E_{10}, BE_{10} and Arithmetical Chaos in Superstring Cosmology

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It is shown that the never ending oscillatory behaviour of the generic solution, near a cosmological singularity, of the massless bosonic sector of superstring theory can be described as a billiard motion within a simplex in 9-dimensional hyperbolic space. The Coxeter group of reflections of this billiard is discrete and is the Weyl group of the hyperbolic Kac-Moody algebra E\textsubscript{10} (for type I) or BE\textsubscript{10} (for type I or heterotic), which are both arithmetic. These results lead to a proof of the chaotic (“Anosov”) nature of the classical cosmological oscillations, and suggest a “chaotic quantum billiard” scenario of vacuum selection in string theory.

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One crucial problem in string theory is the problem of vacuum selection. It is reasonable to believe that this problem can be solved only in the context of cosmology, by studying the time evolution of generic, inhomogeneous (non-SUSY) string vacua. In this vein, it has been recently found \cite{1,2} that the general solution near a spacelike singularity (t \to 0) of the massless bosonic sector of all superstring models (D = 10 IIA, IIB, 1, HE, HO), as well as that of M-theory (D = 11 SUGRA), exhibits a never ending oscillatory behaviour of the Belinskii-Khalatnikov-Lifshitz (BKL) type \cite{3}. In this letter, we analyze in more detail the asymptotic dynamics (as t \to 0) of this oscillatory behaviour. We find that the evolution of the scale factors and the dilaton at each spatial point can be be viewed as a billiard motion in some simplices in hyperbolic space H\textsuperscript{10}, which have remarkable connections with hyperbolic Kac-Moody algebras of rank 10.

The central idea of the BKL approach is that the various points in space approximately decouple as one approaches a spacelike singularity (t \to 0). More precisely, the partial differential equations that control the time evolution of the fields can be replaced by ordinary differential equations with respect to time, with coefficients that are (relatively) slowly varying in space and time. The details of how this is done are explained in \cite{4} for pure gravity, and in \cite{5,6} for the graviton-dilaton-p-form systems relevant to superstring/M-theory.

We shall focus in this letter on the dynamical behaviour of the metric and the string dilaton and recall first the relevant equations from \cite{4,5,6}. To leading order, the metric (in the Einstein frame or the string frame) reads $g_{\mu\nu} \, dx^\mu \, dx^\nu = - N^2(dt)^2 + \sum_{i=1}^D a_i^2(\omega^i)^2$, where D = 1 denotes the spatial dimension, and where $\omega^i(x) = e^i(x) \, dx^i$ is a d-bein whose time-dependence is neglected compared to that of the local scale factors a; it is convenient to work with the 10 field variables $\beta^\mu$, $\mu = 1, \ldots, 10$, with, in the superstring (Einstein-frame) case, $\beta^I \equiv - \ln a_i$ ($i = 1, \ldots, 9$), and $\beta^{10} \equiv - \varphi$ where $\varphi$ is the Einstein-frame dilaton. [In M-theory there is no dilaton but $\mu \equiv i = 1, \ldots, 10$. In the string frame, we define $\beta^\mu_S \equiv - \ln(\sqrt{g} e^{-2\beta})$ and label $\mu = 0, \ldots, 9$.]

We consider the evolution near a past (big-bang) or future (big-crunch) spacelike singularity located at $t = 0$, where $t$ is the proper time from the singularity. In the gauge $N = - \sqrt{g}$ (where $g$ is the determinant of the Einstein-frame spatial metric), i.e. in terms of the new time variable $d\tau = -dt/\sqrt{g}$, the action (per unit comoving volume) describing the asymptotic dynamics of $\beta^\mu$ as $t \to 0^+$ or $\tau \to +\infty$ has the form

$$S = \int d\tau \left[ G_{\mu\nu} \frac{d\beta^\mu}{d\tau} \frac{d\beta^\nu}{d\tau} - V(\beta^\mu) \right],$$

$$V(\beta) \approx \sum_A C_A e^{-2w_A(\beta)}. \quad (2)$$

In addition, the time reparametrization invariance (i.e., the equation of motion of N in a general gauge) imposes the usual “zero-energy” constraint $E = G_{\mu\nu}(d\beta^\mu/d\tau)(d\beta^\nu/d\tau) + V(\beta^\mu) = 0$. The metric $G_{\mu\nu}$ in field-space is a 10-dimensional metric of Lorentzian signature $- + \cdots +$. Its explicit expression depends on the model and the choice of variables. In M-theory,

$$G_{\mu\nu}^{M} \beta^\mu_M \beta^\nu_M = \sum_{i=1}^{10} (d\beta^\mu_M)^2 - \sum_{i=1}^{10} (d\beta^\nu_M)^2,$$

while in the string models,

$$G_{\mu\nu}^{S} d\beta^\mu_S d\beta^\nu_S = \sum_{i=1}^{10} (d\beta^\mu_S)^2 - (d\beta^\nu_S)^2$$

are Lorentzian solutions appearing in Eq. (2), represents the effect, on the evolution of $(g_{\mu\nu}, \varphi)$, of either (i) the spatial curvature of $g_{ij}$ (“gravitational walls”), (ii) the energy density of some electric-type components of some p-form $A_{\mu_1 \ldots \mu_p}$ ("electric p-form wall"), or (iii) the energy density of some magnetic-type components of some p-form $A_{\mu_1 \ldots \mu_p}$ ("magnetic p-form wall"). The coefficients $C_A$ are all found to be positive, so that all the exponential walls in Eq. (2) are repulsive. The $C_A$‘s vary in space and time, but we neglect their variation compared to the asymptotic effect of $w_A(\beta)$ discussed below. Each exponent $-2w_A(\beta)$ appearing in Eq. (2) is a linear form in the $\beta^\mu : w_A(\beta) = w_{A\mu} \beta^\mu$. The complete list of “wall forms” $w_A(\beta)$, was given in \cite{5} for each string model. The number of walls is enormous, typically of the order of 700.

At this stage, one sees that the $\tau$-time dynamics of the variables $\beta^\mu$ is described by a Toda-like system in a
Lorentzian space, with a zero-energy constraint. But it seems daunting to have to deal with $\sim 700$ exponential walls! However, the problem can be greatly simplified because many of the walls turn out to be asymptotically irrelevant. To see this, it is useful to project the motion of the variables $\beta^\mu$ onto the 9-dimensional hyperbolic space $H^9$ (with curvature $-1$). This can be done because the motion of $\beta^\mu$ is always time-like, so that, starting (in our units) from the origin, it will remain within the 10-dimensional Lorentzian light cone of $G_{\mu\nu}$. This follows from the energy constraint and the positivity of $V$. With our definitions, the evolution occurs in the future light-cone. The projection to $H^9$ is performed by decomposing the motion of $\beta^\mu$ into its radial and angular parts (see the generalization \[\Box\] and the recent comments on the covariance of the chaos obtained in the billiard approximation). One writes $\beta^\mu = +\rho \gamma^\mu$ with $\rho^2 \equiv -G_{\mu\nu} \beta^\mu \beta^\nu$, $\rho > 0$ and $G_{\mu\nu} \gamma^\mu \gamma^\nu = -1$ (so that $\gamma^\mu$ runs over $H^9$, realized as the future, unit hyperboloid) and one introduces a new evolution parameter: $dT = k d\tau/\rho^2$. The action \[\Box\] becomes

$$S = k \int d\tau \left[ -\left( \frac{d\ln \rho}{d\tau} \right)^2 + \left( \frac{d\gamma}{d\tau} \right)^2 - V_T(\rho, \gamma) \right]$$

(3)

where $d\gamma^2 = G_{\mu\nu} d\gamma^\mu d\gamma^\nu$ is the metric on $H^9$, and where $V_T = k^{-2} \rho^2 V = \sum_A k^{-2} C_A \rho^2 \exp(-2 \rho w_A(\gamma))$. When $t \to 0^+$, i.e. $\rho \to +\infty$, the transformed potential $V_T(\rho, \gamma)$ becomes sharper and sharper and reduces in the limit to a set of $\rho$-independent impenetrable walls located at $w_A(\gamma) = 0$ on the unit hyperboloid (i.e. $V_T = 0$ when $w_A(\gamma) > 0$, and $V_T = +\infty$ when $w_A(\gamma) < 0$). In this limit, $d\ln \rho/d\tau$ becomes constant, and one can choose the constant $k$ so that $d\ln \rho/d\tau = 1$. The (approximately) linear motion of $\beta^\mu(\tau)$ between two “collisions” with the original multi-exponential potential $V(\beta^\mu)$ is thereby mapped onto a geodesic motion of $\gamma(T)$ on $H^9$, interrupted by specular collisions on sharp hyperplanar walls. This motion has unit velocity $(d\gamma/d\tau)^2 = 1$ because of the energy constraint. In terms of the original variables $\beta^\mu$, the motion is confined to the convex domain (a cone in a 10-dimensional Minkowski space) defined by the intersection of the positive sides of all the wall hyperplanes $w_A(\beta) = 0$ and of the interior of the future light-cone $G_{\mu\nu} \beta^\mu \beta^\nu = 0$.

A further, useful simplification is obtained by quotienting the dynamics of $\beta^\mu$ by the natural permutation symmetries inherent in the problem, which correspond to “large diffeomorphisms” exchanging the various proper directions of expansion and the corresponding scale factors. The natural configuration space is therefore $\mathbb{R}^d/S_d$, which can be parametrized by the ordered multi-plets $\beta^1 \leq \beta^2 \leq \cdots \leq \beta^d$. This quotienting is standard in most investigations of the BKL oscillations \[\Box\] and can be implemented in $\mathbb{R}^d$ by introducing further sharp walls located at $\beta^\mu = \beta^\mu + 1$. Note that the natural permutation symmetry group is different in $M$-theory (where it is $S_{10}$), and in the $D = 10$ string models ($S_9$), and would be still smaller in the successive dimensional reductions of these theories. However, there is a natural consistency in quotienting each model by its natural permutation symmetry. Indeed, one finds that, upon dimensional reduction, there arise new (exponential) walls, which replace the missing permutation symmetries in lower dimensions \[\Box\]. Finally the dynamics of the models is equivalent, at each spatial point, to a hyperbolic billiard problem. The specific shape of this model-dependent billiard is determined by the original walls and the permutation walls. Only the “innermost” walls (those which are not “hidden” behind others) are relevant.

We have determined the set of innermost walls for all string models. The analysis is straightforward \[\Box\] and we report here only the final results, which are remarkably simple. Instead of the $O(700)$ original walls we find, in all cases, that there are only 10 relevant walls. In fact, the seven string theories M, IIA, IIB, I, HO, HE and the closed bosonic string in $D = 10$ \[\Box\], split into three separate blocks of theories, corresponding to three distinct billiards. The first block (with 2 SUSY’s in $D = 10$) is $B_2 = \{ M, IIA, IIB \}$ and its ten walls are (in the natural variables of $M$-theory $\beta^\mu = \beta^\mu(M)$):

$$B_2 : w_i^{[2]}(\beta) = -\beta^i + \beta^{i+1}(i = 1, \ldots, 9),$$

$$w_{10}^{[2]}(\beta) = \beta^3 + \beta^2 + \beta^3.$$ 

(4)

The second block is $B_1 = \{ I, HO, HE \}$ and its ten walls read (when written in terms of the string-frame variables of the heterotic theory $\alpha^{i+1} = \beta^i_{\perp}$, $\alpha^9 = \beta^9_{\perp}$)

$$B_1 : w_i^{[1]}(\alpha) = \alpha^i, \quad w_i^{[1]}(\alpha) = -\alpha^{i-1} + \alpha^i(i = 2, \ldots, 9),$$

$$w_{10}^{[1]}(\alpha) = \alpha^0 - \alpha^7 - \alpha^8 - \alpha^9.$$ 

(5)

The third block is simply $B_0 = \{ D = 10 \}$ closed bosonic and its ten walls read (in string variables)

$$B_0 : w_i^{[0]}(\alpha) = \alpha^1 + \alpha^2, \quad w_i^{[0]}(\alpha) = -\alpha^{i-1} + \alpha^i(i = 2, \ldots, 9),$$

$$w_{10}^{[0]}(\alpha) = \alpha^0 - \alpha^7 - \alpha^8 - \alpha^9.$$ 

(6)

In all cases, these walls define a simplex of $H^9$ which is non-compact but of finite volume, and which has remarkable symmetry properties.

The most economical way to describe the geometry of the simplices is through their Coxeter diagrams. This diagram encodes the angles between the faces and is obtained by computing the Gram matrix of the scalar products between the unit normals to the faces, say $V_{ij} = w_i \cdot w_j$ where $w_i \equiv w_i/\sqrt{w_i \cdot w_i}$, $i = 1, \ldots, 10$ labels the forms defining the (hyperplanar) faces of a simplex, and the dot denotes the scalar product (between co-vectors) induced by the metric $G_{\mu\nu} : w_i \cdot w_j \equiv G_{\mu\nu} w_i^\mu w_j^\nu$ for $w_i(\beta) = w_i(M) \beta^\mu$. This Gram matrix does not depend on the normalization of the forms $w_i$ but actually, all the wall forms $w_i$ listed above are normalized in a natural way, i.e. have a natural length. This is clear for
the forms which are directly associated with dynamical walls in $D = 10$ or 11, but this can also be extended to all the permutation-symmetry walls because they appear as dynamical walls after dimensional reduction [3].

When the wall forms are normalized accordingly (i.e. such that $V^i_{\text{dynamical}} \propto \exp(-2 w_i(\beta))$, they all have a squared length $w^{[n]}_i \cdot w^{[n]}_i = 2$, except for $w^{[1]}_i, w^{[5]}_i = 1$ in the $B_1$ block. We can then compute the “Cartan matrix”, $a^{[n]}_{ij} \equiv 2 w^{[n]}_i \cdot w^{[n]}_j / w^{[n]}_i \cdot w^{[n]}_j$, and the corresponding Dynkin diagram. One finds the diagrams given in Fig. 1.

![Dynkin diagrams](image)

FIG. 1. Dynkin diagrams defined (for each $n = 2, 1, 0$) by the ten wall forms $w^{[n]}_i(\beta^\mu), i = 1, \ldots, 10$ that determine the billiard dynamics, near a cosmological singularity, of the three blocks of theories $B_0 = \{M, IIA, IIB\}$, $B_1 = \{I, HO, HE\}$ and $B_0 = \{D = 10 \text{ closed bosonic}\}$. The node labels 1, \ldots, 10 correspond to the form label $i$ used in the text.

The corresponding Coxeter diagrams are obtained from the Dynkin diagrams by forgetting about the norms of the wall forms, i.e., by deleting the arrow in $BE_{10}$. As can be seen from the figure, the Dynkin diagrams associated with the billiards turn out to be the Dynkin diagrams of the following rank-10 hyperbolic Kac-Moody algebras (see [4]): $E_{10}$, $BE_{10}$ and $DE_{10}$ (for $B_2$, $B_1$ and $B_0$, respectively). It is remarkable that the three billiards exhaust the only three possible simplex Coxeter diagrams on $H^9$ with discrete associated Coxeter group (and this is the highest dimension where such simplices exist) [1]. The analysis suggests to identify the 10 wall forms $w^{[n]}_i(\beta), i = 1, \ldots, 10$ of the billiards $B_2$, $B_1$ and $B_0$ with a basis of simple roots of the hyperbolic Kac-Moody algebras $E_{10}$, $BE_{10}$ and $DE_{10}$, while the 10 dynamical variables $\beta^\mu, \mu = 1, \ldots, 10$, can be considered as parametrizing a generic vector in the Cartan subalgebra of these algebras. It was conjectured some time ago [12] that $E_{10}$ should be, in some sense, the symmetry group of SUGRA$_{11}$ reduced to one dimension (and that $DE_{10}$ be that of type I SUGRA$_{10}$, which has the same bosonic spectrum as the bosonic string). Our results, which indeed concern the one-dimensional reduction, à la BKL, of $M$/string theories exhibit a clear sense in which $E_{10}$ lies behind the one-dimensional evolution of the block $B_2$ of theories: their asymptotic cosmological evolution as $t \rightarrow 0$ is a billiard motion, and the group of reflections in the walls of this billiard is nothing else than the Weyl group of $E_{10}$ (i.e. the group of reflections in the hyperplanes corresponding to the roots of $E_{10}$, which can be generated by the 10 simple roots of its Dynkin diagram).

It is intriguing – and, to our knowledge, unanticipated (see, however, [13]) – that the cosmological evolution of the second block of theories, $B_1 = \{I, HO, HE\}$, be described by another remarkable billiard, whose group of reflections is the Weyl group of $BE_{10}$. The root lattices of $E_{10}$ and $BE_{10}$ exhaust the only two possible unimodular even and odd Lorentzian 10-dimensional lattices [4].

A first consequence of the exceptional properties of the billiards concerns the nature of the cosmological oscillatory behaviour. They lead to a direct technical proof that these oscillations, for all three blocks, are chaotic in a mathematically well-defined sense. This is done by reformulating, in a standard manner, the billiard dynamics as an equivalent collision-free geodesic motion on a hyperbolic, finite-volume manifold $\text{manifold}$ (without boundary) $M$ obtained by quotienting $H^9$ by an appropriate torsion-free discrete group. These geodesic motions define the “most chaotic” type of dynamical systems. They are Anosov flows [14], which imply, in particular, that they are “mixing”. In principle, one could (at least numerically) compute their largest, positive Lyapunov exponent, say $\lambda^{[n]}$, and their (positive) Kolmogorov-Sinai entropy, say $h^{[n]}$. As we work on a manifold with curvature normalized to $-1$, and walls given in terms of equations containing only numbers of order unity, these quantities will also be of order unity. Furthermore, the two Coxeter groups of $E_{10}$ and $BE_{10}$ are the only reflective arithmetic groups in $H^9$ [4] so that the chaotic motion in the fundamental simplices of $E_{10}$ and $BE_{10}$ will be of the exceptional “arithmetical” type [4]. We therefore expect that the quantum motion on these two billiards, and in particular the spectrum of the Laplacian operator, exhibits exceptional features (Poisson statistics of level-spacing,\ldots), linked to the existence of a Hecke algebra of mutually commuting, conserved operators. Another (related) remarkable feature of the billiard motions for all these blocks is their link, pointed out above, with Toda systems. This fact is probably quite significant, both classically and quantum mechanically, because Toda systems whose walls are given in terms of the simple roots of a Lie algebra enjoy remarkable properties. We leave to future work a study of our Toda systems which involve infinite-dimensional hyperbolic Lie algebras.

The present investigation a priori concerned only the “low-energy” ($E \ll (\alpha')^{-1/2}$), classical cosmological behaviour of string theories. In fact, if (when going toward the singularity) one starts at some “initial” time $t_0 \sim (d\beta/dt)^{-1}$ and insists on limiting the application of our results to time scales $|t| \gtrsim (\alpha')^{1/2} \equiv t_*$, the total number of “oscillations”, i.e. the number of collisions on the walls of our billiard will be finite, and will not be very large. The results above show that the
number of collisions between $t_0$ and $t \to 0$ is of order $N_{\text{coll}} \sim \ln \tau \sim \ln(\ln(t_0/t))$. This is only $N_{\text{coll}} \sim 5$ if $t_0$ corresponds to the present Hubble scale and $t$ to the string scale $t_s$. However, the strongly mixing properties of geodesic motion on hyperbolic spaces make it large enough for churning up the fabric of spacetime and transforming any, non particularly homogeneous at time $t_0$, patch of space into a turbulent foam at $t = t_s$. Indeed, the mere fact that the walls associated with the spatial curvature and the form fields repeatedly rise up (during the collisions) to the same level as the “time” curvature terms $\sim t^{-2}$, means that the spatial inhomogeneities at $t \sim t_s$ will also be of order $t_s^{-2}$, corresponding to a string scale foam.

Our results on the $B_2$ theories probably involve a deep (and not a priori evident) connection with those of Ref. [10] on the structure of the moduli space of $M$-theory compactified on the ten torus $T^{10}$, with vanishing 3-form potential. In both cases the Weyl group of $E_{10}$ appears. In our case it is (partly) dynamically realized as reflections in the walls of a billiard, while in Ref. [10] it is kinematically realized as a symmetry group of the moduli space of compactifications preserving the maximal number of supersymmetries. In particular, the crucial E-type node of the Dynkin diagram of $E_{10}$ (Fig. 1) comes, in our study and in the case of $M$-theory, from the wall form $w_{[2]}^{[10]}(\beta) = \beta_1^M + \beta_2^M + \beta_3^M$ associated with the electric energy of the 3-form. By contrast, in [10] the 3-form is set to zero, and the reflection in $w_{[2]}^{[10]}$ comes from the 2/5 duality transformation (which is a double $T$ duality in type II theories), which exchanges (in $M$-theory) the 2-brane and the 5-brane. As we emphasized above, dimensional reduction transforms kinematical (permutation) walls into dynamical ones. This suggests that there is no difference of nature between our walls, and that, viewed from a higher standpoint (12-dimension?), they would all look kinematical, as they are in [10]. By analogy, our findings for the $B_2$ theories suggest that the Weyl group of $BE_{10}$ is a symmetry group of the moduli space of $T^9$ compactifications of $\{I, HO, HE\}$.

Perhaps the most interesting aspect of the above analysis is to provide hints for a scenario of vacuum selection in string cosmology. If we heuristically extend our (classical, low-energy, tree-level) results to the quantum, stringy ($t \sim t_s$) and/or strongly coupled ($g_s \sim 1$) regime, we are led to conjecture that the initial state of the universe is equivalent to the quantum motion in a certain finite volume chaotic billiard. This billiard is (as in a hall of mirrors game) the fundamental polytope of a discrete symmetry group which contains, as subgroups, the Weyl groups of both $E_{10}$ and $BE_{10}$ [13]. We are here assuming that there is (for finite spatial volume universes) a non-zero transition amplitude between the moduli spaces of the two blocks of superstring “theories” (viewed as “states” of an underlying theory). If we had a description of the resulting combined moduli space (orbifolded by its discrete symmetry group) we might even consider as most probable initial state of the universe the fundamental mode of the combined billiard, though this does not seem crucial for vacuum selection purposes. This picture is a generalization of the picture of Ref. [14] and, like the latter, might solve the problem of cosmological vacuum selection in allowing the initial state to have a finite probability of exploring the subregions of moduli space which have a chance of inflating and evolving into our present universe.

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[1] T. Damour and M. Henneaux, Phys. Rev. Lett. 85, 920 (2000).
[2] T. Damour and M. Henneaux, Phys. Lett. B488, 108 (2000).
[3] V.A. Belinskii, E.M. Lifshitz and I.M. Khalatnikov, Adv. Phys. 31, 639 (1982).
[4] E.B. Bogomolny, B. Georgeot, M.J. Giannoni and C. Schmit, Phys. Rep. 291, 219-326 (1997).
[5] D.M. Chitre, Ph. D. thesis, University of Maryland, 1972.
[6] C.W. Misner, in D. Hobill et al. (Eds), "Deterministic chaos in general relativity," (Plenum, 1994) pp. 317-328.
[7] T. Banks, W. Fischler and L. Motl, JHEP 01, 019 (1999).
[8] T. Banks, W. Fischler and L. Motl, JHEP 01, 019 (1999).
[9] With $F$-theory [13] in mind, it is tempting to look for a Kac-Moody algebra with ultra-hyperbolic signature $(-+++\cdots)$ containing both $E_{10}$ and $BE_{10}$. According to V. Kac (private communication) the smallest such algebra is the rank-20 algebra whose Dynkin diagram is obtained by connecting, by a simple line, the $w_{[1]}^{[3]}$ and $w_{[1]}^{[9]}$ nodes in Fig. 1.
[10] V.G. Kac, "Infinite dimensional Lie algebras," third edition (Cambridge University Press, 1990).
[11] B. Julia, in "Lectures in Applied Mathematics, AMS-SIAM, vol. 21, (1985), p. 335.
[12] B. Julia, in "Lectures in Applied Mathematics, AMS-SIAM, vol. 21, (1985), p. 335.
[13] E. Cremmer, B. Julia, H. Lu and C.N. Pope, hep-th/9909009.