CONFORMAL RICCI FLOW ON ASYMPTOTICALLY HYPERBOLIC MANIFOLDS

PENG LU, JIE QING, AND YU ZHENG

ABSTRACT. In this article we study the short-time existence of conformal Ricci flow on asymptotically hyperbolic manifolds. We also prove a local Shi’s type curvature derivative estimate for conformal Ricci flow.

1. INTRODUCTION

The geometry and analysis on asymptotically hyperbolic (AH in short) manifolds attracted significant research interest from both mathematics and theoretical physics communities, particularly after the introduction of AdS/CFT correspondence in theoretical physics (cf. [Ma98, GKP, Wi98]). In this paper we prove the short time existence on AH manifolds and a local Shi’s type curvature derivative estimates for conformal Ricci flow (CRF in short).

Ricci flow is known to be a powerful geometric and analytic tool in differential geometry and topology. CRF was introduced by Fischer [Fi04] as the modified Ricci flow that maintains scalar curvature constant. It is so named because the constancy of scalar curvature is achieved by the conformal deformation of metrics at each time. Fischer [Fi04] observed that, on compact manifolds, Yamabe constant is strictly increasing along CRF. Later, in [LQZ], the short-time existence of CRF on asymptotically flat manifolds was established. Interestingly, it is observed that ADM mass is strictly decreasing unless the initial metric is Ricci-flat [LQZ, Theorem 1.4] (in contrast to the fact that ADM mass stays constant along Ricci flow [DM]). CRF is considered to possibly be an efficient way to search for Einstein metrics because of the nature of the Einstein-Hilbert action (cf. [Be87, Fi04]), that is, Einstein metrics on a manifold $\mathcal{M}^n$ of dimension $n$ may be associated with

$$\sup_{\{[g]: \text{conf classes}\}} \inf_{\{g \in [g]: \text{Riem metrics}\}} \frac{\int_M R_g d\mu_g}{\text{vol}(M, g)^{\frac{n-2}{n}}}.$$ 

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As introduced in [Fi04] (see also [LQZ]), a family of metrics \( \{g(t) : t \in [0, T]\} \) on a smooth manifold \( X^{m+1} \) is said to be CRF if it satisfies:

\[
\begin{align*}
\partial_t g(t) + 2 \left( R_{g(t)} + mg(t) \right) &= -2p(t)g(t) \quad \text{in } X \times (0, T), \\
(-\Delta g(t) + (m + 1))p(t) &= \frac{1}{m} \left| R_{g(t)} + mg(t) \right|_{g(t)}^2 \quad \text{on } X \times [0, T), \\
g(0) &= g_0,
\end{align*}
\]

where the initial metric \( g_0 \) has constant scalar curvature \(-m(m+1)\), \( p(t) \) is an auxiliary function and is named as the pressure function in [Fi04] when comparing CRF with the Navier-Stokes equations, and \( T \) is a positive constant. The scalar curvature \( R_{g(t)} \) remains as \(-m(m+1)\) for all \( t \in [0, T) \).

Before stating the results, let us first briefly introduce AH manifolds. Let \( \bar{X} \) be a compact smooth manifold with nonempty smooth boundary \( \partial X \) and let \( X \) be the interior. A smooth function \( x : \bar{X} \to [0, \infty) \) is called a defining function for the boundary \( \partial X \) if it satisfies:

1) \( x > 0 \) on \( X \); 2) \( x = 0 \) on \( \partial X \); 3) \( dx \neq 0 \) on \( \partial X \).

A metric \( g \) on \( X \) is \( C^{l+\beta} \) conformally compact if \( x^2g \) extends to be a \( C^{l+\beta} \) metric on \( \bar{X} \) for a boundary defining function \( x \), where \( l \geq 2 \) is an integer and \( \beta \in (0, 1) \). \( x^2g \) induces a metric \( \hat{g} \) on the boundary and, in fact, \( g \) induces a conformal structure \( [\hat{g}] \) on the boundary when defining functions vary. \( (X, g) \) is said to be AH if it is conformally compact and the sectional curvature of \( g \) goes to \(-1\) asymptotically at the infinity.

It is often convenient to use geodesic defining functions for AH manifolds. A geodesic defining function \( x \) is a defining function such that \( |dx|_{x^2g} = 1 \) in a collar neighborhood of the boundary not just on the boundary. Hence

\[
g = x^{-2}(dx^2 + g_x)
\]

where \( g_x \) is a family of metrics on \( \partial X \) depending on \( x \), i.e. the metric \( g \) splits orthogonally in \( x \) and the tangential to \( \partial X \).

The central feature of AH manifolds is the association of the AH Riemannian metrics on \( X \) to the conformal structures on its boundary \( \partial X \), which is fundamental to a mathematical theory of AdS/CFT correspondence in theoretical physics. Similar to the explorations in [Fi04, LQZ], one expects that CRF is significant in the search for appropriately canonical AH metrics for a given conformal structure at the boundary to enrich the mathematical theory for AdS/CFT correspondence.

The first result of this article is the short time existence of CRF on AH manifolds (see §2.1 for the description of Hölder spaces used).
Theorem 1.1. Let \((X^{m+1}, g_0), m \geq 3\), be a \(C^{4+\alpha}\) AH manifold with constant scalar curvature \(-m(m+1)\) and let \(x\) be a geodesic defining function. Assume that \(Rc_{g_0} + mg_0 \in x^2C^{2+\alpha}_e(X)\). Then, for some \(T > 0\), there is a family of metrics \(g(t) = g_0 + u(\cdot, t)\) which solves CRF (1.1) such that \(g(t)\) is \(C^{1+\alpha}\) AH with constant scalar curvature \(-m(m+1)\) and \(u \in x^2C^{2+\alpha}e(X_T)\).

We would like to note that one can always conformally deform a given AH metric into an AH metric with constant scalar curvature, say \(-m(m+1)\), thanks to [ACF]. Let \(\wedge\) be the Kulkarni-Nomizu product. The conditions that Riemannian curvature \(Rm + g \wedge g = O(x^2)\), Ricci curvature \(Rc + mg = O(x^2)\), and the boundary \(\partial X\) is totally geodesic in \(X\) under the metric \(x^2g\) for a geodesic defining function \(x\), are all equivalent and preserved under the conformal deformations taken in [ACF] (cf. Lemma 2.1). And the property that \(Rm + g \wedge g = O(x^2)\) on an AH manifold is intrinsic and independent of the choice of the geodesic defining functions.

Ricci flows on complete noncompact manifolds were studied by many people. The most notable early work is [Sh89] by Shi in 1989. Ricci flows on AH manifolds were also studied in [QSW, Ba11], where the existence in [QSW] was based on the maximum principle argument and the existence result in [Sh89], while the existence in [Ba11] is based on asymptotic analysis on AH manifolds in [Mz91, Le06, Al07]. In this paper, we prove Theorem 1.1 using the framework similar to that in [LQZ] for parabolic-elliptic systems based on the analysis on AH manifolds from [Mz91, Le06, Al07, Ba11].

The second result is a local Shi’s type estimate for CRF. In Ricci flow Shi’s estimates on derivatives of Riemannian curvature is crucial for compactness results, and therefore, are essential for the later developments in Ricci flow.

For CRF on smooth manifold \(M^n\) with initial metric \(g_0\) of constant scalar curvature \(R_{g_0} = 2nc\)

\[
\begin{align*}
\partial_t g(t) &= -2(Rc_{g(t)} - 2cg(t)) - 2p(t)g(t) \quad \text{on } M \times (0, T], \\
((n - 1)\Delta_{g(t)} + 2nc)p(t) &= -|Rc_{g(t)} - 2cg(t)|^2_{g(t)} \quad \text{on } M \times [0, T],
\end{align*}
\]

we have

Theorem 1.2. Fix constants \(\alpha \in (0, 1), K \geq 1, \tilde{K} > 0, c, r > 0\), and integer \(n \geq 2\), we have the following.

(i) There exists a constant \(C_1 = C_1(\alpha, n, \sqrt{Kr}, \tilde{K}, |c|)\) depending only on \(\alpha, n, \sqrt{Kr}, \tilde{K}, \) and \(|c|\), such that the following property holds. Let \((M^n, g(t), p(t)), t \in [0, T],\) be a solution to the CRF (1.2). Assume that closed ball \(\bar{B}_{g_0}(O, r) \subset M\)
is compact and that
\[
|\text{Rm}| \leq K \quad \text{on } \bar{B}_{g(0)}(O, r) \times [0, T_*],
\]
(1.4) \[ \max_{i=0,1,2,3} |\nabla^i p| \leq \tilde{K} \quad \text{on } \bar{B}_{g(0)}(O, r) \times [0, T_*], \]
where constant \( T_* \leq \min\{T, \alpha/K\} \), then we have
\[
(1.5) \quad |\nabla \text{Rm}(x, t)|_{g(t)} \leq \frac{C_1 K}{\sqrt{t}}
\]
for all \((x, t) \in B_{g(0)}(O, r/2) \times (0, T_*).\)

(ii) If \((M^n, g(t), p(t)), t \in [0, T], \) in (i) is a complete solution to the CRF. Suppose assumptions (1.3) and (1.4) hold on \( M \times [0, T_*] \), then there is a constant \( C_2 = C_2(\alpha, n, \tilde{K}, |c|) \) such that
\[
(1.6) \quad |\nabla \text{Rm}(x, t)|_{g(t)} \leq \frac{C_2 K}{\sqrt{t}}
\]
for all \((x, t) \in M \times (0, T_*).\)

Here is the outline of the rest of this article. In \( \S 2 \) we discuss basic analysis results on AH manifolds from [Le06, Ba11], which are needed to prove the short time existence. \( \S 3 \) is devoted to the proof of Theorem 1.1 using Banach’s contraction mapping theorem. In \( \S 4 \) we give a proof Theorem 1.2 using the parabolic maximum principle. Also we will outline a proof of high order derivative estimate of the curvature tensor of CRF (see Theorem 4.2).

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2. Preliminaries for AH manifolds

In this section we recall some basic analysis on AH manifolds. We mostly rely on [Mz91, Le06, Al07, Ba11], and readers are referred to them for detailed accounts.

2.1. Notations and function/tensor Hölder spaces. Let \((X^{m+1}, g)\) be a \( C^{l+\beta} \) AH manifold with \( l \geq 2 \) and \( \beta \in (0, 1) \). Let \( x \) be a geodesic defining function such that, in a collar neighborhood \( \{x < \varepsilon\} \) near the infinity, we have \( g = x^{-2}(dx^2 + g_x) \), where \( g_x \) is a family of metrics on \( \partial X \). Let \( X_T = X \times [0, T] \). We will consider the function/tensor Hölder spaces
\[
x^{\mu}C_e^{\kappa+\alpha}(X), \quad x^{\mu}C_e^{\kappa+\alpha, \frac{\lambda}{2}}(X_T)
\]
as described in [Ba1] Section 3.1, for $k + \alpha \leq l + \beta$, which is assumed throughout this paper.

We are concerned with the operators $-\Delta + (m+1)$ on functions and $\Delta_L + 2m$ on 2-tensors on $X$, where the Lichnerowicz operator $\Delta_L = \Delta^g$ acting on 2-tensor $w$ is given by

$$(\Delta^g w)_{ij} = g^{kl} \nabla_k \nabla_l w_{ij} + 2g^{kl} g^{pq} R_{ikpq} w_{lj} - g^{kl} R_{iklj} w_{ij} - g^{kl} R_{jkli} w_{ij}.$$

Throughout this article when we use local coordinates $(x^0, x^1, \cdots, x^m)$ near the boundary of $X$, we choose $x^0 = x$ and $(x^1, \cdots, x^m)$ to be some local coordinates on $\partial X$. Our convention for Riemann curvature tensor components $R_{ijkl}$ is such that $g^{kl}R_{iklj} = R_{ij}$.

Let $Rm$ denote the curvature $(4,0)$-tensor and let $Rc$ denote the Ricci curvature. First by some straightforward calculation we get

**Lemma 2.1.** Suppose that $(X^{m+1}, g)$ is $C^{l+\beta}$ AH and that $x$ is a geodesic defining function. Let $\bar{g} = x^2 g$. Then near the boundary of $X$ we may write Riemannian curvature tensor components as

$$R_{ijkl}(g) = -(g_{ik}g_{jl} - g_{ij}g_{kl}) + x^{-3}T_{ijkl} + x^{-2}R_{ijkl}(\bar{g})$$

for some $(4,0)$-tensor $T$ defined in (2.2) below. It follows that $Rm + g \otimes g \in x^2 C^{l-2+\beta}_e(X)$, $Rc + mg \in x^2 C^{l-2+\beta}_e(X)$, and the condition that boundary $\partial X$ is totally geodesic in $X$ under metric $\bar{g}$, are equivalent. Note that the last condition is independent of the choice of the geodesic defining function $x$.

**Proof.** Note that $Rm + g \otimes g \in x^2 C^{l-2+\beta}_e(X)$ is equivalent to that $T$ vanishes at the boundary $\partial X$, also note that the condition that boundary $\partial X$ is totally geodesic in $X$ under $\bar{g}$ is equivalent to $\partial_x \bar{g}_{ab} |_{x=0} = 0$, where $a, b \in \{1, 2, \cdots, m\}$.

After a long but simple calculation, we get

$$2T_{ijkl} = \bar{g}_{ik} \partial_x \bar{g}_{jl} + \bar{g}_{ij} \partial_x \bar{g}_{kl} - \bar{g}_{ij} \partial_x \bar{g}_{ijkl} - \bar{g}_{jk} \partial_x \bar{g}_{il}.$$

Therefore $T$ vanishes at the boundary $\partial X$ if and only if $\partial_x \bar{g}_{ab} |_{x=0} = 0$.

Taking the trace of (2.1) we have

$$R_{ij}(g) = -mg_{ij} + \frac{1}{2} x^{-1} \left( \bar{g}_{ij} \bar{g}^{kl} \partial_x \bar{g}_{kl} + (m - 1) \partial_x \bar{g}_{ij} \right) - x^2 R_{ij}(\bar{g}).$$

Hence $Rc + mg \in x^2 C^{l-2+\beta}_e(X)$ is equivalent to $\partial_x \bar{g}_{ab} |_{x=0} = 0$. The lemma follows from the equivalences proved above.

Let $(X^{m+1}, g_0)$ be a $C^{l+\beta}$ AH manifold. As a consequence of Lemma 2.1 for time-dependent cases, we have

$$Rm_{g_u(t)} + g_u(t) \otimes g_u(t) \in x^2 C^{k-2+\alpha, \frac{m-4+\alpha}{2}}_e(X_T)$$
Lemma 2.2. ([Le06, Lemma 3.7]) Let \((X^{m+1}, g)\) be a \(C^{l+\beta}\) AH manifold. For 2-tensors, we have continuous embedding

\[
\Delta_{\phi} g_{\alpha}(t) - \Delta_{\phi} \tilde{g}_{\alpha}(t) \|_{x^\alpha C^{k^2+\alpha}(X_T)} \leq C \| u - \tilde{u} \|_{x^\alpha C^{k^2+\alpha}(X_T)}
\]

(2.5) for symmetric 2-tensors \(u, \tilde{u} \in x^\alpha C^{k^2+\alpha}(X_T)\). The other very useful fact for us is the following:

Lemma 2.2. ([Le95, Lemma 3.3]) Let \((X^{m+1}, g)\) be a \(C^{l+\beta}\) AH manifold. For 2-tensors, we have continuous embedding

\[
x^\alpha C^{k^2+\alpha}(X) \hookrightarrow x^{l+2} C^{k+\alpha}(X)
\]

(2.6)

\[
x^{k^2+2} C^{k+\alpha}(X) \hookrightarrow C^{k+\alpha}(X)
\]

(2.7) for \(k + \alpha \leq l + \beta\).

Consequently, we have that \(g_{\alpha}(t)\) is of \(C^{k+\alpha}\) AH if \(g_0\) is \(C^{k+\alpha}\) AH and \(u \in x^\alpha C^{k^2+\alpha}(X_T)\) for \(\mu \geq k + \alpha\).

2.2. Elliptic Schauder estimates on AH manifolds. To treat the pressure function \(p\) when solving equation (1.1), we recall the isomorphism property of \(-\Delta + (m+1)\) on spaces of functions. The following is an immediate consequence of [Le95, Lemma 3.3].

Lemma 2.3. ([Le95, Lemma 3.3]) Let \((X^{m+1}, g)\) be a \(C^{l+\beta}\) AH manifold. Then, for \(k - 1 + \alpha \leq l + \beta\) and \(\mu \in (-1, m+1)\),

\[
-\Delta_{\phi} + (m+1) : x^\mu C^{k^2+\alpha}(X) \to x^\mu C^{k^2-2+\alpha}(X)
\]

is an isomorphism.

For our purpose we need to solve the pressure equation \(p(\cdot, t)\) in (1.1) for each \(t \in [0, T]\), in other words, we need Schauder estimates uniform in the time variable like [LQZ, Lemma 3.11]. To apply Lemma 2.3 to metric \(g_{\alpha}(t) = g_0 + u(\cdot, t)\) at each \(t \in [0, T]\), we need \(g_{\alpha}(t)\) to be at least \(C^{k^2-1+\alpha}\) AH and close to \(g_0\) in some appropriate sense. In fact, as a consequence of Lemma 2.2 and Lemma 2.3, we have

Lemma 2.4. Suppose that \((X^{m+1}, g_0)\) is \(C^{l+\beta}\) AH. Then for \(k - 1 + \alpha \leq l + \beta\) and \(\mu \in (-1, m+1)\), there exist \(\delta > 0\) and \(C > 0\) such that

\[
\| (-\Delta_{\phi} + (m+1))^{-1} \phi \|_{x^\alpha C^{k^2+\alpha}(X_T)} \leq C \| \phi \|_{x^\mu C^{k^2-2+\alpha}(X_T)},
\]

(2.8) provided that

\[
\| u \|_{x^\mu C^{k^2+\alpha}(X_T)} \leq \delta
\]

and \(\nu \geq k - 1 + \alpha\).
For a general initial metric $g_0$ with $C^{l+\beta}$ AH regularity, according to [Ba11], one may expect to work with $u \in xC_e^{2+\alpha,\frac{2}{2+\alpha}}(X_T)$ with $2+\alpha \leq l+\beta$ to prove the short time existence of solutions of form $g_u(t)$ by applying the contraction mapping theorem. This is due to, initially, \( Rc_{g_0} + mg_0 \in xC_e^{l-2+\beta}(X) \) (cf. (2.3)). In the light of (2.7), this only provides $C^{\alpha}$ AH regularity for the metrics $g_u(t)$ after initial time, which is not enough. We need the minimal $C^{1+\alpha}$ AH regularity for $g_u(t)$ in order to apply Lemma 2.4, and Lemma 2.5 and 2.6 below. Therefore one needs, at least for this technical reason, assume that for the initial metric $g_0$

\begin{equation}
(2.9) \quad Rc_{g_0} + mg_0 \in xC_e^{l-2+\beta}(X),
\end{equation}

i.e., $\partial X$ is total geodesic in $(\bar{X}, x^{2\mu}g)$. Later we will choose $k = 2$ and $\nu = 2 > 1 + \alpha$ in applying Lemma 2.4. In the proof of Lemma 3.4, we will need $\|Rc_{g_0} + mg_0\|_{x^2C_e^{2+\alpha}(X)}$ to be bounded, this leads us to choose $l = 4$ and $\beta = \alpha$ in the proof of short time existence.

Based on Lemma 2.3 and Lemma 2.4, we may define

\begin{equation}
(2.10) \quad \mathcal{P}(g) = \frac{1}{m}(-\Delta_g + (m+1))^{-1}(|Rc_g + mg|^2),
\end{equation}

for any $C^{1+\alpha}$ AH metric $g$. Then we can easily derive the following.

**Lemma 2.5.** Let $(X^{m+1}, g_0)$ be a $C^{l+\beta}$ AH manifold. Let $k \in \mathbb{N}, \alpha \in (0, 1)$, and $k + \alpha \leq l + \beta$. Then there are small positive constants $T$ and $\epsilon$ such that the following hold. Let $g_u(t) = g_0 + u(\cdot, t)$ with

\[ u \in x^\mu C_e^{k+\alpha,\frac{2}{2+\alpha}}(X_T) \bigcap x^\nu C_e^{k-1+\alpha,\frac{2}{2+\alpha}}(X_T). \]

(i) For $\|u\|_{x^\nu C_e^{k-1+\alpha,\frac{2}{2+\alpha}}(X_T)} \leq \epsilon$, we have $\mathcal{P}(g_u) \in x^{2\mu}C_e^{k+\alpha,\frac{2}{2+\alpha}}(X_T)$, provided that $2\mu \in (0, 2)$, $\nu > k - 1 + \alpha$; and

(ii) For $u, \tilde{u}$ in some given ball in $x^\mu C_e^{k+\alpha,\frac{2}{2+\alpha}}(X_T)$ which satisfy

\[ \|u\|_{x^\mu C_e^{k-1+\alpha,\frac{2}{2+\alpha}}(X_T)} < \epsilon, \text{ and } \|\tilde{u}\|_{x^\mu C_e^{k-1+\alpha,\frac{2}{2+\alpha}}(X_T)} < \epsilon, \]

we have

\[ \|\mathcal{P}(g_u) - \mathcal{P}(g_{\tilde{u}})\|_{x^\mu C_e^{k+\alpha,\frac{2}{2+\alpha}}(X_T)} \leq C\|u - \tilde{u}\|_{x^\mu C_e^{k+\alpha,\frac{2}{2+\alpha}}(X_T)}, \]

provided that $\mu \in (0, m+1)$, $\nu > k - 1 + \alpha$.

We will use Lemma 2.5 with $\nu = 2$ and $k = 2$ below.
2.3. Parabolic Schauder estimates on AH manifolds. We will need the following basic parabolic Schauder estimate [Ba11, Theorem 3.2], which covers the case when $L$ is $\Delta_L + 2m$ on 2-tensors.

**Lemma 2.6.** ([Ba11, Theorem 3.2]) Suppose that $(X^{m+1}, g_0)$ is $C^{l+\beta}$ AH. Suppose $L$ is a second-order linear uniformly degenerate elliptic operator with time-independent coefficients. Let $k+\alpha \leq l+\beta$. Then for every $f \in \mathcal{C}^{k+\alpha, \frac{k+2+\alpha}{2}}(X_T)$ there is a solution $v$ in $\mathcal{C}^{k+2+\alpha, \frac{k+2+\alpha}{2}}(X_T)$ to equation

$$ (\partial_t - L)v(x,t) = f(x,t) \quad \text{and} \quad v(x,0) = 0. $$

Moreover, $v$ satisfies the parabolic Schauder estimate

$$ \|v\|_{\mathcal{C}^{k+2+\alpha, \frac{k+2+\alpha}{2}}(X_T)} \leq C\|f\|_{\mathcal{C}^{k+\alpha, \frac{k+\alpha}{2}}(X_T)}, $$

where constant $C = C(T)$ is bounded when $T$ is small.

2.4. AH metrics of constant scalar curvature. In this subsection we recall the existence of a unique conformal deformation on a given AH manifold to make the scalar curvature constant due to [ACF, Theorem 1.2 and 1.3]. This is significant because such a conformal deformation does not alter the conformal infinity of the AH manifold.

**Lemma 2.7.** Suppose that $(X^{m+1}, g)$ is smooth AH. Then there exists a unique conformal deformation $w \frac{4}{m-1} g$ such that

- $w$ is positive and in $C^{l+\beta}(\overline{X})$ for $l + \beta < m + 1$;
- metric $w \frac{4}{m-1} g$ is $C^{l+\beta}$ AH with constant scalar curvature $-m(m+1)$;
- $w(p) \to 1$ when $p$ approaches $\partial X$.

In the light of Lemma 2.1 we have

**Lemma 2.8.** Let $(X^{m+1}, g)$ be a smooth AH manifold. Suppose that $\text{Rc}_g + mg \in x^2 C^{l-2+\beta}_e(X)$ with $l + \beta < m + 1$. Then the conformal metric $w \frac{4}{m-1} g$ given in Lemma 2.7 is $C^{l+\beta}$ AH, and its traceless Ricci curvature is in $x^2 C^{l-2+\beta}_e(X)$.

3. Short time existence for CFR on AH manifolds

Based on our discussion of (2.9) and the discussion of [ACF] in §2.4, from now on, we assume that the initial metric $g_0$ is $C^{4+\alpha}$ AH with constant scalar curvature $-m(m+1)$ and $\text{Rc}_{g_0} + mg_0 \in x^2 C^{2+\alpha}_e(X)$.

In this section we give a proof of Theorem 1.1 in three steps. We will first introduce and solve for a short time DCRF (short for DeTurck conformal Ricci flow). Then we convert the solution of DCRF to a solution of CRF.
3.1. DeTurk CRF and linearization. Now we prepare for the application of the contraction mapping theorem to prove the short time existence of DCRF. Let $h_0 = g_0$. The DCRF corresponding to CRF (1.1) on a $C^{4+\alpha}$ AH manifold $(X^{m+1}, h_0)$ is

$$
\begin{aligned}
\partial_t h(t) &= -2 \left( Rc_{h(t)} + mh(t) \right) + \mathcal{L}_{W(t)} h(t) - 2\pi(t) h(t) \quad \text{on } X_T, \\
(\Delta h(t) - (m+1))\pi(t) &= -\frac{1}{m} \left| Rc_{h(t)} + mh(t) \right|_h^2 \quad \text{on } X_T, \\
h(0) &= h_0 
\end{aligned}
$$

(3.1)

where vector field $W(t)$ is defined by

$$W^k(t) := h_0^{ij} \left( \Gamma^k_{ij}(h(t)) - \Gamma^k_{ij}(h_0) \right),$$

$\mathcal{L}_{W(t)}$ is the Lie derivative, and $\Gamma^k_{ij}$ is the Christoffel symbol of the corresponding metric.

For simplicity, in this section we use $\nabla$, $R_m$, and $Rc$ to denote the Levi-Civita connection, Riemann curvature, and Ricci curvature of the metric $h_0$ respectively. From [Ba11, (4.2)] and [Ba11, Lemma 4.1, 4.2, 4.3] we have the following decomposition. Let $h(t) = h_0 + u(\cdot,t)$ and let operator $L = \Delta^h_{L_0} + 2m$, where $\Delta^h_{L_0}$ is the Lichnerowicz operator of $h_0$. We rewrite

$$
-2 \left( Rc_h + mh \right) + \mathcal{L}_W h = Lu + \mathcal{Q}(u) + \mathcal{E},
$$

(3.3)

where operator

$$(\mathcal{Q}(u))_{ij} = ((h_0 + u)^{kl} - h_0^{kl}) \nabla_k \nabla_i u_{ij} + (h_0 + u)^{kl}(h_0 + u)_{ip} (h_0 + u)^{pq} R_{ijklq}$$

$$+ (h_0 + u)^{kl} (h_0 + u)_{jp} (h_0)^{pq} R_{ijklq} - 2R_{ij} - 2R_{ijkl} u_{kl}$$

$$- R_{ik} u_{kj} - R_{jk} u_{ki} + (h_0 + u)^{-1} (h_0 + u)^{-1} \nabla u \ast \nabla u,$$

$$\mathcal{E} = 2(Rc_{h_0} + mh_0) \in x^2 C^{2+\alpha}_e(X).$$

Note that $\mathcal{E}$ only depends of the initial metric $h_0$. The following is from [Ba11 (4.4) and (4.5)].

**Lemma 3.1.** We have

$$\mathcal{Q}(\cdot) : x^2 C^{2+\alpha,\frac{\alpha}{2}}(X_T) \to x^2 C^{\alpha,\frac{\alpha}{2}}(X_T).$$

Moreover

$$\|\mathcal{Q}(u)\|_{x^2 C^{\alpha,\frac{\alpha}{2}}(X_T)} \leq C \|u\|^2_{x^2 C^{2+\alpha,\frac{\alpha}{2}}(X_T)}, \quad \text{and}$$

$$\|\mathcal{Q}(u) - \mathcal{Q}(\tilde{u})\|_{x^2 C^{\alpha,\frac{\alpha}{2}}(X_T)} \leq C \max\{\|u\|_{x^2 C^{2+\alpha,\alpha/2}(X_T)}, \|\tilde{u}\|_{x^2 C^{2+\alpha,\alpha/2}(X_T)}\} \cdot \|u - \tilde{u}\|_{x^2 C^{2+\alpha,\alpha/2}(X_T)}. $$
Next we want to introduce the analogous decomposition for the pressure function \( \pi(t) \) in (3.1). Let \( \mathcal{P} \) be the operator defined in (2.10) and let
\[
(3.4) \quad -2\pi(t)h(t) = -2\mathcal{P}(h_0 + u) \cdot (h_0 + u) = -2\mathcal{P}(h_0)u + \mathcal{Q}(u) + \mathcal{M}(u) + \mathcal{E},
\]
where
\[
\mathcal{E} = -2\mathcal{P}(h_0)h_0,
\]
\[
\mathcal{M}(u) = -2(\mathcal{P}(h_0 + u) - \mathcal{P}(h_0))h_0,
\]
\[
\mathcal{Q}(u) = -2(\mathcal{P}(h_0 + u) - \mathcal{P}(h_0))u.
\]

Note that \( \mathcal{E} \) depends only on the initial metric \( h_0 \). Based on Lemma 2.3 we have the following property of \( \mathcal{E} \) which is analogous to that of \( \mathcal{E} \).

**Lemma 3.3.** Suppose that \( \text{Rc}_{h_0} + mh_0 \in x^2 C^{2+\alpha}_e(X) \) and there is a constant \( C > 0 \) such that
\[
\|\mathcal{E}\|_{x^2 C^{2+\alpha}_e(X)} \leq C \|\text{Rc}_{h_0} + mh_0\|_{x^2 C^{2+\alpha}_e(X)}^2.
\]

Based on Lemma 2.3, we have the following properties of \( \mathcal{M}(u) \), and properties of \( \mathcal{Q}(u) \) which are analogous to that of \( \mathcal{Q}(u) \).

**Lemma 3.4.** Suppose that \( \text{Rc}_{h_0} + mh_0 \in x^2 C^{2+\alpha}_e(X) \) and that \( u \) and \( \tilde{u} \) are in some small balls centered at \( 0 \) in \( x^2 C^{2+\alpha}_e(X_T) \). Then we have
(i) \( \mathcal{M}(u) \in x^2 C^{2+\alpha}_e(\tilde{u}) \) and there is a constant \( C_1 > 0 \) such that
\[
\|\mathcal{M}(u)\|_{x^2 C^{2+\alpha}_e(X_T)} \leq C_1 \|u\|_{x^2 C^{2+\alpha}_e(X_T)}\]
(ii) \( \mathcal{Q}(u) \in x^2 C^{2+\alpha}_e(\tilde{u}) \) and there is a constant \( C_2 > 0 \) such that
\[
\|\mathcal{Q}(u)\|_{x^2 C^{2+\alpha}_e(X_T)} \leq C_2 \|u\|_{x^2 C^{2+\alpha}_e(X_T)}^2\]
(iii) There is a constant \( C_3 \) such that
\[
\|\mathcal{Q}(u) - \mathcal{Q}(\tilde{u})\|_{x^2 C^{2+\alpha}_e(X_T)} \leq C_3 \max\{\|u\|_{x^2 C^{2+\alpha}_e(X_T)}, \|\tilde{u}\|_{x^2 C^{2+\alpha}_e(X_T)}\}
\cdot \|u - \tilde{u}\|_{x^2 C^{2+\alpha}_e(X_T)}\]

3.2. **Contraction mapping theorem for DCRF.** In this subsection we will first prove the short time existence of solutions for DCRF that are \( C^{1+\alpha} \) AH. The Banach space we consider for the contraction mapping theorem is \( x^2 C^{2+\alpha}_e(X_T) \). For two positive parameters \( \epsilon \) and \( T \) to be specified below, we define a closed subset \( Z_{\epsilon,T} \) as
\[
Z_{\epsilon,T} = \{u \in x^2 C^{2+\alpha}_e(X_T): u(x,0) = 0 \text{ and } \|u\|_{x^2 C^{2+\alpha}_e(X_T)} \leq \epsilon\}. 
\]
Let \( h(t) = h_0 + v(\cdot, t) \) in (3.1). In the light of (3.3) and (3.4), we rewrite the DCRF as
\[
\begin{cases}
(\partial_t - \hat{L})v = \mathcal{E} + Q(v) + \hat{E} + \hat{Q}(v) + \hat{M}(v) \\
v(\cdot, 0) = 0,
\end{cases}
\]
where \( \hat{L} = \Delta_h^0 + 2m - 2\mathcal{P}(h_0) \). Since \( \mathcal{P}(h_0) \in x^4C_4^{4+\alpha}(X) \subset xC^{2+\alpha}(\bar{X}) \), \( \hat{L} \) is uniformly degenerate elliptic. We may apply Lemma 2.6 to \( \partial_t - \hat{L} \).

Let us define the mapping as follows: given \( u \in Z_{\epsilon,T} \), from the discussions in the previous subsection, we know
\[
Q(u) + \hat{Q}(u) + \mathcal{E} + \hat{E} + \hat{M}(u) \in x^2C_\epsilon^{2+\alpha}(\tilde{X}_T).
\]
We may solve the linear equations by Lemma 2.6
\[
(3.5) \quad \begin{cases}
(\partial_t - \hat{L})v = Q(u) + \hat{Q}(u) + \mathcal{E} + \hat{E} + \hat{M}(u), \\
v(\cdot, 0) = 0.
\end{cases}
\]
By the parabolic Schauder estimates in Lemma 2.6, we may define a map
\[
\Psi : Z_{\epsilon,T} \subseteq x^2C_\epsilon^{2+\alpha}(\tilde{X}_T) \to x^2C_\epsilon^{2+\alpha}(\tilde{X}_T), \quad \Psi(u) = v_u = v.
\]
For appropriate choices of \( \epsilon \) and \( T \), we claim that the map
\[
(3.6) \quad \Psi : Z_{\epsilon,T} \to Z_{\epsilon,T},
\]
and that \( \Psi \) is contractive. In the following lemmas, we verify the above claim and then apply the contraction mapping theorem to prove the short time existence for DCRF (3.1).

**Lemma 3.4.** The mapping \( \Psi \) maps \( Z_{\epsilon,T} \) into \( Z_{\epsilon,T} \) when \( \epsilon \) and \( T \) are small enough.

**Proof.** We may write \( v = v_u = v_{u1} + v_{u2} + v_{u3} = v_1 + v_2 + v_3 \), where
\[
\begin{cases}
(\partial_t - \hat{L})v_1 = Q(u) + \hat{Q}(u), \\
v_1(\cdot, 0) = 0,
\end{cases}
\]

\[
\begin{cases}
(\partial_t - \hat{L})v_2 = \mathcal{E} + \hat{E}, \\
v_2(\cdot, 0) = 0,
\end{cases}
\]

and
\[
\begin{cases}
(\partial_t - \hat{L})v_3 = \hat{M}(u), \\
v_3(\cdot, 0) = 0.
\end{cases}
\]
Due to the quadratic nature of the operators \( \mathcal{Q} \) and \( \tilde{\mathcal{Q}} \), as in the proof of [Ba11, Lemma 4.5], we have
\[
\|v_1\|_{x^2C^{2+\alpha,\frac{4+\alpha}{2}}(X_T)} \leq \frac{1}{3} \epsilon,
\]
where \( T \) is arbitrarily given and small and \( \epsilon \) is small enough.

For \( v_2 \), by Lemma 3.2 and 2.6 we have
\[
\|v_2\|_{x^{2+\alpha,\frac{4+\alpha}{2}}(X_T)} \leq C\|Rc_0 + mh_0\|_{x^2C^{2+\alpha}(X)},
\]
this implies that
\[
(3.7) \quad \|\hat{L}v_2\|_{x^{2+\alpha,\frac{4+\alpha}{2}}(X_T)} \leq C\|Rc_0 + mh_0\|_{x^2C^{2+\alpha}(X)}.
\]
As in the proof of [Ba11, Lemma 4.5], we may write
\[
v_2(\cdot, t) = \int_0^t (\mathcal{E} + \hat{\mathcal{E}} + \hat{L}v_2)ds,
\]
by (3.7) and Lemma 3.2 we get
\[
\|v_2\|_{x^2C^{2+\alpha,\frac{4+\alpha}{2}}(X_T)} \leq C T^{1-\frac{\alpha}{2}} \|Rc_0 + mh_0\|_{x^2C^{2+\alpha}(X)} \leq \frac{1}{3} \epsilon
\]
when \( T \) is chosen sufficiently small.

For \( v_3 \), the argument is similar to that of \( v_2 \). By Lemma 3.3(i) and 2.6, we have
\[
\|v_3\|_{x^{2+\alpha,\frac{4+\alpha}{2}}(X_T)} \leq C\|u\|_{x^2C^{2+\alpha,\frac{4+\alpha}{2}}(X_T)},
\]
hence
\[
(3.8) \quad \|\hat{L}v_3\|_{x^{2+\alpha,\frac{4+\alpha}{2}}(X_T)} \leq C\|u\|_{x^2C^{2+\alpha,\frac{4+\alpha}{2}}(X_T)}.
\]
We write
\[
v_3(\cdot, t) = \int_0^t (\hat{\mathcal{M}}(u) + \hat{L}v_3)ds,
\]
and get
\[
\|v_3\|_{x^{2+\alpha,\frac{4+\alpha}{2}}(X_T)} \leq C T^{1-\frac{\alpha}{2}} \|u\|_{x^2C^{2+\alpha,\frac{4+\alpha}{2}}(X_T)} \leq \frac{1}{3} \epsilon
\]
when \( T \) is chosen sufficiently small. Thus the proof is complete.

**Lemma 3.5.** The map \( \Psi : Z_{\epsilon,T} \to Z_{\epsilon,T} \) is contractive when \( \epsilon \) and \( T \) are small.

**Proof.** Adopt the notations used in the proof of Lemma 3.4, for \( u, \tilde{u} \in Z_{\epsilon,T} \) it is easy to see that \( v_{u_2} - v_{\tilde{u}_2} = 0 \).

\( v_{u_1} - v_{\tilde{u}_1} \) satisfies
\[
\begin{cases}
(\partial_t - \hat{L})(v_{u_1} - v_{\tilde{u}_1}) = \mathcal{Q}(u) - \mathcal{Q}(\tilde{u}) + \hat{\mathcal{Q}}(u) - \hat{\mathcal{Q}}(\tilde{u}), \\
(v_{u_1} - v_{\tilde{u}_1})(\cdot, 0) = 0.
\end{cases}
\]
As in the proof of [6, Lemma 4.6] and in the light of Lemma 3.1 and 3.3(iii), we have
\[
\|v_{u_1} - v_{\tilde{u}_1}\|_{x^2C^{2+\alpha}e(\mathbf{X}_T)} \leq \frac{1}{3}\|u - \tilde{u}\|_{x^2C^{2+\alpha}e(\mathbf{X}_T)} ,
\]
Note that \(v_{u_3} - v_{\tilde{u}_3}\) satisfies
\[
\begin{cases}
(\partial_t - \tilde{L})(v_{u_3} - v_{\tilde{u}_3}) = \dot{M}(u) - \dot{M}(\tilde{u}), \\
(v_{u_3} - v_{\tilde{u}_3})(*, 0) = 0.
\end{cases}
\]
Since
\[
\dot{M}(u) - \dot{M}(\tilde{u}) = -2(\mathcal{P}(h_0 + u) - \mathcal{P}(h_0 + \tilde{u}))h_0,
\]
using the estimate in Lemma 2.5(ii) we can adopt the argument for the estimate of \(v_3\) in the proof of Lemma 3.4 to get
\[
\|v_{u_3} - v_{\tilde{u}_3}\|_{x^2C^{2+\alpha}e(\mathbf{X}_T)} \leq CT^{1-\frac{\alpha}{2}}\|u - \tilde{u}\|_{x^2C^{2+\alpha}e(\mathbf{X}_T)} \leq \frac{1}{3}\|u - \tilde{u}\|_{x^2C^{2+\alpha}e(\mathbf{X}_T)},
\]
when \(T\) is chosen sufficiently small.

It follows that map \(\Psi\) is contractive when \(\epsilon\) and \(T\) are small enough. \(\Box\)

Now let us summarize and state a short time existence theorem for DCRF.

**Theorem 3.6.** Suppose that \((X^{m+1}, h_0)\) is \(C^{4+\alpha}\) AH with constant scalar curvature \(-m(m+1)\) and that \(x\) is a geodesic defining function. Assume that \(R_{ch_0} + mh_0 \in x^2C^{2+\alpha}(X)\). Then, for some small \(T\), DCRF (3.1) has a solution \(h(t) = h_0 + v(\cdot, t)\) such that \(h(t)\) is \(C^{1+\alpha}\) AH with constant scalar curvature \(-m(m+1)\) and \(v \in x^2C^{2+\alpha, \frac{2+\alpha}{2}}(\mathbf{X}_T)\).

**Proof.** The only thing we need to point out is that, for a fixed point \(v = u \in x^2C^{2+\alpha, \frac{2+\alpha}{2}}(\mathbf{X}_T)\), it is automatic that \(v \in x^2C^{2+\alpha, \frac{2+\alpha}{2}}(\mathbf{X}_T)\) due to the equation (3.5). \(\Box\)

3.3. **Short time existence for CRF on AH manifolds.** To construct CRF from DCRF, we considers the family of diffeomorphisms \(\varphi(t)\) generated by the vector field \(W(t)\):
\[
\frac{d}{dt}\varphi(t) = W(t) \quad \text{and} \quad \varphi(0) = \text{id},
\]
where \(W(t)\) is defined in (3.2). Note that \(W \in xC^{\alpha}(\mathbf{X}_T) \cap x^2C^{1+\alpha, \frac{1+\alpha}{2}}(\mathbf{X}_T)\). Let \(h(t)\) be the solution of DCRF from Theorem 3.6. Then \(g(t) = (\varphi(t))^*h(t) = g_0 + u(\cdot, t)\) solves the CRF for short time and \(u \in x^2C^{2+\alpha, \frac{2+\alpha}{2}}(\mathbf{X}_T)\) (cf. [LQZ], Lemma 3.1, for instance). Moreover, \(g(t)\) is \(C^{1+\alpha}\) AH with constant scalar curvature \(-m(m+1)\).
4. Shi’s curvature derivative estimates for CRF

In this section we will give a proof of Theorem 1.2 and Shi’s estimates of high order derivative of curvature tensor for CRF (Theorem 4.2). The following lemma will be used in the proof which is the CRF analog of Lemma 14.3 in [CC2] for Ricci flow. The lemma can be proved by a straightforward modification of the Lemma 14.3 where the comparison of Christoffel symbol $\Gamma^k_{ij}(g(t))$ with $\Gamma^k_{ij}(g(0))$ requires the bound of $|\nabla p|$.

Lemma 4.1. Let $(M^n, g(t), p(t)), t \in [0, T]$, be a solution to CRF (1.2) with $R_{g(0)} = 2nc$. Assume that closed ball $\overline{B}_{g(0)}(O, r) \subset M$ is compact and that

$$|Rm| \leq K, \quad \max_{i=0,1} |\nabla' p| \leq \tilde{K} \quad \text{on} \quad B_{g(0)}(O, r) \times \{0, T_*\},$$

where $T_* \leq \min\{T, \alpha/K\}$ for some positive constants $\alpha, K, \tilde{K}$. Note that $|c| \leq n(n-1)K$. Let

$$\Theta(x, t) = tK^2 |\nabla Rc(x, t)|^2.$$

Then there exist constants $C_1 = C_1\left(\alpha, n, \tilde{K}, |c|\right), C_2 = C_2\left(\alpha, n, \sqrt{K}r, \tilde{K}, |c|\right)$, and a cutoff function $\eta : M \to [0, 1]$ with support in $B_{g(0)}(O, r)$ such that for $(x, t) \in B_{g(0)}(O, r) \times [0, T_*]$ we have

$$\eta = 1 \quad \text{on} \quad B_{g(0)}(O, r/2)$$

$$|\nabla \eta(x)|_{g(t)}^2 \leq \frac{C_1}{r^2} \eta(x), \quad (4.1)$$

$$- \Delta_{g(t)} \eta(x) \leq \frac{C_2}{r^2} + \frac{C_1}{K^{3/2} r} \sup_{s \in [0, t]} (\eta \Theta)^{1/2}(x, s). \quad (4.2)$$

Proof of Theorem 1.2. We will use the formula of $\partial_t Rm$ in [LQZ, p.417] to compute

$$\partial_t |Rm|^2 = \frac{\partial}{\partial t} \left( g^{ri} g^{sj} g^{pk} g^{ql} R_{rspb} R_{ijkl} \right).$$

Note that the terms, which contain factor $Rc - 2cg$ and arise when we differentiate $g^{-1}$, cancel the corresponding terms which arise when we differentiate $R_{ijkl}$. We get the evolution equation for the norm of the curvature (compare [CK, p.225])

$$(\partial_t - \Delta) |Rm|^2 \leq -2 |\nabla Rm|^2 + 16 |Rm|^3 + 4(|p| + 2|c|) |Rm|^2$$

$$+ 8 |Rm| |\nabla^2 p|.

Actually $|Rm| |\nabla^2 p|$ term can be written as $|Rc| |\nabla^2 p|$.
It follows by a standard computation using the formula of $\partial_t \text{Rm}$ in [LQZ, p.417] that the covariant derivative $\nabla \text{Rm}$ satisfies (compare [CK, p.227])

$$(\partial_t - \Delta) \nabla \text{Rm} = 40 \text{Rm} \ast \nabla \text{Rm} + 8(\text{Rc} - 2c) \ast \nabla \text{Rm}$$

$$-2(p - 2c) \nabla \text{Rm} + 5\nabla p \ast \text{Rm} + 4g \ast \nabla^3 p.$$ 

Here if $A$ and $B$ are tensors, $A \ast B$ means some contraction of the tensor product $A \otimes B$. If $k$ are natural number, then $kA \ast B$ denotes a tensor consisting of $k$ terms of $A \ast B$. Then it follows that

$$(4.4) \quad (\partial_t - \Delta) |\nabla \text{Rm}|^2$$

$$= -2 |\nabla^2 \text{Rm}|^2 + 90 \text{Rm} \ast \nabla \text{Rm} \ast \nabla \text{Rm} + 16(\text{Rc} - 2c) \ast \nabla \text{Rm} \ast \nabla \text{Rm}$$

$$+ 14(p - 2c) \nabla \text{Rm} \ast \nabla \text{Rm} + 10\nabla p \ast \text{Rm} \ast \nabla \text{Rm} + 8\nabla \text{Rm} \ast \nabla^3 p.$$ 

Actually term $\nabla \text{Rm} \ast \nabla^3 p$ can be written as $\nabla \text{Rc} \ast \nabla^3 p.$

Below $C(n)$ is a constant depending only on dimension $n$. Using (4.3) and (4.4) we compute

$$(\partial_t - \Delta) \left( (16K^2 + |\text{Rm}|^2) |\nabla \text{Rm}|^2 \right)$$

$$= (\partial_t - \Delta) |\text{Rm}|^2 \cdot |\nabla \text{Rm}|^2 + (16K^2 + |\text{Rm}|^2) \cdot \left( \frac{\partial}{\partial t} - \Delta \right) |\nabla \text{Rm}|^2$$

$$- 2\nabla |\text{Rm}|^2 \cdot \nabla |\nabla \text{Rm}|^2$$

$$\leq -2 |\nabla \text{Rm}|^4 + 16 |\text{Rm}|^3 |\nabla \text{Rm}|^2 - 2 \left( 16K^2 + |\text{Rm}|^2 \right) |\nabla^2 \text{Rm}|^2$$

$$+ C(n) (16K^2 + |\text{Rm}|^2) |\text{Rm}| |\nabla \text{Rm}|^2 + 8 |\text{Rm}| |\nabla \text{Rm}|^2 |\nabla^2 \text{Rm}|^2$$

$$+ C(n) |\nabla p| (16K^2 + |\text{Rm}|^2) |\text{Rm}| |\nabla \text{Rm}| + C(n) |\nabla^2 p| |\text{Rm}| |\nabla \text{Rm}|^2$$

$$+ C(n) |\nabla^3 p| (16K^2 + |\text{Rm}|^2) |\nabla \text{Rm}|.$$ 

Using the assumption $|\text{Rm}| \leq K$ with $K \geq 1$ and $\max_{i=0,1,2,3} |\nabla^i p| \leq \tilde{K}$ on $\tilde{B}_{g(0)} (O, r) \times [0, T_\ast]$ we get that on $\tilde{B}_{g(0)} (O, r) \times [0, T_\ast]$

$$(\partial_t - \Delta) \left( (16K^2 + |\text{Rm}|^2) |\nabla \text{Rm}|^2 \right)$$

$$\leq -2 |\nabla \text{Rm}|^4 + C(n, \tilde{K}, |c|)K^3 |\nabla \text{Rm}|^2 - 32K^2 |\nabla^2 \text{Rm}|^2 + C(n, \tilde{K})K^6$$

$$+ 8K |\nabla \text{Rm}|^2 |\nabla^2 \text{Rm}|.$$
Since
\[ -\frac{1}{2} |\nabla Rm|^4 + 8K |\nabla Rm|^2 |\nabla^2 Rm|^2 - 32K^2 |\nabla^2 Rm|^2 \leq 0, \]
\[ -\frac{1}{2} |\nabla Rm|^4 + C(n, \tilde{K}, |c|)K^3 |\nabla Rm|^2 \leq \tilde{C}(n, \tilde{K}, |c|)K^6, \]
we have established
\[
(4.5) \quad (\partial_t - \Delta) \left( (16K^2 + |Rm|^2) |\nabla Rm|^2 \right) \leq - |\nabla Rm|^4 + C(n, \tilde{K}, |c|)K^6.
\]

Inequality (4.5) is of the same form as the inequality on the top of [CC2, p.238]. The remaining proof of estimate (1.5) can be finished by the same argument as the proof given in [CC2, pp.238–239] for the Ricci flow. Roughly speaking this is done in two steps. First we localize the inequality (4.5) by multiplying \((16K^2 + |Rm|^2) |\nabla Rm|^2\) by \(t\eta\) where \(\eta\) is the cutoff function given in Lemma 4.1. Then we apply the parabolic maximum principle to the resulting inequality of the localized quantity. We omit the details. \(\square\)

Now we turn to Shi’s high order derivative estimates. First we compute the evolution equation of high derivatives of the curvature tensor, here we follow closely the calculation for Ricci flow (see [CK, p.228], for example).

\[
\partial_t \nabla^k Rm = \nabla^k (\partial_t Rm) + \sum_{j=0}^{k-1} \nabla^j \left( \nabla (Rc + (p - 2c)) \ast \nabla^{k-j-1} Rm \right)
\]
\[= \nabla^k (\Delta Rm + Rm \ast Rm + (Rc - 2cg) \ast Rm - 2(p - 2c) Rm + g \ast \nabla \nabla p)
\]
\[+ \sum_{j=0}^{k-1} \nabla^j (\nabla Rc \ast \nabla^{k-j-1} Rm) + \sum_{j=0}^{k-1} \nabla^j (\nabla p \ast \nabla^{k-j-1} Rm)
\]
\[= \nabla^k \Delta Rm + \sum_{j=0}^{k} \nabla^j Rm \ast \nabla^{k-j} Rm + cg \ast \nabla^k Rm
\]
\[+ \sum_{j=0}^{k} \nabla^j p \ast \nabla^{k-j} Rm + g \ast \nabla^{k+2} p.
\]

Since for any tensor \(A\) we have that the commutator
\[
[\nabla^k, \Delta] A = \sum_{j=0}^{k} \nabla^j Rm \ast \nabla^{k-j} A,
\]
we conclude
(4.6) \((\partial_t - \Delta) \nabla^k Rm\)

\[
= \sum_{j=0}^{k} \nabla^j Rm \ast \nabla^{k-j} Rm + cg \ast \nabla^k Rm + \sum_{j=0}^{k} \nabla^j p \ast \nabla^{k-j} Rm + g \ast \nabla^{k+2} p.
\]

Note that
\[
\partial_t \left| \nabla^k Rm \right|^2 = 2 \langle \partial_t (\nabla^k Rm), \nabla^k Rm \rangle + (Rc + (p - 2c)) \ast \nabla^k Rm \ast \nabla^k Rm.
\]

Using the following equality for tensors
\[
2 \langle \Delta A, A \rangle = \Delta |A|^2 - 2 |\nabla A|^2,
\]
we get
\[
(\partial_t - \Delta) \left| \nabla^k Rm \right|^2 = -2 \left| \nabla^{k+1} Rm \right|^2 + \sum_{j=0}^{k} \nabla^j Rm \ast \nabla^{k-j} Rm \ast \nabla^k Rm + c \nabla^k Rm \ast \nabla^k Rm
\]
\[+ \sum_{j=0}^{k} \nabla^j p \ast \nabla^{k-j} Rm \ast \nabla^k Rm + \nabla^{k+2} p \ast \nabla^k Rm.\]

Using the above evolution equation we can prove the following local estimate of the high order derivative of curvature tensors for CRF by applying maximum principle to the evolution inequality of the localized quantity
\[
\eta \left( C + t^n |\nabla^n Rm|^2 \right) t^{n+1} |\nabla^{n+1} Rm|^2.
\]
We omit the detail of the proof.

**Theorem 4.2.** (i) There exists a constant \(C_1 = C_1(\alpha, n, m, r, K, \tilde{K})\) depending only on \(\alpha, n, m, r, K,\) and \(\tilde{K},\) such that the following property holds. Let \((M^n, g(t), p(t)), t \in [0, T],\) be a solution to CRF \((1.2)\) with \(R_g(0) = 2nc.\) Assume that closed ball \(\bar{B}_{g(0)}(O, r) \subset M\) is compact and that

\[
|Rm| \leq K \text{ on } \bar{B}_{g(0)}(O, r) \times [0, T_*]
\]

\[
\max_{i=0,1,\ldots,m+2} |\nabla^i p| \leq \tilde{K} \text{ on } \bar{B}_{g(0)}(O, r) \times [0, T_*]
\]

where \(T_* \leq \min\{T, \alpha/K\},\) then we have

\[
|\nabla^m Rm(x, t)| \leq \frac{C_1}{t^{m/2}}
\]
for all \((x, t) \in B_{g(0)}(O, r/2) \times (0, T_*).\)
(ii) If \((M^n, g(t), p(t)), t \in [0, T]\), in (i) is a complete solution to CRF. Suppose assumption (4.7) and (4.8) holds on \(M \times [0, T_\ast]\), then there is a constant \(C_2 = C_2(\alpha, n, m, K, |c|)\) depending only on \(\alpha, n, m, \tilde{K}\), and \(|c|\), such that

\[
|\nabla^m Rm(x, t)| \leq \frac{C_2 K}{t^{m/2}}
\]

for all \((x, t) \in M \times (0, T_\ast]\).

**Remark 4.3.** (i) In Theorem 4.2(i) if we further assume

\[
|\nabla^k Rm(x, 0)| \leq K
\]

for \(k = 1, \cdots, l\) and \(x \in B_{g(0)}(O, r)\), then we have

\[
|\nabla^m Rm(x, t)| \leq \frac{C_3}{t^{(m-l)/2}}
\]

for all \((x, t) \in B_{g(0)}(O, r/2) \times (0, T_\ast]\). Here \(C_3 = C_3(\alpha, n, m, r, K, \tilde{K})\) is a constant. A similar generalization of Theorem 4.2(ii) also holds. This is the analog of the so-called modified Shi’s local derivative estimates in Ricci flow ([CC2, Theorem 14.16]).

(ii) Using Theorem 4.2 we can prove a compactness theorem for CRF. Let \(\{(M^n_k, g_k(t), p_k(t), O_k)\}, t \in (-\alpha, \beta)\) with \(\alpha, \beta > 0\), be a sequence of pointed complete solutions of CRF with constant scalar curvature. Assume that for some constants \(m \in \mathbb{N}, K, \tilde{K}\)

\[
(4.11)
|Rm_{g_k}| \leq K \quad \text{on } M_k \times (-\alpha, \beta),
\]

\[
(4.12)
\max_{i=0,1,\cdots,m+2} |\nabla^i p| \leq \tilde{K} \quad \text{on } M_k \times (-\alpha, \beta).
\]

Further assume that the injectivity radius \(\text{inj}_{g_k(0)}(O_k) \geq \delta\) for some \(\delta > 0\). With the aid of Theorem 4.2(ii) we may apply the Cheeger–Gromov compactness theorem to the sequence and conclude the following. There exists a subsequence of \(\{(M_k, g_k(t), p_k(t), O_k)\}, t \in (-\alpha, \beta)\), which converges in the pointed \(C^{m+1}\)-Cheeger–Gromov topology to a pointed complete solution of CRF \((M_\infty^n, g_\infty(t), p_\infty(t), O_\infty)\), \(t \in (-\alpha, \beta)\).

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