Crystallization of deformed Virasoro algebra, Ding-Iohara-Miki algebra and 5D AGT correspondence

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Abstract

In this paper, we consider the $q \to 0$ limit of the deformed Virasoro algebra and that of the level 1, 2 representation of Ding-Iohara-Miki algebra. Moreover, 5D AGT correspondence at this limit is discussed. This specialization corresponds to the limit from Macdonalds functions to Hall-Littlewood functions. Using the theory of Hall-Littlewood functions, some problems are solved. For example, the simplest case of 5D AGT conjectures is proven at this limit, and we obtain a formula for the 4-point correlation function of a certain operator.

1 Introduction

Macdonald symmetric functions are orthogonal functions with many good properties and applications to various fields. Recently, they have played an important role in 5-dimensional AGT conjectures.

AGT conjectures [AGT] are a sort of dualities between 2-dimensional conformal field theories and 4-dimensional gauge theories. Their $q$-deformations are dualities between the deformed Virasoro/W algebra or the Ding-Iohara-Miki algebra [DI, Mi] and 5-dimensional gauge theories [1]. In the simplest case, the norm of the Whittaker vector of the deformed Virasoro algebra coincides with the Nekrasov formula for the 5-dimensional pure gauge theory [AY1, Ya3]. The deformed Virasoro algebra is very closely related to Macdonald functions. The singular vectors of its highest weight representation correspond to Macdonald functions with rectangular Young diagrams [SKAO, AKOS1]. The Whittaker vector can be explicitly written in terms of Macdonald functions [Ya2]. Moreover, as one of the indication of AGT correspondence, there exists an good orthogonal basis called AFLT basis [AFLT, FL], by which conformal blocks can be expanded so that their each factor reproduce Nekrasov factor. The AFLT basis in the case of 5D AGT conjectures is in the representation space of the Ding-Iohara-Miki algebra and can be expressed by generalized Macdonald functions [AFHKSY]. By using it, it is shown that $q$-Dotsenko-Fateev integrals are also decomposed into the form of Nekrasov formula [Z, MZ]. The relationship between the Ding-Iohara-Miki algebra and 5D Nakrasov formula is made clear in [AFS].

Macdonald functions, which contain two parameters $q$ and $t$, are a generalization of orthogonal functions called Jack functions and Hall-Littlewood functions. The degenerate

\[ \text{The correspondence between the elliptic Virasoro algebra and 6-dimensional theory is also proposed [N, IKY].} \]
limit to Jack functions, \( q \to 1 \) with \( t = q^\beta \), agree with the limit from 5D AGT conjectures to 4D one. Then the deformed Virasoro algebra reduces to the ordinary Virasoro algebra, and generalized Macdonald functions are specialized to Morozov-Smirnov’s generalized Jack functions \([O]\). In this limit, there is a scenario of the proof of 4D AGT conjectures as Hubbard-Stratanovich duality with help of these functions \([MS, MMZ]\), and a combinatorial formula of generalized Jack functions for the expansion in the basis of Schur functions is discovered in \([Sm]\).

In this paper, we investigate behavior at the limit to Hall-Littlewood functions, \( q \to 0 \), of the deformed Virasoro algebra and algebras \( \langle X_n^{(i)} \rangle \) generated by certain operators \( X_n^{(i)} \) \((i = 1, \ldots, N)\), which is obtained by the level \( N \) representation of the Ding-Iohara-Miki algebra. Also 5D AGT conjectures are studied at this limit. In this case, phenomena turn simple and some problems can be solved. In particular, by virtue of theories of Hall-Littlewood functions, we can solve the conjecture that the PBW type vectors of the algebra \( \langle X_n^{(i)} \rangle \) form a basis over its representation space when \( N = 2 \). By this fact, it is also proven that in the generic \( q \) case the PBW type vector forms a basis. Furthermore, we can obtain and prove an explicit formula (Theorem 3.34) for the 4-point correlation function of a certain operator \( \tilde{\Phi}(z) \):

\[
\langle \vec{w} | \tilde{\Phi}(z_2) \tilde{\Phi}(z_1) | \vec{u} \rangle = \sum_{\lambda} \left( \frac{u_1 u_2 z_1}{w_1 w_2 z_2} \right)^{N(\lambda)} \prod_{k=1}^{\ell(\lambda)} \frac{1 - t^{k-1} w_1 w_2}{t^{n(\lambda)} b_\lambda(t^{-1})}.
\]

Here for a partition \( \lambda \), \( n(\lambda) := \sum_{i \geq 1} (i-1) \lambda_i \) and \( b_\lambda(t) \) is defined in Appendix \([A.1]\). The function \( \langle \vec{w} | \tilde{\Phi}(z_2) \tilde{\Phi}(z_1) | \vec{u} \rangle \) can be calculated by generalized Hall-Littlewood functions in the same way as \([AFHKSY]\). However, this formula is obtained by inserting the identity with respect to PBW type vectors.

We call works at this limit ”crystallization” after one of the quantum groups \([K]\) since the parameter \( q \) also represents the temperature in the RSOS model \([LP]\) which has symmetry of the deformed Virasoro algebra, and the limit \( q \to 0 \) can be associated with the zero temperature. Although our studies are mathematically different from the notion of the original crystal base of quantum groups, the physical meaning and the motivation to simplify phenomena are same. To investigate their mathematical relationship is interesting further studies. On the other hand, the 5D Nekrasov formula express the instanton partition function of theories on \( \mathbb{R}^4 \times S^1 \). The limit \( q \to 0 \) correspond to the limit \( R \to \infty \), where \( R \) means the radius of \( S^1 \). Thus under the naive concept that the circle of radius infinity converge to a strait line \( \mathbb{R} \), the Nekrasov function at this limit \( \tilde{Z}^\text{inst} \) \((2.27)\) may express a partition function of some theory on \( \mathbb{R}^5 \).

Incidentally, Macdonald functions are reduced to Uglov functions \([U]\) at the root of unity limit of parameters \( q \) and \( t \). AGT conjectures at this limit have been also studied. For example, the super Virasoro algebra are generated from the deformed Virasoro algebra at the limit \( q, t \to -1 \), and it correspond to theories on the ALE space \( \mathbb{R}^4 / \mathbb{Z}_2 \) \([IOY1, IOY2]\). Moreover, certain conformal algebras \( A(r, k) \) are introduced in \([BBFLT]\) and applied to AGT correspondence. The construction of these algebra is also investigated via the root of unity limit from the Ding-Iohara-Miki algebra in \([BBT, Sp]\). Integral formulas for the solutions of the KZ equation can also be constructed from the limit \( q \to 1 \), \( t \to -1 \) of the deformed Virasoro algebra \([Yo]\).
This paper is organized as follows. We give a brief review on the deformed Virasoro algebra and the simplest 5D AGT correspondence in subsection 2.1. In subsection 2.2, we discuss its crystallization and prove the simplest 5D AGT correspondence at the $q \to 0$ limit.

In subsection 3.1, we reargue the approach of AGT correspondence with the help of the level $N$ representation of the Ding-Iohara-Miki algebra and generalized Macdonald functions. In subsection 3.2, we give a conjecture at the $q \to 0$ limit in the case of $N = 1$. In subsection 3.3, we consider a crystallization of the algebra $\langle X_n^{(i)} \rangle$ in the $N = 2$ case. Next, we define generalized Hall-Littlewood functions and formulate conjectures of AGT correspondence at $q \to 0$. Moreover, we prove the formula for the correlation function obtained by PBW type vectors. In subsection 3.4, we describe different types of crystal limit from the subsections 3.2 and 3.3.

In the appendix, we describe the basic notion of Macdonald functions and Hall-Littlewood functions (subsection A.1). Next, we explain several orderings which are required to state the existence theorem of generalized Macdonald functions (subsection A.2), and present some proofs and checks of conjectures in the text (subsection A.3-A.5). Finally, we compare two formulas for the correlation function $\langle \vec{w} | \tilde{\Phi}(z_2) \tilde{\Phi}(z_1) | \vec{u} \rangle$ obtained in subsection 3.3 and provide a strange factorization formula arising from their comparison (subsection A.6).

2 Crystallization of the deformed Virasoro algebra and its Whittaker vector

2.1 Review of the simplest 5D AGT correspondence

We start with recapitulating the result of the Whittaker vector of the deformed Virasoro algebra and the simplest 5-dimensional AGT correspondence.

**Definition 2.1.** Let $q$ and $t$ be independent parameters and $p := q/t$. The deformed Virasoro algebra is the associative algebra over $\mathbb{Q}(q, t)$ generated by $T_n \ (n \in \mathbb{Z})$ with the commutation relation

$$[T_n, T_m] = -\sum_{l=1}^{\infty} f_l (T_{n-l} T_{m+l} - T_{m-l} T_{n+l}) - \frac{(1-q)(1-t^{-1})}{1-p} (p^n - p^{-n}) \delta_{n+m,0}. \quad (2.1)$$
where the structure constant $f_i$ is defined by

$$f(z) = \sum_{l=0}^{\infty} f_l z^l := \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} \frac{(1 - q^n)(1 - t^{-n})}{1 + p^n} z^n \right). \quad (2.2)$$

The relation (2.1) can be written in terms of the generating function $T(z) := \sum_{n \in \mathbb{Z}} z^n$ as

$$f \left( \frac{w}{z} \right) T(z) T(w) - T(w) T(z) f \left( \frac{z}{w} \right) = -\frac{(1 - q)(1 - t^{-1})}{1 - p} \left[ \delta \left( \frac{pw}{z} \right) - \delta \left( \frac{p^{-1}w}{z} \right) \right], \quad (2.3)$$

where $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$.

Let $|h\rangle$ be the highest weight vector such that $T_0 |h\rangle = h |h\rangle$, $T_n |h\rangle = 0$ ($n > 0$), and $M_h$ be the highest weight module generated by $|h\rangle$. Similarly, $\langle h |$ is the vector satisfying the condition that $\langle h | T_n = h \langle h |$, $\langle h | T_n = 0$ ($n < 0$). $M^*_h$ is the highest weight module generated by $|h\rangle$. The PBW theorem cannot be used because the deformed Virasoro algebra isn’t Lie algebra. However, the PBW type vector $|T_{-\lambda}\rangle := T_{-\lambda_1} T_{-\lambda_2} \cdots |h\rangle$ form a basis over $M_h$. Also, $\langle T_{\lambda} | := \langle h | \cdots T_{\lambda_2} T_{\lambda_1}$ form a basis over $M^*_h$. Here $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition or an Young diagram, i.e., a finite sequence of positive integers $\lambda_i$ such that $\lambda_i \geq \lambda_{i+1}$. Moreover, $\ell(\lambda)$ denote the length of $\lambda$, and $|\lambda\rangle := \sum_i \lambda_i$. The Whittaker vector $|G\rangle$ is defined as follows.

**Definition 2.2.** Define the Whittaker vector $|G\rangle$ by the relations

$$T_1 |G\rangle = \Lambda^2 |G\rangle, \quad T_n |G\rangle = 0 \quad (n > 1). \quad (2.4)$$

Similarly, the dual Whittaker vector $\langle G | \in M^*_h$ is defined by the condition that

$$\langle G | T_{-1} = \Lambda^2 \langle G |, \quad \langle G | T_n = 0 \quad (n < -1). \quad (2.5)$$

This vector is in the form $|G\rangle = \sum_{\lambda} \Lambda^{|\lambda|} B^{(\lambda)}(\langle \lambda \rangle) T_{-\lambda} |h\rangle$ and its norm is calculated as

$$\langle G | G \rangle = \sum_{n=0}^{\infty} \Lambda^{2|n|} B^{(\langle n \rangle)}(\langle n \rangle),$$

where for partitions $\lambda$ and $\mu$, $B^{(\lambda)}(\langle \mu \rangle)$ is the inverse matrix element of the Kac matrix $B_{\lambda, \mu} = \langle T_{\lambda} | T_{-\mu} \rangle$.

It is useful to consider the free field representation of the deformed Virasoro algebra. By the Heisenberg algebra generated by $a_n$ ($n \in \mathbb{Z}$) and $Q$ with the relations

$$[a_n, a_m] = n \frac{1 - q^{\delta_{n+m,0}}}{1 - q^{\delta_{m,0}}} \delta_{n+m,0}, \quad [a_n, Q] = \delta_{n,0}, \quad (2.6)$$

the generating function $T(z)$ can be represented as

$$T(z) = \Lambda^+(z) + \Lambda^-(z), \quad (2.7)$$

$$\Lambda^\pm(z) := \exp \left\{ \mp \sum_{n=1}^{\infty} \frac{1 - t^n}{n(t^n + q^n)} (q/t)^{\mp \frac{1}{2}} a_{-n} z^n \right\} \exp \left\{ \mp \sum_{n=1}^{\infty} \frac{1 - t^n}{n} (q/t)^{\mp \frac{1}{2}} a_n z^{-n} \right\} K^\pm. \quad (2.8)$$

Here $K^\pm := e^{\pm a_0}$. Let $|0\rangle$ be the highest weight vector such that $a_n |k\rangle = 0$ ($n \geq 0$), and $|k\rangle := k^Q |0\rangle$. Then $K |k\rangle = k |k\rangle$. Furthermore, $|k\rangle$ can be regarded as the highest weight

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2The vector $|G\rangle$ is also called Gaiotto state.
vector $|h\rangle$ of the deformed Virasoro algebra with highest weight $h = k + k^{-1}$. In [Ya2], Yanagida proved an explicit formula for $|G\rangle$ in terms of Macdonald functions under the free field representation. The simplest 5-dimensional AGT conjecture is that the Norm $\langle G|G\rangle$ corresponds to the 5-dimensional (K-theoretic) Nekrasov formula for pure gauge theory [AK] [NY1] [NY2]:

$$Z_{\text{inst}}^{\text{pure}} := \sum_{\lambda, \mu} \frac{(\Lambda^4 t/q)^{\lambda[\mu]+[\mu]}}{N_{\lambda\lambda}(1)N_{\lambda\mu}(Q)N_{\mu\mu}(1)N_{\mu\lambda}(Q^{-1})},$$

$$N_{\lambda\mu}(Q) := \prod_{(i,j)\in \lambda} (1 - Q q^{A_{\lambda}(i,j)} t^{L_{\lambda}(i,j)+1}) \prod_{(i,j)\in \mu} (1 - Q q^{-A_{\mu}(i,j)-1} t^{-L_{\lambda}(i,j)}),$$

where $A_{\lambda}(i, j) := \lambda_i - j$ and $L_{\mu}(i, j) := \lambda'_j - i$ are the arm length and the leg length of Young diagram, and $\lambda'$ is the conjugate of $\lambda$.

**Fact 2.3.** For $k = Q^{1/2}$,

$$\langle G|G\rangle = Z_{\text{inst}}^{\text{pure}}.$$

This fact is conjectured in [AY1] and proven in [Ya1] [Ya3] for the generic $q$ case.

### 2.2 Crystallization of the deformed Virasoro algebra and AGT correspondence.

Next, we consider a crystallization of the results of the last subsection, that is behavior in the $q \to 0$ limit of the deformed Virasoro algebra and the simplest 5D AGT correspondence. In this limit, the scaled generators

$$\tilde{T}_n := (q/t)^{\frac{1}{2}} T_n$$

satisfy the commutation relation

$$[\tilde{T}_n, \tilde{T}_m] = - (1 - t^{-1}) \sum_{\ell=1}^{n-m} \tilde{T}_{n-\ell} \tilde{T}_{m+\ell} \quad (n > m > 0 \text{ or } 0 > n > m),$$

$$[\tilde{T}_n, \tilde{T}_0] = - (1 - t^{-1}) \sum_{\ell=1}^{n} \tilde{T}_{n-\ell} \tilde{T}_{\ell} - (t - t^{-1}) \sum_{\ell=1}^{\infty} t^{-\ell} \tilde{T}_{\ell} \tilde{T}_{n+\ell} \quad (n > 0),$$

$$[\tilde{T}_0, \tilde{T}_m] = - (1 - t^{-1}) \sum_{\ell=1}^{m} \tilde{T}_{\ell} \tilde{T}_{m+\ell} - (t - t^{-1}) \sum_{\ell=1}^{\infty} t^{-\ell} \tilde{T}_{m-\ell} \tilde{T}_{\ell} \quad (0 > m),$$

$$[\tilde{T}_n, \tilde{T}_m] = - (1 - t^{-1}) \tilde{T}_m \tilde{T}_n - (t - t^{-1}) \sum_{\ell=1}^{\infty} t^{-\ell} \tilde{T}_{m-\ell} \tilde{T}_{n+\ell}$$

$$+ (1 - t^{-1}) \delta_{n+m,0} \quad (n > 0 > m).$$

In [AKOS2], the above algebra is introduced and its free field representation is given. Let the bosons $b_n$ ($n \in \mathbb{Z}$) satisfy the relations $[b_n, b_m] = n \frac{1}{1 - t^{-1}} \delta_{n+m,0}$, $[b_n, Q] = \delta_{n,0}$. These
bosons can be regarded as the limit of the bosons \( a_n \), i.e., \( b_n = \lim_{q \to 0} a_n \), \( Q = \lim_{q \to 0} Q \).

Then \( \tilde{T}_n \) is represented as

\[
\tilde{T}_n = \oint \frac{dz}{2\pi \sqrt{-1} z} (\theta[n < 0] \tilde{\Lambda}^+(z) + \theta[n \geq 0] \tilde{\Lambda}^-(z)) z^n, \tag{2.17}
\]

where

\[
\tilde{\Lambda}^\pm(z) := \exp \left\{ \pm \sum_{n=1}^{\infty} \frac{1 - t^{-n}}{n} b_{-n} z^n \right\} \exp \left\{ \mp \sum_{n=1}^{\infty} \frac{1 - t^n}{n} b_n z^{-n} \right\} K^\pm = \lim_{q \to 0} \Lambda^\pm(z), \tag{2.18}
\]

and \( \theta[P] \) is 1 or 0 if the proposition \( P \) is true or false, respectively. By this free field representation, we can write the PBW type vectors in terms of Hall-Littlewood functions \( Q_\lambda \) defined in Appendix A.1:

\[
\tilde{T}_{\lambda} |\tilde{G}\rangle = k^{\ell(\lambda)} Q_\lambda(b_{-n}; t^{-1}) |\tilde{G}\rangle, \tag{2.19}
\]

\[
\langle h | \tilde{T}_{\lambda} = k^{-\ell(\lambda)} t^{|\lambda|} \langle h | Q_\lambda(-b_n; t^{-1}). \tag{2.20}
\]

Here \( Q_\lambda(b_{-n}; t^{-1}) \) is an abbreviation for \( Q_\lambda(b_{-1}, b_{-2}, \ldots; t^{-1}) \). This expression is shown by the theory of Jing’s operator (Fact A.1). Because of (A.7), they are diagonalized as

\[
\tilde{B}_{\lambda,\mu} := \langle \tilde{T}_{\lambda} | \tilde{T}_{\mu} \rangle = \frac{1}{b_\lambda(t^{-1})} \delta_{\lambda,\mu}, \tag{2.21}
\]

where \( b_\lambda(t) \) is defined in Appendix A.1. Since \( \tilde{B}_{\lambda,\mu} \) is non-degenerate, there is no singular vector at \( q \to 0 \) limit. The disappearance of singular vectors can be understood by the fact that the highest weight which have singular vectors diverge at \( q = 0 \). The Whittaker vector of this algebra is similarly defined.

**Definition 2.4.** Define the Whittaker vector \( |\tilde{G}\rangle \) by the relation

\[
\tilde{T}_1 |\tilde{G}\rangle = \tilde{\Lambda}^2 |\tilde{G}\rangle, \quad \tilde{T}_n |\tilde{G}\rangle = 0 \quad (n > 1). \tag{2.22}
\]

Similarly, the dual Whittaker vector \( \langle \tilde{G} | \in M^*_h \) is defined by

\[
\langle \tilde{G} | \tilde{T}_{-1} = \tilde{\Lambda}^2 \langle \tilde{G} |, \quad \langle \tilde{G} | \tilde{T}_n = 0 \quad (n < -1). \tag{2.23}
\]

Then the crystallized Whittaker vector is in the simple form

\[
|\tilde{G}\rangle = \sum_\lambda \tilde{\Lambda}^2 |\lambda\rangle \tilde{B}_{\lambda,\mu} |\tilde{\lambda}\rangle = \sum_{n=0}^{\infty} \tilde{\Lambda}^{2n} \frac{1}{b_{(1^n)}(t^{-1})} |\tilde{T}_{-1(1^n)}\rangle, \tag{2.24}
\]

and its inner product is

\[
\langle \tilde{G} | \tilde{G} \rangle = \sum_{n=0}^{\infty} \tilde{\Lambda}^{4n} \tilde{B}_{(1^n),(1^n)} = \sum_{n=0}^{\infty} \tilde{\Lambda}^{4n} \frac{1}{b_{(1^n)}(t^{-1})}. \tag{2.25}
\]

On the other hand, we can take the limit of the Nekrasov formula as follows.
Proposition 2.5. The renormalization \( \tilde{\Lambda}^2 := \Lambda^2(q/t)^{1/2} \) controls divergence at the \( q \to 0 \) limit (\( \Lambda \to \infty, \tilde{\Lambda} \) : fixed):

\[
\lim_{\Lambda^2 = \tilde{\Lambda}^2(t/q)^{1/2}} Z_{\text{inst}}^{\text{pure}} = \tilde{Z}_{\text{pure}}^{\text{inst}},
\]

(2.26)

\[
\tilde{Z}_{\text{pure}}^{\text{inst}} := \sum_{n,m \geq 0} \frac{\tilde{\Lambda}^{4(n+m)}}{\prod_{s=1}^{n}(1-t^{-s})(1-Q^{-1}t^{n-m-s}) \prod_{s=1}^{m}(1-t^{-s})(1-Q^{m-n-s})}.
\]

(2.27)

Proof. Removing parts which have singularity in the Nekrasov factor, we have

\[
N_{\lambda \mu}(Q) = q^{-\sum(i,j) \in J} N'_{\lambda \mu}(Q),
\]

(2.28)

\[
N'_{\lambda \mu}(Q) := \prod_{(i,j) \in \lambda} (1 - Qq^{A \lambda(i,j)}t^{L_{\lambda(i,j)}+1}) \prod_{(i,j) \in \mu} (q^{A_{\mu(i,j)}+1} - Qt^{-L_{\mu(i,j)}}).
\]

(2.29)

Hence,

\[
Z_{\text{inst}}^{\text{pure}} = \sum_{\lambda, \mu} \frac{(\tilde{\Lambda}^4 t^2)^{\lambda+|\mu|} q^{E_{\lambda \mu}}}{N_{\lambda \mu}(1) N'_{\lambda \mu}(Q) N'_{\mu \lambda}(1) N'_{\mu \lambda}(Q^{-1})},
\]

(2.30)

\[
E_{\lambda \mu} := 2 \left( \sum_{(i,j) \in \lambda} j + \sum_{(i,j) \in \mu} j - |\lambda| - |\mu| \right).
\]

(2.31)

If \( \lambda \neq (1^n) \) or \( \mu \neq (1^m) \) for any integer \( n, m \), then \( q^{E_{\lambda \mu}} \to 0 \) at \( q \to 0 \). Therefore, the sum with respect to partitions \( \lambda, \mu \) can be rewritten as the sum with respect to integers \( n, m \), i.e.,

\[
\tilde{Z}_{\text{pure}}^{\text{inst}} = \sum_{n,m \geq 0} \frac{(\tilde{\Lambda}^4 t^2)^{n+m}}{N'_{n,m}(1) N'_{n,m}(Q) N'_{m,n}(1) N'_{m,n}(Q^{-1})},
\]

(2.32)

\[
\tilde{N}'_{n,m}(Q) = (-1)^m Q^{-m-n+1} \prod_{s=1}^{n} (1 - Qt^{-m+s+1}).
\]

(2.33)

After some simple algebra we get (2.27).

Using these algebra and the Nekrasov function, we can get theorem which is an analog of Fact 2.3 in the \( q \to 0 \) limit and prove it more easily than the generic case.

Theorem 2.6.

\[
\langle \tilde{G} | \tilde{G} \rangle = \tilde{Z}_{\text{pure}}^{\text{inst}}.
\]

(2.34)

Note that the l.h.s. is independent of \( k \).

Proof. \( \tilde{Z}_{\text{pure}}^{\text{inst}} = \tilde{Z}_{\text{pure}}^{\text{inst}}(Q) \) can be rewrite as

\[
\tilde{Z}_{\text{pure}}^{\text{inst}}(Q) = \sum_{n,m \geq 0} \frac{\tilde{\Lambda}^{4(n+m)} Q^{n+m}}{\prod_{s=1}^{n}(1-t^{-s})(Q^{-1}t^{n-m-s}) \prod_{s=1}^{m}(1-t^{-s})(Q^{-1}t^m-n-s)},
\]

(2.35)

which has simple poles at \( Q = t^M \) with \(-m \leq M \leq n, M \neq n-m \) and \( M \in \mathbb{Z} \). Then

\[
\text{Res}_{Q=t^M} \tilde{Z}_{\text{pure}}^{\text{inst}}(Q) = \sum_{n,m \geq 0} \tilde{\Lambda}^{4(n+m)} Z_{(n,m)}^{(M)},
\]
To take the limit and apply the AGT conjecture for the deformed $W_{3.1}$ Reargument of Ding-Iohara-Miki algebra and AGT correspondence studied by [AY2].

Crystallization of the representation of Ding-Iohara-Miki algebra 

This limit singular vectors disappear. Hence it may be difficult to apply the AGT correspondence along [AFHKSY]. In this section, we use the AFLT basis in the 5D studies.

We now turn to the Ding-Iohara-Miki algebra. First, we reargue the AFLT basis in the 5D studies.

Expected that the limit can be taken for the general deformed $W$ is an odd function in $Q$.

Thus

\[ Z^{(M)}_{(n,m)} = 0. \quad (2.37) \]

Residues at every singularities in $Q$ of $Z^\text{inst}_{\text{pure}}(Q)$ vanish, but $|Z^\text{inst}_{\text{pure}}(\infty)| < \infty$. Hence $Z^\text{inst}_{\text{pure}}(Q)$ is independent of $Q$. Therefore,

\[ Z^\text{inst}_{\text{pure}}(Q) = Z^\text{inst}_{\text{pure}}(0) = \sum_{m \geq 0} \tilde{\Lambda}^{4m} \frac{1}{\prod_{s=1}^{m} (1 - t^{-s})}. \quad (2.40) \]

In this paper, we discuss the crystallization only of the deformed Virasoro algebra. It is expected that the limit can be taken for the general deformed $W_N$ algebra. However in the case of $W_3$, a pole like essential singularity appears and it isn’t easy to control the singularity. To take the limit and apply the AGT conjecture for the deformed $W_N$ algebra [Tak] is further studies.

In the crystallized case, the screening current diverges, which is one of the reason why in this limit singular vectors disappear. Hence it may be difficult to apply the AGT correspondence studied by [AY2].

3 Crystallization of the representation of Ding-Iohara-Miki algebra

3.1 Reargument of Ding-Iohara-Miki algebra and AGT correspondence

We now turn to the Ding-Iohara-Miki algebra. First, we reargue the AFLT basis in the 5D AGT correspondence along [AFHKSY]. In this section, we use $N$ kinds of bosons $a_n^{(i)} (n \in \mathbb{Z}, i = 1, 2, \ldots, N)$ and $Q^{(i)}$ with the relations

\[ [a_n^{(i)}, a_m^{(j)}] = n \frac{1 - q^{\min}}{1 - t^{\min}} \delta_{i,j} \delta_{n+m,0}, \quad (3.1) \]
\[ [a^{(i)}_n, Q^{(j)}] = \delta_{i,j}\delta_{n,0}, \quad [Q^{(i)}, Q^{(j)}] = 0, \quad (\forall i, j, n). \tag{3.2} \]

Here the number \( N \in \mathbb{N} \) corresponds to one of the \( SU(N) \) gauge theory or the \( W_N \) algebra. Let us define the vertex operators \( \eta^{(i)} \) and \( \varphi^{(i)} \) by

\[
\eta^{(i)}(z) = \exp\left( \sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} z^n a^{(i)}_n \right) \exp\left( -\sum_{n=1}^{\infty} \frac{(1-t^n)}{n} z^{-n} a^{(i)}_n \right), \tag{3.3} \\
\varphi^{(i)}(z) = \exp\left( \sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} (1-p^{-n}) z^n a^{(i)}_n \right), \tag{3.4} 
\]

and introduce the following generators

\[
X^{(1)}(z) := \sum_{k=1}^{N} \Lambda^k(z) =: \sum_n X^{(1)}_n z^{-n}, \tag{3.5} \\
X^{(i)}(z) := X^{(1)}(p^{i-1}z) \cdots X^{(1)}(pz) X^{(1)}(z) =: \sum_n X^{(i)}_n z^{-n},
\]

where

\[
\Lambda^i(z) = \varphi^{(1)}(zp^{\frac{N-1}{2}}) \varphi^{(2)}(zp^{\frac{N-3}{2}}) \cdots \varphi^{(i-1)}(zp^{\frac{N-2i+3}{2}}) \eta^{(i)}(zp^{\frac{N-2i+1}{2}}) U_i, \quad U_i := e^{\phi^0}. \tag{3.6} 
\]

The algebra generated by \( X^{(i)}_n \) is obtained by the level \( N \) representation of Ding-Iohara-Miki algebra \[ \text{FHHSY, FHSSY}. \]

**Proposition 3.1.** If \( N = 2 \), the commutation relations of the generators are

\[
f^{(1)}\left( \frac{w}{z} \right) X^{(1)}(z) X^{(1)}(w) - X^{(1)}(w) X^{(1)}(z) f^{(1)}\left( \frac{z}{w} \right) = \frac{(1-q)(1-t^{-1})}{1-p} \left\{ \delta\left( \frac{w}{pz} \right) X^{(2)}(z) - \delta\left( \frac{wu}{z} \right) X^{(2)}(w) \right\}, \tag{3.8} \\
f^{(2)}\left( \frac{w}{z} \right) X^{(2)}(z) X^{(2)}(w) - X^{(2)}(w) X^{(2)}(z) f^{(2)}\left( \frac{z}{w} \right) = 0, \tag{3.9} \\
f^{(1)}\left( \frac{wu}{z} \right) X^{(1)}(z) X^{(2)}(w) - X^{(2)}(w) X^{(1)}(z) f^{(1)}\left( \frac{z}{w} \right) = 0, \tag{3.10} 
\]

where \( \delta(x) = \sum_{n \in \mathbb{Z}} x^n \) is multiplicative delta function and the structure constant \( f^{(i)}(z) = \sum_{n=0}^{\infty} f^{(i)}_n z^n \) is defined by

\[
f^{(1)}(z) := \exp\left\{ \sum_{n>0} \frac{(1-q^n)(1-t^{-n})}{n} z^n \right\}, \tag{3.11} 
\]

\[ X^{(i)} \text{ can be identified with one of \[ \text{AFHKSY} \] by the equation} \]

\[
p^{-\frac{N-3}{2}} a^{(k)}_n = 1 \otimes \cdots \otimes a_n \otimes \cdots \otimes 1. \tag{3.7} 
\]

In the \( N = 1, 2 \) case, singularity can be controled by this renormalization. Note that the commutation relations of \( X^{(i)}(z) \) are not chainged.
\[ f^{(2)}(z) := \exp \left\{ \sum_{n>0} \frac{(1-q^n)(1-t^{-n})(1+p^n)}{n} z^n \right\}. \]  

(3.12)

These relations are equivalent to

\[ [X_n^{(1)}, X_m^{(1)}] = -\sum_{l=1}^{\infty} f_l^{(1)} (X_{n-l}^{(1)} X_{m+l}^{(1)} - X_{m-l}^{(1)} X_{n+l}^{(1)}) + \frac{(1-q)(1-t^{-1})}{1-p} (p^m - p^n) X_{n+m}^{(2)}, \]  

(3.13)

\[ [X_n^{(2)}, X_m^{(2)}] = -\sum_{l=1}^{\infty} f_l^{(2)} (X_{n-l}^{(2)} X_{m+l}^{(2)} - X_{m-l}^{(2)} X_{n+l}^{(2)}), \]  

(3.14)

\[ [X_n^{(1)}, X_m^{(2)}] = -\sum_{l=1}^{\infty} f_l^{(1)} (p^l X_{n-l}^{(1)} X_{m+l}^{(2)} - X_{m-l}^{(2)} X_{n+l}^{(1)}). \]  

(3.15)

In the formula (3.8) we use

\[ f^{(1)}(x) - f^{(1)}(1/px) = \frac{(1-q)(1-t^{-1})}{1-p} (\delta(x) - \delta(px)). \]  

(3.16)

Let \(|0\rangle\) and \(\langle 0|\) be the vacuum state and its dual vector such that \(a_n^{(i)} |0\rangle = 0\) (\(n \geq 0, \forall i\)) and \(\langle 0| a_n^{(i)} = 0\) (\(n \leq 0, \forall i\)). For an N-tuple of parameters \(\vec{u} = (u_1, \ldots, u_N)\), define \(|\vec{u}\rangle := \prod_{i=1}^{N} u_i^{Q(i)} |0\rangle\) and \(\langle \vec{u}| := \langle 0| \prod_{i=1}^{N} u_i^{-Q(i)}\). Then \(U_i |\vec{u}\rangle = u_i |\vec{u}\rangle\) and \(\langle \vec{u}| U_i = u_i \langle \vec{u}|\). In this paper, let parameters \(u_i\) be independent of \(q\). \(\mathcal{F}_{\vec{u}}\) is the highest weight module generated by \(|\vec{u}\rangle\), and \(\mathcal{F}_{\vec{u}}^+\) is the dual space generated by \(\langle \vec{u}|\). The PBW theorem can’t be used because the algebra generated by \(X_n^{(i)}\) isn’t Lie algebra, but the following conjecture is given in [AFHKSY].

**Conjecture 3.2.** The PBW type vectors

\[ X^{(1)}_{-\lambda_1^{(1)}} X^{(1)}_{-\lambda_2^{(1)}} \cdots X^{(2)}_{-\lambda_1^{(2)}} X^{(2)}_{-\lambda_2^{(2)}} \cdots X^{(N)}_{-\lambda_1^{(N)}} X^{(N)}_{-\lambda_2^{(N)}} \cdots |\vec{u}\rangle \]  

(3.17)

is a basis over \(\mathcal{F}_{\vec{u}}\) (resp. \(\mathcal{F}_{\vec{u}}^+\)), where \(\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(N)})\) is an N-tuple of partitions.

In the \(N = 2\) case, we can solve this problem with respect to another PBW type vector by considering its crystallization. The \(N = 1\) case is also solved in the same way.

**Definition 3.3.** For \(\vec{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(N)})\), set

\[ |X_{\vec{\lambda}}\rangle := X^{(N)}_{-\lambda_1^{(N)}} X^{(N)}_{-\lambda_2^{(N)}} \cdots X^{(2)}_{-\lambda_1^{(2)}} X^{(2)}_{-\lambda_2^{(2)}} \cdots X^{(1)}_{-\lambda_1^{(1)}} X^{(1)}_{-\lambda_2^{(1)}} \cdots |\vec{u}\rangle, \]  

(3.19)

\[ \langle X_{\vec{\lambda}}| := \langle \vec{u}| \cdots X^{(1)}_{\lambda_1^{(1)}} X^{(1)}_{\lambda_2^{(1)}} \cdots X^{(2)}_{\lambda_1^{(2)}} X^{(2)}_{\lambda_2^{(2)}} \cdots X^{(N)}_{\lambda_1^{(N)}} X^{(N)}_{\lambda_2^{(N)}}. \]  

(3.20)

**Proposition 3.4.** If \(N = 2\), then \(|X_{\vec{\lambda}}\rangle\) (resp. \(\langle X_{\vec{\lambda}}|\)) form a basis over \(\mathcal{F}_{\vec{u}}\) (resp. \(\mathcal{F}_{\vec{u}}^+\)).

The proof is given in the subsection 3.3 (Remark 3.24).

Let us review the AFLT basis in \(\mathcal{F}_{\vec{u}}\), which is also called generalized Macdonald functions. In order to state its existence theorem, let us prepare the following ordering.
Definition 3.5. For N-tuple of partitions \( \vec{\lambda} \) and \( \vec{\mu} \),

\[
\vec{\lambda} > \vec{\mu} \iff |\vec{\lambda}| = |\vec{\mu}|, \quad \sum_{i=k}^{N} |\lambda^{(i)}| > \sum_{i=k}^{N} |\mu^{(i)}| \quad (\forall k)
\]

(3.21)

\(|\vec{\lambda}| := |\lambda^{(1)}| + \ldots + |\lambda^{(N)}|\). Note that the second condition can be replaced with

\[
\sum_{i=1}^{k-1} |\lambda^{(i)}| < \sum_{i=1}^{k} |\mu^{(i)}| \quad (\forall k).
\]

In the basis of products of Macdonald functions, we can state the existence theorem of
generalized Macdonald functions.

Proposition 3.6. For each N-tuple of partitions \( \vec{\lambda} \), there exist an unique vector \( |P_{\vec{\lambda}}\rangle \in F_{\vec{u}} \)
such that

\[
|P_{\vec{\lambda}}\rangle = \prod_{i=1}^{N} P_{\lambda^{(i)}(a^{(i)}_{-n}; q, t)} |\vec{u}\rangle + \sum_{\vec{\mu} > \vec{\lambda}} c_{\vec{\lambda}, \vec{\mu}} \prod_{i=1}^{N} P_{\mu^{(i)}(a^{(i)}_{-n}; q, t)} |\vec{u}\rangle,
\]

(3.22)

\[
X^{(1)}_{0} |P_{\vec{\lambda}}\rangle = e_{\vec{\lambda}} |P_{\vec{\lambda}}\rangle,
\]

(3.23)

where \( c_{\vec{\lambda}, \vec{\mu}} = c_{\vec{\lambda}, \vec{\mu}}(u_{i}, q, t) \) is a constant, \( e_{\vec{\lambda}} = e_{\vec{\lambda}}(u_{i}, q, t) \) is an eigenvalue of \( X^{(1)}_{0} \) and \( P_{\lambda^{(i)}(a^{(i)}_{-n}; q, t)} \)
are Macdonald symmetric functions defined in Appendix A.1 with substituting the bosons
\( a^{(i)}_{-n} \) for power sum symmetric functions \( p_{n} \). Similarly, there exist an unique vector \( \langle P_{\vec{\lambda}} \| \rangle \in F_{\vec{u}}^{*} \)
such that

\[
\langle P_{\vec{\lambda}} | X^{(1)}_{0} \rangle = e_{\vec{\lambda}}^{*} \langle P_{\vec{\lambda}} | \rangle.
\]

(3.24)

(3.25)

Although the ordering of Definition 3.5 is different from one of [AFHKSY], \( |P_{\vec{\lambda}}\rangle \) coincides with one of [AFHKSY] under the relation (3.7) \(^4\). The proof is similar to the one in the
Appendix A.2 which follows from triangulation of \( X^{(1)}_{0} \). In Appendix A.2 a more elaborated
ordering is introduced and relationship between these orderings is explained.

To use generalized Macdonald functions in the AGT correspondence, we need to consider
its integral form. In this paper, we adopt following renormalization, which is slightly different
from one of [AFHKSY].

Definition 3.7. Define the vector \( |K_{\vec{\lambda}}\rangle \) and \( \langle K_{\vec{\lambda}} | \rangle \), called the integral form, by the condition that

\[
|K_{\vec{\lambda}}\rangle = \sum_{\vec{\mu}} \alpha_{\vec{\lambda}, \vec{\mu}} |X_{\vec{\mu}}\rangle \propto |P_{\vec{\lambda}}\rangle, \quad \alpha_{\vec{\lambda}, (\emptyset, \ldots, \emptyset, (1^{(i)})}) = 1
\]

(3.26)

and

\[
\langle K_{\vec{\lambda}} | \rangle = \sum_{\vec{\mu}} \beta_{\vec{\lambda}, \vec{\mu}} \langle X_{\vec{\mu}} | \rangle \propto \langle P_{\vec{\lambda}} | \rangle, \quad \beta_{\vec{\lambda}, (\emptyset, \ldots, \emptyset, (1^{(i)})}) = 1.
\]

(3.27)

\(^4\)If \( N = 2 \), transformation \( a^{(1)}_{-n} |\vec{u}\rangle \mapsto p_{n} \), \( a^{(2)}_{-n} |\vec{u}\rangle \mapsto (q/t)^{n} \hat{p}_{n} \) and \( \frac{\vec{w}}{n^{2}} = Q \) maps \( |P_{\vec{\lambda}}\rangle \) to \( (q/t)^{|\lambda^{(2)}|} M_{\vec{\lambda}} \) in \( \mathbb{Z} \).
Conjecture 3.8. The coefficients $\alpha_{\lambda \mu}$ and $\beta_{\lambda \mu}$ are polynomials in $q^{\pm 1}$, $t^{\pm 1}$ and $u_i$ with integer coefficients.

Example 3.9. If $N = 2$, the transition matrix $\alpha_{\lambda \mu}$ is as follows:

| $\lambda \setminus \mu$ | $(\emptyset, (1))$ | $(1, \emptyset)$ | $(1, (1))$ | $(1, (2))$ | $(2, (\emptyset))$ | $(2, (1))$ | $(2, (2))$ |
|----------------------|------------------|-----------------|----------------|-----------------|----------------|----------------|----------------|
| $(\emptyset, (1))$   | $1$              | $\frac{q u_2}{t}$ | $1$            | $\frac{q u_2}{t}$ |
| $(1, \emptyset)$     | $\frac{q(t-1)u_2}{t}$ | $1$              | $\frac{q(t+1)u_2}{t}$ |
| $(1, (1))$           | $\frac{(q-1)q(t-1)(u_2^2 + u_2 u_1 + u_1^2)}{t^2}$ | $1$              | $\frac{q(u_1 + u_2)}{t}$ |
| $(1, (2))$           | $\frac{(q-1)q(t-1)(u_2^2 + u_2 u_1 + u_1^2)}{t^2}$ | $1$              | $\frac{q(u_1 + u_2)}{t}$ |
| $(2, \emptyset)$     | $\frac{q(t-1)u_1}{t^2}$ | $1$              | $\frac{q(t+1)u_1}{t^2}$ |
| $(2, (1))$           | $\frac{q(t-1)u_1}{t^2}$ | $1$              | $\frac{q(t+1)u_1}{t^2}$ |
| $(2, (2))$           | $\frac{(q-1)q(t-1)(u_2^2 + u_2 u_1 + u_1^2)}{t^2}$ | $1$              | $\frac{q(u_1 + u_2)}{t}$ |

Then the norm of $\| K_\chi \|$ reproduces the Nekrasov factor.

Conjecture 3.10.

$$
\langle K_\chi | K_\chi \rangle \equiv (-1)^N e_N(\vec{u}) \prod_{i=1}^{N} f_{Nn(\lambda(i))} q^{Nn(\lambda(i))} u_i^{N|\lambda(i)|} \prod_{i,j=1}^{N} N_{\lambda(i), \lambda(j)} (qu_i / tu_j),
$$

where $e_N(\vec{u}) = u_1 u_2 \cdots u_N$.

Definition 3.11. Define the intertwining operator $\Phi(z) = \Phi_\vec{v}^i(z) : \mathcal{F}_{\vec{u}} \to \mathcal{F}_{\vec{v}}$ by

$$
(1 - e_N(\vec{v})w / z) X^{(i)}(z) \Phi(w) = (1 - p^{-i} e_N(\vec{v})w / z) \Phi(w) X^{(i)}(z)
$$

and $\langle \vec{v} | \Phi(w) | \vec{u} \rangle = 1$. Then the relations for the Fourier components are

$$
(X_n^{(i)} - e_N(\vec{v}) w X_{n-1}^{(i)}) \Phi(w) = \Phi(w)(X_n^{(i)} - (t/q)^i e_N(\vec{v}) w X_{n-1}^{(i)}
$$

for $i = 1, 2, \ldots, N$.

Example 3.12. If $N = 1$, it is known that $\Phi(z)$ has explicit form

$$
\Phi(z) = \exp \left\{ - \sum_{n=1}^{\infty} \frac{1}{n} \frac{u^{-n}}{1 - q^n} a_n z^n \right\} \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{n} \frac{v^{-n}}{1 - q^n} a_n z^{-n} \right\} (v/u)^Q.
$$

(3.31)
Conjecture 3.13. The matrix elements of \( \Phi(w) \) with respect to generalized Macdonald functions are
\[
\langle \tilde{K}_{\lambda} | \Phi_{\mu}(w) | \tilde{K}_{\mu} \rangle \equiv (-1)^{\ell} (N - 1) (q/t) N \langle \lambda | h_{\mu} | w \rangle e_{N(\lambda)} e_{N(\mu)} e_{N(\mu)} e_{N(\lambda)} (u/w) (v/w) (z/w)
\]
\[
\times \prod_{i=1}^{N} u_i^{\mu_i(\ell)} q N n(\mu) t^{-N n(\mu)} \times \prod_{i,j=1}^{N} \chi_{\mu}(\lambda, \mu) (q_v/t u_j).
\]

Under these conjectures, we can obtain the formula for multi-point correlation functions of \( \Phi(z) \), and the formula for 4-point functions agree with the 5D \( U(N) \) Nekrsov formula with 4 matters. This M-theoretic derivation is also given by [Tan].

3.2 Crystallization of \( N = 1 \) case

Next, we discuss a crystallization of the results of the last subsection. At first, let us demonstrate the \( q \to 0 \) limit in the \( N = 1 \) case. In this subsection, let us use the same bosons \( b_n \) and \( Q \) as subsection 2.2. Since singularity in \( \Phi(z) \) can be removed by normalization \( \Phi(pz) \), define the vertex operator \( \Phi(z) \) by
\[
\tilde{\Phi}(z) := \lim_{q \to 0} \Phi(pz) = \exp \left\{ \sum_{n=1}^{\infty} \frac{u^n b_n z^n}{n} \right\} \exp \left\{ \sum_{n=1}^{\infty} \frac{1}{n} \frac{u^n - v^n}{t^n} b_n z^n \right\} (v/u)^{Q}.
\]
If \( N = 1, |P_{\lambda} \rangle \) are ordinary Macdonald functions, and their integral form \( |K_{\lambda} \rangle \) have, at \( q = 0 \), the relation
\[
|\tilde{K}_{\lambda} \rangle := \lim_{q \to 0} |K_{\lambda} \rangle = (-u/t)^{|\lambda|} t^{-n(\lambda)} Q_{\lambda}(b_n; t) |u \rangle,
\]
\[
\langle \tilde{K}_{\lambda} | := \lim_{q \to 0} \langle K_{\lambda} | = (-u)^{|\lambda|} t^{-n(\lambda)} \langle u | Q_{\lambda}(b_n; t).
\]
Hence, the matrix elements \( \langle \tilde{K}_{\lambda} | \tilde{\Phi}(x) | \tilde{K}_{\mu} \rangle \) can be written in terms of integrals by virtue of the theory of Jing’s operator \( H(z) \) and \( H^{\dagger}(z) \) defined in (A.8) and (A.9). Using the usual normal ordering product \( : \), with respect to the bosons \( b_n \) and \( Q \), we have
\[
H^{\dagger}(w_{\ell}(\lambda)) \cdots H^{\dagger}(w_1) \tilde{\Phi}(x) H(z_1) \cdots H(z_{\ell(\mu)})
\]
\[
= \mathcal{J}(w, x, z) : H^{\dagger}(w_{\ell}(\lambda)) \cdots H^{\dagger}(w_1) \tilde{\Phi}(x) H(z_1) \cdots H(z_{\ell(\mu)}) :,
\]
\[
\mathcal{J}(w, x, z) := \prod_{1 \leq \ell < i \leq \ell(\lambda)} \frac{w_{i} - w_{j}}{w_{i} - t w_{j}} \prod_{1 \leq i < j \leq \ell(\mu)} \frac{z_{i} - z_{j}}{z_{i} - t z_{j}} \prod_{1 \leq i \leq \ell(\mu)} \frac{w_{i} - t z_{j}}{w_{i} - z_{j}}
\]
\[
\times \prod_{1 \leq i \leq \ell(\mu)} \frac{x - (t/v) z_{i}}{x - (t/u) z_{i}} \prod_{1 \leq i \leq \ell(\lambda)} \frac{w_{i}}{w_{i} - u x}.
\]

Let \( \mathcal{H} \) be the Heisenberg algebra generated by the bosons \( b_n \) \( (n \in \mathbb{Z}) \), \( Q \) and 1. \( \mathcal{H}_c \) is the algebra obtained by making \( \mathcal{H} \) commutative. The normal ordering product \( : \) is defined to be the linear map from \( \mathcal{H}_c \) to \( \mathcal{H} \) such that for \( P \in \mathcal{H}_c \),
\[
: P b_n : = \begin{cases} : P : & n \geq 0, \\ b_n : P : & n < 0, \end{cases}
\]
\[
: P Q : = Q : P :.
\]
and \( : 1 : = 1 \). In the next subsection, the same symbol \( : \) denotes the normal ordering product with respect to the bosons \( b^{(i)}_n \) which is defined similarly.
Thus
\[
\langle \tilde{K}_\lambda | \tilde{\Phi}(x) | \tilde{K}_\mu \rangle = (-u)^{|\lambda|+|\mu|} t^{-n(\lambda)-n(\lambda)} | \mu | \int \frac{dz}{2\pi \sqrt{-1}z} \frac{dw}{2\pi \sqrt{-1}w} \mathcal{I}(w, x, z) z^{-\mu} w^{\lambda},
\]
(3.39)
where \(\int \frac{dz}{2\pi \sqrt{-1}z} \frac{dw}{2\pi \sqrt{-1}w} := \int \prod_{i=1}^{\ell(\mu)} \frac{dz_i}{2\pi \sqrt{-1}z_i} \prod_{i=1}^{\ell(\lambda)} \frac{dw_i}{2\pi \sqrt{-1}w_i},\) \(z^{-\mu} := z_1^{-\mu_1} \cdots z_{\ell(\mu)}^{-\mu_{\ell(\mu)}},\) \(w^{\lambda} := w_1^\lambda \cdots w_{\ell(\lambda)}^\lambda,\) and the integration contour is \(|w_{\ell(\lambda)}| > \cdots > |w_1| > |x| > |z_1| > \cdots |z_{\ell(\mu)}|.\)
This integral reproduce the \(q \to 0\) limit of the Nekrasov factor.

**Definition 3.14.** Set
\[
\tilde{N}_{\lambda, \mu}(Q) := \lim_{q \to 0} q^{n(\mu')} N_{\lambda, \mu}(q/t) Q
\]
(3.40)
\[
= (-Qt^{-1})^{\hat{\mu}_j} t^{-\sum_{(i,j) \in \hat{\mu}} L_{\lambda}(i,j)} \prod_{(i,j) \in \mu - \hat{\mu}} (1 - Qt^{-L_{\lambda}(i,j) - 1}),
\]
where \(\hat{\mu}\) is the set of boxes in \(\mu\) whose arm-length \(A_{\mu}(i, j)\) is not zero. For example, if \(\mu = (5, 3, 3, 1), \hat{\mu} = (4, 2, 2).\) This Nekrasov factor has the property \(\tilde{N}_{\lambda\emptyset}(Q) = 1\) for any \(\lambda.\)

Therefore, the conjecture in the crystallized case of \(N = 1\) is
\[
\int \frac{dz}{2\pi \sqrt{-1}z} \frac{dw}{2\pi \sqrt{-1}w} \mathcal{I}(w, x, z) z^{-\mu} w^{\lambda} \equiv \tilde{N}_{\lambda, \mu}(v/u) x^{[\lambda] - [\mu]} u^{[\lambda]} (-v)^{[\mu]} |u| |v| n + n(\lambda).
\]
(3.41)
The case of some particular partitions can be checked by calculating the contour integral (Appendix 3.5).

### 3.3 Crystallization of \(N = 2\) case
In this subsection, let us use the bosons \(b_n^{(i)} (n \in \mathbb{Z}, i = 1, 2)\) and \(Q^{(i)}\) with the relation
\[
[b_n^{(i)}, b_m^{(j)}] = n \frac{1}{1 - t^{[n]}} \delta_{i,j} \delta_{n+m,0}, \quad [b_n^{(i)}, Q^{(j)}] = 0,
\]
(3.42)
and regard \(b_n^{(i)} = \lim_{q \to 0} a_n^{(i)}, Q^{(i)} = \lim_{q \to 0} Q^{(i)}\). Let us define the generator at \(q \to 0\).

**Definition 3.15.** Set
\[
\tilde{X}_n^{(1)} := \lim_{q \to 0} q^{[n]} X_n^{(1)}.
\]
(3.43)

**Proposition 3.16.** Definition 3.15 is well-defined, i.e., \(q^{[n]} X_n^{(1)}\) has no singularity at \(q = 0,\) and its free field representation is
\[
\tilde{X}_n^{(1)} = \int \frac{dz}{2\pi \sqrt{-1}z} \left\{ \theta[n \geq 0] \tilde{\Lambda}^1(z) + \theta[n \leq 0] \tilde{\Lambda}^2(z) \right\} z^n,
\]
(3.44)
where \(\theta\) is defined in Section 2.2 and
\[
\tilde{\Lambda}^1(z) := \exp \left\{ \sum_{n>0} \frac{1-t^{-n}}{n} z^n b_n^{(1)} \right\} \exp \left\{ - \sum_{n>0} \frac{1-t^n}{n} z^{-n} b_n^{(1)} \right\} U_1,
\]
(3.45)
\[
\tilde{\Lambda}^2(z) := \exp \left\{ - \sum_{n>0} \frac{1-t^{-n}}{n} z^n b_n^{(1)} \right\} \exp \left\{ \sum_{n>0} \frac{1-t^n}{n} z^{-n} b_n^{(1)} \right\} \exp \left\{ - \sum_{n>0} \frac{1-t^n}{n} z^{-n} b_n^{(2)} \right\} U_2.
\]
(3.46)
Proof. Define $\Lambda^1_n$ and $\Lambda^2_n$ by

$$
\Lambda^1(z) := \sum_{n \in \mathbb{Z}} \Lambda^1_n p^{-n/2} z^{-n}, \quad \Lambda^2(z) := \sum_{n \in \mathbb{Z}} \Lambda^2_n p^{n/2} z^{-n},
$$

we can see $\Lambda^i_n$ is well-behaved in the limit $q \to 0$ by the form of $\Lambda^i(z)$. If $n > 0$,

$$
\tilde{X}^{(1)}_n = \lim_{q \to 0} (\Lambda^1 + \Lambda^2_n) = \lim_{q \to 0} \Lambda^1_n = \oint \frac{dz}{2\pi \sqrt{-1}z} \tilde{\Lambda}^1(z) z^n,
$$

if $n < 0$,

$$
\tilde{X}^{(1)}_n = \lim_{q \to 0} (\Lambda^1_n p^{-n} + \Lambda^2_n) = \lim_{q \to 0} \Lambda^2_n = \oint \frac{dz}{2\pi \sqrt{-1}z} \tilde{\Lambda}^2(z) z^n,
$$

and if $n = 0$,

$$
\tilde{X}^{(1)}_n = \lim_{q \to 0} (\Lambda^1_n + \Lambda^2_n) = \oint \frac{dz}{2\pi \sqrt{-1}z} (\tilde{\Lambda}^1(z) + \tilde{\Lambda}^2(z)).
$$

Thus $\tilde{X}^{(1)}_n$ is well-defined and (3.34) is the natural free field representation. □

For the second generator, the following rescale is suitable.

Definition 3.17. Set

$$
\tilde{X}^{(2)}_n := \lim_{q \to 0} p^q X^{(2)}_n.
$$

Proposition 3.18. The free field representation of $\tilde{X}^{(2)}_n$ is obtained by

$$
\tilde{X}^{(2)}_n = \oint \frac{dz}{2\pi \sqrt{-1}z} \tilde{X}^{(2)}(z) z^n,
$$

where

$$
\tilde{X}^{(2)}(z) = (z \tilde{\Lambda}^1(\tilde{z}) \tilde{\Lambda}^2(z));
$$

$$
= \exp \left\{ \sum_{n>0} \frac{1-t^{-n}}{n} z^n b^{(2)}_n \right\} \exp \left\{ - \sum_{n>0} \frac{1-t^n}{n} z^{-n} b^{(1)}_n \right\} \exp \left\{ - \sum_{n>0} \frac{1-t^n}{n} z^{-n} b^{(2)}_n \right\} U_1 U_2.
$$

This proposition is easily seen by checking $\lim_{q \to 0} X^{(2)}(p^{-1/2}z)$. We can calculate the commutation relation of these generators as follows.

Proposition 3.19. The generators $\tilde{X}^{(1)}_n$ and $\tilde{X}^{(2)}_n$ satisfy the relations

$$
[\tilde{X}^{(1)}_n, \tilde{X}^{(1)}_m] = -(1-t^{-1}) \sum_{l=1}^{n-m} \tilde{X}^{(1)}_{n-l} \tilde{X}^{(1)}_{m+l} \quad (n > m > 0 \text{ or } 0 > n > m),
$$

$$
[\tilde{X}^{(1)}_n, \tilde{X}^{(1)}_0] = -(1-t^{-1}) \sum_{l=1}^{n-1} \tilde{X}^{(1)}_{n-l} \tilde{X}^{(1)}_l - (1-t^{-1}) \sum_{l=1}^{\infty} \tilde{X}^{(1)}_{n-l} \tilde{X}^{(1)}_{n+l} + (1-t^{-1}) \tilde{X}^{(1)}_n \quad (n > 0),
$$

$$
[\tilde{X}^{(1)}_n, \tilde{X}^{(1)}_m] = -(1-t^{-1}) \sum_{l=0}^{\infty} \tilde{X}^{(1)}_{m-l} \tilde{X}^{(1)}_{n+l} + (1-t^{-1}) \tilde{X}^{(1)}_{n+m} \quad (n > 0 > m),
$$

$$
[\tilde{X}^{(1)}_0, \tilde{X}^{(1)}_m] = -(1-t^{-1}) \sum_{l=1}^{n-1} \tilde{X}^{(1)}_{l} \tilde{X}^{(1)}_{m+l} - (1-t^{-1}) \sum_{l=1}^{\infty} \tilde{X}^{(1)}_{m-l} \tilde{X}^{(1)}_{l} + (1-t^{-1}) \tilde{X}^{(1)}_m \quad (0 > m),
$$

(3.57)
\[ [\tilde{X}_n^{(1)}, \tilde{X}_m^{(2)}] = (1 - t^{-1}) \sum_{l=1}^{\infty} \tilde{X}_{m-l}^{(2)} \tilde{X}_{n+l}^{(1)} \quad (n > 0, \ \forall m), \quad (3.58) \]

\[ [\tilde{X}_0^{(1)}, \tilde{X}_m^{(2)}] = -(1 - t^{-1}) \sum_{l=1}^{\infty} (\tilde{X}_l^{(1)} \tilde{X}_{m+l}^{(2)} - \tilde{X}_{m-l}^{(2)} \tilde{X}_l^{(1)}) \quad (\forall m), \quad (3.59) \]

\[ [\tilde{X}_n^{(1)}, \tilde{X}_m^{(2)}] = -(1 - t^{-1}) \sum_{l=1}^{\infty} \tilde{X}_{n-l}^{(1)} \tilde{X}_{m+l}^{(2)} \quad (n < 0, \ \forall m), \quad (3.60) \]

\[ [\tilde{X}_n^{(2)}, \tilde{X}_m^{(2)}] = -(1 - t^{-1}) \sum_{l=1}^{\infty} (\tilde{X}_{n-l}^{(2)} \tilde{X}_{m+l}^{(2)} - \tilde{X}_{m-l}^{(2)} \tilde{X}_l^{(1)}) \quad (\forall n, m). \quad (3.61) \]

**Proof.** These are obtained by the following relation of generating functions:

\[ g \left( \frac{w}{z} \right) \tilde{\Lambda}^1(z) \tilde{\Lambda}^1(w) - g \left( \frac{z}{w} \right) \tilde{\Lambda}^1(w) \tilde{\Lambda}^1(z) = 0, \quad (3.62) \]

\[ g \left( \frac{w}{z} \right) \tilde{\Lambda}^2(z) \tilde{\Lambda}^2(w) - g \left( \frac{z}{w} \right) \tilde{\Lambda}^2(w) \tilde{\Lambda}^2(z) = 0, \quad (3.63) \]

\[ \tilde{\Lambda}^1(z) \tilde{\Lambda}^2(w) + \left( g \left( \frac{z}{w} \right) - 1 - t^{-1} \right) \tilde{\Lambda}^2(w) \tilde{\Lambda}^1(z) = (1 - t^{-1}) \delta \left( \frac{w}{z} \right) ; \tilde{\Lambda}^1(z) \tilde{\Lambda}^2(w) ; \quad (3.64) \]

\[ \tilde{\Lambda}^1(z) \tilde{\tilde{X}}^{(2)}(w) - g \left( \frac{z}{w} \right) \tilde{\tilde{X}}^{(2)}(w) \tilde{\Lambda}^1(z) = 0, \quad (3.65) \]

\[ g \left( \frac{w}{z} \right) \tilde{\Lambda}^2(z) \tilde{\tilde{X}}^{(2)}(w) - \tilde{\tilde{X}}^{(2)}(w) \tilde{\Lambda}^2(z) = 0, \quad (3.66) \]

\[ g \left( \frac{w}{z} \right) \tilde{\tilde{X}}^{(2)}(z) \tilde{\tilde{X}}^{(2)}(w) - g \left( \frac{z}{w} \right) \tilde{\tilde{X}}^{(2)}(w) \tilde{\tilde{X}}^{(2)}(z) = 0, \quad (3.67) \]

where

\[ g(x) = \exp \left\{ \sum_{n>0} \frac{1 - t^{-1}}{n} x^n \right\} = 1 + (1 - t^{-1}) \sum_{l=1}^{\infty} x^l, \quad (3.68) \]

and for (3.64) we used the formula

\[ g(x) + g(x^{-1}) - 1 - t^{-1} = +(1 - t^{-1}) \delta(x). \quad (3.69) \]

The algebra generated by \( X^{(1)}_n \) and \( X^{(2)}_n \) is closely related to Hall-Littlewood functions. In particular, the PBW type vectors can be written as the product of two Hall-Littlewood functions.

**Definition 3.20.** For a pair of partitions \( \tilde{\lambda} = (\lambda^{(1)}, \lambda^{(2)}) \), set

\[ |\tilde{X}_{\tilde{\lambda}}| := \tilde{X}_{-\lambda^{(2)}_{-l_2}}^{(2)} \cdots \tilde{X}_{-\lambda^{(1)}_{-l_1}}^{(1)} \tilde{X}_{\lambda^{(1)}_{l_1}}^{(1)} \cdots |\tilde{u}|, \quad (3.70) \]

\[ \langle \tilde{X}_{\tilde{\lambda}} \rangle := \langle \tilde{u} \rangle \cdots \tilde{X}_{\lambda^{(1)}_{l_1}}^{(1)} \tilde{X}_{-\lambda^{(1)}_{-l_1}}^{(1)} \cdots \tilde{X}_{\lambda^{(2)}_{l_2}}^{(2)} \tilde{X}_{-\lambda^{(2)}_{-l_2}}^{(2)}. \quad (3.71) \]

We have the expression of these vectors in terms of Hall-Littlewood polynomials.
Proposition 3.21.

\[ |\tilde{X}_{\lambda,\mu}\rangle = (u_1 u_2)^{\ell(\lambda)} u_2^{\ell(\mu)} Q_\mu(b_\mu(+) ; t^{-1}) Q_\lambda(b_\lambda(-) ; t^{-1}) |\bar{u}\rangle, \]

\[ \langle \tilde{X}_{\lambda,\mu} | = u_1^{\ell(\lambda)} (u_1 u_2)^{\ell(\mu)} Q_\mu(b_\mu(-) ; t^{-1}) Q_\lambda(b_\lambda(+) ; t^{-1}), \]

where

\[ b_\mu^+(n) := b_\mu^{(1)} + b_\mu^{(2)}, \quad b_\mu^-(n) := b_\mu^{(2)} \quad (n > 0), \]

\[ b_\lambda^+(n) := b_\lambda^{(1)}, \quad b_\lambda^-(n) := -b_\lambda^{(1)} + b_\lambda^{(2)} \quad (n > 0). \]

The vectors \( \tilde{X}_{-\lambda_1^{(1)}}, \tilde{X}_{-\lambda_2^{(1)}}, \cdots, \tilde{X}_{-\lambda_1^{(2)}}, \tilde{X}_{-\lambda_2^{(2)}}, \cdots |\bar{u}\rangle \) don’t have such a good expression. This proposition is proven by the theory of Jing’s operator. Then the vectors \( |\tilde{X}_\lambda\rangle \) are partially diagonalized as the following proposition. Furthermore, with the help of Hall-Littlewood functions, we can calculate the Shapovalov matrix \( S_{\lambda,\mu} := \langle \tilde{X}_\lambda | \tilde{X}_\mu \rangle \) and its inverse \( S^{\tilde{X}_{\lambda,\mu}} \).

Proposition 3.22. We can represent \( S_{\lambda,\mu} \) by the inner product \( \langle -, - \rangle_{0,t} \) of Hall-Littlewood functions defined in Appendix A.1:

\[ S_{\lambda,\mu} = (u_1 u_2)^{\ell(\lambda^{(2)}) + \ell(\mu^{(2)})} u_1^{\ell(\lambda^{(1)})} u_2^{\ell(\mu^{(1)})} \]

\[ \times \frac{1}{b_\lambda^{(1)}(t^{-1})} \langle Q_\lambda^{(2)}(p_n; t^{-1}), Q_\mu^{(2)}(-p_n; t^{-1}) \rangle_{0,t-1} \delta_{\lambda^{(1)},\mu^{(1)}}, \]

\[ S^{\tilde{X}_{\lambda,\mu}} = (u_1 u_2)^{-\ell(\mu^{(2)}) - \ell(\lambda^{(2)})} u_1^{-\ell(\mu^{(1)})} u_2^{-\ell(\lambda^{(1)})} \]

\[ \times \frac{b_\mu^{(1)}(t^{-1})}{b_\lambda^{(2)}(t^{-1}) b_\mu^{(2)}(t^{-1})} \langle Q_\lambda^{(2)}(-p_n; t^{-1}), Q_\mu^{(2)}(p_n; t^{-1}) \rangle_{0,t-1} \delta_{\lambda^{(1)},\mu^{(1)}}. \]

Proof. (3.76) follows from Proposition 3.21. (3.77) can be checked by the equation

\[ \sum_{\mu} \langle Q_\lambda(p_n; t), Q_\mu(-p_n; t) \rangle_{0,t} \langle Q_\mu(-p_n; t), Q_\nu(p_n; t) \rangle_{0,t} \]

\[ = b_\lambda(t) \delta_{\lambda,\nu}, \]

which is given by inserting the complete system with respect to \( Q_\mu(-p_n; t) \) into the equation

\[ \langle Q_\lambda(p_n; t), Q_\nu(p_n; t) \rangle_{0,t} = b_\lambda(t) \delta_{\lambda,\nu}. \]

Existence of the inverse matrix \( S^{\tilde{X}_{\lambda,\mu}} \) leads linear independence of \( |\tilde{X}_\lambda\rangle \). Since there are the same number of linear independent vectors as the dimension of each level of \( F_\bar{u} \), we can see that \( |\tilde{X}_\lambda\rangle \) forms a basis over \( F_\bar{u} \).

Proposition 3.23. If \( t \) is not a root of unity and \( u_1, u_2 \neq 0 \), \( |\tilde{X}_\lambda\rangle \) (resp. \( \langle \tilde{X}_\lambda \rangle \)) is a basis over \( F_\bar{u} \) (resp. \( F^*_\bar{u} \)).

Remark 3.24. Since vectors which are linear dependent for the generic \( q \) are not linear independent in any limit, we can show that the PBW type vectors \( X_\lambda \) and \( \langle X_\lambda \rangle \) in generic \( q \) case form a basis over \( F_\bar{u} \) and \( F^*_\bar{u} \), respectively. (Proposition 3.4)

---

6 In this paper, the parameters \( u_i \) is independent of \( q \). If not so, \( |X_\lambda\rangle \) can be linear dependent. To clarify when \( |X_\lambda\rangle \) form a basis in the case that \( u_i \in \mathbb{Q}(q,t) \), we need to investigate their Kac determinant.
Next, let us introduce generalized Hall-Littlewood functions which are specialization of generalized Macdonald functions and give conjectures of AGT correspondence at $q = 0$.

**Definition 3.25.** Define the vector $|\tilde{P}_\lambda\rangle$ and $\langle\tilde{P}_\lambda|$ as the $q \to 0$ limit of generalized Macdonald functions, i.e.,

$$|\tilde{P}_\lambda\rangle := \lim_{q \to 0} |P_{\lambda}\rangle, \quad \langle\tilde{P}_\lambda| := \lim_{q \to 0} \langle P_{\lambda}|.$$ (3.79)

We call the vectors $|\tilde{P}_\lambda\rangle$ generalized Hall-Littlewood functions.

These are the eigenvectors of $\tilde{X}_0^{(1)}$:

$$\tilde{X}_0^{(1)}|\tilde{P}_\lambda\rangle = \tilde{e}_\lambda^s|\tilde{P}_\lambda\rangle, \quad \langle\tilde{P}_\lambda|\tilde{X}_0^{(1)} = \tilde{e}_\lambda^s\langle\tilde{P}_\lambda|$$ (3.80)

Moreover the eigenvalues are

$$\tilde{e}_\lambda^s = \tilde{e}_\lambda^s = \sum_{k=1}^{2} u_k \left( 1 + (1 - t) \sum_{i \geq 1} t^{-i} \right).$$ (3.81)

However there are too many degenerate eigenvalues to ensure the existence of generalized Hall-Littlewood functions. It is difficult to characterize $|\tilde{P}_\lambda\rangle$ as the eigenfunction of only $\tilde{X}_0^{(1)}$.

For example, $\lambda = ((1), (2))$ and $\mu = ((2), (1))$ have the relation $\lambda^s > \mu$, but $\tilde{e}_\lambda = \tilde{e}_\mu$.

**Example 3.26.** The transition matrix $\tilde{c}_{\lambda,\mu}$ is as follows, where

$$|\tilde{P}_\lambda\rangle = \sum_{\mu} \tilde{c}_{\lambda,\mu} \prod_{i=1}^{2} P_{\mu(i)} (a_n^{(i)}, t) |\tilde{u}\rangle.$$ (3.82)

```
| $\lambda \setminus \mu$ | ($\emptyset$, (1)) | ((1), $\emptyset$) |
|------------------|------------------|------------------|
| ($\emptyset$, (1)) | 1                | $\frac{u_2 \bar{u}_2}{u_1 - u_2}$ |
| ((1), $\emptyset$) | 0                | 1                |
| ($\emptyset$, (2)) | 1                | 0                |
| (0, (1))          | 0                | 1                |
| (1, (1))          | 0                | 0                |
| (2, $\emptyset$)  | 0                | 0                |
```

The transition matrix $\tilde{c}_{\lambda,\mu}^s$ is as follows, where

$$\langle\tilde{P}_\lambda| = \sum_{\mu} \tilde{c}_{\lambda,\mu}^s \langle\tilde{u}| \prod_{i=1}^{2} P_{\mu(i)} (a_n^{(i)}, t).$$

```
| $\lambda \setminus \mu$ | ((1), $\emptyset$) | (0, (1)) |
|------------------|------------------|-----------|
| ($\emptyset$, (1)) | 1                | $-\frac{u_2 \bar{u}_2}{u_1 - u_2}$ |
| ((0, (1))        | 0                | 1                |
```

Definition 3.25 is given under the hypothesis that the vector $|P_{\lambda}\rangle$ has no singularity in the limit $q \to 0$. If we can show the existence theorem of both generalized Macdonald and generalized Hall-Littlewood functions by using the same partial ordering and the same basis, this hypothesis is guaranteed.
Definition 3.27. The integral form $|\tilde{K}_\lambda\rangle$ and $\langle \tilde{K}_\lambda|$ are defined by

$$|\tilde{K}_\lambda\rangle = \sum_{\tilde{\mu}} \tilde{\alpha}_{\tilde{\lambda}\tilde{\mu}} X_{\tilde{\mu}} \propto |\tilde{P}_\lambda\rangle, \quad \tilde{\alpha}_{\tilde{\lambda},(0,(1^{\tilde{\lambda}}))} = 1,$$

$$\langle \tilde{K}_\lambda| = \sum_{\tilde{\mu}} \tilde{\beta}_{\tilde{\lambda}\tilde{\mu}} X_{\tilde{\mu}} \propto \langle \tilde{P}_\lambda|, \quad \tilde{\beta}_{\tilde{\lambda},(0,(1^{\tilde{\lambda}}))} = 1.$$  (3.84)

Note that the coefficients $\tilde{\alpha}_{\tilde{\lambda},(1^{\tilde{\lambda}}),0}$ and $\tilde{\beta}_{\tilde{\lambda},(1^{\tilde{\lambda}}),0}$ can be zero at $q = 0$.

Conjecture 3.28.

$$\langle \tilde{K}_\lambda| \tilde{K}_\lambda \rangle = (u_1 u_2)^2 \sum_{\tilde{\lambda}} \tilde{\alpha}_{\tilde{\lambda},(1^{\tilde{\lambda}})} \tilde{\beta}_{\tilde{\lambda},(1^{\tilde{\lambda}})} \prod_{i,j=1}^2 \tilde{N}_{\lambda(i),\lambda(j)} (u_i/u_j).$$  (3.85)

Next, let us define the vertex operator at crystal limit.

Definition 3.29. The vertex operator $\tilde{\Phi}(z) : F_{\tilde{\mu}} \rightarrow F_{\tilde{\nu}}$ is defined by the relations

$$\tilde{X}_{n}^{(1)} \tilde{\Phi}(z) = \tilde{\Phi}(z) \tilde{X}_{n-1}^{(1)} - v_1 v_2 z \tilde{\Phi}(z) \tilde{X}_{n-1}^{(1)} \quad (n \leq 0),$$  (3.86)

$$\tilde{X}_{n}^{(1)} \tilde{\Phi}(z) = \tilde{\Phi}(z) \tilde{X}_{n}^{(1)} \quad (n \geq 1),$$  (3.87)

$$\tilde{X}_{n}^{(2)} \tilde{\Phi}(z) = \tilde{\Phi}(z) \tilde{X}_{n-1}^{(2)} - v_1 v_2 z \tilde{\Phi}(z) \tilde{X}_{n-1}^{(2)} \quad (\forall n),$$  (3.88)

$$\langle \tilde{\nu} | \tilde{\Phi} | \tilde{\mu} \rangle = 1.$$  (3.89)

This definition is obtained by the renormalization $\tilde{\Phi}(z) = \Phi(p^{\tilde{\lambda}} z)$. We give some simple properties of the vertex operator $\tilde{\Phi}(z)$.

Proposition 3.30.

$$\langle \tilde{X}_\lambda | \tilde{\Phi}(z) | \tilde{\nu} \rangle = \begin{cases} (-v_1 v_2 u_1 u_2 z)^n, & \tilde{\lambda} = (0, (1^n)) \text{ for some } n, \\ 0, & \text{otherwise.} \end{cases}$$  (3.90)

For any $n \geq 1$,

$$\langle \tilde{\nu} | \tilde{\Phi}(z) \tilde{X}_n^{(i)} | \tilde{\mu} \rangle = \begin{cases} \left( \frac{1}{v_1 v_2 z} \right)^n (v_1 + v_2 + u_1 + u_2), & i = 1, \\ \left( \frac{1}{v_1 v_2} \right)^n (u_1 u_2 - v_1 v_2), & i = 2. \end{cases}$$  (3.91)
Proof. These follow from the commutation relations of $\hat{\Phi}(z)$. \hfill $\Box$

**Conjecture 3.31.** The matrix elements of $\hat{\Phi}(z)$ with respect to the integral form $|\bar{K}_\lambda\rangle$ are

$$
\frac{\langle \bar{K}_\lambda | \hat{\Phi}(z) | \bar{K}_{\bar{\mu}} \rangle}{\langle \bar{K}_\lambda | \bar{K}_{\bar{\mu}} \rangle} \overset{2}{=} (-1)^{\bar{\lambda} + \bar{\mu}} (u_1 u_2 v_1 v_2)_{\bar{\lambda} + \bar{\mu}} \frac{2^{\mu(1)} u_1^{2 \mu(1)} u_2^{2 \mu(2)} (u_1 u_2)_{\bar{\mu} - 2(n(\mu(1)) + n(\mu(2)))}}{N_{(1), \mu} (v_i / u_i)}. \tag{3.92}
$$

Under these conjectures 3.28 and 3.31, we obtain the formula for correlation functions of the vertex operator $\hat{\Phi}(z)$. For example, the function corresponding to 4-point conformal blocks is

$$
\langle \bar{w} | \Phi_\theta^\varnothing (z_2) | \Phi_\theta^\varnothing (z_1) | \bar{u} \rangle = \sum_{\lambda} \frac{\langle \bar{w} |\Phi(z) | \bar{K}_\lambda \rangle \langle \bar{K}_\lambda |\Phi(z) | \bar{K}_{\lambda} \rangle}{\langle \bar{K}_\lambda | \bar{K}_{\lambda} \rangle} \overset{2}{=} \sum_{\lambda} \frac{u_1 u_2 z_1}{w_1 w_2 z_2} \prod_{i,j=1}^{2} \frac{\bar{N}_{\theta, \bar{\lambda}} (w_i / v_j) \bar{N}_{\bar{\lambda}, 0} (v_i / u_j)}{\bar{N}_{\bar{\lambda}, \lambda} (v_i / v_j)}. \tag{3.93}
$$

(3.93) is AGT conjecture at $q \to 0$ limit with help of the AFLT basis. However, in the crystallized case, we can prove another formula for this 4-point correlation function by using the PBW type basis. At first, let us show following two lemmas.

**Lemma 3.32.** The matrix elements with respect to PBW type vector $|X_{0, \lambda}\rangle$ and $\langle \bar{v}|$ are

$$
\langle \bar{v}| \Phi(z) | X_{0, \lambda} \rangle = (-1)^{\ell(\lambda)} \left( \frac{1}{v_1 v_2 z} \right)^{|\lambda|} t^{-n(\lambda)} \prod_{k=1}^{\ell(\lambda)} (t^{k-1} v_1 v_2 - u_1 u_2). \tag{3.94}
$$

Proof. For $i \geq 2$, by (3.88) and the relation $X_{-n+1} (2) X_{-n} (2) = t^{-1} X_{-n} (2) X_{-n+1} (2)$,

$$
\langle \bar{v}| \Phi(z) \left( X_{-i} (2) \right)^m = \left( \frac{1}{v_1 v_2 z} \right)^{m} \langle \bar{v}| \Phi(z) \left( X_{-i+1} (2) \right)^{m-1} \tag{3.95}
$$

$$
= \left( \frac{1}{v_1 v_2 z} \right)^{m-1} \langle \bar{v}| \Phi(z) \left( X_{-i} (2) \right)^{m-1} X_{-i+1} (2)
$$

$$
= \left( \frac{1}{v_1 v_2 z} \right)^{m} \left( t^{-\frac{1}{2} m(m-1)} \langle \bar{v}| \Phi(z) \left( X_{-i+1} (2) \right)^m \right). \tag{3.96}
$$

Repeating this calculation,

$$
\langle \bar{v}| \Phi(z) \left( X_{-i} (2) \right)^m = \left( \frac{1}{v_1 v_2 z} \right)^{m k} t^{-\frac{1}{2} m(m-1) k} \langle \bar{v}| \Phi(z) \left( X_{-i+k} (2) \right)^m \tag{3.96}
$$

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where $0 \leq k \leq i - 1$. When $i = 1$, by similar calculation

\[
\langle \vec{v} \mid \Phi(z) (X_{-1}^{(2)})^m \mid \vec{u} \rangle = \left( -\frac{1}{v_1 v_2 z} \right)^{m_1 (1-i_2)} (u_1 u_2)^{-t^{m-1} + 1} \langle \vec{v} \mid \Phi(z) (X_{-1}^{(2)})^{m-1} \mid \vec{u} \rangle
\]

\[
= \left( -\frac{1}{v_1 v_2 z} \right)^{m_1 (1-i_2)} \prod_{k=1}^m (v_1 v_2 - t^{-k+1} u_1 u_2). \quad (3.97)
\]

By using above two formulas (3.96) and (3.97), if we write $\lambda = (i_1, i_2, \ldots, i_l)$, we have explicit form of parts of the inverse Shapovalov matrix.

\[
\langle \vec{v} \mid \Phi(z) \mid \tilde{X}_{0, \lambda} \rangle = \left( \frac{1}{v_1 v_2 z} \right)^{m_1 (1-i_2)} t^{-\frac{1}{2} m_1 (m_1-1) (1-i_2)}
\]

\[
\times \langle \vec{v} \mid \Phi(z) (X_{-1}^{(2)})^{m_1+m_2} (X_{-2}^{(2)})^{m_3} \cdots (X_{-l}^{(2)})^{m_l} \mid \vec{u} \rangle
\]

\[
= \left( \frac{1}{v_1 v_2 z} \right)^{m_1 (1-i_2)+m_2 (1-i_3)} t^{-\frac{1}{2} m_1 (m_1-1) (1-i_2) - \frac{1}{2} m_2 (m_2-1) (1-i_3)}
\]

\[
\times \langle \vec{v} \mid \Phi(z) (X_{-1}^{(2)})^{m_1+m_2+m_3} (X_{-2}^{(2)})^{m_4} \cdots (X_{-l}^{(2)})^{m_l} \mid \vec{u} \rangle
\]

\[
= (-1)^{\ell(\lambda)} \left( \frac{1}{v_1 v_2 z} \right)^{|\lambda|} t^{-n(\lambda)} \prod_{k=1}^{\ell(\lambda)} \left( t^{-k-1} v_1 v_2 - u_1 u_2 \right). \quad (3.98)
\]

We have explicit form of parts of the inverse Shapovalov matrix.

**Lemma 3.33.**

\[
S^{(\theta, (1^{[\lambda]})), (\theta, \lambda)} = S^{(\theta, (1^{[\lambda]}))} = \frac{(-1)^{|\lambda|} t^{-n(\lambda)} (u_1 u_2)^{-|\lambda|-\ell(\lambda)}}{b_\lambda(t^{-1})}. \quad (3.99)
\]

**Proof.** In this proof, we put $s = |\lambda|$. Hall-Littlewood function $Q_{(1^s)}(p_n; t)$ is the elementary symmetric function $e_s(p_n)$ times $b_{(1^s)}(t)$. Elementary symmetric functions have the generating function

\[
\sum_{k=0}^\infty z^k e_k(p_n) = \exp \left\{ - \sum_{n>0} \frac{(-z)^n}{n} p_n \right\}. \quad (3.100)
\]

Hence by the $r = 0$ case of Fact [A.2],

\[
\langle e_s(-p_n), Q_\lambda(p_n; t) \rangle_{0,t} = (-1)^s \exp \left\{ \sum_{n>0} \frac{z^n}{1-t^n} \partial \right\} Q_\lambda(p_n; t) \bigg|_{\text{coefficient of } z^s}
\]

\[
= (-1)^s Q_\lambda(p_n; t) \bigg|_{p_n \to \frac{1}{1-t^n}}
\]

\[
= (-1)^s t^n(\lambda). \quad (3.101)
\]

Therefore, the lemma follows from Proposition 3.22. \(\square\)

We give other proofs of this lemma in Appendix [A.3] and the form of $S^{(\theta, (1^{[\lambda]})), (\theta, \lambda)}$ can be seen by Appendix [A.4]. By the property (3.90), Proposition 3.22 and Lemmas 3.32 and 3.33, we can show the following main theorem.
Theorem 3.34.

\[ \langle \bar{w} | \bar{\Phi}(z_2) \bar{\Phi}(z_1) | \bar{u} \rangle = \sum_{\lambda} \langle \bar{w} | \bar{\Phi}(z_2) | \tilde{X}_\lambda \rangle S^{\lambda \bar{\lambda}} \langle \tilde{X}_{\bar{\lambda}} | \bar{\Phi}(z_1) | \bar{u} \rangle \]

\[ = \sum_{\lambda} \langle \bar{w} | \bar{\Phi}(z_2) | \tilde{X}_{\bar{\lambda}, \lambda} \rangle S^{(\bar{\lambda}, \lambda), (\bar{\lambda}, (1^{\lambda}))} \langle \tilde{X}_{\bar{\lambda}, (1^{\lambda})} | \bar{\Phi}(z_1) | \bar{u} \rangle \]

\[ = \sum_{\lambda} \left( \frac{u_1 u_2 z_1}{w_1 w_2 z_2} \right)^{\frac{|\lambda|}{2}} \prod_{k=1}^{t_{\lambda}} \frac{1 - \frac{t}{2} - 1}{t^{2n(\lambda)} b_\lambda(t^{-1})}. \]  

(3.102)

In this way, the explicit formula for the correlation function can be obtained, where we don’t use any conjecture. The formulas (3.93) and (3.102) are compared in Appendix A.6. We expect that these works will be generalized to \( N \geq 3 \) case.

3.4 Other types of limit

Finally, let us present other types of the crystal limit. In this paper, we investigated the crystal limit while the parameters \( u_i, v_i \) and \( w_i \) are fixed. However, it is also important to study the cases that these parameters depend on \( q \). For example, let us consider the case that \( u_i = p^{-M_i} u'_i, v_i = p^{-A_i} v'_i, w_i = p^{-M_i+2} w'_i \) (\( M_i, A_i \in \mathbb{R} \)) and \( u'_i, v'_i, w'_i \) are independent of \( q \) or fixed in the limit \( q \to 0 \). Let \( M_i + 1 > A_i > M_{j+2} \) for all \( i, j \in \{1, 2\} \) and \( A_1 = A_2 \). Then the Nekrasov formula for generic \( q \) case (\( N = 2 \))

\[ Z_{N_f=4}^{\text{inst}} := \sum_{\lambda} \frac{u_1 u_2 z_1}{w_1 w_2 z_2}^{\frac{|\lambda|}{2}} \prod_{i,j=1}^{2} \frac{N_{\lambda, \lambda}(q v_i/t v_j) N_{\lambda, 0}(q v_i/t v_j)}{N_{\lambda, \lambda}(q v_i/t v_j)} \]  

(3.103)

depend only on the partitions of the shape \( \tilde{\lambda} = ((1^n), (1^m)) \) at the limit \( q \to 0 \), where \( \left( \frac{u_1 u_2 z_1}{w_1 w_2 z_2} \right)^{\frac{|\lambda|}{2}} \) is fixed, and coincides with the partition function of the pure gauge theory (2.27):

\[ Z_{N_f=4}^{\text{inst}} \xrightarrow{q \to 0} Z_{\text{pure}}^{\text{inst}}. \]  

(3.104)

where \( \bar{Q} = v'_1/v'_2 \). Hence, we are sure that the vector

\[ \bar{\Phi}(z) | \bar{u} \rangle \]  

(3.105)

corresponds to the Whittaker vector in the section 2.2 at this limit, though we were not able to properly explain it. In this way, by considering the various other values of \( M_i \) and \( A_i \), we can find special behavior of \( Z_{N_f=4}^{\text{inst}} \) and the conformal block \( \langle \bar{w} | \bar{\Phi}(z_2) \bar{\Phi}(z_1) | \bar{u} \rangle \) and may prove the relation. These are our future studies.

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A Appendix

A.1 Macdonald functions and Hall-Littlewood functions

In this subsection, we briefly review some properties of Hall-Littlewood functions and Macdonald functions.

Let $\Lambda_N := \mathbb{Q}(q, t)[x_1, \ldots, x_N]^{S_N}$ be the ring of symmetric polynomials of $N$ variables and $\Lambda := \lim_{N \to \infty} \Lambda_N$ be the ring of symmetric functions. The inner product $\langle -, - \rangle_{q,t}$ over $\Lambda$ is defined such that for power sum symmetric functions $p_\lambda = \prod_{k \geq 1} P_{\lambda_k}$ ($p_n = \sum_{i \geq 1} x_i^n$),

$$\langle p_\lambda, p_\mu \rangle_{q,t} = z_\lambda \prod_{k=1}^{\ell(\lambda)} \frac{1 - q^k}{1 - t^k} \delta_{\lambda,\mu}, \quad z_\lambda := \prod_{i \geq 1} i^{m_i} m_i!,$$

(A.1)

where $m_i = m_i(\lambda)$ is the number of entries in $\lambda$ equal $i$. For a partition $\lambda$, Macdonald functions $P_\lambda \in \Lambda$ are uniquely determined by the following two conditions:

$$\lambda \neq \mu \quad \Rightarrow \quad \langle P_\lambda, P_\mu \rangle_{q,t} = 0; \quad (A.2)$$

$$P_\lambda = m_\lambda + \sum_{\mu < \lambda} c_{\lambda\mu} m_\mu. \quad (A.3)$$

Here $m_\lambda$ is a monomial symmetric function and $<$ is the ordinary dominance partial ordering, which is defined as follows:

$$\lambda \geq \mu \quad \iff \quad \sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i \quad (\forall k) \quad \text{and} \quad |\lambda| = |\mu|. \quad (A.4)$$

In this paper, we regard power sum symmetric functions $p_n$ ($n \in \mathbb{N}$) as the variables of Macdonald functions, i.e., $P_\lambda = P_\lambda(p_n; q, t)$. Here $P_\lambda(p_n; q, t)$ is an abbreviation for $P_\lambda(p_1, p_2, \ldots; q, t)$. In this paper, we often use the symbol $P_\lambda(a_n; q, t)$, which is the polynomial of bosons $a_n$ obtained by replacing $p_n$ in Macdonald functions with $a_n$.

Next, let the Hall-Littlewood function $P_\lambda(p_n; t)$ be given by $P_\lambda(p_n; t) := P_\lambda(p_n; 0, t)$. If $x_{N+1} = x_{N+2} = \cdots = 0$, then for a partition $\lambda$ of length $\leq N$, Hall-Littlewood polynomials $P_\lambda(p_n; t)$ with $p_n = \sum_{i=1}^{N} x_i^n$ is expressed by

$$P_\lambda(p_n; t) = \frac{1}{v_\lambda(t)} \sum_{w \in S_n} w \left( x_1^{\lambda_1} \cdots x_N^{\lambda_N} \prod_{i<j} \frac{x_i - tx_j}{x_i - x_j} \right), \quad (A.5)$$

where $v_\lambda(t) = \prod_{i \geq 0} \prod_{k=1}^{m_i(\lambda)} \frac{1 - t^k}{1 - t}$. Note that $m_0 = N - \ell(\lambda)$. The action of the symmetric group $S_N$ of degree $N$ is defined by $w(x_1^{\alpha_1} \cdots x_N^{\alpha_N}) = x_{w(1)}^{\alpha_1} \cdots x_{w(N)}^{\alpha_N}$ for $w \in S_N$.

It is convenient to introduce functions $Q_\lambda(p_n; t)$, which are defined by scalar multiples of $P_\lambda$ as follows:

$$Q_\lambda(p_n; t) := b_\lambda(t) P_\lambda(p_n; t), \quad (A.6)$$

where $b_\lambda(t) := \prod_{i \geq 1} \prod_{k=1}^{m_i(\lambda)} (1 - t^k)$. They are diagonalized as

$$\langle Q_\lambda, Q_\mu \rangle_{0,t} = b_\lambda(t) \delta_{\lambda,\mu}, \quad (A.7)$$
and they are expressed by Jing’s operator

\[ H(z) := \exp \left\{ \sum_{n \geq 1} \frac{1 - t^n}{n} b_n z^n \right\} \exp \left\{ - \sum_{n \geq 1} \frac{1 - t^n}{n} b_n z^{-n} \right\} =: \sum_{n \in \mathbb{Z}} H_n z^{-n}, \quad (A.8) \]

\[ H^\dagger(z) := \exp \left\{ - \sum_{n \geq 1} \frac{1 - t^n}{n} b_n z^n \right\} \exp \left\{ \sum_{n \geq 1} \frac{1 - t^n}{n} b_n z^{-n} \right\} =: \sum_{n \in \mathbb{Z}} H_n^\dagger z^{-n}, \quad (A.9) \]

where \( b_n \) is the bosons realized by

\[ b_{-n} = p_n, \quad b_n = \frac{n}{1 - t^n} \partial \frac{\partial}{\partial p_n} \quad (n > 0). \quad (A.10) \]

**Fact A.1** (\[J\]). Let \( |0\rangle \) be the vector such that \( b_n |0\rangle = 0 \quad (n > 0) \). Then for a partition \( \lambda \), we have

\[ H_{-\lambda_1} H_{-\lambda_2} \cdots |0\rangle = Q_\lambda(b_n; t) |0\rangle, \quad (A.11) \]

\[ \langle 0 | \cdots H_{\lambda_2}^\dagger H_{\lambda_1}^\dagger = \langle 0 | Q_\lambda(b_n; t). \quad (A.12) \]

Furthermore, the following specialization formula is known.

**Fact A.2** (\[Ma\]). Let \( r \) be indeterminate. Under the specialization

\[ p_n = \frac{1 - r^n}{1 - t^n}, \quad (A.13) \]

Hall-Littlewood functions \( Q_\lambda \in \Lambda \) is specialized as

\[ Q_\lambda(\frac{1 - r^n}{1 - t^n}; t) = t^{n(\lambda)} \prod_{i=1}^{\ell(\lambda)} (1 - t^{1-i} r). \quad (A.14) \]

### A.2 Partial orderings

The existence theorem of generalized Macdonald functions can be stated by the ordering \( > \) in Definition 3.5. In this appendix, we introduce a more elaborated ordering. This ordering point out more elements which is 0 in the transition matrix \( c_{\bar{\lambda}, \bar{\mu}} \), where \( |P_\lambda\rangle = \sum_{\bar{\mu}} c_{\bar{\lambda}, \bar{\mu}} \prod_i P_{\bar{\mu}^{(i)}} (a_{-n}) |\bar{\mu}\rangle \), and give more strict condition to existence theorem.

**Definition A.3.** For \( N \)-tuples of partitions \( \bar{\lambda} \) and \( \bar{\mu} \),

\[ \bar{\lambda} \succ \bar{\mu} \iff \bar{\lambda} > \bar{\mu} \quad \text{and} \quad \{ \nu \mid \nu \supset \lambda^{(\alpha)}, \nu \supset \mu^{(\alpha)}, |\nu| = |\lambda^{(\alpha)}| + \sum_{\beta=\alpha+1}^{N} (|\lambda^{(\beta)}| - |\mu^{(\beta)}|) \} \neq \emptyset \]

for all \( \alpha \). Here \( \lambda \supset \mu \) denote that \( \lambda_i \geq \mu_i \) for all \( i \).
Example A.4. If $N = 3$ and the number of boxes is 3, then

\[
\begin{array}{ccc}
(\emptyset, \emptyset, (3)) & (\emptyset, \emptyset, (2, 1)) & (\emptyset, \emptyset, (1, 1, 1)) \\
(\emptyset, (1), (2)) & (\emptyset, (1), (1, 1)) & \\
(\emptyset, (2), (1)) & ((1), \emptyset, (2)) & ((1), \emptyset, (1, 1)) \\
(\emptyset, (3), \emptyset) & ((1), (1), (1)) & (0, (2, 1), \emptyset) \\
((1), (2), \emptyset) & ((2), \emptyset, (1)) & ((1, 1), \emptyset, (1)) \\
((1, 2), \emptyset) & ((1, 1), (1), \emptyset) & \\
((3), \emptyset, \emptyset) & ((2, 1), \emptyset, \emptyset) & ((1, 1, 1), \emptyset, \emptyset).
\end{array}
\]

Here $\bar{\lambda} \rightarrow \bar{\mu}$ stands for $\bar{\lambda} \preceq \bar{\mu}$.

By using the following conjecture, we can state the existence theorem.

Conjecture A.5. Let $\eta_n^{(i)} := \int \frac{dz}{2\pi \sqrt{-1}z} \eta^{(i)}(z)z^n$. If $\eta_n^{(i)}$ ($n \geq 1$) acting on Macdonald functions $P_\lambda(a^{(i)}_{-n}; q, t) |u\rangle$, then there only appear partitions $\mu$ contained in $\lambda$, i.e.,

\[
\eta_n^{(i)} P_\lambda(a^{(i)}_{-n}; q, t) |u\rangle = \sum_{\mu \subset \lambda} c_{\lambda, \mu} P_\mu(a^{(i)}_{-n}; q, t) |\bar{\mu}\rangle.
\]  \hspace{1cm} \text{(A.15)}

Theorem A.6. Under the Conjecture \text{[A.5]} for an $N$-tuple of partitions $\bar{\lambda}$, there exist an unique vector $|P_\lambda\rangle \in \mathcal{F}_\bar{\lambda}$ such that

\[
|P_\lambda\rangle = \prod_{i=1}^{N} P_\lambda^{(i)}(a^{(i)}_{-n}; q, t) |\bar{\mu}\rangle + \sum_{\bar{\mu} \preceq \bar{\lambda}} c_{\bar{\lambda}, \bar{\mu}} \prod_{i=1}^{N} P_{\mu^{(i)}}(a^{(i)}_{-n}; q, t) |\bar{\mu}\rangle,
\]  \hspace{1cm} \text{(A.16)}

\[
X_{0}^{(1)} |P_\lambda\rangle = e_{\bar{\lambda}} |P_\lambda\rangle.
\]  \hspace{1cm} \text{(A.17)}

Proof. At first, $\eta_n^{(i)}$ satisfies

\[
\eta_n^{(i)} P_\lambda(a^{(i)}_{-n}) |\bar{\mu}\rangle = \epsilon_{\lambda} P_\lambda(a^{(i)}_{-n}) |\bar{\mu}\rangle, \quad \eta_n^{(j)} P_\lambda(a^{(i)}_{-n}) |\bar{\mu}\rangle = P_\lambda(a^{(i)}_{-n}) |\bar{\mu}\rangle, \quad \eta_n^{(j)} P_\lambda(a^{(i)}_{-n}) |\bar{\mu}\rangle = 0, \quad i \neq j, \quad n \geq 1.
\]  \hspace{1cm} \text{(A.18)}
If we act $\Lambda_0^i := \oint \frac{dz}{2\pi \sqrt{-1} z} \Lambda^i(z)$ on the product of the Macdonald functions, then

$$\Lambda_0^i \prod_{j=1}^N P_{\lambda^{(j)}}(a_{-n}^{(j)}) |\vec{u}\rangle = \epsilon_{\lambda^{(i)}} \prod_{j=1}^N P_{\lambda^{(j)}}(a_{-n}^{(j)}) |\vec{u}\rangle$$

(A.19)

$$+ \sum_{\mu \leq \lambda^{(i)}} c_{\lambda^{(i)}, \mu}^i (a_{-n}^{(1)}, \ldots, a_{-n}^{(i-1)}) P_{\mu}(a_{-n}) \prod_{j \neq i} P_{\lambda^{(j)}}(a_{-n}^{(j)}) |\vec{u}\rangle,$$

where $c_{\lambda^{(i)}, \mu}^i (a_{-n}, \ldots, a_{-n}^{(i-1)})$ is a polynomial of degree $|\lambda^{(i)}| - |\mu|$ of $a_{-n}, \ldots, a_{-n}^{(i-1)}$. Hence

$$X_0^{(1)} \prod_{j=1}^N P_{\lambda^{(j)}}(a_{-n}^{(j)}) |\vec{u}\rangle = e_{\bar{\lambda}} \prod_{j=1}^N P_{\lambda^{(j)}}(a_{-n}^{(j)}) |\vec{u}\rangle + \sum_{\bar{\mu} \geq \bar{\lambda}} c_{\bar{\lambda}, \bar{\mu}}^i \prod_{j=1}^N P_{\mu^{(j)}}(a_{-n}^{(j)}) |\vec{u}\rangle.$$  

(A.20)

Therefore one can easily diagonalize it and we have this theorem.

In the basis of monomial symmetric functions $|m_{\lambda^{(i)}}\rangle := \prod_{i=1}^N m_{\lambda^{(i)}}(a_{-n}^{(i)}) |\vec{u}\rangle$, we have

$$X_0^{(1)} |m_{\lambda^{(i)}}\rangle = X_0^{(1)} \sum_{\bar{\lambda} \geq \bar{\mu}} d_{\bar{\lambda}, \bar{\mu}}^i \prod_{i=1}^N P_{\mu^{(i)}}(a_{-n}^{(i)}) |\vec{u}\rangle$$

(A.21)

$$= \sum_{\bar{\lambda} \geq \bar{\mu}} d_{\bar{\lambda}, \bar{\mu}}^i \sum_{\bar{\mu} \geq \bar{\nu}} d_{\bar{\lambda}, \bar{\mu}}^i \prod_{i=1}^N P_{\mu^{(i)}}(a_{-n}^{(i)}) |\vec{u}\rangle$$

$$= \sum_{\bar{\lambda} \geq \bar{\mu}} \sum_{\bar{\mu} \geq \bar{\nu}} \sum_{\bar{\nu} \geq \bar{\rho}} d_{\bar{\lambda}, \bar{\mu}}^i d_{\bar{\mu}, \bar{\nu}}^i d_{\bar{\mu}, \bar{\rho}}^i |m_{\bar{\rho}}\rangle,$$

where

$$\bar{\lambda} \geq \bar{\mu} \overset{\text{def}}{\Leftrightarrow} (|\lambda^{(1)}|, \ldots, |\lambda^{(N)}|) = (|\mu^{(1)}|, \ldots, |\mu^{(N)}|) \text{ and } \lambda^{(\alpha)} \geq \mu^{(\alpha)}$$

(A.22)

(1 \leq \forall \alpha \leq N). Thus the partial ordering $\geq^\ast$ defined as follows also triangulates $X_0^{(1)}$.

$$\bar{\lambda} \geq^\ast \bar{\rho} \overset{\text{def}}{\Leftrightarrow} \text{ there exist } \bar{\mu} \text{ and } \bar{\nu} \text{ such that } \bar{\lambda} \geq \bar{\mu} \geq \bar{\nu} \geq \bar{\rho}.$$  

(A.23)

It can be shown that the partial ordering $\geq^\ast$ is equivalent to the ordering $\geq^L$ introduced in [AFHKSY]. Therefore Theorem [A.6] support the existence theorem in [AFHKSY, Proposition3.8].

### A.3 Other proofs of Lemma 3.33

In this subsection, let us explain other proofs of Lemma 3.33 by the method of contour integrals.

---

8 That is to say, for the operator $\mathcal{O}$ such that $[\mathcal{O}, a_{-n}^{(j)}] = na_{-n}^{(j)}$, the polynomial $c_{\lambda^{(i)}, \mu}^{i}$ satisfy $[\mathcal{O}, c_{\lambda^{(i)}, \mu}^{i}] = (|\lambda^{(i)}| - |\mu|) c_{\lambda^{(i)}, \mu}^{i}$.
The generating function of elementary symmetric functions and Jing’s operator makes the equation

\[ (-1)^s \langle \varepsilon_s(-p_n), \lambda(p_n; t) \rangle_{0,t} = \int_0^1 \frac{dz}{2\pi i} \frac{dw}{2\pi i} \prod_{i=1}^l \left( \frac{1}{1-zw} \right) \prod_{1 \leq i < j \leq \ell(\lambda)} \left( \frac{w_i-w_j}{w_i-tw_j} \right) z^{-s} w^{-\lambda} \]

where we put \( s = |\lambda|, t = \ell(\lambda), \) and \( |1/z| > |w_1| > \cdots > |w_l| \). It suffices to show that \( F_{\lambda_1,\lambda_2,\ldots,\lambda_l} = t^n(\lambda) \), which is proven by a recursive relation of \( F \) as follows. The contour integral \( \oint \frac{dw}{2\pi i} \) surrounding origin is represented as that surrounding \( \infty \). Since \( \lambda_1 > 0 \), the residue of \( w_1 \) at \( w_1 = \infty \) is 0. Hence, the only residue at \( w_1 = \frac{1}{z} \) is left, and it is

\[ F_{\lambda_1,\lambda_2,\ldots,\lambda_l} = \int_0^1 \frac{dz}{2\pi i} \frac{dw}{2\pi i} \prod_{i=2}^l \left( \frac{1}{1-tw_i} \right) \prod_{2 \leq i < j \leq l} \left( \frac{w_i-w_j}{w_i-tw_j} \right). \]

By change of variable \( tz \mapsto z \),

\[ F_{\lambda_1,\lambda_2,\ldots,\lambda_l} = \prod_{i=2}^l t^{\lambda_i} \cdot F_{\lambda_2,\ldots,\lambda_l} \]

Therefore

\[ F_{\lambda_1,\lambda_2,\ldots,\lambda_l} = \left( \prod_{i=2}^l t^{\lambda_i} \right) \left( \prod_{i=3}^l t^{\lambda_i} \right) \cdots t^{\lambda_l} \oint \frac{dz}{2\pi i} \frac{dw}{2\pi i} = t^n(\lambda). \]

Thus the Lemma 3.3 is proven.

Although it is slightly hard, one can also prove this lemma by reversing the order of integration, i.e., first perform over a variable \( w_l \) surrounding origin. Indeed \( w_l \) has the pole only at \( w_l = 0 \), and its residue satisfies

\[ F_{\lambda_1,\ldots,\lambda_l} = \sum_{\alpha_0+\alpha_1+\cdots+\alpha_l = \lambda} \left( \prod_{i=1}^{l-1} A_{\alpha_i} \right) F_{\lambda_1+\alpha_1,\ldots,\lambda_l+\alpha_l}, \]

where for \( n > 0 \), \( A_n := (t-1)t^{n-1} \) and \( A_0 := 1 \). By the assumption that \( F_{\beta} = t^n(\beta) \) for \( \beta = (\beta_1, \beta_2, \ldots) \) with \( \ell(\beta) = l - 1 \), we inductively get

\[ F_{\lambda_1,\ldots,\lambda_l} = t^n(\lambda,\ldots,\lambda_l-1)) \sum_{0 \leq k \leq \lambda_l} \sum_{\alpha_0+\cdots+\alpha_l = \lambda} \left( \prod_{i=1}^{l-1} A_{\alpha_i} \right) t^{n(\alpha)}. \]

By virtue of the equation

\[ \sum_{\alpha_0+\cdots+\alpha_l = \lambda} \left( \prod_{i=1}^{l} A_{\alpha_i} \right) = \begin{cases} t^{l(k-1)} - t^{l(k-1)}, & k \geq 1, \\ 1, & k = 0, \end{cases} \]

which is also proven by induction with respect to \( l \), it can be seen that \( F_{\lambda_1,\ldots,\lambda_l} = t^n(\lambda) \).
A.4 Explicit form of $\langle Q_{(s)}(-p_n), Q\lambda(p_n; t) \rangle_{0,t}$

The formula for $S^{\lambda,(1^r)}$ is given in the Lemma 3.33 and the last subsection. We also have an explicit form of $S^{\lambda,(s)}$ and $S_{\lambda,(s)}$. By the Proposition 3.22 it suffices to give the explicit form of $\langle Q_{(s)}(-p_n), Q\lambda(p_n; t) \rangle_{0,t}$.

Proposition A.7.

$$\langle Q_{(s)}(-p_n; t), Q\lambda(p_n; t) \rangle_{0,t} = t|\lambda|+n(\lambda) \prod_{k=1}^{\ell(\lambda)} (1 - t^{-k}). \quad (A.31)$$

Proof. The proof is similar to the previous subsection. Set

$$G_{k_1,\ldots,k_l} := \oint \frac{dz}{2\pi \sqrt{-1}z} \frac{dw}{2\pi \sqrt{-1}w} \prod_{i=1}^{l} \left( \frac{z - w_i}{t^k z - tw_i} \right) \frac{w_i - w_j}{w_i - tw_j} \prod_{1 \leq i < j \leq \ell(\lambda)} (w_i - w_j)^{|\lambda|}. \quad (A.32)$$

Then $\langle Q_{(s)}(-p_n), Q\lambda(p_n; t) \rangle_{0,t} = G_{0,\ldots,0}$ by Jing’s operator. Integration of $w_i$ around $\infty$ makes recursive relation

$$G_{k_1,\ldots,k_l} = t^{\lambda_1(k+1)-\ell(\lambda)} (t^{k+1} - 1) G_{k_2,\ldots,k_l}, \quad (A.33)$$

and leads this Proposition. \hfill $\square$

For general partitions $\lambda$ and $\mu$, we can get the integral representation of $\langle Q(-p_n), Q\mu(p_n; t) \rangle_{0,t}$. However, it is very hard to give their explicit formula.

A.5 Check of (3.41)

The integral formula (3.41) can be checked by the similar way to subsections A.3 and A.4.

Let us set

$$\mathcal{F}_{\lambda_1,\ldots,\lambda_l}(u) := \oint \prod_{i=1}^{l} \frac{dw_i}{2\pi \sqrt{-1}w_i} w_i^{\lambda_i} \prod_{i=1}^{l} \left( \frac{w_i}{w_i - ux} \right) \prod_{1 \leq i < j \leq l} \frac{w_i - w_j}{w_i - tw_j}, \quad (A.34)$$

$$\mathcal{G}_{\mu_1,\ldots,\mu_m}(u) := \oint \prod_{i=1}^{m} \frac{dz_i}{2\pi \sqrt{-1}z_i} z_i^{-\mu_i} \prod_{1 \leq i < j \leq m} \left( \frac{z_i - z_j}{z_i - tz_j} \right) \prod_{1 \leq i \leq m} \left( \frac{x - (t/v) z_i}{x - (t/v) z} \right). \quad (A.35)$$

Then $\mathcal{F}_{\lambda_1,\ldots,\lambda_l}(u) = (-u)^{|\lambda|+n(\lambda)} \langle \tilde{K}_\lambda \Phi(z) | \tilde{K}_\theta \rangle$, $\mathcal{G}_{\mu_1,\ldots,\mu_m}(u) = (-u)^{-|\mu|+n(\mu)+|\mu|} \langle \tilde{K}_\theta \Phi(z) | \tilde{K}_\mu \rangle$.

The integration of $w_1$ around $0$ give the relation

$$\mathcal{F}_{\lambda_1,\ldots,\lambda_l}(u) = (ux)^{\lambda_1} \mathcal{F}_{\lambda_2,\ldots,\lambda_l}(tu). \quad (A.36)$$

On the other hand, the integration of $z_1$ around $\infty$ makes

$$\mathcal{G}_{\mu_1,\ldots,\mu_m}(u) = (1 - (u/v)) (ux/v)^{\mu_1} \mathcal{G}_{\mu_2,\ldots,\mu_m}(u/t). \quad (A.37)$$

Thus

$$\mathcal{F}_{\lambda_1,\ldots,\lambda_l}(u) = (ux)^{\lambda_1}(utx)^{\lambda_2} \cdots (ut^{l-1}x)^{\lambda_l},$$

$$\mathcal{G}_{\mu_1,\ldots,\mu_m}(u) = \left( 1 - \frac{u}{v} \right) \left( 1 - \frac{u}{vt} \right) \cdots \left( 1 - \frac{u}{v t^{m-1}} \right) \left( \frac{u}{x t} \right)^{-\mu_1} \left( \frac{u}{x t^2} \right)^{-\mu_2} \cdots \left( \frac{u}{x t^m} \right)^{-\mu_m}.$$ 

These agree with the r.h.s. of (3.41).
A.6 Comparison of formulas (3.93) and (3.102)

In this subsection, we compare two formulas (3.93) and (3.102) which are obtained by the other basis.

Comparison of the coefficients of $z_1 z_2$ gives the equation

$$
\sum_{|\lambda|=n} \prod_{i,j=1}^2 \frac{\tilde{N}_{\lambda,i,j}}{N_{\lambda,i,j}(v_i/v_j)} = \sum_{|\lambda|=n} \prod_{k=1}^{\ell(\lambda)} \left( 1 - t^{k-1} \frac{w_1 w_2}{v_1 v_2} \right). \tag{A.38}
$$

Note that the l.h.s. is the summation with respect to pairs of partitions $\tilde{\lambda} = (\lambda^{(1)}, \lambda^{(2)})$ and the r.h.s. is the summation with respect to single partitions $\lambda$. The r.h.s. depend only on the ratio $\frac{w_1 w_2}{v_1 v_2}$ though the l.h.s. doesn’t look that way.

For a single partition $\lambda$, let us define $\langle \lambda \rangle$ to be the set of all pairs of partitions $(\lambda^{(1)}, \lambda^{(2)})$ such that a permutation of the sequence $(\lambda^{(1)}_1, \lambda^{(1)}_2, \ldots, \lambda^{(2)}_1, \lambda^{(2)}_2, \ldots)$ coincides with $\lambda$. For example, if $\lambda = (2, 1, 1)$,

$$
\langle \lambda \rangle = \{((2, 1, 1), \emptyset), ((2, 1), (1)), ((2), (1, 1)), ((1), (2)), ((1, 1), (2)), ((1), (2, 1)), (\emptyset, (2, 1, 1))\}. \tag{A.39}
$$

Then we obtain the strange factorization formula with respect to the partial summation of l.h.s. in (A.38)

$$
\sum_{\tilde{\lambda} \in \langle \lambda \rangle} \prod_{i,j=1}^2 \frac{\tilde{N}_{\lambda,i,j}}{N_{\lambda,i,j}(v_i/v_j)} = \frac{\prod_{k=1}^{\ell(\lambda)} \left( 1 - t^{k-1} \frac{w_1 w_2}{v_1 v_2} \right)}{t^{2n(\lambda)b_\lambda(t-1)}} \left( \frac{w_1 w_2}{v_1 v_2} \right)^{|\lambda|-\ell(\lambda)} t^{2n(\lambda)-I_\lambda}, \tag{A.40}
$$

where $I_\lambda := \sum_{s \in \lambda}(L_\emptyset(s) - L_\emptyset(s)) = \sum_{(i,j) \in \tilde{\lambda}} \lambda_j$. If we prove that l.h.s. of (A.40) depend only on $\frac{w_1 w_2}{v_1 v_2}$, (A.40) is easily seen by checking the case of $w_2 = v_2$. This equation almost reproduce each term of the r.h.s. in (A.38). Hence (A.38) may be proven by this equation. If (A.38) holds, the AGT conjecture at $q \to 0$ with the help of the AFLT basis (3.93) is completely proven.

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