Absolutely singular dynamical foliations

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Introduction

Let $A_2$ be the automorphism of the 2-torus, $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, given by \[
\begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix}.
\]

Let $A_3$ be the automorphism of the 3-torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$ given by \[
\begin{pmatrix}
A_2 & 0 \\
0 & 1
\end{pmatrix}.
\]

Let $\text{Diff}^2_{\mu}(T^3)$ be the set of $C^2$ diffeomorphisms of $T^3$ that preserve Lebesgue-Haar measure $\mu$.

In [SW1], M. Shub and A. Wilkinson prove the following theorem.

Theorem: Arbitrarily close to $A_3$ there is a $C^1$-open set $U \subset \text{Diff}^2_{\mu}(T^3)$ such that for each $g \in U$,

1. $g$ is ergodic.

2. There is an equivariant fibration $\pi : T^3 \to T^2$ such that $\pi g = A_2 \pi$

   The fibers of $\pi$ are the leaves of a foliation $W^c_g$ of $T^3$ by $C^2$ circles. In particular, the set of periodic leaves is dense in $T^3$.

3. There exists $\lambda^c > 0$ such that, for $\mu$-almost every $w \in T^3$, if $v \in T_w T^3$ is tangent to the leaf of $W^c_g$ containing $w$, then

   \[
   \lim_{n \to \infty} \frac{1}{n} \log \|T_w g^n v\| = \lambda^c.
   \]

4. Consequently, there exists a set $S \subseteq T^3$ of full $\mu$-measure that meets every leaf of $W^c_g$ in a set of leaf-measure 0. The foliation $W^c_g$ is not absolutely continuous.
Additionally, it is shown that the diffeomorphisms in $U$ are nonuniformly hyperbolic and Bernoullian. In this note, we prove:

**Theorem I:** Let $g$ satisfy conclusions 1.–3. of the previous theorem. Then there exist $S \subseteq T^3$ of full $\mu$-measure and $k \in \mathbb{N}$ such that $S$ meets every leaf of $W^c_g$ in exactly $k$ points. The foliation $W^c_g$ is absolutely singular.

**Remark:** In A. Katok’s example of an absolutely singular foliation in [Mi], the leaves of the foliation meet the set of full measure in one point. In the [SW] examples, the set $S$ may necessarily meet leaves of $W^c_g$ in more than one point, as the following argument of Katok’s shows.

It follows from Theorem II in [SW2] that for $k \in \mathbb{Z}^+$ and for small $a, b > 0$, the map $g = j_{a,k} \circ h_b$ satisfies the hypotheses of Theorem I, where

$$h_b(x, y, z) = (2x + y, x + y, x + y + z + b \sin 2\pi y),$$

and

$$j_{a,k}(x, y, z) = (x, y, z) + a \cos(2\pi k z) \cdot (1 + \sqrt{5}, 2, 0).$$

For $k \in \mathbb{N}$, let $\rho_k$ be the vertical translation that sends $(x, y, z)$ to $(x, y, z + \frac{1}{k})$. Note that $h_b \circ \rho_k = \rho_k \circ h_b$ and $j_{a,k} \circ \rho_k = \rho_k \circ j_{a,k}$. Thus $g \circ \rho_k = \rho_k \circ g$.

The fibration $\pi : T^3 \to T^2$ was obtained in [SW] by using the persistence of normally hyperbolic submanifolds under perturbations. In the present case the symmetries $\rho_k$ preserve the fibers of the trivial fibration $P : T^3 \to T^2$ from which one starts, and also the maps $g$. Therefore the fibers of $\pi : T^3 \to T^2$ (i.e., the leaves of center foliation $W^c_g$) are invariant under the action of the finite group $< \rho_k >$.

Let $S$ be the (full measure) set of points in $T^3$ for which the center direction is a positive Lyapunov direction (i.e. for which conclusion 3 holds). Since $\rho_k(W^c_g) = W^c_g$, it follows that $\rho_k S = S$. If $p \in S \cap W^c(p)$, then $\rho_k(p) \in \rho_k(S) \cap \rho_k(W^c(p)) = S \cap W^c(p)$; that is, $S \cap W^c(p)$ contains at least $k$ points.

Thus Theorem I is “sharp” in the sense that we cannot say more about the value of $k$ in general. We see no reason why $k = 1$ should hold even for a residual set in $U$.

Theorem I has an interesting interpretation. Recall that a $G$-extension of a dynamical system $f : X \to X$ is a map $f : X \times G \to X \times G$, where $G$ is a compact group, of the form $(x, y) \mapsto (f(x), \varphi(x)y)$. If $f$ preserves
\(\nu\), and \(\varphi : X \to G\) is measurable, then \(f_x\) preserves the product of \(\nu\) with Lebesgue-Haar measure on \(G\). A \(\mathbb{Z}/k\mathbb{Z}\)-extension is also called a \(k\)-point extension.

Let \(\lambda\) be an invariant probability measure for a \(k\)-point extension of \(f : X \to X\), and \(\{\lambda_x\}\) the family of conditional measures associated with the partition \(\{{x}\} \times G\). We remark that if \(\lambda\) is ergodic, then each atom of \(\lambda_x\) must have the same weight \(1/k\) (up to a set of \(\lambda\)-measure 0).

Now take \(g \in U\). Choose a coherent orientation on the leaves of \(\{\pi^{-1}(x)\}_{x \in T^2}\). Take \(h : T^3 \to T^2 \times T\) to be any continuous change of coordinates such that \(h\) restricted to \(\pi^{-1}(x)\) is smooth and orientation preserving to \(\{x\} \times T\). We may then write \(F = h \circ g \circ h^{-1} : T^2 \times T \to T^2 \times T\) in the form

\[F(x, p) = (A_2 x, \varphi_x(p))\]

where \(\varphi_x : T \to T\) is smooth and orientation preserving. If \(P : T^2 \times T \to T^2\) is the projection on the first factor of the product, we have \(P \circ h = \pi\). Therefore, writing \(\lambda = h^* \mu\), we have \(P^* \lambda = \pi^* \mu\). Let \(\{\lambda_x\}\) be the disintegration of the measure \(\lambda\) along the fibers \(\{x\} \times T\). By a further measurable change of coordinates, smooth along each \(\{x\} \times T\) fiber, we may assume that \(\lambda\)-almost everywhere, the atoms of \(\lambda_x\) are at \(l/k\), for \(l = 0, \ldots, k - 1\). But then \(\varphi_x\) permutes the atoms cyclically, and we obtain the following corollary.

**Corollary:** For every \(g \in U\) there exists \(k \in \mathbb{N}\) such that \((T^3, \mu, g)\) is isomorphic to an (ergodic) \(k\)-point extension of \((T^2, \pi^* \mu, A_2)\).

M. Shub has observed that if \(g = j_{a,k} \circ h_b\), then \(\pi^* \mu\) is actually Lebesgue measure on \(T^2\).

## 1 Proof of Theorem I

The proof of Theorem I follows from a more general result about fibered diffeomorphisms. Before stating this result, we describe the underlying setup and assumptions.

Let \(X\) be a compact metric space with Borel probability measure \(\nu\), and let \(f : X \to X\) be invertible and ergodic with respect to \(\nu\). Let \(M\) be a closed Riemannian manifold and \(\varphi : X \to \text{Diff}^{1+\alpha}(M)\) a measurable map. Consider
the skew-product transformation $F : X \times M \to X \times M$ given by

$$F(x, p) = (f(x), \varphi_x(p)).$$

Assume further that there is an $F$-invariant ergodic probability measure $\mu$ on $X \times M$ such that $\pi_* \mu = \nu$, where $\pi : X \times M \to X$ is the projection onto the first factor.

For $x \in X$, let $\varphi_x^{(0)}$ be the identity map on $M$ and for $k \in \mathbb{Z}$, define $\varphi_x^{(k)}$ by

$$\varphi_x^{(k+1)} = \varphi_{f^k(x)} \circ \varphi_x^{(k)}.$$

Since the tangent bundle to $M$ is measurably trivial, the derivative map of $\varphi$ along the $M$ direction gives a cocycle $D\varphi : X \times M \times \mathbb{Z} \to GL(n, \mathbb{R})$, where $n = \dim(M)$:

$$(x, p, k) \mapsto D_p \varphi_x^{(k)}.$$

Assume that $\log^+ \|D\varphi\|_\alpha \in L^1(X \times M, \mu)$, where $\| \cdot \|_\alpha$ is the $\alpha$-Hölder norm. Let $\lambda_1 < \lambda_2 \cdots < \lambda_l$ be the Lyapunov exponents of this cocycle; they exist for $\mu$-a.e. $(x, p)$ by Oseledec’s Theorem and are constant by ergodicity. We call these the fiberwise exponents of $F$. Under the assumptions just described, we have the following result.

**Theorem II:** Suppose that $\lambda_l < 0$. Then there exists a set $S \subseteq X \times M$ and an integer $k \geq 1$ such that

- $\mu(S) = 1$
- For every $(x, p) \in S$, we have $\#(S \cap \{x\} \times M) = k$.

This has the immediate corollary:

**Corollary:** Let $f \in \text{Diff}^{1+\alpha}(M)$. If $\mu$ is an ergodic measure with all of its exponents negative, then it is concentrated on the orbit of a periodic sink.

The corollary has a simple proof using regular neighborhoods. Our proof is a fibered version. Theorem I is also a corollary of Theorem II. For this, the argument is actually applied to the inverse of $g$, which has negative fiberwise exponents, rather than to $g$ itself, whose fiberwise exponents are positive. As we described in the previous remarks, there is a measurable change of
coordinates, smooth along the leaves of $\mathcal{W}_g$ in which $g^{-1}$ is expressed as a skew product of $T^2 \times T$.

**Remark:** Without the assumption that $f$ is invertible, Theorem II is false. An example is described by Y. Kifer [Ki], which we recall here. Let $f : T \to T$ be a $C^{1+\alpha}$ diffeomorphism with exactly two fixed points, one attracting and one repelling. Consider the following random diffeomorphism of $T$: with probability $p \in (0, 1)$, apply $f$, and with probability $1 - p$, rotate by an angle chosen randomly from the interval $[-\epsilon, \epsilon]$.

Let $X = (\{0, 1\} \times T)^N$. To generate a sequence of diffeomorphisms $f_0, f_1, \ldots$, according to the above rule, we first define $\varphi : X \to \text{Diff}^{1+\alpha}(T)$ by

$$
\varphi(\omega) = \begin{cases} 
  f & \text{if } \omega(0) = (0, \theta), \\
  R_\theta & \text{if } \omega(0) = (1, \theta),
\end{cases}
$$

where $R_\theta$ is rotation through angle $\theta$. Next, we let $\nu_\epsilon$ be the product of $p, 1 - p$-measure on $\{0, 1\}$ with the measure on $T$ that is uniformly distributed on $[-\epsilon, \epsilon]$. Then corresponding to $\nu_\epsilon^N$-almost every element $\omega \in X$ is the sequence $\{f_k = \varphi(\sigma^k(\omega))\}_{k=0}^\infty$, where $\sigma : X \to X$ is the one-sided shift $\sigma(\omega)(n) = \omega(n+1)$.

Put another way, the random diffeomorphism is generated by the (noninvertible) skew product $\tau : X \times T \to X \times T$, where $\tau(\omega, x) = (\sigma(\omega), \varphi(\omega)(x))$. An ergodic $\nu_\epsilon$-stationary measure for this random diffeomorphism is a measure $\mu_\epsilon$ on $T$ such that $\mu_\epsilon \times \nu_\epsilon^N$ is $\tau$-invariant and ergodic. Such measures always exist ([Ki], Lemma I.2.2), but, for this example, there is an ergodic stationary measure with additional special properties.

Specifically, for every $\epsilon > 0$, there exists an ergodic $\nu_\epsilon$-stationary measure $\mu_\epsilon$ on $T$ such that, as $\epsilon \to 0$, $\mu_\epsilon \to \delta_{x_0}$, in the weak topology, where $\delta_{x_0}$ is Dirac measure concentrated on the sink $x_0$ for $f$. From this, it follows that, as $\epsilon \to 0$, the fiberwise Lyapunov exponent for $\mu_\epsilon$ approaches $\log |f'(x_0)| < 0$, which is the Lyapunov exponent of $\delta_{x_0}$. Thus, for $\epsilon$ sufficiently small, the fiberwise exponent for $\tau$ with respect to $\mu_\epsilon$ is negative. Nonetheless, it is easy to see that $\mu_\epsilon$ for $\epsilon > 0$ cannot be uniformly distributed on $k$ atoms; if $\mu_\epsilon$ were atomic, then $\tau$-invariance of $\mu_\epsilon \times \nu_\epsilon^N$ would imply that, for every $x \in T$,

$$
\mu_\epsilon(\{x\}) = p\mu_\epsilon(\{f^{-1}(x)\}) + (1 - p) \int_{-\epsilon}^\epsilon \mu_\epsilon(\{R_\theta(x)\}) d\theta
$$

$$
= p\mu_\epsilon(\{f^{-1}(x)\}),
$$

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which is impossible if \( \mu_\epsilon \) has finitely many atoms. In fact, \( \mu_\epsilon \) can be shown to be absolutely continuous with respect to Lebesgue measure (see [3], p. 173ff and the references cited therein). Hence invertibility is essential, and we indicate in the proof of Theorem II where it is used.

**Proof of Theorem II:** We first establish the existence of fiberwise “stable manifolds” for the skew product \( F \). A general theory of stable manifolds for random dynamical systems is worked out in ([3], Theorem V.1.6; see also [1]); since we are assuming that all of the fiberwise exponents for \( F \) are negative, we are faced with the simpler task of constructing fiberwise regular neighborhoods for \( F \) (see the Appendix by Katok and Mendoza in [4]). We outline a proof, following closely [4].

**Theorem 1.1 (Existence of Regular Neighborhoods)** There exists a set \( \Lambda_0 \subseteq X \times M \) of full measure such that for \( \epsilon > 0 \):

- There exists a measurable function \( r : \Lambda_0 \to (0, 1] \) and a collection of embeddings \( \Psi_{(x,p)} : B(0, q(x,p)) \to M \) such that \( \Psi_{(x,p)}(0) = p \) and \( \exp(-\epsilon) < r(F(x,p))/r(x,p) < \exp(\epsilon) \).
- If \( \varphi_{(x,p)} = \Psi_{F(x,p)}^{-1} \circ \varphi_x \circ \Psi_{(x,p)} : B(0, r(x,p)) \to \mathbb{R}^n \), then \( D_0 \varphi_{(x,p)} \) satisfies
  \[
  \exp(\lambda_1 - \epsilon) \leq \|D_0 \varphi_{(x,p)}^{-1}\|^{-1}, \|D_0 \varphi_{(x,p)}\| \leq \exp(\lambda_1 + \epsilon).
  \]
- The \( C^1 \) distance \( d_{C^1}(\varphi_{(x,p)}, D_0 \varphi_{(x,p)}) < \epsilon \) in \( B(0, r(x,p)) \).
- There exist a constant \( K > 0 \) and a measurable function \( A : \Lambda_0 \to \mathbb{R} \) such that for \( y, z \in B(0, r(x,p)) \),
  \[
  K^{-1}d(\Psi_{(x,p)}(y), \Psi_{(x,p)}(z)) \leq \|y - z\| \leq A(x)d(\Psi_{(x,p)}(y), \Psi_{(x,p)}(z)),
  \]
  with \( \exp(-\epsilon) < A(F(x,p))/A(x,p) < \exp(\epsilon) \).

**Proof:** See the proof of Theorem S.3.1 in [4]. \( \Box \)

Decompose \( \mu \) into a system of fiberwise measures \( d\mu(x,p) = d\mu_x(p)d\nu(x) \). Invariance of \( \mu \) with respect to \( F \) implies that, for \( \nu \)-a.e. \( x \in X \),

\[
\varphi_{x*}\mu_x = \mu_{f(x)}.
\]
Corollary 1.2 There exists a set \( \Lambda \subseteq X \times M \), and real numbers \( R > 0 \), \( C > 0 \), and \( c < 1 \) such that

1. \( \mu(\Lambda) > .5 \), and, if \((x, p) \in \Lambda\), then \( \mu_x(\Lambda_x) > .5 \), where \( \Lambda_x = \{p \in M|(x, p) \in \Lambda\} \);

2. If \((x, p) \in \Lambda\) and \( d_M(p, q) \leq R \), then
\[
d_M(\varphi_x^{(m)}(p), \varphi_x^{(m)}(q)) \leq Cc^m d_M(p, q),
\]
for all \( m \geq 0 \).

Proof: This follows in a standard way from the Mean Value Theorem and Lusin’s Theorem. □

To prove Theorem II, it suffices to show that there is a positive \( \nu \)-measure set \( B \subseteq X \), such that for \( x \in B \), the measure \( \mu_x \) has an atom, as the following argument shows. For \( x \in X \), let \( d(x) = \sup_{p \in M} \mu_x(p) \). Clearly \( d \) is measurable, \( f \)-invariant, and positive on \( B \). Ergodicity of \( f \) implies that \( d(x) = d > 0 \) is positive and constant for almost all \( x \in X \). Let \( S = \{(x, p) \in X \times M | \mu_x(p) \geq d\} \). Observe that \( S \) is \( F \)-invariant, has measure at least \( d \), and hence has measure 1. The conclusions of Theorem II follow immediately.

Let \( \Lambda \), \( R > 0 \), \( C > 0 \), and \( c < 1 \) be given by Corollary 1.2, and let \( B = \pi(\Lambda) \). Let \( N \) be the number of \( R/10 \)-balls needed to cover \( M \). We now show that for \( \nu \)-almost every \( x \in B \), the measure \( \mu_x \) has at least one atom.

For \( x \in X \), let
\[
m(x) = \inf \sum \text{diam } (U_j),
\]
where the infimum is taken over all collections of closed balls \( U_1, \ldots, U_k \) in \( M \) such that \( k \leq N \) and \( \mu_x(\bigcup_{j=1}^k U_j) \geq .5 \). Let \( m = \text{ess sup } x \in B m(x) \).

We now show that \( m = 0 \). If \( m > 0 \), then there exists an integer \( J \) such that
\[
C \Delta c^J N < m/2,
\]
where \( \Delta \) is the diameter of \( M \). Let \( \mathcal{U} \) be a cover of \( M \) by \( N \) closed balls of radius \( R/10 \). For \( x \in B \), let \( U_1(x), \ldots, U_{k(x)}(x) \) be those balls in \( \mathcal{U} \)
that meet $\Lambda_x$. Since these balls cover $\Lambda_x$, and $\mu_x(\Lambda_x) > .5$, it follows that $\mu_x(\bigcup_{j=1}^{k(x)} U_j(x)) \geq .5$. But $\varphi_x^{(i)} \mu_x = \mu_{f_i(x)}$, and so it’s also true that
\[
\mu_{f_i(x)}\left(\bigcup_{j=1}^{k(x)} \varphi_x^{(i)}(U_j(x))\right) \geq .5,
\] (2)
for all $i$.

We now use the fact that $\varphi_x^{(i)}$ contracts regular neighborhoods to derive a contradiction. The balls $U_j(x)$ meet $\Lambda_x$ and have diameter less than $R/10$, and so by Corollary 1.2, (2), we have
\[
\text{diam} \left(\varphi_x^{(i)}(U_j(x))\right) \leq C \Delta c_i.
\] (3)

Let $\tau : B \to \mathbb{N}$ be the first-return time of $f^I$ to $B$, so that $f^{\tau(x)}(x) \in B$, and $f^{\tau_i(x)}(x) \notin B$, for $i \in \{1, \ldots, \tau(x) - 1\}$. Decompose the set $B$ according to these first return times:
\[
B = \bigcup_{i=1}^{\infty} B_i \ (\text{mod } 0),
\]
where $B_i = \tau^{-1}(i)$. Because $f$ is invertible and $f^{-1}$ preserves measure, we also have the mod 0 equivalence:
\[
B' := \bigcup_{i=1}^{\infty} f^{\tau_i}(B_i) = B \ (\text{mod } 0).
\]

Let $y \in B'$. Then $y = f^{\tau_i}(x)$, where $x \in B_i \subseteq B$, for some $i \geq 1$. It follows from the definition of $m(y)$ and inequalities (2), (3) and (1) that
\[
m(y) \leq \sum_{j=1}^{k(x)} \text{diam} \left(\varphi_x^{(j_i)}(U_j(x))\right)
\leq C k(x) \Delta c^{\tau_i}
\leq C N \Delta c^J
< m/2.
\]

But then
\[
m = \text{ess sup } x \in B m(x)
= \text{ess sup } y \in B' m(y)
< m/2.
\]
contradicting the assumption \( m > 0 \).

Thus \( m = 0 \), and, for \( \nu \)-almost every \( x \in B \), we have \( m(x) = 0 \). If
\( m(x) = 0 \), then there is a sequence of closed balls \( U^1(x), U^2(x), \ldots \) with
\( \lim_{i \to \infty} \text{diam} (U^i(x)) = 0 \) and \( \mu_x(U^i(x)) \geq \frac{.5}{N} \), for all \( i \). Take \( p_i \in U^i(x) \); any accumulation point of \( \{p_i\} \) is an atom for \( \mu_x \). Since we have shown that
\( \mu_x \) has an atom, for \( \nu \)-a.e. \( x \in B \), the proof of Theorem II is complete. \( \square \)

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