PT Symmetry and Renormalization in Pomeron Model

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A novel perturbative analysis for the 2 + 1 local supercritical field theory of pomerons is developed. It is based on the PT symmetry of the model which allows to study a similar Hamiltonian with the same real perturbative spectrum. In the lowest non trivial order of perturbation theory the pomerons interactions are shown to lead to the renormalization of the slope. The appearance of a non local interaction for two pomeron states is such that at small coupling only scattering states are present and the spectrum of two particle states is not affected.

1 Introduction

The high energy behavior of strong interaction in the Regge limit has been being studied since more than forty years, at the beginning in the so called S-matrix theory approach and subsequently using field theory models for the object describing the leading behavior of the cross section, the Pomeron. After Quantum Chromodynamics has been found to describe perturbatively many aspects of strong interactions, the Regge limit of this theory has been investigated leading to the so called BFKL physics [2] and in general small $x$ physics, mainly analyzed in deep inelastic scattering experiments, as the one at HERA.

Theoretically in small $x$ QCD one can observe the emerging of an effective theory, which in its more simple form can be seen as an interacting theory (due to the appearance of a triple pomeron vertex [3] [4] [5] [6] [7] [8]) of non local Pomerons in 2+1 dimensions. Such field theory models, even if oversimplified, are too complicated to be analyzed analytically. What have been proposed in the past as even simpler toy models to be analyzed and improve the understanding were models in 0+1 dimensions, a kind of quantum mechanics of pomerons, and a local 2+1 dimensional theory. They were introduced before the QCD analysis [9] [10], but nevertheless are characterized by some common features implied by it. The former was analyzed many years ago and reconsidered recently [11] [12] [13] from different points of view. The latter was also studied many years ago but most of the questions remained open. We report here some results of a recent work [14] devoted to formulate a novel perturbative approach useful to analyze this 2 + 1 QFT model.

2 PT Symmetry in QM and QFT

It is well known that a partial effective description of a system can be associated to a non hermitian Hamiltonian which is characterized by a non unitary evolution. Nevertheless several attempts to analyzed such systems and to try even to formulate some consistent non hermitian quantum mechanics has been done. The first interesting result was found by Bender [15] who noted that there exist non hermitian Hamiltonians having a real spectrum bounded from below. This was shown to be possible if also specific boundary conditions for the wave functions of the associated Sturm-Liouville differential problem were properly

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defined. The main point at the base of these properties is that such Hamiltonians can be formally obtained by a similarity transformation acting on well defined hermitian ones \[16\]. Note that the same considerations formally apply to systems with a finite (QM) or infinite (QFT) \[17\] degrees of freedom.

A special class of Hamiltonians having these property was found to have an unbroken \textit{PT} symmetry, so that the Hamiltonian \( H \) and the operator \( PT \) have common eigenstates. In such a case it is possible to define a special scalar product, leading to a norm conserved in time, as follows \[ f,g \] = \( \int dx [PTf]^\dagger(x)g(x) \), \( t \) denoting the transpose operation. Such \textit{PT}-norm is not positive and there exist an operator \( C \), commuting with both \( H \) and \( PT \), which select the two possible signs. By constructing the operator \( C \), which depends on the Hamiltonian \( H \) itself, it is therefore possible to construct another scalar product \( \langle f|g \rangle = \int dx [CPTf]^\dagger(x)g(x) \) and define an Hilbert space with a positive norm conserved in time.

The observables are defined to satisfy \( O^t = CTOCTP \), which reduces to the hermiticity condition for the usual case \( C = P \) in conventional QM.

In our case the local effective theory for interacting pomeron (\textit{PT} symmetric and with an Hamiltonian with real eigenvalues) is associated to the evolution in rapidity and the scalar product is the conventional one with the norm of pomeron states, which interact by the triple pomeron vertex, not conserved in rapidity. Nevertheless the operators introduced above are very useful to develop a perturbative analysis.

Indeed let us consider a system with the Hamiltonian

\[ H = H_0 + \lambda H_I, \quad (1) \]

where \( H_0 \) (the free part) is hermitian and \( H_I \) (the interaction part) is anti-hermitian. We define the parity operator to transform \( H \) into \( H^\dagger \) so that \( [H_0, P] = 0 \), \( \{H_I, P\} = 0 \) and \( P^2 = 1 \). One has to look for the operator \( C \) such that

\[ [C, H] = [C, PT] = 0. \quad (2) \]

It is convenient to assume the general form \( C = e^Q P \), where \( Q \) is an hermitian operator, that together with the previous commutation relations imply

\[ 2\lambda e^QH_I = [e^Q, H]. \quad (3) \]

Moreover one obtains easily the relation \( e^{-Q}He^Q = H^\dagger \), which also implies

\[ h = e^{-Q/2}He^{Q/2} = e^{Q/2}H^\dagger e^{-Q/2} = h^\dagger. \quad (4) \]

Therefore we have found an hermitian Hamiltonian \( h \) which is similar to \( H \) by means of the similarity transformation induced by the operator \( e^{Q/2} \).

This general relations can be studied perturbatively for a small coupling \( \lambda \). We start by looking for a perturbative expansion of \( Q = \lambda Q_1 + \lambda^2 Q_3 + ... \) by solving eq. (3) which gives:

\[ [H_0, Q_1] = -2H_I, \quad [H_0, Q_3] = -\frac{1}{6} [H_I, Q_1]Q_1 \quad (5) \]

and so on. From these relations one obtains, as we shall see, an explicit form for the \( Q_i \). Once \( Q \) is known as a power series in \( \lambda \), the Hamiltonian \( h \) can also be found in the same form: \( h = h^{(0)} + \lambda^2 h^{(2)} + \lambda^4 h^{(4)} + ... \) with the first terms given by

\[ h^{(0)} = H_0, \quad h^{(2)} = \frac{1}{4} [H_I, Q_1], \quad h^{(4)} = \frac{1}{4} [H_I, Q_3] + \frac{1}{32} [H_0, Q_3]Q_1. \quad (6) \]
3 Analysis of the LRFT model

Let us start by defining the LRFT as a theory of two fields φ(y, x) and \phi^{\dagger}(y, x) depending on rapidity y and transverse coordinates x with a Lagrangian density

\[ \mathcal{L} = \phi^{\dagger}(\partial_y - \mu - \alpha' \nabla_x^2)\phi + i\lambda \phi^{\dagger}(x)\left[\phi^{\dagger}(x) + \phi(x)\right]\phi(x), \]  

(7)

where \( \mu > 0 \) is the intercept minus unity and \( \alpha' \) is the slope of the pomeron trajectory. Note that if \( \mu > 0 \) the corresponding functional integral is divergent and the only way to define the theory beyond the set of perturbative Feynman diagram is the analytic continuation from \( \mu < 0 \) when the theory is well defined. Such a continuation is automatic in the the Hamiltonian approach, where a quasi-Schrödinger equation for the wave function \( \Psi \) is defined:

\[ \frac{d\Psi(y)}{dy} = -H\Psi(y), \quad H = H_0 + \lambda H_I \]  

(8)

with the free part given by

\[ H_0 = \int d^2x(-\mu \phi^{\dagger}(x)\phi(x) + \alpha' \nabla \phi^{\dagger}(x)\nabla \phi(x)) \]  

(9)

and the interaction part by

\[ H_I = i \int d^2x \phi^{\dagger}(x)\left[\phi^{\dagger}(x) + \phi(x)\right]\phi(x). \]  

(10)

Standard commutation relations are valid between \( \phi \) and \( \phi^{\dagger} \): \([\phi(x), \phi^{\dagger}(x')] = \delta^2(x - x')\). The scattering amplitude with the target ("initial") state \( \Psi_i(y_1) \) at rapidity \( y_1 \) and the projectile ("final") state \( \Psi_f(y_2) \) at rapidity \( y_2 > y_1 \) is defined as

\[ iA_{f,i}(y_2 - y_1) = \langle \Psi_f(y_2) | e^{-\mathcal{H}(y_2-y_1)} | \Psi_i(y_1) \rangle. \]  

(11)

One can demonstrate that the perturbation expansion in powers of \( \lambda \) of this expression reproduces the standard Reggeon diagrams of the LRFT and also that \[ \text{[11]} \] satisfies the requirement of symmetry between the target and projectile (see [2]) Parity transformation \( P \) is defined by \( \phi(y, x) \rightarrow -\phi(y, -x) \) and \( \phi^{\dagger}(y, x) \rightarrow -\phi^{\dagger}(y, -x) \) so that \( \mathcal{P}\mathcal{H}\mathcal{P} = H^\dagger \), while \( T \) is the complex conjugation, so that \( [H, PT] = 0 \). Any state can be written as \( F(i\phi^{\dagger})|0\rangle \), where \( F \) is a real function and \( \phi(0) = 0 \) so that \( \langle \Psi|H|\Psi\rangle = \langle \Psi_0|F(-i\phi)HF(i\phi^\dagger)|\psi_0\rangle \). We shall look for a perturbatively constructed similarity transformation, as illustrated in the previous section, in order to write any transition amplitude as

\[ iA_{f,i}(y_2 - y_1) = \langle e^{\mathcal{Q}/2}\Psi_f(y_2) | e^{-\mathcal{H}(y_2-y_1)} | e^{-\mathcal{Q}/2}\Psi_i(y_1) \rangle. \]  

(12)

In particular the simplest object one can imagine is the full pomeron Green function at rapidity y and momentum k will be given as as

\[ \delta^2(k - k')G(y, k) = \langle 0 | \phi(k) e^{\mathcal{Q}/2} e^{-yh} e^{-\mathcal{Q}/2} \phi^\dagger(k') | 0 \rangle. \]  

(13)

On restricting to the first non trivial order in perturbation theory one is looking for

\[ Q_1 = -2i\mu \int d^2x_1d^2x_2d^2x_3 \left(f_1(x_1, x_2, x_3)\phi_1^\dagger \phi_2 \phi_3 - h.c.\right), \]  

(14)

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and obtains for the Fourier transform of $f_1$:

$$
\hat{f}_1(k_1, k_2, k_3) = \mu \frac{(2\pi)^2 \delta(k_1 + k_2 + k_3)}{\mu - \alpha'(k_2^2 + k_3^2 - k_1^2)}.
$$

(15)

The corresponding correction of order $\lambda^2$ to the Hamiltonian $h$ similar to $H$ is given by

$$
h^{(2)} = \frac{1}{4}[H_1, Q_1] = h_{\text{single}}^{(2)} + h_{\text{pair}}^{(2)} + h_{\text{NC}}^{(2)}.
$$

(16)

The first contribution is a correction to the single pomeron propagation.

$$
h_{\text{single}}^{(2)} = \int d^2 k \delta^3(k) \phi(k) \Delta^{(2)}(k), \quad \Delta^{(2)}(k) = -\frac{2}{(2\pi)^2} \text{Re} \int \frac{d^2 k_2 d^2 k_3 \delta^2(k_2 + k_3 - k)}{\mu - \alpha'(k_2^2 + k_3^2 - k^2)}.
$$

(17)

This term gives a correction to the pomeron energy $\epsilon(k) = -\mu + \alpha'k^2 + \lambda^2 \Delta^{(2)}(k)$. A renormalization is needed and choosing the condition $\epsilon(0) = -\mu$ one finds $\Delta^{(2)}_{\text{reg}}(k) = -\frac{1}{8\pi\mu} k^2$ which leads to the renormalization of the pomeron slope

$$
\alpha' \rightarrow \alpha'_{\text{ren}} = \alpha' - \lambda^2 \frac{1}{8\pi\mu}.
$$

(18)

The last term in eq. (16), $h_{\text{NC}}^{(2)}$, is not conserving the pomeron number and therefore contributes to order $\lambda^2$ to the eigenvalues, going beyond our approximations. We neglect it here.

The second term of eq. (16) $h_{\text{pair}}^{(2)}$ has a complicated structure associated to the interaction of two pomerons

$$
h_{\text{pair}}^{(2)} = \int d^2 k_1 d^2 k_2 d^2 q_1 d^2 q_2 \delta^2(q_1 + q_2 - k_1 - k_2) V^{(2)}(q_1, q_2 | k_1, k_2) \phi^d(q_1) \phi^d(q_2) \phi(k_1) \phi(k_2),
$$

(19)

with an interaction potential $V^{(2)}$ being non local and with some degenerate terms depending only on the incoming or outgoing momenta. In such a case one may be interested in studying the scattering states, not changing the spectrum, and in the presence of bound states which instead could deeply affect the spectrum. Let us note that due to the fact that to order $\lambda^2$ in the spectrum of $h$ there are no transition in the number of pomeron states, one can really solve the problem with quantum mechanical techniques. In the analysis of the two pomeron potential we have considered for simplicity the forward direction $q_1 + q_2 = k_1 + k_2 = 0$ so that $V^{(2)}(q_1, q_2 | k_1, k_2) = V(q, k) = v(q) + v(k) + V_1(q, k)$ with $v(q) = \frac{1}{8\pi\mu - 2\alpha'_{\text{ren}}}$ and $V_1(q, k) = \frac{1}{2\pi\mu - 2\alpha'_{\text{ren}}} (k^2 + q^2) + (k \leftrightarrow q)$. Therefore the Schrödinger equation to be solved reads in momentum space reads

$$
(\epsilon(q) - E)\psi(q) = -\int d^2 k V(q | k) \psi(k).
$$

(20)

Omitting the technical details derived in [1], we simply present here our findings. Solving the associated Lippman-Schwinger equation for the scattering matrix one obtains to order $\lambda^2$, after performing a regularization to handle divergent quantities,

$$
T(q | l) = V_1(q | l) - \frac{v(l)}{I_2} \chi_2(q) - \frac{v(q) v(l)}{I_2},
$$

(21)

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where \( \epsilon(q) = -2\mu + 2\alpha'_{ren} q^2 \), \( I_n = \int d^2k \frac{\epsilon_n(k)}{\epsilon(l)-\epsilon(k)} \) and \( \chi_2(q) = \int d^2k \frac{V(q,k)\epsilon(k)}{\epsilon(l)-\epsilon(k)} \). This also gives the solution of the scattering states \( \psi_l(q) = \delta^2(q-l) + \frac{\epsilon_l}{\epsilon(q)} \pm i\theta \).

In order to investigate the existence of bound states of energy \( E \) we consider the associated equation

\[
t_E(q) = \int d^2k \frac{V(q,k)t_E(k)}{E-\epsilon(k)}, \quad \psi_E(q) = \frac{t_E(q)}{E-\epsilon(q)}.
\]

(22)

The condition of the existence of bound states can be reduced to the existence of the solution of a secular equation of a finite algebraic problem. in a perturbative sense, i.e. for small values of \( \lambda \) one can show that there are no solutions. They may appear at larger values of \( \lambda \), a case for which nevertheless some higher order terms in the perturbative expansion may be also important. We stress that the results obtained are valid for an evolution along rapidity intervals \( \Delta y \sim 1/\lambda^2 \).

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