Homoclinic orbits in a piecewise linear Rössler-like circuit

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Abstract: In addition to the well-known Rössler funnel that consists in near-homoclinic orbits, perfect homoclinic orbits have been found numerically and experimentally in a simplest piecewise linear Rössler-like electronic circuit. The evolution of the system in the homoclinic range exhibits period-bubbling and period-adding cascades when a control parameter is changed. A scaling law in the period-adding cascade between the period of a homoclinic orbit and the bifurcation parameter is evaluated. Other phenomena, such as the coexistence of two homoclinic orbits, homoclinic chaos, symmetry breaking and phase bistability are also demonstrated. The results of numerical simulations are in a good agreement with experiments.

1. Introduction

Piecewise linear systems deserve much attention due to their natural extensions to linear systems to capture nonlinear phenomena observed in practice [1]. It seems that piecewise linear systems should have a simple behaviour because they are made up of various linear systems matching with continuity at some boundaries. However, the lack of differentiability gives rise to very complicated dynamics, almost the same or even richer than that of nonlinear systems: limit cycles, homoclinic and heteroclinic orbits, strange attractors and chaos. Furthermore, some of phenomena inherent to discontinuous systems cannot be observed in smooth systems, for example, period-adding bifurcations, period multiplication and jumps to chaos or destruction of all periodic orbits. These bifurcations referred to as grazing bifurcations, are associated with portions of trajectory being tangent to surfaces where the system is discontinuous.

The theoretical investigation of piecewise linear systems was started by Andronov and his co-workers [2]. Since the discovering of chaos in such systems by Chua and his colleagues (see e.g. [3]), the interest to these systems has rose significantly in view of incredible potential applications of chaos theory to engineering study and in particular to secure communications [4]. Considering a synchronisation problem of chaotic oscillators, Heagy, Pecora and Carrol [5] constructed a piecewise linear electronic circuit based on the Rössler system. They have shown that this system displays Rössler-type chaos in a certain parameter range.

In this work we study dynamics of the piecewise linear Rössler-like system in a range of parameters different from those explored by Pecora and Carrol [6]. We will show that this system displays very rich dynamics, richer than the conventional Rössler system. Many complex phenomena, such as homoclinic orbits and homoclinic chaos, a period-doubling route to chaos in homoclinic orbits,
coexistence of two homoclinic orbits, period-adding and period-bubbling cascades, change of symmetry in the period doubling and phase bistability are observed in this system.

Homoclinicity was initially studied by Rössler who identified “spiral-” and “screw-” types of homoclinic scenarios [7]. However, systematic characterization was provided by Shil’nikov who derived a condition for homoclinic chaos around a saddle focus [8]. Homoclinic chaos was previously observed in many systems, including lasers, chemical reactions, a superconducting quantum interface device, a glow discharge plasma, an optical bistable device, a bubble formation, magnetoconvection, neural and cardiac rhythms, food chains, a van der Pol oscillator, coupled Chua’s oscillators, Taylor-Couette flows, nematic liquid crystals, rabbit arteries, human brain activity, the respiratory cycle of rats, cosmological models, vacuum Rabi oscillations of moving atoms, as well as in some discrete systems like the double-well map and dissipative Hénon map (see, for example, [9,10] and references therein).

A homoclinic orbit usually changes its period when the number of loops of the trajectory around a saddle focus increases or decreases by adding or resting one loop, while a control parameter is varied. This rather old phenomenon discovered by van der Pol and van der Mark in 1927 [11] and known as a period-adding cascade [12] has been observed in many real systems, such as the Chua’s circuit [13], firing neurons [14], lasers [15] and bubble formation [16]. In this work we demonstrate the period-adding cascade of homoclinic orbits in a piecewise linear Rössler-like system, for the first time to our knowledge, and derive a scaling relation between the orbit period and the control parameter.

Another interesting phenomenon under investigation in this work is the coexistence of two homoclinic orbits. This phenomenon has been previously observed in a bubble formation [14]. Here we will show that the generalized bistability in homoclinic orbits also occurs in a piecewise linear Rössler-like system. Moreover, we demonstrate, for the first time to our knowledge, not only the coexistence of two orbits but also the coexistence of two different routes to chaos, namely, the direct and inverse period-doubling cascades, in the homoclinic orbits.

This paper is organized as follows. In section 2 we describe the model and show the experimental circuit correspondent to this model. We also analyse stability of fixed points. In section 3 we demonstrate diverse complicity in dynamical behaviour of homoclinic orbits. Finally, the main conclusions are given in section 4.

2. Model and experimental setup

We use the Carroll’s model [14] of the electronic circuit shown in figure 1. This model represents a small modification of the Rössler system [7] by replacing the nonlinear element by a piecewise linear function. The equations describing the experimental system can be written in the following dimensionless form

\[
\begin{align*}
\frac{dx}{dt} &= -\alpha(\delta x + \beta y + z), \\
\frac{dy}{dt} &= \alpha(x + \gamma y - \epsilon y), \\
\frac{dz}{dt} &= \alpha[g(x) - z], \\
g(x) &= \begin{cases} 
0, & x \leq 3 \\
\mu(x-3), & x > 3
\end{cases}.
\end{align*}
\]  

(2)

The time factor \( \alpha = 10^4 \, \text{s}^{-1} \) and the other circuit parameters are \( \delta = 0.05, \beta = 0.5, \epsilon = 0.02, \mu = 15, \gamma = r/R, \) where \( r = 10 \, \text{k}\Omega \) and \( R \) is a control parameter varied from 27 k\Omega to 200 k\Omega. The piecewise linear function \( g(x) \) is determined by the diode in the operational amplifier A4. The amplifier is switched on when the voltage \( X \) exceeds 3V.
As distinct from the classical Rössler system [7] with a multiplicative first-order nonlinearity \((xz)\) in \(z\) direction, which has a single saddle-node equilibrium point, our system has two saddle nodes corresponding to different values of variable \(x\). Below the threshold value \((x \leq 3)\) there exists fixed point \(P_0 = \{x_0, y_0, z_0\} = \{0,0,0\}\), whereas above the threshold the fixed point \(P_1 = \{x_1, y_1, z_1\} = \{3a(a-1), 3(\varepsilon\mu - a)(a-1), 3/(a-1)\}\), where \(a = \mu[\varepsilon + (\varepsilon\delta + 1)/(\delta\gamma - \beta)]\). The both points are saddle spirals for our set of the parameters.

For point \(P_0\) the saddle-focus index \(|\text{Re}(\lambda_2^0)/\lambda_1^0| > 1\), where \(\lambda_1^0\) and \(\lambda_2^0\) are the leading eigenvalues (\(\lambda_1^0\) determining the rate of approaching and \(\lambda_2^0\) determining the rate of leaving this point). The Shil’nikov condition for point \(P_0\) is not satisfied and hence homoclinic orbits cannot be created for this point. Indeed, both numerical simulations and experiments confirm that the evolution of the trajectory around this point displays a period-doubling route to Rössler chaos when the control parameter \(R\) is decreasing. The different situation occurs with fixed point \(P_1\) for which \(|\lambda_2^1/\text{Re}(\lambda_1^1)| < 1\), i.e. the characteristic time following the focus direction is the largest one and hence a homoclinic connection is possible.

3. From Rössler to homoclinic chaos

At large values of the control parameter the system displays a period-doubling route to Rössler chaos with decreasing the control parameter from \(R = 140 \text{k}\Omega\) to \(R = 68 \text{k}\Omega\). Periodic windows appear inside the chaotic range for some parameter values. For \(R < 30 \text{k}\Omega\) Rössler chaos is replaced by homoclinic chaos and for \(R < 28.5\) period-doubling cascades in homoclinic orbits gives rise. The evolution of the system from periodic orbits to Rössler chaos, homoclinic chaos and periodic homoclinic orbits with decreasing control parameter is shown in figure 2.

Figure 1. Electronic scheme and experimental setup.

Figure 2. Phase space trajectories corresponding to different dynamical regimes.
The coexistence of the direct and inverse period-doubling routes to chaos in homoclinic orbits is shown in figure 3(a). The generalized bistability is discovered by increasing and decreasing the control parameter. The period-adding cascade is shown in figure 3(b). All these features are observed in both numerical simulations and experiments. As seen from figure 3(b), there exist critical points at which the homoclinic orbit changes its period and by such a way forming the period-adding cascade in homoclinic orbits. We find empirically a specific scaling relation between the number of loops of the trajectory in the $xy$-projection (period of the homoclinic orbit), while approaching saddle focus $P_1$, and the critical value of the control parameter (bifurcation point). The control parameter at which the number of the loops changes satisfies the following relation

$$R_n = R_\infty + 10^{-(2/3)n},$$

where $n$ is the number of loops and $R_\infty$ is the critical parameter value of the control parameter at which $n \to \infty$.

![Figure 3](image1)

Figure 3. (a) Generalized bistability in homoclinic orbits and (b) period-adding cascade of homoclinic orbits.

![Figure 4](image2)

Figure 4. Scaling behaviour of period-adding cascade in (a) normal scale and (b) semilog scale. For our parameters $R_\infty \approx 27.99$ kΩ.

Other interesting phenomena observed in the homoclinic region are a change in symmetry and phase bistability. We demonstrate these phenomena in figure 5 through the bifurcation diagrams. One
can see the exchange between the branches of the period doubling. However, the ordinary regime, when the branches follow their usual ways and the phase does not change, is also coexists. Thus, there exists phase bistability in the period-doubling range.

**Figure 5.** Phase bistability and change of symmetry in period doubling. The arrows indicate the direction of the parameter change.

4. Conclusions

In conclusion, we have demonstrated that the piecewise linear Rössler-like circuit displays a very rich behavior ranging from a period-doubling route to Rössler chaos to homoclinic chaos and perfect homoclinic orbits with coexistence of direct and inverse period-doubling routes to chaos in the homoclinic orbits. Furthermore, in the homoclinic range the system exhibits period-adding cascade. We have derived a scaling relation between the period of a homoclinic orbit and the bifurcation parameter. The period decreases with the $-3/2$ scaling exponent when the control parameter (control resistance) approaches its critical value at which period $n \to \infty$. A good agreement between the theory and experiment is obtained.

We should note that the homoclinic behavior inherent many piecewise systems, such as for instance, lasers with saturable absorber, bursting neurons, etc. The complex dynamical features described in this paper may be useful for application in secure communications with two coupled chaotic electronic circuits. The coexistence of two different chaotic attractors may be utilized for increasing security if appropriate synchronization of coupled circuits is organized [15,16].

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