would make sense only when we control \( M \) as well as \( D \), which is, however, not a mediation analysis.

## 5 Two Estimators

There are many ways to estimate various effects identified with conditional mean differences. The arguably best-known approaches are matching, inverse probability weighting (IPW), and regression adjustment. Among these, matching is most intuitive, but finding its standard error is hard despite advances in Abadie and Imbens (2016). IPW specifies only \( E(D|X) \), but it has the “too small denominator” problem, which remains even when IPW is generalized for ‘doubly robustness’. For our goal, regression adjustment specifying outcome models as in Vansteelandt and Daniel (2014) is well suited.

In regression adjustment for the effect of \( D \) on \( Y \), \( E(Y|D, X) \) is specified to render the mean difference \( E(Y|D = 1, X) - E(Y|D = 0, X) \), from which \( X \) is averaged out.
If the effect is constant, then the averaging step is unnecessary. Hence we explore two estimators. The first is based on the constant-effect linear models (4.1) and (4.2), the main attraction of which is its simplicity. Even when the effect is heterogeneous, the misspecified constant-effect models tend to give weighted versions of the heterogeneous effect. The second estimator essentially replaces the constant effect specifications with functions of $X$, and it relaxes the uniform error assumption for $M^d$ in (4.1) and allows almost any form of $Y$ as is explained next.

Generalizing the approaches in Lee (2018, 2021) without $M$, take $E(\cdot|D, M, X)$ on

$$Y = (1 - D)(1 - M)Y^{00} + (1 - D)MY^{01} + D(1 - M)Y^{10} + DMY^{11}$$

and rearrange the resulting conditional means to obtain

$$E(Y|D, M, X) = E(Y^{00}|D, M, X) + E(Y^{10} - Y^{00}|D, M, X) \cdot D$$

$$+ E(Y^{01} - Y^{00}|D, M, X) \cdot M + E(Y^{11} - Y^{10} - Y^{01} + Y^{00}|D, M, X) \cdot DM$$

$$= \mu_0(X) + \mu_1(X)D + \mu_4(X)M + \mu_3(X)DM \quad \text{(due to C(e))},$$

$$\mu_0(X) \equiv E(Y^{00}|X), \quad \mu_1(X) \equiv E(Y^{10} - Y^{00}|X),$$

$$\mu_4(X) \equiv E(Y^{01} - Y^{00}|X), \quad \mu_3(X) \equiv E(Y^{11} - Y^{10} - Y^{01} + Y^{00}|X);$$

the reason for the subscript 4 will be seen later. Then $U_0 \equiv Y - E(Y|D, M, X)$ gives

$$Y = \mu_0(X) + \mu_1(X)D + \mu_4(X)M + \mu_3(X)DM + U_0. \quad (5.1)$$

We employed two “linearization devices”: the uniform error for $M^d$ in (4.1) to obtain the linear model in (4.2), and the approach for the linear-in-$(D, M, DM)$ representation in (5.1). The first subsection below introduces ordinary least squares estimator (OLS) for the former, and the second subsection for the latter.

### 5.1 OLS for Constant Effects

From the $M^d$ RF in (4.2) and $Y^{dm}$ SF in (4.1), we obtain the observed variables:

$$M = (1 - D)M^0 + DM^1 = \alpha_1 + \alpha_d D + X'\alpha_x + \varepsilon, \quad \varepsilon \equiv (1 - D)\varepsilon^0 + D\varepsilon^1,$$

$$Y = (1 - D)(1 - M)Y^{00} + (1 - D)MY^{01} + D(1 - M)Y^{10} + DMY^{11}$$

$$= \beta_1 + \beta_d D + \beta_m M + \beta_{dm} DM + X'\beta_x + U. \quad (5.2)$$
Since \((\varepsilon^0, \varepsilon^1)\) are parts of \(M^0\) and \(M^1\), \(D \Pi (\varepsilon^0, \varepsilon^1)|X\) holds due to \(C(a)\). Hence
\[
E(\varepsilon|D, X) = (1 - D)E(\varepsilon^0|X) + DE(\varepsilon^1|X) = 0,
\]
and we can obtain
\[
\text{OLS } \hat{\alpha} \text{ of } M \text{ on } W \equiv (1, D, X)' \text{ for } \alpha \equiv (\alpha_1, \alpha_d, \alpha_x)'.
\]
Since \(Y^{dm}|X\) is determined by \(U\), \(C(e)\) implies \(E(U|D, M, X) = 0\), and we obtain
\[
\text{OLS } \hat{\beta} \text{ of } Y \text{ on } Z \equiv (1, D, M, DM, X)' \text{ for } \beta \equiv (\beta_1, \beta_d, \beta_m, \beta_{dm}, \beta_x)'.
\]
Using (4.7), a three-effect decomposition estimator is \((\bar{X}\) is the sample average of \(X)\)
\[
\hat{\beta}_d + (\hat{\beta}_m + \hat{\beta}_{dm}) \hat{\alpha}_d + \hat{\beta}_{dm}(\hat{\alpha}_1 + \bar{X}'\hat{\alpha}_x).
\]
For the direct effect estimator \(\hat{\beta}_d\), with \(X\) being of dimension \(k_x \times 1\) and \(0_{a \times b}\)
denoting the null vector of dimension \(a \times b\),
\[
\sqrt{N}(\hat{\beta}_d - \beta_d) \rightarrow^d N(0, \Omega_1), \quad \hat{\Omega}_1 \equiv \frac{1}{N} \sum_i \hat{\eta}_{1i}^2 \rightarrow^p \Omega_1,
\]
\[
\hat{\eta}_{1i} \equiv C_{11}'\left(\frac{1}{N} \sum_i Z_iZ_i^{-1}Z_i\hat{U}_i\right), \quad \hat{U}_i \equiv Y_i - Z_i\hat{\beta}, \quad C_{11} \equiv (0, 1, 0, 0, 0_{1 \times k_x})'.
\]
For the indirect effect estimator \((\hat{\beta}_m + \hat{\beta}_{dm})\hat{\alpha}_d\), the appendix proves
\[
\sqrt{N}\{(\hat{\beta}_m + \hat{\beta}_{dm})\hat{\alpha}_d - (\beta_m + \beta_{dm})\alpha_d\} \rightarrow^d N(0, \Omega_2), \quad \hat{\Omega}_2 \equiv \frac{1}{N} \sum_i \hat{\eta}_{2i}^2 \rightarrow^p \Omega_2,
\]
\[
\hat{\eta}_{2i} \equiv C_{21}'\left(\frac{1}{N} \sum_i Z_iZ_i^{-1}Z_i\hat{U}_i\right) + C_{22}'\left(\frac{1}{N} \sum_i W_iW_i^{-1}W_i\hat{\varepsilon}_i\right), \quad C_{21} \equiv (0, 0, \hat{\alpha}_d, \hat{\alpha}_d, 0_{1 \times k_x})', \quad C_{22} \equiv (0, \hat{\beta}_m + \hat{\beta}_{dm}, 0_{1 \times k_x})', \quad \hat{\varepsilon}_i \equiv M_i - W_i\hat{\alpha}.
\]
For the interaction effect estimator \(\hat{\beta}_{dm}(\hat{\alpha}_1 + \bar{X}'\hat{\alpha}_x)\), the appendix also proves
\[
\sqrt{N}[\hat{\beta}_{dm}(\hat{\alpha}_1 + \bar{X}'\hat{\alpha}_x) - \beta_{dm}\{\alpha_1 + E(X')\alpha_x\}] \rightarrow^d N(0, \Omega_3), \quad \hat{\Omega}_3 \equiv \frac{1}{N} \sum_i \hat{\eta}_{3i}^2 \rightarrow^p \Omega_3,
\]
\[
\hat{\eta}_{3i} \equiv C_{31}'\left(\frac{1}{N} \sum_i Z_iZ_i^{-1}Z_i\hat{U}_i\right) + C_{32}'\left(\frac{1}{N} \sum_i W_iW_i^{-1}W_i\hat{\varepsilon}_i\right) + \hat{\beta}_{dm}\hat{\alpha}'_x(X_i - \bar{X}), \quad C_{31} \equiv (0, 0, 0, \hat{\alpha}_1 + \bar{X}'\hat{\alpha}_x, 0_{1 \times k_x})', \quad C_{32} \equiv (\hat{\beta}_{dm}, 0, \hat{\beta}_{dm}\bar{X})'.
\]
Finally, for the total effect, its asymptotic variance can be estimated with
\[
\hat{\Omega}_{123} \equiv \frac{1}{N} \sum_i (\hat{\eta}_{1i} + \hat{\eta}_{2i} + \hat{\eta}_{3i})^2.
\]
5.2 OLS for Varying Effects

For $X_0, X_1, X_4, X_3, X_m$ consisting of elements of $X$ and their functions, with $X_j$ being of dimension $k_j \times 1$, consider for (5.1) and for indirect effect:

\[
Y = \beta'_{00}X_0 + \beta'_{1x}X_1D + \beta'_{4x}X_1M + \beta'_{3x}X_3DM + U_0 = \beta'_{0}Q_0 + U_0,
\]

For indirect:

\[
DY = D(\beta'_{20}X_2 + \beta'_{2x}X_2M + U_2) = D(\beta'_{2}Q_2 + U_2),
\]

\[
M = \alpha'_mX_m + \alpha'_{mx}X_mD + U_m = \alpha'_mQ_m + U_m, \quad \alpha_m \equiv (\alpha'_m, \alpha'_{mx})',
\]

\[
Q_0 = (X'_0, X'_1D, X'_4M, X'_3DM)', \quad \beta_0 \equiv (\beta'_{00}, \beta'_{1x}, \beta'_{4x}, \beta'_{3x})',
\]

\[
Q_2 \equiv (X'_2, X'_2M)', \quad Q_m \equiv (X'_m, X'_mD)', \quad \beta_j \equiv (\beta'_{i0}, \beta'_{ijx})', \quad j = 2, m,
\]

$U_j, j = 0, 2, m$, are error terms, and $DY$ is for the indirect effect because $E(Y|D = 1, M = m), m = 0, 1$, are needed. The “irrelevant” subscript 4 appears in $\beta'_{4x}X_4$, because $\beta'_{4x}X_4$ is not used in estimating the three effects.

The linear models here, which differ much from those in (5.2) based on (4.1), are to approximate the $X$-conditional intercept and slopes in (5.1), and other than this, there is no restriction imposed on the data generating process. That is, $\beta'_{00}X_0, \beta'_{1x}X_1, \beta'_{4x}X_4$ and $\beta'_{3x}X_3$ are to approximate $\mu_0(X), \mu_1(X), \mu_4(X)$ and $\mu_3(X)$ in (5.1), and they consist of functions of elements in $X$. Note that $\beta'_{1x}X_1 = \mu_1(X) \equiv E(Y^{10} - Y^{00}|X)$ is the conditional direct effect, $\beta'_{3x}X_3 = \mu_3(X) \equiv E(Y^{11} - Y^{10} - Y^{01} + Y^{00}|X)$ is part of the conditional interaction effect. However, $\beta'_{4x}X_4 = \mu_4(X) \equiv E(Y^{01} - Y^{00}|X)$ is not the desired indirect effect $E\{(Y^{11} - Y^{10})(M^1 - M^0)|X\}$, for which the $DY$ model in (5.4) should be used. The total effect is to be obtained as the sum of the three effects.

For varying effects, we condition the inference on $\bar{X}$. This is not to account for errors of the form $\bar{X}_2X'_m - E(X_2X'_m)$ relevant for the indirect and interaction effects, as accounting for such errors requires vectorizing matrices of the form $\bar{X}_2X'_m - E(X_2X'_m)$, resulting in unnecessary complications. What is gained by conditioning on $\bar{X}$ is ease in doing asymptotic inference, as terms like $X_i - \bar{X}$ in (5.3) drop out. What is lost is losing some ‘external validity’, as the findings apply only to the subpopulation with their $\bar{X}$ values being the same as those in the sample. Our simulation study will show that not accounting for errors of the form $\bar{X}_2X'_m - E(X_2X'_m)$ makes little difference.
For the direct effect $X'_1 \beta_{1x}$, doing the OLS of $Y$ on $Q_0$, we have

$$
\sqrt{N}X'_1(\hat{\beta}_{1x} - \beta_{1x}) \rightarrow^d N(0, \Lambda_1), \quad \hat{\Lambda}_1 \equiv \frac{1}{N} \sum_i X'_i X_{1i} \rightarrow^p \Lambda_1, \quad \hat{U}_{0i} \equiv Y_i - \hat{\beta}'_0 Q_{0i}.
$$

$$G_1 \equiv (0_{1 \times k_0}, X'_1, 0_{1 \times (k_1 + k_3)})', \quad \hat{\lambda}_{1i} \equiv G'_1(\frac{1}{N} \sum_i Q_{0i} Q'_{0i})^{-1} Q_{0i} \hat{U}_{0i}. \quad (5.5)
$$

For the indirect effect $E\{ (Y^{11} - Y^{10})(M^1 - M^0)|X \}$, due to C(e),

$$E\{ (Y^{11} - Y^{10})(M^1 - M^0)|X \} = E(Y^{11} - Y^{10}|X) \cdot E(M^1 - M^0|X) = \beta'_{2x} X_2 \cdot X'_m \alpha_{mx};$$

$E(Y^{11} - Y^{10}|X) = \beta'_{2x} X_2$ is obtained from the DY model in (5.4), and $E(M^1 - M^0|X) = X'_m \alpha_{mx}$ from the M model in (5.4). For the product $\beta'_{2x} X_2 X'_m \alpha_{mx}$, with $X_2 X'_m$ being the sample average of $X_2 X'_m$, it holds up to an $o_p(1)$ term that

$$\sqrt{N}(\beta'_{2x} X_2 X'_m \alpha_{mx} - \beta'_{2x} X_2 X'_m \alpha_{mx}) = \alpha'_m X_2 X'_m \sqrt{N}(\hat{\beta}_{2x} - \beta_{2x}) + \beta'_{2x} X_2 X'_m \sqrt{N}(\hat{\alpha}_m - \alpha_{mx}).$$

Then, the appendix proves that

$$\sqrt{N}(\hat{\beta}'_{2x} X_2 X'_m \hat{\alpha}_{mx} - \beta'_{2x} X_2 X'_m \alpha_{mx}) \rightarrow^d N(0, \Lambda_2), \quad \hat{\Lambda}_2 \equiv \frac{1}{N} \sum_i \hat{\lambda}_{2i}^2 \rightarrow^p \Lambda_2,$$

$$\hat{\lambda}_{2i} \equiv \hat{G}'_{21}(\frac{1}{N} \sum_i D_i Q_{2i} Q'_{2i})^{-1} D_i Q_{2i} \hat{U}_{2i} + \hat{G}'_{22}(\frac{1}{N} \sum_i Q_{mi} Q'_{mi})^{-1} Q_{mi} \hat{U}_{mi},$$

$$\hat{G}_{21} \equiv (0_{1 \times k_2}, \alpha'_m X_2 X'_m)'', \quad \hat{G}_{22} \equiv (0_{1 \times k_m}, \beta'_{2x} X_2 X'_m)'', \quad (5.6)$$

$$\hat{U}_{2i} \equiv Y_i - \beta'_{2x} Q_{2i}, \quad \hat{U}_{mi} \equiv M_i - \alpha_{m} Q_{mi}.$$

For the interaction effect, due to C(e), we need

$$E\{ (Y^{11} - Y^{10} - Y^{01} + Y^{00})M^0|X \} = \mu_3(X) \cdot E(M^0|X) = \beta'_{3z} X_3 \cdot X'_m \alpha_{m0};$$

$\beta'_{3z} X_3$ and $X'_m \alpha_{m0}$ are obtained from the $Y$ and $M$ models in (5.4). This gives

$$\sqrt{N}(\beta'_{3z} X_3 X'_m \hat{\alpha}_{m0} - \beta'_{3z} X_3 X'_m \alpha_{m0}) \rightarrow^d N(0, \Lambda_3), \quad \hat{\Lambda}_3 \equiv \frac{1}{N} \sum_i \hat{\lambda}_{3i}^2 \rightarrow^p \Lambda_3,$$

$$\hat{\lambda}_{3i} \equiv \hat{G}'_{31}(\frac{1}{N} \sum_i Q_{0i} Q'_{0i})^{-1} Q_{0i} \hat{U}_{0i} + \hat{G}'_{32}(\frac{1}{N} \sum_i Q_{mi} Q'_{mi})^{-1} Q_{mi} \hat{U}_{mi}, \quad (5.7)$$

$$\hat{G}_{31} \equiv (0_{1 \times (k_0 + k_1 + k_2)}, \alpha'_m X_3 X'_3)'', \quad \hat{G}_{32} \equiv (\beta'_{3z} X_3 X'_m, 0_{1 \times k_m})''.$$

The total effect is the sum of the three effects: $X'_1 \hat{\beta}_{1x} + \beta'_{2x} X_2 X'_m \hat{\alpha}_{mx} + \beta'_{3z} X_3 X'_m \hat{\alpha}_{m0}$. The asymptotic variance can be estimated with $N^{-1} \sum_i (\hat{\lambda}_{1i} + \hat{\lambda}_{2i} + \hat{\lambda}_{3i})^2$. 

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6 Simulation and Empirical Analyses

6.1 Simulation Study

Recalling (4.1), we use four designs in our simulation study with $D$ randomized, $P(D = 0) = P(D = 1) = 0.5$, $N = 250, 1000$, and 5000 simulation repetitions:

Design 1: $M^d = 1[1 < \alpha_1 + \alpha_d d + X' \alpha_x + Uni[0, 1]]$, $X \sim Uni[0, 1]$, continuous $Y^{dm}$;

Design 2: $M^d = 1[1 < \alpha_1 + \alpha_d d + X' \alpha_x + Uni[0, 1]]$, $X \sim Uni[0, 1]$, probit $Y^{dm}$;

Design 3: $M^d = 1[0 < \alpha_1 + \alpha_d d + X' \alpha_x + N(0, 1)]$, $X \sim N(0, 4)$, continuous $Y^{dm}$;

Design 4: $M^d = 1[0 < \alpha_1 + \alpha_d d + X' \alpha_x + N(0, 1)]$, $X \sim N(0, 4)$, probit $Y^{dm}$.

“Probit $Y^{dm}$” means $Y^{dm} = 1[0 < \text{continuous } Y^{dm}]$ with $U \sim N(0, 1)$ in $Y^{dm}$. In Designs 1 and 2, $E(M^d|X)$ is linear. As for the parameter values, we set

$$\alpha_1 = 0, \alpha_d = \alpha_x = 0.5; \beta_1 = 0, \beta_d = \beta_m = \beta_{dm} = 0.5, \beta_x = -1;$$

$\beta_x = -1$ is to prevent $Y^{11}$ from having too many zeros. We generate $M$ and $Y$ with

$$M = (1-D)M^0 + DM^1, \quad Y = (1-D)(1-M)Y^{00} + (1-D)MY^{01} + D(1-M)Y^{10} + DMY^{11}.$$  

For design 1, the true effects are in (4.6), and for the other designs, the true effects are found numerically. We use three OLS’s: the constant-effect OLS (“OLS$_c$”) for (5.2), the varying effect OLS with $X_0 = X_1 = X_2 = X_3 = X_4 = X_m = X$ (“OLS$_{v1}$”), and the varying effect OLS with $X_0 = X_1 = X_2 = X_3 = X_4 = X_m$ consisting of $X$ and $X^2$ (“OLS$_{v2}$”). OLS$_c$ is consistent for Design 1, but there is no guarantee for the consistency of OLS$_{v1}$ and OLS$_{v2}$ for any design, because they approximate unknown functions of $X$ as in (5.1) with linear functions of $X$ (and $X^2$). Since OLS$_{v2}$ uses more components than OLS$_{v1}$, OLS$_{v2}$ is likely to be less biased but more variable than OLS$_{v1}$. Only in Design 4, we use “OLS$_{v3}$” that uses one more component $\Phi(X)$ than OLS$_{v2}$ does to improve the approximation.

Table 1 presents the Design 1 (left half) and Design 2 (right half) results. Each entry has four numbers: $|\text{Bias}|$, standard deviation (Sd), root mean squared error (Rmse), and the average of the 5000 asymptotic Sd’s; the last is to see how accurate the asymptotic
variance formulas are in comparison with the simulation Sd. Since the effects vary across the designs, we divide all four numbers by the absolute effect magnitude.

Table 1. |Bias/effect|, Sd/|effect|, (Rmse/|effect|) and Asymtotic-Sd/|effect|

|          | Design 1, N=250 | Design 1, N=1000 | Design 2, N=250 | Design 2, N=1000 |
|----------|----------------|------------------|-----------------|------------------|
|          | OLS<sub>c</sub> |                  |                 |                  |
| tot      | 0.00 0.12 (0.12) 0.12 | 0.00 0.06 (0.06) 0.06 | 0.00 0.14 (0.14) 0.14 | 0.00 0.07 (0.07) 0.07 |
| dir      | 0.01 0.42 (0.42) 0.41 | 0.00 0.21 (0.21) 0.21 | 0.06 0.56 (0.56) 0.54 | 0.05 0.27 (0.28) 0.27 |
| ind      | 0.00 0.24 (0.24) 0.24 | 0.00 0.12 (0.12) 0.12 | 0.03 0.32 (0.32) 0.31 | 0.02 0.15 (0.16) 0.15 |
| int      | 0.01 0.60 (0.60) 0.60 | 0.01 0.30 (0.30) 0.30 | 0.17 0.79 (0.81) 0.77 | 0.13 0.40 (0.42) 0.39 |
|          | OLS<sub>v1</sub> |                  |                 |                  |
| tot      | 0.00 0.12 (0.12) 0.12 | 0.00 0.06 (0.06) 0.06 | 0.00 0.14 (0.14) 0.14 | 0.00 0.07 (0.07) 0.08 |
| dir      | 0.01 0.50 (0.50) 0.47 | 0.00 0.25 (0.25) 0.24 | 0.01 0.67 (0.67) 0.63 | 0.01 0.32 (0.32) 0.32 |
| ind      | 0.00 0.28 (0.28) 0.26 | 0.00 0.14 (0.14) 0.13 | 0.00 0.39 (0.39) 0.36 | 0.00 0.18 (0.18) 0.18 |
| int      | 0.01 0.84 (0.84) 0.80 | 0.01 0.41 (0.41) 0.40 | 0.04 1.10 (1.10) 1.05 | 0.01 0.54 (0.54) 0.54 |
|          | OLS<sub>v2</sub> |                  |                 |                  |
| tot      | 0.00 0.12 (0.12) 0.12 | 0.00 0.06 (0.06) 0.06 | 0.00 0.14 (0.14) 0.14 | 0.00 0.07 (0.07) 0.08 |
| dir      | 0.01 0.64 (0.64) 0.51 | 0.00 0.27 (0.27) 0.26 | 0.01 0.85 (0.85) 0.68 | 0.01 0.35 (0.35) 0.34 |
| ind      | 0.00 0.36 (0.36) 0.29 | 0.00 0.15 (0.15) 0.14 | 0.00 0.51 (0.51) 0.40 | 0.01 0.20 (0.20) 0.19 |
| int      | 0.01 1.13 (1.13) 0.91 | 0.01 0.47 (0.47) 0.45 | 0.02 1.46 (1.46) 1.20 | 0.03 0.62 (0.63) 0.60 |
| tru      | 1.125, 0.500, 0.500, 0.125 |                  | 0.395, 0.184, 0.166, 0.045 |                  |

OLS<sub>c</sub> for constant-effect (5.2); OLS<sub>v1</sub> & OLS<sub>v2</sub> for X-heterogeneous effect approximations; tot for total effect; dir for direct; ind for indirect; int for interaction; tru for true effect.

In Design 1 with N = 250, all biases are almost zero, and OLS<sub>c</sub> does best, followed by OLS<sub>v1</sub> and then OLS<sub>v2</sub> that is more variable than OLS<sub>v1</sub>. With N = 1000, all OLS’s improve, and the performance differences narrow. In Design 2 with binary Y, although OLS<sub>c</sub> still does the best followed by OLS<sub>v1</sub> and OLS<sub>v2</sub>, OLS<sub>c</sub> is biased, particularly for the interaction effect, and the biases decrease little even when N increases to 1000. Although omitted from Table 1, due to the bias, OLS<sub>c</sub> is dominated eventually, as N
increases beyond 1000. The second and fourth numbers in each entry of Table 1 are mostly the same, showing that the asymptotic variance formulas are accurate. For this, not accounting for the errors of the form $\bar{X} - E(X)$ in OLS$_{v1}$ and OLS$_{v2}$ hardly matters.

| Table 2. | | | |
| --- | --- | --- | --- |
| | Design 3, N=250 | Design 3, N=1000 | Design 4, N=250 | Design 4, N=1000 |
| | OLS$ _c $ | OLS$ _v1 $ | OLS$ _v2 $ | OLS$ _v3 $ |
| tot | 0.00 0.15 (0.15) 0.15 | 0.00 0.07 (0.07) 0.07 | 0.01 0.27 (0.27) 0.26 | 0.00 0.13 (0.13) 0.13 |
| dir | 0.01 0.39 (0.39) 0.39 | 0.01 0.20 (0.20) 0.20 | 0.18 0.73 (0.75) 0.72 | 0.16 0.36 (0.40) 0.36 |
| ind | 0.01 0.44 (0.44) 0.43 | 0.01 0.21 (0.21) 0.21 | 0.10 0.60 (0.61) 0.59 | 0.09 0.29 (0.30) 0.29 |
| int | 0.00 0.53 (0.53) 0.52 | 0.01 0.26 (0.26) 0.26 | 0.24 0.81 (0.85) 0.81 | 0.23 0.41 (0.47) 0.40 |

**OLS$ _c $**

| tot | 0.00 0.15 (0.15) 0.15 | 0.00 0.07 (0.07) 0.07 | 0.01 0.27 (0.27) 0.26 | 0.00 0.13 (0.13) 0.13 |
| dir | 0.01 0.52 (0.52) 0.50 | 0.00 0.26 (0.26) 0.25 | 0.71 1.20 (1.40) 1.17 | 0.73 0.59 (0.94) 0.59 |
| ind | 0.01 0.46 (0.46) 0.46 | 0.01 0.22 (0.22) 0.22 | 0.24 0.67 (0.71) 0.67 | 0.21 0.31 (0.37) 0.32 |
| int | 0.01 0.83 (0.83) 0.79 | 0.00 0.40 (0.40) 0.39 | 1.05 1.49 (1.82) 1.45 | 1.07 0.73 (1.30) 0.72 |

**OLS$ _v1 $**

| tot | 0.00 0.15 (0.15) 0.15 | 0.00 0.07 (0.07) 0.07 | 0.02 0.26 (0.26) 0.25 | 0.00 0.13 (0.13) 0.13 |
| dir | 0.02 0.71 (0.72) 0.62 | 0.00 0.32 (0.32) 0.31 | 0.12 1.60 (1.61) 1.36 | 0.15 0.72 (0.73) 0.68 |
| ind | 0.01 0.49 (0.49) 0.49 | 0.01 0.22 (0.22) 0.23 | 0.03 0.81 (0.81) 0.77 | 0.01 0.34 (0.34) 0.34 |
| int | 0.02 1.24 (1.24) 1.07 | 0.00 0.55 (0.55) 0.52 | 0.22 2.17 (2.18) 1.82 | 0.23 0.97 (0.99) 0.91 |

**OLS$ _v2 $**

| tot | 0.01 0.25 (0.25) 0.25 | 0.01 0.13 (0.13) 0.12 |
| dir | 0.09 1.98 (1.98) 1.35 | 0.04 0.55 (0.55) 0.52 |
| ind | 0.07 0.87 (0.87) 0.81 | 0.01 0.31 (0.31) 0.32 |
| int | 0.12 2.93 (2.93) 1.98 | 0.07 0.77 (0.78) 0.71 |
| tru | 0.888, 0.500, 0.138, 0.250 | 0.169, 0.089, 0.024, 0.055 |

OLS$ _c $ for constant-effect (5.2); OLS$ _v1 $, OLS$ _v2 $, OLS$ _v3 $ for $X$-heterogeneous effect approximations; tot for total effect; dir for direct; ind for indirect; int for interaction; tru for true effect

Table 2 presents the Design 3 (left half) and Design 4 (right half) results. In Design
3 with continuous $Y$, OLS$_c$ does best followed by OLS$_{v1}$ and OLS$_{v2}$ as in Table 1, showing that, when $Y$ is continuous, the binary model with uniform error in (4.1) is not a bad specification. In Design 4 with binary $Y$, however, OLS$_c$ has large biases that do not decrease even when $N$ goes up. Hence it is hard to recommend OLS$_c$ although it is still the best in terms of Rmse. OLS$_{v1}$ is even more biased, which is also hard to recommend. OLS$_{v2}$ has the smallest biases, which, however, do not drop as $N$ increases. Since OLS$_c$, OLS$_{v1}$ and OLS$_{v2}$ do poorly in terms of bias in Design 4, to see if bias can be reduced further, we use $\Phi(X)$ in addition to $X$ and $X^2$ in $X_0 = X_1 = X_2 = X_3 = X_4 = X_m$ to get OLS$_{v3}$. Indeed, OLS$_{v3}$ has biases much smaller than the other OLS’s.

In summary, first, the OLS for the constant effect model (5.2) does surprisingly well, despite its uniform error specification for $M$. Second, the OLS’s approximating unknown heterogeneous effects with linear functions do slightly worse, but their performances catch up as $N$ goes up. Third, the OLS’s using more extensive specifications to approximate unknown heterogeneous effects tend to be more variable, but they are well worth trying due to the lower biases. Fourth, the asymptotic Sd formulas of our estimators work well, and not accounting for the errors of the form $X - E(X)$ in the varying-effect estimators hardly matters.

6.2 Empirical Analysis

Our empirical analysis uses the National Longitudinal Survey data in Card (1995), which are downloadable from [http://davidcard.berkeley.edu/data_sets.html](http://davidcard.berkeley.edu/data_sets.html) as of this writing; the data have been used also in Tan (2010) and Wang et al. (2017), among others. In our empirical analysis with $N = 3010$, $Y$ is ln(wage in 1976), $D$ is the dummy for black, $M$ is the dummy for college education (i.e., schooling years being 12 or greater), and $X$ consists of age, dummies (“r1, r2, ...”) for 8 residence regions in 1966, dummy for living in a standard metropolitan statistical area (SMSA) in 1966, dummy for living in SMSA in 1976 (“SMSA$_{76}$”), and dummy for living in South in 1976 (“south”). In the original data, there were 9 residence region dummies, but the dummy for region 8 was
dropped due to a singularity problem in our OLS’s. That is, we set

\[ X = (1, \text{age}, r1, r2, r3, r4, r5, r6, r7, r9, \text{SMSA}, \text{SMSA}_{76}, \text{South})' \]

The data set is old, but this suits well our purpose of finding racial discrimination effect on wage, which consists of the direct effect, the indirect effect through college education, and the interaction effect of black and college education. When gender discrimination cases were argued in court in the past, often the counter-argument was that females were less educated/qualified, but lower education/qualification itself might have been due to gender discrimination. Hence it is important to account for the indirect discrimination through missed education opportunities, but doing so with recent data would be difficult because discrimination due to denied education opportunities is unlikely to be present. For this reason, using an old data set as ours is advantageous.

Table 3 presents the estimation results, where OLS$_c$ and OLS$_{v1}$ are the same as those in the simulation study, but OLS$_{v2}$ is different because only age is continuous in $X$ with all the other covariates being binary. For OLS$_{v2}$, we use additionally all interaction terms between age and the other components of $X$ for $X_0 = X_1 = X_2 = X_3 = X_4 = X_m$.

|                  | OLS$_c$ (t-value) | OLS$_{v1}$ (t-value) | OLS$_{v2}$ (t-value) |
|------------------|-------------------|----------------------|----------------------|
| total effect     | -0.243 (-13)      | -0.223 (-9.0)        | -0.220 (-8.5)        |
| direct effect    | -0.272 (-12)      | -0.242 (-7.7)        | -0.248 (-8.0)        |
| indirect effect  | -0.054 (-6.3)     | -0.075 (-4.9)        | -0.075 (-4.5)        |
| interaction effect| 0.083 (4.4)       | 0.094 (3.9)          | 0.103 (4.3)          |

OLS$_c$ for the constant-effect model (5.2) with above $X$
OLS$_{v1}$ approximates $X$-heterogeneous effects with linear functions of $X$
OLS$_{v2}$ additionally uses the interactions between age and the other $X$ elements

Regardless of the estimator in use, the estimates are similar, and all effects are statistically significant. The total effect of being black on wage is about $-22 \sim -24\%$, which consists of the direct effect $-24 \sim -27\%$, the indirect effect of $-5.4 \sim -7.5\%$ through missed college education opportunities, and the interaction effect $8.3 \sim 10\%$. 
That is, had it not been for the indirect effect through college education, the wage
discrimination would have been lesser by $-5.4 \sim -7.5\%$, and college education alleviated
the racial discrimination by $8.3 \sim 10\%$.

7 Conclusions

A treatment $D$ can affect an outcome $Y$ indirectly through a mediator $M$, as well as
directly. $D$ can also interact with $M$ to affect $Y$. In the literature of mediation analysis
decomposing the total effect, this interaction effect has been part of the indirect effect,
which seems however inappropriate, because $D$ and $M$ in the interaction term $DM$ are
on an equal footing, differently from the indirect effect where $D$ precedes $M$.

In this paper, we proposed decomposing the total effect into three effects: direct,
indirect and interaction effects. In addition to the advantage of separating the interaction
effect from the indirect effect, our decomposition is “path-free”, in the sense that there
is no other sensible way to carry out three-effect decomposition. This is in contrast to
the existing path-dependent two-way (direct and indirect) decompositions.

After presenting our three-way decomposition, we showed how to identify them,
which was then followed by two OLS-based estimators. The first OLS assumes an
uniform-distributed error for $M$, which essentially linearizes $E(M|X)$ for covariates $X$.
The second OLS does not make such an assumption; instead, it establishes a linear-
in-$(D, M, DM)$ representation for almost any form of $Y$, with $(D, M, DM)$ carrying
$X$-heterogeneous slopes/effects. The second OLS then approximates the unknown $X$-
heterogeneous effects with linear functions of $X$.

We carried out a simulation study to demonstrate that the two OLS’s work well.
Then we applied the estimators to a data set with $D$ being the dummy for black, $M$
being college education dummy, and $Y = \ln(\text{wage})$. We found out that the total effect
of black dummy on wage is about $-23\%$ consisting of $-25\%$ (direct effect), $-7\%$ (indirect
effect through missed college education opportunities), and $9\%$ (interaction effect).
Proof for (4.5) and (4.6)

Observe

\[ \delta(d) = Y^{1,M^d} - Y^{0,M^d} = Y^{10} + (Y^{11} - Y^{10})M^d - \{Y^{00} + (Y^{01} - Y^{00})M^d\} \]
\[ = Y^{10} - Y^{00} + (Y^{11} - Y^{10} - Y^{01} + Y^{00})M^d = \beta_d + \beta_{dm}M^d; \]
\[ \mu(d) = Y^{d,M^1} - Y^{d,M^0} = Y^{d0} + (Y^{d1} - Y^{d0})M^1 - \{Y^{d0} + (Y^{d1} - Y^{d0})M^0\} \]
\[ = (Y^{d1} - Y^{d0})(M^1 - M^0) = (\beta_d + \beta_m + \beta_{dm}d - \beta_d)d(M^1 - M^0) \]
\[ = \beta_m(M^1 - M^0) + \beta_{dm}d(M^1 - M^0). \]

Since \( M^1 - M^0 = \alpha_d + \varepsilon^1 - \varepsilon^0 \) from (4.2), (1.4) is

\[ \mu(1) + \delta(0) = (\beta_m + \beta_{dm})(M^1 - M^0) + \beta_d + \beta_{dm}M^0 = \beta_d + \beta_m(M^1 - M^0) + \beta_{dm}M^1 \]
\[ = \beta_d + \beta_m(\alpha_d + \varepsilon^1 - \varepsilon^0) + \beta_{dm}(\alpha_1 + \alpha_d + X'\alpha_x + \varepsilon^1). \]

Taking \( E(\cdot) \) removes \( \varepsilon^1 \) and \( \varepsilon^0 \), and gives \( E(\tau) \) in (4.6).

Proof for Indirect-Effect Distribution for Constant-Effect Model

With \( E^{-1}(\cdot) \equiv \{E(\cdot)^{-1}\} \), it holds up to \( o_p(1) \) terms that

\[ \sqrt{N}(\hat{\alpha} - \alpha) = \frac{1}{\sqrt{N}} \sum_i E^{-1}(WW'W_i)\varepsilon_i, \quad \sqrt{N}(\hat{\beta} - \beta) = \frac{1}{\sqrt{N}} \sum_i E^{-1}(ZZ'Z_iU_i); \]
\[ \sqrt{N}\{(\hat{\beta}_m + \hat{\beta}_{dm})\hat{\alpha}_d - (\beta_m + \beta_{dm})\alpha_d\} \]
\[ = \alpha_d\sqrt{N}(\hat{\beta}_m - \beta_m) + \alpha_d\sqrt{N}(\hat{\beta}_{dm} - \beta_{dm}) + (\beta_m + \beta_{dm})\sqrt{N}(\hat{\alpha}_d - \alpha_d) \]
\[ = C'_{21}\sqrt{N}(\hat{\beta} - \beta) + C'_{22}\sqrt{N}(\hat{\alpha} - \alpha) \]
\[ = \frac{1}{\sqrt{N}} \sum_i \{C'_{21}E^{-1}(ZZ'Z_iU_i) + C'_{22}E^{-1}(WW'W_i)\varepsilon_i\} \]

where \( C_{21} \equiv (0, 0, \alpha_d, \alpha_d, 0_{1 \times k_x})' \) and \( C_{22} \equiv (0, \beta_m + \beta_{dm}, 0_{1 \times k_x})'. \)

Defining \( \eta_{2i} \equiv C'_{21}E^{-1}(ZZ'Z_iU_i) + C'_{22}E^{-1}(WW'W_i)\varepsilon_i \), this is asymptotically normal with variance \( \Omega_2 \equiv E(\eta_{2i}\eta_{2i}') \).

Proof for (5.3)
With $\xi \equiv E(X)$, it holds up to $o_p(1)$ terms that

$$
\sqrt{N}\{\hat{\beta}_{dm}(\hat{\alpha}_1 + \tilde{X}'\hat{\alpha}_x) - \beta_{dm}(\alpha_1 + \xi'\alpha_x)\} = (\alpha_1 + \xi'\alpha_x)\sqrt{N}(\hat{\beta}_{dm} - \beta_{dm})
$$

$$
+ \beta_{dm}\sqrt{N}(\hat{\alpha}_1 - \alpha_1) + \beta_{dm}\xi'\sqrt{N}(\hat{\alpha}_x - \alpha_x) + \beta_{dm}\alpha_x'(\tilde{X} - \xi)
$$

$$
= \frac{1}{\sqrt{N}} \sum_i \{C_{31}'E^{-1}(ZZ')Z_iU_i + C_{32}'E^{-1}(WW')W_i\varepsilon_i + \beta_{dm}\alpha_x'(X_i - \xi)\},
$$

where $C_{31} \equiv (0, 0, 0, \alpha_1 + \xi'\alpha_x, 0_{1 \times k_x})'$, $C_{32} \equiv (\beta_{dm}, 0, \beta_{dm}\xi')'$.

Defining $\eta_{3i} \equiv C_{31}'E^{-1}(ZZ')Z_iU_i + C_{32}'E^{-1}(WW')W_i\varepsilon_i + \beta_{dm}\alpha_x'(X_i - \xi)$, this is asymptotically normal with variance $\Omega_3 \equiv E(\eta_{3i}\eta_{3i}')$.

**Proof for (5.6)**

Given $D = 1$, we have $Y = (Y^{11} - Y^{10})M + Y^{10}$. Take $E(\cdot|D = 1, M, X)$ on this $Y$:

$$
E(Y|D = 1, M, X) = E(Y^{11} - Y^{10}|D = 1, M, X)M + E(Y^{10}|D = 1, M, X)
$$

$$
= E(Y^{11} - Y^{10}|X)M + E(Y^{10}|X) \quad (\text{due to C(e)}).
$$

Substitute $U_2 \equiv Y - E(Y|D = 1, M, X)$ into this $E(Y|D = 1, M, X)$ model:

$$
Y = \mu_{20}(X) + \mu_{21}(X)M + U_2, \quad E(U_2|D = 1, M, X) = 0,
$$

$$
\mu_{20}(X) = E(Y^{10}|X) \quad \text{and} \quad \mu_{21}(X) = E(Y^{11} - Y^{10}|X).
$$

With $\mu_{20}(X) = \beta_{20}'X_2$ and $\mu_{21}(X) = E(Y^{11} - Y^{10}|X) = \beta_{2x}'X_2$, we obtain

$$
DY = D(\beta_{20}'X_2 + \beta_{2x}'X_2M + U_2) = D(Q_2\beta + U_2).
$$

The estimand of the OLS of $DY$ on $DQ_2$ is

$$
E^{-1}(DQ_2Q'_2)E(DQ_2Y) = E^{-1}(DQ_2Q'_2)E\{DQ_2(Q'_2\beta + U_2)\} = \beta_2,
$$

as $E(DQ_2U_2) = E[E\{Q_2E(U_2|D = 1, M, X)|D = 1, M, X\} \cdot P(D = 1|M, X)] = 0.$

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