Frontiers in Mathematics

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The Geometry of Filtering

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Introduction

Filtering is the science of finding the law of a process given a partial observation of it. The main objects we study here are diffusion processes. These are naturally associated with second-order linear differential operators which are semi-elliptic and so introduce a possibly degenerate Riemannian structure on the state space. In fact, much of what we discuss is simply about two such operators intertwined by a smooth map, the “projection from the state space to the observations space”, and does not involve any stochastic analysis.

From the point of view of stochastic processes, our purpose is to present and to study the underlying geometric structure which allows us to perform the filtering in a Markovian framework with the resulting conditional law being that of a Markov process which is time inhomogeneous in general. This geometry is determined by the symbol of the operator on the state space which projects to a symbol on the observation space. The projectible symbol induces a (possibly non-linear and partially defined) connection which lifts the observation process to the state space and gives a decomposition of the operator on the state space and of the noise. As is standard we can recover the classical filtering theory in which the observations are not usually Markovian by application of the Girsanov-Maruyama-Cameron-Martin Theorem.

This structure we have is examined in relation to a number of geometrical topics. In one direction this leads to a generalisation of Hermann’s theorem on the fibre bundle structure of certain Riemannian submersions. In another it gives a novel description of generalised Weitzenböck curvature. It also applies to infinite dimensional state spaces such as arise naturally for stochastic flows of diffeomorphisms defined by stochastic differential equations, and for certain stochastic partial differential equations.

A feature of our approach is that in general we use canonical processes as solutions of martingale problems to describe our processes, rather than stochastic differential equations and semi-martingale calculus, unless we are explicitly dealing with the latter. This leads to some new constructions, for example of integrals along the paths of our diffusions in Section 4.1, which are valid more generally than in the very regular cases we discuss here.
Those whose interest is mainly in filtering rather than in the geometry should look at Chapter 1, most of Chapter 2, especially Section 2.3, but omitting Section 2.6. Then move to Chapter 4 where Section 4.11 can be ignored. They could then finish with Chapter 5, though some of the Appendices may be of interest.

A central role is played by certain generalised connections determined by the principal symbols of the operators involved. To describe this in more detail let $M$ be a smooth manifold. Consider a smooth second-order semi-elliptic differential operator $L$ such that $L1 \equiv 0$. In a local chart, such an operator takes the form

$$L = \frac{1}{2} \sum_{i,j=1}^{n} a^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} + \sum_{i} b^{i} \frac{\partial}{\partial x^i}$$

where the $a^{ij}$'s and $b^{i}$'s are smooth functions and the matrix $(a^{ij})$ is positive semi-definite.

Such differential operators are called diffusion operators. An elliptic diffusion operator induces a Riemannian metric on $M$. In the degenerate case we shall have to assume that the “symbol” of $L$ (essentially the matrix $[a^{ij}]$ in the representation (1)) has constant rank and so determines a subbundle $E$ of the tangent bundle $TM$ together with a Riemannian metric on $E$. In Elworthy-LeJan-Li [35] and [36] it was shown that a diffusion operator in Hörmander form, satisfying this condition, induces a linear connection on $E$ which is adapted to the Riemannian metric induced on $E$, but not necessarily torsion free. It was also shown that all metric connections on $E$ can be constructed by some choice of Hörmander form for a given $L$ in this way. The use of such connections has turned out to be instrumental in the decomposition of noise and calculation of covariant derivatives of the derivative flows.

A related construction of connections can arise with principal fibre bundles $P$. An equivariant differential operator on $P$ induces naturally a diffusion operator on the base manifold. Conversely, given an equivariant or “principal” connection on $P$, one can lift horizontally a diffusion operator on the base manifold of the form of sum of squares of vector fields by simply lifting up the vector fields. It still needs to be shown that the lift is independent of choices of its Hörmander form. Consider now a diffusion operator not given in Hörmander form. Since it has no zero-order term we can associate with it an operator $\delta$ which sends differential one-forms to functions. In Proposition 1.2.1, members of a class of such operators are described, each of which determines a diffusion operator. Horizontal lifts of diffusion operators can then be defined in terms of the $\delta$ operator. This construction extends to situations where there is no equivariance and we have only partially defined and non-linear connections. We show that given a smooth $p : N \to M$: a diffusion operator $B$ on $N$ which lies over a diffusion operator $A$ on $M$ satisfying a “cohesiveness” property gives rise to a semi-connection, a partially defined, non-linear, connection which can be characterised by the property that, with respect to it, $B$ can be written as the direct sum of the horizontal lift of its
induced operator and a vertical diffusion operator. Of particular importance are examples where $p : N \to M$ is a principal bundle. In that case the vertical component of $\mathcal{B}$ induces differential operators on spaces of sections of associated vector bundles: we observe that these are zero-order operators, and can have geometric significance.

This geometric significance, and the relationship between these partially defined connections and the metric connections determined by the Hörmander form as in [35] and [36], is seen when taking $\mathcal{B}$ to be the generator of the diffusion given on the frame bundle $GLM$ of $M$ by the action of the derivative flow of a stochastic differential equation on $M$. The semi-connection determined by $\mathcal{B}$ is then equivariant and is the adjoint of the metric connection induced by the SDE in a sense extending that of Driver [25] and described in [36]. The zero-order operators induced by the vertical component of $\mathcal{B}$ acting on differential forms turn out to be generalised Weitzenböck curvature operators, in the sense of [36], reducing to the classical ones when $M$ is Riemannian for particular choices of stochastic differential equations for Brownian motion on $M$. Our filtering then reproduces the conditioning results for derivatives of stochastic flows in [38] and [36].

Our approach is also applied to the case where $M$ is compact and $N$ is its diffeomorphism group, $\text{Diff}(M)$, with $P$ evaluation at a chosen point of $M$. The operator $\mathcal{B}$ is taken to be the generator of the diffusion process on $\text{Diff}(M)$ arising from a stochastic flow. However our constructions can be made in terms of the reproducing Hilbert space of vector fields on $M$ defined by the flow. From this we see that stochastic flows are essentially determined by a class of semi-connections on the bundle $p : \text{Diff}(M) \to M$ and smooth stochastic flows whose one-point motions have a cohesive generator determine semi-connections on all natural bundles over $M$. Apart from these geometrical aspects of stochastic flows we also obtain a skew-product decomposition which, for example, can be used to find conditional expectations of functionals of such flows given knowledge of the one-point motion from our chosen point in $M$.

The plan of the book is as follows: In Chapter 1 we describe various representations of diffusion operators and when they are available. We also define the notion of such an operator being along a distribution. In Chapter 2 we introduce the notion of semi-connection which is fundamental for what follows, and we show how these are induced by certain intertwined pairs of diffusion operators and how they relate to a canonical decomposition of such operators. We also have a first look at the topological consequences on $p : N \to M$ of having $\mathcal{B}$ on $N$ over some $\mathcal{A}$ on $M$ which possesses hypo-ellipticity type properties. This is a minor extension of part of Hermann’s theorem, [51], for Riemannian submersions. In Chapter 3 we specialise to the case of principal bundles, introduce the example of derivative flow, and show how the generalised Weitzenböck curvatures arise.

It is not really until Chapter 4 that stochastic analysis plays a major role. Here we describe methods of conditioning functionals of the $\mathcal{B}$-process given information about its projection onto $M$. We also use our decomposition of $\mathcal{B}$ and resulting decomposition of the $\mathcal{B}$-process to describe the conditional $\mathcal{B}$-process.
In the equivariant case of principal bundles the decomposition of the process can be considered as a skew-product decomposition. In Chapter 5 we show how our constructions can apply to classical filtering problems, where the projection of the $\mathcal{B}$-process is non-Markovian by an appropriate change of probability measure. We can follow the classical approach, illustrated for example in the lecture notes of Pardoux [85], and obtain, in Theorem 5.9, a version of Kushner’s formula for non-linear filtering in somewhat greater generality than is standard. This requires some discussion of analogues of innovations processes in our setting.

We return to more geometrical analysis in Chapter 6, giving further extensions of Hermann’s theorem and analysing the consequences of the horizontal lift of $\mathcal{A}$ commuting with $\mathcal{B}$, thereby extending the discussion in [9]. In particular we see that such commutativity, plus hypo-ellipticity conditions on $\mathcal{A}$, gives a bundle structure and a diffusion operator on the fibre which is preserved by the trivialisations of the bundle structure. This leads to an extension of the “skew-product” decomposition given in [33] for Brownian motions on the total space of Riemannian submersions with totally geodesic fibres. In fact the well-known theory for Riemann submersions, and the special case arising from Riemannian symmetric spaces is presented in Chapter 7.

Chapter 8 is where we describe the theory for the diffeomorphism bundle $p : \text{Diff}(M) \to M$ with a stochastic flow of diffeomorphism on $M$. Initially this is done independently of stochastic analysis and in terms of reproducing kernel Hilbert spaces of vector fields on $M$. The correspondence between such Hilbert spaces and stochastic flows is then used to get results for flows and in particular skew-product decompositions of them.

In the Appendices we present the Girsanov Theorem in a way which does not rely on having to use conditions such as Novikov’s criteria for it to remain valid. This has been known for a long time, but does not appear to be as well known as it deserves. We also look at conditions for degenerate, but smooth, diffusion operators to have smooth Hörmander forms, and so to have stochastic differential equation representations for their associated processes. We also discuss semi-martingales and $\Gamma$-martingales along a subbundle of the tangent bundle with a connection. One section of the Appendix is a very brief exposition of the differential geometry of submanifolds, defining second fundamental forms and shape operators. This is used in the final section which analyses the situation of intertwined stochastic flows or essentially equivalent of diffusion operators which are not only intertwined but also have Hörmander forms composed of intertwined vector fields. It is shown that the Hörmander forms determine a decomposition of the operator $\mathcal{B}$, which is not generally the same as the canonical decomposition described in Chapter 2. Here it is not necessary to make the constant rank condition on the symbol of $\mathcal{A}$ which plays an important role in Chapter 2. At the end of this section we show that having intertwined Brownian flows which both induce Levi-Civita connections can only occur given severe restrictions on the geometry of the submersion $p : N \to M$.

For Brownian motions on the total spaces of Riemannian submersions much of our basic discussion, as in the first two and a half chapters, of skew-product
decompositions is very close to that in [33] which was taken further by Liao in [69]. A major difference from Liao’s work is that for degenerate diffusions we use the semi-connection determined by our operators rather than an arbitrary one, so obtaining canonical decompositions. The same holds for the very recent work of Lazaro-Cami & Ortega, [62] where they are motivated by the reduction and reconstruction of Hamiltonian systems and consider similar decompositions for semi-martingales. An extension of [33] in a different direction, to shed light on the Fadeev-Popov procedure for gauge theories in theoretical physics, was given by Arnaudon & Paycha in [1]. Much of the equivariant theory presented here was announced with some sketched proofs in [34].

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