COMPLEXITY OF CHESS DOMINATION PROBLEMS

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Abstract. We study different domination problems of attacking and non-attacking rooks and queens on polyominoes and polycubes of all dimensions. Our main result proves that maximal domination is NP-complete for non-attacking queens and for non-attacking rooks on polycubes of dimension three and higher. We also analyse these problems for polyominoes and convex polyominoes, conjecture the complexity classes and provide a computer tool for investigation. We have also computed new values for classical queen domination problems on chessboards (square polyominoes). For our computations, we have translated the problem into an integer linear programming instance. Finally, using this computational implementation and the game engine Godot, we have developed a video game of minimal domination of queens and rooks on randomly generated polyominoes.

Keywords: Art Gallery Theorem, NP-Completion, NP-Hardness, Polyomino, Computational Geometry, Visibility Coverage, Guard Number, Domination Problem, N-Queens Problem, Linear Programming

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1. Introduction

One of the most ancient and famous enumeration problems involving a chessboard and chess pieces is the 8-queens problem that was first stated by Max Bezzel in 1848—see [3, 6, 16] for accounts of its history. This problem asks to find the number of different ways of placing 8 queens on a chessboard without attacking each other (queens can attack vertically, horizontally, and in diagonals). Note that 8 is the maximal number of non-attacking queens that can be placed on an 8×8 chessboard. Many problems stemmed from this puzzle, some of them are: the n-queens problem [1], that is, the 8-queens problem generalized to n×n chessboards for n ≥ 1; the completion problem [9, 10, 23], which asks if it is possible to complete a given set of non-attacking queens on a n×n chessboard to a non-attacking set with n queens; and the minimal domination problem [14, Appendix], which consists in finding the minimal number of queens necessary to guard or dominate a chessboard—the known values of this sequence can be found in [26, A075324].

In this paper, we study domination problems on polyominoes and polycubes1 by rooks and queens. This problem is also known as the art gallery problem on polyominoes [2, 4, 25]. In recent work, the NP-hardness of the minimal domination by rooks and queens was proven for d-polycubes for d ≥ 2 [2]—see Table 3. In the same paper, the authors studied the maximal non-attacking rooks set problem on polyominoes, and they proved that it is in P.

We note that the queen problem complexity has been considered on walled chessboards, equivalent to edge-polyomino [21].

Our first main results will be to prove the NP-completeness of the minimal domination of attacking, and non-attacking, rooks and queens on polycubes, extending [2, Thms 3, 4]. To give a perspective, let us note that the minimal domination problem of rooks on a square chessboard is trivially polynomial: a rook is needed in each row and each column so filling the diagonal solves it for any chessboard; and the minimal domination problem for queens on the chessboard has been studied for the last 150 years, and yet we still do not know whether there exists a polynomial-time algorithm to find the minimal number of queens needed to dominate a chessboard.

We also study the problem of finding maximal independent sets of queens or rooks on polyominoes, that is, the maximal number of non-attacking rooks or queens that can be placed on a polyomino. In one of our main results, Theorem 2 which answers Question 3 in [2], we prove that the maximal independent rook domination problem on d-polycubes is NP-complete for d ≥ 3. We also

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1In 1954, Solomon W. Golomb defined a polyomino as a finite rook-connected subset of squares of the infinite checkerboard [11]. A d-polycube is its extension to dimension d.
Max Independent Domination Problems

| Boards  | Rooks (non-attacking) | Queens (non-attacking) |
|---------|-----------------------|------------------------|
| square completion | P (trivial) | NP-complete [9] |
| all polyominoes | P [2, Thm 12], completion P [2, Thm 13] | NP-complete? (Conjecture 17) |
| $d$–polycubes $d \geq 3$ | NP-complete (Thm 2) | NP-complete (Thm 1) |

Table 1. Maximal independent domination problems.

Min Rook Domination Problems

| Boards  | attacking | non-attacking (independent) |
|---------|-----------|-----------------------------|
| square completion | P (trivial) | P (trivial) |
| all polyominoes | NP-hard [2, Thm 3], NP-complete (Thm 3) | NP-hard [2, Lem. 14], NP-complete (Thm 3) |
| $d$–polycubes $d \geq 3$ | NP-hard [2, Thm 3], NP-complete (Thm 3) | NP-complete (Thm 3) |

Table 2. Minimal domination problems for rooks.

Min Queen Domination Problem

| Boards  | attacking | non-attacking (independent) |
|---------|-----------|-----------------------------|
| square completion | NP-complete? (Conjecture 23) | NP-complete (Corollary 24) |
| all polyominoes | NP-hard [2, Thm 4], NP-complete (Thm 4) | NP-complete (Thm 4) |
| $d$–polycubes $d \geq 3$ | NP-hard [2, Thm 4], NP-complete (Thm 4) | NP-complete (Thm 4) |

Table 3. Minimal domination problems for queens.

answer the same question for queens in Theorem 1, proving that the problem is NP-complete for $d \geq 3$.

To put our results in context, in Tables 1, 2 and 3, we collect what is known and what is conjectured for the problems of minimal (independent and non-independent) and maximal (independent) domination for rooks and queens. We hope that the information in these tables will help researchers in the field to avoid imprecision on statements about the complexity of these problems and that it will also inspire further research. In what follows, we give precise definitions and statements of our main results.

1.1. Main Results. We now review the main results of this paper and the relevant definitions.

Definition. Let $d \geq 2$. A $d$–polycube is a finite union of unit cubes of the regular cubic tessellation of $\mathbb{R}^d$ with an interior that is connected (notice that polycubes could have holes or cavities). A 2–polycube is also known as a polyomino.

The problems we are considering resemble art gallery problems on polyominoes (see [4]) with guards having special guarding vision, or attacking powers, related to chess pieces. We now give a precise definition of the attacking powers of rooks and queens on polycubes. For this purpose, we can imagine the $d$–cubes of the $d$–polycube centred at the points of $\mathbb{Z}^d$.

Definition (Rook attacking powers). Suppose a rook is at $(0, \ldots, 0)$ in a $d$–polycube $P$. It guards this point. In addition, for each point that has all its coordinates 0 except for one coordinate $\pm 1$, we say that the $d$–dimensional rook guards or attacks tiles which have coordinates given by all natural-number multiples of this point such that all the smaller natural-number multiples are tiles of $P$. 
Figure 1. Possible movement (in green) of a rook and a queen centred in a $3 \times 3 \times 3$ cube. For the queen, a $5 \times 5$ level is put at the top of the cube. Find 3D models corresponding to these structures at https://skfb.ly/oz8tJ and https://skfb.ly/oz8tn, respectively.

**Definition** (Queen attacking powers [2]). A $d$-dimensional queen placed in a $d$-polycube $P$ at the origin in the cubic tessellation of $\mathbb{R}^d$ can attack all tiles of $P$ with coordinates points equal to 0 or ±1 and all natural-number multiples of such points, as long as all the smaller natural-number multiples of the points are tiles of $P$.

Let us note that there are $d$ directions for rooks and $(3^d - 1)/2$ for queens, as can be seen by placing the piece at the center of a $d$-hypercube. Figure 1 illustrates these definitions for $d = 3$.

Another important remark is that the rook’s or queen’s line of attack ends when it crosses outside the polycube.

Our first two main results study the computational complexity of a class of maximal independent domination problems on polycubes.

**Theorem 1.** Solving the maximal non-attacking queen domination problem on $d$-polycubes is NP-complete for $d \geq 3$.

**Theorem 2.** Solving the maximal non-attacking rook domination problem on $d$-polycubes is NP-complete for $d \geq 3$.

It was proven that, for $d = 2$, there is a polynomial-time algorithm that solves maximal domination for non-attacking rooks [2, Thm 12]. The passage to three dimensions is the crucial point at which the complexity of this problem changes.

We also prove NP-completeness of the minimal domination problem for non-attacking and attacking queens and rooks, extending [2, Thms 3, 4].

**Theorem 3.** Solving the minimal domination problem for attacking and for non-attacking rooks on $d$-polycubes for $d \geq 2$ is NP-complete.

**Theorem 4.** Solving the minimal domination problem for attacking and for non-attacking queens on $d$-polycubes for $d \geq 2$ is NP-complete.

We review the structure of the paper. In Section 2, we present the relevant definitions and prove Theorems 3 and 4. In Section 3, we prove Theorems 1 and 2. In Section 4, we study domination problems polyominoes and convex polyominoes, and state a series of open questions and conjectures. In Section 5, we present a computational model, and with it, we compute some new values of domination numbers for classical problems. Finally, also in Section 5, we give a brief description of the video game on chess domination on polyominoes that we have created using the Godot game engine [20]. The game can be played at this link: https://www.erikaroldan.net/queensrooksdomination. The software developed and implemented in the course of this research is publicly available on GitHub [17].

2. Preliminaries

We now define the two problems we will consider.
Definition (Independent rook domination). We say that an instance of the non-attacking rook set problem for $d$–polycubes is a pair $(P, m)_d^R$, where $P$ is a $d$–polycube and $m$ is a positive integer. The problem asks whether there exists a non-attacking configuration of $m$ rooks placed in $P$ that dominates $P$.

Definition (Independent queen domination). We say that an instance of the non-attacking queen set problem for $d$–polycubes is a pair $(P, m)_d^Q$, where $P$ is a $d$–polycube and $m$ is a positive integer. The problem asks whether there exists a non-attacking configuration of $m$ queens placed in $P$ that dominates $P$.

We will study the complexity class of the maximal and minimal $m$ possible for both instances $(P, m)_d^R$ and $(P, m)_d^Q$. To prove NP-complexity, we follow the usual scheme. To help readers follow our proofs, we briefly review the steps here. We first show the problem is in the class NP by showing that verifying a solution is done in polynomial time. It is done in Lemma 6 below.

We then exhibit a polynomial reduction from a known NP-complete problem. This will be the purpose of Section 3. In our case, we will begin from the following restriction of planar sat.

Definition (P3SAT $3^3$ [7]). Given a set of Boolean variables, or literals, $x_i$ satisfying a set of clauses of the form $x_i \lor x_j$ or $x_i \lor x_j \lor x_k$, we say it is planar if the bipartite graph constructed by two sets of vertices given by the variables and the clause, and by edges drawn by linking a literal $x$ and a clause $C$ if $C$ contains $x$ or $\overline{x}$ is planar. The problem of determining if all the clauses are satisfied is called planar 3sat. If, furthermore, each $x_i$ appears exactly in three clauses we call the problem planar 3sat with exactly 3 occurrences per variable, and it is denoted P3SAT$_3$.

Examples of such instance are given in Figures 2 and 14.

Remark. It was proven to be NP-complete [7] with the further restriction that each variable appears once positively and twice negatively. This was the problem used for the proofs in [2]. We do not require this additional restriction.

Proposition 5 ([7]). The problem P3SAT$_3$ is NP-complete.

The reduction will proceed by introducing gadgets to encode variables, connections and clauses as maximal independent rooks or queens set problems on polycubes. The main difficulty will reside in showing that the size of the polycubes created is polynomially bounded, and that the algorithm to do so runs in polynomial time.

Let us begin our proof of complexity by showing that the verification of a candidate for domination and independence can be done in polynomial time.
Lemma 6. Verifying that a placement of rooks or a placement of queens dominate a $d$-polycube and verifying that a placement of queens or rooks that are not attacking each other can be done in polynomial time.

Proof. Taking each piece and checking all the cubes it guards can be done in polynomial time. This will then tell if the pieces are independent and if all the cubes of the polycube are guarded. 

We are now ready to end this section by giving the proofs of Theorem 3 and Theorem 4.

Proof of Theorem 3. By Lemma 6, we know that verifying the validity of a given set of rooks for the minimal domination problem for attacking and non-attacking rooks can be checked in polynomial time, and so the problems are in the class NP. The proof of [2, Thm 3] for attacking rooks gives that it is NP-hard, so it is in fact NP-complete.

Finally, we note that the gadgets used in the proof of [2, Thm 3] also hold true for non-attacking rooks, thus the domination problem for non-attacking rooks is also NP-complete.

Proof of Theorem 4. That a set of queens guards a polycube and that they do not attack each other can be checked in polynomial time by Lemma 6, so the minimal domination problem for attacking and non-attacking queens is in the class NP. The proof of [2, Thm 4] also holds for non-attacking queens since the gadgets only use non-attacking setups, and so it proves that both problems are NP-hard. Hence, they are NP-complete.

3. Complexity of the domination problems

This section is dedicated to prove Theorems 1 and 2. We will begin with the rooks and proceed to the queens.

3.1. NP-Completion of maximal independent domination of rooks on polycubes. We first introduce the variable gadget for non-attacking rooks in Figure 3. There are two ways to place the maximal number of rooks on the gadget.

![Variable gadget with rooks](https://skfb.ly/oz8tZ)

**Figure 3.** Variable gadget with rooks; when true, 2 additional rooks go on the dark cubes (T), and when false, 2 go on the light red cubes (F). Find a 3D model corresponding to this structure at [https://skfb.ly/oz8tZ](https://skfb.ly/oz8tZ).

Lemma 7. The maximal number of dominating non-attacking rooks for the polycube of Figure 3 is 6, and there are only two ways to do so.

Proof. The middle holed section can be guarded either by four non-attacking rooks in a cross pattern, which would dominate the full polyomino, or by two rooks in opposite corners, which then require four rooks on the remaining unguarded cubes. Thus at most 6 non-attacking rooks can guard the polyomino.

Four rooks are placed on the protruding corners, and two additional ones can be placed either both on the dark T cubes, and then we say the variable has truth value true; or both on the red F cubes, and then we say the variable has truth value false.

To transmit the signal, we will need to use the third dimension. Note, however, that the gadgets will never need to exceed 9 cubes in height. Let us introduce the connection gadgets in Figure 4. It has a maximal covering number by rooks of 6. It is placed on the variable gadget on one of the NW or SW corners of the variable gadget, with the protruding T and F tiles on F and T tiles respectively. According to the variable gadget truth value, the maximal independent covering of rooks on the connection gadget will add rook on the T or F tiles.
Lemma 8. The signal of the variable gadget is propagated by the connection gadget of Figure 4. Furthermore, the connection gadget can be used to turn corners.

Proof. As we connect to the variable gadget placing a red F tile on a dark T tile, and a dark T tile to a red F tile, the truth value of the variable gadget will influence where we can place the maximal 6 rooks on the connection gadget. First, three rooks are placed on the corners. Then if the variable has value true, the remaining rooks are placed on T tiles. Precisely, one on the T tile above the variable gadget, one on the cube just behind the F cube placed above the variable gadget and one on the top T cube of the connection gadget. This process can be repeated on the connection gadget to place a variable tile at the top. One can also keep on the same level by doing the process another time. What we described is viewed on the YZ-axis in Figure 5.

Finally, we can go around corners by rotating in space the connection gadget, always propagating the signal, and duplicating it if needed, as shown in Figure 6. The height we need for this is 9 cubes.

We introduce the clause gadgets in Figures 7. The clause gadget is created as follows. Let $c$ be a clause and suppose the literal gadgets are aligned on the NE axis. We place a connector on top of the NW red F tile in the variable $x$ if $x \in c$ and on the NW gray T tile if it is negated. The connectors are joined together by alternating top and left nodes in a line. We denote $\ell_R(c)$ the length of the clause gadget associated with $c$

$$\ell_R(c) := ||\text{nodes and connectors in } c||.$$ (1)

Figure 8 presents one example.

Lemma 9. Let $c$ be a clause and $K(c)$ be the clause gadget associated with it. A maximal rook placement on $K(c)$ contains $\ell_R(c)$ rooks if $c$ evaluates to false and $\ell_R(c) + 1$ if it evaluates to true.

Proof. Construct the clause gadget $K(c)$ from the clause $c$. On each top and left nodes there can be one rook, and there can be at most two rooks on the connectors. However, the connectors are all placed on a line and so only one can have two rooks. There are one node or connector for each tile, thus
(a) Turning the corner with connection gadgets.  
https://skfb.ly/ozUTt

(b) Duplicating the signal with connection gadgets.  
https://skfb.ly/ozUSO

Figure 6. Connection gadgets propagating and duplicating the signal.

(a) Clause gadget linked to the clauses with two positive or two negated literals.  
https://skfb.ly/ozVqt

(b) Clause gadget linked to the clauses with one positive and one negated literals.  
https://skfb.ly/ozVqH

Figure 7. The clause gadgets with two literals, all light green cubes are guarded, and one additional rook might be placed on one of the two dark T cubes.

ℓ_R(c). By construction as K is placed atop the variable, there can only be an extra rook if the clause evaluates to true. For example, Figure 8 presents a clause gadget K(c) of length 13 for the clause \( c = x_1 \lor x_2 \lor x_3 \). No extra rook can be placed if all literals are false.  

□

Figure 8. The clause gadget of the clause \( x_1 \lor x_2 \lor x_3 \).  
https://skfb.ly/ozV8L
Proposition 10. From an instance \( C \) of P3SAT\(_3\), it is possible to construct a polycube \( P^R_3(C) \) of polynomial size in polynomial time.

Proof. The polycube \( P^R_3(C) \) is constructed from \( C \) by replacing each variable of \( C \) by the variable gadget of Figure 3, and by using an amount of connection gadgets of Figure 4 to transmit their signal or its inverse to clause gadgets that will combine their outputs.

The construction of connection gadget has a height of at most 9 for a corner or a split, and the clause gadget has a height of 3, so the size of the constructed polycube is inside a rectangular prism of height 12 at most, hence still of order \( O(n^2) \). We can thus obtain a polynomial-time algorithm to convert an instance \( C \) to a polycube \( P^R_3(C) \) whose size is bounded polynomially again by following the idea of the proof of [2, Lem. 8].

Lemma 11. Let \( C \) be an instance of P3SAT\(_3\), and let \( P^R_3(C) \) be its associated polycube. Let

\[
m_R := 6n_{\text{var}} + 6n_{\text{connect}} + \sum_{c \in \{\text{clauses of } C\}} \ell_R(c),
\]

where \( n_{\text{var}} \) is the number of variables, \( n_{\text{connect}} \) is the total number of connection gadgets, \( \ell_R(c) \) is the length of the clause \( c \), and \( n_{\text{clause}} \) is the total number of clauses. There exists a non-attacking rook set of size \( m_R + n_{\text{clause}} \) for the polycube \( P^R_3(C) \) if and only if \( C \) is satisfiable.

Proof. Consider a truth assignment that satisfies all the clauses in \( C \). We transfer the assignment on the polycube \( P^R_3(C) \) of Proposition 10. Each variable can have at most 6 rooks placed on them regardless of their value, so there are 6 rooks per variable gadget.

We connect the variable gadgets to the clause gadgets and each of them have a maximal number of 6 rooks regardless of the signal they transfer, so there can be 6 rooks per connection gadget.

Since it is a truth assignment, all clauses are satisfied. Lemma 9 implies that one additional non-attacking rook per clause can be placed.

The total number of non-attacking rooks that can be placed on \( P^R_3(C) \) is then \( m_R + n_{\text{clause}} \), proving that an instance \( C \) is transformed into an independent rook domination problem \((P^R_3(C), m_R + n_{\text{clause}})_3\).

If a set of \( m_R + n_{\text{clause}} \) non-attacking rooks dominating \( P^R_3(C) \) exists and since it is not possible to increase the number of rooks in the variable nor in the connection gadgets, there must be \( \ell_R(c) + 1 \) rooks placed on each clause \( c \) gadget. From Lemma 9, it follows that all the corresponding clauses evaluate to true, meaning \( C \) is satisfied.

We can now prove Theorem 2.

Proof of Theorem 2. Lemma 6 implies that the problem is in the class NP since verifying a solution can be done in polynomial time.

Let \( C \) be an instance of P3SAT\(_3\). We obtain a polynomially-sized polycube \( P^R_3(C) \) in polynomial time by Proposition 10. Lemma 11 implies that the non-attacking rook set problem for \((P^R_3(C), m_R + n_{\text{clause}})_3\) is equivalent to the P3SAT\(_3\) problem for \( C \), which is NP-complete by Proposition 5, and so the non-attacking rook set problem for 3-polycube is NP-complete. As 3-polycubes are contained in \( d \)-polycubes for \( d \geq 3 \), the non-attacking rook set problem for \( d \)-polycubes is also NP-complete.

3.2. NP-completion of maximal independent domination of queens on polycubes. We now turn to the proof of Theorem 1. It will proceed in a similar fashion as the previous section, with different gadgets. The first component of the proof is the literal gadget obtained as a concatenation of four smaller elements identical to the rook variable gadgets; see Figure 9.

Lemma 12. The gadget of Figure 9b has two states of maximal domination by 12 queens.

Proof. The proof proceeds in a similar fashion as for the rook case. It can be checked by the solver [18] introduced in Tool 25 since the gadget is two-dimensional.

Contrary to the rook gadgets, the number of queens will not remain constant on the connected literal gadget. This means that we must analyse carefully the propagation of the signal via the connection gadget presented in Figure 10.
Figure 9. Construction of the literal gadget for 3D queens. Maximally 12 queens can be placed in two different ways.

Figure 10. Top view of the connection gadget (in the dotted teal box) propagating the signal of a variable gadget (in the full orange box). There is one pair of elements with each 4 neighbours in the middle (in a dashed black box).

**Lemma 13.** The signal of the variable gadget is propagated by the connection gadget of Figure 10. The number of queens needed to guard the gadget is given by $4n_{4\text{neigh}} + 5n_{3\text{neigh}} + 6n_{2\text{neigh}}$, where $n_{i\text{neigh}}$ is the number of pairs of tiles with $i$ neighbours. Furthermore, the connection gadget can be used to turn corners.

**Proof.** We begin by counting the number of queens on the gadgets. We notice first that the basic elements come in pairs by construction. The members of the pair can either both have 2, 3 or 4 neighbours. Then a local analysis shows that the pair will have maximally 6, 5 or 4 queens respectively. The three situations can be seen in Figure 10, where the one pair with 4 neighbours has been highlighted.

Finally, to evaluate clauses, gadgets similar to those of Figures 7 and 8 are used. Their construction proceeds as follows. Let $c$ be a clause with three literals (respectively two). Suppose there are three literal gadgets $x_1, x_2, x_3$ (respectively two $x_1, x_2$) on a line. Then the extremal nodes form a sequence of red (false) and gray (true) tiles. If the literal $x_i$ appears as $x_i$ in $C$, we place a connector on the red (F) tile, and if it appears as $\overline{x_i}$ in $C$, we place the connector on the gray (T) tile. We then
join the connector by alternating top $\textbullet$ and left $\textcircled{1}$ nodes. They form a clause gadget—see Figure 12. The length of the clause gadget is denoted $\ell_Q(c)$ and is defined similarly as the length $\ell_R$ of the rook gadget in (1). The process is illustrated in Figure 11.

Figure 11. Top view of the procedure to construct two clause gadgets on four literal gadgets $x_1, x_2, x_3, x_4$. The clause gadgets are in teal and are on top of the brown polycubes.

Figure 12. The 3D queen clause gadget for the clauses $x_1 \lor x_2$ if the literal gadgets are at distance one of each other.

Figure 13. The clause gadget on top of the two variable gadgets for the clause $x_1 \lor x_2$.

**Lemma 14.** Let $c$ be a clause. The clause gadget associated with $c$ when placed on top of literal gadgets can take $\ell_Q(c) + 1$ queens if it evaluates to true, or $\ell_Q(c)$ if it evaluates to false, where the function $\ell_Q$ is the length of the clause gadget.
Proof. Let \( c \) be a clause and \( K(c) \) be the associated clause gadget. The clause gadget is joined with its literals placed on a line by placing it on top of the extremal nodes; see the example of Figure 13. It covers the leftmost to the right most variable and is placed on F tiles if \( x_j \) appears as \( x_j \) in \( c \), and on T if it appears as \( \overline{x}_j \) in \( c \). Then a maximal placement of queens will have all queens on the protruding nodes and maybe a queen on the gray T tiles if the clause is satisfied. There are one queen per protruding nodes and one more for each connector, and maximally one extra in one of the connectors, thus showing that there are \( \ell_Q(c) \) or \( \ell_Q(c) + 1 \) queens in a maximal dominating position. \( \square \)

**Proposition 15.** From an instance \( C \) of P3SAT, it is possible to construct a polycube \( P^Q_3(C) \) of polynomial size in polynomial time.

**Proof.** The proof is similar to that of Proposition 10 and is omitted. \( \square \)

We illustrate the process of Proposition 15 on a small example by taking the instance \( C \) of Figure 14 and constructing the polycube \( P^Q_3(C) \) associated with it in Figure 15.

\[
\begin{align*}
\overline{x}_4 & = x_1 \lor x_3 \\
\overline{x}_2 & = \overline{x}_1 \lor x_2 \\
x_2 & = x_1 \lor x_2 \lor x_3 \\
\overline{x}_4 & = x_1 \lor x_3 \\
x_3 & = x_2 \lor \overline{x}_3 \\
c_2 & = x_1 \lor \overline{x}_2 \\
c_1 & = x_1 \lor x_2 \lor x_3 \\
c_3 & = x_2 \lor \overline{x}_3 \\
c_4 & = \overline{x}_1 \lor x_3
\end{align*}
\]

**Figure 14.** An instance of P3SAT with three literals \( x_1, x_2, x_3 \) and four clauses \( c_1 = x_1 \lor x_2 \lor x_3, c_2 = x_1 \lor \overline{x}_2, c_3 = \overline{x}_2 \lor x_3 \) and \( c_4 = \overline{x}_1 \lor x_3 \). The only solution is \((x_1, x_2, x_3) = (1, 1, 1)\). We now show how to translate the instance of P3SAT into a domination problem.

**Lemma 16.** Let \( C \) be an instance of P3SAT and \( P^Q_3(C) \) be its associated polycube. Let

\[
m_Q := 4n_{1\text{neigh}} + 5n_{2\text{neigh}} + 6n_{3\text{neigh}} + \sum_{c \in \text{clauses of } C} \ell_Q(c),
\]

where \( n_{1\text{neigh}} \) is the number of pairs of elements with 1 neighbours, \( \ell_Q \) is the length of the queen clause gadget. Then one can place \( m_Q + n_{\text{clause}} \) non-attacking queens on \( P^Q_3(C) \) if and only if \( C \) is satisfiable.

**Proof.** Let \( C \) be an instance, and construct the polycube \( P^Q_3(C) \). If \( C \) is satisfiable, there is an assignment of values ensuring all clauses are satisfied. For each satisfied clause, there will be a queen added to the clause gadget. Then from Lemmas 12–14 we get that there are \( m_Q + n_{\text{clause}} \) queens.

If the polycube \( P^Q_3(C) \) has a maximum placement of \( m_Q + n_{\text{clause}} \) queens, it means all clauses are satisfied, since they are the only way to add more than \( n_Q \) queens. Thus \( C \) is satisfiable. \( \square \)

With this last lemma, everything is in place to prove Theorem 1.

**Proof of Theorem 1.** Lemma 6 shows that verifying a solution is done in polynomial time, thus that the problem is in the class NP.

Let \( C \) be an instance of P3SAT and \( P^Q_3(C) \) the polycube constructed from it. From Proposition 15, we know \( P^Q_3(C) \) is polynomially-sized and was constructed in polynomial time. Lemma 16 then tells it is equivalent to finding a guarding set of \( m_Q + n_{\text{clause}} \) queens. As P3SAT is NP-complete by Proposition 5, this means the non-attacking queen set problem for 3–polycubes is NP-complete, and then it is for \( d \)-polycubes when \( d \geq 3 \). \( \square \)

4. INVESTIGATION ON THE DOMINATION OF POLYOMINOES

In this section, we consider open questions for polyominoes and specifically for the class of convex polyominoes. The interested reader can use Tool 25 to get the number of maximal dominating queen positions on any gadget on a polyomino. The underlying algorithm is explained in the next section.
4.1. Maximal queen domination on polyominoes. For the maximal rook domination problem, Theorem 2 and [2, Thm 12] present the whole picture: there is a polynomial algorithm to solve the instance on polyominoes and it is NP-complete for higher-dimensional polycubes. The argument of [2, Thm 12] translates this into a bipartite graph matching problem, for which known polynomial algorithms exist. We present another argument that it is in P, and extend this discussion at the end of Section 5.

In the case of maximal queen domination, Theorem 1 leaves the question of whether maximal queens domination on polyominoes is NP-complete as in higher dimension or in P as the rook domination on polyomino unanswered. Using a similar approach to the rook case proves ineffective: the problem translates into a 4-hypergraph matching problem, which is NP-complete [15], thus leaving us unable to conclude. However, we strongly suspect the problem to be NP-complete and put forward the following conjecture.

**Conjecture 17.** The maximal independent queen domination problem on polyomino is NP-complete.

As a support to this conjecture, we first consider a slight generalisation of the problem. It is equivalent to what is considered by Martin [21], and we provide another proof using n-queens completion.

**Definition.** A path-connected polyomino is a finite collection of unit squares connected by their edges or by their vertices.

The NP-completeness of the maximal independent queen domination on path-connected polyominoes was proven by Martin by proving the NP-completeness of maximal independent queen domination on “walled chessboards”.

**Theorem 18 ([21, Thm 1]).** Maximal independent queen domination is NP-complete on path-connected polyominoes.
Proof. A walled chessboard in the definition of Martin is a chessboard with a set of tiles that stops the queen ray of attack. This is an equivalent definition of path-connected polyomino. □

Of course, this does not say anything about the complexity of the instance on the smaller subset of polyominoes, but along with extensive computations, it encourages us to believe the conjecture, and we think this problem might be an interesting candidate to find a manageable reduction.

4.2. Domination problems on convex polyominoes.

Definition (Convex polyominoes). Let $P$ be a polyomino. A polyomino is called row-convex if each row of $P$ has at most one connected component; it is called column-convex if each column of $P$ has at most one connected component, and it is called convex if it is both row- and column-convex.

![Figure 16. A convex polyomino.](image)

The problem of minimal domination for rooks on $n \times n$ square polyominoes is trivial: $n$ rooks are needed and simply putting them on the diagonal gives a solution. However, the same problem for polyominoes is NP-complete (Theorem 3). Convex polyominoes somehow lie in the middle of those two classes of polyominoes.

It is tempting to think that a simple process mimicking the diagonal domination on $n \times n$ chessboards would work for convex polyominoes; however, there are convex polyominoes where such processes would never find the optimal solution, such as the one presented in Figure 17. We will present in the next section the general solver that we used to generate the solution. We therefore conjecture that the minimal domination problem for rooks on convex polyominoes is NP-complete.

Conjecture 19. The minimal domination problem for attacking or non-attacking rooks on convex polyominoes is NP-complete.

Note that [2, Lem. 14] implies that proving the conjecture for attacking rooks is equivalent to proving the non-attacking case.

We also conjecture the same complexity for the minimal domination problems of queens on convex polyominoes.

Conjecture 20. The minimal domination problem for attacking, or non-attacking, queens on convex polyominoes is NP-complete.

On the other hand, as a consequence of [2, Thm 12] that states that finding a maximal independent dominating set for rooks on polyominoes is in P, we get that the maximal independent domination problem for rooks on convex polyominoes also is in P.

Corollary 21. Maximal domination by non-attacking rooks on convex polyominoes is in P.

Proof. Since it is in P for all polyominoes [2, Thm 12], it is also in P for the convex ones. □

The maximal domination problem on convex polyominoes for non-attacking queens, on the other hand, has yet to be studied. The gadgets used in the proofs on [2] are non-convex; thus a direct proof cannot come from this method. However, the proof of the NP-completeness of the completion for the chessboard in [9] has gadgets that are non-convex only due to the constraint of completing the square. We suspect that an adaptation of their proof could lead to a proof of the result for convex polyominoes, albeit the reduction could be very subtle.
Conjecture 22. The maximal domination problem for non-attacking queens on convex polyominoes is NP-complete.

One subfamily of convex polyominoes is the square polyominoes. There is no known polynomial time algorithm for finding the minimum number of queens needed to guard a $n \times n$ chessboard. We define the minimal chessboard domination queen completion problem $(n, Q, l)$. Given a set $Q$ of $k$ queens placed on the $n \times n$ chessboard, is there a set $Q'$ of size $k + l$ such that $Q \subset Q'$ and that it dominates (guards) the chessboard.

Conjecture 23. The minimal chessboard domination queen completion problem is NP-complete.

Analogously, we define the minimal independent chessboard domination queen completion problem $(n, Q, l)$. Given a set $Q$ of $k$ non-attacking queens placed on the $n \times n$ chessboard, is there a set $Q'$ of size $k + l$ such that $Q \subset Q'$, that it guards the chessboard, and no pair of queens in $Q'$ attack each other. Notice that the difference with the well-known $n$-queens completion problem is that with $n$ queens the $n \times n$ chessboard is trivially dominated. As it has already been proven that this problem is NP-complete [9, Thm 37], we can prove the following corollary.

Corollary 24. The minimal independent chessboard domination queen completion problem is NP-complete.

Proof. Let $n \in \mathbb{N}$ and $0 \leq k \leq n$. An instance, $(n, Q)$ of the $n$-queens completion problem can be reduced in polynomial time to an instance of the minimal independent chessboard domination queen completion problem $(n, Q, n - k)$. Since $n$-queens completion is NP-complete [9, Thm 37], it is NP-hard and minimal independent domination queen completion problem is NP-hard. Finally, verifying a solution is done in polynomial time by Lemma 6, so the proof is completed. 

5. An integer linear programming solver

In this section, we present a general solver using integer linear programming (ILP) for domination problem. This allows us to make use of efficient solvers such as [12]. We illustrate the performance of this general scheme by comparing it against heavily optimized solvers [5] created specifically for the queen domination problem on $n \times n$ chessboards.

Given a polycube, let $m$ be its number of tiles. We start by enumerating the tiles of the polycube in arbitrary order with an index $i \in \{1, \ldots, m\}$. We introduce $m$ binary variables $x_i$, with $x_i = 1$ encoding that a chess piece is placed on this tile and $x_i = 0$ encoding that no chess piece is placed on this tile.

Let us denote by $v$ the attack direction of a selected chess piece $F$. For example, for a queen on the tile $i$ on a given polyomino there are four attacking directions: (1) the row on which it is; (2) the column (see Figure 18); (3) the ascending diagonal, and (4) the descending diagonal. For a three-dimensional queen, there will be 13 attack directions.
For each tile $i$ of the polycube and each attack direction $v$ of the piece $F$, we construct a set $I_{v,i}^F$ consisting of all the labels of the tiles that can be attacked from the tile $i$ using only the attack direction $v$; remark that $i \in I_{v,i}^F$. Then we add the following constraint to our integer linear programming problem for each tile $i$ and each attack direction $v$:

$$\sum_{j \in I_{v,i}^F} x_j \leq 1.$$ 

This ensures that at most one of the $x_j$ for $j \in I_{v,i}^F$ can be 1, meaning that for all pairs of tiles attacking each other, there is only one chess piece placed on one of them. This construction leads in many cases to duplicate constraints, which we can simply remove at the end to optimize the program.

If we now maximize with the objective function $\sum_{i=1}^{m} x_i$, which is the number of chess pieces, we have a translation of the maximal domination problem for non-attacking rook or queen into an integer linear programming problem. This is one of the classical approaches to solve chessboard domination problems and is described in more detail in \cite{8}.

To translate the minimal independent domination problem, we need additional constraints to guarantee the domination of the polycube. For each tile of the polycube, we construct a set $A_i^F$ consisting of the tile $i$ and all tiles that can attack the tile $i$ using one movement of the piece $F$; note that $A_i^R \subset A_i^Q$. We add the following constraint to our integer linear programming problem for each tile $i$:

$$\sum_{j \in A_i^F} x_j \geq 1.$$ 

This ensures that each tile is guarded, since for each tile at least one of the $x_j$ in $A_i^F$ must be 1, meaning a chess piece is either placed on the tile or one can attack the tile.

Combining the two sets of constraints, we get the minimal independent domination integer linear programming problem for queens or rooks.

\begin{align}
(\text{ILP}) \quad \text{minimize} & \quad \sum_{i=1}^{m} x_i \\
\text{subject to} & \quad x \in \{0,1\}^m, \\
(\text{Independence}) & \quad \sum_{j \in I_{v,i}^F} x_j \leq 1, \quad \text{for all } I_{v,i}^F, \\
(\text{Domination}) & \quad \sum_{j \in A_i^F} x_j \geq 1, \quad \text{for all } A_i^F.
\end{align}

\textbf{Tool 25.} As a tool to help further research, we designed a small helper program based on the Julia library. It uses the exhaustive search provided by the proprietary solver Gurobi on the previously discussed ILP model to find all possible combinations of placing $d$ non-attacking queens on a polyomino \cite{18}.
The translation to an ILP process was used to calculate the previously unknown minimal numbers of non-attacking queens guarding the $n \times n$ chessboards for $n = 26, \ldots, 31$, which allowed us to extend the OEIS sequence A075324 [26]. To model the general minimal domination problem (sequence A075458), one only needs to remove the independence constraints.

Furthermore, we tested the performance of our solver on the problem of finding the maximal number of non-attacking queens that can be placed on a $d$–hypercube of size $n$.

| $d/n$ | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 8     | 1  | 1  | 52 |    |    |    |    |    |    |    |    |    |    |
| 7     | 1  | 1  | 32 | 128|    |    |    |    |    |    |    |    |    |
| 6     | 1  | 1  | 19 | 64 |    |    |    |    |    |    |    |    |    |
| 5     | 1  | 1  | 11 | 32 |    |    |    |    |    |    |    |    |    |
| 4     | 1  | 1  | 6  | 16 | 38 | 80 | 145|    |    |    |    |    |    |
| 3     | 1  | 1  | 4  | 7  | 13 | 21 | 32 | 48 | 67 | 91 | 121| 133| 169|
| 2     | 1  | 1  | 2  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 |

**Table 4.** Values of maximal queen domination problem on a $n^d$ hypercube. In blue (line $d = 3$), on a cube; in green (line $d = 4$), on a tesseract, and in orange (column $n = 3$), on a $d$–dimensional hypercube of side length 3. The red numbers in bold are previously unknown values that we have calculated using [19].

Table 4 presents for reference many sequences of maximal independent queen domination problems on $n^d$ hypercube. Some of the sequences are known and are indexed in OEIS [26]: the queens on a cube of size $n$ A068940; queens on a tesseract A068941; queens on a $d$–dimensional hypercube of side length 3 A115992. We have extended the sequences of domination numbers for $d$–dimensional tesseracts of side length 4 and of 4–dimensional tesseracts of side length $n$ using our model—the program is available at [19].

From the data we found with our computations, we conjecture the following formula for the maximal number of non-attacking queens on the hypercube.

**Conjecture 26.** The maximal non-attacking queen set problem on a $d$–dimensional hypercube of side length 4 has the solution size $2^d$ for $d \geq 4$.

To complement the exact algorithm, we now comment on the possibilities to approximate the domination problems we consider. First, the matrix of the rook maximal domination problem is unimodular, providing an alternate proof or [2, Thm 12]. Yet another proof is provided by remarking that the rook graph is claw-free, so finding an independent set is polynomial [22].

For queens however, or even rooks on polycube, the graph is not claw free. However, we can still say something by extending the notion. A $m$-claw is the complete bipartite graph $K_{1,m}$ shown in Figure 19. The normal claw is the bipartite graph $K_{1,3}$. We say a connected graph is $m$–claw free if it does not contain the $m$–claw as an induced subgraph. We construct a chess graph on polycube associated with a piece $p$ by putting a vertex for each unit polycube and an edge between two vertices if the piece $p$ can travel from one to another. Chess graphs are $m$–claw free for a certain $m$ related to the piece $p$. 

![Figure 19. A m–claw.](image-url)
Proposition 27. Let $p$ be a chess piece with $m$ attack directions. Then the associated chess graph is $(m+1)$-claw free.

Proof. Suppose there is a $(m+1)$-claw as a subgraph. This means that a piece can reach $m+1$ independent tiles. As it has $m$ attack direction, by the pigeonhole principle, two of the tiles are in the same attack direction and there must be an edge between them.

We remark briefly that the converse of Proposition 27 is not true as can be seen by the following example of a claw-free graph and a 4-claw free graph that do not correspond to rook and queen domination problem respectively.

![Image](image.png)

Figure 20. A 4-claw free graph and a claw free graph. There are no tetramino with a corresponding queen graph for the first graph, and no tetramino with a corresponding rook graph for the second graph.

Independent set is in P for claw-free graphs. For $m$-claw-free graphs with $m > 3$, there is a constant factor approximation algorithm.

Theorem 28 ([24, Thm 13]). Let $G$ be a $m$-claw-free graph. Then an independent set of $G$ can be approximated in polynomial time with factor $m/2$.

The class of $m$-claw-free graphs has an interesting relation between the minimal independent domination and the maximal independent domination problems.

Proposition 29. Let $G$ be a $m$-claw-free graph. Then if we denote by $\min(G)$ the minimal size of an independent dominating set on $G$ and by $\max(G)$ the maximal size of an independent dominating on $G$, we have

\[(m-1)\min(G) \geq \max(G)\]

Proof. Let $M$ be a minimal independent dominating set on $G$. Then each vertex in $M$ divides in at most $m-1$ fully connected sets and there are thus at most $(m-1)\min(G)$ disjoint induced fully connected subgraphs covering the whole graph.

As a corollary with thus have a constant-factor approximation algorithm for the maximal independent domination problem and an upper bound for the minimal independent domination problem.

Corollary 30. The maximal independent domination on a $m$-claw-free graph can be approximated polynomially with factor $m/2$. Minimal independent domination can be bounded in polynomial time by a constant factor $m(m-1)/2$. In particular, this applies to chess graphs.

Proof. This is a consequence of Theorem 28 and Proposition 29. The last statement follows directly from Proposition 27.

This last corollary gives constant factor approximation and bound for any given dimension. However, the constant factor grows with the dimension. It is linear for the rooks as a $d$-dimensional rook has $d$ attack directions, and it is exponential for the queens as the number of attack directions of a $d$-dimensional queen is $(3^d-1)/2$. To see this last claim, place a queen at the middle of a $d$-hypercube of side 3: it can go in one move in all the $3^d-1$ remaining unit cubes and each cube as a cube opposite in a straight line; see Figure 1 for the example in dimension 3.

To conclude, we now describe the video game we created with the Godot game engine [20] on minimal rook and queen domination on polyominoes. The readers are invited to try the game online at https://www.erikaroldan.net/queensrooksdomination.

The game challenges the player to dominate a random polyomino either with rooks or queens. When the player submits their solution, the minimal number necessary is then given to them. The
polyominoes chosen for the game are all 50-tile polyominoes generated via a shuffling algorithm with a percolation parameter empirically chosen to offer interesting challenges to the player. The optimal solutions for the generated polyominoes were found using the solver described above, which can in fact easily manage polyominoes with thousands of tiles.

We plan to develop further the game to include minimal independent domination, maximal independent domination, create a two-player game and port it to different platforms.

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Availability of data and software

The software developed and implemented in the course of this research is publicly available on GitHub [17], the polyomino verification tool on [18], and the program for the computations done in the last section is publicly available on GitHub [19].

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