Scaling determination of the nonlinear \emph{I-V} characteristics for 2D superconducting networks

Petter Minnhagen,¹,² Beom Jun Kim,³ and Andreas Grönlund²

¹NORDITA, Blegdamsvej 17, Dk 2100, Copenhagen, Denmark
²Department of Theoretical Physics, Umeå University, 901 87 Umeå, Sweden
³Department of Molecular Science and Technology, Ajou University, Suwon 442-749, Korea

It is shown from computer simulations that the current-voltage (\emph{I-V}) characteristics for the two-dimensional \emph{XY} model with resistively-shunted Josephson junction dynamics and Monte Carlo dynamics obeys a finite-size scaling form from which the nonlinear \emph{I-V} exponent \emph{a} can be determined to good precision. This determination supports the conclusion \emph{a} = \emph{z} + 1, where \emph{z} is the dynamic critical exponent. The results are discussed in light of the contrary conclusion reached by Tang and Chen [Phys. Rev. B 67, 024508 (2003)] and the possibility of a breakdown of scaling suggested by Bormann [Phys. Rev. Lett. 78, 4324 (1997)].

I. INTRODUCTION

The physics of the static Kosterlitz-Thouless (KT) transition for quasi-two-dimensional (2D) superconductors is well understood in terms of vortex pair unbinding, since long time. However, in spite of this and in spite of the large interest in quasi-2D superconductors over recent years spurred by the event of high-\emph{Tc} superconductors, the dynamics of 2D vortices in the vicinity of the KT transition is not completely settled. In the present paper we address the specific question of the value of the nonlinear \emph{I-V} exponent \emph{a} in the low-temperature superconducting phase, which has been brought up by Tang et al. in two recent papers.

The first attempt to describe the dynamics close to the KT transition was made by Ambegaokar-Halperin-Nelson-Siggia (AHNS). This attempt was based on the phenomenological assumption that the vortices could be separated into two distinct categories, i.e., free individual vortices and bound pairs of vortices and antivortices. This reasoning led to the AHNS value \emph{a} = \emph{a}_{AHNS} for the nonlinear \emph{I-V} exponent in \emph{V} \propto \emph{I}^\emph{a} in the low-temperature phase. An alternative value was later suggested by Minnhagen et al. based on the observation that the low temperature phase is quasicritical and as a consequence the exponent \emph{a} should follow from critical scaling. This gave the exponent \emph{a} = \emph{a}_{\text{scale}} = \emph{z} + 1 where \emph{z} is the dynamic critical exponent. However, these two alternative values are different \emph{a}_{\text{scale}} > \emph{a}_{AHNS} for \emph{T} < \emph{T}_{KT}. The question is then which one is correct. This is the question discussed in the present paper.

A series of attempts to settle this issue was based on numerical simulations for various models displaying a KT transition in 2D: The Coulomb gas with Langevin dynamics, the lattice Coulomb gas with Monte Carlo (MC) dynamics, the \emph{XY} model with resistively-shunted Josephson junction (RSJ) dynamics. These simulations all suggested that \emph{a}_{\text{scale}} was the correct exponent.

However, for the 2D \emph{XY} model with RSJ dynamics in Ref. Simkin and Kosterlitz argued that their data were more consistent with \emph{a}_{AHNS}. This discrepancy between the result of this simulation and the result obtained in the others cited above reflects the general difficulty in the determination of \emph{a}: In a log-log plot the power law form of the nonlinear \emph{I-V} characteristics should give a straight line with the slope \emph{a}. However, this form is only valid for small enough \emph{I} and the smaller the \emph{I} the larger system size is needed in order to get a size independent results. Thus great care is required to ensure that the data used in the determination of the slope are really size independent, which in turn makes the simulations demanding.

The experimental determination of \emph{a} is faced with the same type of finite size problems, as recently discussed in e.g. Ref. However, in order to experimentally distinguish between the two different predictions of \emph{a} one needs in addition to analyze the temperature dependence of \emph{a} as described in Ref. This makes the experimental route to settle the issue somewhat difficult.

On the other hand, the size dependence of the data can also be turned into an advantage when determining \emph{a} from simulations. The logic here is that by carefully choosing the boundary condition the data should obey a size scaling from which \emph{a} can be determined provided the scaling assumption leading to \emph{a}_{\text{scale}} is indeed correct. This strategy was used in Ref. and provided strong evidence in favor of the critical scaling and \emph{a}_{\text{scale}}, as will be discussed in more detail in the present paper.

In an attempt to resolve the issue of the two different results for the value of \emph{a}, Bormann in Ref. re-investigated the AHNS reasoning and concluded that it contains both of the results: For small enough current \emph{I} the result \emph{a} = \emph{a}_{AHNS} should be correct but as \emph{I} is increased there should be a crossover to a distinct region where \emph{a} = \emph{a}_{\text{scale}}. In the \emph{a} = \emph{a}_{\text{scale}} regime the scaling should hold, so according to this analysis one should see a breakdown of scaling into an AHNS regime for small enough \emph{I}. As found in Ref. and which will be further discussed here, the scaling assumption holds for all the
data obtained and there is no sign of a crossover to an AHNS regime.

The most recent attempt to settle this issue is by Chen, Tang, and Tong in Refs. [2,3]. They again try to estimate the nonlinear $I-V$ exponent $a$, $(V \propto I^a)$ below the KT transition for a 2D Josephson junction array from the slope of $\ln V$ versus $\ln I$ for small $I$. The basic claim made is that an anomalous finite-size effect for small $I$ (meaning that in a certain parameter range the voltage for a fixed small $I$ increases with size instead of decreases) gives an overestimation of $a$. As a consequence it was concluded that $a$ is in better agreement with the AHNS prediction instead of the dynamical scaling predictions concluded in Ref. [11,12]. However, as shown in Ref. [11], such an anomalous finite-size effect is a feature consistent with and emerging from the dynamical scaling and does consequently not affect the reliability of a determination based on finite-size scaling.

Our strategy to settle the issue is based on the observation that the dynamical scaling alternative is very amenable to testing by computer simulations. This is because the dynamical scaling makes direct predictions of the data obtained for finite size systems and does thus not hinge on any estimate of the asymptotic slope in the limit of small $I$ and large size $L$. Using this approach, a result consistent with AHNS requires that the dynamical scaling does in fact fail to describe the data. Alternatively, if one wants to verify a crossover to AHNS from a scaling regime, as suggested by Bormann in Ref. [13], then one needs to demonstrate that the dynamical scaling breaks down as one passes over into the AHNS regime. As shown in the present paper and in Ref. [11] direct tests of the dynamical scaling through simulations of the 2D XY model with RSJ dynamics and MC dynamics give excellent agreement with dynamical scaling without any sign of the breakdown required for the AHNS alternative to be valid.

The content of the paper is as follows: In Sec. II we briefly recapitulate the finite-size dynamical scaling and in Sec. III the results of the simulations and the data analysis are presented. In Sec. IV we discuss our results in the context of other attempts to settle the issue and Sec. V, finally, contains some concluding remarks.

II. THE DYNAMICAL SCALING APPROACH

The method used to determine $a$ directly from finite-size scaling is described in Ref. [11]. It is based on the usual scaling form by Fischer et al. in Ref. [11] adopted to two dimensions and to finite-size scaling at criticality. Since the low temperature phase is quasicritical the scaling form applies at and below the KT transition. In order to take maximum advantage of the size scaling we reduce the number of large scales to one by considering a quadratic system with side $L$. The size scaling form is then given by

$$\nu = i^{z+1} \left[ \frac{f(L_i)}{L_i} \right]^z \quad (1)$$

where $\nu = V/L$ is the voltage per length across the sample, $i = I/L$ is the current density, $z$ is dynamic critical exponent and $f(x)$ is a scaling function which goes to a positive constant for small $x$ and is proportional to $x$ for large $x$. This means that $\nu \propto i^{z+1}$ for a given small enough $i$ in the limit of large $L$. Consequently, the scaling prediction for the nonlinear $I-V$ exponent $a$ ($V \propto I^a$) is $a = z + 1$.

The use of this scaling approach has several advantages when trying to determine the exponent $a$ in model simulations: First of all, it does not hinge on the accuracy to calculate the small $i$-limit of voltages where at the same time $Li >> 1$, which is notoriously difficult. Secondly, the existence of a scaling function $f(x)$ can be determined from a data collapse using data from all system sizes simultaneously. Thirdly, $z$ can be related to and obtained from the equilibrium properties of the system. This gives an independent consistency check on the scaling given by Eq. (1), since this $z$-value has to agree with the one obtained directly from the data collapse.

III. RESULTS FOR THE 2D $XY$ MODEL

Our main results are for the 2D XY model with RSJ dynamics and finite temperatures $T$. This model undergoes a KT transition at $T_c \approx 0.89$ ($T$ is measured in units of the Josephson coupling and current in terms of the critical current $i_c$ of a single Josephson junction). The method used is described in Ref. [11]. We use a fluctuating twist boundary condition (FTBC) and a square lattice with size $L$. The data presented here are well converged data for sizes up to $L = 256$. We here analyze data obtained in the low temperature quasicritical phase somewhat below $T_c$.

In Fig. 1 we present results for $T = 0.8$ in the alternative scaling form $\nu/i^{z+1} = F(L_i)$ where $F(x)$ is related to $f(x)$ in Eq. (1) by $F(x) = [f(x)/x]^z$. In this plot we use the value $z = 3.4$ obtained from equilibrium simulations using the relation $\nu = 1/\tilde{c} T^{CG} - 2$ with the dielectric constant $\tilde{c}$ and the Coulomb gas temperature $T^{CG}$, as described in Ref. [11] and given in Table I of Ref. [11]. As seen from Fig. 1(a) a very good data collapse is obtained. At higher values of $Li$ there is a notable systematic deviation from scaling. In order to determine the cause of this deviation the same data are plotted in Fig. 1(b) as a function of $i$. As seen the data which in Fig. 1(a) deviate from the scaling curve [filled symbols in Fig. 1(a) and (b)] now instead collapse and furthermore the collapse for all sizes starts approximately at the same value of $i \approx 0.3$. This means that the data for values $i > 0.3$ are independent of $L$. The Josephson junctions become completely resistive for $i = i_c$ with a resistance given by normal state $R_N$ of the junction and which also means...
that \( V = R_N I \) for the complete array. This large current relation is given by the full line in Fig. 1(b). It means that the deviation from the scaling curve in Fig. 1(a) is just the trivial crossover to the normal state resistance which always occurs for large enough \( i \) because the small scale \( 1/i_c \) becomes relevant and breaks the scaling when \( i \) becomes of the same order as \( i_c \). The data in Fig. 1(a) show no deviation from finite-size scaling apart from the trivial crossover to normal state resistance as \( i \) approaches \( i_c \).

Figure 2 shows the scaling function \( f(Li) \) in Eq. 11 obtained from the same data. In addition, we have in this figure also included data from the 2D XY model with Monte Carlo (MC) dynamics. As demonstrated in Ref. 11, MC dynamics gives for the same \( T \) the same \( I-V \) characteristics up to a constant factor and the same method as in Ref. 11 is used here. The MC data are also for \( T = 0.8 \) but also include the larger size \( L = 512 \). The data for \( i \) in the crossover towards \( v \propto i \) have been excluded in Fig. 2. As seen both sets of data collapse on the scaling curve with \( z = 3.4 \). The full line (\( \propto Li \)) corresponds to \( v \propto i^{3+z} \). Note that the data for a given size \( L \) which fall on this line have the scaling slope \( a_{\text{scale}} \) in a log-log plot of \( v \) versus \( i \). This fact also gives an idea of the difficulty with a direct determination for a single system size: The larger system the fewer points can in practice be obtained. So although the three data points for \( L = 512 \) fall nicely on the scaling function, a determination of \( a \) based on these three points alone would not have the same reliability.

The inset in Fig. 2 shows how the voltage \( v \) depends on the size \( L \) for a given fixed current. As seen the voltage for this fixed current increases for larger \( L \). This is just a reflection of the fact that the scaling curve falls below the full line in Fig. 2 for \( Li \) between approximately 1 and 10. This means that the increase of the voltage with increasing \( L \) is a property of the scaling regime where the behavior is controlled by a single relevant large scale. Consequently, it is not connected to the appearance of a second relevant scale, as suggested in Ref. 6 in case of the seemingly very similar increase of the voltage with increasing \( L \) found in this paper.
IV. COMPARISONS AND DISCUSSION

Our main result from the 2D XY model with RSJ and MC dynamics is that the data collapse on a scaling curve for a certain value of $z$. This value of $z$ is in agreement with the scaling prediction which connects the value of $z$ to equilibrium quantities. The data regime at higher current which do not fall on the scaling curve is controlled by the Josephson junction critical current $i_c$ and belong to the crossover to the normal state linear relation $V = R_N I$. The data show no crossover to an AHNS behavior.

According to Bormann in Ref. 15 such a crossover should be expected for small enough $i$. Thus, according to this theory, AHNS should be valid for low enough $i$ in the limit of large $L$ and there should be a crossover to the scaling result for larger $i$. As $i$ approaches the critical Josephson junction current there should be a second crossover towards the normal state linear $I$-$V$ characteristics. As seen from Fig. 2 the data give evidence of a crossover from the scaling regime to the linear Ohmic regime but there is no sign of a crossover towards the AHNS regime for smaller $i$. In Fig. 2 such a crossover would mean that the data for larger $L$ and smaller $i$ would fall above the scaling curve. The crossover current $i_{\text{cross}}$ is proportional to $1/n^{1/(z/2-1)}$ where $n$ is the vortex density. An attempt towards a more quantitative estimate of the crossover is given in Fig. 1 of Ref. 15. Our data for $i = 0.01$ correspond to a point in this figure $(x, y) = (z + 2, i^{-1}) = (6.4, 10^2)$ which should be far inside the AHNS regime. In fact all the data up to the crossover to the linear regime (at $i \approx 0.3$) should, according to this estimate, belong to the AHNS regime. However, our data show no deviation from scaling for any size or current, indicating that all the data are instead in the scaling regime and that there is no crossover to an AHNS regime at small currents. However, one can, of course, not entirely rule out that such a crossover might exist for even lower currents and larger sizes than could be reached in the simulations.

Tang et al. in Ref. 20 use a rectangular sample with sides $L_x$ and $L_y$ and the current in the $x$ direction. The idea is to use a small ratio $L_y/L_x$ in order to minimize the dependence on the scale $L_x$ and they in practice use $0.004 \leq L_y/L_x \leq 0.25$ corresponding to $L_y = 8$ and $L_y = 512$, respectively. The current injection method used introduces a nonuniform vortex density in the current direction. However this is compensated by skipping a boundary region with the length $b$ at both boundaries when measuring the voltage. In contrast the use of FTBC is designed to make the vortex density uniform. Nevertheless, it was shown in Ref. 12 that the method adopted by Tang et al. for PBC in the transverse direction gives the same result as the FTBC method provided care is taken to avoid any influence from the two additional length scales $L_x$ and $b$. Ref. 12 confirms the scaling prediction for sizes up to $L_y = 64$. Tang et al. argues that in the case of a periodic boundary condition (PBC) in the direction transverse to the current one should get a crossover from a scaling regime at lower currents towards an AHNS regime at higher. Note that this is precisely the opposite to the Bormann prediction. According to Tang et al. such a crossover is supported by their Fig. 6 in Ref. 20. This figure for $T = 0.8$ corresponds precisely to our Fig. 1(a) for the same transverse sizes. The scaling is clearly visible although the quality in their Fig. 6 is not as good as in our Fig. 1a. However, whereas our method is specifically designed to test the scaling and only introduces a single scale $L$, the method by Tang et al. have three large scales and in particular the fact that the ratio $L_y/L_x$ varies from 0.004 to 0.25 instead of being constant, might well cause deviations from scaling. In addition, there is a crossover for higher currents towards the linear $I$-$V$ characteristics. However, whereas, as shown above, the crossover in our case can unambiguously be attributed to the trivial crossover to the linear regime starting at around $i \approx 0.3$ for all sizes [compare Fig. 1(b)] the situation in Fig. 6 of Tang et al. is slightly different. At small values of $L_y$ the crossover comes at roughly the same value $i \approx 0.5$ (up to $L \approx 32$) and then it decreases to roughly 0.06 at $L_y = 512$. This decrease of the crossover current with $L_y$ indicates that there is another length scale in the problem. Tang et al. suggest that this new length scale is associated with the vortex physics and is given $L_x \propto i^{-1/(z-2)/4}$ and that the crossover is to an AHNS regime occurring for $L_y > L_x$. The problem with this interpretation is, in the light of Fig. 1(a) and (b), that there is no such size dependent crossover for a square lattice. Since the bulk properties of vortex physics cannot depend on the shape of the sample, the complete absence of the appearance of an additional vortex length in Fig. 1(a) and (b) strongly suggests that the explanation offered by Tang et al. cannot be the correct one. An obvious candidate is instead the additional ratio $L_y/L_x$ introduced by Tang et al. and which is not kept fixed when $L_y$ is varied, as required by a proper scaling analysis. The point to note is that it is not the absolute value of $L_x$ per se that matters, but the fact that the ratio $L_y/L_x$ has to be constant, and as pointed out above it varies from very small to 0.25 in Ref. 20. This could explain both why the quality of the size scaling is not as good as for the scaling with fixed $L_y/L_x = 1$ shown in our Fig. 1a, as well as the size dependence of the crossover.

Tang et al. also obtained the data for an open boundary conditions (OBC) in the transverse direction. The difference with the PBC result is the different finite-size dependence of the voltage. The open boundary causes stronger boundary effects and thus changes the details of the size convergence. The data are plotted as $v/L^{2_{\text{AHNS}}}$ versus $i L_y$ in Fig. 8 in Ref. 20. In Fig. 3(a) our data for FTBC and square lattice are plotted in the same way. As seen in Fig. 3(a) the data over a large region apparently fall on a horizontal line. This line corresponds to $v \propto L^{2_{\text{AHNS}}}$ and consequently one might be tempted to interpret this as evidence of an AHNS exponent. The fallacy here is that the data which constitute this line are
the size converged data in the crossover region towards \( i_c \) as shown in Fig. 3(b): The two data points connected with a line in Fig. 3(b) forms a spurious AHNS line by repetition in Fig. 3(a). When going from the steeper slope in the scaling region through the crossover to the linear \( I-V \) dependence there will always be some current region where the slope has the AHNS value. However, this continuous crossover to a linear \( I-V \) characteristics has nothing to do with the AHNS vortex physics. The important thing to notice in Fig. 3(a) is that the scaling systematically fails for the larger sizes at lower \( i \) where the data for each size deviate to lower values with decreasing \( i \) for a fixed size. The same deviation is observed in Fig. 8 in Ref. 6 where the data on the horizontal line are a repetition of the two data points in Fig. 7 from Ref. 6 for the current sizes \( i = 0.3 \) and 0.4 (same currents as in Fig.3a), respectively. Thus for the larger system sizes, which are relevant for the bulk properties of vortex physics, the change of boundary condition to OBC makes little difference. This is of course expected because the boundary condition should not matter at all for large enough samples. For smaller samples the results are different, as apparent when comparing our Fig. 3(a) with Fig. 8 in Ref. 6. This is also expected because here the fluctuation associated with the open boundary increases the voltage for a given \( i \) relative to the PBC boundary. However, the vortex physics in the bulk remains the same and it is these properties which dominate for large samples.

To summarize: Neither the shape of the sample nor the details of the boundary condition can change the vortex physics in the bulk for large enough systems. We find that there is no compelling evidence that the data by Tang et al. are in contradiction with this statement. Proper scaling analysis with a fixed ratio \( L_y/L_x \), as shown in the present paper, gives no evidence of an extra scale in addition to \( L \) and \( i_c \).

V. CONCLUDING REMARKS

As discussed in the present paper the \( I-V \) characteristics for the 2D XY model with RSJ and MC dynamics obey size scaling for a quadratic sample (scaling in \( L_y \) with fixed at \( L_y/L_x = 1 \)). The fact that the data obey size scaling means that \( a = z + 1 \) where \( z \) is the dynamic critical exponent. The exponent \( z \) can be calculated from the equilibrium properties of the system which together with the relation \( a = z + 1 \) gives the scaling prediction for \( a \). As shown this scaling prediction is in agreement with the \( z \) value determined from the data collapse. The only deviation from the scaling can, as shown here, be linked to the trivial high current crossover to the linear \( I-V \) relation.

A prediction for \( a \) which is different from the scaling prediction can only be correct if the scaling breaks down. As shown in the present paper, there is no such breakdown in the scaling except for the one at high currents associated with the critical current of a single Josephson junction. This does in principle not rule out the possibility that a breakdown could occur at sizes larger and currents smaller than reached in the simulations. However, whereas the scaling prediction has been verified over a large parameter range, the breakdown of scaling, which would have to occur if the AHNS prediction was correct in the ultimate small current limit, has not been sup-
ported by any simulation data so far.

In our opinion, the only convincing way to verify that AHNS, or some other alternative prediction is correct in some parameter regime, is to demonstrate that the scaling breaks down in the same regime.

**Acknowledgements**

Support from the Swedish Research Council is gratefully acknowledged. B.J.K acknowledges the support from the Korea Science and Engineering Foundation through Grant No. R14-2002-062-01000-0, and the support from Ajou University.

---

1. V.L. Berezinskii, Zh.kps. Teor. Fiz. **61**, 1144 (1971) [Sov. Phys. JETP **34**, 610 (1971)]; J.M. Kosterlitz and D.J. Thouless, J. Phys. C **6**, 1181 (1973); J.M. Kosterlitz, *ibid* **7**, 1046 (1974).
2. P. Minnhagen, Rev. Mod. Phys. **59**, 1001 (1987).
3. G. Blatter, M.V. Feigelman, V.B. Geshkenbein, A.I. Larkin, and V.M. Vinokur, Rev. Mod. Phys. **66**, 1125 (1994).
4. P. Minnhagen in Models and Phenomenology for Conventional and High-temperature Superconductivity Enrico Fermi Course CXXXVI, eds G Iadonisi, J.R. Schrieffer and M.L. Chiofalo (IOS Press, Amsterdam, 1998) p.451.
5. Q.-H. Chen, L.-H. Tang, and P. Tong, Phys. Rev. Lett. **87**, 067001 (2001).
6. L.-H. Tang and Q.-H. Chen, Phys. Rev. B **67**, 024508 (2003).
7. V. Ambegaokar, B.I. Halperin, D.R. Nelson, and E.D. Siggia, Phys. Rev. Lett. **40**, 783 (1978); Phys. Rev. B **21**, 1806 (1980).
8. P. Minnhagen, O. Westman, A. Jonsson, and P. Olsson, Phys. Rev. Lett. **74**, 3672 (1995).
9. K. Holmlund and P. Minnhagen, Phys. Rev. B **54**, 523 (1996); Physica C **292**, 255 (1997).
10. H. Weber, M. Wallin, and H.J. Jensen, Phys. Rev. B **53**, 8566 (1996).
11. B.J. Kim, P. Minnhagen, and P. Olsson, Phys. Rev. B **59**, 11506 (1999); B.J. Kim, Phys. Rev. B **63**, 024503 (2000).
12. M.Y. Choi, G.S. Jeon, and M. Yoon, Phys. Rev. B **62**, 5357 (2000).
13. M.V. Simkin and J.M. Kosterlitz, Phys. Rev. B **55**, 11646 (1997).
14. D.R. Strachan, C.J. Lobb, and R.S. Newrock **67**, 174517 (2003).
15. D. Bormann, Phys. Rev. Lett **78**, 4324 (1997).
16. M.P.A. Fisher, Phys. Rev. Lett. **62**, 1415 (1989); D.S. Fisher, M.P.A. Fisher, and D.A. Huse, Phys. Rev. B **43**, 130 (1991).
17. A.T. Dorsey, Phys. Rev. B **43**, 7575 (1991).
18. In order to obtain optimal consistency the equilibrium value of $z$ is calculated with the same boundary condition as the dynamical calculation. Note that for this boundary condition the helicity modulus is identically zero. See Ref. [11].
19. K. Medvedyeva, B.J. Kim, and P. Minnhagen, Physica C **355**, 6 (2001).
20. It should be noted that there is a difference in scale between our Fig.1 and Fig.6 of Ref.6, which makes the data for $T = 0.8$ appear more scattered in the latter case. However, even when taking this into account our data collapse is of higher quality. We also stress that in our data analysis we do not find any systematic deviation from the scaling except the one associated with the cross over associated with $i_c$. 
