Action Minimizing Orbits in the 2-Center Problems with Simple Choreography Constraint

Abstract The aim of this paper is to study the motion of a $2 + n$-body problem where two equal masses are assumed to be fixed. We assume that the value of each fixed mass is equal to $M > 0$ and the remaining $n$ moving particles have equal masses $m > 0$. According to Newton’s second law and the universal gravitation law, the $n$ particles move under the interaction of each other and the affection of the two fixed particles. Also, this motion has a natural variational structure. Under the simple choreography constraint, we show that the Lagrangian action functional attains its absolute minimum on a uniform circular motion.

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1 Introduction and Main Results

The N-center problem describes the motion of a test particle in the field of N space fixed Newtonian centers of attraction. The 1-center problem is actually equivalent to a Newtonian 2-body problem which has already been solved by Newton. The 2-center problem was first investigated by Euler in 1760 and he proved the integrability of such system. While for $N \geq 3$, based on the works of Bolotin [6], Bolotin and Negrini [7], Bolotin and Negrini [8], Klein and Knauf [18], Knauf [19], we know that the N-center problem is not completely integrable. Castelli [9], Castelli [10] used the variational approach to study the N-center problem. In this paper, we also use the variational minimization method to study a $2 + n$-body problem where two particles with equal masses of value $M > 0$ are fixed at positions $C_1 = (1, 0, 0)$ and $C_2 = (-1, 0, 0)$, while the remaining $n(\geq 2)$ particles, with their masses equal to $m > 0$, move under the interaction of each other and the affection of the two fixed particles.

Mathematically, we suppose the force of the fixed center and the moving particles generated by a potential of $\frac{1}{r\alpha}, \frac{1}{r\beta}, \alpha, \beta > 0$, respectively. These equations of the motion for the $n$ moving particles can be represented as

$$m\ddot{q}_i(t) = \nabla_{q_i} V(q_1, \ldots, q_n), \quad q_i \in \mathbb{R}^3, \quad i = 1, \ldots, n,$$

(1.1)
where the potential function is given by

\[ V(q_1, \ldots, q_n) = \sum_{1 \leq i \neq j \leq n} \frac{m^2}{|q_i - q_j|^2} + \sum_{i=1}^{n} \left( \frac{mM}{|q_i - C_1|^3} + \frac{mM}{|q_i - C_2|^3} \right), \]

and \( q(t) = (q_1(t), \ldots, q_n(t)) \) are orbits of the \( n \) moving particles. We will analyse the problem from a variational point of view to find the \( 2\pi \)-periodic solutions of the dynamical system (1.1). Looking for periodic solutions of (1.1) is equivalent to seeking the critical points of the associated action functional \( \mathcal{A} : \mathcal{H} \rightarrow \mathbb{R} \cup \{ +\infty \} \)

\[ \mathcal{A}(q(t)) = \int_{0}^{2\pi} \left[ \frac{m}{2} \sum_{i=1}^{n} |\dot{q}_i(t)|^2 + V(q_1, \ldots, q_n) \right] dt \]

on the set

\[ \mathcal{H} = \{ q(t) \in H^{1,\infty}(\mathbb{R}, \mathbb{R}^{3n}) \mid q_i(t) \neq q_j(t), \ \forall i \neq j, \ \forall t \in \mathbb{R}; q_k(t) \neq C_1, q_k(t) \neq C_2, k = 1, 2, \ldots, n, \ \forall t \in \mathbb{R} \}. \]

In order to find collisionless periodic solution, many methods have been exploited in the last decades (see [1–5, 9–17, 21, 23–27]). As well as for the usual Newtonian N-body problem, the main difficulties are that in principle critical points there might be trajectories with collisions and the action functional is not coercive. In this paper, we consider that the motion of the principle critical points there might be trajectories with collisions and the action functional is not coercive. In this paper, we consider that the motion of the \( n \) moving particles is a simple choreography in which the bodies lie on the same curve and exchange their mutual positions after a fixed time; i.e.,

\[ q_i(t) = q_1 \left( t + (i - 1) \frac{2\pi}{n} \right) \quad i = 1, 2, \ldots, n \quad \forall t \in \mathbb{R}. \]

Barutello and Terracini [4] study the \( n \)-body problem with the only constraint to be a simple choreography, but without fixed centers. They prove that the absolute minimum of the corresponding functional is attained on a relative equilibrium motion associated with the regular polygon, which is a solution where the \( n \) particles lie at the vertices of a regular \( n \)-gon and do a uniform circular motion around the center of the regular \( n \)-gon. It is natural to ask if we can get the same result when we consider the \( n \) particles attracting each others and attracted by two fixed centers, with a simple choreographic constraint.

We establish our main theorem.

**Theorem 1.1** For given \( \alpha, \beta, m, M > 0 \), the absolute minimum of \( \mathcal{A} \) on

\[ \Lambda = \left\{ q(t) \in \mathcal{H} \left| \int_{0}^{2\pi} q_i(t) dt = 0, q_1(t) = q_1 \left( t + (i - 1) \frac{2\pi}{n} \right) \quad i = 1, 2, \ldots, n \right\} \]

is attained on a relative equilibrium motion \( \tilde{q}(t) = (\tilde{q}_1(t), \tilde{q}_2(t), \ldots, \tilde{q}_n(t)) \). Moreover, at any instant \( t_0, \tilde{q}_1(t_0), \tilde{q}_2(t_0), \ldots, \tilde{q}_n(t_0) \) lie at the vertices of a regular \( n \)-gon centered at origin in \( yoz \)-plane and the circumradius of the regular \( n \)-gon is an increasing function of \( m \).

The loop space \( \Lambda \) is the subset of the space \( \mathcal{H} \) with the following two conditions which are basically needed in our proof:

(H1) \( \forall q(t) \in \Lambda, \int_{0}^{2\pi} q_i(t) dt = 0, \ i = 1, 2, \ldots, n; \)

(H2) \( \forall q(t) \in \Lambda, q_i(t) = q_1 \left( t + (i - 1) \frac{2\pi}{n} \right) \quad i = 1, 2, \ldots, n. \)

Condition (H2) is the simple choreographic constraint, and condition (H1) is also used in Barutello and Terracini [4] but not emphasized because the functional in the usual \( n \) body problem (which is equivalent to our functional \( \mathcal{A} \) in the case \( M = 0 \) is invariant under translations. It is a classic result that the relative equilibrium motion associated with the regular polygon is a trivial solution in the usual \( n \)-body problem. However, is the action minimizer \( \tilde{q}(t) \) on the subspace \( \Lambda \) a genuine solution of dynamical system (1.1)? Now we give a positive answer through the following lemma.
Lemma 1.1 (Palais principle of symmetric criticality, [22]) Let $G$ be an orthogonal group acting on a Hilbert space $H$. Define the fixed point space: $H_G = \{x \in H | g \cdot x = x, \forall g \in G\}$; if $f \in C^1(H, \mathbb{R})$ and satisfies $f(g \cdot x) = f(x)$ for any $g \in G$ and $x \in H$, then the critical point of $f$ restricted on $H_G$ is also a critical point of $f$ on $H$.

Let $G = \langle g_1, g_2 \rangle$ be the finite group acting on the Sobolev space $H$ by isometries with the following representations $\rho : G \to O(3)$, $\tau : G \to O(2)$ and $\sigma : G \to \Sigma_n$ such that
\[
g_j \cdot q(t) = (\rho(g_j)q_{\sigma(g_j^{-1})}(\tau(g_j^{-1})t), \ldots, \rho(g_j)q_{\sigma(g_j^{-1})}(\tau(g_j^{-1})t)), \quad j = 1, 2, \quad (1.2)
\]
where
\[
\rho(g_1) = id, \quad \sigma(g_1^{-1})i = i - 1, \quad \tau(g_1^{-1})t = t + \frac{2\pi}{n}, \quad (1.3)
\]
\[
\rho(g_2) = -id, \quad \sigma(g_2^{-1})i = i, \quad \tau(g_2^{-1})t = t + \pi. \quad (1.4)
\]

Theorem 1.2 The minimizer of $A|_{H_G}$ is $\tilde{q}(t)$ and thus a uniform circular solution of (1.1).

Proof First we will see the fixed point space $H_G \subset \Lambda$. Precisely, the action of $g_1$ (1.3) is the simple choreography constraint (H2), and the coercive condition (H1) can be deduced by (1.4) which is equivalent to the so-called $T/2$-antiperiodic constraint
\[
q(t) = -q(t + \pi), \quad \forall t \in \mathbb{R}.
\]

Then we can see that
\[
H_G = \left\{ q(t) \in H \mid q(t) = -q(t + \pi), \quad q_i(t) = q_1 \left( t + (i - 1) \frac{2\pi}{n} \right), \quad i = 1, 2, \ldots, n \right\} \subset \Lambda.
\]

And it is not difficult to see that the relative equilibrium described in Theorem 1.1 $\tilde{q}(t) \in H_G$. By Theorem 1.1, $\tilde{q}(t)$ must be the unique minimum of $A$ on $H_G$. On the other hand, (1.2) is the general way to define the group action $G$ for classical equal masses N-body problem that makes the functional invariant (see [15]). In our problem, with two fixed centers, we should be careful. We have $A(g \cdot q(t)) = A(q(t))$, $\forall g \in G$, because the two centers are symmetric and have the same mass. Then Lemma 1.1 implies $\tilde{q}(t)$, the minimum of $A$ on $H_G$, is also a critical point of $A$ on the whole space $H$, thus a solution of dynamical system (1.1).

In the end of this section, we reduce our functional $A$ due to the choreographic condition (H2),
\[
A(q(t)) = \frac{2\pi}{\mu} \left[ \frac{m}{2} \sum_{i=1}^{n} |\dot{q}_i(t)|^2 + V(q_1(t), \ldots, q_n(t)) \right] dt
\]
\[
= \frac{2\pi}{\mu} \left[ \frac{m}{2} |\dot{q}_1(t)|^2 dt + n \int_0^{2\pi} \left[ \frac{mM}{|q_1(t) - C_1|^\beta} + \frac{mM}{|q_1(t) - C_2|^\beta} \right] dt \right]
\]
\[
+ \frac{n}{2} \sum_{j=1}^{n-1} \int_0^{2\pi} \frac{m^2}{|q_1(t) - q_1(t + j \frac{2\pi}{n})|^\alpha} dt.
\]

Dividing by $mn$, we see that seeking the critical points of $A(q(t))$ on $\Lambda$ is equivalent to finding the critical points of $\tilde{A}(x(t))$ on $\widetilde{\Lambda}$, where
\[
\tilde{A}(x(t)) = \frac{2\pi}{\mu} \left[ \frac{1}{2} |\dot{x}(t)|^2 + \frac{M}{|x(t) - C_1|^\beta} + \frac{M}{|x(t) - C_2|^\beta} + \frac{1}{2} \sum_{j=1}^{n-1} \frac{m}{|x(t) - x(t + j = \frac{2\pi}{n})|^\alpha} \right] dt.
\]
\[
\tilde{\Lambda} = \left\{ x(t) \in H^1_{2\pi} (\mathbb{R}, \mathbb{R}^3) \mid \int_0^{2\pi} x(t) dt = 0, \quad x(t) \neq x \left( t + i \frac{2\pi}{n} \right), \quad \forall t \in \mathbb{R}, \quad i = 1, 2, \ldots, n - 1 \right\}.
\]

In the following, we work on $\tilde{A}|_{\tilde{\Lambda}}$ and if $\tilde{x}(t)$ is the minimizer of $\tilde{A}|_{\tilde{\Lambda}}$, then $\tilde{q}(t) = (\tilde{x}(t), \tilde{x}(t + \frac{2\pi}{n}), \ldots, \tilde{x}(t + \frac{n-1}{n}2\pi))$ is a minimum of $A|_{\Lambda}$. 
2 Some Inequalities

We recall the famous Poincaré–Wirtinger inequality. Let \( x \in H_{2\pi}^1(\mathbb{R}, \mathbb{R}^d) \) such that \( \int_0^{2\pi} x(t)dt = 0 \), then

\[
\int_0^{2\pi} |\dot{x}(t)|dt \geq \int_0^{2\pi} |x(t)|^2dt,
\]

where the equality holds if and only if \( x(t) = a \cos t + b \sin t, a, b \in \mathbb{R}^d \). The following lemma is some kind of generalization.

**Lemma 2.1** Let \( x \in H_{2\pi}^1(\mathbb{R}, \mathbb{R}^d) \) such that \( \int_0^{2\pi} x(t)dt = 0 \), then for every \( \theta \in (0, 2\pi) \), we have

\[
\int_0^{2\pi} |\dot{x}(t)|^2dt \geq \mu_\theta^2 \int_0^{2\pi} |x(t) - x(t + \theta)|^2dt,
\]

where \( \mu_\theta = (2 \sin \frac{\theta}{2})^{-1} \); the equality holds if and only if \( x(t) = a \cos t + b \sin t, a, b \in \mathbb{R}^d \).

**Proof** Consider the Fourier representation of \( x(t) \),

\[
x(t) = \sum_{k \in \mathbb{Z}} c_k e^{jkt},
\]

where \( J = \sqrt{-1} \) and \( c_k \in \mathbb{C}^d \). Then \( x(t) \in H_{2\pi}^1(\mathbb{R}, \mathbb{R}^d) \) and \( \int_0^{2\pi} x(t)dt = 0 \) imply that \( c_k = \bar{c}_{-k} \) and \( c_0 = 0 \). We notice that \( \int_0^{2\pi} e^{jk\theta}e^{-j\theta t}dt = 0 \) for integers \( k \neq h \), so we have

\[
\mu_\theta^2 \int_0^{2\pi} |x(t) - x(t + \theta)|^2dt = \mu_\theta^2 \int_0^{2\pi} \left| \sum_{k \in \mathbb{Z}} c_k e^{jk(t+\theta)}(1 - e^{j\theta}) \right|^2 dt
\]

\[
= \mu_\theta^2 \cdot 2\pi \sum_{k \in \mathbb{Z}} |c_k|^2 |1 - e^{j\theta}|^2 = 2\pi \sum_{k \in \mathbb{Z}} |c_k|^2 \left| \frac{1 - e^{jk\theta}}{1 - e^{j\theta}} \right|^2
\]

\[
= 2\pi \sum_{k \in \mathbb{Z}} |c_k|^2 \left| \frac{1 - e^{jk\theta}}{1 - e^{j\theta}} \right|^2 = 2\pi \sum_{k \in \mathbb{Z}} |c_k|^2 \left( 1 + e^{j\theta} + e^{j2\theta} + \cdots + e^{j(k-1)\theta} \right)^2
\]

\[
\leq 2\pi \sum_{k \in \mathbb{Z}} |c_k|^2 k^2,
\]

the equality holds if and only if \( c_k = 0 \) for every \( k \in \mathbb{Z} \setminus \{1, -1\} \).

We also notice that \( \dot{x}(t) = \sum_{k \in \mathbb{Z}} c_k je^{jk(t+\theta)} \), then

\[
\int_0^{2\pi} |\dot{x}(t)|^2dt = 2\pi \sum_{k \in \mathbb{Z}} |c_k|^2 k^2,
\]

which finish the proof. \( \square \)

**Remark 1** Poincaré–Wirtinger inequality can be seen as an average result of our lemma. Because our lemma implies

\[
4 \sin^2 \frac{\theta}{2} \int_0^{2\pi} |\dot{x}(t)|^2dt \geq \int_0^{2\pi} |x(t) - x(t + \theta)|^2dt, \quad \forall \theta \in [0, 2\pi],
\]
and integrating by $\theta$ from 0 to $2\pi$, we have

\[
4\pi \int_0^{2\pi} |\dot{x}(t)|^2 dt \geq 2\pi \int_0^{2\pi} \left| x(t) - x(t + \theta) \right|^2 dt.
\]

Exchanging the order of integration and using the fact that $\int_0^{2\pi} x(t + \theta) d\theta = 0$, we can get Poincaré–Wirtinger inequality.

Let $\theta_j = j \frac{2\pi}{n}$ and $\mu_j = \mu_{\theta_j} = \left(2 \sin \frac{j \pi}{n}\right)^{-1}$, we have the following corollary.

**Corollary 2.2** For every $x(t) \in \hat{X}$ and $\mu_j' > 0$, $j = 1, \ldots, n - 1$, we have

\[
\int_0^{2\pi} |\dot{x}(t)|^2 dt \geq \sum_{j=1}^{n-1} v_j \int_0^{2\pi} \left| x(t) - x \left(t + \frac{j \cdot 2\pi}{n}\right) \right|^2 dt,
\]

where $v_j = \frac{\mu_j'^2}{\sum_{k=1}^{n} \mu_k'^2}$, the equality holds if and only if $x(t) = a \cos t + b \sin t$, $a, b \in \mathbb{R}^3$.

If we take $\mu_j' = (\mu_j)^n$, this corollary is equivalent to the Corollary 2 in Barutello and Terracini [4] which took several pages but played a very important role in their proof. Another inequality we need is Jensen’s inequality,

\[
f \left( \frac{1}{2\pi} \int_0^{2\pi} g(t) dt \right) \leq \frac{1}{2\pi} \int_0^{2\pi} f(g(t)) dt,
\]

for any convex function $f$. Applying it to $f(z) = z^{-\frac{\alpha}{2}}$ with $\alpha > 0$ and $g(t) = |x(t) - x(t + j \frac{2\pi}{n})|^2$, we have

\[
\left[ \frac{1}{2\pi} \int_0^{2\pi} |x(t) - x \left(t + \frac{j \cdot 2\pi}{n}\right)|^2 dt \right]^{-\frac{\alpha}{2}} \leq \frac{1}{2\pi} \int_0^{2\pi} \left| x(t) - x \left(t + \frac{j \cdot 2\pi}{n}\right) \right|^\alpha dt,
\]

then

\[
\int_0^{2\pi} \frac{1}{\left| x(t) - x \left(t + \frac{j \cdot 2\pi}{n}\right) \right|^\alpha} dt \geq (2\pi)^{1+\frac{\alpha}{2}} \left[ \int_0^{2\pi} \left| x(t) - x \left(t + \frac{j \cdot 2\pi}{n}\right) \right|^2 dt \right]^{-\frac{\alpha}{2}}, \tag{2.1}
\]

where the equality holds if and only if $|x(t) - x(t + j \frac{2\pi}{n})|^2 \equiv \text{const}$. Moreover, we notice that if the equalities hold simultaneously for Poincaré–Wirtinger inequality and Jensen’s inequality, $x(t)$ is a uniform circular motion by the following lemma.

**Lemma 2.3** Suppose $x(t) = a \cos t + b \sin t$, $a, b \in \mathbb{R}^d$, if for some $\theta \in (0, 2\pi)$, $t \mapsto |x(t) - x(t + \theta)|^2$ is constant, we have $a \cdot b = 0$ and $|a| = |b|$, where $\cdot$ is the standard inner product in $\mathbb{R}^d$. That is to say, $x(t)$ is a uniform circular motion in the plane spanned by $a$ and $b$.

**Proof** This lemma is similar to Proposition 3 in Barutello and Terracini [4]. Since

\[
|x(t) - x(t + \theta)|^2 = |x(t) - x(t + \theta)| \cdot [x(t) - x(t + \theta)]
\]

\[
= |a|^2[\cos t - \cos(t + \theta)]^2 + |b|^2[\sin t - \sin(t + \theta)]^2 + 2a \cdot b[\cos t - \cos(t + \theta)][\sin t - \sin(t + \theta)]
\]

\[
= 4 \sin^2 \frac{\theta}{2} \left( |a|^2 \sin^2 \left(t + \frac{\theta}{2}\right) + |b|^2 \cos^2 \left(t + \frac{\theta}{2}\right) - \sin(2t + \theta)a \cdot b \right)
\]

is constant, differentiating by $t$, we have

\[
4 \sin^2 \frac{\theta}{2} \left( (|a|^2 - |b|^2) \sin(2t + \theta) - 2 \cos(2t + \theta)a \cdot b \right) \equiv 0, \quad \forall t \in \mathbb{R}.
\]

That is to say $|a|^2 - |b|^2 = 0$ and $a \cdot b = 0$, which finish the proof. $\square$
3 The Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1 by seeking the minimizer of $\tilde{A}(x(t)) = \tilde{A}_1(x(t)) + \tilde{A}_2(x(t))$ on $\tilde{\Lambda}$, where we set

$$\tilde{A}_1(x(t)) = \int_0^{2\pi} \left[ \frac{1 + \lambda}{4} |\dot{x}(t)|^2 + \frac{M}{|x(t) - C_1|^\beta} + \frac{M}{|x(t) - C_2|^\beta} \right] dt,$$

$$\tilde{A}_2(x(t)) = \int_0^{2\pi} \left[ \frac{1}{2} \sum_{j=1}^{n-1} \frac{m}{|x(t) - x(t + j \frac{2\pi}{n})|^\alpha} \right] dt,$$

with the parameter $\lambda = \lambda(m) \in [-1, 1]$ to be determinate later. The idea is to choose $\lambda$ in such a way that $\tilde{A}_1(x(t))$ and $\tilde{A}_2(x(t))$ attain an absolute minimum on the same element $x(t) \in \tilde{\Lambda}$. Thus $x(t)$ will also be a minimizer of $\tilde{A}_1|_{\tilde{\Lambda}}$.

**Lemma 3.1** If $\lambda \in (-1, 4\beta M - 1)$, then $\tilde{x}_1(t) \in \tilde{\Lambda}$ is a minimizer of $\tilde{A}_1$ if and only if it is a uniform circular motion of radius

$$R_1(\lambda) = \left( \frac{4\beta M}{1 + \lambda} \right)^{\frac{1}{2\beta - 1}} - 1,$$

contained in the perpendicular bisector plane between $C_1$ and $C_2$.

**Proof** Inspired by the work of Long and Zhang [20], applying Poincaré–Wirtinger inequality and Jensen’s inequality, we have

$$\tilde{A}_1(x(t)) = \int_0^{2\pi} \left[ \frac{1 + \lambda}{4} |\dot{x}(t)|^2 + \frac{M}{|x(t) - C_1|^\beta} + \frac{M}{|x(t) - C_2|^\beta} \right] dt,$$

$$\geq \int_0^{2\pi} \frac{1 + \lambda}{4} |x(t)|^2 dt + \int_0^{2\pi} \frac{M}{|x(t) - C_1|^\beta} + \frac{M}{|x(t) - C_2|^\beta} dt \tag{3.1}$$

$$\geq \int_0^{2\pi} \frac{1 + \lambda}{4} |x(t)|^2 dt + M(2\pi)^{\frac{\beta}{2} + 1} \left[ \int_0^{2\pi} |x(t) - C_1|^2 dt \right]^{-\frac{\beta}{2}}$$

$$+ M(2\pi)^{\frac{\beta}{2} + 1} \left[ \int_0^{2\pi} |x(t) - C_2|^2 dt \right]^{-\frac{\beta}{2}}$$

$$= \frac{1 + \lambda}{4} \int_0^{2\pi} (|x(t)|^2 + 1) dt + 2M(2\pi)^{\frac{\beta}{2} + 1} \left[ \int_0^{2\pi} (|x(t)|^2 + 1) dt \right]^{-\frac{\beta}{2}} - \frac{1 + \lambda}{2\pi}$$

$$= \Psi(s(x)) \geq \min_{s \geq \sqrt{2\pi}} \Psi(s), \tag{3.2}$$

where $\Psi(s) = \frac{1 + \lambda}{4} s^2 + 2M(2\pi)^{\frac{\beta}{2} + 1} s^{-\beta} - \frac{1 + \lambda}{2} \pi$ and $s(x) = \left[ \int_0^{2\pi} (|x(t)|^2 + 1) dt \right]^{\frac{1}{2}} \geq \sqrt{2\pi}$. Since $\Psi''(s) > 0$ for $s > 0$, $\Psi(s)$ possesses a unique minimum at $s_0 = \sqrt{2\pi \left( \frac{4\beta M}{1 + \lambda} \right)^{\frac{1}{2\beta - 1}}}$; i.e.,

$$\tilde{A}_1(x(t)) \geq \Psi(s_0), \quad \forall x(t) \in \tilde{\Lambda}.$$
Lemma 3.2 If $\lambda \leq 1$, then $\tilde{\chi}_2(t) \in \Lambda$ is a minimizer of $\tilde{A}_2$ if and only if it is a uniform circular motion of radius

$$R_2(\lambda) = 2^{\frac{n}{n+2}} \left( \frac{\alpha m}{1 - \lambda} \sum_{k=1}^{n-1} \sin^{-\alpha} \frac{k}{n} \right)^{\frac{1}{n+2}}.$$ 

Proof We use Jensen’s inequality and our Lemma 2.1 instead of Poincaré–Wirtinger Inequality to give the estimates. By Corollary 2.2 and (2.1), we have

$$\tilde{A}_2(x(t)) = \frac{2\pi}{4} \left[ \frac{(1-\lambda)}{4} |\dot{x}(t)|^2 dt + \frac{n-1}{2} \sum_{j=1}^{n-1} \left| x(t) - x(t + j \frac{2\pi}{m}) \right|^\alpha \right] dt,$$

$$\geq \frac{1-\lambda}{4} \sum_{j=1}^{n-1} v_j \int_0^{2\pi} \left| x(t) - x(t + j \frac{2\pi}{m}) \right|^\alpha dt + \frac{n-1}{2} \sum_{j=1}^{n-1} \frac{m}{2} \int_0^{2\pi} \left| x(t) - x(t + j \frac{2\pi}{m}) \right|^\alpha dt \tag{3.3}$$

$$\geq \frac{1-\lambda}{4} \sum_{j=1}^{n-1} v_j \int_0^{2\pi} \left| x(t) - x(t + j \frac{2\pi}{m}) \right|^\alpha dt + \frac{1}{2} (2\pi)^{1+\frac{\alpha}{2}} \sum_{j=1}^{n-1} \left[ \int_0^{2\pi} \left| x(t) - x(t + j \frac{2\pi}{m}) \right|^\alpha dt \right]^{-\frac{\alpha}{2}} \tag{3.4}$$

where

$$\Phi_j(s_j) = \frac{1-\lambda}{4} v_j s_j + \frac{1}{2} (2\pi)^{1+\frac{\alpha}{2}} m s_j^{-\frac{\alpha}{2}}, \quad j = 1, 2, \ldots, n - 1,$$

$$s_j = \int_0^{2\pi} \left| x(t) - x(t + j \frac{2\pi}{m}) \right|^\alpha dt.$$

In order to make all the $\Phi_j$’s attain their minimum simultaneously, we choose $v_j = \frac{\mu_j^{2+\alpha}}{\sum_{j=1}^{n-1} m_j}$ which is obtained by replacing $\mu_j = (\mu_j)^{\alpha}$ in the expression of $v_j$ given in Corollary 2.2. Let $\tilde{\chi}_2(t)$ be the minimizer of $\tilde{A}_2(x)$ on $\tilde{\Lambda}$, we claim that $\tilde{\chi}_2(t)$ makes all the inequalities above in the infimum estimate become equalities. Because Corollary 2.2 applied in (3.3) yields $\tilde{\chi}_2(t) = a_2 \cos(t) + b_2 \sin(t)$, $a_2, b_2 \in \mathbb{R}^3$. The Jensen’s inequality applied
in (3.4) implies \( |\tilde{x}_2(t) - \tilde{x}_2(t + j \frac{2\pi}{n})| \equiv \text{const.} \) By Lemma 2.3 we conclude that \( \tilde{x}_2(t) \) is a circular motion centered at the origin. Moreover, we see that \( \Phi_j(s) \) attains its minimum at

\[
\tilde{s}_j = 2\pi \left[ \frac{am}{v_j(1 - \lambda)} \right]^{\frac{1}{1 + \lambda}} = 2\pi \left( \frac{4am}{1 - \lambda} \right)^{\frac{1}{1 + \lambda}} \sin^2 j n \pi \left( \sum_{k=1}^{n} \sin^2 \frac{k}{n} \pi \right)^{\frac{1}{1 + \lambda}}.
\]

Now we pick the radius of the circular motion \( R_2(\lambda) = |\tilde{x}_2(t)| \) appropriately, such that for every \( j = 1, 2, \ldots, n - 1 \), \( \xi_j = \tilde{s}_j \), i.e.,

\[
8\pi R_2^2(\lambda) \sin^2 j n \pi = 2\pi \left( \frac{4am}{1 - \lambda} \right)^{\frac{2}{1 + \lambda}} \sin^2 j n \pi \left( \sum_{k=1}^{n} \sin^{-2} \frac{k}{n} \pi \right)^{\frac{2}{1 + \lambda}}.
\]

Then we get

\[
R_2(\lambda) = 2\pi^{\frac{1}{1 + \lambda}} \left( \frac{am}{1 - \lambda} \sum_{k=1}^{n} \sin^{-2} \frac{k}{n} \pi \right)^{\frac{1}{1 + \lambda}}, \quad (3.5)
\]

which is independent with \( j \). That is to say the circular motion centered at the origin with the radius \( R_2(\lambda) \) is the absolutely minimum of \( \tilde{A}_2|_\lambda \).

If \( \lambda \) is chosen so that \( R_1(\lambda) = R_2(\lambda) \), then an element \( x \in \tilde{A} \) is a minimizer both for \( \tilde{A}_1 \) and \( \tilde{A}_2 \) (and therefore for \( \tilde{A} \) too) if and only if it is a uniform circular motion with radius \( R_1(\lambda) = R_2(\lambda) \) contained in the perpendicular bisector plane between \( C_1 \) and \( C_2 \). For \( \lambda \in (-1, 4\beta M - 1) \cap (-1, 1] \), let

\[
F(\lambda, m) = R_2(\lambda) - R_1(\lambda) = 2\pi^{\frac{1}{1 + \lambda}} \left( \frac{am}{1 - \lambda} \sum_{j=1}^{n} \sin^{-2} j n \pi \right)^{\frac{1}{1 + \lambda}} - \sqrt{\left( \frac{4\beta M}{1 + \lambda} \right)^{\frac{2}{1 + \lambda}} - 1}, \quad (3.6)
\]

it is obvious that, for given \( m \) and \( M \), \( F \) is strictly increasing with respect to \( \lambda \), moreover \( \lim_{\lambda \to -1} F = -\infty \) and

\[
\begin{align*}
\lim_{\lambda \to 1-} F &= +\infty, \quad \text{if } 4\beta M \geq 2 \\
F(4\beta M - 1, m) &= 0 \quad \text{if } 4\beta M < 2.
\end{align*}
\]

So there exists only one \( \lambda = \lambda(m) \in (-1, 1) \) such that \( R_1 = R_2 \), which implies \( |\tilde{x}_1(t)| = |\tilde{x}_2(t)| \). Now we investigate the dependence of \( \lambda \) on the parameter \( m \), differentiating (3.6) by \( m \), we have

\[
F_m(\lambda(m)) + F_\lambda(\lambda(m)) \frac{d\lambda(m)}{dm} = 0.
\]

Since \( F_\lambda, F_m > 0 \), we have \( \frac{d\lambda(m)}{dm} < 0 \). Then \( \frac{dR_1}{dm} < 0 \) implies that the radius of the circle increases as \( m \) increases. This completes the proof of Theorem 1.1.

Our method also works for the limiting case \( M = 0 \), where there is no center force and it is just the \( n \)-body problem with simple choreography constraint. We set \( \lambda = -1 \), and we only consider the minimum of \( \tilde{A}_2 = \tilde{A} \).

**Corollary 3.3** In the case of \( M = 0 \), the absolute minimum of \( \tilde{A} \) on \( \Lambda \) is attained on a relative equilibrium motion associated with a regular \( n \)-gon centered at origin with its radius \( R_2 = 2^{-\frac{a+1}{2}} \left( am \sum_{j=1}^{n} \sin^{-2} j n \pi \right)^{\frac{1}{2}} \).

This corollary coincides the result in the work of Barutello and Terracini [4].

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Declarations

Conflict of interest  The authors declare that they have no conflict of interest.

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