Chaos in Hamiltonians with a Cosmological Time Dependence

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This paper summarises a numerical investigation of how the usual manifestations of chaos and regularity for flows in time-independent Hamiltonians can be altered by a systematic time-dependence of the form arising naturally in an expanding Universe. If the time-dependence is not too strong, the observed behaviour can be understood in an adiabatic approximation. One still infers sharp distinctions between regular and chaotic behaviour, even though “regular” does not mean “periodic” and “chaotic” will not in general imply strictly exponential sensitivity towards small changes in initial conditions. However, these distinctions are no longer absolute, as it is possible for a single orbit to change from regular to chaotic and/or vice versa. If the time-dependence becomes too strong, the distinction between regular and chaotic can disappear so that no sensitive dependence on initial conditions is manifest.

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I. MOTIVATION

In time-independent Hamiltonian systems sharp, qualitative distinctions can be made between two different types of behaviour, namely regular and chaotic. Regular orbits are multiply periodic; chaotic orbits are aperiodic. Chaotic orbits exhibit an exponentially sensitive dependence on initial conditions, which is manifested by the existence of at least one positive Lyapunov exponent; regular orbits exhibit at most a power law dependence on initial conditions. This distinction is, moreover, absolute in the sense that it holds for all times: an orbit that starts chaotic will remain chaotic forever; a regular orbit remains regular. This distinction is important physically because it implies very different sorts of behaviour (although topological obstructions like cantori can make a chaotic orbit “nearly regular” for very long times).

An obvious question then is to what extent this distinction persists in cosmology where, oftentimes, one is confronted with a Hamiltonian manifesting a systematic time-dependence reflecting the expansion of the Universe. If the Hamiltonian acquires an explicit secular time-dependence, one would expect that periodic orbits no longer exist; and one might anticipate further that “chaotic” need not imply a strictly exponential dependence on initial conditions. Given, moreover, that the form of the Hamiltonian could change significantly over the course of time, one might anticipate the possibility that a single orbit could shift in behaviour from “chaotic” to “regular” and/or vice versa. In other words, the distinction between regular and chaotic need not be absolute.

Of especial interest is what happens to the gravitational N-body problem for a collection of particles of comparable mass in the context of an expanding Universe. It is by now well known that, when formulated for a system of compact support which exhibits no systematic expansion, the N-body problem is chaotic in the sense that small changes in initial conditions tend to grow exponentially. Largely independent of the details (at least for $N \gg 1$), a small initial perturbation typically grows exponentially on a time scale $t_\ast \sim R/v$ comparable to the natural crossing time for the system. Does this exponential instability persist in the context of an expanding Universe, or does the expansion vitiate the chaos?

To better understand various phenomena in the early Universe, attention has also focused on the behaviour of systems modeled as a small number of interacting low frequency modes (and, perhaps, an external environment, typically visualised as a stochastic bath). The resulting description is usually Hamiltonian (albeit possibly perturbed by the external environment), but, because of the expansion of the Universe, which induces a systematic redshifting of the modes, the Hamiltonian is typically time-dependent. Were this time-dependence completely ignored, as might be appropriate in flat space, the resulting solutions would divide naturally into “regular” and “chaotic.” The obvious question is to what extent such appellations continue to make sense when one allows properly for an expanding Universe? One might be concerned that such systems are intrinsically quantum, and that there is no such thing as “quantum chaos”. However, at least in flat space classical distinctions between regular and chaotic behaviour are typically manifested in the semi-classical behaviour of true quantum systems, so that one might anticipate that any diminution of chaos associated with an expanding Universe could have important implications for such phenomena as decoherence and the classical-to-quantum transition.

Section II discusses the onset of chaos in time-independent Hamiltonian systems as resulting from parameteric instability and, by generalising the discussion to time-dependent Hamiltonians, makes specific predictions as to how chaos should be manifested in the context of an expanding Universe. Section III confirms and extends these predictions with numerical simulations performed for a simple class of models, namely two-
three-degree-of-freedom generalisations of the dihedral potential \[\text{[3]}\]. Section IV concludes with speculations on
the implications of these results for real systems in the early Universe.

II. CHAOS AND PARAMETRIC INSTABILITY

To facilitate sensible predictions as to the manifestations of chaos in time-dependent Hamiltonian systems, it is worth recalling why, in a time-independent Hamiltonian system, chaos implies an exponential dependence
on initial conditions.

A time-independent Hamiltonian of the form

\[H(r, p) = p^2/2 + V(r)\]  
leads immediately to the evolution equation

\[\frac{d^2 r^a}{dt^2} = -\frac{\partial V(r)}{\partial r^a}. \]  

Whether or not an orbit generated as a solution to this equation is chaotic depends on how the orbit responds
everywhere positive.

To understand whether or not the orbit is chaotic is formalised in the average co-moving frame, the evolution equation for a particle moving in the Matthieu plane, but a generic curve in this plane will typically intersect both stable and unstable regions, i.e., regions
where \(\delta r^A\) remains bounded and regions where \(\delta r^A\) grows exponentially.

Eq. (6) provides a simple way to understand physically, if one probes a curve of initial conditions in the phase space of some Hamiltonian system, one finds generically that that curve decomposes into disjoint regular and chaotic regions. Moving along the phase space curve corresponds to motion through the \(\alpha - \beta\) Matthieu plane, but a generic curve in this plane will typically intersect both stable and unstable regions, i.e., regions
where \(\delta r^A\) remains bounded and regions where \(\delta r^A\) grows exponentially.

Given this logic, it is easy to see what changes might arise if the unperturbed evolution equation is altered to allow for the expansion of the Universe. Suppose that one is considering a Universe idealised as a spatially flat Friedmann cosmology, for which

\[ds^2 = -dt^2 + a^2(t)\delta_{ab}dx^a dx^b.\]

The natural starting point for most field theories is then an action

\[S = \int d^4 x a^3(t) \left[ (\partial_r \Phi)^2 - a^{-2} \delta^{ab} \partial_a \Phi \partial_b \Phi - V(\Phi) \right].\]

which leads to mode equations of the form

\[\frac{d^2 \phi_k}{dt^2} + 3 \frac{\dot{a}}{a} \frac{d\phi_k}{dt} = -\frac{k^2}{a^2} \phi_k - \frac{\partial V}{\partial \phi_k}.\]

Similarly, when formulated in the average co-moving frame, the evolution equation for a particle moving in a fixed potential yields a proper peculiar velocity satisfying

\[\frac{d v^a}{dt} + \frac{\dot{a}}{a} v^a = -\frac{\partial V(r, a(t))}{\partial r^a}.\]

In either case, eq. (2) has been changed in two significant ways, namely through the introduction of a frame-dragging term \(\propto \dot{a}/a\) and the explicit time-dependence now entering into the right hand side. The frame-dragging contribution contribution can always be scaled away by a redefinition of the basic field variable, but the time-dependence on the right hand side cannot in general

be eliminated. It follows that, generically, eq. (2) will be replaced by
\[ \frac{d^2 r^a}{dt^2} = -\frac{\partial V(r, a(t))}{\partial r^a}. \] (11)

The simplest case arises when the time-dependence enters only as an overall multiplicative factor, so that
\[ \frac{d^2 r^a}{dt^2} = -R[a(t)] \frac{\partial V(r)}{\partial r^a}. \] (12)

To the extent that perturbations of a solution to eq. (2) can be represented qualitatively by the Matthieu eq. (6), it is not unreasonable to suppose that perturbations of its time-dependent generalisation (12) can be represented by a generalised Matthieu equation of the form
\[ \frac{d^2 \delta r^A}{dt^2} = -R(t) (\alpha + \beta \cos 2t) \delta r^A. \] (13)

or, perhaps, some generalisation thereof in which $2t$ is replaced by some $T(t)$.

The qualitative character of the solutions to this equation are easy to predict theoretically and corroborate numerically, at least when the time-dependence of $R$ is not too strong and this time variability can be treated in an adiabatic approximtion: There is still a sharp distinction between stable and unstable behaviour, but “unstable” does not in general correspond to strictly exponential growth. For the true Matthieu equation (6), unstable solutions do grow exponentially, with $\delta r^A \propto \exp(\omega t)$, but allowing for a nontrivial $R(t)$ leads instead to solutions of the form
\[ \delta r^A \sim \exp\left[ \int R^{1/2}(t) \omega dt \right] \] (14)

If, e.g., $R(t) \propto t^p$, chaos should correspond to the existence of perturbations that grow as
\[ \delta r^A \sim \exp[t^{1+p/2}]. \] (15)

In other words, the evolution of $\delta r^A$ will be sub- or super-exponential.

Two other points should also be clear. First and foremost, the distinction between regular and chaotic should no longer be absolute. Equation (13) can be interpreted as an ordinary Matthieu equation with time-dependent “constants” $\tilde{\alpha} = R^{1/2} \alpha$ and $\tilde{\beta} = R^{1/2} \beta$. However, the fact that these “constants” change in time means that the equation satisfied by $\delta r^A$ is drifting through the $\alpha-\beta$ plane, so that the solutions can drift into and/or out of resonance. In other words, the behaviour of a small perturbation can in principle change from stable to unstable and/or visa versa, which corresponds to the possibility of transitions between regular and chaotic behaviour.

The other point is that, if the time-dependence is too strong, the adiabatic approximation could fail and the expected behaviour could be quite different. In particular, for $R \propto t^{-p}$, with $p \to -2$, eq. (15) implies that the subexponential evolution expected for $p$ somewhat smaller than zero will degenerate into a simple power law evolution where, seemingly, all hints of chaos are lost.

An important question would seem to be how negative $p$ must become before the distinctions between chaos and regularity are completely obliterated.

Corroboration of the behaviour predicted in eq. (15) and a partial answer to this last question are provided by the numerical computations described in the following section.

### III. NUMERICAL SIMULATIONS

The experiments described here were performed for time-dependent extensions of the potential
\[ V_0(x, y, z) = -(x^2 + y^2 + z^2) + \frac{1}{4} (x^2 + y^2 + z^2)^2 \]
\[ - \frac{1}{4} (y^2 z^2 + b z^2 x^2 + c x^2 y^2), \] (16)

with variable parameters $b$ and $c$ of order unity, which constitutes a three-dimensional analogue of the so-called dihedral potential [3]. For $b = c = 1$, the case treated in greatest detail, this corresponds to a slightly cubed Mexican hat potential. At high energies, the potential is essentially quartic, although the quadratic couplings ensure that there are significant measures of both regular and chaotic orbits. At relatively low energies, somewhat less than zero, orbits are confined to a three-dimensional trough where, qualitatively, they behave like orbits in the “stadium problem,” scattering off the walls in a fashion that renders them largely chaotic.

Some of the experiments involved allowing for a time-dependence of the form
\[ V(x, y, z, t) = R(t) V_0(x, y, z) \] (17)

with $R(t) \propto t^p$. Others involved the alternative time-dependence
\[ V(x, y, z, t) = V_0[R(t)t, R(t)y, R(t)z], \] (18)

again with $R(t) \propto t^p$.

Individual sets of experiments involved freezing the energy at some fixed value, usually $E[V_0] = 10.0$, and selecting an ensemble of some 1000 to 5000 initial conditions. Three-degree-of-freedom initial conditions were generated by setting $x = 0$, $z = \text{const}$, uniformly sampling the energetically allowed regions of the $y-z-p_y-p_z$ hypercube, and then solving for $p_x = p_x(y, z, p_y, p_z, E)$. Two-degree-of-freedom initial conditions were generated by setting $x = z = p_z = 0$, uniformly sampling the energetically allowed regions of $y - p_y$ plane, and solving for $p_x = p_x(y, p_y, E)$. The computations were started at $t = 1$, $t = 10$, or $t = 100$ and ran for a total time $T = 256$ or $T = 512$, this corresponding in the absence of
any time-dependence to $100 - 200$ characteristic crossing times. The orbital data were recorded at intervals $\Delta t = 0.25$. An estimate of the largest short time Lyapunov exponent was computed in the usual way by simultaneously evolving $\delta x = 1.0 \times 10^{-8}$ which was periodically renormalised at intervals $\Delta t = 1.0$ to keep the perturbation small. This led for each orbit to a numerical approximation to the quantity

$$\chi(t) = \lim_{\delta Z(0) \to 0} \frac{1}{t} \ln \left( \frac{||\delta Z(t)||}{||\delta Z(0)||} \right)$$

(19)

for a phase space perturbation $\delta Z = (\delta r)^2 + (\delta p)^2)^{1/2}$. Since interest focuses on the probability that “chaos” does not correspond to a purely exponential growth, the data were reinterpreted by partitioning the Lyapunov data into bits of length $\Delta t = 1.0$, each of which probed the growth of the perturbation during the interval $\Delta t$:

$$\chi(\Delta t_i) = \frac{\chi(t_i + \Delta t_i)(t_i + \Delta t_i) - \chi(t_i)t_i}{\Delta t}.$$  

(20)

These bits were recombined to generate partial sums

$$\xi(t_i) = \frac{1}{\Delta t} \sum_{j=1}^{i-1} \chi(\Delta t_j) = \frac{1}{\Delta t} \ln \left( \frac{||\delta Z(t_i + t_0)||}{||\delta Z(t_0)||} \right),$$

(21)

which capture the net growth of the logarithm of the perturbation. These partial sums were then fit to a polynomial growth law

$$\xi = A + Bt^q$$

(22)

A purely exponential growth would yield $q = 1$; sub- and super-exponential growth correspond, respectively, to $q < 1$ and $q > 1$. If the perturbation only grows as a power law, eq. (22) should not yield a reasonable fit.

The principal conclusion of the computations is that, overall, the adiabatic approximation works very well so that, for a broad range of values of $p$ in the potential eq. (17), eq. (15) appears satisfied. This is illustrated in Fig. 1 (a), which exhibits the best fit exponent $q$ as a function of $p$. The data in this plot combine experiments for both two- and three-degree-of-freedom chaotic orbits which allowed for several different values of $b$ and $c$. Changing the values of $b$ and/or $c$ can significantly change the rate at which a small initial perturbation grows, thus altering the characteristic value of $B$ in eq. (22), but the exponent $q$ seems independent of these details.

As a concrete example, consider the special case of two-degree-of-freedom orbits. For $p = 0$, i.e., no time-dependence, initial conditions with $E[V_0] = 10.0$ evolved in (17) exhibit a clean split into regular and chaotic, with some $72\%$ of the orbits chaotic and the remaining regular. Moreover, for this initial energy cantori are relatively unimportant, so that there are few “sticky” chaotic orbits trapped near regular islands. $N[\xi]$, the distribution of the final values of $\xi$, is strongly bimodal, and it is almost always easy to distinguish visually between regularity and chaos. For the chaotic orbits one finds (as must be the case) excellent agreement with eq. (15).

As $p$ increases to assume small positive values, the final $\xi$’s continue to yield a bimodal distibution, $N[\xi]$, which indicates that, in terms of their sensitive dependence on initial conditions, the orbits still divide into two distinct classes. However, there is a systematic decrease in the abundance of regular orbits, so much so that, for $p > 0.3$, there are few if any regular orbits. Moreover, a detailed examination of individual orbits show that they can exhibit abrupt changes in behaviour between chaotic intervals where $\xi$ grows as $t^{1+p/2}$ and regular intervals where $\xi$ exhibits little if any growth. For $0.3 < p < 0.6$ (almost) every orbit in the evolved ensembles of initial conditions seems a member of a single chaotic population with $q = 1 + p/2$.

However, for larger values of $p$ orbits which began as chaotic subsequently exhibit abrupt transitions to regularity and remain regular for the duration of the integration. This behaviour reflects that fact that, at sufficiently late times, orbits with $p > 0$ become trapped in one of the four global minima of the potential with $V_b(x_0, y_0) = -4/3$, located at $x_0 = \pm \sqrt{4/3}, y_0 = \pm \sqrt{4/3}$, and oscillates in what is essentially an integrable quadratic potential. Indeed, when the orbit becomes trapped close enough to one of these minima, so that $V$ is strictly negative, the orbit quickly loses energy until it comes to sit almost exactly at rest. This implies that, eventually, $\xi$ decreases with time, this reflecting the fact that all orbits in the given basin of attraction are driven towards the same final point.

As $p$ decreases from zero to somewhat negative values, two more or less distinct populations again appear to persist, at least initially, although now the relative abundance of chaotic orbits decreases rapidly. For relatively small values of $p$, say $p < -0.5$ or so, $\xi$ grows so slowly that the chaotic and regular contributions to $N[\xi]$ exhibit some considerable overlap, and it is no longer easy in every case to distinguish regular from chaotic. For $p < -1.0$ the relative measure of chaotic orbits appears to be extremely small, and for $p < -1.8$ it is unclear whether any “chaotic” orbits exist at all. To the extent that “chaotic orbits” do exist, they are nearly indistinguishable from “regular” orbits.

These general trends are manifested in Figs. 2 (a) - (f), which exhibits the final $N[\xi]$ at $t = 266$ for integrations with, respectively, $p = -1.8$, $p = -0.5$, $p = 0.0$, $p = 0.2$, $p = 0.5$, and $p = 2.5$. Further insights into the behaviour of small perturbations can be inferred from Figs. 3 (a) - (f), which plot $\xi(t)$ for 150 randomly chosen orbits from each of these integrations. This behaviour can be contrasted with what obtains for two-degree-of-freedom orbits for the same potential $V_b$ if one now allows for the time-dependence given by eq. (18). Here once again one finds that changing $p$ leads to sub- or super-exponential sensitivity, and, as is evident
from Fig. 1 (b), that increasing $p$ tends to yield larger values of $q$. Moreover, there is often tangible evidence for two distinct types of orbits, seemingly chaotic and regular. However, the details change appreciably.

In this case, one finds that increasing $p$ from zero to slightly positive values tends to increase the overall abundance of regular orbits, and that even those orbits which seems chaotic overall often exhibit regular intervals during which $\xi$ exhibits little, if any, growth. For values as alrge as $p = 0.4$, it seems that most – albeit not all – of the orbits are regular nearly all the time. However, for somewhat larger values of $p$ the relative importance of chaos again begins to increase, although one continues to observe regular intervals. For sufficiently high values of $p$, one finds that, as for the potential (17), all the orbits eventually get trapped near one of the four minima of the potential, at which point the orbits become completely regular and $\xi$ becomes negative.

Alternatively, if one passes from zero to negative values of $p$, one finds that the relative abundance of regular orbits rapidly decreases. Indeed, for $p < -0.1$ or so there appear to be virtually no regular orbits. However, the chaos is vitiated in the sense that the growth of a small perturbation is decidedly subexponential. Indeed, as $p$ decreases, one quickly reaches a point where the dependence on initial conditions is so weak that, even though it seems reasonable to term the orbits chaotic, that chaos must be viewed as extremely weak.

This behaviour is exhibited in Figs. 4 (a) - (f), which plot $N[\xi]$ at $t = 266$ for integrations with, respectively, $p = -0.6, p = -0.3, p = -0.05, p = 0.5, p = 0.9$ and $p = 1.25$. Figs. 3 (a) - (f) plot $\xi(t)$ for 150 randomly chosen orbits from each of these integrations.

In terms of their sensitive dependence on initial conditions orbits in these time-independent Hamiltonian systems tend to divide relatively cleanly into two distinct classes; and, for the case of a time-dependence of the form given by eq. (17), those that exhibit such a sensitive dependence are well fit overall by the scaling predicted by eq. (15). However, to justify designating these orbit classes “regular” and “chaotic,” it is important to verify that these distinctions also coincide with the general shapes of the orbits, as manifested visually or through the computation of a Fourier transform.

This is in fact easily done. Orbits that are chaotic in terms of their sensitive dependence on initial conditions tendy systematically to look “more irregular” and to have “messier” Fourier spectra than do those which do not manifest a sensitive dependence on initial conditions. For time-independent Hamiltonian systems, one can distinguish between regular and chaotic by determining the extent to which most of the power is concentrated in a few special frequencies: in a $t \to \infty$ limit, the power spectrum for a regular orbit will only have support at a countable set of frequencies whereas a chaotic orbit will typically have power for a continuous range of frequencies. However, this is not the case for a time-dependent Hamiltonian system! As the energy changes in time, the characteristic frequencies of an orbit must change so that a long time integration necessarily yields broadband power. However, what is true for a regular orbit is that this power tends to vary comparatively smoothly with frequency.

This can be understood easily in the context of an adiabatic approximation. At any given instant, it makes sense to speak of the two principal frequencies that define (say) a two-degree-of-freedom loop orbit; but, as the energy changes, the values of these two frequencies will change continuously with time. Indeed, the phase space can eventually evolve to the point that an orbit that starts as a loop becomes unstable and is transformed into a chaotic orbit or, perhaps, a different type of regular orbit.

Representative examples of regular and chaotic orbits in time-dependent Hamiltonians are given in Figs. 6 and 7 which, respectively, exhibit data generated for the potential (17) with $p = -0.4$ and the potential (18) with $p = 0.3$. In each Figure, the left hand panels correspond to two-degree-of-freedom chaotic orbits; the right hand sides correspond to two-degree-of-freedom regular orbits. The top panels display the orbits in the $x - y$ plane. The middle panels exhibit the power spectra, $|x(\omega)|$ and $|y(\omega)|$. The bottom panels show the total $\xi(t)$.

### IV. PHYSICAL IMPLICATIONS

At least for the time-dependent potentials described in this paper, it seems possible to make sharp distinctions between regularity and and chaos, even if these distinctions are not absolute: as its energy changes, an orbit can evolve from regular to chaotic and/or visa versa. However, the time-dependence alters the exponential dependence on initial conditions manifested in the absence of any time-dependence, yielding instead a sub- or superexponential sensitivity. For the models described in Section III, $p < 0$ corresponds physically to an expanding Universe; and, as such, the computations corroborate the physical expectation that the expansion should suppress chaos. Whether or not in the real Universe this expansion is strong enough to kill the chaos completely depends on the details of the assumed behaviour of both the scale factor and the dynamical model.

But why should one care? Why would the presence or absence of chaos matter? Perhaps the most important implication of chaos for the bulk evolution of self-gravitating systems is its potential role in violent relaxation, the coarse-grained evolution of a non-dissipative system towards a well-mixed state. As formulated originally by Lynden-Bell, violent relaxation is a phase mixing process whereby a generic blob of collisionless matter, be it localised in space or characterised by any other phase space distribution, tends to disperse until it approaches some equilibrium or steady state. The cru-
cial question is one of efficiency. How quickly, and how efficiently, will the matter disperse? At a given level of resolution, how long must one wait before the matter constitutes a reasonable sampling of a near-equilibrium? The important point, then, is that the answer to these questions depends crucially on whether the flow be regular or chaotic. If, in the absence of an expanding Universe, the matter executes regular, nonchaotic trajectories, the approach towards equilibrium will be comparatively inefficient. As probed by a coarse-grained distribution function and/or lower order phase space moments, there is a coarse-grained evolution towards equilibrium, but it proceeds only as a power law in time. If, however, the matter executes chaotic orbits, this approach will be exponential in time, proceeding at a rate that correlates with the values of the positive Lyapunov exponents.

The obvious inference is that to the extent that the expansion of the Universe weakens chaos, it should also weaken this chaotic mixing. This suggests that phase mixing will be a comparatively inefficient mechanism at early times, when the dynamics is dominated by the overall expansion, and that it can only begin to play an important role at later times, once an overdense region has “pinched off” from the overall expansion and begun to evolve more or less independently.

ACKNOWLEDGMENTS

The author would like to acknowledge arguments with himself, most of which he lost.

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[14] The regular examples in Figs. 6 and 7 both correspond to loop orbits, which are characterised by a net sense of rotation. However, with or without a time-dependence these potentials also admit large numbers of so called box orbits which exhibit no net sense of rotation and which, topologically, resemble Lissajous figures.

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[28] FIG. 1. (a) Best fit exponent $q$ for a growth rate $\propto \exp(\alpha t^q)$ for two-degree-of-freedom chaotic orbits evolving in the potential $V = \beta^2 V_0(\rho)$ with $a = b = 1$. The line yields the prediction of an adiabatic approximation. (b) The same, except for the potential $V = V_0(\rho r)$.
FIG. 2. (a) Normalised distribution $N[\xi(t = 522)]$ for two-degree-of-freedom orbits with $a = b = 1$ evolved for the interval $10 < t < 266$ in the potential $V = V_0(t^p r)$ with $p = -1.8$. (b) The same for $p = -0.5$. (c) $p = 0.0$. (d) $p = 0.2$. (e) $p = 0.5$ (f) $p = 2.5$.

FIG. 3. (a) $\xi(t)$ for 150 representative two-degree-of-freedom orbits evolved in the potential $V = t^p V_0(r)$ with $p = -1.8$ (b) The same for $p = -0.5$. (c) $p = 0.0$. (d) $p = 0.2$. (e) $p = 0.5$ (f) $p = 2.5$. 
FIG. 4. (a) Normalised distribution \( N[\xi(t = 522)] \) for two-degree-of-freedom orbits with \( a = b = 1 \) evolved for the interval \( 10 < t < 266 \) in the potential \( V = V_0(t^p r) \) with \( p = -0.6 \). (b) The same for \( p = -0.3 \). (c) \( p = -0.05 \). (d) \( p = 0.5 \). (e) \( p = 0.9 \) (f) \( p = 1.25 \).

FIG. 5. (a) \( \xi(t) \) for 150 representative two-degree-of-freedom orbits evolved in the potential \( V = V_0(t^p r) \) with \( p = -0.6 \) (b) The same for \( p = -0.3 \). (c) \( p = -0.05 \). (d) \( p = 0.3 \). (e) \( p = 0.9 \) (f) \( p = 1.25 \).
FIG. 6. (a) A chaotic two-degree-of-freedom orbit evolved in the potential (17) with $p = -0.4$. (b) A regular orbit evolved with the same value of $p$. (c) The Fourier transformed quantities $10^{-3}|x(\omega)|$ and $10^{-3}|y(\omega)|$ (the latter translated upwards by 0.5) generated from the orbit in (a). (d) The same for the orbit in (b). (e) $\xi(t)$ for the orbit in (a). (f) The same for the orbit in (b).

FIG. 7. (a) A chaotic two-degree-of-freedom orbit evolved in the potential (18) with $p = 0.3$. (b) A regular orbit evolved with the same value of $p$. (c) The Fourier transformed quantities $|x(\omega)|$ and $|y(\omega)|$ (the latter translated upwards by 40) generated from the orbit in (a). (d) The same for the orbit in (b). (e) $\xi(t)$ for the orbit in (a). (f) The same for the orbit in (b).