Riemannian and Finslerian spheres with fractal cut loci

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Abstract

The present paper shows that for a given integer \( k \geq 2 \) it is possible to construct an at least \( k \)-differentiable Riemannian metric on the sphere of a certain dimension such that the cut locus of a point of it becomes a fractal. Moreover, we show that this construction can be extended to the case of Finsler sphere as well.

1 Introduction

The cut locus \( C(p) \) of a point \( p \) in a Riemannian or Finsler manifold is, roughly speaking, the set of all other points for which there are multiple minimizing geodesics connecting them from \( p \). Of course, in some special cases, it may contain additional points where the minimizing geodesic is unique.

The notion of cut locus was introduced and studied for the first time by H. Poincaré in 1905 for the Riemannian case. Later on, in the case of a two dimensional analytical sphere, S. B. Myers has proved in 1935 that the cut locus of a point is a finite tree (11) in both Riemannian and Finslerian cases. Moreover, in the case of an analytic Riemannian manifold, M. Buchner has shown the triangulability of \( C(p) \) (8), and has determined its local structure for the low dimensional case (4) in 1977 and 1978, respectively.

Despite of the vast literature existing for the Riemannian case, the investigations of the cut locus of a Finsler manifolds are scarce (see [1], [13], [10]).

Recently, it was shown that the cut locus of a closed subset \( \mathcal{N} \) of a Finsler surface has the structure of a local tree being a union of rectifiable Jordan arcs (15). Even though the results are similar to the Riemannian case, showing that there is nothing special about the metric structure to be Riemannian, one should pay always attention to the fact that, unlike its Riemannian correspondent, the Finslerian distance is not symmetric, so the proofs and arguments are quite different.

Returning to the Riemannian case, in the case of an arbitrary metric, the cut locus of a point can have a very complicated structure. For example, H. Gluck and D. Singer have constructed a \( C^\infty \) Riemannian manifold that has a point whose cut locus is not triangulable (14).

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There is a closed relationship between the complexity of the cut locus and the regularity of the metric, regardless it is Riemannian or Finslerian. Indeed, if the metric has a certain degree of regularity, then the cut locus of a point may enjoy a simple structure. However, if the metric loses its regularity, the cut locus might become a very complicated set, for example a fractal. Recall that, roughly speaking, a fractal is a set whose Hausdorff dimension is not an integer (see [5] for alternative definitions and examples), fact that make fractals typical examples of what we call “complicated sets”.

Let us mention that the cut locus of any point on a $C^\infty$ Riemannian manifold cannot be a fractal (see [9]). However, there is a $C^{1,1}$ Riemannian metric on the two dimensional sphere $S^2$ and a point $p \in S^2$ such that the total length of $C(p)$ is infinite (see [7]).

Motivated by all these, in the present paper, we are going to study the following two questions:

1. There exists Riemannian metrics having points whose cut locus is a fractal?
2. There are more general metric structures, for example Finsler metrics, with the same property?

The answer to both questions above is affirmative. Indeed, in the present paper we construct an at least $k$ differentiable, $2 \leq k < \infty$, Riemannian metric on a topological sphere $S^n$, provided the dimension $n$ is high enough, namely, we prove

**Theorem 1.1** For any integer $2 \leq k < \infty$ there is an at least $k$-differentiable Riemannian metric on the $n(k)$-dimensional sphere $S^{n(k)}$ and a point $p \in S^{n(k)}$ such that the Hausdorff dimension of $C(p)$ is a real number between 1 and 2, where $n(k) := \frac{3k+1}{2} + 1$.

Moreover, we show that there is a Finsler metric of Randers type on this sphere with the same property. Indeed, if we use the same notations as in Theorem 1.1 we have

**Theorem 1.2** For any integer $2 \leq k < \infty$, under the influence of a suitable magnetic field $\beta$ defined on $S^{n(k)}$, there is an at least $k$-differentiable non-Riemannian Finsler metric of Randers type on $S^{n(k)}$ such that the cut locus of the point $p$ with respect to this Finsler metric coincides with $C(p)$.

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## 2 Basic construction

In this section, we will construct:

- an infinite tree $IT$ in $\mathbb{R}^n$ with the end points set $E$,
- a closed, convex ball $H$ in $\mathbb{R}^n$ with $C^1$-boundary that contains $IT$ and $E \subset \partial H$.

The set $E$ is actually a fractal whose Hausdorff dimension is a value between 1 and 2 (see [5] for definitions).
We begin by defining three infinite series of numbers
\[ t_i := \left( \frac{1}{3^{k-1}} \right)^i, \quad i \in \{0, 1, \ldots\} \]
\[ l_i := \frac{t_i}{\sin\left( \frac{\phi}{3^i} \right)}, \quad i \in \{0, 1, \ldots\} \]
\[ r_i := \sum_{\nu=i+1}^{\infty} l_{\nu} \cos\left( \frac{\phi}{3^\nu} \right), \quad i \in \{-1, 0, 1, \ldots\}, \]
(2.1)
where \( \phi \in \left(0, \frac{\pi}{2}\right) \) is an arbitrary fixed angle and \( k \geq 2 \) a fixed integer.

One can easily see that \((t_i), (l_i)\) and \((r_i)\) are monotone decreasing series that converge to zero, for \( i \to \infty \).

We will use these in order to construct a fractal set in \( \mathbb{R}^n \). For the moment we do not assume any relation between \( n \) and \( k \).

Let us consider in \( \mathbb{R}^n \) the points \( o, q, q_0, \ldots, q_{j_1j_2\ldots j_m} \), where \( m \in \{1, 2, \ldots\} \), \( i \in \{1, 2, \ldots, m\} \), and \( j_i \in \{-n-1, \ldots, -1, 0, 1, \ldots, n-1\} \), defined as follows.

\[ o := (0, 0, \ldots, 0), \quad q := (l_0, 0, \ldots, 0), \]
(2.2)

\[ m = 1 \]
\[ q_0 := (l_0 + l_1, 0, \ldots, 0) \]
\[ q_{j_1} := (l_0 + l_1 \cos\left( \frac{\phi}{3} \right), 0, \ldots, 0, l_1 \sin\left( \frac{\phi}{3} \right) a_{1}, 0, \ldots, 0), \]
(2.3)

\[ m > 1 \]
\[ q_{0\ldots 0} := \sum_{i=0}^{m-1} l_i, 0, \ldots, 0 \]
\[ q_{0\ldots j_m} := \sum_{i=0}^{m-1} l_i + l_m \cos\left( \frac{\phi}{3^m} \right), 0, \ldots, 0, l_1 \sin\left( \frac{\phi}{3^m} \right) a_m, 0, \ldots, 0, \]
(2.4)

\[ q_{j_1j_2\ldots j_m} := R_{|j_1|+1}^{l_1}(q, \frac{\phi}{3} a_1) \circ R_{|j_2|+1}^{l_2}(q, \frac{\phi}{3^2} a_2) \circ \ldots \]
\[ \circ R_{|j_m|+1}^{l_m}(q_{j_1j_2\ldots j_{m-1}}, \frac{\phi}{3^{m-1}} a_{m-1}) \circ q_{00\ldots j_m}, \]
where
\[ a_i := \begin{cases} 
-1, & \text{if } j_i < 0 \\
0, & \text{if } j_i = 0 \\
1, & \text{if } j_i > 0 
\end{cases} \]
(2.5)
and $R(x, \theta)$ is the rotation of angle $\theta$ around the affine subspace that is orthogonal to the $<e_1, e_j>$ plane and contains $x$. Here $e_j$ is the unit vector with all components zero except the $j$-th component which is 1, namely 

$$e_j = (0, \ldots, 1, \ldots, 0),$$

$x \in \{q_{j_1j_2\cdots j_m} : j_1, j_2, \ldots j_m \text{ as above}\}$ and $\theta \in \{\frac{\phi}{3^i} : i = 1, \ldots, m - 1\}$.

One can easily see that the points $q_{j_1j_2\cdots j_m}$ are all on the sphere with center $q_{j_1j_2\cdots j_{m-1}}$ and radius $l_m$, for any fixed $m > 0$ and $j_i, i \in \{1, 2, \ldots, m\}$.

For later use we denote the segment between $o$ and $q$ by $s$, the segment between $q$ and $q_{j_1}$ by $s_{j_1}$ and so on inductively, such that the segment between $q_{j_1j_2\cdots j_{m-1}}$ and $q_{j_1j_2\cdots j_m}$ will be denoted by $s_{j_1j_2\cdots j_{m-1}j_m}$.

Likely, we denote the ray from $o$ that contains $q$ by $\gamma$, the ray from $q$ that contains $s_{j_1}$ by $\gamma_{j_1}$ and so on inductively, such that the ray from $q_{j_1j_2\cdots j_{m-1}}$ that contains $s_{j_1j_2\cdots j_{m-1}j_m}$ will be denoted by $\gamma_{j_1j_2\cdots j_{m-1}j_m}$.

Let $I_T^0 := \{s\}$, $I_T^m := \bigcup_{i=1}^m \{s_{j_1j_2\cdots j_{m-1}j_m}\}$ be the union of segments $s_{j_1j_2\cdots j_{m-1}j_m}$, $I_T := \lim_{m \to \infty} I_T^m$ be the infinite tree and let $Q := \bigcup_{i=1}^\infty \{q_{j_1j_2\cdots j_{m-1}j_m}\}$ be the set of points $q_{j_1j_2\cdots j_{m-1}j_m}$.

Taking into account (2.1) and the construction above, one can see that $\lim_{i \to \infty} l_i = 0$ implies $L := \sum_{i=0}^\infty l_i$ is finite, therefore the edges can not prolong to infinity, so the set $Q$ must have a subset of limit points $E \subset Q$.

The set $E$ is in fact the set of end points of the infinite tree $I_T$, except the root point $o$ (see Figure 1). Moreover, one can see that $I_T$ is completely contained in the ball with center $o$ and radius $L$. 

![Diagram](image-url)
Figure 1. The tree $IT_m$ in $\mathbb{R}^2$ for $m = 1, 2, 3, 4, 5$, respectively.

Remark 2.1 Remark that for the infinite tree $IT$ in $\mathbb{R}^n$, there are $2^n - 1$ branches that ramify from each node. The maximum depth level in the tree is given by $m$ and the branches and nodes at a given depth level $i$ are specified by $j_i$. Figure 1 shows the growth of the tree $IT$ in the case $\mathbb{R}^2$, namely a tree with three branches that ramify from each node at each level.

Next, we will construct a closed, convex ball $H$ in $\mathbb{R}^n$ with $C^1$-boundary as follows:

- take a point $\hat{q}$ on the straight line $\gamma$ such that $d(0, \hat{q}) = \frac{r-1}{\cos \phi}$, where $d$ is the usual Euclidean distance;
- consider the right circular cone $C$ with vertex $\hat{q}$, axis $\gamma$ and vertex angle $\pi - 2\phi$;
- consider the $(n - 1)$-spheres $S_0 = S(q, r_0)$ and $S = S(o, r_{-1})$ in $\mathbb{R}^n$ of center $q$ and $o$, and radii $r_0$ and $r_{-1}$, respectively.

Then, it can be verified by simple trigonometric computations that the right circular cone $C$ is exterior tangent to the spheres $S_0$ and $S$ (see Figure 2). The intersection of $C$ with the spheres $S_0$ and $S$ is made of the $(n - 2)$-spheres $c$ and $c'$, respectively (in the case $IT \subset \mathbb{R}^3$ these are circles).

Figure 2. A longitudinal section in the cone $C$ in the case $IT_1 \subset \mathbb{R}^3$.

Remark that the $(n - 2)$-spheres $c$ and $c'$ cut out:
• a truncated cone \( A \subset C \), from the cone \( C \);
• a small spherical cap \( P_0 \) with boundary \( c \cap S_0 \) from the sphere \( S_0 \);
• a large spherical cap \( P \) with boundary \( c' \cap S \) from the sphere \( S \).

Gluing at the both ends of the truncated cone \( A_0 \) the spherical caps mentioned above we obtain a closed, convex ball \( H_0 \subset \mathbb{R}^n \) whose boundary is a \( C^1 \)-hypersurface \( \partial H_0 \subset \mathbb{R}^n \) (see Figure 3).

**Figure 3.** The segment \( s = oq \) is the inward cut locus of the \( C^1 \)-hypersurface \( \partial H_0 \) in the case \( IT_0 \subset \mathbb{R}^3 \).

We remark that the (inward) cut locus of \( H_0 \) endowed with the induced Euclidean norm from \( \mathbb{R}^n \) is exactly the segment \( s \). Indeed, the inward geodesic rays from \( \partial H_0 \) concentrates at a point \( \hat{u} \in s \). The same length inward geodesic rays orthogonal to the cloth of the truncated cone \( A_0 \), emanating from an arbitrary point of the \((n-2)\)-sphere

\[
c_u := \{ x \in A | d(x, \hat{q}) = (d(o, \hat{q}) - u) \sin \phi \},
\]

concentrate at a point \( \hat{u} := (u, 0, \ldots, 0) \in s \setminus \{ o, q \} \), the inward geodesic rays of same length from the small spherical cap of \( S_0 \) concentrate at \( q \) and similarly the geodesic rays from the large spherical cap of \( S \) concentrate at \( o \).

Therefore, given a segment \( s = oq \subset IT \) we obtain a \( C^1 \)-hypersurface \( \partial H_0 \subset \mathbb{R}^n \) whose (inward) cut locus and conjugate locus is exactly the segment \( s \).

Let us see how this construction looks like at next step. Consider \( m = 1 \) and therefore our tree becomes \( IT_1 = \{ s, s_{j_1} | j_1 \in \{- (n - 1), \ldots, 0, 1, \ldots, n - 1 \} \} \). The construction reads now:
• take a set of points $\hat{q}_{j_1}$ on the straight line $\gamma_{j_1}$ such that $d(q, \hat{q}_{j_1}) = \frac{r_0}{\cos \frac{\phi}{m}}$;

• consider the right circular cones $C_{j_1}$ with vertices $\hat{q}_{j_1}$, axes $\gamma_{j_1}$ and vertex angles $\pi - 2\frac{\phi}{m}$;

• consider the $(n-1)$-spheres $S_{j_1} = S(q_{j_1}, r_1)$ and $S_0 = S(q, r_0)$ in $\mathbb{R}^n$.

The right circular cones $C_{j_1}$ are exterior tangent to the spheres $S_{j_1}$ and $S_0$. The intersection of $C_{j_1}$ with the spheres $S_{j_1}$ and $S_0$ is made of $2n-1$ spheres $c_{j_1} := C_{j_1} \cap S_{j_1}$ and $c_{j_1}^{0} = C_{j_1} \cap S_0$, respectively, that cut off

• $2n-1$ truncated cones $A_{j_1} \subset C_{j_1}$, from the cone $C_{j_1}$;

• $2n-1$ small spherical caps, from the spheres $S_{j_1}$, whose boundaries are $C_{j_1} \cap S_{j_1}$;

• $2n-1$ large spherical caps, from the sphere $S_0$, whose boundaries are $C_{j_1} \cap S_0$.

Consider now the small spherical cap on $S_0$ cut off by the cone $C$ on which the $2n-1$ truncated cones $A_{j_1}$ sit. We denote by $B_0 \subset S_0$ the region left from this small spherical cap after cutting off the new appeared small spherical cups $\bigcup_{j_1} (C_{j_1} \cap S_0)$.

Then

$$\partial H_1 := \bigcup_{j_1} (A_{j_1} \cup B_{j_1}) \cup P$$ \tag{2.6}

is the convex $C^1$-hypersurface and define $H_1$ to be the closed, convex ball whose boundary is $\partial H_1$. Obviously $H_1$ contains $IT_1$ and the inner cut locus of $\partial H_1$ is precisely $IT_1$.

This construction can be repeated for each segment $s_{j_1j_2\cdots j_m} \subset IT$ obtaining in this way a closed, convex ball $H_m \subset \mathbb{R}^n$, which contains $IT_m$, with $C^1$-boundary $\partial H_m$ whose (inward) cut locus coincides with the tree $IT_m$.

Indeed, we construct as follows:

• take a point $\hat{q}_{j_1j_2\cdots j_m}$ on the straightline $\gamma_{j_1j_2\cdots j_m}$ such that

$$d(q_{j_1j_2\cdots j_m-1}, \hat{q}_{j_1j_2\cdots j_m}) = \frac{r_{m-1}}{\cos\frac{\phi}{m}} \tag{2.7}$$

• consider the right circular cone $C_{j_1j_2\cdots j_m}$ with vertex $\hat{q}_{j_1j_2\cdots j_m}$, axis $\gamma_{j_1j_2\cdots j_m}$ and vertex angle $\pi - 2\frac{\phi}{m}$;

• denote the $(n-1)$-sphere $S_{j_1j_2\cdots j_m} := S(q_{j_1j_2\cdots j_m}, r_m)$ and consider the spheres $S_{j_1j_2\cdots j_m}$ and $S_{j_1j_2\cdots j_m-1}$ with centers at the ends of the segment $s_{j_1j_2\cdots j_m}$.

It follows again that the right circular cone $C_{j_1j_2\cdots j_m}$ is exterior tangent to the spheres $S_{j_1j_2\cdots j_m-1}$ and $S_{j_1j_2\cdots j_m}$, The intersection of $C_{j_1j_2\cdots j_m}$ with the spheres $S_{j_1j_2\cdots j_m}$ and $S_{j_1j_2\cdots j_m}$ is made of the $(n-2)$-spheres $c_{j_1j_2\cdots j_m-1}$ and $c_{j_1j_2\cdots j_m}$, respectively.

Similarly as above, the $(n-2)$-spheres $c_{j_1j_2\cdots j_m-1}$ and $c_{j_1j_2\cdots j_m}$ cut out:

• a truncated cone $A_{j_1j_2\cdots j_m} \subset C_{j_1j_2\cdots j_m}$;

• a small spherical cap of $S_{j_1j_2\cdots j_m-1}$ whose boundary is $c_{j_1j_2\cdots j_m-1} \cap S_{j_1j_2\cdots j_m-1}$. 


• a large spherical cap \( S_{j_1j_2\ldots j_m} \) whose boundary is \( c_{j_1j_2\ldots j_m} \cap S_{j_1j_2\ldots j_m} \).

Consider now the small spherical cap on \( S_{j_1j_2\ldots j_m} \) cut off by the cone \( C_{j_1j_2\ldots j_m} \) on which the \( 2n - 1 \) truncated cones \( A_{j_1j_2\ldots j_m} \) sit. We denote by \( B_{j_1j_2\ldots j_m} \subset S_{j_1j_2\ldots j_m} \) the region left from this small spherical cap after cutting off the new appeared small spherical cups \( \bigcup_{j_1j_2\ldots j_m} (C_{j_1j_2\ldots j_m} \cap S_{j_1j_2\ldots j_m}) \).

Then
\[
\partial H_m := \bigcup_{i=0}^{m} \bigcup_{j_1j_2\ldots j_m} (A_{j_1j_2\ldots j_m} \cup B_{j_1j_2\ldots j_m}) \cup P \tag{2.8}
\]
is the convex \( C^1 \)-hypersurface that defines the closed, convex ball \( H_m \) in \( \mathbb{R}^n \). By a similar argument as above one can see that \( \bigcup_{i=0}^{m} \bigcup_{j_1j_2\ldots j_m} s_{j_1j_2\ldots j_m} \) is the inward cut locus of \( \partial H_m \).

One can remark that in our construction the conjugate locus coincides with the focal locus.

Indeed, the equal length inward geodesic rays orthogonal to the cloth of the truncated cone \( A_{j_1j_2\ldots j_m} \), emanating from an arbitrary point of the \((n - 2)\)-sphere \( \{x \in A_{j_1j_2\ldots j_m} : d(x, \hat{q}_{j_1j_2\ldots j_m}) = \text{constant} \} \), concentrate at a point on the open segment \( s_{j_1j_2\ldots j_m} \).

The inward geodesic rays of same length orthogonal to the region \( B_{j_1j_2\ldots j_m} \subset S_{j_1j_2\ldots j_m} \) concentrate at \( q_{j_1j_2\ldots j_m} \) and similarly the geodesic rays from the large spherical cap of \( S \) concentrate at \( o \).

Finally, we define
\[
H := \lim_{m \to \infty} H_m \tag{2.9}
\]
that have the required properties.

Therefore we obtain

**Proposition 2.2** Let \( \text{IT} \) be the infinite tree in \( \mathbb{R}^n \) and \( H \) the closed, convex ball with \( C^1 \)-boundary constructed above. Then \( H \subset \mathbb{R}^n \) contains \( \text{IT} \) and \( E \subset \partial H \) by construction.

### 3 The Hausdorff measure of \( E \)

In this section we will investigate the Hausdorff dimension of the set \( E \). The idea is to construct a set of points \( \hat{E} \) whose Hausdorff dimension \( \dim \hat{E} \) can be easily computed and such that \( \dim H \hat{E} = \dim \hat{E} \).

We start by defining an infinite series of numbers \((\alpha_i), i \in \{0, 1, \ldots\} \) by
\[
\alpha_0 = \frac{3^{k-2}}{3^{k-2} - 1}, \quad \alpha_{i+1} = \frac{1}{3^{k-1}} \alpha_i, \tag{3.1}
\]
where \( t_i \) is defined in (2.1).

Remark that, for any \( i \), we have
\[
\alpha_i = 3\alpha_{i+1} + t_i. \tag{3.2}
\]

In \( \mathbb{R}^{n-1} \) we consider

- the concentric \((n - 2)\)-balls \( \hat{C} = B(o, \alpha_0) \) and \( \hat{S} = B(o, \alpha_0 - t_0) \) with the center \( o \) at the origin of \( \mathbb{R}^{n-1} \) and radii \( \alpha_0 \) and \( \alpha_0 - t_0 \), respectively;
• $\hat{A} := \hat{C} \setminus \hat{S}$ the annulus obtained by removing $\hat{S}$ from $\hat{C}$.

We will construct a set of points $y_{j_1,j_2,\ldots,j_m} \in \hat{S}$, where $m \in \{1,2,\ldots\}$ and $j_m \in \{-n + 1,\ldots,-1,0,1,\ldots,n-1\}$ are as in Section 2.

For $m = 1$ we consider the points

$$(y_{j_1}) = (y_{-(n-1)}, y_{-(n-2)}, \ldots, y_2, y_1, y_0, y_1, y_2, \ldots, y_{n-1}) \in \mathbb{R}^{n-1},$$

(3.3)

where

$$(y_1, y_2, \ldots, y_{n-1})^t = 2\alpha_1 I_{n-1}, \quad y_0 = o,$$

$$(y_{-(n-1)}, y_{-(n-2)}, \ldots, y_2, y_1)^t = -2\alpha_1 I_{n-1},$$

(3.4)

where $^t$ represents the transposed of a vector, and $I_{n-1}$ the identity matrix.

We can construct iteratively, for arbitrary $m$, the set of points $y_{j_1,j_2,\ldots,j_m} \in \mathbb{R}^{n-1}$ defined by

$$y_{j_1,j_2,\ldots,j_m} = (\sum_{i=1}^{m} 2b_1^i \alpha_i, \sum_{i=1}^{m} 2b_2^i \alpha_i, \ldots, \sum_{i=1}^{m} 2b_{n-1}^i \alpha_i),$$

(3.5)

where $(b_j^i)$ is an $m \times (n-1)$-real matrix defined by

$$b_j^i = \begin{cases} 1, & \text{if } j_i = j \\ -1, & \text{if } j_i = -j \\ 0, & \text{otherwise} \end{cases}$$

(3.6)

for all $i \in \{1,2,\ldots,m\}$ and $j \in \{1,2,\ldots,n-1\}$. One can easily see that for $m = 1$ we get (3.4).
Let us denote the set of all points $y$ by $Y := \bigcup_{m=1}^{\infty} y_{jj_2\ldots j_m}$, and let $\hat{E}$ be the set of accumulation points of $Y$. Since $Y \subset \hat{S}$ the set of points $\hat{E} \subset \hat{S}$ exists and it is not empty.

Let $\hat{R}_1 : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be the contraction function that maps $\hat{S}$ into $\hat{S}_1$. It can be seen that $\hat{R}_1$ is a similarity map, i.e. there exists a constant $c_1 \in (0, 1)$ such that

$$d(\hat{R}_1(x), \hat{R}_1(y)) = c_1 d(x, y),$$

(3.7)

for any $x, y \in \mathbb{R}^{n-1}$. The constant $c_1$ is called the ratio of $\hat{R}_1$ (see [5], p. 128). Indeed, taking into account our definitions, one obtains $c_1 = \frac{1}{3^{k-1}}$.

Iteratively, for a given $m$, we define the mapping

$$\hat{R}_j : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}, \quad \hat{S}_{jj_2\ldots j_m} \to \hat{S}_{jj_2\ldots j_m j},$$

(3.8)

for all $j \in \{- (n - 1), \ldots, -1, 0, 1, \ldots, n - 1\}$. Obviously $\hat{R}_j$ is a similarity map with ratio $c_j = \frac{1}{3^{k-1}}$. Thus each $\hat{R}_j$ transform subsets of $\mathbb{R}^{n-1}$ into geometrically similar sets.
The attractor of such a collection of similarity maps values domains, in our case $\hat{E}$, is a self-similar set, being a union of smaller similar copies of itself.

We recall from [5] that the mappings $\hat{R}_j$ satisfy the open set condition if there exists a non-empty bounded open set $V$ such that

$$V \supset \bigcup_{j=1}^{2n-1} \hat{R}_j(V).$$

(3.9)

By taking $V := \hat{S}$ one can see that the mappings $\hat{R}_j$ considered in (3.8) satisfy the open set condition.

It follows from Theorem 9.3 in [5] that the Hausdorff dimension $s := \dim_H \hat{E}$ can be computed from the formula

$$\sum_{j=1}^{2n-1} (c_j)^s = 1,$$

(3.10)

where $c_j$ are the ratios of the similarity maps $\hat{R}_j$. Therefore, in our case we have

$$\sum_{j=1}^{2n-1} \left(\frac{1}{3^{k-1}}\right)^s = 1,$$

(3.11)

and by an elementary computation we obtain

$$s = \frac{\log(2n - 1)}{(k - 1) \log 3}.$$

(3.12)

Imposing now the condition $1 < s < 2$ we get

$$\frac{3^{k-1}}{2} < n < \frac{3^{2(k-1)} + 1}{2},$$

(3.13)

where $n$ and $k$ are positive integers. A moment of thought convinces that

$$n = \frac{3^{k-1} + 3}{2}$$

(3.14)

is a positive integer that satisfies this condition ($n$ is obviously integer because all powers of an odd number are odd and sum of two odd numbers is even).

Therefore, we have

**Proposition 3.1** Let $k \geq 2$ be a fixed integer and let $n := \frac{3^{k-1} + 3}{2}$. Then

$$\dim_H \hat{E} = \frac{\log(2n - 1)}{(k - 1) \log 3} \in (1, 2),$$

(3.15)

where $\hat{E}$ is the accumulation points set of $Y$ constructed as above.

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Remark that the minimal admitted value for \( n \) is \( n = 3 \) for \( k = 2 \). In this case we have \( \dim_{H} \hat{E} = \frac{\log 5}{\log 3} = 1.46497 \).

Let us point out that the sets \( \hat{S} \) and \( H = \bigcup_{m=0}^{\infty} \bigcup_{j_{1},j_{2},\ldots,j_{m}} A_{j_{1},j_{2},\ldots,j_{m}} \cup B_{j_{1},j_{2},\ldots,j_{m}} \) are bi-Lipschitz, i.e. one can define a map \( \Phi : \hat{S} \to H \) and there are positive constants \( c \) and \( C \) such that
\[
c \cdot d(x, y) \leq d(\Phi(x), \Phi(y)) \leq C \cdot d(x, y), \tag{3.16}
\]
for all \( x, y \in \hat{S} \). Indeed, since \( \phi \) has been chosen arbitrary, if we take a small \( \phi < \varepsilon \), for any \( \varepsilon > 0 \), then \( \frac{\pi}{2} - \phi \to \frac{\pi}{2} \), i.e. the small spherical caps \( S_{j_{1},j_{2},\ldots,j_{m}} \) are almost included in the orthogonal planes to the rays \( \gamma_{j_{1},j_{2},\ldots,j_{m}} \). This means that \( \hat{S} \) and \( H \) are bi-Lipschitz.

On the other hand, it is known that the Hausdorff dimensions of two bi-Lipschitz sets coincide (see [5] for example), and therefore we obtain

**Corollary 3.2** Let \( k \geq 2 \) be a fixed integer and let \( n := \frac{2^{k-1} + 3}{2} \). Then
\[
\dim_{H} E = \frac{\log(2n - 1)}{(k - 1) \log 3} \in (1, 2), \tag{3.17}
\]
where \( E \) is the set of endpoints of \( IT \) constructed in Section 2.

We point out that the dimension \( n(k) \) increases exponentially with \( k \).

## 4 Proof of Theorem 1.1

In this section we construct a closed, convex ball in \( \mathbb{R}^{n} \) such that the inward cut locus of \( \partial D \) coincides with the infinite tree \( IT \). Then we will smooth out the truncated cones \( A_{j_{1},j_{2},\ldots,j_{m}} \) at each depth level \( m \).

Firstly, for each depth level \( m \), let us consider an “\( \varepsilon \)-dilatation” of the closed, convex ball \( H_{m} \), namely we define
\[
D_{m} := \{ \text{the convex ball whose boundary is } \partial H_{m} \} \cup \{ x \in \mathbb{R}^{n} \mid d(x, \partial H_{m}) \leq \varepsilon \}, \tag{4.1}
\]
for any positive constant \( \varepsilon > 0 \). Denote the limit
\[
D := \lim_{m \to \infty} D_{m} \tag{4.2}
\]
that is the \( \varepsilon \)-dilatation of convex ball \( H \).

Note that the boundary \( \partial D \in \mathbb{R}^{n} \) is the convex \( C^{1} \)-hypersurface
\[
\partial D = \lim_{m \to \infty} \left[ \bigcup_{i=0}^{m} \bigcup_{j_{1},j_{2},\ldots,j_{m}} \left( \tilde{A}_{j_{1},j_{2},\ldots,j_{m}} \cup \tilde{B}_{j_{1},j_{2},\ldots,j_{m}} \right) \right] \cup \tilde{P} \cup \tilde{E}, \tag{4.3}
\]
where the tilde notation means the \( \varepsilon \)-dilatation of the corresponding geometrical objects of \( H \).

One can easily see that the geodesics orthogonal to \( \partial D \) are in fact geodesics to \( H \) extended by \( \varepsilon \) at one of their ends, and therefore the inward cut locus of \( \partial D \) coincides with the infinite tree \( IT \). Obviously the set of end points \( E \) are now in the interior of the ball \( D \) and not on its boundary as in the case of \( H \). This allows us to realize the infinite tree \( IT \) as the inward cut locus of a hypersurface in \( \mathbb{R}^{n} \).
Secondly, we are going to smooth out the truncated cones $\tilde{A}_{j_1,j_2 \ldots j_m}$ on $\partial D$ such that the inward cut locus does not change.

Let us refer to Figure 6 restricted to the upper half Euclidean plane $\mathbb{H} := \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0 \}$. Recall that $S$ and $S_0$ are circles of centers $o$ and $q$ with radii $r_1$ and $r_0$, respectively. We further denote:

- the intersection points of $\partial D_0$ with the horizontal coordinate axis by $A(a, 0)$ and $B(b, 0)$, $a < b$, respectively;
- the $x_1$ coordinates of the points $\tilde{c}'$ and $\tilde{c}$ by $t_1$ and $t_2$;
- the straightline segment determined by the points $\tilde{c}'$ and $\tilde{c}$ by $d = \{ d(x_1) \mid t_1 < x_1 < t_2 \}$.
- the large circular segment on $\partial D_0$ from $A$ to $\tilde{c}'$ by $\{ f_1(x_1) \mid a < x_1 < t_1 \}$, and the small circular segment on $\partial D_0$ from $\tilde{c}$ to $B$ by $\{ f_2(x_1) \mid t_2 < x_1 < b \}$.

We have the following smoothing Proposition:

**Proposition 4.1** With the notations above, there is a $C^\infty$-function $F(x_1)$ defined on $[a, b]$ that takes values in $\mathbb{H}$ such that

- $F(x_1) = f_1(x_1)$ for $x_1 \in [a, t_1]$;
- $F(x_1) = f_2(x_1)$ for $x_1 \in [t_2, b]$;
- $d(x_1) \geq F(x_1) \geq \max(f_1(x_1), f_2(x_1))$, for $t_1 < x_1 < t_2$.

Figure 6. The $\varepsilon$-dilatation of the convex ball $H_0$. 
• all the inward straight lines (geodesic rays) orthogonal to $F(x_1)$, for $t_1 < x_1 < t_2$, do not intersect each other in the region $\{x_2 < d(x_1)\} \cap \mathbb{H}$.

In other words, we will contract a smooth curve that joins points $\tilde{c}'$ and $\tilde{c}$. By the same operation in the lower half plane $\{(x_1, x_2) \mid x_2 \leq 0\}$ we obtain a smooth plane curve.

![Figure 7. Smoothing of $\partial D_0$. Before smoothing (left), after smoothing (right).](image)

**Proof.** For the sake of simplicity we rotate counterclockwise with $\frac{\pi}{2} - \phi$ the coordinates system used in Section 2 (compare with Figure 2) obtaining Figure 7 (left). The new coordinate system is denoted with $\{x, y\}$ while $\tilde{c}'$, $\tilde{c}$, $f_1$, $f_2$, $d$ have the same meaning as above. We denote the coordinates of points $\tilde{c}'$ and $\tilde{c}$ by $(x_{\tilde{c}'}, y_{\tilde{c}'})$ and $(x_{\tilde{c}}, y_{\tilde{c}})$, respectively. Moreover, we denote $\{R\} = f_1 \cap f_2$ with coordinates $(x_R, y_R)$. In Figure 7 we have expressed by vertical dots the fact that $y_R$ is actually quite far from the origin $o$ and quite closed to $\tilde{c}'$, i.e. $d(o, y_R) >> d(y_R, y_{\tilde{c}'})$.

With these notations, we consider the function

$$F_1(x) := \int_0^x [1 - g_\sigma(t)] f'_1(t)dt|_{\sigma=x} + y_{\tilde{c}'}, \quad 0 \leq x \leq x_R, \quad (4.4)$$
where $0 < \sigma \leq x_R$,

$$
g_\sigma(t) := \begin{cases} 
0, & t \leq 0 \\
\frac{\varphi(t)}{\varphi(t) + \varphi(\sigma-t)}, & 0 < t < \sigma \\
1, & t \geq \sigma
\end{cases} \quad (4.5)
$$

and

$$
\varphi(t) := \begin{cases} 
0, & t \leq 0 \\
e^{-\frac{t}{\tau}}, & t > 0
\end{cases} \quad (4.6)
$$

\textbf{Figure 8.} The graph of the function $g := g_\sigma$ for $\sigma = 1$.

It is elementary to see that $F_1$ is a smooth monotone non increasing function on $[0, x_R]$, $F_1(x) \geq f_1(x)$, $F_1(0) = x'_c$, $F_1(x_R) = y_R + \delta_1$, $\delta_1 > 0$. Moreover $F_1'(0) = 0 = F_1'(x_R)$.

Similarly, we define

$$
F_2(x) := \int_x^1 h_\rho(t)f_2'(t)dt|_{\rho=x} + y_\rho, \quad x_R \leq x \leq x'_c, \quad (4.7)
$$

where $x_R < \sigma \leq x'_c$,

$$
h_\rho(t) := \begin{cases} 
1, & t \leq 0 \\
\frac{\varphi(t-\rho)}{\varphi(t-\rho) + \varphi(1-t)}, & 0 < t < \rho \\
0, & t \geq \rho
\end{cases} \quad (4.8)
$$

and $\varphi$ same as above.

One can easily see that $F_2$ is a smooth monotone non decreasing function on $[x_R, x_c]$, $F_2(x) \geq f_2(x)$, $F_2(x_R) = y_R + \delta_2$, $\delta_2 > 0$, $F_2(x'_c) = y_c$. Moreover $F_2'(x_R) = 0 = F_2'(x_c)$.

It is obvious that we can choose now a constant $y_b \in (\max\{y_R + \delta_1, y_R + \delta_2\}, y_c)$, in practice $y_b$ will be taken as close as possible to $y_c$, such that there exists $x_Q \in (0, x_R)$ and $x_S \in (x_R, x'_c)$ that satisfies $F_1(x_Q) = F_2(x_S) = y_b$ (see Figure 7).
Figure 9. The curvature of $F_1(x)$.

With these, we redefine

$$F_1(x) := \int_0^x \left[ 1 - g_{xQ}(t) \right] f_1'(t) dt + y_c, \quad 0 \leq x \leq x_Q,$$

and

$$F_2(x) := \int_x^1 h_{xS}(t) f_2'(t) dt + y_c, \quad x_S \leq x \leq x_c.$$  

Then the function

$$F(x) := \begin{cases} F_1(x), & 0 \leq x \leq x_Q \\ y_b, & x_Q < x < x_S \\ F_2(x), & x_S \leq x \leq x_c \end{cases}$$

will have the required properties.

One can see that the curvature of $F_1$, namely $k_{F_1}(x) := \frac{F_1''}{[1 + (F_1')^2]^{3/2}}$, satisfies $k_{f_1} \leq k_{F_1}$ on $[0, x_c]$ (see Figure 9) and similar for $F_2$. This guarantees that $F$ smoothly joins points $\hat{c}'$ and $\hat{c}$ and that the inner normals to $F$ cannot intersect each other in the region $[0, x_c] \cap \mathbb{H}$.

We denote by $\tilde{D}_0$ the ball obtained from $D_0$ after smoothing out the truncated cones making use of the function $F$ introduced in Proposition 4.1. $D_0$ is actually a surface of revolution obtained by rotating the profile curve $F_1([a, t_1]) \cup F_2([t_2, b]) \cup d((t_1, t_2))$ around the $\gamma$ axis. This is a $C^\infty$-surface whose inward cut locus is the segment $oq$.

This construction can be repeated in any dimension, we apply the above smoothing procedure to each $\partial D_m$, inductively. In this way we end up with a ball $\tilde{D}_m \subset \mathbb{R}^n$ with $C^\infty$ boundary.

By putting

$$\tilde{D} := \lim_{m \to \infty} \tilde{D}_m$$

(4.12)
we obtain a closed, convex ball in \( \mathbb{R}^n \) with boundary \( \partial \tilde{D} \) such that the inward cut locus of \( \partial \tilde{D} \) is the infinite tree \( IT \), actually a fractal as shown in Section 3.

One should remark that the set of end points \( E \) of the fractal \( IT \) are in the interior of \( \tilde{D} \) at distance \( \varepsilon \) from the hypersurface \( \partial \tilde{D} \). However, taking into account that these end points actually lie on the tree branches \( \gamma_{j_1 ... j_m} \), by extending these branches, they will intersect \( \partial \tilde{D} \) giving a set of points \( \tilde{E} \) on \( \partial \tilde{D} \). Such a point is the point \( B \) in Figure 6, for \( m = 0 \). We point out that the points \( \tilde{E} \) are on \( \partial \tilde{D} \), but they do not belong to the fractal \( IT \), they are only an \( \varepsilon \)-displacement of the end points \( E \).

We will study in the following the differentiability of \( \partial \tilde{D} \). We have

**Proposition 4.2** The hypersurface \( \partial \tilde{D} \) in \( \mathbb{R}^n \) is:

1. at least \( k \)-differentiable at points \( \tilde{E} \),
2. \( C^\infty \) at any point \( \partial \tilde{D} \setminus \tilde{E} \).

**Proof.** Let us denote

\[
\tilde{q}_{j_1 j_2 ... j_m} := \partial \tilde{D}_m \cap \gamma_{j_1 j_2 ... j_m}, \quad \tilde{Q} := \bigcup_{m=1}^{\infty} \{ q_{j_1 j_2 ... j_m} \}.
\] (4.13)

It can be seen that \( \tilde{E} \) is the set of accumulation points of \( \tilde{Q} \). A moment of thought shows that the set \( \partial \tilde{D}_m \cap \partial \tilde{D}_{m-1} \) is a circle on the sphere \( S(q_{j_1 j_2 ... j_m}, r_{m-1} + \varepsilon) \), namely the base of the spherical cap cut off by the smoothed truncated cone \( \tilde{A}_{j_1 j_2 ... j_m} \), and that \( \tilde{q}_{j_1 j_2 ... j_{m-1}} = S(q_{j_1 j_2 ... j_m}, r_{m-1} + \varepsilon) \cap \gamma_{j_1 j_2 ... j_m} \). Obviously \( \tilde{E} \) is not countable.

We define the function

\[
\zeta : \bigcup_{m=1}^{\infty} \partial \tilde{D}_{m-1} \to (0, \infty), \quad \zeta(\tilde{q}_{j_1 j_2 ... j_{m-1}}) = d(\tilde{q}_{j_1 j_2 ... j_{m-1}}, \tilde{q}_{j_1 j_2 ... j_{m-1}}),
\] (4.14)

where \( \tilde{q}_{j_1 j_2 ... j_{m-1}} \in \tilde{E} \) (see Figure 10).

One can see that the differentiability of \( \partial \tilde{D} \) at points of \( \tilde{E} \) can be expressed in terms of the differentiability of \( \zeta \). Indeed, let \( A \in \partial \tilde{D} \) be a point such that there is an \( m \) for which \( A \in \partial \tilde{D}_{m-1} \) and denote by \( \mathbf{r}(u^1, u^2, \ldots, u^{n-1}) \) its position vector in \( \mathbb{R}^n \), when we denote by \( (u^1, u^2, \ldots, u^{n-1}) \) the local coordinates on \( \tilde{D} \). Likely, let \( B \in \partial \tilde{D} \) be a point such that \( B \in \partial \tilde{D}_m \) with position vector \( \Phi(u^1, u^2, \ldots, u^{n-1}) \in \mathbb{R}^n \). Then it is clear that

\[
\Phi(u^1, u^2, \ldots, u^{n-1}) = h(u^1, u^2, \ldots, u^{n-1}) \cdot \mathbf{c}(u^1, u^2, \ldots, u^{n-1}) + \mathbf{r}(u^1, u^2, \ldots, u^{n-1}),
\] (4.15)

where \( \mathbf{c}(u^1, u^2, \ldots, u^{n-1}) \) is the outward pointing unit normal vector to \( \partial \tilde{D}_{m-1} \) at \( A \), and \( h \) is the height function \( h(u^1, u^2, \ldots, u^{n-1}) = d(A, B) \). Here \( d \) is the usual Euclidean distance. Then we can see that \( h|_{\tilde{E}} = \zeta|_{\tilde{E}} \).

Obviously \( \Phi \) is \( C^\infty \) at any point \( \tilde{D} \setminus \tilde{E} \) by construction, so we need to investigate only the differentiability of \( \zeta|_{\tilde{E}} \).

Let us recall that the \( r \) differential of \( \zeta \) can be written using finite differences as follows

\[
\frac{d^r \zeta}{dx^r} = \lim_{h \to 0} \frac{\delta_r \zeta}{h^r}, \quad r \geq 1
\] (4.16)
where

\[
\delta^r_h[\zeta] := \sum_{i=0}^{r} (-1)^i \binom{r}{i} \zeta \left( x + \left( \frac{r}{2} - i \right) h \right)
\]  \hspace{1cm} (4.17)

and \( h = d(\partial \tilde{D}_m \cap \partial \tilde{D}_{m-1}, \tilde{q}_{j_1j_2...j_{m-1}}) \).

\[ \partial \tilde{D}_{m-1} \]

\[ \partial \tilde{D}_m \]

\[ \tilde{q} \]

\[ \zeta(\tilde{q}) \]

\[ \tilde{e} \in \tilde{E} \subset \partial \tilde{D} \]

**Figure 10.** The function \( \zeta \). For simplicity we have denoted \( \gamma := \gamma_{j_1j_2...j_{m-1}}, \tilde{q} := \tilde{q}_{j_1j_2...j_{m-1}} \) and \( \tilde{e} := \tilde{q}_{j_1j_2...j_{m-1}} \) (the smoothing function \( F \) is not drawn).

Let us remark that

\[
\frac{\delta^r_h[\zeta]}{h^r} \leq \frac{2^{k-1}}{h^r} d(\max \zeta, \min \zeta) = \frac{2^{k-1}}{h^r} d(\tilde{q}_{j_1j_2...j_{m-1}}, \tilde{q}_{j_1j_2...j_m})
\]

\[
= \frac{2^{k-1}}{h^r} \sum_{i=0}^{\infty} d(\tilde{q}_{j_1j_2...j_m}, \tilde{q}_{j_1j_2...j_{m-1}})
\]  \hspace{1cm} (4.18)

On the other hand, we compute

\[
h = d(\partial \tilde{D}_m \cap \partial \tilde{D}_{m-1}, \tilde{q}_{j_1j_2...j_{m-1}}) = \varphi \left( \frac{1}{3} \right)^{m-1} (r_{m-1} + \varepsilon) > \phi \left( \frac{1}{3} \right)^m (r_{m-1} + \varepsilon),
\]  \hspace{1cm} (4.19)

and

\[
d(\tilde{q}_{j_1j_2...j_m}, \tilde{q}_{j_1j_2...j_{m-1}}) = (r_m + \varepsilon) - (r_{m-1} + \varepsilon) + l_m = l_m \left( 1 - \cos \phi \left( \frac{1}{3} \right)^m \right)
\]

\[
= \frac{1}{(3^{k-1})^m} \tan \left( \frac{1}{2} \phi \left( \frac{1}{3} \right)^m \right).
\]  \hspace{1cm} (4.20)
where we have used \( r_m - r_{m-1} = -l_m \cos \phi \left( \frac{1}{3} \right)^m \), and the well known trigonometric formula \( \frac{1 - \cos \eta}{\sin \eta} = \tan \frac{\eta}{2} \).

One can easily see that there are infinitely many terms in the sum in the right hand side of the last equality in (4.18), therefore, there is no harm in leaving out a finite number of them, say the first \( m-1 \) terms. It follows

\[
d(\tilde{q}_{i_1i_2\ldots i_m\ldots \infty}, \tilde{q}_{j_1j_2\ldots j_m\ldots}) = \sum_{i=m}^{\infty} \frac{1}{(3k-1)^i} \tan \left( \frac{1}{2} \phi \left( \frac{1}{3} \right)^i \right) = \sum_{i=m}^{\infty} \frac{1}{(3k-1)^i} \left( \frac{1}{2} \phi \left( \frac{1}{3} \right)^i \right) = \frac{\phi}{2} \frac{1}{3^{km-1}}
\]

(4.21)

where we have used the well-known formula \( \lim_{\tau \to 0} \frac{\tan \tau}{\tau} = 1 \).

Let us remark now that \( h \to 0 \) when \( m \to \infty \) by construction, so there is no harm in regarding \( \lim_{h \to 0} \) as \( \lim_{m \to \infty} \).

It results

\[
\lim_{h \to 0} \frac{\delta_r \zeta}{h} \leq \lim_{m \to \infty} \left[ \phi \left( \frac{1}{2} \right)^m \left( r_{m-1} + \varepsilon \right) \right] = \frac{1}{2} \lim_{m \to \infty} \left[ \phi^{r-1} \left( r_{m-1} + \varepsilon \right)^r \times \frac{1}{3^{m-r(m-1)}} \right]
\]

(4.22)

One can now easily see that in the case \( r = k \) the limit in the right hand side of (4.22) takes the finite value \( \frac{1}{2} \frac{\phi}{3^{k-1}\varepsilon^k} \) and therefore \( \lim_{h \to 0} \frac{\delta_r \zeta}{h} \) is finite at a point of \( \tilde{E} \), hence the function \( \zeta \) is at least \( k \)-differentiable on \( \tilde{E} \). Remark that for \( r < k \), (4.22) implies \( \frac{d^r \zeta}{dx^r} = 0 \), namely \( \partial D \) is \( C^{k-1} \) everywhere, while for \( r > k \) we cannot say anything about the convergence of the limit in left hand side of (4.22).

The proposition is proved.

\( \Box \)

In order to finish our construction, we adapt an idea of A. Weinstein from [10], namely the technique of making any disc a unit disc. We have

**Proposition 4.3** Let \( \Delta \) be an \( n \)-dimensional ball embedded in a \( C^r \) manifold \( M \) of dimension \( n \). For any Riemannian metric on \( M \setminus \text{interior of } \Delta \), there is a new Riemannian metric on \( M \), agreeing with the original metric on \( M \setminus \text{interior of } \Delta \), such that for some point \( p \in \Delta \) the exponential map \( \exp_p \) is a \( C^{k-1} \) diffeomorphism of the unit ball around the origin in \( T_pM \) onto \( \Delta \).

This proposition can be proved by exactly the same method as Theorem C in [10].

We reach the final stage of our construction.

Recall that an \( n \)-sphere \( S^n \) can be always constructed topologically by gluing together the boundaries of a pair of \( n \)-balls \( (\Delta_1, \Delta_2) \) provided they have opposite orientations. The boundary of an \( n \)-ball \( \Delta_i \) is an \( (n-1) \)-sphere \( \partial \Delta_i \simeq S^{n-1} \), and these two \( (n-1) \)-spheres are to be identified, for \( i = 1, 2 \).

In other words, if we have a pair of \( n \)-balls of the same size, we superpose them so that their \( (n-1) \)-spherical boundaries match, and let matching pairs of points on the
pair of \((n - 1)\)-spheres be identically equivalent to each other \(\partial \Delta_1 \equiv \partial \Delta_2\). In analogy with the case of the 2-sphere, the gluing surface, that is an \((n - 1)\)-sphere subset of \(\mathbb{S}^n\), can be called “equatorial sphere”. Obviously, the interiors of the original \(n\)-balls are not glued to each other but they cover the “exterior” surface of \(\mathbb{S}^n = \Delta_1 \cup \Delta_2\).

Keeping this topological construction in mind, we will glue together the \(n\)-ball \(\tilde{D}\) constructed above with a new \(n\)-dimensional ball \(\Delta\) by identifying \(\partial \Delta\) with the hypersurface \(H\) through a \(C^k\) diffeomorphism. By this construction we obtain an \(n\)-sphere \(\mathbb{S}^n\) whose equatorial sphere is \(\partial \Delta \equiv \partial \tilde{D} \simeq \mathbb{S}^{n-1}\). Moreover, the infinite tree \(IT \subset \tilde{D}\) lies down on the surface \(\mathbb{S}^n\), more precisely, in the open region cut off by the equatorial sphere and covered by the interior of \(\tilde{D}\). By construction, all the cut points of \(IT\) are at a certain distance (the \(\varepsilon\)-dilatation) from the equatorial sphere. Therefore, \(IT \subset \mathbb{S}^n\), but \(IT \cap \bar{\Delta} = \emptyset\), where \(\bar{\Delta}\) is the closure of \(\Delta\).

Finally, by applying the Proposition \([4,3]\) to this \(n\)-sphere \(M = \mathbb{S}^n\), in the case \(r = k - 1\) with the supplementary condition of \(k\)-differentiability (see Proposition \([4,2]\)), and asking \(n = \frac{3k+1}{2} + 1\), we obtain a new Riemannian metric \(g\) on \(\mathbb{S}^n\), metric that coincides on \(\mathbb{S}^n \setminus \text{(interior of } \Delta\text{)}\) with the initial flat metric defined on \(\tilde{D}\), and a point \(p \in \text{(interior of } \Delta\text{)} \subset \mathbb{S}^n\) whose cut locus is \(IT\). Obviously, the geodesics of \((M, g)\) starting from \(p\) must cross the equatorial sphere before hitting the cut locus \(IT\).

Of course, this Riemannian structure can not be \(C^\infty\) because this would contradict with the main result in \([9]\), namely that the Hausdorff dimension of the cut locus of any point on a \(C^\infty\) Riemannian manifold must be an integer.

Also we remark that \(k \to \infty\) would lead to \(n \to \infty\) and hence our \(n\)-sphere must be infinite dimensional, but this is not allowed (see Proposition \([5,1]\) and Corollary \([3,2]\)).

The Theorem \([4,1]\) is now proved.

5 Randers metrics: a ubiquitous family of Finsler structures

Let us recall that a Finsler manifold \((M, F)\) is a \(n\)-dimensional differential manifold \(M\) endowed with a norm \(F : TM \to [0, \infty)\) such that

1. \(F\) is positive and differentiable;

2. \(F\) is \(1\)-positive homogeneous, i.e. \(F(x, \lambda y) = \lambda F(x, y), \lambda > 0, (x, y) \in TM;\)

3. the Hessian matrix \(g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}\) is positive definite on \(\overline{TM} := TM \setminus \{0\}\).

The Finsler structure is called absolute homogeneous if \(F(x, -y) = F(x, y)\) because this leads to the homogeneity condition \(F(x, \lambda y) = |\lambda|F(x, y), \text{ for any } \lambda \in \mathbb{R}^\star\).

By means of the Finsler fundamental function \(F\) one defines the indicatrix bundle (or the Finslerian unit sphere bundle) by \(SM := \cup_{x \in M} S_x M\), where \(S_x M := \{y \in M : F(x, y) = 1\}\).

On a Finsler manifold \((M, F)\) one can easily define the integral length of curves as follows. Let \(\gamma : [a, b] \to M\) be a regular piecewise \(C^\infty\) curve in \(M\), and let \(a := t_0 < t_1 < \cdots < t_m = b\) be the \(t\)-partition of \([a, b]\) into \(m\) subintervals. The length of \(\gamma\) is defined as

\[ \ell(\gamma) := \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \sqrt{g_{ij}(x, \gamma') \gamma'^i \gamma'^j} \, dt, \]

where \(\gamma' := \frac{d\gamma}{dt}\).
\[
\gamma := \text{a partition of } [a, b] \text{ such that } \gamma|_{[t_{i-1}, t_i]} \text{ is smooth for each interval } [t_{i-1}, t_i], \quad i \in \{1, 2, \ldots, k\}. \text{ The integral length of } \gamma \text{ is given by }
\]
\[
L(\gamma) := \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} F(\gamma(t), \dot{\gamma}(t))dt,
\]
(5.1)
where \( \dot{\gamma} = \frac{d\gamma}{dt} \) is the tangent vector along the curve \( \gamma|_{[t_{i-1}, t_i]} \).

For such a partition, let us consider the regular piecewise \( C^\infty \) map
\[
\bar{\gamma} : (-\varepsilon, \varepsilon) \times [a, b] \to M, \quad (u, t) \mapsto \bar{\gamma}(u, t)
\]
(5.2)
such that \( \bar{\gamma}|_{(-\varepsilon, \varepsilon) \times [t_{i-1}, t_i]} \) is smooth for all \( i \in \{1, 2, \ldots, k\} \), and \( \bar{\gamma}(0, t) = \gamma(t) \). Such a curve is called a regular piecewise \( C^\infty \) variation of the base curve \( \gamma(t) \), and the vector field \( U(t) := \frac{\partial \bar{\gamma}}{\partial u}(0, t) \) is called the variational vector field of \( \bar{\gamma} \). The integral length of \( \bar{\gamma}(u, t) \) will be a function of \( u \), defined as in (5.1).

By a straightforward computation one obtains
\[
L'(0) = g_{\gamma(0)}(\dot{\gamma}(0), U) + \sum_{i=1}^{k} \left[ g_{\gamma(t_i^-)}(\dot{\gamma}(t_i^-), U(t_i)) - g_{\gamma(t_i^+)}(\dot{\gamma}(t_i^+), U(t_i)) \right]
\]
\[- \int_{a}^{b} g_{\gamma}(D_{\gamma} \dot{\gamma}, U)dt,
\]
(5.3)
where \( D_{\gamma} \) is the covariant derivative along \( \gamma \) with respect to the Chern connection and \( \gamma \) is arc length parametrized (see [1], p. 123, or [13], p. 77 for details of this computation as well as for the basis on Finslerian connections).

A regular \( C^\infty \) piecewise curve \( \gamma \) on a Finsler manifold is called a geodesic if \( L'(0) = 0 \) for all piecewise \( C^\infty \) variations of \( \gamma \) that keep its end points fixed. In terms of Chern connection a unit speed geodesic is characterized by the condition \( D_{\gamma} \dot{\gamma} = 0 \).

Using the integral length of a curve, one can define the Finslerian distance between two points on \( M \). For any two points \( p, q \) on \( M \), let us denote by \( \Omega_{p,q} \) the set of all piecewise \( C^\infty \) curves \( \gamma : [a, b] \to M \) such that \( \gamma(a) = p \) and \( \gamma(b) = q \). The map
\[
d : M \times M \to [0, \infty), \quad d(p, q) := \inf_{\Omega_{p,q}} L(\gamma)
\]
(5.4)
gives the Finslerian distance on \( M \). It can be easily seen that \( d \) is in general a quasi-distance, i.e. it has the properties
\begin{enumerate}
\item \( d(p, q) \geq 0 \), with equality if and only if \( p = q \);
\item \( d(p, q) \leq d(p, r) + d(r, q) \), with equality if and only if \( p, q, r \) are on the same geodesic segment (triangle inequality).
\end{enumerate}

In the case when \( (M, F) \) is absolutely homogeneous, the symmetry condition \( d(p, q) = d(q, p) \) holds and therefore \( (M, d) \) is a genuine metric space. We do not assume this symmetry condition in the present paper.
A Randers metric on an \( n \)-differential manifold \( M \) is a special Finsler metric \((M, F := \alpha + \beta)\) obtained by a deformation of a Riemannian metric \( a := a_{ij}(x)dx^i \otimes dx^j \) by a one-form \( \beta := b_i(x)dx^i \). The resulting Finslerian norm \( F : TM \to [0, \infty) \) is given by

\[
F(x, y) := \alpha(x, y) + \beta(x, y) = \sqrt{a_{ij}(x)y^i y^j + b_i(x)y^i}, \quad (x, y) \in TM. \tag{5.5}
\]

It is well known that imposing the condition \( b^2 := a(b, b) < 1 \) is enough to ensure that this \( F \) is a positive definite Finsler structure in the usual sense (see [1] for details).

The geodesic equations of \( F \) can be expressed in terms of \( \alpha \) and \( \beta \), but we don’t need to do this.

The following results are well known (see [1]):

**Proposition 5.1** Let \((M, F = \alpha + \beta)\) be a Randers space. The 1-form \( \beta \) is closed if and only if the geodesics of the Randers metric \( F \) coincide with the geodesics of the underlying Riemannian structure as point sets.

In other words, the Randers space \((M, F = \alpha + \beta)\) has reversible geodesics (see [12] for details).

**Corollary 5.2** Let \((M, \alpha)\) be a Riemannian space form and \( \beta \) a 1-form on \( M \). The 1-form \( \beta \) is closed if and only if the Randers metric \((M, F = \alpha + \beta)\) is projectively flat.

**Remark 5.3** The Randers metrics can be described as the deformation of the Riemannian metric \( a \) by means of a magnetic field specified by the 1-form \( \beta \).

### 5.1 A Randers metric on the \( n \)-ball \( B^n \)

Let us consider the 2-ball \( B^2 := \{(x, y) \mid (x, y) \leq 1\} \subset \mathbb{R}^2 \), where \( |\cdot| \) is the usual Euclidean norm, and introduce polar coordinates \((r, \theta)\), where \( r = \sqrt{x^2 + y^2} \) and

\[
x = r \cos \theta \quad y = r \sin \theta. \tag{5.6}
\]

We will denote the coordinates of that tangent space at a point to \( B^2 \) by \((r, \theta; x, y) \in TB^2 \) regarding \((r, \theta)\) as coordinates on the base manifold \( B^2 \) and \((x, y) \in T_{(r,\theta)}B^2 \) the fiber coordinates.

The inward geodesics with the start point on the 2-ball \( B^2 \) endowed with the Euclidean metric are rays that gather in the origin \( O \in \mathbb{R}^2 \) and since are all Euclidean unit length, the inward cut locus of the boundary \( \partial B^2 = S^1 \) is \( O \).

In polar coordinates, a fixed inward geodesic of \( B^2 \), namely a ray \( \rho \) through the origin \( O \), is given by \( \rho(t) = (1 - r(t), \theta_0) \) where \( r : [0, t_0] \to [0, 1] \) are the usual rays from the origin, \( \theta_0 \) is a constant, namely the angle of the ray with \( Ox \) axis. The tangent vector to the inward ray is \( \dot{\rho} = -(\dot{r}(t), 0) \) and the ray \( \rho(t) \) satisfies the geodesic equation for the flat Euclidean metric

\[
a_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{5.7}
\]
The Riemannian length of the tangent vector $\dot{\rho}$ is $\alpha(\dot{\rho}(t)) = \dot{r}(t)$, and the Riemannian length of the ray is
\[
\int_{0}^{t_0} \alpha(\rho(t), \dot{\rho}(t)) dt = \int_{0}^{t_0} \dot{r}(t) dt = r(t_0) - r(0) = 1 - 0 = 1 \quad (5.8)
\]
as expected.

We are going to construct a magnetic field $\beta = b_1(r, \theta) dr + b_2(r, \theta) d\theta$ acting along the inward rays $\alpha$-orthogonal to the boundary $\partial B^2$. Moreover, we will ask for this magnetic field to vanish at both ends of the inward rays.

We start with a Lemma.

**Lemma 5.4** There exists an even, non constant $C^\infty$ function $h : [-1, 1] \to [0, 1)$ such that $h(-1) = 0 = h(1)$.

*Proof.* Choose any constants $c \in (0, 30.05)$ and $\delta \in (0, 1)$.

The function
\[
h(x) = \begin{cases} 
0, & -1 \leq t \leq -\delta \\
c g(\frac{t}{\delta} + 2) g(-\frac{t}{\delta} + 2), & -\delta < t < \delta \\
0, & \delta \leq t \leq 1
\end{cases} \quad (5.9)
\]
has the desired properties, where $g$ is the function
\[
g : (-3, 3) \setminus \{0, 1\} \to [0, 1], \quad g(t) := \frac{\varphi(t)}{\varphi(t) + \varphi(1 - t)}. \quad (5.10)
\]

Remark that the maximum of $h$ is $h(0) = 0.033c < 1$, for $c$ chosen as above.

![Figure 11](image.png)

**Figure 11.** The graph of function $h$ for $c = 10$ and $\delta = \frac{1}{2}$. \qed
Remark 5.5 The function $h$ in Lemma 5.4 can be actually defined on any finite interval $[a, b] \subset \mathbb{R}$ and extended to a function $h : [a, b] \to [0, 1]$ with $h(a) = h(b) = 0$.

We consider the magnetic field defined by the 1-form $\beta = b_1(r, \theta)dr + b_2(r, \theta)d\theta$ given by

$$b_1 = h(r), \quad b_2 = 0$$

where $h : [0, 1] \to [0, 1]$ is a function constructed as shown in Lemma 5.4.

Then we have

Proposition 5.6

1. The fundamental function $F = \alpha + \beta$, where $\alpha$ is given by (5.7) and $\beta = h(r)dr$, is a positive definite Randers metric on $\mathcal{B}^2$.

2. The Randers metric $(\mathcal{B}^2, F)$ is projectively flat.

3. The cut locus of the boundary $\partial \mathcal{B}^2$ with respect to the Randers metric is the origin $O$.

Proof. 1. Remark that $b = h(r) < 1$ and therefore $F$ is a positive definite Randers metric.

2. Moreover, the 1-form $\beta$ is closed by construction, therefore the Randers geodesics coincide with the underlying Riemannian geodesics as set of points and the second statement follows.

3. For the last statement, we remark that near the center $o$ and the boundary $\partial \mathcal{B}^2$ the magnetic field is zero, therefore the geodesic rays $\rho$ are orthogonal to $\partial \mathcal{B}^2$ with respect to the Randers metric $F$ as well.

Taking into account that the Riemannian length of the inward geodesic rays $\rho$ from $\partial \mathcal{B}^2$ with respect to the Riemannian metric $a$ is the same, then it can be seen that all inward geodesic rays emanating orthogonal to the boundary $\partial \mathcal{B}^2$ gather in the origin $O$ and they have the same Finslerian length.

Indeed, let $\rho : [0, t_0] \to \mathbb{R}^2$ a ray through the origin $o \in \mathbb{R}^2$. Then the Finslerian length of the ray $\rho$ is

$$\mathcal{L}_F(\rho|_{[0,t_0]}) = 1 + \int_0^1 h(\rho)d\rho$$

and it depends only on the Riemannian length of the ray $\rho$.

In order to see this remark that

$$\beta(\rho(t), \dot{\rho}(t)) = -h(1 - r(t))\dot{r}(t).$$

Then the integrals (5.8) and

$$\int_0^{t_0} \beta(\rho(t), \dot{\rho}(t))dt = -\int_0^{t_0} f(r)\dot{r}(t)dt = -\int_0^{r(t_0)} h(r)dr = -\int_0^1 h(r)dr$$

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together with the relation

$$L_F(\rho) = \int_0^{t_0} (\alpha(\rho(t), \dot{\rho}(t)) + \beta(\rho(t), \dot{\rho}(t)))dt$$  \hspace{1cm} (5.15)$$

imply (5.12). In other words, the cut locus of the boundary with respect to the Randers metric $F$ is $O$.

This construction can be easily extended to the case of an $n$-dimensional ball $B^n \in \mathbb{R}^n$ with the Euclidean metric $a$. In this case, we can construct by the same procedure as above a Randers metric $(B^n, F = \alpha + \beta)$, whose inward geodesics coincide to the geodesic rays from the boundary $\partial B^n = S^{n-1} \subset \mathbb{R}^n$ and whose inward cut locus is the center of the sphere $S^{n-1}$.

6 Proof of Theorem 1.2

As explained already the inward cut locus from the $C^1$-boundary $\partial D_0$ with respect to the usual Euclidean metric in $\mathbb{R}^2$ coincides with the segment $oq$. We are going to construct a Randers metric on the 2-ball $D_0$ whose inward cut locus of the boundary $\partial D_0$ is the same segment $oq$.

![Figure 12. The magnetic field defined within the convex ball $H_0$.](image)

In order to write our formulas explicitly in polar coordinates $(r, \theta)$, for the sake of clarity and simplicity, we will consider the circles $S$ and $S'$ to be centred at origin with radius $\sqrt{2}$, and at the point $A(1, 0)$ with radius $\sqrt{2}/2$, respectively. We also take $\phi = \pi/4$.

Remark that the 2-ball $D_0$ is made of three regions (compare with Figure 2 and notations in Section 2):

1. $(x, y) = (r_1 \cos \theta, r_1 \sin \theta) \in B(a, \sqrt{2})$, where $r_1 \in [0, \sqrt{2}]$, $\theta \in \left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$, i.e. the large circular sector whose boundary is $P$;
(2) \((x, y) = (u + \frac{\sqrt{2}}{2} \rho + \frac{1 - u}{2}, \pm \frac{\sqrt{2}}{2} \rho \pm \frac{1 - u}{2})\), where \(\rho \in [0, \frac{\sqrt{2}}{2}], u \in [0, 1]\), i.e. a part of the central region whose boundary is \(A\);

(3) \((x, y) = (r_2 \cos \theta, r_2 \sin \theta) \in B((1, 0), \frac{\sqrt{2}}{2})\), where \(r_2 \in [0, \frac{\sqrt{2}}{2}], \theta \in [-\frac{\pi}{4}, \frac{\pi}{4}]\), i.e. the small circular sector whose boundary is \(P_0\).

We point out that \((r_1, \theta)\) and \((r_2, \theta)\) are polar coordinates in the balls \(B(o, \sqrt{2})\) and \(B((1, 0), \frac{\sqrt{2}}{2})\), while \((\rho, u)\) are coordinates on the interior of the squares defined by the points \((\frac{1}{2}, \frac{1}{2}), (1, 1), (\frac{3}{2}, \frac{1}{2}), (1, 0)\) and \((\frac{1}{2}, -\frac{1}{2}), (1, -1), (\frac{3}{2}, -\frac{1}{2}), (1, 0)\), respectively. Nevertheless, we have \(r_1 = \rho + \frac{\sqrt{2}}{2}\).

We are going to define a magnetic field \(\beta\) acting on the (inner) straight rays from the boundaries of each regions as follows.

\[
\beta = \begin{cases} 
\beta_1 := -h_1(r_1)dr_1, & x \in \text{region (1)} \\
\beta_2 := -h_2(\rho)d\rho, & x \in \text{region (2)} \\
\beta_3 := -h_3(r_2)dr_2, & x \in \text{region (3)}
\end{cases}
\]

defined on \(D_0\), where \(h_1 : [\frac{\sqrt{2}}{2}, \sqrt{2}] \to [0, 1), h_2 : [0, \frac{\sqrt{2}}{2}] \to [0, 1)\) and \(h_3 : [0, \frac{\sqrt{2}}{2}] \to [0, 1)\) are constructed as shown in Lemma 5.4 and Remark 5.5.

Figure 13. The magnetic field extended in \(\tilde{D}_0\).

By naturally extending \(\beta\) to the ball \(\tilde{D}_0\) with smooth boundary, then it can be easily seen that the Randers metric \((\tilde{D}_0, F_0 = \alpha + \beta_0)\) is projectively flat and the cut locus of the hypersurface \(\partial\tilde{D}_0\) with respect to this Finsler metric is the segment \(oq\). This can be easily done by extending the definition domain of the functions \(h_i\) such that \(h_{i|\partial\tilde{D}_0} = 0\),
namely $h_1 : [\sqrt{2}, \sqrt{2} + \varepsilon] \to [0, 1)$, $h_2 : [0, \sqrt{2}] \to [0, 1)$ and $h_3 : [0, \sqrt{2} + \varepsilon] \to [0, 1)$, respectively.

This construction can be extended to arbitrary dimension and repeated iteratively for each depth level $m$ such that we obtain

**Proposition 6.1** For each depth level $m$, there is a Randers metric $(\tilde{D}_m, F_m = \alpha + \beta_m)$ on the smooth ball $\tilde{D}_m \subset \mathbb{R}^n$ with the following properties

1. $F_m$ is projectively flat,

2. the inner cut locus of the boundary $\partial \tilde{D}_m$ with respect to $F_m$ is the tree $s \cup s_{j_1} \cup \ldots \cup s_{j_1j_2 \ldots j_m}$.

This result is interesting in itself because it gives a simple example of Randers metric whose cut locus of a closed hypersurface is a tree (compare [15]).

At limit we obtain

**Proposition 6.2** There is a Randers metric $(\tilde{D}, F = \alpha + \beta)$ on the smooth ball $\tilde{D} = \lim_{m \to \infty} \tilde{D}_m \subset \mathbb{R}^n$ with the following properties

1. $F$ is projectively flat,

2. the inner cut locus of the boundary $\partial \tilde{D}$ with respect to $F$ is the infinite tree $IT$,

where $F = \lim_{m \to \infty} F_m$.

We remark that, for any finite $m$, the magnetic field $\beta_m$ defined here is initially defined inside regions (1), (2), (3) and extended by $\varepsilon$-dilatation. However, one can easily see that the regions (2) and (3), considered inside $H_m$, become smaller as $m$ increases. At limit $m \to \infty$, the regions (2), (3) shrink to domains of Hausdorff dimension one and zero, respectively, in other words the intensity of magnetic field $\beta$ inside $H_m$ decreases to zero as $m$ approaches infinity. Nevertheless, $\beta$ is unchanged in region (1) and in the $\varepsilon$-dilatation of (2) and (3).

**Proof. (Proof of Theorem 1.2)** Let us consider again the construction of the $n$-sphere $M = \mathbb{S}^n$ endowed with a Riemannian metric $g$ such that $g$ restricted to $M \setminus (\text{interior of } \Delta)$ coincides with with the initial metric $a_{ij}$ defined on $\tilde{D}$ (see the Proof of Theorem 1.1 in Section 4).

On the other hand, we have the magnetic field $\beta$ defined above on $\tilde{D}$. For the sake of simplicity we assume here that $\beta$ actually acts only on the $\varepsilon$-dilatation region bounded by $\tilde{H}$ and $\partial \tilde{D}$.

From the discussion above it follows that the Randers metric $(M, g + \beta)$ is a Finsler metric on the $n$-sphere $M = \mathbb{S}^n$ whose cut locus is $IT$ having the same order of differentiability with $(M, g)$.

This magnetic field is acting only on the tropical region of the North Hemisphere.
Figure 14. The Randers metric on $M$ with tropical magnetic field.

Of course $\beta$ is a closed 1-form and therefore the geodesics of the Randers metric $F = g + \beta$ coincide with the Riemannian geodesics of $(M, g)$ as point sets.

The cut locus of $p$ with respect to the Randers metric $F = g + \beta$ coincides with the infinite tree $\IT$ and since $(M, g)$ is $C^k$, but not $C^\infty$ this property is inherited by $F$ as well and hence Theorem 1.2 is proved.

**Remark 6.3** Actually, this magnetic field $\beta$ can be extended in the interior of $\Delta$. In this way we obtain a Randers metric whose magnetic field acts in all $\Delta$.

Let us denote by $\gamma : [0, a] \to M$, $\gamma(0) = p$, $\gamma(a) = q$ a minimizing geodesic segment of $(M, g)$ that joins the point $p$ with a point $q \in C(p)$. Using for example Riemannian geodesic coordinates, any geodesic segment from $p$ to its cut point is a straight line. Then using Lemma 5.4 and Remark 5.5 we can extend the tropical magnetic field defined on the $\varepsilon$-dilatation region to entire hemisphere $\Delta$.

Obviously, our magnetic field is zero at $p$, increases in strength and decreases again to zero after crossing the equator when moving from south to north such that $\beta$ vanishes on $\partial D$.

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