LORENTZIAN POLYNOMIALS ON CONES AND THE HERON-ROTA-WELSH CONJECTURE

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ABSTRACT. We give a short proof of the log-concavity of the coefficients of the reduced characteristic polynomial of a matroid. The proof uses an extension of the theory of Lorentzian polynomials to convex cones, and reproves the Hodge-Riemann relations of degree one for the Chow ring of a matroid.

1. Introduction

Over the past decade, methods originating in algebraic geometry have been developed to solve long-standing conjectures on unimodality and log-concavity in matroid theory, see [4]. In [1], a Hodge theory of matroids was developed by Adiprasito, Huh and Katz to prove the Heron-Rota-Welsh conjecture on the log-concavity of the characteristic polynomial of a matroid.

A different approach to log-concavity problems in matroid theory originates in the works of Choe, Oxley, Sokal, Wagner, Gurvits and the first author, see [13]. This approach uses convexity properties of multivariate polynomials rather than Hodge theory. Recently breakthroughs in this approach were made by Huh and the first author [6], and by Anari, Liu, Oveis Gharan and Vinzant [2]. The theory of Lorentzian polynomials was developed in [2, 6, 8], and used in [2, 6] to prove the strongest of Mason’s conjectures on the log-concavity of the number of independent sets of a matroid.

In [3], Backman, Eur and Simpson combined Lorentzian polynomials and Hodge theory to give a shorter proof of the Hodge-Riemann relations of degree one for the Chow ring of a matroid. The proof in Section 6 of [3] assumes Poincaré duality for the Chow ring of a matroid, which is proved in [1] and in [3, Section 4]. However, until now a purely “polynomial proof” of the Heron-Rota-Welsh conjecture – using the theory of Lorentzian polynomials and avoiding Hodge theory altogether – has been missing. The main purpose of this paper is to give such a proof.

In Section 2 we extend the theory of Lorentzian polynomials so that it applies to cones other than the positive orthant (see Proposition 2.4), and use this extension to give a very short proof of the Heron-Rota-Welsh conjecture in Sections 3 and 4 (see Theorem 4.1), which does not rely on Hodge theory. In fact we give a self-contained proof of the Hodge-Riemann relations of degree one for the Chow ring of a matroid, see Theorem 3.7 and Theorem 5.3.

2. Lorentzian polynomials on cones

Let $\partial_i$ (or $\partial_{x_i}$) denote the partial derivative with respect to $x_i$, and for $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ let $D_u = u_1\partial_1 + \cdots + u_n\partial_n$. 


Definition 2.1. Let $\mathcal{C}$ be an open convex cone in $\mathbb{R}^n$. A homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree $d$ is called $\mathcal{C}$-Lorentzian if for all $v_1, \ldots, v_d \in \mathcal{C}$,

(P) $D_{v_1} \cdots D_{v_d} f > 0$, and

(H) the symmetric bilinear form

$$(x, y) \mapsto D_x D_y D_{v_1} \cdots D_{v_d} f$$

has exactly one positive eigenvalue.

By convention, we also say that the identically zero polynomial is $\mathcal{C}$-Lorentzian.

Recall that the Hessian of $f$ at $x$ is the matrix $\nabla^2 f(x) = (\partial_i \partial_j f(x))_{i,j=1}^n$. Hence (H) asserts that the Hessian of $D_{v_1} \cdots D_{v_d} f$ has exactly one positive eigenvalue.

Remark 2.1. It follows from [6, Thm. 2.25] that Definition 2.1 is equivalent to that for all positive integers $m$ and for all $v_1, \ldots, v_m \in \mathcal{C}$, the polynomial

$$(y_1, \ldots, y_m) \mapsto f(y_1 v_1 + \cdots + y_m v_m)$$

is Lorentzian (in the sense of [6]) and has positive coefficients only. It also follows that Lorentzian polynomials are the same as $\mathbb{R}_{\geq 0}$-Lorentzian polynomials.

From Remark 2.1 and [6, Cor. 2.32] we deduce

Proposition 2.2. Suppose $f$ and $g$ are $\mathcal{C}$-Lorentzian, then so is $fg$.

A matrix $A = (a_{ij})_{i,j=1}^n$ whose off-diagonal entries are nonnegative is called irreducible if for all distinct $i, j$ there is a sequence $i = i_0, i_1, i_2, \ldots, i_\ell = j$ such that $i_{k-1} \neq i_k$ for all $1 \leq k \leq \ell$, and $a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{\ell-1} i_\ell} > 0$. By translating such a matrix with a positive multiple of the identity matrix, the Perron-Frobenius theory [5, Chapter 1] guarantees that $A$ has a unique eigenvector (up to multiplication by positive scalars) whose entries are all positive. Moreover the corresponding eigenvalue is simple and is the largest eigenvalue of $A$.

If $A$ and $B$ are symmetric matrices of the same size, we write $A \preceq B$ if $B - A$ is positive semidefinite.

The following lemma is, in essence, taken from [7, Prop. 3].

Lemma 2.3. Let $f$ be a homogeneous polynomial of degree $d \geq 3$, and let $x \in \mathbb{R}_{> 0}^n$. If

(1) $\partial_i f(x) > 0$ for all $i$, and

(2) the Hessian of $\partial_i f$ at $x$ has exactly one positive eigenvalue for all $i$, and

(3) the Hessian of $f$ at $x$ is irreducible, and its off-diagonal entries are nonnegative,

then the Hessian of $f$ at $x$ has exactly one positive eigenvalue.

Proof. If $g$ is a homogeneous polynomial of degree $d$ and $g(x) > 0$, then the following three statements are equivalent

(a) the Hessian of $g$ at $x$ has exactly one positive eigenvalue,

(b) the Hessian of $g^{1/d}$ is negative semidefinite at $x$,

(c) the matrix $d \cdot g \cdot \nabla^2 g - (d - 1) \cdot \nabla g(\nabla g)^T$ is negative semidefinite at $x$,

see e.g. [6, Prop. 2.33].

Suppose $x$ and $f$ are as in the hypotheses of the lemma. Then, by (c),

$$(d - 1) \cdot \partial_i f \cdot \nabla^2 \partial_i f \leq (d - 2) \cdot \nabla \partial_i f(\nabla \partial_i f)^T.$$
Euler’s identity, \( d \cdot f(x) = \sum_{i=1}^{n} x_i \cdot \partial_i f(x) \), yields
\[
(d - 2) \cdot \nabla^2 f = \sum_{i=1}^{n} x_i \nabla^2 \partial_i f \leq \sum_{i=1}^{n} \frac{x_i}{\partial_i f} (d - 1) \nabla \partial_i f (\nabla \partial_i f)^T.
\]
Rewrite the above inequality as \((d - 1) \cdot \nabla^2 f \geq (\nabla^2 f) \Lambda (\nabla^2 f)\), where \( \Lambda \) is the diagonal matrix \( \text{diag}(x_1/\partial_1 f, \ldots, x_n/\partial_n f) \). For the matrix \( B = \Lambda^{1/2} (\nabla^2 f) \Lambda^{1/2} \), this implies \( B^2 - (d - 1)B \geq 0 \). Hence no eigenvalue of \( B \) lies in the open interval \((0, d - 1)\). The matrix \( B \) is irreducible and has nonnegative off-diagonal entries, so the Perron-Frobenius theorem applies to \( B \). Notice that \( \Lambda^{-1/2}x \) is a positive eigenvector of \( B \), and the corresponding eigenvalue is \( d - 1 \). Hence \( d - 1 \) is the unique largest eigenvalue of \( B \) afforded by the Perron-Frobenius theorem. We conclude that \( B \), and thus also \( \nabla^2 f(x) \), has exactly one positive eigenvalue. \( \square \)

Recall that the lineality space of an open convex cone \( \mathcal{C} \) in \( \mathbb{R}^n \) is \( L_\mathcal{C} = \overline{\mathcal{C}} - \mathcal{C} \), i.e., the largest linear space contained in the closure of \( \mathcal{C} \). We say that \( \mathcal{C} \) is effective if \( \mathcal{C} = \mathcal{C} \cap \mathbb{R}_{\geq 0} + L_\mathcal{C} \).

**Proposition 2.4.** Let \( f \in \mathbb{R}[x_1, \ldots, x_n] \) be a homogeneous polynomial of degree \( d \geq 3 \), and let \( \mathcal{C} \) be an open, convex and effective cone in \( \mathbb{R}^n \). If

1. \( f(x + w) = f(x) \) for all \( x \in \mathbb{R}^n \) and \( w \in L_\mathcal{C} \), and
2. \( D_{v_1} \cdots D_{v_d} f > 0 \) for all \( v_1, \ldots, v_d \in \mathcal{C} \), and
3. the Hessian of \( D_{v_1} \cdots D_{v_{d-2}} f \) is irreducible and its off-diagonal entries are nonnegative for all \( v_1, \ldots, v_{d-2} \in \mathcal{C} \), and
4. \( \partial_i f \) is \( \mathcal{C} \)-Lorentzian for all \( i \),

then \( f \) is \( \mathcal{C} \)-Lorentzian.

**Proof.** Let \( v_1, \ldots, v_{d-3} \in \mathcal{C} \), and consider the cubic \( g = D_{v_1} \cdots D_{v_{d-3}} f \). Since \( \partial_i g \) is \( \mathcal{C} \)-Lorentzian, it follows from Definition 2.1 that so is \( \partial_i g \). By choosing \( v_{d-2} = x \in \mathcal{C} \), it follows from (3) that the Hessian \( \nabla^2 g(x) \) is irreducible and its off-diagonal entries are nonnegative. Since
\[
g(x) = \frac{\partial^{d-3}}{\partial t_1 \cdots \partial t_{d-3}} f \left( x + \sum_{i=1}^{d-3} t_i v_i \right), \quad t_1 = \cdots = t_{d-3} = 0,
\]
it follows from (1) that \( g(x + w) = g(x) \) for all \( x \in \mathbb{R}^n \) and \( w \in L_\mathcal{C} \). Since \( \mathcal{C} \) is effective, we may assume \( x \in \mathcal{C} \cap \mathbb{R}_{\geq 0} \). Lemma 2.3 then implies that the Hessian of \( g \) at \( x \) has exactly one positive eigenvalue. The lemma now follows since the Hessian of \( D_{v_1} \cdots D_{v_{d-3}} f \) at \( x \) is equal to the Hessian of \( D_{v_1} \cdots D_{v_{d-3}} f \).

\( \square \)

### 3. Polynomials associated to graded sub-posets of Boolean lattices

Let \( \mathcal{B}(E) = \{ S : S \subseteq E \} \) denote the Boolean lattice of subsets of a finite set \( E \). In this section \( \mathcal{P} = (X, \leq) \) will be any sub-poset of \( \mathcal{B}(E) \), for which each finite closed interval \([K, L]_\mathcal{P} = \{ F \in \mathcal{P} : K \leq F \leq L \} \) in \( \mathcal{P} \) is graded. We write \( K \prec L \) if \( K < L \) and there is no \( F \in \mathcal{P} \) for which \( K < F < L \).

For \( \emptyset \subseteq K \subseteq L \), let \( \mathcal{E}_K^L = \{ (yS)_{K \subseteq S \subseteq L} : yS \in \mathbb{R} \} = \mathbb{R}^m \), where \( m = 2^{|L \setminus K|} - 2 \). Denote by \( \mathcal{M}_K^L \), the subspace of modular elements \( y \) in \( \mathcal{E}_K^L \), i.e.,
\[
y_S + y_T = y_{S \cap T} + y_{S \setminus T}, \quad \text{for all } S, T,
\]
where \( y_K = y_L = 0 \). It follows that \( y \in \mathcal{M}_K \) if and only if there are real numbers \( y_e \in L \setminus K \), for which \( \sum_{e \in L \setminus K} y_e = 0 \) and \( y_S = \sum_{e \in S \setminus K} y_e \), for all \( K \subset S \subset L \).

If \( K \subset F \subset G \subset L \), define a linear projection \( \pi_F^G : \mathcal{E}_K^G \to \mathcal{E}_F^G \) by

\[
\pi_F^G(t) = \left( t_S - t_G \left( \frac{|S \setminus F|}{|G \setminus F|} - t_F \left( \frac{|G \setminus S|}{|G \setminus F|} \right) \right) \right)_{F \subset S \subset G},
\]
where \( t_K = t_L = 0 \).

Let \( r(K, L) \) be the rank of the interval \( [K, L]_\mathcal{P} \), and let \( d(K, L) = r(K, L) - 1 \). We define a polynomial \( \text{pol}^L_K(t) \), of degree \( d(K, L) \), in the variables \( \{ t_F : K < F < L \} \), for each \( K < L \) in \( \mathcal{P} \). In Section 5 we will prove that \( \text{pol}^L_K(t) \) is the volume polynomial of the Chow ring of a matroid, as defined in [1]. Notice that while \( \text{pol}^L_K(t) \) is defined on variables indexed by \( F \in \mathcal{P} \) such that \( K < F < L \), we will often consider it as a polynomials in the larger set of variables \( \{ t_S : K \subset S \subset L \} \).

**Definition 3.1.** The polynomial \( \text{pol}^L_K(t) \), associated to \( K < L \) in \( \mathcal{P} \) is defined recursively as follows.

If \( d(K, L) = 0 \), then \( \text{pol}^L_K(t) = 1 \). If \( d(K, L) \geq 1 \), then

\[
d(K, L) \cdot \text{pol}^L_K(t) = \sum_{K < F < L} t_F \cdot \text{pol}^F_K(\pi_F^G(t)) \cdot \text{pol}^L_F(\pi_F^G(t)). \tag{3.1}
\]

**Example 3.1.** If \( d(K, L) = 1 \), then \( \text{pol}^L_K(t) = \sum_{K < F < L} t_F \). If \( d(K, L) = 2 \), then

\[
2 \cdot \text{pol}^L_K(t) = \sum_{K < F < G < L} \left( 2 \cdot t_F t_G - t_F^2 \cdot \frac{|L \setminus G|}{|L \setminus F|} - t_G^2 \cdot \frac{|F \setminus K|}{|G \setminus K|} \right). \tag{3.2}
\]

Let \( \mathcal{E}_K^L \) denote the open convex cone in \( \mathcal{E}_K^L \) consisting of all strictly sub-modular \( y \in \mathcal{E}_K^L \), i.e.,

\[
y_S + y_T > y_{S \cap T} + y_{S \cup T}
\]
for all incomparable \( S \) and \( T \), where \( y_K = y_L = 0 \). Hence the lineality space of \( \mathcal{E}_K^L \) is \( \mathcal{M}_K^L \).

**Lemma 3.1.** If \( |L \setminus K| \geq 2 \), then \( \mathcal{E}_K^L \) is effective.

**Proof.** It is plain to see that \( v = (|S \setminus K| : |L \setminus S|)_{K \subset S \subset L} \in \mathcal{E}_K^L \cap \mathbb{R}_{>0}^{(K, L)} \), where \( (K, L) = \{ S : K \subset S \subset L \} \). Suppose \( y \in \mathcal{E}_K^L \). Then \( z := y - \epsilon v \in \mathcal{E}_K^L \), for \( \epsilon > 0 \) sufficiently small. By e.g. [10, Prop. 4.4], there exists \( w \in \mathcal{M}_K^L \) such that \( z + w \in \mathbb{R}_{>0}^{(K, L)} \). But then \( y + w = z + w + \epsilon v \in \mathbb{R}_{>0}^{(K, L)} \), as desired. \( \square \)

**Lemma 3.2.** Suppose \( f \in \mathbb{R}[t_1, \ldots, t_n] \) is a homogeneous polynomial of degree \( d \), and that \( df = \sum_{i=1}^n t_i Q_i \), where \( Q_1, \ldots, Q_n \) are homogeneous polynomials of degree \( d - 1 \) for which \( \partial_i Q_j = \partial_j Q_i \) for all \( i, j \). Then \( Q_i = \partial_i f \), for all \( i \).

**Proof.** By Euler’s identity,

\[
d\partial_i f = \partial_i (t_j Q_j) - t_j \partial_i Q_j + \sum_{i=1}^n t_i \partial_j Q_i = Q_j + \sum_{i=1}^n t_i \partial_i Q_j = Q_j + (d - 1)Q_j = dQ_j.
\]

**Lemma 3.3.** If \( K < F < L \), then

\[
\frac{\partial}{\partial t_F} \text{pol}^L_K(t) = \text{pol}^F_K(\pi_F^L(t)) \cdot \text{pol}^L_F(\pi_F^L(t)). \tag{3.3}
\]
Proof. By induction it follows from (3.1) that \( \partial_{t_F} \partial_{t_G} \text{pol}_K^L(t) = 0 \), unless \( F \) and \( G \) are comparable. We now prove (3.3) by induction over \( d = d(K, L) \), the case when \( d = 1 \) being clear. Suppose \( d > 1 \), and let \( Q_F(t) = \text{pol}_K^L(\pi_K^F(t)) \cdot \text{pol}_K^L(\pi_K^F(t)) \). By Lemma 3.2, it remains to prove \( \partial_{t_F} Q_G(t) = \partial_{t_G} Q_F(t) \), for all \( G < F \). By induction

\[
\frac{\partial}{\partial t_F} Q_G(t) = \text{pol}_K^L(\pi_K^G(t)) \cdot \text{pol}_G^F(\pi_G^F(t)) \cdot \text{pol}_F^L(\pi_F^G(t)),
\]

and

\[
\frac{\partial}{\partial t_G} Q_F(t) = \text{pol}_K^L(\pi_K^G(t)) \cdot \text{pol}_G^F(\pi_G^F(t)) \cdot \text{pol}_F^L(\pi_F^G(t)).
\]

Now \( \partial_{t_F} Q_G(t) = \partial_{t_G} Q_F(t) \) follows since \( \pi_G^F \pi_K^L = \pi_F^G \) whenever \( K \subseteq G \subseteq F \subseteq L \).

The proof of the next lemma is left to the reader.

Lemma 3.4. If \( K \subseteq G < F \subseteq L \), then \( \pi_G^F : \mathcal{M} \to \mathcal{M} \) and \( \pi_F^G : \mathcal{E} \to \mathcal{E} \).

We call a sub-poset \( \mathcal{P} \) of \( \mathcal{P}(E) \) balanced if for each \( K < L \) in \( \mathcal{P} \) such that \( d(K, L) = 1 \),

\[
|\{K < F < L : F \ni i\}| = |\{K < F < L : F \ni j\}|, \quad \text{for all } i, j \in L \setminus K.
\]

In particular, \( \mathcal{P} \) is balanced if \( \{A \setminus K\} \cap A \leq L \) partitions \( L \setminus K \), whenever \( d(K, L) = 1 \).

If so, we say that \( \mathcal{P} \) is 1-balanced.

Lemma 3.5. Suppose \( \mathcal{P} \) is balanced, and let \( K < L \). Then \( \text{pol}_K^L(x + y) = \text{pol}_K^L(x) \), for all \( x \in \mathcal{E}_K \) and \( y \in \mathcal{M}_K \).

Proof. The proof is by induction on \( d(K, L) \). Suppose \( d(K, L) = 1 \). Since \( y \) is modular and \( \text{pol}_K^L \) is linear,

\[
\text{pol}_K^L(x + y) = \sum_{i \in L \setminus K} y_i \cdot |\{K < F < L : F \ni i\}|.
\]

The case when \( d = 1 \) thus follows since \( \mathcal{P} \) is balanced and \( \sum_{i \in L \setminus K} y_i = 0 \).

Consider the space \( \mathcal{W} \) of all homogeneous polynomials \( f(t) \) in \( \mathbb{R}[t_S : F \subseteq S \subseteq L] \) for which \( f(t + y) = f(t) \), for all \( y \in \mathcal{M}_K \). Then \( f \in \mathcal{W} \) if and only if \( \partial^\alpha f(y) = 0 \) for all \( y \in \mathcal{W} \) and \( |\alpha| < \text{deg}(f) \), since \( f(t + y) = f(t) \) is supposed to be homogeneous. By induction it follows from (3.1) that \( \text{pol}_K^L(y) = 0 \) for all \( y \in \mathcal{M}_K \), whenever \( d(K, L) \geq 1 \). Suppose \( d(K, L) > 1 \). Then \( \partial_{t_F} \text{pol}_K^L \in \mathcal{W} \) by Lemma 3.3, Lemma 3.4 and induction. Hence \( \partial^\alpha \text{pol}_K^L(y) = 0 \) for all \( |\alpha| < d(K, L) \), so that \( \text{pol}_K^L \in \mathcal{W} \).

We call \( \mathcal{P} \) interval connected if for each \( K < L \) with \( d(K, L) \geq 2 \) and \( F, G \in (K, L)_\mathcal{P} = \{H \in \mathcal{P} : K < H < L\} \), there exists elements \( F_i \in (K, L)_\mathcal{P} \) such that \( F \subseteq F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_k = G \), where \( F_i \leq F_{i+1} \) means \( F_i < F_{i+1} \) or \( F_i = F_{i+1} \).

Lemma 3.6. Suppose \( \mathcal{P} \) is balanced, and that \( K < L \). If \( v_1, \ldots, v_d \in \mathcal{E}_K \), then \( D_{v_1} \cdots D_{v_d} \text{pol}_K^L > 0 \). Moreover if \( d(K, L) \geq 2 \) and \( \mathcal{P} \) is interval connected, then the Hessian of the quadratic \( D_{v_1} \cdots D_{v_d} \text{pol}_K^L \) is irreducible, and its off-diagonal entries are nonnegative.
Proof. We start by proving the first assertion by induction on \( d = d(K, L) \). We need to prove that all coefficients of the polynomial \( \text{pol}_K^L(s_1v_1 + \cdots + s_dv_d) \) are positive. By Lemmas 3.1 and 3.5, we may assume that all entries of \( v_i \in \mathcal{C}_K^L \) are positive for all \( i \). The first assertion now follows by induction using (3.1) and Lemma 3.4.

For \( K < F_1 < F_2 < L \), let

\[
g(t) = \frac{\partial^2}{\partial t_{F_1} \partial t_{F_2}} \text{pol}_K^L(t) = \text{pol}_{F_1}^L(\pi_{F_1}^L(t)) \cdot \text{pol}_{F_1}^L(\pi_{F_2}^L(t)) \cdot \text{pol}_{F_2}^L(\pi_{F_2}^L(t)).
\]

As above, it follows that all coefficients of \( g(s_1v_1 + \cdots + s_dv_d) \) are positive, whenever \( v_1, \ldots, v_{d-2} \in \mathcal{C}_K^L \). Hence \( \partial_{t_{F_1}} \partial_{t_{F_2}} D_{v_1} \cdots D_{v_{d-2}} \text{pol}_K^L > 0 \). Also the entries corresponding to non-comparable \( F_1 \) and \( F_2 \) are zero. Since \( \mathcal{P} \) is interval connected it follows that the Hessian of \( D_{v_1} \cdots D_{v_{d-2}} \text{pol}_K^L \) is irreducible. \( \square \)

Recall, see e.g. [11], that a sub-poset \( \mathcal{P} \) of \( \mathcal{B}(E) \) is the lattice of flats of a matroid on \( E \) if and only if

(F1) \( E \in \mathcal{P} \),
(F2) If \( F, G \in \mathcal{P} \), then \( F \cap G \in \mathcal{P} \),
(F3) For each \( K \in \mathcal{P}, \{A \setminus K\}_{K \subset A} \) partitions \( E \setminus K \).

Hence, the lattice of flats of a matroid is 1-balanced. Also lattices of flats of matroids are graded and semimodular, see [11, Chapter 1.7]. Recall [12, Prop. 3.3.2] that a finite lattice \( \mathcal{L} \) is semimodular if and only if for all \( a, b \in \mathcal{L} \),

if \( a \) and \( b \) cover \( a \wedge b \), then \( a \vee b \) covers \( a \) and \( b \).

From semimodularity it follows that lattices of flats of matroids are interval connected. From (F1)–(F3) it follows that each closed interval of the lattice flats of a matroid is again the lattice of flats of a matroid.

**Theorem 3.7.** If \([K, L]_{\mathcal{P}} \) is the lattice of flats of a matroid, then \( \text{pol}_K^L \) is \( \mathcal{C}_K^L \)-Lorentzian.

Proof. We shall apply Proposition 2.4 for \( \mathcal{C} = \mathcal{C}_K^L \) and \( \mathcal{W} = \mathcal{M}_K^L \). The proof is by induction on \( d(K, L) \). The case when \( d(K, L) \leq 1 \) is clear. Also, by Lemma 3.5 and Lemma 3.6 and the discussion preceding the lemma, (1)–(3) of Proposition 2.4 are satisfied. Suppose \( d(K, L) \geq 2 \). By Lemma 3.3, Lemma 3.4, Proposition 2.2 and induction, \( \partial_{t_{F}} \text{pol}_K^L \) is \( \mathcal{C}_K^L \)-Lorentzian for each \( F \in (K, L)_{\mathcal{P}} \). Hence it remains to prove the case when \( d(K, L) = 2 \).

Suppose \( d(K, L) = 2 \). We need to prove that the Hessian of \( \text{pol}_K^L \) has exactly one positive eigenvalue. Denote flats of rank one in \([K, L]_{\mathcal{P}} \) by \( F \), and flats of rank two by \( G \).

Since \([K, L]_{\mathcal{P}} \) is 1-balanced,

\[
\sum_{F:F \subset G} \frac{|F \setminus K|}{|G \setminus K|} = 1 \quad \text{and} \quad \sum_{G:G > F} \frac{|L \setminus G|}{|L \setminus F|} = |\{G : G > F\}| - 1.
\]

We deduce from (3.2) that \( 2 \text{pol}_K^L = 2 \sum_{F < G} t_F t_G - \sum_{F < G} t_{F}^2 + \sum_{F \neq G} t_{F}^2 - \sum_{G} t_{G}^2 \).

Notice that

\[
\sum_{G} \left(t_G - \sum_{F < G} t_F \right)^2 = \sum_{G} t_G^2 - 2 \sum_{F < G} t_F t_G + \sum_{F < G} t_F^2 + \sum_{F \neq G} \sum_{F_1 \neq F_2} t_{F_1} t_{F_2}.
\]
Since \([K, L]_\mathcal{P}\) is semimodular,
\[\sum_G \sum_{F_1 \neq F_2 < G} t_{F_1} t_{F_2} = \sum_{F_1 \neq F_2} t_{F_1} t_{F_2} = \left(\sum_F t_F\right)^2 - \sum_F t_F^2.\]
Combining the equations above, we conclude
\[2 \text{pol}_K^L = \left(\sum_F t_F\right)^2 - \sum_G \left(t_G - \sum_{F < G} t_F\right)^2,\]
which proves that the Hessian of \(\text{pol}_K^L\) has exactly one positive eigenvalue. \(\square\)

4. Log-concavity of the reduced characteristic polynomial

The characteristic polynomial of a matroid \(M\) is
\[\chi_M(t) = \sum_{F \in \mathcal{F}(M)} \mu(\emptyset, F) t^{e(F, E)}. \tag{4.1}\]
If \(M\) has rank at least one, then \(\chi_M(t)\) is divisible by \(t - 1\), see [14, Section 7]. The reduced characteristic polynomial of a matroid \(M\) is then \(\chi_M(t) = \chi_M(t)/(t - 1)\). Recall that a sequence \(\{a_k\}_{k=0}^d\) is log-concave if \(a_k^2 \geq a_{k-1} a_{k+1}\) for all \(0 < k < d\). The next theorem, which was first proved by Adiprasito, Huh and Katz [1], solved the Heron-Rota-Welsh conjecture. We provide an alternative proof below.

**Theorem 4.1.** The absolute values of the coefficients of the reduced characteristic polynomial of a matroid form a log-concave sequence.

Consider the following two elements in the closure of \(\mathcal{P}_K^L\):
\[\alpha_K^L = \left(\frac{|S \setminus K|}{|L \setminus K|}\right)_{K \subset S \subset L} \quad \text{and} \quad \beta_K^L = \left(\frac{|L \setminus S|}{|L \setminus K|}\right)_{K \subset S \subset L}.\]
For \(i \in L \setminus K\), let \(\alpha_{K,i}^L = (a_s)_{K \subset S \subset L}\) and \(\beta_{K,i}^L = (b_s)_{K \subset S \subset L}\) be the 0/1-valued vectors defined by \(a_S = 1\) if and only if \(i \in S\) and \(b_S = 1\) if and only if \(i \not\in S\). Then
\[\alpha_K^L - \alpha_{K,i}^L \in \mathcal{M}_K^L \quad \text{and} \quad \beta_K^L - \beta_{K,i}^L \in \mathcal{M}_K^L, \tag{4.2}\]
for all \(i \in L \setminus K\).

The elements \(\alpha_K^L\) and \(\beta_K^L\) behave well under the projections considered Section 3. We leave the proof to the reader.

**Lemma 4.2.** If \(K < F < L\), then
\[\pi_K^F(\alpha_K^L) = 0, \quad \pi_K^F(\beta_K^L) = \beta_K^L, \quad \pi_F^L(\alpha_K^L) = \alpha_F^L, \quad \pi_F^L(\beta_K^L) = 0.\]

**Lemma 4.3.** If \(\mathcal{P}\) is 1-balanced, then \(\text{pol}_K^L(\alpha_K^L) = 1/d(K, L)!\) for all \(K < L\).

**Proof.** Let \(i \in L \setminus K\), and let \(\alpha_{K,i}^L = (a_S)_{K \subset S \subset L}\). Since \(\mathcal{P}\) is 1-balanced, there is a unique \(H \in [K, L]_\mathcal{P}\) containing \(i\) for which \(K < H < L\). By (3.1), (4.2) and Lemma 4.2,
\[d(K, L) \cdot \text{pol}_K^L(\alpha_K^L) = \sum_{K < F < L} a_F \cdot \text{pol}_F^L(0) \cdot \text{pol}_F^L(\alpha_K^L) = \text{pol}_H(\alpha_H^L),\]
from which the lemma follows by induction over \(d(K, L)\). \(\square\)
Recall the Möbius function of a locally finite poset [12, Section 3.7]. The next theorem is usually stated as a consequence of Weisner’s theorem, see [12, p. 277].

**Theorem 4.4** (Weisner’s theorem). If \( x \prec a \prec y \) are elements in a semimodular lattice, then \( \mu(x,y) = -\sum_b \mu(x,b) \) where the sum is over all \( b \) for which \( x < b < y \) and \( a \not\prec b \).

A consequence of Weisner’s theorem is that the Möbius function of a semimodular lattice \( \mathcal{L} \) alternates in sign, i.e., \((-1)^{\rho(b) - \rho(a)} \mu(a,b) \geq 0\), where \( \rho \) is the rank function of \( \mathcal{L} \).

**Lemma 4.5.** If \( \mathcal{P} \) is the lattice of flats of a matroid and \( K < L \) in \( \mathcal{P} \), then \( \text{pol}_K^L(\beta_K^L) = |\mu(K,L)|/d(K,L)! \).

**Proof.** Let \( i \in L \setminus K \), and let \( \beta_{K,i}^L = (b_S)_{K \subset S \subset L} \). Then, by (3.1), (4.2) and Lemma 4.2,

\[
d(K,L) \cdot \text{pol}_K^L(\beta_K^L) = \sum_{K < F < L} b_F \cdot \text{pol}_K^F(\beta_K^F) \cdot \text{pol}_F^L(0) = \sum_{K < F < L} \text{pol}_K^F(\beta_K^F).
\]

The lemma now follows by induction and Weisner’s theorem.

**Theorem 4.6.** Suppose \( [K,L]_{\mathcal{P}} \) is the lattice of flats of a matroid. If \( i \in L \setminus K \), then

\[
d(K,L)! \cdot \text{pol}_K^L(s\alpha^L_K + t\beta^L_K) = \sum_{K < F < L} \binom{d(K,L)}{r(K,F)} |\mu(K,F)| \cdot t^{r(K,F)} d^{d(F,L)}.
\]

**Proof.** Let \( f(t) = \text{pol}_K^L(s\alpha^L_K + t\beta^L_K) \). Then by Lemmas 3.3, 4.2, 4.3 and 4.5, (4.2) and the chain rule,

\[
f'(t) = \sum_{K < F < L} |\mu(K,F)| \cdot \frac{d^{d(F,L)}}{d(K,F)!}.
\]

The lemma follows since by Lemma 4.3, \( f(0) = s^{d(K,L)}/d(K,L)! \).

**Lemma 4.7** (Cor. 7.27, [14]). If \( i \in E \) is not a loop, then

\[
\chi_M(t) = \sum_{F \not\ni i} \mu(E,F)t^{d(F,E)}.
\]

If \( f \) is \( \mathcal{C} \)-Lorentzian and \( v_1 \) and \( v_2 \) are in the closure of \( \mathcal{C} \), then the bivariate polynomial \( f(sv_1 + tv_2) = \sum_{k=0}^d \binom{d}{k} a_k s^{d-k} t^k \) is Lorentzian by Remark 2.1 and the fact that the space of Lorentzian polynomials of degree \( d \) is closed. By [6, Example 2.26], the sequence \( \{a_k\}_{k=0}^d \) is log-concave.

**Proof of Theorem 4.4.** Let \( b_k \) denote the absolute value of the coefficient in front of \( t^k \) in the reduced characteristic polynomial of \( M \). Then

\[
\sum_{k=0}^{r-1} \binom{r-1}{k} \cdot b_k \cdot s^{r-1-k} t^k = (r-1)! \cdot \text{pol}_E^E (s\alpha^E_E + t\beta^E_E),
\]

by Theorem 4.6 and Lemma 4.7. The theorem now follows from Theorem 3.7 and the discussion above.
5. The volume polynomial of the Chow ring of a matroid

Although it is not used in or proofs of Theorems 3.7 and 4.1, we prove in this section that if $K < L$ are flats of a matroid $M$, then the polynomial $\text{pol}_K^L$ is the volume polynomial of the Chow ring of the matroid obtained from $M$ by restricting to $L$, and then contracting $K$. The Chow ring of a matroid was introduced by Feichtner and Yuzvinsky in [9], and Adiprasito, Huh and Katz [1] developed a Hodge theory for it. For our purposes it will be convenient to define the Chow ring of a nonempty open interval of the lattice of flats of a matroid, although this definition is easily seen to be equivalent to the Chow ring of a matroid as defined in [1].

Let $K < L$ be flats of a matroid $M$. The Chow ring, $A_K^L$ is defined as the quotient

$$
\frac{\mathbb{R}[x_F : K < F < L]}{I + J},
$$

where $I = I_K^L$ is the ideal generated by all quadratic monomial $x_F x_G$, where $F$ and $G$ are any two non-comparable flats in $(K, L)$, and $J = J_K^L$ is the ideal generated by the linear forms

$$
\sum_{F \ni i} x_F - \sum_{F \ni j} x_F, \text{ where } i \text{ and } j \text{ are any distinct elements of } L \setminus K. \quad (5.1)
$$

The sums above are over $F$ such that $K < F < L$. The Chow ring is graded $A_K^L = \oplus_{d=0}^\infty (A_K^L)_d$, where $d = d(K, L)$, and $(A_K^L)_d$ is isomorphic to $\mathbb{R}$. Indeed, there is a well-defined isomorphism $\deg : (A_K^L)_d \to \mathbb{R}$,

$$
\deg(x_F x_1 x_2 \cdots x_d) = 1, \text{ for any flag } K \prec F_1 \prec F_2 \prec \cdots \prec F_d \prec L, \quad (5.2)
$$

see [1].

The volume polynomial, $\text{vol}_K^L(t)$, of $A_K^L$ is defined by

$$
\text{vol}_K^L(t) = \frac{1}{d!} \deg \left( \left( \sum_{K < F < L} x_F t_F \right)^d \right), \text{ where } d = d(K, L). \quad (5.3)
$$

**Lemma 5.1.** If $w \in \mathcal{M}_K^L$, then $\text{vol}_K^L(t + w) = \text{vol}_K^L(t)$ for all $t \in \mathcal{E}_K^L$.

**Proof.** The space $\mathcal{M}_K^L$ is generated by vectors of the form $\alpha_{K,i}^j - \alpha_{K,j}^i$, $i, j \in L \setminus K$. Hence, by the definition of the ideal $J$,

$$
\sum_{K < F < L} w_F x_F = 0 \in A_K^L, \text{ for any } w = (w_S) \in \mathcal{M}_K^L.
$$

Thus $\text{vol}_K^L(t + w) = \text{vol}_K^L(t)$, by (5.3). \hfill \Box

**Lemma 5.2.** If $K < F < L$ are flats, then there is a ring homomorphism $\phi_F : A_K^F \otimes A_F^L \to A_K^L$, such that

$$
\deg(\phi_F(\xi \otimes \eta)) = \deg(\xi) \cdot \deg(\eta),
$$

for all $\xi \in A_K^F$ and $\eta \in A_F^L$.

**Proof.** First define $\phi_F : \mathbb{R}[x_G : K < G < F] \otimes \mathbb{R}[x_G : F < G < L] \to A_K^L$, by

$$
\phi_F(\xi \otimes \eta) = \xi x_F \eta. \text{ Then } \phi_F(\xi \otimes \eta) = \xi x_F \eta.
$$

Then $\xi \in I_K^F + J_K^F$ implies $\xi x_F \in I_K^F + J_K^F$, and $\eta \in I_F^L + J_F^L$ implies $x_F \eta \in I_K^F + J_K^F$, so that $\phi_F$ defines a homomorphism $\phi_F : A_K^F \otimes A_F^L \to A_K^L$. 

The identity for the degrees now follows from (5.2).

Theorem 5.3. For any two flats $K < L$, $\Vol^L_F = \pol^L_K$.

Proof. We prove that $\Vol^L_F$ satisfies the recursion (3.3). Since $\Vol^L_F = \pol^L_K$ whenever $d(K, L) = 0$, the theorem will follow by induction and Euler’s identity.

Assume $d = d(K, L) > 1$. Then

$$\frac{\partial}{\partial t} \Vol^F_K(t) \bigg|_{t=0} = \frac{1}{(d-1)!} \sum_{k=0}^{d-1} \binom{d-1}{k} \cdot \deg(\xi_x^F \eta_x^{d-1-k}),$$

where $\xi = \sum_{K<G<F} \alpha_{K,G} x_G$ and $\eta = \sum_{F<G<L} \beta_{F,G} x_G$. If $k > d(K, F)$, then we have $\xi_x^F \eta_x^{d-1-k} = 0$ since $\phi_F$ is a homomorphism. Similarly $\xi_x^{d-1-j} \eta_x^j = 0$ if $j > d(F, L)$. Thus

$$\frac{\partial}{\partial t} \Vol^L_K(t) \bigg|_{t=0} = \frac{\deg(\xi_x^{d(F,L)} \eta_x^{d(F,L)})}{d(F, L)!} = \frac{\deg(\xi_x^{d(F,F)})}{d(F,F)!} \cdot \frac{\deg(\eta_x^{d(F,L)})}{d(F,L)!},$$

by Lemma 5.2. Hence $\partial_t \Vol^L_K(t) \bigg|_{t=0} = \Vol^E_K(t) \cdot \Vol^L_F(t)$. Let $i \in F \setminus K$ and $j \in L \setminus F$. The $F$-entry of $t - t_F(\alpha_{K,i}^L - \alpha_{K,j}^L)$ is zero. Hence by Lemma 5.1,

$$\frac{\partial}{\partial t} \Vol^L_K(t) = \frac{\partial}{\partial t} \Vol^L_F(t - t_F(\alpha_{K,i}^L - \alpha_{K,j}^L)) = \Vol^E_K(t - t_F \phi_{K,i}^L) \cdot \Vol^L_F(t - t_F \beta_{K,j}^L),$$

which proves that $\Vol^L_F(t)$ satisfies the desired recursion. □

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