Negative time splitting is stable

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Abstract

For high order (than two) in time operator-splitting methods applied to dissipative systems, a folklore issue is the appearance of negative-time/backward-in-time linear evolution operators such as backward heat operators interwoven with nonlinear evolutions. The stability of such methods has remained an ensuing difficult open problem. In this work we consider a fourth order operator splitting discretization for the Allen-Cahn equation which is a prototypical high order splitting method with negative time-stepping, i.e. backward in time integration for the linear parabolic part. We introduce a new theoretical framework and prove uniform energy stability and higher Sobolev stability. This is the first strong stability result for negative time stepping operator-splitting methods.

1 Introduction

We consider the Allen-Cahn equation

\[
\begin{aligned}
&\partial_t u = \nu \Delta u - f(u), \quad (t, x) \in (0, \infty) \times \Omega, \\
&u\big|_{t=0} = u_0,
\end{aligned}
\]  

(1.1)

where the unknown \( u = u(t, x) : [0, \infty) \times \Omega \to \mathbb{R} \). The parameter \( \nu > 0 \) is called the mobility coefficient and we fix it as a constant for simplicity. The nonlinear term takes the form \( f(u) = u^3 - u = F'(u) \), where \( F(u) = \frac{1}{4}(u^2 - 1)^2 \) is the standard double well. To minimize technicality, we take the spatial domain \( \Omega \) in (1.1) as the 2\( \pi \)-periodic torus \( \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} = [-\pi, \pi] \). With some additional work our analysis can be extended to physical dimensions \( d = 2, 3 \). The system (1.1) arises as a \( L^2 \)-gradient flow of a Ginzburg-Landau type energy functional \( E(u) \), where

\[
E(u) = \int_{\Omega} \left( \frac{1}{2} \nu |\nabla u|^2 + F(u) \right) dx = \int_{\Omega} \left( \frac{1}{2} \nu |\nabla u|^2 + \frac{1}{4} (u^2 - 1)^2 \right) dx. 
\]  

(1.2)

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The basic energy conservation law takes the form

\[ \frac{d}{dt} E(u(t)) + \| \partial_t u \|_2^2 = \frac{d}{dt} E(u(t)) + \int_\Omega |\nu \Delta u - f(u)|^2 dx = 0. \quad (1.3) \]

It follows that

\[ E(u(t_2)) \leq E(u(t_1)), \quad \forall \, 0 \leq t_1 < t_2. \quad (1.4) \]

Besides the \( L^2 \)-type conservation law, there is also \( L^\infty \)-type control. Due to special form of the nonlinearity, for smooth solutions we have the maximum principle

\[ \| u(t, \cdot) \|_{L^\infty} \leq \max \{ 1, \| u_0 \|_{L^\infty} \}, \quad \forall \, t \geq 0. \quad (1.5) \]

As a consequence the long-time wellposedness and regularity is not an issue for (1.1).

The objective of this work is to establish strong stability of a fourth order in time operating splitting algorithm applied to the Allen-Cahn equation (1.1). This is a part of our on-going program to develop a new theory for the rigorous analysis of stability and convergence of operator-splitting methods applied to dissipative-type problems. Due to various subtle technical obstructions, there were very few rigorous results on the analysis of the operator-splitting type algorithms for the Allen-Cahn equation, the Cahn-Hilliard equation and similar models. Prior to our recent series of works [17, 18], most existing results in the literature are conditional one way or another. To put things into perspective, we briefly review a few closely related representative works and more recent developments.

- **The work of Gidey-Reddy.** Gidey and Reddy considered in [24] a convective Cahn-Hilliard model of the form

  \[ \partial_t u - \gamma \nabla \cdot h(u) + \epsilon^2 \Delta^2 u = \Delta(f(u)), \quad (1.6) \]

  where \( h(u) = \frac{1}{2}(u^2, u^2) \). They adopted an operator-splitting of (1.6) into the hyperbolic part, nonlinear diffusion part and diffusion part respectively. Several conditional results concerning certain weak solutions were obtained.

- **The work of Weng-Zhai-Feng.** In [25], Weng, Zhai and Feng studied a viscous Cahn-Hilliard model:

  \[ (1 - \alpha) \partial_t u = \Delta(-\epsilon^2 \Delta u + f(u) + \alpha \partial_t u), \quad (1.7) \]

  where the parameter \( \alpha \in (0, 1) \). The authors employed a fast explicit Strang-type operator splitting and proved the stability and the convergence (see Theorem 1 on pp. 7 of [25]) under the assumption that \( A = \| \nabla u_{\text{num}} \|_{L^\infty}^2, \quad B = \| u_{\text{num}} \|_{L^\infty}^2 \) stay bounded, and satisfy a technical condition \( 6A + 8 - 24B > 0 \). Here \( u_{\text{num}} \) denotes the numerical solution.
The work of Cheng-Kurganov-Qu-Tang. In [23], Cheng, Kurganov, Qu and Tang considered the Cahn-Hilliard equation

\[ \partial_t u = -\nu \Delta^2 u - \Delta u + \Delta(u^3) \]  

(1.8)

and the MBE equation

\[ \partial_t \phi = -\delta \Delta^2 \phi - \nabla \cdot ((1 - |\nabla \phi|^2) \nabla \phi). \]  

(1.9)

Concerning the Cahn-Hilliard equation, the authors considered a Strang-type splitting approximation of the form

\[ u(t + \tau) \approx S_L^{(1)}(\frac{\tau}{2})S_N^{(1)}(\tau)S_L^{(1)}(\frac{\tau}{2})u(t), \]  

(1.10)

where

\[ S_L^{(1)}(\frac{\tau}{2}) = \exp(\frac{1}{2}\tau(-\nu \Delta^2 - \Delta)); \]  

(1.11)

and \( w = S_N^{(1)}(\tau)a \) is the nonlinear propagator

\[
\begin{aligned}
\partial_t w &= \Delta(w^3), \\
|w|_{t=0} &= a.
\end{aligned}
\]  

(1.12)

Various conditional results were given in [23] but the rigorous analysis of energy stability was a long-standing open problem. This problem and several related open problems were settled in our recent proof [18].

In recent [17], we carried out the first energy-stability analysis of a first order operator-splitting approximation of the Cahn-Hilliard equation. More precisely denote \( u^{CH} \) as the exact PDE solution to the Cahn-Hilliard equation \( \partial_t u = -\nu \Delta^2 u - \Delta u + \Delta(u^3) \). We considered a splitting approximation of the form:

\[ u^{CH}(t + \tau) \approx S_L^{(2)}(\tau)S_N^{(2)}(\tau)u^{CH}(t), \]  

(1.13)

where \( S_L^{(2)}(\tau) = \exp(-\tau \nu \Delta^2) \) and \( w = S_N^{(2)}(\tau)a \) solves

\[ \frac{w - a}{\tau} = \Delta(a^3 - a). \]  

(1.14)

We introduced a novel modified energy and rigorously proved monotonic decay of the new modified energy which is coercive in \( H^1 \)-sense. Moreover we obtained uniform control of higher Sobolev regularity and rigorously justified the first order convergence of the method on any finite time interval.

In [18], we settled the difficult open problem of energy stability of the Strang-type algorithm applied to the Cahn-Hilliard equation which was introduced in the work.
of Cheng, Kurganov, Qu and Tang [23]. One should note that the new theoretical framework developed in [18] is completely different from the first order case [17]. In the second-order case, for most generic data one no longer has strict energy-monotonicity at disposal and several new ideas such as dichotomy analysis, an absorbing-set approach were introduced in [18] in order to settle long time unconditional (i.e. independent of time-step or time interval) Sobolev bounds of the numerical solution.

In this work we develop further the program initiated in [17, 18] and turn to the analysis of higher order in time splitting methods. A folklore issue is that operator-splitting methods with order higher than two require negative time-stepping, i.e. backward time integration for each time step. In [33], Sheng considered dimensional splitting for the two-dimensional parabolic problem \( \partial_t u = a \partial_{xx} u + b \partial_{yy} u \). Using semi-discretization one obtains the ODE system \( \partial_t u = A u + B u \), where the matrices \( A \) and \( B \) correspond to the discretization of \( a \partial_{xx} \) and \( b \partial_{yy} \) respectively. One is then naturally led to the approximation of \( u = e^{t(A+B)} u_0 \approx e^{\frac{1}{2}tA} e^{\frac{1}{2}tB} e^{\gamma_1 tA} e^{\alpha_{1,k} tA} e^{\beta_{1,k} B} \cdot \cdot \cdot e^{\gamma_K tA} e^{\alpha_{K,k} tA} e^{\beta_{K,k} B} \), \( (1.15) \)

\[ \gamma_k > 0, \ \alpha_{i,k} \geq 0, \ \beta_{i,k} \geq 0, \] \( (1.16) \)

then the highest order of a stable approximation is two even if \( K \) is chosen to be large. Put it differently, Sheng’s fundamental result states that for an \( N^{th} \)-order \( (N \geq 3) \) partitioned split-step schemes, at least one of the solution operators must be applied with negative time step, i.e. backwardly. In [34] Suzuki adopted an elegant time-ordering principle from quantum mechanics and proved a general nonexistence theorem of positive decomposition for high order splitting approximations. In [35], Goldman and Kaper strengthened these results further and showed that even with partitioned schemes, each solution operator within a convex partition must be performed with at least one negative/backward fractional time step.

For deterministic Hamiltonian-type systems, backward time stepping in general does not create instabilities and high-order operating splitting methods have shown promising effectiveness [32]. On the other hand, due to the negative time stepping, there are some arguments that higher-than-three operator splitting methods cannot be applied to parabolic equations with diffusive terms [33, 36, 37]. As we shall explain momentarily, these concerns are not unsubstantiated and even turn up in the well-defined-ness of these algorithms.

Although rigorous analysis of these issues were not available before, recently Cervi and Spiteri [31] considered three third-order operating-splitting methods and demonstrated via extensive numerical simulations the effectiveness of higher order splitting methods for representative cardiac electrophysiology simulations. These compelling numerical evidences propel us to re-examine in detail the stability property of general negative/backward time-stepping methods in parabolic problems. Indeed, the very
purpose of this work is to break these aforementioned technical barriers and establish a new stability theory for these problems.

To set the stage and minimize technicality, we consider the Allen-Cahn equation (1.1) posed on the $2\pi$-periodic torus $T = [-\pi, \pi]$. To build some intuition, let $\tau > 0$ and consider

$$S_L(\tau) = \exp(\nu \tau \partial_{xx}),$$

and let $w = S_N(\tau) a$ solve the equation

$$\begin{cases}
\partial_t w = w - w^3; \\
|w|_{t=0} = a.
\end{cases}$$

(1.18)

An explicit formula for $S_N(\tau)$ is readily available, indeed it is not difficult to check that

$$S_N(\tau) a = \frac{e^{2\tau} a}{\sqrt{1 + (e^{2\tau} - 1)a^2}}.$$ 

(1.19)

Define a second-order Strang-type propagator

$$S_2^{(o)}(\tau) = S_L(\frac{\tau}{2}) S_N(\tau) S_L(\frac{\tau}{2}).$$

(1.20)

Following Yoshida [29], we consider a 4th order integrator obtained by a symmetric repetition (product) of the 2nd order integrator:

$$S_4^{(o)}(\tau) = S_2^{(o)}(x_1 \tau) S_2^{(o)}(-x_0 \tau) S_2^{(o)}(x_1 \tau),$$

(1.21)

where

$$x_0 = \frac{2^{\frac{1}{4}}}{2 - 2^{\frac{1}{4}}}, \quad x_1 = \frac{1}{2 - 2^{\frac{1}{4}}}.$$ 

(1.22)

Somewhat surprisingly, we first show that the propagator $S_4^{(o)}(\tau)$ is ill-defined if one does not introduce a judiciously chosen spectral cut-off.

**Proposition 1.1** (Ill-definedness of $S_4^{(o)}(\tau)$ with a spectral cut-off). The propagator $S_4^{(o)}(\tau)$ is ill-defined in general.

The proof of Proposition 1.1 is given in Section 2. Armed with this important observation, we are led to introduce a spectral cut-off condition for the propagators. Let $M \geq 2$ be an integer and we shall consider the projection operator $\Pi_M$ defined for $f : T \to \mathbb{R}$ via the relation

$$\Pi_M f = \frac{1}{2\pi} \sum_{|k| \leq M} \hat{f}(k) e^{ikx},$$

(1.23)
where \( \hat{f}(k) \) denotes the Fourier coefficient of \( f \). We introduce the following spectral cut-off condition.

**Definition** (Spectral condition). Let \( M \geq 2 \) be the spectral truncation parameter. We shall say \( \tau > 0 \) satisfy the spectral condition if \( \tau \leq l_0 M^{-2} \) for some \( l_0 > 0 \).

**Remark 1.1.** This constraint in \( \tau \) is reminiscent of the CFL condition in hyperbolic problems. Here in the parabolic setting we require that the operator \( \tau \partial_{xx} \) to remain bounded when restricted to the spectral cut-off \( |k| \leq M \).

Let \( M \geq 2 \). We now consider the following modified propagators:

\[
S^{(2)}(\tau) = \Pi_M S_L(\frac{\tau}{2}) S_N(\tau) \Pi_M S_L(\frac{\tau}{2});
S^{(4)}(\tau) = S^{(2)}(x_1 \tau) S^{(2)}(-x_0 \tau) S^{(2)}(x_1 \tau).
\]  

(1.24)

**Theorem 1.1** (Stability of negative-time splitting methods). Let \( \nu > 0 \), \( M \geq 2 \) and consider the Allen-Cahn equation (1.1) on the one-dimensional \( 2\pi \)-periodic torus \( \mathbb{T} = [-\pi, \pi] \). Assume the spectral cut-off condition \( \tau \leq l_0 M^{-2} \) for some \( l_0 > 0 \). Assume the initial data \( u^0 = \Pi_M a \in H^{k_0}(\mathbb{T}) \) (\( k_0 \geq 1 \) is an integer). Let \( \tau > 0 \) and define

\[ u^{n+1} = S^{(4)} u^n, \quad n \geq 0. \]  

(1.25)

There exists a constant \( \tau_* > 0 \) depending only on \( l_0, \|a\|_{H^1} \) and \( \nu \), such that if \( 0 < \tau \leq \tau_* \), then

\[ \sup_{n \geq 0} \|u^n\|_{H^{k_0}} \leq A_1 < \infty, \]  

(1.26)

where \( A_1 > 0 \) depends on \( \|u^0\|_{H^{k_0}}, \nu, l_0, k_0 \).

**Remark 1.2.** One can also show the fourth-order convergence of the operator splitting approximation. Namely if \( u^0 \in H^{80}(\mathbb{T}) \) and let \( u \) be the exact PDE solution corresponding to initial data \( a \). Let \( 0 < \tau < \tau_* \) bet the same as in Theorem 1.1. Then for any \( T > 0 \), we have

\[ \sup_{n \geq 1, n\tau \leq T} \|u^n - u(n\tau, \cdot)\|_{L^2(\mathbb{T})} \leq C(\tau^4 + M^{-10}), \]  

(1.27)

where \( C > 0 \) depends on \( (\nu, l_0, \|u^0\|_{H^{80}}, T) \). The regularity assumption on initial data can be lowered. We shall refrain from stating such a pedestrian result here and leave its justification to interested readers as exercises.

The rest of this paper is organized as follows. In Section 2 we set up the notation and collect some preliminary lemmas. In Section 3 we give the proof of Theorem 1.1.
2 Notation and preliminaries

For any two positive quantities $X$ and $Y$, we shall write $X \lesssim Y$ or $Y \gtrsim X$ if $X \leq CY$ for some constant $C > 0$ whose precise value is unimportant. We shall write $X \sim Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold. We write $X \lesssim_{\alpha} Y$ if the constant $C$ depends on some parameter $\alpha$. We shall write $X = O(Y)$ if $|X| \lesssim Y$ and $X = O_{\alpha}(Y)$ if $|X| \lesssim_{\alpha} Y$.

We shall denote $X \ll Y$ if $X \leq cY$ for some sufficiently small constant $c$. The smallness of the constant $c$ is usually clear from the context. The notation $X \gg Y$ is similarly defined. Note that our use of $\ll$ and $\gg$ here is different from the usual Vinogradov notation in number theory or asymptotic analysis.

For any $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, we denote $|x| = |x|_2 = \sqrt{x_1^2 + \cdots + x_d^2}$, and $|x|_\infty = \max_{1 \leq j \leq d} |x_j|$. Also occasionally we use the Japanese bracket notation: $\langle x \rangle = (1 + |x|^2)^{1/2}$.

We denote by $\mathbb{T}^d = [-\pi, \pi]^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ the usual $2\pi$-periodic torus. For $1 \leq p \leq \infty$ and any function $f : x \in \mathbb{T}^d \to \mathbb{R}$, we denote the Lebesgue $L^p$-norm of $f$ as

$$
\|f\|_{L^p(\mathbb{T}^d)} = \|f\|_{L^p(\mathbb{T}^d)} = \|f\|_p.
$$

If $(a_j)_{j \in I}$ is a sequence of complex numbers and $I$ is the index set, we denote the discrete $l^p$-norm as

$$
\|(a_j)\|_{l^p(I)} = \|(a_j)\|_{l^p(I)} = \left\{ \left( \sum_{j \in I} |a_j|^p \right)^{1/p} \right\}, \quad 0 < p < \infty, \quad p = \infty.
$$

For example, $\|\hat{f}(k)\|_{l^p(\mathbb{Z}^d)} = \left( \sum_{k \in \mathbb{Z}^d} |\hat{f}(k)|^2 \right)^{1/2}$. If $f = (f_1, \cdots, f_m)$ is a vector-valued function, we denote $|f| = \sqrt{\sum_{j=1}^m |f_j|^2}$, and $\|f\|_p = \|\sum_{j=1}^m f_j^2\|_p$. We use similar convention for the corresponding discrete $l^p$ norms for the vector-valued case.

We use the following convention for the Fourier transform pair:

$$
\hat{f}(k) = \int_{\mathbb{T}^d} f(x) e^{-ik \cdot x} dx, \quad f(x) = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} \hat{f}(k) e^{ik \cdot x},
$$

and denote for $0 \leq s \in \mathbb{R}$,

$$
\|f\|_{H^s} = \|f\|_{H^s(\mathbb{T}^d)} = \|\nabla^s f\|_{L^2(\mathbb{T}^d)} \sim \|k|^s \hat{f}(k)\|_{l^2(\mathbb{Z}^d)}, \quad (2.3a)
$$

$$
\|f\|_{H^s} = \sqrt{\|f\|_2^2 + \|f\|_{H^s}^2} \sim \|k|^s \hat{f}(k)\|_{l^2(\mathbb{Z}^d)}, \quad (2.3b)
$$

Proof of Proposition 1.1. To prove Proposition 1.1 it suffices for us to examine the following statement. Consider $u = e^{\partial_t + \delta_0}$ where $\delta_0$ is the periodic Dirac comb on $\mathbb{T}$. Let $0 < \tau \ll 1$ and consider

$$
u = w(1 + \tau w^2)^{-\frac{1}{2}}.
$$
Claim: $e^{-c_0\partial_x^2}w \notin L^2$ for any $c_0 > 0$.

Proof of Claim. Observe that the Fourier coefficients of $u$ are all positive and

$$\hat{u}(k) \geq \tau^m \frac{w^{2m+1}(k)}{k}.$$  \hspace{1cm} (2.5)

Clearly the desired conclusion follows. □

3 Proof of Theorem 1.1

Lemma 3.1 (One-step $H^k$ stability). Let $\nu > 0$, $M \geq 2$ and assume the spectral condition $\tau M^{-2} \leq l_0$ for some constant $l_0 > 0$. Suppose $a \in H^k(\mathbb{T})$, $k \geq 1$ and $\|a\|_{H^k} \leq A_0$. Let $0 \leq y_1 \leq y_2 < \infty$. There exists $\tau_1 = \tau_1(\nu, k, y_1, y_2, A_0, l_0) > 0$ sufficiently small such that if $0 < \tau \leq \tau_1$, then

$$\|\Pi_M S_L(-y_1\tau)S_N(\tau)S_L(y_2\tau)a\|_{H^k} \leq e^{c_1\tau}\|a\|_{H^k},$$

$$\|\Pi_M S_L(-y_1\tau)S_N(-\tau)S_L(y_2\tau)a\|_{H^k} \leq e^{c_1\tau}\|a\|_{H^k},$$

$$\|\Pi_M S_L(y_2\tau)S_N(\tau)\Pi_M S_L(-y_1\tau)a\|_{H^k} \leq e^{c_1\tau}\|a\|_{H^k},$$

$$\|\Pi_M S_L(y_2\tau)S_N(-\tau)\Pi_M S_L(-y_1\tau)a\|_{H^k} \leq e^{c_1\tau}\|a\|_{H^k},$$  \hspace{1cm} (3.1)

where $c_1 > 0$ depends on $(\nu, k, y_1, y_2, A_0, l_0)$.

Remark 3.1. Define

$$T_1 a = \Pi_M S_L(-y_1\tau)S_N(\tau)S_L(y_2\tau);$$  \hspace{1cm} (3.2)

$$T_2 a = \Pi_M S_L(-y_1\tau)S_N(-\tau)S_L(y_2\tau)a;$$  \hspace{1cm} (3.3)

$$T_3 a = \Pi_M S_L(y_2\tau)S_N(\tau)\Pi_M S_L(-y_1\tau)a;$$  \hspace{1cm} (3.4)

$$T_4 a = \Pi_M S_L(y_2\tau)S_N(-\tau)\Pi_M S_L(-y_1\tau)a.$$  \hspace{1cm} (3.5)

Later we shall apply Lemma 3.1 $n$-times. In particular we need to bound the expression

$$\|T_{i_1} \cdots T_{i_n} a\|_{H^k}.$$  \hspace{1cm} (3.6)

Since the constant $c_1$ depends on the $H^k$-norm of the iterates, it is of importance to give uniform control of the $H^k$-norm. To resolve this issue, we first choose $A_0, \tau_1$ and $c_1$ such that

$$\|a\|_{H^k} \leq \frac{1}{10}A_0, \quad \tau_1 = \tau_1(\nu, k, y_1, y_2, k, A_0, l_0), \quad c_1 = c_1(\nu, k, y_1, y_2, A_0, l_0).$$  \hspace{1cm} (3.7)

We choose $n$ such that

$$n\tau \leq \frac{1}{c_1}.$$  \hspace{1cm} (3.8)

Clearly in the course of iteration, the $H^k$-norm of the iterates never exceeds $A_0$. In particular

$$\|T_{i_1} \cdots T_{i_n} a\|_{H^k} \leq e^{c_1n\tau}\|a\|_{H^k} \leq e \cdot \frac{1}{10}A_0 \leq A_0.$$  \hspace{1cm} (3.9)
Proof of Lemma 3.1. We begin by noting that for the ODE
\[ \partial_t w = \pm w \pm w^3; \] (3.10)
we have the estimate (note that \( H^k(\mathbb{T}) \) is an algebra for \( k \geq 1 \))
\[ \frac{d}{dt} \| w \|_{H^k} \lesssim \| w \|_{H^k} + \| w \|_{H^k}^3. \] (3.11)

It follows that for \( \tau > 0 \) sufficiently small,
\[ \sup_{0 \leq t \leq \tau} \| S_N(\pm t)b \|_{H^k} \leq \| b \|_{H^k}; \] (3.12)

\[ \| S_N(\tau)b - b \|_{H^k} \leq \tau \cdot O(\| b \|_{H^k} + \| b \|_{H^k}^3). \] (3.13)

The desired estimates then easily follows from the above using the spectral condition. \( \square \)

Lemma 3.2 (Multi-step \( H^1 \) stability and higher regularity). Let \( \nu > 0 \), \( M \geq 2 \) and assume the spectral condition \( \tau M^{-2} \leq l_0 \) for some constant \( l_0 > 0 \). Suppose \( a \in H^1(\mathbb{T}) \) and \( \| a \|_{H^1} \leq B_1 \) for some constant \( B_1 > 0 \). Define
\[ S^{(2)}(\tau) = \Pi_M S_L(\frac{\tau}{2}) S_N(\tau) \Pi_M S_L(\frac{\tau}{2}); \]
\[ S^{(4)}(\tau) = S^{(2)}(x_1\tau) S^{(2)}(-x_0\tau) S^{(2)}(x_1\tau), \] (3.14)
where \( x_0 = \frac{2^{1}}{2-2^2} \approx 1.7, x_1 = \frac{1}{2-2^2} \approx 1.35 \). For \( n \geq 1 \), define
\[ u^n = S^{(4)}(\tau) u^{n-1}, \] (3.15)
where \( u^0 = a \). There exist \( c_1 = c_1(\nu, \ell_0, B_1) > 0 \) and \( \tau_2 = \tau_2(\nu, \ell_0, B_1) > 0 \) such that if \( 0 < \tau \leq \tau_2 \) (we may assume \( c_1 \tau_2 \leq 0.01 \)), then
\[ \| u^n \|_{H^1} \leq e^{c_1 \tau} \| u^{n-1} \|_{H^1}, \quad 1 \leq n \leq \frac{1}{c_1 \tau}; \]
\[ \sup_{1 \leq n \leq \frac{1}{c_1 \tau}} \| u^n \|_{H^1} \leq 3B_1; \]
\[ \sup_{\frac{10c_1 \tau \leq n \leq \frac{1}{c_1 \tau}}{10c_1 \tau \leq n \leq \frac{1}{c_1 \tau}}} \| u^n \|_{H^80} \leq B_2, \] (3.16)
where \( B_2 > 0 \) depends on \( (\nu, \ell_0, B_1) \).

Proof of Lemma 3.2. Observe that
\[ S^{(4)}(\tau) = \Pi_M S_L(\frac{x_1}{2}\tau) S_N(x_1\tau) \Pi_M S_L(-\frac{x_0 - x_1}{2}\tau) S_N(-x_0\tau) \Pi_M S_L(-\frac{x_0 - x_1}{2}\tau) S_N(x_1\tau) S_L(\frac{x_1}{2}\tau) \]
\[ = \Pi_M S_L(\frac{x_1}{2}\tau) S_N(x_1\tau) S_L(-\frac{x_1 - \epsilon_1}{2}\tau) \Pi_M S_L(\frac{2x_1 - x_0 - \epsilon_1}{2}\tau) S_N(-x_0\tau) S_L(-\frac{x_0 - x_1}{4}\tau) \]
\[ \quad =: T_A \]
\[ \Pi_M S_L(-\frac{x_0 - x_1}{4}\tau) S_N(x_1\tau) S_L(\frac{x_1}{2}\tau), \]
\[ =: T_C. \] (3.17)
where $\epsilon_1 = 0.01$. Clearly the operators $T_A$, $T_B$, $T_C$ fulfill the conditions of Lemma 3.1. The $H^1$ estimates then follow easily from the computations outlined in Remark 3.1 with some necessary adjustment of the constants.

To establish (3.16), we note that for $k \geq 1$,

$$\|S_N(\tau) f - f\|_{H^k} \lesssim O(\tau)\|f\|_{H^k},$$

provided $\tau \|f\|_{H^1}^2 \ll 1$. We then rewrite $S^{(4)}(\tau)f$ as

$$S^{(4)}(\tau)f = S_L((2x_1 - x_0)\tau)f + \tau \tilde{f} = S_L(\tau)f + \tau \tilde{f},$$

where $\|\tilde{f}\|_{H^1} \lesssim \|f\|_{H^1}$. One can then bootstrap the higher regularity from this. We omit further details.

**Remark 3.2.** It is not difficult to check that under the assumption of uniform high Sobolev regularity, we have (below we assume $f = \Pi_M f$)

$$S^{(4)}(\tau)f = S_L((f + \tau(f - \Pi_M(f^3))) + O(\tau^2)$$

$$= (1 - \nu \tau \partial_{xx})^{-1}f + \tau(1 - \nu \tau \partial_{xx})^{-1}(f - \Pi_M(f^3)) + O(\tau^2).$$

It follows that

$$E(S^{(4)}(\tau)f) = E(\tilde{S}(\tau)f) + O(\tau^2).$$

**Lemma 3.3** (Control of the energy flux). Let $\nu > 0$ and $M \geq 2$. Suppose $f \in H^2(\mathbb{T})$ satisfies $f = \Pi_M f$ and

$$\|\nu \partial_{xx}f - \Pi_M(f^3) + f\|_2 \leq 1.$$  \hspace{1cm} (3.22)

Then

$$\|f\|_{H^8(\mathbb{T})} \leq C_{\nu}^{(o)},$$  \hspace{1cm} (3.23)

where $C_{\nu}^{(o)} > 0$ depends only on $\nu$. Furthermore if $\tau \leq l_0 M^{-2}$ and $0 < \tau \leq \tau^{(0)}(\nu, l_0)$ where $\tau^{(0)}(\nu, l_0) > 0$ is a sufficiently small constant depending on $\nu, l_0$), then

$$E(S^{(4)}(\tau)f) \leq C_{\nu, l_0}^{(U)},$$  \hspace{1cm} (3.24)

where $C_{\nu, l_0}^{(U)} > 0$ depends only on $(\nu, l_0)$.

**Proof.** The estimate (3.23) follows from energy estimates using (3.22). Note that the condition $f = \Pi_M f$ is used in the identity $\int \Pi_M(f^3)f\,dx = \int f^4\,dx$. The estimate (3.24) follows from (3.23) and Lemma 3.1.
Lemma 3.4 (One-step strict energy dissipation with nontrivial energy flux). Let $\nu > 0$, $M \geq 2$ and $0 < \tau \leq M^{-2} l_0$. Suppose $f \in H^{80}(\mathbb{T})$ with $f = \Pi_M f$ and satisfies
\begin{align*}
\|\nu \partial_{xx}f - \Pi_M (f^3) + f\|_2 \geq 1, \\
\|f\|_{H^{80}(\mathbb{T})} \leq B_0 < \infty,
\end{align*}
where $B_0 > 0$ is a given constant. There exists $\tau_3 = \tau_3(\nu, l_0, B_0) > 0$ sufficiently small such that if $0 < \tau \leq \tau_3$, then
\begin{equation}
E(S^{(4)}(\tau)f) < E(f).
\end{equation}

Proof of Lemma 3.4. By using Remark 3.2, it suffices for us to show
\begin{equation}
E(\tilde{S}(\tau)f) + c_1 \tau \leq E(f),
\end{equation}
for some $c_1 > 0$. Denote $w = \tilde{S}(\tau)f$ and observe that
\begin{equation}
\frac{w - f}{\tau} = \nu \partial_{xx}w + f - \Pi_M (f^3).
\end{equation}

It follows that for $\tau > 0$ sufficiently small,
\begin{equation}
E(w) - E(f) + \frac{c_2}{\tau} \|w - f\|_2^2 \leq 0,
\end{equation}
where $c_2 > 0$ is a constant. Now we clearly have
\begin{equation}
\frac{1}{\tau} - \nu \partial_{xx} (w - f) = \nu \partial_{xx} f + f - \Pi_M (f^3).
\end{equation}
The desired result clearly follows.

Theorem 3.1. Let $\nu > 0$, $M \geq 2$ and $0 < \tau \leq l_0 M^{-2}$. Assume $a \in H^1(\mathbb{T})$ and $\|a\|_{H^1} \leq \gamma_1$. Define $u^0 = \Pi_M a$ and
\begin{equation}
u u^{n+1} = S^{(4)}(\tau) u^n, \quad n \geq 0.
\end{equation}

There exists $\tau_* = \tau_*(\nu, l_0, \gamma_1) > 0$ sufficiently small such that if $0 < \tau \leq \tau_*$, then
\begin{equation}
\sup_{n \geq 1} \|u^n\|_{H^1(\mathbb{T})} \leq F^{(0)}_{\nu, l_0, \gamma_1},
\end{equation}
where $F^{(0)}_{\nu, l_0, \gamma_1} > 0$ depends only on $(\nu, l_0, \gamma_1)$.

Proof of Theorem 3.1. Denote
\begin{equation}
G = 10(1 + \gamma_1 + C^{(U)}_{\nu, l_0} + C^{(o)}_\nu),
\end{equation}

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where the constants $C_{\nu,\gamma_1}^{(U)}, C_{\nu}^{(o)}$ are the same as in Lemma 3.3 In the argument below we shall assume $\tau > 0$ is sufficiently small. The needed smallness (i.e. the existence of $\tau_*$) can be easily worked out by fulfilling the conditions needed in Lemma 3.2, Lemma 3.3 and Lemma 3.4.

By Lemma 3.2 we can find $c_1 = c_1(\nu, l_0, G) > 0$ such that

$$\sup_{1 \leq n \leq c_1} E(u^n) \leq \frac{1}{2} G. \quad (3.34)$$

Claim: We have

$$\sup_{n \geq c_1} E(u^n) \leq G. \quad (3.35)$$

To prove the claim we argue by contradiction. Suppose $n_0 \geq \frac{c_1}{\tau}$ is the first integer such that

$$E(u^{n_0}) \leq G, \quad E(u^{n_0+1}) > G. \quad (3.36)$$

By Lemma 3.3 we must have

$$\|\nu \partial u^{n_0} - \Pi_M ((u^{n_0})^3) + u^{n_0}\|_2 > 1. \quad (3.37)$$

Since $n_0 \geq \frac{c_1}{\tau}$, we have $E(u^{n_0-j_0}) \leq G$ for some integer $\frac{n_0}{10\tau} \leq j_0 \leq \frac{n_0}{10\tau} + 2$. By using smoothing estimates we obtain

$$\|u^{n_0}\|_{H^8(\Omega)} \leq C_{\nu,\gamma_0,G}; \quad (3.38)$$

where $C_{\nu,\gamma_0,G} > 0$ depends on $(\nu, l_0, G)$. Since $G$ depends on $(\nu, l_0, \gamma_1)$, we have $C_{\nu,\gamma_0,G}$ depends only on $(\nu, l_0, \gamma_1)$. By (3.37), (3.38) and Lemma 3.4 we obtain for sufficiently small $\tau$ that

$$E(u^{n_0+1}) < E(u^{n_0}) \quad (3.39)$$

which is clearly a contradiction to (3.36). Thus we have proved the claim. \(\square\)

Proof of Theorem 1.1. The $H^1$ estimate follows from Theorem 3.1. Higher order estimates follow from the smoothing estimates. \(\square\)

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