Large permutations and parameter testing*

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Abstract

A classical theorem of Erdős, Lovász and Spencer asserts that the densities of connected subgraphs in large graphs are independent. We prove an analogue of this theorem for permutations and we then apply the methods used in the proof to give an example of a finitely approximable permutation parameter that is not finitely forcible. The latter answers a question posed by two of the authors and Moreira and Sampaio.

1 Introduction

Computer science applications that involve large networks form one of the main motivations to develop methods for the analysis of large graphs. The theory of graph limits, which emerged in a series of papers by Borgs, Chayes, Lovász, Sós, "The work leading to this invention has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013)/ERC grant agreement no. 259385. Hoppen acknowledges the support of CNPq (Proc. 486108/2012-0 and 304510/2012-2) and FAPESP (Proc. 2013/03447-6). Kohayakawa was partially supported by FAPESP (2013/03447-6, 2013/07699-0), CNPq (459335/2014-6, 310974/2013-5, 477203/2012-4) and the NSF (DMS 1102086). Hoppen and Kohayakawa acknowledge the support of the University of São Paulo, through NUMEC/USP (Project MaCLinC/USP).

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Szegedy and Vesztergombi [4–6, 18], gives analytic tools to cope with problems related to large graphs. It also provides an analytic view of many standard concepts, e.g. the regularity method [19] or property testing algorithms [12, 20]. In this paper, we focus on another type of discrete objects, permutations, and we give permutation counterparts of some of classical results on large graphs. It is worth noting that not all results on large graphs have permutation analogues and vice versa as demonstrated, for example, by the finite forcibility of graphons and permutons [15] (vaguely speaking, finite forcibility means that a global structure is determined by finitely many substructure densities).

Both our main results are related to the dependence of possible densities of (small) substructures. In the case of graphs, Erdős, Lovász and Spencer [8] considered three notions of substructure densities: the subgraph density, the induced subgraph density and the homomorphism density. They showed that these types of densities in a large graph are strongly related and that the densities of connected graphs are independent. The result has a natural formulation in the language of graph limits, which are called graphons: the body of possible densities of any \( k \) connected graphs in graphons, which is a subset of \([0, 1]^k\), has a non-empty interior (in particular, it is full dimensional).

Our first result asserts that the analogous statement is also true for permutations (Theorem 8). As in the case of graphs, it is natural to cast our result in terms of permutation limits, called permutons. The theory of permutation limits was initiated in [13, 14] (also see [21]) and successfully applied e.g. in [12, 17]. To state our first result, we use the notion of a connected permutation, which is an analogue of graph connectivity in the sense that a connected permutation cannot be split into independent parts. Our first result says that the body of possible densities of any \( k \) connected permutations in permutons has a non-empty interior.

Our second result is related to algorithms for large permutations. Such algorithms are counterparts of extensively studied graph property testing, see e.g. [1, 2, 9, 10, 22]. In the case of permutations, two of the authors and Moreira and Sampaio [11, 12] established that every hereditary permutation property is testable with respect to the rectangular distance and two of the other authors [16] strengthened the result to testing with respect to Kendall’s tau distance. In addition to property testing, a related notion of parameter testing was also considered in [12] where testable bounded permutation parameters were characterized.

However, the interplay between testing and the finite forcibility of permutation parameters was not fully understood in [12]. In particular, the authors asked [12, Question 5.5] whether there exists a testable bounded permutation parameter that is not finitely forcible. Our second result (Theorem 12) gives a positive answer to this question. Informally speaking, we utilize the proof methods used in the proof of the first of our results and we construct a permutation parameter that oscillates on connected permutations and the level of oscillation is bounded, so the parameter is testable though it fails to be finitely forcible.
2 Preliminaries

In this section, we introduce the notions used throughout the paper. Most of our
notions are standard but we include all of them for the convenience of the reader.

2.1 Permutations

A permutation of order $n$ is a bijective mapping from $[n]$ to $[n]$, where $[n]$ denotes
the set $\{1, \ldots, n\}$. The order of a permutation $\sigma$ is denoted by $|\sigma|$. We say
a permutation is non-trivial if it has order greater than 1. We denote by $S_n$
the set of all permutations of order $n$ and let $\Theta = \bigcup_{n \in \mathbb{N}} S_n$. An inversion
of a permutation $\sigma$ is a pair $(i, j)$, $i, j \in [|\sigma|]$, such that $i < j$ and $\sigma(i) > \sigma(j)$.

An interval $I$ in $[m]$ is a set of integers of the form $\{k \mid a \leq k \leq b\}$ for some
$a, b \in [m]$. An interval $I$ is proper if $a < b$ and $I \neq [m]$.

We say that a permutation $\sigma$ of order $n$ is connected if there is no $m < n$
such that $\sigma([m]) = [m]$. Note that

$$\Pr_{\sigma \in S_n}(\sigma \text{ is not connected}) \leq \frac{\sum_{m=1}^{n-1} m!(n-m)!}{n!} = \sum_{m=1}^{n-1} \left(\frac{n}{m}\right)^{-1}$$

$$\leq \frac{2}{n} + \sum_{m=2}^{n-2} \left(\frac{n}{m}\right)^{-1} \leq \frac{2}{n} + (n-3) \frac{2}{n(n-1)}.$$

Thus, $\lim_{n \to \infty} \Pr_{\sigma \in S_n}(\sigma \text{ is connected}) = 1.$

We say that a permutation $\sigma$ is simple if it does not map any proper interval
onto an interval. For example the permutation $(\sigma(1), \ldots, \sigma(4)) = (2, 4, 1, 3)$ is
simple.

Albert, Atkinson and Klazar [3] showed that a random permutation is simple with a probability bounded away from zero. Specifically, they proved the following.

$$\lim_{n \to \infty} \mathbb{P}_{\sigma \in S_n}(\sigma \text{ is simple}) = e^{-2}.$$ (2)

Let $\pi$ be a permutation of order $k$ and $\sigma$ a permutation of order $n$. We
introduce three ways in which $\pi$ can appear in $\sigma$: as a subpermutation, through a
monomorphism and through a homomorphism. We say that $\pi$ is a subpermutation
of $\sigma$ if there exists a strictly increasing function $f : [k] \to [n]$, such that $\pi(i) > \pi(j)$ if and only if $\sigma(f(i)) > \sigma(f(j))$ for every $i, j \in [k]$. Let $\text{Sub}(\pi, \sigma)$ be the
set of all such functions $f$ from $[k]$ into $[n]$ and let $\Lambda(\pi, \sigma) = |\text{Sub}(\pi, \sigma)|$. The density of $\pi$ in $\sigma$ is defined as

$$t(\pi, \sigma) = \begin{cases} \Lambda(\pi, \sigma)(\frac{n}{k})^{-1} & \text{if } k \leq n \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$
A non-decreasing function \( f : [k] \to [n] \) is a \textit{homomorphism} of \( \pi \) to \( \sigma \) if \( \sigma(f(i)) > \sigma(f(j)) \) for every \( i, j \in [k] \) such that \( i < j \) and \( \pi(i) > \pi(j) \), that is, \( f \) preserves inversions. A \textit{monomorphism} is a homomorphism that is injective.

Let \( \text{Hom}(\pi, \sigma) \) and \( \text{Mon}(\pi, \sigma) \) be the sets of homomorphisms and monomorphisms of \( \pi \) to \( \sigma \), respectively, and let \( \Lambda_{\text{hom}}(\pi, \sigma) \) and \( \Lambda_{\text{mon}}(\pi, \sigma) \) denote the sizes of the respective sets. Note that \( \text{Sub}(\pi, \sigma) \subseteq \text{Mon}(\pi, \sigma) \subseteq \text{Hom}(\pi, \sigma) \). The \textit{homomorphism density} \( t_{\text{hom}} \) and \textit{monomorphism density} \( t_{\text{mon}} \) are defined as follows:

\[
t_{\text{mon}}(\pi, \sigma) = \begin{cases} \Lambda_{\text{mon}}(\pi, \sigma) \binom{n}{k}^{-1} & \text{if } k \leq n \\ 0 & \text{otherwise}, \end{cases}
\]

\[
t_{\text{hom}}(\pi, \sigma) = \Lambda_{\text{hom}}(\pi, \sigma) \binom{n + k - 1}{k}^{-1}.\]

The three densities that we have just introduced are analogues of the induced subgraph density, homomorphism density and subgraph density for graphs studied in [8].

Let \( q \) be an integer and let \( \{\tau_1, \ldots, \tau_r\} \) be the set of all non-trivial connected permutations of order at most \( q \). We consider the following three vectors

\[
t^q(\sigma) = (t(\tau_1, \sigma), \ldots, t(\tau_r, \sigma)),
\]

\[
t^q_{\text{mon}}(\sigma) = (t_{\text{mon}}(\tau_1, \sigma), \ldots, t_{\text{mon}}(\tau_r, \sigma)), \text{ and}
\]

\[
t^q_{\text{hom}}(\sigma) = (t_{\text{hom}}(\tau_1, \sigma), \ldots, t_{\text{hom}}(\tau_r, \sigma)).
\]

Our aim is to understand possible densities of subpermutations in large permutations. This leads to the following definitions, which reflect the possible asymptotic densities of the connected permutations of order at most \( q \) in permutations:

\[
T^q = \{ v \in \mathbb{R}^r \mid \exists (\sigma_n)_{n=1}^{\infty} \text{ such that } t^q(\sigma_n) \to v \text{ and } |\sigma_n| \to \infty \},
\]

\[
T^q_{\text{mon}} = \{ v \in \mathbb{R}^r \mid \exists (\sigma_n)_{n=1}^{\infty} \text{ such that } t^q_{\text{mon}}(\sigma_n) \to v \text{ and } |\sigma_n| \to \infty \}, \text{ and}
\]

\[
T^q_{\text{hom}} = \{ v \in \mathbb{R}^r \mid \exists (\sigma_n)_{n=1}^{\infty} \text{ such that } t^q_{\text{hom}}(\sigma_n) \to v \text{ and } |\sigma_n| \to \infty \}.
\]

Now we give three observations on how the sets \( T^q \), \( T^q_{\text{mon}} \) and \( T^q_{\text{hom}} \) relate to each other.

**Observation 1.** The sets \( T^q_{\text{mon}} \) and \( T^q_{\text{hom}} \) are equal for every \( q \in \mathbb{N} \).

**Proof.** Observe that, for every fixed integer \( k \),

\[
\Lambda_{\text{hom}}(\tau, \sigma) - \Lambda_{\text{mon}}(\tau, \sigma) \leq \binom{k}{2} n^{k-1} = O(n^{k-1}),
\]

for every \( \sigma \) of order \( n \) and \( \tau \) of order \( k \).

Hence, for every permutation \( \tau \) and every real \( \varepsilon > 0 \) there exists \( n_0 \) such that \( |t_{\text{mon}}(\tau, \sigma) - t_{\text{hom}}(\tau, \sigma)| < \varepsilon \) for every permutation \( \sigma \) with \( |\sigma| > n_0 \). The statement now follows. \( \square \)
In view of Observation 1 we will discuss only $T^q_{\text{mon}}$ in the rest of the paper.

**Observation 2.** For every $q \in \mathbb{N}$, the set $T^q_{\text{mon}}$ is closed.

**Proof.** Consider a convergent sequence $(w_n)_{n \in \mathbb{N}} \subseteq T^q_{\text{mon}}$ and let $w = \lim_{n \to \infty} w_n$. For each $n$, choose $\sigma_n$ such that $\|t^q_{\text{mon}}(\sigma_n) - w_n\| \leq 1/n$. Observe that $t^q_{\text{mon}}(\sigma_n)$ converges to $w$. \qed

**Observation 3.** The set $T^q$ is a non-singular linear transformation of $T^q_{\text{mon}}$ for every $q \in \mathbb{N}$.

**Proof.** Note that $t_{\text{mon}}(\pi, \sigma) = \sum_{\pi' \in \mathcal{P}} t(\pi', \sigma)$, where $\mathcal{P}$ is a set of permutations $\pi'$ of the same order as $\pi$ such that the identity mapping is a monomorphism from $\pi$ to $\pi'$. Consequently, $t_{\text{mon}}(\pi, \sigma) = \sum_{\pi' \in \mathcal{P}} t(\pi', \sigma)$. This gives that $T^q_{\text{mon}}$ is a linear transformation of $T^q$. Observe that if we order $\tau_1, \ldots, \tau_r$ by the number of inversions, the coefficient matrix of the induced linear mapping is upper triangular with diagonal entries equal to 1. We conclude that the linear transformation of $T^q$ is non-singular. \qed

### 2.2 Permutation limits

In this subsection, we survey the theory of permutation limits, which was introduced in \[13\][14] (a similar representation was used in \[21\]). We follow the terminology used in \[17\]. An infinite sequence $(\sigma_i)_{i \in \mathbb{N}}$ of permutations with $|\sigma_i| \to \infty$ is **convergent** if $t(\tau, \sigma_i)$ converges for every permutation $\tau \in \mathfrak{S}$. Observe that every sequence of permutations has a convergent subsequence. A convergent sequence can be associated with an analytic limit object, a **permuton**. A permuton is a probability measure $\Phi$ on the $\sigma$-algebra of Borel sets of the unit square $[0, 1]^2$ such that $\Phi$ has uniform marginals, i.e., $\Phi([\alpha, \beta] \times [0, 1]) = \Phi([0, 1] \times [\alpha, \beta]) = \beta - \alpha$ for every $0 \leq \alpha \leq \beta \leq 1$. We denote the set of all permutons by $\mathcal{P}$. Given a permuton $\Phi$, a **$\Phi$-random permutation of order $n$** is a permutation $\sigma_{\Phi,n}$ obtained in the following way. Sample $n$ points $(x_1, y_1), \ldots, (x_n, y_n)$ in $[0, 1]^2$ at random with the distribution given by $\Phi$. Note that the values of $x_i$ are pairwise distinct with probability one and the same holds for the values of $y_i$. Let $i_1, \ldots, i_n \in [n]$ be such that $x_{i_1} < x_{i_2} < \cdots < x_{i_n}$. Then the permutation $\sigma_{\Phi,n}$ is the unique bijective mapping from $[n]$ to $[n]$ satisfying that $\sigma_{\Phi,n}(j) < \sigma_{\Phi,n}(j')$ if and only if $y_{i_j} < y_{i_{j'}}$ for every $j, j' \in [n]$. Informally speaking, the values $x_i$ determine the ordering of the points and the relative order of the values $y_i$ determines the relative order of the elements of the permutation.

If $\Phi$ is a permuton and $\sigma$ is a permutation of order $n$, then $t(\sigma, \Phi)$ is the probability that a $\Phi$-random permutation of order $n$ is $\sigma$. We say that a permuton $\Phi$ is a **limit** of a convergent sequence of permutations $(\sigma_i)_{i \in \mathbb{N}}$ if $\lim_{i \to \infty} t(\tau, \sigma_i) = t(\tau, \Phi)$ for every $\tau \in \mathfrak{S}$. Every convergent sequence of permutations has a limit.
Figure 1: The permuton \( \Phi_\sigma^v \) for \( \sigma = (2, 4, 3, 1) \) and \( v = (1/6, 1/4, 1/12, 1/4) \).

and the permuton representing the limit of a convergent sequence of permutations is unique.

Likewise, we can define the monomorphism density of \( \tau \) as the probability that the identity mapping to a random \( \Phi \)-permutation is a monomorphism of \( \tau \). Since we view permutons as representing large permutations, if we defined homomorphism densities in a natural way, they would coincide with monomorphism densities. So, we restrict our study to subpermutation densities and monomorphism densities in permutons. By analogy to the finite case, we define the vectors

\[
\mathbf{t}(\tau, \Phi) = \left( t(\tau_1, \Phi), \ldots, t(\tau_r, \Phi) \right)
\]

and

\[
\mathbf{t}_{\text{mon}}(\tau, \Phi) = \left( t_{\text{mon}}(\tau_1, \Phi), \ldots, t_{\text{mon}}(\tau_r, \Phi) \right),
\]

where \( q \in \mathbb{N} \) and \( \{\tau_1, \ldots, \tau_r\} \) is the set of all non-trivial connected permutations of order at most \( q \).

If \( \Phi \) is a permuton and \( \sigma_i \) is a \( \Phi \)-random permutation of order \( i \), then the sequence \( (\sigma_i)_{i \in \mathbb{N}} \) is convergent with probability one and \( \Phi \) is its limit. In particular, this means that for every finite set of permutations \( \mathcal{P} \) and every \( \varepsilon > 0 \), there exists a permutation \( \varphi \) such that \( |t(\pi, \Phi) - t(\pi, \varphi)| < \varepsilon \) for every \( \pi \in \mathcal{P} \). This yields an alternative description of \( T^q \) as the set \( \{t^q(\Phi) \mid \Phi \in \mathcal{P} \} \). Similarly, \( T_{\text{mon}}^q = \{t_{\text{mon}}^q(\Phi) \mid \Phi \in \mathcal{P} \} \).

Let \( \sigma \) be a permutation of order \( n \) and let \( v = (v_1, \ldots, v_n) \in \mathbb{R}^n_+ \) be such that \( \sum_{i=1}^n v_i \leq 1 \), where \( \mathbb{R}_+ \) is the set of positive reals. The step-up permuton of \( \sigma \) and \( v \) is the permuton \( \Phi^v_\sigma \) such that the support of the measure \( \Phi^v_\sigma \) is formed by the segments between the points \( (\sum_{j<i} v_j, \sum_{j<i} \sigma(j) v_j) \) and \( (\sum_{j<i} v_j, \sum_{j<i} \sigma(j) v_j) \) for \( i \in [n] \) and the segment between the points \( (\sum_{j=1}^n v_j, \sum_{j=1}^n \sigma(j) v_j) \) and \( (1, 1) \). Note that this uniquely determines the permuton \( \Phi^v_\sigma \) because it must have uniform marginals. See Figure 1 for an example.

Let \( \Phi_1, \ldots, \Phi_k \) be permutons and let \( p = (p_1, \ldots, p_k) \in \mathbb{R}_+^k \) be such that \( \sum_{i=1}^k p_i \leq 1 \). Let \( \Phi_{k+1} \) be the permuton with support consisting of the segment between \( (0, 0) \) and \( (1, 1) \) and let \( p_{k+1} = 1 - \sum_{i=1}^k p_i \). We define the composed
permuton $\Phi^p = \bigoplus_{i=1}^{k} (p_i, \Phi_i)$ to be the permuton such that

$$\Phi^p(S) = \sum_{i=1}^{k+1} p_i \Phi_i(\theta_i(S \cap C_i))$$

for every Borel set $S$, where

$$C_i = \left[ \sum_{j=1}^{i-1} p_j, \sum_{j=1}^{i} p_j \right]^2$$

and $\theta_i$ is a map from $C_i$ to $[0,1]^2$ defined as

$$\theta_i((x,y)) = \left( \frac{x - \sum_{j=1}^{i-1} p_j}{p_i}, \frac{y - \sum_{j=1}^{i-1} p_j}{p_i} \right)$$

for every $i \in [k+1]$. See Figure 2 for an example.

For a permutation $\tau$ of order $k$, we call a mapping $\kappa : [k] \to [k']$, for $k' \leq k$, $\tau$-compressive if it is surjective non-decreasing and for every $i \in [k']$, $\kappa^{-1}(i)$ is an interval and $\tau(j') - \tau(j) = j' - j$ for every $j, j' \in \kappa^{-1}(i)$. This means, in particular, that $\tau(\kappa^{-1}(i))$ is an interval for every $i \in [k']$. We denote the set of all $\tau$-compressive mappings by $R(\tau)$. Note that for every permutation $\tau$, there exist at least one $\tau$-compressive mapping: the identity mapping.

For a permutation $\tau$ of order $k$ and a $\tau$-compressive mapping $\kappa : [k] \to [k']$, let $\tau \downarrow \kappa$ be a subpermutation of $\tau$ of order $k'$ satisfying $\kappa' \in \text{Sub}(\tau \downarrow \kappa, \tau)$ for a mapping $\kappa' : [k'] \to [k]$ such that $\kappa \circ \kappa'$ is the identity mapping. Observe that $\tau \downarrow \kappa$ is unique (although $\kappa'$ is not) and that if the permutation $\tau$ is connected, then $\tau \downarrow \kappa$ is connected for every $\tau$-compressive mapping $\kappa$.

Informally speaking, the permutation $\tau \downarrow \kappa$ is a permutation that can be obtained from $\tau$ as follows: we choose a set of pairwise disjoint intervals in the domain of $\tau$ that are monotonically mapped by $\tau$ onto intervals and shrink each interval in the set and its image into single points, without changing the relative order of the elements of the permutation.
Observation 4. Let $\tau$ be a non-trivial connected permutation of order $k$, $\sigma$ a permutation of order $n \geq k$ and let $p = (p_1, \ldots, p_n) \in \mathbb{R}_+^n$ be such that $\sum_{i \in [n]} p_i \leq 1$. It follows that

$$t(\tau, \Phi^p_\sigma) = k! \sum_{\kappa \in R(\tau)} \sum_{\psi \in \text{Sub}(\tau \downarrow \kappa, \sigma)} \prod_{i=1}^{k} p_{\psi \circ \kappa}(i).$$

Informally speaking, Observation 4 holds because for a fixed connected permutation $\tau$ of order $k$, $k$ random points chosen based on the distribution $\Phi^p_\sigma$ induce $\tau$ if and only if none of the $k$ points lies on the last segment of the support of $\Phi^p_\sigma$ and there is a $\tau$-compressive mapping $\kappa$ such that two points lie on the same segment of the support of $\Phi^p_\sigma$ whenever $\kappa$ maps the corresponding elements of $\tau$ to the same value.

Observation 5. Let $\tau$ be a non-trivial connected permutation of order $k$ and let $m$ be a positive integer. Let $\Phi_1, \ldots, \Phi_m$ be permutons and let $x = (x_1, \ldots, x_m) \in \mathbb{R}_+^m$ be such that $\sum_{i \in [m]} x_i \leq 1$. The composed permuton $\Phi^x = \bigoplus_{i \in [m]} (x_i, \Phi_i)$ satisfies

$$t(\tau, \Phi^x) = \sum_{i=1}^{m} x_i^k t(\tau, \Phi_i).$$

Observation 5 is based on the fact that if $k$ random points with distribution $\Phi^p_\sigma$ induce a connected permutation $\tau$, then all the points lie in the same square corresponding to one of the permutons $\Phi_i$.

Analogues of Observations 4 and 5 for densities of monomorphisms also hold.

2.3 Testing permutation parameters

A permutation parameter $f$ is a function from $\mathcal{S}$ to $\mathbb{R}$. A parameter $f$ is \textit{finitely forcible} if there exists a finite family of permutations $\mathcal{A}$ such that for every $\varepsilon > 0$ there exist an integer $n_0$ and a real $\delta > 0$ such that if $\sigma$ and $\pi$ are permutations of order at least $n_0$ satisfying $|t(\tau, \sigma) - t(\tau, \pi)| < \delta$ for every $\tau \in \mathcal{A}$, then $|f(\sigma) - f(\pi)| < \varepsilon$. The set $\mathcal{A}$ is referred to as a \textit{forcing family} for $f$.

A permutation parameter $f$ is \textit{finitely approximable} if for every $\varepsilon > 0$ there exist $\delta > 0$, an integer $n_0$ and a finite family of permutations $\mathcal{A}_\varepsilon$ such that, if $\sigma$ and $\pi$ are permutations of order at least $n_0$ satisfying $|t(\tau, \sigma) - t(\tau, \pi)| < \delta$ for every $\tau \in \mathcal{A}_\varepsilon$, then $|f(\sigma) - f(\pi)| < \varepsilon$.

A permutation parameter $f$ is \textit{testable} if for every $\varepsilon > 0$ there exist an integer $n_0$ and $\tilde{f} : S_{n_0} \to \mathbb{R}$ such that for every permutation $\sigma$ of order at least $n_0$, a randomly chosen subpermutation $\pi$ of $\sigma$ of size $n_0$ satisfies $|f(\sigma) - \tilde{f}(\pi)| < \varepsilon$ with probability at least $1 - \varepsilon$. The following was given in [12].

Lemma 6. A bounded permutation parameter $f$ is testable if and only if it is finitely approximable.
3 Properties of the sets $T^q$ and $T_{\text{mon}}^q$

In this section, we show that densities of non-trivial connected permutations are mutually independent and, more generally, that $T^q$ contains a ball. We start by considering the linear span of $T^q$.

**Lemma 7.** For every $q \in \mathbb{N}$, $\text{span}(T^q) = \mathbb{R}^r$, where $r$ is the number of non-trivial connected permutations of order at most $q$.

**Proof.** Let $\{\tau_1, \ldots, \tau_r\}$ be the set of all non-trivial connected permutations of order at most $q$. For a contradiction, suppose that $\text{span}(T^q)$ has dimension less than $r$, i.e., there exist reals $c_1, \ldots, c_r$, not all of which are zero, such that

$$\sum_{i=1}^r c_i v_i = 0$$

for every $(v_1, \ldots, v_r) \in \text{span}(T^q)$. Therefore,

$$\sum_{i=1}^r c_i t(\tau_i, \Phi) = 0$$

for every permuton $\Phi \in \mathcal{P}$.

Consider the permutations $\tau_i$ such that $c_i \neq 0$. Among these pick a $\tau_k$ of maximum order. Observation 4 yields that the following holds for $s = |\tau_k|$ and every $x = (x_1, \ldots, x_s) \in \mathbb{R}_+^s$ such that $\sum_{i=1}^s x_i \leq 1$:

$$\sum_{i=1}^r c_i t(\tau_i, \Phi^x_{\tau_k}) = \sum_{i=1}^r c_i |\tau_i|! \sum_{\kappa \in \mathcal{R}(\tau_i)} \sum_{\psi \in \text{Sub}(\tau_i, \kappa, \tau_k)} \prod_{j=1}^{|	au_i|} x_{\psi(k_j)} = p(x_1, \ldots, x_s),$$

where $p$ is a polynomial. We now argue that $p$ is a polynomial of degree $s$ (and therefore it is a non-zero polynomial). Clearly, the polynomial $p$ has degree at most $s$. Since $\text{Sub}(\tau', \tau_k) = \emptyset$ for every $\tau'$ of order $s$ such that $\tau' \neq \tau_k$, $c_k! x_1 x_2 \cdots x_s$ is the only term of $p$ containing the monomial $x_1 x_2 \cdots x_s$ with nonzero coefficient. Therefore, there exists $x$ such that $\sum_{i=1}^r c_i t(\tau_i, \Phi^x_{\tau_k}) \neq 0$, which is a contradiction.

The following theorem is the main result of this section. It shows that the interior of $T^q$ is non-empty. Observation 3 yields the same conclusion for $T_{\text{mon}}^q$. In the statement of the following theorem and its proof, we write $B(w, \varepsilon)$ for the ball of radius $\varepsilon$ around $w$ in $\mathbb{R}^r$.

**Theorem 8.** For every integer $q \geq 2$, there exist a vector $w \in T^q$ and $\varepsilon > 0$ such that $B(w, \varepsilon) \subseteq T^q$. 
Proof. Let \( \{\tau_1, \ldots, \tau_r\} \) be the set of all non-trivial connected permutations of order at most \( q \) and let \( \Phi_1, \ldots, \Phi_r \) be permutons such that \( \{t^q(\Phi_i) \mid i = 1, \ldots, r\} \) spans \( \mathbb{R}^r \). Consider the matrix \( V = (v_{i,j})_{i,j=1}^r \), where \( v_{i,j} = t(\tau_j, \Phi_i) \). Observe that the matrix \( V \) is non-singular.

Consider a vector \( x = (x_1, \ldots, x_r) \in (0, r^{-1})^r \) and let \( \Phi^x = \bigoplus_{i \in [r]} (x_i, \Phi_i) \). By Observation 5, we have

\[
t(\tau_j, \Phi^x) = \sum_{i=1}^r x_i^{|\tau_j|} t(\tau_j, \Phi_i) = \sum_{i=1}^r x_i^{|\tau_j|} v_{i,j}.
\]

Let \( \Psi \) be a map from \( \mathbb{R}^r \) to \( \mathbb{R}^r \) such that

\[
\Psi_j(x) = \sum_{i=1}^r x_i^{|\tau_j|} v_{i,j} \text{ for all } j \in [r].
\]

Since we have \( \Psi(x) = t^q(\Phi^x) \), we get that

\[
\Psi((0, r^{-1})^r) = \{ \Psi(x) \mid x \in (0, r^{-1})^r \} \subseteq T^q.
\]

The Jacobian \( \text{Jac}(\Psi)(x) \) is a polynomial in \( x_1, \ldots, x_r \). Since for \( x_1 = \cdots = x_r = 1 \) we have

\[
\text{Jac}(\Psi) = \det(v_{i,j} \cdot |\tau_j|)_{i,j=1}^r = \left( \prod_{j=1}^r |\tau_j| \right) \det V \neq 0,
\]

\( \text{Jac}(\Psi) \) is a non-zero polynomial.

Hence, there exists \( x \in (0, r^{-1})^r \) for which \( \text{Jac}(\Psi)(x) \neq 0 \). Consequently, \( T^q \) contains a ball around \( w = \Psi(x) \).

Theorem 8 implies that for every finite family \( A \) of connected permutations, there exist permutons \( \Phi \) and \( \Phi' \) and a connected permutation \( \tau \) such that \( t(\pi, \Phi) = t(\pi, \Phi') \) for every \( \pi \in A \) and \( t(\tau, \Phi) \neq t(\tau, \Phi') \). The following lemma shows that an analogous statement holds for any finite family of permutations, not only for connected permutations.

Lemma 9. For every finite set of permutations \( A = \{\tau_1, \ldots, \tau_k\} \), there exists a permutation \( \tau \) and permutons \( \Phi \) and \( \Phi' \) such that \( t(\tau_i, \Phi) = t(\tau_i, \Phi') \) for every \( i \in [k] \) and \( t(\tau, \Phi) \neq t(\tau, \Phi') \).

Proof. Let \( B = \{\pi_1, \ldots, \pi_{k+1}\} \) be a family of connected permutations each of order \( n \) with \( n > |\tau_i| \) for every \( i \in [k] \), such that for every \( \pi_j \in B \), there is no \( \ell < n \) satisfying \( \pi_j(\ell + 1) = \pi_j(\ell) + 1 \). We call permutations with this property thorough. By (1) in Section 2.1, a random permutation of order \( n \) is connected with probability tending to one as \( n \) tends to infinity. Moreover, by (2) in Section 2.1 such permutations are thorough with probability bounded away
from zero, because every simple permutation is thorough. Therefore, a family $B$ of $k + 1$ connected thorough permutations exists for $n$ sufficiently large.

Let $\Phi^u = \bigoplus_{i \in [k+1]} (u_i, \Phi^n_{\pi_i})$ for $u = (u_1, \ldots, u_{k+1}) \in (0, \frac{1}{k+1})^{k+1}$ where $n = (1/n, \ldots, 1/n)$.

Observe that for a thorough permutation $\pi$, the identity mapping is the only $\pi$-compressive mapping. Hence, by Observations $[3]$ and $[4]$, $t(\pi_i, \Phi^u) = n!(u_i/n)^n$ for every $i \in [k+1]$. For every $j \in [k]$, the function $u \mapsto t(\tau_j, \Phi^u)$ is continuous for every $j \in [k]$. We consider the continuous map $\Gamma$ from $(0, 1/(k + 1)]^{k+1}$ to $\mathbb{R}^k$ such that

$$\Gamma(u) = (t(\tau_1, \Phi^u), \ldots, t(\tau_k, \Phi^u)).$$

Now, consider any $k$-dimensional sphere in $(0, 1/(k + 1)]^{k+1}$. The Borsuk-Ulam Theorem $[7]$ yields the existence of two distinct points on its surface that are mapped by $\Gamma$ to the same point in $[0, 1]^k$. Hence, there exist distinct $v = (v_1, \ldots, v_{k+1})$ and $v' = (v'_1, \ldots, v'_{k+1})$ such that $t(\tau_j, \Phi^v) = t(\tau_j, \Phi^{v'})$ for every $j \in [k]$. However, if, say $v_i \neq v'_i$, then $t(\pi_i, \Phi^v) = n!(v_i/n)^n \neq n!(v'_i/n)^n = t(\pi_i, \Phi^{v'})$. Therefore, we may take $\tau = \pi_i$, $\Phi = \Phi^v$, and $\Phi' = \Phi^{v'}$. \hfill $\square$

## 4 Non-forcible approximable parameter

For this section, we fix a sequence $(\tau_i)_{i \in \mathbb{N}}$ of permutations of strictly increasing orders that satisfies the following: For every $k > 1$, there exist permutons $\Phi_k$ and $\Phi'_k$ such that $t(\sigma, \Phi_k) = t(\sigma, \Phi'_k)$ for every permutation $\sigma$ of order at most $|\tau_{k-1}|$, and $t(\tau_k, \Phi_k) > t(\tau_k, \Phi'_k)$. Such a sequence $(\tau_i)_{i \in \mathbb{N}}$ exists by Lemma $[5]$. We fix such $\Phi_k$ and $\Phi'_k$ for all $k \in \mathbb{N}$ for the rest of this section. Let $\gamma_k = t(\tau_k, \Phi_k) - t(\tau_k, \Phi'_k)$ for every $k \in \mathbb{N}$.

Let $(\alpha_i)_{i \in \mathbb{N}}$ be a sequence of positive reals satisfying $\sum_{i \in \mathbb{N}} \alpha_i < 1/2$ and $\sum_{i > k} \alpha_i < \alpha_k \gamma_k / 4$ for every $k$. The main result of this section is that the permutation parameter

$$f_*(\sigma) = \sum_{i \in \mathbb{N}} \alpha_i t(\tau_i, \sigma)$$

is finitely approximable but not finitely forcible.

**Lemma 10.** The permutation parameter $f_*$ is finitely approximable.

**Proof.** Let $\varepsilon > 0$ be given. Since the sum $\sum_{i \in \mathbb{N}} \alpha_i$ converges, there exists $k$ such that $\sum_{i > k} \alpha_i < \varepsilon / 2$. Set $A = \{\tau_1, \ldots, \tau_k\}$ and $\delta = \varepsilon$. Consider two permutations
σ and π that satisfy |t(τ, σ) − t(τ, π)| < δ for every τ ∈ A. We obtain that

\[ |f.(σ) − f.(π)| = \left| \sum_{i \in \mathbb{N}} \alpha_i (t(τ_i, σ) − t(τ_i, π)) \right| \]

\[ \leq \sum_{i \in \mathbb{N}} \alpha_i |t(τ_i, σ) − t(τ_i, π)| \]

\[ < \sum_{i \leq k} \alpha_i δ + \sum_{i > k} \alpha_i |t(τ_i, σ) − t(τ_i, π)| \]

\[ < \delta/2 + \sum_{i > k} \alpha_i \cdot 1 < ε. \]

It follows that the parameter \( f \) is finitely forcible.

\[ \square \]

In the following lemma, we show that \( f \) is not finitely forcible.

**Lemma 11.** The permutation parameter \( f \) is not finitely forcible.

**Proof.** Suppose that \( f \) is finitely forcible and that \( A \) is a forcing family for \( f \). Let \( τ, γ, Φ_i \) and \( Φ'_i \) be as in the definition of \( f \), and let \( k \) be such that maximum order of a permutation in \( A \) is at most \( |τ_{k-1}| \). We have \( t(ρ, Φ_k) = t(ρ, Φ'_k) \) for every \( ρ \in A \), \( t(τ_i, Φ_k) = t(τ_i, Φ'_k) \) for every \( i < k \), and \( t(τ_k, Φ_k) = t(τ_k, Φ'_k) = γ_k \).

Let \( ε = α_k γ_k/4 \). Let \( δ > 0 \) be as in the definition of finite forcibility of \( f \). Without loss of generality we may assume that \( δ < ε \).

There exist a \( Φ_k \)-random permutation \( σ \) and a \( Φ'_k \)-random permutation \( σ' \) such that \( |t(ρ, σ) − t(ρ, σ')| < δ \) for every \( ρ \in A \), \( |t(τ_i, σ) − t(τ_i, σ')| < δ \) for every \( i < k \) and \( t(τ_k, σ) − t(τ_k, σ') > γ_k − δ > 3γ_k/4 \). Let us estimate the sum in the definition of \( f \) with the \( k \)-th term missing,

\[ \left| \sum_{i \in \mathbb{N}, i \neq k} \alpha_i (t(τ_i, σ) − t(τ_i, σ')) \right| \]

\[ = \left| \sum_{i < k} \alpha_i (t(τ_i, σ) − t(τ_i, σ')) + \sum_{i > k} \alpha_i (t(τ_i, σ) − t(τ_i, σ')) \right| \]

\[ < \sum_{i < k} \alpha_i δ + \sum_{i > k} \alpha_i < \frac{α_k γ_k}{8} + \frac{α_k γ_k}{4} < \frac{α_k γ_k}{2} \]

This leads to the following

\[ |f.(σ) − f.(σ')| = \left| \sum_{i \in \mathbb{N}} \alpha_i (t(τ_i, σ) − t(τ_i, σ')) \right| \]

\[ ≥ \alpha_k (t(τ_k, σ) − t(τ_k, σ')) \]

\[ > \frac{3}{4} α_k γ_k − \frac{α_k γ_k}{2} = \frac{α_k γ_k}{4} = ε. \]
This contradicts our assumption that \( f \) is finitely forcible.

Lemmas 10 and 11 yield the main theorem of this section. Recall that, by Lemma 6 the testable bounded permutation parameters are precisely the finitely approximable ones.

**Theorem 12.** There exists a bounded permutation parameter \( f \) that is finitely approximable but not finitely forcible.

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