On Certain Quantum Deformations of $gl(N,R)$.*

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Abstract: In this paper all deformations of the general linear group, subject to certain restrictions which in particular ensure a smooth passage to the Lie group limit, are obtained. Representations are given in terms of certain sets of creation and annihilation operators. These creation and annihilation operators may belong to a generalisation of the $q$-quark type or $q$-hadronic type, of $q$-boson or $q$-fermion type. We are also led to a natural definition of $q$-direct sums of $q$-algebras.

* Accepted for publication in JMP.
1. Introduction.

In the last years there have been numerous articles extending the study of Hopf algebras with their comultiplication and trying to see if these algebras can be of any use in certain corners of physical theories. Another related approach puts less emphasis on the comultiplication, which is sometimes dropped completely, but insists on realizing deformations of classical algebras in terms of deformed creation and annihilation operators.

This article, as will be explained shortly in more detail, belongs to the second approach. One of its possible aims would be the construction of a new field theory with, obviously, new types of statistics. We will soon make our assumptions precise and justify them. But let us emphasise immediately that, to replace the comultiplication, we will be led to the new and, we believe, important notion of \( q \)-direct sum of representations built out of the deformed creation and annihilation operators.

One of the main characteristic features of operators in Hilbert spaces, which underly quantum mechanics, is the associative law of composition. In a series of papers\(^1\)\(^,\)\(^2\)\(^,\)\(^3\), we have studied various aspects of quadratic algebras which permit the reversal of ordering of any pair of operators and have obtained restrictions on the braiding parameters which arise from the requirements of associativity. For example, while normally creation and annihilation operators belonging to different particles are assumed to satisfy (deformed) commutation or anticommutation relations with no central terms, we have investigated in \(^2\) the possibility of the addition of central terms, which we call neutral operators. As an extension of the same nature with different closure properties has recently been investigated by Polychronakos and others\(^4\)\(^,\)\(^5\)\(^,\)\(^6\). The purpose of this article is to study certain deformations of the general linear group in \( N \) dimensions and their realisation in terms of sets of creation and annihilation operators. The constraint which we shall impose to limit our search and which we make more precise below is that as the deformation parameters approach their undeformed limits no operator relations drop out of the algebra. This rules out deformations into superalgebras.

The algebra \( gl(N, R) \) of the group \( GL(N, R) \), i.e. the group of linear transformations in \( N \)-dimensions on the reals, has \( N^2 \) generators. We will study its deformations, which we will call by definition \( al(N, R)_q \), subject to the two restrictions

1) the commutator terms are deformed into quocommutators in the usual basis of \( gl(N, R) \).

2) the structure constants which are zero in the usual basis for \( gl(N, R) \) remain zero for \( al(N, R)_q \). Note that this assumption is not true for the deformation of \( SU(n + m) \) into the superalgebra \( SU(n|m) \).

The generic form of these deformations is made explicit in (1.1).

The \( N^2 \) generators of \( al(N, R)_q \) are denoted by \( E(j, k) \), \( j = 1, \ldots, N \), \( k = 1, \ldots, N \). These operators are supposed to obey a product law which is in general non commutative but associative.
In agreement with the usual commutation relations of \( gl(N, R) \) and subject to the restrictions above, the general form of the quommutation relations are postulated to be of the restricted form

\[
[E(j, k), E(l, m)]_{p(j, k, l, m)} = \delta_{kl}(1 - \delta_{jm})g_1(j, k, m)E(j, m) \\
- \delta_{mj}(1 - \delta_{kl})g_2(l, m, k)E(l, k) \\
+ \delta_{kl}\delta_{mj}(g_3(j, k)E(j, j) - g_4(j, k)E(k, k))
\]  

(1.1a)

where the symbol \([X, Y]_p\) is a shorthand for

\[
[X, Y]_p = p^{-1}X * Y - p Y * X
\]

(1.1b)

and \( p, g_1, g_2, g_3 \) and \( g_4 \) are indexed sets of complex numbers which are supposed to be essentially non-zero. For the normal \( gl(N, R) \) algebra all the parameters \( p, g_1, g_2, g_3 \) and \( g_4 \) are 1. If some \( p \)'s are equal to \( i \) we have anticommutation type rules.

On these quommutation relations, the following braiding or associativity requirements should be imposed

\[
(E(j, k) * E(l, m)) * E(p, q) = E(j, k) * (E(l, m) * E(p, q))
\]

(1.2)

where the parentheses specify in which order the products have to be performed.

It is clear that the generalized structure constants in (1.1) have to fulfill quite a number of conditions resulting from the associativity and the symmetry requirements. One way to take these into account is to define a “normal” product of the operators \( E(j, k) \) for example by

- \( E(j, k) \) is on the right of \( E(l, m) \) if \( l < j \),
- \( E(j, k) \) is on the right of \( E(j, m) \) if \( m < k \).

The rule (1.1) then allows the rewriting of any product of operators as a sum of normal ordered products and the associativity requirements guarantee that the resulting expression does not depend on the order in which the required transpositions are performed.

We begin by a general discussion about what should be changed for quantum algebras of the form we are considering as compared to normal commutation relations customary for Lie algebras and their representation. In particular the notion underlying the usual concept of a direct sum of representations will be discussed and generalized.

We shall write explicitly the solutions for the generalized structure constants for \( al(N, R)_q \) for all \( N \), and focus on \( al(3, R)_q \) and its realisation in terms of creation and annihilation operators forming triplets or nonets, and which satisfy quantum type quommutation relations. It is suggested that for these representations the usual notion of coproduct be replaced by a more natural newly defined \( q \)-direct sum of representations.
2. General considerations on quantum algebras.

In this section we quote a few results relevant to quantum algebras and in particular discuss some representations including matrix representations as well as representations in terms of quantum creation and annihilation operators of what we will call boson or fermion type. Some consideration will also be paid to the all important notion of $q$-direct sum of representations.

2.a Structure constants. Adjoint representation.

Suppose we start with a general quantum algebra of the type which we have been considering in the introduction i.e. of the chosen form for $i \neq j$

$$V_i \ast V_j = q_{ij} V_j \ast V_i + f_{ij}^k V_k$$  \hspace{1cm} (2.1)

where we will call (see (1.1)) the $q_{ij} \equiv p_{ij}^2$ the quantum parameters and $f_{ij}^k \equiv p_{ij} g_{ij}^k$ the structure constants. These parameters satisfy the trivial symmetry requirements always for $i \neq j$

$$q_{ij} = \frac{1}{q_{ji}}, \quad f_{ij}^k = -q_{ij} f_{ji}^k.$$  \hspace{1cm} (2.2)

Let us remark that these relations could be supplemented by generalized quommutators of the form

$$V_i^2 = f_{ii}^k V_k$$  \hspace{1cm} (2.3)

which we will not consider here. Another way of stating this choice is to postulate in (2.1) that

$$q_{ii} = 1, \quad f_{ii}^k = 0.$$  \hspace{1cm} (2.4)

The associativity requirements

$$(V_i \ast V_j) \ast V_k = V_i \ast (V_j \ast V_k)$$  \hspace{1cm} (2.5)

lead to restrictions on the quantum parameters and on the structure constants. They replace the Jacobi identities customary for the structure constants of the Lie algebras. They are of the form

$$(1 - q_{ij} q_{ik} q_{mi}) f_{jk}^m = 0$$  \hspace{1cm} (2.6a)

$$(1 - q_{jm} q_{km}) f_{jk}^m = 0$$  \hspace{1cm} (2.6b)

$$(1 - q_{im}) f_{jn}^m - (1 - q_{im}) f_{im}^m = 0$$  \hspace{1cm} (2.6c)

$$f_{ij}^p f_{pk}^m - q_{jk} f_{ik}^p f_{pj}^m - f_{jk}^p f_{ip}^m + (q_{ij} - 1) f_{ik}^k f_{jk}^m = 0$$  \hspace{1cm} (2.6d).

It should be stressed that in (2.6) all the indices appearing (except the summation index $p$ and the index $m$ in (2.6d)) should assume different values. Indeed, though
(2.6b) is reducible to (2.6a) when \( m = i \) this is not so for (2.6c) which is a weaker condition than the condition which would follow from (2.6a) if the indices are allowed to take equal values (say \( m = j \)).

In view of (2.6d) it is tempting to introduce what we will call the adjoint representation of the abstract q-algebras, in analogy with what is done for Lie algebras. In other words we ask what are the further restrictions which enable the \( N \) matrices \( V_i \) whose elements are defined in terms of the structure constants by

\[
(V_i)_j^k = f_{ji}^k
\]  

(2.7)

to form a representation of the qualgebra.

It is easy to prove that these \( V \)'s satisfy the starting algebra provided that (2.6d) is replaced by the stronger conditions:

\[
f_{ij}^p f_{pk}^m - q_{jk} f_{ik}^p f_{pj}^m - f_{jk}^p f_{ip}^m = 0
\]  

(2.8)

To be more explicit these conditions mean that

a) First the two last terms of (2.6d) should not be present, i.e. for \( m \neq n \)

\[
(1 - q_{im}) f_{in}^n = 0
\]  

(2.9)

This is an non trivial extension of (2.6a) when indices are allowed to take certain equal values.

b) Moreover these are not the only conditions. Indeed (2.6d) has been derived for unequal indices \( i \neq j \neq k \neq i \) only and (2.8) has to hold for all indices.

If the stronger conditions in a) and b) above do not hold, it seems that an analogous adjoint representation cannot be defined. We will see later that for the general \( al(2, R)_q \) the conditions for the existence of the adjoint representation do not follow from the associativity relations while for \( al(N, R)_q \) for \( N > 2 \) these conditions are automatically satisfied. This will be the essential point allowing the representation of these q-algebras in terms of creation and annihilation operators of q-boson or q-fermion type. On the other hand we shall prove that the existence of the requirement of the existence of the adjoint representation for \( al(2, R)_q \) reduces in an essential way the allowed values of the quantum parameters.

2.b q-direct sums of q-algebras.

A generalized notion of q-direct sum of isomorphic q-algebras can also be defined as follows.

Let \( V_i \) and \( W_i \) two isomorphic q-algebras with the same quantum parameters and structure constants of the form (2.1). We will define the \( Z_i \) operators of the q-direct sum of the two algebras by

\[
Z_i = V_i + W_i
\]  

(2.10a)
provided that $V_i$ and $W_j$ satisfy the quantum relations

$$V_i \ast W_j = q_{ij} W_j \ast V_i \quad . \quad (2.10b)$$

With these definitions and conditions it is obvious that the $Z_i$ satisfy the same qualgebra as $V_i$ and $W_i$ without any further condition. It should be stressed that within a qualgebra structure $(2.10b)$ is a more natural condition than simple commutation and that it reduces to simple commutation in the Lie algebra case, as it should.

With this notion we can now define $q$-direct sums of representations. Contrary to the Lie case where the direct sum can always be defined, using direct products with the unit matrix, this is not the case here. We shall however show that for representations built out of creation and annihilation operators this definition is natural.

3. The qualgebra $al(2, R)_q$.

If we limits ourselves to the case of $al(2, R)_q$, the result of the braiding relations leads to the general deformation, with a suitable rescaling of the diagonal operators $E(k, k)$ given by

$$E(1, 1) \ast E(2, 2) = E(2, 2) \ast E(1, 1)$$
$$E(1, 1) \ast E(1, 2) = \alpha^2 E(1, 2) \ast E(1, 1) + \alpha E(1, 2)$$
$$E(2, 1) \ast E(1, 1) = \alpha^2 E(1, 1) \ast E(2, 1) + \alpha E(2, 1)$$
$$E(2, 2) \ast E(1, 2) = \beta^2 E(1, 2) \ast E(2, 2) + \beta E(1, 2)$$
$$E(2, 1) \ast E(2, 2) = \beta^2 E(2, 2) \ast E(2, 1) + \beta E(2, 1)$$
$$E(2, 1) \ast E(1, 2) = \gamma^2 E(1, 2) \ast E(2, 1) + \gamma \lambda E(1, 1) + \gamma \mu E(2, 2) \quad ,$$

or using

$$V_{i+2(j-1)} = E(i, j) \quad , \quad (3.2)$$

this is written as

$$[V_1, V_2]_\pi = -V_2$$
$$[V_1, V_3]_\alpha = V_3$$
$$[V_1, V_4] = 0$$
$$[V_2, V_3]_\gamma = \lambda V_1 + \mu V_4$$
$$[V_2, V_4]_\beta = V_2$$
$$[V_3, V_4]_\pi = -V_3 \quad (3.3)$$

in the notation of (2.1).

We see that that four meaningful parameters survive since one of the five parameters above can be rescaled away (say $\lambda = 1$) by multiplying (rescaling) $E(1, 2)$ by a well-chosen factor (or equivalently rescaling $E(2, 1)$).
The conditions for the existence of the adjoint representation restrict drastically the number of free parameters. Indeed this implies
\[
\alpha^2 = \beta^2 = 1 \\
(\gamma^2 - 1)(\alpha \mu + \beta \lambda) = 0.
\] (3.4)

Representations of certain if these q-algebras can be constructed using sets of \(N\) creation operators \(a_i^\dagger, i = 1, \ldots, N\) and \(N\) annihilation operators \(a_i, i = 1, \ldots, N\). These operators satisfy
\[
\begin{align*}
a_i a_j &= x_{ij} a_j a_i \\
a_i^\dagger a_j^\dagger &= x_{ij} a_j^\dagger a_i^\dagger \\
a_i a_j &= x_{ji} a_j a_i \quad \text{for} \ i \neq j \\
a_i a_i^\dagger &= y_i a_i + 1
\end{align*}
\] (3.5a)

and, as usual,
\[
x_{ij} = \frac{1}{x_{ji}}.
\] (3.5b)

This set is associative as can be checked easily. The \(a_i^\dagger\) can be interpreted as the conjugates of the \(a_i\) if the \(x_{ij}\) are chosen to be of modulus one. For the case at hand and for \(N = 2\), the natural operators
\[
E(i, j) = z_{ij} a_i^\dagger a_j
\] (3.6)
satisfy (3.1) provided that the following relations hold
\[
\begin{align*}
\gamma^2 &= \alpha^2 \beta^2 \\
\lambda &= \alpha \beta \mu \\
z_{1,1} &= \alpha \\
z_{2,2} &= -\frac{1}{\beta} \\
z_{1,2} z_{2,1} &= -\frac{\mu \gamma}{\beta} \\
y_1 &= \alpha^2 \\
y_2 &= \frac{1}{\beta^2}
\end{align*}
\] (3.7)

With four creation and annihilation operators satisfying the same equations (3.5) for \(N = 4\) we may try to construct
\[
V_i = \sum_{j=1}^{4} \sum_{k=1}^{4} g_{ij} k^a_j a_k^a, \quad (3.8)
\]
where the $g_{ji}^{k}$'s are non zero when the corresponding $f_{ji}^{k}$'s do not vanish but also for non zero $g_{11}^{1}$, $g_{14}^{4}$, $g_{41}^{1}$ and $g_{44}^{4}$ in order to allow for a singlet. The result is more constrained than for two operators. In fact these cases are not interesting: the algebra involves commutators and/or anticommutators only.

4. The qualgebra $al(3, R)_{q}$.

Applying the same technique, one can easily find the most general $al(3, R)_{q}$. This algebra can be represented in terms of creation and annihilation operators either of triplets of $q$-quarks or of nonets of $q$-fermions or $q$-bosons.

4.a The quommutation relations of $al(3, R)_{q}$

It is remarkable that the very presence of the third dimension reduces drastically the number of free parameters to just one $q$, once arbitrary rescalings have been taken into account.

As for the $al(2, R)_{q}$ case, it is again convenient to rename the $E(i, j), i, j = 1, 2, 3$ operators as $V(\alpha), \alpha = 1, \ldots, 9$ in the following way

$$V_{i+3(j-1)} = E(i, j)$$  \hspace{1cm} (4.1)

The general quommutation relation are explicitly given in Appendix A in the form (1.1).

It should be stressed that this algebra has only one free parameter once the rescalings ($V_{i}' = \rho_{i} V_{i}$, no summation over $i$) have been taken into account. These rescalings do not affect the quantum parameters but renormalize the structure constants. The particular rescalings where $\rho_{2}$ and $\rho_{6}$ are arbitrary while $\rho_{1} = \rho_{5} = \rho_{3} = 1, \rho_{3} = \rho_{2} \rho_{6}, \rho_{4} = \rho_{2}^{-1}, \rho_{7} = \rho_{3}^{-1}, \rho_{8} = \rho_{6}^{-1}$ leaves the algebra completely invariant.

Since $V_{d} = V_{1} + V_{5} + V_{6}$ commutes with all the $V$'s, the algebra $al(3, r)_{q}$ can be reduced to the direct sum of the algebra of $sl(3, r)_{q}$ with eight generators and the algebra $u(1)$ with generator $V_{d}$.

The amusing point about these results is that $al(3, r)_{q}$ (resp. $sl(3, r)_{q}$) possesses three undeformed $gl(2, r)$ (resp. $sl(2, r)_{q}$) subalgebras, formerly called, for $U(3)$, the $i$-spin ($V_{1}, V_{2}, V_{4}, V_{5}$), the $u$-spin ($V_{1}, V_{3}, V_{7}, V_{9}$) and the $v$-spin ($V_{5}, V_{6}, V_{8}, V_{9}$). These three subalgebras have normal Lie algebra commutation relations. The only place where the quantum parameter $q$ shows up is in the interplay between the three unbroken $gl(2, r)$.

It should also be stressed that no diagonal $so(3)$ (or $so(3)_{q}$) algebra (the algebra generated by say $\lambda_{2}, \lambda_{5}, \lambda_{7}$ in Gell-Mann’s notation) with three generators can be constructed within the $V_{2}, V_{3}, V_{4}, V_{6}, V_{7}, V_{8}$ subspace as a subalgebra of $al(3, R)_{q}$ (except when $q = 1$).

The associativity relations for $al(3, R)_{q}$ belong automatically to the strong case such that the adjoint representation (2.7) holds.

The quommutation relation of $al(3, R)_{q}$ can be obtained starting from suitable sets of creation and annihilation operators. In the following we present the results
for triplets or octets of boson or fermion type operators obeying commutation relations of quantum type.

4.b Representation of $al(3, R)_q$ with $q$-quark triplets.

Let us take one collection of annihilation operators $a_i, i = 1, 2, 3$ and their corresponding creation operators $a_i^\dagger, i = 1, 2, 3$ and impose upon them the generic relations (3.5a). The $a_i^\dagger$ can be interpreted as the hermitian conjugates of the $a_i$ if the $x_{ij}$ are chosen to be of modulus one. If the $x_{ij}$ are not of modulus one we have to define an involution $\dagger$ between the creation and annihilation operators. For this involution the conjugates of products of two operators have to be redefined and, for example, the adjoints of products of two operators can be defined in the following way

\[(a_i a_j)^\dagger = m_{ij} a_j^\dagger a_i^\dagger\]
\[(a_i^\dagger a_j^\dagger) = m_{ij} a_j a_i\]
\[(a_i a_j^\dagger)^\dagger = m_{ji} a_j a_i^\dagger\]
\[(a_i^\dagger a_j)^\dagger = m_{ji} a_j^\dagger a_i\]

with $m_{ij}$ real given by

\[m_{ij} = \sqrt{x_{ij} x_{ij}^*} .\]  

We now construct, in a natural way, the $al(3, R)_q$ operators by

\[E(i, j) = z_{ij} a_i^\dagger a_j .\]  

Inserting this ansätz into (A.2) we find that there are two solutions for which (4.4) is a representation.

Either

\[y_i = 1 \quad i = 1, 2, 3\]  
\[x_{12} x_{23} x_{31} = p^2 ,\]  

or

\[y_i = -1 \quad i = 1, 2, 3\]  
\[x_{12} x_{23} x_{31} = -p^2 .\]

Moreover the $z$’s have to be chosen as follows

\[z_{ii} = 1 \quad i = 1, 2, 3\]
\[z_{ij} = \frac{1}{z_{ji}} \quad i \neq j\]
\[z_{12} z_{23} z_{31} = p .\]
At this point it is perhaps useful to remark that, for \( p = 1 \), the usual Lie algebra of \( GL(3, R) \) can be represented with \( q \)-quarks i.e. with creation and annihilation operators which are not of pure bosonic or fermionic nature.

From this representation another (the conjugate representation) can be constructed as follows

\[
\hat{E}(i, j) = -\hat{z}_{ij} a_j^\dagger a_i
\]

where

\[
\hat{z}_{ii} = 1 \quad i = 1, 2, 3
\]

\[
\hat{z}_{ij} = \frac{1}{\hat{z}_{ji}}
\]

\[
\hat{z}_{12}\hat{z}_{23}\hat{z}_{31} = \frac{1}{p}
\]

It is easily checked that the \( E(i, j) \)'s and the \( \hat{E}(i, j) \)'s satisfy the quommutation relations (4.2). These results are consistent with the usual relation between a representation and its conjugate \( \hat{E}(i, j) = -E^\dagger(i, j) \) if the adjoint is taken as the involution defined above (see (4.2-3).

We now make a very important point. Suppose that we have a second set of creation and annihilation operators \( b_i, b_i^\dagger, i = 1, 2, 3 \) which satisfy exactly the same quommutation relations as the \( a, a^\dagger \) and hence with the obvious definitions, the \( F \)'s

\[
F(i, j) = z_{ij} b_i^\dagger b_j
\]

satisfy the same \( al(3, R)_q \) algebra.

Let us we try to impose between the \( a \)'s and the \( b \) quommutation relations of the general form

\[
a_i b_j = \alpha_{ij} b_j a_i
\]

\[
a_i^\dagger b_j^\dagger = \alpha_{ij} b_j^\dagger a_i^\dagger
\]

\[
a_i b_j^\dagger = \beta_{ij} b_j a_i
\]

\[
a_i^\dagger b_j = \beta_{ij} b_j a_i^\dagger
\]

where the indices obviously run from 1 to 3.

The naive sums of the operators \( E \) and \( F \)

\[
G(i, j) = E(i, j) + F(i, j)
\]

satisfy the same \( al(3, R)_q \) qualgebra relations if the following restrictions upon the braiding relations (4.7) hold.

1) The diagonal elements \( \alpha_{ii}, \beta_{ii}, \tilde{\alpha}_{ii}, \tilde{\beta}_{ii} \) are essentially free parameters except that

\[
\alpha_{ii}\beta_{ii}\tilde{\alpha}_{ii}\tilde{\beta}_{ii} = 1 \quad i = 1, 2, 3
\]
2) The three first sets are related to the fourth by

\[
\bar{\alpha}_{ij} = \frac{1}{\beta_{ii} \beta_{jj}} \alpha_{ji}, \\
\bar{\beta}_{ij} = \frac{\beta_{ii}}{\alpha_{jj}} \alpha_{ji} \quad i \neq j. \\
\beta_{ij} = \frac{\beta_{jj}}{\alpha_{ii}} \alpha_{ji}.
\]

(4.9b)

3) Within the fourth set the following relations hold

\[
\alpha_{ij} = \alpha_{ii} \alpha_{jj} \frac{1}{\alpha_{ji}} \quad i > j \\
\alpha_{23} = \frac{p^2 \alpha_{22} \alpha_{13}}{\alpha_{12}}.
\]

(4.9c)

In total there are thus nine parameters of a diagonal type and two parameters \(\alpha_{12}, \alpha_{13}\) of a non-diagonal type. A particularly simple solution, the interesting one in fact, is obtained when all the free diagonal elements are chosen to be equal to \(y_1\) i.e. +1 or -1 when the triplets are boson-like or fermion-like respectively while the \(\alpha\)'s are identified with the \(x\)'s

\[
\alpha_{ij} = \alpha_{ii} = \beta_{ij} = \bar{\beta}_{ji} = x_{ij} \quad i \neq j \\
\alpha_{ii} = \beta_{ii} = \bar{\alpha}_{ii} = \bar{\beta}_{ii} = y_1.
\]

(4.10)

For this solution the quommutation relations between two copies of creation and annihilation operators become

\[
a_i b_j = x_{ij} b_j a_i \\
a_i^\dagger b_j^\dagger = x_{ij} b_j^\dagger a_i^\dagger \\
a_i b_j^\dagger = x_{ji} b_j^\dagger a_i \\
a_i^\dagger b_j = x_{ji} b_j a_i^\dagger.
\]

(4.11)

One might hope that this remark would allow the construction of a \(q\)-field theory with quommutation relations. This appears to be not the case however since, when a field of a given \(i\)-index at one space time point is constructed as a sum of creation and annihilation operators, there is no overall quommutation relation with another field of \(j\)-index at a different space time point unless \(x_{ij}^2 = 1\). This is a specific illustration of a difficulty common to the problem of constructing a coherent field theory out of \(q\)-operators.

4.c Representation of \(al(3, R)_q\) with \(q\)-quarks nonets.

In a completely analogous way the qualgebra \(al(3, R)_q\) can be represented using nonets of \(q\)-quarks. Without going into the details let us write explicitly one
such solution, using the $q$’s and the $f$’s obtained from (A.2) once the corresponding
equations have been written in the form (2.1)

$$V_i = f_{ji}^k a_j^k a_k$$  \hspace{1cm} (4.12)

where the commutation relations of the nine creation and nine annihilation operators follow (4.3) with the indices running from 1 to 9. All the $y_i$ are equal to $y_1$ of square 1 and

$$x_{ij} = y_1 q_{ij} \frac{m_i}{m_j}$$ \hspace{1cm} (4.13)

where the $m$’s are arbitrary factors present even when $p = 1$. The square of the creation and annihilation operators have to be equal to zero if $y_1 = -1$, i.e. in the case of nonets of $q$-operators of the fermion type.

The arbitrariness for the braiding coefficients $x_{ij}$ as seen in (4.4) and (4.13) demonstrate that even the usual Lie algebras (i.e. when the $q_{ij}$’s are equal to one) can be represented with operators which do not fully commute nor anticommute.

5. The $q$-algebras $al(N, R)_q$.

A canonical form of the general qualgebra $al(N, R)_q$ for any $N$ can easily be constructed along the same lines as we have done for $al(3, R)$ and under the same restrictions as defined in the introduction. Let us present it in the notation of equation (1.1). The parameters $p(j, k, l, m)$ can be constructed from a set of arbitrary parameters $q$.

Indeed let $q_{ij}$ with $i < j = 1, \ldots, N$ be an arbitrary set of complex numbers and let

$$q_{1j} = 1 \hspace{1cm} j = 2, \ldots, n$$

$$q_{jj} = 1 \hspace{1cm} j = 1, \ldots, n$$

$$q_{ij} = \frac{1}{q_{ji}} \hspace{1cm} i > j$$  \hspace{1cm} (5.1a)

There are thus $N(N - 1)/2$ arbitrary complex parameters $q$. We then obtain

$$p(j, k, l, m) = \frac{q_{jl} q_{km}}{q_{kl} q_{jm}}$$  \hspace{1cm} (5.1b)

Moreover the right hand side in (1,1) differs very little from those of the undeformed $gl(N, R)$ algebra. Indeed, one can chose a specific normalisation of the $V_i$’s such that $g_1, g_2, g_3$ and $g_4$ all retain their undeformed value.

The $q$-algebras $al(N, R)$ were already written down\[7\] in a slightly different form, starting from $N$ creation and $N$ annihilation operators. We would like to stress here that, except for renormalisations of the operators $V_i$, this is the most general solution compatible with our starting hypothesis.

It is easy to see that representation of these operators can be written in terms of a set of basic $N$ creation and $N$ annihilation operators but also in terms of $N^2$ creation and $N^2$ annihilation operators by generalising in an obvious fashion what we have done for $al(N, 3)_q$. 

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6. Conclusions.

One objective of this investigation was to find all the deformations of $gl(N, R)$ subject to natural restrictions. We succeeded and were led, in the process, to realize the deformed algebras in terms of deformed sets of creation and annihilation operators and to introduce a generalisation of the direct sum of algebras and of their representations. In his studies of braided tensor categories Majid has applied a similar concept to the $q$-direct sum to define an addition law for momentum in a braided Lorentz group\cite{8}.

In order to proceed further in the direction of application of these ideas and results to physics\cite{9}, it would seem necessary to construct local fields out of the creation and annihilation operators. These fields should satisfy generalized canonical commutation relations. Unfortunately the rules for commuting different sets of copies of the operators corresponding to different momenta or different space-time points turn out to be incompatible with a naive extension in which fields are linear combinations involving both creation and annihilation operators. Hence the construction of a genuine extension of the usual field theory remains a problem.
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Appendix A. The algebra $\text{al}(3,R)$.

With suitable renormalisations of the operators $V_i$ by coefficients $\rho_i$ the most general $\text{al}(3,R)$, obtained by deformations of $\text{gl}(3,R)$ subject to the conditions written in the first paragraph of the introduction, is given by the set of commutators given in (A.1) below.

By simply looking at (A.1) it is easy to see that the three $\text{al}(3,R)$ subalgebras with the three sets of generators $\{V_1, V_2, V_4, V_5\}$, $\{V_1, V_3, V_7, V_9\}$ and $\{V_5, V_6, V_8, V_9\}$ are undeformed corresponding to $i$-spin, $u$-spin and $v$-spin. On the other hand an $o(3)$, deformed or not, built out of three linear combinations of the generators $\{V_2, V_3, V_4, V_6, V_7, V_8\}$, cannot be constructed.

\[
\begin{array}{ccc}
[V_1, V_2] = -V_2 & [V_2, V_7]_p = V_8 & [V_4, V_8]_p = V_7 \\
[V_1, V_3] = -V_3 & [V_2, V_8]_p = 0 & [V_4, V_9] = 0 \\
[V_1, V_4] = V_4 & [V_2, V_9] = 0 & [V_5, V_6] = -V_6 \\
[V_1, V_5] = 0 & [V_3, V_4]_p = V_6 & [V_5, V_7] = 0 \\
[V_1, V_6] = 0 & [V_3, V_5] = 0 & [V_5, V_8] = V_8 \\
[V_1, V_7] = V_7 & [V_3, V_6]_p = 0 & [V_5, V_9] = 0 \\
[V_1, V_8] = 0 & [V_3, V_7] = V_9 - V_1 & [V_6, V_7]_p = -V_4 \\
[V_1, V_9] = 0 & [V_3, V_8]_p = -V_2 & [V_6, V_8] = V_9 - V_5 \\
[V_2, V_3]_p = 0 & [V_3, V_9] = -V_3 & [V_6, V_9] = -V_6 \\
[V_2, V_4] = V_5 - V_1 & [V_4, V_5] = V_4 & [V_7, V_8]_p = 0 \\
[V_2, V_5] = -V_2 & [V_4, V_6]_p = 0 & [V_7, V_9] = V_7 \\
[V_2, V_6]_p = -V_3 & [V_4, V_7]_p = 0 & [V_8, V_9] = V_8
\end{array}
\]

If $p$ is chosen to be $+1$ we have the undeformed $\text{gl}(3,R)$. If $p$ is chosen to be $+i$ and $\rho_3 = -\rho_7 = i$ one obtains an algebra with commutators and anticommutators only. The anticommutators are specifically given in (A.2).

\[
\begin{array}{ccc}
[V_2, V_3]_+ = 0 & [V_2, V_6]_+ = -V_3 & [V_2, V_7]_+ = -V_8 \\
[V_2, V_9]_+ = 0 & [V_3, V_4]_+ = -V_6 & [V_3, V_6]_+ = 0 \\
[V_3, V_8]_+ = -V_2 & [V_4, V_6]_+ = 0 & [V_3, V_8]_+ = 0 \\
[V_4, V_7]_+ = 0 & [V_4, V_9]_+ = 0 & [V_4, V_8]_+ = -V_7 \\
[V_6, V_7]_+ = -V_4 & [V_7, V_8]_+ = 0
\end{array}
\]

(A.2)