Spin-orbit interaction effect on transport of Dirac fermions in graphene

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We study theoretically the quantum transport properties of the Dirac fermions with spin-orbit interactions (SOIs) in graphene by using the method of Schwinger proper time together with decomposition over Landau level poles and Kubo formula. The analytical expressions for both longitudinal and Hall conductivities are derived explicitly. It is found that, from some numerical examples, when the Rashba SOI is taken into account the Shubnikov-de Haas (SdH) oscillation peaks of the longitudinal conductivity versus the chemical potential are split, while the SdH oscillation of the longitudinal conductivity versus a external magnetic field exhibits a beating pattern. Furthermore, the Rashba SOI tends to suppress the quantum Hall effect in graphene.

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I. INTRODUCTION

Graphene has attracted a lot of attention because of its appealing properties. At low energy, owing to the specific band structure with the unique valley and neutrality separating the hole states from the electron states, graphene has led to the emergence of a paradigm of the relativistic condensed matter physics, where the relativistic quantum phenomena, some of which are unobservable in high energy physics, can be tested in the tabletop experiment. The recent advances in fabrication technique have made it possible to produce a big, and they have given some explicit expressions of the SOIs in graphene. In this work, we investigate the transport of Dirac fermions in graphene. The main purpose is focused on the effect of the SOIs on the transport on the basis of estimation of Yao et al. and Min et al. for the SOIs in graphene. Using the Schwinger proper-time method and Kubo formula, we obtain some analytical expressions for both longitudinal and Hall conductivities. It is found that when the Rashba SOI is considered, the longitudinal conductivity as a function of the chemical potential deviates the linear relation at zero magnetic field. For nonzero magnetic field, the SdH oscillations are observed and the oscillation peaks in the longitudinal conductivity versus the chemical potential are split when the Rashba SOI is applied, while the oscillation in the longitudinal conductivity versus the magnetic field exhibits a beating pattern. It is also shown that the Rashba SOI tends to suppress the quantum Hall effect in graphene.

II. MODEL FORMALISM

The graphene is a flat monolayer of carbon atoms tightly packed into a honeycomb lattice. At low energy, it can be described by a 2 + 1 dimensional relativistic field theory model. When the SOIs are included, the Lagrangian density of the system is given by

$$\mathcal{L} = h v_F \overline{\Psi}(i\gamma^\mu D_\mu + H_\mu) \Psi, \quad (1)$$

where $\Psi = (\Psi_{K+}, \Psi_{K-})$ is the eight-component Dirac spinors with $\Psi_{K+} = (\Psi_{A\uparrow}, \Psi_{A\downarrow}, \Psi_{B\uparrow}, \Psi_{B\downarrow})$ which describes the spin-related Bloch states residing on the atoms of the A, B sub-lattice at momentum $K(K^\prime)$. $D_\mu = \gamma^\mu (\partial_\mu - ieA_\mu)$ with $\gamma^\mu (\mu = 0, 1, 2)$ being $4 \times 4$ γ matrices, $e$ is the electron charge, $v_F$ is the Fermi velocity, the external magnetic field $B = \nabla \times A$ is applied perpendicular to the $x - y$ plane and the corresponding vector potential is taken in the symmetric gauge $A = (-By/2, Bx/2)$. In Eq. (1), $H_\mu$ describes the SOIs that read

$$H_\mu = \lambda_{SO}(1 - \gamma^0 s_\mu) + \lambda_R(\gamma^1 s_\mu - \gamma^0 \gamma^1 s_\mu), \quad (2)$$
where $\lambda_{SO}$ is the intrinsic SOI parameter, $\lambda_R$ is the Rashba SOI parameter, and $s$ is the spin variable. For $B = 0$, the corresponding energy spectrum are given by

$$
\begin{align*}
\varepsilon_1 & = \pm \sqrt{k^2 + (\lambda_R - \lambda_{SO})^2} + \lambda_R - \lambda_{SO}, \\
\varepsilon_2 & = \pm \sqrt{k^2 + (\lambda_R + \lambda_{SO})^2} - \lambda_R - \lambda_{SO}.
\end{align*}
(3)
$$

For $\lambda_{SO} > \lambda_R > 0$, the system includes an energy gap of $2(\lambda_{SO} - \lambda_R)$. For $0 < \lambda_{SO} < \lambda_R$, the energy gap closes.

The Green’s function of Dirac fermions described by the Lagrangian (1) in an external magnetic field can be expressed as

$$
G(x, y) = \int_0^\infty d\epsilon \frac{e^{-\gamma \epsilon}}{8(\pi \hbar)^3/2} e^{i\epsilon \nu x} \epsilon^{\mu\nu}_{\alpha\beta} \left[ \frac{1}{2\epsilon} \gamma^\mu C^\alpha_{\beta\gamma} e^{-\gamma \epsilon} - \frac{1}{2} (e\epsilon B x_2 - e\epsilon^2 B x_1) + \lambda_{SO}(1 - \gamma^0 s_z) \right] \left[ e^{i\epsilon x} e^{i\epsilon y} \right] \frac{e^{i\epsilon x} e^{i\epsilon y}}{\sin(eB\epsilon)} e^{i(\epsilon B - A(x-y))},
(6)
$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $C^\alpha_{\beta\gamma} = \gamma^\alpha \epsilon^{\beta\gamma}_{\alpha\beta\gamma}$, with $g^{\alpha\beta} = \text{diag}(1, -1, -1)$, and

$$
B_\mu = i2\lambda_{SO}\sigma^{s\alpha} s_z + 2\lambda_R(\sigma^{s\alpha} s_y - \delta_{\alpha\beta} \gamma^1 s_x - \delta_{\alpha\beta} \gamma^0 s_x),
(7)
$$

$$
\Delta = 2(\lambda_{SO}^2 + \lambda_R^2)(1 - \gamma^0 s_z) + 4\lambda_{SO}\lambda_R(\gamma^1 s_x - \gamma^0 \gamma^1 s_x),
(8)
$$

$$
\begin{align*}
e^{i(\epsilon B - A(x-y))} & = e^{-i\left[\frac{1}{2}(6\lambda_{SO}^2 - 6\lambda_R^2)\right] \left[ \cos(\zeta s_z) + i\gamma^0 s_z \sin(\zeta s_z) \right]} |\cos(eB) + \gamma^1 \gamma^2 \sin(eB) + \frac{1}{2} (1 - \gamma^0 s_z) \sin(\xi s_z) - \cos(eB)| \\
& \quad - i\left[\frac{1}{2}(12\lambda_{SO} \lambda_R \sin(\xi s_z)) \left( i\gamma^1 s_x - \gamma^0 \gamma^1 s_x \right) + \frac{1}{2} (1 - \gamma^0 s_z) \gamma^2 \left[ \frac{eB}{\xi} \sin(\xi s_z) - \sin(eB) \right] \right].
\end{align*}
(9)
$$

The expressions of $G(i\omega_n, k)$ ($i=1,2,3$) are very complicated and will be given in the appendix A. Whence, we can further obtain the retarded and advanced Green’s functions by the analytic continuation $G^{(R)}(\omega + i0, k) = G(i\omega_n \rightarrow \omega + i0, k)$ and $G^{(A)}(\omega - i0, k) = G(i\omega_n \rightarrow \omega - i0, k)$. When considering the influence of impurities, it is assumed that the scattering rate $\Gamma$ on impurity is described phenomenologically by a constant, and then the Green’s functions acquire the form

$$
G^{(R,A)}(\omega, k) = G_{1}^{(R,A)}(\omega \pm i\Gamma, k) + G_{2}^{(R,A)}(\omega \pm i\Gamma, k) + G_{3}^{(R,A)}(\omega \pm i\Gamma, k).
(11)
$$

In general, the scattering rate $\Gamma$, which is defined by $\Gamma(\omega) = -\text{Im} \Sigma^{R}(\omega)$, is a frequency-dependent quantity. It needs to be determined self-consistently from the Schwinger-Dyson equations. The exact form of this equation actually depends on the impurity scattering fashion, such as short- or long-range scatterers. This kind of consideration have been made for graphene in Ref.24. But in this paper, we mainly focus on the SOI effect on transport, and neglect the exact form of in-
teractions between impurities and electrons.

III. ELECTRONIC CONDUCTIVITY

The Kubo formula concerning the frequency-dependent electrical conductivity as a linear response function to an external field can be written as:

$$\sigma_{ij}(\Omega) = \frac{\text{Im} \Pi_{ij}^R(\Omega + i0)}{\Omega},$$

(12)

where $i, j$ are the component indexes of coordinates, and $\Pi_{ij}^R(\omega)$ is the retarded current-current correlation function obtained by analytical continuation of the Matsubara function

$$\Pi_{ij}(i\omega_n) = \frac{1}{V} \int_0^\beta d\tau e^{i\omega_n \tau} \langle T_r J_i(\tau) J_j(0) \rangle, \quad \omega_n = 2\pi n T,$$

(13)

where $V$ is the volume of the system, $\beta$ is the inverse temperature, and $J_i(\tau) = \int d^2 r j_i(\tau, \mathbf{r})$ with $j_i = -e\nabla \Psi(\tau, \mathbf{r})$. Neglecting the impurity vertex corrections, the calculation of the conductivity reduces to evaluation of the bubble diagram.

Then Eq. (12) can be rewritten as

$$\sigma_{ij}(\Omega) = \frac{e^2 v_F^2}{2\pi} \text{Re} \int_{-\infty}^{\infty} d\omega \frac{1}{4 T \cosh \frac{\omega}{2 T}} A_L(\omega),$$

(15)

and the Hall conductivity

$$\sigma_{xy} = -\frac{e^2 v_F^2}{2 \pi} \text{Im} \int_{-\infty}^{\infty} d\omega \frac{1}{4 T \cosh \frac{\omega}{2 T}} A_H(\omega),$$

(16)

where all the quantities on the right-hand side are calculated in the Appendix B. Eqs. (15) and (16) establish the fundamental basis for investigating the SOI effect on the quantum transport properties of the Dirac fermions in graphene.

In the limit of zero field, the Hall conductivity becomes zero. While for the longitudinal conductivity, using the asymptotic expansions

$$\psi(z) = \ln z - \frac{1}{2z} - \frac{1}{12z^3} + \frac{1}{120z^5} + O\left(\frac{1}{z^7}\right),$$

(17)

we arrive at

$$A_L(\omega) = \left[\frac{\omega^2 - T^2}{6 T^2} - \frac{6 T^2 (\omega^2 + T^2)}{(2 T^2 + \omega^2)^2}\right] \text{ln} \left[\frac{6 T^2 (\omega^2 - T^2)^2 + (2 T^2)^2}{6 T^2 (\omega^2 - T^2)^2 + (2 T^2)^2}\right] + \frac{\Gamma \omega^2}{4 T^2 + \omega^2} \left[\arctan \frac{\omega + 2 T}{\sqrt{\omega^2 + 4 T^2}} + \arctan \frac{\omega - 2 T}{\sqrt{\omega^2 + 4 T^2}}\right]$$

(18)

for $\lambda_{SO} = 0$. When $T \rightarrow 0, |\mu| >> \lambda_R, \Gamma$, the longitudinal conductivity can be further expressed as

$$\sigma_{xx} = \frac{4 e^2}{\pi} \frac{|\mu|/\Gamma}{1 + (3 \lambda_R^2 / \Gamma \mu)^2},$$

(19)

IV. RESULTS AND DISCUSSION

To investigate numerically the behavior of electrical conductivity, we need to restore the whole model parameters in Eqs. (15) and (16). Thus, one should carry out the replacements: $T \rightarrow k_B T, eB \rightarrow e B v_F^2$. In the following, we mainly discuss the Rashba SOI effect on the transport properties since the intrinsic SOI is very small, while the Rashba SOI can be tunable by a perpendicular electric field. In Figs. 1 and 2, we show the chemical potential $\mu$ dependence of the longitudinal conductivity for the different Rashba spin orbit parameter $\lambda_R$ at zero or nonzero field. For zero field (see Fig. 1), one can see that when $\lambda_R = 0$, the conductivity is proportional to $|\mu|$ and tends to the known quantum-limited minimal value $4e^2/h$ at zero chemical potential. For $\lambda_R \neq 0$, there exists a threshold chemical potential $\mu_c$ which increases with increasing $\lambda_R$. 


FIG. 1: The longitudinal conductivity $\sigma_{xx}$ measured in $2e^2/h$ units as a function of the chemical potential $\mu$ for the different values of $\lambda_R$. We take $B = 0T$, $T = 3K$, $\Gamma = 5K$, and $\lambda_{SO} = 0.001K$.

When the chemical potential is smaller than $\mu_c$, the longitudinal conductivity becomes almost independent of $\mu$; while for $\mu > \mu_c$, the $\sigma_{xx}$-$\mu$ curves recover the linear relation. This tendency agrees with Eq. (19). For nonzero field case in Fig. 2, we observe SdH oscillations of the conductivity due to the Landau-level crossing of the Fermi level. From Fig. 2, it is clearly seen that when $\lambda_R \neq 0$, each oscillation peak is split into two implicit peaks, and the splitting peaks shift by $\lambda_R$. This is due to the spin-orbit splitting of the Landau levels.

The longitudinal conductivity as a function of the magnetic field $B$ for the different $\lambda_R$ is shown in Fig. 3. For $\lambda_R = 0$, the longitudinal conductivity decreases and intervals between the neighboring SdH oscillation peaks become large with increasing $B$, which reflects the fact that in the presence of the magnetic field only the transitions between neighboring Landau levels contribute to electrical conductivity, while a further increase of the magnetic field leads to increasing of the distance between neighboring Landau level, thus suppresses the transitions between them. When $B$ is large enough, the conductivity becomes independent of $B$ since the lowest Landau level is filled which is always below the Fermi level. These observations are quite consistent with the previous studies. In particular, it is interesting to note that when the Rashba SOI presents, the longitudinal conductivity exhibits the characteristic feature that the SdH oscillations are enhanced largely at certain positions, however damped at other positions. Such SdH as a beating pattern have been observed in two dimensional electron gas. From Fig. 3, one can find that the enhanced positions and amplitudes of the SdH oscillations can be tuned by the Rashba SOI due to shift of one set of Landau level by $\lambda_R$.

FIG. 2: The longitudinal conductivity measured in $2e^2/h$ units as a function of the chemical potential $\mu$ for the different values of $\lambda_R$ at $B = 1T$. The other parameters are taken the same as Fig. 1.

FIG. 3: The magnetic field dependence of the longitudinal conductivity measured in $2e^2/h$ units for the different values of $\lambda_R$ at $\mu = -600K$. The other parameters are taken as Fig. 1.

FIG. 4: The longitudinal conductivity measured in $2e^2/h$ units as a function of $\lambda_R$ for the different magnetic field $B$ at $\mu = -300K$. The other parameters are taken the same as Fig. 1.
Figure 4 shows the longitudinal conductivity versus the Rashba SOI parameter $\lambda_R$ for the different magnetic field $B$. It is found that when the magnetic field is applied, the longitudinal conductivity as a function of $\lambda_R$ behaves as the oscillation. It is because the Rashba SOI leads to the shift of landau level, the longitudinal conductivity shows a maximum each time a Landau level passes through the Fermi level of system, and a minimum when the Fermi level is situated between two Landau levels.

Figure 5 shows Hall conductivity as a function of the chemical potential $\mu$ for the different $\lambda_R$. When $\lambda_R = 0$, the Hall conductivity has a steplike structure as a function of $\mu$, which reflect the quantum Hall effect. While the Rashba SOI opens, the Hall steps become narrow and the step near $\mu = 0$ is split into two steps. It is observed that the Hall conductivity displays peaks instead of a plateau at larger $\lambda_R$. There is no Hall plateau in the cases of sufficiently strong Rashba SOI. This result suggests that the Rashba SOI tends to suppress the quantum Hall effect in graphene.

V. SUMMARY

We have investigated the effect of the SOIs on transport of Dirac fermions in graphene on the basis of amplitude estimation of Yao et al. and Min et al. for the SOIs. Using the Schwinger proper-time method, decomposition over Landau level poles and Kubo formula, we obtain the analytical expressions for both longitudinal and Hall conductivities. It has been found that when the Rashba SOI is applied, the longitudinal conductivity versus the chemical potential deviates the linear relation at zero magnetic field. For nonzero magnetic field, the SdH oscillation in the longitudinal conductivity is observed, and each SdH oscillation peak is split into two peaks as the Rashba SOI is applied. While the oscillation in the longitudinal conductivity as a function of the magnetic field exhibits a beating pattern with the Rashba SOI turned on. It is also shown that the Rashba SOI tends to suppress the quantum Hall effect in graphene.

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APPENDIX A

The Green functions $G_i^{(R)}$ in Eq. (11) are given by

\begin{align}
G_1^{(R)}(\omega) &= (H_s + \gamma_0(\omega \pm i\Gamma))A_1^{(R)}(\omega) - i\gamma_1\gamma_2\text{sgn}(eB)A_2^{(R)}(\omega) - i(\gamma_1 k_2 - \gamma_2 k_1)\text{sgn}(eB)B_1^{(R)}(\omega) - A_2^{(R)}(\omega) + (\gamma_1 k_1 + \gamma_2 k_2)B_2^{(R)}(\omega) - A_1^{(R)}(\omega), \\
G_2^{(R)}(\omega) &= (H_s + \gamma_0(\omega \pm i\Gamma))A_3^{(R)}(\omega) - i\gamma_1\gamma_2\text{sgn}(eB)A_4^{(R)}(\omega) - i(\gamma_1 k_2 - \gamma_2 k_1)\gamma_3 s_z\text{sgn}(eB)B_3^{(R)}(\omega) - A_4^{(R)}(\omega) + (\gamma_1 k_1 + \gamma_2 k_2)\gamma_3 s_z[B_4^{(R)}(\omega) - A_3^{(R)}(\omega)], \\
G_3^{(R)}(\omega) &= -\frac{12\lambda_S\lambda_R}{\xi}(H_s + \gamma_0(\omega \pm i\Gamma))(\gamma_1^2 s_y - \gamma_0\gamma_1 s_x)(f_4^{(R)}(\omega) + \gamma_0 s_z f_8^{(R)}(\omega)) + \frac{12\lambda_S\lambda_R}{\xi}(\gamma_1 k_1) \\
&+ \gamma_1^2 s_y^2\text{sgn}(eB)f_4^{(R)}(\omega)(\gamma_1^2 s_y - \gamma_0\gamma_1 s_x) + \frac{12\lambda_S\lambda_R}{\xi}(\gamma_1 k_1 + \gamma_2 k_2)\gamma_3 s_z f_8^{(R)}(\omega) \\
&+ i\gamma_1\gamma_2\text{sgn}(eB)f_4^{(R)}(\omega)(\gamma_1^2 s_y - \gamma_0\gamma_1 s_x)\gamma_3 s_z, \tag{A3}
\end{align}
where \( A_1 = I_1 + I_3 + I_5 - I_7, \ A_2 = I_2 + I_4 + I_6 - I_8, \ A_3 = I_1 - I_3 + I_5 + I_7, \ A_4 = I_2 - I_4 + I_6 + I_8, \) \( B_1 = I_2 + I_6 - I_5, \) \( B_2 = I_1 + I_3 + I_5 - I_7, \) \( I_1 = I_2 - I_4 + I_6 + I_8, \) \( I_3 = I_1 - I_3 + I_5 + I_7, \) \( I_4 = I_2 - I_4 + I_6 + I_8, \) \( I_6 = I_2 + I_6 - I_5, \) \( I_5 = I_1 + I_3 + I_5 - I_7, \) \( I_7 = I_1 - I_3 + I_5 + I_7, \) \( I_8 = I_2 - I_4 + I_6 + I_8, \) in which

\[
J_1^{(RA)}(\omega) = \frac{1}{2} e^{-\sigma} \sum_{n=0}^{\infty} (-1)^n \left[ \frac{L_n(2\omega) - L_{n-1}(2\omega)}{\omega (n+1)^2 + 10 \omega_i^2 + 2 \omega_i + 2 \lambda |eB|} + \frac{L_n(2\omega) - L_{n+1}(2\omega)}{\omega (n+1)^2 + 10 \omega_i^2 + 2 \omega_i + 2 \lambda |eB|} \right],
\]

\[
J_2^{(RA)}(\omega) = \frac{1}{2} e^{-\sigma} \sum_{n=0}^{\infty} (-1)^n \left[ \frac{L_n(2\omega) + L_{n-1}(2\omega)}{\omega (n+1)^2 + 10 \omega_i^2 + 2 \omega_i + 2 \lambda |eB|} + \frac{L_n(2\omega) + L_{n+1}(2\omega)}{\omega (n+1)^2 + 10 \omega_i^2 + 2 \omega_i + 2 \lambda |eB|} \right],
\]

\[
J_3^{(RA)}(\omega) = \frac{1}{2} e^{-\sigma} \sum_{n=0}^{\infty} (-1)^n \left[ \frac{L_n(2\omega) - L_{n-1}(2\omega)}{\omega (n+1)^2 + 10 \omega_i^2 + 2 \omega_i + 2 \lambda |eB|} + \frac{L_n(2\omega) - L_{n+1}(2\omega)}{\omega (n+1)^2 + 10 \omega_i^2 + 2 \omega_i + 2 \lambda |eB|} \right],
\]

\[
J_4^{(RA)}(\omega) = \frac{1}{2} e^{-\sigma} \sum_{n=0}^{\infty} (-1)^n \left[ \frac{L_n(2\omega) + L_{n-1}(2\omega) - L_{n+1}(2\omega)}{\omega (n+1)^2 + 10 \omega_i^2 + 2 \omega_i + 2 \lambda |eB|} + \frac{L_n(2\omega) + L_{n-1}(2\omega) - L_{n+1}(2\omega)}{\omega (n+1)^2 + 10 \omega_i^2 + 2 \omega_i + 2 \lambda |eB|} \right],
\]

\[
J_5^{(RA)}(\omega) = \frac{1}{2} e^{-\sigma} \sum_{n=0}^{\infty} (-1)^n \left[ \frac{L_n(2\omega) + L_{n-1}(2\omega) + L_{n+1}(2\omega)}{\omega (n+1)^2 + 10 \omega_i^2 + 2 \omega_i + 2 \lambda |eB|} + \frac{L_n(2\omega) + L_{n-1}(2\omega) + L_{n+1}(2\omega)}{\omega (n+1)^2 + 10 \omega_i^2 + 2 \omega_i + 2 \lambda |eB|} \right],
\]

\[
J_6^{(RA)}(\omega) = \frac{1}{2} e^{-\sigma} \sum_{n=0}^{\infty} (-1)^n \left[ \frac{L_n(2\omega) + L_{n-1}(2\omega) - L_{n+1}(2\omega)}{\omega (n+1)^2 + 10 \omega_i^2 + 2 \omega_i + 2 \lambda |eB|} + \frac{L_n(2\omega) + L_{n-1}(2\omega) - L_{n+1}(2\omega)}{\omega (n+1)^2 + 10 \omega_i^2 + 2 \omega_i + 2 \lambda |eB|} \right],
\]

with \( L_n(z) \) being the generalized Laguerre polynomials, \( \sigma = -k^2/|eB| \), and \( \alpha = \xi/|eB| \).
APPENDIX B

Substituting Eq. (11) into the trace Eq. (14), then it is evaluated after a somewhat tedious calculation

\[
tr[\gamma^i S^{(RA)}(\omega', k)\gamma^j S^{(RA)}(\omega, k)] = \delta_{ij}(\omega' + i\Gamma)(\omega + i\Gamma)\left[A_1^{(RA)}(\omega')A_1^{(RA)}(\omega) - A_2^{(RA)}(\omega')A_2^{(RA)}(\omega) + A_3^{(RA)}(\omega')A_3^{(RA)}(\omega)\right] + (2k_j + i\delta_{ij}k^2)\left(B_1^{(RA)}(\omega') - A_2^{(RA)}(\omega')\right) + (B_2^{(RA)}(\omega') - A_1^{(RA)}(\omega'))(B_2^{(RA)}(\omega') - A_1^{(RA)}(\omega')) + \ldots
\]

where \(\epsilon_{ij}\) is antisymmetric tensor (\(\epsilon_{12} = 1\)). Integrating over momenta in Eq. (14), we obtain the longitudinal conductivity

\[
\sigma_{xx} = \sigma_{xx}(\Omega \to 0) = \frac{e^2\gamma^2_{\text{perpendicular}}}{\pi^2} Re \int_0^\infty d\omega \frac{1}{4\epsilon \cos k_B z_L} [X_L + \frac{1}{2\pi^2}Y_L - \frac{\delta_{ij}\omega^4}{\epsilon}Z_L],
\]

where

\[
X_L = \frac{2(\omega^2 + \Gamma^2)}{(1 - \frac{1}{2})(\omega + \Gamma)^2} \left\{ \frac{(\omega + \Gamma)^2}{(\omega + \Gamma)^2 - (1 - \frac{1}{2})(\omega^2 + \Gamma^2)} - \frac{1}{(\omega + \Gamma)^2 - (1 - \frac{1}{2})(\omega^2 + \Gamma^2)} \right\} + \ldots
\]

\[
Y_L = \psi_{13}(\omega + \Gamma)^2 - (10\omega^2 + 2\Gamma^2) + \ldots
\]

\[
Z_L = \psi_{13}(\omega + \Gamma)^2 - (10\omega^2 + 2\Gamma^2) + \ldots
\]

with \(\psi(z)\) being the digamma function and

\[
W_1 = \frac{2(\omega^2 + \Gamma^2)}{(1 - \frac{1}{2})(\omega^2 + \Gamma^2)} - \frac{2(\omega + \Gamma)^2}{(1 - \frac{1}{2})(\omega + \Gamma)^2} + \ldots
\]

\[
W_2 = \frac{2(\omega^2 + \Gamma^2)}{(1 - \frac{1}{2})(\omega^2 + \Gamma^2)} - \frac{2(\omega + \Gamma)^2}{(1 - \frac{1}{2})(\omega + \Gamma)^2} + \ldots
\]

\[
W_3 = \frac{2(\omega^2 + \Gamma^2)}{(1 - \frac{1}{2})(\omega^2 + \Gamma^2)} - \frac{2(\omega + \Gamma)^2}{(1 - \frac{1}{2})(\omega + \Gamma)^2} + \ldots
\]

\[
W_4 = \frac{2(\omega^2 + \Gamma^2)}{(1 - \frac{1}{2})(\omega^2 + \Gamma^2)} - \frac{2(\omega + \Gamma)^2}{(1 - \frac{1}{2})(\omega + \Gamma)^2} + \ldots
\]
\[z_1 = \frac{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2) + \xi + \xi - [-\epsilon + i \epsilon]}{(4 \Gamma_0 + 2 \epsilon_1 + 2 \epsilon_2 + 2 \epsilon_3 + 2 \epsilon_4 + 2 \epsilon_5 + 2 \epsilon_6 + 2 \epsilon_7 + 2 \epsilon_8)} - \frac{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2) + \xi + \xi - [-\epsilon + i \epsilon]}{(4 \Gamma_0 - 2 \epsilon_1 - 2 \epsilon_2 - 2 \epsilon_3 - 2 \epsilon_4 - 2 \epsilon_5 - 2 \epsilon_6 - 2 \epsilon_7 - 2 \epsilon_8)} \]

\[z_2 = \frac{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2) - \xi - \xi + [-\epsilon + i \epsilon]}{(4 \Gamma_0 + 2 \epsilon_1 + 2 \epsilon_2 + 2 \epsilon_3 + 2 \epsilon_4 + 2 \epsilon_5 + 2 \epsilon_6 + 2 \epsilon_7 + 2 \epsilon_8)} - \frac{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2) - \xi - \xi + [-\epsilon + i \epsilon]}{(4 \Gamma_0 - 2 \epsilon_1 - 2 \epsilon_2 - 2 \epsilon_3 - 2 \epsilon_4 - 2 \epsilon_5 - 2 \epsilon_6 - 2 \epsilon_7 - 2 \epsilon_8)} \]

\[z_3 = \frac{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2) + \xi - \xi + [-\epsilon + i \epsilon]}{(4 \Gamma_0 + 2 \epsilon_1 + 2 \epsilon_2 + 2 \epsilon_3 + 2 \epsilon_4 + 2 \epsilon_5 + 2 \epsilon_6 + 2 \epsilon_7 + 2 \epsilon_8)} - \frac{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2) + \xi - \xi + [-\epsilon + i \epsilon]}{(4 \Gamma_0 - 2 \epsilon_1 - 2 \epsilon_2 - 2 \epsilon_3 - 2 \epsilon_4 - 2 \epsilon_5 - 2 \epsilon_6 - 2 \epsilon_7 - 2 \epsilon_8)} \]

The Hall conductivity is then given by

\[\sigma_{xy} = \frac{2 e^2 \mathcal{L} \rho(E)}{\pi} \int_{-\infty}^{\infty} dw \frac{1}{4 \cosh \frac{w}{2 T}} \left[ X_H + \frac{1}{2 \epsilon_0} Y_H + \frac{6 \epsilon_0 \rho \lambda^2}{\epsilon} Z_H \right] \]

with

\[X_H = a_1 / 2 - a'_1, \quad Y_H = b_1 / 2 - (b'_1 + b'_2), \quad Z_H = c_1 / 2 - c'_1, \]

where

\[a_1 = \frac{2(1+\phi)}{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2)} - \frac{\omega^2 + 1}{(4 \Gamma_0 + 2 \epsilon_1 + 2 \epsilon_2 + 2 \epsilon_3 + 2 \epsilon_4 + 2 \epsilon_5 + 2 \epsilon_6 + 2 \epsilon_7 + 2 \epsilon_8)} + \frac{2(1+\phi)}{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2)} - \frac{\omega^2 + 1}{(4 \Gamma_0 - 2 \epsilon_1 - 2 \epsilon_2 - 2 \epsilon_3 - 2 \epsilon_4 - 2 \epsilon_5 - 2 \epsilon_6 - 2 \epsilon_7 - 2 \epsilon_8)} \]

\[b_1 = \mu_1 \psi\left(-\frac{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2)}{2 \epsilon_0} + \phi\right) - \mu_3 \psi\left(-\frac{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2)}{2 \epsilon_0} - \phi\right) - \mu_5 \psi\left(-\frac{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2)}{2 \epsilon_0} + \phi\right) + \mu_4 \psi\left(-\frac{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2)}{2 \epsilon_0} - \phi\right) \]

\[c_1 = v_1 \psi\left(-\frac{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2)}{2 \epsilon_0} + \phi\right) - v_2 \psi\left(-\frac{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2)}{2 \epsilon_0} - \phi\right) + v_4 \psi\left(-\frac{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2)}{2 \epsilon_0} + \phi\right) - v_6 \psi\left(-\frac{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2)}{2 \epsilon_0} - \phi\right) \]

\[+ v_7 \psi\left(-\frac{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2)}{2 \epsilon_0} + \phi\right) + v_3 \psi\left(-\frac{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2)}{2 \epsilon_0} - \phi\right) + v_5 \psi\left(-\frac{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2)}{2 \epsilon_0} + \phi\right) - v_6 \psi\left(-\frac{(\omega+i \gamma)^2 - (10 \epsilon_0^2, 2 \epsilon_1^2, 2 \epsilon_2^2)}{2 \epsilon_0} - \phi\right) \]
\[ a'_1 = a_0 + a_1 \ln((\omega + i\Gamma)^2 - (10.1^2 + 2.1^2) - \zeta - x|eB|) + a_2 \ln((\omega + i\Gamma)^2 - (10.1^2 + 2.1^2) - \zeta + x|eB|) + a_3 \ln((\omega + i\Gamma)^2 - (10.1^2 + 2.1^2) + \zeta), \]

(B11)

\[ b'_1 = \beta_1 \ln \Gamma(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) + \zeta}{2|eB|}) + \beta_2 \psi(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) + \zeta}{2|eB|}) + \beta_3 \phi(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) + \zeta}{2|eB|}), \]

(B12)

\[ b'_2 = \beta_4 \ln \Gamma(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) - \zeta - x|eB|}{2|eB|}) + \beta_5 \psi(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) - \zeta - x|eB|}{2|eB|}) - \beta_6 \ln \Gamma(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) - \zeta - x|eB|}{2|eB|}) - \beta_7 \ln \Gamma(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) - \zeta - x|eB|}{2|eB|}) + \beta_8 \psi(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) - \zeta - x|eB|}{2|eB|}) + \beta_9 \phi(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) - \zeta - x|eB|}{2|eB|}), \]

(B13)

\[ c'_1 = \gamma_1 \ln \Gamma(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) + \zeta + x|eB|}{2|eB|}) + \gamma_2 \ln \Gamma(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) + \zeta + x|eB|}{2|eB|}) + \gamma_9 \psi(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) + \zeta + x|eB|}{2|eB|}) + \gamma_10 \psi(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) + \zeta + x|eB|}{2|eB|}) + \gamma_11 \psi(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) + \zeta + x|eB|}{2|eB|}) + \gamma_12 \psi(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) + \zeta + x|eB|}{2|eB|}) + \gamma_13 \psi(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) + \zeta + x|eB|}{2|eB|}) + \gamma_14 \psi(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) + \zeta + x|eB|}{2|eB|}) + \gamma_15 \psi(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) + \zeta + x|eB|}{2|eB|}) + \gamma_16 \psi(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) + \zeta + x|eB|}{2|eB|}) + \gamma_17 \psi(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) + \zeta + x|eB|}{2|eB|}) + \gamma_18 \psi(-\frac{(\omega+i\Gamma)^2 - (10.1^2 + 2.1^2) + \zeta + x|eB|}{2|eB|}), \]

(B14)

with \( \phi(z) = \int_0^z \ln \Gamma(x) \, dx \), and

\[ \mu_1 = \frac{2(1 + \frac{1}{2}(\omega^2 + \Gamma^2))}{-4\omega^2 - 2\zeta + x|eB| - 2|eB|} - \frac{2(1 - \frac{1}{2}(\omega^2 + \Gamma^2))}{-4\omega^2 - 2\zeta - x|eB| - 2|eB|} - \frac{2(1 - \frac{1}{2}(\omega^2 + \Gamma^2))}{-4\omega^2 - 2\zeta + x|eB| - 2|eB|}, \]

(B15)

\[ \mu_2 = \frac{2(1 + \frac{1}{2}(\omega^2 + \Gamma^2))}{-4\omega + 2\zeta + x|eB| + 2|eB|}, \]

(B16)

\[ \mu_3 = \frac{2(1 - \frac{1}{2}(\omega^2 + \Gamma^2))}{-4\omega + 2\zeta - x|eB| + 2|eB|}, \]

(B17)

\[ \mu_4 = \frac{2(1 + \frac{1}{2}(\omega^2 + \Gamma^2))}{-4\omega + 2\zeta - x|eB| + 2|eB|}, \]

B18)

\[ \mu_5 = \frac{2(1 - \frac{1}{2}(\omega^2 + \Gamma^2))}{-4\omega + 2\zeta + x|eB| - 2|eB|}, \]

(B19)

\[ v_1 = \frac{2(\omega^2 + (10.1^2 + 2.1^2)^2 + \zeta + x|eB|)}{-4\omega + 2\zeta - 2|eB|}, \]

(B20)

\[ v_2 = \frac{2(\omega^2 + (10.1^2 + 2.1^2)^2 + \zeta - x|eB|)}{-4\omega + 2\zeta + 2|eB|}, \]

(B21)
\[ \begin{align*}
V_5 &= \frac{2(\omega+\Gamma)^2-(10\lambda+\frac{2\lambda}{\zeta}+\zeta\chi+eB)}{-4\lambda \omega^2-2\zeta^2-2\zeta\xi}, \\
V_6 &= \frac{2(\omega+\Gamma)^2-(10\lambda+\frac{2\lambda}{\zeta}+\zeta\chi+eB)}{-4\lambda \omega^2-2\zeta^2-2\zeta\xi}, \\
V_7 &= \frac{2(\omega+\Gamma)^2-(10\lambda+\frac{2\lambda}{\zeta}+\zeta\chi+eB)}{-4\lambda \omega^2-2\zeta^2-2\zeta\xi}, \\
V_8 &= \frac{2(\omega+\Gamma)^2-(10\lambda+\frac{2\lambda}{\zeta}+\zeta\chi+eB)}{-4\lambda \omega^2-2\zeta^2-2\zeta\xi}.
\end{align*} \]
\[
\begin{align*}
\gamma_1 &= \frac{4eB[(\omega+\Gamma)^2-(10\lambda_2+2\lambda_4)+(2\xi-2\xi_e)B]}{(-2\xi+2\xi_e B)^2}, \\
\gamma_2 &= \frac{(\omega+\Gamma)^2-(10\lambda_2+2\lambda_4)+(2\xi-2\xi_e)B}{(-2\xi+2\xi_e B)^2}, \\
\gamma_3 &= \frac{8eB^2}{(-2\xi+2\xi_e B)^2}, \\
\gamma_4 &= \frac{4eB[(\omega+\Gamma)^2-(10\lambda_2+2\lambda_4)+(2\xi-2\xi_e)B]}{(-2\xi+2\xi_e B)^2}, \\
\gamma_5 &= \frac{(\omega+\Gamma)^2-(10\lambda_2+2\lambda_4)+(2\xi-2\xi_e)B}{(-2\xi+2\xi_e B)^2}, \\
\gamma_6 &= \frac{8eB^2}{(-2\xi+2\xi_e B)^2}, \\
\gamma_7 &= \frac{4eB[(\omega+\Gamma)^2-(10\lambda_2+2\lambda_4)+(2\xi-2\xi_e)B]}{(-2\xi+2\xi_e B)^2}, \\
\gamma_8 &= \frac{(\omega+\Gamma)^2-(10\lambda_2+2\lambda_4)+(2\xi-2\xi_e)B}{(-2\xi+2\xi_e B)^2}, \\
\gamma_9 &= \frac{8eB^2}{(-2\xi+2\xi_e B)^2}, \\
\gamma_{10} &= \frac{4eB[(\omega+\Gamma)^2-(10\lambda_2+2\lambda_4)+(2\xi-2\xi_e)B]}{(-2\xi+2\xi_e B)^2}, \\
\gamma_{11} &= \frac{8eB^2}{(-2\xi+2\xi_e B)^2}, \\
\gamma_{12} &= \frac{(\omega+\Gamma)^2-(10\lambda_2+2\lambda_4)+(2\xi-2\xi_e)B}{(-2\xi+2\xi_e B)^2}, \\
\gamma_{13} &= \frac{4eB[(\omega+\Gamma)^2-(10\lambda_2+2\lambda_4)+(2\xi-2\xi_e)B]}{(-2\xi+2\xi_e B)^2}, \\
\gamma_{14} &= \frac{8eB^2}{(-2\xi+2\xi_e B)^2}, \\
\gamma_{15} &= \frac{(\omega+\Gamma)^2-(10\lambda_2+2\lambda_4)+(2\xi-2\xi_e)B}{(-2\xi+2\xi_e B)^2}, \\
\gamma_{16} &= \frac{4eB[(\omega+\Gamma)^2-(10\lambda_2+2\lambda_4)+(2\xi-2\xi_e)B]}{(-2\xi+2\xi_e B)^2}, \\
\gamma_{17} &= \frac{8eB^2}{(-2\xi+2\xi_e B)^2}, \\
\gamma_{18} &= \frac{(\omega+\Gamma)^2-(10\lambda_2+2\lambda_4)+(2\xi-2\xi_e)B}{(-2\xi+2\xi_e B)^2},
\end{align*}
\]
The Eqs. (B2) and (B6) are further rewritten as Eqs. (15) and (16).