Bernstein-gamma functions and exponential functionals of Lévy processes

M. Savov\textsuperscript{1}

joint work with P. Patie\textsuperscript{2}

ICLP8, Angers, France

\textsuperscript{1}Marie Sklodowska Curie Individual Fellowship at IMI, BAS, 2015-2017-project MOCT
\textsuperscript{2}Cornell University
\( \phi \) is a Bernstein function that is \( \phi \in \mathcal{B} \) iff

\[
\phi(z) = m + \delta z + \int_0^\infty (1 - e^{-zy}) \mu(dy),
\]

where \( m, \delta \geq 0; \int_0^\infty (1 \wedge y) \mu(dy) < \infty. \)
\( \phi \) is a Bernstein function that is \( \phi \in \mathcal{B} \) iff

\[
\phi(z) = m + \delta z + \int_0^{\infty} (1 - e^{-zy}) \mu(dy),
\]

where \( m, \delta \geq 0; \int_0^{\infty} (1 \wedge y) \mu(dy) < \infty \).

The unique solution to

\[
W_{\phi}(z + 1) = \phi(z)W_{\phi}(z) \text{ on } z \in \mathbb{C}_{(0,\infty)} = \{ z \in \mathbb{C} : \Re(z) > 0 \} \tag{0.1}
\]

in the space of Mellin transforms of positive random variables we call a Bernstein-gamma function.
Bernstein-gamma functions

\( \phi \) is a Bernstein function that is \( \phi \in \mathcal{B} \) iff

\[
\phi(z) = m + \delta z + \int_0^\infty (1 - e^{-yz}) \mu(dy),
\]

where \( m, \delta \geq 0; \int_0^\infty (1 \wedge y) \mu(dy) < \infty. \)

The unique solution to

\[
W_\phi(z + 1) = \phi(z)W_\phi(z) \text{ on } z \in \mathbb{C}_{(0,\infty)} = \{z \in \mathbb{C} : \Re(z) > 0\}
\]

in the space of Mellin transforms of positive random variables we call a Bernstein-gamma function.

Recall that the Mellin transform of a positive random variable \( Y \) is formally given by \( \mathcal{M}_Y(z) = \mathbb{E} \left[ Y^{z-1} \right] \).
Main goals - Bernstein-gamma functions

1. Understanding of $W_\phi$ as a meromorphic/holomorphic function

2. Development of Stirling type of asymptotic
Motivation - Bernstein-gamma functions

1. $W_\phi$ appears crucially in the spectral studies of the generalized Laguerre semigroups and the positive self-similar Markov processes as instances of non-selfadjoint Markov semigroups. The quantification of its analytic properties offers explicit information about eigen- and coeigen-functions, their norms, etc.

2. $W_\phi$ are related to the “phenomenon of self-similarity” the same way the Gamma function appears in some diffusions.

3. Amongst $W_\phi$ are some well-known special functions, e.g. the Barnes-Gamma function, the q-gamma function.

4. $W_\phi$ appears in exponential functionals of Lévy processes.
Motivation - Bernstein-gamma functions

1. $W_\phi$ appears crucially in the spectral studies of the generalized Laguerre semigroups and the positive self-similar Markov processes as instances of non-selfadjoint Markov semigroups. The quantification of its analytic properties offers explicit information about eigen- and coeigen-functions, their norms, etc.

2. $W_\phi$ are related to the “phenomenon of self-similarity” the same way the Gamma function appears in some diffusions.

3. Amongst $W_\phi$ are some well-known special functions, e.g. the Barnes-Gamma function, the q-gamma function.

4. $W_\phi$ appears in exponential functionals of Lévy processes.
W_\phi appears crucially in the spectral studies of the generalized Laguerre semigroups and the positive self-similar Markov processes as instances of non-selfadjoint Markov semigroups. The quantification of its analytic properties offers explicit information about eigen- and coeigen-functions, their norms, etc.

W_\phi are related to the “phenomenon of self-similarity” the same way the Gamma function appears in some diffusions.

Amongst W_\phi are some well-known special functions, e.g. the Barnes-Gamma function, the q-gamma function.

W_\phi appears in exponential functionals of Lévy processes.
Motivation - Bernstein-gamma functions

1. $W_\phi$ appears crucially in the spectral studies of the generalized Laguerre semigroups and the positive self-similar Markov processes as instances of non-selfadjoint Markov semigroups. The quantification of its analytic properties offers explicit information about eigen- and coeigen-functions, their norms, etc.

2. $W_\phi$ are related to the “phenomenon of self-similarity” the same way the Gamma function appears in some diffusions.

3. Amongst $W_\phi$ are some well-known special functions, e.g. the Barnes-Gamma function, the q-gamma function.

4. $W_\phi$ appears in exponential functionals of Lévy processes.
Denote by
\[
\mathcal{N} = \left\{ \psi : \psi(z) = \frac{\sigma^2}{2}z^2 + bz + \int_{-\infty}^{\infty} (e^{zr} - 1 - zr \mathbf{1}_{|r|<1}) \Pi(dr) - q \right\}
\]
the set of all Lévy-Khintchine exponents of possibly killed at exponential random time of parameter \(q \geq 0\) Lévy processes.
Denote by

\[ \mathcal{N} = \left\{ \psi : \psi(z) = \frac{\sigma^2}{2}z^2 + bz + \int_{-\infty}^{\infty} \left( e^{zr} - 1 - zr \mathbf{1}_{|r|<1} \right) \Pi(dr) - q \right\} \]

the set of all Lévy-Khintchine exponents of possibly killed at exponential random time of parameter \( q \geq 0 \) Lévy processes.

The random variables

\[ I_\psi = \int_0^{e^q} e^{-\xi_s} ds, \quad e_q \sim \text{Exp}(q); \quad e_0 = \infty \]

are called exponential functionals of Lévy processes and

\[ I_\psi < \infty \iff \Psi \in \mathcal{N} = \left\{ \Psi \in \mathcal{N} : q > 0 \text{ or } \lim_{s \to \infty} \xi_s = \infty \right\} \subset \mathcal{N}. \]
Main goals: Exponential functionals of Lévy processes

1. For any $\Psi \in \mathcal{N}$ to solve and characterize the solutions of

$$f(z + 1) = \frac{-z}{\Psi(-z)} f(z) \text{ on } \{z \in i\mathbb{R} : \Psi(-z) \neq 0\} \quad (0.2)$$

in terms of the global quantities of $\Psi$

2. Use that $\mathcal{M}_{I_\Psi} (z + 1) = \mathbb{E} [I_\Psi^z]$ solves in some sense (0.2) to obtain information about $I_\Psi$
For any \( \Psi \in \mathcal{N} \) to solve and characterize the solutions of

\[
f(z + 1) = \frac{-z}{\Psi(-z)} f(z) \quad \text{on} \quad \{ z \in i\mathbb{R} : \Psi(-z) \neq 0 \}
\]  

(0.2)

in terms of the global quantities of \( \Psi \)

Use that \( \mathcal{M}_{I_{\Psi}} (z + 1) = \mathbb{E} [I_{\Psi}^z] \) solves in some sense (0.2) to obtain information about \( I_{\Psi} \)
Exponential functionals of Lévy processes

1. Appear in financial and insurance mathematics; branching with immigration; fragmentation; self-similar growth fragmentation; etc.

2. We also use it in our work on the spectral theory of positive self-similar semigroups.
Exponential functionals of Lévy processes

1. Appear in financial and insurance mathematics; branching with immigration; fragmentation; self-similar growth fragmentation; etc.

2. We also use it in our work on the spectral theory of positive self-similar semigroups.
Background and motivation- Exponential functionals of Lévy processes

1. $I_\Psi$ introduced and studied by Urbanik when $\xi$ is a subordinator
2. Further studied by Carmona, Petit and Yor who have in special cases
   $M_{I_\Psi}(z+1) = \frac{-z}{\psi(-z)} M_{I_\Psi}(z)$
3. Maulik and Zwart derive this key recurrent equation in some generality
   and utilize it
4. Kuznetsov solves this recurrent relation for some classes of Lévy processes
5. There are various other contributions relying on different approaches
Background and motivation- Exponential functionals of Lévy processes

1. $I_\Psi$ introduced and studied by Urbanik when $\xi$ is a subordinator
2. Further studied by Carmona, Petit and Yor who have in special cases $M_{I_\Psi}(z+1) = \frac{-z}{\Psi(-z)} M_{I_\Psi}(z)$
3. Maulik and Zwart derive this key recurrent equation in some generality and utilize it
4. Kuznetsov solves this recurrent relation for some classes of Lévy processes
5. There are various other contributions relying on different approaches
Background and motivation—Exponential functionals of Lévy processes

1. $I_\psi$ introduced and studied by Urbanik when $\xi$ is a subordinator
2. Further studied by Carmona, Petit and Yor who have in special cases $M_{I_\psi}(z + 1) = \frac{-z}{\psi(-z)} M_{I_\psi}(z)$
3. Maulik and Zwart derive this key recurrent equation in some generality and utilize it
4. Kuznetsov solves this recurrent relation for some classes of Lévy processes
5. There are various other contributions relying on different approaches
Background and motivation- Exponential functionals of Lévy processes

1. $I_\Psi$ introduced and studied by Urbanik when $\xi$ is a subordinator
2. Further studied by Carmona, Petit and Yor who have in special cases $M_{I_\Psi}(z + 1) = \frac{-z}{\Psi(-z)} M_{I_\Psi}(z)$
3. Maulik and Zwart derive this key recurrent equation in some generality and utilize it
4. Kuznetsov solves this recurrent relation for some classes of Lévy processes
5. There are various other contributions relying on different approaches
I\(\Psi\) introduced and studied by Urbanik when \(\xi\) is a subordinator

Further studied by Carmona, Petit and Yor who have in special cases
\(\mathcal{M}_{I\Psi}(z+1) = \frac{-z}{\Psi(-z)} \mathcal{M}_{I\Psi}(z)\)

Maulik and Zwart derive this key recurrent equation in some generality and utilize it

Kuznetsov solves this recurrent relation for some classes of Lévy processes

There are various other contributions relying on different approaches
Key quantities of $\phi \in B$ in relation to $W_\phi (z + 1) = \phi(z)W_\phi (z)$

We use $A_{(a,b)}$ (resp. $M_{(a,b)}$) to denote the holomorphic (resp. meromorphic) functions on the complex strip $\mathbb{C}_{(a,b)} = \{ z \in \mathbb{C} : \text{Re}(z) \in (a, b) \}$.
Key quantities of $\phi \in \mathcal{B}$ in relation to $W_\phi(z + 1) = \phi(z)W_\phi(z)$

We use $A_{(a,b)}$ (resp. $M_{(a,b)}$) to denote the holomorphic (resp. meromorphic) functions on the complex strip $\mathbb{C}_{(a,b)} = \{z \in \mathbb{C} : \text{Re}(z) \in (a, b)\}$.

For any $\phi \in \mathcal{B}$ set

\[
\begin{align*}
a_\phi &= \inf_{u < 0} \{ \phi \in A_{(u,\infty)} \} \in [-\infty, 0] \\
u_\phi &= \sup_{u \leq 0} \{ \phi(u) = 0 \} \in [-\infty, 0] \\
d_\phi &= \sup_{u \leq 0} \{ \phi(u) = 0 \text{ or } \phi(u) = -\infty \} \in [a_\phi, 0].
\end{align*}
\]
Main representation of the solution to $W_\phi (z + 1) = \phi(z)W_\phi (z)$

Theorem

For any $\phi \in \mathcal{B}$

$$W_\phi (z) = \frac{1}{\phi(z)}e^{-\gamma_\phi z} \prod_{k=1}^{\infty} \frac{\phi(k)}{\phi(k + z)} e^{\frac{\phi'(k)}{\phi(k)}z} \in A_{(d_\phi, \infty)} \cap M(a_\phi, \infty),$$

is a solution to $f(z + 1) = \phi(z)f(z), f(1) = 1.$
Main representation of the solution to $W_{\phi}(z + 1) = \phi(z)W_{\phi}(z)$

**Theorem**

For any $\phi \in \mathcal{B}$

$$W_{\phi}(z) = \frac{1}{\phi(z)} e^{-\gamma_{\phi}z} \prod_{k=1}^{\infty} \frac{\phi(k)}{\phi(k + z)} e^{\frac{\phi'(k)}{\phi(k)}z} \in A(d_{\phi}, \infty) \cap M(a_{\phi}, \infty),$$

is a solution to $f(z + 1) = \phi(z)f(z)$, $f(1) = 1$. Moreover, $W_{\phi}$ is zero-free on $\mathbb{C}(a_{\phi}, \infty)$ and $W_{\phi}(z + 1) = \mathbb{E}\left[Y_{\phi}^{z}\right]$ for some positive random variable $Y_{\phi}$. 

M. Savov and P. Patie

Bernstein-gamma functions and exponential functionals
Main representation of the solution to $W_\phi (z + 1) = \phi(z)W_\phi (z)$

Theorem

For any $\phi \in B$

$$W_\phi (z) = \frac{1}{\phi(z)}e^{-\gamma_\phi z} \prod_{k=1}^{\infty} \frac{\phi(k)}{\phi(k + z)}e^{\frac{\phi'(k)}{\phi(k)}z} \in A(d_\phi, \infty) \cap M(a_\phi, \infty)$$

is a solution to $f(z + 1) = \phi(z)f(z)$, $f(1) = 1$. Moreover, $W_\phi$ is zero-free on $\mathbb{C}(a_\phi, \infty)$ and $W_\phi (z + 1) = \mathbb{E} \left[Y_\phi^z\right]$ for some positive random variable $Y_\phi$.

When $\phi(z) = z$, $d_\phi = 0$, $a_\phi = -\infty$, $W_\phi (z) = \Gamma(z)$. 

M. Savov and P. Patie

Bernstein-gamma functions and exponential functionals
Stirling asymptotic for $|W_\phi|$

Theorem

If $a, b > 0$, $z = a + ib$. Then

$$|W_\phi (z)| = \frac{\sqrt{\phi(1)}}{\sqrt{\phi(a)\phi(1+a)|\phi(z)|}} e^{G_\phi(a)-A_\phi(z)} e^{-E_\phi(z)-R_\phi(a)} ,$$

error term
Stirling asymptotic for $|W_{\phi}|$

**Theorem**

If $a, b > 0$, $z = a + ib$. Then

$$
|W_{\phi}(z)| = \frac{\sqrt{\phi(1)}}{\sqrt{\phi(a)\phi(1 + a)|\phi(z)|}} e^{G_{\phi}(a) - A_{\phi}(z)} e^{-E_{\phi}(z) - R_{\phi}(a)},
$$

where

$$
A_{\phi}(z) = \int_0^b \arg(\phi(a + iu)) \, du,
$$
$$
\Theta_{\phi}(z) = \int_{\frac{a}{b}}^\infty \ln \left( \frac{|\phi(bu + ib)|}{\phi(bu)} \right) \, du = \frac{1}{b} \int_a^\infty \ln \left( \frac{|\phi(u + ib)|}{\phi(u)} \right) \, du
$$
$$
G_{\phi}(z) = G_{\phi}(a) = \int_1^{1+a} \ln \phi(u) \, du
$$

and $\Theta_{\phi}(a + ib) = \frac{1}{b} A_{\phi}(a + ib) \in [0, \frac{\pi}{2}]$. 

\[ A_\phi(z) = \left( \frac{1}{b} \int_0^b \arg \phi(iu) \, du \right) \times b \]

\[ |W_\phi(z)| \leq e^{G_\phi(a) - A_\phi(z)} \]
Discussion \(|W_\phi(z)| = \frac{\sqrt{\phi(1)}}{\sqrt{\phi(a)\phi(1+a)|\phi(z)|}} e^{G_\phi(a) - A_\phi(z)} e^{-E_\phi(z) - R_\phi(a)}\)

1. If \(a \to \infty\)

\[G_\phi(a) \sim a \ln \phi(a) + \ln \phi(a) - (a + 1)O(1)\]

2. For \(B_P = \{\phi \in B : \delta > 0\}\) then the asymptotic along \(a + i\mathbb{R}\) is

\[A_\phi(a + ib) \sim \frac{\pi}{2} |b| - \left(a + \frac{m}{\delta}\right) \ln |b| + o(|b|)\]

3. For \(B_\alpha = \{\phi \in B : \delta = 0; \mu(dy) \sim y^{-\alpha-1}dy, \alpha (0, 1)\}\)

\[A_\phi(a + ib) \sim \frac{\pi}{2} \alpha |b| + o(|b|)\]
Discussion

\[ |W_\phi(z)| = \frac{\sqrt{\phi(1)}}{\sqrt{\phi(a)\phi(1+a)|\phi(z)|}} e^{G_\phi(a)-A_\phi(z)} e^{-E_\phi(z)-R_\phi(a)} \]

1. If \( a \to \infty \)

\[ G_\phi(a) \sim a \ln \phi(a) + \ln \phi(a) - (a + 1)O(1) \]

2. For \( B_\mathcal{P} = \{\phi \in \mathcal{B} : \delta > 0\} \) then the asymptotic along \( a + i\mathbb{R} \) is

\[ A_\phi(a + ib) \sim \frac{\pi}{2} |b| - \left( a + \frac{m}{\delta} \right) \ln |b| + o(|b|) \]

3. For \( B_\alpha = \left\{ \phi \in \mathcal{B} : \delta = 0; \mu(dy) \sim y^{-\alpha-1}dy, \alpha (0, 1) \right\} \)

\[ A_\phi(a + ib) \sim \frac{\pi}{2} \alpha |b| + o(|b|) \]
Discussion \( |W_\phi(z)| = \frac{\sqrt{\phi(1)}}{\sqrt{\phi(a)\phi(1+a)|\phi(z)|}} e^{G_\phi(a) - A_\phi(z)} e^{-E_\phi(z) - R_\phi(a)} \)

1. If \( a \to \infty \)
   \[ G_\phi(a) \approx a \ln \phi(a) + \ln \phi(a) - (a + 1)O(1) \]

2. For \( B_\mathcal{P} = \{ \phi \in \mathcal{B} : \delta > 0 \} \) then the asymptotic along \( a + i\mathbb{R} \) is
   \[ A_\phi(a + ib) \approx \frac{\pi}{2} |b| - \left( a + \frac{m}{\delta} \right) \ln |b| + o(|b|) \]

3. For \( B_\alpha = \left\{ \phi \in \mathcal{B} : \delta = 0; \mu(dy) \approx y^{-\alpha-1}dy, \alpha (0,1) \right\} \)
   \[ A_\phi(a + ib) \approx \frac{\pi}{2} \alpha |b| + o(|b|) \]
The celebrated factorization Wiener-Hopf

\[ \Psi(-z) = -\phi_+(z)\phi_-(z), \text{ at least for } z \in i\mathbb{R} \]

with

\[ \phi_{\pm}(z) = m_{\pm} + \delta_{\pm}z + \int_0^\infty (1 - e^{-zy}) \mu_{\pm}(dy), z \in \mathbb{C}_{(0,\infty)}, \]

are Bernstein functions then yields

\[ f(z + 1) = -\frac{-z}{\Psi(-z)}f(z) = \frac{z}{\phi_+(z)} \frac{1}{\phi_-(z)}f(z), \quad (0.3) \]

on \( \{z \in i\mathbb{R} : \Psi(-z) \neq 0\} \).
Strategy to solve \( f(z + 1) = \frac{z}{\phi_+(z)} \frac{1}{\phi_-(-z)} f(z) \)

The product of the solutions to the independent system

\[
\begin{align*}
\phi_1(z+1) &= \frac{z}{\phi_+(z)} \phi_1(z) \\
\phi_2(z+1) &= \frac{1}{\phi_-(z)} \phi_2(z)
\end{align*}
\]

on a **common** complex domain is a solution to \( f(z + 1) = \frac{-z}{\psi(-z)} f(z) \) on this domain.
Strategy to solve \( f(z + 1) = \frac{z}{\phi_+(z)} \frac{1}{\phi_-(−z)} f(z) \)

The product of the solutions to the independent system

\[
\begin{align*}
f_1(z + 1) &= \frac{z}{\phi_+(z)} f_1(z) \\
f_2(z + 1) &= \frac{1}{\phi_-(−z)} f_2(z)
\end{align*}
\]

on a common complex domain is a solution to \( f(z + 1) = \frac{−z}{Ψ(−z)} f(z) \) on this domain.

These can be extracted from the general solution to \( f_±(z + 1) = \phi_±(z)f_±(z) \) that is \( W_{\phi_±} \).
Solution to $f(z + 1) = -\frac{z}{\psi(-z)} f(z)$ and representation of $M_{I\psi}(z) = E \left[ I_{\psi}^{z-1} \right]$

**Theorem**

Let $\psi \in \mathcal{N}$. Then

$$M_{\psi}(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-} (1 - z) \in A \left( a_{\phi_+} \{d_{\phi_+} = 0\}, 1 - d_{\phi_-} \right) \cap M(a_{\phi_+}, 1 - a_{\phi_-})$$

solves $f(z + 1) = -\frac{z}{\psi(-z)} f(z)$. 

M. Savov and P. Patie

Bernstein-gamma functions and exponential functionals
Solution to \( f(z + 1) = -\frac{z}{\psi(-z)}f(z) \) and representation of \( M_{I\psi}(z) = E[I_{\psi}^{z-1}] \)

**Theorem**

Let \( \psi \in \widetilde{\mathcal{N}} \). Then

\[
M_{\psi}(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1 - z) \in A \left( a_{\phi_+} \{d_{\phi_+} = 0\}, 1 - d_{\phi_-} \right) \cap M(a_{\phi_+}, 1 - a_{\phi_-})
\]

solves \( f(z + 1) = \frac{-z}{\psi(-z)}f(z) \). Also if \( \psi \in \mathcal{N} \) then

\[
M_{I\psi}(z) = \phi_-(0) M_{\psi}(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} \phi_-(0) W_{\phi_-}(1 - z).
\]
Solution to \( f(z + 1) = -\frac{z}{\psi(-z)} f(z) \) and representation of \( \mathcal{M}_{I\psi}(z) = \mathbb{E}[I_{\psi}^{-1}] \)

**Theorem**

Let \( \psi \in \mathcal{N} \). Then

\[
\mathcal{M}_{\psi}(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-} (1 - z) \in A \left( a_{\phi_+}, 1 - a_{\phi_-} \right) \cap M \left( a_{\phi_+}, 1 - a_{\phi_-} \right)
\]
solves \( f(z + 1) = \frac{-z}{\psi(-z)} f(z) \). Also if \( \psi \in \mathcal{N} \) then

\[
\mathcal{M}_{I\psi}(z) = \phi_-(0) \mathcal{M}_{\psi}(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} \phi_-(0) W_{\phi_-} (1 - z).
\]

As a consequence of the Weierstrass product representations of \( W_{\phi_\pm}, \Gamma \)

\[
I_{\psi} \overset{d}{=} I_{\phi_+} \times X_{\phi_-} \overset{d}{=} \bigotimes_{k=0}^{\infty} C_k Y_k,
\]

where \( \mathbb{E}[f(Y_k)] = \frac{\mathbb{E}[Y_0^k f(Y_0)]}{\mathbb{E}[Y_0^k]} \).
Decay of $|\mathcal{M}_\Psi(z)| = \left| \frac{\Gamma(z)}{W_{\phi_+}(z)} \right| |W_{\phi_-}(1-z)|$ along complex lines

**Theorem**

Let $\Psi \in \mathcal{N}$. Then exists $N_\Psi \in (0, \infty]$ such that for any $a \in (0, 1 - d_{\phi_-})$

$$\lim_{|b| \to \infty} |b|^\eta |\mathcal{M}_\Psi(a + ib)| = 0 \iff \eta \in (0, N_\Psi).$$

Therefore if $\Psi \in \mathcal{N}$, $p_\Psi \in C_0^{\lfloor N_\Psi \rfloor - 1}(\mathbb{R}^+)$ if $N_\Psi > 1$. 
Decay of $|\mathcal{M}_\psi(z)| = \left| \frac{\Gamma(z)}{W_{\phi_+}(z)} \right| |W_{\phi_-}(1-z)|$ along complex lines

**Theorem**

Let $\psi \in \mathcal{N}$. Then exists $N_\psi \in (0, \infty]$ such that for any $a \in (0, 1 - d_{\phi_-})$

$$
\lim_{|b| \to \infty} |b|^\eta |\mathcal{M}_\psi(a + ib)| = 0 \iff \eta \in (0, N_\psi).
$$

Therefore if $\psi \in \mathcal{N}$, $p_\psi \in C_0^{[N_\psi]-1}(\mathbb{R}^+)$ if $N_\psi > 1$.

$$
N_\psi = \frac{m_-(0)}{\bar{\mu}_-(0) + \phi_-(0)} + \frac{\phi_+(0) + \bar{\mu}_+(0)}{\delta_+} \in (0, \infty)
$$

if and only if $\psi$ corresponds to $\xi_t = \delta_+ t + \sum_{j=1}^{N_t} X_j$, $\delta_+ > 0$. 

M. Savov and P. Patie  Bernstein-gamma functions and exponential functionals
Decay of $|\mathcal{M}_\Psi(z)| = \left| \frac{\Gamma(z)}{W_{\phi_+}(z)} \right| |W_{\phi_-}(1 - z)|$ along complex lines

**Theorem**

Let $\Psi \in \mathcal{N}$. Then exists $N_\Psi \in (0, \infty]$ such that for any $a \in (0, 1 - d_{\phi_-})$

$$\lim_{|b| \to \infty} |b|^\eta |\mathcal{M}_\Psi(a + ib)| = 0 \iff \eta \in (0, N_\Psi).$$

Therefore if $\Psi \in \mathcal{N}$, $p_\Psi \in C_0^{[N_\Psi]^{-1}}(\mathbb{R}^+) \text{ if } N_\Psi > 1$.

$$N_\Psi = \frac{m_-(0)}{\bar{\mu}_-(0) + \phi_-(0)} + \frac{\phi_+(0) + \bar{\mu}_+(0)}{\delta_+} \in (0, \infty)$$

if and only if $\Psi$ corresponds to $\xi_t = \delta_+ t + \sum_{j=1}^{N_t} X_j$, $\delta_+ > 0$.

$N_\Psi$ is a measure for the polynomial decay of $|\mathcal{M}_\Psi|$ along complex lines.
Ideas for the proof

For fixed $a \in (0, 1 - d_{\phi_-})$

$$|\mathcal{M}_\psi(z)| = \left| \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1 - z) \right|$$

$$= C \left| \frac{\sqrt{\phi_+(z)}}{\sqrt{\phi_-(z)}} \Gamma(z) \right| e^{-A_{\phi_-}(1-z)+A_{\phi_+}(z)}$$

$$\sim C |b|^{a-\frac{1}{2}} e^{-\frac{\pi}{2}|b|-A_{\phi_-}(1-a-ib)+A_{\phi_+}(a+ib)}$$

The hardest case is when $\delta_+ > 0$, $\delta_- = 0$. Depending on $\Pi_-(y)$ as $y \to 0$ we use different techniques-reducing to $\Pi_+(0) = 0$, using the alternative representation $\Theta_\phi$ for $A_\phi$, etc.
Ideas for the proof

1. For fixed $a \in (0, 1 - d\phi_-)$

$$|\mathcal{M}_\psi(z)| = \left| \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1 - z) \right|$$

$$= C \left| \frac{\sqrt{\phi_+(z)}}{\sqrt{\phi_-}(z)} \Gamma(z) \right| e^{-A_{\phi_-}(1-z)+A_{\phi_+}(z)}$$

$$\sim C |b|^{a-\frac{1}{2}} e^{-\frac{\pi}{2} |b|} e^{-A_{\phi_-}(1-a-ib)+A_{\phi_+}(a+ib)}$$

2. The hardest case is when $\delta_+ > 0$, $\delta_- = 0$. Depending on $\overline{\Pi}_-(y)$ as $y \to 0$ we use different techniques—reducing to $\overline{\Pi}_+(0) = 0$, using the alternative representation $\Theta_\phi$ for $A_\phi$, etc.
Large behaviour of $I_\Psi = \int_0^e e^{-\xi_s} \, ds$: the role of $M_{I_\Psi}(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} \phi_-(0) W_{\phi_-}(1 - z)$

**Theorem**

Let $\Psi \in \mathcal{N}$ that is $I_\Psi < \infty$. Then

$$\lim_{x \to \infty} \frac{\ln \mathbb{P}(I_\Psi > x)}{\ln(x)} = d_{\phi_-} = \sup_{u \leq 0} \{ \phi_-(u) = 0 \text{ or } \phi_-(u) = -\infty \} \in [-\infty, 0],$$

where recall that $\Psi(z) = -\phi_+(-z) \phi_-(z)$.
Theorem

Let $\Psi \in \mathcal{N}$ that is $I_\Psi < \infty$. Then

$$\lim_{x \to \infty} \frac{\ln \mathbb{P}(I_\Psi > x)}{\ln(x)} = d_{\phi_-} = \sup_{u \leq 0} \{ \phi_-(u) = 0 \text{ or } \phi_-(u) = -\infty \} \in [-\infty, 0],$$

(0.4)

where recall that $\Psi(z) = -\phi_+(-z)\phi_-(z)$.

If $\exists \theta_\Psi < 0 : \Psi(\theta_\Psi) = 0$ and $|\Psi(\theta_\Psi^+)| < \infty$ then

$$\lim_{x \to \infty} x^{-\theta_\Psi+n+1} \mathbb{P}^{(n)}(\Psi)(x) = C > 0$$

(0.5)

provided $N_\Psi > n + 1$ and a weak non-lattice condition when $n \geq 1$. 
Small behaviour of $I_{\Psi}$: the role of $M_{I_{\Psi}}(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} \phi_-(0)W_{\phi_-}(1-z)$

The small behaviour depends on the poles of $\frac{\Gamma(z)}{W_{\phi_+}(z)}$ on $\mathbb{C}(a_{\phi_+},1)$.

- If $\phi_+(0) = 0$ then $\frac{\Gamma(z)}{W_{\phi_+}(z)} \in A(a_{\phi_+},\infty)$ and then
  $$\mathbb{P}(I_{\Psi} \leq x) = o(x^{-a}), \forall a \in (a_{\phi_+}, \infty)$$
  and $\mathbb{P}(I_{\Psi} \leq x) = o(1)$ if $a_{\phi_+} = 0$

- If $\phi_+(0) > 0$, then $\frac{\Gamma(z)}{W_{\phi_+}(z)} \in M(a_{\phi_+},\infty)$ and $\Psi(0) = -q$ with
  $$\mathbb{P}(I_{\Psi} \leq x) = q \sum_{j=1}^{N} \frac{\Pi_{k=1}^{j-1} \psi(k)}{j!} x^j - \frac{\phi_-(0)}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^{-z} M_{\psi}(z) dz,$$
The small behaviour depends on the poles of \( \frac{\Gamma(z)}{W_{\phi_+}(z)} \) on \( \mathbb{C}(a_{\phi_+}, 1) \).

- If \( \phi_+(0) = 0 \) then \( \frac{\Gamma(z)}{W_{\phi_+}(z)} \in A(a_{\phi_+}, \infty) \) and then
  \[
  P(I_\Psi \leq x) = o(x^{-a}), \quad \forall a \in (a_{\phi_+}, \infty) \text{ and } P(I_\Psi \leq x) = o(1) \text{ if } a_{\phi_+} = 0
  \]

- If \( \phi_+(0) > 0 \), then \( \frac{\Gamma(z)}{W_{\phi_+}(z)} \in M(a_{\phi_+}, \infty) \) and \( \Psi(0) = -q \) with

\[
P(I_\Psi \leq x) = q \sum_{j=1}^{N} \prod_{k=1}^{j-1} \frac{\Psi(k)}{k!} x^j - \frac{\phi_-(0)}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^{-z} M_\Psi(z) dz,
\]
Large time behaviour of $I_{\Psi}(t) = \int_0^t e^{-\xi s} \, ds$ when $\Psi \in \overline{N} \setminus N$

Let $\Psi \in \overline{N}$ and set $I_{\Psi}(t) = \int_0^t e^{-\xi s} \, ds$. Clearly with

$\psi_q(z) = \psi(z) - q = -\phi_+^q(-z)\phi_-^q(z) \in N$ we have with real $a$ that

$$
\frac{1}{q} \mathcal{M}_{I_{\Psi}^q}(a) = \frac{1}{q} \mathbb{E} \left[ I_{\Psi}^{a-1} \right] = \int_0^\infty e^{-qt} \mathbb{E} \left[ I_{\Psi}^{a-1}(t) \right] \, dt
$$

Laplace transform

$$
= \frac{\Gamma(a)}{W_{\phi_+^q}(a)} \frac{\phi_-^q(0)}{q} W_{\phi_-^q}(1 - a)
$$
Large time behaviour of $I_\psi(t) = \int_0^t e^{-\xi_s} ds$ when $\psi \in \overline{\mathcal{N}} \setminus \mathcal{N}$

**Theorem**

Let $\psi \not\in \mathcal{N}$ with $\lim \sup_{t \to \infty} \xi_t = \lim \sup_{t \to \infty} -\xi_t = \infty$ and

$\lim_{t \to \infty} \mathbb{P}(\xi_t < 0) = \rho \in [0, 1)$. Set $\psi_r(\cdot) = \psi(\cdot) - r = -\phi^r_+(\cdot)\phi^r_-(\cdot) \in \mathcal{N}$ and

$\kappa_-(r) = \phi^r_-(0)$. Then $\kappa_- \in \text{RV}(\rho)$ at zero and for any $a \in (0, 1)$, $f \in C_b(\mathbb{R}^+)$

$$
\lim_{t \to \infty} \frac{\mathbb{E}[I^{-a}_{\psi}(t)f(I_{\psi}(t))]}{\kappa_-(\frac{1}{t})} = \int_0^\infty f(x)\vartheta_a(dx).
$$

If $\mathbb{E}[\xi_1] = 0$, $\mathbb{E}[\xi_2^2] < \infty$ then $\kappa_-(r) \sim Cr^1/2$.
Large time behaviour of \( I_\Psi(t) = \int_0^t e^{-\xi_s} \, ds \) when \( \Psi \in \overline{N} \setminus N \)

Theorem

Let \( \Psi \notin N \) with \( \limsup_{t \to \infty} \xi_t = \limsup_{t \to \infty} -\xi_t = \infty \) and
\( \lim_{t \to \infty} \mathbb{P}(\xi_t < 0) = \rho \in [0, 1) \). Set \( \Psi_r(\cdot) = \Psi(\cdot) - r = -\phi_+^r(-\cdot)\phi_-^r(\cdot) \in N \) and \( \kappa_-^r(\cdot) = \phi_-^r(0) \). Then \( \kappa_- \in RV(\rho) \) at zero and for any \( a \in (0, 1) \), \( f \in C_b(\mathbb{R}^+) \)

\[
\lim_{t \to \infty} \frac{\mathbb{E}[I_\Psi^{-a}(t)f(I_\Psi(t))]}{\kappa_-(\frac{1}{t})} = \int_0^\infty f(x)\vartheta_a(dx).
\]

If \( \mathbb{E}[\xi_1] = 0, \mathbb{E}[\xi_1^2] < \infty \) then \( \kappa_-(r) \sim Cr^{1/2} \).
Thank you!