COMPOSITIONAL ABSTRACTION-BASED SYNTHESIS FOR INTERCONNECTED SYSTEMS: AN APPROXIMATE COMPOSITION APPROACH

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Abstract. In this paper, we consider the problem of abstraction-based controller synthesis for interconnected control systems. In general, the conventional methods for the construction of discrete abstractions and synthesis become computationally expensive due to the state and inputs spaces dimensions while dealing with large interconnected systems. The results in this paper focus on relaxing this issue by providing a compositional framework for the construction of abstractions for interconnected systems. First, we propose a notion of approximate composition which makes it possible to compute an abstraction of the global interconnected system from the abstractions of its components. Second, we propose an incremental procedure for the synthesis of controllers enforcing safety specifications. Finally, we demonstrate the effectiveness of the proposed results on two case studies (viz., DC microgrid and traffic network) by comparing them with different abstraction and controller synthesis schemes.

1. Introduction

Control and verification of dynamical systems using discrete abstractions (a.k.a. symbolic models) and formal methods have been an ongoing research area in recent years (see Tab09 BYG17 and the references therein). In such approaches, a discrete abstraction (i.e., a system with the finite number of states and inputs) is constructed from the original system. When the concrete and abstract systems are related by some relations such as simulation, alternating simulation or their approximate versions, the discrete controller synthesized for the abstraction can be refined into a hybrid controller for the original system. The use of discrete abstractions principally enables the use of techniques developed in the areas of supervisory control of discrete event systems CL09 and algorithmic game theory BJPS+12. The construction of such discrete abstractions is often based on a discretization of the state and input sets. Due to those sets discretization, symbolic control techniques suffer severely from the curse of dimensionality (i.e., the computational complexity for synthesizing abstractions and controllers grows exponentially with the state and input spaces dimension).

To tackle this problem, several compositional approaches were recently proposed. The authors in TI08 proposed a compositional approach for finite-state abstractions of a network of control systems based on the notion of interconnection-compatible approximate bisimulation. The results in PPD16 provide compositional construction of approximately bisimilar finite abstractions for networks of discrete-time control systems under some incremental stability property. In MSSM18, the notion of (approximate) disturbance simulation was used for the compositional synthesis of continuous-time systems, where the states of the neighboring components were modeled as disturbance signals. In ZA17, authors provide compositional abstraction using dissipativity approach. The authors in DT15 KAS17 SGF18a SGF18b SGF19 use contract-based design and assume-guarantee reasoning to provide compositional construction of symbolic controllers.

In this paper, we provide a compositional abstraction-based controller synthesis framework for a composition of $N$ control systems. The main contributions of the work are divided into three parts. First, we introduce a
notion of approximate composition, while the classical exact composition of components requires the inputs and outputs of neighboring components to be equal, we propose a notion of approximate composition allowing the distance between inputs and outputs of neighboring components to be bounded by a given parameter. The proposed notion enables the composition of control systems (possibly of different types) which allows for more flexibility in the design of the overall symbolic model because each component may be suitable for a particular type of abstraction. Second, with the help of the aforementioned notion, we provide results on the compositional construction of abstractions for interconnected systems. Indeed, given a collection of components, where each concrete component is related to its abstraction by an approximate (alternating) simulation relation, we show how the parameter of the composition of the abstractions needs to be chosen in order to ensure an approximate (alternating) simulation relation between the interconnection of concrete components and the interconnection of discrete ones. Third, we propose an incremental procedure for symbolic safety controller synthesis for the composition of \( N \) control systems. Finally, we demonstrate the applicability and effectiveness of the results using two case studies (viz., DC microgrid and traffic network) and compare them with different abstraction and controller synthesis schemes in the literature.

Related works: First attempts to compute compositional abstractions has been proposed for exact simulation relation [Fre05, KvdS10] and simulation maps [TPL04], for which the construction of abstraction exists for restricted class of systems. A first compositionality result using (bi)simulation function have been proposed in [Gir13]. In [TI08], the first approach to provide compositionality result for approximate relationships was proposed using the notion of interconnection-compatible approximate bisimulation. Different approaches have then been proposed recently using small-gain (or relaxed small-gain) like conditions [RZ18, PPD16, NSWZ18, SZ19a] and dissipativity property [ZA17, AZ17, SGZ18]. In [HAT17], a compositional construction of symbolic abstractions was proposed for the class of partially feedback linearizable systems, where the proposed approach relies on the use of a particular type of abstractions proposed in [ZPMT12]. The authors in [KAZ18] present a compositional abstraction procedure for a discrete-time control system by abstracting the interconnection map between different components.

In parallel, other different approaches have been proposed for compositional controller synthesis. In [DT15] the authors propose a compositional approach to deal with persistency specifications using Lyapunov-like functions. The authors in [LFM16] use reachability analysis to provide a compositional controller synthesis for discrete-time switched systems and persistency specifications. In [MGW17, MD18, PPB18, MSSM18] symbolic approaches were proposed for compositional controller synthesis for safety, lasso-shaped, regular language, and more general LTL specifications. All these approaches are based on assume-guarantee reasoning [SGF18c] and generally suffer from the underlying conservatism.

In comparison with existing approaches in the literature, our framework presents the following advantages:

- It allows the use of different types of abstractions for individual components such as abstractions based on state-space quantization [Tab09], partition [MGW15], covering [Rei11], or without any state-space discretization [Gir13];
- We do not need any particular structure of the components such as incremental stability or monotonicity. Moreover, we do not rely on the use of small-gain or dissipativity like conditions;
- The proposed approach allows us to develop an incremental procedure for controller synthesis which helps to reduce the computational complexity while ensuring completeness with respect to the monolithic synthesis.

A preliminary version of this work has been presented in the conference [SJZG18]. The current paper extends the approach in three directions: First, the approach is generalized from cascade interconnections to any composition structure. Second, while in [SJZG18] we showed how to incrementally build a safety controller, in this paper we show also the completeness of this incremental controller with respect to the monolithic one. Third, while in [SJZG18], we only presented a simple numerical example, here the theoretical framework is applied to more realistic case studies: DC microgrids and road traffic networks.
2. Transition Systems and Behavioral Relations

Notations: The symbols \( \mathbb{N}, \mathbb{N}_0, \) and \( \mathbb{R}_0^+ \) denote the set of positive integers, non-negative integers, and non-negative real numbers, respectively. Given sets \( X \) and \( Y \), we denote by \( f : X \to Y \) an ordinary map from \( X \) to \( Y \), whereas \( f : X \Rightarrow Y \) denotes set valued map. For any \( x_1, x_2, x_3 \in X \), the map \( d_X : X \times X \to \mathbb{R}_0^+ \) is a pseudometric if the following conditions hold: (i) \( x_1 = x_2 \) implies \( d_X(x_1, x_2) = 0 \); (ii) \( d_X(x_1, x_2) = d_X(x_2, x_1) \); (iii) \( d_X(x_1, x_3) \leq d_X(x_1, x_2) + d_X(x_2, x_3) \). We identify a relation \( R \subseteq A \times B \) with the map \( R : A \Rightarrow B \) defined by \( b \in R(a) \) if and only if \( (a, b) \in R \). We use notation \( \| \cdot \| \) to denote the infinity norm. The null vector of dimension \( N \in \mathbb{N}_0 \) is denoted by \( 0_N := (0, \ldots, 0)^T \).

First, we introduce the notion of transition systems similar to the one provided in \[Tab09\].

**Definition 2.1.** A transition system is a tuple \( S = (X, X^0, U^\text{ext}, U^\text{int}, \Delta, Y, H) \), where \( X \) is the set of states (possibly infinite), \( X^0 \subseteq X \) is the set of initial states, \( U^\text{ext} \) is the set of external inputs (possibly infinite), \( U^\text{int} \) is the set of internal inputs (possibly infinite), \( \Delta \subseteq X \times U^\text{ext} \times U^\text{int} \times X \) is the transition relation, \( Y \) is the set of outputs, and \( H : X \to Y \) is the output map.

We denote \( x' \in \Delta(x, u^\text{ext}, u^\text{int}) \) as an alternative representation for a transition \( (x, u^\text{ext}, u^\text{int}, x') \in \Delta \), where state \( x' \) is called an \((u^\text{ext}, u^\text{int})\)-successor (or simply successor) of state \( x \), for some input \((u^\text{ext}, u^\text{int}) \in U^\text{ext} \times U^\text{int}\). Given \( x \in X \), the set of enabled (admissible) inputs for \( x \) is denoted by \( U^\text{int} \) and defined as \( U^\text{int}(x) = \{(u^\text{ext}, u^\text{int}) \in U^\text{ext} \times U^\text{int} \mid \Delta(x, u^\text{ext}, u^\text{int}) \neq \emptyset \} \). The transition system is said to be:

- **pseudometric**, if the input sets \( U^i, i \in \{\text{ext}, \text{int}\} \) and the output set \( Y \) are equipped with pseudometrics \( d_U^i : U^i \times U^i \to \mathbb{R}_0^+ \) and \( d_Y : Y \times Y \to \mathbb{R}_0^+ \) respectively.
- **finite** (or symbolic), if sets \( X, U^\text{int} \), and \( U^\text{ext} \) are finite.
- **deterministic**, if there exists at most a \((u^\text{ext}, u^\text{int})\)-successor of \( x \), for any \( x \in X \) and \((u^\text{ext}, u^\text{int}) \in U^\text{ext} \times U^\text{int} \).

In the sequel, we consider the approximate relationship for transition systems based on the notion of approximate (alternating) simulation relation to relate abstractions to concrete systems. We start by introducing the notion of approximate simulation relation adapted from \[JDDBP09\].

**Definition 2.2.** Let \( S_1 = (X_1, X_1^0, U_1^\text{ext}, U_1^\text{int}, \Delta_1, Y_1, H_1) \) and \( S_2 = (X_2, X_2^0, U_2^\text{ext}, U_2^\text{int}, \Delta_2, Y_2, H_2) \) be two transition systems such that \( Y_1 \) and \( Y_2 \) are subsets of the same pseudometric space \( Y \) equipped with a pseudometric \( d_Y \) and \( U^\text{ext} \) (respectively \( U^\text{int} \)), \( j \in \{1, 2\} \), are subsets of the same pseudometric space \( U^\text{ext} \) (respectively \( U^\text{int} \)) equipped with a pseudometric \( d_{U^\text{ext}} \) (respectively \( d_{U^\text{int}} \)). Let \( \varepsilon, \mu \geq 0 \). A relation \( R \subseteq X_1 \times X_2 \) is said to be an \((\varepsilon, \mu)\)-approximate simulation relation from \( S_1 \) to \( S_2 \), if the following hold:

\[
\begin{align*}
(i) & \quad \forall x_1^0 \in X_1^0, \exists x_2^0 \in X_2^0 \text{ such that } (x_1^0, x_2^0) \in R; \\
(ii) & \quad \forall (x_1, x_2) \in R, d_Y(H_1(x_1), H_2(x_2)) \leq \varepsilon; \\
(iii) & \quad \forall (x_1, x_2) \in R, \forall (u_1^\text{ext}, u_1^\text{int}) \in U_1^\text{ext}(x_1), \forall x'_1 \in \Delta_1(x_1, u_1^\text{ext}, u_1^\text{int}), \exists (u_2^\text{ext}, u_2^\text{int}) \in U_2^\text{int}(x_2) \\
& \quad \text{with } \max(d_{U^\text{ext}}(u_1^\text{ext}, u_2^\text{ext}), d_{U^\text{int}}(u_1^\text{int}, u_2^\text{int})) \leq \mu \text{ and } \exists x'_2 \in \Delta_2(x_2, u_2^\text{ext}, u_2^\text{int}) \text{ satisfying } (x'_1, x'_2) \in R.
\end{align*}
\]

We denote the existence of an \((\varepsilon, \mu)\)-approximate simulation relation from \( S_1 \) to \( S_2 \) by \( S_1 \searrow^{\varepsilon, \mu} S_2 \). We can see that when \( \mu = 0 \), we recover the classical notion of approximate simulation relation introduced in \[GP07\] and when \( \mu = \infty \), we get the definition of approximate simulation relation given in \[Tab09\]. Approximate simulation relations are generally used for verification problems. If the objective is to synthesize controllers, the notion of approximate alternating simulation relation introduced in \[Tab09\] is suitable. Interestingly, the notions of approximate simulation and approximate alternating simulation coincide in the case of deterministic transition systems.

**Definition 2.3.** Let \( S_1 = (X_1, X_1^0, U_1^\text{ext}, U_1^\text{int}, \Delta_1, Y_1, H_1) \) and \( S_2 = (X_2, X_2^0, U_2^\text{ext}, U_2^\text{int}, \Delta_2, Y_2, H_2) \) be two transition systems such that \( Y_1 \) and \( Y_2 \) are subsets of the same pseudometric space \( Y \) equipped with a pseudometric \( d_Y \) and \( U^\text{ext} \) (respectively \( U^\text{int} \)), \( j \in \{1, 2\} \), are subsets of the same pseudometric space \( U^\text{ext} \) (respectively \( U^\text{int} \)), respectively.
One can readily see that when \( \mu \) we denote the existence of an \((\varepsilon, \mu)\)-approximate alternating simulation relation from \( S_2 \) to \( S_1 \), if it satisfies the following conditions:

(i) \( \forall x_2^0 \in X_2^0, \exists x_1^0 \in X_1^0 \text{ such that } (x_1^0, x_2^0) \in \mathcal{R} \);

(ii) \( \forall (x_1, x_2) \in \mathcal{R}, \mathcal{d}_V(H_1(x_1), H_2(x_2)) \leq \varepsilon \);

(iii) \( \forall (x_1, x_2) \in \mathcal{R}, \mathcal{d}(u_1^{\text{ext}}, u_2^{\text{int}}) \in U_{S_2}^0(x_2), \exists (u_1^{\text{ext}}, u_1^{\text{int}}) \in U_{S_1}^0(x_1) \)

with \( \max(\mathcal{d}(u_1^{\text{ext}}, u_2^{\text{ext}}), \mathcal{d}(u_1^{\text{int}}, u_2^{\text{int}})) \leq \mu \) such that \( \forall x_1' \in \Delta_1(x_1, u_1^{\text{ext}}, u_1^{\text{int}}), \exists x_2' \in \Delta_2(x_2, u_2^{\text{ext}}, u_2^{\text{int}}) \)

satisfying \( (x_1', x_2') \in \mathcal{R} \).

We denote the existence of an \((\varepsilon, \mu)\)-approximate alternating simulation relation from \( S_2 \) to \( S_1 \) by \( S_2 \sim_{\mathcal{A}_S}^{\varepsilon, \mu} S_1 \).

One can readily see that when \( \mu = \infty \) we recover the classical notion of approximate alternating simulation relation as introduced in [Tab09], and when \( \mu = 0 \) the approximate alternating simulation relation coincides with strong alternating simulation relation given in [BPDB18].

**Remark 2.4.** Note that the definitions of approximate (alternating) simulation relations used in this paper are slightly different from the classical ones. Unlike classical definitions, the choice of inputs in our definitions is constrained by some distance property. However, these constraints over inputs are not restrictive and the proposed notions of \((\varepsilon, \mu)\)-approximate (alternating) simulation relations are compatible for different abstraction techniques presented in the literature.

### 3. Networks of Transition Systems and Approximate Composition

Given a system made of interconnected components, the computation of a direct abstraction of the whole system is computationally expensive. For this reason, we rely here on the notion of approximate composition allowing us to construct a global abstraction of the system from the abstractions of its components. To analyze the necessity of approximate composition, let us start with the simplest interconnection structure, a cascade composition of two concrete components, where the output of the first system is an input to the second one. When going from concrete (infinite) to abstract (finite) components, the output of the first system and the input to the second system do not coincide any more, since abstractions generally involve quantization of variables. To mitigate this mismatch, we introduce a notion of approximate composition, by relaxing the notion of the exact composition and allowing the distance between the output to the first component and input to the second one to be bounded by some given precision.

A network of systems consists of a collection of \( N \in \mathbb{N} \) systems \( \{S_1, \ldots, S_N\} \), a set of vertices \( I = \{1, \ldots, N\} \) and a binary connectivity relation \( \mathcal{I} \subseteq I \times I \) where each vertex \( i \in I \) is labeled with the system \( S_i \). For \( i \in I \), we define \( \mathcal{N}(i) = \{j \in I \mid (j, i) \in \mathcal{I}\} \) as the set of neighbouring components from which the incoming edges originate. An illustration of a network of interconnected components is given in Figure 1.

**Definition 3.1.** Given a collection of transition systems \( \{S_i\}_{i \in I} \), where \( S_i = (X_i, X_i^0, U_i^{\text{ext}}, U_i^{\text{int}}, \Delta_i, Y_i, H_i) \) such that for all \( i \in I \), \( \prod_{j \in \mathcal{N}(i)} Y_j \) and \( U_i^{\text{int}} \) are subsets of the same pseudometric space equipped with the
following pseudometric:

\[(3.1) \quad \text{for } u^{1,\text{int}} = (y^{1}_1, \ldots, y^{1}_{j_k})^T, l = \{1, 2\}, \text{with } N(i) = \{j_1, \ldots, j_k\}, d_{U^{1,\text{int}}}(u^{1,\text{int}}, u^{2,\text{int}}) = \| \prod_{j \in N(i)} \{ d_{Y_j}(y^{1}_j, y^{2}_j) \} \|.
\]

Let \( M := (\mu_1, \ldots, \mu_N)^T \in (\mathbb{R}^+)^N \). We say that \( \{S_i\}_{i \in I} \) is compatible for \( M \)-approximate composition with respect to \( I \), if for each \( i \in I \) and for each \( \prod_{j \in N(i)} \{ y_j \} \in \prod_{j \in N(i)} Y_j \), where the term \( \prod_{j \in N(i)} \{ y_j \} \) can be formally defined as \( \prod_{j \in N(i)} \{ y_j \} = (y_{j_1}, y_{j_2}, \ldots, y_{j_p})^T \) with \( N(i) = \{j_1, j_2, \ldots, j_p\} \), there exists \( u^{i,\text{int}} \in U^{i,\text{int}} \) such that \( d_{U^{i,\text{int}}}(u^{i,\text{int}}, \prod_{j \in N(i)} \{ y_j \}) \leq \mu_i \). We denote \( M \)-approximate composed system by \( \langle S_i \rangle^{M,\text{T}}_{i \in I} \) and is given by the tuple \( \langle S_i \rangle^{M,\text{T}}_{i \in I} = (X, X^0, U^{\text{ext}}, \Delta_M, Y, H) \), where:

- \( X = \prod_{i \in I} X_i \);
- \( X^0 = \prod_{i \in I} X^0_i \);
- \( U^{\text{ext}} = \prod_{i \in I} U^{\text{ext}}_i \);
- \( Y = \prod_{i \in I} Y_i \);
- \( H(x) = H(x_1, \ldots, x_N) = (H_1(x_1), \ldots, H_N(x_N))^T \);
- for \( x = (x_1, \ldots, x_N)^T \), \( x' = (x'_1, \ldots, x'_N)^T \) and \( u^{\text{ext}} = (u^{\text{ext}}_1, \ldots, u^{\text{ext}}_N)^T \) with \( u^{\text{ext}} \in U^{\text{ext}}_1 \), \( x' \in \Delta_M(x, u^{\text{ext}}) \) if and only if for all \( i \in I \), and for all \( \prod_{j \in N(i)} \{ y_j \} = \prod_{j \in N(i)} \{ H_j(x_j) \} \in \prod_{j \in N(i)} Y_j \), there exists \( u^{i,\text{int}} \in U^{i,\text{int}} \) with \( d_{U^{i,\text{int}}}(u^{i,\text{int}}, \prod_{j \in N(i)} \{ y_j \}) \leq \mu_i \), \( u^{i,\text{ext}}, u^{i,\text{int}} \in U^{i,\text{ext}}_1 \) and \( x'_i \in \Delta_i(x_i, u^{i,\text{ext}}, u^{i,\text{int}}) \).

For the sake of simplicity of notations, we use \( S_M \) instead of \( \langle S_i \rangle^{M,\text{T}}_{i \in I} \) throughout the paper. Note that since all the internal inputs of a component are outputs of other components we do not have internal inputs in the tuple of \( S_M \). If \( M = 0_N \), we say that collection of systems \( \{S_i\}_{i \in I} \) is compatible for exact composition. Let us remark that, for the composed system, the set of enabled inputs will be defined with respect to the set \( U^{\text{ext}} \). We equip the composed output space with the metric:

\[(3.2) \quad \text{for } y \in Y \text{ with } y^j = (y^j_1, \ldots, y^j_N)^T, j \in \{1, 2\}, d_{Y}(y^1, y^2) = \| \prod_{i \in I} \{ d_{Y_i}(y^1_i, y^2_i) \} \|.
\]

Similarly, we equip the composed input space with metric:

\[(3.3) \quad \text{for } u \in U^{\text{ext}} \text{ with } u^j = (u^j_1, \ldots, u^j_N)^T, j \in \{1, 2\}, d_{U^{\text{ext}}}(u^1, u^2) = \| \prod_{i \in I} \{ d_{U^{\text{ext}}_i}(u^1_i, u^2_i) \} \|.
\]

Let us remark that the parameter of the composition, i.e., \( M \), affects the conservativeness of the composed transition system. The following result shows that by increasing the parameter of the composition, the composed transition system allows for more nondeterminism in transitions and hence becomes more conservative. This result is straightforward and is stated without any proof.

**Claim 3.2.** Consider a collection of systems \( \{S_i\}_{i \in I} \) and \( M = (\mu_1, \ldots, \mu_N)^T \in (\mathbb{R}^+)^N \). If \( \{S_i\}_{i \in I} \) is compatible for \( M \)-approximate composition with respect to \( I \), then it is also compatible for \( \overline{M} \)-approximate composition with respect to \( I \), for any \( \overline{M} = (\overline{\mu}_1, \ldots, \overline{\mu}_N)^T \in (\mathbb{R}^+)^N \) such that \( \overline{M} \geq M \) (i.e., \( \overline{\mu}_i \geq \mu_i \), \( i \in I \)). Moreover, the relation \( \mathcal{R} = \{(x, x') \in X \times X \mid x = x'\} \) is a \((0, 0)\)-approximate simulation relation from \( S_M \) to \( S_{\overline{M}} \), where \( S_M = \langle S_i \rangle^{M,\text{T}}_{i \in I} \) and \( S_{\overline{M}} = \langle S_i \rangle^{\overline{M},\text{T}}_{i \in I} \).

4. Compositionality Results

In this section, we provide relations between interconnected systems based on the relations between their components.

**Theorem 4.1.** Let \( \{S_i\}_{i \in I} \) and \( \{\overline{S}_i\}_{i \in I} \) be two collection of transition systems with \( S_i = (X_i, X^0_i, U^{\text{ext}}_i, U^{\text{int}}_i, \Delta_i, Y_i, H_i) \) and \( \overline{S}_i = (X_i, X^0_i, \overline{U}^{\text{ext}}_i, \overline{U}^{\text{int}}_i, \overline{\Delta}_i, \overline{Y}_i, H_i) \). Consider non-negative constants \( \varepsilon_i, \mu_i, \delta_i \geq 0, i \in I \), with \( \varepsilon = \max_{i \in I} \varepsilon_i \) and \( \mu = \max_{i \in I} \mu_i \). Let the following conditions hold:
• for all $i \in I$, $S_i \preceq^{\varepsilon_i, \mu_i} \hat{S}_i$ with a relation $R_i$;
• $\{S_i\}_{i \in I}$ are compatible for $M$-approximate composition with respect to $I$, with $M = (\delta_1, \ldots, \delta_N)^T$; 
• $\{\hat{S}_i\}_{i \in I}$ are compatible for $M$-approximate composition with respect to $I$, with $\hat{M} = (\mu_1 + \delta_1 + \varepsilon, \ldots, \mu_N + \delta_N + \varepsilon)^T$.

Then the relation $R \subseteq X \times \hat{X}$ defined by

$$R = \{(x_1, \ldots, x_N, \hat{x}_1, \ldots, \hat{x}_N)^T \in X \times \hat{X} | \forall i \in I, (x_i, \hat{x}_i) \in R_i\}$$

is an $(\varepsilon, \mu)$-approximate simulation relation from $S_M$ to $\hat{S}_M$ (i.e., $S_M \preceq^{\varepsilon, \mu} \hat{S}_M$), where $S_M = \langle S_i \rangle_{i \in I}$ and $\hat{S}_M = \langle \hat{S}_i \rangle_{i \in I}$.

**Proof.** The first condition of Definition 2.2 is directly satisfied. Let $(x, \hat{x}) \in R$ with $x = (x_1, \ldots, x_N)$ and $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_N)$. We have $d_Y(H(x), \hat{H}(\hat{x})) = d_Y((H_1(x_1), \ldots, H_N(x_N))^T, (\hat{H}_1(\hat{x}_1), \ldots, \hat{H}_N(\hat{x}_N))^T) = \max_{i \in I} d_Y(H_i(x_i), \hat{H}_i(\hat{x}_i)) \leq \max \varepsilon_i = \varepsilon$, where the first equality comes from the definition of the output map for approximate composition, the second equality follows from (3.2) and the inequality comes from the second condition of Definition 2.2.

Consider $(x, \hat{x}) \in R$ with $x = (x_1, \ldots, x_N)^T$ and $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_N)^T$ and any $u^\text{ext} \in U_M^a(x)$ with $u^\text{ext} = (u^\text{ext}_1, \ldots, u^\text{ext}_N)^T$. Consider the transition $x' \in \Delta_M(x, u^\text{ext})$. This implies that for all $i \in I$, and for all $\prod_{j \in N(i)} \{y_j\} = \prod_{j \in N(i)} \{H_j(x_j)\} \in \prod_{j \in N(i)} \{y_j\}$, there exists $u^\text{int}_i \in U_i^a$ with $d_{U_i^a}(u^\text{int}_i, \prod_{j \in N(i)} \{y_j\}) \leq \delta_i, (u^\text{int}_1, u^\text{int}_2) \in U^a_{\hat{S}_i}(x_i)$ and $x'_i \in \Delta_s(x_i, u^\text{ext}_i, u^\text{int}_i)$. Let us prove the existence of an input $\hat{u}^\text{ext} \in U^a_{\hat{S}_M}(\hat{x})$ such that $d_{U^a}(u^\text{ext}, \hat{u}^\text{ext}) \leq \mu$ and a transition $\hat{x}' \in \Delta_{\hat{S}_M}(\hat{x}, \hat{u}^\text{ext})$ such that $(x', \hat{x}') \in R$.

From the definition of the relation $R$, we have for all $i \in I, (x_i, \hat{x}_i) \in R_i$, $(u^\text{ext}_i, u^\text{int}_i) \in U^a_{\hat{S}_i}(x_i)$ and $x'_i \in \Delta_s(x_i, u^\text{ext}_i, u^\text{int}_i)$; then from the third condition of the Definition 2.2, there exists $\hat{u}^\text{int}_i \in \hat{U}_{\hat{S}_i}(\hat{x}_i)$ with $d_{U^a}(u^\text{int}, \hat{u}^\text{int}) \leq \mu_i$ and $d_{U_i^a}(u^\text{int}_i, \hat{u}^\text{int}_i) \leq \mu_i$ and there exists $\hat{x}'_i \in \Delta_{\hat{S}_i}(\hat{x}_i, \hat{u}^\text{int}, \hat{u}^\text{int})$ such that $(x'_i, \hat{x}'_i) \in R_i$.

Let us show that the input $\hat{u}^\text{int} = (\hat{u}^\text{int}_1, \ldots, \hat{u}^\text{int}_N)^T$ satisfies the requirement of the $M$-approximate composition of the components $\{\hat{S}_i\}_{i \in I}$. The condition $d_{U_i^a}(u^\text{int}_i, \hat{u}^\text{int}_i) \leq \mu_i$ implies that

$$d_{U_i^a}(u^\text{int}_i, \prod_{j \in N(i)} \{y_j\}) \leq d_{U_i^a}(u^\text{int}_i, \prod_{j \in N(i)} \{y_j\}) \leq d_{U_i^a}(u^\text{int}_i, \prod_{j \in N(i)} \{y_j\}) + d_{U_i^a}(\prod_{j \in N(i)} \{y_j\}) \leq \mu_i + \delta_i + \varepsilon_j$$

Hence, the $M$-approximate composition with respect to $\mathcal{I}$ of $\{\hat{S}_i\}_{i \in I}$ is well defined in the sense of Definition 3.1. Thus, condition (iii) in Definition 2.2 holds with $\hat{u}^\text{ext} = (\hat{u}^\text{ext}_1, \ldots, \hat{u}^\text{ext}_N)$ satisfying $d_{U^a}(u^\text{ext}, \hat{u}^\text{ext}) = ||\prod_{i \in I} d_{U_i^a}(u^\text{ext}_i, \hat{u}^\text{ext}_i)|| = ||\prod_{i \in I} (\mu_i)|| = \mu$ and $\hat{x}' = (\hat{x}_1, \ldots, \hat{x}_N)$, and one obtains $S_M \preceq^{\varepsilon, \mu} \hat{S}_M$. 

**Theorem 4.2.** Let $\{S_i\}_{i \in I}$ and $\{\hat{S}_i\}_{i \in I}$ be two collection of transition systems with $S_i = (X_i, X^0_i, U^\text{ext}_i, U^\text{int}_i, \Delta_i, Y_i, H_i)$ and $\hat{S}_i = (X_i, X^0_i, \hat{U}^\text{ext}_i, \hat{U}^\text{int}_i, \hat{\Delta}_i, \hat{Y}_i, \hat{H}_i)$. Consider non-negative constants $\varepsilon_i, \mu_i, \delta_i \geq 0, i \in I$, with $\varepsilon = \max_{i \in I} \varepsilon_i$ and $\mu = \max_{i \in I} \mu_i$. Let the following conditions hold:

• for all $i \in I$, $S_i \preceq^{\varepsilon_i, \mu_i} S_i$ with a relation $R_i$;
• $\{S_i\}_{i \in I}$ are compatible for $M$-approximate composition with respect to $I$, with $M = (\delta_1, \ldots, \delta_N)^T$;
• \( \{ \hat{S}_i \}_{i \in I} \) are compatible for \( \hat{M} \)-approximate composition with respect to \( I \), with \( \hat{M} = (\mu_1 + \delta_1 + \varepsilon, \ldots, \mu_N + \delta_N + \varepsilon) \).

Then the relation \( \mathcal{R} \subseteq X \times \hat{X} \) defined by
\[
\mathcal{R} = \{(x_1, \ldots, x_N, \hat{x}_1, \ldots, \hat{x}_N)^T \in X \times \hat{X} | \forall i \in I, (x_i, \hat{x}_i) \in \mathcal{R}_i \}
\]
is an \( (\varepsilon, \mu) \)-approximate alternating simulation relation from \( \hat{S}_{HC} \) to \( S_M \) (i.e., \( \hat{S}_{HC} \preceq_{\mathcal{AS}} S_M \)), where \( S_M = \langle S_i \rangle_{i \in I}^{M,T} \) and \( \hat{S}_{HC} = \langle \hat{S}_i \rangle_{i \in I}^{M,T} \).

Proof. The first condition of Definition 2.3 is directly satisfied. Let \( (x, \hat{x}) \in \mathcal{R} \) with \( x = (x_1, \ldots, x_N)^T \) and \( \hat{x} = (\hat{x}_1, \ldots, \hat{x}_N)^T \). We have
\[
d_Y(H(q), \hat{H}(\hat{q})) = d_Y((H_1(q_1), \ldots, H_N(q_N))^T, (\hat{H}_1(\hat{q}_1), \ldots, \hat{H}_N(\hat{q}_N))^T) = \max_{i \in I} d_Y(H_i(q_i), \hat{H}_i(\hat{q}_i)) \leq \varepsilon_i = \varepsilon, \]
where the first equality comes from the definition of the output map for approximate composition, the second equality follows from (3.2) and the inequality comes from the second condition of Definition 2.3.

Consider \( (x, \hat{x}) \in \mathcal{R} \) with \( x = (x_1, \ldots, x_N)^T \) and \( \hat{x} = (\hat{x}_1, \ldots, \hat{x}_N)^T \) and any \( \hat{u}^{\text{ext}} \in \hat{U}_M \) \( (x) \) with \( \hat{u}^{\text{ext}} = (\hat{u}_1^{\text{ext}}, \ldots, \hat{u}_N^{\text{ext}})^T \). Let us prove the existence of \( u^{\text{ext}} \in U_{SM}(x) \) with \( d_{\text{ext}}(u^{\text{ext}}, \hat{u}^{\text{ext}}) \leq \mu \) and such that for any \( x' \in \Delta_M(x, u^{\text{ext}}) \), there exists \( \hat{x}' \in \Delta_{\hat{M}}(\hat{x}, \hat{u}) \) satisfying \( (x', \hat{x}') \in \mathcal{R} \). From the definition of relation \( \mathcal{R} \), we have for all \( i \) \( (x_i, \hat{x}_i) \in \mathcal{R}_i \), then from the third condition of Definition 2.3, we have for all \( (\hat{u}_i^{\text{ext}}, \hat{u}_i^{\text{int}}) \in \hat{U}_M(\hat{x}_i) \), the existence of \( (u_i^{\text{ext}}, u_i^{\text{int}}) \in U_M(x_i) \) with \( d_{\text{ext}}(u_i^{\text{ext}}, \hat{u}_i^{\text{ext}}) \leq \mu_i \) and \( d_{\text{int}}(u_i^{\text{int}}, \hat{u}_i^{\text{int}}) \leq \mu_i \) such that for any \( x'_i \in \Delta_i(x_i, u_i^{\text{ext}}, u_i^{\text{int}}) \) there exists \( \hat{x}'_i \in \Delta(\hat{x}_i, \hat{u}_i^{\text{ext}}, \hat{u}_i^{\text{int}}) \) such that \( (x'_i, \hat{x}'_i) \in \mathcal{R}_i \).

Let us show that the input \( u_i^{\text{int}} = (u_1^{\text{int}}, \ldots, u_N^{\text{int}})^T \) satisfies the requirement of the \( \hat{M} \)-approximate composition of the components \( \{ \hat{S}_i \}_{i \in I} \). The condition \( d_{\text{int}}(u_i^{\text{int}}, \hat{u}_i^{\text{int}}) \leq \mu_i \) implies that
\[
d_{\text{int}}(u_i^{\text{int}}, \hat{u}_i^{\text{int}}) \leq d_{\text{int}}(u_i^{\text{int}}, \hat{u}_i^{\text{int}}) + \prod_{j \in N(i)} \{ y_j \} \leq d_{\text{int}}(u_i^{\text{int}}, \hat{u}_i^{\text{int}}) + \prod_{j \in N(i)} \{ \hat{y}_j \} \leq \mu_i + \delta_i + \max_{j \in N(i)} \varepsilon_j \leq \mu_i + \delta_i + \max_{j \in I} \varepsilon_j = \mu_i + \delta_i + \varepsilon.
\]

Hence, from (iii) the \( \hat{M} \)-approximate composition with respect to \( I \) of \( \{ \hat{S}_i \}_{i \in I} \) is well defined in the sense of Definition 3.1. Thus, condition (iii) in Definition 2.3 holds with \( u^{\text{ext}} = (u_1^{\text{ext}}, \ldots, u_N^{\text{ext}})^T \) satisfying \( d_{\text{ext}}(u^{\text{ext}}, \hat{u}^{\text{ext}}) = \| \prod_{i \in I} \{ d_{\text{ext}}(u_i^{\text{ext}}, \hat{u}_i^{\text{ext}}) \} \| = \| \prod_{i \in I} \{ \mu_i \} \| = \mu_i \), and one obtains \( \hat{S}_{HC} \preceq_{\mathcal{AS}} S_M \).

Intuitively, the results of the previous theorems can be interpreted as follows: the result in Theorem 4.1 can be used for compositional verification. Given a collection of systems \( \{ S_i \}_{i \in I} \), if each component approximately satisfies a specification \( Q_i \), then the composed system \( S_M = \langle S_i \rangle_{i \in I}^{M,T} \) approximately satisfies a composed specification \( Q = \langle Q_i \rangle_{i \in I}^{M,T} \). Note that for constructing controllers compositionally, the results of Theorem 4.2 is more suitable. Given a collection of components \( \{ S_i \}_{i \in I} \), for \( i \), let \( \hat{S}_i \) an abstraction for \( S_i \), then the composed system \( \hat{S}_{HC} = \langle \hat{S}_i \rangle_{i \in I}^{M,T} \) is an abstraction of the system \( S_M = \langle S_i \rangle_{i \in I}^{M,T} \). Figure 2 illustrates these results.
Figure 2. Illustration of compositionality results for a collection of transition systems using the notion of approximate composition and approximate (alternating) simulation relations as formalized in Theorems 4.1 and 4.2.

5. Incremental safety controller synthesis

In this section, we start by introducing notions of controlled systems and safety controllers. Then, we show how the proposed notion of approximate composition enables the incremental synthesis of controllers enforcing safety specifications.

5.1. Controlled systems. Consider a system $S = (X, X^0, U_{\mathrm{ext}}, U_{\mathrm{int}}, \Delta, Y, H)$ and a memoryless controller $C : X \Rightarrow U_{\mathrm{ext}} \times U_{\mathrm{int}}$ such that for all $x \in X$, $C(x) \subseteq U^0_{\mathrm{ext}}(x)$. Let $\text{dom}(C)$ be the domain of controller defined by $\text{dom}(C) = \{x \in X \mid C(x) \neq \emptyset\} \subseteq X$. We define a controlled transition system by a tuple $S|C = (X^C, X^0_C, U^\mathrm{ext}_C, U^\mathrm{int}_C, \Delta_C, Y_C, H_C)$, where:

- $X^C = X \cap \text{dom}(C)$ is the set of states;
- $X^0_C = X^0 \cap \text{dom}(C)$ is the set of initial states;
- $U^\mathrm{ext}_C = U_{\mathrm{ext}}$ is the set of external inputs;
- $U^\mathrm{int}_C = U_{\mathrm{int}}$ is the set of internal inputs;
- $Y_C = Y$ is the set of outputs;
- $H_C = H$ is the output map;
- a transition relation: $x'_C \in \Delta_C(x_C, u^\mathrm{ext}_C, u^\mathrm{int}_C)$ if and only if $x'_C \in \Delta(x_C, u^\mathrm{ext}_C, u^\mathrm{int}_C)$ and $(u^\mathrm{ext}_C, u^\mathrm{int}_C) \in C(x_C)$.

We first introduce the following auxiliary lemma relating the system $S$ and the controlled system $S|C$.

Lemma 5.1. Given the systems $S$ and $S|C$ defined above, we have that $S|C \preceq^{0,0}_{AS} S$. 

Remark 4.3. In symbolic control literature, different approaches have been presented to compute (in)finite abstractions for different classes of systems including linear systems [BYG17, GP09], monotone (or mixed-monotone) systems [CA15, MGW15], time-delay systems [PPDBT10], switched systems [GPT09], incrementally stable (or stabilizable) systems [PGT08], incrementally stable stochastic (switched) systems [ZTA17, ZAG15] and incrementally stable time-delayed (stochastic) control systems [PPDB15, PPDBT10, JZ20]. Let us point out that the proposed compositional framework in this paper is suitable for different types of (in)finite abstractions which allows for modularity and flexibility in the construction of the symbolic models.
Proof. Define the relation \( \mathcal{R} = \{ (x, \pi) \in X \times X_C \mid x = \pi \} \). We have that \( X^0 = X^0 \cap \text{dom}(C) \subseteq X^0 \), hence the first condition of Definition 2.3 is satisfied. Let \( (x, \pi) \in \mathcal{R} \). We have that \( d_Y(H(x), H_C(\pi)) = d_Y(H(x), H(x)) = 0 \) which shows condition (ii) of Definition 2.3. Now consider \( (x, \pi) \in \mathcal{R} \) and any \( (\pi^{\text{ext}}, \pi^{\text{int}}) \in U^0_{\mathcal{R}}(\pi) \). We choose \( u^{\text{ext}} = \pi^{\text{ext}} \) and \( u^{\text{int}} = \pi^{\text{int}} \) with \( (u^{\text{ext}}, u^{\text{int}}) \in U^a(x) \) and \( (u^{\text{ext}}, u^{\text{int}}) \in C(\pi) \). Then for all \( x' \in \Delta(x, u^{\text{ext}}, u^{\text{int}}) \) we have \( x' \in \Delta_C(x, u^{\text{ext}}, u^{\text{int}}) = \Delta_C(\pi, u^{\text{ext}}, u^{\text{int}}) \). Since \( x' \in \Delta_C(\pi, u^{\text{ext}}, u^{\text{int}}) \), we have the existence of \( \pi' \in \Delta_C(\pi, u^{\text{ext}}, u^{\text{int}}) \) satisfying \( \pi' = x' \). This implies \( (x', \pi') \in \mathcal{R} \), which concludes the proof. \( \square \)

5.2. Safety controller.

Definition 5.2. A safety controller \( C \) for the transition system \( S \) and the safe set \( \mathcal{S} \subseteq X \) satisfies:

(i) \( \text{dom}(C) \subseteq \mathcal{S} \);

(ii) \( \forall x \in \text{dom}(C) \) and \( \forall (u^{\text{ext}}, u^{\text{int}}) \in C(x), \Delta(x, u^{\text{ext}}, u^{\text{int}}) \subseteq \text{dom}(C) \).

There are in general several controllers that solve the safety problem. A suitable solution is a controller that enables as many actions as possible. Such controller \( C^* \) is referred to as a maximal safety controller, in the sense that for any other safety controller \( C \) and for all \( x \in X \), we have \( C(x) \subseteq C^*(x) \). In order to define carefully the maximal safety controller, we introduce the concept of a controlled invariant set.

Definition 5.3. Consider a transition system \( S \) and a safe set \( \mathcal{S} \subseteq X \). A subset \( A \subseteq \mathcal{S} \) is said to be a controlled invariant if for all \( x \in A \) there exists \( (u^{\text{ext}}, u^{\text{int}}) \in U^{\text{ext}} \times U^{\text{int}} \) such that \( \Delta(x, u^{\text{ext}}, u^{\text{int}}) \subseteq A \).

It was shown in Tabuada [Tab09] that there exists a maximal controlled invariant \( \text{Cont}(\mathcal{S}) \) which is the union of all controlled invariants. The maximal safety controller can be defined as follows:

- for all \( x \notin \text{Cont}(\mathcal{S}) \), \( C^*(x) = \emptyset \);
- for all \( x \in \text{Cont}(\mathcal{S}) \), \( C^*(x) = \{(u^{\text{ext}}, u^{\text{int}}) \in U^{\text{ext}}(x) \mid \Delta(x, u^{\text{ext}}, u^{\text{int}}) \subseteq \text{Cont}(\mathcal{S})\} \).

Let us remark that for any safety controller \( C \) we have that \( \text{dom}(C) \subseteq \text{Cont}(\mathcal{S}) \), while for the maximal safety controller \( C^* \), we have \( \text{dom}(C^*) = \text{Cont}(\mathcal{S}) \).

5.3. Incremental synthesis of controllers. The size of transition systems is crucial for computational efficiency of discrete safety controller synthesis algorithms. As the size of transition systems grows, the classical safety synthesis suffers from the curse of dimensionality. In this subsection, we show how to incrementally synthesize safety controllers for interconnected systems. Consider a global system \( S \) made of \( N \) interconnected components \( S_i, i \in I \), and a global decomposable safety specification \( \mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_N \). We start by synthesizing a local safety controller \( C_i \) for each component \( S_i \) and safety specification \( \mathcal{S}_i \), compose the local controlled components (by computing \( \langle S_i|C_i|_{i \in I}^{M,I} \)), and then synthesize a global safety controller for \( \langle S_i|C_i|_{i \in I}^{M,I} \) against the safety specification \( \mathcal{S} \). We first give an example illustrating the idea of incremental safety synthesis.

Example 5.4. Consider the transition systems \( S_1 = (\{a, b\}, \{1, 2\}, \{a, b\}, \{1\}, \{a, b\}, \Delta_1, \{a, b\}, I_d) \) and \( S_2 = (\{a, b, \gamma\}, \{a, b, \gamma\}, \{2\}, \{a, b\}, \Delta_2, \{a, b, \gamma\}, I_d) \) as shown in Figure 3, where \( I_d \) is the identity map. Let the interconnection relation be \( I = \{(1, 2), (2, 1)\} \). Since \( Y_1 \subseteq U_2^{\text{int}} \) and \( Y_2 \subseteq U_1^{\text{int}} \), the components \( S_1 \) and \( S_2 \) are compatible for exact composition with respect to \( I \). Let \( \langle S_i\rangle_{i \in I}^{0,2,I} \) be the composed system. Let the global safety specification for the system \( S \) define by \( \mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 \), where \( \mathcal{S}_1 = \{a, b\} \) and \( \mathcal{S}_2 = \{\beta\} \). The classical safety approach directly synthesizes a maximal safety controller for the system \( \langle S_i\rangle_{i \in I}^{0,2,I} \). An illustration of the controlled system \( \langle S_i\rangle_{i \in I}^{0,2,I}(C^*) \) is given in Figure 3. In the proposed incremental approach, see Figure 4, we first start by synthesizing a local controller \( C_i^* \) for the component \( S_i \) against the safety specification \( \mathcal{S}_i \) to obtain \( S_i|C_i^* \). We then compose the local controlled components by computing \( \langle S_i|C_i|_{i \in I}^{0,2,I} \). Finally we synthesize a global safety controller \( C_I \) for the system \( \langle S_i|C_i|_{i \in I}^{0,2,I} \) against the safety specification \( \mathcal{S} \). In the classical safety synthesis, we need to compute the safety controller for the system \( \langle S_i\rangle_{i \in I}^{0,2,I} \), which is made of \( 6 \) states and \( 6 \) transitions. In the proposed incremental synthesis, we need to apply the global safety synthesis for the reduced
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Figure 3. Two transition systems $S_1$ and $S_2$, the composed system $\langle S_i \rangle$ and the controlled system $\langle S_i \rangle | C^*$.

Figure 4. The controlled components $S_1 | C_1$ and $S_2 | C_2$, the composition of controlled components $\langle S_i | C_i \rangle$ which is finally controlled by $C_1$ ($\langle S_i | C_i \rangle | C_1$).

The composed system $\langle S_i | C_i \rangle$ is made of 2 states and 1 transition. Hence, one can notice in this toy example the benefits of the proposed incremental synthesis while ensuring completeness with respect to the classical safety synthesis ($C^* = C_1$).

We start by providing the following auxiliary lemma showing how the maximal safety controller $C^*$ for the composed system $S = \langle S_i \rangle$ is related to the maximal controllers $C_i^*$ synthesized for the components $S_i$, $i \in I$.

Lemma 5.5. Let $\{S_i\}_{i \in I}$ be a collection of transition systems compatible for $M$-approximate composition, with $M = (\delta_1, \ldots, \delta_N)^T$. Let $S_M = \langle S_i \rangle$ be the composed system. Let $\mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_N$ be a safety specification for the composed system and let us assume the following:

- $C_i^*$, $i \in I$, is the maximal safety controller for $S_i$ enforcing the specification $\mathcal{S}_i$;
- $C^*$ is the maximal controller for $S_M$ enforcing the safety specification $\mathcal{S}$.

If $(u_1^{ext}, \ldots, u_N^{ext}) \in C^*(x_1, \ldots, x_N)$ and $d_{U_i^{int}}(u_i^{int}, \prod_{j \in N(i)} H_j(x_j)) \leq \delta_i$ for some $i \in I$ and some $u_i^{int} \in U_i^{int}$, then we have $(u_1^{ext}, u_i^{int}) \in C_i^*(x_i)$.

\footnote{Let us mention that computational complexity to compute the local controllers $C_i^*$ for components $S_i$ is imperceptible with comparison to the safety synthesis on the global reduced composed system $\langle S_i | C_i \rangle$.}
Proof. For $i \in I$, let us define the controller $C_i : X_i \Rightarrow U^\text{ext}_i \times U^\text{int}_i$ as follows: $(u^\text{ext}_{i}, u^\text{int}_{i}) \in C_i(x_i)$ if and only if there exists $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)^T \in X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_N$ and there exists $(u^\text{ext}_1, \ldots, u^\text{ext}_{i-1}, u^\text{int}_{i}, u^\text{int}_{i+1}, \ldots, u^\text{ext}_N)^T \in U^\text{ext}_1 \times \cdots \times U^\text{ext}_{i-1} \times U^\text{int}_i \times U^\text{int}_{i+1} \times \cdots \times U^\text{ext}_N$ such that $(u^\text{ext}_1, \ldots, u^\text{ext}_N)^T \in C^\ast(x_1, \ldots, x_N)$ and $d^\gamma_{U^\text{ext}_i}(u^\text{int}_{i}, \prod_{j \in N \setminus \{i\}} H_j(x_j)) \leq \delta_i$. Let us prove that $C_i$ is a safety controller for $S_i$ and safety specification $\mathcal{G}_i$.

Let $x_i \in \text{dom}(C_i)$, then by construction of $C_i$ we have the existence of $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)^T \in X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_N$ such that $(x_1, \ldots, x_N)^T \in \text{dom}(C^\ast) \subseteq \mathcal{G} = \mathcal{G}_1 \times \cdots \times N$. Hence, $x_i \in \mathcal{G}_i$ and $\text{dom}(C_i) \subseteq \mathcal{G}_i$, then condition (i) of Definition 5.2 is satisfied. Now let $x_i \in \text{dom}(C_i)$ and $(u^\text{ext}_i, u^\text{int}_i) \in C_i(x_i)$. We have the existence of $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)^T \in X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_N$ and $(u^\text{ext}_1, \ldots, u^\text{ext}_{i-1}, u^\text{int}_{i}, u^\text{int}_{i+1}, \ldots, u^\text{ext}_N)^T \in U^\text{ext}_1 \times \cdots \times U^\text{ext}_{i-1} \times U^\text{int}_i \times U^\text{int}_{i+1} \times \cdots \times U^\text{ext}_N$ such that $(u^\text{ext}_1, \ldots, u^\text{ext}_N)^T \in C^\ast(x_1, \ldots, x_N)$ and $d^\gamma_{U^\text{ext}_i}(u^\text{int}_i, \prod_{j \in N \setminus \{i\}} H_j(x_j)) \leq \delta_i$. Since $C^\ast$ is the maximal safety controller for the system $S$ and safety specification $\mathcal{G}$, we have that for all $(x_1', \ldots, x_N') \in \Delta C^\ast(x_1, \ldots, x_N, u^\text{ext}_1, \ldots, u^\text{ext}_N)$, $(x_1', \ldots, x_N') \in \text{dom}(C^\ast)$. Hence, for $i \in I$, $(u^\text{ext}_i, u^\text{int}_i) \in C_i(x_i)$ for all $x_i \in \Delta_i(x_i, u^\text{ext}_i, u^\text{int}_i)$ where $u^\text{int}_i \in U^\text{int}_i$ is such that $d^\gamma_{U^\text{ext}_i}(u^\text{int}_i, \prod_{j \in N \setminus \{i\}} H_j(x_j)) \leq \delta_i$. Then, $C_i$ is a safety controller for the component $S_i$ and safety specification $\mathcal{G}_i$.

Next, we provide theorem showing the completeness of the proposed incremental controller synthesis procedure with respect to the maximal monolithic safety controller $C^\ast$.

**Theorem 5.6.** Let $\{S_i\}_{i \in I}$ be a collection of transition systems compatible for $M$-approximate composition, with $M = (\delta_1, \ldots, \delta_N)^T$. Let $S_M = \langle \{S_i\}_{i \in I} \rangle^M$ be the composed system. Let $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_N$ be a safety specification for the composed system and let us assume the following:

- $C^\ast_i, i \in I$, is the maximal safety controller for $S_i$ enforcing the specification $\mathcal{G}_i$;
- $C^\ast$ is the maximal controller for $S_M$ enforcing the safety specification $\mathcal{G}$;
- $C_I$ is the maximal controller for $\langle \{S_i\}^M \rangle_{i \in I}$ enforcing the safety specification $\mathcal{G}$.

Then, for all $x \in X$, $C^\ast(x) = C_I(x)$.

**Proof.** From Lemma 5.3, we have for all $i \in I$, $S_i|C^\ast_i \preceq_{\mathcal{AS}} S_i$, then it follows from Theorem 4.2 that $\langle S_i|C^\ast_i \rangle_{i \in I}^M \preceq_{\mathcal{AS}} S_M = \langle S_i \rangle_{i \in I}^M$. Since $C_I$ is the maximal safety controller for $\langle S_i|C^\ast_i \rangle_{i \in I}^M$ and safety specification $\mathcal{G}$ and from definition of the alternating simulation relation $\mathcal{AS}$, we have that $C_I$ is a safety controller for $S_M$ and specification $\mathcal{G}$. Then from maximality of $C^\ast$, we have that $C_I(x) \subseteq C^\ast(x)$ for all $x \in X$.

To prove the second inclusion, we at first show that $S_M|C^\ast \preceq_{\mathcal{AS}} \langle S_i|C^\ast_i \rangle_{i \in I}$, with $S_M|C^\ast = (X_C, X^0_C, U^\text{ext}_C, U^\text{int}_C, \Delta_C, Y_C, H_C)$ and $\langle S_i|C^\ast_i \rangle_{i \in I} = (X_I, X^0_I, U^\text{ext}_I, U^\text{int}_I, \Delta_I, Y_I, H_I)$. Let the relation $\mathcal{R}$ defined by $\mathcal{R} = \{ (x, \pi) \in X_I \times X_C | x = \pi \}$.

From Lemma 5.5, we have that $\text{dom}(C^\ast) \subseteq \text{dom}(C_1) \times \cdots \times \text{dom}(C_N)$, hence, $X^0_C = X^0 \cap \text{dom}(C^\ast) \subseteq X^0 = X^0_N \cap \text{dom}(C_N)$ and the first condition of Definition 2.3 is satisfied. Let $(x, \pi) \in \mathcal{R}$, we have that $x = \pi$, hence, $H_C(x) = H_I(\pi) = H(x)$ and condition (ii) of Definition 2.3 is satisfied. Now, let $(x, \pi) \in \mathcal{R}$ and $(u^\text{ext}_1, \ldots, u^\text{ext}_N)^T \in U^\text{ext}_N$. Then $(u^\text{ext}_1, \ldots, u^\text{ext}_N)^T \in U^\text{ext}_N$ and $(u^\text{ext}_1, \ldots, u^\text{ext}_N)^T \in C^\ast(x) = C^\ast(x_1, \ldots, x_N)$. We have from Lemma 5.5 that for all $i \in I$, $(u^\text{ext}_i, u^\text{int}_i) \in C^\ast_i(x_i)$ for any $u^\text{int}_i \in U^\text{int}_i$ satisfying $d^\gamma_{U^\text{ext}_i}(u^\text{int}_i, \prod_{j \in N \setminus \{i\}} H_j(x_j)) \leq \delta_i$. Then, by construction of the transition systems $S_M|C^\ast$ and $\langle S_i|C^\ast_i \rangle_{i \in I}$, we have for all $\pi = (\pi_1', \ldots, \pi_N') \in \Delta_I(\pi_1, \ldots, \pi_N, u^\text{ext}_1, \ldots, u^\text{ext}_N)$, $(\pi_1', \ldots, \pi_N') \in \Delta_C(\pi_1, \ldots, \pi_N, u^\text{ext}_1, \ldots, u^\text{ext}_N)$. Hence there exists $x' \in$...
Δ_{C^*}(x_1, ..., x_N, u_1^{\text{ext}}, ..., u_N^{\text{ext}}) \text{ such that } x' = x'. \quad \text{Then, } (x', \overline{x}') \in \mathcal{R} \text{ and condition (iii) of definition 2.3 is satisfied.}

Since \( C^* \) is the maximal safety controller for \( S_M|C^* \) and safety specification \( \mathcal{S} \) and from the definition of the alternating simulation relation [Tab09], we have that \( C^* \) is a safety controller for the system \( \langle S_i|C_i \rangle_{i \in I} \) and specification \( \mathcal{S} \). Then from maximality of \( C_I \), we have that \( C^*(x) \subseteq C_I(x) \) for all \( x \in X \). Then for all \( x \in X \), \( C^*(x) = C_I(x) \).

\[ \square \]

6. Numerical examples

In this section, we demonstrate the effectiveness of the proposed approach on two control problems: a DC microgrid and a road traffic control problem. The objective of the first example is to illustrate the speed-up that can be attained using the compositional abstraction framework proposed in Section 4. In the second example, we show how the proposed framework can be applied to a more complex example, on which different abstraction techniques are used for different components. Moreover, we will also show the benefits of the incremental safety synthesis approach proposed in Section 5. In the following, the numerical implementations have been done in MATLAB and a computer with processor 2.7 GHz Intel Core i5, Memory 8 GB 1867 MHz DDR3.

6.1. DC microgrids. In the following, we use the DC microgrid model proposed in [ZSGF19b].

6.1.1. Model description and control objective. We represent a microgrid as a directed graph \( G(N, E, B) \), where: \( N \) is the set of nodes, with cardinality \( n \); \( E \) is the set of edges, with cardinality \( t \) and \( B \in \mathbb{R}^{n \times t} \) is the incidence matrix capturing the graph topology. The edges correspond to the transmission lines, while the nodes correspond to the buses where the power units are interfaced. The weighted interconnection topology is equivalently captured by the Laplacian matrix \( L := B G_T B^T \in \mathbb{R}^{n \times n} \), with \( G_T := \text{diag}(G_e) \in \mathbb{R}^{t \times t} \), where \( G_e \) denotes the conductance associated to the edge \( e \in E \). We further define \( N_S \) as the subset of nodes associated to controllable power units (sources), i.e. the generation and energy storage units, with cardinality \( m \), and \( N_L \) as the subset of nodes associated to non-controllable power units (loads), with cardinality \( n - m \). The interconnected dynamics of the voltage buses are:

\[
CV = -(L + G)V + \sigma,
\]

where \( V := \text{col}(v_i) \in \mathbb{R}^{n}_{>0} \) denotes the collection of (positive) bus voltages, \( \sigma := \text{col}(\sigma_i) \in \mathbb{R}^n \) denotes the collection of input currents and \( C := \text{diag}(C_i) \in \mathbb{R}^{n \times n} \), \( G := \text{diag}(G_i) \in \mathbb{R}^{n \times n} \) are matrices denoting the bus capacitances and conductances. Input currents are given by:

\[
\sigma_i = ((1 - b_i)P_i + b_i u_i)/v_i, \quad i \in N,
\]

with: control input \( u_i \in \mathcal{U}_i \), where \( \mathcal{U}_i := [u_i^l, u_i^u] \subset \mathbb{R}_{>0} \); \( b_i \in \{0, 1\} \), where \( b_i = 1 \), if \( i \in N_S \) and \( b_i = 0 \) otherwise; and \( P_i \) is a bounded time-varying demand \( P_i \in \mathcal{P}_i = [P_i^{\text{min}}, P_i^{\text{max}}] \). By replacing (6.2) into (6.1), the overall system can be rewritten in compact form via the following ordinary differential inclusion:

\[
\dot{V} \in f(V, u) = -C^{-1} \left[ (L + G)V + \left[ \begin{array}{c} u \\ \mathcal{P} \end{array} \right] \right] \odot V
\]

with state vector \( V \in \mathbb{R}^{n}_{>0} \); control input \( u \in \mathcal{U} \), where \( \mathcal{U} := \prod_i \mathcal{U}_i \); disturbance input \( \mathcal{P} := \prod_i \mathcal{P}_i \); and where \( \odot \) denotes the element-wise division of matrices.

The safe set is given by \( S = [V^{\text{nom}} - \delta, V^{\text{nom}} + \delta]^n \) and means that the voltage \( V \) of the system need to be kept sufficiently near to the nominal value \( V_{\text{nom}} > 0 \) up to a given precision \( \delta > 0 \).
6.1.2. Abstraction and controller synthesis. We consider a five-terminal DC microgrid as the one depicted in Figure 5. We assume that two units, namely Units 2 and 3, are equipped with a primary control layer, while the remaining three units, Units 1, 4, and 5 correspond to loads with demand varying steadily around a constant power reference. The latter can be thus interpreted as constant power loads affected by noise. The considered bus parameters are $C_1 = 2.2 \ \mu F$, $C_2 = 1.9 \ \mu F$, $C_3 = 1.5 \ \mu F$, $C_4 = C_5 = 1.7 \ \mu F$ and the network parameters are $G_{12} = 5.2 \ \Omega^{-1}$, $G_{13} = 4.6 \ \Omega^{-1}$, $G_{14} = 4.5 \ \Omega^{-1}$, $G_{24} = 6 \ \Omega^{-1}$, $G_{25} = 3.1 \ \Omega^{-1}$, $G_{34} = 5.6 \ \Omega^{-1}$, $G_{15} = G_{23} = G_{35} = G_{45} = 0 \ \Omega^{-1}$. The system is supposed to operate within a region with grid nominal voltage $V_{\text{nom}} = 450 \ \text{V}$ and $\delta = 0.025V_{\text{nom}}$. We use the symbolic approach presented in [MGW15], while exploiting the monotonicity property of the DC grid [ZSGF19a], we select sampling period for the abstractions $\tau = 0.1$ milliseconds, which corresponds to the clock of the controller to be designed. Discretization parameters are $n_d = 5$ and $n_u = 5$ denoting the number of discrete states and inputs, respectively, for each dimension.
We consider two scenarios. In the first case, we assume that Unit 5 is disconnected from the grid and the grid is made of 4 units $I = \{1, 2, 3, 4\}$. We compute local abstraction $\hat{S}_i$ for each Unit $S_i$, $i \in I$, each abstraction $\hat{S}_i$ is related to the original system $S_i$, $i \in I$, by an $(\varepsilon_i, \mu_i)$-approximate alternating simulation relation, with $\varepsilon_i = 4.5$ and $\mu_i = 0$. We then compose the local abstractions in order to compute the global abstraction using an $\hat{M}$-approximate composition, with $\hat{M} = (4.5, 4.5, 4.5, 4.5, 4.5)^T$. Hence, in view of Theorem 4.2, we have that $\hat{S} \approx_{\hat{M}} S$, where $S = (S_i)_{i \in I}$ and $\hat{S} = (\hat{S}_i)_{i \in I}$.

The computation time of the abstractions of the four components $\{1, 2, 3, 4\}$ are given by 5 seconds, 9 seconds, 8 seconds and 4 seconds, respectively, and the composition of the global abstraction from local ones using an approximate composition takes 15 seconds. This resulted in 41 seconds to compute an abstraction compositionally. Constructing an abstraction for the full model monolithically, using the same discretization parameters, took 154 seconds. Hence, the proposed compositional approach was three times faster in this scenario.

In the second scenario, Unit 5 is connected to the grid, we use the same numerical parameters as in the first scenario. In this case, the computation time of the abstraction of the five components $\{1, 2, 3, 4, 5\}$ are given by 5 seconds, 43 seconds, 8 seconds, 4 seconds, and 3 seconds, respectively, and the composition of the global abstraction from local ones using an approximate composition takes 32 minutes. Let us mention that with comparison to the previous scenario (where only Units 1 to 4 are considered), only the computation time of Unit 2 is modified, since it is the only Unit connected to Unit 5 (see Figure 5). Using the same numerical values, the direct computation of the monolithic abstraction takes 13 hours, which shows the practical speedups that can be attained using the compositional approach.

We then synthesize a safety controller for the computed abstraction. The synthesis of the symbolic controller takes 30 seconds. To validate our controller, we assume that the load power demands for Unit 1, Unit 4 and Unit 5 are as follows. Unit 1 is demanding 0.3 kW from 0 to 250 milliseconds, immediately after stepping up to 1 kW. Unit 4 on the other hand is supposed to be characterized by a demand of 0.3 kW from 0 to 250 milliseconds, then a constant demand of 1 kW from 250 milliseconds to 750 milliseconds. Finally, Unit 5 is characterized by a demand of 0.4 kW from 0 to 250 milliseconds, then a constant demand of 1 kW from 250 milliseconds to 750 milliseconds. All demands are affected by small noise. Source power injections are positive and both limited at 8 kW. The controller is implemented via a microprocessor of clock period $\tau = 0.1$ milliseconds. Voltage responses for different units are illustrated in Figure 6. As expected, the controller guarantees that voltages are kept sufficiently near the nominal value.

6.2. Road traffic model.

6.2.1. Model description and control objective. Consider the road traffic model adapted from [SZ19b] as shown schematically in Figure 7 and described by

$$
\begin{align*}
x_1(k+1) &= (1 - \frac{T_v}{1.6l})x_1(k) + 5u_1(k), \\
x_2(k+1) &= \frac{T_v}{l}x_1(k) + (1 - \frac{T_v}{l} - q)x_2(k) + \frac{T_v}{l}x_4(k), \\
x_3(k+1) &= \frac{T_v}{l}x_2(k) + (1 - \frac{T_v}{l} - q)x_3(k) + 8u_2(k), \\
x_4(k+1) &= \frac{T_v}{l}x_3(k) + (1 + \frac{T_v}{l} - q)x_4(k) + 8u_3(k),
\end{align*}
$$

(6.4)

where the state $x_i$, $i \in \{1, 2, 3, 4\}$, represents the density of traffic in $i^{th}$ section of road given in vehicles per section, $l = 0.25$ km is the length of road, $v = 70$ km/hr is the flow speed, $T = \frac{10}{3600}$ hours is the discrete time interval, and $q = 0.25$ is the ratio representing the percentage of vehicles leaving the section of road. The control inputs $u_1, u_2, u_3 \in U = \{0, 1\}$, where 0 represents red signal and 1 represents green signal in the traffic model. We consider the compact state-space $X = [0, 30]^4$. The control objective is to synthesize controller to keep states in a safe region given by $\hat{S} = [2, 25] \times [5, 25]^3$. 

6.2.2. Abstraction and controller synthesis. We consider four subsystems $S_i, i \in I = \{1, 2, 3, 4\}$, corresponding to four sections in traffic network model. To demonstrate the effectiveness of the proposed result on incremental safety controller synthesis, we compare results obtained using monolithic safety synthesis and incremental safety synthesis on compositional abstractions. For constructing abstractions, we construct an $\varepsilon$-approximate bisimilar abstraction of $S_1$ called $\hat{S}_1$ using state-space discretization-free abstraction techniques as discussed in [Gir14] and [ZTA17] using tool QUEST [JZ17]. The abstractions $\hat{S}_i, i \in \{2, 3, 4\}$, are computed by utilizing partitions of the state-space as shown in [MGW15, RWR17], each abstraction $\hat{S}_i$ is related to the original component $S_i$ by an $(\varepsilon_i, \mu_i)$-approximate alternating simulation relation, with $\varepsilon_i = 0.1$ and $\mu_i = 0$. Note that since the subsystem $S_1$ is incrementally input-state stable and do not have any internal input, the input set of the component $S_1$ is much smaller compared to other components. In such a scenario, state-space discretization-free abstractions are efficient compared to state-space discretization based abstractions (see Section 4.D in [LCGG13] and Section 5.4 in [ZTA17] for detailed discussion).

Monolithic and incremental approaches to safety synthesis are then compared. In the first one, we compute the global compositional abstraction $\tilde{S} = \langle S_i \rangle_{i \in I}^{M, \mathcal{I}}$ by composing local abstractions with a composition parameter $M = (0.0016, 0.1, 0.1, 0.1)^T$. We then monolithically synthesize a safety controller for the global abstraction $\tilde{S}$ with the safe set $\mathcal{S}$ using maximal fixed point algorithm [Tab09]. The total computation time required for obtaining the monolithic safety controller is 9 hours and 20 minutes.

In the second approach, we start from the local abstractions $\hat{S}_i, i \in I$, and first compute safety controllers $C^*_i$ for each abstraction $\hat{S}_i, i \in I$, with local safety specification $\mathcal{S}_i$. Then we compose the local controlled components $S_i|C^*_i, i \in I$, with an $M$-approximate composition with $M = (0.0016, 0.1, 0.1, 0.1)^T$ given as $\langle S_i|C^*_i \rangle_{i \in I}^{M, \mathcal{I}}$. Then, as a final step we synthesize a safety controller for $\langle S_i|C^*_i \rangle_{i \in I}^{M, \mathcal{I}}$ against the safety specification $\mathcal{S}$. The total computation time required for obtaining the safety controller using incremental safety synthesis is 2 hours and 25 minutes which is almost 4 times faster than the monolithic synthesis case. The Figure 8 shows the evolution of traffic densities in each section of the road starting from the initial condition $x = [14, 15, 20, 16]^T$ using controller obtained by proposed incremental approach. One can readily see that all the trajectories evolve within the safe region.

![Figure 7. Traffic network model schematic.](image-url)
In this paper, we proposed a compositional abstraction-based synthesis approach for interconnected systems. We introduce a notion of approximate composition that allows composing different types of abstractions. Moreover, we provided compositional results based on approximate (alternating) simulation relation and showed how these results can be used for incremental safety controller synthesis. Two case studies are given to show the effectiveness of our approach. In future work, we plan to extend the incremental synthesis approach from safety to other types of specifications, such as reachability, stability, or more general properties described by temporal logic formulae. Another direction is to go from deterministic relationships to probabilistic ones \cite{Aba13}, which are more suitable to use when dealing with stochastic systems.

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