NON-PARAMETRIC INVERSE CURVATURE FLOWS IN THE ADS-SCHWARZSCHILD MANIFOLD

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ABSTRACT. We consider the inverse curvature flows in the anti-de Sitter-Schwarzschild manifold with star-shaped initial hypersurface, driven by the 1-homogeneous curvature function. We show that the solutions exist for all time and the principle curvatures of the hypersurface converges to 1 exponentially fast.

Keywords: Inverse curvature flows, AdS-Schwarzschild manifold, homogeneous curvature function.
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1. Introduction

During the past decades, geometric flows have been studied intensively. Following the groundbreaking work of Huisken [15], who considered the mean curvature flow, several authors started to investigate inverse, or expanding curvature flows of star-shaped closed hypersurfaces in ambient spaces of constant or asymptotically constant sectional curvature. Gerhardt [7] and Urbas [24] independently considered flows of the form

\[ \frac{d}{dt}X = \frac{1}{F^\nu} \]

in \( \mathbb{R}^{n+1} \), where \( F \) is a curvature function homogeneous of degree 1, and proved that the flow exists for all time and converges to infinity. After a proper rescaling, the rescaled flow will converge to a sphere.

The equation (1.1) has the property that it is scale-invariant which seems to be the underlying reason why expanding curvature flows in Euclidean space do not develop singularities contrary to contracting curvature flows which will contract to a point in finite time (see [15]). Similar convergence results for inverse curvature flows in the hyperbolic space were estimated by Ding [1] and Gerhardt [8], and in the sphere by Gerhardt [11] and Makowski-Scheuer [18]. In [1], Ding also get similar results in rotationally symmetric spaces of Euclidean volume growth except the hyperbolic space. Compared with scale-invariant flows, there may be some difference for non-scale-invariant inverse curvature flows (see [25], [12] and [22]).

It is a natural question, whether one can prove long-time existence and the flow hypersurfaces become umbilic as in case of more general ambient spaces. Recently, Brendle-Hung-Wang [2] investigated the inverse mean curvature flow (IMCF for short) in anti-de Sitter-Schwarzschild manifold which is asymptotically hyperbolic at the infinity, and applied the convergence result to prove a sharp Minkowski inequality for strictly mean convex and star-shaped hypersurface in anti-de Sitter-Schwarzschild manifold. Similar applications can be found in the works [4] and [16], in which the IMCF was used to prove a Minkowski type inequality in the anti-de Sitter-Schwarzschild manifold and in the Schwarzschild manifold respectively. Other geometric inequalities, e.g., Aleksandrov-Fenchel inequalities in hyperbolic space as in [5] [6] have been

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proven also using inverse 1-homogeneous curvature flows [8] (also compare with [18], where additional isoperimetric type problems have been treated).

In the present work, we investigate the convergence of the flow (1.1) in some asymptotically hyperbolic space. More precisely, we consider the convergence of the flow (1.1) in anti-de Sitter-Schwarzschild manifold which is asymptotically hyperbolic at the infinity. Recently, Lu [17] considered the inverse hessian quotient curvature flow with star-shaped initial hypersurface in the anti-de Sitter-Schwarzschild manifold and proved that the solution exists for all time, and the second fundamental form converges to identity exponentially fast.

Let us first recall the definition of the anti-de Sitter-Schwarzschild manifold (see also [2]).

Fixed a real number $m > 0$, and let $s_0$ denote the unique positive solution of the equation

$$1 + s_0 - ms_0^{1-n} = 0.$$ 

The anti-de Sitter-Schwarzschild manifold is an $(n+1)$-dimensional manifold $M = [s_0, +\infty) \times S^n$ equipped with the Riemannian metric

$$\bar{g} = \frac{1}{1 - ms_0^{1-n} + s^2} ds \otimes ds + s^2 g_{S^n},$$

where $g_{S^n}$ is the standard round metric on the unit sphere $S^n$. Clearly, $\bar{g}$ is asymptotically hyperbolic, since the sectional curvatures of $(M, \bar{g})$ approach -1 near infinity.

The anti-de Sitter-Schwarzschild manifold are examples of the static spaces. If we define $f = \sqrt{1 - ms_0^{1-n} + s^2} > 0$, then it satisfies the equation

$$(\Delta f)_{\bar{g}} - \nabla^2 f + fRic = 0.$$ 

In general, a Riemannian metric is called static if it satisfies (1.2) for some positive function $f$. The condition (1.2) guarantees the Lorentzian warped product $-f^2 dt \otimes dt + \bar{g}$ is a solution of the Einstein equation.

In order to formulate the main result, we need a definition below (see also [22]).

**Definition 1.1.** Let $\Gamma \subset \mathbb{R}^n$ be an open, symmetric and convex cone and $F \in C^{\infty}(\Gamma)$ be a symmetric function. A hypersurface $\Sigma_0$ in the anti-de Sitter-Schwarzschild manifold $(M, \bar{g})$ is called $F$-admissible, if at any point $x \in \Sigma_0$ the principal curvatures of $\Sigma_0$, $\kappa_1, \ldots, \kappa_n$, are contained in the cone $\Gamma$.

We mainly get the following result

**Theorem 1.2.** Let $\Gamma \subset \mathbb{R}^n$ be an open, symmetric and convex cone that satisfies $\Gamma_+ = \{ (\kappa_i) \in \mathbb{R}^n : \kappa_i > 0, \ \forall 1 \leq i \leq n \} \subset \Gamma$ and $F \in C^{\infty}(\Gamma) \cap C^0(\Gamma)$ be a monotone, 1-homogeneous and concave curvature function, such that $F|_\Gamma > 0$ and $F|_{\partial \Gamma} = 0$.

We usually normalized $F$ such that $F(1, \ldots, 1) = n$.

Let $\Sigma_0$ be a smooth, star-shaped and $F$-admissible embedded closed hypersurface in AdS-Schwarzschild manifold $(M, \bar{g})$, and $\Sigma_0$ can be written as a graph over a geodesic sphere $S^n$,

$$\Sigma_0 = \text{graph } r(0, \cdot).$$

Then

(1) There is a unique smooth curvature flow

$$X : [0, \infty) \times \Sigma \to M,$$
which satisfies the flow equation

\[
\begin{align*}
\frac{d}{dt} X &= \frac{1}{F} \nu, \\
X(0) &= \Sigma_0.
\end{align*}
\]

where \( \nu(t, \xi) \) is the outward normal to \( \Sigma_t = X(t, \Sigma) \) at \( X(t, x) \), \( F \) is evaluated at the principle curvatures of \( \Sigma_t \) at \( X(t, \xi) \) and the leaves \( \Sigma_t \) are graphs over \( \mathbb{S}^n \),

\( \Sigma_t = \text{graph } r(t,.) \).

(2) The leaves \( \Sigma_t \) become more and more umbilic, namely

\[
|h_i^j - \delta_i^j| \leq C e^{-\frac{2}{n} t}.
\]

(3) Furthermore, the function \( \tilde{r}(t, \theta) = r(t, \theta) - \frac{t}{n} \) converges to a well defined function \( f(\theta) \in C^2(\mathbb{S}^n) \) in \( C^{2,\alpha} \) as \( t \to +\infty \), which implies that the limit of the rescaled induced metric of \( \Sigma_t \) is the conformal metric \( e^{2f} g_{\mathbb{S}^n} \) on \( \mathbb{S}^n \), where \( g_{\mathbb{S}^n} \) is the round metric \( \mathbb{S}^n \).

**Remark 1.1.** Similar to [14] and [19], in general, the function \( f(\theta) \) in Theorem 1.2 may not be constant in the sense that the limit shape of the rescaled flow hypersurfaces does not have to be a round sphere.

The main techniques employed here were from [8] and later were developed by Scheuuer in [22].

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## 2. Graphic hypersurfaces in the AdS-Schwarzschild manifold and a reformulation of the problem

First, we state some general facts about the AdS-Schwarzschild manifold and the graphic hypersurfaces in it. We basically follow the description in [2, Section 2]. Denote the AdS-Schwarzschild manifold by \((M, \mathcal{G})\) and \( \nabla \) by the Levi-Civata connection with respect to the metric \( \mathcal{G} \). By a change of variable, the AdS-Schwarzschild metric can be rewritten as

\[
\mathcal{G} = dr \otimes dr + \lambda(r)^2 g_{\mathbb{S}^n},
\]

where \( \lambda(r) \) satisfies the ODE

\[
\lambda'(r) = \sqrt{1 + \lambda^2 - m\lambda^{1-n}}
\]

and the asymptotic expansion

\[
\lambda(r) = \sinh(r) + \frac{m}{2(n+1)} \sinh^{-n}(r) + O(\sinh^{-n-2}(r)).
\]

We can calculate the asymptotic expansion of Riemannian curvature tensors. Let \( e_\alpha, \alpha = 1, 2, \ldots, n+1 \), be an orthonormal frame and \( R_{\alpha\beta\gamma\mu} \) denote the Riemannian curvature tensor of the AdS-Schwarzschild metric. Then

\[
R_{\alpha\beta\gamma\mu} = -\delta_{\beta\mu}\delta_{\alpha\gamma} + \delta_{\beta\gamma}\delta_{\alpha\mu} + O(e^{-(n+1)r})
\]

and

\[
\nabla_\mu R_{\alpha\beta\gamma\mu} = O(e^{-(n+1)r}).
\]
Since $\Sigma \subset M$ is a graphic hypersurface in $M$, it can be parametrized by
$$\Sigma = \{(r(\theta), \theta) : \theta \in S^n\}$$
for some smooth function $r$ on $S^n$. Let $\theta = \{\theta^i\}_{i=1,\ldots,n}$ be a local coordinate system on $S^n$ and let $\partial_i$ be the corresponding coordinate vector fields on $S^n$. Let $\varphi_i = D_i\varphi$, $\varphi_{ij} = D_jD_i\varphi$ and $\varphi_{ijk} = D_kD_jD_i\varphi$ denote the covariant derivatives of $\varphi$ with respect to the round metric $g_{S^n}$ and $\nabla$ be the Levi-Civata connection of $\Sigma$ with respect to the induced metric $g$ from $(M, \overline{g})$. Let $X = (r(\theta), \theta)$, the tangential vectors on $\Sigma$ take the form
$$X_i = \partial_i + D_i r \partial_r.$$ 

The induced metric on $\Sigma$ is
$$g_{ij} = r_i r_j + \lambda^2 \sigma_{ij},$$
and the outward unit normal vector of $\Sigma$
$$\nu = \frac{1}{v} \left( \partial_r - \lambda^2 D^2 r \partial_j \right).$$

Define a new function $\varphi : S^n \to \mathbb{R}$ by
\begin{equation}
(2.5) \quad \varphi(\theta) = \int_{c}^{r(\theta)} \frac{1}{\lambda(s)} ds.
\end{equation}

Then the induced metric on $\Sigma$ takes the form
$$g_{ij} = \lambda^2 (\varphi_i \varphi_j + \sigma_{ij})$$
with the inverse
$$g^{ij} = \lambda^{-2} (\sigma^{ij} - \varphi^i \varphi_j \nu^2),$$
where $(\sigma^{ij}) = (\sigma_{ij})^{-1}$, $\varphi^i = \sigma^{ij} \varphi_j$ and
\begin{equation}
(2.6) \quad v^2 = 1 + \sigma^{ij} \varphi_i \varphi_j \equiv 1 + |D\varphi|^2 = 1 + \frac{|Dr|^2}{\lambda^2},
\end{equation}

$|.|$ is the norm corresponding to the metric $g_{S^n}$. Let $h_{ij}$ be the second fundamental form of $\Sigma \subset M$ in term of the coordinate $\theta^i$. So
$$h_{ij} = \frac{\lambda}{v} \left( \lambda' (\varphi_i \varphi_j + \sigma_{ij}) - \varphi_{ij} \right)$$
and
\begin{equation}
(2.7) \quad h^i_j = \frac{1}{\lambda v} (\lambda' \delta^i_j - \overline{g}^{ik} \varphi_k),
\end{equation}

where $\overline{g}^{ij} = \sigma^{ij} - \varphi^i \varphi_j \nu^2$.

To calculate some curvature terms, we need the following result from Appendix A in [21].

**Lemma 2.1.**
$$R(\partial_i, \partial_j, \partial_k, \partial_l) = \lambda^2 (1 - (\lambda')^2) (\sigma_{ik} \sigma_{jl} - \sigma_{il} \sigma_{jk})$$
$$R(\partial_i, \partial_j, \partial_k, \partial_r) = -\lambda \lambda'' \sigma_{ij}$$

Then, we can calculate some curvature terms by using the above lemma.
\begin{equation}
(2.8) \quad R(X_i, \nu, X_j, \nu) \equiv R_{\nu ij} = -\frac{1}{v^2} \left[ \lambda \lambda'' + ((\lambda')^2 - 1) |D\varphi|^2 \right] \sigma_{ij} - \frac{1}{v^2} \frac{2 \lambda''}{\lambda} + \frac{1 - (\lambda')^2}{\lambda^2} + \frac{\lambda''}{\lambda} |D\varphi|^2 r_i r_j.
\end{equation}
and

\[
(2.9) \overline{R}(\nu, X_i, X_k, X_j) \equiv \overline{R}_{\nu} = (-\lambda \lambda'' - (\lambda')^2) \frac{r_k \sigma_{ij}}{v} + (\lambda \lambda'' + (\lambda')^2) \frac{r_j \sigma_{ik}}{v}.
\]

Thus,

\[
(2.10) \quad v \overline{R}_{\nu} = \frac{1}{v^2 \lambda^2} (\lambda \lambda'' + (1 - \lambda')^2) [r_i r_j - \lambda^2 |D\varphi|^2 \sigma_{ij}]
\]

The geodesic spheres \( S_r \) in the AdS-Schwarzschild manifold \((M, \overline{g})\) are totally umbilic, their second fundamental form is given by

\[
\overline{h}_{ij} = \overline{h}_{ij}(r) = \frac{\lambda'}{\lambda} \sigma_{ij},
\]

where

\[
\overline{g}_{ij} = \lambda^2 \sigma_{ij}.
\]

Thus \( \overline{h}_j = \frac{\lambda'}{\lambda} \delta_j \), \( \kappa_i = \frac{\lambda'}{\lambda} \) and the mean curvature \( \overline{H} \) of \( S_r \) is given by

\[
\overline{H} = \overline{H}(r) = \frac{n \lambda'}{\lambda}.
\]

For the evolution of graphic hypersurfaces, we can reform the equation (1.1). Let \( \Sigma_0 \) be a graphic hypersurface in AdS-Schwarzschild manifold \((M, \overline{g})\) which is given by an embedding \( X_0 : \mathbb{S}^n \to M \).

Let \( X_t : \mathbb{S}^n \to M, \ t \in [0, T) \), be the solution of inverse curvature flow with initial data given by \( X_0 \). In other word,

\[
(2.11) \quad \frac{\partial X}{\partial t} = \frac{1}{F} \nu,
\]

where \( \nu \) is the outward unit normal vector and \( F \) is a monotone, 1-homogeneous and concave curvature function. We shall call \( (2.11) \) the parametric form of the flow. We can write the initial hypersurface \( \Sigma_0 \) as the graph of a function \( r_0 \) defined on the unit sphere:

\[
\Sigma_0 = \{(r_0(\theta), \theta) : \theta \in \mathbb{S}^n\}.
\]

If each \( \Sigma_t \) is graphic, it can be parametrized as follows

\[
\Sigma_t = \{(r(\theta, t), \theta) : \theta \in \mathbb{S}^n\}.
\]

Then the evolution equation \( (2.11) \) now yields

\[
\frac{dr}{dt} = \frac{1}{F v} \quad \text{and} \quad \frac{d\theta^i}{dt} = -\frac{D^i r}{\lambda^2 F v},
\]

from which we deduce

\[
(2.12) \quad \frac{\partial r}{\partial t} = \frac{v}{F},
\]

where \( v \) is given by \( (2.6) \). Therefore, as long as the solution of \( (1.3) \) exists and remain graphic, it is equivalent to a parabolic PDE \( (2.12) \) for \( r \). The equation \( (2.12) \) is also referred as the non-parametric form of the inverse mean curvature flow. Notice that the velocity vector of \( (1.3) \) is always normal, while the velocity vector of \( (2.12) \) is in the direction of \( \partial_r \). To go from one to the other, we take the difference which is a (time-dependent) tangential vector field and compose the flow of the reparametrization associated with the tangent vector field.
The proof of the short time existence of the flow (1.3) is standard, see Remark 3.4 in [22] and Remark 2.1 in [23]. For completeness, we describe it easily here. We can get the short time existence of the flow on a maximal interval \([0, T^*)\), \(0 < T^* \leq \infty\), and

\[
X \in \mathcal{C}^\infty(\mathbb{R}^n \times \Sigma, M).
\]

Moreover, all the leaves \(M(t) = X(t, M), 0 \leq t < T^*,\) are admissible and can be written as graphs over \(\mathbb{R}^n\). Furthermore, the flow \(X\) exists as long as the scalar flow (2.12) does, where \(r : [0, T^*) \times \mathbb{R}^n \to \mathbb{R}\).

Thus, we will mainly investigate the long time existence of (1.3) in the following chapters.

3. The long-time existence

The proof of the long-time existence of (1.3) is standard which mainly relies on the following \(C^0\) estimates, \(C^1\) estimates and curvature estimates. Before proceeding, we give some notation. Covariant differentiation will usually be denoted by indices, e.g. \(r_{ij}\) for a function \(r : \Sigma \to \mathbb{R}\); or, if ambiguities are possible, by a semicolon, e.g. \(h_{ij;k}\). Usual partial derivatives will be denoted by a comma, e.g. \(u_{,ij}\).

\(C^0\) estimates

First, we recall the \(C^0\) estimates whose proof is standard, see Lemma 3.1 in [8] and Section 4 in [1].

**Lemma 3.1.** The solution \(r\) of (2.12) satisfies

\[
\lambda(\inf r(0, \cdot)) \leq \lambda(r(t, \theta))e^{-\frac{t}{n}} \leq \lambda(\sup r(0, \cdot)), \quad \forall \theta \in \mathbb{R}^n, t \in [0, T^*).
\]

**Remark 3.1.** Noticing the asymptotic expansion (2.2) of \(\lambda(r)\), we have from the above lemma

\[
r(t, \theta) - \frac{t}{n} = o(t).
\]

\(C^1\) estimates

To get the \(C^1\) estimate, we using the the evolution equation of \(\varphi\) instead of \(r\) by noticing the relation (2.2). From (2.12), we get

\[
\frac{\partial \varphi}{\partial t} = \frac{v}{\lambda F(h^i_j)} = \frac{v}{F(\lambda h^i_j)} = \frac{v}{F(h^i_j)}.
\]

Let \((\bar{g}_{ij}) = (\bar{g}^{ij})^{-1}\), clearly, \(g_{ij} = \lambda^2 \bar{g}_{ij}\). Defining

\[
\tilde{h}_{ij} = \bar{g}_{ik}\bar{h}^k_j,
\]

we see that in (3.3) we are considering the eigenvalues of \(\tilde{h}_{ij}\) with respect to \(\bar{g}_{ij}\) and thus we define

\[
F^{ij} = \frac{\partial F}{\partial \tilde{h}_{ij}} \quad \text{and} \quad F^j_i = \frac{\partial F}{\partial \tilde{h}^j_i}.
\]

By a straightforward computation, it is easy to get the following relations.

**Lemma 3.2.**

\[
\tilde{h}^l_{ki} = -\frac{v_i}{v} \tilde{h}^l_k - v^{-1}(g^{lm}g_{mk} + \tilde{g}^{lm}\tilde{g}_{mki} - \lambda\lambda''D_i\varphi^l_k),
\]

\[
\tilde{g}^{kl}_{,i} = \frac{2v_i\varphi^k\varphi^l}{v^3} - \frac{1}{v^2}\left(\varphi^k_i\varphi^l + \varphi^k\varphi^l_i\right),
\]
\[ v_i = v^{-1} \phi_{ki} \phi^k, \]

where the covariant derivatives as well as index raising are performed with respect to \( \sigma_{ij} \).

**Lemma 3.3.** Let \( \phi \) be a solution of (3.3), we have

\[ |D\phi|^2 \leq \sup_{S^n} |D\phi(0, \cdot)|^2. \]

Moreover, if \( F \) is bounded from above \( F \leq C \), then there exists \( 0 < \mu = \mu(C) \) such that

\[ |D\phi|^2 \leq e^{-\mu t} \sup_{S^n} |D\phi(0, \cdot)|^2. \]

**Proof.** Let

\[ w = \frac{1}{2} |D\phi|^2. \]

By differentiating (3.3) with respect to the operator \( D^k \phi D_k \), we obtain

\[ \frac{\partial}{\partial t} w = -\frac{v}{F^2} F_{kl} \tilde{h}^l_{ki} \phi^i + \frac{v_i \phi^i}{F}. \]

Fix \( 0 < T < T^* \) and suppose

\[ \sup_{[0, T] \times S^n} w = w(t_0, \xi_0), \quad t_0 > 0. \]

Then at \((t_0, \xi_0)\), there holds

\[ 0 \leq \frac{\partial}{\partial t} w = -\frac{1}{F^2} \left( -\tilde{g}^{lm}_{ij} \phi_{mki} \phi^i - \tilde{g}^{lm}_{ik} \phi_{mki} + \lambda \lambda'' |D\phi|^2 \delta^i_k \right) + \frac{2}{v^2} \phi_{ki} \phi^k \phi^i \]

where we use Lemma 3.2 and the fact \( \phi_{ik} \phi^i = 0, \forall k \) at \((t_0, \xi_0)\). Then, we apply the rule for exchanging derivatives

\[ \phi_{kli} = \phi_{ikl} + R_{iklm} \varphi^m \]

and notice the fact on \( S^n \)

\[ R_{iklm} = \sigma_{ik} \sigma_{lm} - \sigma_{im} \sigma_{lk}, \]

we can obtain

\[ 0 \leq \frac{\partial}{\partial t} w = \frac{1}{F^2} \left( -2 \lambda \lambda'' F^{kl} \tilde{g}_{kl} \varphi^l + F^{kl} (\phi_{k} \varphi_{l} - |D\phi|^2 \sigma_{kl}) + F^{kl} w_{kl} - F^{kl} \phi_{ik} \phi^i \right) < 0, \]

where we use the assumption that \( F \) is a monotone, 1-homogeneous and concave curvature function and \( F^{kl} w_{kl} \leq 0 \) at \((t_0, \xi_0)\). Hence, the estimate (3.4) follows by the arbitrariness of \( T \).

To prove (3.5), we define

\[ \tilde{w} = w e^{-\mu t}, \]

where \( \mu \) is a positive constant which will be chosen later. Then \( \tilde{w} \) satisfies the same equation as \( w \) with an additional term \( \mu \tilde{w} \) at the right-hand side. Assume \( \tilde{w} \) attains a positive maximum at a point \((t_0, \xi_0)\), \( t_0 > 0 \), by applying the maximum principle as before, there holds

\[ 0 \leq -\frac{2}{F^2} \lambda \lambda'' F^{kl} \tilde{g}_{kl} \tilde{w} + \mu \tilde{w}. \]

Then, since \( \frac{\lambda''}{\lambda} = 1 + \frac{1}{2} m(n-1) \lambda^{1-n} \) is bounded by some constant \( C_1 \) from Lemma 3.1 \( F(\tilde{h}_j^l) \lambda^{-1} = F(h_j^l) \) is bounded from above and \( F^{kl} \tilde{g}_{kl} \geq n \), we can obtain

\[ w e^{\mu t} \leq \sup_{S^n} w(0) \]
for all
\[ 0 < \mu \leq \frac{C_1 n}{C^2}. \]

Remark 3.2. In Theorem 3.13 below, we will estimate the optimal decay rate \( \mu \).

Curvature estimates

In this section, for convenience, we let \( \Phi = \Phi(F) = -\frac{1}{F}, \) \( \Phi' = \frac{\partial \Phi}{\partial F} \) and
\[ \chi = \langle \lambda \frac{\partial}{\partial r}, \nu \rangle = \frac{\lambda}{v}. \]

Lemma 3.4. Under the flow (1.3), the following evolution equations hold true
\[
\begin{align*}
\frac{\partial}{\partial t} \Phi - \Phi' F^{ij} \Phi_{ij} &= \Phi' F^{ij} h^i_k h^k_j \Phi + \Phi' F^{ij} \overline{R}_{\nu i \nu j} \Phi, \\
\frac{\partial}{\partial t} r - \Phi' F^{ij} r_{ij} &= 2 \Phi' F v^{-1} - \Phi' F_{ij} \overline{h}_{ij}, \\
\frac{\partial}{\partial t} \chi - \Phi' F^{ij} \chi_{ij} &= \Phi' F^{ij} h^i_k h^k_j - \Phi' F^{ij} \overline{R}(\nu, X_i, (\lambda \partial_r)^T, X_j), \\
\frac{\partial}{\partial t} h^i_j &= \Phi^i_j + \Phi h^i_k h^k_j + \Phi \overline{R}_{\nu j \nu k} g^{ki},
\end{align*}
\]
where \( \partial_r = \frac{\partial}{\partial r}, X_i = \frac{\partial X}{\partial x^i} \) and \( (\lambda \partial_r)^T = \lambda \partial_r - (\lambda \partial_r, \nu) \nu. \)

Proof. This is a straightforward computation in any case by using the flow equation (1.3). For details, we can see the similar results in [8] for the flow in hyperbolic space. \( \square \)

Proposition 3.5. Let \( X \) be a solution of the inverse curvature flow (1.3). Then the curvature function is bounded from above, i.e. there exists \( C = C(n, \Sigma_0) \) such that
\[ F(t, \xi) \leq C(n, \Sigma_0) < \infty \quad \forall (t, \xi) \in [0, T^*) \times \Sigma. \]

Proof. The proof proceeds similarly to that in Lemma 4.2 in [3]. Let
\[ w = -\log(-\Phi) + \beta(r - \frac{t}{n}), \]
where \( \beta \) is supposed to be large. Fix \( 0 < T < T^* \) and suppose
\[ \sup_{[0, T] \times \mathbb{S}^{n-1}} w = w(t_0, \xi_0), \quad t_0 > 0. \]
Then at \( (t_0, \xi_0) \), there holds
\[ 0 = w_i = -\frac{\Phi_i}{\Phi} + cr_i \]
and
\[
0 \leq \frac{\partial}{\partial t} w - \Phi' F^{ij} w_{ij} = -\Phi' F^{ij} h^i_k h^k_j - \Phi' F^{ij} \overline{R}_{\nu i \nu j} - \Phi' F^{ij} \frac{\Phi_i \Phi_j}{\Phi^2} + 2 \beta \Phi' F v^{-1} - \beta \Phi' F_{ij} \overline{h}_{ij} - \frac{1}{n}. \]
Thus, we have
\[ 0 \leq \Phi' F^{ij} \left( -\mathcal{R}_{\nu i \nu j} - \beta^2 r_i r_j - \beta \frac{\lambda'}{\lambda} \lambda^2 \sigma_{ij} \right) + \beta \left( \frac{2}{F^v} - \frac{1}{n} \right). \]

It is easy to see from (2.1), (2.2) and Lemma 3.1
\[
\frac{\lambda'}{\lambda} = 1 + O(e^{-\frac{n+1}{n} t}), \quad (3.12)
\]
\[
\frac{\lambda''}{\lambda} = 1 - \frac{1}{2} m(1-n)\lambda^{-n-2} = 1 + O(e^{-\frac{n+2}{n} t}), \quad (3.13)
\]
and
\[
\frac{1 - (\lambda')^2}{\lambda^2} = -1 + m\lambda^{-n-1} = -1 + O(e^{-\frac{n+1}{n} t}). \quad (3.14)
\]

Combing the above three estimates, as $\beta$ is supposed to be large, we can get from (2.8)
\[
\Phi' F^{ij} \left( -\mathcal{R}_{\nu i \nu j} - \beta^2 r_i r_j - \beta \frac{\lambda'}{\lambda} \lambda^2 \sigma_{ij} \right) \leq 0.
\]

Therefore, we can obtain
\[
0 \leq \beta \left( \frac{2}{F^v} - \frac{1}{n} \right).
\]

Then,
\[
F(t_0, \xi_0) \leq C(n, \Sigma_0),
\]
which leads to
\[
w \leq C(n, \Sigma_0).
\]

Therefore, the inequality
\[
F \leq C(n, \Sigma_0)
\]
holds.

\[\square\]

**Proposition 3.6.** Let $X$ be a solution of the inverse curvature flow (1.3). Then the curvature function is bounded from below, i.e., there exists $C = C(n, \Sigma_0)$ such that
\[
0 < C(n, \Sigma_0) \leq F(t, \xi), \quad \forall (t, \xi) \in [0, T^*) \times \Sigma.
\]

**Proof.** The proof proceeds similarly to that of [3, Lemma 4.1]. Let
\[
w = \log(-\Phi) - \log(\chi e^{-\frac{T}{n}}).
\]

Fix $0 < T < T^*$ and suppose
\[
\sup_{[0,T] \times \mathbb{S}^n} w = w(t_0, \xi_0), \quad t_0 > 0.
\]

Then at $(t_0, \xi_0)$, there holds
\[
0 = w_i = \frac{\Phi_i}{\Phi} - \frac{\chi_i}{\chi},
\]
which leads to
\[
0 \leq \frac{\partial}{\partial t} w - \Phi' F^{ij} w_{ij} = \Phi' \chi^{-1} F^{ij} \mathcal{R}(\nu, X_i, \lambda \partial_r, X_j) + \frac{1}{n}.
\]
Then, we can have by using (2.8) and (2.10)
\[
(3.16) \quad \chi^{-1} F_{ij} R(\nu, X_i, \lambda \partial_r, X_j) = F_{ij} R(\nu, X_i, \nu, X_j) + \nu F_{ij} R(\nu, X_i, X_k, X_j) r_l g^{kl} - \frac{\chi''}{\lambda} F_{ij} g_{ij}.
\]
Therefore,
\[
0 \leq \frac{\partial}{\partial t} w - \Phi' F_{ij} w_{ij} = -\frac{\chi''}{\lambda} \Phi' F_{ij} g_{ij} + \frac{1}{n},
\]
Since \(F_{ij} g_{ij} \geq F(1, \ldots, 1) = n\), we have from the estimate (3.13)
\[
0 < C(n, \Sigma_0) \leq F(t_0, \xi_0).
\]
Thus,
\[
w \leq w(t_0, \xi_0) \leq C(n, \Sigma_0).
\]
From (3.11), we know there exists \(C(n, \Sigma_0) > 0\) such that
\[
C^{-1} \leq \chi e^{-n} \leq C.
\]
Therefore, the inequality
\[
0 < C(n, \Sigma_0) \leq F
\]
holds.

Now we begin to estimate the second fundamental form which is the most difficult part of the proof of the long-time existence. The proof is similar to that of [8, Lemma 4.4], but due to the non-vanishing term \(\nabla_i R_{ijklm}\) in non-constant curvature manifolds, our case is more complicated and needs a far more delicate treatment.

**Proposition 3.7.** Let \(X\) be a solution of the inverse curvature flow (1.3). Then, the principal curvatures of the flow hypersurfaces are uniformly bounded from above, i.e., there exists \(C = C(n, \Sigma_0)\) such that
\[
\kappa_i(t, \xi) \leq C(n, \Sigma_0), \quad \forall (t, \xi) \in [0, T^*) \times \Sigma.
\]

**Proof.** First, we need the evolution equation of \(h^i_j\). From (3.10) we can get
\[
(3.17) \quad \frac{\partial}{\partial t} h^i_j = \Phi' F^{kl} \nabla_i \nabla_j h_{kl} + \Phi'' F^i F^j + F^{kl, pq} h^i_k h^j_l + \Phi h^i_k h^j_l + \Phi R_{ijkm} g^{kl}.
\]
Using Gauss equation and Codazzi equation, we have
\[
(3.18) \quad F^{kl} \nabla_k \nabla_i h^i_j = F^{kl} \nabla_i \nabla_j h_{kl} + F^{kl} (R_{kijp} h^p_j + R_{kjp} h^p_i) + 2 F^{kl} R_{kijp} h^p_i + F^{kl} R_{ijkl} h^i_j - F^{kl} R_{ijkl} h^i_j + F^{kl} (\nabla_k R_{ijpq} + \nabla_i R_{kljq}) + F^{kl} h^i_k h^j_l p_{ij} - F^{kl} h^i_k h^j_l p_{pj} + F^{kl} h^i_k h^j_l p_{pl} - F^{kl} h^i_k h^j_l p_{lj}.
\]
Then, we get the evolution equation of \(h^i_j\) by combing (3.17) and (3.18)
\[
(3.19) \quad \frac{\partial}{\partial t} h^i_j - F^{kl} \nabla_k \nabla_i h^i_j = -\Phi' (F^{kl} (R_{kijp} h^p_j + R_{kjp} h^p_i) + 2 F^{kl} R_{kijp} h^p_i + F^{kl} R_{ijkl} h^i_j - F^{kl} R_{ijkl} h^i_j + F^{kl} (\nabla_k R_{ijpq} + \nabla_i R_{kljq}) + F^{kl} h^i_k h^j_l p_{ij} - F^{kl} h^i_k h^j_l p_{pj} + F^{kl} h^i_k h^j_l p_{pl} - F^{kl} h^i_k h^j_l p_{lj})
\]
\[
= -\Phi' (F^{kl} (R_{kijp} h^p_j + R_{kjp} h^p_i) + 2 F^{kl} R_{kijp} h^p_i + F^{kl} R_{ijkl} h^i_j - F^{kl} R_{ijkl} h^i_j + F^{kl} (\nabla_k R_{ijpq} + \nabla_i R_{kljq}) + F^{kl} h^i_k h^j_l p_{ij} - F^{kl} h^i_k h^j_l p_{pj} + F^{kl} h^i_k h^j_l p_{pl} - F^{kl} h^i_k h^j_l p_{lj})
\]
Using the estimates (3.1) and (3.4), there exists a constant \( \vartheta > 0 \) such that
\[
2\vartheta \leq \tilde{\chi} \equiv \chi e^{-\alpha}.
\]
Setting
\[
\rho = -\log(\tilde{\chi} - \vartheta),
\]
By using the equation (3.9), we get the evolution of \( \rho \) as follows
\[
\partial_t \rho - \Phi F_{kl} \rho_{kl} = (\tilde{\chi} - \vartheta)^{-1} \left( -\Phi F_{kl} h_k^p h_p^l \tilde{\chi} + \frac{\tilde{\chi}}{n} + \tilde{\chi} \Phi F_{ij} F^{ij}(\nu, X_i, (\lambda \partial_r)^T, X_j) \right) - \Phi F_{kl} \tilde{\chi} \rho_{kl}.
\]
Next, we define the functions
\[
\zeta = \sup \{ h_{ij} \eta_i \eta_j : g_{ij} \eta_i \eta_j = 1 \}
\]
and
\[
w = \log \zeta + \rho + \beta(\tau - \frac{t}{n}),
\]
where \( \beta > 0 \) is supposed to be large. We claim that \( w \) is bounded, if \( \beta \) is chosen sufficiently large. Fix \( 0 < T < T^* \), suppose \( w \) attains a maximal value at \((t_0, \xi_0)\)
\[
\sup_{[0,T] \times S^n} w = w(t_0, \xi_0), \quad t_0 > 0.
\]
Choose Riemannian normal coordinates at \((t_0, \xi_0)\) such that at this point we have
\[
g_{ij} = \delta_{ij}, \quad h_{ij} = \kappa_i \delta_{ij}, \quad \kappa_1 \leq \kappa_2 \leq \ldots \leq \kappa_n,
\]
then
\[
(3.20) \quad F_{kl,pq} \eta_{kl} \eta_{pq} \leq \sum_{k \neq l}^n \frac{F_{kk} - F_{ll}}{\kappa_k - \kappa_l} (\eta_{kl})^2 \leq \frac{2}{\kappa_n - \kappa_1} \sum_{i=1}^n (F^{nn} - F^{ii})(\eta_{ni})^2
\]
and
\[
(3.21) \quad F^{nn} \leq \ldots \leq F^{11}.
\]
For details, see, e.g., [10, Lemma 1.1] and [3, Lemma 2].

Since \( \zeta \) is only continuous in general, we need to find a differential version instead. Set
\[
\tilde{\zeta} = \frac{h_{ij} \eta^i \eta^j}{g_{ij} \eta_i \eta^j},
\]
where \( \eta = (0, \ldots, 0, 1) \). There holds at \((t_0, \xi_0)\),
\[
h_{nn} = h_n^2 = \kappa_n = \zeta = \tilde{\zeta}
\]
By a simple calculation, we find
\[
\frac{\partial}{\partial t} \tilde{\zeta} = \frac{(\partial_t h_{ij}) \eta^i \eta^j}{g_{ij} \eta^i \eta^j} - \frac{h_{ij} \eta^i \eta^j}{(g_{ij} \eta^i \eta^j)^2} (\frac{\partial}{\partial t} g_{ij}) \eta^i \eta^j
\]
and
\[
\frac{\partial}{\partial t} h_n = \frac{\partial}{\partial t} (h_{nk} g^{kn}) = (\frac{\partial}{\partial t} h_{nk}) g^{kn} - g^{ki} (\frac{\partial}{\partial t} g_{ij}) g^{jn} h_{nk}.
\]
Clearly, there holds in a neighborhood of \((t_0, \xi_0)\)
\[
\tilde{\zeta} \leq \zeta
\]
and we find at \((t_0, \xi_0)\)

\[
\frac{\partial}{\partial t} \tilde{\zeta} = \frac{\partial}{\partial t} h^n_{\xi_0}
\]

and the spatial derivatives do also coincide. This implies that \(\tilde{\zeta}\) satisfies the same evolution (3.17) as \(h^n_{\xi_0}\). Without loss of generality, we treat \(h^n_{\xi_0}\) like a scalar and pretend that \(w\) is defined by

\[
w = \log h^n_{\xi_0} + \rho + \beta (r - \frac{t}{n}).
\]

Using the asymptotic expansion of Riemannian curvature tensors (2.4), the non-vanishing terms \(\nabla_i R_{jkln}\) which appear in (3.19) can be fortunately controlled by

\[
|F^{kl}(\nabla_k R_{ijpq}g^{pi} + g^{pm}\nabla_p R_{ijkl})| \leq CF^{pq}g_{pq}.
\]

Then, we get the evolution equation of \(h^n_{\xi_0}\) from (3.19)

\[
\begin{align*}
(3.22) \quad \frac{\partial}{\partial t} \log h^n_{\xi_0} - \Phi' F^{kl} \nabla_k \nabla_l \log h^n_{\xi_0} & = \frac{1}{\kappa_n} \left( \frac{\partial}{\partial t} h^n_{\xi_0} - \Phi' F^{kl} \nabla_k h^n_{\xi_0} \right) + \Phi' \frac{1}{\kappa_n} F^{kl} h^n_{\xi_0,k} h^n_{\xi_0,l} \\
& \leq \frac{1}{\kappa_n} \Phi' \left( F^{kl} h^{\beta p}_{\xi_0} (\kappa_n - 2 F R_{kl} + 2 F^{kl} \overline{R}_{knln} - 2 F^{kl} \overline{R}_{kmnp} h^p_n \\
+ F \overline{R}_{knln} + F^{kl} \overline{R}_{knln} + C F^{kl} g_{kl} - F \overline{R}_{mnk} \right) + \Phi' \frac{1}{\kappa_n} F^{kl} h^n_{\xi_0,k} h^n_{\xi_0,l} \\
& \quad + F^{kl,mn} h^{\beta}_{kl,\xi_0} + \Phi'' F^{ij} F_{ij}.
\end{align*}
\]

Together with the evolution equations of \(\rho\) and \(r\), we infer at \((t_0, \xi_0)\), the following inequality

\[
0 \leq \Phi' F^{kl} h^{\beta p}_{\xi_0} (1 - \frac{\tilde{\chi}}{\chi - \vartheta}) - 2 \Phi' F h^n_{\xi_0} + 2 \beta \Phi' F v^{-1} - \beta \Phi' F^{ij} \overline{R}_{ij} - \frac{\beta}{n} + \frac{1}{n} \frac{\tilde{\chi}}{\chi - \vartheta}
\]

\[
(3.23) \quad + \Phi' F^{kl} (\log h^n_{\xi_0,k} (\log h^n_{\xi_0,l}))_l - \Phi' F^{kl} \rho_{kl} + \frac{2}{\kappa_n - \kappa_1} \Phi \sum_{i=1}^n (F_{mn} - F_{ii})(h_{ni} \nu_{i})^2 (h^n_{\xi_0})^{-1}
\]

\[
+ \frac{1}{\kappa_n} \Phi' \left(- 2 F^{kl} \overline{R}_{knln} - 2 F^{kl} \overline{R}_{knln} + F^{kl} \overline{R}_{knln} + C F^{kl} g_{kl} - 2 F \overline{R}_{mnk} \right)
\]

\[
+ \frac{\tilde{\chi}}{\chi - \vartheta} F^{ij} \overline{R}(\nu, X_i, (\lambda \partial_r)^T, X_j) + \Phi'' F^{ij} F_{ij}
\]

holds. We can estimate the curvature terms by using (3.23)

\[
| - 2 F^{kl} \overline{R}_{knln} - 2 F^{kl} \overline{R}_{kmnp} h^p_n - F^{kl} \overline{R}_{knln} + C F^{kl} g_{kl} | \leq C(1 + \kappa_n) F^{kl} g_{kl}
\]

and

\[
| \overline{R}_{knln} | \leq CF.
\]

Then, using the inequalities (3.20) and (3.21), \(\Phi'' < 0\) and

\[
(\log h^n_{\xi_0})_{\xi_0} = -\rho_{\xi_0} - \beta r_{\xi_0}
\]

at \((t_0, \xi_0)\), we can get from the above inequality

\[
(3.24) \quad 0 \leq \Phi' F^{kl} h^{\beta p}_{\xi_0} \frac{\partial}{\partial t} \frac{1}{\chi - \vartheta} + \Phi' F^{kl} \left(C g_{kl} (1 + \kappa_n^{-1}) - \beta h^n_{\xi_0} \right) - 2 \Phi' F h^n_{\xi_0}
\]

\[
+ 2 \beta \Phi' F v^{-1} - \frac{\beta}{n} + \left( \frac{1}{n} + F^{ij} \overline{R}(\nu, X_i, (\lambda \partial_r)^T, X_j) \right) \frac{\tilde{\chi}}{\chi - \vartheta}
\]
the first line converges to $-\infty$. Hence, after abandoning the negative term $-\beta\Phi'F kl r_k r_l$, we get
\[ + \Phi'F kl r_k r_l - 2\beta\Phi'F^k l \rho_k r_l + \frac{2}{\kappa_n - \kappa_1} \Phi' \sum_{i=1}^n (F_{ni} - F^{ii})(h_{ni}^{-1}h_n^{-1})^{-1} \]
\[ + C(\kappa_n)^{-1}\Phi'F. \]

Now, we estimate the left curvature term in the above inequality. Clearly, we can get from (2.10)
\[ (3.25) \quad \mathcal{R}(\nu, X, (\lambda\partial_r)^T, X_j) = \lambda \mathcal{R}(\nu, X, X_k, X_j)g_{kl} \]
\[ = \frac{1}{v^2\lambda} (\lambda\lambda'' + (1 - (\lambda')^2)) [r_i r_j - \lambda^2 | D\varphi |^2 \sigma_{ij}] . \]

From (3.13) and (3.14), we can get
\[ (3.26) \quad \frac{1}{\lambda} (\lambda\lambda'' + (1 - (\lambda')^2)) = (1 + \frac{n-1}{2\lambda}) \frac{m}{\lambda''}, \]
which is clearly bounded. Therefore,
\[ F^{ij}\mathcal{R}(\nu, X, (\lambda\partial_r)^T, X_j) \leq CF^{ij}g_{ij} \]

Moreover, we know
\[ F^{ij}\mathcal{H}_{ij} = \frac{\lambda'}{\lambda} F^{ij}g_{ij} = \frac{\lambda'}{\lambda} F^{ij}(g_{ij} - r_i r_j) \geq \frac{\lambda'}{\lambda} F^{ij}g_{ij}(1 - g_{kl} r_k r_l) \]
\[ = \frac{\lambda'}{\lambda} v^{-2} F^{ij}g_{ij} \geq C_0 F^{ij}g_{ij}, \]
where we use (3.4) and (3.12) in the last inequality. Furthermore, it is easy to check
\[ v_i = \sqrt{\frac{\rho}{n} - v^2 h_l r_k} \]
(see (5.29) in [13]), and thus
\[ | \nabla \rho | \leq C_2 | \nabla v | + C_2 | \nabla r | \leq C_2 | \rho | | \nabla r | + C_2 | \nabla r | , \]
where $| \nabla \rho | = \sqrt{g^{ij}\nabla_i \rho \nabla_j \rho}$. We distinguish two cases.

Case 1. If $\kappa_1 < -\epsilon_1 \kappa_n$, $0 < \epsilon_1 < 1$, then
\[ F^{kl}h_{kp}h_{q} \geq \frac{1}{n} F^{kl}g_{kp} \epsilon_1^2 \kappa_n^2 . \]

Hence, after abandoning the negative term $-2\Phi'F \kappa_n$, (3.24) becomes
\[ 0 \leq \Phi'F^{kl}g_{kl} \left( - \frac{1}{n} \epsilon_1^2 \kappa_n^2 \frac{\partial}{\partial \chi} + C(1 + \kappa_n^{-1}) - \beta C_0 + C \frac{\chi}{\chi - \vartheta} \right) \]
\[ + 2\beta C_2 (\kappa_n + 1) | \nabla r |^2 + \beta^2 | \nabla r |^2 \]
\[ - \frac{2}{n} \Phi'Fv^{-1} - \frac{\beta}{n} \frac{1}{\chi} + \frac{\chi}{\chi - \vartheta} + C\kappa_n^{-1}\Phi'F . \]

Since $F$ bounded from above and below, $F^{ij}g_{ij} \geq F(1, ..., 1) = n$ and $| \nabla r | = | D\varphi | \leq C(n, \Sigma_0)$, the first line converges to $-\infty$ if $\kappa_n \to +\infty$. Moreover, the last line is uniformly bounded by some $C = C(n, \Sigma_0)$. Hence, in this case we conclude that
\[ \kappa_n \leq C(n, \Sigma_0) \]
for any choice of $\beta$. 

Case 2. If $\kappa_1 \geq -\epsilon_1 \kappa_n$, $0 < \epsilon_1 < 1$, then
\[
\frac{2}{\kappa_n - \kappa_1} \sum_{i=1}^{n} \left( F^{ii} - F^{n i} \right) (h_{ni}^n)^2 (h_n^n)^{-1} \leq \frac{2}{1 + \epsilon_1} \sum_{i=1}^{n} (F^{nn} - F^{ii}) (h_{ni}^n)^2 (h_n^n)^{-1} + C \Phi' F^{ij} g_{ij} \kappa_n^{-2},
\]
where we use $h_{ni}^n = h_{nn;i} + \tilde{R}_{ni}^n$, in view of the Codazzi equation and the boundedness of the curvature $(1.1)$. Thus, the terms in $(3.28)$ containing the derivatives of $h_n^n$ can therefore be estimated from above by
\[
\Phi' F^{ij} (\log h_n^n)_i (\log h_n^n)_j + \frac{2}{\kappa_n - \kappa_1} \sum_{i=1}^{n} (F^{nn} - F^{ii}) (h_{ni}^n)^2 (h_n^n)^{-1}
\]
\[
\leq \frac{2}{1 + \epsilon_1} \Phi' \sum_{i=1}^{n} F^{nn} (\log h_n^n)_i^2 - \frac{1 - \epsilon_1}{1 + \epsilon_1} \Phi' \sum_{i=1}^{n} F^{ii} (\log h_n^n)_i^2 + C \Phi' F^{ij} g_{ij} \kappa_n^{-2}
\]
\[
\leq \frac{2}{1 + \epsilon_1} \Phi' \sum_{i=1}^{n} F^{nn} (\log h_n^n)_i^2 - \frac{1 - \epsilon_1}{1 + \epsilon_1} \Phi' \sum_{i=1}^{n} F^{nn} (\log h_n^n)_i^2 + C \Phi' F^{ij} g_{ij} \kappa_n^{-2}
\]
\[
= \Phi' F^{nn} \langle \nabla \rho + \beta \nabla r \rangle^2 + C \Phi' F^{ij} g_{ij} \kappa_n^{-2}
\]
\[
= \Phi' F^{nn} (|\nabla \rho|^2 + 2 \beta (\nabla \rho, \nabla r) + \beta^2 |\nabla r|^2) + C \Phi' F^{ij} g_{ij} \kappa_n^{-2}.
\]
Hence, taking the above inequality into the estimate $(3.28)$ yields
\[
0 \leq -\Phi' F^{nn} \kappa_n^2 \frac{\partial}{\partial \tilde{\rho}} + \Phi' F^k l g_{kl} \left( 1 - \beta C_0 + C (1 + \kappa_n^{-1} + \kappa_n^{-2} + \frac{\tilde{x}}{\chi - \tilde{\rho}}) \right)
\]
\[
-2 \Phi' F \kappa_n + 2 \beta \Phi' F v^{-1} - \frac{\beta}{n} + \frac{1}{n} \frac{\tilde{x}}{\chi - \tilde{\rho}} + \Phi' F^{nn} (2 \beta |\nabla \rho||\nabla r| + \beta^2 |\nabla r|^2)
\]
\[
+C \kappa_n^{-1} \Phi' F < 0
\]
for large $\kappa_n$ if $\beta$ is chosen large enough. Thus we obtain
\[
\kappa_n(t_0, \xi_0) \leq C(n, \Sigma_0).
\]
Since $\rho$ and $\tilde{r}$ are bounded from above, we conclude our claim. 

**Corollary 3.8.** Under the hypothesis of Proposition 3.7, there exists a compact set $K \subset \mathbb{R}^n$ such that $(\kappa_i) \subset K \subset \Gamma$.

**Proof.** Noticing that $F$ is bounded from below and $F^{ij} h_{ij} = F$, Proposition 3.7 implies the result. 

**Theorem 3.9.** Under the hypothesis of Theorem 1.2, we conclude
\[
T^* = +\infty.
\]

**Proof.** Recalling that $\varphi$ satisfies the equation $(3.3)$
\[
\frac{\partial \varphi}{\partial t} = \frac{v}{\lambda F(h_j^i)} = G(x, \varphi, D\varphi, D^2 \varphi).
\]
By a simple calculation, we get
\[
\frac{\partial G}{\partial \varphi} = \frac{1}{\lambda^2 F^2} F^{-k} g_{kj}^i,
\]
where $\tilde{g}_k^i$ and $\delta_k^i$ are equivalent norms, since $v \leq C$. Therefore, we can conclude the equation (3.3) is uniformly parabolic on finite intervals from Proposition 3.5, Proposition 3.6 and Corollary 3.8. Recalling that $h^i_j = \frac{1}{\lambda} (\lambda' \delta^j_i - \tilde{g}^{ik} \varphi_{kj})$, where $\tilde{g}^{ij} = \sigma^{ij} - \frac{\varphi_i \varphi_j}{v^2}$, we have

(3.27) \[ |\varphi|_{C^2(S^n)} \leq C(n, \Sigma_0, T^*) \]

by using the estimate (3.4) and Corollary 3.8. Then by Krylov-Safonov estimate [20], we have

\[ |\varphi|_{C^{2, \alpha}(S^n)} \leq C(n, \Sigma_0, T^*) \]

which implies the maximal time interval is unbounded, i.e., $T^* = +\infty$. 

Optimal decay estimates

First, we recall [22, Lemma 4.2] which will be used in the next lemma.

Lemma 3.10. Let $f \in C^{0,1}(\mathbb{R}_+)$ and let $D$ be the set of points of differentiability of $f$. Suppose that for all $\epsilon > 0$ there exist $T_\epsilon > 0$ and $\delta_\epsilon > 0$ such that

\[ A_\epsilon = \{ t \in [T, +\infty) \cap D : f(t) \geq \epsilon \} \subset \{ t \in [T_\epsilon, +\infty) \cap D : f'(t) \geq -\delta_\epsilon \}. \]

Then there holds

\[ \lim_{t \to \infty} \sup_{\Sigma} |\kappa_i(t, \cdot) - 1| = 0, \quad t \to \infty, \quad \forall 1 \leq i \leq n. \]

Proof. We use the method which first appears in [22]. Define the functions

\[ \zeta = \sup \{ h_i \eta_i \eta_j : g_{ij} \eta_i \eta_j = 1 \} \]

and

\[ w = (\log \zeta - \log \bar{\chi} + \bar{r} - \log 2)t, \]

where $\bar{\chi} = \chi e^{-\frac{t}{n}}$ and $\bar{r} = r - \frac{t}{n}$. We claim that $w$ is bounded. Fix $0 < T < +\infty$, suppose $w$ attains a maximal value at $(t_0, \xi_0)$,

\[ \sup_{[0,T] \times S^{n-1}} w = w(t_0, \xi_0), \quad t_0 > 0. \]

Choose Riemannian normal coordinates at $(t_0, \xi_0)$ such that at this point we have

\[ g_{ij} = \delta_{ij}, \quad h_{ij} = \kappa_i \delta_{ij}, \quad \kappa_1 \leq \kappa_2 \leq \ldots \leq \kappa_n. \]

Then it follows

\[ w = (\log h_{ii}^n - \log \bar{\chi} + \bar{r} - \log 2)t. \]

First, we claim that

\[ (\log \bar{\chi} + \bar{r} - \log 2)t = (\log v - \log \lambda + r - \log 2)t = (\log v - \log 2\lambda + r)t \]

is bounded. On the one hand, using the estimate (3.5),

\[ t \log v = \log (1 + v)^t \leq \log (1 + Ce^{-\mu t})t \]

is bounded. On the other hand, the asymptotic expansions (2.2) and (3.2) imply

\[ e^{(-\log 2\lambda + r)t} = (1 - e^{-2r} + o(e^{-2r}))^{-t} \leq (1 - Ce^{-\frac{2r}{n}})^{-t} \]
is also bounded. Therefore, we prove our claim.

Using the evolution equations of \( h^a_n \), \( \bar{\chi} \) and \( \bar{r} \), as (3.28), we can obtain the following evolution equation of \( w \)

\[
\frac{\partial}{\partial t} w - \Phi' F^{ij} w_{ij} = -2\Phi' F h^a_n + 2\Phi' F v^{-1} - \Phi' F^{ij} \bar{h}_{ij} - \Phi' \kappa_n F^{kl} (\log h^a_n)_{k}(\log h^a_n)_{l} \\
- \Phi' F^{kl} (\log \bar{\chi})_{k}(\log \bar{\chi})_{l} + \Phi' F^{kl, pq} h_{kl;n} h_{pq;n} (h^a_n)^{-1} \\
+ \frac{1}{\kappa_n} \Phi' (-2 F^{kl} \bar{R}_{knln} \kappa_n - 2 F^{kl} \bar{R}_{kmpq} h^p_l + F^{kl} \bar{R}_{nkvl} \kappa_n + F^{kl} (\nabla_k \bar{R}_{mnld} + \nabla_n \bar{R}_{vlnk}) - 2 F \bar{R}_{vnnn}) \\
+ F^{ij} \bar{R}(\nu, X_i, (\lambda \partial_r)^T, X_j) + \Phi'' F^{ij} F_{ij} t_0 \\
+ \log h^a_n - \log \bar{\chi} + \bar{r} - \log 2.
\]

Using the asymptotic expansion of Riemannian curvature tensors (2.4) and (2.3), we have

\[
| F^{kl}(\nabla_k \bar{R}_{vimi} g^{mj} + \nabla^{mi} \bar{R}_{vlnk}) | \leq Ce^{-\frac{n+1}{2}t}.
\]

and

\[
-2 F^{kl} \bar{R}_{knln} \kappa_n - 2 F^{kl} \bar{R}_{kmpq} h^p_l - F^{kl} \bar{R}_{nkvl} \kappa_n - 2 F \bar{R}_{vnnn} = F^{kl} g_{kl} \kappa_n + O(e^{-\frac{n+1}{2}t}).
\]

Moreover, we can get from (3.25) and (3.26)

\[
| F^{ij} \bar{R}(\nu, X_i, (\lambda \partial_r)^T, X_j) | \leq Ce^{-t}.
\]

Therefore, we have

\[
\frac{\partial}{\partial t} w - \Phi' F^{ij} w_{ij} \leq \left( \Phi' F^{kl} g_{kl} - 2\Phi' F h^a_n + 2\Phi' F v^{-1} - \Phi' F^{ij} \bar{h}_{ij} + \Phi'' F^{ij} F_{ij} (h^a_n)^{-1} \\
+ \Phi' F^{kl} (\log h^a_n)_{k}(\log h^a_n)_{l} - \Phi' F^{kl} (\log \bar{\chi})_{k}(\log \bar{\chi})_{l} + \Phi' F^{kl, pq} h_{kl;n} h_{pq;n} (h^a_n)^{-1} \right) t_0 \\
+ (\log h^a_n - \log \bar{\chi} + \bar{r} - \log 2) + O(1) \\
\leq \Phi(2 h^a_n - 2 v^{-1}) t_0 + \Phi' F^{kl} (r_k r_l + (1 - \frac{\lambda'}{\lambda}) \lambda^2 \sigma_{ij}) t_0 \\
+ \Phi' \left( (\log h^a_n)_{k}(\log h^a_n)_{l} - (\log \bar{\chi})_{k}(\log \bar{\chi})_{l} \right) t_0 + O(1) \\
+ \log h^a_n - \log \bar{\chi} + \bar{r} - \log 2.
\]

Using inequalities (3.20) and (3.21), \( \Phi'' < 0 \) and \( (\log h^a_n)_{i} = -(\log \bar{\chi})_{i} - \bar{r}_i \) at \((t_0, \xi_0)\), we can get from the above inequality

(3.28)

\[
0 \leq \Phi(2 h^a_n - 2 v^{-1}) t_0 + \Phi' F^{kl} r_k r_l t_0 \\
+ \Phi' F^{kl} (\log \bar{\chi})_{k} r_l t_0 + O(1) + \log h^a_n - \log \bar{\chi} + \bar{r} - \log 2.
\]

From (2.7), we have

\[
v_k = \frac{\varphi_j \varphi_{jk}}{v} = \lambda v \varphi_k - \lambda v^2 h^i_k \varphi_i = \frac{\lambda'}{\lambda} v r_k - v^2 h^i_k r_i.
\]
Then, we obtain
\[(\log \tilde{\chi})_k = \frac{\chi_k}{\chi} = \frac{v^2 \lambda' r_k v - \lambda v_k}{v^2} = vh^i_k r_k.\]
Since the principal curvatures are bounded by Corollary 3.8 and \(F\) is also bounded by Propositions 3.5 and 3.6, the following two terms in (3.28) are controlled by
\[(3.29) \quad \Phi' F^{kl} r_k r_l t_0 + \Phi' F^{kl} (\log \tilde{\chi})_k r_l t_0 \leq C g_{ij} r_i r_j t_0.\]
However, \(g_{ij} r_i r_j t_0 = |D\phi|^2 \leq C e^{-\mu} t_0 \leq C(n, \Sigma_0)\) by Lemma 3.3. Therefore, from (3.28) at \((t_0, \xi_0)\), we get
\[0 \leq \Phi(2h^i_n - 2v^{-1})t_0 + C\]
for some \(C = C(n, \Sigma_0)\), which implies
\[h^i_n \leq 1 + \frac{CF}{t_0}.\]
Thus, we have
\[w \leq t_0 \log(1 + \frac{CF}{t_0}) + t_0(-\log \tilde{\chi} + \tilde{r} - \log 2) \leq C(n, \Sigma_0),\]
which means \(w\) has a priori boundness. Hence,
\[(3.30) \quad \limsup_{t \to \infty} \sup_M \kappa_i(t, \cdot) \leq 1, \quad \forall 1 \leq i \leq n.\]
Now we define the function
\[\psi = \log(-\Phi) - \log \tilde{\chi} + \tilde{r} - \log 2 - \log \frac{1}{n}.\]
By a similar computation to that in the proofs of Propositions 3.5 and 3.6, we know that \(\psi\) satisfies
\[\frac{\partial}{\partial t} \psi - \Phi' F^{ij} \psi_{ij} = \Phi' F^{ij} (\log(-\Phi))_i (\log(-\Phi))_j - \Phi' F^{ij} (\log \tilde{\chi})_i (\log \tilde{\chi})_j + \frac{2}{Fv} - \Phi' F^{ij} \tilde{\eta}_{ij}.\]
Then the Lipschitz function
\[\tilde{\psi} = \sup_{\xi \in \Sigma} \psi(\cdot, \xi)\]
satisfies
\[\frac{\partial}{\partial t} \tilde{\psi} \leq Ce^{-\mu t} - \Phi' F^{ij} g_{ij} (1 + O(e^{-\frac{\mu+\frac{1}{2}}{t}})) + \frac{2}{Fv} - \Phi' F^{ij} \tilde{\eta}_{ij}\]
where we use a similar argument which has been done to (3.29) to get the first inequality by noticing (3.16) and (3.13). Setting
\[A_\epsilon = \{t \in [T, +\infty) \cap D : \tilde{\psi}(t) \geq \epsilon\},\]
where \(D\) is the set of points of differentiability of \(\tilde{\psi}\). Let \(\epsilon > 0\) and choose \(T > 0\) such that for all \((t, \xi) \in [T, \infty) \times \Sigma,\)
\[-\log \tilde{\chi} + \tilde{r} - \log 2 < \frac{\epsilon}{2}.\]
Then we have
\[
\left( \log(-\Phi) - \log \frac{1}{n} \right)(t, \xi_t) > \frac{\epsilon}{2}
\]
for \( t \in A_\epsilon \), where \( \tilde{\psi}(t) = \psi(t, \xi_t) \). Thus there exists \( 0 < \gamma = \gamma(\epsilon) = n(1 - e^{-\frac{\epsilon}{2}}) \) such that
\[
F(t, \xi_t) < n - \gamma,
\]
which implies
\[
\Phi'(\frac{2F}{v} - 2F^i g_{ij}) \leq -\frac{2n\gamma}{v}.
\]
Therefore, if \( T \) is chosen large enough, we have
\[
\frac{\partial}{\partial t} \tilde{\psi} \leq -\frac{1}{2} \left( \inf \Phi' \right) \frac{2n\gamma}{v} \equiv \delta_\epsilon.
\]
Now it follows from Lemma 3.10,
\[
\lim_{t \to \infty} \sup_{\Sigma} \log(-\Phi) - \log \frac{1}{n} \leq \lim_{t \to \infty} \sup_{\Sigma} \tilde{\psi}(t) + \lim_{t \to \infty} \sup_{\Sigma} \left( \log \tilde{\chi} - \tilde{r} + \log 2 \right) \leq 0,
\]
which leads to
\[
\lim_{t \to \infty} \inf_{M} F \geq n.
\]
Then, together with (3.30), we conclude that the following fact
\[
\sup_{\Sigma} |\kappa_i(t, \cdot) - 1| \to 0, \quad t \to \infty, \quad \forall 1 \leq i \leq n
\]
is true. \( \square \)

**Theorem 3.12.** Under the assumptions of Theorem 1.2, the principle curvatures of the flow hypersurfaces of (1.3) converge to 1 exponentially fast. There exists \( C = C(n, \Sigma_0) \) such that for all \( (t, \xi) \in [0, \infty) \times \Sigma \), the estimate
\[
|h^i_j - \delta^i_j| \leq Ce^{-\frac{\epsilon t}{n}}
\]
holds.

**Proof.** Define the function
\[
G(t, \xi) = \frac{1}{2} |h^i_j - \delta^i_j|^2(t, \xi), \quad \forall (t, \xi) \in [T, \infty) \times \Sigma.
\]
Using the evolution equation (3.22) of \( h^i_j \), we can get the evolution equation of \( G(t, \xi) \) as follows
\[
\frac{\partial}{\partial t} G(t, \xi) - \Phi'F^{kl} \nabla_k \nabla_l G(t, \xi) = (h^i_j - \delta^i_j) \left( \Phi'F^{kl} h_{kp} h^p_i h^j_k - 2\Phi'F h^i p h^j_p + \Phi'' F^i F_j 
+ \Phi'F^{kl,pq} h_{kl} h^i_p h^j_q + O(e^{-\frac{(n+1)\epsilon}{n}t}) \right)
\]
\[
- \Phi'F^{kl} h^i_{j,k} h^j_{i,l}.
\]
Set
\[
G(t) = G(t, \xi_t) = \sup_{\xi \in \Sigma} G(t, \xi).
\]
Thus we have
\[ F^{kl}h_{j;k}^i h_{j;l}^i \geq C|\nabla A|^2 \]
and
\[ |h_{j}^i - \delta_{j}^i| \to 0, \]
so for large \( t \) we can absorb the terms involving the derivatives of \( h_{j}^i \) by \( \Phi' F^{kl}h_{j;k}^i h_{j;l}^i \). There holds the following identity
\[ h_{k}^l h_{j}^k = (h_{k}^l - \delta_{k}^l)(h_{j}^k - \delta_{j}^k) + 2(h_{j}^l - \delta_{j}^l) + \delta_{j}^l. \]
Thus we have
\[
\frac{\partial}{\partial t} G(t) = (h_{j}^i - \delta_{j}^i)
\left( \Phi' F^{kl}h_{k;p}^i h_{j}^p - 2\Phi' F(h_{p}^i - \delta_{p}^i)(h_{j}^p - \delta_{j}^p) \right)
- 4\Phi' F(h_{j}^i - \delta_{j}^i) - 2\Phi' F + \Phi' F^{kl}g_{kl}h_{j}^i + O(e^{-\frac{(n+1)}{n}t})
\]
(3.31)
\[
= (h_{j}^i - \delta_{j}^i)
\left( \Phi' F^{kl}(h_{k;p}^i - 2h_{kl} + g_{kl})h_{j}^i - 2\Phi' F(h_{p}^i - \delta_{p}^i)(h_{j}^p - \delta_{j}^p) \right)
- 2\Phi' F(h_{j}^i - \delta_{j}^i) + O(e^{-\frac{(n+1)}{n}t})
\]
Choose Riemannian normal coordinates at \((t, \xi)\) such that at this point we have
\[ g_{ij} = \delta_{ij}, \quad h_{ij} = \kappa_{i}\delta_{ij}, \quad \kappa_{1} \leq \kappa_{2} \leq \ldots \leq \kappa_{n}. \]
For \( t \) large enough, we can find \( \epsilon < \frac{4}{\sup_{x \in M}} \) such that
\[
\frac{d}{dt} G(t) \leq \left( -\frac{4}{F} + 2\Phi' \sum_{j=1}^{n} |\kappa_{j}| |\kappa_{j} - 1| \sum_{k=1}^{n} F^{kk} + \frac{4}{F^2} \max_{1 \leq j \leq n} |\kappa_{j} - 1| \right) G(t) + \max_{1 \leq j \leq n} |\kappa_{j} - 1| O(e^{-\frac{n+1}{n}t})
\]
\[
\leq (-\frac{4}{F} + \epsilon)G(t) + \max_{1 \leq j \leq n} |\kappa_{j} - 1| O(e^{-\frac{n+1}{n}t}).
\]
Therefore, we have
\[ G(t) \leq Ce^{-\mu_{1}t}, \]
where \( \mu_{1} = \min\{\frac{4}{\sup_{x \in M} F} - \epsilon, \frac{n+1}{n}\} > 0. \) Thus,
\[
(3.32) \quad | -\frac{4}{F} + \frac{4}{n} | \leq C \max_{i} |\kappa_{i} - 1| \leq Ce^{-\frac{1}{2}\mu_{1}t}.
\]
Now we define
\[ \overline{G} = \sup_{x} \frac{1}{2} |h_{j}^i - \delta_{j}^i|^2 e^{\frac{4}{n}t}. \]
Similar to the process of getting (3.32), we can obtain
\[
\frac{d}{dt} \overline{G} \leq \left( -\frac{4}{F} + \frac{4}{n} + 2\Phi' \sum_{j=1}^{n} |\kappa_{j}| |\kappa_{j} - 1| \sum_{k=1}^{n} F^{kk} + \frac{4}{F^2} \max_{1 \leq j \leq n} |\kappa_{j} - 1| \right) \overline{G} + O(e^{-\frac{n+1}{n}t + \frac{1}{2}t - \frac{1}{2}\mu_{1}t})
\]
\[
\leq Ce^{-\frac{1}{2}\mu_{1}t} \overline{G} + O(e^{-\frac{n+1}{n}t - \frac{1}{2}t + \frac{1}{2}\mu_{1}t}),
\]
where we use (3.32) to get the last inequality. Thus,
\[ \overline{G} \leq C(n, \Sigma_{0}), \]
which implies our result.
Theorem 3.13. The estimate (3.5) in Lemma 3.3 is true for $\mu = \frac{2}{n}$.

Proof. Define

$$\tilde{w} = \sup_{x \in \mathbb{S}^n} \frac{1}{2} |D\varphi(\cdot, x)|^2 e^{-\frac{2}{n}t}.$$ 

The same calculation as in (3.6) implies

$$\frac{d}{dt} \tilde{w} \leq -\frac{2}{F^2} \lambda'' \frac{\lambda}{\lambda} F^{kl} \tilde{g}_{kl} \tilde{w} + \frac{2}{n} \tilde{w} \leq -\frac{2n}{F^2} \lambda'' \tilde{w} + \frac{2}{n} \tilde{w}.$$

Recalling the estimate (3.13)

$$\frac{\lambda''}{\lambda} = 1 - \frac{1}{2} m \lambda (1 - n) \lambda^{-n-2} = 1 + O(e^{-\frac{2n+2}{n}t}).$$

Together with (3.32), we have

$$\frac{d}{dt} \tilde{w} \leq Ce^{-\frac{2}{n} \mu_1 t} \tilde{w},$$

which implies $\tilde{w}$ is bounded from above. Therefore, the theorem holds.

Theorem 3.14. Under the assumptions of Theorem 1.2. There exists a constant $C = C(n, \Sigma_0)$ such that

$$\|D^2 \varphi\| \leq Ce^{-\frac{2}{n}t}.$$

Proof. Recalling (2.7), we have

$$\varphi^j = v^{-2} \varphi^j \varphi^k \varphi_{k,j} + \lambda' \delta^j_i - v \lambda h^j_i.$$

From Lemma 3.1, we get

$$|\lambda' - \lambda| = \frac{1 - m \lambda^{1-n}}{\lambda \sqrt{1 + \lambda^2 - m \lambda^{1-n}}} \leq \frac{1}{\lambda} \leq Ce^{-\frac{2}{n}t}.$$ 

Together with Theorems 3.13 and 3.12, we obtain

$$|D^2 \varphi| \leq C|D\varphi|^2 |D^2 \varphi| + |\lambda' \delta^j_i - \lambda \delta^j_i| + |\lambda \delta^j_i - v \lambda \delta^j_i| + |v \lambda \delta^j_i - v \lambda h^j_i| \leq Ce^{-\frac{2}{n}t} |D^2 \varphi| + Ce^{-\frac{1}{n}t}.$$

Choosing $T$ large enough $(Ce^{-\frac{2}{n}t} < \frac{1}{2})$, we know that the estimate

$$|D^2 \varphi| \leq Ce^{-\frac{2}{n}t}$$

holds for $t > T$. 

Clearly, from Theorem 3.14, we can show that there exists a constant $C = C(n, \Sigma_0)$ such that

$$\|D^2 r\|_{\mathbb{S}^n} \leq C.$$

Then by Krylov-Safonov estimate [20], we have

$$\|r\|_{C^{2,\alpha}(\mathbb{S}^n)} \leq C(n, \Sigma_0),$$

which implies the following conclusion.

Theorem 3.15. Under the assumptions of theorem 1.2. The function

$$\tilde{r}(t, \theta) = r(t, \theta) - \frac{t}{n}$$

converge to a well-defined $C^2$ function $f(\theta)$ in $C^{2,\alpha}$. 

Proof. Because of the boundedness of $\tilde{r} = r - \frac{t}{n}$ in $C^2(S^n)$, we only have to show the pointwise limit
\[
\lim_{t \to \infty} (r - \frac{t}{n})
\]
exists for all $x \in S^n$. We have
\[
\frac{\partial}{\partial t} \tilde{r} = \frac{v}{F} - \frac{1}{n} = \frac{v - 1}{F} + \frac{n - F}{nF} \geq -C(n, \Sigma_0)e^{-\frac{t}{n}}.
\]
Thus,
\[
(\tilde{r} - nCe^{-\frac{t}{n}}) \geq 0,
\]
which implies the result. 

Remark 3.3. Following the techniques in [8, Section 6] and [22, Section 5], we may also get estimates of high order for $\tilde{r}$
\[
\|\tilde{r}\|_{C^k(S^n)} \leq C(n, \Sigma_0), \quad \forall k \in \mathbb{N}.
\]
Therefore, the $C^\infty$ convergence in the above theorem may be obtained.

References

[1] Q. Ding, The inverse mean curvature ow in rotationally symmetric spaces, Chin. Ann. Math., Ser. B 32 (2011), No. 1, 27-44.
[2] Brendle, Simon; Hung, Pei-Ken; Wang, Mu-Tao, A Minkowski inequ ality for hypersurfaces in the anti-de Sitter-Schwarzschild manifold, Comm. Pure Appl. Math. 69 (2016), no. 1, 124-144.
[3] K. Ecker; G. Huisken, nmersed hypersurfaces with constant Weingarten curvature, Math. Ann. 283 (1989), No. 2, 329-332.
[4] Ge, Yuxin; Wang, Guofang; Wu, Jie; Xia, Chao, A Penrose inequality for graphs over Kottler space, Calc. Var. Partial Differential Equations 52 (2015), no. 3-4, 755-782.
[5] Ge, Yuxin; Wang, Guofang; Wu, Jie, The GBC mass for asymptotically hyperbolic manifolds, Math. Z. 281 (2015), no. 1-2, 257-297.
[6] Ge, Yuxin; Wang, Guofang; Wu, Jie, Hyperbolic Alexandrov-Fenchel quermassintegral inequalities II, J. Differential Geom. 98 (2014), no. 2, 237-260.
[7] Claus Gerhardt, Flow of nonconvex hypersurfaces into spheres, J. Differ. Geom. 32 (1990), 299-314.
[8] Claus Gerhardt, Inverse curvature flows in hyperbolic space, J. Differ. Geom. 89 (2011), 487-527.
[9] Claus Gerhardt, Curvature Problems, Ser. in Geom. and Topol., vol. 39, International Press, Somerville, MA, (2006).
[10] C. Gerhardt, Curvature estimates for Weingarten hypersurfaces in Riemannian manifolds, Adv. Calc. Var. 1 (2008), 123-132.
[11] Claus Gerhardt, Curvature flows in the sphere, J. Differential Geom. 100 (2015), no. 2, 301-347.
[12] Claus Gerhardt, Non-scale-invariant inverse curvature flows in Euclidean space, Calc. Var. Partial Differential Equations 49 (2014), no. 1-2, 471-489.
[13] Claus Gerhardt, Closed Weingarten hypersurfaces in space forms, Geom. Anal. and the Calc. of Var. (Jurgen Jost, ed.), International Press, Boston, (1996).
[14] Pei-Ken Hung and Mu-Tao Wang, Inverse mean curvature flows in the hyperbolic 3-space revisited, Calc. Var. Partial Differential Equations 54 (2015), no. 1, 119-126.
[15] Gerhard Huisken, Flow by mean curvature of convex surfaces into spheres., J. Differ. Geom. 20 (1984), 237-266.
[16] Haizhong Li; Yong Wei, On inverse mean curvature flow in Schwarzschild space and Kottler space, available online at arXiv:1212.4218.
[17] Siyuan Lu, Inverse curvature flow in anti-de sitter-schwarzschild manifold, available online at arXiv:1609.09733v1

[18] M. Makowski; Julian Scheuer, Rigidity results, inverse curvature flows and Alexandrov-Fenchel-type inequalities in the sphere, (2013), to appear in Asian J. Math., and available online at arXiv:1307.5764.

[19] A. Neves, Insufficient convergence of inverse mean curvature flow on asymptotically hyperbolic manifolds, J. Differential Geom. 84 (2010), no. 1, 191-229.

[20] N.V. Krylov, Nonlinear elliptic and parabolic equations of the second order, Reidel, Dordrecht, (1987).

[21] Li. P, Harmonic functions and applications to complete manifolds, University of California, Irvine, 2004, preprint.

[22] Julian Scheuer, Non-scale-invariant inverse curvature flows in hyperbolic space, Calc. Var. Partial Differential Equations 53 (2015), no. 1-2, 91-123.

[23] Julian Scheuer, The inverse mean curvature flow in warped cylinders of non-positive radial curvature, available online at arXiv:1312.5662.

[24] Urbas J., On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures, Math. Z. 205 (1990), no. 3, 355-372.

[25] Urbas J., An expansion of convex hypersurfaces, J. Differential Geom. 33 (1991), no. 1, 91-125.

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