Mean Field Behaviour in a Local Low-Dimensional Model

Hans-Martin Bröker and Peter Grassberger
Physics Department, University of Wuppertal
42097 Wuppertal, Germany

September 21, 2018

Abstract

We point out a new mechanism which can lead to mean field type behaviour in nonequilibrium critical phenomena. We demonstrate this mechanism on a two-dimensional model which can be understood as a stochastic and non-conservative version of the abelian sandpile model of Bak et al. [1]. This model has a second order phase transition whose critical behaviour seems at least partly described by the mean field approximation for percolation, in spite of the low dimension and the fact that all interactions are of short range. Furthermore, the approximation obtained by replacing the lattice by a Bethe tree is very precise in the entire range of the control parameter.
Although mean field theory is one of the most useful and most commonly applied approximations for spatially extended complex systems, it has an important drawback: near critical points it is usually not only quantitatively but also qualitatively wrong, in the sense that it predicts wrong critical exponents and scaling functions. There are only few known exceptions to this. They involve systems with infinitely long range interactions, systems in high dimensions, and systems defined on “lattices” without loops (“Bethe tree” lattices).

The reason why mean field theory gives wrong results is easily understood qualitatively. By neglecting correlations, it also neglects the feedback of the local order parameter value $s(x)$ onto itself. Since a system at its critical point is very sensitive to any influence, these feedbacks are enhanced and modify its behaviour in an essential way. The above exceptions avoid this problem by either disallowing feedback loops (Bethe lattices) or by suppressing the effect of short loops in favour of long loops which cover essentially the whole lattice.

In this note we want to point out a novel mechanism for generating mean field behaviour. This mechanism can only be effective in non-equilibrium systems, but we conjecture that it should occur there rather commonly. In equilibrium systems, the above-mentioned feedback is always positive due to detailed balance. In non-equilibrium systems, however, the feedback can also be negative. If the system reacts to the feedback in a strongly non-linear way, the effects of individual loops will thus not be additive, but might cancel. In general, this cancellation will of course not be complete, and the net effect of the feedback will be similar to random noise. Otherwise said, the order parameter $s(x)$ will be “confused” by the influences of the different feedback loops. This confusion will lead to a fast loss of memory which in turn implies that the mean field approximation can become correct.

Specifically, we shall illustrate this with a model which is inspired by the sandpile model [1]. It is also abelian in the sense of [2], but the evolution of avalanches is stochastic and non-conservative. As a consequence, its critical behaviour is not self-organized, but it has a critical point in the ordinary (co-dimension 1) sense. A model which gives identical spatial structures was first introduced by one of us in [3] (model nr. 5 in the appendix of that paper). Similar (but somewhat more complicated) models were also studied in [4]. Although we cannot prove analytically that the critical behaviour of our model is of mean field type, our numerical results suggest very strongly that at least two of the three independent critical exponents have their mean field value, and that scaling functions are identical.

Our model is defined on a two-dimensional square lattice (generalizations to higher dimensions are obvious), and time is discrete. At each lattice site we have a “spin” $z_i$, which can take any non-negative integer value, but only the values $z_i = 0$ and $z_i = 1$ are ”stable”. If $z_i$ becomes $> 1$ during the evolution, it “topples” at the
next time step. The toppling rule is

\[ z_i \rightarrow z_i - 2 \]  

for the site which topples, and

\[ z_j \rightarrow \begin{cases} z_j + 1 & \text{with probability } p \\ z_j & \text{with probability } 1 - p \end{cases} \]

for each of its four neighbours. Notice that each neighbour \( j \) has the same chance \( p \) to get its spin increased, independently of what happens at the other neighbours. Thus the sum \( \sum_i z_i \) fluctuates (during each toppling, it can change by any value between -2 and +2), but in the average each toppling event causes \( \sum_i z_i \) to decrease by \( 4p - 2 \). The critical point is exactly where this vanishes, \( p_c = 1/2 \). For later use we define \( \epsilon = 1/2 - p \), and \( \rho = \langle z \rangle \).

As in the sandpile model, the dynamics consists of two rules:
(i) If all sites are stable, a site \( i \) is chosen randomly, and \( z_i \) as well as time \( t \) are increased by one unit. In the following, we call this an “event”.
(ii) If at least one site is unstable, the above toppling rule is applied simultaneously at all unstable sites. If some \( z_i \geq 4 \) so that it has to topple repeatedly to become stable, then all these topplings are also done simultaneously. After this, \( t \) is increased by 1. This is repeated until all sites are stable again, after which rule (i) is applied again.

Our most extensive studies were done for \( p < p_c \), where all avalanches are finite (with probability one), even on an infinite lattice. In this case we can use periodic boundary conditions, which greatly reduces finite-size effects. Some simulations were done also at \( p = p_c \). There \( \rho \) would not decrease during an avalanche, leading to problems with infinite avalanches when periodic boundary conditions are used. In this case, we used the same kind of open b.c. as in the sandpile model: if a neighbour of site \( i \) is outside the boundary, we just disregard it in eq. (2).

In a mean field theory for this model, we assume that a toppling site ”forgets” after the next time step that it has toppled, and \( z_i \) is replaced by an identically and independently distributed random variable which takes the value 1 with probability \( \rho \), and 0 with probability \( 1 - \rho \). This implies in particular that there are no correlations, \( c(i - j) \equiv \langle z_i z_j \rangle - \rho^2 = 0 \). Correlations measured in simulations are shown in fig.1. We see that they are indeed small and, what is more important, they decay very rapidly with distance. A similarly rapid decay was observed also in the sandpile model. In that model it was indeed proven that the correlations decay as \( ||i - j||^{-4} \).

Notice that we assume that this loss of memory due to confusion by the topplings of the other neighbours happens only after the next time step. So each toppling site

\[1\] Serious problems arise only when \( p > p_c \), but we expect very slow behaviour and numerical instabilities already at the critical point.
(except the very first of the avalanche which we call the root) has three neighbours
which it can cause to become unstable with probability \( a = \rho p \), while the fourth
neighbour (its father) will be induced to topple with a different probability \( a' \).
The simplest assumption would be \( a' = 0 \). More precise estimates can be obtained
from self-consistency arguments or from comparison with simulations. Both give
\( a'/a \approx 0.1 \). In the following we shall compare simulation data with predictions
for the two extreme choices \( a' = 0 \) and \( a' = 0.1a \), mainly to show that the results
depend very little on \( a' \).

For \( a' = 0 \) our mean field treatment would be completely equivalent to percola-
tion on a Bethe lattice with coordination number four, the sites of which are occupied
with probability \( a \) (except for the root, which is occupied with probability \( \rho \)). This
problem is exactly soluble and well discussed in the literature [7]. The extension
to \( a' \neq 0 \) is a straightforward application of the theory of branching processes [8].

But first we have to compute \( \rho \) and \( a \) as functions of the control parameter \( p \). For
this we use stationarity and the fact that each toppling decreases \( \langle \sum_i z_i \rangle \) by \( 4\epsilon \),
while each event (whether it actually triggers off an avalanche or not) increases
it by one unit. Thus the average number of topplings per event is \( \langle s(p) \rangle = 1/4\epsilon \).
On the other hand, the average number of topplings during the first update of an
avalanche is \( 4a \), while it is multiplied by \( 3a + a' \) during each successive update.
Taking into account that each event triggers an avalanche with probability \( \rho \), we
find
\[
\langle s(p) \rangle = \rho [1 + 4a \sum_{i=0}^{\infty} (3a + a')^i] = \frac{\rho (1 + a - a')}{1 - 3a - a'}. \tag{3}
\]
Combining these two estimates, we find a somewhat complicated expression for \( \rho \) as
a function of \( p \). Instead of writing it down we just give inversely \( \rho \) and \( p \) as
functions of \( a \) and \( a' \):
\[
\rho = \frac{1}{2} + 2a \frac{a - a'}{1 + a - a'}, \quad p = a/\rho. \tag{4}
\]
This gives always \( p \leq 1/2 \), as we should expect since no stationary solution exists
for \( p > 1/2 \). For \( p \to 1/2^- \), we find \( \rho = \frac{2}{3 + (a'/a)} [1 - e^{2(1-a'/a)(7+a'/a)/(3+a'/a)^2}] \). Fig.2
shows the very reasonable agreement of this prediction with values obtained by
simulations. We point out in particular that the data show that \( d\rho/dp \) is finite at
\( p \to 1/2 \), \( \rho \approx 0.6483 - 0.73\epsilon \), as predicted by eq.(4).

Figure 2: Density \( \rho(p) \) against \( p \). The continuous line is the mean field prediction of
eq.(4) with \( a' = 0 \), the dashed line is the prediction with \( a' = 0.1a \), and the points show
the numerical data obtained from simulations.
The quantity easiest to compute is the survival probability $P_t$, defined as the probability that an event triggered an avalanche which lasts for $\geq t$ time steps. We denote by $Q_t$ the probability that all sites are again stable at time $t$, provided that the event started with an unstable root at time 0. Then obviously $P_t = \varrho(1 - Q_t)$. Similarly, we call $\tilde{Q}_t$ the probability that an unstable site different from the root will not create any unstable offspring $t$ time steps later.

Furthermore, we denote by $p_k = \binom{4}{k}(1 - a)^{4-k}a^k$ the probability that the first toppling triggers off $k$ topplings at the next time step, and by $\tilde{p}_k$ the analog distribution for a later toppling. Their generating functions are given by

$$g(s) = \sum_{k=0}^{4} s^k p_k = (1 - a + sa)^4$$

and

$$\tilde{g}(s) = \sum_{k=0}^{4} s^k \tilde{p}_k = (1 - a' + sa')(1 - a + sa)^3.$$  

Then we have obviously

$$\tilde{Q}_t = \sum_k \tilde{p}_k [\tilde{Q}_{t-1}]^k = \tilde{g}(\tilde{Q}_{t-1})$$

and

$$P_t = \varrho(1 - Q_t) = \varrho g(\tilde{Q}_{t-1}).$$

In the vicinity of the critical point, $P_t$ obeys the scaling law

$$P_t \approx \frac{1}{t^\delta} \Phi(t\epsilon^{\nu_t})$$

with $\delta = \nu_t = 1$.

Figure 3: Avalanche survival probabilities $P_t$ for different values of $\epsilon$. From left to right $\epsilon = 0.1, 0.0316, 0.01, 0.00316, 0.001$, and $0.000316$. Solid lines are from mean field theory with $a' = 0$, dashed lines from mean field theory with $a'/a = 0.1$, and dots are from simulations.

Predictions for $P_t$ are compared to simulation data in fig.3. Notice that we used there the exact predictions from the recursion relation eq.(7) which can be made numerically stable by some minor rearrangements. For small values of $t$ we see perfect agreement, while there are deviations at very large $t$. They indicate that $\nu_t$ is somewhat larger than its mean field value, $\nu_t = 1.024 \pm 0.008$, while $\delta = 1.00 \pm 0.01$ is in exact agreement with the prediction.

The other quantity we studied is the avalanche size distribution $P_n$, defined as the probability that an event involves exactly $n$ topplings. For $n = 0$ and $n = 1$ we have $P_0 = 1 - \varrho$ and $P_1 = \varrho(1 - a)^4$. To calculate $P_n$ for $n > 1$ we use a theorem due to Dwass[9]. Consider a branching process where the distribution of offsprings
of a single individuum is given by \( p_k \), with generating function \( g(s) \). Its *progeny* is the total number of descendants, including itself. Then the probability that the total progeny of \( n_0 \) individua is \( n \) is given by

\[
P_{n|n_0} = \frac{n_0}{n} p_{n-n_0}, \quad n \geq n_0,
\]

where \( p_k^{(n)} \) is the \( k \)-th Taylor coefficient of \([g(s)]^n\), i.e. \([g(s)]^n = p_0^{(n)} + p_1^{(n)} s + p_2^{(n)} s^2 + \ldots\).

Denoting again quantities referring to side branches by tilde’s, we have in the present case

\[
\hat{P}_{n|n_0} = \hat{\varrho} \sum_{k=0}^{4} p_k \tilde{P}_{n-1|k} = \hat{\varrho} \sum_{k=0}^{4} k \frac{k}{n-1} p_{k-1}^{(n-1)},
\]

while \( \tilde{p}_k^{(n)} \) can be computed in several ways. The most straightforward is to use binomial expansions in \([\tilde{g}(s)]^n\). For large values of \( n \) we can also use the asymptotic behaviour. From the cumulant expansion of \( \tilde{g}(s) \) we see that \( \tilde{p}(n) \) tends for \( n \to \infty \) towards a Gaussian probability distribution with mean value \( n \langle k \rangle \) and variance \( n \sigma \), where \( \langle k \rangle = 3a + a' \) and \( \sigma = 3a(1-a) + a'(1-a') \) are the first two cumulants of \( \tilde{p} \). This gives us

\[
\hat{P}_{n|n_0} \approx \frac{n_0}{\sqrt{2\pi n^2 \sigma}} e^{-\frac{[1-(\langle k \rangle)-n-n_0]^2}{2n\sigma}}.
\]

Figure 4: Integrated avalanche size distribution \( D_n \) for the same values of \( \epsilon \) as in fig.3. Solid lines are again predictions with \( a = 0 \), dashed lines are from predictions with \( a' = 0.1a \), and points are from simulations.

The integrated distribution \( D_n = \sum_{m=n}^{\infty} \mathcal{P}_m \) is shown in fig.4. Again, the continuous and dashed lines give the predictions with \( a' = 0 \) and \( a' = 0.1a \), while the dots are results from simulations. This time the agreement is essentially perfect. This proves also that the simulation data satisfy the mean field scaling behaviour obtained from eq.(12),

\[
D_n = \frac{1}{\sqrt{n}} \Psi(ne^2).
\]

But again we found that the agreement with the detailed prediction of the Bethe lattice model is much better than that with the scaling form.

Finally, we mention that we also performed some simulations at \( p = 1/2 \), on lattices of size \( L \times L \) with open boundary conditions. We found that the average avalanche size \( \langle n \rangle \) scales as \( L^2 \), as expected for any model of this type in which \( \langle \sum_i z_i \rangle \) is conserved [4, 10].

In summary, we have shown numerically that a non-equilibrium model inspired by sandpile models shows a critical behaviour which is extremely close to mean field type. Indeed it seems that all exponents except one are identical to those for
mean field percolation. This is so in spite of the fact that the model involves only short range interactions and lives on a 2-\(d\) lattice. We have argued that this is due to the “confusion” brought about by the non-positive and non-linear effect of feedback loops.

We should mention that similar behaviour was observed earlier in similar models \([4]\). But apart from being more complicated, these models were still closer in spirit to the original sandpile model of \([1]\). In particular, in \([4]\) models were studied where the avalanche evolution was either stochastic or non-conservative, but not both together. It was the combination of both aspects which allowed us to draw a very close connection to percolation, and to compare in detail with percolation on a Bethe lattice.

Finally, we should mention that our model is very similar – superficially seen – to the Ising model. Indeed, only very minor modifications were needed in the computer program given in \([3]\) to switch from one model to the other. The main difference with the Ising model is that the present model does not satisfy detailed balance and is thus an inherently non-equilibrium system. We believe that the latter is necessary to observe mean-field type behaviour generated by the above mechanism.

P.G. wants to thank L. Pietronero for interesting discussions, and A. Vulpiani for hospitality at the University of Rome where part of this work was done. It was also supported by the DFG (Sonderforschungsbereich 237).

References

[1] P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. Lett. 59 (1987) 381
[2] D. Dhar, Phys. Rev. 64, 1613 (1990)
[3] P. Grassberger, J. Phys. A 26, 1023 (1992)
[4] S.S. Manna, L.B. Kiss and J. Kertesz, J. Stat. Phys. 61 (1990) 923
[5] P. Grassberger and S.S. Manna, J. Physique (Paris) 51, 1077 (1990)
[6] S.N. Majumdar and D. Dhar, Physica A 185, 129 (1992)
[7] D. Stauffer, Introduction to Percolation Theory (Taylor & Francis, London 1985)
[8] T.E. Harris, The Theory of Branching Processes (Springer, Berlin 1969)
[9] M. Dwass, J. Appl. Prob. 6, 682 (1969)

[10] L.P. Kadanoff, S.R. Nagel, L. Wu and S.-M. Zhou, Phys. Rev. A 39, 6524 (1989)