NON-COMMUTATIVE LAURENT PHENOMENON FOR
TWO VARIABLES

ALEXANDR USNICH

Abstract. We prove the non-commutative Laurent phenomenon for two variables

1. Introduction

Let us consider an automorphism of the field $K = \mathbb{C}(x, y)$ given by the formula:

$$F : (x, y) \mapsto \left( \frac{H(x)}{y}, x \right),$$

where $H(x) = 1 + h_1 x + \cdots + h_{n-1} x^{n-1} + x^n$ is a reversible polynomial, i.e. $h_i = h_{n-i}$.

The iterations of $F$ are actually given by Laurent polynomials [7]. It means that for any integer $k$ we have:

$$F^k : (x, y) \mapsto (L_1(x, y), L_2(x, y)),$$

where $L_1, L_2 \in \mathbb{C}[x, x^{-1}, y, y^{-1}]$ are Laurent polynomials.

We introduce a non-commutative analog of this transformation: consider

$$F_{nc} : (x, y) \mapsto (y^{-1} H(x), y^{-1} x y).$$

We view $x, y$ as elements freely generating the non-commutative algebra $A$ by addition, multiplication and taking inverses of some elements. Namely, we have a ring morphism $\phi : A \to \mathbb{C}(x, y)$, and we can invert elements $a$ which don’t belong to the kernel of $\phi$. Then $F_{nc}$ is an automorphism of the algebra $A$. If we allow to invert only elements $x, y$, then we will obtain the non-commutative subalgebra $\mathbb{C} < x, x^{-1}, y, y^{-1} > \subset A$ which we call the ring of non-commutative Laurent polynomials.

We will prove the following result, conjectured by M.Kontsevich:

**Theorem 1.1.** For any integer $k$ and for any reversible polynomial $H(x)$, the transformation $F_{nc}^k$ is given by non-commutative Laurent polynomials.

We call this the non-commutative Laurent phenomenon.

Observe that multiplicative commutator $q = x^{-1} y^{-1} x y$ is preserved by $F_{nc}$. In the light of deformation quantization, people often consider algebra, where $q$ is a central element. We’d like to emphasize that we impose no such condition.

A special case of this transformation, where $H(x) = 1 + x^n$, turns up in the study of cluster mutations. In the article [5] we prove the special case
of the Laurent phenomenon for $n = 2$ using explicite computations with matrices. In [6] an alternative proof of the Laurent phenomenon for $n = 2$ is given, via a combinatorial path-counting argument. It is moreover proved that the coefficients of Laurent polynomials are positive.

The main idea of our proof of Theorem (1.1) is as follows. First we resolve birational map $F^k$, namely we construct a sequence of surfaces $Y_i$ and morphisms $\pi_i : Y_i \to \mathbb{P}^1 \times \mathbb{P}^1$, such that the induced birational maps $F_i = \pi_{i+1}^{-1} \circ F \circ \pi_i : Y_i \to Y_{i+1}$ extend to natural biregular isomorphisms. Here $F$ is as defined previously in affine coordinates $x, y$ on $\mathbb{P}^1 \times \mathbb{P}^1$. The surface $Y_i$ is constructed as a blow-up of a toric surface $Y_0^i$, which is a toric weighted blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$, at $2n$ points situated on the chain of toric divisors. We denote by $D_i$ the chain of rational curves on $Y_i$ which is the strict transform of toric divisors on $Y_0^i$. In fact the isomorphism $F_i$ sends the chain $D_i$ to the chain $D_{i+1}$.

Next, we construct quotient triangulated category $\tilde{C}(Y_i) = \tilde{D}(Y_i)/\tilde{D}^1(Y_i)$, where $\tilde{D}(Y_i)$ is a full subcategory of $D(Y_i)$ the derived category of coherent sheaves on $Y_i$ consisting of objects, which are left orthogonal to $O_{Y_i}$. $\tilde{D}^1(Y_i)$ is a full subcategory of $\tilde{D}(Y_i)$ consisting of objects which restrict to 0 at the generic point. We use some properties of the category $\tilde{C}(Y_i)$, which are proved in [4]. Namely this category is generated by one object $Q_i$, which is the image of the line bundle $\pi_i^*O(1, 1) \in \tilde{D}(Y_i)$. Moreover, we have:

$$\text{Hom}_{\tilde{C}(Y_i)}(Q_i, Q_i) = A,$$

where $A$ is the non-commutative algebra, containing distinguished elements $x, y$. The functor $L F_i^*$ descends to an equivalence of quotient categories $L F_i^* : \tilde{C}(Y_{i+1}) \to \tilde{C}(Y_i)$. In [3.1] we write down a specific isomorphism between $Q_i$ and $L F_i^* Q_{i+1}$ in $\tilde{C}(Y_i)$. This gives us an automorphism $F_{nc}$ of $A$, which doesn’t depend on $i$:

$$F_{nc} : A = \text{Hom}_{\tilde{C}(Y_{i+1})}(Q_{i+1}, Q_{i+1}) \xrightarrow{L F_i^*} \text{Hom}_{\tilde{C}(Y_i)}(L F_i^* Q_{i+1}, L F_i^* Q_{i+1}) \xrightarrow{\text{Hom}_{\tilde{C}(Y_i)}(Q_i, Q_i)} A.$$

In the Lemma 3.1 we compute this automorphism explicitly:

$$F_{nc} : (x, y) \mapsto (y^{-1} H(x), y^{-1} xy).$$

Therefore we see, that the functor

$$L \Phi^* = L F_0^* \circ \cdots \circ L F_{k-1}^* : \tilde{C}(Y_k) \to \tilde{C}(Y_0)$$

1by shifts and taking cones
together with the appropriate isomorphism of objects $\Phi^*(Q_k)$ and $Q_0$ in $\tilde{C}(Y_0)$ induces an automorphism $F_{nc}^k$ of $A$:

$$A = \text{Hom}_{\tilde{C}(Y_k)}(Q_k, Q_k) \to \text{Hom}_{\tilde{C}(Y_0)}(Q_0, Q_0) = A.$$  

Then we observe, that morphisms $F_{nc}^k(x), F_{nc}^k(y) : Q_0 \to Q_0$ descend from morphisms in the quotient category $\tilde{D}(Y_0)/\tilde{D}_D(Y_0)$, where $\tilde{D}_D(Y_0)$ is the subcategory of objects supported on the chain of rational curves $D_0$. Therefore $F_{nc}^k(x), F_{nc}^k(y)$ can be viewed also as elements in the endomorphism algebra of $\pi_0^*O(1,1)$ in the quotient category $\tilde{D}(Y_0)/\tilde{D}_B(Y_0)$, where $B$ is the union of $D_0$ and $2n$ exceptional curves of the blow-up of $Y_0$.

Finally we prove in the Lemma 3.3 that the image of the natural functor

$$\text{Hom}_{\tilde{D}(Y_0)/\tilde{D}_B(Y_0)}(Q_0, Q_0) \to \text{Hom}_{\tilde{C}(Y_0)}(Q_0, Q_0) = A$$

is the subalgebra of non-commutative Laurent polynomials

$$C < x, x^{-1}, y, y^{-1} > \subset A.$$  

As we observed, $F_{nc}^k(x), F_{nc}^k(y)$ belong to this subalgebra, so they are non-commutative Laurent polynomials.

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2. Resolution of automorphism

The results of this section appear in [3] in greater generality. We summarize them here for the convenience of our reader.

Consider the birational automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ given by the formula:

$$F : (x, y) \mapsto \left( \frac{H(x)}{y}, x \right),$$

where $H(y) = 1 + h_1 x + \ldots + h_{n-1} x^{n-1} + x^n$ is a polynomial of degree $n$. In the homogeneous coordinates $(X : Z) \times (Y : W)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ the affine coordinates are expressed as $x = \frac{X}{Z}$, $y = \frac{Y}{W}$.

We are interested in constructing explicitly rational surfaces $Y_0, \ldots, Y_k$ equipped with morphisms $\pi_i : Y_i \to \mathbb{P}^1 \times \mathbb{P}^1$ and with biregular isomorphisms $F_i : Y_i \to Y_{i+1}$, such that the following diagrams commute:

$$\begin{array}{ccc}
Y_i & \xrightarrow{F_i} & Y_{i+1} \\
\pi_i \downarrow & & \pi_{i+1} \\
\mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{F} & \mathbb{P}^1 \times \mathbb{P}^1
\end{array}$$

(2.1)

Let us define two series of vectors in $\mathbb{Z}^2$ by a recursive relation:

$$p_0 = (0, 1), p_1 = (-1, 0), p_{i+1} = np_i - p_{i-1};$$
Consider toric surfaces $Y_0^i$ given by the fan spanned by vectors:

$$\{p_i, \ldots, p_0, t_0, t_1, t_2, \ldots, t_{n+2-i}\}.$$ 

Surface $Y_i$ is constructed as a blow-up of the surface $Y_0^i$ in $2n$ points. Fans of surfaces $Y_0^i$ contain sub-fan $\{p_1, p_0, t_0, t_1\}$, which defines a surface $\mathbb{P}^1 \times \mathbb{P}^1$, so they admit natural toric projections to it. We can actually think of them as weighted blow-ups of $\mathbb{P}^1 \times \mathbb{P}^1$. We use standard notations $(x, y)$ for the coordinates on toric surfaces. Namely if a vector $(a, b)$ corresponds to a toric divisor, then rational function $\frac{b}{y}$ induces a canonical (up to taking an inverse) rational coordinate on this divisor. By a canonical coordinate on a divisor $D$ we will mean a rational function, which induces an isomorphism of $D$ with $\mathbb{P}^1$. On each surface $Y_0^i$ toric divisors form a chain of rational curves. Their strict transforms form a chain of rational curves on the blow-up $Y_i$ and the canonical rational coordinates lift from each curve to its strict transform.

The toric divisors corresponding to vectors $t_i$ and $p_i$ will be denoted $T_i$ and $P_i$ respectively. Let $x$ be the canonical coordinate on $P_0$, and $y$ the canonical coordinate on $T_0$. Note that intersection points with other toric divisors have coordinates $0$ and $\infty$.

We begin with lemma, which shows how to resolve birational transformation: $(x, y) \mapsto \left(\frac{H(x)}{y}, x\right)$. Let $Z_1^0$ be the toric surface corresponding to the fan: $\{p_1, p_0, t_0, t_1, t_2\}$, and let $Z_2^0$ be the toric surface corresponding to the fan $\{p_2, p_1, p_0, t_0, t_1\}$.

(2.2)

\[
\begin{array}{ccc}
Z_1 & \xrightarrow{G} & Z_2 \\
\downarrow & & \downarrow \\
Z_1^0 & \xrightarrow{r_1} & Z_2^0 \\
\downarrow & & \downarrow \\
P^1 \times P^1 & \xrightarrow{F} & P^1 \times P^1
\end{array}
\]

The surface $Z_1$ is a blow-up of $Z_1^0$ in the $n$ points on curve $P_0$, where $H(x) = 0$. The surface $Z_2$ is a blow-up of $Z_2^0$ in the $n$ points on $T_0$ where $H(y) = 0$.

**Lemma 2.1.** For any reversible polynomial $H$ with distinct roots, the induced map $G$ is a regular isomorphism of surfaces $Z_1$, $Z_2$. Moreover it preserves the canonical coordinates on the chain of strict transforms of toric divisors.

**Proof.** We denote by $C_{ab}$ the cone in $\mathbb{R}^2$ spanned by vectors $a, b$. Such cones correspond to toric points, and we have coordinates in the neighbourhood of these points on $Z_1^0$. 

\[
t_0 = (1, 0), t_1 = (0, -1), t_{i+1} = nt_i - t_{i-1}.
\]
The coordinates near toric point $C_{p_1,p_0}$ on $Z_1^0$ are $(x^{-1}, y)$; near $C_{p_0,t_0}$ are $(x, y)$; near $C_{t_0,t_1}$ are $(x, y^{-1})$; near $C_{t_1,t_2}$ are $(x^{-1}, x^n y^{-1})$; and near $C_{t_2,p_1}$, which is a singular toric point, are $(x^{-1}, y^{-1}, x^{-n} y)$.

When we blow-up surface $Z_1^0$ at $n$ points on $P_0$, we pull-back coordinates near toric points to $Z_1$, so the coordinates near pull-back of $C_{p_1,p_0}$ on $Z_1$ are $(x^{-1}, \frac{y}{H(x)})$; near pull-back of $C_{p_0,t_0}$ are $(x, \frac{y}{H(x)})$.

The coordinates near other pull-backs are the same as on $Z_1^0$.

Birational transformation $F$ of $P_1 \times P_1$ lifts to a birational map $F^0 : Z_1^0 \to Z_2^0$. Under this map toric divisors $P_1, P_0, T_0, T_1, T_2$ go to divisors $P_2, P_1, P_0, T_0, T_1$ respectively. We now prove that this map is regular everywhere except at $n$ points on the divisor $P_0$ where $H(x) = 0$. Because $H$ has distinct roots, all these points are different.

To avoid confusion, we denote by $(u, v)$ the rational coordinates on $Z_2^0$ and on $Z_2$, so that $G^* u = \frac{H(x)}{y}, G^* v = x$. The map $G$ sends the neighbourhood of the point $C_{p_2,p_1}$ to the neighbourhood of the point $C_{p_1,p_0}$:

$$G^* : \mathbb{C}[u^{-1} v^n, v^{-1}] \to \mathbb{C}[x^{-1}, \frac{y}{H(x)}],$$

$$G^* (u^{-1} v^n, v^{-1}) = (\frac{x^n y}{H(x)}, x^{-1}) = (\frac{y}{H(x)}, x^{-1}).$$

It is an isomorphism of affine neighbourhoods. The canonical coordinate $u^{-1} v^n$ of $P_2$ on $Z_2$ goes to $\frac{y}{H(x)}$, which is equal to $y$ on $P_1$, because divisor $P_2$ is defined by $x^{-1} = 0$. The canonical coordinate $v^{-1}$ on $P_1$ goes to the canonical coordinate $x^{-1}$ on $P_0$.

We do similar verifications for other pull-backs of toric points. For the neighbourhood of $G^*(C_{p_1,p_0}) = C_{p_0,t_0}$ we have:

$$G^* : \mathbb{C}[u^{-1}, v] \to \mathbb{C}[x, \frac{y}{H(x)}],$$

$$G^* (u^{-1}, v) = (\frac{y}{H(x)}, x).$$

It is again an isomorphism of affine neighbourhoods, and the canonical coordinate $u^{-1}$ on $P_0$ goes to $\frac{y}{H(x)}$, which is equal to $y$ on $T_0$, because $T_0$ is defined by $x = 0$ in this neighbourhood.

For the neighbourhood of $G^*(C_{p_0,t_0}) = C_{t_0,t_1}$ we have:

$$G^* : \mathbb{C}[\frac{u}{H(v)}, v] \to \mathbb{C}[x, y^{-1}],$$

$$G^* (u, v) = (\frac{H(x)}{y H(x)}, x) = (y^{-1}, x).$$
It is an isomorphism of affine neighbourhoods. The canonical coordinate \( v \) on \( T_0 \) goes to \( x \) on \( T_1 \). For the neighbourhood of \( G^*(C_{t_0,t_1}) = C_{t_1,t_2} \) we have:

\[
G^*: \mathbb{C}[\frac{u}{H(v^{-1})}, v^{-1}] \to \mathbb{C}[x^{-1}, x^n y^{-1}],
\]

\[
G^*(\frac{u}{H(v^{-1})}, v^{-1}) = (\frac{H(x)}{yH(x^{-1})}, x^{-1}) = (x^n y^{-1}, x^{-1}).
\]

It is an isomorphism of affine neighbourhoods. The canonical coordinate \( u \) on \( T_1 \) goes to \( \frac{H(x)}{y} \) on \( T_2 \), but \( T_2 \) is defined by \( x^{-1} = 0 \), so we can write \( \frac{H(x)}{y} = (x^n y^{-1})H(x^{-1}) = x^n y^{-1} \). This proves that map \( G \) preserves canonical coordinates on toric divisors.

The four neighbourhoods that we considered provide the covering of \( Z_1 \) except at the point \( C_{t_2,p_1} \) and at \( n \) points, each lying on the exceptional curve of the blow-up. We verify, that at these points \( G \) is also an isomorphism.

For the neighbourhood of \( G^*(C_{t_1,t_2}) = C_{t_2,p_1} \) we have:

\[
F^*: \mathbb{C}[u^{-1}, v^{-1}, uv^{-n}] \to \mathbb{C}[x^{-1}, y^{-1}, x^{-n} y],
\]

\[
F^*(u^{-1}, v^{-1}, uv^{-n}) = (\frac{y}{H(x)}, x^{-1}, \frac{H(x)}{x^n y}) = ((x^{-n} y)H(x^{-1})^{-1}, H(x^{-1})y^{-1}).
\]

This map is well defined outside the divisor \( H(x^{-1}) = 0 \). The point \( C_{t_2,p_1} \) doesn’t belong to this divisor, so \( G \) is well defined at this point.

If \( \lambda \) is a root of polynomial \( H \), then we have coordinates \((\frac{x}{y}, y)\) near the point on the exceptional curve, where we have to verify that \( G \) is regular. The coordinates near the corresponding point on \( Z_2 \) are \((u, \frac{x^{-1} y}{u}^{-1})\). It is straightforward to see that \( G^* \) defines an isomorphism of local rings.

Recall that we defined the toric surface \( Y_i^0 \) as given by the fan

\[
\{p_i, \ldots, p_1, p_0, t_0, t_1, \ldots, t_{n+1-i}\}.
\]

Let’s blow it up at \( n \) points where \( P_0 \) intersects \( H(x) = 0 \), and at \( n \) points where \( T_0 \) intersects \( H(y) = 0 \). Here \( x \) and \( y \) are the canonical coordinates on \( P_0 \) and \( T_0 \) respectively. The canonical coordinate are defined up to an inverse, so the polynomial \( H \) needs to be reversible, for the blow up not to depend on the choice of a coordinate. Let us denote this blow-up by \( Y_i \). Let \( D_i \subset Y_i \) be the strict transform of toric divisors under this blow-up.

As a corollary of Lemma 2.1 we have:

**Lemma 2.2.** If the polynomial \( H \) has distinct roots and is reversible, then the map \( F \) induces a regular automorphism \( F_i \) between \( Y_i \) and \( Y_{i+1} \). Moreover \( F_i(D_i) = D_{i+1} \).
Proof. Note that the surface $Y_i$ can be obtained from the surface $Z_1$ of previous lemma, by making two kinds of blow-ups. First, we perform weighted blow-ups to introduce toric divisors $P_{i+1}, \ldots, P_2, T_3, \ldots, T_{n+1-i}$. Then we to blow-up $n$ points on the divisor $T_0$, defined by the equation $H(y) = 0$. We blow-up $Z_2$ in a similar fashion to obtain $Y_{i+1}$. By Lemma 2.1 the map $F$ lifts to regular map $G$ from $Z_1$ to $Z_2$. Divisor $P_3$ on $Z_2$ is a weighted blow-up at the point $C_{t_1,p_2}$. The weights are determined using expression of the vector $p_3$ as the linear combination of $t_1$ and $p_2$. But this expression is the same as the expression of the vector $p_2$ as the linear combination of $t_2$ and $p_1$. So $G$ sends the weighted blow-up corresponding to $P_3$ to the weighted blow-up corresponding to $P_2$. The same argument works for other toric divisors $P_a, T_b$. The toric divisors $P_{i+1}, \ldots, P_2$ are then mapped to $P_{i+2}, \ldots, P_3$, as well as $T_3, \ldots, T_{n+1-i}$ are mapped to $T_2, \ldots, T_{n-i}$.

Also $n$ points on $T_0$ where $H(y) = 0$ are mapped to $n$ points on $P_0$ where $H(x) = 0$, because the canonical coordinates are preserved by $G$ by the previous lemma. Therefore, the blow-ups we do to $Z_1$ to produce $Y_i$ correspond under isomorphism $G$ precisely to the blow-ups we do to $Z_2$ to produce $Y_{i+1}$, and hence $G$ lifts to an isomorphism $F_i : Y_i \rightarrow Y_{i+1}$. The last statement of lemma is also clear.

This lemma implies, that we have a regular isomorphism of surfaces:

$$\Phi = F_{k-1} \circ \cdots \circ F_0 : Y_0 \rightarrow Y_k.$$  

3. DG-CATEGORY ASSOCIATED TO A RATIONAL SURFACE

Let $D(Y_i)$ denote the bounded derived category of coherent sheaves on $Y_i$. By Lemma 2.2 we have a functor

$$\mathbb{L} F_i^* : D(Y_{i+1}) \rightarrow D(Y_i),$$

which is an equivalence of triangulated categories.

In [4] we’ve introduced the notion of $\widetilde{D}(Y_i)$ full triangulated subcategory of $D(Y_i)$ which consists of objects $E$ for which $\text{RHom}_{Y_i}(E, O_{Y_i}) = 0$. As

$$\mathbb{L} F_i^* O_{Y_{i+1}} = O_{Y_i},$$

$\mathbb{L} F_i^*$ restricts to an equivalence

$$\mathbb{L} F_i^* : \widetilde{D}(Y_{i+1}) \rightarrow \widetilde{D}(Y_i).$$

Let $\pi_i : Y_i \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the natural projections. Recall that $\widetilde{D}(\mathbb{P}^1 \times \mathbb{P}^1)$ is generated by three objects:

$$\widetilde{D}(\mathbb{P}^1 \times \mathbb{P}^1) = \langle O(1,0), O(0,1), O(1,1) \rangle.$$

Denote by $Q_i = \pi_i^* O(1,1) \in \widetilde{D}(Y_i)$ the pull-back of the line bundle $O(1,1)$ by $\pi_i$.

Let $D^1(Y_i)$ be the full subcategory of $D(Y_i)$ consisting of objects whose support is at most a divisor, in other words we take objects of $\widetilde{D}(Y_i)$ which
restrict to 0 at the generic point of $Y_i$. Let $D_{D_i}^1(Y_i)$ be the subcategory of $D^1(Y_i)$ consisting of objects supported on $D_i$, the union of all divisors $T_a$ and $P_b$.

Observe that $\mathbb{L} F^*_i$ takes subcategories $D_{D_{i+1}}^1(Y_{i+1})$ and $D^1(Y_{i+1})$ to subcategories $D_{D_i}(Y_i)$ and $D^1(Y_i)$ respectively. This is because $F_i$ is a regular isomorphism, and $F(D_i) = D_{i+1}$. Let also:

$$\tilde{D}^1(Y_i) = D^1(Y_i) \cap \tilde{D}(Y_i),$$
$$\tilde{D}_{D_i}(Y_i) = D_{D_i}^1(Y_i) \cap \tilde{D}(Y_i).$$

The non-commutative cluster mutations appear, when we look at the factor category

$$\tilde{C}(Y_i) = \tilde{D}(Y_i)/\tilde{D}^1(Y_i).$$

It is proved in [4], that this category is a birational invariant of a variety. For rational surfaces it is generated by one object $Q_i$, and moreover

$$\text{Hom}_{\tilde{C}(Y_i)}(Q_i, Q_i) = A,$$

where $A$ is a non-commutative algebra. This algebra is a natural setting for non-commutative cluster mutations. Let us recall some properties of this algebra $A$. First of all there is an embedding $i : \mathbb{C} < x, y > \hookrightarrow A$, and there is a natural map $\phi : A \rightarrow \mathbb{C}(x, y)$. Moreover the kernel of the map $\phi$ is a commutator ideal of $A$:

$$\ker(\phi) = A[A, A].$$

We also have the following property: any $a \in A$ with $\phi(a) \neq 0$ is invertible.

We now choose a way to identify objects $Q_i$ and $F^* Q_{i+1}$ in $\tilde{C}(Y_i)$. This will induce a map on endomorphism ring of object, so we will get a map $F_{nc} : A \rightarrow A$, which we will compute explicitly.

Recall, that $F$ induces a regular map from $Z_1$ to $Z_2$, and we lift it after making some blow-ups to a regular map from $Y_i$ to $Y_{i+1}$. So we can choose an identification of $O_{Z_1}(1, 1)$ and $G^* O_{Z_2}(1, 1)$ on $Z_1$ in $\tilde{C}(Z_1)$, and then lift this identification to $\tilde{C}(Y_i)$.

The surface $Z_2$ is the blow-up of toric surface $Z_2^n$ at $n$ points on the toric divisor $T_0$. We identify this divisor with its strict transform. Denote by $E$ the exceptional curve of this blow-up. It is the union of $n$ rational curves. Also $Z_2$ has a chain of rational curves $P_2, P_1, P_0, T_0, T_1$. And we have linear equivalences of divisors:

$$O_{Z_2}(0, 1) = P_0 = T_1 + P_2,$$
$$O_{Z_2}(1, 0) = T_0 + E = P_1 + nP_2.$$

The divisor $O_{Z_2}(1, 1)$ is therefore linearly equivalent to $T_1 + P_1 + (n+1)P_2$. Then we compute its pull-back by $G$ to $Z_1$:

$$G^* O_{Z_2}(1, 1) = G^* (T_1 + P_1 + (n+1)P_2) = T_2 + P_0 + (n+1)P_1.$$

On $Z_1$ we have a chain of rational curves $P_1, P_0, T_0, T_1, T_2$, and we have the exceptional curve $C$ of the blow-up of $P_0$ at $n$ points. Note that the
effective divisor \( O_{Z_1}(1, 1)(-C) = P_0 + P_1 + T_2 \) is dominated by \( G^* O_{Z_2}(1, 1) = T_2 + P_0 + (n + 1)P_1 \), so we have a natural morphism of line bundles

\[
i_0 : O_{Z_1}(1, 1)(-C) \xrightarrow{nP_1} G^* O_{Z_2}(1, 1).
\]

There is also a unique up to scalar multiplication map of line bundles \( i_1 : O_{Z_1}(1, 0) \xrightarrow{P_0} O_{Z_1}(1, 1)(-C) \), which lifts the map \( O(1, 0) \xrightarrow{} O(1, 1) \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \), which vanishes along the divisor \( P_0 \). Finally there is a map \( i_3 : O_{Z_1}(1, 0) \xrightarrow{T_1 + nT_2} O_{Z_1}(1, 1) \), which vanishes along the divisor \( T_1 \). All in all, we have the following sequence of maps of line bundles on \( Z_1 \):

\[
(3.1) \quad G^* O_{Z_2}(1, 1) \xrightarrow{i_1} O_{Z_1}(1, 1)(-C) \xrightarrow{i_2} O_{Z_1}(1, 0) \xrightarrow{i_3} O_{Z_1}(1, 1).
\]

If we consider the line bundles in this diagram as objects of the derived category of coherent sheaves \( D(Z_1) \) then they belong to \( \overline{D}(Z_1) \). If we pull (3.1) back to \( Y_i \), the objects will belong to \( \overline{D}(Y_i) \). We now claim that the cones of the morphisms in (3.1) belong to \( D^1_{D_i}(Y_i) \). Indeed, on the surface \( Z_1 \) the object \( \text{Cone}(i_1) \) is supported on \( P_1 \), \( \text{Cone}(i_2) \) is supported on \( P_0 \), \( \text{Cone}(i_3) \) is supported on \( T_1 \cup T_2 \). Blowing up \( n \) points on the curve \( T_0 \) doesn't change this. Therefore on \( Y_i \) all these cones belong to \( D_{D_i}(Y_i) \). Consequently, the morphisms in (3.1) become isomorphisms in the quotient categories \( \overline{D}(Y_i)/\overline{D}_{D_i}(Y_i) \) and \( \overline{C}(Y_i) = \overline{D}(Y_i)/\overline{D}^1(Y_i) \). In the later category we thus obtain a particular isomorphism \( j_k : Q_i \xrightarrow{} F_i^* Q_{i+1} \). This isomorphism allows us to define an automorphism of the ring \( A \):

\[
(3.2) \quad F_{nc} : A = \text{Hom}_{\overline{C}(Y_i+1)}(Q_{i+1}, Q_{i+1}) \xrightarrow{\text{L}F_i^*} \text{Hom}_{\overline{C}(Y_i)}(F_i^* Q_{i+1}, F_i^* Q_{i+1}) \xrightarrow{j_k} A = \text{Hom}_{\overline{C}(Y_i)}(Q_i, Q_i).
\]

**Lemma 3.1.** The map \( F_{nc} \) is given by

\[
F_{nc} : (x, y) \mapsto (y^{-1}H(x), y^{-1}xy)
\]

**Proof.** We first explain, how we identify the algebra \( A \) with the endomorphism ring \( \text{Hom}_{\overline{C}(Y_i)}(Q_i, Q_i) \). If \( \pi : X \xrightarrow{} Y \) is a blow-up of a surface \( Y \) at the smooth point, then \( \text{L} \pi^* \) induces a fully faithful embedding, and we have a semiorthogonal decomposition:

\[
D(X) = \langle O_E, \text{L} \pi^* D(Y) \rangle,
\]

where \( O_E \) is a structure sheaf of the exceptional curve \( E \) of the blow-up. The similar decomposition works for weighted blow-ups. Functor \( \text{L} \pi^* \) then induces equivalences between quotient categories \( \overline{C}(X) \) and \( \overline{C}(Y) \). In particular, for a surface \( Y_i \) we use sequence of blow-ups \( \pi_i : Y_i \xrightarrow{} \mathbb{P}^1 \times \mathbb{P}^1 \) to identify \( \overline{C}(Y_i) \) with \( \overline{C}(\mathbb{P}^1 \times \mathbb{P}^1) \).
If \((X : Z) \times (Y : W)\) are homogeneous coordinates on \(\mathbb{P}^1 \times \mathbb{P}^1\), then diagram
\[
(3.3) \quad O(1, 1) \overset{Z}{\hookleftarrow} O(0, 1) \overset{X}{\to} O(1, 1)
\]
defines an element in \(\text{Hom}_{\tilde{C}}(O(1, 1), O(1, 1))\), which we denote by \(x\). Similarly the diagram
\[
O(1, 1) \overset{W}{\hookleftarrow} O(1, 0) \overset{Y}{\to} O(1, 1)
\]
defines an element in \(\text{Hom}_{\tilde{C}}(O(1, 1), O(1, 1))\), which we denote by \(y\).

In the article [4] we computed, that \(\text{Hom}_{\tilde{C}}(\mathbb{P}^2)(O(2)) = A\). If \((X : Y : Z)\) are homogeneous coordinates on \(\mathbb{P}^2\), then denote by \(x, y\) the elements represented by diagrams \(O_{\mathbb{P}^2}(2) \overset{Z}{\leftarrow} O_{\mathbb{P}^2}(1) \overset{X}{\to} O_{\mathbb{P}^2}(2)\) and \(O_{\mathbb{P}^2}(2) \overset{Y}{\leftarrow} O_{\mathbb{P}^2}(2)\) respectively. Consider the toric surface \(T\), given by the fan \((1, 0), (0, -1), (-1, -1), (-1, 0), (0, 1)\). It admits toric projections to both \(\mathbb{P}^2\) and \(\mathbb{P}^1 \times \mathbb{P}^1\). We can therefore pull-back both \(D(\mathbb{P}^2)\) and \(D(\mathbb{P}^1 \times \mathbb{P}^1)\) to \(D(T)\) and compare the diagrams there. We observe that on surface \(T\) the divisors \(O_T(1, 0), O_T(0, 1)\) embed into \(O_T(1)\), and the divisor \(O_T(1, 1)\) embeds into \(O_T(2)\). Moreover, the diagrams that define \(x, y\) in \(\tilde{C}(\mathbb{P}^1 \times \mathbb{P}^1)\) and \(\tilde{C}(\mathbb{P}^2)\) give the same morphisms in \(\tilde{C}(T)\), thus identifying \(\text{Hom}_{\tilde{C}(\mathbb{P}^1 \times \mathbb{P}^1)}(O(1, 1), O(1, 1))\) with \(A\).

We now compute the action of \(F_{nc}\) on \(A\). It is enough to compute the action on elements \(x, y\). First we compute the preimages of line bundles on \(Z_1\):

\[
G^*O_{Z_2}(1, 0) = G^*(P_1 + nP_2) = P_0 + nP_1,
\]
\[
G^*O_{Z_2}(0, 1) = G^*(T_1 + P_2) = P_1 + T_2,
\]
\[
G^*O_{Z_2}(1, 1) = G^*(P_1 + nP_2 + T_1) = P_0 + (n + 1)P_1 + T_2.
\]

Next recall that to represent \(x, y\) on \(Z_2\) we need the following maps:

\[
X, Z : O_{Z_2}(0, 1) \to O_{Z_2}(1, 1),
\]

\[
Y, W : O_{Z_2}(1, 0) \to O_{Z_2}(1, 1).
\]

The map \(X\) defines an inclusion of line bundles \(O_{Z_2}(0, 1) \overset{T_0 + E}{\to} O_{Z_2}(1, 1)\), given by divisor \(T_0 + E\). Consequently we will write the equality, where both sides are understood as inclusions of line bundles:

\[
X = T_0 + E.
\]

In a similar way we compute:

\[
Z = P_1 + nP_2,
\]

\[
Y = P_0,
\]

\[
W = T_1 + P_2.
\]

Therefore we can compute the pull-backs:

\[
G^*X = T_1 + G^*(E),
\]

\[
G^*Z = P_0 + nP_1.
\]
\[ G^*Y = T_0, \]
\[ G^*W = T_2 + P_1. \]

Let \( \pi'_1 \) be the projection of \( Z_1 \) to the toric surface \( Z_0^0 \), and \( C \) is the exceptional curve of the blow-up. As before \( \pi_1 \) is the toric projection from \( Z_0^0 \) to \( \mathbb{P}^1 \times \mathbb{P}^1 \). We can write, using the same notation for the toric divisor \( P_0 \) on \( Z_0^0 \) and for its strict transform to \( Z_1 \):

\[ P_0 + C = \pi'_1 P_0. \]

Moreover for any line bundle \( L \) on \( Z_0^0 \) we have

\[ (\pi'_1)^* L_C = O_C. \]

This implies that there are exact sequences of coherent sheaves on \( Z_1 \):

\[ 0 \to G^*O_{Z_2}(1, 0) \to (\pi'_1)^*(P_0 + nP_1) \to O_C \to 0, \]
\[ 0 \to G^*O_{Z_2}(1, 1) \to (\pi'_1)^*(P_0 + (n + 1)P_1 + T_2) \to O_C \to 0. \]

Observe that \( O_C \) is an object of \( \tilde{D}(Z_1) \). The terms on the left and on the right in both sequences belong to the category \( \tilde{D}(Z_1) \), therefore so do the terms in the middle.

On the surface \( Z_0^0 \) we have:

\[ O_{Z_0^0}(1, 0) = P_1 + T_2. \]

Therefore we have inclusions of line bundles on \( Z_0^0 \), which are isomorphisms outside \( T_2 \):

\[ P_0 + nP_1 \xrightarrow{nT_2} P_0 + nP_1 + nT_2 = O_{Z_1}(n, 1), \]
\[ P_0 + (n + 1)P_1 + T_2 \xrightarrow{nT_2} P_0 + (n + 1)P_1 + (n + 1)T_2 = O_{Z_1}(n + 1, 1). \]

We use objects \( O(n, 1), O(n + 1, 1) \in \tilde{D}(\mathbb{P}^1 \times \mathbb{P}^1) \), and their pullbacks \( O_{Z_1}(n, 1), O_{Z_1}(n + 1, 1) \) to \( Z_1 \). We have inclusions of line bundles on surface \( Z_1 \):

\[ i : G^*O_{Z_2}(1, 0) \xrightarrow{C + nT_2} O(n, 1), \]
\[ j : G^*O_{Z_2}(1, 1) \xrightarrow{C + nT_2} O(n + 1, 1). \]

And therefore we have compositions

\[ j \circ \mathbb{L} G^* X, j \circ \mathbb{L} G^* Z : G^*O_{Z_2}(0, 1) = O_{Z_1}(1, 0) \to O_{Z_1}(n + 1, 1). \]

Now recall that \( G^* X = T_1 + G^* E, j = C + nT_2 \), therefore \( j \circ F^* X = C + T_1 + G^* E + nT_2 \). But the morphism \( j \circ G^* X \) is a lift of a morphism from \( \mathbb{P}^1 \times \mathbb{P}^1 \), where it is given by

\[ (\pi_1) \circ (\pi'_1)(C + T_1 + G^* E + nT_2) = T_1 + \pi_1(\pi'_1(G^* E)). \]

The divisor \( \pi_1(\pi'_1(G^* E)) \) is given by \( H(x) = 0 \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \). If we introduce homogeneous polynomial \( H(X, Z) \) defined by the condition that \( \frac{H(X, Z)}{Z^n} = \)
$H(\frac{X}{Z})$, then the inclusion of line bundles $j \circ G^*X : O_{Z_1}(1, 0) \to O_{Z_1}(n+1, 1)$ is a pullback of the map $H(X, Z)W$. We can write

$$j \circ G^*X = Z^nH(\frac{X}{Z})W.$$ 

By the similar argument $j \circ G^*Z = (\pi_1 \circ \pi_1)^*(P_0 + nP_1)$. It follows that

$$j \circ G^*Z = Z^nY.$$ 

Both inclusions $i$ and $j$ are given by the same divisor $C + nT_2$, so

$G^*Y, G^*W \in \text{Hom}(G^*O_{Z_2}(1, 0), G^*O_{Z_2}(1, 1)) = \text{Hom}(O(n, 1), O(n + 1, 1)).$

Inclusion $G^*Y$ is given by $T_0$, so $j \circ G^*Y = X \circ i$. Inclusion $G^*W$ is given by $T_2 + P_1$, so $j \circ G^*W = Z \circ i$.

We also need to know the map $j \circ i_1 \circ i_2 : O(1, 0) \to O(n + 1, 1)$, where $i_1, i_2$ are used in \([5.1]\) to identify $O_{Z_1}(1, 0)$ and $G^*O_{Z_2}(1, 1)$ in $\tilde{C}(Z_1)$.

In our notations $i_3 = W$. By using the similar techniques, we see that on the surface $\mathbb{P}^1 \times \mathbb{P}^1$ we have $j \circ i_1 \circ i_2 = (\pi_1 \circ \pi_1)^*(P_0 + nP_1)$. It implies that

$$j \circ i_1 \circ i_2 = Z^nY.$$ 

We use map $Z$ to identify $O_{Z_1}(l, 1)$ and $O_{Z_1}(l + 1, 1)$ in $\tilde{C}(Z_1)$, and the map $W$ to indentify $O_{Z_1}(1, l)$ and $O_{Z_1}(1, l + 1)$ in $\tilde{C}(Z_1)$.

Let us denote by

$$\alpha \in \text{Hom}_{\tilde{C}(Z_1)}(O_{Z_1}(1, 1), O_{Z_1}(n + 1, 1))$$

the following morphism

$$\alpha = j \circ i_1 \circ i_2 \circ i_3^{-1}.$$

Then we can write the element $F_{nc}(x)$ in the category $\tilde{C}(Z_1)$ as:

$$F_{nc}(x) = \alpha^{-1} \circ j \circ F^*X \circ (F^*Z)^{-1} \circ j^{-1} \circ \alpha = (Z^nY)^{-1} Z^nH(\frac{X}{Z})W(Z^nY)^{-1} (Z^nY) = y^{-1}H(x)y^{-1}y = y^{-1}H(x).$$

Similarly

$$F_{nc}(y) = (\alpha)^{-1} \circ j \circ F^*Y \circ (F^*W)^{-1} \circ j^{-1} \circ \alpha = (Z^nY)^{-1} XZ^{-1} (Z^nY) = y^{-1}xy.$$ 

The claim of the lemma follows from the observation, that the pull-back along the map $Y_i \to Z_1$ induces equivalence of categories $\tilde{C}(Z_1)$ and $\tilde{C}(Y_i)$. In particular we note, that the formula for $F_{nc}$ doesn’t depend on $i$. \(\square\)

We can now proceed to the final argument.

**Theorem 3.1.** $F_{nc}^k(x), F_{nc}^k(y)$ are non-commutative Laurent polynomials.

**Proof.** First note that elements $F_{nc}^k(x), F_{nc}^k \in A$ are represented by elements of $\text{Hom}_{\tilde{C}(Y_0)}(Q_0, Q_0)$. Let $D_i$ be the chain of strict transforms of toric divisors from $Y_i^0$ to $Y_i$. We have natural functor

$$\mathbb{K}_i : \tilde{D}(Y_i)/\tilde{D}_{D_i}(Y_i) \to \tilde{D}(Y_i)/\tilde{D}^1(Y_i) = \tilde{C}(Y_i).$$
In particular, we have the induced map

\[ \mathbb{K}_i : \text{Hom}(\tilde{D}(Y)/\tilde{D}_0(Y))(Q_i, Q_i) \rightarrow \text{Hom}(\tilde{C}(Y))(Q_i, Q_i). \]

**Lemma 3.2.** Elements \( F_{nc}^k(x), F_{nc}^k(y) \) belong to the image of \( \mathbb{K}_0 \).

**Proof.** By definition (3.2) of \( F_{nc} \) we have:

\[ F_{nc}^k = j_0^* \circ \mathbb{L} F_0^* \circ \cdots \circ j_{k-1}^* \circ \mathbb{L} F_{k-1}^*. \]

If \( \Phi = F_{k-1} \circ \cdots \circ F_0 : Y_0 \rightarrow Y_k \), and \( \sigma : \Phi^* O_{Y_k}(1, 1) \xrightarrow{\sim} O_{Y_0}(1, 1) \) is an appropriate identification in the category \( \tilde{C}(Y_0) \), then \( F_{nc}^k \) is the composition

\[ \text{Hom}_{\tilde{C}(Y_0)}(Q_k, Q_k) \xrightarrow{\mathbb{L} F^*} \text{Hom}_{\tilde{C}(Y_0)}(\Phi^* Q_k, \Phi^* Q_k) \xrightarrow{\sigma} \text{Hom}_{\tilde{C}(Y_0)}(Q_0, Q_0). \]

Observe, that \( x = X \circ Z^{-1} \) as defined in (3.3) is well-defined morphism in \( \tilde{D}(Y_k)/<\text{Cone}(Z)> \), because it uses the inverse of morphism \( Z \). But \( \text{Supp}(\text{Cone}(Z)) = P_1 \cup P_2 \cup \cdots \cup P_{k+1} \subset D_k \), so in particular it is an element of \( \tilde{D}(Y_k)/\tilde{D}_0(Y_k) \). Similarly \( y = Y \circ W^{-1} \) is well-defined morphism of \( \tilde{D}(Y_k)/\tilde{D}_0(Y_k) \), because \( \text{Supp}(\text{Cone}(W)) = T_1 \cup P_2 \cup \cdots \cup P_{k+1} \subset D_k \).

Lemma (2.2) implies, that \( \Phi^{-1}(D_k) = D_0 \). So \( \mathbb{L} \Phi^*(x), \mathbb{L} \Phi^*(y) \) are well-defined in the category \( \tilde{D}(Y_0)/\tilde{D}_0(Y_0) \).

Morphism \( \sigma \) is a composition of morphisms of the kind \( \mathbb{L}(F_{i-1} \cdots F_0)^* \circ j_i \). Recall that \( j_i \) is defined in (3.2) using identification \( i_3 \circ i_2^{-1} \circ i_1^{-1} \) of \( F_i^* Q_{i+1} \) and \( Q_i \) as in (3.1). Observe, that \( i_1, i_2, i_3 \) are invertible isomorphisms in \( \tilde{D}(Y_i)/\tilde{D}_D(Y_i) \), therefore \( \sigma \) is invertible in \( \tilde{D}(Y_0)/\tilde{D}_0(Y_0) \). This proves the lemma.

Let us take a curve \( B = \pi_0^{-1}(X^2 Y Z W = 0) \subset Y_0 \), which is the preimage of all toric divisors on \( \mathbb{P}^1 \times \mathbb{P}^1 \). It is the union of strict transform of toric divisors \( D_0 \) and \( 2n \) exceptional curves of blow-up of \( Y_0^0 \). Then we have:

**Lemma 3.3.** In the quotient category \( C = \tilde{D}(Y_0)/\tilde{D}_B(Y_0) \) we have:

\[ \text{Hom}(C(O(1, 1), O(1, 1)) = C < x, x^{-1}, y, y^{-1} >. \]

**Proof.** By construction \( \pi_0 \) is a composition of regular maps: \( Y_0 \rightarrow Y_0^0 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \), where first arrow is a blow-up at \( 2n \) distinct smooth points, and \( Y_0^0 \) is a toric surface. For the blow-up \( \pi : Y_0 \rightarrow Y_0^0 \) with exceptional divisor \( E \) we have a semiorthogonal decomposition \([1], \ [2]):\]

\[ \tilde{D}(Y_0) = \mathbb{L} \pi^*(\tilde{D}(Y_0^0)), O_E >. \]

So we have an equivalence of categories

\[ \tilde{D}(Y_0)/\tilde{D}_B(Y_0) \rightarrow \tilde{D}(Y_0^0)/\tilde{D}(Y_0^0)_{\text{tor}}. \]

In the last formula \( \tilde{D}(Y_0^0)_{\text{tor}} \) is the full subcategory of objects supported on toric divisors. Because of the semiorthogonal decomposition of the blow-up, this quotient category is the same for any toric surface. Even though \( Y_0^0 \) is not smooth, we can consider a smooth toric surface \( T \) with an toric
morphism \( f : T \to Y_0^0 \), and we can speak about the quotient \( \tilde{D}(T)/\tilde{D}(T)_{tor} \) instead. We didn’t do it in order to avoid cumbersome formulas.

As a consequence we have:

\[
C = \tilde{D}(Y_0)/\tilde{D}_0(Y_0) = \tilde{D}(P^1 \times P^1)/\tilde{D}(XYZW=0)(P^1 \times P^1) =< O(0,1), O(1,0), O(1,1) > / < Cone(X), Cone(Y), Cone(Z), Cone(W) >= D(C < x, y > -mod) / < Cone(x), Cone(y) > .
\]

In the last category we have:

\[
\text{Hom}(O(1,1), O(1,1)) = C < x, x^{-1}, y, y^{-1} >.
\]

We have the following maps

\[
\text{Hom}_{\tilde{D}(Y_0)/\tilde{D}_0(Y_0)}(Q_0, Q_0) \to \text{Hom}_{\tilde{D}(Y_0)/\tilde{D}_0(Y_0)}(Q_0, Q_0) \to \text{Hom}_{\tilde{C}(Y_0)}(Q_0, Q_0) = A.
\]

Lemma 3.2 implies that \( F^k_{nc}(x), F^k_{nc}(y) \) belong to the image of the composition of these maps. In particular, they belong to the image of the second map, which is the subalgebra \( C < x, x^{-1}, y, y^{-1} > \subset A \) by Lemma 3.3. This proves the theorem.

\[
\square
\]

References

[1] A.Bondal and D.Orlov. Reconstruction of a variety from the derived category and groups of autoequivalences. Compositio Math., 125(3):327–344, 2001.
[2] A.Bondal and M.Kapranov. Representable functors, Serre functors, and reconstructions. Math. USSR-Izvestiya, 35(3):519–541, 1990.
[3] A.Usnich. Symplectic automorphisms of CP2 and the Thompson group T. preprint arXiv:math/0611604, 2006.
[4] A.Usnich. Action of the Cremona group on a non-commutative ring. preprint arXiv:math/0710.4561, 2007.
[5] A.Usnich. Non-commutative cluster mutations. Doklady of the National Academy of Sciences of Belarus, 53(4):27–29, 2009.
[6] P.Di Francesco and R.Kedem. Discrete non-commutative integrability: the proof of a conjecture by M.Kontsevich, preprint arXiv:math/0909.0615v1, 2009.
[7] S.Fomin and A.Zelevinsky. Cluster algebras. I. Foundations. J. Amer. Math. Soc., 15(2):497–529, 2002.