OBSTRUCTIONS FOR SYMPLECTIC LIE ALGEBROIDS

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Abstract. Several generically-nondegenerate Poisson structures can be effectively studied as symplectic structures on naturally associated Lie algebroids. Relevant examples of this phenomenon include log-, elliptic, $b^k$-, scattering and elliptic-log Poisson structures.

In this paper we discuss topological obstructions to the existence of such Poisson structures (obtained through their symplectic Lie algebroids) of several different flavors, namely coming from cohomology, characteristic classes, and Seiberg–Witten theory.

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1. Introduction

Generically-nondegenerate Poisson structures have recently seen an intense increase in interest. The main reason for this has been the ability to effectively study them using Lie algebroids. Namely, in several instances it is possible to, given a Poisson structure $\pi \in \text{Pois}(X)$, define a Lie algebroid $\mathcal{A} \to X$ adhering to the same mild degeneracies as $\pi$, such that $\pi$ is in a precise sense dual to a symplectic structure in $\mathcal{A}$, i.e. a closed nondegenerate $\mathcal{A}$-two-form.

Symplectic Lie algebroids were first considered in [36], and have more recently been studied especially when the anchor map $\rho_A: \mathcal{A} \to TX$ is generically an isomorphism. This class includes log- [10, 17, 19, 28, 29], elliptic [8, 11], $b^k$- [18, 34, 35, 38] and scattering symplectic structures [26]. Through the use of symplectic Lie algebroids, powerful symplectic techniques can be brought to bear to study the associated Poisson structures, leading to various results.

In this paper we are interested in obtaining obstructions to the existence of a symplectic structure on a Lie algebroid, and thus to their underlying Poisson structures. The obstructions we present here have three different flavors: they are of cohomological, homotopical, and gauge theoretical nature. We use the language of divisors as we developed in [22] (see also [8, 11]) to describe such Poisson structures and their associated symplectic Lie algebroids. While we focus in this paper primarily on Lie algebroids and their symplectic structures, these should be thought of as tools to make statements about interesting classes of Poisson structures.

The remainder of the introduction describes each of these types of obstructions, including the known cohomological obstructions, and followed by stating the results of this paper.

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1.1. **Cohomological obstructions.** A simple obstruction to the existence of a symplectic structure is of cohomological nature. There are analogues of this for two symplectic Lie algebroids [10, 29, 11]. We recall in Section 3 these results (and their methods of proof) for log- and elliptic symplectic structures. Let $X^{2n}$ be an oriented compact manifold. Then:

- a symplectic structure leads to a class $c \in H^2(X; \mathbb{R})$ such that $c^n \neq 0$.
- a log-symplectic structure leads to classes $a, b \in H^2(X; \mathbb{R})$ such that $a^{n-1}b \neq 0$, $b^2 = 0$;
- elliptic structures give $a \in H^2(X; D; \mathbb{R})$ and $b \in H^2_c(X; D; \mathbb{R})$ with $a^{n-1}b \neq 0$, $b^2 = 0$.

The first of these is standard, the second combines [29, 10], and the third is due to [11], and requires that the elliptic symplectic structure has zero elliptic residue and $D \subseteq X$.

Note that the latter cannot be stated without also specifying the degeneracy locus $D \subseteq X$.

1.2. **Homotopical obstructions.** A symplectic manifold inherits a natural orientation, and is further almost-complex. An analogous statement holds for any symplectic Lie algebroid $\mathcal{A} \rightarrow X$ (see Proposition 4.1), or indeed any symplectic vector bundle, where there need not be an integrability condition. The existence of an orientation and complex structure on $\mathcal{A}$ is determined by the underlying vector bundle, and is obstructed by its characteristic classes.

Indeed, a vector bundle $\mathcal{A} \rightarrow X$ is orientable if and only if $w_1(\mathcal{A}) = 0$. Making use of this we obtain the following obstructions for $\mathcal{A}$-symplectic structures, for concrete Lie algebroids:

**Theorem A.** Let $\mathcal{A} \rightarrow X^n$ be a symplectic Lie algebroid. Then in $H_1(X; \mathbb{Z}_2)$ we have:

- $w_1(TX) + kPD_{\mathbb{Z}_2}[Z] = 0$ if $\mathcal{A} = A^k_Z$, the $b^k$-tangent bundle;
- $w_1(TX) + nPD_{\mathbb{Z}_2}[Z] = 0$ if $\mathcal{A} = B_Z$, the zero tangent bundle;
- $w_1(TX) + (n+1)PD_{\mathbb{Z}_2}[Z] = 0$ if $\mathcal{A} = C_Z$, the scattering tangent bundle;
- $w_1(TX) = 0$ if $\mathcal{A} = A_D$, the elliptic tangent bundle;
- $w_1(TX) + PD_{\mathbb{Z}_2}[Z] = 0$ if $\mathcal{A} = A_W$, the elliptic-log tangent bundle.

Here $w_1$ is the first Stiefel–Whitney class, and $PD_{\mathbb{Z}_2}$ is the Poincaré dual with $\mathbb{Z}_2$-coefficients.

This result can be found in the main text as Theorem 4.15, and the Lie algebroids that are mentioned, as well as our notation for them, are discussed in Section 2.2. Due to explicit use of the Lie algebroid as a vector bundle, these depend on the hypersurface $Z \subseteq X$. Note moreover that this result provides the full obstruction for a surface to be $\mathcal{A}$-symplectic. This latter statement is because the integrability condition (closedness) is immediate, so that only a nondegenerate $\mathcal{A}$-two-form is required, which exists if and only if $\mathcal{A}$ satisfies $w_1(\mathcal{A}) = 0$.

We further determine a more intricate obstruction for four-dimensional $b^k$-symplectic manifolds, making use of the required complex structure on the Lie algebroid (see Theorem 4.19).

**Theorem B.** Let $(X^4, Z)$ be a compact oriented $b^k$-symplectic four-manifold. Then:

$$b^+_2(X) + b_1(X) + kf(X, Z) \text{ is odd.}$$

Here $b_1(X)$ is the first Betti number of $X$, $b^+_2(X)$ is the dimension of a maximal positive definite subspace on $H^2(X; \mathbb{R})$, and $f(X, Z)$ is the discrepancy of the oriented pair $(X, Z)$.

In the above, a $b^k$-symplectic structure is a symplectic structure for $A^k_Z$, the $b^k$-tangent bundle [38], so that a $b^1$-symplectic structure is log-symplectic. Moreover, $kf(X, Z) \in \mathbb{Z}$ is (essentially) the discrepancy of $A^k_Z$ and $TX$, which we will introduce here in Definition 4.16. There is a similar result for scattering-symplectic manifolds (see Theorem 4.24).

**Theorem C.** Let $(X^4, Z)$ be a compact oriented scattering-symplectic manifold. Then:

$$b^+_2(X) + b_1(X) + f(X, Z) \text{ is odd.}$$
Here \( b_1(X) \) is the first Betti number of \( X \), \( b_2^+(X) \) is the dimension of a maximal positive definite subspace on \( H^2(X;\mathbb{R}) \), and \( f(X,Z) \) is the discrepancy of the oriented pair \((X,Z)\).

This obstruction in the scattering-symplectic case is identical to the log-symplectic case.

1.3. **Obstructions from Seiberg–Witten theory.** A useful remark of Osorno–Torres ([37, Lemma 4.2.6]) allows us to transform questions regarding oriented log-symplectic manifolds into those for symplectic manifolds whose boundaries are of cosymplectic-type (see Section 5). Consequently, powerful results from Seiberg-Witten theory can be used ([25, 39, 40], and [5, 3]) to obstruct their existence. We use these techniques to prove the following (see Theorem 5.6).

**Theorem D.** Let \((X^4, Z)\) be an oriented log pair, splitting along \(X_1 \sqcup Z X_2\). If \(b_2^+(X_1) > 0\) and each connected component of \(Z_i\) admits a metric of positive scalar curvature (e.g. \(S^1 \times S^2\)), then \((X,Z)\) cannot be log-symplectic. Here it is allowed that \(X_2 = \emptyset\), so that \(X\) has boundary.

Similar results obtained using holomorphic curves for log-symplectic manifolds are in [1].

**Organization of the paper.** This paper is built up as follows. In Section 2 we recall some required background material, namely divisors capturing mild degeneracy, Lie algebroids built using them, and Poisson and symplectic structures having such degeneracy. In Section 3 we then discuss cohomological obstructions for these Lie algebroids to be symplectic. In Section 4 we turn to homotopical obstructions, computing various characteristic classes. Finally, in Section 5 we discuss obstructions coming from Seiberg–Witten theory in dimension four.

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2. Background material

In this section we very briefly recall required background material. Let \(X\) be a manifold.

2.1. **Divisors on smooth manifolds.** We provide a very brief primer on the language of (real) divisors on smooth manifolds. For more information, see [8, 22].

**Definition 2.1.** A (real) divisor on \(X\) is a pair \((U, \sigma)\) consisting of a real line bundle with a section \(\sigma \in \Gamma(U)\) that has nowhere dense zero set \(Z_\sigma = \sigma^{-1}(0)\). Evaluation via the map \(\sigma: \Gamma(U^+) \rightarrow C^\infty(X)\) specifies a divisor ideal \(I_\sigma \subseteq C^\infty(X)\) with support \(Z_\sigma \subseteq X\).

A divisor ideal determines a divisor up to line bundle isomorphism and multiplication by a nonvanishing smooth function, allowing us to mostly work with divisor ideals. In this paper we will use the following three examples of divisors (see [8, 11, 22]):

- **Log divisors** \((L,s)\) where \(s\) has transverse zeroes. Here \(Z := Z_s\) is a codimension-one hypersurface, and \(I_Z := I_s\) is its vanishing ideal, locally \(I_Z = \langle z \rangle\) with \(Z = \{z = 0\}\).
- **Elliptic divisors** \((R,q)\), where \(q\) has definite Hessian \(\text{Hess}(q) \in \Gamma(D;\text{Sym}^2 N^* D \otimes R)\) along its smooth codimension-two zero set \(D := Z_q\). Its divisor ideal is \(I_{|D|} := I_q\), and is locally given by \(I_{|D|} = \langle r^2 \rangle\) with \(r\) a radial distance in \(ND\);
- **Elliptic-log divisors** \((L,s) \otimes (R,q)\), obtained as the product of a log and elliptic divisor such that \(D \subseteq Z\). Its divisor ideal is \(I_W = I_Z \cdot I_{|D|}\), and locally \(I_W = \langle r^3 \cos \theta \rangle\).

We write \((X,Z)\), \((X,|D|)\) and \((X,W)\) for log, elliptic, and elliptic-log pairs respectively. We denote by \(L_Z\) the line bundle of the log divisor associated to the manifold pair \((X,Z)\). An immediate consequence of Definition 2.1 is the following, which we will use later.
Lemma 2.2. Let \((X, Z)\) be a log pair. Then we have \(w_1(L_Z) = PD_{\mathbb{Z}_2}[Z] \in H^1(X; \mathbb{Z}_2)\).

Here \(w_1\) is the first Stiefel–Whitney class, and \(PD_{\mathbb{Z}_2}\) is the Poincaré dual with \(\mathbb{Z}_2\)-coefficients. This follows because the section \(s \in \Gamma(L_Z)\) can be used to determine the Euler class of \(L_Z\).

2.2. Lie algebroids from divisors. Recall that a Lie algebroid is a vector bundle \(A \to X\) equipped with an anchor map \(\rho_A : A \to TX\) and a Lie bracket \([\cdot, \cdot]_A\) on \(\Gamma(A)\) which satisfies \([fv, w]_A = f[v, w]_A + \mathcal{L}_{\rho_A(v)} f : w\) for all \(v, w \in \Gamma(A)\) and \(f \in C^\infty(X)\). Divisor ideals are an effective tool to construct Lie algebroids generically isomorphic to \(TX\), as we now explain.

Let \(I \subseteq C^\infty(X)\) be a divisor ideal and \(\Gamma(TX)_I = \{ V \in \Gamma(TX) : \mathcal{L}_V I \subseteq I \} \subseteq \Gamma(TX)\) be the involutive submodule of vector fields preserving \(I\). When \(\Gamma(TX)_I\) is projective it specifies uniquely a Lie algebroid \(A_I \to X\) such that \(\Gamma(A_I) \cong \Gamma(TX)_I\) by the Serre–Swan theorem.

Definition 2.3 ([22]). Let \(I \subseteq C^\infty(X)\) be a divisor ideal for which \(\Gamma(TX)_I\) is projective. Then the Lie algebroid \(A_I \to X\) with \(\Gamma(A_I) \cong \Gamma(TX)_I\) is called the ideal Lie algebroid of \(I\).

In [22] these Lie algebroids are denoted by \(TX_I\). Examples of this construction include:

- The log-tangent bundle \(A_Z = TX(-\log Z)\) associated to \(I_Z\) (see [31]);
- The elliptic tangent bundle \(A_{|D|} = TX(-\log |D|)\) associated to \(I_{|D|}\) (see [11]);
- The elliptic-log tangent bundle \(A_W = TX(-\log W)\) associated to \(I_W\) (see [22]).

Note that the latter has natural morphisms onto \(A_D\) and \(A_Z\) via the section module inclusion. These Lie algebroids all have the property that their anchor \(\rho_A : A \to TX\) is an isomorphism on a dense open set, which is the complement of their degeneracy locus. In fact, the anchor map of such a Lie algebroid specifies a divisor \(\text{div}(A) = (\det(TX) \otimes \det(A^*)/\det(\rho_A))\) with divisor ideal \(I_A\). We say that a divisor ideal \(I\) is standard if its ideal Lie algebroid satisfies \(I_{A_I} = I\). Studying the above examples we see that all three given divisor ideals are standard.

The above Lie algebroids admit residue maps, which can be used to extract important information about their degeneracy loci (see [11] and the discussions in [8, 22]). We mention:

- The logarithmic residue \(\text{Res}_Z : \Omega^*(A_Z) \to \Omega^{*-1}(Z)\), given by \(d \log z \wedge \alpha + \beta \mapsto i^*_Z(\alpha)\);
- The radial residue \(\text{Res}_r : \Omega^*(A_{|D|}) \to \Omega^{*-1}(\text{At}(S^1|ND))\), given by \(d \log r \wedge d\theta \wedge \alpha + d\log r \wedge \beta + d\theta \wedge \gamma + \eta \mapsto (d\theta \wedge \alpha + \beta)|_D\), where \(\text{At}(S^1|ND) \to D\) is the Atiyah algebroid of \(S^1|ND \to D\);
- The elliptic residue \(\text{Res}_q : \Omega^*(A_{|D|}) \to \Omega^{*-2}(D; \mathfrak{t}^*)\), given by \((\iota_{\theta r} \circ \text{Res}_r)\), i.e. \(d \log r \wedge d\theta \wedge \alpha + d\log r \wedge \beta + d\theta \wedge \gamma + \eta \mapsto i^*_D(\alpha)\),

where \(\mathfrak{t}^* \cong \vee^2 ND\) is the determinant of the dual of the isotropy bundle over \(D\).

One can also obtain Lie algebroids by modifying a given Lie algebroid using a Lie subalgebroid supported on a hypersurface \(Z\). This process is called (lower) elementary modification [17, 27] or rescaling [31, 26]. This can be extended to divisor ideals \(I \subseteq C^\infty(X)\) supported on smooth submanifolds other than log ideals \(I_Z\) (see [22]), but we will not have use for this here.

Definition 2.4. Let \((X, Z)\) be a log pair and \((B, Z) \subseteq (A, X)\) a Lie subalgebroid. The lower elementary modification or \((B, Z)\)-rescaling of \(A\) along \(B\) is the Lie algebroid \([A; B]\) defined by \(\Gamma([A; B]) \equiv \{ v \in \Gamma(A) : v|_Z \in \Gamma(B)\}\).

Remark 2.5. Given a Lie algebroid \(A \to X\) one can always perform \((0, Z)\)-rescaling. The resulting Lie algebroid \([A; 0]\) is isomorphic to the tensor product \(A \otimes L_Z\) as a vector bundle.

Example 2.6. Let \((X, Z)\) be a log pair. The following are examples of modifications ([31, 32]).
allow us to define:

- The log-tangent bundle $A_Z = \{TX: TZ\}$, locally given by $\Gamma(A_Z) = \langle z\partial_z, \partial_{x_i} \rangle$;
- The zero tangent bundle $B_Z = \{TX: 0\}$, locally given by $\Gamma(B_Z) = \langle z\partial_z, z\partial_{x_i} \rangle$;
- The scattering tangent bundle $C_Z = \{A_Z: 0\}$, locally given by $\Gamma(C_Z) = \langle z^2\partial_z, z\partial_{x_i} \rangle$.

Finally, given $k \geq 1$, by fixing a $(k-1)$-jet $j \in \Gamma(X; \mathcal{L}_j^{-1}(C^\infty(X)/I_Z^k))$, where $\mathcal{L}_j : Z \to X$ is the inclusion, we can define a Lie algebroid $A_Z^j \to X$ by $\Gamma(A_Z^j) \equiv \{V \in \Gamma(TX) : L_V j \in I_Z^k\}$. This is the $b^k$-tangent bundle [38], and is locally given by $\Gamma(A_Z^j) = \langle z^k\partial_z, \partial_{x_i} \rangle$ for a local $z \in j$.

Note that when $k = 1$ the jet data is vacuous, so that $A_Z^j = A_Z$, the log-tangent bundle.

2.3. Poisson structures on Lie algebroids. Poisson structures are readily linked to divisors and the Lie algebroids built from them. Let $\pi \in \text{Poiss}(X^{2n})$ be a Poisson structure, and consider its Pfaffian, $\wedge^n \pi \in \Gamma(\text{det}(TX))$. If $\pi$ is generically nondegenerate this defines a divisor $(\text{det}(TX), \wedge^n \pi)$ and hence a divisor ideal $I_\pi$. We say $\pi$ is of $I$-divisor-type if $I_\pi = I$ (see [22]). We thus obtain, for example, the following classes of Poisson structures:

- Log-Poisson structures, which are of $I_Z$-divisor-type ([19, 29], and see below);
- Elliptic Poisson structures, which are of $I_{Z^{(1)}}$-divisor-type ([11], also [8, 23, 24]);
- Elliptic-log Poisson structures, which are of $I_{W}$-divisor-type ([22, 23]).

Poisson structures on a Lie algebroid $A \to X$ are defined as those sections $\pi_A \in \Gamma(\wedge^2 A)$ such that $[\pi_A, \pi_A] = 0$. These specify underlying Poisson structures $\pi = \rho_A(\pi_A) \in \text{Poiss}(X)$. In [22] we showed that if $\pi \in \text{Poiss}(X)$ is of $I$-divisor-type, and $I$ is such that its ideal Lie algebroid $A_I$ exists, then $\pi$ admits an $A_I$-lift: there exists a (unique) $A_I$-Poisson structure $\pi_{A_I}$ such that $\pi = \rho_{A_I}(\pi_{A_I})$. Moreover, if the divisor ideal $I$ is standard, then $\pi_{A_I}$ is nondegenerate.

2.4. Symplectic Lie algebroids. A Lie algebroid $A \to X$ of even rank is symplectic if it carries a nondegenerate closed $A$-two-form $\omega_A$ (after [36]). Such an $A$-symplectic structure corresponds to a nondegenerate $A$-Poisson structure $\pi_A$ via the relation $\pi_A^x = (\omega_A^x)^{-1}$. Due to this, in order to study Poisson structures of $I$-divisor-type, we must study $A_I$-symplectic geometry. For clarity, the Lie algebroids of Section 2.2 allow us to define:

- Log-symplectic structures, associated to $A_Z$ ([19, 29], also [10, 17, 28] and others);
- Elliptic symplectic structures, associated to $A_{Z^{(1)}}$ ([11], also [8, 23, 24]);
- Elliptic-log symplectic structures, associated to $A_W$ ([22, 23]);
- Zero symplectic structures, associated to $B_Z$ (cf. [26], and Remark 4.9);
- Scattering symplectic structures, associated to $C_Z$ ([26]);
- $b^k$-symplectic structures, associated to $A_Z^j$ ([38], also [18]).

Each of these has an underlying Poisson structure, which can often be characterized intrinsically. While the remainder of this note uses Lie algebroids and Lie algebroid objects, these are viewed as tools to make statements about generically-nondegenerate Poisson structures.

3. Cohomological obstructions

In this section we discuss cohomological obstructions for the existence of $A$-symplectic structures. Such results can be found in the literature, but are included here to provide contrast. These results are analogous to the fact that a symplectic structure on a compact $2n$-dimensional manifold forces the existence of a cohomology class $c \in H^2(X; \mathbb{R})$ with $\omega^c \neq 0$.

For log-symplectic manifolds, cohomological obstructions were obtained by Mărcut–Osorno Torres [29] (also [37]) and Cavalcanti [10] (for simplicity we assume $X$ is compact oriented).

Theorem 3.1 ([29, 10]). Let $(X^{2n}, Z)$ be compact oriented log-symplectic. Then there exists:
• a class \(a \in H^2(X; \mathbb{R})\) such that \(a^{n-1} \neq 0\);
• a class \(b \in H^2(X; \mathbb{R})\) such that \(b^2 = 0\) and \(a^{n-1} \wedge b \neq 0\), if \(n \geq 2\) and \(Z \neq \emptyset\).

Consequently, if \(\dim X = 4\) and \(Z \neq \emptyset\), then \(X\) must have indefinite intersection form.

**Proof.** We reproduce the proofs of [37, Theorem 4.3.1] and [10, Theorem 4.2] respectively.

Let \(\omega_Z \in \text{Symp}(\mathcal{A}_Z)\) be a log-symplectic structure. If \(Z = \emptyset\), then set \(a = [\omega_Z]\). If \(Z \neq \emptyset\), consider the residue one-form \(\alpha = \text{Res}_Z(\omega_Z) \in \Omega^1_{\text{hol}}(Z)\). By the semi-global normal form for \(\omega_Z\) around \(Z\), we have that \(\omega_Z = d\log|z| \wedge p^*(\alpha) + p^*(\beta)\), where \(p:\mathbb{N}Z \to Z\) is a tubular neighbourhood, \(\beta \in \Omega^2_{\text{hol}}(Z)\), and \(|z|\) is a distance function on \(\mathbb{N}Z\). Now let \(\lambda: X \to [0, 1]\) be such that \(\lambda(z) = 1\) for \(z \in \mathbb{N}Z\) with \(|z| \leq \frac{1}{2}\), and \(\lambda(z) = \log|z|\) on \(X\backslash \{z \in \mathbb{N}Z : |z| < 1\}\). Then \(\omega := d\lambda(z) \wedge p^*(\alpha) + p^*(\beta)\) is a smooth closed two-form agreeing with \(\omega_Z\) away from \(Z\), hence can be extended to a global two-form on \(X\), and whose pullback to \(Z\) is \(\beta\). Note that:

\[\omega_Z^2 \neq 0\] on \(X\), so that \(\text{Res}_Z(\omega_Z) = \alpha \wedge \beta^{n-1} \neq 0\) on \(Z\).

Consequently, the class \(a := [\omega] \in H^2(X; \mathbb{R})\) satisfies \(a^{n-1} \neq 0\), as on the level of forms we have \(i_Z^*(\omega_Z) = \beta^{n-1} \neq 0\), and the two-form \(\beta\) cannot be exact due to compactness of \(Z\).

For the second point, if \(Z \neq \emptyset\), by compactness of \(X\) we can perturb \(\omega_Z\) slightly to ensure that \(\alpha\) has a cohomology class that is a real multiple of an integer class \([10, 37]\). Then \(\alpha\) exhibits \(Z\) as a symplectic mapping torus with fiber \(F\), and \(\omega_Z\) is said to be proper. The two-form \(\omega\) constructed above can then be chosen such that its restriction to \(F\) is symplectic. Consider \([F] \in H_{2n-2}(X; \mathbb{R})\) which is necessarily nonzero, and set \(b := \text{PD}[F] \in H^2(X; \mathbb{R})\). As \(F\) is the fiber of a fibration, we have \(b^2 = 0\). Moreover, \([a^{n-1} \wedge b, [X]] = [a^{n-1}, [F]] \neq 0\).

There is a similar result for elliptic symplectic manifolds with zero elliptic residue by Cavaletti–Gualtieri, contained in a preliminary version of [11] (we state the compact case).

**Theorem 3.2 ([11]).** Let \((X^{2n}, D)\) be a compact elliptic symplectic manifold with zero elliptic residue and cooriented degeneracy locus \(D\). Then there exists:

• a class \(a \in H^2(X\backslash D; \mathbb{R})\) such that \(a^{n-1} \neq 0\);
• a class \(b \in H^2_c(X\backslash D; \mathbb{R})\) such that \(b^2 = 0\) and \(a^{n-1} \wedge b \neq 0\).

**Proof.** We reproduce the proof by Cavaletti–Gualtieri [11] (found in its preliminary version).

Let \(\omega \in \text{Sym}^p(\mathcal{A}_D)\) have zero elliptic residue. By [11, Theorem 1.8], decompose \([\omega] = a + c\) for \(a = i^*[\omega] \in H^2(X\backslash D; \mathbb{R})\) with \(i: X\backslash D \to X\) the inclusion of the divisor complement, and \(c = \text{Res}_r([\omega]) \in H^1(S^1 ND; \mathbb{R})\). As \(\omega^n \neq 0\), we see that \(\text{Res}_r(\omega^n) \neq 0\) is a volume form in \(\Omega^{2n-1}(D; S^1 ND; \mathbb{R})\), so that \(\text{Res}_r([\omega^n]) \neq 0\) in cohomology by compactness of \(D\). Using [11, Theorem 1.9] we compute (with \(r: H^2(X\backslash D; \mathbb{R}) \to H^1(S^1 ND; \mathbb{R})\) the restriction):

\[0 \neq \text{Res}_r([\omega^n]) = \text{Res}_r([\omega^n]) = nr(i^*[\omega]) - \text{Res}_r[\omega] = nr(a^{n-1}) + c\]

We conclude that \(a^{n-1} \neq 0\) as desired. Next choose a small tubular neighbourhood for \(D\) and identify \(S^1 ND\) with its boundary, so that it includes via a map \(j: S^1 ND \to X\backslash D\). Above we showed that \(j^*(a^{n-1}) + c \neq 0\) in \(S^1 ND\). Define \(b := j_*(c) \in H^2_c(X\backslash D; \mathbb{R})\), and note that \(a^{n-1} \wedge b \neq 0\) in \(H_{2n-1}(X\backslash D; \mathbb{R})\). As cochains, \(b\) and \(c\) are related by wedging with a Thom form for the normal bundle \(N_X(S^1 ND)\), which is thus seen to square to zero, hence \(b^2 = 0\).

**Remark 3.3.** Theorems 3.1 and 3.2 are related as follows (due to Cavaletti–Gualtieri, in a preliminary version of [11]; see [24] for more details): given an elliptic pair \((X, |D|)\), one can perform real oriented blow-up along \(D\) and obtain a log pair with boundary, \((X', S^1 ND)\). The blow-down map \(p: X' \to X\) satisfies \(p^* I_{|D|} = I_Z\), where \(Z = S^1 ND\), and induces a Lie
algebroid morphism \((\varphi, p): A_Z \to A_{|D|}\) where \(\varphi \equiv dp\) on sections. The elliptic-symplectic structure on \((X, D)\) pulls back via \(\varphi\) to a log-symplectic structure on \((X', Z)\). One can then readily relate the cohomology classes from Theorem 3.1 to those of Theorem 3.2.

One can also obtain obstructions on log and elliptic pairs \((X, Z)\) and \((X, |D|)\) to admit Lie algebroid symplectic structures (i.e. log-, or zero elliptic residue elliptic) by considering the geometric structures induced on \(Z\) and \(D\) respectively. Namely, \(Z\) becomes cosymplectic \((|19|)\), while \(D\) inherits a \(2\)-cosymplectic structure if it is coorientable (see \([23]\)). Both structures can be after slight perturbation assumed to be proper \((|10, 28|, and \([2, 8]\))\), resulting in induced fibration maps \(Z \to S^1\) and \(D \to T^2\) (which themselves also form obstructions).

If we consider the cohomology classes of a \(k\)-cosymplectic structure, we see that:

**Proposition 3.4** (c.f. \([18]\)). Let \((X, Z)\) be a proper log-symplectic manifold. Then \(Z^{2n-1}\) has classes \(a \in H^1(Z; \mathbb{R})\) and \(b \in H^2(Z; \mathbb{R})\) such that \(a \wedge b^{n-1} \neq 0\);

**Proposition 3.5** (c.f. \([23, 2]\)). Let \((X, |D|)\) be a proper zero residue elliptic symplectic manifold. Then \(D^{2n}\) has classes \(a, b \in H^1(D; \mathbb{R})\) and \(c \in H^2(D; \mathbb{R})\) such that \(a \wedge b \wedge c^{n-1} \neq 0\).

We stress again that our notion of being proper implies that \(D\) is coorientable.

**Remark 3.6.** It is natural to ask whether the above results can be extended to other symplectic Lie algebroids. Based on the above proofs, this would require an understanding of the ring structure on Lie algebroid cohomology, or by having semi-global normal form results.

**Remark 3.7.** Instead of adapting the strategies used in Theorem 3.1 and Theorem 3.2 to other Lie algebroids \(A \to X\), one can try to prove directly that given \(\omega_A \in \text{Symp}(A^n)\), the class \([\omega^n_A] \in H^n(A)\) is nontrivial, by understanding the canonical Evens–Lu–Weinstein pairing \([13]\), i.e. \(H^n(A) \times H^0(A; Q_A) \to \mathbb{R}\), where \(Q_A = \det(A) \otimes \det(T^*X)\) is the canonical \(A\)-module.

Note that the divisor \(\text{div}(A)\) of \(A\) uses the dual bundle \(Q_A^*\), which is also an \(A\)-module.

### 4. Homotopical obstructions

In this section we discuss homotopical obstructions to the existence of \(A\)-symplectic structures on a given closed manifold \(X\). More precisely, we focus on the following simple facts.

**Proposition 4.1.** Let \(A \to X\) be a symplectic Lie algebroid. Then:

- \(A\) must be orientable, i.e. it must satisfy \(w_1(A) = 0 \in H_1(X; \mathbb{Z}_2)\);
- \(A\) must be complex, i.e. there must exist a \(J_A \in \text{End}(A)\) with \(J_A^2 = -\text{id}\).

These properties both follow from the linear algebra of having a nondegenerate \(A\)-two-form. Indeed, they hold for any symplectic vector bundle (e.g. \([30]\)), as they do not use integrability.

**Proof.** Let \(\omega_A\) be an \(A\)-symplectic structure. Then \(\text{rank}(A) = 2m\) is necessarily even, and \(\omega_A^n \in \Gamma(\det(A^*))\) is nonvanishing. Thus \(\det(A^*)\) is trivial, and \(w_1(A) = w_1(\det(A^*)) = 0\). To see \(A\) must admit a complex structure, follow the standard proof for \(A = TX\) (e.g. \([30]\)). \(\square\)

Note that when both \(A\) and \(X\) are four-dimensional, a classical result by Wu \([41]\) (see also \([20]\)) can be used, characterizing when an oriented vector bundle admits a complex structure.

---

\(^1\) A \textit{k-cosymplectic structure} on a manifold \(Z^{2k+k}\) is a tuple \((\alpha_i, \beta)\) of \(k\) closed one-forms and a single closed two-form such that \(\alpha_1 \wedge \cdots \wedge \alpha_k \wedge \beta^2 \neq 0\). These have a foliation of rank \(2\ell\) with tangencies \(\text{ker}(\alpha_i)\). See \([37]\).
Theorem 4.2 ([41]). Let $E^4 \to X^4$ be an oriented Euclidean rank-four vector bundle over a compact oriented four-manifold. Then $E$ admits a complex structure if and only if there exists a class $c \in H^2(X; \mathbb{Z})$ such that $c \mod 2 \equiv v_2(E) \in H^2(X; \mathbb{Z}_2)$ and $c^2 = p_1(E) + 2e(E)$.

To make effective use of these observations, it is clear that we must determine the relevant characteristic classes of the bundle $\mathcal{A} \to X$. We do this via stable bundle isomorphisms.

4.1. Stable bundle isomorphisms. We determine how a vector bundle changes when performing lower elementary modification. Denote by $\mathbb{R} \to X$ the trivial real line bundle.

Proposition 4.3. Let $\mathcal{A} \to X$ be a Lie algebroid and $Z \subseteq X$ a hypersurface. Consider a $(\mathcal{B}, Z)$-rescaling $[\mathcal{A}: \mathcal{B}]$ of $\mathcal{A}$ with corank($\mathcal{B}$) = $k$. Then using $k$ copies of $L_Z$ and $\mathbb{R}$ we have

$$[\mathcal{A}: \mathcal{B}] \oplus L_Z \oplus \cdots \oplus L_Z \cong \mathcal{A} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R}.$$

We emphasize that is a vector bundle isomorphism, and not one of Lie algebroids.

Remark 4.4. When $\mathcal{A} = TX$ and $\mathcal{B} = TZ$, Proposition 4.3 reduces to the statement that $\mathcal{A}_Z \oplus L_Z \cong TX \oplus \mathbb{R}$, as $\mathcal{B}$ has corank one. This was noted without proof in [6, 7] when $X$ is orientable, although they inaccurately state that in general one has $\mathcal{A}_Z \oplus \mathbb{R} \cong TX \oplus L_Z$. However, when $Z$ is separating so that $L_Z$ is trivial (see Lemma 2.2), these two above statements are equivalent. In particular this is the case when $(X, Z)$ is log-symplectic and $X$ is orientable (see Corollary 4.14).

Proof. We first consider the case when $k = 1$. Set $n = \text{rank}(\mathcal{A})$. Over $X \setminus Z$, the bundle $L_Z$ is trivial and $[\mathcal{A}: \mathcal{B}] \cong \mathcal{A}$ via the morphism induced by the inclusion on sections, so that there we have an obvious isomorphism. Near $Z$ we define a bundle isomorphism as follows.

Choose a tubular neighbourhood embedding of $Z$ and let $\{U_\alpha\}_\alpha$ be a simultaneous trivializing cover of $L_Z$ and of $\mathcal{B}$ extended to $\mathcal{A}$ (and hence of $[\mathcal{A}: \mathcal{B}]$ and $NZ$, as $L_Z|_Z \cong NZ$), with transverse vanishing sections $s_\alpha \in \Gamma(U_\alpha; L_Z)$ and transition maps $g_\beta$, and moreover

$$\Gamma(U_\alpha; \mathcal{A}) = \langle v_{a,1}, v_{a,2} \rangle, \quad \text{and} \quad \Gamma(U_\alpha; [\mathcal{A}: \mathcal{B}]) = \langle z_\alpha v_{a,1}, v_{a,2} \rangle,$$

with $z_\alpha$ the associated normal bundle coordinates. Choose a metric on $NZ$ and consider the disk bundle $D_{NZ}$ in $NZ$ with radius $\pi/2$. We now explicitly define a bundle isomorphism $\varphi: [\mathcal{A}: \mathcal{B}] \oplus L_Z \to \mathcal{A} \oplus \mathbb{R}$ on $D_{NZ} \cap U_\alpha$.

Generic sections of $[\mathcal{A}: \mathcal{B}] \oplus L_Z$ and $\mathcal{A} \oplus \mathbb{R}$ can be expressed respectively as the tuples

$$\left(\lambda_1 \cdot z_\alpha v_{a,1} + \sum_{2 \leq i \leq n} \lambda_i \cdot v_{a,i}, \lambda_{n+1} \cdot s_\alpha \right) \quad \text{and} \quad \left(\mu_1 \cdot v_{a,1} + \sum_{2 \leq i \leq n} \mu_i \cdot v_{a,i}, \mu_{n+1} \cdot 1 \right),$$

where $\lambda_i, \mu_i \in C^\infty(U_\alpha)$. The map $\varphi$ is defined to be in matrix form given by

$$\left(\begin{array}{ccc}
\mu_1 & \mu_2 & \cdots & \mu_n
\end{array}\right) = \begin{pmatrix}
\alpha \sin |z_\alpha| & \cos |z_\alpha| & 0 \\
\cos |z_\alpha| & \alpha \sin |z_\alpha| & 0 \\
0 & 0 & I_{n-1}
\end{pmatrix}\begin{pmatrix}
\lambda_1 & \lambda_{n+1} & \lambda_2 & \cdots & \lambda_n
\end{array}.$$

\[19, \text{Proposition 4}\]
where $I_{n-1}$ is the $(n-1) \times (n-1)$ identity matrix. We see the top left part of $\varphi$ is given by:

$$z_\alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ when $|z_\alpha| = \pi/2$, and

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$ when $|z_\alpha| = 0$, i.e. on $Z$.

Thus $\varphi$ can be extended smoothly to $U_0$ as being $\pi/2$ times the identity after introducing a radial bump function. It is well-defined because on the intersections $U_\alpha \cap U_\beta$ we have that:

$$z_\alpha v_{\alpha,1} = z_\beta v_{\beta,1},$$

so that $v_{\alpha,1} = \frac{z_\beta}{z_\alpha} v_{\beta,1} = g_{\alpha}^{\beta} v_{\beta,1},$ and also $s_\alpha = \frac{z_\beta}{z_\alpha} s_\beta = g_{\alpha}^{\beta}.$

To see that $\varphi$ is an isomorphism, we merely note that in matrix form it has determinant equal to $z_\alpha^2 \sin^2 |z_\alpha| + \cos^2 |z_\alpha|$, which is positive for all values of $z_\alpha$, showing invertibility.

In the general case when $k \geq 2$, repeat this argument for each generator of the complement of $\mathcal{B}$ inside $\mathcal{A}|_Z$. This gives a bundle isomorphism with block-diagonal matrix form. \qed

A similar result holds for the $b^k$-tangent bundles $\mathcal{A}^k_Z$ constructed using a $(k-1)$-jet $j$ at $Z$.

**Proposition 4.5.** Let $(X, Z)$ be a log pair with $(k-1)$-jet $j$ at $Z$. Then $\mathcal{A}^k_Z \otimes L^k_Z \cong TX \oplus \mathbb{R}$.

**Proof.** Follow the same strategy as for Proposition 4.3, except take $z_\alpha \in j$, replace $\lambda_1 z_\alpha v_{\alpha,1}$ by $\lambda_1 z_\alpha^k v_{\alpha,1}$ (where $v_{\alpha,1} = \partial z_\alpha$), and define the map $\varphi$ near $Z$ to be given in matrix form by

$$\begin{pmatrix} \mu_1 & \mu_{n+1} & \mu_2 & \cdots & \mu_n \end{pmatrix} = \begin{pmatrix} \frac{z_\alpha^k \sin |z_\alpha| \cos |z_\alpha|}{|z_\alpha|} & 0 & 0 & \lambda_1 & \lambda_{n+1} & \lambda_2 & \cdots & \lambda_n \end{pmatrix}.$$

This map has determinant $z_\alpha^2 \sin^2 |z_\alpha| + \cos^2 |z_\alpha|$, again showing invertibility of the map $\varphi$. \qed

4.2. **Computing characteristic classes.** In this section we compute relevant characteristic classes of the Lie algebroids we have introduced. We will mainly be interested in the first and second Stiefel–Whitney classes $w_1, w_2 \in H^1(X; \mathbb{Z}_2)$, and in the first Pontryagin class $p_1 \in H^4(X; \mathbb{Z})$. We recall several properties of these characteristic classes (see e.g. [33]).

**Proposition 4.6.** Let $E^n, F^n \to X$ be real vector bundles. Denote the full Stiefel–Whitney and Pontryagin classes by $w: \text{Vect}(X) \to H^*(X; \mathbb{Z}_2)$ and $p: \text{Vect}(X) \to H^*(X; \mathbb{Z})$. Then:

i) $w(E \oplus F) = w(E) \cup w(F)$, and $w_1(E \otimes F) = nw_1(E) + mw_1(F)$;

ii) $w_2(E \otimes F) = w_2(E) + w_1(F) \cup w_1(E)$ if $m = 4$ and $n = 1$.

iii) $2p(E \otimes F) = 2p(E) \cup p(F)$, and $p(E \otimes F) = p(E)$ if $n = 1$.

We now determine the relevant characteristic classes for the Lie algebroids $\mathcal{A}^k_Z$, $\mathcal{B}_Z$, and $\mathcal{C}_Z$.

**Proposition 4.7.** Let $(X^n, Z)$ be a log pair with Lie algebroids $\mathcal{A}^k_Z$, $\mathcal{B}_Z$, and $\mathcal{C}_Z$. Then:

For $\mathcal{A}^k_Z$: \quad $w_1(\mathcal{A}^k_Z) = w_1(TX) + kw_1(L_Z),$

$w_2(\mathcal{A}^k_Z) = w_2(TX) + kw_1(L_Z) \cup w_1(TX),$

$p_1(\mathcal{A}^k_Z) = p_1(TX)$ if $X$ is orientable and four-dimensional;

For $\mathcal{B}_Z$: \quad $w_1(\mathcal{B}_Z) = w_1(TX) + nw_1(L_Z),$

$w_2(\mathcal{B}_Z) = w_2(TX) + w_1(L_Z) \cup w_1(TX)$ if $X$ is four-dimensional;

For $\mathcal{C}_Z$: \quad $w_1(\mathcal{C}_Z) = w_1(TX) + (n+1)w_1(L_Z),$

$w_2(\mathcal{C}_Z) = w_2(TX)$ if $X$ is four-dimensional,

$p_1(\mathcal{C}_Z) = p_1(TX)$ if $X$ is orientable and four-dimensional.

**Remark 4.8.** The fact that $w(\mathcal{A}_Z) = w(TX)(1 + \text{PD} \mathbb{Z}_2[Z])$ can be found in [7], as this would also be what follows from the stable isomorphism relation noted by them, see Remark 4.4.
Remark 4.9. We will have no direct use for \( w_2(B_Z) \) and \( p_1(B_Z) \) (when \( X \) is four-dimensional), as by [26, Proposition 2.21] we know that \( B_Z \) does not admit Lie algebroid symplectic structures when \( \dim X \geq 4 \). This follows from studying the ring structure of the space \( \Omega^*(B_Z) \).

Proof. For \( A_L^x \): By Proposition 4.5 we have \( A_L^x \oplus L^k \cong TX \oplus \mathbb{R} \), hence due to Proposition 4.6.i) we get \( w(A_L^x) = (1+kw_1(L)) \cup w(TX) \). In degree one this gives \( w_1(A_L^x) = w_1(TX) + kw_1(L) \) as desired. In degree two it follows that \( w_2(A_L^x) = w_2(TX) + kw_1(L) \cup w_1(TX) \). We see that \( 2p_1(A_L^x) = 2p_1(TX) \), as \( p = 1 \) for line bundles. If \( X \) is orientable and four-dimensional we know that \( H^4(X; \mathbb{Z}) \cong \mathbb{Z} \), which in particular has no two-torsion, so that \( p_1(A_L^x) = p_1(TX) \).

For \( B_Z \): This follows from Proposition 4.6 after using Remark 2.5 that \( B_Z \cong TX \otimes L_Z \).

For \( C_Z \): By Remark 2.5 we have \( C_Z \cong A_Z \otimes L_Z \), so that from Proposition 4.3 for \( A = A_Z \) we obtain \( C_Z \oplus L^2_Z \cong B_Z \oplus L_Z \). As \( L^2_Z \) is canonically trivial, using Proposition 4.6.i) this gives \( w(C_Z) = (1 + w_1(L)) \cup w(B_Z) \). In degree one this results in (using the case of \( B_Z \) above):

\[
w_1(C_Z) = w_1(B_Z) + w_1(L) = w_1(TX) + (n + 1)w_1(L).
\]

In degree two we see similarly that if \( X \) is four-dimensional that

\[
w_2(C_Z) = w_2(B_Z) + w_1(L) \cup w_1(B_Z)
= w_2(TX) + w_1(L) \cup w_1(TX) + w_1(L) \cup (w_1(TX) + 4w_1(L)) = w_2(TX).
\]

Assuming also orientability of \( X \), Proposition 4.6 and the case of \( A_Z \) determine \( p_1(C_Z) \). \( \square \)

We can compute these characteristic classes somewhat more generally for rescalings.

Proposition 4.10. Let \( [A:B] \to X \) be a corank-\( k \) \((B,Z)\)-rescaling of \( A \to X \). Then:

- \( w_1([A:B]) = w_1(A) + kw_1(L) \);
- \( w_2([A:B]) = w_2(A) + kw_1(L) \cup w_1(A) + \frac{k(k-1)}{2}w_1(L)^2 \);
- \( p_1([A:B]) = p_1(A) \), if \( X \) is orientable and four-dimensional.

Proof. By Proposition 4.3 we have that \( [A:B] \oplus kW \cong A \oplus kL_Z \) using the shorthand notation \( kL = L + \cdots + L \) with \( k \) copies. This implies using Proposition 4.6.i) by \( k \)-fold induction that \( w([A:B]) \cup 1^k = w(A) \cup (1 + w_1(L))^k \). We have \((1 + w_1(L))^k \equiv 1 + kw_1(L) + \frac{k(k-1)}{2}w_1(L)^2 \) up to degree two. In degree one this gives \( w_1([A:B]) = w_1(A) + kw_1(L) \) as desired, while in degree two it instead gives \( w_2([A:B]) = w_2(A) + kw_1(L) \cup w_1(A) + \frac{k(k-1)}{2}w_1(L)^2 \). The last property follows because by Proposition 4.6.iii) we have \( 2p_1([A:B]) = 2p_1(A) \in H^4(X; \mathbb{Z}) \), and the hypothesis ensures that \( H^4(X; \mathbb{Z}) \cong \mathbb{Z} \) has no two-torsion (c.f. Proposition 4.7). \( \square \)

We can further determine the first Stiefel–Whitney class of the bundles \( A_{[D]} \) and \( A_W \).

Proposition 4.11. Let \((X, |D|)\) and \((X', W)\) be an elliptic and elliptic-log pair. Then:

- \( w_1(A_D) = w_1(TX) \);
- \( w_1(A_W) = w_1(A_Z) = w_1(TX) + w_1(L_Z) \), if \( I_W = I_Z \otimes I_{[D]} \).

To prove this we first turn to an auxilliary lemma regarding triviality of line bundles.

Lemma 4.12. Let \( L \to X \) be a real line bundle with a section vanishing only on a submanifold of codimension at least two. Then \( L \) is trivial, i.e. \( w_1(L) = 0 \in H_1(X; \mathbb{Z}) \). Consequently, if \( (\varphi, \text{id}_X): E \to F \) is a base-preserving vector bundle morphism which is an isomorphism outside a submanifold of codimension at least two in \( X \), then \( w_1(E) = w_1(F) \).
Proof. Let $N \subseteq X$ be that submanifold and consider the Mayer–Vietoris sequence in cohomology for $(X,N)$, giving (for $\mathbb{D}(N)$ the unit disk bundle of the normal bundle to $N$):

$$H^0(X) \to H^0(X\setminus N) + H^0(N) \to H^0(\mathbb{D}(N)) \to$$

$$\to H^1(X) \to H^1(X\setminus N) \oplus H^1(N) \to H^1(\mathbb{D}(N)) \to \ldots .$$

Assume that $X$ is connected, so that $X\setminus N$ is still connected by the codimension assumption. Then the map $H^0(\mathbb{D}(N)) \to H^1(X)$ is zero, so that the map $H^1(X) \to H^1(X\setminus N) \oplus H^1(N)$ is injective. As $L$ is trivial on $X\setminus N$ by hypothesis, we have $w_1(L|_{X\setminus N}) = 0$, so that $w_1(L) = 0$.

The condition on $\varphi$ being generically an isomorphism implies that $\text{rank}(F) = \text{rank}(L)$. Equivalently, using $\det(\varphi) : \det(E) \to \det(F)$, the pair $(\det(F) \otimes \det(E)^*, \det(\varphi))$ is a divisor, and $\det(\varphi)$ vanishes only on a submanifold of codimension at least two by hypothesis. The first part then implies that $w_1(\det(F) \otimes \det(E)^*) = 0$, from which the conclusion follows. $\square$

Proof of Proposition 4.11. The natural maps $\rho_{A|D|} : A|D| \to TX$ and $\varphi_{A,W} : A_W \to A_Z$ are both isomorphisms outside of $D$ and $D'$ respectively, both of which are of codimension two. Consequently Lemma 4.12 applies, hence the result follows (using Proposition 4.7 for $A_Z$). $\square$

4.3. Orientability of Lie algebroids. In this section we discuss orientability for the Lie algebroids $A^k_2$, $B_2$ and $C_2$ associated to log pairs $(X,Z)$, and for the Lie algebroids $A|D|$ and $A_W$ given elliptic and elliptic-log pairs $(X,|D|)$ and $(X,W)$. This further settles when these Lie algebroids admit symplectic structures in dimension two, and gives an obstruction to their existence in arbitrary dimensions, noting Proposition 4.1. They moreover characterize the existence of $A$-Nambu structures of highest degree (i.e. nonvanishing sections $\Pi \in \Gamma(\det(A))$).

Given a Lie algebroid $A \to X$, we say $X$ is $A$-orientable if the vector bundle $A$ admits an orientation. As this does not depend on the Lie algebroid structure, it is clear that this happens if and only if $w_1(A) = 0$. Consequently, Proposition 4.7 shows the following (where we use Lemma 2.2 for the fact that $w_1(L_Z) = PD_{\mathbb{Z}_2}[Z] \in H^1(X;\mathbb{Z}_2)$).

**Proposition 4.13.** Let $(X^n,Z)$ be a log pair. Then $X$ is $A$-orientable if and only if:

- $w_1(TX) + kPD_{\mathbb{Z}_2}[Z] = 0$, in case $A = A^k_2$;
- $w_1(TX) + nPD_{\mathbb{Z}_2}[Z] = 0$, in case $A = B_2$;
- $w_1(TX) + (n+1)PD_{\mathbb{Z}_2}[Z] = 0$, in case $A = C_2$.

For later convenience, let us make explicit what happens in the $A$-orientable case.

**Corollary 4.14.** Let $(X,Z)$ be an $A$-orientable log pair. Then $X$ is orientable if and only if:

- $k$ is even or $|Z| = 0$, in case $A = A^k_2$ (c.f. [34, 35]);
- $n$ is even or $|Z| = 0$, in case $A = B_2$;
- $n$ is odd or $|Z| = 0$, in case $A = C_2$.

We can further use Proposition 4.11 to see that $A|D| \to X$ is orientable if and only if $X$ is, and $A_W \to X$ is orientable if and only if the associated log-tangent bundle $A_Z \to X$ is. Consequently, elliptic-log Poisson structures exist on any surface $X$, as is true for log-Poisson.

Thus, the consequences of Proposition 4.1 and $A$-orientability are as follows (Theorem A):

**Theorem 4.15.** Let $A \to X^n$ be a symplectic Lie algebroid. Then in $H_1(X;\mathbb{Z}_2)$ we have:

- If $A = A^k_2$, then $w_1(TX) + kPD_{\mathbb{Z}_2}[Z] = 0$;
- If $A = B_2$, then $w_1(TX) + nPD_{\mathbb{Z}_2}[Z] = 0$;
- If $A = C_2$, then $w_1(TX) + (n+1)PD_{\mathbb{Z}_2}[Z] = 0$;
• If $A = A|_D$, then $w_1(TX) = 0$;
• If $A = A_W$, then $w_1(TX) + \text{PD}_{Z^2}[Z] = 0$.

Proof. If $A$ is symplectic, by Proposition 4.1 we must have that $A$ is $A$-orientable, so that we see that $w_1(A) = 0$. The result then follows from Proposition 4.7 and Proposition 4.11. □

4.4. Existence of $A$-almost-complex structures. In this section we discuss when some Lie algebroids of interest can admit a complex structure. For this we use Theorem 4.2 together with our earlier computations of characteristic classes (see Proposition 4.7). Due to Proposition 4.1 this provides obstructions to when these Lie algebroids can be symplectic.

Let $(X^4, Z)$ be a four-dimensional log pair, and assume that $X$ is oriented. Consider a $(k - 1)$-jet for $Z$ and its $b^k$-tangent bundle $A^k_Z$, which recall includes the log-tangent bundle if $k = 1$. Assume further that an orientation for $A^k_Z$ is given. Then we can define the following:

**Definition 4.16.** Given orientations on the bundles $A^k_Z$ and $TX$, the $k$-discrepancy $f_k(X, Z)$ of $Z$ is defined as the difference $2f_k(X, Z) := e(A^k_Z) - e(TX) \in H^4(X; Z) \cong \mathbb{Z}$.

**Lemma 4.17.** In the situation above, the $k$-discrepancy for $Z$ is well-defined, i.e. the difference in Euler classes of $A^k_Z$ and $TX$ is even. Further, we have $f_k(X, Z) \equiv kf_1(X, Z)$ (mod 2).

Write $f(X, Z) := f_1(X, Z)$, so that the second statement of the lemma is shorthand for the facts that $f_k(X, Z) \equiv 0 \pmod{2}$ if $k$ is even, and $f_k(X, Z) \equiv f(X, Z) \pmod{2}$ if $k$ is odd.

Proof. Recall that the Euler class of an oriented vector bundle reduces mod 2 to its top Stiefel–Whitney class. Because both $A^k_Z$ and $TX$ are oriented, we have $w_1(A^k_Z) = w_1(TX) = 0$, hence $w_1(L^k_Z) = kw_1(L_Z) = 0$ by Proposition 4.7. This means that $L^k_Z$ is trivial, so that by Proposition 4.5 we have that $A^k_Z \oplus \mathbb{R} \cong TX \oplus \mathbb{R}$. Using Proposition 4.6(i) this implies that $w(A^k_Z) = w(TX)$, so that in particular $e(A^k_Z) \equiv e(TX) \pmod{2}$ as desired.

For the second statement, we remark that there is a more geometric description of the $k$-discrepancy. If $A^k_Z$ is oriented and $X$ is orientable, any choice of orientation for $TX$ does not agree with the orientation on the isomorphism locus $X \setminus Z$ induced by $A_Z$ if and only if $k$ is odd. This follows from the local description of the bundle $A^k_Z$ (c.f. [9] for when $k = 1$). If $k$ is odd, then $Z$ is separating due to Proposition 4.7 for $k = 1$. Then the $k$-discrepancy is given by

$$f_k(X, Z) = -(e(TX), [X_-]) = -\chi(X_-).$$

We see here that the right-hand side does not depend on $k$, nor does the decomposition of $X \setminus Z$ into $X_\pm$. It follows that in fact $f_k(X, Z) = f_1(X, Z)$ if $k$ is odd. On the other hand, if $k$ is even, the orientation on $X \setminus Z$ induced by $A^k_Z$ can be made compatible with the one on $TX$, so that their Euler classes are equal: any generically vanishing section of $TX$ can be assumed to not vanish on $Z$, so that its zero set can compute both $e(TX)$ and $e(A^k_Z)$ (we use here that $Z$ has codimension one in $X$), and its zeros are counted for $TX$ and $A^k_Z$ do or do not agree. Then the $k$-discrepancy is given by

$$f_k(X, Z) = -(e(TX), [X_-]) = -\chi(X_-).$$

We can now state our obstruction to the existence of an $A^k_Z$-almost-complex structure.

**Theorem 4.18.** Let $(X^4, Z)$ be a compact oriented $A^k_Z$-almost-complex log pair. Then we have $[\langle c^2_1(A^k_Z) \rangle, [X]] = 3\sigma(X) + 2\chi(X) + 4f_k(X, Z)$, and $b^+_Z(X) = b^+_Z(X) + b_1(X) + f_k(X, Z)$ is odd.

Here $\chi(X)$ is the Euler characteristic, and $\sigma(X) = b^+_Z(X) - b^-_Z(X)$ is the signature of $X$. The following proof is similar to the case when $Z = \emptyset$, see [16, Theorem 1.4.13].
Theorem 4.2. From Proposition 4.7 Section 3 together with we have, using the definition of the. Due to again Lemma 4.17 to replace we get (as is an obstruction for the log pair classes depend on for C Using the other part of Proof. Because the fact that the divisor ideal of Definition 4.21. Given orientations on the bundles A and assume that an orientation for C is given. There is an analogous definition of discrepancy here, similar to Definition 4.16. Definition 4.21. Given orientations on the bundles C and TX, the scattering discrepancy of Z is defined as the difference 2f(X, Z) := e(C) − e(TX) ∈ H4(X; Z) ≃ Z.

In fact, we can quickly relate the scattering discrepancy to the usual discrepancy of Z.

Lemma 4.22. Let (X2n, X) be a log pair, and choose orientations on C and TX. Then A is naturally oriented, and we have the equality fsc(X, Z) ≡ f(X, Z) (mod 2).

Proof. The natural Lie algebroid morphism φ: C → A can be used to orient A. Note that because the dimension of X is even, we have that w1(C) = w1(A) by Proposition 4.7. From the fact that the divisor ideal of φ is given by Iφ = I2n, or by the local description of C, it readily follows that as for A, the orientation on X\Z induced by C similarly cannot match the one induced from TX everywhere, from which the result follows.

Using this we can obtain an obstruction to the existence of a C-almost-complex structure.

Theorem 4.23. Let (X4, Z) be a compact oriented C-almost-complex log pair. Then we have \(|c^1sc(Z)|, [X]| = 3σ(X) + 2χ(X) + 4fsc(X, Z), and b^4(X) + b_1(X) + f(X, Z) is odd.\)
Theorem 4.18. Theorem 4.18 can be used effectively for other sym-
... Due to the man-
Theorem 4.23
Lemma 4.17 we see that... if both
Corollary 4.26.
Proposition 4.7.
10
81x131
81x392
Corollary 4.26.
(apply-
separating and decompose X
It seems somewhat nontrivial to determine the discrepancy
Remark 4.28.
A
complex while
Z
X
2
Remark 4.25.
One wonders whether Proposition 4.1 can be used effectively for other sym-
plectic Lie algebroids in dimension four, for example the elliptic tangent bundle \(A_{|D|}\). Note that elliptic symplectic structures (of zero elliptic residue) can exist on \(A_{|D|}\) both in cases when \(X\) is and is not almost-complex (c.f. [4]), depending on the coorientability of \(D\) as measured by \(w_1(ND) \in H^1(X;\mathbb{Z}_2)\). We see there is nontrivial dependence on the locus \(D\) in this case.

To illustrate Theorem 4.18, we determine the parity of \(f(X, Z)\) in the following situation. As in explained in the proof of Lemma 4.17, if both \(X\) and \(A_Z\) are oriented, then \(Z\) must be separating and decompose \(X \setminus Z = X_+ \sqcup X_-\) according to whether the orientations agree.

Corollary 4.26. Let \((X, Z)\) be a compact oriented four-dimensional log pair which is \(A_Z\)-almost-complex, such that \(X\) is not almost-complex. Then \(f(X, Z)\) is odd, and the log pair \((X_\neg \cup (X_+ \# \mathbb{CP}^2), Z)\) after connected summing does not admit an \(A_Z\)-symplectic structure.

Proof. If \(X\) is not almost-complex, then \(b_2^+(X) + b_1(X) \equiv 0 \pmod{2}\), while because \((X, Z)\) is \(A_Z\)-almost-complex we obtain from Theorem 4.18 that \(b_2^+(X) + b_1(X) + f(X, Z) \equiv 1 \pmod{2}\). We conclude that \(f(X, Z) \equiv 1 \pmod{2}\). If we perform a connected sum with \(\mathbb{CP}^2\) in the subset \(X_+\) to form the manifold \(X' = X_\neg \cup (X_+ \# \mathbb{CP}^2)\), we see that \(b_2^+(X') = b_2^+(X) + 1\) while \(f(X', Z) = f(X, Z)\). Hence then \(b_2^+(X') + b_1(X') + f(X', Z) \equiv 0 \pmod{2}\), so that by applying Theorem 4.19 we see that \((X', Z)\) does not admit an \(A_Z\)-symplectic structure.

We finish by giving a simple example of how to apply the above results.

Example 4.27 \((3\mathbb{CP}^2 \# \overline{\mathbb{CP}^2})\). The manifold \(X = 2\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}\) admits a log-symplectic structure with \(Z = S^1 \times S^2\) (see [10]), and \(b_2^+(X) = 2\) and \(b_1(X) = 0\). Hence \(X\) is not almost-complex while \((X, Z)\) is \(A_Z\)-almost-complex, and \(f(X, Z)\) is odd. By Corollary 4.26 the manifold \(X' = 3\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}\) does not admit a log-symplectic structure with the given \(Z = S^1 \times S^2\).

Remark 4.28. It seems somewhat nontrivial to determine the discrepancy \(f(X, Z)\) of a separating log pair, even just its parity. We are only able to do so indirectly, c.f. Corollary 4.26. Nevertheless, the question of how \(Z\) distributes the Euler characteristic of \(X\) seems classical.
5. Obstructions from Seiberg–Witten theory

In this section we discuss obstructions for \( \mathcal{A} \)-symplectic four-manifolds coming from more sophisticated techniques. Namely, they rely on the fact that their existence implies that \( X \) symplectically fills \( Z \) in a certain sense, which is then obstructed by Seiberg–Witten theory.

Given a log pair \((X, Z)\) with a symplectic structure \( \omega \), we say it is of cosymplectic-type at \( Z \) if the pair \((\iota_v \omega, \iota^* \omega)\) is a cosymplectic structure on \( Z \), where \( v \in \Gamma(TX) \) is transverse to \( Z \), and \( \iota : Z \to X \) the inclusion. The following is a remark of Osorno Torres ([37, Lemma 4.2.6]).

**Lemma 5.1** ([37]). Let \( X \) be a manifold with boundary. Then the log pair \((X, \partial X)\) admits a log-symplectic structure if and only if it admits a symplectic structure of cosymplectic-type.

**Proof.** Assume first that \((X, Z = \partial X)\) admits a log-symplectic structure \( \omega_Z \in \text{Symp}(A_Z) \). Fix a collar neighbourhood \( U \cong \partial X \times [0, 1) \) where \( \omega_Z = \alpha \log z + p^*(\alpha) + p^*(\beta) \). Choose a positive function \( \lambda : [0, 1] \to \mathbb{R} \) such that \( \lambda(t) = 1/t \) near \( t = 1 \), and \( \lambda(t) = 1 \) near \( t = 0 \). Then \( \omega := \lambda(z)d\alpha + p^*(\alpha) + p^*(\beta) \) is a smooth closed nondegenerate two-form on \( U \) which can be extended to \( X \) as it agrees with \( \omega_Z \) away from \( Z \). It readily follows that \( \omega \) is a symplectic structure of cosymplectic-type at \( \partial X \) given by \((\alpha, \beta)\). The converse is similar and omitted. \( \square \)

By combining this result with Corollary 4.14, which gives that \( Z \) is separating, we get:

**Corollary 5.2.** Let \((X, Z)\) be log-symplectic with \( X \) oriented, so that \( X \setminus Z = X_+ \sqcup X_- \). Then the log pairs with boundary \((X_\pm, Z)\) admit symplectic structures of cosymplectic-type.

This result can obstruct the existence of log-symplectic structures, as specific symplectic fillings of \( Z \) as a cosymplectic manifold need not always exist. This contrasts with results of [14] and [15, Theorem 2], [12, Theorem 3.1], which describe which cosymplectic manifolds admit (log-)symplectic fillings. We are crucially fixing the diffeomorphism type of the filling.

**Remark 5.3.** Note that Corollary 5.2 does not say that \((X, Z)\) admits a symplectic structure of cosymplectic-type. Often \( X \) cannot admit symplectic structures at all (e.g. \( 2\mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \)).

Let us say a manifold is \( psc \) if it admits a positive scalar curvature metric. Then an almost direct consequence of results by Taubes [39, 40] and Kronheimer–Mrowka [25] is the following:

**Proposition 5.4** ([3]). Let \((X^4, Z)\) be an oriented log pair, splitting into \( X_1 \sqcup_Z X_2 \), where \( b_2^+(X_1) > 0 \). Then if each component \( Z_i \) of \( Z \) is psc (e.g. \( S^1 \times S^2 \)), then \((X, Z)\) cannot be symplectic.

**Remark 5.5.** This result appeared first as [5, Proposition 1] and then [3, Lemma 3.4], where the authors remark that any psc three-manifold is a connected sum of spherical three-manifolds and copies of \( S^1 \times S^2 \), and show how to allow \( Z \) to be disconnected (by performing tubing). Note here that the total space of any spherical symplectic mapping torus is given by \( S^2 \times S^1 \).

We now state our obstruction result for orientable log-symplectic manifolds (Theorem D).

**Theorem 5.6.** Let \((X^4, Z)\) be an oriented log pair, splitting along \( X_1 \sqcup_Z X_2 \). If \( b_2^+(X_1) > 0 \) and each connected component of \( Z_i \) is psc (e.g. \( S^1 \times S^2 \)), then \((X, Z)\) cannot be log-symplectic.

**Proof.** If it can, then by Lemma 5.1, the log pair with boundary \((X_1, Z)\) is symplectic of cosymplectic-type. In dimension three any cosymplectic manifold, such as \( Z \), automatically has a taut foliation. Hence by [25, Theorem 41.3.1] we can find another symplectic manifold \( X' \) with \( b_2^+(X') > 0 \) such that \( X_1 \sqcup_Z X' \) is symplectic and splits along \( Z \) (compare this with [10, Theorem 6.1]). This contradicts Proposition 5.4, so that \((X, Z)\) cannot be log-symplectic. \( \square \)
Remark 5.7. This result is similar to those in [1, Section 5.2], where instead of Seiberg–Witten theory, techniques from pseudo-holomorphic curves adapted to log pairs are used.

Remark 5.8. Conceivably something similar can be done for other symplectic Lie algebroids, such as the scattering tangent bundle $C_Z$, and the $b^k$-tangent bundles $A^k_Z$ (in light of [18]).

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