n-BLOCKS COLLECTIONS ON FANO MANIFOLDS AND SHEAVES WITH REGULARITY $-\infty$

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ABSTRACT. Let $X$ be a smooth Fano manifold equipped with a “nice” $n$-blocks collection in the sense of [3] and $\mathcal{F}$ a coherent sheaf on $X$. Assume that $X$ is Fano and that all blocks are coherent sheaves. Here we prove that $\mathcal{F}$ has regularity $-\infty$ in the sense of [3] if $\text{Supp}(\mathcal{F})$ is finite, the converse being true under mild assumptions. The corresponding result is also true when $X$ has a geometric collection in the sense of [2].

1. Introduction

Let $X$ be an $n$-dimensional smooth projective variety over $\mathbb{C}$. Let $\mathcal{D} := \mathcal{D}^b(\mathcal{O}_X - \text{mod})$ denote the bounded category of $\mathcal{O}_X$-sheaves. Let $\mathcal{F}$ be a coherent sheaf on $X$. Assume that $X$ has a geometric collection in the sense of [2] or an $n$-blocks collection in the sense of [3]. L. Costa and R. M. Miró-Roig defined the notion of regularity for $\mathcal{F}$ and asked a characterization of all $\mathcal{F}$ whose regularity is $-\infty$ ([2], Remark 3.3). In section 2 we will recall the definitions contained in [2] and [3] and used in our statements below. After the statements we will discuss our motivations and give a very short list of interesting varieties to which these results may be applied.

We prove the following results.

**Theorem 1.** Assume that $X$ is Fano and that it has an $n$-blocks collection $\mathcal{B}$ whose members are coherent sheaves. Let $\mathcal{F}$ be a coherent sheaf on $X$. If $\mathcal{F}$ has regularity $-\infty$ with respect to $\mathcal{B}$, then $\text{Supp}(\mathcal{F})$ is finite. If all right mutations of all elements of $\mathcal{B}$ are locally free and $\text{Supp}(\mathcal{F})$ is finite, then $\mathcal{F}$ has regularity $-\infty$ with respect to $\mathcal{B}$. [1]

**Corollary 1.** Assume that $X$ is Fano and that it has a geometric collection $\mathcal{G}$ whose members are coherent sheaves. Let $\mathcal{F}$ be a coherent sheaf on $X$. If $\mathcal{F}$ has regularity $-\infty$ with respect to $\mathcal{G}$, then $\text{Supp}(\mathcal{F})$ is finite. If all right mutations of all elements of $\mathcal{G}$ are locally free and $\text{Supp}(\mathcal{F})$ is finite, then $\mathcal{F}$ has regularity $-\infty$ with respect to $\mathcal{G}$.

We recall that any projective manifold with a geometric collection is Fano ([2], part (2) of Remark 2.16). Any $n$-dimensional smooth quadric $Q_n \subset \mathbb{P}^{n+1}$ has an $n$-block collection whose members are locally free ([3], Example 3.2 (2)). It has a geometric collection if and only if $n$ is odd. Any Grassmannian $G$ has an $n$-block collection (with $n := \dim(G)$) whose members are locally free sheaves ([3], Example 3.2 (2))

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3.7 (4)). For the Fano 3-folds $V_5$ and $V_{22}$ D. Faenzi found a geometric collection whose members are locally free ([4], [3]).

Castelnuovo-Mumford regularity was introduced by Mumford in [8], Lecture 14, for a coherent sheaf $\mathcal{F}$ on $\mathbf{P}^n$. He ascribed the idea to Castelnuovo for the following reason. Let $C \subset \mathbf{P}^n$ be a closed subvariety and $H \subset \mathbf{P}^n$ be a general hyperplane. Then we have an exact sequence

$$0 \to \mathcal{I}_C(t-1) \to \mathcal{I}_C(t) \to \mathcal{I}_{C\cap H}(t) \to 0$$

(1)

Castelnuovo used the corresponding classical (pre-sheaves) concepts of linear systems to get informations on $C$ from informations on $C \cap H$ plus other geometrical or numerical assumptions on $C$. The key properties of Castelnuovo-Mumford regularity is that if $\mathcal{F}$ is $m$-regular, then it is $(m+1)$-regular and $\mathcal{F}(m)$ (or $\mathcal{I}_C(m)$) is spanned. Since [8] several hundred papers studied this notion, which is now also a key property in computational algebra. Let $X$ be a projective scheme, $H$ an ample line bundle on $X$ and $\mathcal{F}$ a coherent sheaf on $X$. The definition in [8], Lecture 14, apply verbatim, just writing $\mathcal{F} \otimes \mathcal{O}^{\otimes t}$ instead of $\mathcal{F}(t)$. This is also called Castelnuovo-Mumford regularity with respect to the polarized pair $(X, H)$. $X$ may have several non-proportional polarizations. It is better to collect all informations for all polarizations in a single integer (the regularity) not in a string of integers, one for each proportional class of polarizations on $X$. This is the reason for the definitions given by Hoffman-Wang for products of projective varieties ([3]) and by Maclagan and Smith for toric varieties ([7]). Even when $X$ has only one polarization the search for generalizations of Beilinson’s spectral sequence from $\mathbf{P}^n$ to $X$ gave a strong motivation to introduce the notions of regularities for geometric collections ([2], Th. 2.21) and $n$-block collections ([3], Th. 3.10). The reader will notice that to prove Theorem 1 and Corollary 1 we will use neither the main definitions of [2] and [3] nor the machinery of derived categories. We will only use the formal properties (like ”spannedness” or ”$m$-regularity implies $(m+1)$-regularity”) proved in [2] and [3] (see eq. 2 in section §2 for an explanation of the word ”spannedness”). We hope that our results will be extended and used if other notions of regularity will appear in the literature.

### 2. The main definitions and the proofs

Let $X$ be an $n$-dimensional smooth projective variety over $\mathbb{C}$. Let $\mathcal{D} := \mathcal{D}^b(\mathcal{O}_X - \text{mod})$ denote the bounded category of $\mathcal{O}_X$-sheaves. For all objects $A, B \in \mathcal{D}$ set $\text{Hom}^*(A, B) := \oplus_{k \in \mathbb{Z}} \text{Ext}^k_{\mathcal{D}}(A, B)$. An object $A \in \mathcal{D}$ is said to be exceptional if $\text{Hom}^*(A, A)$ is a 1-dimensional algebra generated by the identity. An ordered collection $(A_0, \ldots, A_m)$ of objects of $\mathcal{D}$ will be called an exceptional collection if each $A_i$ is exceptional and $\text{Ext}^*_{\mathcal{D}}(A_i, A_j) = 0$ for all $0 \leq j < k \leq m$. A collection $(A_0, \ldots, A_m)$ is said to be strongly exceptional if it is exceptional and $\text{Ext}^k_{\mathcal{D}}(A_j, A_k) = 0$ for all $(i, j, k)$ such that $i \neq 0$ and $j \leq k$. A collection $(A_0, \ldots, A_m)$ is said to be full if it generates $\mathcal{D}$. This implies $\mathcal{D} \cong \mathbb{Z}^{\oplus (m+1)}$. Now assume that $X$ admits a fully exceptional collection $\sigma = (A_0, \ldots, A_n)$. For any $A, B \in \mathcal{D}$ the right mutation $R_B A$ of $A$ and the left mutation $L_A B$ of $B$ are defined by the following distinguished triangles

$$R_B A[-1] \to A \to \text{Hom}^*(A, B) \otimes B \to R_B A$$

$$L_A B \to \text{Hom}^*(A, B) \otimes A \to L_A B[1]$$
(2, Definition 2.4). For every integer \( i \) such that \( 1 \leq i \leq n \), define the \( i \)-th right mutation \( R_i \sigma \) and the \( i \)-th left mutation \( L_i \sigma \) of \( \sigma \) by the formulas

\[
R_i \sigma := (A_0, \ldots, A_{i-2}, A_i, R_{A_i} A_{i+1}, \ldots, A_n)
\]

\[
L_i \sigma := (A_0, \ldots, A_{i-2}, L_{A_{i-1}} A_i, A_{i-1}, A_{i+1}, \ldots, A_n)
\]

(a switch of two elements of \( \sigma \) and the application to one of them of a right or left mutation) (2, Definition 2.6). For any \( j \geq 2 \), set \( R^{(j)} \sigma := R A_{i-j} \cdots \circ R_{A_{i+1}} A_i \in \mathcal{D} \) and define in a similar way the iterated left mutations \( L^{(i)} \) (2, Nota-

tion 2.7). Set \( A_{n+i} := R^{(n)} A_{i-1} \) for all \( 0 \leq i \leq n \) and \( A_{-i} := L^{(n)} A_{n-i+1} \) for all \( 1 \leq i \leq n \). Iterating the use of \( R^{(n)} \) and \( L^{(n)} \), we get the elix \( \{A_i\}_{t \in \mathbb{Z}} \) with \( A_i \in \mathcal{D} \) for all \( i \) (2, Definition 2.12). For instance, if \( X = \mathbb{P}^n \), then \( (A_0, \ldots, A_n) := (\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \ldots, \mathcal{O}_{\mathbb{P}^n}(n)) \) is a geometric collection and \( \{\mathcal{O}_{\mathbb{P}^n}(t)\}_{t \in \mathbb{Z}} \) is the corre-

sponding elix. Let \( \mathcal{F} \) be a coherent sheaf on \( X \). \( \mathcal{F} \) is said to be \( m \)-regular with respect to the geometric collection \( \sigma = (A_0, \ldots, A_n) \) if \( \text{Ext}^q(R^{(n)} A_{m+p}, \mathcal{F}) = 0 \) for all integers \( q, p \) such that \( q > 0 \) and \( -n \leq p \leq 0 \). The regularity of \( \mathcal{F} \) is the minimal integer \( m \) such that \( \mathcal{F} \) is \( m \)-regular (or \( -\infty \) if it is \( m \)-regular for all \( m \in \mathbb{Z} \)). An exceptional collection \( (A_0, \ldots, A_n) \) of elements of \( \mathcal{D} \) is called a block if \( \text{Ext}_{\mathcal{D}}^i(A_j, A_k) = 0 \) for all \( i, j, k \) such that \( k \neq j \). An \( m \)-block collection of elements of \( \mathcal{D} \) is an exceptional collection which may be partitioned into \( m+1 \) consecutive blocks. Assume that \( X \) has an \( n \)-block collection whose elements generate \( \mathcal{D} \). Let \( \mathcal{F} \) be a coherent sheaf on \( X \). In 3, Definition 4.5, there is a definition of regularity of \( \mathcal{F} \); it requires only technical modifications with respect to the simpler case of a geometric collection: they gave similar definitions of left and right mutations and elices. Then the definition of \( m \)-regularity is again given by certain Ext-vanishings.

If a coherent sheaf \( \mathcal{F} \) is \( m \)-regular with respect to a geometric collection \( \sigma \) or an \( n \)-block collection \( \sigma \), then it gives a resolution

\[
0 \to \mathcal{L}_{-n} \to \cdots \to \mathcal{L}_{-1} \to \mathcal{L}_0 \to \mathcal{F} \to 0
\]

in which each \( \mathcal{L}_i \in \mathcal{D} \) is constructed from \( \mathcal{F} \) and the elements of \( \sigma \) taking tensor products (2, between 3.1 and 3.2 for geometric collections, 3, eq. (4.2), for \( n \)-blocks). If the elements of \( \sigma \) are coherent sheaves (resp. locally free coherent sheaves), then each \( \mathcal{L}_i \) is a coherent sheaf (resp. a locally free coherent sheaf). In the case of Castelnuovo-Mumford regularity the corresponding result is true. It shows how the Castelnuovo-Mumford regularity bounds the degrees of the syzygies. This is the key reason for its use in computational algebra.

The following well-known result answers the corresponding problem for Castelnuovo-

Mumford regularity.

**Lemma 1.** Let \( X \) be a projective scheme, \( L \) an ample line bundle on \( X \) and \( \mathcal{F} \) a coherent sheaf on \( X \). The following conditions are equivalent:

(a) \( \mathcal{F} \) is supported by finitely many points of \( X \);
(b) \( \mathcal{F} \otimes L^\otimes t \) is spanned for all \( t \ll 0 \);
(c) \( \text{h}^i(X, \mathcal{F} \otimes L^\otimes t) = 0 \) for all \( i > 0 \) and all \( t \in \mathbb{Z} \).

**Proof.** Obviously, (a) implies (b) and (c). Now assume that (b) holds, but that \( \text{dim}(\text{Supp}(\mathcal{F})) > 0 \). Take an integral projective curve \( C \subseteq \text{Supp}(\mathcal{F}) \). Since the restriction of a spanned sheaf is spanned, \( \mathcal{F}|C \) satisfies (c) with respect to the ample line bundle \( R := L|C \). Let \( f : D \to C \) be the normalization. Set \( M := f^*(R) \). \( M \) is an ample line bundle on \( D \). Since \( D \) is a smooth curve, the coherent sheaf \( f^*(\mathcal{F}) \)
is either a torsion sheaf or the direct sum of a torsion sheaf $T$ and a vector bundle $E$ with positive rank. To prove (a) we must check that $f^*(F)$ is torsion. Assume $E \neq 0$. Since the pull-back of a spanned sheaf is spanned, $E \otimes M^\otimes t$ is spanned for all $t \in \mathbb{Z}$. Since $\deg (E \otimes R^\otimes t) = \deg (E) + t \cdot \operatorname{rank}(E) \cdot \deg (M) < 0$ for $t \ll 0$, $E \otimes R^\otimes t$ is not spanned for $t \ll 0$, contradiction. Let $x \geq 1$ be an integer such that $L^\otimes x$ is very ample. If $F$ satisfies (c) for the line bundle $L$, then it satisfies the same condition for the line bundle $L' := L^\otimes x$. Hence to check that (c) implies (a) we may assume that $L$ is very ample. Fix an integer $t$. Since $h^i(X, F \otimes L^{\otimes (t-1-1)}) = 0$ for all $i > 0$, $F \otimes L^{\otimes t}$ is spanned ([8], p. 100). Thus (b) holds and hence (a) holds. \qed

Proof of Theorem [7]. Fix a coherent sheaf $F$. Let $E$ be the helix of blocks generated by $B$ ([3], Definition 4.1). All elements of $E$ are coherent sheaves, not just complexes ([3], Corollary 4.4) and their elements satisfy a periodicity modulo $n + 1$: $\mathcal{E}_i = \mathcal{E}_{i+n+1} \otimes \omega_X$ ([3], lines between 4.3 and 4.4). First assume that $F$ has regularity $-\infty$ with respect to $B$, i.e. that it is $m$-regular with respect to $B$ for all $m \ll 0$. Fix $m \in \mathbb{Z}$. The $m$-regularity of $F$ implies that it is a quotient of a finite sum $L_0$ of sheaves of the form $E_{-m}^r$ appearing in the blocks of $B$ ([3], Definition 4.5). Since $F$ is $t$-regular for all $t \ll 0$, the periodicity property of $E$ shows that for all integers $t \leq 0$, $F$ is a quotient of a finite direct sum of sheaves of the form $L_0 \otimes \omega_X^{\otimes (-t)}$. Since $X$ is Fano, $\omega_X^\ast$ is ample. Take $\mathcal{L} := \omega_X^\ast$ and copy the proof that (b) implies (a) in Lemma [1]. We get that $\text{Supp}(F)$ is finite.

Now assume that $\text{Supp}(F)$ is finite and that all right mutations of elements of $B$ are locally free. Let $A$ be any of these mutations. Since $A$ is locally free, the local Ext-functors $\operatorname{Ext}^t(A, F)$ vanish for all $i > 0$. Hence the local-to-global spectral sequence for the Ext-functors gives $\operatorname{Ext}^i(A, F) \cong H^i(X, \operatorname{Hom}(A, F))$ for all $i \geq 0$. Since $\text{Supp}(F)$ is finite, we get $\operatorname{Ext}^i(A, F) = 0$ for all $q > 0$. Hence for every integer $m$ the sheaf $F$ satisfies the definition of $m$-regularity given in [3], Definition 4.5. Since $F$ is $m$-regular with respect to $B$ for all $m$, its regularity is $-\infty$. \qed

Proof of Corollary [1]. This result is a particular case of Theorem [1] because the definition of regularity for geometric collections given in [2] agrees with the definition of regularity for $n$-blocks collections given in [3] (see [3], Remark 4.7). It may be proved directly, just quoting [2], Remark 2.14, to get the periodicity property $\mathcal{E}_i = \mathcal{E}_{i+n+1} \otimes \omega_X$ and [2], Proposition 3.8, to get the surjection $\mathcal{L}_0 \rightarrow \mathcal{F}$. \qed

Remark 1. In [1] J. V. Chipalkatti defined a notion of regularity for a coherent sheaf $F$ on a Grassmannian. He remarked that $F$ have regularity $-\infty$ (according to his definition) if and only if its support is finite ([1], part 4) of Remark 1.2.

Remark 2. Let $F$ be a coherent sheaf on $\mathbb{P}^n \times \mathbb{P}^m$. Hoffman and Wang introduced a bigraded definition of regularity ([6]). The definition of ampleness and [7], Prop. 2.8, imply that if $F$ is $(a,b)$-regular in the sense of Hoffman-Wang for all $(a,b) \in \mathbb{Z} \times \mathbb{Z}$, then $\text{Supp}(F)$ is finite. The converse is obvious. As remarked in [3], Remark 5.2, Hoffman-Wang definition and its main properties may be extended verbatim to arbitrary multiprojective spaces $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_s}$.

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